The cost of approximate controllability for semilinear heat equations in one space dimension

Kim Dang Phung
17 rue Léonard Mafrand 92320 Châtillon, France
phung@cmla.ens-cachan.fr

Abstract: This note deals with the approximate controllability for the semilinear heat equation in one space dimension. Our aim is to provide an estimate of the cost of the control.

Keywords: Cost of approximate controllability, Semilinear heat equation.

1 Introduction and main result

In this paper, we apply a successful combination of three key tools which allows to get a measure of the cost of the approximate controllability for semilinear heat equation. The first tool consists to get enough information about the approximate control for the linear heat equation with a potential depending on space-time variable. Then a fixed point method is applied. The fixed point technique described here was previously used in [Z] to prove the exact controllability for semilinear wave equation in one dimension. The last tool, usually used for control problem (see [FCZ2, p.589] e.g.), consists to choose adequately the time of controllability.

Many results exist by now concerning the approximate controllability for semilinear heat equation in a bounded domain \( \Omega \subset \mathbb{R}^n, n \geq 1 \) when the control acts in a non-empty subdomain \( \omega \subset \Omega, \omega \neq \Omega \) (see [FPZ, IK] or [FCZ2] and references therein). In particular, it is proved in [FCZ2] that for any time \( T > 0 \), if the system

\[
\begin{aligned}
\partial_t u - \Delta u + f(u) &= h \cdot 1_{\omega} \quad \text{in } \Omega \times (0, T), \\
 u &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
 u(\cdot, 0) &= u_0 \quad \text{in } \Omega,
\end{aligned}
\]  

with \( f : \mathbb{R} \to \mathbb{R} \) locally lipschitz-continuous, admits at least one globally defined and bounded solution \( u^* \), corresponding to the data \( u_0^* \in L^2(\Omega) \) and \( h^* \in L^\infty(\omega \times (0, T)) \), and further if the function \( f \) satisfies

\[ |f'(s)| \leq c(1 + |s|^p) \quad \text{a.e., with } p \leq 1 + 4/n \quad \text{and } c > 0, \]

and

\[
\lim_{|s| \to \infty} \frac{f(s)}{|s| \ln^{3/2} (1 + |s|)} = 0,
\]

then for any \( u_0 \in L^2(\Omega), \ u_d \in L^2(\Omega) \) and \( \varepsilon > 0 \), there exists a control \( h \in L^\infty(\omega \times (0, T)) \) such that the solution of (1.1) is globally defined in \([0, T]\) and satisfies

\[ \| u(\cdot, T) - u_d \|_{L^2(\Omega)} \leq \varepsilon. \]
However, in [FCZ2], no information was given about a measure of the control with respect to \( \varepsilon \). In this paper, we provide an estimate of the control but under more restrictive hypothesis. Our result is

\[\text{Theorem .- Let } \Omega = (0,1) \text{ and } T > 0. \text{ Assume } f \in C^1(\mathbb{R}) \text{ and} \]

\[
\lim_{|s| \to \infty} \frac{f(s)}{|s| \sqrt{\ln (1 + |s|)}} = 0 ,
\]

then, for any \((u_0, u_d) \in H_0^1(\Omega) \times H_0^1(\Omega)\) and any \(\varepsilon \in (0, 1]\), there exist a control \(h_\varepsilon \in L^2(\omega \times (0,T))\) and a function \(u = u(x,t) \in L^\infty (\Omega \times (0,T))\) such that

\[
\|h_\varepsilon\|_{L^2(\omega \times (0,T))} \leq \exp\left\{ e^{C/\varepsilon} \right\} ,
\]

\[
\|u(\cdot, T) - u_d\|_{L^2(\Omega)} \leq \varepsilon ,
\]

and

\[
\begin{cases}
\partial_t u - \partial_{xx} u + f(u) = h_\varepsilon \cdot 1_\omega \quad \text{in } \Omega \times (0,T) , \\
u = 0 \quad \text{on } \partial \Omega \times (0,T) , \\
u \big(\cdot,0\big) = u_0 \quad \text{in } \Omega .
\end{cases}
\]

Here, \(C\) is a positive constant independent on \(\varepsilon\).

Remark .- Notice that we do not assume \(f(0) = 0\). If \(f(0) = 0\) (which correspond to the case \(u^* = 0\)), we can use the following control strategy to provide an estimate of the control when \(u_0 \in L^2(\Omega)\): we divide the time interval \((0, T)\) in two subintervals. During the first time interval \((0, T/2)\), we use a null control to steer the semilinear heat equation starting from \(u_0\) to zero (see [FCZ2]). In the second time interval \((T/2, T)\), we apply the above Theorem with null initial data.

The rest of this note is devoted to the proof of Theorem.

### 2 Proof of Theorem

We proceed in three steps.

**Step 1 .- Preliminary on the cost of the approximate controllability for the linear heat equation with a potential.** We first recall some results from [P] concerning the cost of the approximate controllability for the heat equation with a potential \(a = a(x,t) \in L^\infty (\Omega \times (0,T))\). We denote \(\|a\|_\infty = \|a\|_{L^\infty (\Omega \times (0,T))}\). In the sequel, \(c_1 > 1\) and \(c_2 > 1\) are two constants only depending on \(\Omega\) and \(\omega\). Let \(T' \in (0,T]\) called time of controllability of the linear system. We introduce the operator \(C\) given by

\[
C : \vartheta \in L^2 (\omega \times (0,T')) \longrightarrow w(\cdot, 0) \in L^2 (\Omega) ,
\]

where \(w \in C\left([0,T'] ; H_0^1 (\Omega) \right) \cap W^{1,2} (0,T ; L^2 (\Omega))\) is the solution of

\[
\begin{cases}
-\partial_t w - \Delta w + aw = E \vartheta \cdot 1_\omega \quad \text{in } \Omega \times (0,T') , \\
w = 0 \quad \text{on } \partial \Omega \times (0,T') , \\
w(\cdot,T') = 0 \quad \text{in } \Omega ,
\end{cases}
\]

with \(a \in L^\infty (\Omega \times (0,T))\) and \(E = \exp\left\{ c_2 \left( 1 + T' \|a\|_\infty \left( 1 + e^{c_2 T' \|a\|_\infty^2} + \|a\|_\infty^{2/3} \right) \right) \right\} .\) We define \(\mathcal{F} = \text{Im } C\) the space of exact controllability initial data with the following norm:

\[
\|w_0\|_{\mathcal{F}} = \inf \left\{ \|\vartheta\|_{L^2(\omega \times (0,T'))} \ \big| \ C \vartheta = w_0 \right\} .
\]
Denote $C^*$ the adjoint of $C$. It has been proved (see \[T]\) that the operator $B = CC^*$ is non-negative, compact and self-adjoint on $L^2(\Omega)$ which allows us to associate the Hilbert basis with eigenfunctions $\xi_n$ of $B$ and eigenvalues $\mu_n > 0$ where $\mu_n$ is non-increasing and tends to zero. Furthermore, let the sets $S_n = \{m > 0 / \alpha_n < \mu_m \leq \alpha_n\}$ where

$$\alpha_n = e^{\mu_n + e^{-n}},$$  \hspace{1cm} (2.2)

for all $n > 0$, then each function $\phi \in L^2(\Omega)$ can be represented in the form $\phi = \sum_{n \geq 0} \phi_n$ where $\phi_n = \sum_{m \in S_n} \langle \phi, \xi_m \rangle \xi_m$. Finally, let $N > 0$ and $z \in H^1_0(\Omega)$, then we can write, in $L^2(\Omega)$:

$$z = \sum_{n \leq N} z_n + \sum_{n > N} z_n \text{ with } z_n = \sum_{m \in S_n} \langle z, \xi_m \rangle \xi_m,$$

with the properties

$$\left\| \sum_{n \leq N} z_n \right\|_{L^2(\Omega)} \leq c_3 \frac{1}{\sqrt{N+1}} \| z \|_{L^2(\Omega)} ,$$

$$\left\| \sum_{n \leq N} z_n - z \right\|_{L^2(\Omega)} \leq c_3 \frac{D}{\ln(2+\sqrt{N+1})} \| z \|_{H^1_0(\Omega)} ,$$ \hspace{1cm} (2.3)

for some constant $c_3 > 0$ independent on $N$, $z$, $T$ and $a$ and where $D = c_1 \left( T' e^{c_1 T'} a, \| a \|_{\infty}^2 + \frac{1}{\omega} \right) > 1$ (see \[T]\). Here, $\sum_{n \leq N} z_n \in F$ and precisely

$$\sum_{n \leq N} z_n = \sum_{n \leq N} \sum_{m \in S_n} \langle z, \xi_m \rangle \xi_m = C \left( \sum_{n \leq N} \sum_{m \in S_n} \langle z, \xi_m \rangle \frac{1}{\mu_m} C^* \xi_m \right).$$

On another hand, let $\chi \cdot 1_{\omega}$ be the null-control function which steers to zero at time $T'$ the solution of the heat equation with potential $a(x, T' - t)$ and initial data $\pi_0 \in L^2(\Omega)$. It is known (see \[FCZ1\]) that

$$\| \chi \|_{L^2(\omega \times (0,T'))} \leq G \| \pi_0 \|_{L^2(\Omega)},$$ \hspace{1cm} (2.4)

where $G = \exp \left( c_0 \left( 1 + \frac{1}{T'} + T' \| a \|_{\infty} + \| a \|_{\infty}^{2/3} \right) \right)$ for some constant $c_0 > 0$ only depending on $\Omega$ and $\omega$.

Therefore, for all $T' \in (0,T]$, $a \in L^\infty(\Omega \times (0,T))$, $\pi_0 \in L^2(\Omega)$, $z \in H^1_0(\Omega)$, if we choose $\ell(x, T' - t) = E \sum_{n \leq N} \sum_{m \in S_n} \langle z, \xi_m \rangle \frac{1}{\mu_m} C^* \xi_m$

then from \[T, P, Q, R\] and \[T\], the solution $v_1 \in C \left( [0, T'] ; H^1_0(\Omega) \right) \cap W^{1,2} \left( 0, T' ; L^2(\Omega) \right)$ of

$$\begin{cases}
\partial_t v_1 - \Delta v_1 + a(x, T' - t) v_1 = (\chi + \ell) \cdot 1_{\omega} & \text{in } \Omega \times (0, T'), \\
v_1 = 0 & \text{on } \partial \Omega \times (0, T'), \\
v_1 (\cdot, 0) = \pi_0 & \text{in } \Omega,
\end{cases}$$

satisfies

$$\| v_1 (\cdot, T') - z \|_{L^2(\Omega)} \leq c_4 D e^{-N} \| z \|_{H^1_0(\Omega)} ,$$ \hspace{1cm} (2.5)

and moreover

$$\| \chi + \ell \|_{L^2(\omega \times (0,T'))} \leq G \| \pi_0 \|_{L^2(\Omega)} + c_4 D e^{c_4 T'} \| z \|_{L^2(\Omega)} ,$$ \hspace{1cm} (2.6)

for any $N \geq N_o$ where $N_o > 0$ and $c_4 \geq e^{N_o}$. Clearly, the approximate-control function $\ell$ depends on $N$, $z$ and $a$ coming from $E$ and the Hilbert basis $(\xi_n, \mu_n)$.

Next, let us introduce the operator $S$ given by

$$S : \lambda \in \mathbb{R} \rightarrow v_2 (\cdot, T') \in H^1_0(\Omega),$$
where $v_2 \in C \left([0, T'] ; H^1_0 (\Omega) \cap W^{1,2} (0, T' ; L^2 (\Omega)) \right)$ is the unique solution of

$$
\begin{cases}
\partial_t v_2 - \Delta v_2 + a (x, T' - t) v_2 = \lambda \text{ in } \Omega \times (0, T') , \\
v_2 = 0 \text{ on } \partial \Omega \times (0, T') , \\
v_2 (\cdot, 0) = 0 \text{ in } \Omega ,
\end{cases}
$$

One can easily check that

$$
\| S (\lambda) \|_{H^1_0 (\Omega)} = \| \nabla v_2 (\cdot, T') \|_{L^2 (\Omega)} \leq | \lambda | \sqrt{T'} e^{c_5 T' |a|_\infty^2} ,
$$

for some constant $c_5 > 0$ only depending on $\Omega$ and $\omega$.

Consequently, for all $T' \in (0, T]$, $a \in L^\infty (\Omega \times (0, T))$, $\pi_o \in L^2 (\Omega)$, $z_d \in H^1_0 (\Omega)$, if we choose $z = z_d - S (\lambda)$ the solution $v_3 = v_1 + v_2 \in C \left([0, T'] ; H^1_0 (\Omega) \cap W^{1,2} (0, T' ; L^2 (\Omega)) \right)$ of

$$
\begin{cases}
\partial_t v_3 - \Delta v_3 + a (x, T' - t) v_3 = \lambda + (\chi + \ell) \cdot 1_\omega \text{ in } \Omega \times (0, T') , \\
v_3 = 0 \text{ on } \partial \Omega \times (0, T') , \\
v_3 (\cdot, 0) = \pi_o \text{ in } \Omega ,
\end{cases}
$$

satisfies, taking into account (2.6), (2.7) and (2.8),

$$
\| v_3 (\cdot, T') - z_d \|_{L^2 (\Omega)} \leq c_4 D e^{-N} \left( \| z_d \|_{H^1_0 (\Omega)} + | \lambda | \sqrt{T'} e^{c_5 T' |a|_\infty^2} \right) ,
$$

and

$$
\| \chi + \ell \|_{L^2 (\omega \times (0, T'))} \leq G (\pi_o) + c_4 E e^{c_5} \left( \| z_d \|_{L^2 (\Omega)} + | \lambda | \sqrt{T'} e^{c_5 T' |a|_\infty^2} \right) .
$$

Finally, let $q \in L^\infty (\Omega \times (0, T))$. Now, we conclude with the construction of a solution $v$ of the heat equation with a potential and a second member and with a control acting on the interval $(T - T', T)$. Precisely, we divide the time interval $(0, T)$ in two subintervals. During the first time interval $(0, T - T')$, we let the system

$$
\begin{cases}
\partial_t v - \Delta v + q v = \lambda \text{ in } \Omega \times (0, T - T') , \\
v = 0 \text{ on } \partial \Omega \times (0, T - T') , \\
v (\cdot, 0) = u_o \text{ in } \Omega ,
\end{cases}
$$

to evolve freely without control. In the second time interval $(T - T', T)$, we choose $a (\cdot, t) = q (\cdot, T - t)$, $\pi_o = v (\cdot, T - T')$ and the control function such that

$$
\begin{cases}
\partial_t v - \Delta v + q v = \lambda + [(\chi + \ell) (x, T' - T + t)] \cdot 1_{\omega \times (T - T', T)} \text{ in } \Omega \times (0, T) , \\
v = 0 \text{ on } \partial \Omega \times (0, T) , \\
v (\cdot, 0) = u_o \text{ in } \Omega ,
\end{cases}
$$

satisfies

$$
\| v (\cdot, T) - z_d \|_{L^2 (\Omega)} \leq c_4 D e^{-N} \left( \| z_d \|_{H^1_0 (\Omega)} + | \lambda | \sqrt{T'} e^{c_5 T' |q|_\infty^2} \right) ,
$$

and moreover

$$
\| \chi + \ell \|_{L^2 (\omega \times (0, T'))} \leq G (v (\cdot, T - T')) + c_4 E e^{c_5} \left( \| z_d \|_{L^2 (\Omega)} + | \lambda | \sqrt{T'} e^{c_5 T' |q|_\infty^2} \right) ,
$$

for any $N \geq N_o$ where $N_o > 0$ and $c_4 \geq e^{N_o}$. Notice that one can easily check that

$$
\| v (\cdot, T - T') \|_{L^2 (\Omega)} \leq e^{c_6 T' |q|_\infty^2} \left( \| u_o \|_{L^2 (\Omega)} + c_6 | \lambda | \sqrt{T} \right) ,
$$

for some constant $c_6 > 0$ only depending on $\Omega$ and $\omega$.

Choosing

$$
N \leq \ln \left( c_4 D e^{\frac{1 + \varepsilon}{\varepsilon}} \left( 1 + \| z_d \|_{H^1_0 (\Omega)} + | \lambda | \sqrt{T'} e^{c_5 T' |q|_\infty^2} \right) \right) < N + 1
$$

for some $c_4 > 0$ and $\varepsilon > 0$. The proof is complete.
then one has
\[ \| u ( \cdot, T ) - z_d \|_{L^2(\Omega)} \leq \varepsilon , \]
and moreover,
\[
\| x + \ell \|_{L^2(\omega \times (0, T'))} \leq Ge^{c\varepsilon T} \|q\|^2 \left( \| u_0 \|_{L^2(\Omega)} + c_6 |\lambda| \sqrt{T} \right) + c_4 E \exp \left( q \frac{1 + \| z_d \|_{H^1_0(\Omega)} + |\lambda| \sqrt{T} \varepsilon^{c_5 T'\|q\|^2}}{e} \right) .
\]

Step 2 .- Introduction of \( g \) and choice of \( T' \). We begin to fix \( \varepsilon \in (0, 1) \) and \((u_0, u_d) \in H^1_0(\Omega) \times H^1_0(\Omega) \). Next, we introduce
\[
g (s) = \begin{cases} \frac{f(s) - f(0)}{f'(0)} & \text{for } s \neq 0 \\ 0 & \text{at } s = 0 \end{cases}
\]
which satisfies, from our hypothesis on \( f \), the following assertion
\[
\forall \delta > 0 \quad \exists C_{\delta} > 0 \quad \forall s \in \mathbb{R} \quad |g(s)| \leq C_{\delta} + \delta \sqrt{\ln (1 + |s|)} ,
\]
and consequently, for any \( u \in L^\infty(\Omega \times (0, T)) \), \( g(u) \in L^\infty(\Omega \times (0, T)) \) and one has
\[
\forall \delta > 0 \quad \exists C_{\delta} > 0 \quad \|g(u)\|_\infty \leq C_{\delta} + \delta \sqrt{\ln (1 + u})_\infty .
\]
Hence, we easily deduce that
\[
\forall \delta > 0 \quad \exists C_{\delta} > 0 \quad \exp \left( \frac{1}{\delta} \|g(u)\|_\infty^2 \right) \leq C_{\delta} + \|u\|_\infty . \quad (2.8)
\]
Now, we take \( T' \in (0, T] \) depending on \( \varepsilon \) and \( \|g(u)\|_\infty \) as follows
\[
T' = \begin{cases} T - \varepsilon \|g(u)\|_\infty & \text{if } \varepsilon \|g(u)\|_\infty^2 \leq 1 \\ \varepsilon \|g(u)\|_\infty & \text{if } \varepsilon \|g(u)\|_\infty^2 > 1 \end{cases} \quad (2.9)
\]

Step 3 .- The fixed point method thanks to the homotopy invariance of the Leray-Schauder degree. In order to prove Theorem, we will apply the homotopical version of the Leray-Schauder fixed point theorem.

Theorem (Leray-Schauder) .- Let \( E \) be a Banach space and \( H : E \times [0, 1] \to E \) be a compact continuous mapping such that \( H(u, 0) = 0 \) for every \( u \in E \). If there exists a constant \( K \) such that \( \|u\|_E < K \) for every pair \((u, \sigma) \in E \times [0, 1]\) satisfying \( u = H(u, \sigma) \), then the mapping \( H(\cdot, 1) : E \to E \) has a fixed point.

We introduce the following mapping \( H \)
\[
H : (u, \sigma) \in L^\infty(\Omega \times (0, T)) \times [0, 1] \to \sigma y \in L^\infty(\Omega \times (0, T))
\]
where \( y \in C ([0, T]; H^1_0(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)) \) is the solution of
\[
\begin{cases}
\partial_t y - \Delta y + \sigma g(u) y = -\sigma f(0) + h \cdot 1_{\omega} & \text{in } \Omega \times (0, T) , \\
y = 0 & \text{on } \partial \Omega \times (0, T) , \\
y(\cdot, 0) = u_0 & \text{in } \Omega ,
\end{cases}
\]

}\}

\[
\begin{cases}
\partial_t y - \Delta y + \sigma g(u) y = -\sigma f(0) + h \cdot 1_{\omega} & \text{in } \Omega \times (0, T) , \\
y = 0 & \text{on } \partial \Omega \times (0, T) , \\
y(\cdot, 0) = u_0 & \text{in } \Omega ,
\end{cases}
\]
when the control function \( h \) depends on \((u, \sigma)\) as follows: from \( q = \sigma g (u) \in L^\infty (\Omega \times (0, T)) \), we take \( a (\cdot, t) = q (\cdot, T - t) \) and generate the eigencouple \((\xi_n, \mu_n)\), next we choose the control function

\[
h (x, T - t) = \begin{cases} 0 & \text{for } T' \leq t < T \\ \chi (x, T' - t) + E \sum_{n \leq N} \sum_{m \in S_n} (u_d - S (-\sigma f (0), \xi_n) \frac{1}{\mu_n} C^* \xi_m) & \text{for } 0 < t < T'
\end{cases}
\]

where \( N \geq N_0 \) is such that \( N \leq \ln \left( c_4 D e^{\frac{1 + T'}{2} \left( 1 + \| u_d \|_{L^2 (\Omega)} + \| \sigma g (u) \|_{\infty} \right) + \| \sigma g (u) \|_2^2 / 3 \right) \).

with

\[
\begin{align*}
G &= \exp \left( c_0 \left( 1 + \frac{1}{T'} + T' \| \sigma g (u) \|_{\infty} + \| \sigma g (u) \|_{2/3} / 2 \right) \right), \\
D &= c_1 \left( T' e^{c T' \| \sigma g (u) \|_{2/3}^2} \right) + \frac{1}{T'} > 1, \\
E &= \exp \left( c_2 \left( 1 + T' \| \sigma g (u) \|_{\infty} + \| \sigma g (u) \|_{2/3} / 2 \right) \right).
\end{align*}
\]

Clearly, the control function \( h \) depends on \( \varepsilon, u, u_d \) and \((u, \sigma)\) coming from \( E \) and the eigencouple \((\xi_n, \mu_n)\).

From now, we use the letter \( c \) to denote a positive constant only depending on \( \Omega \) and \( \omega \), whose value can change from line to line. From (2.10) and (2.11), the control function is bounded as follows:

\[
\| h \|_{L^2 (\omega \times (0, T))} \leq \left( \| u_o \|_{L^2 (\Omega)} + \| \sigma f (0) \| \sqrt{T'} \right) \exp \left( c \left( 1 + T' \| \sigma g (u) \|_{\infty}^2 + \frac{1}{T'} + T' \| \sigma g (u) \|_{\infty} + \| \sigma g (u) \|_{2/3} / 2 \right) \right) \]

and therefore

\[
\| h \|_{L^2 (\omega \times (0, T))} \leq \left( \| u_o \|_{L^2 (\Omega)} + \| \sigma f (0) \| \sqrt{T'} \right) \exp \left( c \left( 1 + T' \| \sigma g (u) \|_{\infty}^2 + \sqrt{T'} \| \sigma g (u) \|_{\infty} + \| \sigma g (u) \|_{2/3} / 2 \right) \right) \]

\[
\exp \left( \frac{1}{T} \left( 1 + \| u_d \|_{H^1_0 (\Omega)} + \| \sigma f (0) \|^2 \right) \right) e^{T' \| \sigma g (u) \|_{2/3}^2} \right) \right).
\]

The continuity and compactness property of \( H \) comes from the following embedding

\[
W^{1,2} (0, T; L^2 (\Omega)) \cap L^\infty (0, T; H^1_0 (\Omega)) \subset L^\infty (\Omega \times (0, T))
\]

which is compact in one dimension of space. It remains to prove that

\[
\| u \|_{\infty} < K,
\]

for every pair \((u, \sigma) \in L^\infty (\Omega \times (0, T)) \times [0, 1] \) satisfying \( u = H (u, \sigma) \).
The solution \( u \) of the nonlinear system \( H(u, \sigma) = u \) is also solution of the linear system

\[
\begin{align*}
\partial_t \psi - \partial_{xx} \psi + q(x, t) \psi &= b(x, t) \quad \text{in } \Omega \times (0, T), \\
\psi &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
\psi(\cdot, 0) &= \sigma u_o \quad \text{in } \Omega,
\end{align*}
\]

by substituting \( q = \sigma^2 g(u) \) and \( b = \sigma (-\sigma f(0) + h \cdot \mathbf{1}_\omega) \). But such solution \( \psi \) satisfies, in one space dimension, the following inequality

\[
\|\psi\|_{L^\infty(0, T)}^2 \leq c e^{cT\|g\|_{L^\infty(0, T)}^2} (\|\sigma u_o \|_{H^1_0(\Omega)}^2 + \|b\|_{L^2(\Omega \times (0, T))}^2).
\]

Consequently, the later inequality and (2.13) imply that

\[
\|u\|_{L^2(\Omega \times (0, T))}^2 \leq c e^{cT\|g(u)\|_{L^\infty(0, T)}^2} (\|u_o\|_{H^1_0(\Omega)}^2 + \|u_d\|_{L^2(\Omega)}^2 + |f(0)| T^2)
\]

\[
\cdot \exp \left( c \left( 1 + T^2 + T \left( \|u_d\|_{H^1_0(\Omega)} + |f(0)|^2 \right) + \|g(u)\|_{L^\infty(0, T)} \right) \right)
\]

which gives

\[
\|u\|_{L^2(\Omega \times (0, T))}^2 \leq \left( \|u_o\|_{H^1_0(\Omega)}^2 + \|u_d\|_{L^2(\Omega)}^2 + |f(0)| T^2 \right) \exp \left( C_T \left( 1 + \|u_d\|_{H^1_0(\Omega)} + |f(0)|^2 \right) e^{cT/\varepsilon} \right),
\]

where \( C_T > 0 \) is a constant only depending on \( T, \Omega \) and \( \omega \).

Now if \( \varepsilon \|g(u)\|_{L^\infty(0, T)}^2 > 1 \) then by the choice of \( T' \) given by (2.40), we have

\[
\|u\|_{L^2(\Omega \times (0, T))}^2 \leq \left( \|u_o\|_{H^1_0(\Omega)}^2 + \|u_d\|_{L^2(\Omega)}^2 + |f(0)| T^2 \right)
\]

\[
\cdot \exp \left( c \left( 1 + T^2 + T \left( \|u_d\|_{H^1_0(\Omega)} + |f(0)|^2 \right) + \|g(u)\|_{L^\infty(0, T)} \right) \right)
\]

which gives

\[
\|u\|_{L^2(\Omega \times (0, T))}^2 \leq \left( \|u_o\|_{H^1_0(\Omega)}^2 + \|u_d\|_{L^2(\Omega)}^2 + |f(0)| T^2 \right)
\]

\[
\cdot \exp \left( c \left( 1 + T^2 + T \left( \|u_d\|_{H^1_0(\Omega)} + |f(0)|^2 \right) \right) \right)
\]

and finally, using (2.48), there exists a constant \( C' > 0 \) only depending on \( \left( \|u_d\|_{H^1_0(\Omega)} + |f(0)|^2 \right) \), \( T, \Omega \) and \( \omega \) such that

\[
\|u\|_{L^2(\Omega \times (0, T))}^2 \leq \left( \|u_o\|_{H^1_0(\Omega)}^2 + \|u_d\|_{L^2(\Omega)}^2 + |f(0)| T^2 \right) \exp \left( C_T \left( 1 + \|u_d\|_{H^1_0(\Omega)} + |f(0)|^2 \right) e^{cT/\varepsilon} \right) (C' + \|u\|_{L^\infty(\Omega \times (0, T))}^2).
\]
where $C_T > 0$ is a constant only dependent on $T$, $\Omega$ and $\omega$.

We conclude that any solution $(u, \sigma) \in L^\infty(\Omega \times (0, T)) \times [0, 1]$ of $u = H(u, \sigma)$ satisfies the following estimate: there is a constant $C > 0$ independent of $(u, \sigma)$ such that for any $\varepsilon \in (0, 1]$,  
\[ \|u\|_\infty^2 \leq \exp\left(\frac{C}{\varepsilon}\right), \]
which allows us to get to the existence of a fixed point for $H(\cdot, 1)$. Furthermore, by (2.13), the control is then bounded as follows: for any $\varepsilon \in (0, 1]$,  
\[ \|h\|_{L^2(\omega \times (0, T))} \leq \exp\left(\frac{C}{\varepsilon}\right). \]

This completes the proof.

Remark. Notice that the measure of the cost of the control of the semilinear heat equation (1.2) can be improved and become of order $e^{C/\varepsilon^2}$ by adding the following more restrictive hypothesis $f(0) = 0$ and  
\[ \lim_{|s| \to \infty} \frac{f(s)}{|s| \sqrt{\ln \ln (1 + |s|)}} = 0. \]
Indeed, the minimization of the second member of (2.12) with respect to the quantity $\|g(u)\|_\infty$ suggests us our choice (2.9) of the time of controllability $T'$. But the minimization of the second member of (2.12) when $f(0) = 0$ with respect to $\varepsilon \in (0, 1]$, suggests to take $T' = \varepsilon T$ in order to get an estimate of the cost of order $e^{C/\varepsilon^2}$.

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