Parameterized Objectives and Algorithms for Clustering Bipartite Graphs and Hypergraphs

Nate Veldt  
Cornell University  
nveldt@cornell.edu  

Anthony Wirth  
The University of Melbourne  
awirth@unimelb.edu.au  

David F. Gleich  
Purdue University  
dgleich@purdue.edu  

ABSTRACT

Graph clustering objective functions with tunable resolution parameters make it possible to detect different types of clustering structure in the same graph. These objectives also provide a unifying view of other non-parametric objectives, which in many cases can be captured as special cases. Previous research has largely focused on parametric objectives for standard graphs, in which all nodes are of the same type, and edges model pairwise relationships. In our work, we introduced parameterized objective functions and approximation algorithms specifically for clustering bipartite graphs and hypergraphs, based on the correlation clustering problem. This enables us to develop principled approaches for clustering datasets with different node types (bipartite graphs) or multiway relations (hypergraphs). For both our hypergraph and bipartite objectives, we identify new equivalence relationships and approximation guarantees. We show our hypergraph objective is related to higher-order notions of modularity and normalized cut, and is amenable to approximation algorithms via two different hypergraph expansion techniques. Our bipartite objective generalizes the standard bipartite correlation clustering objective, and in a certain parameter regime is equivalent to bicluster deletion, the task of removing a minimum number of edges to separate a bipartite graph into disjoint bicliques. Although the problem in general is NP-hard, we show that in a certain parameter regime it is equivalent to a bipartite matching problem, meaning that it is polynomial time solvable in this regime. For other regimes, we provide approximation guarantees based on rounding a linear programming relaxation. Our results include the first constant factor approximation algorithm for bicluster deletion. We illustrate the flexibility of our framework and algorithms in several experiments on a diverse array of datasets. This includes clustering a food web and an email network based on parametric objectives often make it possible to interpolate between other existing and commonly studied graph clustering objectives. Recently, we showed that a number of popular graph clustering objectives such as modularity, normalized cut, and cluster deletion can be captured as special cases of a parametric variant of correlation clustering.

Nearly all existing techniques for parametric graph clustering focus on a simple graph setting, where all nodes are of the same type and are inter-related by pairwise connections, represented by edges. However, graph and complex network datasets are often endowed with additional structure, which can be exploited for the purpose of more in depth data analysis. As an example, there has been a recent surge of interest in higher-order methods for clustering bipartite graphs, which are often used as a first step in exploring and understanding a new dataset. When the data to be clustered is represented by a graph or network, the task is referred to as graph clustering or community detection. A good graph clustering is one in which nodes in the same cluster share many edges with each other, but nodes in different clusters share few edges. While these basic principles are shared by nearly all graph clustering techniques, there are many ways to formalize the notion of a graph cluster. However, no one method or objective function is capable of solving all graph clustering tasks.

1 INTRODUCTION

Finding sets of related objects in a large dataset, i.e., clustering, is one of the fundamental tasks in data mining and machine learning, and is often used as a first step in exploring and understanding a new dataset. When the data to be clustered is represented by a graph or network, the task is referred to as graph clustering or community detection. A good graph clustering is one in which nodes in the same cluster share many edges with each other, but nodes in different clusters share few edges. While these basic principles are shared by nearly all graph clustering techniques, there are many ways to formalize the notion of a graph cluster. However, no one method or objective function is capable of solving all graph clustering tasks.
exhibit clustering structure at different resolutions. However, these existing methods for clustering hypergraphs and bipartite graphs largely ignore parametric clustering objectives. Thus, in this paper we present a rigorous framework for parametric clustering in these settings. Our objectives are based on parameterized versions of correlation clustering and we show how, in certain parameter regimes, our objectives are related to a number of these previous objectives for bipartite and hypergraph clustering. Furthermore, our methods come with new approximation results. In summary,

1. We present HyperLAM, a parametric hypergraph clustering objective that we prove is related to hypergraph generalizations of the normalized cut and modularity objectives.
2. We present a parametric bipartite correlation clustering objective (PBCC), which captures standard bipartite correlation clustering and bicluster deletion \cite{9} as special cases. We also prove that in certain parameter regimes it is equivalent to a variant of our HyperLAM objective.
3. We prove that HyperLAM admits an $O(\log n)$ approximation by combining certain expansion techniques with approximation algorithms for correlation clustering in graphs. We also consider faster heuristic approaches based on applying greedy agglomeration methods.
4. While PBCC is NP-hard in general, we prove that in a certain parameter regime it is equivalent to bipartite matching and can thus be solved in polynomial time.
5. Using linear programming relaxation techniques, we show a number of approximation algorithms that apply to different parameter settings of PBCC, including the first constant factor approximation for bicluster deletion, the problem of partitioning a bipartite graph into disjoint bicliques by removing a minimum number of edges.

As a brief overview of our paper, we begin with small technical preliminaries on correlation clustering, graph clustering, and hypergraph clustering. Then we state our two new objectives for parametric hypergraph and bipartite clustering in Sections 3 and 4, as well as prove their equivalence with existing objectives. We discuss algorithms and heuristics in Section 5 before showing how these algorithms work in a variety of scenarios (Section 7).

2. PRELIMINARIES

We begin with technical preliminaries on correlation clustering, graph clustering, and hypergraph clustering.

2.1. Correlation Clustering

A standard weighted instance of correlation clustering is given by a graph $G = (V, W^+, W^-)$, where each pair of nodes $(i, j) \in V \times V$ with $i \neq j$ is associated with positive and negative weights $w_{ij}^+ \in W^+$ and $w_{ij}^- \in W^-$. Given this input, the objective is to minimize the weight of mistakes or disagreements. If nodes $i$ and $j$ are clustered together, they incur a mistake with penalty $w_{ij}^-$, and if they are separated, they incur a mistake with penalty $w_{ij}^+$. For instances where at most one of $(w_{ij}^+, w_{ij}^-)$ is non-zero, this can be viewed as a clustering problem in a signed graph. The objective can formally be stated as a binary linear program (BLP):

$$\begin{align*}
\text{minimize} & \quad \sum_{i < j} w_{ij}^+ x_{ij} + w_{ij}^- (1 - x_{ij}) \\
\text{subject to} & \quad x_{ij} \leq x_{ik} + x_{jk} \quad \text{for all } i, j, k \\
& \quad x_{ij} \in \{0, 1\} \quad \text{for all } i < j.
\end{align*}$$

The objective was first presented for signed graphs, by Bansal et al. \cite{9} and by Shamir et al. \cite{46}. Since its introduction, numerous variants on the objective have been presented for different weighted cases and graph types \cite{2, 3, 14, 17, 34, 41, 54}. In bipartite correlation clustering \cite{2, 5, 8, 15}, nodes can be organized into two different sets, in such a way that $w_{ij}^+ = w_{ij}^- = 0$ for any pair of nodes $i$ and $j$ in the same set. In the complete, unweighted bipartite signed graph case, the best known approximation factor is 3 \cite{15}.

2.2. Graph Clustering

Graph clustering is the task of separating the nodes of a graph into clusters in such a way that nodes inside a cluster share many edges with each other, but few with the rest of the graph. For an overview of graph clustering and community detection, we refer to surveys by Fortunato and Hric \cite{19}, and Schaeffer \cite{44}. Given a graph $G = (V, E)$, we let $C = \{S_1, S_2, \ldots, S_k\}$ represent a disjoint clustering of $V$, with $S_i \cap S_j = \emptyset$ for $i \neq j$, and $\bigcup_j S_j = V$. Given a set of nodes $S \subseteq V$, let $\bar{S} = V \setminus S$ denote the complement set, and cut$(S)$ be the weight of edges between $S$ and $\bar{S}$. One of the most common approaches to graph clustering is to set up and solve (or approximate) a combinatorial objective function that encodes some notion of clustering structure. One common objective used for bipartitioning a graph is the normalized cut objective, defined for a set $S \subseteq V$ to be

$$\phi(S) = \frac{\text{cut}(S)}{\text{vol}(S)} + \frac{\text{cut}(\bar{S})}{\text{vol}(\bar{S})},$$

where $\text{vol}(S) = \sum_{i \in S} d_i$, with $d_i$ being the degree of node $i$. Another very popular approach is to maximize the modularity objective \cite{39}, which measures the difference between the number of edges inside a cluster, and the expected number of edges in the cluster, where expectation is defined by some underlying graph null model.

Flexible parametric frameworks for graph clustering. Recently, we introduced a framework for graph clustering based on correlation clustering called LambdaCC \cite{54}. Given a graph $G = (V, E)$, the LambdaCC framework replaces an edge $(i, j) \in E$ with positive edges of weight $1 - \lambda d_i d_j$. For any pair $(i, j) \notin E$, a negative edge of weight $\lambda d_i d_j$ is introduced. The resulting signed graph can then be partitioned with respect to the correlation clustering objective. LambdaCC generalizes several other objectives including normalized cut \cite{47}, modularity \cite{39}, and cluster deletion \cite{46}.

2.3. Hypergraph clustering

We let $\mathcal{H} = (V, E)$ denote a hypergraph, where $V$ is a set of nodes, and $E$ is a set of hyperedges, which involve two or more nodes. In hypergraphs, the notion of cuts and clustering becomes even more complex, as there can be numerous ways to partition the nodes of a hyperedge, and numerous ways to generalize a graph-based objective. We say that a hyperedge $e \in E$ is cut if it spans more than two clusters of a clustering. In many clustering applications, any way of separating the nodes of a hyperedge is associated with
a penalty equal to the weight of the hyperedge, though other more general notions of hyperedge cuts have also been considered [13, 22, 35, 36]. Given a set of nodes \( S \subseteq V \) in a hypergraph \( \mathcal{H} \), we let \( \partial S = \{ e \in E : S \cap e \neq \emptyset, \bar{S} \cap e \neq \emptyset \} \) denote the boundary of \( S \), and use \( \text{cut}_{\mathcal{H}}(S) \) to denote the hypergraph cut penalty for \( S \). The most basic type of cut penalty is to simply count the number of edges on the boundary: \( \text{cut}_{\mathcal{H}}(S) = |\partial S| \). In this paper we also will consider the linear cut penalty, defined as follows:

\[
\text{cut}_{\mathcal{H}}(S) = \sum_{e \in \mathcal{E}} \min\{|S \cap e|, |\bar{S} \cap e|\} .
\]  

(3)

Hypergraph generalizations of the normalized cut objective have also been introduced in practice [35, 36, 58]. Here we consider the following definition, first introduced for generalized hypergraph cut functions by Li et al. [35]:

\[
\phi_{\mathcal{H}}(S) = \frac{\text{cut}_{\mathcal{H}}(S)}{\text{vol}_{\mathcal{H}}(S)} + \frac{\text{cut}_{\mathcal{H}}(\bar{S})}{\text{vol}_{\mathcal{H}}(\bar{S})} .
\]  

(4)

where \( \text{cut}_{\mathcal{H}} \) is any hypergraph cut function (e.g. \(|\partial S| \) or \((3)\)), and \( \text{vol}_{\mathcal{H}}(S) = \sum_{e \in \mathcal{E}} d_e \) is the hypergraph volume of \( S \). In this paper we will always consider the hypergraph degree \( d_e \) of a node to be the number of hyperedges a node participates in, though other definitions are possible [35, 36]. We also note that a number of higher-order generalizations of the modularity objective have been considered in different contexts [25, 28, 32, 45].

3 PARAMETRIC HYPERGRAPH CLUSTERING

Our first contribution is a hypergraph clustering objective that differentially treats hyperedges and pairwise edges in a parametric fashion. We further develop equivalence results with existing fixed-parameter objectives; algorithms are discussed in Section 5. Given a hypergraph \( \mathcal{H} = (V, E) \) and a resolution parameter \( \lambda \in (0, 1) \), we introduce a negative edge between each pair of nodes \((i, j) \in V \times V\), with weight \( \lambda w_i w_j \), where \( w_i \) is a weight associated with node \( i \). We consider either unit node weights (\( w_i = 1 \) for all nodes), or degree-based weights: \( w_i = d_i \) for each \( i \in V \). We treat each original hyperedge in \( \mathcal{H} \) as a positive edge of weight 1. Typically in higher-order correlation clustering, if a positive hyperedge is cut by a clustering, the same penalty is applied regardless of how its nodes are separated [21, 30, 34]. However, in order to accommodate a broader range of possible hyperedge cuts we use the following general abstraction: let \( P_V \) be the set of all clusterings, and define \( \xi : E \times P_V \to \mathbb{R} \) to be a function that outputs a penalty for the way in which clustering \( C \in P_V \) separates the nodes of a hyperedge \( e \in E \). We then define the HYPERLAM objective for a clustering \( C \) of \( \mathcal{H} \) as follows:

\[
\text{HYPERLAM}(C, \lambda) = \sum_{e \in E} \xi(e, C) + \sum_{i < j} \lambda w_i w_j (1 - z_{ij}) .
\]  

(5)

where \( z_{ij} \) is a binary indicator for whether nodes \( i \) and \( j \) are separated (\( z_{ij} = 1 \)) or clustered together (\( z_{ij} = 0 \)). The inspiration for this form of the objective is the parametric LambdaCC objective for graphs [54].

In practice, there may be many meaningful cut functions \( \xi \) to consider—here we focus mostly on two. The first is the standard all-or-nothing penalty, typically considered in the higher-order correlation clustering literature, which assigns a penalty proportional to the weight of the hyperedge if and only if the hyperedge is cut (at least two of its nodes are separated). Formally, this is defined as

\[
\xi(e, C) = \begin{cases} 
0 & \text{if } e \subseteq S \text{ for some } S \subseteq C \\
1 & \text{otherwise}.
\end{cases}
\]  

(6)

The other cut function we consider is a multiway generalization of the linear hypergraph cut penalty (3), defined by

\[
\xi(e, C) = |e| - \max_{S \subseteq C} |e \cap S| .
\]  

(7)

Given a clustering \( C \), this function assigns a penalty equal to the minimum number of nodes of a hyperedge \( e \) that must be moved in order for \( e \) to be contained in a single cluster. This has the advantage that for large hyperedges, it assigns a smaller penalty if only a small subset of nodes from a hyperedges are separated from the others.

3.1 HYPERLAM with Linear Cuts

If we use the linear hypergraph cut penalty (7), the HYPERLAM objective is equivalent to a clustering problem in a bipartite graph obtained by applying a so-called star expansion [59] to \( \mathcal{H} = (V, E) \). In more detail, for each \( e \in E \) we can introduce a new node \( v_e \), and attach each \( e \in e \) with a unit weight edge. Let \( V_e \) be the set of new hyperedge-nodes introduced via this procedure. This defines a graph \( G_{\mathcal{H}} = (V \cup V_e, E) \), where \( V = V \cup V_e \) is the set of new nodes, and \( E \) is the set of edges between \( V \) and \( V_e \). The goal is then to solve the following objective on \( G_{\mathcal{H}} \):

\[
\text{minimize} \sum_{(i, v_e) \in E} (1 - \delta_{i, v_e}) + \lambda \sum_{i < j} w_i w_j \delta_{ij} ,
\]  

(8)

where \( \delta_{i, v_e} = 1 \) if node \( i \in S \) is clustered with \( v_e \in V_e \), but is zero otherwise, and \( \delta_{ij} \) is similarly a clustering indicator vector for nodes in \( V \). This objective is equivalent to introducing a negative edge of weight \( \lambda w_i w_j \) between each pair of nodes in \( V \), and optimizing the correlation clustering objective.

**Lemma 3.1.** Objective (8) is equivalent to optimizing HYPERLAM with the linear cut penalty. 

**Proof.** Given any fixed clustering \( C = \{S_1, \ldots, S_k\} \) of the node set \( V \), we can define a clustering on all of \( G_{\mathcal{H}} \) by clustering nodes in \( V_e \) in a way that leads to a minimum penalty subject to \( C \). This is accomplished by putting each \( v_e \in V_e \) in the cluster \( S = \arg\max_{S \subseteq C} |e \cap S| \). In other words, we put \( v_e \) in the cluster where the largest number of nodes in \( e \) have been placed, as this minimizes the number of edges adjacent to \( v_e \) that are cut. This is the same as applying the linear hypergraph cut penalty (7) to any way of separating nodes in a hyperedge \( e \in E \). 

□

3.2 Relationship to Normalized Cut

Given any hyperedge cut function \( \xi \), the goal is to optimize (5) over all possible clusterings of nodes \( V \). The following theoretical result shows that our new objective captures a hypergraph generalization of normalized cut [35], just as the LambdaCC graph clustering framework generalizes normalized cut [54]. With unit node weights (\( w_i = 1 \) for all \( i \)), Theorem 3.2 can be adapted to a statement about a hypergraph variant of the sparsest cut clustering objective. Many aspects of our proof mirror our previous results for the LambdaCC framework [54]. We have expanded these results to apply to the
hypergraph setting. In particular, the second statement regarding the linear hyperedge cut requires significant extra treatment.

**Theorem 3.2.** For degree-weighted HyperLAM, there exists some value of $\lambda \in (0, 1)$, such that optimizing (5) over all bipartitions of the form $C = (S, \tilde{S})$ for some $S \subseteq V$, will produce the minimum hypergraph normalized cut partition (4). Furthermore, if the linear penalty (7) is used and we optimize over an arbitrary number of clusters, there exists some $\lambda'$ such that objective (5) will be minimized by the minimum hypergraph normalized cut objective under the linear hypergraph cut function (3).

**Proof.** Let $S^* \subseteq V$ be a set of nodes that minimizes

$$\tilde{\psi}(S) = \frac{\text{cut}_{\mathcal{H}}(S)}{\text{vol}_{\mathcal{H}}(S)\text{vol}_{\mathcal{H}}(S)}, \quad (9)$$

and define $\lambda^* = \tilde{\psi}(S^*)$. Observe that this is simply a scaled version of the hypergraph normalized cut solution: for all $S \subseteq V$, $\psi(S) = \phi(S)/\text{vol}_{\mathcal{H}}(V)$. Thus, $S^*$ is in fact the optimal hypergraph normalized cut solution as well.

For a bipartition $C = (S, \tilde{S})$, the penalty for positive hyperedges given in (5) reduces to

$$\sum_{e \in \mathcal{E}} \xi(e, \mathcal{C}) = \text{cut}_{\mathcal{H}}(S), \quad (10)$$

where $\text{cut}_{\mathcal{H}}$ can be any generalized notion of hypergraph cut function [35, 36, 52]. When we use degree weights $w_i = d_i$, the second term in objective (5) can be expressed in terms of the volume of $S$, so that we can write the objective for a bipartition $C = (S, \tilde{S})$ as

$$\text{HyperLAM}(S, \lambda) = \text{cut}_{\mathcal{H}}(S) - \lambda \sum_{i \in S} \sum_{i \in V \setminus S} d_id_j + \sum_{i < j} \lambda d_id_j \quad (11)$$

$$= \text{cut}_{\mathcal{H}}(S) - \lambda \text{vol}_{\mathcal{H}}(S)\text{vol}_{\mathcal{H}}(S) + \sum_{i < j} \lambda d_id_j . \quad (12)$$

Since the last term is a constant, if we fix $\lambda$ and minimize (12), we can check whether the minimizer $S$ satisfies:

$$\text{cut}_{\mathcal{H}}(S) - \lambda \text{vol}_{\mathcal{H}}(S)\text{vol}_{\mathcal{H}}(S) < 0 \implies \frac{\text{cut}_{\mathcal{H}}(S)}{\text{vol}_{\mathcal{H}}(S)\text{vol}_{\mathcal{H}}(S)} < \lambda .$$

This means that minimizing the HyperLAM objective over bipartitions is equivalent to solving the decision version of hypergraph normalized cut, i.e., given a fixed $\lambda$, find whether there is any bipartition $(S, \tilde{S})$ such that $\tilde{\psi}(S) < \lambda \implies \psi(S) < \text{vol}_{\mathcal{H}}(S)\lambda$. Thus, for a small enough value of $\lambda$, which will be slightly larger than $\lambda^*$, the optimal solutions to HyperLAM and $\tilde{\psi}$ will coincide.

**Using the linear cut penalty.** Next we prove that for the linear cut penalty (7), the HyperLAM objective generalizes hypergraph normalized cut objective with linear penalty (3), even if we do not restrict to considering only bipartitions of $V$. To prove this, we use the characterization of the HyperLAM objective given in Section 3.1. Specifically, optimizing HyperLAM with the linear cut penalty on a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is equivalent to optimizing objective (8) on a bipartite graph $G_{\mathcal{H}} = (V \cup V_{\mathcal{C}}, \mathcal{E})$. For this relationship, every clustering $C$ of $V$ induces a clustering $\tilde{C}$ of $\tilde{V} = V \cup V_{\mathcal{C}}$, obtained by arranging nodes of $V_{\mathcal{C}}$ in a way that minimizes cut edges between $V$ and $V_{\mathcal{C}}$. We will call $\tilde{C}$ the natural extension of $C$. Similarly, for $\tilde{S} \in \tilde{C}$ and $S = \tilde{S} \cap V$, we call $\tilde{S}$ the natural extension of $S$ in $\tilde{C}$.

For any two disjoint subsets $\tilde{S} \subseteq \tilde{V}$ and $\tilde{T} \subseteq \tilde{V}$ with $\tilde{S} \cap \tilde{T} = \emptyset$, let $\text{cut}(\tilde{S}, \tilde{T})$ denote the number of edges between these two sets in $G_{\mathcal{H}}$, and define $\text{cut}(S, \tilde{V} \setminus \tilde{S})$. The HyperLAM objective (with linear penalty) for this clustering is given by

$$\frac{1}{2} \sum_{S \in \mathcal{C}} \text{cut}(\tilde{S}) + \sum_{i < j} \lambda d_id_j - \frac{1}{2} \sum_{S \in \mathcal{C}} \text{vol}_{\mathcal{H}}(S)\text{vol}_{\mathcal{H}}(V \setminus S) . \quad (13)$$

We can interpret (13) in terms of positive and negative edge mistakes in the underlying instance of correlation clustering. The first term corresponds to positive edge mistakes, made by cutting edges between $V$ and $V_{\mathcal{C}}$. The second term is the weight of all negative edges, and the third term subtracts the weight of negative edges between two different clusters, since these are the negative edges where the clustering *does not* make a negative edge mistake. The first and third terms are multiplied by $1/2$, since these terms account for edges that are adjacent to exactly two clusters.

Mirroring our proof for the bipartition case, we can see that for a fixed $\lambda$, minimizing (13) over arbitrary clusterings allows us to check whether there exists a clustering $C$ such that

$$\Psi(C) = \frac{\sum_{S \in \mathcal{C}} \text{cut}(\tilde{S})}{\sum_{S \in \mathcal{C}} \text{vol}_{\mathcal{H}}(S)\text{vol}_{\mathcal{H}}(V \setminus S)} < \lambda . \quad (14)$$

where $\tilde{C}$ is the natural extension of $C$. Therefore, for a certain value of $\lambda$, the solution to HyperLAM with the linear hyperedge cut penalty will be the same as the minimizer for $\Psi$. Observe that for a bipartition $C = (S, \tilde{V} \setminus S)$, $\Psi(C) = \tilde{\psi}(S)$. Although $\Psi$ is defined for clusterings of arbitrary size, we will show that it is minimized by a bipartition. First, we prove a relationship between the cut function in $G_{\mathcal{H}} = (V \cup V_{\mathcal{C}}, \mathcal{E})$ (denoted by $\text{cut}$) and the two-way linear cut function (3) in the original hypergraph $\mathcal{H} = (V, \mathcal{E})$ (denoted by $\text{cut}_{\mathcal{H}}$). Let $\tilde{S}$ be a cluster in an arbitrary clustering $\tilde{C}$ (not necessarily a bipartition), and let $\tilde{S}$ and $\tilde{C}$ be their natural extensions so that $S = \tilde{S} \cap V$. Then

$$\text{cut}(\tilde{S}) = \text{cut}(\tilde{S} \cap V \setminus S) + \text{cut}(\tilde{S} \cap V \setminus \tilde{S} \setminus S)$$

$$= \sum_{e \in \mathcal{E} \cap (V \setminus S \setminus \tilde{S})} \text{cut}(\tilde{S} \cap V \setminus S) + \sum_{e \in V \setminus (S \setminus \tilde{S})} \text{cut}(\tilde{S} \cap V \setminus S)$$

$$= \sum_{e \in \mathcal{E}} |S \cap e| + \sum_{e \in \mathcal{E}} |V \setminus S \cap e| \geq 2 \min_{e \in \mathcal{E}} |S \cap e|, |V \setminus S \cap e| = \text{cut}_{\mathcal{H}}(S) .$$

Finally, let $C'$ be an arbitrary minimizer for $\Psi$. Then,

$$\Psi(C') = \frac{\sum_{\tilde{S} \in \tilde{C}} \text{cut}(\tilde{S})}{\sum_{\tilde{S} \in \tilde{C}} \text{vol}_{\mathcal{H}}(S)\text{vol}_{\mathcal{H}}(V \setminus S)} \quad (15)$$

$$\geq \min_{\tilde{S} \in \tilde{C}} \frac{\text{cut}(\tilde{S})}{\text{vol}_{\mathcal{H}}(S)\text{vol}_{\mathcal{H}}(V \setminus S)} \quad (16)$$

$$\geq \frac{\text{cut}_{\mathcal{H}}(S^*)}{\text{vol}_{\mathcal{H}}(S^*)\text{vol}_{\mathcal{H}}(V \setminus S^*)} \quad (17)$$

$$= \tilde{\psi}(S^*) = \Psi(C'). \quad (18)$$

This confirms that for a certain value of $\lambda$, $C^* = (S^*, V \setminus S^*)$ is optimal for HyperLAM with a linear hyperedge cut penalty. \qed
We illustrate an instance of the problem in Figure 1. As an objective, we have the following

$$\mu_1 = \mu_2 = 1$$

Table 1: Equivalence and approximation results for PBCC; $\varepsilon$ represents a small, graph dependent number.

| Parameters | Equivalence | Approx. |
|------------|-------------|---------|
| $\beta = \mu_1 = \mu_2 = \lambda$ | LambdaCC | see [54], [21] |
| $\mu_1 \geq \mu_2 \geq (1 - \beta)$ | Bip. Matching | 1 (Thm 4.1) |
| $\mu_1 = 2, \beta \geq 1 - \varepsilon$ | Bicluster deletion | 4 (Thm 5.2) |
| $\mu_1 = 0, \beta \geq \frac{1}{2}$ | Generalized deletion | $6 - \frac{1}{\beta}$ (Thm 5.3) |
| $\mu_1 = 2 \in [0, 1], \beta \geq \frac{1}{2}$ | - | 5 (Thm 5.4) |
| $\mu_1 = \lambda, \mu_2 = 0, \beta = 0$ | HYPERLAM | $O(\log n)$ |

### 4 PARAMETRIC BIPARTITE CLUSTERING

Next we present a parameterized variant of bicluster correlation clustering (PBCC) in graphs, which we prove generalizes a number of other clustering objectives in bipartite graphs, and comes with several novel approximation guarantees. Let $G = (V_1, V_2, E)$ be a bipartite graph in which $V_1$ and $V_2$ are node sets and $E$ is a set of edges between nodes in $V_1$ and $V_2$. In order to define an instance of Parametric Bipartite Correlation Clustering (PBCC), we first define parameters $\mu_1$, $\mu_2$, and $\beta$, all in the interval $[0, 1]$. We then associate each $e \in E$ with a positive edge of weight $1 - \beta$, and every $e \in (V_1 \times V_2) - E$ with a negative edge with weight $\beta$. Additionally, each pair of nodes in $V_1$ is given a negative edge of weight $\mu_1$, and each pair of nodes in $V_2$ is given a negative edge of weight $\mu_2$. The result is a complete, weighted instance of correlation clustering, where the underlying positive edge structure is a bipartite graph. We illustrate an instance of the problem in Figure 1. As an objective, we have the following

$$\text{PBCC}(C) = \sum_{i \in V_1, j \in V_2} \beta(1 - A_{ij})\delta_{ij} + (1 - \beta)A_{ij}(1 - \delta_{ij})$$

$$+ \sum_{(i, j) \in V_1 \times V_1} \mu_1 \delta_{ij} + \sum_{(i, j) \in V_2 \times V_2} \mu_2 \delta_{ij},$$

where $A_{ij} = 1$ if $(i, j) \in E$ and is zero otherwise, and $\delta_{ij}$ is the clustering indicator for nodes $(i, j)$, i.e., $\delta_{ij} = 1$ if they are clustered together in $C$, and $\delta_{ij} = 0$ otherwise. The PBCC objective is closely related to several other well-studied problems. We summarize a list of equivalence results and approximation algorithms for PBCC in Table 1, based on the results we prove in this section and the next.

When $\mu_1 = \mu_2 = 0$ and $\beta = 1/2$, the problem corresponds to the standard unweighted bipartite correlation clustering problem (BCC) [2, 5]. When $\mu_1 = \mu_2 = \beta$, it is equivalent to applying the LambdaCC framework [54] to a bipartite graph.

If $\beta > \frac{|V_1||V_2|/(|V_1||V_2| + 1)}$ and $\mu_1 = \mu_2 = 0$, then making a mistake at a single negative edge of weight $\beta$ introduces a greater weight of disagreements than placing each node into a singleton cluster. Therefore, the objective will be optimized by making a minimum number of positive-edge mistakes, subject to all clusters being bicliques. Thus, in this parameter regime, PBCC is equivalent to bicluster deletion, the problem of removing a minimum number of edges from a bipartite graph to partition it into disjoint bicliques.

### 4.1 Relationship with Bipartite Matching

Although PBCC is NP-hard in general, our next theorem shows that in a certain parameter regime, PBCC is equivalent to solving a bipartite matching problem on $G = (V_1, V_2, E)$. Therefore, the problem can be solved in polynomial time in this regime.

**Theorem 4.1.** If parameters $\mu_1, \mu_2$, and $\beta$ satisfy $\min\{\mu_1, \mu_2\} \geq (1 - \beta)$, then the optimal solution to PBCC for these parameters is the same as finding a maximum bipartite matching on $G = (V_1, V_2, E)$.

**Proof.** First consider $\mu = \mu_1 = \mu_2 > 1 - \beta$, and let $C$ denote the optimal clustering in this case. Let $S = \{S_1 \cup S_2\}$ be an arbitrary cluster in $C$, where $S_i \subseteq V_i$ for $i = 1, 2$. Assume without loss of generality that $|S_1| \leq |S_2|$. In three steps, we will prove that $S$ contains at most one node from each of $V_1$ and $V_2$, and thus $C$ is in fact just a matching.

**Step 1.** Observe that $S$ must be a biclique in terms of positive edges between $S_1$ and $S_2$. If we assume not, then there exists a node $s \in S_2$ that does not share a positive edge with every node in $S_1$. By removing $s$ from $S$, we no longer make negative edge mistakes between $s$ and the rest of $S_2$, which decreases the objective by $\mu(|S_2| - 1)$. At the same time, this introduces new errors weighing at most $(1 - \beta)(|S_1| - 1)$, due to positive edge mistakes between $s$ and $S_1$. This decreases the overall objective score by at least

$$\mu(|S_2| - 1) - (1 - \beta)(|S_1| - 1) > 0,$$

since $|S_2| > |S_1|$ and $(1 - \beta) < \mu$. In other words, the weight of mistakes strictly decreases if we removed $s$. This would be a contradiction to the optimality of $C$. Thus, no such $s$ exists, and $S$ is a biclique of positive edges.

**Step 2.** If $|S| > 1$, then $|S_1| = |S_2|$. If we assume instead that $|S_1| \geq |S_2| + 1$, then removing any $s \in S_2$ would decrease the objective by $\mu(|S_2| - 1)$ and increase the objective by $(1 - \beta)|S_1|$, since $S$ is a biclique. Since $|S_1| \leq |S_2| - 1$, this again leads to an overall decrease in the weight of mistakes:

$$\mu(|S_2| - 1) - (1 - \beta)|S_1| \geq \mu(|S_2| - 1) - (1 - \beta)|S_2| > 0,$$

so a contradiction is shown.

**Step 3.** If $|S| > 1$, then $|S_1| = |S_2| = 1$. By Step 2, $S_1$ and $S_2$ have the same size $k$. We will prove that $k = 1$. By Step 1, every node in $S_1$ shares a positive edge with every node in $S_2$. Note that this implies there is a perfect matching between the two sides. Thus, we can split up $S$ into multiple subclusters where each node in $S_1$
We begin by reviewing a general strategy for obtaining approximations when we use this penalty, the objective is equivalent to an instance of correlation clustering defined by performing a star expansion. This results in an instance of correlation clustering where $V_1 = V$ is the set of original nodes, each pair of which has a negative edge of weight $\mu_1 = \lambda$. The auxiliary nodes in $V_2$, with $\mu_2 = 0$, and edges between $V$ and $V_2$ all have weight $1 - \beta = 1$.

4.2 Relationship with HyperLam

When $\mu_1 = \lambda, \mu_2 = 0$, and $\beta = 0$, PBCC is equivalent to a special instance of HyperLam with a linear hyperedge cut penalty (7). As we noted in Section 3.1, when we use this penalty, the HyperLam objective is equivalent to an instance of correlation clustering defined by performing a star expansion. This results in an instance of PBCC where $V_1 = V$ is the set of original nodes, each pair of which has a negative edge of weight $\mu_1 = \lambda$. The auxiliary nodes in $V_2$, with $\mu_2 = 0$, and edges between $V$ and $V_2$ all have weight $1 - \beta = 1$.

5 APPROXIMATIONS AND HEURISTICS

We now turn our attention to specific approximation guarantees that can be obtained by our objectives in different parameter regimes. We begin by reviewing a general strategy for obtaining approximation guarantees for variants of correlation clustering, through which we prove approximation guarantees for PBCC. In order to approximate HyperLam, we combine existing approximation algorithms for correlation clustering with techniques for reducing a hypergraph to a related graph. We conclude with some heuristic approaches for HyperLam.

5.1 General LP Rounding Algorithm for CC

Pivot (aka Algorithm 1) is a simple algorithm unweighted correlation clustering. When pivots are chosen uniformly at random, Ailon et al. [3] showed that this algorithm returns a 3-approximation for complete unweighted correlation clustering. Later, van Zuylen and Williamson [51] produced a de-randomized 3-approximation. We review a generic algorithm for correlation clustering from the work of van Zuylen and Williamson, which we apply later in developing approximation algorithms for new parametric correlation clustering variants. Pseudocode for this method, which we call GenRound, is given in Algorithm 2. One can prove approximation results for special input problems using the following theorem. We have adapted this result from the work of van Zuylen and Williamson [51] to match our notation and presentation. The theorem and its proof rely on a careful consideration of so-called bad triangles in the rounded unweighted graph. A bad triangle is a triplet of nodes in the graph which contains two positive edges and one negative edge.

**Theorem 5.1.** (Theorem 3.1 in [51]). Given a weighted instance of correlation clustering $G = (V, W^+, W^-)$, let $c_{ij} = w_{ij}^w x_{ij} + w_{ij}(1 - x_{ij})$. GenRound returns an $\alpha$-approximation for the min-disagree objective (1) if the threshold parameter, $\delta$, is chosen so that the graph $\hat{G} = (V, \hat{E}^+, \hat{E}^-)$ satisfies the following conditions:

1. For all $(i, j) \in \hat{E}^+$, we have $w_{ij}^\alpha \leq c_{ij}$, and for all $(i, j) \in \hat{E}^-$, we have $w_{ij}^\alpha \leq c_{ij}$.

2. For every bad triangle $(i, j, k)$ in $\hat{G}$, with $\{(i, j), (j, k)\} \subseteq \hat{E}^+$ and $(i, k) \in \hat{E}^-$, we have $w_{ij} + w_{jk} + w_{ik} \leq \alpha \{c_{ij} + c_{jk} + c_{ik}\}$.

When applying Pivot in Algorithm 2, selecting the pivot node uniformly at random gives an expected $\alpha$-approximation. A deterministic algorithm with the same approximation factor $\alpha$ can be obtained via a careful selection of pivot nodes [51].

5.2 Graph Reductions for HyperLam

Although HyperLam is NP-hard to optimize, we can obtain approximation algorithms for the objective using two different techniques for converting hypergraphs to graphs.

**Weighted clique expansion:** Replace each hyperedge $e \in E$ with a clique on $e$ where each edge has weight $1/|e| - 1$. If two nodes appear together in multiple hyperedges, assign a weight equal to the sum of weights from each such clique expansion.

**Star expansion:** As outlined in Section 3.1, replace each hyperedge $e \in E$ with an auxiliary node $v_e$ and an edge from each $v \in e$ to $v_e$. If we use weights $w_i = 1$ for all $i \in V$, this is equivalent to an instance of PBCC with $\mu_1 = \lambda, \mu_2 = 0$, and $\beta = 0$.

For each expansion technique, we still include a negative edge of weight $\lambda w_{ij}$ between each pair $(i, j) \in V \times V$, where $w_i$ is the weight for node $i$. Either way, the result is an instance of weighted correlation clustering, which we can solve with existing approximation algorithms.

The weighting scheme for the clique expansion is chosen specifically to approximately model the all-or-nothing hyperedge cut penalty (6). For three-uniform hypergraphs, the relationship is exact [26]. For a $k$-node hyperedge, with $k > 3$, the minimum penalty
for splitting the clique comes from placing all but one node in the same cluster, giving a penalty equal to \((k-1)/(k-1) = 1\). The maximum possible penalty, when all \(k\) nodes are placed in different clusters, is \(\left(\frac{k}{2}\right)^2 = \frac{k}{2}\). Thus, the penalty at each positive hyperedge in the resulting reduced graph will be within a factor \(k/2\) of the original all-or-nothing penalty for any clustering \(C\). Meanwhile, the star expansion enables us to exactly model the linear cut penalty \((7)\), as shown in Lemma 3.1.

Thus, applying existing approximation algorithms for correlation clustering \([14, 17]\), we get an \(O(k \log n)\) approximation for HyperLAM with all-or-nothing penalty via the weighted clique expansion, where \(k\) is the maximum size hyperedge. We also obtain \(\text{an} O(\log n)\) approximation for HyperLAM with linear hyperedge penalty via the star expansion.

5.3 A Four-Approx for Bicluster Deletion

We now show how GenRound and Theorem 5.1 combine to develop a 4-approximation for bicluster deletion: the first constant-factor approximation for this problem. Rather than the edge weights presented in the last section, we view bicluster deletion as a general weighted correlation clustering problem with the following weights

\[
(w^+_{ij}, w^-_{ij}) = \begin{cases} 
(0, 0) & \text{if } i \text{ and } j \text{ are in the same partition of } G \\
(1, 0) & \text{if } (i, j) \in E^+ \\
(0, \infty) & \text{if } (i, j) \in E^- .
\end{cases}
\]

Above, \(E^+\) and \(E^-\) denote positive and negative edges between the two sides of the bipartite graph. To ensure no mistakes are made at negative edges, we add the constraint \(x_{ij} = 1\) to \(\text{BLP} (1)\), for every \((i, j) \in E^-\). The LP-relaxation of this problem is given by

\[
\begin{align*}
& \text{minimize } \sum_{(i,j) \in E^+} x_{ij} \\
& \text{subject to } \\
& \quad x_{ij} = 1 \quad \text{for all } (i, j) \in E^- \\
& \quad x_{ij} \leq x_{ik} + x_{jk} \quad \text{for all } i, j, k \\
& \quad 0 \leq x_{ij} \leq 1 \quad \text{for all } i < j.
\end{align*}
\]

Theorem 5.2. Applying GenRound to \(\text{LP} (21)\), with \(\delta = 1/2\), returns a 4-approximation to bicluster deletion.

Proof. First of all note that GenRound applied to \(\text{LP} (21)\) with \(\delta = 1/2\) will indeed form only complete biclues. Applying a pivot step around node \(k\) will form a cluster \(S\) in which \(x_{kj} < \delta = 1/2\) for every \(i \in S\). For any two non-pivot nodes \(i\) and \(j\) in \(S\), \(x_{ij} \leq x_{ki} + x_{kj} < 1\). This means that \((i, j) \notin E^-\), since the LP relaxation forces all negative edges to have distance one. It remains to check that the conditions of Theorem 5.1 are satisfied for \(\alpha = 4\).

For the first condition, if \((i,j) \in E^+\), this means \(x_{ij} < 1/2 \implies (i,j) \in E^+\), which means \(w^-_{ij} = 0\) so the inequality \(w^-_{ij} \leq \alpha c_{ij}\) is trivially satisfied. If \((i,j) \in E^- \cap E^+\), then \(w^+_{ij} = 1\) and \(c_{ij} = x_{ij}\), so \(w^+_{ij} = 1 < 2 = 4(1/2) = 4x_{ij} = ax_{ij}\).

If \((i,j) \in \tilde{E}^- \cap E^-\), then \(w^-_{ij} = 0\) and again the inequality is trivial. Thus, the first condition is satisfied for all cases.

For the second condition, consider a bad triangle \((i,j,k) \in G\) with \((i,k) \in \tilde{E}^- \implies x_{ik} \geq 1/2\). Since \((i,j)\) and \((j,k)\) are in \(E^+\), \(x_{ij} < 1/2\) and \(x_{jk} < 1/2\), so \(x_{ik} \leq x_{ij} + x_{jk} < 1\). If \((i,j,k)\) are all on the same side of the graph in \(G\), then \(w^+_{ij} + w^+_{jk} + w^-_{ik} = 0\) and the inequality in condition two of Theorem 5.1 is trivial. If \(i\) and \(j\) are on the same side, but not \(k\), then \(a(c_{ij} + c_{jk} + c_{ik}) = 4(x_{ij} + x_{jk} + x_{ik}) \geq 2 > 1 = w^+_{ij} + w^+_{jk} + w^-_{ik}\).

If \(i\) and \(k\) are on the same side of the graph but not \(j\) (which is symmetric to considering \(j, k\) together and \(i\) on the other side), then \(w^+_{ij} + w^+_{jk} + w^-_{ik} = 2\) and

\[a(c_{ij} + c_{jk} + c_{ik}) = 4(x_{ij} + x_{jk} + 0) \geq 2 > 2 = w^+_{ij} + w^+_{jk} + w^-_{ik} = 2\]

Since all the conditions of Theorem 5.1 hold in all cases, GenRound is a 4-approximation for bicluster deletion when \(\delta = 1/2\).

5.4 Generalized Results for PBCC

We now turn to approximation algorithms for a wider range of parameter settings. In the remainder of the section, we specifically consider \(\mu = \mu_1 = \mu_2\). As we did for bicluster deletion, our goal is to find a threshold parameter \(\delta\) and an approximation factor \(\alpha\) such that the two conditions of Theorem 5.1 hold. To find the best choice of \(\delta\) in different settings, we first set up a system of inequalities that are sufficient to guarantee the assumptions of Theorem 5.1. In these inequalities, \(\mu\) and \(\beta\) are treated as constants, and \(\alpha\) and \(\delta\) are variables we optimize over to obtain the best approximation results. We wish to find \(\delta\) and \(\alpha\) such that these sufficient constraints are satisfied and the approximation factor \(\alpha\) is minimized.

Sufficient constraints for first condition. The first condition of Theorem 5.1 requires that for all \((i,j) \in E^+\), we have \(w^-_{ij} \leq \alpha c_{ij}\), and for all \((i,j) \in \tilde{E}^-\), we have \(w^+_{ij} \leq \alpha c_{ij}\). If \((i,j) \in E^- \cap \tilde{E}^+\) or \((i,j) \in E^- \cap \tilde{E}^-\), then the left hand side of the inequality is zero and the inequality is trivially satisfied.

If \((i,j) \in E^- \cap \tilde{E}^+\), then \(x_{ij} \geq \delta, w^-_{ij} = (1 - \beta), \) and \(c_{ij} = (1 - \beta)x_{ij} \geq (1 - \beta)\delta\). Thus the inequality is satisfied as long as

\[\alpha(1 - \delta) \geq 1.\]

On the other hand, if \((i,j) \in E^- \cap \tilde{E}^-\), then \(x_{ij} < \delta, w^+_{ij}\) is either \(\mu\) or \(\beta\), and \(c_{ij} = w^+_{ij}(1 - x_{ij}) \geq w^+_{ij}(1 - \delta)\). In order for the inequality \(w^+_{ij} \leq \alpha c_{ij}\) to be satisfied, it is sufficient to choose \(\alpha\) and \(\delta\) satisfying

\[\alpha(1 - \delta) \geq 1.\]

Sufficient constraints for second condition. The second condition is defined for a triangle \((i,j,k)\) with \((i,k) \in \tilde{E}^-\) and \((i,j), (j,k) \in \tilde{E}^+\). We refer to this as a “bad triangle,” since at least one of the edges must be violated by any clustering. The requirement is:

\[L = w^+_{ij} + w^+_{jk} + w^-_{ik} \leq \alpha(c_{ij} + c_{jk} + c_{ik}) = R.\]

Following the approach of Ailon et al. [2] for standard BCC, and our 4-approximation for bicluster deletion, the analysis is split into three cases:

- Case 1: \((i,j)\) are on the same side of \(G\), but not \(k\).
- Case 2: \((i,k)\) are on the same side of \(G\), but not \(j\).
- Case 3: \(i, j, k\) are all on the same side of \(G\).

For Case 3, we know all edges in the triangle are negative in \(G\), so there are no subcases to consider. However, Case 1 and Case 2 both require we consider four different subcases, since they both involve two edges crossing from one side to \(G\) to the other, which could be
we consider nine different types of triangles that could be mapped to a bad triangle in $G$. We will handle inequality (24) differently depending on the case.

Table 2: We compute $L = w_{ij}^+ + w_{ik}^+ + w_{ik}^-$ and lower bound on $R/\alpha$ where $R = \alpha(c_{ij} + c_{jk} + c_{ik})$ for each type of bad triangle displayed in Figure 2. Note that the second condition of Theorem 5.1 is to check that $L \leq R$ in all cases. For Case 1a, we include two bounds, one that works for any $\beta$, and another bound that is tighter when $\beta \geq 1/2$.

| Case          | $L$  | Bound $f(\delta) \leq R/\alpha$ |
|---------------|------|----------------------------------|
| Case 1a (any $\beta$) | 1    | $\mu(1 - \delta) + \beta(1 - \delta)$ |
| Case 1a ($\beta \geq 1/2$) | 1    | $\mu(1 - \delta) + \beta + \delta(1 - 3\beta)$ |
| Case 1b       | 1 - $\beta$ | $\mu(1 - \delta) + \delta(1 - \beta)$ |
| Case 1c       | 0    | $\mu(1 - \delta) + \delta(1 - \beta)$ |
| Case 1d       | $\beta$ | $\mu(1 - \delta) + \beta(2 - 3\delta)$ |
| Cases 2a, 2b  | $(1 - \beta) + \mu$ | $\beta(1 - \delta) + \mu(1 - 2\delta)$ |
| Case 2c       | $2(1 - \beta) + \mu$ | $(1 - \beta)\delta + \mu(1 - 2\delta)$ |
| Case 2d       | $\mu$ | $2\beta(1 - \delta) + \mu(1 - 2\delta)$ |
| Case 3        | $\mu$ | $\mu(3 - 4\delta)$ |

positive or negative. Figure 2 illustrates all the ways a triangle in $G$ can be mapped to a bad triangle in $\tilde{G}$.

In Table 2, we display the value of $L$, and a lower bound on $c_{ij} + c_{jk} + c_{ik}$ for each bad triangle displayed in Figure 2. Let $f_t(\delta)$ denote the lower bound determined for bad triangle of type $t$. Full details for computing $L$ and deriving these bounds are presented in the appendix. Once we have such a lower bound $\alpha f_t(\delta) \leq R$ for each type of bad triangle, in order to show that the second condition of Theorem 5.1 is always satisfied, it suffices to prove

$$L \leq \alpha f_t(\delta)$$

for every bad triangle type $t$.

An approximation for $\mu = 0$. Ailon et al. [2] proved a 4-approximation for unweighted bipartite correlation clustering, which is equivalent to PBCC with $\mu = 0$ and $\beta = 1/2$. We show how to select $\delta$ in GenRound so that not only can we recover this same approximation guarantee when $\mu = 0$ and $\beta = 1/2$, but also obtain guarantees for all $\beta \in \left[ \frac{1}{2}, 1 \right]$.

**Theorem 5.3.** When $\mu = \mu_1 = \mu_2 = 0$ and $\beta \geq 1/2$, Algorithm 2 with $\delta = \frac{2\beta}{6\beta - 1}$ returns a $(6 - 1/\beta)$-approximation for PBCC.

**Proof.** When $\mu = 0$, the system of inequalities in Table 2 greatly simplifies to the following set of conditions:

$$1 \leq \alpha(1 - \delta)$$
$$1 \leq \alpha[\beta + \delta(1 - 3\beta)]$$
$$1 \leq \alpha(2 - 3\delta)$$
$$2 \leq a\delta$$

The first of these is a repeat of inequality (23), and the remaining three are derived from Case 1a (the second bound designed specifically for $\beta \geq 1/2$), Case 1d, and Case 2c from Table 2. One can check to see that all other inequalities we must satisfy are less strict and can be subsumed into one of these four bounds.

For inequality (26):

$$\alpha(1 - \delta) = (6 - 1/\beta) \left( 1 - \frac{2\beta}{6\beta - 1} \right) = \left( \frac{6\beta - 1}{\beta} \right) \left( \frac{4\beta - 1}{6\beta - 1} \right)$$

$$\alpha(1 - \delta) = 4\beta - 1 \geq 2 > 1.$$ 

For inequality (27):

$$\alpha[\beta + \delta(1 - 3\beta)] = \left( \frac{6\beta - 1}{\beta} \right) \left( \beta + \frac{2\beta}{6\beta - 1} (1 - 3\beta) \right)$$

$$\alpha[\beta + \delta(1 - 3\beta)] = (6\beta - 1) + 2(1 - 3\beta) = 1.$$ 

For inequality (28):

$$\alpha(2 - 3\delta) = \frac{6\beta - 1}{\beta} \left( 2 - \frac{6\beta}{6\beta - 1} \right)$$

$$\alpha(2 - 3\delta) = 12 - \frac{2}{\beta} - 6 = \frac{6}{\beta} \geq 2.$$ 

For inequality (29):

$$\alpha\delta = \frac{6\beta - 1}{\beta} \left( \frac{2\beta}{6\beta - 1} \right) = 2.$$ 

All cases are satisfied, and the proof is complete. $\square$

**A 5-approx for a generalized parameter regime.** Considering a more general parameter regime, where $\mu = \mu_1 = \mu_2 \in [0, 1]$, we obtain a 5-approximation for all $\beta \geq 1/2$.

**Theorem 5.4.** When $\mu_1 = \mu_2$ and $\beta \geq 1/2$, Algorithm 2 with $\delta = 2/5$ returns a 5-approximation to PBCC.
Proof. In order to prove the result, it is sufficient to show that the following set of inequalities holds when $\delta = 2/5$ and $\alpha = 5$:

\[
\begin{align*}
1 & \leq a\delta & (30) \\
1 & \leq a(1 - \delta) & (31) \\
1 & \leq a[\mu(1 - \delta) + \beta(1 - \delta)] & (32) \\
(1 - \beta) & \leq a[\mu(1 - \delta) + \delta(1 - \beta)] & (33) \\
\beta & \leq a[\mu(1 - \delta) + \beta(2 - 3\delta)] & (34) \\
(1 - \beta) + \mu & \leq a[\beta(1 - \delta) + \mu(1 - 2\delta)] & (35) \\
2(1 - \beta) + \mu & \leq a[2\beta(1 - \delta) + \mu(1 - 2\delta)] & (36) \\
1 & \leq a[3 - 4\delta] & (37) \\
1 & \leq a(3 - 4\delta). & (38)
\end{align*}
\]

These are taken from (22), (23), and all inequalities of the form (25) obtained from the different cases in Table 2. The first two inequalities and the last inequality are easy to show just by plugging in $\alpha = 5$ and $\delta = 2/5$. For inequalities (32), (33), and (34), we will drop the term $a\mu(1 - \delta)$ on the right hand side and prove a more simple set of inequalities that are more strict:

\[
\begin{align*}
1 & \leq a\beta(1 - \delta) & (39) \\
1 & \leq a\delta & (40) \\
1 & \leq a(2 - 3\delta). & (41)
\end{align*}
\]

Inequalities (40) and (41) follow directly from plugging in $\alpha = 5$ and $\delta = 2/5$. For inequality (39), note that

\[
a\beta(1 - \delta) = 5\beta \frac{3}{5} = 3\beta \geq \frac{3}{2} > 1.
\]

Finally, note that inequalities (35), (36), and (37) all have a term $\mu$ on the left and a term $\mu(1 - 2\delta)$ on the right. So all of these inequalities will be satisfied if we prove the more strict conditions

\[
\begin{align*}
\mu & \leq a\mu(1 - 2\delta) & (42) \\
(1 - \beta) & \leq a\beta(1 - \delta) & (43) \\
2(1 - \beta) & \leq a(1 - \beta)\delta & (44)
\end{align*}
\]

Note that (42) holds tightly for $\alpha = 5$ and $\delta = 2/5$. Inequality (43) is less strict than inequality (40), which we already proved. Finally, after canceling $(1 - \beta)$ from both sides, (44) becomes $2 \leq a\delta$, which holds for our choices of $\alpha$ and $\delta$. Thus, all necessary constraints are satisfied and we know that Algorithm 2 will yield a 5-approximate solution for any $\mu$ and whenever $\beta \geq 1/2$. \hfill $\Box$

5.5 Modularity Connections and Heuristics

Returning to the HyperLAM objective, applying our weighted clique expansion and introducing a negative edge of weight $\lambda \delta d_i d_j$ for node pair $(i, j)$ is equivalent to solving a weighted variant of the LambdaCC graph clustering objective [54]. Since LambdaCC is equivalent to a generalization of modularity with a resolution parameter [39, 54], we can also approximately optimize the HyperLAM objective by applying our weighted clique expansion and then running heuristic algorithms for modularity such as the Louvain algorithm [11] or, more appropriately, generalizations of Louvain with a resolution parameter [27]. A similar approach will also work for the star expansion: we set the weight of a node in $V$ to be its hyperedge degree $\omega_e = d_e$, and the weight of an auxiliary node $v_e$ (obtained from expanding a hyperedge) to be $w_{v_e} = 0$. This also corresponds to a weighted variant of LambdaCC, since each pair of nodes $(i, j)$ in the graph share a negative edge of weight $\lambda \omega_i \omega_j$. In many cases this weight will be zero, but we can still apply generalized Louvain-style heuristics to optimize the objective.

Kumar et al. [32] previously considered a modularity-based approach for hypergraph clustering based on the same type of clique expansion. These authors applied the same weight $1/(|e| - 1)$ to each edge in a clique expansion of a hypergraph $|e|$, as this preserves the degree distribution of nodes in the original hypergraph. They then considered applying the modularity objective [39] to the resulting graph. Their approach corresponds to applying a weighted clique expansion to an instance of HyperLAM, and setting $\lambda = 1/(\text{vol}(V))$. Thus, this approach can be viewed as a special case of our hyperedge expansion procedure for HyperLAM. The connection to correlation clustering we show, along with the resulting approximation algorithms for the all-or-nothing hypergraph cut, provide further theoretical motivation for this choice of weighted clique expansion. Despite this connection to a previous clique expansion technique for modularity, we note that our original hypergraph objective (5) nevertheless differs from generalizations of modularity defined directly for hypergraphs [28], as opposed to modularity objectives applied to clique expansions of hypergraphs.

6 RELATED WORK

To anchor our work, we highlight related results on algorithms for correlation clustering, techniques for parametric clustering in standard graphs, and recent results on clustering hypergraphs.

Correlation Clustering Bansal et al. [9] first introduced the problem of correlation clustering, providing a constant factor approximation for the complete unweighted case. Amit was the first to consider the problem in the bipartite setting [5], providing an 11-approximation for the complete unweighted setting. Later, Ailon et al. [2] presented a 4-approximation. Most recently, Chawla et al. [15] improved the best approximation factor to 3.

Higher-order correlation clustering was first considered by Kim et al. [30] in the context of image segmentation. Li et al. [34] were the first to develop approximation algorithms for the complete 3-uniform case, giving a 9-approximation. We later gave a 4$(k - 1)$-approximation for the $k$-uniform setting, which was then improved to 2$k$ by Li et al. [37]. For weighted hypergraphs, Fukunaga [20] presented an $O(k \log n)$ approximation algorithm, where $k$ is the maximum size of negative hyperedges.

Parametric Graph Clustering Our introduction of the LambdaCC framework situating graph clustering within correlation clustering [54]. We proved equivalence results with modularity, normalized cut, and sparsest cut, and gave a 3-approximation when $\lambda \geq 1/2$, based on LP-rounding. We were later able to show that the LP relaxation has an integrality gap of $O(\log n)$ for some small values of $\lambda$ [21]. LambdaCC is in turn related to other graph parametric clustering objectives, such as stability [16], various Potts models [42, 49], and generalizations of modularity [7].

Hypergraph Clustering Several different higher-order generalizations of modularity have been previously developed [6, 28, 32, 45], along with higher-order variants of conductance [10] and normalized cut [35, 58]. In hypergraph clustering, the most common
penalty for a cut hyperedge is the weight of that hyperedge, regardless of how the hyperedge is cut. However, other penalties have also been considered in the context of hypergraph partitioning and clustering [13, 35, 36]. A more comprehensive overview of generalized hypergraph cut functions is included in recent work by one of the authors [52].

7 EXPERIMENTS

We demonstrate our parametric objective functions and algorithms in analyzing an assortment of different types of datasets. Our primary goal is to highlight the diversity of results we can achieve.

### Implementation Details and Runtime Considerations

We implement our algorithms in the Julia programming language, using Gurobi software to solve LP relaxations. Code for all algorithms and experiments is available online at [https://github.com/nveldt/ParamCC](https://github.com/nveldt/ParamCC). In our experiments we focus on studying the differences among the objective functions rather than optimizing implementations. Running large instances with Louvain-style algorithms was not a bottleneck and these always finished in a few minutes or less. On the bipartite graphs we consider, running our PBCC algorithms typically took a few seconds or a few minutes. Solving the correlation clustering LP relaxation for larger graphs is often very expensive; although this is an active research area [12, 43, 53] and solvers have been produced for around 20,000 node graphs. This leaves us with a theory practice gap between the effective Louvain-based heuristics and more principled approximations that we intend to study in the future.

### 7.1 PBCC on Real Bipartite Graphs

We run our PBCC approximation algorithms on five bipartite graphs constructed from real data[^1], with a range of parameter settings. 

- **The Cities graph** encodes which set of 46 global firms (nodes on side \( V_1 \)) have offices in 55 different major cities (nodes on side \( V_2 \)).
- **Newsgroups100** is made up of a set of 100 documents (\( V_1 \)) and 100 words (\( V_2 \)), with edges indicating which documents use which word.

[^1]: Cities data: [https://www.ihmc.edu/gmc/datasets/cities.html](https://www.ihmc.edu/gmc/datasets/cities.html); Newsgroups data: [www.cs.nyu.edu/~roweis/data/Newsgroups100.zip](http://www.cs.nyu.edu/~roweis/data/Newsgroups100.zip); Zoo data: [https://archive.ics.uci.edu/ml/datasets/zoo](https://archive.ics.uci.edu/ml/datasets/zoo); Amazon graphs constructed from the 5-core Amazon datasets at [https://nijiamm.github.io/amazon/index.html](https://nijiamm.github.io/amazon/index.html).

We have extracted a random subset of 100 documents (25 from each of four document categories: sci, comp, rec, and talk) from a larger dataset, which is often used as a benchmark in hypergraph clustering applications [24, 36, 58].

- The **Zoo** dataset encodes 100 animals and their associations with 15 different binary attributes (e.g., “hair”, “feathers”, “eggs”).
- The last two bipartite graphs are constructed from reviewers on Amazon (\( V_1 \)) that have reviewed products (\( V_2 \)) within certain categories. The **Fashion** category has 404 reviewers and 31 products, and **Appliances** has 44 reviewers for 48 products.

We summarize graph sizes (including edge count) in Table 3.

| Graph       | \( |V_1| \) | \( |V_2| \) | \( |E| \) |
|-------------|---------|---------|-------|
| Cities      | 55      | 46      | 1342  |
| News100     | 100     | 100     | 429   |
| Zoo         | 15      | 100     | 654   |
| Appliances  | 44      | 48      | 180   |
| Fashion     | 404     | 31      | 3040  |

### Figure 3: A posteriori approximation ratios for running our LP-based PBCC algorithms on real-world bipartite graphs, first varying \( \beta \), then varying \( \mu = \mu_1 = \mu_2 \).

We have extracted a random subset of 100 documents (25 from each of four document categories: sci, comp, rec, and talk) from a larger dataset, which is often used as a benchmark in hypergraph clustering applications [24, 36, 58].

- The **Zoo** dataset encodes 100 animals and their associations with 15 different binary attributes (e.g., “hair”, “feathers”, “eggs”).
- The last two bipartite graphs are constructed from reviewers on Amazon (\( V_1 \)) that have reviewed products (\( V_2 \)) within certain categories. The **Fashion** category has 404 reviewers and 31 products, and **Appliances** has 44 reviewers for 48 products.

We summarize graph sizes (including edge count) in Table 3.

| Graph       | \( |V_1| \) | \( |V_2| \) | \( |E| \) |
|-------------|---------|---------|-------|
| Cities      | 55      | 46      | 1342  |
| News100     | 100     | 100     | 429   |
| Zoo         | 15      | 100     | 654   |
| Appliances  | 44      | 48      | 180   |
| Fashion     | 404     | 31      | 3040  |

### Figure 3: A posteriori approximation ratios for running our LP-based PBCC algorithms on real-world bipartite graphs, first varying \( \beta \), then varying \( \mu = \mu_1 = \mu_2 \).

We have extracted a random subset of 100 documents (25 from each of four document categories: sci, comp, rec, and talk) from a larger dataset, which is often used as a benchmark in hypergraph clustering applications [24, 36, 58].

- The **Zoo** dataset encodes 100 animals and their associations with 15 different binary attributes (e.g., “hair”, “feathers”, “eggs”).
- The last two bipartite graphs are constructed from reviewers on Amazon (\( V_1 \)) that have reviewed products (\( V_2 \)) within certain categories. The **Fashion** category has 404 reviewers and 31 products, and **Appliances** has 44 reviewers for 48 products.

We summarize graph sizes (including edge count) in Table 3.
hyperedge splitting function of Li et al. [35], we find clusterings with higher correlation with biological classifications of species in a food web.

It has frequently been observed that triangles are important motifs for identifying community structure in networks [31, 50]. We therefore apply the HyperLam framework to cluster the Email-EU dataset [33, 57] based on triangles. Each edge in the graph (which we treat as undirected) represents an email sent between members of a European research institution. A metadata label indicating each researcher’s department comes with each node. We use three techniques to find clusters at different resolutions in the graph. The first is to apply the Lambda-Louvain [54] method to the graph, which makes greedy local node moves just like the Louvain method [11], but greedily optimizes the LambdaCC framework for different values of $\lambda$, rather than optimizing modularity. Second, we run the Graclus graph clustering algorithm [18], forming different numbers of clusters $k$, to detect communities at different resolutions. Finally, we run HyperLam by applying a clique expansion based on triangle motifs, and cluster the resulting weighted graph with a weighted version of Lambda-Louvain. Since the motif has three nodes, the all-or-nothing cut is the same as the linear penalty, and the clique expansion perfectly models both. After forming multiple clusterings with each method for many parameter values ($k$ or $\lambda$), we measure the Adjusted Rand Index score between each clustering and the known department metadata labels. Scores for each cluster size are displayed in Figure 4a. Although the department labels do not exactly match with community structure in the network, there is a strong correlation between the two, and the higher ARI scores obtained by running HyperLam with the triangle motif indicate that our method is best able to detect this relationship.

We perform a similar experiment on the Florida Bay food web, in which nodes indicate species (e.g., Isopods, Eels, Meroplankton), and directed edges indicating carbon exchange [10, 35]. Following the approach of Li and Milenkovic [35], we consider the bifan motif, in which two nodes $\{v_1, v_2\}$ have uni-directional edges to two other nodes $\{v_3, v_4\}$, and any edge combination within sets $\{v_1, v_2\}$ and $\{v_3, v_4\}$ is allowed. We identify each instance of the motif as a hyperedge. Li and Milenkovic specifically use an inhomogeneous hyperedge cutting penalty, which can be modeled by simply adding undirected edges $(v_1, v_2)$ and $(v_3, v_4)$. Thus, we convert the input graph into a new graph, and cluster with a weighted version of Lambda-Louvain, to optimize the HyperLam objective. Figure 4b demonstrates that clustering based on this motif structure leads to much higher ARI clustering scores with the biological classifications identified by Li et al. [35] (e.g. producers, fish, mammals), than applying Lambda-Louvain or Graclus to the undirected graph.

### 7.3 Clustering Amazon Products Categories

In our last experiment we illustrate differences that arise when applying the HyperLam framework with different hyperedge cut functions. In order to do so, we apply our framework to a hypergraph constructed from Amazon review data, similar to the Fashion and Appliances hypergraphs in the first experiment. This time, we extract nine product categories, associating each product in these categories with a node, and defining a hyperedge to be a set of all products that are reviewed by the same person. This results in a hypergraph with 13,156 nodes, 31,544 hyperedges, with the maximum and mean hyperedge sizes being 219 and 8.1, respectively. Each node is associated with exactly one category label.

As outlined in Section 5, we apply a weighted clique expansion and a star expansion to the Amazon review hypergraph. Recall that the star expansion models the linear hyperedge cut, while the weighted clique expansion penalizes each pair of nodes from a hyperedge that are separated. We scale the graphs so that they share the same total volume, then cluster them both with Lambda-Louvain, using various values of $\lambda$. The hypergraph has a single large connected component, indicating that reviewers do review products across different categories. At the same time, 95% of all hyperedges in the hypergraph are completely contained inside one of the sets of nodes defining a product category. Thus, we expect that clustering the hypergraph based on hyperedge structure will yield clusters that correlate highly with product categories. We confirm this by computing ARI scores between category labels and the clusterings returned by optimizing HyperLam for both graph expansions (Figure 5). We find that the star expansion in particular detects clusterings with an ARI score of nearly 0.6 in the best case.

In order to better understand the structure of clusters formed by our methods, and their relationship with product categories, we measure how well each clustering detects individual product-category node sets in the hypergraph. For each category (e.g., “Appliances”), we measure how well a HyperLam clustering “tracks” that category by taking the best F1 score between any of the HyperLam clusters and the product-category node set in question. For example, if one of the clusters returned by HyperLam exactly matches the “Appliances” node set, then we have perfectly “tracked” this category, and we report an F1 score of 1. Figure 5b illustrates that in general, the star expansion is able to better track the two largest categories, “Prime Pantry” and “Industrial & Scientific”, each of which has roughly 5000 nodes. This helps explain why the star expansion obtains higher ARI scores in general. We note, on the other hand, that the clique expansion tracks the “Software” category (802 nodes) better. This illustrates that using different hyperedge cut functions can lead to noticeably different types of clusters in practice. In the future, we wish to better understand what features
of these expansions enable each approach to detect different types of clusters.

8 DISCUSSION

We have presented a new, flexible, and general framework for parametric clustering of hypergraph and bipartite graph datasets. This framework has deep connections to existing objective functions in the literature and there exist polynomial time approximation results as well as heuristic algorithms. While such frameworks are extremely useful to expert practitioners to engineer and investigate datasets, they are often challenging for less sophisticated users who have a tendency to rely on default parameters. Towards that end, there is a general need for statistical and automated techniques to help guide users to the most successful use of these methods, which is something we hope to design in the future.

Another challenge with the methods involves scaling of the parameters. In our experiments, we often scale these by the volume of the graph (the total sum of edge-weighted degrees) as that has proven to be successful in practice. However, it is unclear if this is the best approach in all circumstances, or whether there are situations in which absolute values of the parameters should be preferred. Finally, as our experiments highlight, there are distinct phase transitions in the behavior among these different regimes; finding ways to identify these characteristic regimes would also make these parametric objectives useful to automatically find characteristically different clusterings.

ACKNOWLEDGMENTS

This research was supported by NSF IIS-1546488, CCF-1909528, NSF Center for Science of Information STC, CCF-0939370, DOE DESC0014543, NASA, the Sloan Foundation, and the Melbourne School of Engineering.

REFERENCES

[1] Sameer Agarwal, Jongwoon Lim, Lihi Zelnik-Manor, Pietro Perona, David Kriegman, and Serge Belongie. Beyond pairwise clustering. In Proceedings of the 2005 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR’05) - Volume 2, Volume 02, CVPR ’05, pages 838–845, Washington, DC, USA, 2005. IEEE Computer Society.

[2] Nir Ailon, Noa Avidgor-Efragbi, Edo Liberty, and Anike van Zaylen. Improved approximation algorithms for bipartite correlation clustering. In Camil Demetrescu and Magnús M. Halldórsson, editors, Algorithms – ESA 2011, pages 25–36, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg.

[3] Nir Ailon, Moses Charikar, and Alanthia Newman. Aggregating inconsistent information: ranking and clustering. Journal of the ACM (JACM), 55(5):23, 2008.

[4] Ilya Ambrus, Nate Veldt, and Austin R. Benson. Hypergraph clustering with categorical edge labels. arXiv preprint arXiv:1910.09943, 2019.

[5] Noga Amit. The bicluster graph edge editing problem. Master’s thesis, Tel Aviv University, 2004.

[6] A Arenas, A Fernández, S Fortunato, and S Gómez. Motif-based communities in complex networks. Journal of Physics A: Mathematical and Theoretical, 41(22):224001, may 2008.

[7] A Arenas, A Fernández, and S Gómez. Analysis of the structure of complex networks at different resolution levels. New Journal of Physics, 10(5):053039, may 2008.

[8] Megathenis Asteris, Anastasios Krykillis, Dimitris Papaioannou, and Alexander Dimakis. Bipartite correlation clustering: Maximizing agreements. In Proceedings of the 19th International Conference on Artificial Intelligence and Statistics (AISTATS), volume 51, 2016.

[9] Nikhil Bansal, Avrim Blum, and Shuchi Chawla. Correlation clustering. Machine Learning, 56:89–113, 2004.

[10] Austin R. Benson, David F. Gleich, and Jure Leskovec. Higher-order organization of complex networks. Science, 353(6295):163–166, 2016.

[11] Vincent D Blondel, Jean-Loup Guillaume, Renaud Lambiotte, and Etienne Lefebvre. Fast unfolding of communities in large networks. Journal of Statistical Mechanics: Theory and Experiment, 2008(10):P10008, oct 2008.

[12] Justin Brickell, Inderjit S. Dhillon, Suvrit Sra, and Joel A. Tropp. The metric nearness problem. SIAM Journal on Matrix Analysis and Applications, 30(1):375–396, 2008.

[13] Ümit V. Catalyurek and Cevdet Aykanat. Hypergraph-partitioning based decomposition for parallel sparse-matrix vector multiplication. IEEE Transactions on Parallel and Distributed Systems, 10(7):673–690, 1999.

[14] Moses Charikar, Venkatesan Guruswami, and Anthony Wirth. Clustering with qualitative information. Journal of Computer and System Sciences, 71(3):360 – 383, 2005. Learning Theory 2003.

[15] Shuchi Chawla, Konstantin Makarychev, Tselil Schramm, and Grigory Yaroslavtsev. Near optimal lp rounding algorithm for correlation clustering on complete and complete l-paritite graphs. In Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing, pages 219–228. ACM, 2015.

[16] J-C. Delvenne, S N. Yaliraki, and M. Barahona. Stability of graph communities across time scales. Proceedings of the National Academy of Sciences, 107(29):12755–12760, 2010.

[17] Erik D. Demaine, Dotan Emanuel, Amos Fiat, and Nicole Immorlica. Correlation clustering in general weighted graphs. Theoretical Computer Science, 361(2):172 – 187, 2006. Approximation and Online Algorithms.

[18] Inderjit S. Dhillon, Yuqiang Guan, and Brian Kulis. Weighted graph cuts without eigenvectors a multilevel approach. IEEE Transactions on Pattern Analysis and Machine Intelligence, 29(11):1934–1957, 2007.

[19] Santo Fortunato and Marc Barthélemy. Resolution limit in community detection. Proceedings of the National Academy of Sciences, 104(1):36–41, 2007.

[20] Takuro Fukunaga. Lp-based pivoting algorithm for higher-order correlation clustering. In Ludvig Wang and Damning Zhu, editors, Computing and Combinatorics, pages 51–62, Cham, 2018. Springer International Publishing.

[21] David F. Gleich, Nate Veldt, and Anthony Wirth. Correlation Clustering Generalized. In 29th International Symposium on Algorithms and Computation, volume 123 of ISAAC 2018, pages 44:1–44:13, Dagstuhl, Germany, 2018. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

[22] J. Gong and Sung Kyu Lim. Multiway partitioning with pairwise movement. In 1998 IEEE/ACM International Conference on Computer-Aided Design. Digest of Technical Papers (IEEE Cat. No.98CB36287), pages 512–516, Nov 1998.

[23] S. W. Hadley, B. L. Mark, and A. Vannelli. An efficient eigenvector approach for finding netlist partitions. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 11(7):885–892, July 1992.

[24] Matthias Hein, Simon Setzer, Leonardo Jost, and Syama Sundar Rangapuram. The total variation on hypergraphs - learning on hypergraphs revisited. In Proceedings of the 26th International Conference on Neural Information Processing Systems - Volume 2, NIPS’13, pages 2427–2435, USA, 2013. Curran Associates Inc.

[25] J. Huang, C. Wang, and H. Chao. A harmonic motif modularity approach for multi-layer network community detection. In 2018 IEEE International Conference on Data Mining (ICDM), pages 1043–1048, Nov 2018.
[26] Edmund Iliner, Dorothea Wagner, and Frank Wagner. Modeling hypergraphs by graphs with the same mincut properties. *Inf. Process. Lett.*, 45(4):171–175, March 1993.

[27] Lucas G. S. Jeub, Marya Bazzi, Inderjit S. Jutla, and Peter J. Mucha. A generalized louvain method for community detection implemented in MATLAB. http://netwiki.amath.unc.edu/GenLouvain, 2011-2017.

[28] Bogumil Kamińska, Valérie Poudin, Pawel Pralat, Przemysław Szulc, and François Theberge. Clustering via hypergraph modularity. *PhD Thesis*, 14(11), 2019.

[29] George Karystos and Bipin Kumar. Multilevel k-way hypergraph partitioning. In *Proceedings of the 36th Annual ACM/IEEE Design Automation Conference*, DAC '99, pages 343–348, New York, NY, USA, 1999. ACM.

[30] Sungwoong Kim, Sebastian Nowozin, Pushmeet Kohli, and Chang D. Yoo. Higher-order clustering for image segmentation. In J. Shawe-Taylor, R. S. Zemel, P. L. Bartlett, F. Pereira, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 24*, pages 1530–1538. Curran Associates, Inc., 2011.

[31] Christine Klymko, David F. Gleich, and Tamara G. Kolda. Using triangles to improve community detection in directed networks. In *The Second ASE International Conference on Big Data Science and Computing, BigDataScience*, 2014.

[32] Tarun Kumar, Sankaran Vaidyanathan, Harini Ananthapadmanabhan, Shrivas Parthasarathy, and Balaraman Ravindran. A new measure of modularity in hypergraphs: Theoretical insights and implications for effective clustering. In Hocine Cherifi, Sabrina Gaito, Jose Fernando Mendes, Esteban Moro, and Luis Mateus Rocha, editors, *Complex Networks and Their Applications VIII*, pages 286–297, Cham, 2020. Springer International Publishing.

[33] Jure Leskovec, Jon Kleinberg, and Christos Faloutsos. Graph evolution: Densification and shrinking diameters. *ACM Transactions on Knowledge Discovery from Data (TKDD)*, 1(1):2, 2007.

[34] Pan Li, H. Dau, Gregory J. Puleo, and Olgica Milenkovic. Motif clustering and overlapping clustering for social network analysis. In *IEEE INFOCOM 2017 - IEEE Conference on Computer Communications*, pages 1–9, May 2017.

[35] Pan Li and Olgica Milenkoiv. Inhomogeneous hypergraph clustering with applications. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, *Advances in Neural Information Processing Systems 30*, pages 2308–2318. Curran Associates, Inc., 2017.

[36] Pan Li and Olgica Milenkoiv. Submodular hypergraphs: p-laplacians, cheeger inequalities and spectral clustering. In *Proceedings of the 35th International Conference on Machine Learning, ICML 2018*, Stockholm, Stockholm, Sweden, July 10-15, 2018, pages 3020–3029, 2018.

[37] Pan Li, Gregory J. Puleo, and Olgica Milenkoiv. Motif and hypergraph correlation clustering. *IEEE Transactions on Information Theory*, pages 1–1, 2019.

[38] Tom Michoel and Bruno Nachtergaele. Alignment and integration of complex networks by hypergraph-based spectral clustering. *Physical Review E*, 86:056111, Nov 2012.

[39] Mark EJ Newman and Michelle Girvan. Finding and evaluating community structure in networks. *Physical Review E*, 69(066133), 2004.

[40] Leto Peel, Daniel B. Larremore, and Aaron Clauset. The ground truth about metadata and community detection in networks. *Science Advances*, 3(5), 2017.

[41] Gregory J. Puleo and Olgica Milenkoiv. Correlation clustering and biclustering with locally bounded errors. *IEEE Transactions on Information Theory*, 64(6):4105–4119, June 2018.

[42] Jing Reichardt and Stefan Bornholdt. Detecting fuzzy community structures in complex networks by hypergraph-based spectral clustering. *Phys. Rev. Lett.*, 93:218701, Nov 2004.

[43] Cameron Ruggles, Nate Veldt, and David Gleich. A parallel projection method for metric constrained optimization. In *Proceedings of the 23rd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 555–564, 2017.

[44] Dengyong Zhou, Jiayuan Huang, and Bernhard Schölkopf. Learning with hypergraphs: Clustering, classification, and embedding. In *Proceedings of the 19th International Conference on Neural Information Processing Systems*, NIPS’06, pages 1601–1608, Cambridge, MA, USA, 2006. MIT Press.

[45] J. Y. Zien, M. D. P. Schlag, and P. K. Chan. Multilevel spectral hypergraph partitioning with arbitrary vertex sizes. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 18(9):1389–1399, Sep. 1999.
Table 2 in the main text summarizes the bounds we compute here. For the second condition, we consider each triangle from Figure 2 in turn, each with its own accompanying figure. In each case, we state the left hand side $L = w_{ij}^+ + w_{jk}^+ + w_{ik}^-$ in terms of $\mu$ and $\beta$, and then bound $c_{ij} + c_{jk} + c_{ik}$ below by some linear function $f(\delta)$. We know then that for each case we must satisfy

$$ \frac{f(\delta)}{L} \geq \frac{1}{\alpha}. $$

Table 2 in the main text throughout that $x_{ij} < \delta$, $x_{jk} < \delta$, and $x_{ik} \geq \delta$.

For any $\beta > 1/2$:

\[ c_{ij} + c_{jk} + c_{ik} = \mu(1 - x_{ij}) + (1 - \beta)x_{jk} + \beta(1 - x_{ik}) \]

\[ = \mu(1 - x_{ij}) + (2\beta - 1)(1 - x_{ij}) + \beta(1 - \delta) \]

\[ \geq \mu(1 - \delta) + (2\beta - 1)(1 - 2\delta) + (1 - \beta)(1 - x_{ik}) \]

\[ \geq \mu(1 - \delta) + (2\beta - 1)(1 - 2\delta) + (1 - \beta)(1 - \delta) \]

\[ = \mu(1 - \delta) + \beta + \delta(1 - 3\beta). \]

For any $\beta$: $c_{ij} + c_{jk} + c_{ik} \geq \mu(1 - \delta) + \beta(1 - \delta)$.

\[ L = w_{ij}^+ + w_{jk}^+ + w_{ik}^- = (1 - \beta). \]

Case (1a)

\[ c_{ij} + c_{jk} + c_{ik} = \mu(1 - x_{ij}) + (1 - \beta)(x_{jk} + x_{ik}) \]

\[ \geq \mu(1 - \delta) + (1 - \beta)\delta. \]

Case (1b)

\[ L = w_{ij}^+ + w_{jk}^+ + w_{ik}^- = 0. \]

The inequality is trivial since left hand side is zero.

\[ c_{ij} + c_{jk} + c_{ik} = \mu(1 - x_{ij}) + \beta(1 - x_{jk} + 1 - x_{ik}) \]

\[ \geq \mu(1 - \delta) + \beta(1 - \delta + 2\delta) \]

\[ = \mu(1 - \delta) + \beta(2 - 3\delta). \]

Case (1d)