A note on homotopic versus isomorphic topological phases

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Abstract

Two gapped Hamiltonians compatible with given symmetry constraints may be isomorphic, but not homotopic to each other. We illustrate this with a simple model for Class AIII insulators in one spatial dimension, where a winding number measures the failure of homotopy between the gapped Hamiltonians for two insulators. This suggests that the notion of phases, up to homotopy, should be a relative one rather than an absolute one. We also analyse these issues in the context of $K$-theory.

Keywords: Topological phases, $K$-theory

1. Introduction

The study of quantum phases of matter has benefited greatly from numerous insights and techniques from topology. A recent idea proposes that the homotopy groups of the stable classical groups $[1]$ can equally well classify topological phases of free-fermions. Together with the symmetry class determined by the information of charge-conjugation and time reversal symmetries $[2, 3]$, one is led to a Periodic Table of gapped topological phases $[4]$, provided the $K$-theory groups in such a table are properly interpreted. A number of inconsistencies in the mathematics and physical interpretation of these $K$-theory groups have been pointed out and rectified in $[5]$. More generally, any classification scheme must refer to a well-defined family of physical systems, as well as clearly stated equivalence relations defining the classes.

A common definition of a topological phase is one which cannot be smoothly or continuously deformed into a “trivial phase” while maintaining certain constraints, but this leaves open the question of where this deformation takes place in, and indeed, the precise object being deformed. Furthermore, algebraic operations on phases are not included in such a definition, whereas useful topological invariants usually have extra structure such as that of a group or a ring. For example, $\mathbb{Z}$ sometimes appears as the object which classifies phases in certain symmetry classes.

Whether this means that the phases form a free abelian group or merely a countably infinite set is an important question to address.

Independently of the $K$-theory approach to the classification problem, there is a more basic mismatch between the desire to classify topological phases per se, and taking their equivalence classes up to homotopy. For the former to make sense, the classification object (typically an abelian group) should provide invariants of the
isomorphism classes in an appropriate category of objects modelling the physical phases. The simplest example is the modelling of the valence bands of Class A band insulators (i.e. no charge-conjugation or time reversal symmetries) using vector bundles over the Brillouin torus. There are natural topological invariants which can distinguish between non-isomorphic vector bundles. A prominent example in the two-dimensional case is the first Chern number, which has been linked to the quantized Hall conductivity in many analyses. This is an invariant of the isomorphism class of the valence vector bundle, or indeed, its virtual class in $K$-theory. It may also be construed as a homotopy invariant, as we explain in Section 3.

On the other hand, the general paradigm of a homotopy classification is the following: there is a collection of allowed Hamiltonians (gapped or otherwise) compatible with certain pre-specified constraints, typically arising from symmetry considerations and possibly a gapped condition. These Hamiltonians are presumed to form a topological space $Y$, in which two Hamiltonians are considered to be equivalent up to homotopy, if they are connected via a continuous (or even smooth) path in $Y$. It is then natural to declare that the set of (allowed) phases, up to homotopy, is the set of path-components $\pi_0(Y)$. However, it is not clear that isomorphic Hamiltonians in $Y$ (where “isomorphism” is assumed to be defined in some appropriate way) must be homotopic, i.e. in the same path-component. This presents a problem for the notion of an absolute phase up to homotopy. If $H_1, H_2 \in Y$ are isomorphic but not homotopic, and $H$ describing some other system is also isomorphic to $H_1, H_2$, then there is an ambiguity in assigning the element of $\pi_0(Y)$ which corresponds to the “absolute” phase of $H$. In Section 2, we construct a simple example where such an ambiguity does arise.

2. Isomorphic but non-homotopic Class AIII band insulators

Gapped Hamiltonians in the symmetry class AIII are characterised by the presence of a sublattice (also called chiral) symmetry $S$, which is unitary, squares to the identity, and anticommutes with the Hamiltonian. As is usual in the literature, we adjust the energy scale so that 0 lies in the energy gap, and regard a gapped Hamiltonian $H$ to be homotopic to its spectral flattening into a self-adjoint grading operator $\Gamma = \text{sgn}(H)$. For the purposes of a homotopy classification, we need only concern ourselves with the latter. Then a simplified mathematical model of a Class AIII band insulator in $d$ spatial dimensions (with $\mathbb{Z}^d$ translational symmetry) is a $\mathbb{Z}_2$-graded complex hermitian vector bundle $E$ over the Brillouin torus $\mathbb{T}^d$, equipped with an odd bundle automorphism $S$ commuting with the bundle projection. The $\mathbb{Z}_2$-grading corresponds to the spectrally-flattened Hamiltonian, and distinguishes the conduction bands from the valence bands.

Let us fix $d = 1$ and consider only two-band models, so $E$ is a rank-two bundle over $\mathbb{T}^1 \cong S^1$. The conduction and valence bands are each line bundles, comprising the $+1$ and $-1$ eigenspaces of $S$ respectively. Choosing a global basis of eigenvectors of $S$, we can identify $E$ with the trivial bundle $S^1 \times \mathbb{C}^2$. Thus, the fibrewise matrices
for $S$ are $S(k) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for all $k \in S^1 = [0, 2\pi]$, and a compatible $\Gamma$ has Bloch Hamiltonians $\Gamma(k)$ which are off-diagonal in this basis. Since $\Gamma(k)^2 = 1$ and $\Gamma(k)\dagger = \Gamma(k)$, we must have

$$\Gamma(k) = \begin{pmatrix} 0 & q(k) \\ \dagger q(k) & 0 \end{pmatrix}$$

for some continuous function $q : S^1 \to U(1) \cong S^1$. A homotopy between two such functions $q$ and $q'$ corresponds exactly to a homotopy between the $\Gamma$ and $\Gamma'$ (in the space of operators) which they determine.

Therefore, the set of phases (compatible with the $S$-action on $E$) up to homotopy is $\pi_1(S^1) \cong Z$. Note, however, that the canonical reference Hamiltonian $\Gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which corresponds to the constant function $q_0 : k \mapsto e^{i0k} = 1$, depends on the initial choice of eigenvectors of $S$ used to trivialise $E$.

### 2.1. Winding numbers and odd Chern characters

Each element $n \in Z = \pi_1(S^1)$ has a representative function $q_n(k) = e^{-ink}$ with winding number $n$, and associated compatible Hamiltonian $\Gamma_n(k) := \begin{pmatrix} 0 & e^{-ink} \\ e^{ink} & 0 \end{pmatrix}$. Different $n$ correspond to non-homotopic $q_n$ and non-homotopic $\Gamma_n$. However, the $\Gamma_n$ are all isomorphic: define the bundle maps $U_n$ by the fibrewise operators $U_n(k) = \begin{pmatrix} q_n(k) & 0 \\ 0 & 1 \end{pmatrix}$, which are unitary and commute with $S(k)$, then

$$U_n(k)\Gamma_0(k)U_n(k)^{-1} = \begin{pmatrix} e^{-ink} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{ink} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & e^{-ink} \\ e^{ink} & 0 \end{pmatrix} = \Gamma_n(k).$$

Thus, $\Gamma_n = U_n\Gamma_0 U_n^{-1}$ describes a class AIII band insulator isomorphic to $\Gamma_0$ in the natural sense. More generally, we have $U_n\Gamma_m U_n^{-1} = \Gamma_{m+n}$. Note that the columns of $U_n(k)$ are $\pm1$ eigenvectors of $S(k)$, so what we initially called $\Gamma_m$ with respect to one trivialisation would have been $\Gamma_0$ in another trivialisation.

We conclude that the winding number $n \in Z$ is really a homotopy invariant of the map $q_n$. It can be interpreted as a label for the relative obstruction, in a homotopy sense, between $\Gamma_n$ and a reference phase $\Gamma_0$. Indeed, these obstructions inherit the group structure of $\pi_1(S^1)$. This relative point of view is more meaningful as it does not depend on the choice of trivialisation or reference phase.

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1This can be made more precise in the language of $C^*$-algebras and projections, see Section 3.

2Fortuitously $U(1) \cong S^1$ is itself a group, so the group structure on the homotopy classes of (based) maps $[S^1, U(1)]$ can be taken in two ways: (1) by concatenating loops, or (2) pointwise multiplication of loops. Both choices lead to the group $Z$.  

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More generally, we can consider rank-$2N$ models for Class AIII insulators in $d > 1$. There are homotopy invariants, such as generalisations of the above winding number, which can be associated to continuous maps $Q : \mathbb{T}^d \to U(N)$ \cite{6}. If we assume that the rank-$N$ valence and conduction bands combine to form a trivial bundle $E \cong \mathbb{T}^d \times \mathbb{C}^{2N}$, and that $S(k) = \text{diag}(1_N, -1_N), k \in \mathbb{T}^d$ with respect to some trivialisation, then each $Q$ determines a flattened compatible Hamiltonian

$$\Gamma_Q(k) = \begin{pmatrix} 0 & Q(k) \\ Q(k)^\dagger & 0 \end{pmatrix}.$$  \hfill (3)

This is the construction found in \cite{2, 6}. Non-homotopic $Q$ determine non-homotopic $\Gamma_Q$.

In the $d = 1 = N$ example, we had $[S^1, U(1)] \cong \pi_1(S^1) \cong \mathbb{Z}$, and the explicit computation of the winding number for $q_n$ is

$$\frac{i}{2\pi} \int_0^{2\pi} q_n(k) - 1 \, dq_n(k) = \frac{i}{2\pi} \int_0^{2\pi} e^{ink}(-in)e^{-ink} \, dk = n.$$  \hfill (4)

If we regard the smooth map $q_n : S^1 \to U(1)$ as a unitary element of $C^\infty(S^1)$, the differential 1-form in the integrand in (4) is, up to a constant factor, the odd Chern character of $q_n$ (see \cite{7} for explicit formulae and generalisations). The odd Chern character is perhaps better understood in terms of $K$-theory. An element of $K^{-1}(S^1) \cong \mathbb{Z}$ can be represented by a smooth unitary $Q$ in the matrix algebra $M_N(C^\infty(S^1))$ for some $N \geq 1$. The Chern character of such a $Q$ is $\text{tr}(Q^{-1}dQ) = d \log \det Q$, which integrates over $S^1$ to give $-2\pi i$ times the winding number of $\det(Q)$.

More generally, $S^1$ can be replaced by a higher dimensional compact manifold $X$. The odd Chern character map takes a unitary $Q \in M_N(C^\infty(X))$ into a class in $H^{\text{odd}}_{\text{deRham}}(X)$. It is insensitive to the (smooth) homotopy class of $Q$ and is even a homomorphism from $K^{-1}(X)$ to $H^{\text{odd}}_{\text{deRham}}(X)$ \cite{7}. In our $d = 1 = N$ example, the (class of) the unitary $q_1$ (or $q_{-1}$) is actually the (Bott) generator for $K_1(C(S^1)) \cong K^0(\mathbb{R}^2) \cong \mathbb{Z}$ (see Chapter 3.7 of \cite{8} for details), and the Chern character maps $q_1$ to $dk$. Actually, it is not necessary to define the Chern character in de Rham cohomology; indeed, an application of the noncommutative odd Chern character can be found in \cite{9}.

3. Connection between isomorphism and homotopy in the ungraded case

Class A band insulators in $d$ dimensions modelled by vector bundles can be classified by $K^0(\mathbb{T}^d)$, where a formal difference $E \ominus F$ of bundles has at least two different physical interpretations. On the one hand, $E$ can be regarded as the

\footnote{This is a simplifying assumption, and globally non-trivial $E$ may have further interesting features, see Section 4}
conduction bands and $F$ as the filled valence bands. On the other hand, it may simply be regarded as a formal difference between two valence bands, with the data of the conduction band deemed to be irrelevant. There are a number of features of the $K^0$ functor which differs from $K^{-1}$. Most important for our purposes is the availability of pictures of $K^0$ in terms of both isomorphism and homotopy classes of vector bundles, where the latter needs to be defined carefully. In particular, we are certainly not interested in homotopies of bundles as topological spaces, since they always contract onto their base space (which is fixed as $\mathbb{T}^d$). The relevant notion of homotopy is that of projections in the stabilised algebra $M_\infty(C(\mathbb{T}^d))$, as we explain below. The $K^{-1}$ functor is more about automorphisms of bundles and their homotopies, although it can be linked to $K^0$ by taking suspensions or through Karoubi triples $[10]$.

Complex vector bundles over a compact Hausdorff space $X$ correspond, by the Serre–Swan theorem, to (left) finitely-generated projective (f.g.p.) modules for the $C^*$-algebra of continuous functions $C(X)$. The latter are of the form $(C(X)^N)p$ for some projection $p$ in some matrix algebra $M_N(C(X))$. Here, $(C(X)^N)$ is the free $C(X)$-module of continuous sections of the trivial rank-$N$ bundle over $X$. Note that $p$ may also be considered to be the projection $p \oplus 0_{N' - N} \in M_{N'}(C(X))$ for any $N' \geq N$. For a unital $C^*$-algebra $A$ (e.g. $A = M_N(C(X))$ for a fixed $N$), there are a number of equivalence relations which may be imposed on its projections. There is unitary equivalence, where $p_0 \sim p_1$ if there exists a unitary $u \in A$ such that $up_0u^{-1} = p_1$. There is also homotopy equivalence, where $p_0 \sim_h p_1$ if there is a norm-continuous path of projections in $A$ from $p_0$ to $p_1$.

While $p_0 \sim_h p_1$ implies $p_0 \sim p_1$, the converse is not generally true $[11]$. Nevertheless, the converse does hold when the $p_i$ are regarded as projections in $M_2(A)$. To see this, we first note that for any two unitaries $u, v \in A$, we can construct a path of unitaries $U_t$ in $M_2(A)$ between $U_0 = \text{diag}(uv, 1)$ and $U_1 = \text{diag}(u, v)$, via

$$U_t = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \frac{\pi}{2}t & -\sin \frac{\pi}{2}t \\ \sin \frac{\pi}{2}t & \cos \frac{\pi}{2}t \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \frac{\pi}{2}t & \sin \frac{\pi}{2}t \\ -\sin \frac{\pi}{2}t & \cos \frac{\pi}{2}t \end{pmatrix}. \quad (5)$$

Taking $v = u^{-1}$, we see that there is a homotopy in $M_2(A)$ between $U_0 = 1$ and $U_1 = \text{diag}(u, u^{-1})$. If $p_0 \sim p_1$ is implemented through $u$, then $P_t := U_t \text{diag}(p_0, 0)U_t^{-1}$ is a homotopy in $M_2(A)$ between $P_0 = \text{diag}(p_0, 0)$ and $P_1 = \text{diag}(p_1, 0)$.

When posing the question of whether two vector bundles are “homotopic”, a fixed background bundle for which the two bundles are subbundles is implicitly fixed. Typically, this ambient bundle is taken to be a trivial bundle $X \times \mathbb{C}^N$, and the homotopy in question is between the projections in $M_N(C(X))$ corresponding to the two subbundles. In this sense, homotopy and isomorphism are not necessarily equivalent. Nevertheless, the constructions in the previous paragraph show that isomorphic bundles can always be considered to be homotopic when placed within a larger ambient bundle. Indeed $K^0(X) \cong K_0(C(X))$ can be defined as the the Grothendieck group of the monoid of equivalence classes of projections in $M_\infty(C(X)) = \varprojlim M_N(C(X))$, where either unitary equivalence or homotopy equivalence may be used (see Chapter 5 of $[11]$).
3.1. A relative view of $K^0(X)$

There is an alternative picture of $K^0(X)$ due to Karoubi [10], which makes this idea of homotopy within an ambient bundle more explicit. The detailed construction can be found in the reference, so we simply illustrate it with the simplest example of $X = \{ \text{pt} \}$. We have $K^0(\text{pt}) \cong \mathbb{Z}$ generated by the vector space $\mathbb{C}$. In the relative picture, this generator is represented by a triple $[\mathbb{C}, 1, -1]$, which represents the (ordered) difference between the purely even grading 1 and the purely odd grading $-1$. Note that the rank of the $-1$ eigenspace of the grading operator increases by one. The general triple is of the form $[\mathbb{C}^N, \Gamma_1, \Gamma_2]$ representing the difference between grading operators $\Gamma_1, \Gamma_2$ on $\mathbb{C}^N$. We are allowed to replace $\Gamma_i$ by a grading operator homotopic to it, so it suffices to consider triples of the form $[\mathbb{C}^N, 1_{N-n} \oplus -1_n, 1_{N-n'} \oplus -1_{n'}]$ for some $0 \leq n, n' \leq N$. Suppose $n' \geq n$, then the algebraic rules for triples allow us to write

$$[\mathbb{C}^N, \Gamma_1, \Gamma_2] = [\mathbb{C}^N, 1_{N-n} \oplus -1_n, 1_{N-n'} \oplus -1_{n'}]$$

$$= [\mathbb{C}^{N-n'} + 1_{N-n'} \oplus -1_n, 1_{N-n'} \oplus -1_{n}]$$

$$= [\mathbb{C}^{n-n'} - 1_{n'} - 1_{n} - 1_{n-n}] = (n' - n)[\mathbb{C}, 1, -1],$$

so the “difference class” of the triple $[\mathbb{C}^N, \Gamma_1, \Gamma_2]$ counts the change in the rank of the $-1$ eigenspaces modulo homotopy.

For general compact Hausdorff spaces $X$, the triples generating the group $K^0(X)$ are of the form $[E, \Gamma_1, \Gamma_2]$, with $E$ a vector bundle over $X$ and $\Gamma_i$ gradings on $E$. We may assume $E$ to be trivial by augmenting $[E, \Gamma_1, \Gamma_2]$ by a trivial triple $[E^\perp, 1, 1]$, where $E \oplus E^\perp \cong X \times \mathbb{C}^N$. A virtual class $[E \oplus F]$ in a more usual Grothendieck group definition of $K^0(X)$ corresponds to the difference-class of the triple $[E \oplus F, 1_E \oplus -1_F, -1_E \oplus 1_F]$.

The advantage of this picture of $K^0$, is that it works just as well for the other symmetry classes, which usually involve higher degree $K$-theory groups. We give a brief account of this in Section 3.

3.2. Aside: Homotopies of classifying maps

There is a classifying space $BU(N)$ for complex rank-$N$ bundles, which may be realised as the Grassmannian of $N$-planes in an infinite-dimensional Hilbert space. Every rank-$N$ bundle over $X$ can be realised as the pullback of the universal tautological bundle under some map $f : X \to BU(N)$, and homotopic maps yield isomorphic bundles. For example, when $N = 1$, the classifying space for complex line bundles is $\mathbb{C}P^\infty$. In fact, $[X, \mathbb{C}P^\infty] \cong H^2(X)$ where the right hand side is the ordinary (second) cohomology group of $X$ where the first Chern classes live.

Imposing symmetry constraints means that only bundles with some extra structure are allowed. One can imagine that a suitable classifying space exists for such bundles, and this is the point of view taken up in [12]. There, the relation of homotopy is imposed on the space of “classifying maps” which determine subbundles of an ambient trivial bundle of fixed rank $2N$, with the bundles compatible with the symmetry constraints. This can be understood in terms of projections in $M_{2N}(C(X))$. 

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which are required to satisfy some additional conditions. As before, the relations of isomorphism (of bundles) and homotopy (of classifying maps) are not the same, and their homotopy classification is finer than the isomorphism classification of subbundles. It should be noted that the authors require symmetries to commute with the Hamiltonian, so their conventions differ from those of other authors. In particular, the $S$-symmetry in Class AIII systems are regarded as pseudo-symmetries.

4. Homotopic Hamiltonians and $K$-theory

In Section 2, we explained how the notion of homotopic Hamiltonians is ambiguous when taken in an absolute sense. The “primitive” topological invariant there is the homotopy class of the map $Q$ from the base space to a unitary group. Alternatively, $Q$ determines a class in a $K^{-1}$-group, and is detected by integrating its Chern character over the base space, yielding a numerical winding number invariant.

Let us rephrase our analysis of the $d = 1 = N$ case in algebraic language. The sections of the ambient bundle $E$ form a free $C(S^1)$-module $W = (C(S^1))^2$, and the bundle map $S$ translates into an operator $S$ on $W$. We can think of $W$ as an ungraded module for the graded algebra $C(S^1) \hat{\otimes} C \mathcal{L}_1$, where $C \mathcal{L}_1$ is the complex Clifford algebra on one odd generator, represented on $W$ by $S$. Then the compatible flattened Hamiltonians on $W$ are precisely grading operators which turn $W$ into a graded $C(S^1) \hat{\otimes} C \mathcal{L}_1$-module. As explicit examples, the $\Gamma_n$ associated with the maps $q_n = e^{-ink}$ are compatible Hamiltonians which are non-homotopic for different $n$. Furthermore, $U_n = \text{diag}(q_n, 1)$ implements a unitary equivalence between $\Gamma_m$ and $\Gamma_{m+n}$ for each $m$, and satisfies $U_{n+n'} = U_n U_{n'}$.

We abstract these properties in terms of formal triples $(W, \Gamma, \Gamma')$ representing the obstruction in passing from $\Gamma$ to $\Gamma'$ within the space of compatible grading operators on $W$. Replacing $\Gamma$ or $\Gamma'$ by homotopic compatible grading operators should not change the class $[W, \Gamma, \Gamma']$ of the triple. Our analysis suggests the correspondence $\pi_1(S^1) \ni n \mapsto [W, \Gamma_m, \Gamma_{m+n}]$ independently of $m$, since $U_n$ applied to $\Gamma_m$ yields $\Gamma_{m+n}$. The zero element is $[W, \Gamma_m, \Gamma_n]$, where any $\Gamma_m$ may be used. The group operation on $\pi_1(S^1)$, namely $(n, n') \mapsto n + n'$ translates into

$$[W, \Gamma_m, \Gamma_{m+n}] + [W, \Gamma_{m+n}, \Gamma_{m+n+n'}] = [W, \Gamma_m, \Gamma_{m+n+n'}],$$

so obstructions may be added consistently. The inverse map $n \mapsto -n$ becomes

$$-[W, \Gamma_m, \Gamma_{m+n}] = [W, \Gamma_m, \Gamma_{m-n}] = [W, \Gamma_{m+n}, \Gamma_m],$$

which is just the obstruction taken in the opposite order. What we have described is precisely a model for $K_1(C(S^1)) \cong K^{-1}(S^1)$ using Karoubi’s triples $[10]$, and adapted for the classification of obstructions between topological phases in $[5]$. It is the Class AIII version of our presentation of $K^0(X)$ for Class A systems using similar triples, as outlined in Section 3.1.

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*The authors provide an example of this, see Example 3.1 of their paper.*
For a general ungraded unital $C^*$-algebra $A$, the group $K_1(A)$ may be defined as the abelian group generated by homotopy classes of unitaries in the matrix algebras $M_N(A), N \geq 1,$ with $[1] = 0$ and $[u] + [v] = [u \oplus v]$ \cite{5}. In fact, equation (5) says that $[u \oplus v] = [uv \oplus 1] = [uv]$, so composition of classes in $K_1$ may be realised on representative unitaries in a number of ways. Unitaries in $M_N(A)$ may also be interpreted as obstructions between Type AIII gapped phases as follows. An ungraded f.g.p. $A \otimes C\mathbb{I}_1$-module $W$ may be written as $W_+ \oplus W_-$, where the f.g.p. $A$-modules $W_\pm$ are the $\pm 1$ eigenspaces of the operator $S$ representing the Clifford generator. We may assume that $W_\pm = A^N p_{\pm}$ for some projections $p_{\pm}$. If there is any compatible grading operator at all, it must be of the form $\Gamma = R_{\frac{1}{2}} \oplus R_{\frac{3}{2}}$ for $A$-module maps $R_{\frac{1}{2}} : W_+ \to W_-$ and $R_{\frac{3}{2}} : W_- \to W_+$. Since $\Gamma$ is self-adjoint and involutory, it follows that $R_{\frac{1}{2}}$ is unitary and $R_{\frac{3}{2}}$ is its adjoint map. Given any unitary $Q \in M_N(A)$, we can construct another compatible grading operator $\Gamma_Q := R_{\frac{1}{2}} Q^{-1} \oplus Q R_{\frac{3}{2}}$. Thus, the original grading $\Gamma$ plays the role of a reference grading operator, and non-homotopic unitaries $Q \in M_N(C(X))$ lead to non-homotopic $\Gamma_Q$. This construction generalises \cite{3}, which is the special case where $A = C(\mathbb{T}^d), W = C(\mathbb{T}^d)^{2N}$ and $S = \text{diag}(1_{C(\mathbb{T}^d)^N}, -1_{C(\mathbb{T}^d)^N})$. There are corresponding triples $[W, \Gamma_Q, \Gamma_Q']$ representing the obstruction between $\Gamma_Q$ and $\Gamma_Q'$, and such triples generate $K_1(A)$ in Karoubi’s model.

4.1. A unified picture of homotopic Hamiltonians using difference-groups

The classes $A$ and AIII are the so-called complex symmetry classes and do not have antiunitary symmetry constraints. In the presence of antiunitary charge-conjugation or time-reversal symmetries, there are eight real symmetry classes, each of which is associated to a Morita class of real Clifford algebras \cite{1, 2, 4, 5}. Compatibility of a grading operator (gapped Hamiltonian) entails graded commutation with the Clifford algebra action. More generally, the symmetry constraints determine a graded symmetry algebra $A$ \cite{3}. On an ungraded $A$-module $W$, which we can understand as an ambient representation space hosting the symmetries, there is a (possibly empty) set $\text{Grad} A(W)$ of compatible grading operators turning $W$ into a graded $A$-module. Such grading operators are precisely the spectrally-flattened symmetry-compatible gapped Hamiltonians, and homotopic Hamiltonians are precisely those whose grading operators are homotopically in $\text{Grad} A(W)$.

For a graded $C^*$-algebra $A$ (such as the symmetry algebra), the $K$-theoretic difference group $K_0(A)$ as defined in \cite{3} is generated by triples $[W, \Gamma_1, \Gamma_2]$ where $W$ is an ungraded f.g.p. $A$-module and $\Gamma_i \in \text{Grad} A(W)$. Such a triple represents the obstruction in passing from $\Gamma_1$ to $\Gamma_2$ in a homotopic manner. The difference-group has a uniform interpretation which works for all the symmetry classes and also for more general symmetry algebras. Triples may be added by taking direct

\footnote{In more detail, $A$ is a $C^*$-algebra, and $W$ can be made a Banach space. Then the space of $A$-module maps on $W$, which contains the compatible grading operators, can be topologised naturally.}
soms, and they satisfy the properties of path-independence and existence of inverses as in (7) and (8). These difference groups reduce to ordinary $K$-theory groups in special cases. For instance, we sketched a model of $K^{-1}(S^1) \cong K_1(C(S^1))$ using such triples at the beginning of this section. Thus, we view $K$-theory as a way to obtain groups of obstructions between gapped phases, allowing us to measure one phase relative to another. This is in contrast with the idea of a topological classification of gapped Hamiltonians in an absolute sense up to homotopy, which is inherently problematic. Nevertheless, the relative viewpoint is not completely new and certainly not controversial; it was mentioned in Kitaev’s seminal work [4], and related notions of relative index and charge deficiency had been defined and applied to the Integer Quantum Hall Effect in [15, 16].

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