Classes of Delay-Independent Multimessage Multicast Networks under the Generalized Model

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Abstract

In a network, a node is said to incur a delay if its encoding of each transmitted symbol involves only its received symbols obtained before the time slot in which the transmitted symbol is sent (hence the transmitted symbol sent in a time slot cannot depend on the received symbol obtained in the same time slot). A node is said to incur no delay if its received symbol obtained in a time slot is available for encoding its transmitted symbol sent in the same time slot. Under the classical model, the discrete memoryless multimessage multicast network (MMN) is characterized by one channel and every node in the network incurs a delay. On the other hand, under the generalized model, the MMN can be characterized by multiple channels and some nodes in the network may incur no delay. In this paper, we obtain the capacity regions for three classes of MMN under the generalized model, namely the deterministic network dominated by product distribution, the MMN consisting of independent DMCs and the wireless erasure network. In addition, we show that for any MMN belonging to one of the above three classes, the set of achievable rate tuples under the generalized model and under the classical model are the same, which implies that the set of achievable rate tuples for the MMN does not depend on the amount of delay incurred by each node in the network.

Index Terms

capacity region, classical model, delay-independent, generalized model, multimessage multicast network (MMN).

I. INTRODUCTION

In a multimessage multicast network (MMN), each source sends a message and each destination wants to decode all the messages. The set of source nodes and the set of destination nodes may not be disjoint. A node in the network is said to incur a delay if its encoding of each transmitted symbol involves only its received symbols obtained before the time slot in which the transmitted symbol is sent. In constrast, a node is said to incur no delay if its received symbol obtained in a time slot is available for encoding its transmitted symbol sent in the same time slot. The discrete memoryless MMN (DM-MMN) under the classical model [1, Chapter 18.4], which we will abbreviate as classical MMN, is characterized by one channel and every node in the network incurs a delay. On the other hand, the DM-MMN under our generalized model [2], which we will abbreviate as generalized MMN, can be characterized by multiple channels and some nodes in the network may incur no delay. For the generalized...
MMN, the input symbols of each channel may consist of some symbols transmitted by the nodes as well as some symbols output from the other channels while for the classical MMN, the input symbols of the channel consist of only symbols transmitted by the nodes.

The capacity region of the generalized MMN is defined to be the set of rate tuples achievable by all feasible schemes that do not include deadlock loops, and the positive-delay region [3] is defined to be the set of rate tuples achievable by all classical schemes under the constraint that each node incurs a delay (and hence deadlock loops are automatically excluded). By this definition, the capacity region always contains the positive-delay region as a subset. In particular, the capacity region is strictly larger than the positive-delay region for the relay-without-delay channel studied by El Gamal et al. [4], the causal relay network studied by Baik and Chung [5] and the BSC with correlated feedback studied by Fong and Yeung [3]. Therefore, we are motivated to classify the set of generalized MMNs into the following two categories:

(i) **Delay-independent MMNs** whose capacity regions coincide with their positive-delay regions.

(ii) **Delay-dependent MMNs** whose capacity regions are strictly larger than their positive-delay regions.

For each generalized MMN in Category (i), the set of achievable rate tuples does not depend on the amount of delay incurred by each node in the network. On the other hand, for each generalized MMN in Category (ii), the set of achievable rate tuples shrinks if we impose the additional constraint that each node incurs a positive delay. It is important to decide which category a given generalized MMN belongs to because the category of the MMN affects how the delays should be handled and how the transmissions in the network should be synchronized to achieve optimal performance.

A. Main Contribution

The main contribution of this work is the identification of three classes of delay-independent MMNs and the complete characterization of their capacity regions. The first class is called the deterministic MMN dominated by product distribution. Being a subclass of MMNs consisting of deterministic channels, the deterministic MMN dominated by product distribution is a generalization of the deterministic relay network with no interference in [6] and the finite-field linear deterministic network in [7,8]. The second class is called the MMN consisting of independent DMCs, which is a generalization of the classical MMN in [9]. The third class is called the wireless erasure network, which is a generalization of the classical wireless erasure network in [10]. We successfully evaluate the capacity regions for the above classes of generalized MMNs and show that the capacity regions coincide with the positive-delay regions, which implies that the above classes of MMNs belong to the category of delay-independent MMNs. A natural consequence of our result is that for any generalized MMN belonging to one of the above three classes, using different methods for handling delays and synchronization in the network does not affect the capacity region.

Given a generalized MMN belonging to one of the above three classes, in order to show its delay-independence, we first evaluate an achievable rate region for the generalized MMN by utilizing the noisy network coding (NNC) inner bound [11, Theorem 1]. The achievable rate region is contained in the positive-delay region because the NNC
inner bound was proved in [11] for classical MMNs. Then, we evaluate an outer bound on the capacity region of
the generalized MMN by simplifying the cut-set outer bound in [2, Theorem 1] and show that the cut-set outer
bound coincides with the NNC inner bound (which is within the positive-delay region), implying that the MMN is
delay-independent.

This work should not be confused with the work by Effros [12] under the classical model, which shows that
the set of achievable rate tuples for any MMN does not depend on the amount of positive delay incurred by each
node. Here, we prove a different result for the above classes of MMNs that their capacity regions and positive-
delay regions are the same under the generalized model. Our result is meaningful given the fact that for some
generalized MMN, the capacity region is strictly larger than the positive-delay region (see the second paragraph of
the introduction for more detail).

B. Paper Outline

This paper is organized as follows. Section II presents the notation used in this paper. Section III presents the
formulation of the generalized MMN. Section IV recapitulates the NNC inner bound and the cut-set outer bound
for the capacity region of the generalized MMN. In Section V, we use the two bounds obtained in Section IV to
identify the three classes of delay-independent MMNs – the deterministic MMN dominated by product distributions,
the MMN consisting of independent DMCs and the wireless erasure network, whose problem formulations and
proofs for delay-independence are contained in Section V-A, Section V-B and Section V-C respectively. Section VI
concludes this paper.

II. Notation

We use \( \Pr \{ E \} \) to represent the probability of an event \( E \). We use a capital letter \( X \) to denote a random variable
with alphabet \( \mathcal{X} \), and use the small letter \( x \) to denote the realization of \( X \). We use \( X^n \) to denote a random tuple
\((X_1, X_2, \ldots, X_n)\), where the components \( X_k \) have the same alphabet \( \mathcal{X} \). We let \( p_X \) and \( p_Y|X \) denote the probability
mass distribution of \( X \) and the conditional probability mass distribution of \( Y \) given \( X \) respectively for any discrete
random variables \( X \) and \( Y \). We let \( p_X(x) \) be the evaluation of \( p_X \) at \( X = x \) and \( p_Y|X(y|x) \) be the evaluation
of \( p_Y|X \) at \( Y = y \) and \( X = x \). We let \( p_X p_Y|X(x, y) \) denote the joint distribution of \((X, Y)\), i.e.,
\( p_X p_Y|X(x, y) = p_X(x) p_Y|X(y|x) \) for all \( x \) and \( y \). If \( X \) and \( Y \) are independent, their joint distribution is simply
\( p_X p_Y \). We will take all logarithms to base 2. For any discrete random variable \((X, Y, Z)\) distributed according to
\( p_{X,Y,Z} \), we let \( H_{P_{X,Z}}(X|Z) \) and \( I_{p_{X,Y,Z}}(X;Y|Z) \) be the entropy of \( X \) given \( Z \) and mutual information between
\( X \) and \( Y \) given \( Z \) respectively. For simplicity, we drop the subscript of a notation if there is no ambiguity. If \( X \), \( Y \)
and \( Z \) are distributed according to \( p_{X,Y,Z} \) and they form a Markov chain, we write \((X \rightarrow Y \rightarrow Z)_{p_{X,Y,Z}} \) or more
simply, \((X \rightarrow Y \rightarrow Z)_{p} \). The sets of natural and real numbers are denoted by \( \mathbb{N} \) and \( \mathbb{R} \) respectively. The closure
of a set \( S \) is denoted by \( \text{closure}(S) \).
III. DISCRETE MEMORYLESS MULTIMESSAGE MULTICAST NETWORK

We consider a DM-MMN that consists of $N$ nodes. Let

$$\mathcal{I} \triangleq \{1, 2, \ldots, N\}$$

be the index set of the nodes, and let $\mathcal{V} \subseteq \mathcal{I}$ and $\mathcal{D} \subseteq \mathcal{I}$ be the sets of sources and destinations respectively. We call $(\mathcal{V}, \mathcal{D})$ the multicast demand on the network. The sources in $\mathcal{V}$ transmit information to the destinations in $\mathcal{D}$ in $n$ time slots (channel uses) as follows. Node $i$ transmits message

$$W_i \in \{1, 2, \ldots, M_i\}$$

for each $i \in \mathcal{V}$ and node $j$ wants to decode $\{W_i : i \in \mathcal{V}\}$ for each $j \in \mathcal{D}$. We assume that each message $W_i$ is uniformly distributed over $\{1, 2, \ldots, M_i\}$ and all the messages are independent. For each $k \in \{1, 2, \ldots, n\}$ and each $i \in \mathcal{I}$, node $i$ transmits $X_{i,k} \in \mathcal{X}_i$ and receives $Y_{i,k} \in \mathcal{Y}_i$ in the $k$th time slot where $\mathcal{X}_i$ and $\mathcal{Y}_i$ are some alphabets that depend on $i$. After $n$ time slots, node $j$ declares $\hat{W}_{i,j}$ to be the transmitted $W_i$ based on $(W_j, Y^n_j)$ for each $(i, j) \in \mathcal{V} \times \mathcal{D}$.

To simplify notation, we use the following conventions for each $T \subseteq \mathcal{I}$: For any random tuple

$$(X_1, X_2, \ldots, X_N) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_N,$$

we let

$$X_T \triangleq (X_i : i \in T)$$

be a subtuple of $(X_1, X_2, \ldots, X_N)$. Similarly, for any $k \in \{1, 2, \ldots, n\}$ and any random tuple

$$(X_{1,k}, X_{2,k}, \ldots, X_{N,k}) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_N,$$

we let

$$X_{T,k} \triangleq (X_{i,k} : i \in T)$$

be a subtuple of $(X_{1,k}, X_{2,k}, \ldots, X_{N,k})$. For any $N^2$-dimensional random tuple $(\hat{W}_{1,1}, \hat{W}_{1,2}, \ldots, \hat{W}_{N,N})$, we let

$$\hat{W}_{T \times T'} \triangleq (\hat{W}_{i,j} : (i,j) \in T \times T')$$

be a subtuple of $(\hat{W}_{1,1}, \hat{W}_{1,2}, \ldots, \hat{W}_{N,N})$.

Similar to the formulation of the generalized discrete memoryless network in [2], we formally define the generalized DM-MMN in the following six definitions.

**Definition 1:** An $\alpha$-dimensional tuple $(S_1, S_2, \ldots, S_\alpha)$ consisting of subsets of $\mathcal{I}$ is called an $\alpha$-partition of $\mathcal{I}$ if

$$\bigcup_{h=1}^{\alpha} S_h = \mathcal{I} \text{ and } S_i \cap S_j = \emptyset \text{ for all } i \neq j.$$

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For any \((S_1, S_2, \ldots, S_\alpha)\) which is an \(\alpha\)-partition of \(I\), we let
\[
S^h \triangleq \bigcup_{i=1}^h S_i
\]
for each \(h \in \{1, 2, \ldots, \alpha\}\) to facilitate discussion.

**Definition 2:** The discrete network consists of \(N\) finite input sets \(X_1, X_2, \ldots, X_N\), \(N\) finite output sets \(Y_1, Y_2, \ldots, Y_N\) and \(\alpha\) channels characterized by conditional distributions \(q^{(1)}_{Y_1} | X_{S_1}, q^{(2)}_{Y_2} | X_{S_2}, Y_{G_1}, \ldots, q^{(\alpha)}_{Y_\alpha} | X_{S_\alpha}, Y_{G_{\alpha-1}}\), where
\[
S \triangleq (S_1, S_2, \ldots, S_\alpha)
\]
and
\[
G \triangleq (G_1, G_2, \ldots, G_\alpha)
\]
are two \(\alpha\)-dimensional partitions of \(I\). We call \(S\) and \(G\) the input partition and the output partition of the network respectively. The discrete network is denoted by \((X_I, Y_I, \alpha, S, G, q)\) where
\[
q \triangleq (q^{(1)}, q^{(2)}, \ldots, q^{(\alpha)}).
\]

**Definition 3:** A delay profile is an \(N\)-dimensional tuple \((b_1, b_2, \ldots, b_N)\) where \(b_i \in \{0, 1, 2, \ldots\}\) for each \(i \in I\). The delay profile is said to be positive if its elements are all positive.

When we formally define a code on the discrete network later, a delay profile \(B = (b_1, b_2, \ldots, b_N)\) will be associated to the code and \(b_i\) represents the amount of delay incurred by node \(i\) for the code. Under the classical model, \(B\) can only be positive, meaning that the amount of delay incurred by each node is positive. In contrast, under our generalized model some elements of \(B\) can take 0 as long as deadlock loops do not occur. Therefore our model is a generalization of the classical model. The essence of the following definition is to characterize delay profiles which will not cause deadlock loops for the transmissions in the network.

**Definition 4:** Let \((X_I, Y_I, \alpha, S, G, q)\) be a discrete network. For each \(i \in I\), let \(h_i\) and \(m_i\) be the two unique integers such that \(i \in S_{h_i}\) and \(i \in G_{m_i}\). Then, a delay profile \((b_1, b_2, \ldots, b_N)\) is said to be feasible for the network if the following holds for each \(i \in I\): If \(b_i = 0\), then \(h_i > m_i\).

Under the classical model, Definition 4 is trivial because any delay profile is positive and hence feasible for the network. We are ready to define codes that use the network \(n\) times as follows.

**Definition 5:** Let \(B \triangleq (b_1, b_2, \ldots, b_N)\) be a delay profile feasible for \((X_I, Y_I, \alpha, S, G, q)\), and let \((V, D)\) be the multicast demand on the network. A \((B, n, M_I)\)-code, where \(M_I \triangleq (M_1, M_2, \ldots, M_N)\) denotes the tuple of message alphabets, for \(n\) uses of the network consists of the following:

1) A message set
\[
W_i \triangleq \{1, 2, \ldots, M_i\}
\]
at node \(i\) for each \(i \in I\), where \(M_i = 1\) for each \(i \in \mathcal{V}^c\). Message \(W_i\) is uniformly distributed on \(W_i\).
2) An encoding function \( f_{i,k} : \mathcal{X}_i \times \mathcal{Y}_i^{k-b_i} \rightarrow \mathcal{X}_i \) for each \( i \in \mathcal{I} \) and each \( k \in \{1, 2, \ldots, n\} \), where \( f_{i,k} \) is the encoding function at node \( i \) in the \( k \)-th time slot such that \( X_{i,k} = f_{i,k}(W_i, Y_i^{k-b_i}) \).

3) A decoding function \( g_{i,j} : \mathcal{X}_j \times \mathcal{Y}_j^n \rightarrow \mathcal{W}_i \) for each \( (i, j) \in \mathcal{V} \times \mathcal{D} \), where \( g_{i,j} \) is the decoding function for \( W_i \) at node \( j \) such that

\[
\hat{W}_{i,j} \triangleq g_{i,j}(W_j, Y_j^n).
\]

Given a \((B, n, M)\)-code, it follows from Definition 5 that for each \( i \in \mathcal{I} \), node \( i \) incurs a delay if \( b_i > 0 \), where \( b_i \) is the amount of delay incurred by node \( i \). If \( b_i = 0 \), node \( i \) incurs no delay, i.e., for each \( k \in \{1, 2, \ldots, n\} \), node \( i \) needs to receive \( Y_{i,k} \) before encoding \( X_{i,k} \). The feasibility condition of \( B \) in Definition 4 ensures that the operations of any \((B, n, M)\)-code are well-defined for the subsequently defined discrete memoryless network; the associated coding scheme is described after the network is defined.

**Definition 6**: A discrete network \((\mathcal{X}_I, \mathcal{Y}_I, \alpha, \mathcal{S}, \mathbf{G}, \mathbf{q})\) with multicast demand \((\mathcal{V}, \mathcal{D})\), when used multiple times, is called a discrete memoryless multimeasure multicast network (DM-MMN) if the following holds for any \((B, n, M)\)-code:

Let \( U_{k-1} \triangleq (W_I, X_I^{k-1}, Y_I^{k-1}) \) be the collection of random variables that are generated before the \( k \)-th time slot. Then, for each \( k \in \{1, 2, \ldots, n\} \) and each \( h \in \{1, 2, \ldots, \alpha\} \),

\[
\Pr\{U_{k-1} = u_{k-1}^{(h)}, X_{\mathbf{S}^h,k} = x_{\mathbf{S}^h,k}, Y_{\mathbf{G}^h,k} = y_{\mathbf{G}^h,k}\} = \Pr\{U_{k-1} = u_{k-1}^{(h)}, X_{\mathbf{S}^h,k} = x_{\mathbf{S}^h,k}, Y_{\mathbf{G}^h-1,k} = y_{\mathbf{G}^h-1,k}\} q^{(h)}_{\mathbf{G}^h/k}|x_{\mathbf{S}^h,k}, y_{\mathbf{G}^h-1,k} \tag{1}
\]

for all \( u_{k-1}^{(h)} \in U_{k-1}, x_{\mathbf{S}^h,k} \in \mathcal{X}_{\mathbf{S}^h,k} \) and \( y_{\mathbf{G}^h,k} \in \mathcal{Y}_{\mathbf{G}^h,k} \).

Following the notation in Definition 6, consider any \((B, n, M)\)-code on the DM-MMN. In the \( k \)-th time slot, \( X_{\mathcal{I},k} \) and \( Y_{\mathcal{I},k} \) are generated in the order

\[
X_{\mathbf{S}^1,k}, Y_{\mathbf{G}^1,k}, X_{\mathbf{S}^2,k}, Y_{\mathbf{G}^2,k}, \ldots, X_{\mathbf{S}^n,k}, Y_{\mathbf{G}^n,k} \tag{2}
\]

by transmitting on the channels in this order \( q^{(1)}, q^{(2)}, \ldots, q^{(n)} \) using the \((B, n, M)\)-code (as prescribed in Definition 5). Specifically, \( X_{\mathbf{S}^h,k}, Y_{\mathbf{G}^h-1,k} \) and channel \( q^{(h)} \) together define \( Y_{\mathbf{G}^h,k} \) for each \( h \in \{1, 2, \ldots, \alpha\} \). It is shown in [2, Section IV] that the encoding of \( X_{\mathbf{S}^h,k} \) before the transmission on \( q^{(h)} \) and the generation of \( Y_{\mathbf{G}^h,k} \) after the transmission on \( q^{(h)} \) for each \( h \in \{1, 2, \ldots, \alpha\} \) are well-defined.

We are now ready to formally define the capacity region and the positive-delay region through the following three standard definitions.

**Definition 7**: For a \((B, n, M)\)-code on the DM-MMN, the average probability of decoding error \( P_{\text{err}}^n \) is defined as

\[
P_{\text{err}}^n = \Pr\left\{ \bigcup_{i,j \in \mathcal{D}} \bigcup_{i \in \mathcal{V}} \{\hat{W}_{i,j} \neq W_i\} \right\}.
\]
Definition 8: Let $B$ be a feasible delay profile for the network. A rate tuple $(R_1, R_2, \ldots, R_N)$, denoted by $R_B$, is $B$-achievable for the DM-MMN if there exists a sequence of $(B, n, M_I)$-codes such that
\[
\lim_{n \to \infty} \frac{\log M_i}{n} \geq R_i
\]
for each $i \in \mathcal{I}$ and
\[
\lim_{n \to \infty} P_{err}^n = 0.
\]

Definition 9: The $B$-capacity region, denoted by $C_B$, of the DM-MMN is the set consisting of every $B$-achievable rate tuple $R_B$ with $R_i = 0$ for all $i \in \mathcal{V}^c$. The capacity region $C$ is defined as
\[
C \triangleq \bigcup_{B: B \text{ is feasible}} C_B
\]
and the positive-delay region $C_+$ is defined as
\[
C_+ \triangleq \bigcup_{B: B \text{ is positive}} C_B.
\]

If $C = C_+$, the DM-MMN is said to be delay-independent. If $C \supseteq C_+$, the DM-MMN is said to be delay-dependent.

Definitions 3, 4 and 9 imply that $C \supseteq C_+$, which implies that each DM-MMN is either delay-independent or delay-dependent.

IV. INNER AND OUTER BOUNDS FOR THE CAPACITY REGION

We start this section by stating an achievability result for classical DM-MMNs in the following theorem, which is a specialization of the main result of noisy network coding by Lim, Kim, El Gamal and Chung [11].

**Theorem 1:** Let $(\mathcal{X}, \mathcal{Y}, \alpha, \mathcal{S}, \mathcal{G}, q)$ be a DM-MMN, and let
\[
R_{in} \triangleq \bigcup_{p: p_{X_{\mathcal{I}}, Y_{\mathcal{I}}}} \left\{ R_I \mid \sum_{i \in \mathcal{I}} R_i \leq I_{p_{X_{\mathcal{I}}, Y_{\mathcal{I}}}}(X_{\mathcal{I}}; Y_{\mathcal{I}}|X_{\mathcal{I}}), R_i = 0 \text{ for all } i \in \mathcal{V}^c \right\}.
\]

Then,
\[
R_{in} \subseteq C_+.
\]

**Proof:** For every (classical) $(1, 1, \ldots, 1, n, M_I)$-code, $(\mathcal{X}, \mathcal{Y}, \alpha, \mathcal{S}, \mathcal{G}, q)$ is equivalent to $(\mathcal{X}, \mathcal{Y}, 1, \mathcal{I}, \mathcal{I}, \mathcal{I}, \prod_{h=1}^n \mathcal{q}_{\mathcal{Y}_h|\mathcal{X}_h, \mathcal{Y}_h}^{(h)})$ by Lemma 4 in [3], which implies that any achievable rate region of $(\mathcal{X}, \mathcal{Y}, 1, \mathcal{I}, \mathcal{I}, \mathcal{I}, \prod_{h=1}^n \mathcal{q}_{\mathcal{Y}_h|\mathcal{X}_h, \mathcal{Y}_h}^{(h)})$ lies in $C_+$. Consequently, (4) follows from applying Theorem 1 in [11] on $(\mathcal{X}, \mathcal{Y}, 1, \mathcal{I}, \mathcal{I}, \mathcal{I}, \prod_{h=1}^n \mathcal{q}_{\mathcal{Y}_h|\mathcal{X}_h, \mathcal{Y}_h}^{(h)})$ by taking $\hat{Y} = Y$. □

Following similar procedures for proving Theorem 1 in [2], we can prove an outer bound on $C$ stated in the following theorem.
Theorem 2: Let \((X, Y, \alpha, \mathcal{S}, G, q)\) be a DM-MMN, and let

\[
R_{\text{out}} \triangleq \bigcup_{p_{X,Y} \in \mathcal{P}_{X,Y}} \bigcap_{T \subseteq \mathcal{I} : T \cap \mathcal{D} \neq \emptyset} \prod_{h=1}^{\alpha} (p_{X|h} | x_{s|h-1}, y_{g|h-1}^{(h)} | x_{s|h}, y_{g|h-1})
\]

\[
\left\{ \begin{array}{l}
R_{\mathcal{I}} \sum_{i \in \mathcal{I}} R_i \leq \sum_{h=1}^{\alpha} I_{p_{X,Y}}(X_{T \cap S^h}, Y_{T \cap G^h-1} | Y_{T \cup \cap G^h}, X_{T \cup \cap S^h}, Y_{T \cup \cap G^h-1}), \\
R_i = 0 \text{ for all } i \in \mathcal{Y}^c
\end{array} \right.
\]

(5)

Then,

\[
\mathcal{C} \subseteq R_{\text{out}}.
\]

Proof: Let \(R_{\mathcal{I}}\) be an achievable rate tuple for the DM-MMN denoted by \((X, Y, \alpha, \mathcal{S}, G, q)\). By Definitions 8 and 9, there exists a sequence of \((B, n, M_{\mathcal{I}})\)-codes on the DM-MMN such that

\[
\lim_{n \to \infty} \frac{\log M_i}{n} \geq R_i
\]

(6)

and

\[
\lim_{n \to \infty} P_{\text{err}}^n = 0
\]

(7)

for each \(i \in \mathcal{I}\). Fix any \(T \subseteq \mathcal{I}\) such that \(T^c \cap \mathcal{D} \neq \emptyset\), and let \(d\) denote a node in \(T^c \cap \mathcal{D}\). For each \((B, n, M_{\mathcal{I}})\)-code, since the \(N\) messages \(W_1, W_2, \ldots, W_N\) are independent, we have

\[
\sum_{i \in T} \log M_i = H(W_T | W_{T^c})
\]

\[
= I(W_T; Y_{T^c}^n | W_{T^c}) + H(W_T | Y_{T^c}^n, W_{T^c})
\]

\[
\leq I(W_T; Y_{T^c}^n | W_{T^c}) + H(W_T | Y_d^n, W_d)
\]

\[
\leq I(W_T; Y_{T^c}^n | W_{T^c}) + 1 + P_{\text{err}}^n \sum_{i \in T} \log M_i,
\]

(8)

where the last inequality follows from Fano’s inequality (cf. Definition 7). Following similar procedures for proving Theorem 1 in [2], we can show by using (6), (7) and (8) that there exists a joint distribution \(p_{X,Y}\) which depends on the sequence of \((B, n, M_{\mathcal{I}})\)-codes but not on \(T\) such that

\[
p_{X,Y} = \prod_{h=1}^{\alpha} (p_{X|h} | x_{s|h-1}, y_{g|h-1}^{(h)} | x_{s|h}, y_{g|h-1})
\]

and

\[
\sum_{i \in T} R_i \leq \sum_{h=1}^{\alpha} I_{p_{X,Y}}(X_{T \cap S^h}, Y_{T \cap G^h-1} | Y_{T \cup \cap G^h}, X_{T \cup \cap S^h}, Y_{T \cup \cap G^h-1}).
\]

(9)

Since \(p_{X,Y}\) depends on only the sequence of \((B, n, M_{\mathcal{I}})\)-codes but not on \(T\), (9) holds for all \(T \subseteq \mathcal{I}\) such that \(T^c \cap \mathcal{D} \neq \emptyset\). This completes the proof.
V. Classes of Delay-Independent MMNs

In this section, we will use our inner and outer bounds developed in the previous section to calculate the capacity regions for some classes of generalized MMNs and then show that the MMNs are delay-independent, i.e., $C = C_+$. In the process of calculating their capacity regions, we will use the following proposition extensively to characterize an important property of Markov chains.

**Proposition 1:** Suppose there exist two probability distributions $r_{X,Y}$ and $q_{Z|Y}$ such that

$$p_{X,Y,Z}(x, y, z) = r_{X,Y}(x, y)q_{Z|Y}(z|y)$$

for all $x$, $y$ and $z$ whenever $p_Y(y) > 0$. Then

$$(X \rightarrow Y \rightarrow Z)_{p_{X,Y,Z}}$$

forms a Markov chain. In addition,

$$p_{Z|Y} = q_{Z|Y}.$$  \hspace{1cm} (12)

**Proof:** The proof of (11) is contained [13, Proposition 2.5]. It remains to show (12). Summing $x$ and then $z$ on both sides of (10), we have $p_{Y,Z}(y, z) = r_Y(y)q_{Z|Y}(z|y)$ and $p_Y(y) = r_Y(y)$ for all $x$, $y$ and $z$ whenever $p_Y(y) > 0$, which implies (12).

A. Deterministic MMN Dominated by Product Distribution

1) Problem Formulation and Main Result:

**Definition 10:** A conditional distribution $q_{Y|X}$ is said to be deterministic if for each $x^* \in \mathcal{X}$, there exists a unique $y^* \in \mathcal{Y}$ such that $q_{Y|X}(y^*|x^*) = 1$.

**Definition 11:** The MMN $(\mathcal{X}_I, \mathcal{Y}_I, \alpha, \mathcal{S}, \mathcal{G}, q)$ is said to be deterministic if $q_{Y_h|X_{S_h},Y_{G_h-1}}^{(h)}$ is deterministic for each $h \in \{1, 2, \ldots, \alpha\}$.

With the help of the following definition, we can completely characterize the capacity region for a class of generalized deterministic MMNs.

**Definition 12:** The deterministic MMN $(\mathcal{X}_I, \mathcal{Y}_I, \alpha, \mathcal{S}, \mathcal{G}, q)$ is said to be dominated by product distributions if the following holds for each distribution $p_{X_Z}$:

Define $s_{X_i}$ to be the marginal distribution of $p_{X_i}$ for each $i \in \mathcal{I}$, i.e., $s_{X_i}(x_i) = \sum_{x_j:j \in \mathcal{I} \setminus \{i\}} p_{X_i}(x_i)$ for all $x_i$.

In addition, define

$$p_{X_I,Y_Z} \triangleq p_{X_Z} \prod_{h=1}^\alpha q_{Y_h|X_{S_h},Y_{G_h-1}}^{(h)}$$

and

$$s_{X_I,Y_Z} \triangleq \left(\prod_{i=1}^N s_{X_i}\right)\left(\prod_{h=1}^\alpha q_{Y_h|X_{S_h},Y_{G_h-1}}^{(h)}\right).$$

Then for any $T \subseteq \mathcal{I}$, $H_{p_{X_I,Y_Z}}(Y_T|X_{T^c}) \leq H_{s_{X_I,Y_Z}}(Y_T|X_{T^c})$.

The following is our main result in this section.
Theorem 3: Let \((\mathcal{X}, \mathcal{Y}, \alpha, \mathcal{S}, \mathcal{G}, q)\) be a deterministic MMN dominated by product distributions, and let

\[
\mathcal{R}_{\text{in}}^{\text{det}} = \bigcup_{p_{XZ}, YZ = \prod_{i=1}^{N} p_{X_i} \prod_{h=1}^{\alpha} q_{Y_{G_h}|X_{G_h}Y_{G_h}^{h-1}}^{(h)}} \left\{ R_{i} \mid \sum_{i \in T} R_{i} \leq H_{p_{XZ}, YZ}(Y_{T^c}|X_{T^c}), \quad R_{i} = 0 \text{ for all } i \in V^c \right\}.
\]

Then,

\[
\mathcal{C} = \mathcal{C}_{+} = \mathcal{R}_{\text{in}}^{\text{det}}
\]

and hence the network is delay-independent. In particular, (14) holds for the deterministic relay network with no interference in [6] and the finite-field linear deterministic network in [7, 8], which implies that they are delay-independent.

In the following, we provide the proof of Theorem 3. Since the last statement of the theorem follows from the fact that the deterministic relay network with no interference and the finite-field linear deterministic network are dominated by product distributions [11, Section II.A], it suffices to prove (14). To this end, it suffices to prove the achievability statement

\[
\mathcal{R}_{\text{in}}^{\text{det}} \subseteq \mathcal{C}_{+}
\]

and the converse statement

\[
\mathcal{C} \subseteq \mathcal{R}_{\text{in}}^{\text{det}}.
\]

2) Achievability: In this subsection, we would like to show (15) by using Theorem 1 and Definition 11. Since \(\mathcal{R}_{\text{in}} \subseteq \mathcal{C}_{+}\) by Theorem 1 (cf. (3)), it suffices to prove that

\[
\mathcal{R}_{\text{in}} = \mathcal{R}_{\text{in}}^{\text{det}}.
\]

Fix any \(p_{XZ}, YZ\) that satisfies

\[
p_{XZ, YZ} = \left( \prod_{i=1}^{N} p_{X_i} \right) \left( \prod_{h=1}^{\alpha} q_{Y_{G_h}|X_{G_h}Y_{G_h}^{h-1}}^{(h)} \right).
\]

Marginalizing (18), we have

\[
p_{XZ} = \prod_{i=1}^{N} p_{X_i}.
\]

Therefore,

\[
p_{YZ|XZ} = \prod_{h=1}^{\alpha} q_{Y_{G_h}|X_{G_h}Y_{G_h}^{h-1}}^{(h)}
\]

by (18) and (19). Since \(q_{Y_{G_h}|X_{G_h}Y_{G_h}^{h-1}}^{(h)}\) is deterministic for each \(h \in \{1, 2, \ldots, \alpha\}\), it follows from (20) that \(p_{YZ|XZ}\) is deterministic and hence

\[
H_{p_{YZ|XZ}}(Y_{T^c}|X_{T^c}) = 0,
\]

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where \( p_{W_{I}X_{I},Y_{I}} \) is the joint distribution induced by the code according to Definitions 5 and 6.

**Proof:** Fix a \((B, n, M_{I×I})\)-code and let \( p_{W_{I}X_{I},Y_{I}} \) be the joint distribution induced by the code according to Definitions 5 and 6. For each \( k \in \{1, 2, \ldots, n\} \), let \( U_{k}^{k-1} \triangleq (W_{I}X_{I}^{k-1}, Y_{I}^{k-1}) \) be the collection of random variables that are generated before the \( k \)th time slot for the \((B, n, M_{I×I})\)-code. In order to prove (22), it suffices to show that

\[
H_{p_{U_{k}^{k-1},X_{I}^{k},Y_{I}^{k}}}(X_{I}^{k}, Y_{I}^{k}|U_{k}^{k-1}) = 0
\]

holds for each \( k \in \{1, 2, \ldots, n\} \) and each \( h \in \{1, 2, \ldots, \alpha\} \), which will then imply that

\[
H_{p_{W_{I}X_{I},Y_{I}}}(X_{I}^{n}, Y_{I}^{n}|W_{I}) = \sum_{k=1}^{n} H_{p_{U_{k}^{k-1},X_{I},Y_{I}}}(X_{I}, Y_{I}|U_{k}^{k-1}) \overset{(21)}{=} 0.
\]

Fix a \( k \in \{1, 2, \ldots, n\} \). We prove (23) by induction on \( h \) as follows. For \( h = 1 \), the LHS of (23) is

\[
\begin{align*}
&H_{p_{U_{k}^{k-1},X_{I}^{k},Y_{I}^{k}}}(X_{I}^{k}, Y_{I}^{k}|U_{k}^{k-1}) \\
&= H_{p_{U_{k}^{k-1},X_{I}^{1},Y_{I}^{1}}}(X_{I}^{1}, Y_{I}^{1}|U_{k}^{k-1}) + H_{p_{U_{k}^{k-1},X_{I}^{k},Y_{I}^{k}}}(Y_{I}^{k}|U_{k}^{k-1}, X_{I}^{1}) \\
&\overset{(a)}{=} H_{p_{U_{k}^{k-1},X_{I}^{1},Y_{I}^{1}}}(Y_{I}^{1}|U_{k}^{k-1}, X_{I}^{1}) \\
&\overset{(b)}{=} 0,
\end{align*}
\]

where

(a) follows from Definitions 4 and 5 that \( X_{I}^{1} \) is a function of \( U_{k}^{k-1} \) for the code.

(b) follows from the fact that \( q^{(1)} \) is deterministic (cf. Definition 11).
If (23) holds for $h = m$, i.e.,
\[ H_{p_{U:k-1},X_{G:m:k},Y_{G:m:k}}(X_{S:m:k},Y_{G:m:k}|U^{k-1}) = 0 \] (25)
then for $h = m + 1$ such that $m + 1 \leq \alpha$, the LHS of (23) is
\[
H_{p_{U:k-1},X_{G:m+1:k},Y_{G:m+1:k}}(X_{S:m+1:k},Y_{G:m+1:k}|U^{k-1}) \\
= H_{p_{U:k-1},X_{G:m:k},Y_{G:m:k}}(X_{S:m:k},Y_{G:m:k}|U^{k-1}) + H_{p_{U:k-1},X_{G:m:k},Y_{G:m,k}}(X_{S:m+1:k}|U^{k-1},X_{S:m:k},Y_{G:m,k}) \\
+ H_{p_{U:k-1},X_{G:m+1:k},Y_{G:m+1:k}}(X_{S:m+1:k}|U^{k-1},X_{S:m:k},Y_{G:m,k}) \\
\overset{(a)}{=} H_{p_{U:k-1},X_{G:m+1:k},Y_{G:m+1:k}}(Y_{G:m+1:k}|X_{S:m+1:k},Y_{G:m,k}) \\
\overset{(b)}{=} 0, \tag{26}
\]
where
(a) follows from Definitions 4 and 5 that $X_{S:m+1:k}$ is a function of $(U^{k-1}, Y_{G:m,k})$ for the code.
(b) follows from the fact that $q^{(m+1)}$ is deterministic (cf. Definition 11).

For $h = 1$, it follows from (24) that (23) holds. For all $1 \leq m \leq \alpha - 1$, it follows from (25) and (26) that if (23) is assumed to be true for $h = m$, then (23) is also true for $h = m + 1$. Consequently, it follows by mathematical induction that (23) holds for all $1 \leq h \leq \alpha$.

In order to show that the capacity region of the generalized deterministic MMN lies within the classical cut-set bound, we will prove in Theorem 4, the theorem following the proposition below, that the generalized deterministic MMN is equivalent to some classical deterministic MMN. The following proposition is an important step for proving Theorem 4.

**Proposition 3:** Let $(X_I, Y_I, \alpha, S, G, q)$ be a deterministic MMN. For any $(B, n, M_{I,S})$-code on the network, if some $u, x_I$ and $y_I$ satisfy
\[ \Pr\{U^{k-1} = u^{k-1}, X_{I,k} = x_I\} > 0 \] (27)
and
\[ \Pr\{U^{k-1} = u^{k-1}, X_{I,k} = x_I, Y_{I,k} = y_I\} = 0, \] (28)
then there exists some $h \in \{1, 2, \ldots, \alpha\}$ such that $q^{(h)}(y_{G_h}|x_{S_h}, y_{G_{h-1}}) = 0$ (where $x_{S_h}$ is a subtuple of $x_I$, and $y_{G_h}$ is a subtuple of $y_I$).
Proof: Suppose there exist \( u, x_T \) and \( y_T \) that satisfy (27) and (28). We prove the proposition by assuming the contrary. Assume
\[
q^{(h)}(y_{\mathcal{G}_h}, y_{\mathcal{G}^{h-1}}) > 0
\]  
for all \( h \in \{1, 2, \ldots, \alpha\} \). We now prove by induction on \( h \) that
\[
\Pr\{U^{k-1} = u^{k-1}, X_{\mathcal{G}^{h,k}} = x_{\mathcal{G}^h}, Y_{\mathcal{G}^{h,k}} = y_{\mathcal{G}^h}\} > 0
\]  
for each \( h \in \{1, 2, \ldots, \alpha\} \). For \( h = 1 \), the LHS of (30) is
\[
\Pr\{U^{k-1} = u^{k-1}, X_{\mathcal{S}^{1,k}} = x_{\mathcal{S}^1}, Y_{\mathcal{G}^{1,k}} = y_{\mathcal{G}^1}\}
\]
(a) follows from Definitions 6 and 11.
(b) follows from (27) and (29).
If (30) holds for \( h = m \), i.e.,
\[
\Pr\{U^{k-1} = u^{k-1}, X_{\mathcal{S}^{m,k}} = x_{\mathcal{S}^m}, Y_{\mathcal{G}^{m,k}} = y_{\mathcal{G}^m}\} > 0,
\]
then for \( h = m + 1 \) such that \( m + 1 \leq \alpha \),
\[
\Pr\{U^{k-1} = u^{k-1}, X_{\mathcal{S}^{m+1,k}} = x_{\mathcal{S}^{m+1}}, Y_{\mathcal{G}^{m+1,k}} = y_{\mathcal{G}^{m+1}}\}
\]
(a) follows from Definitions 6 and 11.
(b) follows from Lemma 2 that \( (X_{\mathcal{S}^{m,k}}, Y_{\mathcal{G}^{m,k}}) \) is a function of \( U^{k-1} \).
(c) follows from (32), (29) and (27).
Consequently, it follows from (31), (32) and (33) that (30) holds for \( h = \alpha \) by mathematical induction, which then implies that \( \Pr\{U^{k-1} = u^{k-1}, X_{\mathcal{S}^\alpha,k} = x_{\mathcal{S}^\alpha}, Y_{\mathcal{G}^{\alpha,k}} = y_{\mathcal{G}^\alpha}\} > 0 \), which contradicts (28).
Surprisingly, each generalized deterministic MMN is equivalent to some classical deterministic MMN, which is proved as follows.

**Theorem 4:** For any \((B, n, M_{I\times I})\)-code, the deterministic MMN specified by \((X_I, Y_I, \alpha, S, G, q)\) is equivalent to the deterministic MMN specified by \((X_I, Y_I, 1, I, I, q^{(1)} q^{(2)} \ldots q^{(\alpha)})\).

**Proof:** Fix a \((B, n, M_{I\times I})\)-code, and let \(U^{k-1} \triangleq (W_I, X^{k-1}_I, Y^{k-1}_I)\) be the collection of random variables that are generated before the \(k^{th}\) time slot. To prove the theorem statement for this code, it suffices to show that the following two statements are equivalent for each \(k \in \{1, 2, \ldots, n\}\) (cf. (1)):

**Statement 1:** For each \(h \in \{1, 2, \ldots, \alpha\},\)

\[
\Pr\{U^{k-1} = u^{k-1}, X^{h, k} = x^{h, k}, Y^{h, k} = y^{h, k}\} = \Pr\{U^{k-1} = u^{k-1}, X^{h, k} = x^{h, k}, Y^{h-1, k} = y^{h-1, k}\} q^{(h)}(y^{h, k} | x^{h, k}, y^{h-1, k}).
\]  

(34)

**Statement 2:**

\[
\Pr\{U^{k-1} = u^{k-1}, X^{k, k} = x^{k, k}, Y^{k, k} = y^{k, k}\} = \Pr\{U^{k-1} = u^{k-1}, X^{k, k} = x^{k, k}\} \prod_{h=1}^{\alpha} q^{(h)}(y^{h, k} | x^{h, k}, y^{h-1, k}).
\]  

(35)

Fix a \((B, n, M_{I\times I})\)-code and a \(k \in \{1, 2, \ldots, n\}\). We first show that (34) implies (35). Suppose (34) holds for each \(h \in \{1, 2, \ldots, \alpha\}\). Consider the following three mutually exclusive cases:

**Case** \(\Pr\{U^{k-1} = u^{k-1}, X^{k, k} = x^{k, k}\} = 0\):

Both the LHS and the RHS of (35) equal zero.

**Case** \(\Pr\{U^{k-1} = u^{k-1}, X^{k, k} = x^{k, k}\} > 0\) and \(\Pr\{U^{k-1} = u^{k-1}, X^{k, k} = x^{k, k}, Y^{k, k} = y^{k, k}\} = 0\):

For this case, the LHS of (35) equals zero. By Proposition 3, there exists some \(h \in \{1, 2, \ldots, \alpha\}\) such that \(q^{(h)}(y^{h, k} | x^{h, k}, y^{h-1, k}) = 0\), which implies that the RHS of (35) equals zero.

**Case** \(\Pr\{U^{k-1} = u^{k-1}, X^{k, k} = x^{k, k}, Y^{k, k} = y^{k, k}\} > 0\):

For this case,

\[
\Pr\{U^{k-1} = u^{k-1}, X^{k, k} = x^{k, k}, Y^{k, k} = y^{k, k}\} = p_{U^{k-1}, X^{k, k} = x^{k, k}} \prod_{h=1}^{\alpha} p_{Y^{h, k} | U^{k-1}, X^{h, k} = x^{h, k}}(y^{h, k} | u^{k-1}, x^{h, k})
\]

\[
= p_{U^{k-1}, X^{k, k} = x^{k, k}} \prod_{h=1}^{\alpha} p_{Y^{h, k} | U^{k-1}, X^{h, k}, Y^{h-1, k}}(y^{h, k} | u^{k-1}, x^{h, k})
\]

\[
\overset{(a)}{=} p_{U^{k-1}, X^{k, k} = x^{k, k}} \prod_{h=1}^{\alpha} p_{Y^{h, k} | U^{k-1}, Y^{h-1, k}} \prod_{h=1}^{\alpha} q^{(h)}(y^{h, k} | x^{h, k}, y^{h-1, k}),
\]

where (a) follows from Lemma 2 that for the \((B, n, M_{I\times I})\)-code, \(X^{k, k}\) is a function of \(U^{k-1}\). Therefore, the LHS and the RHS of (35) are equal.
Combining the three mutually exclusive cases, we obtain that (34) implies (35). We now show that (35) implies (34). Suppose (35) holds. Then for each \( h \in \{1, 2, \ldots, \alpha\} \) and each \( m \in \{1, 2, \ldots, h\} \),

\[
\Pr\{U^{k−1} = u^{k−1}, X_{S_h,k} = x_{S_h}, Y_{G^m,k} = y_{G^m}\} = \sum_{x_{S_h+1}, \ldots, x_{S\alpha}} p_{U^{k−1}, X_{Z,k}}(u^{k−1}, x_I, y_I) = \sum_{x_{S_h+1}, \ldots, x_{S\alpha}} p_{U^{k−1}, X_{Z,k}}(u^{k−1}, x_I) \prod_{\ell=1}^{\alpha} q^{(\ell)}(y_{G^\ell}|x_{S^\ell}, y_{G^{\ell−1}})
\]

(35)

\[
= \sum_{x_{S_h+1}, \ldots, x_{S\alpha}} p_{U^{k−1}, X_{Z,k}}(u^{k−1}, x_I) \prod_{\ell=1}^{m} q^{(\ell)}(y_{G^\ell}|x_{S^\ell}, y_{G^{\ell−1}})
\]

(36)

where (a) follow from the fact that \( m \leq h \). Then, for each \( h \in \{1, 2, \ldots, \alpha\} \), the equality in (34) can be verified by substituting (36) into the LHS and the RHS.

The following lemma simplifies the outer bound in Theorem 2 for the deterministic MMN.

**Lemma 4:** Let \((X_I, Y_I, \alpha, S, G, q)\) be a deterministic MMN. Define

\[
R_{\text{det}} \triangleq \bigcup_{P_{X_I} Y_I \mathcal{P}_{X_I} Y_I = P_{X_I}} \bigcap_{T \subseteq I : T \cap \mathcal{D} \neq \emptyset} \left\{ R_Z \left| \sum_{i \in T} R_i \leq H_{p_{X_I} Y_I}(Y_T^c | X_T^c), R_i = 0 \text{ for all } i \in \mathcal{V}^c \right. \right\}.
\]

(37)

Then,

\[
\mathcal{C} \subseteq R_{\text{det}}.
\]

**Proof:** Suppose \( R_T \) is an achievable rate tuple for \((X_I, Y_I, \alpha, S, G, q)\). It follows from Definition 8 that there exists a sequence of \((B, n, M_I)\)-codes on \((X_I, Y_I, \alpha, S, G, q)\) such that \( \lim_{n \to \infty} \frac{\log M_I}{n} \geq R_i \) for each \( i \in I \) and \( \lim_{n \to \infty} P^n_{\text{err}} = 0 \). Since \((X_I, Y_I, \alpha, S, G, q)\) is equivalent to \((X_I, Y_I, 1, I, I, q^{(1)} q^{(2)} \ldots q^{(\alpha)})\) for any \((B, n, M_{I \times I})\)-code on the deterministic DMN by Theorem 4, it follows from Definition 5 that \( R_{I} \) is achievable for \((X_I, Y_I, 1, I, I, q^{(1)} q^{(2)} \ldots q^{(\alpha)})\), which then implies from Theorem 2 that there exists some \( p_{X_I, Y_I}^* \) satisfying

\[
p_{X_I, Y_I}^* = p_{X_I} \prod_{h=1}^{\alpha} q_{G_h}^{(h)}|x_{S_h}, y_{G_{h-1}}
\]

(38)

such that for any \( T \subseteq I \) such that \( T^c \cap \mathcal{D} \neq \emptyset \),

\[
\sum_{i \in T} R_i \leq I_{p_{X_I, Y_I}}(X_T; Y_T^c | X_T^c)
\]

\[
= H_{p_{X_I, Y_I}}(Y_T^c | X_T^c) - H_{p_{X_I, Y_I}}(Y_T^c | X_T)
\]

(a)

\[
= H_{p_{X_I, Y_I}}(Y_T^c | X_T^c)
\]

(39)
where (a) follows from the fact that
\[ p_{Y|X}^* \overset{(38)}{=} \prod_{h=1}^{\alpha} q_{Y_h|X_{gh},Y_{gh-1}} \]

is deterministic. Consequently, it follows from (37), (38) and (39) that \( R \in R_{out}^{det} \).

We are now ready to prove (16) as follows. Using Lemma 4, we obtain \( C \subseteq R_{out}^{det} \) where \( R_{out}^{det} \) is defined in (37).

In addition, it follows from (13), (37) and Definition 12 that \( R_{out}^{det} \subseteq R_{in}^{det} \). Consequently, \( C \subseteq R_{in}^{det} \).

B. MMN Consisting of Independent DMCs

1) Problem Formulation and Main Result: Consider a DM-MMN \((\mathcal{X}_I, \mathcal{Y}_I, \alpha, \mathcal{S}, \mathcal{G}, q)\) defined as follows: The edge set of the network is characterized by
\[ \Omega \triangleq \bigcup_{h=1}^{\alpha} S^h \times \mathcal{G}_h, \]

and a DMC denoted by \( q_{Y_{i,j}|X_{i,j}} \) is associated to every edge \((i, j) \in \Omega\), where \( X_{i,j} \) and \( Y_{i,j} \) are the input and output alphabets of the DMC carrying information from node \( i \) to node \( j \). The definition of \( \Omega \) in (40) ensures that \( q_{Y_h|X_{gh},Y_{gh-1}} \) can be well-defined for each \( h \in \{1, 2, \ldots, \alpha\} \). For each \((i, j) \in \Omega\), the capacity of channel \( q_{Y_{i,j}|X_{i,j}} \), denoted by \( C_{i,j} \), is attained by some \( \bar{p}_{X_{i,j}} \) due to the channel coding theorem, i.e.,
\[ C_{i,j} \triangleq \max_{p_{X_{i,j}}} I_{p_{X_{i,j}} q_{Y_{i,j}|X_{i,j}}} (X_{i,j}; Y_{i,j}) \]
\[ = I_{\bar{p}_{X_{i,j}} q_{Y_{i,j}|X_{i,j}}} (X_{i,j}; Y_{i,j}). \]

(41)

For all the other \((\tilde{i}, \tilde{j}) \in \Omega^c\), we assume without loss of generality that
\[ X_{\tilde{i},\tilde{j}} = Y_{\tilde{i},\tilde{j}} = \{0\} \]
and \( C_{\tilde{i},\tilde{j}} = 0 \). Then, we define the input and output alphabets for each node \( i \) in the following natural way:
\[ X_i \triangleq X_{i,1} \times X_{i,2} \times \ldots \times X_{i,N} \]
(43)
and
\[ Y_i \triangleq Y_{1,i} \times Y_{2,i} \times \ldots \times Y_{N,i} \]
(44)
for each \( i \in I \). In addition, we define
\[ q_{Y|h|X_{gh},Y_{gh-1}} \overset{(h)}{=} \prod_{(i,j) \in S^h \times \mathcal{G}_h} q_{Y_{i,j}|X_{i,j}} \]
for each \( h \in \{1, 2, \ldots, \alpha\} \), i.e., the random transformations (noises) from \( X_{i,j} \) to \( Y_{i,j} \) are independent and each channel \( q_{Y_{i,j}|X_{i,j}} \) is in a product form. We call the network described above the DM-MMN consisting of independent DMCs. The classical MMN consisting of independent DMCs studied in [9] is a special case of this network model when \( \alpha = 1 \) and \( \Omega = I \times I \). The following is the main result in this section.
**Theorem 5:** For the DM-MMN consisting of independent DMCs, define

\[
\mathcal{R}_{\text{DMCs}} \triangleq \bigcap_{T \subseteq I : T \cap D \neq \emptyset} \{ R_T \mid \sum_{i \in T} R_i \leq \sum_{(i,j) \in T \times T^c} C_{i,j}, \quad R_i = 0 \text{ for all } i \in S^c \}. \tag{46}
\]

Then,

\[
\mathcal{C} = \mathcal{C}_+ = \mathcal{R}_{\text{DMCs}}
\]

and hence the network is delay-independent.

In the following, we provide the proof of Theorem 5. To this end, it suffices to prove the achievability statement

\[
\mathcal{R}_{\text{in}} \subseteq \mathcal{C}_+ \tag{47}
\]

and the converse statement

\[
\mathcal{C} \subseteq \mathcal{R}_{\text{DMCs}}. \tag{48}
\]

2) **Achievability:** In this subsection, we would like to prove (47). Since the DMCs \( q_Y \mid X \) are all independent and each of the DMC can carry information at a rate arbitrarily close to the capacity, it is intuitive that \( \mathcal{R}_{\text{in}} \) lies in the positive-delay region of the DM-MMN consisting of independent DMCs, which is proved as follows.

Let \( (X_I, Y_I, \alpha, S, G, q) \) be the DM-MMN consisting of independent DMCs whose positive-delay region is denoted by \( \mathcal{C}_+ \), and construct an \( L \)-copied counterpart of \( (X_I, Y_I, \alpha, S, G, q) \) as follows: Let \( (\bar{X}_L^I, \bar{Y}_L^I, \alpha, \bar{S}, \bar{G}, \bar{q}) \) be a noiseless DM-MMN consisting of independent DMCs with multicast demand \( (V, D) \) such that for each \( (i, j) \in I \times I \), the DMC carrying information from node \( i \) to node \( j \) is an error-free (noiseless) channel, denoted by \( \bar{q}_{\bar{X}_L^i \mid \bar{X}_L^i} \), with capacity \( \lfloor LC_{i,j} \rfloor \) (cf. (41)). To be more precise, \( \bar{q}_{\bar{X}_L^i \mid \bar{X}_L^i} \) is defined to be a channel that can carry

\[
| \bar{X}_L^i | = \lfloor LC_{i,j} \rfloor \tag{49}
\]

error-free bits for each \( (i, j) \in I \times I \). Let \( \bar{C}_L^+ \) denote the positive-delay region of \( (\bar{X}_L^I, \bar{Y}_L^I, \alpha, \bar{S}, \bar{G}, \bar{q}) \). Since every channel \( q_Y \mid X \) in \( (X_I, Y_I, \alpha, S, G, q) \) can carry information from node \( i \) to node \( j \) at a rate arbitrarily close to \( C_{i,j} \), it follows that for every \( R_T \) that lies in the interior of \( \bar{C}_L^+ \), there exists a sequence of \( (B, nL, M_T) \)-codes for the original channel \( (X_I, Y_I, \alpha, S, G, q) \) such that \( \lim_{n \to \infty} \log \frac{M_T}{nL} \geq R_i \) for each \( i \in I \) and \( \lim_{n \to \infty} P_{\text{err}}^{nL} = 0 \). Consequently, \( \bar{C}_L^+/L \subseteq \mathcal{C}_+ \) for all \( L \in \mathbb{N} \), which implies

\[
\text{closure} \left( \bigcup_{L=1}^\infty \bar{C}_L^+/L \right) \subseteq \mathcal{C}_+. \tag{50}
\]

Therefore, it remains to show that

\[
\mathcal{R}_{\text{in}} \subseteq \text{closure} \left( \bigcup_{L=1}^\infty \bar{C}_L^+/L \right). \tag{51}
\]
Define
\[
\mathcal{R}^L_{\text{in}} \triangleq \bigcup_{P_{X^L_k}, P_{Y^L_k} \in (\Pi_{h=1}^\infty P_{X^L_k}^\infty)} \bigcap_{T \subseteq \mathcal{I} \cap \mathcal{D} \neq \emptyset} \left\{ R_x \left\vert \sum_{i \in T} R_i \leq H_{P_{X^L_k}, P_{Y^L_k}}(\bar{Y}^L_{T^c} | \bar{X}^L_{T^c}) \right. \right\}
\]
(52)
for each \( L \in \mathbb{N} \). Straightforward verification reveals that \((\bar{X}^L_i, \bar{Y}^L_i, \alpha, \bar{S}, \bar{G}, \bar{q})\) is a deterministic MNM dominated by product distributions, which implies from Theorem 3 that
\[
\mathcal{C}^L_+ = \mathcal{R}^L_{\text{in}}.
\]
Combining (51) and (53), it remains to show
\[
\text{closure} \left( \bigcup_{L=1}^{\infty} \mathcal{R}^L_{\text{in}} / L \right) \supseteq \mathcal{R}^\text{DMC}_{\text{in}}.
\]
To this end, we let \( u_{X^L_i}^L \) be the uniform distribution on \( \bar{X}^L_i \) for each \( i \in \mathcal{I} \) and let
\[
u_{X^L_i, Y^L_i}^L = \left( \prod_{i=1}^{N} u_{X^L_i}^L \right) \left( \prod_{h=1}^{\alpha} q_{\bar{Y}^L_{Sh}, \bar{Y}^L_{Sh-1}}^{(h)} \right)
\]
and consider the following chain of equalities for each \( L \in \mathbb{N} \):
\[
\begin{align*}
H_{u_{X^L_k}, \nu_{Y^L_k}}^{(L)}(\bar{Y}^L_{T^c} | \bar{X}^L_{T^c}) & = \sum_{h=1}^{\alpha} H_{u_{X^L_k}, \nu_{Y^L_k}}^{(L)}(\bar{Y}^L_{T^c \cap \mathcal{G}^h}, \bar{X}^L_{T^c \cap \mathcal{G}^h-1}) \\
& \overset{(a)}{=} \sum_{h=1}^{\alpha} H_{u_{X^L_k}, \nu_{Y^L_k}}^{(L)}(\{\bar{X}^L_{i,j}\}_{i \in \mathcal{I}, j \in \mathcal{T^c \cap \mathcal{G}^h}}, \bar{X}^L_{T^c \cap \mathcal{G}^h-1}) \\
& \overset{(b)}{=} \sum_{h=1}^{\alpha} H_{u_{X^L_k}, \nu_{Y^L_k}}^{(L)}(\{\bar{X}^L_{i,j}\}_{i \in \mathcal{I}, j \in \mathcal{T^c \cap \mathcal{G}^h}}, \bar{Y}^L_{T^c \cap \mathcal{G}^h-1}) \\
& = \sum_{h=1}^{\alpha} \sum_{k \in \mathcal{T^c \cap \mathcal{G}^h}} H_{u_{X^L_k}, \nu_{Y^L_k}}^{(L)}(\bar{X}^L_{k,\ell} | \bar{X}^L_{T^c}, \bar{Y}^L_{T^c \cap \mathcal{G}^h-1}, \{\bar{X}^L_{i,j}\}_{i<k, j<\ell, j \in \mathcal{T^c \cap \mathcal{G}^h}}) \\
& \overset{(c)}{=} \sum_{h=1}^{\alpha} \sum_{k \in \mathcal{T^c \cap \mathcal{G}^h}} H_{u_{X^L_k}, \nu_{Y^L_k}}^{(L)}(\bar{X}^L_{k,\ell} | \bar{X}^L_{T^c}, \bar{Y}^L_{T^c \cap \mathcal{G}^h-1}, \{\bar{X}^L_{i,j}\}_{i<k, j<\ell, j \in \mathcal{T^c \cap \mathcal{G}^h}}) \\
& \overset{(d)}{=} \sum_{h=1}^{\alpha} \sum_{k \in \mathcal{T^c \cap \mathcal{G}^h}} H_{u_{X^L_k}, \nu_{Y^L_k}}^{(L)}(\bar{X}^L_{k,\ell} | \bar{X}^L_{i,j} \cap \mathcal{T^c}, \{\bar{X}^L_{i,j}\}_{i \in \mathcal{I} \times \mathcal{T^c \cap \mathcal{G}^h-1}, \{\bar{X}^L_{i,j}\}_{i<k, j<\ell, j \in \mathcal{T^c \cap \mathcal{G}^h}}) \\
& \overset{(e)}{=} \sum_{h=1}^{\alpha} \sum_{k \in \mathcal{T^c \cap \mathcal{G}^h}} H_{u_{X^L_k}, \nu_{Y^L_k}}^{(L)}(\bar{X}^L_{k,\ell}) \\
& \overset{(f)}{=} \sum_{h=1}^{\alpha} \sum_{k \in \mathcal{T^c \cap \mathcal{G}^h}} |L_{C_{k,\ell}}| \\
& = \sum_{(k, \ell) \in \mathcal{T^c \cap \mathcal{T^c}}} |L_{C_{k,\ell}}|,
\end{align*}
\]
(a) follows from applying (44) to \( (\bar{\mathcal{X}}_T^L, \bar{\mathcal{Y}}_T^L, \alpha, \bar{\mathcal{S}}, \bar{\mathcal{G}}, \bar{q}) \).

(b) follows from the fact that \( (\bar{\mathcal{X}}_T^L, \bar{\mathcal{Y}}_T^L, \alpha, \bar{\mathcal{S}}, \bar{\mathcal{G}}, \bar{q}) \) is a deterministic MMN.

(c) follows from the fact that for all \( k \in T^c \) and \( \ell \in T^c \cap \mathcal{G}_h, \)

\[
H_{u_{\bar{\mathcal{X}}_T^L, \bar{\mathcal{Y}}_T^L}} (\bar{\mathcal{X}}_{k,\ell}^L | \bar{\mathcal{X}}_T^L, \bar{\mathcal{Y}}_T^L \cap \mathcal{G}_{h-1}, \{ \bar{\mathcal{X}}_{i,j}^L \}_{i,j} \neq k, \ell, j \in T^c \cap \mathcal{G}_h) \\
\leq H_{u_{\bar{\mathcal{X}}_T^L, \bar{\mathcal{Y}}_T^L}} (\bar{\mathcal{X}}_{k,\ell}^L | \bar{\mathcal{X}}_k^L) \\
= 0.
\]

(d) follows from applying (43) and (44) to \( (\bar{\mathcal{X}}_T^L, \bar{\mathcal{Y}}_T^L, \alpha, \bar{\mathcal{S}}, \bar{\mathcal{G}}, \bar{q}) \).

(e) follows from the fact that \( \{ \bar{\mathcal{X}}_{i,j}^L \}_{i,j} \in \mathcal{I} \times \mathcal{I} \) are independent under the uniform distribution \( \prod_{i=1}^N u_i \).

(f) follows from the fact that \( \bar{\mathcal{X}}_{i,j}^L \) is uniformly distributed on \( \bar{\mathcal{X}}_{i,j}^L \) whose cardinality is \( |L\bar{C}_{i,j}| \) by (49) for each \( (i, j) \in \mathcal{I} \times \mathcal{I} \).

Using (46), (52) and (54), we have

\[
\text{closure} \left( \bigcup_{i=1}^\infty \bar{R}_i^L / L \right) \supseteq \mathcal{R}_m^{\text{DMCs}},
\]

which implies from (53) that (51) holds, which then implies from (50) that (47) holds.

3) Converse of Theorem 5: In this subsection, we would like to prove (48). Define

\[
\mathcal{R}_m^{\text{DMCs}} \triangleq \mathcal{R}_m^{\text{DMCs}} \supseteq \mathcal{R}_m^{\text{DMCs}}.
\]

It follows form Theorem 2 that \( \mathcal{C} \subseteq \mathcal{R}_m^{\text{DMCs}} \). Therefore, it remains to show that

\[
\mathcal{R}_m^{\text{DMCs}} \subseteq \mathcal{R}_m^{\text{DMCs}}.
\]

For any \( p_{X,T} = \prod_{h=1}^n \left( p_{X_{S_h},X_{G_{h-1}},Y_{G_{h-1}} q_{Y_{G_{h-1}|X_{S_h},Y_{G_{h-1}}}}^{(h)} \right) \), it follows from (45) that

\[
p_{X,T} = \prod_{h=1}^\alpha \left( p_{X_{S_h},X_{G_{h-1}},Y_{G_{h-1}} \prod_{(i,j) \in \mathcal{S} \times \mathcal{G}_h} q_{Y_{i,j}|X_{i,j}}}^{(h)} \right).
\]

Marginalizing (57), we have

\[
p_{X_{S_m},Y_m} = \prod_{h=1}^m \left( p_{X_{S_h},X_{G_{h-1}},Y_{G_{h-1}} \prod_{(i,j) \in \mathcal{S} \times \mathcal{G}_h} q_{Y_{i,j}|X_{i,j}}}^{(h)} \right).
\]
for each \(m \in \{1, 2, \ldots, \alpha\}\). Relabeling symbols in (58) and using (43) and (44), we have

\[
px_{\mathcal{S}^h \times \mathcal{Y}, \mathcal{Y}_{\mathcal{G}^h}} = \prod_{\ell=1}^{h} \left( \prod_{(i,j) \in \mathcal{S}^h} q_{Y_{i,j},X_{i,j}} \right)
\]

for each \(h \in \{1, 2, \ldots, \alpha\}\), which implies from Proposition 1 that for each \(h \in \{1, 2, \ldots, \alpha\}\) and each \((i, j) \in \mathcal{S}^h \times \mathcal{G}^h\) (cf. (40)),

\[
\left( \{(X_{k,t}, Y_{k,t}) : (k, t) \in \mathcal{S}^h \times \mathcal{G}^h, (k, t) \neq (i, j)\} \right) \rightarrow X_{i,j} \rightarrow Y_{i,j} \right)_{px_{\mathcal{S}^h \times \mathcal{Y}}, \mathcal{Y}_{\mathcal{G}^h}}
\]

forms a Markov chain. Following (55), we consider the following chain of inequalities for a fixed \(px_{\mathcal{S}^h \times \mathcal{Y}}, \mathcal{Y}_{\mathcal{G}^h} = \prod_{h=1}^{\alpha} (px_{\mathcal{S}^h \mid X_{\mathcal{S}^h-1}, Y_{\mathcal{G}^h-1}} q_{Y_{\mathcal{G}^h}, X_{\mathcal{S}^h}})\) and a fixed \(T \subseteq \mathcal{I}\):

\[
\sum_{h=1}^{\alpha} I_{\mathcal{P}_X, \mathcal{Y}}(X_{T \cap \mathcal{S}^h}, Y_{T \cap \mathcal{G}^h-1} ; Y_{T \cap \mathcal{G}^h-1}, Y_{T \cap \mathcal{G}^h}) = \sum_{h=1}^{\alpha} \sum_{j \in T \cap \mathcal{G}^h} I_{\mathcal{P}_X, \mathcal{Y}}(X_{T \cap \mathcal{S}^h}, Y_{T \cap \mathcal{G}^h-1} ; Y_j | X_{T \cap \mathcal{S}^h}, Y_{T \cap \mathcal{G}^h-1}, \{Y_{i} \} \in T \cap \mathcal{G}^h, i < j, \{Y_{m_j} \} \in m_i, m_i \leq i) \\
\overset{(a)}{=} \sum_{h=1}^{\alpha} \sum_{i \in T \cap \mathcal{G}^h} I_{\mathcal{P}_X, \mathcal{Y}}(X_{T \cap \mathcal{S}^h}, Y_{T \cap \mathcal{G}^h-1} ; \{Y_{i} \} \in T \cap \mathcal{G}^h, i < j, \{Y_{m_j} \} \in m_i, m_i \leq i) \\
\overset{(b)}{=} \sum_{h=1}^{\alpha} \sum_{i \in T \cap \mathcal{G}^h} I_{\mathcal{P}_X, \mathcal{Y}}(X_{T \cap \mathcal{S}^h}, Y_{T \cap \mathcal{G}^h-1} ; \{Y_{i} \} \in T \cap \mathcal{G}^h, i < j, \{Y_{m_j} \} \in m_i, m_i \leq i) \\
\overset{(c)}{=} \sum_{h=1}^{\alpha} \sum_{i \in T \cap \mathcal{G}^h} I_{\mathcal{P}_X, \mathcal{Y}}(X_{i,j} ; Y_{i,j}) \\
\overset{(d)}{=} \sum_{(i,j) \in T \times T^c} C_{i,j} \\
\overset{(e)}{=} \sum_{h=1}^{\alpha} \sum_{i \in T \cap \mathcal{G}^h} C_{i,j}
\]

where

(a) follows from the fact that for each \(h \in \{1, 2, \ldots, \alpha\}\) and each \((i, j) \in (\mathcal{I} \setminus \mathcal{S}^h) \times \mathcal{G}^h\), \((i, j) \) lies in \(\Omega^c\) (cf. (40)) and hence

\[
I_{\mathcal{P}_X, \mathcal{Y}}(X_{T \cap \mathcal{S}^h}, Y_{T \cap \mathcal{G}^h-1} ; \{Y_{i} \} \in T \cap \mathcal{G}^h, i < j, \{Y_{m_j} \} \in m_i, m_i \leq i) \leq H_{\mathcal{P}_X, \mathcal{Y}}(Y_{i,j}) \overset{(f)}{=} 0.
\]
(b) follows from the fact that for each \( h \in \{1, 2, \ldots, \alpha \} \) and each \((i,j) \in (T^c \cap S^h) \times (T^c \cap G_h) \subseteq \Omega, \)

\[
I_{p_{X_i}Y_j} (X_{T^c \cap S^h}, Y_{T^c \cap G^{h-1}}; Y_{i,j} | X_{T^c \cap S^h}, Y_{T^c \cap G^{h-1}}, \{Y_t\}_{t \in T^c \cap G_h, t \neq j}, \{Y_{m,j}\}_{m \in I, m \neq i}) \\
= H_{p_{X_i}Y_j} (Y_{i,j} | X_{T^c \cap S^h}, Y_{T^c \cap G^{h-1}}, \{Y_t\}_{t \in T^c \cap G_h, t \neq j}, \{Y_{m,j}\}_{m \in I, m \neq i}) \\
- H_{p_{X_i}Y_j} (Y_{i,j} | X_{S^h}, Y_{T^c \cap G^{h-1}}, \{Y_t\}_{t \in T^c \cap G_h, t \neq j}, \{Y_{m,j}\}_{m \in I, m \neq i}, Y_{T^c \cap G^{h-1}}) \\
= H_{p_{X_i}Y_j} (Y_{i,j} | X_{i,j}) - H_{p_{X_i}Y_j} (Y_{i,j} | X_{i,j}) \\
= 0.
\]

(c) follows from the fact that for each \( h \in \{1, 2, \ldots, \alpha \} \) and each \((i,j) \in (T \cap S^h) \times (T^c \cap G_h) \subseteq \Omega, \)

\[
I_{p_{X_i}Y_j} (X_{T^c \cap S^h}, Y_{T^c \cap G^{h-1}}; Y_{i,j} | X_{T^c \cap S^h}, Y_{T^c \cap G^{h-1}}, \{Y_t\}_{t \in T^c \cap G_h, t \neq j}, \{Y_{m,j}\}_{m \in I, m \neq i}) \\
\leq H_{p_{X_i}Y_j} (Y_{i,j}) - H_{p_{X_i}Y_j} (Y_{i,j} | X_{S^h}, Y_{T^c \cap G^{h-1}}, \{Y_t\}_{t \in T^c \cap G_h, t \neq j}, \{Y_{m,j}\}_{m \in I, m \neq i}, Y_{T^c \cap G^{h-1}}) \\
= I_{p_{X_i}Y_j} (X_{i,j}; Y_{i,j}).
\]

Consequently, it follows from (46), (55) and (60) that (56) holds.

C. Wireless Erasure Network

1) Problem Formulation and Main Result: Consider a DM-MMN \((X_I, Y_I, \alpha, S, G, q)\) defined as follows: Similar to the MMN consisting of independent DMCs discussed in the previous section, we let

\[
\Omega \triangleq \bigcup_{h=1}^{\alpha} S^h \times G_h
\]

characterize the edge set of the network so that \( q_{Y_{i,h_i}}^{h_i}(X_{s_{gh}}, Y_{gh} = 1) \) can be well-defined for each \( h \in \{1, 2, \ldots, \alpha \}. \)

To simulate the broadcast nature of wireless networks, we assume that in every time slot, each node \( i \) broadcasts a symbol \( X_i \) for each \( i \in I \) and we let \( X_i \) denote the finite alphabet of \( X_i \). For each \((i,j) \in \Omega\), we assume that node \( j \) receives \( X_i \) with erasure probability \( \varepsilon_{i,j} \in [0, 1] \), and we let \( Y_{i,j} \) denote the received symbol and its alphabet respectively where \( \varepsilon \) denotes the erasure symbol. For every edge \((i', j')\) that is not in \( \Omega \), we set its erasure probability \( \varepsilon_{i',j'} \) to be 1 and

\[
Y_{i',j'} \triangleq \{ \varepsilon \}, \tag{62}
\]

indicating that no information can be transmitted from \( i' \) to \( j' \). We let \( q_{Y_{i,j}}(X_i) \) characterize the channel corresponding to edge \((i,j)\) such that for each \( x_i \in X_i \) and each \( y_{i,j} \in Y_{i,j}, \)

\[
q_{Y_{i,j}}(X_i | y_{i,j}, x_i) = \begin{cases} 1 - \varepsilon_{i,j} & \text{if } y_{i,j} = x_i, \\ \varepsilon_{i,j} & \text{if } y_{i,j} = \varepsilon. \end{cases} \tag{63}
\]
The symbols transmitted on the edges in $\Omega$ are assumed to be erased independently, i.e.,

$$
\Pr \left\{ \bigcap_{(i,j) \in \Omega} \{ Y_{i,j} = y_{i,j} \} \bigg| X_{I} = x_{I} \right\} = \prod_{(i,j) \in \Omega} q_{Y_{i,j} | X_{i}}(y_{i,j} | x_{i})
$$

for each $x_{I} \in \mathcal{X}_{I}$ and each $\Omega$-dimensional tuple $(y_{i,j} : (i,j) \in \Omega) \in \prod_{(i,j) \in \Omega} \mathcal{Y}_{i,j}$.

For each $(i,j) \in I \times I$, let $E_{i,j}$ and $E_{i,j}$ be the indicator random variable for the erasure occurred on edge $(i,j)$ and its alphabet respectively such that

$$
E_{i,j} \triangleq 1(\{ Y_{i,j} = \varepsilon \}) = \begin{cases} 
1 & \text{if } Y_{i,j} = \varepsilon, \\
0 & \text{if } Y_{i,j} \neq \varepsilon.
\end{cases}
$$

(65)

Let

$$
E_{I \times I} \triangleq (E_{1,1}, E_{1,2}, \ldots, E_{N,N})
$$

be the $N^2$-dimensional random tuple containing all the $E_{i,j}$’s so that $E_{I \times I}$ characterizes the network erasure pattern, and let $E_{I \times I}$ denote the alphabet of $E_{I \times I}$. Recalling that $(V, D)$ is the multicast demand, we assume that the following two statements hold:

(i) All the destinations are contained in $G_{\alpha}$, i.e., $D \subseteq G_{\alpha}$.

(ii) The network erasure pattern $E_{I \times I}$ in each time slot is accessible by each destination node in $D$.

Since $E_{I \times I}$ is a function of $Y_{I}$ by (65), there exists some conditional distribution $\chi_{E_{I \times I} | Y_{I}}$ such that

$$
\chi_{E_{I \times I} | Y_{I}}(e_{I \times I} | y_{I}) = \begin{cases} 
1 & \text{if } e_{i,j} = 1(\{ y_{i,j} = \varepsilon \}) \text{ for all } (i,j) \in I \times I, \\
0 & \text{otherwise}
\end{cases}
$$

(66)

for all $y_{I}$ and $e_{I \times I}$. We are now ready to formally define $X_{I}$, $Y_{I}$ and $q$ as follows: For each $i \in I$, recalling that $X_{i}$ is the finite alphabet of the symbol $X_{i,k}$ broadcast by node $i$, we define

$$
X_{I} \triangleq X_{1} \times X_{2} \times \ldots \times X_{N}.
$$

(67)

Recalling that $Y_{i,j} = X_{i} \cup \{ \varepsilon \}$ is the alphabet of the noisy version of $X_{i,k}$ that is received by node $j$ in each time slot for each $(i,j) \in \Omega$, we define for each $h \in \{ 1, 2, \ldots, \alpha \}$ and each $m \in G_{h}$

$$
Y_{m} \triangleq \begin{cases} 
(\prod_{i \in I} Y_{i,m}) \times E_{I \times I} & \text{if } m \text{ is an element in } D \subseteq G_{\alpha}, \\
\prod_{i \in S_{h}} Y_{i,m} & \text{otherwise}.
\end{cases}
$$

(68)

The definition of $Y_{m}$ in (68) is divided into two cases because we assume according to Statements (i) and (ii) that the destination nodes in $D \subseteq G_{\alpha}$ have access to the network erasure pattern. After defining $Y_{m}$ for each $m \in I$ in (68), we define

$$
Y_{I} \triangleq Y_{1} \times Y_{2} \times \ldots \times Y_{N}.
$$

(69)
Based on the definitions of $X_I$ and $Y_I$ in (67) and (69) respectively and recalling (63) and (66), we define $q_{Y_{(h)}}^{(h)}(x, y_{(h-1)})$ for each $h \in \{1, 2, \ldots, \alpha\}$ as

$$
q_{Y_{(h)}}^{(h)}(x, y_{(h-1)}) = \begin{cases} 
\prod_{i \in I} \prod_{m \in G_i} q_{y_{(m),i}}(y_{(m),i} | x_{(m)}) \chi_{E_{x,I}} | y_{(I \times I)}(e_{I \times I} | y_{(I \times I)}) & \text{if } h = \alpha, \\
\prod_{i \in S^h} \prod_{m \in G_i} q_{y_{(m),i}}(y_{(m),i} | x_{(m)}) & \text{otherwise}
\end{cases}
$$

(70)

for all $x_{(h)} \in X_{(h)}$, $y_{(h)} \in Y_{(h)}$ and $e_{I \times I} \in E_{I \times I}$, where for each $m \in G_h$

$$
y_m = \begin{cases} 
(y_{(m),i} : i \in I, e_{I \times I}) & \text{if } m \text{ is an element in } D \subseteq G_{\alpha}, \\
(y_{(m),i} : i \in S^h) & \text{otherwise}.
\end{cases}
$$

We call the network described above the wireless erasure network. The random variables $X_I$ and $Y_I$ in the wireless erasure network are generated according to this order

$$X_{S_1}, Y_{G_1}, X_{S_2}, Y_{G_2}, \ldots, X_{S_{\alpha}}, Y_{G_{\alpha}}$$

(cf. (2)), which implies from (68), (69) and (70) that $X_I, \{Y_{i,j}\}_{(i,j) \in \Omega}$ and $E_{I \times I}$ are generated according to this order

$$X_{S_1}, \{Y_{i,j}\}_{(i,j) \in S^1 \times G_1}, X_{S_2}, \{Y_{i,j}\}_{(i,j) \in S^2 \times G_2}, \ldots, X_{S_{\alpha}}, \{Y_{i,j}\}_{(i,j) \in S_{\alpha} \times G_{\alpha}}, E_{I \times I}.$$  

(71)

It may not be obvious from (71) that $X_I$ and $E_{I \times I}$ are always independent, but it follows from (63), (64) and (66) that for any $e_{I \times I} \in E_{I \times I}$ and $x_I \in X_I$,

$$\Pr\{E_{I \times I} = e_{I \times I} | X_I = x_I\} = \prod_{(i,j) \in \Omega} \left( \varepsilon_{i,j}^{1(\{e_{i,j} = 1\})} (1 - \varepsilon_{i,j})^{1(\{e_{i,j} = 0\})} \right),$$

(72)

which implies the independence between $X_I$ and $E_{I \times I}$, i.e.,

$$\Pr\{E_{I \times I} = e_{I \times I} | X_I = x_I\} = \Pr\{E_{I \times I} = e_{I \times I}\}$$

(73)

for any $e_{I \times I} \in E_{I \times I}$ and $x_I \in X_I$. The classical wireless erasure network studied in [10] is a special case of our model when $\alpha = 1$ and $\Omega = I \times I$. The following theorem is the main result in this section.

**Theorem 6**: For the wireless erasure network, let

$$R_{W_{\text{m}}} \triangleq \bigcap_{T \subseteq I: I^c \cap D \neq \emptyset} \left\{ R_T \left| \sum_{i \in T} R_i \leq \sum_{i \in T^c} \left( 1 - \prod_{j \in T^c} e_{i,j} \right) |X_i|, |R_i| = 0 \text{ for all } i \in V^c \right. \right\},$$

(74)

Then,

$$C = C_{+} = R_{W_{\text{m}}}$$

and hence the network is delay-independent.
In the following, we provide the proof of Theorem 6. To this end, it suffices to prove the achievability statement

\[ \mathcal{R}_m^{\text{WEN}} \subseteq \mathcal{C}^{+} \]  

(75)

and the converse statement

\[ \mathcal{C} \subseteq \mathcal{R}_m^{\text{WEN}}. \]  

(76)

2) Achievability: In this subsection, we would like to prove (75). Let \( u_{X_i} \) be the uniform distribution on \( \mathcal{X}_i \) for each \( i \in \mathcal{I} \) and let

\[ u_{X_\mathcal{I}, Y_\mathcal{I}} = \left( \prod_{i=1}^{N} u_{X_i} \right) \left( \prod_{h=1}^{\alpha} q_{Y_h | X_h, Y_h^{-1}}^{(h)} \right) \]  

(77)

Fix any \( T \subseteq \mathcal{I} \) such that

\[ T^c \cap \mathcal{D} \neq \emptyset. \]  

(78)

In order to apply Theorem 1, we consider

\[ H_{u_{X_\mathcal{I}, Y_\mathcal{I}}} ( Y_{\mathcal{T}} | X_\mathcal{I}, Y_{\mathcal{T}^c} ) \]

\[ \equiv (a) \ H_{u_{X_\mathcal{I}, Y_\mathcal{I}}} ( Y_{\mathcal{T}} | X_\mathcal{I}, Y_{\mathcal{T}^c}, E_{\mathcal{I} \times \mathcal{I}} ) \]

\[ \equiv (b) \ 0 \]  

(79)

and

\[ I_{u_{X_\mathcal{I}, Y_\mathcal{I}}} ( X_{\mathcal{T}}; Y_{\mathcal{T}^c} | X_{\mathcal{T}^c} ) \]

\[ \equiv (c) \ I_{u_{X_\mathcal{I}, Y_\mathcal{I}}} ( X_{\mathcal{T}}; Y_{\mathcal{T}^c}, E_{\mathcal{I} \times \mathcal{I}} | X_{\mathcal{T}^c} ) \]

\[ \equiv (d) \ I_{u_{X_\mathcal{I}, Y_\mathcal{I}}} ( X_{\mathcal{T}}; Y_{\mathcal{T}}, E_{\mathcal{I} \times \mathcal{I}} | X_{\mathcal{T}^c} ) \]

\[ = H_{u_{X_\mathcal{I}, Y_\mathcal{I}}} ( Y_{\mathcal{T}^c} | X_{\mathcal{T}^c}, E_{\mathcal{I} \times \mathcal{I}} ) - H_{u_{X_\mathcal{I}, Y_\mathcal{I}}} ( Y_{\mathcal{T}^c} | X_{\mathcal{I}}, E_{\mathcal{I} \times \mathcal{I}} ) \]

\[ \equiv (79) \ H_{u_{X_\mathcal{I}, Y_\mathcal{I}}} ( Y_{\mathcal{T}^c} | X_{\mathcal{T}^c}, E_{\mathcal{I} \times \mathcal{I}} ) \]

\[ \equiv (68) \ H_{u_{X_\mathcal{I}, Y_\mathcal{I}}} ( \{ Y_{i,j} \} (i,j) \in \mathcal{I} \times \mathcal{I}^c | X_{\mathcal{T}^c}, E_{\mathcal{I} \times \mathcal{I}} ) \]

\[ \equiv (e) \ H_{u_{X_\mathcal{I}, Y_\mathcal{I}}} ( \{ Y_{i,j} \} (i,j) \in T \times T^c | X_{\mathcal{T}^c}, E_{\mathcal{I} \times \mathcal{I}} ) \]  

(80)

where

(a) follows from Statements (i) and (ii) in the previous subsection and (78) that \( Y_{\mathcal{T}^c} \) contains the random variable \( E_{\mathcal{I} \times \mathcal{I}} \).

(b) follows from (63) and (65) that \( Y_{\mathcal{T}} \) is a function of \( (X_{\mathcal{I}}, E_{\mathcal{I} \times \mathcal{I}}) \).

(c) follows from Statements (i) and (ii) in the previous subsection and (78) that \( Y_{\mathcal{T}^c} \) contains the random variable \( E_{\mathcal{I} \times \mathcal{I}} \).
(d) follows from (73) that $X_I$ and $E_{IX}$ are independent, i.e.,

$$I_{uX_2Y_2}(X_I; E_{IX}) = 0. \quad (81)$$

(e) follows from (63) and (65) that $\{Y_{i,j}\}_{(i,j) \in T^c \times T^c}$ is a function of $(X_{T^c}, E_{IX})$.

In order to further simplify (80), consider the following chain of inequalities for any $T_1, T_2 \subseteq I$ such that $T_1 \cap T_2 = \emptyset$:

$$I_{uX_2Y_2}(\{Y_{i,j}\}_{(i,j) \in T_1 \times T_2}; X_{T_2}|E_{IX})$$

$$\leq I_{uX_2Y_2}(X_{T_1}; \{Y_{i,j}\}_{(i,j) \in T_1 \times T_2}; X_{T_2}|E_{IX})$$

$$(a) \quad I_{uX_2Y_2}(X_{T_1}; X_{T_2}|E_{IX})$$

$$(b) \quad I_{uX_2Y_2}(X_{T_1}; X_{T_2})$$

$$(c) \quad 0$$

where

(a) follows from the fact that $\{Y_{i,j}\}_{(i,j) \in T_1 \times T_2}$ is a function of $(X_{T_1}, E_{IX})$.

(b) follows from (73) that $X_I$ and $E_{IX}$ are independent.

(c) follows from (77) that $X_{T_1}$ and $X_{T_2}$ are independent.

Following (80), consider the following chain of inequalities:

$$H_{uX_2Y_2}(\{Y_{i,j}\}_{(i,j) \in T \times T^c}; X_{T^c}, E_{IX})$$

$$(a) \quad H_{uX_2Y_2}(\{Y_{i,j}\}_{(i,j) \in T \times T^c}; X_{T^c}, E_{IX}) + I_{uX_2Y_2}(X_{T^c}; \{Y_{i,j}\}_{(i,j) \in T \times T^c}; X_{T^c}|E_{IX})$$

$$= H_{uX_2Y_2}(\{Y_{i,j}\}_{(i,j) \in T \times T^c}; X_{T^c}, E_{IX})$$

$$= \sum_{i \in T} H_{uX_2Y_2}(Y_{i}\times T^c|E_{IX}; \{Y_{m,\ell}\}_{m \in T, m \prec i, \ell \in T^c})$$

$$(b) \quad \sum_{i \in T} H_{uX_2Y_2}(Y_{i}\times T^c|E_{IX}; \{X_{m}\}_{m \in T, m \prec i})$$

$$(82) \quad \sum_{i \in T} H_{uX_2Y_2}(Y_{i}\times T^c|E_{IX})$$

$$(c) \quad \sum_{i \in T}(H_{uX_2Y_2}(Y_{i}\times T^c|E_{i}\times T^c) - I_{uX_2Y_2}(E_{i}\times T^c; Y_{i}\times T^c|X_{i}, E_{i}\times T^c))$$

$$(d) \quad H_{uX_2Y_2}(Y_{i}\times T^c|E_{i}\times T^c)$$

where

(a) follows from (82) by letting $T_1 = T$ and $T_2 = T^c$.

(b) follows from (63) and (65) that $\{Y_{m,\ell}\}_{m \in T, m \prec i, \ell \in T^c}$ is a function of $(\{X_{m}\}_{m \in T, m \prec i}, E_{IX})$. 

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(c) follows from the fact that
\[ I_{uX,Y} (E_{\{i\} \times T^c}; Y_{\{i\} \times T^c} | E_{\{i\} \times T^c} ) \]
\[ \leq I_{uX,Y} (E_{\{i\} \times T^c}; X_i, Y_{\{i\} \times T^c} | E_{\{i\} \times T^c} ) \]
\[ (\text{81}) = I_{uX,Y} (E_{\{i\} \times T^c}; Y_{\{i\} \times T^c} | X_i, E_{\{i\} \times T^c} ). \]

(d) follows from (63) and (65) that \( Y_{\{i\} \times T^c} \) is a function of \( (X_i, E_{\{i\} \times T^c} ) \).

Following (83) and letting \( 1^{T^c} \) denote the \(|T^c|\)-dimensional all-1 tuple, we consider the following chain of equalities for each \( i \in T \):

\[ H_{uX,Y} (Y_{\{i\} \times T^c} | E_{\{i\} \times T^c} ) = \Pr \{ E_{\{i\} \times T^c} = 1^{T^c} \} \cdot H_{uX,Y} (Y_{\{i\} \times T^c} | E_{\{i\} \times T^c} = 1^{T^c} ) \]
\[ + \Pr \{ E_{\{i\} \times T^c} \neq 1^{T^c} \} \cdot H_{uX,Y} (X_i | E_{\{i\} \times T^c} \neq 1^{T^c} ) \]
\[ (\text{a}) \cdot \Pr \{ E_{\{i\} \times T^c} \neq 1^{T^c} \} \cdot H_{uX,Y} (X_i ) \]
\[ (\text{b}) \cdot \Pr \{ E_{\{i\} \times T^c} \neq 1^{T^c} \} \cdot |X_i| \]
\[ (\text{c}) \cdot \left( 1 - \prod_{j \in T^c} e_{i,j} \right) |X_i| \]
\[ (84) \]

where

(a) follows from (65) that for each \( j \in I \),
\[ Y_{i,j} = \begin{cases} \varepsilon & \text{if } E_{i,j} = 1, \\ X_i & \text{otherwise}. \end{cases} \]

(b) follows from (73) that \( X_{\mathcal{I}} \) and \( E_{\mathcal{I} \times \mathcal{I}} \) are independent.

(c) follows from (77) that \( X_i \) is uniform on \( |X_i| \).

(d) follows from (72) and (73).

Combining (80), (83) and (84), we have

\[ I_{uX,Y} (X_{T^c}; Y_{T^c} | X_{T^c}) \geq \sum_{i \in T} \left( 1 - \prod_{j \in T^c} e_{i,j} \right) |X_i| \]
\[ (85) \]

Using Theorem 1, (74), (79) and (85), we have \( R_{\text{WEN}}^{\text{in}} \subseteq C_+ \).

3) Converse: In this subsection, we would like to prove (76). We will first prove the following counterpart of Theorem 2 to show an outer bound on \( C \), and then show that the outer bound is contained in \( R_{\text{WEN}}^{\text{in}} \).
Then, there exists a sequence of $E$ where consider the following chain of inequalities:

\[ \begin{align*}
\exists (\mathcal{E}, \mathcal{S}, \mathcal{G}, q) \quad &\text{be a wireless erasure network, and let} \\
\mathcal{R}_{\text{out}}^{\text{WEN}} &\triangleq \\
\bigcup_{p_{X_T,Y_T}} \bigcap_{h=1}^{\alpha} \prod_{h=1}^{\alpha} (p_{X_h | X_{h-1}, Y_{h-1}} q_{Y_h | X_{gh}, Y_{gh-1}}) \\
\end{align*} \]

where $E_{T \times T}$, the network erasure pattern, is a function of $Y_T$ defined by (65). Then,

\[ \mathcal{C} \subseteq \mathcal{R}_{\text{out}}^{\text{WEN}}. \]

**Proof:** Let $R_T$ be an achievable rate tuple for the wireless erasure network denoted by $(\mathcal{X}_T, \mathcal{Y}_T, \alpha, \mathcal{S}, \mathcal{G}, q)$. Then, there exists a sequence of $(B, n, M_T)$-codes on the network such that

\[ \lim_{n \to \infty} \frac{\log M_i}{n} \geq R_i \]  

(87)

and

\[ \lim_{n \to \infty} P_{\text{err}}^n = 0 \]  

(88)

for each $i \in T$. Fix any $T \subseteq I$ such that $T^c \cap D \neq \emptyset$, and let $d$ denote a node in $T^c \cap D$. Fix a $(B, n, M_T)$-code and let $E_{T \times T, k}$ denote the network erasure pattern occurred in time slot $k$ for each $k \in \{1, 2, \ldots, n\}$. Then, we consider the following chain of inequalities:

\[ \sum_{i \in T} \log M_i \overset{(a)}{=} H(W_T | W_{T^c}) \]

\[ \overset{(b)}{=} H(W_T | W_{T^c}, E_{T \times T}^n) \]

\[ = I(W_T; Y_{T^c}^n | W_{T^c}, E_{T \times T}^n) + H(W_T | Y_{T^c}^n, W_{T^c}, E_{T \times T}^n) \]

\[ \leq I(W_T; Y_{T^c}^n | W_{T^c}, E_{T \times T}^n) + H(W_T | Y_{d}^n, W_d, E_{T \times T}^n) \]

\[ \leq I(W_T; Y_{T^c}^n | W_{T^c}, E_{T \times T}^n) + 1 + P_{\text{err}}^n \sum_{i \in T} \log M_i, \]  

(89)

where

(a) follows from the fact that the $N$ messages $W_1, W_2, \ldots, W_N$ are independent.

(b) follows from the fact that $W_T$ and $E_{T \times T}^n$ are independent.

(c) follows from Fano’s inequality.

Following similar procedures for proving Theorem 1 in [2], we can show by using (87), (88) and (89) that there exists a joint distribution $p_{X_T, Y_T}$ which depends on the sequence of $(B, n, M_T)$-codes but not on $T$ such that

\[ p_{X_T, Y_T} = \prod_{h=1}^{\alpha} (p_{X_h | X_{gh-1}, Y_{gh-1}} q_{Y_h | X_{gh}, Y_{gh-1}}) \]
and
\[ \sum_{i \in T} R_i \leq \sum_{h=1}^{\alpha} I_{p_{X,Y}}(X_{T \cap S^h}, Y_{T \cap G^h-1}; Y_{T \cap G^h-1}, X_{T \cap S^h}, Y_{T \cap S^h}, E_{I \times I}). \] (90)

Since \( p_{X,Y} \) depends on only the sequence of \((B, n, M_z)\)-codes but not on \( T \), (90) holds for all \( T \subseteq I \) such that \( T^c \cap D \neq \emptyset \). This completes the proof.

Since
\[ C \subseteq R_{w_{\text{out}}} \] (91)
by Theorem 2 and our goal is to prove \( C \subseteq R_{w_{\text{in}}} \), it remains to show that
\[ R_{w_{\text{out}}} \subseteq R_{w_{\text{in}}}. \] (92)

For any \( p_{X,Y} = \prod_{h=1}^{\alpha} (p_{X_{S_h}}|X_{S_h-1}, Y_{G_h-1}, q_{Y_{G_h}}|X_{S_h}, Y_{G_h-1}) \), it follows from (63) and (65) that for each \( h \in \{1, 2, \ldots, \alpha\} \) and each \((i, j) \in S^h \times G_h\), \( Y_{i,j} \) is a function of \((X_i, E_{I \times I})\) and hence
\[ \{(X_i, Y_{i,j}) : (k, \ell) \in (S^h \setminus \{i\}) \times G_h \} \rightarrow (X_i, E_{I \times I}) \rightarrow Y_{i,j} \] (93)
forms a Markov chain. Following (92) and (86), we fix \( p_{X,Y} = \prod_{h=1}^{\alpha} (p_{X_{S_h}}|X_{S_h-1}, Y_{G_h-1}, q_{Y_{G_h}}|X_{S_h}, Y_{G_h-1}) \) and \( T \subseteq I \) such that \( T^c \cap D \neq \emptyset \), and we consider
\[ \sum_{h=1}^{\alpha} I_{p_{X,Y}}(X_{T \cap S^h}, Y_{T \cap G^h-1}; Y_{T \cap G^h-1}, X_{T \cap S^h}, Y_{T \cap S^h}, E_{I \times I}) \]
\[ \overset{(a)}{=} \sum_{h=1}^{\alpha} \sum_{i \in S^h} I_{p_{X,Y}}(X_{T \cap S^h}, Y_{T \cap G^h-1}; Y_{i}(T \cap G_h)|X_{T \cap S^h}, X_{T \cap G^h-1}, \{Y_{i}(T \cap G_h)\} \ell<i, E_{I \times I}) \]
\[ \overset{(b)}{=} \sum_{h=1}^{\alpha} \sum_{i \in T \cap S^h} I_{p_{X,Y}}(X_i; Y_{i}(T \cap G_h)|Y_{i}(T \cap G^h-1), E_{I \times I}) \]
\[ \overset{(c)}{=} \sum_{h=1}^{\alpha} \sum_{i \in T \cap S^h} H_{p_{X,Y}}(Y_{i}(T \cap G_h)|Y_{i}(T \cap G^h-1), E_{I \times I}) \]
\[ = \sum_{i \in T \cap S^h} H_{p_{X,Y}}(Y_{i}(T \cap G_h)|E_{I \times I}) \]
\[ \leq \sum_{i \in T \cap S^h} H_{p_{X,Y}}(Y_{i}(T \cap G_h)|E_{I \times I}) \] (94)
where
(a) follows from the fact that for each \( h \in \{1, 2, \ldots, \alpha\} \) and each \( i \in T^c \cap S^h \),
\[ I_{p_{X,Y}}(X_{T \cap S^h}, Y_{T \cap G^h-1}; Y_{i}(T \cap G_h)|X_{T \cap S^h}, Y_{T \cap G^h-1}, \{Y_{i}(T \cap G_h)\} \ell<i, E_{I \times I}) \]
\[ = H_{p_{X,Y}}(Y_{i}(T \cap G_h)|X_{T \cap S^h}, Y_{T \cap G^h-1}, \{Y_{i}(T \cap G_h)\} \ell<i, E_{I \times I}) \]
\[
- H_{P_{X_Z^T Y Z}}(Y_{(i)\times(T^c\cap S^h)}|X_{S^h}, Y_{G^h-1}, \{Y_{(t)\times T^c\cap S^h}\}_{t<i}, E_{T^c\times I})
\]
\[
= H_{P_{X_Z^T Y Z}}(Y_{(i)\times(T^c\cap S^h)}|X_i, E_{T^c\times I})
- H_{P_{X_Z^T Y Z}}(Y_{(i)\times(T^c\cap S^h)}|X_i, E_{T^c\times I})
\]
\[
= 0.
\]

(b) follows from the fact that for each \(h \in \{1, 2, \ldots, \alpha\}\) and each \(i \in T \cap S^h\),
\[
I_{P_{X_Z^T Y Z}}(X_{T^c\cap S^h}, Y_{T^c\cap G^h-1}; Y_{(i)\times(T^c\cap S^h)}|X_{T^c\cap S^h}, Y_{T^c\cap G^h-1}, \{Y_{(t)\times T^c\cap S^h}\}_{t<i}, E_{T^c\times I})
\]
\[
\leq H_{P_{X_Z^T Y Z}}(Y_{(i)\times(T^c\cap S^h)}|Y_{(i)\times T^c\cap S^h}, E_{T^c\times I})
- H_{P_{X_Z^T Y Z}}(Y_{(i)\times(T^c\cap S^h)}|X_{S^h}, Y_{G^h-1}, \{Y_{(t)\times T^c\cap S^h}\}_{t<i}, E_{T^c\times I})
\]
\[
\leq I_{P_{X_Z^T Y Z}}(X_i; Y_{(i)\times(T^c\cap S^h)}|Y_{(i)\times T^c\cap S^h}, E_{T^c\times I})
- H_{P_{X_Z^T Y Z}}(Y_{(i)\times(T^c\cap S^h)}|X_i, E_{T^c\times I})
\]
\[
(93)
\]
(c) follows from (63) and (65) that \(Y_{(i)\times(T^c\cap S^h)}\) is a function of \((X_i, E_{T^c\times I})\).

Following (94) and letting \(1^{T^c}\) denote the \(|T^c|\)-dimensional all-1 tuple, we consider the following chain of inequalities for each \(h \in \{1, 2, \ldots, \alpha\}\) and each \(i \in T \cap S^h\):
\[
H_{P_{X_Z^T Y Z}}(Y_{(i)\times T^c}|E_{(i)\times T^c})
\]
\[
= \Pr\{E_{(i)\times T^c} = 1^{T^c}\} H_{P_{X_Z^T Y Z}}(Y_{(i)\times T^c}|E_{(i)\times T^c} = 1^{T^c})
\]
\[
+ \Pr\{E_{(i)\times T^c} \neq 1^{T^c}\} H_{P_{X_Z^T Y Z}}(Y_{(i)\times T^c}|E_{(i)\times T^c} \neq 1^{T^c})
\]
\[
(94)
\]
\[
\leq \Pr\{E_{(i)\times T^c} \neq 1^{T^c}\} |X_i| \quad (95)
\]
where

\[
(95)
\]

(a) follows from (65) that for each \(j \in I\),
\[
Y_{i,j} = \begin{cases} 
\varepsilon & \text{if } E_{i,j} = 1, \\
X_i & \text{otherwise}. 
\end{cases}
\]

(b) follows from (72) and (73).

Combining (94) and (95), we have
\[
\sum_{h=1}^{\alpha} I_{P_{X_Z^T Y Z}}(X_{T^c\cap S^h}, Y_{T^c\cap G^h-1}; Y_{T^c\cap S^h}, Y_{T^c\cap G^h-1}, E_{T^c\times I})
\]
\[
\leq \sum_{i \in T \cap S^h} \left(1 - \prod_{j \in T^c} E_{i,j}\right) |X_i|
\]
\[ \leq \sum_{i \in T} \left( 1 - \prod_{j \in T} e_{i,j} \right) |X_i| . \] (96)

Consequently, it follows from (74), (86) and (96) that (92) holds, which implies from (91) that \( C \subseteq R^{\text{WEN}} \).

VI. Conclusion

We have investigated under the generalized model three classes of delay-independent multimeasure multicast networks (MMNs), namely the deterministic MMN dominated by product distributions, the MMN consisting of independent DMCs and the wireless erasure network respectively. We are able to evaluate the capacity regions for the above classes of generalized MMNs and demonstrate that their capacity regions coincide with the positive-delay regions, which implies that the above classes of MMNs belong to the category of delay-independent MMNs. In other words, for each generalized MMN which belongs to one of the above three classes, the set of achievable rate tuples does not depend on the amount of delay incurred by each node in the network. This is in contrast to the fact that for some generalized MMNs, the set of achievable rate tuples shrinks if we impose the additional constraint that each node incurs a positive delay. An important implication of our result is that for each MMN belonging to one of the above three classes, using different methods for handling delay and synchronization does not affect the network capacity.

One direction for future research is continuing the theme of this work – to identify other important classes of delay-independent and delay-dependent MMNs under the generalized model. Another direction is exploring more general connection types such as multiple-unicast and multiple-multicast demands for different networks under the generalized model.

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