Inductive limit violates quasi-cocommutativity

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Abstract
We show that the inductive limit of a certain inductive system of
quasi-cocommutative C^*-bialgebras is not quasi-cocommutative. This
implies that the category of quasi-cocommutative C^*-bialgebras is not
closed with respect to the inductive limit.

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1 Introduction

A C^*-bialgebra is a C^*-algebra with several extra structures on it, which was
introduced in the operator algebra approach to quantum groups as a locally
compact quantum semigroup [16] (see also [6, 17]). A quasi-cocommutative
C^*-bialgebra is defined as a C^*-bialgebra with a universal R-matrix which
is modified to focus on C^*-bialgebra [9, 15]. In this paper, we prove the
following statement.

Theorem 1.1 The category of quasi-cocommutative C^*-bialgebras is not
closed with respect to the inductive limit.

In this section, we roughly explain our motivation and the significance of
Theorem 1.1. Explicit mathematical definitions will be shown after §2.

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1.1 Motivation

We have constructed various non-commutative and non-cocommutative $C^*$-bialgebras by using sets of $C^*$-algebras and $*$-homomorphisms among them \cite{11,12,14}. These studies are motivated by a discovery of a certain non-symmetric tensor product of representations of Cuntz algebras \cite{10}, without reference to the study of quantum groups.

On the other hand, the inductive limit is an important tool to construct new $C^*$-algebras, which often changes properties of $C^*$-algebras \cite{7,18}. For a given subcategory of the category of $C^*$-algebras, the inductive limit on it is an interesting subject of research.

Our interests are to define the inductive limit of $C^*$-bialgebras and to study its property. Especially, we consider the inductive limit of quasi-cocommutative $C^*$-bialgebras in this paper.

In order to prove Theorem 1.1, we construct an inductive system of quasi-cocommutative $C^*$-bialgebras such that its inductive limit is not quasi-cocommutative.

1.2 Comparison with quantum enveloping algebras

We explain the significance of Theorem 1.1 in comparison to cases of quantum enveloping algebras in this subsection.

We start with a brief history of quantum groups. At the beginning, quantum groups were introduced as one-parameter deformations of universal enveloping algebras $U_q(g)$ (= quantum enveloping algebra \cite{9}) of semisimple complex Lie algebras $g$ \cite{5,8}. A motivation of the study is to construct solutions of Yang-Baxter equations \cite{2,20}, which are called $R$-matrices. The fundamental structure of a quantum group is a Hopf algebra, but it is not sufficient for the original purpose. In order to define its universal $R$-matrix as an infinite series in the tensor square of the completion of $U_q(g)$, the $h$-adic topology is introduced (\cite{9}, Chap. XVI) where $h$ is related to $q$ as $q = e^h$. In this case, the topology is an inverse limit topology, and it is necessary to define the universal $R$-matrix in the theory of quantum groups in general. The quasi-cocommutativity is acquired by taking the limit in this case.

On the other hand, Theorem 1.1 means that inductive limit violates the quasi-cocommutativity of $C^*$-bialgebras in general. This is a new phenomenon which is quite a contrast to the case of quantum enveloping algebras. Both cases give an indication of relations between topology and quasi-cocommutativity.
In § 2 we will recall basic definitions of quasi-cocommutative $C^*$-bialgebras and consider categories. In § 3 we will prepare a general method to construct quasi-cocommutative $C^*$-bialgebras. In § 4 we will construct an example of inductive system of quasi-cocommutative $C^*$-bialgebras and prove Theorem 1.1.

2 Definitions

In this section, we recall basic definitions.

2.1 Quasi-cocommutative $C^*$-bialgebra and categories

In this subsection, we recall definitions of quasi-cocommutative $C^*$-bialgebra and related notions [14, 15]. In addition, we consider categories of $C^*$-bialgebras in Remark 2.1.

For a $C^*$- algebra $A$, let $\mathcal{M}(A)$ denote the multiplier algebra of $A$. For two $C^*$-algebras $A$ and $B$, let $\text{Hom}(A,B)$ and $A \otimes B$ denote the set of all $\ast$-homomorphisms from $A$ to $B$ and the minimal $C^*$-tensor product of $A$ and $B$, respectively. We state that $f \in \text{Hom}(A,\mathcal{M}(B))$ is nondegenerate if $f(A)B$ is dense in $B$. In this case, $f$ is called a morphism from $A$ to $B$ [19]. If $f$ is a nondegenerate $\ast$-homomorphism from $A$ to $B$, then we can regard $f$ as a morphism from $A$ to $B$ by using the canonical embedding of $B$ into $\mathcal{M}(B)$. Each morphism $f$ from $A$ to $B$ can be extended uniquely to a homomorphism $\tilde{f}$ from $\mathcal{M}(A)$ to $\mathcal{M}(B)$ such that $\tilde{f}(m)f(b)a = f(mb)a$ for $m \in \mathcal{M}(B)$, $b \in B$, and $a \in A$.

A $C^*$-bialgebra is a pair $(A, \Delta)$ of a $C^*$- algebra $A$ and a morphism $\Delta$ from $A$ to $A \otimes A$ which satisfies $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$. We call $\Delta$ the comultiplication of $A$. A $C^*$-bialgebra $(A, \Delta)$ is counital if there exists $\varepsilon \in \text{Hom}(A, \mathcal{C})$ such that $(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta$. We call $\varepsilon$ the counit of $A$ and write $(A, \Delta, \varepsilon)$ as the counital $C^*$-bialgebra $(A, \Delta)$ with the counit $\varepsilon$. Remark that we do not assume $\Delta(A) \subset A \otimes A$. Furthermore, $A$ has no unit for a $C^*$-bialgebra $(A, \Delta)$ in general.

Define the extended flip $\tilde{\tau}_{A,A}$ from $\mathcal{M}(A \otimes A)$ to $\mathcal{M}(A \otimes A)$ as $\tilde{\tau}_{A,A}(X) \equiv \tau_{A,A}(X(y \otimes x))$ for $X \in \mathcal{M}(A \otimes A)$, $x, y \in A$ where $\tau_{A,A}$ denotes the flip of $A \otimes A$. The map $\Delta^\text{op}$ from $A$ to $\mathcal{M}(A \otimes A)$ defined as $\Delta^\text{op} = \tilde{\tau}_{A,A} \circ \Delta$ is called the opposite comultiplication of $\Delta$. A $C^*$-bialgebra $(A, \Delta)$ is cocommutative if $\Delta = \Delta^\text{op}$. An element $R$ in $\mathcal{M}(A \otimes A)$ is called a (unitary) universal $R$-matrix of $(A, \Delta)$ if $R$ is a unitary and

$$R\Delta(x)R^* = \Delta^\text{op}(x) \quad (x \in A). \quad (2.1)$$
In this case, we state that \((A, \Delta)\) is quasi-cocommutative (or almost cocommutative \([4]\)). We write \((A, \Delta, R)\) as a quasi-cocommutative \(C^*\)-bialgebra \((A, \Delta)\) with a universal \(R\)-matrix \(R\). If \(A\) is unital, then \(\mathcal{M}(A \otimes A) = A \otimes A\) and \(\tilde{\tau}_{A,A} = \tau_{A,A}\). In addition, if \((A, \Delta)\) is quasi-cocommutative with a universal \(R\)-matrix \(R\), then \(R \in A \otimes A\). We state that a quasi-cocommutative \(C^*\)-bialgebra \((A, \Delta, R)\) is quasi-triangular (or braided \([9]\)) if the following holds:

\[
(\Delta \otimes id)(R) = R_{13}R_{23}, \quad (id \otimes \Delta)(R) = R_{13}R_{12}
\]  

(2.2)

where we use the leg numbering notation \([1]\): \((A, \Delta, R)\) is triangular if \((A, \Delta, R)\) is quasi-triangular and the following holds:

\[
R \tilde{\tau}_{A,A}(R) = I
\]  

(2.3)

where \(I\) denotes the unit of \(\mathcal{M}(A \otimes A)\). The cocommutativity is the dual notion of the commutativity. Since a cocommutative \(C^*\)-bialgebra is always quasi-cocommutative (furthermore, it is triangular), the quasi-cocommutativity is a generalization of the cocommutativity.

We consider categories of \(C^*\)-bialgebras as follows.

**Remark 2.1** Let \((A_i, \Delta_i)\) be a \(C^*\)-bialgebra for \(i = 1, 2\).

(i) A map \(f\) is a \(C^*\)-bialgebra morphism from \((A_1, \Delta_1)\) to \((A_2, \Delta_2)\) if \(f\) is a nondegenerate \(*\)-homomorphism from \(A_1\) to \(M(A_2)\) such that \((f \otimes f) \circ \Delta_1 = \Delta_2 \circ f\). The category of \(C^*\)-bialgebras is defined as the category with \(C^*\)-bialgebra morphisms as morphisms among objects. In addition, if \(R_1\) is a universal \(R\)-matrix of \((A_1, \Delta_1)\), then \(R' \equiv (f \otimes f)(R_1)\) is also a universal \(R\)-matrix of the \(C^*\)-bialgebra \((f(A_1), \Delta_2|_{f(A_1)})\).

(ii) Define \(C \equiv A_1 \otimes A_2\) and \(\Delta \equiv (id \otimes \tau \otimes id) \circ (\Delta_1 \otimes \Delta_2)\) where \(\tau\) denotes the flip from \(A_1 \otimes A_2\) to \(A_2 \otimes A_1\) and \(id \otimes \tau \otimes id\) is extended on \(\mathcal{M}(A_1 \otimes A_1) \otimes \mathcal{M}(A_2 \otimes A_2)\). Then \(C\) is a \(C^*\)-bialgebra with the comultiplication \(\Delta\). We see that the tensor product of two \(C^*\)-bialgebra morphisms is also a \(C^*\)-bialgebra morphism. In addition, if \(R_i\) is a universal \(R\)-matrix of \(A_i\) for \(i = 1, 2\), then \(R \equiv (id \otimes \tau \otimes id)(R_1 \otimes R_2)\) is also a universal \(R\)-matrix of \((C, \Delta)\). From this, we see that the tensor product of quasi-cocommutative \(C^*\)-bialgebras is also quasi-cocommutative. Furthermore, we can verify that the tensor product of quasi-triangular (resp. triangular) \(C^*\)-bialgebras is also quasi-triangular (resp. triangular). In this way, the category of (quasi-cocommutative, quasi-triangular, triangular) \(C^*\)-bialgebras is closed with respect to the tensor product.
2.2 Direct product, direct sum and inductive limit of C*-algebras

We recall direct product, direct sum and inductive limit of C*-algebras \[3\]. For an infinite set \( \{ A_i : i \in \Omega \} \) of C*-algebras, we define two C*-algebras \( \prod_{i \in \Omega} A_i \) and \( \bigoplus_{i \in \Omega} A_i \) as follows:

\[
\prod_{i \in \Omega} A_i \equiv \{ (a_i) : \| (a_i) \| = \sup_i \| a_i \| < \infty \}, \tag{2.4}
\]

\[
\bigoplus_{i \in \Omega} A_i \equiv \{ (a_i) : \| (a_i) \| \to 0 \text{ as } i \to \infty \} \tag{2.5}
\]

in the sense that for every \( \epsilon > 0 \) there are only finitely many \( i \) for which \( \| a_i \| > \epsilon \). We call \( \prod_{i \in \Omega} A_i \) and \( \bigoplus_{i \in \Omega} A_i \) the direct product and the direct sum of \( A_i \)'s, respectively. The algebra \( \bigoplus_{i \in \Omega} A_i \) is a closed two-sided ideal of \( \prod_{i \in \Omega} A_i \). The algebraic direct sum \( \bigoplus_{i \in \Omega} A_i \) is a dense \( \ast \)-subalgebra of \( \bigoplus_{i \in \Omega} A_i \). Since \( \mathcal{M}(\bigoplus_{i \in \Omega} A_i) \cong \prod_{i \in \Omega} \mathcal{M}(A_i) \) (\[3\], II.8.1.3), if \( A_i \) is unital for each \( i \), then

\[
\mathcal{M}\left( \bigoplus_{i \in \Omega} A_i \right) \cong \prod_{i \in \Omega} A_i. \tag{2.6}
\]

An inductive system of C*-algebras is a collection \( \{(A_i, f_{ij}) : i, j \in \Omega, i \leq j \} \), where \( \Omega \) is a directed set, the \( A_i \) are C*-algebras, and \( f_{ij} \) is a \( \ast \)-homomorphism from \( A_i \) to \( A_j \) with \( f_{ik} = f_{jk} \circ f_{ij} \) for \( i \leq j \leq k \). With respect to the seminorm \( \| a \| \equiv \lim_{j \uparrow i} \| f_{ij}(a) \| \) for \( a \in A_i \), the completion of the algebraic direct limit with elements of seminorm 0 divided out is a C*-algebra called the inductive limit of the system, written \( \varprojlim A_i \). Clearly, if \( A_i \) is unital for each \( i \), then \( \varprojlim A_i \) is also unital.

We introduce the inductive limit of C*-bialgebras as follows.

**Definition 2.2** A data \( \{(A_i, \Delta_i, f_{ij}) : i, j \in \Omega \} \) is an inductive system of C*-bialgebras if \( \{(A_i, f_{ij}) : i, j \in \Omega \} \) is an inductive system of C*-algebras such that \( A_i \) is a C*-bialgebra and \( f_{ij} \) is a C*-bialgebra morphism from \( A_i \) to \( A_j \).

For an inductive system \( \{(A_i, \Delta_i, f_{ij}) : i, j \in \Omega \} \) of C*-bialgebras, let \( A \) denote the inductive limit of the inductive system \( \{(A_i, f_{ij}) : i, j \in \Omega \} \) of C*-algebras. Let \( \mu_i \) denote the canonical map from \( A_i \) to \( A \). Define the map \( \Delta^{(0)} \) on \( \bigcup_i \mu_i(A_i) \) as

\[
\Delta^{(0)}(\mu_i(x)) \equiv \{(\mu_i \otimes \mu_i) \circ \Delta_i\}(x) \quad (x \in A_i). \tag{2.7}
\]
Let $\Delta$ denote the unique extension of $\Delta^{(0)}$ on $A$. Then $(A, \Delta)$ is a C*-bialgebra. We call $(A, \Delta)$ the inductive limit of $\{(A_i, \Delta_i, f_{ij}) : i, j \in \Omega\}$. If $(A_i, \Delta_i)$ is cocommutative for each $i$, then the inductive limit $(A, \Delta)$ is also cocommutative. In this way, the inductive limit preserves both the commutativity and the cocommutativity.

We prepare a lemma for the proof of Theorem 1.1 in § 4.2.

**Lemma 2.3** ([14], Lemma 2.1) Let $(A, \Delta)$ be a C*-bialgebra. If $(A, \Delta)$ is quasi-cocommutative, then for any two nondegenerate representations $\pi_1$ and $\pi_2$ of the C*-algebra $A$, $(\pi_1 \otimes \pi_2) \circ \Delta$ and $(\pi_2 \otimes \pi_1) \circ \Delta$ are unitarily equivalent where we write the extension of $\pi_i \otimes \pi_j$ on $M(A \otimes A)$ as the same notation $\pi_i \otimes \pi_j$ for $i, j = 1, 2$.

### 3 C*-weakly coassociative system

In this section, we recall a general method to construct C*-bialgebras and develop it.

#### 3.1 Definition

According to [11, 15], we recall C*-weakly coassociative system in this subsection. We call $M$ a monoid if $M$ is a semigroup with unit.

**Definition 3.1** Let $M$ be a monoid with the unit $e$. A data $\{(A_a, \varphi_{a,b}) : a, b \in M\}$ is a C*-weakly coassociative system (= C*-WCS) over $M$ if $A_a$ is a unital C*-algebra for $a \in M$ and $\varphi_{a,b}$ is a unital $*$-homomorphism from $A_{ab}$ to $A_a \otimes A_b$ for $a, b \in M$ such that

(i) for all $a, b, c \in M$, the following holds:

$$
(id_a \otimes \varphi_{b,c}) \circ \varphi_{a,b,c} = (\varphi_{a,b} \otimes id_c) \circ \varphi_{ab,c}
$$

(3.1)

where $id_x$ denotes the identity map on $A_x$ for $x = a, c$,

(ii) there exists a counit $\varepsilon_e$ of $A_e$ such that $(A_e, \varphi_{e,e}, \varepsilon_e)$ is a counital C*-bialgebra,

(iii) $\varphi_{e,a}(x) = I_e \otimes x$ and $\varphi_{a,e}(x) = x \otimes I_e$ for $x \in A_a$ and $a \in M$.

Then the following holds.
Theorem 3.2 ([11], Theorem 3.1) Let \( \{(A_a, \varphi_{a,b}) : a, b \in M\} \) be a C*-WCS over a monoid \( M \). Assume that \( M \) satisfies
\[
\#N_a < \infty \text{ for each } a \in M \tag{3.2}
\]
where \( N_a \equiv \{(b, c) \in M \times M : bc = a\} \). Define C*-algebras
\[
A_* \equiv \bigoplus\{A_a : a \in M\}, \quad C_a \equiv \bigoplus\{A_b \otimes A_c : (b, c) \in N_a\} \quad (a \in M), \tag{3.3}
\]
and define *-homomorphisms \( \Delta_{(a)} \varphi \in \text{Hom}(A_a, C_a) \) and \( \Delta \varphi \in \text{Hom}(A_*, A_* \otimes A_*) \) by
\[
\Delta_{(a)} \varphi(x) \equiv \sum_{(b, c) \in N_a} \varphi_{b,c}(x) \quad (x \in A_a), \quad \Delta \varphi \equiv \bigoplus\{\Delta_{(a)} \varphi : a \in M\}. \tag{3.4}
\]
Then \( (A_*, \Delta \varphi) \) is a C*-bialgebra.

We call \( (A_*, \Delta \varphi) \) in Theorem 3.2 the C*-bialgebra associated with \( \{(A_a, \varphi_{a,b}) : a, b \in M\} \). In this paper, we always assume the condition (3.2).

Definition 3.3 Let \( \{(A_a, \varphi_{a,b}) : a, b \in M\} \) be a C*-WCS.

(i) For \( a, b \in M \), define \( \varphi_{a,b}^{\text{op}} \in \text{Hom}(A_{ab}, A_b \otimes A_a) \) by
\[
\varphi_{a,b}^{\text{op}} \equiv \tau_{a,b} \circ \varphi_{a,b} \tag{3.5}
\]
where \( \tau_{a,b} \) denotes the flip from \( A_a \otimes A_b \) to \( A_b \otimes A_a \).

(ii) \( \{(A_a, \varphi_{a,b}) : a, b \in M\} \) is locally quasi-cocommutative if there exists \( \{R^{(a,b)} : a, b \in M\} \) such that \( R^{(a,b)} \) is a unitary in \( A_a \otimes A_b \) and
\[
R^{(a,b)} \varphi_{a,b}(x) R^{(a,b)^*} = \varphi_{b,a}^{\text{op}}(x) \quad (x \in A_{ab}) \tag{3.6}
\]
for each \( a, b \in M \). In this case, we write \( \{(A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in M\} \) as a locally quasi-cocommutative C*-WCS.

(iii) A locally quasi-cocommutative C*-WCS \( \{(A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in M\} \) is locally quasi-triangular if the following holds for each \( a, b, c \in M \):
\[
(\varphi_{a,b} \otimes \text{id}_c)(R^{(ab,c)}) = R^{(a,c)}_{13} R^{(b,c)}_{23} \tag{3.7}
\]
\[
(\text{id}_a \otimes \varphi_{b,c})(R^{(a,bc)}) = R^{(a,c)}_{13} R^{(b,c)}_{12} \tag{3.8}
\]

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(iv) A locally quasi-cocommutative C*-WCS \{ (A, \varphi_{a,b}, R^{(a,b)} ) : a, b \in M \} is locally triangular if \{ (A, \varphi_{a,b}, R^{(a,b)} ) : a, b \in M \} is locally quasi-triangular and the following holds:

\[ R^{(a,b)} \tau_{b,a} (R^{(b,a)}) = I_a \otimes I_b \quad (a, b \in M) \]  

where \( I_x \) denotes the unit of \( A_x \) for \( x = a, b \).

For a C*-WCS \{ (A, \varphi_{a,b}) : a, b \in M \}, we see that \( \mathcal{M}(A_+ \otimes A_+) \cong \prod_{a, b \in M} A_a \otimes A_b \) from (2.6). Hence we identify an element in \( \mathcal{M}(A_+ \otimes A_+) \) with that in \( \prod_{a, b \in M} A_a \otimes A_b \). Then the following holds.

**Lemma 3.4** ([15], Lemma 2.4) Assume that a monoid \( M \) is abelian.

(i) If a C*-WCS \{ (A, \varphi_{a,b}) : a, b \in M \} is locally quasi-cocommutative with respect to \( \{ R^{(a,b)} : a, b \in M \} \) in (3.6), then the unitary \( R \in \mathcal{M}(A_+ \otimes A_+) \) defined by

\[ R \equiv (R^{(a,b)})_{a, b \in M} \]  

is a universal R-matrix of \( (A_+, \Delta_\varphi) \) in Theorem 3.2.

(ii) If a locally quasi-cocommutative C*-WCS \{ (A, \varphi_{a,b}, R^{(a,b)}) : a, b \in M \} is locally quasi-triangular (resp. locally triangular), then \( (A_+, \Delta_\varphi, R) \) is quasi-triangular (resp. triangular) for \( R \) in (3.10).

### 3.2 Componentwise tensor power of C*-weakly coassociative system

In this subsection, we give a new method to construct C*-weakly coassociative systems (=C*-WCSs) from a given C*-WCS. Assume that \{ (A, \varphi_{a,b}) : a, b \in M \} is a C*-WCS. Fix \( n \geq 1 \). Let \( A_{a,n} \) denote the \( n \)-times tensor power of \( A_a \) for \( a \in M \). For \( a, b \in M \), define \( \varphi_{a,b}^{(n)} \in \text{Hom}(A_{ab} \otimes A_{a}^{\otimes n} \otimes A_{b}^{\otimes n}) \) by

\[ \varphi_{a,b}^{(n)} \equiv T_{a,b}^{(n)} \circ (\varphi_{a,b})^{\otimes n} \]  

where \( T_{a,b}^{(n)} \in \text{Hom}((A_a \otimes A_b)^{\otimes n} , A_{a}^{\otimes n} \otimes A_{b}^{\otimes n}) \) is defined as

\[ T_{a,b}^{(n)}(x_1 \otimes y_1 \otimes x_2 \otimes y_2 \otimes \cdots \otimes x_n \otimes y_n) \equiv x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_n \]  

for \( x_1, \ldots, x_n \in A_a \) and \( y_1, \ldots, y_n \in A_b \). Then we see that \( (\varphi_{a,b} \otimes id_c^{\otimes n}) \circ \varphi_{a,b}^{(n)} = (id_a^{\otimes n} \otimes \varphi_{b,c}^{(n)}) \circ \varphi_{a,b}^{(n)} \) for each \( a, b, c \in M \). Hence we can verify that \{ (A, \varphi_{a,b}^{(n)} : a, b \in M \} is a C*-WCS.
Definition 3.5 The $C^*$-WCS $\{(A_a^{\otimes n}, \varphi_{a,b}^{(n)}) : a, b \in \mathcal{M}\}$ is called the componentwise $n$-times tensor power of $\{(A_a, \varphi_{a,b}) : a, b \in \mathcal{M}\}$.

Clearly, $(\bigoplus_a A_a)^{\otimes n}$ and $(\bigoplus_a A_a^{\otimes n})$ are not isomorphic as a $C^*$-algebra when $n \geq 2$ in general. Hence the $C^*$-bialgebra associated with $\{(A_a^{\otimes n}, \varphi_{a,b}^{(n)}) : a, b \in \mathcal{M}\}$ is not isomorphic to a tensor power of the $C^*$-bialgebra associated with $\{(A_a, \varphi_{a,b}) : a, b \in \mathcal{M}\}$ in general.

Lemma 3.6 Assume that $\{(A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in \mathcal{M}\}$ is a locally quasi-cocommutative $C^*$-WCS. Fix $n \geq 1$.

(i) For $a, b \in \mathcal{M}$, $(\varphi_{b,a}^{(n)})^{\text{op}} = (\varphi_{b,a}^{(n)})_{\text{op}}$.

(ii) For $a, b \in \mathcal{M}$, define $R^{(a,b,n)} \in A_a^{\otimes n} \otimes A_b^{\otimes n}$ by

$$R^{(a,b,n)} = T_{a,b}^{(n)}((R^{(a,b)})^{\otimes n}).$$  \hspace{1cm} (3.13)

Then $\{(A_a^{\otimes n}, \varphi_{a,b}^{(n)}, R^{(a,b,n)}) : a, b \in \mathcal{M}\}$ is a locally quasi-cocommutative $C^*$-WCS.

(iii) In addition to (ii), if $\{(A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in \mathcal{M}\}$ is locally quasi-triangular (resp. locally triangular), then $\{(A_a^{\otimes n}, \varphi_{a,b}^{(n)}, R^{(a,b,n)}) : a, b \in \mathcal{M}\}$ is also locally quasi-triangular (resp. locally triangular).

Proof. (i) Let $\tau_{b,a}^{(n)}$ denote the flip from $A_b^{\otimes n} \otimes A_a^{\otimes n}$ to $A_a^{\otimes n} \otimes A_b^{\otimes n}$. Then we can verify that

$$T_{a,b}^{(n)} \circ (\tau_{b,a}^{(n)})^{\otimes n} = \tau_{b,a}^{(n)} \circ T_{b,a}^{(n)}.$$  \hspace{1cm} (3.14)

On the other hand, we see that $(\varphi_{b,a}^{(n)})^{\text{op}} = \tau_{b,a}^{(n)} \circ T_{b,a}^{(n)} \circ (\tau_{b,a}^{(n)})^{\otimes n}$ and $(\varphi_{b,a}^{(n)})_{\text{op}} = T_{a,b}^{(n)} \circ (\tau_{b,a}^{(n)})^{\otimes n} \circ (\varphi_{b,a}^{(n)})^{\otimes n}$. From these, the statement holds.

(ii) Let $x = x_1 \otimes \cdots \otimes x_n \in A_{ab}^{\otimes n}$. Then

$$R^{(a,b,n)}(\varphi_{a,b}^{(n)})(x)(R^{(a,b,n)})^*$$

$$= T_{a,b}^{(n)}(R^{(a,b)})^{\otimes n}(\varphi_{a,b})(R^{(a,b)})\otimes (R^{(a,b)})^*$$

$$= T_{a,b}^{(n)}(R^{(a,b)}(\varphi_{a,b}^{(n)})(x_1) \otimes \cdots \otimes (R^{(a,b)})^*)(\varphi_{b,a}^{(n)}(x_n))$$

$$= T_{a,b}^{(n)}(\varphi_{b,a}^{(n)}(x) \otimes \cdots \otimes (\varphi_{b,a}^{(n)}(x_n))_{\text{op}})$$

$$= (\varphi_{b,a}^{(n)})_{\text{op}}(x)$$  \hspace{1cm} (from (i)).

This implies the statement.
(iii) Assume the local quasi-triangularity for \(\{ (A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in M \} \).
For \(a, b, c \in M\), define \(T_{(a,b),c}^{(n)} \in \text{Hom}((A_a \otimes A_b \otimes A_c)^{\otimes n}, (A_a \otimes A_b)^{\otimes n} \otimes A_c^{\otimes n})\)
and \(T_{a,b,c}^{(n)} \in \text{Hom}((A_a \otimes A_b \otimes A_c)^{\otimes n}, A_a^{\otimes n} \otimes A_b^{\otimes n} \otimes A_c^{\otimes n})\)
by
\[
T_{(a,b),c}^{(n)}(x_1 \otimes y_1 \otimes z_1 \otimes \cdots \otimes x_n \otimes y_n \otimes z_n) = x_1 \otimes y_1 \otimes \cdots \otimes x_n \otimes y_n \otimes z_1 \otimes \cdots \otimes z_n,
\]
\[
T_{a,b,c}^{(n)}(x_1 \otimes y_1 \otimes z_1 \otimes \cdots \otimes x_n \otimes y_n \otimes z_n) = \equiv x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_n \otimes z_1 \otimes \cdots \otimes z_n,
\]
for \(x_1, \ldots, x_n \in A_a, y_1, \ldots, y_n \in A_b\) and \(z_1, \ldots, z_n \in A_c\). Then
\[
(\varphi_{a,b}^{(n)} \otimes \text{id}_c^{(n)})(R^{(a,b,c)})
\]
\[
= \{ (T_{a,b}^{(n)} \otimes \text{id}_c^{(n)}) \circ (\varphi_{a,b}^{(n)} \otimes \text{id}_c^{(n)}) \}(T_{a,b,c}^{(n)}(R^{(a,b,c)}))
\]
\[
= (T_{a,b}^{(n)} \otimes \text{id}_c^{(n)})(T_{a,b,c}^{(n)}(\{ \varphi_{a,b}^{(n)} \circ \text{id}_c^{(n)} \}(R^{(a,b,c)})))
\]
\[
= (T_{a,b}^{(n)} \otimes \text{id}_c^{(n)})(T_{a,b,c}^{(n)}(\{ R^{(a,c)}_{13} \otimes R^{(b,c)}_{12} \}) \otimes n)) \quad \text{(from (3.7))}
\]
\[
= T_{a,b,c}^{(n)}(\{ R^{(a,c)}_{13} \otimes R^{(b,c)}_{12} \}) \otimes n)
\]
\[
= T_{a,b,c}^{(n)}(\{ R^{(a,c)}_{13} \otimes R^{(b,c)}_{12} \}) \otimes n).
\]
By the same reasoning, we obtain \((\text{id}_a^{\otimes n} \otimes \varphi_{b,c}^{(n)})(R^{(a,b,c)}) = R^{(a,b,c)}_{13} R^{(a,b,c)}_{12} \).
Hence the statement about the local quasi-triangularity holds.

Assume the local triangularity for \(\{ (A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in M \} \). It is sufficient to show (3.9) for \(\{ (A_a^{\otimes n}, \varphi_{a,b}^{(n)}, R^{(a,b)}) : a, b \in M \} \) here. For \(a, b \in M\), let \(\tau_{b,a}^{(n)}\) be as in the proof of (i). Then
\[
R^{(a,b)}(\tau_{b,a}^{(n)})(R^{(b,a)})
\]
\[
= T_{a,b}^{(n)}((R^{(a,b)})^{\otimes n}) \tau_{b,a}^{(n)}(T_{b,a}^{(n)}((R^{(b,a)})^{\otimes n}))
\]
\[
= T_{a,b}^{(n)}((R^{(a,b)})^{\otimes n}) (T_{a,b}^{(n)} \circ (\tau_{b,a}^{(n)})((R^{(b,a)})^{\otimes n}) \quad \text{(from (3.4))}
\]
\[
= T_{a,b}^{(n)}((R^{(a,b)})^{\otimes n}((R^{(b,a)})^{\otimes n}))
\]
\[
= T_{a,b}^{(n)}((R^{(a,b)})^{\otimes n}((R^{(b,a)})^{\otimes n}))
\]
\[
= T_{a,b}^{(n)}((R^{(a,b)})^{\otimes n}((R^{(b,a)})^{\otimes n}))
\]
\[
= T_{a,b}^{(n)}((I_a \otimes I_b)^{\otimes n}) \quad \text{(from (3.9))}
\]
\[
= I_a^{\otimes n} \otimes I_b^{\otimes n}.
\]
Hence the statement about the local triangularity holds.
3.3 Componentwise infinite tensor power of $C^*$-weakly coassociative system

In this subsection, we define the componentwise infinite tensor power of $C^*$-weakly coassociative system (=$C^*$-WCS) from a given $C^*$-WCS as the inductive limit of componentwise tensor powers of $C^*$-WCS. Let $\{(A_a, \varphi_{a,b}) : a, b \in M\}$ be a $C^*$-WCS and let $\{(A_a^{\otimes n}, \varphi_{a,b}^{(n)}) : a, b \in M\}$ be the componentwise $n$-times tensor power of $\{(A_a, \varphi_{a,b}) : a, b \in M\}$ in Definition 3.5 for $n \geq 1$. With respect to the embedding

$$\psi_a^{(n)} : A_a^{\otimes n} \ni x \mapsto x \otimes I_a \in A_a^{\otimes n} \otimes A_a = A_a^{\otimes (n+1)},$$

(3.17)

we regard $A_a^{\otimes n}$ as a $C^*$-subalgebra of $A_a^{\otimes (n+1)}$ for each $a \in M$. Let $A_a^{\otimes \infty}$ denote the inductive limit of the inductive system $\{(A_a^{\otimes n}, \psi_{a,b}^{(n)}) : n \geq 1\}$ of $C^*$-algebras:

$$A_a^{\otimes \infty} \equiv \varinjlim_n (A_a^{\otimes n}, \psi_{a,b}^{(n)}).$$

(3.18)

The $C^*$-algebra $A_a^{\otimes \infty}$ is called the infinite tensor product of $A_a$ (3.11 §II.9.8). The map $\psi_a^{(n)}$ in (3.17) satisfies

$$(\psi_a^{(n)} \otimes \psi_{b}^{(n)}) \circ \varphi_{a,b}^{(n)} = \varphi_{a,b}^{(n+1)} \circ \psi_{a,b}^{(n)} \quad (a, b \in M, \ n \geq 1).$$

(3.19)

From $\{\varphi_{a,b}^{(n)} : n \geq 1\}$, we can define the $*$-homomorphism $\varphi_{a,b}^{(\infty)}$ from $A_{ab}^{\otimes \infty}$ to $A_a^{\otimes \infty} \otimes A_b^{\otimes \infty}$ such that

$$(\varphi_{a,b}^{(\infty)})|_{A_{ab}^{\otimes n}} = \varphi_{a,b}^{(n)}$$

(3.20)

for each $n$ where we identify $A_a^{\otimes \infty} \otimes A_b^{\otimes \infty}$ with the inductive limit of the system $\{(A_a^{\otimes n} \otimes A_b^{\otimes n}, \psi_{a,b}^{(n)}) : n \geq 1\}$. Then the following holds:

$$(\varphi_{a,b}^{(\infty)} \otimes id_c) \circ \varphi_{ab,c}^{(\infty)} = (id_a \otimes \varphi_{b,c}^{(\infty)}) \circ \varphi_{a,bc}^{(\infty)} \quad (a, b, c \in M)$$

(3.21)

where $id_x$ denotes the identity map on $A_x^{\otimes \infty}$ for $x = a, c$. From this, we see that $\{(A_a^{\otimes \infty}, \varphi_{a,b}^{(\infty)}) : a, b \in M\}$ is a $C^*$-WCS.

**Definition 3.7** The $C^*$-WCS $\{(A_a^{\otimes \infty}, \varphi_{a,b}^{(\infty)}) : a, b \in M\}$ is called the componentwise infinite tensor power of $\{(A_a, \varphi_{a,b}) : a, b \in M\}$.

By Theorem 3.2 the following direct sum

$$(A^{\otimes \infty})_* \equiv \bigoplus_{a \in M} A_a^{\otimes \infty}$$

(3.22)
is a C*-bialgebra. From here, we write \( A_*^{\otimes \infty} \) as \( (A_*^{\otimes \infty})_* \) for simplicity of description.

Let \( \psi(n) \equiv \bigoplus_{a \in M} \psi_a^{(n)} \) and \( A_*^{\otimes n} \equiv \bigoplus_{a \in M} A_a^{\otimes n} \). Then \( \{(A_*^{\otimes n}, \psi(n)) : n \geq 1\} \) is an inductive system of C*-bialgebras. We see that \( A_*^{\otimes \infty} \) in (3.22) is the inductive limit of the inductive system \( \{(A_*^{\otimes n}, \psi(n)) : n \geq 1\} \) of C*-algebras. Furthermore, we can verify that the C*-bialgebra associated with the C*-WCS \( \{(A_a^{\otimes \infty}, \varphi_{a,b}^{(\infty)}) : a, b \in M\} \) coincides with the inductive limit \( \lim_{\rightarrow}(A_*^{\otimes n}, \psi(n)) \) of C*-bialgebras \( \{(A_*^{\otimes n}, \psi(n)) : n \geq 1\} \). Hence we obtain the following equation of C*-bialgebras:

\[
A_*^{\otimes \infty} = \lim_{\rightarrow}(A_*^{\otimes n}, \psi(n)).
\] (3.23)

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Procedures are as follows: In § 4.1, we recall the example of locally triangular C*-weakly coassociative system (=C*-WCS) in [15]. From this, an inductive system of triangular (especially, quasi-cocommutative) C*-bialgebras is constructed by using the method in § 3.3. In § 4.2, it is proved that its inductive limit is not quasi-cocommutative.

4.1 An example of locally triangular C*-weakly coassociative system

Let \( N \equiv \{1, 2, 3, \ldots\} \). We regard \( N \) as an abelian monoid with respect to the multiplication. For \( n \in N \), let \( M_n \) denote the (finite-dimensional) C*-algebra of all \( n \times n \)-complex matrices where we define \( M_1 = \mathbb{C} \). We recall a locally triangular C*-WCS \( \{(M_n, \varphi_{n,m}, R^{(n,m)}) : n, m \in N\} \) in [15] as follows.

Let \( \{E_{i,j}^{(n)}\}_{i,j=1}^n \) denote the set of standard matrix units of \( M_n \). For \( n, m \in N \), define the \( \ast \)-isomorphism \( \varphi_{n,m} \) from \( M_{nm} \) onto \( M_n \otimes M_m \) by

\[
\varphi_{n,m}(E_{m(i-1)+j,m(i'-1)+j'}) = E_{i,i'}^{(n)} \otimes E_{j,j'}^{(m)}
\] (4.1)

for \( i, i' \in \{1, \ldots, n\} \) and \( j, j' \in \{1, \ldots, m\} \). For \( n \in N \), let \( \{e_i^{(n)}\}_{i=1}^n \) denote the standard basis of the finite dimensional Hilbert space \( \mathbb{C}^n \). Define the unitary transformation \( R^{(n,m)} \) on \( \mathbb{C}^n \otimes \mathbb{C}^m \) by

\[
R^{(n,m)}(e_i^{(n)} \otimes e_j^{(m)}) = e_i^{(n)} \otimes e_{j'}^{(m)}
\] (4.2)
for \((i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}\) where the pair \(i, j\) is uniquely defined as the following integer equation:

\[
m(i - 1) + j = n(j - 1) + i.
\]

By the natural identification \(\text{End}_C(C^n \otimes C^m) \cong M_n \otimes M_m\), \(R^{(n,m)}\) is regarded as a unitary element in \(M_n \otimes M_m\) for each \(n, m \in \mathbb{N}\). Then \(\{(M_n, \varphi_{n,m}, R^{(n,m)}): n, m \in \mathbb{N}\}\) is a locally triangular WCS \((15, \S \ 3)\).

For \(i \geq 1\), let \(M_n^{\otimes i}\) denote the \(i\)-times tensor power of \(M_n\). From Lemma \(3.6(iii)\), we obtain the locally triangular \(C\)-WCS \(\{(M_n^{\otimes i}, \varphi^{(i)}_{n,m}, R^{(n,m)i)}): n, m \in \mathbb{N}\}\) associated with \(\{(M_n, \varphi_{n,m}, R^{(n,m)}): n, m \in \mathbb{N}\}\). By Lemma \(3.4(ii)\),

\[
M_n^{\otimes i}(\ast) \equiv \bigoplus_{n \in \mathbb{N}} M_n^{\otimes i}
\]

is the triangular \(C\)-bialgebra associated with \(\{(M_n^{\otimes i}, \varphi^{(i)}_{n,m}, R^{(n,m)i)}): n, m \in \mathbb{N}\}\). Especially, \(M_n^{\otimes i}(\ast)\) is quasi-cocommutative for each \(1 \leq i < \infty\).

Define the \(C\)-algebra \(M_n^{\otimes \infty}(\ast)\) by

\[
M_n^{\otimes \infty}(\ast) \equiv \bigoplus_{n \in \mathbb{N}} M_n^{\otimes \infty}
\]

where \(M_n^{\otimes \infty}\) denotes the infinite tensor product of the \(C\)-algebra \(M_n\). By definition, \(M_n^{\otimes \infty}\) is a uniformly hyperfinite algebra of Glimm’s type \(\{n^i\}_{i \geq 1}\) \((7, 13)\). From \(\S 3.3\), \(M^{\otimes \infty}(\ast)\) is the inductive limit of the inductive system \(\{(M_n^{\otimes i}(\ast), \psi^{(i)}_n): i \geq 1\}\) of \(C\)-bialgebras:

\[
M^{\otimes \infty}(\ast) = \lim_{\longrightarrow} (M_n^{\otimes i}(\ast), \psi^{(i)}_n).
\]

We illustrate relations among algebras as follows:
4.2 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1.

Lemma 4.1 Let \( \{ (M_n^{\otimes \infty}, \varphi_{n,m}) : n, m \in \mathbb{N} \} \) be the componentwise infinite tensor power of \( \{ (M_n, \varphi_{n,m}) : n, m \in \mathbb{N} \} \) in § 4.1.

(i) For two representations \( \pi_1 \) and \( \pi_2 \) of the C*-algebra \( M_2^{\otimes \infty} \), define the representation of \( M_2^{\otimes \infty} \) by

\[
\pi_i \ast \pi_j \equiv (\pi_i \otimes \pi_j) \circ \varphi_2^{(\infty)} (i, j = 1, 2).
\]

Then there exist two unital representations \( \pi_1 \) and \( \pi_2 \) of \( M_2^{\otimes \infty} \) such that \( \pi_1 \ast \pi_2 \) and \( \pi_2 \ast \pi_1 \) are not unitarily equivalent.

(ii) Let \( \Delta_\varphi \) denote the comultiplication of \( M^{\otimes \infty}(\ast) \) in (4.3) with respect to the C*-weakly coassociative system \( \{ (M_n^{\otimes \infty}, \varphi_{n,m}^{(\infty)}) : n, m \in \mathbb{N} \} \). Then the C*-bialgebra \( (M^{\otimes \infty}(\ast), \Delta_\varphi) \) is not quasi-cocommutative.

Proof. (i) For \( i = 1, 2 \), define the (pure) state \( \omega_i^{(0)} \) of \( M_2 \) by

\[
\omega_i^{(0)}(x) \equiv x_{ii} \quad (x = (x_{ij})_{i,j=1}^{2} \in M_2)
\]

where \( x_{ij} \)'s denote standard matrix units of the \( 2 \times 2 \) matrix \( x \). Let \( \omega_i \) denote the product state \( (\omega_i^{(0)})^{\otimes \infty} \) of \( M_2^{\otimes \infty} \) for \( i = 1, 2 \). Let \( \pi_i \) denote the Gel’fand-Naimark-Segal representation of \( M_2^{\otimes \infty} \) by \( \omega_i \) and let \( P[i] \) denote its unitary equivalence class. Then \( P_2[1] \ast P_2[2] \neq P_2[2] \ast P_2[1] \) from (2.6) and (3.2) in [13] where we remark that \( \ast \) is well-defined on unitary equivalence classes of representations. Hence the statement is proved.

(ii) Let \( p_n \) denote the projection from \( M^{\otimes \infty}(\ast) \) to \( M_n^{\otimes \infty} \) for \( n \in \mathbb{N} \). From this, any representation of \( M_n^{\otimes \infty}(\ast) \) lifts on \( M_2^{\otimes \infty}(\ast) \). Let \( \pi_1 \) and \( \pi_2 \) be unital representations of \( M_2^{\otimes \infty}(\ast) \) in (i). Then \( (\pi_1 \ast \pi_j) \circ p_4 \) is a nondegenerate representation of \( M_2^{\otimes \infty}(\ast) \) such that \( (\pi_1 \ast \pi_j) \circ p_4 = (\pi_1 \circ p_2 \otimes \pi_j \circ p_2) \circ \Delta_\varphi \). From this and (i), \( (\pi_1 \circ p_2 \otimes \pi_2 \circ p_2) \circ \Delta_\varphi \) and \( (\pi_2 \circ p_2 \otimes \pi_1 \circ p_2) \circ \Delta_\varphi \) are not unitarily equivalent. From this and Lemma 2.3 \( (M^{\otimes \infty}(\ast), \Delta_\varphi) \) is not quasi-cocommutative.

Proof of Theorem 1.1. By Remark 2.1 the category of (quasi-cocommutative) C*-bialgebras makes sense. From this and Lemma 4.1(ii) and (4.6), \( \{ M^{\otimes i}(\ast) \}_{i \geq 1} \) is an example of inductive system of quasi-cocommutative C*-bialgebras such that its inductive limit is not quasi-cocommutative. This example implies the statement.
From Remark 2.1, Lemma 4.1(ii) and (4.6), the following is automatically proved.

**Corollary 4.2**

(i) The category of quasi-triangular $C^*$-bialgebras is not closed with respect to the inductive limit.

(ii) The category of triangular $C^*$-bialgebras is not closed with respect to the inductive limit.

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