RELATIVISTIC DYNAMICS OF VORTEX DEFECTS IN SUPERFLUIDS

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Abstract: Superfluid condensates are known to occur in contexts ranging from laboratory liquid helium to neutron stars, and are also likely to occur in cosmological phenomena such as axion fields. In the zero temperature limit, such condensates are describable at a mesoscopic level by irrotational configurations of simple relativistic perfect fluid models. The general mechanical properties of such models are presented here in an introductory review giving special attention to the dynamics of vorticity flux 2-surfaces and the action principles governing both individual flow trajectories and the evolution of the system as a whole. Macroscopic rotation of such a condensate requires the presence of a lattice of quantised vortex defects, whose averaged tension violates perfect fluid isotropy. It is shown that for any equation of state (relating the mass density \( \rho \) to the pressure \( P \)) the mesoscopic perfect fluid model can be extended in a uniquely simple and natural manner to a corresponding macroscopic model (in a conformally covariant category) that represents the effects of the vortex fibration anisotropy. The limiting case of an individual vortex defect is shown to be describable by a (“global”) string type model with a variable tension \( T \) (obtained as a function of the background fluid density) whose “vorton” (i.e. closed loop equilibrium) states are derived as an exercise.
1 Introduction.

These lectures offer an introduction to the dynamics of vortices in simple superfluid models of a general category that includes the kind appropriate in the context of laboratory condensed matter for the representation of Helium-4 in the zero temperature limit (for which no "normal" entropy carrying constituent is present), as well as the ultrarelativistic "stiff" kind that has been widely used [1] for the representation of a massless axion field in cosmology. Between these extremes, this category also includes the kind of model suitable as a simple approximation for the description of the neutron-pair condensation superfluid that is generally believed (for both theoretical and observational reasons [2]) to occur in the intermediate layers of neutron stars. (A more sophisticated treatment would include allowance for the presence of an interpenetrating ionic crust lattice or of an independently superconducting proton pair condensate, not to mention complications such as spin.)

The declared purpose of this school for which these lectures are intended is to study analogies between low temperature laboratory phenomena and high energy cosmological phenomena. However it is desirable, when possible, to reinforce mere analogy by interpolation. I would therefore like to emphasize the particular interest for this purpose of the intermediate regime of neutron star interiors, about which a great deal of admittedly indirect information is available from analysis of pulsar frequency variations. In contrast with most other areas of astrophysics, but in common with typical laboratory applications of condensed matter physics, the temperatures in typical neutron star interiors can usually be considered to be very low compared with the relevant energy scales. On the other hand, in contrast with typical laboratory applications of condensed matter physics, but in common with high energy cosmological scenarios, the treatment of neutron star interiors requires allowance for significant relativistic effects.

While evidently indispensible for an accurate treatment of neutron star matter, a relativistic treatment is commonly – but wrongly – considered to be an unnecessary complication for low temperature laboratory applications. For a reasonably accurate treatment of laboratory liquid Helium a relativistic treatment is indeed unnecessary, but what is wrong is to suppose that it is more complicated. On the contrary, as I hope these lectures will make clear, the relativistic treatment, in so far as it is available, is mathematically simpler. (The essential reason for this is that the Lorentz group is, in the technical sense "semi-simple" whereas the Galilei group is not.)
The work, in collaboration with David Langlois [3], on which these lectures are based, was originally inspired by the observation by Rick Davis and Paul Shellard [4] (in precisely the spirit that is the official raison d’être for the present school) of a strong analogy between the behaviour of vortices in the “stiff” (ultrarelativistic) massless axion model at one extreme, and in the incompressible superfluid model that is commonly used for the description of zero temperature Helium-4 at the opposite extreme.

What is shown here is the way to carry out the analogous, but not quite so trivially simple, treatment that is needed for the intermediate category of generically compressible superfluid models. At a mesoscopic level these models will be of irrotational perfect fluid type, and will be characterised by a subluminal speed, $c_I$ say, of ordinary “first” sound, that will be determined – by an equation of state specifying the pressure $P$ as a function of the mass density $\rho$ – according to the familiar formula

$$c_I = \left(\frac{dP}{d\rho}\right)^{1/2}.$$  \hfill (1)

Rotation at a macroscopic level will entail the presence of an Abrikosov type lattice of quantised vortex defects whose averaged tension produces a deviation from perfect fluid isotropy. Analysis of individual vortex tubes [5] indicates that, in the limit for which their relative “drift” or “flight” velocity is highly subsonic, their averaged effect can be described by the inclusion of an extra isotropy violating term of a uniquely simple and natural kind [3] in the action for a corresponding macroscopic model. Despite their violation of isotropy, such models (for different equations of state) will be shown to belong to a category that is conserved by conformal transformations. More particularly, it will be shown that the specific model corresponding to the high pressure limit of a gas of negligibly interacting particles as characterised by $c_I^2 = c^2/3$ (where $c$ is the speed of light) has the same kind of conformal invariance property as the well known example of Maxwell’s equations.

The scope of the present review does not extend to the recently perfected relativistic analogue [6, 7] of Landau’s original non-dissipative two constituent superfluid model (which has recently been shown to be expressible in a very elegant Galilean covariant form [8]) for the purpose of allowing for the dynamical effect of the “normal” entropy flux current that would be present at a non zero temperature). Such effects are not the ones that are most important in the context of neutron stars, the main application for which the work described here is intended, where the temperature will typically be so low (compared with the relevant M.e.v. range energy scales) that
thermal corrections can for most purposes be neglected. For application to neutron stars, a more important kind of generalisation is to a relativistic models capable of describing neutron superfluid penetration of the solid material forming the crust, and of describing the protonic superconductivity that is expected within the neutron superfluid at a deeper level below the crust.

2 Canonical treatment of relativistic flow trajectories.

Before proceeding it is desirable to recall some essentials the relativistic kinematics and dynamics, particularly in view of the regrettable tradition in non-relativistic fluid theory – and most notably in non-relativistic superfluid theory – of obscuring the essential distinction between velocity (which formally belongs in a tangent bundle) and momentum (which formally belongs in a cotangent bundle) despite the fact that the distinction is generally respected in other branches of non-relativistic condensed matter theory, such as solid state physics, where the possibility of non-alignment between the 3-velocity $v^a$, and the effective 3-momentum $p_a$ of an electron travelling in a metallic lattice is well known.

In a non relativistic treatment it is only in strictly Cartesian (rather than e.g. cylindrical or comoving) that the distinction between contravariant entities such as the velocity $v^a$ and covariant entities such as the momentum $p_a$ can be ignored. In a relativistic treatment, even using coordinates $x^\mu \leftrightarrow \{t, x^a\}$ of Minkowski type, with a flat spacetime metric $g_{\mu\nu}$ whose components are of the fixed standard form $\text{diag}\{-c^2, 1, 1, 1\}$, (where $c$ is the speed of light) the necessity of distinguishing between raised and lowered indices is inescapable. Thus for a trajectory parametrised by proper time $\tau$, the corresponding unit tangent vector

$$ u^\mu = \frac{dx^\mu}{d\tau} $$

is automatically, by construction a contravariant vector: its space components, $u^a = \gamma v^a$ with $\gamma = (1-v^2/c^2)^{-1/2}$ will be unaffected by the index lowering operation $u^\mu \rightarrow u_\mu = g_{\mu\nu} u^\nu$, but its time component $u^0 = dt/d\tau = \gamma$ will differ in sign from the corresponding component $u_0 = -\gamma c^2$ of the associated covector $u_\mu$. On the other hand the 3-momentum $p_a$ and energy $E$ determine a 4-momentum covector $\pi_\nu$ that is intrinsically covariant, with
components $\pi_a = p_a$, $\pi_a = -E$. The covariant nature of the momentum can be seen from the way it is introduced by the defining equation,

$$\pi_\nu = \frac{\partial L}{\partial u^\nu},$$

in terms of the relevant position and velocity dependent Lagrangian function $L$, from which the corresponding equation of motion is obtained in the well known form

$$\frac{d\pi_\nu}{d\tau} = \frac{\partial L}{\partial x^\nu}.$$

In the case of a free particle trajectory, and more generally for fluid flow trajectories in all the simple “barotropic” perfect fluid models with which the present lectures will be concerned, the Lagrangian function will have the familiar standard form

$$L = \frac{1}{2} \mu g_{\mu\nu} u^\mu u^\nu - \frac{1}{2} \mu c^2,$$

in which (unlike what is needed for more complicated chemically inhomogeneous models\textsuperscript{[11, 9]} it is the same scalar spacetime field $\mu$ that plays the role of mass in the first term and that provides the potential energy contribution in the second term. The momentum will thus be given by the simple proportionality relation

$$\pi_\nu = \mu u_\nu,$$

so that one obtains the expressions $E = \gamma \mu c^2$, $p_a = \mu \gamma v_a$, in which the field $\mu$ is interpretable as the relevant effective mass.

In the case of a free particle model, the effective mass $\mu$ will of course just be a constant, $\mu = m$. This means that if, as we have been supposing so far, the metric $g_{\mu\nu}$ is that of flat Minkowski type, the resulting free particle trajectories will be obtainable trivially as straight lines. However the covariant form of the equations (2) to (6) means that they will still be valid for less trivial cases for which, instead of being flat, the metric $g_{\mu\nu}$ is postulated to have a variable form in order to represent the effect of a gravitational field, such as that of a Kerr black hole (for which, as I showed in detail in a much earlier Les Houches school\textsuperscript{[12]}, the resulting non-trivial geodesic equations still turn out to be exactly integrable).

In the case of the simple perfect fluid models with which we shall be concerned here, the effective mass field $\mu$ will be generically non-uniform. In these models the equation of state giving the pressure $P$ as a function of the mass density $\rho$ can most conveniently be specified by first giving $\rho$
in terms of the corresponding conserved number density \( n \) by an expression that will be decomposable in the form
\[
\rho = mn + \frac{\epsilon}{c^2}, \tag{7}
\]
in which \( m \) is a fixed “rest mass” characterising the kind of particle (e.g. a Cooper type neutron pair) under consideration, while \( \epsilon \) represents an extra compression energy contribution. The pressure will then be obtainable using the well known formula
\[
P = (n\mu - \rho)c^2, \tag{8}
\]
in which the effective dynamical mass \( \mu \) (or equivalently the “specific enthalpy” \( \mu c^2 \)) is given by
\[
\mu = \frac{d\rho}{dn} = m + \frac{1}{c^2} \frac{d\epsilon}{dn}. \tag{9}
\]
It is this that is to be taken as the effective mass function appearing in the specification (5) of the relevant Lagrangian.

When one is dealing not just with a single particle trajectory but a spacefilling fluid flow, it is possible and for many purposes desirable to convert the Lagrangian dynamical equation (4) from particle evolution equation to equivalent field evolution equations \[11, 9\]. Since the momentum covector \( \pi_\nu \) will be obtained as a field over spacetime, it will have a well defined gradient tensor \( \nabla_\rho \pi_\nu \) that can be used to rewrite the left hand side of (4) in the form \( d\pi_\nu/d\tau = u^\rho \nabla_\rho \pi_\nu \). Since the value of the Lagrangian will also be obtained as a scalar spacetime field \( L \), it will also have a well defined gradient which will evidently be given by an expression of the form \( \nabla_\nu L = \partial L/\partial x^\nu + (\partial L/\partial \pi_\rho) \nabla_\nu \pi_\rho \). We can thereby rewrite the Lagrangian dynamical equation (4) as a field equation of the form
\[
u^\rho \nabla_\rho \pi_\nu + \pi_\rho \nabla_\nu \nu^\rho = \nabla_\nu L. \tag{10}
\]

An alternative approach is of course to start from the corresponding Hamiltonian function, as obtained in terms of the position and momentum variables (so that formally it should be considered as a function on the spacetime cotangent bundle) via the Legendre transformation
\[
H = \pi_\nu u^\nu - L. \tag{11}
\]
In this approach the velocity vector (2) and the dynamical equation (4) are recovered using the familiar formulae
\[
\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial \pi_\nu}, \quad \frac{d\pi_\nu}{d\tau} = -\frac{\partial H}{\partial x^\nu}. \tag{12}
\]
The consideration that we are concerned not just with a single trajectory but with a spacefilling fluid means that, as in the case of the preceding Lagrangian equations, so in a similar way this familiar Hamiltonian dynamical equation can also be converted to a field equation which takes the form

$$2u^\rho \nabla_{[\rho,\pi_\nu]} = -\nabla_\nu H,$$

(13)

with the usual convention that square brackets are used to indicate index antisymmetrisation. On contraction with $u_\nu$, the left hand side will evidently go out, leaving the condition

$$u^\nu \nabla_\nu H = 0,$$

(14)

expressing the conservation of the value of the Hamiltonian along the flow lines.

The actual form of the Hamiltonian function that is obtained from the particularly simple kind of Lagrangian function (5) with which we are concerned will evidently be given by

$$H = \frac{1}{2\mu} g^{\nu\rho} \pi_\nu \pi_\rho + \frac{\mu c^2}{2}.$$

(15)

In order to ensure the proper time normalisation for the parameter $\tau$ the equations of motion (in whichever of the four equivalent forms (11), (10), (12), (13) may be preferred) are to be solved subject to the constraint that – in order for $u^\mu$ to be correctly normalised – the numerical value of the Hamiltonian should vanish,

$$H = 0 \quad \Rightarrow \quad u^\mu u_\mu = -c^2,$$

(16)

initially, and hence also by (14) at all other times. In the more general systems that are needed for some purposes the Hamiltonian may be constrained in a non uniform manner [11, 9] so that the term on the right of (13) will be non zero, but in the simpler systems that suffice for our present purpose the restraint (16) ensures that this final term will drop out, leaving a Hamiltonian equation of the very elegant and convenient form

$$u^\mu \nabla_{[\mu,\pi_\nu]} = 0.$$

(17)
3 Vorticity conservation and flux 2-surfaces.

The preceding form (17) of the dynamical equations is particularly handy for the analysis of symmetries and the derivation of conservation laws according to which physically interesting quantities are preserved by various kinds of continuous displacement [11, 9]. The variation induced by an infinitesimal displacement generated by an arbitrary vector field \( k^\mu \) say, will be given by the corresponding Lie derivative operator, whose effect on a scalar field, \( \mu \) say, will evidently be given simply by

\[
\bar{k} \mathcal{L}_\mu = k^\nu \nabla_\nu \mu ,
\]

while its effect on the metric will be given by the well known though not quite so immediately obvious formula

\[
\bar{k} \mathcal{L} g_{\mu \nu} = 2 \nabla_{(\mu} k_{\nu)} ,
\]

using round brackets to denote index antisymmetrisation. If \( k^\mu \) is a Killing vector field, i.e. if the displacement generated by \( k^\mu \) is a symmetry of the spacetime metric, then the right hand side of (19) will vanish. In curved space such symmetries are rare, but in ordinary flat spacetime there is of course a ten parameter family of such Killing vector fields generating the Poincaré group, whose algebra has as its basis the four independent generators of uniform spacetime translations and the six independent generators of the Lorentz group. The effect of the Lie differentiation operation on another vector field, \( u^\rho \), say will be given simply by their mutual commutator bracket,

\[
\bar{k} \mathcal{L} u^\rho = [\bar{k}, \bar{u}]^\rho = -\bar{u} \mathcal{L} k^\rho , \quad [\bar{k}, \bar{u}]^\rho = k^\nu \nabla_\nu u^\rho - k^\rho \nabla_\nu k_\nu .
\]

Such Lie differentiation is an example of an operation for which the distinction between covectors and ordinary contravariant vectors is important: except when performed with respect to a Killing vector, Lie differentiation does not commute with index raising and lowering. The rule that applies to the covector \( u_\mu = g_{\mu \nu} u^\nu \) can evidently be obtained by combining (19) and (20), so that for \( \pi_\nu \) one obtains a formula that can be conveniently expressed in terms of exterior (antisymmetrised) derivatives in the form

\[
\bar{k} \mathcal{L} \pi_\nu = 2 k^\rho \nabla_{(\rho} \pi_{\nu)} + \nabla_\nu (\pi_\rho k^\rho) .
\]

Although it does not commute with index raising and lowering, Lie differentiation does commute with exterior differentiation. Thus for the relativistic
vorticity tensor, which is defined to be the antisymmetrised derivative of the momentum covector, i.e.

$$w_{\mu\nu} = 2\nabla_{[\mu}w_{\nu]}$$, \hspace{1cm} (22)

so that its own exterior derivative will automatically vanish, i.e.

$$\nabla_{[\mu}w_{\nu\rho]} = 0$$, \hspace{1cm} (23)

it follows that its Lie derivative will be obtainable just by taking the exterior derivative of (21) which gives

$$\vec{k}\mathcal{L}w_{\mu\nu} = -2\nabla_{[\mu}(w_{\nu]}w^k\rho)$$.

The preceding general formula (21) can immediately be used to rewrite the Lagrangian dynamical equation (10) in the expressive form \[11, 9\]

$$\vec{u}\mathcal{L}\pi_{\nu} = \nabla_{\nu}L$$, \hspace{1cm} (25)

from which by taking the exterior derivative, one can immediately derive the dynamical vorticity conservation law that is the key to superfluidity theory (and much else) in the form

$$\vec{u}\mathcal{L}w_{\mu\nu} = 0$$.

(26)

The interpretation of this crucially important result is that for any flow governed by the Lagrangian equations (4) or their Hamiltonian equivalent (12) (which have so far simply been postulated ex cathedra, but whose validity for any “barotropic” perfect fluid be made clear in the following section) the vorticity field (22) will simply be convected onto itself by the flow field $u^\mu$, with the implication that if it vanishes initially $w_{\mu\nu}$ will remain zero throughout the flow, which in this case will be describable as “irrotational”.

The foregoing results can be considerably strengthened in cases such as those of the simple “barotropic” perfect fluids considered here for which the proper time normalisation is ensured by the Hamiltonian constraint \[12\] (which have so far simply been postulated ex cathedra). This has the effect of reducing the dynamical equations to the particularly simple form \[17\], whose interpretation is that the flow vector $u^\mu$ must be an zero eigenvalue eigenvector of the vorticity tensor $w_{\mu\nu}$. The possession of a zero eigenvalue requires that $w_{\mu\nu}$ should satisfy the degeneracy condition

$$w_{\mu[\nu}w_{\rho]} = 0$$, \hspace{1cm} (27)
which excludes the possibility of it having matrix rank 4, with the implication that unless it actually vanishes it must have rank 2 (since an anti-symmetric tensor can never have odd integer rank). This means that the flow vector \( u^\mu \) is just a particular case within a whole 2-dimensional tangent subspace of eigenvectors \( e^\mu \) satisfying

\[
e^\mu w_{\mu\nu} = 0. \tag{28}
\]

This subspace will be spanned by a unit worldsheet element tangent bivector \( \mathcal{E}^{\mu\nu} \) of the kind whose use was developed by Stachel \[13\], and that is definable, wherever the vorticity magnitude

\[
w = \left( \frac{1}{2} w_{\mu\nu} u^{\mu\nu} \right)^{1/2} \tag{29}
\]
does not vanish, as being proportional to the dual vorticity tensor \( W^{\mu\nu} \), i.e.

\[
\mathcal{E}^{\mu\nu} = \frac{1}{w} W^{\mu\nu}, \quad W^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} w_{\rho\sigma}, \tag{30}
\]

(note that sign convention used for the worldsheet orientation here is the opposite of what was used in the preceding article \[3\]) where \( \varepsilon^{\mu\nu\rho\sigma} \) is the totally antisymmetric tensor normalised by the convention that its non-zero components are equal to 1 or -1 (depending on whether the index ordering is an even or odd permutation) with respect to locally Minkowskian coordinates with \( g_{\mu\nu} \leftrightarrow \text{diag}\{ -c^2, 0, 0, 0 \} \), so that with respect to an arbitrary coordinate system the non-zero component values of \( \varepsilon^{\mu\nu\rho\sigma} \) will be given by \( \pm c ||g||^{-1/2} \).

It can be seen that this worldsheet tangent bi-vector will satisfy

\[
\mathcal{E}^{\mu\nu} \mathcal{E}_{\mu\nu} = -2c^2, \quad \mathcal{E}^{\mu\nu} w_{\nu\rho} = 0. \tag{31}
\]

It follows from this last equation that the contraction of any covector with \( \mathcal{E}^{\mu\nu} \) will provide a solution of the vortex worldsheet tangentiality condition \( \mathcal{E}^{\mu\nu} w_{\mu\nu} = 0 \). A noteworthy example is the helicity \( h^\mu \) as defined \[11, 6\] by

\[
h^\mu = w \mathcal{E}^{\mu\nu} \pi_{\nu}, \tag{32}
\]

which can be seen from \( \mathcal{P} \) and \( \mathcal{P} \) to satisfy the helicity current conservation law \[11, 9\]

\[
\nabla_\nu h^\nu = 0. \tag{33}
\]

Again using the degeneracy condition \( \mathcal{P} \) and the condition that the vorticity also satisfies the Poincaré closure condition \( \mathcal{P} \), it can be shown \[4\]
that the tangent elements characterised by $E_{\mu\nu}$ and generated by solutions of (28) (or more specifically by $u^\nu$ and $h^\nu$) will automatically satisfy the relevant Frobenius condition for integrability, meaning that they will mesh together to form well behaved timelike 2-dimensional worldsheets. This makes it possible to extend the flow line conservation laws resulting from any continuous symmetries that may be present.

The simplest example is the Bernouilli type theorem that applies – even if the Hamiltonian does not satisfy the constraint (14) – whenever $k^\rho$ is a symmetry generator of the system, so that in particular $\tilde{k}LH$ and $\tilde{k}L\pi_\nu$ both vanish: it can be seen from (13) and (21) that $u^\nu\nabla_\nu(\pi_\rho k^\rho)$ will vanish, so $\pi_\rho k^\rho$ will be constant along each flow line. In the most obvious application, the Killing vector is just the generator of time translations in flat space with the Minkowski coordinates, $k^\rho \leftrightarrow \{1, 0, 0, 0\}$, so that the Bernouilli constant will simply be identifiable with the negative of the effective energy per particle, i.e we shall have $\pi_\rho k^\rho = -E$ with $E = \gamma\mu c^2$ which automatically includes allowance for both compression energy and kinetic energy contributions.

This conclusion can be greatly strengthened when the Hamiltonian constraint (16) holds: one then gets $e^\nu \nabla_\nu(\pi_\rho k^\rho) = 0$ for any vector $e^\nu$ satisfying (28) which evidently means that $\pi_\rho k^\nu$ will not just be constant along each flow trajectory but that it will be constant throughout each of the 2-dimensional vorticity flux worldsheets. In the irrotational case

$$w_{\mu\nu} = 0,$$  \hspace{1cm} (34)

for which (28) is satisfied trivially by any vector $e^\nu$ at all, one can draw the even stronger conclusion that $\pi_\rho k^\rho$ will be constant throughout the fluid:

$$\tilde{k}L\pi_\rho = 0 \Rightarrow \nabla_\nu(\pi_\rho k^\rho) = 0.$$  \hspace{1cm} (35)

(A similar conclusion of global uniformity of $\pi_\rho k^\rho$ would also be obtainable immediately from (16) and (21) in the alternative, more widely familiar, case of motion that is rigid in the sense of having a flow vector that is aligned with the Killing vector, i.e. for which $u^{[\nu}k^{\rho]} = 0$.) It is to be emphasised that the applicability of this kind of generalised Bernouilli theorem is not limited to the case of ordinary stationarity, for which $k^\rho$ is a time translation generator, but is just as well applicable to cases of axisymmetry for which the Killing vector is a rotation generator, and it has recently been found to be very useful [15] for a hybrid case (involving a non-rigidly rotating binary pair of tidally deformed and thus non-axisymmetric neutron stars...
which, from the point of view of a distant observer, are not stationary but periodically evolving, but which are nevertheless stationary from the point of view of a local observer with respect to a suitably rotating frame).

An alternative way of obtaining the local vorticity conservation theorem (26) is as the differential limit of the global Kelvin-Helmoltz theorem to the effect that, as is manifest from the form (25) of the Lagrangian dynamical equation, the action integral

$$S = \int \pi_{\nu} dx^{\nu}$$

will be preserved if taken round a closed circuit that is convected by the flow field $u^\mu$. In the irrotational case (34), the Jacobi type action integral (36) between any two fixed endpoints will be unaffected by continuous displacements of the path between them, and hence can be used to construct a locally well defined field $S$ that is not only such that one has

$$\pi_{\nu} = \nabla_{\nu} S,$$

but that will also be a solution of the Hamilton Jacobi equation specified by setting the Hamiltonian function to zero with the gradient of $S$ substituted in place of the momentum.

The special case of a (simple, zero temperature limit) superfluid is specified by the existence of a well defined mesoscopic phase factor $e^{i\varphi}$ (representing the phase factor of an underlying bosonic condensate that might consist of Helium-4 atoms or Cooper type neutron pairs) in which the phase angle $\varphi$ is given according to the usual correspondence principle by

$$\varphi = \frac{S}{\hbar}.$$  

In a multiconnected configuration of a classical irrotational fluid the Jacobi action field $S$ obtained from (37) might have an arbitrary periodicity, but in a superfluid there will be a $U(1)$ quantisation requirement that the periodicity of the phase angle $\varphi$ should be a multiple of $2\pi$, and thus that the periodicity of the Jacobi action $S$ should be a multiple of $2\pi\hbar$. The simplest configuration for any such superfluid is a uniform stationary state in a flat Minkowski background, for which the phase will have the standard plane wave form

$$S/\hbar = k_a x^a - \omega t,$$

from which one obtains the correspondence $\pi_{\nu} \leftrightarrow \{-\hbar\omega, \hbar k_a\}$, which means that the effective energy per particle will be given by $E = \gamma \mu c^2 = \hbar \omega$ and that the 3-momentum will be given by $p_a = \mu \gamma v_a = \hbar k_a$. 

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It is to be remarked that for ordinary timelike superfluid particle trajectories the corresponding phase speed $\omega/k$ of the wave characterised by (39) will always be superluminal, a fact which people working with liquid Helium-4 in the laboratory can blithely ignore, since what matters for most practical purposes is not the phase speed but the group velocity of perturbation wave packets. Our present discussion will be limited to the strict zero temperature limit for which no such packets are excited, but it is easy to extend the relativistic analysis to low but non-zero temperatures for which the relevant excitations are phonons [7]. Although their phase speed and group velocity are the same, both being given by the formula (1) for the ordinary (“first”) soundspeed which will of course be sublumina, phonons do nevertheless have a tachyonic aspect of their own: their 4-momentum covector is always spacelike, in contrast with that of an ordinary fluid or superfluid particle which is timelike. This means that whereas the effective energy $E$ of an ordinary fluid or superfluid particle is always positive, the effective energy $E$ of a phonon may be positive or negative, depending on whether the frame of reference with respect to which it is measured is moving subsonically or supersonically. The well known implication is that if the superfluid is in contact with a supersonically moving boundary there will inevitably be an instability giving rise to dissipative phonon creation.

4 Conventional formulation of perfect fluid and simple superfluid theory.

Although sufficient for the derivation of many important properties of the flow, the dynamical equations on which the preceding sections have been based contain only part of the information needed for a complete determination of the perfect fluid evolution. The most usual way of presenting the complete set of equations of motion of a simple perfect fluid of the barotropic type we are considering – meaning one whose intrinsic local physical state is characterised just by a single independent scalar field – is in the form of a conservation law of the standard form

$$\nabla_\nu T^{\mu\nu} = 0,$$

(40)

for a stress momentum energy density tensor that is specified as a function of the timelike unit flow tangent vector $u^\nu$ and a single independent scalar field variable, such as the conserved particle number density $n$, on which the
other relevant quantities, such as the effective mass $\mu$ given by (8), will be functionally dependent.

For a system of this simple kind, the 4 independent components of the energy momentum conservation law (40) provide all that is needed to determine the evolution of the 4 independent components that characterise the local state of the system, which can be taken to be the scalar $n$ and the 3 space components $u^a$ of $u^\mu$ (since the remaining component $u^0$ is not dynamically independent but determined by the unit normalisation condition (16) as an algebraic function of the 3 other components).

In any perfect fluid model the mass density $\rho$ and pressure $P$ are physically characterised by their role in the specification of the stress momentum energy density tensor, for which the standard expression is

$$ T^{\mu\nu} = (\rho + \frac{P}{c^2}) u^\mu u^\nu + Pg^{\mu\nu}. \quad (41) $$

In the simple barotropic case, the relation (8) between the dependences of $P$ and $\rho$ on $n$ can be seen to be necessitated by the requirement that the dynamical system (40) should ensure conservation of the number current

$$ n^\mu = nu^\mu. \quad (42) $$

It is easy to check, using (8) that contraction of (42) with $u_\mu$ does indeed lead to the required result, namely

$$ \nabla_\mu n^\mu = 0. \quad (43) $$

One can also verify the not quite so well known result [14] that the remaining independent equations (40) can be reorganised in the canonical uniformly Hamiltonian form (17) on which the work [11, 9] of the preceding subsection was based, and which is expressible succinctly as

$$ n^\mu w_{\mu\nu} = 0. \quad (44) $$

Thus in addition to the formula (44) that has been used so far, the only additional information needed for the complete specification of the dynamics of a barotropic fluid system is the obvious particle conservation law (43).

Given a dynamical system, one of the first things any physicist is inclined to ask is whether it is derivable from a Lagrangian type variation principle. We have already seen in the previous sections that (44) by itself is obtainable from Lagrangian equations of motion for the individual trajectories, which
are of course obtainable from a one dimensional action integral of the form

\[ \int L \, d\tau \] with \( L \) as given by (5). The question to be addressed now is how to obtain the complete set of dynamical equations (40), including (43) as well as (44), from an action integral over the 4-dimensional background manifold \( S^{(4)} \) of the form

\[ \mathcal{I} = \int \mathcal{L} \, dS^{(4)}, \quad dS^{(4)} = \frac{\|g\|^{1/2}}{c} \, d^4x, \tag{45} \]

for some suitable scalar Lagrangian functional \( \mathcal{L} \).

Several radically different procedures are available for doing this. Although ultimately equivalent “on shell”, they involve variation over “off shell” bundles that differ not just in structure but even in dimension. The oldest and most economical from a dimensional point of view is the worldline variation procedure developed by Taub \[16\], followed Clebsch type variation procedure developed by Schutz \[17\], but for our present purpose it will be more convenient to employ the more recently developed Kalb-Ramond type method \[18\] that has been specifically designed for dealing with problems of macroscopic superfluidity.

The problem is greatly simplified if, to start off with, one restricts oneself to the purely irrotational case \( (34) \), which is all that is needed for the description of zero temperature superfluidity at a mesoscopic level. For this case independent variable can be taken to be just the Jacobi action \( S \), or equivalently in a superfluid context, the phase \( \varphi \) as given by (38), and the action is simply taken to be the pressure \( P \) expressed as a function of the effective mass \( \mu \), with the latter constructed as proportional to the amplitude of the 4-momentum, according to the prescription

\[ \mu^2 c^2 = -\pi_\nu \pi^\nu \tag{46} \]

with the 4-momentum itself given by the relation (37) that applies in the irrotational case, i.e.

\[ \pi_\nu = \hbar \nabla_\nu \phi. \tag{47} \]

Thus setting

\[ \mathcal{L} = P, \tag{48} \]

and using the standard pressure variation formula \( \delta P = c^2 n \delta \mu \) one sees that the required variation of the Lagrangian will be given by

\[ \delta \mathcal{L} = -n^\nu \delta \pi_\nu = -\hbar n^\nu \nabla_\nu (\delta \varphi). \tag{49} \]
Demanding that the action integral (45) be invariant with respect to infinitesimal variations of $\varphi$ then evidently leads to the required conservation law (43).

5 Introduction of the dilatonic amplitude field $\Phi$.

For an equation of state such that $P \propto \mu^2$, the Lagrangian (48) will, as it stands, have the quadratic field gradient dependence that is typical of simple physical field theories. This occurs in the special limit case, relevant to the massless axion field in cosmology [1, 4], of the “stiff” model, characterised by (62) as discussed below, for which the “first” sound speed (1) is equal to that of light. Except in this idealised limit case, a feature of the preceding Lagrangian function (48) that is widely considered to be undesirable is its generically non quadratic dependence on $\nabla \varphi$, an apparent drawback that is not uncommonly dealt with by recourse to approximation [19].

What this section will show however is that that, not just for $P \propto \mu^2$ but for a quite general equation of state, the Lagrangian (48) can be reformulated in the much more desirable form

$$\mathcal{L} = -\frac{\hbar^2}{2} \Phi^2 (\nabla_\nu \varphi) \nabla^\nu \varphi - V\{\Phi\}$$

(50)

—in which the potential energy density term $V$ is some suitably chosen algebraic function of the amplitude $\Phi$ — by a transformation of variables that is absolutely exact [18], without any need for recourse to approximation provided one adopts the correct definition for the auxiliary field variable $\Phi$.

What would involve an approximation would be to implement a further step whereby $\Phi$ is “promoted” from the status of an auxiliary variable to that of an extra dynamical variable by adding in a supplementary kinetic term,

$$\Delta \mathcal{L}_{(a)} = -a^2 \hbar^2 (\nabla_\nu \Phi) \nabla^\nu \Phi,$$

(51)

for some constant value of $a$. Various kinds of gradient term, of which this is the most obvious, might indeed be added for the purpose of improving the physical precision of the model in regimes of rapid amplitude variation, where deviations from the strictly isotropic perfect fluid form (41) might be expected to become significant. However for a strongly interacting liquid like Helium-4 (as opposed to a weakly interacting gas) I know of no reliable procedure for the quantitative evaluation of such a term on the basis of an underlying many-particle quantum theory. The most commonly
used [19] ansatz, namely to take \( a = 1 \), is not automatically guaranteed to provide an improvement but might even bring about a deterioration of physical precision in some circumstances. The standard choice \( a = 1 \) does not have theoretical or empirical foundations but is merely based on the purely mathematical consideration that it provides an adjusted Lagrangian \( \mathcal{L}_{\{1\}} = \mathcal{L} + \Delta_{\{1\}} \mathcal{L} \) of the “relativistic Ginzburg Landau” form

\[
\mathcal{L}_{\{1\}} = -\frac{\hbar^2}{2} (\nabla_\nu \Psi) \nabla^\nu \bar{\Psi} - V\{|\Psi|\}, \tag{52}
\]

in which the phase variable \( \varphi \) and the amplitude variable \( \Phi \) have been combined to form a complex variable

\[
\Psi = e^{i\varphi}, \quad \bar{\Psi} = e^{-i\varphi}, \quad \Phi = |\Psi|. \tag{53}
\]

This feature is useful for some purposes – notably for providing a smoothed out treatment [20, 21] of the vortex defects that arise where the phase \( \varphi \) becomes indeterminant – but not so convenient for deriving exact results such as the uniformity of the Bernoulli constant \( \pi_\nu k_\nu \) that was shown above to be valid for a model of the original kind whenever \( k_\rho \) is a symmetry generator.

As far as physical accuracy (as opposed to mathematical convenience) is concerned it will usually not matter very much whether one uses the exact perfect fluid model given by (50) or the associated Ginzburg Landau model given by (52) since their difference, as given by the extra kinetic term (51) can be expected to be very small except in the immediate neighbourhood of a vortex core where neither model can be expected to be physically accurate. What does matter for physical accuracy is the choice of the functional form for the amplitude \( \Phi \).

When it comes to the point, without even attempting to provide any serious microscopic derivation, many introductory presentations start by assuming the validity of the Landau Ginzburg framework, and then resort to crude guesswork for the specification of \( \Phi \), typically taking \( \Phi \propto \sqrt{n} \), or even more commonly

\[
\Phi \propto \sqrt{\rho}. \tag{54}
\]

Such a choice just happens to provide a quantitatively acceptable result in the non-relativistic limit, but the flimsiness of its theoretical basis is exposed by the fact that for a generic relativistic model it gives the wrong answer in the regime of slow amplitude variation where the (correctly calibrated) Landau Ginzburg model (52) should agree with the relevant perfect fluid
model (50). The correct choice – the only way to get exact agreement between (48) and (50) – is to take

\[ \Phi = \frac{n}{\sqrt{\rho + P/c^2}} = \left(\frac{n}{\mu}\right)^{1/2}, \]

(55)

the corresponding characterisation for the potential energy density function \( V \) being that it should be given by

\[ V = \frac{\rho c^2 - P}{2}. \]

(56)

When characterised in this manner, – with the correct identification (55) rather than (54) – the perfect fluid model given by the Lagrangian (50) provides a uniquely canonical model for the representation at zero temperature of a simple superfluid such as Helium-4, at least in the regime where the rate of variation of the amplitude \( \Phi \) is not too rapid compared with that of the phase. It is to be emphasised that for this regime the justification for this perfect fluid model is actually on a sounder footing than that of the corresponding Ginzburg Landau type model (52), since apart from the postulate of the isotropic perfect fluid form (41) for the stress momentum energy density tensor, whose applicability in the slow amplitude variation regime is hard to doubt, and the invocation of the generally valid conservation laws (40) and (42), the only assumption on which this fluid model is based, and for which it relies on an underlying microscopic quantum analysis, is that of the rather well established existence of the mesoscopic phase \( \varphi \).

Having evaluated \( V \) as a function of \( \Phi \) one can recover the effective mass \( \mu \), number density \( n \), mass density \( \rho \) and pressure \( P \) of the fluid using the formulae

\[ \mu^2 = \frac{1}{c^2} \frac{dV}{d\Phi}, \quad n = \Phi^2 \mu, \]

(57)

\[ \rho = \frac{1}{2} \Phi^2 \mu^2 + \frac{V}{c^2}, \quad P = \frac{1}{2} \Phi^2 \mu^2 c^2 - V, \]

(58)

which are derivable from (8) and (9).

Since the preceding relations entail the variation rule

\[ \delta V = \mu^2 c^2 \Phi \delta \Phi, \]

(59)

it can be seen that the full variation of the reformulated perfect fluid Lagrangian (50) will be given by

\[ \delta \mathcal{L} = -\left(\pi^\nu \pi^\nu + \mu^2 c^2\right) \Phi \delta \Phi - \hbar \Phi^2 \pi^\nu \nabla_\nu (\delta \varphi), \]

(60)
with the momentum covector $\pi_\nu$ as specified before by (17). Thus in this new formulation, instead of imposing (16) as a defining relation, we obtain it as a field equation from the requirement that the action integral should be invariant with respect to local variations of the auxiliary field $\Phi$, while as before the analogous requirement for variations of the phase variable $\varphi$ gives back the particle conservation law (42).

5.1 Special equations of state

It is to be remarked that whereas for a generic compressible fluid equation of state giving $\rho$ as a function of $n$, and hence giving $P$ as a function of $\mu$, the formula (59) will provide a corresponding function $V$ that provides a reformulated action function of the form (50) from which the required perfect fluid dynamical equations are obtainable by treating $\Phi$ and $\varphi$ as independent. There is however an exceptional case that works somewhat differently, namely that of the “stiff” Zel’dovich model – pertaining to the massless axion field in cosmology [1, 4] – which is characterized by

$$P = \rho c^2,$$

(61)

corresponding by (1) to sound propagation at the speed of light. This case is obtained from a primary equation of state of the form $\rho \propto n^2$ or equivalently (as observed at the beginning of Section 5) $P \propto \mu^2$, which gives $V = 0$ and $\mu \propto n$. This last relation means that the amplitude $\Phi$ will simply be a constant, and so will not act as an auxiliary variable in the usual way. In this case what one has to do to obtain the relevant “stiff” dynamical equations is simply to treat $\varphi$ as the only independent variable in the Lagrangian (50).

There is another important special limit for which it is not the new formulation but the original variational formulation (18) in terms of the pressure function that fails to work, namely the extreme case – which often useful as an approximation – of a “dust type” model characterized by $\rho \propto n$ for, which there is no pressure, i.e. $P = 0$. This low pressure limit model (the only model for which the naive identification (54) is exactly valid) is immediately obtainable within the framework of the new formulation (50) simply by taking $V \propto \Phi^2$.

The simplest non trivial potential energy function that can be used in the new formulation (50) is provided by the “radiation gas” model (61) characterized by

$$c_1^2 = \frac{c^2}{3} \quad \Rightarrow \quad V \propto \Phi^4,$$

(62)
which is obtainable from a primary equation of state of the form \( \rho \propto n^{4/3} \). This is the model that is appropriate for representing the cosmological black body radiation (which is to a good approximation irrotational, though of course it is not a superfluid). It also applies to a degenerate Fermi gas of massless (or due to high compression effectively massless) non interacting particles, and for that reason was used as a first crude approximation in some of the pioneering studies of neutrons stars, whose intermediate layers are indeed believed to be superfluid. As will be explained below, this particular model is characterised by conformal invariance of the same kind as is familiar in the well known case of Maxwellian electromagnetism.

For more general purposes, as an approximation that can be usually be expected to be reasonably accurate within a limited density range, one can of course use the obvious generalisation of (62) to the standard form

\[
V = \frac{m^2 c^2}{2} \Phi^2 + a^2 \Phi^4, \quad (63)
\]

for suitably adjusted constants \( m \) and \( a \). For such a model the fluid mass density and pressure will be given according to (58) by

\[
\rho = m^2 \Phi^2 + \frac{3a^2}{c^2} \Phi^4, \quad P = a^2 \Phi^4, \quad (64)
\]

from which it can be seen that the sound speed (the quantity one would probably want to use in practice to fix the appropriate value of \( a \)) will be given according to (1) by

\[
c_1^2 = \left( \frac{3}{c^2} + \frac{m^2}{2a^2 \Phi^2} \right)^{-1} \quad (65)
\]

6 Introduction of the Kalb Ramond gauge field

The treatment given in the preceding section is ideal for the description of a superfluid at a mesoscopic (i.e. intervortex) scale, but for the treatment of an ordinary perfect fluid with rotation, or for the treatment of a superfluid on a macroscopic scale (allowing for the averaged effect of a large number of vortices) more general models are required.

A first step towards the kind of generalisation that is needed is to formulate the current in terms of an antisymmetric Kalb Ramond type tensor field \( B_{\mu\nu} = -B_{\nu\mu} \) whose exterior derivative

\[
N_{\mu\nu\rho} = 3\nabla_{[\mu} B_{\nu\rho]}, \quad (66)
\]
whose (Hodge type) dual

\[ \eta^\mu = \frac{1}{3!} \varepsilon^{\mu\nu\rho\sigma} N_{\nu\rho\sigma} \]  

(67)
is to be identified with the particle number current [12], which will evidently be invariant under the effect of Kalb Ramond gauge transformations \( B_{\mu\nu} \mapsto B_{\mu\nu} + 2\nabla_{[\mu} \chi_{\nu]} \). It can then be seen that the consequent closure condition,

\[ \nabla_{[\mu} N_{\nu\rho\sigma]} = 0 , \]  

(68)
is equivalent to the usual form [13] of the particle conservation law.

The idea now is to perform a Legendre type transformation \( \mathcal{L} \mapsto \Lambda \) whereby the independent scalar field \( \varphi \) of the preceding formulation based on \( \mathcal{L} \) is replaced by the antisymmetric gauge tensor \( B_{\mu\nu} \) in a new formulation of the same model in terms of a different dually related Lagrangian function \( \Lambda \) which takes the form

\[ \Lambda = - \frac{e^2}{12\Phi^2} N^{\mu\rho\sigma} N_{\mu\nu\rho} - V\{\Phi\} . \]  

(69)

In the preceding formulation based on \( \mathcal{L} \) as given by (50) the irrotationality property was kinematically imposed in advance while the particle current conservation was obtained from the variational principle as a dynamical equation. However in the reformulated version based on the dual Lagrangian (69) it is the particle current conservation law that is obtained in advance as we have seen via the kinematic identity (68), while on the other hand the irrotationality condition (34), i.e.

\[ \nabla_{[\mu} \pi_{\nu]} = 0 , \]  

(70)
is obtained directly from (69) in the equivalent dual form

\[ \nabla_\nu (\Phi^{-2} N^{\nu\rho\sigma}) = 0 . \]  

(71)
from the requirement of invariance with respect to independent variations of the gauge 2-form \( B_{\mu\nu} \).

The purpose of replacing the simple scalar field \( \varphi \) by the tensorial field \( B_{\mu\nu} \) is to enable the extension of the model to the general perfect fluid case, in which the particle conservation law (68) is retained, but the irrotationality condition (70) is abandoned. The way to do this [18] is to introduce a complete Lagrangian of the form

\[ \mathcal{L} = \Lambda - \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} B_{\mu\nu} w_{\rho\sigma} , \]  

(72)
where, in analogy with what has already been done for the current 3-form, the vorticity 2-form is constructed from independent gauge fields in such a way that its conservation property, (23) is automatically ensured in advance as a kinematic identity. For this purpose the requisite independent gauge fields can be taken to be a pair of independent scalars, $\chi^\pm$ say, in terms of which an identically conserved vorticity flux will be given by

$$w_{\nu\sigma} = 2(\nabla_{[\nu}\chi^{+})\nabla_{\sigma]}\chi^{-}. \quad (73)$$

When the Lagrangian (72) is substituted in the action integral (45), the requirement of invariance with respect to local variations of the (non-physical) dynamical gauge fields $B_{\mu\nu}$ and $\chi^\pm$, and of the (physical) auxiliary amplitude $\Phi$, can be seen to lead back to our original dynamical momentum transport equation (44). Thus (since the particle conservation law (43) has been imposed kinematically in advance) it provides the complete system of generic perfect fluid equations of motion as given by the standard stress momentum energy conservation law (40).

7 Macroscopic allowance for vortex quantisation.

All we have done so far is to reformulate ordinary barotropic perfect fluid theory in such a way that the vorticity $w_{\mu\nu}$ comes in as an independent dynamical variable determined by the pair of scalar gauge fields $\chi^\pm$. The model describing simple zero temperature superfluidity is obtained within this framework simply by taking the scalars $\chi^\pm$ to be constants (e.g. zero) so as to get $w_{\mu\nu} = 0$.

For a macroscopic treatment of a bulk superfluid fibred by a dense congruence of discrete vortex defects, the need for an extra term in the action to allow for the extra tension and energy in the vortices was recognised long ago [22] in the laboratory context of Helium-4, and has more recently been taken into account in the Newtonian mechanical analysis of neutron star matter [23]. In an analogous manner, following earlier work [24] on the corresponding relativistic formulation needed for a more accurate treatment of neutron star matter, the more complete treatment [3] to be described here takes account of the averaged effect of quantised vortices aligned in an Abrikosov type lattice by summing over contributions of individual vortex cells as estimated [5] using the usual (very good) approximation in which the hexagonal cells are treated as if they were cylindrically symmetric. What
this analysis suggests is that the averaged effect of such vortices can be represented rather well – provided any relative flow is highly subsonic – by a remarkably simple and mathematically elegant modification of the generic potential function introduced by (56): all that seems to be necessary is to make an adjustment of the very simple form

\[ V\{\Phi\} \mapsto V\{\Phi\} + \Phi^2 \Upsilon\{w\}, \tag{74} \]

for some function \( \Upsilon \) depending just on the scalar magnitude \( w \) of the vorticity as given by (29). Furthermore, as was observed in analogous investigations in a non relativistic framework [22], the dependence on the vorticity magnitude is approximately linear, having the form

\[ \Phi^2 \Upsilon = \lambda w , \tag{75} \]

in which

\[ \lambda = K \Phi^2 , \quad \Upsilon = Kw , \tag{76} \]

with a coefficient \( K \) that can be taken to be a constant of the order of the Planck value, \( K \approx \hbar \).

A remarkable consequence of the ansatz (74) in conjunction with the linearity postulate (76) is that, as will be described below, the extended model retains the noteworthy, though little known, conformal covariance property of the simple perfect fluid model.

A more specific quantitative estimate of the value of the coefficient in (76) is given by

\[ K = \frac{\hbar \hat{l}}{4} , \quad \hat{l} = \ln \left\{ \frac{\Delta^2}{\delta^2} \right\} , \tag{77} \]

where \( \delta \) is an inner cut off length representing the microscopic vortex core radius, and \( \Delta \) is a long range cut of length typically representing the mean intervortex separation distance.

For a very precise treatment one would of course need to allow for a weak logarithmic dependence of \( K \) on \( w \) since \( \Delta^2 \) will be inversely proportional to \( w \), but so long as \( \Delta \) is very large compared with \( \delta \), as will be the case in typical macroscopic applications, the effect of such a refinement will in practice be negligible. To obtain very high precision when the Mach (flow to sound speed) ratio is non-negligible [25] one might also have to allow for some sort of tensorial, not just scalar dependence on the vorticity tensor.

Assuming that a sufficiently accurate treatment is obtainable without the need to take account of tensorial vorticity dependence, it follows that
the complete macroscopic superfluid Lagrangian will be given \(^3\) by

\[
\mathcal{L} = -\frac{c^2}{12\Phi^2} N^\mu\nu\rho N_{\mu\nu\rho} - V\{\Phi\} + \frac{1}{2} \left( \frac{\lambda}{c^2} E_{\mu\nu} - B_{\mu\nu} \right) W^{\mu\nu},
\]

where \(W^{\mu\nu}\) is the dual vorticity vector as defined by \(^30\).

In order to obtain the equations of motion for this system, one needs to evaluate the corresponding variation which will be expressible by

\[
\delta\mathcal{L} = \pi_\nu \delta n^\nu - \frac{1}{2} w_{\mu\nu} \delta b^{\mu\nu} - \frac{1}{2} (b^{\mu\nu} + \lambda^{\mu\nu}) \delta w_{\mu\nu} + \left( \frac{c^2 n^2}{\Phi^2} - \frac{dV}{d\Phi} - \frac{2\lambda w}{\Phi} \right) \delta\Phi \tag{79}
\]

using the notation

\[
b^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} B_{\rho\sigma}, \quad \lambda^{\mu\nu} = \frac{\lambda}{w} w^{\mu\nu}. \tag{80}
\]

Requiring invariance of the corresponding integral with respect to \(\delta B_{\mu\nu}\), or equivalently to \(\delta b^{\mu\nu}\), just leads back again to the usual relation \(^22\) specifying the vorticity 2-form \(w_{\mu\nu}\) as the exterior derivative of the momentum 1-form \(\pi_\nu\), so there will be a conserved helicity vector given by exactly the same formula \(^32\) as before. The equation of state relation \(^57\) is however modified by the addition of an extra term proportional to the vorticity: requiring invariance with respect to \(\delta\Phi\) gives

\[
\Phi^2 \mu^2 c^2 = \frac{n^2 c^2}{\Phi^2} = \Phi \frac{dV}{d\Phi} + 2\lambda w. \tag{81}
\]

The final requirement is invariance with respect to variations of the gauge scalars determining the vorticity field according to \(^74\), which is equivalent to the gauge independent requirement of invariance with respect to Lie transportation, \(\delta w_{\mu\nu} = \vec{k} \mathcal{L} w_{\mu\nu}\), as given by \(^24\) for an arbitrary displacement vector field \(k^\mu\). The resulting dynamical equation is expressible in the form

\[
\delta_n^\mu w_{\mu\nu} = 0, \tag{82}
\]

which differs from its analogue in the perfect fluid limit only by the replacement of the conserved particle current \(n^\mu\) by an “augmented” current \(\delta_n^\mu\) that is also automatically conserved

\[
\nabla_\mu \delta_n^\mu = 0, \tag{83}
\]

whose specification is given by

\[
\delta_n^\mu = n^\mu + \nabla_\nu \lambda^{\mu\nu}, \tag{84}
\]
which is equivalent, by (80), to the condition that it be obtainable from a corresponding “augmented” Kalb-Ramond gauge 2-form given by

\[ tB_{\mu\nu} = B_{\mu\nu} - \frac{\lambda}{e^2} \epsilon_{\mu\nu}. \] (85)

The geometrical interpretation of the modified dynamical equation (82) is facilitated by the observation that it is the same as the single vortex equation of motion (128) that will be derived below.

8 The conformal covariance property

It is well known that – when expressed in the usual way in terms of an electromagnetic gauge potential \( A_\nu \) and charge current \( j^\mu = e n^\mu \), where \( e \) is a charge coupling constant and \( n^\mu \) is an automatically conserved current vector such as is obtainable from a corresponding closed 3-form \( N_{\mu\nu\rho} \) according to the Hodge type duality formula (67) – the equations of ordinary Maxwellian electromagnetism are preserved by any conformal transformation of the form

\[ g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}, \] (86)

for an arbitrary scalar field \( \phi \), on the understanding that the transformation affects neither the gauge 1-form, which obeys \( A_\nu \mapsto A_\nu \), nor the closed current 3-form, which obeys \( N_{\mu\nu\rho} \mapsto N_{\mu\nu\rho} \). This last condition means that, since (86) implies

\[ \epsilon^{\mu\nu\rho\sigma} \mapsto \tilde{\epsilon}^{\mu\nu\rho\sigma} = e^{-4\phi} \epsilon^{\mu\nu\rho\sigma}, \] (87)

the current vector itself will undergo a conformal transformation of the form

\[ n^\mu \mapsto \tilde{n}^\mu = e^{-4\phi} n^\mu. \] (88)

In the framework of the Kalb-Ramond representation (66) this is evidently equivalent to the requirement of preservation of the gauge 2-form,

\[ B_{\mu\nu} \mapsto \tilde{B}_{\mu\nu} = B_{\mu\nu}. \] (89)

What is not so well known (since the utility of the auxiliary field \( \Phi \), as correctly defined by (55), is not yet widely appreciated) is that the ordinary barotropic perfect fluid equations (43) and (44) are also preserved by such a conformal transformation, subject to the understanding that the auxiliary
amplitude field undergoes a corresponding conformal transformation of the form

$$\Phi \mapsto \tilde{\Phi} = e^{-\phi} \Phi,$$

which is what is required to ensure the preservation,

$$\pi_\nu \mapsto \tilde{\pi}_\nu = \pi_\nu,$$

of the momentum 1-form $\pi_\mu$ (whose role in the perfect fluid case is analogous to that of the gauge 1-form $A_\mu$ in the electromagnetic case).

The preservation of the form of the dynamical equations (43) and (44) is in general not quite sufficient for preservation of the complete system, because it is also necessary to satisfy the algebraical equation of state relation (57) specified by the function $V\{\Phi\}$ which governs the relation between the amplitude $\Phi$ and the 4-momentum magnitude $\mu = (-\pi_\nu \pi^\nu)^{1/2}$. However provided the conformal scalar $\phi$ is chosen to depend only the amplitude $\Phi$, making the later a function of new variable $\tilde{\Phi}$, then the system will be formally covariant in the sense that the new system will also behave as an ordinary barotropic fluid but with a modified equation of state function $\tilde{V}\{\tilde{\Phi}\}$ in place of the original potential $V\{\Phi\}$.

My original discussion of the perfect fluid case [18] envisaged a scenario involving gravitational coupling in the framework of a Brans-Dicke-Jordan type generalisation of Einstein’s theory, with the dimensionless field $\phi = \ln\{\tilde{\Phi}/\Phi\}$ acting as a dilatonic coupling scalar. Conformal covariance of this scheme was found to require that the transformation law for $V$ should be given by an algebraic relation of the form $\tilde{V}/\tilde{\Phi}^4 = V/\Phi^4$. (For readers interested in this scenario I should warn that it is necessary to the correct the final sentence, concerning a special case for which there is a transformation to a form in which “the dilatonic field is genuinely absent”: the term “genuinely” should be replaced by “apparently”, since although it disappears from the fluid sector of the Lagrangian it effectively turns up again in the gravitational sector instead.)

In present discussion I wish to describe a kind of conformal covariance like that of the well known Maxwellian example, having nothing to do with any particular kind of gravitational coupling theory, whether it be that of Einstein or anyone else. In order to preserve the formal structure of the fluid system by itself (without involving anything to do with active gravitational coupling) it can be seen from (57) that since we shall have

$$\mu \mapsto \tilde{\mu} = e^{-\phi} \mu,$$
it is necessary and sufficient that the effective potential should transform in such a way as to satisfy the differential condition

\[ \tilde{\Phi}^{-3} \frac{d\tilde{V}}{d\tilde{\Phi}} = \Phi^{-3} \frac{dV}{d\Phi}. \]  

(93)

This means, for example, that if the new system is to be characterised by a pressure free “dust” type equation of state of the form \( \tilde{V} = m^2 c^2 \tilde{\Phi}^2 \) for some constant \( m \) (which has the convenient feature that the flow trajectories will simply be geodesics with respect to the new metric \( \tilde{g}_{\mu\nu} \)) then the desired transformation giving \( \tilde{\Phi} \) as a function of \( \Phi \) will be obtainable immediately by taking \( m^2 c^2 \tilde{\Phi}^{-2} = \Phi^{-3} dV/d\Phi \) except in the exceptional case for which the right hand side of (93) is constant.

Whereas a perfect fluid solution for a generic equation of state can thus be conformally transformed to a solution for any other generic equation of state, there is an exceptional case for which such a procedure fails, namely the case with a constant value for right hand side of (93), which is that of the “radiation gas” model for which the dependence on \( \Phi \) of the effective potential \( V \) is of the homogeneously quartic form (62). It is apparent that the particular form

\[ P = \frac{\rho c^2}{3} \Rightarrow \mu \propto \Phi \]  

(94)

of the equation of state relation for this particular model will automatically be preserved by (90) and (92), not just when the conformal scalar \( \phi \) is chosen as a function of \( \Phi \) but for any field \( \phi \) whatsoever.

In summary, just as solutions of Maxwell’s equation are well known to be mapped conformally onto other solutions of Maxwell’s equations, solutions of the equations of the “radiation gas” model are analogously mapped conformally onto other solutions of the “radiation gas” model, while generically solutions of the barotropic perfect fluid equations for any equation of state can be conformally mapped onto solutions for another different equation of state. In view of the intrinsic isotropy of a perfect fluid these conformal properties are not very startling. However we conclude this subsection by the rather more surprising observation that these properties will still hold, subject to the same condition (93), when the intrinsic isotropy is violated by the inclusion of the term (75) that allows for the energy and tension in quantised vortices. Since we shall have

\[ \mathcal{E}_{\mu\nu} \rightarrow \tilde{\mathcal{E}}_{\mu\nu} = e^{2\phi} \mathcal{E}_{\mu\nu}, \]  

(95)
and
\[ w \mapsto \tilde{w} = e^{-2\phi}w, \]  
(96)
it can be seen that in order to have invariance of \( \mathring{\mathcal{B}}_{\mu\nu} \) as given by (85) and of the extra term \( \lambda w \) in the action integral (remembering that the measure \( dS^{(4)} \) will transform proportionally to \( e^{4\phi} \)) we must have
\[ \lambda \mapsto \tilde{\lambda} = e^{-2\phi}\lambda. \]  
(97)

Rather remarkably, this condition just happens to be satisfied by the formula (76), but it would fail for a more complicated vorticity dependent term such as might be needed for an accurate treatment of relative flow at a speed comparable with that of sound.

9 The thin vortex string limit

Let us now consider the limiting case for which instead of being 4-dimensionally extended, the vorticity distribution is concentrated in the neighbourhood of some particular vorticity flux 2-surface \( \mathcal{S}^{(2)} \) say, which might conveniently be characterised by zero values for the scalar gauge fields \( \chi^\pm \). Such a 2-surface will be describable in terms of internal coordinates \( \sigma^0 \) and \( \sigma^1 \), which might conveniently be specified by the values on the worldsheet of the spacetime coordinates \( t \) and \( x^1 \).

However the internal coordinates may be chosen, the worldsheet embedding \( \{\sigma^0, \sigma^1\} \mapsto x^\mu = \bar{x}^\mu(\sigma) \) will induce (in the technical sense “pull back”) a two surface metric with components \( \gamma_{ab} \) on the worldsheet that will be given by
\[ \gamma_{ab} = g_{\mu\nu}\bar{x}^\mu_{,a}\bar{x}^\nu_{,b}, \]  
(98)
using a comma to denote partial differentiation with respect to the internal coordinates \( \sigma^a \) \((a = 0, 1)\), and this induced metric in turn will specify the worldsheet measure
\[ dS^{(2)} = \frac{\|\gamma\|^{1/2}}{c} d\sigma^0 d\sigma^1, \]  
(99)
of the timelike 2-surface element spanned by the coordinate variations \( d\sigma^0 \) and \( d\sigma^1 \).

Let us start by supposing that the vorticity distribution is confined within a small but finite range specified by displacements \( \delta\chi^+, \delta\chi^- \) of the comoving scalar fields, and then let us take the thin string limit as the size of these displacements tends to zero. In this limit it can be seen that the dual
vorticity \( W^{\mu\nu} \) will take the form of a two dimensional Dirac distribution that will be expressible in Dirac’s notation as an integral over the 2-dimensional world sheet \( S^{(2)} \) by the formula

\[
W^{\mu\nu} = \frac{c}{\|y\|^{1/2}} \int \overline{W}^{\mu\nu} \delta^4 \left[ x^{\mu} - \bar{x}^{\mu} \{ \sigma \} \right] dS^{(2)}, \tag{100}
\]

in which \( \overline{W}^{\mu\nu} \) is a well behaved antisymmetric tensor on the worldsheet representing the concentrated 2 surface vorticity flux.

In the case of the continuous vorticity distribution considered in the preceding section, the vorticity conservation law (23) is expressible in its dual version in the form

\[
\nabla_{\nu} W^{\mu\nu} = 0. \tag{101}
\]

When one goes over to the singular limit in which the distribution \( W^{\mu\nu} \) is concentrated in the form (100) in the infinitesimal neighbourhood of a single flux worldsheet, the conservation law (101) will translate into a corresponding condition on the regular worldsheet supported 2-surface vorticity flux tensor \( \overline{W}^{\mu\nu} \). Using the abbreviation

\[
\overline{\nabla}_\mu = \eta^\nu_{\mu} \nabla_\nu \tag{102}
\]

for the operator of worldsheet tangential covariant differentiation, where \( \eta^\nu_{\mu} \) is the “first” fundamental tensor \([23, 27]\) of the worldsheet (i.e. the rank-2 projection operator whose contraction with an arbitrary vector at a point on the worldsheet projects it onto its tangential part within the worldsheet) which will be given by

\[
\eta^\nu_{\mu} = c^{-2} \mathcal{E}^{\nu\rho} \mathcal{E}_{\rho\mu}, \tag{103}
\]

the condition expressing the conservation of the vorticity flux distribution in the worldsheet will expressible simply as

\[
\overline{\nabla}_\mu \overline{W}^{\mu\nu} = 0. \tag{104}
\]

Since the canonically normalised worldsheet tangential bivector \( \mathcal{E}^{\mu\nu} \) automatically satisfies a conservation condition of the same form

\[
\overline{\nabla}_\mu \mathcal{E}^{\mu\nu} = 0 \tag{105}
\]

as a kinematic identity, it follows that (104) is interpretable as implying that the worldsheet supported surface vorticity flux tensor \( \overline{W}^{\mu\nu} \) must have the form

\[
\overline{W}^{\mu\nu} = \kappa \mathcal{E}^{\mu\nu}, \tag{106}
\]

\[
\eta^\nu_{\mu} \nabla_\nu = \eta^\nu_{\mu} \nabla_\nu.
\]
where $\kappa$ is a constant on the worldsheet, i.e. $\nabla_\nu \kappa = 0$.

It can be seen that the constant $\kappa$ defined by (106) will be interpretable as the value in the thin string limit of the 2-surface integral of the vorticity across any small spacelike section through the world tube. By Stoke’s theorem, using the defining relation (22), this will be equal to the value of the Jacobi action around the boundary circuit of the section, which can be taken to be any closed curve encircling the vortex string. Thus for any such surrounding circuit we shall have

$$\kappa = \oint dS = \oint \pi_\nu dx^\nu = 2\pi \hbar \nu , \quad (107)$$

where $\nu$ is an integer representing the winding number of the phase $\varphi$, i.e. $\nu$ represents the number of individual quantised vortices carrying the flux under consideration. This means that if we are considering a string representing just a single such quantised vortex, with orientation chosen so that $\nu = +1$ we shall simply have

$$\kappa = 2\pi \hbar . \quad (108)$$

Substituting (78) in formula (45) it can be seen that one obtains the total action integral in the form

$$I = I^{(4)} + I^{(2)} , \quad (109)$$

in which the first part is just the 4-dimensional contribution from the irrotational superfluid outside the vortex, which by (69) is

$$I^{(4)} = \int \mathcal{L}^{(4)} dS^{(4)} , \quad \mathcal{L}^{(4)} = -\frac{1}{12\Phi^2} N^{\mu\rho\sigma} N_{\mu\rho\sigma} - V\{\Phi\} , \quad (110)$$

while by (100) and (106) the second part reduces to a 2-dimensional string worldsheet integral given by

$$I^{(2)} = \int \mathcal{L}^{(2)} dS^{(2)} , \quad \mathcal{L}^{(2)} = -\frac{\pi \hbar}{2} \hbar^2 \Phi^2 - \pi \hbar B_{\mu\nu} \mathcal{E}^{\mu\nu} . \quad (111)$$

The first term in (111) is like the corresponding action density for a simple Nambu Goto (internally structureless) string, as characterised by a string worldsheet stress-energy density tensor $\mathcal{T}_{\mu\nu}$ that is proportional to the fundamental tangential projection tensor $\eta^{\mu}_{\nu}$ of the worldsheet as defined by (103), except that for an ordinary Nambu-Goto model the effective string
tension $T$ is fixed. In the special limit case of the “stiff” fluid model $\text{[61]}$ which describes the massless axion case, it is well known $[1]$ that the vortices will have an effective tension $T$ will also be fixed, but for a generic superfluid model one obtains

$$T^\mu_\nu = -T^\mu_\nu, \quad T = \frac{\pi i}{2} \kappa^2 \Phi^2, \quad (112)$$

which shows that the effective tension $T$ will vary proportionally to the square of the auxiliary field amplitude, which according to $[53]$ will be given on shell by

$$\Phi^2 = \frac{n}{\mu}. \quad (113)$$

The second term in $\text{(111)}$ has the form of what is known in the context of superstring theory as a “Wess-Zumino” coupling. When such a term turned up in the “stiff” massless axion case $\text{(61)}$ it was at first interpreted $\text{(1)}$ as a “very unusual interaction”. However it soon came to be recognised $\text{(4)}$ as a manifestation of an ordinary aerodynamic “lift” type force (arising from the Magnus effect) of the kind first evaluated for an aerofoil in the long thin (i.e. string type) limit by the Russian theorist Joukowski in the pioneering days of subsonic flight a hundred years ago.

10 \text{ The vortex string dynamical equations.}

To work out the effects of the pair of terms in $\text{(111)}$, one must obtain the condition for the string type action $I^{(2)}$ to be invariant with respect to an infinitesimal displacement of the worldsheet generated by an arbitrary vector field $k^\mu$ say. Any such displacement will give rise to corresponding “Lagrangian” variations $\delta \Phi, \delta B_{\mu\nu}, \delta g_{\mu\nu}$ (meaning variations as measured with respect to local coordinates that are comoving with the displacement) of the relevant background fields. Such “Lagrangian” variations will be given by corresponding Lie differentiation formulae of the kind introduced above for the respective cases of a scalar $\text{(18)}$, a closed antisymmetric covariant field $\text{(24)}$, and last but in importance not least, the metric itself $\text{(19)}$. Thus in the trivial case of the scalar $\Phi$ we shall simply have

$$\delta \Phi = k^\nu \nabla_\nu \Phi, \quad (114)$$

while in the case of the metric we have the familiar expression

$$\delta g_{\mu\nu} = 2 \nabla_{(\mu} k_{\nu)}, \quad (115)$$
(which would of course vanish if $k^\mu$ were not arbitrary but restricted to be a Killing vector generating a background spacetime isometry, i.e. in flatspace if $k^\mu$ were restricted to be a generator of some Poincaré combination of translations and Lorentz transformations). Since the antisymmetric field $B_{\mu\nu}$ is not closed (i.e. since its exterior derivative, the current 3-form $N_{\mu\nu\rho}$ is non vanishing) the formula for its Lie derivative is not quite so simple as that of its analogue (24) for the vorticity but contains an extra term: the relevant formula is

$$\delta B_{\mu\nu} = k^\mu N_{\mu\nu\rho} - 2\nabla_\mu (B_{\nu\rho} k^\rho) .$$

(116)

The corresponding variation of the string type action integral specified by (111) takes the standard form

$$\delta I^{(2)} = \int \left( \frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{2} W^{\mu\nu} \delta B_{\mu\nu} + \mathcal{T} \delta \Phi \right) dS^{(2)} ,$$

(117)

in which it can be seen, taking account of the variation of the surface element (99) due to the induced variation

$$\delta \| \gamma \|^{1/2} = \frac{1}{2} \| \gamma \|^{\eta^{\mu\nu}} \delta g_{\mu\nu} ,$$

(118)

that the relevant surface stress energy momentum tensor $\mathcal{T}^{\mu\nu}$ can be read out in the form (112), and that the relevant surface vorticity flux bi-vector will evidently be as given by (106) and (108), while finally the dynamical dual of the amplitude $\Phi$ can be read out simply as

$$\mathcal{T} = -\pi \hat{l} \tilde{h}^2 \Phi .$$

(119)

Since we are only concerned with local variations, there will be no boundary contribution when we perform the usual operation of integration by parts using Green’s theorem so as to eliminate gradients of the arbitrary displacement vector $k^\mu$. The result is thus obtained in the standard form

$$\delta I_{(2)} = \int k^\mu (\mathcal{T}_\mu - \nabla_\nu \mathcal{T}_\nu) dS^{(2)} ,$$

(120)

in which, after taking account of the surface vorticity flux conservation law (104), the effective surface force density acting in the ensuing dynamical equation,

$$\nabla_\nu \mathcal{T}_\nu = \mathcal{T}_\mu ,$$

(121)
can be read out as
\[ \overline{f}_\mu = \frac{1}{2} N_{\mu\nu\rho} \overline{W}^{\nu\rho} + \overline{F}_\nu \Phi. \] (122)

In a model of this particular type (as a consequence of the fact that the vortex string has no internal degrees of freedom of its own) the tangentially projected part of the force balance equation (121), i.e. the part that is obtained by contracting it with the fundamental tensor \( \eta^\mu_\rho \) given by (103), will automatically be satisfied as a kinematic identity. The only dynamical information in (121) is the part obtained by contracting it with the complement of the tangential projection tensor \( \eta^\mu_\rho \), namely the orthogonal projection tensor defined by
\[ \perp^\mu_\rho = g^\mu_\rho - \eta^\mu_\rho. \] (123)

It can be seen that the remaining, purely orthogonal part of the left hand side of (121) can be evaluated in terms of the extrinsic geometry of the imbedding, without knowledge of the gradient of \( \overline{T}^{\mu\nu} \), using the fact that, since the surface stress momentum energy tensor \( \overline{T}^{\mu\nu} \) is purely tangential, i.e. since its contraction with \( \perp^\mu_\rho \) vanishes, an integration by parts leads to the identity
\[ \perp^\rho_\mu \nabla_\nu \overline{T}^{\nu\mu} = \overline{T}^{\mu\nu} K^\mu_\nu \rho, \] (124)
where the \( K^\mu_\nu \rho \) is the second fundamental tensor as defined [26] in terms of tangential differentiation of the first fundamental tensor \( \eta^\rho_\sigma \) by
\[ K^\mu_\nu \rho = \eta^\sigma_\nu \nabla_\mu \eta^\rho_\sigma. \] (125)

One thus obtains an expression of the standard form [27]
\[ \overline{T}^{\mu\nu} K^\mu_\nu \rho = \perp^\rho_\mu \overline{f}_\mu, \] (126)
for the extrinsic part of the dynamical equation (121), i.e. the part that governs the evolution of the worldsheat itself, which is the only part there is in the present case.

Due to the Nambu-Goto like form (112) obtained for \( \overline{T}^{\mu\nu} \) in the simple vortex model under consideration here, it will not be necessary to work out the complete second fundamental tensor \( K^\mu_\nu \rho \) in the present case, but only its trace, the curvature vector
\[ K^\rho = K^\nu_\nu \rho = \nabla_\nu \eta^\rho_\nu, \] (127)
in terms of which it can be seen that the equation of motion (126) will reduce to the form
\[ \overline{T} K^\rho = \perp^\rho_\nu \nabla_\nu \overline{T} + \pi h \mathcal{E}^{\mu\nu} N_{\mu\nu\rho}, \] (128)
with the effective tension \( T \) given by (112), according to which it is proportional to the square of the auxiliary field \( \Phi \) in the ambient superfluid background. The first term on the right of (128), allowing for the effect of any non-uniformity of this field, was not needed in the special limit of the “stiff” fluid model (61) characterised by a fixed value of \( \Phi \), which describes the massless axion case to which earlier studies of relativistic vortex string dynamics [1, 4] were restricted.

11 Vorton equilibrium in stationary background

As a simple exercise, let us apply the foregoing theory to the problem of a vorton, i.e. an equilibrium state of a vortex ring in a uniform background. As in the well known non relativistic version of this problem, relative motion is needed to provide the Joukowsky type force (due to the Magnus effect) that supports the string against its own tension. A superfluid vorton differs from local cosmic vorton [28, 27] in that the latter is supported not by a Magnus effect, but by the centrifugal effect of circulating current.

By definition, an equilibrium state is invariant with respect to the action of a time displacement vector, \( k^\mu \) say, which, to preserve the background metric must satisfy the Killing equation given according to (19) by

\[
2 \nabla_{(\mu} k_{\nu)} = 0.
\]

(129)

Invariance requires that the vortex worldsheet be tangent to this vector, whose components in a corresponding Minkowski coordinate system will be given by \( k^\mu \leftrightarrow \{1, 0, 0, 0\} \) with normalisation \( k^\mu k_\mu = -c^2 \).

The condition that the superfluid background is not just stationary but uniform with velocity \( v \) relative to such a frame means that for suitably aligned space axes its 4-velocity will be given by \( u^\mu \leftrightarrow \gamma \{1, v, 0, 0\} \) with \( \gamma = (1 - v^2/c^2)^{-1/2} \). The assumption that the ambient fluid is uniform means that the gradient term will drop out of the dynamical equation (128), with the consequence that the background number density \( n \) will also cancel out, leaving an equilibrium condition that is expressible just in terms of the 4-momentum covector \( \pi_\nu = \mu u_\nu \) in the form

\[
\hbar \bar{K}_\rho = 2 \mathcal{E}^{\mu\nu} e_{\mu\rho\sigma} \pi^\sigma.
\]

(130)

In such a stationary case it is easy to evaluate the tangent bivector \( \mathcal{E}^{\mu\nu} \) and the worldsheet curvature vector \( K^\mu \) in terms of the unit tangent
vector, $e^\mu$ say, that is orthogonal to $k^\mu$ within the worldsheet (which to be uniquely specified requires a choice of orientation). In terms of this vector, as characterised by $e^\mu k_\mu = 0$, $e^\mu e_\mu = 1$, we shall simply have

$$\mathcal{E}^{\mu\nu} = 2 k^{[\mu} e^{\nu]}, \quad (131)$$

and (independently of the choice of orientation) the corresponding fundamental tensor will be given by

$$\eta^{\mu\nu} = -\frac{1}{c^2} u^\mu u^\nu + e^\mu e^\nu. \quad (132)$$

It can be seen from the defining relations \[(102)\] and \[(127)\] that when \[(129)\] is satisfied we shall simply have

$$K^\mu = e^\nu \nabla_\nu e^\mu. \quad (133)$$

The vorton equilibrium states in which we are interested will be symmetric about an axis aligned with the relative flow, which we take to be the $x_1$ axis of our Minkowski coordinate system. The worldsheet will be circular, with fixed radius $r$ say, meaning that any point on it will be located relative to the axis by a radial displacement vector that can be parametrised in terms of the corresponding axial angle $\theta$ in the form $r^\mu \leftrightarrow r\{0, 0, \cos \theta, \sin \theta\}$ with $r^\mu r_\mu = r^2$. The corresponding unit tangent vector will be given by $e^\mu \leftrightarrow \{0, 0, -\sin \theta, \cos \theta\}$ and thus the corresponding curvature vector is easily seen to be given by $K^\rho = r^{-2} r^\rho \leftrightarrow r^{-1}\{0, 0, \cos \theta, \sin \theta\}$.

From this familiar result, namely that the curvature vector $K^\rho$ of a circle of radius $r$ is inwardly directed with magnitude $r^{-1}$, it can be seen that the solution to the equilibrium condition \[(130)\] will be obtained for a vorton radius given by

$$r = \frac{\hbar \hat{l}}{4 \mu v \gamma}, \quad (134)$$

in which it is to be recalled that, according to \[(77)\], $\hat{l}$ is an order of unity factor that can be estimated as the (natural) logarithm of the ratio of the vorton area to the sectional area of the vortex core.

The vagueness of the prescription for the $\hat{l}$ symptomises an inherent limitation of any attempt (whether in a relativistic or in a Newtonian framework) to treat a "global" vortex as if it were a locally confined string type phenomenon, despite the "infra-red divergence" of its energy and tension, which can only be made finite by a long range cut off. This problem does
not arise for the vortex defects in an ordinary metallic type II superconductor, in which the local gauge coupling to the electromagnetic field ensures an exponential fall off ensuring convergence without any need for a cut off. In the cosmic string context, there are locally gauged examples for which a string type description is highly accurate there are also examples of non-local vortex defects, such as the axionic case (and cases involving electromagnetic or gravitational coupling) for which the thin string limit description is less satisfactory. If the vortex core radius $\delta$ is relatively small, the problem of sensitivity to the long range cut off $\Delta$ is mitigated by the fact that the dependence is only logarithmic. Thus a formula such as (134) (or its well known non-relativistic anologue, in which the $\gamma$ factor is omitted, and $\mu$ is replaced by a fixed mass $m$) can be usefully applied to macroscopic vorton configurations, but becomes quantitatively unreliable for describing the microscopic vortons, known as “rotons”, that are important in the analysis of thermal excitations.

Whereas the problem of specification of the cut off can limit the utility of the string type description (111) for application to the dynamics of individual vortices, this reservation does not apply to the macroscopically averaged description for an Abrikosov type lattice of aligned vortices as given by (78), which is on a much sounder footing: in this case the relevant cut off is unambiguously determined by the lattice spacing. Although its accuracy is more questionable, the single string picture provides useful insight into the interpretation of the – at first sight rather mysterious – dynamical equation (82) for the more robust macroscopically averaged model, which is in fact formally identical to the more intuitively interpretable string dynamical equations (128).
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