UNIPOTENT ORBITS AND LOCAL $L$-FUNCTIONS

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ABSTRACT. In a previous article (Orbites unipotentes et pôles d’ordre maximal de la fonction $\mu$ de Harish-Chandra, to appear in Canad. J. Math.), we have assumed the existence of the local Langlands correspondence for supercuspidal representations and deduced from this a local Langlands correspondence for discrete series representations and beyond (without going into the structure of the $L$-packets). The aim of the present article is to show that this extension of the local Langlands correspondence for supercuspidal representations (and some of the assumptions in the article above) is compatible with the theory of $L$-functions due to Langlands-Shahidi.

Let $G$ be the group of $F$-points of a connected reductive group defined over a non archimedean local field. In [H2] we have assumed the local Langlands correspondence for supercuspidal representations of $F$-Levi-subgroups of $G$ and deduced from this a local Langlands correspondence for discrete series representations of $G$ and beyond (without going into the structure of the $L$-packets). The aim of this note is to show that the results and assumptions in [H2] are compatible with the theory of $L$-functions of Langlands-Shahidi. This theory applies at this moment to generic representations of $F$-points of quasi-split connected reductive groups. It has been established until now only for $F$ of characteristic 0. So we have to make this assumption, too, and suppose in the sequel that $G$ is quasi-split.

Let us be more precise. Let $P = MU$ be an $F$-parabolic subgroup of $G$. Denote by $\Sigma_{\text{red}}(P)$ the set of reduced roots in $\text{Lie}(U)$ of the maximal split torus $A_M$ in the center of $M$. Recall that to each $\alpha \in \Sigma_{\text{red}}(P)$ corresponds a semi-standard $F$-Levi subgroup $M_{\alpha}$ of $G$, which contains $M$ as maximal Levi subgroup. One identifies $\Sigma_{\text{red}}(P)$ to a subset of the dual $a_{M}^*$ of the real Lie algebra of $A_M$. There is a natural way to attach to an element $\lambda$ of the complexification of $a_{M}^*$ a character $\chi_{\lambda}$ of $M$ [H2, 0.6]. If $\lambda = sa$, $s \in \mathbb{C}$, and $m \in A_M$, one has $\chi_{sa}(m) = |\alpha(m)|_F$, where $| \cdot |_F$ denotes the normalized absolute value of $F$.

Let $\sigma$ be an irreducible unitary supercuspidal generic representation of $M$ and $W_F$ the Weil group of $F$. In [H2] we have assumed that one can attach to $\sigma$ an admissible homomorphism $\psi_\sigma : W_F \times \text{SL}_2(\mathbb{C}) \to \hat{L}M$ (see [H2] for the precise definition of the Langlands $L$-group and an admissible homomorphism used here),
verifying some properties, coming from the conjectural local Langlands correspondence. As in particular it is believed that \( \psi_{\sigma}|_{\text{SL}_2(\mathbb{C})} \) is trivial, when \( \psi_{\sigma} \) is attached to a generic supercuspidal representation, we will assume this here, too. The assumption [H2,4.3] simplifies then considerably and reads (with \( q \) the number of elements in the residue field of \( F \), referring to [H2, 3.5] for the notion of "\( q \)-distinguished")

\[ (LM) \text{ For each root } \alpha \in \Sigma_{\text{red}}(P), \text{ the Harish-Chandra } \mu \text{-function } s \mapsto \mu^{M_{\alpha}}(\sigma \otimes \chi_{s,\alpha}) \text{ (see [W] for the definition of this function) has a pole in a real number } s_0 > 0, \text{ if and only if } \alpha(q)^{s_0} \text{ is } \mu \text{-distinguished in the connected centralizer of the image of } \psi_{\sigma} \text{ and this group is not a torus.} \]

Fix a non trivial additive character \( \psi_F \) of \( F \). In [Sh] Shahidi (proving a conjecture of Langlands) has associated to an irreducible smooth generic representation \( \sigma \) of \( M \) a set of complex functions \( \{ s \mapsto \gamma(s, \sigma, r_i, \psi_F), 1 \leq i \leq m \} \). From them he deduces canonically \( L \)-functions \( L(s, \sigma, r_i) \) and \( \epsilon \)-factors \( \epsilon(s, \sigma, r_i, \psi_F) \) (see also 1.3 for more details). As the maps \( r_i \circ \psi_{\sigma} \) are representations of the Weil-Deligne group, the Artin \( L \)-functions \( L(s, r_i \circ \psi_{\sigma}) \) and \( \epsilon \)-functions \( \epsilon(s, r_i \circ \psi_{\sigma}, \psi_F) \) are defined and one derives from them \( \gamma(s, r_i \circ \psi_{\sigma}, \psi_F) \) as above (see 1.4 - 1.5 for more details).

Our first result is, that the assumption \((LM)\) is satisfied, if \( \sigma \) and \( \psi_{\sigma} \) have the same \( L \)-functions w.r.t. each \( M_{\alpha}, \alpha \in \Sigma_{\text{red}}(P) \). We get also a converse under some condition on the \( L \)-functions attached to \( \psi_{\sigma} \).

Under the assumption \((LM)\) we have in [H2] associated to each elliptic admissible homomorphism \( \psi : W_F \times \text{SL}_2(\mathbb{C}) \rightarrow \mathbb{T}G \) an irreducible square-integrable representation \( \pi \) of \( G \), and vice-versa. Our next result is that \( \psi \) and \( \pi \) have same \( \gamma \)-functions if they correspond to each other by this correspondence. We show also that this property remains true, if one extends the correspondence to arbitrary admissible homomorphisms \( \psi \) and arbitrary smooth irreducible representations \( \pi \) of \( G \), as done in the last section of [H2].

We finish by a discussion of the general case of non generic representations and non quasi-split groups, in taking into account the conjectural framework in [Sh, 9.].

We refer to the introduction of [H2] for information of the actual state of the local Langlands conjectures.

1. Notations and preliminaries:

1.1. We denote by \( I_F \) the inertial subgroup of \( W_F \), by \( Fr \) a geometric Frobenius automorphism of \( F \) [De] and normalize the reciprocity map in local class field theory so that \( |Fr|_F = q^{-1} \).
To simplify the notations, we will denote by $\Im(s)$ the imaginary part of a complex number $s$ multiplied by $\sqrt{-1}$.

1.2 We fix a minimal $F$-parabolic subgroup $P_0 = M_0 U_0$ of $G$ and a maximal $F$-split torus $A_0$ contained in $M_0$. We denote by $\Sigma$ the set of roots of $A_0$ in $\lie{L}$ and by $\Delta$ the set of simple roots with respect to $P_0$. If $P = MU$ is a standard parabolic of $G$ (i.e. $P \supseteq P_0$), $\alpha \in \Sigma_{\text{red}}(P)$, we note $P_\alpha$ the standard parabolic $P \cap M_\alpha$ of $M_\alpha$ and $U_\alpha = U \cap M_\alpha$.

1.3 Let $P = MU$ be a maximal standard $F$-parabolic subgroup of $G$, $\rho$ half of the sum of the roots in $\Sigma$ whose root space spans $\lie{L}$. We denote by $\gamma$ the unique root in $\Delta$ which does not lie in the root subsystem of $\Sigma$ corresponding to $M$. Put $\tilde{\alpha} = (\rho, \alpha^\vee)^{-1}\rho$.

Denote by $r$ the adjoint action of $L M$ on $\lie{L}$ and

$$V_i = \{X_{\beta^\vee} \in \lie{L} \mid \langle \tilde{\alpha}, \beta^\vee \rangle = i\}.$$  
(Here $\lie{L}$ has been decomposed into weight spaces relative to the roots with respect to the action of the connected center of $L T$, which equals $L A_0$.) The spaces $V_i$ are invariant by $r$. Denote by $r_i$ the restriction of $r$ to $V_i$. One has a decomposition $r = \oplus_{i=1}^{m} r_i$ with some integer $m \geq 1$, called the length of $r$. The components $r_i$, $1 \leq i \leq m$, are irreducible [Sh].

Let $\sigma$ be a smooth irreducible generic representation of $M$. Fix a non trivial additive character $\psi_F$ of $F$. In [Sh] Shahidi (proving a conjecture of Langlands) has associated to $\sigma$ a set of complex functions $\{s \mapsto \gamma(s, \sigma, r_i, \psi_F), 1 \leq i \leq m\}$. If $\sigma$ is tempered, he deduces from them canonically $L$-functions $L(s, \sigma, r_i)$ and $\epsilon$-factors $\epsilon(s, \sigma, r_i, \psi_F)$ in the following way: Denote by $P_{\sigma,i}$ the unique polynomial satisfying $P_{\sigma,i}(0) = 1$ such that $P_{\sigma,i}(q^{-s})$ is the numerator of $\gamma(s, \sigma, r_i, \psi_F)$ (in particular $P_{\sigma,i}(q^{-s})$ has the same zeros than $\gamma(s, \sigma, r_i, \psi_F)$). Then

$$L(s, \sigma, r_i) := P_{\sigma,i}(q^{-s})^{-1}, \quad L(s, \sigma, \tilde{r}_i) := P_{\sigma,i}(q^{-s})^{-1}$$

and

$$\epsilon(s, \sigma, r_i, \psi_F) := (L(s, \sigma, r_i)/L(1-s, \sigma, \tilde{r}_i))\gamma(s, \sigma, r_i, \psi_F)$$

for $1 \leq i \leq m$, where $\tilde{\sigma}$ and $\tilde{r}_i$ are the contragredient representations.

As $\gamma(s, \sigma \otimes \chi_{s', \sigma, r_i}, \psi_F) = \gamma(s + s', \sigma, r_i, \psi_F)$ by [Sh, (3.12)], $L(s, \sigma, r_i)$ and $\epsilon(s, \sigma, r_i, \psi_F)$ are also defined, if $\sigma$ is only quasi-tempered.

The following properties hold:

1.3.1 $L(s, \sigma, r_i) = 1$ for $3 \leq i \leq m$, if $\sigma$ is supercuspidal [Sh, 7.5];

1.3.2 Suppose that $P$ is associated to its opposite parabolic subgroup $\overline{P}$ and that $\sigma$ is unitary and supercuspidal. (We will later say that $P$ is self-conjugated.) Denote by $w$ a representative of an element of the Weyl group that conjugates $P$.
and $\mathcal{T}$. Then the Harish-Chandra $\mu$-function (see [W] for the definition of this function) verifies (with $\sim$ meaning equality up to a monomial in $q^{-s}$)

$$
\mu(\sigma \otimes \chi_{s,\tilde{\alpha}}) \sim \frac{P_{\sigma,1}(q^{-s})P_{\sigma,2}(q^{-2s})P_{w_\sigma,1}(q^s)P_{w_\sigma,2}(q^{2s})}{P_{\sigma,1}(q^{-(1-s)})P_{\sigma,2}(q^{-(1-2s)})P_{w_\sigma,1}(q^{-(1+s)})P_{w_\sigma,2}(q^{-(1+2s)})}
$$

$$
= \frac{L(1-s,\sigma,\tilde{\tau}_1)L(1-2s,\sigma,\tilde{\tau}_2)L(1+s,w_\sigma,\tilde{\tau}_1)L(1+2s,w_\sigma,\tilde{\tau}_2)}{L(s,\sigma,\tau_1)L(2s,\sigma,\tau_2)L(-s,w_\sigma,\tau_1)L(-2s,w_\sigma,\tau_2)}
$$

[Sh, 1.4 and 7.6].

(1.3.3) \[ L(s,\sigma,\tau_1) = L(\bar{\sigma},\bar{\tau}_1) = L(\bar{\pi},\bar{\tau}_1) \quad [\text{Sh, 7.8 and p. 308}]. \]

(1.3.4) \[ L(s,\sigma \otimes \chi_{s,\tilde{\alpha}},\tau_1) = L(s+s',\sigma,\tau_1) \quad [\text{Sh, (3.12)}]. \]

(1.3.5) If $\sigma$ is unitary and supercuspidal, all poles of $L(\cdot,\sigma,\tau_1)$ have real part 0 [Sh, 7.3].

(1.3.6) If $\sigma$ is supercuspidal, the poles of $L(\cdot,\sigma,\tau)$ are simple. (This is because of (1.3.2) and the simplicity of the poles of Harish-Chandra’s $\mu$-function [H1, remark in the proof of 4.1].)

If $(r',V')$ is a sub-representation of $r$, one defines $\gamma(\cdot,\sigma,r',\psi_F) = \prod_{i,V_i \subseteq V'} \gamma(\cdot,\sigma,\tau_i,\psi_F)$ and in the same way $L(\cdot,\sigma,r')$ and $\epsilon(\cdot,\sigma,r',\psi_F)$.

If $\pi$ is a general generic smooth irreducible representation of $M$, then the $L$-functions $L(\cdot,\pi,\tau_1)$ are defined in the following way [Sh, p. 308]: by Langlands’ classification there is a standard $F$-parabolic subgroup $P_1 = M_1U_1$ of $G$ with $M_1 \subseteq M$ and an irreducible quasi-tempered representation $\tau$ of $M_1$, such that $\pi$ is the unique sub-representation of $\pi_{\tau_1 \cap M}$. By [R, Theorem 2], the quasi-tempered representation $\tau$ is generic. Denote by $\kappa_1$ the inclusion $L^\circ M_1 \to L^\circ M$, by $r_{1,i}$ the composition $r \circ \kappa_1$ and, for $\alpha \in \Sigma_{\text{red}}(P_1) - \Sigma_{\text{red}}(P_1 \cap M)$, by $r_{1,i,\alpha}$ the restriction $L^\circ M_1 \to L(\kappa U_{1,\alpha})$ of $r_{1,i}$. The $L$-functions $L(\cdot,\tau,\tau_{1,i,\alpha})$ w.r.t. $M_{1,\alpha}$ are defined by analytic continuation from the tempered case. The $L$-function associated to $\pi$ and $r_1$ is

(1.3.7) \[ L(\cdot,\pi,\tau_1) = \prod_{\alpha \in \Sigma_{\text{red}}(P_1) - \Sigma_{\text{red}}(P_1 \cap M)} L(\cdot,\tau,\tau_{1,i,\alpha}). \]

The corresponding $\epsilon$-factor is deduced from $L(\cdot,\pi,\tau_1)$ and $\gamma(\cdot,\pi,\tau_1,\psi_F)$ by the same equation as in the tempered case.
Consider finally an arbitrary standard parabolic subgroup $P = MU$ of $G$. Denote still by $\rho$ half of the sum of the roots in $\Sigma$ that generate $U$. For each $\beta$ with $X_{\beta} \in \text{Lie}(U)$, $(2\rho, \beta) = 1$ is a positive integer. Let $1 \leq a_1 < a_2 < \cdots < a_m$ be the different values. Following [Sh], we define

$$V_i := \{ X_{\beta} \in \text{Lie}(U) | (2\rho, \beta) = a_i \}$$

and denote by $r_i$ the restriction of the adjoint representation $r : L M \rightarrow \text{Lie}(U)$ to $V_i$. This definition agrees with the one above for $P$ maximal. For $\alpha \in \Sigma_{\text{red}}(P)$, let $r_{i,\alpha} : L M \rightarrow \text{Lie}(U_{\alpha})$ be the restriction of $r_i$. If $r_{\alpha}$ denotes the adjoint representation $L M \rightarrow \text{Lie}(U_{\alpha})$, then it follows from elementary properties of root systems that $r_{i,\alpha} = r_{\alpha,i}$ (with $r_{\alpha,i}$ defined relative to the maximal parabolic subgroup $P \cap M_\alpha$ of $M_\alpha$ as above). Let $\pi$ be a general generic irreducible smooth representation of $M$. For $\alpha \in \Sigma_{\text{red}}(P)$, denote by $\gamma(\cdot, \pi, r_{i,\alpha}, \psi_F)$ the $\gamma$-function of $\pi$ defined relative to $P \cap M_\alpha$. Then, by definition,

$$\gamma(\cdot, \pi, r_{i,\alpha}, \psi_F) = \prod_{\alpha \in \Sigma_{\text{red}}(P)} \gamma(\cdot, \pi, r_{i,\alpha}, \psi_F)$$

[Sh, p. 307, 1.15 -20]. The $L$- and $\epsilon$-factors of $\pi$ relative to $P$ are defined in the same way as product of $L$- and $\epsilon$-factors attached to $\alpha \in \Sigma_{\text{red}}(P)$.

If $r'$ is an arbitrary sub-representation of $r$, then one defines local factors for $r'$ in the same way than for maximal $P$.

1.4 Recall the definition of the Artin $L$-function [De]. An admissible homomorphism $\psi : W_F \times \text{SL}_2(\mathbb{C}) \rightarrow \text{GL}_n(\mathbb{C})$ can be written as direct sum of twists of elliptic admissible homomorphisms. As the Artin $L$-functions are additive and behave well under unramified twists (i.e. $L(s + s', \psi) = L(s, \psi | \det(F^s))$, it is enough to give the definition for $\psi$ elliptic. Let $N$ be the nilpotent $n \times n$-matrix, such that

$$\psi(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = \exp(N).$$

Identify $N$ with the corresponding nilpotent endomorphism of $V := \mathbb{C}^n$. As $\psi$ is elliptic, the restriction $\psi_0$ of $\psi$ to $W_F$ is a multiple of an irreducible representation and the subspace $\ker(N)$ is an irreducible component. If $m$ is the multiplicity of $\psi_0$ in $\psi$, one has

$$L(s, \psi) = \det(I - \psi(Fr)q^{-s-m+1} | \ker(N)^{Fr})^{-1}.$$

Remark that the action of $\psi_0$ on $V^{Fr}$ is an unramified character. As $\ker(N)$ is an irreducible component of $\psi_0$, either the representation $\psi_0$ is itself an unramified character or $V^{Fr} = 0$. So $L(s, \psi) = 1$, if $\psi$ is ramified. Otherwise $\dim(\ker(N)) = 1$. So, if $q^{s_0}$ is the proper value for the action of $\psi(Fr)$ on $V^{Fr}$, then we have

$$L(s, \psi) = (1 - q^{s_0-n+1-s})^{-1}$$

in this case. Remark that $\Re(s_0) = 0$, if $\psi(W_F)$ is relatively compact.
1.5 The $\gamma$, $L$- and $\epsilon$-factors should be preserved by the (in general) conjectural local Langlands correspondence. More precisely, let $\psi_\sigma : W_F \times \text{SL}_2(\mathbb{C}) \to L^1 M$ be the conjectural admissible homomorphism attached to the generic irreducible smooth representation $\sigma$. (It is in particular assumed that $\psi_\sigma(W_F)$ is relatively compact, when $\sigma$ is tempered.)

Then we should have

$$L(s, \sigma, r_i) = L(s, r_i \circ \psi_\sigma) \quad \text{and} \quad \epsilon(s, \sigma, r_i, \psi_F) = \epsilon(s, r_i \circ \psi_\sigma, \psi_F).$$

Here $L$- and $\epsilon$-factors on the Galois side are the Artin $L$- and $\epsilon$-functions defined by Deligne [De]. If one defines $\gamma(s, r_i \circ \psi_\sigma, \psi_F)$ by the corresponding equation on the Galois side, one gets also

$$\gamma(s, \sigma, r_i, \psi_F) = \gamma(s, r_i \circ \psi_\sigma, \psi_F).$$

Remark that, as $\epsilon(s, r_i \circ \psi_\sigma, \psi_F)$ is a monomial in $q^{-s}$ [De], $L(s, r \circ \psi_\sigma)^{-1}$ is the unique polynomial in $z = q^{-s}$, which takes value 1 in $z = 0$ and which is the numerator of $\gamma(s, r_i \circ \psi_\sigma, \psi_F)$.

So, in particular, if $\sigma$ is tempered, the equality of $\gamma$-factors implies the equality of $L$- and $\epsilon$-factors.

2. We will now start to prove that in the generic case the assumption $(LM)$ in [H2] is implied by an equality of $L$-functions (referring to [H2, 3.5] for the notion of ”$q$-distinguished”), establishing also a kind of converse.

The lemma below is a reformulation of results in [Sh].

2.1 Lemma: Let $P = MU$ be a maximal standard $F$-parabolic subgroup of $G$ and let $\sigma$ be a unitary irreducible generic supercuspidal representation of $M$. Then, for any $s \in \mathbb{C}$, $\mu(\sigma \otimes \cdot)$ has a pole in $\chi_s \tilde{\alpha}$, if and only if $\Im(s)$ is a pole of $L(\cdot, \sigma, r_i)$ with $i \Re(s) = \pm 1$.

Proof: Suppose $\mu(\sigma \otimes \cdot)$ has a pole in $\chi_s \tilde{\alpha}$. Then, by results of Harish-Chandra [Si], $\sigma$ is ramified, $P$ is self-conjugated, $i \hat{\sigma}\sigma$ is irreducible and $\mu(\sigma \otimes \chi_{\Im(s)\tilde{\alpha}}) = 0$. Write $\sigma_0 = \sigma \otimes \chi_{\Im(s)\tilde{\alpha}}$. By [Sh, 7.6], there exists a unique $i = 1, 2$ such that 0 is a pole of $L(\cdot, \sigma_0, r_i) = L(\Im(s) + \cdot, \sigma, r_i)$. This proves the first assertion and, by (1.3.3), 0 is then also a pole of $L(\cdot, \sigma_0, r_i)$.

As $\chi_{\Im(s)\tilde{\alpha}}$ is a pole of $\mu(\sigma_0 \otimes \cdot)$, it follows from the expression (1.3.2) for the $\mu$-function and (1.3.5), that $1 - i \Re(s)$ or $1 + i \Re(s)$ is a pole of $L(\cdot, \sigma_0, r_i)$, i.e. one of them must be 0. This concludes the proof of the first implication.

Conversely, choose $i$ such that $i \Re(s) = \pm 1$ and assume that $\Im(s)$ is a pole of $L(\cdot, \sigma, r_i)$ (so that, in particular, $i$ is an integer $\geq 1$). As $L(\cdot, \sigma, r_i)$ is regular for
by trivial, there exists is \(i = 1\). We will first prove that (so that in particular $N$ action of the Frobenius on $N$ then $\exp(\tilde{\alpha})$ contains a unipotent element cannot be a torus. So (a unipotent element. But, the connected component of a reductive group which by (1.3.4). By [Sh, 7.4] this can only happen if $P$ is self-conjugated. As the poles of $L(\cdot, \sigma_0, r_i)$ have real part 0, it follows from (1.3.2), (1.3.3) and (1.3.5) that $\mu(\sigma_0 \otimes \cdot)$ has a pole in $\chi_{\mathcal{R}(s)\tilde{\alpha}}$.

2.2 Lemma: Let $P = MU$ be a maximal standard $F$-parabolic subgroup of $G$ and let $\psi_\sigma: W_F \to L M$ be an elliptic admissible homomorphism.

Then, for any complex number $s$, the following two properties are equivalent:

(i) $\Im(s)$ is a pole of $L(\cdot, r_i \circ \psi_\sigma)$ for some positive integer $i$ verifying $i\mathcal{R}(s) = \pm 1$;

(ii) $\tilde{\alpha}(q)^{\mathcal{R}(s)}$ is $q$-distinguished in the connected centralizer of the image of the map $W_F \to L M$, $\gamma \mapsto \tilde{\alpha}(q)^{v_F(\gamma)\Im(s)}\psi_\sigma(\gamma)$, and this connected centralizer is not a torus.

Proof: Replacing $\sigma$ by $\sigma \otimes \chi_{\Im(s)\tilde{\alpha}}$ and $\psi_\sigma$ by $\gamma \mapsto \tilde{\alpha}(q)^{v_F(\gamma)\Im(s)}\psi_\sigma(\gamma)$, we can suppose by (1.3.4) that $\Im(s) = 0$.

Denote by $\tilde{M}^\sigma$ the centralizer of $\psi_\sigma(W_F)$ in $L G$ and by $(\tilde{M}^\sigma)^0$ its connected component.

Suppose $\tilde{\alpha}(q)^s$ $q$-distinguished in $\tilde{M}^\sigma$ and that the connected component $(\tilde{M}^\sigma)^0$ is not a torus. So there is a nilpotent element $N$ in the Lie algebra of the connected component of $\tilde{M}^\sigma$, such that $(\text{Ad}(\tilde{\alpha}(q)^s))(N) = qN$. Then $N \in V_{+i}$ for some integer $i$, $1 \leq i \leq m$, and it follows that $is = \pm 1$. Consider $r_i \circ \psi_\sigma$. As $N \in V_{+i}^{W_F}$, we have $V_{+i}^{W_F} \neq 0$. So the $L$-function $L(\cdot, r_i \circ \psi_\sigma)$ is non trivial. As the Frobenius acts trivial on $N$, it has a pole in $0$ by the above discussion of the Artin $L$-function.

Conversely, choose $i$ such that $is = \pm 1$ and assume that $0$ is a pole of $L(\cdot, r_i \circ \psi_\sigma)$ (so that in particular $i$ is a positive integer). Replacing $s$ by $|s|$, we can assume $is = 1$. We will first prove that $(\tilde{M}^\sigma)^0$ is not a torus. As $L(\cdot, r_i \circ \psi_\sigma)$ is non trivial, there exists $N \in V_i \subseteq \text{Lie}(\text{Lie}(U))$, which is invariant under the action of $I_F$ by $r_i \circ \psi_\sigma$. As $L(\cdot, r_i \circ \psi_\sigma)$ has a pole in $0$, by 1.4 we can choose $N$ such that the action of the Frobenius on $N$ by $r_i \circ \psi_\sigma$ is trivial, i.e. $N$ is invariant by $W_F$. But then $\exp(N)$ lies in the centralizer of $(r_i \circ \psi_\sigma)(W_F)$. So this centralizer contains a unipotent element. But, the connected component of a reductive group which contains a unipotent element cannot be a torus. So $(\tilde{M}^\sigma)^0$ is not a torus.

Remark that $(\text{Ad}(\tilde{\alpha}(q)^s))(N) = qN$. As $\text{rk}_s(\tilde{M}^\sigma)^0 = 1$, because $T_{\tilde{M}}$, the maximal torus in the center of $\tilde{M}$, is by [H2, 4.2] a maximal torus of $(\tilde{M}^\sigma)^0$ and because $P$ (and consequently $\tilde{M}$) is maximal, it follows that $\tilde{\alpha}(q)^s$ is $q$-distinguished in $(\tilde{M}^\sigma)^0$.

2.3 Theorem: Let $G$ be the set of $F$-points of a reductive connected quasi-split group defined over $F$, $P = MU$ a maximal standard $F$-parabolic subgroup of $G$ and $\sigma$ a unitary irreducible cuspidal generic representation of $M$.

Let $\psi_\sigma: W_F \to L M$ be an admissible elliptic homomorphism. Suppose that one
has the equalities of $L$-functions

$$L(\cdot, \sigma, r_i) = L(\cdot, r_i \circ \psi_\sigma), \quad 1 \leq i \leq m.$$ 

Then the following property is true:

For any complex number $s$, $\mu(\sigma \otimes \cdot)$ has a pole in $\chi_{\tilde{\alpha}}$, if and only if $\tilde{\alpha}(q)^{\Re(s)}$ is $q$-distinguished in the connected centralizer of the image of the map $W_F \to LM$, $\gamma \mapsto \tilde{\alpha}(q)^{\rho_F(\gamma)^{2g}(s)}\psi_\sigma(\gamma)$, and this connected centralizer is not a torus.

Conversely, if this property is fulfilled and if all the poles of the $L$-functions $L(\cdot, r_i \circ \psi_\sigma), 1 \leq i \leq m$, are simple, then the above equalities of $L$-functions are true.

Proof: If one has the equalities of $L$-functions, the property in the theorem is a direct consequence of the lemmas 2.1 and 2.2. Conversely, if the property in the theorem is true, the $L$-functions $L(\cdot, \sigma, r_i)$ and $L(\cdot, r_i \circ \psi_\sigma)$ have the same poles on the imaginary axes. So by (1.3.5) and 1.4 they have same poles in $C$. As $L(\cdot, \sigma, r_i)^{-1}$ and $L(\cdot, r_i \circ \psi_\sigma)^{-1}$ are both polynomials in $q^{-s}$ which take value 1 in 0, we conclude from the simplicity of their zeroes (by (1.3.6) and by assumption) that they must be equal. $\Box$

2.4 Corollary: In the notations and under the assumptions of the preceding theorem assume that one has the equality of $\gamma$-functions

$$\gamma(\cdot, \sigma, r_i, \psi_F) = \gamma(\cdot, r_i \circ \psi_\sigma, \psi_F), \quad 1 \leq i \leq m.$$ 

Then $\sigma$ verifies the assumption (LM) in [H2, 4.3] relative to $G$.

3. In this section we will show that the correspondence derived in [H2] from the (conjectural) local Langlands correspondence for supercuspidal representations preserves $L$- and $\epsilon$-functions for generic representations of quasi-split groups.

The following lemma is contained, but not explicitly stated in [Sh].

3.1 Lemma: Let $G$ be the set of $F$-points of a reductive connected quasi-split group, $P = MU$ and $P_1 = M_1U_1$ standard $F$-parabolic subgroups of $G$, $M \supseteq M_1$, $\tau$ an irreducible smooth generic representation of $M_1$ and $\pi$ an irreducible smooth generic representation of $M_1$ which is a sub-representation of $i_{P_1}^{LM}$. Then, for any component $r_i$ of the adjoint representation $r : LM \to \text{Lie}(L)$, we have

$$\gamma(\pi, r_i, \psi_F) = \prod_{\alpha \in \Sigma_{\text{red}}(P_1), U_1, \alpha \subseteq U} \gamma(\sigma, r_{1,i,\alpha}, \psi_F), \quad (3.1.1)$$

where $r_{1,i,\alpha} : LM_1 \to \text{Lie}(LU_1, \alpha)$ denotes the restriction of $r_i$. 


Proof: Denote by $\kappa_1$ the inclusion $^LM_1 \to ^L M$ and define $r_{1,i} = r_i \circ \kappa_1$. For any root $\alpha \in \Sigma_{\text{red}}(P_1)$ verifying $U_{1,\alpha} \subseteq U$, the space $\text{Lie}(^LU_{1,\alpha})$ is invariant by $r_{1,i}$. Denote this representation by $r_{1,i,\alpha}$. Then we have

$$r_{1,i} = \bigoplus_{\alpha \in \Sigma_{\text{red}}(P_1), U_{1,\alpha} \subseteq U} r_{1,i,\alpha}.$$ 

Assume first $\tau$ supercuspidal. The product formula for the $\gamma$-function (cf. [Sh, (3.13)]) gives an expression for $\gamma$ any root $\tau$. Assume first $\tau$ and write $\gamma$ and tell us that this $\gamma$-factor attached to $U_{1,\alpha}$. Then we get that $\gamma$-factor and write $\gamma$ and $\tau$ attached to $U_{1,\alpha}$. The unicity of that $\gamma$-factor and the identity [Sh, p. 305] tell us that this $\gamma$-factor must be equal to $\prod_{\alpha} \gamma(\tau, r_{1,i,\alpha}, \psi_F)$ with $\alpha \in \Sigma_{\text{red}}(P_1)$, $U_{1,\alpha} \subseteq U$. The equality $\gamma(\tau, r_{1,i,\alpha}, \psi_F)$ stated in the lemma follows.

If $\tau$ is no more supercuspidal, then there exist a standard $F$-parabolic subgroup $P_2 = M_2 U_2$ of $G$, $M_2 \subseteq M_1$, and an irreducible supercuspidal representation $\sigma$ of $M_1$, such that $\tau$ is a sub-representation of $i_{P_2 \cap M_1}^M \sigma$. By Theorem 2 of [R], $\sigma$ is generic. Denote by $\kappa_2$ the inclusion $^L M_2 \to ^L M$, by $\kappa_2$ the inclusion $^L M_2 \to ^L M_1$ and write $r_{2,i} = r_i \circ \kappa_2$ and $r_{21,i} = r_{1,i} \circ \kappa_2$. Of course, $r_{2,i} = r_{21,i}$. By, what we have just proved, we get that

$$\gamma(\tau, r_i, \psi_F) = \prod_{\beta \in \Sigma_{\text{red}}(P_2), U_{2,\beta} \subseteq U} \gamma(\sigma, r_{2,i,\beta}, \psi_F) \quad (\ast)$$

and, for $\alpha \in \Sigma_{\text{red}}(P_1)$, that

$$\gamma(\tau, r_{1,i,\alpha}, \psi_F) = \prod_{\beta \in \Sigma_{\text{red}}(P_2), U_{2,\beta} \subseteq U_{1,\alpha}} \gamma(\sigma, r_{21,i,\beta}, \psi_F). \quad (\ast\ast)$$

Substituting $(\ast\ast)$ in $(\ast)$ proves the identity (3.1.1) in the general case. 

\[ \Box \]

3.2 Theorem: Let $G$ be the set of $F$-points of a reductive connected quasi-split group defined over $F$. Let $\pi$ be a generic discrete series representation of a standard Levi subgroup $M$ of $G$. Fix a standard parabolic subgroup $P_1 = M_1 U_1$ of $G$ with $M_1 \subseteq M$ and a generic supercuspidal representation $\sigma$ of $M_1$ such that $\pi$ is a sub-representation of $i_{P_1 \cap M}^M \sigma$.

Suppose that there is an admissible homomorphism $\psi_\sigma : W_F \to ^L M_1$ such that for any $\alpha \in \Sigma_{\text{red}}(P_1)$ and any irreducible component $r_{\alpha,i}$ of $r_\alpha$ : $^L M_{1,\alpha} \to \text{Lie}(^LU_{1,\alpha})$, we have

$$\gamma(r_{\alpha,i} \circ \psi_\sigma, \psi_F) = \gamma(\sigma, r_{\alpha,i}, \psi_F).$$

Then $\sigma$ verifies the assumption $(LM)$ in [H2]. Let $\psi_\pi$ be the admissible homomorphism $W_F \times \text{SL}_2(\mathbb{C}) \to ^L M$ attached to $\pi$ in [H2,5.3]. Then, for any component $r_i$ of the adjoint representation $r : ^L M \to \text{Lie}(^LU)$, we have

$$\gamma(r_i \circ \psi_\pi, \psi_F) = \gamma(\pi, r_i, \psi_F).$$
Proof: It is a direct consequence of the corollary 2.4 that $\sigma$ verifies the assumption $(LM)$ in [H2] under our hypothesis. Denote by $P = MU$ the standard parabolic of $G$ with Levi factor $M$ and by $r_{1,i} : L M_1 \rightarrow \text{Lie}(L U)$ the restriction of $r_1$ to $L M_1$. We have a decomposition $r_{1,i} = \bigoplus_{\alpha \in \Sigma(P_1), U_1, \alpha \subseteq U} r_{1,i,\alpha}$ with $r_{1,i,\alpha} : L M_1 \rightarrow \text{Lie}(L U_{1,\alpha})$. Inserting our assumptions $\gamma(\sigma, r_{\alpha,i}, \psi_F) = \gamma(r_{\alpha,i} \circ \psi_\sigma, \psi_F)$ in the identity (3.1.1) and using the multiplicity of Artin $L$- and $\epsilon$-functions, we get

$$\gamma(\pi, r_i, \psi_F) = \gamma(r_{1,i} \circ \psi_\sigma, \psi_F). \quad \text{(\#)}$$

Define $\psi_\pi^{gal} : W_F \rightarrow L M_1$ by $w \mapsto \psi_\pi(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix})$. (If one considers $\psi_\pi$ as defined on the Weyl-Deligne group, then $\psi_\pi^{gal}$ is the restriction of $\psi_\pi$ to $W_F$.) It is proved in [Sh, 3.4] that $\gamma(r_i \circ \psi_\pi, \psi_F) = \gamma(r_i \circ \psi_\pi^{gal}, \psi_F)$. As by construction $\psi_\pi^{gal} = \psi_\sigma$, it follows that $\gamma(r_{1,i} \circ \psi_\sigma, \psi_F) = \gamma(r_i \circ \psi_\pi, \psi_F)$, which implies with the equality (\#) the theorem. \hfill \Box

3.3 Corollary: Let $\pi$ be an irreducible smooth generic representation of a standard Levi subgroup $M$ of $G$. Fix a standard parabolic subgroup $P_1 = M_1 U_1$ of $G$ with $M \supseteq M_1$ and an irreducible generic supercuspidal representation $\sigma$ of $M_1$ such that $\pi$ is a sub-representation of $i_{1\cap M}^M \sigma$.

Suppose that there is an admissible homomorphism $\psi_\sigma : W_F \rightarrow L M_1$ such that for any $\alpha \in \Sigma_{\text{red}}(P_1)$ and any irreducible component $r_{\alpha,i}$ of $r_\alpha : L M_{1,\alpha} \rightarrow \text{Lie}(L U_{1,\alpha})$, we have

$$\gamma(r_{\alpha,i} \circ \psi_\sigma, \psi_F) = \gamma(\sigma, r_{\alpha,i}, \psi_F).$$

Then $\sigma$ verifies the assumption $(LM)$ in [H2]. Let $\psi_\pi$ be the admissible homomorphism $W_F \times \text{SL}_2(\mathbb{C}) \rightarrow L M$ attached to $\pi$ in [H2, 5.5]. Then, for any component $r_i$ of the adjoint representation $r : L M \rightarrow \text{Lie}(L U)$, we have

$$L(\gamma, r_i \circ \psi_\pi) = L(\gamma, \pi, r_i) \quad \text{and} \quad \epsilon(r_i \circ \psi_\pi, \psi_F) = \epsilon(\pi, r_i, \psi_F).$$

Proof: We will first consider the case, when $\pi$ is tempered. Then it is by 1.5 enough to show that

$$\gamma(r_i \circ \psi_\pi, \psi_F) = \gamma(\pi, r_i, \psi_F)$$

for any $i$. After possibly changing $\sigma$ (and consequently $\psi_\sigma$) by an unramified character twist (which conserves by (1.3.4) and 1.4 the equalities of $\gamma$-functions), we can find a standard parabolic subgroup $P_2 = M_2 U_2$, $M \supseteq M_2 \supseteq M_1$, and a generic irreducible discrete series representation $\tau$ of $M_2$ which is a sub-representation of $i_{M_2 \cap P_2}^{M_2} \sigma$, such that $\pi$ is a sub-representation of $i_{M_1 P_2}^{M_1} \tau$. Denote by $\kappa_2$ the inclusion
\( L^{M_2} \to L^M \) and put \( r_{2,i} = r_i \circ \kappa_2 \). By the identity (3.1.1) and Theorem 3.2 we have
\[
\gamma(\pi, r_i, \psi_F) = \prod_{\alpha \in \Sigma_{\text{red}}(P_2), U_{2,\alpha} \subseteq U} \gamma(\tau, r_{2,i,\alpha}, \psi_F)
\]
\[
= \prod_{\alpha \in \Sigma_{\text{red}}(P_2), U_{2,\alpha} \subseteq U} \gamma(r_{2,i,\alpha} \circ \psi_\tau, \psi_F)
\]
\[
= \gamma(r_{2,i} \circ \psi_\tau, \psi_F).
\]

As by construction \( \psi_\tau \) and \( \psi_\pi \) take the same values, it follows that
\[
\gamma(r_i \circ \psi_\pi, \psi_F) = \gamma(\pi, r_i, \psi_F).
\]

Let now \( \pi \) be an arbitrary generic smooth representation of \( M \). Then, after possibly changing \( \sigma \) and \( \psi_\sigma \) by an unramified character twist, using Langlands’ classification, there is a semi-standard parabolic subgroup \( P_2 = M_2 U_2 \) of \( G \) with \( M \supseteq M_2 \supseteq M_1 \) and a generic quasi-tempered representation \( \tau \) of \( M_2 \) such that \( \tau \) is a sub-representation of \( i_{P_2 \cap M_2}^{M_2} \sigma \) and \( \pi \) is a sub-representation of \( i_{P_2 \cap M} \tau \). By (1.3.7), \( L(\cdot, \tau, r_i) \) is a product of \( L \)-functions attached to \( \tau \) with respect to simple reflections of \( P_2 \). As \( L(\cdot, r_i \circ \psi_\pi) \) is obtained in the same way from the \( L \)-functions of \( \psi_\tau \), the equality of the \( L \)-functions of \( \pi \) and \( \psi_\pi \) follows from the tempered case proved just before. The proof of the equality of \( \gamma \)-functions is literally the same as for \( \pi \) tempered. The identity for \( \epsilon \)-factors follows from this (cf. 1.3 and 1.5). \( \square \)

4. We will now finish with remarks on the general case, i.e. we will consider representations which are not generic and later also groups which are not quasi-split.

4.1 So suppose first that \( G \) is still the set of \( F \)-points of a quasi-split connected reductive group. In order to define \( L \)-functions and \( \epsilon \)-factors for non generic representations, two assumptions are made in [Sh] (and justified by other more basic assumptions).

(4.1.1) Each tempered \( L \)-packet of a standard Levi subgroup contains a generic representation.

(4.1.2) Harish-Chandra’s \( \mu \)-function defined on discrete series depends only on \( L \)-packets.

Let \( P = MU \) be a standard \( F \)-parabolic subgroup of \( G \) and \( \pi \) a non generic irreducible tempered representation of \( M \). Let \( r_i \) be a component of the adjoint representation \( r : L^M \to \text{Lie}(L^U) \). By assumption (4.1.1) there exists a generic irreducible representation \( \pi' \) in the \( L \)-packet of \( \pi \). One defines \( L(\cdot, \pi, r_i), \epsilon(\cdot, \pi, r_i, \psi_F) \) and \( \gamma(\cdot, \pi, r_i, \psi_F) \) to be \( L(\cdot, \pi', r_i), \epsilon(\cdot, \pi', r_i, \psi_F) \) and \( \gamma(\cdot, \pi', r_i, \psi_F) \) respectively.

Let now \( \pi \) be an arbitrary irreducible smooth non generic representation of \( M \).
By Langlands’ classification, there is a standard $F$-parabolic subgroup $P_1 = M_1U_1$ of $G$, $M \supseteq M_1$, and an irreducible quasi-tempered representation $\tau$ of $M_1$, such that $\pi$ is the unique sub-representation of $i_{P_1}^M \tau$. By assumption (4.1.1) there is a generic quasi-tempered representation $\tau'$ in the $L$-packet of $\tau$ such that $i_{P_1}^M \tau'$ has a unique sub-representation $\pi'$. This representation $\pi'$ may not be generic, but we define $L(\cdot, \pi', r_i)$, $\epsilon(\cdot, \pi', r_i, \psi_F)$ and $\gamma(\cdot, \pi', r_i, \psi_F)$ by the same formulas (see (1.3.7) and following) as in the generic case. The local factors for $\pi'$ are by definition those for $\pi'$.

4.2 To extend our discussion of the results in [H2] to non generic representations of $G$, we have in order to use the results in section 2 and 3 to make the following assumption.

(4.2.1) If $\sigma$ is an irreducible generic supercuspidal representation of an $F$-Levi subgroup $M$ of $G$, then there is an admissible homomorphism $\psi_\sigma : W_F \to L^M$, which has the same local factors with respect to the adjoint action of $L^M$ then $\sigma$.

4.3 We are now able to deduce from this the general version of the assumption (PM) in [H2]:

Lemma: Let $G$ be the set of $F$-points of a quasi-split group, $P = MU$ a standard parabolic subgroup of $G$ and $\sigma$ a unitary irreducible supercuspidal representation of $M$. Suppose that (4.1.1) and (4.2.1) hold. Then there is a discrete series representation $\tau$ in the $L$-packet of $\sigma$, a standard parabolic subgroup $P_1 = M_1U_1$ of $G$, $M \supseteq M_1$, and an irreducible unitary supercuspidal representation $\sigma_1$ of $M_1$, $\tau \subseteq i_{P_1}^M (\sigma_1 \otimes \chi_\lambda)$ for some $\lambda \in a_{M_1}^*$, with the following property with respect to any root $\alpha \in \Sigma(P_1)$:

Let $s$ be a real number $s > 0$. Then $\mu(\sigma_1 \otimes \cdot)$ has a pole in $\chi_{s\tilde{\alpha}}$, if and only if $\tilde{\alpha}(q)^s$ is $q$-distinguished in the connected centralizer of $\psi_\sigma(W_F)$ and this connected centralizer is not a torus.

In addition, one can choose for $\sigma_1$ a generic representation.

Proof: By (4.1.1), there is a generic representation $\tau$ in the $L$-packet of $\sigma$, which must be a discrete series. One can choose $P_1 = M_1U_1$ as in the statement and an irreducible supercuspidal representation $\sigma_1$ of $M_1$ such that $\tau$ is a sub-representation of $i_{P_1}^M \sigma_1$. The representation $\sigma_1$ must be generic by [R, Theorem 2]. So, using the assumption (4.2.1), the corollary 2.4 applies and proves the theorem.

4.4 Theorem: Let $G$ be the set of $F$-points of a quasi-split group, $P = MU$ a standard parabolic subgroup of $G$ and $\pi$ an irreducible smooth representation of $M$. If the assumptions (4.1.1), (4.1.2) and (4.2.1) are verified, then the construction in [H, 5.5], that associates to $\pi$ an admissible homomorphism $\psi_\pi : W_F \times SL_2(C) \to$
\[ L(M, \psi) \circ r_{i} = L(\pi, r_{i}) \quad \text{and} \quad \epsilon(\psi \circ r_{i}, \psi_{F}) = \epsilon(\pi, r_{i}, \psi_{F}). \]

**Proof:** By Langlands’ classification, there is a standard \( F \)-parabolic subgroup \( P_{1} = M_{1}U_{1} \) of \( G \), \( M \supseteq M_{1} \), and an irreducible quasi-tempered representation \( \tau \) of \( M_{1} \), such that \( \pi \) is the unique sub-representation of \( i_{P_{1} \cap M}^{M} \tau \). By assumption (4.1.1) there is a generic representation \( \tau' \) in the \( L \)-packet of \( \tau \). The unique sub-representation \( \pi' \) of \( i_{P_{1} \cap M}^{M} \tau' \) is in the same \( L \)-packet than \( \pi \). By 4.1 it has the same local factors than \( \pi \). The proof of corollary 3.3 generalizes to \( \pi' \), showing that the local factors for \( \pi' \) and \( \psi_{\pi'} \) are the same. As one can choose \( \psi_{\pi} = \psi_{\pi'} \), this proves the theorem. \( \dashv \)

4.5 Consider now that \( G \) is the set of \( F \)-points of an arbitrary connected reductive group defined over \( F \) which may not be quasi-split. It is believed that Harish-Chandra’s \( \mu \)-function is invariant for inner forms (cf. [Sh, 9]). The constructions in [H2] are also invariant for inner forms. Local factors for representations of Levi subgroups of \( G \) are defined by the ones for the corresponding representations for the quasi-split inner form of \( G \). So it is clear that the correspondence must conserve the local factors.

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