GALOIS EMBEDDING OF K3 SURFACE
– ABELIAN CASE –

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Abstract. We study Galois embeddings of K3 surfaces in the case where the
Galois groups are abelian. We show several properties of K3 surfaces concerning
the Galois embeddings. In particular, if the Galois group \( G \) is abelian, then
\( G \cong \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z} \) or \( (\mathbb{Z}/2\mathbb{Z})^3 \) and \( S \) is a smooth complete intersection of hypersurfaces.
Further, we state the detailed structure of such surfaces.

1. Introduction

The purpose of this article is to study Galois embeddings of K3 surfaces, where
the Galois groups are abelian. The non-abelian case will be treated later. Before
going into the study on K3 surfaces, we recall the definition of Galois embeddings
of algebraic varieties and their properties.

Let \( k \) be the ground field of our discussion, we assume it to be the field of complex
numbers, however most results hold also for an algebraically closed field of characteristic zero. Let \( V \) be a nonsingular projective algebraic variety of dimension \( n \) with a
very ample divisor \( D \), we denote this by a pair \((V, D)\). Let \( f = f_D : V \hookrightarrow \mathbb{P}^N \)
be the embedding of \( V \) associated with the complete linear system \( |D| \), where
\( N + 1 = \dim H^0(V, \mathcal{O}(D)) \). Suppose that \( W \) is a linear subvariety of \( \mathbb{P}^N \) satisfying
\( \dim W = N - n - 1 \) and \( W \cap f(V) = \emptyset \). Consider the projection \( \pi_W \) from \( W \)
to \( \mathbb{P}^n \), \( \pi_W : \mathbb{P}^N \rightarrow \mathbb{P}^n \). Restricting \( \pi_W \) onto \( f(V) \), we get a surjective morphism
\( \pi = \pi_W \circ f : V \rightarrow \mathbb{P}^n \).

Let \( K = k(V) \) and \( K_0 = k(\mathbb{P}^n) \) be the function fields of \( V \) and \( \mathbb{P}^n \) respectively.
The morphism \( \pi \) induces a finite extension of fields \( \pi^*: K_0 \hookrightarrow K \) of degree \( d = \deg f(V) = D^n \), which is the self-intersection number of \( D \). We denote by \( K_W \)
the Galois closure of this extension and by \( G_W = Gal(K_W/K_0) \) the Galois group
of \( K_W/K_0 \). By [11] we see that \( G_W \) is isomorphic to the monodromy group of the
covering \( \pi : V \rightarrow \mathbb{P}^n \). Let \( V_W \) be the \( K_W \)-normalization of \( V \) (cf. [2, Ch.2]). Note
that \( V_W \) is determined uniquely by \( V \) and \( W \).

Definition 1.1. In the above situation we call \( G_W \) and \( V_W \) the Galois group and
the Galois closure variety at \( W \) respectively (cf. [12]). If the extension \( K/K_0 \) is
Galois, then we call \( f \) and \( W \) a Galois embedding and a Galois subspace for the
embedding respectively.
**Definition 1.2.** A nonsingular projective algebraic variety $V$ is said to have a Galois embedding if there exist a very ample divisor $D$ satisfying that the embedding associated with $|D|$ has a Galois subspace. In this case the pair $(V, D)$ is said to define a Galois embedding.

If $W$ is the Galois subspace and $T$ is a projective transformation of $\mathbb{P}^N$, then $T(W)$ is a Galois subspace of the embedding $T \cdot f$. Therefore the existence of Galois subspace does not depend on the choice of the basis giving the embedding.

**Remark 1.3.** If a smooth variety $V$ exists in a projective space, then by taking a linear subvariety, we can define a Galois subspace and Galois group similarly as above. Suppose that $V$ is not normally embedded and there exists a linear subvariety $W$ such that the projection $\pi_W$ induces a Galois extension of fields. Then, taking $D$ as a hyperplane section of $V$ in the embedding, we infer readily that $(V, D)$ defines a Galois embedding with the same Galois group in the above sense.

By this remark, for the study of Galois subspaces, it is sufficient to consider the case where $V$ is normally embedded.

We have studied Galois subspaces and Galois groups for hypersurfaces in [8], [9] and [10] and space curves in [11] and [13]. The method introduced in [12] is a generalization of the ones in these studies.

Hereafter we use the following notation and convention:

- $\text{Aut}(V)$: the automorphism group of a variety $V$
- $|G|$: the order of a group $G$
- $\sim$: the linear equivalence of divisors
- $1_m$: the unit matrix of size $m$
- $[\alpha_1, \ldots, \alpha_m]$: the diagonal matrix with entries $\alpha_1, \ldots, \alpha_m$

The organization of this article is as follows: In Section 2 we review the results of Galois embeddings, which will be used in the sequel. We devote the remainder sections to the study of the Galois embedding of $K3$ surfaces.

## 2. Results on Galois Embeddings

We state several properties concerning Galois embedding without proofs, for the details see [12]. By definition, if $W$ is a Galois subspace, then each element $\sigma$ of $G_W$ is an automorphism of $K = K_W$ over $K_0$. Therefore it induces a birational transformation of $V$ over $\mathbb{P}^n$. This implies that $G_W$ can be viewed as a subgroup of $\text{Bir}(V/\mathbb{P}^n)$, the group of birational transformations of $V$ over $\mathbb{P}^n$. Further we can say the following:

**Representation 1.** Each birational transformation belonging to $G_W$ turns out to be regular on $V$, hence we have a faithful representation

$$\alpha : G_W \to \text{Aut}(V).$$

(1)

Therefore, if the order of $\text{Aut}(V)$ is smaller than the degree $d$, then $(V, D)$ cannot define a Galois embedding. In particular, if $\text{Aut}(V)$ is trivial, then $V$ has no Galois embedding. On the other hand, in case $V$ has infinitely many automorphisms, we
have examples such that there exist infinitely many distinct Galois embeddings, see Example 4.1 in [12].

When \((V, D)\) defines a Galois embedding, we often identify \(f(V)\) with \(V\). Let \(H\) be a hyperplane of \(\mathbb{P}^N\) containing \(W\). Let \(D'\) be the intersection divisor of \(V\) and \(H\). Since \(D' \sim D\) and \(\sigma^*(D') = D'\), for any \(\sigma \in G_W\), we see that \(\sigma\) induces an automorphism of \(H^0(V, O(D))\). This implies the following.

**Representation 2.** We have a second faithful representation

\[
\beta : G_W \hookrightarrow PGL(N, \mathbb{C}).
\]  

(2)

In the case where \(W\) is a Galois subspace we identify \(\sigma \in G_W\) with \(\beta(\sigma) \in PGL(N, \mathbb{C})\) hereafter. Since \(G_W\) is a finite subgroup of \(\text{Aut}(V)\), we can consider the quotient \(V/G_W\) and let \(\pi G\) be the quotient morphism, \(\pi_G : V \twoheadrightarrow V/G_W\).

**Proposition 2.1.** If \((V, D)\) defines a Galois embedding with the Galois subspace \(W\) such that the projection is \(\pi_W : \mathbb{P}^N \rightarrow \mathbb{P}^n\), then there exists an isomorphism \(g : V/G_W \rightarrow \mathbb{P}^n\) satisfying \(g \cdot \pi_G = \pi\). Hence the projection \(\pi\) turns out to be a finite morphism and the fixed loci of \(G_W\) consist of only divisors.

Therefore, \(\pi\) is a Galois covering in the sense of Namba [6]. We have a criterion that \((V, D)\) defines a Galois embedding.

**Theorem 2.2.** The pair \((V, D)\) defines a Galois embedding if and only if the following conditions hold:

1. There exists a subgroup \(G\) of \(\text{Aut}(V)\) satisfying that \(|G| = D^n\).
2. There exists a \(G\)-invariant linear subspace \(L\) of \(H^0(V, O(D))\) of dimension \(n + 1\) such that, for any \(\sigma \in G\), the restriction \(\sigma^*|L\) is a multiple of the identity.
3. The linear system \(L\) has no base points.

It is easy to see that \(\sigma \in G_W\) induces an automorphism of \(W\), hence we obtain another representation of \(G_W\) as follows. Take a basis \(\{f_0, f_1, \ldots, f_N\}\) of \(H^0(V, O(D))\) satisfying that \(\{f_0, f_1, \ldots, f_n\}\) is a basis of \(L\) in Theorem 2.2. Then we have the representation

\[
\beta_1(\sigma) = \begin{pmatrix}
\lambda_\sigma & \cdots & \\
\vdots & \ddots & * \\
\cdots & \cdots & \lambda_\sigma \\
0 & \cdots & M'_\sigma
\end{pmatrix}.
\]  

(3)

Since the representation is completely reducible, we get another representation using a direct sum decomposition:

\[
\beta_2(\sigma) = \lambda_\sigma \cdot 1_{n+1} \oplus M'_\sigma.
\]

Thus we can define

\[
\gamma(\sigma) = M'_\sigma \in PGL(N - n - 1, \mathbb{C}).
\]

Therefore \(\sigma\) induces an automorphism on \(W\) given by \(M'_\sigma\).
Representation 3. We get a third representation

\[ \gamma : G_W \to PGL(N - n - 1, \mathbb{C}). \]  

Let \( G_1 \) and \( G_2 \) be the kernel and image of \( \gamma \) respectively.

Theorem 2.3. We have an exact sequence of groups

\[ 1 \to G_1 \to G \xrightarrow{\gamma} G_2 \to 1, \]

where \( G_1 \) is a cyclic group.

Corollary 2.4. If \( N = n + 1 \), i.e., \( f(V) \) is a hypersurface, then \( G \) is a cyclic group.

This assertion has been obtained in \([10]\). Moreover we have another representation.

Suppose that \((V, D)\) defines a Galois embedding and let \( G \) be a Galois group for some Galois subspace \( W \). Then, take a general hyperplane \( W_1 \) of \( \mathbb{P}^n \) and put \( V_1 = \pi^*(W_1) \). The divisor \( V_1 \) has the following properties:

(i) If \( n \geq 2 \), then \( V_1 \) is a smooth irreducible variety.
(ii) \( V_1 \sim D \).
(iii) \( \sigma^*(V_1) = V_1 \) for any \( \sigma \in G \).
(iv) \( V_1/G \) is isomorphic to \( W_1 \).

Put \( D_1 = V_1 \cap H_1 \), where \( H_1 \) is a general hyperplane of \( \mathbb{P}^N \). Then \((V_1, D_1)\) defines a Galois embedding with the Galois group \( G \) (cf. Remark \([3]\)). Iterating the above procedures, we get a sequence of pairs \( (V_i, D_i) \) such that

\[ (V, D) \supset (V_1, D_1) \supset \cdots \supset (V_{n-1}, D_{n-1}). \]

These pairs satisfy the following properties:

(a) \( V_i \) is a smooth subvariety of \( V_{i-1} \), which is a hyperplane section of \( V_{i-1} \), where \( D_i = V_{i+1}, V = V_0 \) and \( D = V_1 (1 \leq i \leq n-1) \).
(b) \( (V_i, D_i) \) defines a Galois embedding with the same Galois group \( G \).

In particular, letting \( C \) be the curve \( V_{n-1} \), we get the following fourth representation.

Representation 4. We have a fourth faithful representation

\[ \delta : G_W \hookrightarrow \text{Aut}(C), \]

where \( C \) is a smooth curve in \( V \) given by \( V \cap L \) such that \( L \) is a general linear subvariety of \( \mathbb{P}^N \) with dimension \( N - n + 1 \) containing \( W \).

Note that in some cases there exist several Galois subspaces and Galois groups for one embedding (see, for example \([13]\)). Generally we have the following.

Proposition 2.5. Suppose that \((V, D)\) defines a Galois embedding and let \( W_i \) (\( i = 1, 2 \)) be Galois subspaces such that \( W_1 \neq W_2 \). Then \( G_1 \neq G_2 \) in \( \text{Aut}(V) \), where \( G_i \) is the Galois group at \( W_i \).

Corollary 2.6. If \( V \) is a smooth projective algebraic variety of general type, then there are at most finitely many Galois subspaces.
Remark 2.7. It may happen that there exist infinitely many Galois subspaces for one embedding if the Kodaira dimension of $V$ is small. For example, if $V = \mathbb{P}^1$ and $\deg D = 3$, i.e., $f(V)$ is a twisted cubic, then the Galois lines form two dimensional locally closed subvariety of the Grassmannian $G(1, 3)$, parametrizing lines in projective three space (cf. [11]).

3. K3 Surfaces

We apply the methods developed in the previous sections to the study of K3 surfaces. For each abelian surface with a Galois embedding, we have studied in detail in [12]. In particular, we have given the complete list of the complex representation of every possible group and shown that the surface is isogenous to the square of an elliptic curve.

A curve and surface will mean a nonsingular projective algebraic curve and surface respectively. In addition to the notation listed in Section 1, we use the following hereafter:

- $\langle a_1, \cdots, a_m \rangle$: the subgroup generated by $a_1, \cdots, a_m$
- $\mathbb{Z}_m$: the cyclic group of order $m$
- $e_m := \exp(2\pi \sqrt{-1}/m)$
- $D_1, D_2$: the intersection number of two divisors $D_1$ and $D_2$ on a surface
- $D^2$: the self-intersection number of a divisor $D$ on a surface
- $(X_0 : \cdots : X_m)$: a set of homogeneous coordinates on $\mathbb{P}^m$
- $g(C)$: the genus of a smooth curve $C$
- $\text{Supp } D$: the support of a divisor $D$
- $X(4)$: a smooth quartic surface in $\mathbb{P}^3$
- $X(23)$: a smooth $(2, 3)$-complete intersection of hypersurfaces in $\mathbb{P}^4$
- $X(222)$: a smooth $(2, 2, 2)$-complete intersection of hypersurfaces in $\mathbb{P}^5$

Suppose that $S$ is a K3 surface such that $(S, D)$ defines a Galois embedding with the Galois group $G \subset \text{Aut}(S)$. Let $\omega_S$ be a nowhere vanishing holomorphic two form of $S$ Then, let $\varepsilon : G \rightarrow \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ be the character of the natural representation of $G$ on the space $H^{2,0}(S) = \mathbb{C}\omega_S$, i.e., $\varepsilon(\sigma) = \lambda$ for $\sigma \in G$ if $\sigma^*(\omega_S) = \lambda \omega_S$. There exists a multiplicative group $\Gamma_m$ of the $m$-th roots of unity and the following exact sequence of groups:

$$1 \rightarrow G_s \rightarrow G \stackrel{\varepsilon}{\rightarrow} \Gamma_m \rightarrow 1, \quad (6)$$

where $G_s$ is a symplectic group [5]. Let $\pi : S \rightarrow \mathbb{P}^2$ be the projection, which is a Galois covering defined in Section 2. Let $W$ be the center of the projection and $H$ a general hyperplane containing $W$. Put $C = S \cap H$. Then $C$ is an irreducible smooth curve and $C \sim D$.

Lemma 3.1. The representation $r : G \rightarrow \text{Aut}(C)$ given by $r(\sigma) = \sigma|_C$ is injective.

Proof. Note that $\sigma \in G$ is an automorphism of $S$ over $\mathbb{P}^2$ and $\sigma(C) = C$. If $\sigma|_C$ is identity, then $C$ is a component of the ramification divisor of the covering. Since $C$ is given by $H$ which is general, $\sigma$ must be identity. \qed

The restriction $\pi|_C : C \rightarrow \mathbb{P}^1$ turns out to be a Galois covering, where the Galois group is isomorphic to $G$. Since $H^1(S, \mathcal{O}) = 0$ and the canonical divisor on $S$ is
trivial, the restriction of \( f_D \) to \( C \) gives the canonical embedding of \( C \). Therefore \( C \) has a Galois embedding given by its canonical divisor.

**Lemma 3.2.** The group \( G \) is non-symplectic, i.e., \( \Gamma_m \neq \{1\} \).

**Proof.** Suppose \( \Gamma_m = \{1\} \). Then, \( G = G_2 \). This means that the fixed loci of each element of \( G \) is at most finitely many points. This contradicts to Proposition 2.1. \( \Box \)

Let \( R \) be the ramification divisor for \( \pi \).

**Lemma 3.3.** We have \( R \sim 3D \) and \( \text{Supp } R \) is connected. Each irreducible component of \( \text{Supp } R \) is smooth.

**Proof.** Since the canonical divisor on \( S \) is trivial, using the adjunction formula, we get \( \pi^*(-3\ell) + R \sim 0 \) for a line \( \ell \) in \( \mathbb{P}^2 \). Since \( \pi^*(\ell) \sim D \), we have \( R \sim 3D \), hence \( R \) is very ample. \( \Box \)

**Example 3.4.** Let \( S \) be the Fermat quartic surface:
\[
X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0
\]
and \( P \) be one of the points \((1 : 0 : 0 : 0)\), \((0 : 1 : 0 : 0)\), \((0 : 0 : 1 : 0)\) and \((0 : 0 : 0 : 1)\).

The projection from \( P \) to the hyperplane \( \mathbb{P}^2 \) defines a cyclic Galois covering (such \( P \) is called a Galois point \([10]\)). Note that \( S \) is a Kummer surface \( K_{m}(E \times E) \), where \( E = \mathbb{C}/(1, e_4) \). Further, it is a singular \( K3 \) surface, i.e., \( \rho(S) = 20 \) (cf. \([3]\)).

4. ABEILIAN CASE

In the case of Galois embeddings of abelian surfaces, the group cannot be abelian. However, in the case of \( K3 \) surfaces, the group can be a cyclic group as in Example 3.4. Nikulin \([7]\) shows that there exist many abelian automorphism groups for \( K3 \) surfaces. So let us consider the Galois embedding where the Galois group \( G \) is abelian. Hereafter we assume \( G \) is abelian if not otherwise mentioned.

**Theorem 4.1.** If the Galois group \( G \) is abelian, then \( G \cong Z_4 \), \( Z_6 \) or \( Z_2^3 = Z_2 \times Z_2 \times Z_2 \) and \( S \) is isomorphic to \( S(4) \), \( S(23) \) or \( S(222) \) respectively.

We will give concrete examples for the three surfaces in Section 5. We note that for the proof of Theorem 4.1 we do not use the property of Galois embedding, but only use that the covering \( \pi : S \to \mathbb{P}^2 \) is Galois (except in the proof of Claim 4.16). So the result may be known, but for the sake of completeness, we give the proof in this article.

Before proceeding with the proof, we fix the notation. Let \( \pi : S \to \mathbb{P}^2 \) be the Galois covering induced by the projection. Put \( |G| = n \) and assume that \( R = (n_1 - 1)C_1 + \cdots + (n_r - 1)C_r \), where \( C_i \) are irreducible components. For \( \sigma \in G \) put \( F(\sigma) = \{ x \in S \mid \sigma(x) = x \} \).

**Lemma 4.2.** For each point \( x \in \text{Supp } R \) the stabilizer of the point \( G_x = \{ \sigma \in G \mid \sigma(x) = x \} \) is generated by at most two elements.

**Proof.** There exists an open neighbourhood \( U_x \) and coordinates on it such that \( G_x \) has a representation in \( GL(2, \mathbb{C}) \). Since \( G \) is abelian, we can assume each element of \( G_x \) is generated by one or two diagonal matrices \([\alpha, 1]\) and \([1, \beta]\), where \( \alpha^n = \beta^n = 1 \). \( \Box \)

**Lemma 4.3.** The following assertions hold true.
(1) $\text{Supp } R$ is connected.
(2) Each irreducible component is a smooth curve.
(3) $\text{Supp } R$ has normal crossings.

Proof. Since $R \sim 3D$, we have $R$ is ample, hence $\text{Supp } R$ is connected. Each component $C_i$ is given by \{ $x \in S \mid \sigma(x) = x$ \} for some $\sigma \in G \setminus \{\text{id}\}$, where locally $\sigma$ can be expressed as a diagonal matrix $[\alpha, 1]$. Thus $C_i$ is smooth. Suppose $x \in \text{Supp } R$ is an intersection point of some components $C_i$ ($1 \leq i \leq r$). As we have seen in the proof of Lemma 4.2, there exist two elements $[\alpha, 1]$ and $[1, \beta]$ which are generators of the stabilizer $G_x$. Thus there exist just two irreducible components meeting normally.

\[ \square \]

**Lemma 4.4.** For each irreducible component $C$ of $R$, we have $\text{Supp } \pi^*(\pi(C)) = C$, i.e., $\tau(C) = C$ for any $\tau \in G$. In particular $C$ is an ample divisor.

Proof. Let $\sigma \in G$ satisfy $\sigma \neq \text{id}$ and $\sigma|_C = \text{id}$. Since $\pi(C)$ is ample and $\pi : S \rightarrow \mathbb{P}^2$ is a finite morphism, $\pi^*(\pi(C))$ is ample and hence $\text{Supp } \pi^*(\pi(C))$ is connected. Suppose $\text{Supp } \pi^*(\pi(C))$ is reducible. Then, there exists another irreducible component $C'$ of $\pi^*(\pi(C))$ such that $C' = \pi(C)$ for some $\sigma' \in G$ and $C \cap C' \neq \emptyset$. Since $\sigma\sigma' = \sigma'\sigma$, we have $\sigma(\sigma'(y)) = \sigma'(y)$ for any $y \in C$. This means $\sigma|_{C'} = \text{id}$. Take $x \in C \cap C'$ Then $C$ and $C'$ have a normal crossing at $x$ by Lemma 4.3. However, looking at $\sigma$ near $x$, the $\sigma$ can be expressed as one of the diagonal matrices $[\alpha, 1]$ and $[1, \beta]$, where $\alpha \neq 1$ and $\beta \neq 1$. This contradicts to that $\sigma|_{C'} = \text{id}$.

\[ \square \]

**Corollary 4.5.** With the same notation as in Lemma 4.4, we have $C^2 > 0$, hence $g(C) \geq 2$.

Proof. Since $\pi^*(\pi(C))$ can be expressed as $mC$, we have $m^2C^2 = n(\pi(C))^2 \geq n$. Since $2g(C) - 2 = C^2$ we have the assertion.

Put $G_i = \{ \sigma \in G \mid \sigma|_{C_i} = \text{id} \}$ ($1 \leq i \leq r$). Then $G_i$ is determined uniquely by $C_i$ and not a trivial subgroup of $G$. 

**Lemma 4.6.** The group $G_i$ is cyclic.

Proof. For a general point $x \in C_i$, taking a suitable local coordinates, we can express each $\sigma \in G_i$ as $[\alpha, 1]$. We have a monomorphism $\rho : G_i \rightarrow \mathbb{C}^\times$, where $\rho(\sigma) = \alpha$. Since $\rho(G_i)$ is a cyclic group, so is $G_i$.

\[ \square \]

**Lemma 4.7.** The surface $S_i = S/G_i$ ($1 \leq i \leq r$) is a smooth rational surface.

Proof. Since near each point $x \in C_i$, the $\sigma \in G_i$ can be expressed as a diagonal matrix. Hence $S_i$ is smooth. Let $K_i$ be a canonical divisor on $S_i$. Then, we have $\pi_i^*(K_i) + R_i \sim 0$, where $\pi_i : S \rightarrow S_i$ and $R_i$ is a ramification divisor for $\pi_i$. Since $R_i$ is effective, we infer that $\dim H^0(S_i, \Omega^1_{S_i}) = 0$. Clearly we have $\dim H^0(S_i, \Omega^1_i) = 0$, where $\Omega^1_i$ is the sheaf of holomorphic 1-forms on $S_i$. Therefore $S_i$ is rational by Castelnuovo’s Rationality Criterion.

\[ \square \]

**Lemma 4.8.** There does not exist $\tau \in G$ such that $\tau \neq \text{id}$ and $F(\tau) = \emptyset$.

Proof. Suppose otherwise. Since $G$ is abelian, expressing $G \cong \langle \tau \rangle \times G'$, we put $S' = S/G'$. Then $S'$ is a smooth rational surface. Because, as we see in the proof of Lemma 4.2, $G_x$ is generated locally by reflections. Hence $S'$ is smooth. Since there exists a covering $S_i \rightarrow S'$ (or $S_i = S'$) and $S_i$ is rational, we see that $S'$
is rational. The $\pi': S' \to \mathbb{P}^2 = S/G'$ is an unramified double covering, this is a contradiction. □

**Lemma 4.9.** The group $G_i$ ($1 \leq i \leq r$) determines $C_i$ uniquely and $G_i \cap G_j$ consists of identity if $i \neq j$. Therefore, there exists a one to one correspondence between the set $\{ G_i \mid 1 \leq i \leq r \}$ and $\{ C_i \mid 1 \leq i \leq r \}$.

**Proof.** If $C_i \neq C_j$, then we have $C_i \cap C_j \neq \emptyset$ by Lemma 4.4. Take $x \in C_i \cap C_j$. Consider $G_i$ and $G_j$ in a neighbourhood of $x$. Since $G$ is abelian, there exist generators $[a, 1]$ and $[1, \beta]$ of $G_i$ and $G_j$ respectively. If $\sigma \in G_i \cap G_j$, then $\sigma|_{C_i} = \sigma|_{C_j} = id$. This implies that $\sigma = id$. □

**Lemma 4.10.** The group $G$ can be expressed as a direct product $G_1 \times \cdots \times G_r$, where each $G_i$ is cyclic $(1 \leq i \leq r)$.

**Proof.** For each element $\sigma \in G$, there exists a fixed point of $\sigma$ by Lemma 4.8. If $F(\sigma)$ contains a curve, then there exist $i$ such that $\sigma|_{C_i} = id$. This means that $\sigma \in G_i$. On the other hand, if $F(\sigma)$ consists of only points, then take $x \in F(\sigma)$. It is easy to see that there exist two curves $C_i$ and $C_j$ containing $x$. Then $\sigma$ can be expressed as a product of elements of $G_i$ and $G_j$. Therefore, we conclude the assertion from Lemma 4.9. □

Let $\Delta_i$ be the plane curve $\pi(C_i)$ and put $\Delta = \Delta_1 + \cdots + \Delta_r$.

**Lemma 4.11.** Each $\Delta_i$ is smooth $(1 \leq i \leq r)$ and $\Delta$ has normal crossings.

**Proof.** In the proof of Lemma 4.4 we have shown that $\pi(C_i) = C_i$ for each $\tau \in G$. Therefore $G$ acts on $C_i$ and we can consider $C_i/G$. We denote it by $\Delta_i$. Hence $\Delta_i$ is smooth. For a point $x \in C_i \cap C_j$ we have $\sigma_i(x) = \sigma_j(x) = x$ and $\sigma_k(C_i) = C_i$ and $\sigma_k(C_j) = C_j (1 \leq i, j, k \leq r)$. Hence $\Delta$ has normal crossings. □

Put $n_i = |G_i|$. Then we have $n = \prod_{i=1}^r n_i$ by Lemma 4.10. Denote by $\chi(V)$ the topological Euler characteristic of a curve or a surface $V$.

**Lemma 4.12.** We have $n_i = 2, 3$ or 4 for each $i$.

**Proof.** Put $\mathcal{C}_i = \pi_i(C_i)$ where $\pi_i : S \to S_i = S/G_i$. Compare $\chi(S)$ and $\chi(S_i)$. Since $G_i = \langle \sigma_i \rangle$ and $\sigma_i|_{C_i} = id$, the $C_i$ is isomorphic to $\mathcal{C}_i$. Hence we have

$$
\chi(S) = \chi(S - C_i) + \chi(C_i) = n_i \chi(S_i - \mathcal{C}_i) + \chi(\mathcal{C}_i) = n_i \chi(S_i) + (1 - n_i) \chi(\mathcal{C}_i)
$$

We have $\chi(C_i) = 2 - 2g(C_i) = \chi(\mathcal{C}_i)$. Therefore we have

$$
24 = n_i \chi(S_i) + (n_i - 1)(2g(C_i) - 2). \quad (7)
$$

Since $S_i$ is a smooth rational surface by Lemma 4.7 we have $\chi(S_i) \geq 3$. Further, we have $g(C_i) \geq 2$ by Corollary 4.5. Thus, clearly we have $n_i \leq 5$. In case $n_i = 5$, we have $24 = 5\chi(S_i) + 8(2g(C_i) - 2)$, but this cannot hold. Whence we conclude $n_i \leq 4$. □

Next we consider the branch divisor for $\pi$. Put $d_i = \deg \Delta_i (1 \leq i \leq r)$.
Lemma 4.13. We have the equality
\[ d_1 \left( 1 - \frac{1}{n_1} \right) + \cdots + d_r \left( 1 - \frac{1}{n_r} \right) = 3. \]  
In particular, we have \( r \leq 6 \).

Proof. Letting \( \ell \) be a line in \( \mathbb{P}^2 \), we have \( \pi^*(3\ell), \pi^*(\ell) = 3n \). Since \( \pi^*(\Delta_i) = n_iC_i \), we have \( n_iC_i, \Gamma = \pi^*(\Delta_i), \Gamma = nd_i \), where \( \Gamma = \pi^*(\ell) \). Since \( R \sim 3D \) by Lemma 3.3 and \( D \sim \pi^*(\ell) \), we get
\[(n_1 - 1)\Gamma C_1 + \cdots + (n_r - 1)\Gamma C_r = 3n, \]
hence
\[(n_1 - 1)\frac{d_1}{n_1} + \cdots + (n_r - 1)\frac{d_r}{n_r} = 3n. \]
This proves the equation. Since \( n_i \geq 2 \) and \( d_i \geq 1 \), we have \( r \leq 6 \).

Lemma 4.14. We have \( d_i \geq 2 \) for each \( i \).

Proof. Put \( \hat{G}_i = G/G_i \) and consider the coverings
\[ p_i : S \longrightarrow S/\hat{G}_i = \hat{S}_i \quad \text{and} \quad q_i : \hat{S}_i \longrightarrow \mathbb{P}^2 = S/G. \]
By Lemma 4.4 \( \hat{G}_i \) acts on \( C_i \), hence put \( \hat{C}_i = p_i(C_i) = C_i/\hat{G}_i \). By repeating the similar arguments as in the proof of Lemma 4.7, we conclude \( \hat{S}_i \) is a smooth rational surface. Suppose \( d_i = 1 \), then \( q_i(\hat{C}_i) \sim \Delta_i \) is a line \( \ell \). Hence we get
\[ q_i^*(\ell) = n_i\hat{C}_i. \]
This means \( q_i^*(\ell)^2 = n_i = n_i^2\hat{C}_i^2 \). Hence \( n_i\hat{C}_i^2 = 1 \), i.e., \( n_i = 1 \). This is a contradiction.

Making use of Lemmas 3.12, 4.13 and 4.14 we determine \( r, n_i \) and \( d_i \) \((1 \leq i \leq r)\).

Claim 4.15. If there exists \( i \) such that \( n_i = 4 \), then \( G \cong Z_4 \).

Proof. We may assume \( i = 1 \). We prove the claim by examining the cases:

(i) \( r = 1 \).

(ii) There exists \( j \geq 2 \) such that \( n_j = 4 \).

(iii) \( n_j \leq 3 \) for all \( j \geq 2 \) and there exists for some \( j \geq 2 \) such that \( n_j = 3 \)

(iv) \( n_j = 2 \) for all \( j \geq 2 \)

In case of (i) we observe the equality (7). We have \( 24 = 4\chi(S_1) + 6(g(C_1) - 1) \).

Since \( g_1 = g(C_1) \geq 2 \), we have \( g_1 = 3 \) and \( \chi(S_1) = 3 \). Since \( S_1 \) is a smooth rational surface, we have \( S_1 \cong \mathbb{P}^2 \), hence \( d_1 = 4 \). Therefore we have \( G \cong Z_4 \).

The case (ii) does not occur. Suppose otherwise. Then, since \( d_j \geq 2 \), we infer from (8) that \( r = 2, d_1 = d_2 = 2 \) and \( n_1 = n_2 = 4 \). Note that \( \sigma_1|C_i = id \) and \( \sigma_1 \) acts on \( C_2 \). Thus we have \( \chi(\Delta_i) = 2 \) and \( \chi(C_i) = -4 \) \((i = 1, 2)\). Then we get
\[
\begin{align*}
\chi(S) &= \chi(S - (C_1 \cup C_2)) + \chi(C_1 \cup C_2) \\
&= 16\chi(\mathbb{P}^2 - (\Delta_1 \cup \Delta_2)) + \chi(\Delta_1) + \chi(\Delta_2) - \chi(\Delta_1 \cap \Delta_2) \\
&= 36.
\end{align*}
\]
which is a contradiction.

The case (iii) does not occur. Suppose otherwise. Then, from (8) we have
\[
3 \geq d_1 \left( 1 - \frac{1}{4} \right) + d_2 \left( 1 - \frac{1}{3} \right) \geq \frac{3}{2} + \frac{2}{3}d_j.
\]
Claim 4.16. The case \( r = 2 \) and \( n_1 = n_2 = 2 \) does not occur.

Proof. We show that \( r = 2 \) and \( 3d_2/4 = 5/3 \), which is a contradiction.

The case (iv) does not occur. Suppose otherwise. Then, from (8) we have that \( 3d_1 + 2(d_2 + \cdots + d_r) = 12 \), which implies that \( r = 2 \) and \( d_2 = 3 \), i.e., \( n_1 = 4 \), \( d_1 = 2 \) and \( n_2 = 2 \). Note that \( \sigma_1|_{C_1} = \text{id} \) and \( \sigma_1 \) acts on \( C_2 \). We infer readily that \( C_2 := C_2/G \) is a smooth curve in \( S_1 := S/G_1 \cong \mathbb{P}^2 \). We have \( \pi = q_1 \cdot p_1 \), where \( p_1 : S \rightarrow S_1 \) and \( q_1 : S_1 \rightarrow S/G \cong \mathbb{P}^2 \). Then \( q_1 : S_1 \rightarrow \mathbb{P}^2 \) is a double covering branched along just \( \Delta_2 \), which is cubic. This is a contradiction. □

Therefore we assume \( n_i = 2 \) or 3. Since \( d_i \geq 2 \), we have \( r \leq 3 \).

(1) In case \( r = 3 \), it is easy to see that \( d_i = n_i = 2 \) for \( i = 1, 2, 3 \). Then, \( G \cong Z_2^3 \).

(2) In case \( r = 2 \), we have

\[
d_1 \left(1 - \frac{1}{n_1}\right) + d_2 \left(1 - \frac{1}{n_2}\right) = 3.
\]

Then we have \( 5 \leq d_1 + d_2 \leq 6 \). We assume \( d_1 \geq d_2 \) and find the solutions. Here we use the notation \( (a, b; c, d) \), which means \( a = d_1, b = n_1 \) and \( c = d_2, d = n_2 \).

(b-1) In the case \( d_1 + d_2 = 6 \), we have \((4, 2; 2, 2)\) or \((3, 2; 3, 2)\).

(b-2) In the case \( d_1 + d_2 = 5 \), we have \((3, 3; 2, 2)\) or \((3, 2; 2, 4)\).

Claim 4.17. The case \( (3, 2; 2, 4) \) does not occur.

Proof. Suppose otherwise. Then, there exists a smooth surface \( S_2 = S/G_2 \), which is a double covering of \( S/G \cong \mathbb{P}^2 \) branched along \( \Delta_1 \). However, \( \text{deg} \Delta_1 \) is odd, hence the double covering cannot exist. This is a contradiction. □

Thus only the case \( n_1 = 3 \) and \( n_2 = 2 \) remains, which corresponds to \( G \cong Z_3 \times Z_2 \cong Z_6 \). Combining the results above, we complete the proof. The last assertion \( S \cong S_{(222)} \) will be proved in Theorem 5.12 below.

5. PARTICULARS

In this section we describe all the surfaces in Theorem 4.11. We study the Galois embeddings of \( S \) in detail for \( D^2 = 2m \), where \( m = 2, 3 \) and 4. Let \( C \) be a general member of the complete linear system \( |D| \). Then we have \( g(C) = m + 1 \), where \( g = g(C) \) is the genus of \( C \) and \( \text{dim} \, \mathcal{H}^0(S, \mathcal{O}(D)) = g + 1 \). So \( f_D(S) \) is assumed to be embedded in \( \mathbb{P}^g \).

CASE 1. \( g = 3 \)

Assume \( |G| = D^2 = 4 \). Then \( G \cong Z_4 \). Taking suitable homogeneous coordinates such that the Galois point be \( X_0 = X_1 = X_2 = 0 \), then \( \beta_1(\sigma) \), which is the projective transformation (3) defined in Section 2, can be expressed as a diagonal matrix \( [1, 1, 1, e_4] \). Since the defining equation of \( f_D(S) \) is invariant by this transformation, we infer readily the following.
**Theorem 5.1.** We have that \( g = 3 \) if and only if \( G \cong \mathbb{Z}_4 \). In this case the defining equation of \( f_D(S) \) can be given by \( X_3^4 + F_4(X_0, X_1, X_2) = 0 \), where \( F_4(X_0, X_1, X_2) \) is a form of degree four.

**Remark 5.2.** The maximal number of Galois points for the surface in Theorem 5.1 is four. And it is four if and only if it is the Fermat quartic (Example 3.4).

We can show a relation between the possibility of Galois embedding and the Picard number \( \rho(S) \) for \( S \).

**Lemma 5.3.** For the surface \( S \) in Theorem 5.1 we have \( \rho(S) \geq 2 \).

Before proceeding with the proof we note the following.

**Remark 5.4.** A smooth quartic plane curve \( \Delta \) has at least 16 bitangent lines.

**Proof.** Let \( \hat{\Delta} \) be the dual curve of \( \Delta \). Then we have \( \deg \hat{\Delta} = 12 \) and the genus of smooth model of \( \hat{\Delta} \) is 3. Let \( T_P \) be the tangent line to \( \Delta \) at \( P \). If the intersection number of \( T_P \) and \( \Delta \) at \( P \) is \( i + 2 \geq 3 \), then \( P \) is said to be an \( i \)-flex. Letting \( a_i \) be the number of \( i \)-flexes of \( \Delta \) \( (i = 1, 2) \), we see that \( \hat{\Delta} \) has \( a_1 \)-pieces of \((2, 3)\) cusps and \( a_2 \)-pieces of \((3, 4)\) cusps. Referring to [2, Theorem 6.11], we get \( a_1 + 2a_2 = 24 \). If \( b \) is the number of nodes of \( \hat{\Delta} \), then, applying the genus formula [2 Theorem 9.1], we get \( b \geq 16 \). Since a node of \( \hat{\Delta} \) corresponds to a bitangent line of \( \Delta \), the proof is complete. \( \square \)

Let \( \Delta \) be the branch locus of \( \pi : S \rightarrow \mathbb{P}^2 \), which is a smooth quartic curve. Let \( \ell \) be a bitangent line to \( \Delta \) and we consider \( \pi^*(\ell) \).

**Claim 5.5.** The curve \( \pi^*(\ell) = \Gamma \) is a sum of two \((-2)\)-curves.

**Proof.** Let \( P_1 \) and \( P_2 \) be \( \pi^{-1}(\ell \cap \Delta) \). Suppose \( \Gamma \) is irreducible. Then, it is not difficult to see by local consideration that it has two singular points \( P_i \) \( (i = 1, 2) \). Each \( P_i \) is locally isomorphic to the singularity defined by \( y^2 = x^4 \). Let \( \mu : \hat{\Gamma} \rightarrow \Gamma \) be the resolution of singularities. Then \( \pi|_{\hat{\Gamma}} : \hat{\Gamma} \rightarrow \ell \) is a cyclic Galois covering of degree 4. Then, by Riemann-Hurwitz formula we have \( 2g(\hat{\Gamma}) - 2 = 4(-2) + 4 = -4 \), which is a contradiction. Hence we have \( \pi^*(\ell) = \Gamma_1 + \Gamma_2 \), where \( \Gamma_i \) \( (i = 1, 2) \) is a \((-2)\)-curve and \( \Gamma_1, \Gamma_2 = 4 \).

From this claim Lemma 5.3 is clear.

**CASE 2.** \( g = 4 \)

In this case \( |G| = 6 \). So \( G \cong \mathbb{Z}_6 \) and put \( G = \langle \sigma \rangle \).

**Theorem 5.6.** If \( G \cong \mathbb{Z}_6 \), then \( f_D(S) \) is a \((2, 3)\)-complete intersection, furthermore the defining equation of \( f_D(S) \) can be given by \( F_2(X_0, X_1, X_2) + X_3^2 = F_3(X_0, X_1, X_2) + X_4^3 = 0 \), where \( F_i(X_0, X_1, X_2) \) is a form of degree \( i \) \( (i = 2, 3) \) such that each curve \( F_i \) is 0 in \( \mathbb{P}^2 \) has no singular points.

**Proof.** Since the embedding is given by \( |D| \), where \( D^2 = 6 \), the surface \( f_D(S) \) is the smooth complete intersection. In the proof of Theorem 4.3 in case \( |G| = 6 \), we have shown that \( d_1 = n_1 = 3 \) and \( d_2 = n_2 = 2 \). So that \( \sigma^2 \) (resp. \( \sigma^3 \)) is identity on \( C_1 \) (resp. \( C_2 \)). We have two covering maps \( f_i : S_i := S/(\sigma^i) \rightarrow \mathbb{P}^2 \), where \( i = 2 \) and 3. The \( f_i \) is a Galois covering of degree \( i \) branched along \( \Delta_i \). Put
gi : S → Si. Then we have f2g2 = f3g3. Since Δ2 and Δ3 have normal crossings, the fiber product S2 ×P2 S3 is smooth. Since S is also given by the double covering of S3 branched along g3(C2), we see that S is isomorphic to the fiber product S2 ×P2 S3. Furthermore, by taking a suitable coordinates on P2, we can assume S2 is defined by X32 + F2(X0, X1, X2) = 0 and S3 by X34 + F3(X0, X1, X2) = 0. This proves the theorem.

There is some relation between a Galois embedding and the trivilality of the symplectic group Gs. From the following Corollary 5.7 to Corollary 5.11 we do not assume that G is abelian.

Corollary 5.7. Suppose S has a Galois embedding. Then, Gs is trivial if and only if the embedding is given by a divisor D such that D2 = 4 or 6.

Proof. If Gs is trivial, then G is cyclic, hence G ∼= Z4 or Z6. Conversely, if G ∼= Z4 or Z6, then the defining ideal of fD(S) and the generator σ are given in Theorems 5.3 or 5.6. Referring to [5, Lemma 2.1], we conclude |Γm| = 4 or 6, where Γm is the cyclic group in (6). Thus Gs is trivial.

We consider the Picard number ρ(S) for the surfaces S in Theorem 5.1 and 5.6.

Lemma 5.8. If S is the surface in Theorem 5.7 then ρ(S) ≥ 2.

Proof. Let σ be a generator of G and consider S/(σ3), which is a rational surface containing (−1)-curve. In fact, it is a smooth cubic in P3. Then we infer readily that S has a (−2)-curve.

Let T_X be the transcendental lattice for a K3 surface X. Machida and Oguiso [4] prove the following:

Lemma 5.9. Let X be a K3 surface and G be a finite automorphism group of X. Assume that rank T_X ≥ 14. Then Gs = {1}, or equivalently, G ∼= Γm.

As we expect, a “general K3 surface” does not have a Galois embedding. Indeed, combining the results above, we deduce the following assertion.

Theorem 5.10. If ρ(S) = 1, then S has no Galois embeddings.

Corollary 5.11. If S has a Galois embedding and ρ(S) ≤ 8, then it is isomorphic to S(4) or S(23). Hence G ∼= Z4 or Z6.

Then, what can we say about a Galois embedding when ρ(S) is large? Can we say that S has the Galois embedding in the case where ρ(S) is the maximal possible 20?

CASE 3. g = 5
In this case |G| = 8. So G ∼= Z2^3.

Theorem 5.12. If G ∼= Z2^3, then S is a double covering of S(22), where S(22) is a rational surface of (2, 2)-complete intersection in P4. Furthermore we have the following sequence of surfaces:

\[ S \xrightarrow{\pi_1} S(22) \xrightarrow{\pi_2} S(2) \xrightarrow{\pi_3} P^2, \]

which have the following properties.
The author expresses his gratitude to Hiroyasu Tsuchihashi, who gave him useful information.

Problems.

(1) $\pi_i$ ($i = 1, 2, 3$) is a double covering and $\pi = \pi_3 \cdot \pi_2 \cdot \pi_1$.
(2) $S_{(22)}$ is a surface of $(2,2)$-complete intersection of $\mathbb{P}^4$.
(3) $S_{(2)}$ is a smooth conic in $\mathbb{P}^3$.

Further more, the defining equation of $f_D(S)$ can be given by $X_0^2 + F_{23}(X_0, X_1, X_2) = X_1^2 + F_{24}(X_0, X_1, X_2) = X_2^2 + F_{25}(X_0, X_1, X_2) = 0$, where $F_{2i}(X_0, X_1, X_2)$ is a form of $X_0, X_1, X_2$ with degree 2, such that each curve $F_{2i} = 0$ ($i = 3, 4, 5$) in $\mathbb{P}^2$ has no singular points. In particular $S$ is isomorphic to $S_{(222)}$.

Proof. The proof is done by the same way as the one of Theorem 5.9. In this case we have $d_i = n_i = 2$ ($i = 1, 2, 3$). Let $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ and $G_i = \langle \sigma_i \rangle$. Put $S_i = S/\langle \sigma_i \rangle$ and $S_{ij} = S/\langle \sigma_i, \sigma_j \rangle$, where $i \neq j$. Then $S$ is a double covering of $S_i$ and so is $S_i$ of $S_{ij}$, and $S_{ij}$ is a double covering of $\mathbb{P}^2$ branched along $\Delta_k$, where $i, j, k$ are mutually distinct. It is easy to see that $S_i$ is isomorphic to the fiber product $S_{ij} \times_{\mathbb{P}^2} S_{ik}$ and hence $S$ is isomorphic to $(S_{12} \times_{\mathbb{P}^2} S_{13}) \times_{\mathbb{P}^2} S_{23}$. In particular $S$ is isomorphic to $S_{(222)}$. \qed

Remark 5.13. In the case where $G$ is not abelian and $g = 4$ or $g = 5$, we can show that $G$ is isomorphic to the dihedral group. Furthermore such a K3 is obtained as a Galois closure surface of some rational surface. The research for non-abelian case will be done in the forthcoming paper.

There are a lot of problems concerning our theme, we pick up some of them.

Problems.

(1) How many Galois subspaces do there exist for one Galois embedding and how is their arrangement? In the case of a smooth quartic surface in $\mathbb{P}^3$, see Remark 5.2. Then, how is the case for $(2,3)$-complete intersection or $(2,2,2)$-complete intersection?
(2) Does there exist a K3 surface $S$ on which there exist two divisors $D_i$ ($i = 1, 2$) such that they give Galois embeddings and $D_1^2 \neq D_2^2$?
(3) Does each singular K3 surface have a Galois embedding?

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