Catalan numbers and power laws in cellular automaton rule 14

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Abstract

We discuss example of an elementary cellular automaton for which the density of ones decays toward its limiting value as a power of the number of iterations n. Using the fact that this rule conserves the number of blocks 10 and that preimages of some other blocks exhibit patterns closely related to patterns observed in rule 184, we derive expressions for the number of n-step preimages of all blocks of length 3. These expressions involve Catalan numbers, and together with basic properties of iterated probability measures they allow us to compute the density of ones after n iterations, as well as probabilities of occurrence of arbitrary block of length smaller or equal to 3.

1. Introduction

A question of interest in the theory of cellular automata (CA) is the action of the global function of a given cellular automaton on an initial probability measure. Typically, one starts with a Bernoulli measure and wants to determine the measure after n applications of the CA rule. More informally, we start with an initial configuration where each site is (in the case of binary rules) equal to 1 with probability p and equal to 0 with probability 1 − p, independently for all sites. Then we ask: after n iterations of the CA rule, what is the probability of occurrence of a given block b in the resulting configuration? The simplest question of this type is often phrased informally as follows: what is the density of ones ρn

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after $n$ iterations of the CA rule over a “random” initial configuration with 50% of zeros and 50% of ones? Here by “density” one understands the probability of occurrence of 1.

It is usually very hard to answer questions like this rigorously, and quite often one has to resort to numerical experiments. Such experiments reveal that for many cellular automata, $\rho_n$ exponentially decays toward some limiting value $\rho_\infty$. In physics this is sometimes called “exponential relaxation to equilibrium”. Other types of relaxation to equilibrium, such as power law decay toward $\rho_\infty$, are less common in CA, and to our knowledge, no such case has ever been discussed rigorously in the CA literature.

In this paper we demonstrate that in the elementary CA rule 14 the decay of $\rho_n$ toward its limiting value follows a power law. We show that the origin of this power law is the scaling of the numbers of $n$-step preimages of certain finite blocks. The precise numbers of preimages of these blocks can be expressed in terms of Catalan numbers, similarly as previously reported for the rule 184 [1].

For large $n$, one can approximate Catalan numbers via Stirling formula, obtaining as a result the power law $\rho_n - \rho_\infty \sim n^{-1/2}$.

The paper is organized as follows. Following basic definitions, we present a theorem on enumeration of preimages of basic blocks in rule 14. The proof of this theorem requires four technical lemmas, which are proved in sections following the proof of the main theorem. The formula for density of ones is then proved using the enumeration theorem, and the asymptotic power law is derived.

2. Basic definitions

Let $\mathcal{G} = \{0, 1, \ldots, N - 1\}$ be called a symbol set, and let $\mathcal{S}(\mathcal{G})$ be the set of all bisequences over $\mathcal{G}$, where by a bisequence we mean a function on $\mathbb{Z}$ to $\mathcal{G}$. Set $\mathcal{S}(\mathcal{G})$, which is a compact, totally disconnected, perfect, metric space, will be called the configuration space. Throughout the remainder of this text we shall assume that $\mathcal{G} = \{0, 1\}$, and the configuration space $\mathcal{S}(\mathcal{G}) = \{0, 1\}^{\mathbb{Z}}$ will be simply denoted by $\mathcal{S}$.

A block of length $r$ is an ordered set $b_0b_1\ldots b_{n-1}$, where $n \in \mathbb{N}$, $b_i \in \mathcal{G}$. Let $n \in \mathbb{N}$ and let $\mathcal{B}_n$ denote the set of all blocks of length $n$ over $\mathcal{G}$. The number of elements of $\mathcal{B}_n$ (denoted by $\text{card} \mathcal{B}_n$) equals $2^n$.

For $r \in \mathbb{N}$, a mapping $f : \{0, 1\}^{2r+1} \mapsto \{0, 1\}$ will be called a cellular automaton rule of radius $r$. Alternatively, the function $f$ can be considered as a mapping of $\mathcal{B}_{2r+1}$ into $\mathcal{B}_0 = \mathcal{G} = \{0, 1\}$.

Corresponding to $f$ (also called a local mapping) we define a global mapping $F : \mathcal{S} \mapsto \mathcal{S}$ such that $(F(s))_i = f(s_{i-r}, \ldots, s_i, \ldots, s_{i+r})$ for any $s \in \mathcal{S}$. The composition of two rules $f, g$ can be now defined in terms of their corresponding global mappings $F$ and $G$ as $(F \circ G)(s) = F(G(s))$, where $s \in \mathcal{S}$.

A block evolution operator corresponding to $f$ is a mapping $f : \mathcal{B} \mapsto \mathcal{B}$ defined as follows. Let $r \in \mathbb{N}$ be the radius of $f$, and let $a = a_0a_1\ldots a_{n-1} \in \mathcal{B}_n$ where $n \geq 2r + 1 > 0$. Then

$$f(a) = \{f(a_i, a_{i+1}, \ldots, a_{i+2r})\}_{i=0}^{n-2r-1}.$$  \hfill (1)

Note that if $b \in \mathcal{B}_{2r+1}$ then $f(b) = f(b)$.
We will consider the case of \( G = \{0, 1\} \) and \( r = 1 \) rules, i.e., elementary cellular automata. In this case, when \( b \in B_3 \), then \( f(b) = f(b) \). The set \( B_3 = \{000, 001, 010, 011, 100, 101, 110, 111\} \) will be called the set of basic blocks.

The number of \( n \)-step preimages of the block \( b \) under the rule \( f \) is defined as the number of elements of the set \( f^{-n}(b) \). Given an elementary rule \( f \), we will be especially interested in the number of \( n \)-step preimages of basic blocks under the rule \( f \).

### 3. Rule 14

Local function of the elementary cellular automaton rule 14 is defined as

\[
\bar{f}(x_0, x_1, x_2) = x_1 + x_2 + x_1x_0x_2 - x_1x_2 - x_0x_2 - x_1x_0.
\]  

(2)

This means that \( f(0, 0, 1) = f(0, 1, 0) = f(0, 1, 1) = 1 \), and \( f(x_0, x_1, x_2) = 0 \) for all other triples \((x_0, x_1, x_2) \in \{0, 1\}^3\).

Preimage sets of basic blocks, that is, sets \( f^{-n}(b) \) for \( b \in B_3 \), have rather complex structure for this rule. Figure 1 shows, as an example, preimages \( f^{-n}(b) \) for \( b = 101 \) for \( n = 1, 2, 3 \). Preimages are represented in the form of a tree rooted at 101. Preimages \( f^{-1}(101) \) are 01010, 01001, and 01101, and they are shown as the first level of the tree, \( f^{-2}(101) \) as the second level, and \( f^{-3}(101) \) as the third level. When one block \( a \) is the image of another block \( b \), that is, \( f(b) = a \), an edge from \( a \) to \( b \) is drawn.

In spite of the apparent complexity of sets \( f^{-n}(b) \) for basic blocks \( b \), it is possible to determine cardinalities of these sets.

**Theorem 3.1.** Let \( f \) be the block evolution operator for CA rule 14. Then for any positive integer \( n \) we have

\[
\begin{align*}
\text{card } f^{-n}(000) & = (4n + 3)C_n, \\
\text{card } f^{-n}(001) & = 2^{2n+1} - (2n + 1)C_n, \\
\text{card } f^{-n}(010) & = 2(n + 1)C_n, \\
\text{card } f^{-n}(011) & = 2^{2n+1} - 2(n + 1)C_n, \\
\text{card } f^{-n}(100) & = 2^{2n+1} - (2n + 1)C_n, \\
\text{card } f^{-n}(101) & = (2n + 1)C_n, \\
\text{card } f^{-n}(110) & = 2^{2n+1} - 2(n + 1)C_n, \\
\text{card } f^{-n}(111) & = 0,
\end{align*}
\]

where \( C_n \) is the \( n \)-th Catalan number

\[
C_n = \frac{1}{2n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}.
\]  

(3)

Proof of this theorem will be based on the following four lemmas.
Lemma 3.2. Let

\[ f_{184}(x_0, x_1, x_2) = x_0 + x_1 x_2 - x_1 x_0, \]  
\[ f_{195}(x_0, x_1, x_2) = 1 - x_1 - x_0 + 2x_1 x_0 \]  

for \( x_0, x_1, x_2 \in \{0, 1\} \). Then if \( x_0, x_1, x_2, x_3, x_4 \in \{0, 1\} \) and at least of one of \( x_1, x_2 \) is equal to zero, we have

\[ f_{195}(f_{14}(x_0, x_1, x_2), f_{14}(x_1, x_2, x_3), f_{14}(x_2, x_3, x_4)) = \]
\[ f_{184}(f_{195}(x_0, x_1, x_2), f_{195}(x_1, x_2, x_3), f_{195}(x_2, x_3, x_4)) \]  

Lemma 3.3. For any \( n \in \mathbb{N} \), the number of \( n \)-step preimages of 101 under the rule 14 is the same as the number of \( n \)-step preimages of 000 under the rule 184, that is,

\[ \text{card} f_{14}^{-n}(101) = \text{card} f_{184}^{-n}(000), \]
where subscripts 184 and 14 indicate block evolution operators for, respectively, CA rules 184 and 14. Moreover, the bijection $M_n$ from the set $f_{184}^{-n}(000)$ to the set $f_{14}^{-n}(101)$ is defined by

$$M_n(x_0x_1\ldots x_m) = \left\{ n + j + 1 + \sum_{i=0}^{j} x_i \mod 2 \right\}_{j=0}^{m}$$

for $m \in \mathbb{N}$ and for $x_0x_1\ldots x_m \in \{0,1\}^m$.

**Lemma 3.4.** For any $n \in \mathbb{N}$, we have

$$\text{card } f_{14}^{-n}(10) = \text{card } f_{14}^{-n}(01) = 4^n. \quad (9)$$

**Lemma 3.5.** For any $n \in \mathbb{N}$,

$$f_{14}^{-n}(101) = R[R[f_{14}^{-n-1}(010)]], \quad (10)$$

where $R$ is the right truncation operator defined as $R(b_0b_1\ldots b_n) = b_0b_1\ldots b_{n-1}$.

### 4. Proof of the main theorem

We will prove the theorem first assuming validity of all four lemmas. Proofs of lemmas will be presented in section 5.

Since Lemma 3.3 implies that sets $f_{184}^{-n}(000)$ and $f_{14}^{-n}(101)$ have the same cardinality, and since it has been shown in [1] that

$$\text{card } f_{184}^{-n}(000) = (2n+1)C_n, \quad (11)$$

we immediately obtain the desired result $\text{card } f_{14}^{-n}(101) = (2n+1)C_n$.

Lemma 3.5 immediately yields

$$\text{card } f^{-n}(010) = 2(n+1)C_n. \quad (12)$$

Since by definition of $f$, $\text{card } f^{-n}(101) + \text{card } f^{-n}(100) = 2 \text{card } f^{-n}(10)$, we obtain

$$\text{card } f^{-n}(100) = 2 \text{card } f^{-n}(10) - f^{-n}(101), \quad (13)$$

and using Lemma 3.4

$$\text{card } f^{-n}(100) = 2^{2n+1} - (2n+1)C_n. \quad (14)$$

Similar argument leads to

$$\text{card } f^{-n}(001) = 2^{2n+1} - (2n+1)C_n. \quad (15)$$

The formula for $\text{card } f^{-n}(011)$ can be obtained by observing, again from the definition of $f$, that $\text{card } f^{-n}(011) + \text{card } f^{-n}(010) = 2 \text{card } f^{-n}(01)$, and by using Lemma 3.4. Proof of the formula for $\text{card } f^{-n}(110)$ is similar.
Thus, the only cases which we are missing are 000 and 111. The case of 111 is a direct consequence of the fact that the block 111 has no preimages under CA rule 14, which can be easily verified by direct computation.

The last case, formula for \( \text{card} f^{-n}(000) \), follows from the fact that
\[
\sum_{b \in B_3} \text{card} f^{-n}(b) = \text{card} B_{2n+3} = 2^{2n+3}.
\]  
This yields
\[
\text{card} f^{-n}(000) = 2^{2n+3} - \sum_{i=1}^{7} \text{card} f^{-n}(\beta_i) = (4n + 3)C_n.
\]  

5. Proofs of Lemmas

5.1. Proof of Lemma 3.2

Using definitions of \( f_{195}, f_{14}, \) and \( f_{184} \), we obtain after simplification and after using \( x^2 = x \) for \( x \in \{0, 1\} \) the following expressions:
\[
f_{195}(f_{14}(x_0, x_1, x_2), f_{14}(x_1, x_2, x_3), f_{14}(x_2, x_3, x_4)) =
1 - x_1 + x_1x_0x_2 - x_2x_1x_3 - x_0x_2 + x_1x_0 + x_2x_3 + x_1x_3 - x_3
\]  
and
\[
f_{184}(f_{195}(x_0, x_1, x_2), f_{195}(x_1, x_2, x_3), f_{195}(x_2, x_3, x_4)) =
1 + x_1x_0 - x_0x_2 + x_2x_3 + x_1x_3 - x_3 - x_1.
\]  
Subtracting the right hand side of eq (19) from the right hand side of eq (18) we obtain
\[
x_1x_0x_2 - x_2x_1x_3, \text{ which vanishes if one of } x_1, x_2 \text{ is equal to zero.} \]

5.2. Proof of Lemma 3.3

We need to prove that the mapping \( M_n \) is indeed a bijection from \( A := f_{184}^{-n}(000) \) to \( B := f_{14}^{-n}(101) \). It will be enough to show that \( M_n \) is injective, that is, it has the left inverse, and that \( M_n \) is onto.

Before we proceed, let us make two observations. First, note that \( f_{14}^{-1}(101) = \{01001, 01010, 01011\} \) and \( f_{14}^{-1}(010) = \{10100, 010101, 10110, 10111\} \). This can be verified by direct computations. As an immediate result, we see that all elements of \( f_{14}^{-n}(101) \) begin with 101 when \( n \) is even, and with 010 when \( n \) is odd.

Secondly, again by direct computation, note that \( f_{184}^{-1}(000) = \{00000, 00001, 00010\} \), which implies that elements of \( f_{184}^{-1}(000) \) always begin with 000, for any \( n \in \mathbb{N} \).

For \( m \in \mathbb{N} \), define the block transformation \( T(x_0x_1 \ldots x_{m-1}) = y_0y_1 \ldots y_{m-1} \) such that
\[
[T(x)]_i = \begin{cases} 
0 & \text{if } i = 0, \\
1 + x_{i-1} + x_i \mod 2 & \text{if } i > 0.
\end{cases}
\]  

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We claim that $T$ is the required left inverse of $M_n$, that is, $(T \circ M)(x) = x$ for every $x \in A$. By definition of $T_n$ we have

$$[T(M(x))]_i = \begin{cases} 0 & \text{if } i = 0, \\ 1 + [M(x)]_{i-1} + [M(x)]_i \mod 2 & \text{if } i > 0. \end{cases} \tag{21}$$

Since, as observed at the beginning of this proof, all preimages of 000 start with 0, we know that $x_0 = 0$ for all $x \in A$, and (21) yields $[T(M(x))]_i = x_i$ for $i = 0$, as required. We need to deal with $i > 0$ case separately. By definition of $M_n$, for $i > 0$

$$[T(M(x))]_i = 1 + n + i - 1 + 1 + \sum_{l=0}^{i-1} x_l + n + i + 1 + \sum_{l=0}^{i} x_l \mod 2. \tag{22}$$

Since $a + a \mod 2 = 0$ for any $a \in \{0, 1\}$, eq. (22) reduces to $[T(M(x))]_i = x_i$, as necessary for $T$ to be the left inverse of $M_n$. $M_n$ is therefore injective.

To show that $M_n$ is onto, we need to prove that for every $b \in B$ there exists $a$ such that $a \in A$ and $M_n(a) = b$. Let us choose some $b \in B$ and take $a = T(b)$. Using Lemma 3.2, the fact that

$$1 + x_0 + x_1 \mod 2 = f_{195}(x_0, x_1, x_2), \tag{23}$$

and knowing that $b$ starts with either 101 or 010, we can easily show that

$$T \circ f_{14}(b) = f_{184} \circ T(b). \tag{24}$$

Therefore, $f_{184}(T(b)) = T(f_{14}(b)) = T(101) = 000$, which shows that $a = T(b)$ is indeed a member of $A$. Now we need to show that $M_n(a) = b$, or equivalently that $M_n(T(b)) = b$. By definition of $M_n$,

$$[M_n(T(b))]_j = n + j + 1 + \sum_{i=0}^{j} b_i \mod 2, \tag{25}$$

hence

$$[M_n(T(b))]_j = \begin{cases} n + 1 \mod 2 & \text{if } j = 0, \\ n + j + 1 + \sum_{i=1}^{j} (1 + b_{i-1} + b_i) \mod 2 & \text{if } j > 0. \end{cases} \tag{26}$$

Basic properties of $\mod 2$ operation reduce this to

$$[M_n(T(b))]_j = \begin{cases} n + 1 \mod 2 & \text{if } j = 0, \\ n + 1 + b_0 + b_j \mod 2 & \text{if } j > 0. \end{cases} \tag{27}$$

As remarked at the beginning, preimages of 101 begin with 010, while preimages of 010 begin with 101. This means that $n$-step preimages of 101 under $f_{14}$ start either with 0 for odd $n$ or with 1 for even $n$, or equivalently, for $b \in f_{14}^{-n}(101)$, $b_0 = n + 1 \mod 2$. Using this fact, we can see that eq. (27) reduces to $[T(M(b))]_j = b_j$, which proves that $M_n(a) = b$ and concludes the proof of surjectivity of $M_n$. □
5.3. Proof of Lemma 3.4

Let $n \in \mathbb{N}$, and let $x \in \{0, 1\}^\mathbb{Z}$ be a periodic binary bisequence of period $T \in \mathbb{N}$ which has the property that all possible blocks $b$ of length $2n+2$ appear in one period of $x$ exactly once. We say that a block $b$ of length $m$ appears in one period of $x$ if there exists integer $i$, $0 \leq i < T$, such that $b_k = x_{i+k}$ for all $k = 0, 1, \ldots, m-1$.

Existence of $x$ with the aforementioned property is guaranteed by the existence of a Hamiltonian cycle in de Bruijn graph of dimension $2n+1$ [2, 3]. Note that the total number of blocks of length $2n+2$ in $x$ is $2^{2n+2} = 4^{n+1}$.

Let us now iterate rule 14 $n$ times starting with $x$, obtaining $x'$. To be more precise, if $F_{14}$ is the global function corresponding to $f_{14}$ defined in eq. (2), we take $x' = F_{14}^n(x)$.

Since rule 14 conserves the number of blocks 10 [4], the number of blocks 10 in one period of $x'$ is the same as in one period of $x$.

Among all possible blocks of length $2n+2$, exactly $\text{card} f_{14}^{-n}(10)$ produced block 10 in one period of $x'$. Thus, the number of blocks 10 in one period of $x'$ is $\text{card} f_{14}^{-n}(10)$. Because the number of blocks 10 is conserved, the number of blocks 10 in one period of $x$ is also $\text{card} f_{14}^{-n}(10)$. Yet among all possible blocks of length $2n+1$, exactly $1/4$ of all of them begin with 10 (the remaining ones begin with 01, 00, or 11). Thus, $\text{card} f_{14}^{-n}(10) = \frac{1}{4}4^{n+1} = 4^n$.

Proof for the block 01 is similar. □

5.4. Proof of Lemma 3.5

Let $b \in f_{14}^{-(n+1)}(01).$ Then $f_{14}^{n}(b) \in f_{14}^{1}(01) = \{10100, 10101, 10110, 10111\}$, and therefore $f_{14}^{n}(R^2(b)) = 101.$ This implies $R^2(b) \in f_{14}^{-n}(101)$, and therefore

$$R^2 \left( f_{14}^{-(n+1)} \right) \subset f_{14}^{-n}(101).$$

(28)

Now take a block $a \in f_{14}^{-n}(101)$, which implies that $f_{14}^{-n}(a) = 101$. For any $u, v \in \{0, 1\}$ there exist $u', v' \in \{0, 1\}$ such that $f_{14}^{-n}(auv) = 101u'v'$. This implies that $f_{14}^{-(n+1)}(auv) = 010$, and therefore $auv \in f_{14}^{-(n+1)}(010), sO that a \in R^2(f_{14}^{-(n+1)}(010)).$ This demonstrates that $f_{14}^{-n}(101) \subset R^2 \left( f_{14}^{-(n+1)} \right),$ which together with eq. (28) leads to the conclusion that sets $f_{14}^{-n}(101)$ and $R^2 \left( f_{14}^{-(n+1)} \right)$ are equal, as required. □

6. Evolution of measures

We will now come back to the question posed in the introduction. The appropriate mathematical description of an initial distribution of configurations is a probability measure $\mu$ on $S$. Such a measure can be formally constructed as follows. If $b$ is a block of length $k$, i.e., $b = b_0 b_1 \ldots b_{k-1}$, then for $i \in \mathbb{Z}$ we define a cylinder set

$$C_i(b) = \{s \in S : s_i = b_0, s_{i+1} = b_1, \ldots, s_{i+k-1} = b_{k-1}\}.$$

The cylinder set is thus a set of all possible configurations with fixed values at a finite number of sites. Intuitively, measure of the cylinder set given by the block $b = b_0 \ldots b_{k-1}$, denoted
by $\mu[C_i(b)]$, is simply a probability of occurrence of the block $b$ starting at $i$. If the measure $\mu$ is shift-invariant, than $\mu(C_i(b))$ is independent of $i$, and we will therefore drop the index $i$ and write simply $\mu(C(b))$.

The Kolmogorov consistency theorem states that every probability measure $\mu$ satisfying the consistency condition

$$\mu[C_i(b_1 \ldots b_k)] = \mu[C_i(b_1 \ldots b_k, 0)] + \mu[C_i(b_1 \ldots b_k, 1)]$$

extends to a shift invariant measure on $\mathcal{S}$. For $p \in [0, 1]$, the Bernoulli measure defined as $\mu_p[C(b)] = p^j(1-p)^{k-j}$, where $j$ is a number of ones in $b$ and $k - j$ is a number of zeros in $b$, is an example of such a shift-invariant (or spatially homogeneous) measure. It describes a set of random configurations with the probability that a given site is in state 1 equal to $p$.

Since a cellular automaton rule with global function $F$ maps a configuration in $\mathcal{S}$ to another configuration in $\mathcal{S}$, we can define the action of $F$ on measures on $\mathcal{S}$. For all measurable subsets $E$ of $\mathcal{S}$ we define $(F\mu)(E) = \mu(F^{-1}(E))$, where $F^{-1}(E)$ is an inverse image of $E$ under $F$.

If the initial configuration was specified by $\mu_p$, what can be said about $F^n\mu_p$ (i.e., what is the probability measure after $n$ iterations of $F$)? In particular, given a block $b$, what is the probability of the occurrence of this block in a configuration obtained from a random configuration after $t$ iterations of a given rule?

In the simplest case, when $b = 1$, we will define the density of ones as

$$\rho_n = (F^n\mu_p)(C(1)).$$

In what follows, we will assume that the initially measure is symmetric Bernoulli measure $\mu_{1/2}$, so that the initial density of ones is $\rho_0 = 1/2$.

Assume now that for a given block $b$, the set of $n$-step preimages is $f_1^{-n}(b)$. Then by the definition of the action of $F_{14}$ on the initial measure, we have

$$\mu(F_{14}^{-n}(C(b))) = (F^n\mu_p)(C(b)),$$

and consequently

$$\sum_{a \in f_1^{-n}(b)} \mu(a) = (F^n\mu_p)(C(b)).$$

If $b = 1$, and $\mu = \mu_{1/2}$, the above reduces to

$$\sum_{a \in f_1^{-n}(1)} \mu_{1/2}(C(a)) = \rho_n.$$

Note that all blocks $a \in f_1^{-n}(1)$ have length $2n + 1$, and under the symmetric Bernoulli measure, the probability of their occurrence (that is, $\mu_{1/2}(C(a))$) is the same for all of them, and equal to $1/2^{2n+1}$. This is because we have $2^{2n+1}$ of all possible blocks, and each of them is equally probable, so a single one has probability $1/2^{2n+1}$. As a result, we obtain

$$\rho_n = 2^{-2n-1}\text{card }f_1^{-n}(1).$$
In rule 14, preimages of 1 are 001, 010, and 011. We can therefore write
\[
\rho_n = 2^{-2n-1} \left( \text{card} \ f_{14}^{-n}(001) + f_{14}^{-n}(010) + f_{14}^{-n}(011) \right). \tag{34}
\]

Using formulas of Theorem 3.1, this yields
\[
\rho_n = 2^{-2n-1} (4^n - (2n - 1)C_{n-1}) = \frac{1}{2} \left( 1 - \frac{2n - 1}{4n} C_{n-1} \right). \tag{35}
\]

Using Stirling’s formula to approximate factorials in the definition of \( C_{n-1} \), after elementary calculations one obtains asymptotic approximation valid for \( n \to \infty \),
\[
\rho_n \approx \frac{1}{2} - \frac{1}{4\sqrt{\pi n}} n^{-\frac{1}{2}}. \tag{36}
\]

Very similar calculations can be performed for other block probabilities. Defining the probability of occurrence of a block \( b \) in a configuration obtained from the random initial configuration after \( n \) iterations by \( P_n(b) = (F^n \mu_{1/2})(C(b)) \) we obtain
\[
P_n(b) = \sum_{a \in f_{14}^{-n}(b)} \mu_{1/2}(C(a)). \tag{37}
\]

If the length of block \( b \) is denoted by \( |b| \), its \( n \)-step preimage has length \( |b| + 2n \). Again, since all blocks are equally probable in \( \mu_{1/2} \), a single block of length \( |b| + 2n \) has probability \( 2^{-|b|-2n} \), and we obtain
\[
P_n(b) = 2^{-|b|-2n} \text{card} \ f_{14}^{-n}(b). \tag{38}
\]

Since cardinalities of blocks of length up to 3 are known, the above result together with formulas of Theorem 3.1 can be used to derive exact probabilities of blocks of length up to 3.

7. Conclusions

In closing, let us remark that rule 14 belongs to class 3 rules in informal Wolfram’s classification. Its dynamics, while not as complicated as other class 3 rules, is far from simple. It is therefore encouraging to find meaningful regularities in preimage sets of this rule, and to be able to compute probabilities of small-length cylinder sets exactly, without any approximations.

One hopes that systematic analysis of properties of preimage trees of other elementary rules reveals more regularities of this type, perhaps leading to more general results. To speculate a bit, one may conjecture that rules having additive invariants (like rule 14) are likely to possess regularities in their preimage trees. Further investigation of this issue is currently ongoing.
References

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