On the component structure of one-loop effective actions in $6D$, $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ supersymmetric gauge theories

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Abstract

We study the six-dimensional $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ supersymmetric Yang-Mills (SYM) theories in the component formulation. The one-loop divergencies of effective action are calculated. The leading one-loop low-energy contributions to bosonic sector of effective action are found. It is explicitly demonstrated that the contribution to effective potential for the constant background scalar fields are absent in the $\mathcal{N} = (1, 1)$ SYM theory.

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1 Introduction

The various six-dimensional supersymmetric gauge theories attract a certain interest in context of string/brane dynamics. It is known that the superconformal $\mathcal{N} = (2, 0)$ theory of self-dual tensor multiplet is closely related with low-energy dynamics of M5-branes (see, e.g., for a review [1, 2]). The study of the maximally extended $\mathcal{N} = (1, 1)$ SYM theory in six-dimensions is motivated by the connection with D5-branes (see [3] for a review). Both the $\mathcal{N} = (2, 0)$ superconformal field theory and the $\mathcal{N} = (1, 1)$ SYM theory were also considered as low energy limits for the little string theories with corresponding $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (1, 1)$ supersymmetries (see [4, 5]).

The $\mathcal{N} = (1, 1)$ SYM theory is a maximally extended supersymmetric gauge theory of vector multiplet in six dimensions. This theory can be treated as the $\mathcal{N} = (1, 0)$ vector multiplet theory coupled to hypemultiplet which transforms under the adjoint representation of gauge group. Although the $\mathcal{N} = (1, 1)$ theory is non-renormalizable by power counting, it possesses the remarkable properties in quantum domain and it is the subject of comprehensive research. It was proved that the theory under consideration is on-shell finite at one- and two loops [6–12] and the first divergences here can appear only starting from three loops (see e.g. [13] and the references therein). Moreover it was shown, using the 6D harmonic superspace approach [14–21], that the one- and some two-loop divergent contributions to effective action in $\mathcal{N} = (1, 1)$ SYM theory are finite even off-shell [22–24] in Fermi-Feynman gauge. The finite leading low energy contributions to one-loop effective action were recently calculated in superfield approach as well [35]. Also we point out that the essential progress in studying of the $\mathcal{N} = (1, 1)$ theory was achieved in analysis of four-point on-shell scattering amplitudes [27–31]. In particular, the leading and subleading divergences in all loops are obtained in the framework of the spinor-helicity and on-shell supersymmetric formalism (the recent results are presented in the review [32]).

In the present paper we study the quantum aspects of the $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ gauge theories in the framework of component approach. As well known, in supersymmetric quantum field theory the component and superfield approaches complement each other. The component formulation of SUSY theories, although does not possess the manifest supersymmetry, very closely relates with conventional field theory and allows to analyze and test independently the results which where obtained in superfield approach. Besides, the component approach allows to use efficiently in SUSY theories the different special methods well developed in conventional quantum field theory. In four dimensional SUSY theories the component and superfield approaches were developed in parallel. As to six-dimensional SUSY theories, the component analysis of off-shell quantum effective action is not well worked out in many details.

In this paper we are going to focus on the component calculations of the one-loop divergences in $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ gauge theories and derive the leading low-energy contribution to one-loop effective action in $\mathcal{N} = (1, 1)$ theory in bosonic sector. Also we will demonstrate the absence of the one-loop contribution to effective potential for constant background scalar fields in the $\mathcal{N} = (1, 1)$ SYM theory.

The paper is organized as follows. In section 2 we briefly discuss some details of the supersymmetry in six dimensions. Also we present an independent derivation of the action for six-dimensional $\mathcal{N} = (1, 0)$ SYM theory in terms of physical component fields and the corresponding supersymmetry transformations of these fields. In section 3, using the background field method, we evaluate the one-loop effective action. Then we calculate the logarithmic divergent contributions to one-loop effective action in the $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ SYM theories. Section 4 includes the details of calculation of

\footnote{The study of the gauge dependence of the one-loop divergent contributions to one-loop effective actions in $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ SYM was recently done in [25, 26, 33]}

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leading low-energy contributions to one-loop effective action and one-loop contribution to the effective potential. The last section contains discussion of obtained results and some possible further directions of the work.

2 Six-dimensional $\mathcal{N} = (1, 0)$ SYM theory

We begin with discussing the classical six-dimensional $\mathcal{N} = (1, 0)$ SYM theory. The $\mathcal{N} = (1, 0)$ SYM theory in six dimensions describes the interaction of vector multiplet with a set of hypermultiplets. The six-dimensional on-shell vector multiplet consists of real vector field $A_M$ and a left-handed $\lambda^i$ pseudoreal Weyl spinor one, both in adjoint representation of gauge group. The on-shell hypermultiplet contains complex scalar $\varphi^A$ and right-handed $\psi^A$ pseudoreal Weyl spinor fields. We denote the indices of spinor representation by the small latin letters $a, b, \ldots = 1, 2, 3, 4$ and for Minkowski space-time indices stand the capital ones $M, N, \ldots = 0, 1, \ldots, 5$. The indices $i, j, \ldots = 1, 2$ correspond to $SU(2)$ group of $R$-symmetry of $\mathcal{N} = (1, 0)$ superalgebra in six dimensions. By the calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$ we denote the Pailu-Gürsey $SU(2)$ indexes. Both $i, j, \ldots$ and $\mathcal{A}, \mathcal{B}, \ldots$ are lowered and raised by antisymmetric quantities $\epsilon_{ij}$ and $\epsilon_{\mathcal{AB}}$ correspondingly.

We use the antisymmetric representation of six-dimensional Weyl matrices

$$ (\gamma_M)^{ab} = - (\gamma_M)^{ba}, \quad (\tilde{\gamma}_M)^{ab} = \frac{1}{2} \varepsilon^{abcd} (\gamma_M)^{cd}, \quad (2.1) $$

where $\varepsilon^{abcd}$ is the totally antisymmetric tensor. The matrices $\gamma_M$ and $\tilde{\gamma}_M$ subject to basic relations for Weyl matrices

$$ (\gamma_M)^{ac} (\tilde{\gamma}_N)^{cb} + (\gamma_N)^{ac} (\tilde{\gamma}_M)^{cb} = -2 \delta^b_a \eta_{MN}, \quad (\gamma_M)^{ac} (\gamma_M)^{cb} = 2 \varepsilon^{abcd}. \quad (2.2) $$

We choose the Minkowski metric $\eta_{MN}$ with a mostly negative signature. The generators of the spinor representation $\sigma_{MN}$ are real, where

$$ (\sigma_{MN})^a_b = \frac{1}{2} (\gamma^M \gamma^N - \gamma^N \gamma^M)^a_b. \quad (2.3) $$

The action of the theory can be derived as a sum of actions for 6D vector multiplet and hypermultiplet with specific scalar and Yukawa interactions and reads\(^2\)

$$ S^{(1,0)}_{\text{SYM}} = \frac{1}{2 f^2} \text{tr} \int d^6 x \left( - F_{MN} F^{MN} + i \lambda^a (\gamma_M)^{ab} \nabla_M \lambda^b \right) $$

$$ + \frac{1}{2 f^2} \int d^6 x \left( - \varphi^A \nabla^2 \varphi_A + i \psi^A (\tilde{\gamma}^M)^{ab} \nabla_M \psi_A^b - \frac{1}{4} (\varphi^A \varphi^A)^2 - 2 \psi^A \lambda^a \varphi_A \right), \quad (2.4) $$

where the hypermultiplet fields are taken in some representation of the gauge group and $\nabla_M$ is a covariant derivative $\nabla_M = \partial_M - i A_M$. The coupling constant $f$ has a dimension of inverse mass, $[f] = -1$. The action (2.4) is invariant under the gauge transformation

$$ \delta_A A_M = \nabla_M \Lambda, \quad \delta_A \varphi_A = i \Lambda \varphi^A, \quad (2.5) $$

$$ \delta_A \lambda^i = i [\Lambda, \lambda^i], \quad \delta_A \psi^A = i \Lambda \psi^A, \quad (2.6) $$

with the gauge transformation parameter $\Lambda = \Lambda(x)$ which takes value in the Lie algebra of gauge group of the theory in the vector multiplet and hypermultiplet representations respectively.

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\(^2\)Some comments about derivation of this action are given at the end of this section.
The action (2.4) possesses $\mathcal{N} = (1, 0)$ supersymmetry. It relates the spinor field $\psi^A_a$ with the complex scalar $\phi^A_i$ and the spinor field $\lambda^a_i$ with the gauge vector field $A_M$ and scalar one $\varphi^A_i$. Explicitly we have

$$
\delta \varphi^A_i = -i\epsilon^a_i \psi^A_a, \quad \delta \psi^A_a = \epsilon^b (\gamma^M)_{ab} \nabla_M \varphi^A_i, \\
\delta A_M = \frac{i}{2} \epsilon^{ia} (\gamma_M)_{ab} \lambda^b_i, \quad \delta \lambda^a_i = -\frac{1}{2} F^{MN} (\sigma_{MN})^a_{b} \epsilon^{bi} + \frac{i}{2} \epsilon^{aj} \varphi^A_i \varphi^A_j,
$$

where $\epsilon^a_i$ is a left-handed parameter of $\mathcal{N} = (1, 0)$ supersymmetry transformation.

If the hypermultiplet component fields $\varphi^A_i$ and $\psi^A_a$ align in the adjoint representation of gauge group, the action (2.4) possesses an additional implicite $\mathcal{N} = (0, 1)$ supersymmetry and describes the six-dimensional $\mathcal{N} = (1, 1)$ SYM theory

$$
S^{(1,1)}_{\text{SYM}} = \frac{1}{2f^2} \int d^6x \left( -F_{MN} F^{MN} + i \lambda^a_i (\gamma^M)_{ab} \nabla_M \lambda^b_i - \varphi^A_i \nabla^2 \varphi^A_i + \\
+ i \psi^A_a (\gamma^M)_{ab} \nabla_M \psi^A_b - \frac{1}{4} [\varphi^A_i, \varphi^A_j]^2 - 2 \psi^A_a [\lambda^a_i, \varphi^A_i] \right).
$$

Indeed the action (2.8) is invariant under both $\mathcal{N} = (1, 0)$ supersymmetry transformations

$$
\delta \varphi^A_i = -i\epsilon^a_i \psi^A_a, \quad \delta \psi^A_a = \epsilon^b (\gamma^M)_{ba} \nabla_M \varphi^A_i, \\
\delta A_M = \frac{i}{2} \epsilon^{ia} (\gamma_M)_{ab} \lambda^b_i, \quad \delta \lambda^a_i = -\frac{1}{2} F^{MN} (\sigma_{MN})^a_{b} \epsilon^{bi} + \frac{i}{2} \epsilon^{aj} [\varphi^A_i, \varphi^A_j],
$$

and $\mathcal{N} = (0, 1)$ ones

$$
\delta_0 A_M = \frac{i}{2} \epsilon^{aA} (\gamma_M)_{ab} \psi^A_a, \quad \delta_0 \varphi^A_i = -i \epsilon_{Aa} \lambda^a_i, \\
\delta_0 \lambda^a_i = \epsilon^b (\gamma^M)_{ba} \nabla_M \varphi^A_i, \quad \delta_0 \psi^A_a = -\frac{1}{2} \epsilon_{AB} (\sigma^M)^{ab} \epsilon^b_{Aa} F_{MN} + \frac{i}{2} \epsilon_{Ba} [\varphi^B_i, \varphi^A_i],
$$

mixing the spinor field $\lambda^a_i$ and the scalar $\varphi^A_i$ one and etc. with corresponding right-handed spinor parameter $\epsilon_{Aa}$.

The action (2.8) of the $\mathcal{N} = (1, 1)$ SYM theory was obtained at first by dimensional reduction of the corresponding ten-dimensional SYM theory (see e.g., [7], [8]). As to the action for $\mathcal{N} = (1, 0)$ vector multiplet theory interacting with hypermultiplets, it was known only in superfield form [21] and its complete component form was unknown. It is evident that the component form of such an action can be useful for various aims. The action (2.4) fills this gap. Let us briefly comment on the derivation of this action. We follow to Noether procedure beginning with the free actions for vector multiplet and hypermultiplet and assuming invariance under the linearized supersymmetry transformations. Then we include the appropriate interactions and modify the supersymmetry transformations. As a result, we arrive at the action (2.4) and transformations (2.7). The action of $\mathcal{N} = (1, 1)$ theory (2.8) is obtained from (2.4) if to take the hypermultiplet in the adjoint representation. Then we construct the transformations (2.10) and explicitly check that the action (2.8) is invariant under these transformations. As far as we know, the component actions (2.4) and (2.8) for the $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ SYM theories with corresponding supersymmetry transformation (2.7), (2.10) were not obtained earlier in the framework of component formulation.

### 3 Background field method and one-loop divergences

We are going to study the effective action in the theory with a classical action (2.4). To preserve the manifest gauge invariance we will use the background field method (see e.g., [34], [36]). First of all
we split the initial gauge field $A_M$ and scalar field $\varphi_{Ai}$ to classical background fields $A_M, \Phi_{Ai}$ and quantum ones $a_M, \varphi_{Ai}, \lambda_i^a, \psi_{Ai}$.

$$A_M \rightarrow f a_M + A_M, \quad \varphi_{Ai} \rightarrow f \varphi_{Ai} + \Phi_{Ai},$$

$$\lambda_i^a \rightarrow f \lambda_i^a, \quad \psi_{Ai} \rightarrow f \psi_{Ai}. \tag{3.1}$$

We are going to study the bosonic sector of effective action in the model (2.4). Therefore we introduce the background field only for $A_M$ and $\varphi_{Ai}$ and keep the spinor fields $\lambda_i^a$ and $\psi_{Ai}$ as a quantum ones.

Following Fadeev-Popov method we choose the gauge fixing action in the standard way (see e.g., [34], [36])

$$S_{gf} = - \text{tr} \int d^6x (\nabla^M a_M)^2. \tag{3.2}$$

Here $\nabla_M = \partial_M - i A_M$ are the background dependent covariant derivative. The corresponding Fadeev-Popov ghosts $\tilde{c}$ and $c$ action is

$$S_{gh} = - \text{tr} \int d^6x \tilde{c} \nabla^M \left( \partial_M - i (A + a)_M \right) c. \tag{3.3}$$

The total quantum action $S_{\text{quant}}$ constructs as sum of the classical action (2.4) with shifted field variables (3.1) and the actions (3.2) and (3.3). According to the background field method one has to introduce the corresponding background and quantum gauge transformations for both quantum fields and classical background ones. These transformations are constructed on the base of the initial gauge transformation (2.6) and splitting (3.1) (see for details, e.g., [34], [36]). The background field method provides the procedure of constructing the effective action $\Gamma[A, \Phi]$ for background fields $A_M$ and $\Phi_{Ai}$ which is invariant under the classical gauge transformations.

### 3.1 One-loop divergences in the $\mathcal{N} = (1, 0)$ SYM theory

Let we proceed to the study of the one-loop divergences in the $\mathcal{N} = (1, 0)$ SYM theory, where the hypermultiplet transforms under the arbitrary representation $R$ of the gauge group. We consider the logarithmic divergences in a sector of gauge field $A_M$ and switch off the background scalar field $\Phi_{Ai} = 0$. We assume the background field $A_M$ is completely off-shell. We choose the gauge group to be compact Lie one and use the Hermitian basis for generators

$$[T^I, T^J] = i f^{IJK} T^K, \quad \text{tr} (T^I T^J) = T_R \delta^{IJ}, \quad \text{tr} (T^I T^J T^K) = \frac{i}{2} T_R f^{IJK} + \frac{1}{2} A_R d^{IJK}, \tag{3.4}$$

where $f^{IJK}$ is a totally antisymmetric structure constants, $d^{IJK}$ is a totally symmetric tensor and $A_R$ is the anomaly coefficient of the representation $R$ [36]. The generators of the fundamental representation $T^I_f \equiv \tilde{t}^I$ are normalized in the standard way $\text{tr} (\tilde{t}^I \tilde{t}^J) = \frac{1}{2} \delta^{IJ}$. In this case the one-loop contribution to effective action is

$$\Gamma_{(1, 0)}^{(1)}[A] = \frac{i}{2} \text{Tr}_{\text{Adj}} \ln \left( - \eta_{MN} (\nabla^2)^{IJ} - 2 f^{IJK} F^K_{MN} \right) - \frac{i}{4} \text{Tr}_{\text{Adj}} \ln \left( i \delta_i^j (\gamma^M)_{ab} \nabla M \right)^2$$

$$+ \frac{i}{2} \text{Tr} \ln \left( - \delta_i^j \delta_\alpha^\beta \nabla^2 \right) - \frac{i}{4} \text{Tr} \ln \left( i \delta_\alpha^\beta (\widetilde{\gamma}^M)_{ab} \nabla M \right)^2$$

$$- i \text{Tr}_{\text{Adj}} \ln \left( - \nabla^2 \right), \tag{3.5}$$

where $F^K_{MN}$ is a background vector field strength and we have denoted $\nabla^2 = \nabla^M \nabla_M$. The first two contributions in (3.5) come from vector multiplet. The next two terms in the second line of (3.5) are
the contributions from hypermultiplet which transforms under some irreducible representation \( R \) of the gauge group \( G \). The last term in (3.5) is a contribution from the ghosts fields.

We use proper time method to study one-loop effective action. For differential operators \( \Delta \) associated with action (3.5) we define

\[
\frac{i}{2} \text{Tr } \ln \Delta = \frac{i}{2} \mu^{2\omega} \text{tr} \int d^6x \int_0^\infty \frac{d(is)}{(is)^{1-\omega}} e^{is} \delta^6 (x - x') \Big|_{x=x'},
\]

(3.6)

where \( \mu \) is an arbitrary parameter of mass dimension and trace is taken over all gauge group, spinor and Minkowski space-time indexes. Also we introduce the parameter of dimension regularization \( \omega \). Logarithmic divergent contributions in (3.5) arise as a pole \( \frac{1}{\omega} \) in the limit \( \omega \to 0 \). To calculate such terms we use (3.6) and expand the exponent of each operator up to sixth order over covariant derivative \( \nabla_M \). After that we pass to momentum representation for delta-function and calculate the integral over momentum and proper time \( s \). The result for each contribution in (3.5) is listed below

\[
\frac{i}{2} \text{Tr } \ln \left( -\eta_{MN} (\nabla^2)^{IJ} - 2 f^{IKJ} F_{MN}^K \right) \Big|_{\text{div}} = -C_2 \frac{(3 \pi F_3 + 17}{60 I_3},
\]

(3.7)

\[
\frac{i}{2} \text{Tr } \ln \left( -\delta^{ij} \delta_A \nabla^2 \right) \Big|_{\text{div}} = -T_R \frac{2 F_3}{(4\pi)^3 \omega} \frac{2 F_3}{60 I_3} - \frac{A_R}{(4\pi)^3 \omega} \frac{2}{90} F_3,
\]

(3.8)

\[
\frac{i}{4} \text{Tr } \ln \left( i \delta^{ij} (\gamma^M)_{ab} (\nabla_M)^2 \right) \Big|_{\text{div}} = C_2 \frac{(2 F_3 + 90 I_3)}{(4\pi)^3 \omega} \frac{2}{15} I_3,
\]

(3.9)

\[
\frac{i}{4} \text{Tr } \ln \left( i \delta_A (\gamma^M)_{ab} (\nabla_M)^1 \right) \Big|_{\text{div}} = T_R \frac{2 F_3 + 90 I_3}{(4\pi)^3 \omega} + \frac{A_R}{(4\pi)^3 \omega} \frac{2}{90} F_3,
\]

(3.10)

\[
-\text{Tr } \ln \left( - (\nabla^2)^{11} \right) \Big|_{\text{div}} = C_2 \frac{(1 F_3 + 160 I_3)}{(4\pi)^3 \omega}.
\]

(3.11)

where \( C_2 \) is a second Casimir operator of adjoint representation of gauge group and we have introduced the quantities

\[
F_3 = \int d^6 x f^{IKJ} (F_M^N)^{11} (F_N^L)^{11} (F_L^M)^{11}, \quad I_3 = \int d^6 x (\nabla_M F^{ML})^{11} (\nabla_N F^{NL})^{11},
\]

(3.12)

\[
\bar{F}_3 = \int d^6 x f^{IKJ} (F_M^N)^{11} (F_N^L)^{11} (F_L^M)^{11}.
\]

Summing up all divergent contributions (3.7) - (3.11) to the one-loop effective action (3.5) one can see that all contributions with \( F_3 \) and \( \bar{F}_3 \) cancel each other and the divergent part of effective action is proportional to the equation of motion for background vector field

\[
\Gamma^{(1)}_{\text{div}} [A] = \frac{T_R - C_2}{3(4\pi)^3 \omega} \text{tr} \int d^6 x \nabla_M F^{ML} \nabla^N F_{NL}.
\]

(3.13)

The absence of divergent contributions with \( F_3 \) and \( \bar{F}_3 \) in the \( \mathcal{N} = (1,0) \) SYM theory without hypermultiplet was mentioned earlier [39]. Here we demonstrate absence of these terms by explicit calculation of the divergences in theory including the hypermultiplet in arbitrary representation of the gauge group. The divergent contribution from quantum hypermultiplet (3.8) and (3.10) containing \( F_3 \) and \( \bar{F}_3 \) exactly cancel each other. Also we note that this result is consistent with the superfield calculations [22,23]. The divergent part of one-loop superfield effective action in gauge multiplet sector includes the square of classical superfield equations of motion and the contributions with \( (F_{MN})^3 \) in the components are ruled out.

In the \( \mathcal{N} = (1,1) \) SYM theory the hypermultiplet transforms under adjoin representation of gauge group. In this case \( T_R = C_2 \) and \( A_R = 0 \) and we obtain the cancelation of ultraviolet divergences in (3.13) off-shell [22,23,37]. However result is valid only for Fermi-Feynman gauge and in case of general off-shell background it can be gauge-dependent [25,37]. It means that in general the one-loop divergences of the theory under consideration apparently are absent only on-shell.
3.2 One-loop effective action in $\mathcal{N} = (1,1)$ SYM theory

In this subsection we consider the finite one-loop contribution to the effective action for the $\mathcal{N} = (1,1)$ SYM theory. Unlike the previous section, the background scalar field is also taken into account.

One-loop contribution $\Gamma^{(1)}[A, \Phi]$ is determined by the quadratic over quantum fields part of quantum action

$$S^{(1,1)}_2[A, \Phi] = \frac{1}{2} \text{tr} \int d^6x \left( 2a^M \nabla^2 a_M + 4i a^M [F_{MN}, a^N] - [a^M, \Phi^{Ai}]^2 - 2i[a^M, \Phi^{Ai}] \nabla_M \varphi_{Ai} \right.$$

$$- \varphi^{Ai} \nabla^2 \varphi_{Ai} - [\varphi^{Ai}, \Phi^{jA}]^2 - \frac{1}{2} [\varphi^{Ai}, \varphi^{Aj}] [\Phi^B_i, \Phi^B_j] - 2i[a^M, \varphi^{Ai}] \nabla_M \Phi_{Ai} \right.$$  

$$+ i \lambda^{ia} (\gamma^M)_{ab} \nabla_M \varphi_{ab} - 2 \lambda_2 \Phi [\Phi^{A_1}, \Phi^{A_2}] - 2 \varphi^{A_i} \varphi^{A_j} \right).$$

(3.14)

We note that the quadratic action (3.14) contains terms mixing quantum vector $a_M$ and scalar $\varphi_{Ai}$ fields as well as the quantum spinor $\lambda^{ia}$ and $\psi_a^A$ fields. In order to exclude the mixed $a_M$ and $\varphi_{Ai}$ terms we introduce the gauge fixing action in form of the $R_2$ gauge (see e.g., [34], [36])

$$S_{gf} = - \text{tr} \int d^6x (\nabla_M a^M - \frac{i}{2} [\Phi^{Ai}, \varphi_{Ai}])^2.$$  

(3.15)

Using the gauge fixing action (3.15) instead of (3.2) leads to cancelation of mixed $a_M$ and $\varphi_{Ai}$ terms in quadratic action (3.14). However in that case the additional contribution arises for the ghosts fields.

Integrating over quantum fields in the functional integral we produce the one-loop contribution to the effective action

$$\Gamma^{(1)}[A, \Phi] = \frac{i}{2} \text{Tr} \ln B - \frac{i}{4} \text{Tr} \ln F^2 - i \text{Tr} \ln G,$$

(3.16)

where $G = -(\nabla^2)^I + \frac{i}{2} f^{IKP} f^{JLP} (\Phi^{Ai})^K (\Phi^{Ai})^L$. The matrices $B$ and $F$ depend on the background fields $A_M$ and $\Phi_{Ai}$. To reveal the explicit structure of the matrices $B$ and $F$ it is useful to write down the the gauge group indices. All fields we assumed in adjoint representation of gauge group. We represent the matrices $B$ and $F$ in a block form

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix},$$

(3.17)

where for the $B$ matrix elements we have introduced the notations

$$(B_1)^I_{MN} = - \eta_{MN} (\nabla^2)^I - 2 j^{IKP} f^{JKL} (\Phi^K)_{MN} + \frac{i}{2} \eta_{MN} f^{IKP} f^{JLP} (\Phi^{Ai})^K (\Phi^{Ai})^L,$$

(3.18)

$$(B_2)_{AI}^{ij} = - \frac{i}{2} \delta^i_j (\gamma^M)_{ab} \nabla_M a^M + \frac{i}{2} \delta^i_j f^{IKP} f^{JLP} (\Phi^K)_{MN} (\Phi^{Ai})^L,$$

(3.19)

and for the $F$ elements we have the following expressions

$$(F_1)_{ab}^i = \frac{i}{2} \delta^i_j (\gamma^M)_{ab} \nabla_M a^M,$$

(3.20)

$$(F_2)_{ai}^b = - \frac{i}{2} \delta^a_i f^{iK} (\Phi^K)_{ai},$$

(3.21)

$$(F_3)_{ai}^{ij} = - \frac{i}{2} \delta^a_i f^{iK} (\Phi^K)_{ai},$$

(3.22)

$$(F_4)_{ab} = \frac{i}{2} \delta^a_i f^{iK} (\Phi^K)_{ab}.$$  

(3.23)

Equations of motion for background fields has the following form

$$\nabla^M F_{MN} + \frac{i}{2} [\Phi^{Ai}, \nabla_N \Phi_{Ai}] = 0,$$

$$\nabla^2 \Phi^{Ai} - \frac{i}{2} \Phi^A_i [\Phi^{B_1}, \Phi^{B_2}] = 0.$$  

(3.24)
The one-loop divergences in the theory under consideration were studied in many details both in component and superfield approaches [6–12, 22, 23, 37, 39]. The absence of one-loop divergences in the six-dimensional $\mathcal{N}=(1,1)$ SYM theory on-shell was known [6–8] and, as we discussed in the previous section, it is immediately following from the (3.13).

4 Leading low-energy contribution to one-loop effective action of $\mathcal{N}=(1,1)$ SYM theory

Next we proceed to the evaluation of leading low-energy finite contribution to effective action (3.16). In what follows we assume the gauge group of the theory to be $SU(2)$. We choose standard Hermitian basis for generators $\tau^I = \frac{1}{2} \sigma^I$ using Pauli matrices

$$
\sigma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Generators $\tau^I$, $I = 1, 2, 3$, satisfy the $su(2)$ algebra, $[\tau^I, \tau^J] = i \epsilon^{IJK} \tau^K$, with totally antisymmetric symbol $\epsilon^{IJK}$ and they are normalized as, $\text{tr}(\tau^I \tau^J) = \frac{1}{2} \delta^{IJ}$.

We also suppose that background fields $A_M$ and $\Phi_{Ai}$ align into Cartan subalgebra of $su(2)$ generated by $\tau^3$ matrix

$$
A_M = A_M \tau^3, \quad \Phi_{Ai} = \Phi_{Ai} \tau^3.
$$

We have denoted by $A_M$ and $\Phi_{Ai}$ the third components of background fields $A_M$ and $\Phi_{Ai}$. Our choice of background corresponds to the case when gauge symmetry group $SU(2)$ is broken to $U(1)$. Equations of motion for background fields (3.24) in accordance with restriction (4.2) take a free form

$$
\partial^M F_{MN} = 0, \quad \partial^2 \Phi_{Ai} = 0,
$$

where $F_{MN} = \partial_M A_N - \partial_N A_M$ is an Abelian field strength corresponding to $F_{MN} = F_{MN} \tau^3$. Also we assume that background fields are slowly varying in space-time

$$
\partial_M F_{NL} = 0, \quad \partial_M \Phi_{Ai} = 0.
$$

The last condition means that we systematically neglect all terms with derivatives of the background field strength $F_{MN}$ and the scalar one $\Phi_{Ai}$.

Gauge transformations for background fields in case of (4.2) are simplified and take the form

$$
\delta_{\Lambda} A_M = \partial_M \Lambda, \quad \delta_{\Lambda} \Phi_{Ai} = 0,
$$

where $\Lambda = \Lambda(x)$ is an Abelian gauge parameter. In Abelian case the gauge field strength $F_{MN}$ is known to be invariant under gauge transformation (4.5), $\delta_{\Lambda} F_{MN} = 0$. Hence the effective action as a polynomial function of $F_{MN}$ is a gauge invariant under construction.
4.1 Leading low-energy contribution

First of all let we rewrite the matrices $B$ and $F$ in accordance with our choice of background fields (4.2) and (4.4). We consider non-vanishing background for scalar field $Φ_A$, hence we have

$$\Gamma^{(1)}[A, Φ] = \frac{i}{2} Tr \left( -\eta_{MN} (\nabla^2)^{IJ} + \frac{1}{2} η_{MN} δ^{[I} Φ^2 - 2ε^{I3J} F_{MN} \right)$$

$$+ \frac{i}{4} Tr \left( -δ_{I}^{J} δ_{A}^{B} (\nabla^2)^{IJ} + \frac{1}{2} δ_{I}^{j} δ_{A}^{B} δ^{IJ} Φ^2 \right)$$

$$- \frac{i}{4} Tr \left( -δ_{I}^{J} (\nabla^2)^{IJ} - \frac{i}{2} δ_{I}^{i} ε^{I3J} Φ^2 \right)$$

$$- \frac{i}{4} Tr \left( -δ_{A}^{B} (\nabla^2)^{IJ} - \frac{i}{2} δ_{A}^{B} ε^{I3J} Φ^2 \right)$$

$$- iTr \left( - (\nabla^2)^{IJ} + \frac{1}{2} δ^{IJ} Φ^2 \right),$$  \hspace{1cm} (4.6)

where the operator $(\nabla^2)^{IJ}$ is constructed using the Abelian covariant derivative $(\nabla_M)^{IJ} = δ^{IJ} η_M + ε^{I3J} A_M$ and we introduced $Φ^2 = Φ^4 Φ_{AI}$. In order to obtain the leading low-energy contribution to one-loop effective action (4.6) one should follow the same method as in study of divergences in previous section. We use the proper time technique (3.6) to study the one-loop effective action (4.6). We are interested in the leading low-energy contributions to one-loop effective action which have not higher than fourth power of gauge field strength $F_{MN}$. Thus we have to expand the exponent in the (3.6) and collect all terms up to the eighth power of covariant derivative or the forth power of background gauge field strength. In order to determine the structure of the leading contribution to effective action we consider the contribution of forth power over background gauge field strength from the determinant of spinor field $λ^{\mu}$.

$$- \frac{i}{4} Tr \ln \frac{Δ_{1/2}}{i} = \frac{i}{4} μ^{2ω} Tr \int d^6 x \int_{0}^{∞} \frac{d(iσ)}{(iσ)^{1-ω}} e^{isΔ_{1/2} δ^0(x-x')} |_{x=x'}$$  \hspace{1cm} (4.7)

where $Δ_{1/2} = -δ_{I}^{J} (\nabla^2)^{IJ} - \frac{i}{2} δ_{I}^{ij} ε^{I3J} Φ^2$. We assume the constant background scalar field, thus the covariant d’Alembertian in the exponent is comute with the $Φ^2$ term

$$- \frac{i}{4} Tr \ln \frac{Δ_{1/2}}{i} = \frac{i}{4} Tr \int d^6 x \int_{0}^{∞} \frac{d(iσ)}{(iσ)^{1-ω}} e^{isΔ_{1/2} δ^0(x-x')} |_{x=x'}$$

$$= \frac{i}{24} \int d^6 x Tr \left( \frac{1}{2} Σ_{MN} F^{MN} \right)^4 \int_{0}^{∞} d(iσ) (iσ)^{3} e^{isΔ_{1/2} δ^0(x-x')} |_{x=x'} + ...,$$

where the trace is taken over spinor indexes and dots mean over contributions of the forth power of $F_{MN}$. After that we pass to momentum space for delta-function and calculate the integral over proper-time. We have

$$- \frac{i}{4} Tr \ln Δ_{1/2} \sim \frac{1}{(4π)^7} \int d^6 x \frac{1}{Φ^2} \left( 4F^4 - 3F^2 F^2 \right),$$  \hspace{1cm} (4.8)

where we denote $F^4 = F_M^N F_N^P F_P^Q F_Q^M$ and $F^2 = F_M^N F_N^M$. The same strategy we apply to the remaining terms in (4.6). We omit tedious details of calculations and the result is

$$Γ^{(1)}_{\text{lead}}[A, Φ] = \frac{1}{(4π)^3 360} \int d^6 x \frac{1}{Φ^2} \left( 4F^4 - 3F^2 F^2 \right).$$  \hspace{1cm} (4.9)

Recently [35] the leading low-energy contribution to the one-loop effective action in $N = (1, 1)$ SYM theory was analyzed using superfield approach. The effective action (4.9) coincides with bosonic sector the superfield effective action obtained in [35]. The component calculation, carried out here, can be considered as an independent test of the superfield result.
4.2 Effective potential

In order to evaluate the one-loop contribution to the effective action for scalar field $\Phi_{Ai}$ we switch off the vector background field $A_M = 0$ in the effective action (4.6). Then we obtain

$$V^{(1)}[\Phi] = \frac{i}{2} \text{Tr} \ln \left( -\eta_{MN} \delta^{ij} \partial^2 + \frac{i}{2} \eta_{MN} \delta^{ij} \Phi^2 \right)$$

$$+ \frac{i}{2} \text{Tr} \ln \left( -\delta^{ij} \delta_a \delta^b \partial^2 + \frac{i}{2} \delta^{ij} \delta_a \delta^b \Phi^2 \right)$$

$$- \frac{i}{4} \text{Tr} \ln \left( -\delta^{ij} \delta^a \delta^b \partial^2 + \frac{i}{2} \delta^{ij} \delta^a \delta^b \Phi^2 \right)$$

$$- \frac{i}{4} \text{Tr} \ln \left( -\delta^{ij} \delta^a \delta^b \partial^2 + \frac{i}{2} \delta^{ij} \delta^a \delta^b \Phi^2 \right)$$

$$- i \text{Tr} \ln \left( -\delta^{ij} \partial^2 + \frac{i}{2} \delta^{ij} \Phi^2 \right).$$

(4.10)

Note that indices of adjoin representation $I, J = 1, 2$ in the expression (4.10).

The direct evaluation of the above functional determinants on the base of proper-time technique leads to cancelation of all contributions in effective potential (3.5)

$$V^{(1)}[\Phi] = 0.$$  

(4.11)

The absence of the one-loop effective potential for scalar fields is in consistence with the $N = (1,1)$ supersymmetry. The same result can be explained explicitly using $N = (1,0)$ harmonic superfield consideration in the $N = (1,1)$ SYM theory [35]. In terms of superfields the effective potential for background scalar superfields corresponds to the contributions which depend only on constant background hypermultiplet. But all such contributions are forbidden by the on-shell hidden $N = (0,1)$ supersymmetry [35]. The absence of the one-loop contribution to the effective potential for scalar field in $N = (1,1)$ SYM theory is also dictated by the absence of the Coulomb branch in the theory under consideration [38]. However, as far as we know it was never confirmed before by the direct calculation.

5 Summary

In this paper we have studied the bosonic sector of the low-energy effective action in the six-dimensional $N = (1,0)$ and $N = (1,1)$ SYM theories (2.4) and (2.8) formulated in terms of physical components. First of all we have derived the component action of 6D, $N = (1,0)$ SYM theory and the corresponding supersymmetry transformations. The one-loop effective action is considered on the base the background field method with help of $R_\xi$-type gauge. We provided the explicit calculation of the divergent contributions to the one-loop effective action in $N = (1,0)$ SYM theory interacting with hypermultiplets. We have demonstrated by direct calculation of corresponding contributions to the one-loop effective action that all $F^3$-type divergent contributions (3.12) cancel each other. The result is in an agreement with the earlier component calculations [6–12,37,39] without hypermultiplet contributions and with the superfield analysis [22–24]. We also demonstrated that divergent contributions in the $N = (1,1)$ theory cancel each other and the theory is one-loop off-shell finite as expected.

Studying the finite contributions to the $N = (1,1)$ SYM theory we have considered the gauge group $SU(2)$ and have assumed that the background fields take the value in Cartan subalgebra of $su(2)$. To construct the finite low-energy contribution to the effective action of the theory we assume the non-zero values for both vector and scalar background fields and using the $R_\xi$ gauge. In case of slowly varying on-shell background fields we calculated the leading low-energy contribution to the one-loop
effective action. It is determined by the contributions of the fourth power of the gauge field strength $F_{MN}$. The obtaining result being in agreement with the bosonic part of the leading contribution to the one-loop harmonic superfield effective action [35] and it can be considered as an independent test of the superfield calculation. We also demonstrate the absence of the one-loop contribution to the effective potential for the background scalar field, which as we know was not mentioned before by direct calculations.

As a further development of the approach under consideration we plan to study the divergences in the fermionic sector and investigate the two-loop counterterms in the theory (2.4). Also it would be interesting to study the finite low-energy contributions to the effective action in high derivative SYM theory [19, 40].

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