A DOUBLE RETURN TIMES THEOREM

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ABSTRACT. We prove that for any bounded functions \( f_1, f_2 \) on a measure-preserving dynamical system \((X, T)\) and any distinct integers \( a_1, a_2 \), for almost every \( x \) the sequence
\[
f_1(T^{a_1n}x)f_2(T^{a_2n}x)
\]
is a good weight for the pointwise ergodic theorem.

1. INTRODUCTION

Bourgain’s bilinear pointwise ergodic theorem [Bou90] (see also [Dem07; DOP17]) is one of the hardest known convergence results for multiple ergodic averages. It can therefore be considered surprising that Assani, Duncan, and Moore [ADM16] have been able to extend it to a Wiener–Wintner type result by relatively simple means, using Bourgain’s result as a black box. We strengthen their result further to a double return times theorem, similarly assuming Bourgain’s result as a black box.

Theorem 1.1. Let \((X, \mu, T)\) be a (not necessarily ergodic) invertible measure-preserving dynamical system and \( a_1, a_2 \) distinct non-zero integers. Then for any \( f_1, f_2 \in L^\infty(X) \) there exists a full measure subset \( X' \subset X \) such that for every \( x \in X' \) the sequence
\[
c_n = f_1(T^{a_1n}x)f_2(T^{a_2n}x)
\]
is a good weight for the pointwise ergodic theorem in the sense that for every further measure-preserving dynamical system \((Y, \nu, S)\) and every \( g \in L^\infty(Y) \) the limit
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_n g(S^ny)
\]
exists for \(\nu\)-almost every \( y \in Y \).

This was previously known for weakly mixing systems \((X, \mu, T)\) [Ass00, Theorem 2]. Convergence in \(L^2(Y)\) in Theorem 1.1 follows from the result of Assani, Duncan, and Moore [ADM16].

Our proof relies on the following description of a class of good weights for pointwise convergence of ergodic averages to zero that is implicit in the Bourgain–Furstenberg–Katznelson–Ornstein orthogonality criterion [Bou+89].

Theorem 1.2 ([LMM94, Theorem 4.1], see also [Zor14, Theorem 1.2]). Let \((c_n)_{n \in \mathbb{Z}}\) be a bounded sequence. For \( \delta > 0 \) and \( 0 < L < R < \infty \) define
\[
S_{\delta, L, R}(c) := \bigcap_{N=L}^{R} S_{\delta, N}(c), \quad S_{\delta, N}(c) := \{ h \mid \frac{1}{N} \sum_{n=1}^{N} c(n)c(n+h) < \delta \}.
\]
Suppose
\[
(1.3) \quad \inf_{\delta > 0} \lim_{L \to \infty} \inf_{R \geq L} d(S_{\delta, L, R}(c)) = 1,
\]
where the lower density of a set is defined by \( d(S) := \liminf_{N \to \infty} |S \cap \{1, \ldots, N\}|/N \). Then for every measure-preserving system \((Y, S)\) and every \( g \in L^\infty(Y) \) we have
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_n g(S^ny) = 0
\]
pointwise almost everywhere.
The main technical result is that nilfactors of order 2 are characteristic for the double return times theorem in the following sense.

**Theorem 1.4.** Let \((X,T,\mu)\) be a (not necessarily ergodic) invertible measure-preserving dynamical system and \(a_1, a_2\) distinct non-zero integers. Suppose \(f_1, f_2 \in L^\infty(X)\) with \(\|f_i\|_{U^3(T)} = 0\) for some \(i \in \{1, 2\}\). Then there exists a full measure set \(X' \subset X\) such that for every \(x \in X'\) the sequence 
\[
(f_1(T^{a_1 n} x) f_2(T^{a_2 n} x))_n
\]
satisfies the orthogonality criterion \((1.3)\).

Given this result, Theorem 1.1 follows from the known structural theory for \(U^3\) seminorms.

**Proof of Theorem 1.1 assuming Theorem 1.4.** Decompose the functions \(f_1, f_2\) according to Theorem 2.3. The uniform parts contribute universally good weights for pointwise convergence to zero by Theorems 1.4 and 1.2. The error terms can be controlled by the usual 1-linear maximal inequality. It remains to handle the structured parts, and here convergence follows from the nilsequence Wiener–Wintner theorem [HK09]. □

The basic idea for verification of the orthogonality criterion is to use Bourgain’s bilinear ergodic theorem to convert the lower density in \((1.3)\) to an integral, see Section 4. The integral is then evaluated using Bourgain’s theorem on a certain extension of \(X\) that is constructed in Section 3. A necessary uniformity seminorm estimate is proved in Section 2.

### 2. Uniformity seminorms

**Definition 2.1.** Let \((X,\mu,T)\) be a (not necessarily ergodic) invertible measure-preserving dynamical system and \(c\) a non-zero integer. We define uniformity seminorms by 
\[
\|f\|_{U^l(X,\mu,T,c)} := \|\mathbb{E}(f|I_T^c)\|_{L^2},
\]
where \(I_T^c\) is the invariant factor of \(T^c\), and
\[
\|f\|_{U^{l+1}(X,\mu,T,c)} := \limsup_{H \to \infty} \frac{1}{2H + 1} \sum_{h=-H}^{H} \|fT^h f\|_{U^l(X,\mu,T,c)}^2, \quad l \geq 1.
\]

We omit the parameter \(c\) if \(c = 1\):
\[
\|f\|_{U^l(X,\mu,T)} := \|f\|_{U^l(X,\mu,T,1)}.
\]

In this case our definition specializes to the standard (non-ergodic) definition of uniformity seminorms in [CFH11].

Using the well-known fact that the limit in the definition of uniformity seminorms exists (even in the uniform Cesàro sense) and induction on \(l\) one can show that
\[
(2.2) \quad \|f\|_{U^l(X,\mu,T,c)}^2 = \int_X \|f\|_{U^l(X,\mu,T,c)}^2 d\mu,
\]
for any disintegration \(\mu = \int_X \mu_x\) over a factor contained in the invariant factor \(I_T\).

We will omit some or all of the subscripts \(X,\mu,T\) from the uniformity seminorms when there is no potential for confusion. We will not verify subadditivity of the functionals \(U^l(T,c)\) for \(c \neq 1\) because it will not be used.

The main structural result about uniformity seminorms in the non-ergodic case is the following.

**Theorem 2.3 ([CFH11, Proposition 3.1]).** Let \((X,\mu,T)\) be a (not necessarily ergodic) measure-preserving dynamical system. For every \(l \geq 0\), every function \(f \in L^\infty(X,\mu)\) bounded by 1, and every \(\epsilon > 0\) there exists a decomposition
\[
f = f_s + f_e + f_u
\]
with \(\|f_s\|_\infty, \|f_e\|_\infty, \|f_u\|_\infty < 2\) such that
(1) for almost every $x \in X$ the sequence $(f_s(T^nx))$ is an $l$-step nilsequence,
(2) $\|f_c\|_{L^1} < \epsilon$, and
(3) $\|f_a\|_{U^{l+1}(X,\mu,T)} = 0$.

Note that
\[ c' | c \implies \|f\|_{U^t(T,c')} \leq \|f\|_{U^t(T,c)} \quad \text{for all } f \in L^\infty(X). \]

The main reason to use the $U^t(T,c)$ seminorms instead of the (smaller) $U^t(T)$ seminorms is the following estimate, originally proved in [HK05b, Theorem 12.1] in the case $a_1 = i$.

**Lemma 2.4.** Let $(X,T)$ be a measure-preserving system, $f_1, \ldots, f_k \in L^\infty(X)$ functions bounded by 1, and $a_1, \ldots, a_k$ distinct integers. Then for every Følner sequence $(\Phi_N)$ in $\mathbb{Z}$ and every $i$ we have
\[ \limsup_{N \to \infty} \left| \sum_{n \in \Phi_N} T^{a_1n}f_1 \cdots T^{a_kn}f_k \right|_2 \leq_c a_1, \ldots, a_k,i,c \|f_i\|_{U^k(T,c)}, \]
where we can choose $c = a_1$ if $k = 1$ and $c = a_i - a_{i'}$ for any $i' \neq i$ if $k > 1$.

If one insists on an estimate with $c = 1$, then $U^{k+1}(T,1)$ norms have to be used for general $a_i$’s, see [HK05a, Proposition 2].

**Proof.** By induction on $k$. If $k = 1$, then $c = a_1$, and by the mean ergodic theorem and the definition of uniformity seminorms the limit equals
\[ \|\mathbb{E}(f_1|T^{a_1})\|_{L^2} = \|f_1\|_{U^1(T,c)}. \]
Suppose that the result is known for some $k$ and consider the case of $k+1$ functions. Let
\[ u_n := T^{a_1n}f_1 \cdots T^{a_kn}f_k. \]
By the van der Corput lemma we have
\[ \limsup_{N \to \infty} \left| \sum_{n \in \Phi_N} \langle u_n, u_{n+h} \rangle \right| \leq \liminf_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \limsup_{N \to \infty} \left| \sum_{n \in \Phi_N} \langle u_n, u_{n+h} \rangle \right| \]
Fix distinct indices $i, j$ if $k = 1$ or $i, i', j$ if $k > 1$. We have
\[ \left| \sum_{n \in \Phi_N} \langle u_n, u_{n+h} \rangle \right| = \left| \sum_{n \in \Phi_N} \int T^{a_1n}(f_i T^{a_1h} f_i) \cdots T^{a_kn}(f_{k+1} T^{a_{k+1}h} f_{k+1}) \right| \]
\[ = \left| \sum_{n \in \Phi_N} \int f_j T^{a_jh} f_j \prod_{i \neq j} T^{(a_i-a_j)n}(f_i T^{a_ih} f_i) \right| \]
\[ \leq \left| \sum_{n \in \Phi_N} \prod_{i \neq j} T^{(a_i-a_j)n}(f_i T^{a_ih} f_i) \right|_{L^2}. \]
By the inductive hypothesis the $\limsup_{N \to \infty}$ of this is bounded by
\[ \|f_i T^{a_ih} f_i\|_{U^k(T,c)}, \]
where we can choose $c = a_i - a_j$ if $k = 1$ and $c = (a_i - a_j) - (a_{i'} - a_j)$ if $k > 1$, so that in both cases $c = a_i - a_{i'}$ with $i' \neq i$. It follows that the right-hand side of (2.5) is bounded by
\[ \liminf_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|f_i T^{a_ih} f_i\|_{U^k(T,c)} \lesssim a_i \liminf_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|f_i T^{h} f_i\|_{U^k(T,c)} \]
by Cauchy–Schwarz \[ \leq \left( \liminf_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|f_i T^{h} f_i\|_{U^k(T,c)}^{2} \right)^{1-k} \]
as required. \[ \square \]
The next result is a hybrid between two well-known facts. Firstly, the nilfactor of step $k$ of a product of two systems is contained in the product of their nilfactors of step $k+1$. Secondly, the nilfactor of order $k$ of $T^c$ is contained in the nilfactor of order $k+1$ of $T$, see [HK05a, Proposition 2] and [CFH11, §2.2]. By proving these two facts simultaneously we lose only one nilpotency step instead of two.

**Lemma 2.6.** Let $(X,T)$ and $(S,Y)$ be measure-preserving systems and $f \in L^\infty(X)$, $g \in L^\infty(Y)$ be measurable functions. Then for any non-zero integers $a, b, c$ and $l \geq 1$ we have

$$\|f \otimes g\|_{U^{l+1}(T^a \times S^b,\mu)} \leq |ab|^{1/4}|c|^{1/2} \|f\|_{U^{l+1}(T)} \|g\|_{U^{l+1}(S)}.$$ 

**Proof.** Consider first the case $l = 1$. By the mean ergodic theorem we have

$$\|f \otimes g\|_{U^1(T^a \times S^b,\mu)}^2 = \lim_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \langle T^c g \otimes S^b f, f \otimes g \rangle$$

by Cauchy–Schwarz $\leq \lim_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|f \|_{U^1(T)}^2 \|g \|_{U^1(S)}^2$.

$$\leq (\lim_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|f T^c f \|_{U^1(T)}^2)^{1/2} \cdot (\lim_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|g S^b g \|_{U^1(S)}^2)^{1/2}$$

$$\leq |ac|^{1/2} (\lim_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|f T^c f \|_{U^1(T)}^2)^{1/2} \cdot (\lim_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|g S^b g \|_{U^1(S)}^2)^{1/2}$$

$$\leq |ab|^{1/2} \|f\|_{U^2(T)} \|g\|_{U^2(S)}$$

as required. Suppose now that the claim is known for some $l \geq 1$, we will show that it holds for $l + 1$. We have

$$\|f \otimes g\|_{U^{l+1}(T^a \times S^b)}^2 = \lim_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|(f \otimes g)(T^ah \otimes S^bh g)\|_{U^1(T^a \times S^b)}^2$$

by inductive hypothesis $\leq |ab|^{2l/4} |c| \lim_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|f T^c g \|_{U^{l+1}(T)}^2 \|g S^b g \|_{U^{l+1}(S)}^2$.

by Cauchy–Schwarz $\leq |ab|^{2l-2} |c| (\lim_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|f T^c g \|_{U^{l+1}(T)}^2)^{1/2}$

$$\cdot (\lim_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|g S^b g \|_{U^{l+1}(S)}^2)^{1/2}$$

$$\leq |ab|^{2l-2} \|f\|_{U^2(T)} \|g\|_{U^2(S)}$$
\[
\left(\limsup_{H \to \infty} \frac{1}{2H+1} \sum_{h=-H}^{H} \|gS^h g\|_{U^{l+1}(S)}^{2l+1}\right)^{1/2}
= |ab|^{2l+1/2} |c|^{l+2/2} \|f\|_{U^{l+2}(T)} \|g\|_{U^{l+2}(S)}^{2l+2/2}
\leq |ab|^{2l-1} |c|^{l} \|f\|_{U^{l+2}(T)} \|g\|_{U^{l+2}(S)}^{2l+1}
\]
as required. \hfill \Box

3. An extension

Let \((X, \mu, T)\) be an ergodic measure-preserving dynamical system and let \(\pi : (X, T) \to (Z, \alpha)\) be the projection onto the Kronecker factor, where \((Z, \alpha)\) is a compact monothetic group. Let

\[
\mu = \int_{Z} \mu_{z} \, dz
\]
be the corresponding disintegration, where the integral over \(Z\) is taken with respect to the Haar measure. Fix distinct non-zero integers \(a_1, a_2\) and let

\[Z := \{(z_1, z_2) \in Z^3 | (z_1 - z, z_2 - z) \in \mathbb{Z}'\}, \quad Z' := \{(a_1n\alpha, a_2n\alpha), n \in \mathbb{Z}\}.
\]
Then \(\tilde{Z}\) is a closed \((\alpha, \alpha, \alpha)\)-invariant subgroup of the compact commutative group \(Z^3\). Consider the space \(\tilde{X} := \{(x, \xi_1, \xi_2) \in X^3 | (\pi x, \pi \xi_1, \pi \xi_2) \in \tilde{Z}\}\)
with the measure

\[
\tilde{\mu} := \int_{\tilde{Z}} \mu_{z_0} \otimes \mu_{z_1} \otimes \mu_{z_2} \, dz_0 \otimes dz_1 \otimes dz_2,
\]
where the integral is taken with respect to the Haar measure on \(\tilde{Z}\). With the coordinate projections \(\pi_0, \pi_1, \pi_2\) the space \(\tilde{X}\) becomes a 3-fold self-joining of \(X\) conditionally independent over \(\tilde{Z}\) and invariant under \(\tilde{T} = (T, T, T)\).

**Lemma 3.1.** Let \(i \in \{1, 2\}\) and \(f_i \in L^{\infty}(X)\). Define a function on \(\tilde{X}\) by

\[F_i(x, \xi_1, \xi_2) := f_i(x) f_i(\xi_i).
\]
Then for every \(l \geq 1\) and \(c \in \mathbb{N}_{>0}\) we have

\[\|F_{i}\|_{U^{l}(\tilde{X}, \tilde{\mu}, \tilde{T}, c)} \lesssim_{a_i, l, c} \|f_{i}\|_{U^{l+1}(X, \mu, T)}^{2l+1}.
\]

**Proof.** Since the uniformity seminorm does not change upon passing to an extension, it suffices to estimate the seminorm on the factor \(\pi_0 \lor \pi_i\) of \(\tilde{X}\). This factor is in fact the 2-fold relatively independent self-joining of \(X\) over \(I_{T^a_i}\). Since \(T\) is ergodic on \(X\), the invariant factor \(I_{T^a_i}\) is finite, and it follows that the factor \(\pi_0 \lor \pi_i\) is isomorphic to a positive measure invariant subset of the product system \(X \times X\). It follows from (2.2) that

\[\|F_{i}\|_{U^{l}(\tilde{T}, c)} \lesssim_{a_i} \|f_{i} \otimes f_{i}\|_{U^{l}(T \times T, c)}.
\]
The latter quantity can be estimated by Lemma 2.6. \hfill \Box

**Corollary 3.2.** Let \(f_1, f_2 \in L^{\infty}(X)\) and suppose \(\|f_{1}\|_{U^{3}(T)} = 0\) or \(\|f_{2}\|_{U^{3}(T)} = 0\). Then for \(\tilde{\mu}\)-almost every point \((x, \xi_1, \xi_2) \in \tilde{X}\) we have

\[\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_1(T^{a_1n} x) f_2(T^{a_2n} x) f_1(T^{a_1n} \xi_1) f_2(T^{a_2n} \xi_2) = 0.
\]

**Proof.** The left-hand side of the conclusion can be written as

\[\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \tilde{T}^{a_1n} f_1(x, \xi_1, \xi_2) \tilde{T}^{a_2n} F_2(x, \xi_1, \xi_2).
\]
By Bourgain’s bilinear pointwise ergodic theorem this limit exists pointwise almost everywhere on \(\tilde{X}\). On the other hand, by Lemma 2.4 (with \(k = 2\)) and Lemma 3.1 (with \(l = 2\)) the \(L^2\) limit is zero. \hfill \Box
4. Fully generic points

Recall that a measure-preserving system \((X, \mu, T)\) is called regular if \(X\) is a compact metric space, \(T : X \to X\) is a homeomorphism, and \(\mu\) is a Radon probability measure.

**Definition 4.1.** Let \(f_1, f_2 \in C(X)\), where \((X, T)\) is a regular ergodic dynamical system. Let \(D_i \subset C(X), i = 1, 2\), be the minimal \(T\)-invariant sub-\(\mathbb{Q}\)-algebras containing \(f_i\). Fix distinct non-zero integers \(a_1, a_2\). We call a point \(x \in X\) fully generic for \((f_1, f_2)\) if for every function \(F \in A(f_1, f_2) := \lim D_1 \otimes D_2 \subset C(X \times X)\) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F(T^{a_1n}x, T^{a_2n}x) = \int F \, d\nu_{\pi x}, \quad \text{where } \nu_{\pi} = \int \mu_{z_1} \otimes \mu_{z_2} \, d(z_1, z_2),
\]

the latter integral being taken with respect to the Haar measure on \(Z\).

In other words, \(\nu_{\pi x}\) is the natural measure on the set of pairs \((\xi_1, \xi_2)\) with

\[
(\pi \xi_1 - \pi x, \pi \xi_2 - \pi x) \in Z.
\]

Note that \(A(f_1, f_2)\) is a closed sub-\(\mathbb{R}\)-algebra of \(C(X \times X)\).

**Lemma 4.3.** For any regular ergodic system \((X, T)\) and any \(f_1, f_2 \in C(X)\) the set of fully generic points has full measure.

**Proof.** Recall that the Kronecker factor is characteristic for bilinear ergodic averages in the sense that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^{a_1n}g_1 T^{a_2n}g_2 = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^{a_1n}E(g_1|Z)T^{a_2n}E(g_2|Z)
\]

in \(L^2(X)\) for any functions \(g_1, g_2 \in L^\infty(X)\). One way to see this is to show that the limit vanishes if \(g_1 \perp Z\), say. To this end by Lemma 2.4 it suffices to write \(g_1\) as a finite linear combination of functions with vanishing \(U^2(T, c)\) norm, \(c = a_1 - a_2\). Since \(T\) is ergodic, the factor \(I_{T^c}\) consists of finitely many atoms, say \(B_1, \ldots, B_c\). These atoms lie in \(Z\), and it follows that \(g_{1,j} := 1_{B_j} g_1 \perp Z\) for every \(j\). On the other hand, any product of \(T\)-translates of \(g_{1,j}\) is supported on an atom of \(I_{T^c}\), so the \(L^2\) norm of its projection onto \(I_{T^c}\) is comparable with the \(L^2\) norm of its projection onto \(I_T\). It follows that \(\|g_{1,j}\|_{U^2(T, c)} \approx_c \|g_{1,j}\|_{U^2(T)} = 0\).

On the product of Kronecker factors the ergodic averages converge to the integral over the orbit closure, so we obtain

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g_1(T^{a_1n}x)g_2(T^{a_2n}x) = \int_{(\pi x, \pi x) + Z'} E(g_1|Z) \otimes E(g_2|Z) = \int g_1 \otimes g_2 \, d\nu_{\pi x}
\]

in \(L^2\), and by Bourgain’s theorem also pointwise almost everywhere.

Hence there is a full measure set \(X'\) on which the equality (4.2) holds for functions of the form \(F = g_1 \otimes g_2, g_i \in D_i\). The claim follows by linearity and density. \(\square\)

We can now proceed with the verification of the orthogonality criterion.

**Proof of Theorem 1.4.** The condition (1.3) is measurable (since it suffices to consider rational \(\delta\)), so in view of (2.2) and by the ergodic decomposition we may assume that \((X, T, \mu)\) is ergodic. Passing to a suitable topological model we may assume that \((X, \mu, T)\) is a regular ergodic system and \(f_1, f_2 \in C(X)\).

It suffices to show that

\[
\int \lim_{L \to \infty} \inf_{R \geq L} d((S_{\delta L, R}(f_1(T^{a_1n}x)f_2(T^{a_2n}x))_n)) \, d\mu(x) = 1
\]

holds for a fixed \(\delta > 0\). Let \(\eta_\delta\) be a smooth function with \(1_{[-\delta/2, \delta/2]} \leq \eta_\delta \leq 1_{[-\delta, \delta]}\). Then for every \(x \in X\) we have

\[
\lim_{L \to \infty} \inf_{R \geq L} d((S_{\delta L, R}(f_1(T^{a_1n}x)f_2(T^{a_2n}x))_n))
\]
\[
\begin{align*}
&= \lim_{L \to \infty} \inf_{R \geq L} \liminf_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \prod_{n=1}^{R} \\
&\quad \{ \frac{1}{N} \sum_{n=1}^{N} f_1(T^{a_1 n} x)f_2(T^{a_2 n} x)f_1(T^{a_1(n + h)} x)f_2(T^{a_2(n + h)} x) < \delta \} \\
&\geq \lim_{L \to \infty} \inf_{R \geq L} \liminf_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \prod_{n=1}^{R} \\
&\quad \eta_\delta \left( \frac{1}{N} \sum_{n=1}^{N} f_1(T^{a_1 n} x)f_2(T^{a_2 n} x)f_1(T^{a_1(n + h)} x)f_2(T^{a_2(n + h)} x) \right)
\end{align*}
\]

Now, by the Stone–Weierstrass theorem the function

\[
F_{x,L,R}(\xi_1, \xi_2) := \prod_{N=L}^{R} \eta_\delta \left( \frac{1}{N} \sum_{n=1}^{N} f_1(T^{a_1 n} x)f_2(T^{a_2 n} x)f_1(T^{a_1 n} \xi_1)f_2(T^{a_2 n} \xi_2) \right)
\]

lies in \( A(f_1, f_2) \). Hence for every \( x \in X \) that is fully generic for \( (f_1, f_2) \) we obtain

\[
\lim_{L \to \infty} \inf_{R \geq L} d(S_{\delta, L, R}((f_1(T^{a_1 n} x)f_2(T^{a_2 n} x)))_{n}) \geq \lim_{L \to \infty} \inf_{R \geq L} \int F_{x,L,R}(\xi_1, \xi_2) \, d\nu_{\pi x}
\]

\[
\geq \lim_{L \to \infty} \int \inf_{R \geq L} F_{x,L,R}(\xi_1, \xi_2) \, d\nu_{\pi x}
\]

Since the sequence \( (\inf_{R \geq L} F_{x,L,R}(\xi_1, \xi_2)) \) is monotonically increasing in \( L \) for every \( x, \xi_1, \xi_2 \in X \) and by the monotone convergence theorem we obtain for the left-hand side of (4.4) the lower bound

\[
\lim_{L \to \infty} \int \inf_{R \geq L} F_{x,L,R}(\xi_1, \xi_2) \, d\nu_{\pi x}(\xi_1, \xi_2) \, d\mu(x).
\]

The double integral above is taken with respect to the measure

\[
\int_X \delta_x \otimes \nu_{\pi x} \, d\mu(x) = \int_Z \mu_z \otimes \nu_z \, dz = \mu.
\]

On the other hand, by Corollary 3.2 we have

\[
\frac{1}{N} \sum_{n=1}^{N} f_1(T^{a_1 n} x)f_2(T^{a_2 n} x)f_1(T^{a_1 n} \xi_1)f_2(T^{a_2 n} \xi_2) \to 0
\]

as \( N \to \infty \) for \( \bar{\mu} \)-almost every \( (x, \xi_1, \xi_2) \in \bar{X} \). It follows that

\[
\inf_{R \geq L} F_{x,L,R}(\xi_1, \xi_2) \to 1
\]

as \( L \to \infty \) for \( \bar{\mu} \)-almost every \( (x, \xi_1, \xi_2) \in \bar{X} \), and (4.4) follows by the monotone convergence theorem. \( \square \)

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