The Analogue of Dedekind Eta Functions for Calabi-Yau Manifolds II.
(Algebraic, Analytic Discriminants and the Analogue of Baily-Borel Compactification of the Moduli Space of CY Manifolds.)

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Abstract

In this paper we construct the analogue of Dedekind $\eta$–function on the moduli space of polarized CY manifolds. We prove that the $L^2$ norm of $\eta(\tau)$ is the regularized determinants of the Laplacians of the CY metric on $(0, 1)$ forms.

We construct the analogue of the Baily-Borel Compactification of the moduli space of polarized CY and prove that it has the same properties as the Baily-Borel compactification of the locally symmetric Hermitian spaces. We proved that the compactification constructed in the paper is the minimal.

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1 Introduction

This is the second part of the paper [4]. In this paper we will study the algebraic and analytic discriminants of Calabi-Yau manifolds. We will generalize the notion of algebraic and analytic discriminants of elliptic curves to the case of CY manifolds. Next we will review the notion of algebraic and analytic discriminants for elliptic curves and their relation.

1. Algebraic Discriminant on the Moduli Space of Marked Elliptic Curves.

The algebraic discriminant of an elliptic curve is defined on the moduli space of elliptic curves which is the quotient of the Teichmüller space by the mapping class group. The Teichmüller space is the upper half plane \( \mathfrak{h} := \{ \tau \in \mathbb{C} | \text{Im} \tau > 0 \} \) and the mapping class group of the elliptic curve is \( \text{SL}_2(\mathbb{Z}) \). Thus the moduli space of elliptic curves is \( \text{PSL}_2(\mathbb{Z}) \backslash \mathfrak{h} \). The elliptic curves corresponding to lattices spanned by \( (1, \sqrt{-1}) \) and \( (1, \rho) \), where \( \rho^3 = 1 \) and \( \rho \neq 1 \) have automorphisms of order four and six. So the moduli space \( \text{PSL}_2(\mathbb{Z}) \backslash \mathfrak{h} \) is a stack. If we consider the moduli space of elliptic curves with a fixed basis in \( H_1(E, \mathbb{Z}/2\mathbb{Z}) \), then it is isomorphic to \( \Gamma(2) \backslash \mathfrak{h} \), where

\[
\Gamma(2) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | \det A = 1; \ a, b, c, d \in \mathbb{Z}, \ A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \right\}.
\]

The group \( \Gamma(2) \) acts on \( \mathfrak{h} \) without fixed points. Thus over \( \Gamma(2) \backslash \mathfrak{h} \) we have a universal family of elliptic curves with a fixed basis in \( H_1(E, \mathbb{Z}/2\mathbb{Z}) \). This
universal family can be represented by

\[ y^2 = x(x-1)(x-\lambda). \]  

(1)

So \( \Gamma(2) \setminus \mathfrak{h} = \mathbb{CP}^1 - \{0, 1, \infty\} \). The algebraic discriminant \( \Delta_{\text{alg}}(\tau) \) of the family (1) is given by

\[ \Delta_{\text{alg}}(\lambda) = ((1 - \lambda) \lambda)^2. \]  

(2)

2. Analytic Discriminant.

The analytic discriminant of the elliptic curve \( E_\tau := \mathbb{C} / \{ m + n\tau \} \) is just the regularized determinant of the Laplacian \( \Delta_{(0, 1)}(\tau) \) of the flat metric acting on \((0, 1)\) forms. It is defined as follows: Let \( 0 < \lambda_1 \leq \ldots \leq \lambda_k \leq \ldots \) be the spectrum of \( \Delta_{(0, 1)}(\tau) \). Let \( \zeta_{\Delta_{(0, 1)}}(\tau, s) := \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s} \). It is a well known fact that \( \zeta_{\Delta_{(0, 1)}}(\tau, s) \) is a meromorphic function on \( \mathbb{C} \) well defined at \( 0 \).

\[ \Delta_{(0, 1)}(\tau) := \exp \left( \left( -\frac{d}{ds} \zeta_{\Delta_{(0, 1)}}(\tau, s) \right) \bigg|_{s=0} \right). \]  

(3)

It is easy to see that the regularized determinant \( \Delta_{(0, 1)}(\tau) \) is a function on \( \mathcal{M}_2 \). The Kronecker limit formula gives an explicit expression of the regularized determinant:

**Theorem 1** Let \( \det \Delta_\tau \) be the regularized determinant of the flat metric on the elliptic curve \( E_\tau := \mathbb{C} / \{ m + n\tau \} \). Then \( \det \Delta_\tau = \text{Im} \tau |\eta(\tau)|^2 \), \( \eta(\tau) \) is defined as: \( \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \), where \( q = e^{2\pi i \tau} \).

3. Relations between the Analytic and Algebraic Discriminants of the Elliptic Curves.

On the moduli space \( \mathcal{M}_2 := \Gamma(2) \setminus \mathfrak{h} \) of marked \( H_1(E, \mathbb{Z}/2\mathbb{Z}) \) elliptic curves, the algebraic discriminant is a section of the line bundle \( \mathcal{L} \) on \( \mathcal{M}_2 := \Gamma(2) \setminus \mathfrak{h} \), associated with the principle bundle

\[ \text{U}(1) \rightarrow \text{SL}_2(\mathbb{R}) \rightarrow \text{SL}_2(\mathbb{R}) / \text{U}(1) = \mathfrak{h}. \]

It is well known that \( \mathcal{M}_2 \) is isomorphic to \( \mathbb{CP}^1 - \{0, 1, \infty\} \). The universal cover of \( \mathbb{CP}^1 - \{0, 1, \infty\} \) is the upper half plane \( \mathfrak{h} \). The fundamental group \( \pi_1(\mathbb{CP}^1 - \{0, 1, \infty\}) \) of \( \mathbb{CP}^1 - \{0, 1, \infty\} \) is the free group with two generators isomorphic to \( \Gamma(2) \). There is a universal family of marked \( H_1(E, \mathbb{Z}/2\mathbb{Z}) \) elliptic curves

\[ \pi : \mathcal{E} \rightarrow \mathbb{CP}^1 - \{0, 1, \infty\}, \]  

(4)

given by \( y^2 = x(x-1)(x-\lambda) \). The line bundle \( \mathcal{L} \) on \( \mathcal{M}_2 := \Gamma(2) \setminus \mathfrak{h} \) is isomorphic to the dual of \( \pi_* \omega_{E/\mathcal{M}_2} \), where \( \omega_{E/\mathcal{M}_2} \) is the relative dualizing sheaf of...
the family (4) and on $\pi_* \omega_{E/M^2}$ we have a natural metric. The local sections of $\pi_* \omega_{E/M^2}$ are families of holomorphic one forms $\omega_{\tau}$. Then we define $L^2$ metric

$$\|\omega_{\tau}\|^2_{L^2} = \frac{-\sqrt{-1}}{2} \int_{E_{\tau}} \omega_{\tau} \wedge \overline{\omega_{\tau}}.$$ 

There is a natural compactification $\mathbb{CP}^1 - \{0, 1, \infty\}$ to $\mathbb{CP}^1$. We can prolong $\pi_* \omega_{E/M^2}$ to a line bundle $\pi_* \omega_{E/M^2}$ over $\mathbb{CP}^1$ by extending the holomorphic sections with finite $L^2$ norm.

It is a well known fact that any elliptic curve $E_{\tau} := \mathbb{C}/\Lambda_{\tau}$ where $\Lambda_{\tau}$ is the lattice $\{m + n\tau | m, n \in \mathbb{Z}, \tau \in \mathbb{C} \text{ and } \text{Im} \tau > 0\}$ can be embedded in $\mathbb{CP}^2$ and the equation in the standard affine open set of $E_{\tau}$ is given by $y^2 = 4x^3 - g_2(\tau) - g_3(\tau)$, where

$$g_2(\tau) = 60 \sum_{(n, m) \neq (0, 0)} \frac{1}{(n + m\tau)^4} \quad \text{and} \quad g_3(\tau) = 140 \sum_{(n, m) \neq (0, 0)} \frac{1}{(n + m\tau)^6}.$$ 

The algebraic discriminant $\Delta(\tau)$ of the elliptic curve $E_{\tau}$ is defined as the discriminant of the polynomial $4x^3 - g_2(\tau) - g_3(\tau)$. Thus we get the explicit formula for the algebraic discriminant

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2.$$ 

(5)

The relation between the algebraic discriminants given by (2) and (5) is given by the following; It is a well known fact that $\Gamma(2)$ is a normal subgroup in $\text{PSL}_2(\mathbb{Z})$ and $\text{PSL}_2(\mathbb{Z}) / \Gamma(2) = S_3$, where $S_3$ is the symmetric group. Thus we have a finite Galois covering

$$\pi_2 : \Gamma(2) \backslash \mathfrak{h} \to \text{PSL}_2(\mathbb{Z}) \backslash \mathfrak{h}.$$ 

(6)

Thus $\Delta_{alg}(\tau) = \pi_2^* (g_2(\tau)^3 - 27g_3(\tau)^2)$ is a section $\Delta$ of $\pi_* \omega_{E/M^2}$ which vanishes on $\{0, 1, \infty\}$. We will call $\Delta_{alg}(\tau)$ the algebraic discriminant of the elliptic curve with fixed basis in $H^1(E, \mathbb{Z}/2\mathbb{Z})$.

Let us consider the function $\pi^* (\Delta)$ on $\mathfrak{h}$. Then $\pi^* (\Delta)$ will be up to a constant equal to the cusp form of weight 12. $\pi^+ (\Delta)$ will be called the analytic discriminant. The relations between the analytic and algebraic discriminants is given by the following interpretation of the Kronecker limit formula:

**Theorem 2** $\det \Delta_{\tau} = \|\pi^+(\Delta)\|^2_{L^2}.$

4. The Analogue of Baily-Borel Compactification of the Moduli Space of Polarized CY Manifolds

Baily and Borel constructed a compactification of locally symmetric spaces quotient by an arithmetic group by using cusp forms, i.e. automorphic forms which vanish at the cusps. In case of polarized CY manifolds it was proved in [17] that the completion of the Teichmüller space with respect to the Hodge metric is a domain of holomorphy. We proved in this paper that sections of the
power of the relative dualizing sheaf with finite $L^2$ norm are the analogue of
cusp forms. We show that some power of the relative dualizing sheaf and its
holomorphic sections with finite $L^2$ norm define a holomorphic embedding of
the moduli space $\mathcal{M}_L(M)$ of polarized CY manifolds into $\mathbb{P}^{m_0}$. The projective
closure $\overline{\mathcal{M}}_L(M)$ of the image of $\mathcal{M}_L(M)$ in $\mathbb{P}^{m_0}$ will be the analogue of the Baily-
Borel compactification. We also prove the analogue of Borel extension Theorem,
namely any map of $(D^*)^k \times D^{h^n-1,1-k}$ into $\mathcal{M}_L(M)$ can be analytically prolonged
to a holomorphic map of $D^k \times D^{h^n-1,1-k}$ to the Baily-Borel compactification $\overline{\mathcal{M}}_L(M)$ of $\mathcal{M}_L(M)$. This implies that the compactification we constructed is
minimal model.

5. Relations between Algebraic and Analytic Discriminants

The generalization of the relation between the analytic and algebraic dis-
criminant on elliptic curves to CY three folds is done in the last section of
this paper and it follows from the proof that the regularized determinants of
the Laplacians of CY metrics acting on $(0,1)$ forms on a CY manifold $M$ are
bounded. The proof of the boundedness is based on the computa-
tion of the short term asymptotic of the trace $Tr(\exp(-t\Delta_{\tau,q}))$ of the heat kernel of a CY
metric for a CY threefold. We established that

$$Tr(\exp(-t\Delta_{\tau,q})) = \frac{a_{-3}}{t^3} + \frac{a_{-2}}{t^2} + \frac{a_{-1}}{t} + a_0 + O(t),$$

where

$$a_{-3} = \int_M L^3 = \text{vol}(g_\tau), \quad a_{-2} = \int_M c_1(M) \wedge L^2 = 0,$$

$$a_{-1}(g) = -\frac{1}{720\pi} \int_M c_2(M) \wedge L$$

and $a_0$ are constants. As a consequence of this formula we get that there exists
a non-zero holomorphic section $\eta^{\otimes N}$ of the $N^{th}$ power $\omega^{\otimes N}_{X/\mathcal{M}_L(M)}$ of the relative
dualizing sheaf such that its $L^2$ norm $\|\eta^{\otimes N}\|_{L^2}$ is $(\det(\Delta_{0,1}(\tau)))^N$, where

$$N = \#(\Gamma/\Gamma_G).$$

Recall that the moduli space $\mathcal{M}_L(M)$ of polarized CY manifolds is obtained
from the Teichmüller space $\mathcal{T}(M)$ by the action of some arithmetic group $\Gamma$ of rank at least two. From a Theorem proved by Kazhdan it follows that the abelian group $\Gamma/\Gamma_G$ is finite.

Conjecture 3 Suppose that $M$ is a CY manifold of complex dimension $n$ with
fixed polarization class $L$. Suppose that $g$ is a CY metric such that the cohomol-
ogy class of the imaginary part is $L$. Then the coefficients $a_{-k}(g)$ for $0 \leq k \leq n$
of the short term asymptotic expansion of the trace of the heat kernel are given
by the formula:

$$Tr(k_t(x,y)) = \frac{a_{-n}(g)}{t^n} + \cdots + \frac{a_{-k}(g)}{t^k} + \cdots + a_0(g) + \cdots,$$
where \(a_k = b_k \int_M c_n - k(M) \wedge L_k^k\) for \(k = 1, \ldots, n\), \(b_k\) are some constants which depend on the dimension of \(M\), and \(c_k(M)\) are the Chern classes of \(M\).

Conjecture 3 implies that the regularized determinant \(\det_{\sigma(p,q)}(\tau)\) of the Laplacian of a CY metric with a fixed class of cohomology of its imaginary part and acting on \((p, q)\) forms is a bounded function on the moduli space \(\mathcal{M}_L(M)\) of polarized CY manifolds.

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2 Basic Definitions and Notions

2.1 Kuranishi Space and Flat Local Coordinates

The following Theorem is proved in [24]:

**Theorem 5** Let \(M\) be a CY manifold and let \(\{\phi_i\}\) for \(i = 1, \ldots, N\) be a basis of harmonic \((0, 1)\) forms with coefficients in \(T^{1,0}\), i.e. \(\{\phi_i\} \in \mathbb{H}^1(M, T^{1,0})\). Then the equation \(\overline{\partial} \phi(\tau) = \frac{1}{2} [\phi(\tau), \phi(\tau)]\) has a solution in the form:

\[
\phi(\tau) = \sum_{i=1}^N \phi_i \tau^i + \sum_{|I_N| \geq 2} \phi_{I_N} \tau^{I_N} = \\
\sum_{i=1}^N \phi_i \tau^i + \frac{1}{2} \overline{\partial}^* G[\phi(\tau^1, \ldots, \tau^N), \phi(\tau^1, \ldots, \tau^N)],
\]

(7)

where \(I_N = (i_1, \ldots, i_N)\) is a multi-index,

\[
\overline{\partial}^* \phi(\tau^1, \ldots, \tau^N) = 0, \quad \phi_{I_N} \omega_M = \partial \psi_{I_N}
\]

\(\phi_{I_N} \in C^\infty(M, \Omega^{0,1} \otimes T^{1,0}), \tau^{I_N} = (\tau^i_{i_1} \ldots (\tau^N_{i_N})^{i_N}\)

and there exists \(\varepsilon > 0\) such that for \(|\tau^i| < \varepsilon\), \(\phi(\tau) \in C^\infty(M, \Omega^{0,1} \otimes T^{1,0})\).

(See [23] and [24]).

It is a standard fact from Kodaira-Spencer-Kuranishi deformation theory that for each \(\tau = (\tau^1, \ldots, \tau^N) \in \mathcal{K}\) as in Theorem 5 the Beltrami differential \(\phi(\tau^1, \ldots, \tau^N)\) defines a new integrable complex structure on \(M\). This means that the points of \(\mathcal{K}\), define a family of integrable in the sense of Newlander-Nirenberg operators \(\overline{\partial}_\tau\) on the \(C^\infty\) family \(\mathcal{K} \times M \rightarrow M\). Moreover, it was proved by Kodaira, Spencer and Kuranishi that over \(\mathcal{K}\) there exists a complex analytic family of CY manifolds \(\pi : \mathcal{X} \rightarrow \mathcal{K}\). The family \(\pi : \mathcal{X} \rightarrow \mathcal{K}\) is called the Kuranishi family. The operators \(\overline{\partial}_\tau\) are defined as follows: Let \(\{U_i\}\) be an open covering of \(M\),
with a local coordinate system in $\mathcal{U}_i$ given by $\{z^k_i\}$ with $k = 1, \ldots, n = \dim_{\mathbb{C}} M$. Assume that $\phi(\tau^1, \ldots, \tau^N)|_{\mathcal{U}_i}$ is given by:

$$
\phi(\tau^1, \ldots, \tau^N) = \sum_{j,k=1}^{n} (\phi(\tau^1, \ldots, \tau^N))_{j}^{k} \, dz^j \otimes \frac{\partial}{\partial z^k}.
$$

Then the operators $\overline{\partial}_i$ are given by the following explicit formulas:

$$
(\overline{\partial})_{\tau, j} = \frac{\partial}{\partial z^j} - \sum_{k=1}^{n} (\phi(\tau^1, \ldots, \tau^N))_{j}^{k} \frac{\partial}{\partial z^k}.
$$

**Definition 6** The coordinates $\tau = (\tau^1, \ldots, \tau^N)$ defined in Theorem 2 will be fixed from now on and will be called the flat coordinate system in $\mathcal{K}$.

### 2.2 The Definition of Hodge and Weil-Petersson Metrics

It is a well-known fact from Kodaira-Spencer-Kuranishi theory that the tangent space $T_{\tau, K}$ at a point $\tau \in \mathcal{K}$ can be identified with the space of harmonic (0,1) forms with values in the holomorphic vector fields, which we will denote by $\mathbb{H}^1(M_{\tau}, T)$. We will view each element $\phi \in \mathbb{H}^1(M_{\tau}, T)$ as a point wise linear map from $\phi: \Omega^{1,0}(M_{\tau}) \to \Omega^{0,1}(M_{\tau})$.

Given $\phi_1$ and $\phi_2 \in \mathbb{H}^1(M_{\tau}, T)$, the trace of the map $\phi_1 \circ \phi_2: \Omega^{1,0}(M_{\tau}) \to \Omega^{0,1}(M_{\tau})$ at the point $m \in M_{\tau}$ with respect to the metric $g$ is simply:

$$
\text{Tr}(\phi_1 \circ \phi_2)(m) = \sum_{k,l,m=1}^{n} (\phi_1^k_{\bar{m}})(\phi_2^l_{m}) \cdot g^{l,k} g_{m, \bar{m}}.
$$

We will define the Weil-Petersson metric on $\mathcal{K}$ as:

$$
\langle \phi_1, \phi_2 \rangle = \int_M \text{Tr}(\phi_1 \circ \phi_2) \text{vol}(g).
$$

**Definition 7** Let $G$ be a semi-simple Lie group, $K$ be a maximal compact group and $K_1$ be a proper subgroup in $K$. Let us consider the homogeneous space $G/K_1$ and its projection: $\mathbb{P}r: G/K_1 \to G/K$. Let us consider the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$:

$$
\mathfrak{g} = K \oplus \mathcal{P},
$$

where $K$ is the Lie algebra of $K$ and (11) is orthogonal decomposition of the Lie algebra $\mathfrak{g}$ with respect to the Killing form $K(x, y)$. Then the Killing form is non degenerate negative bilinear on $K$ and positive non-degenerate form on $\mathcal{P}$. The tangent space $T_{id, G/K}$ is isomorphic to $K/K_1 \oplus \mathcal{P}$, where $K_1$ is the Lie algebra of $K_1$. Then if $(\alpha, \beta) \in K_1^\perp \oplus \mathcal{P}$, where $K_1^\perp \subset K$ is the perpenicular space to $K_1$ in $K$, we define the Killing norm of $(\alpha, \beta)$ as follows:

$$
\| (\alpha, \beta) \|^2_K = -K(\alpha, \alpha) + K(\beta, \beta).
$$

Thus (12) define an invariant metric on $G/K_1$. We will call this metric the Hodge metric on $G/K_1$.
Remark 8 It is easy to see that the Hodge metric on $\mathbb{G}/\mathbb{K}_1$ is a complete metric.

Definition 9 It is a well known and easy fact that the moduli space of Variations of Hodge Structures of given weight is isomorphic $\mathbb{G}/\mathbb{K}_1$, where $\mathbb{G}$ is a semi-simple Lie group and $\mathbb{K}_1$ is a compact subgroup. The period map:

$$p : \mathcal{M}_L(M) \to \mathbb{G}/\mathbb{K}_1$$

is a well defined and it is a local isomorphism when $M$ is a CY manifold. Then the restriction of the Hodge metric on $p(\mathcal{M}_L(M))$ defines the Hodge metric on the moduli space $\mathcal{M}_L(M)$ of polarized CY manifolds.

In [17] the following Theorem was proved:

Theorem 10 The Hodge metric on the moduli space $\mathcal{M}_L(M)$ of polarized CY manifolds has non-positive curvature and bounded from above holomorphic sectional curvature by a negative constant.

2.3 Review of the Results in [18]

Definition 11 We will define the Teichmüller space $\mathcal{T}(M)$ of a CY manifold $M$ as follows: $\mathcal{T}(M) := \mathcal{I}(M)/\text{Diff}_0(M)$, where

$$\mathcal{I}(M) := \{\text{all integrable complex structures on } M\}$$

and $\text{Diff}_0(M)$ is the group of diffeomorphisms isotopic to identity. The action of the group $\text{Diff}_0(M)$ is defined as follows; Let $\phi \in \text{Diff}_0(M)$ then $\phi$ acts on integrable complex structures on $M$ by pull back, i.e. if

$$I \in C^\infty(M, \text{Hom}(T(M), T(M)))$$

then we define $\phi(I) = \phi^*(I)$.

We will call a pair $(M; \gamma_1, ..., \gamma_n)$ a marked CY manifold where $M$ is a CY manifold and $\{\gamma_1, ..., \gamma_n\}$ is a basis of $H_n(M, \mathbb{Z})/\text{Tor}$.

Remark 12 Let $\mathcal{K}$ be the Kuranishi space. It is easy to see that if we choose a basis of $H_n(M, \mathbb{Z})/\text{Tor}$ in one of the fibres of the Kuranishi family $M \to \mathcal{K}$ then all the fibres will be marked, since as a $C^\infty$ manifold $\mathcal{K} \cong M \times \mathcal{K}$.

Theorem 13 There exists a family of marked polarized CY manifolds

$$\mathcal{Z}_L \to \mathcal{T}(M), \quad (13)$$

which possesses the following properties: a) It is effectively parametrized, b) For any marked CY manifold $M$ of fixed topological type for which the polarization class $L$ defines an imbedding into a projective space $\mathbb{CP}^N$, there exists an isomorphism of it (as a marked CY manifold) with a fibre $M_s$ of the family $\mathcal{Z}_L$. c) The base has dimension $h^n - 1, 1$. 8
Corollary 14 Let \( Y \rightarrow X \) be any family of marked CY manifolds, then there exists a unique holomorphic map \( \phi : X \rightarrow \tilde{T}(M) \) up to a biholomorphic map \( \psi \) of \( M \) which induces the identity map on \( H_n(M, \mathbb{Z}) \).

From now on we will denote by \( T(M) \) the irreducible component of the Teichmüller space that contains our fixed CY manifold \( M \).

2.4 Construction of the Moduli Space of Polarized CY Manifolds

Definition 15 We will define the mapping class group \( \Gamma_1 \) of any compact \( C^\infty \) manifold \( M \) as follows: \( \Gamma_1 = \text{Diff}_+ (M)/\text{Diff}_0 (M) \), where \( \text{Diff}_+ (M) \) is the group of diffeomorphisms of \( M \) preserving the orientation of \( M \) and \( \text{Diff}_0 (M) \) is the group of diffeomorphisms isotopic to identity.

Definition 16 Let \( L \in H^2(M, \mathbb{Z}) \) be the imaginary part of a Kähler metric. Let \( \Gamma_L := \{ \phi \in \Gamma_1 | \phi(L) = L \} \).

It is a well know fact that the moduli space of polarized algebraic manifolds \( \mathcal{M}_L(M) = \Gamma_L \setminus \tilde{T}(M) \).

Theorem 17 There exists a subgroup of finite index \( \Gamma \) of \( \Gamma_L \) such that \( \Gamma \) acts freely on \( T(M) \) and \( \Gamma \setminus T(M) = \mathcal{M}_L(M) \) is a non-singular quasi-projective variety.

Remark 18 Theorem 17 implies that we constructed a family of non-singular CY manifolds \( \pi : X \rightarrow \mathcal{M}_L(M) \) over a quasi-projective non-singular variety \( \mathcal{M}(M) \). Moreover it is easy to see that \( X \subset \mathbb{CP}^N \times \mathcal{M}_L(M) \). So \( X \) is also quasi-projective. From now on we will work only with this family.

2.5 Metrics on Vector Bundles with Logarithmic Growth

In Theorem 17 we constructed the moduli space \( \mathcal{M}_L(M) \) of CY manifolds. From the results in [26] and Theorem 17 we know that \( \mathcal{M}_L(M) \) is a quasi-projective non-singular variety. Using Hironaka’s resolution theorem, we may suppose that \( \mathcal{M}_L(M) \subset \mathcal{M}_L(M) \), where \( \mathcal{M}_L(M) - \mathcal{M}_L(M) = D \) is a divisor with normal crossings. We need now to show how we will extend the determinant line bundle \( L \) to a line bundle \( L' \) on \( \mathcal{M}_L(M) \). For this reason we are going to recall the following definitions and results from [14]. We will look at polydisks \( D^N \subset \mathcal{M}_L(M) \), where \( D \) is the unit disk, \( N = \dim \mathcal{M}_L(M) \) and such that \( D^N \cap \mathcal{O}_\infty = \{ \text{union of hyperplanes}; \tau_1 = 0, ..., \tau_k = 0 \} \).

Hence, \( D^N \cap \mathcal{M}(M) = (D^*)^k \times D^{N-k} \). On \( D^* \) we have the Poincare metric

\[
\frac{|dz|^2}{|z|^2 (\log |z|)^2}
\]
and on $D$ we have the simple metric $|dz|^2$, giving us a product metric on $(D^*)^k \times D^{N-k}$ which we call $\omega^{(P)}$.

A complex-valued $C^\infty$ p-form $\eta$ on $\mathcal{M}(M)$ is said to have Poincaré growth on $\overline{\mathcal{M}}_L (M) - \mathcal{M}_L (M)$ if there is a set of if polydisks $\mathcal{U}_\alpha \subset \overline{\mathcal{M}}_L (M)$ covering $\overline{\mathcal{M}}_L (M) - \mathcal{M}_L (M)$ such that in each $\mathcal{U}_\alpha$ an estimate of the following type holds:

$$|\eta(\tau_1, ..., \tau_N)| \leq C_\alpha \omega^{(P)}_{\mathcal{U}_\alpha} (\tau_1, ..., \tau_N, \overline{\tau_1}, ..., \overline{\tau_N}).$$

This property is independent of the covering $\mathcal{U}_\alpha$ of $\overline{\mathcal{M}}_L (M) - \mathcal{M}_L (M)$ but depends on the compactification $\overline{\mathcal{M}}_L (M)$. If $\eta_1$ and $\eta_2$ both have Poincaré growth on $\overline{\mathcal{M}}_L (M) - \mathcal{M}_L (M)$, then so does $\eta_1 \wedge \eta_2$. The basic property of the Poincaré growth is the following:

**Theorem 19** A p-form $\eta$ with a Poincaré growth on $\overline{\mathcal{M}}_L (M) - \mathcal{M}_L (M)$ = $\mathcal{D}$ has the property that for every $C^\infty (r-p)$ form $\psi$ on $\overline{\mathcal{M}}_L (M)$ we have:

$$\int_{\overline{\mathcal{M}}_L (M) - \mathcal{M}_L (M)} |\eta \wedge \psi| < \infty.$$  

Hence, $\eta$ defines a current $[\eta]$ on $\overline{\mathcal{M}}_L (M)$.

**Proof:** For the proof see [14].

A complex valued $C^\infty$ p-form $\eta$ on $\overline{\mathcal{M}}_L (M)$ is good on $M$ if both $\eta$ and $d\eta$ have Poincaré growth. Let $\mathcal{E}$ be a vector bundle on $\mathcal{M}(M)$ with a Hermitian metric $h$. We will call $h$ a good metric on $\overline{\mathcal{M}}_L (M)$ if the following holds:

1. If for all $x \in \overline{\mathcal{M}}_L (M) - \mathcal{M}_L (M)$, there exists sections $e_1, ..., e_m$ of $\mathcal{E}$ which form a basis of $\mathcal{E}|_{D^r \cap \mathcal{D}_\infty}$.

2. In a neighborhood $D^r$ of $x$ in which $\overline{\mathcal{M}}_L (M) - \mathcal{M}_L (M)$ is given by $z_1 \times ... \times z_k = 0$.

3. The metric $h_{ij} = h(e_i, e_j)$ has the following properties:

$$h_{ij} = C \left( \sum_{i=1}^k \log |z_i| \right)^{2m}, \quad (\det (h))^{-1} \leq C \left( \sum_{i=1}^k \log |z_i| \right)^{2m}$$

for some $C > 0$, $m \geq 0$. b. The 1-forms $((dh^{-1}))_{ij}$ are good forms on $\overline{\mathcal{M}}_L (M) \cap D^N$.

It is easy to prove that there exists a unique extension $\overline{\mathcal{E}}$ of $\mathcal{E}$ on $\overline{\mathcal{M}}_L (M)$, i.e. $\overline{\mathcal{E}}$ is defined locally as holomorphic sections of $\mathcal{E}$ which have a finite norm in $h$.

**Theorem 20** Let $(\mathcal{E}, h)$ be a vector bundle with a good metric on $\overline{\mathcal{M}}_L (M)$, then the Chern classes $c_k(\mathcal{E}, h)$ are good forms on $\overline{\mathcal{M}}_L (M)$ and the currents $[c_k(\mathcal{E}, L^2)]$ represent the cohomology classes $c_k(\mathcal{E}, L^2) \in H^{2k}(\overline{\mathcal{M}}_L (M), \mathbb{Z})$.

**Proof:** For the proof see [14].

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2.6 Applications of Mumford’s Results to the Moduli of CY

In [19] and [25] the following Theorem was proved:

**Theorem 21** Let \( \pi : X \to \mathcal{M} \) be the flat family of non-singular CY manifolds. Let the relative dualizing sheaf \( \omega_{X/\mathcal{M}(M)} := \pi_* \Omega_{X/\mathcal{M}(M)}^{n,0} \) be equipped with the metric \( \| \omega \|_{L^2}^2 := \left( \frac{\sqrt{-1}}{2} \right)^n \int_M \omega \wedge \overline{\omega} \) defined by

\[
\| \omega \|_{L^2}^2 := (-1)^{\frac{n(n-1)}{2}} \left( \frac{\sqrt{-1}}{2} \right)^n \int_M \omega \wedge \overline{\omega} \tag{14}
\]

Then \( L^2 \) is a good metric.

3 The Analogue of Baily-Borel Compactification of the Moduli Space of CY Manifolds

3.1 Construction of the Analogue of the Dedekind \( \eta \) Function for CY Manifolds

**Theorem 22** Let \( M \) be a CY manifold. Let \( \mathcal{N} = \# \Gamma / \Gamma, \Gamma \). Then \( \omega_{X/\mathcal{M}(M)} \) is a trivial complex analytic line bundle over \( \mathcal{M}(M) \).

**Proof:** The proof of Theorem 22 is based on the following Theorem proved in [24]:

**Theorem 23** The Chern class of the relative dualizing sheaf \( \omega_{X/\mathcal{M}(M)} \) is the imaginary part of the Weil-Petersson metric on \( \mathcal{M}(M) \).

According to the results proved in [4] we have

\[
\dd c \log (\det \Delta_{(0,1)}(\tau)) = \text{Im} W - P. \tag{15}
\]

So from (15), the fact that the \( L^2 \) metric is good, Theorem 20 and Theorem 22 we get that the Chern class of the relative dualizing sheaf \( \omega_{X/\mathcal{M}(M)} \) is zero in \( H^2(\mathcal{M}(M), \mathbb{Z}) \). This means that \( \omega_{X/\mathcal{M}(M)} \) is a trivial \( C^\infty \) line bundle on \( \mathcal{M}(M) \). Let us denote by \( \sigma : T(M) \to \mathcal{M}(M) = \Gamma \setminus T(M) \) the natural projection map. So the line bundle \( \sigma^* (\omega_{X/\mathcal{M}(M)}) \) will be trivial on \( T(M) \), i.e.

\[
\omega_{X/\mathcal{M}(M)} \cong T(M) \times \mathbb{C} \text{ and }
\]

\[
\omega_{X/\mathcal{M}(M)} \cong \Gamma \setminus \mathbb{C} \times T(M), \tag{16}
\]

where \( \Gamma \) acts in a natural way on the Teichmüller space and it acts by a character

\[
\chi \in \text{Hom}(\Gamma, \mathbb{C}^*_1) \cong \text{Hom}(\Gamma/[\Gamma, \Gamma], \mathbb{C}^*_1)
\]
of the group Γ on the fibre C. A Theorem of Kazhdan states that Γ/[Γ, Γ] is a finite group if the rank of Γ is bigger or equal to 2. For CY manifolds Γ is an arithmetic group of rank ≥ 2 according to [22]. From here we deduce that ω^N_X/\mathfrak{M}_L(M) will be a trivial complex analytic bundle on \mathfrak{M}_L(M), where N = # Γ/[Γ, Γ]. Theorem 22 is proved. ■

**Theorem 24** Let M be a CY manifold. Let \mathfrak{M}_L(M) be the moduli space of polarized CY manifolds such that τ_0 ∈ \mathfrak{M}_L(M) corresponds to M. Let \omega^N_X/\mathfrak{M}_L(M) be the relative dualizing sheaf of the family X → \mathfrak{M}_L(M). Then there exists a non zero section η^N ∈ H^0(\mathfrak{M}_L(M),\omega^N_X/\mathfrak{M}_L(M)) such that \|η(τ)\|^2_{L^2} = \det\Delta_{(0,1)}(τ).

**Proof:** Let \{φ_i\} be a basis of harmonic Dolbault representatives with respect to the CY metric corresponding to the polarization class \(L \in H^1(M, T^1, -1, M)\).

As it was proved in [24] \{φ_i\} defines a coordinate system (τ_1, ..., τ_N) in the local deformation space K which we will call the Kuranishi space. We know that the CY metric with a fixed polarization class depends real analytically on the coordinates (τ_1, ..., τ_N, \overline{τ}_1, ..., \overline{τ}_N). From here it follows that the regularized determinants \(\Delta_{(0, q)}(τ)\) depend also real analytically on the coordinates (τ_1, ..., τ_N, \overline{τ}_1, ..., \overline{τ}_N). The main result in [14] is the following Theorem:

**Theorem 25** We have: \(\frac{∂^2}{∂τ_i ∂τ_j} \log (\det \Delta_{(0, 1)}(τ)) \bigg|_{τ=0} = \langle φ_i, φ_j \rangle\).

In [24] we proved the following Theorem:

**Theorem 26** We have \(\frac{∂^2}{∂τ_i ∂τ_j} (\log (⟨ω_τ, ω_τ⟩)) \bigg|_{τ=0} = \langle φ_i, φ_j \rangle\).

Theorems 25 and 26 imply that for each point τ ∈ \mathfrak{M}_L(M) there exists an open set \(U_τ\) such that for τ ∈ \(U_τ\) and

\[
\|η(τ)\|^2_{L^2} = \frac{|f_{U_τ}(τ)|^2}{⟨ω_τ, ω_τ⟩} = \frac{|f_{U_τ}(τ)|^2}{∥ω_τ∥^2_{L^2}} \tag{17}
\]

where \(f_{U_τ}(τ)\) is a holomorphic function in \(U_τ\). Let \(\{U_α\}\) be a covering of \(\mathfrak{M}_L(M)\) by polydisks. Then (17) implies that

\[
\|η(τ)\|^2_{L^2}_{U_α} = \frac{|f_α(τ)|^2}{⟨ω_τ, ω_τ⟩} = \frac{|f_α(τ)|^2}{∥ω_τ∥^2_{L^2}} \tag{18}
\]

Theorem 22 and (18) imply that there exists a global section η^N ∈ H^0(\mathfrak{M}_L^N_X/\mathfrak{M}_L(M)) such that \(η^N|_{U_α} = f_α^N(τ)\). Then (18) implies Theorem 24. ■

We proved the following Theorem in [17]:

**Theorem 27** Let \(\hat{T}(M)\) be the completion of the Teichmüller space \(T(M)\) with respect to the Hodge metric. Then \(\hat{T}(M)\) is a domain of holomorphy, \(T(M)\) is an open and everywhere dense subset in \(\hat{T}(M)\) and \(\hat{T}(M) – T(M)\) is a countable union of complex subspaces.
Definition 28 The arithmetic group $\Gamma$ acts on the completion $\widehat{T(M)}$ of the Teichmüller space $T(M)$. Let us denote by $\mathcal{M}_L(M) := \Gamma \backslash \widehat{T(M)}$. Let $\overline{\mathcal{M}_L(M)}$ be the projective compactification of $\mathcal{M}_L(M)$ such that

$$\overline{\mathcal{M}_L(M)} - \mathcal{M}_L(M) = \mathcal{D}$$

is a divisor of normal crossings. We will define $\mathcal{D}_\infty$ as follows: A point $\tau_\infty \in \mathcal{D}_\infty$ if and only if 1. around $\tau_\infty$ we can find a disk $D$ such that

$$\tau_\infty \in D, \ D - \tau_\infty \subset \mathcal{M}_L(M)$$

over $D - \tau_\infty$ the restriction of the family $X \to \mathcal{M}_L(M)$ of the family of polarized CY manifolds on $D - \tau_\infty$ has a monodromy group of infinite order in $H^n(M, \mathbb{Q})$ or 2. $D_f$ is the codimension one component of $\overline{\mathcal{M}_L(M)} - \mathcal{M}_L(M)$ that is fixed point set by a subgroup $G \subset \Gamma$.

Theorem 29 Let $M$ be a CY manifold. Then there exists a holomorphic section $\eta^N$ of $\omega \otimes X^{\mathcal{M}_L(M)}$ on $\mathcal{M}_L(M)$ such that it can be prolonged to a holomorphic section $\eta^N$ of the line bundle $\omega_X^{\mathcal{M}_L(M)}$ such that for each point $m \in \mathcal{M}_L(M)$, $\eta^N(m) \neq 0$, and the support of the zero set of $\eta^N$ is the divisor $D_\infty$.

Proof: The proof of Theorem 29 is based on the following Lemmas:

Lemma 30 $\frac{\omega_X^{\mathcal{M}_L(M)}}{\mathcal{M}_L(M)} \cong \mathcal{O}_{\mathcal{M}_L(M)} \left( \sum_j k_j D_j \right)$, where $k_j \geq 0$ and $D_j$ are components of $\overline{\mathcal{M}_L(M)} - \mathcal{M}_L(M)$.

Proof: Let $\mathcal{D} = \bigcup_i D_i$ be the decomposition of the divisor $\mathcal{D}$ on irreducible components on $\overline{\mathcal{M}_L(M)}$. Theorem 21 implies that the line bundle $\omega_X^{\mathcal{M}_L(M)}$ is holomorphic trivial bundle on $\mathcal{M}_L(M)$ and $N = \# (\Gamma/ [\Gamma, \Gamma])$. So we can conclude that

$$\frac{\omega_X^{\mathcal{M}_L(M)}}{\mathcal{M}_L(M)} \cong \mathcal{O}_{\mathcal{M}_L(M)} \left( \sum_j k_j D_j \right), \tag{19}$$

where $D_j$ are the components of $\mathcal{D}$. We will prove that the multiplicities $k_j$ are non-negative integers. Indeed we know from Theorem 21 that the $L^2$ metric defined on the line bundle $\omega_X^{\mathcal{M}_L(M)}$ is a good one in the sense of Mumford. So the Chern form $c_1(\omega_X^{\mathcal{M}_L(M)}, L^2)$ of the good metric $h$ defined by (14) is a positive current on $\overline{\mathcal{M}_L(M)}$. The Poincare dual of the cohomology of the current

$$[c_1(\omega_X^{\mathcal{M}_L(M)}, L^2)] \in H^2 \left( \overline{\mathcal{M}_L(M)}, \mathbb{Z} \right)$$

is

$$\mathcal{P} \left( [c_1(\omega_X^{\mathcal{M}_L(M)}, L^2)] \right) = \sum_j k_j [D_j] \in H_{2n-2} \left( \overline{\mathcal{M}_L(M)}, \mathbb{Z} \right). \tag{20}$$
where the coefficients $k_i$ are defined as in (19). The positivity of the current $c_1(\omega_{X/\mathbb{M}_L(M)}, \mathbb{L}^2)$ implies that its Poincare dual current $\sum_j k_j [D_j]$ is positive. From here we can conclude that the coefficients $k_i$ are positive integers. Indeed, let $[\omega_{D_i}] \in H^{2n-2}(M, \mathbb{Z})$ be such classes of cohomology that:

$$\int_{D_j} [\omega_{D_i}] = \delta_{ij}.$$  \hspace{1cm} (21)

Since the current $\sum_j k_j [D_j]$ is positive (21) implies

$$\left\langle \sum_j k_j [D_j], [\omega_{D_i}] \right\rangle = k_i \geq 0.$$  \hspace{1cm} (22)

Lemma 30 is proved. 

**Corollary 31** There exists a section $\eta^N$ of $\omega_{X/\mathbb{M}_L(M)}^\otimes N$ over $M_L(M)$ such that it vanishes on components of $M_L(M) - M_L(M)$.

**Lemma 32** The zero set of the section $\eta^N$ constructed in Corollary 31 of the line bundle $\omega_{X/\mathbb{M}_L(M)}^\otimes N$ is a non zero divisor with the same support as $D_{\tau_\infty}$.

**Proof:** The proof of Lemma 32 is based on the followig two Propositions:

**Proposition 33** Let $M$ be a CY manifold. Let $\mathcal{U}_{\tau_\infty}$ be an open polydisc containing $\tau_\infty \in \mathcal{D}_f$. Let $\mathcal{D}_{\tau_\infty} \subset \mathcal{U}_{\tau_\infty} \subset \mathcal{M}_L(M)$ be an open disk containing the point $\tau_\infty \in \mathcal{D}_f$. Then the monodromy operator of the family of polarized CY manifolds over $\mathcal{D}_{\tau_\infty}^\ast = \mathcal{D}_{\tau_\infty} - \tau_\infty$ is non-trivial group.

**Proof:** The subgroup $\Gamma$ of the mapping class group $\Gamma(M)$ acts on the completion $\widehat{T}(M)$ of the Teichmüller space $T(M)$ with respect to the Hodge metric. Then each of the components $D_i$ of $\sigma^{-1} \left( \Gamma \backslash \widehat{T}(M) - \Gamma \backslash T(M) \right)$ is a fixed point set of some subgroup $G_i$ of $\Gamma$. Let

$$\mathcal{X}_{\mathcal{D}_{\tau_\infty}} \rightarrow \mathcal{D}_{\tau_\infty}$$  \hspace{1cm} (23)

be a family of CY manifolds such that the $p(0) = \tau_\infty \in \mathcal{D}_\infty$. Then the above arguments show that monodromy group of the family is the stab ilizer of the point $\tau_\infty \in \mathcal{D}_i$. So Proposition 33 is proved. 

**Proposition 34** Let $M$ be a CY manifold. Let $\mathcal{U}_{\tau_\infty}$ be an open polydisc containing $\tau_\infty \in \mathcal{D}_f$. Let $\mathcal{D}_{\tau_\infty} \subset \mathcal{U}_{\tau_\infty} \subset \mathcal{M}_L(M)$ be an open disk containing the point $\tau_\infty \in \mathcal{D}_f$. Then then $L^2$ metric on the relative dualizing sheaf has growth

1. $|\tau - \tau_\infty|^{1/k}$ if the monodoromy operator $T$ is of finite order on the family $\mathcal{X}|_{\mathcal{D}_{\tau_\infty} - \tau_\infty}$,
2. $(\log |\tau - \tau_\infty|)^k$ if $T$ has an infinite order and $k > 1$. 

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Proof: Let $\omega_\tau$ be a family of holomorphic $n$–forms over the disk punctured disk $D_{\tau_\infty} - \tau_\infty$. We need to consider two cases:

Case 1. Suppose that the monodromy operator $T$ acting on $H^n(M, \mathbb{Z})$ is of infinite order. Let $\{\gamma_i\}$ be a basis of $H_n(M, \mathbb{Z})$ for $\tau \neq \tau_\infty$. Let us consider

$$\left(..., a_j(\tau) = \int_{\gamma_i} \omega_\tau, ...\right).$$

Then the components of (24) are solutions of a ordinary differential equation with regular singular points. Let $\{\gamma_i^*\}$ be the Dirac dual basis of $H_n(M, \mathbb{Z})$, i.e. $\langle \gamma_i^*, \gamma_j \rangle = \delta_{ij}$. Then $\omega_\tau = \sum_{j=1}^{b_n} a_j(\tau) \gamma_j^*$. It follows from Poincare duality that the $L^2$ metric is given by

$$(-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \int_M \omega_\tau \wedge \overline{\omega_\tau} =$$

$$\left(..., a_j(\tau) = \int_{\gamma_i} \omega_\tau, ...\right) \left(..., a_j(\tau) = \int_{\gamma_i} \overline{\omega_\tau}, ...\right).$$

The assumption that the monodromy operator is of infinite order implies that if $\gamma_0$ and $\gamma_k$ are cycles such that $T(\gamma_0) = \gamma_0$ and $(T^m - id)^k \gamma_k = \gamma_0$ we get that

$$a_k(\tau) = \int_{\gamma_k} \omega_\tau \wedge (\log(\tau - \tau_\infty))^k.$$

From the assumption that the monodromy operator is of infinite order, and from (25) and (30) we can conclude that

$$\lim_{\tau \to \tau_\infty} \left| \frac{\|\omega_\tau\|^2}{\log |\tau - \tau_\infty|} \right| = c < \infty.$$

So we proved Proposition 34 in the case the monodromy is of infinite order. So we need to consider the second case:

Case 2. Suppose that the monodromy operator $T$ acting on the middle cohomology $H^n(M, \mathbb{Z})$ for $\tau \in D_{\tau_\infty} - \tau_\infty$ has an order $k > 1$. Let $\gamma_0$ and $\gamma_1 \in H_n(M, \mathbb{Z})$, $\langle \gamma_0, \gamma_k \rangle \neq 0$, $\gamma_0$ is an invariant vanishing cycle of $T$, i.e. $T(\gamma_0) = \gamma_0$. Then according to [?] we have $T(\gamma_k) \neq \gamma_k$ and $T^k(\gamma_k) = \gamma_k$. Then according to the general theory of the monodromy we have that:

$$\lim_{\tau \to \tau_\infty} \int_{\gamma_0} \omega_\tau = 0 \quad \text{and} \quad \lim_{\tau \to \tau_\infty} \left(\int_{\gamma_0} \omega_\tau \wedge (\log(\tau - \tau_\infty))^k \right)^{1/k} = c_1.$$
Formulas (26) and (28) imply that

$$\lim_{\tau \to \tau_\infty} \frac{\|\omega_\tau\|^2}{|\tau - \tau_\infty|^{1/k}} = c_2.$$  

(29)

So (27) and (29) imply Proposition 34.

Proposition 35 Let $M$ be a CY manifold. Let $U_{\tau_\infty}$ be an open polydisc containing $\tau_\infty \in D_\infty$. Let $D_{\tau_\infty} \subset U_{\tau_\infty}$ be an open disk containing the point $\tau_\infty \in D_\infty$.

Then $\eta_{\tau_\infty}(\tau_\infty) = 0$.

Proof: Let $X_D \to D$ be a family of CY manifolds such that the $p(0) = \tau_\infty \in D_\infty$. We need to consider two cases:

Case 1. Suppose that the monodromy operator is of infinite order. Let $\eta^N$ be the section constructed in Theorem 22 and $\eta^N|_{U_{\tau_\infty}} = f_{U_{\tau_\infty}}(\tau)$. From the explicit expression for the $L^2$ norm of the section $\eta^N$ we see that

$$\|\eta^N(\tau)\|^2_{L^2|_{U_\tau}} = \frac{|f_{U_{\tau_\infty}}(\tau)|^2}{\|\omega_\tau\|^2_{L^2}} \leq c_0,$$

(30)

where $\omega_\tau$ is a family of holomorphic $n$-forms defined over the family of polarized CY manifolds obtained from the restriction of the versal family $X \to M$. Then Formulas (30), (29), and $\|\eta^N\|^2_{L^2} < \infty$ imply that $\lim_{\tau \to \tau_\infty} f_{U_{\tau}}(\tau) = 0$. Proposition 34 is proved.

Case 2. Suppose that the monodromy operator is of finite order. Let $\eta^N$ be the section constructed in Theorem 22 and $\eta^N|_{U_{\tau_\infty}} = f_{U_{\tau_\infty}}(\tau)$. Then Formulas (30), (29), and $\|\eta^N\|^2_{L^2} < \infty$ imply that $\lim_{\tau \to \tau_\infty} f_{U_{\tau}}(\tau) = 0$. Proposition 34 is proved.

Propositions 33 and 34 imply Lemma 32. Lemma 32 implies Theorem 29.

Problem 36 We constructed in Theorem 22 a holomorphic section $\eta^N$ of the prolonged line bundle $\omega^N_{X/\mathfrak{M}_L(M)}$ for any dimension to some compactification $\mathfrak{M}_L(M)$ of $\mathfrak{M}_L(M)$ such that $\mathfrak{M}_L(M) - \mathfrak{M}_L(M)$ is a divisor with a normal crossings. Is it true that the function defined by the regularized determinants $\det(\Delta_{0,1})$ of the Laplacians of the CY metrics with imaginary parts cohomological to the polarization class $L$ is bounded and $\|\eta^N\|^2_{L^2} = (\det(\Delta_{0,1}))^N$ and $N = \#\Gamma/|\Gamma, \Gamma|$.  

3.2 The Analogue of Cusp Automorphic Forms

Definition 37 Let $M$ be a projective variety. A line bundle $L$ on $M$ will be called big and nef if the Chern class $c_1(L)$ of $L$ satisfies the following conditions:
1. For every irreducible curve $C$ on $M$ we have
   \[ \int_C c_1 (L) \geq 0. \] (32)

2. The following inequality holds
   \[ \int_M \wedge^n c_1 (L) > 0. \] (33)

**Definition 38** A line bundle $L$ on a complete scheme $M$ is semi ample if $L^\otimes m$ is globally generated for some integer $m > 0$.

**Theorem 39** Let $M$ be a CY manifold. Let $\mathcal{M}_L(M)$ be the moduli space of polarized CY manifolds such that $\tau_0 \in \mathcal{M}_L(M)$ corresponds to $M$. Let $\omega_{X/\mathcal{M}_L(M)}$ be the relative dualizing sheaf of the family $X \to \mathcal{M}_L(M)$. Let
   \[ H^0_\infty \left( \mathcal{M}_L(M), \omega_{X/\mathcal{M}_L(M)}^\otimes N \right) := \left\{ s \in H^0_\infty \left( \mathcal{M}_L(M), \omega_{X/\mathcal{M}_L(M)}^\otimes N \right) \mid \|s\|_{L^2} < \infty \right\}. \]

Then the space $H^0_\infty \left( \mathcal{M}_L(M), \omega_{X/\mathcal{M}_L(M)}^\otimes N \right)$ is finite dimensional for any $N$ and for large enough $N$ the linear system $H^0_\infty \left( \mathcal{M}_L(M), \omega_{X/\mathcal{M}_L(M)}^\otimes N \right)$ defines a holomorphic inclusion
   \[ \phi_\infty : \mathcal{M}_L(M) \subset \mathbb{CP}^N_0 \]

such that $\phi_\infty (\mathcal{M}_L(M))$ is a Zariski open set.

**Proof:** Let $\overline{\mathcal{M}_L(M)}$ be any compactification of $\mathcal{M}_L(M)$ such that
   \[ \mathcal{D}_\infty := \overline{\mathcal{M}_L(M)} - \mathcal{M}_L(M) \]

is a divisor of normal crossing. Let $\omega_{X/\mathcal{M}_L(M)}$ be the line bundle on $\overline{\mathcal{M}_L(M)}$ such that
   \[ \omega_{X/\mathcal{M}_L(M)} \mid \mathcal{M}_L(M) = \omega_{X/\mathcal{M}_L(M)} \]

and for any $\tau_\infty \in \mathcal{D}_\infty$ and $U_{\tau_\infty}$ polydisk containing $\tau_\infty$ we have
   \[ \omega_{X/\mathcal{M}_L(M)} \mid _{U_{\tau_\infty}} := \left\{ s \in \Gamma \left( U_{\tau_\infty} - U_{\tau_\infty} \cap \mathcal{D}_\infty, \omega_{X/\mathcal{M}_L(M)} \right) \mid \|s\|_{L^2} < \infty \right\}. \]

We will need the following Lemma:

**Lemma 40** The line bundle $\omega_{X/\mathcal{M}_L(M)}$ on $\overline{\mathcal{M}_L(M)}$ is a big, nef and semi-ample.

**Proof:** According to the Definition 37 we need to show that the following Proposition:
Proposition 41. Let $C$ be any irreducible curve in $\mathcal{M}_L(M)$. Then we have:

$$\int_C c_1(\omega_{X/\mathcal{M}_L(M)}, h) \geq 0.$$ 

Proof: We proved in [24] that the Chern form of $\omega_{X/\mathcal{M}_L(M)}$ is the imaginary part of the Weil-Petersson metric of $\mathcal{M}_L(M)$. In [25] we proved that the Weil-Petersson metric is a good metric in the sense of Mumford. See [14]. According to [14] the Chern forms $c_k(E, g)$ of a good metric $g$ on a vector bundle $E$ define currents which are elements of $H^{2k}(\mathcal{M}_L(M), \mathbb{Z})$. Let us denote by $h_L^2$ the $L^2$ metric on $\omega_{X/\mathcal{M}_L(M)}$. Since the Chern form $c_1(\omega_{X/\mathcal{M}_L(M)}, h_L^2)$ of the $L^2$ metric $h_L^2$ on $\omega_{X/\mathcal{M}_L(M)}$ is the imaginary part of the Weil-Petersson metric and since $h_L^2$ is a good metric, then the current $c_1(\omega_{X/\mathcal{M}_L(M)}, h_L^2)$ is positive. This fact implies that for any irreducible curve $C$ on $\mathcal{M}_L(M)$ we have

$$\int_C c_1(\omega_{X/\mathcal{M}_L(M)}, h_L^2) \geq 0.$$ 

The last inequality implies Proposition 41. □

Theorem 2.1.27 proved on page 129 of [16] implies Theorem 39. Before formulating Theorem 2.1.27 we will introduce some notions. Let $L$ be some semi-ample line bundle. We will denote by $\mathcal{M}(X, L)$ the semi group $\mathcal{M}(X, L) = \{ m \in \mathbb{N} \mid L^\otimes m \text{ has no fixed points} \}$.

$f(\mathcal{L})$ will be the exponent of $M(X, \mathcal{L})$, i.e. the largest natural number such that every element of $M(X, \mathcal{L})$ is a multiple of $g(\mathcal{L})$. Given $m \in M(X, \mathcal{L})$ we will denote by $Y_m = \phi_m(X)$ the image of $X$ by the holomorphic map

$$\phi_m : X \rightarrow \phi_m(X) = Y_m \subset \mathbb{P}(H^0(X, \mathcal{L}^\otimes m)),$$

determined by the linear system $|\mathcal{L}^\otimes m|$.

Theorem 2.1.27 (semi-ample fibrations). Let $X$ be a normal projective space and let $L$ be a semi-ample line bundle on $X$. Then there is a fibre space $\phi : X \rightarrow Y$ having the property that for any sufficiently large $m \in M(X, L)$, $Y_m = Y$ and $\phi_m = \phi$. Moreover there is an ample line bundle $A$ on $Y$ such that $\mathcal{L} = \phi^*(A)$. Theorem 39 is proved. □
3.3 Extension Properties

**Theorem 42** Let $\phi : (D^*)^k \times D^m \to \mathcal{M}_L(M)$ be a holomorphic map. Then $\phi$ can be extended to a holomorphic map $\overline{\phi} : D^k \times D^m \to \mathcal{M}_L(M)$, where $\mathcal{M}_L(M)$ is the Baily-Borel compactification of $\mathcal{M}_L(M)$.

**Proof:** The proof of Theorem 42 is based on the following generalization of Schwarz Lemma due to S. -T. Yau;

**Theorem 43** Let $M$ be a complete Kähler manifold with a Ricci curvature bounded from below by a constant, and $N$ be another Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant. Then any holomorphic mapping from $M$ into $N$ decreases distances up to a constant depending only on the curvature of $M$ and $N$. See [28].

Theorem 42 follows from the following Lemma:

**Lemma 44** Let $\phi : (D^*)^k \times D^m \to \mathcal{M}_L(M)$ be a holomorphic map. Let $(z_k, w_k) \in (D^*)^k \times D^m$ such 
\[ \lim_{k \to \infty} (z_k, w_k) = (0, w_0) \in D^k \times D^m. \]

Then $\lim_{k \to \infty} \phi(z_k, w_k)$ exists and 
\[ \lim_{k \to \infty} \phi(z_k, w_k) \in \overline{\mathcal{M}_L(M)} \]

$\overline{\mathcal{M}_L(M)}$, where $\overline{\mathcal{M}_L(M)}$ is the Baily-Borel compactification of $\mathcal{M}_L(M)$.

**Proof:** Let us consider on $(D^*)^k \times D^m$ the Poincare metric:
\[ \sum_{i=1}^{k} \frac{dz^i \otimes d\overline{z}^i}{|z^i|^2 (\log |z^i|)^2} + \sum_{j=1}^{m} \frac{dw^j \otimes d\overline{w}^j}{1 - |w^j|^2}. \]

Then to apply Theorem 43 we replace $M$ with $(D^*)^k \times D^m$ with Poincare metric and replace $N$ with $\overline{\mathcal{M}_L(M)}$ with Hodge metric. Then Lemma 44 follows from Theorem 43 directly since the holomorphic map $\phi$ is a distance decreasing up to a constant, with respect to the Hodge metric on $\overline{\mathcal{M}_L(M)}$. Theorem 10 implies that the Hodge metric satisfies the conditions of Theorem 43. See [17].

Lemma 44 implies that the map $\phi : (D^*)^k \times D^m \to \mathcal{M}_L(M)$ can be extended to a continuous map:
\[ \overline{\phi} : D^k \times D^m \to \overline{\mathcal{M}_L(M)}. \]

Then according to Riemann extension Theorem (See Theorem 44.42 on p. 420 in [1]) $\overline{\phi}$ is a holomorphic map. Theorem 42 is proved. ■
Corollary 45 Let $Z$ be a quasi-projective variety. Suppose that 
\[ \phi : Z \rightarrow \mathcal{M}_L(M) \]
be a non trivial holomorphic map. Let $\overline{Z}$ be a projective manifold such that $\overline{Z} - Z$ is a divisor of normal crossings. Then $\phi$ can be extended to a holomorphic map 
\[ \tilde{\phi} : \overline{Z} \rightarrow \mathcal{M}_L(M). \]

Remark 46 Theorem 42 is a generalization of a Theorem of A. Borel proved for locally symmetric Hermitian spaces in [5]. Theorem 42 implies that $\overline{\mathcal{M}_L(M)}$ is a minimal model among all possible compactifications of the moduli space $\mathcal{M}_L(M)$ with a boundary divisors with normal crossings.

4 CY Threefolds

4.1 Invariants of the Short Term Asymptotic Expansion of the Heat Kernel

Theorem 47 Suppose that $M$ is a three dimensional CY manifold and $g$ is a CY metric. Then the coefficients $a_{2k}$ for $k = 3, 2, 1$ and 0 in the expression (34) are constants which depend only on the CY manifolds and the fixed class of cohomology of the CY metric.

Proof: We know that the Heat kernel has the following asymptotic expansion:

\[ \text{Tr}(\exp(-t\Delta_0)) = \frac{a_{-n}(g)}{t^{n}} + \frac{a_{-n+1}(g)}{t^{n-1}} + \frac{a_{-n+2}(g)}{t^{n-2}} + ... + a_0(g) + h(t, \tau, \overline{\tau}). \] (34)

(See [20].) We will apply (34) for three dimensional CY manifolds. In [11] on page 118 one can find the following formulas for $a_{-3}(g), a_{-2}(g)$ and $a_{-1}(g)$:

\[ \alpha_{-3}(g) = \frac{\text{vol}(g)}{4\pi}, \quad a_{-2}(g) = \frac{-\int k(g) \text{vol}(g)}{24\pi} \]
and

\[ a_{-1}(g) = \frac{-12\left( \int_M \Delta_g(k(g)) \text{vol}(g) \right) + 5\|Ric(g)\|^2 - 2\|R(g)\|^2}{1440\pi} \] (35)

where $k(g)$ is the scalar curvature of the metric $g$, $\|Ric(g)\|^2$ is the $L^2$ norm of the Ricci tensor of $g$ and $\|R(g)\|^2$ is the $L^2$ norm of the curvature of the metric $g$. Using the fact that $g$ is a Calabi-Yau metric, i.e. $Ric(g) = k(g) = 0$, we obtain:

\[ \alpha_{-n}(g) = \frac{\text{vol}(g)}{4\pi}, \quad a_{-1}(g) = 0 \text{ and } a_{-2}(g) = \frac{-\|R(g)\|^2}{720\pi}. \] (36)

In [6] Calabi proved on page 264 the following Proposition:
**Proposition 48** The following formula holds on a complex Kähler manifold $M$ with a fixed cohomology class $L$ of the imaginary part of a Kähler metric:

$$2 \| \text{Ric}(g) \|^2 - \| R(g) \|^2 - \int_M k(g)^2 \text{vol}(g) = - \int_M c_2(M) \wedge \omega_g^{n-2}, \quad (37)$$

where $c_2(M)$ is the second Chern class of $M$.

Applying formula (37) to a CY metric, we obtain that on a three dimensional CY manifold with a Calabi-Yau metric $g$ we have:

$$a_{-n}(g) = \frac{1}{4\pi} \int_M L^n, \quad a_{n-1}(g) = 0 \quad \text{and} \quad a_{n-2}(g) = -\frac{1}{720\pi} \int_M c_2(M) \wedge L^{n-2}. \quad (38)$$

Theorem 47 follows directly from (38) since (38) implies that $a_{-1}(g)$, $a_{-2}(g)$ and $a_{-3}(g)$ are topological invariants. We need to prove that $a_{0}(g)$ is a constant in order to deduce Theorem 47.

**Lemma 49** Let

$$T_r(\exp(-t\Delta_0) = \frac{a_{-n}}{t^n} + \frac{a_{-1}}{t} + a_0 + O(t)$$

be the asymptotic expansion of $T_r(\exp(-t\Delta_0)$ with respect to a CY metric with a fixed class of cohomology of its imaginary part. Then the real coefficient $a_0$ is a constant, i.e.

$$\frac{\partial}{\partial \tau} a_0(\tau, \tau) = \frac{\partial}{\partial \tau} a_0(\tau, \tau) = 0.$$

**Proof:** According to [7] the following equality is true:

$$\zeta_0,\tau(0) = a_0(\tau), \quad (39)$$

where $a_0$ is a real valued function on the moduli space of polarized CY manifolds. If we prove that:

$$\frac{\partial}{\partial \tau} (\zeta_0,\tau(s)) \bigg|_{s=0} = 0, \quad (40)$$

then Lemma 49 will follow directly from (39). In the paper [4] on page 85 the following formula was proved:

$$\frac{\partial}{\partial \tau^s} (\zeta_0,\tau(s)) =$$

$$\frac{1}{\Gamma(s)} \int_0^\infty Tr \left( \exp(-t\Delta_{\tau,0}) \circ \Delta_{\tau}^{-1} \circ F'(1, \frac{\partial}{\partial \tau^s} \phi(\tau)) \circ \Delta_{\tau} \right) t^s dt.$$

repeating word by word the proof the above formula from [4] we get:

$$\frac{\partial}{\partial \tau^s} (\zeta_0,\tau(s)) =$$
\[
\frac{1}{\Gamma(s)} \int_0^\infty Tr \left( \frac{d}{dt} \left( \exp \left( -t \triangle_{\tau,0} \right) \right) \circ (\partial_\tau)^{-1} \circ F'(q, \partial_{\tau^1}) \circ \partial_{\tau} \right) t^s dt. \tag{41}
\]

By integrating by parts the expressions in (41), we obtain:

\[ \frac{\partial}{\partial_{\tau^i}} (\zeta_{0,\tau}(s)) = \]

\[ \frac{s}{\Gamma(s)} \int_0^\infty Tr \left( \exp \left( -t \triangle_{\tau,0} \right) \circ (\partial_\tau)^{-1} \circ F'(q, \partial_{\tau^1}) \circ \partial_{\tau} \right) t^{s-1} dt. \tag{42} \]

We can rewrite the integral in the right hand side of (42) as follows:

\[ \frac{s}{\Gamma(s)} \int_0^\infty Tr \left( \exp \left( -t \triangle_{\tau,0} \right) \circ (\partial_\tau)^{-1} \circ F'(q, \partial_{\tau^1}) \circ \partial_{\tau} \right) t^{s-1} dt = \]

\[ \frac{s}{\Gamma(s)} \int_0^1 Tr \left( \exp \left( -t \triangle_{\tau,0} \right) \circ (\partial_\tau)^{-1} \circ F'(q, \partial_{\tau^1}) \circ \partial_{\tau} \right) t^{s-1} dt + \]

\[ \frac{s}{\Gamma(s)} \int_1^\infty Tr \left( \exp \left( -t \triangle_{\tau,0} \right) \circ (\partial_\tau)^{-1} \circ F'(q, \partial_{\tau^1}) \circ \partial_{\tau} \right) t^{s-1} dt. \tag{43} \]

From the short term asymptotic expansion

\[ Tr \left( \exp \left( -t \triangle_{\tau,0} \right) \circ (\partial_\tau)^{-1} \circ F'(q, \partial_{\tau^1}) \circ \partial_{\tau} \right) = \]

\[ \frac{c_{-k}(\tau)}{t^k} + \ldots + \frac{c_{-1}(\tau)}{t} + c_0(\tau) + O(t) \tag{44} \]

we obtain that

\[ \frac{s}{\Gamma(s)} \int_0^1 Tr \left( \exp \left( -t \triangle_{\tau,0} \right) \circ (\partial_\tau)^{-1} \circ F'(q, \partial_{\tau^1}) \circ \partial_{\tau} \right) t^{s-1} dt = \]

\[ \frac{s}{\Gamma(s)} \left( \frac{c_0(\tau)}{s} + \gamma_0(\tau) + O(s) \right). \tag{45} \]

From (45) and the fact that \( \frac{s}{\Gamma(s)} = s^2 + O(s^3) \) we get that

\[ \frac{\partial}{\partial_{\tau^1}} (\zeta_{0,\tau}(s)) = s^2 \left( \frac{c_0(\tau)}{s} + \gamma_0(\tau) + O(s) \right). \]

From the last formula we obtain that

\[ \left( \frac{\partial}{\partial_{\tau^1}} (\zeta_{0,\tau}) \right)(0) = 0. \]

Lemma 49 is proved. ■

Lemma 49 implies Theorem 47. ■
4.2 The Regularized CY Determinants are Bounded

**Theorem 50** For CY threefolds the regularized determinants of the Laplacians $\Delta_{\tau}^{q}$ of the Calabi Yau metrics $g(\tau, \tau)$ with a fixed cohomology class $L$ for $0 \leq q \leq n = \dim_{\mathbb{C}} M$ are bounded as functions on the moduli space, i.e. we have: $0 \leq \det(\Delta_{\tau,q}) \leq C_{q}$.

**Proof:** We will outline the main ideas of the proof of Theorem 50. In order to prove Theorem 50 it is enough to bound $\det(\Delta_{0})$. The bound of $\det(\Delta_{0})$ is based on the following expression for the zeta function of the Laplacian acting on functions:

$$\zeta_{0}(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} (\text{Tr}(\exp(-t\Delta_{0}))t^{s-1}dt = b_{0} + b_{1}s + O(s^{2}).$$

From the definition of $\det(\Delta_{0})$ it follows that

$$\det(\Delta_{0}) = \exp(-b_{1}). \quad (46)$$

So if $b_{1}$ is bounded from below, i.e.

$$c_{1} \leq b_{1} \quad (47)$$

then Theorem 50 will be proved. The bound of $b_{1}$ is based on several facts.

1. The following explicit formula for $b_{1}$ is proved in [2]:

$$b_{1} = \gamma a_{0} + \sum_{k=1}^{3} \frac{a_{-k}}{k} + \psi_{1} + \psi_{2}, \quad (48)$$

where $\gamma$ is the Euler constant, $\psi_{1}$ is given by the formula

$$\psi_{1}(t, \tau, \tau) = \int_{0}^{1} \left( \text{Tr}(\exp(-t\Delta_{0})) - \sum_{k=0}^{3} \frac{a_{-k}}{tk} \right) \frac{dt}{t}$$

and $\psi_{2}$ by

$$\psi_{2}(t, \tau, \tau) = \int_{1}^{\infty} \text{Tr}(\exp(-t\Delta_{0})) \frac{dt}{t}.$$  

2. Theorem 47 implies that the expression: $\gamma a_{0} + \sum_{k=1}^{3} \frac{a_{-k}}{k}$ in (48) is a constant. Clearly $\psi_{2}(t, \tau, \tau) > 0$.

3. The third fact is that $\psi_{1}(t, \tau, \tau) \geq c_{0}$, where $c_{0}$ is a constant.

Combining all these facts we get [17], [17] and the explicit formula [18] will imply that $0 \leq \det(\Delta_{0}) \leq C < \infty$. So we need to prove the following Lemma:
Lemma 51 The following inequality holds: $\psi_1(t, \tau, \overline{\tau}) \geq c_0$.

**Proof:** Let

$$h(t, \tau, \overline{\tau}) = \text{Tr}(\exp(-t\triangle_0)) - \sum_{k=0}^{3} \frac{a_{-k}}{t^k}$$

We also know that $h(t, \tau, \overline{\tau}) = th_1(t, \tau, \overline{\tau})$. According to Theorem 47, the expression $\sum_{i=1}^{3} \frac{a_{-k}}{t^k}$ is a function which does not depend on $\tau$ and $\overline{\tau}$. We also know that $\text{Tr}(\exp(-t\triangle, 0))$ is a strictly positive function for $t > 0$ which depends on $t, \tau$ and $\overline{\tau}$. Thus $\inf_{\tau} (\text{Tr}(\exp(-t\triangle, 1)))$ exists for $t > 0$. Let

$$\phi(t) := \inf_{\tau} (\text{Tr}(\exp(-t\triangle, 1))) - \sum_{k=0}^{3} \frac{a_{-k}}{t^k}$$

for $0 < t \leq 1$. Then

$$\phi(t) = \inf_{\tau} (\text{Tr}(\exp(-t\triangle, 1))) - \sum_{k=0}^{3} \frac{a_{-k}}{t^k} =$$

$$\inf_{\tau} \left(\text{Tr}(\exp(-t\triangle, 1)) - \sum_{k=0}^{3} \frac{a_{-k}}{t^k}\right) = \inf_{\tau} (h(t, \tau, \overline{\tau})).$$

**Proposition 52** $\phi(t)$ is a continuous function for $0 \leq t \leq 1$.

**Proof:** Let $0 \leq t_0 \leq 1$ be a fixed number. Let $\{t_n\}$ be a sequence of numbers such that $0 < t_n < 1$ such that $\lim_{n \to \infty} t_n = t_0$ then we need to prove that

$$\lim_{n \to \infty} \phi(t_n) = \lim_{n \to \infty} \left(\inf_{\tau} (h(t_n, \tau, \overline{\tau}))\right) = \inf_{\tau} (h(t_0, \tau, \overline{\tau})) = \phi(t_0).$$

The definition of the function $h(t, \tau, \overline{\tau})$ show that $h(t, \tau, \overline{\tau})$ is a continuous function such that $h(0, \tau, \overline{\tau}) = 0$. The definition of $\inf_{\tau} h(t, \tau, \overline{\tau})$ implies that there exists a sequence $\{\tau_n\}$ of points $\tau_n \in \mathcal{M}_L(M)$ such that

$$\lim_{n \to \infty} (h(t, \tau_n, \overline{\tau})) = \inf_{\tau} (h(t, \tau, \overline{\tau})).$$

Since $h(t, \tau, \overline{\tau})$ is a continuous function we get that

$$\lim_{n \to \infty} \left(\lim_{k \to \infty} (h(t_k, \tau_n, \overline{\tau}))\right) = \lim_{n \to \infty} (h(t_0, \tau_n, \overline{\tau})) = \inf_{\tau} (h(t, \tau, \overline{\tau})).$$

On the other hand we have

$$\lim_{n \to \infty} \left(\lim_{k \to \infty} (h(t_k, \tau_n, \overline{\tau}))\right) = \lim_{k \to \infty} \left(\lim_{n \to \infty} (h(t_k, \tau_n, \overline{\tau}))\right) = \lim_{k \to \infty} \left(\inf_{\tau} (h(t, \tau, \overline{\tau}))\right).$$
Combining (52) and (53) we get that
\[
\lim_{k \to \infty} \left( \inf_{\tau} (h(t_k, \tau, \tau)) \right) = \lim_{k \to \infty} \phi(t_k) = \inf_{\tau} (h(t_0, \tau, \tau)) = \phi(t_0).
\]
Proposition 52 is proved.

Since \( \phi(t) \) is a continuous function on the closed interval \([0, 1]\) it has a minimum. Let \( c_0 = \min_{0 \leq t \leq 1} \phi(t) \). So we have
\[
\int_0^1 (h(t, \tau, \tau) - c_0) dt \geq 0.
\]
On the other hand, we have
\[
\int_0^1 (h(t, \tau, \tau) - c_0) dt = \int_0^1 h(t, \tau, \tau) dt - c_0 = \psi_1(t, \tau, \tau) - c_0 \geq 0.
\]
So \( \psi_1(t, \tau, \tau) \geq c_0 \). Lemma 51 is proved.

Theorem 50 is proved.

4.3 The Existence of Global Section \( \eta^{\otimes N} \) of the Relative Dualizing Sheaf with Finite \( \mathbb{L}^2 \) Norm \( (\det(\Delta_{0,1}))^N \)

**Theorem 53** Let \( M \) be a three dimensional CY manifold and let \( N = \#\Gamma / [\Gamma, \Gamma] \). Then there exists a section \( \eta^N \) of the line bundle \( \left( \omega^{\otimes N}_{X/\mathfrak{M}_L(M)} \right) \) such that the norm of \( \eta^N \) with respect to the \( N \) tensor power of the \( \mathbb{L}^2 \) metric on \( \omega^{\otimes N}_{X/\mathfrak{M}_L(M)} \) is given by: \( \|\eta^N\|_{\mathbb{L}^2}^2 = (\det(\Delta_{0,1}))^N \). The zero set of \( \eta^N \) is a non zero effective divisor whose support contains or is equal to the support of \( \mathcal{D}_\infty \), where \( \mathcal{D}_\infty \) is defined in Definition 28.

**Proof:** Theorem 24 implies that there exists a the section \( \eta^{\otimes N} \) of the relative dualizing line bundle \( \omega^{\otimes N}_{X/\mathfrak{M}_L(M)} \) on \( \mathfrak{M}_L(M) \) such that \( \|\eta^N\|_{\mathbb{L}^2}^2 = \det(\Delta_{0,1}(\tau)) \) and \( \|\eta^N\|_{\mathbb{L}^2}^2 > 0 \) for \( \tau \in \mathfrak{M}_L(M) \). Now we can apply Propositions 33 34 and 35 to conclude the zero set \( \left( \eta^{\otimes N} \right)_0 \) is the effective divisor \( \mathcal{D}_\infty \) defined by Definition 28. Theorem 53 is proved.

**References**

[1] S. Abhyankar, "Local Analytic Geometry", Academic Press, New York, 1964.

[2] D. Abramovich, J. -F. Burnol, J. Kramer and C. Soulé, "Lectures on Arakelov Geometry", Cambridge Studies In Advanced Mathematics Volume 33, Cambridge University Press, 1992.
[3] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitude", Comm. Math. Phys. 165 (1994), 311-428.

[4] J. Bass, A. Todorov, "The Analogue of the Dedekind Eta Function for CY Manifolds I", Journal fur die reine und angewandte Mathematik (Crelles Journal), v. 599, 61-96 (2006).

[5] A. Borel, "Some Metric Properties of Arithmetic Quotients of Symmetric Spaces and Extension Theorem", J. Diff. Geometry, vol. 6 (1972), 543-560.

[6] E. Calabi, "Extremal Kähler Metrics", Seminar on Differential Geometry, ed. S.-T. Yau, Annals of Mathematical Studies, vol. 102, Princeton University Press.

[7] N. Berline, E. Getzler and M. Vergne, "Heat Kernels and Dirac Operators", Springer Verlag, 1991.

[8] A. A. Kirillov and Cl. Delaroche, "Sur les Relations entre l’espace dual d’un group et la struture de ses sous-groupes fermés" (d’après D. A. Kazhdan), Bourbaki No. 343, 1967/68.

[9] S. Donaldson, "Infinite Determinants, Stable Bundles and Curvature", Duke Mathematical Journal, vol. 54, Number 1 (1987), 231-248.

[10] S. Donaldson and P. Kronheimer, "The Geometry of Four Manifolds", Oxford Mat. Monog., Oxford Science Publications, Oxford University Press, New York 1990.

[11] P. Gilkey, "Invariance Theory, The Heat Equation, And the Atiyah-Singer Index Theorem", Mathematics Lecture Series vol. 11, Publish or Parish, Inc. Wilmington, Delaware (USA) 1984.

[12] J. Jorgenson and A. Todorov, "A Conjectural Analogue of Dedekind Eta Function for K3 Surfaces", Math. Research Lett. 2(1995) 359-360.

[13] J. Jorgenson and A. Todorov, "Analytic Discriminant for Polarized Algebraic K3 Surfaces", Mirror Symmetry III, ed. S-T. Yau and Phong, AMS, p. 211-261.

[14] D. Mumford, "Hirzebruch's Proportionality Principle in the Non-Compact Case", Inv. Math. 42(1977), 239-272.

[15] K. Kodaira and Morrow, "Complex Manifolds".

[16] R. Lazarsfeld, "Positivity in Algebraic geometry I and II", Springer, 2004.

[17] K. Liu, X. Sun, A. Todorov, Shing-Tung Yau, "The Moduli Space of CY Manifolds II (Existence of Kähler-Einstein Metrics on the Moduli Space of CY Manifolds)", preprint.
[18] K. Liu, A. Todorov, Shing-Tung Yau and K. Zuo, "Shafarevich Conjecture for CY Manifolds I" (Moduli of CY Manifolds), Quarterly Journal of Pure and Applied Mathematics, vol. 1.

[19] Z. Lu and X. Sun, "The Weil-Petersson Volume of the Moduli Space of Calabi-Yau Manifolds", Comm. Math. Phys., vol. 261, No 2, p. 297–322, 2006.

[20] John Roe, "Elliptic Operators, Topology and Asymptotic Methods" Pitman Research Notes in Mathematics Series 179, Longman Scientific & Technical, 1988.

[21] D. Ray and I. Singer, "Analytic Torsion for Complex Manifolds", Ann. Math. 98 (1973) 154-177.

[22] D. Sullivan, "Infinitesimal Computations in Topology", Publ. Math. IHES, No 47 (1977), 269-331.

[23] G. Tian, "Smoothness of the Universal Deformation Space of Calabi-Yau Manifolds and its Petersson-Weil Metric", Math. Aspects of String Theory, ed. S.-T. Yau, World Scientific (1998), 629-346.

[24] A. Todorov, "The Weil-Petersson Geometry of Moduli Spaces of SU(n ≥ 3) (Calabi-Yau Manifolds) I", Comm. Math. Phys. 126 (1989), 325-346.

[25] A. Todorov, "Weil-Petersson Volumes of the Moduli Spaces of CY Manifolds", Communication in Analysis and geometry, Vol. 15, No2, p. 407-434.

[26] E. Viehweg, "Quasi-Projective Moduli for Polarized Manifolds", Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, Band 30, Springer-Verlag, 1991.

[27] S. T. Yau, "On the Ricci Curvature of Compact Kähler Manifolds and Complex Monge-Ampère Equation I", Comm. Pure and App. Math. 31(1979), 339-411.

[28] Shing-Tung Yau, "A General Schwarz Lemma for Kahler Manifolds", American Journal of Mathematics, Vol. 100, No. 1 (Feb., 1978), pp. 197-203.