The Chow motive of semismall resolutions

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Abstract

We consider proper, algebraic semismall maps $f$ from a complex algebraic manifold $X$. We show that the topological Decomposition Theorem implies a “motivic” decomposition theorem for the rational algebraic cycles of $X$ and, in the case $X$ is compact, for the Chow motive of $X$. The result is a Chow-theoretic analogue of Borho-MacPherson’s observation [3] concerning the cohomology of the fibers and their relation to the relevant strata for $f$. Under suitable assumptions on the stratification, we prove an explicit version of the motivic decomposition theorem. The assumptions are fulfilled in many cases of interest, e.g. in connection with resolutions of orbifolds and of some configuration spaces. We compute the Chow motives and groups in some of these cases, e.g. the nested Hilbert schemes of points of a surface. In an appendix with T. Mochizuki, we do the same for the parabolic Hilbert scheme of points on a surface. The results above hold for mixed Hodge structures and explain, in some cases, the equality between orbifold Betti/Hodge numbers and ordinary Betti/Hodge numbers for the crepant semismall resolutions in terms of the existence of a natural map of mixed Hodge structures. Most results hold over an algebraically closed field and in the Kähler context.

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1 Introduction

Borho and MacPherson [3] first realized that the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber [2] has a simple and explicit statement for complex proper algebraic semismall maps \( f : X \to Y \) from a manifold. See §2 for definitions and Theorem 2.3.2.

Semismall maps appear naturally in a variety of contexts arising in geometry and representation theory, e.g. the Springer resolutions of nilpotent cones, maps from Gieseker to Uhlenbeck moduli spaces, Hilbert-Chow maps for various Hilbert schemes of points on surfaces, maps from Laumon to Drinfeld spaces of maps to flag varieties, holomorphic symplectic resolutions, resolutions of theta divisors for hyperelliptic curves, etc.

Every stratum for \( f \) on \( Y \) gives rise to the local system given by the monodromy action on the irreducible components of maximal dimension of the fibers over the stratum. Associated with this local system there is the corresponding intersection cohomology complex on the closure of the stratum.

The Decomposition Theorem for \( f \) states that the direct image complex \( Rf_* \mathbb{Q}_X [\dim X] \) is isomorphic to the direct sum over the set of relevant strata of these intersection cohomology complexes.

A striking consequence is that, for these maps, the cohomology of the fiber over a point of a stratum gets decomposed in, so to speak, the easiest possible way: all but its top cohomology is a direct sum of the stalks of the cohomology sheaves of the intersection cohomology complexes of the strata “near” the point; the top cohomology comes, naturally, from the stratum itself.

This statement is, in a sense, surprising. It is as if the geometry of the fiber did not really count, but only the irreducible components of the fiber and their monodromy along the stratum.
It is instructive to check how all the above fails for the non-holomorphic map that contracts to a point the zero section in the total space of the trivial line bundle on an algebraic curve.

In this paper we show, using the Decomposition Theorem, that given a map $f : X \to Y$ as above, the picture for the Chow groups and the Chow motive of $X$ is analogous. Here, we freely use the language and properties of correspondences and projectors even if $X$ is not proper, in view of the refined Gysin formalism; see [11] and [8].

The theory of motives, of various kinds, has been conceived by Grothendieck as a candidate for a universal cohomology theory for nonsingular projective varieties. The so-called “standard conjectures” on algebraic cycles are what is needed to realize the above vision. This paper does not touch upon this conjectural part of the theory of motives. Rather, it proves decomposition results in the well-defined category of Chow motives where one considers nonsingular proper varieties $X, Y$ and the operations induced on their associated objects by algebraic cycles in $X \times Y$. Natural maps such as blowing-ups, projections for projective bundles and finite maps induce direct sum decompositions in this category.

We show that new decompositions arise for semismall maps. They imply analogous ones for Chow groups, singular cohomology and mixed Hodge structures.

Every relevant stratum with the associated local system gives rise to a non trivial correspondence in $X \times X$ which is a projector. This correspondence is constructed with the monodromy datum of the irreducible components of the fibers over the stratum. These projectors are orthogonal to each other. This is the Chow-theoretic and motivic analogue of the fact that each relevant stratum contributes a direct summand in the Decomposition Theorem.

The sum of these projectors is the diagonal of $X$. This corresponds to the fact that the direct summands above add up to the direct image complex $Rf_*\mathbb{Q}_X[\dim X]$.

The precise formulation is given in Theorem 2.4.1 which one may call the Motivic Decomposition Theorem for semismall maps.

In general, even though the formulation of the Decomposition Theorem may be quite explicit, as it is in the case of semismall maps, it may still be difficult in practice to compute the intersection cohomology complexes involved. It is important to know that the decomposition occurs, but it may be difficult to extract information from that fact alone. The geometry of the particular situation is instrumental in making this tool one of the most effective methods of computation in the homology of algebraic maps.

This situation presents itself in the case of algebraic cycles and motives. On the one hand, Theorem 2.4.1 allows to talk about intersection Chow motives and intersection Chow groups in the context of semismall resolutions; see [8]. Let us remark that Chow motives whose realization is an intersection cohomology group have been also defined in a special but highly significant case, that of parabolic cohomology for the $k$-fold product of the universal elliptic curve, in [23] by A. Scholl.

On the other hand, it may be difficult to compute with these objects.

The main result of this paper is that, under suitable hypotheses on the strata, which are verified in many cases of interest, one can recover the algebraic cycles and
the motive of $X$ in terms of the corresponding objects for the strata. The precise statement is given by Theorem 4.0.4, whose proof is independent of Theorem 2.4.1.

A strengthening of this result is given by Theorem 5.0.2, though it applies to more general situations, e.g. we use it to prove Theorem 6.0.7; its formulation involves a projector that in practice may be difficult to compute.

Let us state the following easy consequence of Theorem 4.0.4. It is sufficient for many applications and may give the reader a better idea of the results of this paper and of its computational consequences.

Let $f : X \to Y$ be a semismall map of complex projective varieties with $X$ nonsingular. For every connected relevant stratum $Y_a$, let $t_a := (1/2)(\dim X - \dim Y_a)$, $y_a \in Y_a$ and consider the $\pi_1(Y_a, y_a)$ -- set of irreducible maximal dimensional components of $f^{-1}(y_a)$. Let $\nu_a : Z_a \to Y_a$ the corresponding not necessarily connected covering. Assume that every $Z_a$ has a projective compactification $\overline{Z}_a$ which is the disjoint union of quotients of nonsingular varieties such that the map $\nu_a$ extends to a finite map $\overline{\nu}_a : \overline{Z}_a \to \overline{Y}_a$. We denote the Chow motive of a nonsingular (or quotient) proper variety $T$ by $[T]$, and the $n$-th twist of it by the Lefschetz motive by $[T](n)$; e.g. $[\mathbb{P}^1] \cong [pt] \oplus [pt](1)$.

We construct a correspondence $\Gamma \subseteq \bigsqcup_a Z_a \times X$ and prove

**Theorem 1.0.1** There are natural isomorphisms of Chow motives:

$$\Gamma : \oplus_a [Z_a](t_a) \cong [X].$$

The analogous statement for Chow groups follows easily. The one for Hodge structures, though not new, as it follows from Saito’s [24], is given here a direct proof using correspondences, i.e. without relying on the theory of mixed Hodge modules.

We provide the following applications of Theorem 4.0.4 and Theorem 5.0.2.

For every nonsingular complex surface, we determine the Chow groups, the mixed Hodge structures and the Chow motive of Hilbert schemes (Theorem 7.1.1), of nested Hilbert schemes (Theorem 7.2.1) and, in an appendix with T. Mochizuki (§8), of parabolic Hilbert schemes of points. All these results hold over any field and in the analytic context. The former is not new; see [6] and also [14]. The second refines Göttsche’s result [14]. The third is new.

Our results allow to verify that for some semismall and crepant resolutions of orbifolds, the mixed Hodge structure of the resolution is canonically isomorphic to the corresponding orbifold structure. See [8]; This refines the known fact that one had an equality for the corresponding Hodge numbers. See [10] and [27].

Another interesting case, which we plan to consider in the near future, in which the methods of this paper seem to be relevant, is when a group acts by symplectic reflections on a nonsingular quasi projective variety and the quotient is given a symplectic resolution, cfr. Kaledin [18], Verbitski [26].

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2 A motivic decomposition theorem for semismall maps

We work with algebraic varieties over the field of complex numbers and consider Chow groups with rational coefficients. The results of this paper hold, with minor changes, left to the reader, over an algebraically closed field $K$ and in the complex analytic setting when the map $f : X \to Y$ is Kähler, for the Decomposition Theorem holds for $f$ in those contexts. See [2] and [24]; see also [9] for a necessary and sufficient condition for the Decomposition Theorem to hold for $f$ in the analytic setting. Over the field $K$ one must work with $\mathbb{Q}_l$-coefficients, $l \neq \text{char} K$, and with étale homology (cfr. [17]). However, with the exception of Theorem 2.4.1, the relevant cycles have $\mathbb{Q}$-coefficients and the results for cycles and motives remain valid as stated. The results for (parabolic) Hilbert schemes hold over any field $F$ and in the complex analytic setting.

2.1 Fibred products of semismall maps

Throughout this section, $f : X \to Y$ is a proper surjective and semismall morphism of complex algebraic varieties with $n := \dim X$.

Recall that, in this context, the semismallness of $f$ is equivalent to the following condition: let $\delta \in \mathbb{N}$ and $Y_\delta := \{ y \in Y \mid \dim f^{-1}(y) = \delta \}$, then $2\delta \leq \dim Y - \dim Y_\delta$, $\forall \delta \geq 0$.

On the other hand, the smallness of $f$ is equivalent to requesting that $2\delta < \dim Y - \dim Y_\delta$, $\forall \delta > 0$.

By checking the condition on $\delta > 0$, one sees immediately that $f$ must be generically finite so that $\dim X = \dim Y = n$.

The following elementary fact plays an important role in what follows.

Proposition 2.1.1 Let $f' : X' \to Y$ be a proper surjective and semismall map. Then

$$\dim X' \times_Y X = n.$$  

Proof. Let $\delta, \delta' \in \mathbb{N}$ and define the locally closed subspaces $Y_{f'}^{\delta\delta'} := Y_\delta \cap Y_{\delta'}$. We have that $Y = \bigsqcup_{\delta \geq 0, \delta' \geq 0} Y_{f'}^{\delta\delta'}$. Note that

$$X' \times_Y X = \bigsqcup_{\delta \geq 0, \delta' \geq 0} f'^{-1}(Y_{f'}^{\delta\delta'}) \times_{Y_{f'}} f^{-1}(Y_{f'}^{\delta\delta'}).$$

where all the subspaces on the right hand side are locally closed in the left hand side.

We have $\dim f'^{-1}(Y_{f'}^{\delta\delta'}) \times_{Y_{f'}} f^{-1}(Y_{f'}^{\delta\delta'}) = \delta + \delta' + \dim Y_{f'}^{\delta\delta'} \leq n - \frac{\dim Y_{f'}^{\delta\delta'}}{2} + \dim Y_{f'}^{\delta\delta'}$.

Remark 2.1.2 A map $f : X \to Y$ is semismall if and only if the irreducible components of $X \times_Y X$ have dimension at most equal to $\dim X$. Similarly, a map $f : X \to Y$ is small if and only if the $n$-dimensional irreducible components of $X \times_Y X$ dominate $Y$.  

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Remark 2.1.3 An argument analogous to the proof of Proposition 2.1.1 shows that if we merely assume that $f$ and $f'$ are small over their image and $\dim X = \dim X'$, then $\dim X \times_Y X' \leq \dim X = \dim X'$ and equality holds if and only if $f(X) = f'(X')$.

There exists a stratification for $f : X \to Y$. By this we mean that $Y = \bigsqcup_{b \in B} Y_b$, where every space $Y_b$ is a locally closed and smooth subvariety of $Y$ and the induced maps $f_i : f^{-1}(Y_b) \to Y_b$ are locally topologically trivial over $Y_b$. See [12].

Definition 2.1.4 Let $A := \{a \in B \mid 2 \dim f^{-1}(y) = \dim Y - \dim Y_a, \forall y \in Y_a\}$. We call $A$ the set of relevant strata for $f$.

Remark 2.1.5 While there may be many different stratifications for $f$, the set of subvarieties $\{Y_a\}_{a \in A}$ is uniquely determined by $f$.

Remark 2.1.6 A small map has only one relevant stratum, i.e. the dense one.

2.2 The intersection form on the local system associated with a relevant stratum

In this section we assume that $X$ is a quotient variety, i.e. that $X \simeq X'/G$ is the quotient of a nonsingular variety $X'$ by the action of a finite group $G$ of automorphisms of $X'$.

The goal is to define certain intersection numbers which can be defined when $X$ is nonsingular and therefore also when $X$ is a quotient variety. For more details on what follows see [3] and [2].

Let $S$ be a relevant stratum for $f$ and $s \in S$. We have $2 \dim f^{-1}(s) = n - \dim S$. Let $N$ be a contractible euclidean neighborhood of $s$ and $N' = f^{-1}(N)$. Let $\{F_i\}$ be the set of irreducible components of maximal dimension $(1/2)(n + \dim S)$ of $f^{-1}(S \cap N)$, and let $\{f_i\}$ be the set of irreducible components of maximal dimension $(1/2)(n - \dim S)$ of $f^{-1}(s)$. We denote by $f_i \cdot F_j$ the rational valued refined intersection product defined in $N'$; see [1]. It is independent of the choice of $s \in S \cap N$ and is a monodromy invariant, i.e. it is $\pi(S,s)$-invariant.

Remark 2.2.1 The integer-valued intersection form is defined similarly over any algebraically closed field, provided an étale neighborhood is used to trivialize the monodromy of the components of the fibre.

We denote Borel-Moore homology (cfr. [3] for example) with rational coefficients by $H^{BM}(-)$. As $s$ varies on $S$, the Borel-Moore classes $[f_i] \in H^{BM}_{n-\dim S}(f^{-1}(s))$ form a local system $L_S^*$ and the intersection pairing above forms a pairing $L_S^* \otimes L_S^* \to \mathbb{Q}_S$ on this local system. Since the monodromy is finite, $L_S^*$ is abstractly, i.e. not necessarily via the pairing above, isomorphic to its dual which can be identified with $L_S := (R^{n-\dim S} f_* \mathbb{Q}_X)|_{S}$. 

\[6\]
2.3 The Decomposition Theorem and homological correspondences

In the case of constant coefficients, the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber has a simple statement for semismall maps from a quotient variety. For a different proof for projective semismall maps from projective complex manifolds see [9].

Given a variety \( Z \) and a local system \( L \) on a Zariski-dense open subset of its regular part \( Z_{\text{reg}} \), we denote by \( IC_Z(L) \) the Goresky, MacPherson, Deligne intersection cohomology complex associated with \( L \). We denote \( IC_Z(\mathbb{Q}Z_{\text{reg}}) \) simply by \( IC_Z \).

Remark 2.3.1 If \( Z \) is a quotient variety, then \( IC_Z \simeq \mathbb{Q}[\dim Z] \).

The following is a special case of the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber. See [2], [24], [3], [9].

Theorem 2.3.2 Let \( f : X \to Y \) be a proper, semismall algebraic map of algebraic varieties with \( X \) a quotient variety. There is a canonical isomorphism

\[
Rf_* \mathbb{Q}_X[n] \simeq D_{\text{cc}}(Y) \bigoplus_{a \in A} IC_{Y_a}(L_a)
\]

where \( L_a \) are the semisimple local systems on \( Y_a \) given by the monodromy action on the maximal dimensional irreducible components of the fibers of \( f \) over \( Y_a \), and \( D_{\text{cc}}(Y) \) is the bounded derived category of constructible complexes of sheaves of rational vector spaces on \( Y \).

Lemma 2.3.3 Let \( g' : U' \to V \), \( g : U \to V \), \( g'' : U'' \to V \) be three proper maps of analytic varieties with \( U' \), \( U \) and \( U'' \) quotients of smooth varieties by the actions of finite groups, and \( i, j \) and \( k \) be integers. There are canonical isomorphisms

\[
\varphi_{U'U} : Hom_{D_{\text{cc}}(V)}(Rg'_U^*[i], Rg_U^*[j]) \simeq H_{2\dim V-i-j}(U' \times_V U).
\]

Given morphisms \( u : Rg'_U^*[i] \to Rg_U^*[j] \), \( v : Rg_U^*[j] \to Rg''_U^*[k] \), one has \( \varphi_{U'U''}(v \circ u) = \varphi_{U''U}(v) \bullet \varphi_{UU}(u) \), where \( \bullet \) denotes the refined composition of homological correspondences.

Proof. [6], Lemma 2.21, Lemma 2.23. \( \square \)

2.4 A motivic decomposition theorem for semismall maps

Let \( X \) be a nonsingular algebraic variety of dimension \( n \). Given an algebraic variety \( X' \), a cycle \( \gamma \in Z_i(X \times X') \), whose support is proper over \( X' \), defines, via the refined Gysin formalism, a correspondence and a graded map of cycles \( \gamma_* : A_*(X) \to A_{*+i-n}(X') \).

This calculus is compatible with composition, provided the appropriate “properness” conditions are imposed. In addition, one may merely assume that \( X \) is a quotient variety by a finite group. See [11], [8].

We employ the standard notation for Chow motives; see [19] and [11]. Later we shall switch to a more compact one.

The following is a formal consequence of Theorem 2.3.2 and Lemma 2.3.3.
Theorem 2.4.1 Let \( f : X \to Y \) be a proper, semismall, algebraic map of complex algebraic varieties with \( X \) a quotient variety and \( A \) be the set of relevant strata for \( f \). There exist algebraic cycles with \( \mathbb{Q} \)-coefficients \( C_a \in Z_n(X \times_Y X), a \in A \), such that

i) \( C_a \neq 0, \forall a \in A \),

ii) \( \sum_a C_a = \Delta_X \), in \( Z_n(X \times_Y X) \),

iii) \( C_a \circ C_a' = \delta_{aa'}C_a \), as refined correspondences.

In particular, there are canonical isomorphisms of Chow groups

\[
A_*(X) = \bigoplus_{a \in A} C_a(A(X))
\]

and, when \( X \) is proper, of Chow motives:

\[
(X, \Delta_X) \simeq \bigoplus_{a \in A} (X, C_a).
\]

The cycles \( C_a \) are supported on \( f^{-1}(V_a) \times_{V_a} f^{-1}(V_a) \).

Proof. By Proposition 2.1.1, the class map \( cl : Z_n(X \times_Y X) \to H_{2n}^BM(X \times_Y X) \) is an isomorphism. By Lemma 2.3.3 we get isomorphisms of \( \mathbb{Q} \)-algebras:

\[
\text{End}_{D^b_c(Y)}(Rf_*\mathbb{Q}_X \langle n \rangle) \xrightarrow{\varphi} H_{2n}^BM(X \times_Y X) \xrightarrow{cl^{-1}} Z_n(X \times_Y X),
\]

where the algebra structures are given by composition in \( D^b_c(Y) \) for the first term and by the topological and algebraic/analytic Gysin formalism of correspondences (which are compatible via the class map) for the second and third.

Let \( \phi := cl^{-1} \circ \varphi \).

By Theorem 2.3.2 we have, \( \forall a \in A \), canonical maps \( p_a : Rf_*\mathbb{Q}_X \langle n \rangle \to IC_{V_a}(L_a) \) considered as elements of \( \text{End}_{D^b_c(Y)}(Rf_*\mathbb{Q}_X \langle n \rangle) \).

Define \( C_a := \phi(p_a), \forall a \in A \).

The three properties i), ii) and iii) for the \( C_a \) follow from the analogous obvious properties of the \( p_a \)'s and by the compatibility of \( \varphi \) with compositions stated in Lemma 2.3.3.

The isomorphisms of Chow groups and Chow motives follow formally.

The statement on the supports follows from the fact that the isomorphism of Lemma 2.3.3 is compatible with restricting to open subsets of \( Y \). If we take as an open set the complement of \( V_a \), then \( p_a \) restricts to the zero map so that \( C_a \) restricts to the zero cycle. \( \square \)

2.5 (Symmetric) products of semismall maps

Let \( f : X \to Y \) be semismall with \( X \) a quotient variety. It is easy to check that the induced maps \( f^l : X^l \to Y^l \), \( f^{(l)} : X^{(l)} \to Y^{(l)} \) and \( f^{(v)} : X^{(v)} \to Y^{(v)} \) are semismall.

Let \( A \) be the set of relevant strata for \( f \). Then the sets \( A^l, A^{(l)} \) and \( A^{(v)} \) are naturally labeling the relevant strata for \( f^l, f^{(l)} \) and \( f^{(v)} \).

One has \( Rf_*\mathbb{Q}_X \langle n \rangle \simeq \bigoplus_{a \in A} IC_{V_a}(L_a) \) and \( Rf_*\mathbb{Q}_X \langle m \rangle \simeq \bigoplus_{a \in A} IC_{V_a}(L_a) \), etc.

Let \( P \in Z_n(X \times X) \) have support that maps properly onto either factor. Note that we are not assuming that \( X \) is proper, thus the assumption on the support. Let \( P^2 = P \) as a refined correspondence, i.e. \( P \) is a projector on \( X \). It induces projectors \( P^l, P^{(l)} \) and \( P^{(v)} \) on \( X^l, X^{(l)} \) and \( X^{(v)} \). See [19], §2.
If $P_a$ denotes the projector in $Z_n(X \times_Y X)$ corresponding to $IC_{\mathbb{T}_a}(L_a)$, then $P_a$ corresponds to $IC_{\mathbb{T}_a}(L_a)$. Similarly, for $f^{(i)}$ and for $f^{(v)}$.

We leave to the reader the task of making explicit, using the set of indices $A^i$, $A^{(l)}$ and $A^{(v)}$ the analogues of Theorem 2.4.1, 4.0.4 and 5.0.2 for the maps $f^i$, $f^{(l)}$ and $f^{(v)}$.

3 Intersection Chow motives and groups in the presence of semismall resolutions

What follows is a generalization of the well known construction of Mumford [22], which we now recall: Let $X$ be a normal surface with a singular point $x$ and let $r : \tilde{X} \to X$ be a resolution. Denote by $E_i$ the exceptional divisors in $r^{-1}(x)$ and by $A$ the inverse of the intersection matrix $(E_i, E_j)$. Given a curve $C$ on $X$ one has a unique lifting to a curve $\tilde{C}$ on $\tilde{X}$ with the condition that $\tilde{C} \cdot E_i = 0$ for all exceptional curves $E_i$. In other words, one is considering the cycle $P = \Delta_{\tilde{X}} - \sum \Lambda_{i,j} E_i \times E_j$ in $\tilde{X} \times_X \tilde{X}$, which is easily seen to be a projector. A direct computation shows that the Chow motive $(X, P)$ has the intersection cohomology of $X$ as its Betti realization. In a completely analogous way, whenever a singular variety $Y$ is given a semismall resolution $f : X \to Y$, it is possible to construct a projector $P \in A(X \times_Y X)$ whose Betti realization is the intersection cohomology of $Y$. This construction is meaningful because the Chow motive turns out to be independent of the semismall resolution chosen, as we prove in Proposition 3.0.1.

One can therefore talk about intersection Chow motives and groups, in the presence of semismall resolutions.

Let $f : X \to Y \leftarrow X' : f'$ be two proper surjective semismall maps, with $X$ and $X'$ $n$-dimensional proper nonsingular algebraic varieties. Assume that there exists a Zariski-dense open subet $U$ of $Y$ and an isomorphism $g : f^{-1}(U) \to f'^{-1}(U)$ such that $f|_U = f' \circ g$ and that $f^{-1}(U) \to U \leftarrow f'^{-1}(U)$ are topological coverings with associated isomorphic local systems $L_f$ and $L_{f'}$.

The projections $p_f : Rf_*\mathbb{Q}_X[n] \to IC_Y(L_f) = IC_Y(L_{f'}) \leftarrow Rf'_*\mathbb{Q}_{X'}[n] : p_{f'}$ induce, as in the proof of Theorem 2.4.1, algebraic cycles $C_f \in Z_n(X \times_Y X)$ and $C_{f'} \in Z_n(X' \times_Y X')$ which give Chow motives $(X, C_f)$ and $(X', C_{f'})$, direct summands of $(X, \Delta_X)$ and $(X', \Delta'_{X'})$, respectively.

**Proposition 3.0.1** There is a canonical isomorphism of Chow motives

$$C_{f'f} : (X', D_{f'}) \to (X, D_f)$$

defined by an algebraic cycle $C_{f'f} \in Z_n(X' \times_Y X)$.

**Proof.** Let $p_{f'f} : Rf'_*\mathbb{Q}_{X'}[n] \to Rf_*\mathbb{Q}_X[n]$ be the map obtained as the composition of the natural maps: $Rf'_*\mathbb{Q}_{X'}[n] \to IC_Y(L_{f'}) = IC_Y(L_f) \to Rf_*\mathbb{Q}_X[n]$. Consider the isomorphism of $\mathbb{Q}$-vector spaces $\psi_{X'/X} : \text{Hom}_{\overline{\text{DM}}_{\mathbb{Q}}(Y)}(Rf_*\mathbb{Q}_X[n], Rf_*\mathbb{Q}_X[n]) \simeq H^B_{\text{BM}}(X' \times_Y X) \simeq Z_n(X' \times_Y X)$ (see Lemma 2.3.3 and Proposition 2.1.1) and define $C_{f'f} := \psi(p_{f'f})$.

Analogously, define $p_{ff'}$ and $C_{ff'}$. 9
Clearly, $p_{ff'} \circ p_{f'} = p_f'$ and $p_{f'} \circ p_{ff'} = p_f$. Since $\psi$ is compatible with compositions, see Lemma 2.3.3, the same equations hold for the corresponding cycles, i.e. $D_{ff'} \circ D_{f'} = D_f'$ and $D_{f'} \circ D_{ff'} = D_f$. The result follows by noting that $D_{f'}$ and $D_f$ induce the identity on the motives $(X, D_{f'})$ and $(X, D_f)$, respectively. \qed

Remark 3.0.2 Any two small resolutions of an algebraic variety have isomorphic motives, cycles and mixed Hodge structures. This can be shown directly using the graph of the corresponding birational map.

Remark 3.0.3 If $X \rightarrow Y \leftarrow X'$ are two semismall resolutions (in general there is none), then $(X, D_f)$ and $(X', D_{f'})$ are both motivic counterparts of the topological $IC_Y$. Proposition 3.0.1 states that they are canonically isomorphic and therefore independent of the semismall resolution.

4 An explicit motivic decomposition theorem

The purpose of this section is to prove Theorem 4.0.4 which, though less general than Theorem 5.0.2, is quite explicit and affords many applications.

From now on we denote the Chow motive of a nonsingular (or quotient) proper variety $T$ by $[T]$ and $[T](n)$ will denote the $n$-th twist by the Lefschetz motive.

The set-up is as follows. Let $f : X \rightarrow Y$ be a proper, surjective and semismall map of complex algebraic varieties with $X$ a quotient variety and $A$ be the set of relevant strata for $f$. Let $a \in A, Y_a$ be the corresponding relevant stratum, $t_a = (1/2)(\dim X - \dim Y_a), y_a \in Y_a, E_a$ be the $\pi_1(Y_a, y_a)$-set given by the irreducible components of maximal dimension of $f^{-1}(y_a)$ and $\nu_a : Z_a = \coprod_i Z_{a,i} \rightarrow Y_a$ be the not necessarily connected étale covering associated with $E_a$. For every $a, i$, let $Z_{a,i}$ be a complex algebraic variety containing $Z_{a,i}$ as a Zariski-dense open subset and such that there is a proper surjective map $\nu_{a,i} : Z_{a,i} \rightarrow Y_a$ extending $\nu_a := \nu_a|_{Z_{a,i}}$, i.e. such that $\nu_{a,i}|_{Z_{a,i}} = \nu_{a,i}$.

Theorem 4.0.4 Assume that, for every $a, i$, $\overline{Z_{a,i}}$ is a quotient variety and $\nu_{a,i}$ is small. Then there exists a correspondence $\Gamma \in Z_*(\coprod_a \overline{Z_a}) \times X)$ inducing isomorphisms of Chow groups

$$\Gamma_* = \bigoplus_{a \in A} A_*(\overline{Z_a}) \rightarrow A_*(X) \quad \text{(with $A_*(\overline{Z_a}) \rightarrow A_{*+t_a}(X)$),}$$

of mixed Hodge structures

$$\Gamma_*^H = \bigoplus_{a \in A} H^*(\overline{Z_a})(t_a) \rightarrow H^*(X)$$

and, if $X$ is proper, of Chow motives.

$$\Gamma_* = \bigoplus_{a \in A} [\overline{Z_a}] (t_a) \rightarrow [X].$$
We shall prove the theorem in this section by constructing $\Gamma$ and its inverse explicitly.

**Remark 4.0.5** Note that if $g : U \to V$ is a small proper map of analytic varieties, then $Rf_!IC_U \simeq IC_V(L)$, where $L$ is the local system given by the monodromy action on the points of a general fiber of $f$.

**Remark 4.0.6** The assumptions on the varieties $Z_{a,i}$ and the maps $\nu_{a,i}$ imply, by 2.3.1 and 4.0.5, that the Decomposition Theorem 2.3.2 reads as follows:

$$Rf_!Q_X[n] \simeq \bigoplus_{a \in A} IC_{\mathbb{Z}_a}[\dim Y_a].$$

### 4.1 Construction of correspondences

We now introduce the correspondences that define the isomorphism of motives. We shall construct a set of correspondences $\Gamma_{a,i,I}$ associated with a relevant stratum $Y_a$. Their closures will define the wanted isomorphisms.

In order to simplify the notation, we temporarily work on one stratum at the time. Let $S$ be a relevant stratum, $s \in S$, $E$ be the right $G := \pi_1(S,s)$-set of maximal dimensional irreducible components of $f^{-1}(s)$. Denote by $\nu_E : S_E \to S$ the not necessarily connected étale cover of $S$ corresponding to $E$.

By assumption, $S_E$ is a Zariski-dense open subset of a disjoint union of global quotient varieties $S_{E_i}$, and $\nu_E : S_E \to S$ extends to a small map $\overline{\nu}_E : \overline{S_E} \to \overline{S} \subseteq Y$. Note that $S_E$ is of pure dimension $\dim S$. The correspondences we will use are supported on the closures in $S_E \times_Y X$ of $S_E \times_Y X$. We introduce notation in order to deal with their top-dimensional irreducible components.

From now on we use the term “component” to refer to a top-dimensional one. This should create no confusion.

Fix representatives $o_1, \ldots, o_r$ of the $G$-orbits of $E$; this corresponds to fixing base points $s_1, \ldots, s_r$ in the various connected components $S_1, \ldots, S_r$ of $S_E$. Let $G_i$ be the stabilizer of $o_i$. The following is a well-known and elementary fact on $G$-sets:

**Lemma 4.1.1** The irreducible components of $S_i \times_Y X$ are in 1-1 correspondence with the $G_i$ orbits of $E$.

It follows that in order to specify an irreducible component $\Gamma_{i,o_i}$ of $S_i \times_Y X$ it suffices to specify a $G_i$-orbit $I = \{f_{i_1}, \ldots, f_{i_l}\}$ of components of $f^{-1}(s)$.

Let $\Gamma_i$ denote the correspondence $\Gamma_{i,o_i}$ associated with the $G_i$ orbit $\{o_i\}$, $i = 1, \ldots, r$.

Note that the fiber over $s_i$ of the natural map $\Gamma_i \to S_i$ is naturally identified with the corresponding irreducible component of $f^{-1}(s)$ and that $\Gamma_i$ is an irreducible variety. Let $\Lambda = (\Lambda_{i,j})_{i,j \in E}$ be the inverse of the intersection matrix associated with the pair $(S,s)$. Clearly, $\Lambda_{o_i,o_j} = \Lambda_{i,j}$, $\forall g \in G$, for the intersection numbers are monodromy invariant. From this follows that, for every $i \in E$, $\sum_j \Lambda_{i,j} f_j$ is a rational linear combination of $G_i$-orbits in the set of maximal dimensional components of $f^{-1}(s)$, and thus defines, by 4.1.1, a correspondence $\Gamma'_i \in Z(S_i \times_Y X)$. Note that in general its support is not irreducible.
Let us summarize what we have done. In general, the pre-image of the stratum $S$ is not irreducible. In addition, each irreducible component may fail to be irreducible over small euclidean neighborhoods of $s \in S$. Each irreducible component of the pre-image of a stratum corresponds to a connected étale cover $S_i$ and the pull-back $\Gamma_i$ of this irreducible component to $S_i$ is irreducible locally in the euclidean topology over $S_i$ and in fact has irreducible fibers over $S_i$. The intersection forms being non-degenerate allows to define $\Gamma'_i$.

We need the following simple

**Lemma 4.1.2** Let $T \in Z_{\dim S}(S_i \times_S S_j)$ be a correspondence, $p_i : S_i \to S \leftarrow S_j : p_j$ be the natural projections. Assume that $T_* : A_0(\{s_i\}) \longrightarrow A_0(p_j(p_i^{-1}(s_i)))$ is the zero map. Then $T = 0$.

**Proof.** Let $E_i$ and $E_j$ be the corresponding $\pi_1(S, s)$-sets. The nonsingular space $S_i \times_S S_j$ is of pure dimension $\dim S$ and has one connected component for every $G_{ij}$-orbit inside $E_j$. Note that $T$ is a rational linear combination of these components, $T = \sum q_i T_i$, and that it defines a map $T_*$ as in the statement purely by set theory, i.e. without recourse to the refined Gysin formalism. We have that $T_*(s_i) = \sum q_i (T_i \cap p_i^{-1}(s_i))$. Since the $G_{ij}$-orbits in $E_2$ are disjoint, the vanishing of the left hand side implies that $r_i = 0$, $\forall i$. The statement follows.

**Lemma 4.1.3** $\iota_{\Gamma i} \circ \Gamma_i = \delta_{ij} \Delta_{s_i}$.

**Proof.** The composition of the two correspondences is represented by a cycle of dimension equal to $\dim S$ supported on $S_i \times_Y S_j$. By Lemma 1.1.2, it is enough to prove that $(\iota_{\Gamma i} \circ \Gamma_i)_*(s_i) = \delta_{ij}(s_i)$. With the notation introduced in §2.2 $\Gamma_i(s_i) = f_i$ and $\iota_{\Gamma i}(f_i) = (\sum A_{i,k} F_k \cdot f_i)(s_i) = \delta_{ij} s_i$.

In what follows we need to emphasize the stratum with which the correspondences are associated so we will revert to the complete notation: we have $Z_a = \bigsqcup_a Z_{a,i}, \Gamma_{a,i}$ and $\Gamma'_{a,i}$.

Define $\Gamma_{a,i}$ to be the closure of $\Gamma_{a,i} \subseteq Z_{a,i} \times_Y X$ in $\overline{Z_{a,i}} \times_Y X$. Similarly for $\Gamma'_{a,i}$.

Let $\Gamma_a := \bigsqcup_a \Gamma_{a,i}, \Gamma := \bigsqcup_a \Gamma_a, \overline{\Gamma_a} := \bigsqcup_a \overline{\Gamma_a}, \Gamma' := \bigsqcup_a \Gamma'_{a,i}$.

**Lemma 4.1.4** $\iota_{\overline{\Gamma_{b,j}} \circ \Gamma_{a,i}} = \delta_{ab} \delta_{ij} \Delta_{Z_{a,i} \times Z_{a,i}}$.

**Proof.** Let $a = b$. For reasons of dimension of supports, see a similar argument in §8, the composition of the two correspondences is represented by a cycle of dimension equal to $\dim S$ supported on $\overline{S_i} \times_Y \overline{S_j}$. By the smallness assumption, the only such cycles are linear combinations of the maximal dimensional irreducible components of the fiber product, each of which is dominant over both factors; see Remark 2.1.2. We conclude by an argument identical to the one in the proof of Lemma 1.1.3.

Let $a \neq b$. The composition is represented by a cycle of dimension $(1/2)(\dim Z_a + \dim Z_b)$ supported on $\overline{Z_{a,i}} \times_Y \overline{Z_{b,j}}$ which, by an argument analogous to the proof of Lemma 2.1.3, has dimension at most equal to $\min\{\dim Z_{a,i}, \dim Z_{b,j}\}$. If $\dim Z_a \neq \dim Z_b$, then the composition is zero because it is supported on too small a set. If $\dim Z_a = \dim Z_b$, but $a \neq b$, then Remark 21.3 shows again that the support is too small and therefore the composition is trivial.

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4.2 End of proof of Theorem 4.0.4.

By Lemma 2.1.4 we see that \( \Gamma \circ \Gamma = \Delta \bigoplus a, i \gamma_a, i \). By Remark 4.0.6, it follows formally that the map induced on \( Rf_*\mathbb{Q}_X[n] \) by the cycle \( \Gamma \circ \Gamma \), via Lemma 2.3.3, is the identity.

The cycle \( \Gamma \circ \Gamma \) is supported on the \( \dim X \)-part of \( X \times Y \), so that, by Lemma 2.3.3, this cycle is the diagonal \( \Delta_X \).

\[\square\]

Remark 4.2.1. The projectors appearing in Theorem 2.4.4) satisfy \( C_a = \Gamma_\gamma \circ \Gamma_\gamma \). See also Remark 5.0.3.

5 A strengthening of Theorem 4.0.4

We place ourselves in the same situation as in §4, but we shall allow the maps \( \nu_a, i \) to be semismall. We can prove a stronger result at the price of not being able to write down the correspondence in an explicit form due to the presence of the projectors stemming from the semismall maps not being small.

Let \( a \in A \), \( L_a \) be the associated local system. Note that if \( L_a, a, i := v_a, i \otimes \mathbb{Q}_Z, a, i \), then \( L_a = \bigoplus a, i \).

If we assume that the maps \( \nu_a, i \) are semismall, then, corresponding to the canonical projection \( R\nu_a, i \otimes \mathbb{Q}_Z \otimes \mathbb{Q}_{Y_a} \to IC_{Y_a, (L_a, a)} \), we get, via Theorem 2.3.2 and Lemma 2.3.3 projectors \( P_{a, i} \in Z_{\dim Y_a}(Z_{a, i} \otimes Y Z_{a, i}) \). Let \( \mathcal{P} = \bigoplus a, i \) and \( P = \bigoplus a, i \).

Note that, as a refined correspondence, \( P \) is a projector in \( Z(Y_a, Z_{a, i} \otimes Y Z_{a, i}) \) with trivial components in \( Z(\mathbb{Z}_{a, i} \otimes Y \mathbb{Z}_{a, i}) \) whenever \( a \neq b \) and/or \( i \neq j \). Similarly for \( \mathcal{P} \).

Theorem 5.0.2 Assume that the maps \( \nu_a, i \) are semismall. Then

\[ P \circ \Gamma \circ \Gamma \circ \Gamma \circ P = P \]

and

\[ \Gamma \circ \Gamma \circ \Gamma \circ \Gamma = \Delta_X. \]

Proof. Let \( p, p_a, \gamma_a, \gamma_a, i, \gamma_a, \gamma_a, i, \gamma_a, \gamma_a, i, \gamma_a, \gamma_a, i \) be the maps in \( D_{Y_a}^\bullet(Y) \) corresponding, via Lemma 2.3.3, to the correspondences labeled by the semismall maps.

By Lemma 2.3.3 we need to show that \( p \circ \gamma_a \circ \gamma_a \circ \gamma_a \circ p = p \).

It is enough to show that \( p_{a, i} \circ \gamma_a, i \circ \gamma_a, i \circ \gamma_a, i \circ p_{a, i} = p_{a, i} \forall a, i \).

Since the first and the last map are \( p_{a, i} \), it is enough to check that the restriction \( \gamma_a, i \circ \gamma_a, i \to IC_{Y_a, (L_a, a, i)} \) is the identity. This is true over \( Y \setminus (Y_a \setminus Y) \) by the equation \( \Gamma_{b, j} \circ \Gamma_{a, i} = \delta_{a, j} \delta_{b, j} \Delta_{Z_{a, i}} \) and it is still true over \( Y \) by the semisimplicity of the perverse sheaf \( IC_{Y_a, (L_a, a, i)} \) in the abelian category of perverse sheaves on \( Y \).

The second statement is proved similarly. It is enough to show that \( \gamma_a \circ p_a \circ \gamma_a = \pi_a \), where \( \pi_a : Rf_*\mathbb{Q}_X[n] \simeq \bigoplus b IC_{Y_a, (L_b, a, b)} \to Rf_*\mathbb{Q}_X[n] \simeq \bigoplus b IC_{Y_a, (L_b, a, b)} \) is the natural projection onto \( IC_{Y_a, (L_a, a)} \).

By reasons of semisimplicity \( \gamma_a \) annihilates \( \bigoplus b IC_{Y_a, (L_b, a, b)} \). It follows that \( p \circ \gamma_a = \gamma_a \) so that \( \gamma_a \circ p_a \circ \gamma_a = \gamma_a \circ \gamma_a \). By reason of semisimplicity again, \( \gamma_a \circ \gamma_a \) maps \( Rf_*\mathbb{Q}_X[n] \)
into \( IC_{\gamma_a}^Y(L_a) \) and in fact is a-priori non-trivial only on the direct summand \( IC_{\gamma_a}^Y(L_a) \).

We have shown above that \( g_a^{\gamma} \circ \gamma_a \) is the identity, when viewed as a self map on \( IC_{\gamma_a}^Y \).

The statement follows

\[ \square \]

The following corollaries and remarks follow.

\textbf{Corollary 5.0.3} 

\[ \Gamma_* : \bigoplus_{a \in A} P_a^*(A_*(Z_a)) = \bigoplus_{a \in A,i} P_a,i^*(A_*(Z_a,i)) \to A_*(X) \]

is an isomorphism with inverse \( (P \circ \Gamma')_* = P_* \circ \Gamma_* \).

\[ \Gamma^H_* : \bigoplus_{a \in A} P_a^*(H^*(Z_a)) = \bigoplus_{a \in A,i} P_a,i^*(H^*(Z_a,i)) \to H^*(X) \]

is an isomorphism of mixed Hodge structures with inverse \( (P \circ \Gamma')^H_* = P_*^H \circ \Gamma^H_* \).

Let \( X \) be proper. Then

\[ \Gamma \circ P : \bigoplus_{a \in A} (Z_a,P_a)(t_a) = \bigoplus_{a,i} (Z_{a,i},P_{a,i})(t_a) \to [X] \]

is an isomorphism of Chow motives with inverse \( P \circ \Gamma \).

\textbf{Remark 5.0.4} While the maps \( P_a^* \) preserve degrees, \( \Gamma_a^* \) increases degrees by \( t_a \).

Similarly for \( P^H_* \) and for \( \Gamma^H_* \).

\textbf{Remark 5.0.5} In the context of Theorem 5.0.2, the projectors \( C_a \) of Theorem 2.4.1 satisfy \( C_a = \Gamma_a^* \circ P_a \circ \Gamma'_a \).

\textbf{Remark 5.0.6} If the map \( \nu_a \) is small, then \( P_{a,i} \) is the diagonal and it can be omitted.

One recovers Theorem 4.0.4 at once.

\section{Maps induced by maps between surfaces}

Let \( X \) be a nonsingular, connected complex algebraic surface, \( Y \) be an algebraic surface and \( f : X \to Y \) be a proper surjective holomorphic map. Note that \( f \) is automatically semismall.

The map \( f \) induces semismall maps \( f_n : X^{[n]} \to Y^{(n)} \). We write down the the relevant strata for \( f_n \) and prove Theorem 6.0.7, an explicit version of Theorem 5.0.2. An application is given in Theorem 7.3.1.

Let \( n \) be a positive integer, \( X^{[n]} \) the Hilbert scheme of \( n \)-points of \( X \) (cfr. [23], [7]), \( S_n \) be the \( n \)-th symmetric group and \( X^{(n)} := X^n/S_n \) be the \( n \)-th symmetric product of \( X \). There is a natural crepant and semismall map \( \pi_n : X^{[n]} \to X^{(n)} \). Let \( P(n) \) be the set of partitions \( \nu \) of \( n \), i.e. \( \nu = \nu_1 \geq \ldots \geq \nu_l \), where every \( \nu_k \) is a positive integer and \( \sum \nu_k = n \). The same partition can also be denoted by \( a = (a_1, \ldots, a_n) \),
where $a_i := \text{the number of times } i \text{ appears in } \nu$. Clearly, $\sum \nu a_i = n$ and the length $l(a) = l(\nu) := l = \sum \nu a_i$.

The morphism $\pi_n$ admits a stratification $X^{[n]} = \coprod_{\nu \in P(n)} X^{[n]}_{\nu} \to \coprod_{\nu \in P(n)} X^{[n]}_\nu = X^{(n)}$, where every $X^{(n)}_\nu := \{ x \in X^{(n)} | x = \sum \nu_k x_k, x_k \in X, x_k \neq x_{k'} \text{ for } k \neq k' \}$, is a connected locally closed smooth subvariety of $X^{(n)}$ of dimension $2l(\nu)$, $X^{[n]}_\nu := (\pi^{-1}(X^{(n)}_\nu))_{\text{red}}$ and the fibers of $\pi$ over $X^{[n]}_\nu$ are irreducible of dimension $n - l(\nu)$.

Let $X^{(n)} := \coprod_{i=1}^n X^{(n)}$. There is a natural map $\pi_\nu : X^{(n)} \to X^{(n)}$ which factors through the normalization of $X^{[n]}_\nu \subseteq X^{(n)}$.

The composition $f_n : X^{[n]} \xrightarrow{\pi_n} X^{(n)} \xrightarrow{f^{(n)}} Y^{(n)}$ is semismall.

A stratification for $f$ induces one for $f^{(n)}$. Using the known one for $\pi_n$ in conjunction with the one for $f^{(n)}$ one gets one for $f_n$. For our purposes, we need only the stratification by the dimension of the fibers which requires less indices.

Let $V := Y^{(n)} \subseteq Y, U := f^{-1}V \subseteq X$ and $Y^{(1)}_j := \{ y_1, \ldots, y_N \}$. Note that $V$ contains the unique dense stratum for $f$, which is relevant for $f$ and that all the $y_k$ are relevant strata, if $Y^{(1)}_j \neq \emptyset$. There are no more relevant strata.

Let $D_k := (f^{-1}(y_k))_{\text{red}}$. $D_k$ is one dimensional, but not necessarily pure-dimensional.

The one-dimensional part of $D_k$ is a not necessarily connected configuration of curves $\{ C_{kh} \}_{h \in H_k}$ on $X$.

Let $N \neq 0$. Define, for every $h \geq 0$, $Q_N(h) = \{ \underline{m} = (m_1, \ldots, m_N) | m_k \in \mathbb{Z}_{\geq 0}, \sum m_k = h \}$.

If $N = 0$, then define $Q_0(0) = \{ * \}$ to be a set with one element and $Q_0(0) = \emptyset$, $\forall h > 0$.

We have a decomposition

$$Y^{(n)} = \coprod_{i=0}^n \coprod_{\nu \in P(i)} \coprod_{\underline{m} \in Q(n-i)} V^{(i)}_\nu \times \ast \underline{m}$$

where $\ast$ denotes a point and we identify $V^{(i)}_\nu \times \ast \underline{m}$ with the locally closed subset of points of type $\sum \nu_j \ast j + \sum_{k=1}^N m_k y_k$.

We have that $f^{-1}(V^{(i)}_\nu \times \ast \underline{m}) = U^{(j)}_\nu \times \prod_{k=1}^N \pi^{-1}(D^{(m_k)}_k)$, where both sides are given the reduced structure.

The reduced fibers over the corresponding stratum for $f_n$ are the product of a fiber of $\pi_i$ over $X^{(i)}$ with the reduced configuration $\prod_{k=1}^N \pi^{-1}(D^{(m_k)}_k)$.

Let $\Sigma$ be a finite set and $M_d(\Sigma)$ be the set of degree $d$ monomials on the set of variables $\Sigma$.

The first factor above is irreducible and the second has irreducible components of top dimension labeled by $M_\underline{m} := \prod_{k=1}^N M_{m_k}(H_k)$. The monodromy on these components is trivial.

Consider the natural map $X^{(\nu)} = X^{(\nu)}_{\underline{a}} \to Y^{(n)}$ obtained by composing $f^{(\nu)}$ with the assignment $(z^1, \ldots, z^1_{a_1}; z^2, \ldots, z^2_{a_2}; \ldots; z^i, \ldots, z^i_{a_i}) \mapsto \sum_{j=1}^i \sum_{l=1}^{a_j} j z^i_j + \sum_{k=1}^N m_k y_k$.

Each one of this will count as a relevant stratum as many times as the numbers of irreducible components over it. At this point we can apply Theorem 5.0.2 to $f_n$ and obtain precise analogues of the results of §3.
Theorem 6.0.7 There are natural isomorphisms of Chow groups

\[ \bigoplus_{i=0}^{n} \bigoplus_{\nu \in P(i)} \bigoplus_{m \in Q(n-i)} P^\nu (A^\nu (X^\nu)) \otimes |M_m| \simeq A^*(X^n), \]

of mixed Hodge structures

\[ \bigoplus_{i=0}^{n} \bigoplus_{\nu \in P(i)} \bigoplus_{m \in Q(n-i)} P^\nu (H^BM_p (X^\nu)) \otimes |M_m| (2n - i - l(\nu)) \simeq H^BM_p (X^n), \]

and, when \( X \) is assumed to be proper, of Chow motives

\[ \bigoplus_{i=0}^{n} \bigoplus_{\nu \in P(i)} \bigoplus_{m \in Q(n-i)} (X^\nu, P(\nu)) \otimes |M_m| (2n - i - l(\nu)) \longrightarrow [X^n]. \]

7 Applications of Theorem 4.0.4

We now give a series of applications of Theorem 4.0.4 to the computation of Chow motives and groups; see §7.1, §7.2, §7.3 and §8. It will be sufficient to determine the relevant strata for the maps in question, the fibers over them, and the monodromy on the components of maximal dimension. Except for the situations in §7.3 and §8, the fibers are irreducible. This fact alone implies trivial monodromy. Theorem 4.0.4 applies to give the Chow motive and groups of Hilbert schemes and of nested Hilbert schemes. In the former case, that determination had been done in §6 using the same correspondences, but additional knowledge was required, such as the affine cellular structure of the fibers, and the fact that the pre-image of each stratum maps locally trivially over the stratum in the étale topology. By way of contrast, here one only needs to know that the fibers have only one component of maximal dimension. For nested Hilbert schemes, the result is new. The fibers of the maps in §7.3 are not irreducible, but have trivial monodromy. Finally, the fibers of the Hilbert-Chow map for parabolic Hilbert schemes (cfr. §8) are not irreducible, but they are over the relevant strata and that is enough.

7.1 The Hilbert scheme of points on a surface

The following result follows from either Theorem 5.0.2, Theorem 4.0.4 or even Theorem 6.0.7 (applied to the map \( Id: X \rightarrow X \)).

Theorem 7.1.1 Let \( X \) be a nonsingular algebraic surface. There are natural isomorphisms of Chow groups

\[ \Gamma_* : \bigoplus_{\nu \in P(n)} A^* (X^\nu) \simeq A^* (X^n), \]

of mixed Hodge structures

\[ \Gamma^H_* : \bigoplus_{\nu \in P(n)} H^* (X^\nu) (n - l(\nu)) \simeq H^* (X^n) \]

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and, if $X$ is assumed to be proper, of Chow motives

$$\Gamma : \bigoplus_{\nu \in P(n)} [X^{(\nu)}](n - l(\nu)) \simeq [X^{[n]}].$$

**Remark 7.1.2** Let $A$ be an abelian surface and $K_n$ be its $n$-th generalized Kummer variety ([1]). An application of Theorem 4.0.4 gives a refinement at the level of Chow motives of the formulae for the Betti and Hodge numbers of $A \times K_n$ given in [15], Theorems 7 and 8. We omit the details.

### 7.2 The motive of nested Hilbert schemes of surfaces $X^{[n,n+1]}$.

We apply Theorem 4.0.4 to nested Hilbert schemes. The results we obtain refine Göttsche’s [14]. Let $X$ be a nonsingular quasi-projective surface, $n$ be a positive integer and define the nested Hilbert scheme of points on $X \times X^{[n,n+1]}$ to be the closed subscheme \{(\eta, \beta) | \eta \subseteq \beta \} \subseteq X^{[n]} \times X^{[n+1]} given its reduced structure. As is well-known (cfr. [4]), it is connected and smooth of dimension $2n + 2$ and admits a semismall map $\pi_{n,n+1} : X^{[n,n+1]} \rightarrow X^{[n]} \times X$ which is semismall. It is not crepant, e.g. $X^{[1,2]} \rightarrow X \times X$ is the blowing up of the diagonal. This map admits a natural stratification which we now describe.

Fix $a \in P(n)$. Define $I_a = \{0\} \coprod \{j \ | \ a_j \neq 0\}$. Clearly, $I_a \setminus \{0\} \neq \emptyset$.

Define subvarieties $X_{a,j} \subseteq X^{(n)} \times X$ as follows

$$X_{a,j} = \begin{cases} \{(\zeta, x) \ | \ \zeta \in X_a^{(n)}, \ x \notin \zeta\} & \text{if } j = 0, \\ \{(\zeta, x) \ | \ \zeta \in X_a^{(n)}, \ x \in \zeta, \ length_x(\zeta) = j\} & \text{otherwise.} \end{cases}$$

The subvarieties $X_{a,j} \subseteq X^{(n)} \times X$ are irreducible, smooth, locally closed of dimension $2l(a) + 2$, if $j = 0$ and $2l(a) + 1$, if $j \neq 0$. All strata are relevant.

We have the following stratification

$$X^{(n)} \times X = \bigsqcup_{a \in P(n)} \bigsqcup_{j \in I_a} X_{a,j}$$

inducing, as in §6, a stratification for the map $\pi_{n,n+1}$ for which the fibers are irreducible. The fibers over the stratum $X_{a,j}$ have dimension $n - l(a)$, if $j = 0$ and $n - l(a) + 1$, if $j \neq 0$. All strata are relevant.

Define, for every $a \in P(n)$ and for every $j \in I_a$, $X^{(a,j)} := X^{a,j}/\Sigma_{a,j}$. There is a natural map $X^{(a,j)} \rightarrow X^{(n)} \times X$ which factors through the normalization of $X_{a,j} \subseteq X^{(n)} \times X$.

We are in the position to apply Theorem 4.0.4 and prove

**Theorem 7.2.1** Let $X$ be a nonsingular algebraic surface. There are isomorphisms of Chow groups

$$\Gamma_* : \bigoplus_{a \in P(n), j \in I_a} A_*(X^{(a,j)}) \simeq A_*(X^{[n,n+1]}),$$

of mixed Hodge structures

$$\Gamma_*^H : \bigoplus_{a \in P(n), j \in I_a} H^*(X^{(a,j)})(m(a,j)) \simeq H^*(X^{[n,n+1]}).$$
and, if $X$ is proper, of Chow motives

$$\Gamma : \bigoplus_{a \in P(n), j \in I_a} [X^{(a,j)}](m(a, j)) \simeq [X^{[n,n+1]}],$$

where $m(a, j) = n - l(a)$ if $j = 0$ and $m(a, j) = n - l(a) + 1$ if $j \neq 0$.

**Remark 7.2.2** [14] computes the class of $X^{[n,n+1]}$ in the Grothendieck ring of motives, when $X$ is defined over a field of characteristic zero. The result affords the algebraic cycles modulo some standard conjectures on algebraic cycles. [4] computed the mixed Hodge algebraic cycles modulo some standard conjectures on algebraic cycles. [15] computed the mixed Hodge structure using Saito’s mixed Hodge polynomials. [15] computed the mixed Hodge numbers using virtual Hodge polynomials. [4] computed the mixed Hodge numbers using virtual Hodge polynomials. [15] computed the mixed Hodge numbers using virtual Hodge polynomials.

### 7.3 Wreath products, rational double points and orbifolds

Let $G$ be a finite group, $n$ be an integer and $G_n$ be the associated Wreath product, i.e., the semidirect product of $G^n$ and $S_n$. For background on what follows see [12], from which all the constructions below are taken. Let $G \subseteq \text{SL}_2(\mathbb{C})$ be a finite group. Their classification is known. Let $\tau : \mathbb{C}^2/G \to \mathbb{C}^2/G$ be the minimal and crepant resolution of the corresponding simple singularity. Let $D := \tau^{-1}(o)$ where $o \in Y$ is the singular point. With the notation of this section, $N = 1$, $y_1 = 0$ and the divisor $D$ is supported on a tree of nonsingular rational curves: $|D| = \cup_{k \in H_1} C_k$.

There is a commutative diagram of semismall and crepant birational morphisms

$$
\begin{array}{ccc}
\mathbb{C}^2/G \quad & \overset{\pi_2}{\longrightarrow} & \text{C}^2/\Gamma(n) \\
\downarrow \tau_n & & \downarrow \tau(n) \\
\text{C}^{2n}/G_n & \simeq & (\text{C}^2/\Gamma(n)).
\end{array}
$$

Let $\text{C}^2_0/G := \mathbb{C}^2/G \setminus \{o\}$. The following is a natural stratification for $\tau_n$ for which every stratum is relevant:

$$\tau_n = \prod_{i=0}^{n} \prod_{\nu \in P(i)} \left( \prod_{i=0}^{n} (\mathbb{C}^2/\Gamma)^{(i)}_{\nu} \times \pi^{-1}_{n-i}(D^{(n-i)}) \right) \longrightarrow \prod_{i=0}^{n} \prod_{\nu \in P(i)} (\mathbb{C}^2_0/\Gamma)^{(i)}_{\nu} \times \{(n-i)o\}.$$

The fibers over each stratum $(\mathbb{C}^2_0/\Gamma)^{(i)}_{\nu}$ are isomorphic to the product of the corresponding irreducible fibers of $\pi_{n-i}$ with $\pi^{-1}_{n-i}(D^{(n-i)})$. The latter factor has dimension $n - i$ and its irreducible components of dimension $n - i$ are naturally labeled by the set $M_{n-i}(H_1)$ of monomials of degree $n - i$ in the set of variables $H_1$.

Let $P(i) \ni \nu = (a_1, \ldots, a_i)$. The stratum $(\mathbb{C}^2_0)^{(i)}_{\nu} \leq \prod_{j=1}^{i} (\mathbb{C}^2/\Gamma)^{(a_j)} = \prod_{j=1}^{i} \mathbb{C}^{2a_j}/G_{a_j}$

$$= \mathbb{C}^{2l(\nu)}/\prod_{j=1}^{i} G_{a_j}.$$ 

There is the natural finite map $\mathbb{C}^{2l(\nu)}/\prod_{j=1}^{i} G_{a_j} \to (\mathbb{C}^2/\Gamma)^{(n)}$ which factors through the normalization of the closure of the stratum $(\mathbb{C}^2_0)^{(i)}_{\nu}$.

Denote by $P = \Delta_{\nu} - \sum h_k \Lambda_{h,k} C_h \times C_k$ be the projector corresponding to the projection $R\tau Q[\mathbb{C}^2/\Gamma(\mathbb{C}^2/G)]^{[2]} \to IC\mathbb{C}^2/G \simeq Q[\mathbb{C}^2/G][2]$. Note that one can compute explicitly the projectors $P^{(\nu)}$ for every partition $\nu$ of every integer. We can apply Theorem 4.0.4 and Theorem 6.0.7 and prove
Theorem 7.3.1 There are canonical isomorphisms of Chow groups

\[ \bigoplus_{i=0}^{n} \bigoplus_{\nu \in P(i) M_{n-i}(G_s \setminus \{1\})} A_*(\mathbb{C}^{2l(\nu)} / \prod_{j=1}^{i} G_{a_j}) \simeq \bigoplus_{i=0}^{n} \bigoplus_{\nu \in P(i) M_{n-i}(G_s \setminus \{1\})} P^*(\nu) A_*(\mathbb{C}^2 / G^{(\nu)}) \simeq A(\mathbb{C}^2 / G^{[n]}) \]

and of mixed Hodge structures

\[ \bigoplus_{i=0}^{n} \bigoplus_{\nu \in P(i) M_{n-i}(G_s \setminus \{1\})} H^*(\mathbb{C}^{2l(\nu)} / \prod_{j=1}^{i} G_{a_j}) \longrightarrow H^*(\mathbb{C}^2 / G^{[n]}) \simeq \bigoplus_{\mu \in P(n)} H^*(\mathbb{C}^2 / G^{(\mu)}).
\]

Let \( G_s \) be the set of conjugacy classes of \( G \). There is a well-known bijection \( b : G_s \setminus \{1\} \to T_G \) (cfr. [16]; see also [20]). Using the bijection \( b : G_s \setminus \{1\} \to T_G \) we get a natural set of bijections \( M_{n-i}(G_s \setminus \{1\}) \to M_{n-i}(T_G) \).

The set of triplets \((i, \nu, x)\) with \(0 \leq i \leq n, \nu \in P(i), x \in M_{n-i}(G_s \setminus \{1\})\) is in natural bijection with the set \( G_n^* \) of conjugacy classes of \( G_n \).

It follows easily that one can identify canonically the orbifold mixed Hodge structure of the orbifold \( \mathbb{C}^{2n} / G_n \) with the mixed Hodge structure of \( \mathbb{C}^2 / G^{[n]} \). The following result refines slightly [27].

Corollary 7.3.2 There is a canonical isomorphism of mixed Hodge structures

\[ (H^*(\mathbb{C}^{2n} / G_n))_{orb} \simeq H^*(\mathbb{C}^2 / G^{[n]}). \]

Remark 7.3.3 More generally, let \( Y' \) be a nonsingular algebraic surface on which a finite group acts as a finite group of automorphism such that the only fixed points are isolated points on \( Y' \). Let \( Y = Y'/G \) and \( f : X \to Y \) be its minimal resolution, which is automatically crepant. One can prove, using the method above and the analysis of twisted sectors in [27], that the orbifold mixed Hodge structure of \( Y''/G_n \) is isomorphic to the mixed Hodge structure of \( X^{[n]} \). This refines slightly results in [10] and [27].

8 Appendix: the Chow groups and the Chow motive of parabolic Hilbert schemes; with T. Mochizuki

M. de Cataldo, L. Migliorini, T. Mochizuki

8.1 The parabolic Hilbert scheme of points on a surface

Let \( X \) be an irreducible nonsingular surface defined over an algebraically closed field and \( D \) be a nonsingular curve on \( X \). Let \( I \) be an ideal sheaf on \( X \). Recall that a
parabolic structure on $I$ of depth $h$ at $D$ is a filtration of the $\mathcal{O}_D$-modules, denoted here using the induced successive quotients

$$I \otimes \mathcal{O}_D \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G}_2 \rightarrow \cdots \rightarrow \mathcal{G}_h \rightarrow \mathcal{G}_{h+1} = 0.$$ 

An ideal sheaf of points with a parabolic structure is called a parabolic ideal of points. The moduli scheme of the ideal sheaves of points with parabolic structure is called the parabolic Hilbert scheme of points. For simplicity, we only consider the case that $\mathcal{G}_1$ is torsion. There is no loss of generality, see [21].

We put the $K_i = \text{Ker}(G_i \rightarrow G_{i+1})$. For a parabolic ideal sheaf, we denote the length of $K_\alpha$ by $l_\alpha$ for $\alpha = 1, \ldots, h$. We denote the $h$-tuple $(l_1, \ldots, l_h)$ by $l_*$. The data $(n, h, l_*)$ is called the type of the parabolic ideal sheaf $(I, \mathcal{G}_*)$.

We denote the parabolic Hilbert scheme of points of $X$ of type $(n, h, l_*)$ by $\text{Hilb}(X, D; n, h, l_*)$. It is irreducible, nonsingular of dimension $2n + \sum_{\alpha=1}^{h} l_\alpha$ (cfr. [21]). Note that if $l_\alpha = 0$ for every $\alpha$, then $\text{Hilb}(X, D; n, h, l_*)$ is isomorphic to $X^{[n]}$.

### 8.2 The corresponding Hilbert-Chow morphism

For any sheaf $\mathcal{G}$ of finite length, we denote its support with multiplicity by $[\mathcal{G}]$, i.e., if the length of $\mathcal{G}$ at $x$ is $l(x)$, then $[\mathcal{G}]$ defined to be $\sum l(x)x$. We denote $X^{(n)} \times \prod_i D^{(l_i)}$ by $X^{(n)} \times D^{(l_*)}$. We have the natural morphism $F_{n,l_*} : \text{Hilb}(X, D; n, h, l_*) \rightarrow X^{(n)} \times D^{(l_*)}$ defined by the assignment $(I, \mathcal{G}_*) \mapsto ([\mathcal{O}_X/Z], [K_\alpha])$. We call it the Hilbert-Chow morphism. [21]. Corollary 3.1 imply the following

**Lemma 8.2.1** The morphism $F_{n,l_*}$ is semismall.

**Remark 8.2.2** The fibers of $F_{n,l_*}$ are not irreducible, in general. However, the ones over relevant strata are; see Lemma 8.3.1.

### 8.3 Stratification of the morphism $F_{(n,l_*)}$

We denote the set of integers larger than $i$ by $\mathbb{Z}_{\geq i}$. We put $A = \prod_{\alpha=0}^{h} \mathbb{Z}_{\geq 0}$. An element $v$ of $A$ is described as $v = (v_0, v_*)$, where $v_0$ is an element of $\mathbb{Z}_{\geq 0}$ and $v_* = (v_1, \ldots, v_k)$ is an element of $\prod_{\alpha=1}^{h} \mathbb{Z}_{\geq 0}$. We denote the set $A - \{(0, \ldots, 0)\}$ by $A'$.

We denote the set $\{ \chi : A' \rightarrow \mathbb{Z}_{\geq 0} | \sum_v \chi(v) < \infty \}$ by $\mathcal{S}(A')$. We have the natural morphism $\Phi : \mathcal{S}(A') \rightarrow A$ defined by $\chi \mapsto \sum_v \chi(v)\cdot v$. For any element $u$ of $A$, we put $\mathcal{S}(A', u) := \Phi^{-1}(u)$.

For any element $\chi \in \mathcal{S}(A')$, we set:

$$X_\chi := \prod_{v \in A', v_* = 0} X^{(\chi(v))} \times \prod_{v \in A', v_* \neq 0} D^{(\chi(v))}.$$ 

A point of $X_\chi$ is denoted $(\sum_{k=1}^{\chi(v)} x_{v,k} | v \in A')$, where $x_{v,k}$ is a point of $X$ (resp. $D$) if $v_* = 0$ (resp. $v_* \neq 0$). The dimension of $X_\chi$ is $2\sum_{v_* = 0} \chi(v) + \sum_{v_* \neq 0} \chi(v)$. Let $X_\chi$ denote the open set of $X_\chi$ determined by the condition that $x_{v,k} \neq x'_{v',k'}$ for any $(v,k) \neq (v',k')$.
For any $n \in \mathbb{Z}_{\geq 0}$ and $l_\ast \in \prod_{i=1}^h \mathbb{Z}_{\geq 0},$ we have the element $l_\ast(n) = (n, l_\ast) \in A$. If $\chi$ is contained in $\mathcal{S}(A', l_\ast(n)),$ we have the finite morphism $G_\chi$ of $X_\chi$ to $X^{(n)} \times D^{(l_\ast)}$ defined as follows:

$$
\left( \sum_{v=0}^{n} x_v \vspace{1mm} \right) \longmapsto \left( \sum_{v=0}^{n} x_v \vspace{1mm} \right) = \left( \sum_{v=0}^{n} v_\alpha x_v \vspace{1mm} \right), \quad \alpha = 0, 1, \ldots, h.
$$

The restriction of $G_\chi$ to $X_{0, \chi}$ is an embedding. We denote the image $G_\chi(X_{0, \chi})$ also by $X_{0, \chi}.$ Then $\{X_{0, \chi} | \chi \in \mathcal{S}(A', l_\ast(n))\}$ gives a stratification for the morphism $F_{(n, l_\ast)}.$

Let $e_\alpha$ denote the element of $A'$ whose $\beta$-th component is defined to be 1 ($\alpha = \beta$) or 0 ($\alpha \neq \beta$). We put $C := \{m \cdot e_0 | m \in \mathbb{Z}_{\geq 0}\} \cup \{m \cdot e_\alpha + e_\alpha | m \in \mathbb{Z}_{\geq 0}, \alpha = 1, \ldots, h\}.$ The following two lemmata follow from [21], Corollary 3.1.

**Lemma 8.3.1** For any point $P \in X_{0, \chi},$ the dimension of the fiber $F_{(n, l_\ast)}^{-1}(P)$ is $(n - \sum_{v=0}^{n} \chi(v)).$ The codimension of the stratum $X_{0, \chi}$ is $2(n - \sum_{v=0}^{n} \chi(v)) + \sum_{\alpha} l_\alpha - \sum_{\chi(v) \neq 0} \chi(v).$ A stratum $X_{0, \chi}$ is a relevant stratum if and only if $\chi(v) = 0$ for any $v$ which is not contained in $C.$

**Lemma 8.3.2** If a stratum $X_{0, \chi}$ is relevant, then the top dimensional part of the inverse image $F_{(n, l_\ast)}^{-1}(X_{0, \chi})$ is irreducible.

Let $\mathcal{S}(A', l_\ast(n))$ denote the subset of $\chi \in \mathcal{S}(A', l_\ast(n))$ given by the relevant strata. The set $\mathcal{S}(A', l_\ast(n))$ can be regarded as the set of $\mathbb{Z}_{\geq 0}$-valued functions $\chi$ on $C$ satisfying $\sum_{v=0}^{n} \chi(v) = l_\ast(n).$

**Example 8.3.3** ($h = 1, n = 1, l_1 = 1$)

In this case, the parabolic Hilbert scheme is isomorphic to the blowing up of $X \times D$ with center $D,$ where $D$ is embedded by the composition of the diagonal embedding $D \subset \times D$ with the natural inclusion $D \times D \subset X \times D.$ The Hilbert-Chow morphism is the blowing up morphism. The set $\mathcal{S}(A', (1, 1))$ has two elements: $\delta_{(0,1)} + \delta_{(1,0)}$ and $\delta_{(1,1)}.$ Here the function $\delta_\alpha : A' \to \mathbb{Z}_{\geq 0}$ is defined by $\delta_\alpha(0) = 0$ ($\alpha \neq 0$) and $\delta_\alpha(w) = 1$ ($\alpha = 0$). In this case $\mathcal{S}(A', (1, 1)) = \mathcal{S}(A', (1, 1)).$ The stratum corresponding to $\delta_{(0,1)} + \delta_{(0,1)}$ (resp. the $\delta_{(1,1)}$) is $X \times D - D$ (resp. the $D$). The dimensions of the fibers corresponding to $\delta_{(0,1)} + \delta_{(0,1)}$ and $\delta_{(1,1)}$ are 0 and 1, respectively.

**Example 8.3.4** ($h = 1, n = 1, l_1 = 2$)

In this case, $\mathcal{S}(A', (1, 1))$ has four elements: $\chi_1 := \delta_{(1,0)} + 2\delta_{(0,1)}$, $\chi_2 := \delta_{(1,0)} + \delta_{(0,1)}$, $\chi_3 := \delta_{(1,1)} + \delta_{(0,1)},$ and $\chi_4 := \delta_{(2,1)}.$ The dimensions $d_i$ of the fibers and the codimensions $c_i$ of the strata corresponding to $\chi_i$ are as follows: $(d_1, c_1) = (0, 0),$ $(d_2, c_2) = (0, 1),$ $(d_3, c_3) = (1, 2),$ $(d_4, c_4) = (1, 3).$ Thus $\mathcal{S}(A', (1, 1))$ has two elements $\chi_1$ and $\chi_3.$

### 8.4 The results

Let $\Gamma$ be the correspondence constructed using the relevant strata above and $\Gamma$. Let $t_\chi = n - \sum_{v=0}^{n} \chi(v).$
Theorem 8.4.1 We have natural isomorphisms of Chow groups
\[
\Gamma_* : \bigoplus_{\chi \in \mathcal{A}'(X, n, l_\alpha)} A_*(X^\chi) \to A_*(\text{Hilb}(X; n, h, l_\alpha)),
\]
of mixed Hodge structures
\[
\Gamma^H_* : \bigoplus_{\chi \in \mathcal{A}'(X, n, l_\alpha)} H^*(X^\chi)(t_\chi) \to H^*(\text{Hilb}(X; n, h, l_\alpha)),
\]
and, if \(X\) is assumed to be proper, of Chow motives
\[
\Gamma : \bigoplus_{\chi \in \mathcal{A}'(X, n, l_\alpha)} [X^\chi](t_\chi) \to [\text{Hilb}(X; n, h, l_\alpha)].
\]

In what follows, for an algebraic variety \(X\), \(b_i(X)\) denotes the \(i\)-th Betti number of \(X\), \(P(Y)(z) = \sum b_i(Y)z^i\) denotes the Poincaré polynomial of \(Y\), \(h^{p,q}(X)\) denote the \((p,q)\)-th Hodge number of \(Y\), \(h(Y)(x,y) = \sum_{x,y} h^{p,q}(Y) x^py^q\) denotes the Hodge polynomial of \(Y\), and we put \(\epsilon(p,q) = (-1)^{p+q-1}\).

Corollary 8.4.2 The generating functions for Chow motives (where we denote a motive \((T, \Delta_T)\) simply by \([T]\)), Betti and Hodge numbers are
\[
\sum_{n,l_\alpha} [\text{Hilb}(X; n, h, l_\alpha)] \cdot t^n \prod_{\alpha=1}^h \prod_{i=0}^\infty s_{\alpha \alpha}^i =
\]
\[
\prod_{m \geq 1} \left( \sum_{i=0}^\infty (X^{(m)})(i-1)m \cdot t^{im} \right) \times \prod_{\alpha=1}^h \prod_{i=0}^\infty \left( \sum_{m=0}^\infty (D^{(m)})(im) \cdot t^{im} s_{\alpha \alpha}^m \right),
\]
\[
\prod_{m \geq 1} \frac{(1 + z^{2m-1}t^{im}) b_1(X) (1 + z^{2m}t^{im}) b_3(X)}{(1 - z^{2m-2t^m}) b_0(X)(1 - z^{2m+2t^m}) b_4(X)} \times
\]
\[
\prod_{\alpha=1}^h \prod_{m \geq 0} \left( \frac{(1 + z^{2m+1m}t^{im} s_{\alpha \alpha}) b_1(D)}{(1 - z^{2m+2t^m} s_{\alpha \alpha}) b_0(D)(1 - z^{2m+2t^m} s_{\alpha \alpha}) b_4(D)} \right),
\]
\[
\sum_{n,l_\alpha} h(\text{Hilb}(X; n, h, l_\alpha))(x,y) t^n \prod_{\alpha=1}^h \prod_{i=0}^\infty s_{\alpha \alpha}^i =
\]
\[
\prod_{m \geq 1} \prod_{0 \leq p,q \leq 2} \left( 1 + \epsilon(p,q) x^{p+m-1}y^{q+m-1}t^m \right) \epsilon(p,q) h^{p,q}(X) \times
\]
\[
\prod_{\alpha=1}^h \prod_{m \geq 0} \prod_{0 \leq p,q \leq 1} \left( 1 + \epsilon(p,q) x^{p+m}y^{q+m}t^m s_{\alpha \alpha} \right) \epsilon(p,q) h^{p,q}(D).
\]

Remark 8.4.3 The generating functions for the Betti and Hodge numbers can be found in [21]. The slightly weaker statement in the Grothendieck ring of motives can also be deduced easily from [21].
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