Modelling and numerical solution of problems of structural mechanics with unilateral constraints and friction

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Abstract. The article deals with the problems of calculation of elastically deformable systems with unilateral constraints and Coulomb friction. For the numerical solution of the problem, the finite element method is used, the contact interaction is modeled by means of the contact finite elements (CFE) of the frame-rod type. The basic relations and statements of static and dynamic contact problems for the interacting elastic bodies are given. To comply with the limitations under the conditions of ultimate friction-sliding, the method of compensating loads is applied. On the basis of the proposed discrete contact model and step-by-step loading schemes, the numerical algorithms are developed, which allows calculating structures with unilateral constraints and Coulomb friction, both under static and dynamic loads, as well as taking into account the different contact conditions that approximate the calculation scheme to the real operating conditions of a structure or construction. With the help of the proposed approach, numerical solutions of the problem of contact of the structure with the base under dynamic load are obtained and analyzed.

1. Introduction

The problems of contact interaction of structures and their parts have a wide range of applications in construction and other fields of engineering. For example, deformation and technological seams can be opened and closed, both with slippage, and with clutch of contacting surfaces, at various combinations of external loadings. The same can happen at interaction of the cracks coasts, on contact of a sole of the construction with the basis or on the supports allowing a separation or slippage of the construction leaning on them. Herewith, it is often the state of the contact zone can be decisive in assessing the stress-strain state of a constructions and structures and their strength and reliability [1-10].

The numerical solution of contact problems is usually realized on the basis of different schemes of the finite element method (FEM). In this case, the continuum problems of contact of elastic bodies are reduced to finite-dimensional problems with discrete unilateral constraints. The development of various numerical methods for calculating systems with unilateral constraints on the basis of FEM is devoted to a large number of studies, among which we can note the work [11-18].

When solving constructively nonlinear contact problems, both iterative (successive approximations) [11-14] and incremental (step-by-step) [15-18] methods are used. The positive side of step-by-step methods is that, based on them, the solution of the contact problem can be obtained at any stage of static or dynamic loading. In addition, this approach can be quite effective with difficult contact conditions and the nature of loading, where the solution depends on the history of the structure deformation [15,16]. The constructive nonlinearity in the step-by-step process will be manifested in the
change of the working schemes of the structure on load value or in time – switching on and off unilateral constraints, both in normal and tangential direction. It is assumed that between two consecutive events on the contact, i.e. within the limits of each fixed working scheme, the character of the structure deformation is linear.

In this paper for the solving problems with unilateral constraints and friction, finite element models and algorithms of their numerical implementation are used directly. To simulate contact interaction, it is proposed to use contact finite elements (CFE) of the frame-rod type \([15, 19]\), which allows to calculate the displacements and forces in the contact zone with the high accuracy. The CFE data provide a discrete contact between the nodes of the finite element mesh located on the boundary surfaces of the interacting elastic bodies. With their help, it is convenient to simulate both the physical properties of the contact surfaces (initial strength, deformability, etc.) and the various conditions and states of contact (gaps, separation, clutch, friction-sliding).

2. Statement and solution of the problem

Let’s consider the plane problem of contact interaction of elastic bodies \(V^+\) and \(V^-\) (which may be, for example, the structure and base) with contacting surfaces, \(S^+_c\) and \(S^-_c\) respectively. Simulating the state of the contact zone is carried out with the help of plane frame-rod contact finite elements (figure 1 a).

![Figure 1. Contact interaction model using frame-rod contact elements.](image)

The boundary conditions on the contact are expressed through the forces and deformations in a separate CFE \([9, 20]\):

\[
\begin{align*}
N_k &\leq 0; \quad u_{nk}N_k = 0; \\
|T_k| &\leq |T_{uk}| = -f_kN_k; \quad T_k u_{tk} \geq 0; \quad (T_k - T_{uk})u_{tk} = 0, \quad k \in S_c.
\end{align*}
\]

here \(N_k, T_k\) are contact force in \(k\) CFE; \(u_{nk}, u_{tk}\) are mutual displacements of opposed nodes on \(S^+_c\) and \(S^-_c\) (\(S_c = S^+_c \cup S^-_c\)) in the normal and tangential direction (figure 1 b); \(T_{uk} = -f_kN_k\) is ultimate Coulomb friction force for the contact \(k\); \(f_k \geq 0\) is the coefficient of friction.

The computational implementation of the conditions (1) is performed using a step-by-step analysis of changes in the state of the contact in the process of sequential application of a given load. In this case, the friction conditions can be satisfied to the best extent, since the solution of the friction problem depends on the loading history of the structure. The moment of transition from one state to another is, respectively, the event of separation or contact for unilateral constraints, slippage or clutching.

Static loading is simulated by step-by-step application of a given load. Assuming that between two consecutive events the system is deformed linearly, within each step the solution of the linearly elastic problem is constructed, and the events at the contact are determined according to the scheme of simple loading. Thus, the solution of constructively nonlinear contact problem will be represented as a solution of series of linear problems, with a consecutive change of working schemes of the structure.
The calculation procedure consists of trial and basic steps of loading. From the analysis of the trial step is determined the moment of the next event on contact. The basic step results in new condition of contact and, in its turn, analysis of change of working schemes results in designating of a new trial step. This allows you to specify the value of not only the current loading step, but also to predict the next steps and, thus, to build the optimal step-by-step loading process [15, 20].

Loading steps, at which slippage, separation or contact events occur for constraint $k$, that was previously in the state of the clenching $(k \in S_{ic})$, sliding $(k \in S_{2c})$ or separation, are determined $(k \in S_{3c})$ by the following expressions:

$$
\Delta \lambda^{s+1}_k = \Delta \lambda^{s+1}_k \left( \frac{T_{uk}^s - T^s_k}{\Delta T_{uk}^s - \Delta T^s_k} \right), k \in S_{ic}; \\
\Delta \lambda^{s+1}_k = \Delta \lambda^{s+1}_k \left( -\frac{N^s_k}{\Delta N^{s+1}_k} \right), k \in S_{1c}, S_{2c}.
$$

(2)

The transition from the sliding state to the clenching state on the step $(s+1)$ is determined by the conditions $\left( \Delta u^{s+1}_n / \Delta u^s_n \right) < 0$, $\Delta u^s_n \neq 0$, $k \in S_{2c}$. Where $\Delta T^{s+1}_k$, $\Delta T^{s+1}_{uk}$, $\Delta N^{s+1}_k$, $\Delta u^{s+1}_n$, $\Delta u^{s+1}_k$, $\Delta N^{s+1}_k$ are increments of contact forces and mutual displacements for $k$ CFE on trial step $\Delta \lambda^{s+1}$.

Of all the values $\Delta \lambda^{s+1}_k$, having been found using the equation (2), the smallest one corresponding to the moment of approach of the nearest event in the contact zone is selected: $\Delta \lambda^{s+1}_k = \min(\Delta \lambda^{s+1}_k)$, $k \in S_{c}$. Recalculation of refined in such a way step with value $\Delta \lambda^{s+1}_k$ is performed. To fulfill the conditions of ultimate friction the method of compensating loads is applied [20]. Changing of the ultimate friction forces on the contact is taken into account by the application of compensating forces to the opposite nodes:

$$
F_{ch} = -\Delta T^{s+1}_c = -\frac{\Delta \lambda^{s+1}_n}{\Delta \lambda^{s+1}_n} (T^{s+1}_uk - T^s_uk), k \in S_{2c}.
$$

The value of the transverse force on the contact $k$ is corrected by the same quantity:

$$
T^{s+1}_k = T^s_k + \Delta T^{s+1}_c.
$$

With the dynamic action of load constructive nonlinearity manifests itself in the change of the working schemes of the structure in time – switching on and off unilateral constraints, both in normal and tangential direction. The time point of changing of the contact state, i.e. the occurrence of the next event, in this case is determined by step-by-time analysis discrete contact model with the use of appropriate approximating expressions for displacements, speeds and accelerations on the time step $\Delta t$.

In this case, the integration step size is corrected and the current step is recalculated. As a result, a new state of contact is established at the given time point and, thus, the current working scheme of the structure is changed.

Let us write down the motion equations of elastic bodies in the form that allows the solution of a constructively nonlinear dynamic contact problem to be reduced to the solution of the sequence of linear dynamic problems on the basis of step-by-step on time analysis of the contact state [21, 22]:

$$
M \ddot{U}^{t+\Delta t} + C \dot{U}^{t+\Delta t} + K U^{t+\Delta t} = P^{t+\Delta t} - K U^t.
$$

(3)

here $M$, $C$ and $K$ are the mass, damping, and stiffness matrices of the finite element system respectively; $U^{t+\Delta t}$, $\dot{U}^{t+\Delta t}$, $\ddot{U}^{t+\Delta t}$ and $P^{t+\Delta t}$ are the vectors of nodal displacements, speeds, accelerations, and the external nodal load at time point $t+\Delta t$; $\Delta U^{t+\Delta t}$ is the increment of displacement at step $\Delta t$. In addition, on the part of the outer boundaries (for $V^+$ and $V^-$ respectively) boundary conditions on the forces, and on $S^{+}_m$ in the displacements should be given. It is assumed that at the initial time $t = 0$ the
vectors of displacements, speeds and accelerations are given and it is necessary to find a solution (3) during the time interval from 0 to some value $T$.

For the numerical integration of the motion equations (3), an implicit Newmark finite difference scheme is used [23], which is based on the assumption of a linear change of accelerations in the $\Delta t$ interval. At any time point $t'$ within the interval $\Delta t$ ($t \leq t' \leq t + \Delta t$), the values of accelerations $\ddot{U}(t')$, speeds $\dot{U}(t')$ and displacements $U(t')$ can be calculated with the following formulas:

$$
\ddot{U}(t') = \ddot{U}^r + \frac{(t'-t)}{\Delta t} \left[ \dot{U}^{r+1} - \ddot{U}^r \right] ; \\
\dot{U}(t') = \dot{U}^r + \frac{(t'-t)}{2} [\ddot{U}^r + \ddot{U}^r] ; \\
U(t') = U^r + (t'-t)\dot{U}^r + \left[ \ddot{U}^r + \ddot{U}^r \right].
$$

(4)

In the presence of unilateral constraints with Coulomb friction, the boundary conditions on the contact, written for the time point $t$, must be satisfied:

$$
\begin{align*}
\left. u_{nk}^l \right|_{t} & \geq 0; \quad N_n^l \leq 0; \quad u_{nk}^l N_n^l = 0; \\
\left| T_k^l \right| & \leq T_{Uk}^l; \quad T_k^l \dot{u}_{nk}^l \geq 0; \quad (T_k^l - T_{Uk}^l) \dot{u}_{nk}^l = 0, \quad k \in S_c.
\end{align*}
$$

(5)

here $u_{nk}^l$, $u_{nk}^l$ are mutual displacements of opposite nodes for unilateral constraint $k$ in the normal and tangential direction; $\dot{u}_{nk}^l = \dot{u}_{nk}^l / \dot{t}$ is the speeds of mutual tangential displacement on the contact $k$; $N_n^l$, $T_k^l$ are contact forces in the normal and tangential direction (forces in CFE $k$); $T_{Uk}^l = -f_k N_k^l$ is ultimate Coulomb friction force for the contact $k$; $f_k \geq 0$ is the coefficient of friction-sliding. The states on the contact $k$ will be determined by the following conditions: when the clutching $u_{nk}^l = 0$, $\dot{u}_{nk}^l = 0$, $\left| T_k^l \right| < T_{Uk}^l$: when sliding $\dot{u}_{nk}^l \neq 0$, $\left| T_k^l \right| = T_{Uk}^l$: when separation $u_{nk}^l > 0$, $N_k^l = 0$.

The numerical solution of the dynamic contact problem, thus, will consist in carrying out the process of step-by-step on time integration of equation (3) when the contact conditions (5). The expressions for determining the time point $\tilde{t}_k$ of occurrence of the nearest event, respectively, slippage, clutch, separation or contact for the constraint $k$ will have the following form here:

$$
\begin{align*}
\tilde{t}_k = t + \Delta t \left( \frac{T_{Uk}^l - T_k^l}{(T_{Uk}^{t+M} - T_k^l) - (T_{Uk}^{t+M} - T_{Uk}^l)} \right), \quad k \in S_{1c}; \\
\tilde{t}_k = t + \Delta t \left( -\left. u_{nk}^l \right|_{t} + \Delta t \left( \frac{u_{nk}^{t+M} - u_{nk}^l}{u_{nk}^{t+M} - u_{nk}^l} \right), \quad k \in S_{2c}; \\
\tilde{t}_k = t + \Delta t \left( -\left. N_n^l \right|_{t} \right), \quad k \in S_{1c}, S_{2c}; \\
\tilde{t}_k = t + \Delta t \left( \frac{-u_{nk}^{t+M}}{u_{nk}^{t+M} - u_{nk}^l} \right), \quad k \in S_{3c}.
\end{align*}
$$

(6)

Since the change of displacements, speeds (and, consequently, forces) within the step $\Delta t$ does not follow a linear law, then, additionally, using the expressions (4), an iterative refinement of the time point $\tilde{t}_k$ can be performed, the time consumption at that increases slightly [21,22].

Of all the values $\tilde{t}_i$ having been found using the formulas (6) and situated in the interval $(t, t + \Delta t)$, the smallest one corresponding to the moment of occurrence of the nearest time event on the contact is selected: $\tilde{t} = \min(\tilde{t}_k)$, $k \in S_c$. In case $\tilde{t} > t + \Delta t$ the next basic integration step $\Delta t$ is executed.

In case $t < \tilde{t} < t + \Delta t$ – recalculation of updated in such a way step with value $\Delta t = \tilde{t} - t$ is performed. Changing of the ultimate friction forces on the contact is taken into account by the application
of compensating forces to the opposite nodes: 
\[ \hat{F}_{ik} = -\Delta \hat{T}_{Uk} = -\frac{\Delta t}{\Delta t} (T_{Uk}^{t+\Delta t} - T_{Uk}^t), \ k \in S_{2c}. \]
The value of the transverse force on the contact \( k \) is corrected by the same quantity: 
\[ T_k^t = T_k^t + \Delta \hat{T}_{Uk} \] [22].

3. Results and discussion

Let us demonstrate the represented approach using the example of calculation of a plane framed system, which, for example, can simulate a pipeline section with a difference of relief [7] under dynamic loading (figure 2 a). It is considered that the system is fixed from lateral displacements (i.e. from the \( xy \) plane). On the supports \( k = 1, 2 \) the conditions of Coulomb friction-sliding with the possibility of separation on the contact operate. The structure is in the state of rest, then a horizontal impulse load \( P(t) \) is applied at the left end. The law of change of the pulse has a triangular shape with the duration of 0.1 s, the amplitude of 100 kN (figure 2 b).

The longitudinal stiffness of the rods \( EA = 9 \cdot 10^6 \) kN, bending stiffness \( EI = 2 \cdot 10^6 \) kN·m², the linear mass \( m = 0.4 \) t/m. The damping matrix of the system here was calculated as \( C = \alpha \cdot M \) (the coefficient \( \alpha \) was taken to be 0.1 s\(^{-1} \)). Contact interaction was modelled with frame-rod CEF [19, 20], connecting the support nodes of the framed system with fixed supports.

![Figure 2. Framed system under the action of pulsed load \( P(t) \).](image)

At the initial time point \( t = 0 \) on all supports the clutch state was set. The normal forces of interaction \( N_k^0 \) on the contact of the structure nodes with the corresponding supports were taken to be equal to the reactions from the own weight of the structure. In the future, as a result of the action of the dynamic load, slippage, and the subsequent clutch on the contact is possible, as well as the switching off (separation) and the switching on of unilateral constraints during the considered time period. At the same time, due to the geometry of the framed system, the normal forces of interaction \( N_k^0 \) also change over time.

Numerical calculations were performed using the computer program [19] having been developed by the authors. The purpose of the calculations was to estimate the effect of the friction coefficient (the value \( f \) in the calculations varied from 0 to 1.5) on the behavior of the framed structure under dynamic loading. Figures 3 and 4 shows the horizontal and vertical displacements of the frame on support 1 depending on the time at different values of the coefficient \( f \) (integration step here was taken 0.0002 s).

As can be seen from the graphs, with an increase in the coefficient \( f \), the separation of the structure from the supports as a result of the dynamic loading decreases significantly. Moreover, in the example considered here, there is a certain threshold value of the coefficient \( f \) (0.6–0.7), in which the effect of contact friction on the value of maximum separation is extreme.

In order to study the dependence of the solution on the value of the integration step over time, the behavior of the framed structure for different values of \( \Delta t \) in the range from 0.0001 to 0.0064 s (with successive doubling of the step length) was calculated. The comparison of the obtained results allows us to conclude that the proposed numerical approach shows satisfactory internal convergence in a rather wide range of integration steps on time. Thus, the values of horizontal and vertical displacements on the supports do not differ much when assigning the basic step in the range from 0.0001 to 0.0008 s. With a further increase in \( \Delta t \), there is some deterioration in the convergence, especially at large values of the friction coefficient.
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Note that, in the general case, the choice of the optimal integration step is a rather complex problem [21,23]. In this regard, basing on conducted numerical studies, it can be recommended to choose the value of the basic integration step in such a way that when it is increased (for example, twice), the change in the results does not exceed some specified error of calculation.

4. Conclusion
The finite element model of contact interaction of constructions and structures has been proposed, taking into account the various types of loading and contact conditions. For the modeling of unilateral

![Figure 3. Horizontal displacements (slippage) on support 1 depending on time.](image1)

![Figure 4. Vertical displacements (separation) on support 1 depending on time.](image2)
constraints with Coulomb friction, the contact finite elements of the frame-rod type has been used, that allows you to calculate forces and displacement in the contact zone with the same accuracy, to apply an inconsistent finite element grids, to take into account the physical properties of contact layer. The statements of static and dynamic contact problems for the interacting elastic bodies is given.

The numerical algorithms developed on the basis of the proposed discrete contact model and of the stepping method provide the possibility of step-by-step analysis of the contact interaction and have advantages in cases where the solution of the problem depends on the loading history, namely, when taking into account friction-sliding on the contact, the sequence of erection of the structure, dynamic action of loading and etc. It is not difficult to extend the presented models and methods of their calculation to solving the problems with unilateral constraints taking into account the physical and strength properties of the contact layer, as well as other complicating factors [4,10,16,19,20].

The results of the calculations allow us to conclude about the efficiency and reliability of the proposed algorithm, taking into account the complicated contact conditions and dynamic loading, which is essential for solving applied problems of structural mechanics. In conclusion, based on the example considered here, we note that taking into account the contact friction forces contributes to the approximation of the calculation scheme to the real picture of the interaction and, thus, allows to obtain more accurate and complete information about the stress-strain state of the structure, and, consequently, about its strength and reliability.

References

[1] Laursen T A 2002 Computational Contact and Impact Mechanics (Berlin: Springer) p 454
[2] Ajzikovich S M et al 2003 Mechanics of contact interaction (Moscow: Fizmatlit) p 672
[3] Tolstikov V V 2006 Vestnik MG SU 2 123–32
[4] Lukashevich A A 2009 Bulletin of civil engineers 3 18–23
[5] Kravchuk A S 2009 Applied mathematics and mechanics 3 492–502
[6] Bukhartsev V N and Lukashevich A A 2012 Power technology and engineering 45(5) 346–50
[7] Yavarov A V, Kolosova G S and Kuroyedov V V 2013 Construction of Unique Buildings and Structures 1 1–6
[8] Bukhartsev V N and Vu Man Khuan 2013 Magazine of civil engineering 1 57–64
[9] Lukashevich A A and Rozin L A 2014 Applied mechanics and materials. Advances in civil and industrial engineering IV 583 2932–35
[10] Lukashevich A A, Lukashevich N K and Timohina E I 2018 IOP Conf. Ser.: Mater. Sci. Eng. 463 042054
[11] Puso M A and Laursen T A 2004 Computer Methods in Applied Mechanics and Engineering 193 601–29
[12] Wriggers P, Schroder J and Schwarz A 2013 Computational Mechanics 52 837–47
[13] Galanin M P, Lukin V V, Rodin A S and Stankevich I V 2015 Computational Mathematics and Mathematical Physics 55(8) 1393–406
[14] Sofonea M and Souleiman Y 2017 Mathematics and Mechanics of Solids 22 324–42
[15] Lukashevich A A 2008 Nauchno-tekhnicheskiye vedomosti SPbGPU 4 233–37
[16] Bukhartsev V N and Lukashevich A A 2010 Gidrotekhnicheskiye stroitelstvo 4 52–55
[17] Wriggers P, Rust W T and Reddy B D 2016 Computational Mechanics 58 1039–50
[18] Ignatyev A V, Ignatyev V A and Gamzatova E A 2018 Izvestiya vuzov. Stroitelstvo 8 5–14
[19] Lukashevich A A 2007 Bulletin of Pacific National University 1 69–82
[20] Lukashevich A A 2018 Magazine of civil engineering 5 149–59
[21] Rozin L A and Lukashevich A A 2009 Nauchno-tekhnicheskiye vedomosti SPbGPU 3 195–99
[22] Rozin L A and Lukashevich A A 2010 Nauchno-tekhnicheskiye vedomosti SPbGPU 4 288–94
[23] Bathe K-J and Wilson E L 1976 Numerical methods in finite element analysis (Englewood Cliffs: Prentice Hall) p 544