On the minimal normal compactification of a polynomial in two variables

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1 Introduction

Let $C$ be an integral affine curve over a field $\kappa$, $\alpha, \beta : C \hookrightarrow \mathbb{A}^2$ two closed embeddings. We say that $\alpha$ and $\beta$ are equivalent when there is an automorphism $\phi$ of $\mathbb{A}^2$ with $\phi \circ \alpha = \beta$. It was stated by B. Segre ([Se]) and proved by Suzuki in [Su] that when $\kappa = \mathbb{C}$ any embedding of $\mathbb{A}^1$ into $\mathbb{A}^2$ is equivalent to the standard embedding $t \mapsto (t, 0)$. This was generalized to the case that $\kappa$ is arbitrary, and the degree of $f$ is prime to the characteristic of $\kappa$, by Abhyankar and Moh ([AM]). This is what is usually called the Abhyankar–Moh theorem. On the other hand, there are many affine curves with an infinite number of non equivalent embeddings into $\mathbb{A}^2$: for example, $\mathbb{A}^1 \setminus \{0\}$.

Suzuki in [Su] also proves a very nice result: if $C$ is smooth and has only one branch at infinity (that is, it is the complement of a point in a smooth projective curve) and $f$ is a generator of the ideal of $C$ in $\mathbb{A}^2$, then $C$ is an ordinary fiber of $f$, that is, $f$ is a topological fibration in a neighborhood of 0. In their important, and arduous, article [AS] Abhyankar and Singh carry the study of this case much further, over arbitrary fields; in particular, for example, such a curve $C$ has at most finitely many nonequivalent embeddings, with appropriate conditions on the characteristic of the field.

Another proof of Suzuki’s theorem was given by Artal Bartolo, in [AB], based on the results of [EN], relating knot theory with the theory of polynomials in two variables.

Now, let $f : \mathbb{A}^2 \to \mathbb{A}^1$ be a polynomial in two variables defined over an algebraically closed field $\kappa$. We shall always assume that $f$ is primitive, that is, that the generic fiber of $f$ is integral. We consider the minimal normal compactification $\overline{f} : X \to \mathbb{A}^1$ of $f$, namely the only normal irreducible surface $X$ containing $\mathbb{A}^2$ as an open subset, together with a proper morphism $\overline{f} : X \to \mathbb{A}^1$ extending $f$, with the property that each fiber of $\overline{f}$ is dense in the corresponding fiber of $f$. It is often singular.

Let $E_1, \ldots, E_r$ be the horizontal components of $X \setminus \mathbb{A}^2$, namely the irreducible components of $X \setminus \mathbb{A}^2$ that dominate $\mathbb{A}^1$. By standard results, $E_1, \ldots, E_r$ are isomorphic to $\mathbb{A}^1$ and do not intersect (Proposition 1). To each $E_i$ we associate two integers. The first is the degree $e_i$ of $E_i$ over $\mathbb{A}^1$; one can think of $e_1, \ldots, e_r$ as the orders of the orbits of the monodromy group acting on the branches at infinity of a general fiber of $f$. The second is the least positive integer $\delta_i$ such that $\delta_i E_i$ is a Cartier divisor on $X$; since $E_i$ is smooth, we have that $\delta_i = 1$ if and only if $X$ has no singularities along $E_i$.

Our result (Theorem 1) says that if the characteristic of $\kappa$ is 0, the greatest common divisor of $\delta_1 e_1, \ldots, \delta_r e_r$ is 1. In particular, if there only one component $E_1$, this maps isomorphically onto $\mathbb{A}^1$, and $\overline{f}$ is smooth along $E_1$. So, if one of the fibers of $f$ has only one branch at infinity, then there is a simultaneous resolution of singularities at infinity of $f$. This easily implies the Suzuki–Abhyankar–Moh embedding theorem.

One can show that the integers $e_i \delta_i$ coincide with the integers $m_i$ defined by Eisenbud and Neumann (see [AB], p. 102). So in characteristic 0 our result follows from [EN], section 4, although our proof is shorter.

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In characteristic $p$ we only get that the greatest common divisor of $\delta_1 e_1, \ldots, \delta_r e_r$ is $1$ when the degree of the polynomial is prime to $p$ (see Theorem 2). This implies the Suzuki–Abhyankar–Moh embedding theorem over a perfect field.

The proof of the Theorem 1 is entirely straightforward, and very short; it uses standard topological methods, plus some elementary facts about rational surface singularities. If one substitutes ordinary topological cohomology with étale cohomology with $\mathbb{Z}_\ell$ coefficients, where $\ell$ is a prime different from the characteristic of $\kappa$, one gets a proof of Theorem 2. We do not include the proof of this general case, but anyone who is familiar with étale cohomology will be able to reconstruct the details.

2 Acknowledgments

I am grateful to Pierrette Cassou-Noguès for several helpful comments about the history of the results I quote. In particular she pointed out reference [AB] to me.

3 The results

Consider a complex polynomial in two variables, i.e., a morphism $f: \mathbb{A}^2 \to \mathbb{A}^1$ defined over $\mathbb{C}$. We shall always assume that $f$ is primitive, that is, that $f$ is not constant, and not obtained by composition with a polynomial in one variable of degree greater than $1$. This the same as saying that the generic fiber of $f$ is integral, or that the subfield $\mathbb{C}(f)$ is algebraically closed in $\mathbb{C}(x, y)$.

We consider the minimal normal compactification $\overline{f}: X \to \mathbb{A}^1$ of $f$, obtained by taking the closure $\Gamma$ of the graph of $f$ in $\mathbb{P}^2 \times \mathbb{P}^1$, considering its normalization $X'$, and then calling $X$ the inverse image of $\mathbb{A}^1$ in $X'$. Then $X$ is a normal integral complex quasiprojective scheme over $\kappa$, containing $\mathbb{A}^2$ as an open subscheme. Furthermore the morphism $f$ extends to a morphism $\overline{f}_*: X \to \mathbb{A}^1$, which has the useful property that every fiber of $f$ is dense inside the corresponding fiber of $\overline{f}$. Let us call $E_1, \ldots, E_r$ the irreducible components of the complement $E$ of $\mathbb{A}^2$ in $X$. Each of the $E_1, \ldots, E_r$ is an affine integral curve dominating $\mathbb{A}^1$: we will call $e_1, \ldots, e_r$ the degrees of $E_1, \ldots, E_r$ over $\mathbb{A}^1$.

Furthermore, the divisor class groups of the local rings of $X$ are finite, because $X$ has rational singularities ([Li], Proposition 17.1.) We will call $\delta_i$ the least common multiple of the orders of $E_i$ in each of the divisor class groups of the local rings of $X$ at points of $E_i$; clearly $\delta_i E_i$ is a Cartier divisor on $X$, while $\delta E_i$ is not a Cartier divisor for any integer $\delta$ with $0 < \delta < \delta_i$.

**Theorem 1.** Each of the $E_1, \ldots, E_r$ is isomorphic to $\mathbb{A}^1$, and they are pairwise disjoint.

Furthermore the greatest common divisor of the products $\delta_i e_1, \ldots, \delta_r e_r$ is $1$.

The first statement in the theorem is quite standard. It has an important consequence; if $\delta_i$ is $1$, that is, if $E_i$ is a Cartier divisor on $X$, then $X$ is smooth at all point of $E_i$.

**Corollary 1.** Assume that $X$ has only one component at infinity. Then all the fibers of $\overline{f}$ are integral and smooth at infinity.

In particular, this happens when one of the fibers of $f$ has only one branch at infinity.

This follows immediately from the theorem, because the hypothesis implies that $\delta_1 = 1$, i.e., $X$ is smooth, and $e_1 = 1$, i.e., $E_1$ maps isomorphically onto $\mathbb{A}^1$.

From the corollary we get a new proof of the renowned Suzuki–Abhyankar–Moh theorem. For this we only need to assume that $\kappa$ is perfect.

**The Suzuki–Abhyankar–Moh theorem over $\mathbb{C}$.** Any embedding of $\mathbb{A}^1$ into $\mathbb{A}^2$ defined over $\mathbb{C}$ is equivalent to the standard embedding $t \mapsto (t, 0)$.

**Proof.** Let $C$ be a curve in $\mathbb{A}^2$ isomorphic to $\mathbb{A}^1$, $f \in \mathbb{C}[x, y]$ a generator of the ideal of $C$. Because of the corollary, each geometric fiber of $\overline{f}$ is isomorphic to $\mathbb{P}^1$, so $X$ is a $\mathbb{P}^1$-bundle on $\mathbb{A}^1$. If we call $E$ the complement of $\mathbb{A}^2$ in $X$, with its reduced scheme structure, then the projection from $E$ onto $\mathbb{A}^1$ is isomorphic. Hence there is an isomorphism $\phi$ of $\mathbb{P}^1 \times \mathbb{A}^1$ with $X$ carrying $\mathbb{A}^1 \times \infty$ into $E$, and such that $\overline{f} \circ \phi: \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the second projection. The restriction of $\phi$ to $\mathbb{A}^2$ carries the line with equation $y = 0$ into $C$, and this proves the theorem. \[\Box\]
This can be made to work in positive characteristic. Let us fix an algebraically closed field $\kappa$, and call $p$ be the characteristic exponent of $\kappa$, namely the characteristic of $\kappa$ if this is positive, and 1 otherwise.

Consider a primitive polynomial in two variables, i.e., a morphism $f: \mathbb{A}^2 \to \mathbb{A}^1$ defined over $\kappa$ with integral general fiber; as before, $f$ has a minimal normal compactification $\overline{f}: X \to \mathbb{A}^1$. Define $E_1, \ldots, E_r$, $e_1, \ldots, e_r$, and $\delta_1, \ldots, \delta_r$ as before. Then we can not conclude that the $\delta_i e_i$ are relatively prime; however, we have the following.

**Theorem 2.** Each of the $E_1, \ldots, E_r$ is isomorphic to $\mathbb{A}^1$, and they are pairwise disjoint.

Furthermore the greatest common divisor of the products $\delta_1 e_1, \ldots, \delta_r e_r$ is a power of $p$ and divides the degree of $f$.

We still get the corollary, in the following form.

**Corollary 2.** Assume that the degree of $f$ is prime to the $p$, and that $X$ has only one component at infinity. Then all the fibers of $\overline{f}$ are integral and smooth at infinity.

In particular, this happens when one of the fibers of $f$ has only one branch at infinity.

Remarkably, using a different technique one can prove that $X$ is smooth when it has only one component at infinity, without assuming that the degree of $f$ is prime to $p$. Unfortunately, I do not have any interesting application of this.

From Corollary 2 we get a new proof of the Suzuki–Abhyankar–Moh theorem over any perfect field.

**The Suzuki–Abhyankar–Moh theorem over a perfect field.** Any embedding of $\mathbb{A}^1$ into $\mathbb{A}^2$ defined over a perfect field, whose degree is relatively prime to the characteristic of $\kappa$ is equivalent to the standard embedding $t \mapsto (t, 0)$.

**Proof of Theorem 1.** Recall that $\Gamma$ is the closure of the graph of $f$ in $\mathbb{P}^2 \times \mathbb{P}^1$, $X'$ its normalization, $f'$ and $\pi$ the projections of $X'$ onto $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively. Let $L = \mathbb{P}^2 \setminus \mathbb{A}^2$ be the line at infinity, $L'$ its proper transform in $X'$.

Let $E_i'$ be the closure of $E_i$ in $X'$: the first statement of the theorem is a consequence of the following fact.

**Lemma 1.** The curves $L'$ and $E_i'$, for each $i = 1, \ldots, r$, are isomorphic to $\mathbb{P}^1$, and any two of them do not intersect in more than one point. Furthermore, if $E_i'$ and $E_j'$, with $i \neq j$, intersect in a closed point $p \in X'$, then $p \in L'$.

Assuming Lemma 1, and keeping in mind that that $L'$ is the fiber of $f'$ over the point at infinity $\infty \in \mathbb{P}^1(\kappa)$, we see that each of the $E_i'$ can have only one point over $\infty$, and therefore the inverse image $E_i$ of $A^1$ in $E_i'$ is isomorphic to $\mathbb{A}^1$. Also from Lemma 1 we get that the $E_i$ do not intersect.

**Proof.** The natural morphism $\pi: X' \to \mathbb{P}^2$ is birational and $\mathbb{P}^2$ is smooth, so $R^1 \pi_* \mathcal{O}_{X'} = 0$. Let $\tilde{L} = \pi^{-1}(L)_{\text{red}}$. Since $\mathcal{O}_{\tilde{L}}$ is a quotient of $\mathcal{O}_{X'}$, so $R^1 \pi_* \mathcal{O}_{\tilde{L}} = 0$. We have $\pi_* \mathcal{O}_{\tilde{L}} = \mathcal{O}_{L}$, so from the Leray spectral sequence

$$E_2^{ij} = H^i(\mathbb{P}^1, R^j \pi_* \mathcal{O}_{\tilde{L}}) \implies H^{i+j}(\tilde{L}, \mathcal{O})$$

we get that $H^1(\tilde{L}, \mathcal{O}) = 0$. If $Z$ is subscheme of $\tilde{L}$, the sheaf $\mathcal{O}_Z$ is a quotient of $\mathcal{O}_{\pi^{-1}(L)}$, and if $I$ is the ideal of $Z$ in $\pi^{-1}(L)$ we have $H^2(\pi^{-1}(L), I) = 0$, hence $H^1(Z, \mathcal{O}) = 0$. This in particular applies to any of the curves $L'$ and $E_i'$. Any integral projective curve with arithmetic genus 0 is isomorphic to $\mathbb{P}^1$.

Also, if $C_1$ and $C_2$ are two of these curve, from the fact that $H^1(C_1 \cup C_2, \mathcal{O}) = 0$ we see that $C_1$ and $C_2$ have at most one common point. Analogously, the fact that $H^1(L' \cup E_i' \cup E_j', \mathcal{O}) = 0$ implies that $E_i'$ and $E_j'$ cannot meet outside of $L'$, because $L'$ meets both $E_i'$ and $E_j'$.

Now consider the group $\text{Pic} X$ of Cartier divisors on $X$, and the natural map $\text{Pic} X \to \text{Cl} X$ into the group of Weil divisors. Since $\mathbb{A}^2$ is factorial, and all of its invertible regular functions are constant, it follows that $\text{Cl} X$ is a free abelian group with basis $E_1, \ldots, E_r$. Since the map $\text{Pic} X \to \text{Cl} X$ is injective, because $X$ is normal, this proves the following.

**Lemma 2.** The group $\text{Pic} X$ is free, with basis $\delta_1 E_1, \ldots, \delta_r E_r$.

The fact that the $\delta_i e_i$ are relatively prime is easily proved, after having established the following two facts.
Lemma 3. The first Chern class map $\text{Pic} X \to \mathbb{H}^2(X, \mathbb{Z})$ is an isomorphism.

Lemma 4. Let $C$ be a general fiber of $\overline{f}$. The restriction map $\mathbb{H}^2(X, \mathbb{Z}) \to \mathbb{H}^2(C, \mathbb{Z})$ is surjective.

Proof of Lemma 4. Let $\rho: \tilde{X} \to X$ be a resolution of the singularities of $X$, $F_1, \ldots, F_s$ the exceptional divisors, $E_i$ the proper transforms of the $E_i$. Then the complement of the $F_j$ and the $E_i$ in $X$ is $\mathbb{A}^2$; therefore the Picard group of $\tilde{X}$ is freely generated by the $F_j$ and the $E_i$. Likewise, $\mathbb{H}^2(\tilde{X}, \mathbb{Z})$ is freely generated by the cohomology classes of the $E_i$ and $F_j$; so the first Chern class map $\text{Pic} \tilde{X} \to \mathbb{H}^2(\tilde{X}, \mathbb{Z})$ is an isomorphism.

The pullback map $\text{Pic} X \to \text{Pic} \tilde{X}$ is clearly injective, and a divisor class in $\text{Pic} \tilde{X}$ is in the image of $\text{Pic} X$ if and only if its restriction to each of the $F_j$ has degree 0. The reason is that $X$ has rational singularities ([Li], Theorem 12.1)

Now take cohomology. We have that $R^i \rho_* \mathbb{Z}_{\overline{X}} = 0$, while $\rho_* \mathbb{Z}_{\overline{X}} = \mathbb{Z}$, and $R^2 \rho_* \mathbb{Z}_{\overline{X}}$ is a sheaf concentrated in the singular points of $X$, whose stalk over $p \in X$ is a direct sum of one copy of $\mathbb{Z}$ for each exceptional divisor over $p$. By considering the Leray spectral sequence of the map $\rho: \tilde{X} \to X$, one deduces that the restriction map $\mathbb{H}^2(X, \mathbb{Z}) \to \mathbb{H}^2(\tilde{X}, \mathbb{Z})$ is injective, and its image consists exactly of the classes in $\mathbb{H}^2(\tilde{X}, \mathbb{Z})$ which have degree 0 on each $F_j$.

By putting these two statements together, we see that $\text{Pic} X$ and $\mathbb{H}^2(X, \mathbb{Z})$ are identified with two subgroups of $\text{Pic} \tilde{X}$ and $\mathbb{H}^2(\tilde{X}, \mathbb{Z})$ which correspond under the isomorphism $\text{Pic} \tilde{X} \to \mathbb{H}^2(\tilde{X}, \mathbb{Z})$ given by the first Chern class. This proves Lemma 3.

Proof of Lemma 4. Consider the Leray spectral sequence

$$E_2^{i,j} = H^i(\mathbb{A}^1, R^j \overline{f}_* \mathbb{Z}_X) \implies H^{i+j}(X, \mathbb{Z});$$

since $H^2(\mathbb{A}^1, R^1 \overline{f}_* \mathbb{Z}_X) = 0$, because $\mathbb{A}^1$ is an affine curve and $R^1 \overline{f}_* \mathbb{Z}_X$ a constructible sheaf, we get that the map

$$H^2(X, \mathbb{Z}) \to H^0(\mathbb{A}^1, R^2 \overline{f}_* \mathbb{Z}_X)$$

is surjective. Now take the trace map

$$\text{tr} : R^2 \overline{f}_* \mathbb{Z}_X \to \mathbb{Z}_{\mathbb{A}^1}.$$

Because the general fiber of $\overline{f}$ is integral, the trace map is generically an isomorphism. Let $F = \overline{f}^{-1}(t)$ be a fiber of $\overline{f}$ over a closed point $t \in \mathbb{A}^1(\kappa)$, $F_1, \ldots, F_s$ the irreducible components of $F$, $m_1, \ldots, m_s$ the lengths of the local rings of $F$ at $F_1, \ldots, F_s$. By proper base change the stalk $(R^2 \overline{f}_* \mathbb{Z}_X)_t$ is canonically isomorphic to

$$H^2(F, \mathbb{Z}) \simeq \bigoplus_{i=1}^s H^2(F_i, \mathbb{Z}) \simeq \mathbb{Z}^s;$$

with this identification, the trace map on the stalks over $t \in \mathbb{A}^1(\kappa)$ is identified with the map from $\mathbb{Z}^s$ to $\mathbb{Z}$ that sends $(k_1, \ldots, k_s)$ to $k_1 m_1 + \cdots + k_s m_s$. But $m_1, \ldots, m_s$ are relatively prime, because $f: S \to \mathbb{A}^1$ does not have multiple fibers, so the trace map is surjective, and its kernel is concentrated on a finite number of points. By taking global sections we see that the global trace map

$$\text{tr} : H^2(X, \mathbb{Z}) \to \mathbb{Z}$$

is surjective. But $\text{tr} : H^2(X, \mathbb{Z}) \to \mathbb{Z}$ coincides with the restriction map $H^2(X, \mathbb{Z}) \to H^2(C, \mathbb{Z}) \simeq \mathbb{Z}$. Hence this restriction map is surjective. This proves the lemma, and hence the theorem.

Note. From the spectral sequence of the map $X \to \mathbb{A}^1$ one deduces that $H^3(X, \mathbb{Z}) = 0$; furthermore, from the spectral sequence of a resolution $\tilde{X} \to X$ one sees that the restriction map $H^2(\tilde{X}, \mathbb{Z}) \to H^2(F, \mathbb{Z})$ is surjective. From this one can deduce that the class of $E_i$ generates the product $\prod_{p \in E_i} \text{Cl} \tilde{O}_{X,p}$; this means that $\delta_i$ can also be defined as the product of the orders of the group $\text{Cl} \tilde{O}_{X,p}$ for $p \in E_i$.

To prove Theorem 2 one follows the steps in the proof of Theorem 1, substituting étale cohomology with $\mathbb{Z}_l$ coefficients to classical cohomology, where $l$ is a prime different from the characteristic of $\kappa$; in this
way one shows that ℓ does not divide the greatest common divisor of the $e_i \delta_i$. We leave the details to the interested reader. The only thing that does not follow is that the greatest common divisor of the $\delta_i e_i$ divides the degree $d$ of $f$.

To show this, call $C$ the closure in $\mathbf{P}^2$ of a general fiber of $f$, $C'$ the proper transform of $C$ in $X'$. Then $C'$ is a general fiber of $f'$, hence it is a Cartier divisor on $X'$; the intersection number $(C' \cdot L')$ is 0, and $(C' \cdot E'_i) = e_i$ for each $i = 1, \ldots, r$. We have a decomposition of $\pi^*(L)$ as a Weil divisor

$$\pi^*[L] = [L'] + \sum_{i=1}^r m_i E_i$$

for certain positive integers $m_1, \ldots, m_r$. Since the restriction of the $m_i E_i$ to $X$ must be a Cartier divisor, we see that $\delta_i$ divides $m_i$, so we write

$$\pi^*[L] = [L'] + \sum_{i=1}^r n_i \delta_i E_i.$$ 

But

$$d = (C \cdot L) = (C' \cdot \pi^*[L]) = (C' \cdot L') + \sum_{i=1}^r n_i \delta_i (C' \cdot E'_i) = \sum_{i=1}^r n_i \delta_i e_i,$$

by the projection formula, and this completes the proof.

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