Wave formation in a two-layer band of liquid from a source located in the lower layer

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Abstract. The article is devoted to the construction of an analytical algorithm for solving the problem of wave formation in a two-layer liquid from a source located in the lower layer of the liquid. In the process of solving the problem, it is reduced to a boundary value problem for a system of differential equations with respect to the velocity potentials of the disturbed flow of liquids. Next, the problem is divided into three simpler edge subtasks. Expressions describing the evolution of the free boundary and the separation line of the liquid layers are obtained. This evolution is also represented graphically. Conclusions are made about the influence of the ratio of densities, layer width and source power on the process of wave formation.

1. Introduction

The study of wave movements in a liquid is caused by the needs of various fields of science and industry, such as geophysics, hydrodynamics, shipbuilding and operation of hydraulic structures. In this paper, we will consider a stratified liquid [1] in which the physical characteristics (density, dynamic viscosity, and others) in a stationary state change only along the vertical direction, that is, in depth. These changes can be caused by various reasons, such as gravity or temperature and different water composition (in natural reservoirs). Due to stratification, the liquid thickness can be divided into several layers of different densities.

One of the first L Euler obtained the equation of an ideal fluid [2] in 1752. Lagrange formulated and solved the problem of the motion of an incompressible fluid with a free boundary [3]. Currently, many works are devoted to the formation of waves in single- and multi-layer media. The results of these studies can be found, for example, in articles [4-8].

In this paper, we consider the problem of the appearance and evolution of a wave in a two-layer ideal liquid from a source located in the lower layer. The appearance of a wave causes deformation of the free surface and the interface between the media. This article is a logical continuation of the work [9], which considered the occurrence of a wave in a two-layer liquid caused by a source located in the upper layer. Recently, when solving such problems, preference is often given to numerical [10, 11] or variation [12,13] methods, since it is believed that they can solve even complex configuration problems. However, when using these methods, calculation errors accumulate in the calculations, which lead to inaccurate results. Therefore, when initially solving mathematical problems, preference should be given to analytical methods, and only if the analytical solution is too time-consuming or cumbersome, it is worth switching to numerical methods.
In this paper, we propose a solution to the problem of wave formation in a two-layer liquid from a source located in the lower layer by an analytical method that allows us to find an exact solution. Based on the obtained solution, we propose a graphical representation of the evolution of the free boundary and the interface between environments. Based on the obtained solution, conclusions are made about the influence of densities, band thickness and source intensity on the wave formation process.

2. Mathematical statement of the problem
A two-layer weighty liquid of finite depth is considered. The thickness and density of each layer are respectively equal $h_1, \rho_1$ and $h_2, \rho_2$ (fig. 1). The liquid is at rest at the initial moment of time, and at some point in time $t = t_0$ in the lower layer at a distance $l$ from the media partition boundary, a source of intensity $Q(t)$ is turned on. Thus, the source is located at a point $(0, -l)$.

![Figure 1. The double layer strip of the liquid.](image)

We will solve the problem under the following conditions:

- the liquid is ideal and incompressible, the current is flat, potential and the condition $\text{rot}(\text{grad } \varphi) = 0$

- is met, where $\varphi$ is the velocity potential;

- the liquid is affected by gravity, acting along the $Oy$ axis;

- at the initial time, the medium is at rest;

- the pressure on the free surface $y = h_1$ is zero;

- on the interface of media $y = 0$ pressure and normal components of the particle velocity liquids are equal;

- the lower boundary $y = -h_2$ is impenetrable;

- the intensity of the source, acting on the interval $t \in (t_0, t_1)$, must be such that the relationship $\varepsilon = a/\lambda \leq 1$ ($a$ is the wave amplitude, $\lambda$ is the wavelength) is fulfilled [14]; the presence of a source at a point $(0, -l)$ is modeled by the equation $\text{div}(\text{grad } \varphi) = \delta(y + l)\delta(x)Q(t)$,

where $\delta$ is Dirac function.

Let's solve the following boundary value problem: we need to find the functions of the velocity potentials $\varphi_1 = \varphi_1(x, y, t), \varphi_2 = \varphi_2(x, y, t)$, acting in each band respectively and satisfying the system of differential equations.
\[
\begin{cases}
\Delta \phi_1 = 0, & 0 < y < h_1, \\
\Delta \phi_2 = \delta(y + l)\delta(x)Q(t), & -h_2 < y < 0
\end{cases}
\]  

with the boundary conditions

\[
\begin{align*}
\frac{\partial^2 \phi_1}{\partial t^2} & \bigg|_{y=-h} = g \frac{\partial \phi_1}{\partial y} \bigg|_{y=-h}, \\
\rho_1 \frac{\partial^2 \phi_2}{\partial t^2} & \bigg|_{y=0} = \rho_2 \frac{\partial^2 \phi_2}{\partial t^2} \bigg|_{y=0} + g (\rho_2 - \rho_1) \frac{\partial \phi_2}{\partial y} \bigg|_{y=0}, \\
\frac{\partial \phi_1}{\partial y} & \bigg|_{y=0} = \frac{\partial \phi_2}{\partial y} \bigg|_{y=0}, \\
\frac{\partial \phi_2}{\partial y} & \bigg|_{y=-h} = 0,
\end{align*}
\]

\[\phi_1, \phi_2 \to 0 \text{ for } \sqrt{x^2 + y^2} \to \infty.\]  

To determine the deformation function of the free boundary \( \xi(x, t) \) and the media partition boundary \( \xi_0(x, t) \), two more equations must be added to the system (1)-(2):

\[
\frac{\partial \phi_1}{\partial t} = g (\xi - h_1), \quad y = h_1, \\
\rho_1 \frac{\partial \phi_1}{\partial t} = \rho_2 \frac{\partial \phi_2}{\partial t} + g (\rho_2 - \rho_1) \xi_0, \quad y = 0.
\]

3. **Analytical solution method**

We will look for potentials in the form:

\[
\phi_i = \phi^0_i + \phi^x_i, \quad i = 1, 2,
\]

where \( \phi^0_i \) are the potentials of the flow of a weightless two-layer liquid under the influence of a source, \( \phi^x_i \) are the additional potentials of the disturbed flow, taking into account the influence of weight. In turn, the functions \( \phi^0_i \) can also be decomposed for simplification into two components \( \phi^0_i = \tilde{\phi}^0_i + \hat{\phi}^0_i \), where \( \tilde{\phi}^0_i \) are the flow potentials in a layer of width \( h_i \), \( \hat{\phi}^0_i \) are the additional potentials, due to which potentials \( \phi^0_i \) are «glued» together at the interface of media according to the condition:

\[
\rho_1 \phi^0_1 \bigg|_{y=0} = \rho_2 \phi^0_2 \bigg|_{y=0}, \quad \frac{\partial \phi^0_1}{\partial y} \bigg|_{y=0} = \frac{\partial \phi^0_2}{\partial y} \bigg|_{y=0}.
\]

Then expression (4) will take the form

\[
\phi_i = \phi^0_i + \phi^x_i = \tilde{\phi}^0_i + \hat{\phi}^0_i + \phi^x_i, \quad i = 1, 2,
\]

The boundary value problem is divided into three subtasks.
For the functions \( \tilde{\varphi}_1^0, \tilde{\varphi}_2^0 \), the system
\[
\begin{align*}
\Delta \tilde{\varphi}_1^0 &= 0, \\
\Delta \tilde{\varphi}_2^0 &= \delta(y + I)\delta(x)Q(t),
\end{align*}
\]
is solved with the boundary conditions
\[
\left. \tilde{\varphi}_1^0 \right|_{y=b} = \left. \tilde{\varphi}_2^0 \right|_{y=0} = 0 \quad \text{and} \quad \left. \frac{\partial \tilde{\varphi}_2^0}{\partial y} \right|_{y=-b} = \left. \tilde{\varphi}_2^0 \right|_{y=0} = 0. \tag{6}
\]

For the functions \( \tilde{\varphi}_1^0, \tilde{\varphi}_2^0 \) the system
\[
\begin{align*}
\Delta \tilde{\varphi}_1^0 &= 0, \\
\Delta \tilde{\varphi}_2^0 &= 0
\end{align*}
\]
with the boundary conditions
\[
\begin{align*}
\left. \tilde{\varphi}_1^0 \right|_{y=b} &= 0, \\
\rho_1 \left. \tilde{\varphi}_1^0 \right|_{y=0} &= \rho_2 \left. \tilde{\varphi}_2^0 \right|_{y=0}, \\
\left. \frac{\partial \tilde{\varphi}_1^0}{\partial y} \right|_{y=0} &= \left. \frac{\partial \tilde{\varphi}_2^0}{\partial y} \right|_{y=0} = 0, \\
\left. \frac{\partial \tilde{\varphi}_2^0}{\partial y} \right|_{y=-b} &= 0.
\end{align*}
\]  \tag{9}

For the functions \( \varphi_1^g, \varphi_2^g \) the system
\[
\begin{align*}
\Delta \varphi_1^g &= 0, \\
\Delta \varphi_2^g &= 0
\end{align*}
\]
is solved with the boundary conditions
\[
\begin{align*}
\left. \frac{\partial^2 \varphi_1^g}{\partial t^2} \right|_{y=b} &= g \left. \left( \frac{\partial \varphi_1^g}{\partial y} + \frac{\partial \varphi_2^g}{\partial y} \right) \right|_{y=b}, \\
\rho_1 \left. \frac{\partial^2 \varphi_1^g}{\partial t^2} \right|_{y=0} &= \rho_2 \left. \frac{\partial^2 \varphi_2^g}{\partial t^2} \right|_{y=0} + g \left( \rho_2 - \rho_1 \right) \left. \left( \frac{\partial \varphi_1^0}{\partial y} + \frac{\partial \varphi_2^0}{\partial y} \right) \right|_{y=0}, \\
\left. \frac{\partial \varphi_1^g}{\partial y} \right|_{y=0} &= \left. \frac{\partial \varphi_2^g}{\partial y} \right|_{y=0},
\end{align*}
\]
\[
\left. \frac{\partial \varphi_1^g}{\partial y} \right|_{y=-b} = 0. \tag{10}
\]
The conditions are also met at infinity \( \phi_i^0, \phi_i^0, \phi_i^0 \rightarrow 0 \) for \( x^2 + y^2 \rightarrow \infty \).

Since the problem is symmetric with respect to the \( O_x \)-axis, we will look for the solution in the half-band \( 0 \leq x < +\infty \). In this case, you can use half the intensity of the source \( 0.5Q \).

Let's move on to the solution of each task. System solutions can be written as

\[
\tilde{\phi}_i^0 = 0,
\]

\[
\phi_i^0 = \int_0^\infty A_i(\lambda,t) \frac{\sinh(\lambda h_i - \lambda y)}{\sinh(\lambda h_i)} \cos(\lambda x) d\lambda,
\]

\[
\overline{\phi}_i^0 = \frac{1}{4\pi} \int_D \delta(\eta + l) \delta(\zeta) Q(t) G_2(\zeta,\eta,x,y) d\zeta d\eta = \frac{Q(t)}{4\pi} G_2(0, -l, x, y),
\]

\[
\phi_i^0 = \int_0^\infty A_i(\lambda,t) \frac{\cosh(\lambda h_i + \lambda y)}{\cosh(\lambda h_i)} \cos(\lambda x) d\lambda,
\]

where \( D : -\infty < \zeta < +\infty, -h_2 < \eta < 0, \quad G_2 = G_2(\zeta,\eta,x,y) \) - Green function for half strip. Then,

\[
\phi_i^0 = \int_0^\infty A_i(\lambda,t) \frac{\sinh(\lambda h_i - \lambda y)}{\sinh(\lambda h_i)} \cos(\lambda x) d\lambda,
\]

\[
\phi_i^0 = \frac{Q(t)}{4\pi} G_2(0, -l, x, y) + \int_0^\infty A_i(\lambda,t) \frac{\cosh(\lambda h_i + \lambda y)}{\cosh(\lambda h_i)} \cos(\lambda x) d\lambda.
\]

From expressions (12), taking into account the conditions (5), we get an expression

\[
\frac{\partial G_2}{\partial y} \bigg|_{y=0} = -4 \int_0^\infty \frac{\cosh(\lambda h_i - \lambda l)}{\cosh(\lambda h_i)} \cos(\lambda x) d\lambda.
\]

That we will need later.

To determine the function \( G_2 \), we used the mapping of half a band to half a square on the complex plane using the mapping function obtained by the Christophe-Schwartz formula [15]:

\[
G_2(0, -l, x, y) = \ln \left[ \frac{(\cosh(ax)\sin(ay) + \sin(al))^2 + \sinh^2(ax)\cos^2(ay)}{(\cosh(ax)\sin(ay) - \sin(al))^2 + \sinh^2(ax)\cos^2(ay)} \right], \quad a = \frac{\pi}{2h_2}.
\]

The functions \( A_i(\lambda,t) \) can be defined from conditions (5), i.e. the following equality

\[
\rho_i \lambda_i = \rho_i A_2
\]

holds. Thus, these functions will take the form

\[
A_i(\lambda,t) = \rho_i / \rho_i \cdot A_i(\lambda,t),
\]

\[
A_2(\lambda,t) = \frac{Q(t)}{\lambda \pi} \frac{\cosh(\lambda h_i - \lambda l)}{\cosh(\lambda h_i)} \left[ \cosh(\lambda h_i) + \rho_i / \rho_1 \cdot \cosh(\lambda h_i) \right].
\]

The solution of the problem (10)-(11) is obtained as
\begin{equation}
\phi_1^t = \int_0^\infty \left[ B(\lambda, t) \text{ch}(\lambda y) + C(\lambda, t) \text{sh}(\lambda y) \right] \cos(\lambda x) \, d\lambda,
\end{equation}
\begin{equation}
\phi_2^t = \int_0^\infty \left[ D(\lambda, t) \frac{\text{ch}(\lambda(y + h_2))}{\text{ch}(\lambda h_2)} \right] \cos(\lambda x) \, d\lambda.
\end{equation}

Taking into account the boundary conditions (11), a system of differential equations (10) has the form
\begin{equation}
\begin{aligned}
\dot{B} + C \text{th}(\lambda h_1) - \lambda g \left[ B \text{th}(\lambda h_1) + C \right] &= gF_1, \\
\rho_1 \dot{B} - \rho_2 \dot{B} - \lambda g (\rho_2 - \rho_1) \text{th}(\lambda h_2) D &= g(\rho_2 - \rho_1)F_2, \\
C &= D \text{th}(\lambda h_2),
\end{aligned}
\end{equation}

where
\begin{equation}
F_1 = \frac{-\lambda A_1}{\text{sh}(\lambda h_1) \cdot \text{ch}(\lambda h_1)}, \quad F_2 = \lambda A_1 \text{th}(\lambda h_2) - \frac{Q(t) \text{ch}(\lambda(h_2 - l))}{\pi \text{ch}(\lambda h_2)}.
\end{equation}

At time \( t = t_0 \), the functions \( B, C, D \) and their time derivatives are equal to zero. To solve (15), we can first go to the 4th-order equation with respect to the function \( D \). Due to the cumbersome nature of the decision; we will not give it here.

4. The deformation of the free surface and the interface

The formula for estimating free surface deformation can be obtained from the boundary conditions (2) at the boundary \( y = h_1 \):
\begin{equation}
\xi = h_1 + \frac{1}{g} \frac{\partial \phi_1^t}{\partial t} (x, y, t) \bigg|_{y = h_1}.
\end{equation}

At the initial moment the liquid is at rest, and at each subsequent perturbed motion is determined by the function \( Q(t) \). Consider the special case when \( Q(t) = \delta(t - t_1)Q \), \( Q = \text{const} \). That is, at time \( t = t_1 \), the liquid is instantly ejected by a source with intensity \( Q \).

Similarly, you can write a formula for the media partition line:
\begin{equation}
\xi_0 = \frac{1}{g(\rho_2 - \rho_1)} \left[ \frac{\partial \phi_1^t}{\partial t} (x, y, t) - \rho_2 \frac{\partial \phi_2^t}{\partial t} (x, y, t) \right].
\end{equation}

A detailed algorithm for analytical transformation of formulas (16)-(17) is presented in the paper [9].

Figure 2 shows graphs of the dimensionless value \( \bar{\xi} = \frac{\xi - h_1}{0.5Q \frac{T}{h_1}} \) and figure 3 shows graphs of the dimensionless value \( \bar{\xi}_0 = \frac{\xi_0}{0.5Q \frac{T}{h_1}} \) for \( \bar{\xi}_0 \)

\begin{align*}
h_1 &= 1, \quad h_2/h_1 = 0.5, \quad l/h_2 = 0.5, \quad \rho_1/\rho_2 = 0.1, \quad F = gT^2/h_1 = 0.25.
\end{align*}
5. Conclusion
According to the results of the study, the following conclusions can be drawn:

- The change in the parameter $\rho_1/\rho_2$ affects wave generation. The amplitude of the wave at the free boundary decreases, the amplitude of the wave at the interface does not change significantly, but the speed of propagation of the wave decreases for $\rho_1/\rho_2 \to 1$.
- Changing the parameter $h_1/h_2$ significantly affects only the amplitude of the wave at the free boundary.
- When the parameter $F$ is increased, the wave propagation speed increases at both boundaries.

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