Weyl Neutrinos On A Lattice: An Explicit Construction

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ABSTRACT

Introducing a new and universally applicable discretizing technique, I construct a class of local and unitary lattice theories of Weyl neutrinos; this solves a longstanding and allegedly unsolvable problem in quantum field theory. En route, I also prove a general “go” theorem that all Lagrangian-density based continuum quantum field theories can be lattice-regularized.

(Informal Abstract. You didn’t study the Nielsen-Ninomiya theorem, only trusted the authors to have proven the “absence of neutrinos on a lattice”. Well, they didn’t. Nor can anyone else: every continuum theory can be lattice-regularized. A proof of that, plus an explicit construction of lattice neutrinos: if you read only one paper this year, here it is! From now on, this is how chiral fermions should be latticized. All else is gaslight.)
The primary aim of this letter is to solve a longstanding and allegedly unsolvable problem in quantum field theory, namely, the local and unitary lattice-regularization of chiral Weyl (“neutrino”) fields. Towards that, I shall construct a specific class of such theories, varying in the steepness of regularization. En route, I shall also prove that every Lagrangian-density based continuum theory can be lattice-regularized.

The problem needs very little introduction: The lattice is a powerful nonperturbative platform. Neutrino-like fields, the most fundamental of all, are indispensable to our models. So is locality to practical work. However, the problem of local lattice neutrinos has mainly attracted “no-go” theorems and is commonly regarded as unsolvable.

That conventional wisdom is untenable for deep mathematical and physical reasons: (i) If you say that something exists in the continuum but not on the lattice, what you are claiming in the momentum representation is that it exists in \([-\infty, \infty]^4\) but not in \([-\pi/\ell, \pi/\ell]^4\). That is impossible since the latter two spaces are isomorphic, they are the same space labeled differently. (ii) Neutrinos are observed in Nature. If such particles cannot exist on lattices, it stands proven that spacetime is not a lattice at any scale. Yet, at any untested scale, that is an experimentally open question; a particle observed in an apparently continuous spacetime should be describable on lattices.

Failure to identify such theories mainly implicates our assumptions as faulty; perhaps, as with the rethinking of paradigms needed to consistently “cut off” velocities (relativity) or phase-space volumes (quantum mechanics), the spacetime cutoff too requires thinking “out of the box”. But what should we change, where, and how? A methodological contribution of this paper is to identify one viewpoint in which an exact solution is easily apparent.

I shall translate the existence of isomorphisms between the two momenta into a discretizing technique. It is not an ad hoc approach to the lattice neutrino problem, but will apply to a variety of structures, including all theories that develop species-doubling upon usual discretization. However, I shall limit this discussion to quantum field theories, and the construction itself to ultraviolet-regularized lattice theories of neutrinos (unregularized lattice theories can exist; please note the perimeter of regularization when I identify it). I shall not discuss the doubling: the goal here is not so much to understand that unwanted phenomenon as to simply avoid it. This work confirms that problems with lattice fermions are due to overlocalization which needs “smearing out” to paraphrase Einstein, physics should be made as local as possible but not any more local. This work is independent of the Nielsen-Ninomiya theorem whose critique has been given separately.

Here is the arena. A physical theory predicts certain observables. The choice of a working parameterization is called a representation. A lattice quantum field theory consists of instructions like

$$Q^A(u') \equiv N \int [\mathcal{D}\chi]_\rho \quad q^A_\rho [\chi(u')] \exp iS_\rho [\chi(u)],$$

(1)
where $\chi$ denotes all fields and $N \equiv 1 / \int [\mathcal{D}\chi] \rho \exp i S_\rho[\chi]$. Both $\chi$ and $u$ are dummy variables: $\chi$ parameterizes the path integral and defines the “outer” or “functional” representation $\rho$ of the theory; $u$ parameterizes the domain of $\chi$, over which the ordinary integral (or sum) $S_\rho[\chi] = \int du \mathcal{L}_\rho[\chi(u)]$ is evaluated, and defines the “inner” representation. Being dummy, both can be changed. Changing path variables yields representations differing in both the action $S[\chi]$ and the path-transcription $q^A(\chi)$ of observables—notice that one has no physical relevance without the other. A representation is (i) local if it preserves the locality of the continuum action and observables; I shall call it (ii) “hamiltonian” if the Lagrangian-density has a continuous-time limit of the form $\chi[i \frac{\text{PhyS. Lett.}}{\text{PhyS. Lett.}}, \ell \hat{H}]$; and (iii) “canonical” if the expressions $q^A(\chi)$ coincide with their continuum counterparts. In the compact formulation of the theory,

$$W[J] = N \int [\mathcal{D}\chi] \rho \exp i S_\rho[\chi; J],$$

where the integrals (1) arise as the coefficients in the functional Taylor-Berezin series in $J$, a representation $\rho$ is (i) local if $S_\rho[\chi; J]$ is local, (ii) hamiltonian if $S_\rho[\chi; 0]$ obeys the above mentioned continuous-time limit, and (iii) canonical if the source-field coupling is $J\chi$. (Thus a theory is determined not by a spectrum, an equation, or an action, but by its observables, the functional derivatives of $W[J]$ at $J = 0$. Just as in transitions to relativity and quantum mechanics, it is necessary to alter the mathematical transcription of the observables along with the dynamics: the essential result here will be that regularized local unitary lattice theories of neutrinos exist, but the representations which manifest this locality are neither hamiltonian nor canonical.)

After a decade of “no-go” theorems, you may find it refreshing to encounter a “go” theorem: Every Lagrangian-density based continuum quantum field theory can be lattice-regularized. (That covers all fundamental theories presently envisioned.) Here is a terse but complete proof. Let $J$ denote all sources, $q_\alpha \in [-\infty, \infty]$ the coordinates, and $p_\alpha \in [-\infty, \infty]$ the momenta. A theory would be defined by a functional $W_0[J(q)]$ in the coordinate and $W_0[J(p)]$ in the momentum representation. Take the latter. By assumption, it can be written as $W_0 = N \int \mathcal{D}\chi \exp i \int \mu_0 \mathcal{L}_0$ where $\mu_0 \equiv d^4p/(2\pi)^4$. It can be modified to obtain others, $W_\ell = N \int \mathcal{D}\chi \exp i \int \mu_\ell \mathcal{L}_\ell$, where $\ell$ is a parameter, $\mu_\ell = \xi(\ell; p) \mu_0$, and $W_\ell \to W_0$ as $\ell \to 0$; in particular, a regulator $\xi(\ell; p)$ can always be found to suppress the measure in the ultraviolet (or any other) region to any specified degree. From the regularized functionals $W_\ell$, take a specific $W[J(p)]$. Any one of the infinitely many isomorphisms $\lambda_\ell$ between the real line and Brillouin Zone yields a $W[J(k)]$ where $k_\alpha \in [-\pi/\ell, \pi/\ell]$. By Fourier analyzing to the coordinate representation conjugate to $k$, we obtain $W[J(x)]$, i.e., a lattice theory.

This proof shows that each continuum theory possesses lattice representations (equivalent to it), as well as transcriptions (equivalent in the $\ell \to 0$ limit); typically one has a divergent continuum theory and seeks a regularized lattice transcription.
I note a few practical aspects of this admittedly spartan existence argument: (i) Even if \( W[J(p)] \) can be studied ("is defined") only perturbatively, \( W[J(x)] \) can be subjected to nonperturbative lattice tools. (ii) Besides regulating the measure \( (\mu_0 \to \mu_\ell) \), you can use the modification \( \mathcal{L}_0 \to \mathcal{L}_\ell \) to fine-tune other properties if desired. (iii) The theorem can be made stronger: unitarity- and locality-preserving transcriptions always exist. I shall present the general argument elsewhere; for neutrinos, unitary local theories are obtained below by explicit construction. (iv) The theorem suggests a practical discretizing technique: use an isomorphism \( \lambda_\ell \) to associate a continuum-like space with the lattice; write theories in it; choose convenient field variables; change the inner variables back to \( x \).

One such construction of lattice neutrinos is presented below in four steps.

First, I associate a continuum with the lattice:

\[
p_\alpha = \frac{2}{\ell} \tan \frac{1}{2} \ell k_\alpha, \quad p_\alpha \in [-\infty, \infty], \quad k_\alpha \in [-\pi/\ell, \pi/\ell]. \tag{3}
\]

It is convenient to introduce the functions:

\[
\eta(p) = \prod_{\alpha=0}^{3} \frac{1 - \frac{1}{4} \ell p_\alpha}{1 + \frac{1}{4} \ell^2 p_\alpha^2}, \quad \vartheta(p) = \prod_{\alpha=0}^{3} \frac{1}{1 + \frac{1}{4} \ell^2 p_\alpha^2}, \quad \delta_\alpha(p) = \frac{p_\alpha}{1 + \frac{1}{4} \ell^2 p_\alpha^2} \prod_{\beta \neq \alpha} \frac{1}{1 + \frac{1}{4} \ell^2 p_\beta^2}; \tag{4a}
\]

their conventional representations:

\[
\eta(k) = \prod_{\alpha=0}^{3} \frac{1}{2} \left(1 + e^{-i \ell k_\alpha}\right), \quad \vartheta(k) = \prod_{\alpha=0}^{3} \cos^2 \frac{1}{2} \ell k_\alpha, \quad \delta_\alpha(k) = \frac{1}{\ell} \sin \ell k_\alpha \prod_{\beta \neq \alpha} \cos^2 \frac{1}{2} \ell k_\beta; \tag{4b}
\]

and their operator representations:

\[
\hat{\eta} = \prod_{\alpha=0}^{3} \frac{1}{2} \left(1 + \hat{t}_\alpha\right), \quad \hat{\vartheta} = \prod_{\alpha=0}^{3} \frac{1}{4} \left(\hat{t}_\alpha + 2 + \hat{t}_\alpha^*\right), \quad \hat{\delta}_\alpha = \frac{i}{2\ell} \left(\hat{t}_\alpha - \hat{t}_\alpha^*\right) \prod_{\beta \neq \alpha} \frac{1}{4} \left(\hat{t}_\beta + 2 + \hat{t}_\beta^*\right). \tag{4c}
\]

Second, I choose a Lagrangian-density in \( \{p\} \)-space:

\[
\mathcal{L}_\ell = \mathcal{L}_0 = \bar{\phi} \sigma^\alpha p_\alpha \phi + \bar{J} \phi + \bar{\phi} J. \tag{5}
\]

It determines a classical theory identical with the continuum Weyl theory. There is no question of doubling and, using (3), the spectrum \( p^\alpha p_\alpha = 0 \) can be written as

\[
\tan^2 \frac{1}{2} \ell \omega - \sum_{i=1}^{3} \tan^2 \frac{1}{2} \ell k_i = 0; \tag{6}
\]

all quantum theories to follow possess this spectrum.

Third, I quantize and regularize the theory. That requires an action \( S = \int \mu_\ell \mathcal{L}_\ell \) where the measure \( \mu_\ell \) vanishes suitably fast as \( p_\alpha \to \pm \infty \). While the standard lattice measure \( d^4k = \theta^4 d^4p \) is certainly adequate, consider more generally:

\[
\mu_{\ell,M} = \theta^M d^4p/(2\pi)^4, \quad M = 1, 2, 3, \ldots \tag{7}
\]
where \( M \geq 1 \) is a positive-definite integer (\( M=1 \) gives the standard case). This yields

\[
W_{\ell,M} [\bar{J}, J] = N \int D\phi D\bar{\phi} \exp i \int \cdots \int \left\{ \prod_{\alpha=0}^{3} \frac{dp_{\alpha}/2\pi}{(1 + \frac{1}{4} \ell^{2} p_{\alpha}^{2})^{M}} \right\} \left( \bar{\phi} \sigma^{\beta} p_{\beta} \phi + \bar{J} \phi + \bar{\phi} J \right). \tag{8}
\]

The regularization here improves with increasing \( M \) since \( \mu_{\ell,M} \propto p_{\alpha}^{-2M} \) as \( p_{\alpha} \to \pm \infty \); along a generic direction in the euclideanized space, the measure \( \mu_{\ell,M} \) falls as \( |p|^{-8M} \). (In \( n \) dimensions it falls as \( |p|^{-2nM} \); overcoming a classic continuum limitation, the theory now remains regularized in higher dimensions.)

Before proceeding further, you should satisfy yourself—the familiar continuum tools suffice—that for \( M \geq 1 \), Eq. (8) does define ultraviolet-regularized quantum field theories of neutrinos; from now on, it is only change of variables.

Fourth and the last, I bring forth the hidden lattice parameterization. The following yields the simplest local and manifestly hermitian representations: First, writing \( \ell^{2} \psi_{\sigma} \), change of variables (outer representation) to

\[
\psi = \vartheta^{m-1} \eta^{\varepsilon} \phi, \quad \bar{\psi} = \vartheta^{m-1} \eta^{\varepsilon} \bar{\phi}; \tag{9}
\]

the Jacobian \( D\phi D\bar{\phi} / D\psi D\bar{\psi} \) is a functional constant. Next, change the inner representation to \( x \); this means using Eq. (3) to revert the momentum space parameterization to \( k \) (the Jacobian is \( d^{4}p/d^{4}k=\vartheta^{-1} \)) and going to the coordinate representation. The result is

\[
W_{\ell,M} [\bar{J}, J] = N \int D\psi D\bar{\psi} \exp i \sum_{x} \ell^{4} \left[ \bar{\psi} \sigma^{\alpha} \delta_{\alpha} \psi + \bar{J} \hat{\tau}_{M} \psi + \hat{\bar{\tau}}_{M} \bar{\psi} J \right], \tag{10}
\]

where \( \hat{\tau}_{M} = \vartheta^{m} \tilde{\eta} \) (said differently, \( \hat{\tau}_{2m} = \vartheta^{m} \) and \( \hat{\tau}_{2m+1} = \vartheta^{m} \tilde{\eta} \); the first few \( \hat{\tau} \)’s are \( \hat{\tau}_{1} = \tilde{\eta} \), \( \hat{\tau}_{2} = \vartheta \), \( \hat{\tau}_{3} = \vartheta \tilde{\eta} \), and so on).

We now have a class of lattice theories of neutrinos: Each positive-definite integer \( M \geq 1 \) defines a distinct theory. In the chosen representation, all \( M \)-dependence is in the source-field coupling, the action is \( M \)-independent. The ultraviolet regularization improves with increasing \( M \).

The formalism (10) lattice-transcribes a continuum \( q^{A}(\bar{\psi}, \psi) \) as \( q^{A}(\hat{\tau}_{M} \psi, \hat{\bar{\tau}}_{M} \bar{\psi}) \)\(^{1}\) and then averages it with the action \( \sum_{x} \ell^{4} \bar{\psi} \sigma^{\alpha} \delta_{\alpha} \psi \): i.e., each continuum path-average

\[
Q_{0}^{A} = \frac{\int D\psi D\bar{\psi} q^{A}(\bar{\psi}, \psi) \exp i \int d^{4}x \bar{\psi} \sigma^{\alpha} \partial_{\alpha} \psi}{\int D\psi D\bar{\psi} \exp i \int d^{4}x \bar{\psi} \sigma^{\alpha} \partial_{\alpha} \psi} \tag{11a}
\]

is lattice-regularized as

\[
Q_{\ell,M}^{A} = \frac{\int D\psi D\bar{\psi} q^{A}(\hat{\tau}_{M} \psi, \hat{\bar{\tau}}_{M} \bar{\psi}) \exp i \sum_{x} \ell^{4} \bar{\psi} \sigma^{\alpha} \delta_{\alpha} \psi}{\int D\psi D\bar{\psi} \exp i \sum_{x} \ell^{4} \bar{\psi} \sigma^{\alpha} \delta_{\alpha} \psi} \tag{11b}
\]
Equation (10), or its restatement (11), is the main result of this paper. It defines a class of manifestly local, unitary, translationally invariant lattice theories; that they are regularized theories of neutrinos is manifest in the representation (8).

These formalisms possess two unusual features: (i) Derivatives $\hat{\delta}_\alpha$ incorporate transverse smearing; this gives the action a non-hamiltonian structure. (ii) The ultraviolet regularization $[M \geq 1, \text{Eq. (8)}]$ further requires non-canonical source-field coupling [Eq. (10)], i.e., smeared transcription of observables [Eqs. (11)]. In other words, the formalisms do not incorporate the continuum Born interpretation but a smeared version of it. This smearing involves quantum interference of nearby fields—notice that Eq. (10), or (11b), automatically generates the Schwinger “point-splitting” which upon gauging leads to chiral anomaly—rather than classical averaging of observables. Both of these features are simultaneously needed.

Incidentally, the only relevance that the attempted “no-go” arguments offered in the Nielsen-Ninomiya theorem have for this subject is as follows: If in Eq. (10) or (11b) you change path variables so as to make the action hamiltonian, it will also become nonlocal in that parameterization. Since parameterizations do not affect observables and are chosen for convenience, this has no importance beyond telling us that such parameterizations should be avoided in practical computations.

Equations (8) and (10) yield a useful rule: with the action held fixed, greater smearing of $\psi$ in the source-field coupling, i.e., in the lattice transcription of the observables, improves the ultraviolet regularization. The same pattern holds in the interaction with a dynamical field: For example, the simplest ultraviolet-admissible Yukawa interaction (with a scalar $\Phi$ possessing the standard kinetic term) is $\Phi \bar{\eta} \hat{\psi} \hat{\psi}$; the interaction $\Phi \bar{\hat{\psi}} \hat{\psi}$ leads to better regularization; and so on.

The simplest one of these formalisms is

$$W_{\ell,1} [\bar{J}, J] = N \int \mathcal{D} \psi \mathcal{D} \bar{\psi} \exp i \sum_x \ell^4 \left[ \bar{\psi} \sigma^\alpha \hat{\delta}_\alpha \psi + \bar{J} \bar{\eta} \psi + \bar{\bar{\psi}} \eta J \right],$$

(12)

$$Q_{\ell,1}^A = \frac{\int \mathcal{D} \psi \mathcal{D} \bar{\psi} q^A (\bar{\eta} \psi, \bar{\bar{\psi}}) \exp i \sum_x \ell^4 \bar{\psi} \sigma^\alpha \hat{\delta}_\alpha \psi}{\int \mathcal{D} \psi \mathcal{D} \bar{\psi} \exp i \sum_x \ell^4 \bar{\psi} \sigma^\alpha \delta_\alpha \psi};$$

(13)

$\hat{\eta}$ and $\hat{\delta}_\alpha$ are defined in Eq. (4c). Although I would recommend the manifestly hermitian representation (12)–(13), the representation can be simplified further by giving up the manifestness of hermiticity (I omit the details):

$$W_{\ell,1} [\bar{J}, J] = N \int \mathcal{D} \psi \mathcal{D} \bar{\psi} \exp i \sum_x \ell^4 \left[ \bar{\psi} \sigma^\alpha \hat{\delta}_\alpha^{+} \psi + \bar{J} \bar{\bar{\psi}} \eta + \bar{\bar{\psi}} J \right],$$

(14)

$$Q_{\ell,1}^A = \frac{\int \mathcal{D} \psi \mathcal{D} \bar{\psi} q^A (\bar{\psi}, \bar{\bar{\psi}}) \exp i \sum_x \ell^4 \bar{\psi} \sigma^\alpha \hat{\delta}_\alpha^{+} \psi}{\int \mathcal{D} \psi \mathcal{D} \bar{\psi} \exp i \sum_x \ell^4 \bar{\psi} \sigma^\alpha \delta_\alpha \psi},$$

(15)
\[
\delta^+_{\alpha} = \frac{i}{\ell} (i_{\alpha} - 1) \prod_{\beta \neq \alpha} \frac{1}{2} (i_\beta + 1); \tag{16}
\]

if you opt for this simplification, expressly ensure unitarity when adding interactions.

Explicitly, the Lagrangian-density in Eq. (12) has 248 terms contributed by 81 sites constituting the \(3^4\) hypercube centered at \(x\); that in (14) has 81 terms from 16 sites constituting the \(2^4\) hypercube with a corner at \(x\). You could complain that these formalisms will strain today’s computers, but one could just as reasonably have faulted Maxwell’s theory or general relativity for being too complex for slide rules. Physical problems often have an irreducible complexity we must face; I shall show elsewhere that there is no simpler local unitary latticization of neutrinos than \(W_{\ell,1}\).

While the focus here was on the lattice regularization, you can cast these theories in local \(\{q\}\)-space representations as well (\(q\) being the coordinate conjugate to \(p\)); that gives a continuum regularization of neutrinos. [However, the measure (7) was specifically optimized for the lattice. If it is the continuum regularization that you want, other choices will be more appropriate.]

I promised to identify the perimeter of the ultraviolet-regularization. From Eq. (8), we need \(M \geq 1\). Basically, all that is asked of us is not to sleepwalk into the \(M = 0\) case: if you merely latticize the Lagrangian-density as \(\bar{\psi} \sigma^\alpha \delta^+_{\alpha} \psi\) but retain the continuum transcription \(q(\bar{\psi}, \psi)\) of observables, you get a lattice theory which, though free from the doubling, is not regularized. You would not expect it to yield the anomaly, for example. I cannot possibly overemphasize the following: if you want locality, unitarity, chirality, and the regularization all at once, then don’t just latticize the action but follow the prescription (11) completely. [By the same token, in judging the proposal you should examine its final observables (11b), not some part of some path integrand.]

As for the latticizing technique, I began with a continuum-like parameterization of the lattice momentum space because that places us in the only manifold where we know how to avoid the doubling. It lets us explicitly control the ultraviolet regularization, and separates the regularization from the latticization. Locality could be preserved here because the regulator is a rational algebraic function of \(p_\alpha\) [Eq. (8)] which in turn is a rational algebraic function of \(\exp \pm i\ell k_\alpha\) [Eq. (3)]; this is a general theorem. Similar construction can discretize any continuum theory; I shall state the equivalent final prescription elsewhere.

The construction and the go-theorem given here are but elementary applications of a deeper principle which may be called “Continuum-Lattice Duality”: those two mathematically distinct spaces are interchangeable as domains of physical theories. This duality is analogous to wave-particle duality (which too asserts that two mathematically distinct concepts are interchangeable as vehicles for physical laws); it too has similarly deep consequences for physics which I hope to discuss elsewhere.

I conclude by recalling what I did here. I showed that all Lagrangian-density based continuum theories can be lattice-regularized; I stated that unitarity and locality can be
preserved in the process. I introduced a new latticizing technique; I promised to spell out the equivalent final algorithm elsewhere. For neutrinos, I constructed a class of manifestly local, unitary, and ultraviolet-regularized lattice theories. They do come with two unfamiliar features (the action is non-hamiltonian, and observables are transcribed differently from the continuum), but you can hardly insist that introducing a new fundamental constant in physics should not change anything.

1. While a lattice theory was first considered by G. Wentzel, Helv. Phys. Acta 13, 269 (1940), their modern revival is due to K. G. Wilson, Phys. Rev. D 10, 2455 (1974). Recognizing the problem with fermions, Wilson suggested a compromise scheme in Erice lectures (1975); it retains locality but sacrifices chirality. Though differently, so does the “staggered” formalism: J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975); T. Banks, J. Kogut and L. Susskind, Phys. Rev. D 13, 1043 (1976). A nonlocal scheme, similar to Wentzel’s, is due to S. D. Drell, M. Weinstein and S. Yankielowicz, Phys. Rev. D 14, 1627 (1976). None of these schemes can handle local Weyl neutrinos.

Non-existence statements include: L. H. Karsten and J. Smit, Nucl. Phys. B183, 103 (1981); L. H. Karsten, Phys. Lett. B104, 315 (1981); J. M. Rabin, Phys. Rev. D 24, 3218 (1981); Nucl. Phys. B201, 315 (1982); F. Wilczek, Phys. Rev. Lett. 59, 2397 (1987); A. Pelissetto, Ann. Phys. (N.Y.) 182, 177 (1988). The best-known of the genre is the Nielsen-Ninomiya “no-go” theorem: H. B. Nielsen and M. Ninomiya, Nucl. Phys. B185, 20 (1981); B195(E), 54 (1982); B193, 173 (1981); D. Friedan, Commun. Math. Phys. 85, 481 (1982).

2. Technical preliminaries: $\hbar = c = 1$; the spacing = $\ell$; the metric = $(1, -1, -1, -1)$; $\sigma^\alpha \equiv (1, \sigma^i)$. The summation convention is automatically suspended for indices not explicitly balanced: $\alpha$ is summed in $A^\alpha B_\alpha$, but not in $A_\alpha B_\alpha$ or $B_\alpha^2$. A lattice operator $\hat{\lambda}$ is a function of $\hat{t}_\alpha \equiv \hat{t}_\alpha^+$ and $\hat{t}_\alpha^*$ whose right- and left-actions are $\hat{t}_\alpha^+ f(x) \equiv f(x + \ell e_\alpha)$ and $f(x) \hat{t}_\alpha^\pm \equiv f(x \mp \ell e_\alpha)$. Identity $\sum_x (f_1 \hat{\lambda}) f_2 = \sum_x f_1 (\hat{\lambda} f_2)$, the lattice analog of integration by parts, shows that under sum, operators may be taken as acting in either direction. The conjugate (*) of an operator is defined by exchanging $\hat{t}_\alpha \leftrightarrow \hat{t}_\alpha^*$ and replacing $i \rightarrow -i$. An operator is local if it is a polynomial in the $\hat{t}$’s. By “smearing operator” I mean a nonconstant local operator that reduces to 1 in the continuum limit ($\ell \rightarrow 0$).

3. The usual convention of viewing the two spaces through different topologies is not relevant here.

4. A. K. Trivedi, to be published.

5. These range from the time-dependent Schrödinger equation to solitons [J. Govaerts, J. Mandula, and J. Weyers, Phys. Lett. 112B, 465 (1982)] to gravity [P. Menotti and A. Pelissetto, Phys. Rev. D 35, 173 (1981)].
6. A. K. Trivedi, Phys. Rev. Lett. 61, 907 (1988).
7. A. K. Trivedi, Phys. Lett. B230, 113 (1989).
8. A. K. Trivedi, hep-lat/9309012.
9. While this is undoubtedly a “physicist’s proof”, its mathematical ingredients—Fourier analysis; existence of functions \( \xi(\ell;p) \) which decrease arbitrarily fast for large \( p \); mappings between real intervals and the real line—are all well established, and it can be made as rigorous as you desire.
10. This may be referred to as “Continuum-Lattice Duality”.
11. For example, the continuum \( \bar{\psi}\sigma^\alpha\psi \) will be lattice-transcribed as \( \hat{\bar{\tau}}_M\hat{\bar{\psi}}\sigma^\alpha(\hat{\tau}_M\psi) \). Indeed, formalisms (10) admit no observable like \( \bar{\psi}\sigma^\alpha\psi \). This should not disturb you: just as quantum mechanics can alter mathematical representation of classical observables, a lattice theory too need only transcribe, not retain, continuum expressions.
12. This derives the suggestion of Ref. 7 that the ultraviolet problems plaguing lattice fermions can be solved by smearing out two instances of overlocalization, each incompatible with the limited spacetime resolution a fundamental length mandates: (i) the infinitely sharp transverse localization of differentiation, and (ii) the possibility of observations limited to a single point.
13. You may wish to compare this approach with that of Drell et al. (Ref. 1) who also start out in the momentum space. Those authors set the measure \( \mu = 0 \) outside a box (Brillouin Zone), which regularizes and latticizes at the same time, but also destroys locality. In my approach, ultraviolet contributions are first regularized smoothly with \( \mu \propto \vartheta_M \) [Eq. (7)] and the resulting theory, already regularized, is latticized not by cutting off the momentum space, but by mapping it to Brillouin Zone [Eq. (3)].

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*I do not know what I may appear to the world; but to myself I seem to have been only like a boy who kept getting showered with smooth and rough pebbles on the seashore, whilst the great ocean of truth lay undiscovered before us all.*

— ANONYMOUS (circa 1675)