RECONSTRUCTING OBSTACLES USING CGO SOLUTIONS FOR THE BIHARMONIC EQUATION

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Abstract. In this article, we study an inverse problem for detecting unknown obstacle by the enclosure method using the Dirichlet to Neumann map as measurements. We justify the method for the impenetrable obstacle case involving the biharmonic equation. We use complex geometrical optics solutions with logarithmic phase to reconstruct some non-convex part of the obstacle. The proof is based on the global $L^p$-estimates for the gradient and Laplacian of the solutions of the biharmonic equation for $p$ near 2.

1. Introduction

The inverse problem in this paper is to determine an unknown obstacle or jump of the inclusions embedded in a known background medium from the near field measurement. Several methods have been proposed in the literature to detect this jump based on mainly two types of special solutions. The one is the Green’s type or singular solutions of the elliptic equations, which were introduced by Isakov, see [16] and subsequent methods have been developed, for example linear sampling method in [3], [4], and factorization method in [11], [22]. Then another set of solutions is the complex geometrical optics solutions, which were considered by Ikehata, [12], to develop enclosure method to reconstruct the unknown inclusions. The main interest in this paper is to discuss the enclosure method for the first order perturbation of the biharmonic type operator.

Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded smooth domain. We assume $D(\subset \subset \Omega)$ to be an unknown obstacle with Lipschitz regular boundary $\partial D$ such that $\Omega \setminus D$ is connected. As a model problem, we consider the first order perturbation of the biharmonic equation with Navier boundary condition

$$
\begin{align*}
\Delta (\tilde{\gamma} \Delta u) + \tilde{A} \cdot Du + \kappa^2 \tilde{n} u &= 0 \text{ in } \Omega \\
u &= f_1 \text{ on } \partial \Omega \\
\Delta u &= f_2 \text{ on } \partial \Omega,
\end{align*}
$$

where $D := -i^{-1} \nabla$ and the physical quantities $\tilde{\gamma}$, $\tilde{A}$, and $\tilde{n}$ satisfy the following assumption:

Assumption 1.

(i) We assume that $\tilde{\gamma}(x) = 1 + \gamma_D(x) \chi_D(x)$ and $\tilde{n}(x) = 1 + q_D(x) \chi_D(x)$, for all $x \in \Omega$ such that $\tilde{\gamma} \in L^\infty(\Omega, \mathbb{C})$ and $\tilde{q} \in L^\infty(\Omega, \mathbb{C})$. Here $\chi_D$ is characteristic function of $D$. Let us also assume that $\gamma_D \in C^2(D)$, $q_D \in L^\infty_+(D)$, where $L^\infty_+(D) := \{f \in L^\infty(D); f \geq C > 0 \text{ for some } C \in \mathbb{R}\}$.

(ii) We also assume that $\tilde{A} \in L^\infty_+(\Omega, \mathbb{C}^n)$ such that

$$
\tilde{A}(x) = \begin{cases} A_D(x) & \text{when } x \in D \\ 0 & \text{when } x \in \Omega \setminus D, \end{cases}
$$

where $A_D \in C^1(D)$. 

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Moreover, we need to choose $\frac{1}{\tilde{\gamma}}, \tilde{n}$ and $\tilde{A}$ so small that estimates (5.7) and (5.8) must satisfy. Such an assumption is required due to the fact that the sesquilinear form in Lemma 5.1 holds to be coercive.

Here, $\kappa > 0$ denote the wave number corresponding to the problem (1.1) in the domain $\Omega$. In practice, the biharmonic equation arises in many areas of physics, in particular, to study the elasticity theory, plate plasma and stokes flow, see for instance [8]. An operator similar to $\Delta(\gamma \Delta u)$ has been considered in [18] to study the inverse boundary value problems for the thin elastic plates. In [18], they posed the problem in the planner domain for the system $\text{div} \div (C \nabla^2 u) = 0$, where $C$ is the 4th order tensor. Inspired by [18], we consider the scalar equation of the type (1.1), where the coefficient $\tilde{\gamma}$ is a scalar valued function. The other parameters $\tilde{n}$ and $\tilde{A}$ in equation (1.1) correspond to the index of refraction and the magnetic field of the medium respectively. We denote $\mathcal{L}_{\tilde{\gamma}, \tilde{A}, \tilde{n}}$ as the operator

$$\mathcal{L}_{\tilde{\gamma}, \tilde{A}, \tilde{n}} := \Delta(\tilde{\gamma} \Delta u) + \tilde{A} \cdot Du + \kappa^2 \tilde{n} u,$$

where $D := -i^{-1} \nabla$. The domain of definition of the operator $\mathcal{L}_{\tilde{\gamma}, \tilde{A}, \tilde{n}}$ is $\mathcal{D}(\mathcal{L}_{\tilde{\gamma}, \tilde{A}, \tilde{n}}) := \{ u \in H^4(\Omega); u|_{\partial \Omega} = (\Delta u)|_{\partial \Omega} = 0 \}$, which is a closed operator on $L^2(\Omega)$ with purely discrete spectrum, see [10]. We assume that 0 is not an eigenvalue of the operator $\mathcal{L}_{\tilde{\gamma}, \tilde{A}, \tilde{n}} : \mathcal{D}(\mathcal{L}_{\tilde{\gamma}, \tilde{A}, \tilde{n}}) \to \mathcal{L}^2(\Omega)$, related to the problem (1.1). By the standard well-posedness of the boundary value problem for the forth-order elliptic equation, the problem (1.1) has a unique solution $u \in H^4(\Omega)$ for any $f_1 \in H^{7/2}(\partial \Omega)$ and $f_2 \in H^{3/2}(\partial \Omega)$. We define the Dirichlet-to-Neumann map corresponding to the above biharmonic problem is as follows:

$$\mathcal{N}_{\tilde{\gamma}, \tilde{A}, \tilde{n}} : H^{7/2}(\partial \Omega) \times H^{3/2}(\partial \Omega) \to H^{5/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)$$

by

$$\mathcal{N}_{\tilde{\gamma}, \tilde{A}, \tilde{n}} (f_1, f_2) = \left( \frac{\partial u}{\partial \nu}|_{\partial \Omega}, \frac{\partial}{\partial \nu}(\Delta u)|_{\partial \Omega} \right),$$

where $u$ is the solution to the equation (1.1) and $\nu$ denote the outward unit normal vector to $\partial \Omega$. We now state the following theorem.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n, (n \geq 3)$ be a bounded smooth domain and let $D$ be a Lipschitz sub-domain contained in $\Omega$. Furthermore, we assume that $\Omega \setminus D$ is connected and coefficients satisfy Assumption 1. Then, the Dirichlet to Neumann map $\mathcal{N}_{\tilde{\gamma}, \tilde{A}, \tilde{n}}$ determines the shape and location of $D$.

To prove this Theorem, we apply the enclosure method, proposed by Ikehata [15], see also [12], [13] and [14]. In [15], he considered the inverse problem for conductivity equation as a model problem and used complex geometrical optics solution with linear phase to detect the convex hull of the obstacle. The main idea behind this method is as follows: We first define an indicator function, $I_{\tilde{\gamma}_0}(h, t)$, see (3.1), via the difference of Dirichlet to Neumann map $\mathcal{N}_{\tilde{\gamma}, \tilde{A}, \tilde{n}} - \mathcal{N}_{1,0} \tilde{A}$ and CGO solutions on the boundary. The indicator function represents the energy difference when there is an obstacle inside $\Omega$ and there is no obstacle in $\Omega$. We only consider CGOs with logarithmic phase. In Theorem 3.1 we show an asymptotic estimate of the indicator function for a small parameter $h > 0$ and this indicates whether or not a level set of the phase function touches the obstacle surface. Finally, intersection of all the level sets touching the interface determines not only the convex hull of the obstacle.
but also some non-convex part of it. A lot of work has been done to detect the unknown obstacle using this enclosure method for various other kinds of partial differential equations, see for example in the case of Maxwell system [20], elasticity equation [19], and Helmholtz equation [28]. In [28], authors considered the problem of determining unknown obstacle for the divergence form elliptic equations with lower order terms from the knowledge of Dirichlet to Neumann map. The result has been proved by using Meyers $L^p$-estimates, see [27], to remove some geometrical assumption on the obstacle surface. Ideas of using $L^p$-estimates to prove the enclosure method for the reconstruction, have been implemented also in [19] and [20]. Calderón problem corresponding to the biharmonic operator has been first studied by Krupchyk-Lassas-Uhlmann [23] to prove unique determination of the first order perturbation $A(x) \cdot D + q$ of the biharmonic equation from the Dirichlet to Neumann map measured on the part of the boundary. Further results on the inverse problems for the biharmonic equation can be found, see for example in [2], [24], [25] and the references therein.

We now explain some ideas to prove Theorem 1.1. Since the method is based on the asymptotic behavior of $I_{s,p}(h,t)$, our goal in this paper is to prove Theorem 3.1 Due to Lemma 4.1, it is enough to provide a lower and upper bound of the indicator function when $t = h_D(x)$. In view of Lemma 4.9 one needs an appropriate estimate of the corresponding reflected solution $w := u - v$, where $u$ satisfies (1.1) and $v$ is the CGO solution of the equation (3.3). Precisely, the reflected solution $w$ of the equation (4.3) satisfies
\[ \|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} \leq C \left( \|\Delta v\|_{L^p(D)} + \|\nabla v\|_{L^p(D)} + \|v\|_{L^p(D)} \right), \] (1.2)
where $p \leq 2$ and near 2, see Proposition 4.2. To prove estimate (1.2) we first reduce our problem to the system of second order linear elliptic equations. Then we use $W^{s,p}$-estimate, see for example the work of Jerison and Kenig, [17], for the solutions of equation $\Delta \Phi = F$ of the form
\[ \|\Phi\|_{W^{1-s,p'}(\Omega)} \leq \|F\|_{W^{-1-s,p'}(\Omega)}, \]
for $p' > 2$ and $s < 1 - \frac{1}{p'}$ and the Sobolev embedding $L^2(\Omega) \hookrightarrow W^{-1-s,p'}(\Omega)$ for all $p' > 2$ and $\frac{3}{2} - \frac{s}{p} \leq s < \frac{3}{2}$, where $\frac{1}{p} + \frac{1}{p'} = 1$. A detailed explanation of the proof of (1.2) is given in Section 4. To justify the enclosure method, we need to construct CGO solutions with an appropriate decay estimate in the correction term. In [23], authors constructed CGO solutions of the form
\[ v(x; h) = e^{\frac{\phi(x)}{h}} (a_0(x) + ha_1(x) + r(x, h)), \] (1.3)
where $\phi \in C^\infty(\bar{\Omega}, \mathbb{R})$ is a limiting Carleman weight for the semiclassical Laplacian on $\bar{\Omega}$, where $\Omega \subset \subset \bar{\Omega}$. The functions $a_0, a_1$ are smooth and the correction term $r$ satisfies $\|r\|_{H^2(\Omega)} = O(h^2)$. However, to deal with our shape reconstruction problem, CGOs of the form (1.3) are not enough. Instead, we construct solutions of the form $v(x; h) = e^{\frac{\phi(x)}{h}} (a_0(x) + ha_1(x) + h^2a_2(x) + r(x, h))$, for sufficiently small $h > 0$, where the correction term $r$ satisfies $\|r\|_{H^2(\Omega)} = O(h^3)$. Finally we deduce an asymptotic estimate for the indicator function using this CGO solutions for small enough $h > 0$.

The paper is organized as follows: In Section 2 we discuss the construction of CGO solution. In Section 3 we sate our main result, and finally the proof of the result has been provided in Section 4.
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2. Complex geometric optics

In this section, we construct CGO solutions for the following forth order elliptic equation

$$\mathcal{L}_{A,q} v := (\Delta^2 + A \cdot D + \kappa^2 q(x))v = 0 \text{ in } \Omega,$$

where $A = (A_j)_{1 \leq j \leq n} \in C^4(\bar{\Omega}, \mathbb{C}^n)$ and $q \in L^\infty(\Omega, \mathbb{C})$ with $q(x) \geq C$, for some constant $C > 0$. Note that $D = -i^{-1} \nabla$. We mainly follow the work of [23], see also, [9], [21], to construct such solutions using the Carleman estimate method. CGO solutions of the form $v(x; h) = e^{\frac{-i \xi \cdot \rho}{h}}(a_0(x) + ha_1(x) + r(x; h))$, were derived in [23, Proposition 2.4] to determine the first order perturbation of the biharmonic operator, are not enough to justify the enclosure method, see Lemma [6.1]. So, we need solutions with stronger estimate in the correction term $r$. In particular, we are looking for the solution of the form

$$v(x; h) = e^{\frac{-i \xi \cdot \rho}{h}}(a_0(x) + ha_1(x) + h^2 a_2(x) + r(x; h)),$$

for sufficiently small $h > 0$. Here $\phi \in C^\infty(\bar{\Omega}, \mathbb{R})$ is a limiting Carleman weight for semiclassical Laplacian on $\bar{\Omega}$ and $\bar{\Omega}$ is an open set in $\mathbb{R}^n$ such that $\Omega \subset \subset \bar{\Omega}$. Recall that $\phi$ is a limiting Carleman weight for the semiclassical Laplacian $-h^2 \Delta$ in $\bar{\Omega}$, if $\nabla \phi \neq 0$ in $\bar{\Omega}$ and the Poisson bracket of Re $p_\phi$ and Im $p_\phi$ satisfies $\{\text{Re } p_\phi, \text{Im } p_\phi\}(x, \xi) = 0$ when $p_\phi(x, \xi) = 0$, $(x, \xi) \in \bar{\Omega} \times \mathbb{R}^n$, where $p_\phi(x, \xi) = \xi^2 + 2i \nabla \phi \cdot \xi - |\nabla \phi|^2$, for all $x \in \bar{\Omega}$, $\xi \in \mathbb{R}^n$ is the semiclassical principal symbol corresponding to the conjugated operator $P_\phi = e^{\frac{i}{\hbar}(-h^2 \Delta)}e^{-\frac{i}{\hbar}}$. The function $\psi \in C^\infty(\bar{\Omega}, \mathbb{R})$ satisfies the eikonal equation

$$\begin{cases}
|\nabla \psi|^2 = |\nabla \phi|^2 \\
\nabla \phi \cdot \nabla \psi = 0 \text{ in } \bar{\Omega}.
\end{cases}$$

We note that the amplitude $a_0 \in C^\infty(\bar{\Omega})$, $a_1 \in C^4(\bar{\Omega})$, $a_2 \in C^4(\bar{\Omega})$ are solutions of the first (2.8), second (2.9) and third (2.10) transport equations respectively and the correction term $r$ has the following behavior $\|r\|_{H^s_{\text{cl}}(\Omega)} \leq O(h^3)$, where the semiclassical Sobolev norm is defined as $\|r\|_{H^s_{\text{cl}}(\Omega)} = \|\langle h D \rangle^s r\|_{L^2}$ for $s \in \mathbb{R}$ with $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. It is well known that, CGOs with different phase functions $\phi + i\psi$ play a crucial role in enclosure method to reconstruct different shapes of the obstacle. For instance, using the linear phase $\phi = x \cdot \rho$, where $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$, one can reconstruct the convex hull of the obstacle, see [15]. In this work, we are interested in CGO solutions with nonlinear phase. In particular, we use solutions with logarithmic phases. Such kind of solutions derived by Kenig-Sjöstrand-Uhlmann for the stationary Schrödinger equation [21] to solve Calderón uniqueness problem for the partial data case. In our case Carleman weight is of the form of $\phi(x) = \frac{1}{2} \log |x - x_0|^2$ and

$$\psi(x) = \frac{\pi}{2} - \tan^{-1} \left( \frac{w \cdot (x - x_0)}{\sqrt{(x - x_0)^2 - (w \cdot (x - x_0))^2}} \right) = \text{dist}_{S^{n-1}} \left( \frac{x - x_0}{|x - x_0|}, w \right).$$
where \( w \in S^{n-1} \) is chosen so that \( \psi \) is smooth near \( \Omega \) and \( x_0 \) is a fixed point outside the convex hull of \( \Omega \). Recall that, for a given \( \phi \) the function \( \psi \) satisfies the eikonal equation \((2.3)\) on \( \Omega \). We next use Carleman estimate and the WKB method to construct complex geometric optics solutions for the equation \((2.1)\). Iterating the Carleman estimate for the semiclassical Laplacian \(-h^2\Delta\), we can obtain the estimate for the semiclassical biharmonic operator for sufficiently small \( h \). If we define \( \mathcal{L}_\phi = e^{\frac{\phi}{h}}h^4\mathcal{L}_{A,q}e^{-\frac{\phi}{h}} \), then the Carleman estimate for the first order perturbation of the biharmonic equation is as follows
\[
\|u\|_{H_{scl}^{s+\frac{1}{2}}} \leq \frac{C_{s,\Omega,A,q}}{h^2} \|\mathcal{L}_\phi u\|_{H_{scl}^{s}},
\]
for all \( u \in C_0^\infty(\Omega) \) with \( h > 0 \) small enough and \( s \in [-4,0] \), see [23], Proposition 2.2, for a detailed derivation of the estimate \((2.5)\). Finally, using the estimate \((2.5)\), we state the following solvability result of the conjugated operator equation
\[
\mathcal{L}_\phi u = v \in \Omega.
\]

**Proposition 2.1** ([23], Proposition 2.3). Assume that \( A \in C^4(\Omega,\mathbb{C}^n) \), \( g \in L_\infty^2(\Omega,\mathbb{C}) \) and also let \( \phi \) be a limiting Carleman weight for the semiclassical Laplacian on \( \Omega \). Then for any \( v \in L^2(\Omega) \), there exists a solution \( u \in H^4(\Omega) \) of the equation \((2.6)\) such that
\[
\|u\|_{H_{scl}^{s+\frac{1}{2}}} \leq \frac{C}{h^2} \|\mathcal{L}_\phi u\|_{L^2},
\]
holds for sufficiently small \( h > 0 \).

Now we use WKB method for the conjugated operator \( \mathcal{L}_\phi \). Consider the operator
\[
e^{-\frac{\phi}{h}}h^4\mathcal{L}_{A,q}e^{\frac{\phi}{h}} = (h^2\Delta + 2hT)^2 + h^3A \cdot hD + h^3A \cdot (D\phi + iD\psi) + h^4\kappa^2q(x)
\]
where \( T = (\nabla\phi + i\nabla\psi) \cdot \nabla + \frac{1}{2}(\Delta\phi + i\Delta\psi) \). Substituting \((2.2)\) in \((2.1)\), we obtain
\[
\mathcal{L}_{A,q}e^{\frac{\phi}{h}}(a_0 + ha_1 + h^2a_2 + r) = 0,
\]
which we write as
\[
e^{-\frac{\phi}{h}}h^4\mathcal{L}_{A,q}e^{\frac{\phi}{h}}(a_0 + ha_1 + h^2a_2 + r) = 0. \tag{2.7}
\]

We compute
\[
[(h^2\Delta + 2hT)^2 + h^3A \cdot hD + h^3A \cdot (D\phi + iD\psi)
+ h^4\kappa^2q(x)](a_0 + ha_1 + h^2a_2)
= [h^4\Delta^2 + 2h^3(\Delta \circ T + T \circ \Delta) + 4h^2T^2 + h^3A \cdot hD + h^3A \cdot (D\phi + iD\psi)
+ h^4\kappa^2q(x)](a_0 + ha_1 + h^2a_2)
= h^6 \left[ \Delta^2a_2 + A \cdot Da_2 + \kappa^2q(x)a_2 \right]
+ h^5[\Delta^2a_1 + A \cdot Da_1 + \kappa^2q(x)a_1 + 2(\Delta \circ T + T \circ \Delta)a_2
+ A \cdot (D\phi + iD\psi)a_2]
+ h^4[\Delta^2a_0 + 2(\Delta \circ T + T \circ \Delta)a_1 + 4T^2a_2
+ (A \cdot Da_0 + A \cdot (D\phi + iD\psi)a_1) + \kappa^2q(x)a_0]
+ h^3[2(\Delta \circ T + T \circ \Delta)a_0 + A \cdot (D\phi + iD\psi)a_0 + 4T^2a_1] + h^2 \left[ 4T^2a_0 \right].
\]
We choose \( \psi, a_0, a_1 \) and \( a_2 \) so that, \( \psi \) solves the eikonal equation. The function \( a_0 \) solves the first transport equation

\[
T^2 a_0 = 0 \text{ in } \Omega. \tag{2.8}
\]

The function \( a_1 \) solves the second transport equation

\[
T^2 a_1 = -\frac{1}{2}(\Delta \circ T + T \circ \Delta) a_0 - \frac{1}{4} A \cdot (D\phi + iD\psi) a_0 \text{ in } \Omega, \tag{2.9}
\]

and the function \( a_2 \) solves the third transport equation

\[
\Delta^2 a_0 + 2(\Delta \circ T + T \circ \Delta) a_1 + 4T^2 a_2 + A \cdot Da_0 + A \cdot (D\phi + iD\psi) a_1 + \kappa^2 q(x) a_0 = 0. \tag{2.10}
\]

The correction term \( r \) satisfies

\[
e^{-\frac{a_1 + i\psi}{h}} h^4 \mathcal{L}_{A,q}(e^{\frac{\phi + i\psi}{h}} r) = -e^{-\frac{a_1 + i\psi}{h}} h^4 \mathcal{L}_{A,q}(e^{\frac{\phi + i\psi}{h}} (a_0 + a_1 h + a_2 h^2))
\]

\[
= -h^6 [\Delta^2 a_2 + A \cdot Da_2 + \kappa^2 q(x) a_2]
\]

\[
- h^5 [\Delta^2 a_1 + A \cdot Da_1 + \kappa^2 q(x) a_1 + 2(\Delta \circ T + T \circ \Delta) a_2]
\]

\[
+ A \cdot (D\phi + iD\psi) a_2]. \tag{2.11}
\]

We now follow \cite{6}, \cite{23}, to show that, the above equations \((2.8), (2.9), (2.10)\) and \((2.11)\) are solvable. We choose co-ordinates in \( \mathbb{R}^n \) so that \( x_0 = 0 \) and \( \Omega \subset \{ x_n > 0 \} \). We set \( w = e_1 = (1, 0, \cdots, 0) \) and introduce the cylindrical coordinate \((x_1, r\theta)\) on \( \mathbb{R}^n \) with \( r > 0 \) and \( \theta \in \mathbb{S}^{n-2} \). We also consider the change of coordinate in the complex plane \( x \mapsto (z, \theta) \) where \( z = x_1 + ir \). In this coordinate system, the limiting Carleman weights are of the form \( \phi = \log |z| = \text{Re} \log z, \psi = \frac{z}{2} - \tan^{-1} \left( \frac{\text{Re} z}{\text{Im} z} \right) = \text{Im} \log z \), when \( \text{Im} z > 0 \). So we have \( \phi + i\psi = \log z \). Moreover, \( \nabla(\phi + i\psi) = \frac{1}{z}(e_1 + ie_r) \), where \( e_r = (0, \theta) \) is the unit vector pointing in the direction of the \( r \)-axis. We also have

\[
\nabla(\phi + i\psi) \cdot \nabla = \frac{\partial \log z}{\partial x_1} \frac{\partial}{\partial x_1} + \frac{\partial \log z}{\partial r} \frac{\partial}{\partial r} = \frac{2}{z} \frac{\partial}{\partial z} \text{ and } \Delta(\phi + i\psi) = -\frac{2(n - 2)}{z(z - \overline{z})}.
\]

In the cylindrical coordinate system the operator \( T \) can be written as \( T = \frac{2}{z} \left( \frac{\partial}{\partial z} - \frac{n - 2}{2(z - \overline{z})} \right) \).

Therefore, the first transport equation reduces to

\[
\left( \frac{\partial}{\partial z} - \frac{n - 2}{2(z - \overline{z})} \right)^2 a_0 = 0 \text{ in } \Omega. \tag{2.12}
\]

Now, if we take \( a_0 = (z - \overline{z})^\frac{n}{2-n} g_0 \), where \( g_0 \in C^\infty(\overline{\Omega}) \) satisfying \( \frac{\partial g_0}{\partial z} = 0 \), i.e., \( g_0 \) is a holomorphic function of \( z = x_1 + ir \). Then, \( a_0 \) satisfies \( \left( \frac{\partial}{\partial z} - \frac{n - 2}{2(z - \overline{z})} \right) a_0 = 0 \) and hence \( a_0 \) solves \((2.12)\).

In this coordinate system, the second transport equation has the form

\[
\left( \frac{\partial}{\partial z} - \frac{n - 2}{2(z - \overline{z})} \right)^2 a_1 = F \text{ in } \Omega, \tag{2.13}
\]

where \( F = -\frac{n}{2} [\frac{1}{2}(\Delta \circ T + T \circ \Delta) a_0 + \frac{1}{4} A \cdot (D\phi + iD\psi) a_0] \). Observe that the regularity of \( a_1 \) is same as the regularity of \( F \) in equation \((2.13)\). Also notice that, to get a meaningful equation \((2.11)\) we require \( a_1 \) and \( a_2 \) to be \( C^2 \) and in that case we need to assume \( A \) to be...
Finally, choosing the amplitudes $a$ following two system of equations:

\[
\left( \frac{\partial}{\partial \xi} - \frac{n - 2}{2(z - \overline{z})} \right) V = F \quad \text{and} \quad \left( \frac{\partial}{\partial \xi} - \frac{n - 2}{2(z - \overline{z})} \right) a_1 = V \quad \text{in} \quad \Omega.
\]

Take $V = e^g v_0$ with $g \in C^\infty(\Omega)$ such that $\frac{\partial g}{\partial \xi} = \frac{n - 2}{2(z - \overline{z})}$. Here, $v_0 \in C^4(\Omega)$ can be obtained by solving $\frac{\partial}{\partial x} v_0 = e^{-g} F$. Therefore, the second transport equation has a solution $a_1 \in C^4(\Omega)$. However, the third transport equation can be written as

\[
\left( \frac{\partial}{\partial \xi} - \frac{n - 2}{2(z - \overline{z})} \right)^2 a_2 = \tilde{F} \quad \text{in} \quad \Omega, \quad \text{(2.14)}
\]

where

\[
\tilde{F} = -\frac{1}{4} \left( \frac{\partial}{\partial \xi} \right)^2 \left[ \Delta^2 a_0 + 2(\Delta \circ T + T \circ \Delta) a_1 + A \cdot D a_0 + A \cdot (D \phi + i D \psi) a_1 + \kappa^2 q(x) a_0 \right].
\]

Using the similar approach as in (2.13), we obtain that (2.14) has solution $a_2 \in C^4(\Omega)$. Finally, choosing the amplitudes $a_0 \in C^\infty(\Omega), a_1, a_2 \in C^4(\Omega)$, we obtain from (2.11) that, the correction term satisfies

\[
e^{-\frac{\phi}{h}} h^4 L_{A,q} e^{\frac{\phi}{h}} r = O(h^5). \quad \text{(2.15)}
\]

Finally, using Proposition 2.2 we obtain that there exists a solution $r \in H^4(\Omega)$ of (2.15) such that $\|r\|_{H^4_{sc}} \leq O(h^3)$. Hence the above discussion summarizes to the following proposition.

**Proposition 2.2.** Let $A \in C^4(\Omega, \mathbb{C}^n), q \in L^\infty_+(\Omega, \mathbb{C})$. Then, for $h > 0$ sufficiently small enough, there exist solutions $v(x; h) \in H^4(\Omega)$ to the equation

\[
\Delta^2 v + A \cdot D v + \kappa^2 q v = 0 \quad \text{in} \quad \Omega \quad \text{(2.16)}
\]

of the form $v(x; h) = e^{\frac{\phi}{h} + i \overline{\phi}} (a_0(x) + h a_1(x) + h^2 a_2(x) + r(x; h))$, where $\phi$ is a limiting Carleman weight for the semi-classical Laplacian on $\Omega$ and $\psi$ is defined as (2.4). The amplitudes $a_0 \in C^\infty, a_1, a_2 \in C^4(\Omega)$ satisfy (2.8), (2.9) and (2.10) respectively and the correction term $r$ satisfies $\|r\|_{H^4_{sc}} = O(h^3)$.

Let $t$ be a constant and $h > 0$ be a small parameter. Keeping the same notation as in Proposition 2.2 we define

\[
v(x, h, t) = e^{\frac{\phi}{h}} (t^{-\frac{1}{2}} \log |x - x_0|^2) - v(x) (a_0(x) + h a_1(x) + h^2 a_2(x) + r(x; h))
\]

to be the complex geometric optics solution with spherical phases for the equation (2.16).

3. Main result

In this section, we state our main result. Let us first introduce the distance function as $h_D(x_0) := \inf_{x \in D} \log |x - x_0|$, where $x_0 \in \mathbb{R}^n \setminus \text{ch}(\Omega)$, and $\text{ch}(\Omega)$ denotes the convex hull of the domain $\Omega$. Note that, $e^{h_D(x_0)}$ actually measures the distance from $x_0$ to $D$. By using the CGO solutions with spherical phases, we define an indicator function as follows:

\[
I_{x_0}(h, t) := \langle (N_{\gamma, \overline{\alpha}, 0} - N_{1, 0, 1}) f, f \rangle \quad \text{(3.1)}
\]

\[
= \int_{\partial \Omega} (N_{\gamma, \overline{\alpha}, 0} - N_{1, 0, 1}) f \cdot \overline{T} dS, \quad \text{(3.2)}
\]
where $dS$ denote the surface measure of $\partial \Omega$. Here $\mathcal{N}_{\tilde{\gamma}, \tilde{A}, \tilde{n}}$ is the Dirichlet to Neumann map corresponding to the solution $u$ of the equation (1.1), where the material parameters $\tilde{\gamma}, \tilde{A}, \tilde{n}$ satisfy Assumption 1, and $\mathcal{N}_{1,0,1}$ denote the Dirichlet to Neumann map corresponding to the solution $v$ which satisfies

$$\Delta^2 v + \kappa^2 v = 0 \text{ in } \Omega.$$  

(3.3)

We note that $f$ is a vector valued function defined as $f = (f_1, f_2) = (u|_{\partial \Omega}, (\Delta u)|_{\partial \Omega}) = (v|_{\partial \Omega}, (\Delta v)|_{\partial \Omega})$, as the boundary values corresponding to (1.1) and (3.3) are the same. Also we mention that the solution $v$ is taken to be CGO solution with spherical phase functions as described in Section 2. The inner product $\langle \mathcal{N}_{\tilde{\gamma}, \tilde{A}, \tilde{n}}(f), f \rangle$ can be defined as follows:

$$\langle \mathcal{N}_{\tilde{\gamma}, \tilde{A}, \tilde{n}}(f), f \rangle = \int_{\partial \Omega} \langle (\frac{\partial u}{\partial \nu}|_{\partial \Omega}, \frac{\partial}{\partial \nu} (\Delta u)|_{\partial \Omega}), (\tilde{T}_1, \tilde{T}_2) \rangle dS = \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \tilde{T}_2 + \frac{\partial}{\partial \nu} (\Delta u) \tilde{T}_1 \right) dS.$$

We now state our main result.

**Theorem 3.1.** Let $x_0 \in \mathbb{R}^n \setminus \overline{ch(\Omega)}$. We have the following characterization of the distance function $h_D(x_0)$. There exist constants $c, C > 0$ independent of $t$ and $h$ such that

1. When $t < h_D(x_0)$,

$$|I_{x_0}(h, t)| \leq Ce^{-\frac{c}{h}}, \ h \ll 1.$$

In particular, $\lim_{h \to 0} |I_{x_0}(h, t)| = 0.$

2. When $t = h_D(x_0)$,

$$ch^{n-2} \leq |I_{x_0}(h, t)| \leq C h^{-4}, \ h \ll 1.$$

3. When $t > h_D(x_0)$,

$$|I_{x_0}(h, t)| \geq Ce^{\frac{c}{h}}, \ h \ll 1.$$

In particular, $\lim_{h \to 0} |I_{x_0}(h, t)| = 0.$

Moreover,

$$t - h_D(x_0) = \lim_{h \to 0} \frac{1}{2} h \log |I_{x_0}(h, t)|.$$

From the above Theorem, we obtain a certain asymptotic behavior of the indicator function which is needed to reconstruct the unknown obstacle from the boundary data. Precisely, let us fix a point $x_0 \in \mathbb{R}^n \setminus \overline{ch(\Omega)}$. Then observe that the complex geometric optics solution, see Proposition 2.2, has an asymptotic behavior, that is growing exponentially inside the sphere $S = \{ x \in \mathbb{R}^{n-1}; |x - x_0| = e^t \}$ for a sufficiently small $h > 0$ and it decays exponentially faster outside the sphere. Using this feature of CGO solution, we can see that, when $t < h_D(x_0)$, the indicator function $I_{x_0}(h, t)$ vanishes exponentially for sufficiently small $h > 0$. Now we can expand the sphere so that when the time $t \geq h_D(x_0)$, the obstacle intersects the sphere and by Theorem 3.1 the indicator function becomes large for small $h$. Finally, moving the point $x_0$ around $\overline{ch(\Omega)}$, we can enclose the unknown obstacle by the spheres. In this way, we can recover not only the convex hull of the obstacle but also some non-convex part of it.

In the following section, we proceed to prove our main result.
We start this section with the following lemma.

**Lemma 4.1.** Prove that

\[ I_{x_0}(h, t) = e^{\frac{2}{\pi} (\sqrt{\frac{t}{h}} \cdot h_D(x_0))} I_{x_0}(h, h_D(x_0)). \]

The lemma follows directly from the definition of the indicator function and the complex geometric optics solutions.

Due to the Lemma 4.1, it is enough to give a proof of the second part (2) in Theorem 3.1. We only need to prove a lower bound of the indicator function at \( t = h_D(x_0) \), since the upper bound is easy due to the well-posedness of the forward problem (1.1). Let us recall the integration by parts formula which is often useful in the estimates. For any \( \phi \in H^4(\Omega) \) and \( \psi \in H^2(\Omega) \), the following Green’s theorem holds:

\[
\int_{\Omega} \nabla \cdot \nabla (\tilde{\gamma} \Delta \phi) \psi dx = -\int_{\Omega} \nabla (\tilde{\gamma} \Delta \phi) \cdot \nabla \psi dx + \int_{\partial \Omega} \frac{\partial}{\partial \nu}(\tilde{\gamma} \Delta \phi) \psi dS
\]

\[ = \int_{\Omega} (\tilde{\gamma} \Delta \phi) \Delta \psi dx - \int_{\partial \Omega} \Delta \phi \frac{\partial \psi}{\partial \nu} dS + \int_{\partial \Omega} \frac{\partial}{\partial \nu}(\tilde{\gamma} \Delta \phi) \psi dS. \quad (4.1) \]

4.1. **Lower bound of** \( I_{x_0}(h, h_D(x_0)) \). Let \( v \) be a solution of the following biharmonic equation

\[
\begin{aligned}
\Delta^2 v + \kappa^2 v &= 0 \quad \text{in } \Omega \\
v &= f_1 \quad \text{on } \partial \Omega \\
\Delta v &= f_2 \quad \text{on } \partial \Omega.
\end{aligned} \quad (4.2)
\]

Here \( v \) is taken to be CGO solutions, discussed in Sec 2 in presence of zero magnetic field in the medium. Let \( w := u - v \) be the reflected solution and \( u \) a solution to (1.1). Then \( w \) satisfies the following boundary value problem

\[
\begin{aligned}
\Delta(\tilde{\gamma} \Delta w) + \tilde{A} \cdot Dw + \kappa^2 \tilde{n}(x) w &= -\Delta((\tilde{\gamma} - 1) \Delta v) - \tilde{A} \cdot Dv - \kappa^2 (\tilde{n} - 1) v \quad \text{in } \Omega \\
w &= 0 \quad \text{on } \partial \Omega \\
\Delta w &= 0 \quad \text{on } \partial \Omega.
\end{aligned} \quad (4.3)
\]

The main step to prove the lower bound of \( I_{x_0} \) for \( t = h_D(x_0) \) lies in proving the following Proposition.

**Proposition 4.2.** Let \( \Omega \) be a smooth domain in \( \mathbb{R}^n \) and the inclusion \( \mathcal{D} \) strictly embedded inside \( \Omega \), then there exists \( C > 0, 1 \leq p_0 < 2 \) such that for all \( p \in (p_0, 2] \), we have

\[
\|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} \leq C(\|\Delta v\|_{L^p(D)} + \|\nabla v\|_{L^p(D)} + \|v\|_{L^p(D)}).
\]

Before entering into the proof of this Proposition, we provide first some elementary estimates. Let us define a function space \( X := \{ \phi \in H^4(\Omega) ; \phi = \Delta \phi = 0 \text{ on } \partial \Omega \} \). Suppose \( \Phi \in X \) be a weak solution of the equation

\[
\Delta(\tilde{\gamma} \Delta \Phi) - \tilde{D} \cdot (\tilde{A} \Phi) + \kappa^2 \tilde{n}\Phi = w - \Delta w \quad \text{in } \Omega,
\]

where \( w \) satisfies (4.3).
Lemma 4.3. Let $v, w$ and $\Phi$ satisfy (4.2), (4.3) and (4.4) respectively. Then the following estimate holds true.

$$
\|w\|_{L^2(\Omega)}^2 + \|\nabla w\|_{L^2(\Omega)}^2 \leq C(\|\Delta v\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}) \times (\|\Delta \Phi\|_{L^p(\Omega)} + \|\nabla \Phi\|_{L^p(\Omega)} + \|\Phi\|_{L^p(\Omega)}),
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $p < 2$.

Proof. Multiplying $w$ in the equation (4.4), and integrating by parts, see (4.1), we obtain,

$$
\int_{\Omega} |w(x)|^2 dx - \int_{\Omega} w(x) \Delta w(x) dx
= \int_{\Omega} \nabla \cdot (\gamma \Delta \Phi) w(x) dx + \kappa^2 \int_{\Omega} \frac{\nabla w(x)}{\partial \nu} w(x) dx - \int_{\Omega} D \cdot (\Phi w(x) dx
= \int_{\Omega} \gamma \Delta \Phi \Delta w dx - \int_{\partial \Omega} \gamma \Delta \Phi \frac{\partial w}{\partial \nu} dS + \kappa^2 \int_{\Omega} \nabla w(x) \Delta \Phi dx
$$

$$
+ \int_{\partial \Omega} \Phi \cdot Dw(x) dx + \int_{\partial \Omega} \frac{\partial}{\partial \nu} (\gamma \Delta \Phi) w(x) dS.
$$

Once again, applying the integration by parts formula on the left hand side of the above identity, we obtain

$$
\int_{\Omega} |w(x)|^2 dx + \int_{\Omega} |\nabla w(x)|^2 dx - \int_{\partial \Omega} \frac{\partial w(x)}{\partial \nu} w(x) dS
= \int_{\Omega} \gamma \Delta \Phi \Delta w dx - \int_{\partial \Omega} \gamma \Delta \Phi \frac{\partial w}{\partial \nu} dS + \kappa^2 \int_{\Omega} \nabla w(x) \Delta \Phi dx
$$

$$
+ \int_{\partial \Omega} \Phi \cdot Dw(x) dx + \int_{\partial \Omega} \frac{\partial}{\partial \nu} (\gamma \Delta \Phi) w(x) dS.
$$

Due to $w = 0, \Delta \Phi = 0$ on $\partial \Omega$, the above identity becomes

$$
\int_{\Omega} |w(x)|^2 dx + \int_{\Omega} |\nabla w(x)|^2 dx
= \int_{\Omega} \gamma \Delta \Phi \Delta w dx + \kappa^2 \int_{\Omega} \nabla w(x) \Delta \Phi dx + \int_{\partial \Omega} \Phi \cdot Dw(x) dx.
$$

(4.5)

Multiplying by $\Phi$ in (4.3) and integrating by parts, we have

$$
\int_{\Omega} \gamma \Delta w \Delta \Phi dx + \kappa^2 \int_{\Omega} \nabla w(x) \Phi(x) dx + \int_{\partial \Omega} A \cdot Dw \Phi dx
= - \int_{\Omega} [\Delta((\gamma - 1)\Delta v) + \kappa^2 (\nabla - 1)v \Phi] dx - \int_{\partial \Omega} A \cdot Dv \Phi dx
$$

$$
= - \int_{\Omega} [(\gamma - 1)\Delta v \Phi + \kappa^2 (\nabla - 1)v \Phi] dx - \int_{\partial \Omega} (\gamma - 1)\Delta v \frac{\partial \Phi}{\partial \nu} dS
$$

$$
+ \int_{\partial \Omega} \frac{\partial}{\partial \nu} ((\gamma - 1)\Delta v) \Phi dS - \int_{\partial \Omega} A \cdot Dv \Phi dx.
$$

(4.6)
Combining (4.5) and (4.6) together with taking real part of (4.5) and (4.6), we get
\[ \|w\|_{L^2(\Omega)}^2 + \|
abla w\|_{L^2(\Omega)}^2 = -\text{Re} \int_D XD\gamma D \Delta v \Delta \Phi dx - k^2 \text{Re} \int_D q D v \Phi dx \]
\[ - \text{Re} \int_D XD A D \cdot D v \Phi dx. \]
Using the Cauchy-Schwarz inequality, we obtain
\[ \|w\|_{L^2(\Omega)}^2 + \|
abla w\|_{L^2(\Omega)}^2 \leq C(\|\Delta v\|_{L^p(D)} \|
abla \Phi\|_{L^{p'}(\Omega)} + \|v\|_{L^{p'}(D)} \|
abla \Phi\|_{L^p(\Omega)}) \leq C(\|\Delta v\|_{L^p(D)} + \|
abla v\|_{L^p(D)} + \|v\|_{L^p(D)}) \times (\|
abla \Phi\|_{L^{p'}(\Omega)} + \|\nabla \Phi\|_{L^{p'}(\Omega)} + \|
abla \Phi\|_{L^p(\Omega)}), \]
where \( \frac{1}{p} + \frac{1}{p'} = 1, p < 2 \) and this ends the proof.

We now state the following lemma and give a detailed proof of it.

**Lemma 4.4.** Let \( \Omega \) be a bounded Lipschitz domain. Suppose \( p' > 2 \). Then for any \( \frac{n}{2} - \frac{n}{p'} \leq s < \frac{n}{2} \), we have
\[ \|(I - \Delta)w\|_{W^{-1,s,p'}(\Omega)} \leq C(\|w\|_{L^2(\Omega)} + \|
abla w\|_{L^2(\Omega)}). \]

**Proof.** Observe that
\[ \|(I - \Delta)w\|_{W^{-1,s,p'}(\Omega)} = \sup_{\|\phi\|_{W^{1+s,p}(\Omega)} \leq 1, \phi \in W_0^{1+s,p}(\Omega)} |\langle (I - \Delta)w, \phi \rangle|, \]
where \( 1/p + 1/p' = 1 \). Since \( \phi = 0 \) on \( \partial \Omega \), using integration by parts we compute the duality product
\[ \langle (I - \Delta)w, \phi \rangle = \int_\Omega (I - \Delta)w \phi dx = \int_\Omega w \phi dx + \int_\Omega \nabla w \cdot \nabla \phi dx. \]
Using Cauchy Schwarz inequality and the estimate
\[ \|\phi\|_{L^2(\Omega)} \leq C \|\phi\|_{W^{s,p}(\Omega)}, \forall p < 2 \leq \frac{np}{n - sp} \text{ and } sp < n, \]
see [Theorem 6.7 in [3]], we obtain
\[ \|\langle (I - \Delta)w, \phi \rangle\| \leq (\|w\|_{L^2(\Omega)} + \|
abla w\|_{L^2(\Omega)}) (\|\phi\|_{L^2(\Omega)} + \|\nabla \phi\|_{L^2(\Omega)}) \leq C (\|w\|_{L^2(\Omega)} + \|
abla w\|_{L^2(\Omega)}) (\|\phi\|_{W^{s,p}(\Omega)} + \|\nabla \phi\|_{W^{s,p}(\Omega)}) \leq C (\|w\|_{L^2(\Omega)} + \|
abla w\|_{L^2(\Omega)}) \|\phi\|_{W^{1+s,p}(\Omega)}. \]
Taking supremum over \( \phi \in W_0^{1+s,p}(\Omega) \) with \( \|\phi\|_{W_0^{1+s,p}(\Omega)} \leq 1 \) on the both side of the above inequality, we obtain the required result.

We recall the following lemma.

**Lemma 4.5** (Theorem 0.3 in [17]). Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^n, n \geq 3 \). Suppose \( p' > 2 \). Then for every \( f \in W^{-1,s,p'}(\Omega) \) there exists a unique solution \( u \in W^{1-s,p'}(\Omega) \) to the **inhomogeneous Dirichlet problem**
\[ \begin{align*}
\left\{ \begin{array}{ll}
\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{array} \right.
\end{align*} \]
provided $\frac{1}{p} < \frac{1}{2}$ and $s < 1 - \frac{1}{p'}$. Moreover, we have the following estimate
\[ \|u\|_{W^{1-s,p'}(\Omega)} \leq C \|f\|_{W^{1-s,p'}(\Omega)} \]
for all $f \in W^{-1-s,p'}(\Omega)$.

**Lemma 4.6.** Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be a bounded Lipschitz domain. Then the inclusion
\[ i : L^2(\Omega) \rightarrow W^{-1-s,p'}(\Omega), \]
for all $p' > 2$ and for any $\frac{n}{2} - \frac{n}{p'} \leq s < \frac{n}{2}$, is a bounded linear operator.

**Proof.** Remark that, for $s > 0$ and $1 < p < n$, if $n > sp$, then the inclusion $i : W^{s,p}_0(\Omega) \subset W^{s,p}(\Omega) \rightarrow L^r(\Omega)$ is a bounded linear map for $p \leq r \leq \frac{np}{n-sp}$, see [1], Theorem 7.57. Combining (4.8) and (4.9), we get
\[ i^* : L^r(\Omega) \rightarrow (W^{s,p}_0(\Omega))^* = W^{-s,p'}(\Omega), \]
where $1/r + 1/r' = 1$ and $1/p + 1/p' = 1$. Also we have the usual inclusion
\[ W^{-s,p'}(\Omega) \subset W^{-s-1,p'}(\Omega), \]
see for example [Proposition 2.1, Corollary 2.3, [5]]. Combining (4.8) and (4.9), we get
\[ i^* : L^{r'}(\Omega) \rightarrow W^{-s-1,p'}(\Omega), \]
is a bounded linear map that holds for all $s > 0, 1 < p < n, n > sp$ and $p \leq r \leq \frac{np}{n-sp}$. Finally we replace $r' = 2$ in (4.10) to obtain that the map $i^* : L^2(\Omega) \rightarrow W^{-1-s,p'}(\Omega)$, is bounded for all $s > 0, 1 < p < n, n > sp$ and $p \leq 2 \leq \frac{np}{n-sp}$ with $1/p + 1/p' = 1$. If we assume that $p' > 2$ then the range of $s$ will be $\frac{n}{2} - \frac{n}{p'} \leq s < \frac{n}{2}$, and the Lemma follows. \hfill \Box

We now state the following $L^p$-estimate and provide a detailed proof of it.

**Lemma 4.7.** [*$L^p$-estimate*] Let the coefficients $\overline{\gamma}, \overline{\nu}, \overline{n}$ satisfy the Assumption [1] and also assume that 0 is not an eigen value of the operator $\Delta(\overline{\gamma}\Delta) - D \cdot \overline{A} + \kappa^2 \overline{n}$ corresponding to the Navier boundary condition. Let $\Phi$ satisfies
\[ \begin{cases} \Delta(\overline{\gamma}\Delta \Phi) - D \cdot (\overline{A}\Phi) + \kappa^2 \overline{n}(x)\Phi(x) = (I - \Delta)w & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial \Omega \\ \Delta \Phi = 0 & \text{on } \partial \Omega. \end{cases} \tag{4.11} \]
where $w$ is the reflected solution satisfies (4.3). Then for all $2 < p' < \frac{2n}{n-1}$, we have
\[ \|\Delta \Phi\|_{L^{p'}(\Omega)} + \|\nabla \Phi\|_{L^{p'}(\Omega)} + \|\Phi\|_{L^{p'}(\Omega)} \leq C(\|\nabla w\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}), \]
where $C > 0$ is a constant independent of the data.

**Proof.** Define $\Psi := \overline{\gamma}\Delta \Phi$, Then the equation (4.11) will be reduced to
\[ \begin{cases} \Delta \Psi - D \cdot (\overline{A}(x)\Phi) + \kappa^2 \overline{n}(x)\Phi = (I - \Delta)w & \text{in } \Omega \\ \Psi = 0 & \text{on } \partial \Omega \end{cases} \tag{4.12} \]
and
\[ \begin{cases} \Delta \Phi = \frac{1}{\overline{\gamma}}\Psi & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial \Omega. \end{cases} \tag{4.13} \]
By the assumption on \( \tilde{A} \), we can rewrite the equation (4.12) as follows:

\[
\begin{align*}
\Delta \Psi &= F \quad \text{in } \Omega \\
\Psi &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]  

(4.14)

where \( F \) is defined as \( F := (I - \Delta)w + A_D \cdot D\Phi + (D \cdot A_D - \kappa^2 \tilde{n}(x))\Phi \). We first show that \( F \in W^{-1-s,p'}(\Omega) \) for all \( p' > 2 \) where \( \Omega \) is a bounded smooth domain. Using Lemma 4.6, we obtain

\[
\|\nabla \Phi\|_{W^{-1-s,p'}(\Omega)} \leq C \|\nabla \Phi\|_{L^2(\Omega)},
\]  

(4.15)

and

\[
\|\Phi\|_{W^{-1-s,p'}(\Omega)} \leq C \|\Phi\|_{L^2(\Omega)},
\]  

(4.16)

for all \( p' > 2 \) and \( \frac{n}{2} - \frac{n}{p'} \leq s < \frac{n}{2} \). It is worth noticing that, the following estimate holds true.

\[
\|\Phi\|_{L^2(\Omega)} + \|\nabla \Phi\|_{L^2(\Omega)} \leq C(\|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)}).
\]  

(4.17)

To see this fact precisely, we first define \( W := \Phi - w_0 \), where \( \Phi \) and \( w_0 \) satisfy (4.11) and

\[
\begin{align*}
\Delta (\tilde{\gamma} \Delta w_0) &= (I - \Delta)w \quad \text{in } \Omega \\
w_0 &= 0 \quad \text{on } \partial \Omega \\
\Delta w_0 &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]  

(4.18)

respectively. Therefore \( W \) satisfies

\[
\begin{align*}
\Delta (\tilde{\gamma} \Delta W) - D \cdot \tilde{A}W + \kappa^2 \tilde{n}W &= D \cdot \tilde{A}w_0 - \kappa^2 \tilde{n}w_0 \quad \text{in } \Omega \\
W &= 0 \quad \text{on } \partial \Omega \\
\Delta W &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]  

(4.19)

We replace \( a = \tilde{\gamma}, b = -\tilde{A}, c = \kappa^2 \tilde{n} - D \cdot \tilde{A} \) and \( f = D \cdot (\tilde{A}w_0) - \kappa^2 \tilde{n}w_0 \) in Lemma 5.1, and choose \( \frac{1}{2}, \tilde{n} \) and \( \tilde{A} \) so small that equations (5.7) and (5.8) must satisfy. By the Lemma 5.1 in (4.19), we obtain

\[
\|W\|_{H^1(\Omega)} \leq C(\|w_0\|_{L^2(\Omega)} + \|w_0\|_{L^2(\Omega)}).
\]  

(4.20)

Define \( U := \tilde{\gamma} \Delta w_0 + w \) then \( U \) satisfies

\[
\begin{align*}
\Delta U &= w \quad \text{in } \Omega \\
U &= w \quad \text{on } \partial \Omega,
\end{align*}
\]  

(4.21)

and

\[
\begin{align*}
\Delta w_0 &= \frac{1}{2}(U - w) \quad \text{in } \Omega \\
w_0 &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]  

(4.22)

Using \( L^2 \)-estimates for the solutions of the equations (4.21) and (4.22) together with trace theorem, we obtain

\[
\|U\|_{H^1(\Omega)} \leq C(\|w\|_{L^2(\Omega)} + \|w\|_{H^{1/2}(\partial \Omega)}) \leq C(\|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)}),
\]  

and

\[
\|w_0\|_{H^1(\Omega)} \leq C(\|U\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}) \leq C(\|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)}).
\]  

(4.23)
Combining above two estimates together with (4.20), we prove the estimate (4.17). We now have

\[ \|F\|_{W^{-1-s,p'}(\Omega)} \leq \|(I - \Delta)w\|_{W^{-1-s,p'}(\Omega)} + \|A_D\|_{L^\infty(D)} \|\nabla \Phi\|_{W^{-1-s,p'}(\Omega)} \]

\[ + \left\| D \cdot A_D - \kappa \mathcal{N}(x) \right\|_{L^\infty(\Omega)} \|\Phi\|_{W^{-1-s,p'}(\Omega)} \]

Use (4.15) and (4.16),

\[ \leq \|(I - \Delta)w\|_{W^{-1-s,p'}(\Omega)} + C \left( \|\nabla \Phi\|_{L^2(\Omega)} + \|\Phi\|_{L^2(\Omega)} \right) \]

Apply Lemma 4.4

\[ \leq C \left( \|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} \right) + C \left( \|\nabla \Phi\|_{L^2(\Omega)} + \|\Phi\|_{L^2(\Omega)} \right) \]

Use (4.17),

\[ \leq C \left( \|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} \right) \]

Apply Lemma 5.1 in (4.3),

\[ \leq C \left( \|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} + \|\Delta w\|_{L^2(\Omega)} + \|\Delta^2 w\|_{L^2(\Omega)} \right), \quad (4.23) \]

where \( \frac{n}{2} - \frac{n}{p'} \leq s < \frac{n}{2} \) and \( p' > 2 \). In the last inequality, we used the fact that \( \gamma_D \in C^2(D) \) and \( v \in H^2(\Omega) \), where \( v \) is the CGO solution described in Proposition 2.2. Therefore \( F \in W^{-1-s,p'}(\Omega) \). Applying Lemma 4.5 we deduce an \( L^p \)-estimate for the solution of equation (4.14). Precisely, for any \( (s,p') \) satisfying \( \frac{1}{p'} < \frac{1}{2}, \frac{n}{2} - \frac{n}{p'} \leq s < \frac{n}{2} \) and \( s < 1 - \frac{1}{p'} \), it holds the implication \( F \in W^{-1-s,p'}(\Omega) \implies \Phi \in W_{0}^{1-s,p'}(\Omega) \), which is basically the estimate

\[ \|\Phi\|_{W_{0}^{1-s,p'}(\Omega)} \leq C \|F\|_{W^{-1-s,p'}(\Omega)}. \quad (4.24) \]

Further, we assume that \( s < 1 \) so that \( W^{3-s,p'}(\Omega) \subset W^{2,p'}(\Omega) \), then we get

\[ \|\Phi\|_{L^p(\Omega)} + \|\nabla \Phi\|_{L^{p'}(\Omega)} + \|\Phi\|_{L^{p'}(\Omega)} \leq C \|\Phi\|_{W_{0}^{2,p'}(\Omega)} \leq C \|\Phi\|_{W_{0}^{3-s,p'}(\Omega)}. \quad (4.25) \]

Finally, we use the regularity estimates, see Lemma 4.5 or [Theorem 0.3, [17]], to the equation (4.13) to obtain,

\[ \|\Phi\|_{W_{0}^{3-s,p'}(\Omega)} \leq C \|\Psi\|_{W^{1-s,p'}(\Omega)}, \quad (4.26) \]

for all \( \frac{1}{p'} < \frac{1}{2} \) and \( s < 3 - \frac{1}{p'} \). Hence, the estimates (4.23), (4.24), (4.25) together with (4.26), we obtain

\[ \|\Delta \Phi\|_{L^{p'}(\Omega)} + \|\nabla \Phi\|_{L^{p'}(\Omega)} + \|\Phi\|_{L^{p'}(\Omega)} \leq C(\|\nabla w\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}), \quad (4.27) \]

where \((s,1/p')\) satisfy

\[ \begin{cases} \frac{1}{p'} < \frac{1}{2}, & \frac{n}{2} - \frac{n}{p'} \leq s < \frac{n}{2} \text{ and } s < 1 - \frac{1}{p'}, \\ s < 1, & \\ s < 3 - \frac{1}{p'}. \end{cases} \quad (4.28) \]

Now, we will take all those \( p' > 2 \) such that \( \frac{n}{2} - \frac{n}{p'} < \frac{1}{2} \). Then the set of all \( s \) satisfying \( \frac{n}{2} - \frac{n}{p'} < s < \frac{1}{2} \) must satisfy (4.28). Therefore the estimate (4.27) holds for all \( 2 < p' < \frac{2n}{n-1} \). This ends the proof.
Proof. (Proof of Proposition 4.2) The proof is a simple consequence of Lemma 4.7 and the estimate (4.7). □

Remark 4.8. It is essential that we need to assume Ω to be a smooth domain, see Lemma 4.5. For instance, if one applies [Theorem 1.1, [17]], where the sharp regularity estimate has been done for Lipschitz domain, then it seems unlikely to get a uniform range of s so that similar to (4.28) holds. The main reason is that, due to the lower bound of s in Theorem 1.1, [17], it is difficult to get a uniform range of s for which (4.28) is valid.

Lemma 4.9. Assume that the functions v and w are the solutions of (4.2) and (4.3) respectively. Then

1. We have a lower bound of the indicator function

\[ |I_{x_0}(h, t)| \geq C \left[ \int_{\Omega} |\Delta v|^2 dx - \int_{\partial \Omega} |w(x)|^2 dx - \int_{\Omega} |v(x)|^2 dx \right] - C (\|\nabla w\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(D)}) \times (\|w\|_{L^2(\Omega)} + \|v\|_{L^2(D)}), \]

2. and an upper bound of the form

\[ |I_{x_0}(h, t)| \leq C \left[ \|\Delta^2 v\|^2_{L^2(D)} + \|\nabla \Delta v\|^2_{L^2(D)} + \|\Delta v\|^2_{L^2(D)} + \|\nabla v\|^2_{L^2(D)} + \|v\|^2_{L^2(D)} \right] \]

where C > 0 is a constant.

Proof. Let us recall that \( N_{\gamma, \tilde{\lambda}, \tilde{n}} \) represents the Dirichlet to Neumann map which encodes the current measurement on the boundary \( \partial \Omega \) corresponding to the boundary voltage \( u = f \) prescribed on \( \partial \Omega \), when there is an obstacle \( D \) embedded inside the domain \( \Omega \). We write the weak form of \( N_{\gamma, \tilde{\lambda}, \tilde{n}} \) as

\[ \langle N_{\gamma, \tilde{\lambda}, \tilde{n}} f, f \rangle = \int_{\partial \Omega} \left( \frac{\partial f}{\partial \nu} - \frac{\partial \nu}{\partial \nu} \Delta u \right) \bar{f} dS = \int_{\partial \Omega} \left( \frac{\partial f}{\partial \nu} + \frac{\partial \nu}{\partial \nu} \Delta u \right) \bar{f} dS, \] (4.29)

where \( u \) satisfies (1.1). On the other hand, we denote \( N_{1,0,1} \) as the Dirichlet to Neumann map, when there is no obstacle present inside \( \Omega \) and it has the following weak form

\[ \langle N_{1,0,1} f, f \rangle = \int_{\partial \Omega} \left( \frac{\partial f}{\partial \nu} + \frac{\partial \nu}{\partial \nu} \Delta u \right) \bar{f} dS, \]

where \( v \) solves (4.2). Multiplying by \( \bar{v} \) in the equation (1.1) and integrating by parts gives,

\[ 0 = \int_{\Omega} \nabla \cdot \nabla (\gamma \Delta u) \bar{v} dx + \kappa^2 \int_{\Omega} n(x) u \bar{v} dx + \int_{\Omega} (\bar{A} \cdot Du) \bar{v} dx \]

\[ = \int_{\Omega} \gamma \Delta u \Delta v dx - \int_{\partial \Omega} f_2 \frac{\partial \nu}{\partial \nu} dS + \int_{\partial \Omega} \frac{\partial}{\partial \nu} (\Delta u) \bar{f}_1 dS + \kappa^2 \int_{\Omega} n(x) u \bar{v} dx + \int_{\Omega} (\bar{A} \cdot Du) \bar{v} dx. \] (4.30)

Using (4.30), the Dirichlet to Neumann map can be written as

\[ \langle N_{\gamma, \tilde{\lambda}, \tilde{n}} f, f \rangle = \int_{\partial \Omega} \left( \frac{\partial f}{\partial \nu} + \frac{\partial \nu}{\partial \nu} \bar{f}_2 \right) dS - \int_{\bar{\Omega}} \gamma \Delta u \Delta v dx - \int_{\Omega} (\bar{A} \cdot Du) \bar{v} dx - \kappa^2 \int_{\Omega} n u \bar{v} dx. \] (4.31)
On the other hand, multiplying by $u$ in equation (4.2), and integrating by parts, we obtain
\begin{equation}
0 = \int_\Omega (\Delta^2 v) u dx + \kappa^2 \int_\Omega \nabla v dx \\
= \int_\Omega \Delta \nabla v dx - \int_{\partial \Omega} f_2 \frac{\partial u}{\partial \nu} dS + \int_{\partial \Omega} \frac{\partial}{\partial \nu} (\Delta v) f_1 dS + \kappa^2 \int_\Omega uv dx. \tag{4.32}
\end{equation}

We now take the real part of (4.32) and compute the following DN map.
\begin{align*}
\Re \langle N_{1,0,1} f, f \rangle &= \Re \left[ \int_{\partial \Omega} \frac{\partial}{\partial \nu} f_2 dS + \int \Delta u \Delta v dx + \kappa^2 \int \nabla v dx \right] \\
&= \Re \left[ \int_{\partial \Omega} \frac{\partial}{\partial \nu} f_2 dS - \kappa^2 \Re \int \nabla v dx \right]. \tag{4.33}
\end{align*}

Therefore, using (4.31) and (4.33), we have
\begin{align*}
-I_{x_0}(h, t) &= - \Re \int_{\partial \Omega} \langle (N_{1,0,1} f, f) dS \\
&= \Re \left[ \int_{\partial \Omega} (\nabla \bar{\gamma} - 1) \Delta u \Delta \bar{v} dx + \kappa^2 \Re \int_{\partial \Omega} (\bar{n} - 1) u \bar{v} dx \right] \\
&= \int_{\Omega} \Re (\bar{\gamma} - 1) |\Delta v|^2 dx + \kappa^2 \Re \int_{\Omega} (\bar{n} - 1) |v|^2 dx - \Re \int_{\Omega} (\bar{A} \cdot Du) (u - v) dx \\
&\quad + \int_{\Omega} (\bar{A} \cdot Du) v dx - \int_{\Omega} \bar{\gamma} |\Delta w|^2 dx - \kappa^2 \int_{\Omega} |\bar{n} - 1| w dx - \int_{\Omega} |\bar{A} \cdot Dw| w dx. \tag{4.34}
\end{align*}

Due to the fact that $\bar{A} = 0$ in $\Omega \setminus D$ together with Cauchy Schwarz inequality in (4.35), and applying the elliptic regularity estimate from Lemma 5.1 for the equation (4.3) we obtain an upper bound of $-I_{x_0}(h, t)$.

To estimate the lower bound of $-I_{x_0}(h, t)$, we multiply the identity
\[ \Delta^2 w + \kappa^2 w + \Delta((\bar{\gamma} - 1) \Delta u) + \bar{A} \cdot Du + \kappa^2 (\bar{n} - 1) u = 0 \]
by $\bar{w}$ and doing integration by parts over $\Omega$, we have
\begin{align*}
\int_{\Omega} |\Delta w|^2 dx + \kappa^2 \int_{\Omega} |w|^2 dx + \int_{\Omega} (\bar{\gamma} - 1) |\Delta u|^2 dx \\
- \int_{\Omega} (\bar{\gamma} - 1) \Delta u \Delta \bar{v} + \int_{\Omega} (\bar{A} \cdot Du) \bar{w} dx + \kappa^2 \int_{\Omega} (\bar{n} - 1) u \bar{w} dx = 0. \tag{4.36}
\end{align*}

Using the formula
\[ |\Delta w|^2 + (\Re \bar{\gamma} - 1) |\Delta u|^2 = \Re \bar{\gamma} |\Delta u|^2 - 2 \Re \Delta u \Delta \bar{v} + |\Delta v|^2 \]
\[ = (\Re \bar{\gamma}) |\Delta u| - \frac{1}{\Re \bar{\gamma}} |\Delta v|^2 + (1 - \frac{1}{\Re \bar{\gamma}}) |\Delta v|^2 \]
together with the real part of the equation (4.36) we obtain
\[
\text{Re} \int_{\Omega} (\tilde{\gamma} - 1) \Delta u \Delta v = \int_{\Omega} (\text{Re} \tilde{\gamma}) |\Delta u - \frac{1}{\text{Re} \tilde{\gamma}} \Delta v|^2 + \int_{\Omega} (1 - \frac{1}{\text{Re} \tilde{\gamma}}) |\Delta v|^2 + \kappa^2 \int_{\Omega} |v|^2 \, dx \\
+ \text{Re} \int_{\Omega} (\tilde{A} \cdot Du) w \, dx + \kappa^2 \text{Re} \int_{\Omega} (\tilde{n} - 1) u w \, dx.
\]

Therefore using (4.34) we have
\[
-I_{x_0}(h, t) = \int_{\Omega} \text{Re} \tilde{\gamma} |\Delta u - \frac{1}{\text{Re} \tilde{\gamma}} \Delta v|^2 + \int_{\Omega} (1 - \frac{1}{\text{Re} \tilde{\gamma}}) |\Delta v|^2 + \kappa^2 \int_{\Omega} |w|^2 \, dx \\
+ \text{Re} \int_{\Omega} (\tilde{A} \cdot Du) w \, dx + \kappa^2 \text{Re} \int_{\Omega} (\tilde{n} - 1) u(\overline{w} + \overline{v}) \, dx + \text{Re} \int_{\Omega} (\tilde{A} \cdot Du) \overline{v} \, dx \\
\geq C_1 \int_D |\Delta v|^2 - C_2 \int_{\Omega} |w|^2 \, dx - C_3 \int_D |v|^2 \, dx + I,
\]
where \(C_1, C_2, C_3 > 0\) are positive constants and
\[
I := \text{Re} \int_{\Omega} (\tilde{A} \cdot Du) \overline{w} \, dx + \text{Re} \int_{\Omega} (\tilde{A} \cdot Du) \overline{v} \, dx.
\]

Since, \(\tilde{A} = 0\) in \(\Omega \setminus \overline{D}\), using Cauchy-Schwarz inequality we can estimate \(I\) as
\[
|I| \leq \left\| \tilde{A} \right\|_{L^\infty(D)} \left( \|\nabla u\|_{L^2(D)} \left( \|w\|_{L^2(D)} + \|v\|_{L^2(D)} \right) \right) \\
\leq C(\|\nabla w\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(D)}) \times (\|w\|_{L^2(\Omega)} + \|v\|_{L^2(D)})
\]

Hence, we obtain the required estimate. \(\square\)

**Remark 4.10.** The result also holds true even if we assume the background magnetic field to be non-zero constant vector field. However, for simplicity we assumed that the background magnetic field is a zero vector throughout the paper.

4.1.1. End of the proof of Theorem 3.1

**Lemma 4.11.** For an \(\epsilon > 0\) small enough so that, we have following estimates for \(2 - \epsilon < p < 2\),

\[
\begin{align*}
\text{(1)} & \quad \frac{\|v\|^2_{L^2(D)}}{\|\Delta v\|^2_{L^2(D)}} \leq C h^2, \quad \frac{\|w\|^2_{L^2(\Omega)}}{\|\Delta v\|^2_{L^2(D)}} \leq C h^{3/2 - 1}, \quad h \ll 1. \\
\text{(2)} & \quad \left( \|\nabla w\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(D)} \right) \times (\|w\|_{L^2(\Omega)} + \|v\|_{L^2(D)}) \rightarrow 0 \quad \text{as} \ h \rightarrow 0. \\
\text{(3)} & \quad \int_D |\Delta v(x)|^2 \, dx \geq C h^{n-2}, \quad h \ll 1.
\end{align*}
\]
Proof. Since \( \partial D \) is Lipschitz, we have \( l_j(y') \leq C |y'| \) and we obtain the following estimate

\[
\sum_{j=1}^{N} \int_{|y'|<\delta} e^{-\frac{2}{\kappa}l_j(y')} dy' \geq C \sum_{j=1}^{N} \int_{|y'|<\delta} e^{-\frac{2}{\kappa}|y'|} dy' \geq C h^{n-1} \sum_{j=1}^{N} \int_{|y'|<\frac{\delta}{h}} e^{-2|y'|} dy' \geq Ch^{n-1}. \quad (4.37)
\]

Using Lemma 6.1 and (4.37) we obtain

\[
\frac{\|v\|_{L^2(D)}^2}{\|\Delta v\|_{L^2(D)}^2} \leq C \frac{\|\Delta v\|_{L^2(D)}^2 + \|\nabla v\|_{L^2(D)}^2 + \|v\|_{L^2(D)}^2}{\|\Delta v\|_{L^2(D)}^2} \leq C h^{\frac{2}{p}-1}, \quad h \ll 1.
\]

Similarly, using the Proposition 4.2 for \( p < 2 \), we obtain

\[
\frac{\|w\|_{L^2(\Omega)}^2}{\|\Delta v\|_{L^2(D)}^2} \leq C \frac{\|\Delta v\|_{L^2(D)}^2 + \|\nabla v\|_{L^2(D)}^2 + \|v\|_{L^2(D)}^2}{\|\Delta v\|_{L^2(D)}^2} \leq C h^{\frac{2}{p}-1}, \quad h \ll 1.
\]

(2) Combining Lemma 6.1 and Proposition 4.2 together with the estimate

\[
\sum_{j=1}^{N} \int_{|y'|<\delta} e^{-\frac{2}{\kappa}l_j(y')} dy' \leq C \left( \sum_{j=1}^{N} \int_{|y'|<\delta} e^{-\frac{2}{\kappa}l_j(y')} dy' \right)^{\frac{n}{2}},
\]

we obtain

\[
\frac{(\|\nabla w\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(D)}) \times (\|w\|_{L^2(\Omega)} + \|v\|_{L^2(D)})}{\|\Delta v\|_{L^2(D)}} \leq C \frac{1}{\|\Delta v\|_{L^2(D)}^2} \left( \|\Delta v\|_{L^2(D)} + \|\nabla v\|_{L^2(D)} + \|v\|_{L^2(D)} + \|\nabla v\|_{L^2(D)} \right) \times (\|\Delta v\|_{L^2(D)} + \|\nabla v\|_{L^2(D)} + \|v\|_{L^2(D)} + \|v\|_{L^2(D)}) \leq C h^{\frac{\kappa}{\tau} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}} \rightarrow 0, \quad (h \rightarrow 0).
\]

3. This is a consequence of (4.37) and the estimate (5) of Lemma 6.1.

Using the above two lemmas, we prove our main result.

5. \( L^2 \) Estimate

In this section we give a detailed proof of \( L^2 \) regularity estimate of the solutions of the bi-Laplace equation with nonsmooth coefficients, which will be useful to prove an \( L^p \)-estimates of the bi-Laplace problem with Navier boundary data discussed in Section 4.

Lemma 5.1. Assume \( \Omega \subset \mathbb{R}^n, n \geq 3 \) to be an open bounded set with sufficiently smooth regular boundary. Let \( u \) be a solution of the following forth order elliptic equation

\[
\begin{cases}
\Delta(a(x)\Delta u) + b(x) \cdot Du + \kappa^2 c(x) u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega \\
\Delta u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(5.1)
where coefficients $1/a(x), b(x), c(x) \in L^\infty(\Omega)$ are so small that they must satisfy \eqref{5.7} and \eqref{5.8}, and $0$ is not an eigen value of $\Delta a(x)\Delta + b(x) \cdot D + \kappa^2 c(x)$. Then for any $f \in L^2(\Omega)$ the problem \eqref{5.1} is wellposed in $\{(u, \Delta u) \in (H^1_0(\Omega))^2\}$. Moreover
\[
\|u\|_{H^1(\Omega)} + \|\Delta u\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)},
\]
where $C > 0$ be a constant independent of the data.

Proof. We use Lax-Milgram lemma to prove this result. See for example, \cite[Chapter 6]{7}, where the similar approach was followed for the second order elliptic equation with real valued coefficient. Since the coefficients in our bi-harmonic equation are complex valued, the solution will be complex valued and in this case we follow \cite[Chapter 2]{26}. For the notational simplicity we denote $(H^1_0(\Omega, \mathbb{C}))^2$ by $(H^1_0(\Omega))^2$. Let us transform the above bi-harmonic equation \eqref{5.1} to the system of second order elliptic equations. Define $a(x)\Delta u := v$, then the equation \eqref{5.1} reduces to
\[
\begin{cases}
Lw = F & \text{in } \Omega \\
w = 0 & \text{on } \partial \Omega
\end{cases}
\]
where $w := \begin{pmatrix} u \\ v \end{pmatrix}$ and $F := \begin{pmatrix} 0 \\ -f \end{pmatrix}$, and the operator is defined as a $2 \times 2$ matrix
\[
L := \begin{pmatrix}
\frac{1}{i}b(x) \cdot \nabla - \kappa^2 c(x) & \frac{1}{a(x)} \\
\frac{1}{i}b(x) \cdot \nabla & -\Delta 
\end{pmatrix}.
\]
We now define
\[
B(U, V) := \int_{\Omega} \nabla u_1 \cdot \nabla v_1 dx + \int_{\Omega} \nabla u_2 \cdot \nabla v_2 dx + \frac{1}{i} \int_{\Omega} (b \cdot \nabla u_1)v_2 dx \\
- \kappa^2 \int_{\Omega} c(x)u_1v_2 dx + \int_{\Omega} \frac{1}{a(x)}u_2v_1 dx,
\]
for $U := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in (H^1_0(\Omega))^2$, $V := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in (H^1_0(\Omega))^2$. Observe that $B : (H^1_0(\Omega))^2 \times (H^1_0(\Omega))^2 \to \mathbb{C}$ is a sesquilinear form, i.e., $B(U, V)$ is conjugate-linear in $U$ and linear in $V$. Using the Cauchy-Schwarz inequality, we obtain
\[
|B(U, V)| \leq \|\nabla u_1\|_{L^2(\Omega)} \|\nabla v_1\|_{L^2(\Omega)} + \|\nabla u_2\|_{L^2(\Omega)} \|\nabla v_2\|_{L^2(\Omega)} \\
+ \|b\|_{L^\infty(\Omega)} \|\nabla u_1\|_{L^2(\Omega)} \|v_2\|_{L^2(\Omega)} + \kappa^2 \|c\|_{L^\infty(\Omega)} \|u_1\|_{L^2(\Omega)} \|v_2\|_{L^2(\Omega)} \\
+ \|\frac{1}{a}\|_{L^\infty(\Omega)} \|u_2\|_{L^2(\Omega)} \|v_1\|_{L^2(\Omega)} \\
\leq \alpha \|u_1\|_{H^1_0(\Omega)} + \|u_2\|_{H^1_0(\Omega)} \times (\|v_1\|_{H^1_0(\Omega)} + \|v_2\|_{H^1_0(\Omega)}) \\
= \alpha \|U\|_{(H^1_0(\Omega))^2} \|V\|_{(H^1_0(\Omega))^2},
\]
where $\alpha$ is chosen to be $\alpha := \max\{1, \|b\|_{L^\infty(\Omega)}, \kappa^2 \|c\|_{L^\infty(\Omega)}, \|\frac{1}{a}\|_{L^\infty(\Omega)} \}$. We now want to show that $B$ is coercive sesquilinear form, i.e., there exists a constant $\beta > 0$ such that
\[
\text{Re } B(U, U) \geq \beta \|U\|_{(H^1_0(\Omega))^2}^2,
\]
for all $(H^1_0(\Omega))^2$. Let us recall the celebrated Poincaré inequality which will be frequently used in the later estimates. For any function $u \in W^{1, p}_0(\Omega), 1 \leq p < \infty$, we have the Poincaré
inequality
\[ \|u\|_{L^p(\Omega)} \leq \left( \frac{1}{\omega_n |\Omega|} \right)^{1/n} \|\nabla u\|_{L^p(\Omega)}, \tag{5.5} \]
where \(\omega_n\) is the Lebesgue measure of the unit ball in \(\mathbb{R}^n\), see equation (7.44) in \([9]\). Note that
\[
B(U,U) := \int_{\Omega} |\nabla u_1|^2 dx + \int_{\Omega} |\nabla u_2|^2 dx + \frac{1}{i} \int_{\Omega} (b \cdot \nabla u_1)\overline{u_2} dx - \kappa^2 \int_{\Omega} c(x) u_1 \overline{u_2} dx + \int_{\Omega} \frac{1}{a(x)} u_2 \overline{u_1} dx. \tag{5.6}
\]
We now estimate
\[
\text{Re} \left[ \frac{1}{i} \int_{\Omega} (b \cdot \nabla u_1)\overline{u_2} \right] \leq \left| \frac{1}{i} \int_{\Omega} (b \cdot \nabla u_1)\overline{u_2} \right| \leq \int_{\Omega} |b \cdot \nabla u_1| |u_2| \leq \int_{\Omega} |b| |\nabla u_1| |u_2|
\]
Use Cauchy’s inequality with \(\epsilon\), see \([7]\) page 622,
\[
\leq \left[ \int_{\Omega} \frac{1}{4\epsilon} |\nabla u_1|^2 |b|^2 \right] + \epsilon \int_{\Omega} |u_2|^2
\]
Use Poincaré inequality (5.5),
\[
\leq \int_{\Omega} \frac{1}{4\epsilon} |\nabla u_1|^2 |b|^2 dx + \epsilon \left( \frac{1}{\omega_n |\Omega|} \right)^{2/n} \int_{\Omega} |\nabla u_2|^2 dx
\]
\[
\leq \frac{1}{4\epsilon} \|b\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla u_1|^2 dx + \epsilon \left( \frac{1}{\omega_n |\Omega|} \right)^{2/n} \|\nabla u_2\|_{L^2(\Omega)}^2.
\]
Similarly, we apply Cauchy Schwarz inequality and Poincaré inequality (5.5) to get the following estimate.
\[
\text{Re} \left[ \kappa^2 \int_{\Omega} cu_1 \overline{u_2} dx \right] \leq \left| \kappa^2 \int_{\Omega} cu_1 \overline{u_2} dx \right|
\leq \kappa^2 \|c\|_{L^\infty(\Omega)} \|u_1\|_{L^2(\Omega)} \|u_2\|_{L^2(\Omega)}
\leq \frac{1}{2} \kappa^2 \|c\|_{L^\infty(\Omega)} \left[ \|u_1\|_{L^2(\Omega)}^2 + \|u_2\|_{L^2(\Omega)}^2 \right]
\leq \frac{1}{2} \kappa^2 \|c\|_{L^\infty(\Omega)} \left( \frac{1}{\omega_n |\Omega|} \right)^{2/n} \left[ \|\nabla u_1\|_{L^2(\Omega)}^2 + \|\nabla u_2\|_{L^2(\Omega)}^2 \right].
\]
Similarly,
\[
\text{Re} \left[ \int_{\Omega} \frac{1}{a} u_2 \overline{u_1} dx \right] \leq \left| \int_{\Omega} \frac{1}{a} u_2 \overline{u_1} dx \right| \leq \int_{\Omega} \left| \frac{1}{a} \right| \ |u_2| \ |u_1| \ dx
\leq \left| \frac{1}{a} \right| \|L^\infty(\Omega) \int_{\Omega} |u_1| |u_2| dx
\]
Use Cauchy Schwarz inequality,
\[
\leq \left| \frac{1}{a} \right| \|L^\infty(\Omega) \|u_1\|_{L^2(\Omega)} \|u_2\|_{L^2(\Omega)} \leq \frac{1}{2} \left| \frac{1}{a} \right| \|L^\infty(\Omega) \left[ \|u_1\|_{L^2(\Omega)}^2 + \|u_2\|_{L^2(\Omega)}^2 \right]
\]
Use Poincaré inequality (5.5).
Therefore,

\[
\text{Re } B(U, U) = \int_{\Omega} |\nabla u_1|^2 \, dx + \int_{\Omega} |\nabla u_2|^2 \, dx + \text{Re} \left[ \frac{1}{i} \int_{\Omega} (b \cdot \nabla u_1) \overline{u_2} \, dx \right] \\
- \text{Re} \left[ \kappa^2 \int_{\Omega} cu_1 \overline{u_2} \, dx \right] + \text{Re} \left[ \int_{\Omega} \frac{1}{a} u_2 \overline{u_1} \, dx \right]
\]

\[
\geq \int_{\Omega} |\nabla u_1|^2 \, dx + \int_{\Omega} |\nabla u_2|^2 \, dx - \frac{1}{4} \|b\|_{L^\infty(\Omega)}^2 \|\nabla u_1\|_{L^2(\Omega)}^2 + \epsilon \left( \frac{1}{\omega_n} |\Omega| \right)^{2/n} \|\nabla u_2\|_{L^2(\Omega)}^2
\]

\[
- \frac{1}{2} \kappa^2 \|c\|_{L^\infty(\Omega)}^2 \left( \frac{1}{\omega_n} |\Omega| \right)^{2/n} \left[ \|\nabla u_1\|_{L^2(\Omega)}^2 + \|\nabla u_2\|_{L^2(\Omega)}^2 \right]
\]

\[
- \frac{1}{2} \left( \frac{1}{\omega_n} |\Omega| \right)^{2/n} \left( \kappa^2 \|c\|_{L^\infty(\Omega)}^2 + \|a\|_{L^\infty(\Omega)} \right) \|\nabla u_1\|_{L^2(\Omega)}^2
\]

\[
+ \left[ 1 - \left( \frac{1}{\omega_n} |\Omega| \right)^{2/n} \left( \epsilon + \frac{1}{2} \kappa^2 \|c\|_{L^\infty(\Omega)}^2 + \|a\|_{L^\infty(\Omega)} \right) \right] \|\nabla u_2\|_{L^2(\Omega)}^2.
\]

Choose \( \epsilon = \frac{1}{2} \left( \frac{1}{\omega_n} |\Omega| \right)^{-2/n} \) so that \( 1 - \epsilon \left( \frac{1}{\omega_n} |\Omega| \right)^{2/n} = \frac{1}{2} \). We also choose all the coefficients \( 1/\alpha, b, c \) so small that there exist uniform constants \( C_0 > 0 \) and \( \widetilde{C}_0 > 0 \) such that

\[
\frac{1}{2} \left[ 1 - \left( \frac{1}{\omega_n} |\Omega| \right)^{2/n} \left( \kappa^2 \|c\|_{L^\infty(\Omega)}^2 + \frac{1}{\alpha} \|\nabla u_1\|_{L^\infty(\Omega)} \right) \right] \geq C_0 > 0 \quad (5.7)
\]

and

\[
\left[ 1 - \frac{1}{2} \left( \frac{1}{\omega_n} |\Omega| \right)^{2/n} \left( \|b\|_{L^\infty(\Omega)}^2 + \kappa^2 \|c\|_{L^\infty(\Omega)}^2 + \frac{1}{\alpha} \|\nabla u_1\|_{L^\infty(\Omega)} \right) \right] \geq \widetilde{C}_0 > 0. \quad (5.8)
\]

Therefore, we finally using Poincaré inequality (5.5) obtain that

\[
\text{Re } B(U, U) \geq C_0 \|\nabla u_1\|_{L^2(\Omega)}^2 + \widetilde{C}_0 \|\nabla u_2\|_{L^2(\Omega)}^2
\]

\[
\geq C \left[ \|u_1\|_{H_0^2(\Omega)}^2 + \|u_2\|_{H_0^2(\Omega)}^2 \right] = C \|U\|_{(H_0^2(\Omega))^2}^2,
\]

holds for all \( U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in (H_0^1(\Omega))^2 \). Finally using the Lax-Milgram lemma, see for example [Lemma 2.32, 26], we conclude the lemma.

\[ \square \]

6. Appendix

In the Appendix, we provide a detailed estimate for the complex geometrical optics solutions related to bi-Laplace equation. Let \( B(\alpha, \delta) \) denote the ball of radius \( \delta \) centered at \( \alpha \). We
define $K := \partial D \cap \{ x \in \mathbb{R}^n : \frac{1}{2} \log |x - x_0|^2 = h_D(x_0) \}$, where $x_0 \in \mathbb{R}^n \setminus \overline{ch(\Omega)}$. For $\alpha \in K$, the set $K$ can be covered by countable number of balls, precisely, $K \subset \bigcup_{\alpha \in K} B(\alpha, \delta)$. Since, $K$ is compact, there exist $\alpha_1, \ldots, \alpha_N \in K$ such that $K \subset \bigcup_{j=1}^N B(\alpha_j, \delta)$. Then we define $D_{j,\delta} := D \cap B(\alpha_j, \delta)$ and $D_\delta = \bigcup_{j=1}^N D_{j,\delta}$. Note that, $\int_{D_j \cap \Omega} e^{-\frac{1}{\log|\frac{1}{2} \log|x - x_0|^2 - h_D(x_0)|}} dx = O(e^{-\frac{\delta}{h}})$ as $h \to 0$. We introduce a change of co-ordinates as in [28], $y' = x'$, $y_n = \frac{1}{2} \log|x - x_0|^2 - h_D(x_0)$, where $x' = (x_1, \ldots, x_{n-1})$, $y' = (y_1, \ldots, y_{n-1})$, $x = (x', x_n)$, $y = (y', y_n)$. We also denote the parametrization $\partial D$ near $\alpha_j$ by the smooth function $l_j(y')$. We now have following Lemmas.

**Lemma 6.1.** The following estimates hold:

1. For $1 \leq q \leq 2$, we have

   $$\int_D |v(x)|^q dx \leq Ch \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q_j(y')}{h}} dy' + \text{exponentially decaying terms}.$$  

2. $$\int_D |v(x)|^2 dx \geq Ch \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2q_j(y')}{h}} dy' + \text{exponentially decaying terms}.$$  

3. For $1 \leq q \leq 2$, we have

   $$\int_D |
abla v(x)|^q dx \leq Ch^{1-q} \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q_j(y')}{h}} dy' + \text{exponentially decaying terms}.$$  

4. For $1 \leq q < 2$, we have

   $$\int_D |\Delta v(x)|^q dx \leq Ch^{1-q} \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q_j(y')}{h}} dy' + \text{exponentially decaying terms}.$$  

5. $$\int_D |\Delta v(x)|^2 dx \geq Ch^{-1} \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2q_j(y')}{h}} dy' + \text{exponentially decaying terms}.$$  

**Proof.** We follow the approach of [28]. Recall that, when $t = h_D(x_0)$, the complex geometrical optics solution $v$ is of the form $v(x, h) = e^{\frac{x + \psi}{h}} (a_0(x) + ha_1(x) + h^2 a_2(x) + r(x, h))$ where $\phi = h_D(x_0) - \frac{1}{2} \log |x - x_0|^2$. We have following computations:

$$\nabla v(x, h) := \frac{\nabla \phi + i \nabla \psi}{h} e^{\frac{x + \psi}{h}} (a_0(x) + ha_1(x) + h^2 a_2(x) + r(x, h)) + e^{\frac{x + \psi}{h}} (\nabla a_0(x) + h \nabla a_1(x) + h^2 \nabla a_2(x) + \nabla r(x, h)).$$
Also
\[
\Delta v(x, h) = \frac{\Delta \phi + i \Delta \psi}{h} e^{\frac{\phi + i \psi}{h}} (a_0(x) + ha_1(x) + h^2 a_2(x) + r(x, h))
\]
\[
+ \left( \frac{\nabla \phi + i \nabla \psi}{h} \right)^2 e^{\frac{\phi + i \psi}{h}} (a_0(x) + ha_1(x) + h^2 a_2(x) + r(x, h))
\]
\[
+ 2 \frac{\nabla \phi + i \nabla \psi}{h} e^{\frac{\phi + i \psi}{h}} (\nabla a_0(x) + h \nabla a_1(x) + h^2 \nabla a_2(x) + \nabla r(x, h))
\]
\[
+ e^{\frac{\phi + i \psi}{h}} (\Delta a_0(x) + h \Delta a_1(x) + h^2 \Delta a_2(x) + \Delta r(x, h)).
\]

We also recall that, the correction term \( r \) satisfies
\[
\|r\|_{H^3_{scl}(\Omega)} \leq O(h^3),
\]
where for a given \( h > 0 \) and \( k \in \mathbb{N} \), the semiclassical norm of \( r \) is defined as
\[
\|r\|_{H^k_{scl}(\Omega)} := \left[ \sum_{|\alpha| \leq k} \int_{\Omega} |(hD)^{\alpha} u|^2 dx \right]^{1/2},
\]
see [29, Chapter 7], for an extensive literature of these spaces and its properties. Simplifying we get
\[
\left\{ \begin{array}{l}
\|r\|_{L^2(\Omega)} \leq h^3 \\
\|\nabla r\|_{L^2(\Omega)} \leq h^2 \\
\|\Delta r\|_{L^2(\Omega)} \leq h \\
\|\nabla \Delta r\|_{L^2(\Omega)} \leq C.
\end{array} \right. \tag{6.1}
\]

(1) Let us first compute the following integral.
\[
\int_D |v(x)|^2 dx \leq C \int_D e^{-\frac{a}{h^2}(\frac{1}{2} \log |x-x_0|^2 - h \cdot D(x_0))} (a_0^q + h^q a_1^q + h^2 q a_2^q + r^q) dx
\]
\[
= \left( \int_{D_1} + \int_{D_2 \setminus D_1} \right) e^{-\frac{a}{h^2}(\frac{1}{2} \log |x-x_0|^2 - h \cdot D(x_0))} (a_0^q + h^q a_1^q + h^2 q a_2^q + r^q) dx
\]
\[
\leq C(1 + h^q + h^{2q}) \sum_{j=1}^N \int_{|y'| < \delta} dy' \int_{L_j} e^{\frac{-q}{h} \rho_1} dy_1
\]
\[
+ C \left( \int_{D_2} e^{-\frac{a}{h^2}(\frac{1}{2} \log |x-x_0|^2 - h \cdot D(x_0))} dx \right)^{\frac{1}{p}} \left( \int_{D_2} r^q dx \right)^{\frac{q}{2} - p} + Ce^{-\frac{q}{h}}
\]
where \( p = \frac{6}{6 - q} \).
\[
\leq C(1 + h^q + h^{2q}) \left[ h \sum_{j=1}^N \int_{|y'| < \delta} e^{-\frac{q}{h} \rho_1(y')} dy' - \frac{ch}{q} e^{-\frac{q}{h}} \right]
\]
\[
+ ch^q \left[ h \sum_{j=1}^N \int_{|y'| < \delta} e^{-\frac{q}{h} \rho_1(y')} dy' \right]^{\frac{1}{p}} + ch^q h^{\frac{1}{p}} e^{-\frac{q}{h}} + Ce^{-\frac{q}{h}}.
\]
Here we used the fact that \( \|r\|_{L^p(\Omega)} \leq h^2 \) and using the Hölder inequality we obtain

\[
\int_D |v(x)|^q dx \leq C(h + h^{q+1} + h^{2q+1}) \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{qj(y')}{h}} dy' + h^{2q+\frac{6-q}{\alpha}} \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{qj(y')}{h}} dy' \\
- C \frac{(1 + h^{q} + h^{2q})h^{-\frac{q\delta}{h}}}{q} e^{-\frac{q\delta}{h}} + Ch^{2q+\frac{6-q}{\alpha}} e^{-\frac{q\delta}{h}} + Ce^{-\frac{q\delta}{h}}.
\]

The second, third and forth terms are absorbed by the first term since

\[
h^{q+1} \leq o(h), \ h^{2q+1} \leq o(h) \text{ and } h^{2q+\frac{6-q}{\alpha}} \leq o(h)
\]

for \( h \) sufficiently small and \( 1 \leq q \leq 2 \), whereas the other terms are exponentially decaying. Here \( o(\cdot) \) is the small \( o \) notation. Therefore, we have

\[
\int_D |v(x)|^q dx \leq C h \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{qj(y')}{h}} dy' + \text{exponentially decaying terms}.
\]

(2) We next compute the lower bound estimate of the \( L^2 \)-norm of \( v \).

\[
\int_D |v(x)|^2 dx \geq C \int_D e^{-\frac{2}{h}(\frac{1}{2} \log |x-x_0|^2 - hD(x_0))} dx - Ch^2 \int_D e^{-\frac{2}{h}(\frac{1}{2} \log |x-x_0|^2 - hD(x_0))} dx \\
- C h^4 \int_D e^{-\frac{2}{h}(\frac{1}{2} \log |x-x_0|^2 - hD(x_0))} dx - C \int_D e^{-\frac{2}{h}(\frac{1}{2} \log |x-x_0|^2 - hD(x_0))} r^2 dx \\
:= I_1 + I_2 + I_3 + I_4,
\]

where

\[
I_1 := \int_D e^{-\frac{2}{h}(\frac{1}{2} \log |x-x_0|^2 - hD(x_0))} dx \geq \int_{D_{\delta}} e^{-\frac{2}{h}(\frac{1}{2} \log |x-x_0|^2 - hD(x_0))} dx \\
\geq C \sum_{j=1}^N \int_{|y'|<\delta} dy' \int_{I_j(y')} e^{-\frac{2y_{1n}}{h}} dy_n \geq Ch \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2j(y')}{h}} dy' - \frac{C}{2} he^{-\frac{2\delta}{h}}.
\]

\[
I_2 := h^2 \int_D e^{-\frac{2}{h}(\frac{1}{2} \log |x-x_0|^2 - hD(x_0))} dx = h^2 \left( \int_{D_{\delta}} + \int_{D\setminus D_{\delta}} \right) e^{-\frac{2}{h}(\frac{1}{2} \log |x-x_0|^2 - hD(x_0))} dx \\
\leq ch^3 \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2j(y')}{h}} dy' + \text{exponentially decaying terms}.
\]

Similarly,

\[
I_3 \leq ch^5 \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2j(y')}{h}} dy' + \text{exponentially decaying terms}.
\]
Finally, we estimate the remainder term using the Hölder inequality and Sobolev embedding $H^1(D_δ) \hookrightarrow L^6(D_δ)$.

\[ I_4 := \int_D e^{-\frac{q}{8} \frac{1}{2} \log |x-x_0|^2 - h_D(x_0)} r^2 \, dx \leq \left( \int_{D_δ} e^{-\frac{q}{8} \frac{1}{2} \log |x-x_0|^2 - h_D(x_0)} \, dx \right)^{\frac{2}{3}} \left( \int_{D_δ} r^6 \, dx \right)^{\frac{1}{3}} + Ce^{-\frac{2q}{h}} \]

\[ \leq \|r\|_{H^1(D_δ)}^2 \left( h \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q}{8} \frac{1}{2} l_j(y')} \, dy' \right)^{\frac{2}{3}} + \text{exponentially decaying terms} \]

\[ \leq (h^6 + h^4)h^\frac{q}{6} \left( \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q}{8} \frac{1}{2} l_j(y')} \, dy' \right) + \text{exponentially decaying terms}. \]

The first term $I_1$ is the dominating term and it dominates rest of the terms, so finally we have

\[ \int_D |v(x)|^2 \, dx \geq Ch \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{2q}{8} \frac{1}{2} l_j(y')} \, dy' + \text{exponentially decaying terms}. \]

(3) For $1 \leq q < 2$, we write

\[ \int_D |\nabla v|^q \, dx \leq C \int_D e^{-\frac{q}{8} \frac{1}{2} \log |x-x_0|^2 - h_D(x_0)} \times \left[ \frac{1}{h^q} (1 + h^q + h^{2q} + r^q) + (1 + h^q + h^{2q} + |\nabla r|^q) \right]. \]

Using the estimates below

\[ \int_D e^{-\frac{q}{8} \frac{1}{2} \log |x-x_0|^2 - h_D(x_0)} \, dx \leq Ch \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q}{8} \frac{1}{2} l_j(y')} \, dy' + \text{exponentially decaying terms}, \]

\[ \int_D e^{-\frac{q}{8} \frac{1}{2} \log |x-x_0|^2 - h_D(x_0)} r^q \, dx \leq Ch^{2q + \frac{6q}{h}} \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q}{8} \frac{1}{2} l_j(y')} \, dy' + \text{exponentially decaying terms}, \]

\[ \int_D e^{-\frac{q}{8} \frac{1}{2} \log |x-x_0|^2 - h_D(x_0)} |\nabla r|^q \, dx \leq Ch^{2q + \frac{6q}{h}} \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q}{8} \frac{1}{2} l_j(y')} \, dy' + \text{exponentially decaying terms}, \]

we finish the proof of this part.

(4)

\[ \int_D |\Delta v(x)|^q \, dx \leq C \int_D e^{-\frac{q}{8} \frac{1}{2} \log |x-x_0|^2 - h_D(x_0)} \left[ \frac{1}{h^q} (a_0^q + h^q a_1^q + h^{2q} a_2^q + r^q) \right. \]

\[ + \frac{1}{h^q} (|\nabla a_0|^q + h^q |\nabla a_1|^q + h^{2q} |\nabla a_2|^q + |\nabla r|^q) \]

\[ + \left. (|\Delta a_0|^q + h^q |\Delta a_1|^q + h^{2q} |\Delta a_2|^q + |\Delta r|^q) \right] \, dx \]

\[ \leq C \int_D e^{-\frac{q}{8} \frac{1}{2} \log |x-x_0|^2 - h_D(x_0)} \left[ 1 + h^q + h^{-q} + h^{2q} + (h^{-q} r^q + h^{-q} |\nabla r|^q + |\Delta r|^q) \right] \, dx. \]
We now estimate the terms involving $r, \nabla r$ and $\Delta r$. Therefore, applying Hölder inequality and Sobolev estimate, as in (1), we estimate

$$h^{-q} \int_D e^{-\frac{q}{6}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} r^q dx \leq Ch^{q+\frac{6-q}{q}} \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q}{6}I_j(y')} dy' + \text{exponentially decaying terms.}$$

To estimate

$$h^{-q} \int_D e^{-\frac{q}{6}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} |\nabla r|^q dx,$$

we again apply Hölder inequality and Sobolev embedding $H^1(D_\delta) \to L^6(D_\delta)$. Precisely,

$$h^{-q} \int_D e^{-\frac{q}{6}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} |\nabla r|^q \leq h^{-q} \left( \int_{D_\delta} e^{-\frac{pq}{6}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} \right)^{\frac{1}{p}} \times \left( \int_{D_\delta} |\nabla r|^6 \right)^{\frac{2}{6}} \leq Ch^{q+\frac{6-q}{q}} \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q}{6}I_j(y')} dy' + \text{exponentially decaying term.}$$

Here we use the fact that, $||\nabla r||_{L^6(D_\delta)}^q \leq h^q$. Finally, to estimate the term involving $\Delta r$, we again use Hölder inequality and Sobolev embedding. Therefore,

$$\int_D e^{-\frac{q}{6}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} |\Delta r|^q dx \leq C \left( \int_{D_\delta} e^{-\frac{pq}{6}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} dx \right)^{\frac{1}{p}} \times \left( \int_{D_\delta} |\Delta r|^6 \right)^{\frac{2}{6}} \leq h^{q+\frac{6-q}{6}} \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q}{6}I_j(y')} dy' + \text{exponentially decaying terms.}$$

Finally, combining all the above estimate, we thus obtain,

$$\int_D |\Delta v(x)|^q dx \leq Ch^{1-q} \left( \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q}{6}I_j(y')} dy' \right) (1 + h^q + h^{2q} + h^{3q} + h^{\frac{11q}{6}} + h^{\frac{7q}{6}}) \leq Ch^{1-q} \left( \sum_{j=1}^N \int_{|y'|<\delta} e^{-\frac{q}{6}I_j(y')} dy' \right) + \text{exponentially decaying terms.}$$

(5) We start with the inequality

$$\int_D |\Delta v(x)|^2 dx \geq C \int_D e^{-\frac{4}{6}(\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} \times (h^{-2} - 1 - h^2 - (h^{-2} r^2 + h^{-2} |\nabla r|^2 + |\Delta r|^2)) dx.$$
We already have
\[
\int_D e^{-\frac{2}{\kappa} (\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} \geq Ch \sum_{j=1}^{N} \int_{|y'|<\delta} e^{-\frac{2}{\kappa} l_j(y')} dy' \geq Ch^n,
\]
\[
\int_D e^{-\frac{2}{\kappa} (\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} r^2 dx \leq Ch^{14/3} \sum_{j=1}^{N} \int_{|y'|<\delta} e^{-\frac{2}{\kappa} l_j(y')} dy' + \text{exponentially decaying terms},
\]
\[
\int_D e^{-\frac{2}{\kappa} (\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} |\nabla r|^2 dx \leq Ch^{8/3} \sum_{j=1}^{N} \int_{|y'|<\delta} e^{-\frac{2}{\kappa} l_j(y')} dy' + \text{exponentially decaying terms},
\]
\[
\int_D e^{-\frac{2}{\kappa} (\frac{1}{2} \log |x-x_0|^2 - h_D(x_0))} |\Delta r|^2 dx \leq Ch^{2/3} \sum_{j=1}^{N} \int_{|y'|<\delta} e^{-\frac{2}{\kappa} l_j(y')} dy' + \text{exponentially decaying terms}.
\]
Using above all estimates we get
\[
\int_D |\Delta v|^2 dx \geq C \sum_{j=1}^{N} \int_{|y'|<\delta} e^{-\frac{2}{\kappa} l_j(y')} dy' (h^{-1} - h - h^{3} - h^{8/3} - h^{2/3})
\geq Ch^{-1} \sum_{j=1}^{N} \int_{|y'|<\delta} e^{-\frac{2}{\kappa} l_j(y')} dy' + \text{exponentially decaying terms}.
\]

\[\square\]

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