On string theory in $AdS_3$ backgrounds

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Abstract

We discuss the string theory on $AdS_3$. In the first half of this talk, we review the $SL(2, R)$ and the $SL(2, C)/SU(2)$ WZW models which describe the strings on the Lorentzian and Euclidean $AdS_3$ without RR backgrounds, respectively. An emphasis is put on the fundamental issues such as the unitarity, the modular invariance and the closure of the OPE. In the second half, we discuss some attempts at clarifying such problems. In particular, we discuss the modular invariance of the $SL(2, R)$ WZW model and the calculation of the correlation functions of the $SL(2, C)/SU(2)$ WZW model using the path-integral approach.

* Talk given at YITP workshop ‘Developments in Superstring and M-theory’, Kyoto, Japan, October 27-29, 1999.
1 Introduction: why string theory on $AdS_3$

In this talk, I would like to discuss the string theory on $AdS_3$, namely, $SL(2,R)$ or its Euclidean analog $SL(2,C)/SU(2)$. Besides the recent intensive studies, this string theory has in fact been investigated for more than a decade from various interests [1]-[8].

First of all, string theory in backgrounds with curved time is not well understood. There are several such models which are relatively well studied [9], but the analysis is essentially reduced to that of the free theory. In this respect, the string theory on $SL(2,R)$ seems to give the simplest truly interacting model. This is because $SL(2,R)$ is a very simple space-time with maximal symmetries and the corresponding model is described by the $SL(2,R)$ WZW model when there are no RR charges.

Second, related to the above, little is known about non-compact (non-rational) CFTs [7]. Again, the $SL(2,R)$ WZW model or its Euclidean analog, $SL(2,C)/SU(2)$ WZW model, gives the simplest one.

Third, it is known that the $SL(2,R)$ WZW model is closely related to the string models in various black hole backgrounds. For instance, the $SL(2,R)/U(1)$ WZW model, which is obtained by a coset, describes the strings in two-dimensional black hole backgrounds [11]. An orbifold of the $SL(2,R)$ WZW model [11, 12] gives the string model in the three-dimensional BTZ black hole geometry [13]. When a five-dimensional black hole corresponding to the D1-D5 system is lifted to six dimensions, its near-horizon geometry becomes $AdS_3 \times S^3$ [14] (precisely speaking (BTZ black hole)$\times S^3$). By further taking an S-dual, the system is described by the $SL(2,R) \times SU(2)$ WZW [15]. Similarly, $AdS_3$ or the BTZ black hole appears quite generally as the near-horizon geometry of the black strings obtained by lifting charged black holes in generic dimension [16].

Finally, closely related to the above D1-D5 system, the string theory on $AdS_3$ gives the simplest case of the AdS/CFT correspondence [17]. This aroused the renewed interest and many works have been devoted to the study of the strings on $AdS_3$ in the cases both with [18, 19] and without [20]-[25] RR charges.

However, in spite of recent progress, it seems that there still remain open questions about the string theory on $AdS_3$ itself at the fundamental level. Such a state of the problem was recently discussed in [26]. In this talk, we will focus on the cases without RR charges. In the next two sections, we will review the $SL(2,R)$ WZW model and its Euclidean analog, the $SL(2,C)/SU(2) = H^+_3$ WZW model. We will see that our understanding is still incomplete on the fundamental consistency conditions of string theory such as the unitarity, the modular invariance and the closure of the operator product expansions. Hence we will discuss some attempts towards better understanding in the following sections. In section 4, we will discuss modular invariance of the $SL(2,R)$ WZW model and obtain some important information about the spectrum [27]. In section 5, we will discuss the calculation of the correlation...
functions of the $H^+_{3}$ WZW model using a path-integral approach \[28\]. We will conclude with a brief summary.

\section{SL(2, R) WZW model}

Let us start with the discussion of Lorentzian AdS$_3$. It is defined by the following metric and the embedding equation,

\[ ds^2 = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2, \]
\[ -l^2 = -x_0^2 - x_1^2 + x_2^2 + x_3^2. \]  

(2.1)

This is a maximally symmetric space with negative constant curvature and a solution to the three-dimensional Einstein’s equations with a negative cosmological term $-l^{-2}$,

\[ R_{\mu\nu} = -2l^{-2}g_{\mu\nu}. \]  

(2.2)

The space-time defined in the above is the same as the group manifold $SL(2, R)$. Hence without RR charges the (bosonic) string theory in this background is described by the $SL(2, R)$ WZW model. Its action is given by

\[ S = -\frac{k}{8\pi} \int_{\Sigma} \text{Tr}(dg d^{-1}g) + \frac{iK}{12\pi} \int_{B} \text{Tr}(g^{-1}dg)^3, \]  

(2.3)

where $g(z) \in SL(2, R)$, $k$ is the level, $\Sigma$ is a two-dimensional surface (world-sheet) and $B$ is a three-dimensional manifold satisfying $\partial B = \Sigma$. The action has the $\hat{sl}(2, R)_L \times \hat{sl}(2, R)_R$ current algebra symmetry. The corresponding currents are

\[ J_a(z) = \frac{ik}{2} \partial g g^{-1}, \quad \tilde{J}_a(\bar{z}) = \frac{ik}{2} g^{-1} \partial \bar{g}. \]  

(2.4)

Here we have denoted the quantity in the right sector by tilde. In the following we will omit the expressions in the right sector unless they are necessary. The model has the conformal symmetry and its energy-momentum tensor is given by the Sugawara form,

\[ T(z) = \frac{1}{k-2} \eta_{ab} J^a(z) J^b(z), \]  

(2.5)

where $\eta_{ab} = \text{diag} (-1, 1, 1)$. $J^a(z)$ are defined through $J(z) = \eta_{ab} \tau^a J^b(z)$ with $\tau^a \in sl(2, R)$. In terms of the modes of $J^a(z)$, those of $T(z)$ are written as

\[ L_n = \frac{1}{k-2} \sum_{m \in \mathbb{Z}} \left( \frac{1}{2} J^+_m J^{-}_{m-n} + \frac{1}{2} J^-_{m-n} J^+_m - J^0_{n-m} J^0_m \right). \]  

(2.6)
These currents and the energy momentum tensor satisfy the following commutation relations

\[
\begin{align*}
& [J^0_n, J^0_m] = -\frac{1}{2} kn \delta_{n+m}, \quad [J^0_n, J^\pm_m] = \pm J^\pm_{n+m}, \\
& [J^+_n, J^-_m] = -2 J^0_{n+m} + kn \delta_{n+m}, \\
& [L_n, J^a_m] = -m J^a_{n+m}, \\
& [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m},
\end{align*}
\]

with \(c\) being the central charge given by

\[
c = \frac{3k}{k-2}.
\]

Because of the symmetry, the states at the lowest grade are classified by the representations of \(SL(2, R)\). They are labeled by the values of \(J^0_0\) and \(\vec{J}^2 = \frac{1}{2} J^+_0 J^-_0 + \frac{1}{2} J^-_0 J^+_0 - J^0_0 J^0_0\). These operators act as

\[
\vec{J}^2 | j, m \rangle = -j(j+1) | j, m \rangle, \quad J^0_0 | j, m \rangle = m | j, m \rangle.
\]

Since \(-j(j+1)\) is invariant under \(j \rightarrow -j - 1\), one can always bring the values of \(j\) into the region \(\text{Re } j \leq -1/2\) and \(\text{Im } j \geq 0\). We will take this convention. A generic state in the left sector is obtained by acting on \(| j, m \rangle\) with \(J^a_{-n}\) \((n \geq 0)\) and takes the form

\[
(J^a_{-n_1} J^a_{-n_2} \cdots) | j, m \rangle.
\]

A generic states in the model is obtained by tensoring (2.10) and a similar expression of the right sector.

Since we expect the model to be unitary, we choose the unitary \(SL(2, R)\) representations for the zero-mode part. There are five classes of such representations. For the universal covering group of \(SL(2, R)\), they are

1. Identity representation \(D_{id}\): the trivial representation with \(\vec{J}^2 = J^0_0 = 0\).
2. Principal continuous series \(D_{pc}\): representations with \(m = m_0 + n, 0 \leq m_0 < 1, n \in \mathbb{Z}\) and \(j = -1/2 + i\rho, \rho > 0\).
3. Supplementary series \(D_{sup}\): representations with \(m = m_0 + n, 0 \leq m_0 < 1, n \in \mathbb{Z}\) and \(\min\{-m_0, m_0 - 1\} < j \leq -1/2\).
4. Highest weight discrete series \(D_{hw}\): representations with \(m = M_{\text{max}} - n, n = 0, 1, 2, \ldots, j = M_{\text{max}} \leq -1/2\) and the highest weight state satisfying \(J^+_0 | j, j \rangle = 0\).
5. Lowest weight discrete series \(D_{lw}\): representations with \(m = M_{\text{min}} + n, n = 0, 1, 2, \ldots, j = -M_{\text{min}} \leq -1/2\) and the lowest weight state satisfying \(J^-_0 | j, -j \rangle = 0\).

If we do not take the universal covering group, the parameters are restricted to \(m_0 = 0, 1/2\) in (2), \(m_0 = 0\) in (3) and \(j = \) (half integers) in (4) and (5).
The harmonic analysis on $SL(2, R)$ shows that the square-integrable functions are decomposed into the representations of $D_{pc}$, $D_{hw}$ and $D_{lw}$. Schematically,

$$L^2(SL(2, R)) \sim \sum_{j < -1/2} (-j - \frac{1}{2})(D^j_{hw} \oplus D^j_{lw}) \oplus \int_0^\infty d\rho f(\rho) D_{pc}^{-1/2+i\rho},$$

where $f(\rho)$ is a certain measure (for details, see, e.g., [29]).

**Ghost problem**

Soon after the study of the string theory on $SL(2, R)$ was initiated, it turned out that the model contains negative-norm physical states, namely, ghosts [1]. In the flat case, the original model (in the conformal gauge) also contains negative-norm states because of the time direction. However, the physical state conditions $(L_n - \delta_n) | \Psi \rangle = 0$ ($n \geq 0$) are sufficient to remove such states. The result in [1] indicates that this does not work in the $SL(2, R)$ case. In fact, it is easy to find the ghosts.

To see this, we first note the on-shell condition

$$(L_0 - 1) | \Psi \rangle = 0, \quad L_0 = -\frac{j(j+1)}{k-2} + N,$$

where $N$ is the grade. This means that the spin at the zero-mode part, $|j, m\rangle$, should be

$$j = j(N) \equiv -\frac{1}{2} \left(1 + \sqrt{1 + 4(k-2)(N-1)}\right),$$

which corresponds to $D_{hw}$ or $D_{lw}$ for $k > 2$ and $N > 1$. Next, we consider a set of states

$$\left\{ J^a_0 \cdots J^b_0 | E_N \rangle \right\}, \quad | E_N \rangle = (J^{+}_{-1})^N | j(N), j(N) \rangle .$$

We then find that all the above states are physical but form a non-unitary representation of $SL(2, R)$ for a sufficiently large $N$. This is because $|E_N \rangle$ behaves like a highest weight state of an $SL(2, R)$ representation with $j = m = j(N) + N > 0$. Thus we have found the ghosts.

Having found that the model contains ghosts, one might think that the $SL(2, R)$ WZW model is sick. However, there are several pieces of evidence that the model should make sense. First of all, in the weak curvature limit the model becomes the flat model and hence one should be able to get a sensible model at least at weak curvature. Second, the authors of [1] studied the particle limit of the model but did not find any pathologies. Third, the effective action of the bosonic $\sigma$-model was studied in [30]. There it was found that the effective action has an extremal point corresponding to $AdS_3$ and the model is unitary at one-loop. Finally, as discussed in the introduction, the near-horizon geometry of the D1-D5 system is described by the $SL(2, R) \times SU(2)$ WZW model after an S-dual transformation. Since the D1-D5 system is unitary (at least at weak coupling), we expect that the $SL(2, R) \times SU(2)$ WZW model should also be unitary.
Resolution of the ghost problem

The above argument implies that we might be missing something important and it might be possible to get a sensible theory by finding out an appropriate treatment. There are actually two types of the proposals for the resolution of the ghost problem.

In one proposal [4], the discrete series $D_{hw}$ and $D_{lw}$ are used and the claim is that if we truncate the spectrum so that the spin and the level are restricted to

$$\frac{1}{2} \leq -j < \frac{k}{2}, \quad k > 2,$$

one can remove the ghosts. We call this the unitarity bound. This bound seems natural if we recall the argument of the $SU(2)$ WZW model. In that case, to maintain the unitarity or the modular invariance, one needs to truncate the $SU(2)$ spin so that

$$0 \leq j \leq \frac{k}{2}.$$  

Such a truncation is compatible with the closure of the OPE and the Ward identities. Thus it is completely sensible.

However, in the $SL(2, R)$ case, it is not clear if the truncation (2.15) is compatible with other consistency conditions of string theory. This is because such consistency conditions are not well understood either. (Regarding the discussion of the OPE in the Euclidean case, see [24].) Moreover, from the on-shell condition (2.12), the unitarity bound means the truncation of the string excitation $N$. This seems physically unnatural. In addition, the dimensions of the primaries $L_0 = -j(j + 1)/(k - 2)$ are negative for the discrete series when $k > 2$.

In the other proposal [5, 6], the principal continuous series $D_{pc}$ is used. One way to understand this argument is to start with a Wakimoto-like representation of $\hat{sl}(2, R)$ [33] using one free boson $\phi$ and the $\beta$-$\gamma$ system. We then bosonize the $\beta$-$\gamma$ by two free bosons. One of the points there is that an additional zero-mode is introduced through this bosonization. Here, it may be useful to recall that the primary states $|j, m\rangle$ have only two zero-modes whereas those in the three-dimensional flat theory have three as $|p^0, p^1, p^2\rangle$ (though the total zero-modes in the left and the right sectors are three in both cases). Thus it seems natural to incorporate another zero-mode if we expect that the $SL(2, R)$ model smoothly leads to the flat model in the weak curvature limit. The added zero-mode turn out to give the sector satisfying

$$\int \frac{\partial \gamma}{\gamma} \neq 0,$$

which is called the long string sector [20]. Then by carefully treating the zero-mode part, we find that the on-shell condition picks up the spins $j = -1/2 + i\rho$ which precisely correspond to $D_{pc}$. Finally, using the expressions in terms of the free bosons, the no-ghost theorem is
shown similarly to the flat case. In this proposal, the smooth flat limit is achieved by taking $k \to \infty$. In addition, the applications to the black holes discussed in the introduction appear to be straightforward [6].

Nevertheless, as in the previous case, it is not clear if this proposal is compatible with the other consistency conditions. (For the discussions of the OPE and the modular invariance in this case, see [23] and [4] respectively.)

In fact, we must say that there is no agreement about how to construct the sensible theory of the $SL(2, R)$ strings. Therefore, to clarify this issue it is very important to further investigate the fundamental problems such as the modular invariance, the closure of the OPE, how to choose the spectrum and how to calculate the correlators.

3 $SL(2, C)/SU(2)$ WZW model

In the previous section, we discussed the fundamental open questions about the $SL(2, R)$ WZW model. Now let us turn to the discussion of the $SL(2, C)/SU(2) = H^+_3$ WZW model. The precise formulation of the AdS/CFT correspondence requires Euclidean anti-de Sitter spaces [17]. Euclidean $AdS_3$ is called $H^+_3$ and given by

$$
\begin{align*}
    ds^2 &= -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2, \\
    -l^2 &= -x_0^2 + x_1^2 + x_2^2 + x_3^2.
\end{align*}
$$

Note the sign-flips compared with the Lorentzian case (2.1). This space is also a maximally symmetric space with negative constant curvature.

To get a string background, one needs to introduce the NS $B_{\mu \nu}$ field. In some parametrization, the action takes the form,

$$
S = \frac{k}{\pi} \int d^2 \sigma \left( \partial \phi \partial \phi + e^{2\phi} \partial \bar{\gamma} \partial \bar{\gamma} \right). \tag{3.2}
$$

Here $\bar{\gamma} = \gamma^*$ and $\phi \to +\infty$ corresponds to the boundary of $H^+_3$. If $\gamma$ and $\bar{\gamma}$ are independent, the geometry becomes Lorentzian $AdS_3$. This action is obtained also by substituting $g(z) = h(z)h^\dagger(z)$ with

$$
    h = \begin{pmatrix}
        1 & \gamma \\
        0 & 1
    \end{pmatrix}
    \begin{pmatrix}
        e^{-\phi/2} & 0 \\
        0 & e^{\phi/2}
    \end{pmatrix} \in SL(2, C) \tag{3.3}
$$

into (2.3) [7]. From this construction, the coset structure $SL(2, C)/SU(2)$ is obvious and one finds that the model is actually a WZW model. In terms of $\phi$, $\gamma$ and $\bar{\gamma}$, the functional measure takes a non-trivial form

$$
    D\phi D(\phi \gamma)D(e^\phi \bar{\gamma}). \tag{3.4}
$$
The action has the current algebra symmetry $\hat{sl}(2, C) \times \hat{sl}(2, C)^\ast$. In this case, the left and the right symmetries are the complex conjugate to each other. The currents of the global symmetry acting on the zero-mode part are realized by

\[
\begin{align*}
J^{-}_0 &= \partial_\gamma, \\
J^{0}_0 &= \gamma \partial_\gamma - \frac{1}{2} \partial_\phi, \\
J^{+}_0 &= \gamma^2 \partial_\gamma - \gamma \partial_\phi - e^{-2\phi} \partial_\gamma.
\end{align*}
\]

Note that the last term in the second line. This does not affect the commutation relations but is necessary to assure the invariance of the action.

A convenient way to generate the primary fields is to use the following functionals [8]

\[
V^j = \left[ (\gamma - x)(\bar{\gamma} - \bar{x}) e^\phi + e^{-\phi} \right]^{2j},
\]

where $x$ and $\bar{x}$ are some parameters. By expanding $V^j$ in terms of $x^j + m$ and $\bar{x}^{j + \bar{m}}$, one gets the primary fields $V^j_{m, \bar{m}}$ with the definite eigenvalues of $J^{-}_0, J^{0}_0$ and the left and the right Casimirs. It turns out that $x$ and $\bar{x}$ are interpreted as the coordinates of the boundary CFT [21].

Similarly to the Lorentzian case, the Hilbert space is decomposed into the representations of $SL(2, C)$. Schematically [7],

\[
L^2(H^+_3) \sim \int_0^\infty d\rho \rho^2 D^{-1/2+i\rho}_{pc}.
\]

We remark that only the principal continuous series $D_{pc}$ appear and there are no discrete series [1].

Furthermore, by (i) introducing auxiliary fields $\beta$ and $\bar{\beta}$, (ii) taking into account the non-trivial measure (3.4) and (iii) rescaling $\phi$, one obtains the following action,

\[
S = \frac{1}{2\pi} \int d^2\sigma \left( \partial_\phi \partial_\phi + \beta \bar{\beta} \partial_\gamma + \beta \bar{\beta} \partial_\gamma - \beta \bar{\beta} e^{-2\phi/\alpha} - \frac{2}{\alpha_+} \phi \sqrt{\hat{g} \hat{R}} \right),
\]

where $\alpha_+ = \sqrt{2(k - 2)}$ and $\hat{g}$ and $\hat{R}$ are the background metric and curvature of the world-sheet, respectively. In this expression, the interaction term $\beta \bar{\beta} e^{-2\phi/\alpha}$ drops out in the limit $\phi \to \infty$. Thus we obtain a free theory in that limit, namely, near the boundary of $H^+_3$. The last term in $J^+_0$ in (3.5) also drops out and we get the free-field expression in that limit.

**Puzzles**

This WZW model has been intensively studied recently [20]-[24] in relation to the AdS/CFT correspondence. We may need to be careful about to what extent and how the string theory

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1In section 2, we considered the representations of $SL(2, R)$. Here we are considering the corresponding representations of $SL(2, C)$. In this case, the spectrum for $j = -1/2 + i\rho$ is given by $m = (ip + n)/2, \bar{m} = (ip - n)/2$ with $p \in \mathbb{R}, n \in \mathbb{Z}$. 


without RR charges is relevant to the AdS/CFT correspondence (see, for example, [34]). However, if the correspondence is naively taken, one finds some puzzles. They are summarized in the following table of the correspondence,

| string (WZW model)          | supergravity | CFT           |
|-----------------------------|--------------|---------------|
| discrete series (non-normalizable) | KK mode      | chiral primary |
| continuous series (normalizable) | ??           | ??            |

Namely, although the Hilbert space of the $H_3^+$ WZW model consists of the principal continuous series, we do not find the corresponding objects on the supergravity and the CFT sides. Moreover, following the argument in [20], the scaling dimension of a boundary CFT operator and the $sl_2$ spin of the corresponding operator of the $H_3^+$ WZW model are related by $h = -j$. If this is valid also for the principal continuous series, the dimension of the corresponding boundary CFT operator becomes complex. Thus it is hard to interpret the correspondence for the continuous series even if it exists.

These puzzles might not lead to an immediate contradiction because, as discussed in [21], the $H_3^+$ WZW model is a non-compact CFT and hence there might not be the state-operator correspondence as in the Liouville theory. Nevertheless, in order to complete the correspondence, it seems necessary to further investigate these puzzles. To this end, we may need to study the $H_3^+$ WZW model in detail. Again, the fundamental consistency conditions play the role of the guideline there.

How are then the precise discussions of the $H_3^+$ WZW model possible? One way is to use the free field approximation. By this approach, we can get much information [20, 35] but this is valid only near the boundary $\phi \to \infty$. Another way is to use the generating functionals of the primary fields (3.6) following [8, 21, 23, 24]. In this approach, the full analysis beyond the free field approximation is possible but it tends to be semi-classical (see, however, [8, 24] regarding the full quantum analysis based on the bootstrap). Therefore it would be nice to have a description beyond the free field or the semi-classical treatment. We will return to this point later.

4 Modular invariance

In the preceding sections, we put an emphasis on the importance of the further investigations of the fundamental problems. Here we would like to discuss some attempts at clarifying the modular invariance of the $SL(2, R)$ WZW model. Although the modular invariance in
the non-compact case is not well understood, there are several arguments in the $SL(2, R)$ case. For example, the modular invariants are discussed in \cite{3} by using the $\hat{sl}(2, R)$ characters based on the discrete series $D_{hw}$ and $D_{lw}$ and by incorporating some new sectors corresponding to winding modes. They are also discussed in \cite{4} using the characters for $D_{pc}$ along the line of \cite{3}.

In this section, we will focus on the possibility of constructing the modular invariants from the characters for $D_{hw}$ and $D_{lw}$ without incorporating any additional sectors as in \cite{3}. For details, see \cite{27}. This issue is also discussed in \cite{26}.

Let us start with the definition of the characters. For the current algebras based on compact Lie groups, the characters are naturally defined using three variables. With this in mind, we define the characters for the discrete series by

$$
\chi_j(z, \tau, u) \equiv e^{2\pi i k u} \sum e^{-2\pi i p_{J_0}(z, \tau)} e^{2\pi i (L_0 - \frac{j}{2})}.
$$

The summation is taken over the entire module of the current algebra representations. The plus sign in the first factor $e^{+2\pi i k u}$ is due to the change $k \rightarrow -k$ compared with the compact case. To calculate these characters, one needs to know about singular vectors. For a generic highest (or lowest) weight representation, which is not necessarily $D_{hw}$ or $D_{lw}$, the current module has singular vectors when one of the following conditions is satisfied \cite{36}:

1. $2j + 1 = s + (k - 2)(r - 1)$,
2. $2j + 1 = -s - r(k - 2)$,
3. $k - 2 = 0$,

where $r, s$ are positive integers.

The characters $\chi_j$ in generic cases seem unknown. However, when there are no singular vectors, they are given by \cite{37, 3}:

$$
\chi_{\mu}^{lw}(z, \tau, u) = e^{2\pi i k u} e^{-2\pi i \mu z} q^{\frac{\mu^2}{8}} i \vartheta^{-1}(z|\tau),
$$

for $D_{hw}$ and $\chi_{\mu}^{lw}(z, \tau, u) = \chi_{\mu}^{lw}(-z, \tau, u)$ for $D_{lw}$. Here $q = e^{2\pi i \tau}$, $\mu \equiv j + 1/2$ and

$$
\vartheta_1(z|\tau) = 2q^{1/8} \sin(\pi z) \prod_{n=1}^{\infty} (1 - q^n) (1 - q^n e^{2\pi i z}) (1 - q^n e^{-2\pi i z}).
$$

Since $\chi_{\mu}^{lw}(z, \tau, u) = -\chi_{-\mu}^{lw}(z, \tau, u)$, $\chi_{\mu}^{lw}$ with $\mu \geq 0$ ($j \geq -1/2$) are regarded as $-\chi_{\mu}^{lw}$ with $\mu \leq 0$. Thus we will use only $\chi_{\mu}^{lw}$ and drop the superscript hw. We remark that one cannot consider the specialized characters $\chi_{\mu}(0, \tau, 0)$ since they diverge in the limit $z \rightarrow 0$ because of the infinite degeneracy with respect to $L_0$.

In our normalization of $(z, \tau, u)$, the modular transformations are generated by

$$
S: \quad (z, \tau, u) \rightarrow \left( \frac{z}{\tau}, -\frac{1}{\tau}, u + \frac{z^2}{4\tau} \right),
$$

$$
T: \quad (z, \tau, u) \rightarrow (z, \tau + 1, u).
$$

(4.5)
Under $T$-transformation, the characters just get phases,

$$\chi_\mu(z, \tau + 1, u) = e^{-2\pi i \left( \frac{\mu^2}{k-2} + \frac{1}{8} \right)} \chi_\mu(z, \tau, u).$$  \hspace{1cm} (4.6)

For $k-2 < 0$, the $S$-transformation of $\chi_\mu(z, \tau, 0)$ is given in [20]. In our case with three variables, it reads as

$$\chi_\mu\left( \frac{z}{\tau}, -\frac{1}{\tau}, u + \frac{z^2}{4\tau} \right) = \sqrt{-\frac{2}{2-k}} \int_{-\infty}^{\infty} d\nu e^{4\pi i \frac{\mu\nu}{k-2}} \chi_\nu(z, \tau, u).$$ \hspace{1cm} (4.7)

For $k-2 > 0$, the right-hand side of (4.7) does not converge on the upper half plane of $\tau$. Instead, after some calculation, we get a slightly different result,

$$\chi_\mu\left( \frac{z}{\tau}, -\frac{1}{\tau}, u + \frac{z^2}{4\tau} \right) = \sqrt{-\frac{2}{k-2}} \int_{-\infty}^{\infty} d\nu e^{-4\pi i \frac{\mu
u}{k-2}} \chi_{i\nu}(z, \tau, u).$$ \hspace{1cm} (4.8)

Note that an imaginary $\mu = j + 1/2$ corresponds to a spin of the principal continuous series but $\chi_{i\mu}$ are not the characters for those representations.

As a simpler case, we will first discuss the possibility of constructing modular invariants using finite number of the discrete series characters. Given the explicit forms of $\chi_\mu$, we can then show that it is impossible to make modular invariants from finite number of the discrete series characters without singular vectors, i.e., from $\chi_\mu$. Similarly, since the characters with singular vectors are obtained by subtracting states from $\chi_\mu$, the above statement is extended to some cases including singular vectors. In fact, we can show that, for $k > 2$, it is impossible to construct modular invariants from finite number of the characters based on either $D_{hw}$ or $D_{lw}$. The arguments are simple applications of Cardy’s for $c > 1$ CFT [38] and we will omit them. To further extend the latter statement to the cases including both $D_{hw}$ and $D_{lw}$, the explicit forms of the characters with singular vectors seem to be necessary. In addition, we notice that a similar statement does not hold for $k < 2$. To see this, we note that the arguments do not use any special properties of the discrete unitary series and hence it is the same as for a generic highest (or lowest) weight $\hat{sl}(2, R)$ representations. However, when $k < 2$, modular invariants using finite number of the characters are actually known for the so-called admissible representations [39].

Next, let us move on to the case in which infinitely many characters are allowed. For the time being, we will discuss the modular invariants using $\chi_\mu$ only. In such a case, using their modular properties we can show that it is impossible to construct modular invariants only from $\chi_\mu$ with $\mu$ belonging to a finite interval $\mu \in [\mu_1, \mu_2]$ even if infinitely many $\chi_\mu$ are used. This might seem obvious from the $S$-transformation of $\chi_\mu$ since the right-hand sides of (4.7) and (4.8) does not close within $\chi_\mu$ with $\mu \in [\mu_1, \mu_2]$. However, we need to be careful because we are considering an infinite dimensional space of the characters $\chi_\mu$. For instance, it is not clear which $\chi_\mu$ are independent and whether or not the expressions (4.7) and (4.8) are
unique. In fact, it may be possible to get different expressions by deforming the integration contours in (4.7) and (4.8). In any case, the detailed argument is given in [27].

Since, for $k > 2$, $\chi_\mu$ become divergent for $\text{Im}\tau > 0$ as $|\mu| \to \infty$, the above statement means that for $k > 2$ it is impossible to construct modular invariants only from $\chi_\mu$. Thus the possibility of constructing modular invariants from $\chi_\mu$ is limited to the case where $k < 2$ and $\chi_\mu$ with $|\mu| \to \infty$ are included. In this case, we can actually construct a modular invariant,

$$Z_{\text{diag}}(z, \tau, u) = \int_{-\infty}^{\infty} d\mu \ |\chi_\mu|^2 = \int_{-\infty}^{0} d\mu \ (|\chi_{\mu}^{\text{hw}}|^2 + |\chi_{\mu}^{\text{lw}}|^2)$$

$$= \frac{1}{2} e^{-4\pi k \text{Im} u} e^{(2-k)\pi \text{Im} z^2} \sqrt{\frac{2 - k}{\text{Im} \tau}} |\vartheta^{-2}(z|\tau)| .$$

(4.9)

The diagonal partition function with $u = 0$, i.e., $Z_{\text{diag}}(z, \tau, 0)$, was discussed in [26]. In our case, it is straightforward to check that $Z_{\text{diag}}(z, \tau, u)$ is modular invariant owing to the presence of $u$. Although it may be interpreted as a kind of a twisted partition function, its physical meaning is still unclear (recall that we cannot set $u = z = 0$).

As pointed out also in [26], $Z_{\text{diag}}(z, \tau, 0)$ was discussed in [7] in the context of a path-integral approach to the $H_3^+$ WZW model. Since this model has the $\hat{sl}(2, C) \times \hat{sl}(2, C)^*$ symmetry, the diagonal partition function may be understood also as the partition function of this model. However, in [7] different spectrum seems to be summed up. It is interesting to consider the precise relationship between the approach here and the one in [7].

In order to discuss a generic case including the characters with singular vectors, we may again need the explicit forms of such characters. Nevertheless, it turns out that the case without singular vectors covers physically interesting cases and gives important implication to the unitarity bound (2.13). This is because the condition of the singular vectors (4.2) implies that there are no singular vectors within (2.13). Furthermore, since the spins $j$ in that bound belong to a finite interval, our results indicate that one cannot construct modular invariants only from the discrete series characters based on the representations satisfying the unitarity bound (2.13). This means that one cannot make a consistent string theory on $SL(2, R) = AdS_3$ only from the spectrum within (2.13). This was already discussed in [26], but we believe that at least we have refined the argument a little.

Since there exist ghosts for the discrete series outside the unitarity bound, simply adding such spectrum may not give a consistent theory. Therefore, the possibilities for a consistent theory seem (a) to use the discrete series satisfying (2.13) but include some new sectors with different characters from $\chi_\mu$ as in [3], and/or (b) to use the spectrum of other representations as in [3, 8]. To settle down this problem, further investigations are necessary.
5 Correlation functions

In the previous section, we discussed the modular invariance and saw that it gives an important information about the spectrum of the strings on $SL(2, R)$. Finally in this section, we will discuss the calculation of the correlations functions of the $H_3^+$ WZW model [28]. This is important not only by itself but also for studying the OPE and hence the spectrum of the model.

The outstanding feature of the $H_3^+$ WZW model is that it allows us the Lagrangian approach [40, 7]. It is alternative and complementary to the current algebra approach which is often used. Actually in the Lagrangian approach we may be able to get a description beyond the free field and the semi-classical approximations.

To see this, let us first recall the action (3.2) and the functional measure (3.4). Surprisingly, with (3.2) and (3.4) it is possible to carry out the path-integrals for some correlators [40, 7]. For this purpose, we need (i) the ‘partition function’ obtained after integrating out $\gamma$ and $\bar{\gamma}$,

$$
\exp[-S(\phi)] = \exp\left[\frac{1}{\pi} \int d^2\sigma (k - 2) \partial \phi \partial \phi - \frac{1}{4} \phi \sqrt{\hat{g} \hat{R}}\right],
$$

(5.1)

and (ii) the propagator,

$$
\langle \gamma(z) \bar{\gamma}(\bar{w}) \rangle = \frac{1}{k\pi} \int d^2 y \frac{e^{-2\phi(y)}}{(z - y)(\bar{w} - y)}.
$$

(5.2)

The ‘partition function’ implies that the resulting effective theory of $\phi$ is a free theory with a background charge. Since the the propagator contains the factor $e^{-2\phi(y)}$, it plays a similar role to the screening charges in the free field approach. Using (5.1) and (5.2), one can calculate the correlators of the form

$$
\left\langle \prod_i e^{a_i \phi(\mathbf{z}_i)} \gamma^{b_i}(\mathbf{z}_i) \bar{\gamma}^{c_i}(\bar{\mathbf{z}}_i) \right\rangle.
$$

(5.3)

We are interested in the correlation functions of the primary fields in the discrete series. In the free field approximation, the primaries are given by

$$
V^{j(\text{free})}_{m,\bar{m}} = e^{2j\phi_{j+m} \gamma_{j+m}}.
$$

(5.4)

In the full theory, they are obtained by expanding the functionals (3.6) and take the form

$$
V^{j}_{m,\bar{m}} = \sum\limits_{j',\bar{m}',\bar{m}'} C^{j'}_{m',\bar{m}'} V^{j'(\text{free})}_{m',\bar{m}'} ,
$$

(5.5)

where $C^{j'}_{m',\bar{m}'}$ are some coefficients.
We would like to calculate the correlators among the above primary fields. A similar calculation was actually carried out in the case of the finite dimensional representations [7]. In our infinite dimensional case, it seems that we need to carefully choose the ‘conjugate’ fields paired with the above primaries. Once we have obtained the correlation functions, we can extract important information about the $H^+_3$ WZW model and in turn this gives useful insights into the AdS/CFT correspondence. We would like to report progress in this direction elsewhere [28].

6 Summary

In this talk, we first noted that the string theory on $AdS_3$ is important in various respects besides in relation to the AdS/CFT correspondence. We then reviewed the $SL(2, R)$ and the $H^+_3$ WZW models which describe the string propagations on Lorentzian and Euclidean $AdS_3$, respectively. We saw that in spite of the recent intensive studies there still remain open questions at the fundamental level. An emphasis was put on the importance of further investigating the fundamental problems such as the modular invariance, the closure of the OPE, the issue of the spectrum and the calculation of the correlation functions. With this in mind, we discussed some attempts at clarifying such problems. First, we discussed the modular invariance of the $SL(2, R)$ WZW model and showed that it gives important information of the spectrum. Next, we discussed the correlation functions of the $H^+_3$ WZW model using the full Lagrangian approach.

Although our attempts were quite incomplete, some of the problems seem still tractable. Thus further investigations will lead us to a deeper understanding of the string theory on $AdS_3$. We expect that such investigations also shed some light on the AdS/CFT correspondence.

Note added

Some of the questions raised in this talk have been discussed also in recent papers [41].
Acknowledgements

I would like to thank N. Ishibashi, A. Kato and K. Okuyama for the collaborations on the subjects discussed here. I would also like to thank I. Bars, J. de Boer, A. Giveon, H. Ishikawa, K. Ito, M. Kato, P.M. Petropoulos and S.-K. Yang for useful discussions and correspondences. Finally, I am grateful to the organizers of the workshop ‘Developments in Superstring and M-theory’ held at YITP, Kyoto, 27-29 October, 1999, for giving a chance to deepen my understanding on the string theory on $AdS_3$.

References

[1] J. Balog, L. O’Raifeartaigh, P. Forgács and A. Wipf, Nucl. Phys. B325 (1989) 225.

[2] P.M. Petropoulos, Phys. Lett. 236B (1990) 151;
   N. Mohameddi, Int. J. Mod. Phys. A5 (1990) 3201;
   S. Hwang, Nucl. Phys. B354 (1991) 100;
   J. Evans, M. Gaberdiel and M. Perry, Nucl. Phys. B535 (1998) 152, hep-th/9806024.

[3] M. Henningson, S. Hwang, P. Roberts and B. Sundborg, Phys. Lett. 267B (1991) 350.

[4] I. Bars and D. Nemeschansky, Nucl. Phys. B348 (1991) 89.

[5] I. Bars, Phys. Rev. D53 (1996) 3308, hep-th/9503205.

[6] Y. Satoh, Nucl. Phys. B513 (1998) 213, hep-th/9705208.

[7] K. Gawędzki, Nucl. Phys. B328 (1989) 733; Non-compact WZW conformal field theories, hep-th/9110076, Proceedings of the Cargèse Summer Institute.

[8] J. Teschner, Nucl. Phys. B546 (1999) 390, hep-th/9712256; Nucl. Phys. B546 (1999) 369, hep-th/9712258.

[9] J.G. Russo and A.A. Tseytlin, Nucl. Phys. B449 (1995) 91; E. Kiritsis, C. Kounnas and D. Lüst, Phys. Lett. 331B (1994) 321.

[10] E. Witten, Phys. Rev. D44 (1991) 314; G. Mandal, A. Senguputa and S. Wadia, Mod. Phys. Lett. A6 (1991) 1685.

[11] G.T. Horowitz and D.L. Welch, Phys. Rev. Lett. 71 (1993) 328, hep-th/9302126; N. Kaloper, Phys. Rev. D48 (1993) 2598, hep-th/9303007; A. Ali and A. Kumar, Mod. Phys. Lett. A8 (1993) 2045, hep-th/9303032.

[12] M. Natsuume and Y. Satoh, Int. J. Mod. Phys. A13 (1998) 1229, hep-th/9611041.
[13] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849 (1992), hep-th/9204099;
M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48 (1993) 1506, gr-qc/9302012.

[14] S. Hyun, hep-th/9704005, U duality between three-dimensional and higher dimensional black holes;
K. Sfetsos and K. Skenderis, Nucl. Phys. B 517 (1998) 179, hep-th/9711138.

[15] M. Cvetić and A.A. Tseytlin, Phys. Lett. 366B (1996) 95.

[16] Y. Satoh, Phys. Rev. D 59 (1999) 084010, hep-th/9810135.

[17] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200;
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. 428B (1998) 105, hep-th/9802109;
E. Witten, Adv. Theor. Math. Phys. 2 (1998) 2253, hep-th/9802150.

[18] I. Pesando, JHEP 9902 (1999) 7, hep-th/9809145;
J. Rahmfeld and A. Rajaraman, Phys. Rev. D 60 (1999) 064014, hep-th/9809164;
J. Park and S.-J. Rey, JHEP 9901 (1999) 1, hep-th/9812062.

[19] N. Berkovits, C. Vafa and E. Witten, JHEP 9903 (1999) 018, hep-th/9902098.

[20] A. Giveon, D. Kutasov and N. Seiberg, Adv. Theor. Math. Phys. 2 (1998) 733, hep-th/9806194.

[21] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, JHEP 9812 (1998) 26, hep-th/9812046.

[22] K. Ito, Phys. Lett. 449B (1999) 48, hep-th/9811002;
K. Hosomichi and Y. Sugawara, JHEP 9901 (1999) 013, hep-th/9812100;
S. Yamaguchi, Y. Ishimoto and K. Sugiyama, JHEP 9902 (1999) 026, hep-th/9902079.

[23] D. Kutasov and N. Seiberg, JHEP 9904 (1999) 8, hep-th/9903219.

[24] J. Teschner, Operator product expansion and factorization in the $H_3^+$ WZNW model, hep-th/9906215.

[25] I. Bars, C. Deliduman and D. Minic, String theory on AdS$_3$ revisited, hep-th/9907087.

[26] P.M. Petropoulos, String theory on ADS$_3$: some open questions, hep-th/9908189.

[27] A. Kato and Y. Satoh, to appear (Modular invariance of string theory on AdS$_3$, hep-th/0001063).

[28] N. Ishibashi, K. Okuyama and Y. Satoh, work in progress (Path integral approach to string theory on AdS$_3$, hep-th/0005152).
[29] N. Ja. Vilenkin and A.U. Klimyk, *Representations of Lie Groups and Special Functions*, (Kluwer Academic Publishers, Dordrecht, 1991).

[30] E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. B271 (1986) 333.

[31] P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303.

[32] D. Gepner and E. Witten, Nucl. Phys. B278 (1986) 493.

[33] M. Wakimoto, Comm. Math. Phys. 104 (1986) 605.

[34] N. Seiberg and E. Witten, JHEP 9904 (1999) 017, hep-th/9903224.

[35] G. Giribet and C.A. Núñez, JHEP 9911 (1999) 031, hep-th/9909149.

[36] V.G. Kač and D.A. Kazhdan, Adv. Math. 34 (1979) 97;
F.G. Malikov, B.L. Feigin and D.B. Fuks, Funct. Anal. and Appl. 20 (1986) 103.

[37] P.A. Griffin and O.F. Hernandez, Nucl. Phys. B356 (1991) 287;
K. Sfetsos, Phys. Lett. 271B (1991) 301;
I. Bakas and E. Kiritsis, Int. J. Mod. Phys. A7 (1992) 55, hep-th/9109029;
P.M. Petropoulos, Thèse de doctorat, Ecole Polytechnique, 1991.

[38] J. Cardy, Nucl. Phys. B270 (1986) 186.

[39] V. Kač and M. Wakimoto, Proc. Nat. Acad. Sci. USA 85 (1988) 4956;
I.G. Koh and P. Sorba, Phys. Lett. 215B (1988) 723.

[40] Z. Haba, Int. J. Mod. Phys. A4 (1989) 267.

[41] J. Maldacena and H. Ooguri, *Strings in AdS3 and SL(2,R) WZW model: I*, hep-th/0001053;
A.L. Larsen and N. Sánchez, *Quantum coherent string states in AdS3 and SL(2,R) WZWN model*, hep-th/0001180;
I. Pesando, *Some remarks on the free fields realization of the bosonic string on AdS3*, hep-th/0003036;
Y. Hikida, K. Hosomichi, Y. Sugawara, *String theory on AdS3 as discrete light-cone Liouville theory*, hep-th/0005065.