Counting on the variety of modules over the quantum plane

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Background

Given a field \( F \) and \( n \geq 0 \), define the \( n \)-th \textbf{commuting variety} over \( F \) as

\[
K_{1,n}(F) := \{(A, B) \in \text{Mat}_n(F) \times \text{Mat}_n(F) : AB = BA\}.
\]

(The meaning of the notation will be clear later.)

What’s known:

- When \( F = \mathbb{C} \), the commuting variety \( K_{1,n}(\mathbb{C}) \) is a complex algebraic variety. Motzkin and Taussky (1955) and Gerstenhaber (1961) showed that \( K_{1,n}(\mathbb{C}) \) is irreducible.

- When \( F = \mathbb{F}_q \), the finite field of \( q \) elements, the set \( K_{1,n}(\mathbb{F}_q) \) is a finite set. Feit and Fine (1960) gave its cardinality by the formula:

\[
\sum_{n=0}^{\infty} \frac{|K_{1,n}(\mathbb{F}_q)|}{(q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})} x^n = \prod_{i,j \geq 1} \frac{1}{1 - x^i q^{2-j}}. \tag{1}
\]
We now consider a quantum deformation of the commuting variety. Let $\zeta$ be a nonzero element of $\mathbb{F}$, define the $n$-th $\zeta$-commuting variety as

$$K_{\zeta,n}(\mathbb{F}) := \{(A, B) \in \text{Mat}_n(\mathbb{F}) \times \text{Mat}_n(\mathbb{F}) : AB = \zeta BA\}.$$ 

If $\zeta = 1$, then it simply becomes the commuting variety, hence the notation $K_{1,n}$ for the commuting variety.

Efforts have been spent to extend the work of Motzkin, Taussky and Gerstenhaber to the $\zeta$-commuting variety:

- Chen and Wang (2018) described the irreducible components of the anti-commuting variety $K_{-1,n}(\mathbb{C})$. There are more than one, unlike the $\zeta = 1$ case.
- Chen and Lu (2019) further extended the above result to general $\zeta$. 
Main result

We give a direct generalization of Feit–Fine’s formula.

Main Theorem (H., 2021).

Let \( \zeta \) be a nonzero element of \( \mathbb{F}_q \), and let \( m \) be the smallest positive integer such that \( \zeta^m = 1 \). Then

\[
\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{(q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})} x^n = \prod_{i=1}^{\infty} F_m(x^i; q),
\]

where

\[
F_m(x; q) := \frac{1 - x^m}{(1 - x)(1 - x^m q)} \cdot \frac{1}{(1 - x)(1 - x q^{-1})(1 - x q^{-2})\cdots}.
\]

When \( \zeta = 1 \), we have \( m = 1 \), so \( F_1(x^i; q) = \prod_{j \geq 1} \frac{1}{1 - x^i q^{2-j}} \) and we recover Feit–Fine.
The commuting variety $K_{1,n}(\mathbb{F})$ parametrizes and classifies finite-$\mathbb{F}$-dimensional modules over the polynomial ring $\mathbb{F}[X, Y]$. So $K_{1,n}(\mathbb{F})$ is also called the **variety of modules** over $\mathbb{F}[X, Y]$. To specify an $\mathbb{F}[X, Y]$-module with underlying space $\mathbb{F}^n$, it suffices to specify the $x$-action $A : \mathbb{F}^n \to \mathbb{F}^n$ and the $y$-action $B : \mathbb{F}^n \to \mathbb{F}^n$ under the constraint $AB = BA$. This constraint is because $x$ and $y$ commmute in $\mathbb{F}[X, Y]$.

Similarly, the $\zeta$-commuting variety parametrizes finite-$\mathbb{F}$-dimensional modules over the associative algebra $\mathbb{F}\{X, Y\}/(XY - \zeta YX)$. This algebra is called the **quantum plane**, and is considered as a quantum deformation of $\mathbb{F}[X, Y]$. 
Remarks on Main Theorem

\[
\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{(q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})} x^n = \prod_{i=1}^{\infty} F_m(x^i; q),
\]

\[
F_m(x; q) := \frac{1 - x^m}{(1-x)(1-x^m q)} \cdot \frac{1}{(1-x)(1-xq^{-1})(1-xq^{-2}) \ldots}.
\]

- The cardinality of \( K_{\zeta,n}(\mathbb{F}_q) \) depends only on the order of \( \zeta \) as a root of unity of \( \mathbb{F}_q \). This is expected.
- The denominator \((q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})\) is precisely the size of \( \text{GL}_n(\mathbb{F}_q) \). This is the natural denominator in this type of generating function. In fact, the coefficient \( |K_{\zeta,n}(\mathbb{F}_q)|/|\text{GL}_n(\mathbb{F}_q)| \) is the number of \( n \)-dimensional modules over the quantum plane up to isomorphism, each measured with a weight of \( 1/(\text{size of automorphism group}) \).
- Bavula (1997) classified simple modules over the quantum plane; Main Theorem should encode some statistical information about this classification.
Further breakdown

We now state a refinement of Main Theorem. Let

\[ U_{\zeta,n}(\mathbb{F}_q) := \{(A, B) \in \text{GL}_n(\mathbb{F}_q) \times \text{Mat}_n(\mathbb{F}_q) : AB = \zeta BA\}, \]

and

\[ N_{\zeta,n}(\mathbb{F}_q) := \{(A, B) \in \text{Nilp}_n(\mathbb{F}_q) \times \text{Mat}_n(\mathbb{F}_q) : AB = \zeta BA\}. \]

It turns out that the varieties \( U_{\zeta,n}(\mathbb{F}_q) \) and \( N_{\zeta,n}(\mathbb{F}_q) \) are building blocks of \( K_{\zeta,n}(\mathbb{F}_q) \), in the sense that

\[
\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \left( \sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n \right) \left( \sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n \right)
\]

Recall that the left-hand side is the content of Main Theorem.
Refined Theorem (H., 2021)

Let $m$ be the order of $\zeta$. Then

$$
\sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|GL_n(\mathbb{F}_q)|} x^n = \prod_{i=1}^{\infty} G_m(x^i; q),
$$

where

$$
G_m(x; q) := \frac{1 - x^m}{(1 - x)(1 - x^m q)}.
$$

$$
\sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|GL_n(\mathbb{F}_q)|} x^n = \prod_{i=1}^{\infty} H(x^i; q),
$$

where

$$
H(x; q) := \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2}) \ldots}.
$$
Remarks on Refined Theorem

Refined Theorem can be interpreted as that in the formula

\[ F_m(x; q) := \frac{1 - x^m}{(1 - x)(1 - xmq)} \cdot \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2})\ldots} \]

related to the count of \( \{(A, B) : AB = \zeta BA\} \), the factor

\[ \frac{1 - x^m}{(1 - x)(1 - xmq)} \]

is the contribution of invertible \( A \), while the factor

\[ \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2})\ldots} \]

is the contribution of nilpotent \( A \).

Note that the latter does not depend on \( m \), so \( |N_{\zeta,n}(\mathbb{F}_q)| \) does not depend on \( \zeta \).
Ideas of proof: decomposition

- Given $A, B \in \text{Mat}_n(F_q)$ such that $AB = \zeta BA$, by Fitting’s lemma, there is a unique direct sum decomposition $F_q^n = V \oplus W$ such that $A(V) \subseteq V, A(W) \subseteq W$, $A|_V$ is invertible, and $A|_W$ is nilpotent.
- It turns out that $B$ must satisfy $B(V) \subseteq V, B(W) \subseteq W$. All we need in the proof is that $\zeta \neq 0$.
- This allows $K_{\zeta,n}(F_q)$ to be “decomposed” into $U_{\zeta,n}(F_q)$ (requiring invertible $A$) and $N_{\zeta,n}(F_q)$ (requiring nilpotent $A$), in the sense of

$$
\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(F_q)|}{|\text{GL}_n(F_q)|} x^n = \left( \sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(F_q)|}{|\text{GL}_n(F_q)|} x^n \right) \left( \sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(F_q)|}{|\text{GL}_n(F_q)|} x^n \right)
$$
Ideas of proof: nilpotent part

- To compute $|N_{\zeta,n}(\mathbb{F}_q)| = |\{(A, B) : AB = \zeta BA, A \text{ nilp}\}|$, we first fix $A$ and count the number of $B$.
- The number of $B$ only depends on the similarity class of $A$, so we may assume $A$ is in the Jordan canonical form.
- The general form of $B$ can then be determined entry-wise.
- In particular, the number of $B$ does not depend on $\zeta$ (this works even for $\zeta = 0$).
Ideas of proof: invertible part

- To compute $|U_{\zeta,n}(\mathbb{F}_q)| = |\{(A, B) : AB = \zeta BA, A \text{ invertible}\}|$, we first fix $B$ and count the number of $A$. (Opposite to the nilpotent case!!)

- Not every $B$ contributes. In order for the number of $A$ to be nonzero, we must have that $B$ is similar to $\zeta B$ (by the definition of similarity).

- Using the standard orbit-stabilizer argument, it suffices to count the number of similarity classes of $B$ such that $B$ is similar to $\zeta B$.

- **This is where $m$, the order of $\zeta$, matters.** The similarity class corresponds to a finite sequence $(g_1, g_2, \ldots)$ of monic polynomials over $\mathbb{F}_q$ such that $g_i$ divides $g_{i+1}$. Requiring $B$ to be similar to $\zeta B$ is equivalent to requiring every $g_i$ in the sequence of polynomials associated to $B$ to be of the following form:

  $$t^d + c_1 t^{d-m} + c_2 t^{d-2m} + \ldots$$
Pause

Second half: Cohen–Lenstra series
A broader framework

The result can be put in a general framework which I call the Cohen–Lenstra series. Let $R$ be an $\mathbb{F}_q$-algebra with some reasonable finiteness assumption (e.g., $\mathbb{F}_q[X]$, $\mathbb{F}_q[[X,Y]]/(XY)$, or the quantum plane). Define the Cohen–Lenstra series of $R$ as

$$\hat{Z}_R(x) = \sum_{M/R} \frac{1}{|\text{Aut} M|} x^{\dim_{\mathbb{F}_q} M},$$

where $M$ runs over all isomorphism classes of finite-cardinality modules over $R$. This framework has the advantages of:

- can unify many matrix enumeration problems of very distinct flavors, by varying $R$;
- can easily tell new problems from old.
- has local-global if $R$ is commutative: $\hat{Z}_R(x) = \prod_{p\in \text{Spec} R} \hat{Z}_{R_p}(x)$.

(follows naturally from the definition)
The concept of Cohen–Lenstra series has been considered before in various formulations.

- **Cohen and Lenstra, 1984**: considered the series for Dedekind domain $R$, in the same formulation.
- **Feit and Fine, 1960**: The generating function they gave for counting commuting matrices matches the series for $\mathbb{F}_q[X, Y]$.
- **Bryan and Morrison, 2015**: reinterpreted Feit–Fine in motivic Donaldson–Thomas theory, where they considered a generating series for motivic classes of the stack of coherent sheaves over $R$, which is a refined version of $\hat{Z}_R(x)$. 
Matrix formulation

If \( R = \mathbb{F}_q[X_1, \ldots, X_m]/(f_1, \ldots, f_r) \), consider the variety of modules over \( R \):

\[
K_n(R)(\mathbb{F}_q) = \{ A_1, \ldots, A_m \in \text{Mat}_n(\mathbb{F}_q) : [A_i, A_j] = 0, f_k(A) = 0 \}.
\]

Then \( K_n(R) \) parametrizes (with some nonuniqueness) modules over \( R \) that are \( n \)-dimensional over \( \mathbb{F}_q \), and

\[
\widehat{Z}_R(x) = \sum_{n \geq 0} \frac{|K_n(R)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n.
\]

**Punctual version:** If \( R = \mathbb{F}_q[[X_1, \ldots, X_m]]/(f_1, \ldots, f_r) \) where \( f_k(0) = 0 \), the variety of modules over \( R \) is

\[
K_n(R)(\mathbb{F}_q) = \{ A_1, \ldots, A_m \in \text{Nilp}_n(\mathbb{F}_q) : [A_i, A_j] = 0, f_k(A) = 0 \}.
\]
Matrix enumeration interpreted as Cohen–Lenstra

By varying $R$, many generating series of matrix enumeration problems can be viewed as $\hat{Z}_R(x)$:

- $R = \mathbb{F}_q[[X]]$: $\hat{Z}_R(x)$ is a generating series for $|\text{Nilp}_n(\mathbb{F}_q)|$ ($= q^{n^2-n}$ by Fine–Herstein 1958)
- $R = \mathbb{F}_q[X, Y]$: $\hat{Z}_R(x)$ is the series counting commuting matrices computed in Feit–Fine 1960.
- $R = \mathbb{F}_q[X, Y]/(XY)$: $\hat{Z}_R(x)$ counts mutually annihilating matrices $AB = BA = 0$ (computed in H., 2021).
- $R$ is the quantum plane $\mathbb{F}_q\{X, Y\}/(XY - \zeta YX)$: $\hat{Z}_R(x)$ is the series in Main Theorem.
Behaviors of the Cohen–Lenstra series

The status of knowledge of $\hat{Z}_R(x)$ is best summarized in terms of the algebraic geometry of $R$:

- $R$ noncommutative: mostly unknown, except Main Theorem.

For the below, $R$ is commutative and $X = \text{Spec } R$:

- $X$ a smooth curve: $\hat{Z}_R(x) = \prod_{j \geq 1} Z_X(xq^{-j})$, where $Z_X$ is the Hasse–Weil zeta.
- $X$ a smooth surface: $\hat{Z}_R(x) = \prod_{i,j \geq 1} Z_X(x^i q^{-j})$.

The two above follow from classical formulas for $R = \mathbb{F}_q[X], \mathbb{F}_q[X, Y]$ and local-global.

- $X$ a nodal singular curve: has an explicit formula with mysterious combinatorics (H., 2021)
- $X$ other singular curve: has conjectural patterns (H., 2021)
- Fun fact: $R = \mathbb{F}_q$ (a point) is not a trivial case; in fact $\hat{Z}_R(x)$ is the Rogers–Ramanujan series! ($1, 4 \mod 5...$)
- All other cases: wide open.
Since the combinatorial behavior of $\hat{Z}_R(x)$ depends heavily on the algebraic geometry of $R$, any observation about $\hat{Z}_R(x)$ may suggest interesting geometry of $R$.

In the rest of the talk, we will discuss an elementary observation about our Refined Theorem that seems to owe a higher-level explanation. It might inspire interesting noncommutative geometry of the quantum plane.
Let’s consider the following series:

\[ \hat{Z}_{F_q[X]}(x) = \sum_{n \geq 0} \frac{|\text{Mat}_n(F_q)|}{|\text{GL}_n(F_q)|} x^n \]

\[ \hat{Z}_{F_q[X,X^{-1}]}(x) = \sum_{n \geq 0} \frac{|\text{GL}_n(F_q)|}{|\text{GL}_n(F_q)|} x^n = \frac{1}{1 - x} \]

\[ \hat{Z}_{F_q[[X]]}(x) = \sum_{n \geq 0} \frac{|\text{Nilp}_n(F_q)|}{|\text{GL}_n(F_q)|} x^n \]

Local-to-global \implies \hat{Z}_{F_q[X]}(x) = \hat{Z}_{F_q[X,X^{-1}]}(x) \hat{Z}_{F_q[[X]]}(x).

Essentially the cut-and-paste \( \mathbb{A}^1 = (\mathbb{A}^1 \setminus \{0\}) \sqcup \{0\} \).

Since \( \text{Mat}_n(F_q) \) is easy to count (there are \( q^{n^2} \) matrices), this computes \( |\text{Nilp}_n(F_q)| \) (an alternative proof of Fine–Herstein).
The decomposition above is essentially the same phenomenon as

$$\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \left( \sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n \right) \left( \sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n \right).$$

It appears that $\hat{Z}_{\mathbb{F}_q[X,X^{-1}]}(x)$ and $\hat{Z}_{\mathbb{F}_q[[X]]}(x)$ are “disjoint and independent” building blocks, just like $|U_{\zeta,n}(\mathbb{F}_q)|$ vs $|N_{\zeta,n}(\mathbb{F}_q)|$. But the story doesn’t end here – $\hat{Z}_{\mathbb{F}_q[X,X^{-1}]}(x)$ and $\hat{Z}_{\mathbb{F}_q[[X]]}(x)$ are amazingly interconnected!

Key: $\mathbb{A}^1 \setminus \{0\}$ is composed of closed points, each of which “looks like” the origin.
Global-to-local

Recall local-to-global for $R = \mathbb{F}_q[X, X^{-1}]$:

$$\hat{Z}_{\mathbb{F}_q[X,X^{-1}]}(x) = \prod_p \hat{Z}_{R_p}(x).$$

Now, each $R_p$ is smooth of dimension one, so its completion is isomorphic to a power series ring over some finite field. Therefore, all $\hat{Z}_{R_p}(x)$ is essentially $\hat{Z}_{\mathbb{F}_q[[X]]}(x)$ (though a substitution is needed), so $\hat{Z}_R(x)$ is determined by $\hat{Z}_{\mathbb{F}_q[[X]]}(x)$.

Moreover, because $\hat{Z}_R(x)$ is determined by $\hat{Z}_{\mathbb{F}_q[[X]]}(x)$ alone, we can reverse this process ("global-to-local") and write down a formula that recovers $\hat{Z}_{\mathbb{F}_q[[X]]}(x)$ from $\hat{Z}_R(x) = 1/(1 - x)$. (Yet another proof of Fine–Herstein!)

This idea is due to Bryan and Morrison (2015) in a more refined language of motivic classes. They actually did the case for $R = \mathbb{F}_q[X, Y]$ (namely, $K_{1,n}$).
How about the quantum plane?

Noncommutative Question.

Can you recover $|U_{\zeta,n}(\mathbb{F}_q)|$ from $|N_{\zeta,n}(\mathbb{F}_q)|$ (or vice versa), using the geometry of the quantum plane?

- The commutative analogue requires the notion of “localization at a prime ideal”. Does this notion exist for the quantum plane?
- Recall that $|U_{\zeta,n}(\mathbb{F}_q)|$ depends on $\zeta$ (or the order of $\zeta$), while $|N_{\zeta,n}(\mathbb{F}_q)|$ doesn’t. So if $|N_{\zeta,n}(\mathbb{F}_q)|$ recovers $|U_{\zeta,n}(\mathbb{F}_q)|$, it must be because it also takes account of some geometry of the quantum plane that depends on $\zeta$. 
Final takeaway

- We extend a formula that counts matrix pairs $AB = BA$ to the case $AB = \zeta BA$ where $\zeta$ is nonzero. The answer depends on the order of $\zeta$ as a root of unity.
- The count of $AB = \zeta BA$ encodes statistical information about modules over the quantum plane.
- The count in question has two seemingly independent building blocks that turn out to be interdependent in the $\zeta = 1$ case, using ingredients from (commutative) algebraic geometry. I hope that the study of a possible interdependence in the case of general $\zeta$ will inspire interesting noncommutative geometry.