MR-RePair: Grammar Compression based on Maximal Repeats

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Abstract

We analyze the grammar generation algorithm of the RePair compression algorithm and show the relation between a grammar generated by RePair and maximal repeats. We reveal that RePair replaces step by step the most frequent pairs within the corresponding most frequent maximal repeats. Then, we design a novel variant of RePair, called MR-RePair, which substitutes the most frequent maximal repeats at once instead of substituting the most frequent pairs consecutively. We implemented MR-RePair and compared the size of the grammar generated by MR-RePair to that by RePair on several text corpus. Our experiments show that MR-RePair generates more compact grammars than RePair does, especially for highly repetitive texts.

1 Introduction

Grammar compression is a method of lossless data compression that reduces the size of a given text by constructing a small context free grammar that uniquely derives the text. While the problem of generating the smallest such grammar is NP-hard [1], several approximation techniques have been proposed. Among them, RePair [2] is known as an off-line method that achieves a high compression ratio in practice [3, 4, 5], despite its simple scheme. There have been many studies concerning RePair, such as extending it to an online algorithm [6], improving its practical working time or space [7, 8], applications to other fields [9, 10], and analyzing the generated grammar size theoretically [1, 11, 12].

Recently, maximal repeats have been considered as a measure for estimating how repetitive a given string is: Belazzougui et al. [13] showed that the number of extensions of maximal repeats is an upper bound on the number of runs in the Burrows-Wheeler transform and the number of factors in the Lempel-Ziv parsing. Also, several index structures whose size is bounded by the number of extensions of maximal repeats have been proposed [13, 14].

In this paper, we analyze the properties of RePair with regard to its relationship between maximal repeats. As stated above, several works have studied RePair, but, to the best of our knowledge, none of them associate RePair with maximal repeats. Moreover, we propose a grammar compression algorithm, called MR-RePair, that focuses on the property of maximal repeats. Ahead of this work, several offline grammar compression schemes focusing on (non-maximal) repeats have been proposed [15, 16]. Very recently, Gauczczor and Jež addressed to heuristically improve the compression ratio of RePair with regard to the grammar size [19]. However, none of these techniques use the properties of maximal repeats. We show that, under a specific condition, there is a theoretical guarantee that the size of the grammar generated by MR-RePair is smaller than or equal to that generated by RePair. We also confirmed the effectiveness of MR-RePair compared to RePair through computational experiments.

Contributions: The primary contributions of this study are as follows.
1. We analyze RePair and show the relation between a grammar generated by RePair and maximal repeats.

2. We design a novel variant of RePair called MR-RePair, which is based on substituting the most frequent maximal repeats.

3. We implemented our MR-RePair algorithm and experimentally confirmed that MR-RePair reduces the size of the generated grammar compared to RePair; in particular, the size decreased to about 55% for a highly repetitive text that we used in our experiment.

The remainder of this paper is organized as follows. In Section 2, we review the notations of strings and the definitions of maximal repeats, grammar compression, and RePair. In Section 3, we analyze RePair and show the relation between a grammar generated by RePair and maximal repeats. In Section 4, we define MR-RePair, compare it with RePair, and describe the implementation of it. In Section 5, we present experimental results of comparison to RePair. Finally, we conclude the paper in Section 6.

2 Preliminaries

In this section, we provide some notations and definitions to be used in the following sections. In addition, we recall grammar compression and review RePair.

2.1 Strings

Let $\Sigma$ be an alphabet, which is an ordered finite set of symbols. An element $T = t_1 \cdots t_n$ of $\Sigma^*$ is called a string, where $|T| = n$ denotes its length. We denote the empty string by $\epsilon$ which is the string of length 0, namely, $|\epsilon| = 0$. Let $\Sigma^* = \Sigma^* \setminus \{\epsilon\}$. A string is also called a text. Let $T = t_1 \cdots t_n \in \Sigma^*$ be any text of length $n$. If $T = usw$ with $u, s, w \in \Sigma^*$, then $s$ is called a substring of $T$. Then, for any $1 \leq i \leq j \leq n$, let $T[i..j] = t_i \cdots t_j$ denote the substring of $T$ that begins and ends at positions $i$ and $j$ in $T$, and let $T[i] = t_i$ denote the $i$th symbol of $T$. For a finite set $S$ of texts, text $T$ is said a superstring of $S$, if $T$ contains all texts of $S$ as substrings. We call the number of occurrences of $s$ in a text as a substring, the frequency of $s$, and denote it by $\#occ(s)$. Texts $\Sigma^*$ and $\hat{\Sigma}^*$ are said to be isomorphic for alphabet $\Sigma$ and $\hat{\Sigma}$, if there exists an isomorphism from $\Sigma$ to $\hat{\Sigma}$.

2.2 Maximal repeats

Let $s$ be a substring of text $T$. If the frequency of $s$ is greater than 1, $s$ is called a repeat. A left (or right) extension of $s$ is any substring of $T$ with the form $ws$ (or $sw$), where $w \in \Sigma^*$. We say that $s$ is left (or right) maximal if left (or right) extensions of $s$ occur strictly fewer times in $T$ than $s$, and call $s$ a maximal repeat of $T$ if $s$ is left and right maximal. We call $s$ a maximal repeat of $T$ if both left- and right-extensions of $s$ occur strictly fewer times in $T$ than $s$. In this paper, we consider only such strings with length more than 1 as maximal repeats. For example, substring $abra$ of $T = \text{abracadabra}$ is a maximal repeat, while $br$ is not.

2.3 Grammar compression

A context free grammar (CFG or grammar, simply) $G$ is defined as a 4-tuple $G = \{V, \Sigma, S, R\}$, where $V$ is an ordered finite set of variables, $\Sigma$ is an ordered finite alphabet, $R$ is a finite set of binary relations called production rules (or rules) between $V$ and $(V \cup \Sigma)^*$, and $S \in V$ is a special variable called start variable. A production rule represents the manner in which a variable is substituted and written in a form $v \rightarrow w$ with $v \in V$ and $w \in (V \cup \Sigma)^*$. Let $X, Y \in V \cup \Sigma^*$. If there are $x_1, x, x_r, y \in (V \cup \Sigma)^*$ such that $X = x_1xx_r, Y = xyx_r$, and $x \rightarrow y \in R$, we write $X \Rightarrow Y$, and denote the reflexive transitive closure
where $n$ is the length of the source text, $k$ is the cardinality of the source alphabet, and $k'$ is the cardinality of the final dictionary.

3 Analyzing RePair

In this section, we analyze RePair with regard to its relationship to maximal repeats, and introduce an important concept, called MR-order.

The following theorem shows an essential property of RePair. That is, RePair recursively replaces the most frequent maximal repeats.

Theorem 1. Let $T$ be a given text, and assume that every most frequent maximal repeat of $T$ does not appear with overlaps with itself. Let $f$ be the frequency of the most frequent pairs of $T$, and $t$ be a text

| $v_a \to \alpha$ $(\alpha = a, b, r, c, d)$ | $v_a$ | $v_b$ | $v_c$ | $v_d$ | $v_a$ | $v_b$ | $v_c$ | $v_d$ |
|---|---|---|---|---|---|---|---|---|
| $v_1 \to v_a v_b$ | $v_1$ | $v_a$ | $v_c$ | $v_a$ | $v_2$ | $v_c$ | $v_a$ | $v_3$ |
| $v_2 \to v_1 v_c$ | $v_2$ | $v_1$ | $v_c$ | $v_a$ | $v_3$ | $v_c$ | $v_a$ | $v_4$ |
| $v_3 \to v_2 v_c$ | $v_3$ | $v_c$ | $v_a$ | $v_3$ | $v_c$ | $v_a$ | $v_3$ | $v_4$ |
| $S \to v_3 v_c v_a v_4 v_3$ | $S$ |

Figure 1: An example of the grammar generation process of RePair for text abracadabra. The generated grammar is $\{\{v_a, v_b, v_c, v_d, v_1, v_2, v_3, S\}, \{a, b, r, c, d\}, S, \{v_a \to a, v_b \to b, v_c \to r, v_c \to c, v_d \to d\}$, and the grammar size is 16.

of $\Rightarrow$ by $\Rightarrow$. Let $val(v)$ be the string derived from $v$, i.e., $v \Rightarrow val(v)$, and let $[w]$ denote a variable that derives $w$, i.e. $[w] \Rightarrow w$. Note that $[w]$ is not necessarily unique. We define grammar $G = \{V, \Sigma, \hat{S}, \hat{R}\}$ as a subgrammar of $G$ if $\hat{V} \subseteq V$, $\hat{\Sigma} \subseteq \Sigma$, and $\hat{R} \subseteq R$.

Given a text $T$, grammar compression is a method of lossless text data compression that constructs a restricted CFG, which uniquely derives the given text $T$. For $G$ to be deterministic, a production rule for each variable $v \in V$ must be unique. In what follows, we assume that any grammars are deterministic and each production rule is $v_i \rightarrow expr_i$, where $expr_i$ is an expression either $expr_i = a$ ($a \in \Sigma$) or $expr_i = v_j v_k \cdots v_n$ ($i > j$, for all $1 \leq k \leq j_n$).

We estimate the effectiveness of compression by the size of generated grammar, which is counted by the total length of the right-hand-side of all production rules of the generated grammar.

2.4 RePair

RePair is a grammar compression algorithm proposed by Larsson and Moffat [2]. For input text $T$, let $G = \{V, \Sigma, S, R\}$ be the grammar generated by RePair. RePair constructs $G$ by the following steps:

Step 1. Replace each symbol $\alpha \in \Sigma$ with a new variable $v_\alpha$ and add $v_\alpha \rightarrow \alpha$ to $R$.

Step 2. Find the most frequent pair $p$ in $T$.

Step 3. Replace every occurrence (or, as many occurrences as possible, when $p$ is a pair consisting of the same symbol) of $p$ with a new variable $v$, then add $v \rightarrow p$ to $R$.

Step 4. Reevaluate the frequencies of pairs for the renewed text generated in Step 3. If the maximum frequency is 1, add $S \rightarrow \text{(current text)}$ to $R$, and terminate. Otherwise, return to Step 2.

Figure 1 shows an example of the grammar generation process of RePair.

Lemma 1 ([2]). RePair works in $O(n)$ expected time and $5n + 4k^2 + 4k' + [\sqrt{n + 1}] - 1$ words of space, where $n$ is the length of the source text, $k$ is the cardinality of the source alphabet, and $k'$ is the cardinality of the final dictionary.

3 Analyzing RePair

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Theorem 1. Let $T$ be a given text, and assume that every most frequent maximal repeat of $T$ does not appear with overlaps with itself. Let $f$ be the frequency of the most frequent pairs of $T$, and $t$ be a text
obtained after all pairs with frequency \( f \) in \( T \) are replaced by variables. Then, there is a text \( s \) such that \( s \) is obtained after all maximal repeats with frequency \( f \) in \( T \) are replaced by variables, and \( s \) and \( t \) are isomorphic to each other.

We need two lemmas and a corollary to prove Theorem 1. The following lemma shows a fundamental relation between the most frequent maximal repeats and the most frequent pairs in a text.

**Lemma 2.** A pair \( p \) of variables is most frequent in text \( T \) if and only if \( p \) occurs once in exactly one of the most frequent maximal repeats of \( T \).

**Proof.** (\( \Rightarrow \)) Let \( r \) be a most frequent maximal repeat that contains \( p \) as a substring. It is clear that \( p \) can only occur once in \( r \), since otherwise, \( \text{\#occ}(p) > \text{\#occ}(r) \) would hold, implying the existence of a frequent maximal repeat that is more frequent than \( r \), contradicting the assumption that \( r \) is most frequent. Suppose there exists a different most frequent maximal repeat \( r' \) that contains \( p \) as a substring. Similarly, \( p \) occurs only once in \( r' \). Furthermore, since \( r \) and \( r' \) can be obtained by left and right extensions to \( p \), \( \text{\#occ}(r) = \text{\#occ}(r') = \text{\#occ}(p) \), and any occurrence of \( p \) is contained in an occurrence of both \( r \) and \( r' \). Since \( r' \) cannot be a substring of \( r \), there exists some string \( w \) that is a superstring of \( r \) and \( r' \), such that \( \text{\#occ}(w) = \text{\#occ}(r) = \text{\#occ}(r') = \text{\#occ}(p) \). However, this contradicts that \( r \) and \( r' \) are maximal repeats.

(\( \Leftarrow \)) Let \( r \) be the most frequent maximal repeat such that \( p \) occurs once in. By definition, \( \text{\#occ}(r) = \text{\#occ}(p) \). If \( p \) is not the most frequent symbol pair in \( T \), there exists some symbol pair \( p' \) in \( T \) such that \( \text{\#occ}(p') > \text{\#occ}(p) = \text{\#occ}(r) \). However, this implies that there is a maximal repeat \( r' \) with \( \text{\#occ}(r') = \text{\#occ}(p') > \text{\#occ}(r) \), contradicting that \( r \) is most frequent. \( \square \)

The following corollary is directly derived from Lemma 2.

**Corollary 1.** For a given text, the frequency of the most frequent pairs and that of the most frequent maximal repeats are the same.

The following lemma shows an important property of the most frequent maximal repeats.

**Lemma 3.** The length of overlap between any two occurrences of most frequent maximal repeats is 1.

**Proof.** Let \( xwy \) and \( wy \) be most frequent maximal repeats that have an overlapping occurrence \( xwyz 

From the above lemmas and a corollary, now we can prove Theorem 1.

**Proof of Theorem 1.** By Corollary 1, the frequency of the most frequent maximal repeats in \( T \) is \( f \). Let \( p \) be one of the most frequent pairs in \( T \). By Lemma 2, there is a unique maximal repeat that is most frequent and contains \( p \) once. We denote such maximal repeat by \( r \). Assume that there is a substring \( xwy \) in \( T \), where \( z, w \in \Sigma, x, y \in \Sigma^* \), and \( xpy = r \). We denote \( r' \) by \( \hat{x} \) and \( \hat{y} \), respectively. There are 2 cases to consider:

(i) \( \text{\#occ}(\hat{x}) < f \) and \( \text{\#occ}(\hat{y}) < f \). If \( |r| = 2 \), the replacement of \( p \) directly corresponds to the replacement of the most frequent maximal repeat, since \( p = r \). If \( |r| > 2 \), after \( p \) is replaced with a variable \( v \), \( r \) is changed to \( xwy \). This occurs \( f \) times in the renewed text, and by Lemma 2, the frequency of every pair occurring in \( xwy \) is still \( f \). Because the maximum frequency of pairs does not increase, \( f \) is still the maximum frequency. Therefore, we replace all pairs contained in \( xwy \) in the following steps, and \( \hat{x} \) and \( \hat{y} \) are not replaced. This holds for every occurrence of \( p \), implying that replacing the most frequent pairs while the maximum frequency does not change, corresponds to replacing all pairs contained

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(old and new) in most frequent maximal repeats of the same frequency until they are replaced by a single variable. Then, we can generate \( s \) by replacement of \( r \).

(iii) \( \#\text{occ}(z\dot{x}) = f \) or \( \#\text{occ}(yw) = f \). We consider the case where \( \#\text{occ}(z\dot{x}) = f \). Note that \( \#\text{occ}(zxy) < f \) by assumption that \( zxy \) is a maximal repeat. Suppose RePair replaces \( z\dot{x} \) by a variable \( v \) before \( p \) is replaced. Note that by Lemma 2, there is a maximal repeat occurring \( f \) times and including \( z\dot{x} \) once (we denote the maximal repeat by \( r' \)), and \( r' \neq r \) by assumption. By Lemma 3, the length of overlap of \( r \) and \( r' \) is at most 1, then only \( \dot{x} \) is a symbol contained both \( r \) and \( r' \). After that, \( xpy = r \) is no longer the most frequent maximal repeat because some of its occurrences are changed to \( vr[2..|r|] \). However, \( r[2..|r|] \) still occurs \( f \) times in the renewed text. Since \( \#\text{occ}(zxy) < f \) and \( \#\text{occ}(xy) = f \), \( \#\text{occ}(vr[2]) < f \) and \( r[2..|r|] \) is a maximal repeat. Then, \( r[2..|r|] \) will become a variable in subsequent steps, similarly to (i). Here, \( r' \) would also become a variable. Thus, we can generate \( s \) in the way that we replace \( r' \) first, then we replace \( r[2..|r|] \). This holds similarly for \( yw \) when \( \#\text{occ}(yw) = f \), and when \( \#\text{occ}(z\dot{x}) = \#\text{occ}(yw) = f \).

From Theorem 1, if the most frequent maximal repeat is unique in the current text, then all the occurrences of it are replaced step by step by RePair. However, it is a problem if there are two or more the most frequent maximal repeats and some of them overlap. In this case, which maximal repeat is first summarized up depends on the order in which the most frequent pairs are selected. Note, however, if there are multiple the most frequent pairs, which pair is first replaced depends on the implementation of RePair. We call this order of selecting (summarizing) maximal repeats maximal repeat selection order (or MR-order, simply).

For instance, consider a text \( \text{abcdeabcde} \). \( \text{abc} \) and \( \text{cde} \) are the most frequent maximal repeats occurring 2 times. There are 2 MR-order, depending on which one is attached priority to the other. The results after replacement by RePair with the MR-order are (i) \( \text{xyzxcx} \) with variables \( x \) and \( y \) such that \( x \Rightarrow \text{abc} \) and \( y \Rightarrow \text{de} \), and (ii) \( \text{zwcwz} \) with variables \( z \) and \( w \) such that \( z \Rightarrow \text{ab} \) and \( w \Rightarrow \text{cde} \). More precisely, there are 12 possible ways in which RePair can compress the text, and the generated rule sets are:

1. \( \{ v_1 \rightarrow \text{ab}, v_2 \rightarrow v_1c, v_3 \rightarrow \text{de}, S \rightarrow v_2v_3v_1cv_3 \} \),
2. \( \{ v_1 \rightarrow \text{ab}, v_2 \rightarrow \text{de}, v_3 \rightarrow v_1c, S \rightarrow v_3v_2v_1cv_2 \} \),
3. \( \{ v_1 \rightarrow \text{bc}, v_2 \rightarrow \text{av_1}, v_3 \rightarrow \text{de}, S \rightarrow v_2v_3v_2cv_3 \} \),
4. \( \{ v_1 \rightarrow \text{bc}, v_2 \rightarrow \text{de}, v_3 \rightarrow \text{av_1}, S \rightarrow v_3v_2v_1cv_2 \} \),
5. \( \{ v_1 \rightarrow \text{ed}, v_2 \rightarrow \text{ab}, v_3 \rightarrow v_2c, S \rightarrow v_3v_1v_3cv_1 \} \),
6. \( \{ v_1 \rightarrow \text{ed}, v_2 \rightarrow \text{bc}, v_3 \rightarrow \text{av_2}, S \rightarrow v_3v_1v_3cv_1 \} \),
7. \( \{ v_1 \rightarrow \text{ab}, v_2 \rightarrow \text{cd}, v_3 \rightarrow v_2e, S \rightarrow v_1v_3v_1cv_3 \} \),
8. \( \{ v_1 \rightarrow \text{ab}, v_2 \rightarrow \text{de}, v_3 \rightarrow cv_2, S \rightarrow v_1v_3v_1cv_3 \} \),
9. \( \{ v_1 \rightarrow \text{cd}, v_2 \rightarrow \text{ab}, v_3 \rightarrow v_1e, S \rightarrow v_2v_3v_2cv_3 \} \),
10. \( \{ v_1 \rightarrow \text{cd}, v_2 \rightarrow v_1e, v_3 \rightarrow \text{ab}, S \rightarrow v_3v_2v_1cv_3 \} \),
11. \( \{ v_1 \rightarrow \text{ed}, v_2 \rightarrow \text{ab}, v_3 \rightarrow cv_1, S \rightarrow v_2v_3v_2cv_3 \} \),
12. \( \{ v_1 \rightarrow \text{ed}, v_2 \rightarrow cv_1, v_3 \rightarrow \text{ab}, S \rightarrow v_3v_2v_3cv_2 \} \).
There are 3 most frequent pairs, \( \text{ab} \), \( \text{bc} \), and \( \text{da} \) with 3 occurrences. If RePair takes \( \text{ab} \) first, the rule set of generated grammar may become \( \{ v_1 \rightarrow \text{ab}, v_2 \rightarrow \text{bc}, v_3 \rightarrow \text{dv}_1, S \rightarrow v_2x_13cyv_1zv_13v_2uda \} \) and the size of it is 19. On the other hand, if RePair takes \( \text{da} \) first, the rule set of generated grammar may become \( \{ v_1 \rightarrow \text{da}, v_2 \rightarrow \text{bc}, S \rightarrow v_2x_11v_2yabzv_1bvuv_2u1 \} \) and the size of it is 18.

**Remark 1.** If there are several distinct pairs with the same maximum frequency, the size of the grammar generated by RePair depends on the replacement order of them.

However, the following theorem states that MR-order rather than the replacement order of pairs is essentially important for the size of the grammar generated by RePair.

**Theorem 2.** The sizes of the grammars generated by RePair are the same if they are generated in the same MR-order.

**Proof.** Let \( T \) be a variable sequence appearing in the grammar generation process of RePair, and \( f \) be the maximum frequency of pairs in \( T \). Suppose that \( T' \) is a variable sequence generated after RePair replaces every pair occurring \( f \) times. By Theorem 1 all generated \( T' \) are isomorphic each other then the length of them are common, regardless of the replacement order of pairs. Let \( r_1 \) be the most frequent maximal repeats of \( T \) such that \( r_1 \) is prior to all other ones in this MR-order. \( r_1 \) is converted into a variable as a result, and by Lemma 2 all pairs included in \( r_1 \) are distinct each other. Then, the size of the subgrammar which exactly derives \( r_1 \) is \( 2|f_1| - 1 + 1 = 2|f_1| - 1 \). This holds for the next prioritized maximal repeat (we denote it by \( r_2 \)) with a little difference; the pattern actually replaced would be a substring of \( r_2 \) excluding the beginning or the end of it, if there are occurrences of overlap with \( r_1 \). However, these strings are common in the same MR-order, then the sizes of generated subgrammars are the same, regardless of the selecting order of pairs. This similarly holds for all of the most frequent maximal repeats, for every maximum frequency of pairs, through the whole the process of RePair. \( \square \)

Next, we consider the problem of how large the difference between possible outcomes of RePair can be.

**Definition 1** (Greatest Size Difference). Let \( g \) and \( g' \) be sizes of any two possible grammars that can be generated by RePair for a given text. Then, the Greatest Size Difference of RePair (GSDRP) is \( \max(|g - g'|) \).

We show a lower bound of above GSDRP in the following theorem.

**Theorem 3.** Given a text with length \( n \), a lower bound of GSDRP is \( \frac{1}{6}(\sqrt{n}+1+13) \).

**Proof.** Let \( B, L, \) and \( R \) be strings such that

\[
B = l_1xyr_1l_2xyr_2\cdots l_{f-1}xyr_f-l_1xyr_f,
L = \Diamond l_1x\Diamond l_2x\cdots \Diamond lyx, \quad R = \Diamond yr_1\Diamond yr_2\cdots \Diamond yrf,
\]

where \( x, y, l_1, \ldots, l_f, r_1, \ldots, r_f \) denote distinct symbols, and each occurrence of \( \Diamond \) denotes a distinct symbol. Consider text \( T = BL^{f-1}R^{f-1} \). Here, \( xy, l_fx, \ldots, l_fy, yr_f, \ldots, yrf \) are the most frequent maximal repeats with frequency \( f \) in \( T \). Let \( G \) and \( G' \) be grammars generated by RePair for \( T \) in different MR-order, such that (i) \( xy \) is prior to all other maximal repeats, and (ii) \( xy \) is posterior to all other maximal repeats, respectively. We denote the sizes of \( G \) and \( G' \) by \( g \) and \( g' \), respectively.
First, we consider \( G \) and how RePair generates it. The first rule generated by replacement is \( v_1 \rightarrow xy \) because of the MR-order. After replacement, \( L \) and \( R \) is unchanged but \( B \) becomes the following text:

\[
B_1 = l_1v_1r_1l_2v_1r_2 \cdots l_{-1}v_1r_{-1}l_1v_1r_f.
\]

Each pair in \( B_1 \) occurs only once in the whole text \( B_1L^{-1}R^{-1} \). This means that \( B_1 \) is never be shortened from the current length, \( 3f \). In the remaining steps, each \( l_i x \) and \( yr_i \) (for \( i = 1, \cdots, f \)) is replaced, and that is all. \( L \) and \( R \) changed to texts whose length are both \( 2f \). Hence, the following holds:

\[
g = 3f + 2 \cdot 2f + 2(1 + 2f) = 11f + 2.
\]

Next, we consider \( G' \) and how RePair generates it. By its MR-order, \( l_1 x, \cdots, l_f x, yr_f, \cdots, yr_f \) are replaced before \( xy \) is selected. They do not overlap with each other, and after they are replaced, \( xy \) does not occur in the generated text. Therefore, in \( G' \), there are \( 2f \) rules which derive each \( l_i x \) and \( yr_i \) (for \( i = 1, \cdots, f \)), respectively, but the rule which derives \( xy \) is absent. \( L \) and \( R \) changed to texts whose length are both \( 2f \), and \( B \) changed to a text with length \( 2f \). Hence, the following holds:

\[
g' = 2f + 2 \cdot 2f + 2 \cdot 2f = 10f.
\]

Let us denote the length of the original text \( T = BL^{-1}R^{-1} \) by \( n \). Then, the following holds:

\[
n = 4f + 2(3f)(f - 1) = 6f^2 - 2f. \quad \therefore f = \frac{1}{6}(\sqrt{6n + 1} + 1).
\]

By (1), (2), and (3), \( g' = 11f - 2 - 10f = f + 2 = \frac{1}{6}(\sqrt{6n + 1} + 13) \) holds and the theorem follows. \( \square \)

4 MR-RePair

The main strategy of our proposed method is to recursively replace the most frequent maximal repeats, instead of the most frequent pairs.

In this section, first, we explain the naïve version of our method called Naïve-MR-RePair. While it can have bad performance in specific cases, it is simple and helpful to understand our main result. Then, we describe our proposed MR-RePair.

**Definition 2 (Naïve-MR-RePair).** For input text \( T \), let \( G = \{ V, \Sigma, S, R \} \) be the grammar generated by Naïve-MR-RePair. Naïve-MR-RePair constructs \( G \) by the following steps:

**Step 1.** Replace each symbol \( a \in \Sigma \) with a new variable \( v_a \) and add \( v_a \rightarrow a \) to \( R \).
**Step 2.** Find the most frequent maximal repeat \( r \) in \( T \).
**Step 3.** Replace every occurrence (or, as many occurrences as possible, when there are overlaps) of \( r \) in \( T \) with a new variable \( v \), and then add \( v \rightarrow r \) to \( R \).
**Step 4.** Reevaluate the frequencies of maximal repeats for the renewed text generated in **Step 3.** If the maximum frequency is 1, add \( S \rightarrow (\text{current text}) \) to \( R \), and terminate. Otherwise, return to **Step 2**.

We can easily extend the concept of MR-order to this Naïve-MR-RePair.

We show an example of the grammar generation process of Naïve-MR-RePair in Figure 2. Figure [1] and [2] illustrates the intuitive reason why the strategy using maximal repeats is effective compared to that using pairs. When compressing text \( v_a v_b v_c v_d v_a v_b v_d v_a \), RePair and Naïve-MR-RePair both generate subgrammars which derive the most frequent maximal repeat \( v_a v_b v_d v_a \). The rule set of the subgrammar of RePair is \( \{ v_1 \rightarrow v_a v_b, v_2 \rightarrow v_1 v_c, v_3 \rightarrow v_2 v_a \} \), and the size is 6. On the other hand, the rule set of subgrammar of Naïve-MR-RePair is \( \{ v_1 \rightarrow v_a v_b, v_2 v_a \} \), and the size is 4.

However, the following theorem indicates that the size of the grammar generated by Naïve-MR-RePair is larger than that by RePair in particular cases, even when they work in the same MR-order.
Figure 2: An example of the grammar generation process of Naïve-MR-RePair for text abracadabra. The generated grammar is \{\{v_a, v_b, v_r, v_c, v_d, v_1\}, \{a, b, r, c, d\}, S, \{v_a \rightarrow a, v_b \rightarrow b, v_r \rightarrow r, v_c \rightarrow c, v_d \rightarrow d, v_1 \rightarrow v_a v_b v_r v_c, S \rightarrow v_1 v_a v_d v_1\}\}, and the grammar size is 14.

**Theorem 4.** Given a text $T$ with length $n$, and assume that RePair and Naïve-MR-RePair work in the same MR-order. Let $g_{rp}$ and $g_{nmr}$ be sizes of grammars generated by RePair and Naïve-MR-RePair for $T$, respectively. Then, there is a case where $g_{nmr} = g_{rp} + \mathcal{O}(\log n)$ holds.$^1$

**Proof.** Assume that $G_{rp} = \{V_{rp}, \Sigma_{rp}, S_{rp}, R_{rp}\}$ and $G_{nmr} = \{V_{nmr}, \Sigma_{nmr}, S_{nmr}, R_{nmr}\}$ are grammars generated by RePair and Naïve-MR-RePair, respectively. Let $T' = v_1 \cdots v_n$ such that $v_i \in V_{rp} \cap V_{nmr}$ and $v_i \rightarrow T'[i] \in R_{rp} \cap R_{nmr}$ (for $i = 1, \cdots, n$), and $\hat{G}_{rp} = \{V_{rp}, \hat{\Sigma}_{rp}, \hat{S}_{rp}, R_{rp}\}$ (or $\hat{G}_{nmr} = \{V_{nmr}, \hat{\Sigma}_{nmr}, \hat{S}_{nmr}, R_{nmr}\}$) be a subgrammar of $G_{rp}$ (or $G_{nmr}$) which derives $T'$. Assume that $T' = (uw)^{2^{m+1}-1}u$, where $u \in V_{rp} \cap V_{nmr}$, $w \in (V_{rp} \cap V_{nmr})^+$ such that $uwu$ is the most frequent maximal repeat of $T'$, and $m \in \mathbb{N}^+$. Note that $2^{m+1} - 1 = \sum_{i=0}^{2^m} 2^i$. Here $R_{rp}$ and $\hat{R}_{nmr}$ consist as follows:

\[ \hat{R}_{rp}: \text{Assume that } x_i \in \hat{V}_{rp} \text{ for } 1 \leq i \leq m \text{ and } y_j \in \hat{V}_{rp} \cup \hat{\Sigma}_{rp} \text{ for } 1 \leq j \leq |w|, \text{ then} \]

- $|w|$ rules $y_j \rightarrow y_i y_w$ with $\text{val}(y_w) = uw$.
- One rule $x_1 \rightarrow y_i w | y_i w$ and $\log_2 (2^{m+1} - 1) = m - 1$ rules $x_i \rightarrow x_{i-1} x_{i-1}$ for $2 \leq i \leq m$.
- One rule $\hat{\Sigma}_{rp} \rightarrow x_m x_{m-1} \cdots x_1 y_w$.

\[ \hat{R}_{nmr}: \text{Assume that } d = |\hat{V}_{nmr}| = |\hat{R}_{nmr}| \text{ and } z_i \in \hat{V}_{nmr} \text{ for } 1 \leq i \leq d, \text{ then} \]

- One rule $z_1 \rightarrow uwu$.
- $d - 1$ rules $z_i \rightarrow z_{i-1} w z_{i-1}$ for $2 \leq i \leq d$ and $z_d = \hat{S}_{nmr}$.

Let $\hat{g}_{rp}$ and $\hat{g}_{nmr}$ be sizes of $\hat{G}_{rp}$ and $\hat{G}_{nmr}$, respectively. Then, the following holds:

\[ \hat{g}_{rp} = 2|w| + 2m + (m + 2) = 3m + 2|w| + 2 \]

(4)

\[ \hat{g}_{nmr} = |w| + 2 + (|w| + 2)(d - 1) = (|w| + 2)d \]

(5)

Here, with regard to the length of $T''$, the following holds:

\[ (2^d - 1)|w| + 2^d = n = (2(2^m - 1) + 1)(|w| + 1) + 1. \]

Since the right-side results in $2^{m+1}|w| + 2^{m+1}$, $d = m + 1$ follows it. Hence, by equation (4) and (5), the following holds:

\[ \hat{g}_{nmr} - \hat{g}_{rp} = (m - 1)(|w| - 1) - 1. \]

Therefore, $\hat{g}_{nmr} > \hat{g}_{rp}$ holds for some $(m, |w|)$, and the proposition holds.

\[ \square \]

The reason why the grammar size of Naïve-MR-RePair becomes larger than that of RePair as seen in Theorem 4 is that Naïve-MR-RePair cannot replace all occurrences of the most frequent maximal repeats if it overlaps with another occurrence of itself. In the remainder of this section, we describe MR-RePair, which is an improved version of the above Naïve-MR-RePair.

\[ ^1 \text{We show a concrete example of this theorem in Appendix.} \]
Proof. Assume that \( \{v_a, v_b, v_c, v_d, v_1, S\}, \{a, b, r, c, d\}, S, \{v_a \to a, v_b \to b, v_c \to r, v_d \to c, v_1 \to d, v_1 \to v_2 v_b v_r, v_2 \to v_1 v_4, S \to v_2 v_c v_d v_2\} \) is generated by RePair and MR-RePair, respectively, for a given text. Let \( \{v_a, v_b, v_c, v_d, v_1, S\}, \{a, b, r, c, d\}, S, \{v_a \to a, v_b \to b, v_r \to r, v_c \to c, v_1 \to d, v_1 \to v_2 v_b v_r, v_2 \to v_1 v_4, S \to v_2 v_c v_d v_2\} \) be the size of the generated grammar in Figure 3 is larger than that of Na¨ ıve-MR-RePair shown in Figure 2, it extends the concept of MR-order to this MR-RePair. We do not care if it treats the maximum frequency of pairs in \( T \) is still smaller than that of RePair shown in Figure 1.

Definition 3 (MR-RePair). For input text \( T \), let \( G = \{V, \Sigma, S, R\} \) be the grammar generated by MR-RePair. MR-RePair constructs \( T \) by the following steps:

Step 1. Replace each symbol \( a \in \Sigma \) with a new variable \( v_a \) and add \( v_a \to a \) to \( R \).

Step 2. Find the most frequent maximal repeat \( r \) in \( T \).

Step 3. Check if \( |r| > 2 \) and \( r[1] = r[|r|] \), and if so, replace \( r \) with \( r[2..|r|] \).

Step 4. Replace every occurrence of \( r \) with a new variable \( v \), then add \( v \to r \) to \( R \).

Step 5. Reevaluate the frequencies of maximal repeats for the renewed text generated in Step 4. If the maximum frequency is 1, add \( S \to (\text{current text}) \) to \( R \), and terminate. Otherwise, return to Step 2.

We show an example of the grammar generation process of MR-RePair in Figure 3. We can easily extend the concept of MR-order to this MR-RePair. We do not care if it treats the maximum frequency of pairs in \( T \). MR-RePair can replace all occurrences of \( r \) even if overlaps with itself in some occurrences, since by Lemma 3, the length of overlaps of the most frequent maximal repeats is at least 1. If \( r[1] = r[|r|] \) but \( r \) does not overlap with itself, then \( r[1]v \) becomes the most frequent maximal repeat after \( r[2..|r|] \) is replaced by \( v \), and \( r[1]v \) would be replaced immediately. MR-RePair still cannot replace all of them if \( |r| = 2 \), but the same is said to RePair.

We show an example of the grammar generation process of MR-RePair in Figure 3. Although the size of the generated grammar in Figure 3 is larger than that of Na¨ ıve-MR-RePair shown in Figure 2, it is still smaller than that of RePair shown in Figure 1.

Theorem 5. Assume that RePair and MR-RePair work based on the same MR-order for a given text. Let \( g_{rp} \) and \( g_{mr} \) be sizes of grammars generated by RePair and MR-RePair, respectively. Then, \( \frac{1}{2} g_{rp} < g_{mr} \leq g_{rp} \) holds.

Proof. Assume that \( G_{rp} = \{V_{rp}, \Sigma_{rp}, S_{rp}, R_{rp}\} \) and \( G_{mr} = \{V_{mr}, \Sigma_{mr}, S_{mr}, R_{mr}\} \) are grammars generated by RePair and MR-RePair, respectively, for a given text \( T \) with length \( n \). Let \( T' = v_1 \cdots v_n \) such that \( v_i \in V_{rp} \cap V_{mr} \) and \( v_i \to T[i] \in R_{rp} \cap R_{mr} \) (for \( i = 1, \cdots, n \)).

We start with first \( T' \). Let \( f_1 \) be the maximum frequency of maximal repeats in \( T' \). By Corollary 1, the maximum frequency of pairs in \( T' \) is also \( f_1 \). Let \( G'_{rp}(f_1) \) (or \( G'_{mr}(f_1) \)) be a subgrammar of \( G_{rp} \) (or \( G_{mr} \)) which is generated while RePair (or MR-RePair) replaces pairs (or maximal repeats) with frequency \( f_1 \), \( \hat{g}'_{rp}(f_1) \) (or \( \hat{g}'_{mr}(f_1) \)) be the size of it, and \( T'_{rp}(f_1) \) (or \( T'_{mr}(f_1) \)) be the renewed text after all pairs (or maximal repeats) with frequency \( f_1 \) are replaced. Let \( f_1^{(1)}, \cdots, f_1^{(m_1)} \) be maximal repeats with frequency \( f_1 \) in \( T' \), and assume that they are prioritized in this order by the MR-order. Let each \( l_i^{(f_1)} \) (for \( i = 1, \cdots, m_1 \)) be the length of the longest substring of \( f_1^{(i)} \) such that there are variables that derive the substring in both \( G'_{rp}(f_1) \) and \( G'_{mr}(f_1) \). Note that this substring is common to RePair and MR-RePair, and each \( l_i^{(f_1)} \) is at least 2. Then, by Lemma 2, the following holds:

\[
\hat{g}'_{rp} = \sum_{i=1}^{m_1} 2(l_i^{(f_1)} - 1), \quad \hat{g}'_{mr} = \sum_{i=1}^{m_1} l_i^{(f_1)}, \quad \vdots \quad \frac{1}{2} g_{rp} < \hat{g}'_{mr} \leq \hat{g}'_{rp}.
\]
The renewed texts $T^{(f_{1})}_{rp}$ and $T^{(f_{1})}_{mr}$ are isomorphic for $V_{rp}$ and $V_{mr}$. Let $f_{2}$ be the maximum frequency of maximal repeats in $T^{(f_{1})}_{rp}$ (and this is the same in $T^{(f_{1})}_{mr}$). Then, the similar discussion holds for $g^{(f_{2})}_{rp}$ and $g^{(f_{2})}_{mr}$. Hence, $\frac{1}{2}g^{(f_{1})}_{rp} < g^{(f_{2})}_{mr} \leq g^{(f_{1})}_{rp}$ holds similarly to (6), and the renewed texts $T^{(f_{2})}_{rp}$ and $T^{(f_{2})}_{mr}$ are isomorphic. Inductively, for every maximum frequency of maximal repeats $f_{i}$, $\frac{1}{2}g^{(f_{i})}_{rp} < g^{(f_{i})}_{mr} \leq g^{(f_{i})}_{rp}$ holds and the renewed texts $T^{(f_{i})}_{rp}$ and $T^{(f_{i})}_{mr}$ are isomorphic. Let $k$ be a natural number such that $f_{k} > 1$ and $f_{k+1} = 1$, which is the number of decreasing of maximum frequency through the whole process of RePair and MR-RePair. Then,

$$g_{rp} = \sum_{j=1}^{k} g^{(f_{j})}_{rp} + |\Sigma| + |T^{(f_{k})}_{rp}| = \sum_{j=1}^{k} \sum_{i=1}^{m_{j}} 2(l^{(f_{j})}_{i} - 1) + |\Sigma| + |T^{(f_{k})}_{rp}| ,$$  

(7)

$$g_{mr} = \sum_{j=1}^{k} g^{(f_{j})}_{mr} + |\Sigma| + |T^{(f_{k})}_{mr}| = \sum_{j=1}^{k} \sum_{i=1}^{m_{j}} l^{(f_{j})}_{i} + |\Sigma| + |T^{(f_{k})}_{mr}| .$$  

(8)

holds. Because every $l^{(f_{j})}_{i} \geq 2$ and $|T^{(f_{k})}_{rp}| = |T^{(f_{k})}_{mr}|$, $\frac{1}{2}g_{rp} < g_{mr} \leq g_{rp}$ follows (7) and (8) and the proposition holds. $g_{mr} = g_{rp}$ holds when every length of $l^{(f_{i})}_{i}$ is 2.

The following theorem shows that unless the MR-order of RePair and MR-RePair are the same, the grammar generated by MR-RePair might be larger than that by RePair.

**Theorem 6.** Unless the MR-order of RePair and MR-RePair are the same, there is a case where the size of the generated grammar by MR-RePair becomes larger than that by RePair.

**Proof.** As shown in Theorem[4], the size of grammar generated by MR-RePair would be strictly equal to that by RePair with the same MR-order. Thus, we can reduce this problem to the problem that there is a difference between sizes of possible grammars generated by RePair as stated in Remark[1]. Hence, there are the cases stated in the proposition if the MR-order of MR-RePair matches with a MR-order of RePair which does not generate the smallest RePair grammar.

We can implement MR-RePair by extending the original implementation of RePair stated in[2], holding the same complexity.

**Theorem 7.** Let $G = \{V, \Sigma, S, R\}$ be the generated grammar by MR-RePair for a given text with length $n$. Then, MR-RePair works in $O(n)$ expected time and $5n + 4k^{2} + 4k' + |\sqrt{n} + 1| - 1$ word space, where $k$ and $k'$ are the cardinalities of $\Sigma$ and $V$, respectively.

**Proof.** Compared with RePair, the additional operations which MR-RePair does in our implementation are (i) it extends the selected pair to left and right until it becomes a maximal repeat, and (ii) it checks and excludes either of the beginning or the end of the obtained maximal repeat if they are the same. They can be realized by only using the same data structures as that of RePair. Then, the space complexity of MR-RePair follows Lemma[1].

We can clearly execute operation (ii) in constant time. So we consider how the time complexity is affected by operation (i). Let $l$ be the length of the maximal repeat containing the focused pair, and $f$ be the frequency of the pair. Then, when MR-RePair checks the left- and right-extensions for all occurrences of the focused pair, $O(fl)$ excessive time is required compared with RePair. However, the length of entire text is shortened at least $f(l-1)$ by the replacement. Therefore, according to possible counts of replacement through the entire steps of the algorithm, MR-RePair works in $O(n)$ expected time.

**Remark 2.** A reader may wonder that we can convert a grammar of RePair to that of MR-RePair by repeating the following transform: If a variable $v$ appears only once on the right-hand side of other
Table 1: Text files used in our experiments.

| texts        | size (byte) | \(\Sigma\) | contents                                                                   |
|--------------|-------------|-------------|---------------------------------------------------------------------------|
| rand77.txt   | 2,097,152   | 77          | 32 copies of 1024 random patterns of length 64                           |
| fib41        | 267,914,296 | 2           | Fibonacci string from Pizza\&Chili Corpus                                |
| einstein.de.txt | 92,758,441  | 117         | Edit history of Wikipedia for Albert Einstein                            |
| E.coli       | 4,638,690   | 4           | Complete genome of the E. Coli bacterium                                 |
| bible.txt    | 4,047,392   | 63          | The King James version of the bible                                      |
| world192.txt | 2,473,400   | 94          | The CIA world fact book                                                  |

rules, remove the rule for \(v\) and replace the one occurrence of \(v\) with the right-hand side of the removed rule. However, time and space complexity stated in Theorem\(^7\) cannot be achieved in this manner, since additional operation and memory for searching and storing such variables are required.

5 Experiments

We implemented MR-RePair and measured the number of generated rules and the execution time in order to compare it to existing RePair implementations and Re-Pair\textsuperscript{improved} proposed by Gańczorz and Jeż\(^10\).

As stated in Remark\(^1\) the size of a generated grammar depends on the MR-order. In practice, the MR-order varies how we implement the priority queue managing symbol pairs. To see this, we used five RePair implementations in the comparison; they were implemented by Maruyama\(^3\), Navarro\(^4\), Prezza\(^5\), Wan\(^6\), and Yoshida\(^7\). Table 1 summarizes the details of the texts we used in the comparison. We used three texts as highly repetitive texts; one is a randomly generated text (rand77.txt), and the others are a Fibonacci string (fib41) and a German text (einstein.de.txt) which were selected from Repetitive Corpus of Pizza\&Chili Corpus\(^8\). The randomly generated text, rand77.txt, consists of alphanumeric symbols and some special symbols; and it is generated by concatenating 32 copies of a block that includes 1024 random patterns of length 64, i.e., the size is \(64 \times 1024 \times 32 = 2,097,152\) byte. In addition, we used three texts (E.coli, bible.txt, world192.txt) for real data selected from Large Corpus\(^9\). We executed each program seven times for each text and measured the elapsed CPU time only for grammar generation process. We calculated the average time of the five results excluding the minimum and maximum values among seven. We ran our experiments on a workstation equipped with an Intel(R) Xeon(R) E5-2670 2.30GHz dual CPU with 64GB RAM, running on Ubuntu 16.04 LTS on Windows 10. All the programs are compiled by gcc version 7.3.0 with “-O3” option.

Table 2 lists the experimental results. Here, we excluded the number of rules that generate a single terminal symbol from the number of rules because it is the same between MR-RePair and RePair. As shown in the table, for all texts except for fib41, the size of rules generated by each RePair implementation differs from each other\(^11\). In any cases, MR-RePair is not inferior to RePair in the size of rules. For rand77.txt in particular, the number of rules decreased to about 11% and the size of rules decreased to about 55%. For einstein.de.txt, moreover, the number of rules decreased to about 44% and the size of

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11 We found that the results of Yoshida were the same as those of Maruyama because Yoshida utilized the code of Maruyama.

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[https://bitbucket.org/IguanaBen/repairimproved](https://bitbucket.org/IguanaBen/repairimproved)
[https://code.google.com/archive/p/re-pair/](https://code.google.com/archive/p/re-pair/)
[https://github.com/nicolaprezza/Re-Pair](https://github.com/nicolaprezza/Re-Pair)
[https://github.com/rwanwork/Re-Pair](https://github.com/rwanwork/Re-Pair)
[https://github.com/syoshid/Re-Pair-VF](https://github.com/syoshid/Re-Pair-VF)
[http://pizzachili.dcc.uchile.cl/repcorpus.html](http://pizzachili.dcc.uchile.cl/repcorpus.html)
[http://corpus.canterbury.ac.nz/descriptions/#large](http://corpus.canterbury.ac.nz/descriptions/#large)
rules decreased to about 72%. On the other hand, for the texts of Large Corpus, which are not highly repetitive, it turned out that the effect of improvement was limited. Note that fib41 does not contain any maximal repeats longer than 2 without overlaps. Therefore, MR-RePair generates the same rules as RePair. Also note that MR-RePair runs at a speed comparable to the fastest implementation of RePair.

6 Conclusion

In this paper, we analyzed RePair and showed that RePair replaces step by step the most frequent pairs within the corresponding most frequent maximal repeats. Motivated by this analysis, we designed a novel variant of RePair, called MR-RePair, which is based on substituting the most frequent maximal repeats at once instead of substituting the most frequent pairs consecutively. Moreover, we implemented MR-RePair and compared the grammar generated by it to that by RePair for several texts, and confirmed the effectiveness of MR-RePair experimentally especially for highly repetitive texts.

We defined the greatest size difference of any two possible grammars that can be generated by RePair for a given text, and named it as GSDRP. Moreover, we showed that a lower bound of GSDRP is $\frac{1}{6}(\sqrt{6n} + 1 + 13)$ for a given text of length $n$. Improving the lower bound and showing a upper bound of GSDRP are our future works.

Although we did not discuss how to encode grammars in this time, it is a very important issue from a practical point of view. For MR-RePair, if we simply use delimiters to store the rule set, the number of rules may drastically effect on the final compressed data size. To develop an efficient encoding method for MR-RePair is one of our future works.

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Table 2: The sizes of generated grammars and the execution times. Each cell in the table represents the number of generated rules, the total lengths of the right side of all the rules except for the start variable, the length of the right side of the start variable, and the total grammar size in order from the top row. The fifth row separated by a line represents the execution time with seconds.

| text file     | RePair Maruyama | Navarro | Prezza | Wan | Yoshida | Re-PairImp | MR-RePair |
|---------------|----------------|---------|--------|-----|---------|------------|-----------|
| **rand77.txt**| 41,651         | 41,642  | 41,632 | 41,675 | 41,651  | 41,661     | 4,492     |
|               | 83,302         | 83,384  | 83,264 | 83,350 | 83,302  | 83,322     | 46,143    |
|               | 9              | 2       | 7      | 2    | 9       | 2          | 9         |
|               | 83,311         | 83,286  | 83,271 | 83,352 | 83,311  | 83,324     | 46,152    |
|               | 0.41           | 0.37    | 4.76   | 4.27 | 0.40    | 3.95       | 0.42      |
| **fib41**     | 38             | 38      | 38     | 38   | 38      | 37         | 38        |
|               | 76             | 76      | 76     | 76   | 76      | 74         | 76        |
|               | 3              | 3       | 3      | 3    | 3       | 23         | 3         |
|               | 79             | 79      | 79     | 79   | 79      | 97         | 79        |
|               | 26.75          | 23.94   | 96.05  | 483.86 | 25.04   | 1360.40    | 33.62     |
| **einstein.de.txt** | 49,968          | 49,949  | 50,218 | 50,057 | 49,968  | 49,933     | 21,787    |
|               | 99,936         | 99,898  | 100,436 | 100,114 | 99,936  | 99,866     | 71,709    |
|               | 12,734         | 12,665  | 13,419 | 12,610 | 12,734  | 12,672     | 12,683    |
|               | 112,670        | 112,563 | 113,855 | 112,724 | 112,670  | 112,538    | 84,392    |
|               | 30.08          | 43.45   | 216.74 | 213.15 | 30.76   | 452.56     | 29.63     |
| **E.coli**    | 66,664         | 66,757  | 66,660 | 67,368 | 66,664  | 66,739     | 62,363    |
|               | 133,328        | 133,514 | 133,320 | 134,736 | 133,328  | 133,478    | 129,138   |
|               | 651,875        | 649,660 | 650,538 | 652,664 | 651,875  | 650,209    | 650,174   |
|               | 785,203        | 783,174 | 783,858 | 787,400 | 785,203  | 783,687    | 779,312   |
|               | 1.20           | 1.02    | 14.67  | 10.37 | 1.56    | 27.04      | 1.33      |
| **bible.txt** | 81,193         | 81,169  | 80,999 | 81,229 | 81,193  | 81,282     | 72,082    |
|               | 162,386        | 162,338 | 161,998 | 162,458 | 162,386  | 162,564    | 153,266   |
|               | 386,514        | 386,381 | 386,992 | 386,094 | 386,514  | 385,989    | 386,516   |
|               | 548,900        | 548,719 | 548,990 | 548,552 | 548,900  | 548,553    | 539,782   |
|               | 1.33           | 1.21    | 13.00  | 9.12  | 1.47    | 24.38      | 1.27      |
| **world192.txt** | 55,552          | 55,798  | 55,409 | 55,473 | 55,552  | 55,437     | 48,601    |
|               | 111,104        | 111,596 | 110,812 | 110,946 | 111,104  | 110,874    | 104,060   |
|               | 213,131        | 213,962 | 213,245 | 212,647 | 213,131  | 212,857    | 212,940   |
|               | 324,235        | 325,558 | 324,057 | 323,593 | 324,235  | 323,731    | 317,000   |
|               | 0.59           | 0.80    | 7.57   | 4.89  | **0.56** | 12.35      | 0.66      |
References

[1] Moses Charikar, Eric Lehman, Ding Liu, Panigrahy Ring, Manoj Prabhakaran, Amit Sahai, and abhi shelat, “The smallest grammar problem,” IEEE Transactions on Information Theory, vol. 51, no. 7, pp. 2554–2576, July 2005.

[2] N. Jesper Larsson and Alistair Moffat, “Off-line dictionary-based compression,” Proceedings of the IEEE, vol. 88, no. 11, pp. 1722–1732, Nov 2000.

[3] Francisco Claude and Gonzalo Navarro, “Fast and compact web graph representations,” ACM Transactions on the Web (TWEB), vol. 4, no. 4, pp. 16, 2010.

[4] Rodrigo González and Gonzalo Navarro, “Compressed text indexes with fast locate,” in Annual Symposium on Combinatorial Pattern Matching. Springer, 2007, pp. 216–227.

[5] Raymond Wan, Browsing and searching compressed documents, Ph.D. thesis, 2003.

[6] Takuya Masaki and Takuya Kida, “Online grammar transformation based on re-pair algorithm,” in 2016 Data Compression Conference (DCC), March 2016, pp. 349–358.

[7] Philip Bille, Inge Li Grøtz, and Nicola Prezza, “Space-efficient re-pair compression,” in 2017 Data Compression Conference (DCC), April 2017, pp. 171–180.

[8] Kei Sekine, Hirohito Sasakawa, Satoshi Yoshida, and Takuya Kida, “Adaptive dictionary sharing method for re-pair algorithm,” in Data Compression Conference (DCC), 2014. IEEE, 2014, pp. 425–425.

[9] Markus Lohrey, Sebastian Maneth, and Roy Mennicke, “Xml tree structure compression using repair,” Information Systems, vol. 38, no. 8, pp. 1150–1167, 2013.

[10] Yasuo Tabei, Hiroto Saigo, Yoshihiro Yamanishi, and Simon J. Puglisi, “Scalable partial least squares regression on grammar-compressed data matrices,” in Proceedings of the 22nd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining. ACM, 2016, pp. 1875–1884.

[11] Gonzalo Navarro and Luís MS Russo, “Re-pair achieves high-order entropy,” in DCC, 2008, p. 537.

[12] C. Ochoa and G. Navarro, “Repair and all irreducible grammars are upper bounded by high-order empirical entropy,” IEEE Transactions on Information Theory, pp. 1–5, 2018.

[13] Djamal Belazzougui, Fabio Cunial, Travis Gagie, Nicola Prezza, and Mathieu Raffinot, “Composite repetition-aware data structures,” in Combinatorial Pattern Matching, Ferdinando Cicalese, Ely Porat, and Ugo Vaccaro, Eds., Cham, 2015, pp. 26–39, Springer International Publishing.

[14] Djamal Belazzougui and Fabio Cunial, “Fast label extraction in the cdawg,” in International Symposium on String Processing and Information Retrieval. Springer, 2017, pp. 161–175.

[15] Takuya Takagi, Keisuke Goto, Yuta Fujishige, Shunsuke Inenaga, and Hiroki Arimura, “Linear-size cdawg: New repetition-aware indexing and grammar compression,” in String Processing and Information Retrieval, Gabriele Fici, Marinella Sciortino, and Rossano Venturini, Eds., Cham, 2017, pp. 304–316, Springer International Publishing.

[16] Alberto Apostolico and Stefano Lonardi, “Off-line compression by greedy textual substitution,” Proceedings of the IEEE, vol. 88, no. 11, pp. 1733–1744, 2000.
[17] Shunsuke Inenaga, Takashi Funamoto, Masayuki Takeda, and Ayumi Shinohara, “Linear-time off-line text compression by longest-first substitution,” in *International Symposium on String Processing and Information Retrieval*. Springer, 2003, pp. 137–152.

[18] Ryosuke Nakamura, Hideo Bannai, Shunsuke Inenaga, and Masayuki Takeda, “Simple linear-time off-line text compression by longest-first substitution,” in *Data Compression Conference, 2007. DCC ’07*. March 2007, pp. 123–132.

[19] Michał Gańczorz and Artur Jeż, “Improvements on re-pair grammar compressor,” in *Data Compression Conference (DCC), 2017*. IEEE, 2017, pp. 181–190.
A Appendix

We show Figures 4, 5, and 6 to help for understanding Proof of Theorem 4.

Let $G_{rp}$, $G_{nmr}$, and $G_{mr}$ be the grammars generated by RePair, Na¨ıve-MR-RePair, and MR-RePair, respectively. For a given text $T = a_1 \cdots a_n$ ($a_i \in \Sigma$, $1 \leq i \leq n$) of length $|T| = n$, let $g_{rp}$, $g_{nmr}$, and $g_{mr}$ be the sizes of $G_{rp}$, $G_{nmr}$, and $G_{mr}$, respectively. Here, assume that $T = (aw)^2(2^m-1)+1a$, where $w \in \Sigma^+$ such that $awa$ is the most frequent maximal repeat of $T$, and $m \in \mathbb{N}^+$. Then, by Proof of Theorem 4 $g_{nmr} > g_{rp}$ holds with some $m$ and $w$ such that $(m-1)(|w| - 1) > 1$.

We show a concrete example of the grammar generation process of RePair and $G_{rp}$ for $T = (abcd)^7a$ with $m = 2$ and $|w| = 3$ in Figure 4. The size $g_{rp}$ is 18 for this example. We also show an example of the process of Na¨ıve-MR-RePair and $G_{nmr}$ for the same $T$ in Figure 5. As we see, the size $g_{nmr}$ is 19, and thus $g_{nmr} > g_{rp}$ holds. As shown in Figure 5, in particular cases, Na¨ıve-MR-RePair may fail to extract repetitive patterns (like $abcd$ of $(abcd)^7a$ for the running example). However, this problem is solved by using MR-RePair. We show an example of the process of MR-RePair and $G_{mr}$ for the same $T = (abcd)^7a$ in Figure 6. The size $g_{mr}$ is 16 and this is smaller than $g_{rp} = 18$. While the most frequent maximal repeat at the second replacement step is $v_a v_b v_c v_d v_a$, MR-RePair replaces $v_a v_b v_c v_d$ with new variable $v_1$ because of the additional Step 3 of Definition 3.
Figure 4: Grammar generation process of RePair and its generated grammar for text (abcd)\(^7\)a. The grammar size is 18.

Figure 5: Grammar generation process of Naïve-MR-RePair and its generated grammar for text (abcd)\(^7\)a. The grammar size is 19.

Figure 6: Grammar generation process of MR-RePair and its generated grammar for text (abcd)\(^7\)a. The grammar size is 16.