Pion Electromagnetic Current in the Light-Cone Formalism *

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ABSTRACT

The electromagnetic form factor of the pion is calculated in a pseudoscalar field theoretical model which constituent quarks. We extract the form factor using the “+” component of the electromagnetic current in the light-cone formalism. For comparison, we also compute the form factor in the covariant framework and we obtain perfect agreement. It is shown that the pair terms do not contribute in this pseudoscalar model. This explains why a naive light-cone calculation, i.e., omitting pair terms from the onset, also yields the same results.

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I. INTRODUCTION

The pion, as quark-antiquark bound state, is an appropriate system to study aspects of QCD in low and intermediate energies regions. In the nonperturbative regime of QCD, the pion has indeed received much attention, e.g. using the constituent quark model in the light-cone formalism [1–3]. In these studies, where the pion is described by light-cone wave functions, the electromagnetic form factor has been calculated for low and high \(q^2\) and a fairly good agreement with experiment has been obtained.

In general, the null-plane is invariant under kinematical transformations (see [4] and references therein). The description in term of wave functions on the null-plane, however, violates covariance [5]. In a recent work [6], the structure of the pion is formulated in terms of the triangle diagram (Fig.1). By means of the Feynman approach on the light-cone the pion form factor and charge radius have been calculated. For the actual extraction of these observables the “+” component of the electromagnetic current has been utilized. In this case, pseudoscalar particles and coupling, full covariance is respected.

In several light-cone studies [7,8] the role of the so-called pair terms, \(i.e.,\) particle–antiparticle pair creation by the photon (Fig.2), has been extensively discussed. In ref. [8], we studied pair terms in an explicit computation of the electromagnetic current of scalar and vector mesons. It was shown that these pair terms are essential for retaining full covariance. Recently [9], we also demonstrated in a boson model the relevance of pair terms for the Ward-Takahashi identity. The latter expresses local gauge invariance for (off-shell) Greens functions.

In this work we calculate the pion form factor in a similar light-cone model as in [6], however carefully addressing the issue of pair terms. We restrict ourselves to the study of on-shell current matrix elements. It is shown that pair term contributions eventually disappear. We argue, however, that this is a coincidence and in general not true. In other words, this cancellation presumably is a property of the pseudoscalar model under consideration.
II. MATRIX ELEMENTS

As in earlier applications \[6\], we use an effective Lagrangian approach with pion and quark degrees of freedom. We choose pseudoscalar coupling:

\[
L_I = -i \frac{m}{f_\pi} \bar{\pi} \gamma^5 \pi q ,
\]

(1)

where \(m\) denotes the constituent quark mass and \(f_\pi\) the pion decay constant. The electromagnetic field is coupled in the usual minimal way, ensuring gauge invariance. The light-cone coordinates are defined as \(k^+ = k^0 + k^z\), \(k^- = k^0 - k^z\), \(k_\perp = (k^x, k^y)\). For the “+” component of the electromagnetic current of the \(\pi^+\), we get the expression corresponding to the Feynman triangle diagram (Fig.1):

\[
J^+ = 2e \frac{m^2}{f_\pi^2} N_c \int \frac{d^4k}{(2\pi)^4} Tr[S(k)\gamma^5 S(k - P')\gamma^+ S(k - P)\gamma^5 \Lambda(k, P')\Lambda(k, P)] ,
\]

(2)

with \(S(p) = \frac{1}{p - m + i\epsilon}\). \(N_c = 3\) is the number of colors and the factor 2 stems from isospin algebra. We will work in the Breit-frame, where \(P^0 = P'^0\) and \(\vec{P}'_\perp = -\vec{P}_\perp = \frac{q}{2}\).

The function \(\Lambda(k, p) = \frac{N}{((p - k)^2 - m_R^2 + i\epsilon)}\) is our choice for regularizing the divergent integral. The normalization constant \(N\) is found by imposing the condition \(F_\pi(0) = 1\) on the pion form factor.

The trace in light-cone coordinates is given by

\[
Tr_\pi = -4k^-(k^+ - P^+)^2 + 4(k_\perp^2 + m^2)(k^+ - 2P^+) + k^+ q^2 .
\]

(3)

Rewriting Eq.(2) in light-cone coordinates yields

\[
J^+ = 2ie \frac{m^2 N_c^2}{f_\pi^2} \int \frac{d^2k_\perp dk^+ dk^-}{2(2\pi)^4} \frac{4k^-(k^+ - P^+)^2 + 4(k_\perp^2 + m^2)(k^+ - 2P^+) + k^+ q^2}{k^+(P^+ - k^+)^2(P'^+ - k^+)^2(k^- - \frac{f_1 - i\epsilon}{k^+})}
\]

\[
\times \frac{1}{(P^- - k^- - \frac{f_2 - i\epsilon}{P' + k^+})(P'^- - k^- - \frac{f_3 - i\epsilon}{P' + k^+})(P'^- - k^- - \frac{f_4 - i\epsilon}{P' + k^+})(P'^- - k^- - \frac{f_5 - i\epsilon}{P' + k^+})} ,
\]

(4)

where \(f_1 = k_\perp^2 + m^2\), \(f_2 = (P - k)_\perp^2 + m^2\), \(f_3 = (P' - k)_\perp^2 + m^2\), \(f_4 = (P - k)_\perp^2 + m_R^2\) and \(f_5 = (P' - k)_\perp^2 + m_R^2\).
According to ref. [8], the “bad terms” \( (Tr^B_π) \) in the matrix element are terms proportional to \( k^- \), the light-cone energy in the loop integral and the “good terms” depend on \( k_\perp \) and \( k^+ \). Therefore we define the traces

\[
Tr^G_π = 4(k^2_\perp + m^2)(k^+ - 2P^+) + k^+ q^2,
\]

\[
Tr^B_π = -4k^-(k^+ - P^+)^2.
\]

For the good terms, only momenta in the interval \( 0 < k^+ < P^+ \) contribute to the Cauchy integration; this means that the spectator particle is on mass-shell and the pole contribution is \( k^- = (k_\perp + m^2)/k^+ \). We also construct the bad terms for the pion current matrix elements:

\[
\Delta^{\pi} = \int \frac{d^2k^\perp dk^-}{2(2\pi)^4} \frac{k^-(k^+ - P^+)^2}{k^+(P^+ - k^-)^2(P^+ - k^- + f^3_{P^+ - k^-}) + k^+(P^+ - k^- + f^3_{P^+ - k^-} + f^5_{P^+ - k^-}) (P^+ - k^- - f^3_{P^+ - k^-} - f^5_{P^+ - k^-})}.
\]

This integral has contributions in two nonzero intervals:

I) \( 0 < k^+ < P^+ \) and

II) \( P^+ < k^+ < P'^+ \), where \( P'^+ = P^+ + \delta \).

Note that in the Breit frame \( P^+ = P'^+ \), which implies that there appear coinciding poles in Eq.(6). As in [3], we have dislocated them by shifting \( P'^+ \) with \( \delta \). The interval (I) corresponds to a spectator particle on mass-shell. The other interval (II) corresponds to a pair term contribution (Fig. 2). Eventually, we take the limit \( \delta \to 0 \), i.e., \( P'^+ = P^+ \), and the exact kinematics of the Breit frame is recovered.

Let us consider interval II, \( P^+ < k^+ < P'^+ \); after integration in \( k^- \) we obtain

\[
\Delta^{II}_π = i \int \frac{d^2k^\perp dk^+}{2(2\pi)^3} \frac{(P'^+ - P^+ + f^3_{P^+ - k^+})}{k^+(P^+ - k^-)^2(P^+ - k^- + f^3_{P^+ - k^-} + f^5_{P^+ - k^-}) + k^+(P^+ - k^- + f^3_{P^+ - k^-} + f^5_{P^+ - k^-} + f^5_{P^+ - k^-})} \theta(P^+ - k^-)\theta(k^+ - P^+).
\]

The limit to the Breit frame is performed, after the momentum fraction is used as integration variable; \( x = (k^+ - P^+)/(P'^+ - P^+) \). In the limit of \( \delta \to 0 \) the integration becomes:
\[ \Delta^{II}_{\pi} = \frac{i \delta}{p^+} \int \frac{d^2k_\perp dx}{2(2\pi)^3} \frac{\theta(x)\theta(1-x)}{(1-x)^2 \left( \frac{f_3}{1-x} + \frac{f_2}{x} \right) \left( \frac{f_3}{1-x} + \frac{f_2}{x} \right)} \to 0 ; \]  

which vanishes linearly with \( \delta \) when the Breit frame is recovered. Thus we see that in this model, pseudoscalar coupling between the constituent fermions and the pseudoscalar pion, the pair terms in the electromagnetic current disappear. Rather than generalizing this result to other models in the light-cone formalism, we consider it to be a coincidence for this particular case. We therefore get contributions of the good terms and only of the interval \( I (\Delta^{I}_{\pi}) \). As a consequence, one obtains agreement between naive light-cone and covariant calculations.

**III. FORM FACTOR**

In general the form factor is extracted from the covariant expression:

\[ J^\mu = e (P^\mu + P'^\mu) F_\pi(q^2) . \]  

We only use the “+” component of the current. The equation for the form factor written in light-cone coordinates for this model is

\[ F_\pi(q^2) = 2te \frac{m^2 N^2}{2P + f_2^2} \int \frac{d^2k_\perp dk^- d\theta^+}{2(2\pi)^3} \frac{-4k^- (k^+ - P^+)^2 + 4(k_\perp + m^2)(k^+ - 2P^+) + k^+ q^2}{k^+ (P^+ - k^+)^2 (P'^+ - k^+)^2 (k^- - f_2 \epsilon_k)} \]  

\[ \theta(p'^+ - k^+) \theta(k^+ - P^+) \]  

\[ \frac{1}{(P^- - k^- - f_2 \epsilon_P - f_2 \epsilon_{P'^+ - k^+})(P'^- - k^- - f_2 \epsilon_{P'^+ - k^+})(P'^+ - k^- - f_2 \epsilon_{P'^+ - k^+})} . \]  

One can verify that only the on-shell pole \( k^- = (k_\perp + m^2)/k^+ \) contributes to the \( k^- \) integration:

\[ F_\pi(q^2) = \frac{m^2 N^2}{P + f_2^2} \int \frac{d^2k_\perp dk^- d\theta^+}{2(2\pi)^3} \frac{-4(k_\perp + m^2)(k^+ - 2P^+) + k^+ q^2}{k^+ (P^+ - k^+)^2 (P'^+ - k^+)^2} \]  

\[ \theta(p'^+ - k^+) \theta(k^+ - P^+) \]  

\[ \frac{1}{(P^- - f_2 \epsilon_k - f_2 \epsilon_{P'^+ - k^+})(P'^- - f_2 \epsilon_{P'^+ - k^+})(P'^+ - f_2 \epsilon_{P'^+ - k^+})} . \]  

In this way the null-plane (light-cone) wave function for the \( \pi \) meson appears.
\[ \Phi_i(x, k_\perp) = \frac{1}{(1 - x)^2 (m_i^2 - M_0^2)(m_i^2 - M_R^2)} . \]  

where \( x = k^+/P^+ \). \( M_R^2 \) is given by the function

\[ M_R^2 = M^2(m^2, m_R^2) = \frac{k_\perp^2 + m^2}{x} + \frac{(P - k_\perp)^2 + m_R^2}{1 - x} - P_\perp^2 . \]  

Recall the free quark-antiquark mass squared: \( M_0^2 = M^2(m^2, m^2) \). The form factor is finally written as:

\[
F_\pi(q^2) = \frac{m_\pi^2}{P_+ f_\pi^2} N_c \int \frac{d^2k_\perp dx}{2(2\pi)^3} \frac{-4(k\cdot P) (x P^+ - P^+)^2 + 4(k^2_\perp + m^2)(x P^+ - 2P^+) + k^+ q^2}{x} \\
\theta(x) \theta(1 - x) \Phi^*_i(x, k_\perp) \Phi_i(x, k_\perp) .
\]  

The remaining integrals are evaluated numerically and the result is presented in Fig.3. The two free parameters in this model, the constituent quark mass \( m_q \) and the regulator mass \( m_R \) were fixed as: \( m_q = 0.220 \text{ GeV}, \ m_R = 0.946 \text{ GeV} \); for the pion mass we take \( m_\pi = 0.140 \text{ GeV} \). This model form factor calculated in the light-cone framework agrees with the one obtained in the covariant formalism (see also Fig.3). In the covariant calculation the energy integral, \( i.e. \), the \( k^0 \) integral, is obtained analytically via Cauchy’s theorem. Again, the remaining part is computed numerically. Furthermore, in Fig.3 we compare the calculated model pion form factors to experimental data [11] and find good agreement.

IV. PION DECAY CONSTANT

The pion decay constant \( f_\pi \) is measured in the weak leptonic decay of the charged pion and appears in the matrix element of the partially conserved axial vector current:

\[ P_\mu < 0 |A_\mu^i|_{\pi_j} > = m_\pi^2 f_\pi \delta_{ij} . \]  

Following ref. [3], we take \( A_\mu^i = \bar{q} \gamma^\mu \gamma^5 \frac{\gamma_\nu}{2} q \) and use the interaction Lagrangian Eq.(1) for the pion-\( \bar{q}q \) vertex function. In this way we obtain
\[ iP^2 f_\pi = \frac{m}{f_\pi} N_c \int \frac{d^4k}{(2\pi)^4} Tr \left\{ P\gamma^5 S(k)\gamma^5 S(k-P)\Lambda(k,P) \right\}. \] (16)

After calculating the trace in the pion center of mass system, this equation is rewritten in light-cone coordinates as

\[ m^2 f_\pi = \frac{m}{f_\pi} N_c \int \frac{d^2k_\perp dk^-}{2(2\pi)^4} \frac{-4m^2}{k^+(P^++k^+)(P^-+k^-+\frac{f_1}{p^+-k^+})(P^-+k^-+\frac{f_2}{p^+-k^+})}. \] (17)

Performing the Cauchy integration in \( k^- \) results in:

\[ f^2_\pi = NN_c \int \frac{d^2k_\perp dk^+}{(2\pi)^3} \frac{2m^2}{k^+(P^++k^+)(P^-+\frac{f_1}{p^+-k^+})(P^-+\frac{f_1}{p^+-k^+})}. \] (18)

In terms of the model light-cone wave function we get as final expression for the weak decay constant \( f_\pi \):

\[ f^2_\pi = N_c \int \frac{d^2k_\perp dx}{(2\pi)^3} \frac{2m^2}{x} \Phi(x,k_\perp). \] (19)

Numerically, this yields \( f_\pi = 101 \text{ MeV} \) to be compared with the experimental value \( f_\pi = 93 \text{ MeV} \). Similar discrepancies were found in refs. [1,6] and appear to be a property of these models.

**V. SUMMARY**

In a pseudoscalar constituent quark model, we calculated the pion form factor in light-cone as well as covariant field theory. The results are in perfect agreement with each other and also describe the experimental form factor well in the \( q^2 \)–range considered. In the light-cone formalism we have explicitly shown that the contribution of the pair terms in the bad matrix elements vanishes. Thus, the remaining terms –good terms and the on mass-shell spectator bad term– exactly yield the covariant result. Since this is a peculiar property of the model under consideration, this does not justify to omit the pair terms in general. It merely explains the success of such a naive light-cone computation in this particular case.
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FIG. 1. Feynman triangle diagram with the corresponding momenta. We use $P$ for the initial state and $P'$ for final state of the pion, $q = P' - P$ is the momentum transfer.

FIG. 2. Pair creation diagram
FIG. 3. Pion form factor as a function of $Q^2 = -q^2$. 