Continuous-Variable Bell Inequalities in Phase Space

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We propose a variation of Bell inequalities for continuous variables that employs the Wigner function and Weyl symbols of operators in phase space. We present examples of Bell inequality violation which beat Cirel’son’s bound.

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Since its discovery in 1964 [1] the Bell inequality (BI) has triggered an enormous interest in the differences between classical and quantum correlations. Bell inequalities are now commonly referred to as equations that relate correlation measurements which are fulfilled by any local hidden variable (LHV) theory, but are violated within the framework of quantum mechanics (QM). The original inequality was formulated for dichotomic variables [2]. Recently, a new approach for CV Bell inequalities has been developed using Weyl symbols of operators in phase space. Clauser, Horne, Shimony, and Holt (CHSH) [2] presented a BI that was more amenable for experimental tests and is nowadays widely used.

The original BI was inspired by the Einstein-Podolsky-Rosen paradox [3,4] for continuous variables (CV). BIs for CV systems were first developed using dichotomic variables that have eigenvalues ±1 [5–7]. In this work we propose a variation of Bell inequalities for CV observables and Weyl symbols of operators, which is based on the assumption of a generalization of the CHSH inequality for CV observables [11–14]. Here we derive a BI that is based on Weyl symbols for a large class of observables \(\hat{A}\) with dichotomic symbols. As there is no general relation between bounds of operators and bounds of their symbols, the resulting BI is conceptually different than other BIs.

In the context of CV observables, the assumption LHV1 is problematic. To see this, consider the case of a single 1D particle. If we choose \(\hat{B}\) to be the position \(\hat{q}\) or momentum \(\hat{p}\) of the particle, then the spectrum of \(\hat{B}\) is the set of real numbers. Let us instead choose \(\hat{B} = \hat{H} = \frac{\hbar}{2}(\hat{q}^2 + \hat{p}^2)\), which is the energy of a harmonic oscillator. If, in a LHV theory, \(q\) and \(p\) can take any real value, then the LHV spectrum of \(\hat{B}\) should include all positive real numbers. However, the quantum mechanical energy spectrum is discrete. Hence, if we tried to describe position and energy simultaneously, we could run into a contradiction about the spectra of the two observables.

The origin of this contradiction is of course that, in quantum mechanics, \(\hat{H}\) and \(\hat{q}\) do not commute and thus cannot be measured simultaneously. On the other hand, in LHV models the observables must be commutative for at least a very large class of models [15]. The derivation of our generalized BI is based on the assumption of commutativity.

Re-derivation of the CHSH inequality. — Our method to derive BIs is related to the proof of the CHSH inequality presented by Cirel’son [8], which employs that the variance of a (generally complex) observable \(B\) must be
positive, \( |\langle B \rangle|^2 \leq \langle |B|^2 \rangle \). The expectation values in this expression may either be evaluated within QM, then denoted by \( \langle \hat{B} \rangle \), or in the framework of LHV models where we use the notation \( \langle B \rangle_{\text{LHV}} \). Within each theory this inequality is always fulfilled. However, the maximum value of \( \langle |B|^2 \rangle_{\text{LHV}} \) in all LHV theories provides an upper bound on local realism: if the quantum mechanical expectation value does not fulfill the inequality

\[
|\langle \hat{B} \rangle|^2 \leq \max_{\text{LHV}} \langle |B|^2 \rangle_{\text{LHV}},
\]

then the predictions of QM are inconsistent with the assumptions behind LHV models.

To derive the CHSH inequality we choose the observable \( B = X_1 X_2 + X_1 Y_2 + Y_1 X_2 - Y_1 Y_2 \), with \( X_i, Y_i \) four dichotomous observables (so that \( X_i^2 = Y_i^2 = 1 \)) for two particles \( i = 1, 2 \). In quantum mechanics one finds \( \langle \hat{B}^2 \rangle = 4 + \langle \{X_1, X_2 \} \rangle \). CFRD then argue that in any LHV model the commutators must vanish, which leads to the CHSH inequality \( |\langle \hat{B} \rangle| \leq 2 \). QM violates this inequality for a suitable choice of states and observables. Because of the above-mentioned commutativity of many LHV models, this seems to be a suitable approach to deriving BIs.

**Bell inequalities in phase space.** — In Ref. [8], CFRD used the method of commuting observables to find BIs for a more general form of the Bell observable \( B \). Here we employ this approach to derive BIs for a system of \( n \) degrees of freedom (e.g., \( n \) one-dimensional particles) in phase space. We consider an operator \( \hat{B} \) that is a function of position and momentum operators \( \hat{q}_i, \hat{p}_i \) (\( i = 1, \cdots n \)) and will establish an upper bound for \( \langle \hat{B} \rangle \) in LHV models by ignoring the commutators between position and momentum. We represent \( \hat{B} \) by a sum of Weyl-ordered products,

\[
\hat{B} = \sum_{\mu_1, \cdots, \mu_{2n}} B_{\mu_1 \cdots \mu_{2n}} (\hat{q}_{\mu_1}^{\mu_2} \cdots \hat{q}_{\mu_{2n-1}}^{\mu_2} \hat{p}_{\mu_{2n}})_{\text{Weyl}},
\]

with complex coefficients \( B_{\mu_1 \cdots \mu_{2n}} \). Weyl ordering of a product of operators \( \hat{A}_1, \hat{A}_2, \cdots \) corresponds to the completely symmetric sum

\[
(\hat{A}_1 \hat{A}_2 \cdots \hat{A}_k)_{\text{Weyl}} := \frac{1}{k!} \sum \hat{A}_{\pi_1} \hat{A}_{\pi_2} \cdots \hat{A}_{\pi_k},
\]

where \( \sum_{\pi} \) stands for the sum over all permutations of the \( k \) operators. Any operator that possesses a Taylor expansion can be brought into the form (4). For instance, the operator \( \hat{B} = \hat{q} \hat{p} \) can also be written as \( \hat{B} = \frac{i}{\hbar}(\hat{q} \hat{p} - \hat{p} \hat{q}) + \frac{1}{2} \hat{p}^2 \), which corresponds to a single degree of freedom (\( n = 1 \)) and \( B_{1,1} = 1, B_{0,0} = i \frac{\hbar}{2} \) in Eq. (4).

In phase space, the left-hand side (l.h.s.) of Eq. (3) can be evaluated using relation (2). The Weyl symbol of a Weyl-ordered operator \( \hat{B} \) can be evaluated by replacing the operators \( \hat{q}_i, \hat{p}_i \) by the respective phase space variables, \( \text{Sm}\{\hat{B}\}(q_i, p_i) = B(q_i, p_i) \). For the operator (4), the symbol \( B(q_i, p_i) \) then has the explicit form

\[
B(q_i, p_i) = \sum_{\mu_1, \cdots, \mu_{2n}} B_{\mu_1 \cdots \mu_{2n}} q_{\mu_1}^{\mu_2} p_{\mu_2}^{\mu_3} \cdots p_{\mu_{2n-1}}^{\mu_2} p_{\mu_{2n}}. \tag{6}
\]

To find the right-hand side (r.h.s.) of Eq. (3) we evaluate \( \langle \hat{B} \hat{B}^\dagger \rangle \) in QM and eliminate the contributions of all commutators between position and momentum operators. In quantum phase space one has

\[
\langle \hat{B} \hat{B}^\dagger \rangle = \int \prod_{j=1}^{n} dp_j \, \|B(q_i, p_i) \| (B \ast B^*)(q_i, p_i), \tag{7}
\]

where the star product between two operator symbols captures the non-commutativity and non-locality of quantum observables and is given by [16–18]

\[
f \ast g(x) = \frac{1}{(\pi \hbar)^n} \int \int d^{2n}y \, d^{2n}z \, f(y) g(z) \exp \frac{2i}{\hbar} (y \cdot Jz + z \cdot Jx + x \cdot Jy). \tag{8}
\]

Here, \( x = (q, p) \in \mathbb{R}^{2n} \) and \( J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \), where \( I \) denotes the \( n \)-dimensional identity matrix.

We now employ the central assumption that the LHV upper bound can be found by eliminating all commutators between position and momentum operators. In the Weyl symbol representation this can easily be accomplished by taking the limit \( \hbar \to 0 \) in the star product \( B \ast B^* \) of Eq. (7). For smooth, \( h \)-independent \( f, g \) the star product has the expansion \( f \ast g(x) = f(x)g(x) + \frac{i}{\hbar} \{ f, g \}(x) + O(\hbar^2) \), where the \( h \)-linear term is a Poisson bracket [16–18]. Consequently, \( B \ast B^* \) in Eq. (7) is replaced by the algebraic product \( |B|^2 \) so that a tentative new BI is given by

\[
|\langle \hat{B} \rangle|^2 \leq \int \prod_{j=1}^{n} dp_j \, |B(q_i, p_i)|^2 W(q_i, p_i). \tag{9}
\]

The r.h.s. of Eq. (9) does not yet provide the correct upper bound for LHV theories because it depends on the Wigner function of QM. Intuitively one could fix this by replacing the quantum state by a suitable phase space distribution \( W_{\text{LHV}}(q_i, p_i) \) which is compatible with the assumptions of LHV models. However, general LHV models do not necessarily possess such a phase space distribution.

Instead, one has to consider suitable Bell operators for which the r.h.s. of Eq. (9) can be evaluated without referring to a specific state. A simple but relevant example is the case when \( |B(q_i, p_i)|^2 = |B_0|^2 \) is constant. This happens if \( B(q_i, p_i) \) is a phase factor or a sign function, for instance. Because of the normalization of the Wigner function the BI then becomes

\[
|\langle B(q_i, p_i) \rangle|^2 \leq |B_0|^2. \tag{10}
\]
This is the main result of our paper. We remark that the dependence on the state has been removed here because the integral of the Wigner function over the entire phase space equals to the probability to find any measurement result, which is unity both in QM and LHV theories. Consequently, $|B_0|^2$ can be interpreted as the upper bound for commutative LHV theories.

We emphasize that the BI (10) is generally different from CFRD or CHSH type Bell inequalities. First, the upper bound is independent of the quantum state, while for the CFRD inequality the state dependence needs to be addressed. This can be a rather subtle issue: some entangled quantum states do have a positive Wigner function [19] and positivity of the Wigner function is not sufficient to ensure consistency with LHV models [20]. In fact, Revzen et al. [13] have shown that a dichotomic CV BI can be violated with a non-negative Wigner function. Second, the upper bound is determined by the symbol of the operator $\hat{B}$ rather than its spectrum. This can be exploited to find new examples of BI violation.

**Example of Bell inequality violation.** — We consider a system of two harmonic oscillators, which are prepared in the Bell state $|\psi_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$. We set $\hbar = 1$ and measure lengths in units of the ground state width of the oscillators so that $\langle q|m\rangle = e^{-\frac{x^2}{2}}H_m(q)/(\sqrt{2^m m! \sqrt{\pi}})$, with $H_m(q)$ the Hermite polynomials. If we introduce a phase space variable $\mathbf{x} := (q, p)$ with $\mathbf{x}^2 = q^2 + p^2$, and employ center-of-mass variables $\delta \mathbf{x} := (x_1 - x_2)/\sqrt{2}$ and $\mathbf{x} := (x_1 + x_2)/\sqrt{2}$, the Wigner function of $|\psi_{\text{Bell}}\rangle$ is given by

$$W_{\text{Bell}}(x_c, \delta \mathbf{x}) = \pi^{-2} e^{-(x_c^2 + \delta \mathbf{x}^2)} (2\delta \mathbf{x}^2 - 1). \quad (11)$$

This corresponds to the product $W_{00}(x_c)W_{11}(\delta \mathbf{x})$ of the ground state in the center-of-mass coordinates and first excited state in the relative coordinates. To exploit the negativity of the Wigner function we consider a Bell operator that has the dichotomic Weyl symbol $B(x_1, x_2) = \text{sgn}(2\delta \mathbf{x}^2 - 1)$, which implies that the bound $|B_0|^2$ in BI (10) is unity. We remark that the assumption that the Weyl symbol is dichotomic is different from the assumption that the operator is dichotomic. There is no general relationship between the bounds for an operator and the bounds for its symbol.

The L.H.S. of BI (10) can be evaluated using Eq. (2), so that the BI turns into

$$|\langle \hat{B} \rangle|^2 = \left| \int d^2 \delta \mathbf{x} d^2 \mathbf{x}_c \, W(\delta \mathbf{x}, \mathbf{x}_c) \text{sgn} (2\delta \mathbf{x}^2 - 1) \right|^2 \leq 1. \quad (12)$$

For the Bell state (11) we find $|\langle \hat{B} \rangle| = \frac{1}{\sqrt{2}} - 1 \approx 1.426$, so that the generalized BI is violated in quantum mechanics. This violation is slightly larger than Cirel’son’s bound $\sqrt{2}$ for the ratio between the maximal quantum mechanical value of $|\langle \hat{B} \rangle|$ and the bound of the CHSH inequality for LHV theories [21].

**Generalization.** — The previous example of BI violation exhibits two special features: (i) in center-of-mass variables, the Wigner function $W$ takes a product form. The negativity of $W$ depends on just the relative phase space variable $\delta \mathbf{x}$. (ii) The Bell operator’s symbol $B(\delta \mathbf{x})$ exactly cancels the negativity of $W$, so that $\langle \hat{B} \rangle = \int d^2 \delta \mathbf{x} \, |W(\delta \mathbf{x})|$. These features can be readily generalized.

Consider a Wigner function $W(\mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^{2n}$, that is negative on a measurable subset $\mathcal{E}_- \subset \mathbb{R}^n$ and positive on its complement $\mathcal{E}_+$. We introduce characteristic symbols $\chi_{\pm}(\mathbf{x})$ that are unity for $\mathbf{x} \in \mathcal{E}_\pm$ and zero otherwise. The symbol of the Bell operator is then given by $B(\mathbf{x}) = \chi_+(\mathbf{x}) - \chi_-(\mathbf{x})$. The general relation between an operator on Hilbert space, $\hat{H} = L^2(R^n, \mathbb{C})$, and its Weyl symbol is given by

$$\hat{A} = \int d^2 x \, \hat{\Delta}(x) \text{Smb}[\hat{A}](x), \quad (13)$$

with the quantizer [18] defined via its Dirac kernel

$$\langle q'|\hat{\Delta}(x)|q''\rangle = (\pi\hbar)^{-n/2} \delta(q' + q'' - 2q) e^{\pm i\pi q' - q''}. \quad (14)$$

The two operators corresponding to the characteristic symbols are therefore given by $\hat{\chi}_\pm = \int_{\mathcal{E}_\pm} d^2 x \, \hat{\Delta}(x)$. Together, they form a partition of unity, $\hat{1} = \hat{\chi}_\mp + \hat{\chi}_\pm$, and commute. The non-local character of the star product (8) causes $\hat{\chi}_\pm \hat{\chi}_\mp \neq 0$ and as a result the Bell operator fulfills $\hat{B}^2 \neq \hat{1}$. This implies that $\hat{B}$ is not a dichotomic operator, even though its symbol is dichotomic, in the sense that $B(\mathbf{x})^2 = 1$.

As a concrete example, we consider the special case that $\mathcal{E}_-$ consists of a disk of (dimensionless) radius $R$ around the origin of a 2D phase space. Since $\chi_-(\mathbf{x}) \in L^2(R^2, \mathbb{C})$ the corresponding operator $\hat{\chi}_-$ is Hilbert-Schmidt and so has a discrete spectrum. Furthermore, because of its spherical symmetry $\hat{\chi}_-$ commutes with the Hamiltonian of the harmonic oscillator, so that the eigenstates of $\hat{\chi}_-$ are the harmonic energy eigenstates $|m\rangle$.

The eigenvalues $\lambda_m(R)$ of $\hat{\chi}_-$ are conveniently computed from the fact that they are the expectation values with respect to the state $|m\rangle$ [22]

$$\lambda_m(R) = \frac{1}{2\pi \hbar} \int d^2 x \, \chi_-(x) \text{Smb}[|m\rangle\langle m|](x). \quad (15)$$

The integral above is the evaluation of the trace $\text{Tr} \chi_- |m\rangle\langle m|$ via the corresponding Weyl symbol. Using $\text{Smb}[|m\rangle\langle m|](x) = 2(-1)^m L_m(2x^2) \exp(2x^2)$ where $L_m$ is the Laguerre function gives

$$\lambda_m(R) = \int_0^R (-1)^m L_m(r^2) e^{-r^2} dr^2. \quad (16)$$

By replacing the Laguerre polynomials with their generating function (Eq. (22.9.15) of Ref. [23]), we can derive
a generating function for the eigenvalues of $\hat{\chi}_-$,

$$G(t) = (t-1)^{-1}\left(e^{R^2\frac{t-1}{t+1}} - 1\right).$$  \hspace{1cm} (17)$$

The $m$th eigenvalue $\lambda_m$ is given by the $m$th coefficient of the Taylor expansion of $G(t)$ around $t = 0$.

For the Bell operator $\hat{1} - 2\hat{\chi}_-$ and the relative coordinate prepared in an eigenstate of $\hat{\chi}_-$, the BI then turns into the condition $|1 - 2\lambda_m|^2 \leq 1$. In Fig. 1 we display the eigenvalues $1 - 2\lambda_m$ as a function of the excitation number $m$ for different choices of $R$. The example of the Bell state above corresponds to the case $m = 1$ and $R = 1/\sqrt{2}$. For larger $R$ and higher excitation numbers, a similar degree of BI violation can be obtained.

To increase the degree of BI violation, one can choose $\mathcal{E}_-$ to agree with the area in phase space where the Wigner function $W_m$ of state $|m\rangle$ is negative, so that $\langle \hat{B} \rangle = \int d^2x |W_m(x)|$. The result for this expression for $m \leq 30$ is shown in Fig. 2. BI violations that are much larger than in the CHSH case are possible. However, because of the normalization and exponential decay of the Wigner function we conjecture that the maximal BI violation for arbitrary $m$ will be bounded.

In conclusion, we have proposed generalized BIs for which the bound is determined by the Weyl symbol of the Bell operator. Examples of states for which the BI is strongly violated have been presented for bi-partite systems. Extensions of this work for multi-partite or interacting systems may reveal further insight into the differences between classical and quantum mechanics.

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