On Lattice Barycentric Tetrahedra

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Abstract

A lattice tetrahedron $T \subset \mathbb{R}^3$ is a tetrahedron whose four vertices are all in the lattice $\mathbb{Z}^3$. Lattice tetrahedra are preserved by those affine linear maps of the form $\vec{v} \mapsto A\vec{v} + \vec{b}$, such that $A$ is an element of $GL(3, \mathbb{Z})$ and $\vec{b}$ is an element of the lattice $\mathbb{Z}^3$. Such affine linear maps are called unimodular maps. We say that a lattice tetrahedron whose barycentre is its only non-vertex lattice point is lattice barycentric. The notation $T(a, b, c)$ describes that lattice tetrahedron with vertices $\{0, e_1, e_2, ae_1 + be_2 + ce_3\}$. Our result is then that all such lattice barycentric tetrahedra $T(a, b, c)$ are unimodularly equivalent to $T(3, 3, 4)$ or $T(7, 11, 20)$.

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1 Introduction

Definition 1.1 For \( v_1,v_2,v_3,v_4 \in \mathbb{R}^3 \), the tetrahedron \( T = T(v_1,v_2,v_3,v_4) \) is

\[
T(v_1,v_2,v_3,v_4) = \left\{ v = \sum_{i=1}^{4} \lambda_i v_i : 0 \leq \lambda_i \leq 1, \sum_{i=1}^{4} \lambda_i = 1 \right\}
\]  

(1)

A lattice tetrahedron \( T \) is a tetrahedron such that \( \{v_1,v_2,v_3,v_4\} \subset \mathbb{Z}^3 \), where \( \mathbb{Z}^3 = \{ae_1 + be_2 + ce_3 : a,b,c \in \mathbb{Z}\} \).

A lattice tetrahedron \( T \) is said to be fundamental if \( \partial T \cap \mathbb{Z}^3 = \{v_1,v_2,v_3,v_4\} \).

By \( \text{int}(T) \) we mean those points \( v \) with \( 0 < \lambda_i < 1, 1 \leq i \leq 4 \). A lattice tetrahedron \( T \) is said to be primitive if it is fundamental and \( \text{int}(T) \cap \mathbb{Z}^3 = \emptyset \).

The barycentre, or centroid, of any tetrahedron \( T(v_1,v_2,v_3,v_4) \) is the point \( BC[T(v_1,v_2,v_3,v_4)] \) formed by an equal weighting of the vertices, that is,

\[
BC[T(v_1,v_2,v_3,v_4)] = \frac{1}{4}v_1 + \frac{1}{4}v_2 + \frac{1}{4}v_3 + \frac{1}{4}v_4
\]  

(2)

Definition 1.2 A fundamental \( T(v_1,v_2,v_3,v_4) \) with \( \{v_1,v_2,v_3,v_4\} \subset \mathbb{Z}^3 \) and \( \text{int}(T) \cap \mathbb{Z}^3 = \{BC[T(v_1,v_2,v_3,v_4)]\} \) is called lattice barycentric.

For unimodular equivalence we will work with elements of \( GL(3,\mathbb{Z}) \), which is defined as follows:

\[
GL(3,\mathbb{Z}) = \left\{ A = (A_{ij}) \text{ a } 3 \times 3 \text{ matrix : } A_{ij} \in \mathbb{Z} ; \det(A) = \pm 1 \right\}
\]  

(3)

We say two tetrahedra \( T_1, T_2 \) are unimodularly equivalent if there exists an \( A \in GL(3,\mathbb{Z}) \) and \( \bar{b} \in \mathbb{Z}^3 \) such that the map \( \bar{v} \mapsto A\bar{v} + \bar{b} \) carries \( T_1 \) bijectively to \( T_2 \). Such \( \bar{v} \mapsto A\bar{v} + \bar{b} \) are called (affine) unimodular maps. These maps are precisely the maps which are lattice and volume preserving. All lattice tetrahedra are unimodularly equivalent to some \( T(a,b,c) = T(0,e_1,e_2,ae_1 + be_2 + ce_3) \) [Rez86, Thm. 5.2]. We now state the main result.

Proposition 1.3 Every lattice barycentric tetrahedron \( T \) is unimodularly equivalent to either \( T(3,3,4) \) or \( T(7,11,20) \).

2 Outline Of The Proof And Basic Notation

2.1 Tetrahedra

Definition 2.1 A grounded tetrahedron \( T \) has vertices \( 0, e_1, e_2, \) and \( ze_1 + ye_2 + ze_3 \). Such grounded tetrahedra are denoted \( T(x,y,z) \).

We start the proof by placing certain greatest common divisor conditions (GCD) on \( (a,b,c) \) such that \( T(a,b,c) \) is fundamental per Definition 1.1. Second we recall further GCD conditions requiring that \( T(a,b,c) \) be primitive.
Ae

\begin{itemize}
\item Fact 1: \( T(a, b, c) \) is fundamental if and only if \( \gcd(a, c) = \gcd(b, c) = \gcd(a + b - 1, c) = 1 \). \[\text{Rez86 Thm. 5.2}\]
\item Fact 2: \( T(a, b, c) \) is primitive if and only if \( T(a, b, c) \) is fundamental, and also \( a \equiv 1 \mod c \) or \( b \equiv 1 \mod c \) or \( a + b \equiv 0 \mod c \). \[\text{Rez86 Thm. 5.5 (Reeve-White-Howe-Scarf)}\]
\end{itemize}

Introducing new notation, put the barycentre, \( BC[T(a, b, c)] = (\alpha, \beta, \gamma) \), where \( \alpha = \frac{a + 1}{4}, \beta = \frac{b + 1}{4}, \) and \( \gamma = \frac{c}{4} \). For lattice barycentric tetrahedron, \( (\alpha, \beta, \gamma) \) is required to be a lattice point. This forces \( a \equiv 3 \mod 4, b \equiv 3 \mod 4, \) and \( c \equiv 0 \mod 4 \). Using the barycentre as a common vertex, one may cone over the triangular faces of the tetrahedron \( T \) to produce four sub-tetrahedra, \( T_j, j = 1, 2, 3, 4 \). We would then like to analyse these sub-tetrahedra for primitiveness. For if we know each sub-tetrahedra \( T_j \) is primitive, then we know that \( T(a, b, c) \) is lattice barycentric. The conditions for primitiveness, however, may only be applied on the grounded sub-tetrahedra, that is, the unique sub-tetrahedron with vertices \( 0, e_1, e_2, BC[T(a, b, c)] \). We define this to be \( T_0 \) of \( T \). One may find unimodular maps which bring a sub-tetrahedra \( T_j \) of \( T \) into the ground position. These maps move the entire tetrahedron \( T \) such that the apex, \( (a, b, c) \), is sent to another point \( (\bar{a}, \bar{b}, \bar{c}) \in \mathbb{Z}^3 \). The primitivity conditions may then be applied to \( T(\bar{a}, \bar{b}, \bar{c}) \) for \( \bar{\alpha} = \frac{\bar{a} + 1}{4}, \bar{\beta} = \frac{\bar{b} + 1}{4}, \) and \( \bar{\gamma} = \frac{\bar{c}}{4} = \frac{c}{4} \).

### 2.2 Construction Of Unimodular Maps

We construct unimodular maps, \( h_j(v) \), which carry the respective sub-tetrahedra into the grounded position as follows:

\[
\begin{array}{cccc}
\h_1(v) & \h_2(v) & \h_3(v) & \h_4(v) = id(v) \\
0 \mapsto (\bar{a}, \bar{b}, \bar{c}) & 0 \mapsto 0 & 0 \mapsto 0 & 0 \mapsto 0 \\
e_1 \mapsto e_1 & e_1 \mapsto (\bar{a}, \bar{b}, \bar{c}) & e_1 \mapsto e_1 & e_1 \mapsto e_1 \\
e_2 \mapsto e_2 & e_2 \mapsto e_2 & e_2 \mapsto e_2 & (a, b, c) \mapsto (a, b, c) \\
(a, b, c) \mapsto 0 & (a, b, c) \mapsto e_1 & (a, b, c) \mapsto e_2 & (a, b, c) \mapsto (a, b, c)
\end{array}
\]

For the hypothesised maps, we must have \( \bar{c} = \pm c \) for unimodular maps preserve volume. We note that \( \text{vol}[T(a, b, c)] = |c|/6 \) and \( \text{vol}(T(\bar{a}, \bar{b}, \bar{c})) = |\bar{c}|/6 \), therefore, \( |c| = |\bar{c}| \). Without loss of generality, \( c = \bar{c} \) since \( (x, y, z) \leftrightarrow (x, y, -z) \) is unimodular.

Also, the maps force certain congruences on \( \bar{a} \) and \( \bar{b} \), specifically:

\[
\begin{array}{ccc}
\h_1(v) & \h_2(v) & \h_3(v) \\
\bar{a} \equiv (a + b - 1)^{-1}a \mod c & \bar{a} \equiv a^{-1} \mod c & \bar{a} \equiv -b^{-1}a \mod c \\
\bar{b} \equiv (a + b - 1)^{-1}b \mod c & \bar{b} \equiv -a^{-1}b \mod c & \bar{b} \equiv b^{-1} \mod c
\end{array}
\]

How these congruences were derived will be explained via a sample calculation for \( h_1(v) \), as they are all similar. Label \( h_3(v) = Av + b \), and we note that \( h_3(0) = 0 \). Then \( Ae_1 = e_1 \) determines the first column of \( A \). Similarly, \( Ae_2 = \bar{a}e_1 + \bar{b}e_2 + \bar{c}e_3 \), provides the second column. Column three is determined by \( Ae_3 \). To compute this, we look at
the action of $A$ on $ae_1 + be_2 + ce_3$.

$$A(ae_1 + be_2 + ce_3) = aAe_1 + bAe_2 + cAe_3$$
$$= ae_1 + b(\tilde{a}e_1 + \tilde{b}e_2 + \tilde{c}e_3) + cAe_3$$
$$= (a + b\tilde{a})e_1 + b\tilde{b}e_2 + b\tilde{c}e_3 + cAe_3$$
$$= e_2$$

(6)

Recalling from the above argument that $c = \tilde{c}$, we find $Ae_3$ is:

$$Ae_3 = -\frac{(a + b\tilde{a})}{c}e_1 + \frac{(1 - \tilde{b})}{c}e_2 - be_3$$

(7)

We have now found explicitly the matrix $A$ with $\det(A) = 1$.

$$A = \begin{bmatrix}
    Ae_1 & Ae_2 & Ae_3 \\
    A(\tilde{a}) & A(\tilde{b}) & A(\tilde{c})
\end{bmatrix} = \begin{bmatrix}
    1 & \tilde{a} & -\frac{(a + b\tilde{a})}{c} \\
    0 & \tilde{b} & \frac{(1 - \tilde{b})}{c} \\
    0 & c & -b
\end{bmatrix}$$

(8)

We recall that $A = A_{ij}$ is an element of $GL(3, \mathbb{Z})$, and therefore require that $A_{13}$ and $A_{23}$ be integers. This, therefore, forces congruence relations on $\tilde{a}$ and $\tilde{b}$. The congruence relations are as follows:

$$\tilde{a} \equiv -b^{-1}a \mod c \quad \tilde{b} \equiv b^{-1} \mod c$$

(9)

Finally, note all congruences of (4) may be solved by Fact 1 of the introduction. Thus, maps $h_1(v), h_2(v), h_3(v),$ and $h_4(v)$ exist.

2.3 Naming Cases For Sub-Tetrahedra

We now define the notation that will allow us to search for such lattice barycentric tetrahedra. We begin by defining the sub-tetrahedra, $T_j$. We do this so that the $j^{th}$ sub-tetrahedron is brought into the ground position by map $h_j(v)$.

$$T_1 = T(e_1, e_2, BC[T(a, b, c)], (a, b, c))$$
$$T_2 = T(0, e_2, BC[T(a, b, c)], (a, b, c))$$
$$T_3 = T(0, e_1, BC[T(a, b, c)], (a, b, c))$$
$$T_4 = T(\alpha, \beta, \gamma)$$

(10)

We now define the notation for primitivity. Any tetrahedron $T(l, m, n)$ can, from Fact 2, be primitive in three different ways. They are $l \equiv 1 \mod \gamma$, or $m \equiv 1 \mod \gamma$, or $l + m \equiv 0 \mod \gamma$. These conditions are labeled cases $a, b, c$ respectively. Thus saying a tetrahedron is lattice barycentric by case $(1a, 2c, 3b, 4a)$ means that

$$e_1 \cdot h_1(\alpha, \beta, \gamma) \equiv 1 \mod \gamma, \ e_1 \cdot h_2(\alpha, \beta, \gamma) + e_2 \cdot h_2(\alpha, \beta, \gamma) \equiv 0 \mod \gamma$$

$$e_2 \cdot h_3(\alpha, \beta, \gamma) \equiv 1 \mod \gamma, \alpha \equiv 1 \mod \gamma.$$
Table 1: Congruences with $a \equiv 3 \mod{c}$

| Case 1a | $b \equiv -1 \mod{\gamma}$ |
|---------|-----------------------------|
| Case 1b | $b \equiv -3 \mod{\gamma}$ |
| Case 1c | $3b \equiv -7 \mod{\gamma}$ |
| Case 2a | $1 \equiv 9 \mod{\gamma} \Rightarrow \gamma = 8$ |
| Case 2b | $b \equiv -9 \mod{\gamma}$ |
| Case 2c | $b \equiv 7 \mod{\gamma}$ |
| Case 3a | $b \equiv -1 \mod{\gamma}$ |
| Case 3b | $3b \equiv 1 \mod{\gamma}$ |
| Case 3c | $b \equiv 1 \mod{\gamma}$ |

3 Analysis in the case $a \equiv 3 \mod{c}$ or $b \equiv 3 \mod{c}$

When we begin to look for such possible lattice barycentric tetrahedra, $T(a, b, c)$, the following lemma greatly reduces the set of possible configurations. The author thanks his R.E.U. adviser, Stephen Bullock, for this observation.

Lemma 3.1 If $T(a, b, c)$ is lattice barycentric, then $4 | c$, however, $8 \not| c$.

Proof: Assume by way of contradiction, that $T(a, b, 8\tau), \tau \in \mathbb{Z}$ is lattice barycentric. Then $T(\frac{a+1}{4}, \frac{b+1}{4}, 2\tau)$ is primitive, and in particular fundamental. Therefore by Fact 1, $a \equiv 3 \mod{8}$, else $GCD(\frac{a+1}{4}, 2\gamma)$ is even, implying boundary points on $T(\frac{a+1}{4}, \frac{b+1}{4}, 2\tau)$. Similarly for $b \equiv 3 \mod{8}$.

Now, transform unimodularly $a \rightarrow -a^{-1}b \mod{8}$ and $b \rightarrow b^{-1} \mod{8}$ by the map $h_{2}(\nu)$ of equation 4. Note that $a^{-1} = 3 \mod{8}$, so $\tilde{a} = -3 \cdot 3 = -9 = 7 \mod{8}$. Contradiction.

With the above lemma, we begin to look at all such cases of the form $a \equiv 3 \mod{c}$. We note that looking at $b \equiv 3 \mod{c}$ is equivalent, as $(x, y, z) \rightarrow (y, x, z)$ is unimodular. We thus seek configurations of the form $T(3, b, 4\gamma)$. Our computation will show that $a \equiv 3 \mod{c}$ or $b \equiv 3 \mod{c}$ implies $\gamma = 1$, so $T(3, 3, 4)$. Otherwise, we would arrive at inconsistencies in the congruences. This section allows us to ignore the case of $a \equiv 3 \mod{c}$ or $b \equiv 3 \mod{c}$, in the more general section 4, that is, 4a and 4b.

For $T(3, b, 4\gamma)$, each case of $(*, *, 4a)$ implies a certain congruence on $b$. We provide an example of the congruence which results on $b$ from the case 2a of Table 1. Label $h_{2}(\alpha, \beta, \gamma) = (\alpha, \beta, \gamma)$.

\[
\begin{align*}
\alpha & \equiv 1 \mod{\gamma} \\
& \implies \frac{3^{-1} + 1}{4} \equiv 1 \mod{\gamma} \\
& \implies 3^{-1} = 1 \equiv 4 \mod{\gamma} \\
& \implies 3^{-1} \equiv 3 \mod{\gamma} \\
& \implies 1 \equiv 9 \mod{\gamma}
\end{align*}
\]

As an aside, the congruence for Case 2a is a contradiction by lemma 3.1, given 4a.. We shall denote these cases by a *. Whilst it is implied that $\gamma = 8$ by case 2a, we shall say explicitly for the other cases that $\gamma = 8$, as they result from two non-obvious
### Table 2: Case study for $a \equiv 3 \mod c$

| Configuration | Conclusion |
|---------------|------------|
| 1a, 2a, 3a, 4a | *          |
| 1a, 2a, 3b, 4a | *          |
| 1a, 2a, 3c, 4a | *          |
| 1a, 2b, 3a, 4a | $\gamma = 8$ |
| 1a, 2b, 3b, 4a | $\gamma = 8$ |
| 1a, 2b, 3c, 4a | $\gamma = 8$ |
| 1a, 2c, 3a, 4a | $\gamma = 8$ |
| 1a, 2c, 3b, 4a | $\gamma = 8$ |
| 1a, 2c, 3c, 4a | $\gamma = 8$ |
| 1b, 2a, 3a, 4a | *          |
| 1b, 2a, 3b, 4a | *          |
| 1b, 2a, 3c, 4a | *          |
| 1b, 2b, 3a, 4a | $\gamma = 8$ |
| 1b, 2b, 3b, 4a | $\gamma = 8$ $\gamma = 8$ $\gamma = 8$ $\gamma = 8$ $\gamma = 8$ |
| 1b, 2b, 3c, 4a | $\gamma = 8$ $\gamma = 8$ $\gamma = 8$ $\gamma = 8$ $\gamma = 8$ $\gamma = 8$ |
| 1b, 2c, 3a, 4a | $\gamma = 3$ or $\gamma = 5$ |
| 1b, 2c, 3b, 4a | $\gamma = 3$ or $\gamma = 5$ |
| 1c, 2a, 3a, 4a | *          |
| 1c, 2a, 3b, 4a | *          |
| 1c, 2a, 3c, 4a | *          |
| 1c, 2b, 3a, 4a | $\gamma = 8$ |
| 1c, 2b, 3b, 4a | $\gamma = 8$ |
| 1c, 2b, 3c, 4a | $\gamma = 8$ |
| 1c, 2c, 3a, 4a | $\gamma = 8$ |
| 1c, 2c, 3b, 4a | $\gamma = 8$ |
| 1c, 2c, 3c, 4a | $\gamma = 6$ |

Congruences. The results of this exercise are found in Table 2. The conclusion of the case study is that any lattice barycentric tetrahedron, $T(3, b, 4\gamma)$ is in fact $T(3, 3, 4)$. 

### 4 Remaining Cases

#### 4.1 Congruences For Primitive Sub-Tetrahedra

We shall now look at a sample calculation for the conditions on primitivity and the resulting congruences on $a$ and $b$ in the general case $a \not\equiv 3 \mod c$ and $b \not\equiv 3 \mod c$. Let us take for example, the map $h_1(v)$. Map $h_1(v)$ takes $BC[T(a, b, c)] = (\alpha, \beta, \gamma) \mapsto (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$. We know that $\tilde{a} = \frac{a+1}{4} \mod \gamma$ and $\tilde{b} = \frac{b+1}{4} \mod \gamma$. We shall look at the first condition for primitiveness 1a, as 1b and 1c are similar.
Upon checking all cases, we arrive at Table 3.

### 4.2 Remaining Case Study With Examples

We will now look at the general case, \( a \not\equiv 3 \mod \gamma \) and \( b \not\equiv 3 \mod \gamma \). Recall section 3 showed \( T(3,3,4) \) is the only lattice barycentric tetrahedron \( T \), of the form \( T(3,b,4\gamma) \). Two sample calculations from the generic case study will now be shown. The first will use a configuration in which a unimodular equivalence class is found, the other will use a configuration that leads to an inconsistency.

**Example Of Calculation Resulting In A Unimodular Equivalence Class**

Let us look at the case 1b, 2a, 3a, 4c, from Table 3. This forces the following congruences:

\[
3a + 2b \equiv 3 \mod \gamma \quad 3a \equiv 1 \mod \gamma \quad a + 3b \equiv 0 \mod \gamma \quad a + b \equiv -2 \mod \gamma
\]

One may solve this system of linear congruences by row reductions. However, since we seek congruence relations on \( a \) and \( b \) modulo \( \gamma = 3 \), we do not divide by any integers except 2. Division by 2 is allowed modulo \( \gamma \) by lemma 3.1. This point in the reduction is indicated by !!.

\[
\begin{bmatrix}
3 & 2 & 3 \\
3 & 0 & 1 \\
1 & 3 & 0 \\
1 & 1 & -2
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 0 & 0 \\
0 & -3 & 7 \\
0 & 2 & 2 \\
1 & 1 & -2
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 0 & 0 \\
0 & -3 & 7 \\
0 & 1 & 1 \\
1 & 1 & -2
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & -3
\end{bmatrix}
\]
We see that the second line demands that 10 ≡ 0 mod γ, so γ = 0, 2, 5, 10. By lemma 3.1 and γ > 0, γ = 5. Then by Chinese Remainder Theorem, a ≡ −3 mod γ and a ≡ 3 mod 4 means a ≡ 7 mod 20. Similarly, by CRT b ≡ 1 mod 5 and b ≡ 3 mod 4 means b ≡ 11 mod 20. We note also that (7, 11, 20) satisfies the conditions for primitivity and fundamentality, so this case produces T(7, 11, 20).

Example Of Calculation Not Resulting In A Unimodular Equivalence Class

Let us look now at the case 1a, 2a, 3a, 4c from Table 3. This forces the following congruences:

\[ 2a + 3b \equiv 3 \pmod{\gamma} \quad 3a \equiv 1 \pmod{\gamma} \quad a + 3b \equiv 0 \pmod{\gamma} \quad a + b \equiv -2 \pmod{\gamma} \]

We apply the same solution technique as in the last example.

\[
\begin{pmatrix}
2 & 3 & 3 \\
3 & 0 & 1 \\
1 & 3 & 0 \\
1 & 1 & -2 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
0 & 1 & 7 \\
0 & -3 & 7 \\
0 & 2 & 2 \\
1 & 1 & -2 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
0 & 1 & 7 \\
0 & 0 & 28 \\
0 & 1 & 1 \\
1 & 1 & -2 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
0 & 0 & 2 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & -3 \\
\end{pmatrix}
\]

We see that the second line demands that 2 ≡ 0 mod γ. By lemma 3.1 γ is odd, and therefore we have an inconsistency.

Results Of The Case Study

We now present the results of an exhaustive search for the equivalence classes in Table 4. The final conclusion is that γ = 1 or γ = 5, producing T(3, 3, 4) and T(7, 11, 20) consecutively. This concludes the proof of proposition 1.3.

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| Combination        | Conclusion |
|-------------------|------------|
| 1a, 2a, 3a, 4c    | γ even     |
| 1a, 2a, 3b, 4c    | γ even     |
| 1a, 2a, 3c, 4c    | γ even     |
| 1a, 2b, 3a, 4c    | γ even     |
| 1a, 2b, 3b, 4c    | γ = 5 → (7, 11, 20) |
| 1a, 2b, 3c, 4c    | γ even     |
| 1a, 2c, 3a, 4c    | γ = 3 → (3, 7, 12)(3, 12) ≠ 1 |
| 1a, 2c, 3b, 4c    | γ even     |
| 1a, 2c, 3c, 4c    | γ = 3 → (3, 7, 12)(3, 12) ≠ 1 |
| 1b, 2a, 3a, 4c    | γ = 5 → (7, 11, 20) |
| 1b, 2a, 3b, 4c    | γ even     |
| 1b, 2a, 3c, 4c    | γ even     |
| 1b, 2b, 3a, 4c    | γ even     |
| 1b, 2b, 3b, 4c    | γ even     |
| 1b, 2b, 3c, 4c    | γ = 3 → (3, 7, 12)(3, 12) ≠ 1 |
| 1b, 2c, 3a, 4c    | γ even     |
| 1b, 2c, 3b, 4c    | γ even     |
| 1b, 2c, 3c, 4c    | γ even     |
| 1c, 2a, 3a, 4c    | γ even     |
| 1c, 2a, 3b, 4c    | γ even     |
| 1c, 2a, 3c, 4c    | γ even     |
| 1c, 2b, 3a, 4c    | γ even     |
| 1c, 2b, 3b, 4c    | γ even     |
| 1c, 2b, 3c, 4c    | γ even     |
| 1c, 2c, 3a, 4c    | γ even     |
| 1c, 2c, 3b, 4c    | γ even     |
| 1c, 2c, 3c, 4c    | γ even     |

Table 4: Results of generic case study