The fibred density property and the automorphism group of the spectral ball

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Abstract We generalize the notion of the density property for complex manifolds to holomorphic fibrations, and introduce the notion of the fibred density property. We prove that the natural fibration of the spectral ball over the symmetrized polydisc enjoys the fibred density property and describe the automorphism group of the spectral ball.

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1 Introduction and results

The density property for complex manifolds has been introduced by Varolin [25,26] in 2000 and has its origin in the work of Andersén [2] and Andersén–Lempert [3] in the early 1990s. It characterizes complex manifolds with large automorphism groups. Since then the so-called Andersén–Lempert theory has developed rapidly. It has many applications for geometric questions in Several Complex Variables and has contributed a lot to a better understanding of large holomorphic automorphism groups. For a recent overview of the theory we refer to [12].

In this paper we introduce a parametrized version of the density property for holomorphic fibrations where the density property holds only in the direction of possibly

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singular fibres which are parametrized by the base space. For a comparison we restate
the definition of the density property.

**Definition 1.1** Let $X$ be a complex manifold. We say that $X$ has the *density property*
if the Lie algebra generated by all holomorphic $\mathbb{C}$-complete vector fields on $X$ is dense
(w.r.t. compact-open topology) in the Lie algebra of all holomorphic vector fields on $X$.

**Definition 1.2** We call a holomorphic surjection $\pi : X \to Y$ between complex mani-

folds a *holomorphic fibration*. We say that the fibration has the *fibred density property*
if the Lie algebra generated by all holomorphic $\mathbb{C}$-complete vector fields on $X$ tangent
to the fibres of $\pi$ is dense (w.r.t. compact-open topology) in the Lie algebra of all
holomorphic vector fields on $X$ tangent to $\pi$.

**Example 1.3** The fibred density property is known for a trivial fibration: consider the
projection $\pi : U \times V \to V$ for Stein manifolds $U$ and $V$ where $U$ has the density
property. Then this fibration has the fibred density property, as a consequence of [26,
Lemma 3.5]. For the case $\mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^k$ with fibers $\mathbb{C}^n$, $n \geq 2$,
see also [14, Corollary 2.2].

In this paper we will first develop the general theory for the fibred density property
(Sect. 2) and then prove the fibred density property for a well-known quotient map
from classical invariant theory. This enables us to describe the automorphism group
of the spectral ball.

Concerning the general theory the following theorem is a fibred version of the
classical Andersén–Lempert theorem [2,3,10,11,25,26], for a special case see also
[14, Theorem 2.3].

**Theorem 1.4** Let $X$ be a Stein manifold and let $\pi : X \to Y$ be a holomorphic fibration
with the fibred density property. Let $\Omega \subseteq X$ be an open subset and $\varphi_t : \Omega \to X$, $t \in
[0, 1]$, be a fibre-preserving $C^1$-homotopy of injective holomorphic maps such that $\varphi_0$
is the natural embedding $\Omega \hookrightarrow X$ and $\varphi_t(\Omega)$ is Runge in $X$ for all $t \in [0, 1]$. Then
there exists a fibre-preserving homotopy $\Phi_t : X \to X$ of holomorphic automorphisms
such that $\Phi_0 = \text{id}_X$ and $\Phi_t$ is arbitrarily close to $\varphi_t$ on $\Omega$ in the compact-open
topology for every $t \in [0, 1]$.

Moreover, $\Phi_t$ can be chosen as a composition of flow maps corresponding to com-
plete fibre-preserving vector fields which generate a dense Lie subalgebra of the Lie
algebra of the fibre-preserving holomorphic vector fields.

By a *fibre-preserving homotopy* $\varphi_t : \Omega \to X$ we mean a homotopy such that $\pi \circ
\varphi_t = \pi$. A vector field $\Theta$ is called *fibre-preserving* if it is tangential to the fibres, i.e.
$d\pi(\Theta) = 0$.

By $\text{Aut}^{\pi}(X)$ we denote the group of fibre-preserving holomorphic automorphisms
of $X$, i.e.

$$\text{Aut}^{\pi}(X) := \{ f : X \to X \text{ holomorphic automorphism, } \pi \circ f = \pi \}$$

and by $\text{Aut}^{\pi}_0(X)$ its path-connected component of identity.

By choosing $\Omega = X$ and $\varphi_t$ a path in $\text{Aut}^{\pi}(X)$ starting at $\text{id}_X$, we obtain:
Corollary 1.5 Let $X$ be a Stein manifold and let $\pi : X \to Y$ be a holomorphic fibration with the fibred density property. Then all fibre-preserving holomorphic automorphisms $\text{Aut}^0_\pi(X)$ path-connected to the identity can be approximated by compositions of time-1 maps of flows of complete fibre-preserving vector fields that generate a dense Lie subalgebra of the Lie algebra of fibre-preserving holomorphic vector fields.

Next we present an application of our general result concerning the fibred density to a concrete example, the so-called spectral ball. It appears naturally in Control Theory [5,6], but is also of theoretical interest in Several Complex Variables.

Definition 1.6 The spectral ball of dimension $n \in \mathbb{N}$ is defined to be

$$\Omega_n := \{ A \in \text{Mat}(n \times n; \mathbb{C}) : \rho(A) < 1 \}$$

where $\rho$ denotes the spectral radius, i.e. the modulus of the largest eigenvalue.

The study of the group of holomorphic automorphisms of the spectral ball started with the work of Ransford and White [20] in 1991 and continued by various authors [4,21,22]. Recently, Kosiński [16] described a dense subgroup of the $2 \times 2$ spectral ball $\Omega_2$. We generalize this result to $\Omega_n$ for $n \geq 2$ with another approach using the fibred density property.

The spectral ball $\Omega_n$ can also be understood in the following way: Denote by $\sigma_1, \ldots, \sigma_n : \mathbb{C}^n \to \mathbb{C}$ the elementary symmetric polynomials in $n$ complex variables. Let $\text{Eig} : \text{Mat}(n \times n; \mathbb{C}) \to \mathbb{C}^n$ assign to each matrix a vector of its eigenvalues. Then we denote by $\pi_1 := \sigma_1 \circ \text{Eig}, \ldots, \pi_n := \sigma_n \circ \text{Eig}$ the elementary symmetric polynomials in the eigenvalues. By symmetrizing we avoid any ambiguities of the order of eigenvalues in the definition of $\text{Eig}$ and obtain a polynomial map $\pi_1, \ldots, \pi_n$, symmetric in the entries of matrices in $\text{Mat}(n \times n; \mathbb{C})$, actually

$$\chi_A(\lambda) = \lambda^n + \sum_{j=1}^n (-1)^j \cdot \pi_j(A) \cdot \lambda^{n-j}$$

where $\chi_A$ denotes the characteristic polynomial of $A$.

Now we can consider the fibration $\pi := (\pi_1, \ldots, \pi_n) : \Omega_n \to \mathbb{G}_n$ of the spectral ball over the symmetrized polydisc $\mathbb{G}_n := (\sigma_1, \ldots, \sigma_n)(\mathbb{D}^n)$. A generic fibre, i.e. a fibre above a base point with no multiple eigenvalues, consists exactly of one equivalence class of similar matrices. Therefore it is natural to study the action of $\text{SL}_n(\mathbb{C})$ on $\Omega_n$ by conjugation.

A generic fibre is obviously a homogeneous space and hence smooth. A fibre above a base point with multiple eigenvalues decomposes into several strata of $\text{SL}_n(\mathbb{C})$ orbits where the largest orbit is the orbit of a matrix with the largest possible Jordan blocks. The structure of these fibres is well-known in classical invariant theory, see e.g. [18].

Because a generic fibre is a homogeneous space of the complex Lie group $\text{SL}_n(\mathbb{C})$, it has the density property (according to [8]). This does however not imply the fibred density property, since the dependence on the base point and the role of singular fibres.
is a priori not clear. However it motivates the investigation of fibre-preserving automorphisms by exploiting the homogeneity of the generic fibres. The most difficult part of our paper is to prove the fibred density property for \( \pi : \Omega_n \to \mathbb{C}^n \) (see Theorem 4.6) that enables us to determine a dense subgroup of the holomorphic automorphism group \( \text{Aut}(\Omega_n) \).

**Theorem 1.7** The \( \text{SL}_n(\mathbb{C}) \)-shears and the \( \text{SL}_n(\mathbb{C}) \)-overshears together with matrix transposition and Möbius transformations generate a dense subgroup (in compact-open topology) of the holomorphic automorphism group \( \text{Aut}(\Omega_n) \).

The precise definitions of shears and overshears will be given in Sect. 3. They are obtained as time-1 maps of certain re-parametrizations of flow maps of complete vector fields; in case of \( \text{SL}_n(\mathbb{C}) \)-shears and \( \text{SL}_n(\mathbb{C}) \)-overshears they are certain re-parametrizations of 1-parameter subgroups of \( \text{SL}_n(\mathbb{C}) \) which act on \( \text{Mat}(n \times n; \mathbb{C}) \) by conjugation. For the definition of Möbius transformations of matrices we refer to Eq. (8) on page 16.

Similar to the situation for the holomorphic automorphism group of \( \mathbb{C}^n, n \geq 2 \), it seems impossible to give an explicit set of algebraic generators for \( \text{Aut}(\Omega_n) \), since we prove the following in the last section:

**Theorem 1.8** The dense subgroup of \( \text{Aut}(\Omega_n) \) generated by the automorphisms in Theorem 1.7 is a meagre subset of \( \text{Aut}(\Omega_n) \).

Since many homogeneous spaces of complex Lie groups enjoy the density property, it is natural to ask the following question.

**Question 1.9** For which holomorphic actions of a reductive group \( G \) on a Stein manifold \( X \) does the map \( \pi : X \to X // G \) to the categorical quotient admit the fibred density property?

## 2 Andersén–Lempert theory for fibrations

We follow the original idea of the Andersén–Lempert Theorem, see [2,3] and also the survey article [12] and the textbook [9, Sec. 4] for a more recent presentation.

**Definition 2.1** ([1, p. 254] and [9, Def. 4.8.1]) Let \( \Theta \) be a vector field on a complex manifold \( X \), and let \((t, x) \mapsto A_t(x)\) be a continuous map to \( X \), defined on an open subset of \( \mathbb{R} \times X \) containing \( \{0\} \times X \) such that its \( t \)-derivative exists and is continuous. We say that \( A \) is algorithm for \( \Theta \) if we have for all \( x \in X \) that

\[
A_0(x) = x \\
\left. \frac{d}{dt} A_t(x) \right|_{t=0} = \Theta_x
\]

Obviously, a flow map is always an algorithm whereas the converse does not need to be true. However, the following variant of Euler’s method for solving an ODE works:

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Proposition 2.2 ([1, Thm. 4.1.26] and [9, Thm. 4.8.2]) Let \( \Theta \) be a locally Lipschitz continuous vector field with flow \( \varphi \) on a complex manifold \( X \). Let \( \Omega \) be the fundamental domain of \( \Theta \) and \( \Omega_+ := \Omega \cap (\mathbb{R}_+ \times X) \). If \( A_t \) is an algorithm for \( \Theta \), then for all \((t, x) \in \Omega_+\) the \( n \)-th iterate \( A_{t/n}^n(x) \) of the map \( A_{t/n} \) is defined for sufficiently large \( n = n(t, x) \in \mathbb{N} \) and we have

\[
\lim_{n \to \infty} A_{t/n}^n(x) = \varphi_t(x)
\]

The convergence is uniform on compacts in \( \Omega_+ \). Conversely, if \( t_0 > 0 \) is such that \( A_{t/n}^n(x) \) is defined for all \( t \in [0, t_0] \) and all sufficiently large \( n \in \mathbb{N} \), and \( \lim_{n \to \infty} A_{t/n}^n(x) \) exists, then \((t_0, x) \in \Omega_+\).

For simplicity, we focus on the situation \( t \geq 0 \), but the same results hold for negative times by replacing \( \Theta \) with \( -\Theta \). The following lemma can be verified easily in local coordinates by Taylor series expansion, see e.g. [9, Prop. 4.7.3].

Lemma 2.3 Let \( X \) be a complex manifold and let \( \Theta \) and \( \Xi \) be holomorphic vector fields on \( X \) with flow maps \( \varphi \) and \( \psi \). Then

1. \( \varphi_t \circ \psi_t \) is an algorithm for \( \Theta + \Xi \)
2. \( \psi_{-\sqrt{t}} \circ \varphi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}} \) is an algorithm for \([\Theta, \Xi]\).

Proof of Theorem 1.4 By \( \mathcal{A} \) we denote a Lie subalgebra of complete fibre-preserving vector fields which is dense in the Lie algebra of fibre-preserving holomorphic vector fields. This Lie subalgebra exists by assumption (fibred density property).

We define a time-dependent vector field

\[
\Theta_z^t := \varphi_t(\varphi_t^{-1}(z))
\]

which is still tangent to the fibres of \( \pi : X \to Y \). For any \( n \in \mathbb{N} \) we can partition the interval \([0, 1]\) in \( n \) intervals \([k/n, (k + 1)/n]\) of length \( 1/n \) and consider the piecewise constant vector field

\[
\Theta_z^t := \Theta_z^{k/n} \text{ for } t \in [k/n, (k + 1)/n]
\]

Let \( \varphi_z^k \) denote the flow map of \( \Theta_z^{k/n} \). Because \( \Omega \subseteq X \) is Runge and \( X \) is Stein, we can approximate any fibre-preserving vector field on \( \Omega \) by a fibre-preserving vector field on \( X \), uniformly on compacts of \( \Omega \). We remark that the sheaf of germs of fibre-preserving vector fields is (as the kernel of the map induced by \( \pi \) between the coherent sheafs of sections of the tangent bundles) a coherent sheaf of \( \mathcal{O}(X) \) modules. A standard application of Cartan’s Theorems A and B implies that the sections of any coherent sheaf over a Runge subset in a Stein space can be approximated by global sections.

By assumption we know that every such vector field can be approximated by vector fields from the Lie algebra \( \mathcal{A} \). Proposition 2.2 and Lemma 2.3 then show that we are able to approximate the flows of all the vector fields in the closure of \( \mathcal{A} \) in the compact-open topology by the flows of the complete vector fields which generate \( \mathcal{A} \).
The composition of the flows $\phi^k \circ \cdots \circ \phi^1$ is the flow of $\hat{\Theta}^t$. It remains only to show that in the limit $n \to \infty$ the flow of $\hat{\Theta}^t$ converges uniformly on compacts to the flow of $\Theta^t$ which follows from the fact that these flows are tangent to each other at the times $k/n$. \hfill \Box

### 3 Shears and overshears for $\text{SL}_n(\mathbb{C})$

As a preparation for proving a fibred density property for the spectral ball, we need to study a special type of automorphisms, the so-called shears and overshears. They will serve us as building blocks for general automorphisms.

The following notion of generalized shears and overshears has been introduced by Varolin [24, Section 3].

**Definition 3.1** Let $X$ be a complex manifold and let $\Theta$ be a $\mathbb{C}$-complete vector field on $X$, i.e. such that its flow-map exists for all complex times. A vector field $f \cdot \Theta$, $f \in \mathcal{O}(X)$, is called a $\Theta$-shear vector field if $\Theta(f) = 0$. It is called a $\Theta$-overshear vector field if $\Theta^2(f) = 0$.

**Example 3.2** Let $X = \mathbb{C}^2$ with coordinates $(z, w)$. Then $\Theta = \partial_z$ is obviously a $\mathbb{C}$-complete vector field, with flow map $\phi_t(z, w) = (z + t, w)$. A $\partial_z$-shear vector field is of the form $f(w) \cdot \partial_z$, and a $\partial_z$-overshear vector field is of the form $(f(w) \cdot z + g(w)) \cdot \partial_z$ where $f, g \in \mathcal{O}(\mathbb{C})$.

The following $\mathbb{C}$-completeness result can be found also in [24, Section 3], but without an explicit formula for the flow map. Our proof gives an explicit formula which will be needed in the applications.

**Lemma 3.3** Let $X$ be a complex manifold and let $\Theta$ be a $\mathbb{C}$-complete vector field on $X$, then all $\Theta$-overshear vector fields are $\mathbb{C}$-complete as well. In fact, if $\phi_t$ denotes the flow map of $\Theta$, the flow map $\psi_t$ of $f \cdot \Theta$ is given by

$$
\psi_t(z) = \phi_t \left( f(z) \cdot t \cdot \frac{f(z)}{\Theta(z)} \right)
$$

where $\varepsilon : \mathbb{C} \to \mathbb{C}$ is given by

$$
\varepsilon(\zeta) = \sum_{k=1}^{\infty} \frac{\zeta^{k-1}}{k!} = \frac{\varepsilon - 1}{\zeta}
$$

**Remark 3.4** The flow map of a $\Theta$-shear takes the form

$$
\psi_t(z) = \phi_t f(z) (z)
$$

In particular, if $\phi_t$ and $f$ are polynomial, then $\psi_t$ is polynomial as well.

**Proof** We calculate the time derivative of the given $\psi_t$

$$
\frac{d}{dt} \psi_t(z) = \frac{d}{dt} \left( \phi_t(\exp(t\Theta \cdot f) - 1) \cdot f(z)/\Theta(z) \right)
$$
\[
\begin{align*}
&= \Theta_z f \cdot \exp(t \Theta_z f) \cdot f(z) / \Theta_z f \cdot \Theta_z f \\
&= f(z) \cdot \exp(t \Theta_z f) \cdot \Theta_{\psi_t(z)}
\end{align*}
\]

and consider now

\[
\frac{d}{dt} f(\psi_t(z)) = d_{\psi_t(z)} \circ \hat{\phi}(\exp(t \Theta_z f) \cdot f(z)) \cdot f(z) \cdot \exp(t \Theta_z f) = f(z) \cdot \exp(t \Theta_z f) \cdot \Theta_{\psi_t(z)} f
\]

Note that \(\frac{d}{dt} \Theta_{\psi_t(z)} f = 0\) because of \(\Theta^2 f = 0\). We can compare the higher order derivatives:

\[
\frac{d^m}{dt^m} f(\psi_t(z)) = f(z) \cdot \exp(t \Theta_z f) \cdot (\Theta_z f)^{m-1} \cdot \Theta_{\psi_t(z)} f \quad (1)
\]

\[
\frac{d^m}{dt^m} f(z) \cdot \exp(t \Theta_z f) = f(z) \cdot \exp(t \Theta_z f) \cdot (\Theta_z f)^m \quad (2)
\]

For \(t = 0\) and for all \(m \in \mathbb{N}_0\) the values of (1) and (2) agree, hence \(f(\psi_t(z)) = f(z) \cdot \exp(t \Theta_z f)\) and \(\frac{d}{dt} \psi_t(z) = f(z) \cdot \Theta_{\psi_t(z)}\).

We will call the time-1 maps of such \(\mathbb{C}\)-complete vector fields \(\Theta\)-shears resp. \(\Theta\)-overshears.

Observe that the action of \(\text{GL}_n(\mathbb{C})\) on \(\text{Mat}(n \times n; \mathbb{C})\), by conjugation, is not effective, its center is the ineffectivity, and we get an effective action of \(\text{SL}_n(\mathbb{C})\). Moreover, as a linear representation this is the direct sum of the adjoint representation and a trivial one-dimensional representation, i.e. \(\text{Mat}(n \times n; \mathbb{C}) \cong \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}\), where the second summand is the subspace of scalar matrices.

In our context here we will focus on shears and overshears arising from the \(\text{SL}_n(\mathbb{C})\)-action on \(\Omega_n\) and on \(\text{Mat}(n \times n; \mathbb{C})\) by conjugation.

A \(\Theta\)-shear of a vector field \(\Theta\) arising from the \(\text{SL}_n(\mathbb{C})\)-action will be called a \(\text{SL}_n(\mathbb{C})\)-shear and a \(\Theta\)-overshear of such a vector field will be called a \(\text{SL}_n(\mathbb{C})\)-overshear.

By \(E_{ab}\) with \(a, b \in \{1, \ldots, n\}\) we denote the elementary matrices in \(\text{Mat}(n \times n; \mathbb{C})\), i.e.

\[
E_{ab} = (\delta_{ak} \delta_{bt})_{k, \ell = 1}^n
\]

We denote the following commutators as \(H_a := [E_{a,a+1}, E_{a+1,a}] = (\delta_{ak} \delta_{at} - \delta_{a+1,k} \delta_{a+1,t})_{k, \ell = 1}^n\) for \(a = 1, \ldots, n - 1\). It is well-known that the \(E_{ab}\) with \(a \neq b\) together with the \(H_a\) span the matrix Lie algebra \(\mathfrak{sl}_n(\mathbb{C})\) as vector space over \(\mathbb{C}\). We need to write down explicitly the adjoint representation of \(\text{SL}_n(\mathbb{C})\) with vector fields and determine the action of these vector fields on polynomials.

For a matrix \(V \in \mathfrak{sl}_n(\mathbb{C})\) and \(X \in \text{Mat}(n \times n; \mathbb{C})\) it is well known that

\[
\left. \frac{d}{dt} \exp(tV) \cdot X \cdot \exp(-tV) \right|_{t=0} = [V, X]
\]
The entries $x_{k\ell}$ of a matrix $X \in \text{Mat}(n \times n; \mathbb{C})$ will serve as coordinates on $\text{Mat}(n \times n; \mathbb{C}) \cong \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}$. We denote the fundamental vector fields of the adjoint representation of $\mathfrak{sl}_n(\mathbb{C})$ corresponding to $E_{ab}$ resp. $H_a$ by $\Theta_{ab}$ resp. $\Xi_a$. They are

$$\Theta_{ab} := \sum_{k=1}^{n} \left( x_{bk} \frac{\partial}{\partial x_{ak}} - x_{ka} \frac{\partial}{\partial x_{kb}} \right), \quad a \neq b \quad (3)$$

$$\Xi_a := \sum_{k=1}^{n} \left( x_{ak} \frac{\partial}{\partial x_{ak}} - x_{a+1,k} \frac{\partial}{\partial x_{a+1,k}} - x_{ka} \frac{\partial}{\partial x_{ka}} + x_{k,a+1} \frac{\partial}{\partial x_{k,a+1}} \right) \quad (4)$$

We will frequently refer to the vector fields $\Xi_a$ as hyperbolic vector fields.

The vector fields $\Theta_{ab}$ and their commutators $\Xi_a = [\Theta_{a,a+1}, \Theta_{a+1,a}]$ span the adjoint representation of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ written as vector fields and obey of course the same commutation relations as the $E_{ab}$ and $H_a$. In particular,

$$[\Theta_{ab}, \Theta_{cd}] = 0 \iff a = c \lor b = d \quad (5)$$

**Example 3.5** Let $n \geq 2$. Since $\Theta_{12}(x_{21}) = 0$, the vector field $x_{21}\Theta_{12}$ is a shear vector field. And its flow map is given by

$$X \mapsto \exp(tx_{21}E_{12}) \cdot X \cdot \exp(-tx_{21}E_{12}) = (\text{id} + tx_{21}E_{12}) \cdot X \cdot (\text{id} - tx_{21}E_{12})$$

The semi-group property is satisfied because $x_{21}$ is conjugation invariant under the action of the one-parameter subgroup generated by $\Theta_{12}$.

Now we consider the overshear vector field $x_{11}\Theta_{12}$ with $\Theta_{12}(x_{11}) = x_{21}$ and $\Theta_{12}^2(x_{11}) = 0$. Using the function $\varepsilon$ from Lemma 3.3, the flow map is given by

$$X \mapsto \exp(\varepsilon(tx_{21}) \cdot x_{11} \cdot E_{12}) \cdot X \cdot \exp(-\varepsilon(tx_{21}) \cdot x_{11} E_{12}) = \exp \left( (e^{tx_{21}} - 1) \frac{x_{11}}{x_{21}} \cdot E_{12} \right) \cdot X \cdot \exp \left( -(e^{tx_{21}} - 1) \frac{x_{11}}{x_{21}} \cdot E_{12} \right)$$

The semi-group property is less obvious, but can be verified by direct calculation or the more abstract argument in Lemma 3.3.

### 4 Fibred density property for the spectral ball

In this section we prove the fibred density property for the spectral ball and determine its automorphism group. The crucial technical part is Proposition 4.5, and the following lemmas will be needed for the induction in the proof of this proposition.

**Lemma 4.1** For $\Theta_{ab}$, $a \neq b$ and $x_{cd}$ with $a, b, c, d \in \{1, \ldots, n\}$ we have

$$\Theta_{ab}(x_{cd}) = \delta_{ac}x_{bd} - \delta_{bd}x_{ca}$$
and for \( a < n \) we have
\[
\Xi_a(x_{cd}) = (\delta_{ac} - \delta_{a+1,c} - \delta_{ad} + \delta_{a+1,d}) x_{cd}
\]

**Proof** The proof is a straightforward calculation. \( \square \)

**Corollary 4.2**

\[
\begin{align*}
\Theta_{ab}(x_{cd}) &= 0 & \iff a \neq c \land b \neq d \\
\Theta_{ab}^2(x_{cd}) &= 0 & \iff a \neq c \lor b \neq d \\
\Theta_{ab}(x_{ab}) &= x_{bb} - x_{aa} \\
\Theta_{ab}^2(x_{ab}) &= -2x_{ba} & \Theta_{ab}^3(x_{ab}) &= 0 \\
\Xi_a(x_{cd}) &= 0 & \iff c = d \lor \{c, d\} \cap \{a, a + 1\} = \emptyset
\end{align*}
\]

We illustrate these results by summarizing them for \( \mathfrak{sl}_3(\mathbb{C}) \) in Tables 1 and 2.

**Lemma 4.3** Let \( n \geq 3 \). The \( \text{span}_\mathbb{C} \{\Theta_{ab}(x_{cd}) : \Theta_{12}(x_{cd}) = 0\} \) contains all linear monomials except \( x_{12} \).

**Table 1** Vector fields for the adjoint representation of \( \mathfrak{sl}_3(\mathbb{C}) \)

| \( \Theta_{12} \) | \( \Theta_{13} \) | \( \Theta_{21} \) | \( \Theta_{23} \) | \( \Theta_{31} \) | \( \Theta_{32} \) | \( \Xi_1 \) | \( \Xi_2 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( +x_{11}\partial_{11} - x_{11}\partial_{12} + x_{22}\partial_{12} + x_{23}\partial_{13} - x_{21}\partial_{22} - x_{31}\partial_{32} \) | \( +x_{31}\partial_{11} + x_{32}\partial_{12} - x_{11}\partial_{13} + x_{33}\partial_{13} - x_{21}\partial_{23} - x_{31}\partial_{33} \) | \( -x_{12}\partial_{11} + x_{11}\partial_{21} - x_{22}\partial_{21} + x_{12}\partial_{22} + x_{13}\partial_{23} - x_{32}\partial_{31} \) | \( -x_{12}\partial_{13} + x_{31}\partial_{21} + x_{32}\partial_{22} - x_{22}\partial_{23} + x_{33}\partial_{23} - x_{32}\partial_{33} \) | \( -x_{13}\partial_{11} - x_{23}\partial_{21} + x_{11}\partial_{31} - x_{33}\partial_{31} + x_{12}\partial_{32} + x_{13}\partial_{33} \) | \( +x_{12}\partial_{12} + x_{13}\partial_{13} - 2x_{21}\partial_{21} - x_{23}\partial_{23} - x_{31}\partial_{31} + x_{32}\partial_{32} \) | \( -x_{12}\partial_{12} + x_{13}\partial_{13} + x_{21}\partial_{21} + 2x_{23}\partial_{23} - x_{31}\partial_{31} - 2x_{32}\partial_{32} \) |

**Table 2** Action of vector fields on linear monomials for \( \mathfrak{sl}_3(\mathbb{C}) \)

| \( x_{11} \) | \( x_{21} \) | \( x_{31} \) | \( -x_{12} \) | \( 0 \) | \( -x_{13} \) | \( 0 \) | \( 0 \) | \( 0 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( x_{12} \) | \( -x_{11} + x_{22} \) | \( x_{32} \) | \( 0 \) | \( 0 \) | \( 0 \) | \( -x_{13} \) | \( 2x_{12} \) | \( -x_{12} \) |
| \( x_{13} \) | \( x_{23} \) | \( -x_{11} + x_{33} \) | \( 0 \) | \( -x_{12} \) | \( 0 \) | \( 0 \) | \( x_{13} \) | \( x_{13} \) |
| \( x_{21} \) | \( 0 \) | \( 0 \) | \( x_{11} - x_{22} \) | \( x_{31} \) | \( -x_{23} \) | \( 0 \) | \( -2x_{21} \) | \( x_{21} \) |
| \( x_{22} \) | \( -x_{21} \) | \( 0 \) | \( x_{12} \) | \( x_{32} \) | \( 0 \) | \( -x_{23} \) | \( 0 \) | \( 0 \) |
| \( x_{23} \) | \( 0 \) | \( -x_{21} \) | \( x_{13} \) | \( -x_{22} + x_{33} \) | \( 0 \) | \( 0 \) | \( -x_{23} \) | \( 2x_{23} \) |
| \( x_{31} \) | \( 0 \) | \( 0 \) | \( -x_{32} \) | \( 0 \) | \( x_{11} - x_{33} \) | \( x_{21} \) | \( -x_{31} \) | \( -x_{31} \) |
| \( x_{32} \) | \( -x_{31} \) | \( 0 \) | \( 0 \) | \( 0 \) | \( x_{12} \) | \( x_{22} - x_{33} \) | \( x_{32} \) | \( -2x_{32} \) |
| \( x_{33} \) | \( 0 \) | \( -x_{31} \) | \( 0 \) | \( -x_{32} \) | \( x_{13} \) | \( x_{23} \) | \( 0 \) | \( 0 \) |
Proof (1) We consider first the monomials of the form \( x_{aa} \): Let \( x_{ab} \in \ker \Theta_{12} \), i.e. \( a \neq 1 \) and \( b \neq 2 \).

\[
\Theta_{ab}(x_{ab}) = x_{bb} - x_{aa}
\]

Thus, we obtain all such differences except \( x_{22} - x_{11} \), but instead the scalar multiple \( x_{11} - x_{22} \). For the coordinate functions on \( \mathfrak{sl}_n(\mathbb{C}) \), i.e. elements of \( (\mathfrak{sl}_n(\mathbb{C}))^* \), we further have the trace condition \( x_{11} + x_{22} + \cdots + x_{nn} = 0 \). Therefore the span contains all functions \( x_{aa} \) on \( \mathfrak{sl}_n(\mathbb{C}) \).

(2) For \( a \neq 1, d \neq 2 \) and \( b \neq d \) we obtain

\[
\Theta_{ab}(x_{ad}) = x_{bd}
\]

hence all \( x_{k\ell} \) with \( \ell \neq 2, k \neq \ell \) are in the span. Here, we need \( n \geq 3 \). For \( c \neq 1, b \neq 2 \) and \( c \neq a \) we obtain

\[
\Theta_{ab}(x_{cb}) = -x_{ca}
\]

hence \( x_{k\ell} \) with \( k \neq 1, k \neq \ell \) are in the span. We again need \( n \geq 3 \).

We are now prepared to prove our main proposition. The proof is by induction over the degree. To understand the proof and its structure it might be helpful to look first at the induction step which starts after degree two.

Definition 4.4

\[
\mathcal{L}_n := \text{span}_{\mathbb{C}}\{f \cdot \Theta : f \text{ polynomial on } \mathfrak{sl}_n(\mathbb{C}), \Theta \in \langle \Theta_{k\ell}, \Xi_m \rangle \}
\]

By \( A_n \) we denote the Lie algebra generated by all vector fields which are \( \text{SL}_n(\mathbb{C}) \)-overshears with monomial coefficients of degree at most 2.

Proposition 4.5 Let \( n \geq 2 \), then \( \mathcal{L}_n = A_n \).

Proof We only need to show the inclusion \( \mathcal{L}_n \subseteq A_n \). The proof is by induction on the degree \( d \) of the polynomial coefficients. The induction hypothesis for \( d = 0 \) is true by assumption.

We treat the case \( d = 1 \) separately: The only missing vector fields are \( x_{k\ell} \Theta_{k\ell} \) and the linear monomials in front of the hyperbolic vector fields. For \( (k, \ell) = (1, 2) \) a short calculation shows:

\[
[x_{22} \Theta_{12}, \Theta_{21}] = x_{22}[\Theta_{12}, \Theta_{21}] - (\Theta_{21} x_{22}) \Theta_{12} = x_{22} \Xi_1 + x_{12} \Theta_{12}
\]

Because \( x_{22} \Xi_1 \) is a shear vector field, we can conclude that \( x_{12} \Theta_{12} \) is in \( A_n \). By symmetry (index permutation in \( \Theta_{k\ell} \)), this is true for all \( x_{k\ell} \Theta_{k\ell} \). For the hyperbolic vector fields, see the general case, step 3.
We also need to treat the case \( d = 2 \) separately:

1. Let \( a \) and \( f \) be monomials of degree one. We first remark that if \( \Theta(a) = 0 \) and \( \Theta^2(f) = 0 \), then \( \Theta^2(af) = 0 \), i.e. under these assumptions \( af \cdot \Theta \) is a \( \Theta \)-overshear vector field and hence in \( \mathcal{A}_q \). Consider the following difference of Lie brackets:

\[
[af \Theta, \Lambda] - [f \Theta, a \Lambda] = af[\Theta, \Lambda] - \Lambda(af)\Theta - af[\Theta, \Lambda] - f \Theta(a)\Lambda + a \Lambda(f)\Theta = -f \Lambda(a) \cdot \Theta
\]

For the quadratic terms in front of \( \Theta = \Theta_{k\ell} \) we can restrict ourselves without loss of generality to the case \( \Theta = \Theta_{12} \). By Lemma 4.3 we obtain all terms of the form \( f \Lambda(a) \cdot \Theta \otimes x_{ab} x_{cd} \Theta_{12} \) with \( x_{ab} \neq x_{12} \neq x_{cd} \).

2. We focus on the most difficult term, i.e. \( x_{k\ell}^2 \Theta_{k\ell} \). It is sufficient to consider \( \Theta_{12} \): We make a detour to a hyperbolic vector field and aim to obtain \( x_{12}^2 \Xi_1 \). Note that \( x_{12}^2 \cdot \Theta_{21} \) is a shear vector field.

\[
\left[ x_{12}^2 \cdot \Theta_{21}, \Theta_{12} \right] = 2x_{12} \Theta_{12}(x_{12}) \Theta_{21} - x_{12}^2 \Xi_1
\]

It remains to check that \( 2x_{12} \Theta_{12}(x_{12}) \Theta_{21} \) is an overshear vector field: \( \Theta_{21} (2x_{12} \Theta_{12}(x_{12})) = 2x_{12} \Theta_{21}(-x_{11} + x_{22}) = 4x_{12}^2 \) and \( \Theta_{21}(x_{12}^2) = 0 \).

Now we calculate the following Lie brackets of already obtained terms:

\[
[x_{12} \cdot \Xi_1, x_{12} \cdot \Theta_{12}] = x_{12} \Xi_1(x_{12}) \Theta_{12} - x_{12} \Theta_{12}(x_{12}) \Xi_1 + x_{12}^2 \Xi_1(\Xi_1, \Theta_{12}) = 4x_{12}^2 \Theta_{12} - x_{12} \Theta_{12}(x_{12}) \Xi_1
\]

\[
[x_{12}^2 \cdot \Xi_1, \Theta_{12}] = x_{12}^2 [\Xi_1, \Theta_{12}] - 2x_{12} \Theta_{12}(x_{12}) \Xi_1 = 2x_{12}^2 \Theta_{12} - 2x_{12} \Theta_{12}(x_{12}) \Xi_1
\]

Now, a linear combination of these Lie brackets yields

\[
2[x_{12} \cdot \Xi_1, x_{12} \cdot \Theta_{12}] - [x_{12}^2 \cdot \Xi_1, \Theta_{12}] = 6x_{12}^2 \Theta_{12}
\]

3. After having obtained the term \( x_{12}^2 \) in front of \( \Theta_{12} \) we get the other terms by letting \( \mathfrak{sl}_n(\mathbb{C}) \) act on it and subtracting already obtained terms:

\[
\left[ x_{12}^2 \Theta_{12}, \Lambda \right] - [x_{12} \Theta_{12}, x_{12} \Lambda] = x_{12}^2 [\Theta_{12}, \Lambda] - \Lambda(x_{12}^2) \Theta_{12} - x_{12} \Theta_{12}(x_{12}) \Lambda + x_{12} \Lambda(x_{12}) \Theta_{12} - x_{12} x_{12} [\Theta_{12}, \Lambda] = -x_{12} \Lambda(x_{12}) \cdot \Theta_{12} + \Theta_{12}(x_{12}) \cdot \Lambda
\]

By Lemma 4.3 we obtain all terms of the form \( x_{12} x_{cd} \Theta_{12} \) if we manage to subtract the terms \( x_{12} \Theta_{12}(x_{12}) \cdot \Lambda = -x_{12} x_{11} \Lambda + x_{12} x_{22} \Lambda \). For this we only need to
see that for a $\Lambda = \Theta_{k\ell}$, $k \neq \ell$, it is always true that both $\Lambda^2(x_{12}) = 0$ and $\Lambda^2(x_{11}) = 0$ as well as $\Lambda^2(x_{22}) = 0$ which follows from Corollary 4.2.

(4) For other hyperbolic vector fields, see again the general case, step 3.

**Induction step** $d \mapsto d + 1$, $d \geq 2$:

(1) Let $f$ and $g$ be monomials of degree $d - 1$ (or less) and let $a$ be a monomial of degree one.

$$[af \cdot \Theta, g \cdot \Lambda] - [f \cdot \Theta, ag \cdot \Lambda] = (af \Theta(g)\Lambda - g\Lambda(af)\Theta + afg[\Theta, \Lambda])$$

$$- (f\Theta(ag)\Lambda - ag\Lambda(f)\Theta + afg[\Theta, \Lambda])$$

$$= af\Theta(g)\Lambda - ga\Lambda(f)\Theta - gf\Lambda(a)\Theta$$

$$- af\Theta(g)\Lambda + fg\Theta(a)\Lambda + ag\Lambda(f)\Theta$$

$$= - fg \cdot (\Theta(a)\Lambda + \Lambda(a)\Theta)$$

To obtain the coefficients in front of $\Theta_{k\ell}$ it is by symmetry sufficient to consider $\Theta = \Theta_{12}$. Choose $a \in \ker \Theta_{12}$. For $\Lambda$ we can choose any other vector field in $\mathfrak{s}_{kn}$.

From Lemma 4.3 we know that all linear monomials except $x_{12}$ are obtained as $\Lambda(a)$ in case of dimension $n \geq 3$. In dimension $n = 2$ we only have $\Theta_{12}(x_{21}) = 0$, but – using also the hyperbolic vector field – still obtain $\Theta_{21}(x_{21}) = x_{11} - x_{22}$ and $\Xi_1(x_{21}) = 2x_{21}$; note that $x_{11} + x_{22} = 0$. We therefore obtain all monomial coefficients $fg\Lambda(a)$ of degree $d + 1$ (actually, up to $2d - 1$) in front of $\Theta_{12}$ except $x_{12}^{d+1}\Theta_{12}$.

(2) To obtain $x_{12}^{d+1}\Theta_{12}$ we calculate:

$$[x_{12} \cdot \Xi_1, x_{12}^d \cdot \Theta_{12}] = x_{12}^d \Xi_1(x_{12})\Theta_{12} - x_{12}^d \Theta_{12}(x_{12})\Xi_1 + x_{12}^{d+1}[\Xi_1, \Theta_{12}]$$

$$= d \cdot x_{12}^d \Xi_1(x_{12})\Theta_{12} - x_{12}^d \Theta_{12}(x_{12})\Xi_1 + x_{12}^{d+1}[\Xi_1, \Theta_{12}]$$

$$= d \cdot x_{12}^d \cdot 2x_{12}\Theta_{12} - x_{12}^d(-x_{11} + x_{22})\Xi_1 + x_{12}^{d+1}\Theta_{12}$$

$$= (2d + 2)x_{12}^{d+1}\Theta_{12} - x_{12}^d(-x_{11} + x_{22})\Xi_1$$

and

$$[x_{12}^d \cdot \Xi_1, x_{12} \cdot \Theta_{12}] = x_{12}^d \Xi_1(x_{12})\Theta_{12} - x_{12}^d \Theta_{12}(x_{12})\Xi_1 + x_{12}^{d+1}[\Xi_1, \Theta_{12}]$$

$$= x_{12}^d \Xi_1(x_{12})\Theta_{12} - d \cdot x_{12}^d \Theta_{12}(x_{12})\Xi_1 + x_{12}^{d+1}[\Xi_1, \Theta_{12}]$$

$$= x_{12}^d \cdot 2x_{12}\Theta_{12} - d \cdot x_{12}^d(-x_{11} + x_{22})\Xi_1 + x_{12}^{d+1}\Theta_{12}$$

$$= 4x_{12}^{d+1}\Theta_{12} - d \cdot x_{12}^d(-x_{11} + x_{22})\Xi_1$$

A linear combination of these two Lie brackets yields

$$d \cdot [x_{12} \cdot \Xi_1, x_{12}^d \cdot \Theta_{12}] - [x_{12}^d \cdot \Xi_1, x_{12} \cdot \Theta_{12}] = 2(d^2 + d - 2)x_{12}^{d+1}\Theta_{12}$$

and we have found all monomial coefficients of degree $d + 1$ in front of the $\Theta_{k\ell}$. 

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(3) Now we turn to the hyperbolic vector fields. Again by symmetry it is sufficient to consider \( \Xi_1 = [\Theta_{12}, \Theta_{21}] \).

\[
[\Theta_{21}, f \cdot \Theta_{12}] = f \cdot \Xi_1 - \Theta_{21}(f) \cdot \Theta_{12}
\]

Hence, we obtain all polynomials \( f \) of degree \( d + 1 \) in front of \( \Xi_1 \) which are such that both \( f \cdot \Theta_{12} \) and \( \Theta_{21}(f) \cdot \Theta_{12} \) are already known to be in \( A_n \). Since \( \Theta_{21}(f) \) does not increase the degree of \( f \), we are done. \( \square \)

**Theorem 4.6** The fibration \( \pi : \text{Mat}(n \times n; \mathbb{C}) \to \mathbb{C}^n \) has the fibred density property as has any restriction of \( \pi \) to a Runge domain \( \Omega \subseteq \text{Mat}(n \times n; \mathbb{C}) \) with \( \pi^{-1}(\pi(\Omega)) = \Omega \).

**Corollary 4.7** The natural fibration \( \pi : \Omega_n \to G_n \) has the fibred density property.

**Proof** The condition \( \pi^{-1}(\pi(\Omega_n)) = \Omega_n \) is clear, because \( \Omega_n = \pi^{-1}(G_n) \). The domain \( \Omega_n \subseteq \text{Mat}(n \times n; \mathbb{C}) \) is balanced since obviously \( \lambda z \in \Omega_n \) for all \( z \in \Omega_n \) and all \( \lambda \in \mathbb{D} \). Therefore it is Runge and then by the preceding remark it enjoys the fibred density property. \( \square \)

We will need the following terminology from representation theory and a result of Dixmier [7].

**Definition 4.8** Let \( \mathfrak{g} \) be a complex Lie algebra and \( f \) a holomorphic function on \( \mathfrak{g} \). Then \( f \) is called invariant if

\[
\forall x_0 \in \mathfrak{g} \quad \forall x \in \mathfrak{g} \quad [x, x_0]f(x) = 0.
\]

Let \( U \subseteq \mathfrak{g} \) be an open subset and \( f \) a function defined on \( U \). A function \( f \) is called locally invariant if this holds \( \forall x_0 \in \mathfrak{g} \quad \forall x \in U \).

**Theorem 4.9** ([7, Théorème 2.4]) Let \( \mathfrak{g} \) be a complex semi-simple Lie algebra. Let \( U \subseteq \mathfrak{g} \) be an open Stein subset and \( \Theta \) a holomorphic vector field on \( U \). Then the following conditions are equivalent:

1. \( \Theta \) annihilates the locally invariant functions on \( U \).
2. There exists a holomorphic map \( g : U \to \mathfrak{g} \) such that \( \Theta(x) = [x, g(x)] \) for all \( x \in U \).

**Corollary 4.10** Every fibre-preserving holomorphic vector field on an \( \Omega \subseteq \text{Mat}(n \times n; \mathbb{C}) \) which is Stein and satisfies \( \Omega = \pi^{-1}(\pi(\Omega)) \) can be written as a holomorphic linear combination of the vector fields \( \Theta_{ab}, \Xi_c \).

**Proof** Each fibre in \( \Omega \) is also a fibre in \( \text{Mat}(n \times n; \mathbb{C}) \) due to \( \Omega = \pi^{-1}(\pi(\Omega)) \). It follows from Theorem 4.9 that the vector fields \( \Theta_{ab}, \Xi_c \) (which form a basis of the Lie algebra) generate the stalk of the sheaf of germs of fibre-preserving vector fields for the categorical quotient map \( \pi|_{\mathfrak{sl}_n(\mathbb{C})} : \mathfrak{sl}_n(\mathbb{C}) \to \mathbb{C}^{n-1} \) at every point, including the singular points. (This result is in fact due to Kostant [17], and Dixmier’s contribution in proving the above Theorem lies in the application of Cartan’s Theorem B.) If we...
denote by $T$ the tangent sheaf and by $T^\pi$ its subsheaf of sections tangent to $\pi$, this is equivalent to the following exact sequence of sheaves:

$$\mathcal{O}_{\mathfrak{sl}_n(\mathbb{C})}^{n^2-1} \to T^\pi_{\mathfrak{sl}_n(\mathbb{C})} \to 0$$

(6)

Remember that as linear representations $\text{Mat} (n \times n; \mathbb{C}) \cong \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}$ and thus $\pi : \text{Mat} (n \times n; \mathbb{C}) \to \mathbb{C}^n$ is equal to $\pi (\mathfrak{sl}_n(\mathbb{C}) \oplus \text{id}_\mathbb{C}$.

The exactness of the sequence (6) implies the exactness of the following sequence:

$$\mathcal{O}_{\mathfrak{gl}_n(\mathbb{C})}^{n^2-1} \to T^\pi_{\mathfrak{gl}_n(\mathbb{C})} \to 0$$

(7)

Now a standard application of Cartan’s Theorem B implies the claim of the Corollary.

To see the exactness of the sequence (7), let $U \subset \mathfrak{sl}_n(\mathbb{C})$ and $V \subset \mathbb{C}$ be open sets, biholomorphic to (poly-)discs. Then we have

$$\mathcal{O}_{\mathfrak{gl}_n(\mathbb{C})}(U \times V) \cong \mathcal{O}_{\mathfrak{sl}_n(\mathbb{C})}(U) \hat{\otimes} \mathcal{O}(V)$$

canonically. For $U$ and $V$ so small that the involved tangent spaces are trivial, we have moreover that $T_{\mathfrak{gl}_n(\mathbb{C})}(U \times V) \cong T_{\mathfrak{sl}_n(\mathbb{C})}(U) \hat{\otimes} \mathcal{O}(V) \oplus \mathcal{O}(U) \hat{\otimes} T_{\mathbb{C}}(V)$ canonically, hence $T^\pi_{\mathfrak{gl}_n(\mathbb{C})}(U \times V) \cong T^\pi_{\mathfrak{sl}_n(\mathbb{C})}(U) \hat{\otimes} \mathcal{O}(V)$.

Here, $\hat{\otimes}$ stands for the completed tensor product in the sense of Grothendieck. It does not matter with respect to which tensor norm this completion is taken, because spaces of holomorphic functions are nuclear [19, Section 6.4]. Cartan’s Theorem B implies that

$$\mathcal{O}_{\mathfrak{sl}_n(\mathbb{C})}^{n^2-1}(U) \to T^\pi_{\mathfrak{sl}_n(\mathbb{C})}(U)$$

induced from the sequence (6) is surjective, and by [23, Proposition 43.9] the map

$$\mathcal{O}_{\mathfrak{sl}_n(\mathbb{C})}^{n^2-1}(U) \hat{\otimes} \mathcal{O}(V) \to T^\pi_{\mathfrak{sl}_n(\mathbb{C})}(U) \hat{\otimes} \mathcal{O}(V)$$

is also surjective. This shows that that the sequence (7) is indeed exact.

We are now able to prove the fibred density property for the spectral ball and to determine its automorphism group.

**Proof of Theorem 4.6** Let $\Theta$ be a fibre-preserving holomorphic vector field on $\Omega_n \subseteq \text{Mat} (n \times n; \mathbb{C})$. According to Corollary 4.10 we can write $\Theta$ as a holomorphic linear combination of the vector fields $\Theta_{ab}$ and $\Xi_c$. In $\text{Mat} (n \times n; \mathbb{C}) \cong \mathbb{C}^{n^2}$ we can approximate these holomorphic linear combinations by polynomial linear combinations. This works as well for any Runge domain $\Omega \subseteq \text{Mat} (n \times n; \mathbb{C})$. Note that all holomorphic linear combinations of such vector fields are automatically tangent to the fibres of $\sigma$.

By Proposition 4.5 the Lie algebra $L_n$ is generated by the vector fields in $\mathcal{A}_n$ which correspond to $\text{SL}_n(\mathbb{C})$-shears and $\text{SL}_n(\mathbb{C})$-overshears with degree $d \leq 2$. Since we are interested in vector fields with polynomial coefficients not only on $\mathfrak{sl}_n(\mathbb{C})$, but also
on \( \mathfrak{gl}_n(\mathbb{C}) \), we need to adjoin the trace as additional variable which gives an additional trivial fibre that hence enjoys the fibred density property as well, see Example 1.3. □

**Proof of Theorem 1.7** Let \( f : \Omega_n \to \Omega_n \) be a holomorphic automorphism. By [20, Thm. 4] we find a Möbius transformation \( h : \Omega_n \to \Omega_n \)

\[
A \mapsto \gamma \cdot (A - \alpha \cdot \text{id}) \cdot (\text{id} - \alpha A)^{-1}, \quad \alpha \in \mathbb{D}, \gamma \in \partial \mathbb{D}
\]

such that the composition \( g := f \circ h^{-1} : \Omega_n \to \Omega_n \) is fibre-preserving, i.e. \( \pi \circ g = \pi \), and in addition \( g(0) = 0 \). By [20, Thm. 2] we know that under these circumstances \( g'(0) \) is a linear automorphism of \( \Omega_n \). According to [20, Thm. 4, Cor.] it is of the form

\[
A \mapsto G \cdot A \cdot G^{-1} \quad \text{or} \quad A \mapsto G \cdot A^t \cdot G^{-1}, \quad \text{with} \ G \in \text{SL}_n(\mathbb{C})
\]

We can connect \( g \) to the identity, indeed we connect it to its linear part \( g'(0) \) by a \( C^1 \)-homotopy \( [0, 1] \ni s \mapsto \frac{g(sA)}{s} \to g'(0)(A) \) and then use the fact that any conjugation by a \( G \in \text{SL}_n(\mathbb{C}) \) can be connected to the identity or to the matrix transposition.

Therefore we may assume that \( f \), after composition with a Möbius transformation and possibly a transposition is in the identity component of \( \text{Aut}^\pi(\Omega_n) \). By Theorem 4.6 the spectral ball has the fibred density property and we can apply Theorem 1.4 with \( \mathcal{A} = A_n \) generated by \( \text{SL}_n(\mathbb{C}) \)-overshears (with coefficients of at most quadratic degree) as the dense Lie subalgebra generated by complete vector fields. □

We would like to compare our result to a question asked by Ransford and White [20] when they started the study of holomorphic automorphisms of the spectral ball. They asked whether the automorphisms of the spectral ball are compositions of the Möbius transformations, the transposition and conjugations of the following form \( X \mapsto \exp(f(X)) \cdot X \cdot \exp(f(X)) \) with \( f : \Omega_n \to \mathfrak{sl}_n(\mathbb{C}) \) such that it is \( \text{SL}_n(\mathbb{C}) \)-invariant, i.e. \( f(GXG^{-1}) = f(X) \) for all \( X \in \Omega_n \) and all \( G \in \text{SL}_n(\mathbb{C}) \). A counterexample was already given in [15].

It is however easy to see that the invariance condition is too restrictive: Choose a generic fibre \( X_\lambda \). It is a \( \text{SL}_n(\mathbb{C}) \)-homogeneous complex manifold and hence isomorphic to \( \text{SL}_n(\mathbb{C})/H \) for a reductive subgroup \( H \) of \( \text{SL}_n(\mathbb{C}) \). Hence there exists a (up to a scalar constant) unique algebraic volume form \( \omega' \) on \( X_\lambda \) which is invariant under the action of \( \text{SL}_n(\mathbb{C}) \) by left-multiplication, see [13, Appendix]. This volume form \( \omega' \) corresponds to a \( \text{SL}_n(\mathbb{C}) \)-conjugation invariant volume form \( \omega \). The invariance condition for the automorphisms of Ransford and White is equivalent to saying that \( G(X) \) depends only on \( \sigma(X) \). Therefore it necessarily preserves the volume form \( \omega \). However, the overshears that are not shears never preserve this volume form, since

\[
\text{div}(f \Theta) = f \text{ div } \Theta + \Theta(f).
\]

The divergence of a vector field with algebraic flow map (e.g. all the \( \Theta_{ab} \)) necessarily vanishes, and \( \Theta(f) \neq 0 \) for such overshears. Since all the automorphisms of Ransford and White however have vanishing \( \omega \)-divergence, no finite or infinite composition of them can yield such an overshear. In the next section we will prove a much stronger result.
5 Dense, but meagre

In Theorem 1.7 we have determined a dense subgroup of Aut($\Omega$). In this section we will prove that this dense subgroup is not the whole Aut($\Omega$), in fact a meagre subset. We follow the original strategy of Andersén and Lempert [3, Sec. 7]

**Definition 5.1** By $P_m$ we denote the vector space of polynomials in $\mathfrak{gl}_n(\mathbb{C})^*$ of total degree at most $m \in \mathbb{N}_0$. By $\tilde{P}_m$ we denote the subspace of homogeneous polynomials of total degree $m$.

The polynomial vector fields $\Theta_{ab}$, $a \neq b$, and $\Xi_a$ act as $\mathbb{C}$-linear derivations on the polynomials $\tilde{P}_m$ into $\tilde{P}_m$, preserving their total degree. In the following, we want to estimate the dimension of the kernels of $\Theta_{2ab}$ and $\Xi_a$.

**Proposition 5.2** Let $D : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ be a non-zero polynomial differential operator of degree $d$. By $K_m = \ker D|\tilde{P}_m \subset \mathbb{C}[z_1, \ldots, z_n]$ we denote the vector space of homogeneous polynomials of degree $m$ that are annihilated by $D$. We assume that $d \leq m$. Then we have the following estimate:

$$\dim_{\mathbb{C}} K_m \leq \left( \binom{m+n-1}{m} - \binom{m-d+n-1}{m-d} \right)$$  \hspace{1cm} (9)

**Proof** After a suitable linear change of coordinates of the form

$$(z_1, \ldots, z_n) \mapsto (z_1, z_2 + \alpha_2 z_1, z_3 + \alpha_3 z_1, \ldots, z_n + \alpha_n z_1)$$

with $\alpha_2, \ldots, \alpha_n \in \mathbb{C}$ we may assume that

$$D = a(z) \frac{\partial^d}{\partial z_1^d} + \sum_{k=0}^{d-1} D_k \frac{\partial^k}{\partial z_1^k}$$  \hspace{1cm} (10)

where the $D_k$ are polynomial differential operators of degree at most $d - k$ involving only partial derivatives w.r.t. $z_2, \ldots, z_n$ and where $a \in \mathbb{C}[z_1, \ldots, z_n]$ is such that $a|\{z_1 = 0\}$ is not vanishing identically.

Let $p \in \mathbb{C}[z_1, \ldots, z_n]$ be a homogeneous polynomial of degree $m$ such that $Dp = 0$. We may write $p$ as a polynomial in $z_1$ with polynomial coefficients in $z_2, \ldots, z_n$:

$$p(z) = \sum_{j=0}^{m} \frac{\partial^j p}{\partial z_1^j}(0, z_2, \ldots, z_n) \cdot z_1^j$$  \hspace{1cm} (11)

Hence, $p$ is uniquely determined by the values of $\frac{\partial^j p}{\partial z_1^j}(0, z_2, \ldots, z_n)$ for $j = 0, \ldots, m$. However, using the relation $\frac{Dp}{a}|\{z_1 = 0\} = 0$ recursively, we see that already the values of $\frac{\partial^j p}{\partial z_1^j}(0, z_2, \ldots, z_n)$ for $j = 0, \ldots, d - 1$ determine such $p$ completely.
The vector space of the homogeneous polynomials of degree in \( m \) in \( n \) variables has dimension \( \binom{n+m-1}{m} \). Since \( p \) is homogeneous, also each of these restrictions of the derivatives is homogeneous and we obtain the following estimate:

\[
\dim_{\mathbb{C}} K_m \leq \sum_{j=0}^{d-1} \binom{n-1 + m - j - 1}{m - j} = \binom{m + n - 1}{m} - \binom{m - d + n - 1}{m - d}
\]

Remark 5.3 This estimate is optimal, consider e.g. the differential operator \( D = \frac{\partial^d}{\partial z_1^d} \) on \( \mathbb{C}[z_1, \ldots, z_n] \) whose kernel consists exactly of the polynomials in the variables \( z_1, z_2, \ldots, z_n \) such that the degree in \( z_1 \) is at most \( d - 1 \).

Corollary 5.4 We have the following estimates:

1. \( \ker (\Theta_{ab}^2|P_m) \) grows at most polynomially of degree \( n^2 - 1 \) in \( m \).
2. \( \ker (\Xi_a^2|P_m) \) grows at most polynomially of degree \( n^2 - 1 \) in \( m \).

Proof We apply Proposition 5.2 with \( d = 2 \) and obtain in both cases the estimates

\[
\dim_{\mathbb{C}} K_{m'} \leq \binom{n^2 + m' - 2}{m'} + \binom{n^2 + m' - 1}{m' - 1} \sim 2 \frac{(m')^{n^2-2}}{(n^2 - 2)!}
\]

Summation over \( m' = 0, 1, \ldots, m \) leads to a polynomial growth estimate of degree \( n^2 - 1 \) in \( m \). \qed

Remark 5.5 (1) If you compare Corollary 5.4 to [3, Sec. 7] and their notation, note that \( \ker (\Theta_{ab}^2|P_m) \) resp. \( \ker (\Xi_a^2|P_m) \) correspond to the vector space of polynomials \( \Theta_0^n \times \Theta^{m-1} \) in their notation.

(2) Note one subtlety about the degree: For a \( p \in \ker (\Theta_{ab}|P_m) \subset \ker (\Theta_{ab}^2|P_m) \) the polynomial vector field \( p \cdot \Theta_{ab} \) is of the degree \( m + 1 \), but the corresponding matrix \( \exp(p \cdot E_{ab}) \) will contain polynomials of degree \( m \), since \( E_{ab}^2 = 0 \) for \( a \neq b \). However, the polynomial shear automorphism arising by conjugation with this matrix will be of degree \( 2m + 1 \).

It is convenient to introduce the notion of density property also for Lie algebras.

Definition 5.6 ([25, Definition 0.1]) Let \( \mathfrak{g} \) be a Lie algebra of holomorphic vector fields on a complex manifold \( X \). We say that \( \mathfrak{g} \) has the density property if the Lie subalgebra of \( \mathfrak{g} \) generated by the complete holomorphic vector fields is dense in \( \mathfrak{g} \).

Definition 5.7 Following [25, Section 0] we recall the definition of jet spaces:

1. Let \( J^k_\mathfrak{g}(X) \) be the space of \( k \)-jets of local biholomorphisms of the form \( \varphi_{t_1}^{(t)} \circ \cdots \circ \varphi_{t_m}^{(t)} \) where \( \varphi_t^{(t)} \) is the local flow map at time \( t \in \mathbb{C} \) of the vector field \( \Theta \) for vector fields \( \Theta_1, \ldots, \Theta_m \in \mathfrak{g} \) and times \( t_1, \ldots, t_m \) small enough. For a jet \( \gamma \in J^k_\mathfrak{g}(X) \) we denote by \( \sigma(\gamma) \) its source point and denote \( J^k_\mathfrak{g},x = \{ \gamma \in J^k_\mathfrak{g} : \sigma(\gamma) = x \} \).
(2) By $\text{Aut}_\mathfrak{g}(X)$ we denote the subgroup of $\text{Aut}(X)$ generated by the time-1-maps of flows of complete holomorphic vector fields.

(3) For a holomorphic map $f : X \to X$, its $k$-jet at $x \in X$ is denoted by $j^k_x(f)$.

**Theorem 5.8** ([25, Theorem 0.1]) Let $\mathfrak{g}$ be the Lie algebra of holomorphic vector fields on a complex manifold $X$ with the density property. Then for each jet $\gamma \in J^k_\mathfrak{g}(X)$ there exists $f \in \text{Aut}_\mathfrak{g}(X)$ such that

$$j^k_{\sigma(\gamma)}(f) = \gamma$$

**Remark 5.9** From the proof of Theorem 0.1 in [25, Section 3] we can in fact deduce a slightly stronger statement, namely that for each jet $\gamma \in J^k_\mathfrak{g}(X)$ there exists an open neighborhood $U$ in $J^k_\mathfrak{g}(X)$ and a continuous section of $j^k$ on $U$. Hence, the map $j^k$ is open.

**Lemma 5.10** Let $\mathfrak{g}$ be the Lie algebra of all fibre-preserving vector fields of $\pi : \Omega_n \to \mathbb{G}_n$. The jet space $J^{2m, 0}_\mathfrak{g}(\Omega_n)$ is a finite-dimensional complex-affine space and

$$\dim J^{2m+1, 0}_\mathfrak{g}(\Omega_n) \geq \left( \frac{m+n^2}{n^2} \right)$$

**Proof** For this estimate, it is enough to consider all matrix conjugations of the form

$$X \mapsto X + a(X) \cdot [E_{12}, X] - a^2(X) \cdot E_{12} X E_{12}$$

where $a$ is a polynomial of degree at most $m$ in $\text{gl}(\mathbb{C})^*$. The vector space of these polynomials has dimension $\binom{m+n^2}{m} = \sum_{k=0}^{n^2} \binom{n^2-k-1}{m}$. Each such polynomial gives rise to a different conjugation (consider e.g. the matrix entry $(2, 2)$) which in turn defines a $(2m+1)$-jet that preserves the fibres of $\pi$ and is locally invertible. The map $\phi(X) := A(X) \cdot X \cdot A^{-1}(X)$ can be connected to its linear part by a path $1/s \cdot \phi(s \cdot X)$ and letting $s \to 0$, and can then be connected further to the identity, as shown already in the proof of Theorem 1.7 on page 16. By the fibred density property, the vector field can then be approximated by linear combinations and Lie brackets of complete vector fields, hence the jet of $\phi$ indeed belongs to $J^{2m+1, 0}_\mathfrak{g}(\Omega_n)$.

**Corollary 5.11** For fixed $k \in \mathbb{N}$ and $m \in \mathbb{N}$ large enough:

$$\dim J^{2m+1, 0}_\mathfrak{g}(\Omega_n) \geq k \cdot \max \left\{ \dim \ker \left( \Theta_{ab}^2 \vert P_m \right), \dim \ker \left( \Xi_a^2 \vert P_m \right) \right\}$$

**Proof** The l.h.s. is a polynomial in $m$ of degree $n^2$ and the r.h.s. is of degree at most $n^2 - 1$ according to Corollary 5.4.

**Definition 5.12** For the vector fields, we define a truncation map $\text{tr}_m : \mathcal{O} \Omega_n \to P_m$ which sends a holomorphic function to its Taylor polynomial about 0 of degree $m$. 

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Lemma 5.13 The following diagram commutes:

\[
\begin{array}{ccc}
\ker \left( \Lambda_1^2 \times \cdots \times \Lambda_k^2 \right) & \xrightarrow{\Psi_k} & \text{Aut}^\pi(\Omega_n) \\
\downarrow \text{tr}_m \times \cdots \times \text{tr}_m & & \downarrow j_0^{2m+1} \\
\ker \left( \Lambda_1^2 | P_m \times \cdots \times \Lambda_k^2 | P_m \right) & \xrightarrow{\psi_k} & J_0^{2m+1,0}(\Omega_n)
\end{array}
\]

where \( \Lambda_1, \ldots, \Lambda_k \in \{\Theta_{ab} : 1 \leq a \neq b \leq n\} \cup \{\Xi_a : 1 \leq a \leq n - 1\} \) are acting as derivations on \( O(\Omega_n) \), and

\[
\begin{align*}
\Psi_k(f_1, \ldots, f_k) &:= \exp(f_1 \Lambda_{a_1}) \cdots \exp(f_k \Lambda_{a_k}) \\
\psi_k &:= j_0^{2m+1} \circ \Psi_k.
\end{align*}
\]

Proof We first need to show that the maps really map into the given targets. This is clear for \( \Psi_k \). Since the \( \Lambda_1^2, \ldots, \Lambda_k^2 \) preserve the total degree of polynomials, the truncation \( \text{tr}_m \) actually maps the kernel into itself. The diagram commutes by definition of \( \psi_k \).

Proof of Theorem 1.8 We can restrict ourselves to the case of the subgroup \( \text{Aut}^\pi(\Omega_n) \) of fibre-preserving automorphisms, since \( \text{Aut}(\Omega_n) \) is generated by \( \text{Aut}^\pi(\Omega_n) \) together with Möbius transformations and matrix transposition. We follow the idea of the proof of [3, Theorem 7.1].

The topology on \( \text{Aut}^\pi(\Omega_n) \) shall be the topology of local uniform convergence for both the automorphisms and their inverses, which is a completely metrizable space. We denote by \( C_k, k \in \mathbb{N} \), the set of automorphisms obtained by the composition of \( k \) overshears of \( \Theta_{ab} \) resp. \( \Xi_a \). Using the notation of Lemma 5.13 above, we see that \( C_k \) is the image of the map \( \Psi_k \). We claim that the set \( C_k \) is meagre in \( \text{Aut}^\pi(\Omega_n) \) for all \( k \in \mathbb{N} \). By Baire’s theorem it would then follow that \( \cup_{k \in \mathbb{N}} C_k \neq \text{Aut}^\pi(\Omega_n) \) since \( \cup_{k \in \mathbb{N}} C_k \) would be meagre too.

Assume now by contradiction that \( C_k \) is non-meagre in \( \text{Aut}^\pi(\Omega_n) \) for some \( k \in \mathbb{N} \). Then we set \( V := (\Psi_k)^{-1}(\text{Aut}^\pi(\Omega_n)) \) and \( V_m := (\text{tr}_m \times \cdots \times \text{tr}_m)(V) \). Since \( j_0^{2m+1} \) is an open mapping by Remark 5.9, also \( j_0^{2m+1}(\Psi_k(V)) \) is non-meagre in \( J_0^{0,2m+1}(\Omega_n) \).

By Lemma 5.13 also \( \psi_k(V_m) \) is non-meagre in \( J_0^{0,2m+1}(\Omega_n) \). However, the mapping \( \psi_k \) is differentiable. The inequality of Corollary 5.11 forces \( \psi_k(V_m) \) to be meagre, a contradiction.

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