Quark Number Susceptibility: Revisited with Fluctuation-Dissipation Theorem in mean field theories

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Fluctuations of conserved quantum numbers are associated with the corresponding susceptibilities because of the symmetry of the system. The underlying fact is that these fluctuations as defined through the static correlators become identical to the direct calculation of these susceptibilities defined through the thermodynamic derivatives, due to the fluctuation-dissipation theorem. Through a rigorous exercise we explicitly show that a diagrammatic calculation of the static correlators associated with the conserved quark number fluctuations and the corresponding susceptibilities are possible in case of mean field theories, if the implicit dependence of the mean fields on the quark chemical potential are taken into account appropriately. As an aside we also give an analytical prescription for obtaining the implicit dependence of the mean fields on the quark chemical potential.

I. INTRODUCTION

In relativistic heavy-ion collisions the fluctuations and correlations of conserved charges like the baryon number, electric charge etc. are considered to carry promising signals for the formation of the exotic Quark Gluon Plasma (QGP). Close to any continuous phase transition region these are supposed to exhibit critical behavior [1–4]. Therefore the characteristics of quark-hadron phase transition can be understood by analyzing the fluctuations of the system. The fluctuations are often calculated theoretically through the respective susceptibilities. For comparison with experimental data various combinations of the ratios of these susceptibilities constitute important phenomenological observables [5–7].

Here we shall discuss the quark number susceptibility (QNS) which provides the response of the net quark number density to the change in quark chemical potential. Several first principle studies have been done to calculate the QNS in various approaches. These include the numerical simulation of QCD on a space-time lattice (LQCD) [8–16] as well as Hard Thermal Loop (HTL) calculations [17–26]. In the present manuscript we intend to revisit the calculations of QNS through the phenomenological models namely the Nambu–Jona-Lasinio (NJL) model and its Polyakov Loop extended version, the PNJL model. We begin with a brief introduction for the various studies already carried out within the various QCD inspired models to understand the thermodynamic properties of strongly interacting matter.

QCD phase transitions for vanishing and non-vanishing baryon chemical potential have been studied in a great detail and the possible phases that may arise in the phase diagram have been addressed [27–45]. Similar studies were also carried out for imaginary chemical potential [46–48] and the well-known Roberge-Weiss periodicity is discussed in that context. Although hadrons are not present as dynamical degrees of freedom in NJL or PNJL model, mesonic modes at real as well as imaginary chemical potential are studied as collective excitations within Random Phase Approximation [49–56] and formation of baryons composed of quarks and diquarks has also been studied by solving the Dyson-Schwinger equation [57, 58]. Mesons involving heavy quarks have been studied recently in NJL and PNJL models [59]. The average phase factor of QCD determinant is evaluated through PNJL model in Ref. [60], where it is argued that since CEP lies within the region of vanishing phase factor, location of CEP cannot be determined by LQCD alone. NJL model is explored in the context of CP restoring phase transition [61], where it is shown that nontrivial vacuum term of NJL model can always alter the qualitative aspects of the high temperature phase transition. The

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issue of color neutrality is crucially investigated in Ref. [62, 63]. Interplay between chiral and deconfinement transition is also investigated with U(1) valued boundary condition for fermionic fields [64–66], where quantities such as quark condensate or dressed Polyakov loop seem to be very effective in those studies. Effects of theta vacuum on QCD phase structure is investigated using PNJL model in Ref. [67]. NJL and PNJL model are also studied within the background magnetic field [68–71]. Another interesting phenomenon namely chiral magnetic effect, which could possibly explain the observed charge separation observed in the STAR experiment at RHIC, Brookhaven, is also investigated through PNJL model [72, 73]. Existence of conjectured chirally symmetric but confined phase in QCD phase diagram which is popularly named as quarkyonic phase is discussed in terms of PNJL model [74–77]. Role of axial anomaly and vector interaction determining the phase diagram of QCD is studied in Ref. [78–82]. For three degenerate or non-degenerate flavored system, it can be shown that NJL vacuum is unstable unless one incorporates finite volume effect which is relevant for studying a system created in heavy-ion collisions, has been studied very recently in Ref. [93]. Various interesting features of Polyakov loop [94–97] have encouraged people to study within different formalisms. Inclusion of gluon Polyakov loop is studied in various aspects [98–100]. Interestingly NJL model has also been studied within Monte-Carlo framework also [101, 102]. Studies of various transport coefficients in NJL model framework have also been reported recently [103, 104].

The quark number susceptibility has been studied extensively within the framework of NJL and PNJL models [30, 34, 105–110]. These studies revealed the order parameter like behavior of QNS, similar to that obtained in LQCD at vanishing baryon chemical potential. Recently it has been shown by some of us [111] that when isospin symmetry is broken explicitly, the baryon-isospin correlations exhibit an almost linear scaling with the scale of isospin breaking over the entire temperature-baryon chemical potential phase plane.

Another well-known formulation to study the strongly interacting matter in nonperturbative regime is the Quark Meson (QM) model and its Polyakov loop extended version (PQM), which is used to explore phase transition and phase diagram of QCD [112–117] as well as quark number susceptibility [118–124].

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The QNS is the response of the conserved number density, \( \rho \), with infinitesimal variation in the chemical potential \( \mu + i \beta \) as an external source. It is then defined as the second order derivative of pressure, \( P \), with respect to \( \mu \). On the other hand, according to the fluctuation-dissipation theorem (FDT), the QNS may also be obtained from the time-time component of the current-current correlator in the vector channel [19, 125, 126]. The QNS is then expressed as

\[
\chi_q = \frac{\partial \rho}{\partial \mu} = \frac{\partial^2 P}{\partial \mu^2} = \int d^4x \langle J_0(0, \vec{x})J_0(0, \vec{0}) \rangle = -\lim_{l \to 0} \text{Re} \Pi_{00}(0, l) = \lim_{l \to 0} \beta \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{-2}{1 - e^{-\beta \omega}} \text{Im} \Pi_{00}(\omega, l),
\]

(1)

where \( J_0 \) is the temporal component of the vector current and \( \Pi_{00}(\omega, l) \) is the time-time component of the vector correlation function or self-energy with external four momenta \( L \equiv (\omega, l = |l|) \). Because of the symmetry of the system the FDT guarantees that the thermodynamic derivative with respect to the external source, \( \mu \), is related to the time-time component of static correlation function in the vector channel. This is known as thermodynamic sum rule [19, 123, 126].

In usual perturbation theory the loop expansion and coupling expansion are symmetric and the thermodynamic consistency is automatic, for a given order of coupling \( \alpha_s \). So, it really does not matter which of the equivalent definition is used in (1) to compute QNS for a given order of \( \alpha_s \). For resummed approach like Hard Thermal Loop perturbation theory, the loop expansion and coupling expansion are not symmetric because higher loops contribute to the lower order in \( \alpha_s \). Unlike usual perturbation theory, in resummed case an appropriate measure is to be employed [19, 22, 25, 27] if one desires to compute QNS in a given order in \( \alpha_s \) correctly using (1). In effective approaches like NJL or PNJL models the QNS is usually obtained [30, 34, 105–110] as the second order Taylor coefficient of pressure when it is Taylor expanded in the direction of the quark chemical potential, \( \mu \), with an approximation \( \mu < T \). In model calculations any response of a thermodynamic quantity to some external parameters should also account for the fact that the mean fields also depend implicitly on those external parameters [30]. Therefore, a proper care has to be taken to relate the thermodynamic derivatives (viz., QNS) with the fluctuation associated with the conserved density. One of the purposes of the present work is to demonstrate whether the effective models like NJL and PNJL explicitly obey the FDT vis-a-vis thermodynamic sum rule.

Furthermore, the implicit dependence of the mean fields are usually obtained numerically but this may contain numerical errors which will increase with the order of derivatives. This is true for either a direct numerical derivative as well as the method of using a fitting function for a polynomial expansion of pressure in terms of chemical potential to corroborate with the Taylor expanded pressure. Recently a numerical technique based on algorithmic differentiation [129, 131] has been developed to solve this shortcoming. In the present work we also propose an alternative
analytical formalism to calculate the derivatives of the mean fields with respect to the external parameters. Using these derivatives we are going to calculate the QNS in a consistent manner.

The paper is organized as follows. In the Sec. II we discuss the formalism for obtaining the derivatives of the mean fields in our alternative approach in the NJL and the PNJL models. In Sec. III the calculations of QNS exploiting the FDT for a toy model and in effective approaches like NJL as well as in PNJL models are presented. Finally we draw our conclusions in Sec. IV.

II. CALCULATING THE DERIVATIVES OF THE MEAN FIELDS

A. NJL model

The Lagrangian for the 2 flavor NJL model at finite quark chemical potential ($\mu_q$) is given as \[ \mathcal{L}_{NJL} = \bar{\psi} (i\slashed{\partial} - m_0 + \gamma_0 \mu_q) \psi + \frac{G}{2} (\bar{\psi} \gamma^5 \bar{\psi})^2. \] (2)

In the mean field approximation the pion condensate $\langle \bar{\psi} \gamma^5 \bar{\psi} \rangle = 0$ and the Lagrangian can be rewritten in terms of the chiral condensate $\sigma = \langle \bar{\psi} \psi \rangle = \langle \bar{u} u + \bar{d} d \rangle = \sigma_u + \sigma_d$ as,

\[ \mathcal{L}_{MF} = \bar{\psi} (i\slashed{\partial} - m_0 + \gamma_0 \mu_q + G\sigma) \psi - \frac{G}{2} \sigma^2. \] (3)

In the mean field approach, the thermodynamic potential is a functional of the mean field $\sigma(m_0, T, \mu_q)$, and is given as,

\[ \Omega[\sigma, m_0, T, \mu_q] = -i \text{Tr} \ln S_1^{-1} + \frac{G}{2} \sigma^2. \] (4)

Here and unless stated, ‘Tr’ denotes the sum over color, flavor and Dirac indices as well as the four-momentum and any other notation involving trace operation will be clarified accordingly. The first term on the right hand side of (4) is the fermionic contribution related to the dressed propagator $S_1$ where,

\[ S_1^{-1} = \not{p} - m_0 + \gamma_0 \mu_q + G\sigma = S_0^{-1} + G\sigma, \] (5)

and $S_0$ is the bare propagator with current quark mass $m_0$. The second term in $\Omega$ may be considered as the background contribution of the mean field $\sigma$. To utilize thermodynamic relations, derivatives of $\Omega$ with respect to the various parameters as well as their presence through the mean fields are often necessary. In this regard care has to be taken for the explicit appearances of the parameters as well as their presence through the mean fields. For example, the computation of quark number susceptibility requires derivatives of $\Omega$ with respect to $\mu_q$ that appear explicitly in $\Omega$ as well as their implicit effects through $\sigma$. This is an important observation that we want to revisit in the present manuscript. Therefore we set the notation for the explicit derivatives by $\partial/\partial x$ and that for the total derivatives with $d/dx$ for some parameter $x$.

Let us start with the computation of the mean field $\sigma$. One way to obtain $\sigma$ is to use the stationarity condition $\partial \Omega/\partial \sigma = 0$ in the mean field approximation, which gives,

\[ \sigma = i \text{Tr} (S_1). \] (6)

On the other hand, we may also use the defining equation $\sigma = \partial \Omega/\partial m_0$, which gives the same result as in Eq.(6). However as discussed above, for the derivative with respect to $m_0$ one should also consider the implicit dependence on $m_0$ of $\Omega$ through $\sigma$. In that case we should rather use the relation,

\[ \sigma \frac{d\Omega}{dm_0} = \frac{\partial \Omega}{\partial m_0} + \frac{\partial \Omega}{\partial \sigma} \frac{d\sigma}{dm_0}. \] (7)

Using the stationarity condition for the mean field, the second term vanishes and therefore in this case we get back to the original defining equation of $\sigma$. Interestingly, if we straightway calculate $\frac{d\Omega}{dm_0}$ from Eq.(4) and demand it to be equal to $\sigma$, without imposing the stationarity condition (like in Eq.(7)), then Eq.(6) will emerge as a consistency condition.

The transcendental nature of the solutions of (6) is apparent. Thus a closed form analytical expression of $\sigma$ as a function of $m_0$, $T$ and $\mu_q$ cannot be obtained from this equation. One has to solve the equations numerically and obtain $\sigma(T, \mu_q)$. 

The implicit derivatives, in general do not disappear, as can be seen from the chiral susceptibility, which is the second order derivative of thermodynamic potential with respect to $m_0$,

$$
\chi_\sigma = \frac{d^2\Omega}{dm_0^2} = \frac{\partial^2\Omega}{\partial m_0^2} + 2 \frac{\partial^2\Omega}{\partial \sigma \partial m_0} \cdot \frac{d\sigma}{dm_0} + \frac{\partial^2\Omega}{\partial \sigma^2} \left( \frac{d\sigma}{dm_0} \right)^2 + \frac{\partial \Omega}{\partial \sigma} \cdot \frac{d^2\sigma}{dm_0^2}.
$$

The last term in Eq. (8) again vanishes due to stationarity condition of the mean field, but the second and third terms will remain and give the implicit contributions. This implicit dependence is what makes life a little difficult in the mean field calculations which are otherwise quite straightforward. While the explicit derivatives can be systematically obtained up to any desired order in a closed analytic form, the implicit contributions are usually obtained through numerical derivatives. Normally one has to resort to numerical evaluation of the total derivatives like $\chi_\sigma$, or the implicit part like $d\sigma/dm_0$. Such derivatives may either be done by direct difference approximations or by a Taylor series method as proposed by us in Ref. [30]. Unfortunately both these methods tend to give either large errors or become quite insensitive as the order of the derivatives are increased.

A possible alternative to these numerical techniques has been explored in Refs. [129–131] through the method of algorithmic differentiation. Derivatives up to very high orders may be computed in this technique. Though very efficient and less error prone even for obtaining very high derivatives, the method is algorithmically involved.

One of our main focuses here is to obtain the implicit contribution in a semi-analytic approach so that numerical uncertainties are minimized. Here we shall outline a simple algorithm for obtaining derivatives in a straightforward efficient and less error prone even for obtaining very high derivatives, the method is algorithmically involved.

For this purpose we shall discuss the derivatives with respect to $\mu_q$, though the methodology would be identical for any other similar derivatives. The only numerics involved will be the momentum integrals, and to this end all the methods of differentiation would have identical efficiency and accuracy.

The derivatives of $\Omega$ with respect to $\mu_q$ would give the quark number and its susceptibilities. Since the quark number is exactly conserved one may use the Ward-Takahashi identity to derive the corresponding three-point functions for the bare and effective theories. The identity is given as,

$$
q^\mu \Gamma_\mu(p, p + q) = S^{-1}_1(p + q) - S^{-1}_1(p).
$$

Here $p$ and $q$ denote the four-momenta of the fermion and the boson respectively. Now, $q_\mu \rightarrow 0$ limit of Eq. (9) yields the Ward Identity,

$$
\frac{dS^{-1}_1}{dp^\mu} = \Gamma_\mu(p, p),
$$

which in differential form, gives the insertion factor corresponding to the zero momentum boson line into an internal fermion line. In the imaginary time formalism, at finite temperature and chemical potential, the fourth component of momentum becomes $p_0 = i(2n + 1)\pi T + \mu_q$ and thus Eq. (10) can be written as

$$
\frac{dS^{-1}_1}{d\mu_q} = \Gamma_0(p, p).
$$

Using Eq. (9) in the above relation we obtain,

$$
\frac{dS^{-1}_2}{d\mu_q} = \gamma_0 + \left( G \frac{d\sigma}{d\mu_q} \right) \cdot \mathbb{I}_D \equiv \Gamma_0,
$$

where $\mathbb{I}_D$ is the identity matrix in Dirac space. For the bare propagator we get the expected insertion factor for non-interacting quarks as,

$$
\frac{dS^{-1}_0}{d\mu_q} = \gamma_0.
$$

Let us now consider the derivative of $\sigma$ from Eq. (3) w.r.t. $\mu_q$ which gives,

$$
\frac{d\sigma}{d\mu_q} = -i\mathrm{Tr} [S_i \Gamma_0 S_1] = -i\mathrm{Tr} (S_1 \gamma_0 S_i) - i\mathrm{Tr} \left( S_i G \frac{d\sigma}{d\mu_q} S_1 \right).
$$
where the effective three-point function from Eq. (12) is used. For the bare propagator $S_0$ one can easily check that

$$\frac{dS_0}{d\mu_q} = -S_0 \gamma_0 S_0,$$

which is basically another form of Ward identity for bare three-point function. The corresponding relation for the dressed propagator $S_1$ with the effective three-point function will be,

$$\frac{dS_1}{d\mu_q} = -S_1 \Gamma_0 S_1.$$

Rearranging terms in (14) it is possible to write $\frac{d\sigma}{d\mu_q}$ in a closed form in terms of $m_0$, $T$, $\mu_q$ and $\sigma$ only:

$$\frac{d\sigma}{d\mu_q} = \frac{-i \text{Tr}(S_1 \gamma_0 S_1)}{1 + iG \text{Tr}(S_1^2)}.$$

For the second order derivative, one may start from Eq.(14) to get,

$$\frac{d^2\sigma}{d\mu_q^2} = \frac{2i \text{Tr}[S_1 \Gamma_0 S_1 \Gamma_0 S_1] - i \text{Tr}(S_1 G \frac{d^2\sigma}{d\mu_q^2} S_1)}.\)

Again rearranging terms, one can write a closed form expression for $\frac{d^2\sigma}{d\mu_q^2}$ as a function of $m_0$, $T$, $\mu_q$, $\sigma$ and $\frac{d\sigma}{d\mu_q}$ as,

$$\frac{d^2\sigma}{d\mu_q^2} = \frac{2i \text{Tr}(S_1 \Gamma_0 S_1 \gamma_0 S_1) + 2(G \frac{d\sigma}{d\mu_q}) \text{Tr}(S_1^3 \gamma_0) + (G \frac{d\sigma}{d\mu_q})^2 \text{Tr}(S_1^3)}{1 + iG \text{Tr}(S_1^2)}.\)

We have plotted the first and second order derivatives of $\sigma$ w.r.t. $\mu_q$ as function of $T$ in Fig. 1(a) and Fig. 1(b) respectively, where we have compared the results of semi-analytical approach presented here to that of numerical methods like Taylor expansion or finite difference.

![Figure 1](image)

**FIG. 1:** First (a) and second (b) derivatives of the mean field $\sigma$ with respect to $\mu_q$ in NJL model at $\mu_q = T_c$. Here the points represent the result from numerical differentiation and the lines are from the semi-analytical approach described in the text.

In the same way all the higher order derivatives may be obtained systematically as a function of $m_0$, $T$, $\mu_q$ and derivatives up to one lower order.

This method is certainly more accurate than a direct numerical differentiation of $\sigma$ w.r.t. $\mu_q$, or fitting Taylor coefficients in an expansion w.r.t. $\mu_q$. No numerical approximations or uncertainties are introduced, except for the numerical integration of the fermionic momentum integrals. Therefore the question of insensitivity at higher orders also does not arise.
B. PNJL model

We now discuss the Polyakov loop enhanced NJL model. The situation here is similar to that of the NJL model except that we now have a couple of mean fields more in the form of the expectation value of the Polyakov loop $\Phi$ and that of its conjugate $\bar{\Phi}$. The Lagrangian for the 2 flavor PNJL model is given by,

$$L_{PNJL} = \bar{\psi} \left( i \slashed{D} - m_0 + \gamma_0 \mu \right) \psi + \frac{G}{2} \left[ (\bar{\psi} \gamma_5 \psi)^2 + (\bar{\psi} \gamma_5 \tau \psi)^2 \right] - \mathcal{U}(\Phi, \bar{\Phi}, T),$$

(20)

where $D^\mu = \partial^\mu - ig A^\mu_0 \lambda_3 / 2$, $A^\mu_0$ being the SU(3) background fields and $\lambda_3$ are the Gell-Mann matrices. Here the effective Polyakov loop potential is given by,

$$\frac{\mathcal{U}(\Phi, \bar{\Phi}, T)}{T^4} = \frac{-b_2(T)}{2} \Phi \bar{\Phi} - \frac{b_3}{6} (\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4} (\Phi \bar{\Phi})^2,$$

(21)

with

$$b_2(T) = a_0 + a_1 \left( \frac{T_0}{T} \right) + a_2 \left( \frac{T_0}{T} \right)^2 + a_3 \left( \frac{T_0}{T} \right)^3.$$

$\Phi$ is Polyakov loop and $\bar{\Phi}$ is its charge conjugate [31]. Values of coefficients $a_0, a_1, a_2, a_3, b_3, b_4$ have been taken from Ref.[29]. To take into account the effect of SU(3) Haar measure in the PNJL model, we consider the modified thermodynamic potential defined as [34],

$$\Omega' = \Omega - \kappa T^4 \ln[J(\Phi, \bar{\Phi})],$$

(22)

where $J(\Phi, \bar{\Phi})$ is the Vandermonde (VdM) determinant given as,

$$J[\Phi, \bar{\Phi}] = \left( \frac{27}{24 \pi^2} \right) (1 - 6 \Phi \bar{\Phi} + 4(\Phi^3 + \bar{\Phi}^3) - 3(\Phi \bar{\Phi})^2).$$

Following Ref.[34], the pressure in PNJL model is defined as $P = -\Omega$.

FIG. 2: (a) $d\sigma/d\mu_q$ and (b) $d\Phi/d\mu_q$ and $d\bar{\Phi}/d\mu_q$ in the PNJL model at $\mu_q = T_c$. Here the points represent the result from numerical differentiation, and lines are from the semi-analytical approach described in the text.

In NJL model we obtained $d\sigma/d\mu_q$ by differentiating the gap equation (6) w.r.t. $\mu_q$. In the PNJL model we do not have such gap equations for the $\Phi$ and $\bar{\Phi}$ fields. We therefore differentiate the stationarity conditions $\partial \Omega'/\partial X = 0$ for $X = \Phi, \bar{\Phi}, \sigma$ directly with the $\mu_q$ derivatives to get,

$$\frac{d}{d\mu_q} \left( \frac{\partial \Omega'}{\partial X} \right) = 0.$$

(23)

Note that this equation is valid only if we insert the mean field values in $\partial \Omega'/\partial X$ before taking the $\mu_q$ derivatives. This immediately gives us,

$$\frac{\partial}{\partial \mu_q} \left( \frac{\partial \Omega'}{\partial X} \right) + \frac{\partial}{\partial \Phi} \left( \frac{\partial \Omega'}{\partial X} \right) \cdot \frac{d\Phi}{d\mu_q} + \frac{\partial}{\partial \bar{\Phi}} \left( \frac{\partial \Omega'}{\partial X} \right) \cdot \frac{d\bar{\Phi}}{d\mu_q} + \frac{\partial}{\partial \sigma} \left( \frac{\partial \Omega'}{\partial X} \right) \cdot \frac{d\sigma}{d\mu_q} = 0.$$

(24)
So we have the matrix equation of the form $A \cdot x = B$, where $A$ is the coefficient matrix of the variables $x = (\frac{d\Phi}{d\mu_q}, \frac{d\bar{\Phi}}{d\mu_q}, \frac{d\sigma}{d\mu_q})^T$ and $B$ matrix has the form $B = \begin{pmatrix} -\frac{\partial}{\partial\mu_q}(\frac{\partial\Omega'}{\partial\Phi}), & -\frac{\partial}{\partial\mu_q}(\frac{\partial\Omega'}{\partial\bar{\Phi}}), & -\frac{\partial}{\partial\mu_q}(\frac{\partial\Omega'}{\partial\sigma}) \end{pmatrix}^T$. The above matrix equation has the solutions of the form:

\[
\frac{d\Phi}{d\mu_q} = \frac{\Delta_1}{\Delta}, \quad \frac{d\bar{\Phi}}{d\mu_q} = \frac{\Delta_2}{\Delta}, \quad \frac{d\sigma}{d\mu_q} = \frac{\Delta_3}{\Delta},
\] (25)

where the Cramer’s determinants are given by,

\[
\Delta = \det(A) = \begin{vmatrix}
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma}
\end{vmatrix},
\] (26)

\[
\Delta_1 = -\begin{vmatrix}
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma}
\end{vmatrix},
\] (27)

\[
\Delta_2 = -\begin{vmatrix}
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma}
\end{vmatrix},
\] (28)

\[
\Delta_3 = -\begin{vmatrix}
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma} \\
\frac{\partial}{\partial\mu_q} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial\bar{\Phi}} & \frac{\partial}{\partial\sigma}
\end{vmatrix}.
\] (29)

The elements of the determinants can be obtained from the expression of $\Omega'$. Instead of using Cramer’s rule, the solutions can also be obtained through Gaussian Elimination method. In Fig. we have plotted the first order derivatives of the mean fields along $T$ and compared the results from two different methods as in case of NJL in Fig.1.

One may similarly obtain the higher derivatives of the mean fields with respect to $\mu_q$ by sequentially increasing the order of derivatives to act upon $d\Phi$. The derivatives at any order will depend upon the various thermodynamic parameters as well as the derivatives up to one lower order. For these higher orders also we shall have to solve the similar matrix equations of the form $A \cdot x = B$. The most interesting part is that while the column matrix $B$ changes for every order, the coefficient matrix $A$ will remain same as in first order. We expect that this semi-analytical prescription for obtaining field derivatives will certainly give better results specifically in higher order, compared to numerical derivatives. The detailed study in that direction will be presented elsewhere. Here we shall use this elegant formulation to gain some insight into the physics aspects of susceptibilities in the context of fluctuation-dissipation theorem.
III. QNS FROM FDT

A. A Toy Model

\[ K = P + Q \]

We consider a Lagrangian with an effective mass

\[ \mathcal{L} = \bar{\psi} \left( i \frac{\partial}{\partial t} - \hat{M} + \gamma_0 \hat{\mu} \right) \psi. \]  

(30)

where \( \hat{M} \) is effective mass matrix and \( \hat{\mu} \) is the matrix of chemical potential and both are diagonal in flavor space. \( \gamma_0 \) is the three point function \(^1\) and the corresponding two-point function for flavor \( f \) is

\[ S_f(L) = \frac{1}{L - M_f + \mu_f \gamma_0}, \]

where \( L \) is the four momentum. With this simple consideration, it is obvious that Fig. 3 is the relevant diagram in one loop that would contribute to the time-time component of vector correlator \( \Pi_{00} \),

\[ \Pi_{00}(q_0 = \omega, q = |\vec{Q}|) = -i \sum_{f=u,d} \int \frac{d^4P}{(2\pi)^4} \text{Tr}_{D,c}[\gamma_0 S_f(K)\gamma_0 S_f(P)], \]

(31)

with \( K = P + Q \). Here trace \( \text{Tr}_{D,c} \) is over Dirac and color indices only. Replacing

\[ \int \frac{dP_0}{2\pi} \rightarrow \frac{i}{\beta} \sum_{\omega_n}, \]

and performing the Dirac trace, Eq. (31) becomes,

\[ \Pi_{00}(\omega, q) = \sum_{f=u,d} \sum_n \frac{4}{\beta} \int \frac{d^3p}{(2\pi)^3} \text{Tr}_c \left\{ \frac{(i\omega_n + \omega + \hat{\mu}_f)(i\omega_n + \hat{\mu}_f) + \vec{P} \cdot \vec{K} + M_f^2}{[(i\omega_n + \omega + \hat{\mu}_f)^2 - E_{fK}^2][(i\omega_n + \hat{\mu}_f)^2 - E_{fp}^2]} \right\}. \]

(32)

Now the remaining trace operation is over color space only. Breaking into partial fractions R.H.S. of last equation

\(^1\) Equivalent to a massive free theory.
can be written as \[132\],

\[
\Pi_{00}(\omega, q) = \sum_{f=u,d} \sum_{n} \frac{1}{\beta} \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{fp}E_{fk}} \text{Tr}_{c} \left\{ \frac{1}{(i\omega_n + \omega + \tilde{\mu}) - E_{fk}} \frac{1}{(i\omega_n + \tilde{\mu}) - E_{fp}} [E_{fp}E_{fk} + M_f^2 + \tilde{P} \cdot \tilde{K}] \right. \\
+ \frac{1}{(i\omega_n + \omega + \tilde{\mu}) - E_{fk}} \frac{1}{(i\omega_n + \tilde{\mu}) + E_{fp}} [E_{fp}E_{fk} - M_f^2 - \tilde{P} \cdot \tilde{K}] \\
+ \frac{1}{(i\omega_n + \omega + \tilde{\mu}) + E_{fk}} \frac{1}{(i\omega_n + \tilde{\mu}) - E_{fp}} [E_{fp}E_{fk} + M_f^2 - \tilde{P} \cdot \tilde{K}] \\
+ \frac{1}{(i\omega_n + \omega + \tilde{\mu}) + E_{fk}} \frac{1}{(i\omega_n + \tilde{\mu}) + E_{fp}} [E_{fp}E_{fk} + M_f^2 + \tilde{P} \cdot \tilde{K}] \right\}. 
\] (33)

Now performing the Matsubara summation over the discrete frequencies, \(\omega_n = (2n + 1)\pi T\), we are left with,

\[
\Pi_{00}(\omega, q) = \sum_{f=u,d} \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{fp}E_{fk}} \text{Tr}_{c} \left\{ \frac{E_{fp}E_{fk} + M_f^2 + \tilde{P} \cdot \tilde{K}}{\omega + E_{fp} - E_{fk}} [\mathcal{F}_1(E_{fp} - \tilde{\mu}) - \mathcal{F}_1(E_{fk} - \tilde{\mu} - \omega)] \\
+ \frac{E_{fp}E_{fk} - M_f^2 + \tilde{P} \cdot \tilde{K}}{\omega - E_{fp} - E_{fk}} [1 - \mathcal{F}_1(E_{fp} + \tilde{\mu}) - \mathcal{F}_1(E_{fk} - \tilde{\mu} - \omega)] \\
+ \frac{E_{fp}E_{fk} - M_f^2 - \tilde{P} \cdot \tilde{K}}{\omega + E_{fp} + E_{fk}} [\mathcal{F}_1(E_{fp} - \tilde{\mu}) - 1 + \mathcal{F}_1(E_{fk} + \tilde{\mu} + \omega)] \\
+ \frac{E_{fp}E_{fk} + M_f^2 + \tilde{P} \cdot \tilde{K}}{\omega - E_{fp} + E_{fk}} [-\mathcal{F}_1(E_{fp} + \tilde{\mu}) + \mathcal{F}_1(E_{fk} + \tilde{\mu} + \omega)] \right\}, 
\] (34)

where \(\mathcal{F}\) is the Fermi-Dirac distribution function. If we make a change of variable \(\tilde{P} \rightarrow -\tilde{P} - \tilde{Q}\), in the third and fourth term then dot product of 3-vectors remains unchanged and the momentum label of quasiparticle energy just interchanges. Moreover keeping in mind that \(e^{i\omega} = 1\), after simplification Eq.\[132\] becomes,

\[
\Pi_{00}(\omega, q) = \sum_{f=u,d} \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{fp}E_{fk}} \text{Tr}_{c} \left\{ \frac{E_{fp}E_{fk} + M_f^2 + \tilde{P} \cdot \tilde{K}}{\omega + E_{fp} - E_{fk}} [\mathcal{F}(E_{fp} - \mu_f) + \mathcal{F}(E_{fp} + \mu_f) - \mathcal{F}(E_{fk} - \mu_f) - \mathcal{F}(E_{fk} + \mu_f)] \\
+ \frac{E_{fp}E_{fk} - M_f^2 + \tilde{P} \cdot \tilde{K}}{\omega - E_{fp} - E_{fk}} \left[ \frac{1}{\omega - E_{fp} - E_{fk}} - \frac{1}{\omega + E_{fp} + E_{fk}} \right] [1 - \mathcal{F}(E_{fp} + \mu_f) - \mathcal{F}(E_{fk} - \mu_f)] \right\}. 
\] (35)

1. Calculation of \(\chi_q\) from real part of \(\Pi_{00}\)

After taking the real part of \(\Pi_{00}(\omega, q)\) when we put \(\omega = 0\), we are left with;

\[
\text{Re}\Pi_{00}(\omega = 0, q) = \sum_{f=u,d} \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{fp}E_{fk}} \text{Tr}_{c} \left\{ \frac{[E_{fp}E_{fk} + M_f^2 + \tilde{P} \cdot \tilde{K}]}{E_{fp} - E_{fk}} \right. \\
\times \mathcal{F}(E_{fp} - \mu_f) + \mathcal{F}(E_{fp} + \mu_f) - \mathcal{F}(E_{fk} - \mu_f) - \mathcal{F}(E_{fk} + \mu_f) \\
+ \left. \frac{E_{fp}E_{fk} - M_f^2 - \tilde{P} \cdot \tilde{K}}{E_{fp} + E_{fk}} \right\} [1 - \mathcal{F}(E_{fp} + \mu_f) - \mathcal{F}(E_{fk} - \mu_f)]. 
\] (36)
Now we are going to use the FDT as in Eq. (1). In the limit $q \to 0$ the second term of (36) vanishes and for the first term taking care of the $\frac{1}{q}$ form using L'Hospital rule we get,

$$\chi_q = 2\beta \sum_{f=u,d} \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left\{ \frac{e^{\beta(E_{fp}-\mu_f)}}{1 + e^{\beta(E_{fp}-\mu_f)}} \right\}.$$  \hspace{1cm} (37)

2. Calculation of $\chi_q$ from Imaginary part of $\Pi_{00}$

The imaginary part of the retarded correlator can be calculated from the discontinuity in the following way:

$$\text{Im} \Pi_{00}(\omega, q) = \frac{1}{2\pi} \text{Im} \Pi_{00}(\omega + i\eta, q) - \Pi_{00}(\omega - i\eta, q)$$

$$= -\pi \sum_{f=u,d} \int \frac{d^3p}{(2\pi)^3} E_{fp} E_{fk} \text{Tr} \left\{ [\mathcal{F}(E_{fp} - \mu_f) + \mathcal{F}(E_{fp} + \mu_f) - \mathcal{F}(E_{fk} + \mu_f) - \mathcal{F}(E_{fk} - \mu_f)] \right\}.$$

$$\delta(\omega + E_{fp} - E_{fk})$$

$$+ [E_{fp} E_{fk} - M_f^2 - \vec{P} \cdot \vec{K}] [1 - \mathcal{F}(E_{fp} + \mu_f) - \mathcal{F}(E_{fk} - \mu_f)]$$

$$\delta(\omega + E_{fp} + E_{fk}) - \delta(\omega + E_{fp} - E_{fk})].$$  \hspace{1cm} (38)

The delta function in the first term of R.H.S. of the above equation represents the contribution from the scattering process and the first delta function of the second term represents the pair creation process for $\omega > 0$ [135, 136]. The prefactors containing Fermi-Dirac distributions to both of the above-mentioned terms can be rearranged to show that they basically account for the statistical weights of corresponding processes. Similarly for $\omega < 0$, one can realize processes [136] corresponding to the second delta function in the second term. Although that will clearly violates energy conservation for $\omega > 0$, since quasiparticle energies are always positive and therefore hereinafter this term will be dropped.

As an intermediate but important step we want to show that, the first term of R.H.S. of the above equation can be written as [137],

$$\mathcal{F}(E_{fp} - \mu_f) + \mathcal{F}(E_{fp} + \mu_f) - \mathcal{F}(E_{fp} - \mu_f + \omega) - \mathcal{F}(E_{fp} + \mu_f + \omega)$$

$$\approx -\mathcal{F}(E_{fp} - \mu_f) \left\{ 1 - \mathcal{F}(E_{fp} - \mu_f) \right\} + \mathcal{F}(E_{fp} + \mu_f) \left\{ 1 - \mathcal{F}(E_{fp} + \mu_f) \right\}.$$  \hspace{1cm} (39)

Here, for the time being we have omitted the integration and prefactors. Proportionality of this term to $\omega \delta(\omega)$ is important because it is related to number conservation [18, 13, 22, 126, 127, 135]. Apart from that, this kind of zero mode contribution in the spectral functions is significant and gives rise to a constant contribution in finite temperature Euclidean correlator [138, 142].

From the FDT as in Eq. (1) and using (38) we get,

$$\chi_q = \lim_{q \to 0} \beta \sum_{f=u,d} \int \frac{d^3p}{(2\pi)^3} E_{fp} E_{fk} \text{Tr} \left\{ \frac{1}{e^{\beta(E_{fp} - E_{fk})}} \right\}$$

$$- \mathcal{F}(E_{fp} - \mu_f) + \mathcal{F}(E_{fp} + \mu_f) - \mathcal{F}(E_{fk} - \mu_f) - \mathcal{F}(E_{fk} + \mu_f)$$

$$\frac{1}{1 - e^{\beta(E_{fp} - E_{fk})}}$$

$$+ (E_{fp} E_{fk} - M_f^2 - \vec{P} \cdot \vec{K}) [1 - \mathcal{F}(E_{fp} + \mu_f) - \mathcal{F}(E_{fk} - \mu_f)] 1 - e^{-\beta(E_{fp} + E_{fk})}.$$  \hspace{1cm} (39)

For the limit of vanishing external momentum second term vanishes. Using L'Hospital rule for the first term we are left with,

$$\chi_q = 2\beta \sum_{f=u,d} \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left\{ \frac{e^{\beta(E_{fp} - \mu_f)}}{1 + e^{\beta(E_{fp} - \mu_f)}} + \frac{e^{\beta(E_{fp} + \mu_f)}}{1 + e^{\beta(E_{fp} + \mu_f)}} \right\}.$$  \hspace{1cm} (40)
3. Calculation of $\chi_q$ from Thermodynamic derivative

One can write the partition function corresponding to (30) as

$$Z(\beta, \{\mu_f\}) = \int \mathcal{D}[\bar{\psi}] \mathcal{D}[\psi] e^{-i\int d^4x \mathcal{L}(\psi, \bar{\psi}; \{\mu_f\})},$$

where $\beta$ is the inverse temperature. The pressure can be written as

$$P = \frac{1}{V} \ln Z(\beta, \{\mu_f\}),$$

where the four-volume, $V = \beta V$ with $V$ is the three-volume. One can straight away compute the pressure and show that the $\chi_q$ obtained from it through thermodynamic derivative with respect to $\mu_f$ is exactly the same as those obtained in (37) and (40).

Nevertheless, we now demonstrate this in a very general perspective. $P'$ can be obtained from (42) as

$$\frac{\partial P}{\partial \mu_f} = \frac{-i}{VZ} \int \mathcal{D}[\bar{\psi}] \mathcal{D}[\psi] \int d^4x \bar{\psi}^f(x) \psi^f(x) e^{-i\int d^4x \mathcal{L}(\psi, \bar{\psi}; \{\mu_f\})}. \tag{43}$$

The quark propagator in a hot and dense medium is defined as

$$S^f_{\alpha\sigma}(x, x') = \frac{\int \mathcal{D}[\bar{\psi}] \mathcal{D}[\psi] \bar{\psi}^f_{\alpha}(x) \psi^f_{\sigma}(x') \exp \left(-i \int d^4x \mathcal{L}(\psi, \bar{\psi}; \{\mu_f\})\right)}{\int \mathcal{D}[\bar{\psi}] \mathcal{D}[\psi] \exp \left(-i \int d^4x \mathcal{L}(\psi, \bar{\psi}; \{\mu_f\})\right)}. \tag{44}$$

From Eq. (43) one can write the quark number density as

$$n_q = \sum_f n_f = \sum_f \frac{\partial P}{\partial \mu_f} = -i \sum_f \frac{d^4P}{(2\pi)^4} \text{Tr}_{D,c} [S_f(P)\gamma_0]. \tag{45}$$

Likewise, one can also obtain QNS as

$$\chi_q = \sum_f \frac{\partial^2 P}{\partial \mu_f^2} = i \sum_f \frac{d^4P}{(2\pi)^4} \text{Tr}_{D,c} [S_f(P)\gamma_0 S_f(P)\gamma_0] = -i \text{Tr}[S(P)\gamma_0 S(P)\gamma_0] \tag{46}$$

where the relation in (15) is used. Now one can see that the (46) corresponds to the temporal correlator in Fig. 3 but at the external momentum $Q = (\omega_q, 0) = 0$ or amputated external legs. This static correlator has been computed as $C_{00}$ in Appendix A, which is exactly equal to those obtained in (37) and (40). This shows that the FDT vis-a-vis the thermodynamic sum rule is satisfied in a toy model.

B. NJL Model

So far our discussions are based on the naive consideration of a toy model with a free massive propagator, but this has set the stage for any realistic model calculations. Here, we consider the NJL Lagrangian of Ref. (3), in which the explicit interaction term through chiral condensate $\sigma$ is present and this would contribute to the physical quantities one would like to compute. The relevant diagrams that would contribute to the correlation function are shown in Fig. 4. We note that the effective propagator $S_1$, three-point vertex $\Gamma_0$ and the chiral condensates for NJL model have already been defined in Sec. IIA and one can easily compute these diagrams, but we purposefully avoid this and the reason for which will be clear later.

Let us now first concentrate on the calculation of QNS from the thermodynamic derivative of pressure with respect to the chemical potential. As discussed earlier in Sec. IIA, the mean fields have implicit dependences on chemical potential, thus the thermodynamic derivatives are to be considered appropriately. This implies that one needs to consider the total derivative of pressure rather than the explicit one.

With these considerations, we write the pressure $P = -\Omega$ from (41). The quark number density is then given as,

$$n_q = \frac{dP}{d\mu_q} = \frac{\partial P}{\partial \mu_q} + \frac{\partial P}{\partial \sigma} \cdot \frac{d\sigma}{d\mu_q} = i \text{Tr}(\gamma_0 S_1) + G \frac{d\sigma}{d\mu_q} [i \text{Tr}(S_1) - \sigma] = i \text{Tr}(\Gamma_0 S_1) - G \sigma \frac{d\sigma}{d\mu_q}. \tag{47}$$
where we have used
\[ \frac{\partial P}{\partial \mu_q} = i \text{Tr}(\gamma_0 S_1), \]
and
\[ \frac{\partial P}{\partial \sigma} = G[i \text{Tr}(S_1) - \sigma]. \]

It is interesting to note that for number density, \( n_q \), the partial derivative alone gives the full contribution in the mean field theory as \( \frac{\partial P}{\partial \sigma} = 0 \) if one uses (6) in (49).

One can similarly obtain the second order derivative of pressure w.r.t. \( \mu_q \) from (47) to get the QNS as,
\[ \chi_q = \frac{d^2 P}{d \mu_q^2} = -i \text{Tr}(\gamma_0 S_1 \gamma_0 S_1) = -C_{00}, \]
\[ \frac{\partial}{\partial \mu_q} \left( \frac{\partial P}{\partial \sigma} \right) = -iG \text{Tr}(S_1 \gamma_0 S_1) = -C_{01}, \]
\[ \frac{\partial}{\partial \sigma} \left( \frac{\partial P}{\partial \mu_q} \right) = -iG \text{Tr}(S_1 \gamma_0 S_1) = -C_{10}, \]
\[ \frac{\partial^2 P}{\partial \sigma^2} = G[-iG \text{Tr}(S_1^2) - 1] = -C_{11} - G. \]

The detailed calculations of the correlators \( C_{00}, C_{01} \) and \( C_{11} \) are presented in Appendix A. Here we have used the relation \( \frac{\partial S_1}{\partial \sigma} = -GS_1^2 \), which can be easily obtained from (4).

Furthermore, the last term of (50) vanishes due to (6) and using the first equality of (14) we can finally write,
\[ \chi_q = \frac{d^2 P}{d \mu_q^2} = -i \text{Tr}(\Gamma_0 S_1 \Gamma_0 S_1) - G(-i \text{Tr}[S_1 \Gamma_0 S_1])^2. \]

The right hand side of (55) can be viewed in terms of diagrammatic topology as displayed in Fig 5. It is evident that these are equivalent to the vector correlator in NJL model in static limit or amputated legs as given in Fig 4.
note that in a mean field approach, where the mean fields are sensitive to external source, an appropriate measure is to be taken to satisfy the FDT vis-a-vis thermodynamic sum rule. This implies that the inclusion of implicit $\mu_q$ dependences of the mean fields is not ad hoc, rather it enables us, from the field theoretic point of view, to compute the correlators associated with the conserved density fluctuation through diagrammatic way in NJL model.

Another interesting as well as relevant point we would like to demonstrate below. Putting the results of (52) and (54) into (17), we can rewrite it as

$$\frac{d\sigma}{d\mu_q} = \frac{\partial}{\partial \sigma} \left( \frac{\partial P}{\partial \mu_q} \right)$$

(56)

Now replacing this $\frac{d\sigma}{d\mu_q}$ in the first line of Eq. (50) and keeping in mind that the last but one term will vanish due to the mean field condition, one can have

$$\frac{d^2 P}{d\mu_q^2} = \frac{\partial^2 P}{\partial \mu_q^2} - 2 \left[ \frac{\partial}{\partial \sigma} \left( \frac{\partial P}{\partial \mu_q} \right) \right]^2 + \frac{\partial^2 P}{\partial \sigma^2} \cdot \left[ \frac{\partial}{\partial \sigma} \left( \frac{\partial P}{\partial \mu_q} \right) \right]^2$$

$$= \frac{\partial^2 P}{\sigma_q^2} \cdot \frac{\partial^2 P}{\sigma_q^2} - \left[ \frac{\partial}{\partial \sigma} \left( \frac{\partial P}{\partial \mu_q} \right) \right]^2 = \frac{\partial^2 P}{\sigma_q^2} \cdot \frac{\partial^2 P}{\sigma_q^2} - \frac{\partial^2 P}{\sigma_q^2} \cdot \left( \frac{\partial^2 P}{\sigma_q^2} \right)^{-1} \cdot \frac{\partial^2 P}{\sigma_q^2}$$

(57)

This mixing pattern is already established in Refs. [126, 143] and is similar to the mixing in susceptibilities when computed from the inverse of curvature matrix [27, 37].

The behavior of QNS in NJL model is shown in Fig. 6. Here the study has been done for 2 flavor NJL model at only at non-zero $\mu_q$. This is because at $\mu_q = 0$ we have $\frac{d\sigma}{d\mu_q} = 0$ due to CP symmetry, and thus all implicit
contributions vanish. The two curves in Fig. 4 represent explicit and total contributions to the QNS respectively. Important contributions from the implicit $\mu_q$ dependent terms arise close to the transition region where the change in the mean fields is most significant. It is needless to mention that the QNS obtained here from (55) comes out to be the same as that obtained from any of the numerical derivative methods.

C. PNJL Model

![Graph](image)

FIG. 7: (a) Quark number susceptibility in 2 flavor PNJL model at $\mu_q = 0$. Lattice data are taken from Ref. [10]. (b) QNS in PNJL model for non-zero chemical potential.

In the present form of the PNJL model, the gluon physics is contained only effectively in a static background field that comes through the inclusion of Polyakov loop. We have no gluon-like quasi particles in PNJL model. We treat such bosonic fields as purely classical ones unlike the fermionic fields. One can only obtain their mean values through the minimization of the thermodynamic potential. Obviously the Polyakov loop fields are also to be determined from a set of transcendental equations. The mean fields $\Phi$ and $\bar{\Phi}$ so obtained also depend on $T$ and $\mu_q$ in a similar way as $\sigma$. Here one can intuitively write as in the case of the NJL model that,

$$dP/d\mu_q = \frac{\partial P}{\partial \mu_q} + \sum_{X=\sigma,\phi,\bar{\phi}} \frac{\partial P}{\partial X} \frac{dX}{d\mu_q}. \quad (58)$$

Like in the NJL model, here also the second term of (58) will vanish due to $\partial P/\partial \sigma = 0$. But a crucial difference lies in the fact that for the PNJL model, $\partial P/\partial \phi \neq 0$ and $\partial P/\partial \bar{\phi} \neq 0$. This is due to the fact that $P = -\Omega \neq -\Omega'$. So, in case of the PNJL model even for first order derivative, we shall have a finite contribution from implicit $\mu_q$ dependences through $\Phi$ and $\bar{\Phi}$, i.e. $dP/d\mu_q \neq \frac{\partial P}{\partial \mu_q}$.

Differentiating (58) w.r.t. $\mu_q$ we have,

$$\frac{d^2 P}{d\mu_q^2} = \frac{\partial^2 P}{\partial \mu_q^2} + 2 \sum_{X=\sigma,\phi,\bar{\phi}} \frac{\partial^2 P}{\partial \mu_q \partial X} \frac{dX}{d\mu_q} + \sum_{X=\sigma,\phi,\bar{\phi}} \frac{\partial P}{\partial X} \frac{d^2 X}{d\mu_q^2} + \sum_{X,Y=\sigma,\phi,\bar{\phi}} \frac{\partial^2 P}{\partial X \partial Y} \frac{dX}{d\mu_q} \frac{dY}{d\mu_q}, \quad (59)$$

where, the first term is from the explicit appearances of $\mu_q$ and the other three terms contains the contributions coming from the implicit $\mu_q$ dependences of pressure through the mean fields.

In the left panel of Fig. 7 the plots of $\chi_q$ at $\mu_q = 0$ are presented for the PNJL model. The contribution from the explicit $\mu_q$ dependence and the total contribution are shown separately. The latter contribution comes out to be same as the QNS obtained from numerical derivatives of pressure. Again due to CP symmetry, the non-vanishing implicit contributions come through the $\mu_q$ dependence of $\Phi$ and $\bar{\Phi}$ only. Our result is compared to that of Lattice QCD[10]. The QNS in the PNJL model for non-zero $\mu_q$ is presented in the right panel of Fig. 7. Here again the explicit contribution is shown separately. As is clearly evident, the presence of the implicit contributions are significant close to the transition. The most notable feature is that the peak in the susceptibility arises solely due to the implicit chemical potential dependence of pressure. Location of any critical point is therefore crucially dependent on the proper evaluation of chemical potential dependence of the mean fields.
IV. CONCLUSION

In QCD inspired model the fluctuation-dissipation theorem is usually assumed to be applicable, and susceptibilities which are associated with fluctuations are calculated from the derivatives of pressure. In the present work through an extensive exercise we have shown the fluctuation-dissipation theorem holds true within the framework of NJL and PNJL models. In mean field approaches, the mean fields are sensitive to the external source like quark chemical potential, there should be additional contributions coming from the implicit dependence of the mean fields on the chemical potential. On the other hand, the temporal vector correlator associated with the fluctuations is modified due to the effective interaction in these model Lagrangians. Here we have given an elegant formalism and shown that the inclusion of implicit dependent terms through the mean fields is actually consistent with the field theoretic point of view and consolidates the fluctuation-dissipation theorem. For the NJL model a complete analysis through diagrammatics could be found. While such elegant exercise did not result for the PNJL model due to the classical nature of the Polyakov loop, the essence of the modification required has been clearly presented.

We have also described an analytical method for calculating the derivative of the mean fields with respect to the chemical potential, which forms the essential part of the modifications in the fluctuation-dissipation theorem. This approach is essential if one intends to study higher order derivatives for which numerical differentiation becomes unreliable. Further studies in higher order derivatives will be presented elsewhere. We expect these studies to play an important role in understanding the nature of the critical region in the phase diagram of strongly interacting matter.

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Appendix A: Calculation of traces in main text

As we are working in isospin symmetric limit, the current and constituent masses of $u$ and $d$ flavors are equal. So, in all the trace calculations of this section we will not carry the flavor index of propagators and chemical potentials and whenever there is trace or sum over flavors that will give only a factor of $N_f$.

1. Calculation of R.H.S. of Eq. (51)

Here we discuss the calculation of the time-time component of the amputated correlator with bare vertices which is shown in Fig. 8 and discussed in the main text.

\[
C_{00} = i \text{Tr}[\gamma_0 S_1(P)\gamma_0 S_1(P)] \\
= -i \int \frac{d^4P}{(2\pi)^4} \text{Tr}_{D,f,c}[\gamma_0 S_1(P)\gamma_0 S_1(P)],
\]

(A1)
with $S_1(P)$ being the quark propagator of momentum $P$ which is given by, $S_1(P) = \frac{1}{P^2 - M^2 + \gamma \mu_q}$, where, $\mu_q$ is the chemical potential and $M$ is constituent quark mass. $M = m_0 - G\sigma$. For NJL $\tilde{\mu}_q = \mu_q$ and for PNJL $\tilde{\mu}_q = \mu_q - iA_4$.

In last equation ‘Tr$_{D,f,c}$’ represents trace over Dirac, flavor and color indices only. Making the replacement

$$\int \frac{dP_0}{2\pi} \rightarrow \frac{i}{\beta} \sum_n, \quad \text{where} \quad \omega_n = (2n + 1)\frac{\pi}{\beta},$$

$$C_{00} = \frac{4N_f}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} \operatorname{Tr}_c \left[ \frac{(i\omega_n + \tilde{\mu}_q)^2 + E_p^2}{[(i\omega_n + \tilde{\mu}_q)^2 - E_p^2]^2} \right]$$

$$= \frac{2N_f}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} \operatorname{Tr}_c \left[ \frac{1}{(i\omega_n + \tilde{\mu}_q - E_p)^2} + \frac{1}{(i\omega_n + \tilde{\mu}_q + E_p)^2} \right]. \quad (A2)$$

where, $E_p^2 = p^2 + M^2$. For Matsubara summation we use:

$$\frac{1}{\beta} \sum_n \frac{1}{(i\omega_n \pm \beta)^2} = -\frac{\beta e^{\beta z}}{(1 + e^{\beta z})^2}. \quad \text{(A3)}$$

So finally,

$$C_{00} = -2N_f\beta \int \frac{d^3p}{(2\pi)^3} \operatorname{Tr}_c \left[ \frac{e^{\beta(E_p - \tilde{\mu}_q)}}{(1 + e^{\beta(E_p - \tilde{\mu}_q)})^2} + \frac{e^{\beta(E_p + \tilde{\mu}_q)}}{(1 + e^{\beta(E_p + \tilde{\mu}_q)})^2} \right]. \quad (A4)$$

For NJL model color trace is trivial, only the number of color $N_c$ will be factored out. We have,

$$C_{00} = -2N_fN_c\beta \int \frac{d^3p}{(2\pi)^3} \operatorname{Tr}_c \left[ \frac{e^{\beta(E_p - \tilde{\mu}_q)}}{(1 + e^{\beta(E_p - \tilde{\mu}_q)})^2} + \frac{e^{\beta(E_p + \tilde{\mu}_q)}}{(1 + e^{\beta(E_p + \tilde{\mu}_q)})^2} \right]. \quad (A5)$$

Now we are going to evaluate the color trace in PNJL model. Let us assume, $\mathcal{F}_2(x) = \frac{e^x}{(1 + e^x)^2}$. Note the fact that,

$$\frac{\partial^2}{\partial E_p^2} \ln[1 + e^{-\beta(E_p - \tilde{\mu}_q)}] = \beta^2 \mathcal{F}_2(E_p - \tilde{\mu}_q).$$

So, following Ref.51 we can write,

$$\operatorname{Tr}_c \mathcal{F}_2(E_p - \tilde{\mu}_q) = \frac{1}{\beta^2} \frac{\partial^2}{\partial E_p^2} \ln[1 + e^{-\beta(E_p - \tilde{\mu}_q)}]$$

$$= \frac{1}{\beta^2} \frac{\partial^2}{\partial E_p^2} \ln[1 + \mathcal{L} e^{-\beta(E_p - \tilde{\mu}_q)}]$$

$$= \frac{3}{\beta^2} \left[ \Phi e^{-\beta(E_p - \tilde{\mu}_q)} + 4\Phi e^{-2\beta(E_p - \tilde{\mu}_q)} + 3\Phi e^{-3\beta(E_p - \tilde{\mu}_q)} \right]$$

$$- \left( \sum_{n=1}^{\infty} \left( \frac{e^{-\beta(E_p - \tilde{\mu}_q)}}{1 + e^{-\beta(E_p - \tilde{\mu}_q)}} + \frac{e^{-2\beta(E_p - \tilde{\mu}_q)}}{1 + e^{-2\beta(E_p - \tilde{\mu}_q)}} + \cdots \right) \right)^2. \quad (A5)$$

Similar expression can be found for $\operatorname{Tr}_c \mathcal{F}_2(E_p + \tilde{\mu}_q)$. Finally putting altogether these expressions we got,

$$C_{00} = -6N_f\beta \int \frac{d^3p}{(2\pi)^3} \left[ \frac{\Phi e^{-\beta(E_p - \tilde{\mu}_q)} + 4\Phi e^{-2\beta(E_p - \tilde{\mu}_q)} + 3\Phi e^{-3\beta(E_p - \tilde{\mu}_q)}}{1 + 3\Phi e^{-\beta(E_p - \tilde{\mu}_q)} + 3\Phi e^{-2\beta(E_p - \tilde{\mu}_q)} + e^{-3\beta(E_p - \tilde{\mu}_q)}} \right]$$

$$- \left( \sum_{n=1}^{\infty} \left( \frac{e^{-\beta(E_p - \tilde{\mu}_q)}}{1 + e^{-\beta(E_p - \tilde{\mu}_q)}} + \frac{e^{-2\beta(E_p - \tilde{\mu}_q)}}{1 + e^{-2\beta(E_p - \tilde{\mu}_q)}} + \cdots \right) \right)^2. \quad (A6)$$

2 For a toy model $M = m_0$ (say).
2. Calculation of R.H.S. of Eq. (53) and Eq. (52)

\[ C_{10} = C_{01} = iG \text{Tr}[S_1(P) \gamma_0 S_1(P)] \]

\[ = -iG \int \frac{d^4P}{(2\pi)^4} \text{Tr}_{D,f,c}[S_1(P) \gamma_0 S_1(P)] \]

\[ = 4N_f G \sum_n \int \frac{d^3p}{(2\pi)^3} \text{Tr}_\epsilon \left[ \frac{2M(i\omega_n + \mu_q)}{(i\omega_n + \mu_q)^2 - E_p^2} \right] \]

\[ = 2N_f \sum_n \int \frac{d^3p}{(2\pi)^3} \left( \frac{GM}{E_p} \right) \text{Tr}_\epsilon \left[ \frac{1}{(i\omega_n + \mu_q - E_p)^2} - \frac{1}{(i\omega_n + \mu_q + E_p)^2} \right]. \quad (A7) \]

Using the result of (A5) and similar one for anti-particle part we arrive at,

\[ C_{10} = C_{01} = -2N_f \sum_n \int \frac{d^3p}{(2\pi)^3} \left( \frac{GM}{E_p} \right) \text{Tr}_\epsilon \left[ \frac{e^{\beta(E_p - \mu_q)}}{(1 + e^{\beta(E_p - \mu_q)})^2} - \frac{e^{\beta(E_p + \mu_q)}}{(1 + e^{\beta(E_p + \mu_q)})^2} \right]. \quad (A8) \]

Calculation of color trace in PNJL model is already shown in appendix (A1). Using the result of (A8) and similar one for anti-particle part we arrive at,

\[ C_{10} = C_{01} = -6N_f \sum_n \int \frac{d^3p}{(2\pi)^3} \left( \frac{GM}{E_p} \right) \text{Tr}_\epsilon \left[ \frac{e^{\beta(E_p - \mu_q)}}{(1 + e^{\beta(E_p - \mu_q)})^2} - \frac{e^{\beta(E_p + \mu_q)}}{(1 + e^{\beta(E_p + \mu_q)})^2} \right]. \]

3. Calculation of R.H.S. of Eq. (54)

\[ C_{11} = iG^2 \text{Tr}[S_1(P) \cdot S_1(P)] \]

\[ = -iG^2 \int \frac{d^4P}{(2\pi)^4} \text{Tr}_{D,f,c}[S_1(P) \cdot S_1(P)] \]

\[ = G^2 \frac{4N_f}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} \text{Tr}_\epsilon \left[ \frac{\Lambda}{((i\omega_n + \mu_q)^2 - p^2 + M^2)} \right]. \quad (A11) \]

Expressing the integrand in partial fractions we arrive at,

\[ C_{11} = G^2 \frac{N_f}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} \text{Tr}_\epsilon \left[ \frac{2M^2}{E_p^2} \left( \frac{1}{(i\omega_n + \mu_q - E_p)^2} + \frac{1}{(i\omega_n + \mu_q + E_p)^2} \right) \right. \]

\[ + \left. \frac{2p^2}{E_p^3} \right] \left[ \frac{1}{i\omega_n + \mu_q - E_p} - \frac{1}{i\omega_n + \mu_q + E_p} \right]. \quad (A12) \]

So finally;

\[ C_{11} = -N_f G^2 \text{Tr}_\epsilon \left[ \int \frac{d^3p}{(2\pi)^3} \left( \frac{2M^2}{E_p^2} \right) \frac{e^{\beta(E_p - \mu_q)}}{(1 + e^{\beta(E_p - \mu_q)})^2} + \frac{e^{\beta(E_p + \mu_q)}}{(1 + e^{\beta(E_p + \mu_q)})^2} \right] \]

\[ - \frac{2p^2}{E_p^3} \left[ \frac{1}{1 + e^{\beta(E_p - \mu_q)}} + \frac{1}{1 + e^{\beta(E_p + \mu_q)}} \right] \] \quad (A13)
For NJL model this reduces to,

\[ C_{11} = -N_c N_f G^2 \left[ \int \frac{d^3 p}{(2\pi)^3} \left( \frac{2 M^2}{E_p} \beta \left[ \frac{e^{\beta(E_p - \mu_q)}}{1 + e^{\beta(E_p - \mu_q)}} + \frac{e^{\beta(E_p + \mu_q)}}{1 + e^{\beta(E_p + \mu_q)}} \right] \right] 
\]

\[ - \frac{2 \rho^2}{E_p^3} \left[ \frac{1}{1 + e^{\beta(E_p - \mu_q)}} + \frac{1}{1 + e^{\beta(E_p + \mu_q)}} \right] \right] + \int \frac{d^3 p}{(2\pi)^3} \frac{2 \rho^2}{E_p^3}. \]  

(A14)

Color trace for PNJL model can be done following [51] and using (A5) which gives,

\[ C_{11} = -N_c N_f G^2 \left[ \int \frac{d^3 p}{(2\pi)^3} \left( \frac{2 M^2}{E_p} \beta \left[ \frac{\Phi e^{-\beta(E_p - \mu_q)}}{1 + 3 \Phi e^{-\beta(E_p - \mu_q)} + 3 e^{-3\beta(E_p - \mu_q)}} + \frac{\Phi e^{-\beta(E_p - \mu_q)}}{1 + 3 \Phi e^{-\beta(E_p - \mu_q)} + 3 e^{-3\beta(E_p - \mu_q)}} \right] \right] 
\]

\[ - \frac{3 \rho^2}{E_p^3} \left[ \frac{\Phi e^{-\beta(E_p + \mu_q)}}{1 + 3 \Phi e^{-\beta(E_p + \mu_q)} + 3 e^{-3\beta(E_p + \mu_q)}} + \frac{\Phi e^{-\beta(E_p + \mu_q)}}{1 + 3 \Phi e^{-\beta(E_p + \mu_q)} + 3 e^{-3\beta(E_p + \mu_q)}} \right] \right] 
\]

\[ + \frac{2 \rho^2}{E_p^3} \left[ \frac{\Phi e^{-\beta(E_p - \mu_q)}}{1 + 3 \Phi e^{-\beta(E_p - \mu_q)} + 3 e^{-3\beta(E_p - \mu_q)}} + \frac{\Phi e^{-\beta(E_p - \mu_q)}}{1 + 3 \Phi e^{-\beta(E_p - \mu_q)} + 3 e^{-3\beta(E_p - \mu_q)}} \right] \right] + \int \frac{d^3 p}{(2\pi)^3} \frac{2 \rho^2}{E_p^3}. \]  

(A15)

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