A Direct Proof of a Theorem Concerning Singular Hamiltonian Systems

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Abstract

This technical report presents a direct proof of Theorem 1 in [1] and some consequences that also account for (20) in [1]. This direct proof exploits a state space change of basis which replaces the coupled difference equations (10) in [1] with two equivalent difference equations which, instead, are decoupled.

I. INTRODUCTION

Theorem 1 in [1] provides in (19) the set of the admissible solutions \((x_k, p_k, u_k)\) of the singular Hamiltonian system (10) defined on the discrete-time interval \(0 \leq k \leq k_f - 1\). The proof therein presented is twofold: sufficiency is shown by direct replacement of (19) in (10); necessity relies on maximality of the involved structural invariant subspaces, as it is deducible from Properties 1 and 2. In the following, it will be shown that a direct proof, which does not distinguish between the if and the only-if part, but extensively uses relations pointed out in [1], is also feasible. The main point of the direct proof is replacing the coupled difference equations (10) in [1] with two decoupled difference equations by means of a suitable state space basis transformation. The direct proof herein presented can also be used to prove (20) in [1], that expresses the set of the admissible solutions \((x_k, p_k)\) of the same Hamiltonian system in the extended time interval \(0 \leq k \leq k_f\).

II. DIRECT PROOF OF RELATION (20) AND THEOREM 1 IN [1]

The direct proof is based on the following lemma.

**Lemma 1:** The problem of finding the sequences \(x_k, p_k, u_k\), with \(0 \leq k \leq k_f - 1\), that solve the equations (10) of [1], or, equivalently,

\[
\begin{align*}
x_{k+1} &= A x_k + B u_k, \\
-A^T p_{k+1} &= Q x_k - p_k + S u_k, \\
-B^T p_{k+1} &= S^T x_k + R u_k,
\end{align*}
\]

with \(0 \leq k \leq k_f - 1\), can be reduced to that of finding the sequences \(v_k\) and \(w_k\) that solve the pair of decoupled difference equations

\[
\begin{align*}
v_{k+1} &= A_+ v_k, \\
A_+^T w_{k+1} &= w_k,
\end{align*}
\]
with \(0 \leq k \leq k_f - 1\), where \(A_+\) is defined by (14) in [I], provided that the following correspondences are set up

\[
x_k = v_k + W w_k,
\]

\[
p_k = P_+ v_k + (-I + P_+ W) w_k,
\]

\[
u_k = -K_+ v_k + \bar{K}_+ w_{k+1},
\]

where \(P_+\) is the positive semidefinite symmetric solution of (11)–(12) in [I], \(W\) is the solution of the symmetric discrete Lyapunov equation (15), \(K_+\), and \(\bar{K}_+\) are defined by (13) and (17).

**Proof:** First, the following relation will be shown:

\[
-W + B\bar{K}_+ = -AW A_+^T.
\]  

(9)

Use of (17) in [I] yields the identity

\[
-W + B\bar{K}_+ = -W + B(R + B^TP_+B)^{-1}(B^T - B^TP_+AWA_+^T - S^TWA_+^T) =
\]

and, by applying distributivity of the product with respect to the sum,

\[
= -W + B(R + B^TP_+B)^{-1}B^T - B(R + B^TP_+B)^{-1}B^TP_+AWA_+^T
\]

\[-B(R + B^TP_+B)^{-1}S^TWA_+^T =
\]

and, by collecting \(WA_+^T\) in the last two terms,

\[
=W + B(R + B^TP_+B)^{-1}B^T - B(R + B^TP_+B)^{-1}(B^TP_+A + S^T)WA_+^T =
\]

and, by the definition (13) of \(K_+\) in [I], and summing and subtracting the term \(AWA_+^T\)

\[
=W + B(R + B^TP_+B)^{-1}B^T - B K_+WA_+^T + AWA_+^T - AWA_+^T =
\]

and, by reordering,

\[
=(A - BK_+)WA_+^T - W + B(R + B^TP_+B)^{-1}B^T - AWA_+^T =
\]

and, by using (14) in [I],

\[
=A_+WA_+^T - W + B(R + B^TP_+B)^{-1}B^T - AWA_+^T =
\]

and, eventually, tacking (15) in [I] into account,

\[
= -AWA_+^T.
\]

Thus, (9) is proven. Now we are ready to obtain the difference equation in the unknowns \(v_k\) and \(w_k\). By using (6) and (8) in [I], it follows that:

\[
v_{k+1} + W w_{k+1} = Av_k + AW w_k - BK_+ v_k + B \bar{K}_+ w_{k+1},
\]

or also

\[
v_{k+1} = (A - BK_+) v_k + (-W + B\bar{K}_+) w_{k+1} + AW w_k ,
\]

or, by the definition (14) in [I],

\[
v_{k+1} = A_+ v_k + (-W + B\bar{K}_+) w_{k+1} + AW w_k,
\]  

(10)

or, equivalently because of (9),

\[
v_{k+1} = A_+ v_k - AW A_+^T w_{k+1} + AW w_k .
\]  

(11)
Similarly, by using (6)–(8) in (2), the following is obtained:

\[-A^T(P_+v_{k+1} + (P_+W - I)w_{k+1}) =
= Q(v_k + Ww_k) - (P_+v_k - (P_+W - I)w_k) + S(-K_+v_k + \bar{K}_+w_{k+1}),\]

or

\[-A^T P_+v_{k+1} - A^T(P_+W - I)w_{k-1} =
= Qv_k + QWw_k - P_+v_k - (P_+W - I)w_k - SK_+v_k + S\bar{K}_+w_{k+1}.\]

By the identity \(-A^T(P_+W - I) = QWA^T_+ - (P_+W - I)A^T_+ + S\bar{K}_+\) (see the proof of Property 2 in [1] – second row block), the following holds:

\[-A^T P_+v_{k+1} + (QWA^T_+ - (P_+W - I)A^T_+ + S\bar{K}_+)w_{k+1} =
= Qv_k + QWw_k - P_+v_k - (P_+W - I)w_k - SK_+v_k + S\bar{K}_+w_{k+1},\]

and, by doing away with the terms \(SK_+w_{k+1}\) at the right of both members,

\[-A^T P_+v_{k+1} + (Q - (P_+W - I))A^T_+w_{k+1} =
= (Q - P_+ - SK_+)v_k + (QW - (P_+W - I))w_k.\]

Recall the identity \(Q - P_+ - SK_+ = -A^TP_+A_+\) (see the proof of Property 1 in [1] – second row block), the following is obtained:

\[-A^T P_+v_{k+1} + (QW - (P_+W - I))A^T_+w_{k+1} = -A^TP_+A_+v_k + (QW - (P_+W - I))w_k.\] (12)

Let us multiply both members of (11) by \(A^TP_+\), thus obtaining

\[A^TP_+v_{k+1} = A^TP_+A_+v_k - A^TP_+AW A^T_+w_{k+1} + A^TP_+AW w_k,\] (13)

and, by summing both members of (12) and (13), it follows that

\[(QW - (P_+W - I))A^T_+w_{k+1} =
= (QW - (P_+W - I))w_k - A^TP_+AW A^T_+w_{k+1} + A^TP_+AW w_k.\]

By collecting \(w_{k+1}\) on the left and \(w_k\) on the right, it follows that

\[(QW - (P_+W - I)) + A^TP_+AW) A^T_+w_{k+1} = (QW - (P_+W - I) + A^TP_+AW)w_k,\]

or \(A^T_+w_{k+1} = w_k\), that is (5). Taking into account this latter equation in (11) one gets

\[v_{k+1} = A_+v_k - AW w_k + AW w_k,\]

or \(v_{k+1} = A_+v_k\), that is (4).

Now we are ready to conclude the direct proof of both (20) and (19) in [1]. Refer to the pair of decoupled difference equations (4), (5), defined in the time interval \(0 \leq k \leq k_f - 1\). Their solutions can be expressed as

\[v_k = A^k_+\alpha,\]

\[w_k = (A^T_+)^{k_f-k}\beta,\]

where \(\alpha, \beta \in \mathbb{R}^n\) are parameters. Substitution of (14) in (6), (7) yields

\[x_k = A^k_+\alpha + W (A^T_+)^{k_f-k}\beta,\]

\[p_k = P_+ A^k_+\alpha + (P_+W - I)(A^T_+)^{k_f-k}\beta,\]

\[0 \leq k \leq k_f,\]

\[\]

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that, re-written in compact notation as
\[
\begin{bmatrix}
  x_k \\
  p_k \\
  u_k
\end{bmatrix} =
\begin{bmatrix}
  I \\
  P_+ \\
  -K_+
\end{bmatrix}
\begin{bmatrix}
  A_+^k \alpha \\
  (P_+ W - I) A_+^k \\
  (P_+ W - I)
\end{bmatrix}
\begin{bmatrix}
  W A_+^T \\
  (A_+^T)^{k_f - k} \beta \\
  (A_+^T)^{k_f - k} \beta
\end{bmatrix},
\]
\[0 \leq k \leq k_f,
\]
coincides with equation (20) in [1].

To prove equation (19) in [1], let us substitute (5), i.e.,
\[w_k = A_+^T w_{k+1}, \quad 0 \leq k \leq k_f - 1,
\]
in (6), (7), thus obtaining
\[
\begin{align*}
x_k &= v_k + W A_+^T w_{k+1}, \quad 0 \leq k \leq k_f - 1, \\
p_k &= (P_+ W - I) A_+^T w_{k+1},
\end{align*}
\]
(15)

Using (14) in (15) yields
\[
\begin{align*}
x_k &= A_+^k \alpha + W A_+^T (A_+^T)^{k_f - k - 1} \beta, \\
p_k &= P_+ A_+^k \alpha + (P_+ W - I) A_+^T (A_+^T)^{k_f - k - 1} \beta, \quad 0 \leq k \leq k_f - 1,
\end{align*}
\]
(16)

while using (14) in (8) provides
\[
\begin{align*}
u_k &= -K_+ A_+^k \alpha + \bar{K}_+ (A_+^T)^{k_f - k - 1} \beta, \quad 0 \leq k \leq k_f - 1.
\end{align*}
\]
(17)

Equations (16), (17) can be re-written in compact form as
\[
\begin{bmatrix}
  x_k \\
  p_k \\
  u_k
\end{bmatrix} =
\begin{bmatrix}
  I \\
  P_+ \\
  -K_+
\end{bmatrix}
\begin{bmatrix}
  A_+^k \alpha \\
  (P_+ W - I) A_+^k \\
  (P_+ W - I)
\end{bmatrix}
\begin{bmatrix}
  W A_+^T \\
  (A_+^T)^{k_f - k - 1} \beta \\
  (A_+^T)^{k_f - k - 1} \beta
\end{bmatrix},
\]
\[0 \leq k \leq k_f - 1,
\]
that coincides with (19) in [1]. Thus, Theorem 1 in [1] has been directly proven by using the correspondences stated in Lemma 1.

**REFERENCES**

[1] E. Zattoni, “Structural invariant subspaces of singular Hamiltonian systems and nonrecursive solutions of finite-horizon optimal control problems,” *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1279–1284, June 2008.