We study the local indistinguishability problem of quantum states. By introducing an easily calculated quantity, non-commutativity, we present a criterion which is both necessary and sufficient for the local indistinguishability of a complete set of pure orthogonal product states. A constructive distinguishing procedure to obtain the concrete local measurements and classical communications is given. The non-commutativity of ensembles can also be used to characterize the quantumness for classical-quantum or quantum-classical correlated states.
orthogonal, the ensemble becomes classical since different states of classical information can be thought of merely as orthogonal quantum states. Non-complete overlap of the two states, \( x = \langle \psi | \phi \rangle, 0 < x < 1 \), induces quantumness for the ensemble. In fact the ensemble is most ‘quantum’ when \( x = 1/\sqrt{2} \), i.e. when the two states have an angle 45° between them. Taking the above observation into account, we introduce non-commutativity to characterize the quantumness for a quantum ensemble. Let \( A_1, A_2, \ldots, A_n \) be a set of operators. We define the total non-commutativity for this set,

\[
N(A_1, A_2, \ldots, A_n) = \sum_{i,j=1,i<j}^n \|[A_i, A_j]\|,
\]

where \( [A, B] = AB - BA, \|A\| \) is the trace norm of the operator \( A \). The non-commutativity is a natural measure of ‘quantumness’ for a quantum ensemble \( \varepsilon = \{p, |\psi_i\rangle\} \), where \( p_i \) are density operators with probability \( p_i \). Denote \( A_i = p_i \rho_i \), then \( N(A_1, A_2, \ldots, A_n) \) is the measure of quantumness for the ensemble \( \varepsilon \). For the problem of local distinguishability of states \( \rho_1, \rho_2, \ldots, \rho_n \), the prior probabilities \( p_i \) are irrelevant. One only needs to concern the quantity \( N(\varepsilon) = N(p_1, p_2, \ldots, p_n) \) to judge the local distinguishability of the set of states \( \rho_1, \rho_2, \ldots, \rho_n \). The prior probabilities \( p_i \) do make sense when one concerns the quantumness of a state given by the ensemble \( \varepsilon = \{p, \rho_i\} \) (see section “The quantumness of semi-classical states”).

The non-commutativity \( N \) has the following properties, which make it a well defined measure for quantumness of an ensemble:

1. \( N \) is non negative;
2. \( N \) is unitary invariant, \( N(\varepsilon) = N(U\varepsilon U^\dagger) \) where \( U \varepsilon U^\dagger = \{p, U|\psi_i\rangle U^\dagger\} \) with \( U \) being any unitary matrix;
3. For an ensemble only containing two pure states \( A_1 = |\psi\rangle \langle \psi| \) and \( A_2 = |\phi\rangle \langle \phi| \), without considering their prior probabilities, \( N(A_1, A_2) \) is zero only when the overlap \( x = \langle \phi | \psi \rangle = 0 \) or 1. \( N(A_1, A_2) \) gets to a maximum of 1 when \( x = 1/\sqrt{2} \) (see Methods for the proof), which coincides with the above analysis of quantumness for two pure states;
4. The sum of two sets’ non-commutativity is equal to or less than the non-commutativity of the sum of the two sets: \( N(|\psi\rangle \langle \psi|, |\phi\rangle \langle \phi|) \leq N(|\psi\rangle \langle \psi|) + N(|\phi\rangle \langle \phi|) \). One can easily verify the inequality from the definition of \( N \). The equality holds if \( |\psi\rangle \langle \phi| \) or \( |\phi\rangle \langle \psi| \) are either mutually orthogonal or identical.

A set of quantum states corresponds to a quantum ensemble. Nevertheless, one density operator may have many quantum ensemble decompositions. For instance, consider the density operator in a 3 \( \otimes \) 3 system, \( \rho = \frac{1}{9} = \frac{1}{9} \sum_{i=1}^9 |\psi_i\rangle \langle \psi_i| = \frac{1}{9} \sum_{s,t} |st\rangle \langle st| \),

where \( |\psi_1\rangle = |1\rangle |1\rangle, |\psi_2\rangle = |0\rangle |1\rangle, |\psi_3\rangle = |1\rangle |0\rangle, |\psi_4\rangle = |1\rangle |2\rangle, |\psi_5\rangle = |2\rangle |1\rangle, |\psi_6\rangle = |2\rangle |2\rangle, |\psi_7\rangle = |0\rangle |0\rangle, |\psi_8\rangle = |0\rangle |2\rangle, |\psi_9\rangle = |1\rangle |0\rangle \), \( |s\rangle \) and \( |t\rangle \) are the computational basis. Both \( \{\psi_i\} \) and \( \{st\} \) are pure-state decompositions of \( \rho \). However, the set \( \{st\} \) can be locally distinguished while \( \{\psi_i\} \) cannot. Hence that \( \rho \) having zero quantum correlation is not an indicator for the local distinguishability of the states in \( \rho \)’s pure-state decomposition. On the other hand, if we change the probability of \( |\psi_i\rangle \), \( \rho \) is no longer an identity and can have nonzero quantum correlation, but the nine states \( |\psi_i\rangle \) still remain locally indistinguishable. Usually, the local indistinguishability of a set of pure states has no simple relations with the properties of the related density operator.

To study the local indistinguishability, and its relations with the quantumness of a quantum ensemble in terms of non-commutativity, in the following we call an ensemble \( \varepsilon = \{p_i, |\psi_i\rangle\} \) classical if \( N(\varepsilon) = 0 \), and quantum if \( N(\varepsilon) > 0 \). If \( N(\varepsilon) = 0 \), from the properties of non-commutativity, \( \{|\psi_i\rangle\} \) must be either mutually orthogonal or identical and the states form a set of classical signals from infinitesimal point of view. If \( N(\varepsilon) > 0 \), among \( \{\psi_i\} \) there must be at least one pair of states that are neither orthogonal nor identical. Consider a set of bipartite POPS \( \{|\psi_i\rangle = |a_i\rangle \otimes |b_i\rangle\} \), where \( |a_i\rangle \) are all mutually orthogonal, \( |a_i\rangle \) and \( |b_i\rangle \) are associated to part A and B respectively. Obviously, this set forms a classical ensemble \( \varepsilon = \{p_i, |\psi_i\rangle\} \) since \( N(\varepsilon) = 0 \), where \( p_i \) is the probability with respect to \( |\psi_i\rangle \). Correspondingly one has ensembles \( \varepsilon^2 = \{p_i, |a_i\rangle\} \) and \( \varepsilon^2 = \{p_i, |b_i\rangle\} \) respectively. If \( N(\varepsilon') = N(\varepsilon^2) = 0 \), we call \( \varepsilon \) a classical-quantum ensemble. Analogously we can define quantum-classical and quantum-quantum ensembles.

Local distinguishability for a set quantum states. A set of states is said to be reliably distinguished locally or distinguished locally if all the states in the set can be distinguished by finite rounds of LOCC protocols. If there are at least two states in a set which can not be distinguished by finite rounds of LOCC protocols, we say that the set can not be distinguished locally. The arguably more operational “asymptotic local operations and classical communications discrimination problem” is not under our consideration.

Theorem 1 If a set of bipartite POPs, not necessarily complete, \( \{\psi_i\} = \{a_i\} \otimes \{b_i\} \) with \( \langle \psi_i | \psi_j \rangle = 0, \forall i \neq j \). Suppose these states form a classical-quantum or classical-classical ensemble \( \varepsilon = \{p_i, |a_i\rangle \otimes |b_i\rangle\} \) with non zero \( p \). Since \( N(|\{a_i\}|) = 0 \), \( \{a_i\} \) must be either mutually orthogonal or identical. To distinguish \( |\psi_i\rangle \) locally, we can first take projective measurement on \( A \) side to distinguish those orthogonal states in \( \{a_i\} \). For those identical states \( |a_i\rangle \), for example, \( a_1 = a_2 \), we can take a projective measurement on \( B \) to distinguish state \( |a_1\rangle \) from state \( |a_2\rangle \) since \( |b_1\rangle \) and \( |b_2\rangle \) must be mutually orthogonal to ensure that \( \langle \psi_1 | \psi_2 \rangle = 0 \). In this way we can distinguish the states in the set \( \varepsilon \) reliably. The analysis is similar if \( \varepsilon \) is a quantum-classical ensemble. Therefore if a set of bipartite POPs forms a classical-classical, quantum-classical or classical-quantum ensemble, the set can be locally distinguished. And if the set of POPs states cannot be locally distinguished, the ensemble composed of these states must be a quantum-quantum one.

Since it is impossible to form a quantum-quantum ensemble for a set of POPs in a \( 2 \otimes 2 \) system, Theorem 1 is both necessary and sufficient for \( 2 \otimes 2 \) systems. For higher dimensional cases, there are quantum-quantum ensembles whose states can be locally distinguished. In fact, for \( 2 \otimes n \) systems, any set of POPs is locally distinguishable. Theorem 1 reveals a relation between local distinguishability and quantumness of a set of states. In the following, we give a necessary and sufficient criterion for local distinguishability for all POPs from the view of quantumness of ensembles.

We say that two sets \( \{a_i\} \) and \( \{a'_j\} \) are orthogonal if and only if \( \langle a_i | a'_j \rangle = 0, \forall i, j \). If a set of states cannot be divided into subsets such that those subsets are mutually orthogonal, then we call the set a single set. Consider a set of states \( \varepsilon = \{a_1, a_2, \ldots, a_n\} \) with \( a_1 = \{a_1\} \), \( a_2 = \{a_2\} \) and \( a_3 = \{a_3\} \) which are orthogonal and each subset cannot be split further. We call this partition the direct sum decomposition and denote it as \( \varepsilon = a_1 \oplus a_2 \oplus a_3 \). A single set means that the set cannot be decomposed into such direct sums. If all the subsets are single sets in a direct sum decomposition, we say that the decomposition is a single set decomposition. It can verified that the single set decomposition of a non-single set is unique. For a single set, adding some states in the vector space spanned by itself will keep the set a single one. Nevertheless, taking away some states from a single set, the set could...
become a non-single one. In the following, by a set of states’ decomposition we mean the single set decomposition.

Lemma For a set of states ε = |{a_i}⟩, m = dim(span|a_i⟩), the following statements are equivalent: (a) ε is a single set. (b) There are m linear independent states {⟨a_i⟩}^m_i=1 in ε satisfying the following relations:

\[ 0 < N(|a_i⟩, |a_j⟩) < N(|a_i⟩, |a_j⟩, |a_k⟩) \]

\[ < \ldots < N(|a_i⟩, |a_j⟩, \ldots, |a_n⟩) \]  \hspace{1cm} (2)

(c) A nondestructive projective measurement, a measurement which keeps the quantum state unchanged, can do nothing to distinguish the states in ε.

See Methods for the proof of the Lemma. From the Lemma we have

Theorem 2 For a complete set of m ⊗ n POPS, ε = |{ψ_i}⟩ = |{a_i}⟩ ⊗ |{b_j}⟩ with |ψ_i⟩|ψ_j⟩ = 0, ∀ i ≠ j, the set ε cannot be completely locally distinguished if and only if there exist subsets {⟨ψ_i⟩} in ε, such that {⟨a_i⟩} and {⟨b_j⟩} are all single sets, i.e., there exist m’ = dim(span{⟨a_i⟩}) linear independent {⟨a_i⟩}^m’_i=1 in {⟨a_i⟩} and n’ = dim(span{⟨b_j⟩}) linear independent {⟨b_j⟩}^n’_j=1 in {⟨b_j⟩} satisfying

\[ 0 < N(|a_i⟩, |a_j⟩) < N(|a_i⟩, |a_j⟩, |a_k⟩) \]

\[ < \ldots < N(|a_i⟩, |a_j⟩, \ldots, |a_n⟩) \]  \hspace{1cm} (3)

Proof. Recall that a complete set of POPS can be locally distinguished if and only if the states can be distinguished by local nondestructive projective measurement and classical communication18. If a set of states in ε cannot be completely locally distinguished by local nondestructive projective measurement and classical communication, there must exist a subset {|ψ_i⟩ = |a_i⟩ ⊗ |b_j⟩} of ε such that {⟨a_i⟩} and {⟨b_j⟩} are all single sets (Lemma 1). Obviously, the converse is also right. From Lemma 1 we have that the formula (3) holds if and only if {⟨a_i⟩} and {⟨b_j⟩} are single sets.

From the view of accessible information, the more quantum an ensemble is, the less information one can get from the ensemble19. Hence the quantumness of the two parts of a complete set of POPS must be “large” enough, at least larger than N(|{a_i}⟩) and N(|{b_j}⟩), so that one can only get limited information and the states in the set cannot be locally distinguished. Therefore the local quantumness of an ensemble determines the local indistinguishability.

Theorem 2 gives a constructive distinguishing procedure to check the local distinguishability for a complete set of POPS. Let |{ψ_i}⟩ = |{a_i}⟩ ⊗ |{b_j}⟩ be a complete set of POPS, corresponding to two sets ε^a = |{a_i}⟩ and ε^b = |{b_j}⟩, to verify the inequality (3), one needs to find all the subsets involved. This can be done in the following way.

(i) Decompose the sets ε^a and ε^b into subsets, ε^a = ε^a^1 ⊕ ε^a^2 ⊕ \ldots, ε^b = ε^b^1 ⊕ ε^b^2 ⊕ \ldots with the states in each subset constituting a single set. This process employs the corresponding local measurement M^a = P^a_1 ⊕ P^a_2 ⊕ \ldots and M^b = P^b_1 ⊕ P^b_2 ⊕ \ldots, where P^a_1, P^a_2, P^b_1, P^b_2, \ldots are the projections to the spaces spanned by e^a_1, e^a_2, e^b_1, e^b_2, \ldots, respectively.

(ii) Find the overlapping states between these subsets, |{ψ_i}⟩_{ij} = |e^a_i⟩|e^b_j⟩, where \tildes{\cap} means to find the states with the same subscripts by classical communication. For example, if e^a_i = |{a_i}⟩, |{a_j}⟩, e^b_j = |{b_j}⟩, then e^a_i \tildes{\cap} e^b_j = |{ψ_i}⟩_{ij}.

Each |{ψ_i}⟩ = |{a_i}⟩ ⊗ |{b_j}⟩ corresponds to two new subsets ε^a_i = |{a_i}⟩ and ε^b_j = |{b_j}⟩.

We repeat the above process n rounds for the new subsets until each of those new subsets has only one element or both A and B parts cannot be decomposed further. At last we have subset |{ψ_i}⟩ = |a_{i_1}⟩ ⊗ |b_{j_1}⟩ \ldots (here st, \ldots lj are the measurement outcomes) corresponding to two new sets ε^a_{i_1} = |{a_{i_1}}⟩ and ε^b_{j_l} = |{b_{j_l}}⟩, where ε^a_{i_1} and ε^b_{j_l} are single sets. If all the final single subsets have one element, the set |{ψ_i}⟩ can be locally distinguished.

As an example, there is a complete set of POPS for 3 ⊗ 4 system,

\[ |ψ_1⟩ = |1⟩, |ψ_2⟩ = |0⟩, |ψ_3⟩ = 1/√2, \]

\[ |ψ_4⟩ = |0⟩, |ψ_5⟩ = |1⟩, |ψ_6⟩ = 1/√2, \]

which corresponds to two sets ε^a = |{a_i}⟩, |{a_{i_1}}⟩, |{a_{i_2}}⟩, |{a_{i_3}}⟩, |{a_{i_4}}⟩, |{b_j}⟩, |{b_{j_1}}⟩, |{b_{j_2}}⟩, |{b_{j_3}}⟩, |{b_{j_4}}⟩, |{b_{j_5}}⟩, |{b_{j_6}}⟩, |{b_{j_7}}⟩, |{b_{j_8}}⟩, |{b_{j_9}}⟩, |{b_{j_10}}⟩, |{b_{j_11}}⟩, |{b_{j_12}}⟩. |ψ_5⟩ = |1⟩, |ψ_6⟩ = 1/√2, \]

\[ \begin{align*}
|ψ_9⟩ &= |1⟩ + |2⟩/\sqrt{2}, \\
|ψ_{10}⟩ &= |0⟩ + |1⟩/\sqrt{2}, \\
|ψ_{11}⟩ &= |3⟩, \\
|ψ_{12}⟩ &= |2⟩/\sqrt{2}, \\
&\end{align*} \hspace{1cm} (4)

The quantumness of semi-classical states. The non-commutativity defined by (1) can be also used to describe the quantumness for bipartite semi-classical states. For a given semi-classical state, ρqc = \sum_i p_i |i⟩ ⟨i|, the corresponding ensemble ε = {p_i, ρ_i} is fixed (see Methods). Denote X_i = p_i ρ_i. We have the non-commutativity describing the quantumness for the state ρqc.
Figure 1 | The comparison between the quantum correlations and the ensemble quantumness for a $3 \otimes 3$ quantum-classical state $\rho_s = 1/3 |000\rangle \otimes |000\rangle + |111\rangle \otimes |111\rangle + |\phi_2\rangle \otimes |222\rangle$, where $|\phi_2\rangle = \cos \theta |0\rangle + \sin \theta |2\rangle$ and horizontal ordinate is the overlap $x = \langle 0 | \phi_2 \rangle = |\cos \theta|$. Quantum correlations, quantum discord (circle dotted line) and quantum deficit (square dotted line), and the ensemble quantumness, non-commutativity (solid line), for $\rho_s$ all get minimal at $x = 0$, 1 while get maximal at $x = 1/\sqrt{2}$.

Due to that one semi-classical state corresponds to one quantum ensemble, the non-commutativity has been shown to be able to characterize the ensemble quantumness for classical-quantum or quantum-classical systems.

**Methods**

**Proof of the property (2) of the non-commutativity $N$.** Consider two pure $n$ dimensional states $A = \langle \psi | \psi \rangle$ and $B = |\phi \rangle \langle \phi |$. Denote $|\psi \rangle = x e^\theta$ with $x \in [0, 1]$, $\theta \in [0, 2\pi)$. Then $[A, B] = xx' e^{\theta - \theta'} |\psi \rangle \langle \psi | - xx' e^{\theta - \theta'} |\phi \rangle \langle \phi |$. Under the base $|\phi \rangle = |\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_{n-1}\rangle$, one gets $|\psi \rangle |\phi \rangle = \sum_{i=0}^{n-1} |\psi_i \rangle |\phi_i \rangle$. $|A, B|$ can then be expressed as

$$
\begin{pmatrix}
0 & \psi_{0,1} & \cdots & \psi_{0,n-1} \\
-\psi_{1,0} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\psi_{n-1,0} & 0 & \cdots & 0
\end{pmatrix}
$$

where $\psi_{i,0} = \langle \psi_i | \psi \rangle$ and $\psi_{i,0}^*$ is the conjugate of $\psi_{i,0}$. The eigenvalues of the above matrix are $\{\sqrt{-Z_1}, \sqrt{-Z_2}, \ldots, 0\}$, where $Z = |A, B|^2 = 2 - x^2 - x^2$. One gets $|A, B| = 2 \sqrt{1 - x^2}$. Therefore for two pure states $|\psi \rangle$ and $|\phi \rangle$, $|\langle \psi \rangle \langle \psi | |\phi \rangle \rangle = 0$ if $|\langle \psi \rangle \langle \phi | |\psi \rangle \rangle = 0$ or 1, and 1 when $|\langle \psi | \phi \rangle | = 1/\sqrt{2}$.

**Proof of Lemma 1.** (a) $\Rightarrow$ (b) First choose any vector in $e$, say, $|a_0\rangle$. Then choose the second vector $|a_1\rangle$ in $e$ such that it is not equal to or orthogonal to $|a_0\rangle$, i.e., $N(|a_0\rangle, |a_1\rangle) > 0$. The existence of $|a_0\rangle$ is guaranteed by the fact that $e$ is a single set. Next choose the third vector $|a_1\rangle$ in $e$ such that it is independent of $\{|a_0\rangle, |a_1\rangle\}$, and also not orthogonal to $\{|a_0\rangle, |a_1\rangle\}$, i.e., $N(|a_0\rangle, |a_1\rangle) < N(|a_0|, |a_1\rangle)$. Or else, if $|a_0\rangle$ does not exist, then $e$ can be direct sum decomposed into two parts. Continuing in the above process, we can finally get $m$ linear independent states $\{|a_i\rangle\}_{i=0}^m$ in $e$ satisfying inequality (2).

(b) $\Rightarrow$ (c) A nondestructive projective measurement is described by an observable, $M$, an Hermitian operator such that all $|a_i\rangle$ in $e$ are the eigenvectors of $M$. If $N(|a_0\rangle, |a_1\rangle) > 0$, $|a_0\rangle$ and $|a_1\rangle$ are neither identical nor orthogonal. Hence the corresponding eigenvalues of $|a_0\rangle$ and $|a_1\rangle$ must be equal, since the eigenvectors corresponding to different eigenvalues are orthogonal. If $N(|a_0\rangle, |a_1\rangle) < N(|a_0|, |a_1\rangle)$, $|a_1\rangle$ is at least not orthogonal to one of $|a_0\rangle$ and $|a_1\rangle$. For instance, suppose $N(|a_0\rangle, |a_1\rangle) > 0$, then the eigenvalues corresponding to $|a_0\rangle$ and $|a_1\rangle$ are equal. Therefore we have that if (2) is satisfied, those $m$ linear independent $|a_i\rangle$’s have the same eigenvalues of $M$. Therefore $M$ becomes an identity (apart of real factor) and it cannot be used to distinguish the states in $e$.

(c) $\Rightarrow$ (a) Suppose the set is not a single one and has a decomposition $e = e_1 \oplus e_2$. Then one can use a corresponding nondestructive projective measurement $M = a b_1 \oplus b_2$, where $a \neq b$, $I_1, I_2$ are identity operators, to distinguish the states in $e_1$ from $e_2$.

Discussion

We have proposed non-commutativity as a quantumness measure for an ensemble. It has been shown that the local ensemble quantumness, quantum discord, quantum deficit, etc., which is a function of a density operator, accounts for the local indistinguishability for a complete set of POVS. It implies that the quantumness of local ensembles must satisfy certain conditions so that the states in the ensemble cannot be locally distinguished.

A constructive distinguishing procedure to obtain the concrete local measurements and classical communications has been presented to judge the local indistinguishability for a complete set of POVS. Our approach to judge the local indistinguishability can also be directly extended to distinguish a complete set of multipartite POVS or a non-complete set of POVS, when the local operations are restricted within nondestructive projective measurements.
Hence if $M$ cannot do any in distinguishing the states in $e$, $e$ must be a single set, that is, statement (a) holds.

**The unique expression of semi-classical states.** Let $\rho = \sum p_i \rho_i \otimes |x_i\rangle \langle x_i|$ be an arbitrary given semi-classical state, where $|x_i\rangle$ is an orthogonal base. Suppose there exists another orthogonal base $\{|\beta_i\rangle\}$ such that $\rho = \sigma$, where $\sigma = \sum q_i \sigma_i |\beta_i\rangle \langle \beta_i|$. Let $U = \sum_j |\beta_i\rangle \langle x_i|$ be the unitary operator such that $U |x_i\rangle = |\beta_i\rangle$, $U_0 = u_{ij} = \langle x_i | \beta_i \rangle$, $|\beta_i\rangle = \sum a_{ij} |x_i\rangle$. So $\sigma$ can be reexpressed as $\sum_i q_i \sigma_i = \sum_{ij} q_{ij} |x_i\rangle \langle \beta_j| |\beta_j\rangle \langle x_i|$. Let $q_{ij}$ be the entries of $q_{ij}$. Set $Q_i = g_{ij}|\beta_j\rangle = |\beta_j\rangle$. Then $\rho = \sigma$ implies that

$$0 = \sum_i q_{ij} a_{ij} = \sum_i q_{ij} |x_i\rangle \langle x_i|,$$

and

$$= \langle x_i | \sum_i q_{ij} |\beta_j\rangle \langle \beta_j| |x_i\rangle \langle x_i| = \langle x_i|Q_i|x_i\rangle,$$

for $s \neq t$, which means that both $|x_i\rangle$ and $|\beta_j\rangle$ are the eigenvectors of $Q_i$ and $q_{ij}$ are the corresponding eigenvalues. Note that the set $\{|\sigma_i\rangle\}$ of state $\sigma$ can be divided into degenerate part (in which all the states are identical) and non-degenerate part (in which all the states are mutually different). For non-degenerate part, taking over all $k$, one will find that the intersection of the eigenspaces belonging to the eigenvalues $q_{ij}$ must be one dimensional, namely, $|x_i\rangle = |\beta_j\rangle$ and then $p_{ij} = q_{ij}$. For degenerate part, without loss of generality, assume $q_{ij} = q_{ij'}$, then $\rho = \sigma$ means $p_{ij} = p_{ij'}$ and $2q_{ij} = a_{ij} = a_{ij'}$. Therefore $p_{ij} = p_{ij'} = q_{ij} = q_{ij'}$. Finally we get $p_{ij} = q_{ij}$. \( \Box \)

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**Author contributions**

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