OSTROWSKI TYPE INEQUALITIES FOR HARMONICALLY
s-CONVEX FUNCTIONS

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Abstract. The author introduces the concept of harmonically s-convex functions and establishes some Ostrowski type inequalities and Hermite-Hadamard type inequality of these classes of functions.

1. Introduction

Let \( f : I \to \mathbb{R} \), where \( I \subseteq \mathbb{R} \) is an interval, be a mapping differentiable in \( I^o \) (the interior of \( I \)) and let \( a, b \in I^o \) with \( a < b \). If \( |f'(x)| \leq M \), for all \( x \in [a, b] \), then the following inequality holds

\[
|f(x) - \frac{1}{b-a} \int_a^b f(t)dt| \leq M(b-a) \left[ \frac{1}{4} + \frac{(x-a+b)^2}{(b-a)^2} \right]
\]

for all \( x \in [a, b] \). This inequality is known in the literature as the Ostrowski inequality (see [12]), which gives an upper bound for the approximation of the integral average \( \frac{1}{b-a} \int_a^b f(t)dt \) by the value \( f(x) \) at point \( x \in [a, b] \). For some results which generalize, improve and extend the inequalities(1.1) we refer the reader to the recent papers (see [1, 11]).

In [7], Hudzik and Maligranda considered the following class of functions:

Definition 1. A function \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R} \) where \( \mathbb{R}_+ = [0, \infty) \), is said to be s-convex in the second sense if

\[
f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)
\]

for all \( x, y \in I \) and \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \) and \( s \) fixed in \( (0, 1] \). They denoted this by \( K_2^s \).

It can be easily seen that for \( s = 1 \), s-convexity reduces to ordinary convexity of functions defined on \( [0, \infty) \).

In [5], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the s-convex functions.

Theorem 1. Suppose that \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is an s-convex function in the second sense, where \( s \in [0, 1) \) and let \( a, b \in [0, \infty) \), \( a < b \). If \( f \in L[a,b] \), then the following inequalities hold

\[
2^{s-1} f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s + 1}.
\]
the constant \( k = \frac{1}{s^2 + 1} \) is the best possible in the second inequality in \((1.2)\).

The above inequalities are sharp. For recent results and generalizations concerning \(s\)-convex functions see \[2, 5, 6, 8, 10\].

In \[9\], the author gave harmonically convex and established Hermite-Hadamard’s inequality for harmonically convex functions as follows:

**Definition 2.** Let \( I \subset \mathbb{R} \setminus \{0\} \) be a real interval. A function \( f : I \to \mathbb{R} \) is said to be harmonically convex, if

\[
(1.3) \quad f \left( \frac{xy}{tx + (1-t)y} \right) \leq tf(y) + (1-t)f(x)
\]

for all \( x, y \in I \) and \( t \in [0,1] \). If the inequality in \((1.3)\) is reversed, then \( f \) is said to be harmonically concave.

**Theorem 2.** Let \( f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L[a,b] \) then the following inequalities hold

\[
(1.4) \quad f \left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

The above inequalities are sharp.

The goal of this paper is to introduce the concept of the harmonically \(s\)-convex functions, obtain the similar the inequalities \((1.4)\) for harmonically \(s\)-convex functions and establish some new inequalities of Ostrowski type for harmonically \(s\)-convex functions.

2. **Main Results**

**Definition 3.** Let \( I \subset (0, \infty) \) be an real interval. A function \( f : I \to \mathbb{R} \) is said to be harmonically \(s\)-convex (concave), if

\[
(2.1) \quad f \left( \frac{xy}{tx + (1-t)y} \right) \leq (\geq) \ t^s f(y) + (1-t)^s f(x)
\]

for all \( x, y \in I \), \( t \in [0,1] \) and for some fixed \( s \in (0,1] \).

**Proposition 1.** Let \( I \subset (0, \infty) \) be an real interval and \( f : I \to \mathbb{R} \) is a function, then:

1. if \( f \) is \(s\)-convex and nondecreasing function then \( f \) is harmonically \(s\)-convex.
2. if \( f \) is harmonically \(s\)-convex and nonincreasing function then \( f \) is \(s\)-convex.

**Proof.** Since \( f : (0, \infty) \to \mathbb{R}, \ f(x) = x \), harmonically convex function, we have

\[
(2.2) \quad \frac{xy}{tx + (1-t)y} \leq ty + (1-t)x
\]

for all \( x, y \in (0, \infty) \), \( t \in [0,1] \) (see also \[4\] page 4). The proposition (1) and (2) is easily obtained from the inequality \((2.2)\). \( \square \)

**Example 1.** Let \( s \in (0,1] \) and \( f : (0,1] \to (0,1], \ f(x) = x^s \). Since \( f \) is \(s\)-convex (see \[7\]) and nondecreasing function, \( f \) is harmonically \(s\)-convex.

The following result of the Hermite-Hadamard type holds.
Theorem 3. Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be an harmonically \( s \)-convex function and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \) then the following inequalities hold:

\[
2^{s-1} f \left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{s+1}.
\]

Proof. Since \( f : I \to \mathbb{R} \) is an harmonically \( s \)-convex function, we have, for all \( x, y \in I \) (with \( t = \frac{1}{2} \) in the inequality (2.1))

\[
f \left( \frac{2xy}{x+y} \right) \leq \frac{f(y) + f(x)}{2^s}.
\]

Choosing \( x = \frac{ab}{ta+(1-t)b} \), \( y = \frac{ab}{tb+(1-t)a} \), we get

\[
f \left( \frac{2ab}{a+b} \right) \leq \frac{f \left( \frac{ab}{ta+(1-t)a} \right) + f \left( \frac{ab}{tb+(1-t)b} \right)}{2^s}.
\]

Further, integrating for \( t \in [0,1] \), we have

\[
f \left( \frac{2ab}{a+b} \right) \leq \frac{1}{2^s} \left[ \int_0^1 f \left( \frac{ab}{tb+(1-t)a} \right) dt + \int_0^1 f \left( \frac{ab}{ta+(1-t)b} \right) dt \right]
\]

Since each of the integrals is equal to \( \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \), we obtain the left-hand side of the inequality (2.3) from (2.4).

The proof of the second inequality follows by using (2.1) with \( x = a \) and \( y = b \) and integrating with respect to \( t \) over \([0,1]\). \(\square\)

In order to prove our main theorems, we need the following lemma:

Lemma 1. Let \( f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be a differentiable function on \( I^0 \) and \( a, b \in I \) with \( a < b \). If \( f' \in L[a, b] \) then

\[
f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du = \frac{ab}{b-a} \left\{ (x-a)^2 \int_0^1 \frac{t}{(ta+(1-t)x)^2} f' \left( \frac{ax}{ta+(1-t)x} \right) dt \right. \\
- (b-x)^2 \int_0^1 \frac{t}{(tb+(1-t)x)^2} f' \left( \frac{bx}{tb+(1-t)x} \right) dt \right\}
\]

Proof. Integrating by part and changing variables of integration yields

\[
\frac{ab}{b-a} \left\{ (x-a)^2 \int_0^1 \frac{t}{(ta+(1-t)x)^2} f' \left( \frac{ax}{ta+(1-t)x} \right) dt \\
- (b-x)^2 \int_0^1 \frac{t}{(tb+(1-t)x)^2} f' \left( \frac{bx}{tb+(1-t)x} \right) dt \right\}
\]
\[
\begin{align*}
&= \frac{1}{x(b-a)} \left[ b(x-a) \int_0^1 \frac{ax}{ta+(1-t)x} \, dt + a(b-x) \int_0^1 \frac{bx}{tb+(1-t)x} \, dt \right] \\
&= \frac{1}{x(b-a)} \left[ b(x-a) \left\{ t f \left( \frac{ax}{ta+(1-t)x} \right) \right\}^1_0 - \int_0^1 f \left( \frac{ax}{ta+(1-t)x} \right) \, dt \right] \\
&\quad + \frac{1}{x(b-a)} \left[ a(b-x) \left\{ t f \left( \frac{bx}{tb+(1-t)x} \right) \right\}^1_0 - \int_0^1 f \left( \frac{bx}{tb+(1-t)x} \right) \, dt \right] \\
&= f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} \, du.
\end{align*}
\]

\[\square\]

**Theorem 4.** Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a,b] \). If \( |f'|^q \) is harmonically \( s \)-convex on \((a,b)\) for \( q \geq 1 \), then for all \( x \in (a,b) \), we have

\[(2.5) \quad \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} \, du \right| \leq \frac{ab}{b-a} \left\{ (x-a)^2 \left( \lambda_1(a,x,s,q,q) |f'(x)|^q + \lambda_2((a,x,s,q,q) |f'(a)|^q) \right) \right\}^\frac{1}{q} \]

\[+ (b-x)^2 \left( \lambda_3(b,x,s,q,q) |f'(x)|^q + \lambda_4(b,x,s,q,q) |f'(b)|^q \right)^\frac{1}{q} \right\}, \]

where

\[
\lambda_1(a,x,s,q,q) = \frac{\beta(\rho+s+1,1)}{x^2q} \cdot 2F_1 \left( \begin{array}{c} 2\vartheta, \rho+s+1; \rho+s+2;1 - \frac{a}{x} \end{array} \right),
\]

\[
\lambda_2(a,x,s,q,q) = \frac{\beta(\rho+1,1)}{x^2q} \cdot 2F_1 \left( \begin{array}{c} 2\vartheta, \rho+1; \rho+s+2;1 - \frac{a}{x} \end{array} \right),
\]

\[
\lambda_3(b,x,s,q,q) = \frac{\beta(1,\rho+s+1)}{b^2q} \cdot 2F_1 \left( \begin{array}{c} 2\vartheta, 1; \rho+s+2;1 - \frac{x}{b} \end{array} \right),
\]

\[
\lambda_4(b,x,s,q,q) = \frac{\beta(s+1,\rho+1)}{b^2q} \cdot 2F_1 \left( \begin{array}{c} 2\vartheta, s+1; \rho+s+2;1 - \frac{x}{b} \end{array} \right),
\]

\( \beta \) is Euler Beta function defined by

\[
\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \cdot \int_0^1 (1-t)^{y-1} \, dt, \quad x,y > 0,
\]

and \( 2F_1 \) is hypergeometric function defined by

\[
2F_1 (a,b;c;z) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \, dt, \quad c > b > 0, \ |z| < 1 \ (\text{see } ^3).\]
Proof. From Lemma 1, Power mean inequality and the harmonically $s$-convexity of $|f'|^q$ on $[a, b],$ we have

\[
|f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du| \\
\leq \frac{ab}{b-a} \left\{ (x-a)^2 \int_0^1 \frac{t}{(ta+(1-t)x)^2} |f'(ax+ta+(1-t)x)| dt \\
+ (b-x)^2 \int_0^1 \frac{t}{(tb+(1-t)x)^2} |f'(bx+tb+(1-t)x)| dt \right\}
\]

(2.6) \leq \frac{ab(x-a)^2}{b-a} \left( \int_0^1 dt \right)^{\frac{1-s}{q}}
\times \left( \int_0^1 \frac{t^q}{(ta+(1-t)x)^{2q}} \left[ t^s |f'(x)|^q + (1-t)^s |f'(a)|^q \right] dt \right)^{\frac{1}{q}}
\times \frac{ab(b-x)^2}{b-a} \left( \int_0^1 dt \right)^{\frac{1-s}{q}}
\times \left( \int_0^1 \frac{t^q}{(tb+(1-t)x)^{2q}} \left[ t^s |f'(x)|^q + (1-t)^s |f'(b)|^q \right] dt \right)^{\frac{1}{q}},

where an easy calculation gives

(2.7)
\[
\int_0^1 \frac{t^{q+s}}{(ta+(1-t)x)^{2q}} dt = \beta\left( q+s+1, 1 \right) \frac{1}{x^{2q}} {}_2F_1 \left( 2q, q+s+1; q+s+2; 1 - \frac{a}{x} \right),
\]

(2.8)
\[
\int_0^1 \frac{t^{q+s}}{(tb+(1-t)x)^{2q}} dt = \beta\left( q+s+1, 1 \right) \frac{1}{b^{2q}} {}_2F_1 \left( 2q, 1; q+s+2; 1 - \frac{a}{b} \right),
\]

(2.9)
\[
\int_0^1 \frac{t^q(1-t)^s}{(ta+(1-t)x)^{2q}} dt = \beta\left( q+1, s+1 \right) \frac{1}{x^{2q}} {}_2F_1 \left( 2q+1, q+1; q+s+2; 1 - \frac{a}{x} \right),
\]

(2.10)
\[
\int_0^1 \frac{t^q(1-t)^s}{(tb+(1-t)x)^{2q}} dt = \beta\left( q+1, s+1 \right) \frac{1}{b^{2q}} {}_2F_1 \left( 2q, q+1; q+s+2; 1 - \frac{a}{b} \right).
\]

Hence, if we use (2.7)-(2.8) in (2.6), we obtain the desired result. This completes the proof. \[\square\]
Corollary 1. In Theorem 4 additionally, if $|f'(x)| \leq M$, $x \in [a, b]$, then inequality

$$
\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right|
$$

\leq \frac{ab}{b-a} M \left\{ (x-a)^2 \left( \lambda_1(a, x, s, q, q) + \lambda_2((a, x, s, q, q)) \right)^{\frac{1}{\gamma}} 
+ (b-x)^2 \left( \lambda_3(b, x, s, q, q) + \lambda_4(b, x, s, q, q) \right)^{\frac{1}{\gamma}} \right\}

holds.

Theorem 5. Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I$, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically $s$-convex on $[a, b]$ for $q \geq 1$, then for all $x \in [a, b]$, we have

$$
(2.9)
\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right|
\leq \frac{ab}{b-a} \left( \frac{1}{2} \right)^{1-\frac{1}{\gamma}} \left\{ (x-a)^2 \left( \lambda_1(a, x, s, q, q) + \lambda_2((a, x, s, q, q)) \right)^{\frac{1}{\gamma}} 
+ (b-x)^2 \left( \lambda_3(b, x, s, q, q) + \lambda_4(b, x, s, q, q) \right)^{\frac{1}{\gamma}} \right\}
$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ are defined as in Theorem 4.

Proof. From Lemma 1, Power mean inequality and the harmonically $s$-convexity of $|f'|^q$ on $[a, b]$, we have

$$
(2.10)
\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right|
\leq \frac{ab(x-a)^2}{b-a} \left( \frac{1}{2} \right)^{1-\frac{1}{\gamma}}
\times \left( \int_0^1 \frac{t}{(ta + (1-t)x)^{2q}} \left[ t^s |f'(x)|^q + (1-t)^s |f'(a)|^q \right] dt \right)^{\frac{1}{q}}
+ \frac{ab(b-x)^2}{b-a} \left( \frac{1}{2} \right)^{1-\frac{1}{\gamma}}
\times \left( \int_0^1 \frac{t}{(tb + (1-t)x)^{2q}} \left[ t^s |f'(x)|^q + (1-t)^s |f'(b)|^q \right] dt \right)^{\frac{1}{q}}
\leq \frac{ab}{b-a} \left( \frac{1}{2} \right)^{1-\frac{1}{\gamma}} \left\{ (x-a)^2 \left( \lambda_1(a, x, s, q, q) + \lambda_2((a, x, s, q, q)) \right)^{\frac{1}{\gamma}} 
+ (b-x)^2 \left( \lambda_3(b, x, s, q, q) + \lambda_4(b, x, s, q, q) \right)^{\frac{1}{\gamma}} \right\}
This completes the proof. \[\square\]

**Corollary 2.** In Theorem 4 additionally, if \( |f'(x)| \leq M, x \in [a, b] \), then inequality

\[
\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} \, du \right| 
\leq \frac{ab}{b-a} M \left( \frac{1}{2} \right)^{1-\frac{1}{q}} \left\{ (x-a)^2 \left( \lambda_1(a, x, s, q, 1) + \lambda_2((a, x, s, q, 1)) \right)^{\frac{1}{q}} + (b-x)^2 \left( \lambda_3(b, x, s, q, 1) + \lambda_4(b, x, s, q, 1) \right)^{\frac{1}{q}} \right\}
\]

holds.

**Theorem 6.** Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^o \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'|^q \) is harmonically \( s \)-convex on \( [a, b] \) for \( q \geq 1 \), then for all \( x \in [a, b] \), we have

\[
\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} \, du \right| 
\leq \frac{ab}{b-a} \left\{ \lambda_5^{1-\frac{1}{q}}(a, x) (x-a)^2 \left( \lambda_1(a, x, s, q, 1) |f'(x)|^q + \lambda_2((a, x, s, q, 1)) \right)^{\frac{1}{q}} + \lambda_3^{1-\frac{1}{q}}(b, x) (b-x)^2 \left( \lambda_3(b, x, s, q, 1) |f'(x)|^q + \lambda_4(b, x, s, q, 1) |f'(b)|^q \right)^{\frac{1}{q}} \right\}
\]

where

\[
\lambda_5(\theta, x) = \frac{1}{x-\theta} \left\{ \frac{1}{\theta} - \frac{\ln x - \ln \theta}{x-\theta} \right\},
\]

and \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are defined as in Theorem 4.

**Proof.** From Lemma 1 Power mean inequality and the harmonically \( s \)-convexity of \( |f'|^q \) on \( [a, b] \), we have

\[
\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} \, du \right| 
\leq \frac{ab (x-a)^2}{b-a} \left( \int_0^1 \frac{t}{(ta+(1-t)x)^2} \, dt \right)^{1-\frac{1}{q}} 
\times \left( \int_0^1 \frac{t}{(ta+(1-t)x)^2} \left[ t^s |f'(x)|^q + (1-t)^s |f'(a)|^q \right] \, dt \right)^{\frac{1}{q}} 
\leq \frac{ab (b-x)^2}{b-a} \left( \int_0^1 \frac{t}{(tb+(1-t)x)^2} \, dt \right)^{1-\frac{1}{q}} 
\times \left( \int_0^1 \frac{t}{(tb+(1-t)x)^2} \left[ t^s |f'(x)|^q + (1-t)^s |f'(b)|^q \right] \, dt \right)^{\frac{1}{q}}.
\]
It is easily check that
\[
\int_0^1 \frac{t}{(ta + (1-t)x)^2} dt = \frac{1}{x-a} \left\{ \frac{1}{a} - \frac{\ln x - \ln a}{x-a} \right\},
\]
\[
\int_0^1 \frac{t}{(tb + (1-t)x)^2} dt = \frac{1}{b-x} \left\{ \frac{\ln b - \ln x}{b-x} - \frac{1}{b} \right\},
\]
Hence, If we use (2.7)-(2.8) for \( q \) in (2.13) in (2.12), we obtain the desired result. This completes the proof.

**Corollary 3.** In Theorem 4 additionally, if \( |f'(x)| \leq M, x \in [a,b] \), then inequality
\[
\left\| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right\| \leq \frac{ab}{b-a} M \left\{ \frac{1}{2} (a,x) (x-a)^2 \left( \lambda_1(a,x,s,1,1) + \lambda_2(a,x,s,1,1) \right)^{\frac{1}{q}} + \frac{1}{q} (b,x) (b-x)^2 \left( \lambda_3(b,x,s,1,1) + \lambda_4(b,x,s,1,1) \right)^{\frac{1}{q}} \right\}
\]
holds.

**Theorem 7.** Let \( f : I \subset (0,\infty) \to \mathbb{R} \) be a differentiable function on \( I^s, a,b \in I \) with \( a < b \), and \( f' \in L[a,b] \). If \( |f'|^q \) is harmonically s-convex on \([a,b]\) for \( q > 1, \frac{1}{p} + \frac{1}{q} = 1 \), then
\[
\left\| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right\| \leq \frac{ab}{b-a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ (x-a)^2 \left( \lambda_1(a,x,s,q,0) |f'(x)|^q + \lambda_2(a,x,s,q,0) |f'(a)|^q \right)^{\frac{1}{q}} + (b-x)^2 \left( \lambda_3(b,x,s,q,0) |f'(x)|^q + \lambda_4(b,x,s,q,0) |f'(b)|^q \right)^{\frac{1}{q}} \right\}
\]
where \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are defined as in Theorem 4.

**Proof.** From Lemma 1 Hölder’s inequality and the harmonically convexity of \( |f'|^q \)
on \([a,b]\),we have
\[
\left\| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right\| \leq \frac{ab(x-a)^2}{b-a} \left( \int_0^1 t^p dt \right)^{\frac{1}{p}}
\times \left( \int_0^1 \frac{1}{(ta + (1-t)x)^2q} \left[ t^s |f'(x)|^q + (1-t)^s |f'(a)|^q \right] dt \right)^{\frac{1}{q}}
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\[ \frac{ab(b - x)^2}{b - a} \left( \int_0^1 t^p dt \right)^{\frac{1}{p}} \]

\[ \times \left( \int_0^1 t (tb + (1 - t)x)^{2q} \left[ t^s |f'(x)|^q + (1 - t)^s |f'(b)|^q \right] dt \right)^{\frac{1}{q}} \]

\[ \leq \frac{ab(b - a)}{b - a} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left\{ (x - a)^2 \left( \lambda_1(a, x, s, q, 0) |f'(x)|^q + \lambda_2(a, x, s, q, 0) |f'(a)|^q \right)^{\frac{1}{q}} \right. \]

\[ + (b - x)^2 \left( \lambda_3(b, x, s, q, 0) |f'(x)|^q + \lambda_4(b, x, s, q, 0) |f'(b)|^q \right)^{\frac{1}{q}} \left. \right\} \]

This completes the proof. \[ \square \]

**Corollary 4.** In Theorem 4 additionally, if \(|f'(x)| \leq M, x \in [a, b]\), then inequality

\[ \left| f(x) - \frac{ab}{b - a} \int_a^b \frac{f(u)}{u^{2}} du \right| \]

\[ \leq \frac{ab}{b - a} M \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left\{ (x - a)^2 \left( \lambda_1(a, x, s, q, 0) + \lambda_2(a, x, s, q, 0) \right)^{\frac{1}{q}} \right. \]

\[ + (b - x)^2 \left( \lambda_3(b, x, s, q, 0) + \lambda_4(b, x, s, q, 0) \right)^{\frac{1}{q}} \left. \right\} \]

holds.

**Theorem 8.** Let \( f : I \subset (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I^* \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \(|f'|^q \) is harmonically s-convex on \([a, b]\) for \( q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), then

\[ \left| f(x) - \frac{ab}{b - a} \int_a^b \frac{f(u)}{u^2} du \right| \]

\[ \leq \frac{ab}{b - a} \left\{ \left( \lambda_1(a, x, 0, p, p) \right)^{\frac{1}{p}} (x - a)^2 \left( \frac{|f'(x)|^q + |f'(a)|^q}{s + 1} \right)^{\frac{1}{q}} \right. \]

\[ + (b - x)^2 \left( \frac{|f'(x)|^q + |f'(b)|^q}{s + 1} \right)^{\frac{1}{q}} \left. \right\} \]

where \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are defined as in Theorem 4.
Proof. From Lemma [1] Hölder’s inequality and the harmonically convexity of $|f'|^q$ on $[a, b]$, we have
\[
\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \\
\leq \frac{ab(x-a)^2}{b-a} \left( \int_0^1 \frac{t^p}{(ta + (1-t)x)^2p} dt \right)^\frac{1}{p} \\
\times \left( \int_0^1 [t^s |f'(x)|^q + (1-t)^s |f'(a)|^q] dt \right)^\frac{1}{q} \\
+ \frac{ab(b-x)^2}{b-a} \left( \int_0^1 \frac{t^p}{(tb + (1-t)x)^2p} dt \right)^\frac{1}{p} \\
\times \left( \int_0^1 [t^s |f'(x)|^q + (1-t)^s |f'(b)|^q] dt \right)^\frac{1}{q} \\
\leq \frac{ab}{b-a} \left\{ \left(\lambda_1(a, x, 0, p, p)\right)^\frac{1}{p} (x-a)^2 \left( \frac{|f'(x)|^q + |f'(a)|^q}{s+1} \right)^\frac{1}{q} \\
+ \left(\lambda_3(b, x, 0, p, p)\right)^\frac{1}{p} (b-x)^2 \left( \frac{|f'(x)|^q + |f'(b)|^q}{s+1} \right)^\frac{1}{q} \right\}.
\]
This completes the proof. □

Corollary 5. In Theorem 8 additionally, if $|f'(x)| \leq M$, $x \in [a, b]$, then inequality
\[
\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \\
\leq \frac{ab}{b-a} M \left( \frac{2}{s+1} \right)^\frac{1}{p} \left\{ \left(\lambda_1(a, x, 0, p, p)\right)^\frac{1}{p} (x-a)^2 \\
+ \left(\lambda_3(b, x, 0, p, p)\right)^\frac{1}{p} (b-x)^2 \right\}
\]
holds.

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