Link Quality Control Mechanism for Selective and Opportunistic AF Relaying in Cooperative ARQs: A MLSD Perspective

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Abstract

Incorporating relaying techniques into Automatic Repeat reQuest (ARQ) mechanisms gives a general impression of diversity and throughput enhancements. Allowing overhearing among multiple relays is also a known approach to increase the number of participating relays in ARQs. However, when opportunistic amplify-and-forward (AF) relaying is applied to cooperative ARQs, the system design becomes nontrivial and even involved. Based on outage analysis, the spatial and temporal diversities are first found sensitive to the received signal qualities of relays, and a link quality control mechanism is then developed to prescreen candidate relays in order to explore the diversity of cooperative ARQs with a selective and opportunistic AF (SOAF) relaying method. According to the analysis, the temporal and spatial diversities can be fully exploited if proper thresholds are set for each hop along the relaying routes. The SOAF relaying method is further examined from a packet delivery viewpoint. By the principle of the maximum likelihood sequence detection (MLSD), sufficient conditions on the link quality are established for the proposed SOAF-relaying-based ARQ scheme to attain its potential diversity order in the packet error rates (PERs) of MLSD. The conditions depend on the minimum codeword distance and the average signal-to-noise ratio (SNR). Furthermore, from a heuristic viewpoint, we also develop a threshold searching algorithm for the proposed SOAF relaying and link quality method to exploit both the diversity and the SNR gains in PER. The effectiveness of the proposed thresholding mechanism is verified via simulations with trellis codes.

Keywords

Selective AF relaying, cooperative ARQ, opportunistic AF relaying and SOAF.

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I. Introduction

Packet-oriented services, such as voice over IP or multimedia streaming [1], are a major feature of the next-generation wireless communications. The quality of such kinds of multimedia services is, however, sensitive to the packet loss rate in wireless transmissions. An effective method to reduce the packet loss rate in such a wireless network is to employ Automatic Repeat reQuest (ARQ) mechanisms [2, 3]. In addition to ARQ, cooperative relaying is also considered as a promising technique [4] to increase the transmission reliability without compromising the transmission rate. Cooperative relaying not only can compensate the path loss of signal power in radio propagation but also can exploit the spatial diversities from alternative transmission routes to combat the wireless fading effect.

Since the pioneering work of [5], a host of half-duplex relaying protocols have been presented in [6–23], among which the amplified-and-forward (AF) and the decoded-and-forward (DF) methods are the two mostly studied relaying mechanisms. In spite of the spatial diversity of cooperative relaying, the two-hop nature of half-duplex relaying limits the spectral efficiency of this diversity technique. To circumvent this problem, two derivative techniques have attracted recent attentions; one proposes to use a non-orthogonal data transmission policy that allows the source node to continue sending new packets to its destination at the same time and frequency while the relay nodes forward the old ones, e.g., [6, 7]. This approach typically requires a fairly high complexity in data detection. In contrast, the other approach considers a more straightforward method that incorporates cooperative relaying in ARQ. In other words, relaying is employed on demand for retransmissions only.

In view of the potential of relaying for ARQ, plenty of protocols have been proposed and analyzed on relay-assisted cooperative ARQ or Hybrid-ARQ (HARQ), either in international standards, such as 3GPP LTE-Advanced and IEEE802.16j/16m [24], or in academia [8–16]. Among the works, [8] proposes a relay-assisted HARQ scheme that is shown to outperform the conventional multi-hop scheme in throughput from an information-theoretical point of view. Similar throughput advantages are presented in the HARQ and ARQ schemes of [10, 11] that use multiple relays for retransmissions with distributed space-time coding (DSTC), and also in the HARQ scheme of [14] that uses opportunistic relaying for coop-
gerative retransmissions. On the other hand, from the outage probability point of view, the diversity gains of various relay-assisted HARQ schemes are examined in [9, 12, 13, 16]. Besides, the advantages of relaying on packet delivery delays are shown in [15] for various HARQ schemes that use multiple relays for retransmissions.

Despite the rich results in cooperative relaying, a majority of the researches study DF relaying owing to its attractive performance. In contrast to DF relaying, AF relaying only forwards scaled versions of the received signals, which makes it not necessary to possess complex hardware for decoding and re-encoding information data, hence providing much cost advantages over DF relaying [17]. This simplicity also brings other benefits to the system, such as a simpler system requirement in deployment [25], a higher power efficiency, and less delays in signal processing and transmission. In spite of these operational advantages, AF relaying inevitably causes noise enhancement in relayed signals. To control the effects of noise propagation, a selective AF (SAF) relaying method is proposed in [19] for multi-hop relaying and in [18] to improve the power efficiency of AF relaying. Basically, SAF relaying is a variation of the selective DF relaying method developed in [5]. A relay in the SAF scheme is activated only if the source-to-relay (S-R) channel quality is greater than a predetermined threshold. Motivated by this thresholding mechanism to control noise propagations in AF relaying, we develop and analyze in this work a class of SAF relaying methods to support ARQ from the viewpoint of diversity enhancement.

The basic idea of our proposed methods originates from opportunistic relaying (OR) [22]. To exploit the channel diversity offered by multiple relays without worrying about the synchronization and scalable coding issues of DSTC [20, 21], [22] proposes to choose the relay that has the best channel quality to the destination as the forwarding relay. This simple OR method in principle can make use of the full spatial diversities offered by all participating relays if channel qualities are perfectly known to the destination. In practice, this relaying method can also be implemented in the relay grouping mode of IEEE 802.16j/16m or in the transparent relay mode of LTE-A [26].

Based on the concept of OR, it seems quite reasonable to design ARQ protocols that use opportunistic AF relaying to exploit the spatial and temporal diversities altogether. However, as will be shown soon in this work, this design problem is nontrivial, and indeed
quite involved due to the noise propagation effects in the relay reselection and retransmission processes of ARQ protocols. More specifically, on one hand, the temporal diversity offered by retransmissions is limited by the noise coupled in the amplified and relayed signals. On the other hand, the spatial diversity is dominated by the worst source-to-relay (S-R) channel quality in retransmissions even if OR is employed to reselect the best relay in every ARQ round.

To resolve these problems, one would not only need to overcome the noise propagation effects in AF relaying, but also need to prevent the spatial diversity from being limited by the worst S-R channel quality. The key lies in a delicate screening process to avoid unqualified relays from being selected in each single ARQ, while also in a rejuvenation process to reactivate the unqualified relays for subsequent ARQs. This implies that an effective screening mechanism comes hand in hand with an ARQ scheme that has the potential to provide the full spatial and temporal diversities, and that the screening threshold to some extent depends on the channel coding scheme used in transmission since whether a relayed signal is resistant to the amplified and propagated noise at the destination also relies on the minimum codeword distance of the channel code. To unravel these intertwined issues in the screening, reselection and reactivation processes, we adopt a divide-and-conquer approach to solve each individual problem step by step, starting from an information-theoretical point of view, and extending to the packet error rate (PER) analysis of maximum likelihood sequence detection (MLSD).

To begin with, we provide an outage analysis to show that the thresholding method of SAF relaying also plays a key role for ARQs that use AF relaying to exploit the temporal diversity. If the threshold is not properly set to screen out unqualified relays in advance, then an ARQ will fail to make use of the temporal diversity from channel variations after the first round of AF relaying. This implies that the typical opportunistic AF (OAF) relaying method [22] is not able to leverage the spatial diversity from multiple S-R links after the first round of ARQ, too. To overcome this noise propagation effect, we first provide a thresholding method that allows AF relaying to continue exploiting the temporal diversity through ARQs. Different from OAF relaying, the proposed method employs both the selective and the opportunistic AF (SOAF) relaying mechanisms to control channel qualities.
on the source and forward links of the AF relaying nodes, respectively. Besides, the opportunistic relay selection method studied herein only relies on the relay-to-destination (R-D) channel qualities, which much simplifies the implementation of the proposed scheme.

Extending this result, we then devise an advanced version of the SOAF relaying method for ARQ to exploit the full spatial and temporal diversities offered by multiple relays. The basic idea originates from overhearing, which allows unqualified relays to overhear the forwarded signals sent from qualified relays, as well as a stricter thresholding mechanism to screen relays in every step of the qualification and selection process of the every hop of AF relaying. The quality control on the S-R or relay-to-relay (R-R) links of different hops is served by a set of thresholds designed for each hop, and is done by the relays themselves. Our analysis shows that the advanced SOAF relaying method can attain the full spatial and temporal diversities in outage probabilities if the set of the thresholds is well defined. In contrast, the typical OAF ARQ scheme in [15] suffers from sever diversities losses, nor is its variant [12] able to achieve the full diversities.

The fact that the achievable diversity of the SOAF-relaying-based ARQ schemes is highly dependent on the settings of thresholds makes our outage analysis very different from that of [9,13] in which the diversity and multiplexing tradeoff (DMT) property is studied for the DF-relaying-based ARQ protocols. As a matter of fact, the thresholds will change with the channel codebook when the SOAF schemes are applied in practical ARQ processes. After all, the influences of noise propagations in AF relaying also depend on the minimum codeword distance of the codebook. This practical issue is typically not a concern in DF-relaying-based schemes, while it turns out to be the most crucial factor in our analysis. We will show in the sequel that the thresholds need to increase with the average signal-to-noise ratio (SNR) as well, and the PER of MLSD can achieve the same diversity of outage probability if two sufficient conditions are satisfied by the thresholds. The sufficient conditions explicitly characterize the high-SNR relationship between the thresholds and the minimum codeword distance, and show how the thresholds increase with the SNR in order to maintain the diversity as the SNR goes to infinity. In addition to the diversity property at high SNR, we also try to explore the SNR gain in the low or mid SNR regime. To this end, we further develop a heuristic algorithm to search for
proper thresholds that allow the proposed ARQ schemes to exploit both the diversity and the SNR gains in PER.

The system performances of the proposed ARQ schemes are also verified through simulations with trellis codes. Simulation studies show that the throughputs of the proposed ARQ schemes are superior to the no-relay case and are more robust to the variations in channel qualities than the typical OAF scheme. In addition, to demonstrate the applicability of the proposed thresholding mechanism, we also extend the proposed ARQ schemes to HARQ cases in which the destination performs HARQs with the SNR-weighted maximum ratio combining (MRC). The results show that the problem of diversity recovery is still a crucial issue even in the HARQ cases and the thresholding concept provides us a feasible solution to apply opportunistic AF relaying in cooperative ARQs.

The rest of this paper is organized as follows. In Sections II and III, we provide the outage analysis for relay-assisted ARQ schemes, from which two types of SOAF protocols are developed. In Section IV, we do PER analysis on the proposed ARQ schemes that use practical codebooks, and then develop new requirements on thresholding for the ARQ schemes to attain their potential diversities in PER. In Section V, we further provide a new and heuristic threshold setting method for the ARQ schemes to exploit both the potential diversity and the SNR gains. Section VI mainly shows simulation studies in throughput.

A. Mathematical notations

The notations \( \mathbb{R} \) and \( \mathbb{C} \) stand for the real and the complex field respectively, \( \mathbb{R}^{+} \) represents the positive real field, and \( \mathbb{N} \) denotes the set of natural numbers. \( \mathbb{R}\{\cdot\} \) denotes the real part of a complex number, and \( \bar{a}^H \) represents the conjugate transpose of the vector \( \bar{a} \). The term \( \mathbb{E}_{a>0}[f(a)] \) denotes the expectation value of \( f(a) \) over the region of the random variable \( a > 0 \), and \( 1_A \) is the indicator function for the event \( A \). \( O(\cdot) \) is the big-O notation. The expression \( a := b \) means “\( b \)” is assigned to “\( a \)”.

The diversity order \( d \) of a function \( f(\rho) \) is defined as [27]

\[
d \triangleq - \lim_{\rho \to \infty} \frac{\log f(\rho)}{\log \rho},
\]
and \( f(\rho) \overset{d}{=} g(\rho) \) means they have the same diversity order. Finally, \( f(\rho) \overset{d}{=} g(\rho) \) represents

\[
\lim_{\rho \to \infty} \rho^d f(\rho) = \lim_{\rho \to \infty} \rho^d g(\rho)
\]

with the exponent \( d \) equal to the diversity order of \( f(\rho) \) and \( g(\rho) \). Following this definition, we may define \( \leq \) and \( \geq \) accordingly. Other notations for analysis will be defined in the sequel when needed.

II. Cooperative ARQ with Selective AF Relaying

We introduce in this section the system model and assumption to be used throughout the paper. Under the system setting, we will discuss the issues encountered in using the SAF relaying [19] for ARQs. In particular, we will show from the outage probability point of view that the diversity order of this SAF-based ARQ scheme is limited to two if the threshold on the S-R channel SNR quality is not properly set, regardless of the number of ARQs. On the contrary, if the threshold is set high enough, then the temporal diversity offered by retransmissions can be utilized by ARQs with SAF relaying. Nevertheless, a too high threshold may result in a loss of SNR gain. The notion established in this section will be extended to design ARQ schemes that use both SAF and opportunistic relaying.

A. Basic system model and assumptions

Throughout the paper, the channels between any transmit and receive pairs are considered flat and independently faded. The channel coefficients of the source-to-destination (S-D), source-to-relay (S-R), and relay-to-destination (R-D) links are denoted by \( h_{sd} \), \( h_{sr} \), and \( h_{rd} \), respectively, and are all zero-mean complex Gaussian random variables with variances respectively equal to \( \beta_0 \), \( \beta_1 \) and \( \beta_2 \). Without loss of generality, the average transmit SNR is assumed to be the same at the source and the relay, and is denoted by \( \rho \triangleq \frac{E_s}{N_0} \) where \( E_s \) and \( N_0 \) stand for the symbol energy and the noise variance.

Following the channel model commonly used for outage analysis [27], we also assume that all channel coefficients remain unchanged within the duration of a packet of \( L \) symbols, and change randomly from one packet duration to another. This assumption is reasonable under the following situation [28]: within a network that serves multiple users over time-variant channels, such as TDMA systems, the length of each packet to be trans-
mitted or received by one user is less than the channel coherence interval, and every packet belonging to a certain user would not be processed in continuous phases or resource blocks since each user should be fairly served. Though stated for a TDMA system more than a decade ago, the same operation condition applies to the modern WiMAX or LTE-A systems [29]. Despite the fact that an exact analysis for a large network such as LTE-A is beyond the scope of this paper, the single-user analysis conducted herein under the block-fading assumption can provide many valuable insights into the multi-user systems described above. The same assumption is also made in [8, 9, 14, 16, 17].

Other than the previous assumptions for outage analysis, to perform PER analysis for the different relaying schemes considered in this work, we employ a channel codebook of rate $R$ (in bits/channel use) with the codeword length equal to the packet length, $L$. The total number of codewords, $\bar{x}_j \in \mathbb{C}^{L \times 1}$, is thus equal to $\lceil 2^{RL} \rceil$, and the codewords satisfy an average power constraint of $\frac{1}{D+1} \sum_{j=0}^{D} \|\bar{x}_j\|^2 \leq L$, where $D \triangleq \lceil 2^{RL} \rceil - 1$.

Finally, to focus on the diversity analysis for cooperative ARQs, the errors in signals such as positive acknowledge (ACK) or negative ACK (NAK) in the feedback channels are ignored in the sequel, and there is no full channel state information (CSI) available at the source for instantaneous transmission rate adaption. We also assume coherent detection at the destination.

B. The outage probability of ARQs with SAF relaying

Different from the typical AF relaying function, the relay in ARQ of using SAF first compares the instantaneous S-R channel quality $\rho|h_{sr}|^2$ against a predetermined threshold, $\Delta$, before retransmission. If $\rho|h_{sr}|^2$ is less than or equal to $\Delta$, then the source will be asked to do retransmissions by itself if necessary, and in the mean time, the relay keeps overhearing the signal from the source’s retransmissions. Once $\rho|h_{sr}|^2 > \Delta$, the relay proceeds with the retransmission using the AF relaying method and will continue to use the same quantity for retransmission until the packet is delivered successfully to the destination, or when the maximal number of ARQs is reached, namely, no ARQ is further needed. The corresponding instantaneous received SNR at the destination is given by [5, 30]

$$
\text{SNR}_{\text{inst},\text{rd}} = \frac{\rho^2|h_{sr}|^2|h_{rd}|^2}{\rho|h_{sr}|^2 + \rho|h_{rd}|^2 + 1}. \quad (3)
$$
Following the previous system assumptions and setting for ARQ-SAF, we show in this section that the threshold $\Delta$ for SAF relaying plays a crucial role for an ARQ to achieve its full diversity. In contrast, the ARQ scheme with the typical AF relaying ($\Delta = 0$) is not able to make use of the temporal diversity from retransmissions. The analysis is mainly based on the outage probability of the form

$$\Pr \{ \log_2 (\text{SNR}_{\text{inst}} + 1) < R \} = \Pr \{ \text{SNR}_{\text{inst}} < 2^R - 1 \} \triangleq \Pr \{ \text{SNR}_{\text{inst}} < \delta \}.$$  \hspace{1cm} (4)

The outage probabilities derived in what follows can be intuitively related to the PERs of using a Gaussian random codebook of infinite-length codewords. In other words, they can serve as the performance limitations of the PERs of using practical coding schemes, and may be achieved when using capacity-approaching codes over fading channels of sufficiently large coherence intervals. The outage analysis thus provides us a fundamental approach to examine the potential diversity of the proposed ARQ schemes from an information-theoretical point of view.

For convenience of exposition, we define a number of notations to be used frequently in the analysis. First, we have several exponential random variables defined as $w \triangleq \rho |h_{sd}|^2 \sim \text{Exp}(\rho \beta_0)$, $a \triangleq \rho |h_{sr}|^2 \sim \text{Exp}(\rho \beta_1)$, and $b \triangleq \rho |h_{rd}|^2 \sim \text{Exp}(\rho \beta_2)$ where $x \sim \text{Exp}(y)$ means $x$ is exponentially distributed with a mean equal to $y$. In addition, we define the maximal number of ARQ rounds to be $N$, and denote the $i$-th ARQ round by ARQ$_i$ with ARQ$_0$ standing for the initial transmission from the source. The outage probability after $n$ rounds of ARQs with a relaying scheme $A$ is denoted by $P_{A}^{\text{out},n}$, and the corresponding PER is denoted by $P_{A}^{\text{e},n}$. Finally, we note from (4) that $\delta \triangleq 2^R - 1$.

Let $F(\Delta, \ell)$ stand for the outage probability of $\ell$ consecutive retransmissions with AF relaying. Given the threshold $\Delta$, the outage probability after $n$ rounds of ARQs with the SAF relaying can be expressed as

$$P_{\text{out},n}^{\text{SAF}} = \Pr \{ w < \delta \} \sum_{\ell=0}^{n} \left[ (\Pr \{ a \leq \Delta \} \Pr \{ w < \delta \})^{n-\ell} \times F(\Delta, \ell) \right]$$  \hspace{1cm} (5)

where $(\Pr \{ a \leq \Delta \} \Pr \{ w < \delta \})^{n-\ell}$ is the outage probability after $n-\ell$ consecutive retransmissions by the source, and $F(\Delta, \ell)$ is defined as the joint probability of

$$F(\Delta, \ell) = \Pr \left\{ a > \Delta, \frac{a b_1}{a + b_1 + 1} < \delta, \ldots, \frac{a b_\ell}{a + b_\ell + 1} < \delta \right\}, \ell > 0$$  \hspace{1cm} (6)
with \( F(\Delta, 0) \triangleq 1 \). Since the R-D channel is assumed to fade independently in each ARQ round, the subscript of \( b_t \) is used to distinguish the channel quality in each ARQ round.

Apparently, the retransmission events in \( F(\Delta, \ell) \) are correlated since the S-R channel quality “a” in them are the same even if \( b_t \) are statistically independent. With some mathematical manipulations, it can be further shown that the form of \( F(\Delta, \ell) \) depends on the ratio of \( \Delta \) to \( \delta \), and can be expressed as a formula summarized in the next lemma.

**Lemma 1:** Given \( \Delta \) and \( R \), namely \( \delta \), we have

\[
F(\Delta, \ell) = e^{-\frac{\Delta}{\rho \beta}} + \begin{cases} 
\sum_{i=1}^{\ell} C_i^\ell (-1)^i e^{-\left(\frac{1}{\rho \beta_1} + \frac{1}{\rho \beta_2}\right) \Delta} \Gamma(1, 0, i(\delta^2 + \delta)) / p^\ell \beta_1 \beta_2, & \Delta < \delta \\
\sum_{i=1}^{\ell} C_i^\ell (-1)^i e^{-\left(\frac{1}{\rho \beta_1} + \frac{1}{\rho \beta_2}\right) \Delta} \Gamma(1, \frac{\Delta}{\rho \beta_1}, i(\delta^2 + \delta)) / p^\ell \beta_1 \beta_2, & \Delta \geq \delta 
\end{cases}
\]

where \( \Gamma(\alpha, x; b) \triangleq \int_x^\infty t^{\alpha-1} e^{-t} e^{-t-b} dt \) is the generalized incomplete gamma function \([31]\), and \( C_i^\ell \triangleq \frac{\ell!}{i!(\ell-i)!} \) is the total number of the combinations of picking \( i \) out of \( \ell \) distinct objects.

**Proof:** See Appendix-A.1.

Substituting (7) into (5) gives the exact expression for \( P_{\text{out}, n}^{\text{SAF}} \). Let \( \Delta := \lambda \times \delta \) with \( \lambda \in \mathbb{R}^+ \). Given \( R \), namely \( \delta \), the relation between \( \lambda \) and \( P_{\text{out}, n}^{\text{SAF}} \) is illustrated in Fig. 1. As can be seen in the figure, the required SNR for \( P_{\text{out}, n}^{\text{SAF}} \) equal to the target \( P_t \) reduces dramatically around \( \lambda = 1 \). This in fact results from the diversity variations with respect to (w.r.t.) \( \Delta \) in \( F(\Delta, \ell) \) of \( P_{\text{out}, n}^{\text{SAF}} \), which is analyzed in the next lemma. Before that, we
introduce a useful lower bound for \( F(\Delta, \ell) \), denoted by \( \tilde{F}(\Delta, \ell) \), which is defined as

\[
\tilde{F}(\Delta, \ell) \triangleq \Pr\{a > \Delta\} \Pr\{b_1 < \delta, \ldots, b_\ell < \delta\}, \ \ell \geq 1
\]

with \( \tilde{F}(\Delta, 0) \triangleq 1 \). The inequality of \( F(\Delta, \ell) \geq \tilde{F}(\Delta, \ell) \) holds true for any \( \Delta \) due to the fact that \( \frac{ab}{a+b+1} \leq \min\{a, b\} \leq b, \ \forall a, b \geq 0 \).

**Lemma 2:** Given the rate \( R \), namely \( \delta \), let \( \Delta := \lambda \times \delta \) with \( \lambda \in \mathbb{R}^+ \). For \( \ell \in \mathbb{N} \), we have

\[
F(\Delta, \ell) \triangleq \tilde{F}(\Delta, \ell) \triangleq (\frac{\delta}{\rho^\lambda})^\ell \text{ if } \lambda > 1; \text{ whereas, if } \lambda < 1, \text{ } F(\Delta, \ell) \text{ is of the order of } \rho^{-1}, \text{ and follows } F(\Delta, \ell) \triangleq \frac{\delta - \Delta}{\rho^\lambda} \text{ for } \ell \geq 2.
\]

**Proof:** From (6), for \( \lambda > 1 \), we have

\[
F(\Delta, \ell) = \Pr\left\{ a > \lambda \delta, b_1 < \frac{\delta(a+1)}{(a-\delta)}, \ldots, b_\ell < \frac{\delta(a+1)}{(a-\delta)} \right\} \leq \left( \Pr\left\{ b_1 < \frac{\lambda \delta + 1}{\lambda - 1} \right\} \right)^\ell
\]

since \( (a-\delta) > 0 \) and \( \frac{a+1}{a-\delta} \) is decreasing w.r.t. \( a \) when \( a > \delta \). Apparently, \( F(\Delta, \ell) \) can attain the same order of \( \rho^{-\ell} \). As for the case of \( \lambda < 1 \), due to the fact that \( \frac{ab}{a+b+1} \leq \min\{a, b\} \leq a \), \( F(\Delta, \ell) \) is lower bounded by \( \Pr\{\delta > a > \lambda \delta\} \) whose order is equal to 1, and thus suffers from diversity losses. A rigorous proof is provided in Appendix-A.2.

Intuitively, when \( \lambda > 1 \), \( F(\Delta, \ell) \) can provide the diversity order of \( \rho^{-\ell} \) since the thresholding mechanism prevents “outage events” from being relayed and the ARQ events thus become virtually uncorrelated with the S-R channel quality “a” at high SNR. In addition, \( \tilde{F}(\Delta, \ell) \) is indeed a useful approximation for \( F(\Delta, \ell) \) at high SNR since it does not require numerical integrations for the evaluation of \( \Gamma(\alpha, x; b) \) in (7). In view of the simplicity, making use of the fact that \( F(\Delta, \ell) \geq \tilde{F}(\Delta, \ell) \), we define a lower bound \( \tilde{P}^{SAF}_{out,n} \) for \( P^{SAF}_{out,n} \), which is given by replacing \( F(\Delta, \ell) \) in (5) with \( \tilde{F}(\Delta, \ell) \). We thus obtain

\[
\tilde{P}^{SAF}_{out,n} \leq P^{SAF}_{out,n}.
\]

Since the term \( \Pr\{a \leq \Delta\} \Pr\{w < \delta\} \) in \( P^{SAF}_{out,n} \) is equal to \( (1-e^{-\frac{\delta}{\rho^\lambda}})(1-e^{-\frac{\delta}{\rho^\lambda}}) \leq \frac{\delta}{\rho^\lambda} \) whose diversity order is 2, then by (5) and Lemma 2, we have \( P^{SAF}_{out,n} \equiv \Pr\{w < \delta\} F(\Delta, n) \).

The relationship between \( P^{SAF}_{out,n} \) and \( \Delta \) can thus be easily characterized as follows:

**Proposition 1:** Given \( \lambda \in \mathbb{R}^+ \) such that \( \Delta := \lambda \delta \), if \( \Delta > \delta \), we have \( P^{SAF}_{out,n} \equiv \tilde{P}^{SAF}_{out,n} \equiv \frac{\delta}{\rho_0} (\frac{\delta}{\rho_2})^n \rho^{-(n+1)} \); whereas, if \( \Delta < \delta \), we arrive at \( P^{SAF}_{out,n} \equiv \frac{\delta - \Delta}{\rho_0} \rho^{-(n+1)} \), for \( n \geq 2 \).

The proposition shows that if a basic channel quality is met at the relay before using the AF relaying, the temporal diversity of ARQs can be greatly improved from the viewpoint of outage probability. This gives us an interesting reminiscence of the selective DF relaying in
Fig. 2. Outage probabilities after 3 rounds of ARQ-SAFs ($P_{out,3}^{SAF}$), with different values of $\lambda$ when $\Delta := \lambda \delta$.

[5], even if the source signal here is not decoded before the AF retransmission. Simulation results for $P_{out,3}^{SAF}$ with different $\Delta$s are shown in Fig. 2 to verify our analysis. For $\lambda > 1$, $P_{out,3}^{SAF}$ becomes closer to $\tilde{P}_{out,3}^{SAF}$ as the SNR increases.

On the other hand, Proposition 1 also shows that the diversity order of ARQs with direct AF relaying (ARQ-AF) is equal to two since it is simply a special case of ARQ-SAF with $\Delta = 0$, which is always less than $\delta$ for $R > 0$. According to (6), the corresponding outage probability for ARQ-AF is given by $P_{out,n}^{AF} = \Pr\{w < \delta\} \times F(0,n)$.

III. COOPERATIVE ARQ WITH SELECTIVE AND OPPORTUNISTIC AF RELAYING

The analysis in the previous section recalls the importance and role of quality control on the S-R link to our attempt to improve the system reliability with AF retransmissions. In addition to utilizing the temporal diversity with ARQs, one can also exploit the spatial diversity with multiple relays through the opportunistic relay selection in [22]. Incorporating the spatial diversity scheme of OAF into the SAF ARQ framework allows us to jointly exploit the spatial and temporal diversities in multiple-relay systems with the same and simple AF relaying method. The outage analysis on this selective and opportunistic idea of AF relaying leads to two types of ARQ schemes. More importantly, it provides a new look and method on quality control along each hop of multiple-AF-relay systems.

We assume that there are $m$ neighboring relays available in the system. The channel
coefficient between the source and relay \( j \) is denoted by \( h_{j,sr} \), and the coefficient between the relay \( j \) to the destination is denoted by \( h_{j,rd} \), and that between relay \( i \) and relay \( j \) is by \( h_{i,j} \). All the channel coefficients are assumed to be independent complex Gaussian random variables with zero mean. As for the channel variances, we assume that the smallest channel variance from the source to relays, namely the worst S-R average channel quality, is equal to a number \( \beta_1 \). Also, the worst average channel quality from relays to the destination is \( \beta_2 \), and the worst between any two relays (R-R) is \( \beta_3 \).

Rigorously speaking, in a realistic network, the variances of \( h_{j,sr} \) for different \( j \) might not be identical. However, for the simplicity of analysis, we assume that the variances of \( h_{j,sr}, \forall j \in I_1^m \), are the same and equal to \( \beta_1 \). Similarly, the variances of \( h_{j,rd}, \forall j \in I_1^m \), are \( \beta_2 \), and the variances of \( h_{i,j}, \forall i, j \in I_1^m, i \neq j \), are \( \beta_3 \). Because of this simplified assumption on channel variances, the system performance can be considered as the lowest potential performance of a true one. Nevertheless, the following diversity analysis and channel requirements for relays are still valid even in a true system, and quality control based on this channel variance assumption also makes it simple to apply the proposed schemes to a real system.

A. The outage probability of ARQ with the typical opportunistic AF relaying (ARQ-OAF)

We first investigate the outage probability of the ARQ scheme that uses the opportunistic AF relaying method (ARQ-OAF) in [15]. The ARQ-OAF basically chooses the relay \( i \) in each round of ARQs that satisfies

\[
i = \arg \max_{j \in \{1, \ldots, m\}} \left\{ \frac{\rho^2 |h_{j,sr}|^2 |h_{j,rd}|^2}{\rho |h_{j,sr}|^2 + \rho |h_{j,rd}|^2 + 1} \right\}
\]

(10)
to directly amplify and forward the signal. Following the relay selection rule, we summarize the outage probability after \( n \) rounds of the OAF-based ARQs in the following proposition.

Proposition 2: Given \( R \) and \( m \), the outage probability after \( n \) rounds of ARQs with the typical OAF relaying is given by \( P_{\text{out},n}^{\text{OAF}} = \Pr \{ w < \delta \} \times (F(0,n))^m \), and its diversity order is limited to \( (m + 1) \), \( \forall n \in \mathbb{N} \).

Proof: : See Appendix-B.1 for the formula of \( P_{\text{out},n}^{\text{OAF}} \). As for the diversity order analysis, since \( F(0,n) \) is of the order of \( \rho^{-1} \) by Lemma 2, we thus have \( P_{\text{out},n}^{\text{OAF}} \asymp \rho^{-(m+1)}. \)
In fact, the ARQ-OAF scheme offers the full cooperative diversity only for the first ARQ round. In the subsequent ARQs, similar to the ARQ-AF scheme, the S-R channel gains $\rho|h_{j,sr}|^2$ remain unchanged in the AF signal, which results in the loss of the temporal diversity as will be verified in Fig. 3. This motivates us to develop ARQ schemes that on one hand, require relays to prescreen their incoming signal qualities, like the SAF relaying method, and on the other hand, allow the destination to opportunistically choose a relay only from the set of relays that pass the screening. This idea leads to two types of ARQ schemes to be presented below. Next, we start with the most straightforward one.

B. ARQ with selective and opportunistic AF relaying (ARQ-SOAF)

To implement the idea, we define for SOAF relaying a qualified set $\mathcal{Q}$ of the relays whose $\rho|h_{j,sr}|^2 > \Delta$. In each ARQ, the relay in $\mathcal{Q}$ with the highest $\rho|h_{j,rd}|^2$ gets selected for AF relaying. In case of $\mathcal{Q} = \emptyset$, the source will do the retransmissions until $\mathcal{Q} \neq \emptyset$ or when no ARQ is further needed. Compared to the typical OAF scheme of (10), the opportunistic relay selection method here does not require the information of instantaneous S-R channel gains at the destination, which makes it easier to implement the SOAF scheme in practice.

In the next subsections, we discuss two ARQ protocols based on this relaying strategy, referred to as the type A and B of ARQ-SOAF. Type A forms the set of $\mathcal{Q}$ by overhearing the signals from the source only, while type B continues to enlarge the cardinality of $\mathcal{Q}$ by overhearing the signals forwarded by relays in $\mathcal{Q}$ as well.

B.1 ARQ with the type A of SOAF relaying (SOAF-A)

Under the assumption that all R-D channels have the same statistical property, every relay in $\mathcal{Q}$ has equal probability to be chosen as the active relay for AF relaying. The outage probability after $n$ rounds of ARQs with SOAF-A relaying can thus be expressed in the next compact form of

\begin{equation}
P_{\text{out,SOAF-A}} = \Pr\{w < \delta\} \times \sum_{\ell=0}^{n} \left[ (\Pr\{a \leq \Delta\})^m \times \Pr\{w < \delta\} \right]^{n-\ell} \times G_1(\Delta, \ell)
\end{equation}

Proposition 3:
where \( G_1(\Delta, \ell) \overset{\Delta}{=} 1 \) for \( \ell = 0 \), and for \( \ell > 0 \), it follows

\[
G_1(\Delta, \ell) = \sum_{q=1}^{m} \left[ C_{m-q}^{q} \left( \Pr\{a \leq \Delta\} \right)^{m-q} \times \left( \frac{1}{q} \right)^{\ell} \times F^{(q)}(\Delta, \ell, q) \right]
\]

(12)

in which \( F^{(i)}(\Delta, \ell, q) \) stands for the sum of the outage probabilities of \( \ell \) consecutive retransmissions by permutations of \( i \) different active relays chosen from \( Q \) with \( |Q| = q \). For \( q \geq 2 \) and \( \ell > 0 \), \( F^{(q)}(\Delta, \ell, q) \) with an index \( \zeta_q \overset{\Delta}{=} \ell \) can be recursively expressed as

\[
F^{(q)}(\Delta, \zeta_q, q) = \sum_{\zeta_{q-1}=0}^{\zeta_q} C_{\zeta_q}^{\zeta_{q-1}} (e^{-\frac{\Delta}{\rho}})^{\mu(\zeta_q, \zeta_{q-1})} \times F(\Delta, q \times (\zeta_q - \zeta_{q-1})) \times F^{(q-1)}(\Delta, \zeta_{q-1}, q)
\]

(13)

until \( F^{(2)}(\Delta, \zeta_2, q) \overset{\Delta}{=} \sum_{\zeta_1=0}^{\zeta_2} C_{\zeta_2}^{\zeta_1} (e^{-\frac{\Delta}{\rho}})^{\mu(\zeta_1, \zeta_2)} \times F(\Delta, q \times \zeta_1) \times F(\Delta, q \times (\zeta_2 - \zeta_1))
\]

(14)

in which \( \mu(\zeta_1, \zeta_{i-1}) \overset{\Delta}{=} \delta_f[\zeta_i - \zeta_{i-1}] + \delta_f[\zeta_{i-1} - \delta_f[\zeta_i + \zeta_{i-1}] \text{ such that } \mu(\zeta_i, \zeta_{i-1}) = 1 \text{ if } \zeta_{i-1} = 0 \text{ or } \zeta_i = \zeta_{i-1} = \zeta_i \text{ where } \delta_f[\cdot] \text{ is the delta function; otherwise, } \mu(\zeta_i, \zeta_{i-1}) = 0 \). As for \( q = 1 \), we have \( F^{(1)}(\Delta, \ell, 1) \overset{\Delta}{=} F(\Delta, \ell) \).

\textbf{Proof: } See Appendix-B.2. \hfill \Box

Through replacing the form of \( F(\Delta, \ell) \) in (13)~(14) with its lower bound \( \tilde{F}(\Delta, \ell) \), a lower bound of \( F^{(q)}(\Delta, \ell, q) \) can be obtained and further summarized as \( q^\ell \Pr\{a > \Delta\} q \Pr\{b < \delta\} q^{\times \ell} \). Similar to \( \tilde{P}_{\text{out,tn}}^{\text{SAF}} \) for \( P_{\text{out,tn}}^{\text{SAF}} \), a lower bound for \( P_{\text{out,tn}}^{\text{SOAP-A}} \) is thus characterized as follows:

\textbf{Corollary 1:}

\[
\tilde{P}_{\text{out,tn}}^{\text{SOAP-A}} = \Pr\{w < \delta\} \times \sum_{\ell=0}^{n} \left( \Pr\{a \leq \Delta\}^m \times \Pr\{w < \delta\} \right)^{m-\ell} \times \tilde{G}_1(\Delta, \ell)
\]

(15)

where \( \tilde{G}_1(\Delta, \ell) \overset{\Delta}{=} \sum_{q=1}^{m} C_{m-q}^{m} \Pr\{a \leq \Delta\}^{m-q} \Pr\{a > \Delta\} q \Pr\{b < \delta\} q^{\times \ell} \text{ with } \tilde{G}_1(\Delta, 0) \overset{\Delta}{=} 1 \).

\textbf{Proof: } See Appendix-B.3. \hfill \Box

According to Proposition 1, different thresholds for ARQ-SAF result in different outage probabilities or even diversity losses. Let \( \Delta := \lambda \delta \) for the SOAF-A ARQ scheme. Given \( \lambda \in \mathbb{R}^+ \), the diversity order of \( P_{\text{out,tn}}^{\text{SOAP-A}} \) can be analyzed by Lemma 2. If \( \Delta > \delta \), it follows that \( P_{\text{out,tn}}^{\text{SOAP-A}} \leq \tilde{P}_{\text{out,tn}}^{\text{SOAP-A}} \) since \( F(\Delta, \ell) \leq \tilde{F}(\Delta, \ell) \). In this case, by \( \tilde{P}_{\text{out,tn}}^{\text{SOAP-A}} \), we thus have

\[
P_{\text{out,tn}}^{\text{SOAP-A}} \overset{\Delta}{=} \frac{d}{p} \times \sum_{\ell=0}^{n} \left[ \left( \frac{1}{p} \right)^{n-\ell} \times \left( \sum_{q=1}^{m} \frac{1}{p^{m-q}} \times \frac{1}{p^{\ell q}} \right)^{1-\delta_f[\ell]} \right]
\]

(16)
Fig. 3. Outage probabilities for ARQs with OAF and SOAF-A relaying. For SOAF-A with $\Delta := \lambda \delta > \delta$, the diversity orders increase by 1 in each round of the ARQs. Otherwise, they are limited to 2.

where $\delta_f[\cdot]$ is the delta function. The term $\left(\frac{1}{\rho m} \times \frac{1}{\rho}\right)^{n-\ell}$ in (16) results from $n-\ell$ consecutive retransmissions by the source, which means the diversity order will increase by $m+1$ with every round of the $n-\ell$ ARQs with $|Q| = 0$. In comparison with the cases of $|Q| = q \geq 1$ in (16), the diversity order offered by each round of ARQs through relaying only increases by $m$ at most. As a result, at high SNR, for $n \geq 1$, the $P_{\text{SOAF-A, out}}$ must be dominated by the case of $Q \neq 0$, i.e., when $\ell = n$ in (16), which leads to

$$P_{\text{SOAF-A, out}} = \frac{1}{\rho} \sum_{q=1}^{m} \frac{1}{\rho^{m+q(n-1)}}.$$  \hfill (17)

By (17), the diversity order of $P_{\text{SOAF-A, out}}$ for $n = 1$ is equal to $m + 1$ regardless of $q$. For $n \geq 2$, $P_{\text{SOAF-A, out}}$ is apparently dominated by the case of $q = 1$, namely $|Q| = 1$. As a result, the diversity order of $P_{\text{SOAF-A, out}}$ only increases at a rate of $n$ and is equal to $m + n$ for $n \geq 1$.

The results can be verified with the outage probabilities presented in Fig. 3. Although only R-D channel qualities, $\rho|h_{j,rd}|^2$, are used for relay selection in SOAF-A, the SOAF relaying scheme is able to exploit the temporal diversity through ARQs if $\Delta > \delta$. Nevertheless, the diversity order only increases by 1 in each round after the first ARQ round.

On the other hand, for $\Delta < \delta$, the diversity order is limited to 2 due to the poor S-R channel qualities and the selection rule of SOAF. We leave the proof in Appendix-B.4. In comparison, based on Proposition 2, the diversity order of ARQ-OAF is equal to $m + 1$, but both $\rho|h_{j,rd}|^2$ and $\rho|h_{j,rd}|^2$ are required for the destination to choose the best relay.
Fig. 4. An illustration for ARQs with SOAF-B relaying ($m = 6$). The subscript of $c_i$ is used to indicate the number of hops before reaching the destination, and $\Delta_i$ is the SNR threshold for the $i$-th hop.

According to (10).

**B.2 ARQ with the type B of SOAF relaying (SOAF-B)**

Based on the previous diversity analysis for SOAF-A ARQ, the key to further improve the diversity via ARQs is to increase the cardinality of $Q$, i.e., $|Q|$, through ARQs as well. This cannot be made possible without the unqualified relays being able and continuing to overhear the signals forwarded by relays in $Q$ during the process of ARQs. If proper conditions can be set on the link qualities, $\rho|h_{i,j}|^2$, between the transmitting and receiving relays to qualify and bring new relays into $Q$, then the diversity may no longer be limited to the case of $|Q| = 1$. This type of the SOAF scheme is referred to as the SOAF-B relaying. The functioning of the protocol is illustrated in Fig. 4. For convenience, we define a random variable $c \triangleq \rho|h_{i,j}|^2 \sim \text{Exp}(\rho\beta_3)$ to denote the R-R channel quality.

As shown in this figure, the active relay, R5, of $Q$ receives a signal from R3 in the previous ARQ and is currently forwarding the signal to the destination. The relay R6 in the complement set of $Q$, denoted by $Q^c$, overhears the signal from R5. If $c_4 = \rho|h_{5,6}|^2$ exceeds a threshold, say $\Delta_4$ with 4 indicating the number of hops before reaching the destination, then R6 will be taken out of the set $Q^c$ and put into the qualified set $Q$. In the next round of ARQ, if any, the destination then chooses the relay in the new $Q$ with the highest $\rho|h_{j,rd}|^2$ to forward the signal, even if the signal from R6 has accumulated more
noise through the hops from the source to R2, R3 and then R5.

We define a threshold for each hop to control the channel quality of the entire relaying path. Since the maximal number of hops is limited to \(\min[m, N]\), we thus have an array of thresholds, \(\Delta \triangleq [\Delta_1, \ldots, \Delta_i, \ldots, \Delta_{\min[m, N]}]\) with \(\Delta_i\) corresponding to the threshold for the \(i\)-th hop. In general, for an active relay that forwards a signal which has already gone through \(k\) hops, the instantaneous received SNR at the destination can be given by [30]

\[
\text{SNR}_{\text{inst}, rd} = \left[ (1 + \frac{1}{a_1}) \prod_{i=2}^{k} (1 + \frac{1}{c_i})(1 + \frac{1}{b|q|}) - 1 \right]^{-1} \tag{18}
\]

where \(b|q|\) represents the highest \(\rho|h_{j, rd}|^2\) of relays in \(Q\) with \(|Q| = q\).

We recall from Proposition 1 that the potential diversity of ARQ-SA can be achieved as long as the thresholding mechanism can prevent "outage events" from occurring at the relay, i.e., when \(\Delta := \lambda\delta > \delta\). Thus, to exploit the diversity in SOAF-B ARQs, we also need to define a requirement on the thresholds for relays to qualify the received instantaneous SNR (SNR\(_{\text{inst}}\)). By a form of (18), we may intuitively define the requirement as follows:

**Requirement 1:** Given a number \(\lambda\) with \(\lambda > 1\), for a \(k\)-hop signal received by a relay in \(Q\), the received SNR of the relay satisfies \(\text{SNR}_{\text{inst}} = [(1 + \frac{1}{a_1}) \prod_{i=2}^{k} (1 + \frac{1}{c_i}) - 1]^{-1} \geq \lambda\delta\).

Under Requirement 1, the outage probability corresponding to (18) becomes

\[
\Pr\{\text{SNR}_{\text{inst}, rd} < \delta\} \leq \Pr\left\{ (1 + \frac{1}{a_1})(1 + \frac{1}{b|q|}) > 1 + \frac{1}{\delta} \right\} = \left( \Pr\left\{ b < \frac{1}{1+1/\lambda\delta} \right\} \right)^q \tag{19}
\]

where \(\frac{1+1/\delta}{1+1/(\lambda\delta)} > 1\) owing to \(\lambda > 1\). This probability apparently attains the diversity order \(q\). Therefore, if there is a qualified set \(Q\) with \(|Q| = q > 0\), and every relay chosen from \(Q\) for ARQs satisfies Requirement 1, then the diversity order offered by relaying can increase by \(q\) with every extra ARQ round according to (19).

Following the above requirement and the result of (19), we can finally arrive at a theorem for the SOAF-B relaying to exploit the potential diversity of ARQs.

**Theorem 1:** Given \(R\), if the thresholds of \(\Delta\) are constant with \(\rho\) but sufficiently large for Requirement 1 to be satisfied, then the diversity order of the outage probability after \(n\) rounds of ARQ-SOAF-B is given by \((m \times n + 1)\).

**Proof:** First, the outage probability of ARQ0 can offer diversity order 1. Then, the probability that there are \(q\) relays in \(Q\) after ARQ0 can attain the diversity order of \(\frac{1}{\rho^{q-1}}\).
because the threshold $\Delta_1$ is constant with $\rho$ and thus it follows that

$$\Pr\{a < \Delta_1\}^{m-q} = \left(1 - e^{-\frac{\Delta_1}{\rho \delta}}\right)^{m-q} \frac{d}{\rho^{m-q}}. \quad (20)$$

As a result, the performance of the SOAF-B ARQ scheme will be dominated at high SNR by the case of $q > 0$ due to its lower diversity order. Based on $q \neq 0$, the retransmission by a relay for ARQ1 can offer the extra diversity order $q$ according to (19). Thus, the diversity order of the outage probability after ARQ1 will achieve a total of $m + 1$.

Similarly, the diversity order after ARQ2 will increase by $m$ since the probability that $|Q|$ becomes $q'$ after ARQ1 can achieve the order of $\frac{1}{\rho^{m-q}}$ due to $\Pr\{c < \Delta_2\}^{m-q'} = \frac{d}{\rho^{m-q'}}$, and further the relaying in ARQ2 will contribute the diversity order $q'$ according to (19). Following the same argument for the subsequent ARQs, we can conclude that the diversity order will increase by $m$ with every extra ARQ round. As a result, the outage probability after $n$ round of ARQ-SOAF-B can attain the diversity order $m \times n + 1$.

Requirement 1 can also be applied to the ARQ-SOAF-A case where the relays are only allowed to overhear from the source and hence only the S-R hops need to be qualified. More specifically, we suppose an ARQ-SOAF-A scheme uses a threshold $\Delta_1$ that satisfies Requirement 1 when $k = 1$. With proof steps similar to that of Theorem 1, it can be shown that the lowest diversity order occurs at $q = 1$ since $|Q|$ is fixed after ARQ1, which leads to a total diversity of $m + n$ after $n$ rounds of ARQs. This result matches the diversity order analysis for SOAF-A in Section III-B.1.

B.3 Thresholds assignment for ARQ-SOAF-B

Let $\overline{\Delta} := \delta \times [\lambda_1, \ldots, \lambda_{\min[m,N]}]$ with $\lambda_i \in \mathbb{R}^+, \forall i$. To achieve the diversity order stated in Theorem 1, following Requirement 1, we need to find proper settings for $\lambda_i$ such that $(1 + \frac{1}{\lambda_1 \delta}) \times \cdots \times (1 + \frac{1}{\lambda_{\min[m,N]} \delta}) \leq 1 + \frac{1}{\overline{\Delta}}$, given $\overline{\Delta} > 1$. This is because given $a_1 > \Delta_1 := \lambda_1 \delta$ and $c_i > \Delta_i := \lambda_i \delta, \forall i \in I_k^m$, this condition will meet $(1 + \frac{1}{a_1}) \prod_{i=2}^k (1 + \frac{1}{c_i}) < (1 + \frac{1}{\lambda_1 \delta}) \prod_{i=2}^k (1 + \frac{1}{\lambda_i \delta}) \leq 1 + \frac{1}{\overline{\Delta}}, \forall k \in I_{\min[m,N]}^m$,

Example: Fig. 5 demonstrates the outage probabilities for two different assignments of $\lambda_i$ with $m = 3, N = 3$ and $\delta = 1$. The thresholds are $\delta \times [3.9, 3.9, 3.9]$ and $\delta \times [2, 5, 10]$, respectively, and all satisfy Requirement 1 with $\overline{\Delta} = 1.01$. As characterized by Theorem 1, all lead to the full diversity order while with small offsets among them. Besides, an analytic
Fig. 5. Outage probabilities of ARQ-SOAF-B in comparison with OAF. The analytic lower bounds are evaluated with $\tilde P^\text{SOAF-B}_{\text{out},n}$ (defined in Appendix-B.5). $[\lambda_1, \lambda_2, \lambda_3]$ stands for the thresholds of $\delta \times [\lambda_1, \lambda_2, \lambda_3]$.

| ARQ scheme | SAF | SOAF-A | SOAF-B | AF | OAF |
|------------|-----|--------|--------|----|-----|
| Threshold(s) rule | $\Delta < \delta$ | $\Delta > \delta$ | $\Delta < \delta$ | $\Delta > \delta$ | Requirement 1 | - | - |
| Diversity order | 2 | $n + 1$ | 2 | $m + n$ | $m \times n + 1$ | 2 | $m + 1$ |

**TABLE I**

A summary on the diversity orders of different ARQ schemes after ARQ$n$ from the viewpoint of outage analysis.

bound, $\tilde P^\text{SOAF-B}_{\text{out},n}$, whose derivation is provided in Appendix-B.5, is shown to be a tight lower bound for $P^\text{SOAF-B}_{\text{out},n}$.

*Application:* The OSAF-A relaying method can be used in general two-hop AF relaying networks, which only requires R-D CSIs for opportunistic relaying. In addition, the notion of SOAF-B relaying can also be applied to wireless multi-hop transmission communications. In contrast to the typical sequentially relaying manner [19,32], the SOAF-B relaying protocol allows a source packet to go through a dynamic relaying path before reaching the destination, and thus is able to benefit from more spatial diversity. For instance, given $k$ relays and $k + 1$ hops, namely $m = k$ and $N = k$, the SOAF-B relaying protocol can provide a diversity order $k^2 + 1$ which is greater than the order of $k + 1$ in [19].
As a short summary of Table I, from the outage probability point of view, the temporal and spatial diversities of ARQs can be fully exploited with SOAF-B relaying if the channel quality on each hop of the entire relaying path exceeds its corresponding threshold that depends on the source data rate $R$ as well. Intuitively, if a capacity approaching code with an infinite codeword length is used for transmission, then the PERs of the ARQs should be able to achieve the same diversity orders analyzed with outage probabilities. However, this presupposition is rather optimistic and unrealistic for an AF relaying system that uses typical channel codes. To address this issue, we next reexamine the thresholds requirement from a MLSD point of view in order to exploit the same diversities in PERs.

IV. THE PERs OF ARQs WITH SOAF RELAYING

With outage analysis, we have developed a link quality control mechanism for OSAF-B ARQ in an attempt to exploit the full diversity offered by retransmissions. In this section, we show that the simple threshold setting methods in Table I, derived from outage analysis, are still insufficient for the proposed ARQ schemes to attain their same outage diversities in the PER of using MLSD on practical coding schemes. To resolve this problem, from a PER analysis point of view, we develop new requirements on the received signal qualities of the relays for the ARQ schemes to maintain their diversities in PER. Sufficient conditions for the new requirements are further established to explicitly show the relationships between the requirements, the threshold setting methods, the minimum codeword distance of the employed channel code, and the SNR $\rho$. For clarity, we start the analysis with a system that has one relay only, and then extend the results to systems with multiple relays.

A. The PER of ARQ-SAF

Suppose there is only one relay available in the system. Then, the signals sent from the relay to the destination can be expressed as

$$\bar{y}'_{rd} = \frac{\sqrt{\rho h_{rd}}}{\sqrt{\rho |h_{sr}|^2 + 1}} \left( \frac{\sqrt{\rho h_{sr}} \bar{x}_j + \bar{n}'_r}{\bar{y}'_{sr}} + \bar{n}'_d \right)$$

(21)

where the term $\frac{1}{\sqrt{\rho |h_{sr}|^2 + 1}}$ is the power normalization factor [30] at the relay on its received signal, $\bar{y}'_{sr}$, and besides, $\bar{x}_j \in \mathbb{C}^{L \times 1}$ is the $j$-th codeword in the cookbook and the
entries of the noise vector $\bar{n}_r'$ and $\bar{n}_d'$ are independent and identically distributed (i.i.d.) complex Gaussian random variables with zero mean and unit variance.

To simplify the analysis of MLSD and to focus on reexamining our proposed link quality control mechanism, in this paper, we assume coherent detection at the destination. The assumption in principle requires the destination to have the perfect CSI of a cascaded channel on $\tilde{x}_j$ in its received signal, e.g., the CSI of $\frac{\alpha h_{rd} h_{sr}}{\sqrt{\rho |h_{sr}|^2 + 1}}$ in $\bar{y}_{rd}'$. In contrast, noncoherent detection may be a possible solution [33] to circumvent the difficulty of obtaining non-local CSIs for detecting AF relayed signals at the destination. Nevertheless, the analysis in the sequel still offers a different look on the link quality control of AF relaying from a PER point of view.

For conciseness of presentation, based on coherent detection, without loss of generality, we first equalize the signal phase of $\bar{y}_{rd}'$ with the phase term $\frac{(h_{rd} h_{sr})^H}{|h_{rd} h_{sr}|} \bar{x}_j$. That is, we have

$$\frac{(h_{rd} h_{sr})^H}{|h_{rd} h_{sr}|} \bar{y}_{rd}' = \frac{\sqrt{\rho} |h_{rd}|}{\sqrt{\rho} |h_{sr}|^2 + 1} (\sqrt{\rho} |h_{sr}| \tilde{x}_j + \frac{h_{sr}^H}{|h_{sr}|} \bar{n}_r') + \frac{(h_{rd} h_{sr})^H}{|h_{rd} h_{sr}|} \bar{n}_d' \triangleq \bar{y}_{rd}$$

in which we further denote the noise vector $\frac{h_{sr}^H}{|h_{sr}|} \bar{n}_r'$ by $\bar{n}_r$ and $\frac{(h_{rd} h_{sr})^H}{|h_{rd} h_{sr}|} \bar{n}_d'$ by $\bar{n}_d$. We also note that the entries of $\bar{n}_r$ and $\bar{n}_d$ are still i.i.d. complex Gaussian random variables with zero mean and unit variance. Then, following the notations defined previously for channel gains, we rewrite (22) as

$$\bar{y}_{rd,\ell} = \frac{\sqrt{b_\ell}}{\sqrt{a} + 1} (\sqrt{a} \tilde{x}_j + \bar{n}_r) + \bar{n}_{d,\ell} \triangleq \bar{y}_{sr}$$

(23)

where we specifically use the extra subscript $\ell$ to denote the ARQ index. Notice that $\bar{y}_{sr}$ in (23) remains unchanged throughout the relaying process of a sequence of ARQ.

Suppose the codeword $\tilde{x}_j$ is decoded individually according to the principle of MLSD within each retransmission block. Let $\mathcal{E}_{AB}^n$ be the event of $n$ consecutive MLSD decoding failures in transmissions from node A to node B, and $\text{Pr}\{\mathcal{E}_{AB}^n\}$ be the average PER. Given the threshold $\Delta$, the PER after $n$ rounds of ARQs with SAF relaying is expressed as

$$P_{e,n}^{\text{SAF}} = \text{Pr}\{\mathcal{E}_{sd}\} \times \sum_{\ell=0}^n (\text{Pr}\{a \leq \Delta\} \text{Pr}\{\mathcal{E}_{sd}\})^{n-\ell} \times F_e(\Delta, \ell)$$

(24)

where $F_e(\Delta, \ell) \triangleq \text{Pr}\{a > \Delta, \mathcal{E}_{rd}^\ell\}$ with $F_e(\Delta, 0) \triangleq 1$. 
We will show that $F_e(\Delta, \ell)$ is lower bounded by $\Pr\{a > \Delta, \mathcal{E}_{sr}\}$ at high SNR. For simplicity, we consider that the codewords are equiprobable and uniformly distributed over a complex sphere. The PER of a certain codeword $\bar{x}_j$ is thus equal to the average PER, namely, $\Pr\{a > \Delta, \mathcal{E}_{rd}\} = \Pr\{a > \Delta, \mathcal{E}_{rd} | \bar{x}_j\}$. We therefore assume in the analysis that the codeword $\bar{x}_0$ was sent.

Conditioned on channel states, the PER of $\mathcal{E}_{rd}$ is given by

$$\Pr\{\bar{y}_{rd,1} \in \Lambda_1, \ldots, \bar{y}_{rd,\ell} \in \Lambda_\ell | \bar{x}_0, a > \Delta, b_1, \ldots, b_\ell\}$$

where according to the MLSD principle, the regions of $\Lambda_i$, $i \in \mathcal{I}_\ell$, are defined as

$$\Lambda_i = \left\{ \bar{y} : \|\bar{y} - \sqrt{ab_i/(a+1)}\bar{x}_0\|^2 > \min_{j \in \mathcal{I}_\ell^i} \|\bar{y} - \sqrt{ab_i/(a+1)}\bar{x}_j\|^2 \right\}, \quad i \in \mathcal{I}_\ell.$$  (26)

By substituting (23) into (25) with $\bar{x}_j := \bar{x}_0$, we express the ARQ error event in (25) as

$$\{\bar{y}_{rd,i} \in \Lambda_i\} = \left\{ \min_{j \in \mathcal{I}_\ell^i} \left[ \sqrt{b_i} \left( \sqrt{a} \|d_{0,j}\|^2 + 2\Re\{d_{0,j}^H \bar{n}_r\} \right) + 2\Re\{d_{0,j}^H \bar{n}_{d,i}\} \right] < 0 \right\}.$$  (27)

where $\bar{d}_{0,j} \triangleq \bar{x}_0 - \bar{x}_j$. For conciseness, we have

$$\phi_{1,j} \triangleq \frac{1}{\sqrt{a}+1} \left( \sqrt{a} \|\bar{d}_{0,j}\|^2 + 2\Re\{\bar{d}_{0,j}^H \bar{n}_r\} \right)$$

with the subscript 1 of $\phi_{1,j}$ used here to denote the one-hop signal received by the relay. We also define $\theta_{1,j}(\phi) \triangleq \sqrt{a} \phi + 2\Re\{\bar{d}_{0,j}^H \bar{n}_{d,i}\}$. In Section IV-B, $\theta_{1,j}^{[q]}(\phi)$ will be extended to

$$\theta_{1,j}^{[q]}(\phi) \triangleq \sqrt{b_{1,j}^q} \phi + 2\Re\{\bar{d}_{0,j}^H \bar{n}_{d,i}\}$$

(29)

to indicate that the active relay for ARQ is selected from $q = |\mathcal{Q}|$ relays in $\mathcal{Q}$. The notation $b_{[q]}$ is defined in (18).

Now marginalizing out the channel effects of (25), we obtain

$$F_e(\Delta, \ell) = \Pr\{a > \Delta, \mathcal{E}_{rd}^\ell | \bar{x}_0\} = \Pr\left\{a > \Delta, \bigcap_{i=1}^{\ell} \left( \min_{j \in \mathcal{I}_\ell^i} \theta_{1,j}^{[i]}(\phi_{1,j}) < 0 \right) \right\}.$$  (30)

We notice that $\phi_{1,j}$, $\forall j$, result from the received signal at the relay and thus remain unchanged with $\ell$ in (30). In contrast, $\theta_{1,j}^{[1]}(\phi_{1,j})$ varies for different ARQ rounds, $i$, due to the channel variations of the R-D link in different retransmissions. On the other hand, the PER of the received signal at the AF relay would have been

$$\Pr\{a > \Delta, \mathcal{E}_{sr} | \bar{x}_0\} = \Pr\left\{a > \Delta, \min_{j \in \mathcal{I}_\ell^p} \phi_{1,j} < 0 \right\};$$  (31)
if the source signal were decoded at the relay. For conciseness, we define $\phi_1 \triangleq \min_{j \in I^p} \phi_{1,j}$, and rewrite (31) as $\Pr\{a > \Delta, \phi_1 < 0\}$ or $\mathbb{E}_{a > \Delta, n_r}[1_{\phi_1 < 0}]$

Recall from Lemma 2 that $F(\Delta, \ell) = (\frac{\ell}{\rho \beta_2})^\ell$ if $\Delta := \lambda \delta > \delta$. The thresholding mechanism of ARQ-SAF in fact prescreens the S-R channel qualities to avoid “outage events” from occurring at the relay. The temporal diversity owing to the R-D channel variations in ARQs can thus be fully exploited from the outage probability point of view. Nevertheless, when typical channel encoders of finite code lengths are used in practice, the corresponding packets are not free from errors at the relay even if its received SNR is greater than $\delta$, although the transmitted codeword in fact is not decoded by the AF relay. The implicit error propagation from the relay will reduce the probability of successfully delivering packets to the destination, and may even cause severe diversity loss in PERs of ARQs.

The relations between $F_e(\Delta, \ell)$ and the implicit PER at the relay, namely $\mathbb{E}_{a > \Delta, n_r}[1_{\phi_1 < 0}]$, are characterized in the following lemma. Basically, we use the upper and lower bounds, $\mathbb{E}_{a > \Delta, n_r}[1_{\phi_1 < \tau}]$ and $\mathbb{E}_{a > \Delta, n_r}[1_{\phi_1 < -\tau}]$, of $\mathbb{E}_{a > \Delta, n_r}[1_{\phi_1 < 0}]$ to conduct the analysis, where $\tau := \frac{1}{(\ln \rho)^2}$ is a positive parameter and will converge to zero as $\rho$ increases.

**Lemma 3:** Given $\Delta$ and $\tau := \frac{1}{(\ln \rho)^2}$, for $\ell > 0$, $F_e(\Delta, \ell)$ are bounded in the form of

$$\mathbb{E}_{a > \Delta, n_r}[1_{\phi_1 < -\tau}] \leq F_e(\Delta, \ell) \leq \mathbb{E}_{a > \Delta, n_r}[1_{\phi_1 < \tau}] + \left(\Pr\{\min_{j \in I^p} \theta_{1,j}(\tau) < 0\}\right)^\ell. \quad (32)$$

The second term of the upper bound achieves the diversity order of $\rho^{-\ell}$, namely, it achieves the potential order expected on $F_e(\Delta, \ell)$ in temporal diversity.

**Proof:** We use the event $[\phi_1 < \tau]$ and its complementary set to partition the probability (30), which results in an upper bound for (30) given by

$$\Pr\{a > \Delta, \phi_1 < \tau\} + \Pr\left\{a > \Delta, \phi_1 \geq \tau, \bigcap_{i=1}^{\ell} \left(\min_{j \in I^p} \theta_{i,j}(\phi_{1,j}) < 0\right)\right\}. \quad (33)$$

The first term of (33) is equivalent to $\mathbb{E}_{a > \Delta, n_r}[1_{\phi_1 < \tau}]$. As for the second term of (33), due to the condition of $\phi_{1,j} \geq \phi_1 \geq \tau$, and the fact that $\theta_{i,j}(\phi)$ is monotonically increasing with $\phi$, thus this part can be further upper bounded by

$$\Pr\left\{\bigcap_{i=1}^{\ell} \left(\min_{j \in I^p} \theta_{i,j}(\tau) < 0\right)\right\} = \left(\Pr\{\min_{j \in I^p} \theta_{1,j}(\tau) < 0\}\right)^\ell, \quad (34)$$

where the equality holds since $\theta_{i,j}(\tau)$, for $i \in I^p$, are statistically independent of each other.

The $\tau$ here can be viewed as a special squared codeword distance, and (34) represents the
Fig. 6. The PERs of ARQ-SAF for different $\Delta$s, and ARQ1~ARQ3. The information bits are encoded with a trellis code using the generator polynomial $[5_8, 7_8]$ and the QPSK modulator.

PER for $\ell$ uncorrelated rounds of ARQs. The PER for each round can be shown to achieve the order of $\rho^{-1}$, and thus (34) will attain the order of $\rho^{-\ell}$. The corresponding derivation is shown in Appendix-C.1 together with the proof for the lower bound of (32). For the sake of completeness, in this appendix, we will first show a more general result of

$$\Pr\left\{\min_{j \in I_0^\ell} \theta_{1,j}^{[q]}(\tau) < 0\right\} \approx \rho^{-q}$$

(35)

in the appendix, and then continue to prove the lower bound of (32).

Lemma 3 clearly shows how $F_e(\Delta, \ell)$ is affected by the S-R link. A key observation from (32) is that even though the second term of the upper bound in (32) can attain the potential diversity of $F_e(\Delta, \ell)$, namely the order of $\rho^{-\ell}$, $F_e(\Delta, \ell)$ is still lower bounded by $\mathbb{E}_{a > \Delta, a_r}[1_{\phi_1 < -\tau}]$ of the S-R link at high SNR. Furthermore, we show in Appendix-C.2 that given a rate $R$, if the threshold $\Delta$ remains unchanged with $\rho$, then it follows that

$$\mathbb{E}_{a > \Delta, a_r}[1_{\phi_1 < -\tau}] \approx \frac{1}{\rho^\ell}.$$

(36)

As a result, if $\Delta$ does not change with $\rho$, diversity loss will happen in $F_e(\Delta, \ell)$ for $\ell > 1$. On the other hand, from (24), we know that $P_{e,n}^{\text{SAF}} = \Pr\{E_{sd}\} F_e(\Delta, n)$ owing to the fact of $\Pr\{a \leq \Delta\} \Pr\{E_{sr}\} \approx \frac{1}{\rho^\ell}$. Given that $\mathbb{E}_{a > \Delta, a_r}[1_{\phi_1 < -\tau}] \leq \mathbb{E}_{a > \Delta, a_r}[1_{\phi_1 < \tau}]$, both approach $\mathbb{E}_{a > \Delta, a_r}[1_{\phi_1 < 0}]$ when $\rho$ increases. Thus, according to Lemma 3, for all $n \geq 2$, $P_{e,n}^{\text{SAF}}$ at high
SNR will be limited by a performance bound of

\[ \Pr\{\mathcal{E}_{sd}\} \times \mathbb{E}_{a > \Delta, \bar{n}_r} [1_{\phi_1 < 0}] = \Pr\{\mathcal{E}_{sd}\} \times \Pr\{a > \Delta, \mathcal{E}_{sr}\}. \]  

(37)

We simulate in Fig. 6 the PERs of the ARQ-SAF scheme for data encoded with a trellis code. The results are for a single relay system in three rounds of ARQs when \( \Delta := 1.5\delta \) and \( 3\delta \), both of which satisfy Requirement 1. As the SNR \( \rho \) increases, the PERs of the second and third ARQ rounds are diversity limited as expected from (37) even though the thresholds satisfy Requirement 1 for outage analysis.

In the end of this subsection, we can conclude that increasing threshold \( \Delta \) with respect to \( \rho \) is a necessary condition to achieve the full diversity order in \( \mathcal{P}_{e,n}^{SAF} \) for \( n \geq 2 \), because if \( \Delta \) doesn’t increase with \( \rho \), we can find a finite number \( \Delta_u \geq \Delta \) such that \( \mathbb{E}_{a > \Delta, \bar{n}_r} [1_{\phi_1 < -\tau}] \geq \mathbb{E}_{a > \Delta_u, \bar{n}_r} [1_{\phi_1 < -\tau}] \frac{d}{\rho} \), which leads to the same diversity loss result in \( F_e(\Delta, n) \) for \( n \geq 2 \) according to (32). With this result, a question that arises to us is how fast \( \Delta \) should increase with \( \rho \) to recover the diversity loss. Before hopping into the details of this problem, we first develop a new requirement on the received signal quality at the relay.

A.1 Diversity recovery for ARQ-SAF in PER

To address this issue, \( \Delta \) needs to be adjusted properly with respect to (w.r.t.) the SNR \( \rho \). However, it is worth noticing from the diversity analysis of (15) and (16) on \( \mathcal{P}_{out,n}^{SOAP-A} \) that \( \Pr\{a \leq \Delta\}^m \frac{d}{\rho^m} \). This result is obtained based on the fact that the threshold, \( \Delta := \lambda\delta \), is constant w.r.t. \( \rho \). This constant requirement is also applied to each of the thresholds of \( \overline{\Delta} \) in (20) for \( \mathcal{P}_{out,n}^{SOAP-B} \). Thus, for ARQ schemes that adjust \( \Delta \) or \( \Delta_i \) in \( \overline{\Delta} \) according to \( \rho \), we need to pose an extra requirement on the threshold(s) in order to maintain the same order of \( \frac{1}{\rho^{m-|Q|}} \) in the probability of relays in \( Q^c \) not being brought into \( Q \).

Requirement 2: If overhearing signals from the source or relays is allowed, the probability that \( m - |Q| \) relays in \( Q^c \) cannot be added into \( Q \) has the order of \( \rho^{-(m-|Q|)} \).

Requirement 2 also implies that relays in \( Q^c \) will be brought into \( Q \) at high SNR with probability one. For the ARQ-SAF, if its threshold setting method satisfies Requirement 2 such that \( \Pr\{a \leq \Delta\} \Pr\{\mathcal{E}_{sd}\} \frac{d}{\rho^s} \), then by (24) we still have \( \mathcal{P}_{e,n}^{SAF} = \Pr\{\mathcal{E}_{sd}\} F_e(\Delta, n) \).
Further, according to Lemma 3, $P_{\text{SAF}}^{e,n}$ will be upper bounded by

$$P_{\text{SAF}}^{e,n} \leq \Pr\{E_{sd}\} \mathbb{E}_{a > \Delta, \bar{n}_r} [1_{\phi_1 < \tau}] + \Pr\{E_{sd}\} \left( \Pr\left\{ \min_{j \in J, D} \theta_{1,j}^{[1]}(\tau) < 0 \right\} \right)^{n}, \text{ for } n \in \mathcal{I}_1^N. \quad (38)$$

We notice that the second term of the upper bound achieves the order of $\frac{1}{\rho^{n+1}}$, which attains the potential diversity order predicted with outage analysis, and the term $\mathbb{E}_{a > \Delta, \bar{n}_r} [1_{\phi_1 < \tau}]$ is irrelevant to the ARQ index $n$.

Therefore, to ensure $P_{\text{SAF}}^{e,n} \leq \frac{1}{\rho^{n+1}}, \forall n \in \mathcal{I}_1^N$, we also need to have $\mathbb{E}_{a > \Delta, \bar{n}_r} [1_{\phi_1 < \tau}] = \frac{1}{\rho^{n}},$ in addition to Requirement 2. Though it seems counterintuitive, this condition can actually happen if we have $\Delta$ continuously increase with the SNR $\rho$. In other words, the essence to recover the diversity loss in $P_{\text{SAF}}^{e,n}$ lies in setting a higher signal quality than what is posed by Requirement 1 on the S-R link in order to prevent the performance of relaying from being dominated by the errors in this link. Before proceeding to find the threshold $\Delta$ to make this happen, we continue to reexamine how the PERs of the two proposed SOAF ARQ schemes are affected by multiple relays.

**B. The PER of ARQ-SOAF**

In contrast to a single relay system, there are multiple relays in the SOAF ARQ schemes to participate in retransmission. As illustrated in Fig 7, for the SOAF-B ARQ protocol, the source signal can go through various kinds of paths before reaching the destination. As a result, the inter-relationships among the ARQ events are quite complicated. Analyzing the exact PERs of the ARQs turns out to be a formidable task.

To simplify the complexity in presentation, we mainly study the case of SOAF-B ARQ in the sequel. The result can be similarly extended to the SOAF-A case. To proceed with the diversity analysis on the PERs of SOAF-B ARQs, we first analyze a special sequence of ARQ error events. According to the analysis, we develop a new requirement stricter than Requirement 1 to guarantee that the potential diversity in PER can be fully exploited for the special ARQ events. We then show that this new requirement can be applied to the general case of SOAF-B ARQ as well.

We first extend the two-hop decision statistic model (23) to a multiple-hop case. If a relay in $Q$ with $|Q| = q$ is chosen to be active for ARQ\(\ell\) and forwards a $k$-hop signal, then
Fig. 7. Three simple examples are used to illustrate the possible paths for relaying the source signal to the destination in different SOAF-B ARQs. We assume ARQ_\(j\) occurs after ARQ_\(i\), i.e., \(i < j\). For ARQ_\(i\), the channel qualities of the hops that the source signal has gone through before reaching the destination are marked by a superscript \((i)\). In case (a), the relaying paths for the two ARQs are correlated, while in case (b), the two paths are independent. Case (c) illustrates the two paths of different hops.

the decision statistic at the destination can be expressed as

\[
\bar{y}_{rd,\ell} = \sqrt{\frac{b^{[q]}}{c_k + 1}} \times \bar{y}_{rr,k} + \bar{n}_{d,\ell}
\]

(39)

where \(\sqrt{b^{[q]}}\) is contributed by the link quality from the active relay to the destination, and \(\frac{1}{\sqrt{c_k + 1}}\) is the power normalization factor \([30]\) at the active relay on its received signal, \(\bar{y}_{rr,k}\), given by

\[
\bar{y}_{rr,k} = \frac{\sqrt{c_k}}{\sqrt{c_{k-1} + 1}} \times \bar{y}_{rr,k-1} + \bar{n}_{r,k}, \text{ for } k > 2,
\]

(40)

with \(\bar{y}_{rr,2} = \frac{\sqrt{c_2}}{\sqrt{a_1 + 1}} \times (\sqrt{a_1} \bar{x}_j + \bar{n}_{r,1}) + \bar{n}_{r,2}\). In (40), the link qualities, \(a_1, c_2, \ldots, c_k\), correspond to each hop of the relaying path of the source signal until reaching this active relay, and the entries of \(\bar{n}_{r,k}\) are also i.i.d. complex Gaussian random variables with zero mean and unit variance.

We then consider a sequence of ARQ error events in which a relay received a forwarded signal that had gone through a certain path of \(k\) hops, and this relay have continued to be chosen as the forwarding relay for ARQ_\(k \sim ARQN\), while the relayed source signal still fails to be decoded at the destination. We notice that the case of a relay to be added into \(Q\) by receiving a \(k\)-hop signal is only possible after ARQ_\(k-1\). This implies that ARQ_\(k \sim ARQN\) are the only occasions for this relay to be selected to retransmit its received \(k\)-hop signal.

To distinguish the qualified sets \(Q\) for different ARQs, we denote the qualified set for ARQ_\(i\) by \(Q_i\) and the corresponding cardinality of \(Q_i\) by \(q_i\), for \(i \in \mathbb{I}_N\). Conditioned on
\[ Q_k, \ldots, Q_N, \] according to the MLSD principle, the PER for this special sequence of ARQ events can be expressed in a form similar to (30) as follows

\[
\left( \prod_{i=k+1}^{N} \frac{1}{q_i} \right) \times \Pr \left\{ a_1 > \Delta_1, \ldots, c_k > \Delta_k, \bigcap_{i=k}^{N} \left( \min_{j \in \mathcal{D}_i} \theta_{i,j}^{[q_i]}(\phi_{k,j}) < 0 \right) \right\} \triangleq \left( \prod_{i=k+1}^{N} \frac{1}{q_i} \right) \times P_{e,k} \quad (41)
\]

where \( \left( \prod_{i=k+1}^{N} \frac{1}{q_i} \right) \) is the probability of choosing the same relay in \( Q_{k+1}, \ldots, Q_N \), and \( \phi_{k,j} \) can be recursively expressed as

\[
\phi_{k,j} = \frac{1}{\sqrt{c_k + 1}} \left( \sqrt{c_k \phi_{k-1,j}} + 2\Re\{d_{0,j}^{H}n_{r,k}\} \right) \text{ for } k \geq 2 \quad (42)
\]

with \( \phi_{1,j} \) already defined in (30). Apparently, the value of \( \phi_{k,j} \) is correlated to the channel qualities, \( a_1, c_2, \ldots, c_k \) in (41), that correspond to each hop along this particular path from the source to the relay. In fact, there may exist another possible path from the source to this active relay in the same number of hops. The PER for the corresponding ARQ events, however, still has the same form of (41).

For the sake of conciseness, we have

\[
\mathcal{Z}_k \triangleq (a_1 > \Delta_1) \cap (\cap_{i=2}^{k} c_i > \Delta_i). \quad (43)
\]

Recall (31) for the PER of decoding a one-hop signal. The PER of decoding a \( k \)-hop signal received at a relay can be similarly obtained as \( \Pr\{\mathcal{Z}_k, \phi_k < 0\} \) where \( \phi_k \triangleq \min_{j \in \mathcal{D}_k} \phi_{k,j} \).

Then, following the same procedure to prove (32), we can show \( P_{e,k} \) bounded by

\[
\Pr\{\mathcal{Z}_k, \phi_k < -\tau\} \leq P_{e,k} \leq \Pr\{\mathcal{Z}_k, \phi_k < \tau\} + \prod_{i=k}^{N} \Pr\left\{ \min_{j \in \mathcal{D}_i} \theta_{i,j}^{[q_i]}(\tau) < 0 \right\} \quad (44)
\]

where \( \tau = \frac{1}{(\ln \rho)^2} \). For clarity, a sketch of the proof for (44) is provided in Appendix-C.3. According to (35), the second term of the upper bound can attain the order of \( \rho^{-(q_k + \cdots + q_N)} \).

To determine the maximum possible diversity order that \( P_{e,k} \) can achieve, which is closely related to the diversity order of \( \Pr\{\mathcal{Z}_k, \phi_k < \tau\} \) in (44), we first recall from Section III-B.2 that in comparison with SOAF-A ARQ, SOAF-B ARQ can prevent \(|Q|\) from being limited by the case of \(|Q| = 1\) after ARQ1. Therefore, the maximum possible order in \( P_{e,k} \) should be equal to \( m(N - k + 1) \) when \( q_k = \cdots = q_N = m \). As a result, to avoid diversity loss in \( P_{e,k} \), by (44), the order of \( \Pr\{\mathcal{Z}_k, \phi_k < \tau\} \) has to achieve \( m(N - k + 1) \) at least to allow \( P_{e,k} \) to attain the order of \( \rho^{-(q_1 + \cdots + q_N)} \) for any \( q_i \in \mathcal{I}_1^m \). With this observation, we define a new requirement on \( \Pr\{\mathcal{Z}_k, \phi_k < \tau\} \) for the SOAF-B relaying.
Requirement 3: For each relay in \( Q \), if it receives a \( k \)-hop signal with \( \phi_k \) defined as \( \min_{j \in I} \phi_{k,j} \) in (42), then \( \Pr \{ Z_k, \phi_k < \tau \} = \frac{1}{\rho^{\min_{i \in I} N_i - q_i}}, \forall k \in I_1^{\min[N,m]} \).

Apparently, \( \Pr \{ Z_k, \phi_k < \tau \} \) is strongly related to the PER of decoding a \( k \)-hop signal, namely \( \Pr \{ Z_k, \phi_k < 0 \} \). In other words, the essence of Requirement 3 is to provide different conditions on the received signal qualities at the relays in \( Q \) for signals of different hops. Nevertheless, Requirement 3 is basically derived from a special case in which the ARQs are performed with a same active relay. By (44), it has been shown that under Requirement 3, the diversity order offered by each ARQ in this case can increase by an amount equal to the cardinality of \( Q \) for this ARQ round. As a matter of fact, the same diversity analysis result can also be obtained for the general case that ARQs are done with different active relays chosen from \( Q \), provided that Requirement 3 is satisfied. This case will be discussed in the next theorem.

Theorem 2: If Requirement 2 and 3 are satisfied, then the diversity order of the PER after \( n \) rounds of ARQ-SOAF-B is given by \( (m \times n + 1) \), \( \forall n \in I_1^N \).

Proof: In order to show the general case of ARQ, as illustrated in Fig. 7, we extend the notations \( \phi_{k,j} \) for (41) to \( \phi_{k,j}^{(i)} \) that corresponds to \( Z_k^{(i)} \) for ARQi. When two relays of receiving \( k \)-hop signals are chosen to be the active relays for ARQi and ARQj respectively, the corresponding \( \phi_k^{(i)} \) and \( \phi_k^{(j)} \) may or may not be statistically independent, depending on whether the relaying paths from the source to the two relays are completely different or not. If yes, then they are statistically independent; otherwise, they are correlated, because some events of \( Z_k^{(i)} \) are identical to those of \( Z_k^{(j)} \).

In addition, for convenience, we use \( \kappa_i \) to represent the number of hops that the relayed signal in ARQi had gone through before reaching the destination.

After the settings, we are now ready to analyze the diversity order of the PER after \( n \) rounds of the SOAF-B relaying. Under Requirement 2, the order of the PER after ARQn can be written in the from of

\[
\frac{1}{\rho} \times \left( \prod_{i=1}^{n} \frac{1}{\rho^{\min_{j \in I} N_j - q_i}} \right) \times \Pr \left\{ \bigcap_{i=1}^{n} Z_{\kappa_i}^{(i)}, \min_{j \in I, i} \theta_{i,j}^{[q_i]}(\phi_{k,j}^{(i)}) < 0 \right\}
\]

where \( q_i = |Q_i| \), the first term \( \frac{1}{\rho} \) results from ARQ0, and the product term \( \left( \prod_{i=1}^{n} \frac{1}{\rho^{\min_{j \in I} N_j - q_i}} \right) \) is contributed by the probabilities of relays failing to be added into \( Q_i \) during ARQi \( -1 \), for \( i \in I_1^n \), given that Requirement 2 is satisfied. We notice that \( \kappa_i \leq \min[i, m] \).
Then, we use the event \([\cap_{i=1}^n (\phi_{i}^{(i)} > \tau)]\) and its complement \([\cap_{i=1}^n (\phi_{i}^{(i)} > \tau)]^c\) to partition (45) as a sum of two parts. Further, given that \(\theta_{i,j}^{[q]}(\phi)\) is monotonically increasing with \(\phi\), an upper bound for (45) can thus be given by

\[
\frac{1}{\rho} \times \left( \prod_{i=1}^n \frac{1}{\rho^{m-n_i}} \right) \times \Pr \left\{ \cap_{i=1}^n \min_{j \in I_P^i} \theta_{i,j}^{[q]}(\tau) < 0 \right\} 
+ \frac{1}{\rho} \times \left( \prod_{i=1}^n \frac{1}{\rho^{m-n_i}} \right) \times \Pr \left\{ \cap_{i=1}^n \theta_{i,j}^{[q]}(\phi_{i}^{(i)}) < 0, \cap_{i=1}^n (\phi_{i}^{(i)} > \tau) \right\}.
\]

(46)

According to (35), the first part in (46) can attain the order of \(\frac{1}{\rho^{m+n}}\), which apparently meets the proposed diversity order of this theorem. As for the second part, we first show the fact that the complement event \([\cap_{i=1}^n (\phi_{i}^{(i)} \geq \tau)]^c\) can be expressed as a union of \(n\) mutual events (mutual sets) as follows

\[
[\phi_{i}^{(1)} \geq \tau]^c \cup [\phi_{i}^{(1)} \geq \tau] \cap [\phi_{i}^{(2)} \geq \tau]^c \cup \cdots \cup \left[ \cap_{i=1}^{n-1} \phi_{i}^{(i)} \geq \tau \right] \cap (\phi_{i}^{(n)} \geq \tau)^c. \]

(47)

This immediately makes the second part equal to

\[
\sum_{\ell=1}^n \frac{1}{\rho} \left( \prod_{i=1}^n \frac{1}{\rho^{m-n_i}} \right) \Pr \left\{ \cap_{i=1}^n \theta_{i,j}^{[q]}(\phi_{i}^{(i)}) < 0, \cap_{i=1}^n \min_{j \in I_P^i} \theta_{i,j}^{[q]}(\phi_{i}^{(i)}) < 0 \right\}.
\]

(48)

The \(\ell\)-th term of the above summation can be upper bounded as follows

\[
\frac{1}{\rho} \left( \prod_{i=1}^n \frac{1}{\rho^{m-n_i}} \right) \Pr \left\{ \cap_{i=1}^n \theta_{i,j}^{[q]}(\phi_{i}^{(i)}) < 0, \cap_{i=1}^n \min_{j \in I_P^i} \theta_{i,j}^{[q]}(\phi_{i}^{(i)}) < 0 \right\} \leq \frac{1}{\rho} \times \left( \prod_{i=1}^n \frac{1}{\rho^{m-n_i}} \right) \Pr \left\{ \theta_{i,j}^{[q]}(\phi_{i}^{(i)}) < 0, \cap_{i=1}^n \min_{j \in I_P^i} \theta_{i,j}^{[q]}(\phi_{i}^{(i)}) < 0 \right\}
\]

(49)

where the last equality holds according to (35). Further, if Requirement 3 is satisfied, each term of \(\ell\) in (48) can thus attain the order of

\[
\frac{1}{\rho} \times \left( \prod_{i=\ell}^n \frac{1}{\rho^{m-n_i}} \right) \times \frac{1}{\rho^{\ell-1}} \times \frac{1}{\rho^{N-N-\kappa_\ell+1}} \leq \frac{1}{\rho^{1+mN}}
\]

(50)

in which the inequality follows due to \(\kappa_\ell \leq \ell\), and the corresponding equality holds when \(\ell = \kappa_\ell\) and \(q_\ell = \cdots = q_n = m\). As a result, by (46), the diversity order of the PER after ARQn will be dominated by the first part and attain \(1 + mn\) for \(n \in \mathcal{I}_1^N\).

With a procedure similar to (41)~(49), Requirement 3 can be revised for SOAF-A ARQ. A sketch of the analysis is shown as below. Since the relays in SOAF-A ARQ only receive
one-hop signals, thus just $P_{e,1}$ in (44) needs to be considered, and its maximum possible
diversity order will become $m + N - 1$ when $q_1 = m$ and $q_i = 1, \forall i \in \mathcal{I}_2^V$. That is because
for SOAF-A ARQ, the diversity after ARQ1 will be dominated by the case of $|\mathcal{Q}| = 1$. As a result, Requirement 3 for SOAF-A ARQ is revised as $\Pr\{Z_1, \phi_1 < \tau\} \leq \frac{1}{\rho^{m + N - 1}}$. Further, similar to (45) for SOAF-B ARQ, the order of the PER after $n$ rounds of ARQs with
SOAF-A relaying can be equal to that of
\begin{equation}
\frac{1}{\rho} \times \frac{1}{\rho^{m - q_1}} \times \Pr\left\{ \bigcap_{i=1}^{n} Z_1^{(i)}, \bigcap_{i=1}^{n} \min_{j=1}^{j \in \mathcal{I}_i^D} \theta_i^{(j)}(\phi_1^{(j)}) < 0 \right\}.
\end{equation}

We notice that $|\mathcal{Q}_1| = q_1$ remains the same throughout the ARQs.

We then use $\bigcap_{i=1}^{n} \phi_1^{(i)} \geq \tau$ and its complement to partition (51) into two parts similar
to (46). For the first part, it can be easily shown that it will attain the order of $\frac{1}{\rho} \frac{1}{\rho^{m - q_1}} \frac{1}{\rho^{q_1}}$, and apparently, its diversity order will be limited by the case of $q_1 = 1$ for $n \geq 2$. As for
the second part, by the same arguments for (47)∼(49), it can also be upper bounded by a
summation of $n$ probability functions. Then, the $\ell$-th term in this summation can attain
the same diversity order of $\frac{1}{\rho} \frac{1}{\rho^{m - q_1}} \frac{1}{\rho^{q_1}} \frac{1}{\rho^{q_1 \ell - 1}} \frac{1}{\rho^{m + N - 1}} \leq \frac{1}{\rho^{m + N}}$
for any $\ell \in \mathcal{I}_1^a$ and $q_1 \in \mathcal{I}_1^m$. As a result, the diversity order of the PER after ARQ$n$ will
be dominated by the first part and thus attain $m + n$, for $n \in \mathcal{I}_1^N$.

To show the explicit relation between the thresholds of $\Delta$ and the proposed requirements,
we next derive two sufficient conditions to meet Requirement 2 and 3 respectively, thereby
providing useful insights into a threshold setting method for the SOAF-B ARQ scheme.

C. Sufficient conditions for Requirement 2 and 3

We recall from Requirement 2 that its purpose is to guarantee the diversity in the proba-
bility of unqualified reception at relays. An explicit sufficient condition for Requirement 2
can be given by
\begin{equation}
\lim_{\rho \to \infty} \frac{\ln \Delta_k}{\ln \rho} = 0, \forall k \in \mathcal{I}_1^{\min[m,N]}.
\end{equation}
The reason is as follows. The condition of $\lim_{\rho \to \infty} \frac{\ln \Delta_k}{\ln \rho} = 0$ implies $\lim_{\rho \to \infty} \Delta_k = 0$ owing to the
fact that $\frac{\Delta_k}{\rho} = \rho^{-1 - \ln \Delta_k / \ln \rho}$. Under the result of $\lim_{\rho \to \infty} \Delta_k = 0$, we have the probability
\begin{equation}
\Pr\{c < \Delta_k\} = \left(1 - e^{-\frac{\Delta_k}{\rho^{m-q_1}}}ight)^{m-|\mathcal{Q}|} = \left(\frac{\Delta_k}{\rho^{m-q_1}}\right)^{m-|\mathcal{Q}|} + \mathcal{O}\left(\frac{1}{\rho^{m-|\mathcal{Q}|+1}}\right)
\end{equation}
where \( \Phi(\cdot) \) is the big-O notation, and by definition, the dominant term of this probability apparently attains the order of \( \frac{1}{\rho^{|Q|}} \). Thus, the condition (52) can imply Requirement 2.

In contrast to Requirement 2, the purpose of Requirement 3 is to guarantee the diversity in the PER of a qualified reception at a relay. To this end, we need the following lemma to characterize the impact of \([\Delta_1, \ldots, \Delta_k]\) on the diversity order of \( \Pr\{Z_k, \phi_k < \tau\} \).

**Lemma 4:** Define \( \tau = \frac{1}{(\ln \rho)^2} \), and \( \epsilon \) and \( \varepsilon_0 \) to be fixed and small positive numbers where \( \varepsilon_0 \) is less than the squared minimum codeword distance, \( d_m^2 \). Let a threshold set \([\Delta_1, \ldots, \Delta_k]\) satisfy \( \Delta_i > \epsilon, \forall i \in I_k \). When \( \rho \) increases such that \( (1 + \frac{1}{\epsilon})^2 \tau < \varepsilon_0 \), we have

\[
\Pr\{Z_k, \phi_k < \tau\} \leq \sum_{d \in \{d_m, d_M\}} Q \left( \sqrt{\frac{\psi_k \times (d^2 - \varepsilon_0)^2}{2d^2}} \right) \times \frac{\omega_d}{\rho \times (d^2 - d_0^2) + 1} \left( \frac{\Delta_1}{\beta_1} + \sum_{i=2}^k \frac{\Delta_i}{\beta_i} \right) \tag{54}
\]

where \( \psi_k \equiv (\prod_{i=1}^k (1+1/\Delta_i) - 1)^{-1} \), \( d_M \) is the maximum Euclidean distance in the codebook, and \( \omega_d \) is the number of the codewords with the distance \( d \).

**Proof:** See Appendix-C.4.

We notice that \( \psi_k \) is a function of the thresholds, \([\Delta_1, \ldots, \Delta_k]\). Under the condition (52), according to (54), we can find a simple relation between the diversity order of \( \Pr\{Z_k, \phi_k < \tau\} \) and \( \psi_k \), whereby we are able to define a condition on \( \psi_k \) in order to have \( \Pr\{Z_k, \phi_k < \tau\} \) as stated in Requirement 3. We summarize this result in the next corollary.

**Corollary 2:** Under the condition (52), a sufficient condition to meet Requirement 3 can be given by

\[
\lim_{\rho \to \infty} \frac{\psi_k}{\ln \rho} \geq \frac{4}{d_{m, \varepsilon_0}^2} \times \left[ m \left( N - k + 1 \right) - 1 \right], \forall k \in I_1^{\min[m,N]},
\]

where \( d_{m, \varepsilon_0}^2 \equiv \left( \frac{d_m^2 - \varepsilon_0}{d_m^2} \right)^2 \). In fact, \( d_{m, \varepsilon_0}^2 \) can be very close to \( d_m^2 \) if \( \varepsilon_0 \) is set small enough.

**Proof:** See Appendix-C.5.

Apparently, the conditions, (52) and (55), rely on the average SNR \( \rho \) and the minimum codeword distance \( d_m \), and they also reveal some clues for the thresholds assignment. For instance, given that \( \psi_k \leq \min[\Delta_1, \ldots, \Delta_k] \) if \( \Delta_i \geq 0, \forall i \in I_k \), by (55), we can observe that all the thresholds of \( \overline{\Delta} \) need to increase with \( \rho \) and intuitively, they can be scaled values of \( \ln \rho \). As a result, we may assign the thresholds of \( \overline{\Delta} \) as

\[
\overline{\Delta} = [\Delta_1, \ldots, \Delta_i, \ldots, \Delta_{\min[m,N]}] := [\lambda_1, \ldots, \lambda_i, \ldots, \lambda_{\min[m,N]}] \times \ln \rho \tag{56}
\]
with \( \lambda_i \in \mathbb{R}^+ \), \( \forall i \), being assigned properly to meet (55). More specifically, with the simple thresholds setting method of (56), we first obtain

\[
\lim_{\rho \to \infty} \frac{\psi_k}{\ln \rho} = \lim_{\rho \to \infty} \frac{1}{\ln \rho} \sum_{i=1}^{k} \frac{1}{\lambda_i \ln \rho} = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_k}. \tag{57}
\]

Then, substituting the result of (57) back into (55), we must be able to find a constant set of \([\lambda_1, \ldots, \lambda_{\min[m,N]}]\) to make (55) hold true. On the other hand, this simple thresholds setting method satisfies (52) as well due to \( \lim_{\rho \to \infty} \frac{\ln \Delta_i}{\ln \rho} = \lim_{\rho \to \infty} \frac{\ln \lambda_i + \ln \ln \rho}{\ln \rho} = 0, \forall i \). In addition, according to (55), if the employed codebook has a larger minimum codeword distance, the corresponding effective thresholds can be set smaller, accordingly.

Therefore, a SOAF-B ARQ scheme that uses the log-scale setting method can be guaranteed to exploit the maximum temporal and spatial diversities in PER. In the next section, we further develop a heuristic thresholds searching algorithm for the SOAF-B ARQ scheme in an attempt to exploit both the potential diversity and the SNR gain in PER.

V. Link Quality Control Revisited

From the viewpoint of diversity analysis, we have provided the conditions, (52) and (55), to avoid the diversity loss problem in SOAF-B ARQ. When the SNR goes to infinity, it seems unavoidable to increase the thresholds at a rate of \( \ln \rho \), cf. (56), in order to maintain the potential diversity order. However, within a low or mid SNR regime, the thresholds may not need to obey this rule. In fact, a lower threshold in this regime may give a better PER performance. Nevertheless, to properly set the thresholds for all possible values of SNR is a difficult task since this will need to characterize a finite-SNR result of \( P_{e,n}^{\text{SOAF-B}} \). To circumvent this difficulty, in this section, we reexamine the thresholds setting method from a heuristic point of view and develop a thresholds searching algorithm for SOAF-B ARQ to exploit both the potential diversity and the SNR gains.

For simplicity, we start our investigation from the case of ARQ-SAF. Ideally, one may expect to find a threshold \( \Delta_\ast \triangleq \arg_{\Delta} \min_{e,n} P_{e,n}^{\text{SAF}} \). However, in addition to the difficulty of solving this optimization problem, the analysis result is also hard to be extended to the case of SOAF-B ARQ due to the lack of the exact expression for \( P_{e,n}^{\text{SOAF-B}} \). To alleviate the difficulties, we adopt a rather heuristic approach for the threshold setting method.
We first recall (37) for the performance limitation in $P_{e,n}^{SAF}$ when $n \geq 2$ due to a constant threshold setting on $\Delta$. Then, from Fig. 6, we have the following observations: within the SNR range where the PERs are larger than the limitation (37), the diversity of ARQ-SAF can be clearly seen as what is expected; when $\Delta$ is set as $3\delta$, the PERs at high SNR are closer to their corresponding outage probabilities in comparison with the case of $\Delta = 1.5\delta$, owing to a lower bound of (37).

Based on the observations, to achieve the full diversity, it seems necessary to prevent the PER after the final ARQ round, i.e., $P_{e,N}^{SAF}$, from being limited by (37). Thus, for each $\rho$, a rough idea to properly set a threshold is to continue increasing $\Delta$ until the limitation (37) is less than the corresponding $P_{e,N}^{SAF}$ by a certain amount. Specifically, given a proper number $\alpha < 1$, for each $\rho$, we may correspondingly find the smallest $\Delta$ to satisfy

$$\Pr\{E_{sd}\} \times \Pr\{a > \Delta, \phi_1 < 0\} \leq \alpha \times P_{e,N}^{SAF}(\rho, \Delta)$$

(58)

where we note that $P_{e,N}^{SAF}$ is a function of $\rho$ and $\Delta$. The purpose of this is to bend the curve of (37) versus $\rho$ according to the possibly steepest decreasing trend of $P_{e,N}^{SAF}$, like $P_{out,N}$. We reasonably conjecture that to avoid diversity losses, the limitation (37) should decay at a rate of $\frac{1}{\rho^x\tau}$ in the high SNR regime, i.e., $E_{a>\Delta,\tilde{n}_r}|1_{\phi_1<0}\frac{d}{\rho^x\tau}$. This gives us a clue that the proposed sufficient requirement $E_{a>\Delta,\tilde{n}_r}|1_{\phi_1<\tau}\frac{d}{\rho^x\tau}$ in Section IV-A.1 may be or close to the necessary condition as well. On the other hand, more importantly, the threshold searching method of (58) offers a feasible way to properly assign $\Delta$ in the low and mid SNR regime, as opposed to that of (56). This method will be elaborated later for implementation. The notion of (58) can be similarly extended to the case of SOAF-A ARQ since $P_{e,N}^{SAF}$ and $P_{e,N}^{SOAF-A}$ both meet only one performance limitation which results from improper threshold setting for their S-R links only. In contrast, $P_{e,N}^{SOAF-B}$ suffers from several possible performance limitations due to its multiple thresholds for signals of different hops.

A. Multiple possible performance limitations in $P_{e,N}^{SOAF-A}$

Before showing the multiple limitations, we need a result similar to (36) that for a $k$-hop signal, if a threshold designed to qualify any of its inter hops dose not increase with SNR, then this threshold setting method will lead to

$$\Pr\{\phi_k < -\tau \mid Z_k\} \frac{d}{\rho^x} \frac{1}{\rho^x\tau}$$

(59)
The proof is omitted here, and it basically follows the procedures to prove Lemma 4 and (36) in Appendix C.4 and C.2 respectively. In addition, for conciseness of presentation, we extend the notation $\mathcal{E}_{rd}^n$ to $\mathcal{E}_{rd,k}^{n,q}$ in order to represent an event that a relay of receiving a $k$-hop signal continues to be chosen as the active relay from $\mathcal{Q}$ with $|\mathcal{Q}| = q$ in $n$ consecutive ARQs, while its signal still fails to be decoded at the destination. If (59) and $\Pr\{\mathcal{Z}_k\} \doteq 1$ both hold, then based on the results of (41) and (44), it follows that

$$\Pr\{\mathcal{Z}_k, \mathcal{E}_{rd,k}^{n,q}\} \doteq \frac{1}{q^{-1}} \Pr\{\mathcal{Z}_k, \phi_k < 0\} \doteq \frac{1}{\rho^n}, \forall n \geq 1, q = |\mathcal{Q}| \geq 2. \quad (60)$$

We start with the case of a constant $\Delta_1$ which makes (60) hold when $k = 1$. According to this result, we can know that $\mathcal{P}_{e,1}^{SOAF-B}$ is dominated at high SNR by the case of $|\mathcal{Q}_1| = m$ due to its lowest diversity order. Following the same argument and partitioning the events of ARQ2 into two sets with one of them corresponding to choosing the same relay in two ARQs, and the other for its complements, we express $\mathcal{P}_{e,2}^{SOAF-B}$ at high SNR as

$$\Pr\{\mathcal{E}_{sd}\} \times \left( \frac{1}{m} \Pr\{a > \Delta_1, \mathcal{E}_{rd,1}^{2,m}\} + \frac{m-1}{m} \Pr\{a > \Delta_1, \mathcal{E}_{rd,1}^{1,m}\} \right) \doteq \frac{1}{m} \Pr\{\mathcal{E}_{sd}\} \Pr\{a > \Delta_1, \phi_1 < 0\}. \quad (61)$$

Apparently, $\mathcal{P}_{e,m}^{SOAF-B}$ is dominated at high SNR by the case of a relay being continuously selected as the active relay in the entire ARQs. Thus, $\mathcal{P}_{e,N}^{SOAF-B}$ at high SNR is limited to

$$\frac{1}{m^{N-1}} \Pr\{\mathcal{E}_{sd}\} \Pr\{a > \Delta, \phi_1 < 0\}. \quad (62)$$

In fact, there exist other possible performance limitations than (62). To show this, we assume that $\Delta_1$ satisfies the conditions (52) and (55) with $k = 1$ and $\Delta_2$ is constant w.r.t. $\rho$. By the proof of Theorem 2, this assumption on $\Delta_1$ can imply that $\Pr\{a > \Delta_1, \mathcal{E}_{rd,1}^{n,q}\} \doteq \frac{1}{\rho^n q^n}, \forall n \in \mathbb{N}$ and $q \in \mathbb{N}$. Therefore, the diversity order of $\mathcal{P}_{e,1}^{SOAF-B}$ is no longer limited by the case of $|\mathcal{Q}_1| = m$. We then consider a case that ARQ2 is performed by a relay that was newly added to $\mathcal{Q}$ after ARQ1 by receiving a 2-hop signal. Conditioned on the error event of ARQ1 and $|\mathcal{Q}_2| = m$, the PER for this case can be expressed as

$$\frac{m-q}{m} \times \Pr\left\{ c_2 > \Delta_2, \min_{j \in D} \theta_{2,j}^{[m]}(\phi_{2,j}) < 0 \left| a_1 > \Delta_1, \min_{j \in D} \theta_{1,j}^{[q]}(\phi_{1,j}) < 0 \right. \right\}. \quad (63)$$

As $\rho$ increases, the threshold $\Delta_1$ is set far greater than $\Delta_2$ such that the PER of ARQ2 is deeply affected by the link quality of the second hop of the relayed signal rather than
the link of its first hop. As a result, the correlation between the first ARQ error event and the second one becomes smaller when the SNR increases. We can thus approximate (63) at high SNR to

\[
\frac{m-q_1}{m} \times \Pr \left\{ c_2 > \Delta_2, \mathcal{E}_{rd,2}^{1,m} \mid a_1 > \Delta_1 \right\}
\]

whose diversity order is equal to 1 according to (60). Using the same argument of (61), we can find that the PER for the subsequent ARQs is dominated at high SNR by the case of the same active relay in ARQ2 being continuously chosen till the end of the ARQs. With this result and (60), we come to the second limitation of \( \mathcal{P}_{e,N}^{SOAF-B} \) given by

\[
\Pr \{ \mathcal{E}_{sd} \} \sum_{q_1=1}^{m-1} \sum_{q_1}^{m-1} \frac{1}{m^{n-2}} \left( \frac{m-q_1}{m} \right) \times \Pr \left\{ c_2 > \Delta_2, \phi_2 < 0 \mid a_1 > \Delta_1, \right\} \times \Pr \{ a_1 \leq \Delta_1 \}^{m-q_1} \times \Pr \{ a_1 > \Delta_1, \mathcal{E}_{rd,1}^{1,q_1} \} \times \Pr \left\{ a_1 > \Delta_1 \right\} \times \mathcal{P}_{e,1}^{SOAF-B}. (65)
\]

We extend the result to other limitations. Based on the assumption that the thresholds \([\Delta_1, \ldots, \Delta_{k-1}]\) satisfy the conditions, (52) and (55), and \(\Delta_k\) is constant, similarly, the \(k\)-th limitation of \( \mathcal{P}_{e,N}^{SOAF-B} \) can be approximately upper bounded at high SNR by

\[
\frac{1}{m^{n-k}} \Pr \{ a_1 > \Delta_1, \ldots, c_k > \Delta_k, \phi_k < 0 \} \times \mathcal{P}_{e,k-1}^{SOAF-B}, \text{ for } k \in \mathcal{T}_{2}^{\min[m,N]}. (66)
\]

B. A heuristic thresholds searching algorithm for ARQ-SOAF-B

In contrast to the case of SAF ARQ, it is obviously much harder to find a set of thresholds to approach the optimal SNR gain of \( \mathcal{P}_{e,N}^{SOAF-B} \) since multiple unknown thresholds need to be exhaustively searched. Instead, here we consider a heuristic approach. On one hand, we define the thresholds to have a predetermined nonzero ratio \([v_1, \ldots, v_{\min[m,N]}]\) among themselves for any SNR \(\rho\). As a result, the thresholds \(\Xi := \Delta_e[v_1, \ldots, v_{\min[m,N]}]\) for the SOAF-B relaying can be determined by one unknown variable \(\Delta_e\). Thus, following the notion of (58), for each \(\rho\), we may design the thresholds by increasing \(\Delta_e\) such that all the possible limitations, (62) and (66), can be less than the corresponding \(\mathcal{P}_{e,N}^{SOAF-B}(\rho, \Delta_e)\).

On the other hand, the ratio \( \mathcal{P}_{e,N}^{SOAF-B}(\rho, \Delta_e)/\mathcal{P}_{e,N}^{SOAF-B}(\rho, \Delta_e) \) should be greater than or equal to \( \mathcal{P}_{out,N}^{SOAF-B}(\rho, \Delta_e)/\mathcal{P}_{out,N}^{SOAF-B}(\rho, \Delta_e) \) since the noise enhancement resulting from multiple hops would more deeply affect the PERs of ARQs for any practical coding schemes than their corresponding outage probabilities from the information-theoretical point of view.
In view of the simplicity of outage analysis, the objective of this algorithm turns out to find the smallest $\Delta_e$ that satisfies, $\forall k \in I_{\min[N]}$, 

$$
P_{out,k-1}(\rho, \Delta_e) \times \frac{1}{m} \Pr\{a_1 > v_1 \Delta_e, \ldots, c_k > v_k \Delta_e, \phi_k < 0\} \leq P_{out,N}^{SOAF-B}(\rho, \Delta_e). \quad (67)
$$

Though the system’s performance dependence on $v_i$ is not studied here owing to its high complexity in analysis, according to our experience from simulations, $v_i$ is suggested to set to increase with $i$, e.g., $v_i := i$, in order to obtain a good performance.

Before summarizing the steps of the thresholds searching algorithm, we need to do some simplification procedures for (67). First of all, due to the lack of the exact expression for the outage probability of SOAF-B ARQ, we replace $P_{out,N}^{SOAF-B}$ with the analytical and tight lower bound, $\tilde{P}_{out,n}^{SOAF-B}$, that has been mentioned and defined in Section III-B.3 and Appendix-B.5 respectively. Nevertheless, $\tilde{P}_{out,n}^{SOAF-B}(\rho, \Delta_e)$ in $\Delta_e$ is still difficult to characterize. To further simplify our analysis, we replace all the terms that decrease w.r.t. $\Delta_e$ in $\tilde{P}_{out,n}^{SOAF-B}$ with 1. More specifically, we replace the term $\Pr\{a > \Delta_1\}$ in (97) and $\Pr\{c > \Delta_k\}$ in (98) with 1, and then we denote the resultant one by $P_{out,n}^{SOAF-B}$ which becomes monotonically increasing with $\Delta_e$. This trick in fact will not cause $P_{out,n}^{SOAF-B}$ to be greatly different from $\tilde{P}_{out,n}^{SOAF-B}$ when the SNR $\rho$ increases, if Requirement 2 can be met. This is because under Requirement 2, we have $\Pr\{a > \Delta_1\} \equiv 1$ and $\Pr\{c > \Delta_k\} \equiv 1$ and thus obtain $P_{out,n}^{SOAF-B} \approx \tilde{P}_{out,n}^{SOAF-B}$.

For clarity, following the definition of $\tilde{P}_{out,n}^{SOAF-B}$ in Appendix-B.5, namely (95)~(98), we show the expression of $P_{out,n}^{SOAF-B}$ as follows:

$$
P_{out,n}^{SOAF-B} \triangleq \Pr\{w < \delta\} \sum_{n=0}^{n} \left[ (\Pr\{a_1 \leq v_1 \Delta_e\})^m \Pr\{w < \delta\} \right]^{n-\ell} \times \tilde{G}_2'(\Delta_e, \ell) \quad (68)
$$

in which $\tilde{G}_2'(\Delta_e, 0) \triangleq 1$, and for $\ell \in T_1^n$, $\tilde{G}_2'(\Delta_e, \ell)$ is defined as

$$
\tilde{G}_2'(\Delta_e, \ell) \triangleq \sum_{q_{4,1}=1}^{m-|Q_{4,1}|} F_{1,1}'(q_{4,1}) \times \sum_{q_{4,2}=0}^{m-|Q_{4,2}|} F_{2,2}'(q_{4,2}) \times \left( \sum_{q_{3,2}=0}^{m-|Q_{3,2}|} F_{3,2}'(q_{3,2}) + \sum_{q_{3,3}=0}^{m-|Q_{3,3}|} F_{3,3}'(q_{3,3}) \right) \times \cdots \times \left( \sum_{k=2}^{m-|Q_{4,1}|} \sum_{q_{4,1}=0}^{m-|Q_{4,1}|} F_{\ell,k}'(q_{4,1}) \right) 
$$

(69)

where $F_{i,1}'(q_{4,1})$ and $F_{i,k}'(q_{4,1})$ for $i \in T_2^n$ and $k \in T_2^{\min[i,m]}$ are given by

$$
F_{i,1}'(q_{4,1}) = C_{q_{4,1}}^{m} \Pr\{a \leq v_i \Delta_e\}^{m-q_{4,1}} \times \Pr\{b < \delta\}^{q_{4,1}} \quad (70)
$$

$$
F_{i,k}'(q_{4,1}) = \frac{Q_{i-1}^{1-k-1} q_{4,1}}{|Q_{i-1}|} \times C_{q_{4,1}}^{m-|Q_{i-1}|} \Pr\{c \leq v_k \Delta_e\}^{m-|Q_{i-1}|+q_{4,1}} \times \Pr\{b < \delta\}\{|Q_{i-1}|+q_{4,1}\} \quad (71)
$$
in which \(|Q_i| := \sum_{k=1}^{\min[i,m]} \sum_{j=k}^i q_{j,k}^i\) and \(|Q_{i,k}| := \sum_{j=k}^i q_{j,k}^i\). The details of the relationships between \(Q_i\), \(Q_{i,k}\) and \(q_{i,k}\) are provided in Appendix-B.5 (also in Fig. 15).

In addition to the monotonicity property in \(\Delta_e\), there are other properties of \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}}(\rho, \Delta_e)\) to be used later. For conciseness, we summarize them in the following facts:

- **Fact 1:** Under the condition (52), namely Requirement 2, we have \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}} d_{\text{out,n}} = \frac{1}{\rho^{v + m + r}}\). Fact 2 results from \(\text{Pr}\{a \leq v_1 \Delta_e\} = \text{Pr}\{c \leq v_k \Delta_e\} = 0\) in (68)~(71). With Fact 2, we know that given \(\rho\), when \(\Delta_e\) decreases, \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}}(\rho, \Delta_e)\) will be lower bounded by a positive number.

- **Fact 2:** When \(\Delta_e \leq 0\), we have \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}}(\rho, \Delta_e) = \text{Pr}\{w < \delta\} \text{Pr}\{b < \delta\}^{m \times n}\). In other words, given \(\rho\), \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}}(\rho, \Delta_e)\) for \(\Delta_e \leq 0\) is positive and constant.

- **Fact 3:** Given \(\rho\), \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}}(\rho, \Delta_e)\) is bounded between two nonzero and positive finite numbers, regardless of \(\Delta_e\).

Fact 1 holds true since under Requirement 2, we have \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}} d_{\text{out,n}} = \frac{1}{\rho^{v + m + r}}\). Fact 2 results from \(\text{Pr}\{a \leq v_1 \Delta_e\} = \text{Pr}\{c \leq v_k \Delta_e\} = 0\) in (68)~(71). With Fact 2, we know that given \(\rho\), when \(\Delta_e\) decreases, \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}}(\rho, \Delta_e)\) will be lower bounded by a positive number. On the other hand, it can be shown that \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}}(\rho, \Delta_e)\) must be upper bounded by a finite number, regardless of \(\rho\) and \(\Delta_e\). This is because \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}} \leq \sum_{\ell=0}^{\infty} G_2(\Delta, \ell)\) and each term of this summation is positive and bounded owing to the results of \(F_{i,1}(\varphi_{1,1}) \leq C_m^2\), and \(F_{i,k}(\varphi_{i,k}) \leq C_m^{\rho - |Q_i| - 1}\). Therefore, we obtain Fact 3.

Back to the discussion of (67), in addition to replacing \(\mathcal{P}_{\text{out},k-1}^{\text{SOAF-B}}\) and \(\mathcal{P}_{\text{out},N}^{\text{SOAF-B}}\) in (67) with the analytical forms of \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}}(\rho, \Delta_e)\) and \(\mathbb{P}_{\text{out,n}}^{\text{SOAF-B}}(\rho, \Delta_e)\) respectively, we further use the upper bound in (54) with the dominant term, \(d_m\), to replace the probability \(\text{Pr}\{a_1 > v_1 \Delta_e, \ldots, c_k > v_k \Delta_e, \phi_k < 0\}\) in (67). Finally, our objective turns into finding the smallest \(\Delta_e\), denoted by \(\Delta_e^*\), to satisfy the next conditions:

\[
\frac{\omega_{m-\delta}}{m^{N-\delta}} \times Q \left( \sqrt{\frac{\psi k m \rho}{\delta}} \right) \times \frac{1}{\rho} \times \left( \frac{1}{d_{m-\rho} \psi} \right) \times \left( \frac{v_1}{\beta_1} + \sum_{i=2}^{\infty} \frac{v_i}{\beta_i} \right) \times \mathbb{P}_{\text{out,n}}^{\text{SOAF-B}}(\rho, \Delta_e) \leq \mathbb{P}_{\text{out,n}}^{\text{SOAF-B}}(\rho, \Delta_e), \quad \forall k \in I_{1}^{\min[m,N]}, \tag{72}
\]

where \(d_{m-\rho} = \frac{(d_m - \rho)^2}{d_{m-\rho}}\) is defined in (55), \(\psi_k = [\Pi_{i=1}^{k} (1 + \frac{1}{\Delta_i v_i}) - 1]^{-1} = \frac{1}{v_1 \Delta e + \cdots + v_k \Delta e + O(\Delta_e)^2}\) is monotonically increasing with \(\Delta_e\), and in contrast, \(\frac{1}{d_{m-\rho} \psi} = \frac{1}{\psi k} \frac{\Delta e}{\Delta e} = \frac{4}{d_{m-\rho} \psi k} \frac{1}{\psi k} + \cdots + \frac{1}{\psi k} + O(\Delta_e)^2\) is decreasing. If \(\Delta_e \to 0^+\), it follows that \(\psi_k \to 0^+\) and \(\frac{1}{d_{m-\rho} \psi k} \to \infty\). On the other hand, if \(\Delta_e \to \infty\), then \(\psi_k \to \infty\) and \(\frac{1}{d_{m-\rho} \psi k}\) will approach the fixed number \(\frac{4}{d_{m-\rho} \psi k} (\frac{1}{\psi k} + \cdots + \frac{1}{\psi k})\).

Following those results, we next show that there must be a smallest nonzero \(\Delta_e\), denoted \(\Delta_e^{(k)}\), to satisfy the \(k\)-th inequality condition in (72). To this end, we begin with the case
of $\Delta_e \to 0^+$. Given $\rho$, when $\Delta_e$ is close to $0^+$, the inequality condition in (72) will not be met because its left-hand-side (LHS) will go to the infinity due to $\frac{4\Delta_e}{d_{m,e_0}^2 \psi_k} \to \infty$ and its right-hand-side (RHS) is bounded according to Fact 3. Further, when $\Delta_e$ increases from zero ($\Delta_e \to \infty$), the RHS will monotonically increase with $\Delta_e$. In contrast, the LHS is not monotonic with $\Delta_e$. The LHS has two terms that increase with $\Delta_e$; one is the added term, $\Delta_e$, and the other is $\frac{1}{C_8} SOAF-B_{\text{out},k-1} - 1$ which is bounded. Nevertheless, given that $Q(x) \leq e^{-\frac{x^2}{2}}$, by substituting this upper bound back into the LHS, it can be easily observed that the LHS will eventually decrease to zero when $\Delta_e \to \infty$. In other words, there must be a smallest nonzero $\Delta_e$, denoted by $\Delta_e^*(k)$, to make the $k$-th condition in (72) hold, namely to make its equality hold. The steps to search $\Delta_e^*$ are summarized in Algorithm 1 (Alg. 1).

**Algorithm 1** The procedures to find $\Delta_e^*$ for the thresholds setting of the ARQ-SOAF-B 0. Set nonzero ratios $[v_1, \ldots, v_{\min[m,N]}], a$ sufficiently small number $\varepsilon_0$, and $k := 1$.

1. Increase $\Delta_e$ from 0 until the $k$-th condition in (72) holds. Set $\Delta_e^*(k) := \Delta_e$.

2. $k := k + 1$. If $k > \min[m,N]$, $\Delta_e^* := \max_{1 \leq i \leq \min[m,N]} [\Delta_e^*(i)]$. Else go back to Step 1.

Step 1 of the algorithm can be solved accurately with the Bisection method. Besides, the thresholds derived from Algorithm 1 still meet the proposed conditions (52) and (55), and the proof is provided in the next proposition. We also note that with some mathematical manipulations, a threshold searching algorithm for SOAF-A ARQ can be obtained as well.

**Proposition 4:** The thresholds obtained from Algorithm 1 can satisfy (52) and (55).

**Proof:** We start from finding a lower bound, denoted by $\Delta_e^{(k)}_{L,L}$ of $\Delta_e^{(k)}_{e,L}$, and prove the thresholds $\Delta_e^{(k)}_{e,L}[v_1, \ldots, v_k]$ satisfy (55), which shows that the case of $\Delta_e^{(k)}_{e,*}$ also satisfies (55). To this end, we first find a special lower bound for the LHS of (72) by replacing $Q(x)$ and $P_{\text{out},k-1}(\rho, \Delta_e)$ with their lower bounds, $\frac{1}{\sqrt{2\pi}} \frac{1}{x+1} e^{-\frac{x^2}{2}}$ [34] and $P_{\text{out},k-1}(\rho, 0)$, respectively. Given that $\Delta_e \geq 0$, substituting those lower bounds back into the LHS of (72) results in

$$\frac{\omega_{m,n}}{mN-k} \times \frac{1}{\sqrt{2\pi}} \frac{1}{1+\sqrt{0.5 \times d_{m,e_0}^2 \times \psi_k}} e^{\frac{-d_{m,e_0}^2 \times \psi_k}{4}} \times \frac{1}{\rho} \times \frac{4\Delta_e}{d_{m,e_0}^2 \psi_k} \times \left( \frac{1}{\psi_1} + \sum_{i=2}^{k} \frac{1}{\psi_i} \right) \times P_{\text{out},k-1}(\rho, 0) \leq P_{\text{out},N}(\rho, \Delta_e).$$

(73)

We note that the LHS of (73) is strictly monotonically decreasing w.r.t. $\Delta_e$ due to the facts that $\psi_k = \frac{1}{\varepsilon_1 \Delta_e} + \cdots + \frac{1}{\varepsilon_k \Delta_e} + O\left(\frac{1}{\Delta_e}\right)$ and $\frac{4\Delta_e}{d_{m,e_0}^2 \psi_k} = \frac{4}{d_{m,e_0}^2} \left( \frac{1}{\varepsilon_1} + \cdots + \frac{1}{\varepsilon_k} + O\left(\frac{1}{\Delta_e}\right) \right)$. When
Fig. 8. Given $\rho$, for the $k$-th inequality condition in (72) and its variants, such as that in (73), this figure shows the relationships of the different LHS equations of the conditions with their same RHS, $P_{\text{out},N}^{\text{SOAF-B}}$.

$\Delta_e \to 0^+$, this LHS will approach infinity and thus be greater than the RHS since $P_{\text{out},N}^{\text{SOAF-B}}$ is bounded according to Fact 3. Then, as demonstrated in Fig 8, when $\Delta_e$ increases from zero, we can obtain a threshold $\Delta_e$, denoted by $\Delta_{e,L}^{(k)}$, to satisfy the equality of (73). The solution $\Delta_{e,L}^{(k)}$ must be unique since there is only one chance to let the equality of (73) hold. On the other hand, due to the fact that the LHS of (73) is less than the LHS of (72), when $\Delta_e$ increases, as shown in Fig. 8, the LHS of (73) will first touch the curve of $P_{\text{out},N}^{\text{SOAF-B}}$ in comparison with that of (72), and thus it follows that $\Delta_{e,L}^{(k)} \leq \Delta_{e,*}^{(k)}$.

Next, we show that the thresholds $\Delta_{e,L}^{(k)}[v_1, \ldots, v_k]$ satisfy the condition (55). To do this, we first have $\Delta_e := \lambda \ln \rho$ and substitute it back into (73). We notice that the thresholds, $\Delta_i = v_i \Delta_e = v_i \lambda \ln \rho$, still satisfy the condition (52). To simplify the analysis, according to Fact 1, we approximate $P_{\text{out},N}^{\text{SOAF-B}}$ by a form of $\frac{A_1}{\rho^m N + 1}$ where $A_1$ is the SNR gain that may vary with $\rho$ but satisfies $\lim_{\rho \to \infty} \log A_1 \log \rho = 0$. Similarly, by Fact 2, we have $P_{\text{out},k-1}(\rho, 0)$ to be equal to $\frac{A_2}{\rho^m(k-1) + 1}$, where the SNR gain $A_2$ satisfies $\lim_{\rho \to \infty} \log A_2 \log \rho = 0$. After substituting the two expressions into (73), we then arrive at

$$\frac{\psi_k}{\ln \rho} \geq \frac{4}{d_{\text{SNR},0}} \left[ m(N-k+1) - 1 + \frac{\ln(A_0 A_2/A_1)}{\ln \rho} \right]$$

(74)

where $A_0 \triangleq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{0.5 d_{\text{SNR},0} \times v_k m N-k} + 4 \Delta_e (v_i \beta_i + \sum_{i=2}^k v_i \beta_i)}$, and $\lim_{\rho \to \infty} \frac{\ln A_0}{\ln \rho} = 0$ because $A_0$ is proportional to $\frac{1}{\sqrt{\ln \rho}}$ due to $\Delta_e = \lambda \ln \rho$ and $\psi_k = \frac{1}{\sqrt{2\pi} \times \ln \rho} \frac{1}{\sqrt{\ln \rho} \times \ln \rho} + O\left(\frac{1}{(\ln \rho)^2}\right)$.
Applying Alg. 1 to Algorithm 1, it therefore follows that \( \Delta_{k}^{(e)} \) is not as simple as a monotonic function. Nevertheless, all the solutions \( \Delta_{e,L}^{(k)} \) will approach \( \lambda_{e} \ln \rho \), which means that the thresholds \( \Delta_{e,L}^{(k)}[v_{1}, \ldots, v_{k}] \) can actually satisfy the condition (55) and therefore so does the case of \( \Delta_{e,L}^{(k)} \) due to \( \Delta_{e,L}^{(k)} \geq \Delta_{e,L}^{(k)} \).

On the other hand, to prove that the thresholds, \( v_{i} \Delta_{e,L}^{(k)}(\bar{x}) \), satisfy the condition (52) as well, we need to find an upper bound of \( \Delta_{e,L}^{(k)}(\bar{x}) \). To this end, given \( Q(x) \leq e^{-\frac{x^{2}}{2}} \), we substitute this upper bound into the LHS of (72), which immediately yields a new inequality condition similar to (73). We denote all the possible solutions to the equality of (73), as \( \rho \rightarrow \infty \), \( \Delta_{e,L}^{(k)} \) will approach \( \lambda_{e} \ln \rho \), which means that the thresholds \( \Delta_{e,L}^{(k)}[v_{1}, \ldots, v_{k}] \) can actually satisfy the condition (55) and therefore so does the case of \( \Delta_{e,L}^{(k)} \) due to \( \Delta_{e,L}^{(k)} \geq \Delta_{e,L}^{(k)} \).

After procedures similar to (73) \( \sim \) (74), it can be shown that \( \lambda_{e} \ln \rho \) is one of the solutions \( \Delta_{e,U}^{(k)}(\bar{x}) \) as \( \rho \rightarrow \infty \). We thus conclude that the thresholds \( \Delta_{e,L}^{(k)}[v_{1}, \ldots, v_{k}] \) can also satisfy the inequality \( \Delta_{e,L}^{(k)} \leq \Delta_{e,U}^{(k)} \) and the results of \( \lim_{\rho \rightarrow \infty} \frac{\ln(\nu_{i} \lambda_{e} \ln \rho)}{\ln \rho} = 0, \forall i \).
Example: Fig. 9 shows the results of using Algorithm 1 for the ARQ-SOAF-B scheme that employs different trellis codebooks of $R = 1$ with two kinds of generator polynomials (GPs). Different GPs can result in different minimum codeword distances, e.g., $d_m^2 = 10$ for the GP $[5_8, 7_8]$ and $d_m^2 = 20$ for $[133_8, 171_8]$. The losses of the potential diversities can be fully recovered. In contrast to the ARQ protocols with OAF, the new threshold setting method allows the SOAF ARQ protocols to provide for practical coding systems a spatial and temporal diversity comparable to what is predicted with outage probability.

Remark: With the proof of Proposition 4, we can find that when SNR goes to infinity, by Algorithm 1, the thresholds will be assigned to scaled values of $\ln \rho$, like (56). Nevertheless, within the low or mid SNR regime, this algorithm is still superior to the simple log-scale setting method of (56) because it explicitly incorporates more system parameters into the thresholds setting method according to (72), e.g., the channel variances, $\beta_i, i \in I_3$, the relay number $m$, and the codeword number $\omega_{dm}$. The performance comparison between the two methods is shown in Fig. 11(a) of the next section.

VI. Simulations Studies

We verify our theoretical analysis with extensive and more realistic simulations in this section. In particular, we consider a more practical relay deployment scenario and compare the simulation results with our analysis. We also employ trellis coded modulation (TCM) in transmission to verify our diversity analysis in PER and study the throughput enhancement via relay-assisted ARQ schemes.
A. Simulation settings

We consider an ARQ scheme that employs a total of 3 relays, i.e., $m = 3$, and performs delay-aware data transmissions with the maximum number of ARQ, $N$, equal to 3. The three relays are assumed randomly located in a small circle centered at $(s_1, s_2)$ in the coordinate of Fig. 10. For convenience, the S-D distance is normalized to one. Given the distance between any two nodes, the corresponding channel variance is defined as the inverse of the distance raised to the power of the path loss exponent which in the simulation is set equal to 3. The channel variances for any transmit-and-receive pair in Fig. 10 can thus be obtained according to their geometric relations.

Recall from our system setting in Section III that we define those the parameters, $\beta_1, \beta_2$ and $\beta_3$, to qualify the channel variances of different links in order to simplify our analysis. Under the above simulation setting that relays are actually considered distributed in a circle rather than located altogether at a point, the parameter $\beta_1$ can be related here to the longest distance from the source to the circle, which is equal to $\sqrt{s_1^2 + s_2^2 + s_0}$. As a result, by definition, we have $\beta_1 = (\sqrt{s_1^2 + s_2^2 + s_0})^{-3}$. Similarly, $\beta_2$ is set as $(\sqrt{(1-s_1)^2 + s_2^2 + s_0})^{-3}$ and $\beta_3$ is set as $(2s_0)^{-3}$ to model the worst case that could ever happen in our system. Moreover, in order to perform Algorithm 1, we also need to define the other parameters, $\varepsilon_0 := 10^{-5}$ and $[v_1, v_2, v_3] := [1, 2, 3]$. We note that the thresholds obtained here using Algorithm 1 can be applied to any three relays arbitrarily located in the circle.

Table II shows the parameters of the different TCM codes for rate 1 to 5. All the codes satisfy the Ungerboeck design rules [35]. The rate-1 code has been used in Figs. 6 and 9 already, which is constructed based on the convolutional code of GP $[5_8, 7_8]$. As for other code rates, the encoder structures can be found in Ungerboeck’s paper [35].

In addition to validating the diversity order analysis, we also simulate the throughputs of different ARQ schemes in the sequel. For simplicity, the throughputs are estimated based on the following three assumptions: 1) the ACK/NAK signals and other signalings to implement the relay selection and AF policies are assumed error free and zero propagation delay, 2) the source node always has data to be sent, and 3) it also has capability to adapt the transmission rate $R$ with the average SNR $\rho$.

In our scenario, a packet is dropped only if failing to be sent to the destination after
| Name of States | Type of Mod. (rate) | $R$ | $d_m^2$ | $\omega d_m / L \approx$ |
|----------------|---------------------|-----|--------|------------------|
| rate-1         | 4 QPSK              | 1   | 10.0   | 1                |
| rate-2         | 4 8PSK              | 2   | 4.0    | 1                |
| rate-3         | 8 16QAM             | 3   | 2.0    | 3.656            |
| rate-4         | 8 32QAM             | 4   | 1.0    | 3.656            |
| rate-5         | 8 64QAM             | 5   | 0.4762 | 3.656            |

**TABLE II**

Five types of trellis codes for different rates $R$, and their $d_m$ and $\omega d_m$. Note that $d_m^2$ are estimated under the condition that the average symbol energy is normalized to 1.

ARQN. Given a target $P_t$, we define our rate adaptation strategy subject to a packet-loss-rate constraint of $P^A_{e,N} \leq P_t$. In addition, different deployments of the relays in the circle may lead to different throughputs. To take their randomness into account, we define our throughput metric as follows

$$T_{P_t} \triangleq \mathbb{E}_{\text{Relay locations}} \left[ \max_{R \in I_3, \ P^A_{e,N} \leq P_t} \frac{R \times [1 - P^A_{e,N}]}{NP^A_{e,N-1} + \sum_{\ell=1}^{N-1} \ell [P^A_{e,\ell-1} - P^A_{e,\ell}]} \right],$$

(75)

where the term in the expectation is the long-time averaged throughput derived using the renewal-reward theorem (see [3, 8] for details).

Finally, for the purpose of performance comparison, we introduce a DF counterpart of the SOAF-B relaying, denoted by SODF-B, to serve as a benchmark. The difference between SODF and SOAF is that DF schemes do decoding at relays, and only relays that succeed in decoding are brought into $Q$.

**B. Cooperative ARQs via general relays ($s_1 = 0.5, s_2 = 0$)**

We start by considering a special case of $s_1 = 0.5$ and $s_2 = 0$ to examine the effect of the simplified model on $\beta_1$, $\beta_2$, and $\beta_3$. As stated previously, relays are randomly placed in a circle of radius $s_0$. Fig. 11(a) shows the results of $P^A_{e,3}$ with different relay locations. For the employed relays, their locations have been tried 20 times, each time of which corresponds to a PER point in this figure. In contrast, the dash line stands for the PER
that is evaluated with the simplified model on the channel variances of $\beta_1, \beta_2$, and $\beta_3$. As can be seen, the dash lines are indeed the lower bounds on the performance of the true systems, and a smaller radius $s_0$ leads to a tight bound and smaller performance variations.

Fig. 11(a) also compares the PER of the ARQ scheme using Algorithm 1 with the case of using a simple log-scale setting method inspired by (56). For the latter method, we turn the problem of finding proper $\lambda_i$’s in (56) into searching the smallest value of $\lambda_e \in \mathbb{R}^+$ such that the thresholds, $\lambda_e [v_1, v_2, v_3] \ln \rho$, satisfy the conditions (52) and (55). As discussed in the end of section V, the proposed thresholds setting method performs a much better SNR gain than the simple one at low and mid SNR, and it also has the PER closer to that of SODF-B ARQ. Fig. 11(b) shows that for each rate type of Table II, the PER after ARQ3 can achieve the potential diversity order as expected.

In the remainder of this section, we focus on the case of the circle radius $s_0 = 0.05$ to examine the throughput advantage of relaying. For the S-D distance equal to $10^3$ meters, $s_0 = 0.05$ means that relays are in a circle of the diameter 100 meters. In Fig. 12(a), we show the relationship between $T_{P_t}$ and the packet-loss-rate constraint, $P_t$. Apparently, when a lower packet loss rate is required, the SOAF-B ARQ scheme shows more advantage than the ARQ-OAF in throughput. In Fig. 12(b), we compare the throughput $T_{P_t}$ of different ARQ schemes versus $\rho$ when $P_t = 10^{-3}$. We notice that all the relay-assisted
schemes outperform the no-relay one, even though the ARQ-OAF in fact suffers from a severe diversity loss. In addition, the throughput of the SOAF-B ARQ scheme is 40% ~ 20% higher than the ARQ-OAF in the low to mid SNR regime, which shows the effectiveness of the proposed scheme in supporting users close to the cell coverage boundary.

C. The robustness of SOAF protocols in channel variations

In Fig. 13, we examine the influence of relay group placement on the throughput. In
Fig. 13(a), we fix $s_0 = 0.05$, $s_2 = 0$ and $P_t = 10^{-3}$ while varying $s_1$ from 0 to 1. When $s_1$ is close to 1, the poor S-R channel condition causes the $T_P$ of the ARQ-OAF almost as worse as that of the no-relay one. In contrast, the two proposed ARQ schemes still perform well. The throughput of the ARQ-SOAF-B is very close to that of the ARQ-SODF-B.

On the other hand, we fix $s_1 = 0.5$ and vary $s_2$ from 0 to 1 to see the influence from another geometric viewpoint. As observed in Fig. 13(b), the throughputs of the relay-assisted schemes decrease as $s_2$ increases, and will be worse than that of the no-relay one when $s_2$ is sufficiently large. This is because when $s_2 > 0.87$, the S-R and R-D average link qualities have been almost worse than the S-D one. Nevertheless, the two SOAF ARQ schemes still outperform the ARQ-OAF. Generally speaking, the proposed protocols are more robust than the typical OAF ARQ scheme in throughout.

D. Extend to the case of Hybrid-ARQ with SNR-weighted MRC method

In the end of this section, we demonstrate the applicability of the thresholding mechanism to hybrid-ARQ (HARQ). Making use of the SNR-weighted maximal ratio combining (MRC) method, all previously discussed ARQ schemes can be readily extended to their HARQ cases. We take the SOAF-B HARQ as an example: the relays still use the SOAF-B relaying method, but the destination buffers previously corrupted retransmissions and combines them using the SNR-weighted MRC method. Following the same method, the ARQ-OAF and ARQ-SODF-B schemes can also be extended to the HARQ-OAF and HARQ-SODF-B ones respectively.
We show in Fig. 14 the PERs of the different HARQ and ARQ schemes where all of
the schemes are evaluated based on the same and fixed deployment of relays. Due to our
link quality control mechanism, although Algorithm 1 is not specifically designed for the
HARQ case, the HARQ-SOAF-B scheme offers the same diversity as the HARQ-SODF-B
one and has a SNR gain over the ARQ-SOAF-B. In contrast, HARQ-OAF still suffers
from the diversity loss problem due to the lack of mechanism in link quality control.

VII. Conclusions

From the outage probability point of view, we developed two types of the SOAF ARQ
protocols in our attempt to explore the spatial and temporal diversities with AF relaying.
The outage analysis shows that the two proposed protocols can offer much higher diversities
than the ARQ scheme that uses the typical OAF relaying method if link qualities are
properly controlled for each hop along the relayings. The quality control methods derived
from the outage analysis, however, cannot be directly applied to ARQ schemes that use
general channel codebooks. From the MLSD point of view, we developed two requirements
for link quality control in the proposed ARQ schemes in order to make PERs enjoy the
same diversity orders of the corresponding outage probabilities. Following those results,
we further provided a heuristic thresholds searching algorithm in order to exploit both the
diversity and the SNR gains in PER.

Via extensive simulations with TCM codes, we verified that the proposed thresholding
mechanism indeed allows the ARQ-SOAF protocols to exploit more spatial and temporal
diversities in PER than the ARQ-OAF. The simulation studies also demonstrate that the
throughputs of the proposed ARQ schemes are more robust to the variations of channel
qualities, compared with the ARQ-OAF, and the proposed link quality control method
can be readily extended to HARQ as well.
APPENDICES

A. The outage analysis for one-relay schemes

A.1 Proof of Lemma 1

From (6), if $\Delta \geq \delta$, we have

$$F(\Delta, \ell) = \mathbb{E}_{a > \Delta} \left[ \Pr \{ b < \frac{a \delta - \delta}{a - \delta} \mid a \} \right] = \int_{\Delta}^{\infty} (1 - e^{-\frac{a \delta}{\rho \beta_2} (x - \delta)})^\ell e^{-\frac{\delta}{\rho \beta_1} x} \, dx$$

$$= e^{-\frac{\delta}{\rho \beta_1}} + \sum_{i=1}^{\ell} C_i^\ell (-1)^i e^{-\frac{\delta}{\rho \beta_1}} e^{-\frac{i \delta}{\rho \beta_2}} \times \Gamma(1, \frac{\Delta - \delta}{\rho \beta_1}, i(\delta^2 + \delta)).$$  

On the other hand, for the case of $\Delta < \delta$, we have

$$F(\Delta, \ell) = \Pr \{ \delta \geq a > \Delta, \cap_{i=1}^{\ell} \left( \frac{a b_1}{a + b_1 + 1} < \delta \right) \} + \Pr \{ a > \delta, \cap_{i=1}^{\ell} \left( \frac{a b_1}{a + b_1 + 1} < \delta \right) \}

= \Pr \{ \delta \geq a > \Delta \} + F(\delta, \ell)

= e^{-\frac{\Delta}{\rho \beta_1}} + \sum_{i=1}^{\ell} C_i^\ell (-1)^i e^{-\frac{\delta}{\rho \beta_1}} e^{-\frac{i \delta}{\rho \beta_2}} \times \Gamma(1, 0; i(\delta^2 + \delta)).$$

The equality (a) is based on the fact that $\Pr \{ \delta \geq a > \Delta \} = e^{-\frac{\Delta}{\rho \beta_1}} - e^{-\frac{\delta}{\rho \beta_1}}$ and the result of (76) with $\Delta := \delta$.

A.2 Proof of Lemma 2

Suppose $\Delta := \lambda \delta$, $\lambda > 1$. We use the value of $(\beta_1 \ln \rho)$ to partition the integral region of the random variable “a” in $F(\Delta, \ell)$. When $\rho$ increases such that $(\beta_1 \ln \rho) > \lambda \delta$, we have

$$F(\Delta, \ell) = \mathbb{E}_{\beta_1 \ln \rho \geq \lambda \delta} \left[ \left( 1 - e^{-\frac{\lambda \delta}{\rho \beta_2} (1 + \frac{\lambda \delta}{\rho \beta_2} + \frac{\lambda \delta}{\rho \beta_2})} \right)^\ell \right] + \mathbb{E}_{\beta_1 \ln \rho \geq \lambda \delta} \left[ \left( 1 - e^{-\frac{\lambda \delta}{\rho \beta_2} (1 + \frac{\lambda \delta}{\rho \beta_2} + \frac{\lambda \delta}{\rho \beta_2})} \right)^\ell \right].$$

(78)

The exponent $(1 + \frac{1}{\delta})/(\frac{1}{\delta} - \frac{1}{a})$ is a decreasing function with $a > \delta$, and it will approach $\delta$ as $a \to \infty$. The first expectation term, named $T_1$, in (78) can thus be bounded by

$$\left( 1 - e^{-\frac{\delta}{\rho \beta_2}} \right)^\ell \int_{\lambda \delta}^{\beta_1 \ln \rho} \frac{1}{\rho \beta_1} e^{-\frac{a}{\rho \beta_1}} \, da \leq T_1 \leq \left( 1 - e^{-\frac{\delta}{\rho \beta_2} (1 + \frac{\lambda \delta}{\rho \beta_2} + \frac{\lambda \delta}{\rho \beta_2})} \right)^\ell \int_{\lambda \delta}^{\beta_1 \ln \rho} \frac{1}{\rho \beta_1} e^{-\frac{a}{\rho \beta_1}} \, da$$

where the two integrations are the same and equal to $e^{-\frac{\lambda \delta}{\rho \beta_1}} (1 - e^{-\frac{1}{\rho} (\ln \rho - \lambda \delta/\beta_1)})$ with the same order of $\rho^{-1}$ by definition. In other words, $T_1$ is of the order of $\rho^{-(\ell+1)}$.

As for the second expectation, denoted by $T_2$, in (78), we similarly have

$$\left( 1 - e^{-\frac{\delta}{\rho \beta_2}} \right)^\ell e^{-\frac{\ln \rho}{\rho}} \leq T_2 \leq \left( 1 - e^{-\frac{1}{\rho \beta_2} (1 + \frac{\lambda \delta}{\rho \beta_2} + \frac{\lambda \delta}{\rho \beta_2})} \right)^\ell e^{-\frac{\ln \rho}{\rho}}.$$
We apparently have $T_2 \doteq \left( \frac{\lambda}{\rho \beta} \right)^\ell$ by the definition of (2). As a result, we obtain $F(\Delta, \ell) \doteq T_2$ owing to its smaller diversity order than that of $T_1$. In addition, it can be easily verified from (8) that $\tilde{F}(\Delta, \ell) \doteq \left( \frac{\lambda}{\rho \beta} \right)^\ell$. We thus arrive at the fact that $F(\Delta, \ell) \doteq \tilde{F}(\Delta, \ell) \doteq \left( \frac{\lambda}{\rho \beta} \right)^\ell$.

As for the case of $\Delta := \lambda \delta$, $\lambda < 1$, we first define a special threshold $\Delta' := \lambda' \delta > \delta$ where $\lambda' := (1 + \frac{1}{\ln \rho})$ is a function of $\rho$ such that $\Delta'$ can be arbitrarily close to $\delta$ as $\rho \to \infty$. Using the fact that $\frac{a b}{a_1 + b_1 + 1} \leq \min[a,b] \leq a$, according to (6), we have

$$\Pr\{\Delta < a < \delta\} \leq F(\Delta, \ell) \leq \Pr\{\Delta < a \leq \Delta'\} + F(\Delta', \ell).$$

(81)

As a matter of fact, $F(\Delta', \ell) \doteq \rho^{-\ell}$, and its diversity analysis can be done by replacing the $\lambda$ in (78) and (79) with $\lambda' = (1 + 1/\ln \rho)$. Specifically, we express $F(\Delta', \ell)$ in the form of (78), and an upper bound for the first expectation term can be obtained based on (79) as

$$\left(1 - e^{-\frac{\lambda'}{\rho \beta}((1+\frac{1}{\beta})\ln(\rho) + 1)}\right)^\ell \times \int_{\lambda' \delta}^{(\delta - \frac{\lambda'}{\rho \beta}((1+\frac{1}{\beta})\ln(\rho) + 1))} e^{-\frac{\lambda}{\rho \beta}a}\frac{1}{\rho \beta} da$$

(82)

whose diversity order can be shown equal to $\ell + 1$. As for the second expectation term of $F(\Delta', \ell)$, the same result as (80) is obtained, thus leading to $F(\Delta', \ell) \doteq \rho^{-\ell}$. Further, due to the fact that $\Pr\{\Delta < a < \delta\} \doteq \Pr\{\Delta < a < \Delta'\} \doteq \frac{\delta - \Delta}{\rho \beta}$, by (81), we can find that if $\lambda < 1$, $F(\Delta, \ell)$ is of the diversity order of $\rho^{-1}$ and for $\ell \geq 2$, $F(\Delta, \ell) \doteq \Pr\{\Delta < a < \delta\} \doteq \frac{\delta - \Delta}{\rho \beta}$. 

B. The outage analysis for multiple-relay schemes

B.1 Proof of Proposition 2

For the clarity of representation, we extend the notations $a$ and $b$ to $a_{\ell,j}$ and $b_{\ell,j}$ for the relay $j$ at ARQ$\ell$. According to (6) and (10), we have

$$\mathcal{P}_{out, m}^{OAP} = \Pr\{w < \delta\} \times \Pr\left\{\max\left( \frac{a_{0,0} b_{1,1}}{a_{0,0} + b_{1,1} + 1}, \ldots, \frac{a_{0,m} b_{1,m}}{a_{0,m} + b_{1,m} + 1} \right) < \delta, \ldots, \right.$$  

$$\left. \max\left( \frac{a_{0,0} b_{n,1}}{a_{0,0} + b_{n,1} + 1}, \ldots, \frac{a_{0,m} b_{n,m}}{a_{0,m} + b_{n,m} + 1} \right) < \delta \right\}$$

$$= \Pr\{w < \delta\} \times \Pr\left\{\frac{a_{0,0} b_{1,1}}{a_{0,0} + b_{1,1} + 1} < \delta, \ldots, \frac{a_{0,1} b_{n,1}}{a_{0,1} + b_{n,1} + 1} < \delta \right\} \times \ldots$$

$$\times \Pr\left\{\frac{a_{0,m} b_{1,m}}{a_{0,m} + b_{1,m} + 1} < \delta, \ldots, \frac{a_{0,m} b_{n,m}}{a_{0,m} + b_{n,m} + 1} < \delta \right\}$$

(83)

$$= \Pr\{w < \delta\} \times (F(0, n))^m.$$
B.2 Proofs of Proposition 3

Clearly, \( G_1(\Delta, \ell) \) in (11) is used to characterize the outage probabilities of retransmissions by relaying. We assume without loss of generality that the relays of \( \{r_1, \ldots, r_q\} \) after ARQ0 form the set \( Q \) with \(|Q| = q\) where \( r_q \) denotes the \( q \)-th relay. Since there are \( q \) different possible active relays in each ARQ, the total number of possible permutations of the active relays in \( \ell \) rounds of ARQs is \( q^\ell \). Let \( p_i \) denote the event of choosing the \( i \)-th possible permutation of relays from \( Q \), for \( i \in T_1^\ell \). The outage probability of \( \ell \) consecutive ARQ events can thus be expressed as \( \sum_{i=1}^{q^\ell} \Pr\{p_i\} \Pr\{O_i\} \) where \( O_i \) denotes the outage events of the \( \ell \) ARQs conditioned on the \( i \)-th permutation of relays. Under the assumption of \( h_{j,rd} \) having the same variance, \( \forall j \), we have \( \Pr\{p_i\} = (1/q)^\ell, \forall i \in T_1^\ell \).

Furthermore, we use \( X_{r_q}^n \) to represent \( n \) ARQ rounds through the relay \( r_q \). Given that \( \Pr\{O_i\} = \Pr\{O_j\} \) if permutation \( i \) and \( j \) have the same combination of active relays, we apply the Binomial theorem to obtain the numbers of distinct combinations as follows

\[
(X_{r_1} + \cdots + X_{r_q})^\ell = \sum_{\zeta_{q-1}=0}^{\zeta_q} \binom{\ell}{\zeta_{q-1}} X_{r_q}^{\zeta_q-\zeta_{q-1}}(X_{r_1} + \cdots + X_{r_{q-1}})^{\zeta_{q-1}} \tag{84}
\]

\[
= \sum_{\zeta_{q-1}=0}^{\zeta_q} \binom{\ell}{\zeta_{q-1}} X_{r_q}^{\zeta_q-\zeta_{q-1}} \sum_{\zeta_{q-2}=0}^{\zeta_{q-1}} \binom{\zeta_{q-1}-1}{\zeta_{q-2}} X_{r_{q-1}}^{\zeta_{q-1}-\zeta_{q-2}} \cdots \sum_{\zeta_1=0}^{\zeta_2} \binom{\zeta_2-1}{\zeta_1} X_{r_2}^{\zeta_2-\zeta_1} \times X_{r_1}^{\zeta_1} \tag{85}
\]

with \( \zeta_q \Deltaq \ell \). The term \( X_{r_q}^{\zeta_q-\zeta_{q-1}} \) in (84) denotes \( \zeta_q - \zeta_{q-1} \) ARQ rounds through \( r_q \) during the \( \ell \) ARQ rounds. In addition, the product \( X_{r_q}^{\zeta_q-\zeta_{q-1}} \times \cdots \times X_{r_1}^{\zeta_1} \) in (85) shows one type of the combinations, and its multiplier is the total number of the permutations that belongs to this combination. The outage probability of \( X_{r_q}^{\zeta_q-\zeta_{q-1}} \) in (84) can be expressed as

\[
\Pr \left\{ a > \Delta, \bigcap_{i=1}^{\zeta_q-\zeta_{q-1}} \left( \frac{ab_i^{[q]}}{a+b_i^{[q]+1}} < \delta \right) \right\} \overset{(a)}{=} \Pr \left\{ a > \Delta, \bigcap_{i=1}^{\zeta_q-\zeta_{q-1}} \left( \max_{j \in I_1^q} \frac{ab_{i,j}}{a+b_{i,j}+1} < \delta \right) \right\} \tag{86}
\]

where \( b_{i,j} \) denotes in ARQi the channel gain from the \( j \)-th relay to the destination, namely \( \rho|h_{j,rd}|^2 \), and in contrast, \( b^{[q]} \) denotes the highest \( \rho|h_{j,rd}|^2 \) in \( Q \) with \(|Q| = q\). The equality \((a)\) holds since given \( a > 0 \), \( \frac{ab_i^{[q]}}{a+b_i^{[q]+1}} = \frac{1}{\frac{a}{b_i^{[q]}}+1} \) is monotonically increasing w.r.t. \( b > 0 \). Note that for ARQ rounds through different relays in (85), their outage events are independent.

Define \( F(i) \Deltaq (\Delta, \zeta, q) \) to be the sum outage probability of \( \zeta \) retransmissions by permutations of \( i \) relays in \( Q \) with \(|Q| = q\), i.e., the outage probability of the partial event of (84),
given by \((r_1 + \cdots + r_i)^\ell\). In other words, \(F(q)(\Delta, \ell, q) = \sum_{i=1}^{q} \Pr\{C_i\}\). Based on (84)\textasciitilde(86), for \(q \geq 2\), \(F(q)(\Delta, \ell, q)\) with an index \(\zeta_q \equiv \ell\) can be expressed as a recursive form of

\[
F(q)(\Delta, \zeta_q, q) = \sum_{\zeta_{q-1}=0}^{\zeta_q} \binom{\zeta_q}{\zeta_{q-1}} \times (e^{-\frac{\Delta}{\rho\tau_1}})^{\mu(\zeta_q, \zeta_{q-1})} \\
\times F(\Delta, q \times (\zeta_q - \zeta_{q-1})) \times F(q-1)(\Delta, \zeta_{q-1}, q)
\]

(87)

until \(F(2)(\Delta, \zeta_2, q) \triangleq \sum_{\zeta_1=0}^{\zeta_2} C_{\zeta_1} \times (e^{-\frac{\Delta}{\rho\tau_1}})^{\mu(\zeta_2, \zeta_1)} \times F(\Delta, q \times (\zeta_2 - \zeta_1)) \times F(\Delta, q \times \zeta_1)

where the term \(e^{-\frac{\Delta}{\rho\tau_1}}\) is the probability \(\Pr\{a > \Delta\}\) for a case of a relay in \(Q\) being not selected during the entire ARQs, and we thus define the function \(\mu(\zeta_i, \zeta_{i-1}) \triangleq \delta_f[\zeta_i - \zeta_{i-1}] + \delta_f[\zeta_{i-1} - \delta f[\zeta_i + \zeta_{i-1}]\) such that if \(\zeta_{i-1} = \zeta_i\) or \(\zeta_{i-1} = 0\), then \(\mu(\zeta_i, \zeta_{i-1}) = 1\); otherwise, \(\mu(\zeta_i, \zeta_{i-1}) = 0\). As for \(q = 1\), \(F(1)(\Delta, \ell, 1)\) is defined as \(F(\Delta, \ell)\).

Based on the result of \(\Pr\{p_i\} = (1/q)^\ell\) and (87), \(G_1(\Delta, \ell)\) can finally be expressed as

\[
G_1(\Delta, \ell) = \sum_{q=1}^{m} C_{m-q}^m \Pr\{a \leq \Delta\}^{m-q} \left(\frac{1}{q}\right)^\ell F(q)(\Delta, \ell, q).
\]

(88)

B.3 Proof of Corollary 1

Comparing (11) to (15), we only need to show that \(F(q)(\Delta, \ell, q)\) for \(\ell > 0\) and \(q > 0\) can be summarized as \(\tilde{F}(q)(\Delta, \ell, q) \triangleq q^\ell \Pr\{a > \Delta\}^q \Pr\{b < \delta\}^{q \times \ell}\) after the form of \(F(\Delta, n)\) in (13)\textasciitilde(14) are replaced with that of \(\tilde{F}(\Delta, n)\) in (8).

With \(\delta_f[\cdot]\) defined as the delta function, we first rewrite \(\tilde{F}(\Delta, n)\) as

\[
\tilde{F}(\Delta, n) \triangleq \Pr\{a > \Delta\}^{1 - \delta_f[n]} \Pr\{b < \delta\}^n, \text{ for } n \geq 0.
\]

(89)

Then, substituting (89) into (13) or (87), we can obtain \(\tilde{F}(q)(\Delta, \zeta_q, q)\) given by

\[
\tilde{F}(q)(\Delta, \zeta_q, q) = \sum_{\zeta_{q-1}=0}^{\zeta_q} C_{\zeta_{q-1}}^\zeta \Pr\{a > \Delta\}^{\mu(\zeta_q, \zeta_{q-1})+1 - \delta_f[q \times (\zeta_q - \zeta_{q-1})]} \\
\times \Pr\{b < \delta\}^{q \times (\zeta_q - \zeta_{q-1})} \times \tilde{F}(q-1)(\Delta, \zeta_{q-1}, q).
\]

(90)

By expanding the recursive form of (90) with \(\zeta_q \equiv \ell > 0\) and \(q > 2\), we have

\[
\tilde{F}(q)(\Delta, \ell, q) = \sum_{\zeta_{q-1}=0}^{\ell} \sum_{\zeta_{q-2}=0}^{\zeta_{q-1}} \cdots \sum_{\zeta_1=0}^{\zeta_2} C_{\zeta_{q-1}}^\ell C_{\zeta_{q-2}}^{\zeta_{q-1}} \cdots C_{\zeta_2}^{\zeta_{q-1}} \Pr\{a > \Delta\}^{t_q} \Pr\{b < \delta\}^{q \times \ell}
\]

where \(t_q \triangleq \mu(\ell, \zeta_{q-1})+1 - \delta_f[q \times (\ell - \zeta_{q-1})] + \sum_{j=q-1}^{2} \{\mu(\zeta_j, \zeta_{j-1})+1 - \delta_f[q \times (\zeta_j - \zeta_{j-1})]\} + 1 - \delta_f[q \times \zeta_1]\), and \(\ell \geq \zeta_{q-1} \geq \cdots \geq \zeta_1 \geq 0\).
By the definition of \( \mu(\zeta_i, \zeta_i-1) \), the exponent \( t_q \) can be further simplified as

\[
\begin{align*}
t_q & \overset{(a)}{=} \delta_f[\zeta_{q-1}] + 1 - \delta_f[\zeta_q + \zeta_{q-1}] + \sum_{j=q-1,\ldots,2} \{ \delta_f[\zeta_{j-1}] + 1 - \delta_f[\zeta_j + \zeta_{j-1}] \} + 1 - \delta_f[\zeta_1] \\
& \overset{(b)}{=} -\delta_f[\zeta_q + \zeta_{q-1}] + \sum_{j=q,\ldots,3} \{ \delta_f[\zeta_{j-1}] + 1 - \delta_f[\zeta_{j-1} + \zeta_{j-2}] \} + 1 + 1 \\
& \overset{(c)}{=} 0 + (q - 2) + 2
\end{align*}
\]

where the equality \((a)\) is based on the definition of \( \mu(\zeta_i, \zeta_i-1) \) and the fact that \( \delta_f[q(\zeta_j - \zeta_{j-1})] = \delta_f[\zeta_j - \zeta_{j-1}] \) for \( q > 0 \), and \((b)\) is just to reduce the expression and rearrange its summation index, and finally, \((c)\) results from \( \delta_f[\zeta_q + \zeta_{q-1}] = 0 \) since \( \zeta_q = \ell > 0 \), and the fact that \( \delta_f[\zeta_{j-1}] + 1 - \delta_f[\zeta_{j-1} + \zeta_{j-2}] \) is equal to 1 due to \( \zeta_{j-1} \geq \zeta_{j-2} \geq 0 \).

As a result, for \( \ell > 0 \) and \( q > 2 \), \( \widetilde{F}(q)(\Delta, \ell, q) \) can be reduced as

\[
\begin{align*}
\widetilde{F}(q)(\Delta, \ell, q) &= \Pr\{ a > \Delta \}^q \Pr\{ b < \delta \}^{q\ell} \sum_{\zeta_{q-1}=0}^{\zeta_q-1} \sum_{\zeta_{q-2}=0}^{\zeta_{q-1}} \times \cdots \times \sum_{\zeta_1=0}^{\zeta_2} C_{\zeta_{q-1}}^\ell C_{\zeta_{q-2}}^{q-1} \times \cdots \times C_{\zeta_1}^2 \\
&= \Pr\{ a > \Delta \}^q \Pr\{ b < \delta \}^{q\ell} \times q^\ell.
\end{align*}
\]

As for the cases of \( \widetilde{F}(2)(\Delta, \ell, 2) \) and \( \widetilde{F}(1)(\Delta, \ell, 1) \), by similar steps, we can derive the same result of \((91)\) for \( q \in \mathbb{Z}^2 \) from the original equation \((13)\).

Replacing \( F(q)(\Delta, \ell, q) \) of \((88)\), we obtain a lower bound \( \widetilde{G}_1(\Delta, \ell) \) of \( G_1(\Delta, \ell) \) in \((87)\), which is given by \( \widetilde{G}_1(\Delta, \ell) = \sum_{q=1}^{m} \sum_{l=0}^{m-q} (\Pr\{ a \leq \Delta \})^{m-q} \Pr\{ a > \Delta \}^q \Pr\{ b < \delta \}^{q\ell} \) for \( \ell > 0 \).

**B.4 The diversity analysis for** \( \mathcal{P}_{\text{out},n}^{\text{SOAP-A}} \) **with** \( \Delta := \lambda\delta < \delta \)

Here, we need to show that if \( \Delta := \lambda\delta \) with \( \lambda < 1 \), the diversity order of \( \mathcal{P}_{\text{out},n}^{\text{SOAP-A}} \) will be limited to 2. Recall from Lemma 2 that \( F(\Delta, \ell) \overset{d}{=} (1/\rho)^{1-\delta f[\ell]} \) for \( \ell \geq 0 \) if \( \Delta := \lambda\delta, \lambda < 1 \). Given that \( F(\Delta, \ell) \overset{d}{=} (1/\rho)^{1-\delta f[\ell]} \), the recursive formula of \( F(q)(\Delta, \zeta_q, q) \) with \( \zeta_q = \ell > 0 \) in \((13)\) or \((87)\) can be expanded from the diversity analysis point of view, which results in

\[
\begin{align*}
F(q)(\Delta, \zeta_q, q) & \overset{d}{=} \sum_{\zeta_{q-1}=0}^{\zeta_q} \frac{1}{\rho}^{1-\delta f[\zeta_q \times (\zeta_{q-1} - \zeta_q - 1)]} \times F(q-1)(\Delta, \zeta_{q-1}, q) \\
& \overset{d}{=} \sum_{\zeta_{q-1}=0}^{\zeta_q} \sum_{\zeta_{q-2}=0}^{\zeta_{q-1}} \times \cdots \times \sum_{\zeta_1=0}^{\zeta_2} \frac{1}{\rho}^{t_q'}, \text{ for } q > 2,
\end{align*}
\]

where \( t_q' \overset{d}{=} \sum_{j=q-1}^{2} (1 - \delta f[q \times (\zeta_j - \zeta_{j-1})]) + 1 - \delta f[q \times \zeta_1] \) and \( \zeta_q, \ldots, \zeta_1 \) are integers that satisfy \( \zeta_q = \ell > 0 \) and \( \zeta_q \geq \zeta_{q-1} \geq \cdots \geq \zeta_1 \geq 0 \).
Fig. 15. An illustration for the evolution of $Q$ in 5 ARQs in a system that uses with the help of 4 relays for the SOAF-B relaying scheme. In the figure, the arrow represents $Q_{\ell,k} \subseteq Q_{\ell+1,k'}$ and besides the line between $Q_{\ell,k}$ and $Q_{\ell+1,k+1}$ means when the active relay at ARQ $\ell$ is in $Q_{\ell,k}$, then the relays in $\overline{Q}$ can use $\Delta_k$ to judge if they can be brought into $Q_{\ell+1,k+1}$ or not.

We next show the smallest value of $t_q'$ is equal to 1. Since $q > 0$, $t_q'$ can be reduced as

$$t_q' = 1 - \delta_f[\zeta_q - \zeta_{q-1}] + \sum_{j=q-1}^{2} (1 - \delta_f[\zeta_j - \zeta_{j-1}]) + 1 - \delta_f[\zeta_1].$$

(93)

Apparently, $t_q'$ is a non-negative integer, and the equality for $t_q' = 0$ occurs only if $\zeta_1 = \zeta_2 = \cdots = \zeta_q = 0$. The condition of $\zeta_q = 0$ doesn’t satisfy that $\zeta_q = \ell > 0$. In other words, the smallest value of $t_q'$ will be equal to 1, which occurs if $\zeta_1 = \zeta_2 = \cdots = \zeta_{q-1} = 0$.

As a result, $F^{(q)}(\Delta, \ell, q) \doteq 1/\rho$ for $q > 2$. As for the cases of $F^{(1)}(\Delta, \ell, 1)$ and $F^{(2)}(\Delta, \ell, 2)$, by similar steps, we have the same result of $F^{(q)}(\Delta, \ell, q) \doteq 1/\rho$ for $q \in \mathbb{I}_1^2$. Thus, we know by (88) that $G_1(\Delta, \ell) \doteq 1/\rho$ for $\lambda < 1$, $\ell > 0$. Substituting the result back into (11) yields

$$P^{SOAF-A}_{out,n} \doteq \frac{1}{\rho} \times \sum_{\ell=0}^{n} \left(\frac{1}{\rho^{\ell+1}}\right)^{n-\ell} \times \left(\frac{1}{\rho}\right)^{1-\delta_f[\ell]} \doteq \frac{1}{\rho} \times \frac{1}{\rho}.$$

(94)

B.5 Derivation of $\tilde{P}^{SOAF-B}_{out,n}$ for SOAF-B ARQs

In contrast to the ARQ-SOAF-A, the outage probability of the SOAF-B ARQ scheme, $P^{SOAF-B}_{out,n}$, is more difficult to analyze since there are too many possible inheritance relationships from the source to the final forwarding relay. The outage events of retransmissions will become correlated once the relayed signals had ever commenced from a parent relay. To circumvent this difficulty, instead of directly tackling on $P^{SOAF-B}_{out,n}$, we next show two steps to estimate a lower bound for $P^{SOAF-B}_{out,n}$ by ignoring the effect of noise enhancement on the received signal qualities at relays. We denote the lower bound by $\tilde{P}^{SOAF-B}_{out,n}$.
To begin with, we use $Q_\ell$ with $Q_0 = \{\emptyset\}$ to stand for the qualified set at the beginning of ARQ$\ell$. Then, we divide $Q_\ell$ into $\min[\ell, m]$ subsets, denoted by $Q_{\ell,k}$ for $k = 1, \ldots, \min[\ell, m]$, where the set $Q_{\ell,k}$ only contains relays that receive $k$-hop signals in $Q_\ell$. Finally, we denote $q_{\ell,k} \triangleq |Q_{\ell,k} \setminus Q_{\ell-1,k}|$, namely, we use $q_{\ell,k}$ to represent the number of relays newly brought into $Q_{\ell,k}$ at the end of $\text{ARQ}\ell-1$.

For example, in Fig. 15, the subset $Q_{3,3}$ has contained all the relays in $Q_{3,3}$, and it can be further enlarged at the end of ARQ3, provided that the active relay for ARQ3 is chosen from $Q_{3,2}$ and there are some overhearing relays (in $Q^e$) to be added into $Q_4$ due to their corresponding channel qualities from the active relay exceeding $\Delta_3$. Following the above definitions, the number of the relays newly added into $Q_{4,3}$ is denoted by $q_{4,3}$.

The two steps to find $\tilde{P}^{\text{SOAF-B}}_{\text{out,}n}$ are stated below:

Step 1). Set the variables $q_{\ell,k} = 0$ for $\ell \in \mathcal{I}_1^n$ and $k \in \mathcal{I}_1^{\min[\ell,m]}$. Besides, we define $|Q_\ell| \triangleq \sum_{k=1}^{\min[\ell,m]} \sum_{i=k}^{\ell} q_{i,k}$ and $|Q_{\ell,k}| \triangleq \sum_{i=k}^{\ell} q_{i,k}$.

Step 2). Follow the formula:

$$\tilde{P}^{\text{SOAF-B}}_{\text{out,}n} \triangleq \Pr\{w < \delta\} \sum_{\ell=0}^{n} \left[\Pr\{a_1 \leq \Delta_1\}^m \Pr\{w < \delta\}\right]^{n-\ell} \times \tilde{G}_2(\Delta, \ell) \quad (95)$$

in which $\tilde{G}_2(\Delta, 0) \triangleq 1$ and is otherwise defined for $\ell \in \mathcal{I}_1^n$ as

$$\tilde{G}_2(\Delta, \ell) \triangleq \sum_{q_{1,1}=1}^{m} \sum_{q_{2,2}=0}^{m-|Q_1|} F_{1,1}(q_{1,1}) \times \sum_{q_{3,2}=0}^{m-|Q_1|} F_{2,2}(q_{3,2}) \times \left(\sum_{q_{3,3}=0}^{m-|Q_1|} F_{3,2}(q_{3,3}) \times \sum_{q_{3,3}=0}^{m-|Q_1|} F_{3,3}(q_{3,3}) \right) \times \cdots \times \left(\sum_{q_{k,k}=0}^{m-|Q_{\ell-1}|} F_{\ell,k}(q_{k,k}) \right) \quad (96)$$

where the variables $q_{\ell,k}$, initiated in Step 1, will be updated with the corresponding summation index. Besides, $F_{1,1}(q)$ and $F_{i,k}(q)$ for $i \in \mathcal{I}_2^n$ and $k \in \mathcal{I}_2^{\min[i,m]}$ are given by

$$F_{1,1}(q) = C_q^m \Pr\{a \leq \Delta_1\}^{m-q} \Pr\{a > \Delta_1\}^q \Pr\{b < \delta\}^q \quad (97)$$

$$F_{i,k}(q) = \frac{|Q_{i-1,k-1}|}{|Q_{i-1}|} \times C_q^{m-|Q_{i-1}|} \Pr\{c \leq \Delta_k\}^{m-|Q_{i-1}|-q} \times \Pr\{c > \Delta_k\}^q \Pr\{b < \delta\}^{(|Q_{i-1}|-q)} \quad (98)$$

The $\tilde{G}_2(\Delta, \ell)$ here can be easily programmed via a recursive form. If all the thresholds in $\Delta$ are set constant with $\rho$, it follows that $F_{1,1}(q) \overset{d}{=} F_{1,1}(q) \overset{d}{=} \frac{1}{\rho^m}$, leading to $\tilde{G}_2(\Delta, \ell) \overset{d}{=} \frac{1}{\rho^{m\times\ell}}$. We can thus obtain $\tilde{P}^{\text{SOAF-B}}_{\text{out,}n} \overset{d}{=} \frac{1}{\rho^{m\times\ell}}$. The simulation results of Fig. 5 also show that $\tilde{P}^{\text{SOAF-B}}_{\text{out,}n}$ seems to be a tight lower bound of $P^{\text{SOAF-B}}_{\text{out,}n}$ when Requirement 1 is satisfied.
C. The PER analysis

C.1 Proof of Lemma 3

We continue to analyze the diversity order of (34) and show that \( \Pr\{\min_{j \in \mathcal{D}} \theta^{[q]}_{1,j}(\tau) < 0\} \overset{d}{=} \rho^{-q} \) for \( q \geq 1 \) where \( \tau = 1/(\ln \rho)^2 \). We first use the union bound method to obtain

\[
\Pr\left\{ \min_{j \in \mathcal{D}} \theta^{[q]}_{1,j}(\tau) < 0 \right\} \leq \sum_{j=1}^{\mathcal{D}} \Pr \left\{ \theta^{[q]}_{1,j}(\tau) < 0 \right\} = \sum_{j=1}^{\mathcal{D}} \Pr \left\{ \sqrt{b^{[q]}} \times \tau < -2\Re\{\bar{d}_{0,j} H_{d,1}\} \right\}
\]

(99)

where \( b^{[q]} \) represents the highest \( \rho|h_{r,j,d}|^2 \) of the relays in \( \mathcal{Q} \) with \( |\mathcal{Q}| = q \). For convenience, let \( x \triangleq -2\Re\{\bar{d}_{0,j} H_{d,1}\} \sim \mathcal{N}(0, \sigma_d^2) \) with \( \sigma_d^2 = 2||\bar{d}_{0,j}||^2/\tau^2 = 2||\bar{d}_{0,j}||^2(\ln \rho)^4 \). The union bound in (99) can thus be expressed as

\[
\sum_{j=1}^{\mathcal{D}} \Pr \left\{ \sqrt{b^{[q]}} < x \right\} = \sum_{j=1}^{\mathcal{D}} \Pr \left\{ b^{[q]} < x^2, x \geq 0 \right\}
\]

\[
\overset{(a)}{\leq} \sum_{j=1}^{\mathcal{D}} \frac{1}{\rho \sigma_d^2} \int_0^{\infty} (1 - e^{-\frac{x^2}{\rho \sigma_d^2}})^q \times e^{-\frac{x^2}{\rho \sigma_d^2}} \, dx
\]

\[
\overset{(b)}{=} \frac{(\ln \rho)^{d u}}{(\rho \sigma_d^2)^q} \times \sum_{j=1}^{\mathcal{D}} \frac{[4||\bar{d}_{0,j}||^2q(2q-1)!]}{2^{q+1}}
\]

(100)

where (a) and (b) follow from the facts that \( (1 - e^{-y}) \leq y \) and \( \frac{1}{\sqrt{\pi}} \int_0^{\infty} x^{2q}e^{-x^2/r} \, dx = \frac{r^q(2q-1)!}{2^{q+1}} \) for \( r > 0 \) [36] respectively. Note that given \( q \), \( \lim_{\rho \to \infty} \frac{-\log(\ln \rho)^{d u}(\rho \sigma_d^2)^{-q}}{\log \rho} = q \). Thus, by definition, the upper bound in (100) is of the diversity order of \( \rho^{-q} \) that has attained the potential spatial diversity order conditioned on \( |\mathcal{Q}| = q \). This sufficiently proves (35) and also completes the diversity analysis for the second term of the upper bound of (32).

As for the lower bound of (32), we begin with (30). Conditioned on \( a \) and \( \bar{n}_r \) in \( \phi_{1,j} \), the effects of \( \bar{n}_{d,i} \) and \( b_i \) for \( i \in \mathcal{I}_r \) can be individually marginalized. For clarity, we define some notations in the following. \( \bar{u} \) consists of \( \mathcal{D} \) real Gaussian RVs \( [u_1, \ldots, u_\mathcal{D}]^T \), each of which is defined as \( u_j \triangleq -2\Re\{\bar{d}_{0,j} H_{d,1}\} \). Besides, given a space \( \mathcal{X} \) of \( \bar{u} \), we use \( J(\mathcal{X}) \) to stand for the function of integrating the joint probability density function (p.d.f.) of \( \bar{u} \), \( f_u(\bar{u}) \), over \( \mathcal{X} \). That is, \( J(\mathcal{X}) \triangleq \int \cdots \int_{\mathcal{X}} f_u(\bar{u}) \, du_1 \ldots du_\mathcal{D} \).

Then, conditioned on \( \bar{n}_r \), \( a \), and \( b_i^{[1]} \) for each ARQ round \( i \), the integral region of \( \bar{u} \) in (30) can be characterized by \( \mathcal{S}_1 \triangleq \mathcal{G}_{1,1} \cup \cdots \cup \mathcal{G}_{1,\mathcal{D}} \) where \( \mathcal{G}_{1,j} \triangleq \{ \bar{u} \in \mathbb{R}^\mathcal{D} | u_j > \sqrt{b_i^{[1]} \times \phi_{1,j}} \} \).

Specifically, \( \mathcal{G}_{1,j} \) is a \( \mathcal{D} \)-dimensional bounded real space with a lower bound in the \( j \)-th
dimension, and the subscript 1 of \(G_{1,j}\) is used to indicate that it corresponds to one-hop signals. Since \(b_i^{[1]}\) are statistically independent for different \(i\), thus for \(\ell > 0\), we have

\[
F_{\epsilon}(\Delta, \ell) = \Pr \{ a > \Delta, \mathcal{E}^\ell_{rd} | \bar{x}_0 \} = \mathbb{E}_{a>\Delta, \bar{n}_r} \left[ \left( \int_0^\infty J(S_1) \cdot f_{b^{[1]}(b)} db \right)^\ell \right] \tag{101}
\]

where \(f_{b^{[1]}(b)} \triangleq \frac{1}{\rho \beta_2} e^{-\frac{b}{\beta_2}}\) is the p.d.f. of the channel gain \(b^{[1]}\) and can be extended to the case of \(b^{[2]}\) whose p.d.f. is given by \(f_{b^{[2]}(b)} \triangleq \frac{q}{\rho \beta_2} (1 - e^{-\frac{b}{\beta_2}})^q e^{-\frac{b}{\beta_2}}\).

Next, we define \(\phi_1 < -\tau\) and \(\xi \triangleq (\log \rho)^6 > 0\) to lower bound (101) as

\[
\mathbb{E}_{a>\Delta, \bar{n}_r} \left[ 1_{\phi_1 < -\tau} \times \left( \int_\xi^\infty J(S_1) \cdot f_{b^{[1]}(b)} db \right)^\ell \right]. \tag{102}
\]

Under the condition of \(\phi_1 < -\tau\) and \(b^{[1]} \geq \xi\) in (102), there exists an index \(j^*\) such that \(\phi_{1,j^*} < -\tau\), and the subset \(G_{1,j^*}\) of \(S_1\) contains a smaller subset \(G_{1,j^*} \triangleq \{ \bar{u} \in \mathbb{R}_r^\mathcal{P} | u_{j^*} > -\sqrt{\xi} \times \tau \} = \{ \bar{u} \in \mathbb{R}_r^\mathcal{P} | u_{j^*} > -| \log \rho | \} \) since \(b^{[1]} \phi_{1,j^*} < -\sqrt{\xi} \tau\).

As a result, we have the space \(S_1 \supseteq G_{1,j^*}\), and thus \(J(S_1) \geq J(G_{1,j^*})\). We also note that \(J(G_{1,j^*})\) is unrelated to the channel gain \(b^{[1]}\) and \(\phi_{1,j^*}\), neither is it related to “a” and \(\bar{n}_r\).

As \(\rho \to \infty\), \(G_{1,j^*}\) will eventually converge to the space \(\{ \bar{u} \in \mathbb{R}_r^\mathcal{P} \} \) such that \(J(G_{1,j^*})\) goes to 1. Replacing \(J(S_1)\) in (102) with \(J(G_{1,j^*})\), we obtain a lower bound of (102), given by

\[
\mathbb{E}_{a>\Delta, \bar{n}_r} \left[ 1_{\phi_1 < -\tau} \times \left( \int_\xi^\infty J(G_{1,j^*}) \cdot f_{b^{[1]}(b)} db \right)^\ell \right] = \mathbb{E}_{a>\Delta, \bar{n}_r} \left[ 1_{\phi_1 < -\tau} \times J(G_{1,j^*})^\ell \times e^{-\frac{H}{\beta_2}} \right] \tag{103}
\]

where the equality (c) follows by definition from the fact that \(J(G_{1,j^*})^\ell \times e^{-\frac{H}{\beta_2}} \leq 1\). By (101)~(103), we finally obtain the result that \(F_{\epsilon}(\Delta, \ell) \geq \mathbb{E}_{a>\Delta, \bar{n}_r} [1_{\phi_1 < -\tau}]\) for \(\ell > 0\).

C.2 The diversity analysis for \(\mathbb{E}_{a>\Delta, \bar{n}_r} [1_{\phi_1 < -\tau}]\)

Considering a fixed rate \(\mathcal{R}\), we first have \(\mathbb{E}_{a>\Delta, \bar{n}_r} [1_{\phi_1 < -\tau}] \leq \mathbb{E}_{\bar{n}_r} [1_{\phi_1 < 0}] = \Pr \{ \mathcal{E}_o \} \triangleq \frac{1}{\rho}\) for any \(\Delta\). We then start with the case of \(\Delta > 0\) and show that the setting of a constant \(\Delta > 0\) will result in \(\mathbb{E}_{a>\Delta, \bar{n}_r} [1_{\phi_1 < -\tau}] \triangleq \frac{1}{\rho}\). Basically, we analyze a lower bound of \(\mathbb{E}_{a>\Delta, \bar{n}_r} [1_{\phi_1 < -\tau}]\) to obtain this result. According to \(\phi_1 \triangleq \min_{j \in \mathcal{P}} \phi_{1,j}\) and (28), we have

\[
\Pr \left\{ a > \Delta, \min_{j \in \mathcal{P}} \phi_{1,j} < -\tau \right\} \geq \Pr \{ a > \Delta, \phi_{1,j} < -\tau \} \geq \Pr \{ a > \Delta, \sqrt{a} \left[ \| \bar{d}_{0,j} \|^2 + (1 + \frac{1}{a})^{0.5} \tau \right] + 2 \Re \{ \bar{d}_{0,j}^H \bar{n}_r \} < 0 \} \tag{104}
\]
Then, we define a fixed number $\varepsilon > 0$. As $\rho$ increases, it will follow that $\varepsilon > (1 + \frac{1}{\Delta})0.5\tau \geq (1 + \frac{1}{\Delta})0.5\tau$. For convenience of expression, here we abuse the notations a little and redefine $x \triangleq \frac{2R(\frac{\rho}{\rho_1}, \frac{\rho_1}{\rho})}{\|d_{o,i}\|^2 + \varepsilon} \sim \mathcal{N}(0, \sigma_j^2)$ where $\sigma_j^2 = \frac{2\|d_{o,j}\|^2}{\|d_{o,j}\|^2 + \varepsilon}$. By those definitions, the right-hand-side (RHS) of (104) can be lower bounded at high SNR by

$$Pr\{a > \Delta, \sqrt{a} < x\} = \frac{1}{\sqrt{2\pi\sigma_j}} e^{-\Delta} \times \int_{\Delta}^{\infty} (1 - e^{-\frac{x^2-\Delta}{\rho_{\beta}^2}})e^{-\frac{x^2}{2\sigma_j^2}} \, dx. \quad (105)$$

To analyze the diversity order of (105), we make use of Taylor’s theorem to expand the term $(1 - e^{-\frac{x^2-\Delta}{\rho_{\beta}^2}})$ into a form of $\frac{x^2}{\rho_{\beta}^2} + \mathcal{O}(\frac{1}{\rho_{\beta}^2})$. With substituting this result back to (105), we can observe its first order term which is given by $\frac{1}{\rho_{\beta}^2} e^{-\Delta} \int_{\Delta}^{\infty} (x^2 - \Delta)e^{-x^2/(2\sigma_j^2)} \, dx$. Clearly, the integration of this term is finite and fixed with $\rho$ due to the constant threshold $\Delta > 0$ and the fact that

$$0 < \frac{1}{\sqrt{2\pi\sigma_j}} \int_{\Delta}^{\infty} (x^2 - \Delta)e^{-x^2/(2\sigma_j^2)} \, dx < \frac{1}{\sqrt{2\pi\sigma_j}} \int_{-\infty}^{\infty} x^2 e^{-x^2/(2\sigma_j^2)} \, dx = \sigma_j^2. \quad (106)$$

In other words, the dominant term of (105) only attains the order of $\frac{1}{\rho}$. This implies that $\mathbb{E}_{a > \Delta, \bar{n}_r}[1_{\phi_1 < -\tau}]$ is lower bounded at high SNR by a function that only attains one diversity order. Consequently, we obtain $\mathbb{E}_{a > \Delta, \bar{n}_r}[1_{\phi_1 < -\tau}] \equiv \frac{1}{\rho}$. This result also follows when $\Delta \leq 0$ because in this case, we have $\mathbb{E}_{a > \Delta, \bar{n}_r}[1_{\phi_1 < -\tau}] = \mathbb{E}_{\bar{n}_r}[1_{\phi_1 < -\tau}] \geq \mathbb{E}_{a > 1, \bar{n}_r}[1_{\phi_1 < -\tau}] \equiv \frac{1}{\rho}$.

C.3 The hints for the proof of the result of (44)

The result of (44) can also be obtained based on Appendix-C.1. The steps are described as follows. First, according to (35), the second term of the upper bound in (44) can attain the order of $\rho^{-\{q_k + \cdots + q_x\}}$. As for the lower bound of (44), the proof can be done with a procedure similar to (102)~(103). Essentially, we extend the notations $\phi_k, S_k$ and $f_{b|l}(b)$ to the multiple-relay case of $\phi_k, S_k$ and $f_{b|l}(b)$ respectively, where $S_k \equiv \bigcup_{k=1}^{K} \mathcal{G}_{k,\mathcal{D}}$ with $\mathcal{G}_{k,j} = \{ \bar{u} \in \mathbb{R}^D | u_j > \sqrt{b_{k,j}^2} \phi_{k,j} \}$. After some mathematical manipulations, we can derive the lower bound of $P_{e,k}$ in (41) as

$$a_1 > \Delta_1, \ldots, a_k > \Delta_k \quad \mathbb{E}_{a_{r,1}, \ldots, a_{r,k}} \left[ 1_{\phi_k < -\tau} \right] \times \prod_{i=k}^{N} \left( \int_{\xi}^{\infty} J(\mathcal{G}_{k,i*}) \cdot f_{b|l_i}(b) \, db \right) \equiv a_1 > \Delta_1, \ldots, a_k > \Delta_k \quad \mathbb{E}_{a_{r,1}, \ldots, a_{r,k}} \left[ 1_{\phi_k < -\tau} \right]. \quad (107)$$

C.4 Proof of Lemma 4

Before showing the proof, we first introduce another lemma.
Lemma 5: Given arbitrary positive numbers $g_1, \ldots, g_n$, the harmonic mean of the non-negative numbers $x_1, \ldots, x_n$ is given and then lower bounded by

$$\frac{1}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} \geq \min\left(\frac{x_1}{g_1}, \ldots, \frac{x_n}{g_n}\right) \times \frac{n}{\frac{1}{g_1} + \cdots + \frac{1}{g_n}}. \quad (108)$$

Proof: If $\exists j$ such that $x_j = 0$, then (108) is true. For $x_i > 0, \forall i \in I_1^n$, we have

$$\frac{1}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} = \frac{1}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} \geq \max\left(\frac{g_1}{x_1}, \ldots, \frac{g_n}{x_n}\right) \times \left(\frac{1}{g_1} + \cdots + \frac{1}{g_n}\right). \quad (109)$$

This completes the proof. The equality holds at $g_i = x_i$, $\forall i \in I_1^n$.

To prove Lemma 4, we first investigate the case of $k = 1$ where $\Pr\{a > \Delta_1, \phi_1 < \tau\}$ is equal to $\Pr\{a > \Delta_1, \min_j \in Z_1^n \phi_{1,j} < \tau\}$ by definition and this probability can be shown bounded from above by the union of the pairwise error probabilities. By (28), we have

$$\Pr\{a > \Delta_1, \min_j \in Z_1^n \phi_{1,j} < \tau\} = \Pr\left\{a > \Delta_1, \min_j \in Z_1^n \left[\sqrt{a} \left[\|\tilde{d}_{0,j}\|^2 - (1 + \frac{1}{a})^0.5 \times \tau\right] + 2\Re\{d_{0,j} H_{n_r}\}\right] < 0\right\} \quad (110)$$

where the last inequality follows when $\rho$ increases such that $\varepsilon_0 > (1 + \frac{1}{\varepsilon})^{0.5} \tau > (1 + \frac{1}{a})^{0.5} \tau$ due to $a > \Delta_1 > \varepsilon$. For convenience, let $x = \frac{-2\Re\{d_{0,j} H_{n_r}\}}{\|\tilde{d}_{0,j}\|^2 - \varepsilon_0} \sim N(0, \sigma^2_j)$ with $\sigma^2_j = \frac{2\|d_{0,j}\|^2}{(\|d_{0,j}\|^2 - \varepsilon_0)^2}$.

The upper bound in (110) can be further bounded from above by

$$\sum_{j=1}^D \frac{1}{\sqrt{2\pi\sigma^2_j}} \int_{-\infty}^\infty \left(e^{-\frac{\Delta_1}{\sigma^2_j}} - e^{-\frac{x^2}{2\sigma^2_j}}\right) e^{-\frac{x^2}{2\sigma^2_j}} dx \leq \sum_{j=1}^D \frac{1}{\sqrt{2\pi\sigma^2_j}} \int_{\sqrt{\Delta_1}}^\infty \frac{x^2}{\sigma^2_j} e^{-\frac{x^2}{2\sigma^2_j}} dx$$

$$= \sum_{j=1}^D \left(\sqrt{\Delta_1 \sigma^2_j} \times \frac{\sigma^2}{\sigma^2_j} \times \frac{1}{\sqrt{\Delta_1 \sigma^2_j}} + 1\right)$$

$$\leq \sum_{j=1}^D \left(\sqrt{\Delta_1 \sigma^2_j} \times \frac{\sigma^2}{\sigma^2_j} \times \left(2 + \frac{\Delta_1}{\sigma^2_j}\right)\right) \quad (111)$$

where the inequality (a) follows from the facts that $e^{-\frac{x^2}{\sigma^2_j}} \leq 1$ and $(1 - e^{-y}) \leq y$, and the inequality (b) is due to $Q(y) \geq \frac{y}{1+y} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ [37].

As for the case of $k > 1$, based on (42), $\Pr\{Z_k, \phi_k < \tau\}$ can be shown equivalently to

$$\Pr\left\{a_1 > \Delta_1, \ldots, a_k > \Delta_k, \min_j \in Z_1^n \left[\sqrt{\text{SNR}_{\text{inst}} d_{0,j,k}^2} + 2\Re\{d_{0,j} H_{n_r}\}\right] < 0\right\} \quad (112)$$
where \( d_{0,j,k}^2 \triangleq \| \bar{d}_{0,j} \|^2 - (1 + \frac{1}{\alpha_1})^{0.5} \times \cdots \times (1 + \frac{1}{\alpha_k})^{0.5} \tau \), and \( \text{SNR}_{\text{inst}} \triangleq [(1 + \frac{1}{\alpha_1}) \pi k - 1]^{-1} \) is the instantaneous SNR of a \( k \)-hop signal. Using Lemma 5, we can find a lower bound for \( \text{SNR}_{\text{inst}} \), which is given by

\[
\text{SNR}_{\text{inst}} = \frac{1}{\frac{1}{a_1} \prod_{i=2}^{k} (1 + \frac{1}{\alpha_i}) + \frac{1}{2} \prod_{i=3}^{k} (1 + \frac{1}{\alpha_i}) + \cdots + \frac{1}{k}} \geq \frac{1}{\frac{1}{a_1} \prod_{i=1}^{k} (1 + \frac{1}{\alpha_i}) + \sum_{i=2}^{k} \frac{1}{\alpha_i}} \geq \min \left( \frac{1}{a_1 \prod_{i=2}^{k} (1 + \frac{1}{\alpha_i})} \frac{1}{g_1}, \ldots, \frac{1}{g_k} \right) \frac{1}{g_1 + \cdots + g_k} \quad (113)
\]

where \( \psi_k = [\prod_{i=1}^{k} (1 + \frac{1}{\alpha_i}) - 1]^{-1} \). The inequality (c) follows by Lemma 5, and (d) holds when \( \frac{1}{g_1} := \frac{\psi_k}{\Delta_1} \prod_{i=2}^{k} (1 + \frac{1}{\alpha_i}) \), \( \frac{1}{g_2} := \frac{\psi_k}{\Delta_2} \prod_{i=3}^{k} (1 + \frac{1}{\alpha_i}) \), \ldots, and \( \frac{1}{g_k} := \frac{\psi_k}{\Delta_k} \). Notice that \( \frac{1}{g_1} + \cdots + \frac{1}{g_k} = 1 \).

Substituting the result of (113) back to (112), we have \( \text{Pr} \{ Z_k, \phi_k < \tau \} \) upper bounded by

\[
\text{Pr} \left\{ a_1 > \Delta_1, \ldots, c_k > \Delta_k, \min_{j \in I_k^p} \left[ \min \left( \sqrt{\frac{a_1 \psi_k}{\Delta_1}}, \ldots, \sqrt{\frac{c_k \psi_k}{\Delta_k}} \right) d_{0,j,k}^2 + 2 \text{Re} \{ \bar{d}_{0,j}^H \bar{n}_r \} \right] < 0 \right\} \leq \text{Pr} \left\{ a_1 > \Delta_1, \min_{j \in I_k^p} \left( \sqrt{\frac{a_1 \psi_k}{\Delta_1}} d_{0,j,k}^2 + 2 \text{Re} \{ \bar{d}_{0,j}^H \bar{n}_r \} \right) < 0 \right\} + \cdots + \text{Pr} \left\{ c_k > \Delta_k, \min_{j \in I_k^p} \left( \sqrt{\frac{c_k \psi_k}{\Delta_k}} d_{0,j,k}^2 + 2 \text{Re} \{ \bar{d}_{0,j}^H \bar{n}_r \} \right) < 0 \right\} \quad (114)
\]

\( \equiv \text{Pr} \left\{ a'_1 > \psi_k, \min_{j \in I_k^p} \left( \sqrt{a'_1 d_{0,j,k}^2 + 2 \text{Re} \{ \bar{d}_{0,j}^H \bar{n}_r \} } \right) < 0 \right\} + \cdots + \text{Pr} \left\{ c'_k > \psi_k, \min_{j \in I_k^p} \left( \sqrt{c'_k d_{0,j,k}^2 + 2 \text{Re} \{ \bar{d}_{0,j}^H \bar{n}_r \} } \right) < 0 \right\}
\]

where the equality (e) follows by the change of the variables, \( a'_i \triangleq \frac{a_1 \psi_k}{\Delta_i} \sim \text{Exp}(\frac{a_1 \psi_k}{\Delta_i}) \) and \( c'_i \triangleq \frac{c_i \psi_k}{\Delta_i} \sim \text{Exp}(\frac{c_i \psi_k}{\Delta_i}) \), \( \forall i \in I_k^p \).

Given that \( a > \Delta_1 > \epsilon \) and \( c_i > \Delta_i > \epsilon \), \( i \in I_k^p \), we first have \( d_{0,j,k}^2 \geq \| \bar{d}_{0,j} \|^2 - (1 + \frac{1}{\epsilon})^{\frac{1}{2}} \tau \). As \( \rho \) increase such that \( (1 + \frac{1}{\epsilon})^{\frac{1}{2}} \tau < \epsilon \), we thus have \( d_{0,j,k}^2 \geq \| \bar{d}_{0,j} \|^2 - \epsilon \). Substituting this result back into (114), we obtain another upper bound. Further, by applying the union bound of (110) and the inequality of (111) to the resultant upper bound, we finally obtain

\[
\text{Pr} \{ \mathcal{Z}_k, \phi_k < \tau \} \leq \sum_{j=1}^{D} Q \left( \sqrt{\frac{2 \sigma_j^2}{\rho}} \right) \times \frac{1}{\rho} \left( \frac{2 \sigma_j^2}{\rho} + 1 \right) \times \left( \frac{\Delta_1}{\rho_1} + \sum_{i=2}^{k} \frac{\Delta_i}{\rho_i} \right) \quad (115)
\]

where \( \sigma_j^2 = \frac{2 \| d_{0,j} \|^2}{\| d_{0,j} \|^2 - \epsilon} \). After reformulating this upper bound according to the codeword distances, then (54) follows.
C.5 The Proof of Corollary 2

By replacing the $Q(x)$ in (54) with an upper bound, $e^{-\frac{x^2}{4}}$, and considering the dominant case, $d = d_m$, in the summation, we can obtain a new upper bound for (54) given by

$$ e^{\left(-\frac{\psi_k \times d_m^2 \psi_k}{4}\right)} \times \frac{1}{\rho} \left(\frac{4}{d_{m_e} ^2 \psi_k} + 1\right) \times \left(\frac{k}{\beta_1} + \sum_{i=2}^{N_m} \frac{\Delta_i}{\beta_2}\right) \sum_{d \in \{d_m, \ldots, d_M\}} \omega_d. \quad (116) $$

To simplify (116), without the loss of generality, we assume that $i^* \triangleq \arg \max_{i \in \mathcal{I}_1} \Delta_i$, and $i^*_m \triangleq \arg \min_{i \in \mathcal{I}_1} \Delta_i$. Then, \( \frac{4(d_{m_e} ^2 \psi_k) \psi_k + 1}{d_{m_e} ^2 \psi_k} < \left(\frac{4(1+1/\Delta_i^*)}{d_{m_e} ^2 \psi_k} + 1\right) < \left(1 + \frac{1}{\Delta_i^*}\right)^k \left(\frac{4}{d_{m_e} ^2 \psi_k} + 1\right) \) where the last inequality follows by $(1+1/\Delta_i^*)^k \geq 1$. Thus, (116) can be upper bounded by

$$ e^{\left(-\frac{\psi_k \times d_m^2 \psi_k}{4}\right)} \times \frac{1}{\rho} \times \left(1 + \frac{1}{\Delta_i^*}\right)^k \left(\frac{4}{d_{m_e} ^2 \psi_k} + 1\right) \times \Delta_i^* \times \left(\frac{k}{\beta_1} + \frac{k-1}{\beta_2}\right) \sum_{d \in \{d_m, \ldots, d_M\}} \omega_d. \quad (117) $$

By definition, the diversity order of (117) is equal to

$$ \lim_{\rho \to \infty} \frac{d_m^2 \psi_k \times \psi_k}{\ln \rho} + 1 - \lim_{\rho \to \infty} \frac{\ln \Delta_i^*}{\ln \rho} - \lim_{\rho \to \infty} \frac{k \ln(1+1/\Delta_i^*)}{\ln \rho} - \lim_{\rho \to \infty} \frac{\ln \left(\left(\frac{4}{d_{m_e} ^2 \psi_k} + 1\right)\left(\frac{k}{\beta_1} + \frac{k-1}{\beta_2}\right) \sum_{d \in \{d_m, \ldots, d_M\}} \omega_d\right)}{\ln \rho}, \quad (118) $$

where the last term is zero. Besides, the third and forth terms are also equal to zero due to the fact that $- \ln \Delta_i^* < \ln(1 + \frac{1}{\Delta_i^*}) < \max[\ln 2, \ln \frac{2}{\Delta_i^*}]$, and the condition of (52).

To have $\Pr\{Z_k, \phi_k < \tau\} \triangleq \frac{1}{\rho m(N-k+1)}, \ \forall k \in \mathcal{I}_{1}^{\text{min}[m,N]}$, the thresholds assignment for $\Delta$ therefore at least satisfies $\lim_{\rho \to \infty} \frac{\psi_k}{\ln \rho} \geq \frac{d_m^2 \psi_k}{\ln \rho} \times \left[m (N-k+1) - 1\right], \ \forall k \in \mathcal{I}_{1}^{\text{min}[m,N]}$.

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