ANALYTICAL APPROXIMATION FOR 2-D NONLINEAR PERIODIC DEEP WATER WAVES

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Abstract. A recently developed method [1], [2], [3] has been extended to a nonlocal equation arising in steady water wave propagation in two dimensions. We obtain analytic approximation of steady water wave solution in two dimensions with rigorous error bounds for a set of parameter values that correspond to heights slightly smaller than the critical. The wave shapes are shown to be analytic. The method presented is quite general and does not assume smallness of wave height or steepness and can be readily extended to other interfacial problems involving Laplace’s equation.

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1. Introduction

Recently [1], [2], [3], a method has been developed for study of nonlinear differential equations where strong nonlinearity can be reduced to weakly nonlinear analysis even when the problem has no natural perturbation parameter. The idea is quite natural: consider an equation in the form \( N[u] = 0 \), where \( N \) is some nonlinear operator in some suitable function space. A crucial part of this process is to determine a quasi-solution \( u_0 \) so that \( R = N[u_0] \) is small in an appropriate norm and \( u_0 \) comes close to satisfying appropriate initial and/or boundary conditions. Then, proving that there exists solution \( u \) satisfying \( N[u] = 0 \) is equivalent to showing that \( E = u - u_0 \) satisfies appropriately small initial/boundary conditions and \( \mathcal{L}[E] = -R - N_1[E] \), where the linear operator \( \mathcal{L} \) is the Fréchet derivative \( N_u \) at \( u = u_0 \) and \( N_1[E] = N[u_0 + E] - N[u_0] - \mathcal{L}[E] \) contains only nonlinear terms. When \( \mathcal{L} \) is suitably invertible subject to initial/boundary conditions and the nonlinearity \( N_1 \) sufficiently regular, then standard contraction mapping provides a rigorous proof of existence of solution to the weakly nonlinear problem for \( E \). Thus existence of solution to original problem \( N[u] = 0 \) is shown, while at the same time a rigorous error bound on \( u - u_0 \) is obtained. An added benefit to this method relative to abstract nonconstructive methods for proving solutions is that one obtains a concrete expression for the approximate solution \( u_0 \). The only non-standard part of this program is to come up with good candidates for quasi-solution \( u_0 \). In previous studies [1], [2], [3], this has involved application of classical orthogonal polynomial approximations in finite domains coupled with exponential asymptotic approach in its complement when domains extend to \( \infty \).

In the present paper, we show that the quasi-solution method can be extended to a nonlinear integral equation arising in propagation of steady two dimensional deep water waves of finite amplitude for a set of values in a range of wave heights. We provide accurate efficient representation for water waves and at the same time provide rigorous error bounds for these approximations. The literature for water waves is quite extensive and goes back two centuries involving some of the best known mathematicians Laplace, Lagrange, Cauchy, Poisson, Airy, Stokes and many others (see, for instance, a recent review [5]). There are many aspects of the water wave problem; these include steady state analysis, linear and nonlinear stability of these states, the initial value problem and long time behavior. There is also much interest in finite depth wave propagation and in particular limiting cases when KdV or Boussinesq models are valid. There is also interest in waves in the presence of
shear and other variants that arise in modeling wind-water interaction. The effect of boundaries is also of interest. In principle, the method given here can be extended to every one of these problems.

Here we are concerned only with steady periodic solutions in two dimensions in deep water. Existence of steady two dimensional periodic deep water waves of small amplitudes was shown by Nekrasov [6], Levi-Civita [7]. Larger amplitude waves were also studied more recently [8], [9], [10] culminating in the proof [11] of Stokes’ conjecture of a a 120° angle at the apex of the wave with highest height $h_M$. There have been many numerical calculations as well for water waves including an elucidation of the delicate behavior near highest wave (see for instance [12]-[24] some of which have been proved [25], [26] Further, there is numerical evidence for bifurcation to to periodic waves with multiple crests with unequal heights [27] as well as to non-symmetric waves [28] that is yet to be proved.

It is also interesting to note that the mathematical formulation used in numerical calculations and rigorous analysis have been rather different; one relying on series representation similar in the spirit of Stokes, while the other relies primarily on integral reformulation due to to Nekrasov [6]. The present approach is constructive in that we present approximate solution with rigorous error bounds; hence proof of existence of solution follows as a consequence. In some sense, the approach combines constructive numerical calculations with mathematical rigor. We expect this to be helpful both in the rigorous stability analysis and bifurcation studies where details of the solution are likely to be critical. Another important aspect of the present analysis is that the approach is quite general and may be readily extended to other free boundary problems, particularly ones that involve analytic functions of a complex variable (for e.g. Hele-Shaw Flow, Stokes Bubbles, Vortex patches, just to name a few ). Further, with an eye towards generalization to other interfacial problems, we employ a straight forward series representation and use spaces isometric to a weighted $l^1$ space. A bi-product of the analysis is that analyticity of the boundary follows for waves with a sequence of heights smaller than the critical for which quasi-solutions have been determined, though analyticity also follows from other methods in more general contexts [32], [31].

2. Steady Water Waves Formulations

We non-dimensionalize length and time scales implicit in setting wavelength and gravity constant $g$ to be $2\pi$ and 1 respectively. It is known that the existence of a steady symmetric water wave in two dimensions when vorticity is unimportant is equivalent to showing that there exists analytic function $f$ inside the unit $\zeta$-circle so that $(1 + \zeta f') \neq 0$ for $|\zeta| \leq 1$ and

$$\text{Re} f = -\frac{c^2}{2|1 + \zeta f'|} \quad \text{on} \quad |\zeta| = 1,$$

where $c$ is the non-dimensional wave speed. Further, for symmetric water waves, $f$ is real valued on the real diameter $(-1,1)$, implying real $\ddot{f}_j$ in the following
representation of $f$:

$$f(\zeta) = \sum_{j=0}^{\infty} \hat{f}_j \zeta^j$$  \hspace{1cm} \text{(2)}$$

It is to be noted that $i(\log \zeta + f(\zeta)) + 2\pi$ is the conformal map that maps the interior of a cut unit-circle to a periodic strip in the water-wave domain in a frame where wave profile is stationary, with $\zeta = \pm 1$ corresponding to wave trough and crest, respectively. The condition $1 + \zeta f' \neq 0$ in $|\zeta| \leq 1$ ensures univalency of the conformal map. The formulation (1)-(2) is closely related to those used by others including Stokes himself. Nekrasov\cite{6} integral reformulation also follows directly from it as discussed in the ensuing and involves a parameter

$$\mu = \frac{v_{\text{crest}}^3}{3 \hat{c}} = \frac{c^2}{3 \left| 1 + \zeta f' \right|^{3/4}_{\zeta=-1}}$$  \hspace{1cm} \text{(3)}$$

where $v_{\text{crest}}$ is the dimensional speed of fluid at the crest and $\hat{c}$ is the dimensional wave speed. For efficiency in representation, it is better to represent $f$ in a series in $\eta$:

$$f(\eta) = \sum_{j=0}^{\infty} F_j \eta^j ,$$  \hspace{1cm} \text{(4)}$$

where

$$\eta = \frac{\zeta + \alpha}{1 + \alpha \zeta} ,$$  \hspace{1cm} \text{(5)}$$

for $\alpha \in (0, 1)$, where $\alpha$ will be appropriately chosen. The crest speed parameter (3) in this formulation becomes

$$\mu = \frac{c^2}{3 \left| 1 + \eta q f \eta \right|^{3/4}_{\eta=-1}} ,$$  \hspace{1cm} \text{(6)}$$

where

$$q(\eta) = \frac{(\eta - \alpha)(1 - \alpha \eta)}{\eta(1 - \alpha^2)} .$$  \hspace{1cm} \text{(7)}$$

The non-dimensional wave height\cite{12} is given by

$$h = -\frac{1}{2} \left[ \text{Re} f(1) - \text{Re} f(-1) \right] = -\sum_{j=1, \text{odd}}^{\infty} f_j .$$  \hspace{1cm} \text{(8)}$$

Earlier evidence \cite{14}, \cite{15}, \cite{29} suggests that for deep water waves with one trough and one peak in a period, there is only one 1/2 singularity of $f$ at $\zeta = -\zeta_s$ for $\zeta_s^{-1} \in (0, 1)$ in the finite complex plane and a fixed logarithmic type singularity at $\zeta = \infty$. Evidence suggests that $\zeta_s^{-1}$ increases monotonically with $h \in (0, h_M)$, where $h_M \approx 0.4435 \cdots \cite{22}$ corresponds to the Stokes highest wave that makes a 120° angle at the apex. If $\mu \in \left(0, \frac{1}{3}\right)$ is used as a parameter, $\mu = \frac{1}{3}$ corresponds to $h = 0$.

\hspace{1cm} \text{(1) Some authors define } 2h \text{ as the non-dimensional height, while others present results for the scaled height } \frac{h}{h_M} . \hspace{1cm} \text{(2) Reported values of } \frac{h_M}{h} \text{ from computation differ slightly between \cite{22} and \cite{18} (0.1412 versus 0.141063)
while \( \mu = 0 \) corresponds to Stokes highest wave \( h = h_M \), which has a stagnation point at the crest. The optimal choice of \( \alpha \) that ensures the most rapid decay \( f_j \) with \( j \) is one where \( \zeta = -\zeta_s \), \( \zeta = \infty \) are mapped to equidistant points from the origin in the \( \eta \) plane, i.e. when \( \alpha = \alpha_0 = \zeta_s - \sqrt{\zeta_s^2 - 1} \). Since the relation of \( \zeta_s \) with height (or \( \mu \)) is only known numerically, we choose a simple empirical relation:

\[
\alpha = \frac{2}{237} + \frac{67}{11} \left( \frac{1}{3} - \mu \right) - \frac{113}{3} \left( \frac{1}{3} - \mu \right)^2 + \frac{165}{2} \left( \frac{1}{3} - \mu \right)^3
\]

that appears to be optimal for small \( \mu \) corresponding to large amplitude waves; the choice is is not the best for small \( \frac{1}{3} - \mu \), but it matters little since \( f_j \) decays rapidly in any case for small wave height. Note that any choice of \( \alpha \in (0, 1) \) still ensures a convergent series for \( f \) in \( \eta \); an optimal choice of \( \alpha \) ensures better accuracy in a finite truncation. In the \( \eta \) variable, the boundary condition (1) becomes

\[
\text{Re} f = -\frac{c^2}{2\left|1 + q(\eta)f'(\eta)\right|^2} \text{on } |\eta| = 1
\]

where \( q \) is given by (7). We note that on the unit \( \eta \)-circle, \( q = \frac{|\eta - \alpha|^2}{1 - \alpha^2} \) is real valued.

On \( \eta = e^{i\nu} \) and taking derivative with respect to \( \nu \) of the relation (10) and multiplying through by \( q \) (which is real), we obtain

\[
-\text{Im}(q\nu f') = \frac{c^2 q}{\left|1 + q\eta f'\right|^2} \text{Re} \left\{ \frac{d}{d\nu} \log (1 + \eta q f') \right\}
\]

If we introduce new variable

\[
w = -\frac{2}{3} \log c + \log (1 + \eta q f')
\]

implies \( 1 + \eta q f' = c^{2/3} e^{\text{Rew}} \), then (11) implies \( w \) is analytic in the unit-\( \eta \) circle and that on \( \eta = e^{i\nu} \), \( w \) satisfies

\[
\frac{d}{d\nu} \text{Rew} + q^{-1} e^{2\text{Rew}} \text{Im} w = 0
\]

This is an alternate formulation of the water wave problem. This is equivalent to Nekrasov’s integral formulation. If we define \( \theta = \text{Im} w \), and integrate (13) from \( \nu = \pi \) to a variable \( \nu \) using the Hilbert transform relation between \( \text{Re} w \) to \( \text{Im} w \) on \( |\eta| = 1 \), integration by parts gives the integral equation:

\[
\theta(\nu) = -\frac{1}{3\pi} \int_{-\pi}^{\pi} \log \left| \sin \frac{\nu - \nu'}{2} \right| \sin \left[ \theta(\nu') \right] \mu + \int_{\pi}^{\nu} \sin \frac{\theta(s)}{q(s)} ds \, d\nu'
\]

If we set \( \alpha = 0 \) (in which case \( q = 1 \)), then (14) reduces\(^{(3)}\) to Nekrasov\(^{(2)}\) integral equation, when oddness of \( \phi \) in \( \nu - \pi \) is used. In the \( \nu \) variable, the relation (10) becomes

\[
\mu = \frac{1}{3} \exp \left[ -3w(-1) \right]
\]

using \( w(-1) \) to be real.

\(^{(3)}\) Rather a change of variable \( \nu \rightarrow \nu - \pi \) gives the Nekrasov form
3. Quasi-solution and Transformation to a Weakly Nonlinear Problem

For given $\mu \in (0, \frac{1}{3})$ corresponding to $h \in (0, h_M)$, we define a quasi solution $(f_0, c_0)$ with the property that $f_0$ is analytic inside the unit circle with $1 + q\eta f_0' \neq 0$ in $|\eta| \leq 1$, and that on $\eta = e^{i\nu}$, the residual $R_0(\nu)$, defined below, along with its derivative and the quantity

$$w_0(-1) + \frac{1}{3} \log \frac{3}{\mu} = \frac{1}{3} \log \frac{\mu_0}{\mu},$$

are each small enough for Proposition 21 to hold. Here,

$$R_0(\nu) = \left| 1 + q(\eta)\eta f_0'(\eta) \right|^2 \text{Re} f_0 + \frac{c_0^2}{2},$$

on $\eta = e^{i\nu}$. We note that if $f_0$ is a polynomial in $\eta$ of order $N$, then $R_0(\nu)$ is a polynomial in $\cos \nu$ of order $2N + 1$, which can be computed without errors for if $c_0$ and coefficients of the $f_0$ series are chosen as rational numbers. This can be transformed to a Fourier cosine series with only the first $2N + 2$ possibly non-zero terms.

We note that the representation of the analytic function $w$ inside the unit $\eta$-circle:

$$w(\eta) = \sum_{j=0}^{\infty} b_j \eta^j, \text{ where } b_j \text{ is real}$$

Since $q(\alpha) = 0$, it follows that $w(\alpha) = -\frac{2}{3} \log c$, i.e.

$$-\frac{2}{3} \log c = \sum_{j=0}^{\infty} b_j \alpha^j$$

Corresponding to the quasi-solution $(f_0, c_0)$, we define

$$w_0 = -\frac{2}{3} \log c_0 + \log (1 + \eta q(\eta) f_0')$$

Then, we can check that $w_0$ satisfies

$$\frac{d}{d\nu} \text{Re} w_0 + q^{-1} e^{2\text{Re} w_0} \text{Im} w_0 = R(\nu) := -\frac{R_0'(\nu)}{c_0^2 - 2R_0} - \frac{4A(\nu)R_0(\nu)}{3(c_0^2 - 2R_0)},$$

where

$$2A(\nu) = 3q^{-1} e^{2\text{Re} w_0} \text{Im} \{e^{w_0}\} = \frac{3}{c_0^2} \text{Im} \{\eta f_0'\} \left| 1 + \eta q f_0' \right|^2,$$

It is to be noted that if $f_0$ is a polynomial in $\eta$ of degree $N$, then (22) implies that $A(\nu)$ is a polynomial in $\cos \nu$ of order $2N + 1$, and therefore $A(\nu)$ has a finite Fourier sine series with only the first $2N + 2$ terms that are possibly nonzero. Again, as with $R_0$, if $c_0$ and polynomial coefficients of $f_0$ are given as rationals, the calculation of Fourier sine series coefficient of $A(\nu)$ can be done without round-off errors. We also note that

$$w_0(\alpha) = -\frac{2}{3} \log c_0$$

\footnote{For $n$ not too large, this can be done by hand, though use of symbolic language Maple or Mathematica eases the task}
Corresponding to the given quasi-solution \((f_0, c_0)\), the wave height \(h_0\) and wave crest speed parameter \(\mu_0\) are given by
\[
(24) \quad h_0 = -\frac{1}{2} \left[ f_0(1) - f_0(-1) \right], \quad \mu_0 = \frac{c_0^2}{3[1 + \eta q \partial_\eta f_0]^{3}_{\eta=-1}},
\]
which may be computed without round-off errors for rational \(c_0\) and polynomial representation of \(f_0\) involving rational coefficients.

Now, we seek to prove that there are solutions nearby \(w_0\). For that purpose, we decompose
\[
(25) \quad w = w_0 + W.
\]
It follows from (13) and (21) that \(W\) satisfies
\[
(26) \quad \frac{d}{d\nu} \Re \{W\} + 2A(\nu) \Re \{W\} + 2B(\nu) \Im \{W\} = \tilde{\mathcal{M}}[W] - R(\nu)
\]
where on \(\eta = e^{i\nu}\),
\[
(27) \quad 2B(\nu) = q^{-1} e^{2\Re q \eta} \Re \{e^{q \eta}\} = \frac{1}{q c_0} [1 + \eta q f_0]^2 \Re \{1 + q \eta f'_0\},
\]
and the nonlinear operator \(\tilde{\mathcal{M}}\) is defined so that
\[
(28) \quad \tilde{\mathcal{M}}[W] = -\frac{2}{3} A(\nu) M_1 - 2B(\nu) M_2,
\]
where
\[
(29) \quad M_1 = e^{2\Re W} \Re e^{W} - 1 - 3\Re W, \quad M_2 = e^{2\Re W} \Im e^{W} - \Im W
\]
It is to be noted from (27) that a polynomial \(f_0\) in \(\eta\) of degree \(N\) immediately implies that \(\tilde{B}(\nu) = qB(\nu)\) is a polynomial in \(\cos \nu\) of degree \(2N + 2\) and therefore has a truncated Fourier cosine series representation with at most \(2N + 3\) terms. After changes of variable, the constraint (13) implies
\[
(30) \quad W(-1) = \frac{1}{3} \log \frac{\mu_0}{\mu}, \quad \text{where} \quad \mu_0 = \frac{1}{3} e^{-3w_0(-1)}
\]
which is small from requirement on quasi-solution. Once a solution is found for \(W\), the corresponding height of the water wave is given by
\[
(31) \quad h = h_0 - \frac{1}{2}(1 - \alpha^2) \int_{-1}^{1} \frac{e^{W(\eta) - W(\alpha)} - 1}{(\eta - \alpha)(1 - \alpha \eta)} [1 + \eta q(\eta) f'_0(\eta)] d\eta,
\]
where, noting \(f_0\) to be real valued on the real diameter \([-1, 1]\),
\[
(32) \quad h_0 = -\frac{1}{2} \left[ f_0(1) - f_0(-1) \right]
\]
It is convenient to separate out the linear and nonlinear parts of (31) in the form
\[
(33) \quad h = h_0 + \mathcal{F}[W] + \mathcal{Q}[W]
\]
where the functionals \(\mathcal{F}\) and \(\mathcal{Q}\) are defined by
\[
(34) \quad \mathcal{F}[W] = -\frac{1}{2}(1 - \alpha^2) \int_{-1}^{1} \frac{W(\eta) - W(\alpha)}{(\eta - \alpha)(1 - \alpha \eta)} [1 + \eta q(\eta) f'_0(\eta)] d\eta,
\]
\[
(35) \quad \mathcal{Q}[W] = -\frac{1}{2}(1 - \alpha^2) \int_{-1}^{1} \frac{W(\eta) - W(\alpha) - 1 - W(\eta) + W(\alpha)}{(\eta - \alpha)(1 - \alpha \eta)} [1 + \eta q(\eta) f'_0(\eta)] d\eta.
\]
Once $W$ is determined, the actual wave speed is determined from

\[ W(\alpha) = -\frac{2}{3} \log \frac{c}{c_0} \]

Define

\[ \Phi(\nu) = \text{Re} W(e^{i\nu}) \]

Analyticity of $W$ in the unit circle with sufficient regularity\(^{(5)}\) up to $|\eta| = 1$ implies

\[ \Psi(\nu) = \text{Im} W(e^{i\nu}) = \frac{1}{2\pi} \text{PV} \int_{-\pi}^{\pi} \cot \frac{\nu - \nu'}{2} \Phi(\nu') d\nu' \]

Then (36) may be written abstractly as

\[ L\Phi = M[\Phi] - R(\nu) \]

where $M[\Phi] = M[W]$ where $\Phi(\nu) = \text{Re} W(e^{i\nu})$ for $W$ analytic in $|\eta| < 1$ and suitably regular in $|\eta| \leq 1$, and

\[ L\Phi := \Phi'(\nu) + 2A(\nu)\Phi(\nu) + 2B(\nu)\Psi(\nu) \]

We will first prove that each given $a_1 \in [-\epsilon_0, \epsilon_0] = I$ for sufficiently small $\epsilon_0$, $L$ is invertible in a space of functions defined later, and (39) in that space is equivalent to

\[ \Phi = K'M[\Phi] - KR + a_1 G := N[\Phi] \]

for some function $G$, and $K$ is a bounded linear operator. We will then show that for each $a_1 \in I$, the operator $N$ is contractive in a small ball in some function space if quasi-solution satisfies certain conditions that can be readily checked. This corresponds to a waterwave for which corresponding $\mu$ is in some small neighborhood of $\mu_0$ because of the relation

\[ \frac{1}{3} \log \frac{\mu_0}{\mu} = W(-1) = \Phi(\pi) \]

Using (41), we may rewrite (42) in the form

\[ a_1 = \frac{1}{G(\pi)} \left( \frac{1}{3} \log \frac{\mu_0}{\mu} + KR[\pi] - K'M[\Phi][\pi] \right) =: U[a_1] \]

We will then prove $U : I \rightarrow I \rightarrow I$ is contractive when appropriate smallness conditions are satisfied by quasi-solution and $G(\pi)$ is not small in which case there exists unique $a_1 \in I$ so that (42) is satisfied for the specified $\mu$.

4. Definitions, Space of Functions and main results

**Definition 1.** For fixed $\beta \geq 0$, define $A$ to be the space of analytic functions in $|\eta| < e^\beta$ with real Taylor series coefficient at the origin, equipped with norm:

\[ \| W \|_A = \sum_{l=0}^{\infty} e^{\beta l} |W_l| \]

where

\[ W(\eta) = \sum_{l=0}^{\infty} W_l \eta^l \]

\(^{(5)}\)The regularity requirements will be clear in the definition of space $A$
Remark 1. It is easily seen that $W \in \mathcal{A}$ implies $W$ is continuous in $|\eta| \leq e^\beta$. Further, in the domain $|\eta| \leq e^\beta$, $\|W\|_{\infty} \leq \|W\|_{\mathcal{A}}$.

Definition 2. For $\beta \geq 0$, define $\mathcal{E}$ to be the Banach space of real $2\pi$-periodic even functions $\phi$ so that

\begin{equation}
\phi(\nu) = \sum_{j=0}^{\infty} a_j \cos(j\nu), \text{ with norm } \|\phi\|_{\mathcal{E}} := \sum_{j=0}^{\infty} e^{\beta j} |a_j| < \infty
\end{equation}

Define $\mathcal{S}$ to be Banach space of real $2\pi$-periodic odd functions such that

\begin{equation}
\psi(\nu) = \sum_{j=1}^{\infty} b_j \sin(j\nu), \text{ with norm } \|\psi\|_{\mathcal{S}} := \sum_{j=1}^{\infty} e^{\beta j} |b_j| < \infty
\end{equation}

Remark 2. It is clear that if $\phi \in \mathcal{E}$ if and only if there exists $W \in \mathcal{A}$ so that $\phi(\nu) = \Re W(e^{i\nu})$. Similarly, $\psi \in \mathcal{S}$ if and only if $\psi(\nu) = \Im W(e^{i\nu})$ for some $W \in \mathcal{A}$. We also note that for such $W$, $\|\phi\|_{\mathcal{E}} = \|W\|_{\mathcal{A}}$, while $\|\psi\|_{\mathcal{S}} \leq \|W\|_{\mathcal{A}}$.

Remark 3. The space $\mathcal{A}$ and $\mathcal{E}$ are clearly isomorphic to each other and to $\mathcal{H}$, the space of sequences of real Taylor series coefficients $W = (W_0, W_1, \cdots)$ with weighted $l^1$ norm

\begin{equation}
\|W\|_{\mathcal{H}} = \sum_{l=0}^{\infty} e^{\beta l} |W_l|
\end{equation}

Because of this isomorphism we will move back and forth between spaces $\mathcal{A}$, $\mathcal{E}$ and $\mathcal{H}$ as convenient. Similarly the subspace $\mathcal{H}_0 \subset \mathcal{H}$ that consists of all $W = (0, W_1, W_2, \cdots)$ is isomorphic to $\mathcal{S}$.

Theorem 1. (Main Result) For $\mu \in S$, defined as

\begin{equation}
S := \{ \mu : \mu \in \bigcup_{j=1}^{3} I_{\mu_j}, \text{ where } \mu_1 = 0.0018306, \mu_2 = 0.002, \mu_3 = 0.0023 \}
\end{equation}

where $I_{\mu_j}$ is some sufficiently small interval containing $\mu_j$, the solution $w$ to the water wave problem (13) has the representation

\begin{equation}
w = -\frac{2}{3} \log c_0 + \log (1 + \eta q f_0') + W
\end{equation}

where quasi-solution $(f_0, c_0)$ is specified in $[14]$ for different cases, and $W \in \mathcal{A}$ satisfies error bounds

\begin{equation}
\|W\|_{\mathcal{A}} \leq M_E
\end{equation}

where $M_E$, depending on $\mu$, is specified in $[14]$ and for all cases is less than $2.2 \times 10^{-4}$. The corresponding nondimensional wave speed and heights $(c, h)$ are close to $(c_0, h_0)$ reported in $[14]$ in the sense that

\begin{equation}
|h - h_0| \leq K_3 M_E \left(1 + 2e^{1/4} M_E \right)
\end{equation}

\begin{equation}
\left| \log \frac{c}{c_0} \right| \leq \frac{3}{2} M_E
\end{equation}

for some constant $K_3$ that depends on $\mu$, estimated in $[14]$. In all cases considered $K_3 \leq 5.24$. 

Proof. The proof of the theorem follows by showing that Propositions 21 and 23 in the ensuing apply for each proposed quasi-solution in §9 and determining bounds on solutions \( \Phi \in \mathcal{E} \) satisfying the weakly nonlinear problem (41), where \( \Phi = \text{Re}W \).

Remark 4. In all likelihood, the error estimates for \( M_E \) in the Theorem is an overestimate by a factor of about a thousand or so. This is suggested from comparison with a sequence of numerical calculations with increasing number of modes.

5. Preliminary Lemmas

Lemma 3. If \( W, V \in \mathcal{A}, \) \( WV \in \mathcal{A} \) and

\[
\|WV\|_\mathcal{A} \leq \|W\|_\mathcal{A} \|V\|_\mathcal{A}
\]

Proof. We note that if

\[
W(\eta) = \sum_{l=0}^{\infty} W_l \eta^l, \quad V(\eta) = \sum_{l=0}^{\infty} V_l \eta^l
\]

then using the convolution expression for power series of \( WV \),

\[
\|WV\|_\mathcal{A} \leq \sum_{k=0}^{\infty} e^{\beta k} \sum_{l=0}^{k} |V_l||W_{k-l}| = \sum_{l=0}^{\infty} e^{\beta l} |W_l| \left\{ \sum_{j=0}^{\infty} |W_j| e^{\beta j} \right\} = \|W\|_\mathcal{A} \|V\|_\mathcal{A}
\]

\[
\|WV\|_\mathcal{A} \leq \sum_{k=0}^{\infty} e^{\beta k} \sum_{l=0}^{k} |V_l||W_{k-l}| = \sum_{l=0}^{\infty} e^{\beta l} |W_l| \left\{ \sum_{j=0}^{\infty} |W_j| e^{\beta j} \right\} = \|W\|_\mathcal{A} \|V\|_\mathcal{A}
\]

Corollary 4. If \( W \in \mathcal{A} \), then for any \( m \geq 0 \),

\[
\| \sum_{j=m}^{\infty} \frac{W_j}{j!} \|_\mathcal{A} \leq \sum_{j=m}^{\infty} \frac{1}{j!} \|W\|_\mathcal{A}^j = e^{\|W\|_\mathcal{A}} - \sum_{j=0}^{m-1} \frac{\|W\|_\mathcal{A}}{j!}
\]

Proof. The proof follows immediately by using the Banach algebra property in Lemma 3.

Lemma 5. If \( \phi_1, \phi_2 \in \mathcal{E} \), then \( \phi_1 \phi_2 \in \mathcal{E} \); if \( \psi_1, \psi_2 \in \mathcal{S} \), then \( \psi_1 \psi_2 \in \mathcal{E} \) with

\[
\|\phi_1 \phi_2\|_\mathcal{E} \leq \|\phi_1\|_\mathcal{E} \|\phi_2\|_\mathcal{E}
\]

(59)

\[
\|\psi_1 \psi_2\|_S \leq \|\psi_1\|_S \|\psi_2\|_S
\]

Further if \( \phi \in \mathcal{E} \) and \( \psi \in \mathcal{S} \), \( \phi \psi \in \mathcal{S} \) with

\[
\|\phi \psi\|_S \leq \|\phi\|_S \|\psi\|_S
\]

Proof. Assume

\[
\phi_1(\nu) = \sum_{j=0}^{\infty} a_j \cos(j\nu), \quad \phi_2(\nu) = \sum_{j=0}^{\infty} c_j \cos(j\nu)
\]

We note that if we define \( \hat{a}_j = \frac{a_{-j}}{2} \) and \( \hat{b}_j = \frac{b_{-j}}{2} \) for \( j \in \mathbb{Z} \setminus \{0\} \), and \( \hat{a}_0 = a_0, \hat{b}_0 = b_0 \), then \( \phi_1, \phi_2 \) has complex Fourier representations

\[
\phi_1(\nu) = \sum_{j \in \mathbb{Z}} \hat{a}_j e^{ij\nu}, \quad \phi_2(\nu) = \sum_{j \in \mathbb{Z}} \hat{b}_j e^{ij\nu}
\]
We also note that in the complex Fourier representation, we may write

\[ \| \phi_1 \|_E = \sum_{j \in \mathbb{Z}} |\hat{a}_j| e^{\beta |j|}, \| \phi_2 \|_E = \sum_{j \in \mathbb{Z}} |\hat{b}_j| e^{\beta |j|} \]

Then

\[ \phi_1(\nu)\phi_2(\nu) = \sum_{k \in \mathbb{Z}} e^{ik\nu} \sum_{l \in \mathbb{Z}} |\hat{a}_k - l\hat{b}_l| \]

Therefore,

\[ \| \phi_2\phi_2 \|_E = \sum_{k \in \mathbb{Z}} e^{\beta |k|} \left| \sum_{l \in \mathbb{Z}} |\hat{a}_k - l\hat{b}_l| \right| \leq \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{\beta |k-l|}|\hat{a}_k - l\hat{b}_l| \]

\[ = \sum_{l \in \mathbb{Z}} |\hat{b}_l| e^{\beta |l|} \left[ \sum_{j \in \mathbb{Z}} |\hat{a}_j| e^{\beta |j|} \right] = \| \phi_1 \|_E \| \phi_2 \|_E \]

Assume

\[ \psi_1(\nu) = \sum_{j=1}^{\infty} a_j \sin(j\nu), \psi_2(\nu) = \sum_{j=1}^{\infty} c_j \sin(j\nu) \]

We define \( \hat{a}_j = \frac{1}{2\pi j} a_{|j|}, \hat{b}_j = \frac{1}{2\pi j} b_{|j|}, \) for \( j \in \mathbb{Z} \setminus \{0\}, \) and \( \hat{a}_0 = \hat{b}_0 = 0; \) then \( \psi_1, \psi_2 \) have complex Fourier representations

\[ \psi_1(\nu) = \sum_{j \in \mathbb{Z}} \hat{a}_j e^{j\nu}, \psi_2(\nu) = \sum_{j \in \mathbb{Z}} \hat{b}_j e^{j\nu}. \]

We also note that

\[ \| \psi_1 \|_S = \sum_{j \in \mathbb{Z}} |\hat{a}_j| e^{\beta |j|}, \| \psi_2 \|_S = \sum_{j \in \mathbb{Z}} |\hat{b}_j| e^{\beta |j|} \]

Therefore, using the convolution expression in terms of \( \hat{a}_j \) and \( \hat{b}_j \) it is clear that as for product \( \phi_1\phi_2, \)

\[ \| \psi_1\psi_2 \|_E \leq \| \psi_1 \|_S \| \psi_2 \|_S \]

The third expression follows in a similar manner using a complex Fourier Representation.

**Corollary 6.** If \( W \in A, \) then on \( |\eta| = 1, \) for any \( m \geq 0, \)

\[ \| e^{ReW(\eta)} - \sum_{j=0}^{m-1} \frac{[Re W(\eta)]^j}{j!} \|_E \leq e^{\|Re W(e^{i\nu})\|_E} - \sum_{j=0}^{m-1} \frac{[Re W(e^{i\nu})]^j}{j!} \]

**Proof.** This simply follows from noting that

\[ e^{ReW} - \sum_{j=0}^{m-1} \frac{[Re W]^j}{j!} = \sum_{j=m}^{\infty} \frac{[Re W]^j}{j!} \]

and using the Banach algebra property in the previous Lemma.
Lemma 7. If $R_0 \in E$, $R'_0 \in S$ and $\|R_0\|_E < \frac{\sqrt{3}}{2}$, then $R \in S$ (recall definition in (21)) with

\[ ||R||_S \leq \frac{\|R'_0\|_S}{c_0^2 - 2\|R_0\|_E} + \frac{4\|A\|_S\|R_0\|_E}{3(c_0^2 - 2\|R_0\|_E)} \]

Proof. We use the definition of $R$ in (21) and Banach Algebra properties in the preceding lemmas applied to a series expansion of $(1 - \frac{2}{c_0^2}R_0)^{-1}$ for small $R_0$. The proof readily follows.

Lemma 8. For $0 \leq \beta < \log \alpha^{-1}$, if $\phi \in E$, then $\frac{1}{q}\phi \in E$, where for $\phi = \sum_{\ell=0}^{\infty} b_\ell \cos(\ell\nu)$, $\frac{\phi}{q} = \sum_{\ell=0}^{\infty} d_\ell \cos(j\nu)$ where

\[ d_\ell = \sum_{\ell=0}^{\infty} b_\ell \alpha_\ell^j, \quad d_j = \alpha^j \sum_{\ell=0}^{j} b_\ell (\alpha^{-\ell} + \alpha^\ell) + (\alpha^{-j} + \alpha^j) \sum_{\ell=j+1}^{\infty} b_\ell \alpha_\ell^j \text{ for } j \geq 1 \]

and

\[ \|q^{-1}\phi\|_E \leq C_5\|\phi\|_E \]

where

\[ C_5 = \frac{2}{1 - \alpha e^\beta} + \frac{2\alpha}{e^\beta - \alpha} \]

Proof. We note that for $j \geq 1$,

\[ d_j = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\phi(\nu)}{q(\nu)} \cos(j\nu) d\nu \]

using $\eta = e^{i\nu}$ on a unit circle counter-clockwise contour integral,

\[ d_j = (1 - \alpha^2) \sum_{\ell=0}^{j} b_\ell \int_{|\eta|=1} \frac{d\eta}{(\eta - \alpha)(1 - \alpha\eta)} (\eta^j + \eta^{-j}) (\eta^j + \eta^{-j}) \]

On collecting residues, we obtain the expression

\[ d_j = \alpha^j \sum_{\ell=0}^{j} b_\ell (\alpha^{-\ell} + \alpha^\ell) + \left( \sum_{\ell=j+1}^{\infty} b_\ell \alpha_\ell^j \right) (\alpha^{-j} + \alpha^j) \]

Therefore, it follows that

\[ \sum_{j=0}^{\infty} e^{\beta j} |d_j| \leq \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} |b_\ell| e^{\beta (j-\ell)} \alpha_{\ell-\ell} (1 + \alpha^{2\ell}) + \sum_{j=0}^{\infty} (1 + \alpha^{2j}) \sum_{\ell=j+1}^{\infty} |b_\ell| e^{\beta(j-\ell)} e^{-\beta(l-j)} \]

\[ \leq \frac{2}{1 - \alpha e^\beta} \left( \sum_{\ell=0}^{\infty} |b_\ell| e^{\beta \ell} \right) + 2 \left( e^\beta \alpha^{-1} - 1 \right)^{-1} \sum_{\ell=0}^{\infty} |b_\ell| e^{\beta \ell} (1 - e^{-\beta \ell} \alpha^\ell) \]

The same calculation is valid for $j = 0$, except for a factor of 2.

Remark 5. The above Lemma is very useful in calculating the Fourier cosine coefficients of $B(\nu)$ defined in (27) exactly. When $f_0$ is a degree $N$ polynomial in $\eta$, as mentioned earlier, $qB(\nu)$ is then a polynomial of $\cos \nu$ of degree $2N + 2$, whose coefficients can be determined without round off errors with rational choice of coefficients. The above lemma then gives $B_j$ coefficients.
Lemma 9. For $0 \leq \beta < \log \alpha^{-1}$, if $\psi \in \mathcal{S}$, then $\frac{1}{q} \psi \in \mathcal{S}$, where for $\psi = \sum_{l=1}^{\infty} b_l \sin(l \nu), \frac{\psi}{q} = \sum_{j=1}^{\infty} d_j \sin(j \nu)$ where

\[ d_j = \alpha^j \sum_{l=1}^{j} b_l (\alpha^{-l} - \alpha^l) + (\alpha^{-j} - \alpha^j) \sum_{l=j+1}^{\infty} b_l \alpha^l \text{ for } j \geq 1 \]

and

\[ \|q^{-1} \psi\|_{\mathcal{S}} \leq C_6 \|\psi\|_{\mathcal{S}}, \]

where

\[ C_6 = \frac{1}{1 - \alpha e^{\beta}} + \frac{\alpha}{e^{\beta} - \alpha}. \]

Proof. We note that for $j \geq 1$,

\[ d_j = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\psi(\nu)}{q(\nu)} \sin(j \nu) d\nu \]

using $\eta = e^{i \nu}$ on a unit circle contour integral,

\[ d_j = -\frac{(1 - \alpha^2)}{4 \pi i} \sum_{l=1}^{\infty} b_l \int |\eta| = 1 \frac{d\eta}{(\eta - \alpha)(1 - \alpha \eta)} (\eta^j - \eta^{-j}) \]

On collecting residues, we obtain the expression

\[ d_j = \alpha^j \sum_{l=1}^{j-1} b_l (\alpha^{-l} - \alpha^l) + \left( \sum_{l=j}^{\infty} b_l \alpha^l \right) (\alpha^{-j} - \alpha^j) \]

Therefore, it follows that

\[ \sum_{j=1}^{\infty} e^{\beta j} |d_j| \leq \sum_{j=1}^{\infty} \sum_{l=1}^{j} |b_l| e^{\beta l} e^{\beta (j-l)} \alpha^{j-l} (1 - \alpha^{2l}) + \sum_{j=1}^{\infty} (1 - \alpha^{2l}) \sum_{l=j+1}^{\infty} |b_l| e^{\beta l} \alpha^{-l-j} e^{-\beta (l-j)} \]

\[ \leq \frac{2}{1 - \alpha e^\beta} \left( \sum_{l=0}^{\infty} |b_l| e^{\beta l} \right) + 2 (e^{\beta \alpha^{-1}} - 1)^{-1} \sum_{l=1}^{\infty} |b_l| e^{\beta l} (1 - e^{-\beta l} \alpha^l) \]

Lemma 10. The linear functional $\mathcal{F}$ defined in [34] satisfies

\[ |\mathcal{F}[W]| \leq K_3 \|W\|_{\mathcal{E}}, \]

\[ K_3 = \int_{-1}^{1} \frac{1 + \eta q f_0}{1 - \alpha \eta} d\eta \]

Proof. Since for $|\eta| \leq 1$,

\[ |\eta^{l-1} + \alpha \eta^{l-2} + \alpha^2 \eta^{l-2} + \cdots + \alpha^{l-1}| \leq \frac{1}{1 - \alpha} \]

and $1 + \eta q f_0, (1 - \alpha \eta) > 0$, it follows that

\[ |\mathcal{F}[W]| \leq \left( \int_{-1}^{1} \frac{1 + \eta q f_0(\eta)}{1 - \alpha \eta} d\eta \right) \sum_{l=1}^{\infty} |W_l| \leq \left( \int_{-1}^{1} \frac{1 + \eta q f_0(\eta)}{1 - \alpha \eta} d\eta \right) \|W\|_{\mathcal{A}} =: K_3 \|W\|_{\mathcal{E}} \]
Remark 6. For polynomial \( f_0 \), in which case \( 1 + \eta q f_0' \) is also a polynomial, \( K_3 \) can be computed exactly as a finite sum of closed form definite integrals.

Lemma 11. The nonlinear functional \( Q \) defined in (35) satisfies the following bounds for \( \|W\|_\mathcal{A} \leq \frac{1}{18} \):

\[
Q[W] \leq 2e^{1/4}K_3\|W\|^2_\mathcal{A}
\]

Proof. We note that the functional

\[
Q[W] = \mathcal{F}[e^U - 1 - U], \quad \text{where } U(\eta) = W(\eta) - W(\alpha)
\]

since \( U(\alpha) = 0 \) and therefore \( e^{U(\alpha)} - 1 - U(\alpha) = 0 \). Clearly \( U \in \mathcal{A} \) with \( \|U\|_\mathcal{A} \leq 2\|W\|_\mathcal{A} \). Applying Corollary 6. and using mean value theorem and the fact \( 2\|W\|_\mathcal{A} \leq 2B_0(1 + \epsilon) \leq \frac{1}{\epsilon} \), we obtain

\[
\|e^U - 1 - U\|_\mathcal{A} \leq e^\|U\|_\mathcal{A} - 1 - \|U\|_\mathcal{A} \leq 2e^{1/4}\|W\|^2_\mathcal{A}
\]

from which it follows that

\[
Q[W] \leq 2e^{1/4}K_3\|W\|^2_\mathcal{A}
\]

6. Solving \( \mathcal{L}\Phi = r \) for given \( a_1 \in I, r \in \mathcal{S} \) and bounds on \( \|\Phi\|_\mathcal{E} \)

Consider solving for \( \Phi \in \mathcal{E} \) satisfying the linear problem \( \mathcal{L}\Phi = r \) for given \( r \in \mathcal{S} \) and \( a_1 \in I \). If we use Fourier representation

\[
\Phi(\nu) = \sum_{j=0}^{\infty} a_j \cos(j\nu), \Psi(\nu) = \sum_{j=1}^{\infty} a_j \sin(j\nu)
\]

\[
A(\nu) = \sum_{j=1}^{\infty} A_j \sin(j\nu), B(\nu) = \sum_{j=0}^{\infty} B_j \cos(j\nu), r(\nu) = \sum_{j=1}^{\infty} r_j \sin(j\nu)
\]

Then, equating coefficients of \( \sin(\kappa \nu) \) for \( \kappa \geq 1 \) in the relation \( \mathcal{L}\Phi = r \), where \( \mathcal{L} \) given by (10), we obtain

\[
2a_0 A_k + (-k + 2B_0 + A_{2k} - B_{2k}) a_k + \sum_{l=1}^{k-1} a_l (A_{k-l} + A_{l+k} + B_{k-l} - B_{l+k}) + \sum_{l=k+1}^{\infty} a_l (A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k}) = r_k
\]

We will solve (97) for \((a_0, a_2, a_3, \ldots)\) for given \( a_1 \in I \). For that purpose, it is convenient to re-write (97) in the following form for \( \kappa \geq 2 \):

\[
2A_k \frac{a_k}{l_k} - a_k + \sum_{l=2}^{k-1} a_l \frac{1}{l_k} (A_{k-l} + A_{l+k} + B_{k-l} - B_{l+k}) + \sum_{l=k+1}^{\infty} a_l \frac{1}{l_k} (A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k}) = r_k \frac{a_k}{l_k} \frac{1}{l_k} (A_{k-1} + A_{k+1} + B_{k-1} - B_{k+1})
\]

\[
+ \sum_{l=k+1}^{\infty} a_l \frac{1}{l_k} (A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k}) = r_k \frac{a_k}{l_k} \frac{1}{l_k} (A_{k-1} + A_{k+1} + B_{k-1} - B_{k+1})
\]
where

\begin{align*}
l_k &= k - 2B_0 - A_{2k} + B_{2k}.
\end{align*}

Quasi solution calculations in the range of \( h \) reported here show that \( A_1 < 0 \) and \( l_k > 0 \) for \( k \geq 2 \); this will be assumed in the the ensuing. Setting \( k = 1 \) in (97) leads to

\begin{align*}
a_0 + \sum_{l=2}^{\infty} \frac{a_l}{2A_1} (A_{l+1} - A_{l-1} + B_{l-1} - B_{l+1}) &= \frac{r_1}{2A_1} + \frac{1}{2A_1} (1 - 2B_0 - A_2 + B_2) a_1
\end{align*}

Equations (98) and (100) determine a system of equations for \( a = (a_0, a_1, a_2, \cdots) \in H \) for given

\begin{align*}
\tilde{r} &= \left[ 0, \frac{r_1}{2A_1} + \frac{a_1}{2A_1} (1 - 2B_0 - A_2 + B_2), \left\{ \frac{r_k}{l_k} - \frac{a_1}{l_k} (A_{k-1} + A_{k+1} + B_{k-1} - B_{k+1}) \right\}_{k=2}^{\infty} \right]
\end{align*}

and may be written abstractly as

\begin{align*}
La &= \tilde{r}
\end{align*}

and will consider inversion of \( L \) in the space \( H \) of sequences as above since this is easily seen to determine solution to \( L\Phi = r \) in the space \( E \) for given \( a_1 \).

**Definition 12.** We define \( H_0, H_1 \) to be the subspaces of \( H \) comprising sequences in the form \( a = (0, a_1, a_2, \cdots) \) and \( a = (a_0, 0, a_2, \cdots) \) respectively. We define \( H_F \) to be the (finite) \( K \)-dimensional subspace of \( H_1 \) consisting of all sequences \( a \) in the form

\begin{align*}
a &= (a_0, 0, a_2, \cdots, a_K, 0, 0, \cdots)
\end{align*}

Also, we define \( K \)-dimensional subspace \( H_q \) of \( H_0 \) consisting of all sequences \( q \) in the form

\begin{align*}
q &= (0, q_1, q_2, q_3, \cdots, q_K, 0, 0, \cdots)
\end{align*}

We define \( H_L \) to be infinite dimensional subspace of \( H \) consisting of all sequences \( a \) in the form

\begin{align*}
a &= (0, 0, 0, \cdots, 0, a_{K+1}, a_{K+2}, \cdots)
\end{align*}

It is clear that \( H_L \) is the compliment of \( H_F \) in \( H_1 \), which is the domain of \( L \), while \( H_L \) is the compliment of \( H_q \) in \( H_0 \), the range of \( L \).

It is useful to express \( a = (a_F, a_L) \), \( \tilde{r} = (\tilde{r}_q, \tilde{r}_L) \), where \( a_F = (a_0, 0, a_2, \cdots, a_K, 0, 0, \cdots) \in H_F \), \( \tilde{r}_q = (0, \tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_K, 0, 0, \cdots) \in H_q \), \( a_L = (0, 0, \cdots, 0, a_{K+1}, a_{K+2}, \cdots) \in H_L \), \( \tilde{r}_L = (0, 0, \cdots, 0, \tilde{r}_{K+1}, \tilde{r}_{K+2}, \cdots) \in H_L \). Then, the system of equation (102) may be separated out in the following manner

\begin{align*}
L_{1,1} a_F &= -L_{1,2} a_L + \tilde{r}_q, 
L_{2,2} a_L &= -L_{2,1} a_F + \tilde{r}_L
\end{align*}

where for \( k = 2, \cdots, K \),

\begin{align*}
[L_{1,1} a_F]_k &= \frac{2A_k}{l_k} a_0 - a_k + \sum_{l=2}^{k-1} \frac{a_l}{l_k} (A_{l-t} + A_{l+k} + B_{l-1} - B_{l+k}) \\
&+ \sum_{l=k+1}^{K} \frac{a_l}{l_k} (A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k})
\end{align*}
while

\[(108) \ [L_{1,1}a_F]_0 = 0, \ [L_{1,1}a_F]_1 = a_0 + \sum_{l=2}^{K} \frac{a_l}{2A_1} (A_{l+1} - A_{l-1} + B_{l-1} - B_{l+1}) \ .\]

For \( k = 2, \cdots K, \)

\[(109) \ [L_{1,2}a_L]_k = \sum_{l=K+1}^{\infty} \frac{a_l}{l_k} (A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k}) \ ,\]

while

\[(110) \ [L_{1,2}a_L]_0 = 0, \ [L_{1,2}a_L]_1 = \sum_{l=K+1}^{\infty} \frac{a_l}{2A_1} (A_{l+1} - A_{l-1} + B_{l-1} - B_{l+1}) \ ,\]

and for \( k \geq K + 1, \)

\[(111) \ [L_{2,1}a_F]_k = \frac{2A_k}{l_k} a_0 + \sum_{l=2}^{K} \frac{a_l}{l_k} (A_{k-l} + A_{l+k} + B_{k-l} - B_{l+k}) \ ,\]

\[(112) \ [L_{2,2}a_L]_k = -a_k + \sum_{l=K+1}^{k-1} \frac{a_l}{l_k} (A_{k-l} + A_{l+k} + B_{k-l} - B_{l+k}) \]

\[+ \sum_{l=k+1}^{\infty} \frac{a_l}{l_k} (A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k}) \ ,\]

It is to be noted that \( L_{1,1} : H_F \rightarrow H_q, \) each being a \( K \)-dimensional space. Furthermore, it will be seen that each of \( L_{1,2} : H_L \rightarrow H_q, L_{2,1} : H_F \rightarrow H_L \) is a bounded operator. We will first show that for sufficient large integer \( K, \) \( L_{2,2} : H_L \rightarrow H_L. \) Then, it will follow from (109) that \( a_F \) satisfies the finite dimensional system of \( K \) scalar equations for \( K \) unknowns given by

\[(113) \qquad \left( L_{1,1} - L_{1,2}L_{2,2}^{-1}L_{2,1} \right) a_F = \tilde{r}_q - L_{1,2}L_{2,2}^{-1} \tilde{r}_L \]

When \( L_{1,1}^{-1} \) exists, as may be checked by a finite matrix calculation, (113) implies

\[(114) \qquad \left( I - L_{1,1}^{-1}L_{1,2}L_{2,2}^{-1}L_{2,1} \right) a_F = L_{1,1}^{-1} \tilde{r}_q - L_{1,1}^{-1}L_{1,2}L_{2,2}^{-1} \tilde{r}_L \]

For specific quasi-solution for different \( \mu, \) we use explicit matrix computation of \( L_{1,1}^{-1} \) and estimate \( \| L_{1,1}^{-1}L_{1,2}L_{2,2}^{-1}L_{2,1} \| \) in the finite dimensional subspace of \( H_F \) and demonstrate that it is less than 1, implying

\[(115) \qquad \| a_F \|_{H_F} \leq \left( 1 - \| L_{1,1}^{-1}L_{1,2}L_{2,2}^{-1}L_{2,1} \| \right)^{-1} \left( \| L_{1,1}^{-1} \| \| \tilde{r}_q \|_{H_q} + \| L_{1,1}^{-1}L_{1,2}L_{2,2}^{-1} \| \| \tilde{r}_L \|_{H_L} \right) \]

Using (109), we can also estimate \( \| a_L \|_{H_L}: \)

\[(116) \qquad \| a_L \|_{H_L} \leq \| L_{2,2}^{-1}L_{2,1} \| \| a_F \|_{H_F} + \| L_{2,2}^{-1} \| \| \tilde{r}_L \|_{H_L} \]
6.1. **Bounds on operators.** Consider the system

$$L_{2,2}a_L = \hat{r}_L$$

This is equivalent to the following infinite set of equations for $k \geq K + 1$.

$$a_k = \frac{1}{l_k^2} \sum_{i=K+1}^{k-1} a_i (A_{k-i} + A_{i+k} + B_{k-i} - B_{i+k}) + \frac{1}{l_k^2} \sum_{i=k+1}^{\infty} a_i (A_{i+k} - A_{i-k} + B_{i-k} - B_{i+k}) - \hat{r}_k =: [M a_L]_k - \hat{r}_k$$

**Lemma 13.** The operator $M$ defined in \((118)\) satisfies the following bounds in sub-space $H_L$:

$$\|M a_L\|_{H_L} \leq \gamma \|a_L\|_{H_L}$$

where

$$\gamma = \sup_{l \geq K+1} \left\{ \sum_{k=K+1}^{l-1} l_k^{-1} \left| e^{\beta(k-l)} (A_{l+k} + A_{k-l} + B_{k-l} - B_{l+k}) \right| \right. \right.$$

$$\left. \left. + \left[ \sum_{k=K+1}^{l-1} l_k^{-1} \left| e^{\beta(k-l)} (A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k}) \right| \right] \right\}$$

Further, for $K$ large integer, for $A, B \in H, \gamma$ is small, in which case the operator $L_{2,2} : H_L \rightarrow H_L$ is invertible with

$$\|L_{2,2}^{-1} \hat{r}\|_{H_L} \leq (1 - \gamma)^{-1} \|\hat{r}\|_{H_L} =: \gamma_{2,2}^{-1} \|\hat{r}\|_{H_L}$$

**Proof.** It is convenient to define $m_{k,l}$ so that $m_{k,k} = 0$, while

$$m_{k,l} = l_k^{-1} e^{\beta(k-l)} A_{k-l} + A_{l+k} + B_{k-l} - B_{l+k}$$

for $l < k$

$$m_{k,l} = l_k^{-1} e^{\beta(k-l)} A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k}$$

for $l > k$

Then, it follows from the definition of $M$ that

$$\sum_{k=K+1}^{\infty} e^{\beta k} \|M a_L\|_{H_L} \leq \sum_{k=K+1}^{\infty} \sum_{l=K+1}^{\infty} e^{\beta l} |a_l| m_{k,l} \leq \left\{ \sup_{l \geq K+1} \sum_{k=K+1}^{l-1} m_{k,l} \right\} \|a_L\|_{H_L}$$

from which the first part of the Lemma follows using definition of $m_{k,l}$. It is also clear that since for sufficiently large $K$, $t_k$ is an increasing function of $k$ for $k \geq K + 1$,

$$\sum_{k=K+1}^{l-1} l_k^{-1} e^{\beta(k-l)} A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k} + \sum_{k=l+1}^{\infty} l_k^{-1} e^{\beta(k-l)} A_{k-l} + A_{l+k} + B_{k-l} - B_{l+k}$$

$$\leq l_K^{-1} \sum_{m=1}^{\infty} e^{\beta m} (|A_m| + |B_m|) + e^{-\beta l} \sum_{m=K+1}^{2l-1} e^{\beta m} (|A_m| + |B_m|)$$

$$+ e^{-\beta l} \sum_{m=2l+1}^{\infty} e^{\beta m} (|A_m| + |B_m|) + \sum_{m=1}^{l-K-1} e^{-\beta m} (|A_m| + |B_m|)$$
The supremum of the above expression over all \( l \geq K + 1 \) clearly shrinks to 0 as \( K \to \infty \). Therefore \( \gamma \) which is bounded by the above is small for large \( K \). The second part of the Lemma follows readily from bounds on \( \mathbf{M} \).

**Lemma 14.** The operator \( \mathbf{L}_{1,2} : \mathbf{H}_L \to \mathbf{H}_q \) is bounded and satisfies the uniform bound

\[
\|\mathbf{L}_{1,2} \mathbf{a}_L\|_{\mathbf{H}_q} \leq \gamma_{1,2} \|\mathbf{a}_L\|_{\mathbf{H}_L},
\]

where

\[
\gamma_{1,2} = \sup_{l \geq K+1} \sum_{k=2}^{K} \frac{1}{l_k} \left| e^{\beta (k-1)} (A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k}) \right|
\]

Furthermore, for large \( K \), \( \gamma_{1,2} \) is small.

**Proof.** From definition of \( \mathbf{L}_{1,2} \) in (109), it follows that

\[
\|\mathbf{L}_{1,2} \mathbf{a}_L\|_{\mathbf{H}_q} = \left| e^{\beta} \sum_{l=K+1}^{\infty} \frac{a_l}{2A_1} (A_{l+1} - A_{l-1} + B_{l-1} - B_{l+1}) \right|
\]

\[
+ \sum_{k=2}^{K} e^{\beta k} \sum_{l=K+1}^{\infty} \frac{a_l}{l_k} (A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k})
\]

\[
\leq \left\{ \sum_{l=K+1}^{\infty} |a_l| e^{\beta l} \right\} \left\{ \frac{e^\beta}{2|A_1|} \sup_{l \geq K+1} e^{-\beta l} \left| A_{l+1} - A_{l-1} + B_{l-1} - B_{l+1} \right| \right\}
\]

\[
+ \sup_{l \geq K+1} \sum_{k=2}^{K} \frac{1}{l_k} e^{\beta (k-1)} (A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k})
\]

from which the first part of the Lemma follows. We note that since \( \mathbf{A}, \mathbf{B} \in \mathbf{H}_L \), it is clear that

\[
|A_{l+1} - A_{l-1} + B_{l-1} - B_{l+1}| \to 0 \text{ when } l \geq K + 1 \text{ and } K \to \infty
\]

Also, we note that

\[
\sum_{k=2}^{K} \frac{e^{\beta (k-1)}}{l_k} |B_{l-k} - A_{l-k}| = \sum_{k=2}^{K/2} \frac{e^{\beta (k-1)}}{l_k} |B_{l-k} - A_{l-k}| + \sum_{k=K/2+1}^{K} \frac{e^{\beta (k-1)}}{l_k} |B_{l-k} - A_{l-k}|
\]

\[
\leq \frac{1}{l_2} \sum_{l-K/2}^{l-2} e^{-\beta m} |B_m - A_m| + \frac{1}{l_{K/2+1}} \sum_{l-K}^{l-K/2-1} e^{-\beta m} |B_m - A_m|
\]

Clearly, the above shrinks to zero as \( K \to \infty \) for any \( l \geq K + 1 \). Also,

\[
\sum_{k=2}^{K} e^{\beta (k-1)} |A_{l+k} - B_{l+k}| = e^{-2\beta l} \sum_{m=0}^{K+1} e^{\beta m} |A_m - B_m|
\]

This also shrinks to zero as \( K \to \infty \) for any \( l \geq K + 1 \). Therefore, it follows from expression for \( \gamma_{1,2} \) that it shrinks to zero as \( K \to \infty \).
Lemma 15. The operator $L_{2,1} : H_F \to H_L$ is bounded and satisfies
\begin{equation}
\|L_{2,1} a_F\|_{H_L} \leq \gamma_{2,1} \|a_F\|_{H_F},
\end{equation}
where
\begin{equation}
\gamma_{2,1} = \max \left\{ \sum_{k=K+1}^{\infty} \frac{2|A_k|}{l_k} e^{\beta k}, \sup_{2 \leq l \leq K} \sum_{k=K+1}^{\infty} \left| e^{\beta(k-l)} \right| (A_{k-l} + A_{l+k} + B_{k-l} - B_{l+k}) \right\}
\end{equation}
Furthermore for large $K$, $\gamma_{2,1}$ is small.

Proof. Using (111), we obtain
\begin{equation}
\|L_{2,1} a_F\|_{H_L} \leq |a_0| \sum_{k=K+1}^{\infty} \frac{2|A_k|}{l_k} e^{\beta k} + \sum_{k=K+1}^{\infty} \frac{e^{\beta k}}{l_k} \sum_{l=2}^{K} |a_l| \left| A_{k-l} + A_{l+k} + B_{k-l} - B_{l+k} \right|
\end{equation}
\begin{equation}
\leq |a_0| \sum_{k=K+1}^{\infty} \frac{2|A_k|}{l_k} e^{\beta k} + \left[ \sum_{k=K+1}^{\infty} |a_l| \right] \sum_{k=K+1}^{\infty} \frac{e^{\beta(k-l)}}{l_k} \left| A_{k-l} + A_{l+k} + B_{k-l} - B_{l+k} \right|
\end{equation}
\begin{equation}
\leq \gamma_{2,1} \left\{ |a_0| + \sum_{k=2}^{K} e^{\beta k} |a_k| \right\},
\end{equation}
from which the first part of the lemma follows. We also note that for sufficiently large $K$,
\begin{equation}
\sum_{k=K+1}^{\infty} e^{\beta(k-l)} \left| A_{k-l} + B_{k-l} + A_{l+k} - B_{l+k} \right| \leq \frac{1}{l_{K+1}} \sum_{m=1}^{\infty} e^{\beta m} |A_m + B_m|
\end{equation}
\begin{equation}
+ \frac{e^{-2\beta l}}{l_{K+1}} \sum_{m=K+1}^{\infty} e^{\beta m} |A_m - B_m|,
\end{equation}
which shrinks to zero as $K \to \infty$ for any $2 \leq l \leq K$. Furthermore,
\begin{equation}
\sum_{k=K+1}^{\infty} \frac{2}{l_k} |A_k| e^{\beta k} \leq \frac{2}{l_{K+1}} \sum_{K+1}^{\infty} |A_k| e^{\beta k} \to 0 \text{ as } K \to \infty.
\end{equation}
Therefore $\gamma_{2,1}$ is small for large $K$.

Lemma 16. $L_{1,1} : H_F \to H_q$ is invertible if and only if the $K \times K$ matrix $J = \{ J_{k,l} \}_{k,l}$ with elements determined by:
\begin{equation}
J_{1,l} = \frac{e^{\beta(l-1)}}{2A_1} (A_{l+1} - A_{l-1} + B_{l-1} - B_{l+1}) \text{ for } l = 2, 3, \ldots, K,
\end{equation}
\begin{equation}
J_{k,k} = -1, \quad J_{k,l} = \frac{1}{l_k} e^{\beta k} A_k \text{ for } 2 \leq k \leq K,
\end{equation}
\begin{equation}
J_{k,l} = \frac{e^{\beta(k-l)}}{l_k} (A_{k-l} + A_{k+l} + B_{k-l} - B_{k+l}) \text{ for } 2 \leq l \leq k-1 \leq K - 1
\end{equation}
\begin{equation}
J_{k,l} = \frac{e^{\beta(k-l)}}{l_k} (A_{l+k} - A_{l-k} + B_{l-k} - B_{l+k}) \text{ for } 2 \leq k \leq l-1 \leq K - 1
\end{equation}
Further $\gamma_{1,1}^{-1} := \| L_{1,1}^{-1} \| = \| J^{-1} \|_1$ where $\| \cdot \|_1$ denotes the matrix 1-norm.
Proof. The proof follows from examining the definition of \( L_{1,1} \) in (107)-(108) and noting that both the domain and range of \( L_{1,1} \) is \( K \) dimensional. The factors of \( e^\beta \) in the matrix elements of \( J \) ensure that the \( H \) norm of \( L_{1,1}^{-1} \) is the 1-norm of the matrix \( J^{-1} \), if and when it exists. \( \blacksquare \)

**Proposition 17.** If for some suitably large \( K \), \( L_{1,1}^{-1} \) exists with \( \| L_{1,1}^{-1} \| = \gamma_{1,1}^{-1} \) satisfying

\[
\gamma_{1,2,1}^2 \gamma_{2,1}^{-1} < 1
\]

then \( L^{-1} : H_0 \to H_1 \) exists and satisfies

\[
\| L^{-1} \| \leq \tilde{M},
\]

where

\[
\tilde{M} = \max \left\{ \gamma_{1,1}^{-1} \left( 1 - \frac{\gamma_{1,2,1}}{\gamma_{1,1}^2 \gamma_{2,1}} \right)^{-1} \left( 1 + \frac{\gamma_{2,1}}{\gamma_{2,2}} \right), \right. \\
\left. \gamma_{2,2} + \gamma_{1,1}^{-1} \left( 1 - \frac{\gamma_{1,2,1}}{\gamma_{1,1}^2 \gamma_{2,2}} \right)^{-1} \left( \gamma_{1,2}^2 + \gamma_{2,2}^2 \gamma_{1,2} \right) \right\}
\]

When condition (139) is satisfied, for given \( a_1 \in \mathbb{R}, r \in \mathcal{S} \), the linear system \( \mathcal{L} \Phi = r \) has a unique solution in the form

\[
\Phi(\nu) = K[\nu](\nu) + a_1 G(\nu),
\]

where \( K : \mathcal{S} \to \mathcal{E} \) is a linear operator

\[
\| K \| \leq \tilde{M} \max \left\{ \frac{1}{2|A_1|}, \sup_{k \geq 2} \frac{1}{l_k} \right\} =: M
\]

and

\[
G(\nu) = g_0 + \cos \nu + \sum_{k=2}^{\infty} g_k \cos(k\nu),
\]

where \( g = (g_0, g_2, \cdots) \in H_1 \) is given by \( g = L^{-1} h \) where \( h = (0, h_1, h_2, \cdots) \in H_0 \), where

\[
h_1 = \frac{1}{2A_1} (1 - 2B_0 - A_2 + B_2)
\]

and for \( k \geq 2,

\[
h_k = -\frac{1}{l_k} (A_{k-1} + A_{k+1} + B_{k-1} - B_{k+1})
\]

Furthermore,

\[
\| G \|_{\mathcal{E}} \leq e^\beta + \tilde{M} \| h \|_{H_0}
\]

Proof. The first part follows from applying estimates in Lemmas 13, 15 to (106) and using \( \| F \|_{H_0} = \| F_g \|_{H_0} + \| F_L \|_{H_0}, \| a \|_{H_L} = \| a_F \|_{H_F} + \| a_L \|_{H_L}. \) For the second part, we note that if \( r = 0 \), then \( F = h a_0 \) and therefore in that case \( a = a_1 L^{-1} h \in H_1, a \) is isomorphic to \( a_1 \Phi \in \mathcal{E} \) where \( \mathcal{L} \Phi = -\mathcal{L}[\cos \nu], \) where \( \Phi \) has the form \( a_0 + \sum_{l=2}^{\infty} a_l \cos(l\nu) \). Therefore, \( G = \Phi + \cos \nu \) is the unique solution to \( \mathcal{L}[G] = 0 \).
with unit coefficient of \( \cos \nu \). When \( a_1 = 0 \), but \( \mathbf{r} \neq 0 \), \( \mathbf{r} = \left( \frac{x_{2k}}{2A_1}, \frac{x_k}{I_k} \right) \) for which case

\[
\| \mathbf{r} \|_{\mathbf{H}_0} \leq \max \left( \frac{1}{2|A_1|}, \sup_{k \geq 2} \frac{1}{|I_k|} \right) \| \mathbf{r} \|_{\mathbf{H}_0}
\]

and corresponding \( \mathbf{a} \in \mathbf{H}_1 \) is isomorphic to \( \phi \in \mathcal{E} \) with no \( \cos \nu \) term uniquely satisfying \( L\phi = r \). This is defined to be \( K\mathbf{r} \). Using linear superposition of the two cases: i. \( a_1 \neq 0 \), \( \mathbf{r} = 0 \) and ii. \( a_1 = 0 \), \( \mathbf{r} \neq 0 \) gives the second part of the proposition. The bounds on \( \| G \|_{\mathcal{E}} \) follow from the bounds on \( L^{-1} \mathbf{h} \) and adding to it the contribution from the \( \cos \nu \) term.

**Corollary 18.** For \( a_1 \in I := [-\varepsilon_0, \varepsilon_0] \), define \( \Phi^{(0)}(\nu) = -\kappa R + a_1 G = \mathcal{N}[0] \), where operator \( \mathcal{N} \) is defined in \( (44) \). Then \( \Phi^{(0)} \) satisfies

\[
\| \Phi^{(0)} \|_{\mathcal{E}} \leq M\| R \|_{\mathcal{S}} + \varepsilon_0\| G \|_{\mathcal{E}} =: B_0
\]

**Proof.** The proof follows from bounds on operator \( \mathcal{K} \) in the previous proposition.

**Lemma 19.** Assume \( G_0 \in \mathcal{E} \) is an approximate expression for \( G \) in the sense that \( \| L\mathcal{S}G_0 \| = \varepsilon \) is small and \( \cos \nu \) coefficient of \( G_0 \) is also 1, as for \( G \). If conditions of Proposition 17 hold, then

\[
\| G - G_0 \|_{\mathcal{E}} \leq M\varepsilon_G
\]

In particular,

\[
\| G \|_{\mathcal{E}} \leq \| G_0 \|_{\mathcal{E}} + M\varepsilon_G
\]

and if \( G_0(\pi) \neq 0 \) and \( \varepsilon_G \) is sufficiently small then

\[
\left| G(\pi) \right| > \left| G_0(\pi) \right| - M\varepsilon_G > 0
\]

**Proof.** Since \( L[G - G_0] = -L[G_0] \) and coefficient of \( \cos \nu \) for \( G - G_0 \) is zero, applying Proposition 17 it follows that

\[
G - G_0 = -\mathcal{K}L G_0
\]

which gives the result

\[
\| G - G_0 \|_{\mathcal{E}} \leq M\varepsilon_G
\]

The remaining two parts of the Lemma follow from triangular inequality and the observation \( \left| G(\pi) - G_0(\pi) \right| \leq \| G - G_0 \|_{\mathcal{E}} \).

7. **Nonlinearity bounds and solution to (41) for given \( a_1 \in I \)**

**Proposition 20.** \( \tilde{\mathcal{M}} \) defined in \( (28) \) satisfies \( \tilde{\mathcal{M}} : \mathcal{A} \rightarrow \mathcal{S} \) with

\[
\| \tilde{\mathcal{M}}[W] \|_{\mathcal{S}} \leq \frac{2}{3} \| \mathcal{A} \|_{\mathcal{S}} \left[ e^{2\| W \|_{\mathcal{A}}} \left( e^{\| W \|_{\mathcal{A}}} - 1 - \| W \|_{\mathcal{A}} \right) \right]
\]

\[
+ (1 + \| W \|_{\mathcal{A}}) \left( e^{2\| W \|_{\mathcal{A}}} - 1 - 2\| W \|_{\mathcal{A}} \right) + 2\| W \|_{\mathcal{A}}^2
\]

\[
+ 2\| \mathbf{B} \|_{\mathcal{E}} \left[ e^{2\| W \|_{\mathcal{A}}} \left( e^{\| W \|_{\mathcal{A}}} - 1 - \| W \|_{\mathcal{A}} \right) + \| W \|_{\mathcal{A}} \left( e^{2\| W \|_{\mathcal{A}}} - 1 \right) \right]
\]

\[
\left( e^{2\| W \|_{\mathcal{A}}} - 1 - 2\| W \|_{\mathcal{A}} \right) + 2\| W \|_{\mathcal{A}}^2
\]

\[
+ 2\| \mathbf{B} \|_{\mathcal{E}} \left[ e^{2\| W \|_{\mathcal{A}}} \left( e^{\| W \|_{\mathcal{A}}} - 1 - \| W \|_{\mathcal{A}} \right) + \| W \|_{\mathcal{A}} \left( e^{2\| W \|_{\mathcal{A}}} - 1 \right) \right]
\]
In particular if \( \|W\|_A \leq \frac{1}{16} \),
\[
(156) \quad \|\tilde{M}[W]\| \leq (4\|A\|_S + 6\|B\|_{\varepsilon}) \|W\|^2_A
\]

**Proof.** Recall \( M_1 \) and \( M_2 \) defined in expression (28) defining \( \tilde{M} \). We may rewrite \( (162) \)
\[ M_1 = e^{2ReW}Re(e^W - 1 - W) + Re(1 + W)(e^{2ReW} - 1 - 2ReW) + 2|ReW|^2 \]
Using Corollary 6,
\[
(158) \quad \|M_1\|_E \leq e^{2|W|^2_A} \left( e^{\|W\|^2_A} - 1 - \|W\|_A \right) + (1 + \|W\|_A) \left( e^{2\|W\|^2_A} - 1 - 2\|W\|_A \right) + 2\|W\|^2_A
\]
Also, we have from (28), we may write
\[ M_2 = e^{2ReW}Im(e^W - 1 - W) + Im W (e^{2ReW} - 1) \]
Therefore, using corollaries 4 and 6
\[
(160) \quad \|M_2\|_S \leq e^{2|W|^2_A} \left( e^{\|W\|^2_A} - 1 - \|W\|_A \right) + \|W\|_A (e^{2\|W\|^2_A} - 1)
\]
Therefore from Lemma 5, \( M_1 [W] \in S \) and
\[
(161) \quad \|\tilde{M}[W]\|_S \leq \frac{2}{3} \|A\|_S \|M_1\|_{\varepsilon} + 2\|B\|_{\varepsilon} \|M_2\|_S
\]
from which the first part of the Lemma follows. The second statement can be checked by use of mean value theorem to estimate \( e^{z} - 1 - z \) and \( e^{z} - 1 \).

**Proposition 21.** For given \( a_1 \in I, \mathcal{N} \), defined in (11), satisfies \( \mathcal{N} : B \to B \) and is contractive in the ball \( B \subset E \) of radius \( B_0(1 + \varepsilon) \) about the origin \( (B_0 \text{ defined in (142)}) \) if there exists \( \varepsilon > 0 \) so that \( B_0(1 + \varepsilon) \leq \frac{1}{16} \) and the following conditions hold:
\[
(162) \quad M (4\|A\|_S + 6\|B\|_{\varepsilon}) B_0(1 + \varepsilon)^2 < \varepsilon, 2M (4\|A\|_S + 6\|B\|_{\varepsilon}) B_0(1 + \varepsilon) < 1
\]
When these conditions are satisfied, (44) has unique solution \( \Phi \in B \subset E \). Each such choice of \( a_1 \) corresponds to a symmetric water wave with nondimensional height, wave speed and crest speed \( (h, c, \mu) \) close to \( (h_0, c_0, \mu_0) \) satisfying the following estimates:
\[
(163) \quad |h - h_0| \leq K_3(1 + \varepsilon)B_0 \left( 1 + 2e^{1/4}B_0(1 + \varepsilon) \right)
\]
\[
(164) \quad \left| \log \frac{c}{c_0} \right| \leq \frac{3}{2}B_0(1 + \varepsilon)
\]
\[
(165) \quad \frac{1}{3} \log \left| \frac{\mu}{\mu_0} \right| \leq B_0(1 + \varepsilon)
\]

**Proof.** Applying Propositions 17 and 20 to (11) for for \( \Phi^{(1)}, \Phi^{(2)} \in B \),
\[
(166) \quad \|\mathcal{N}[\Phi^{(1)}] - \mathcal{N}[\Phi^{(2)}]\|_{\varepsilon} = \|\mathcal{K}_M[\Phi^{(1)}] - \mathcal{K}_M[\Phi^{(2)}]\|_{\varepsilon}
\]
\[
\leq M (4\|A\|_S + 6\|B\|_{\varepsilon}) \| \left( \Phi^{(1)} + \Phi^{(2)} \right) \left( \Phi^{(1)} - \Phi^{(2)} \right) \|_{\varepsilon}
\]
Using this and given condition 102,
\[
(167) \quad \|\mathcal{N}[\Phi]\|_{\varepsilon} \leq \|\mathcal{N}[0]\|_{\varepsilon} + \|\mathcal{N}[\Phi] - \mathcal{N}[0]\|_{\varepsilon}
\]
\[
\leq B_0 + M (4\|A\|_S + 6\|B\|) \| B_0^2(1 + \varepsilon)^2 \leq B_0(1 + \varepsilon)
\]
Further, from expression for $M$ from (41), we note that

$$\partial a_1 \Phi \|_{\mathcal{E}} \leq K_1 \|G\|_{\mathcal{E}}$$

and

$$\|\partial a_1 \Phi - G\|_{\mathcal{E}} \leq K_1 M \|G\|_{\mathcal{E}} B_0(1 + \epsilon) \left(\frac{26}{3} \|A\|_{\mathcal{S}} + 18 \|B\|_{\mathcal{E}}\right) =: K_4 \|G\|_{\mathcal{E}}$$

where

$$K_1 = \left(1 - B_0(1 + \epsilon) \left[\frac{26}{3} \|A\|_{\mathcal{S}} + 18 \|B\|_{\mathcal{E}}\right]\right)^{-1}$$

Proof. From (11), we note that $\partial a_1 \Phi$ satisfies

$$\partial a_1 \Phi = \mathcal{K} \partial a_1 \tilde{\mathcal{M}}[W] + G$$

where

$$\partial a_1 \tilde{\mathcal{M}}[W] = -\frac{2}{3} A \partial a_1 M_1 - 2 B \partial a_1 M_2$$

Calculation gives

$$\partial a_1 M_1 = e^{2\Phi} [\partial a_1 \Phi] \left(2 \text{Re} \left[e^W - 1 - W\right] + \text{Re} \left[e^W - 1\right]\right) - \partial a_1 \Psi e^{2\Phi} \text{Im} \left[e^W - 1\right]$$

$$+ \partial a_1 \Phi \left\{\left(e^{2\Phi} - 1 - 2\Phi\right) + 2(1 + \Phi) (e^{2\Phi} - 1) + 4\Phi\right\},$$

where $\partial a_1 \Psi$ is the Hilbert transform of $\partial a_1 \Phi$. Therefore,

$$\|\partial a_1 M_1\|_{\mathcal{E}} \leq \|\partial a_1 \Phi\|_{\mathcal{E}} \left(2 e^{2\|\Phi\|_{\mathcal{E}}} \left[e^{\|\Phi\|_{\mathcal{E}}} - 1 - \|\Phi\|_{\mathcal{E}}\right] + 2 e^{2\|\Phi\|_{\mathcal{E}}} \left(e^{\|\Phi\|_{\mathcal{E}}} - 1\right)\right)$$

$$+ e^{2\|\Phi\|_{\mathcal{E}}} - 1 - 2 \|\Phi\|_{\mathcal{E}} + 2(1 + \|\Phi\|_{\mathcal{E}}) \left[e^{2\|\Phi\|_{\mathcal{E}}} - 1\right] + 4 \|\Phi\|_{\mathcal{E}}$$

Further, from expression for $M_2$,

$$\partial a_1 M_2 = \partial a_1 \Phi \left\{2 e^{2\Phi} \text{Im} \left[e^W - 1 - W\right] + e^{2\Phi} \text{Im} \left[e^W - 1\right] + 2 \Psi e^{2\Phi}\right\}$$

$$+ \partial a_1 \Psi \left\{e^{2\Phi} \text{Re} \left[e^W - 1\right] + e^{2\Phi} - 1\right\}$$
implying

\[ \| \partial_{a_1} M_2 \| \leq \| \partial_{a_1} \Phi \| \left\{ 2 \epsilon^2 \Phi \| \| \Phi \| \| \Phi \| \| - 1 - \| \Phi \| \| \Phi \| \| - 1 \right\} + 2 \| \Phi \| \| \Phi \| \| \Phi \| \| + \epsilon^2 \| \Phi \| \| \Phi \| \| - 1 \right\} + 2 \| \Phi \| \| \Phi \| \| \Phi \| \| - 1 \right\} \]

Therefore, when \( \Phi \in B \) for \( B_0(1 + \epsilon) \leq \frac{1}{19} \),

\[ \| \partial_{a_1} M_1 \| \leq 13 \| \partial_{a_1} \Phi \| \| \Phi \| \| \Phi \| \| + 9 \| \partial_{a_1} \Phi \| \| \Phi \| \| \]

Therefore,

\[ \| K \partial_{a_1} M_1 \| \leq M \| \partial_{a_1} \Phi \| \| \Phi \| \| \Phi \| \| \left( \frac{26}{3} \| A \| \| s + 18 \| B \| \| \right) \]

The lemma readily follows from using above bounds in (171).

**Proposition 23.** Define \( B_0 \) as in (149). If in addition to conditions in Proposition 27 the following two conditions

\[ \left( \frac{1}{G(\pi)} \right) \left[ \frac{1}{3} \log \frac{\mu_0}{\mu} \right] + M \| R \| \| s + M \left( 4 \| A \| \| s + 6 \| B \| \| \right) B_0^2 (1 + \epsilon)^2 \right] < \epsilon_0 \]

\[ \left( \frac{M}{G(\pi)} \right) K_1 \| G \| \| B_0 (1 + \epsilon) \left( \frac{26}{3} \| A \| \| s + 18 \| B \| \| \right) < 1 \]

hold, then there exists unique \( a_1 \in I \) \( = (-\epsilon_0, \epsilon_0) \) so that the solution in Proposition 27 satisfies (72).

**Proof.** From (43), it follows that if \( a_1 \in I \), then

\[ \left| U[a_1] \right| \leq \left( \frac{1}{G(\pi)} \right) \left[ \frac{1}{3} \log \frac{\mu_0}{\mu} \right] + M \| R \| \| s + M \left( 4 \| A \| \| s + 6 \| B \| \| \right) B_0^2 (1 + \epsilon)^2 \right] \]

Condition (179) implies that \( U : I \rightarrow I \). Applying \( \partial_{a_1} \) to (12), and using (178), Propositions 20 and Lemma 22, it follows by applying (180) that

\[ \left| \partial_{a_1} U[a_1] \right| \leq \left( \frac{1}{G(\pi)} \right) K \partial_{a_1} M \| \Phi \| \leq \left( \frac{1}{G(\pi)} \right) MK_1 \| G \| \| B_0 (1 + \epsilon) \left( \frac{26}{3} \| A \| \| s + 18 \| B \| \| \right) < 1 \]

Hence \( U : I \rightarrow I \) is contractive, implying existence of unique \( a_1 \) satisfying (72). 

9. QUASI-SOLUTION AND APPLICATION OF PROPOSITIONS 21, 22

We describe in this section determination of quasi-solutions \( (f_0, c_0) \) and checking conditions for application of Propositions 21 and 23. Though quasi-solutions have been obtained numerically, it has no bearing on the mathematical rigor of Theorem 1 since Propositions 21 and 23 concern the difference \( W = w - w_0 \) and calculation of norms of residual \( R_0 \) and \( R_0' \) based on \( (f_0, c_0) \) are exact.

The process of obtaining quasi-solution is straightforward. As mentioned earlier, a polynomial representation for \( f_0 \) is most suitable for determining exact representation for determination of \( R_0(\nu) \), \( A(\nu) \) and \( B(\nu) \). For that purpose, one can use a numerical truncation of a series representation of \( f \) in \( \eta \) and find the coefficients
through a Newton iteration procedure involving wave speed \( c \) and the series coefficient \( F_0, F_1, F_2, \cdots F_N \) for \( f \) in \( (1) \) by satisfying boundary condition \( (10) \) at \( N \) uniformly spaced out points in the upper-half semi-circle and enforcing constraint \( (3) \) for given \( \mu \). Such procedures are fairly standard and have been used routinely in the past by many investigators. However, such a representation for \( f_0 \) requires more than two hundred modes for \( \| R \|_S \) to be small enough to apply Proposition \( 24 \) for the values of \( \mu \) quoted here. Hence, for efficiency of representation and of presentation, a rational Pade approximant for \( f' \) is found, similar to the one employed earlier by \[15\]; integration and replacement of each coefficient by a ten to twelve digit accurate rational approximation gives rise to the quoted expressions for \( \tilde{f}_0 \) in the following subsections. Note that this requires specification of only upto fifty two numerical coefficients, compared to more than 200 otherwise. With well-known location of singularities, it can be easily proved that the truncated Taylor expansion \( f_0 = \mathcal{P}_N \tilde{f}_0 \) for \( N = 255 \) ensures that \( \| f_0' - \tilde{f}_0' \|_A \) is less than \( 10^{-10} \) in all cases reported. Though \( f_0 \) is still a large order polynomial, we only need to list up to fifty two rational numbers for \( \tilde{f}_0 \) to represent \( f_0 \) exactly. A polynomial quasi-solution allows precise computation of all Fourier sine series coefficients of \( A, B, R_0 \) and \( R_0' \) needed to check conditions of Proposition \( 24 \). Additionally to check conditions in Proposition \( 23 \) one needs lower bounds on \( |G(\pi)| \). This is done by applying Lemma \[19\] to an an approximate quasi-G solution \( G_0 \in \mathcal{E} \) for which \( \mathcal{L}G_0 \) is small. We report the coefficients of \( G_0 \) in the appendix for each of the three cases. Note that a truncated rational Fourier cosine series representation for \( G_0 \) allows an exact computation of all Fourier sine series coefficients of \( \mathcal{L}G_0 \) and by using these, one estimates \( \| \mathcal{L}G_0 \|_S = \epsilon_G \). Since \( G_0(\pi) \) and \( \| G_0 \|_S \) are exactly known, positive upper and lower bounds for \( |G|_S \) and \( |G(\pi)| \) follow from Lemma \[19\] when \( G_0(\pi) \neq 0 \) for sufficiently small \( \epsilon_G \).

Checking univalence condition for \( 1 + \eta q f_0' \neq 0 \) in \( |\eta| \leq e^{\beta} \) for suitably chosen \( \beta \geq 0 \) is fairly simple, since one can determine approximate roots of a polynomial of any order numerically. We can then express

\[
1 + \eta q f_0' = \delta \prod_{j=1}^{N+2} (\eta - \eta_j) + z_{N+2}(\eta)
\]

where \( \delta \) is the coefficient of \( \eta^{N+2} \), \( \eta_j \) are the numerically obtained roots approximated by rational numbers and \( z_{N+2} \) is a polynomial of degree \( N + 2 \) with small coefficients which accomodates any error in the root calculations. In all cases reported, \( |\eta_j| > 1.09 \). Note that though \( \eta_j \) have been computed numerically, \( z_{N+2} \) as a difference of the two polynomials is known exactly. We can then check \( \| z_{N+2} \|_A \) and prove it is small enough for suitably chosen \( \beta \), \( \inf_j |\eta_j| > e^{\beta} \geq 1 \) and on \( |\eta| = e^{\beta} \), \( |z_{N+2}| < |\delta| \sum_{j=1}^{N+2} |\eta - \eta_j| \). By application of Rouche’s theorem, \( 1 + \eta q f_0' \neq 0 \) for \( |\eta| \leq e^{\beta} \). This also ensures analyticity of \( w_0 = -\frac{\beta}{4} \log c_0 + \log (1 + \eta q f_0') \) in \( |\eta| \leq e^{\beta} \). Alternately, we can show \( 1 + \eta q f_0' \neq 0 \) for \( |\eta| \leq e^{\beta} \) by rationalizing the expression and working with the polynomial in the numerator. The closeness of \( f_0' \) and \( \tilde{f}_0' \) implies that the same condition is true for \( 1 + \eta q f_0' \) from Rouche’s theorem.

Recall set \( S \) for which Theorem \[1\] applies:

\[
S := \{ \mu : \mu = \mathcal{I}_{\mu_j}, \mu_1 = 0.0018306, \mu_2 = 0.002, \mu_3 = 0.0023 \}
\]
where $I_{\mu_j}$ are sufficiently small intervals containing $\mu_j$. We will only check in the ensuing that conditions for application of Propositions 21 and 23 apply for $\mu = \mu_j$, $j = 1, 2, 3$. Since these conditions are open set conditions; so they must hold for a sufficient small neighborhood of $\mu = \mu_j$. The maximal sizes of the intervals $I_{\mu_j}$ which still ensures that Theorem 1 applies can also be estimated if desired, though larger size reduces the accuracy of the quasi-solution.

9.1. Case of $\mu = \mu_1 := 0.0018306$. In this case, we choose $\epsilon_0 = \frac{9195}{8113}$. This is close to the empirical maximum wave speed. Wave speed and for $N = 255$, we take $f_0 = P_N f_0$, where

$$\hat{f}_0 = b_0 + \sum_{j=1}^{35} b_j \eta_j + \sum_{m=1}^{8} \lambda_m \gamma_m^{-1} \log \left(1 + \gamma_m \eta \right),$$

where $b = (b_0, b_1, \cdots, b_{35})$ is given by

$$b = \begin{bmatrix}
14947 & 7671 & 3587 & 5489 & 5157 & 1747 & 4211 & 1597 & 1055 \\
69357 & 114751 & 64227 & 240353 & 273887 & 200565 & 590640 & 458477 & 381241 \\
1393 & 587 & 821 & 524 & 221 & 760 & 93 & 151 & 213 \\
978106 & 1397729 & 1174777 & 912265 & 4238347 & 936895 & 2113416 & 5300075 & 922 \\
173 & 193 & 61 & 199 & 9 & 74 & 29 & 22 & 92 \\
6167757 & 11441683 & 5654848 & 31968962 & 2227843 & 31411653 & 19817069 & 2553923 & 3 \\
20 & 7 & 28 & 11 & 7 & 3 & 29313301 & 82781219 & 41644027 & 278473151 & 210993463 & 222912770 & 200969923 & 5 & 1 & 1 & 552578509 & 259446883 & 425548468 \\
\end{bmatrix}
$$

$\gamma_m$, for $m = 1, \cdots, 8$ given by

$$\gamma = \begin{bmatrix}
279593 & 486832 & 29306 & 15231 & 34356 & 53945 & 40025 & 289698 \\
312700 & 53467 & 35053 & 19853 & 45869 & 65058 & 45693 & 322535 \\
\end{bmatrix}$$

$$\lambda = \begin{bmatrix}
213509 & 6866 & 1248 & 5225 & 1284 & 1347 & 1555 & 42283 \\
381372 & 53037 & 13703 & 73982 & 177829 & 224215 & 278171 & 7792157 \\
\end{bmatrix}
$$

The height corresponding to this quasi-solution $(\epsilon_0, f_0)$ is found to be $h_0 = 0.435905237 \cdots$, while corresponding $\epsilon_0 = \frac{1}{10} \log \frac{1}{\rho}$, where simple Taylor series estimates show that $||f_0 - \hat{f}_0||_A \leq 10^{-10}$. Calculations, made simple by use of symbolic language maple, gives bounds $||R_0||_E \leq 2.2 \times 10^{-8}$ and $||R_0^2||_E \leq 1.47 \times 10^{-6}, ||A||_E \leq 6.23, ||B||_E \leq 5.34, ||L||_E \leq 1.39 \times 10^{-6}$. With choice of $K = 80$, one may check we obtained $\gamma_{1,1} \geq 0.095, \gamma_{2,2} \geq 0.82, \gamma_{1,2} \leq 0.096, \gamma_{2,1} \leq 0.123$ and $\gamma_f \leq 1.18$, implying $M \leq 14.3$, and $M \leq 18.3$. Further, based on quasi-G solution $G_0$ in the appendix for this case, we found $\epsilon_G := ||L G_0||_S \leq 0.011$ and therefore from explicit calculations of $||G_0||_E$ and $G_0(\pi)$ and using Lemma 19 as explained earlier, we conclude $||G||_E \leq 34.7, ||G(\pi)||_E \geq 32.3$. Therefore with $\epsilon_0 = 4 \times 10^{-6}, ||\Phi(0)||_E \leq 1.64 \times 10^{-4}$, $B_0$. It may be checked that conditions for applying Proposition 21 hold when $\epsilon = \frac{1}{10} \log \frac{1}{\rho}$ in which case solution $\Phi$ to the weakly nonlinear problem exists in a ball of size $M_E = B_0(1 + \epsilon) \leq 2.2 \times 10^{-4}$ for any $a_1 \in I$. This is the bound $M_E$ in Theorem 1. With estimated $||G||_E \leq 34.7$ and $||G(\pi)||_E \geq 32.3$, we also check that conditions (179) and (180) for for application of
proposition 23 for specified $\epsilon_0$ and hence $\mathcal{U} : I \to I$ is contractive and there exists unique $a_1$ corresponding to given $\mu$. The constant $K_3$, estimated from a finite sum of closed form definite integrals, satisfies $K_3 \leq 5.24$ in this case.

9.2. Case of $\mu = \mu_2 := 0.002$. In this case, $c_0 = \frac{22419}{29062}$ and we take for $N = 255$, $f_0 = P_N \hat{f}_0$, where

$$
\hat{f}_0 = b_0 + \sum_{j=1}^{35} \frac{b_j}{j} \eta^j + \sum_{m=1}^{8} \lambda_m \gamma_m^{-1} \log (1 + \gamma_m \eta),
$$

where $b = (b_0, b_1, \cdots, b_{35})$ is given by

$$
b = \begin{bmatrix}
50693 & 10841 & 3827 & 1833 & 4757 & 5211 & 113 & 1151 & 659 \\
233705 & 160294 & 69041 & 79169 & 255635 & 589261 & 16372 & 325172 & 245124 \\
445 & 443 & 629 & 338 & 215 & 335 & 290 \\
948881 & 1469338 & 1372099 & 1256464 & 3319023 & 4276857 \\
794888 & 250 & 38 & 223 \\
3012359 & 2340033 & 2201739 & 14727322 & 13774187 & 231617276 & 9552156 \\
19 & 8 & 13 & 10 & 8 & 3 \\
21786791 & 80694165 & 26196413 & 83929864 & 98473649 & 167259223 & 94712678 \\
7 & 4 & 1 & 2 \\
513297007 & 438834173 & 3854798571
\end{bmatrix}
$$

$\gamma_m$, for $m = 1, \cdots, 8$ given by

$$
\gamma = \begin{bmatrix}
53379 & 23440 & 7774 & 22168 & 61118 & 66536 & 106753 & 133678 \\
59794 & 26803 & 9313 & 28939 & 82803 & 81045 & 122929 & 150055
\end{bmatrix}
$$

The corresponding $h_0 = 0.4354696138 \cdots$ and $\mu_0 = 0.00199999998 \cdots$. For $\beta = \frac{1}{9} \log \frac{1}{\lambda}$, we use the truncated Taylor series expansion $f_0 = P_N \hat{f}_0$ for $N = 255$ where Taylor series estimates show that $\|f_0' - \hat{f}_0'\|_{\mathcal{A}} \leq 10^{-11}$. Calculations, made simple by use of symbolic language maple, gives bounds $\|R_0\|_{\mathcal{E}} \leq 2.52 \times 10^{-8}$ and $\|R_0\|_{\mathcal{S}} \leq 8.82 \times 10^{-6}$, $\|A\|_{\mathcal{S}} \leq 5.76$, $\|B\|_{\mathcal{E}} \leq 4.95$, $\|R\|_{\mathcal{S}} \leq 9.0 \times 10^{-6}$. With choice of $K = 80$, one may check we obtained $\gamma_{1,1} \geq 0.11$, $\gamma_{2,2} \geq 0.82$, $\gamma_{2,1} \leq 0.095$, $\gamma_{2,1} \leq 0.12$ and $\gamma_f \leq 1.15$, implying $M \leq 12.0$, and $M \leq 15.3$. Further, based on quasi-G solution $G_0$ in the appendix for this case, we found $\epsilon_G := \|LG_0\|_{\mathcal{S}} \leq 0.034$ and therefore from explicit calculations of $\|G_0\|_{\mathcal{E}}$ and $G_0(\pi)$ and using Lemma 19 as explained earlier, we conclude $\|G\|_{\mathcal{E}} \leq 29.4$, $|G(\pi)| \geq 27.3$. Therefore with $\epsilon_0 = 2 \times 10^{-6}$, $|\Phi(0)\|_{\mathcal{E}} \leq 7.23 \times 10^{-5} =: B_0$. It may be checked that conditions for applying Proposition 24 hold when $\epsilon = \frac{1}{36}$ in which case solution $\Phi$ to the weakly nonlinear problem exists in a ball of size $M_E = B_0(1 + \epsilon) \leq 8.4 \times 10^{-5}$ for any $a_1 \in I$. This is the bound $M_E$ in Theorem 1. With estimated $\|G\|_{\mathcal{E}} \leq 29.4$ and $|G(\pi)| \geq 27.3$, we also check that conditions (179) and (180) for for application of proposition 23 for specified $\epsilon_0$ and hence $\mathcal{U} : I \to I$ is contractive and there exists unique $a_1$ corresponding to given $\mu$. The constant $K_3$, estimated from a finite sum of closed form definite integrals, satisfies $K_3 \leq 5.20$ in this case.
9.3. Case of $\mu = \mu_4 := 0.0023$. In this case, $c_0 = \frac{22885}{30921}$ and we take for $N = 255$, $f_0 = P_N\tilde{f}_0$, where $P_N$ is the truncation of the Taylor series of $\tilde{f}_0$ about the origin up to and including $\eta^N$ term, and

$$
\tilde{f}_0 = b_0 + \sum_{j=1}^{37} b_j \eta^j + \sum_{m=1}^{6} \lambda_m \gamma_m^{-1} \log (1 + \gamma_m \eta),
$$

where $b = (b_0, b_1, \cdots, b_{37})$ is given by

$$
b = \begin{bmatrix}
  63307 & 8943 & 7094 & 5467 & 5869 & 2551 & 3360 & 3609 & 2951 \\
  288588 & 92260 & 90177 & 140443 & 191402 & 147538 & 254201 & 449717 & 495374 \\
  5189 & 1541 & 814 & 2661 & 1310 & 465 & 409 & 664 \\
  1363594 & 561523 & 446813 & 2084309 & 1498987 & 781442 & 979023 & 2403679 & \\
  311 & 431 & 355 & 109 & 117 & 104 & 146 & 83 & 49 \\
  1569858 & 3391197 & 3825084 & 1889706 & 2735110 & 4046415 & 7559227 & 7419778 & 5764323 \\
  48 & 82 & 25 & 55 & 67 & 35 & 10 & 5 \\
  10153027 & 22615125 & 12976987 & 36954884 & 89304183 & 60078712 & 36228767 & 23269474 & \\
  211760067 & 95298423 & 170112576 & 308113800 \\
\end{bmatrix}
$$

$\gamma_m$, for $m = 1, \cdots, 6$ given by

$$
\gamma = \begin{bmatrix}
  -155593 & 119606 & 21931 & 64810 & 16039 & 154314 \\
  174744 & 138125 & 271143 & 84503 & 18975 & 175871 \\
\end{bmatrix}
$$

$$
\lambda = \begin{bmatrix}
  184732 & 10272 & 15677 & 2844 & 1136 & 48187 \\
  341273 & 73979 & 156817 & 284831 & 156707 & 7142266 \\
\end{bmatrix}
$$

The corresponding $h_0 = 0.4347167189 \cdots$ and $\mu_0 = 0.00230000015 \cdots$. For $\beta = \frac{1}{20} \log \frac{1}{\alpha}$, a truncated Taylor series expansion of $f_0 = P_N\tilde{f}_0$ to a degree of $N = 255$ gives rise to $\|f_0 - \tilde{f}_0\|_A \leq 10^{-11}$. For $f_0$ as above, $\|R_0\|_\varepsilon \leq 1.065 \times 10^{-7}$ and $\|R_0\|_G \leq 5.33 \times 10^{-6}, \|A\|_G \leq 5.10, \|B\|_G \leq 4.40$. Based on this, $\|R\|_G \leq 5.1 \times 10^{-6}$. With choice of $K = 80$, we obtained $\gamma_{1,1} \geq 0.137, \gamma_{2,2} \geq 0.84, \gamma_{1,2} \leq 0.091, \gamma_{2,1} \leq 0.11$ and $\gamma_f \leq 1.10$, implying $M \leq 8.98$, and $M \leq 11.3$. Based on the quasi-G solution $G_0$ in the appendix for this case, we calculated the Fourier sine coefficient of $LG_0$ and estimated $\|LG_0\|_G \leq 0.0005 =: \varepsilon_G$. Using it in Lemma [14] with explicit calculation of $\|G_0\|_G$ and $|G_0(\pi)|$, we get the bounds $\|G\|_G \leq 23.21, |G(\pi)| \geq 21.87$, and therefore with $\varepsilon_0 = 4 \times 10^{-6}, B_0 = |\Phi(0)| \leq 1.5 \times 10^{-4}$. Contraction mapping argument follows for $\varepsilon = \frac{1}{20}$ giving rise to a ball size $B_0(1 + \varepsilon) \leq 1.65 \times 10^{-4}$ for any $a_1 \in I$, where solution exists for $H$ to the weakly nonlinear problem. This bound $M_E$ in Theorem [1] To prove that there exists $a_1 \in I$ satisfying constraint [12] we checked that both conditions [170] and [180] for contraction of $U : I \to I$, we were valid. The constant $K_3$, estimated from a finite sum of closed form definite integrals, satisfies $K_3 \leq 5.14$ in this case.

10. Discussion

We have shown how, through construction of quasi-solutions $(f_0, c_0)$ obtained through numerical calculations, one can rigorously and constructively prove existence of water wave solution by turning the strongly nonlinear problem into a
weakly nonlinear analysis. Thus far, we have only demonstrated this for a small
set of $\mu$ in the range $(0, \frac{1}{3})$.

The quasi-solution can be determined also with explicit dependence on $\mu$ over
suitably small intervals of $\mu$ by using small order polynomials in $\mu$ for coefficients of
the rational approximant $\tilde{f}_0$. Proving the residuals $R_0$ and $R'_0$ is also not difficult
since the cosine or sine series involving $\cos(n\nu)$ or $\sin(n\nu)$ now involve polynomials
in $\mu$, which can be expressed as a Chebyshev basis in scaled $\mu$ variable. An $l^1$
estimate of these Chebyshev coefficients gives the maximal value of the coefficient
of $\cos(n\nu)$ or $\sin(n\nu)$.

However, the proof thus far is manageable (with help of symbolic manipulation
language MAPLE) for $\mu$ relatively large, which corresponds to modest $h$, where
Stokes original expansion works just as well. Hence we have limited presentations
for small intervals around isolated values of $\mu$. The corresponding wave heights are
somewhat smaller than the critical. When $\mu \to 0$, the accuracy needs for quasi-
solution becomes more stringent since the bound $M$ in our method deteriorates.
The present rational function approximation gets taxed to the limit when $\mu$ be-
comes very small. For more efficiency in these cases, it is better to incorporate
local behavior near the crest as was done earlier in numerical computations [13].
Unfortunately, the simple empirical approximation due to Longuet-Higgins [24] is
not accurate enough to be controlled rigorously. One also needs a closer exami-
nation of of the Nekrasov integral formulation which we suspect will work better
for higher waves than the simple minded, though general, series method employed
here.

Nonetheless, what is also interesting in this approach is that detailed features of
the solution that are difficult to prove in non-constructive methods can be obtained
with relative ease. For instance, a crucial role in the stability of periodic water
waves is played by the empirical fact that wave speed $c$ goes through a maximum
close to $\mu = \mu_1 = 1.8306 \times 10^{-3}$. This can be confirmed in the following manner.
We take two values on either side of $\mu_1$ and compute $\partial W(\alpha)$, which up to nonlinear
correction is given by $G(\alpha)$. Through a more accurate representation of quasi-G
solution $G_0$ than provided here, it can be proved that $\partial W(\alpha)$ which determines
change in $c$ changes in some interval around $\mu = \mu_1$. Control of the the lower bound of
second derivative is also needed to prove that there is only one such maximum of $c$
in some interval. In this context, it is interesting to note that even for our relatively
inaccurate quasi-G solution $G_0$ for $\mu = \mu_1$, we find $G_0(\alpha) = 9.97 \cdots \times 10^{-6}$, which
is significantly smaller than $4.86 \cdots \times 10^{-3}$ at $\mu = \mu_2$, suggesting that $G(\alpha)$ does
change sign for some $\mu$ close to $\mu_1$.

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12. APPENDIX

Here we simply present the quasi-G solution $G_0$ found numerically for different
$\mu$ with coefficients approximated by rationals to 8 digits. It is clear that if $|G_0(-1)|$
is sufficiently large, which it is in all the cases presented, \( LG_0 \) need not be too small to check Proposition 23. In all cases, the quasi-solution is taken to be in the form

\[
G_0 = \sum_{j=0}^{126} g_j \cos(j\nu)
\]

12.1. \textbf{G}_0 \textbf{Fourier cosine coefficients} \( \mu = \mu_1 := 0.0018306 \). \( g = (g_0, g_1, g_2, \cdots g_{126}) \) is given by:

\[
\begin{bmatrix}
997 & 1, & 7874 & 3214 & 6536 & 12795 & 6399 & 6783 & 1337 & 750 & 2585 & 3479 \\
19434 & 3587 & 1445 & 2921 & 6056 & 3214 & 3683 & 786 & 481 & 1811 & 2672 \\
2210 & 1081 & 2909 & 6520 & 4591 & 2598 & 2931 & 3030 & 2033 & 1974 & 1689 & 1146 \\
1863 & 1002 & 3098 & 1508 & 1937 & 617 & 773 & 5259 & 724 & 1042 & 1413 \\
2368 & 9409 & 2019 & 4245 & 6447 & 8874 & 3131 & 4344 & 32743 & 4993 & 7963 & 11963 \\
5905 & 1302 & 692 & 823 & 1574 & 341 & 898 & 457 & 845 & 695 & 637 \\
55408 & 13537 & 7975 & 10511 & 22285 & 5351 & 15623 & 8813 & 18068 & 16474 & 16743 \\
578 & 413 & 146 & 113 & 257 & 169 & 233 & 851 & 572 & 279 & 428 \\
16843 & 13346 & 15231 & 4490 & 11323 & 8258 & 12625 & 51143 & 38121 & 20624 & 35087 \\
699 & 133 & 175 & 257 & 263 & 73 & 687 & 285 & 181 & 219 \\
63562 & 13413 & 10817 & 31887 & 36198 & 11144 & 116342 & 53534 & 37717 & 50620 \\
279 & 106 & 473 & 109 & 41 & 118 & 111 & 125 & 109 & 394 \\
71543 & 10015 & 149263 & 38156 & 15923 & 50837 & 53056 & 66281 & 64125 & 257143 \\
121 & 208 & 19 & 70 & 99 & 17 & 49 & 194 & 75 & 41 \\
87618 & 167093 & 16935 & 62919 & 108619 & 20693 & 66179 & 290695 & 124696 & 75630 \\
50 & 19 & 23 & 93 & 57 & 104 & 37 & 31 & 38 & 302 \\
102339 & 43147 & 57955 & 260003 & 176824 & 357963 & 141313 & 131367 & 178685 & 157565 \\
13 & 38 & 32 & 20 & 47 & 23 & 52 & 13 & 11 & 32 \\
75263 & 244105 & 228103 & 158187 & 412506 & 223988 & 561947 & 155885 & 146370 & 472479 \\
28 & 15 & 9 & 17 & 37 & 58 & 27 & 14 & 7 & 19 \\
458767 & 272711 & 181576 & 380580 & 919192 & 159885 & 825968 & 475243 & 263693 & 794229 \\
31 & 10 & 11 & 7 & 4 & 11 & 25 & 13 & 7 \\
1438034 & 514757 & 628366 & 443727 & 281383 & 858680 & 2165713 & 1249706 & 746770
\end{bmatrix}
\]
12.2. \textbf{Fourier cosine coefficients for} $\mu = \mu_2 := 0.002$. In this case, $g = (g_0, g_1, \cdots, g_{126})$ given by

\begin{equation}
(199) \quad g = \begin{bmatrix}
975, 2174, 6185, 4355, 14321, 1807, 7847, 3627, 2457, 3787
\end{bmatrix}

\begin{bmatrix}
83047, 1053, 3017, 2136, 7532, 1018, 4829, 2438, 1819, 3089
\end{bmatrix}

\begin{bmatrix}
9427, 4444, 1397, 2770, 2254, 10127, 9302, 5913, 1027, 1936, 719
\end{bmatrix}

\begin{bmatrix}
8507, 4439, 1548, 3407, 3081, 15393, 15733, 11135, 2154, 4525, 4783
\end{bmatrix}

\begin{bmatrix}
768, 20423, 11843, 646, 843, 20679, 3268, 146, 1340, 793, 319
\end{bmatrix}

\begin{bmatrix}
2231, 66159, 42804, 1879, 3794, 103865, 18327, 9303, 9369, 6191, 2782
\end{bmatrix}

\begin{bmatrix}
5586, 1289, 2847, 1552, 886, 123, 572, 289, 587, 471, 298
\end{bmatrix}

\begin{bmatrix}
54407, 14027, 34619, 21081, 13445, 2086, 10839, 6121, 13893, 12461, 8811
\end{bmatrix}

\begin{bmatrix}
242, 688, 1083, 977, 799, 445, 272, 119, 293, 371, 501
\end{bmatrix}

\begin{bmatrix}
7999, 25417, 44731, 45105, 41243, 25677, 17549, 8583, 23631, 33452, 50516
\end{bmatrix}

\begin{bmatrix}
56, 173, 396, 247, 395, 213, 163, 58, 113, 180, 317
\end{bmatrix}

\begin{bmatrix}
6313, 21810, 55819, 38937, 69624, 41989, 35930, 14299, 31152, 55501, 109303
\end{bmatrix}

\begin{bmatrix}
56, 95, 553, 118, 39, 50, 187, 93, 229, 46
\end{bmatrix}

\begin{bmatrix}
21597, 40972, 26676, 63659, 23534, 33743, 141162, 78515, 216260, 48585
\end{bmatrix}

\begin{bmatrix}
53, 59, 79, 51, 65, 169, 40, 207, 64, 83
\end{bmatrix}

\begin{bmatrix}
62618, 77963, 116775, 84317, 120213, 349587, 92561, 535768, 185307, 268804
\end{bmatrix}

\begin{bmatrix}
417, 33, 24, 51, 65, 169, 40, 207, 64, 83
\end{bmatrix}

\begin{bmatrix}
1510792, 133732, 108805, 259263, 164519, 158644, 35496, 230292, 177681, 1301845
\end{bmatrix}

\begin{bmatrix}
35, 22, 10, 29, 13, 15, 42, 19, 76
\end{bmatrix}

\begin{bmatrix}
389126, 273607, 139137, 451365, 226368, 292183, 915289, 463190, 2072849
\end{bmatrix}

\begin{bmatrix}
10, 17, 35, 9, 5, 17, 13, 8, 13
\end{bmatrix}

\begin{bmatrix}
305109, 580304, 1336533, 384511, 238971, 909037, 777658, 535421, 973341
\end{bmatrix}

\begin{bmatrix}
5, 10, 6, 5, 17, 8, 6, 8, 5
\end{bmatrix}

\begin{bmatrix}
418846, 937139, 629101, 586492, 3543425, 1174561, 985613, 1470193, 1028078
\end{bmatrix}

\begin{bmatrix}
10, 3, 5, 2
\end{bmatrix}

\begin{bmatrix}
2300319, 772117, 1439682, 644321
\end{bmatrix}
12.3. **Fourier cosine coefficients for** $G_0$ **in case** $\mu_3 := 0.0023$. In this case, $g = (g_0, g_1, g_2, \cdots, g_{126})$ is given by

$$
\begin{bmatrix}
127 & 3652 & 5442 & -4351 & 1839 & 1412 & 1459 & 39833 & 4286 & 5351 \\
18432 & -2490 & 2981 & -2441 & 1129 & 943 & 1083 & 32774 & 3947 & 5508 \\
2569 & 1109 & 1841 & 3146 & 3741 & -2436 & 3291 & 3812 & 1671 & 2186 & 1475 \\
2971 & 1440 & 2692 & 5179 & 6946 & 5101 & 7781 & 10177 & 5041 & 7453 & 5686 \\
954 & 3809 & 883 & 1343 & 557 & 155 & 517 & 683 & 743 & 1389 & 357 \\
4159 & 18784 & -4927 & 8436 & 3981 & 1254 & -4736 & 7084 & 8728 & 18478 & 5380 \\
447 & 9247 & 7591 & 1419 & 1310 & 333 & 494 & 309 & 1118 & 520 \\
7630 & 178337 & 166312 & -35230 & 36849 & -10616 & 17845 & -12652 & 51875 & -27351 \\
493 & 416 & 200 & 289 & 231 & 163 & 721 & 11 & 538 & 215 \\
29388 & 28113 & 15319 & -25097 & 22738 & -18192 & 91216 & -1578 & 87491 & 39648 \\
133 & 214 & 272 & 50 & 57 & 305 & 149 & 125 & 77 & 35 \\
27805 & 50739 & 73109 & -15241 & 19699 & -119544 & 66215 & -63002 & 44004 & -22686 \\
113918 & 215071 & 91689 & 67546 & 355007 & -241333 & 107903 & -101110 & 130752 \\
110 & 155 & 258 & 97 & 63 & 292 & 175 & 69 & 57 & 65 \\
-250999 & 463188 & -8806 & 133153 & 234123 & 64237 & 315774 & 1944853 & 156253 \\
86 & 25 & 29 & 31 & 20 & 20 & 359 & 16 & 17 & 64 \\
609597 & 201033 & 264477 & 320731 & 234681 & 4779007 & 241567 & 291184 & 1243307 \\
17 & 32 & 14 & 53 & 15 & 73 & 49 & 4 & 20 \\
374674 & 799097 & -397035 & 1704772 & 547391 & 3021506 & 2301003 & 213049 & 1208577 \\
12 & 19 & 10 & 21 & 11 & 7 & 13 & 35 & 6 \\
822485 & 1477506 & 882027 & 2101522 & 1248582 & 901486 & 1898967 & 5800724 & 1127929 \\
7 & 1 & 5 & 5 & 7 & 8 & 7 & 6 \\
1493042 & 241932 & -1372493 & 1556798 & 2472921 & 3205731 & 3182644 & 3094345 \\
4 & 1 & 1 & 4 & 3 & 1 & 4 & 3 \\
2340633 & 663748 & 753116 & 3417071 & 2907881 & 1099486 & 4990151 & 4245329 \\
\end{bmatrix}
$$

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