Three dimensional Lifshitz-like black hole in a special class of $F(R)$ gravity

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Regarding a special class of pure $F(R)$ gravity in three dimensions, we obtain, analytically, Lifshitz-like black hole solutions. We check the geometrical properties of the solutions which behave such as charged BTZ black holes in special limit. We also investigate the thermodynamic properties of the solutions and examine the first law of thermodynamics and Smarr formula. In addition, we study thermal stability via the heat capacity and discuss the possibility of criticality in the extended phase space.

I. INTRODUCTION

The generalization of Einstein’s Lagrangian to a more general invariant of the Riemann tensor, an arbitrary function of the Ricci scalar, is considered by Buchdahl in 1970 [1]. Unlike Einstein’s gravity, $F(R)$ theory can explain the accelerated expansion [2, 3] and structure formation of the Universe without considering dark energy or dark matter. Such a theory may avoid the known instability [4] and is coincident with Newtonian and post-Newtonian approximations [4, 5]. So, different solutions of $F(R) = R + f(R)$ gravity with various motivations have been proposed and their properties are investigated [6, 13]. Regardless of the trivial spherical symmetric solutions of $F(R)$ models of gravity, analyzing new solutions of this theory with nontrivial topologies are interesting.

Einstein’s gravity cannot explain the non-relativistic scale-invariant theory, and it should be regarded as an effective theory that does not show Galilean symmetry and brakes down at some scales. In order to overcome such a problem, one can use of Horava-Lifshitz [14, 15] approach introducing a metric which shows an anisotropic scale invariant between time and space

$$t \rightarrow \lambda^z t, \quad x \rightarrow \lambda x,$$

where $z$ is called as dynamic critical exponent of the Horava-Lifshitz theory. This theory provides a violation of the Lorentz symmetry in high energies which is a possible characteristic of quantum gravity theory.

The static vacuum solutions of a Lifshitz model in $(2 + 1)$-dimensions has been investigated in [16]. Three dimensional black hole solution introduced by Banados-Teitelboim-Zanelli (BTZ) [17] is one of the interesting subjects for gravitating systems in recent years [18–23]. These black holes have been used to develop the knowledge of gravitational interaction in low dimensional manifolds and also improvement in quantum theory of gravity, string theory and gauge field theory [24, 25]. The holography of the BTZ black holes has been investigated in details [26]. In addition, thermodynamic properties of BTZ black holes have been studied during the last years [27, 28].

Thermodynamical structure of the black holes has been of great interest [30, 31]. Especially in recent years, considering the cosmological constant as a thermodynamical variable and working in the extended phase space lead to find additional analogy between the black holes and the behavior of the van der Waals liquid/gas system [32, 36].

The phase transition of a black hole plays an important role in exploring its critical behavior near the critical point. In order to study the phase transition, one may adopt different approaches [37–39]. One of these methods is studying the behavior of the heat capacity. It is argued that divergence points of the heat capacity hint us the existence of phase transition. In addition, it is notable that a thermally stable black hole has a non-negative heat capacity. In addition, focusing on the Gibbs free energy and its derivatives, one can find the possibility of the phase transition and its order. Moreover, working on the extended phase space, one can extract some useful information through the $PV$ and $PT$ diagrams. The stability of BTZ black hole has been studied in some papers [40]. Here, we are going to evaluate thermal stability of the Lifshitz black holes in three dimensional $F(R)$ gravity with constant Ricci scalar.

The structure of this paper is as follows: At first, we obtain three dimensional Lifshitz-like black hole in the context of a special class of the $F(R)$ gravity and investigate its geometric and thermodynamic properties. Then, we work in the extended phase space thermodynamics and examine thermal stability. We also show that for the obtained Lifshitz-like solutions there is no critical behavior. In addition, we check the validity of the first law of thermodynamics and the Smarr relation, and find that the Lifshitz parameter modifies the conserved charges. Final section is devoted to concluding remarks.

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II. EXACT SOLUTIONS OF $F(R)$ GRAVITY IN THREE DIMENSIONS

In order to study 3-dimensional black holes in pure $F(R)$ gravity, we employ the following action

$$S = \int_\mathcal{M} d^{2+1}x \sqrt{-g} F(R),$$

(2)

where $\mathcal{M}$ is a three-dimensional bulk manifold. In this action, $R$ and $F(R)$ are, respectively, the Ricci scalar which we regard it as a constant ($R_0$) and an arbitrary function of it. It is note that the action is constructed as a pure geometric (gravitational) theory without matter field.

Using the variational principle, it is straightforward to obtain the following field equation

$$G_{\mu\nu}F_R - \frac{1}{2}g_{\mu\nu}[F(R) - RF_R] - [\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box]F_R = 0,$$

(3)

where $F_R = \frac{dF(R)}{dR}$.

Hereafter, we follow the method of Ref. [41], which is applicable for the special class of $F(R)$-gravity models satisfying two constraints, simultaneously, $F(R_0) = 0$ and $F_R = 0$. Regarding the mentioned constraints, we find that the vacuum field equation (3) are automatically satisfied with arbitrary Ricci scalar $R_0$. It is notable that the mentioned class of $F(R)$-theory does not lead to the usual general relativity since the vacuum field equation of general relativity identically satisfied $F_R = 1$ with vanishing Ricci scalar. As it is mentioned in [41], there are several models for the early-time inflation or late-time accelerated expansion that can satisfy the mentioned constraints.

Here, our main motivation is to study the thermodynamic aspects of black hole solutions in a Lifshitz-like background spacetime. Therefore, we consider the metric of 3-dimensional spacetime as

$$ds^2 = -\left(\frac{r}{r_0}\right)^2 B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\phi^2,$$

(4)

where $z$ is a real number so called Lifshitz -like parameter and $r_0$ is an arbitrary (positive) length scale. Considering the mentioned metric, we can extract the metric function for $R = R_0$, where

$$R_0 = -B'' - \frac{3z + 4}{2r}B' - \frac{z^2}{2r^2}B,$$

(5)

where we used the usual notation $B = B(r), B' = \frac{dB(r)}{dr}$ and $B'' = \frac{d^2B(r)}{dr^2}$ for the sake of brevity. Considering Eq. (5), we can find the following exact solution

$$B(r) = -\frac{m}{r^\gamma} + \frac{q^2}{r^\delta} - \Lambda r^2,$$

(6)

in which $m$ and $q$ are two integration constants of the second order differential equation and $\Lambda$ is (positive/negative or zero) constant that its value is depending on the sign/value of $R_0$ as

$$\Lambda = \frac{2R_0}{z^2 + 6z + 12}.$$

In addition, $m$ and $q$ are two integration constants while $\gamma$ and $\delta$ are defined as

$$\gamma = \frac{1}{4} \left(3z + 2 - \sqrt{z^2 + 12z + 4}\right),$$

(7)

$$\delta = \frac{1}{4} \left(3z + 2 + \sqrt{z^2 + 12z + 4}\right).$$

(8)

In order to interpret the solutions as black holes, we should examine the existence of horizon and singularity for the singular black holes. The presence of singularity could be investigated by studying curvature scalars for which we choose the Kretschmann scalar. It is a matter of calculation to show that for these solutions, the Kretschmann scalar is

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = B'' + \frac{z}{r^2}(3rB' + (z - 2)B)B'' + \frac{9z^2 + 2}{4r^2}B^2 + \frac{z(3z^2 - 6z + 4)}{2r^3}BB' + \frac{z^2(3z^2 - 6z + 4)}{4r^4}B^2.$$

(9)
Regarding Eq. (9) with the obtained metric function, we find that Kretschmann scalar diverges at \( r = 0 \) and is finite for \( r \neq 0 \). In addition, according to the Fig. 1, one finds the metric function has at least one real positive root (with positive slope), and therefore, the mentioned solutions can be interpreted as black holes. In other words, one finds that depending on the values of parameters \((z, m, \Lambda \text{ and } q)\), the metric function has two real positive roots \((r_- \text{ and } r_+)\), one degenerate root \((r_{\text{ext}} = -(\gamma + 2)\Lambda (\delta - \gamma)q^2)\) or it may be positive definite. Therefore, these solutions may be interpreted as black holes with two horizons (one Cauchy horizon at \( r_- \) and one event horizon at \( r_+ \)), extreme black holes \((B(r)|_{r=r_{\text{ext}}} = B'(r)|_{r=r_{\text{ext}}} = 0)\) or naked singularity (see all panels of Fig. 1 for more details).

Regarding \( z = 0 \), one may expect to obtain the usual BTZ black hole solutions. However in this limit the metric function reduces to \( B(r) = -\Lambda r^2 - M + \frac{q^2}{r} \). The first two terms are the usual cosmological constant and mass term while third term differs from the charge-term of BTZ-Maxwell solutions which is logarithmic form. Nonetheless, we find that the last term can be interpreted as a charge-term in power-Maxwell nonlinear electrodynamics with the following total charge

\[
Q = \frac{3}{16} \frac{2^4}{r_0^{\delta - 1}} \sqrt{q}.
\]
In other words, such a metric function is completely in agreement with that of charged black hole solution of Einstein-power Maxwell invariant (Einstein-PMI) gravity when the nonlinearity parameter is chosen $s = \text{dimension}/4$ which is conformally invariant Maxwell source. This result implies an interesting result. Indeed, $F(R)$ gravity provides a framework for putting the gravity and PMI nonlinear electrodynamics in a unified context through the pure geometry. It is notable that the same approach is applied in four dimensional $F(R)$ gravity \[42\] and its solution is in agreement with the above statement ($s = 1$ since $\text{dimension} = 4$). The general form of $d$–dimensional solutions is addressed in the appendix.

III. THERMODYNAMIC BEHAVIOR AND THERMAL STABILITY

Regarding an exact solution of gravitational field equation, we have to check the stability of the solutions. There are two main stability criteria which a known as dynamic and thermodynamic stability. In this paper we investigate the latter one. Considering the black hole as a thermodynamic system, one has to examine the validity of the first law and thermal stability. To do so, we should calculate thermodynamic and conserved quantities.

A. Conserved and thermodynamic quantities

Since the employed metric contains a temporal Killing vector ($\partial/\partial t$), we can use the concept of surface gravity ($\kappa$) to calculate the temperature of black holes at the event horizon $r_+$.

$$T = \frac{\kappa}{2\pi} = \lim_{{r \to r_+}} \frac{B'(r)}{4\pi} \left( \frac{r}{r_0} \right)^{\frac{2}{{s}}}.$$

(11)

Regarding Eq. (11) with the metric function (6), one can find

$$T = \left( \frac{\pi r_0}{4\pi} \right)^{\frac{2}{{s}}} \left[ \frac{q^2(\gamma - \delta)}{r_+^{1+\delta}} - (2 + \gamma)\Lambda r_+ \right].$$

(12)

where $r_+$ is the radius of event horizon determined from $B(r_+) = 0$ due to the fact that the metric function vanishes on the event horizon.

Comparing the solutions with BTZ black holes or using the Ashtekar-Magnon-Das (AMD) formula for a far away observer, we find that the finite total mass can be written as

$$M = \frac{m_0}{8} = \frac{m}{8} r_0^{-\gamma}.$$

(13)

Here, we desire to examine the validity of the first law of thermodynamics. Evaluating the metric function on the event horizon ($B(r_+) = 0$), one can obtain the geometrical mass, $m_0$, as a function of $r_+$. Inserting $m_0(r_+)$ in Eq. (13), one finds

$$M = \frac{1}{8} \left( -\Lambda r_+^2 - \frac{q^2}{r_+^2} \right) \left( \frac{r_+}{r_0} \right)^{\gamma}.$$

(14)

Since we consider a special class of $F(R)$ gravity with $F_R = 0$, calculation of the entropy based on Wald’s formula is problematic. In order to obtain the entropy of black holes in such a class of $F(R)$ gravity, one can respect the validity of the first law of thermodynamics. So we can obtain the entropy as

$$\delta S = \frac{1}{T} \delta M,$$

and therefore, it is a matter of calculation to show that the following relation holds

$$S = \int \frac{dM}{T} = \frac{\pi r_+}{2\gamma - z + 2} \left( \frac{r_0}{r_+} \right)^{\frac{2}{{s}} - \gamma}.$$

(15)

It is notable that the obtained relation for the entropy reduces to the area law for $z = 0$ ($\gamma = 0$).
B. Examine thermal stability and phase transition

Regarding the variation of the cosmological constant as the vacuum expectation value of a quantum field, one may expect to consider it and its conjugate in the first law of thermodynamics. It is notable that the finite mass of black holes interpreted as the enthalpy \( M \equiv H \) of the system rather than the internal energy in the extended phase space.

\[
P = -\frac{\Lambda}{8\pi},
\]
\[
V = \frac{\partial M}{\partial P}_{S,q} = \pi r_+^2 \left( \frac{r_+}{r_0} \right)^\gamma.
\]

Regarding the relation of temperature \( \text{(12)} \) with the mentioned equation of pressure \( \text{(16)} \), we can obtain the so-called equation of state

\[
P = \frac{1}{2r_+^2(\gamma + 2)} \left[ \frac{Tr_+}{\left( \frac{r_+}{r_0} \right)^\gamma} + \frac{q^2(\delta - \gamma)}{4\pi r_+^4} \right].
\]

To obtain the critical point, we use the feature of the inflection point of the \( P-V \) diagram at the critical point. In other words, the first and second derivatives of the pressure with respect to the volume vanish at the critical point, i.e.:

\[
\left( \frac{\partial P}{\partial r_+} \right)_T = 0,
\]
\[
\left( \frac{\partial^2 P}{\partial r_+^2} \right)_T = 0,
\]

in which after some calculations, we find

\[
q^2(\gamma - \delta)(\delta + 2) \left( \frac{r_0}{r_+} \right)^\frac{1+\delta}{\gamma} r_+^\frac{1+\delta}{\gamma} - 2\pi T(z + 2) = 0,
\]
\[
q^2(\delta - \gamma)(\delta + 2)(\delta + 3) \left( \frac{r_0}{r_+} \right)^\frac{1+\delta}{\gamma} r_+^\frac{1+\delta}{\gamma} + \pi T(z + 2)(z + 4) = 0.
\]

According to the equations \( \text{(21)} \) and \( \text{(22)} \), it is clear that these equations cannot admit any critical point for a positive real value of \( r_+ \). To conclude, for the Lifshitz-like black holes in three dimensions addressed in this paper, we could not observe any van der Waals like behavior. So it seems that obtained solutions may be thermally stable. To confirm this statement, we can calculate the heat capacity as

\[
C_{P,Q} = T \left( \frac{\partial S}{\partial T} \right)_{P,Q} = \frac{\pi r_+}{\left( \frac{r_0}{r_+} \right)^\frac{1}{\gamma}} \frac{A}{B},
\]

where

\[
A = r_+^{\delta + 2} P - \frac{(\delta - \gamma)q^2}{8\pi(\gamma + 2)},
\]
\[
B = (z + 2)r_+^{2+\delta} P + \frac{(\delta - \gamma)(2\delta + 2 - z)q^2}{8\pi(\gamma + 2)}.
\]

According to Fig. 2, we find that the heat capacity has no divergence point which is due to the fact that \( B \) is a positive definite function. In addition, one can find a real positive root for \( A = 0 \) called bound point \( r_0 \). We should note that \( r_0 \) is the root of temperature, simultaneously, and both \( T \) and \( C_{P,Q} \) are positive definite functions for \( r_+ > r_0 \). Roughly speaking, the obtained solutions are thermally stable.
The Smarr relation [44] as one of thermodynamical relation in the case of black holes has attracted attention. In this section, we derived this relation for black holes in three dimensional with Lifshitz-like spacetime. Smarr relation, together with the first law of black hole thermodynamics [45] has a main role in black hole physics. The variable cosmological constant is considered as a thermodynamic pressure to explain scaling relation of Smarr formula [46–50]. Taking into account the scaling argument for our Lifshitz like solutions in the extended phase space, one can find the following Smarr relation is hold

\[ \gamma M = (1 + \gamma - \frac{2}{\delta}) TS + \frac{2}{3} \Phi Q - 2PV, \tag{26} \]

where \( T = \left( \frac{\partial M}{\partial S} \right)_Q \) is temperature as calculated in Eq. (12) and \( \Phi \) is a modified potential that calculated at the event horizon of Lifshitz like black hole solutions as

\[ \Phi = \left( \frac{\partial M}{\partial Q} \right)_S = q r_+ \left( \frac{r_+}{r_0} \right)^{1+\gamma-\frac{2}{\delta}}. \tag{27} \]

It is notable that for \( z = 0 \), Eq. (26) reduces to that of nonlinearly charged BTZ black holes in which mass term has no scaling.
IV. CONCLUSION

In this paper, we have obtained a new Lifshitz-like charged black hole solutions in three dimensional pure $F(R)$ gravity. We have discussed geometrical properties of the solutions and found that these solutions reduce to charged BTZ like solutions in Einstein-Λ-PMI gravity.

We have also investigated thermodynamic quantities of black holes and showed that these quantities satisfy the first law of thermodynamics. Next, we look for possible phase transition in the extended phase space thermodynamics and found that the three dimensional Lifshitz-like black holes do not have any critical behavior. Then, we studied thermal stability and calculated the heat capacity of the Lifshitz-like black hole solutions in canonical ensembles and found that regarding the root of the temperature as a bound point ($r_0$), the obtained solutions are thermally stable for $r_+ > r_0$. Finally, we obtain modified Smarr relation in the presence of Lifshitz parameter and found that regardless of the cosmological constant term, the scaling of other thermodynamic quantities is modified.

V. APPENDIX

Here, we obtain the higher dimensional topological black hole solutions in a Lifshitz-like background spacetime. Thus, we consider the metric of $d$-dimensional spacetime as

$$ds^2 = -\left(\frac{r}{r_0}\right)^{2}B(r)dt^2 + \frac{dr^2}{B(r)} + r^2d\Omega^2,$$

where

$$d\Omega^2 = \begin{cases} 
    dx_1^2 + \sum_{i=2}^{d-2} \prod_{j=1}^{i-1} \sin^2 x_j dx_i^2, & k = 1 \\
    dx_1^2 + \sinh^2 x_1 \left( dx_2^2 + \sum_{i=3}^{d-1} \prod_{j=2}^{i-1} \sin^2 x_j dx_i^2 \right), & k = 0 \\
    \sum_{i=1}^{d-2} dx_i^2, & k = -1
\end{cases}.$$

Considering Eq. (28), we can extract the metric function for $R = R_0$, where

$$R_0 = -B'' - \frac{3z + 4d - 8}{2r} B' - \frac{z^2 + 2z(d - 3) + (d - 2)(d - 3)}{2r^2} B + \frac{k(d - 2)(d - 3)}{r^2},$$

with the following exact solutions

$$B(r) = K - \frac{m}{r^\gamma} + \frac{q^2}{r^\delta} - \Lambda r^2,$$

in which $K$ is related to the horizon topology and Λ is a (positive/negative or zero) constant that its value is depending on the sign/value of $R_0$ with the following explicit forms

$$K = \frac{2(d - 2)(d - 3)k}{z^2 + 2z(d - 3) + 2(d - 2)(d - 3)},$$
$$\Lambda = \frac{2R_0}{z^2 + 2dz + 2d(d - 1)}.$$

In addition, $m$ and $q$ are two integration constants while $\gamma$ and $\delta$ are defined as

$$\gamma = \frac{1}{4} \left( 3z + 4d - 10 - \sqrt{z^2 + (8d - 12)z + 4} \right),$$
$$\delta = \frac{1}{4} \left( 3z + 4d - 10 + \sqrt{z^2 + (8d - 12)z + 4} \right).$$
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