Periodic temperature field in a composite plate

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Abstract. The temperature fields in two-layer plates under periodic temperature effects on one of the surfaces are considered. The transition to a steady-state temperature field is investigated. For numerical calculations, a program based on the implementation of the finite element method was used. The same steady-state periodic temperature field was obtained asymptotically and was compared with the results of a numerical experiment. Numerical tests showed that for loads close to explosion during detonation of gas mixtures in thermodynamic treatment measures, the periodic temperature field is established after 5-6 cycles of heat flow exposure. Also after analysis of temperature field in single-layer and double-layer plates it is shown that introduction of thermal protection allows to significantly lower temperature of surface under influence of heat flow.

1. Introduction

On the inner surface of the two-layer plate, a periodic heat flux with period $T$ is set, while the other surface is maintained at a constant temperature. Out of physical considerations, it is obvious that after a certain number of cycles of exposure to the heat flux, the temperature field on the inner surface reaches a steady temperature. Studies of the transition process to a steady-state temperature regime were carried out based on a numerical experiment. From a practical point of view, it is of interest to estimate the time it takes to establish a periodic temperature field.

Further, the same steady-state periodic temperature field was obtained asymptotically. The obtained steady-state periodic temperature field is compared with the results of a numerical experiment.

2. Problem statement

The problem statement is as follows:

\[
\begin{align*}
\frac{\partial t_1}{\partial \tau} &= a_1 \frac{\partial^2 t_1}{\partial x^2}, & 0 \leq x \leq h, \\
\frac{\partial t_2}{\partial \tau} &= a_2 \frac{\partial^2 t_2}{\partial x^2}, & h \leq x \leq H,
\end{align*}
\]

under initial conditions:
and conditions on inner surfaces:

\[
\begin{align*}
\lambda_1 \frac{\partial t_1(0,\tau)}{\partial x} &= -F(\tau), \\
t_1(H,\tau) &= 0,
\end{align*}
\]

when \(F(\tau + nT) = Q(\tau), \quad n = 0, 1, 2, \ldots\) - periodical heat flux.

At the interface \((x = h)\) it is assumed that ideal thermal contact conditions are set:

\[
\begin{align*}
\lambda_1 \frac{\partial t_1(h,\tau)}{\partial x} &= \lambda_2 \frac{\partial t_2(h-0,\tau)}{\partial x}, \\
t_1(h,\tau) &= t_2(h-0,\tau).
\end{align*}
\]

Notations are introduced: \(t_i(x,\tau), i = 1, 2\), - temperature field in the first and second layers respectively, \(\tau\) - time, \(x\) - inner layer thickness, \(H\) - plate thickness, \(a_i, i = 1, 2\) - thermal diffusivity, \(\lambda_i, i = 1, 2\) - coefficients of thermal conductivity of the layers.

### 3. Asymptotic solution

We obtain the steady-state periodic temperature field in an asymptotic way, without considering the transient process. We use a method based on the application of generalized functions [1,2].

Equations (1) are written:

\[
\frac{\partial^2 t}{\partial x^2} = \frac{\partial t}{\partial \tau} \left[ \frac{1}{a_1} + \left( \frac{1}{a_2} - \frac{1}{a_1} \right) S_1(x-h) \right] + \left( 1 - K_x \frac{\partial t}{\partial x} \right) \delta_s(x-h)
\]

(5)

Here, \(t(x,\tau) = \begin{cases} t_2(x,\tau), & x \geq h \\ t_1(x,\tau), & x < h \end{cases}\)

\(K_x = \frac{\lambda_2}{\lambda_1}, \quad S_1(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}\) - asymmetric unit function, \(\delta_s(x)\) - Dirac delta function. On external surfaces, the conditions (3) are set.

In the asymptotic approach, we assume that the process is sufficiently far from the initial one and that the initial conditions do not influence the temperature distribution. We expand the periodic heat flux in the Fourier series by \([0,T]\):

\[
F(\tau) = Q_0 + \sum_{n=1}^{\infty} \left( a_n \cos w_n \tau + b_n \sin w_n \tau \right) = Q_0 + \sum_{n=1}^{\infty} \left( A_n \cos(w_n \tau - \bar{\xi}_n) \right)
\]

(6)

Here, \(A_n = \sqrt{a_n^2 + b_n^2}, \quad \bar{\xi}_n = \arccos \left( \frac{a_n}{b_n} \right), \quad Q_0 = \frac{1}{T} \int_0^T \! Q(\tau) d\tau, \quad w_n = \frac{2\pi}{T}, \quad a_n = \frac{2}{T} \int_0^T \! Q(\tau) \cos w_n \tau d\tau, \quad b_n = \frac{2}{T} \int_0^T \! Q(\tau) \sin w_n \tau d\tau, \quad n = 1, 2, 3, \ldots\)

The solution to problem (5), (3) can be obtained as a superposition of solutions corresponding to the terms in (6), i.e.:

\[
t(x,\tau) = t_0(x) + \sum_{n=1}^{\infty} t_n(x,\tau)
\]

(7)

Here \(t_0(x)\) is the static component of the steady-state temperature field, it is defined as the solution of the static heat conduction problem with boundary conditions (3) and the condition on the interface (4) and has the form:
Temperature waves \( t_n(x, \tau) \) satisfy the heat equation (5) and the next boundary conditions:

\[
\begin{align*}
\lambda_4 \frac{\partial t_n(0, \tau)}{\partial x} & = -A_n \cos(w_n \tau - \xi_n) \\
t_n(H, \tau) & = 0
\end{align*}
\]

The solution to this problem is found as the real part of the expression:

\[
v_n(x, \tau) = X_n e^{i(w_n \tau - \xi_n)}
\]

Substituting (9) into (5), we obtain the following problem for determining \( X_n(x) \):

\[
\frac{\partial^2 X_n}{\partial x^2} = -iw_n \left[ \frac{1}{a_1} + \frac{1}{a_2} \right] S_n(x - h) X_n = \left[ 1 - K_4 \frac{\partial X_n}{\partial x} \right] \cdot \delta_n(x - h)
\]

with boundary conditions:

\[
X_n(0) = A_n, \quad X_n(H) = 0.
\]

With this representation, the obtained solution for the parabolic equation has the character of a boundary layer; therefore, the second boundary condition (11) can be replaced by a simpler one: \( X_n(H) \to 0 \) while \( x \to \infty \).

Solving equation (10) under boundary conditions (11):

\[
X_n = \frac{A_n}{\Delta(0)} \left[ \Delta(x) S_n(x - h) + q_{1n} e^{2\tau(x - h)} S_n(x - h) \right],
\]

where \( \Delta(x) = q_{1n} c_h(q_{1n}(x - h)) - K_4 q_{2n} s_h(q_{2n}(x - h)) \), \( q_{jn} = k_{jn} (1 + i), \quad k_{jn} = \sqrt{\frac{w_n}{2a_j}}, \quad j = 1, 2 \), \( S_n(x - h) = 1 - S_n(x - h) \), \( i \) - imaginary unit.

Substituting (12) into (9) and isolating the material part, we determine the temperature waves. In the first layer:

\[
t_{1n}(x, \tau) = \frac{A_n}{\sqrt{2k_1D}} \left[ (k_{1n} - k_{2n})^2 \left[ e^{k_{1n}(x)} (\cos(y_{1n}(x) + z_{1n}(\tau)) - e^{-k_{1n}(x)} (\cos(y_{1n}(x) - z_{1n}(\tau))) + \\
+ (k_{1n} - k_{2n})^2 \left[ e^{-k_{1n}(x)} (\cos(k_{1n}(x) - z_{1n}(\tau)) - e^{k_{1n}(x)} (\cos(k_{1n}(x) + z_{1n}(\tau))) \right] \right]
\]

In the second layer:

\[
t_{2n}(x, \tau) = \frac{A_n}{\sqrt{2k_1D}} \left[ (k_{1n} + k_{2n})^2 e^{k_{1n}(x) + k_{1n}h \tau + z_{1n}(\tau)} - \\
- (k_{1n} - k_{2n})^2 e^{-k_{1n}(x) + k_{1n}h \tau + z_{1n}(\tau)} \right]
\]

where \( D = e^{2k_{1n}(h)} (k_{1n} + k_{2n})^2 + e^{-2k_{1n}(h)} (k_{1n} - k_{2n})^2 + 2(k_{2n}^2 - k_{2n})^2 \cos(2k_{1n}h), \)

\[
y_{1n}(x) = k_{1n} (x - 2h), \quad y_{2n}(x) = k_{2n}(x - h), \quad z_{1n}(\tau) = w_n \tau - \xi_n - \frac{\pi}{4}.
\]

4. Calculations

For numerical calculations of problem (1)-(4) a program based on the implementation of the finite element method was used. The plate is approximated by a grid consisting of one layer of 100 flat isoparametric 4-nodal finite elements. The first layer \( (h = H / 10) \) accounts for 60 elements. This grid turned out to be the most optimal for the convergence of the numerical solution (the numerical
solution on a coarser grid (the number of elements is less than 100) has a significant discrepancy with the exact solution). The optimal time step, also selected to take into account the rapidly changing temperature process, was $\Delta \tau = 5 \cdot 10^{-3}$ s. Numerical calculations were carried out up to the point in time necessary to establish a periodic temperature field. They showed that this temperature field is set for given loads almost after 5-6 cycles of exposure to heat flux.

The calculation results are presented on figure 1 in the form of time dependences on the temperature of the inner surface of a single-layer (dashed line 1) and double-layer plate (solid line 2). Peaks of the curves for the considered time interval (100s equal 5 cycles) go beyond certain stationary level. The maximum values for a single-layer plate and double-layer plate are $430^\circ C$ and $-220^\circ C$ respectively.

![Figure 1](image)

The expression for the heat flux was taken in the form [3]:

$$Q(\tau) = Q_1 (1 - \theta)e^{-k_1 \tau} + Q_2 e^{-k_2 \tau}.$$  

The frequency of the heat flux $T = 20$s.

Constants used in the calculations:

- $Q_1 = Q_2 = 10^3 \frac{w}{cm^2}$,
- $k_1 = 4.45 s^{-1}$, $k_2 = 132.5 s^{-1}$,
- $a_1 = 1.16 \frac{cm^2}{s}$, $a_2 = 0.16 \frac{cm^2}{s}$,
- $\lambda_1 = 0.96 \frac{cal}{cm \cdot s \cdot ^\circ C}$, $\lambda_2 = 0.13 \frac{cal}{cm \cdot s \cdot ^\circ C}$,
- $\theta = 0.12$, $h = 0.4 cm$, $H = 4 cm$.

Numerical calculations using asymptotic formulas (13) - (14) are presented in figure 2 in the form of time dependences on the temperature of the inner surface of a single-layer (dashed line 1) and double-layer (dashed line 2) plates.
The results of a numerical experiment are also presented here: solid lines 3 and 4 for a single-layer and double-layer plate, respectively. The considered time interval amounted to \([0, T]\). The dashed-dotted line corresponds to the change in temperature of the inner surface of the double-layer plate upon the first exposure to heat flux.

A slight overestimation of the temperature values obtained by formulas (13) - (14) can be explained by the fact that the plate wall thickness was assumed to be infinite in determining the temperature waves.

The disadvantage of the formulas (13) - (14) is due to the poor convergence of the Fourier series (6) in the expansion of the heat flux function \(F(\tau)\), which can be explained by the fact that this function \([0, T]\) does not experience a first-order discontinuity and is rapidly decreasing.

5. Conclusions
Thus, asymptotic expressions have been obtained for determining steady-state periodic temperature fields in double-layer plates when exposed to periodic thermal flux on one of its surfaces. The transition process has been studied numerically using the finite element method. Numerical tests showed that for loads close to explosion during detonation of gas mixtures in thermodynamic treatment measures [3], the periodic temperature field is established after 5-6 cycles of heat flow exposure. Also after analysis of temperature field in single-layer and double-layer plates it is shown that introduction of thermal protection allows to significantly lower temperature of surface under influence of heat flow.

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