Radion-induced gravitational wave oscillations and their phenomenology

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We discuss the theory and phenomenology of the interplay between the massless graviton and its massive Kaluza-Klein modes in the Randall-Sundrum two-brane model. The equations of motion of the transverse traceless degrees of freedom are derived by means of a Green function approach as well as from an effective nonlocal action. The second procedure clarifies the extraction of the particle content from the nonlocal action and the issue of its diagonalization. The situation discussed is generic for the treatment of two-brane models if the on-brane fields are used as the dynamical degrees of freedom. The mixing of the effective graviton modes of the localized action can be interpreted as radion-induced gravitational-wave oscillations, a classical analogy to meson and neutrino oscillations. We show that these oscillations arising in M-theory-motivated braneworld setups could lead to effects detectable by gravitational-wave interferometers. The implications of this effect for models with ultra-light gravitons are discussed.

1 Introduction

In recent years the old Kaluza-Klein idea of a world with additional dimensions got a strong impact from attempts motivated by string theory to resolve the hierarchy problem [1]. Especially fruitful is the idea of implementing warped manifolds, for which the metric of the four-dimensional space-time can depend on the additional dimensions. The most popular models using warped geometries are the Randall-Sundrum (RS) models [2,3] in which our observable Universe is treated as a four-dimensional brane embedded into a five-dimensional anti-de Sitter (AdS) bulk space. The particular properties of the AdS geometry allow to describe the trapping of the gravitational zero-mode on the four-dimensional brane [3]. Displacing the second four-dimensional brane in the bulk, one can propose a resolution to the hierarchy problem [2].

However, warped two-brane models have some important additional features. The most interesting effect is the existence of light massive graviton modes, which have already attracted a lot of attention in the context of higher-spin theories and in a cosmological context [4,5]. A particularly intriguing development is the construction of semi-realistic braneworld models in which the Kaluza-Klein gravitons1 (KK

1 In the following we will denote also a classical linearized gravitational mode by the term graviton.
gravitons) can have an extremely low mass and can be responsible for the mediation of all [6,7] or a large part of the gravitational interaction [8,9]. In these setups, the Veltman-van-Dam-Zakharov discontinuity (VvDZ discontinuity) [10] (for a recent analysis of VvDZ discontinuity at the classical level see [11]) is either unobservable due to the setup [12] or cured by non-linear effects [13], though it is still under dispute if these models exhibit a realistic behavior in the limit of strong gravitational fields [14].

An effect so far overlooked is the mixing between graviton modes of different masses which inevitably occurs in higher-dimensional spacetime models. This effect is of little relevance in models with small extra dimensions and heavy KK modes, because in these models the massive gravitons will rarely be produced and therefore are of no astrophysical significance. In models where gravitons can have masses in the sub-eV scale like in the RS models, the mixing between the different modes can lead to interesting effects in gravitational-wave astrophysics. Gravitational waves (GW's) will then consist of contributions from several graviton modes. This will induce a beat in the gravitational wave which could be detectable with gravitational-wave interferometers. The parameters of the oscillations will crucially depend on the size of the extra dimension — commonly parameterized by the radion-field —, thus suggesting the notion of radion-induced graviton oscillations (RIGO’s). This effect was investigated in [16] and will be discussed in full detail in the present paper.

RIGO’s are particularly interesting because their observability is not limited to a certain minimal value of the extra-dimensional curvature scale. RIGO’s could therefore become the first detectable signature of M-theory phenomenology. From the very beginning, the affinity of the RS two-brane setup to the M-theory compactification of [17] (see [18] for reviews of effective five-dimensional M-theory) was noticed [2]. Actually, the RS two-brane model can be viewed as a simplified version of this M-theory solution which appears five-dimensional in an intermediate energy-range. The RS model simplifies the effective supergravity action of [17] to an Einstein-Hilbert action and neglects the gauge-field part of the M-theory effective action. Due to this relationship, the RS model has been used as an approximation for aspects of M-theory cosmology where a full-fledged M-theory calculation is not feasible (see [19] for an example).

On the other hand, the RS model deviates in major aspects from the requirements of M-theory: First, in contrast to the RS model, M-theory restricts the size of the largest extra dimension to typically $10^{-31}$ m [20,18] (see, however, [22]). This rules out the possibility to detect the extra dimension by measuring short-distance deviations from Newtonian gravity. Second, in M-theory the requirements of having the standard-model group structure and supersymmetry breaking forces one to identify our Universe with the positive-tension brane (cf. [21]). This prevents one from obtaining the simple geometrical solution to the hierarchy problem by locating our Universe on the negative tension brane of the RS model. Therefore it seems that one has to choose between either the attractive phenomenology of the RS model or the appealing theoretical framework of M-theory, although it remains tempting to conjecture that if M-theory realizes large extra-dimensions in nature they should exhibit a similar setup as the Randall-Sundrum two-brane model.

Considering RIGO’s one may, however, be able to probe several additional orders of smallness of the extra-dimensional curvature radius if only the radion acquires suitable values. Although the required values of the radion are still far from the parameter range of typical M-theory models, the search for RIGO’s thus offers a possibility to narrow down the gap between the phenomenologically accessible parameter space of large extra dimensions and the predictions of M-theory.

In our preceding paper [26] we have studied the four-dimensional effective action in the RS two-brane model. This action is a functional of the two induced metrics and radion fields on the two four-dimensional branes. Using the results of [16,27], we shall here extend the study of radion-induced graviton oscillations in the two-brane world of [16] and investigate their phenomenological consequences.

2 There may remain some loopholes though, see for example [23] for a construction which would allow us to live on the positive-tension brane and have the hierarchy inherited from the negative-tension brane or [24,25] for M-theory constructions which would allow the standard-model fields to reside on the negative-tension brane.
The structure of this paper is the following. In the next section we will introduce the RS setup and describe the corresponding nonlocal braneworld effective action and the equations of motion in terms of a nonlocal kernel and Green function, respectively. In Sec. 5 to 7 we will then set up the theoretical framework for studying RIGO’s by deriving the localized equations of motion from the nonlocal theory. In Sec. 3 we will present an explicit expansion of the Green function in terms of its residues which leads to the equations of motion in terms of orthogonal mass eigenmodes of gravitons. In Sec. 4 we will demonstrate how these modes can be extracted directly from the action. We find that there are two sets of eigenmodes of the action and explain how they are interrelated. To further elucidate this finding, we undertake in Sec. 5 a direct diagonalization of the action in a limit in which the two sets of modes coincide. This method was first employed in the preliminary study of RIGO’s in [27]. The rest of the paper will be devoted to phenomenology: In Sec. 6 we discuss the description of RIGO’s and their possible observational consequences in the RS model. After reminding the reader of the analogous effect of quantum oscillations in Sec. 6.2 we first discuss the phenomenology of RIGO’s on the positive-tension brane in Sec. 6.3. This will lead us, in Sec. 6.4 to consider in particular the gravitational waves produced by a network of cosmic strings on the negative-tension brane. In Sec. 6.5 follows the discussion of RIGO’s on the negative-tension brane. Finally, in Sec. 6 the discussion is extended to so-called bi-gravity braneworld models. In the concluding Sec. 8, we summarize our results and argue for the occurrence of RIGO’s in a general class of braneworlds featuring orbifolded compactifications. A detailed presentation of the diagonalization procedure for the kinetic and massive parts of the gravitational effective action used in Sec. 5 is given in the Appendix.

2 The RS two-brane model and the braneworld effective action

In the RS setup [2], the braneworld effective action is induced by the five-dimensional bulk spacetime as follows. We start with the theory having the action of the five-dimensional gravitational field with metric $G_{\alpha\beta}(x,y)$, $\alpha = (x^i; y)$, $i = 0; 1; 2; 3$, propagating in the bulk spacetime $x^\Lambda = (x^i,y)$; $x = x^i; x^5 = y$, and matter fields confined to the two branes, where the bulk-space part of the action is given by

$$S_{\text{5}}[G; g; ] = \frac{1}{16\pi G_5} \int d^5x G^{\Lambda\mu_\nu} \, ^5\mathcal{R}(G) \, 2 \, ;$$

and the brane-part of the action is

$$S_{\text{4}}[G; g; ] = \int d^4x \, \mathcal{L}_m (\, \mathit{\Theta} \, ; g) \, \, ^4\mathcal{R}(g) \, + \frac{1}{8\pi G_5} \, \mathcal{K} \, ;$$

The branes are marked by the index and carry induced metrics $g = g (x)$ and matter-field Lagrangians $\mathcal{L}_m (\, \mathit{\Theta} \, ; g)$. The bulk part of the action is characterized by the five-dimensional gravitational constant $G_5$ and the cosmological constant $\Lambda_5$, while the brane parts carry four-dimensional cosmological constants. The bulk cosmological constant $\Lambda_5$ is negative and, thus, is capable of generating an AdS geometry, while the brane cosmological constants play the role of brane tensions and, depending on the model, can be of either sign. The Einstein-Hilbert bulk action is accompanied by terms in containing the jumps of extrinsic curvatures traces $\mathcal{K}$ associated with both sides of each brane. Solving the five-dimensional Einstein equations with prescribed values of the four-dimensional metric on the branes, we have obtained, in the tree-level effective four-dimensional action.

In the RS two-brane setup, the fifth dimension has the topology of a circle labeled by the coordinate $y$, with an orbifold $\mathbb{Z}_2$-identification of points $y$ and $-y$. The branes are located at antipodal...
fixed points of the orbifold, \( y = y \); \( y_+ = 0 \); \( \dot{y} \); \( y = d \). They are empty, i.e. \( L_m (g^2 \rho g^2) = 0 \), and their tensions are opposite in sign and fine-tuned to the values of \( \xi \) and \( G_5 \),

\[
g = \frac{6}{P^2}; \quad \xi = \frac{3}{4 G_5};
\]

Then this model admits a solution to the Einstein equations with an AdS metric in the bulk (1 is its curvature radius),

\[
ds^2 = dy^2 + e^{2y^2} \, dx \, dx;
\]

\(0 = y_+ \dot{y} \dot{y} \, y = d\), and with a flat induced metric on both branes \([2]\). With the fine tuning \(4\), this solution exists for arbitrary brane separation \(d\).

Now consider small matter sources and metric perturbations \( A_B (x, y) \) on the background of this solution \([3, 29, 30]\),

\[
ds^2 = dy^2 + e^{2y^2} \, dx \, dx + A_B (x, y) \, dx^A \, dx^B;
\]

Then this five-dimensional metric induces on the branes two four-dimensional metrics of the form

\[
g (x) = a^2 + \, (x);
\]

where the scale factors \( a = a (y) \) are given by

\[
a_+ = 1; \quad a = e^{\dot{d} = 1} \, a;
\]

and \( (x) \) are the perturbations by which the brane metrics \( g (x) \) differ from the (conformally) flat metrics \( a^2 \) of the RS solution \([5]\). The variable \( a (y) \) represents the ‘modulus’ — the global homogeneous part of the radion field determining the interbrane separation.

Due to the metric perturbations, the branes no longer stay at fixed values of the fifth coordinate \( y \). Up to four-dimensional diffeomorphisms, their embedding variables consist of two four-dimensional scalar fields, the linearized radions \( (x) \), and the braneworld action can depend on these scalars. Their geometrically invariant meaning is revealed in a special coordinate system where the bulk-metric perturbations \( A_B (x, y) \) satisfy the “Randall-Sundrum gauge conditions” \( A_5 = 0 \), \( i = h = 0 \). In this coordinate system, the brane embeddings are defined by the equations

\[
: \ y = y + \frac{1}{a^2} \, (x); \quad y_+ = 0; \quad y = d;
\]

Therefore the radions describe the bending of the branes in the bulk (cf. \([29, 30]\)).

In \([26]\) we have derived the effective braneworld action in terms of the four-dimensional on-brane metrics \([7]\) and nonlocal matrix-valued form factors. We call the part of the action quadratic in \( h (x) \) — the transverse-traceless parts of the full metric perturbations \( (x) \) on the branes — the graviton sector. It can be written as the following 2 \( \times \) 2 quadratic form in terms of the metric perturbations and the special nonlocal operator \( F (Z) \),

\[
S_{\text{grav}} [h] = \frac{1}{16 G_4} \, dz \, \frac{1}{2} \, h^+ \, h \, \frac{F (Z)}{F} \, h^+ \, h;
\]

The effective gravitational constant \( G_4 \) is given by \( G_4 = G_5 \). As was shown in Ref. \([26]\), the operator \( F (Z) \) is a complicated non-linear function of the D'Alembert operator \( 2 \) expressed by means of Bessel and Neumann functions of the arguments \( \sqrt{P} \) and \( \sqrt{Z} \),

\[
F (Z) = \frac{1}{\sqrt{J_2 y_2} \, \sqrt{J_2 y_2}} \, \frac{3}{4} \, P_{\sqrt{Z} = a} \, u \, \frac{2}{2} \, \frac{P_{\sqrt{Z} = a} \, u}{(P_{\sqrt{Z} = a})^2} \, \frac{2}{2} \, \frac{P_{\sqrt{Z} = a} \, u}{(P_{\sqrt{Z} = a})^2};
\]

\(11\)
where

\[ u(z) = Y_1 J_2(z) - J_2 Y_1(z); \]

\[ J_1^* (\vec{l} \cdot \vec{z} = a); \quad Y_1^* Y_1 (\vec{l} \cdot \vec{z} = a); \]

\[ J_1^* (\vec{l} \cdot \vec{z} = a); \quad Y_1^* Y_1 (\vec{l} \cdot \vec{z} = a); \]

The action (10) should be amended by the standard coupling of the transverse-traceless gravitational modes to the transverse-traceless part of the stress-energy tensors, \( T \), on the two branes,

\[ S_{\text{mat}} = \int \frac{z}{d^4x} [\pi; T^+]; \]

The inverse of the kernel \( F(\zeta) \) is the Green function \( G(\zeta) \) of the problem,

\[ G = F^{-1}; \]

With the abbreviations given in (12) – (14), it reads

\[ G(\zeta) = \frac{a}{\vec{l}^2} \frac{1}{J_1 Y_1^* J_1^* Y_1} \left( \frac{\vec{l} \cdot \vec{z}}{a} \right)^2 \frac{u(\vec{l} \cdot \vec{z})}{u(\vec{l} \cdot \vec{z} = a)} \left( \frac{\vec{l} \cdot \vec{z}}{a} \right)^3 \]

It has first been calculated in [50]. With the help of \( G(\zeta) \), one finds the equations of motion for the transverse-traceless sector

\[ h^+ = 8 G_4 G(\zeta) \frac{T^+}{T}; \]

corresponding to the variation of the combined action of (10) and (15). In this paper, we study in detail the properties of the model in the low-energy limit when \( \vec{l} \cdot \vec{z} \rightarrow 1 \) but when \( \vec{l} \cdot \vec{z} = a \) can take arbitrary values in view of the smallness of the parameter \( a = e^{\pm 1} \) (large interbrane distance).

As we will only be considering the transverse-traceless part of the gravitational dynamics, tensor indices will be omitted for the rest of this paper.

### 3 The eigenmode expansion of the Green function

Both the Green function \( G(\zeta) \) and the kernel of the action \( F(\zeta) \) are highly nonlocal. Nevertheless, it is possible to obtain a conventional interpretation of the equations of motion and the action in terms of an infinite tower of orthogonal Kaluza-Klein modes. We will first discuss the recovery of the KK tower from the Green function because this procedure works in a more direct way than the recovery of the particle spectrum from the action, which will be discussed in the Sec.4. The elements \( G(\zeta) \) of \( G(\zeta) \) are meromorphic functions of \( \zeta \). If we consider a concentric circle \( C_n \) around \( \zeta = 0 \) so that \( C_n \) includes the first \( n \) poles and let its radius \( R_n \rightarrow 1 \) as \( n \rightarrow 1 \), we find the falloff property

\[ |G(\zeta)| < R_n; \quad R_n \rightarrow 1; \]

for any small constant \( \varepsilon \) and all \( \zeta \) on \( C_n \). Therefore we can apply the Mittag-Leffler expansion, which provides an expansion of a meromorphic function in terms of its poles, to the Green function in order to obtain a representation of \( G(\zeta) \) as a sum of scalar propagators.
In the limit $\frac{p^2}{2} \rightarrow 1$ but $\frac{p^2}{2} = a$ arbitrary, the Green function $G_{\varphi}(x)$ is given by

$$G_{\varphi}(x) = \sum_{n=0}^{\infty} \frac{J_n(2n)\cosh a}{J_n(2n)\cosh a} + \ln \frac{p^2}{2} \frac{1}{a} J_1(2 \sqrt{a}) + \ln \frac{p^2}{2} \frac{1}{a} J_1(2 \sqrt{a})^3 + \ln \frac{p^2}{2} \frac{1}{a} J_1(2 \sqrt{a})^4 + \ln \frac{p^2}{2} \frac{1}{a} J_1(2 \sqrt{a})^5 + \cdots. \quad (20)$$

For convenience we use $x$ as the fundamental variable of the expansion and not $\frac{p^2}{2}$. The Mittag-Leffler expansion for an element $G_{\varphi}$ of the Green-function matrix reads

$$G_{\varphi}(x) = \sum_{n=0}^{\infty} \frac{2 \text{Res} [G_{\varphi} (x = m^2_I)]}{m^2_I}; \quad (21)$$

where the first pole is at $x = 0$, i.e. at $m^2 = 0$. One can write

$$G_{\varphi}(x) = \sum_{n=0}^{\infty} \frac{\text{Res} [G_{\varphi} (x = m^2_I)]}{2 m^2_I} + \sum_{n=0}^{\infty} \frac{\text{Res} [G_{\varphi} (x = m^2_I)]}{m^2_I}; \quad (22)$$

As the second sum contributes only a constant, we can drop this part and obtain our final result for the elements of the Green function,

$$G_{\varphi}(x) = \sum_{n=0}^{\infty} \frac{\text{Res} [G_{\varphi} (x = m^2_I)]}{2 m^2_I}; \quad (23)$$

In this way, each nonlocal element of the nonlocal Green function $G_{\varphi}(x)$ can be represented as an infinite sum of scalar propagators with different masses. The mass-squares are given by the positions of the poles of $G_{\varphi}(x)$. From the representation of the Green function (20), we immediately infer that all elements of $G_{\varphi}(x)$ have poles at the same values of $x$. This is a direct consequence of the common prefactor displayed in (17), which contains all poles of $G_{\varphi}(x)$. One pole is located at $x = 0$, which is also an exact pole of (17). An infinite sequence of poles is located at the roots of the Bessel function $J_1$. Denoting the argument at the $\text{th}$ zero of $J_1$ by $j_\text{th}$, we find that these poles are located at

$$x = (j_\text{th}a=1)^2 m^2_I; \quad (24)$$

where the first few values of $j_\text{th}$ are given by

$$j_0 = 3; j_1 = 5; j_2 = 7; j_3 = 10; \ldots; \quad (25)$$

Whereas the first term of the sum (23) from the pole at $x = 0$ will provide us with the effective four-dimensional massless graviton, the infinite series of poles at $m^2_I$ is responsible for the generation of the KK tower of massive gravitons. The matrix of residues of $G_{\varphi}$ at the pole $x = 0$ is given by

$$\text{Res} [G_{\varphi} (x = 0)] = \frac{2}{a_0 (1 - a_0)} \frac{1}{a^2} \frac{a^2}{a^4}; \quad (26)$$

and the residues for the infinite sequence of poles at $x = m^2_I$ are found to be

$$\text{Res} [G_{\varphi} (x = m^2_I)] = \frac{2a^2}{2} \frac{1}{1 - a^2} \frac{1}{1 - a^2} \frac{1}{1 - a^2} \frac{1}{1 - a^2}; \quad (27)$$

Note that the falloff property (19) is not valid for the element $G_{\varphi}$ in the approximation (20) but only for the full expression (17). This does of course not prevent us to use the approximation (20) for an approximate determination of the residues and thus for an approximate determination of the Mittag-Leffler expansion.
The residues can be factorized as
\[ \text{Res} \left[ G ( \epsilon = m^2 ) \right] = \nu_1 \nu_1^T ; \]  
with
\[ \nu_0 = \frac{p^2}{1 - \frac{1}{a^2}} \frac{1}{a^2} ; \quad \nu_1 = \frac{p^2 a}{1 - \frac{1}{a^2}} \frac{1=J_2 [ \ln \frac{1}{a} ]}{1 - \frac{1}{a^2}} \quad ; \quad i \neq 1. \]  
This property reflects the fact that the residues of the poles of the Green function are projectors on the corresponding propagating modes of the theory. These energy eigenmodes will be recovered from the action in the next section. First, however, we should display explicitly our newly acquired representation of the Green function:
\[ G ( \epsilon ) = \frac{2}{\prod_{i=1}^{2} (1 - m_i^2)} \frac{1}{a^2} a^{2^2} \frac{1=J_2 [ \ln \frac{1}{a} ]}{1 - \frac{1}{a^2}} a^{2^2} + \frac{2a^2}{\prod_{i=1}^{2} (2 \frac{m_i^2}{a^2})} \frac{1=J_2 [ \ln \frac{1}{a} ]}{1 - \frac{1}{a^2}} a^{2^2} a^{2^2} \]  

4 The spectrum and the eigenmodes of the effective action

The eigenmodes of the equations of motion can be recovered from the action (10) and its kernel \( F ( \epsilon ) \). We will find that the situation has many similarities with resonance theory, and our discussion will parallel many of the considerations of [31] from which we will also adapt our nomenclature of eigenstates.

In the low-energy limit, \( \frac{p^2}{2} \) arbitrary, the kernel of the action \( F ( \epsilon ) \) can be approximated by
\[ F ( \epsilon ) = \frac{2^{2^2}}{\prod_{i=1}^{2} (1 - m_i^2)} \frac{1=J_2 [ \ln \frac{1}{a} ]}{1 - \frac{1}{a^2}} a^{2^2} a^{2^2} + \frac{2a^2}{\prod_{i=1}^{2} (2 \frac{m_i^2}{a^2})} \frac{1=J_2 [ \ln \frac{1}{a} ]}{1 - \frac{1}{a^2}} a^{2^2} a^{2^2} a^{2^2} \]  

The typical way to extract the particle content from an action with a matrix-valued kernel is its diagonalization in terms of normal modes. However, as found by the application of the Mittag-Leffler expansion of the Green function in the last section, the number of propagating modes enormously exceeds the number of entries in the \( 2 \times 2 \)-matrix \( F ( \epsilon ) \) and thus the modes do not diagonalize the quadratic action (10) in the usual sense. This behavior could also have been anticipated from the nonlocality of \( F ( \epsilon ) \). The propagating modes are the zero modes of \( F ( \epsilon ) \) which solve the matrix-valued nonlocal equation
\[ F ( \epsilon ) h_1 ( \epsilon ) = 0 ; \]  
These modes are conveniently split into a scalar part \( h_1 ( \epsilon ) \), depending on the four-dimensional spacetime coordinates, and a spacetime-independent (isotopic) vector part \( v_1 \).
\[ h_1 ( \epsilon ) = v_1 \nu_1 ; \]  
The eigenvector equation (32) is accompanied by its consistency condition,
\[ \det F ( \epsilon ) = 0 ; \]  
which corresponds to picking the poles of the Green function \( G ( \epsilon ) \). The condition (34) yields the mass spectrum of the theory given by the roots of this equation, i.e. \( 2 = m_i^2 \). Hence the \( h_1 ( \epsilon ) \) above are Klein-Gordon modes and the isotopic vectors of the propagating modes \( v_1 \) are zero eigenvectors of \( F ( m_i^2 ) \),
\[ ( \epsilon - m_i^2 ) h_1 ( \epsilon ) = 0 ; \]  
\[ F ( m_i^2 ) v_1 = 0 ; \]
Thus we obtain the Kaluza-Klein spectrum which contains the massless mode \( i = 0, m_0 = 0 \), and a tower of massive modes. In the low-energy approximation of \( F (z) \), see Eq. (31), its masses \( m_i = a j_i^2 = 1 \) are given by the roots of the first-order Bessel function, \( J_1 (j_i) = 0 \). The isotopic structure of their \( v_i \) is given by the vectors (29) which were found to factorize the residues of the Green function.\(^4\) In view of standard arguments of gauge invariance, the massless graviton \( h_0 \) has two dynamical degrees of freedom, while the massive tensor field \( h_m \) possesses all five polarizations of a generic transverse-traceless tensor field.

The action (10) is not, however, diagonalizable in the basis of these states because under the decomposition \( h (x) = \sum_i h_i (x) v_i \) (with off-shell coefficients \( h_i (x) \)) the cross terms intertwining different \( i \)'s are non-vanishing,

\[
v_i^T F (z) v_j \neq 0:
\]

(37)

A crucial observation is, however, that the diagonal and non-diagonal terms of this expansion are linear and bilinear, respectively, in on-shell operators (41)

\[
v_i^T F (z) v_i = v_i^T \frac{dF (z)}{dz} v_i \quad \text{with} \quad m_i^2;
\]

(38)

\[
v_i^T F (z) v_j = M_{ij} (z) (z) m_i^2 m_j^2 \quad \text{for} \quad i \neq j;
\]

(39)

where higher powers of \( m_i^2 \) have been dropped in (38), and \( M_{ij} (z) \) is non-vanishing at both \( z = m_i^2 \) and \( z = m_j^2 \). Therefore, the non-diagonal terms of the action do not contribute to the residues of the Green function \( G (z) \) of \( F (z) \). The normalization of \( v_i \), as given by (29), automatically yields in (38) a unit coefficient of \( m_i^2 \).

Thus we have found the particle spectrum of the action in terms of modes which turn out to be non-orthogonal off-shell but become orthogonal if one considers the action on the mass-shell. In the theory of atomic and nuclear resonances, such non-orthogonal energy eigenstates are called Siegert states after their introduction in (32) (cf. also (31)).

This particle spectrum can also be recovered from the two orthogonal eigenstates of \( F (z) \). These states which will be energy dependent, i.e. not mass eigenstates, have also been extensively used in resonance theory and are named Kapur-Peierls states according to their first use in (33, 34) (cf. also (31)).

In resonance theory, Kapur-Peierls states \( b_i (z) \) diagonalize the matrix Hamiltonian of the system (the analogue of our \( F (z) \)). For our \( z = 2 \) matrix-valued kernel \( F (z) \), this property reads

\[
F (z) b_s (z) = n (z) b_s (z) \quad n = 1; 2;
\]

(40)

\[
b_s^T (z) b_s (z) = n_s^2;
\]

(41)

where the second equation describes the orthonormality of the Kapur-Peierls states.

In the limit \( \int \frac{d^2 z}{2} = 1 \), the energy-dependent eigenvalues of \( F (z) \) [as determined by Eq. (11)] are found to be

\[
1 (z) = \frac{n_1}{2} z^2 + O (a); \quad 2 (z) = \frac{n_2}{a} J_1 [m_i = a] + O (a);
\]

(42)

The corresponding eigenvectors (normalized to unity to leading order in \( a \)) are

\[
b_1 (z) = \frac{1}{a} \frac{L}{2 J_1 [m_i = a]} 5; \quad b_2 (z) = \frac{1}{a} \frac{1}{L^2} z^2;
\]

(43)

\(^4\) The vector \( v_0 \) is here actually given by \( v_0 = \int \frac{d^2 z}{2} = 1 \) \( a^2 z^2 \), i.e. the leading order in \( a \) of the exact \( v_0 \) as given by (29), because here the \( v_i \)'s are obtained for the approximation (31) of \( F \).
where we have introduced
\[
Z = \frac{P}{a^2} 2 + \frac{1}{2} J_1 [\text{im}_1 = a] \quad \text{(44)}
\]

The eigenvalue \(\lambda_1\) given in (42), has only one root at the value \(Z = 0\), whereas the eigenvalue \(\lambda_2\) has infinitely many zeros at the zeros of the Bessel function \(J_1\), i.e., at \(Z = \text{im}_1^2\). Approximating \(\lambda_2(\lambda)\) around its zeros by the first term of its Taylor expansion, we find for the eigenvalues
\[
\lambda_1(\lambda) = \frac{Z^2}{2 a^2} \quad \lambda_2(\lambda) = \frac{Z^2}{2 a^2} (\text{im}_1^2)
\]

At \(Z = 0\) the eigenvector \(b_1\) takes the value
\[
b_1(0) = \frac{1}{a^2} \quad \text{(46)}
\]

and at energies \(Z = \text{im}_1^2\) the eigenvector \(b_2\) becomes
\[
b_2(\text{im}_1^2) = \frac{1}{\text{im}_1 [\text{im}_1 = a]} \quad \text{(47)}
\]

As expected, the mass levels of the Siegert states equal the zeros of the Kapur-Peierls eigenvalues. The massless graviton Siegert state corresponds to the single root of the eigenvalue \(\lambda_1(\lambda)\), and the tower of massive Siegert states arises from the infinitely many zeros of the eigenvalue \(\lambda_2(\lambda)\). It is easy to find the relation between the eigenvectors and the Kapur-Peierls eigenvectors by taking from Eq. (40) that the vectors \(b_s(\lambda)\) are eigenvectors also of \(F^{-1}(\lambda) = G(\lambda)\) with eigenvalues \(1 = \lambda(\lambda)\). Using the completeness and orthonormality of the \(b_s(\lambda)\)'s, we obtain a representation of \(G(\lambda)\),
\[
G(\lambda) = \sum_{s=1}^{\infty} \frac{b_s(\lambda) b_s^T(\lambda)}{s(\lambda)}
\]

As the poles of the Green function lie at \(s(\lambda) = 0\), we find for the residues of \(G(\lambda)\)
\[
\lim_{2 ! m_1^2 = \text{im}_1^2} G(\lambda) = \frac{b_s(\lambda) b_s^T(\lambda)}{d_s(\lambda) = d2_{m_1^2}} \quad \text{(49)}
\]

where \(s = 1\) for \(m_0 = 0\) and \(s = 2\) for \(m_1\) with \(i = 1\). Comparing this expression for the residues with (28), we obtain
\[
v_0 = \frac{d_1(\lambda)}{d2} \quad v_1 = \frac{d_1(\lambda)}{d2} \quad \text{(50)}
\]

The decomposition of the action into Siegert eigenmodes (33) provides us with the conventional particle interpretation of the propagating modes of the nonlocal operator \(F(\lambda)\). It clarifies their role in its Green function \(G\) that mediates the gravitational effect of matter sources. Amended by the matter action on the branes, the effective action of the graviton sector reads
\[
S = h^T \int d^4x \left[ \frac{1}{32 \pi G_4} h^T \left( \frac{F}{F} \right) h + \frac{1}{2} \right] \quad \text{(51)}
\]

where \(T\) is the column vector of the transverse-traceless part the stress-energy tensors on the branes and \(h\) is now given by the sum of Siegert modes (33),
\[
h = \sum_{i=0}^{\infty} h_i(\lambda) v_i \quad \text{(52)}
\]
Varying this action with respect to each Siegert mode $h_i$ and decomposing the results to recover the equations of motion for $h$, we obtain the linearized equations of motion. Their solution

$$h = 8 G_4 \frac{2}{l} G_{\text{ret}} T$$

is expressed in terms of the retarded version of the Green function (30). We note the explicit expressions for $h^+$ and $h$ obtainable from (50) and (52):

$$h^+ (x) = \begin{pmatrix} \frac{1}{a^2} h_0 (x) + \frac{1}{a} X \frac{1}{J_2 (m_1 = a)} h_1 (x) \\ \frac{1}{a^2} h_0 (x) \end{pmatrix}$$

$$h (x) = \begin{pmatrix} \frac{1}{a^2} h_0 (x) + \frac{1}{a} X \frac{1}{J_2 (m_1 = a)} h_1 (x) \\ \frac{1}{a^2} h_0 (x) \end{pmatrix}$$

It is particularly interesting to study the asymptotic behavior of the expressions (54) and (55). In the long-distance limit $a \to 0$, we have

$$h^+ h_0 + O (a); \quad h \not\approx h_0 a^{-1} h_1;$$

Thus, in this limit, the $h^+$-mode practically coincides with the massless graviton $h_0$, and the mixing with the other modes is suppressed by the factor $a$. On the other hand, the $h$-brane mode $h$ is almost exclusively composed from the massive modes, and mixing with the massless mode is suppressed, whereas the mixing between the massive modes can be called maximal, i.e. $h$ is proportional to the sum of the massive modes. Moreover, the contribution of both the modes $h_1$ and $h_0$ to the $h$-mode in this regime is suppressed by the common factor $a$. There is also a regime of maximal mixing between all modes for the metric perturbation $h$ on the $h$-brane at $a = 1 - \frac{2}{l}$ where

$$h = h_0 + \frac{1}{a} h_1;$$

5 The graviton effective action in the diagonalization approximation

In the last section we have found that the Siegert state representing the massless graviton corresponds to the zero of the eigenvalue $\lambda_1$ of the kernel of the action $F (2)$, whereas the Siegert states corresponding to all of the KK modes stem from the zeros of the second eigenvalue $\lambda_2$. This opens the possibility of a low-energy limit in which the Siegert and Kapur-Peierls states coincide. In the approximation

$$\begin{pmatrix} \frac{1}{a^2} h_0 (x) + \frac{1}{a} X \frac{1}{J_2 (m_1 = a)} h_1 (x) \\ \frac{1}{a^2} h_0 (x) \end{pmatrix}$$

one can use the small-argument expansion for all Bessel and Neumann functions appearing in $F (2)$. If we truncate this expansion at $O (2)$, we obtain an expression for the kernel $F (2)$ which is entirely local. The choice of the second part of the low-energy condition (59) guarantees that the approximation is still valid slightly above the mass of the first KK mode,

$$m_1 = \frac{1}{a} = 0, \quad \text{Im}_1 = a = 3 \pm 3$$

(cf. 25). In fact, at least for Bessel functions of the second kind, the small-argument expansion is considered as the method of choice in numerical calculations for arguments $4$, see [35]. However, we should be prepared for some deviation of the expansion-result for the first KK mode from that of the previous sections due to the low order of truncation of the expansion. Taking this into account, the result will turn out to be surprisingly accurate.
In the low-energy approximation, the kernel \( F = l^2 \) can be represented in the form:

\[
F(2) = M + D + O(2^2);
\]

\[
M = \frac{1}{2} \frac{8}{l^2} \frac{a^4}{d^2} \frac{a^2}{d^2} \frac{1}{d^2} ;
\]

\[
D = \frac{1}{3} \frac{a^2}{1 + a^2} \frac{a^2}{2} \frac{3 + a^2}{2} ;
\]

The operator \( D \) represents the kinetic part of the action, while the operator \( M \) plays the role of the mass term. Calculating the trace and the determinant of the matrix \( D \), it is easy to see that it is positive definite for any value of the parameter \( a \). The determinant of the matrix \( M \) is equal to zero, therefore \( M \) is degenerate. Using the positive-definiteness of the matrix \( D \), one can diagonalize both the matrices \( D \) and \( M \) simultaneously.

Our purpose here is not simply to diagonalize these matrices, but also to make a canonical normalization of the kinetic terms. With this choice, the action for gravity eigenmodes with fixed masses, representing some mixtures of the functions \( h^+ \) and \( h^- \), will have the form:

\[
S_{graviton}[h_0; h M] = \frac{1}{16} \frac{Z}{G_{\mu}} \int d^4 x \left( \frac{1}{4} h_0 h_0 + \frac{1}{4} h M (2 \ M^2) h M \right) ;
\]

where \( M \) is the mass of the massive gravitational mode.

The details of the procedure of diagonalization are presented in the Appendix. Here we shall give the final results. The expression for the mass of the massive graviton mode \( h M \) is:

\[
M^2 = \frac{24a^2 (1 + a^2)}{F (1 \ a^2)^2} ;
\]

At small values of the parameter \( a \), i.e. at large distances between the branes, the squared mass behaves as:

\[
M^2 \sim \frac{24a^2}{F} (1 + a^2) ;
\]

and tends to zero together with \( a^2 \). This result is to be compared with the more accurate expression for the first KK mass in relation (24). The large discrepancy is easily explained from the fact that taking the limit \( a \to 0 \) is strictly speaking not legitimate in our approximation because the second condition of (59) would not be fulfilled anymore.

The relations describing the transition from the old graviton variables \( h^+ \) and \( h^- \) to the modes \( h M \) and \( h_0 \) have the following form:

\[
h_M = \frac{1}{3} \frac{a^2}{1 + a^2} \ h^+ ;
\]

\[
h_0 = \frac{1}{3} \frac{a^2}{1 + a^2} (h^+ + h^-) ;
\]

The inverse transformations are:

\[
h^+ = \frac{1}{3} \frac{a^2}{1 + a^2} (h_0 + \frac{2}{3} a h_M) ;
\]

\[
h = \frac{a}{3} \frac{a^2}{1 + a^2} (ah_0 + \frac{2}{3} a h_M) ;
\]
The corresponding expressions for \( v_0 \) and \( v_M \) are

\[
v_0 = \frac{p}{2} \frac{1}{1} \frac{1}{a^2} \text{; } v_M = \frac{p}{6a} \frac{1}{1} \frac{1}{a^2} \text{;} \tag{71}
\]

which should be compared with the expressions for \( v_0 \) and \( v_1 \) in (29). As might have been anticipated from the fact that the residues depend only on the (1)th term of the Laurent expansion of the function considered, we find exact agreement for \( v_0 \).

Amending the action (64) by the matter action (15), we can obtain the equations of motion for the metric perturbations \( h^+ \) and \( h^0 \) on the brane by varying the total action with respect to \( h_0 \) and \( h_M \) and then combining their equations of motion according to the transformations (69) and (70).

\[
\begin{align*}
h^+ &= 16 \frac{G_4}{2} \left( T^+ + \frac{a}{a^2} T \right) \frac{1}{2 \text{ ret}} \frac{1}{a^2} \frac{1}{a^4} + \frac{3a}{2} \frac{1}{a^2} \frac{1}{a^4} \frac{1}{1} \frac{1}{T} ; \tag{72}
\end{align*}
\]

In the language of the Siegert and Kapur-Perierls eigenmodes of Sec. 4, we have chosen for the diagonalization an approximation in which the Siegert eigenmodes become orthogonal and the Kapur-Perierls eigenvalues and eigenstates energy-independent. Thus, in the approximation (59), the Siegert and Kapur-Perierls descriptions coincide. The first Kapur-Perierls state coincides with the Siegert state of the massless graviton, while the massive Siegert states become a kind of “collective” massive state which coincides with the second Kapur-Perierls mode.

## 6 Graviton oscillations

### 6.1 Gravitational waves on the + -brane

After the extensive theoretical considerations of Secs. 4 and 5 we will now return to the equations of motion (18) and study their phenomenological properties. We will proceed to show how graviton oscillations arise naturally in the propagations of gravitational waves in this model. For studying the properties of graviton oscillations, it is convenient to abandon the condensed matrix notation of Eq. (18) and to study the behavior of metric perturbations on the + -brane and -brane separately, starting with the + -brane.

Using the spectral representation (40), we find for \( h^+ \),

\[
\begin{align*}
h^+ &= \frac{16}{2} \frac{G_4}{\text{ ret}} \left( T^+ + \frac{a}{a^2} T \right) \frac{1}{i = 1} \frac{16}{2} \frac{G_4}{m_i} \frac{a}{J_2^+} \frac{a}{J_2^+} ; \tag{73}
\end{align*}
\]

where we have introduced \( J_2 \quad J_2 \left[ m_i = a \right] \quad 0 \neq 0 \). We consider astrophysical sources of equal intensity at \( x = 0 \) on both branes with a harmonic time dependence

\[
T \left( t; x \right) = e^{i t \frac{r}{x}} ; \tag{74}
\]

where \( r / x \) is the mass quadrupole moment of the source. The sources with an oscillating quadrupole moment will generate gravitational waves. At a distance \( r \) from the source, the waves on each brane are given by a mixture of massless and massive spherical waves. On the + -brane, this superposition is given by the sum of the contributions from the sources on the + and -brane, respectively,

\[
\begin{align*}
h^+ \left[ T^+ \right] &= A e^{i t \frac{r}{x}} e^{i t \frac{r}{x}} + \sum_{i = 1} X \frac{a^2}{J_2 \left[ m_i = a \right]} e^{i t \frac{r}{x}} ; \tag{75}
\end{align*}
\]

\[
\begin{align*}
h^+ \left[ T_0 \right] &= A a^2 e^{i t \frac{r}{x}} e^{i t \frac{r}{x}} + \sum_{i = 1} X \frac{1}{J_2 \left[ m_i = a \right]} e^{i t \frac{r}{x}} ; \tag{76}
\end{align*}
\]
where $A = 4G_4 = r$ is the amplitude of the massless mode generated on the $+$-brane. If the frequency $!$ with which the source is oscillating is smaller than the mass $m_1$ of a KK mode, $! < m_1$, then the exponent of the corresponding mode will be real and negative, instead of imaginary, and the mode will be decaying instead of oscillating. In effect all modes with masses above the frequency of the source will have died off after propagating for a short distance and only those modes with a mass below the frequency of the source will contribute to the gravitational wave in the far field. Thus, as long as we are interested in GW’s, we can content ourselves with including only the terms with $m_1 < !$ into the sums of Eqs. (75) and (76) (cf. [6]).

In order to have a simple situation, we consider a source with a frequency above the mass threshold of the first massive mode but below the threshold of the second KK mode, i.e. $m_1 < ! < m_2$. Then only the massless and the first massive mode are excited and produce long-range gravitational waves. At a distance $r$ from the source, the waves on each brane are given by a mixture of massless and massive spherical waves. On the $+$-brane this superposition is given by the sum of the contributions

$$h^+[T^+] = A e^{i!t} e^{ikr} + \frac{a^2}{J_2} e^{iP_2 k^2 m_1 r};$$

$$h^+[T] = A a^2 e^{i!t} e^{ikr} \frac{1}{J_2} e^{iP_2 k^2 m_1 r};$$

from sources on the $+$- and $-$-brane, respectively.

### 6.2 Quantum oscillations — an analogy

The above results have interesting physical consequences. We have seen that the observable transverse-traceless metric on each brane is indeed a linear combination of different gravitational modes with fixed masses. One of these modes appears to be massless, while the others are massive with their masses depending on the distance between the branes.

This situation is quite analogous to the well-known mixing arising in the context of kaons [36, 37] or other bosons [38] or with neutrino oscillations [39]. As a matter of fact, the phenomenon which we encounter here is not quantum oscillations but classical ones. Nevertheless, the interference effects between the different gravitational modes, arising in the process of their propagation in space (on the brane), are analogous to those of quantum oscillations. This is quite natural because the oscillations of quantum particles are totally conditioned by their wave nature (see, for example, Ref. [37]), and both effects are caused by the superposition of modes with different dispersion relations.

However, there is one important difference: it does not matter for quantum oscillations if one considers the superposition to consist of two components of different energies but with the same momentum (as is usually assumed) or if the components are supposed to have the same energy but different momenta [40]. For a mixture of classical gravitational waves, however, it is natural to assume that they have been generated by the same source oscillating with a frequency $!$. Therefore one should consider oscillations between modes of the same frequency $!$ which have different wave-numbers $k$, and not vice versa.

Let us recall briefly some basic formulas from the theory of quantum oscillations. Suppose that a certain particle can exist in two different observable states $j_{1i}$ and $j_{2i}$, which correspond to a quantity conserved in some interaction but which do not have fixed masses. For example, kaons can be created or detected in states with fixed strangeness, which are not eigenstates of their Hamiltonian, and according to the theory of neutrino oscillations, which is now supported by good experimental evidence [41], the states of the electron, muon and tau neutrinos are also mixtures of different Hamiltonian (i.e. mass) eigenstates. Generally, this situation can be described for the two-state case by

$$j_{1i} = \cos j_M i + \sin j_M i;$$

$$j_{2i} = \sin j_M i + \cos j_M i;$$

(79)
where the states $j_M^i$ and $j_m^i$ are states with masses $M$ and $m$, respectively, and the angle parameterizes the mixing between the states. If at the initial position of an evolution at $x = 0$ somebody observes, say, a particle in a state

$$j^i(0) = j^i_1;$$

(80)

propagating in space with the energy $\omega$, then at the position $x$ the state (80) will become equal to

$$j^i(x) = \cos \left( \omega_0 x + \sin \left( \omega_0 x \right) + \frac{k_0}{k_0^2 + M^2} \right);$$

(81)

where $k_0 = \omega$. Now we are in a position to calculate the probability of observing the particle in the state $j^i_1$ at the position $x$ which is given by the formula

$$P_1(x) = \frac{1}{2} \sin^2 \left( \frac{1}{2} \left( k_0 - \omega_0 \right) x \right);$$

(82)

Correspondingly, the probability of finding the particle in the state $j^i_2$ is

$$P_2(x) = \frac{1}{2} \sin^2 \left( \frac{1}{2} \left( k_0 + \omega_0 \right) x \right);$$

(83)

It is easy to see that the oscillations of the probabilities $P_1$ and $P_2$ are strongest when $\sin^2 \left( \frac{1}{2} \right) = 1$, i.e. when

$$\frac{1}{2} = \frac{k_0}{4};$$

(84)

This situation is called the case of maximal mixing between oscillating particles. Looking at the expressions for the probabilities (82) and (83), one can see that they represent periodic functions with a period

$$L = \frac{2}{k_0^2 + M^2};$$

(85)

This is the oscillation length of a quantum state with the energy $\omega = k_0$. In the case of maximal mixing (84), detecting the particle $j^i_1$ at the distance $L = 2$, one finds with probability $P_2 (L = 2) = 1$ the particle in the state $j^i_2$. In other words, the particle in the state $j^i_1$ is totally transformed into the particle in the state $j^i_2$. The time-dependence of the oscillation can be inferred from this expression by considering semi-classically the propagation speed of the corresponding particles.

The description of quantum oscillations presented above coincides with describing classical oscillations of superpositions of waves with different dispersion relations. In this case, instead of the probability of detecting a particle of a certain type, one can calculate the amplitude of the wave.

6.3 Gravitational-wave oscillations on the $+$-brane

For the superpositions (75) and (76), the amplitudes detected by a gravitational-wave interferometer are given by the absolute values of the waves,

$$h^+ \left[ T^+ \right] = A^+ \left( \frac{4a^2J_2^2}{(J_2^2 + a^2)^2} \sin^2 \frac{r}{L} \right);$$

(86)

$$h^+ \left[ T^- \right] = A \left( \frac{4J_2}{J_2^2} \sin^2 \frac{r}{L} \right);$$

(87)
Here, \( L \) is the oscillation length of the amplitude modulation of the gravitational wave (GW),

\[
L = \frac{2}{m_1^2} \, .
\]  

(88)

This corresponds to Eq. (85) with one mode being massless. The pre-factors of the amplitudes (86) and (87) are given by

\[
A^+ = 1 + \frac{a^2}{J_2^2} A \, A \, ;
\]  

(89)

\[
A = \frac{1}{J_2} \left( 1 - \frac{a^2}{J_2} \right) A \, 15 a A \, ;
\]  

(90)

where the approximations are valid in the limit \( a \rightarrow 1 \). We find oscillations in the amplitudes of the waves from both sources. For a GW generated by \( T^+ \), see Eq. (86), the oscillating part of the amplitude is suppressed by a factor of \( a^2 \) compared to the constant part in the limit of large brane separation, \( a \rightarrow 1 \). The amplitude of the GW produced by \( T^- \), see Eq. (87), is totally oscillating, regardless of the inter-brane distance.

In the limit of a source-frequency just above the mass-threshold \( \ell_1 \sim m_1 \), the oscillation length (88) of these radion-induced gravitational (“graviton”) wave oscillations (RIGO’s) tends to

\[
L \sim \frac{2}{m_1} \, .
\]  

(91)

This limit is mainly useful to set a lower bound on the oscillation lengths as in this limit the propagation speed of the massive mode will tend to zero. Therefore, in this limit, the massless and massive wavetrains from a source would soon become spatially separated from each other and, far from the source, one would detect two separated wavetrains instead of one wavetrain exhibiting RIGO’s. The astrophysically more interesting limit is the case \( \ell_1 \sim m_1 \) in which the expression for the oscillation length becomes

\[
L = 4 \frac{1}{m_1} \, .
\]  

(92)

We can express the minimal oscillation length (91) of RIGO’s in terms of the AdS radius \( \ell_1 \) and the scale factor \( a \),

\[
L_{m i n} = 2 \frac{1}{J_2 a} \left( 1 + \frac{1}{a} \right) \, ;
\]  

(93)

where \( J_2 \) is the first root of \( J_1 \) (cf. (24) and (25)). The oscillation length is inversely proportional to \( a \). Graviton oscillations become observable when the oscillation length is at least of the order of the arm length of a GW detector. For the ground-based interferometric detectors this requirement corresponds to \( L \sim 10^3 \, m \). Combining this with the constraint on the maximal size of the AdS radius \( \ell_1 \) from sub-millimeter tests of gravity, \( \ell_1 \sim 10^{-4} \, m \), we find an upper limit on the warp factor \( a \sim 10^{-7} \) for the oscillation length to be detectable. Inserting this into the ratio of the amplitudes (89) and (90) we find

\[
A = A^+ a^2 \sim 10^{-14} \, ;
\]  

(94)

Therefore, the amplitude of a wave originating from a source on the (“hidden”) \( - \)-brane with oscillations which are sufficiently long to be detectable, is strongly suppressed by a damping factor \( a^2 \) as compared to a GW stemming from a source on the \( + \)-brane itself. A strongly oscillating wave has to be generated by a source of 14 orders of magnitude stronger than that of a weakly oscillating one in order to be of the same magnitude, which at first sight makes the detection of RIGO’s impossible. We will, however, see in the
next subsection that the damping factor could easily be overcome if the RS model is considered as a model for a putative M-theory realization of large extra dimensions.

Beforehand we want to summarize that the main differences between the graviton oscillations considered in this paper and the more traditional neutrino or kaon oscillations:

First, as is readily inferred from Eq. (52) or (54) and (55), respectively, the transformation between the gravitational modes $h_+$ and $h_i$ living on certain branes and the graviton modes $h_0$ and $h_i$ possessing fixed masses is not an orthogonal (and not a unitary) transformation. This is connected with the fact that — in contrast to the mixing for neutrinos or kaons, where only the mass matrix is non-diagonal if written in terms of observable states — in the case of gravity in a two-brane world the whole kernel of the action is non-diagonal and, moreover, is diagonalizable only on-shell (cf. Sec. 4).

The second and, perhaps, more important distinguishing feature of RIGO’s is the dependence of the mixing parameters and the mass of the massive mode on the parameter $a$, i.e. their dependence on the radion. If one considers a model in which the distance between the branes is time-dependent, the mixing parameters and all the oscillatory effects become functions of time depending on the particular features of the model under consideration.

6.4 High-amplitude RIGO’s on the $+,-$-brane from M-theory

The bad prospects for the detection of RIGO’s found at the end of the last section may considerably be improved if one considers the RS two-brane model as a toy model for a yet-to-be-constructed model of the strong-coupling limit of heterotic M-theory with one large extra dimension. Although such a model is lacking, we can extrapolate generic features of M-theory compactifications with small extra dimensions and draw some reasonable conclusions for the hidden sector of the model. These conclusions suggest the possibility that the hidden brane could in fact be the source of GW’s with very large amplitudes.

Cosmological M-theory models consist of an 11-dimensional spacetime which is compactified on an $S^1=\mathbb{Z}_2$ orbifold with a stack of D-branes on each orbifold-fixed plane \[43, 44, 17\]. Further 6 of the 10 dimensions are compactified on a Calabi-Yau three-fold, while the other dimensions parallel to the stacks of branes, including the timelike direction, remain uncompactified. The compactification scale of the Calabi-Yau manifold is generally assumed to be smaller by one or two orders of magnitude than the size of the compact 11th dimension. Each stack of branes hosts one super-multiplet of $E_8$ gauge fields.

This construction can become a suitable model for our universe if one of the set of $E_8$ fields is broken into the subgroup $E_6$ and further into $SU(3)\times SU(2)\times U(1)$. These fields are then called the visible sector of the model and the corresponding stack of branes is identified as our universe, while the other $E_8$ is called the hidden sector. In order to obtain a viable phenomenology it is necessary to break the $E_8$ of the visible sector by gaugino condensation in the hidden sector and then mediate the influence of the gaugino condensate to the visible sector by its coupling to moduli fields on the Calabi-Yau manifold.\[47\] The volume of the Calabi-Yau space $V$ at the visible sector is assumed to be of the size of the standard-model grand-unification scale $V_{\text{1=6}} = (10^6 G_{eV})$. In generic M-theory compactifications, it has been found necessary in this context to choose the volume of the Calabi-Yau space at the hidden sector smaller than at the visible sector, which will also lead to a stronger gauge coupling in the hidden sector than in the visible one \[44\] (cf. also \[21\]). Also the volume of four-dimensional slices parallel to the uncompactified dimensions of the stacks of branes will decrease towards the hidden sector. It has been observed that in order to obtain a viable phenomenology one has to assume that the $E_8$ on the hidden sector is also broken to a smaller group $E_6$\[46\].

In order to make connection of these M-theory setups to the Randall-Sundrum two-brane model, one identifies the visible stack of branes with the $+$-brane and the hidden stack of branes with the $-$-brane: the orbifolded coordinate of M-theory is identified with orbifolded bulk coordinate of the RS model, and

---

\[5\] Cf. \[21\] for a non-technical description of the cosmological aspects of this class of models.

\[6\] Cf. \[45\], Chap. 18.3, p. 366–371, for a pedagogical description of this mechanism in the weak-coupling limit of $E_8\rightarrow E_6$ heterotic string theory.
the Calabi-Yau space is neglected in the effective five-dimensional description due to its small volume \[2\]. The major deviation of the RS model from M-theory ideas is that the size of the orbifolded bulk may be large. In M-theory models the scale factor for the uncompactified coordinates corresponding to the warp factor \( a(y) \) in the RS model is tied to the Calabi-Yau volume. A decreasing warp factor also leads to a decrease of the Calabi-Yau volume. In the M-theory solution of [17] the relation between the four-dimensional scale factor and the Calabi-Yau volume reads, for example,

\[
V(y) \propto a^6(y); \tag{95}
\]

where \( y \) denotes the coordinate of the orbifolded dimension. In scenarios with a large orbifolded dimension, the dependence of the Calabi-Yau volume on the scale factor should be weaker because the string scale sets a lower limit for the size of the Calabi-Yau manifold. Nevertheless, if M-theory can be extrapolated to setups with a large bulk, one has to expect a considerable decrease of the Calabi-Yau volume from the visible to the hidden brane. The crucial observation for our argument is now that the vacuum expectation value of the gaugino condensate located on the hidden brane depends strongly on the inverse size of the Calabi-Yau space [47],

\[
\frac{3}{V^{\frac{3}{2}}} \exp \left( \frac{18V}{b_0} (S + T) \right); \tag{96}
\]

Here, \( b_0 \) is the 10-dimensional gauge-coupling constant, \( b_0 \) is the beta-coefficient of the gauge-coupling renormalization group, \( S \) and \( T \) are moduli of the Calabi-Yau space and \( \beta \) describes loop-corrections to the moduli fields. From the structure of (96) we find that even a moderate decrease of \( V \) with the warp factor leads to a strong increase of the vacuum expectation value of the gaugino condensate, which will therefore be very large in a RS-like model deduced from M-theory.

Turning now to the theory of cosmic strings, it is well known that the quadrupole moment of a cosmic string is proportional to the square of the vacuum expectation value of the corresponding symmetry breaking,

\[
2 = \frac{3}{3}; \tag{97}
\]

where 50 ::::100 is a numerical coefficient depending on the trajectory and shape of the string loop and \( \lambda \) is the characteristic frequency of string oscillations [43]. Therefore a large gaugino vacuum expectation value would lead to a very large quadrupole moment for cosmic strings that are produced during the breaking of the hidden \( \mathbb{E}_8 \), which in turn will generate high-amplitude gravitational waves on the hidden brane. The described effect may easily compensate the damping factor [92] and lead to strong gravitational waves on the hidden brane and, as a consequence, also on the + -brane. These in turn can be distinguished from waves originating from the + -brane by detectable RIGO’s.

Note that the described mechanism is not spoiled by the dependence of the effective gravitational coupling constant on the volume of the Calabi-Yau space, \( G_5 \sim G_{11} \approx V \), because the dependence of the gaugino vacuum expectation value on the size of the Calabi-Yau space is much stronger than that of the effective five-dimensional gravitational coupling.

At first glance our result about a strongly enhanced symmetry-breaking vacuum expectation value on the hidden brane compared to the visible brane seems to be in sharp contrast to the conventional interpretation of the solution to the hierarchy problem in the RS-model (cf. e.g. [2, 49, 23]). This conventional interpretation states that the vacuum expectation value of a symmetry breaking on the hidden brane should be suppressed by two powers of the warp factor compared to the same value on the visible brane,

\[
\sim a^2 + \tag{98}
\]

However, already Randall and Sundrum pointed out that the generation of the hierarchy could equally well be described by considering the gravitational scale on the -brane as the derived scale, whereas
the symmetry breaking scale on both branes is to be taken the same \[2\]. That this interpretation is in fact the correct one has first been demonstrated in \[50\] by considering the fall-off properties of on-brane Green functions in the Euclidean domain and has been confirmed by an analysis of the measuring process for masses on the \(-\)-brane in \[51\]. Therefore, all effects of the hierarchy generation in the RS model are already contained in the damping factor \(94\) and there is no geometrical suppression of symmetry-breaking scales on the hidden brane in the RS model.

### 6.5 Graviton oscillations on the \(-\)-brane

Graviton oscillations on the \(-\)-brane can be treated along the same lines as RIGO’s on the \(+\)-brane have been treated in Secs. \(6.1\) and \(6.3\). Abandoning once again the matrix notation we find for \(h\) with the help of the spectral representation \(30\) the equations of motion

\[
\begin{align*}
\frac{16}{2} \frac{G_4}{r} (a^2 T^+ + a^4 T) \sum_{i=1}^6 \frac{16}{2} \frac{G_4}{m_i^2} (a^4 T^+ + a^2 T^+) ; \quad (99)
\end{align*}
\]

Considering the same set of sources \(74\) we now study the gravitational waves seen by an observer on the negative tension brane,

\[
\begin{align*}
\frac{16}{2} \frac{G_4}{r} (a^2 T^+ + a^4 T) \sum_{i=1}^6 \frac{16}{2} \frac{G_4}{m_i^2} (a^4 T^+ + a^2 T^+) ; \quad (100)
\end{align*}
\]

\[
\begin{align*}
\frac{16}{2} \frac{G_4}{r} (a^2 T^+ + a^4 T) \sum_{i=1}^6 \frac{16}{2} \frac{G_4}{m_i^2} (a^4 T^+ + a^2 T^+) ; \quad (101)
\end{align*}
\]

For simplicity we suppose again that only the massless and the first massive modes are generated, i.e. \(m_1 < \lambda < m_2\). Then, for the waves generated by the source on the \(+\)-brane and the \(-\)-brane we have, respectively,

\[
\begin{align*}
\frac{16}{2} \frac{G_4}{r} (a^2 T^+ + a^4 T) \sum_{i=1}^6 \frac{16}{2} \frac{G_4}{m_i^2} (a^4 T^+ + a^2 T^+) ; \quad (102)
\end{align*}
\]

\[
\begin{align*}
\frac{16}{2} \frac{G_4}{r} (a^2 T^+ + a^4 T) \sum_{i=1}^6 \frac{16}{2} \frac{G_4}{m_i^2} (a^4 T^+ + a^2 T^+) ; \quad (103)
\end{align*}
\]

where \(A = 4 G_4 \frac{r}{L}\) is the amplitude of the massless mode generated on the \(+\)-brane. The amplitudes detected by a gravitational-wave interferometer are given by the absolute values of \(102\) and \(103\),

\[
\begin{align*}
\frac{16}{2} \frac{G_4}{r} (a^2 T^+ + a^4 T) \sum_{i=1}^6 \frac{16}{2} \frac{G_4}{m_i^2} (a^4 T^+ + a^2 T^+) ; \quad (104)
\end{align*}
\]

\[
\begin{align*}
\frac{16}{2} \frac{G_4}{r} (a^2 T^+ + a^4 T) \sum_{i=1}^6 \frac{16}{2} \frac{G_4}{m_i^2} (a^4 T^+ + a^2 T^+) ; \quad (105)
\end{align*}
\]

Here \(L\), the oscillation length of the amplitude modulation of the gravitational wave, is given by \(88\). The pre-factors of the amplitudes \(104\) and \(105\) are

\[
\begin{align*}
A^+ &= \frac{1}{J_2} 1 \frac{a^2 A}{a} \frac{1}{1 + a^2} ; \quad (106)
A &= (1 + a^2) a^2 A \frac{a^2 A}{a} ; \quad (107)
\end{align*}
\]

Therefore, for the ratio of the typical intensities measured on the \(-\)-brane we have,

\[
\frac{A^+}{A} = \frac{1}{1 + a} ; \quad (108)
\]
For waves on the $-$brane we thus find that the intensities of waves produced on this brane and on the other brane are of the same order of magnitude. Therefore at first glance the situation of an observer on the $+$brane seems to be favorable for observing graviton oscillations.

Unfortunately, this effect is spoiled by the dependence of the graviton mass — and therefore of the oscillation length — on the position of the brane. In order to find the oscillation length actually measured by an observer on the brane, we have to use the coordinate system in which an observer does his measurements. The coordinates of an observer will always be chosen with respect to the Lorentz metric (cf. [51]). No rescaling is necessary for studying effects on the $+$-brane because there the induced metric coincides with the Minkowski metric. On the $-$brane the “physical” coordinates used by an observer are related to those given with respect to the induced metric $g = a^2$ by

$$x_{\text{phys}} = ax.$$  

(109)

In the equations of motion (18), the flat space d'Alembertian has to be replaced by the one with respect to the physical coordinates,

$$2 \frac{\partial}{\partial x} \frac{\partial}{\partial x} = a^2 \quad 2 \frac{\partial}{\partial x_{\text{phys}}} \frac{\partial}{\partial x_{\text{phys}}} = a^2 2 \frac{\partial}{\partial x_{\text{phys}}}.$$  

(110)

To be precise, we would also have to rescale the metric perturbation $h$ (with respect to the background metric $g$ ) to the measured one $h_{\text{phys}} = h = a^2 (y)$ (with respect to the Lorentz metric). We will not perform this step because we are comparing amplitudes measured on the same brane. Therefore this rescaling does not change the ratio between the amplitudes. In physical coordinates, the source-free equations of motion on the $-$brane are given by

$$2_{\text{phys}} h_{\text{phys}}^{\varphi} = 0; \quad 2_{\text{phys}} \frac{m_i^2}{a^2} h_{\text{phys}}^{\varphi} = 0;$$  

(111)

whereas the equations of motion on the $+$-brane remain unchanged. Therefore the graviton mass appears different when observed on a different brane,

$$m_i^+ = \frac{m_i}{a} = \frac{3}{1}; \quad m_i^- = \frac{m_i}{a} = \frac{1}{1};$$  

(112)

This rescaling is characteristic of bulk fields in the Randall-Sundrum two-brane model [51][50].

Whereas the measured graviton masses $m_i^+$ on the $+$-brane depend on the warp factor, the masses $m_i^-$ on the $-$brane are independent of the inter-brane distance. From these masses we can derive the oscillation length observed on the branes for a GW with a wave-number $\ell m$. In this limit we have

$$L = \frac{2}{m_i};$$  

(113)

On the $-$brane this becomes

$$L = \frac{2}{m_i^+} = \frac{1}{3};$$  

(114)

By means of this expression, the upper experimental limit on the AdS radius $l_\text{AdS} \approx 10^4 m$ also yields the upper limit on the oscillation length on the $-$brane. However, an oscillation length of sub-millimeter size is clearly unobservable. Therefore graviton oscillations will be hidden to an observer on the $-$brane.

To conclude, the situation is unfavorable for the detection of RIGO’s by an observer on the $-$brane: albeit being connected with strong GW’s, the observer cannot detect the oscillations because their oscillation length is far too short to exhibit itself in interferometric detectors.

---

7 Note that the rescaling does not apply to on-brane fields and therefore does not affect the symmetry breaking scale considered in Sec. 6.4 (cf. [51][50]).
7 RIGO’s in bi-gravity models

Bi-gravity models are braneworld setups in which the effective four-dimensional gravity on the brane which is considered as our Universe is mediated by one massless and one light massive gravitational mode which have approximately equally strong couplings to matter on the visible brane. The rest of the KK spectrum is separated by a mass gap and its coupling to matter on the visible brane is strongly suppressed. Meanwhile there are various realizations of these setups [8, 9], although only the six-dimensional ones do not seem to suffer from a VvDZ discontinuity for the massive mode or from a phenomenologically unrealistic negative curvature of the visible brane. For our consideration we do not need to consider a specific realization of these models but can content ourselves with the general structure of their equations of motion and the assumption that phenomenological constraints from the VvDZ discontinuity are somehow circumvented.

Then the four-dimensional effective equations of motion for the transverse-traceless sector of a generic bi-gravity model reads

\[ \mathbf{h}^{\text{vis}} = 16 \mathcal{G} \left( \frac{c_0}{2_{\text{ret}}} \right)^{m_1} + \frac{c_1}{2_{\text{ret}}} \left( \frac{X}{m_n} \right)^{2_{\text{ret}}} \mathbf{T}^{\text{vis}} ; \]  

(115)

where

\[ c_0, c_1, c_n, m_1, m_2, m_n = m_{n-1}, n 2; \]  

(116)

and

\[ m_n = m_{n-1}, n 2; \]  

(117)

we have only included sources \( \mathbf{T}^{\text{vis}} \) on the visible brane. The effective gravitational constant \( \mathcal{G} \) does not coincide with the usual effective four-dimensional gravitational constant. Rather we have

\[ G_4 \mathcal{G} (c_0 + c_1) ; \]  

(118)

From (116) we infer that we have a strong mixing between the massless mode and the first KK mode. This would make bi-gravity models a natural candidate for strong RIGO’s. However, this expectation is spoiled by phenomenological constraints on this class of models. The major experimental constraint is established by precision measurements of Newton’s law on the orbital motion of the planets. Our treatment will closely follow that of [53] where a purely massive gravitational interaction is considered. It is sufficient to consider only the influence of the first KK mode, since the heavier modes will only contribute to the short-range dynamics. The non-relativistic static gravitational potential \( V (r) = \frac{G_4 M}{r} \) then gets modified by a Yukawa potential to the form

\[ V (r) = \frac{\mathcal{G} M}{r} \left( c_0 + c_1 \exp \left( \frac{r}{1} \right) \right) ; \]  

(119)

where \( M \) is the mass of the central body. The gravitational acceleration of a test body in the modified potential is \( g = \mathbf{e}_r \cdot \nabla V (r) \) where \( \mathbf{e}_r \) is the unit-vector pointing from the central body to the test body and

\[ (r) = \frac{\mathcal{G} M}{r} \left[ c_0 + c_1 \exp \left( \frac{r}{1} \right) \right] ; \]  

(120)

For pure Newtonian gravity \( = G_4 M = \text{constant} \). Its value can be determined to a high precision from the orbit of the earth around the sun. If a massive gravitational mode contributes, the determined for the orbits of other planets will differ from the value of the earth orbit. For a planet with a semi-major
axis $a_p$ and a period $T_p$, Kepler’s third law yields 
\[ 1 = 3 \left( \frac{a_p}{a} \right)^3 = \left( \frac{2}{T_p} \right)^3. \]
Therefore it is convenient to parametrize the deviation of a planet’s 
\[ (a_p) \text{ from the one of the earth } (a) \text{ through a small parameter } \]
\[ p = 1 + \frac{(a_p)}{(a)} \quad (121) \]
Using (120) in (121), we obtain a lower bound on the wavelength of the first KK mode in terms of the 
experimental limits on $p$, 
\[ \frac{1}{12} \left( a_p \right)^{1=2} = \frac{a^2}{12} \quad (122) \]
Restricting to the case with $c_0 = q_1 = 1=2$, we find 
\[ 1 = \frac{a^2}{12} = \frac{a^2}{12} \quad (123) \]
The most restrictive bound on $p$ comes from the measurement of the Mars orbit for which we have 
\[ m < 6.5 \times 10^{-12} \text{ km } \quad (124) \]
for a realistic bi-gravity model. One might try to infer more stringent upper bounds on the mass of the first 
KK mode by considering the motions of galaxies and clusters of galaxies. However, these considerations 
rely heavily on the assumed amount of dark matter in the universe. In view of the fact that the cosmology 
of braneworld models is still in its infancy and can provide us with quite surprising models of dark matter 
(see e.g. [52]), we resist from using such constraints which would lead us to an upper mass for the first 
KK mode close to the bounds on the graviton mass given by the Particle Data Group, $m_g$. 
\[ m_g \approx (10^{19} \text{ km }) \quad (125) \]
However, even the less restrictive solar-system limit (124) on $m_1$ renders RIGO’s in bi-gravity models 
unobservable. Inserting (124) into the formula for the oscillation length (92), we find even for GW’s of a 
frequency of $10^{-4}$ Hz, the lowest frequency observable by LISA, an oscillation length of more than 300 
light years. An amplitude modulation of that length will clearly remain undetectable. The limit on the 
wavlength of the first KK mode does however still allow for the possibility of detecting ultralight KK 
modes by studying the propagation speed of gravitational waves with LISA (cf. [53]). 
In addition to the bounds on the length of RIGO’s from solar-system dynamics, it is interesting to note 
that some models even predict a minimal oscillation length beyond the present-day Hubble radius of our 
Universe. This is in particular the case in the Kogan-Moulopoulos-Papazoglou version [9] of the Karch-
Randall model [7] with two AdS$_4$ branes embedded in AdS$_5$. This model is particularly attractive because 
the longitudinal polarization of the KK gravitons is supressed in the propagator by the geometry of the 
setting. Therefore, the VvDZ discontinuity is absent [9][12]. 
In this model, the strongest constraint on the length of RIGO’s comes from the requirement that the 
observable cosmological constant of our Universe, i.e. the effective cosmological constant $\lambda$ of one of the 
AdS$_4$ branes, should be compatible with the observed magnitude of the cosmological constant, 
\[ j \lambda \leq 10^{-20} \]
where $M_{Pl}$ denotes the four-dimensional Planck mass. This requirement on the geometry of the model is 
reflected in the coefficients of the harmonics of the bulk spacetime and thereby restricts the mass of the 
first KK-mode $m_1$ to a fraction of the inverse size of the Hubble radius of our Universe, 
\[ m_1 = e^{135} = 64 \quad (126) \]
Using the expression for the lower limit of the oscillation length $L_{\text{min}}$, the value for the first KK mass leads to an oscillation length for RIGO's,

$$L_{\text{min}} \sim 1.35 H^{-1} H^{1};$$

which is much larger than the size of the horizon of our Universe and thus renders RIGO's unobservable in principle in this particular model.

8 Conclusions

We have studied the phenomenon of radion induced graviton oscillations in the two-brane world. First, we have established a method to extract the particle content from the nonlocal braneworld effective action of [26] for the Randall-Sundrum two-brane model. This method is easily extendable to arbitrary two-brane models and provides the missing link between nonlocal braneworld actions, which are specifically suited for the treatment of cosmological problems [26, 56, 57], and spectral representations of the effective action particularly suited to particle-phenomenology considerations [58, 51].

From the equations of motion for the graviton sector, we have found a mixing of massless and massive modes which depends parametrically on the radion. This mixing leads to the effect of radion induced graviton oscillations which is in principle observable. RIGO’s are a feature of every higher-dimensional spacetime model, since there will always occur amplitude modulations in GW's which are a mixture of a massless mode and KK modes. However, in traditional models with flat extra dimensions, the mass of the first KK mode is so big that it will neither be produced by astrophysical sources nor lead to oscillation lengths of macroscopic size. In contrast to this, warped geometries allow KK mode masses which are so low that they can lead to oscillations of detectable length. In particular, waves from sources on the hidden brane show strong oscillations on the visible brane. The amplitudes of these waves are strongly suppressed by the warped geometry and thus, at first sight, seem to remain undetectable. However, by using simple M-theory motivated scaling arguments, we have demonstrated that one should expect a network of cosmic strings on the hidden brane which would produce a background of high amplitude gravitational waves. The M-theory scaling properties considered may easily compensate the geometrical damping of GW’s from the hidden brane. Therefore GW’s with RIGO’s stemming from the hidden brane may actually become a relevant effect for gravitational-wave astronomy. The characteristic pattern of RIGO’s may even help to discriminate between noise and the stochastic GW background in GW interferometers.

At first sight another natural candidate for observable RIGO’s are bi-gravity models. In these models the massless graviton and the lightest massive one are coupled to matter with nearly equal strength and, thus, produce strong oscillations. Unfortunately, bi-gravity models which exhibit RIGO’s of experimentally detectable oscillation-length are already ruled out by precision measurements of Newton’s law on solar-system scales.

The effects connected with the mixing and oscillations of quantum states in a multidimensional spacetime have received some attention in the literature [59, 60, 61, 62]. Neutrino mixing and oscillations were reconsidered in [59, 60, 61], while the mixing between quarks and an attempt to explain the origin of CP violation was described in [62]. Nevertheless, to our knowledge, the possibility of oscillations between different graviton states and their potential observability in GW interferometers has not been considered before.

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A Diagonalization of the kinetic and mass terms

Here we describe in detail the diagonalization of the kinetic and massive operators and the calculation of the mixing parameters from Sec. 5. The procedure of the diagonalization of the matrices \( D \) and \( M \) will include three successive operations. First we shall find the eigenvalues \( \lambda_1, \lambda_2 \) of the operator \( D \) and shall construct an orthogonal matrix \( O \) which provides the transition from an old basis to the basis of normalized eigenvectors of the operator \( D \). Application of the transformation \( O \) to the matrix \( D \) transforms it to a diagonal matrix whose elements coincide with the eigenvalues of \( D \),

\[
O DO^T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} : \tag{128}
\]

Secondly, in order to transform this matrix into the unit matrix, we act on the eigenvectors by the generalized dilatation matrix 

\[
= \begin{pmatrix} p & 0 \\ 1 & 0 \\ 0 & p \end{pmatrix} : \tag{129}
\]

Of course, the action of the inverse operators \( O^{-1} \) on the matrix \( D \) yields the unit matrix,

\[
O DO^T O^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \tag{130}
\]

The simultaneous action of the operators \( O \) and \( O^{-1} \) on the matrix \( M \) transforms it into

\[
M' = OMO^T ; \tag{131}
\]

which is still degenerate. The non-zero eigenvalue of this matrix, which gives the squared mass of the massive graviton, is equal to the trace of this matrix,

\[
M^2 = \text{Tr}M' ; \tag{132}
\]

As a third step, in order to finally obtain an expression for the transformation from the initial gravitational modes \( h^+ \) and \( h^- \) to the new modes \( h_m \) and \( h_0 \) corresponding to the massive and massless gravitons, we should use another orthogonal rotation \( Q \), diagonalizing the matrix \( M' \) in such a way that its diagonal elements coincide with its eigenvalues, i.e.

\[
QM'Q^T = \begin{pmatrix} M^2 & 0 \\ 0 & 0 \end{pmatrix} ; \tag{133}
\]

Of course, the application of the orthogonal transformation \( Q \) to the transformed kinetic matrix \( D \) does not change it because it is a unit matrix.

Thus, one can formulate the transformation between the modes \( h^+ \) and \( h^- \) and the eigenmodes of the operator \( \mathcal{L} \) in the form

\[
h_m = Q O \begin{pmatrix} h^+ \\ h^- \end{pmatrix} ; \tag{134}
\]
The inverse transformation is
\[
\frac{h^*}{h} = \mathbf{O}^\top \mathbf{Q}^\top \frac{\mathbf{D}_{\text{et}}}{\mathbf{D}_{\text{et}}} : \tag{135}
\]

The transformations (134) and (135) are not orthogonal. This is connected with the fact that we had to apply the generalized dilatation matrix to normalize properly the kinetic part of the effective action.

The eigenvalue equation for the kinetic matrix \( \mathbf{D} \), defined in Eq. (63), has the following form,
\[
2 \left( a^4 + 6a^2 + 1 \right) + 3 \left( 2a^2 (a^2 + 1)^2 \right) = 0 ; \tag{136}
\]
where we have introduced the abbreviation \( a = (1 - a^2)^{1/2} = 3a^2 (1 + a^2)^{1/2} \). The solutions are
\[
1, 2 = \left( a^4 + 6a^2 + 1 \right) \frac{p}{2} a^8 + 14a^4 + 1 : \tag{137}
\]
Now, we substitute these eigenvalues into an eigenvector equation,
\[
(\mathbf{D} - 1, 2 \mathbf{I}) \mathbf{1}, 2 = 0 ; \tag{138}
\]
where \( \mathbf{I} \) is a unit matrix and \( \mathbf{1}, 2 \) are eigenvectors corresponding to the eigenvalues \( 1, 2 \), respectively. We choose the normalized eigenvectors
\[
1 = \frac{1}{N_1} \begin{pmatrix} D_{12} & D_{11} \end{pmatrix} ; \tag{139}
\]
\[
2 = \frac{1}{N_2} \begin{pmatrix} D_{12} & D_{11} \end{pmatrix} ; \tag{140}
\]
where \( D_{ij} \) are the corresponding elements of the matrix \( \mathbf{D} \), while the normalization factors are equal to
\[
N_1 = \sqrt{D_{12}^2 + (\mathbf{D}_{11} - \mathbf{D}_{11})^2} ; \tag{141}
N_2 = \sqrt{D_{12}^2 + (\mathbf{D}_{11} - \mathbf{D}_{11})^2} ;
\]
Correspondingly, the orthogonal matrix providing the transformation from the old basis \( 1, 2 \) (given by the expressions (139), (140)), has the form
\[
\mathbf{O} = \frac{6}{4} \begin{pmatrix} N_1 & N_2 \end{pmatrix} \begin{pmatrix} D_{12} & D_{11} \end{pmatrix} \begin{pmatrix} 1, 2 \end{pmatrix} : \tag{142}
\]
Using the expressions for the orthogonal matrix \( \mathbf{O} \) and for the generalized dilatation matrix \( \mathbf{D}_{\text{et}} \), one can get an expression for the rotated matrix \( \mathbf{M} \) by substituting Eqs. (142) and (129) into Eq. (131),
\[
\mathbf{M} = \mathbf{O} \mathbf{M} \mathbf{O}^\top = \frac{6}{4} \begin{pmatrix} \mathbf{M}^2 & \mathbf{M}^m \end{pmatrix} \begin{pmatrix} 1, 2 \end{pmatrix} \begin{pmatrix} 1, 2 \end{pmatrix} : \tag{143}
\]
where

\[ M' = a^2 D_{12} \frac{a^2}{N_1} \frac{1}{N_2}; \]  

\[ m' = a^2 (r - s) \frac{1}{N_1} + \frac{r + s}{N_2}; \]  

and

\[ r = \frac{1}{2}; \quad s = \frac{D_{11} D_{22}}{2}; \]

Now we are in a position to calculate the value of the mass of the massive graviton mode by substituting the formulas (143) – (146), together with formula (137) and the explicit values of the matrix elements \( D_{ij} \) from formula (61), into Eq. (132). Straightforward but rather cumbersome calculations lead us to the simple expression

\[ M'^2 = \frac{24 a^2 (1 + a^2)}{l^2 (l - a^2)^2}; \]

To accomplish the task of simultaneous diagonalization of the matrices \( D \) and \( M \), we should find the matrix \( Q \) rotating the matrix \( M' \) to the diagonal form (133). One can find normalized eigenvectors of the matrix \( M' \) by solving the equations

\[ M' \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0; \]

\[ M' \begin{pmatrix} 2 \end{pmatrix} = 0; \]

These eigenvectors have the form

\[ 1 = \frac{1}{N} \begin{pmatrix} p_{1M'} \\ p_{2M'} \end{pmatrix}; \]

\[ 2 = \frac{1}{N} \begin{pmatrix} p_{1m} \\ p_{2m} \end{pmatrix}; \]

where

\[ N = 2M'^2 + m'^2; \]

Correspondingly, the orthogonal matrix \( Q \) reads

\[ Q = \frac{1}{N} \begin{pmatrix} p_{1M'} & p_{1m} \\ p_{2M'} & p_{2m} \end{pmatrix}; \]

Substituting the matrices \( O \) and \( Q \), given by Eqs. (129), (129) and (151), into Eq. (134), we find the expressions (67), (68), describing the transformation from the old graviton modes \( h^+ \) and \( h^- \) to the modes \( h_M \) and \( h_0 \), which have already been given in Sec. 5. Analogously, one can also obtain the expressions (69), (70) presented in Sec. 5 and describing the inverse transition from the eigenmodes of the diagonalized Hamiltonian \( h_M \) and \( h_0 \) to the old graviton modes \( h^+ \) and \( h^- \).
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