Novel Superposed Kink and Pulse Solutions for $\phi^4$, MKdV, NLS and Other Nonlinear Equations

Avinash Khare
Physics Department, Savitribai Phule Pune University
Pune 411007, India

Avadh Saxena
Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

Abstract:
We show that a number of nonlinear equations including symmetric as well as asymmetric $\phi^4$, modified Korteweg de Vries (MKdV), mixed KdV-MKdV, nonlinear Schrödinger (NLS), quadratic-cubic NLS as well as higher order neutral scalar field theories, higher order KdV-MKdV and higher order quadratic-cubic NLS admit superposed periodic kink and pulse solutions and some of them also admit superposed hyperbolic kink solutions.

1 Introduction

During the last few decades several nonlinear equations including $\phi^4$, NLS, KdV and MKdV equations [1] have found application in several different areas of physics. While some of them are integrable, the others are nonintegrable. All these systems admit periodic as well as hyperbolic pulse and kink solutions. Besides, the integrable models such as NLS, KdV, MKdV and some others also admit N-soliton solutions, breather solutions, etc. Many novel properties of the nonlinear equations have been discovered over the years. However, the nonlinear equations are very rich and it is not clear whether we have uncovered all of their novel properties.

Several years ago, in a largely unnoticed paper, Tankeyev, Smagin, Borich and Zhuravlev [2] obtained a novel superposed hyperbolic kink solution for the repulsive MKdV equation. Their work was inspired by the
earlier work of [3]. The obvious question is whether a similar solution also exists for other nonlinear equations. Secondly, are there superposed periodic as well as hyperbolic pulse and kink solutions for MKdV and other nonlinear equations? The purpose of this paper is to answer these questions in the affirmative. In particular, we show that a large number of nonlinear equations, including both symmetric and asymmetric $\phi^4$, MKdV, NLS and several other nonlinear equations, many of which have found wide application, indeed admit superposed periodic and hyperbolic kink and periodic pulse solutions. By superposed here we mean that a solution is a linear combination of two kink solutions or two pulse solutions or the corresponding periodic solutions. Interestingly, such solutions are known to exist in both condensed matter and field theory contexts, e.g. the scalar potential for baryons or polarons in conducting polymers [4, 5, 6, 7].

The plan of the paper is the following. In Sec. II we show that the celebrated symmetric double well $\phi^4$ equation admits superposed periodic kink and pulse solutions. In Sec. III we consider the asymmetric double well $\phi^4$ equation and show that it not only admits the superposed periodic kink and pulse solutions but also admits two distinct types of hyperbolic superposed kink solutions. We also consider a one-parameter family of scalar field theories of the form $\phi^{2m}\phi^{2m+2}\phi^{4m+2}$ and show that even these models admit superposed hyperbolic kink solutions. In Sec. IV we discuss the MKdV equation and show that apart from the superposed hyperbolic kink solution already known [2], like the symmetric $\phi^4$ case, the MKdV equation also admits superposed periodic kink and pulse solutions. Further, using the celebrated Miura transformation [8], from the superposed solutions of the repulsive MKdV equation we immediately obtain the corresponding superposed solutions of the KdV equation. In Sec. V we discuss mixed KdV-MKdV equation [9] and show that like the asymmetric $\phi^4$ case, it also admits superposed periodic kink and pulse solutions as well as superposed hyperbolic kink solutions. We further consider a one-parameter family of generalized mixed KdV-MKdV system characterized by $um_{ux} - u^{2m}u_x$ ($m = 1, 2, 3, ...$) and show that it admits novel superposed hyperbolic kink solutions. In Sec. VI we consider the celebrated integrable NLS equation and show that like the symmetric $\phi^4$ case, it also admits novel superposed periodic kink and pulse solutions. In Sec. VII we study a mixed quadratic-cubic NLS equation [10] and show that like the asymmetric $\phi^4$ case, it not only admits superposed periodic pulse and kink solutions but also admits a superposed hyperbolic kink solution. In addition, we consider a one-parameter family of generalized higher order NLS systems characterized by $|u|^m u - |u|^{2m}u$ ($m = 1, 2, 3, ...$) and show that even such systems admit superposed hyper-
bolic kink solutions. Finally, in Sec. VIII we summarize the main results obtained in this paper and discuss some of the open problems.

2 Superposed Solutions of Symmetric $\phi^4$ Field Theory

In this section we show that the well known $\phi^4$ field equation
\[ \phi_{xx} = a\phi + d\phi^3, \]  
not only admits the periodic kink and the pulse solutions $\text{sn}(x, m)$ $\text{cn}(x, m)$ and $\text{dn}(x, m)$ respectively, but even superposed periodic kink and pulse solutions. Here $\text{sn}(x, m)$, $\text{cn}(x, m)$ and $\text{dn}(x, m)$ denote Jacobi elliptic functions with modulus $m$.

But before that let us recall the three well known periodic solutions of the $\phi^2$-$\phi^4$ field Eq. (1). In particular, it is well known [12, 13] that Eq. (1) admits the pulse-like periodic solutions
\[ \phi(x) = A\text{cn}(\beta x, m), \]  
provided
\[ dA^2 = -2m\beta^2, \quad a = (2m - 1)\beta^2, \]  
and
\[ \phi(x) = A\text{dn}(\beta x, m), \]  
provided
\[ dA^2 = -2\beta^2, \quad a = (2 - m)\beta^2. \]

In the limit $m = 1$, both the solutions (2) and (4) go over to the hyperbolic pulse solution
\[ \phi(x) = A\text{sech}(\beta x), \]  
provided
\[ dA^2 = -2\beta^2, \quad a = \beta^2. \]

Notice that for all three pulse solutions, $a > 0, d < 0$.

The same field Eq. (1) also admits the periodic kink solution
\[ \phi(x) = A\text{sn}(\beta x, m), \]  
provided
\[ dA^2 = 2m\beta^2, \quad a = -(1 + m)\beta^2. \]
In the limit $m = 1$, the periodic kink solution \( \phi(x) \) goes over to the celebrated kink solution
\[
\phi(x) = A \tanh(\beta x),
\]
provided
\[
dA^2 = 2\beta^2, \quad a = -2\beta^2.
\]
Notice that unlike the three pulse solutions, the two kink solutions are valid if $a < 0, d > 0$.

We now show that the symmetric $\phi^4$ field Eq. (11) also admits four novel periodic solutions which can be written as superposition of either the periodic kink or the pulse solutions $\text{sn}(x,m)$ and $\text{dn}(x,m)$, respectively.

**Solution I**

It is easy to check that the $\phi^4$ field Eq. (11) admits the periodic solution
\[
\phi(x) = \frac{A\text{dn}(\beta x,m)\text{cn}(\beta x,m)}{1 + B\text{cn}^2(\beta x,m)}, \quad B > 0,
\]
provided
\[
0 < m < 1, \quad a = [5m - 1 - 6B(1 - m)]\beta^2,
\]
\[
dmA^2 = [2mB(1 - 2B) + 6(1 - m)B^2]\beta^2,
\]
while $B$ satisfies a cubic equation
\[
(1 - m)^2B^3 - 3m(1 - m)B^2 + Bm(3m - 1) + m^2 = 0.
\]
Note that this solution is not valid for $m = 1$, i.e. the symmetric $\phi^4$ field Eq. (11) does not admit a corresponding hyperbolic solution.

This cubic equation is easily solved once one realizes that one solution must be $B = m/(1 - m)$ giving the well known solution $\phi = \frac{A\text{cn}(\beta x,m)}{\text{dn}(\beta x,m)}$. We find that the three roots of the cubic equation are
\[
B = \frac{m}{1 - m}, \quad \pm \frac{\sqrt{m}}{1 - \sqrt{m}},
\]
out of which the acceptable solution is
\[
B = \frac{\sqrt{m}}{1 - \sqrt{m}},
\]
and the corresponding values of $a$ and $dA^2$ turn out to be
\[
a = -[1 + m + 6\sqrt{m}]\beta^2 < 0, \quad dA^2 = \frac{8\sqrt{m}\beta^2}{(1 - \sqrt{m})^2}.
\]
Notice that for this solution \( a < 0, d > 0 \).

At this stage, we recall the well known addition theorem for \( \text{sn}(x, m) \) [11], i.e.

\[
\text{sn}(a + b, m) = \frac{\text{sn}(a, m)\text{cn}(b, m)\text{dn}(b, m) + \text{cn}(a, m)\text{dn}(a, m)\text{sn}(b, m)}{1 - m\text{sn}^2(a, m)\text{sn}^2(b, m)}.
\] (18)

From here it is straightforward to obtain the identity

\[
\text{sn}(y + \Delta, m) - \text{sn}(y - \Delta, m) = \frac{2\text{cn}(y, m)\text{dn}(y, m)\frac{\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}}{1 + B\text{cn}^2(y, m)}, \quad B = \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}.
\] (19)

On comparing Eqs. (12) and (19) and using Eqs. (16) and (17), one can re-express the periodic solution I given by Eq. (12) as superposition of two periodic kink solutions, i.e.

\[
\phi(x) = \sqrt{2m/\beta} \left[ \text{sn}(\beta x + \Delta, m) - \text{sn}(\beta x - \Delta, m) \right],
\] (20)

Here \( \Delta \) is defined by \( \text{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4} \), where use has been made of the identity

\[
\sqrt{m\text{sn}(y, m)} = \text{sn}(\sqrt{m}y, 1/m).
\] (21)

**Solution II**

Remarkably, the symmetric \( \phi^4 \) Eq. (11) also admits another periodic solution

\[
\phi(x) = \frac{A\text{sn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)}, \quad B > 0,
\] (22)

provided

\[
0 < m < 1, \quad dA^2 = [-6B^2(1 - m) + 20mB - 8B + 14m]\beta^2,
\]

\[
a = [(5m - 1) + \frac{6m}{B}]\beta^2,
\] (23)

where \( B \) satisfies a cubic equation

\[
(1 - m)B^3 - (3m - 1)B^2 - 3mB - m = 0.
\] (24)

Note that this solution is also not valid for \( m = 1 \), i.e. the symmetric \( \phi^4 \) field Eq. (11) does not admit a corresponding hyperbolic solution. This cubic equation is easily solved once one realizes that one root must be \( B = -1 \)
giving rise to the well known singular solution \( \phi = A/\text{sn}(\beta x, m) \). We find that the three roots of Eq. (24) are

\[
B = -1, \pm \frac{\sqrt{m}}{1 - \sqrt{m}},
\]

out of which the only acceptable solution is

\[
B = \frac{\sqrt{m}}{1 - \sqrt{m}}.
\]

Using this value of \( B \) one finds that

\[
a = [6\sqrt{m} - (1 + m)]\beta^2, \quad dA^2 = -8\sqrt{m}\beta^2.
\]

Thus for this solution while \( d < 0, \ a > (\leq) 0 \) depending on if \( 6\sqrt{m} > (\leq) 0 \).

Now on using the \( \text{sn}(x, m) \) addition theorem \([18]\), one can derive another novel identity

\[
\text{sn}(y + \Delta, m) + \text{sn}(y - \Delta, m) = \frac{2\text{sn}(y, m)\text{cn}(\Delta, m)}{1 + B\text{cn}^2(y, m)}, \quad B = \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}.
\]

On comparing Eqs. (22) and (28) and using Eqs. (26) and (27), the periodic solution II given by Eq. (22) can be re-expressed as superposition of two periodic kink solutions

\[
\phi(x) = i\sqrt{\frac{2m}{|d|}} \beta \left[ \text{sn}(\beta x + \Delta, m) + \text{sn}(\beta x - \Delta, m) \right].
\]

Here \( \Delta \) is defined by \( \text{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4} \), where use has been made of the identity \([21]\).

It is worth noting that for both the solutions I and II, the value of \( B \) is the same but while \( d > 0 \) for the first solution, \( d < 0 \) for the second solution.

**Solution III**

It is easy to check that the symmetric \( \phi^4 \) field Eq. (1) admits another periodic solution

\[
\phi(x) = \frac{A\text{sn}(\beta x, m)\text{cn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)},
\]

provided

\[
0 < m < 1, \quad dA^2 = [14(1 - m)B^2 + 12B - 20Bm - 6m]\beta^2,
\]

\[
a = [5m - 4 - 6(1 - m)B]\beta^2.
\]
while $B$ satisfies a cubic equation

$$ (1 - m)B^3 + 3(1 - m)B^2 - (3m - 2)B - m = 0. \quad (32) $$

Note that this solution is also not valid for $m = 1$, i.e. the symmetric $\phi^4$ field Eq. (11) does not admit a corresponding hyperbolic solution. The cubic Eq. (32) is easily solved once one realizes that one solution must be $B = -1$ giving rise to the singular solution $\phi = A\text{cn}(\beta x, m)/\text{sn}(\beta x, m)$. It turns out that the three roots of $B$ are

$$ B = -1, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad B = -\frac{1 + \sqrt{1 - m}}{\sqrt{1 - m}}, \quad (33) $$

out of which the only acceptable solution is

$$ B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}. \quad (34) $$

Using this value of $B$ in Eq. (31) we find that

$$ a = (2 - m - 6\sqrt{1 - m})\beta^2, \quad dA^2 = -\frac{8(1 - \sqrt{1 - m})^2\beta^2}{\sqrt{1 - m}}. \quad (35) $$

Notice that for this solution $d < 0$ while $a$ could be positive or negative depending on the value of $m$.

On using the well known addition theorem for $\text{dn}(x, m)$ [11],

$$ \text{dn}(a + b, m) = \frac{\text{dn}(a, m)\text{dn}(b, m) - m\text{sn}(a, m)\text{cn}(a, m)\text{sn}(b, m)\text{cn}(b, m)}{1 - m\text{sn}^2(a, m)\text{sn}^2(b, m)}, $$

one can derive the novel identity

$$ \text{dn}(y - \Delta, m) - \text{dn}(y + \Delta, m) = \frac{2m\text{sn}(\Delta, m)\text{cn}(\Delta, m)\text{sn}(y, m)\text{cn}(y, m)}{\text{dn}^2(\Delta, m)[1 + B\text{cn}^2(y)]}, $$

$$ B = \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}. \quad (36) $$

On comparing solutions (30) and (37) and using Eqs. (34) and (35), we find that the solution III of the symmetric $\phi^4$ field Eq. (1) given by Eq. (30) can be re-expressed as a superposition of two periodic pulse solutions, i.e.

$$ \phi(x) = \beta\sqrt{\frac{2}{|d|}}\left(\text{dn}[\beta x - \frac{K(m)}{2}, m] - \text{dn}[\beta x + \frac{K(m)}{2}, m]\right). \quad (38) $$
Solution IV

Remarkably, the symmetric $\phi^4$ Eq. (11) also admits another periodic solution

$$\phi(x) = \frac{A \text{dn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)},$$

(39)

provided

$$0 < m < 1, \quad mdA^2 = [6B^2(1 - m) - 20mB + 12B - 14m] \beta^2,$$

$$a = [(5m - 4) + 6m/B] \beta^2,$$

(40)

while $B$ satisfies a cubic equation

$$(1 - m)^2B^3 + B^2(1 - m)(2 - 3m) - 3m(1 - m)B + m^2 = 0.$$ 

(41)

Note that this solution is also not valid for $m = 1$, i.e. the symmetric $\phi^4$ field Eq. (11) does not admit a corresponding hyperbolic solution. This cubic equation is easily solved once one realizes that one root must be $B = \frac{m}{1 - m}$.

We find that the three roots of Eq. (41) are

$$B = \frac{m}{1 - m}, \quad \pm \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}},$$

(42)

out of which the only acceptable solution is

$$B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}.$$ 

(43)

Note that the choice $B = \frac{m}{1 - m}$ leads to the well known solution $\phi = \frac{(1 - m)A}{\text{dn}(\beta x, m)}$.

Using $B$ as given by Eq. (43), one finds that

$$a = [2 - m + 6\sqrt{1 - m}] \beta^2, \quad dA^2 = -\frac{8}{\sqrt{1 - m}} \beta^2.$$ 

(44)

Thus for this solution while $d < 0, a > 0$.

Now using the addition theorem for $\text{dn}(x, m)$ as given by Eq. (36), one can derive another novel identity, i.e.

$$\text{dn}(y + \Delta, m) + \text{dn}(y - \Delta, m) = \frac{2\text{dn}(y, m)}{\text{dn}(\Delta, m)[1 + B \text{cn}^2(y, m)]}, \quad B = \frac{msu^2(\Delta, m)}{\text{dn}^2(\Delta, m)}.$$ 

(45)
On comparing Eqs. (39) and (45) and using Eqs. (43) and (44), the periodic solution IV given by Eq. (39) can be re-expressed as superposition of two periodic pulse solutions, i.e.

\[
\phi(x) = \sqrt{\frac{2}{|d|}} \beta \left( \text{dn}[\beta x + K(m)/2, m] + \text{dn}[\beta x - K(m)/2, m] \right).
\] (46)

It is worth noting that for both the superposed periodic pulse solutions III and IV, not only the value of \(B\) is the same but also \(d < 0\) for both the solutions. It is worth reminding that even for the periodic pulse solution \(\phi = A\text{dn}(\beta x, m), d < 0\) (see Eqs. (4) and (5)).

3 Superposed Solutions of Asymmetric \(\phi^4\) Field Theory

We now show that the asymmetric \(\phi^4\) field equation

\[
\phi_{xx} = a\phi - b\phi^2 + d\phi^3,
\] (47)

not only admits the superposed periodic kink and pulse solutions but even the superposed hyperbolic kink solution.

But before we proceed further, let us recall the three well known periodic solutions of the asymmetric field Eq. (47) [14, 15]. In particular, it is well known that Eq. (47) admits the periodic pulse solutions

\[
\phi(x) = A + B\text{cn}(\beta x, m),
\] (48)

provided

\[
b = 3dA, \quad a = 2dA^2, \quad dB^2 - 2m\beta^2, \quad dA^2 = -(2m - 1)\beta^2;
\] (49)

which implies that this solution exists only if

\[
a, b, d < 0, \quad 2b^2 = 9ad, \quad m > 1/2, \quad A = \sqrt{\frac{|a|}{2|d|}}, \quad B = \sqrt{\frac{|a|m}{(2m - 1)|d|}}, \quad \beta = \sqrt{\frac{|a|}{2(2m - 1)}}.
\] (50)

Similarly, Eq. (47) also admits another periodic pulse solution

\[
\phi(x) = A + B\text{dn}(\beta x, m),
\] (51)
\[ b = 3dA, \quad a = 2dA^2, \quad dB^2 = -2\beta^2, \quad dA^2 = -(2 - m)\beta^2, \quad (52) \]

which implies that this solution exists only if

\[ a, b, d < 0, \quad 2b^2 = 9ad, \quad A = \sqrt{\frac{|a|}{2|d|}}, \quad B = \sqrt{\frac{|a|}{(2 - m)|d|}}, \quad \beta = \sqrt{\frac{|a|}{2(2 - m)}}, \quad (53) \]

In the limit \( m = 1 \) both solutions (48) and (51) go over to the hyperbolic pulse solution

\[ \phi(x) = A + B\text{sech}(\beta x), \quad (54) \]

provided

\[ b = 3dA, \quad a = 2dA^2, \quad dB^2 = -2\beta^2, \quad dA^2 = -\beta^2, \quad (55) \]

which implies that this solution exists only if

\[ a, b, d < 0, \quad 2b^2 = 9ad, \quad A = \sqrt{\frac{|a|}{2|d|}}, \quad B = \sqrt{\frac{|a|}{d}}, \quad \beta = \sqrt{\frac{|a|}{2}}. \quad (56) \]

Notice that for all the three pulse solutions, \( a, b, d < 0. \)

The same field Eq. (47) also admits the periodic kink solution

\[ \phi(x) = A + B\text{sn}(\beta x, m), \quad (57) \]

provided

\[ b = 3dA, \quad a = 2dA^2, \quad dB^2 = 2m\beta^2, \quad dA^2 = (1 + m)\beta^2, \quad (58) \]

which implies that this solution exists only if

\[ a, b, d > 0, \quad 2b^2 = 9ad, \quad A = \sqrt{\frac{a}{2d}}, \quad B = \sqrt{\frac{2am}{2d(1 + m)}}, \quad \beta = \sqrt{\frac{a}{2(1 + m)}}, \quad (59) \]
In the limit $m = 1$ the periodic kink solution (57) goes over to the celebrated kink solution

$$\phi(x) = A + B \tanh(\beta x),$$

provided

$$b = 3dA, \quad a = 2dA^2, \quad dB^2 = 2\beta^2, \quad dA^2 = 2\beta^2,$$

which implies that this solution exists only if

$$a, b, d > 0, \quad 2b^2 = 9ad, \quad A = \sqrt{\frac{a}{2d}},$$

$$B = \sqrt{\frac{am}{(1 + m)d}}, \quad \beta = \frac{\sqrt{a}}{2}.$$

Notice that unlike the three pulse solutions, the two kink solutions are valid if $a, b, d > 0$.

We now show that the asymmetric $\phi^4$ field Eq. (47) also admits four novel periodic solutions which can be written as superposition of either the periodic kink or the periodic pulse solution $\text{sn}(x, m)$ or $\text{dn}(x, m)$, respectively.

**Solution I**

It is straightforward to check that the asymmetric $\phi^4$ field Eq. (47) admits the periodic solution

$$\phi(x) = D - \frac{Adn(\beta x, m)cn(\beta x, m)}{1 + Bcn^2(\beta x, m)}, \quad B > 0,$$

provided

$$0 < m < 1, \quad b = 3dD, \quad a = 2dD^2, \quad 2b^2 = 9ad$$

and essentially following the arguments used in obtaining a similar solution for the symmetric $\phi^4$ field Eq. (1), we find that

$$B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \quad dD^2 = [1 + m + 6\sqrt{m}]\beta^2 > 0, \quad dA^2 = \frac{8\sqrt{m}\beta^2}{(1 - \sqrt{m})^2}.$$

Notice that for this solution $a, b, d > 0$ and it is only valid for $0 < m < 1$, i.e. there is no corresponding hyperbolic pulse solution.

On using the identity (19) one can re-express the periodic solution I given by (63) as the superposition of two periodic kink solutions, i.e.

$$\phi(x) = \sqrt{\frac{a}{2d}} - \sqrt{\frac{2m\beta}{\sqrt{d}}} \left[ \text{sn}(\beta x + \Delta, m) - \text{sn}(\beta x - \Delta, m) \right],$$

(66)
Here $\Delta$ is defined by $\text{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4}$, where use has been made of the identity (21). It is worth noting that for the periodic kink solution (57) as well as for the superposed periodic kink solution (63), $a, b, d > 0$.

**Solution II**

Remarkably, the asymmetric $\phi^4$ Eq. (47) also admits another periodic solution

$$
\phi(x) = D - \frac{A\text{sn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)}, \quad B > 0 .
$$

On essentially following the arguments used in obtaining a similar solution for the symmetric $\phi^4$ field Eq. (1), we find that (67) is an exact solution provided Eq. (64) is satisfied and further if

$$
B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \quad dD^2 = [(1 + m) - 6\sqrt{m}]\beta^2, \quad dA^2 = -8\sqrt{m}\beta^2 .
$$

Thus, for this solution $a, b, d < 0$ and only those values of $m$ are allowed satisfying $6\sqrt{m} > 1 + m$. Further it is only valid for $0 < m < 1$, i.e. there is no corresponding hyperbolic pulse solution.

On using the identity (28), the periodic solution II given by Eq. (67) can be re-expressed as the superposition of two periodic kink solutions

$$
\phi(x) = \sqrt{\frac{|a|}{2d}} - i\sqrt{\frac{2m}{|d|}}\beta \left[ \text{sn}(\beta x + \Delta, m) + \text{sn}(\beta x - \Delta, m) \right] ,
$$

Here $\Delta$ is defined by $\text{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4}$, where use has been made of the identity (21).

It is worth noting that while $a, b, d > 0$ for the superposed solution I, for the superposed solution II, the values of $a, b, d < 0$. However, the value of $B$ is the same for both the solutions.

**Solution III**

It is easy to check that the asymmetric $\phi^4$ field Eq. (47) also admits another periodic solution

$$
\phi(x) = D - \frac{A\text{sn}(\beta x, m)\text{cn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)} .
$$

On essentially following the arguments used in obtaining a similar solution for the symmetric $\phi^4$ field Eq. (1), we find that (70) is an exact solution provided Eq. (64) is satisfied and further if

$$
B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad dD^2 = -[(2 - m) - 6\sqrt{1 - m}]\beta^2 , \quad dA^2 = -\frac{8(1 - \sqrt{1 - m})^2\beta^2}{\sqrt{1 - m}} .
$$
Notice that for this solution $a, b, d < 0$ and only those values of $m$ are allowed such that $2 - m > 6\sqrt{1 - m}$. Further, it is only valid for $0 < m < 1$, i.e. there is no corresponding hyperbolic pulse solution.

On using the identity (47), the solution III of the asymmetric $\phi^4$ field Eq. (47) given by Eq. (70) can be re-expressed as a superposition of the two periodic pulse solutions, i.e.

$$\phi(x) = \sqrt{\frac{|a|}{2|d|}} - \beta \sqrt{\frac{2}{|d|}} \left( \text{dn}[\beta x - \frac{K(m)}{2}, m] - \text{dn}[\beta x + \frac{K(m)}{2}, m] \right). \quad (72)$$

**Solution IV**

The asymmetric $\phi^4$ Eq. (47) also admits another periodic solution

$$\phi(x) = D - \frac{A \text{dn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}. \quad (73)$$

On essentially following the arguments used in obtaining a similar solution for the symmetric $\phi^4$ field Eq. (1), we find that (73) is an exact solution provided Eq. (64) is satisfied and further if

$$B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad dD^2 = -(2 - m + 6\sqrt{1 - m})\beta^2, \quad dA^2 = -\frac{8}{\sqrt{1 - m}}\beta^2.$$  

(74)

Thus for this solution $a, b, d < 0$. Further, it is only valid for $0 < m < 1$, i.e. there is no corresponding hyperbolic pulse solution.

On using the identity (41), the periodic solution IV given by Eq. (73) can be re-expressed as the superposition of two periodic pulse solutions, i.e.

$$\phi(x) = \sqrt{\frac{|a|}{2|d|}} - \beta \sqrt{\frac{2}{|d|}} \left( \text{dn}[\beta x + \frac{K(m)}{2}, m] + \text{dn}[\beta x - \frac{K(m)}{2}, m] \right). \quad (75)$$

It is worth noting that for both the superposed periodic pulse solutions III and IV, not only the value of $B$ is the same but also $a, b, d < 0$ for both the solutions. It is worth reminding that even for the periodic pulse solution as given by Eqs. (51) and (52), $a, b, d < 0$.

**3.1 Novel Superposed (Hyperbolic) Kink Solutions**

We now show that unlike the symmetric $\phi^4$ field theory characterized by Eq. (1), the asymmetric field theory characterized by the field Eq. (47) admits the superposition of two tanh-type localized kink solutions.
In particular, we now show that Eq. (47) admits two distinct pulse solutions

$$\phi(x) = D - \frac{A}{B + \cosh^2(\beta x)}, \quad B > 0,$$

i.e. one with $D \neq 0$ and other with $D = 0$.

**Case I: $D \neq 0$**

On using the ansatz (76) in Eq. (47) we find that it is an exact solution provided

$$a - bD + dD^2 = 0, \quad dA^2 = 8B(B + 1)\beta^2, \quad 4\beta^2 = dD^2 - a,$$

$$(3dD - b)A = 6(2B + 1)\beta^2.$$

(77)

Note that for this solution $a, b, d > 0$.

Now starting from the novel identity (19) and taking $m = 1$, we obtain the corresponding trigonometric identity

$$\tanh(y + \Delta) - \tanh(y - \Delta) = \frac{\sinh(2\Delta)}{B + \cosh^2(y)}, \quad B = \sinh^2(\Delta).$$

(78)

On using Eqs. (77) and (78) in Eq. (76) one can re-express the solution (76) as a localized state with nonzero asymptote at $\pm\infty$

$$\phi(x) = b + \sqrt{b^2 - 4ad} \left[ \tanh(\beta x + \Delta) - \tanh(\beta x - \Delta) \right],$$

(79)

where $\sinh(\Delta) = \sqrt{B}$. It is worth noting that while the kink solution (60) exists when $b^2 = 9ad$, the above superposition of two kink solutions exists only when $b^2 > 9ad$.

**Case II: $D = 0$**

We now show that Eq. (47) also admits another pulse solution

$$\phi(x) = \frac{A}{B + \cosh^2(\beta x)}, \quad B > 0,$$

(80)

provided

$$\beta = \frac{\sqrt{a}}{2}, \quad dA^2 = 8B(B + 1)\beta^2, \quad \frac{4(B + 1)}{(B + 2)^2} = \frac{9ad}{2b^2} < 1.$$

(81)

On using the identity (78), one can re-express the pulse solution (80) as a localized state with zero asymptote at $\pm\infty$

$$\phi = \sqrt{\frac{2}{d}} \beta \left[ \tanh(\beta x + \Delta) - \tanh(\beta x - \Delta) \right],$$

(82)
where \(\sinh(\Delta) = \sqrt{B} \). Note that a la kink solution (60), for the solutions (76) and (80) too \(a, b, d > 0\) but whereas the single kink solution exists only when \(2b^2 = 9ad\) while the above superposed solutions (76) and (80) exist only when \(2b^2 > 9ad\).

### 3.2 Novel Solutions of \(\phi^2-\phi^{2m+2}-\phi^{4m+2}\) Field Theory

Remarkably, it turns out that some of the results of the \(\phi^2-\phi^3-\phi^4\) field theory are immediately extended to the \(\phi^2-\phi^{2m+2}-\phi^{4m+2}\) field theory. In particular, we now show that

\[
\phi_{xx} = a\phi - b\phi^{2m+1} + d\phi^{4m+1}, \quad a, b, d > 0, \tag{83}
\]

also admits the superposition of two tanh-type localized solutions. Here \(m\) can take any integer or half-integer value. Notice that for \(m = 1/2\), Eq. (83) goes over to Eq. (47) for the \(\phi^2-\phi^3-\phi^4\) field theory. Before deriving such a solution, let us first note that in case \((2m + 1)b^2 = 4(m + 1)^2ad\), Eq. (83) admits the kink solution [16]

\[
\phi = \left(\frac{(2m + 1)a}{4d}\right)^{1/4m} [1 + \tanh(ax)]^{1/2m}. \tag{84}
\]

As expected, at \(m = 1/2\) this kink solution goes over to the kink solution (60) for the \(\phi^2-\phi^3-\phi^4\) field theory. Note that for the kink solution (60), \(9ad = 2b^2\).

Now we show that in case \(4ad(m + 1)^2 < (2m + 1)b^2\), the field Eq. (83) admits a tanh-type localized superposed solution. In particular, it is easy to show that Eq. (83) also admits the pulse solution

\[
\phi(x) = \frac{A}{B + \cosh^2(\beta x)]^{1/2m}}, \quad B > 0, \tag{85}
\]

provided

\[
\beta = m\sqrt{a}, \quad A = \left[\frac{(2m + 1)B(B + 1)\beta^2}{dm^2}\right]^{1/4m}, \quad \frac{4(B + 1)}{(B + 2)^2} = \frac{(m + 1)^2ad}{(2m + 1)b^2} < 1. \tag{86}
\]

On using the identity (78), one can re-express the pulse solution (85) as a localized state with zero asymptote at \(\pm\infty\)

\[
\phi = \left[\frac{(2m + 1)}{4dm^2}\right]^{1/4m} \beta^{1/2m} \left[\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)\right]^{1/2m}, \tag{87}
\]
where sinh(Δ) = √B. For m = 1/2, all the expressions reduce to those of the φ^2-φ^3-φ^4 field theory. On the other hand, for m = 1 we have the expressions for the celebrated φ^2-φ^4-φ^6 field theory.

It is worth pointing out that for m = (2p + 1)/2, p = 0, 1, 2... we get theories like φ^2-φ^{2p+3}-φ^{4p+4} field theories, (i.e for m = 3/2 we get φ^2-φ^5-φ^8 field theories) which are not invariant under φ → −φ.

4 Novel Superposed Solutions of MKdV Equation

We now show that the MKdV equation also admits superposed periodic kink and pulse solutions in addition to the superposed hyperbolic kink solution obtained in [2]. Before we do that it is worthwhile reminding about the well known periodic kink and pulse solutions of the MKdV equation [1]

\[ u_t + u_{xxx} + 6gu^2u_x = 0, \quad (88) \]

where g > (≤) 0 corresponds to the attractive (repulsive) MKdV equation. Without loss of generality, we will choose g = 1 (-1) for the attractive (repulsive) MKdV.

It is well known that Eq. (88) admits the pulse-like solution

\[ u(x, t) = A \text{cn} (\xi, m), \quad \xi = \beta (x - vt), \quad (89) \]

provided g = 1, i.e. attractive MKdV and further

\[ A^2 = m\beta^2, \quad v = (2m - 1)\beta^2. \quad (90) \]

In addition, it also admits another periodic pulse solution

\[ u(x, t) = A \text{dn} (\xi, m), \quad (91) \]

provided g = 1 and

\[ A^2 = \beta^2, \quad v = (2 - m)\beta^2. \quad (92) \]

In the limit m = 1, both these periodic pulse solutions go over to the (hyperbolic) pulse solution

\[ u(x, t) = A \text{sech} (\xi), \quad (93) \]

provided g = 1 and

\[ A^2 = \beta^2, \quad v = \beta^2. \quad (94) \]

The MKdV Eq. (88) also admits a periodic kink solution

\[ u(x, t) = A \text{sn} (\xi, m), \quad (95) \]
provided $g = -1$ i.e. repulsive MKdV and

$$A^2 = m \beta^2, \quad v = -(1 + m) \beta^2. \quad (96)$$

In the limit $m = 1$, the periodic kink solution goes over to the (hyperbolic) kink solution

$$u(x, t) = A \tanh(\xi), \quad (97)$$

provided $g = -1$ and

$$A^2 = \beta^2, \quad v = -2 \beta^2. \quad (98)$$

Let us now show that like the symmetric $\phi^4$ case, the MKdV Eq. (88) also admits four superposed periodic kink and pulse solutions. Further, as shown already [2], unlike the symmetric $\phi^4$ case, the MKdV Eq. (88) also admits a superposed (hyperbolic) kink solution.

On using the ansatz $u(x, t) = u(\xi)$ where $\xi = \beta(x - vt)$ in the MKdV Eq. (88) and then integrating once, we obtain

$$\beta^2 u_{\xi\xi} = vu(\xi) - 2gv^3(\xi) + C, \quad (99)$$

where $C$ is the constant of integration. Now for those solutions for which $u(\xi)$ as well as $u_{\xi\xi}$ vanish at some value of $\xi = \xi_0$, clearly $C = 0$. Thus in such cases Eq. (99) is essentially similar to the symmetric $\phi^4$ field Eq. (1) provided we identify $v$ with $a$, $-2g$ with $d$ and $\xi$ with $\beta x$. In particular $d > (\leq) 0$ will correspond in our case to the repulsive (attractive) MKdV case. It turns out that all the four superposed solutions of the symmetric $\phi^4$ case are also the solutions of the MKdV Eq. (88). We list these superposed solutions below.

**Solution I**

It is easy to check that the repulsive MKdV Eq. (88) (i.e. $g = -1$) admits the periodic pulse solution

$$u(x, t) = \frac{A \text{dn}(\xi, m) \text{cn}(\xi, m)}{1 + B \text{cn}^2(\xi, m)}, \quad B > 0, \quad \xi = \beta(x - vt), \quad (100)$$

provided

$$0 < m < 1, \quad B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \quad v = -[1 + m + 6\sqrt{m}] \beta^2 < 0, \quad A^2 = \frac{4\sqrt{m}\beta^2}{(1 - \sqrt{m})^2}. \quad (101)$$
Note that this solution is not valid for $m = 1$, i.e. the MKdV Eq. (88) does not admit a corresponding hyperbolic solution. Notice that for this solution $v < 0$.

On using the identity (19) one can then rewrite the periodic pulse solu-
tion (100) as the superposition of the two periodic kink solutions, i.e.

$$u(x, t) = \sqrt{2m} \beta \left[ \text{sn}(\xi + \Delta, m) - \text{sn}(\xi - \Delta, m) \right], \quad \xi = \beta(x - vt). \quad (102)$$

Here $\Delta$ is defined by $\text{sn}(\sqrt{m} \Delta, 1/m) = \pm m^{1/4}$, where use has been made of the identity (21).

**Remark**

At this stage we recall the celebrated Miura transformation [8] which showed a remarkable connection between the solutions of the repulsive MKdV equation and the KdV equation. In particular, it was shown that if $u$ is a solution of the repulsive MKdV equation

$$u_t + u_{xxx} - 6u^2 u_x = 0,$$  \quad (103)

then $w = u^2 + u_x$ is the corresponding solution of the KdV equation

$$w_t + w_{xxx} - 6w w_x = 0.$$  \quad (104)

Hence on using the superposed solution (102) of the repulsive MKdV equation we obtain the following superposed solution of the KdV Equation

$$w(x, t) = 2m\beta^2 \left[ \text{sn}(\xi + \Delta, m) - \text{sn}(\xi - \Delta, m) \right]^2 \sqrt{2m} \beta^2 \left[ \text{cn}(\xi + \Delta, m) \text{dn}(\xi + \Delta, m) - \text{cn}(\xi - \Delta, m) \text{dn}(\xi - \Delta, m) \right], \quad (105)$$

where $\xi = \beta(x - vt)$.

A comment is in order here. Unlike the superposed solution (102), Sma-
gin, Tankeyev and Borich [17] obtained a periodic solution of the repulsive MKdV Eq. (103) which is a product of two periodic kink solutions, i.e.

$$u(x, t) = D + \text{sn}(\xi + \delta, m) \text{sn}(\xi - \delta, m). \quad (106)$$

The nice point of this solution is that in the limit $m = 1$ this solution smoothly goes over to the superposition of two hyperbolic kink solutions obtained earlier [2] (which we have mentioned below for completeness).

**Solution II**

The attractive MKdV Eq. (88) (i.e. $g = 1$) admits the periodic kink solution

$$u(x, t) = \frac{A \text{sn}(\xi, m)}{1 + B \text{cn}^2(\xi, m)}, \quad B > 0, \quad \xi = \beta(x - vt), \quad (107)$$
provided

\[ 0 < m < 1, \ B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \ v = [6\sqrt{m} - (1 + m)]\beta^2, \ A^2 = 4\sqrt{m}\beta^2. \quad (108) \]

Note that this solution does not exist for \( m = 1 \), i.e. the corresponding hyperbolic solution does not exist.

Now on using the novel identity (28) we can rewrite the solution (107) as the superposition of the two periodic kink solutions, i.e.

\[ u(x, t) = i\sqrt{m}\beta \left[ \text{sn}(\xi + \Delta, m) + \text{sn}(\xi - \Delta, m) \right], \ \xi = \beta(x - vt). \quad (109) \]

Here \( \Delta \) is defined by \( \text{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4} \), where use has been made of the identity (21).

It is worth noting that for both the solutions I and II, the value of \( B \) is the same but while the solution I is valid in the case of the repulsive MKdV, the solution II is valid in the case of the attractive MKdV.

**Solution III**

Yet another periodic solution to the attractive MKdV Eq. (88) is

\[ u(x, t) = \frac{A\text{sn}(\xi, m)\text{cn}(\xi, m)}{1 + B\text{cn}^2(\xi, m)}, \ B > 0, \ \xi = \beta(x - vt), \quad (110) \]

provided

\[ 0 < m < 1, \ B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \ v = (2 - m - 6\sqrt{1 - m})\beta^2, \]

\[ A^2 = \frac{4(1 - \sqrt{1 - m})^2\beta^2}{\sqrt{1 - m}}. \quad (111) \]

On using the identity (37), the periodic solution (110) can be re-expressed as a superposition of the two periodic pulse solutions, i.e.

\[ u(x, t) = \beta \left( \text{dn}[\xi - \frac{K(m)}{2}, m] - \text{dn}[\xi + \frac{K(m)}{2}, m] \right), \ \xi = \beta(x - vt). \quad (112) \]

**Solution IV**

Another periodic solution to the attractive MKdV Eq. (88) is

\[ u(x, t) = \frac{A\text{dn}(\xi, m)}{1 + B\text{cn}^2(\xi, m)}, \ B > 0, \ \xi = \beta(x - vt), \quad (113) \]
Thus for this solution \( v > 0 \).

On using the identity (113) the periodic solution IV given by Eq. (113) can be re-expressed as superposition of two periodic pulse solutions, i.e.

\[
u(x) = \beta \left[ \text{dn}(\xi + K(m), m) + \text{dn}(\xi - K(m), m) \right], \quad \xi = \beta(x - vt).
\]  

(115)

### 4.1 Superposed hyperbolic Kink Solution

For completeness, we now give the hyperbolic pulse solution of the repulsive MKdV Eq. (88) which, as has been shown in [2], can be re-expressed as the superposition of the two hyperbolic kink solutions. In particular it is easy to verify that another solution to Eq. (88) is

\[
u(x) = 1 - \frac{A}{B + \cosh^2(\xi)}, \quad B > 0, \quad \xi = \beta(x - vt),
\]  

(116)

provided \( g = -1 \), i.e. repulsive MKdV and further

\[
A = 2\sqrt{B(B + 1)}\beta, \quad \beta^2 = \frac{4(B + 1)}{(B + 2)^2} < 1, \quad v = 4\beta^2 - 6.
\]  

(117)

On using the identity (118), the solution (116) can be written as the superposition of two (hyperbolic) kink solutions

\[
u(x) = 1 - \beta \left[ \tanh(\xi + \Delta) - \tanh(\xi - \Delta) \right], \quad \xi = \beta(x - vt),
\]  

(118)

where \( \sinh(\Delta) = \sqrt{B} \).

**Remark**

On using the superposed solution (118) of the repulsive MKdV Eq. (103) and using the Miura transformation [8], we immediately obtain the corresponding superposed solution of the KdV Eq. (104)

\[
w(x) = \left( 1 - \beta \left[ \tanh(\xi + \Delta) - \tanh(\xi - \Delta) \right] \right)^2
\]

\[-\beta^2 \left[ \text{sech}^2(\xi + \Delta) - \text{sech}^2(\xi - \Delta) \right].
\]  

(119)
5 Superposed Solutions of Mixed KdV-MKdV Equation

We now show that even the mixed KdV-MKdV equation \( \text{[9]} \)

\[
    u_t + u_{xxx} + 2buu_x + 6gu^2u_x = 0, \tag{120}
\]

admits superposed periodic kink and pulse solutions. But before that let us note the well known periodic kink and pulse solutions of the mixed KdV-MKdV Eq. (120). Without loss of generality we choose \( g = \pm 1 \) with \( g = 1 \) (−1) corresponding to the attractive (repulsive) MKdV.

It is well known that Eq. (120) admits the periodic pulse solution

\[
    u(x, t) = A[1 \pm \text{cn}(\xi, m)], \quad \xi = \beta(x - vt), \tag{121}
\]

in the case of attractive MKdV, i.e. \( g = 1 \) and further

\[
    A^2 = m\beta^2, \quad b = -6\sqrt{m}\beta < 0, \quad v = -(4m + 1)\beta^2. \tag{122}
\]

The attractive MKdV admits another periodic pulse solution

\[
    u(x, t) = A[1 \pm \text{dn}(\xi, m)], \tag{123}
\]

provided

\[
    A^2 = \beta^2, \quad b = -6\beta < 0, \quad v = -(4 + m)\beta^2. \tag{124}
\]

In the limit \( m = 1 \), both these periodic pulse solutions go over to the hyperbolic pulse solution

\[
    u(x, t) = A[1 \pm \text{sech}(\xi)], \tag{125}
\]

provided

\[
    A^2 = \beta^2, \quad b = -6\beta < 0, \quad v = -5\beta^2. \tag{126}
\]

On the other hand, the repulsive MKdV \( (g = -1) \) admits the periodic kink solution

\[
    u(x, t) = A[1 \pm \text{sn}(\xi, m)], \tag{127}
\]

provided

\[
    A^2 = m\beta^2, \quad b = 6\sqrt{m}\beta > 0, \quad v = (5 - m)\beta^2. \tag{128}
\]

In the limit \( m = 1 \), the periodic kink solution (127) goes over to the kink solution

\[
    u(x, t) = A[1 \pm \text{tanh}(\xi)], \tag{129}
\]
provided

\[ A^2 = \beta^2, \quad b = 6 \beta > 0, \quad v = 4 \beta^2. \]  

We now show that the mixed KdV-MKdV Eq. (120) also admits periodic superposed kink and pulse solutions which are similar to those given in Sec. III for the $\phi^2$-$\phi^3$-$\phi^4$ field theory and hence we simply list the solutions and omit any details.

On using the ansatz $u(x, t) = u(\xi)$ where $\xi = \beta(x - vt)$ in Eq. (120) and integrating it once we obtain

\[ \beta^2 u_{\xi\xi} = vu - bu^2 - 2gu^3 + C, \]  

where $C$ is the integration constant.

**Solution I**

It is easy to check that Eq. (120) with $g = -1$ admits the periodic pulse solution

\[ u(x, t) = D - \frac{A\text{dn}(\xi, m)\text{cn}(\xi, m)}{1 + B\text{cn}^2(\xi, m)}, \quad B > 0, \]  

provided

\[ C = -vD + bD^2 + 2gD^3, \]  

and further if

\[ 0 < m < 1, \quad B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \quad D = \frac{b}{6}, \]  

\[ v = \frac{b^2}{6} - [1 + m + 6\sqrt{m}]\beta^2, \quad A^2 = \frac{4\sqrt{m}\beta^2}{(1 - \sqrt{m})^2}. \]  

Note that this solution is not valid for $m = 1$, i.e. the mixed KdV-MKdV Eq. (120) does not admit a corresponding hyperbolic solution.

On using the identity (19) one can then rewrite the periodic pulse solution (132) as a superposition of two periodic kink solutions, i.e.

\[ u(x, t) = \frac{b}{6} - \sqrt{2m}\beta \left[ \text{sn}(\xi + \Delta, m) - \text{sn}(\xi - \Delta, m) \right], \quad \xi = \beta(x - vt). \]

Here $\Delta$ is defined by $\text{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4}$, where use has been made of the identity (21).

**Solution II**

It is easy to check that Eq. (120) with $g = 1$ admits the periodic kink solution

\[ u(x, t) = D - \frac{A\text{sn}(\xi, m)}{1 + B\text{cn}^2(\xi, m)}, \quad B > 0, \]  

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provided $C$ is given by Eq. \((133)\) and further if

$$0 < m < 1, \quad B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \quad D = -\frac{b}{6},$$

$$v = -\frac{b^2}{6} + [6\sqrt{m} - (1 + m)]\beta^2, \quad A^2 = 4\sqrt{m}\beta^2. \quad (137)$$

Note that this solution does not exist for $m = 1$, i.e. the corresponding hyperbolic solution does not exist.

Now on using the novel identity (28) we can rewrite the solution (107) as a superposition of two periodic kink solutions, i.e.

$$u(x, t) = -\frac{b}{6} + i\sqrt{m}\beta \left[ \text{sn}(\xi + \Delta, m) + \text{sn}(\xi - \Delta, m) \right], \quad \xi = \beta(x - vt). \quad (138)$$

Here $\Delta$ is defined by $\text{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4}$, where use has been made of the identity (21).

It is worth noting that for both the solutions I and II of the mixed KdV-MKdV Eq. (120), the value of $B$ is the same but while the solution I is valid in case $g = -1$, the solution II is valid in case $g = 1$.

**Solution III**

Yet another periodic solution to Eq. (120) with $g = 1$ is

$$u(x, t) = D - \frac{A \text{sn}(\xi, m) \text{cn}(\xi, m)}{1 + B \text{cn}^2(\xi, m)}, \quad (139)$$

provided $C$ is given by Eq. \((133)\) and further if

$$0 < m < 1, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad D = -\frac{b}{6},$$

$$v = -\frac{b^2}{6} + (2 - m - 6\sqrt{1 - m})\beta^2, \quad A^2 = \frac{4(1 - \sqrt{1 - m})^2\beta^2}{\sqrt{1 - m}}. \quad (140)$$

On using the identity (37), the periodic solution \((139)\) can be re-expressed as a superposition of two periodic pulse solutions, i.e.

$$u(x, t) = -\frac{b}{6} - \beta \left( \text{dn}[\xi - \frac{K(m)}{2}, m] - \text{dn}[\xi + \frac{K(m)}{2}, m] \right). \quad (141)$$

**Solution IV**

Yet another periodic pulse solution to the mixed KdV-MKdV Eq. (120) with $g = 1$ is

$$u(x, t) = D - \frac{A \text{dn}(\xi, m)}{1 + B\text{cn}^2(\xi, m)}, \quad (142)$$
provided $C$ is given by Eq. (133) and further if

$$0 < m < 1, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad D = -\frac{b}{6},$$

$$v = -\frac{b^2}{6} + [2 - m + 6\sqrt{1 - m}]\beta^2, \quad A^2 = \frac{4}{\sqrt{1 - m}}\beta^2. \quad (143)$$

On using the identity (41), the periodic solution IV given by Eq. (142) can be re-expressed as the superposition of two periodic pulse solutions, i.e.

$$u(x, t) = -\frac{b}{6} - \beta \left[ \text{dn}(\xi + \Delta) - \text{dn}(\xi - \Delta) \right], \quad \xi = \beta(x - vt). \quad (144)$$

5.1 Novel Superposed (Hyperbolic) Kink Solutions

We now show that the mixed KdV-MKdV Eq. (120) admits two distinct solutions both of which can be re-expressed as a superposition of two tanh-type localized kink solutions. We discuss these one by one.

**Solution I**

It is easy to check that the mixed KdV-MKdV Eq. (120) admits the solution

$$u(x, t) = 1 - \frac{A}{B + \cosh^2(\xi)}, \quad B > 0, \quad \xi = \beta(x - vt), \quad (145)$$

provided $g = -1, C = -v + b - 2$, and further if

$$v = 4\beta^2 - 6 + 2b, \quad A^2 = 4B(B + 1)\beta^2, \quad b > 0, \quad \beta^2 = \frac{4(6 - b)^2B(B + 1)^2}{36(2B + 1)^2} < \frac{b^2}{36}. \quad (146)$$

On using the identity (78) one can re-express the solution (145) as a localized state with nonzero asymptote at $\pm\infty$

$$u(x, t) = 1 - \beta \left[ \text{tanh}(\xi + \Delta) - \text{tanh}(\xi - \Delta) \right], \quad \xi = \beta(x - vt), \quad (147)$$

where $\sinh(\Delta) = \sqrt{B}$. It is worth noting that while the kink solution (129) exists if $\beta^2 = \frac{b^2}{36}$, the above superposition of the two kink solutions, exists only when $\beta^2 < \frac{b^2}{36}$.  

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Solution II

We now show that the mixed KdV-MKdV Eq. (120) also admits another pulse solution

\[ u(x, t) = \frac{A}{B + \cosh^2(\xi)}, \quad B > 0, \quad \xi = \beta(x - vt), \quad (148) \]

provided \( g = -1, C = 0 \) and further if

\[ v = 4\beta^2, \quad A^2 = 4B(B + 1)\beta^2, \quad \beta^2 = \frac{b^2}{36} \frac{4(B + 1)}{(2B + 1)^2} \leq \frac{b^2}{36}. \quad (149) \]

On using the identity (78), one can re-express the pulse solution (148) as a localized state with zero asymptote at \( \pm\infty \)

\[ u(x, t) = \beta \left[ \tanh(\xi + \Delta) - \tanh(\xi - \Delta) \right], \quad \xi = \beta(x - vt), \quad (150) \]

where \( \sinh(\Delta) = \sqrt{B} \). Note that whereas for the kink solution (129), \( \beta^2 = \frac{b^2}{36} \), the superposed solution (148) exists only when \( \beta^2 < \frac{b^2}{36} \).

5.2 Novel Solutions of \( u^m u_x - u^{2m} u_x \) Mixed KdV-MKdV Equation

Remarkably, it turns out that some of the results of the mixed KdV-MKdV Eq. (120) are immediately extended to the generalized \( u^m u_x - u^{2m} u_x \) mixed KdV-MKdV equation

\[ u_t + u_{xxx} + 2bu^m u_x + 6gu^{2m} u_x = 0, \quad m = 1, 2, 3, \ldots . \quad (151) \]

In particular, we show that Eq. (151) admits a pulse solution which can be re-expressed as a superposition of two tanh-type localized solutions. Notice that for \( m = 1 \), Eq. (151) goes over to Eq. (120) for the mixed KdV-MKdV case.

On using the ansatz \( u(x, t) = u(\xi) \) where \( \xi = \beta(x - vt) \) in Eq. (151) and integrating it once we obtain

\[ \beta^2 u_{\xi\xi} = vu - \frac{2b}{m + 1} u^{m+1} - \frac{6g}{2m + 1} u^{2m+1} + C, \quad (152) \]

where \( C \) is the integration constant. We now consider two solutions for both of which \( u, u_{\xi\xi} = 0 \) at \( \xi = -\infty \) so that \( C = 0 \) for these two solutions.
Before deriving the desired pulse solution, let us first note that Eq. (151) also admits a kink solution

\[ u(x, t) = A[1 + \tanh(\xi)]^{1/m}, \quad \xi = \beta(x - vt), \quad (153) \]

provided \( g = -1 \) and further if

\[ v = \frac{4\beta^2}{m^2}, \quad bA^m = \frac{(m + 1)(m + 2)}{m^2}\beta^2, \]
\[ b^2 = \frac{6(m + 1)(m + 2)^2}{(2m + 1)m^2}\beta^2. \quad (154) \]

As expected, at \( m = 1 \) this kink solution goes over to the kink solution \( (129) \).

It is easy to show that Eq. (151) also admits the pulse solution

\[ u(x, t) = \frac{A}{[B + \cosh^2(\xi)]^{1/m}}, \quad B > 0, \quad \xi = \beta(x - vt), \quad (155) \]

provided

\[ v = \frac{4\beta^2}{m^2}, \quad bA^m = \left[ \frac{(m + 1)(m + 2)(2B + 1)}{m^2} \right] \beta^2, \]
\[ b^2 = \frac{6(m + 1)(m + 2)^2(2B + 1)^2}{4B(B + 1)(2m + 1)m^2}\beta^2. \quad (156) \]

On using the identity \( (78) \), one can re-express the pulse solution \( (155) \) as a localized state with zero asymptote at \( \pm \infty \)

\[ u(x, t) = \left[ \frac{(m + 1)(2m + 1)\beta^2}{6m^2} \right]^{1/2m} \left[ \tanh(\xi + \Delta) - \tanh(\xi - \Delta) \right]^{1/m}, \quad (157) \]

where \( \sinh(\Delta) = \sqrt{B} \). For \( m = 1 \), all the expressions reduce to those of the mixed KdV-MKdV case.

6 Superposed Solutions For the NLS Equation

We now show that even the celebrated NLS equation

\[ iu_t + u_{xx} + g|u|^2u = 0, \quad (158) \]

admits the superposed periodic kink and the pulse solutions. Here \( g = 1 \) \((-1) \) corresponds to the attractive (repulsive) NLS. But before we explore
these solutions let us note the well known periodic kink and pulse solutions of the NLS Eq. (158) [1].

One of the well known periodic pulse solutions of NLS is
\[ u(x, t) = Ae^{i\omega t} \text{dn}(\beta x, m), \]  
provided \( g = 1 \), i.e. attractive NLS and further if
\[ A^2 = 2\beta^2, \quad \omega = (2 - m)\beta^2. \]  
Another well known periodic pulse solution of the NLS Eq. (158) is
\[ u(x, t) = Ae^{i\omega t} \text{cn}(\beta x, m), \]  
provided \( g = 1 \), i.e. attractive NLS and further if
\[ A^2 = 2m\beta^2, \quad \omega = (2m - 1)\beta^2. \]  
In the limit \( m = 1 \) both the periodic pulse solutions go over to the hyperbolic pulse solution
\[ u(x, t) = Ae^{i\omega t} \text{sech} (\beta x), \]  
provided \( g = 1 \) and
\[ A^2 = 2\beta^2, \quad \omega = \beta^2. \]  
On the other hand, the repulsive NLS (i.e. \( g = -1 \)) is known to admit the periodic kink solution
\[ u(x, t) = Ae^{i\omega t} \text{sn}(\beta x, m), \]  
provided
\[ A^2 = 2m\beta^2, \quad \omega = -(1 + m)\beta^2. \]  
In the limit \( m = 1 \), the periodic kink solution (165) goes over to the (hyperbolic) kink solution
\[ u(x, t) = Ae^{i\omega t} \text{tanh} (\beta x), \]  
provided
\[ A^2 = 2\beta^2, \quad \omega = -2\beta^2. \]  
We now show that the NLS Eq. (158) admits three periodic pulse and kink solutions which can be re-expressed as the superposition of the periodic kink and the pulse solutions. This is easily seen if one notices that the NLS Eq. (158) can be mapped to the symmetric \( \phi^4 \) Eq. (1) provided we start with the ansatz
\[ u(x, t) = e^{i\omega t} \phi(x), \]  
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with $\phi(x)$ being real. On substituting the ansatz \[169\] in Eq. \[158\] we obtain

$$\phi_{xx} = \omega \phi - g \phi^3,$$

(170)

which is identical to the symmetric $\phi^4$ field Eq. \[1\] provided we identify $\omega$ with $a$ and $d$ with $-g$. Using this identification we now simply list those three superposed periodic solutions of NLS Eq. \[158\] in which $\phi$ is real.

Solution I

It is easy to check that the NLS field Eq. \[158\] admits the periodic solution

$$u(x,t) = e^{i\omega t} \frac{A \text{dn}(\beta x, m) \text{cn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)} , \quad B > 0 ,$$

(171)

provided $g = -1$, i.e. repulsive NLS and further if

$$0 < m < 1 , \quad B = \frac{\sqrt{m}}{1 - \sqrt{m}} , \quad \omega = \frac{[1 + m + 6 \sqrt{m}] \beta^2}{2} < 0 ,$$

$$A^2 = \frac{8 \sqrt{m} \beta^2}{(1 - \sqrt{m})^2} .$$

(172)

On using the identity \[19\], one can re-express the periodic solution I given by \[171\] as a superposition of the two periodic kink solutions, i.e.

$$u(x,t) = e^{i\omega t} \sqrt{2m} \beta \left[ \text{sn}(\beta x + \Delta, m) - \text{sn}(\beta x - \Delta, m) \right] .$$

(173)

Here $\Delta$ is defined by $\text{sn}(\sqrt{m} \Delta, 1/m) = \pm m^{1/4}$.

Solution II

It is easy to check that the NLS Eq. \[158\] admits another periodic solution

$$u(x,t) = e^{i\omega t} \frac{A \text{sn}(\beta x, m) \text{cn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)} ,$$

(174)

provided $g = 1$, i.e. attractive NLS and further if

$$0 < m < 1 , \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}} , \quad \omega = (2 - m - 6 \sqrt{1 - m}) \beta^2 ,$$

$$A^2 = \frac{8 (1 - \sqrt{1 - m})^2 \beta^2}{\sqrt{1 - m}} .$$

(175)

On using the identity \[37\] one can re-express the solution \[174\] as a superposition of the two periodic pulse solutions, i.e.

$$u(x,t) = e^{i\omega t} \sqrt{2} \beta \left( \text{dn}[\beta(x) - \frac{K(m)}{2}, m] - \text{dn}[\beta(x) + \frac{K(m)}{2}, m] \right) .$$

(176)
Solution III

Another periodic solution to the NLS Eq. (158) is

\[ u(x,t) = e^{i\omega t} \frac{A\text{dn}(\beta x,m)}{1+B\text{cn}^2(\beta x,m)}, \]  \hspace{1cm} (177)

provided \( g = 1 \), i.e. attractive NLS and further

\[ 0 < m < 1, \quad B = \frac{1 - \sqrt{1-m}}{\sqrt{1-m}}, \quad \omega = [2-m+6\sqrt{1-m}]\beta^2, \quad A^2 = \frac{8}{\sqrt{1-m}}\beta^2. \]  \hspace{1cm} (178)

On using the identity (41), the periodic solution (177) can be re-expressed as a superposition of the two periodic pulse solutions, i.e.

\[ u(x,t) = e^{i\omega t}\sqrt{2\beta} \left( \text{dn}[\beta x + \frac{K(m)}{2},m] + \text{dn}[\beta x - \frac{K(m)}{2},m] \right). \]  \hspace{1cm} (179)

It is worth noting that for both the superposed periodic pulse solutions II and III, not only the value of \( B \) is the same but both the solutions are valid for the attractive NLS.

7 Superposed Solutions of the quadratic-cubic NLS

We now show that even the quadratic-cubic NLS equation [10] given by

\[ iu_t + u_{xx} + b|u|u + g|u|^3u = 0, \]  \hspace{1cm} (180)

admits superposed periodic pulse and the hyperbolic kink solutions.

We start with the ansatz

\[ u(x,t) = e^{i\omega t}\phi(x), \]  \hspace{1cm} (181)

where we will only consider those \( \phi(x) \) which are nonnegative so that \(|\phi| = \phi\). On substituting the ansatz (181) in Eq. (180) it is easy to see that \( \phi(x) \) satisfies the field equation

\[ \phi_{xx} = \omega\phi - b\phi^2 - g\phi^3, \]  \hspace{1cm} (182)

which is identical to the asymmetric \( \phi^4 \) field Eq. (17) provided we identify \( \omega \) with \( a \) and \(-g\) with \( d \) while \( b \) is identical in both the cases. It then follows that all those solutions of the asymmetric \( \phi^4 \) Eq. (17) in which \( \phi(x) \) is nonnegative will also be the solutions of the quadratic-cubic Eq. (180). We now simply mention such solutions one by one.
Let us first discuss the periodic kink and pulse solutions. It is easy to show that Eq. (180) admits the periodic pulse solution
\[ u(x, t) = e^{i\omega t}[A + Bdn(\beta x, m)], \tag{183} \]
provided \( g = 1 \) and further if
\[ b = -3A, \quad \omega = -2A^2, \quad B^2 = 2\beta^2, \quad A^2 = (2 - m)\beta^2. \tag{184} \]
In the limit \( m = 1 \), the periodic pulse solution (183) goes over to the hyperbolic pulse solution
\[ \phi(x) = A + B\sech(\beta x), \tag{185} \]
provided \( g = 1 \) and further if
\[ b = -3A, \quad \omega = -2A^2, \quad B^2 = 2\beta^2, \quad A^2 = \beta^2. \tag{186} \]
The quadratic-cubic NLS Eq. (180) also admits a periodic kink solution
\[ u(x, t) = e^{i\omega t}[A + B\sn(\beta x)], \tag{187} \]
provided \( g = -1 \) and further if
\[ b = 3A, \quad \omega = 2A^2, \quad B^2 = 2m\beta^2, \quad A^2 = (1 + m)\beta^2. \tag{188} \]
In the limit \( m = 1 \), the periodic kink solution goes over to the hyperbolic kink solution
\[ u(x, t) = e^{i\omega t}[A + B\tanh(\beta x)], \tag{189} \]
provided \( g = -1 \) and further if
\[ b = 3A, \quad \omega = 2A^2, \quad B^2 = 2\beta^2, \quad A^2 = 2\beta^2. \tag{190} \]

Let us now discuss the superposed solutions of Eq. (180). On comparing with the four periodic superposed solutions for the asymmetric \( \phi^4 \) discussed in Sec. III it is clear that in this case only one superposed periodic solution and one superposed hyperbolic solution are possible which we mention next.

**Solution I: Periodic Superposed Pulse Solution**

In particular, it is easy to show that
\[ u(x, t) = e^{i\omega t} \left[ D + \frac{A\dn(\beta x, m)}{1 + B\cn^2(\beta x, m)} \right], \tag{191} \]
is an exact solution with $A, D, B$ all being positive provided $g = 1$ and further if

$$D^2 = -\omega = 2[2 - m + \sqrt{1 - m}]\beta^2, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}},$$

$$0 < m < 1, \quad A^2 = \frac{8\beta^2}{\sqrt{1 - m}}. \quad (192)$$

On using the identity (11), the periodic solution (191) can be re-expressed as a superposition of the two periodic pulse solutions, i.e.

$$u(x, t) = e^{i\omega t} \beta \left[ \text{dn}(\beta x + K(m), m) + \text{dn}(\beta x - K(m), m) \right]. \quad (193)$$

**Solution II: Hyperbolic Superposed Solution**

It is easy to show that the quadratic-cubic NLS Eq. (180) admits the hyperbolic solution

$$u(x, t) = e^{i\omega t} \beta \frac{A}{B + \cosh^2(\beta x)}, \quad (194)$$

provided $g = -1$ and further if

$$\omega = 4\beta^2, \quad A^2 = 8B(B + 1)\beta^2, \quad b^2 = \frac{18(2B + 1)^2}{4B(B + 1)}\beta^2. \quad (195)$$

On using the identity (18), one can re-express the pulse solution (194) as a localized state with zero asymptote at $\pm \infty$

$$u(x, t) = e^{i\omega t} \sqrt{2}\beta \left[ \tanh(\beta x + \Delta) - \tanh(\beta x - \Delta) \right], \quad (196)$$

where $\sinh(\Delta) = \sqrt{B}$.

### 7.1 Superposed Solutions For $|u|^m u - |u|^{2m} u$ NLS Equation

We now show that some of the results of the quadratic-cubic NLS equation (10) are easily extended to the generalized NLS equation

$$iu_t + u_{xx} + b|u|^m u + g|u|^{2m} u = 0, \quad m = 1, 2, 3, \ldots. \quad (197)$$

Note that for $m = 1$ this reduces to the quadratic-cubic NLS equation (10) while for $m = 2$ it reduces to the cubic-quintic NLS equation (15). Before
we discuss the superposed solution, it is worth pointing out that Eq. \((197)\) admits a kink solution
\[
    u(x, t) = e^{i\omega t} A[1 \pm \tanh(\beta x)]^{1/m},
\]
provided \(g = -1\) and further if
\[
    \omega = \frac{4\beta^2}{m^2}, \quad A^{2m} = \frac{(1 + m)\beta^2}{m^2}, \quad b^2 = \frac{4(m + 2)^2}{(1 + m)m^2} \beta^2.
\]

Finally, it is easy to show that the generalized NLS Eq. \((197)\) also admits a pulse solution
\[
    u(x, t) = e^{i\omega t} \frac{A}{[B + \cosh^2(\beta x)]^{1/m}},
\]
provided \(g = -1\) and further if
\[
    \omega = \frac{4\beta^2}{m^2}, \quad A^{2m} = \frac{4B(B + 1)(m + 1)}{m^2} \beta^2,
\]
\[
    b^2 = \frac{4(m + 2)^2 \beta^2}{(1 + m)m^2} \frac{(2B + 1)^2}{4B(B + 1)}.
\]

On using the identity \((78)\), one can re-express the solution \((91)\) as a superposition of the two kink solutions
\[
    u(x, t) = e^{i\omega t} \left[ \left( \frac{m + 1}{m^2} \right)^{1/2m} \beta^{1/m} \left( \tanh(\beta x + \Delta) - \tanh(\beta x - \Delta) \right) \right]^{1/m},
\]
where \(\sinh(\Delta) = \sqrt{B}\).

8 Conclusion and Open Problems

In this paper we have shown that a large number of nonlinear equations such as the symmetric and asymmetric (double well) \(\phi^4\) equations, MKdV equation \([1]\), mixed KdV-MKdV equation \([9]\), NLS as well as quadratic-cubic NLS \([10]\), which have applications in several areas of physics, admit novel superposed periodic kink and pulse solutions. Besides, some of them also admit hyperbolic superposed kink solutions. We have also shown that even generalized higher order neutral scalar field theory models, generalized mixed KdV-MKdV models as well as generalized mixed higher order NLS models admit superposed hyperbolic kink solutions. We believe that we have only touched the tip of the iceberg and there are many more surprises in store. This paper raises several questions which are still not quite understood. We list some of them below.
1. In this paper, in several different models we have obtained a number of solutions which can be re-expressed as either the sum or the difference of two \text{sn}(x, m) or two \text{dn}(x, m) Jacobi elliptic functions. However, we have not been able to obtain similar superposed solutions in the Jacobi elliptic \text{cn}(x, m) case. It is not clear as to what is the underlying reason. Clearly it would be worthwhile to find \text{cn}(x, m) superposed solutions in some nonlinear models.

2. While we have been able to obtain hyperbolic solutions which can be re-expressed as difference of two \text{tanh}(x) type solutions, so far we have not been able to obtain hyperbolic solutions which can be re-expressed either as sum of two \text{tanh}(x) or sum or difference of two \text{sech}(x) type solutions. It is clearly of interest to look for such solutions.

3. It is not clear what is the physical interpretation of such superposed periodic or hyperbolic solutions. Do they correspond to a bound state of two kink or two pulse solutions as in some field theory and condensed matter contexts [4, 5, 6, 7]? Or do they merely correspond to some excitation of two kink or two pulse solutions? It is worthwhile finding the interpretation of such superposed solutions vis a vis a single kink or pulse solution.

4. It is clearly of interest to discover other nonlinear equations which also admit such or even more unusual superposed solutions.

Hopefully, one can find answers to some of the questions raised above.

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