Nonlinear effects in Pulsations of Compact Stars and Gravitational Waves.

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Abstract. Nonlinear stellar oscillations can be studied by using a multiparameter perturbative approach, which is appropriate for investigating the low and mild nonlinear dynamical regimes. We present the main properties of our perturbative framework for describing, in the time domain, the nonlinear coupling between the radial and nonradial perturbations of spherically symmetric and perfect fluid compact stars. This particular coupling can be described by gauge invariant quantities that obey a system of partial differential equations with source terms, which are made up of product of first order radial and nonradial perturbations. We report the results of numerical simulations for both the axial and polar coupling perturbations, that exhibit in the stellar dynamics and in the associated gravitational wave signal some interesting nonlinear effects, such as combination harmonics and resonances. In particular, we concentrate on the axial case, where the linear axial perturbations describe a harmonic component of a differentially rotating neutron star. The gravitational wave signal of this stellar configuration mirrors at second perturbative order the spectral features of the linear radial normal modes. In addition, a signal amplification appears when one of the radial frequencies is close to the axial w-mode frequencies of the star.

1. Introduction
Neutron stars can undergo oscillating phases during the dynamical evolution of various astrophysical scenarios, such as in a binary system due to the tidal force exerted by the companion during the coalescence, or in a protoneutron star after the core bounce due either to the bounce dynamics or to the fall-back accretion of material, which has not been expelled by the supernova shock. Gravitational radiation is one of the dissipative processes that damp the stellar oscillations carrying away important information about the physical properties of the sources. Several works have been dedicated to the understanding of the spectral properties and wave forms of gravitational waves emitted by neutron stars, e.g. see [1, 7, 8, 3]. Stellar oscillations of compact stars generate high frequency gravitational signal, above 500 – 600 Hz, where the sensitivity curves of Earth-based laser interferometers (LIGO, VIRGO, GEO600 and TAMA [4]) are dominated by the shot noise of laser. In order to increase the chances of a detection, it is the necessary to increase our theoretical understanding of the sources and provide more accurate templates of gravitational signals.
Although strong non-linearity requires a fully numerical approach, the development of the nonlinear perturbative theory could help us in the description and interpretation of phenomena that involve mild nonlinear regimes. To this end we started to investigate the coupling between radial and nonradial perturbations of compact stars, where for simplicity we have considered as equilibrium configuration a perfect fluid and spherically symmetric star. Radial and nonradial oscillations are expected to be prevalently excited after a core bounce. Even though the quadrupole component provides the dominant contribution to the gravitational radiation, the radial pulsations may store a considerable amount of kinetic energy and transfer a part of it to the nonradial perturbations. As a result, this nonlinear interaction could produce a damping of the radial pulsations and an interesting gravitational signal. The strength of this signal depends on the efficiency of the coupling, which is the main effect we want to explore. We have then developed a gauge invariant perturbative formalism for studying in the time domain the coupling between the radial pulsations and both polar and axial nonradial oscillations. We have used the multi-parameter relativistic perturbation theory \cite{12}, which enabled us to correctly address the gauge issues of nonlinear perturbations.

In what follows we outline the main result of our numerical simulations for the axial sector, where the linear axial perturbations describe a differentially rotating star (see \cite{10, 11} for a complete description). For the polar sector, we report only some preliminary results, as further investigations are currently under way. In this paper, we use geometrical units $G = c = 1$.

2. Perturbation Equations

The structure of perturbation methods is intrinsically hierarchical. In this section we shortly describe the main properties of our framework, from the background spacetime up to the coupling perturbations.

As equilibrium configuration we consider a perfect fluid spherically symmetric relativistic star, which is determined by solving the TOV equations for a polytropic equation of state $p = K \rho^\Gamma$ with adiabatic index $\Gamma = 2$ and $k = 100 \text{ km}^2$. For a central mass energy density $\rho_c = 3 \times 10^{15} \text{ g cm}^{-3}$ one obtains $M = 1.26 M_\odot$ and $R = 8.862 \text{ km}$. Due to the spherically symmetry of the background spacetime, the nonradial perturbations can be expanded in tensor harmonics and then separated in two classes with opposite parity, which have an independent dynamics on a spherical star. They are called axial (odd-parity) and polar (even-parity) nonradial perturbations.

At first perturbative order, we study two independent sets of axisymmetric oscillations: i) the radial pulsations, which corresponds to the $l = 0$ harmonic index and ii) the axial and polar nonradial perturbations with $l \geq 2$. The radial perturbations are completely described by a set of four perturbative fields, two metric and two fluid, which obey three first order in time evolution equations and two constraints, as there is a single radial degree of freedom. This allows us to set up a hyperbolic-elliptic formulation, where the Hamiltonian constrain is solved at any time step to get one of the metric variables.

The axial nonradial perturbations have been studied with a system of two equations, the axial master wave equation and a conservation equation. The former describes the only gauge invariant metric variable of the axial sector $\Psi^{NR}$, while the latter the axial velocity perturbation $\beta^{NR}$. At first order, the stationary character of the axial velocity allowed us to study separately the dynamical degree of freedom of the spacetime and its stationary part. In particular, the stationary solution describes the differential rotation induced on the background star by the $l$ harmonic component of the velocity perturbation, and the related metric perturbation the dragging of the inertial frames.

The perturbative equations for the polar nonradial perturbations are studied with a
hyperbolic-elliptic system of three equations. The two hyperbolic equations describe respectively the gravitational waves and the sound waves propagation, and an elliptic equation, i.e. the Hamiltonian constraint \([5, 9, 11]\). We have found that this system of equations is more suitable for numerical integration than others available in literature, as the Hamiltonian constraint is used for updating at any time step one of the unknowns of the problem. Therefore, the errors associated with the violation of this constraint are automatically corrected. In the exterior, the axial Regge-Wheeler and polar Zerilli functions can be constructed for extracting the waveform and the energy emitted in gravitational waves.

The axial and polar coupling oscillations are also described by gauge invariant quantities: the metric \(\Psi^C\) and velocity \(\beta^C\) axial perturbations, and two metric \(S^C, k^C\) and one fluid \(H^C\) polar perturbations. In the stellar interior, they now obey inhomogeneous linear equations, where the associated homogeneous equations are given by the same linear operators as the equations for the first order nonradial perturbations. The source terms are instead quadratic in the first-order radial and nonradial perturbative quantities. In the exterior the source terms vanish and the gravitational waveforms can be described by Regge-Wheeler and Zerilli equations.

With this system of equations in the axial case we can study two independent initial configurations: i) a differentially rotating and radially pulsating star, ii) the scattering of a gravitational wave by a radially pulsating star. We will focus on the former configuration, as the dynamics of coupling oscillations exhibits more interesting results (for a complete description see \([11]\)).

2.1. Initial Data

The initial configuration for the radial pulsations has been excited by selecting specific radial eigenmodes. We have chosen an origin of time such that the radial eigenmodes are described only by the eigenfunctions associated with the radial velocity perturbation \(\gamma^R\). To this end we have first determined the wave equation for this variable, then we have managed it in order to set up a Sturm-Liouville problem. By solving the eigenvalue problem with a numerical code based on the relaxation method we were able to determine the eigenfrequencies of radial modes with accuracy to better than 0.2 percent with respect to the published values. The simulations for any initial radial mode satisfy with high accuracy the Hamiltonian constraint and are stable for very long evolutions. The radial spectrum, which has been determined with a Fast Fourier Transformation of the time evolution, reproduce the published results with an accuracy to better than 0.2 percent.

The axial differential rotation instead has been described by expanding in vector harmonics the relativistic \(j\)-constant rotation law, then taking the first component which is related to the gravitational wave emission, that is \(l = 3\). We can specify two parameters in the initial profile for the axial velocity perturbations, i.e. the differential parameter \(A\) and the angular velocity at the rotation axis \(\Omega_c\). The value for \(A\) has been chosen in order to have a smooth profile and a relatively high degree of differential rotation, as for high \(A\) the rotation tends to be uniform and then the \(l = 3\) component vanishes. We have chosen an angular velocity that corresponds to a 10 ms rotation period at the axis of the star. For other values of the angular velocity, the linearity of the perturbative equations allows us to get the respective gravitational signal with a simple re-scaling.

The polar nonradial oscillations are excited with \(l = 2\) approximated enthalpy eigenfunctions, i.e. sinusoidal curves with a number of nodes equal to the main mode we want to excite. Since these curves are not the correct eigenfunctions, also other modes of the spectrum will be excited with lower amplitudes.

In order to study the effects of the coupling between the linear perturbations we have set
up initial vanishing values for all the coupling perturbations.

3. Numerical Results for Axial Perturbations

Numerical simulations of a radially pulsating and differentially rotating star have shown a new interesting gravitational signal. The wave forms have these properties: i) an excitation of the first \( w \)-mode at the early stage of the evolution ii) a periodic signal which is driven by the radial pulsations through the source terms (see figure 1). The picture is confirmed by the spectra, where we have noticed that the radial normal modes are precisely mirrored in the gravitational signal at nonlinear perturbative order. On the other hand, the excitation of the \( w \)-mode at the early stages of the numerical simulations is an unphysical response of the system to the initial violation of the axial constraint equations for the coupling perturbations.

In figure 1, the wave forms of the axial master metric function \( \Psi^C \) exhibit an interesting amplification when the radial oscillations pulsate at frequencies close to the \( l = 3 \) axial spacetime \( w \)-mode, \( \nu_w = 16.092 \text{ kHz} \). For the stellar model under consideration this effect appears at third and fourth radial overtones, whose frequencies are 13.545 kHz and 16.706 kHz respectively. It is worth to remark that this effect takes place despite the energy and the maximum displacement of the surface of the radial modes decrease proportionally to the order of the radial modes (see Table 1). We can interpret this amplification as a resonance between the radial frequencies of the source, which behave as forcing terms, and the natural frequencies of the axial master wave-like equation.

Our perturbative approach does not include backreaction, in other words, it does not account for the damping of the radial oscillations or the slowing down of the stellar rotation due to contribution of the nonlinear coupling to the energy loss in gravitational waves. Backreaction could be studied by looking at higher perturbative orders. Nevertheless, we can provide a rough estimate of the damping time of the radial pulsations by assuming that the energy emitted is completely supplied by the first-order radial oscillations, and that the power radiated in gravitational waves is constant in time. In this way, the damping time is given by the following expression:

\[
\tau_{lm}^C \equiv \frac{E_n^R}{\langle \dot{E}_{lm}^C \rangle},
\]

where \( E_n^R \) is the energy of a radial eigenmode (see Table 1), and \( \langle \dot{E}_{lm}^C \rangle \) is the averaged value of the nonlinear coupling contribution to the power emitted. The results for \( \tau_{30}^C \) are shown in Table 1. Moreover, in the last row of Table 1 we give an estimation of the damping of the radial pulsations associated with a certain radial eigenmode in terms of the number of oscillation cycles:

\[
N_{\text{osc}} = \frac{\tau_{lm}^C}{T_n},
\]

where \( T_n = \nu_n^{-1} \), with \( \nu_n \) being the eigenfrequency of the radial eigenmode. It is interesting to mention that the number of oscillations required for the damping of the H4 mode is only 12, and hence it would already affect the H4-waveform shown in figure 1. This is not surprising, and shows that the coupling near resonances is a very effective mechanism for extracting energy from the radial oscillations.

It is worthwhile to remark that a possible detection of this gravitational signal could provide new information of the stellar parameters, as the second order gravitational spectra reproduce those of the radial modes of a nonrotating star, which can be easily determined for a large class of equations of state.
4. Numerical Results for Polar Perturbations

Numerical simulations for studying the dynamical properties of polar coupling perturbations are currently under way for various initial perturbative configurations. Here, we discuss some of the typical properties of this nonlinear perturbative class. We have set up an initial perturbative configuration that consists of: i) a linear combination of radial modes up to the fourth overtone for radial pulsations, and ii) an approximate enthalpy eigenfunction with three nodes for polar nonradial oscillations. The amplitudes of these initial perturbations have been chosen in order to have $10^{-6} M_\odot$ energy in radial pulsations and $10^{-7} M_\odot$ in nonradial oscillations. In figure 2 we show the time evolution of the total enthalpy $H_{20}^{total} = H_{20}^{NR} + H_{20}^{C}$ and Zerilli function $Z_{20}^{total} = Z_{20}^{NR} + Z_{20}^{C}$ with their respective Fourier transformations. The total waveform and enthalpy clearly manifest the presence of several oscillation frequencies. In the related spectra we can notice the excitation of a set of linear and nonlinear harmonics, where the latter appear as combination tones, i.e. their frequencies are the sum or difference of the linear mode frequencies. It is worth remarking that linear radial modes appear only in the enthalpy evolution and not in the waveforms. This is what we expect for $l = 0$ polar perturbations on a nonrotating spherical star. For polar coupling perturbations, we have not found any resonance, as the oscillation frequency of the source terms are different from the natural frequencies of the linear differential operator. A more complete description of the second order harmonics must take into account the self-coupling terms between nonradial oscillations. This will be the matter of future investigations.
Table 1. Quantities associated with radial normal modes and their coupling to the first-order axial differential rotation: Energy, $E_R^R$, and maximum stellar surface displacement $\xi_{sf}^R$ of the radial eigenmodes for initial conditions (Sec. 2.1) with velocity amplitude $0.001$; average power, $<E^C_{30}>$, emitted in gravitational waves to infinity from the coupling between the radial eigenmode and the axial differential rotation; estimated values of the damping times, $\tau^C_{30}$; and number of oscillation periods, $N_{osc}$, that takes for the nonlinear oscillations to radiate the total energy initially contained in the radial modes.

| Radial Mode | Frequency [kHz] | $E_R^R$ [10^{-8} km] | $\xi_{sf}^R$ [m] | $<E^C_{30}>$ [$10^{-14}$] | $\tau^C_{30}$ [ms] | $N_{osc}$ |
|-------------|-----------------|------------------------|------------------|-----------------------------|------------------|----------|
| F           | 2.138           | 35.9                   | 12.65            | $1.54 \times 10^{-6}$       | $7.78 \times 10^{10}$ | $1.67 \times 10^{11}$ |
| H1          | 6.862           | 4.2                    | 4.02             | $5.69 \times 10^{-2}$       | $24.59 \times 10^{4}$ | $1.69 \times 10^{6}$ |
| H2          | 10.302          | 1.37                   | 2.66             | $4.04$                      | $11.29 \times 10^{2}$ | $1.16 \times 10^{4}$ |
| H3          | 13.545          | 0.62                   | 2.02             | $46.69$                     | $44.28$           | $5.99 \times 10^{2}$ |
| H4          | 16.706          | 0.34                   | 1.64             | $130.84$                    | $8.64$            | $1.44 \times 10^{2}$ |
| H5          | 19.823          | 0.21                   | 1.38             | $15.90$                     | $44.04$           | $8.73 \times 10^{2}$ |
| H6          | 22.914          | 0.14                   | 1.19             | $3.71$                      | $126.12$          | $2.89 \times 10^{3}$ |

5. Gravitational Strain
The strain of the gravitational signal can be determined by the following expression:

$$h_+ - i h_\times = \frac{1}{r} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left( \frac{Z^{lm}}{2} + \Psi^{lm} \right) \sqrt{\frac{(l+2)!}{(l-2)!}} \sum_{s=-2}^{2} Y^{lm} (\theta, \varphi),$$

(3)

where $-2 \sum_{s} Y^{lm} (\theta, \varphi)$ are the $s = -2$ spin-weighted spherical harmonics and $\Psi^{lm}$ and $Z^{lm}$ are the Regge-Wheeler and Zerilli gauge invariant functions (see [6] for a review). For axisymmetric perturbations ($m = 0$), we have the following expressions for polar $l = 2$ and axial $l = 3$ perturbations:

$$h_+^{20} = 9.63 \times 10^{-18} Z^{20} \left( \frac{10 \text{kpc}}{r} \right) \sin^2 \theta,$$

(4)

$$h_\times^{30} = 3.63 \times 10^{-17} \Psi^{30} \left( \frac{10 \text{kpc}}{r} \right) \cos \theta \sin^2 \theta,$$

(5)

where the distance has been rescaled for a galactic source. For a periodic signal, the effective amplitude of the strain can be increased by statistical methods. This improvement depend on the number of oscillations $N_{osc}$ of the signal and can be estimated as $h_{eff} = \sqrt{N_{osc}} h$. In figure 3 we show the gravitational wave strain for a galactic radial and nonradial oscillating star, and the sensitivity curves of the VIRGO, LIGO and Advanced LIGO laser interferometer detectors. For the polar case we have considered the total Zerilli function $Z^{tot}$ in equation (4), and the axial master function $\Psi^{C}$ in equation (5). For polar nonradial pulsations with $10^{-6} M_\odot$ oscillating energy, the fundamental quadrupolar modes $2f$ can be detected by the current Earth based laser interferometer detectors. On the onther hand, the nonlinear axial signal arisen from a radially pulsating and differentially rotating star is hidden by the detector noise even for Advanced Ligo. However, when the first radial overtone H1 is excited, and the oscillations are
Figure 2. The first column displays, on the upper panel, the time evolution of $H_{20}^{\text{tot}}$, and on the lower panel the Zerilli function $Z_{20}^{\text{tot}}$. In the second column we plot the power spectra of $H_{20}^{\text{tot}}$ (upper panel) and $Z_{20}^{\text{tot}}$ (lower panel) respectively.

not damped by other dissipative processes, the effective strain $h_{\text{eff}}$ could lay into the sensitivity window of the Advanced Ligo detector. It is then necessary to investigate whether viscosity, magnetic braking etc. can drastically reduce the detection chances of this signal.

6. Discussion
The results reported here represent a first analysis of the nonlinear coupling of oscillations of compact relativistic stars and its impact for gravitational wave physics. Future extensions of this research must certainly consider more realistic stellar models, by taking into account the effects of rotation, composition gradients, magnetic fields, dissipative effects, etc. In particular, by including rotation we may find new interesting nonlinear effects due to the different behaviour of radial and nonradial modes in a rotating configuration. While the radial modes are only weakly affected so that their spectrum is essentially the same as in a nonrotating configuration (when scaled by the central density), the non-axisymmetric modes manifest a splitting similar to the Zeeman effect in atomic spectra. The rotation removes the mode degeneracy in the azimuthal harmonic number of the nonrotating case. The details of this frequency separation depend on the stellar compactness and rotation rate. It may then happen that for a given stellar rotation rate and compactness the nonradial frequencies cross the sequence of radial frequencies [2] so that the frequencies of these two kinds of modes are close, and possible resonances or instabilities could influence the spectrum and the gravitational
Figure 3. Gravitational strain for galactic radially and nonradially oscillating stars. The strain of the quadrupolar fundamental polar mode $2f$ is denoted with a circle. The nonlinear axial gravitational strain for radially pulsating and differentially rotating stars is denoted with a triangle, while the effective strain with a diamond. The overtone that dominates the radial pulsations are denoted with the letter $H$. The sensitivity curves of VIGO, LIGO and Advanced LIGO are shown with dashed, solid and dotted lines respectively.

wave signal.

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