Section 5.2

Probability Distributions
Section 5.2 Objectives

- Distinguish between discrete random variables and continuous random variables
- Construct a discrete probability distribution and its graph
- Determine if a distribution is a probability distribution
- Find the mean, variance, and standard deviation of a discrete probability distribution
- Find the expected value of a discrete probability distribution
Random Variables

Random Variable

- Represents a numerical value associated with each outcome of a probability distribution.
- Denoted by $x$
- Examples
  - $x = \text{Number of sales calls a salesperson makes in one day.}$
  - $x = \text{Hours spent on sales calls in one day.}$
Random Variables

Discrete Random Variable

• Has a finite or countable number of possible outcomes that can be listed.

• Example
  - $x = \text{Number of sales calls a salesperson makes in one day.}$
Random Variables

Continuous Random Variable

• Has an uncountable number of possible outcomes, represented by an interval on the number line.

• Example
  - \( x = \) Hours spent on sales calls in one day.
Example: Random Variables

Decide whether the random variable \( x \) is discrete or continuous.

1. \( x = \) The number of Fortune 500 companies that lost money in the previous year.

Solution:

Discrete random variable (The number of companies that lost money in the previous year can be counted.)

\( \{0, 1, 2, 3, \ldots, 500\} \)
Example: Random Variables

Decide whether the random variable $x$ is discrete or continuous.

2. $x =$ The volume of gasoline in a 21-gallon tank.

Solution: Continuous random variable (The amount of gasoline in the tank can be any volume between 0 gallons and 21 gallons.)
Discrete Probability Distributions

Discrete probability distribution

• Lists each possible value the random variable can assume, together with its probability.
• Must satisfy the following conditions:

| In Words                                                                 | In Symbols            |
|--------------------------------------------------------------------------|-----------------------|
| 1. The probability of each value of the discrete random variable is between 0 and 1, inclusive. | $0 \leq P(x) \leq 1$ |
| 2. The sum of all the probabilities is 1.                                | $\sum P(x) = 1$       |
Constructing a Discrete Probability Distribution

Let \( x \) be a discrete random variable with possible outcomes \( x_1, x_2, \ldots, x_n \).

1. Make a frequency distribution for the possible outcomes.
2. Find the sum of the frequencies.
3. Find the probability of each possible outcome by dividing its frequency by the sum of the frequencies.
4. Check that each probability is between 0 and 1, inclusive, and that the sum of all probabilities is 1.
Example: Constructing a Discrete Probability Distribution

An industrial psychologist administered a personality inventory test for passive-aggressive traits to 150 employees. Individuals were given a score from 1 to 5, where 1 was extremely passive and 5 extremely aggressive. A score of 3 indicated neither trait. Construct a probability distribution for the random variable \( x \). Then graph the distribution using a histogram.

| Score, \( x \) | Frequency, \( f \) |
|----------------|-----------------|
| 1              | 24              |
| 2              | 33              |
| 3              | 42              |
| 4              | 30              |
| 5              | 21              |
Solution: Constructing a Discrete Probability Distribution

- Divide the frequency of each score by the total number of individuals in the study to find the probability for each value of the random variable.

\[
P(1) = \frac{24}{150} = 0.16 \quad P(2) = \frac{33}{150} = 0.22 \quad P(3) = \frac{42}{150} = 0.28
\]

\[
P(4) = \frac{30}{150} = 0.20 \quad P(5) = \frac{21}{150} = 0.14
\]

- Discrete probability distribution:

| $x$ | 1  | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|----|
| $P(x)$ | 0.16 | 0.22 | 0.28 | 0.20 | 0.14 |
This is a valid discrete probability distribution since

1. Each probability is between 0 and 1, inclusive, $0 \leq P(x) \leq 1$.

2. The sum of the probabilities equals 1, $\sum P(x) = 0.16 + 0.22 + 0.28 + 0.20 + 0.14 = 1$. 

| $x$ | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|
| $P(x)$ | 0.16 | 0.22 | 0.28 | 0.20 | 0.14 |
Solution: Constructing a Discrete Probability Distribution

• Histogram

Because the width of each bar is one, the area of each bar is equal to the probability of a particular outcome.
Mean of a discrete probability distribution

- \( \mu = \Sigma xP(x) \)
- Each value of \( x \) is multiplied by its corresponding probability and the products are added.
Example: Finding the Mean

The probability distribution for the personality inventory test for passive-aggressive traits is given. Find the mean score.

Solution:

| x  | P(x) |
|----|------|
| 1  | 0.16 |
| 2  | 0.22 |
| 3  | 0.28 |
| 4  | 0.20 |
| 5  | 0.14 |

\[ \mu = \sum xP(x) = 2.94 \]
Variance and Standard Deviation

Variance of a discrete probability distribution

\[ \sigma^2 = \sum (x - \mu)^2 P(x) \]

Standard deviation of a discrete probability distribution

\[ \sigma = \sqrt{\sigma^2} = \sqrt{\sum (x - \mu)^2 P(x)} \]
Example: Finding the Variance and Standard Deviation

The probability distribution for the personality inventory test for passive-aggressive traits is given. Find the variance and standard deviation. \( \mu = 2.94 \)

| x | \( P(x) \) |
|---|---|
| 1 | 0.16 |
| 2 | 0.22 |
| 3 | 0.28 |
| 4 | 0.20 |
| 5 | 0.14 |
Solution: Finding the Variance and Standard Deviation

Recall $\mu = 2.94$

| $x$ | $P(x)$ | $x - \mu$ |
|-----|--------|-----------|
| 1   | 0.16   | 1 – 2.94 = –1.94 |
| 2   | 0.22   | 2 – 2.94 = –0.94 |
| 3   | 0.28   | 3 – 2.94 = 0.06 |
| 4   | 0.20   | 4 – 2.94 = 1.06 |
| 5   | 0.14   | 5 – 2.94 = 2.06 |

Variances: $\sigma^2 = \Sigma(x - \mu)^2P(x) = 1.616$

Standard Deviation: $\sigma = \sqrt{\sigma^2} = \sqrt{1.616} \approx 1.3$
Expected Value

Expected value of a discrete random variable

- Equal to the mean of the random variable.
- \( E(x) = \mu = \sum xP(x) \)
Example: Finding an Expected Value

At a raffle, 1500 tickets are sold at $2 each for four prizes of $500, $250, $150, and $75. You buy one ticket. What is the expected value of your gain?
Solution: Finding an Expected Value

To find the gain for each prize, subtract the price of the ticket from the prize:

- Your gain for the $500 prize is $500 – $2 = $498
- Your gain for the $250 prize is $250 – $2 = $248
- Your gain for the $150 prize is $150 – $2 = $148
- Your gain for the $75 prize is $75 – $2 = $73

If you do not win a prize, your gain is $0 – $2 = –$2
Solution: Finding an Expected Value

• Probability distribution for the possible gains (outcomes)

| Gain, x | $498 | $248 | $148 | $73  | –$2  |
|---------|------|------|------|------|------|
| \( P(x) \) | \( \frac{1}{1500} \) | \( \frac{1}{1500} \) | \( \frac{1}{1500} \) | \( \frac{1}{1500} \) | \( \frac{1496}{1500} \) |

\[
E(x) = \sum xP(x)
\]

\[
= \$498 \cdot \frac{1}{1500} + \$248 \cdot \frac{1}{1500} + \$148 \cdot \frac{1}{1500} + \$73 \cdot \frac{1}{1500} + (-\$2) \cdot \frac{1496}{1500}
\]

\[
= -\$1.35
\]

You can expect to lose an average of $1.35 for each ticket you buy.
Section 5.2 Summary

- Distinguished between discrete random variables and continuous random variables
- Constructed a discrete probability distribution and its graph
- Determined if a distribution is a probability distribution
- Found the mean, variance, and standard deviation of a discrete probability distribution
- Found the expected value of a discrete probability distribution
Section 5.3 & 5.4 Objectives

• Determine if a probability experiment is a binomial experiment
• Find binomial probabilities using the binomial probability formula
• Find binomial probabilities using technology, formulas, and a binomial probability table
• Graph a binomial distribution
• Find the mean, variance, and standard deviation of a binomial probability distribution
Binomial Experiments

1. The experiment is repeated for a fixed number of trials, where each trial is independent of other trials.

2. There are only two possible outcomes of interest for each trial. The outcomes can be classified as a success ($S$) or as a failure ($F$).

3. The probability of a success $P(S)$ is the same for each trial.

4. The random variable $x$ counts the number of successful trials.
# Notation for Binomial Experiments

| Symbol | Description |
|--------|-------------|
| $n$    | The number of times a trial is repeated |
| $p = P(S)$ | The probability of success in a single trial |
| $q = P(F)$ | The probability of failure in a single trial ($q = 1 - p$) |
| $x$    | The random variable represents a count of the number of successes in $n$ trials: $x = 0, 1, 2, 3, \ldots, n$. |
Example: Binomial Experiments

Decide whether the experiment is a binomial experiment. If it is, specify the values of $n$, $p$, and $q$, and list the possible values of the random variable $x$.

1. A certain surgical procedure has an 85% chance of success. A doctor performs the procedure on eight patients. The random variable represents the number of successful surgeries.
Solution: Binomial Experiments

Binomial Experiment

1. Each surgery represents a trial. There are eight surgeries, and each one is independent of the others.

2. There are only two possible outcomes of interest for each surgery: a success ($S$) or a failure ($F$).

3. The probability of a success, $P(S)$, is 0.85 for each surgery.

4. The random variable $x$ counts the number of successful surgeries.
Solution: Binomial Experiments

Binomial Experiment

- $n = 8$ (number of trials)
- $p = 0.85$ (probability of success)
- $q = 1 - p = 1 - 0.85 = 0.15$ (probability of failure)
- $x = 0, 1, 2, 3, 4, 5, 6, 7, 8$ (number of successful surgeries)
Example: Binomial Experiments

Decide whether the experiment is a binomial experiment. If it is, specify the values of $n$, $p$, and $q$, and list the possible values of the random variable $x$.

2. A jar contains five red marbles, nine blue marbles, and six green marbles. You randomly select three marbles from the jar, without replacement. The random variable represents the number of red marbles.
Solution: Binomial Experiments

Not a Binomial Experiment

• The probability of selecting a red marble on the first trial is 5/20.

• Because the marble is not replaced, the probability of success (red) for subsequent trials is no longer 5/20.

• The trials are not independent and the probability of a success is not the same for each trial.
Binomial Probability Formula

The probability of exactly \( x \) successes in \( n \) trials is

\[
P(x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{(n-x)!x!} p^x q^{n-x}
\]

- \( n \) = number of trials
- \( p \) = probability of success
- \( q = 1 - p \) probability of failure
- \( x \) = number of successes in \( n \) trials
Example: Finding Binomial Probabilities

Microfracture knee surgery has a 75% chance of success on patients with degenerative knees. The surgery is performed on three patients. Find the probability of the surgery being successful on exactly two patients.
Solution: Finding Binomial Probabilities

Method 1: Draw a tree diagram and use the Multiplication Rule

\[ P(2 \text{ successful surgeries}) = \binom{3}{2} \left( \frac{9}{64} \right) \approx 0.422 \]
Solution: Finding Binomial Probabilities

Method 2: Binomial Probability Formula

\[ n = 3, \quad p = \frac{3}{4}, \quad q = 1 - p = \frac{1}{4}, \quad x = 2 \]

\[ P(2 \text{ successful surgeries}) = \binom{3}{2} \left( \frac{3}{4} \right)^2 \left( \frac{1}{4} \right)^{3-2} \]

\[ = \frac{3!}{(3-2)!2!} \left( \frac{3}{4} \right)^2 \left( \frac{1}{4} \right)^1 \]

\[ = 3 \left( \frac{9}{16} \right) \left( \frac{1}{4} \right) = \frac{27}{64} \approx 0.422 \]
Binomial Probability Distribution

• List the possible values of $x$ with the corresponding probability of each.

• Example: Binomial probability distribution for Microfracture knee surgery: $n = 3, p = \frac{3}{4}$

| $x$ | 0   | 1   | 2   | 3   |
|-----|-----|-----|-----|-----|
| $P(x)$ | 0.016 | 0.141 | 0.422 | 0.422 |

- Use the binomial probability formula to find probabilities.
Example: Constructing a Binomial Distribution

In a survey, U.S. adults were asked to give reasons why they liked texting on their cellular phones. Seven adults who participated in the survey are randomly selected and asked whether they like texting because it is quicker than calling. Create a binomial probability distribution for the number of adults who respond yes.
Solution: Constructing a Binomial Distribution

- 56% of adults like texting because it is quicker than calling.
- \( n = 7, \ p = 0.56, \ q = 0.44, \ x = 0, 1, 2, 3, 4, 5, 6, 7 \)

\[
P(0) = \binom{7}{0}(0.56)^0(0.44)^7 = 1(0.56)^0(0.44)^7 \approx 0.0032
\]

\[
P(1) = \binom{7}{1}(0.56)^1(0.44)^6 = 7(0.56)^1(0.44)^6 \approx 0.0284
\]

\[
P(2) = \binom{7}{2}(0.56)^2(0.44)^5 = 21(0.56)^2(0.44)^5 \approx 0.1086
\]

\[
P(3) = \binom{7}{3}(0.56)^3(0.44)^4 = 35(0.56)^3(0.44)^4 \approx 0.2304
\]

\[
P(4) = \binom{7}{4}(0.56)^4(0.44)^3 = 35(0.56)^4(0.44)^3 \approx 0.2932
\]

\[
P(5) = \binom{7}{5}(0.56)^5(0.44)^2 = 21(0.56)^5(0.44)^2 \approx 0.2239
\]

\[
P(6) = \binom{7}{6}(0.56)^6(0.44)^1 = 7(0.56)^6(0.44)^1 \approx 0.0950
\]

\[
P(7) = \binom{7}{7}(0.56)^7(0.44)^0 = 1(0.56)^7(0.44)^0 \approx 0.0173
\]
Solution: Constructing a Binomial Distribution

| $x$ | $P(x)$ |
|-----|--------|
| 0   | 0.0032 |
| 1   | 0.0284 |
| 2   | 0.1086 |
| 3   | 0.2304 |
| 4   | 0.2932 |
| 5   | 0.2239 |
| 6   | 0.0950 |
| 7   | 0.0173 |

All of the probabilities are between 0 and 1 and the sum of the probabilities is 1.
Example: Finding Binomial Probabilities Using Technology

The results of a recent survey indicate that 67% of U.S. adults consider air conditioning a necessity. If you randomly select 100 adults, what is the probability that exactly 75 adults consider air conditioning a necessity? Use a technology tool to find the probability. \((Source: \textit{Opinion Research Corporation})\)

Solution:

- Binomial with \(n = 100, \ p = 0.67, \ x = 75\)
Solution: Finding Binomial Probabilities Using Technology

**MINITAB**

Probability Distribution Function

Binomial with n = 100 and p = 0.67

| x | P(X=x) |
|---|---|
| 75 | 0.0201004 |

**TI-83/84 PLUS**

```
binompdf(100, 0.67, 75)
```

```
0.0201004116
```

**EXCEL**

|   | A               | B               | C         | D          |
|---|-----------------|-----------------|-----------|------------|
| 1 | BINOMDIST(75, 100, 0.67, FALSE) |                |           |            |
| 2 |                 |                 |           | 0.020100412 |

From the displays, you can see that the probability that exactly 75 adults consider air conditioning a necessity is about 0.02.
Example: Finding Binomial Probabilities

A survey indicates that 41% of women in the U.S. consider reading their favorite leisure-time activity. You randomly select four U.S. women and ask them if reading is their favorite leisure-time activity. Find the probability that at least two of them respond yes.

Solution:

• $n = 4$, $p = 0.41$, $q = 0.59$
• At least two means two or more.
• Find the sum of $P(2)$, $P(3)$, and $P(4)$.
Solution: Finding Binomial Probabilities

\[ P(2) = \binom{4}{2}(0.41)^2(0.59)^2 = 6(0.41)^2(0.59)^2 \approx 0.351094 \]
\[ P(3) = \binom{4}{3}(0.41)^3(0.59)^1 = 4(0.41)^3(0.59)^1 \approx 0.162654 \]
\[ P(4) = \binom{4}{4}(0.41)^4(0.59)^0 = 1(0.41)^4(0.59)^0 \approx 0.028258 \]

\[ P(x \geq 2) = P(2) + P(3) + P(4) \]
\[ \approx 0.351094 + 0.162654 + 0.028258 \]
\[ \approx 0.542 \]
**Example: Finding Binomial Probabilities Using a Table**

About ten percent of workers (16 years and over) in the United States commute to their jobs by carpooling. You randomly select eight workers. What is the probability that exactly four of them carpool to work? Use a table to find the probability. *(Source: American Community Survey)*

**Solution:**
- Binomial with $n = 8, p = 0.10, x = 4$
Solution: Finding Binomial Probabilities Using a Table

• A portion of Table 2 is shown

| n x | .01 | .05 | .10 | .15 | .20 | .25 | .30 | .35 | .40 | .45 | .50 | .55 | .60 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 2   | .990 | .902 | .810 | .723 | .640 | .563 | .490 | .423 | .360 | .303 | .250 | .203 | .160 |
| 1   | .020 | .095 | .180 | .255 | .320 | .375 | .420 | .455 | .480 | .495 | .500 | .495 | .480 |
| 2   | .000 | .002 | .010 | .023 | .040 | .063 | .090 | .123 | .160 | .203 | .250 | .303 | .360 |
| 3   | 0   | .970 | .857 | .729 | .614 | .512 | .422 | .343 | .275 | .216 | .166 | .125 | .091 | .064 |
| 1   | .029 | .135 | .243 | .325 | .384 | .422 | .441 | .444 | .432 | .408 | .375 | .334 | .288 |
| 2   | .000 | .007 | .027 | .057 | .096 | .141 | .189 | .239 | .288 | .334 | .375 | .408 | .432 |
| 3   | .000 | .000 | .001 | .003 | .008 | .016 | .027 | .043 | .064 | .091 | .125 | .166 | .216 |

The probability that exactly four of the eight workers carpool to work is 0.005.
Example: Graphing a Binomial Distribution

Sixty percent of households in the U.S. own a video game console. You randomly select six households and ask each if they own a video game console. Construct a probability distribution for the random variable $x$. Then graph the distribution. *(Source: Deloitte, LLP)*

**Solution:**

- $n = 6$, $p = 0.6$, $q = 0.4$
- Find the probability for each value of $x$
Solution: Graphing a Binomial Distribution

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|---|
| $P(x)$ | 0.004 | 0.037 | 0.138 | 0.276 | 0.311 | 0.187 | 0.047 |

Histogram:

Owning a Video Game Console
Mean, Variance, and Standard Deviation

- **Mean:** $\mu = np$

- **Variance:** $\sigma^2 = npq$

- **Standard Deviation:** $\sigma = \sqrt{npq}$
Example: Finding the Mean, Variance, and Standard Deviation

In Pittsburgh, Pennsylvania, about 56% of the days in a year are cloudy. Find the mean, variance, and standard deviation for the number of cloudy days during the month of June. Interpret the results and determine any unusual values. *(Source: National Climatic Data Center)*

**Solution:** \( n = 30, \ p = 0.56, \ q = 0.44 \)

**Mean:** \( \mu = np = 30 \cdot 0.56 = 16.8 \)

**Variance:** \( \sigma^2 = npq = 30 \cdot 0.56 \cdot 0.44 \approx 7.4 \)

**Standard Deviation:** \( \sigma = \sqrt{npq} = \sqrt{30 \cdot 0.56 \cdot 0.44} \approx 2.7 \)
Solution: Finding the Mean, Variance, and Standard Deviation

\[ \mu = 16.8 \quad \sigma^2 \approx 7.4 \quad \sigma \approx 2.7 \]

- On average, there are 16.8 cloudy days during the month of June.
- The standard deviation is about 2.7 days.
- Values that are more than two standard deviations from the mean are considered unusual.
  - \[ 16.8 - 2(2.7) = 11.4, \] a June with 11 cloudy days or fewer would be unusual.
  - \[ 16.8 + 2(2.7) = 22.2, \] a June with 23 cloudy days or more would also be unusual.
Section 4.2 Summary

- Determined if a probability experiment is a binomial experiment
- Found binomial probabilities using the binomial probability formula
- Found binomial probabilities using technology and a binomial table
- Graphed a binomial distribution
- Found the mean, variance, and standard deviation of a binomial probability distribution

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Section 5.5

More Discrete Probability Distributions
Section 5.5 Objectives

- Find probabilities using the geometric distribution
- Find probabilities using the Poisson distribution
Geometric Distribution

- A discrete probability distribution.
- Satisfies the following conditions
  - A trial is repeated until a success occurs.
  - The repeated trials are independent of each other.
  - The probability of success $p$ is constant for each trial.
  - $x$ represents the number of the trial in which the first success occurs.
- The probability that the first success will occur on trial $x$ is $P(x) = p(q)^{x-1}$, where $q = 1 - p$. 
Example: Geometric Distribution

Basketball player LeBron James makes a free throw shot about 74% of the time. Find the probability that the first free throw shot LeBron makes occurs on the third or fourth attempt.

Solution:

• \( P(\text{shot made on third or fourth attempt}) = P(3) + P(4) \)
• Geometric with \( p = 0.74, q = 0.26, x = 3 \)
Solution: Geometric Distribution

- \( P(3) = 0.74(0.26)^{3-1} = 0.050024 \)
- \( P(4) = 0.74(0.26)^{4-1} \approx 0.013006 \)

\( P \) (shot made on third or fourth attempt)

\[ = P(3) + P(4) \approx 0.050024 + 0.013006 \approx 0.063 \]
Poisson Distribution

Poisson distribution

• A discrete probability distribution.
• Satisfies the following conditions
  ▪ The experiment consists of counting the number of times $x$ an event occurs in a given interval. The interval can be an interval of time, area, or volume.
  ▪ The probability of the event occurring is the same for each interval.
  ▪ The number of occurrences in one interval is independent of the number of occurrences in other intervals.
Poisson Distribution

Poisson distribution

• Conditions continued:
  - The probability of exactly $x$ occurrences in an interval is

$$P(x) = \frac{\mu^x e^{-\mu}}{x!}$$

where $e$ is an irrational number $\approx 2.71828$ and $\mu$ is the mean number of occurrences per interval unit.
Example: Poisson Distribution

The mean number of accidents per month at a certain intersection is 3. What is the probability that in any given month four accidents will occur at this intersection?

Solution:
- Poisson with $x = 4, \mu = 3$

$$P(4) \approx \frac{3^4(2.71828)^{-3}}{4!} \approx 0.168$$
Section 5.5 Summary

• Found probabilities using the geometric distribution
• Found probabilities using the Poisson distribution