Deforming SW curve

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Abstract

A system of Bethe-Ansatz type equations, which specify a unique array of Young tableau responsible for the leading contribution to the Nekrasov partition function in the $\epsilon_2 \to 0$ limit is derived. It is shown that the prepotential with generic $\epsilon_1$ is directly related to the (rescaled by $\epsilon_1$) number of total boxes of these Young tableau. Moreover, all the expectation values of the chiral fields $\langle \text{tr} \phi^J \rangle$ are simple symmetric functions of their column lengths. An entire function whose zeros are determined by the column lengths is introduced. It is shown that this function satisfies a functional equation, closely resembling Baxter’s equation in 2d integrable models. This functional relation directly leads to a nice generalization of the equation defining Seiberg-Witten curve.
Introduction

The idea of considering $\mathcal{N} = 2$ SYM theories in a specific presently commonly known as $\Omega$- background is proven to be extremely fruitful. The general $\Omega$- background is characterized by two parameters $\epsilon_1, \epsilon_2$ introduced in [1,2] to regularize the integrals over moduli space of instantons. In [3] Nekrasov has shown how the partition function in this background is related to the Seiberg-Witten prepotential. In the same paper he performed explicit calculation of the prepotential up to 5 instantons choosing $h = \epsilon_1 = -\epsilon_2$ and demonstrated that at vanishing $h$ one exactly recovers the results extracted from the Seiberg-Witten curve. In [4] a combinatorial formula which allows one to calculate the Nekrasov partition function for generic $\epsilon_1, \epsilon_2$ up to desired order was proposed. The partition function is represented as a sum over arrays of Young tableau with total number of boxes equal to the number of instantons.

The partition function with generic $\epsilon_1, \epsilon_2$ is essential also from the point of view of the recently established AGT duality [5] relating this partition function to the conformal blocks in 2d Conformal Field Theory. In a parallel very interesting development Nekrasov and Shatashvily [6] show that in the case when $\epsilon_2 = 0$ the prepotential is closely related to the quantum integrable many body systems. This case is also the main subject of consideration of the present paper. Note one more point which to my opinion makes the investigation of $\epsilon_2 = 0$ case even more interesting: namely, due to above mentioned AGT relation this should be related to the quasi-classical ($c \to \infty$) limit of conformal blocks and hence to the classical Liouville field theory.

The paper is organized as follows: In section 1 generalizing the idea of Nekrasov and Okounkov [7] to the case with finite $\epsilon_1$ a system of Bethe-Ansatz type equations determining the shape of a unique, most relevant array of Young tableau is derived.

In section 2 an entire function denoted by $Y(z)$ whose zeros are determined through the column length of the Young tableau is introduced. It is shown that the above mentioned Bethe-Ansatz type equations are equivalent to a certain functional relation for the function $Y$ very much resembling the Baxter equation for the 2 d integrable models.

In section 3 this functional relation is represented in a form which is a direct
generalization of the algebraic equation defining the Seiberg-Witten curve \[8\]. The relation of the generalized Seiberg-Witten "curve" to the prepotential and the expectation values of the chiral fields \(\langle \text{tr} \phi^J \rangle\) are investigated. It is shown that they are given by contour integrals which are close analogues of the period integrals of the non-deformed case. It is emphasised that the prepotential is given by the number of total boxes of the limiting array of Young tableau and that the expectation values of the chiral fields \(\langle \text{tr} \phi^J \rangle\) are simple symmetric functions of their column lengths.

The section 4 is devoted to the simplest case with gauge group \(U(1)\) without matter hypermultiplets. In this case the functional equation and the generalized SW curve equation are solved in analytically in terms of the (generalized) hypergeometric function \(0F1\).

In appendix the solution of Bethe-Ansatz type equations of section 1 in few orders of instanton expansion is presented.

1 Prepotential in the limit \(\epsilon_2 \to 0\)

In this section I derive saddle point equations which determine the \(\epsilon_2 = 0\) limit of the deformed prepotential

\[
W(a, m, \epsilon_1, q) = \lim_{\epsilon_2 \to 0} \log Z_{\text{inst}}(a, m, \epsilon_1, \epsilon_2, q),
\]  

(1)

where \(a\) collectively denotes all VEV’s of the gauge multiplet and \(m\) the masses of possible extra matter hypermultiplets. I will make use both equivalent representations of the Partition function as contour integral \([2]\) and as sum over Young tableau \([3,4]\). The integral representation for the \(k\) instanton contribution in the case of \(U(N)\) gauge group and \(f\ (f \leq 4)\) fundamental hyper-multiplets \([3]\) reads

\[
Z_k = \int \frac{dx_I}{2\pi i} \frac{1}{k!} k^\prime \prod_{I,J=1}^k (x_I - x_J)(x_I - x_J + \epsilon_1 + \epsilon_2).
\]  

(2)

where

\[
\chi_k(x_I) = \frac{1}{k!} \prod_{I,J=1}^k \frac{(x_I - x_J)(x_I - x_J + \epsilon_1 + \epsilon_2)}{(x_I - x_J + \epsilon_1)(x_I - x_J + \epsilon_2)} \times \prod_{I=1}^k \frac{\prod_{a=1}^f (x_I + m_a)}{\prod_{u=1}^N (x_I - a_u)(-x_I + a_u - \epsilon_1 - \epsilon_2)}
\]  

(3)
where the prime on the product symbol indicates that the diagonal \( i = j \) factors \((x_i - x_i)\) should be omitted. The instanton part of the partition function is

\[
Z_{\text{inst}} = 1 + \sum_{k=1}^{\infty} Z_k q^k
\]

(4)

Following the ideology of [7] it is natural to expect that in the limit \( \epsilon_2 \to 0 \) of our interest the main contribution to the \((1)\) will be dominated by certain pole in the integrand of \((2)\) with large \( k \sim 1/\epsilon_2 \). It is possible to show that the poles which contribute to the integral \((2)\) are in one to one correspondence (up to permutations of the variables \( x_i \)) with the arrays of \( N \) Young tableau \( Y_1, \ldots, Y_N \) with total number of boxes equal \( k \). These are the same Young tableau which appear in already mentioned alternative combinatorial representation constructed in [3,4]. It is convenient to arrange the variables \( x_I \) (in some fixed order) over the \( k \) boxes. The specific values assumed by the variables \( x_I \) at a pole are nothing but the eigenvalues of the instanton group \( U(k) \). The rule, how to assign values to \( x_I \) for given set of Young tableau is simple: assign to the \( N \) corner boxes the expectation values \( a_1, \ldots, a_N \), increase the value by \( \epsilon_1 \) (\( \epsilon_2 \)) each time when passing to the next box in horizontal (vertical) direction. Thus the entry of the box \( s = (i, j) \), \( s \in Y_u \) would be

\[
x_{u,i,j} = a_u + (i - 1)\epsilon_1 + (j - 1)\epsilon_2
\]

(5)

Let us estimate \( \log (\chi_k q^k) \) for a very large \( k \sim 1/\epsilon_2 \):

\[
\log(\chi_k q^k) \sim k \log q + \epsilon_2 \sum_{I,J=1}^{k} \left( \frac{1}{x_I - x_J + \epsilon_1} - \frac{1}{x_I - x_J} \right) - \sum_{I=1}^{k} \left( \sum_{u=1}^{N} \log ((x_I - a_u)(-x_I + a_u - \epsilon_1)) - \sum_{\ell=1}^{f} \log (x_I + m_\ell) \right)
\]

(6)

The next step is to evaluate this expression at the values \((5)\) replacing the discrete sums over those indices which are multiplied by small quantity \( \epsilon_2 \) \( (i.e. \text{the indices of type } j \text{ in eq. (5)}) \) by integrals. The assumption made here is very natural, the number of boxes in vertical \( (\epsilon_2) \) direction is very large, but this number multiplied by \( \epsilon_2 \) is expected to be finite and will be denoted as \( \lambda_{u,i} \). The calculation is elementary and leads to the conclusion that \( \log(\chi_k q^k) \sim \mathcal{H}/\epsilon_2 \), where (below the
indices \( u, v \in 1, \ldots, N; \ell \in 1, \ldots, f; i, j \in 1, 2, \ldots \):

\[
\mathcal{H}(x_{u,i} | \epsilon_1) = \sum_{u,i,v,j} \left[ -G(x_{u,i} - x_{v,j} + \epsilon_1) + G(x_{u,i} - x_{v,j}^0 + \epsilon_1) \\
+ G(x_{u,i}^0 - x_{v,j} + \epsilon_1) - G(x_{u,i}^0 - x_{v,j}^0 + \epsilon_1) \right] \\
+ \sum_{u,i,v} \left[ -G(x_{u,i} - a_v) + G(x_{u,i}^0 - a_v) - G(x_{u,i} - a_v + \epsilon_1) + G(x_{u,i}^0 - a_v + \epsilon_1) \right] \\
+ \sum_{u,i,\ell} \left[ G(x_{u,i} + m_\ell) - G(x_{u,i}^0 + m_\ell) \right] + \sum_{u,i} (x_{u,i} - (i-1)\epsilon_1 - a_u) \log q
\] (7)

where

\[
G(x) = x(\log |x| - 1),
\] (8)

and

\[
x_{u,i} = a_u + (i-1)\epsilon_1 + \lambda_{u,i}; \quad x_{u,i}^0 = a_u + (i-1)\epsilon_1
\] (9)

It is useful to regularize the expression (7) assuming that there is an integer \( L \) such that the (scaled) lengths of columns \( \lambda_{u,i} = 0 \) when \( i > L \). It is a particularly nice feature of the expression (7) that its value does not depend on the upper limit of the summation indices \( i, j \) provided this upper limit is chosen to be more or equal to \( L \). This allows one to restrict the sums up to the range \( L \) and pass to the limit of infinite \( L \) at the final stage. In fact we will see below that the column lengths, extremizing the ”action” (7) are of order \( \lambda_{u,i} \sim \mathcal{O}(q^i) \). Here is the extremality conditions for (7)

\[
-q \prod_{u,v}^{N,L} \frac{(x_{u,i} - x_{v,j} - \epsilon_1)(x_{u,i} - x_{v,j}^0 + \epsilon_1)}{(x_{u,i} - x_{v,j} + \epsilon_1)(x_{u,i} - x_{v,j}^0 - \epsilon_1)} \prod_{\ell=1}^f (x_{u,i} + m_\ell) \prod_{v=1}^N (x_{u,i} - a_v + \epsilon_1)(x_{u,i} - a_v) = 1
\] (10)

which, in view of [6] not very surprisingly, closely resembles the Bethe-Ansatz equations of integrable models.

### 2 The functional equation

To investigate the system of equations (10) in the limit of infinitely large \( L \) it is useful to introduce the function

\[
Y(z) = \prod_{u=1}^N \frac{e^{x_{u,i} \psi(x_{u,i})}}{e^{x_{u,i} \psi(x_{u,i}^0)}} \prod_{i=1}^{\infty} \left( 1 - \frac{z}{x_{u,i}} \right) e^{z/x_{u,i}}
\] (11)
where
\[
\psi(x) = \partial_z \log \Gamma(z) \quad (12)
\]
Under the assumption that the column lengths tend to zero (which is equivalent to \(x_{u,i} \to x_{u,i}^0\) at large \(i\)) the product \((11)\) is convergent for arbitrary complex number \(z\) and defines an entire function of \(z\) with zeros located at \(x_{u,i}\). In extreme case when all column lengths are zero the product \((11)\) results in the entire function
\[
Y_0(z) = \prod_{u=1}^{N} \frac{\Gamma(\frac{a_u}{\epsilon_1})}{\Gamma(\frac{a_u - z}{\epsilon_1})}, \quad (13)
\]
whose zeros are located at \(x_{u,i}^0\). In view of these definitions the large \(L\) limit of the eqs. \((10)\) can be represented as
\[
-\frac{q}{\epsilon_1^2} \frac{Y(x_{u,i} - \epsilon_1)}{Y(x_{u,i} + \epsilon_1)} \prod_{\ell=1}^{f} (x_{u,i} + m_\ell) = 1 \quad (14)
\]
Let’s introduce the notation
\[
Q_f(z) = \prod_{\ell=1}^{f} (z + m_\ell) \quad (15)
\]
and consider the function
\[
(-1/\epsilon_1)^N P_N(z) = \frac{Y(z + \epsilon_1) + \frac{q}{\epsilon_1^2} Q_f(z) Y(z - \epsilon_1)}{Y(z)}. \quad (16)
\]
It does not have singularities at finite part of the complex plane since the potential poles \(z = x_{u,i}\) are cancelled due to the equality \((14)\). The behaviour at large \(z\) is also easy to estimate. Indeed at large \(z\) the ratio
\[
\frac{Y(z + \epsilon_1)}{Y(z)} \sim \frac{Y_0(z + \epsilon_1)}{Y_0(z)} = (-z/\epsilon_1)^N + O(z^{N-1}). \quad (17)
\]
Thus the function \(P_N(z)\) is in fact an \(N\)-th order polynomial (provided \(f \leq 2N\)). Taking into account \((17)\) we see that
\[
P_N(z) = z^N + O(z^{N-1}) \quad (18)
\]
for \(f = 1, 2, \ldots, 2N - 1\) and
\[
P_N(z) = (1 + q) z^N + O(z^{N-1}) \quad (19)
\]
for the conformal case \( f = 2N \).

So we finally arrive at the following functional equation for \( Y \):

\[
Y(z + \epsilon_1) + \frac{q}{\epsilon_1^N} Y(z - \epsilon_1) \prod_{\ell=1}^{f} (z + m_\ell) = (-1/\epsilon_1)^N P_N(z) Y(z). \tag{20}
\]

which very much resembles the Baxter’s \( T - Q \) equation well known in the context of 2d integrable statistical systems.

## 3 Deformed Seiberg-Witten curve

It is useful to introduce the notation

\[
w(z) = \frac{q}{(-\epsilon_1)^N Y(z + \epsilon_1)} \tag{21}
\]

and rewrite the functional equation in the following suggestive form

\[
Q_f(z)w(z)w(z - \epsilon_1) - P_N(z)w(z) + q = 0. \tag{22}
\]

This equation supplemented with the large \( z \) asymptotic condition \( w(z) = 1/z^N + \mathcal{O}(1/z^{N+1}) \) (see (17)) generalizes the algebraic equation defining the SW curve to the case with finite \( \epsilon_1 \). The deformation is surprisingly simple. The only difference from the standard case is the shift of one of the arguments by \( \epsilon_1 \). Indeed putting \( \epsilon_1 = 0 \) in (21) and absorbing the polynomial \( Q_f(z) \) by means of redefinition \( \sqrt{Q_f(z)}w(z) \to w(z) \) one gets the standard curve equation. Of course, eq. (22) no longer defines a curve in a usual sense and its geometric interpretation needs to be clarified yet. The question how the information about prepotential and the expectation values of chiral operators are encoded in the deformed ”curve” will be subject of the next two subsections.

### The prepotential

The prepotential \( W(a, m, \epsilon_1) \) defined by (1) should be equal to the critical value of the ”action” (7). To evaluate this critical value it is more convenient first to calculate its derivative with respect to the instanton parameter \( q \). Using (7) and the criticality conditions (10) one easily gets

\[
q \partial_q W(a, m, \epsilon_1, q) = \sum_{u,i} (x_{u,i} - (i - 1)\epsilon_1 - a_u) \equiv \sum_{u,i} \lambda_{u,i} \tag{23}
\]
i.e. the \( q \partial_q W(a, m, q) \) is simply the sum of all (rescaled) column lengths of the "critical" Young tableau! It is instructive to express this quantity in terms of the functions \( Y(z) \), \( Y_0(z) \) introduced before:

\[
q \partial_q W(a, m, q) = \oint_C \frac{dz}{2\pi i} z \partial_z \log \frac{Y(z)}{Y_0(z)},
\]

where the integration contour \( C \) encloses all zeros of \( Y(z) \) and \( Y_0(z) \) i.e. all the points \( x_{u,i}, x^0_{u,i} \).

**Expectation values and the chiral ring**

The technique developed in the previous sections apply as well to the computation of the general chiral correlator \( \text{tr} \phi^J \) in the gauge theory. These correlators constitute the so called chiral ring. It is well known that in 4 d \( \mathcal{N} = 2 \) SYM the chiral correlators \( \langle \text{tr} \phi^J \rangle \) can be represented as \(^7\)[9] [10],

\[
\langle \text{tr} \phi^J \rangle = \langle \text{tr} \phi^J \rangle_{cl} + \frac{1}{Z_{\text{inst}}} \sum_k q^k \int \prod_{I=1}^k \frac{dx_I}{2\pi i} \chi_k(x_I) O_J(\{x_I\})
\]

where the classical part of the expectation value

\[
\langle \text{tr} \phi^J \rangle_{cl} = \sum_{u=1}^N a^J_u,
\]

\( Z_{\text{inst}} \) is the instanton partition function and

\[
O_J(x_I) = -\sum_{I=1}^k \left[ (x_I + \epsilon_1 + \epsilon_2)^J - (x_I + \epsilon_1)^J - (x_I + \epsilon_2)^J - x_I^J \right].
\]

In the small \( \epsilon_2 \) limit of our interest \( O_J(x_I) \) becomes

\[
\lim_{\epsilon_2 \to 0} \epsilon_2 O_J(x_I) = -\sum_{u,i} \left[ (x_{u,i} + \epsilon_1)^J - (x_{u,i}^0 + \epsilon_1)^J - x_{u,i}^J + x_{u,i}^0 \right].
\]

Similar to the case of prepotential the saddle point approximation amounts to keeping one "critical" term in (25). The factors \( 1/Z_{\text{inst}} \) and \( \chi \) cancel out and we get

\[
\langle \text{tr} \phi^J \rangle = \sum_{u=1}^N a^J_u - \sum_{u,i} \left[ (x_{u,i} + \epsilon_1)^J - (x_{u,i}^0 + \epsilon_1)^J - x_{u,i}^J + x_{u,i}^0 \right].
\]
Recall now the definitions of our functions $Y(z)$, $Y_0(z)$ to rewrite the above expression in following three equivalent ways:

$$\langle \text{tr} \phi^J \rangle = \sum_{u=1}^{N} a_u^J - \oint_C \frac{dz}{2\pi i} z^J \partial_z \left( \log \frac{Y(z - \epsilon_1)}{Y_0(z - \epsilon_1)} - \log \frac{Y(z)}{Y_0(z)} \right)$$  \hspace{1cm} (30)

$$= \sum_{u=1}^{N} a_u^J - \oint_C \frac{dz}{2\pi i} ((z + \epsilon_1)^J - z^J) \partial_z \log \frac{Y(z)}{Y_0(z)}$$  \hspace{1cm} (31)

$$= - \oint_C \frac{dz}{2\pi i} z^J \partial_z \log \frac{Y(z - \epsilon_1)}{Y(z)}$$  \hspace{1cm} (32)

Comparing (24) with the second representation (31) specified to $J = 2$ one gets the well known Matone relation [11] between the prepotential and $\langle \text{tr}\phi^2 \rangle$ which holds for generic $\epsilon_1$, $\epsilon_2$ as well [10]. The last representation (32) is also very interesting, it provides a physical interpretation for the function $w(z)$ entering in expression of the deformed SW curve (21)

$$\langle \text{tr} \phi^J \rangle = - \oint_C \frac{dz}{2\pi i} z^J \partial_z \log w(z - \epsilon_1)$$  \hspace{1cm} (33)

which besides the shift by $\epsilon_1$ coincides with the standard non-deformed expression. Thus $\partial_z \log w(z - \epsilon_1)$ is the analogue of the SW differential. It is worth noting that the "classical" expectation value $a_u$ also can be represented in a similar way

$$a_u = - \oint_{C_u} \frac{dz}{2\pi i} z^J \partial_z \log w(z - \epsilon_1),$$  \hspace{1cm} (34)

where the contour $C_u$ encloses only the points $x_{u,i}$, $x_{u,i}^0$ with $i = 1, 2, \ldots$ and fixed $u$. Evidently this is the analogue of the A-cycle integral of the Seiberg-Witten theory.

4 **Explicit solution for $U(1)$**

The simplest case with gauge group $U(1)$ without hyper-multiplets can be analysed in full details. The deformed SW curve (21) is now defined as

$$w(z)w(z - \epsilon_1) - (z - c)w(z) + q = 0,$$  \hspace{1cm} (35)

where $c$ is a constant to be identified later. It is convenient to cast this equation into the form

$$f(x)f(x + 1) - xf(x) - t = 0.$$  \hspace{1cm} (36)
The dictionary is

\[ w(z) = -\frac{1}{\epsilon_1} f \left( -\frac{z - c}{\epsilon_1} \right); \quad t = -\frac{q}{\epsilon_1^2} \quad (37) \]

It is easy to see that the following continued fraction is a solution of (36)

\[ -\frac{f(x)}{t} = \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \ldots}}} \quad (38) \]

Gauss has investigated this continued fraction almost two centuries ago. The answer is given by the ratio of (generalized) hyper-geometric functions

\[ -\frac{f(x)}{t} = \frac{1}{x} \frac{{}_0F_1(x + 1, t)}{{}_0F_1(x, t)} \quad (39) \]

where the function \( {}_0F_1 \) is defined by the power series

\[ {}_0F_1(x, t) = \sum_{k=0}^{\infty} \frac{t^k}{x(x + 1) \cdots (x + k - 1)k!} \quad (40) \]

Thus

\[ w(z) = \frac{q}{z - c} \frac{{}_0F_1 \left( \frac{\epsilon_1 + c - z}{\epsilon_1}, \frac{-q}{\epsilon_1} \right)}{{}_0F_1 \left( \frac{\epsilon_1 - z}{\epsilon_1}, \frac{-q}{\epsilon_1} \right)} \quad (41) \]

To fix the normalization constant \( c \) notice that at large \( x \) the function \( {}_0F_1(x, t) \sim \exp(t/x) \). Hence at large \( z \)

\[ w(z) \sim \frac{1 + \mathcal{O}(z^2)}{z - c} \quad (42) \]

In our case there is no singularity outside of the integration contour in (32) so it can be freely deformed to a circle of a very large radius. Then taking into account the eq. (42) one gets convinced that

\[ \langle \phi \rangle \equiv a = c + \epsilon_1. \quad (43) \]

So, the final answer is

\[ w(z) = \frac{q}{z - a + \epsilon_1} \frac{{}_0F_1 \left( \frac{a - z}{\epsilon_1}, \frac{-q}{\epsilon_1} \right)}{{}_0F_1 \left( \frac{a - z - \epsilon_1}{\epsilon_1}, \frac{-q}{\epsilon_1} \right)} \quad (44) \]
For the sake of completeness let me present here also a closed expression for the entire function $Y(z)$ entering in the definition of $w(z)$ \[21\] and satisfying the functional equation \[20\]:

$$Y(z) = \frac{\Gamma\left(\frac{a_1}{\epsilon_1}\right)}{\Gamma\left(\frac{a_1-\epsilon}{\epsilon_1}\right)} \frac{\text{$_0F_1$}\left(\frac{a_1-z}{\epsilon_1}, -\frac{q}{\epsilon_1^2}\right)}{\text{$_0F_1$}\left(\frac{a_1}{\epsilon_1}, -\frac{q}{\epsilon_1^2}\right)}$$ \[45\]

It is straightforward to expand $w(z)$ around $q = 0$. The result up to 4th order is:

$$w(z) = \frac{q}{-a+z+\epsilon_1} - \frac{q^2}{(a-z)(-a+z+\epsilon_1)^2} - \frac{2q^3}{(a-z)(a-z-\epsilon_1)^3(a-z+\epsilon_1)}$$

$$- \frac{(5a-5z+\epsilon_1)q^4}{(a-z)^4(a-z+\epsilon_1)(-a+z+\epsilon_1)^4(a-z+2\epsilon_1)} + O[q^5]$$

At $\epsilon_1 = 0$ this series coincides with the small $q$ expansion of

$$w_0(z) = \frac{z-a-\sqrt{(z-a)^2-4q}}{2}$$ \[46\]

as expected from \[35\]. We see that in the limit $\epsilon_1 \to 0$ the poles of \[44\] condense around $a$ giving rise to the brunch cut of \[46\]. This is a generic phenomenon, in the case of the gauge group $U(N)$ in small $\epsilon_1$ limit the familiar $N$ brunch cuts around the expectation values would emerge.

## 5 Conclusions

To summarise, a saddle point analysis of the instanton series for the Nekrasov partition function in the limit $\epsilon_2 \to 0$ is performed. The criticality condition can be consistently truncated to a finite system of Bethe-Ansatz type equations considering array of Young tableau with number of columns less or equal to $L$. The truncated system with fixed $L$ determines all quantities up to the instanton order $q^L$. In large $L$ limit this system of algebraic equations is equivalent to a functional equation for an entire function whose zeros carry information about the lengths of columns of the Young tableau. This functional equation resembles Baxter’s equation for 2d integrable systems which also emerges in the context of 2d integrable field theories \[12\]. After a simple transformation it becomes evident that this functional equation represents a direct generalization of the algebraic
equation defining the Seiberg-Witten curve. The analogue of the SW differential and its relation to the prepotential and chiral correlation functions is established. In particular it is shown that the derivative of the \((\epsilon_1\text{ deformed})\) prepotential with respect to the gauge coupling is simply the sum of all column lengths. Finally, the simplest \(U(1)\) case is solved analytically making use of the Gauss’ method of the continued fractions.

It would be interesting to find Thermodynamic Bethe-Ansatz (TBA) like equations \([13–15]\) corresponding to our functional relation thus establishing direct contact with the results of the paper \([6]\).

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**Appendix: Instanton expansion**

In this Appendix I demonstrate how easily one gets instanton expansion directly from the Bethe-Ansatz type equation \([10]\).

A careful analysis shows that the structure of equations is consistent with \(\lambda_{u,i} \sim \mathcal{O}(q^i)\). Having an \(i\)-th order solution with this property the equations \([10]\) with \(L = i + 1\) uniquely determine not only \(\lambda_{u,1}, \ldots, \lambda_{u,i}\) up to the next order \(i + 1\) but also in leading order the length of the next column \(\lambda_{u,i+1}\), which automatically turns out to be of order \(\mathcal{O}(q^{i+1})\). In other words the \(i\)-th columns do not contribute up to \((i - 1)\)-instanton order. Thus one can start with just \(L = 1\) and solve the equation step by step up to desired order. At each stage the problem boils down to a system of linear equations. \(L = 1\) case is simple. Here
is the result

\[ \lambda_{u,1} = \frac{-q \prod_{\ell=1}^{f} (a_1 + m_\ell)}{\prod_{v \neq u} (a_u - a_v)(a_u - a_v + \epsilon_1)} + O(q^2) \]  \hspace{1cm} (47)

Summing this expression over \( u \) one gets the correct 1-instanton prepotential (with arbitrary \( \epsilon_1 \) but \( \epsilon_2 = 0 \)). In view of eq. (29) other non-trivial checks can be performed against known results for \( \langle \text{tr} \phi^J \rangle \) (see e.g. [10]). I have performed higher instanton order computations with various specific choices of \( N \) and \( f \) always finding perfect agreement with known results. As an example below is given 2-instanton result for the case with \( U(2) \) gauge group without extra hypermultiplets:

\[ \lambda_{1,1} = \frac{-q}{2a\epsilon_1 (2a + \epsilon_1)} \left( \frac{-8a^5 + 4a^4\epsilon_1 + 22a^3\epsilon_1^2 + 3a^2\epsilon_1^3 + 3a\epsilon_1^4 + \epsilon_1^5}{8a^3\epsilon_1^3 (2a - \epsilon_1)^2 (a + \epsilon_1) (2a + \epsilon_1)^3} \right) q^2 + O(q^3) \]

\[ \lambda_{1,2} = \frac{-q^2}{8a\epsilon_1^3 (a + \epsilon_1) (2a + \epsilon_1)^2} + O(q^3) \]

\[ \lambda_{2,1} = \frac{-q}{2a\epsilon_1 (2a - \epsilon_1)} \left( \frac{-8a^5 + 4a^4\epsilon_1 - 22a^3\epsilon_1^2 + 3a^2\epsilon_1^3 - 3a\epsilon_1^4 + \epsilon_1^5}{8a^3 (a - \epsilon_1) (2a - \epsilon_1)^3\epsilon_1^4 (2a + \epsilon_1)^2} \right) q^2 + O(q^3) \]

\[ \lambda_{2,2} = -\frac{q^2}{8a (a - \epsilon_1) \epsilon_1^3 (-2a + \epsilon_1)^2} + O(q^3) \]  \hspace{1cm} (48)

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