Tidal synchronization of close-in satellites and exoplanets. III. Tidal dissipation revisited and application to Enceladus.

Abstract This paper deals with the bulk tidal dissipation predicted by the creep tide theory (Ferraz-Mello, Cel. Mech. Dyn. Astron. 116, 109, 2013) and with the general problem of averaging the evolution equations in the case of stiff bodies whose rotation is not synchronous, but is oscillating around the synchronous state with a period equal to the orbital period. We show that this tidally induced libration influences the amount of energy dissipated in the body and the average perturbation of the orbital elements. The resulting dissipation of synchronous homogeneous stiff bodies whose rotation axis is assumed perpendicular to the orbital plane shows a deviation from the Maxwell law reminiscent of the Andrade model, but distinct of it. The results are applied to the rotation of Enceladus and the relaxation factor necessary to explain its observed dissipation ($\gamma = 2 - 6 \times 10^{-8}\text{s}^{-1}$) has the expected order of magnitude for planetary satellites and corresponds to the uniform viscosity $5 - 15 \times 10^{14}$ Pa.s, which agrees with the reference viscosity of water ($10^{15}$ Pa.s) adopted by Běhounková et al. in their modeling of the melting events associated with the activity of Enceladus. The corrections of some mistakes and typos of paper II (Ferraz-Mello, Cel. Mech. Dyn. Astron. 122, 359, 2015) are included at the end of the paper.

1 Introduction

The calculus of the energy dissipation inside a stiff body is generally done by estimating the dissipation resulting from the action of the primary tidal force in deforming the planet (Kopal, 1963; Kaula, 1963, 1964; Peale and Cassen, 1978; Segatz et al., 1988; Wisdom, 2008, Frouard and Efroimsky, 2017). Such approach involves the choice of the physical model of the forces acting on the body at the microscopic level and of the dissipation parameters inside the body. An alternative approach was discussed by Kaula (1964; p.677) in which the bulk dissipation is calculated from the estimation of the total mechanical energy lost by the system. This approach was also used by Yoder
and Peale (1981) to estimate the tidal energy dissipated in a synchronous satellite, by Lissauer et al. (1984) to study the melting of Enceladus, by Ferraz-Mello et al. (2008) and Ferraz-Mello (2013) to estimate the energy dissipated in the frame of Darwin’s and creep tide theories, respectively, and by Correia et al. (2014) in the frame of a Maxwell model. The results obtained in these two approaches show good agreement as long as similar models are considered.

In this paper, we revisit the alternative approach to evaluate the bulk loss of mechanical energy by the system. This approach has the merit of its simplicity. As long as the second body (the body responsible for the tide raised in the stiff body) is considered as a mass point, the energy tidally dissipated in the first body (the stiff body under consideration) may only take origin in its rotation and in the orbit of the system. The secular variations of the semi-major axis and of the rotation of the body are the two gauges allowing us to evaluate the energy lost by the system. No other process exists able to continuously add energy to the system. We thus consider the energy exchanged with the orbit due to the direct attraction of the two bodies, the stored rotational energy in the first body and, for the sake of completeness, several minor contributions that are shown to be negligible.

The other feature important in the estimate of the dissipation (and of the tidal evolution of the orbital elements) is the averaging operation used to overcome the large and rapid variations of the mechanical energy in one period of the system. When the body rotation tends to a stationary state (the so-called spin-orbit resonance), these variations are essentially related to the relative orbital longitude (or anomaly), which, explicitly appears in the equations. The Keplerian elements of the orbit show only a very slow variation and can be assumed as constants in one orbital period. In the Darwinian regime of gaseous bodies this approach is universally adopted. In the case of stiff bodies, however, the approach must be different. The rotation of stiff bodies is not damped to a stationary value but is rather driven to a periodic attractor (see Correia et al., 2014; Ferraz-Mello, 2015; Folonier, 2016) with the same period as the orbital motion. The composition of the orbital motion with the physical libration forced by the tides changes the frequencies of the individual contributions to the net variation of the mechanical energy of the system and forces us to adopt an averaging operation formally different from the usual one and taking into account also the variations due to the forced libration.

The main result of the present investigation is the dissipation curve discussed in section 4.3. The fact that, in the range of frequencies of interest, the dissipation curve departs from a Maxwell law in a way that is reminiscent of the dissipation law of an Andrade body, is the first theoretical justification, from first principles, for the use of

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1 A discrepancy in the results presented by Kaula (1964) and Ferraz-Mello et al. (2008) appears mainly because Kaula considers only the contribution of the terms associated with the tidal argument \((2\varphi - 3\varpi - 2\omega)\) (his multi-index 2201). However, even when this restriction is introduced in Ferraz-Mello et al. (2008), the numerical coefficient obtained is \(-147/16\) instead of \(+147/32\) as given by Kaula. We note that Kaula’s result is retrieved if we consider only half of the orbital energy. In some instances, in Kaula’s paper, only half of the orbital energy is considered, but this is not said at this point of his paper and, indeed, we have found no physical rationale for doing it.

2 This forced libration is related to the asymmetries of the tides raised on the body; it is different from the asymmetries resulting from the assumption of a permanent triaxiality of the body. (See Frouard and Efroimsky, 2017).
such models in the study of the tidal evolution on stiff bodies. It is important to stress
that the only hypothesis done in the theory is that self-gravitation and tidal stress
permanently adjust the surface of the body to an equilibrium surface with speed given
by the Newtonian creep law. The adjustment is ruled by an approximate solution of
the Navier-Stokes equation for the flow of matter in the immediate neighborhood of
the equilibrium surface of the body. No constitutive equation linking strain and stress
is introduced at any point in the creep tide theory. All developments to reach the
conclusion are the solution of the creep differential equation and the use of classical
physics to compute the force and torque acting on the external body due to the tidal
deformation of the considered extended body. The observed dissipation law results
directly from the above described first principles of physics, with no additional ad-hoc
hypotheses.

The results are applied to Enceladus. We know from Cassini’s observations the
second-degree components of the gravitational field (Iess et al., 2014), we know that the
crust of Enceladus presents a forced libration of 0.120 ± 0.014 degrees (Thomas et al.
2016) and also that a huge quantity of heat flows from the satellite (5–16 GW cf Howett
et al. 2011; Spencer et al. 2013; Le Gall et al. 2017). However, the results obtained for
the dissipation and the forced libration are not consistent one with another. Either the
dissipation is much larger than the cited values or the observed forced libration is not
only due to tidal torques. A preliminary analysis using the recent model developed by
Folonier (Folonier, 2016; Folonier and Ferraz-Mello, 2017) for bodies stratified in layers
of uniform density was done. We could expect different results because of the decoupling
of the ice shell and the core, but the results obtained were not enough different from
those of the homogeneous model. The stratified model is now being adapted to consider
the case of one liquid layer below the crust.

This paper is organized as follows: We first proceed, in Section 2, at an inventory
of the main mechanical processes involving the storage of energy in the body and then
discuss the average loss of energy in each one. The averaging procedures used to obtain
the net loss of mechanical energy by the system is introduced in section 4, where the
influence of the tidally induced physical libration (section 3) on the averages is analyzed.
The next sections consider the dissipation law and the average variations of the metrical
elements of the orbit: semi-major axis and eccentricity. Some technical details of the
model used to evaluate the orbital energy are discussed in some appendices.

2 The energy balance

In this section, we examine the various manifestations of the mechanical energy in a
system formed by two mutually attracting bodies: the extended body m and one mass
point M. Let their masses be respectively, m and M. We consider only the case where
M lies on the equatorial plane of m. Our aim is to evaluate the amount of the energy
dissipated by the body.3

Let us first review some known facts of a system formed by the extended body m
and the mass point M (see Scheeres, 2002). Let r be the radius-vector in a system

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3 The dissipation in each of the two bodies may be considered separately. Within the order of approximation
generally adopted (first order in the tidal deformations), the variation of the energy can be split into two parts,
each one associated with the tidal deformation in one of the bodies while the other - source of the tidal potential
- is kept as a mass point. Therefore only the dissipation in one of the two bodies is explicitly considered.
of reference centered on \( m \). Since Darwin (1880), the problem is split into two parts: First, we consider the deformation of one body \((m)\) due to an external body \((M^*)\) – the body called Diana by Darwin. Then, we consider the disturbing potential due to this deformation on a mass point \((M)\) placed at a generic point of coordinates \((r, \varphi, \theta)\) and, eventually, we identify \( M \) and \( M^* \). The result is that actually \( M \) is moving in a time-dependent potential field and the potential energy of the system is

\[
E_{\text{pot}} = MU(r, r^*)
\]

where \( U(r, r^*) \) is the gravitational potential generated by \( m \) in the point \( r \). The vector \( r^* \) is the radius vector of Diana, i.e. the source of tidal potential deforming \( m \). Even if \( M \) and Diana are physically the same body, the formulation may remain valid even if they were different bodies. Thus, the force acting on \( M \) due to the attraction of \( m \) is

\[
f = -M \text{grad}_r U(r, r^*)
\]

where we included the subscript \( r \) to make clear that the derivatives in the evaluation of the gradient are done exclusively with respect to \( r \). The sign in this expression comes from the fact that we are using the conventions of Physics (\( U \) is a potential not a force-function). It is important to stress that in agreement with Newton laws, the force acting on \( m \) due to the attraction of \( M \) is \(-f\). This fact is neglected in many studies where one of the masses is negligible when compared to the other, but its neglect in general problems is an error (see Ferraz-Mello et al. 2003).

2.1 The function \( \delta U(r, r^*) \)

The disturbing potential due to the tidal forces in one point of space whose spherical coordinates are \( r, \varphi, \theta \) is

\[
\delta U = -\frac{GmR^2}{5r^3} \sum_{k \in \mathbb{Z}} \left( 3C_k \cos \sigma_k \sin^2 \theta \cos (2\varphi - 2\omega - (2 - k)\ell^* - \sigma_k) \right.

\left. + C''_k \cos \sigma'_k (3 \cos^2 \theta - 1) \cos (k\ell^* - \sigma''_k) \right)
\]

(cf Ferraz-Mello, 2015; eqns. 21-22) where

\[
C_k = \frac{1}{2} \tau_{\rho} E_{2,k}
\]

\[
C''_k = -\frac{1}{2} \tau_{\rho} E_{0,k} - \delta_{0,k} \tau_z
\]

\( G \) is the gravitational constant, \( E_{p,q}(e) \) are the Cayley coefficients, \( \ell^* \), \( \omega \) and \( e \) are, respectively, the mean anomaly, the argument of the pericenter and the eccentricity of the body source of the tidal potential; \( M^* \) (i.e. Diana) (to stress the origin of \( \ell^* \), we marked this anomaly with one star), \( R \) is the mean radius of \( m \), \( \tau_{\rho} \) is the mean flattening of its equator (i.e. when \( M^* \) is at the mean distance \( a \)), \( \tau_z \) is the polar oblateness of \( m \) due to the rotation of the body, \( \delta_{0,k} \) is the Kronecker delta.
The phases $\sigma_k$ and $\sigma'_k$ are constants introduced by the integration of the creep differential equation (see Ferraz-Mello, 2015) and are such that

$$\tan \sigma_k = \frac{\nu + kn}{\gamma}, \quad \cos \sigma_k = \frac{\gamma}{\sqrt{(\nu + kn)^2 + \gamma^2}}, \quad \sin \sigma_k = \frac{\nu + kn}{\sqrt{(\nu + kn)^2 + \gamma^2}}$$

$$\tan \sigma'_k = \frac{kn}{\gamma}, \quad \cos \sigma'_k = \frac{\gamma}{\sqrt{k^2n^2 + \gamma^2}}, \quad \sin \sigma'_k = \frac{kn}{\sqrt{k^2n^2 + \gamma^2}}$$

(6)

where $\gamma$ is the relaxation factor, $\nu = 2\Omega - 2\lambda$ is the semi-diurnal frequency and $n$ is the mean-motion.

We notice that the first part of Eqn. (3) is associated with sectorial (or, more properly, tesseral) components of the tidal displacement while the second part is associated with zonal terms (terms independent of the longitudes). We emphasize that the correction due to the neglected radial term in the solution of the creep equation in Ferraz-Mello (2015) (see Appendix A) is represented in Eqn. (3) by the use of the mean radius $R$ instead of the mean equatorial radius $R_e$. The corresponding correction is of second order in the flattening.

The variable elements in $\delta U$ are $r$, $\varphi$, $\theta$ and $\ell^*$. Strictly speaking, there are other variable parameters in $\delta U$, as, for instance, $n$, $e$, $\nu$, but their derivatives are first-order infinitesimals and thus, their contribution to the variation of $\delta U$ will be of second-order. The variation of $\nu$ may become important when $k = 0$ and in averaging operations. Its influence will be analyzed in section 4.2.

We may substitute the variables $r$ and $\varphi$ by the Keplerian variables of the motion of $M$ around $m$. Also, since we are only studying the planar case, we adopt $\theta = \pi/2$ and $\varphi = v + \omega$ ($v$ is the true anomaly). If we introduce the two-body expansions via Cayley coefficients, there follows:

$$\delta U = -\frac{GmR^2}{5a^3} \sum_{k \in \mathbb{Z}} \sum_{k+j \in \mathbb{Z}} \left( 3C_k \cos \sigma_k E_{2,j+k} \cos ((2 - k)\ell^* + \sigma_k + (j + k - 2)\ell) 
- C'_k \cos \sigma'_k E_{0,j+k} \cos (-k\ell^* + \sigma'_k + (j + k)\ell) \right)$$

(7)

where the subscripts in the infinite summations were adjusted to mimic those used in Ferraz-Mello (2015).

By construction, the function $U(r, r^*)$ is adequate to compute the forces and torques acting on $M$. However, the considered problem is only pseudo-potential since the internal processes leading to the continuously changing tidal deformation of $m$ are not described by this potential. To take them into account it is necessary to add a dissipation function (see Ragazzo and Ruiz, 2017). It is however worth emphasizing that the numerical example reported in the Appendix D shows that the only expected additional variations of the system energy are periodic and do not contribute to the dissipation of the system energy.

\footnote{For practical formulas using the Cayley coefficients, see the Online Supplement linked to Ferraz-Mello (2015).}
2.2 The work done by the disturbing force

The rate of variation of the work done by the force derived from the potential $\delta U$ is

$$\dot{W} = \delta f \cdot \mathbf{V} = -M \text{grad}_r \delta U \cdot \mathbf{V} = -M n \frac{\partial \delta U}{\partial \ell}$$

We remind that in the calculus of the force arising from the deformation of $m$ via the gradient of $\delta U$, only the derivatives w.r.t. the coordinates of $r$ are considered (see e.g. Kaula, 1964, Efroimsky, 2012a). Hence,

$$\dot{W} = -\frac{G m M n R^2}{5a^3} \sum_{k \in \mathbb{Z}} \sum_{k+j \in \mathbb{Z}} \left( 3(j+k-2)C_k \cos \sigma_k E_{2,j+k} \sin ((2-k)\ell^* + \sigma_k + (j+k-2)\ell) - (j+k)C_k'' \cos \sigma_k'' E_{0,j+k} \sin (j\ell + \sigma_k'' + (j+k)\ell) \right),$$

or, if we do $\ell^* = \ell$ (the body origin of the tidal potential and the moving body are the same):

$$\dot{W} = -\frac{G m M n R^2}{5a^3} \sum_{k \in \mathbb{Z}} \sum_{k+j \in \mathbb{Z}} \left( 3(j+k-2)C_k \cos \sigma_k E_{2,j+k} \sin (j\ell + \sigma_k) - (j+k)C_k'' \cos \sigma_k'' E_{0,j+k} \sin (j\ell + \sigma_k'') \right).$$

2.3 The orbital energy

The variation of the osculating semi-major axis due to the tides raised on $m$ may be obtained using the corresponding Lagrange variational equation:

$$\dot{a} = \frac{2}{na} \frac{\partial R}{\partial \ell}$$
where the disturbing function is $R = -(1 + M/m)\delta U$ (see Brouwer and Clemence, 1961, Chap. XI). The minus sign is included because $\delta U$ is a potential (not a force-function) and the factor $(1 + M/m)$ is introduced to account for the fact that the disturbing force is not of external origin, but an interaction between the two bodies. We thus consider the force per unit mass acting on one body minus its reaction on the other body (see discussion in Ferraz-Mello et al. 2008, Section 18.1).

Hence, taking into account the third Kepler law ($n^2a^3 = G(M + m)$) and comparing to the work calculated above,

$$\dot{a} = \frac{2a^2}{GM} \dot{W}.$$  

This is the same equation obtained when taking the time derivatives of both sides of the two-body classical equation relating the orbital energy and the osculating semi-major axis, $W = -GmM/2a$ (see Brouwer and Clemence, 1961), which has been used in previous papers (see Ferraz-Mello, 2013). It shows that $\dot{W}$ is equal to the time derivative of the Keplerian orbital energy of the system.

The large excursions of the variation of the orbital energy to positive and negative values do not violate energy conservation rules since the considered equations do not include the internal processes leading to the continuously changing tidal deformation of $m$.

2.4 Rotational energy

We have to consider, next, the energy stored in the extended body $m$. In all reported studies (e.g. Ferraz-Mello, 2013), the variation of that energy is given by the averaged derivative of the kinetic energy: $C\Omega \langle \dot{\Omega} \rangle$ where the moment of inertia $C$ is considered as a constant (for the term due to $\dot{C}$, see Section 2.6). Besides, in some of them, only the stationary solution $\langle \dot{\Omega} \rangle = 0$ is considered and the rotational energy of the extended body is neglected. In this discussion, we consider all terms and use the equation for the rotation of the body: $C\dot{\Omega} = M_z$ where $M_z$ is the $z$-component of the torque acting on the extended body. Hence,

$$\dot{W}_{\text{rot}} = -\frac{3GMm\Omega R^2\tau}{5a^3} \sum_{k \in \mathbb{Z}} \sum_{j+k \in \mathbb{Z}} E_{2,k}E_{2,k+j} \cos \bar{\sigma}_k \sin (j\ell + \bar{\sigma}_k)$$

(13)

(see Ferraz-Mello 2015; eq. 36)

The variation of the orbital and rotational energies as given above, in the case of Enceladus, adopting a relaxation factor $\gamma = 6 \times 10^{-8}\text{s}^{-1}$, is shown in figure 1. All parameters used in the simulations were taken from NASA’s Planetary Fact sheets (https://nssdc.gsfc.nasa.gov/planetary/factsheet/saturniansatfact.html)

After the trapping of the body rotation in the quasi-synchronous attractor, the average variation of the rotational energy becomes much smaller than the average variation of the orbital energy and it is often neglected in studies of dissipation in synchronous planetary satellites (e.g. Yoder and Peale, 1981). In the case under study, it is much less than 1 GW and so less than some of the small corrections discussed below.
2.5 Binding gravitational energy

Using the explicit formulas given by Essén (2004), the variation of the binding gravitational energy of the homogeneous ellipsoid is

\[ \dot{W}_{\text{int}} = \frac{3}{5} \frac{Gm^2}{R} \left( \frac{8}{45} \varepsilon_z \dot{\varepsilon}_z + \frac{2}{15} \varepsilon_\rho \dot{\varepsilon}_\rho \right). \] (14)

where \( \varepsilon_z \) and \( \varepsilon_\rho \) are the flattenings of the ellipsoid and \( \dot{\varepsilon}_z, \dot{\varepsilon}_\rho \) their time derivatives (see appendix B).

From the shape of the ellipsoid resulting from the integration of the creep differential equation (See Eqn. 34), we obtain

\[ \varepsilon_\rho = 2 \sum_{k \in \mathbb{Z}} C_k \cos \sigma_k \] (15)
\[ \varepsilon_z = - \sum_{k \in \mathbb{Z}} C_k' \cos(k \ell - \sigma_k') \] (16)

and

\[ \dot{\varepsilon}_\rho = -2 \sum_{k \in \mathbb{Z}} C_k \sin \sigma_k \frac{d\sigma_k}{dt} \] (17)
\[ \dot{\varepsilon}_z = \sum_{k \in \mathbb{Z}} k n C_k'' \cos \sigma_k \sin(k \ell - \sigma_k') \] (18)

where

\[ \frac{d\sigma_k}{dt} = \frac{2 \gamma \dot{\Omega}}{(\nu + kn)^2 + \gamma^2}. \] (19)

Since the coefficients \( C_k, C_k' \) are first-order infinitesimals, this variation is of second order and much smaller than the variation of the orbital energy. The actual application to Enceladus shows an oscillation whose amplitude is one order of magnitude smaller than the orbital energy variation and whose average is quickly damped to much less than 1 GW.

2.6 Moment of Inertia variation

One more non-elastic contribution comes from the fact that the moment of inertia of the body is not constant. The moment of inertia with respect to the polar axis is \( C = m(a^2 + b^2)/5 \) or, introducing the flattenings of the ellipsoid resulting from the integration of the creep differential equation: \( C = \frac{2}{5} m R^2 (1 + \frac{2}{3} \varepsilon_z + O(\varepsilon_\rho^2)) \) and \( \dot{C} \simeq \frac{4}{15} m R^2 \dot{\varepsilon}_z \). We note that, at this order of approximation, the equatorial tidal flattening does not affect the moment of inertia (the deformations inward and outward compensate themselves). The only variations come from the tidal oscillations in the actual polar oblateness of the body.

The contribution of this term to the variation of the rotational kinetic energy of the body is

\[ \delta W_{\text{rot}} = \frac{2}{15} m R^2 \dot{\varepsilon}_z \Omega^2. \] (20)
The result is a small correction to the energy variation. The actual application to Enceladus shows an oscillation whose amplitude is also about two orders of magnitude smaller than the main term of the rotation energy variation calculated in the previous sections and whose average is negligible.

2.7 The static component

We have also calculated the variation in the orbital energy in the pure hydrostatic case. In that case, the surface of the body is approximated by the ellipsoid:

$$\rho = R_e + R_e \sum_{k \in \mathbb{Z}} \left( C_k \sin^2 \hat{\theta} \cos \Theta_k + C_k'' \cos^2 \hat{\theta} \cos k\ell \right)$$  \hspace{1cm} (21)

where $\rho$, $\hat{\theta}$, $\hat{\varphi}$ are, respectively, the radius vector, co-latitude and longitude of the surface points. $R_e$ is the mean equatorial radius of $m$ and the $\Theta_k$ are linear functions of time:

$$\Theta_k = 2\hat{\varphi} + (k - 2)\ell - 2\omega.$$  \hspace{1cm} (22)

If this equation is compared to Eqn. (34) of Appendix A, we see that the equation of the ellipsoid in the hydrostatic case is the same as the solution of the creep equation when we impose $\sigma_k = \sigma_k'' = 0$. The variation of the orbital energy in the static case may then be obtained from the equations of section 2.2 just by imposing $\sigma_k = \sigma_k'' = 0$. That is:

$$\dot{W}_{st} = -\frac{GmMnR^2}{5a^3} \sum_{k \in \mathbb{Z}} \sum_{k+j \in \mathbb{Z}} \left( 3(j+k-2)C_k E_{2,j+k} \sin j\ell - (k+j)C_k'' E_{0,j+k} \sin j\ell \right).$$  \hspace{1cm} (23)

whose average over the orbital period is $\langle \dot{W}_{st} \rangle = 0$.

Since $\dot{W}_{st}$ is a zero-average periodic function, its explicit inclusion in the calculations of Section 4 does not affect the results. However, if we substitute, in the calculations of that section, $\dot{W}$ by $\dot{W} - \dot{W}_{st}$, the numerical determination of the averages become more stable (see fig. 2). Indeed, because of the large amplitude of $\dot{W}$, the average of the functions in that section are 3-4 orders of magnitude smaller than the peak of the functions.
Fig. 3 Tidal forced oscillation of the semi-diurnal frequency of one body like Enceladus in function of the relaxation factor $\gamma$. (N.B. A descending scale is adopted for $\gamma$ to ease the comparison with the other plots in the paper.) The red dashed line is the mean value of $\nu$.

being averaged. The similarity of the functions $\bar{W}$ and $\bar{W}_{st}$ is such that their difference has a much smaller amplitude, and the use of $\bar{W} - \bar{W}_{st}$ instead of $\bar{W}$ makes the averaging operations more numerically stable without affecting the final result.

3 The rotation. Forced libration

When the time variation of the moment of inertia is discarded, the rotation of the body $m$ is ruled by the differential equation $C\dot{\Omega} = M_z$ or, using the expression already written in Sec. 2.2, $\dot{\Omega} = \dot{W}_{rot}/C\Omega$ (see Ferraz-Mello 2015, Sec. 3). The numerical integration of this equation shows that the rotation is tidally damped to a stationary solution which, in the case of stiff bodies, is a forced libration around a nearly synchronous center. That is, a periodic attractor.

In the case of Enceladus, the numerical solution of the exact equations is an almost symmetric oscillation of $\nu$. The semi-amplitude of this oscillation depends on the adopted relaxation factor. If we adopt the $\gamma = 2 - 6 \times 10^{-8}s^{-1}$, which corresponds to the observed dissipation $5 - 16$ GW, the forced oscillation amplitude is very small, less than 0.001 deg/d. The numerical integration shows larger oscillations when larger values of $\gamma$ are adopted (while $\gamma < n$). The maximum amplitude of 0.3 deg/d is reached when the relaxation factor is about $\gamma = 10^{-5}s^{-1}$. For completeness we notice that for relaxation factors higher than this value, the center of the librations is shifted to a clear super-synchronous one (see fig. 3).

These values may be compared to the observed values. Measurements of control points on the surface of Enceladus accumulated over seven years of Cassini’s observations allowed Thomas et al. (2016) to determine the satellite’s rotation state. They found a libration of $0.120 \pm 0.014$ degrees. If we assume that these oscillations follow a harmonic law we obtain, correspondingly, for the oscillation of the velocity of rotation: $0.56 \pm 0.06$ deg/day, and for the semi-diurnal frequency $1.12 \pm 0.12$ deg/day. The immediate conclusion from the comparison of these values is that it is not possible to reproduce the observed forced libration of Enceladus with a homogeneous body model. The predicted oscillations are smaller and are significant only for values of the relaxation factor much larger than those suggested by the observed dissipation.
4 Dissipation and averaging

The actual variation of the main components of the mechanical energy is shown in fig. 1. It is large, indicating a continuous interplay of these processes with some other energy storage processes unaccounted for in this paper (e.g. the mechanical processes associated with the static tide and the interior processes that occur when the body is tidally deformed). In the long run, the only sources for the energy dissipated by the tides are those discussed in the previous section, with predominance of the orbital and the rotational energies.

The usual operation to get rid of these variations is the averaging of the total mechanical energy. The canonical tool to separate conservative and dissipative terms is the analysis of the differential form expressing the variation of the orbital energy. However, in the adopted model, only the attraction of the external body $M$ by the deformed body $m$ is available. The dynamics of the action of $M$ creating the deformation in $m$ is concealed by the creep equation used to determine the shape of $m$. We may, however, assimilate the averaging to the main characteristic of conservative phenomena where the energy variation calculated on a closed path vanishes.

4.1 The standard simplified averaging

The variable characterizing the variation in a short time interval is the mean anomaly $\ell$ and the averaging operation is just $\frac{1}{2\pi} \int_0^{2\pi} \dot{E}_{\text{tot}} \, d\ell$. The other elements are assumed to remain constant along one period. However, in the case of stiff bodies, $\nu$ may have a significant variation during one period and this variation may be accounted for in the averaging. We have thus to distinguish two different approaches. The standard averaging, unanimously adopted in all papers on tidal evolution, in which all orbital elements are assumed as constant in one period, and the exact averaging, identified here when necessary with the name “non-standard averaging”, in which the short-period variation of $\nu$ is taken into account.

The standard average of the variation of the orbital energy over one orbital period is

$$\langle \dot{W} \rangle = -\frac{G m M n R^2}{10a^3} \sum_{k \in \mathbb{Z}} (3(k - 2)C_k E_{2,k} \sin 2\sigma_k - kC_k'' E_{0,k} \sin 2\sigma_k'')$$

and the standard average of the variation of the rotational energy is

$$\langle \dot{W}_{\text{rot}} \rangle = -\frac{3GMm\Omega R^2 \tau_p}{10a^3} \sum_{k \in \mathbb{Z}} E_{2,k}^2 \sin 2\sigma_k.$$  

These two averages are generally added to give the energy dissipated by the system. In the case of Enceladus, the result is shown in fig. 4 (blue line). We emphasize two features of this result: (1) The resulting $\langle \dot{E}_{\text{tot}} \rangle = \langle \dot{W} + \dot{W}_{\text{rot}} \rangle$ is always negative, that is, the system is losing mechanical energy, as expected; (2) The resulting standard average $\langle \dot{E}_{\text{tot}} \rangle$ is not constant because the variation of $\nu$ was not taken into account in the averaging. These variations are observed always when $\gamma$ is much smaller than $n$. Even for very small $\gamma$, when the oscillations of $\nu$ become too small (see the right side of fig. 3), the small resulting $\langle \dot{E}_{\text{tot}} \rangle$ continues to show oscillations because in the equations
the intervening variable is the normalized quantity $\nu/\gamma$ that is almost constant for all values of $\gamma$ below the critical domain where the forced libration is seen (See the red dashed line in Fig. 5).

If the leading terms ($k = 0$) of Eqs. (24) and (25) are combined, we obtain:

$$\langle \dot{E}_{\text{tot}} \rangle_{(k=0)} = -\frac{3GmM(\Omega - n)R^2\tau_\nu E^2_{2,0} \sin 2\sigma_0}{10a^3}$$

which is a classical result (see e.g. Ferraz-Mello 2013). For sake of completeness, it is worth repeating some properties of this result: (1) since $\sin 2\sigma_0$ has the same sign as $\nu = 2(\Omega - n)$, the result is always negative (energy is lost); (2) the variation of the dissipation with the relaxation factor has an inverted V-shape with a maximum when $\nu = \gamma$, characteristic of the Maxwell rheology; (3) in the neighborhood of the stationary solution we know that $\nu \propto e^2$ and so the quantity defined by Eqn. (26) is of the order $O(e^4)$. This means that this result where terms of order $O(e^2)$ were discarded cannot be used in the close neighborhood of the stationary solution.

In order to obtain the energy variation of the stationary solution, we have to consider higher-order terms in the above equations. In this case one shortcut is allowed since, by definition, in this case $\dot{\Omega} = 0$ and so $\langle \dot{W}_{\text{rot}} \rangle = 0$. Thus, it is enough to consider the orbital variation of the energy. If we consider all terms with $|k| \leq 1$ and introduce the simplifications valid in the neighborhood of the stationary solution: $n \gg \nu, \sigma'_k \sim \sigma_k, -\sigma_k \sim -\sigma_k, \sin 2\sigma_0 \sim 12e^2 \sin 2\sigma_1$, after some algebraic manipulation, we obtain

$$\langle \dot{W}_{\text{station}} \rangle = -\frac{21GmMnR^2\tau_\nu e^2 \sin 2\sigma_1}{10a^3} + O(e^4)$$

or, if we adopt the heuristic approximation $Q^{-1} = \frac{1}{2} \sin 2\sigma_1 \sim \frac{2n\gamma}{\pi^2 + \gamma}$,

$$\langle \dot{W}_{\text{station}} \rangle = -\frac{21GmMnR^2\tau_\nu e^2}{5Qa^3} + O(e^4),$$

which is the usual formula used to calculate the dissipation in stationary solutions (see e.g. Ferraz-Mello et al., 2008).

It is worth noting that in the close neighborhood of the synchrony, the main terms depending on $\sigma_0$ in the variation of the orbital and rotational energies almost compensate themselves so that the energy variation becomes almost independent of $\nu$, notwithstanding the important dependence on $\nu$ of its two components.

4.2 The exact averaging

In order to have the averaging correctly done, we have to take into account the forced libration of the body. So, we may substitute $\nu$ by the solution of the differential equations of the body rotation (cf. Section 3.3 of Ferraz-Mello, 2015) before averaging. This can be done using the analytic approximation of the solution, for example, the periodic approximation $\nu = B_0 + B_1 \cos \ell + B_2 \sin \ell$ (see Appendix C). It is however easy to include $\dot{W}$ and $\dot{W}_{\text{rot}}$ in the output of the numerical integration of the differential equation, and compute afterwards the average of $\dot{W} + \dot{W}_{\text{rot}}$ (or $\dot{W} - \dot{W}_{\text{st}} + \dot{W}_{\text{rot}} - \text{cf. discussion in 5}$).
Fig. 4 Averaged variation of the mechanical energy. Red dashed line: Numerical average of $\dot{W} + \dot{W}_{\text{rot}}$ simulated over 2000 days. Blue line: Result obtained using the equations of Sec. 4.1 (simplified or standard average) and the actual values of $\nu$ obtained from the simulation. Blue dashed line: Mean value of the standard average. NB. $\gamma = 6 \times 10^{-8} \text{s}^{-1}$.

Sec. 2.7 The red dashed lines in Fig. 5 and 6 show (in arbitrary units) the amplitude of oscillation of the normalized quantity $\nu/\gamma$ which is the factor actually affecting some significant terms in the expressions of $\dot{W}$ and $\dot{W}_{\text{rot}}$.

4.3 Dissipation laws. An Andrade-like junction

The average loss of energy of the system is given by the sum of the two averages above calculated: $\dot{W}$ and $\dot{W}_{\text{rot}}$. If we neglect the short period librations and assume $\nu = \text{const}$, we obtain the same Maxwell law as Ferraz-Mello (2013), shown in fig. 5 by a dashed line. If, however, we take into account the forced oscillation of $\nu$ and include its actual value in the calculations, the result departs from the Maxwell law for values of the relaxation factor smaller than a critical limit $\gamma_{\text{crit}} \sim n$ and follows a pattern reminiscent of that of the Andrade theoretical model (see Efroimsky, 2012b). In fig. 5 it appears as a plateau which subsists for an important range of values of $n/\gamma$ before changing again to an inverse power law (log-log plot slope = $-1$ w.r.t. $n/\gamma$ or $+1$ w.r.t. $\gamma$). It is noteworthy the similarity of the dissipation law in the log plot, to a sum of two Maxwell laws with vertices, one at $n/\gamma = 1$ and the other at $n/\gamma = 10$.

The pattern seen in fig. 5 is distinct of that of Andrade model found in texts on tides, but, there, the figures result from an arbitrary construction, while here, it is the result of the application of simple physical laws to one model whose only hypothesis is that the anelastic tide is determined by a Newtonian creep. The existence of short-period librations and the influence of these librations on the averaging of the mechanical energy are all immediate consequences of the theory without the involvement of any other hypothesis. The dissipation curve starts departing from the Maxwellian curve near its maximum. In Andrade’s examples, the junction happens for values corresponding to the descending linear branch of the Maxwellian curve, but this difference may be a consequence of the adopted parameters. The presence of the junction near the vertex of the Maxwellian curve is characteristic for small close-in satellites.

In the case of Enceladus, used in this paper as example of application, the estimations of the heat dissipated in the SPT (south polar terrain) area based on the observations with Cassini are in the range $5 - 16 \text{ GW}$ (cf Howett et al. 2011; Spencer et al. 2013; Le Gall et al. 2017). This dissipation corresponds to a relaxation factor $\gamma =$
Fig. 5 Dissipation curves: Solid line: Exact average of the net variation of the mechanical energy; Dashed blue line: Simplified standard average of the energy variation. Red dashed line: Corresponding amplitude of the tidally forced oscillation of the normalized \( \nu/\gamma \) (in arbitrary units). The body and orbital parameters used correspond to a homogeneous Enceladus. The actual range of the observed dissipation values is shown by a red box.

Fig. 6 Left: Solid line: Variation of the exact average of \(-\dot{a}/\dot{t}\) in function of \(\gamma\) (see the top axis) for a body with the same elements as Enceladus. Dashed line blue: Evaluations of the standard average for two fixed values of \(\nu\): The supersynchronous stationary value given in Ferraz-Mello (2013) (A) and \(\nu = 0\) (B). Dashed line red: Corresponding amplitude of the tidally forced oscillation of the normalized quantity \(\nu/\gamma\) (in arbitrary units). Right: Solid line: Variation of the exact average of \(-\dot{e}/\dot{t}\) in function of \(\gamma\) (see top axis). Dashed line: Standard average of \(-\dot{e}/\dot{t}\) for a fixed \(\nu\). In this case, the branches (A) and (B) are indistinguishable one from another.

\[ 2 - 6 \times 10^{-8} \text{ s}^{-1} \text{ (see Figure 5).} \]

The uniform viscosity corresponding to this relaxation is \(0.5 - 1.5 \times 10^{15} \text{ Pa s.}\)

It is not possible to translate the value of the relaxation factor to an equivalent \(Q\), because the heuristic formulas often used for that sake assume a Maxwell dissipation law which is far from the observed law when the forced libration is taken into account in the averaging process.

5 Semi-major axis and eccentricity average perturbations

5.1 Semi-major axis

The variation of the semi-major axis due to the tidal deformations of \(m\) is given by Eqn. 12. The results corresponding to a homogeneous Enceladus are shown in fig. 6 (Left). In that figure the two averages are shown for a wide range of values of \(\gamma\) and \(\nu\). The simplified standard average depends on the value imposed to \(\nu\). If the supersynchronous stationary solution \(\nu = 6ne^2\gamma^2/(n^2 + \gamma^2)\) (cf Ferraz-Mello,

\footnote{According with the relation \(\gamma = wR/2\eta\) given in Ferraz-Mello, 2013, 2015.}
2013, Eqn. 35) is imposed, we obtain the curve labeled as A which coincides with the exact averaging for the values of \( n/\gamma \) corresponding to the stationary solution in the Darwin regime (roughly, for \( n/\gamma < 1 \)) and shows that the standard averaged formulas used in the study of tidal evolution may be used in the case of gaseous planets and stars. The second solution shown, labeled as B, corresponds to imposing \( \nu = 0 \), a stationary solution that in fact does not exist. In that case the disagreement is total and the standard averaged formulas gives results that may be either much larger or much smaller that the actual exact averages obtained taking into account the real values of \( \nu \) after the transient variations due to the initial conditions of the simulations are totally damped. The exact average has the same aspect as the dissipation law shown in fig. 5. This similarity happens because, near the stationary solution, the dissipated energy comes almost totally from the power \( \dot{W} \) (the loss of rotational energy \( \dot{W}_{\text{rot}} \) is several orders of magnitude smaller). When \( \gamma = 6 \times 10^{-8}\text{s}^{-1} \), its value is \(-2.6 \times 10^{-16} \text{AU/day} \) \((- -1.4 \text{ cm/d})\). This value is 3 orders of magnitude larger than the value \( 9 \times 10^{-20} \text{AU/day} \) recently estimated by Efroimsky (2017) and of opposite sign.

It is important to emphasize that this is not the actual rate of change of the semi-major axis. This is only the part of it due to the tides on the satellite. To obtain the actual rate of change of \( a \), it is necessary to consider also the part of it due to the tides raised on the planet, which is given by the same equations, but where the variables are interchanged to express the dissipation on the planet instead of the satellite. In the case of Enceladus, the two effects appear to have similar orders of magnitude; according to Lainey et al. (2012), the variation of the semi-major axis due to the tides raised by Enceladus on Saturn is \( 7.8 \times 10^{-16} \text{AU/day} \). The variation of the semi-major axis of Saturnian satellites is important for some theories of the formation of these satellites (see Lainey et al. 2012). With the actual estimates of the energy dissipated in Enceladus, we get the above values which imply in the present expansion of the satellite’s orbit.

The orbital variations due to the tides raised on the planet are more important than the variations due to the tides raised on the satellite. However, this picture can change if the total energy dissipation in Enceladus is shown to be larger than currently estimated, in which case the two effects can become of same magnitude and even such that the sign of \( \dot{a} \) is reversed.

5.2 Eccentricity

The variation of the eccentricity is given by the corresponding Lagrange variational equation:

\[
\dot{e} = -\frac{\sqrt{1-e^2}}{na^2e} \frac{\partial \mathcal{R}}{\partial \omega} + \frac{1-e^2}{na^2e} \frac{\partial \mathcal{R}}{\partial \ell}
\]

(see Brouwer and Clemence, 1961). This equation is equivalent to

\[
\dot{e} = \frac{1-e^2}{e} \left( \frac{\dot{a}}{2a} - \frac{\dot{L}}{L} \right)
\]

(30)

where \( L = \frac{GMm}{na} \sqrt{1-e^2} \) is the orbital angular momentum. In order to use the averages given in the previous sections, we may introduce a change in this equation reminding
that $\dot{L} = M_z = -\dot{W}_{\text{rot}}/\Omega$. Hence

$$\langle \dot{e} \rangle = \frac{1 - e^2}{e} \left( \frac{\langle \dot{a} \rangle}{2a} + \frac{1}{L} \langle \dot{W}_{\text{rot}} \rangle / \Omega \right)$$

(31)

The close inspection of the above equations allows one to see that the parenthesis in the right-hand side of Eqn. (30) is a quantity of the order of the eccentricity and that its average is of the order $O(e^2)$. In both cases, the parenthesis vanishes when $e \to 0$.

When $\gamma = 6 \times 10^{-8} \text{s}^{-1}$, the averages are very small (of order $10^{-8} \text{yr}^{-1}$). In general, the eccentricity variation may become important in long-term studies but, in the case of Enceladus, the effects of the almost 2:1 resonance between Enceladus and Dione produces a forced eccentricity of 0.00459 that must be taken into account (see Ferraz-Mello, 1985). We remind that the proper eccentricity of Enceladus is only 0.00012.

The variation of the averaged $de/dt$ as a function of the relaxation factor $\nu$ is shown in fig. 6(Right). This figure shows that the simplified standard averaging produces correct results only when $\gamma > n$, but underestimates the actual (exact) averages when $\gamma < n$, that is, in the case of stiff bodies. One may also notice that the standard average gives coincident results no matter in which of the two cases, (A) or (B), of Fig. 6(Left) are considered. This coincidence is of the same nature as the coincidence of these two cases in the dissipation law. There, it was due to the fact that the parts of $\langle \dot{W} \rangle$ and $\langle \dot{W}_{\text{rot}} \rangle$ linear in $\nu$ (i.e. proportional to $\sin \sigma_0$) cancel themselves when the two averages are added (see Eqn. 26). Here, we may easily see that, when terms of the order $O(e^2)$ are neglected, $\langle de/dt \rangle$ also becomes proportional to $\langle \dot{W} \rangle + \langle \dot{W}_{\text{rot}} \rangle$.

6 Conclusions

The main result of the present investigation is the dissipation law for quasi-synchronous homogeneous bodies discussed in section 4.3. The fact that, in the range of frequencies of interest, the dissipation curve departs from a Maxwell law in a way that is reminiscent of the dissipation law of an Andrade body is the first theoretical justification, from first principles of Mechanics, for the use of such models in the study of the tidal evolution on stiff bodies. It is important to stress that the only hypothesis done in the theory is that the surface of the body permanently adjust itself to an equilibrium surface with speed given by the Newtonian creep law. No constitutive equation linking strain and stress is introduced at any point in the creep tide theory. All developments to reach the conclusion are the solution of the creep differential equation and the use of classical physics to compute the force and torque acting on the external body due to the tidal deformation of the considered extended body. The observed dissipation law results directly from the above described first principles of physics, with no additional ad-hoc hypotheses.

In the case of Enceladus, used in this paper as example of application, the estimations of the heat dissipated in the SPT (south polar terrain) area based on the observations with Cassini are in the range $5 - 16 \text{ GW}$ (cf Howett et al. 2011; Spencer et al. 2013; Le Gall et al. 2017). In addition, a recent study by Kamata and Nimmo (2017) showed that a value about ten times higher than the classical estimate of 1.1 GW is necessary if the ice shell is in thermal equilibrium. The given values correspond to a relaxation factor
\[ \gamma = 2 - 6 \times 10^{-8} \text{s}^{-1}. \] It is worth saying that the heuristic formulas for the conversion of the relaxation factor into the quality factor for synchronized homogeneous bodies (Ferraz-Mello, 2013) are based on the similarity of the Maxwell laws followed by models and so cannot be used in the present case because of the large departure of the actual dissipation law from a Maxwell model.

The uniform viscosity corresponding to this relaxation factor is \(5 - 15 \times 10^{14} \text{ Pa.s}\). This value fully agrees with the reference viscosity of water at 255 K (\(10^{15} \text{ Pa.s}\)) adopted by Běhoušková et al. (2012) in their modeling of the melting events at origin of the south-pole activity on Enceladus, but is \(\sim 10\) times larger than the value recently estimated by Efroimsky (2017) and than the value adopted by Roberts and Nimmo (2008) for the viscosity of the ice shell.

We shall mention the difficulty in assessing the averages because of oscillations 2 orders of magnitude larger than the average. Because of this numerical instability, the zero-average static tide energy variation was subtracted (see Section [2.7]) from the bulk variation before averaging; besides, the averages in the Darwin branch were done over some tens of thousand days and discarded the initial sections of the solution before it is damped to the attractor.

On the technical side, this paper raises the question of the usual averaging operations adopted in tidal evolution theories. One hypothesis behind the existing formulas is that the rotation of the body reached a stationary solution synchronous or supersynchronous. The solutions resulting from some recent theories of the anelastic tides (Correia et al., 2014; Ferraz-Mello, 2015) contradicts this possibility in the case of stiff bodies. These theories show that, if the eccentricity is not identically zero, the rotation evolution is damped to a short-period oscillation around a synchronous or quasi-synchronous state. The oscillation of the semi-diurnal frequency \(\nu\) needs to be taken into account in the averaging operation and the results are different when it is not considered. The two approaches are compared showing striking differences for values of the relaxation factor smaller than a critical limit \(\gamma_{\text{crit}} \sim n\).

### Appendices

#### A Inclusion of one neglected radial term

The creep differential equation is

\[
\dot{\zeta} + \gamma \zeta = \gamma R_e + \gamma R_e \sum_{k \in \mathbb{Z}} \left( C_k \sin^2 \hat{\theta} \cos \Theta_k + C''_k \cos^2 \hat{\theta} \cos k\ell \right)
\]

(Ferraz-Mello 2015, Eqn. 5) where \(\zeta, \hat{\theta}, \hat{\phi}\) are, respectively, the radius vector, co-latitude and longitude of the surface point. \(R_e\) is the mean equatorial radius of \(m\), The \(\Theta_k\) are linear functions of time:

\[
\Theta_k = 2\hat{\varphi} + (k - 2)\ell - 2\omega.
\]

The solution of the creep differential equation is

\[
\zeta(\hat{\theta}, \hat{\varphi}) = R_e + R_e \sum_{k \in \mathbb{Z}} \left( C_k \sin^2 \hat{\theta} \cos \Theta_k \cos(\Theta_k - \sigma_k) + C''_k \cos^2 \hat{\theta} \cos \Theta''_k \cos(k\ell - \sigma''_k) \right)
\]
The phases $\sigma_k$ and $\sigma''_k$ are functions of the frequencies introduced by the integration of the first-order differential equation (see Sec. 2.1). In Ferraz-Mello (2015), the variation of the first term ($R_e$) has not been considered. However, this term is not a constant. When the equatorial flattening varies due to a variation in the proximity of $M$, the polar flattening varies accordingly and this should be examined in a more precise way to warrant that the results are not affected.

Let us substitute $R_e$ by its value

$$R_e \simeq R(1 + \frac{1}{3}\varepsilon_z) \simeq R(1 + \frac{1}{3}\varepsilon_z + \frac{1}{6}\varepsilon_\rho)$$

(35)

where $R$ is the mean radius of $m$, $\varepsilon_z$ is the actual polar flattening of the body, $\varepsilon_z$ the part of polar flattening of $m$ due to the rotation of the body only and

$$\varepsilon_\rho = \varepsilon_\rho(a)^3,$$

where $$\varepsilon_\rho = \frac{15}{4} M_m(R_e/a)^3.$$ (36)

$\varepsilon_\rho$ is the equatorial flattening of $m$ due to the tidal force.

If Cayley functions are used to expand the variable part $(\hat{\varphi})^3$ and the same notations used above are introduced, we obtain

$$R_e \simeq R - \frac{1}{3}R \sum_{k \in \mathbb{Z}} C''_k \cos k \ell.$$ (37)

When these terms are included in the differential equation, they introduce, in the solution, the radial terms

$$\delta \zeta_{\text{rad}} = -\frac{1}{3}R \sum_{k \in \mathbb{Z}} C''_k \cos \sigma''_k \cos(k \ell - \sigma''_k).$$

(38)

In Ferraz-Mello (2015), these terms were discarded on the grounds of heuristic reason: Radial terms in $\delta \zeta$ were considered as artifacts resulting from the model, since the volume of the body must remain unchanged. However, these terms are in fact correcting the non-conservation of the volume introduced by the zonal terms in Eqn. (34).

When these terms are added to the given solution of the creep equation, Eqn. (34) becomes

$$\zeta(\hat{\theta}, \hat{\varphi}) = R + R \sum_{k \in \mathbb{Z}} (C_k \sin^2 \hat{\theta} \cos \sigma_k \cos(\Theta_k - \sigma_k) + C_k''(\cos^2 \hat{\theta} - \frac{1}{3}) \cos \sigma''_k \cos(k \ell - \sigma''_k)).$$

(39)

We note that $R_e$ has been substituted by $R$ before the summation. Since the terms in the summation are proportional to the flattenings, the errors resulting of this substitution are of second order and they may be neglected. It is worth noting that the flattening of the spheroid corresponding to the zonal terms remains unchanged and thus the expression of the potential derived in Ferraz-Mello (2015) remains the same with only the change of $R_e$ by $R$ in the factor.

The term added in this way gives rise to a radial force which is torqueless and conservative.
B Gravitational energy of an ellipsoid

The equation of the surface of an ellipsoid, approximated to the first order of the flattenings, is

$$
\zeta = R_e (1 + \frac{1}{2} \varepsilon_\rho \sin^2 \theta \cos 2 \varphi - \varepsilon_z \cos^2 \theta)
$$

(40)

where \( \zeta, \theta \) and \( \varphi \) are the radius vector, the co-latitude and the longitude of the surface points, and \( \varepsilon_\rho \) and \( \varepsilon_z \) are the actual equatorial and polar flattenings of the ellipsoid:

$$
\varepsilon_\rho = \frac{a - b}{R_e}, \quad \varepsilon_z = 1 - \frac{c}{R_e}.
$$

(41)

\( R_e = \sqrt{ab} \) is the equatorial radius. If the mean radius \( R = R_e (1 - \frac{1}{3} \varepsilon_z) \) is used instead of the equatorial radius \( R_e \), the equation of the surface becomes

$$
\zeta = R \left(1 + \frac{1}{2} \varepsilon_\rho \sin^2 \theta \cos 2 \varphi - \varepsilon_z (\cos^2 \theta - \frac{1}{3})\right)
$$

(42)

B.1 Direct calculation

The last equation can be described as corresponding to a sphere with radius \( R \) enveloped by a thin shell of thickness \( h = \zeta - R \). When second-order terms are discarded, the binding gravitational energy of the shell to the sphere is given by

$$
W_{\text{int}} = GmR\mu \int_0^\pi \sin \hat{\theta} d\hat{\theta} \int_0^{2\pi} (\zeta - R) d\hat{\varphi}
$$

(43)

where \( \mu \) is the density of the body (homogeneous). The second integral is obvious: The integral of the sectorial terms (first summation) is equal to zero, since these terms are \( \pi \)-periodic functions of \( \hat{\varphi} \), and the integral of the zonal terms (second summation) are just equal to the product of the terms themselves by \( 2\pi \) because they are independent of \( \varphi \). Hence,

$$
W_{\text{int}} = GmR\mu \int_0^\pi \sin \hat{\theta} d\hat{\theta} \sum_{k \in \mathbb{Z}} 2\pi \varepsilon_z (\cos^2 \theta - \frac{1}{3})
$$

(44)

The remaining integral only involves elementary functions and it is easy to check that it vanishes; the positive contributions near the equator are exactly compensated by the negative contributions near the poles. Hence, the binding energy of the ellipsoid differs from the binding energy of the sphere by quantities of second-order in the flattenings.

Auxiliary first-order relations:

$$
a = R_e (1 + \varepsilon_\rho/2), \quad b = R_e (1 - \varepsilon_\rho/2), \quad c = R_e (1 - \varepsilon_z)
$$
B.2 Gravitational energy of an ellipsoid after Essén (2004)

The gravitational energy of an ellipsoid of mass $m$ and semi-axes $a > b > c$ is explicitly given by

$$ W_{\text{int}} = -\frac{3Gm^2}{R}X(\xi, \tau) $$

(Essén, 2004) where the Taylor expansion of $X$ around $\xi = 1, \tau = 0$ is

$$ X(\xi, \tau) = 1 - \frac{4}{5}(\xi - 1)^2 - \frac{4}{15}\tau^2 + \cdots $$

and $\xi, \tau$ are functions of the flattening such that

$$ a = (\xi + \tau)R $$
$$ b = (\xi - \tau)R $$
$$ c = (\xi^2 - \tau^2)^{-1}R. $$

Hence, to the first-order,

$$ \tau \simeq \frac{1}{2}\varepsilon_\rho $$
$$ \xi - 1 \simeq \frac{1}{3}\varepsilon_z, $$

and

$$ W_{\text{int}} = \frac{3Gm^2}{R}(1 - \frac{4}{45}\varepsilon_z^2 - \frac{1}{15}\varepsilon_\rho^2). $$

The variation of the binding energy then is given by

$$ \dot{W}_{\text{int}} = \frac{3Gm^2}{R}(\frac{8}{45}\varepsilon_z\dot{\varepsilon}_z + \frac{2}{15}\varepsilon_\rho\dot{\varepsilon}_\rho). $$

Eqn. (52) agrees with the result of the direct integration presented at the beginning of this section after which the difference between the binding gravitational energy of an ellipsoid and that of the corresponding sphere is of the order of the square of the flattenings $\varepsilon_\rho, \varepsilon_z$.

It is worth mentioning that the alternative formulation due to Neutsch (1979), does not agree neither with the results obtained by the direct integration nor with those of Essén (2004).

C Short-period librations

The creep tide theory predicts a forced libration due to tides in the case of stiff bodies which may be studied with the differential equation for $\dot{\nu}$:

$$ \dot{\nu} = -\frac{3GMm\sigma}{a^3} \sum_{k\in\mathbb{Z}} \sum_{j+k\in\mathbb{Z}} E_{2,k}E_{2,k+j} \cos\sigma_k \sin(j\ell + \sigma_k) $$

(cf. Ferraz-Mello, 2015. Eq. 36). The solutions of this equation tend to a periodic attractor which may be described using the approximate expression

$$ \nu = \gamma(A_0 + A_1 \cos \ell + A_2 \sin \ell) $$
Fig. 7 Variation of the semi-diurnal frequency of an Enceladus-like satellite over 3 periods, for \( \gamma = 1 \times 10^{-6} \) s\(^{-1} \). *Solid red line:* Variation of \( \nu/\gamma \) as given by Eqn. 55. *Dashed black line:* Variation of \( \nu/\gamma \) as given by the numerical integration of the differential equation. After the initial transitory of the solution obtained by integration, the two curves are almost indistinguishable.

where

\[
A_1 = Ce(\alpha - np) \\
A_2 = Ce(n + \alpha p) \\
A_0 = 12e^2 \frac{p}{(1 + p^2)} - (q_1 A_1 + r_1 A_2)e,
\]

and

\[
p = \frac{n}{\gamma}, \quad \alpha = 3np\overline{e}_p, \\
q_1 = \frac{3(2 + p^2 + p^4)}{2(1 + p^2)^2}, \quad r_1 = \frac{3p}{(1 + p^2)^2}, \\
C = \frac{4p\alpha}{(1 + p^2)(\alpha^2 + n^2)}.
\]

(cf Folonier and Ferraz-Mello, 2017, Online supplement, sec. B) We emphasize that the approximations were done only with respect to \( \nu \) and \( e \) (no approximations w.r.t \( p \) were done.)

D The average of the potential energy \( M \frac{dU}{dt} \)

In the adopted Darwinian model of Sec. 2, \( \delta U(r, r^*) \) is the disturbing potential due to the deformation of the mass \( (m) \) acting on a generic point of coordinates \( (r, \varphi, \theta) \) where the body \( M \) is placed. Thus, actually \( M \) is moving in a time-dependent potential field and the energy of the system is

\[
E_{\text{orb}} = W + M\delta U(r, r^*)
\]

whose time variation is

\[
\dot{E}_{\text{orb}} = \dot{W} + M\frac{d\delta U}{dt} = \dot{W} + Mn \frac{\partial \delta U}{\partial \ell} + Mn \frac{\partial \delta U}{\partial \ell^*} + \sum_{k \in \mathbb{Z}} M \frac{\partial \delta U}{\partial \sigma_k} \frac{d\sigma_k}{dt}
\]

or

\[
\dot{E}_{\text{orb}} = \dot{W}^* + \dot{W}_{\sigma}
\]
Fig. 8 Exact averages of $-\dot{W}$ (black dashed line), $-\dot{W}^*$ (blue dashed line), $-\dot{W}_\sigma$ (red line) and of the total $-M^m$ (green line).

where making $\ell^* = \ell$ in the results,

$$\dot{W}^* = M n \frac{\partial s U}{\partial \ell^*} = -\frac{GmMnR^2}{5a^3} \sum_{k \in \mathbb{Z}} \sum_{k+j \in \mathbb{Z}} \left(3(k - 2)C_k \cos \sigma_k E_{2,k} \sin (j\ell + \sigma_k) \right.$$

$$- kC'_{k} \cos \sigma_k E_{0,k+1} \sin (j\ell + \sigma'_k) \left. \right)$$

(59)

and

$$\dot{W}_\sigma = \frac{3GmMR^2}{5a^3} \sum_{k \in \mathbb{Z}} \sum_{k+j \in \mathbb{Z}} C_k E_{2,k} \sin(j\ell + 2\sigma_k) \frac{d\sigma_k}{dt}.$$  (60)

The term $\dot{W}$ is canceled by an opposite term $-\dot{W}$ coming from the derivative with respect to $\ell$. The variations of the slow variables $a, n$ are not considered in the differentiation because they do not give significant contributions at the order considered.

This definition of the orbital energy variation is the same as adopted by Correia et al (2014). Fig 8 shows that the result of it and that obtained in Sec. 4.2 almost coincide. This allows us to conclude that the internal processes responsible for the deformation of the body do not add any significant secular variation in the energy and is compatible with the secular variation of the semi-major axis of the orbit.

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Corrections to Paper II (Ferraz-Mello, 2015)

1. Typo in equation (2). The right definition is \( \varepsilon_z = 1 - \frac{\sigma}{R_c} \).
2. Typo in Eqn. (31). The argument should be \( k\ell - \sigma''_k \).
3. One radial term is missing in Eqn. (11). The solution including this correction is discussed in the Appendix A in this paper. The changes in the potential are of second order.
4. Mistake in Eqn. (61). In the last line the arguments should be \( v + k\ell - \sigma''_k \) and \( v - k\ell + \sigma''_k \).
5. Mistake in Eqns. (62-68) The sign in front of the zonal part is wrong. The sign in front of \( C_p' \) in Eqns. (62-63) should be +, the sign in front of \( kE_{0,k}^2 \) in Eqns. (64-66) should be – and the sign in in front of \( k\sigma E_{0,k}^2 \) in Eqn. (68) should be +.
6. Typo in Eqn. (69). The sign in front of the right-hand side should be changed to –.
7. Mistake in Eqns. (70-71) The sign in the beginning of the second line should be changed to –.
8. Mistake in Eqn. (B.6) (Online supplement) The sign in front of \( 2\sqrt{1 - e^2E_{2,k}^{(5)}} \) should be changed to –.

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