The Geometric Gauss-Dedekind

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Abstract

Gauss and Dedekind have shown a bijection between the set of $\text{SL}_2(\mathbb{Z})$-equivalence classes of primitive positive definite binary quadratic $\mathbb{Z}$-forms of the discriminant of $\mathbb{Q}(\sqrt{\Delta} < 0)$ and the class group of its ring of integers. Using étale cohomology we show an analogue of this correspondence in the positive characteristic. This leads to the description of the set of genera and to another result analogous to Gauss’ one by which any form composed with itself belongs to the principal genus.

1 Introduction

Let $(M, q)$ be a binary quadratic module defined over a unital ring $R$, namely, a projective $R$-module $M$ of rank 2 equipped with a map $q : M \to R$, such that $q(cx) = c^2q(x)$ for all $x \in M, c \in R$ and such that the induced symmetric map

$$B_q : M \times M \to R : (x, y) \mapsto q(x+y) - q(x) - q(y)$$

is $R$-bilinear ([Con3 §2]). Fix a basis $\{e_1, e_2\}$ for $M$ over $R$. Then $(M, q)$ is primitive if $R \cdot q(M) = R$ ([Knus III p.164]), or, equivalently, if there exist co-prime elements $a, b, c \in R$ such that:

$$\tilde{q}(X, Y) := q(Xe_1 + Ye_2) = aX^2 + bXY + cY^2.$$  

For brevity we denote $q = (a, b, c)$. A change of variables by $h \in \text{GL}(M)$ gives rise to an isomorphic form $q' = q \circ h$. This $h$ is called proper if $\det(h) = 1$. The discriminant of $q$, up to isomorphism, is defined as the coset ([Con3 §2.1])

$$\text{disc}(q) := \det(B_q) \cdot (R^\times)^2 \in R/(R^\times)^2,$$

where $\det(B_q)$ stands for $\det(B_q(e_i, e_j))$. Given $\Delta \in R/(R^\times)^2$, a natural and classical problem is the classification of all primitive binary quadratic $R$-forms sharing the same discriminant, up to proper $R$-isomorphisms:

$$\text{cl}_0(\Delta) := \{q : \text{disc}(q) = \Delta\}/\text{SL}_2(R).$$  \hspace{1cm} (1.1)

This classification extends – when $\text{Pic}(R)$ is not trivial – to $\text{cl}_1(\Delta)$, including quadratic maps that get values in any invertible $R$-module.
Gauss, in his famous *Disquisitiones Arithmeticae* [Gau], defined the composition law of two binary quadratic forms defined over $R = \mathbb{Z}$. Later, Dedekind identified the obtained group structure with the one of an abelian group; if $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field with ring of integers $\mathcal{O}_K$ and discriminant $\Delta_K$ (being just an integer in this case as $(\mathbb{Z}^\times)^2 = 1$), called then a *fundamental discriminant*, then there is a bijection of pointed sets ([FT Thm. 58]):

\[
\text{cl}_1'(\Delta_K) \xrightarrow{\sim} \text{Pic}^+(\mathcal{O}_K) : \left[ \left[ a, b - \frac{\sqrt{\Delta_K}}{2} \right] \right] \mapsto \left[ \left[ a, \frac{b/2 + \sqrt{\alpha}}{a} \right] \right],
\]

where $\text{cl}_1'(\Delta_K)$ is the restriction of $\text{cl}_1(\Delta_K)$ to *positive definite* forms when $d < 0$, and equals $\text{cl}_1(\Delta_K)$ when $d > 0$. The term $\text{Pic}^+(\Delta_K)$ stands for the narrow Picard group of $\mathcal{O}_K$, being equal to $\text{Pic}(\mathcal{O}_K)$ when $K$ is imaginary. So if $K = \mathbb{Q}(\sqrt{d})$ with $d < 0$ with ring of integers $\mathcal{O}_K$, the above bijection is $\text{cl}_1'(\Delta_K) \xrightarrow{\sim} \text{Pic}(\mathcal{O}_K)$.

M. Kneser has shown in ([Kne p. 112, Prop. 2]) (1982), that given a primitive binary quadratic form $(M, q)$ over any base $R$, there exists an exact sequence of abelian groups:

\[
1 \to R^\times / n(M^\times) \to \text{cl}_1(\text{disc}(q)) \to \text{Pic}(M) \to 1,
\]

(1.2)

where $n$ is the norm map associated to $M$. M. M. Wood in [Wood] (2011) has proved, using actions on the symmetric space of a quadratic module, that the Dedekind correspondence holds over $R$. M. A. Knus (1991) has applied étale cohomology in the nondegenerate case ([Knus IV §5]). In this paper, due to a recent result by [APS] regarding the smoothness of the special orthogonal group $\mathbf{SO}_q$, when $\text{det}(B_q)$ is nonsquare, we generalize the cohomological approach to such forms (though maybe degenerate), thus being primitive, which leads to a *constructive* bijection – like the one of Dedekind – in the geometric case. This enables us to divide the classes into genera and to deduce another result analogous to a Gauss’ one for the principal genus.

In more details, assume 2 is invertible in $R$. In Section 2 given a binary quadratic $R$-form $q$ with a nonsquare $\text{det}(B_q)$, thus being primitive, we form a bijection of pointed sets $\text{cl}_0(\text{disc}(q)) \cong H^1_{et}(R, \mathbf{SO}_q)$. This extends to an exact sequence similar to the one of Kneser. In Section 3 $R = \mathcal{O}$ is a Dedekind domain in a global function field $k$ of odd characteristic. After providing a geometric interpretation to the notion of a *positive form* with respect to a given discriminant, we formulate the Dedekind correspondence for the restriction $\text{cl}_1'(\Delta_K)$ to such positive forms, sharing the discriminant $\Delta_K$ of a quadratic imaginary extension $K/k$ (Theorem 3.2 below):

**Theorem 1.1.** Let $K/k$ be a quadratic extension of imaginary fields with discriminant $\Delta_K$. Then $\mathcal{O}_K = \mathcal{O}[\sqrt{\alpha}]$ for some $\alpha \in \mathcal{O}$ and there is an isomorphism of abelian groups:

\[
\bar{i}_* : \text{cl}_1'(\Delta_K) \xrightarrow{\sim} \text{Pic}(\mathcal{O}_K) : \left[ q_L = \left( a, b, \frac{b^2/4 - \alpha}{a} \right) \right] \mapsto \left[ L = \left( a, b/2 + \sqrt{\alpha} \right) \right].
\]

A coarser classification, up to also *improper* $\mathcal{O}$-isomorphisms, is described in Section
by: \( \text{cl}(\Delta) \cong \text{cl}_1(\Delta)/(\{q \sim [q^{op}]\}) \), where for \( q = (a, b, c) \), \( q^{op} = (a, -b, c) \). For any prime \( p \) of \( k \), let \( \hat{k}_p \) be the completion of \( k \) with respect to the induced discrete valuation \( \nu_p \) on it and \( \hat{O}_p \) its ring of integers. The principal genus \( \text{Cl}_\infty(q) \) of \( q \) is the set of classes of \( O \)-forms that are generically and \( \hat{O}_p \)-isomorphic to \( q \) for any \( p \). In Section 5 after dividing the classes in \( \text{cl}_1(\text{disc}(q)) \) into genera, we prove another result analogous to a one of Gauss in characteristic 0: denote by \( \ast \) the operation in the group \( \text{cl}_1(\text{disc}(q)) \). Then (Corollary 5.3 below):

\[ \text{Corollary 1.2.} \quad \text{For any} \quad q' \in \text{cl}_1(\text{disc}(q)), \quad [q' \ast q'] \in \text{Cl}_\infty(q). \]

In Section 6 an application towards elliptic curves is demonstrated.

## 2 Torsors of norm forms

Let \( R \) be a ring in which 2 is invertible and \((M, q)\) a binary quadratic \( R \)-module. Schemes defined over \( \text{Spec} \, R \) are underlined, omitting the underline for the generic fiber.

**Definition 1.** The orthogonal group of \((M, q)\) is the affine \( R \)-group of its self isometries

\[ O_q := \{ h \in \text{GL}(M) : q \circ h = q \} \]

([Con2, §2,p. 6]). As 2 is invertible in \( R \), the special orthogonal subgroup \( \text{SO}_q \) is \( \ker[O_q \xrightarrow{\text{det}} \mathbb{G}_m] \) ([Con1, §1,p. 1]). Since \( \text{det} \) factors through \( \mu_2 = \text{Spec} \, R[t]/(t^2 - 1) \) ([CF, Lemme 4.3.0.21]), we may just write

\[ \text{SO}_q := \ker[O_q \xrightarrow{\text{det}} \mu_2]. \quad (2.1) \]

An \( R \)-algebra \( A \) is quadratic if it is a projective \( R \)-module of rank 2. It admits a standard involution \( \sigma \), thus a norm map \( n_A : A \to R : x \mapsto x\sigma(x) \). A quadratic \( R \)-module \((M, q)\) is said to be of type \( A \), if \( M \) is an \( A \)-module of rank 1 and \( q \) is the associated norm form, i.e., such that \( q(xa) = q(x)n_A(a) \) for all \( x \in M, a \in A \) ([Knus, III p.164]). If \( A \) has a basis over \( R \), then a representation \( \varphi : \text{Aut}(A) \to \text{GL}_2 : h \mapsto H \), is an isomorphism of \( R \)-group schemes. In particular any \( a \in A \) induces an automorphism \( h_a \) via the multiplication by \( a \). This yields a canonical norm for \( A \):

\[ n_A(a) := \text{det}(\varphi(h_a)) \quad ([\text{Bie}, \text{Def}. 2.3]). \]

Let \( \Omega := \{1, \sqrt{\alpha}\} \) where \( \alpha \neq 0 \) is a nonsquare element of \( R \). Then \( A_\alpha := R(\Omega) = R \oplus \sqrt{\alpha}R \) is a quadratic \( R \)-algebra that is closed under multiplication and contains \( R \), hence carries a ring structure. The Weil restriction of scalars \( R := \text{Res}_{A_\alpha/R}(\mathbb{G}_m) \), is a 2-dimensional \( R \)-group whose generic fiber is a \( k \)-torus. The group of points \( R(R) \), via its isomorphism with \( A_\alpha^* \) ([BLR, §7.6], naturally acts on \( A_\alpha \) through its basis \( \Omega \), yielding a canonical embedding of \( R \) in \( \text{Aut}(A_\alpha) \). Then the above norm \( n_\alpha(h) := \text{det}(\varphi(h)) \) fits
into the commutative diagram:

\[
\begin{array}{ccc}
R & \longrightarrow & \operatorname{Aut}(A_\alpha) \\
& & \downarrow{}^{\varphi}
\end{array}
\]

\[
\begin{array}{ccc}
& & GL_2 \\
& & \downarrow{}^{\text{det}}
\end{array}
\]

\[
\begin{array}{ccc}
R & \longrightarrow & \mathbb{G}_m \\
& & \downarrow{}^{n_\alpha}
\end{array}
\]

Consider the \(R\)-group \(N := \ker[R \overset{n_\alpha}{\longrightarrow} \mathbb{G}_m]\). Its generic fiber is a one-dimensional \(k\)-torus \(N\). At any prime \(p\) the map applied to the reductions \(R_p \overset{(n_\alpha)_p}{\longrightarrow} (\mathbb{G}_m)_p\) cannot be trivial as \(R_p\) is two-dimensional, thus the local norm \((n_\alpha)_p\) is surjective, hence \(n_\alpha\) as well. This surjectivity holds even at a ramified prime \(p\) of \(\det(B_q)\) in which \(R_p = \operatorname{Spec} \hat{O}_p[x, y, t]/(t(x^2 - \alpha y^2) - 1)\) is not reductive.

**Lemma 2.1.** The scheme \(N\) is flat over \(\operatorname{Spec} R\).

**Proof.** The schemes in \(R \overset{n_\alpha}{\longrightarrow} \mathbb{G}_m\) are smooth (e.g., \([\text{CGP1, Cor. A.5.4}]\)), hence regular and Cohen-Macaulay, thus it suffices to check that all the geometric fibers of \(N = \ker(n_\alpha)\) are one-dimensional. This is guaranteed by the surjectivity of \(n_\alpha\). \(\square\)

The quadratic module \((A_\alpha, q_\alpha)\) where

\[
q_\alpha(X, Y) = q_\alpha(X + \sqrt{\alpha}Y) := n_\alpha(X + \sqrt{\alpha}Y) = X^2 - \alpha Y^2
\]

i.e., \(q_\alpha = (1, 0, -\alpha)\), being primitive, is up to isomorphism the unique quadratic module associated to \(A_\alpha\) (\([\operatorname{Knus, Prop. 7.3.1}]\)). From now on, just \(n\) and \(q\) will stand for \(n_\alpha\) and \(q_\alpha\), respectively.

**Remark 2.2.** As \(\det(B_q)\) is squarefree and 2 is invertible, according to \([\text{APS, Prop. 2.3.}]\) \(O_q\) and \(SO_q\) are smooth (thus flat). Moreover, by the correspondence between flat closed subschemes of \(O_q\) and closed subschemes of the generic fiber \(O_q\) \([\operatorname{EGAIV, Prop. 2.8.1}]\), \(SO_q\) is the unique flat and closed subgroup of \(O_q\) whose generic fiber is \(SO_q\).

**Lemma 2.3.** \(SO_q = N\).

**Proof.** Recall that \(N \subset \operatorname{SL}(A_\alpha)\). Then:

\[
SO_q = \{a \in \operatorname{SL}(A_\alpha) : q \circ a = q\}
\]

\[
\supset \{a \in N : q \circ a = q\}
\]

\[
= \{a \in N : q(xa) = q(x) \cdot n(a) = q(x) \forall x \in A_\alpha\}
\]

\[
= \{a \in N : n(a) = 1\} = N.
\]

Both \(SO_q\) and \(N\) are \(O\)-flat (see Remark \(\square\) and Lemma \(\square\)) closed subgroups of \(O_q\) with the same generic fiber \(SO_q = N\). Such an object is unique by Remark \(\square\) implying that \(SO_q = N\). \(\square\)
Given an affine $R$-group scheme $G$, a $G$-torsor in the étale topology is a sheaf of sets on $R$ equipped with a (right) $G$-action, which is locally trivial in the étale topology, namely, locally for the étale topology on $R$ this action is isomorphic to the action of $G$ on itself by translation. The pointed set $H^1_{\text{ét}}(R, G)$ classifies these $G$-torsors up to $R$-isomorphisms. The following correspondence is due to Giraud (see [CE] §2.2.4):

**Proposition 2.4.** Let $S$ be a scheme and $X_0$ an object of a fibered category of schemes defined over $S$. Let $\text{Aut}(X_0)$ be its $S$-group of automorphisms. Let $\mathcal{F}$\textit{orms}$(X_0)$ be the category of $S$-forms that are locally isomorphic for some topology to $X_0$, and let $\mathcal{T}$\textit{ors}(\text{Aut}(X_0)) be the category of $\text{Aut}(X_0)$-torsors in that topology.

$$\varphi : \mathcal{F}$\textit{orms}$(X_0) \rightarrow \mathcal{T}$\textit{ors}(\text{Aut}(X_0)) : X \mapsto \text{Iso}(X_0, X)$$

is an equivalence of fibered categories.

In particular the category of torsors of $\mathcal{O}_q = \text{Aut}(q)$ in the étale topology is equivalent to the one of $R$-schemes of the form $\text{Iso}(q, q')$ where $q'$ is a quadratic $R$-form isomorphic to $q$ in that topology. The associated discriminant algebra is $D(q') = (\cap^2 M', \det(q'))$ and its isomorphism class in $H^1_{\text{ét}}(R, \mu_2)$ is its Arf invariant. Being diagonal, $q = (1, 0, -\alpha)$ admits the improper isometry $\text{diag}(1, -1)$ defined over $R$, so $\mathcal{O}_q(R) \xrightarrow{\det} \mu_2(R)$ is surjective. Then étale cohomology applied to the exact sequence of smooth $R$-groups:

$$1 \rightarrow \text{SO}_q \rightarrow \mathcal{O}_q \xrightarrow{\det} \mu_2 \rightarrow 1,$$

(2.2)

provides rise to the exact sequence of pointed-sets:

$$1 \rightarrow H^1_{\text{ét}}(R, \text{SO}_q) \rightarrow H^1_{\text{ét}}(R, \mathcal{O}_q) \xrightarrow{\det} H^1_{\text{ét}}(R, \mu_2),$$

(2.3)

where $\det_*([\text{Iso}(q, q')]) = [D(q')] - [D(q)]$ in $H^1_{\text{ét}}(R, \mu_2)$ (preserving the base point, [Gir IV, Prop.4.3.4]). By the exactness of (2.3), $\text{Iso}(q, q')$ represents a class in $H^1_{\text{ét}}(R, \mathcal{O}_q) = \ker(\text{det}_*)$ if and only if $q'$ is isomorphic to $q$ in the étale topology and shares the Arf invariant of $q$. The class $[D(q')]$ is trivial if and only if $q'$ is isotropic ([Knus V, Prop. 2.1.7]).

**Example 2.5.** The form $q = (1, 0, 1)$ defined over $R = \mathcal{F}_5[x]$ is isotropic, unlike $q' = (2, 0, 1)$, thus $q'$ cannot represent a class in $H^1_{\text{ét}}(R, \text{SO}_q)$. Fixing $a$ such that $a^2 = 2$, $H = \text{diag}(a, 1)$ yields an isomorphism $q \rightarrow q'$ defined over a flat cover of $R$, thus $[q'] \in H^1_{\text{ét}}(R, \mathcal{O}_q)$. The form $q'' = (4, 0, 1)$, however, is $R$-isomorphic by $H = \text{diag}(2, 1)$ thus isotropic as $q$, hence represents a class in $H^1_{\text{ét}}(R, \text{SO}_q)$, but not the one of $q$ as such $H$ cannot be proper (see more details in Lemma 4.1).

**Lemma 2.6.** The map $\varphi : H^1_{\text{ét}}(R, \text{SO}_q) \cong \mathcal{C}_0(\text{disc}(q)) : [\text{Iso}(q, q')] \mapsto [q']$ is a bijection of pointed sets having a structure of an abelian group.
Proof. A representative in $H^1_{\mathrm{et}}(R, \SO_q)$ is of the form $\Iso(q, q')$, where $q' : M' \to R$ is a norm, being isomorphic to $q$ in the étale topology such that $[D(q')] = [D(q)]$ in $H^1_{\mathrm{et}}(R, \mu_q)$. The latter condition is equivalent to $\det(B_{q'}) = a^2 \det(B_q)$ for some $a \in \mathbb{F}^\times$ ([Kma III, §3.3]), i.e., to $\disc(q') = \disc(q)$. Identifying these sets of representatives modulo proper isomorphisms by $\varphi$, $\chi_0(\disc(q))$ inherits the group structure of $H^1_{\mathrm{et}}(R, \SO_q)$ ($\SO_q = \mathbb{N}$ by Lemma 2.3) is commutative. \[\square\]

Given a set $\Omega' = \{\omega_1, \omega_2\}$, set the vector $\overline{\Omega}' = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ and notice that $q' = q \circ H$ can be written as $q'(X, Y) = \tilde{q}((X, Y) \cdot H)$, or, equivalently, as

$$q'(X, Y) \cdot \overline{\Omega} = q((X, Y) \cdot \overline{\Omega})$$

where $\overline{\Omega} = H \cdot \overline{\Omega}$. We denote $\Omega'$ shortly by $H\Omega$.

**Lemma 2.7.** The map $\tilde{i}_*([q \circ h]) = [R(H\Omega)]$ forms an exact sequence of abelian groups:

$$1 \to \mathrm{cok}(\eta(A_\alpha^\times)) \to \chi_1(-\alpha(R^\times)^2) \xrightarrow{\tilde{i}_*} \Pic(A_\alpha) \to 1.$$  (2.4)

**Proof.** Applying étale cohomology to the short exact sequence of smooth $R$-groups:

$$1 \to \mathbb{N} \xrightarrow{\iota_1} \mathbb{R} \xrightarrow{\eta} \mathbb{G}_m \to 1$$  (2.5)

being commutative, gives rise to the short exact sequence of abelian groups

$$1 \to \mathbb{R}^\times / \eta(A_\alpha^\times) \xrightarrow{\delta} H^1_{\mathrm{et}}(R, \mathbb{N}) \xrightarrow{i_*} H^1_{\mathrm{et}}(R, \mathbb{R}) \xrightarrow{n_*} H^1_{\mathrm{et}}(R, \mathbb{G}_m) \cong \Pic(R),$$  (2.6)

in which $H^1_{\mathrm{et}}(R, \mathbb{R})$ is isomorphic by the Shapiro’s Lemma to $H^1_{\mathrm{et}}(A_\alpha, \mathbb{G}_m)$, whose representatives are torsors of $\mathbb{G}_m = \text{Aut}(A_\alpha)$ defined over $\text{Spec} A_\alpha$, namely, of the form $\Iso(A_\alpha, L)$, $L$ is an invertible $A_\alpha$-module. Given $q \circ h$, $h$ is proper defined over an étale cover $S/R$, $[\Iso(q, q \circ h)]$ in $H^1_{\mathrm{et}}(R, \SO_q = \mathbb{N})$ (Lemma 2.3) is mapped by $i_*$ to $[\Iso(A_\alpha, R(H\Omega))]$ in $H^1_{\mathrm{et}}(A_\alpha, \mathbb{G}_m)$ ([Gir V, 3.1.1.1]). Consider the composition:

$$\tilde{i}_* : \chi_0(-\alpha(R^\times)^2) \xrightarrow{\varphi^{-1}} H^1_{\mathrm{et}}(R, \mathbb{N}) \xrightarrow{\iota_*} H^1_{\mathrm{et}}(R, \mathbb{R}) \cong \Pic(A_\alpha) : \{q \circ h\} \mapsto [R(H\Omega)]$$

where $\varphi$ is the bijection in Lemma 2.6. This $\tilde{i}_*$ should not be surjective when $\Pic(R)$ is not trivial. Extending, however, $\chi_0(-\alpha(R^\times)^2)$ to $\chi_1(-\alpha(R^\times)^2)$ (see Section 1), including quadratic maps that get values in any invertible $A_\alpha$-line bundle $L$, since any representative $L$ in $\Pic(A_\alpha)$ can be considered as $R \cdot q'(M')$ for a proper isometry $q'$ of $q$, $\chi_1(\disc(q)) \xrightarrow{\tilde{i}_*} \Pic(A_\alpha)$ is surjective. The added fibers of $\tilde{i}_*$ have a similar kernel. The assertion follows. \[\square\]
3 Dedekind correspondence: the geometric case

Let $C$ be a projective, smooth and geometrically connected curve defined over a finite field $\mathbb{F}$ of odd characteristic. For any closed point $p$ on $C$ let $v_p$ be the induced discrete valuation on the function field $k = \mathbb{F}(C)$ and $\hat{O}_p$ the ring of integers in the completion $\hat{k}_p$ of $k$ with respect to $v_p$. Throughout, $k$ being an extension of the rational function field $\mathbb{F}(x)$ is imaginary, namely, following Artin in [Art], the prime $\infty = (1/x)$ in $\mathbb{F}(x)$ does not split into distinct places in $k$. Let $\infty_k$ be the unique prime of $k$ lying above $\infty$, regarded as a closed point on $C$. Then the ring of regular functions on the affine curve $C^\text{af} := C - \{\infty_k\}$ is a Dedekind domain:

$$\mathcal{O} := \mathbb{F}[C^\text{af}] = \{x \in k : v_p(x) \geq 0 \ \forall p \neq \infty_k\}.$$

**Definition 2.** A binary quadratic $\mathcal{O}$-form $q$ is said to be positive with respect to $\alpha \in \mathcal{O}$ if $B_q = H^t \text{diag}(1, -\alpha)H$ for some $H \in \text{GL}_2(S)$ where $S/\mathcal{O}$ is finite étale, such that $\det(H) \in (\mathbb{F}^\times)^2$. We then set

$$\mathcal{d}_i^\prime(\alpha(\mathbb{F}^\times)^2) := \{q \text{ is positive w.r.t. } \alpha\}/\text{SL}_2(\mathcal{O})$$

(this does not depend on the choice of the representative in $\alpha(\mathbb{F}^\times)^2$).

**Example 3.1.** The form $q' = (4, 0, 1)$ defined over $\mathcal{O} = \mathbb{F}_5[x]$ represents a class in $H^1_\text{ét}(\mathcal{O}, SO_{q=(1,0,1)})$ (Example [2.3]), which coincides with $\mathcal{d}_1((-\alpha = 1) \cdot (\mathbb{F}^\times)^2)$ (Lemma [2.6]). But $q'$ is not positive with respect to $\alpha = -1$, as no isomorphism $h : q \rightarrow q'$ can be such that $\det(H) \in (\mathbb{F}_5^\times)^2$.

This leads to the geometric analogue of the Dedekind correspondence:

**Theorem 3.2.** Let $K/k$ be a quadratic extension of imaginary fields with discriminant $\Delta_K$. Then $\mathcal{O}_K = \mathcal{O}[\sqrt{\alpha}]$ for some $\alpha \in \mathcal{O}$ and there is an isomorphism of abelian groups:

$$\tilde{i}_* : \mathcal{d}_1^\prime(\Delta_K) \xrightarrow{\sim} \text{Pic}(\mathcal{O}_K) : \left[q_L = \left(a, b, \frac{b^2 + 4 - \alpha}{a}\right)\right] \mapsto [L = \langle a, b/2 + \sqrt{\alpha} \rangle].$$

**Proof.** As a (maximal) order over $\mathcal{O}$, $\mathcal{O}_K$ is a free $\mathcal{O}$-module of rank 2, thus admits a basis $\{1, t\}$ over $\mathcal{O}$, where $t$ is an algebraic integer thus a root of a monic quadratic polynomial over $\mathcal{O}$, i.e., there exist $m, n \in \mathcal{O}$ such that $t^2 + mt + n = 0$. Taking $t$ to be the root $\frac{\sqrt{m^2 - 4} - m}{2}$, we see that $\mathcal{O}_K = A_n$ for $\alpha = m^2 - 4n$ (as 2 is invertible). Consider the exact sequence [2.4] for $R = \mathcal{O}$:

$$1 \rightarrow \mathcal{O}^\times/n(A_n^\times) \xrightarrow{\delta} \mathcal{d}_1(\Delta_K) \xrightarrow{\tilde{i}_*} \text{Pic}(A_n) \rightarrow 1.$$

We show that $\mathcal{d}_1(\Delta_K)/\text{Im}(\delta) \cong \mathcal{d}_1^\prime(\mathcal{O}_K)$: As $K$ is imaginary, $\text{cok}(n) \cong \{\pm 1\}$ ([Mor, Example 1]). Notice that $\lambda \in \mathcal{O}^\times$ is not $n(a + \sqrt{\alpha}b) = a^2 - \alpha b^2$ for some $a, b \in \mathcal{O}$ if and only if $\lambda$ is not a square. Each $\lambda \in \mathcal{O}^\times$ is mapped to $[\lambda q] \in \mathcal{d}_1(\Delta_K)$ reached by
the isometry $h = \sqrt{\lambda_1}$:

$$\sqrt{\lambda_1} \cdot \text{diag}(1, -\alpha) \cdot \sqrt{\lambda_1} = \text{diag}(\lambda, -\lambda \alpha).$$

So given $[\lambda q] \in \text{Im}(\delta)$:

$$[q] \neq [\lambda q] \iff \lambda \in \mathbb{F}^\times - (\mathbb{F}^\times)^2$$

therefore $\text{cl}_1(\Delta_K)/\text{Im}(\delta) \cong \text{cl}_1'(\Delta_K)$.

Explicitly, starting by a most general form $q_L = (a, b, b^2/4 - \alpha a)$ of the nonsquare discriminant $\Delta_K = -\alpha(\mathbb{F}^\times)^2$, we have $q_L = q \circ h_L$ where $H_L = \frac{1}{\sqrt{a}} \left( \begin{array}{cc} a & 0 \\ b/2 & 1 \end{array} \right)$. As

$$\det(H_L) = 1, \ [q_L] \in \text{cl}_1'(\Delta_K).$$

By Lemma 2.7:

$$\tilde{i}_*([q_L]) = [\mathcal{O}(H_L \Omega = \frac{1}{\sqrt{a}} \{a, b/2 + \sqrt{\alpha}\})] = [(a, b/2 + \sqrt{\alpha})]$$

in Pic($\mathcal{O}_K$) (the two ideals differ by tensoring with a principal one).

**Remark 3.3.** Given a primitive $\mathcal{O}$-form $q = (a, b, c)$, we call $q^{op} = (a, -b, c)$ its opposite form. When $\mathcal{O}_K = \mathcal{O}(\Omega)$, the tensor product in Pic($\mathcal{O}_K$) induces a group operation in $\text{cl}_1'(\text{disc}(q))$:

$$[q_L_1] \star [q_L_2] = \tilde{i}^{-1}_*([L_1 \otimes L_2]).$$

In particular, $[q]^{-1} = [q^{op}]$. Indeed, let $L^{op}$ be the ideal corresponding to $q^{op}$. Then:

$$I = L \otimes L^{op} = \langle a, \sqrt{a} + b/2 \rangle \otimes \langle a, \sqrt{\alpha} - b/2 \rangle = \langle a^2, a(\sqrt{\alpha} + b/2), a(\sqrt{\alpha} - b/2), b^2/4 - \alpha \rangle.$$

But $b^2/4 - \alpha = ac$, thus $I \subseteq \langle a \rangle$. On the other hand, both $L$ and $L^{op}$ are primitive, thus $\langle a \rangle \subseteq I$, whence $I = \langle a \rangle$ is principal.

4 Not necessarily proper classification

In this section we would like to study a less narrow classification, namely, the one of binary quadratic $\mathcal{O}$-forms of the same discriminant $\Delta$ up to proper and improper $\mathcal{O}$-isomorphisms:

$$\text{cl}(\Delta) := \{ q : \text{disc}(q) = \Delta \}/\text{GL}_2(\mathcal{O}).$$

Given a smooth $\mathcal{O}$-group $\mathcal{G}$ and a representative $P$ in $H^1_{\text{et}}(\mathcal{O}, \mathcal{G})$, the quotient of $P \times \mathcal{G}$ by the $\mathcal{G}$-action, $(p, g) \mapsto (ps^{-1}, sgs^{-1})$, is an affine $\mathcal{O}$-group scheme $P\mathcal{G}$, being an inner form of $\mathcal{G}$, called the twist of $\mathcal{G}$ by $P$ (e.g., [Sko, §2.2]).

**Lemma 4.1.** $\text{cl}(\Delta) \cong \text{cl}_1(\Delta)/([q] \sim [q^{op}]).$
Proof. Since $B_q = \text{diag}(1, -\alpha)$, the short exact sequence of smooth $\mathcal{O}$-groups (2.2)

$$1 \to \text{SO}_q \to \mathcal{O}_q \to \mu_2 \to 1$$

(4.1)
splits by the section mapping the non-trivial element in $\mu_2$ to $\text{diag}(1, -1)$ in $\mathcal{O}_q$, i.e., $\mathcal{O}_q$ is isomorphic to $\text{SO}_q \times \mu_2$. Consequently, according to [Gil, Lemma 2.6.3] we get:

$$H^1_{\text{et}}(\mathcal{O}, \mathcal{O}_q) = \bigoplus_{[P] \in H^1_{\text{et}}(\mathcal{O}, \mu_2)} H^1_{\text{et}}(\mathcal{O}, P \text{SO}_q)/\mu_2(\mathcal{O}),$$

(4.2)
in which $\mu_2(\mathcal{O})$ acts on the set of representatives of $H^1_{\text{et}}(\mathcal{O}, P \text{SO}_q)$ by $\text{diag}(1, \pm 1)$. If $P$ is a trivial $\mu_2$-torsor, then $H^1_{\text{et}}(\mathcal{O}, P \text{SO}_q) = \text{SO}_q$ isomorphic to $\mu_2$. Consequently, according to $\mu_2(\mathcal{O})$ acts on the set of representatives of $H^1_{\text{et}}(\mathcal{O}, \mu_2)$ by $\text{diag}(1, -1)$ in $\mu_2$. Consequently, according to [Gil, Lemma 2.6.3] we get:

$$H^1_{\text{et}}(\mathcal{O}, \mathcal{O}_q) = \bigoplus_{[P] \in H^1_{\text{et}}(\mathcal{O}, \mu_2)} H^1_{\text{et}}(\mathcal{O}, P \text{SO}_q)/\mu_2(\mathcal{O}),$$

(4.2)
in which $\mu_2(\mathcal{O})$ acts on the set of representatives of $H^1_{\text{et}}(\mathcal{O}, P \text{SO}_q)$ by $\text{diag}(1, \pm 1)$. If $P$ is a trivial $\mu_2$-torsor, then $H^1_{\text{et}}(\mathcal{O}, P \text{SO}_q) = \text{SO}_q$ isomorphic to $\mu_2$. Consequently, according to [Gil, Lemma 2.6.3] we get:

$$H^1_{\text{et}}(\mathcal{O}, \mathcal{O}_q) = \bigoplus_{[P] \in H^1_{\text{et}}(\mathcal{O}, \mu_2)} H^1_{\text{et}}(\mathcal{O}, P \text{SO}_q)/\mu_2(\mathcal{O}),$$

(4.2)
in which $\mu_2(\mathcal{O})$ acts on the set of representatives of $H^1_{\text{et}}(\mathcal{O}, P \text{SO}_q)$ by $\text{diag}(1, \pm 1)$. If $P$ is a trivial $\mu_2$-torsor, then $H^1_{\text{et}}(\mathcal{O}, P \text{SO}_q) = \text{SO}_q$ isomorphic to $\mu_2$. Consequently, according to [Gil, Lemma 2.6.3] we get:
Let \([\xi_0] := \varphi([G])\). The principal genus of \(G\) is then \(\varphi^{-1}([\xi_0])\), i.e., the set of classes of \(G\)-torsors that are generically and locally trivial at all primes of \(O\). More generally, a genus of \(G\) is any fiber \(\varphi^{-1}([\xi])\) where \([\xi] \in \text{Im}(\varphi)\). The set of genera of \(G\) is then:

\[
\text{gen}(G) := \{\varphi^{-1}([\xi]) : [\xi] \in \text{Im}(\varphi)\},
\]

whence \(H^1_{\text{ét}}(O, G)\) is a disjoint union of its genera. The left exactness of sequence (5.1) reflects the fact that \(\text{Cl}_\infty(G)\) coincides with the principal genus of \(G\). If there is an embedding of groups 

\[
\hat{O}_p, G_p \hookrightarrow H^1(\hat{k}_p, G_p)
\]

at all primes \(p\), then as in [Nis, Cor. I.3.6], the sequence (5.1) simplifies to

\[
1 \rightarrow \text{Cl}_\infty(G) \rightarrow H^1_{\text{ét}}(O, G) \rightarrow H^1(k, G).
\] (5.3)

More precisely, there is an exact sequence of pointed sets (cf. [GP, Corollary A.8])

\[
1 \rightarrow \text{Cl}_\infty(G) \rightarrow H^1_{\text{ét}}(O, G) \rightarrow B \rightarrow 1
\] (5.4)

in which

\[
B = \left\{ [\gamma] \in H^1(k, G) : \forall p \neq \infty, [\gamma \otimes \hat{O}_p] \in \text{Im}\left( H^1_{\text{ét}}(\hat{O}_p, G_p) \rightarrow H^1(\hat{k}_p, G_p) \right) \right\}.
\]

Let \(K/k\) be a finite Galois extension, \(p\) be a prime of \(k\), and \(\mathfrak{p}\) be a prime of \(K\) dividing \(p\). Write \(\hat{k}_p\) and \(\hat{K}_\mathfrak{p}\) for the completions of \(k\) at \(p\) and of \(K\) at \(\mathfrak{p}\), respectively, noting that \(\hat{K}_\mathfrak{p}\) is independent of the choice of \(\mathfrak{p}\), up to isomorphism. The norm map \(\text{Nr} : K \rightarrow k\) extends the above norm \(n : O_K \rightarrow O\) and induces local maps \(\text{Nr} : K \otimes_k \hat{k}_p \rightarrow \hat{k}_p\); under the isomorphism above this corresponds to the product of the norm maps \(N_{\hat{K}_\mathfrak{p}/k_p}\) on the components. Similarly, \(O_K \otimes_O \hat{O}_p \simeq O_{\hat{K}_\mathfrak{p}}\). Write \(U_p\) and \(U_{\mathfrak{p}}\) for \(\hat{O}_p^\times\) and \(O_{\hat{K}_\mathfrak{p}}^\times\), respectively.

Recall our definition of \(N\) and \(R\) (Section 2). Applying étale cohomology to the short exact sequence of smooth \(\hat{O}_p\)-groups 

\[
1 \rightarrow \mathbb{N}_p \rightarrow \mathbb{R}_p \rightarrow (\mathbb{Z}/m)_{\hat{p}} \rightarrow 1
\]

yields the exact and functorial sequence

\[
1 \rightarrow \mathbb{N}_p(\hat{O}_p) \rightarrow \mathbb{R}_p(\hat{O}_p) \cong U_{\mathfrak{p}}^\times \overset{\text{Nr}}{\rightarrow} U_p \rightarrow H^1_{\text{ét}}(\hat{O}_p, \mathbb{N}_p) \rightarrow 1,
\]

since \(H^1_{\text{ét}}(\hat{O}_p, \mathbb{N}_p)\) is the Picard group of a product of local rings and thus vanishes. We deduce an isomorphism \(H^1_{\text{ét}}(\hat{O}_p, \mathbb{N}_p) \cong U_p/\text{Nr}(U_{\mathfrak{p}}^\times) = U_p/N_{\hat{K}_\mathfrak{p}/k_p}(U_{\mathfrak{p}})\). Applying
Galois cohomology to the short exact sequence of \( \hat{k}_p \)-groups

\[ 1 \rightarrow N_p \rightarrow \mathcal{O}_p \rightarrow (\mathbb{G}_m)_p \rightarrow 1 \]

gives rise to the exact sequence of abelian groups

\[ 1 \rightarrow N_p(\hat{k}_p) \rightarrow \mathcal{O}_p(\hat{k}_p) \cong (\hat{K}_p)^r \xrightarrow{N} \hat{k}_p^\times \rightarrow H^1(\hat{k}_p, N_p) \rightarrow 1, \]

where the rightmost term vanishes by Hilbert’s Theorem 90. Hence we may again deduce a functorial isomorphism \( H^1(\hat{k}_p, N_p) \cong \hat{k}_p^\times / N_{K_p/k_p}(\hat{K}_p^\times) \). Note that \( U_p \) is compact and thus \( N_{K_p/k_p}(U_p) \) is closed in \( k_p^\times \). Only units have norms that are units, so we obtain an embedding of groups:

\[ H^1_{\text{ét}}(\hat{O}_p, \hat{N}_p) \cong U_p/N_{K_p/k_p}(U_p) \hookrightarrow \hat{k}_p^\times / N_{K_p/k_p}(\hat{K}_p^\times) \cong H^1(\hat{k}_p, N_p). \] (5.5)

**Definition 4.** Let \( S \) be a non-empty finite set of primes of \( k \). The **first Tate-Shafarevich set** of \( G \) over \( k \) relative to \( S \) is

\[ \prod^1_S(k, G) := \ker \left( H^1(k, G) \rightarrow \prod_{p \not\in S} H^1(\hat{k}_p, G_p) \right). \]

**Proposition 5.1.** Suppose \([K : k]\) is prime and \( N_{r}(\hat{R}(\hat{O}_p)) = U_p \cap N_{K_p/k_p}(\hat{K}_p^\times) \) for all \( p \). Let \( S_r \) be the set of primes dividing \( \Delta_k \). Then there is an exact sequence of abelian groups (compare with formula (5.3) in [Mor]):

\[ 1 \rightarrow \text{Cl}_\infty(N) \rightarrow H^1_{\text{ét}}(\mathcal{O}, N) \rightarrow \prod^1_{S_r \cup \{\infty\}}(k, N) \rightarrow 1. \]

**Proof.** As \( H^1_{\text{ét}}(\hat{O}_p, \hat{N}_p) \) embeds into \( H^1(\hat{k}_p, N_p) \) for any prime \( p \) by (5.5), the \( \mathcal{O} \)-group \( \hat{N}_p \) admits the exact sequence (4.4), in which the terms are abelian groups as \( N \) is commutative. The pointed set \( \text{Cl}_\infty(N) \) is in bijection with the first Nisnevich cohomology set \( H^1_{\text{Nis}}(\mathcal{O}, N) \) (cf. [Nis 1. Theorem 2.8]), which is a subgroup of \( H^1_{\text{ét}}(\mathcal{O}, N) \) because any Nisnevich cover is flat. Hence the first map is an embedding. Since \( K/k \) has prime degree and so is necessarily abelian, at any prime \( p \) the local Artin reciprocity law implies that

\[ n_p = |\text{Gal}(K_p/k_p)| = |\hat{k}_p^\times : N_{K_p/k_p}(\hat{K}_p^\times)| = |H^1(\hat{k}_p, N_p)|. \]

Furthermore, since \([K : k]\) is a prime number, any ramified place \( p \) is totally ramified, which implies that \([U_p : U_p \cap N_{K_p/k_p}(U_p)] = n_p \) [Haz, Theorem 5.5]. Together with (5.5) this means that \( H^1_{\text{ét}}(\hat{O}_p, \hat{N}_p) \) coincides with \( H^1(Q_p, N_p) \) at ramified primes and vanishes elsewhere. Thus the set \( B \) of (4.4) consists of classes \( [\gamma] \in H^1(k, N) \) whose fibers vanish at unramified places. This means that \( B = \prod^1_{S_r \cup \{\infty\}}(k, N) \), where \( S_r \) is the set of ramified primes of \( K/k \).

[11]
Remark 5.2. The group \( B = \prod_{p \mid (\Delta)} \ker \) embeds in the group \( H^1(k, N) \) by definition. But \( H^1(k, N) \cong k^\times / \text{Nm}(K^\times) \), so \( B \) has exponent dividing \( n = [K : k] \).

Corollary 5.3. For any \([q'] \in \mathfrak{c}_1(\text{disc}(q))\), \([q' \ast q'] \in \text{Cl}_\infty(q) := \text{Cl}_\infty(\mathcal{O}_q)\).

Proof. At a prime \( \mathfrak{p} \) dividing \( \det(B_q) \) the embedding (5.2) follows from \( \det(B_q) \) assumed squarefree thus \( q \) being of simple degeneration and multiplicity one (cf. [APS Cor. 3.8]). Otherwise, if \( \mathfrak{p} \nmid \det(B_q) \), it follows from \( \mathcal{O}_q \) being reductive (see the proof of [CGP2 Prop. 3.14]). As a result, by (5.3) and Lemma 2.3 the principal genus satisfies

\[
\text{Cl}_\infty(\mathcal{O}_q) = \mathbb{N} = \ker[H^1_{et}(\mathcal{O}, \mathbb{N}) \to H^1(k, N)].
\]

The quotient \( H^1_{et}(\mathcal{O}, \mathbb{N}) / \text{Cl}_\infty(\mathbb{N}) = \prod_{p \mid (\Delta)} \ker \) has exponent 2 by Prop. 5.1 and Remark 5.2. As \( H^1_{et}(\mathcal{O}, \mathbb{N}) \) is an abelian group, if \([q'] \in H^1_{et}(\mathcal{O}, \mathbb{N} = \mathcal{O}_q)\), the latter bijection by Lemma 2.3 to \( \mathfrak{c}_1(\text{disc}(q)) \), then \([q' \ast q'] \) lies in \( \text{Cl}_\infty(q) \). \( \square \)

6 Over elliptic curves

Let \( \mathcal{O} = \mathbb{F}[x] \) and so \( k = \mathbb{F}(x) \). Let \( C = \{Y^2Z = X^3 + aXZ^2 + bZ^3\} \) be a (projective) elliptic curve defined over \( \mathbb{F} \). Then \( K = \mathbb{F}(C) \) is quadratic imaginary over \( k \); \( K = k(\sqrt{\alpha}) \) where \( \alpha = x^3 + ax + b \in \mathcal{O} \), as \( \text{char}(k) \) is odd \( K/k \) is separable, and as \( \text{deg}(\alpha) = 3 \), \( \infty_K \) ramifies in \( K \) ([DLB Theorem 1(1)(a)]). Suppose \( \infty_K = (0 : 1 : 0) \) belongs to \( C(\mathbb{F}) \). Then \( \mathcal{O}_K = \mathbb{F}[C^{\text{aff}}] \) where \( C^{\text{aff}} := C - \{\infty_K\} = \{y^2 = \alpha\} \) in affine coordinates is an affine \( \mathbb{F} \)-curve, and one has \( \text{Pic}(\mathcal{O}_K) \cong C(\mathbb{F}) \) (e.g., [Bir Example 4.8]). Let as above \( \mathfrak{c}_1'(\Delta_K) \) be the set of classes of quadratic binary \( \mathcal{O} \)-forms with discriminant \( \Delta_K \) up to proper \( \mathcal{O} \)-isomorphisms. By Proposition 5.3 we get: \( \mathfrak{c}_1'(-\alpha(\mathbb{F}^x)^2) \cong C(\mathbb{F}) \). Explicitly,

Corollary 6.1. Let \( C = \{Y^2Z = X^3 + aXZ^2 + bZ^3\} \) be an elliptic curve defined over \( \mathbb{F} \) such that \( (0 : 1 : 0) \in C(\mathbb{F}) \). There is an isomorphism of abelian groups:

\[
C(\mathbb{F}) \cong \mathfrak{c}_1'((x^3 + ax + b)(\mathbb{F}^x)^2) : [(A : B : C \neq 0)] \mapsto \left( x - \frac{A}{C}, -\frac{B}{C}, \frac{(B/C)^2 - \alpha}{x - \frac{A}{C}} \right),
\]

\([(0 : 1 : 0)] \mapsto [(1, 0, -\alpha)].
\]

Proof. Since \( \infty_K = (0 : 1 : 0) \) is a closed point on \( C \), \( \mathbb{F}[C - \{\infty_K\}] = \mathcal{O}[\sqrt{\alpha}] \) where \( \alpha = x^3 + ax + b \) is the ring of \( \{\infty_K\} \)-integers in \( K = \mathbb{F}(C) \). The above correspondence is then given by:

\[
[(A : B : C \neq 0)] \in C(\mathbb{F}) - \{\infty_K\} \mapsto [(A/C, B/C)] \in C^{\text{aff}}(\mathbb{F})
\]

\( \mapsto [L = \langle x - A/C, y - B/C \rangle] \in \text{Pic}(\mathcal{O}_K) \)

\( \mapsto \left[ q_L = \left( x - A/C, -\frac{B}{C}, \frac{(B/C)^2 - \alpha}{x - A/C} \right) \right] \in \mathfrak{c}_1'(-\alpha(\mathbb{F}^x)^2) \)
and \((0 : 1 : 0) \mapsto (1, 0, -\alpha)\).

\[\text{Example 6.2.} \text{ Let } C = \{Y^2Z = X^3 + XZ^2 + Z^3\} \text{ defined over } \mathbb{F}_3. \text{ Removing } \infty_K = \langle 0 : 1 : 0 \rangle, \text{ we get the affine curve } C^{af} = \{y^2 = x^3 + x + 1\}. \text{ with } \mathcal{O}_K = \mathbb{F}_3[x, y]/(y^2 - x^3 - x - 1). \text{ Then:}\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
i & C(\mathbb{F}_3) & \text{affine support} & \text{order} & L_i & q_i \\
\hline
1 & (1 : 0 : 1) & (1, 0) & 2 & (x - 1, y) & (x - 1, 0, 2x^2 + 2x + 1) \\
2 & (0 : 1 : 2) & (0, 2) & 4 & (x, y - 2) & (x, 2, 2x^2 + 2) \\
3 & (0 : 1 : 1) & (0, 1) & 4 & (x, y - 1) & (x, 1, 2x^2 + 2) \\
4 & (0 : 1 : 0) & O & 1 & \mathcal{O}_K & (1, 0, 2x^3 + 2x + 2) \\
\hline
\end{array}
\]

Here \(q_2\) and \(q_3\) are opposite: \([q_2] * [q_3] = [q_4]\). Indeed: \(\langle x, y - 2 \rangle \otimes \langle x, y - 1 \rangle = \langle x \rangle\), hence:

\[\mathfrak{c}_1'(\langle x^3 + x + 1\rangle(\mathbb{F}_3^\times)^2) \cong \text{Pic}(\mathcal{O}_K) \cong \mathbb{Z}/4,\]

while \(\mathfrak{c}_1'((x^3 + x + 1)(\mathbb{F}_3^\times)^2)\) has no group structure.

According to Proposition \(6.4\) there are \(2^{2 - 1} = 2\) genera, and by Proposition \(6.3\) \([q_2]\) belongs to the principal genus, though not being the trivial class.

Indeed: \([q_2] = [q_1]\) by the group law, and as \(y^2 = (x - 1)(x^2 + x - 1)\), \(q_1\) is isomorphic to \(q_4\) by \(\left(\begin{array}{cc}
\frac{1}{\sqrt{x - 1}} & 0 \\
0 & \sqrt{x - 1}
\end{array}\right)\) locally at any \(p \neq \langle x - 1 \rangle\), and by \(\left(\begin{array}{cc}
\sqrt{x^2 + x - 1}/y & 0 \\
0 & y/\sqrt{x^2 + x - 1}
\end{array}\right)\)

at \(p = \langle x - 1 \rangle\).

\[\text{Example 6.3.} \text{ If } C = \{Y^2Z = X^3 + XZ^2\} \text{ defined over } \mathbb{F}_5 \text{ then removing } \infty_K = \langle 0 : 1 : 0 \rangle, \text{ we get the affine elliptic curve } C^{af} = \{y^2 = x^3 + x\} \text{ with } \mathcal{O}_K = \mathbb{F}_5[x, y]/(y^2 - x^3 - x). \text{ Then:}\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
i & C(\mathbb{F}_5) & \text{affine support} & \text{order} & L_i & q_i \\
\hline
1 & (0 : 0 : 1) & (0, 0) & 2 & (x, y) & (x, 0, 4x^2 + 4) \\
2 & (1 : 0 : 2) & (3, 0) & 2 & (x - 3, y) & (x - 3, 0, x^2 + 3x) \\
3 & (1 : 0 : 3) & (2, 0) & 2 & (x - 2, y) & (x - 2, 0, x^2 + 2x) \\
4 & (0 : 1 : 0) & O & 1 & \mathcal{O}_K & (1, 0, 4x^3 + 4x) \\
\hline
\end{array}
\]

Here we observe no opposite forms and \(\mathfrak{c}_1'((x^3 + x)(\mathbb{F}_5^\times)^2) = \mathfrak{c}_1'((x^3 + x)(\mathbb{F}_5^\times)^2) \cong \text{Pic}(\mathcal{O}_K) \cong \mathbb{Z}_2^2.\)

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