Recursion operators for dispersionless integrable systems in any dimension

M Marvan and A Sergeyev
Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, 746 01 Opava, Czech Republic
E-mail: Michal.Marvan@math.slu.cz and Artur.Sergyeyev@math.slu.cz

Received 25 July 2011, in final form 26 December 2011
Published 2 February 2012
Online at stacks.iop.org/IP/28/025011

Abstract
We present a new approach to construction of recursion operators for multidimensional integrable systems which have a Lax-type representation in terms of a pair of commuting vector fields. It is illustrated by the examples of the Manakov–Santini system which is a hyperbolic system in \( N \) dependent and \((N + 4)\) independent variables, where \( N \) is an arbitrary natural number, the six-dimensional generalization of the first heavenly equation, the modified heavenly equation and the dispersionless Hirota equation.

Introduction

Existence of an infinite hierarchy of local or nonlocal symmetries is a strong sign of integrability of a given partial differential equation (PDE) system and can be even employed as an integrability test, see e.g. [20] and references therein. It is well known that for integrable PDE systems in more than two independent variables, the higher commuting flows and the associated higher symmetries are nonlocal with a fairly nontrivial structure of nonlocalities. This renders the direct search for such symmetries virtually impossible, thus making a recursion operator (or eventually a master symmetry, see e.g. [4] and references therein for more details) the primary tool for systematic generation of higher symmetries for the system under study.

Moreover, if one has an auxiliary linear system and a class of its coefficients for which the inverse scattering problem can be solved, then using a recursion operator it is usually possible to write down the most general nonlinear system whose solutions can be obtained using the inverse scattering problem in question, see e.g. [2, 30, 14, 25], and this is how the recursion operators entered the soliton theory for the first time.

While the recursion operator \textit{per se} generates an infinite hierarchy of symmetries, its formal adjoint does the same for cosymmetries; under certain technical assumptions the latter give rise to conservation laws for the system under study, see e.g. [12] and references therein. The invariant manifolds associated with both symmetries and cosymmetries, as well as higher Bäcklund transformations, see [21, 22], constructed using the (generalized) recursion operator,
can be employed for generation of new solutions for the system under study. These solutions include the famous soliton, multi-soliton and finite-gap solutions and often turn out to be of considerable physical significance.

It is well known that the existence of an infinite hierarchy of conservation laws implies that the dynamics of the system in question is highly regular as it has to preserve all this infinite hierarchy of conserved quantities. While not all higher symmetries and conservation laws admit a physical interpretation, taking them into account is of crucial importance e.g. in discretizing the system under study in order to perform numerical simulations thereof. In fact, the best ways to discretize an integrable system are those preserving integrability, i.e. one should discretize not a system alone but rather the whole associated hierarchy constructed using the recursion operator, cf e.g. [43] and references therein.

Finally, composing the powers of the recursion operator with a Hamiltonian (resp. symplectic) structure, if the latter exists, yields an infinite hierarchy of Hamiltonian (resp. symplectic) structures for the system under study, thus establishing multi-Hamiltonian nature thereof, which plays an important role in the geometry of integrable systems, see e.g. [4, 12, 33].

Recursion operators can be sought for in a variety of ways, depending on the definition used. For instance, a typical recursion operator in (1+1) dimensions is a pseudo-differential operator which maps symmetries into symmetries [32] and can be derived from the Lax pair, see e.g. [32, 21, 19, 44] and references therein. However, these definitions and methods do not immediately extend to higher dimensions [47]. Instead, a higher-dimensional equation may admit a bilocal recursion operator as introduced by Fokas and Santini, see e.g. [13–15], and [22]; a prototypical example here is the Kadomtsev–Petviashvili equation.

From a different perspective, recursion operators are Bäcklund auto-transformations of a linearized equation. They first appeared in this form in several works by Papachristou [34–36]. The idea is that symmetries are essentially the solutions of the linearized equation and Bäcklund transformations are the most general transformations that relate solutions to solutions.

However, as pointed out by one of us [26], the same point of view applies to Guthrie’s [18] generalized recursion operators in (1+1) dimensions. Guthrie introduced them to avoid difficulties connected with the lack of rigorous interpretation of the action of pseudo-differential operators on symmetries.

The recursion operators which are Bäcklund auto-transformations of linearized systems appear to exist only for a certain class of integrable systems, which has yet to be characterized in full generality. Such recursion operators are closely related to zero-curvature representations whenever the latter exist, see e.g. [28] and references therein.

By a zero-curvature representation for a system of PDEs $F = 0$, we mean a one-form $\alpha = \sum A_i d\bar{x}^i$ which satisfies $D_{\bar{x}^i} A_i - D_{\bar{x}^j} A_i + [A_i, A_j] = 0$ on the solution manifold of $F = 0$; here, $\bar{x}^i$ are the independent variables and $A_i$ belong to some matrix Lie algebra. Then the operators $D_{\bar{x}^i} - ad A_i$ commute and we can define a matrix pseudo-potential $\Psi$ by setting

$$\Psi' = [A_i, \Psi] = \partial U A_i,$$

(1)

where $\partial U$ denotes linearization along a symmetry $U$ (see section 2 for details).

For the majority of integrable systems in (1+1) dimensions, the pseudo-potentials $\Psi$ provide nonlocal terms of inverse recursion operators, whereas local terms thereof are, as a rule, limited to zero-order ones at most. This approach applies to multidimensional systems whenever a zero-curvature representation is available. For instance, the recursion operator found by Papachristou in [34] is easily seen to be of this kind.
Now turn to dispersionless multidimensional systems which can be written as a commutativity condition for a pair of first-order linear scalar differential operators with no free terms (i.e. vector fields) and no derivatives with respect to the spectral parameter; see e.g. [48, 24, 5, 25, 9] and references therein for more information on such systems. The systems of this kind are a subject of intense research as they arise in a multitude of important areas of modern physics and mathematics, from self-dual gravity, see e.g. [16, 31, 41, 9], hyper-Kähler [17], symplectic [8] and conformal [7] geometries to fluid dynamics and related fields, cf e.g. [39, 10, 11]. Even though the recursion operators for some of these systems were found, see e.g. [34, 1, 42, 31, 25, 40, 41], they were obtained using either various \textit{ad hoc} methods or the partner-symmetry method, both of which can be applied only under fairly restrictive assumptions.

Below we present a method for finding recursion operators which is based on the generalization of pseudo-potentials (1) using the adjoint representation of the Lie algebra of vector fields. We are convinced that our approach applies to a considerably broader class of dispersionless systems than the methods mentioned in the previous paragraph. Moreover, our method is also more algorithmic: given a Lax-type representation for the system under study, finding a recursion operator of the type described in our paper, if it exists, is an essentially algorithmic task, while e.g. the partner-symmetry approach involves a non-algorithmic subproblem of representing the linearized equation as a two-dimensional divergence.

The paper is organized as follows. In section 1, we present a general construction of recursion operators which are Bäcklund auto-transformations of linearized systems, and sections 2 and 3 illustrate its application on the examples of the Manakov–Santini system and the dispersionless Hirota equation. In section 4, we give a modification of the construction from section 1 for the case of Hamiltonian vector fields and provide some further examples, and in section 5 we briefly discuss our results.

1. The general approach

Let $F = 0$ be a system of PDEs in $n$ independent variables $x_i, \ i = 1, \ldots, n$, for the unknown $N$-component vector function $u = (u^1, \ldots, u^N)^T$; here and below the superscript '$T$' denotes matrix transposition. Denote $u^\alpha_{n_1 \ldots n_l} = \partial^{n_1 + \cdots + n_l} u^\alpha / \partial (x^1)^{n_1} \cdots \partial (x^m)^{n_l}$; in particular, $u^{\alpha}_{0 \ldots 0} \equiv u^\alpha$. As usual in the formal theory of PDEs [33, 6, 4], $x'$ and $u^\alpha_{n_1 \ldots n_l}$ are considered as independent quantities and can be viewed as coordinates on an abstract infinite-dimensional space (a jet space). By a \textit{local function} or a function on the jet space, we shall mean any function of a finite number of $x', u^\alpha$ and derivatives of the latter. We denote

$$D_{\nu} = \frac{\partial}{\partial x^\nu} + \sum_{\alpha=1}^{N} \sum_{n_1=0}^{\infty} \ldots \sum_{n_l=0}^{\infty} u^\alpha_{n_1 \ldots n_l} \frac{\partial}{\partial u^\alpha_{n_1 \ldots n_l}},$$

the usual total derivatives, which can be naturally viewed as vector fields on the jet space. The condition $F = 0$ along with its differential consequences $D_{\nu}^i \cdots D_{\nu}^p F = 0$ determines what is called a \textit{solution manifold}, which in general is an infinite-dimensional submanifold of the jet space. In what follows, we tacitly restrict all our considerations to the solution manifold or its suitable extension, see below; note that the restrictions of total derivatives on the latter are tangent to it. In particular, all equalities below are assumed to hold there rather than on the whole jet space.
As usual, the directional derivative along an \( N \)-component vector \( \mathbf{U} = (U^1, \ldots, U^N) \) is the vector field of the form
\[
\partial U = \sum_{a=1}^{N} \sum_{\ell_1, \ldots, \ell_a=0}^{\infty} (D^{\ell_1}_{x_1} \cdots D^{\ell_a}_{x_\ell_a} U^a) \frac{\partial}{\partial U^{a}_{\ell_1, \ldots, \ell_a}}.
\]
The total derivatives as well as the directional derivative can be applied to (possibly vector or matrix) local functions \( P \). Recall [33, 6] that \( \mathbf{U} \) is (characteristic of a) symmetry for the system \( F = 0 \) if \( \mathbf{U} \) satisfies \( \partial \mathbf{U} \cdot \mathbf{F} = 0 \).

Assume now that the system \( \mathbf{F} = 0 \) can be written as a commutativity condition \( [\mathbf{X}_1, \mathbf{X}_2] = 0 \), where \( \mathbf{X}_i = \sum_{j=1}^{n} \xi^j_i D_i \) are vector fields, \([ \cdot, \cdot]\) is the usual Lie bracket thereof and \( \xi^j_i \) are local functions that may further depend on a spectral parameter \( \lambda \).

Further consider a vector field \( \mathbf{Z} \) of the same form, i.e. \( \mathbf{Z} = \sum_{j=1}^{n} \xi^j D_i \), except that we do not insist that \( \xi^j_i \) are local functions. However, we assume that the total derivatives can be extended to \( \xi^j_i \), see below.

The main idea of this paper is that we look for an \( N \times n \) matrix \( A = (a^{ij}) \) such that
\[
\hat{\mathbf{U}}^a = \sum_{j=1}^{n} a^{ij}_a \xi^j
\]
are components of a symmetry \( \hat{\mathbf{U}} \) whenever \( \mathbf{U} \) is a symmetry and \( \mathbf{Z} \) satisfies
\[
[X_i, Z] = \partial \mathbf{U} X_i, \quad i = 1, 2. \tag{2}
\]
We shall write \( \hat{\mathbf{U}} = \mathfrak{R}_A(\mathbf{U}) \) when such a matrix \( A = (a^{ij}_a) \neq 0 \) exists; this is precisely a recursion operator of the type described in the introduction, i.e. a Backlund auto-transformation for the linearized system \( \partial \mathbf{U} \cdot \mathbf{F} = 0 \). Here and below we assume for simplicity that the entries \( a^{ij}_a \) of \( A \) are local functions; however, in principle, nothing prevents them from being nonlocal.

Note that we do not insist that the vector fields \( \mathbf{X}_i \) necessarily involve any spectral parameter, but we do exclude the case when they involve derivatives with respect to the spectral parameter.

The condition (2) is a system of first-order PDEs in the unknowns \( \xi^j_i \). To show that the system is compatible we check the Jacobi identity
\[
[X_1, [X_2, Z]] + [X_2, [Z, X_1]] + [Z, [X_1, X_2]] = [X_1, \partial \mathbf{U} X_2] - [X_2, \partial \mathbf{U} X_1] = \partial \mathbf{U} [X_1, X_2] = 0
\]
since \( [X_1, X_2] = 0 \) is equivalent to \( \mathbf{F} = 0 \) and \( \mathbf{U} \) is a symmetry.

As a rule, the system (2) is not solvable in terms of local functions. Therefore, strictly speaking, \( \hat{\mathbf{U}} \) are not necessarily local symmetries of the system \( \mathbf{F} = 0 \). Instead, they are nonlocal symmetries (or shadows in the sense of [23, 6]). This naturally leads to introduction of pseudo-potentials (for instance, \( \xi^j_i \) and their derivatives) and subsequent extension of the total derivatives and of the solution manifold to include the terms coming from pseudo-potentials. To simplify notation we shall, however, denote the extended total derivatives by the same symbol \( D_i \). Note that when applied to local functions the original and extended total derivatives coincide.

Note that in a number of examples, where the recursion operators are already known, e.g., the Pavlov equation [38, 25], our method produces the recursion operators which are inverse to the known ones. Moreover, the inverses of our recursion operators often have simpler structure of nonlocal terms; in particular, this holds for all systems discussed below. Thus, it is often appropriate to invert the operator \( \mathfrak{R}_A \) resulting from the above construction in order to obtain a simpler recursion operator; the inversion is an algorithmic process described in [18].
Let us also mention that, in sharp contrast with the case of \((1+1)\)-dimensional systems where one usually can make a clear distinction among positive (local) and negative (nonlocal) hierarchies (see, however, \([3]\)), the multidimensional hierarchies we have been able to generate contain, an eventual inversion of the recursion operator notwithstanding, only a few local symmetries. The same phenomenon occurs for the multidimensional hierarchies generated using bilocal recursion operators, see e.g. \([29, 45]\) and references therein.

2. The Manakov–Santini system

Consider the Manakov–Santini system \([24]\) in \((N + 4)\) independent variables \(x^1, \ldots, x^N, y^1, y^2, z^1, z^2\) and \(N\) dependent variables \(u^i\),

\[
\begin{align*}
u^i_{x^j, x^k} - u^j_{x^i, x^k} + \sum_{j=1}^{N} (u^i_{x^j} u^j_{x^k, x^i} - u^j_{x^k} u^i_{x^j, x^i}) &= 0, \quad i = 1, \ldots, N. \\
\end{align*}
\]

As usual, the subscripts refer to partial derivatives.

System (3) can be written \([24]\) as a commutativity condition of the vector fields

\[
X_i = D_{y^i} + \lambda D_{z^i} + \sum_{k=1}^{N} u^k_{x^i} D_{x^k}, \quad i = 1, 2.
\]

Assume that \(Z\) has the form

\[
Z = \sum_{j=1}^{N} V^j D_{x^j}
\]

(no terms involving \(D_{y^i}\) and \(D_{z^i}\) are actually needed).

It is straightforward to verify that the following assertion holds for any natural \(N\): if \(U = (U^1, \ldots, U^N)^T\) is a characteristic of symmetry for (3), then so is \(V = (V^1, \ldots, V^N)^T\), where \(V^j\) are determined from equation (2) with \(X_i\) and \(Z\) given by (4) and (5), that is,

\[
V^j + \lambda V^j_{x^i} + \sum_{j=1}^{N} u^j_{x^i} V^j = \sum_{j=1}^{N} u^j_{x^i} V^j - U^j_{x^i}, \quad i = 1, \ldots, N, \quad s = 1, 2.
\]

To emphasize the dependence on \(\lambda\), the recursion operator given by formula (6) will be denoted \(R_\lambda\). Applying \(R_\lambda\) to local symmetries yields a highly nonlocal ‘negative’ hierarchy of the Manakov–Santini system. In order to obtain the ‘positive’ hierarchy with simpler nonlocalities, we look for the inverse \(R_\lambda^{-1}\).

Inverting the recursion operator \(R_\lambda\) amounts to solving (6) for \(U^j_{x^i}\). The inverse operator \(R_\lambda^{-1}\) sends \(V = (V^1, \ldots, V^N)^T\) to \(U = (U^1, \ldots, U^N)^T\), where \(U^i\) are determined from the relations

\[
U^i_{x^j} = -V^i_{x^j} - \lambda V^j_{x^i} + \sum_{j=1}^{N} u^j_{x^i} V^j_{x^j} + \sum_{j=1}^{N} u^j_{x^j} V^j, \quad i = 1, \ldots, N, \quad s = 1, 2.
\]

Upon multiplying by \(-1\) and removing the trivial contribution \(\lambda V\) from \(U\), we end up with the recursion operator \(\Delta = -R_\lambda^{-1} - \lambda\) Id, which no longer depends on \(\lambda\). The components of \(U = R(V)\) are defined by the relations

\[
U^i_{x^j} = V^i_{x^j} + \sum_{j=1}^{N} u^j_{x^i} V^j_{x^j} - \sum_{j=1}^{N} u^j_{x^j} V^j, \quad i = 1, \ldots, N, \quad s = 1, 2.
\]

The symmetries generated using this recursion operator are complicated nonlocal expressions; the results of its application to the Lie point symmetries of (3) are given in the appendix.
3. Dispersionless Hirota equation

Consider the equation [46, 7]

\[ a u_t u_y + b u_x u_{xy} + c u_x u_y = 0, \quad a + b + c = 0. \quad (8) \]

It has a Lax pair [46] of the form

\[ \psi_y = \frac{\lambda u_y \psi_x}{u_x}, \quad \psi_t = \frac{\mu u_t \psi_x}{u_x}, \]

where \( \mu = (a + b) \lambda / (a \lambda + b) \).

The vector field \( Z \) now can be chosen in the form \( Z = \zeta D_x \). An easy computation shows that the associated recursion operator is given by the formula

\[ R_{\lambda}(U) = u_x \zeta. \]

Here, \( U \) is a symmetry for (8) and \( \zeta \) is defined by the following equations:

\[ \frac{\zeta_t}{\mu} = \frac{u_t}{u_x} \zeta_x - \left( \frac{u_t}{u_x} \right)_x, \quad \frac{\zeta_y}{\lambda} = \frac{u_y}{u_x} \zeta_x - \left( \frac{u_y}{u_x} \right)_x. \]

The inverse recursion operator \( W = R_{\lambda}^{-1}(U) \) is given by the formulas

\[ W_y = \frac{u_y W_x}{u_x} + \frac{u_x U_y - u_y U_x}{u_x}, \quad W_t = \frac{u_t W_x}{u_x} + \frac{u_x U_t - u_t U_x}{u_x}. \]

If we replace \( W \) by \( W - U \), we obtain a somewhat simpler recursion operator \( \hat{R}(U) \), where \( \hat{R} = R_{\lambda}^{-1} + \text{Id} \), so \( W \) can be determined from the compatible equations

\[ W_y = \frac{u_y W_x}{u_x} + \frac{\lambda - 1}{\lambda} \left( U_y - \frac{u_y U_x}{u_x} \right), \quad W_t = \frac{u_t W_x}{u_x} + \frac{\mu - 1}{\mu} \left( U_t - \frac{u_t U_x}{u_x} \right). \]

Let us apply \( \hat{R} \) to the Lie point symmetries, which are \( X(x) u_x, Y(y) u_y, T(t) u_t \) and \( F(u) \).

To start with, we find \( \hat{R}(0) = F(u) \), where \( F \) is an arbitrary smooth function. Upon having agreed to remove this trivial contribution from the results, we readily find

\[ \hat{R}(X u_x) = 0, \quad \hat{R}(Y u_y) = \frac{\lambda - 1}{\lambda} Y u_y, \quad \hat{R}(T u_t) = \frac{\mu - 1}{\mu} T u_t. \]

Thus, \( X u_x, Y u_y \) and \( T u_t \) are eigenvectors of \( \hat{R} \). To the best of our knowledge, this is a first known example of local eigenvectors for a nontrivial recursion operator.

Finally,

\[ \hat{R}(F(u)) = F p + \frac{\partial F}{\partial u} q, \]

where \( p \) and \( q \) are nonlocal variables defined by the following equations which are compatible by virtue of (8):

\[ p_y = \frac{u_y}{u_x} p_x - \frac{\lambda - 1}{\lambda} u_{xy}, \quad p_t = \frac{u_t}{u_x} p_x - \frac{\mu - 1}{\mu} u_{xt}, \]

\[ q_y = \frac{u_y}{u_x} q_x + \frac{\lambda - 1}{\lambda} u_y, \quad q_t = \frac{u_t}{u_x} q_x - \frac{\mu - 1}{\mu} u_t. \]
4. The case of Hamiltonian vector fields

If the Lax pair for the system under study consists of Hamiltonian vector fields, it is natural to apply the ideas from section 1 to the Lie algebra of functions with respect to Poisson bracket rather than to the Lie algebra of vector fields.

Namely, suppose that the Lax pair for the system under study can (up to the obvious renumbering of independent variables) be written as

\[ \psi_{x_{n-1}} = a\psi_{x_{n-2}} + [H_1, \psi], \quad \psi_{x_{n-3}} = b\psi_{x_{n-4}} + [H_2, \psi]. \] (9)

Here, {·, ·} denotes the Poisson bracket in question (usually w.r.t. the independent variables \( x^1, \ldots, x^{n-4} \) only), and \( a \) and \( b \) are some constants, which are typically proportional to the spectral parameter \( \lambda \).

Instead of \( \zeta_j \) we introduce a single nonlocal variable \( \Omega_1 \) defined by the formulas

\[ \Omega_{x_{n-1}} = a\Omega_{x_{n-2}} + [H_1, \Omega] + \partial_t H_1, \quad \Omega_{x_{n-3}} = b\Omega_{x_{n-4}} + [H_2, \Omega] + \partial_t H_2, \] (10)

where \( a \) and \( b \) are some constants which are often proportional to the spectral parameter \( \lambda \).

Then we shall seek for a recursion operator in the form

\[ \tilde{U} = R(U) = A_0\Omega_1 + \sum_{i=1}^{n-2} A_i\Omega_{x^i}, \] (11)

where now \( A_j = (a_1^j, \ldots, a_N^j) \) and \( j = 0, \ldots, n-2 \) are \( N \)-component vectors whose entries are local functions. If necessary, the terms containing higher order derivatives can also be included; moreover, \( A_j \) can be allowed to depend on nonlocal variables too.

As an example, consider the following six-dimensional generalization of the first heavenly equation, see e.g. \[8, 37\],

\[ u_{xp} = -u_{yq} + u_{xt}u_{yz} - u_{xq}u_{yt}. \] (12)

It admits \[8\] a Lax representation of the form (9), namely

\[ \psi_p = \lambda\psi_y + [u_y, \psi], \quad \psi_q = -\lambda\psi_x + [-u_x, \psi], \]

with the Poisson bracket given by

\[ \{f, g\} = D_z(f)D_t(g) - D_t(f)D_z(g). \]

It is readily verified that (12) possesses a recursion operator of the form \( \tilde{U} \equiv R(U) = \Omega \), where the nonlocal variable \( \Omega \) is defined via (10), that is,

\[ \Omega_p = \lambda\Omega_y + [u_y, \Omega] + U_y \equiv \lambda\Omega_y + u_{yz}\Omega_t - u_{yt}\Omega_z + U_y, \]
\[ \Omega_q = -\lambda\Omega_x + [-u_x, \Omega] - U_x \equiv -\lambda\Omega_x - u_{xz}\Omega_t + u_{zt}\Omega_z - U_x. \]

Upon inversion, we obtain a simpler recursion operator \( \tilde{U} = (R^{-1} + \lambda \text{Id})(U) \), where \( \tilde{U} \) is defined by the formulas

\[ \tilde{U}_x = -u_{xt}U_t + u_{xt}U_z - U_q, \quad \tilde{U}_y = -u_{yt}U_t + u_{yt}U_z + U_p. \] (13)

For another example, consider the modified heavenly equation \[8\]

\[ u_{xz} = u_{xy}u_t - u_{xt}u_y, \] (14)

which has \[8\] a Lax representation of the form (9),

\[ \psi_x = [u_x/\lambda, \psi], \quad \psi_z = \lambda\psi_t + [-u_x, \psi], \]

with the Poisson bracket given by

\[ \{f, g\} = D_z(f)D_t(g) - D_t(f)D_z(g). \]
It is readily seen that (14) admits a recursion operator $\tilde{\mathcal{R}} \equiv \mathcal{R}(U) = \Omega$, where the nonlocal variable $\Omega$ is now defined by the formulas

$$\Omega_i = \{u_i, \Omega\} + U_i/\lambda \equiv u_{\Omega} \Omega_i/\lambda - u_i \Omega_i/\lambda + U_i/\lambda,$$

$$\Omega \equiv \lambda \Omega_i + \{-u_i, \Omega\} - U_i \equiv \lambda \Omega_i + u_{\Omega} \Omega_i - u_i \Omega_i - U_i.$$ 

Inversion again leads to a simpler recursion operator $\widetilde{\mathcal{R}} = (\mathcal{R}^{-1} - \lambda \mathbb{I})(U)$, with $\widetilde{U}$ defined by the formulas

$$\widetilde{U}_i = u_{\Omega} U_i - u_i U_i, \quad \widetilde{U}_t = u_{\Omega} U_t - u_t U_i - U_z.$$ (15)

To the best of our knowledge, the recursion operator (13) has not yet appeared in the literature, while (15) is a special case of the recursion operator for the so-called asymmetric heavenly equation found in [40] using the partner-symmetry approach. Note that (13) also could have been obtained within the partner-symmetry approach [40]. On the other hand, the recursion operators for the second heavenly and Husain equations, which were found in [31, 41], can be easily recovered using the approach of this section.

5. Conclusions

In the preceding sections, we suggested a new construction of recursion operators for multidimensional dispersionless integrable systems which can be written as commutativity conditions for pairs of vector fields not involving derivatives with respect to the spectral parameter.

The construction in question can be summarized as follows. Given any symmetry of the system under study, we introduce a new nonlocal vector field $Z$ using the adjoint action of the Lie algebra of vector fields of the above type, or an appropriate subalgebra thereof, on itself, see (2). The new symmetry resulting from the action of the sought for recursion operator, say $\mathfrak{R}$, on the original one is then assumed to be a linear combination of the components of $Z$; see section 1 for details. A recursion operator resulting from our construction should be viewed as a Bäcklund auto-transformation for the linearized version of the system under study, cf [26].

For the special case of Hamiltonian vector fields, the above procedure should be modified: roughly speaking, one should pass from the Lie algebra of vector fields to that of functions with an appropriate Poisson bracket, as explained in section 4. We stress once again that it is often possible to obtain a simpler recursion operator by inverting $\mathfrak{R}$ or a suitable linear combination thereof with the identity operator.

We succeeded in applying the above approach to a number of integrable dispersionless systems, including those with applications in physics (the six-dimensional generalization of the first heavenly equation and the modified heavenly equation) and geometry (the dispersionless Hirota equation). Last but not least, the recursion operator for the Manakov–Santini system, see (7), provides the first example of a recursion operator for a non-overdetermined multidimensional PDE system (as for overdetermined systems, see [27]) in an arbitrary number of independent variables.

Acknowledgments

AS gratefully acknowledges the discussion of the results of this paper with JDE Grant and with BG Konopelchenko. AS also thanks JDE Grant for bringing the reference [17] to his attention and BG Konopelchenko for pointing out the reference [5] and the fact that auxiliary linear problems involving commuting vector fields were considered for the first time by Zakharov.
Inverse Problems 28 (2012) 025011 M Marvan and A Sergyeyev

and Shabat in [48]. We also thank the referees for their useful suggestions. This research was supported in part by the Ministry of Education, Youth and Sports of Czech Republic (MŠMT ČR) under grant MSM4781305904, and by the Czech Grant Agency (GA ČR) under grant P201/11/0356.

Appendix. Symmetries of the Manakov–Santini system

In this section, we use the standard convention on summation over repeated indices. The indices \( r \) and \( s \) run from 1 to 2, the others run from 1 to \( n \).

The symmetries of the Manakov–Santini system can be routinely computed as solutions \( U \) of the linearized equation \( \partial_t F = 0 \) which must hold only on the solution manifold of (3); here, \( F^i \) stands for the left-hand side of the \( i \)th equation of (3).

The simplest of symmetries, the Lie point ones, are characterized by the property that \( U^i \) are linear in the first derivatives. A computer-aided computation reveals 14 Lie point symmetries for (3), namely

\[ \eta_1, \eta_2, \zeta_1, \zeta_2, \alpha, \beta_1^1, \beta_2^1, \beta_2^2, \gamma, \chi_1, \chi_2, j^j, s^j, \]

where \( j^j \) and \( s^j \) each depend on \( N \) arbitrary functions of the coordinates \( x^i \) and \( y^j \), \( i, j = 1, \ldots, N \); the left-hand side subscripts indicate the arbitrary functions these symmetries depend on.

The generators of these Lie point symmetries read

\[ \eta^k_r = u^k_r, \quad \zeta^k_r = u^k_z, \quad \alpha^k = y^u^k, \]

\[ (\beta^k_r)^1 = y^u^k_r + z^u^k_z, \quad \beta^k_r^2 = u^k + y^u^k, \]

\[ j^j = f^j(x, y), \quad s^j = -g^j(x, y)u^k + \frac{\partial g^j(x, y)}{\partial x^i}u^i + \frac{\partial g^j(x, y)}{\partial y^i}z^i. \]

This notation is to be read as follows: for instance, the symmetry \( s^j \) has a characteristic \( (s^j^1, \ldots, s^j^N) \) and the associated evolutionary vector field is \( \sum_{j=1}^N s^j \partial / \partial u^i \).

Let us investigate the action of the recursion operator defined by (7) on the above symmetries. Obviously, equations (7) determine \( U^k \) uniquely up to adding the Lie symmetry we denoted \( j^j \) (we could also write \( j^j = R(0) \)). Like the integration constants, this term will be omitted in what follows.

Four classical symmetries, namely \( \xi, \zeta_1, \zeta_2 \) and \( \alpha \), are mapped to local symmetries again:

\[ R(\xi) = \phi, \quad R(\zeta_1) = \eta, \quad R(\zeta_2) = \gamma. \]

The others are mapped to nonlocal symmetries, sharing the same set of nonlocal variables \( u^{(2)k}, k = 1, \ldots, N \), subject to the equations

\[ u^{(2)k} = u^k + \sum_{i=1}^N u^i u^k. \]  

The system (A.1) is compatible by virtue of the Manakov–Santini system (3). Equation (A.1) determines a covering in the sense of [6], but this covering is infinite-dimensional. Each successive application of \( R \) requires one more level of nonlocal variables \( u^{(q)k}, \) subject to the compatible equations

\[ u^{(q+1)k} = u^{(q)k} + \sum_{i=1}^N u^i u^{(q)k}. \]  

Thus we have obtained an infinite hierarchy of successive coverings.
Upon denoting $\eta_{r}^{(q)} \equiv \mathcal{R}^{q}(\eta_{r}) \equiv \mathcal{R}^{q+1}(\zeta_{r}) \equiv \zeta_{r}^{(q+1)}$, we routinely generate
\begin{align*}
\eta_{r}^{(1)k} &= \zeta_{r}^{(2)k} = u_{r}^{(2)k} - u^{(2)k}u_{r}
\eta_{r}^{(2)k} &= \zeta_{r}^{(3)k} = u_{r}^{(3)k} - u^{(3)k}u_{r} + u^{(2)k}u_{r}^{(2)i} + u^{(2)k}u_{r}^{(2)i} - u^{(2)k}u_{r}^{(2)i}
\eta_{r}^{(3)k} &= \zeta_{r}^{(4)k} = u_{r}^{(4)k} - u^{(3)k}u_{r} - u^{(2)k}u_{r}^{(2)i} - u^{(2)k}u_{r}^{(2)i} + u^{(2)k}u_{r}^{(2)i} - u^{(2)k}u_{r}^{(2)i} - u^{(2)k}u_{r}^{(2)i} + u^{(2)k}u_{r}^{(2)i} - u^{(2)k}u_{r}^{(2)i}.
\end{align*}

etc. If we assign level $1$ to the local variables, i.e. $u^{k} = u^{(1)k}$, we observe that the sum in the above formulas runs over all homogeneous monomials of the same level. We conjecture that this pattern holds for $\eta_{r}^{(q)}$ with any natural $q$.

We also have the following general formula for $\beta^{(q)} = \mathcal{R}^{q}(\beta)$:
\[
(\beta^{(q)})^{k} = y^{i} \eta_{r}^{(q)k} + z^{i} \zeta_{r}^{(q)k}.
\]

Likewise, for $\gamma^{(q)} = \mathcal{R}^{q}(\gamma) = \mathcal{R}^{q+1}(\alpha) \equiv \alpha^{(q+1)}$, we obtain the expressions
\begin{align*}
\gamma^{(1)k} &= \alpha^{(2)k} = 2u^{(2)k} - u^{(2)k} + y^{i} \eta_{r}^{(1)k},
\gamma^{(2)k} &= \alpha^{(3)k} = 3u^{(3)k} - u^{(2)k}u^{(2)i} + u^{(2)k}u^{(2)i} + y^{i} \eta_{r}^{(2)k},
\gamma^{(3)k} &= \alpha^{(4)k} = 4u^{(4)k} - u^{(3)k}u^{(2)i} - u^{(3)k}u^{(2)i} + y^{i} \eta_{r}^{(3)k},
\gamma^{(4)k} &= \alpha^{(5)k} = 5u^{(5)k} - u^{(4)k}u^{(2)i} + u^{(4)k}u^{(2)i} - u^{(4)k}u^{(2)i} + u^{(4)k}u^{(2)i} + y^{i} \eta_{r}^{(4)k},
\end{align*}

Here, another pattern can be observed. Namely, the sum again runs over all homogeneous monomials with the coefficient at each monomial equal, up to the sign, to the level of the nonlocal variable which is not differentiated. It would be interesting to find out whether this pattern holds for all natural $q$.

In terms of $\gamma^{(q)}$, we have the following general formula for $\chi^{(q)} = \mathcal{R}^{q}(\chi)$:
\[
(\chi^{(q)})^{k} = y^{i} \gamma^{(q)k} + z^{i} \alpha^{(q)k}.
\]

Finally, we have $\phi^{(q)} = \mathcal{R}^{q}(\phi) = \mathcal{R}^{q+1}(\phi) \equiv \phi^{(q+1)}$,
\begin{align*}
\phi^{(1)k} &= \phi^{(2)k} = -u^{(2)k} + \frac{\partial g^{k} \partial x^{i} u^{(2)i} + \partial g^{k}}{\partial y^{i} z^{i}},
\phi^{(1)k} &= \phi^{(2)k} = -u^{(2)k} + \frac{\partial g^{k} \partial x^{i} u^{(2)i} + \partial g^{k}}{\partial y^{i} z^{i}}.
\end{align*}

\begin{align*}
\phi^{(1)k} &= \phi^{(2)k} = -u^{(2)k} + \frac{\partial g^{k} \partial x^{i} u^{(2)i} + \partial g^{k}}{\partial y^{i} z^{i}} (z^{i} - u^{(2)k} u^{(2)i} + u^{(2)k} u^{(2)i} + \frac{\partial g^{k}}{\partial x^{i} u^{(2)i} + \partial g^{k}} + \frac{\partial g^{k}}{\partial x^{i} u^{(2)i} + \partial g^{k}} + \frac{\partial g^{k}}{\partial x^{i} u^{(2)i} + \partial g^{k}} + \frac{\partial g^{k}}{\partial x^{i} u^{(2)i} + \partial g^{k}}).
\end{align*}

References

[1] Ablowitz M J, Chakravarty S and Takhtajan L A 1993 A self-dual Yang–Mills hierarchy and its reductions to integrable systems in 1 + 1 and 2 + 1 dimensions Commun. Math. Phys. 158 289–314
[2] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 The inverse scattering transform—Fourier analysis for nonlinear problems Stud. Appl. Math. 53 249–315
[3] Baran H 2005 Can we always distinguish between positive and negative hierarchies? J. Phys. A: Math. Gen. 38 L301–L6
[37] Plebański J F and Przanowski M 1996 The Lagrangian for a self-dual gravitational field as a limit of the SDYM Lagrangian Phys. Lett. A 212 22–8 (arXiv:hep-th/9605233)
[38] Pavlov M V 2003 Integrable hydrodynamic chains J. Math. Phys. 44 4134–56 (arXiv:nlin/0301010)
[39] Rogers C and Schief W K 2002 Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory (Cambridge: Cambridge University Press)
[40] Sheftel M B and Malykh A A 2009 On classification of second-order PDEs possessing partner symmetries J. Phys. A: Math. Theor. 42 395202 (arXiv:0904.2909)
[41] Sheftel M B and Yazıcı D 2010 Bi-Hamiltonian representation, symmetries and integrals of mixed heavenly and Husain systems J. Nonlinear Math. Phys. 17 453–84 (arXiv:0904.3981)
[42] Strachan I A B 1995 The symmetry structure of the anti-self-dual Einstein hierarchy J. Math. Phys. 36 3566–73 (arXiv:hep-th/9410047)
[43] Suris Y B 2003 The Problem of Integrable Discretization: Hamiltonian Approach (Basel: Birkhäuser)
[44] Wang J P 2002 A list of 1 + 1 dimensional integrable equations and their properties J. Nonlinear Math. Phys. 9 213–33 (Suppl. 1)
[45] Wang J P 2006 On the structure of (2 + 1)-dimensional commutative and noncommutative integrable equations J. Math. Phys. 47 113508 (arXiv:nlin/0606036)
[46] Zakharevich I 2000 Nonlinear wave equation, nonlinear Riemann problem, and the twistor transform of Veronese webs arXiv:math-ph/0006001
[47] Zakharov V E and Konopelchenko B G 1984 On the theory of recursion operator Commun. Math. Phys. 94 483–509
[48] Zakharov V E and Shabat A B 1979 Integration of nonlinear equations of mathematical physics by the method of inverse scattering II Funkt. Anal. Appl. 13 166–74