SIMILAR RELATIVELY HYPERBOLIC ACTIONS OF A GROUP.

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Abstract. We study similarity properties between different non-trivial relatively hyperbolic actions of a fixed group $G$ on topological compacta. We do not impose any restriction on the cardinality of $G$ neither the metrisability of the space on which it acts.

We provide a sufficient condition when such two relatively hyperbolic actions $\mu_i$ imply the existence of an action $\mu$ of $G$, called pullback action, and equivariant continuous maps $\pi_i$ conjugating $\mu$ to $\mu_i$ ($i = 1, 2$). The condition is that the parabolic subgroups for the action $\mu_1$ are quasiconvex with respect to $\mu_2$ and vice versa.

It follows from a recent result of O. Baker and T. Riley that the above pullback action $\mu$ in general does not exist even for a finitely generated $G$ if $\mu_i$ are only assumed to be convergence actions [BR]. We provide an example of two relatively hyperbolic actions of a non-finitely generated countable free group which do not possess a common pullback space neither. We note that such an example is not possible for finitely generated groups.

One of the tools used in the proofs is the first theorem of the paper providing the equivalence between two notions of quasiconvexity for the subgroups of a relatively hyperbolic group $G$.

1. Introduction

The aim of the paper is to describe "similarity" properties of different actions by homeomorphisms of a group on compacta which are nontrivial and relatively hyperbolic.

An action is non-trivial if it does not have a fixed common point. If the opposite is not stated we will always assume that all actions are non-trivial. The relatively hyperbolic action is defined as follows:

Definition 1.1. An action of a discrete group $G$ on a compactum $X$ is 3-discontinuous (convergence action) if the induced action of $G$ on the space of distinct triples $\Theta^3X$ of $X$ is discontinuous.

An action $G \curvearrowright X$ is 2-cocompact if the induced action of $G$ on the space of distinct pairs $\Theta^2X$ is cocompact.

An action $G \curvearrowright X$ is relatively hyperbolic if it is 3-discontinuous and 2-cocompact (shortly (32)-action).

We provide few remarks about this definition. First, there is no any restriction on the cardinality of $G$: it is not supposed to be finitely generated neither even countable. Also we do not suppose that the space $X$ is metrisable. It is shown in [GePo2] that the above definition of relative hyperbolicity is equivalent to the other definitions valid in this general setting [Bo1], [Os].

We start by describing properties of subgroups of relatively hyperbolic groups. Recall that a subgroup $H$ of a group $G$, acting 3-discontinuously on $X$, is called dynamically quasiconvex.
if for every two disjoint closed subsets \( K \) and \( M \) of \( X \) the set \( \{ g \in G : g(\Lambda_X H) \cap K \neq \emptyset \land g(\Lambda_X H) \cap M \neq \emptyset \} \) is at most finite, where \( \Lambda_X H \) denote the limit set of \( H \) for the action on \( X \).

Our first result (section 3) is the following equivalence between two different quasiconvexity properties of \( H \). We stress that no restrictions on the cardinality of \( G \) neither \( H \) are requested here.

**Theorem A.** Let \( G \) be a group which admits a \( 3 \)-discontinuous and \( 2 \)-cocompact non-trivial action on a compactum \( X \). Let \( H \) be a subgroup of \( G \). The following conditions are equivalent.

1. The action \( H \looparrowright \Lambda_X H \) is \( 2 \)-cocompact.

2. \( H \) is dynamically quasiconvex.

\( \square \)

To prove this theorem we need to generalize several facts known for finitely generated groups to the case of non-finitely generated ones. Using the \( (32) \)-action of \( G \) on \( X \) we have constructed in [GePo2] a fine, hyperbolic graph \( \Gamma \) such that \( G \) acts properly and cofinitely on its edges (see Lemma 2.1 below). The following statement plays an important role in the proof of Theorem A and seems to have an independent interest.

**Theorem** (Theorem 3.8). Every two points \( a \) and \( b \) at the boundary of \( \Gamma \) can be joined by an infinite geodesic.

Let now \( G \looparrowright X \) and \( G \looparrowright Y \) be two different convergence actions of a group \( G \). The first similarity property studied in the paper is the existence of a pullback space for these actions. This is a space \( Z \) which admits a convergence action of \( G \) such that there are two equivariant continuous maps \( \pi_X : Z \to X \) and \( \pi_Y : Z \to Y \) [Ge1]. It follows from a recent example of O. Baker and T. Riley [BR] that such a pullback space does not always exist even if \( G \) is finitely generated (see Proposition 4.2 in section 4). However the question about its existence remained intriguing if one supposes that both actions on \( X \) and on \( Y \) are \( 32 \)-actions. We note that if \( G \) is a finitely generated relatively hyperbolic group then \( Z \) exists and coincides with the Floyd boundary of \( G \) [Ge2, Map theorem 3.4.6].

We will show (section 4) that in the general case of a non-finitely generated group the answer is negative.

**Proposition** (Proposition 4.4). A non-finitely generated countable free group \( F_\infty \) admits two distinct \( (32) \)-actions not having a common pullback space. \( \square \)

Recall that if \( G \) acts relatively hyperbolically on a compact \( X \) then every limit point for this action is either bounded parabolic or conical [Ge1]. Denote by \( \text{Par}_X \) the set of bounded parabolic points for the action \( G \looparrowright X \). For every parabolic point \( p \in \text{Par}_X \) its stabilizer \( \text{St}_X p \) is the maximal parabolic subgroup for the action on \( X \).

The next result (section 4) provides a sufficient condition for the existence of a pullback space.

**Theorem B.** Let \( G \) be a group which admits \( (32) \)-actions on compacta \( X \) and \( Y \). Let \( P \) and \( Q \) be the systems of maximal parabolic subgroups for the actions on \( X \) and \( Y \) respectively. Assume that every \( P \in P \) acts \( 2 \)-cocompactly on \( \Lambda_Y P \) and every \( Q \in Q \) acts \( 2 \)-cocompactly on \( \Lambda_X Q \).

Then there there exists a pullback space \( Z \) for the actions on \( X \) and \( Y \). Furthermore the action \( G \looparrowright Z \) is a \( (32) \)-action.
We further consider (section 5) a stronger similarity than the existence of a pullback space. This is the existence of an equivariant continuous map between two (32)-actions. Refining the argument of the proof of Theorem B we obtain.

**Theorem C.** Let $G$ be a group which admits (32)-action on compacta $X$ and $Y$. Let $\mathcal{P}$ and $\mathcal{Q}$ denote the systems of maximal parabolic subgroups for the actions on $X$ and on $Y$ respectively. Suppose that for every $P \in \mathcal{P}$ there exists $Q \in \mathcal{Q}$ such that $P < Q$. Then there exists an equivariant continuous map $f : X \to Y$.

Furthermore $f$ is injective on the set of conical points and for every parabolic point $q \in Y$: $f^{-1}(q) = \Lambda_X(\text{St}_Y q)$ is the limit set for the action of $\text{St}_Y q$ on $X$.

We note that Theorem C has been previously known in several cases. If $G$ is finitely generated then it again follows from the existence of the equivariant map between the Floyd boundary and a compactum admitting (32)-action of $G$ [Ge2, Map theorem 3.4.6] (see also remarks 5.1).

Theorem C was also known when $G$ is countable and both spaces $X$ and $Y$ are metrisable [MOY]. We provide an independent argument valid in the general case of a topological space without assuming the countability of $G$.

As an application of the above results we prove in Section 6 the following:

**Proposition (Proposition 6.1).** Let $G$ be a group which admits (32)-actions on compacta $X$ and $Y$. Let $\mathcal{P}$ and $\mathcal{Q}$ be the systems of maximal parabolic subgroups for the actions on $X$ and $Y$ respectively. Assume that $\forall P \in \mathcal{P} \exists Q \in \mathcal{Q} : P < Q$. Then the induced action of every $Q \in \mathcal{Q}$ on $\Lambda_X Q$ is 2-cocompact.

Concluding the discussion we note that the similarities between two (32)-actions considered in the paper are characterized by the following properties of parabolic subgroups of the actions: a) one system is included into the other one (Theorem C); or b) mutual 2-cocompactness for the actions of the parabolic subgroups (Theorem B).

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2. Preliminaries

2.1. **Entourages and Cauchy-Samuel completions.** Let $X$ be a compactum (i.e. Hausdorff and compact space). We start by recalling few standard notions from the general topology, for further references we refer to [Ke].

We denote by $S^n X$ the set of non-ordered n-tuples which is the quotient of $X^n$ by the action of the permutation group on n symbols.

An entourage is a neighborhood of the diagonal $\Delta^2 X$ in $S^2 X$. The set of all entourages of $X$ is denoted by EntX. We use the bold font to denote the entourages. For $u \in \text{Ent} X$ a pair of points $(x, y) \in X^2$ is called $u$-small (or belongs to $\text{Small}(u)$) if $(x, y) \in u$. Similarly a set $A \subset X$ is $u$-small if $A^2 \subset u$.

For an entourage $u$ we define its power $u^n$ as follows: $(x, y) \in u^n$ if there exists $x_i \in X$ such that $(x_{i-1}, x_i) \in u$ ($x_0 = x, x_n = y, i = 1, ..., n - 1$). We denote by $\sqrt[n]{u}$ an entourage $v$ such that $v^n \subset u$.

A filter $U$ on $X$ whose elements are entourages is called uniformity if $\forall u \in U \exists v \in U : v^2 \subset u$. 
A pair \((X, U)\) of a Hausdorff space \(X\) equipped with an uniformity \(U\) is called uniform space. A Cauchy filter \(F\) on the uniform space \((X, U)\) is a filter such that \(\forall u \in U : F \cap \text{Small}(u) \neq \emptyset\). A space \(X\) is complete if every Cauchy filter on \(X\) contains all \(u\)-small neighborhoods of every \(x \in X\) \((\forall u \in U)\). The uniform space \((X, U)\) admits a completion \((\overline{X}, \overline{U})\) called Cauchy-Samuel completion whose construction is the following. Every point of \(\overline{X}\) is the minimal Cauchy filter \(\xi\) which is the system of all \(u\)-small neighborhoods of a point \(x \in X\). For every \(u \in U\) we define an entourage \(\overline{u}\) on \(\overline{X}\) as follows:

\[
\overline{u} = \{(\xi, \eta) \in S^2 \overline{X} : \xi \cap \eta \cap \text{Small}(u) \neq \emptyset\}.
\]

The uniformity \(\overline{U}\) of \(\overline{X}\) is the filter generated by the entourages \(\{\overline{u} : u \in U\}\). We note that the completion \(\overline{U}\) is exact: \(\forall a, b \in \overline{X} \exists \overline{u} \in \overline{U} : (a, b) \not\in \overline{u}\) [Ge2 2.4].

2.2. Properties of \((32)\)-actions of groups. Let \(G\) be a group acting 3-discontinuously on a compactum \(X\) (convergence action). Recall that the limit set, denoted by \(\Lambda_X G\) (or \(\Lambda G\) if \(X\) is fixed), is the set of accumulation (limit) points of the \(G\)-orbit for the action of \(G\) on \(X\).

The action \(G\) on \(X\) is said to be minimal if \(X = \Lambda G\).

The action \(G \actson X\) is elementary if \(|\Lambda G| \in \{0, 1, 2\}\) in the opposite case \(\Lambda G\) is a perfect set [Tu2]. If \(G\) is non-elementary then \(\Lambda G\) is the minimal closed subset of \(X\) invariant under \(G\).

An elementary action of a group \(G\) on \(X\) is called parabolic (or trivial) if there is unique fixed point \(p \in X\) called parabolic fixed point. We denote by \(\text{St}_X p\) (or \(\text{St}_{X,G} p\)) its stabilizer which is the maximal parabolic subgroup fixing \(p\). The set of parabolic points for the action on \(X\) is denoted by \(\text{Par}_X\).

A parabolic fixed point \(p \in \Lambda G\) is called bounded parabolic if the quotient space \((\Lambda G \setminus \{p\})/\text{St}_{X,G} p\) is compact.

We will use an equivalent reformulation of the convergence property in terms of crosses. Since we do not assume the metrisability of \(X\) our terminology is purely topological [Ge1, Ge2] (compare with [Tu1]). A cross \((r, a)^x \in X \times X\) is the set \(r \times Y \cup X \times a\) where \((r, a) \in X \times X\). The corresponding limit cross (or limit quasihomeomorphism) is the map \((r, a)^x(x) = a\) for all \(x \in X\) \(\setminus \{r\}\) and which is undefined at \(r\).

The points \(a\) and \(r\) are called respectively attractive and repelling points (or attractor and repeller). It is shown in [Ge1, Proposition P, 5.3] that an action \(G \actson X\) is 3-discontinuous if and only if \(\overline{G} \setminus G\) is the set of the limit crosses on \(X \times X\) where \(\overline{G}\) is the compactification of \(G\) in the space of all continuous maps \(X \to X\).

A point \(x \in \Lambda G\) is conical if there is an infinite set \(S \subseteq G\) such that for every \(y \in X \setminus \{x\}\) the closure of the set \(\{(s(x), s(y)) : s \in S\}\) in \(X^2\) does not intersect the diagonal \(\Delta^2 X\).

The convergence actions can be also characterized using the language of entourages [Ge1, GePo2]. A group \(G\) acting on the space \(X\) acts on the set of entourages \(\text{Ent} X\). For \(u \in \text{Ent} X\) we denote by \(g u\) the set \(\{(x, y) \in X^2 : g^{-1}(x, y) \in u\}\) and by \(G u\) the \(G\)-orbit of \(u\). Two entourages \(u\) and \(v\) are said to be unlinked if there exist \(U \in \text{Small}(u)\) and \(V \in \text{Small}(v)\) such that \(X = U \cup V\). We denote this relation by \(u \bowtie v\). If the opposite is true we say that \(u\) and \(v\) are linked, and write \(u \# v\). Dynkin property states that the action \(G \actson X\) is 3-discontinuous if and only if the set \(\{g \in G : g u \# v\}\) is finite [Ge1 Proposition P, 5.3].

Let \(\Gamma\) be a graph. We denote by \(\Gamma^0\) and \(\Gamma^1\) the set of vertices and edges of \(\Gamma\) respectively. Recall that an action \(G\) on \(\Gamma\) is proper on edges if the stabilizer \(\text{St}_{re}\) of every edge \(e\) in \(\Gamma\) is finite. The action \(G \actson \Gamma\) is called cofinite if \(|\Gamma^1/G| < \infty\).

According to B. Bowditch [Bo1] a graph \(\Gamma\) is called fine if for any two vertices the set of arcs of fixed length joining them is finite.
Suppose now that a group $G$ admits a 3-discontinuous and 2-cocompact non-parabolic minimal action $(\Theta^2 X)$-action on a compactum $X$. Then every point of $X$ is either a bounded parabolic or conical [Ge1, Main Theorem]. If $X$ is metrisable then P. Tukia showed that the converse statement is also true [Tu2, Theorem 1C, (b)].

It is shown in [GePo2, Theorem A] that if $G$ admits $(32)$-action on a compactum $X$ then $G$ decomposes as a star graph of groups whose central vertex group is a finitely generated relatively hyperbolic group $G_0$ and all other vertex groups are maximal parabolic subgroups. The group $G_0$ acts on a connected fine hyperbolic graph $\Gamma_0$ properly and cofinitely on its edges (e.g. on the relative Cayley graph). The vertices of $\Gamma_0$ are the elements of $G_0$ and the set of parabolic points for the action on $X$. Then by [GePo2, Proposition 3.43] there also exists a connected fine and hyperbolic graph $\Gamma$ acted upon by $G$ cofinitely and properly on edges. Every vertex of $\Gamma$ is either an element of $G$ and its stabilizer is trivial, or belongs to the set of parabolic points $\text{Par}_X$ and has an infinite stabilizer.

Consider now the augmented space $\tilde{X} = X \cup \Gamma$. By the attractor sum theorem [Ge2, Proposition 8.3.1] it admits a unique compact Hausdorff topology whose restriction on $X$ coincides with the original topologies of $X$ and $\Gamma$. Furthermore the action on $\tilde{X}$ is 3-discontinuous (a direct proof of this fact for a discrete group $G$ is given in [GePo2, Proposition 3.14]).

The action on $\tilde{X}$ is also 2-cocompact. Indeed by assumption the action of $G \actson \Theta^2 X$ is cocompact. So there exists a compact fundamental set $K \subset \Theta^2 X$. Hence $\tilde{K} = K \cup (\{1\} \times (\tilde{X} \setminus \{1\}))$ is a compact fundamental set for the action on $\Theta^2 \tilde{X}$. Therefore the action $G \actson \tilde{X}$ is a $(32)$-action. We recapitulate all these facts in the following lemma.

**Lemma 2.1.** Suppose $G$ is a group which admits a non-trivial $(32)$-action on a compactum $X$. Then there exists a connected, fine and hyperbolic graph $\Gamma$ acted upon by $G$ properly and cofinitely on edges. Furthermore $G$ acts 3-discontinuously and 2-cocompactly on the augmented space $\tilde{X} = X \cup \Gamma$ and $\Gamma^0 = G \cup \text{Par}_X$ is the set of all non-conical points for the action on $\tilde{X}$.

By the lemma the $G \actson \tilde{X}$ is also of type $(32)$ so we admit the following.

**Convention.** We will consider the entourages $u \in \text{Ent}\tilde{X}$ on the augmented space $\tilde{X}$ as well as their restrictions on $\Gamma$ and on $X$.

Following Bowditch [Bo1] we use the term connected $G$-set (or connected $G$-structure) for the set of vertices $M = \Gamma^0$ of a connected graph $\Gamma$ admitting a cofinite and proper action on edges of a group $G$.

We now recall few definitions from [Ge2]. An entourage $u$ given on a connected $G$-set $M$ is called *perspective* if for any pair $(a, b) \in \Gamma^0 \times \Gamma^0$ the set \( \{g \in G \mid g(a, b) \notin u\} \) is at most finite.

As we indicated above a group $G$ acting on a connected $G$-set $M$ acts naturally on the set of entourages $\text{Ent}M$.

An entourage $u$ given on a connected $G$-set $M$ is called *divider* if there exists a finite set $F \subset G$ such that $(u_F)^2 \subset u$ where $u_F = \cap_{f \in F} fu$.

A uniformity $U$ on a compactum $\tilde{X}$ is generated by an entourage $u$ if it is generated as a filter by the orbit $Gu$.

By Lemma [Ge2, Proposition 8.4.1] there exists a uniformity $U$ on $\tilde{X}$ generated by $u$ where $u^0 = u|_M$ is a perspective divider.
Note that one can also go in the opposite way: starting from a perspective divider on a connected $G$-set $M = \Gamma^0$ and then obtain a (32)-action on the compactum $X = X \cup \Gamma$ where $X$ is a "boundary" of $\Gamma$.

**Definition 2.2.** We say that the pair of vertices $(a, b)$ of $\Gamma$ is $u_e$-small for an edge $e \in \Gamma^1$ if there exists a geodesic in $\Gamma$ with endpoints $a$ and $b$ which does not contain $e$.

A uniformity $U^0$ on $M = \Gamma^0$ has a visibility property if for every entourage $u^0 \in U^0$ there exists a finite set of edges $F \subset \Gamma^1$ such that $u^0 = \cap \{u_e \mid e \in F\} \subset u^0$.

The following lemma summarize several propositions of [Ge2].

**Lemma 2.3.** Suppose that a group $G$ acts on a connected graph $\Gamma$ properly and cofinitely on edges. Suppose that the set of vertices $\Gamma^0$ of $\Gamma$ admits a uniformity $W^0$ generated by a perspective divider $w^0$. Then the Cauchy-Samuel completion $(Z, W)$ of the pair $(\Gamma^0, W^0)$ admits a (32)-action of $G$.

**Proof:** As we indicated above the points of $Z$ are the minimal Cauchy filters of the vertices of $\Gamma$ with respect to the uniformity $W^0$.

By [Ge2] Proposition 3.5.1] the uniformity $W^0$ generated by a perspective divider has the visibility property. Then the completion space $(Z, W)$ is a compact Hausdorff space whose topology is consistent with the uniformity $W$ [Ge2 Proposition 4.1.1]. Furthermore the action of $G$ on $Z$ is a (32)-action [Ge2 Propositions 4.2.1, 4.2.2]. The lemma follows. 

### 3. Quasiconvex subgroups for both actions

#### 3.1. The statement of the result

We start by restating the definition of the dynamical quasiconvexity in terms of entourages [GePo3].

**Definition 3.1.** Let $G$ be a discrete group acting 3-discontinuously on a compactum $X$. A subgroup $H$ of $G$ is said to be dynamically quasiconvex if for every entourage $u$ of $X$ the set $G_u = \{gG : g(\Lambda H) \notin \text{Small}(u)\}/H$ is finite.

It is shown in [GePo3] that in case of a finitely generated group the dynamical quasiconvexity is equivalent to the relative quasiconvexity. The aim of this Section is the following theorem generalizing this result to the case of non-finitely generated groups.

**Theorem [A].** Let $G$ be a group which admits 3-discontinuous and 2-cocompact non-trivial action on a compactum $X$. Let $H$ be a subgroup of $G$. The following conditions are equivalent.

1. The action $H \curvearrowright \Lambda X H$ is 2-cocompact.
2. $H$ is dynamically quasiconvex.

#### 3.2. Proof of the implication $2) \Rightarrow 1)$

Let $K$ be a compact fundamental set for the action of $G$ on the set of distinct couples $\Theta^2 X$. Denote by $u$ the entourage $X^2 \backslash \overline{K}$ where $\overline{K}$ denotes the closure of $K$ in $\Theta^2 X$. By the dynamical quasiconvexity it follows that the set $\{i \in I : g_i(\Lambda H) \notin \text{Small}(u)\}$ is finite. So if the set $g(\Lambda H)$ is not $u$-small for some $g \in G$ then there exists $i \in I$ such that $g(\Lambda H) = g_i(\Lambda H)$.

Let $u_1$ denote the entourage $X^2 \backslash \Theta^2 (\Lambda H)$. By the dynamical quasiconvexity applied to $u_1$ we obtain that the index of $H$ in the stabilizer $St_X(\Lambda H)$ of $\Lambda_H$ in $G$ is finite and $St_X(\Lambda H) = \cup_{j \in J} k_j H$ ($|J| < \infty$).
Consider finally the following entourage on $\Lambda H$:

$$v = ( \bigcap_{i \in I, j \in J} k_j^{-1} g_i^{-1} u ) \cap \Lambda^2 H.$$ 

Let $(x, y) \in \Theta^2(\Lambda H)$. Then $\exists g \in G : g(x, y) \in K$. So $g(x, y) \notin u$ implying that $g(\Lambda H) = g_i(\Lambda H)$ for some $i \in I$. Then $g_i^{-1} g(\Lambda H) = \Lambda H$ and so $g_i^{-1} g = k_j h$ for some $j \in J$ and $h \in H$. We have $g(x, y) = g_i k_j h(x, y) \notin u$. Hence $h(x, y) \notin v$. We have proved that for every $(x, y) \in \Theta^2(\Lambda H)$ there exists $h \in H$ such that $(x, y) \notin h^{-1} v$ implying that the orbit $H v$ is separating [Ge1, 6.12]. Hence the action $H \curvearrowright \Lambda H$ is 2-cocompact [Ge1, Section 7.1, Proposition E]. The implication is proved. \hfill \Box

3.3. Proof of the implication 1) $\Rightarrow$ 2). The proof consists of several results which seem to have an independent interest. We subdivide them in several paragraphs.

We will always assume in this subsection that a group $G$ admits a (32)-action on a compactum $X$ and $H$ is a subgroup of $G$.

3.3.1. Topology of the space of eventual geodesics. Since the action $G \curvearrowright X$ is of type (32) then by Lemma 2.1 $G$ acts on the augmented space $\widetilde{X} = X \cup \Gamma$ where $\Gamma$ is a connected, fine, hyperbolic graph and the action $G \curvearrowright \Gamma^1$ is proper and cofinite. Denote by $U$ the uniformity on $\widetilde{X}$ generated by an entourage whose restriction on $\Gamma^0$ is a perspective divider. The uniformity $U$ is exact (see section 2.1).

A path in $\Gamma$ is a map $\gamma : \mathbb{Z} \to \Gamma$ such that $\gamma\{n, n+1\}$ is either an edge of $\Gamma$ or a point $\gamma(n) = \gamma(n+1)$. A path $\gamma$ can contain "stop" subpaths, i.e. a subsets of consecutive integers $J \subset \mathbb{Z}$ such that $\gamma|_J \equiv \text{const}$.

For a finite subset $I \subset \mathbb{Z}$ of consecutive integers we define the boundary $\partial(\gamma|_I)$ to be $\gamma(\partial I)$. We extend naturally the meaning of $\partial \gamma$ over the half-infinite and bi-infinite paths in $\Gamma^0 \subset \widetilde{X}$ in the case if the corresponding half-infinite branches of $\gamma$ converge to points in $\widetilde{X}$. The latter one means that for every entourage $v \in U$ the set $\gamma|_{[n, \infty[}$ is $v$-small for a sufficiently big $n$.

Since the uniformity $U$ is generated by a perspective divider $u$ its restriction $u|_{\Gamma}$ on $\Gamma$ has the visibility property (see the proof of Lemma 2.3). So we admit the following.

Convention 3.2. If the opposite is not stated we will always suppose that the uniformity $U$ on $\widetilde{X}$ has the visibility property.

Lemma 3.3. Every half-infinite geodesic ray $\gamma : [0, \infty[ \to \Gamma$ converges to a point in $\widetilde{X}$.

Proof: Fix an entourage $v \in U$. By the visibility property there exists a finite set of edges $F \subset \Gamma^1$ such that $u_F = \bigcap_{e \in F} u_e \subset v$. Since $\gamma$ is a geodesic, the ray $\gamma|_{[n_0, \infty[}$ does not contain $F$ for some $n_0 \in \mathbb{N}$. So $\gamma|_{[n_0, \infty[}$ is $u_F$-small and therefore $v$-small. \hfill \Box

Definition 3.4. A path $\gamma : I \to \Gamma$ is an eventual geodesic if its stop subpaths can only happen on one of the following half-infinite intervals: $]-\infty, n[\text{, or } [m, +\infty[\text{, or } ]-\infty, n] \cup [m, +\infty[\text{ (m > n), and outside of them } \gamma \text{ is a geodesic.}$

The set of eventual geodesics in $\Gamma$ is denoted by $\text{EG}(\Gamma)$.

Proposition 3.5. The space $\text{EG}(\Gamma)$ is closed in the space of maps $\widetilde{X}^\mathbb{Z}$ equipped with the Tikhonov topology.
Proof: Denote by $\overline{\text{EG}(\Gamma)}$ the closure of $\text{EG}(\Gamma)$ in $\tilde{X}^2$. The aim is to show that every $\gamma \in \overline{\text{EG}(\Gamma)}$ is an eventual geodesic.

Let us first show that $\gamma$ cannot contain stop-subpaths in its interior. Let $a = \gamma(n), b = \gamma(n+1), c = \gamma(m), d = \gamma(m+1)$ such that $a \neq b, c \neq d$ and $m > n + 1$. Suppose by contradiction that $b = c$. Since $U$ is an exact uniformity (see section 2.1) there exists $v \in U$ such that $(a, b)$ and $(c, d)$ are not $v^3$-small. The curve $\gamma$ is a limit of eventual geodesics in the Tikhonov topology. So there exists an eventual geodesic $\lambda \subset \Gamma$ which passes through vertices $a', b', c', d' \in G^0$ which are $v$-close to the points $a, b, c, d$ respectively. Then $(a', b') \notin v$ as otherwise the pair $(a, b)$ would be $v^3$-small. Similarly $(c', d') \notin v$. By the visibility property there exists a finite set $F \subset \Gamma^1$ such that $u_F \subset v$. So $(a', b') \notin u_F$ and $(c', d') \notin u_F$. Thus each part of $\lambda$ between $a'$ and $b'$ and $c'$ and $d'$ contains an edge from $F$. Since $F$ is a fixed finite subset of $\Gamma^1$, choosing $\lambda$ sufficiently close to $\gamma$, we may assume that $(a', b') = (a, b)$ is a fixed edge, and similarly $(c, d) = (c', d')$. So the eventual geodesic $\lambda$ contains an intermediate stop between the points $b' = b$ and $c = c'$ which is impossible.

It remains to prove that $\gamma|_{[n, n+k]}$ is either geodesic if $\gamma(n+k) \neq \gamma(n)$, or $\gamma(n+k) = \gamma(n)$ $(n, k \in \mathbb{N})$. Let us prove it by induction on $k$. Suppose first that $k = 1$ and $(x = \gamma(n)) \neq (\gamma(n + 1) = y)$. Then there exists a finite subset $F \subset \Gamma^1$ such that $(x, y) \notin u_F^3$. By the previous argument $(x, y)$ is an edge in $\Gamma$ and we are done in this case.

Suppose that the statement is true for all $k' < k$. We have two cases.

Case 1. $\gamma(n) = \gamma(n + k) = x$.

Let $z = \gamma(n + 1)$. If first, $z = x$ then by induction $\gamma|_{[n+1, n+k]}$ is eventual geodesic.

If $x \neq z$ then $d(z, x) = 1$ and $[z, x] = \gamma|_{[n+1, n+k]}$ is an eventual geodesic containing an edge starting from $z$. By the above argument $\gamma$ does not contain stop subpaths in the middle so $\gamma([n, n+k]) = [x, z] \cup [z, x]$ is a union of two adjacent edges having both $x$ as an endpoint. This curve cannot be approximated by eventual geodesics.

Case 2. $(a = \gamma(n)) \neq (b = \gamma(n + k))$.

By the above argument the interval $[a, b]$ of $\gamma$ contains an edge $[a_1, b_1] = e \in G^1$. If first $a = a_1$ then $[b_1, b]$ is an eventual geodesic by induction and so is $[a, b]$.

If $a \neq a_1$ then using the diagonal procedure we approximate the part $[a, a_1]$ of $\gamma$ by the curves $\lambda \in \text{EG}(\Gamma)$ which coincide with $\gamma$ on the edge $e$. In this way we find a new edge $e_1 \subset [a, a_1]$ and an eventual geodesic containing the edges $e$ and $e_1$. After finitely many steps we obtain an eventual geodesic, still denoted by $\lambda$, that coincides with $\gamma|_{[n, n+k]}$. The proposition follows. \hfill \Box

Corollary 3.6. For every finite path $l = \{a_1, ..., a_n\} \subset \Gamma$ the set

$$(\text{EG}(\Gamma))_l = \{\gamma : \gamma|_l = l, \ I \subset Z, \ \gamma(-\infty) \neq a_1, \ \gamma(\infty) \neq a_m\}$$

is open.

Proof: By the argument above $\gamma$ admits a neighborhood $U \subset \overline{\text{EG}(\Gamma)}$ such that $\forall \lambda \in U \ \gamma|_l = \lambda|_l$. \hfill \Box

By Lemma 3.3 for a half-infinite geodesic ray $\gamma : \mathbb{Z}_{>0} \to X$ $\lim_{t \to +\infty} \gamma(t)$ exists. The following proposition refines this statement.

Proposition 3.7. The boundary map $\partial : \text{EG}(\Gamma) \to \tilde{X}^2$ where $\partial : \gamma \to \partial \gamma = \lim_{t \to \pm\infty} \gamma(t)$ is continuous.
Proof: Let us fix an eventual geodesic \( \alpha \in \text{EG}(\Gamma) \) such that \( \lim_{t \to \pm \infty} \alpha(t) = \{a, b\} \in \partial(\text{EG}(\Gamma)) \). We need to prove that there exist small neighborhoods of \( a \) and \( b \) all of whose points are the endpoints of eventual geodesics close to \( \alpha \).

Let \( U_a \) and \( U_b \) be closed disjoint neighborhoods of \( a \) and \( b \) in \( \tilde{X} \) respectively. By the exactness of \( \mathcal{U} \) on \( \tilde{X} \) there exists \( v \in \mathcal{U} \) such that \( v \cap (U_a \times U_b) = \emptyset \). Then by the visibility property we have \( \mathbf{u}_F = \cap_{e \in F} \mathbf{u}_e \subset v \) where \( F \) is a finite set of edges of \( \Gamma \). So every eventual geodesic with one endpoint in \( U_a \) and the other one in \( U_b \) must pass through \( F \).

Suppose first that both points \( a \) and \( b \) are not vertices of \( \Gamma \). Let \( U_a^* \subset U_a \) be a closed neighborhood of \( a \) such that \( U_a^* \cap U_b^* = \emptyset \) where \( U_b^* \) is the complement of \( U_a \). By the same reason as above there exists a finite set of edges \( F_a \subset \Gamma^1 \) such that once \( \gamma \in \text{EG}(\Gamma) \) passes from \( U_a^* \) to \( U_b^* \) it passes through a finite set of edges \( F_a \subset \Gamma^1 \). Up to adding a finitely many edges we can assume that \( F_a \) is connected. Let \( d = \text{diam}(F_a) \) denote the diameter of \( F_a \). We claim that if \( \gamma \) penetrates \( U_a^* \) in a distance bigger than \( d \) it cannot re-enter \( U_a^* \). Indeed if not then there exists a subpath \( \gamma' \) of \( \gamma \) inside of \( U_a^* \) of length bigger than \( d \) whose endpoints belong to \( \partial U_a^* \cap F_a \). Since the eventual geodesic \( \gamma \) is non-trivial outside of the both endpoints of \( \gamma' \) we have that \( \gamma' \) is a geodesic. Then \( \text{diam}(\partial \gamma') \leq \text{diam}(F_a) = d \) which is impossible. Choose now a smaller neighborhood \( \tilde{U}_a \subset U_a^* \) of \( a \) such that the graph distance between \( \partial(\tilde{U}_a) \cap \Gamma \) and \( \partial(U_a^*) \cap \Gamma \) is greater than \( d \). Similarly we choose a neighborhood \( \tilde{U}_b \) of \( b \). Then every \( \gamma \in \text{EG}(\Gamma) \) whose one boundary point belongs to \( \tilde{U}_a \) and the other one to \( \tilde{U}_b \) passes through \( F \) only once.

If now the point \( a \) is finite and \( b \notin \Gamma^0 \) then by identifying in the above argument \( \tilde{U}_a \) with \( a \) we obtain that \( \gamma \) paths only once through a finite set \( F \) of edges between \( a \) and \( U_b \). Finally if both points \( a \) and \( b \) are in \( \Gamma^0 \) then there exists a fixed finite set \( F \subset \Gamma^1 \) such that every such \( \gamma \) passes only once through \( F \).

Thus in all cases every eventual geodesic whose endpoints are in \( \tilde{U}_a \) and \( \tilde{U}_b \) respectively passes only once through a finite set of edges \( F \). By Corollary \( \ref{cor:openness} \) this set of geodesics is open.

As a consequence of the previous discussion we have the following.

**Theorem 3.8.** Every two distinct points \( a \) and \( b \) at the boundary of \( \Gamma \) can be joined by an infinite geodesic.

**Proof:** For every two disjoint neighborhoods \( U_a \) and \( U_b \) of \( a \) and \( b \) in \( \Gamma \) we consider a geodesic curve \( \beta \) connecting \( \partial U_a \) and \( \partial U_b \). By Proposition \( \ref{prop:penetration} \) a sequence of geodesics \( \{\beta\} \) penetrating every such pair of neighborhoods \( U_a \) and \( U_b \) converges to an eventual geodesic \( \gamma \). By Proposition \( \ref{prop:boundary} \) the boundary points of the curves \( \beta \)'s converge to the boundary of an eventual geodesic. By definition of an eventual geodesic it is a real geodesic connecting \( a, b \in \partial \Gamma \).

### 3.3.2. Refinement of the dynamical quasiconvexity property.

Let \( B \subset \tilde{X} \) be a closed set. By Theorem \( \ref{thm:connectedness} \) the following set is well-defined:

\[
\text{Hull}(B) = \cup \{ \gamma \in \text{EG}(\Gamma) : \partial \gamma \subset \partial B \}. \tag{1}
\]

We start with few technical lemmas.

**Lemma 3.9.** \( \text{Hull}(B) \) is a closed set in \( \Gamma \).

**Proof:** Since \( B^2 \) is closed in \( \tilde{X}^2 \) by Proposition \( \ref{prop:closets} \) the set \( \partial^{-1}(B^2) = \{ \gamma \in \text{EG}(\Gamma) \mid \partial \gamma \subset B^2 \} \) is closed too. The projection map \( \pi : \gamma \to \gamma(0) \) is continuous. So \( \text{Hull}(B) \) is closed in \( \Gamma \).
Let $H < G$ be a subgroup. Denote by $\Delta H \subset \tilde{X}$ its limit set for the action on $\tilde{X}$. By the lemma above the set $C = \text{Hull}(\Delta H)$ is closed in $\tilde{X}$.

**Lemma 3.10.** $\forall v \in C^0 = C \cap \Gamma^0$ is either a parabolic point or its degree $\deg_C(v)$ is finite.

**Proof:** Let $v \in C^0 \setminus \Delta H$. The set $\Delta H \subset X$ is compact. So by the exactness of $\mathcal{U}$ there exists an entourage $w \in \mathcal{U}$ such that $(v, \Delta H) \cap w = \emptyset$. By the visibility property there exists a finite set $E \subset \Gamma^1$ such that $u_E \subset w$. Hence for some finite sets $E_-$ and $E_+$ of edges every eventual geodesic with endpoints $a, b \in \Delta H$ containing $v$ passes through $E_-$ and $E_+$.

By the finess the number of geodesic arcs between $E_-$ and $E_+$ is finite too. Thus the degree of $v$ in the graph $\Gamma$ is finite too. \qed

**Lemma 3.11.** Let $C = \text{Hull}(\Delta H)$. If $|C^1/H| < \infty$ the subgroup $H$ is dynamically quasiconvex.

**Proof:** We first extend the visibility property defined previously on $\Gamma^0$ to the space $\tilde{X}$ (see Definition 2.2). By Theorem 3.8 for every $(x, y) \in \tilde{X}^2$ there exists a geodesic $\gamma \in \text{EG}(\Gamma)$ whose endpoints are $x$ and $y$. So for an edge $e \in \Gamma^1$ and $(x, y) \in \tilde{X}^2$ put $(x, y) \in u_e$ if there exists such $\gamma$ which does not contain the edge $e \in \Gamma^1$.

By [Ge2] Proposition 3.5.1 the graph $\Gamma$ has the visibility property so for every $u \in \mathcal{U}$ there exists a finite set $E \subset \Gamma^1$ such that $u_E = \bigcap_{e \in E} u_e \subset \sqrt{u}_{\Gamma}$. Choose now $(x', y') \in \Gamma^2$ such that $(x, x')$ and $(y, y')$ are $\sqrt{u}_{\Gamma}$-small. We have $(x, y) \not\in u \Rightarrow (x', y') \not\in \sqrt{u}_{\Gamma} \Rightarrow (x', y') \not\in u_E$. So any $\gamma' \in \text{EG}(\Gamma)$ connecting $x'$ and $y'$ contains an edge from $E$. Since $E$ is finite, by Proposition 3.7 and Corollary 3.6 for $x'$ and $y'$ sufficiently close to $x$ and $y$ we have $\gamma'|_e = \gamma|_e$ for a fixed $e \in E$. So $(x, y) \not\in u$, and the inclusion $u_E \subset u$ is valid on $\tilde{X}^2$.

If now $H$ is not dynamically quasiconvex then by Definition 3.1 the set $G_u = \{g \in G : g(\Delta H) \not\in \text{Small}(u)\}/H$ is infinite for some $u \in \mathcal{U}$. By the above argument there exists a finite $E \subset \Gamma^1$ such that $u_E \subset u$ is valid on $\tilde{X}$. Since $|\Gamma^1/G| < +\infty$ then there exists an edge $e \in E$ for which the set $\{g \in G : g(\Delta H) \not\in u_e\}/H$ is infinite. Therefore the set $\{g \in G : e \in g(C^1)\}/H = \{g \in G : g^{-1}(e) \subset C^1\}/H$ is infinite too. The lemma follows. \qed

3.3.3. **Proof of 1) $\Rightarrow$ 2) of Theorem A.** Suppose that the action $H \curvearrowright \Delta H$ is 2-cocompact. By Lemma 3.11 it is enough to prove that $|C^1/H| < +\infty$ where $C = \text{Hull}(\Delta H)$.

Denote by $K \subset \Theta^2(\Delta H)$ a compact fundamental set for the action $H \curvearrowright \Theta^2(\Delta H)$. So $H(K) = \Theta^2(\Delta H)$. Let $u \in \mathcal{U}$ be a small entourage such that $u^3 \cap K = \emptyset$. By the visibility property there exists a finite set $F \subset \Gamma^1$ such that $u_F \subset u$. Thus $u_F^3 \cap K = \emptyset$. Up to adding a finite number of edges to $F$ we can assume that $F$ is a connected subset of $\Gamma^1$.

The edges of $C^1$ which belong to the $H$-orbit $H(F)$ of $F$ we call red edges and the other edges of $C^1$ are white. Similarly parabolic points of $H$ we call red and all other vertices of $C$ are white.

**Lemma 3.12.** Any infinite ray $\rho : [0, \infty) \to C$ contains at least one red edge. Furthermore every geodesic between two red vertices contains a red edge.

**Proof of the lemma.** We start with the first statement. By Lemma 3.3 the the ray $\rho \subset C$ converges to a point at infinity $x = \rho(\infty) \in A$. Since the action $H \curvearrowright \Delta H$ is 2-cocompact by [Ge1] every point of $\Delta H$ is either conical or bounded parabolic. The parabolic points are finite vertices of $\tilde{\Gamma}$ so $x$ is a conical point.

By the topological definition of the latter one there exists an infinite subset $S \subset H$ such that $\forall y \in \overline{C} \setminus x$ we have $\overline{Sx} \cap \overline{Sy} = \emptyset$. Thus there exists $s \in S$ such that two points of $\partial(s(\rho))$ of $s(\rho)$ belong to two $u_F$-small neighborhoods $U_a$ and $U_b$ of points $a, b \in \Delta H$. There exists $h \in H$...
such that \( h(a, b) \in K \). Hence \( h(a, b) \not\in u_F^3 \) and \( \partial(h \circ s(\rho)) \not\in u_F \). It follows that \( h \circ s(\rho) \) contains a red edge and so is \( \rho \).

Let now \( \gamma \) be a geodesic between two red points in \( C \). Then \( \exists h \in H : h(\partial \gamma) \in K \) so the pair \( h(\partial \gamma) \) is not \( u_F \)-small. Thus every geodesic \( \gamma \) connecting two red points contains at least one red edge. The Lemma is proved. \( \square \)

It remains to show that the set of white edges of \( C^1 \) is \( H \)-finite.

Say that an eventual geodesic ray is pure white if all its edges and vertices are white. We add to the set of edges \( F_1 \) of \( F \) all adjacent pure white rays. The obtained subgraph \( F \) is connected as \( F \) is connected. By the first statement of Lemma 3.12 every geodesic interval containing only white edges has a finite length. Furthermore by Lemma 3.10 the degree of every white vertex is finite. Thus by König Lemma the connected subgraph \( F \) is finite.

We affirm that \( HF = C^1 \). Indeed if \( e = (a, b) \in \Gamma^1 \) is a white edge then by the second statement of 3.12 one of its vertices, say \( a \), is white. Consider maximal pure white segment \( l_1 \) of \( C \) starting from \( a \) and not containing \( e \). It has a finite length and ends either at a red vertex \( c \) or at a red edge. Suppose first that we are not in the second case. Then again by 3.12 the other vertex \( b \) of \( e \) cannot be red. So \( b \) is white and similarly we consider another maximal pure white segment \( l_2 \) starting from \( b \). By the same reason it cannot end up at a red vertex \( d \). So in either case there is a pure white eventual geodesic segment \( l \) starting from \( e \) and terminating in a red edge \( e_1 \). Thus there exists \( h \in H : h(l \cup e) \subset F \). The Theorem is proved. \( \square \)

**Corollary 3.13.** The following conditions are equivalent:

a) \( H \) is dynamically quasiconvex

b) \( |C^1/H| < \infty \) where \( C = \text{Hull}(\Lambda H) \).

**Proof:** By Lemma 3.11 it remains to prove that a) \( \Rightarrow \) b). By the statement 2) \( \Rightarrow \) 1) of Theorem A the dynamical quasiconvexity implies 2-cocompactness of the action \( H \raction \Lambda X \). We have proved above that the latter one implies that \( |C^1/H| < \infty \). \( \square \)

4. **Pullback space for (\( 32 \))-actions of a non-finitely generated group**

In the paper [Ge1, page 142] the following problem was formulated. Let a group \( G \) admit convergence actions on two compacta \( T_i \) does there exist a convergence action on a compactum \( Z \) and two \( G \)-equivariant mappings \( \pi_0 \) and \( \pi_1 \) ?

\[
\begin{array}{ccc}
& Z & \\
\pi_0 & \pi_1 \\
T_0 & \downarrow & T_1
\end{array}
\] (1)

**Definition 4.1.** We call pullback space the space \( Z \) and the problem above pullback problem.

In the paper [BR] O. Baker and T. Riley constructed a hyperbolic group \( G \) containing a free subgroup \( H \) of rank 3 such that the embedding does not induce the equivariant extension map (called “Cannon-Thurston map”) \( \partial H \to \partial G \) where \( \partial \) is the boundary of a hyperbolic group. Denote \( T_0 = \partial H \), and let \( T_1 = \Lambda_{\partial G} H \) be the limit set for the action of \( H \) on the hyperbolic boundary of \( G \). The following proposition shows that Baker-Riley’s example is also a contre-example to the pullback problem in the general (convergence) case.

**Proposition 4.2.** The compacta \( T_i \) \( (i = 0, 1) \) do not admit a common pullback space on which \( H \) acts 3-discontinuously.
Proof: Suppose by contradiction that such a space exists and we have the diagram (1). Consider the spaces \( \tilde{Z} = Z \cup H, \tilde{T}_0 = T_0 \cup H, \tilde{T}_1 = T_1 \cup H \) equipped with the following topology (which we illustrate only for \( \tilde{T}_0 \) and is defined similarly in the other cases). A set \( F \) is closed in \( \tilde{T}_0 \) (\( F \in \text{Closed}(T_0) \)) if

1. \( F \cap T_0 \in \text{Closed}(T_0) \);
2. \( F \cap H \in \text{Closed}(H) \);
3. \( \partial_1(F \cap H) \subset F \) where \( \partial_1 \) denotes the set of attractive limit points.

The topology axioms are easily checked. Since \( H \) is a convergence group, its points are isolated in \( \tilde{T}_0 \) and the condition 2) is automatically satisfied.

By the following lemma the maps \( \pi_i \) can be extended to the continuous maps \( \tilde{\pi}_0 : \tilde{Z} \to \tilde{T}_0 \) and \( \tilde{\pi}_1 : \tilde{Z} \to \tilde{T}_1 \) where \( \tilde{\pi}_0|_Z = \pi_i \) and \( \tilde{\pi}_1|_H = \text{id} \) (\( i = 0, 1 \)).

**Lemma 4.3.** Let \( G \) be a group acting 3-discontinuously on two compacta \( X \) and \( Y \). Suppose that the action on \( Y \) is minimal and \( |Y| > 2 \). If \( f : X \to Y \) is a continuous \( G \)-equivariant map then it extends continuously to an equivariant map \( \tilde{f} : \tilde{X} \to \tilde{Y} \) such that \( \tilde{f}|_X = f \) and \( \tilde{f}|_G \equiv \text{id} \).

Assuming the lemma for the moment let us finish the argument. By hypothesis \( H \rhd Z \) is a convergence action. The map \( \pi_0 \) is equivariant and continuous and the action \( H \rhd T_0 \) is minimal. So the map \( \pi_0 \) is surjective. Since \( H \) is hyperbolic all points of \( T_0 \) are conical \([B33]\). By \([Ge2]\) Proposition 7.5.2] the map \( \pi_0 \) is a homeomorphism. So we have the equivariant continuous map \( \pi = \pi_1 \circ \pi_0^{-1} : T_0 \to T_1 \). By Lemma \([L3]\) it extends equivariantly to the map \( \tilde{\pi} : \tilde{T}_0 \to \tilde{T}_1 \) where \( \tilde{T}_0 = H \cup \partial H \) and \( \tilde{T}_1 = G \cup \partial_\infty G \). This is a Cannon-Thurston map. A contradiction. The Proposition is proved modulo the following.

**Proof of the Lemma:** Let \( F \subset \tilde{Y} \) be a closed set. Denote \( F_Y = F \cap Y \) and \( F_G = F \cap G \). We need to check that the set \( \tilde{f}^{-1}(F) = f^{-1}(F_Y) \cup F_G \) is closed. The conditions 1) and 2) are obvious for \( \tilde{f}^{-1}(F) \cap X \) and for \( \tilde{f}^{-1}(F) \cap G \) respectively.

Let \( z^x = r \times X \cup X \times a \) be a limit cross for the action \( F_G \) on \( X \). To check condition 3) for the set \( f^{-1}(F) \) we need to show that \( b = f(a) \in F_Y \). Suppose not \( b \notin F_Y \) and let \( B \) be a closed neighborhood of \( b \) such that \( B \cap F_Y = \emptyset \). Let \( v \in \text{Ent} Y \) be an entourage such that \( Bv \cap F_Y = \emptyset \) where \( Bv = \{y \in Y : (y, b_1) \in v, b_1 \in B\} \). Set \( A = f^{-1}(B) \supset a \). For a neighborhood \( \partial_1 r \) of the repelling point \( r \in X \) the set \( F_0 = \{g \in F_G : g(X \setminus R) \subset A\} \) is infinite.

Let \( w^x = p \times Y \cup Y \times q \) be a limit cross of \( F_0 \) on \( Y \), and \( P \times Y \cup Y \times Q \) be its neighborhood. Since \( F \subset Y \) (closed by condition 3) we have \( q \in F_Y \). Suppose that \( Q \) is \( v \)-small. Fix three distinct points \( y_i \in Y \) (\( i = 1, 2, 3 \)). Since the set \( Y \) is minimal and \( f \)-equivariant one has \( f^{-1}(y_i) = X_i \neq \emptyset \) and \( X_i \) are mutually disjoint (\( i = 1, 2, 3 \)).

Let us now put some restrictions on \( R \). Suppose that \( R \cap X_i = \emptyset \) for at least two indices \( i \in \{k, j\} \subset \{1, 2, 3\} \) and for one of them, say \( k \), we have \( y_k \notin P \).

If \( g \in G \) is close to \( w^x \) we have \( g(Y \setminus P) \subset Q \) and \( g(y_k) \in Q \). From the other hand \( g(X_k) \subset A \) since \( X_k \cap R = \emptyset \). Thus \( g(y_k) \in Q \cap B \) and so \( (q, g(y_k)) \in v \). Hence \( q \in Bv \) and \( q \notin F_Y \). A contradiction. The lemma is proved.

Since the answer to the pullback problem for general convergence actions is negative, it seems to be rather intriguing to study the pullback problem in a more restrictive case of (32)-actions. The rest of the section is devoted to a discussion of this problem.
If $G$ is a finitely generated group which admits two (32)-actions on compacta $X_1$ and $X_2$ then by the Mapping theorem [Ge2, Proposition 3.4.6] there exist equivariant maps $F_i : \partial G \to X_i$ ($i = 1, 2$) from the Floyd boundary $\partial G$ of $G$. By [Ka] the action on $\partial G$ is convergence. So $\partial G$ is the universal pullback space for any two (32)-actions of $G$.

If $G$ is not finitely generated this argument does not work as the Mapping theorem requires the cofiniteness of a graph on which the group acts and which is not true for the Cayley graphs in this case. An action of such a group on a relative fine hyperbolic graph depends on the system of non-finitely generated parabolic subgroups [GePo2, Proposition 3.43]. Furthermore the completion of the diagonal image of the group in the product space used above does not a priori imply that the group acts 3-discontinuously on this space. We will show in Proposition 4.4 below that it can indeed happen.

However we start by proving a positive result in this direction. The following theorem provides a sufficient condition for the existence of the pullback space for two (32)-actions of a group.

**Theorem** [B]. Let $G$ be a group which admits (32)-actions on compacta $X$ and $Y$. Let $\mathcal{P}$ and $\mathcal{Q}$ be the systems of maximal parabolic subgroups for the actions on $X$ and $Y$ respectively. Assume that every $P \in \mathcal{P}$ acts 2-cocompactly on $\Lambda Y P$ and every $Q \in \mathcal{Q}$ acts 2-cocompactly on $\Lambda X Q$.

Then there there exists a pullback space $Z$ for the actions on $X$ and $Y$. Furthermore the action $G \curvearrowright Z$ is a (32)-action.

**Remark.** Using Theorem A one can reformulate the hypotheses above by requesting that the action of each subgroup $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ is dynamically quasiconvex respectively on $Y$ and on $X$.

**Proof:** By Lemma 2.1 there exist connected and fine graphs $\Gamma_1$ and $\Gamma_2$ such that the action of $G$ on their edges is proper and cofinite. The sets of their vertices are $\Gamma_0^1 = G \cup \text{Par}_X$ and $\Gamma_0^2 = G \cup \text{Par}_Y$ where $\text{Par}_X$ and $\text{Par}_Y$ are the sets of parabolic points for the action on $X$ and on $Y$ respectively. By Lemma 2.1 the actions on the augmented spaces $G \curvearrowright \tilde{X}$ and $G \curvearrowright \tilde{Y}$ are 3-discontinuous and 2-cocompact, where $\tilde{X} = X \cup \Gamma_1$ and $\tilde{Y} = Y \cup \Gamma_2$.

Using the graphs $\Gamma_i$ ($i = 1, 2$) we will now introduce a new graph $\Gamma$. The vertices of $\Gamma$ are of two types. The vertices of the first (group) type are the pairs $(g, g) \in \Delta(G^2)$ which we will identify with $g$. The pairs $(p, q) \in \Gamma^0$ are the vertices of the second (parabolic) type where $p \in \text{Par}_X$ and $q \in \text{Par}_Y$ such that $\text{Sty} p \cap \text{Sty} q$ is infinite.

Using the graph product construction we now join two vertices $(x, y)$ and $(x', y')$ of $\Gamma^0$ by an edge if either $xx' \in \Gamma_1^1$ and $y = y'$ or vice versa $x = x'$ and $yy' \in \Gamma_1^2$. Since both graphs are connected the graph $\Gamma$ is connected. The actions $G \curvearrowright \Gamma^1_i$ ($i = 1, 2$) are proper and cofinite so $G \curvearrowright \Gamma^1$ is proper and cofinite too.

By [Ge2, Proposition 8.4.1] the actions on the spaces $\tilde{X}$ and $\tilde{Y}$ admit uniformities $U_i$ generated by proper dividers $u_i$ ($i = 1, 2$) respectively. Let $\tilde{u}_i^0$ denote their restrictions on the set of vertices $\Gamma_0^i$. We now lift the entourage $u_i$ to $\Gamma^0$ as follows. For a parabolic point $q \in \text{Par}_Y$ denote by $\Lambda_X(\text{St}_Y q)$ the limit set for the action on $X$ of its stabilizer $\text{St}_Y q$ on $Y$. Let $\pi_i : \Gamma^0 \to \Gamma_i^1$ denote the coordinate projections which are injective. By construction $\pi_i$ sends parabolic (respectively group) vertices of $\Gamma$ to parabolic (respectively group) vertices of $G_i$. We say that the pair $(y_1, y_2) \in \Gamma^0 \times \Gamma^0$ is $\tilde{u}_i^0$-small in the following three cases:

1) if $y_i \in G$ for $i = 1, 2$ then $(y_1, y_2) \in \tilde{u}_i^1$ if and only if $(\pi_1(y_1), \pi_1(y_2)) \in u_i^0$;

2) if one of them, say $y_1 \not\in G$ and $y_2 \in G$ then $(y_1, y_2) \in \tilde{u}_1^0 \iff \forall x \in \Lambda_X(\text{St}_Y(\pi_2(y_1)) : (x, \pi_1(y_2)) \in u_1$;
3) if \( y_i \not\in G \) \((i = 1, 2)\) then \((y_1, y_2) \in \tilde{u}_i^0 \Leftrightarrow \forall x_i \in A_X(\text{St}_Y y_i) : (x_1, x_2) \in u_1.\)

Similarly we lift the entourage \( u_2 \) to \( \Gamma^0 \). We introduce now the entourage \( w^0 \) on \( \Gamma^0 \) to be \((\tilde{u}_1^0)^3 \cap (\tilde{u}_2^0)^3 \). So the couples \((x_1, y_1)\) and \((x_2, y_2)\) form a \( w^0 \)-small pair if and only if \((x_1, x_2)\) and \((y_1, y_2)\) are both \((\tilde{u}_0^0)^3\)-small \((i = 1, 2)\).

**Claim 1.** The entourage \( w^0 \) is perspective.

**Proof:** Suppose first that \( y_1 \) and \( y_2 \) are in \( G \) (the case 1) above). Since \( u^0_1 \) are perspective on \( \Gamma^0 \) \((i = 1, 2)\) we have that the set \( \{g \in G : g(y_1, y_2) \not\in w^0\} \) is finite.

In the second case fix \( x_0 \in A_X(\text{St}_Y(\tilde{\pi}_2(y_1))).\) Then by the perspectiveity of \( u^0_1 \) the couple \((g(\pi_1(y_2)), g(x_0))\) is \( u^0_1 \)-small for all but finitely many \( g \in G \). Since every maximal parabolic subgroup \( Q \in \text{Par}_Y \) acts 2-cocompactly on \( \tilde{X} \) by Theorem [A] it is dynamically quasiconvex. So for all but finitely many elements \( g \in G \) and for every \( x \in A_X(\text{St}_Y(\tilde{\pi}_2(y_1))) \) the pair \((g(x_0), g(x))\) is \( u^0_1 \)-small. Hence \((g(\pi_1(y_2)), g(x)) \in (u^0_1)^2\) for all but finitely many \( g \in G \).

Using the same argument in the third case we fix points \( x^i_0 \in A_X(\text{St}_Y(\tilde{\pi}_2(y_i))) \) \((i = 1, 2)\). Then \( \forall x_i \in A_X(\text{St}_Y(\tilde{\pi}_2(y_i))) \) we have \((g(x_i), g(x^i_0)) \in u^0_1\) by the dynamical quasiconvexity; and \((g(x^i_0), g(x)) \in u^0_1\) by the perspectiveity (for almost all \( g \in G \) and \( i = 1, 2 \)). So we obtain \((g(x_i), g(x)) \in (u^0_1)^3\) for almost all \( g \in G \).

We conclude that for all but finitely many \( g \in G \) the pair \((y_1, y_2)\) is \((\tilde{u}^0_1)^3\)-small. By the same argument it is also \((\tilde{u}^0_2)^3\)-small. Therefore \( w^0 \) is perspective.

**Claim 2.** The entourage \( w^0 \) is a divider.

**Proof:** Both \( u^0_1 \) and \( u^0_2 \) are dividers. So there exist finite sets \( F_1 \subset G \) and \( F_2 \subset G \) such that \( (\cap F_1(u^0_1))^2 \subset u^0_1 \) and \( (\cap F_2(u^0_2))^2 \subset u^0_2 \) where we denote \( \cap F(u^0) \) the set \( \cap \{f(u) : f \in F \subset G\} \).

Since \( u^0_1 \) (respectively \( u^0_2 \)) is a divider on \( \tilde{X} \) (respectively \( \tilde{Y} \)) the conditions 1)-3) of the definition of \( \tilde{u}^0_i \) imply that \( \tilde{u}^0_i \) are dividers on \( \Gamma^0 \) \((i = 1, 2)\). Putting \( F = F_1 \cup F_2 \) we obtain:

\[
(\cap F \{w^0\})^2 = ((\cap F \{\tilde{u}^0_1\})^3 \cap (\cap F \{\tilde{u}^0_2\})^3)^2 \subset (\cap F \{\tilde{u}^0_1\})^6 \cap (\cap F \{\tilde{u}^0_2\})^6 \subset (\cap F \{\tilde{u}^0_1\})^6 \cap (\cap F \{\tilde{u}^0_2\})^6 \subset (\tilde{u}^0_1)^3 \cap (\tilde{u}^0_2)^3 = w^0,
\]

Here we used that \((u \cap v)^n \subset u^n \cap v^n\). Claim 2 is proved.

By Claims 1 and 2 the set of vertices \( \Gamma^0 \) of \( \Gamma \) admits the perspective divider \( w^0 \). So by Lemma 2.23 in [cauchy-samuel completion (Z, W)] of \( \Gamma^0, w^0 \) admits a \((32)\)-action of \( G \).

By [bourb] II.23, Proposition 13] the completion \((Z, W)\) is unique and coincides with the closure \( \text{Cl}_{\tilde{X}, Y}(\Gamma^0) \) of \( \Gamma^0 \) embedded diagonally in \( \tilde{X} \times \tilde{Y} \). So the projections \( \pi_i : \Gamma^0 \rightarrow \Gamma_i^0 \) extend continuously to the equivariant maps \( \tilde{\pi}_1 : Z \rightarrow \tilde{X} \) and \( \tilde{\pi}_2 : Z \rightarrow \tilde{Y} \). The Theorem is proved. \( \square \)

The aim of the following Proposition is to provide an example of two \((32)\)-actions of a non-finitely generated group which does not admit a pullback space. We note that it is one of the rare cases when a result known for finitely generated relatively hyperbolic groups is not in general true for non-finitely generated ones.

**Proposition 4.4.** A non-finitely generated countable free group \( F_\infty \) admits two distinct \((32)\)-actions not having a common pullback space.

**Proof:** Let \( G = \langle x_1, \ldots, x_n, y_1, \ldots, y_m, \ldots \rangle \) be a free group freely generated by the finite system \( X = \{x_1, \ldots, x_n\} \) and infinite system \( Y = \{y_1, \ldots, y_m, \ldots\} \). Let \( A = \langle X \rangle \) be the subgroup freely generated by \( X \). Choose a non-finitely generated subgroup \( H = \langle w_1, \ldots, w_m, \ldots \rangle \) of \( A \) freely generated by a system \( W = \{w_i : i \in \mathbb{N}\} \).
Set $Z = \{z_m = y_m w_m : m \in \mathbb{N}\}$, $P = \langle Y \rangle$ and $Q = \langle Z \rangle$. The system $X \cup Z$ is obtained by Nielsen transformations from $X \cup Y$. So $Z$ is also a free basis of $G$, and the map\(\varphi(x_i) = x_i, \varphi(y_k) = z_k \ (i = 1, \ldots, k; k \in \mathbb{N})\) extends to an automorphism of $G$ [LS]. So $G$ is also freely generated by \(\{x_1, \ldots, x_n, z_1, \ldots, z_m, \ldots\}\) too. So we have two splittings of $G$:

$$G = A \ast P, \text{ and } G = A \ast Q.$$  

(1)

Each splittings in (1) gives rise to an action of $G$ on a simplicial tree whose vertex groups are conjugates either to $A$ or to $P$ (respectively to $Q$). We now replace the vertices stabilized by $A$ by the Cayley tree of $A$ as well as all its conjugates. Denote the obtained simplicial $G$-trees by $\mathcal{T}_i$ $(i = 1, 2)$. Their edge stabilizers are trivial and vertex stabilizers are non-trivial if only if they are conjugate to $P$ (respectively to $Q$). So $\mathcal{T}_i$ is a connected fine hyperbolic graph such that the action of $G$ on edges are proper and cofinite. Hence the actions satisfy Bowditch’s criterion of relative hyperbolicity [Bo1]. By [GePo2] Theorem 3.1 both actions on the trees extends to (32)-actions on compacta $\tilde{X}_i = X_i \cup \mathcal{T}_i$ $(i = 1, 2)$ where $X_i$ are the limit sets for the actions.

We now claim that $P \cap g^{-1}Qg = \{1\}$. Indeed consider the endomorphism $f$ such that $f(x_i) = x_i$, $f(z_j) = w_j$ $(i = 1, \ldots, n, j = 1, \ldots, m, \ldots)$. The map $f$ restricted on $Q$ is injective as well as on every conjugate class $g^{-1}Qg$. From the other hand $y_j \in \text{Ker } f$ $(j = 1, \ldots, m, \ldots)$. So $P < \text{Ker } f$. We have proved that 

$$\forall g \in G : P \cap g^{-1}Qg = \{1\}.$$  

(2)

Arguing now by contradiction assume that there exists a pullback space $X$ and equivariant projections $\pi_i : X \to X_i$ $(i = 1, 2)$. The vertex set $\mathcal{T}_i^0$ consists of parabolic vertices (belonging also to $X_i$) and the elements of $G$. So we have $\tilde{X}_i = X_i \cup G$. By lemma [13] the maps $\pi_i$ extend to the continuous equivariant maps $\tilde{\pi}_i : \tilde{X} \to \tilde{X}_i$ where $\tilde{\pi}_i|_{G} = \text{id}$, $\tilde{\pi}_i|_{X_i} = \pi_i$ and $\tilde{X} = X \cup G$.

Without lost of generality we can assume that the spaces $X_i$ coincide with the limit sets i.e. the actions $G \curvearrowright X_i$ are minimal. Let $\tilde{\pi} : \tilde{X} \to \tilde{X}_1 \times \tilde{X}_2$ be the map $\tilde{\pi}(x) = (\tilde{\pi}_1(x), \tilde{\pi}_2(x))$. Consider the spaces:

$$\tilde{T} = \tilde{\pi}(\tilde{X}) = \{(\tilde{\pi}_1(x), \tilde{\pi}_2(x)) \mid x \in \tilde{X}\}.$$  

The action $G \curvearrowright T$ is minimal where $T = \pi(X)$, $\pi = \tilde{\pi}|_{X}$. Indeed since $\tilde{\pi}|_{G} = \text{id}$ then for every $z = \pi(x) = (\pi_1(x), \pi_2(x)) \in X_1 \times X_2$ there exists an infinite subset $S \subset G$ converging to a cross in $X_i$ whose attractive limit point is $\pi_i(x)$ $(i = 1, 2)$. Therefore $S$ converges to $\pi(x) \in T$ and so $G$ is dense in $\tilde{T}$.

Proceeding as in the proof of Theorem [13] we obtain $\tilde{T}$ using the completion procedure. Let $U_i$ be a uniformity on $\tilde{X}_i$ generated by a perspective divider $u_i$ $(i = 1, 2)$. Denote by $U^0$ its restriction on $G$ generated by $u^0 = u_i|_{G}$. Consider the uniformity $W^0$ on $G$ generated by the entourage $w^0 = u^0 \cap u^0$. Then the Cauchy-Samuel completion of $(G, W^0)$ coincides with its topological closure [Bourb] II.23, Proposition 13:

$$(\tilde{T}, W) = (\overline{G}, \overline{W^0}) = \text{Im}(\pi) \cup \{(g, g) \mid g \in G\}.$$  

By assumption the action on $X$ is convergence so by [Ge2] Proposition 8.3.1] the action on $\tilde{X}$ is convergence too. Since the map $\tilde{\pi}$ is equivariant, continuous and surjective the action $G \curvearrowright \tilde{T}$ is convergence too [GePo1] Proposition 3.1].

Denoting by $\tilde{\pi}_i : \tilde{T} \to \tilde{X}_i$ $(i = 3, 4)$ the projections on the factors we obtain the following commutative diagram.
The entourage $w^0$ is perspective on $G$. Indeed if $g(a, b) \not\in w^0$ then $g(\pi_{i+2}(a), \pi_{i+2}(b)) \not\in u_i^0$ for at least one $i \in \{1, 2\}$. So there exist at most finitely many such elements $g \in G$ as $u_i^0$ is perspective ($i = 1, 2$).

Similarly it is a divider on $G$ as if $(\bigcap F_i \{u_i^0\})^2 \subset u_i^0$ for some finite $F_i \subset G$ ($i = 1, 2$) then $(\bigcap F \{w^0\})^2 \subset w^0$ where $F = F_1 \cap F_2$.

Hence by [Ge2 Proposition 4.2.2] the action of $G$ on the completion $\tilde{T} = \overline{(G, W)}$ is 2-cocompact.

So all points of $T$ are either conical or bounded parabolic limit points. If $p \in X$ is a parabolic fixed point then $\pi_{i+2}(p)$ are parabolic points in both $\tilde{X}_i$ ($i = 1, 2$) as the preimage of a conical point by an equivariant map is also conical [Ge2 Proposition 7.5.2]. So $p$ must be fixed by the intersection of some parabolic subgroup $g_1 P g_1^{-1}$ of the first action and a parabolic subgroup $g_2 Q g_2^{-1}$ of the second one ($g_i \in G$). However by (2) this intersection is empty. Thus there are no parabolic points for the $(32)$-action $G \curvearrowright \tilde{T}$. By [GePo2 Corollary 3.40] the group $G$ must be finitely generated. This is a contradiction. □

Remark. Note that we did not use in the above proof that $G$ is the set of vertices of a connected graph admitting the uniformity $W^0$. This condition was used in the proof of Theorem [B] to show that the action of $G$ on the completion is 3-discontinuous (compare with Lemma [2.3]) which we have here by the assumption.

5. Equivariant map between two convergence actions

The goal of this Section is the following.

Theorem [C]. Let $G$ be a group which admits $(32)$-action on compacta $X$ and $Y$. Let $P$ and $Q$ denote the systems of maximal parabolic subgroups for the actions on $X$ and on $Y$ respectively. Suppose that for every $P \in P$ there exists $Q \in Q$ such that $P < Q$. Then there exists an equivariant continuous map $f : X \rightarrow Y$.

Furthermore $f$ is injective on the set of conical points and for every parabolic point $q \in Y : f^{-1}(q) = \Lambda_X(\text{St}_Y q)$ is the limit set for the action of $\text{St}_Y q$ on $X$.

Remarks 5.1. Note that the statement was already known in several partial cases. If first, $G$ is finitely generated then by [Ge2] there exist continuous equivariant (Floyd) maps $F_1 : \partial G \rightarrow X$ and $F_2 : \partial G \rightarrow Y$ where $\partial G$ is the Floyd boundary of the Cayley graph of $G$ (with respect to some admissible scalar function). By [GePo1 Theorem A] for a parabolic point $p \in \Lambda_X G$ the set $F_1^{-1}(p)$ is the limit set $\Lambda_{\partial G} P$ of the stabilizer $P = \text{St}_G p$ for the action $G \curvearrowright \partial G$. The image of a parabolic point is always parabolic by [Ge2 Proposition 7.5.2]. So $q = F_2(\Lambda_{\partial G} P)$ is the fixed point for the parabolic action of the subgroup $Q \in Q$ containing $P$ on $Y$. Furthermore the map $f = F_2 \circ F_1^{-1}$ is 1–to–1 at every conical point of $\Lambda_X G$. So $f$ satisfies the claim in this case.

The statement of the Theorem in the case when $G$ is countable and $X$ and $Y$ are metrisable compacta was proved in [MOY]. We show below that the result remains valid in the more general case when $G$ acts on topological compacta $X$ and $Y$ and there is not any restriction on
the cardinality of $G$. The argument below refines the proof of Theorem $\text{[B]}$. Since the assumptions are different and the statement about the existence of the equivariant map is stronger we repeat some parts of the proof $\text{[B]}$ using new notations.

Proof of the Theorem. By Lemma $\text{[2.1]}$ there exist connected and fine graphs $\Gamma$ and $\Delta$ equipped with proper and cofinite actions of $G$ on their edges such that $\Gamma^0 = G \cup \text{Par}_X$ and $\Delta^0 = G \cup \text{Par}_Y$. The actions extend to $(32)$-actions on the augmented spaces $\tilde{X} = X \cup \Gamma$ and $\tilde{Y} = Y \cup \Delta$.

Denote by $\pi_1$ the identity map on $\Gamma^0$. For every parabolic point $p \in \text{Par}_X$ there exists a unique parabolic point $q \in \text{Par}_Y$ such that $\text{St}_X p < \text{St}_Y q$. So there exists a map $\pi_2 : \Gamma^0 \to \Delta^0$.

Let $v_0^0 = \pi_2^{-1}(v^0)$ be the lifting entourage on $(\Gamma^0)^2$. Set $w^0 = u^0 \cap v_0^0$. Using that $u^0$ and $v_0$ are perspective dividers on $\Gamma^0$ we obtain the following (see the proofs of more general Claims 1 and 2 in the proof of Theorem $\text{[B]}$)

Claim. The entourage $w^0$ is a perspective divider on $\Gamma^0$.

Let now $(Z, W)$ denote the Cauchy-Samuel completion of the pair $(\Gamma^0, W^0)$. Then by lemma $\text{[2.3]}$ the space $Z$ admits a $(32)$-action on $G$. Since the completion $(Z, W)$ is the closure $\text{Cl}_{\tilde{X} \times \tilde{Y}}(\Gamma^0)$ the maps $\pi_1$ and $\pi_2$ extend continuously to the equivariant maps $Z \to \tilde{X}$ and $Z \to \tilde{Y}$ which we denote by the same symbols $\pi_1$ and $\pi_2$ respectively. We will now prove that there exists an equivariant map $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that the following diagram is commutative:

![Diagram](image)

Define $\tilde{f} : \tilde{X} \to \tilde{Y}$ to be $\pi_2 \pi_1^{-1}$. Note that its restriction $\tilde{f} |_{\Gamma^0}$ coincides with the initial correspondence $\Gamma^0 \to \Delta^0$. Every point $x$ of $X$ is either conical or bounded parabolic $\text{[Ge1]}$. Main Theorem, b] so suppose first that $x \in X$ is conical. Then the set $\pi_1^{-1}(x)$ contains one conical point $\tilde{x}$ for the action $G \rhd Z$. So $\pi_2(\tilde{x}) = f(x)$.

Let now $x \in X \cap \Gamma^0$ be a bounded parabolic point and $\text{St}_X p$ denote its stabilizer for the action on $X$. Then by Lemma $\text{[5.3]}$ below $\pi_1^{-1}(p) = \Lambda_Z(\text{St}_X p)$ is the limit set of $\text{St}_X p$ for the action on $Z$. By assumption there exists a parabolic point $q \in Y \cap \Delta^0$ such that $\text{St}_X p < \text{St}_Y q$. Since $\pi_2$ is equivariant we have

$$\pi_2(\Lambda_Z(\text{St}_X p)) \subset \Lambda_Y(\text{St}_X p) \subset \Lambda_Y(\text{St}_Y q) = \{q\}.$$  

So the map $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a well-defined equivariant continuous map between two actions of $G$ on $\tilde{X}$ and on $\tilde{Y}$. Put $f = \tilde{f} |_X$. The theorem is proved modulo the following.

Proposition 5.2. Let $G$ be a group which admits two non-trivial convergence actions on compacta $X$ and $Y$. Suppose that $f : X \to Y$ is an equivariant map. Let $\Lambda_Y H \subset Y$ be the limit set of a subgroup $H$ acting cocompactly on $Y \setminus \Lambda_Y H$. Suppose that for every infinite set $B \subset G \setminus H$ there exist an infinite subset $B_0 \subset B$ and at least two distinct points $r_i \in f^{-1}(\Lambda_Y H)$ such that $\forall g \in B_0 : g(r_i) \notin f^{-1}((\Lambda_Y H) \setminus (i = 1, 2))$. Then $f^{-1}(\Lambda_Y H) = \Lambda_X H$.

The following is the statement needed above.

Lemma 5.3. If $p$ is a bounded parabolic point for the action of $G$ on $Y$ then $f^{-1}(p)$ is the limit set $\Lambda_X(\text{St}_Y p)$ of $\text{St}_Y p$ for the action on $X$. 

Proof: By the equivariance of $f$ we have $\Lambda_X H \subset f^{-1}(\Lambda_Y H)$. So if $f^{-1}(p)$ is a single point then the statement is obviously true. If $f^{-1}(p)$ contains at least two distinct points $r_i (i = 1, 2)$ then we have $g(r_i) \notin f^{-1}(\Lambda_Y H)$ as $\forall g \in G \setminus H : g(p) \neq p$. The lemma follows from Proposition 5.2.

Remark. In the case when $G$ is countable and both $X$ and $Y$ are metrisable compacta the lemma is proved in [MOY, Lemma 2.3, (4)]. Proposition 5.2 is a more general statement whose proof is a direct generalization of the argument of [MOY].

Proof of the Lemma. The Lemma is obvious if $H$ is finite, so we assume that it is not the case. Suppose first that $f^{-1}(\Lambda_Y H)$ is a finite set. Since $f(\Lambda_X H) \subset \Lambda_Y H$ then $f^{-1}(\Lambda_Y H)$ is pointwise fixed under a finite index subgroup of $H$ and so it coincides with $\Lambda_X H$.

Let now $f^{-1}(\Lambda_Y H)$ be an infinite set. Suppose by contradiction that there exists a point $s \in f^{-1}(\Lambda_Y H) \setminus \Lambda_X H$. Then there exist an infinite set $B \subset G \setminus H$ converging to the cross whose attractive limit point is $s$. By our assumption there exists an infinite subset $B_0 \subset B$ and distinct points $r_i \in f^{-1}(\Lambda_Y H)$ such that $\forall g \in B_0 : g(r_i) \notin f^{-1}(\Lambda_Y H) (i = 1, 2)$. Then one of them $z \in \{r_1, r_2\}$ is not a repulsive point of $B_0$. We have $\forall g \in B_0 \ g(z) \in U_s \setminus f^{-1}(\Lambda_Y H)$.

Let $K$ be a compact fundamental set for the action $H \curvearrowright Y \setminus \Lambda_Y H$. Since $X$ is compact and $f$ is equivariant the set $f^{-1}(K) = K_1$ is a compact fundamental set for the action of $H$ on $X \setminus f^{-1}(\Lambda_Y H)$. Therefore for every $g \in B$ there exists $h \in H$ such that $hg(z) \in K_1$. The set

$$A_s = \{ h \in H : h(K_1) \cap U_s \neq \emptyset \}$$

is infinite. Indeed if not by the argument above the orbit $A_s(K_1)$ intersects every neighborhood $U_s$ of $s$. Then by compactness of $K_1$ we would have $h^{-1}(s) \in K_1$ for some $h \in H$, implying that $f(s) \subset h(K)$. This is impossible as $\Lambda_Y H \cap h(K) = \emptyset$ for any $h \in H$. Therefore there exists an infinitely many $h \in H$ such that $h(K_1) \cap U_s \neq \emptyset$ for every neighborhood $U_s$ of $s$. Thus $s \in \Lambda_X H$. A contradiction. The lemma and Theorem C are proved. $\square$

The following Corollary follows directly from the proof of Theorem C.

Corollary 5.4. Let $\mathcal{V}$ be a uniformity on $Y$ generated by a perspective divider $v$. Then for the map $f$ constructed in Theorem C there exists a perspective divider $w$ on $X$ such that $w \subset f^{-1}(v)$.

Corollary 5.5. Suppose that all the assumptions of Theorem C are satisfied. Suppose in addition that $\Delta$ and $\Delta$ are any fine, hyperbolic, connected graphs equipped with cofinite and proper on edges actions of $G$ such that the vertices of infinite degrees of $\Gamma$ and $\Delta$ are respectively $\text{Par}_X$ and $\text{Par}_Y$. Then there exists an equivariant map $\tilde{f} : (\tilde{X} = X \cup \Gamma) \to (\tilde{Y} = Y \cup \Delta)$.

Proof: The proof is the same as before besides that the map $\pi_1$ is not the identity but finite-to-one from $G$ to the set of non-parabolic vertices of $\Gamma^0$. $\square$

6. Some applications

The goal of the section is to prove the following result summarizing the relations between different conditions used previously.

Proposition 6.1. Let $G$ be a group which admits $(32)$-actions on compacta $X$ and $Y$. Let $P$ and $Q$ be the systems of maximal parabolic subgroups for the actions on $X$ and $Y$ respectively. Assume that $\forall P \in P \exists Q \in Q : P < Q$. Then the induced action of every $Q \in Q$ on $\Lambda_X Q$ is $2$-cocompact.
Proof: It is a compilation of the results previously proved in this paper and several known facts. By Theorem [A] it is enough to prove that $Q \in \mathcal{Q}$ is a dynamically quasiconvex subgroup of $G$ for the action on $X$. Let $\Gamma$ and $\Delta$ be fine hyperbolic graphs with cofinite and proper on edges actions of $G$ [GePo2, Theorem 3.1]. By Lemma [2.1] the actions of $G$ on the augmented spaces $\tilde{X} = X \cup \Gamma$ and $\tilde{Y} = Y \cup \Delta$ are also of type (32).

A bounded parabolic subgroup $Q \in \mathcal{Q}$ acts cocompactly on $Y \setminus \{q\}$ where $q = \Lambda_Y Q$. By [GePo3, Corollary of 9.1.3] $Q$ also acts cocompactly on $\tilde{Y} \setminus \{q\}$. By Corollary 5.5 there exists an equivariant map $\tilde{f} : \tilde{X} \to \tilde{Y}$. By Lemma 5.3 $\tilde{f}^{-1}(g) = \Lambda_X Q$. Since $\tilde{X}$ is compact $Q$ acts cocompactly on $Z = \tilde{X} \setminus \Lambda_X Q$ too. By Lemma 3.9 the convex hull $C = \text{Hull}(\Lambda_X Q)$ is a $Q$-invariant closed subset of $\Gamma$. Then $C/Q$ is a closed discrete subset of the compact space $Z/Q$. Therefore $C/Q$ is a finite set. It follows from Lemma 3.11 that the subgroup $Q$ is dynamically quasiconvex for the action on $X$. The proposition is proved. \hfill $\Box$

Remark. One can try to deduce Theorem [C] directly from Theorem [B] using Proposition 6.1. However this is a tautological argument as the proof of 6.1 uses Theorem [C]. Thus we needed to proceed independently with the proofs of Theorems [B] and [C].

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