LOWER BOUNDS FOR UNCENTERED MAXIMAL FUNCTIONS ON METRIC MEASURE SPACE

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ABSTRACT. We show that the uncentered Hardy-Littlewood maximal operators associated with the Radon measure $\mu$ on $\mathbb{R}^d$ have the uniform lower $L^p$-bounds (independent of $\mu$) that are strictly greater than 1, if $\mu$ satisfies a mild continuity assumption and $\mu(\mathbb{R}^d) = \infty$. We actually do that in the more general context of metric measure space $(X, d, \mu)$ satisfying the Besicovitch covering property. In addition, we also illustrate that the continuity condition can not be ignored by constructing counterexamples.

1. Introduction

Let $Mf$ be the uncentered Hardy-Littlewood maximal function of $f$, where $f$ is a $p$-th power Lebesgue integrable function for $p > 1$. Maximal function plays at least two roles. Firstly it provides an important example of a sub-linear operator used in real analysis and harmonic analysis. Secondly, $Mf$ is apparently comparable to the original function $f$ in the $L^p$ sense. Not only that, by Riesz’s sunrise lemma, Lerner [Le10] proved for the real line that

$$\|Mf\|_{L^p(\mathbb{R})} \geq \left( \frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R})}.$$ 

And Ivanisvili et al. found the lower bound of the high dimensional Euclidean space in [IJN17].

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One may suspect whether in general metric measure space \((X, d, \mu)\), for given \(1 < p < \infty\), there exists a constant \(\varepsilon_{p,d,\mu} > 0\) such that

\[
\|M_\mu f\|_{L^p(\mu)} \geq (1 + \varepsilon_{p,d,\mu})\|f\|_{L^p(\mu)}, \quad \text{for all } f \in L^p(\mu),
\]

where \(M_\mu\) (cf. (2.1) below for the definition) is the uncentered Hardy-Littlewood maximal operator associated with \(\mu\). Unfortunately, such a measure must not be finite. In fact, \(1\)\(_X\) is a fixed point of \(M_\mu\) in \(L^p(\mu)\) if \(\mu(X) < \infty\). This suggests that we only need to consider infinite measures.

One of our main results is the following.

**Theorem 1.1.** Let \((X, d, \mu)\) be a metric measure space satisfying the Besicovitch covering property with constant \(L\). Suppose that

\[(1.1) \quad \mu(\{x \in \text{supp}(\mu) : r \mapsto \mu(B(x, r)) \text{ is discontinuous}\}) = 0 \quad \text{and } \mu(X) = \infty.\]

Then

\[(1.2) \quad \|M_\mu f\|_p \geq \left(1 + \frac{1}{(p - 1)L}\right)\|f\|_p, \quad \text{for all } f \in L^p(\mu).\]

Since any normed space \((\mathbb{R}^d, \| \cdot \|)\) has the Besicovitch covering property with a constant which equals its strict Hadwiger number \(H^*(\mathbb{R}^d, \| \cdot \|)\), we derive

**Corollary 1.2.** Let \(\| \cdot \|\) be any norm on \(\mathbb{R}^d\). Let \(\mu\) be a Radon measure on \((\mathbb{R}^d, \| \cdot \|)\) such that \(\mu(\mathbb{R}^d) = \infty\). Suppose that

\[
\mu(\{x \in \text{supp}(\mu) : r \mapsto \mu(B(x, r)) \text{ is discontinuous}\}) = 0.
\]

Then

\[
\|M_\mu f\|_p \geq \left(1 + \frac{1}{(p - 1)H^*(\mathbb{R}^d, \| \cdot \|)}\right)\|f\|_p, \quad \text{for all } f \in L^p(\mu).
\]

Now let us make a few remarks about the condition of (1.1). By Lebesgue decomposition theorem, any Radon measure can be divided into three main parts: the absolutely continuous part; the singular continuous part; the discrete measure part. If its discrete measure part exists (that is, measure is not continuous or has atoms), then it must not satisfy (1.1). But if it contains only the absolutely continuous part, the condition is true.
Indeed, Theorem 1.1 does not hold in general for infinite Radon measures. We can construct a measure containing atoms to show that (1.2) is invalid.

On the other hand, if $d = 1, 2$, then (1.1) is satisfied for every Radon measure on the Euclidean space $(\mathbb{R}^d, \| \cdot \|_2)$ having no atoms. Hence for this metric measure space, we have $\|M_\mu f\|_p > \|f\|_p$ for all $f \in L^p$. More precisely, we prove

**Theorem 1.3.** If $\mu$ is a Radon continuous measure on $(\mathbb{R}^2, \| \cdot \|_2)$ such that $\mu(\mathbb{R}^2) = \infty$, then

$$\|M_\mu f\|_p \geq \left(1 + \frac{1}{5(p - 1)}\right)^{\frac{1}{p}} \|f\|_p, \text{ for all } f \in L^p(\mu).$$

It is worth noting that we formulate and prove Lerner’s theorem in a more general setting which remains true.

**Theorem 1.4.** If $\mu$ is a Radon continuous measure on $(\mathbb{R}, \| \cdot \|_2)$ such that $\mu(\mathbb{R}) = \infty$, then

$$\|M_\mu f\|_p \geq \left(\frac{p}{p - 1}\right)^{\frac{1}{p}} \|f\|_p, \text{ for all } f \in L^p(\mu).$$

Besides, the lower $L^p$-bounds of the maximal operator with respect to a more general measure were discussed and we extended the results of Theorem 1.4 to a class of measures containing only one or two atoms. The proof depends on our adopted approach from a variant of Theorem 1.1, which differs from the traditional one of Theorem 1.4. It also means that the condition of (1.1) is not necessary.

The paper is organized as follows. In Section 2 we give some definitions and basic properties of metric measure spaces $(X, d, \mu)$ and the corresponding maximal operators. In Section 3 we prove Theorem 1.1 and Corollary 1.2. In Section 4 we mainly restrict our attention to the case of Radon measures in Euclidean space; the counterexample and other theorems mentioned above are presented.

## 2. Preliminaries

Throughout the whole paper, we use the same notation as [Al21]. If not declared specifically, the symbol $B^{cl}(x, r) := \{y \in X : d(x, y) \leq r\}$ denotes closed balls, and
\( B^o(x, r) := \{ y \in X : d(x, y) < r \} \) to refer to open balls. Because most of the definitions in this paper do not depend on the selection of open and closed balls, we generally use \( B(x, r) \) to denote it, but it should be noted that all balls are taken to be of the same kind when we utilize \( B(x, r) \) once. When referring to \( B \), suppose that its radius and center have been determined.

**Definition 2.1.** A measure on a topological space is Borel if it is defined on the \( \sigma \)-algebra generated by all open sets. Sometimes we need to study problems on a \( \sigma \)-algebra larger than Borel \( \sigma \)-algebra, so we definite Borel semi-regular measure.

**Definition 2.2.** A measure space \((X, \mathcal{A}, \mu)\) on a topological space is Borel semi-regular measure if for every \( A \in \mathcal{A} \), there exists a Borel set \( B \) such that \( \mu((A \setminus B) \cup (B \setminus A)) = 0 \).

In general, the maximal operator problem studied in this paper has no essential difference between these two kinds of measures. Therefore, although our conclusion is valid for general Borel semi-regular measure, the proofs are always carried out under the assumption of Borel measure. We will explain this reason in the proof of Lemma 3.4.

**Definition 2.3.** \([\text{Bo07}, \text{Definition 7.2.1}]\) A Borel measure is \( \tau \)-additive, if for every collection \( \mathcal{B} = \{ U_\lambda : \lambda \in \Lambda \} \) of open sets,

\[
\mu \left( \bigcup_\lambda U_\lambda \right) = \sup_{\mathcal{F}} \mu \left( \bigcup_{i=1}^n U_{\lambda_i} \right)
\]

where the supremum is taken over all finite subcollections \( \mathcal{F} \subset \mathcal{B} \). A Borel measure on a metric space is locally finite if there exists an \( r > 0 \) such that \( \mu(B(x, r)) < \infty \) for every \( x \in X \).

Recall that the complement of the support of the Borel measures \( (\text{supp}(\mu))^c := \bigcup \{ B(x, r) : x \in X, \mu(B(x, r)) = 0 \} \) is open, and hence measurable. If \( \mu \) is \( \tau \)-additive, we immediately know from its definition that \( \mu \) is full support, i.e., \( \mu(X \setminus \text{supp}(\mu)) = 0 \).

We also note that if a metric space is separable, then it is second countable, and hence \( \tau \)-additivity holds for all Borel measures.
In particular, locally finite Borel regular measures in complete separable metric spaces are equivalent to Radon measures, see e.g. Schwartz [Sc73, Part I, §11.3]. Note that our definition of locally finite is different from that of Aldaz [Al21, Definition 2.1], and we need to identify carefully. Locally finiteness in the sense of Aldaz refers to that if each bounded set has finite measure.

**Definition 2.4.** We say that \((X, d, \mu)\) is a metric measure space if \(\mu\) is a Borel semi-regular measure and its restriction on Borel set is \(\tau\)-additive and locally finite. Such a measure is undoubtedly \(\sigma\)-finite. A function \(f \in L^{1}_{\text{loc}}(\mu)\) is defined to be local integral if \(\int_{B} |f| d\mu < \infty\) for each ball \(B\). For \(1 \leq p < \infty\), we define \(L^{p}_{\text{loc}}(\mu) := \{f : |f|^p \in L^{1}_{\text{loc}}(\mu)\}\).

Recall the uncentered Hardy-Littlewood maximal operator acting on a locally integrable function \(f\) by

\[
M_\mu f(x) := \sup_{x \in B, \mu(B) > 0} \frac{1}{\mu(B)} \int_{B} |f| d\mu, \quad x \in X.
\]

We will denote \(M_\mu f\) briefly by \(Mf\) when no confusion can arise. It may be checked that for any \(f \in L^{1}_{\text{loc}}(\mu)\), \(Mf\) is lower semicontinuous, hence it is Borel measurable. By approximation, it is insignificant in the definition whether one takes the balls \(B(x, r)\) to be open or closed. These were shown by Stempak and Tao (see [ST14, Lemma 3]).

Now we are in the position of the definition of Besicovitch covering property.

**Definition 2.5.** We say that \((X, d)\) satisfies the Besicovitch covering property (BCP) if there exists a constant \(L \in \mathbb{N}^{+}\) such that for every \(R > 0\), every set \(A \subset X\), and every cover \(\mathcal{B}\) of \(A\) given by

\[
\mathcal{B} = \{B(x, r) : x \in A, 0 < r < R\},
\]

then there is a countable subfamily \(\mathcal{F} \subset \mathcal{B}\) such that the balls in \(\mathcal{F}\) cover \(A\), and every point in \(X\) belongs to at most \(L\) balls in \(\mathcal{F}\), that is,

\[
1_{A} \leq \sum_{B(x,r) \in \mathcal{F}} 1_{B(x,r)} \leq L.
\]

Note that unlike [Al19], we require subfamily to be countable. Recall that the validity of BCP is sufficient to imply the validity of the differentiation theorem for every locally finite
3. THE PROOF OF THE MAIN RESULT

We shall always work on a metric measure space \((X, d, \mu)\). We denote by \(\langle f \rangle_A\) the integral average of \(f\) over a measurable set \(A\), namely \(\langle f \rangle_A = \frac{1}{\mu(A)} \int_A |f| d\mu\). If \(\mu(A) = 0\) then we set \(\langle f \rangle_A = 0\).

To prove the main result, we need to establish the following lemmas.

**Lemma 3.1.** Let \(f, f_n \in L^1_{\text{loc}}(\mu)\) be non-negative. If \(\lim \inf_{n \to \infty} \int_B f_n d\mu \geq \int_B f d\mu\) for all balls \(B\) with \(\mu(B) < \infty\), then \(\lim \inf_{n \to \infty} Mf_n(x) \geq Mf(x)\).

**Proof.** Fix a point \(x \in X\). If \(Mf(x) < \infty\), so for every real number \(\varepsilon > 0\), there exists a ball such that \(\mu(B_\varepsilon) > 0\) and \(\langle f \rangle_{B_\varepsilon} > Mf(x) - \frac{\varepsilon}{2}\). By the assumption that \(\lim \inf_{n \to \infty} \int_B f_n d\mu \geq \int_B f d\mu\) for all balls \(B\), so we have \(\lim \inf_{n \to \infty} \langle f_n \rangle_{B_\varepsilon} \geq \langle f \rangle_{B_\varepsilon}\), then for \(\varepsilon > 0\), there exists a natural number \(N_\varepsilon > 0\) such that \(\langle f_n \rangle_{B_\varepsilon} \geq \langle f \rangle_{B_\varepsilon} - \frac{\varepsilon}{2}\) for \(n \geq N_\varepsilon\), hence \(\langle f_n \rangle_{B_\varepsilon} > Mf(x) - \varepsilon\).

Applying the definition of \(Mf_n(x)\), for \(n > N_\varepsilon\), we get \(Mf_n(x) > Mf(x) - \varepsilon\), which prove the lemma in the case of \(Mf(x) < \infty\).

Now suppose \(Mf(x) = \infty\). Thus for every \(M > 0\), we can also find a ball \(B_M\) such that \(\mu(B_M) > 0\) and \(\langle f \rangle_{B_M} > Mf(x)\). The same way shows that there exists a \(N_M > 0\) such that \(\langle f_n \rangle_{B_M} > \frac{M}{2}\) for all \(n \geq N_M\). Hence \(\lim \inf_{n \to \infty} Mf_n(x) = \infty\). This completes the proof. \(\square\)

**Corollary 3.2.** Let \(f, f_n \in L^1_{\text{loc}}(\mu)\) be non-negative. If \(f_n\) is monotonically increasing and converges to \(f\) a.e., then \(\lim_{n \to \infty} \|f_n\|_p = \|f\|_p\) and \(\lim_{n \to \infty} \|Mf_n\|_p = \|Mf\|_p\).

**Proof.** Since strong convergence implies weak convergence, \(\lim_{n \to \infty} \int_B f_n d\mu = \int_B f d\mu\) for all balls \(B\) with \(\mu(B) < \infty\). By Lemma 3.1, we have \(Mf \leq \lim \inf_{n \to \infty} Mf_n\). Further, operator \(M\) is order preserving, then \(Mf_n\) is monotonically increasing and converges to \(Mf\). By monotone convergence theorem, the result follows. \(\square\)

The following approximation theorem is well known (see e.g. [EG92, Theorem 1.1.4], [Fe69, Theorem 2.2.2] or [Si83, Theorem 1.3]).
Lemma 3.3. Let $\mu$ be a Borel semi-regular measure on $(X, d)$, $E$ a $\mu$-measurable set, and $\varepsilon > 0$. If $\mu(E) < \infty$, then there is a bounded closed set $C \subset E$ such that $\mu(E \setminus C) \leq \varepsilon$.

Recall that a finitely simple Borel function has the form $\sum_{i=1}^{N} c_i 1_{E_i}$, where $N < \infty$, $c_i \in \mathbb{R}$, and $E_i$ are pairwise disjoint Borel sets with $\mu(E_i) < \infty$. The support of a measurable function $g$, denoted by $\text{supp}(g)$, is the closure of the set $\{x \in X : g(x) \neq 0\}$.

Lemma 3.4. Let $C$ be a constant. The following are equivalent:

(i) For all $f \in L^p(\mu)$, $\|Mf\|_p \geq C\|f\|_p$.

(ii) For all non-negative finitely simple Borel function $g$, we have $\|Mg\|_p \geq C\|g\|_p$.

(iii) For all non-negative bounded upper semi-continuous functions $g$ whose $\text{supp}(g)$ is bounded and $\mu(\text{supp}(g)) < \infty$, we have $\|Mg\|_p \geq C\|g\|_p$.

Proof. Without loss of generality, the functions that appear in this proof are all non-negative. By restricting the measure on its Borel $\sigma$-algebra to get a new $\nu$, for $f \in L^p(\mu)$, there is a Borel function $g$ such that $\mu$-a.e. $g = f$. Thus

$$M_\mu f = \sup_{x \in B; \mu(B) > 0} \frac{\int_B f d\mu}{\mu(B)} = \sup_{x \in B; \mu(B) > 0} \frac{\int_B g d\mu}{\nu(B)} = \sup_{x \in B; \mu(B) > 0} \frac{\int_B g d\nu}{\nu(B)} = M_\nu g$$

and $\|g\|_{p,\nu} = \|f\|_{p,\mu}$. This means that we only need to focus on Borel functions.

Now we are in the position that (ii) implies (i). Applying Corollary 3.2 it suffices to prove that for every Borel function $f \in L^p(\mu)$, there exists a finitely simple Borel function sequence $\{f_n\}_{n=1}^\infty$ which is monotonically increasing and converges to $f$. In fact, since $f \in L^p(\mu)$, $f$ is a.e. finite. Recall that $\mu$ is $\sigma$-finite and let the pairwise disjoint subsets $A_i$ such that $\bigcup_{i=1}^\infty A_i = X$ and $\mu(A_i) < \infty$. For $k \in \mathbb{N}$, set

$$E_{k,j} = \{x \in \bigcup_{i=0}^{k} A_i : \frac{j-1}{2^k} \leq f(x) < \frac{j}{2^k} \} \ (j = 1, 2, ..., k \cdot 2^k)$$

and

$$f_k(x) = \sum_{j=1}^{k \cdot 2^k} \frac{j-1}{2^k} 1_{E_{k,j}}(x).$$

Clearly, $f_k$ is what we need by simple function approximation theorem.

As we have shown above, it suffices to consider the simple Borel function $f = \sum_{n=1}^{N} c_n 1_{B_n}$, where $B_n$ is a Borel set with $\mu(B_n) < \infty$. By Lemma 3.3 for each $\varepsilon > 0$ and $1 \leq n \leq N$
there exists a bounded closed set $F_{n,ε}$ with $F_{n,ε} ⊂ B_n$ and $c_nμ(B_n \setminus F_{n,ε}) < \frac{ε}{2N}$. Set $ψ_k = ∑_{n=1}^{N} c_n 1_{F_{n,ε}}$. Then $ψ_k$ is upper continuous and supports on a bounded closed set with $μ(supp(ψ_k)) < ∞$, and $ψ_k ↑ f$ as desired. □

**Lemma 3.5.** Let $μ$ be an infinite Borel semi-regular measure on $(X, d)$. If the sequence $x_n ∈ X$ is bounded and $lim_{n→∞} r_n = ∞$, then $lim_{n→∞} μ(B(x_n, r_n)) = ∞$.

*Proof.* Suppose that $x_n ∈ B(x, R)$ for some $x ∈ X$. Applying the continuity from above of measure, we get $lim_{n→∞} μ(B(x, \frac{r_n}{2})) = μ(X) = ∞$. Since $lim_{n→∞} r_n = ∞$, there is a $N$ such that $B(x, \frac{r_n}{2}) ⊂ B(x_n, r_n)$ for every $n ≥ N$. Hence $lim_{n→∞} μ(B(x_n, r_n)) = ∞$ which proves lemma. □

Now we follow methods from [IJN17, Theorem 1.1] to prove Theorem 1.1.

*Proof of Theorem 1.1.* By Lemma 3.4, it remains to prove that (1.2) holds for all non-negative bounded functions $f$ whose $supp(f)$ is bounded and $μ(supp(f)) < ∞$. Suppose first that $f$ is bounded by $C$. Then $∫_X f dμ ≤ C μ(supp(f)) < ∞$, and hence $f ∈ L^1(μ)$.

As $(X, d)$ satisfies the Besicovitch covering property, the differentiation theorem holds, hence $lim_{r→0} f_{B(x,r)} = f(x)$ almost everywhere $x ∈ X$ for every $f ∈ L^1_{loc}(μ)$. If we set $F = \{ x ∈ X : lim_{r→0} f_{B(x,r)} = f(x) \}$, then $μ(X \setminus F) = 0$. Let $E = \{ x ∈ supp(μ) : r → μ(B(x, r)) is continuous \}$. Clearly, $μ(X \setminus E) = 0$ by assumption. Fix $t > 0$ and consider $K_t = \{ x ∈ E ∩ F : f(x) > t \}$. For fixed $x ∈ K_t$, applying the definition of $supp(μ)$ and $F$, now choose a ball $B$ centered at $x$ such that $⟨ f ⟩_B > t$ and $μ(B) > 0$. From the assumption that $μ(X) = ∞$, we obtain $lim_{n→∞} ⟨ f ⟩_{B(x_n,r_n)} = 0$ as $lim_{n→∞} r_n = ∞$. Fix $x ∈ K_t$ and set $p(s) = ⟨ f ⟩_{B(x,s)}$. Then $p(s)$ is a continuous function on $[0, ∞]$ that has the intermediate value property and so $\{ r : ⟨ f ⟩_{B(x,r)} = t \}$ is non-empty for every $x ∈ K_t$. Let $R_t = sup_{x,r} \{ r : x ∈ K_t, ⟨ f ⟩_{B(x,r)} = t \}$.

We show that $R_t < ∞$. To do it, we argue by contradiction. Suppose, if possible, that $R_t = ∞$. Then there exists a sequence $(x_n, r_n) ∈ K_t × (0, ∞]$ such that $lim_{n→∞} r_n = ∞$ and $⟨ f ⟩_{B(x_n,r_n)} = t$. As $supp(f)$ is bounded, Lemma 3.5 gives $lim_{n→∞} sup_{x_n,r_n} ⟨ f ⟩_{B(x_n,r_n)} \leq$
\[
\limsup_{n \to \infty} \frac{\|f\|_1}{\mu(B(x_n, r_n))} = 0. \text{ Thus } \lim_{n \to \infty} \langle f \rangle_{B(x_n, r_n)} = 0 \text{ which obviously contradicts } t > 0. \text{ Note that this number depends only on } t \text{ if } f \text{ is given. Thus for } x \in K_t, \text{ there exists an } r_x \leq R_t \text{ such that } \langle f \rangle_{B(x, r_x)} = t. \text{ Now for cover } C \text{ of } K_t \text{ given by }\]
\[
C = \{B(x, r) : x \in K_t, \langle f \rangle_{B(x, r)} = t\},
\]
applying Besicovitch covering property, we extract a countable subfamily \( B(x_{t,j}, r_{t,j}) \in C \) so that
\[
1_{K_t} \leq \psi(x, t) := \sum_j 1_{B(x_{t,j}, r_{t,j})} \leq L.
\]
We show that \( \psi(x, t) \) satisfies the following properties also:

(1) if \( t > Mf(x) \) then \( \psi(x, t) = 0 \);
(2) if \( f(x) > t \), then \( \psi(x, t) \geq 1 \) almost everywhere;
(3) for every \( t > 0 \), we have \( \int_X t\psi(x, t)d\mu(x) = \int_X \psi(x, t)f(x)d\mu(x) \).

For the first property, we prove it by contradiction. Suppose, if possible, that \( \psi(x, t) > 0 \), then there exists a \( B(x_{t,t}, r_{t,t}) \) containing \( x \). Thus, \( Mf \geq \langle f \rangle_{B(x_{t,t}, r_{t,t})} = t \), contradicting the assumption that \( t > Mf(x) \).

To obtain the second property, let \( x \in K_t \). The selection of \( \psi \) gives \( \psi(x, t) \geq 1_{K_t}(x) \geq 1 \), and the property follows by \( \mu(X \setminus (E \cap F)) = 0 \).

The third property follows immediately.

We now seek to prove the desired inequality. Since \( \mu \) is \( \sigma \)-finite, applying Fubini-Tonelli’s theorem, (3) implies the following equality:
\[
\int_X \int_0^\infty t^{p-1}\psi(x, t)dtd\mu(x) = \int_X \int_0^\infty t^{p-2}\psi(x, t)f(x)dtd\mu(x).
\]
By property (1), we can restrict the integration to \([0, Mf(x)]\), that is
\[
(3.1) \int_X \int_0^{Mf(x)} t^{p-1}\psi(x, t)dtd\mu(x) = \int_X \int_0^{Mf(x)} t^{p-2}\psi(x, t)f(x)dtd\mu(x).
\]
Now, since $\psi \leq L$, and $Mf \geq f$ a.e., it follows from the above equality (3.1) that

$$\frac{L}{p} (\|Mf\|_p^p - \|f\|_p^p) \geq \int_X \int_0^{Mf(x)} t^{p-1} \psi(x,t) dtd\mu(x) \geq \int_X \int_0^{f(x)} t^{p-1} \psi(x,t) dtd\mu(x).$$

Note that property (2) yields

$$\frac{L}{p} (\|Mf\|_p^p - \|f\|_p^p) \geq \int_X \int_0^{f(x)} t^{p-1} \left( \frac{f(x)}{t} - 1 \right) dtd\mu(x) = \frac{\|f\|_p^p}{p(p-1)}.$$  

This finishes the proof. □

**Remark 3.1.** More generally, the same is true for quasi-metric in place of metric if we assume that all balls are Borel semi-regular measurable. It may happen that a ball in quasi-metric space is not a Borel set. To avoid such pathological cases, the assumption must be made to ensure the definition of $M\mu$ is reasonable. In fact, Macías and Segovia showed in [MS79, Theorem 2, p. 259] that there exists an $\alpha \in (0,1)$ and a quasi-metric $d_*$ which is equivalent to original quasi-metric and original topology is metrizable by $(d_*)^\alpha$. Thus we can generalize Lemma 3.3 to quasi-metric space since the bounded closed sets of both are the same. Lemma 3.5 can also be established by quasi-triangle inequality. From the foregoing discussion, the same proof carries over into quasi-metric space.

A set of the form $L^+(f) := \{ x \in \text{supp}(\mu) : f(x) > t \}$ is called a *strict superlevel sets* of $f$. It is rather straightforward to see that the following theorem remains valid by modifying the above proof.

**Theorem 3.6.** Let $f \in L^p(\mu)$ be non-negative. If $\mu(X) = \infty$, and for any $t > 0$, there exists a finite or countable ball-coverings $F_t$ of $L^+(f)$ such that the average value of the function $f$ on each ball $B \in F_t$ is equal to $t$, and almost everywhere every point in $\text{supp}(\mu)$ belongs to at most $L$ balls in $F_t$, then

$$\|Mf\|_p \geq \left( 1 + \frac{1}{(p-1)L} \right)^{\frac{1}{p}} \|f\|_p.$$  

Especially, if the balls are pairwise disjoint, then $\|Mf\|_p \geq \left( \frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_p.$
Whether the theorem is true without Besicovitch covering property seems worthwhile to pursue. However, due to the bottleneck in the extraction of the ball subfamily, the approach of Theorem 1.1 can not help. To the surprise, if we preserve the continuity assumption of \( \mu(B(x_0, r)) \) and impose on \( f \) the additional conditions of being radially decreasing symmetric (with the point \( x_0 \)), we can avoid such difficulties.

**Theorem 3.7.** Let \( f \in L^p(\mu) \) be a radial decreasing function with the point \( x_0 \in X \). If \( \mu(X) = \infty \) and the function \( r \mapsto \mu(B(x_0, r)) \) is continuous on the interval \((0, +\infty)\), then

\[
\|Mf\|_p \geq \left(\frac{p}{p - 1}\right)^{\frac{1}{p}} \|f\|_p.
\]

**Proof.** Note first that the strict superlevel sets of a radial decreasing function are always balls centered on the point \( x_0 \). Then we can choose a larger ball \( B \) centered at \( x_0 \) to contain \( L^+_t(f) \) while making \( \langle f \rangle_B = t \) possible for any \( t \) by the assumption. Thus the theorem follows immediately applying Theorem 3.6. \( \square \)

Given a norm \( \| \cdot \| \) on \( \mathbb{R}^d \), the strict Hadwiger number \( H^*(\mathbb{R}^d, \| \cdot \|) \) is the maximum number of translates of the closed unit ball \( B^{cl}(0, 1) \) that can touch \( B^{cl}(0, 1) \) and such that any two translates are disjoint. See [A121, p. 10] in more detail.

**Proof of Corollary 1.2.** As special cases of Theorem 1.1 when \((X, d) = (\mathbb{R}^d, \| \cdot \|)\), we only need to prove that the Besicovitch constant of \((\mathbb{R}^d, \| \cdot \|)\) equals its strict Hadwiger number. But this fact has been shown in [A121, Theorem 3.2]. \( \square \)

4. Radon measure in Euclidean spaces

Since every finite measure has no such lower bounds greater than 1, a natural attempt to generalize Corollary 1.2 is to consider measures only satisfying \( \mu(X) = \infty \). The following example tells us that condition (1.1) can not be omitted.

**Example 4.1.** Let \( \| \cdot \| \) be any norm on \( \mathbb{R}^d \). For any \( p > 1 \) and \( \varepsilon > 0 \), there exists a discrete measure \( \mu \) on \((\mathbb{R}^d, \| \cdot \|)\) which satisfies the following conditions:

(i) \( \mu(\mathbb{R}^d) = \infty \);

(ii) \( \mu(\{x \in \text{supp}(\mu) : r \mapsto \mu(B(x, r)) \text{ is discontinuous}\}) = \infty \);
(iii) \( \inf_{\|g\|_p=1} \|Mg\|_p \leq 1 + \varepsilon. \)

**Proof.** Given \( x \in \mathbb{R} \), we use \( e_x \) to denote the point \((x, 0, \ldots, 0)\) on \( \mathbb{R}^d \). For \( t > 1 \) and \( i \in \mathbb{N} \), consider \( \mu = \frac{1}{t-1} \delta_{e_0} + \sum_{i=1}^{\infty} t^{-i} \delta_{e_i} \), where \( \delta_y \) is the Dirac measure concentrated at the point \( y \in \mathbb{R}^d \). Since (i) and (ii) follow immediately, we only need to verify (iii). Assume \( i \in \mathbb{N}^+ \).

Then, using the convexity of the ball, we have

\[
M \mu 1_{\{e_0\}}(e_i) = 1_{\{e_0\}} = \|1_{\{e_0\}}\|_{1, \mu} = \frac{1}{t-1} + \sum_{j=1}^{i} j^{-1} = t^{-i}.
\]

Hence, a simple calculation gives

\[
\|M \mu 1_{\{e_0\}}\|_{p, \mu} = \frac{1}{t-1} (M \mu 1_{\{e_0\}}(e_0))^p + \sum_{i=1}^{\infty} t^{-i} (M \mu 1_{\{e_0\}}(e_i))^p
\]

\[
= \frac{1}{t-1} + \sum_{i=1}^{\infty} t^{(1-p)i-1} = \frac{1}{t-1} + \frac{1}{tp - t}.
\]

As \( t \to \infty \), we get \( \frac{\|M \mu 1_{\{e_0\}}\|_{p, \mu}}{\|1_{\{e_0\}}\|_{p, \mu}} = 1 + \frac{1}{p-1} \to 1. \) Thus fixing \( p > 1 \) and \( \varepsilon > 0 \), \( \inf_{\|g\|_p=1} \|Mg\|_p \leq \frac{\|M \mu 1_{\{e_0\}}\|_{p, \mu}}{\|1_{\{e_0\}}\|_{p, \mu}} \leq 1 + \varepsilon \), when \( t \) is large enough. \( \square \)

Set \( c_p = \left( 1 + \frac{1}{(p-1)H^*(\mathbb{R}^d, \|\cdot\|)} \right)^{1/p} \), where \( \|\cdot\| \) is a norm on \( \mathbb{R}^d \). As we mentioned before, (1.1) is satisfied for every absolutely continuous measure with respect to the Lebesgue measure, it is reasonable to further ask whether \( M \mu f \) for every continuous measure must have bound \( c_p \). Surprisingly, for lower-dimensional Euclidean space, the answer is affirmative.

We need the following basic fact:

**Lemma 4.1.** For \( d = 1, 2 \), let \( \mu \) be a Radon measure on \( (\mathbb{R}^d, \|\cdot\|_2) \) that has no atoms, then \{ \( x \in \text{supp}(\mu) : r \mapsto \mu(B(x, r)) \) is discontinuous \} is countable. In particular, \( \mu(\{ x \in \text{supp}(\mu) : r \mapsto \mu(B(x, r)) \) is discontinuous \}) = 0.

**Proof.** We begin by observing that \( \lim_{r \to r_0} \mu(B^o(x, r)) = \mu(B^c(x, r_0)) \) and \( \lim_{r \to r_0} \mu(B^o(x, r)) = \mu(B^o(x, r_0)) \). Thus the continuity of this function at \( r_0 \) is equivalent to measure zero on the spherical boundary \( \{ y : d(y, x) = r_0 \} \). If in the case of one dimension, the result follows
immediately. Next we turn to the case of two dimension. To prove it, we claim that

\[ E = \{ S : \text{ where } S \text{ is an one-dimensional sphere s.t. } \mu(S) > 0 \} \]

is countable. We shall do this by contradiction. Suppose, if possible, that \( E \) is uncountable. Letting \( E = (S_\gamma)_{\gamma \in \Gamma} \) and \( x_\gamma \) be a center of \( S_\gamma \), we obtain \( E = \bigcup_{i \in \mathbb{Z}} \bigcup_{j \in \mathbb{N}} \bigcup_{s \in \mathbb{N}} E_{i,j,s} \), where

\[ E_{i,j,s} \triangleq \{ S_\gamma : 2^{-i-1} \leq \mu(S_\gamma) < 2^{-i}; j \leq \text{radii } S_\gamma < j+1; s \leq d(x_\gamma,0) < s+1 \}. \]

Then there exists at least uncountable spheres meeting requirements listed above for some \( i_0, j_0, s_0 \), that is, \( E_{i_0,j_0,s_0} \) is uncountable. The next observation is that the intersection of two different spheres is at most two points. By continuity of \( \mu \), any two different spheres are disjoint in the sense of measure. Hence we can choose a subfamily \( \{ S_{\gamma_k} \}_{k \in \mathbb{N}} \subset E_{i_0,j_0,s_0} \) so that \( \mu(B_{\mathbb{R}^2}(0,j_0+s_0+2)) \geq \mu(\bigcup_k S_{\gamma_k}) = \sum_{k \in \mathbb{N}} \mu(S_{\gamma_k}) \geq \infty \), which contradicts the assumption that \( \mu \) is a Radon measure. Thus \( E \) is countable. Since every sphere has only one center, \( \{ x \in \mathbb{R}^2 : \exists r \text{ s.t. } \mu(\{ y : d(y,x) = r \}) > 0 \} \) is also countable as desired. \( \square \)

**Remark 4.1.** Notice that the lemma cannot be extended to the high dimension. An example of a singular continuous measure shows the value of the right-hand item of (1.1) can take infinite. Our methods at present are not able to tackle the high dimension problem.

By corresponding results of Sullivan [Su94, Proposition 23], we know that strict Hadwiger numbers in one-dimensional and two-dimensional Euclidean space are 2 and 5. Thus the lower bounds are greater than \( \left( 1 + \frac{1}{2(p-1)} \right)^{1/p} \) and \( \left( 1 + \frac{1}{5(p-1)} \right)^{1/p} \) respectively, and this verifies Theorem 1.3. We ignore the details of this proof since it is the direct inference of Sullivan’s proposition, Corollary 1.2 and Lemma 4.1.

We would like to mention that in one dimension we can improve the lower bound estimation to \( \left( \frac{p}{p-1} \right)^{1/p} \). The core of our proof is exactly consistent with Lerner’s approach under one-dimensional Lebesgue measures. To this end, we introduce the following theorem proved by Ephremidze et al. [EFT07, Theorem 1].
Theorem 4.2. Let $\mu$ be a Borel semi-regular measure on $\mathbb{R}$ that contains no atoms. For $f \geq 0$, we consider the one sided maximal operator

$$M_+f(x) = \sup_{b > x} \frac{1}{\mu([x, b))} \int_{[x, b)} f(s) d\mu.$$ 

If $t > \liminf_{x \to -\infty} M_+ f(x)$, then

$$t\mu(\{x \in \mathbb{R} : M_+ f(x) > t\}) = \int_{\{x \in \mathbb{R} : M_+ f(x) > t\}} f(s) d\mu.$$ 

Proof of Theorem 1.4. If $\mu(\mathbb{R}) = \infty$, we have either $\mu((0, \infty)) = \infty$ or $\mu((-\infty, 0]) = \infty$. Without loss of generality, we assume $\mu((-\infty, 0]) = \infty$. Otherwise we consider $M_- f(x) = \sup_{a < x} \frac{1}{\mu((a, x])} \int_{(a, x]} f(s) d\mu$ instead of $M_+ f(x)$. As $\mu$ is Radon, $\mu((-\infty, -R)) = \mu((\infty, -R)) = \mu((\infty, 0]) - \mu([-R, 0]) = \infty$ for any $R > 0$. Assume that the function $f \in L^1(\mu)$ with $\text{supp}(f) \subseteq [-R, R]$. If $x < -R$, we have

$$M_+ f(x) = \sup_{b > -R} \frac{1}{\mu([x, b])} \int_{[x, b)} f(s) d\mu \leq \sup_{b > -R} \frac{\|f\|_1}{\mu([x, b])} \leq \frac{\|f\|_1}{\mu([x, -R])}.$$ 

From $\mu((-\infty, -R)) = \infty$, we obtain

$$\liminf_{x \to -\infty} M_+ f(x) = \liminf_{x \to -\infty} \frac{\|f\|_1}{\mu([x, -R])} = 0.$$ 

Applying Theorem 4.2, we conclude

(4.1) $$t\mu(\{x \in \mathbb{R} : M_+ f(x) > t\}) = \int_{\{x \in \mathbb{R} : M_+ f(x) > t\}} f(s) d\mu \text{ for all } t > 0.$$ 

Now multiplying both sides of (4.1) by $t^{p-2}$ and integrating with respect to $t$ over $(0, \infty)$, we obtain $\frac{\|M_+ f\|_p^p}{p} = \frac{1}{p-1} \int_{\mathbb{R}} f(M_+ f)^{p-1} d\mu$. Therefore, $\frac{\|Mf\|_p^p}{p} \geq \frac{1}{p-1} \int_{\mathbb{R}} f^p d\mu$ and the desired inequality follows. 

For one-dimensional Banach space, there are more powerful covering theorems to ensure the weak type inequality with respect to arbitrary measures, as was done by Peter Sjogren in [Sj83]. See also [Be83] and [GK98]. Hence the maximal operator $M$ satisfies the strong type $L^p(\mu)$ estimates for $1 < p < \infty$. The function family of the following criterion is easier to process in comparison with Lemma 3.4.
Lemma 4.3. Let \( \mu \) be a Borel semi-regular measure on \( \mathbb{R} \) and let \( C \) be a constant. The following are equivalent:

(i) \( \|Mf\|_p \geq C\|f\|_p \) for all \( f \in L^p(\mu) \).

(ii) \( \|Mg\|_p \geq C\|g\|_p \) for all \( g \) of form \( g = \sum_{i=1}^{N} \beta_i \mathbf{1}_{I_i} \), where the bounded open intervals \( I_i \) are disjoint and \( \beta_i > 0 \).

Proof. Suppose that (ii) is invalid. Let \( f \in L^p(\mu) \). Since the sub-linear operator \( M \) is bounded on \( L^p(\mu) \), we have \( Mf_n \to Mf \) as \( f_n \to f \). Thus (i) follows from the fact that the set of all positive coefficients linear combinations of characteristic functions of bounded open intervals is dense. \( \square \)

Let \( A_\mu \) be the set of all the real numbers \( x \) with \( \mu(\{x\}) > 0 \). If we denote by \( x_n \) the points of \( A_\mu \) and by \( w_n \) the weight of \( x_n \), the decomposition follows: \( \mu = \mu_c + \sum_n w_n \delta_{x_n} \), where the continuous part of \( \mu \): \( \mu_c \) is defined by \( \mu_c(B) = \mu(B \cap \mathbb{R}^c) \). This brings us nicely to consider the case of that \( \mu_c(\mathbb{R}) = \infty \).

Theorem 4.4. Let \( \mu \) be a Borel semi-regular measure on \( \mathbb{R} \) and suppose that the axis of negative and positive reals have measures infinite. If one of the following holds:

(i) the set \( A_\mu \) is one point;
(ii) the set \( A_\mu \) contains only two points and there is no mass between these two points;

then

\[ \|M_\mu f\|_p \geq \left( \frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_p \quad \text{for all } f \in L^p(\mu). \]

Proof. We claim that the ball-coverings family assumed in Theorem [3.6] exists for each step function, then we justify the theorem. Let \( f = \sum_{i=1}^{N} \beta_i \mathbf{1}_{I_i} \), where the bounded open intervals \( I_i \) are disjoint and \( \beta_i > 0 \). If necessary, we rearrange the intervals \( I_i \) to ensure that it is mutually disjoint successively. Meanwhile, we also relabel the series \( \beta_i \), that is \( \beta_{i_1} \leq \beta_{i_2} \leq \cdots \leq \beta_{i_N} \), where \( (i_1, \ldots, i_N) \) is a permutation of \( (1, \ldots, N) \). Suppose first (i) holds and write \( A_\mu = \{y\} \).

We first consider the case when \( y \notin \cup_{1 \leq i \leq N} I_i \). Now suppose \( 0 < t < \beta_{i_N} \), otherwise the corresponding superlevel set is nonempty. We define \( j_t \) to be the first number in \( \{\beta_{i_j}\}_{1 \leq j \leq N} \) skipping over \( t \), then \( \{x \in X : f(x) > t\} = \cup_{j \geq j_t} I_{i_j} \). Now we apply a standard selection
procedure: since \( y \notin \bigcup_{1 \leq i \leq N} I_i \), \((-\infty, y]\) and \((y, +\infty)\) cut apart the intervals family \( \bigcup_{j \geq i} I_{ij} \) into the two parts, then we write \( b_1 \) be the right endpoint of the last interval in the list which lies on the left side of \( y \) (the exception is considered later); thus there exists a ball \( B_1 = (s_1, b_1) \) such that \( \langle f \rangle_{B_1} = t \), since \( \langle f \rangle_{(s, b_1]} \) is continuous, \( \lim_{s \to b_1^-} \langle f \rangle_{(s, b_1]} > t \) and \( \lim_{s \to -\infty} \langle f \rangle_{(s, b_1]} = 0 \) which all stems from the assumption that \( \mu|_{(-\infty, y]} \) is an infinite continuous measure; if \( s_1 \notin \bigcup_{j \geq i} I_{ij} \), we choose it; otherwise, using a while loop, we will find a new ball \( B_s = (s_i, s_{i-1}) \) to meet \( \langle f \rangle_{B_s} = t \) until \( s_i \notin \bigcup_{j \geq i} I_{ij} \), then we select \((s_i, b_1)\) be the ball \( B_1 \) as desired; having disposed of this step, we use induction to find the family of balls which covers the part of \( \bigcup_{j \geq i} I_{ij} \) belonging to \((-\infty, y]\). For the intervals family on the right side of \( y \), we do the same by consider the ball in \((y, +\infty)\). It was clear that the balls chosen was what we needed.

We now turn to the case when \( y \in \bigcup_{1 \leq i \leq N} I_i \). Then we suppose \( I_{iy} \supseteq y \) for some \( i_y \in (1, \ldots, N) \). The proof of this case is quite similar to the former since we can choose the right endpoint of \( I_{iy} \) as the starting point of the selection procedure.

The same reasoning applies to the case (ii).

The following example indicates that a one-sided infinite measure containing only one atom can also lead to the phenomenon of Example 4.1.

**Example 4.2.** For \( t > 1 \), consider \( \mu = t\delta_1 + m|_{(0, \infty)} \), where \( m \) is the Lebesgue measure. Obviously, the examples satisfy \( \mu(\mathbb{R}) = \infty \). Letting \( f = 1_{(0, 1)} \), we will show that we can take large \( t \) such that \( \|M_\mu f\|_{p, \mu} \) gets close enough to 1. If \( x \in (0, 1) \), \( M_\mu f(x) = 1 \); if \( x \in [1, \infty) \), then

\[
M_\mu f(x) = \sup_{a < x} \frac{1}{\mu((a, x])} \int_{(a, x]} f(s) d\mu = \sup_{0 < a \leq 1} \frac{1 - a}{t + x - a} = \frac{1}{t + x}.
\]

Thus, we have

\[
\|M_\mu f\|_{p, \mu}^p = \|M_\mu f\|_{p, m|_{(0, \infty)}}^p + \|M_\mu f\|_{p, t\delta_1}^p = 1 + \int_1^\infty \frac{1}{(t + x)^p} dx + \frac{t}{(t + 1)^p} \leq 1 + \frac{p}{p - 1} (t + 1)^{1-p}.
\]
Since \( \lim_{t \to \infty} \|M_{\mu}f\|_{p,\mu} = 1 \), the result follows.

**Remark 4.2.** Another similar example shows that in general Theorem 4.4 cannot be popularizing to the case where \( A_{\mu} \) contains more than three atoms.

Finally, we comment on a result that the lower \( L^p \)-bound of \( M \) becomes significantly increased with the decreases of \( p \).

Since it is immediate consequence of Hölder’s inequality, the proof is not shown here.

**Proposition 4.5.** Let \((X, d, \mu)\) be a metric measure space and \( M_r = \inf_{\|g\|_r = 1} \|Mg\|_r \), then \( M_r \leq M_p^\frac{p}{r} \) for \( 1 < p < r \).

**References**

[Al19] J. M. Aldaz, *Besicovitch type properties in metric spaces*. Available at the Mathematics ArXiv.

[Al21] J. M. Aldaz, *Kissing numbers and the centered maximal operator*. J. Geom. Anal. 31(2021), no.10, 10194–10214.

[Be83] A. Bernal, *A note on the one-dimensional maximal function*. Proc. Roy. Soc. Edinburgh Sect. A 111(1989), no.3-4, 325–328.

[Bo07] V. I. Bogachev, *Measure theory. Vol. I, II*. Springer-Verlag, Berlin, 2007.

[EFT07] L. Ephremidze, N. Fujii, Y. Terasawa, *The Riesz “rising sun” lemma for arbitrary Borel measures with some applications*. J. Funct. Spaces Appl. 5 (2007), no.3, 319–331.

[EG92] L. C. Evans, R. F. Gariepy, *Measure theory and fine properties of functions*. Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[Fe69] H. Federer, *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer Verlag New York Inc, New York, 1969.

[GK98] L. Grafakos, J. Kinnunen, *Sharp inequalities for maximal functions associated with general measures*. Proc. Roy. Soc. Edinburgh Sect. A 128(1998), no.4, 717–723.

[He01] J. Heinonen, *Lectures on analysis on metric spaces*. Universitext. Springer-Verlag, New York, 2001.

[IJN17] P. Ivanisvili, B. Jaye, F. Nazarov, *Lower bounds for uncentered maximal functions in any dimension*. Int. Math. Res. Not. IMRN (2017), no.8, 2464–2479.

[Le10] A. K. Lerner, *Some remarks on the Fefferman-Stein inequality*. J. Anal. Math. 112(2010), 329–349.

[LR17] E. Le Donne and S. Rigot, *Besicovitch covering property for homogeneous distances on the Heisenberg group*. J. Eur. Math. Soc. 19(2017), no.5, 1589–1617.

[LR19] E. Le Donne, S. Rigot, *Besicovitch covering property on graded groups and applications to measure differentiation*. J. Reine Angew. Math. 750(2019), 241–297.
[Ma95] P. Mattila, *Geometry of sets and measures in Euclidean spaces*. Fractals and rectifiability. Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1995.

[MS79] R. A. Macías, C. Segovia, *Lipschitz functions on spaces of homogeneous type*. Adv. in Math. 33(1979), no.3, 257–270.

[Sc73] L. Schwartz, *Radon measures on arbitrary topological spaces and cylindrical measures*. Tata Institute of Fundamental Research Studies in Mathematics, No.6. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973.

[Si83] L. Simon, *Lectures on geometric measure theory*. Proceedings of the Centre for Mathematical Analysis, Australian National University, vol.3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[Sj83] P. Sjögren, A remark on the maximal function for measures in $\mathbb{R}^n$. Amer. J. Math. 105(1983), no.5, 1231–1233.

[ST14] K. Stempak, X. Tao, *Local Morrey and Campanato spaces on quasimetric measure spaces*. J. Funct. Spaces 2014, Art. ID 172486, 1–15.

[Su94] J. M. Sullivan, *Sphere packings give an explicit bound for the Besicovitch covering theorem*. J. Geom. Anal. 4 (1994), no.2, 219–231.

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