ERDŐS-KO-RADO PROBLEMS FOR PERMUTATION GROUPS

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Abstract. In this paper, we study intersecting sets in primitive and quasiprimitive permutation groups. Let $G \leq \text{Sym}(\Omega)$ be a transitive permutation group, and $S$ an intersecting set. Previous results show that if $G$ is either 2-transitive or a Frobenius group, then $|S| \leq |G_\omega|$ (for some $\omega \in \Omega$). Furthermore, for some 2-transitive groups, $|S| = |G_\omega|$ if and only if $S$ is a coset of a stabilizer. In this paper, we prove that these statements are far from the truth for general transitive groups. In particular, we show that in the case of primitive groups, there is even no absolute constant $c$ such that $|S| \leq c|G_\omega|$. In the case $G$ is a primitive permutation group isomorphic to $\text{PGL}(2, q)$, we characterize the subgroups of $G$ which are intersecting sets.

1. Introduction

The Erdős-Ko-Rado (EKR) theorem is a classical result of Erdős-Ko-Rado in extremal set theory, which states that, for a collection $\mathcal{S}$ of $k$-subsets of a set $\Omega$ of size $n$ with $k < n/2$, if any two members of $\mathcal{S}$ intersect, then

$$|\mathcal{S}| \leq \binom{n-1}{k-1},$$

and equality holds if and only if $\mathcal{S}$ consists of all $k$-subsets of $\Omega$ which contains a common element.

Let $G$ be a permutation group on a set $\Omega$. A subset $S \subset G$ is called an intersecting set if for every $x, y \in S$, the ratio $xy^{-1}$ fixes a point in $\Omega$. We say that an intersecting set is maximum if it is an intersecting set of the largest possible size. Cosets of point stabilizers are some obvious examples of intersecting subsets. Thus the size of a maximum intersecting set is at least $|G_\omega|$ (for some $\omega \in \Omega$). In view of the classical EKR theorem, it is now natural to consider the following questions:

(A) Is the size of every intersecting set bounded above by the order of a point stabilizer?

(B) Is every maximum intersecting set a coset of a point stabilizer?

In 1977, Deza and Frankl [4] showed that, for symmetric group of degree $n$, the answer to question (A) is positive, that is, the size of any intersecting set is at most $(n-1)!$. They conjectured that every maximum intersecting set is a coset of a point stabilizer. This conjecture was proved by Cameron-Ku ([2]) and Larose-Malvernuto ([8]) independently. Later in [5], Godsil and Meagher gave another proof.

A permutation group $G$ on $\Omega$ has the EKR property if its point stabilizers are maximum intersecting sets. The group is further said to have the strict-EKR property if cosets of point stabilizers are the only maximum intersecting sets in $G$. A
few classes of permutation groups satisfying the strict-EKR property are as follows: 
(a) Alternating groups of degree \( n \) (c.f [1]); (b) the groups \( \text{PGL}(2, q) \) ((c.f [11]) and \( \text{PSL}(2, q) \) (c.f [9]) acting on 1-spaces. It was shown in [12] that all 2-transitive permutation groups have the EKR property. Not all 2-transitive permutation groups have the strict-EKR property, for example the affine group \( \text{AGL}(1, p^d) = \mathbb{Z}_{p^d} : \mathbb{Z}_{p^d-1} \) is 2-transitive but does not have the strict-EKR property unless \((p, d) = (3, 1)\). A recent result [10], shows that the characteristic vector of any maximum intersecting set of 2-transitive groups is a linear combination of characteristic vectors of cosets of point stabilizers. Groups with this property are said to have the EKR-Module property. In this paper we show that general transitive groups are “far from satisfying the EKR property”. As means of measuring how “far” a group is from satisfying the EKR property, we define the following quantity.

**Definition 1.1.** Given a transitive permutation group \( G \) on \( \Omega \), by \( \rho(G, \Omega) \), we denote the ratio \( |S|/|G_\omega| \), where \( S \) is a maximum intersecting set and \( \omega \in \Omega \).

Since \( G_\omega \) is an intersecting set, we have \( \rho(G, \Omega) \geq 1 \). The group \( G \) satisfies EKR property if and only if \( \rho(G, \Omega) = 1 \). We show that \( \rho(G, \Omega) \) can be arbitrarily large for general permutation groups.

**Theorem 1.1.** Given \( M > 0 \) and \( 0 < \epsilon < 1 \), there is a transitive permutation group \( G \) on a set \( \Omega \) satisfying \( \rho(G, \Omega) > M \) and \( 1 - \epsilon < \frac{\rho(G, \Omega)}{\sqrt{|\Omega|}} < 1 \).

In section 3, we construct examples of transitive permutation groups satisfying the hypotheses of the above Theorem. Noticing \( |G| = |\Omega||G_\omega| \), we are inclined to believe \( \sqrt{|\Omega||G_\omega|} \) would be the best possible upper bound for the sizes of intersecting sets. The construction in section 3 shows that intersecting set sizes can be very close to this upper bound.

**Conjecture 1.2.** Let \( G \) be a transitive permutation group on \( \Omega \), and \( \omega \in \Omega \). Then \( |S| < \sqrt{|\Omega||G_\omega|} \) for any intersecting subset \( S \) of \( G \).

It is known that any permutation group with a regular subgroup has the EKR property. [6, Lemma 14.6.1] gives a proof of this using some methods from algebraic graph theory. We give a different proof (c.f Lemmas 2.2 and 2.3) using set-theoretic counting arguments. So all Frobenius groups have the EKR property. The following theorem characterizes all intersecting subsets of a Frobenius group and also shows that the generalized dihedral groups are the only Frobenius groups with strict-EKR property.

**Theorem 1.3.** Let \( G \) be a Frobenius group with \( K \leq G \) as its Frobenius kernel and \( H \leq G \) as its Frobenius complement. Then the following are true:

(i) \( S \subset G \) is an intersecting set if and only if there exists a set \( \{H_c \mid c \in K\} \) of pairwise disjoint subsets of \( H \) such that \( S = \bigcup_{c \in K} H_c \); 

(ii) and \( G \) satisfies the strict-EKR property if and only if \( |H| = 2 \).

Part (i) of this theorem may be deduced from [13, Theorem 3.6] and part (ii) is equivalent to [13, Theorem 3.7]. The proofs in [13] rely on techniques from algebraic graph theory and representation theory. We will prove this Theorem as an application of Lemma 2.2.
There are results describing all the maximum intersecting sets of Frobenius groups (c.f Theorem 1.3) and of 2-transitive groups (c.f [12, 10]). A natural next step is to carry similar investigations for primitive permutation groups. A permutation group is called quasiprimitive if each of its non-trivial normal subgroup is transitive (All primitive permutation groups are quasiprimitive but the converse is not true.). In this paper, we will see that although many quasiprimitive groups have the EKR-property, infinitely many of them don’t have the EKR property and moreover have “large” intersecting sets. The O’Nan-Scott-Praeger classification theorem for quasiprimitive groups (c.f [14]) shows that there are 8 types of quasiprimitive groups. For our purposes, we divide them into three types: (a) those with subnormal regular subgroups, (b) almost simple, and (c) quasiprimitive groups of product action type. Corollary 2.2 shows that the quasiprimitive groups of first type satisfy the EKR property. In 4.1 and 4.2, we find examples to show that $\rho(G, \Omega)$ can be arbitrarily large for quasiprimitive groups of the other two types.

**Theorem 1.4.** Let $G$ a quasiprimitive permutation group on a set $\Omega$, and let $\rho(G, \Omega)$ be as defined in 1.1. Then either

(i) $\rho(G, \Omega) = 1$ and $G$ has the EKR property, or
(ii) $G$ is an almost simple group or a group of product action type.

Moreover, in case (ii), there is no upper-bound for $\rho(G, \Omega)$.

Two problems now naturally arise, corresponding to the two parts of Theorem 1.4.

**Problem 1.5.** (a) Determine which quasiprimitive permutation groups with a subnormal regular subgroup have the strict-EKR property.
(b) Determine which quasiprimitive almost simple groups have the EKR property.

The two problems in Problem 1.5 seem difficult. For a possible approach, we propose to study a weaker property defined below.

**Definition 1.2.** Let $G$ be a transitive permutation group on a set $\Omega$, and $\omega \in \Omega$.

(a) A subgroup $S$ is called an intersecting subgroup if it is an intersecting set;
(b) $G$ is said to have the weak-EKR property if each intersecting subgroup has size at most $|G_\omega|$;
(c) $G$ is said to have the strict-weak-EKR property if it has the weak-EKR property and all maximum intersecting subgroups are conjugate to $G_\omega$.

Our proof of Theorem 1.4 is based on constructing groups with large intersecting subgroups. To do so, we will use characterizations of intersecting subgroups given in Lemma 2.6. In view of Theorem 1.4 and Lemma 2.6, it is natural to ask the following problem which is a weaker version of Problem 1.5.

**Problem 1.6.** (a) Classify all almost simple primitive permutation groups $G \leqslant Sym(\Omega)$ such that $|H| \leqslant |G|/|\Omega|$, for any subgroup $H < G$ with $H < \bigcup_{\omega \in \Omega} G_\omega$.
(b) Classify all almost simple primitive permutation groups $G \leqslant Sym(\Omega)$ such that any subgroup $H < G$ with $H < \bigcup_{\omega \in \Omega} G_\omega$ is a point stabilizer.

As a starting point, we solve this problem for primitive permutation groups isomorphic to $PSL(2, p)$. We prove the following theorem in Section 5. Statement (iii) of the Theorem is a hard result that was proved in [9].
Theorem 1.7. Let \( p \geq 5 \) be a prime, \( G = \text{PSL}(2, p) \) be a primitive permutation group on \( \Omega \) and \( \omega \in \Omega \). Then the following are true.

(i) If \( p \neq 11, 13, 23, 29, 31, 59, \) or \( 61 \), then \( G \) has the strict-weak-EKR property.
(ii) If \( p \neq 29 \) or \( 31 \), then \( G \) has the weak-EKR property.
(iii) If \( G\omega \cong \mathbb{Z}_p \mathbb{Z}_{p-1} \), then \( G \) has the strict-EKR property.
(iv) If \( p \equiv 3 \pmod{4} \) and \( G\omega \cong \text{D}_{p+1} \), then \( G \) has the EKR property.
(v) If \( p = 29 \) or \( 31 \), then \( G \) has the strict-weak-EKR property unless \( G\omega \cong \text{D}_{30} \).
(vi) If \( p = 13 \) or \( 61 \), then \( G \) has the strict-weak-EKR property unless \( G\omega \cong \text{D}_{p-1} \).
(vii) If \( p = 11, 23 \) or \( 59 \), then \( G \) has the strict-weak-EKR property unless \( G\omega \cong \text{D}_{p+1} \).

Let \( p \geq 5 \) be a prime and \( G = \text{PGL}(2, p) \) be a primitive permutation group on \( \Omega \), with \( G\omega \cong \text{D}_{p+1} \) (with \( \omega \in \Omega \)). The group \( \tilde{G} = \text{PGL}(2, p) \) also acts primitively on \( \Omega \) with \( \tilde{G}\omega \cong \text{D}_{2(p+1)} \). We show in Example 2.4 that \( \tilde{G} \) has the EKR property. In the same example, we also see that \( G \) also satisfies the EKR property in the case \( p \equiv 3 \pmod{4} \). From Theorem 1.7, we know that \( G \) has the strict-weak-EKR property if and only if \( p \neq 29 \). All maximum intersecting subgroups of \( \text{PSL}(2, 29) \) are isomorphic to \( A_5 \). We wonder if for \( p \neq 29 \), the group \( G \) has the strict-EKR property.

This example leads us to the following general question:

Problem 1.8. Let \( G \leq \text{Sym}(\Omega) \) be a permutation group.

(a) Let \( X \) be the set of maximum intersecting sets in \( G \). Does there exist a subgroup \( H \leq G \) such that \( H \in X \)?
(b) If \( G \) has the weak-EKR property, does it also have the EKR property?

The answer to the above question is positive if \( G \) has the EKR property. We do not study this problem seriously in this paper, but believe that this would be an interesting direction for further research.

2. SOME GENERAL RESULTS AND PROOF OF THEOREM 1.3.

In this section, we will gather some general results about intersecting sets. Let \( G \) be a finite transitive permutation group on \( \Omega \), and let \( \omega \) be some arbitrary but fixed element of \( \Omega \). Consider an intersecting set \( S \) in \( G \). Given any \( r, s \in S \), by definition, \( rs^{-1} \) fixes a point, say \( \alpha \). Now let \( x \in G \), then \((xg)(xh)^{-1} = xgh^{-1}x^{-1}\) fixes the point \( \alpha x^{-1} \). This observation leads us to the following Lemma.

Lemma 2.1. If \( S \) is an intersecting set, then

(a) for all \( x \in G \), the set \( xS \) is also an intersecting set, and
(b) and there is an intersecting set of size \( |S| \) with no fixed point free permutations as its elements.

Proof. Part (a) is true from the observation before the Lemma. Let us pick an element \( s \in S \), then by (a), \( s^{-1}S \) is an intersecting set containing the identity \( e \) of \( G \). Since \( e \in s^{-1}S \), for any \( t \in s^{-1}S \), the ratio \( te^{-1} = t \) fixes some point. \( \square \)

A subset \( C \) of a permutation group \( G \leq \text{Sym}(\Omega) \) is said to be sharply transitive if for any \((\alpha, \beta) \in \Omega \times \Omega \) there is exactly one \( c \in C \) such that \( \alpha^c = \beta \). The following result which is equivalent to [13, Corollary 2.2] shows that permutation groups containing sharply transitive sets satisfy the EKR property. In [13], the
authors give a proof using Linear Algebra and Graph Theory. Our proof is group theoretic. We

Lemma 2.2. Let $G$ be a finite permutation group on $\Omega$ and $\omega \in \Omega$. If $G$ contains a sharply transitive subset $C$, then the following are true:

(a) Every intersecting set of $G$ is of the form $\bigcup_{c \in C} H_c$, where $\{H_c \mid c \in C\}$ is a set of pairwise disjoint subsets of $G_\omega$;

(b) and $G$ has the EKR property

Proof. Consider $c_1, c_2 \in C$ and $\alpha \in \Omega$ such that $\alpha^{c_1c_2^{-1}} = \alpha$. So we have $\alpha^{c_1} = \alpha^{c_2}$. As $C$ is a sharply transitive set, we must have $c_1 = c_2$. Thus for distinct $c_1, c_2 \in C$, the ratio $c_1c_2^{-1}$ is a fixed point free permutation. Moreover for any $\omega \in \Omega$, $C$ is a set of right coset representatives for $G_\omega$ in $G$. We thus have the decomposition

$$G = \bigcup_{c \in C} G_\omega c.$$ 

Let $S$ be any intersection set in $G$ and define $S_c = S \cap G_\omega c$. So $S$ is a disjoint union of $S_c$’s, that is,

$$S = \bigcup_{c \in C} S_c.$$ 

Now we consider the translations of $S_c$’s into $G_\omega$, that is, the sets $H_c := S_c c^{-1}$ for $c \in C$. We then have

$$|S| = \sum_{c \in C} |S_c| = \sum_{c \in C} |H_c|. \quad (2.1)$$

Let $c, d \in C$ such that $H_c \cap H_d \neq \emptyset$. Let $h \in H_c \cap H_d \subset G_\omega$. So we have an element $x \in S_c \subset S$ and an element $y \in S_d \subset S$ such that $xc^{-1} = h = yd^{-1}$. Therefore, $cd^{-1} = xh^{-1}y^{-1}$. Since $x, y$ are elements of an intersecting set, $xy^{-1}$ fixes a point. As $c, d$ are elements of a sharply transitive set, $cd^{-1}$ fixes a point if and only if $c = d$. Thus $\{H_c \mid c \in C\}$ is a set of pairwise disjoint subsets of $G_\omega$ such that $S = \bigcup_{c \in C} H_c c$. So part (a) is true. Now by (2.1), we have $|S| \leq |H|$, and thus $G$ has EKR property.

Applications of this lemma give us the following results.

Corollary 2.3. ([6, Lemma 14.6.1]) A permutation group with a regular subgroup has the EKR property.

Proof. The regular subgroup is a sharply transitive subset and thus this follows from Lemma 2.2

Example 2.4. Let $q$ be the power of an odd prime. Let $t$ be an integer such that $2^t || q - 1$. Consider $\tilde{G} = \text{PGL}(2, q)$ and its subgroup $\tilde{H}$ generated by $\begin{pmatrix} 0 & 1 \\ \beta^{q+1} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -\beta^{(1+q)} & \beta + \beta^q \end{pmatrix}$, where $\beta \in \mathbb{F}_q^\times$ is an element of order $2^t (q + 1)$. We note that $\tilde{H}$ is isomorphic to $D_{2(q+1)}$. 

\[ \text{Erdös-Ko-Rado Problems for Permutation Groups} \]
Let $F$ be the field of order $q$, and $m$ be such that $2^m | (q - 1)$. By $M < F^\times$ be the unique multiplicative subgroup of index $2^m$. Let $\delta$ denote a primitive $2^m$th root of unity and set $\tilde{M} := \bigcup_{i=0}^{2^{m-1}-1} \delta^i M$. Now the set
\[
\tilde{C} = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a \in \tilde{M} \land b \in F \right\}
\]
is a transversal for the cosets of $H$. We also note that for any $c_1, c_2 \in \tilde{C}$, the ratio $c_1c_2^{-1}$ is not conjugate to an element of $H \setminus \{1\}$. Thus $\tilde{C}$ is a sharply transitive subset and thus by Lemma 2.2, the action of $\tilde{G}$ on $[\tilde{G} : \tilde{H}]$ has the EKR property.

We note that this is the primitive action of $\text{PGL}(2, q)$ on pairs of imaginary points (that is, points of $\text{PG}(1, q^2)$ that don’t lie in $\text{PG}(1, q)$).

When $q \equiv 3 \pmod{4}$, we observe that $\tilde{C} \subset G = \text{PSL}(2, q)$. By arguments similar to those above, we see that the primitive action of $G$ on $[G : H]$, where $H = \tilde{H} \cap G \cong D_{q+1}$, also satisfies the EKR property. \hfill $\Box$

As another application of Lemma 2.2, we now prove Theorem 1.3.

**Proof of Theorem 1.3:** We may assume that $G$ is a Frobenius group acting on $K$. In this case, $H = G_1$, the stabilizer of the identity 1 of $G$. Let $\{H_c \mid c \in K\}$ be a set of pairwise disjoint subsets of $H$. Consider $S = \bigcup_{c \in K} H_c$. Consider elements $x, y \in S$ such that $x \in H_ay$ and $y \in H_bb$, for some $a, b \in K$. If $a = b$, then clearly $xy^{-1} \in H_a \subset H$, and thus $xy^{-1}$ fixes a point. Now we assume that $a \neq b$. Then for some $h_1 \in H_a, h_2 \in H_b$, we have
\[
xy^{-1} = h_1ab^{-1}h_2^{-1} = h_1ab^{-1}h_1^{-1}h_1h_2^{-1}.
\]
Now $H_a$ and $H_b$ are disjoint and hence $h_1h_2^{-1} \neq 1$. Also as $K$ is a normal subgroup, $h_1ab^{-1}h_1^{-1}h \in K$. Thus $xy^{-1} = hk$, for some $h \in H \setminus \{1\}$ and $k \in K \setminus \{1\}$. Therefore $xy^{-1} \notin K$. Since $G$ is a Frobenius group, every element not in $K$ fixes a point, and thus $xy^{-1}$ fixes a point. We have proved that $S$ is an intersecting set.

Now application of Lemma 2.2 leads to part (i).

We now prove part (ii). Let us first assume that $|H| = 2$ and $H = \{1, h\}$, it follows from part (i) that every maximum intersecting set of $G$ is of the form $\{k_1, hk_2\}$, for some $k_1, k_2 \in K$. This is a coset of the subgroup $\{1, hk_2k_1^{-1}\}$, which is the point stabilizer as $hk_2k_1^{-1} \notin K$. Therefore $G$ has the strict-EKR property in this case.

Let us now assume that $|H| \geq 3$. Let us pick $c \in K \setminus \{1\}$ and $h \in H \setminus \{1\}$. The by part (i), $S = H \setminus \{h\} \cup \{ch\}$ is an intersecting subset. As $G$ is a Frobenius group, and $ch \notin K$, the element $ch$ fixes only one point. As $ch \notin H$, it does not fix 1. Since every other point of $S$ fixes 1, the set $S$ cannot be a coset of a point stabilizer. \hfill $\Box$

We end this section with an example of a non-Frobenius group, in which we find intersecting sets that are not cosets of subgroup. In order to do so we use the description of intersecting sets given in Lemma 2.2.

**Example 2.5.** Let $G = \text{SL}(2, 3) = Q_8 \rtimes \mathbb{Z}_3$, and $H = \mathbb{Z}_3$. Then $G$ is not a Frobenius group. Let $\Omega = [G : H]$. Then $Q_8$ acting on $\Omega$ is a regular subgroup, and hence $G$ has the EKR-property on $\Omega$ by Corollary 2.3.
Let \( H = \langle h \rangle \) and \( a \in K = Q_8 \) be of order 4. Let \( H = \{1\} \cup \{ h \} \cup \{ h^{-1} \} \), and
\[
S = 1\{1\} \cup a\{ h \} \cup a^{-1}\{ h^{-1} \} = \{1, ah, a^{-1}h^{-1} \}.
\]
Then \( S \) is not a coset of any subgroup of \( G \), and so \( G \) does not have the strict EKR-property.

Now we turn our attention to intersecting subgroups. If \( S' \) is an intersecting set containing the identity element \( e \), then it is clear that every element \( s = se^{-1} \in S' \) must fix a point. As \( G \) acts transitively on \( \Omega \), the set of permutations that fix at least one point is \( \bigcup_{g \in G} G^g_\omega \). Therefore we have the following:

**Lemma 2.6.** Let \( G \) be a transitive permutation group on \( \Omega \), and fix a point \( \omega \in \Omega \). Then, for each subgroup \( S < G \), the following statements are equivalent:

(a) \( S \) is an intersecting subgroup,
(b) every element of \( S \) fixes some point of \( \Omega \),
(c) each element of \( S \) is conjugate to an element of \( G_\omega \).

This Lemma can be applied to construct large intersecting sets.

**Example 2.7.** Let \( p \) be an odd prime, \( V = \mathbb{Z}_p^d \), and \( G = AGL(d, p) = V:GL(d, p) \) be the affine group. Pick an involution \( x \) of the centre \( Z(GL(d, p)) \), and another involution \( y \) in \( (V: \langle x \rangle) \setminus \langle x \rangle \). Then \( G_\omega := \langle x, y \rangle \cong D_{2p} \). We now consider the action of \( G \) on \( [G: G_\omega] \).

By Lemma 2.6, we see that \( S = V: \langle x \rangle \) is an intersecting set of \( G \). We note that
\[
|S| = 2p^d, \quad |G_\omega| = 2p.
\]
So \( |S| \) can be arbitrarily larger than \( |G_\omega| \), dependent on \( d \).

It would be worth to point out that group theoretic properties (solubility, nilpotence etc) of the stabilizer subgroup may not hold for intersecting subgroups. We give the following examples to support this statement.

**Example 2.8.**

(1) Assume \( p \) is a prime with \( p \equiv 1 \pmod{30} \), say \( p = 31 \). Consider \( G = PSL(2, p) \) and a subgroup \( H \) isomorphic to the dihedral group \( D_{p-1} \). In this case, \( G \) has a subgroup \( S \) isomorphic to the alternating group \( A_5 \). It is known that all subgroups of \( G \) of order 2, 3 or 5 are conjugate. As \( |H| \) is divisible by 30, each element of \( S = A_5 \) is conjugate to an element of \( H \). Thus by Lemma 2.6, we see that \( S \) is an intersecting subgroup of \( G \), with respect to its action on \( [G: H] \). Note that \( H \cong D_{p-1} \) is solvable, and \( S \cong A_5 \) is non-solvable.

(2) Let \( G = PSL(2, 2^f) \) where \( f = 2r \) with \( r \geq 2 \). Consider a subgroup \( H \) isomorphic to \( PSL(2, 2^2) \). It is known that all involutions of \( G \) are conjugate. It follows that a Sylow 2-subgroup \( S \cong \mathbb{Z}_2^f \) is an intersecting set of \( G \), with respect to its action on \( [G: H] \). Note that \( H \cong PSL(2, 2^2) \cong A_5 \) is non-solvable, and \( S \) is an elementary abelian group.
3. Proof of Theorem 1.1.

Let $p$ be a prime, $q = p^d$. Let $E = \mathbb{F}_q$ and $F = \mathbb{F}_{q^2}$ be a finite fields of orders $p^d$ and $p^{2d}$ respectively. We consider the action of the affine group $G = \text{AGL}(1, p^{2d}) = F : F^\times$ on the coset space $\Omega = [G : H]$ of the subgroup $H = E : E^\times$. Now consider the subgroup $S = F : E^\times$. Let $x \in F^\times$ and $y \in E^\times$, and consider the element $(x, y) \in S$. We have $(0, x^{-1})(x, y)(0, x) = (1, y) \in E : E^\times = H$. Thus every element of $S$ is conjugate to some element of $H$. So by Lemma 2.6, we see that $S$ is an intersecting subgroup.

We see that $|S| = q < \sqrt{q(q+1)} = \sqrt{|\Omega|}$. Also $\lim_{q \to \infty} \frac{|S|}{\sqrt{|\Omega|/|H|}} = 1$.

Now as $H < S$, every element of $H$ is vacuously conjugate to some element of $S$. Thus $g \in G$ is conjugate to an element $H$ if and only if it is conjugate to an element of $S$. Now $g \in G$ fixes a point in the coset space $[G : S]$ if and only if $g$ is conjugate to an element of $S$. Therefore a subset $X$ is an intersecting set with respect to the action of $G$ on $[G : H]$ if and only if it is an intersecting set with respect to the action of $G$ on $[G : S]$.

Now the $q + 1$ element cyclic subgroup $K$ of $F^\times$ is a regular subgroup with respect to the action of $G$ on $[G : S]$. Thus by Corollary 2.3, this action has EKR property. Therefore $S$ is a maximum intersecting set for the action of $G$ on $[G : S]$ and hence also for the action of $G$ on $[G : H]$.

Our construction proves Theorem 1.1.

4. Proof of Theorem 1.4.

Let $G$ be a quasiprimitive permutation group on $\Omega$. Then from the O’Nan-Scott theorem for quasiprimitive groups (c.f [14]), we see that either (a) $G$ has a subnormal regular subgroup, (b) $G$ is an almost simple group or (c) $G$ is a quasiprimitive group of product type action.

If $G$ has a regular subgroup, by Corollary 2.3, $G$ has the EKR property. So we can shift our focus to quasiprimitive groups that are either almost simple or are of product action type.

4.1. Almost simple case. We now construct quasiprimitive almost simple groups $G \leq \text{Sym}(\Omega)$ with large $\rho(G, \Omega)$. Let $q = 2^f \geq 4$ and let $\mathbb{F}$ be the field of order $q$. We consider the action of $T = \text{PSL}(2, q) = \text{PGL}(2, q)$ on pairs of projective points, that is, on $\Delta = \{\{P, S\} | P, S \in \text{PG}(1, q)\}$. This action is a transitive permutation action. Consider $\alpha = \{[1 : 0], [0 : 1]\}$, we note that $T_\alpha = \frac{|T|}{|\Delta|} = 2(q-1)$. Let $\gamma$ be a generator of the multiplicative group $\mathbb{F}^\times$. We now observe that $r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$ fix $\alpha$ and thus $T_\alpha \cong D_{2(q-1)}$. Since $T_\alpha$ is a maximal subgroup, $T$ is a simple primitive permutation group on $\Delta$. Let us now consider a maximal parabolic subgroup $S = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} | a \in \mathbb{F} \& b \in \mathbb{F}^\times \right\}$.
When \( b \neq 1 \in \mathbb{F} \), the element \( \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \) of \( S \) fixes \( \{ [1 : 0], [a : b - 1] \} \). Let \( a \in \mathbb{F} \), then since characteristic of \( \mathbb{F} \) is 2, we see that \( \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \) fixes the set \( \{ [a : 1], [0 : 1] \} \). Since every element of \( S \) fixes an element, Lemma 2.6 implies that \( S \) is an intersecting subgroup. Thus \( \rho(T, \Delta) \geq \frac{|S|}{|T_\alpha|} = \frac{q}{2} \). Therefore the set

\( \{ \rho(G, \Omega) \mid G \leq Sym(\Omega) \text{ is an almost simple quasiprimitive permutation group} \} \)

is unbounded.

**Remark 4.1.** In the group \( T \) constructed above, every element of \( T_\alpha \) is also conjugate to some element in \( S \). Thus we see that \( X \) is an intersection sets for the permutation action of \( T \) on \( \Delta \) if and only if it is an an intersection sets for the permutation action of \( T \) on \( \Gamma := [T : S] \). The action of \( T \) on \( \Gamma \) is a 2−transitive action. In [11], it was shown that the permutation action of \( T \) on \( \Gamma \) has strict-EKR property. Thus \( S \) is a maximum intersection set for the action of \( T \) on \( \Delta \) and moreover every maximal intersection set is a coset of some conjugates of \( S \). Therefore in this case, \( \rho(T, \Delta) = q \). We observe that \( \rho(T, \Delta) < \frac{\sqrt{|\Delta|}}{\sqrt{2}} \) and that \( \lim_{q \to \infty} \frac{\rho(T, \Delta)}{\sqrt{|\Delta|}} = \frac{1}{\sqrt{2}} \).

**4.2. Product action case.** We now turn our focus to quasiprimitive groups of product type action. Let \( T, \Delta, \alpha, T_\alpha, \) and \( S \) be as constructed in 4.1. Given \( \ell \) a prime, we consider

\[
P = T \wr \mathbb{Z}_\ell = T^\ell : \mathbb{Z}_\ell
\]

\[
\Lambda = \prod_{\ell \in \mathbb{Z}_\ell} \Delta.
\]

Then \( P \) is a quasiprimitive group on \( \Pi \) of product action type. The stabilizer of \( \beta = (\alpha, \alpha, \ldots, \alpha) \) is \( P_\beta = T^\ell_\alpha : \mathbb{Z}_\ell \). Now consider the subgroup \( \mathcal{S} = S^\ell : \mathbb{Z}_\ell \). Given \( [s] = (s_0, s_1, \ldots, s_{\ell-1}) \in T^\ell \) and \( m \neq 0 \in \mathbb{Z}_\ell \). We now show that \( ([s], m) \) fixes a point of \( \Lambda \). Since every element of \( S \) fixes a point in \( \Delta \), we may choose a point \( x_m \in \Delta \) such that it is fixed by \( \prod_{j=1}^{\ell-2} s_{jm} \). For \(-1 \leq i \leq \ell - 2\), we set

\[
x_{im} := \left( \prod_{j=1}^{i-1} s_{jm} \right)^{-1} x_m.
\]

We thus have \( s_{im}x_{(i+1)m} = x_{im} \). Note the \( m \) generates \( \mathbb{Z}_\ell \) and thus \( \{ im \mid -1 \leq i \leq \ell - 2 \} = \mathbb{Z}_\ell \). Let \( x \in \Lambda \) be such that its \( im \)-th coordinate is \( x_{im} \), then \( ([s], m)x = x \). Therefore by Lemma 2.6, \( \mathcal{S} \) is an intersecting subgroup. We now have \( \rho(P, \Lambda) \geq \frac{|\mathcal{S}|}{|P_\beta|} = \left( \frac{q}{2} \right)^\ell \). Therefore the set

\( \{ \rho(G, \Omega) \mid G \leq Sym(\Omega) \text{ is a quasiprimitive permutation of product action type} \} \)

is unbounded.

Now the constructions in 4.1 and 4.2 prove Theorem 1.4.
5. Proof of Theorem 1.7

Let \( p \geq 5 \) be a prime. In this section, we investigate EKR properties of primitive permutation groups isomorphic to \( G = \text{PSL}(2,p) \). Any primitive action of \( G \) is equivalent to an action of \( G \) on the coset space \([G : H]\) of a maximal subgroup \( H \).

The subgroup structure of \( \text{PSL}(2,p) \) is well known. The following result describing the subgroups may be found in many classical texts such as [3, 7].

**Theorem 5.1.** ([3, Chapter 12]) For a prime \( p \geq 5 \), the subgroups of \( G = \text{PSL}(2,p) \) are isomorphic to one of the following groups:

(i) \( \mathbb{Z}_p : \mathbb{Z}_\ell \), where \( \ell \) divides \( \frac{p-1}{2} \);
(ii) \( \mathbb{Z}_\ell \), where \( \ell \) divides \( \frac{p-1}{2} \) or \( \frac{p+1}{2} \);
(iii) \( \text{D}_\ell \), where \( \ell \) divides \( p-1 \) or \( p+1 \);
(iv) subgroups of \( A_5 \), when \( p \equiv \pm 1 \pmod{10} \);
(v) \( A_4 \), provided \( p \equiv \pm 1 \pmod{4} \);
(vi) \( S_4 \) provided \( p \equiv \pm 1 \pmod{8} \).

**Corollary 5.2.** For a prime \( p \geq 5 \), the maximal subgroups of \( G = \text{PSL}(2,p) \) are:

(i) \( \mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}} \);
(ii) \( \text{D}_{p-1} \), for \( p \geq 13 \);
(iii) \( \text{D}_{p+1} \), for \( p \neq 7 \);
(iv) \( A_5 \), for \( p \equiv \pm 1 \pmod{10} \);
(v) \( A_4 \), for \( p \equiv \pm 3 \pmod{8} \);
(vi) \( S_4 \) for \( p \equiv \pm 1 \pmod{8} \).

It is also known (for example, see [7]) that all cyclic subgroups of \( \text{PSL}(2,p) \) of the same order are conjugate. This fact along with Lemma 2.6 gives us the following result.

**Lemma 5.3.** Let \( G = \text{PSL}(2,p) \) be a transitive permutation group on \( \Omega \), and \( \omega \in \Omega \). Then \( S < G \) is an intersecting subgroup if and only if \( \{ |s| \mid s \in S \} \subseteq \{|h| \mid h \in G_\omega \} \).

Theorem 5.1, Corollary 5.2 and the above lemma equip us to find all intersecting subgroups of all primitive group isomorphic to \( G \).

Let \( H \) be one of the maximal subgroups described in Corollary 5.2. We will now determine all intersecting subgroups with respect to the action of \( G \) on \([G : H]\).

**Case 1: When \( H = \mathbb{Z}_p : \mathbb{Z}_{p-1} \).** In this case, the action of \( G \) on \([G : H]\) is 2-transitive. In [9], it was shown that this action satisfies the strict-EKR property.

**Case 2: When \( H = \text{D}_{p-1} \) with \( p \geq 13 \).** In view of Lemma 5.3, a subgroup \( S \) is an intersecting subgroup if and only if for all \( n \in \{ |s| \mid s \in S \} \), either \( n = 2 \) or \( n \mid \frac{p-1}{2} \). From Theorem 5.1 and Lemma 5.3, we can now conclude that

- all cyclic intersecting subgroups are subgroups of \( \mathbb{Z}_{\frac{p-1}{2}} \),
- \( A_4 \) is an intersecting subgroup if and only if \( 6 \mid \frac{p-1}{2} \) (note that 6 is the exponent of \( A_4 \)),
- \( S_4 \) is an intersecting subgroup if and only if \( 12 \mid \frac{p-1}{2} \);
- and \( A_5 \) is an intersecting subgroup if and only if \( 30 \mid \frac{p-1}{2} \).
From this we conclude that in this case, weak-EKR property is satisfied whenever $p \neq 31$. All maximum intersecting subgroups of $\text{PSL}(2, 31)$ are of order 60 are isomorphic to $A_5$. We see that strict-weak-EKR holds whenever $p \neq 13, 31, 61$. We also observe that $A_4$ is a maximal intersecting subgroup of $\text{PSL}(2, 13)$, and that $A_5$ is a maximal intersecting subgroup of $\text{PSL}(2, 61)$.

**Case 2:** When $H = D_{p+1}$ with $p \neq 7$.) Applying Theorem 5.1 and Lemma 5.3, we can now conclude that

- all cyclic intersecting subgroups are subgroups of $\mathbb{Z}_{p+1}$,
- $A_4$ is an intersecting subgroup if and only if $6 \mid \frac{p+1}{2}$,
- $S_4$ is an intersecting subgroup if and only if $12 \mid \frac{p+1}{2}$;
- and $A_5$ is an intersecting subgroup if and only if $30 \mid \frac{p+1}{2}$.

Thus this case, weak-EKR property holds whenever $p \neq 29$; and all maximal intersection subgroups of $\text{PSL}(2, 29)$ are isomorphic to $A_5$. We can also conclude that strict-weak-EKR property holds whenever $p \neq 11, 23, 29, 59$. $A_4$ is a maximal intersecting subgroup of $\text{PSL}(2, 11)$, $S_4$ is a maximal intersecting subgroup of $\text{PSL}(2, 23)$, and $A_5$ is a maximal intersecting subgroup of $\text{PSL}(2, 59)$.

In the case $p \equiv 3 \pmod{4}$, consider the subgroup $R = \mathbb{Z}_p : \mathbb{Z}_{p-1}$. Given $r \in R$, we have $\gcd(|r|, p+1) = 1$, and hence it does not fix any element of $[G : D_{p+1}]$. Since $|[G : D_{p+1}]| = |p(p-1)/2|$, we can see that $R$ is a regular subgroup. Therefore by Corollary 2.2, we see that in this scenario, $G$ has the EKR property.

**Case 3:** When $p \equiv \pm 1 \pmod{10}$ and $H = A_5$. ) By Lemma 5.3, a subgroup $S$ is an intersecting subgroup if and only if its each non-identity element of $S$ is of order 2, 3, or 5. Any dihedral/cyclic group with elements of order both 3 and 5 has an element of order 15. As $p \equiv \pm 1 \pmod{10}$, $D_{10}$ is the only dihedral groups intersecting subgroup. Now using Theorem 5.1, we conclude that $A_5$ is the only maximal intersecting subgroup. Therefore strict-weak-EKR property is true in this case.

**Case 4:** When $p \equiv \pm 3 \pmod{8}$ and $H = A_4$.) By Lemma 5.3, a subgroup $S$ is an intersecting subgroup if and only if its each non-identity element of $S$ is of order 2, or 3. Any dihedral group whose order is bigger than 6 and has an element of order 3, will also have element of bigger than 6. Using similar order arguments to the subgroups described in Theorem 5.1, we can conclude that strict-weak-EKR property holds in this case.

**Case 4:** When $p \equiv \pm 1 \pmod{8}$ and $H = S_4$. ) By Lemma 5.3, a subgroup $S$ is an intersecting subgroup if and only if its each non-identity element of $S$ is of order 2, 3, or 4. Any dihedral/cyclic group with elements of order both 3 and 4 has an element of order 12. Also any dihedral group whose order is $4k$ with $k > 2$ has an element of 8. Using similar order arguments to the subgroups described in Theorem 5.1, we can conclude that strict-weak-EKR property holds in this case.

**References**

[1] Bahman Ahmadi and Karen Meagher. A new proof for the Erdős–Ko–Rado theorem for the alternating group. *Discrete Mathematics*, 324:28–40, 2014.
[2] Peter J Cameron and Cheng Yeaw Ku. Intersecting families of permutations. *European Journal of Combinatorics*, 24(7):881–890, 2003.

[3] Leonard Eugene Dickson. *Linear groups: With an exposition of the Galois field theory*. Courier Corporation, 2003.

[4] Peter Frankl and Mikhail Deza. On the maximum number of permutations with given maximal or minimal distance. *Journal of Combinatorial Theory, Series A*, 22(3):352–360, 1977.

[5] Chris Godsil and Karen Meagher. A new proof of the Erdős–Ko–Rado theorem for intersecting families of permutations. *European Journal of Combinatorics*, 30(2):404–414, 2009.

[6] Christopher Godsil and Karen Meagher. *Erdos-Ko-Rado theorems: algebraic approaches*. Number 149. Cambridge University Press, 2016.

[7] Bertram Huppert. *Endliche gruppen I*, volume 134. Springer-verlag, 2013.

[8] Benoit Larose and Claudia Malvenuto. Stable sets of maximal size in Kneser-type graphs. *European Journal of Combinatorics*, 25(5):657–673, 2004.

[9] Ling Long, Rafael Plaza, Peter Sin, and Qing Xiang. Characterization of intersecting families of maximum size in PSL(2, q). *Journal of Combinatorial Theory, Series A*, 157:461–499, 2018.

[10] Karen Meagher and Peter Sin. All 2-transitive groups have the EKR-module property. *arXiv preprint arXiv:1911.11252*, 2019.

[11] Karen Meagher and Pablo Spiga. An Erdős–Ko–Rado theorem for the derangement graph of PGL(2, q) acting on the projective line. *Journal of Combinatorial Theory, Series A*, 118(2):532–544, 2011.

[12] Karen Meagher, Pablo Spiga, and Pham Huu Tiep. An Erdős–Ko–Rado theorem for finite 2-transitive groups. *European Journal of Combinatorics*, 55:100–118, 2016.

[13] Bahman Ahmadi and Meagher Karen. The Erdos-Ko-Rado property for some permutation groups. *Australasian Journal of Combinatorics*, 61(1):23–41, 2015.

[14] Cheryl E Praeger. An O’Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs. *Journal of the London Mathematical Society*, 2(2):227–239, 1993.

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