Quasi Exactly Solvable Difference Equations

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Abstract

Several explicit examples of quasi exactly solvable ‘discrete’ quantum mechanical Hamiltonians are derived by deforming the well-known exactly solvable Hamiltonians of one degree of freedom. These are difference analogues of the well-known quasi exactly solvable systems, the harmonic oscillator (with/without the centrifugal potential) deformed by a sextic potential and the $\frac{1}{\sin^2 x}$ potential deformed by a $\cos 2x$ potential. They have a finite number of exactly calculable eigenvalues and eigenfunctions.

1 Introduction

Exactly solvable and Quasi Exactly Solvable (QES) quantum mechanical systems have played a very important role in modern physics. The former, the exactly solvable systems, are quite well-known. In the Schrödinger picture, if all the eigenvalues and corresponding eigenfunctions are known, the system is exactly solvable. Plenty of such systems are known, for example, the Pöschl-Teller and the Morse potential on top of the best-known harmonic oscillator and the coulomb potential [1] for degree one cases and the Calogero-Sutherland systems [2 3 4] for many degrees of freedom cases. Recently the exact Heisenberg operator solutions and the corresponding annihilation-creation operators are constructed for most of the degree one exactly solvable quantum mechanics [5] and for the multi-particle Calogero systems [6]. The notion of exact solvability was generalised to the so-called ‘discrete’ quantum mechanics, in which the Schrödinger equation is a difference equation in stead of differential. The
difference analogues of the Calogero-Sutherland systems were constructed by Ruijsenaars-Schneider-van Diejen [7, 8]. The difference equation analogues of the equations determining the Hermite, Laguerre and Jacobi polynomials were derived within the Askey-scheme of hypergeometric orthogonal polynomials [9, 10]. Later they were reformulated as Hamiltonian dynamics with shape-invariance [11] by Odake-Sasaki [12].

In contrast, the latter, Quasi Exactly Solvable (QES) systems have a short history and less known. If a finite number of exact eigenvalues and eigenfunctions are known, the system is QES [13]. Since the number of exactly solvable states can be chosen as large as wanted, a QES system could be used as a good alternative to an exactly solvable system. Several one degree of freedom QES systems are listed in [13, 14] and multi-particle QES systems were first constructed by Sasaki-Takasaki [15] as deformation of Inozemtsev- Calogero-Sutherland systems, which was followed by [16].

In the present paper we derive several explicit examples of QES difference equations as deformation of exactly solvable ‘discrete’ quantum mechanics [12]. They are difference analogues of the well-known quasi exactly solvable systems, the harmonic oscillator (with/without the centrifugal potential) deformed by a sextic potential and the $1/\sin^2 x$ potential deformed by a cos $2x$ potential.

This paper is organised as follows. In the next section, the deformation method to obtain a QES from an exactly solvable system is explained in some detail by taking two well-known examples of the ordinary quantum mechanical QES systems. Then two corresponding difference equation analogues are derived. Section 3 provides three more explicit examples. The final section is for a summary and comments.

2 Quasi Exactly Solvable Deformation

There are many different ways of deriving QES Hamiltonians for ordinary quantum mechanics [13, 14]. However, to the best of our knowledge, a very limited number of explicit examples of QES difference equations are known in connection with $U_q(sl(2))$ [17]. In these examples, quantum wavefunctions are related to those defined on discrete lattice points only.

In the present paper we present several explicit examples of QES ‘discrete’ quantum mechanical Hamiltonians of one degree of freedom, whose wavefunctions are continuous functions of $x$ as in the ordinary quantum mechanics. They are obtained by deforming exactly solvable ‘discrete’ quantum mechanical Hamiltonians [12], which have the Askey-scheme of hyperge-
ometric orthogonal polynomials \cite{9,10} as part of the eigenfunctions; the Meixner-Pollaczek, continuous Hahn, continuous dual Hahn, Wilson and Askey-Wilson polynomials. This deformation method was first applied by Sasaki and Takasaki \cite{15} to derive multi-particle QES based on the Inozemtsev models.

For illustrative purposes, we will explain the deformation method for the two well-known examples of degree one QES systems in ordinary quantum mechanics in the next subsection. These examples are the sextic \((x^6)\) potential added to the harmonic oscillator \((x^2)\) potential, and another \(\cos 2x\) potential added to the exactly solvable \(1/\sin^2 x\) potential. The same method is applied in subsection 2.2 to derive the first two examples of QES Hamiltonian in ‘discrete’ quantum mechanics corresponding to the the sextic potential deformation. The rest of the examples are given in section 3.

2.1 Ordinary Quantum Mechanics

2.1.1 Harmonic Oscillator Deformed by Sextic Potential

A best-known example of QES Hamiltonian, the harmonic oscillator plus a sextic \((x^6)\) potential is given succinctly by

\[
\mathcal{H} \equiv -\frac{d^2}{dx^2} + \left(\frac{dW}{dx}\right)^2 + \frac{d^2W}{dx^2} + \alpha_M(x), \quad \alpha_M(x) \equiv -2aMx^2, \quad M \in \mathbb{N}.
\]  

(2.1)

Here we tentatively call the last term of the above Hamiltonian, \(\alpha_M(x)\), the compensation term. Throughout this paper we adopt the unit system \(2m = \hbar = 1\). The real prepotential \(W\) is a deformation of that for the harmonic oscillator \(W_0 \equiv -bx^2/2\), with \(a\) being the deformation parameter:

\[
W = W(x) \equiv -\frac{a}{4}x^4 + W_0 = -\frac{a}{4}x^4 - \frac{b}{2}x^2, \quad a, b > 0.
\]  

(2.2)

By the similarity transformation in terms of the pseudo ground state wavefunction \(\phi_0(x) \equiv e^W(x)\), we obtain

\[
\tilde{\mathcal{H}} \equiv \phi_0^{-1} \circ \mathcal{H} \circ \phi_0 = -\frac{d^2}{dx^2} - 2\frac{dW(x)}{dx} \frac{d}{dx} - 2aMx^2,
\]  

(2.3)

\[
= -\frac{d^2}{dx^2} + (2ax^3 + 2bx) \frac{d}{dx} - 2aMx^2.
\]  

(2.4)

In the absence of the compensation term \(\alpha_M(x)\), \(\phi_0(x)\) is actually a ground state wavefunction, therefore it has no node and it is square integrable. Another characterisation of
the pseudo ground state wavefunction \( \phi_0 \) is that it is annihilated by the operator \( A \) which factorises the main part of the Hamiltonian (2.1):

\[
A\phi_0(x) = 0, \quad A \overset{\text{def}}{=} -\frac{d}{dx} + \frac{dW(x)}{dx}, \quad A^\dagger \overset{\text{def}}{=} \frac{d}{dx} + \frac{dW(x)}{dx},
\]

\[
\mathcal{H} = A^\dagger A - 2aMx^2.
\] (2.5) (2.6)

The action of the Hamiltonian \( \tilde{\mathcal{H}} \) (2.3) on monomials of \( x \) reads

\[
\tilde{\mathcal{H}} x^n = \begin{cases} 
-n(n-1)x^{n-2} + 2nbx^n + 2a(n-M)x^{n+2}, & n \leq M - 2, \\
-M(M-1)x^{M-2} + 2Mbx^M, & n = M.
\end{cases}
\] (2.7)

Since the parity is conserved, it is now obvious that \( \tilde{\mathcal{H}} \) keeps the polynomial space \( \mathcal{V}_M \) invariant,

\[
\tilde{\mathcal{H}} \mathcal{V}_M \subseteq \mathcal{V}_M,
\] (2.8)

\[
\mathcal{V}_M \overset{\text{def}}{=} \begin{cases} 
\text{Span} \left[ 1, x^2, \ldots, x^{2k}, \ldots, x^M \right], & M: \text{even}, \\
\text{Span} \left[ x, x^3, \ldots, x^{2k+1}, \ldots, x^M \right], & M: \text{odd},
\end{cases}
\] (2.9)

and that \( \tilde{\mathcal{H}} \) is a tri-diagonal matrix. Thus we can obtain a finite number of exact eigenvalues and eigenfunctions of the sextic potential Hamiltonian (2.1) in the form:

\[
\mathcal{H}\phi = \mathcal{E}\phi, \quad \phi(x) = \phi_0(x)P_M(x), \quad P_M \in \mathcal{V}_M, \iff \tilde{\mathcal{H}}P_M = \mathcal{E}P_M,
\] (2.10)

by the diagonalisation of a finite dimensional Hamiltonian matrix \( \tilde{\mathcal{H}} \) (2.3) with

\[
\dim \mathcal{V}_M = \begin{cases} 
M/2 + 1, & M: \text{even}, \\
(M+1)/2, & M: \text{odd}.
\end{cases}
\] (2.11)

Since \( \mathcal{H} \) is obviously hermitian (or self-adjoint), all the eigenvalues are real and eigenfunctions belonging to different eigenvalues are orthogonal with each other. In other words, two polynomial solutions \( P_M \) and \( P'_M \) are orthogonal with respect to the weight function \( \phi^2_0(x) \).

The square integrability of all the eigenfunctions of the above form (2.10) \( \int_{-\infty}^{\infty} \phi^2(x)dx < \infty \) is obvious. The true ground state wave function has the form (2.10) with the lowest eigenvalue, say \( \mathcal{E}_0 \) and it has no node due to the oscillation theorem.

### 2.1.2 1/ sin^2 x Potential Deformed by cos 2x Potential

Another well-known example of quasi exactly solvable system is the exactly solvable \( 1/ \sin^2 x \) potential \((W_0 \overset{\text{def}}{=} g \log \sin x)\) deformed by a \( \cos 2x \) potential. The Hamiltonian has the same
form as (2.1) with only the prepotential \( W(x) \) and the compensation term \( \alpha_M(x) \) different:

\[
\mathcal{H} \overset{\text{def}}{=} -\frac{d^2}{dx^2} + \left( \frac{dW}{dx} \right)^2 + \frac{d^2W}{dx^2} + \alpha_M(x), \quad \alpha_M(x) \overset{\text{def}}{=} 4aM \sin^2 x, \quad M \in \mathbb{N}, \quad (2.12)
\]

\[
W \overset{\text{def}}{=} \frac{a}{2} \cos 2x + W_0 = \frac{a}{2} \cos 2x + g \log \sin x, \quad g > 0, \quad 0 < x < \pi. \quad (2.13)
\]

Again by the similarity transformation in terms of the pseudo ground state wavefunction \( A\phi_0 = 0, \phi_0(x) \overset{\text{def}}{=} e^{W(x)} \), we obtain

\[
\tilde{\mathcal{H}} \overset{\text{def}}{=} \phi_0^{-1} \circ \mathcal{H} \circ \phi_0 = -\frac{d^2}{dx^2} - 2\frac{dW(x)}{dx} \frac{d}{dx} + 4aM \sin^2 x, \quad (2.14)
\]

\[
= -\frac{d^2}{dx^2} + (2a \sin 2x - 2g \cot x) \frac{d}{dx} + 4aM \sin^2 x. \quad (2.15)
\]

Needless to say \( \phi_0 \) has no node or singularity and it is square integrable \( \int_0^\pi \phi_0^2(x)dx < \infty \).

The action of the Hamiltonian \( \tilde{\mathcal{H}} \) (2.14) on monomials of \( \sin x \) reads

\[
\tilde{\mathcal{H}} \sin^n x = \begin{cases} 
-n(n - 1 + 2g) \sin^{n-2} x + n(n + 2g + 4a) \sin^n x + 4a(M - n) \sin^{n+2} x, \quad n \leq M - 2, \\
-M(M - 1 + 2g) \sin^{M-2} x + M(M + 2g + 4a) \sin^M x, \quad n = M.
\end{cases} \quad (2.16)
\]

Since the parity is conserved, it is now obvious that \( \tilde{\mathcal{H}} \) keeps the polynomial space \( \mathcal{V}_M \) invariant,

\[
\tilde{\mathcal{H}} \mathcal{V}_M \subseteq \mathcal{V}_M, \quad (2.17)
\]

\[
\mathcal{V}_M \overset{\text{def}}{=} \begin{cases} 
\text{Span } [1, \sin^2 x, \ldots, \sin^{2k} x, \ldots, \sin^M x], & M : \text{even}, \\
\text{Span } [\sin x, \sin^3 x, \ldots, \sin^{2k+1} x, \ldots, \sin^M x], & M : \text{odd},
\end{cases} \quad (2.18)
\]

\[
\dim \mathcal{V}_M = \begin{cases} 
M/2 + 1, & M : \text{even}, \\
(M + 1)/2, & M : \text{odd},
\end{cases} \quad (2.19)
\]

and that \( \tilde{\mathcal{H}} \) is again a tri-diagonal matrix. Thus we can obtain a finite number (\( \dim \mathcal{V}_M \)) of exact eigenvalues and eigenfunctions in the same way as in the sextic potential Hamiltonian (2.1) case.

\section*{2.2 ‘Discrete’ Quantum Mechanics}

\subsection*{2.2.1 Difference Equation Analogue of Harmonic Oscillator Deformed by Sextic Potential I}

A difference analogue of the sextic potential Hamiltonian (2.1) is

\[
\mathcal{H} \overset{\text{def}}{=} \sqrt{V(x)} e^{-i\theta_x} \sqrt{V(x)^*} + \sqrt{V(x)^*} e^{i\theta_x} \sqrt{V(x)} - (V(x) + V(x)^*) + \alpha_M(x). \quad (2.20)
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Graphical representation of the quantum system.}
\end{figure}
\[ A^\dagger A + \alpha_M(x), \quad \alpha_M(x) \overset{\text{def}}{=} 2Mx^2, \quad (2.21) \]

\[ A^\dagger \overset{\text{def}}{=} \sqrt{V(x)}e^{-\frac{i}{2}\partial_x} - \sqrt{V(x)^*}e^{\frac{i}{2}\partial_x}, \quad A \overset{\text{def}}{=} e^{-\frac{i}{2}\partial_x}\sqrt{V(x)^*} - e^{\frac{i}{2}\partial_x}\sqrt{V(x)}, \quad (2.22) \]

\[ V(x) \overset{\text{def}}{=} (a + ix)(b + ix)V_0(x), \quad V_0(x) \overset{\text{def}}{=} c + ix, \quad a, b, c \in \mathbb{R}_+. \quad (2.23) \]

Here as usual \( V(x)^* \) is the complex conjugate of \( V(x) \). If \( V \) is replaced by \( V_0 \) and the last term in \( (2.20) \), \( \alpha_M(x) \), is removed, \( \mathcal{H} \) becomes the exactly solvable Hamiltonian of a difference analogue of the harmonic oscillator, or the \textit{deformed harmonic oscillator} in ‘discrete’ quantum mechanics \[12\]. Its eigenfunctions consist of the Meixner-Pollaczek polynomial, which is a deformation of the Hermite polynomial \[12\ [18\]. The quadratic polynomial factor \((a + ix)(b + ix)\) can be considered as multiplicative deformation, although the parameters \(a, b \) and \(c\) are on the equal footing. On the other hand one can consider it as a multiplicative deformation by a linear polynomial in \(x\):

\[ V(x) = (a + ix)V_01(x), \quad V_01(x) \overset{\text{def}}{=} (b + ix)(c + ix), \]

with \( V_01 \) describing another difference version of an exactly solvable analogue of the harmonic oscillator \[12\]. Its eigenfunctions consist of the continuous Hahn polynomial.

Next let us introduce the similarity transformation in terms of the \textit{pseudo ground state} wavefunction \( \phi_0(x) \):

\[ \phi_0(x) \overset{\text{def}}{=} \sqrt{\Gamma(a + ix)\Gamma(b + ix)\Gamma(c + ix)\Gamma(e - ix)}, \quad (2.24) \]

\[ \tilde{\mathcal{H}} \overset{\text{def}}{=} \phi_0^{-1} \circ \mathcal{H} \circ \phi_0 = V(x)\left(e^{-i\partial_x} - 1\right) + V(x)^*\left(e^{i\partial_x} - 1\right) + 2Mx^2. \quad (2.25) \]

It is obvious that \( \phi_0 \) has no node and that it is square integrable. As in the ordinary quantum mechanics, \( \phi_0(x) \) is annihilated by the \( A \) operator \( (2.22) \)

\[ 0 = A \phi_0(x) = \left(e^{-\frac{i}{2}\partial_x}\sqrt{V(x)^*} - e^{\frac{i}{2}\partial_x}\sqrt{V(x)}\right)\phi_0(x). \quad (2.26) \]

It is rather trivial to verify the action of the Hamiltonian \( \tilde{\mathcal{H}} \) \( (2.25) \) on monomials of \( x \):

\[ \tilde{\mathcal{H}} x^n = \begin{cases} 
\sum_{j=0}^{[n/2+1]} a_{n,j} x^{n+2-2j}, & n \leq M - 2, \quad a_{n,j} \in \mathbb{R}, \\
\sum_{j=0}^{[M/2]} a'_{n,j} x^{M-2j}, & n = M, \quad a'_{n,j} \in \mathbb{R}.
\end{cases} \quad (2.27) \]

Here \([m]\) is the standard Gauss’ symbol denoting the greatest integer not exceeding or equal to \( m \). Since the parity is conserved, \( \tilde{\mathcal{H}} \) keeps the polynomial space \( \mathcal{V}_M \) invariant,

\[ \tilde{\mathcal{H}} \mathcal{V}_M \subseteq \mathcal{V}_M, \quad (2.28) \]
\[ \mathcal{V}_M \overset{\text{def}}{=} \begin{cases} \text{Span} \left[ 1, x^2, \ldots, x^{2k}, \ldots, x^M \right], & \mathcal{M} : \text{even}, \\ \text{Span} \left[ x, x^3, \ldots, x^{2k+1}, \ldots, x^M \right], & \mathcal{M} : \text{odd}, \end{cases} \]  

\[ \dim \mathcal{V}_M = \begin{cases} \mathcal{M}/2 + 1, & \mathcal{M} : \text{even}, \\ (\mathcal{M} + 1)/2, & \mathcal{M} : \text{odd}, \end{cases} \]  

but \( \tilde{\mathcal{H}} \) is no longer a tri-diagonal matrix, \( (\tilde{\mathcal{H}})_{jk} \neq 0, j \geq k - 1 \). The Hamiltonian \( \mathcal{H} \) is obviously hermitian (self-adjoint) and all the eigenvalues are real and eigenfunctions can be chosen real. We can obtain a finite number of exact eigenvalues and eigenfunctions by sweeping in a similar way as in the sextic potential case \( \text{(2.10), (2.11)} \). The oscillation theorem linking the number of eigenvalues (from the ground state) to the zeros of eigenfunctions does not hold in the difference equations. The square integrability of all the eigenfunctions \( \int_{-\infty}^{\infty} \phi^2(x)dx < \infty \) is obvious.

### 2.2.2 Difference Equation Analogue of Harmonic Oscillator Deformed by Sextic Potential II

Another difference analogue of the sextic potential Hamiltonian \( \text{(2.1)} \) has the same form as \( \text{(2.20), (2.21) and (2.22)} \), with only the potential function \( V(x) \) and the compensation term \( \alpha_M(x) \) are different:

\[ V(x) \overset{\text{def}}{=} (a + ix)(b + ix)V_0(x), \quad V_0(x) \overset{\text{def}}{=} (c + ix)(d + ix), \quad a, b, c, d \in \mathbb{R}_+, \quad (2.31) \]

\[ \alpha_M(x) \overset{\text{def}}{=} \mathcal{M}(\mathcal{M} - 1 + 2(a + b + c + d)) x^2. \quad (2.32) \]

This Hamiltonian can be considered as a deformation by a quadratic polynomial factor \((a + ix)(b + ix)\) of the exactly solvable ‘discrete’ quantum mechanics having the continuous Hahn polynomials as eigenfunctions \[12\], another difference analogue of the harmonic oscillator. See the comments in section 5 of \[21\].

The **pseudo ground state** wavefunction \( \phi_0(x) \) annihilated by the \( A \) operator \( A\phi_0 = 0 \) reads

\[ \phi_0(x) \overset{\text{def}}{=} \sqrt{\Gamma(a + ix)\Gamma(a - ix)\Gamma(b + ix)\Gamma(b - ix)\Gamma(c + ix)\Gamma(c - ix)\Gamma(d + ix)\Gamma(d - ix)}. \]

Again it has no node and it is square integrable. The similarity transformed Hamiltonian acting on the polynomial space is

\[ \tilde{\mathcal{H}} \overset{\text{def}}{=} \phi_0^{-1} \circ \mathcal{H} \circ \phi_0 = V(x) \left( e^{-i\partial_k} - 1 \right) + V(x)^* \left( e^{i\partial_k} - 1 \right) + \mathcal{M}(\mathcal{M} - 1 + 2(a + b + c + d)) x^2. \]
It is straightforward to verify the relationship (2.27) and to establish the existence of the invariant polynomial subspaces of given parity (2.28), (2.29) and (2.30). The hermiticity of the Hamiltonian and the square integrability of the eigenfunctions also hold. Thus another example of quasi exactly solvable difference equation is established.

3 Other Examples

The other two examples are the difference equation analogues of the harmonic oscillator with the centrifugal potential deformed by the sextic potential. There are two types corresponding to the linear and quadratic polynomial deformations. The corresponding exactly solvable difference equation has the Wilson polynomial [12, 9, 10] as the eigenfunctions. The last example is the difference analogue of the model discussed in subsection 2.1.2, /sin^2 x potential deformed by a cos 2x potential. In this case the corresponding exactly solvable difference equation has the Askey-Wilson polynomials [12, 9, 10] as eigenfunctions. The basic idea for showing quasi exact solvability is almost the same as shown above.

3.1 Difference Equation Analogues of Harmonic Oscillator With Centrifugal Potential Deformed by Sextic Potential

The Hamiltonians have the same form as (2.20), (2.21) and (2.22), with only the potential function \( V(x) \) and the compensation term \( \alpha_M(x) \) are different:

**Type I:** \( V(x) \overset{\text{def}}{=} (b + ix)V_0(x), \quad \alpha_M(x) \overset{\text{def}}{=} Mx^2, \) (3.1)

**Type II:** \( V(x) \overset{\text{def}}{=} (a + ix)(b + ix)V_0(x), \) \( \alpha_M(x) \overset{\text{def}}{=} M(M - 1 + (a + b + c + d + e + f))x^2, \) (3.2)

with a common \( V_0(x) \)

\[
V_0(x) \overset{\text{def}}{=} \frac{(c + ix)(d + ix)(e + ix)(f + ix)}{2ix(2ix + 1)}, \quad a, b, c, d, e, f \in \mathbb{R}_+ - \{1/2\}. \] (3.4)

None of the parameters \( a, b, c, d, e \) or \( f \) should take the value 1/2, since it would cancel the denominator. Because of the centrifugal barrier, the dynamics is constrained to a half line; \( 0 < x < \infty \). The type I case can also be considered as a quadratic polynomial deformation of the exactly solvable dynamics with \( V_{01}(x) \):

**Type I:** \( V(x) \overset{\text{def}}{=} (b + ix)(c + ix)V_{01}(x), \quad V_{01}(x) \overset{\text{def}}{=} \frac{(d + ix)(e + ix)(f + ix)}{2ix(2ix + 1)}, \) (3.5)
which has the continuous dual Hahn polynomials \([12, 9, 10]\) as eigenfunctions. This re-
interpretation does not change the dynamics, since the Hamiltonian and \(A\) and \(A^\dagger\) operators
depend on \(V(x)\).

The pseudo ground state wavefunction \(\phi_0(x)\) is determined as the zero mode of the \(A\) operator \(A\phi_0 = 0\):

\[
\begin{align*}
\text{Type I : } \phi_0(x) & \overset{\text{def}}{=} \sqrt{\frac{\prod_{j=2}^{6} \Gamma(a_j + ix) \Gamma(a_j - ix)}{\sqrt{\Gamma(2ix)\Gamma(-2ix)}}}, \\
\text{Type II : } \phi_0(x) & \overset{\text{def}}{=} \sqrt{\frac{\prod_{j=1}^{6} \Gamma(a_j + ix) \Gamma(a_j - ix)}{\sqrt{\Gamma(2ix)\Gamma(-2ix)}}},
\end{align*}
\]

(3.6)

(3.7)

in which the numbering of the parameters

\[
a_1 \overset{\text{def}}{=} a, \quad a_2 \overset{\text{def}}{=} b, \quad a_3 \overset{\text{def}}{=} c, \quad a_4 \overset{\text{def}}{=} d, \quad a_5 \overset{\text{def}}{=} e, \quad a_6 \overset{\text{def}}{=} f,
\]

(3.8)
is used. It is obvious that both \(\phi_0\) have no node in the half line \(0 < x < \infty\).

The similarity transformed Hamiltonian acting on the polynomial space has the same
form as before (2.34)

\[
\tilde{\mathcal{H}} \overset{\text{def}}{=} \phi_0^{-1} \circ \mathcal{H} \circ \phi_0 = V(x) \left(e^{-i\partial_x} - 1\right) + V(x)^* \left(e^{i\partial_x} - 1\right) + \alpha_M(x).
\]

(3.9)

Although the potential \(V(x)\) has the harmful looking denominator \(1/\{2ix(2ix + 1)\}\), it is straightforward to verify that \(\tilde{\mathcal{H}}\) maps a polynomial in \(x^2\) into another:

\[
\tilde{\mathcal{H}}x^{2n} = \begin{cases} 
\sum_{j=0}^{n+1} a_{n,j} x^{2n+2-2j}, & n \leq M - 1, \quad a_{n,j} \in \mathbb{R}, \\
\sum_{j=0}^{M} a'_{n,j} x^{2M-2j}, & n = M, \quad a'_{n,j} \in \mathbb{R}.
\end{cases}
\]

(3.10)

This is because \(V_0\), which has the above denominator, keeps the polynomial subspace of
any even degree invariant, reflecting the exact solvability. This establishes that \(\tilde{\mathcal{H}}\) keeps the polynomial space \(\mathcal{V}_M\) invariant,

\[
\tilde{\mathcal{H}} \mathcal{V}_M \subseteq \mathcal{V}_M, \quad \mathcal{V}_M \overset{\text{def}}{=} \text{Span} \left[1, x^2, \ldots, x^{2k}, \ldots, x^{2M}\right], \quad \dim \mathcal{V}_M = M + 1.
\]

(3.11)

(3.12)
The hermiticity of the Hamiltonians is obvious and the square-integrability of the eigenfunc-
tions \(\int_0^\infty \phi^2(x)dx < \infty\) holds true. This establishes the quasi exact solvability.
The corresponding quantum mechanical system has the prepotential \( W(x) \) and the compensation term \( \alpha_M(x) \) as

\[
W(x) = \frac{a}{4} x^4 - \frac{b}{2} x^2 + g \log x, \quad a, \ b, \ g > 0, \ 0 < x < \infty, \quad (3.13)
\]
\[
\alpha_M(x) = -aM x^2. \quad (3.14)
\]

This and the above two difference analogue systems share the same invariant polynomial subspace \((3.11), (3.12)\). The undeformed exactly solvable system, i.e. \((3.13)\) with \(a = 0\), has the Laguerre polynomials as eigenfunctions. The corresponding undeformed exactly solvable difference equations determined by \(V_0 \) \((3.4)\), \(V_{01} \) \((3.5)\) have the Wilson and the continuous dual Hahn polynomials as eigenfunctions. These are three and two parameter deformation of the Laguerre polynomial \([12, 9, 10]\).

### 3.2 Difference Equation Analogue of \(1/\sin^2 x\) Potential Deformed by \(\cos 2x\) Potential

This system is a quasi exactly solvable deformation of the exactly solvable dynamics which has the Askey-Wilson polynomials \([12, 9, 10]\) as eigenfunctions. Let us first introduce the variables and notation appropriate for the Askey-Wilson polynomials. We use variables \(\theta, x, z\), which are related as

\[
0 < \theta < \pi, \quad x = \cos \theta, \quad z = e^{i\theta}. \quad (3.15)
\]

The dynamical variable is \(\theta\) and the inner product is \((f|g) = \int_0^\pi d\theta f(\theta)^* g(\theta)\). We denote \(D \equiv z \frac{d}{dz}\). Then \(q^D\) is a \(q\)-shift operator, \(q^D f(z) = f(qz)\), with \(0 < q < 1\). The Hamiltonian is obtained by deforming the potential function \(V_0(z)\) by a linear polynomial in \(z\):

\[
\mathcal{H} \overset{\text{def}}{=} \sqrt{V(z)} q^D \sqrt{V(z)}^* + \sqrt{V(z)^*} q^{-D} \sqrt{V(z)} - (V(z) + V(z)^*) + \alpha_M(z), \quad (3.16)
\]
\[
= A^\dagger A + \alpha_M(z), \quad \alpha_M(z) \overset{\text{def}}{=} -abcede^{-1}(1 - q^M)(z + \frac{1}{z}), \quad (3.17)
\]
\[
A^\dagger \overset{\text{def}}{=} -i \left( \sqrt{V(z)} q^\frac{D}{2} - \sqrt{V(z)^*} q^{-\frac{D}{2}} \right), \quad A \overset{\text{def}}{=} i \left( q^\frac{D}{2} \sqrt{V(z)^*} - q^{-\frac{D}{2}} \sqrt{V(z)} \right), \quad (3.18)
\]
\[
V(z) \overset{\text{def}}{=} (1 - az)V_0(z), \quad V_0(z) \overset{\text{def}}{=} \frac{(1 - bj)(1 - cz)(1 - dz)(1 - ez)}{(1 - z^2)(1 - qz^2)}, \quad (3.19)
\]
\[
-1 < a, b, c, d, e < 1. \quad (3.20)
\]

The pseudo ground state wavefunction \(\phi_0(z)\) is determined as the zero mode of the \(A\)
operator $A\phi_0 = 0$:

$$\phi_0(z) \overset{\text{def}}{=} \sqrt{\frac{(z^2, z^{-2}; q)_\infty}{(az, az^{-1}, bz, bz^{-1}, cz, cz^{-1}, dz, dz^{-1}, ez, ez^{-1}; q)_\infty}}, \quad (3.21)$$

where $(a_1, \cdots, a_m; q)_\infty = \prod_{j=1}^{m} \prod_{n=0}^{\infty} (1 - a_j q^n)$. Obviously $\phi_0$ has no node or singularity in $0 < \theta < \pi$. We look for exact eigenvalues and eigenfunctions of the Hamiltonian (3.16) in the form:

$$\mathcal{H}\phi = \mathcal{E}\phi, \quad \phi(z) = \phi_0(z) P_M(x), \quad (3.22)$$

in which $P_M(x)$ is a degree $M$ polynomial in $x$ or in $z + 1/z = 2 \cos \theta = 2x$. The similarity transformed Hamiltonian acting on the polynomial space has the form

$$\tilde{\mathcal{H}} \overset{\text{def}}{=} \phi_0^{-1} \circ \mathcal{H} \circ \phi_0 = V(z) (q^D - 1) + V(z)^* (q^{-D} - 1) - abcdeq^{-1}(1 - q^M)(z + \frac{1}{z}). \quad (3.23)$$

Without the deformation factor $1 - az$ and the compensation term, the above Hamiltonian $\tilde{\mathcal{H}}$ keeps the polynomial subspace in $z + 1/z$ of any degree invariant. It is straightforward to show

$$\tilde{\mathcal{H}} \mathcal{V}_M \subseteq \mathcal{V}_M, \quad (3.24)$$

$$\mathcal{V}_M \overset{\text{def}}{=} \text{Span} \left[1, z + \frac{1}{z}, \ldots, \left(z + \frac{1}{z}\right)^k, \ldots, \left(z + \frac{1}{z}\right)^M\right], \quad \dim \mathcal{V}_M = M + 1. \quad (3.25)$$

The hermiticity of the Hamiltonian is obvious and the square-integrability of the eigenfunctions holds also true. This establishes the quasi exact solvability.

### 4 Summary and Comments

First let us summarise: Five explicit examples of quasi exactly solvable difference equations of one degree of freedom are derived by multiplicatively deforming the the known exactly solvable difference equations for the Meixner-Pollaczeek, continuous Hahn, continuous dual Hahn, Wilson and Askey-Wilson polynomials. The finite dimensional Hamiltonian matrix, no longer tri-diagonal, can be solved exactly by sweeping. All the eigenvalues and eigenfunctions are real, but the oscillation theorem, connecting the excitation level to the number of zeros, does not hold. Similarity and contrast with the known QES examples in ordinary quantum mechanics; the harmonic oscillator (with or without the centrifugal potential) deformed by a sextic potential and the $1/\sin^2 x$ potential deformed by a $\cos 2x$ potential, are demonstrated in some detail.
A few comments are in order. First let us stress that the mere existence of the finite dimensional invariant polynomial subspace \( \tilde{H} V_M \subseteq V_M \) is not sufficient for the quasi exact solvability. The theory must be endowed with a pseudo ground state wavefunction \( \phi_0 \), which must be nodeless and square integrable. Moreover, the reverse similarity transformed Hamiltonian \( H \overset{\text{def}}{=} \phi_0 \circ \tilde{H} \circ \phi_0^{-1} \) must be hermitian, in order to guarantee the real spectrum.

Let us consider an additively deformed potential with a cubic term

\[
V(x) \overset{\text{def}}{=} ax^3 + V_0(x), \quad \alpha_M(x) \overset{\text{def}}{=} a\mathcal{M}(\mathcal{M} - 1)x, \quad a \in \mathbb{R},
\]

with an exactly solvable \( V_0 \), for example,

\[
V_0(x) = b + ix, \quad \text{or} \quad (b + ix)(c + ix), \quad b, c \in \mathbb{R}^+,
\]

corresponding to the Meixner-Pollaczek and the continuous Hahn polynomials. It is rather trivial to verify the existence of the invariant polynomial subspace:

\[
\tilde{H} \overset{\text{def}}{=} V(x) (e^{-i\partial_x} - 1) + V(x)^* (e^{i\partial_x} - 1) + a\mathcal{M}(\mathcal{M} - 1)x,
\]

\[
\tilde{H} V_M \subseteq V_M, \quad V_M \overset{\text{def}}{=} \text{Span} \{1, x, x^2, \ldots, x^k, \ldots, x^M\}.
\]

Although we have not been able to derive the explicit form of the pseudo ground state wavefunction \( \phi_0(x) \) as a solution of \( A\phi_0 = 0 \), it seems rather unlikely that the \( \phi_0 \) satisfies the above mentioned criteria. This is because the corresponding quantum mechanical case, the quartic oscillator,

\[
W(x) = ax^3 - \frac{b}{2}x^2, \quad \alpha_M(x) = 6a\mathcal{M}x,
\]

also have the invariant polynomial subspace (4.4). The square integrability of \( \phi_0(x) = e^{W(x)} \) does not hold whichever sign \( a \) might take.

Let us discuss the hermiticity of the Hamiltonians (2.20) and (3.16). The hermiticity means \( \langle g|Hf \rangle = \langle Hg|f \rangle \) for a given inner product \( \langle g|f \rangle \) for arbitrary elements \( f \) and \( g \) in a certain dense subspace of the appropriate Hilbert space. The obvious choice for such a subspace is spanned by the ‘pseudo ground state’ wavefunction \( \phi_0 \) times polynomials. The types of the polynomials are:

\[\text{(a)}: \text{polynomials in } x \text{ for the Hamiltonians in section 2.2} \]

\[
\langle g|f \rangle = \int_{-\infty}^{\infty} g(x)^* f(x) dx, \quad f(x) = \phi_0(x)P(x), \quad g(x) = \phi_0(x)Q(x),
\]

\[\text{(b)}: \text{polynomials in } x^2 \text{ for the Hamiltonians in section 3.1} \]

(4.1)
\[ \langle g | f \rangle = \int_0^\infty g(x)^* f(x) dx, \quad f(x) = \phi_0(x) P(x^2), \quad g(x) = \phi_0(x) Q(x^2), \quad (4.9) \]

(c) : polynomials in \( x = \cos \theta \) for the Hamiltonians in section 3.2

\[ \langle g | f \rangle = \int_0^\pi g(\theta)^* f(\theta) d\theta, \quad f(\theta) = \phi_0(z) P(\cos \theta), \quad g(\theta) = \phi_0(z) Q(\cos \theta). \quad (4.11) \]

The Hamiltonians (2.20) and (3.16) consist of three parts:

\[ \mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3, \quad \mathcal{H}_3 = \alpha M - (V + V^*), \quad (4.12) \]

and

\[ \mathcal{H}_1 = \sqrt{V(x)} e^{-i\partial_x} \sqrt{V(x)^*}, \quad \mathcal{H}_2 = \sqrt{V(x)^*} e^{i\partial_x} \sqrt{V(x)}, \quad \text{for (2.20)}, \quad (4.13) \]
\[ \mathcal{H}_1 = \sqrt{V(z)} q^D \sqrt{V(z)^*}, \quad \mathcal{H}_2 = \sqrt{V(z)^*} q^{-D} \sqrt{V(z)}, \quad \text{for (3.16)}. \quad (4.14) \]

It is obvious that \( \mathcal{H}_3 \) is hermitian by itself. When \( \mathcal{H}_1 \) acts on \( f \), the argument is shifted from \( x \) to \( x - i \) or from \( \theta \) to \( \theta - i \log q \). With the compensating change of integration variable from \( x \) to \( x + i \) or from \( \theta \) to \( \theta + i \log q \) one can formally show \( \langle g | \mathcal{H}_1 f \rangle = \langle \mathcal{H}_1 g | f \rangle \) in a straightforward way. Similarly we have \( \langle g | \mathcal{H}_2 f \rangle = \langle \mathcal{H}_2 g | f \rangle \) by another change of integration variable. This is the ‘formal hermiticity.’

Actually, the shift of integration variable, to be realised by the Cauchy integral, would involve additional integration contours:

(a) : \((-\infty, \pm i - \infty), \quad (+\infty, \pm i + \infty) \quad \text{for the Hamiltonians in section 2.2} \quad (4.15)\]
(b) : \((0, \pm i), \quad (+\infty, \pm i + \infty) \quad \text{for the Hamiltonians in section 3.1} \quad (4.16)\]
(c) : \((0, \pm i \log q), \quad (\pi, \pi \pm i \log q) \quad \text{for the Hamiltonians in section 3.2} \quad (4.17)\]

It should be noted that all the singularities arising from \( V \) and \( V^* \) in cases (b) and (c) are cancelled by the zeros coming from the pseudo ground state wavefunctions \( \phi_0 \) and \( \phi_0^* \), and the Cauchy integration formula applies in all cases. The contribution of the additional contour integrals (4.15)–(4.17) cancel with each other and the shifts of integration variables is justified.

To be more specific, the contribution from the contours at infinity in (a) and (b) vanish identically due to the strong damping by \( \phi_0 \) and \( \phi_0^* \). The contribution from the two vertical contours in (b) passing the origin cancel with each other due to the evenness of \( \phi_0, \phi_0(x) = \phi_0(-x) \) and the polynomials in \( x^2, P((-x)^2) = P(x^2) \). Likewise the contribution from the two vertical contours in (c) passing the origin cancel with each other. Those in (c) passing
π also cancel with each other due to the $2\pi$ periodicity of $\phi_0$, $\phi_0(\theta) = \phi_0(2\pi + \theta)$. Note that the hermiticity in (b) and (c) cases holds only for the sum $\mathcal{H}_1 + \mathcal{H}_2$. This concludes the comments on hermiticity.

It is now evident that the scope of the present method, deformation of $W$ or $V$ for generating a quasi exactly solvable system from an exactly solvable dynamics, is rather limited. It is highly unlikely to get a quasi exactly solvable system, if $W$ contains a term higher than $x^5$ or $\cos 4x$, or if $V$ has a form $\prod_{j=1}^n (a_j + ix)$, $n \geq 5$ or $\left(\prod_{j=1}^n (a_j + ix)\right) / \{2ix(2ix + 1)\}$, $n \geq 7$. However, generalisation to multi-particle difference equations, the Ruijsenaars-Schneider-van Diejen systems \[7, 8\], is possible \[21\]. The corresponding ordinary quantum mechanical systems, \textit{i.e.} quasi-exactly solvable Calogero-Sutherland systems \[2, 3\] are derived by Sasaki and Takasaki \[15\].

For simplicity of presentation, we have restricted the parameters $a$, $b$, $c$, $d$, $e$ and $f$ in $V(x)$ to be real, positive etc. In most cases the ranges of parameters can be relaxed without losing quasi exact solvability. For example, in the first example of subsection 2.2.1 a configuration $a > 0$ and $b$ and $c$ are complex conjugate, $b = c^*$ with positive real parts, is also possible.

It is a challenge to see if these newly derived quasi exactly solvable difference equations can be understood by the existing ideas of QES; the $\text{sl}(2)$ algebra \[14\] and/or its deformations, various generalised super symmetry ideas \[19\]. Can we use the Bethe ansatz method to solve these problems? Are these QES systems equivalent to some spin systems \[20\]?

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References

[1] See, for example: L. Infeld and T. E. Hull, “The factorization method,” Rev. Mod. Phys. 23 (1951) 21-68; F. Cooper, A. Khare and U. Sukhatme, “Supersymmetry and quantum mechanics,” Phys. Rept. 251 (1995) 267-385, arXiv:hep-th/9405029.

[2] F. Calogero, “Solution of the one-dimensional $N$-body problem with quadratic and/or inversely quadratic pair potentials,” J. Math. Phys. 12 (1971) 419-436.
[3] B. Sutherland, “Exact results for a quantum many-body problem in one-dimension. II,” Phys. Rev. A5 (1972) 1372-1376.

[4] S. P. Khastgir, A. J. Pocklington and R. Sasaki, “Quantum Calogero-Moser models: Integrability for all root systems,” J. Phys. A33 (2000) 9033-9064, arXiv:hep-th/0005277.

[5] S. Odake and R. Sasaki, “Unified theory of annihilation-creation operators for solvable (‘discrete’) quantum mechanics,” J. Math. Phys. 47 (2006) 102102 (33 pages), arXiv:quant-ph/0605215. “Exact solution in the Heisenberg picture and annihilation-creation operators,” Phys. Lett. B641 (2006) 112–117, arXiv:quant-ph/0605221.

[6] S. Odake and R. Sasaki, “Exact Heisenberg operator solutions for multiparticle quantum mechanics,” J. Math. Phys. 48 (2007) 082106 (12 pages), arXiv:0706.0765[quant-ph].

[7] S. N. M. Ruijsenaars and H. Schneider, “A new class of integrable systems and its relation to solitons,” Annals Phys. 170 (1986) 370-405; S. N. M. Ruijsenaars, “Complete integrability of relativistic Calogero-Moser systems and elliptic function identities,” Comm. Math. Phys. 110 (1987) 191-213.

[8] J. F. van Diejen, “The relativistic Calogero model in an external field,” solv-int/9509002. “Multivariable continuous Hahn and Wilson polynomials related to integrable difference systems,” J. Phys. A28 (1995) L369-L374.

[9] G. E. Andrews, R. Askey and R. Roy, “Special Functions,” Encyclopedia of mathematics and its applications, Cambridge, (1999).

[10] R. Koekoek and R. F. Swarttouw, “The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue,” arXiv:math.CA/9602214.

[11] L. E. Gendenshtein, “Derivation of exact spectra of the Schrodinger equation by means of supersymmetry,” JETP Lett. 38 (1983) 356-359.

[12] S. Odake and R. Sasaki, “Shape invariant potentials in ‘discrete’ quantum mechanics,” J. Nonlinear Math. Phys. 12 Suppl. 1 (2005) 507-521, arXiv:hep-th/0410102. “Equilibrium positions, shape invariance and Askey-Wilson polynomials,” J. Math. Phys. 46
(2005) 063513 (10 pages), [arXiv:hep-th/0410109]; “Calogero-Sutherland-Moser Systems, Ruijsenaars-Schneider-van Diejen Systems and Orthogonal Polynomials,” Prog. Theor. Phys. 114 (2005) 1245-1260, [arXiv:hep-th/0512155]; “Equilibrium Positions and Eigenfunctions of Shape Invariant (‘Discrete’) Quantum Mechanics,” Rokko Lectures in Mathematics (Kobe University) 18 (2005) 85-110, [arXiv:hep-th/0505070].

[13] A. G. Ushveridze, Sov. Phys.-Lebedev Inst. Rep. 2, 50, 54 (1988); Quasi-exactly solvable models in quantum mechanics (IOP, Bristol, 1994); A. Y. Morozov, A. M. Perelomov, A. A. Roslyi, M. A. Shifman and A. V. Turbiner, “Quasiexactly solvable quantal problems: one-dimensional analog of rational conformal field theories,” Int. J. Mod. Phys. A 5 (1990) 803-832.

[14] A. V. Turbiner, “Quasi-exactly-soluble problems and sl(2,R) algebra,” Comm. Math. Phys. 118 (1988) 467-474.

[15] R. Sasaki and K. Takasaki, “Quantum Inozemtsev model, quasi-exact solvability and \( \mathcal{N} \)-fold supersymmetry,” J. Phys. A34 (2001) 9533-9553. Corrigendum J. Phys. A34 (2001) 10335.

[16] A. V. Turbiner, “Quasi-exactly soluble Hamiltonian related to root spaces,” J. Nonlinear Math. Phys. 12 Suppl. 1 (2005) 660-675.

[17] P. B. Wiegmann and A. V. Zabrodin, “Bethe-ansatz for Bloch electron in magnetic field,” Phys. Rev. Lett. 72 (1994) 1890-1893; “Algebraization of difference eigenvalue equations related to \( U_q(sl_2) \),” Nucl. Phys. B451 (1995) 699-724.

[18] A. Degasperis and S. N. M. Ruijsenaars, “Newton-equivalent Hamiltonians for the harmonic oscillator,” Ann. of Phys. 293 (2001) 92-109.

[19] See e.g.: A. A. Andrianov, M. V. Ioffe, V. P. Spiridonov, “Higher derivative supersymmetry and the Witten index,” Phys. Lett. A174 (1993) 273-279; V. G. Bagrov and B. F. Samsonov, “Darboux transformation, factorization and supersymmetry in one-dimensional quantum mechanics,” Theor. Math. Phys. 104 (1995) 1051-1060; S. M. Klishevich and M. S. Plyushchay, “Supersymmetry of parafermions,” Mod. Phys. Lett. A14 (1999) 2739-2752; H. Aoyama, H. Kikuchi, I. Okouchi, M. Sato and S. Wada, “Valley views: Instantons, large order behaviors, and supersymmetry,” Nucl. Phys.
B553 (1999) 644-710; H. Aoyama, M. Sato and T. Tanaka, “General forms of a N fold supersymmetric family,” Phys. Lett. B503 (2001) 423-429.

[20] V.V. Ulyanov and O.B. Zaslavskii, “New methods in the theory of quantum spin systems,” Phys. Rep. 216 (1992) 179-251.

[21] S. Odake and R. Sasaki, “Multi-particle quasi exactly solvable difference equations,” arXiv:0708.0716[nlin:SI], YITP-07-44, DPSU-07-3.