New infinite families of near MDS codes holding $t$-designs

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Abstract

In “Infinite families of near MDS codes holding $t$-designs, IEEE Trans. Inform. Theory, 2020, 66(9), pp. 5419-5428”, Ding and Tang made a breakthrough in constructing the first two infinite families of NMDS codes holding 2-designs or 3-designs. Up to now, there are only a few known infinite families of NMDS codes holding $t$-designs in the literature. The objective of this paper is to construct new infinite families of NMDS codes holding $t$-designs. We determine the weight enumerators of the NMDS codes and prove that the NMDS codes hold 2-designs or 3-designs. Compared with known $t$-designs from NMDS codes, ours have different parameters. Besides, several infinite families of optimal locally recoverable codes are also derived via the NMDS codes.

Keywords: Linear code, weight enumerator, $t$-design

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1. Introduction

Let $q$ be a power of a prime $p$. Denote by $\mathbb{F}_q$ the finite field with $q$ elements and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$.

1.1. Linear codes

For a positive integer $n$, a non-empty subset $C$ of $\mathbb{F}_q^n$ is called an $[n, \kappa, d]$ linear code over $\mathbb{F}_q$ provided that it is a $\kappa$-dimensional linear subspace of $\mathbb{F}_q^n$, where $d$ is its minimum distance. Define the dual of an $[n, \kappa]$ linear code $C$ over $\mathbb{F}_q$ by

$$C^\perp = \{ u \in \mathbb{F}_q^n : \langle u, c \rangle = 0 \text{ for all } c \in C \},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product. By definition, $C^\perp$ is an $[n, n - \kappa]$ linear code over $\mathbb{F}_q$. For an $[n, \kappa]$ linear code $C$ over $\mathbb{F}_q$, let $A_i$ represent the number of codewords with weight $i$ in $C$, where $0 \leq i \leq n$. Define the weight enumerator of $C$ as the following polynomial:

$$A(z) = 1 + A_1 z + A_2 z^2 + \cdots + A_n z^n.$$

The weight distribution of $C$ is defined by the sequence $(A_0, A_1, \ldots, A_n)$. In recent years, the weight distribution of linear codes has been widely investigated in the literature [4, 5, 6, 7, 9, 10, 11].

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The weight distribution of a linear code contains crucial information including the error detection and correction capabilities of the code and allows to compute the error probability of its error detection and correction [17].

In coding theory, modifying existing codes may yield interesting new codes. A longer code can be constructed by adding a coordinate. Let \( C \) be an \([n, \kappa, d]\) linear code over \( \mathbb{F}_q \). The extended code \( \overline{C} \) of \( C \) is defined by

\[
\overline{C} = \left\{ (c_1, c_2, \ldots, c_n, c_{n+1}) \in \mathbb{F}_q^{n+1} : (c_1, c_2, \ldots, c_n) \in C \text{ with } \sum_{i=1}^{n+1} c_i = 0 \right\}.
\]

This construction is said to be adding an overall parity check [23]. Note that \( \overline{C} \) has only even-like vectors. Then \( \overline{C} \) is also a linear code over \( \mathbb{F}_q \) with parameters \([n+1, \kappa, d]\), where \( d = d \) or \( d + 1 \). For instance, the extended code of the binary [7, 4, 3] Hamming code has parameters [8, 4, 4]. The extension technique was used in [3, 33] to obtain desirable codes.

An \([n, \kappa, d]\) linear code is said to be good if it has both large rate \( \kappa/n \) and large minimum distance \( d \). However, there is a tradeoff among the parameters \( n, \kappa \) and \( d \). If an \([n, \kappa, d]\) linear code over \( \mathbb{F}_q \) exists, then the following Singleton bound holds:

\[ d \leq n - \kappa + 1. \]

Linear codes achieving the Singleton bound with parameters \([n, \kappa, n - \kappa + 1]\) are called maximum distance separable (MDS for short) codes. Linear codes nearly achieving the Singleton bound are also interesting and have attracted the attention of many researchers. An \([n, \kappa, n - \kappa]\) linear code is said to be almost maximum distance separable (AMDS for short). It is known that the dual of an AMDS code may not be AMDS. AMDS codes whose duals are also AMDS are said to be near maximum distance separable (NMDS for short). NMDS codes are of interest because they have many nice applications in finite geometry, combinatorial designs, locally recoverable codes and many other fields [5, 20, 21, 28, 29]. In general, constructing infinite families of NMDS codes with desirable weight distribution is challenging. In recent years, a few families of NMDS codes were constructed in [5, 20, 21, 29, 33, 34] and their weight distributions were determined.

### 1.2. Combinatorial designs from linear codes

Let \( k, t, n \) be positive integers such that \( 1 \leq t \leq k \leq n \). Let \( \mathcal{P} \) be a set with \( |\mathcal{P}| = n \geq 1 \). Denote by \( \mathcal{B} \) a collection of \( k \)-subsets of \( \mathcal{P} \). For each \( t \)-subset of \( \mathcal{P} \), if there exist exactly \( \lambda \) elements of \( \mathcal{B} \) such that they contain this \( t \)-subset, then the pair \( \mathbb{D} := (\mathcal{P}, \mathcal{B}) \) is referred to as a \( t-(n,k,\lambda) \) design (\( t \)-design for short). The elements in \( \mathcal{P} \) and \( \mathcal{B} \) are called points and blocks, respectively. A \( t \)-design is said to be simple if it contains no repeated blocks. A \( t \)-design with \( k = t \) or \( k = n \) is said to be trivial. In this paper, we are interested in only simple and nontrivial \( t \)-designs with \( n > k > t \). A \( t-(n,k,\lambda) \) design satisfying \( t \geq 2 \) and \( \lambda = 1 \) is called a Steiner system denoted by \( S(t,k,n) \). For a \( t-(n,k,\lambda) \) design, the following equality holds [23]:

\[
\binom{n}{t} \lambda = \binom{k}{t} b,
\]

where \( b \) is the number of blocks in \( \mathcal{B} \). Let \( \mathcal{B}^c \) represent the set of the complements of the blocks in \( \mathcal{B} \). Then \( (\mathcal{P}, \mathcal{B}^c) \) is a \( t-(n, n-k, \lambda_{0,t}) \) design if \((\mathcal{P}, \mathcal{B})\) is a \( t-(n,k,\lambda) \) design, where

\[
\lambda_{0,t} = \lambda \frac{n-t}{k-t}.  \tag{1}
\]
The pair \((\mathcal{P}, \mathcal{B}^c)\) is referred to as the complementary design of \((\mathcal{P}, \mathcal{B})\).

In past decades, the interplay between linear codes and \(t\)-designs has been a very interesting research topic. For one thing, the incidence matrix of a \(t\)-design yields a linear code. See \([2]\) for progress in this direction. For another thing, linear codes may hold \(t\)-designs. The well-known coding-theoretic construction of \(t\)-designs is described as follows. Let \(C\) be an \([n, \kappa, d]\) linear code over \(\mathbb{F}_q\) and the coordinates of a codeword in it be indexed by \((1, 2, \ldots, n)\). Denote by \(\mathcal{P}(C) = \{1, 2, \ldots, n\}\). For a codeword \(c = \{c_1, c_2, \ldots, c_n\} \in C\), define its support by

\[
\text{suppt}(c) = \{1 \leq i \leq n : c_i \neq 0\}.
\]

Denote by

\[
\mathcal{B}_w(C) = \frac{1}{q-1}\{\{\text{suppt}(c) : wt(c) = w \text{ and } c \in C\}\},
\]

where \(\{\{\}\}\) denotes the multiset notation, \(wt(c)\) is the Hamming weight of \(c\) and \(\frac{1}{q-1}S\) denotes the multiset obtained by dividing the multiplicity of each element in the multiset \(S\) by \(q - 1\) \([30]\). If the pair \((\mathcal{P}(C), \mathcal{B}_w(C))\) is a \(t\)-(\(n, w, \lambda\)) design with \(b\) blocks for \(0 \leq w \leq n\), we say that the code \(C\) supports \(t\)-designs, where

\[
b = \frac{1}{q-1}A_w, \quad \lambda = \frac{{w \choose t}}{(q-1){n \choose t}}A_w.
\]

(2)

The following Assmus-Mattson Theorem provides a sufficient condition for a linear code to hold \(t\)-designs.

**Theorem 1.** \([3]\) **Assmus-Mattson Theorem** \(n\) Let \(C\) be an \([n, \kappa, d]\) linear code over \(\mathbb{F}_q\) whose weight distribution is denoted by \((1, A_1, A_2, \ldots, A_n)\). Let \(d^\perp\) be the minimum weight of \(C^\perp\) whose weight distribution is denoted by \((1, A_1^\perp, A_2^\perp, \ldots, A_n^\perp)\). Let \(t\) be an integer satisfying \(1 \leq t < \min\{d, d^\perp\}\). Assume that there are at most \(d^\perp - t\) nonzero weights of \(C\) in the range \(\{1, 2, \ldots, n - t\}\). Then the followings hold:

1. \((\mathcal{P}(C), \mathcal{B}_t(C))\) is a simple \(t\)-design if \(A_i \neq 0\) with \(d \leq i \leq w\), where \(w\) is the largest integer satisfying \(w \leq n\) and

\[
w - \left\lfloor \frac{w + q - 2}{q - 1} \right\rfloor < d.
\]

2. \((\mathcal{P}(C^\perp), \mathcal{B}_t(C^\perp))\) is a simple \(t\)-design if \(A_i^\perp \neq 0\) with \(d^\perp \leq i \leq w^\perp\), where \(w^\perp\) is the largest integer satisfying \(w^\perp \leq n\) and

\[
w^\perp - \left\lfloor \frac{w^\perp + q - 2}{q - 1} \right\rfloor < d^\perp.
\]

The Assmus-Mattson Theorem is a powerful tool for constructing \(t\)-designs from linear codes and has been widely used in \([3, 6, 7, 8]\). Another method to prove that a linear code holds \(t\)-designs is via the automorphism group of the code. Several infinite families of \(t\)-designs were constructed via this method in the literature \([3, 6, 10, 11, 31, 35, 36]\). The third method is directly characterizing the supports of the codewords of fixed weight. See \([5, 29, 35]\) for known \(t\)-designs obtained by this method. Recently, Tang, Ding and Xiong generalized the Assmus-Mattson Theorem and derived \(t\)-designs from codes which don’t satisfy the conditions in the Assmus-Mattson Theorem and don’t admit \(t\)-homogeneous group as a subgroup of their automorphisms \([30]\).
1.3. Motivations and objectives of this work

Constructing \( t \)-designs from special NMDS codes has been an interesting research topic for a long time. The first NMDS code dates back to 1949. Golay discovered the \([11,6,5]\) ternary NMDS code which is called the ternary Golay code. This NMDS code holds 4-designs. In the past 70 years after this discovery, only sporadic NMDS codes holding \( t \)-designs were found. The question as to whether there exists an infinite family of NMDS codes holding \( t \)-designs remained open during this long period. In 2020, Ding and Tang made a breakthrough in constructing the first two infinite families of NMDS codes holding 2-designs or 3-designs \([5]\). Up to now, there are only a few known infinite families of NMDS codes holding \( t \)-designs for \( t = 2, 3, 4 \) in the literature \([5, 29, 34, 38]\). It is challenging to construct new infinite families of NMDS codes holding \( t \)-designs with \( t > 1 \).

The objective of this paper is to construct several new infinite families of NMDS codes holding \( t \)-designs. To this end, some special matrices over finite fields are used as the generator matrices of the NMDS codes. We then determine the weight enumerators of the NMDS codes and prove that the NMDS codes hold 2-designs or 3-designs. Most of the NMDS codes in this paper don’t satisfy the conditions in the Assmus-Mattson Theorem but still hold \( t \)-designs. Compared with known \( t \)-designs from NMDS codes, ours have different parameters. Besides, several infinite families of optimal locally recoverable codes are also derived via the NMDS codes.

2. Preliminaries

In this section, we present some preliminaries on the properties of NMDS codes and oval polynomials, and the number of zeros of some equations over finite fields.

2.1. Properties of NMDS codes

Let \( C \) be an \([n, \kappa]\) linear code and \( C^\perp \) be its dual. Denote by \((1,A_1,\ldots,A_n)\) and \((1,A_1^\perp,\ldots,A_n^\perp)\) the weight distributions of \( C \) and \( C^\perp \), respectively. If \( C \) is an NMDS code, then the weight distributions of \( C \) and \( C^\perp \) are given in the following lemma.

**Lemma 2.** \([3]\) Let \( C \) be an \([n, \kappa, n-K]\) NMDS code over \( \mathbb{F}_q \). If \( s \in \{1,2,\ldots,n-K\} \), then

\[
A_{\kappa+s}^\perp = \left( \frac{n}{\kappa+s} \right) \sum_{j=0}^{s-1} (-1)^j \left( \begin{array}{c} \kappa+s \\ j \\ \end{array} \right) (q^{s-j} - 1) + (-1)^s \left( \frac{n-K}{s} \right) A_{\kappa}^\perp.
\]

If \( s \in \{1,2,\ldots,\kappa\} \), then

\[
A_{n-K+s} = \left( \frac{n}{\kappa-s} \right) \sum_{j=0}^{s-1} (-1)^j \left( \begin{array}{c} n-K+s \\ j \\ \end{array} \right) (q^{s-j} - 1) + (-1)^s \left( \frac{\kappa}{s} \right) A_{n-K}.
\]

Though Lemma \(2\) and the relation \( 1 + \sum_{s=0}^{\kappa} A_{n-K+s} = q^\kappa \) hold, the weight distribution of an \([n, \kappa]\) NMDS code still cannot be totally determined. In \([20]\), some infinite families of NMDS codes with the same parameters but different weight distributions were constructed.

The following lemma establishes an interesting relationship between the minimum weight code-words in \( C \) and those in \( C^\perp \).
Lemma 3. [12] Let $C$ be an NMDS code. For any $c = (c_1, \ldots, c_n) \in C$, its support is defined by $\text{suppt}(c) = \{1 \leq i \leq n : c_i \neq 0\}$. Then for any minimum weight codeword $c$ in $C$, there exists, up to a multiple, a unique minimum weight codeword $c^\perp$ in $C^\perp$ satisfying $\text{suppt}(c) \cap \text{suppt}(c^\perp) = \emptyset$. Besides, the number of minimum weight codewords in $C$ and the number of those in $C^\perp$ are the same.

By Lemma 3, if the minimum weight codewords of an NMDS code hold a $t$-design, then the minimum weight codes of its dual hold a complementary $t$-design.

2.2. Oval polynomials and their properties

The definition of oval polynomial is presented as follows.

Definition 4. [25] Let $q = 2^m$ with $m \geq 2$. If $f \in \mathbb{F}_q[x]$ is a polynomial such that $f$ is a permutation polynomial of $\mathbb{F}_q$ with $\deg(f) < q$ and $f(0) = 0$, $f(1) = 1$, and $g_a(x) := (f(x + a) + f(a))x^{q-2}$ is also a permutation polynomial of $\mathbb{F}_q$ for each $a \in \mathbb{F}_q$, then $f$ is called an oval polynomial.

The following gives some known oval polynomials.

Lemma 5. [26] Let $m \geq 2$ be an integer. Then the followings are oval polynomials of $\mathbb{F}_q$, where $q = 2^m$.

1. The translation polynomial $f(x) = x^{2^h}$, where $\gcd(h, m) = 1$.
2. The Segre polynomial $f(x) = x^6$, where $m$ is odd.

By Lemma 5 it is obvious that $f(x) = x^4$ is an oval polynomial of $\mathbb{F}_q$, where $q = 2^m$ for odd $m$.

By Definition 4 the following also holds for oval polynomials.

Lemma 6. Let $q = 2^m$ with $m \geq 2$. Then $f$ is an oval polynomial of $\mathbb{F}_q$ if and only if the followings simultaneously hold:

1. $f$ is a permutation polynomial of $\mathbb{F}_q$ with $\deg(f) < q$ and $f(0) = 0$, $f(1) = 1$; and
2. $\frac{f(x) + f(y)}{x+y} \neq \frac{f(x) + f(z)}{x+z}$

for all pairwise distinct elements $x, y, z$ in $\mathbb{F}_q$.

2.3. The number of zeros of some equations over finite fields

Let $q = p^m$ with $p$ a prime. The following lemma is very useful for determining the greatest common divisor of some special integers.

Lemma 7. Let $h$ and $m$ be two integers with $\gcd(h, m) = \ell$. Then

$$\gcd(p^h + 1, q - 1) = \begin{cases} 1, & \text{for odd } \frac{m}{\ell} \text{ and } p = 2, \\ 2, & \text{for odd } \frac{m}{\ell} \text{ and odd } p, \\ p^\ell + 1, & \text{for even } \frac{m}{\ell}. \end{cases}$$

The following lemmas present some results on the number of solutions of some equations over $\mathbb{F}_q$. 

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Lemma 8. \[ \text{Let } n \text{ be a positive integer and } q \text{ a prime power such that } \gcd(n, q - 1) = s. \text{ Then } x^n - 1 \text{ has } s \text{ zeros in } \mathbb{F}_q. \]

Lemma 9. \[ \text{Proof of Lemma 4} \] Let \( h \) be a positive integer. Denote by \( N_g \) the number of zeros of \( g(x) = x^{p^h} + 1 + c \) in \( \mathbb{F}_q \), where \( c \in \mathbb{F}_q^* \). Then \( N_g = 0 \) if and only if \( -c \) is not a \( (p^h + 1) \)-th power in \( \mathbb{F}_q^* \). If \( -c \) is a \( (p^h + 1) \)-th power in \( \mathbb{F}_q^* \), then \( N_g = \gcd(p^h + 1, q - 1) \).

Lemma 10. \[ \text{[25]} \text{ The trace function from } \mathbb{F}_q \text{ to } \mathbb{F}_p \text{ is defined by } \]
\[
\Tr_{q/p}(x) = x + x^p + x^{p^2} + \cdots + x^{p^{m-1}}.
\]
Then for \( \alpha \in \mathbb{F}_q \), \( \Tr_{q/p}(\alpha) = 0 \) if and only if \( \alpha = \beta^p - \beta \) for some \( \beta \in \mathbb{F}_q \).

Lemma 11. \[ \text{Let } \mathbb{F}_q \text{ be a finite field of characteristic 2 and let } f(x) = ax^2 + bx + c \in \mathbb{F}_q[x] \text{ be a polynomial of degree 2. Then} \]
\[ \begin{align*}
1. & \text{ } f \text{ has exactly one root in } \mathbb{F}_q \text{ if and only if } b = 0; \\
2. & \text{ } f \text{ has exactly two roots in } \mathbb{F}_q \text{ if and only if } b \neq 0 \text{ and } \Tr_{q/2}(\frac{ac}{b^2}) = 0; \\
3. & \text{ } f \text{ has no root in } \mathbb{F}_q \text{ if and only if } b \neq 0 \text{ and } \Tr_{q/2}(\frac{ac}{b^2}) = 1.
\end{align*}
\]
Let \( m \) and \( h \) be positive integers and \( q = 2^m \). Now we consider the zeros of polynomials
\[ f(x) = ax^{2h} + bx + c, \quad a, b, c \in \mathbb{F}_q \]
and
\[ g(x) = ax^{2h} + bx^{2h} + c, \quad a, b, c \in \mathbb{F}_q \]
in \( \mathbb{F}_q \). If \( a, b, c \in \mathbb{F}_q^* \), then \( f(x) \) can be reduced to
\[ P_\gamma(x) = x^{2h} + x + \gamma, \]
by the substitution \( x \mapsto ux \), where \( u^{2h} = \frac{b}{a} \) and \( \gamma = \frac{c}{ax^{2h} + 1} \in \mathbb{F}_q^* \). If \( a, b, c \in \mathbb{F}_q^* \), then \( g(x) \) can be reduced to
\[ U_\ell(x) = x^{2h} + x^{2h} + \ell, \]
by the substitution \( x \mapsto vx \), where \( v = \frac{b}{a} \) and \( \ell = \frac{c}{ax^{2h} + 1} \in \mathbb{F}_q^* \).

Lemma 12. \[ \text{[12]} \text{ Let } N_\gamma \text{ denote the number of zeros of } P_\gamma(x) \text{ in } \mathbb{F}_q \text{ with } \gamma \in \mathbb{F}_q^*. \text{ If } \gcd(h, m) = 1, \text{ then } N_\gamma = 0, 1 \text{ or } 3. \]

Lemma 13. \[ \text{Let } h \text{ and } m \text{ be positive integers with } \gcd(h, m) = 1 \text{ and } q = 2^m. \text{ Denote by } N_f \text{ the number of zeros of } f(x) = ax^{2h} + bx + c \text{ in } \mathbb{F}_q^*, \text{ where } (a, b, c) \neq (0, 0, 0), a, b, c \in \mathbb{F}_q. \text{ Then } N_f \in \{0, 1, 3\}. \]

Proof. It is obvious that \( N_f \) is equal to 0 or 1 if \( a = 0 \) or \( b = c = 0 \). Now let \( a \neq 0 \). If \( b \neq 0 \) and \( c = 0 \), then \( f(x) = ax^{2h} + bx = ax(x^{2h} + \frac{b}{a}) \). Since \( q \) is even, it is clear that \( f(x) \) has only one zero in \( \mathbb{F}_q^* \). If \( b = 0 \) and \( c \neq 0 \), then \( f(x) = ax^{2h} + c = a(x^{2h} + \frac{c}{a}) \) and \( N_f \in \{0, 1, 3\} \) by Lemmas 7 and 9. If \( b \neq 0 \) and \( c \neq 0 \), by Lemma 12 \( N_f = N_\gamma \in \{0, 1, 3\} \). Then the desired conclusion follows. \qed
**Lemma 14.** Let $N_\ell$ denote the number of zeros of $U_\ell(x)$ in $\mathbb{F}_q^n$ with $\ell \in \mathbb{F}_q^*$. If $\gcd(h,m) = 1$, then $N_\ell$ is equal to 0, 1 or 3.

**Proof.** Let $x_0$ be a zero of $U_\ell(x)$ in $\mathbb{F}_q^n$, then it is easy to prove that $P_\ell(x_0 + 1) = U_\ell(x_0) = 0$. Let $x_0$ be a zero of $P_\ell(x)$ in $\mathbb{F}_q^n$, then we have $U_\ell(x_0 + 1) = P_\ell(x_0) = 0$. Then it is easy to deduce $N_\ell = N_q$. The desired conclusion follows from Lemma [12](#).

**Lemma 15.** Let $h$ and $m$ be positive integers with $\gcd(h,m) = 1$ and $q = 2^m$. Denote by $N_g$ the number of zeros of $g(x) = ax^{2^h+1} + bx^4 + c$ in $\mathbb{F}_q^n$, where $(a,b,c) \neq (0,0,0), a,b,c \in \mathbb{F}_q$. Then $N_g \in \{0, 1, 3\}$.

**Proof.** Similarly to the proof of Lemma [13](#), the desired conclusion follows from Lemmas [7](#) and [9](#) and [14](#).

**Lemma 16.** Let $q = 2^m$, where $m$ is an odd integer with $m \geq 3$. Then for two distinct elements $a, b \in \mathbb{F}_q$, the polynomial $u(x) = x^2 + (a + b)x + a^2 + b^2 + ab$ has no root in $\mathbb{F}_q$. In other words, for any $c \in \mathbb{F}_q$, we have $a^2 + b^2 + c^2 + ab + ac + bc \neq 0$. Particularly, if $c = 0$, then $a^2 + b^2 + ab \neq 0$.

**Proof.** It is obvious that

$$\text{Tr}_{q/2} \left( \frac{a^2 + b^2 + ab}{(a + b)^2} \right) = \text{Tr}_{q/2}(1) + \text{Tr}_{q/2} \left( \frac{ab}{a^2 + b^2} \right).$$

Let $\beta = \frac{a}{a+b} \in \mathbb{F}_q$. It is clear that $\frac{ab}{a^2 + b^2} = \beta^2 - \beta$. By Lemma [10](#),

$$\text{Tr}_{q/2} \left( \frac{ab}{a^2 + b^2} \right) = 0.$$

When $m$ is odd, $\text{Tr}_{q/2} \left( \frac{a^2 + b^2 + ab}{(a + b)^2} \right) = \text{Tr}_{q/2}(1) = 1$. Then by Lemma [11](#), $u(x)$ has no root in $\mathbb{F}_q$. The proof is completed.

### 2.4. Generalized Vandermonde determinant

The following provides a general equation for a generalized Vandermonde determinant with one deleted row in terms of the elementary symmetric polynomial.

**Lemma 17.** [22](#) Lemma 17] [24](#) Page 466] For each $\ell$ with $0 \leq \ell \leq n$, it holds that

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 & 1 \\
1 & u_1 & \cdots & u_{n-1} & u_n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \ell & \ell & \cdots & \ell \\
1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 \\
\end{vmatrix} = \left( \prod_{1 \leq i < j \leq n} (u_j - u_i) \right) \sigma_{n-\ell}(u_1, u_2, \ldots, u_n),
\]

where

$$\sigma_{n-\ell}(u_1, u_2, \ldots, u_n) = \sum_{1 \leq i_1 < \ldots < i_{n-\ell} \leq n} u_{i_1} \cdots u_{i_{n-\ell}}$$

is the $(n-\ell)$-th elementary symmetric polynomial over the set $\{u_1, u_2, \ldots, u_n\}$. 

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3. Two families of 3-dimensional near MDS codes holding 2-designs

In this section, let \( q = 2^m \) with \( m \geq 3 \). Hereafter, let \( \dim(C) \) and \( d(C) \) respectively denote the dimension and minimum distance of a linear code \( C \). Let \( \alpha \) be a generator of \( \mathbb{F}_q^* \) and \( \alpha_i := \alpha^i \) for \( 1 \leq i \leq q - 1 \). Then \( \alpha_{q-1} = 1 \).

Let \( h \) be a positive integer with \( \gcd(m, h) = 1 \). Define

\[
D = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{q-2} & \alpha_{q-1} \\
\alpha_1^{2h+1} & \alpha_2^{2h+1} & \cdots & \alpha_{q-2}^{2h+1} & \alpha_{q-1}^{2h+1}
\end{bmatrix}.
\]

\( D \) is a \( 3 \times (q-1) \) matrix over \( \mathbb{F}_q \). Let \( C_D \) be the linear code over \( \mathbb{F}_q \) generated by \( D \). We will show that \( C_D \) is an NMDS code and both \( C_D \) and its dual \( C_D^\perp \) support 2-designs.

**Theorem 18.** Let \( q = 2^m \) with \( m \geq 3 \), \( h \) be a positive integer with \( \gcd(m, h) = 1 \). Then \( C_D \) is a \([q - 1, 3, q - 4]\) NMDS code over \( \mathbb{F}_q \) with weight enumerator

\[
A(z) = 1 + \frac{(q-1)^2(q-2)}{6} z^{q-4} + \frac{(q-1)^2(q+4)}{2} z^{q-2} + \frac{(q-1)(q^2+8)}{3} z^{q-1}.
\]

Moreover, the minimum weight codewords of \( C_D \) support a \( 2-(q-1, q-4, \frac{(q-4)(q-5)}{6}) \) simple design and the minimum weight codewords of \( C_D^\perp \) support a \( 2-(q-1, 3, 1) \) simple design, i.e. a Steiner system \( S(2, 3, q-1) \). Furthermore, the codewords of weight 4 in \( C_D^\perp \) support a \( 2-(q-1, 4, \frac{(q-4)(q-7)}{2}) \) simple design.

**Proof.** We first prove that \( \dim(C_D) = 3 \). Let \( g_1, g_2 \) and \( g_3 \) respectively represent the first, second and third rows of \( D \). Assume that there exist elements \( a, b, c \in \mathbb{F}_q \) with \( (a, b, c) \neq (0, 0, 0) \) such that \( cg_1 + bg_2 + ag_3 = 0 \). Then

\[
\begin{align*}
& a\alpha_1^{2h+1} + b\alpha_2 + c = 0, \\
& a\alpha_2^{2h+1} + b\alpha_2 + c = 0, \\
& \vdots \\
& a\alpha_{q-1}^{2h+1} + b\alpha_{q-1} + c = 0.
\end{align*}
\]

This contradicts with the fact that the polynomial \( f(x) = ax^{2h+1} + bx + c \) has at most 3 zeros in \( \mathbb{F}_q^* \) by Lemma [13]. Hence \( g_1, g_2 \) and \( g_3 \) are linearly independent over \( \mathbb{F}_q \) and \( \dim(C_D) = 3 \).

We then prove that \( C_D^\perp \) has parameters \([q - 1, q - 4, 3]\). Obviously, \( \dim(C_D^\perp) = (q-1) - 3 = q - 4 \). It is clear that each column of \( D \) is nonzero and any two columns of \( D \) are linearly independent over \( \mathbb{F}_q \). Then \( d(C_D^\perp) > 2 \). Let \( x_1, x_2, x_3 \) be three pairwise different elements in \( \mathbb{F}_q^* \). Consider the following submatrix as

\[
D_1 = \begin{bmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
x_1^{2h+1} & x_2^{2h+1} & x_3^{2h+1}
\end{bmatrix}.
\]

Then \( D \) has 3 columns that are linearly dependent if and only if \( |D_1| = 0 \) for some \((x_1, x_2, x_3)\). Besides, \( \text{rank}(D_1) = 2 \) if \( |D_1| = 0 \). Now we consider the following two cases.
Case 1: Let $m$ be even. By Lemmas 7 and 8, the polynomial $x^{2^h+1} - 1$ has 3 zeros in $\mathbb{F}_q^*$ denoted by $r_1, r_2$ and $r_3$. Let $(x_1, x_2, x_3) = (r_1, r_2, r_3)$. Then

$$D_1 = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{bmatrix}.$$ 

Thus $|D_1| = 0$.

Case 2: Let $m$ be odd and $x_3 = \alpha_{q-1} = 1$. Then

$$D_1 = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & 1 \\ x_1^{2^h+1} & x_2^{2^h+1} & 1 \end{bmatrix}.$$ 

It is easy to deduce that $|D_1| = (1 + x_1)x_2^{2^h+1} + (1 + x_1^{2^h+1})x_2 + x_1 + x_1^{2^h+1}$. Denote by $f(x) = (1 + x_1)x_2^{2^h+1} + (1 + x_1^{2^h+1})x_2 + x_1 + x_1^{2^h+1}$. Note that $1 + x_1 \neq 0$ as $x_1 \neq x_3$. By Lemma 13, $f(x)$ has 0 or 1 or 3 zeros in $\mathbb{F}_q^*$. It is easy to verify that $f(1) = f(x_1) = 0$. Then exists an element $r \in \mathbb{F}_q^*$ which is different from 1 and $x_1$ such that $f(r) = 0$. Let $x_2 = r$ and we have $|D_1| = 0$.

Summarizing the above cases yields that $d(C_D^1) = 3$. Therefore, $C_D^1$ has parameters $[q-1, q-4, 3]$.

By definition, we have

$$C_D = \{ e_{a,b,c} = (ax^{2^h+1} + bx + c)_{x \in \mathbb{F}_q^*}, a, b, c \in \mathbb{F}_q \}.$$ 

To determine the weight $w(e_{a,b,c})$ of a codeword $e_{a,b,c} \in C_D$, it is sufficient to determine the number of zeros of the equation

$$ax^{2^h+1} + bx + c = 0$$

in $\mathbb{F}_q^*$. By Lemma 13, the above equation has 0 or 1 or 3 zeros in $\mathbb{F}_q^*$. Hence, $w(e_{a,b,c}) \in \{ q-1, q-2, q-4 \}$.

Finally, we compute the weight enumerator of $C_D$ by the first three Pless Power Moments in [23] and prove that $C_D$ is a $[q-1, 3, q-4]$ NMDS code. Let $A_{w_1}, A_{w_2}, A_{w_3}$ respectively represent the frequencies of the weights $w_1 = q-4, w_2 = q-2, w_3 = q-1$. Then we have

$$\begin{cases} A_{w_1} + A_{w_2} + A_{w_3} = q^3 - 1, \\
w_1A_{w_1} + w_2A_{w_2} + w_3A_{w_3} = q^2(q-1)^2, \\
w_1^2A_{w_1} + w_2^2A_{w_2} + w_3^2A_{w_3} = q(q-1)^2(q^2 - 2q + 2). \end{cases}$$

Solving the above system of linear equations gives

$$\frac{(q-1)^2(q-2)}{6}, A_{q-2} = \frac{(q-1)^2(q+4)}{2}, A_{q-1} = \frac{(q-1)(q^2+8)}{3}.$$ 

Thus $C_D$ is a $[q-1, 3, q-4]$ NMDS code and the weight enumerator of $C_D$ follows from Lemma 2. It then follows from the Assmus-Mattson Theorem in Theorem 11 and Equation 2 that the minimum weight codewords of $C_D$ support a $2\cdot(q-1, q-4, \frac{(q-4)(q-5)}{6})$ simple design, and the minimum weight codewords of $C_D^1$ support a $2\cdot(q-1, 3, 1)$ simple design.
We finally prove that the codewords of weight 4 in $C_D^\perp$ support a $2-(q-1,4,\frac{(q-4)(q-7)}{2})$ simple design. Thanks to a generalized version of the Assmus-Mattson Theorem (Theorem 2.2 in [30]), the codewords of weight 4 in $C_D^\perp$ support a 2-design. We need to prove that this design is simple. Let $x,y,z$ be three pairwise distinct elements in $\mathbb{F}_q^*$. Define

$$D_2 = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^{h+1} & y^{h+1} & z^{h+1} \end{bmatrix}.$$ 

It is obvious that

$$|D_2| = x^{h+1}(x+y) + z(x^{h+1} + y^{h+1}) + xy^{h+1} + yx^{h+1}.$$ 

Let $f(z) = x^{h+1}(x+y) + z(x^{h+1} + y^{h+1}) + xy^{h+1} + yx^{h+1}$. Note that $f(x) = f(y) = 0$. By Lemma [13], $f(z)$ has 0 or 1 or 3 zeros in $\mathbb{F}_q^*$. Thus there exists an element $r_{(x,y)} \in \mathbb{F}_q^*$ which is different from $x$ and $y$ such that $f(r_{(x,y)}) = 0$. Then we have rank$(D_2) = 3$ if and only if $z \not\in \{x,y,r_{(x,y)}\}$. Next we prove that the rank of the submatrix

$$D_{(x,y,z,w)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & y & z & w \\ x^{h+1} & y^{h+1} & z^{h+1} & w^{h+1} \end{bmatrix}$$

equals 3 for any four pairwise distinct elements $x,y,z,w \in \mathbb{F}_q^*$. It is obvious that at least one of $z,w$ is not equal to $r_{(x,y)}$, which implies $D_{(x,y,z,w)}$ has a three-order non-zero minor. Thus

$$\text{rank}(D_{(x,y,z,w)}) = \text{rank}(D_2) = 3.$$ 

Let $c = (c_1,c_2,\ldots,c_{q-1})$ be a codeword of weight 4 in $C_D^\perp$ with nonzero coordinates in $\{i_1,i_2,i_3,i_4\}$, which means $c_{i_j} \neq 0$ for $1 \leq j \leq 4$ and $c_v = 0$ for all $v \in \{1,2,\ldots,q-1\} \setminus \{i_1,i_2,i_3,i_4\}$. Since $D$ is a parity-check matrix of $C_D^\perp$, there exist four pairwise distinct elements $x,y,z,w \in \mathbb{F}_q^*$ such that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x & y & z & w \\ x^{h+1} & y^{h+1} & z^{h+1} & w^{h+1} \end{bmatrix} \begin{bmatrix} c_{i_1} \\ c_{i_2} \\ c_{i_3} \\ c_{i_4} \end{bmatrix} = 0.$$ 

Since $\text{rank}(D_{(x,y,z,w)}) = 3$, the all nonzero solutions of the above equation are $\{a(c_{i_1},c_{i_2},c_{i_3},c_{i_4}) : a \in \mathbb{F}_q^*\}$. Thus $\{ac : a \in \mathbb{F}_q^*\}$ is a set of all codewords of weight 4 in $C_D^\perp$ whose nonzero coordinates are $\{i_1,i_2,i_3,i_4\}$. Hence, the codewords of weight 4 in $C_D^\perp$ support a $2-(q-1,4,\lambda)$ simple design. Since $C_D^\perp$ is an NMDS code, we have $A_4 = \frac{(q-1)^2(q-2)(q-4)(q-7)}{24}$ by Lemma [2]. By Equation (2), we have $\lambda = \frac{(q-4)(q-7)}{2}$.

Then we have completed the proof. \hfill \Box

Let $h$ be a positive integer with $\gcd(m,h) = 1$. Define

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_{1}^{2h} & \alpha_{2}^{2h} & \cdots & \alpha_{q-2}^{2h} & \alpha_{q-1}^{2h} \\ \alpha_{1}^{2h+1} & \alpha_{2}^{2h+1} & \cdots & \alpha_{q-2}^{2h+1} & \alpha_{q-1}^{2h+1} \end{bmatrix}.$$ 

$H$ is a 3 by $q-1$ matrix over $\mathbb{F}_q$. Let $C_H$ be the linear code over $\mathbb{F}_q$ generated by $H$. We will show that $C_H$ is an NMDS code and both $C_H$ and its dual $C_H^\perp$ support 2-designs.
Theorem 19. Let \( q = 2^m \) with \( m \geq 3 \), \( h \) be a positive integer with \( \gcd(m,h) = 1 \). Then \( C_H \) is a \([q-1,3,q-4]\) NMDS code over \( \mathbb{F}_q \) with weight enumerator

\[
A(z) = 1 + \frac{(q-1)^2(q-2)}{6}z^{q-4} + \frac{(q-1)(q+4)}{2}z^{q-2} + \frac{(q-1)(q^2+8)}{3}z^{q-1}.
\]

Moreover, the minimum weight codewords of \( C_H \) support a 2-\((q-1,q-4,\frac{(q-4)(q-5)}{6})\) simple design and the minimum weight codewords of \( C_H^\perp \) support a 2-\((q-1,3,1)\) simple design, i.e. a Steiner system \( S(2,3,q-1) \). Furthermore, the codewords of weight 4 in \( C_H^\perp \) support a 2-\((q-1,4,\frac{(q-4)(q-7)}{2})\) simple design.

Proof. Similarly to the proof of Theorem 18, we can easily derive this theorem by Equation (2), Lemmas 2 and 15.

Note that \( C_D \) and \( C_H \) have the same parameters and weight enumerator. It is open whether they are equivalent to each other.

4. Two families of 4-dimensional near MDS codes holding 2-designs

In this section, let \( q = 2^m \) with \( m \geq 3 \). Let \( \alpha \) be a generator of \( \mathbb{F}_q^* \) and \( \alpha_i := \alpha^i \) for \( 1 \leq i \leq q-1 \). Then \( \alpha_{q-1} = 1 \). Define a 4 by \( q-1 \) matrix over \( \mathbb{F}_q \) by

\[
G_{(i,j)} = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
\alpha_i & \alpha_j & \cdots & \alpha_{q-2}^j & \alpha_{q-1}^j \\
\alpha_i & \alpha_j & \cdots & \alpha_{q-2}^j & \alpha_{q-1}^j \\
\alpha_i & \alpha_j & \cdots & \alpha_{q-2}^j & \alpha_{q-1}^j
\end{bmatrix},
\]

where \( (i,j) = (1,3) \) or \( (2,3) \). Let \( C_{(i,j)} \) be the linear code over \( \mathbb{F}_q \) generated by \( G_{(i,j)} \). We will prove that \( C_{(i,j)} \) is an NMDS code and the minimum weight codewords of \( C_{(i,j)} \) and its dual \( C_{(i,j)}^\perp \) support 2-designs.

4.1. When \((i,j) = (1,3)\)

The following lemma plays an important role in the proof of our main result.

Lemma 20. Let \( m \) be an odd integer with \( m \geq 3 \), \( q = 2^m \). Let \( x_1, x_2, x_3, x_4 \) be four pairwise distinct elements in \( \mathbb{F}_q^* \) and we define the matrix

\[
M_{(1,3)} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 \\
x_1^3 & x_2^3 & x_3^3 & x_4^3 \\
x_1^4 & x_2^4 & x_3^4 & x_4^4
\end{bmatrix}.
\]

Then for any two different and fixed elements \( x_1, x_2 \), the total number of different choices of \( x_3, x_4 \) such that \( |M_{(1,3)}| = 0 \) is equal to \( \frac{q-8}{2} \) (regardless of the ordering of \( x_3, x_4 \)).
Proof. By Lemma \([17]\), \(|M_{(1,3)}| = 0\) if and only if \(x_1x_2 + x_1x_3 + x_2x_3 + (x_1 + x_2 + x_3)x_4 = 0\). Then we first need to consider whether \(x_1 + x_2 + x_3\) equals 0 or not in the following cases.

Case 1: Let \(x_1 + x_2 + x_3 = 0\). Then \(x_3 = x_1 + x_2\) and

\[
|M_{(1,3)}| = \prod_{1 \leq i < j < 4} (x_j - x_i)(x_j^2 + x_1x_2) \neq 0
\]

by Lemma \([16]\). So there is no \((x_3, x_4)\) such that \(|M_{(1,3)}| = 0\) in this case.

Case 2: Let \(x_1 + x_2 + x_3 \neq 0\). Then \(x_3 \neq x_1 + x_2\) and

\[
|M_{(1,3)}| = 0 \iff x_1x_2 + x_1x_3 + x_2x_3 + (x_1 + x_2 + x_3)x_4 = 0 \iff x_4 = \frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1 + x_2 + x_3}.
\]

Since \(x_1, x_2, x_3, x_4\) are four pairwise distinct elements in \(\mathbb{F}_q\), we have \(x_4 \notin \{0, x_1, x_2, x_3\}\). Note that

\[
x_4 \neq 0 \iff \frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1 + x_2 + x_3} \neq 0 \iff x_1x_2 + x_1x_3 + x_2x_3 \neq 0 \iff x_3 \neq \frac{x_1x_2}{x_1 + x_2}.
\]

Similarly, \(x_4 \notin \{x_1, x_2, x_3\}\) if and only if \(x_3 \notin \{\frac{x_1^2}{x_1}, \frac{x_2^2}{x_2}, a\}\), where \(a^2 = x_1x_2\). We conclude that

\[
|M_{(1,3)}| = 0 \text{ if and only if } x_3 \notin \left\{0, x_1, x_2, x_1 + x_2, \frac{x_1x_2}{x_1 + x_2}, \frac{x_1^2}{x_1}, \frac{x_2^2}{x_2}, a\right\},
\]

where \(a^2 = x_1x_2\), and

\[
x_4 = \frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1 + x_2 + x_3}.
\]

By Lemmas \([8]\) and \([16]\) it is easy to prove that the elements in

\[
\left\{0, x_1, x_2, x_1 + x_2, \frac{x_1x_2}{x_1 + x_2}, \frac{x_1^2}{x_1}, \frac{x_2^2}{x_2}, a\right\}
\]

are pairwise distinct. It is obvious that if \((x_3, x_4)\) is a choice, so is \((x_4, x_3)\). Hence the total number of different choices of \(x_3, x_4 \in \mathbb{F}_q\) such that \(|M_{(1,3)}| = 0\) is equal to \(\frac{q^{2}}{2}\) for any two different fixed elements \(x_1, x_2\).

The proof is completed. \(\square\)

Theorem 21. Let \(m\) be an odd integer with \(m > 3\), \(q = 2^m\). Then \(C_{(1,3)}\) is a \([q - 1, 4, q - 5]\) NMDS code over \(\mathbb{F}_q\) with weight enumerator

\[
A(z) = 1 + \frac{(q - 1)^2(q - 2)(q - 8)}{24}z^{q-5} + \frac{5(q - 1)^2(q - 2)}{6}z^{q-4} + \frac{q(q - 1)^2(q - 2)}{4}z^{q-3}
\]

\[
+ \frac{(q - 1)^2(2q^2 + 7q + 20)}{6}z^{q-2} + \frac{(q - 1)(9q^2 - 14q^2 - 6q + 80)}{24}z^{q-1}.
\]

Moreover, the minimum weight codewords of \(C_{(1,3)}\) support a \(2-(q - 1, q - 5, \frac{(q-5)(q-6)(q-8)}{24})\) simple design and the minimum weight codewords of \(C_{(1,3)}^\perp\) support a \(2-(q - 1, 4, \frac{q^8-8}{2})\) simple design.
Proof. We first prove that \( \dim(C_{1,3}) = 4 \). Let \( g_1, g_2, g_3 \) and \( g_4 \) respectively represent the first, second, third and fourth rows of \( G_{1,3} \). Assume that there exist elements \( a, b, c, d \in \mathbb{F}_q \) with \( (a, b, c, d) \neq (0, 0, 0, 0) \) such that \( ag_1 + bg_2 + cg_3 + dg_4 = 0 \). Then

\[
\begin{aligned}
& a + b\alpha_1 + c\alpha_3^2 + d\alpha_4^2 = 0, \\
& a + b\alpha_2 + c\alpha_3^2 + d\alpha_4^2 = 0, \\
& \vdots \\
& a + b\alpha_{q-1} + c\alpha_{q-1}^2 + d\alpha_{q-1}^2 = 0.
\end{aligned}
\]

Obviously, the polynomial \( f(x) = a + bx + cx^3 + dx^4 \) has at most 4 zeros in \( \mathbb{F}_q^* \), which leads to a contradiction. Hence \( g_1, g_2, g_3 \) and \( g_4 \) are linearly independent over \( \mathbb{F}_q \). Thus \( \dim(C_{1,3}) = 4 \).

We then prove that \( C_{1,3}^\perp \) has parameters \([q - 1, q - 5, 4]\). Obviously, \( \dim(C_{1,3}^\perp) = (q - 1) - 4 = q - 5 \). We need to prove \( d(C_{1,3}^\perp) = 4 \). It is sufficient to prove that any 3 columns of \( G_{1,3} \) are linearly independent and there exist 4 columns of \( G_{1,3} \) that are linearly dependent. Choosing any three columns from \( G_{1,3} \) yields the submatrix

\[
M_{1,1} = \begin{bmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
x_1^3 & x_2^3 & x_3^3 \\
x_1^4 & x_2^4 & x_3^4
\end{bmatrix},
\]

where \( x_1, x_2, x_3 \) are pairwise distinct elements in \( \mathbb{F}_q^* \). Consider the 3 by 3 submatrix of \( M_{1,1} \) as

\[
M_{1,2} = \begin{bmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
x_1^3 & x_2^3 & x_3^3 \\
x_1^4 & x_2^4 & x_3^4
\end{bmatrix}.
\]

Denote by \( f(x) = x^4 \). By Lemma \( \Box \) we have \( |M_{1,2}| = (x_2 + x_1)(f(x_3) + f(x_1)) + (x_3 + x_1)(f(x_2) + f(x_1)) \neq 0 \). Then \( \text{rank}(M_{1,2}) = 3 \) and any 3 columns of \( G_{1,3} \) are linearly independent. Now we consider the submatrix \( M_{1,3} \) of \( G_{1,3} \) in Equation \( \Box \). By the proof of Lemma \( \Box \) there is proper \( \{x_1, x_2, x_3, x_4\} \) such that \( \text{det}(M_{1,3}) = 0 \). This shows that there exist 4 columns of \( G_{1,3} \) that are linearly dependent. To sum up, \( d(C_{1,3}^\perp) = 4 \). Let \( e = (c_1, c_2, \ldots, c_{q-1}) \in C_{1,3}^\perp \) and \( \text{wt}(e) = 4 \). Assume that \( c_{ij} = r_{ij} \in \mathbb{F}_q^* \), \( 1 \leq j \leq 4 \), and \( c_v = 0 \) for all \( v \in \{1, 2, \ldots, q - 1\} \setminus \{i_1, i_2, i_3, i_4\} \), i.e., \( \text{supp}(e) = \{i_1, i_2, i_3, i_4\} \). Set \( x_j = \alpha^{i_j} \). By definition,

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 \\
x_1^3 & x_2^3 & x_3^3 & x_4^3 \\
x_1^4 & x_2^4 & x_3^4 & x_4^4
\end{bmatrix}
\begin{bmatrix}
r_{i_1} \\
r_{i_2} \\
r_{i_3} \\
r_{i_4}
\end{bmatrix} = 0.
\]

Since \( \text{rank}(M_{1,3}) = 3 \), the number of solutions with \( \{r_{i_1}, r_{i_2}, r_{i_3}, r_{i_4}\} \in (\mathbb{F}_q^*)^4 \) is \( q - 1 \). Then \( \{ae : a \in \mathbb{F}_q^*\} \) is the set of all codewords of weight 4 in \( C_{1,3}^\perp \) whose nonzero coordinates are \( \{i_1, i_2, i_3, i_4\} \). Therefore, every codeword of weight 4 and its nonzero multiples in \( C_{1,3}^\perp \) with nonzero coordinates \( \{i_1, i_2, i_3, i_4\} \) must correspond to the set \( \{x_1, x_2, x_3, x_4\} \). By Lemma \( \Box \) the number of choices of
$x_3, x_4$ is independent of $x_1, x_2$. We then deduce that the codewords of weight 4 in $C_{(1,3)}$ support a $2-(q - 1, 4, \frac{q - 8}{2})$ design. Then by Equation (2),

$$A_4 = (q - 1) \binom{q - 1}{2} - \binom{q - 1}{2} = \frac{(q - 1)^2(q - 2)(q - 8)}{2}.$$ 

We next prove $d(C_{(1,3)}) = q - 5$. By definition, we have

$$C_{(1,3)} = \{c_{a,b,c,d} = (a + bx + cx^3 + dx^4)_{x \in \mathbb{F}_q}, a, b, c, d \in \mathbb{F}_q\}.$$ 

To determine the weight $\text{wt}(c_{a,b,c,d})$ of a codeword $c_{a,b,c,d} \in C_{(1,3)}$, it is sufficient to determine the number of zeros of the equation

$$a + bx + cx^3 + dx^4 = 0$$

in $\mathbb{F}_q^*$. The above equation has at most 4 zeros in $\mathbb{F}_q^*$. Hence, $d(C_{(1,3)}) \geq q - 5$. By the Singleton bound, $d(C_{(1,3)}) \leq q - 4$. Then $d(C_{(1,3)}) = q - 4$ or $q - 5$. If $d(C_{(1,3)}) = q - 4$, then $C_{(1,3)}$ is an MDS code. Then $C_{(1,3)}$ is also MDS, which leads to a contradiction. Therefore, $C_{(1,3)}$ is a $[q - 1, 4, q - 5]$ NMDS code. By Lemma 3,

$$A_{q-5} = \frac{(q - 1)^2(q - 2)(q - 8)}{24}.$$ 

Then by Lemma 3 and Equation (2), the minimum weight codewords of $C_{(1,3)}$ support a $2-(q - 1, 4, q - 5, \frac{(q - 5)(q - 6)(q - 8)}{24})$ simple design. Finally, the weight enumerator of $C_{(1,3)}$ follows from Lemma 2. 

4.2. When $(i, j) = (2, 3)$

In this subsection, we consider the case for $(i, j) = (2, 3)$. We will need the following lemma in the proof of our main result.

**Lemma 22.** Let $m$ be an integer with $m \geq 3$, $q = 2^m$. Let $x_1, x_2, x_3, x_4$ be four pairwise distinct elements in $\mathbb{F}_q^*$. Define the matrix

$$M_{(2,3)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 \end{bmatrix}.$$ 

Then for any different fixed elements $x_1, x_2$, the total number of different choices of $x_3, x_4$ such that $|M_{(2,3)}| = 0$ is equal to $2^{q - 4}$. 

**Proof.** Similarly to the proof of Lemma 20 we can easily derive this lemma by Lemma 17. 

**Theorem 23.** Let $m$ be an integer with $m \geq 3$, $q = 2^m$. Then $C_{(2,3)}$ is a $[q - 1, 4, q - 5]$ NMDS code over $\mathbb{F}_q$ with weight enumerator

$$A(z) = 1 + \frac{(q - 1)^2(q - 2)(q - 4)}{24}z^{q - 5} + \frac{(q - 1)^2(q - 2)}{6}z^{q - 4} + \frac{(q - 1)^2(q - 2)(q + 4)}{4}z^{q - 3} + \frac{(q - 1)^2(2q^2 + 3q + 28)}{6}z^{q - 2} + \frac{(q - 1)(9q^3 + 17q^2 - 18q + 88)}{24}z^{q - 1}.$$ 

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Moreover, the minimum weight codewords of $C_{(2,3)}$ support a $2-(q - 1, q - 5, \frac{(q - 4)(q - 5)(q - 6)}{24})$ simple design and the minimum weight codewords of $C_{(2,3)}^\perp$ support a $2-(q - 1, q - 4, \frac{q - 4}{2})$ simple design.

Proof. Similarly to the proof of Theorem 21 we can prove this theorem by Lemma 22.

Note that Theorem 23 works for any $m \geq 3$ as Lemma 22 dose not rely on Lemma 16.

5. Five families of 5-dimensional near MDS codes holding 2-designs or 3-designs

In this section, let $q = 2^m$ with $m > 3$. Let $\alpha$ be a generator of $\mathbb{F}_q^*$ and $\alpha_i := \alpha^i$ for $1 \leq i \leq q - 1$. Then $\alpha_{q-1} = 1$. Define

$$G_{(i,j,k)} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha_1^i & \alpha_2^i & \cdots & \alpha_{q-2}^i & \alpha_{q-1}^i \\ \alpha_1^j & \alpha_2^j & \cdots & \alpha_{q-2}^j & \alpha_{q-1}^j \\ \alpha_1^k & \alpha_2^k & \cdots & \alpha_{q-2}^k & \alpha_{q-1}^k \\ \alpha_1^5 & \alpha_2^5 & \cdots & \alpha_{q-2}^5 & \alpha_{q-1}^5 \end{bmatrix},$$

where $(i,j,k) = (2,3,4), (1,2,3), (1,2,4)$ or $(1,3,4)$. Let $C_{(i,j,k)}$ be the linear code over $\mathbb{F}_q$ generated by $G_{(i,j,k)}$. We will show that $C_{(i,j,k)}$ is an NMDS code and the minimum weight codewords of $C_{(i,j,k)}$ and its dual $C_{(i,j,k)}^\perp$ support 2-designs. Besides, we denote by $\overline{C}_{(1,2,4)}$ the extended code of $C_{(1,2,4)}$. We will also prove that the minimum weight codewords of $\overline{C}_{(1,2,4)}$ and its dual $\overline{C}_{(1,2,4)}^\perp$ support 3-designs.

5.1. When $(i,j,k) = (2,3,4)$

Let $(i,j,k) = (2,3,4)$. We study the linear code $C_{(2,3,4)}$ in this subsection.

The following lemma plays an important role in the proof of our result.

Lemma 24. Let $m$ be a positive integer with $m > 3$ and $q = 2^m$. Let $x_1, x_2, x_3, x_4, x_5$ be five pairwise distinct elements in $\mathbb{F}_q^*$. Define the matrix

$$M_{(2,3,4)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 \\ x_1^5 & x_2^5 & x_3^5 & x_4^5 & x_5^5 \end{bmatrix}.$$

Then for any two different and fixed elements $x_1, x_2$, the total number of different choices of $(x_3, x_4, x_5)$ such that $|M_{(2,3,4)}| = 0$ is equal to $\frac{(q - 4)(q - 8)}{6}$ (regardless of the ordering of $x_3, x_4, x_5$).

Proof. By Lemma 17 $|M_{(2,3,4)}| = 0$ if and only if

$$x_1 x_2 x_3 x_4 + (x_1 x_2 x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3) x_4 x_5 = 0.$$

Let $x_1, x_2$ be two different and fixed elements in $\mathbb{F}_q^*$. Consider the following cases.
**Case 1:** Let \( x_3 = \frac{x_1 x_2}{x_1 + x_2} \), then \( x_1 x_2 + x_1 x_3 + x_2 x_3 = 0 \) and \( x_1 x_2 x_3 + (x_1 x_2 + x_1 x_3 + x_2 x_3) x_4 = x_1 x_2 x_3 \neq 0 \). Then
\[
|M_{(2,3,4)}| = \prod_{1 \leq i < j \leq 5} (x_j - x_i)(x_1 x_2 x_3 (x_4 + x_5)) \neq 0.
\]
So there is no \((x_3, x_4, x_5)\) such that \( |M_{(2,3,4)}| = 0 \) in this case.

**Case 2:** Let \( x_3 \neq \frac{x_1 x_2}{x_1 + x_2} \) and \( x_1 x_2 x_3 + (x_1 x_2 + x_1 x_3 + x_2 x_3) x_4 = 0 \) which implies \( x_4 = \frac{x_1 x_2 x_3}{x_1 x_2 + x_1 x_3 + x_2 x_3} \). Then
\[
|M_{(2,3,4)}| = \prod_{1 \leq i < j \leq 5} (x_j - x_i)(x_1 x_2 x_3 x_4) \neq 0.
\]
So there is no \((x_3, x_4, x_5)\) such that \( |M_{(2,3,4)}| = 0 \) in this case.

**Case 3:** Let \( x_3 \neq \frac{x_1 x_2}{x_1 + x_2} \) and \( x_1 x_2 x_3 + (x_1 x_2 + x_1 x_3 + x_2 x_3) x_4 \neq 0 \) which implies \( x_4 \neq \frac{x_1 x_2 x_3}{x_1 x_2 + x_1 x_3 + x_2 x_3} \). Then
\[
|M_{(2,3,4)}| = 0 \iff x_5 = \frac{x_1 x_2 x_3 x_4}{x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4}.
\]
Since \( x_1, x_2, x_3, x_4, x_5 \) are five pairwise distinct elements in \( \mathbb{F}_q^n \), then \( x_5 \notin \{0, x_1, x_2, x_3, x_4\} \). Similarly to the proof in Lemma [20] we can derive that \( x_4 \notin \{\frac{x_2 x_3}{x_2 + x_3}, \frac{x_1 x_3}{x_1 + x_3}, \frac{x_1 x_2}{x_1 + x_2}\} \). We then conclude that \( |M_{(2,3,4)}| = 0 \) if and only if
\[
x_3 \notin \left\{0, x_1, x_2, \frac{x_1 x_2}{x_1 + x_2}\right\},
\]
\[
x_4 \notin \left\{0, x_1, x_2, x_3, \frac{x_2 x_3}{x_2 + x_3}, \frac{x_1 x_3}{x_1 + x_3}, \frac{x_1 x_2}{x_1 + x_2}, \frac{x_1 x_2 x_3}{x_1 x_2 + x_1 x_3 + x_2 x_3}\right\}
\]
and
\[
x_5 = \frac{x_1 x_2 x_3 x_4}{x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4}.
\]
It is easy to prove that the elements in
\[
\left\{0, x_1, x_2, \frac{x_1 x_2}{x_1 + x_2}\right\}
\]
are pairwise distinct, so is
\[
\left\{0, x_1, x_2, x_3, \frac{x_2 x_3}{x_2 + x_3}, \frac{x_1 x_3}{x_1 + x_3}, \frac{x_1 x_2}{x_1 + x_2}, \frac{x_1 x_2 x_3}{x_1 x_2 + x_1 x_3 + x_2 x_3}\right\}.
\]
Then the total number of different choices of \((x_3, x_4, x_5)\) such that \( |M_{(2,3,4)}| = 0 \) is equal to \( \frac{(q-4)(q-8)}{6} \) regardless of the ordering of \( x_3, x_4, x_5 \).

The desired conclusion follows.

**Theorem 25.** Let \( m \) be a positive integer with \( m > 3 \) and \( q = 2^m \). Then \( C_{(2,3,4)} \) is a \([q - 1, 5, q - 6]\) NMDS code over \( \mathbb{F}_q \) with weight enumerator
\[
A(z) = 1 + \frac{(q-1)^2(q-2)(q-4)(q-8)}{120} z^{q-6} + \frac{5(q-1)^2(q-2)(q-4)}{24} z^{q-5} + \frac{(q-1)^2(q-2)(q^2-2q+2)}{12} z^{q-4} + \frac{(q-1)^2(q-2)(2q^2+9q+28)}{12} z^{q-3} + \ldots
\].
Lemma 26.\ Let $m$ be a positive integer with $m > 3$ and $q = 2^m$. Let $x_1, x_2, x_3, x_4, x_5$ be five pairwise distinct elements in $\mathbb{F}_q^*$. Define a matrix

$$M_{(1,2,3)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 \end{bmatrix}.$$\

Then for any two different and fixed elements $x_1, x_2$, the total number of different choices of $(x_3, x_4, x_5)$ such that $|M_{(1,2,3)}| = 0$ is equal to $\frac{(q-4)(q-8)}{6}$ (regardless of the ordering of $x_3, x_4, x_5$).

**Proof.** Let $x_1, x_2$ be two different and fixed elements in $\mathbb{F}_q^*$. By Lemma 17, $|M_{(1,2,3)}| = 0$ if and only if $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Then

$$|M_{(1,2,3)}| = 0 \iff x_5 = x_1 + x_2 + x_3 + x_4.$$\

Since $x_5 \notin \{0, x_1, x_2, x_3, x_4\}$, we deduce that $x_4 \notin \{x_2 + x_3, x_1 + x_3, x_1 + x_2, x_1 + x_2 + x_3\}$ and $x_3 \neq x_1 + x_2$. We then conclude that $|M_{(1,2,3)}| = 0$ if and only if

$$x_3 \notin \{0, x_1, x_2, x_1 + x_2\},$$

$$x_4 \notin \{0, x_1, x_2, x_3 + x_2 + x_3, x_1 + x_3, x_1 + x_2 + x_3\}$$

and

$$x_5 = x_1 + x_2 + x_3 + x_4.$$\

Obviously, the elements in $\{0, x_1, x_2, x_1 + x_2\}$ and $\{0, x_1, x_2, x_3 + x_2 + x_3, x_1 + x_3, x_1 + x_2 + x_3\}$ are pairwise distinct, respectively. So the total number of different choices of $(x_3, x_4, x_5)$ such that $|M_{(1,2,3)}| = 0$ is equal to $\frac{(q-4)(q-8)}{6}$ regardless of the ordering of $x_3, x_4, x_5$. Then we have completed the proof.\[\square\]
Theorem 27. Let $m$ be a positive integer with $m > 3$ and $q = 2^m$. Then $C_{(1,2,3)}$ is a $[q-1,5,q-6]$ NMDS code over $\mathbb{F}_q$ with weight enumerator

$$A(z) = 1 + \frac{(q-1)^2(q-2)(q-4)(q-8)}{120} z^{q-6} + \frac{5(q-1)^2(q-2)(q-4)}{24} z^{q-5} +$$

$$+ \frac{(q-1)^2(q-2)(q^2-2q+2)}{12} z^{q-4} + \frac{(q-1)^2(q-2)(2q^2+9q+28)}{12} z^{q-3} +$$

$$+ \frac{(q-1)(9q^3+22q^2+12q+176)}{24} z^{q-2} + \frac{(q-1)(44q^4+65q^3+125q^2-170q+536)}{120} z^{q-1}.$$

Moreover, the minimum weight codewords of $C_{(1,2,3)}$ support a $2-(q-1,q-6,\frac{(q-4)(q-6)(q-7)(q-8)}{120})$ simple design and the minimum weight codewords of $C_{(1,2,3)}^*$ support a $2-(q-1-5,\frac{(q-4)(q-8)}{6})$ simple design.

Proof. With a similar proof as that of Theorem 21 we can easily prove this theorem by Lemma 26.

5.3. When $(i,j,k) = (1,2,4)$

In this subsection, let $(i,j,k) = (1,2,4)$ and we study the linear code $C_{(1,2,4)}$.

The following lemma is essential for the proof of our result.

Lemma 28. Let $m$ be an odd integer with $m > 3$ and $q = 2^m$. Let $x_1,x_2,x_3,x_4,x_5$ be five pairwise distinct elements in $\mathbb{F}_q^*$. Define the matrix

$$M_{(1,2,4)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
\xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 \\
\xi_1^2 & \xi_2^2 & \xi_3^2 & \xi_4^2 & \xi_5^2 \\
\xi_1^3 & \xi_2^3 & \xi_3^3 & \xi_4^3 & \xi_5^3 \\
\xi_1^4 & \xi_2^4 & \xi_3^4 & \xi_4^4 & \xi_5^4
\end{bmatrix}.$$

Then for any two different and fixed elements $x_1,x_2$, the total number of different choices of $(x_3,x_4,x_5)$ such that $|M_{(1,2,4)}| = 0$ is equal to $\frac{(q-5)(q-8)}{6}$ (regardless of the ordering of $x_3,x_4,x_5$).

Proof. By Lemma 17 $|M_{(1,2,4)}| = 0$ if and only if

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 + (x_1 + x_2 + x_3 + x_4)x_5 = 0.$$

Let $x_1,x_2$ be two different and fixed elements in $\mathbb{F}_q^*$. Consider the following cases.

Case 1: Let $x_3 = x_1 + x_2$. Then

$$|M_{(1,2,4)}| = \prod_{1 \leq i < j \leq 5} (x_j - x_i)(x_1^2 + x_2^2 + x_1x_2 + x_4x_5) = 0 \iff x_5 = \frac{x_1^2 + x_2^2 + x_1x_2}{x_4}.$$ 

Then $x_5 \notin \{0,x_1,x_2,x_3,x_4\}$ implies $x_4 \notin \{x_1 + x_2 + \frac{x_2^2}{x_1},x_1 + x_2 + \frac{x_1^2}{x_2},\frac{x_1^2 + x_1x_2 + x_2^2}{x_1 + x_2},a\}$, where $a^2 = x_1^2 + x_1x_2 + x_2^2$. In this case, we conclude that $|M_{(1,2,4)}| = 0$ if and only if

$$x_4 \notin \left\{0,x_1,x_2,x_3,x_1 + x_2 + \frac{x_2^2}{x_1},x_1 + x_2 + \frac{x_1^2}{x_2},\frac{x_1^2 + x_1x_2 + x_2^2}{x_1 + x_2},a\right\}.$$
where $a^2 = x_1^2 + x_1x_2 + x_2^2$, and
\[ x_5 = \frac{x_1^2 + x_1x_2 + x_2^2}{x_4}. \]

By Lemmas 8 and 16 we can easily prove that the elements in
\[ \left\{ 0, x_1, x_2, x_1 + x_2, x_1 + x_2 + \frac{x_2^2}{x_1}, \frac{x_1^2 + x_1x_2 + x_2^2}{x_1 + x_2}, a \right\} \]
are pairwise distinct. In this case, the total number of different choices of $(x_3, x_4, x_5)$ such that $|M_{(1,2,4)}| = 0$ is equal to $\frac{(q-5)}{6}$ regardless of the ordering of $x_3, x_4, x_5$.

Case 2: Let $x_3 \neq x_1 + x_2$ and $x_1 + x_2 + x_3 + x_4 = 0$. Then $x_4 = x_1 + x_2 + x_3$. By Lemma 16
\[ |M_{(1,2,4)}| = \prod_{1 \leq i<j \leq 5} (x_j - x_i)(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3) \neq 0. \]
So there is no $(x_3, x_4, x_5)$ such that $|M_{(1,2,4)}| = 0$ in this case.

Case 3: Let $x_3 \neq x_1 + x_2$ and $x_1 + x_2 + x_3 + x_4 \neq 0$. Then $x_4 \neq x_1 + x_2 + x_3$ and
\[ |M_{(1,2,4)}| = 0 \iff x_5 = \frac{x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4}{x_1 + x_2 + x_3 + x_4}. \]
It is easy to deduce that $x_5 \notin \{0, x_1, x_2, x_3, x_4\}$ implies
\[ x_4 \notin \left\{ \frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1 + x_2 + x_3}, \frac{x_1^2 + x_2x_3}{x_2 + x_3}, \frac{x_2^2 + x_1x_3}{x_1 + x_3}, \frac{x_3^2 + x_1x_2}{x_1 + x_2}, b \right\}, \]
where $b^2 = x_1x_2 + x_1x_3 + x_2x_3$. In this case, we conclude that $|M_{(1,2,4)}| = 0$ if and only if
\[ x_3 \notin \{0, x_1, x_2, x_1 + x_2\}, \]
\[ x_4 \notin L := \left\{ 0, x_1, x_2, x_3, x_1 + x_2 + x_3, \frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1 + x_2 + x_3}, \frac{x_1^2 + x_2x_3}{x_2 + x_3}, \frac{x_2^2 + x_1x_3}{x_1 + x_3}, \frac{x_3^2 + x_1x_2}{x_1 + x_2}, b \right\}, \]
where $b^2 = x_1x_2 + x_1x_3 + x_2x_3$, and
\[ x_5 = \frac{x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4}{x_1 + x_2 + x_3 + x_4}. \]

Consider the following subcases of $L$.

Subcase 3.1: If $x_3 = \frac{x_2^2}{x_1}$, then $\frac{x_1^2 + x_2x_3}{x_2 + x_3} = 0$, $\frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1 + x_2 + x_3} = x_1$ and other elements in $L$ are pairwise distinct, which implies $|L| = 8$. If $x_3 = \frac{x_3^2}{x_1}$, then by the symmetry of $x_1$ and $x_2$, we have $|L| = 8$.

Subcase 3.2: If $x_3 = c$ for $c^2 = x_1x_2$, then $\frac{x_1^2 + x_2x_3}{x_1 + x_2} = 0$, $\frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1 + x_2 + x_3} = x_3$ and the other elements in $L$ are pairwise distinct, which implies $|L| = 8$.

Subcase 3.3: If $x_3 = \frac{x_1x_2}{x_1 + x_2}$, then $\frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1 + x_2 + x_3} = b = 0$ and other elements in $L$ are pairwise distinct, which implies $|L| = 8$. 

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Subcase 3.4: Let \( x_3 \notin S := \{0, x_1, x_2, x_1 + x_2, x_1^2, x_2^2, x_1x_2, c\} \), where \( c = x_1x_2 \). By Lemmas 8 and 16 it is easy to prove that the elements in \( S \) are pairwise distinct and \( |S| = 8 \). By Lemmas 8 and 11, the elements in \( L \) are pairwise distinct, which implies \( |L| = 10 \).

In this case, the total number of different choices of \( (x_3, x_4, x_5) \) such that \( |M_{(1,2,4)}| = 0 \) is equal to

\[
\frac{4(q-8)}{3!} + \frac{(q-8)(q-10)}{3!} = \frac{(q-6)(q-8)}{6}
\]

regardless of the ordering of \( x_3, x_4, x_5 \).

Thanks to the above cases, the total number of different choices of \( (x_3, x_4, x_5) \) such that \( |M_{(1,2,4)}| = 0 \) is equal to

\[
\frac{(q-8)}{6} + \frac{(q-6)(q-8)}{6} = \frac{(q-5)(q-8)}{6}
\]

regardless of the ordering of \( x_3, x_4, x_5 \).

Then we have completed the proof. \( \square \)

**Theorem 29.** Let \( m \) be an odd integer with \( m > 3 \) and \( q = 2^m \). Then \( C_{(1,2,4)} \) is a \([q - 1, 5, q - 6]\) NMDS code over \( \mathbb{F}_q \) with weight enumerator

\[
A(z) = 1 + \frac{(q-1)(q-2)(q-5)(q-8)}{120}z^{q-6} + \frac{(q-1)^2(q-2)(3q-14)}{12}z^{q-5} + \frac{(q-1)^2(q-2)(q^2-3q+10)}{12}z^{q-4} + \frac{(q-1)^2(q-2)(q^2+5q+10)}{6}z^{q-3} + \frac{(q-1)^2(9q^3+21q^2+22q+160)}{24}z^{q-2} + \frac{(q-1)(22q^4+33q^3+57q^2-72q+260)}{60}z^{q-1}.
\]

Moreover, the minimum weight codewords of \( C_{(1,2,4)} \) support a \( 2-(q-1, q-6, \frac{(q-5)(q-6)(q-7)(q-8)}{120}) \) simple design and the minimum weight codewords of \( C_{(1,2,4)}^{\perp} \) support a \( 2-(q-1, 5, \frac{(q-5)(q-8)}{6}) \) simple design.

**Proof:** By Lemma 28 we can prove this theorem with a similar proof as that of Theorem 21. The details are omitted here. \( \square \)

### 5.4. The extended code \( \overline{C}_{(1,2,4)} \)

It is obvious that the extended code \( \overline{C}_{(1,2,4)} \) of \( C_{(1,2,4)} \) is generated by the following matrix:

\[
\overline{G}_{(1,2,4)} = \begin{bmatrix}
1 & 1 & \ldots & 1 & 1 \\
\alpha_1 & \alpha_1 & \ldots & \alpha_1 & 0 \\
\alpha_2 & \alpha_2 & \ldots & \alpha_2 & 0 \\
\alpha_3 & \alpha_3 & \ldots & \alpha_3 & 0 \\
\alpha_4 & \alpha_4 & \ldots & \alpha_4 & 0 \\
\alpha_5 & \alpha_5 & \ldots & \alpha_5 & 0 \\
\end{bmatrix}.
\]

We need the following lemma to give our main result in this subsection.
Lemma 30. Let $m$ be an odd integer with $m > 3$ and $q = 2^m$. Let $x_1, x_2, x_3, x_4, x_5$ be five pairwise distinct elements in $\mathbb{F}_q$. Define the matrix

$$M_{(1,2,4)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \\
x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 \\
x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 \\
x_1^5 & x_2^5 & x_3^5 & x_4^5 & x_5^5
\end{bmatrix}. $$

Then for any pairwise different and fixed elements $x_1, x_2, x_3$, the total number of different choices of $(x_4, x_5)$ such that $|M_{(1,2,4)}| = 0$ is equal to $\frac{q^6 - 8}{2}$ (regardless of the ordering of $x_4, x_5$).

Proof. By Lemmas 11, 16 and 17 we can prove this theorem with a similar proof as that of Lemma 20.

\[\square\]

Theorem 31. Let $m$ be an odd integer with $m > 3$ and $q = 2^m$. Then $C_{(1,2,4)}$ is a $[q, 5, q - 5]$ NMDS code over $\mathbb{F}_q$ with weight enumerator

$$A(z) = 1 + \frac{q(q - 1)^2(q - 2)(q - 8)}{12} z^{q - 5} + \frac{5q(q - 1)^2(q - 2)}{24} z^{q - 4} + \frac{q^2(q - 1)^2(q - 2)}{12} z^{q - 3} + \frac{q(q - 1)^2(2q^2 + 7q + 20)}{12} z^{q - 2} + \frac{q(q - 1)(9q^3 + 13q^2 - 6q + 80)}{24} z^{q - 1} + \frac{(q - 1)(44q^4 + 21q^3 + 49q^2 - 114q + 120)}{120} z^q.$$

Moreover, the minimum weight codewords of $C_{(1,2,4)}$ support a $3$-$(q, q - 5, \frac{(q - 5)(q - 6)(q - 7)(q - 8)}{120})$ simple design and the minimum weight codewords of $C_{(1,2,4)}^\perp$ support a $3$-$(q, 5, \frac{q - 8}{2})$ simple design.

Proof. By Lemma 30, we can derive this theorem with a similar proof as that of Theorem 21. The details are omitted here. \[\square\]

5.5. When $(i, j, k) = (1, 3, 4)$

Let $(i, j, k) = (1, 3, 4)$ and we study the linear code $C_{(1,3,4)}$ in this subsection.

The following lemma will be used to prove the main result in this subsection.

Lemma 32. Let $m$ be an odd integer with $m > 3$ and $q = 2^m$. Let $x_1, x_2, x_3, x_4, x_5$ be five pairwise distinct elements in $\mathbb{F}_q$. Define the matrix

$$M_{(1,3,4)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 \\
x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 \\
x_1^5 & x_2^5 & x_3^5 & x_4^5 & x_5^5
\end{bmatrix}. $$

Then for any two different and fixed elements $x_1, x_2$, the total number of different choices of $(x_3, x_4, x_5)$ such that $|M_{(1,3,4)}| = 0$ is equal to $\frac{(q - 5)(q - 8)}{6}$ (regardless of the ordering of $x_3, x_4, x_5$).
Proof. Let $x_1, x_2$ be two different and fixed elements. By Lemma 17, $|M_{(1,3,4)}| = 0$ if and only if

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + (x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4)x_5 = 0.$$ 

Consider the following cases.

**Case 1:** Let $x_3 = x_1 + x_2$. Then $x_1x_2 + x_1x_3 + x_2x_3 + (x_1 + x_2 + x_3)x_4 = x_1^2 + x_1x_2 + x_2^2 \neq 0$ and

$$|M_{(1,3,4)}| = 0 \iff x_5 = \frac{(x_1 + x_2)x_1x_2}{x_1^2 + x_1x_2 + x_2^2} + x_4.$$ 

It is easy to deduce that $x_5 \notin \{0, x_1, x_2, x_3, x_4\}$ if and only if

$$x_4 \notin \left\{ \frac{(x_1 + x_2)x_1x_2}{x_1^2 + x_1x_2 + x_2^2}, \frac{x_1^3}{x_1^2 + x_1x_2 + x_2^2}, \frac{x_2^3}{x_1^2 + x_1x_2 + x_2^2}, \frac{(x_1 + x_2)^3}{x_1^2 + x_1x_2 + x_2^2} \right\}.$$ 

In this case, we conclude that $|M_{(1,3,4)}| = 0$ if and only if

$$x_4 \notin \left\{ 0, x_1, x_2, x_1 + x_2, \frac{(x_1 + x_2)x_1x_2}{x_1^2 + x_1x_2 + x_2^2}, \frac{x_1^3}{x_1^2 + x_1x_2 + x_2^2}, \frac{x_2^3}{x_1^2 + x_1x_2 + x_2^2}, \frac{(x_1 + x_2)^3}{x_1^2 + x_1x_2 + x_2^2} \right\}$$

and

$$x_5 = \frac{(x_1 + x_2)x_1x_2}{x_1^2 + x_1x_2 + x_2^2} + x_4.$$ 

By Lemmas 8 and 16, we can easily prove that the elements in

$$\left\{ 0, x_1, x_2, x_1 + x_2, \frac{(x_1 + x_2)x_1x_2}{x_1^2 + x_1x_2 + x_2^2}, \frac{x_1^3}{x_1^2 + x_1x_2 + x_2^2}, \frac{x_2^3}{x_1^2 + x_1x_2 + x_2^2}, \frac{(x_1 + x_2)^3}{x_1^2 + x_1x_2 + x_2^2} \right\}$$

are pairwise distinct. Then the total number of different choices of $(x_3, x_4, x_5)$ such that $|M_{(1,3,4)}| = 0$ is equal to $\frac{5!}{6}$ regardless of the ordering of $x_3, x_4, x_5$.

**Case 2:** Let $x_3 \neq x_1 + x_2$ and $x_1x_2 + x_1x_3 + x_2x_3 + (x_1 + x_2 + x_3)x_4 = 0$ which implies $x_4 = \frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1 + x_2 + x_3}$. By Lemma 11, it is easy to prove

$$|M_{(1,3,4)}| = \prod_{1 \leq i < j \leq 5} (x_j - x_i) \frac{x_1^2 (x_2^2 + x_2x_3 + x_3^2) + x_1 (x_2^2x_3 + x_2x_3^2) + x_2^3}{x_1 + x_2 + x_3} \neq 0.$$ 

Hence there is no $(x_3, x_4, x_5)$ such that $|M_{(1,3,4)}| = 0$.

**Case 3:** Let $x_3 \neq x_1 + x_2$ and $x_1x_2 + x_1x_3 + x_2x_3 + (x_1 + x_2 + x_3)x_4 \neq 0$ which implies $x_4 \neq \frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1 + x_2 + x_3}$. Then

$$|M_{(1,3,4)}| = 0 \iff x_5 = \frac{(x_1x_2 + x_1x_3 + x_2x_3)x_4 + x_1x_2x_3}{x_1x_2 + x_1x_3 + x_2x_3 + (x_1 + x_2 + x_3)x_4}.$$ 

Let

$$x_5 = \frac{(x_1x_2 + x_1x_3 + x_2x_3)x_4 + x_1x_2x_3}{x_1x_2 + x_1x_3 + x_2x_3 + (x_1 + x_2 + x_3)x_4}, \quad a^2 = \frac{x_1x_2x_3}{x_1 + x_2 + x_3}.$$
Subcase 3.1: Let $x_3 = \frac{x_1 x_2}{x_1 + x_2}$, then $x_3 = \frac{x_1^2 x_2^2}{x_1^2 + x_1 x_2 + x_2^2}$ and $\frac{x_1 x_2 + x_1 x_3 + x_2 x_3}{x_1 + x_2 + x_3} = 0$. It is easy to deduce that $x_5 \notin \{0, x_1, x_2, x_3, x_4\}$ implies $x_4 \notin \left\{ \frac{x_1 x_2}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1^2 x_2}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1 x_2 (x_1 + x_2)}{x_1^2 + x_1 x_2 + x_2^2}, a \right\}$. In this subcase, we conclude that $|M_{(1,3,4)}| = 0$ if and only if

$$x_4 \notin L_1 := \left\{ 0, x_1, x_2, \frac{x_1 x_2}{x_1 + x_2}, \frac{x_1^2 x_2}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1 x_2 (x_1 + x_2)}{x_1^2 + x_1 x_2 + x_2^2}, a \right\},$$

and

$$x_5 = \frac{x_1^2 x_2^2}{x_1^2 + x_1 x_2 + x_2^2}.$$

It is obvious that the elements in $L_1$ are pairwise distinct. Then the total number of different choices of $(x_3, x_4, x_5)$ such that $|M_{(1,3,4)}| = 0$ is equal to $\frac{a-8}{6}$ regardless of the ordering of $x_3, x_4, x_5$.

Subcase 3.2: Let $x_3 = \frac{x^2}{x_1}$, then $x_5 = \frac{x_1 (x_4 (x_1 + x_2)^2 + x_1 x_2 (x_1 + x_4))}{x_4 (x_1 + x_2)^2 + x_1 x_2 (x_1 + x_4) + x_1 (x_1 + x_2)^2}$ and $\frac{x_1 x_3 + x_1 x_4 + x_2 x_3}{x_1 + x_2 + x_3} = x_1$. It is easy to deduce that $x_5 \notin \{0, x_1, x_2, x_3, x_4\}$ if and only if $x_4 \notin \left\{ \frac{x_1 x_2}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1^2}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1 x_2 (x_1 + x_2)}{x_1^2 + x_1 x_2 + x_2^2}, a \right\}$. We conclude that $|M_{(1,3,4)}| = 0$ if and only if

$$x_4 \notin L_2 := \left\{ 0, x_1, x_2, \frac{x_1^2}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1 x_2}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1 x_2 (x_1 + x_2)}{x_1^2 + x_1 x_2 + x_2^2}, a \right\},$$

and

$$x_5 = \frac{x_1 (x_4 (x_1 + x_2)^2 + x_1 x_2 (x_1 + x_4))}{x_4 (x_1 + x_2)^2 + x_1 x_2 (x_1 + x_4) + x_1 (x_1 + x_2)^2}.$$

It is obvious that the elements in $L_2$ are pairwise distinct. In this subcase, the total number of choices of $x_3, x_4, x_5$ such that $|M_{(1,3,4)}| = 0$ is equal to $\frac{a-8}{6}$ regardless of the ordering of $x_3, x_4, x_5$.

Let $x_3 = \frac{x^2}{x_1}$, then by the symmetry of $x_1$ and $x_2$, we can drive the same conclusion that the total number of choices of $x_3, x_4, x_5$ such that $|M_{(1,3,4)}| = 0$ is equal to $\frac{a-8}{6}$ regardless of the ordering of $x_3, x_4, x_5$.

Subcase 3.3: Let $x_3 = b$, where $b^2 = x_1 x_2$. Then $x_5^2 = \frac{x_1 x_2 (x_1^2 + x_1 x_2 + x_2^2) + x_1^2 x_2}{x_1^2 (x_1^2 + x_1 x_2 + x_2^2) + x_1^2 x_2 (x_1 + x_2)^2}$. It is easy to deduce that $x_5 \notin \{0, x_1, x_2, x_3, x_4\}$ implies

$$x_4 \notin \left\{ \frac{x_1 x_2}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1^2 x_2}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1 x_2 (x_1 + x_2)}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1 x_2 b}{x_1 + x_2 + b} \right\}.$$

We conclude that $|M_{(1,3,4)}| = 0$ if and only if

$$x_4 \notin L_4 := \left\{ 0, x_1^2, x_1^2 x_2, \frac{x_1 x_2}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1^2 x_2}{x_1^2 + x_1 x_2 + x_2^2}, \frac{x_1 x_2 (x_1 + x_2)}{x_1^2 + x_1 x_2 + x_2^2}, x_1 x_2 b \right\},$$

and

$$x_5 = \frac{x_1 x_2 (x_1^2 + x_1 x_2 + x_2^2) + x_1^2 x_2}{x_1^2 (x_1^2 + x_1 x_2 + x_2^2) + x_1^2 x_2 (x_1 + x_2)^2}.$$
By Lemma[11] we can verify that the elements in \( L_4 \) are pairwise distinct. In this subcase, the total number of different choices of \( (x_3,x_4,x_5) \) such that \(|M_{(1,3,4)}| = 0\) is equal to \( \frac{q^8-8}{6} \) regardless of the ordering of \( x_3,x_4,x_5 \).

**Subcase 3.4:** Let \( x_3 \notin L_5 := \left\{ 0,x_1,x_2,x_1 + x_2, \frac{x_1 x_2}{x_1 + x_2}, \frac{x_1^2 + x_2^2}{x_1 x_2}, b \right\} \), where \( b^2 = x_1 x_2 \). By Lemmas [8] and [16] the elements in \( L_5 \) are pairwise distinct. It is easy to prove that \( x_5 \notin \{0,x_1,x_2,x_3,x_4\} \) implies \( x_4 \notin \left\{ \frac{x_1^2 (x_2+x_3)}{x_1^2 + x_2^2 + x_3^2}, \frac{x_1 (x_2+x_3)}{x_2 + x_3}, \frac{x_1 x_2}{x_2 + x_3}, \frac{x_1 (x_2+x_3)}{x_1 + x_2} \right\} \). We conclude that \(|M_{(1,3,4)}| = 0\) if and only if \( x_3 \notin L_5 \) and \( x_4 \notin L_6 \), where \( L_6 \) is given by

\[
\left\{ \frac{x_1 x_2 + x_1 x_3 + x_2 x_3}{x_1 + x_2 + x_3}, \frac{x_1 x_2 + (x_1 + x_2) x_3}{x_1 + x_2 + x_3}, \frac{x_1^2 (x_2 + x_3)}{x_1^2 + x_2 x_3}, \frac{x_1 (x_2 + x_3)}{x_2 + x_1 x_3}, \frac{x_1 x_2}{x_3 + x_1 x_2}, \frac{x_1 x_2 x_3}{x_1 + x_2 + x_3} \right\},
\]

and

\[
x_5 = \frac{(x_1 x_2 + x_1 x_3 + x_2 x_3) x_4 + x_1 x_2 x_3}{x_1 x_2 + x_1 x_3 + x_2 x_3 + (x_1 + x_2 + x_3) x_4}.
\]

By Lemma[11] we can verify that the elements in \( L_6 \) are pairwise distinct. In this subcase, the total number of different choices of \( (x_3,x_4,x_5) \) such that \(|M_{(1,3,4)}| = 0\) is equal to \( \frac{(q-8)(q-10)}{6} \) regardless of the ordering of \( x_3,x_4,x_5 \).

In Case 3, the total number of different choices of \( (x_3,x_4,x_5) \) such that \(|M_{(1,3,4)}| = 0\) is equal to

\[
\frac{4(q-8)}{6} + \frac{(q-8)(q-10)}{6} = \frac{(q-6)(q-8)}{6}
\]

regardless of the ordering of \( x_3,x_4,x_5 \).

Thanks to the above cases, the total number of different choices of \( (x_3,x_4,x_5) \) such that \(|M_{(1,3,4)}| = 0\) is equal to

\[
\frac{(q-8)}{6} + \frac{(q-6)(q-8)}{6} = \frac{(q-5)(q-8)}{6}
\]

regardless of the ordering of \( x_3,x_4,x_5 \). Then we complete the proof. \( \square \)

**Theorem 33.** Let \( m \) be an odd integer with \( m > 3 \) and \( q = 2^m \). Then \( C_{(1,3,4)} \) is a \([q-1,5,q-6]\) NMDS code over \( \mathbb{F}_q \) with weight enumerator

\[
A(z) = 1 + \frac{(q-1)^2(q-2)(q-5)(q-8)}{120} z^{q-6} + \frac{(q-1)^2(q-2)(3q-14)}{12} z^{q-5} + \frac{(q-1)^2(q-2)(q^2-3q+10)}{12} z^{q-4} + \frac{(q-1)^2(q-2)(q^2+5q+10)}{6} z^{q-3} + \frac{(q-1)^2(9q^3+21q^2+22q+160)}{24} z^{q-2} + \frac{(q-1)(22q^4+33q^3+57q^2-72q+260)}{60} z^{q-1}.
\]

Moreover, the minimum weight codewords of \( C_{(1,3,4)} \) support a \( 2\)-(\( q-1,6,\frac{(q-5)(q-6)(q-7)(q-8)}{120} \)) simple design and the minimum weight codewords of \( C^+_{(1,3,4)} \) support a \( 2\)-(\( q-1,5,\frac{(q-5)(q-8)}{6} \)) simple design.

**Proof.** By Lemma[32] the desired conclusions can be derived with a similar proof as that of Theorem 21. \( \square \)
6. Constructions of 6-dimensional near MDS codes holding 2-designs or 3-designs

In this section, let $q = 2^m$ with $m > 3$. Let $\alpha$ be a generator of $\mathbb{F}_q^*$ and $\alpha_t = \alpha^t$ for $1 \leq t \leq q - 1$. Then $\alpha_{q-1} = 1$. Define

$$H_{(i,j,k,l)} = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
\alpha_i^l & \alpha_2^l & \cdots & \alpha_{q-2}^l & \alpha_{q-1}^l \\
\alpha_i^k & \alpha_2^k & \cdots & \alpha_{q-2}^k & \alpha_{q-1}^k \\
\alpha_i^j & \alpha_2^j & \cdots & \alpha_{q-2}^j & \alpha_{q-1}^j \\
\alpha_i^6 & \alpha_2^6 & \cdots & \alpha_{q-2}^6 & \alpha_{q-1}^6
\end{bmatrix},$$

where $(i, j, k, l) \in \{(1, 2, 3, 4), (2, 3, 4, 5), (1, 2, 4, 5)\}$. Then $H_{(i,j,k,l)}$ is a 6 by $q - 1$ matrix over $\mathbb{F}_q$. Let $C_{(i,j,k,l)}$ be the linear code over $\mathbb{F}_q$ generated by $H_{(i,j,k,l)}$. We will prove that $C_{(i,j,k,l)}$ is an NMDS code and the minimum weight codewords of both $C_{(i,j,k,l)}$ and its dual $C_{(i,j,k,l)}^\perp$ support 2-designs. Let $\overline{C}_{(i,j,k,l)}$ be the extended code of $C_{(i,j,k,l)}$. We will also prove that $\overline{C}_{(i,j,k,l)}$ is an NMDS code and the minimum weight codewords of both $\overline{C}_{(i,j,k,l)}$ and its dual $\overline{C}_{(i,j,k,l)}^\perp$ support 3-designs for $(i, j, k, l) \in \{(1, 2, 3, 4), (2, 3, 4, 5)\}$.

6.1. When $(i, j, k, l) \in \{(1, 2, 3, 4), (2, 3, 4, 5)\}$

The following lemma plays an important role in the proof of our main result in this subsection.

**Lemma 34.** Let $m$ be an integer with $m > 3$ and $q = 2^m$. Let $x_1, x_2, x_3, x_4, x_5, x_6$ be six pairwise distinct elements in $\mathbb{F}_q^*$. Define the matrix

$$M_{(i,j,k,l)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
x_1^l & x_2^l & x_3^l & x_4^l & x_5^l & x_6^l \\
x_1^k & x_2^k & x_3^k & x_4^k & x_5^k & x_6^k \\
x_1^j & x_2^j & x_3^j & x_4^j & x_5^j & x_6^j \\
x_1^6 & x_2^6 & x_3^6 & x_4^6 & x_5^6 & x_6^6
\end{bmatrix},$$

where $(i, j, k, l) = (1, 2, 3, 4)$ or $(2, 3, 4, 5)$. Then for any two different and fixed elements $x_1, x_2$ and any $(i, j, k, l) \in \{(1, 2, 3, 4), (2, 3, 4, 5)\}$, the total number of different choices of $(x_3, x_4, x_5, x_6)$ such that $|M_{(i,j,k,l)}| = 0$ is equal to $\frac{(q-4)(q-6)(q-8)}{24}$ (regardless of the ordering of $x_3, x_4, x_5, x_6$).

**Proof.** In the following, we only give the proof for $(i, j, k, l) = (1, 2, 3, 4)$ as the proof for $(i, j, k, l) = (2, 3, 4, 5)$ can be similarly given.

Let $x_1, x_2$ be any two different and fixed elements in $\mathbb{F}_q^*$. By Lemma 17, $|M_{(1,2,3,4)}| = 0$ if and only if $x_6 = x_1 + x_2 + x_3 + x_4 + x_5$. It is easy to deduce that $x_6 \notin \{0, x_1, x_2, x_3, x_4, x_5\}$ implies $x_5 \notin \{x_1 + x_2 + x_3 + x_4, x_2 + x_3 + x_4, x_1 + x_3 + x_4, x_1 + x_2 + x_4, x_1 + x_2 + x_3\}$ and $x_4 \notin x_1 + x_2 + x_3$. In conclusion, $|M_{(1,2,3,4)}| = 0$ if and only if $x_3 \notin \{0, x_1, x_2\}$, $x_4 \notin L_1 := \{0, x_1, x_2, x_3, x_1 + x_2 + x_3\}$ and $x_5 \notin L_2 := \{0, x_1, x_2, x_3, x_4, x_1 + x_2 + x_3 + x_4, x_2 + x_3 + x_4, x_1 + x_3 + x_4, x_1 + x_2 + x_4, x_1 + x_2 + x_3\}$.

Consider the following cases.
Case 1: Let $x_3 = x_1 + x_2$. Then $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 + x_3 + x_4 = x_4$. We have $|L_1| = 4$, $|L_2| = 8$. In this case, the total number of different choices of $(x_3, x_4, x_5, x_6)$ such that $|M_{(1,2,3,4)}| = 0$ is equal to $\frac{(q-4)(q-8)}{24}$ regardless of the ordering of $x_3, x_4, x_5, x_6$.

Case 2: Let $x_3 \neq x_1 + x_2$. It is obvious that the elements in $L_1$ are pairwise distinct. Consider the following subcases of $L_2$.

Subcase 2.1: Let $x_4 = x_1 + x_2$. Then $x_1 + x_2 + x_3 + x_4 = x_3, x_1 + x_2 + x_4 = 0$ and elements in $L_2$ are pairwise distinct. Thus $|L_2| = 8$. If $x_4 = x_1 + x_3$ or $x_4 = x_2 + x_3$, then by the symmetry of $x_1, x_2,$ and $x_3$, we also have $|L_2| = 8$.

Subcase 2.2: Let $x_4 \notin L_3 = \{0, x_1, x_2, x_3, x_1 + x_2 + x_3, x_1 + x_2, x_1 + x_3, x_2 + x_3\}$. Then the elements in $L_2$ and $L_3$ are pairwise distinct. Thus $|L_2| = 10$.

By summarizing the four subcases in Case 2, the total number of different choices of $(x_3, x_4, x_5, x_6)$ such that $|M_{(1,2,3,4)}| = 0$ is equal to

$$\frac{3(q-4)(q-8)}{4!} + \frac{(q-4)(q-8)(q-10)}{4!} = \frac{(q-4)(q-7)(q-8)}{24}$$

regardless of the ordering of $x_3, x_4, x_5, x_6$.

By the above cases, the total number of different choices of $(x_3, x_4, x_5, x_6)$ such that $|M_{(1,2,3,4)}| = 0$ is equal to

$$\frac{(q-4)(q-8)}{24} + \frac{(q-4)(q-7)(q-8)}{24} = \frac{(q-4)(q-6)(q-8)}{24}$$

regardless of the ordering of $x_3, x_4, x_5, x_6$. Then the proof is completed. \qed

**Theorem 35.** Let $m$ be a integer with $m > 3$ and $q = 2^m$. Let $(i, j, k, l) = (1, 2, 3, 4)$ or $(2, 3, 4, 5)$. Then $C(i, j, k, l)$ is a $[q-1, 6, q-7]$ NMDS code over $\mathbb{F}_q$ with weight enumerator

$$A(z) = 1 + \frac{(q-1)^2(q-2)(q-4)(q-6)(q-8)}{720}z^{q-7} + \frac{(q-1)^2(q-2)(q-4)(q-11)}{40}z^{q-6} +$$

$$+ \frac{(q-1)^2(q-2)(q-4)(q^2-2q+12)}{48}z^{q-5} + \frac{(q-1)^2(q-2)(2q^3+6q^2-5q-78)}{36}z^{q-4} +$$

$$+ \frac{(q-1)^2(q-2)(q+4)(3q^2-2q+24)}{16}z^{q-3} + \frac{(q-1)^2(44q^4+110q^3+235q^2-110q+1416)}{120}z^{q-2} +$$

$$+ \frac{(q-1)(265q^5+399q^4+400q^3+1200q^2-1880q+3936)}{720}z^{q-1}.$$ 

Moreover, the minimum weight codewords of $C(i, j, k, l)$ support a $2-(q-1, 6, (q-4)(q-6)(q-7)(q-8)^2)$ simple design and the minimum weight codewords of $C(i, j, k, l)$ support a $2-(q-1, 6, (q-4)(q-6)(q-8))$ simple design.

**Proof:** By Lemma 34 we can prove this theorem with a similar proof as that of Theorem 21. \qed

**Remark 1.** Note that $C_{(1,2,3)}$ and $C_{(2,3,4)}$, $C_{(1,2,4)}$ and $C_{(1,3,4)}$, $C_{(1,2,3,4)}$ and $C_{(2,3,4,5)}$ have the same parameters and weight enumerators, respectively. Besides, the minimum weight codewords of each pair of NMDS codes hold $t$-designs with the same parameters, respectively. However, by Magma, the blocks of the $t$-designs are different for each pair of NMDS codes. It is open whether the NMDS codes in each pair are equivalent to each other.
By Theorems 27 and 35 we have the following conjecture.

**Conjecture 36.** Let \( q = 2^m \) with \( m \geq k_1 \geq 3 \), where \( k_1 \) is some proper positive integer. Let \( \alpha_1, \alpha_2, \ldots, \alpha_{q-1} \) be all elements of \( \mathbb{F}_q \). Define

\[
M_k = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{q-2} & \alpha_{q-1} \\
\alpha_1^2 & \alpha_2^2 & \cdots & \alpha_{q-2}^2 & \alpha_{q-1}^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_1^{k-2} & \alpha_2^{k-2} & \cdots & \alpha_{q-2}^{k-2} & \alpha_{q-1}^{k-2} \\
\alpha_1^k & \alpha_2^k & \cdots & \alpha_{q-2}^k & \alpha_{q-1}^k
\end{bmatrix},
\]

where \( 2 < k < q - 1 \). Let \( C_k \) be the linear code over \( \mathbb{F}_q \) generated by \( M_k \). Then \( C_k \) is a \( [q - 1, k, q - 1 - k] \) NMDS code and the minimum weight codewords of both \( C_k \) and its dual \( C_k^\perp \) support 2-designs.

By Magma, Conjecture 36 has been verified to be correct for

\[
(m, k_1, k) \in \{(4, 4, 4), (4, 4, 5), \ldots, (4, 4, 12)\}
\]

and

\[
(m, k_1, k) \in \{(5, 4, 5), (5, 4, 6), (5, 4, 7), (5, 4, 8)\}.
\]

### 6.2. The extended code \( \overline{C}_{(1,2,3,4)} \) of \( C_{(1,2,3,4)} \)

In this subsection, we study the extended code \( \overline{C}_{(1,2,3,4)} \) of \( C_{(1,2,3,4)} \). It is obvious that the linear code \( \overline{C}_{(1,2,3,4)} \) is generated by the following matrix:

\[
H_{(1,2,3,4)} = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{q-1} & 0 \\
\alpha_1^2 & \alpha_2^2 & \cdots & \alpha_{q-1}^2 & 0 \\
\alpha_1^3 & \alpha_2^3 & \cdots & \alpha_{q-1}^3 & 0 \\
\alpha_1^4 & \alpha_2^4 & \cdots & \alpha_{q-1}^4 & 0 \\
\alpha_1^6 & \alpha_2^6 & \cdots & \alpha_{q-1}^6 & 0
\end{bmatrix}.
\]

The following lemma will be used to give our main result in this subsection.

**Lemma 37.** Let \( m \) be an integer with \( m \geq 3 \) and \( q = 2^m \). Let \( x_1, x_2, x_3, x_4, x_5, x_6 \) be six pairwise distinct elements in \( \mathbb{F}_q \). Define the matrix

\[
M_{(1,2,3,4)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 \\
x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 & x_6^3 \\
x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 & x_6^4 \\
x_1^6 & x_2^6 & x_3^6 & x_4^6 & x_5^6 & x_6^6
\end{bmatrix}.
\]

Then for any pairwise different and fixed elements \( x_1, x_2, x_3 \), the total number of different choices of \( (x_4, x_5, x_6) \) such that \( |M_{(1,2,3,4)}| = 0 \) is equal to \( \frac{(q-4)(q-8)}{6} \) (regardless of the ordering of \( x_4, x_5, x_6 \)).
Proof. Similarly to the proof of Lemma \ref{lemma26} we can easily derive this lemma by Lemma \ref{lemma17}.

**Theorem 38.** Let \( m \) be an integer with \( m > 3 \) and \( q = 2^m \). Then \( \overline{C_{(1,2,3,4)}} \) is a \([q, 6, q - 6]\) NMDS code over \( \mathbb{F}_q \) with weight enumerator

\[
A(z) = 1 + \frac{q(q-1)^2(q-2)(q-4)(q-8)}{720} z^{q-6} + \frac{q(q-1)^2(q-2)(q-4)}{24} z^{q-5} + \frac{q(q-1)^2(q-2)(q^2-2q+2)}{48} z^{q-4} + \frac{q(q-1)^2(q-2)(2q^2+9q+28)}{36} z^{q-3} + \frac{q(q-1)(9q^3+22q^2+12q+176)}{48} z^{q-2} + \frac{q(q-1)(44q^4+65q^3+125q^2-170q+536)}{120} z^{q-1} + \frac{(q-1)(53q^5+27q^4+2q^3+90q^2-172q+144)}{144} z^q.
\]

Moreover, the minimum weight codewords of \( \overline{C_{(1,2,3,4)}} \) support a \( 3-(q, q-6, \frac{(q-4)(q-6)(q-7)(q-8)^2}{720}) \) simple design and the minimum weight codewords of \( \overline{C_{(1,2,3,4)}} \) support a \( 3-(q, 6, \frac{(q-4)(q-8)}{6}) \) simple design.

Proof. By Lemma \ref{lemma37} we can derive the desired conclusion with a similar proof as that of Theorem \ref{theorem21}. The details are omitted.

6.3. When \((i, j, k, l) = (1, 2, 4, 5)\)

In this subsection, we consider the case for \((i, j, k, l) = (1, 2, 4, 5)\).

The following lemma plays an important role in the proof of our next main result.

**Lemma 39.** Let \( m \) be an odd integer with \( m > 3 \), \( q = 2^m \). Let \( x_1, x_2, x_3, x_4, x_5, x_6 \) be six pairwise distinct elements in \( \mathbb{F}_q^* \). Define the matrix

\[
M_{(1,2,4,5)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 \\
x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 & x_6^3 \\
x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 & x_6^4 \\
x_1^5 & x_2^5 & x_3^5 & x_4^5 & x_5^5 & x_6^5 \\
x_1^6 & x_2^6 & x_3^6 & x_4^6 & x_5^6 & x_6^6
\end{bmatrix}.
\]

Then for any two different and fixed elements \( x_1, x_2 \), the total number of different choices of \((x_3, x_4, x_5, x_6)\) such that \( |M_{(1,2,4,5)}| = 0 \) is equal to \( \frac{(q-5)(q-8)}{6} \) (regardless of the ordering of \( x_3, x_4, x_5, x_6 \)).

Proof. Similarly to the proof of Lemma \ref{lemma32} we can easily derive this lemma by Lemma \ref{lemma17}.

**Theorem 40.** Let \( m \) be an odd integer with \( m > 3 \) and \( q = 2^m \). Then \( C_{(1,2,4,5)} \) is a \([q-1, 6, q-7]\) NMDS code over \( \mathbb{F}_q \) with weight enumerator

\[
A(z) = 1 + \frac{(q-1)^2(q-2)(q-5)(q-6)(q-8)}{720} z^{q-7} + \frac{(q-1)^2(q-2)(q-5)(7q-36)}{120} z^{q-6} + \frac{(q-1)^2(q-2)(q-4)(q-8)}{720} z^{q-6} + \frac{(q-1)^2(q-2)(q-4)(q-8)}{24} z^{q-5} + \frac{(q-1)^2(q-2)(2q^2+9q+28)}{36} z^{q-3} + \frac{q(q-1)(9q^3+22q^2+12q+176)}{48} z^{q-2} + \frac{q(q-1)(44q^4+65q^3+125q^2-170q+536)}{120} z^{q-1} + \frac{(q-1)(53q^5+27q^4+2q^3+90q^2-172q+144)}{144} z^q.
\]
\[
\frac{(q-1)^2(q-2)(q^3-7q^2+34q-96)}{48}z^{q-5} + \frac{(q-1)^2(q-2)(2q^3+7q^2-19q-30)}{36}z^{q-4} + \\
\frac{(q-1)^2(q-2)(9q^3+29q^2+62q+240)}{48}z^{q-3} + \frac{(q-1)^2(44q^4+111q^3+219q^2-34q+1320)}{120}z^{q-2} \\
+ \frac{(q-1)(265q^5+398q^4+417q^3+1108q^2-1708q+3840)}{720}z^{q-1}.
\]

Moreover, the minimum weight codewords of \( C_{(1,2,4,5)} \) support a \( 2-(q-1, q-7, \frac{(q-5)(q-6)(q-7)(q-8)^2}{720}) \) simple design and the minimum weight codewords of \( C_{(1,2,4,5)} \) support a \( 2-(q-1, 6, \frac{(q-5)(q-8)}{6}) \) simple design.

**Proof.** Similarly to the proof of Theorem 21 we can prove this theorem by Lemma 39.

6.4. The extended code \( \overline{C}_{(1,2,4,5)} \)

It is obvious that the extended code \( \overline{C}_{(1,2,4,5)} \) of \( C_{(1,2,4,5)} \) is generated by the following matrix:

\[
\overline{H}_{(1,2,4,5)} = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{q-2} & \alpha_{q-1} & 0 \\
\alpha_2 & \alpha_3 & \cdots & \alpha_{q-2} & \alpha_{q-1} & 0 \\
\alpha_3 & \alpha_4 & \cdots & \alpha_{q-2} & \alpha_{q-1} & 0 \\
\alpha_4 & \alpha_5 & \cdots & \alpha_{q-2} & \alpha_{q-1} & 0 \\
\alpha_5 & \alpha_6 & \cdots & \alpha_{q-2} & \alpha_{q-1} & 0 \\
\end{bmatrix}.
\]

We need the following lemma to give our main result in this subsection.

**Lemma 41.** Let \( m \) be an odd integer with \( m > 3 \) and \( q = 2^m \). Let \( x_1, x_2, x_3, x_4, x_5, x_6 \) be six pairwise distinct elements in \( \mathbb{F}_q \). Define the matrix

\[
\overline{M}_{(1,2,4,5)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
x_2^4 & x_3^4 & x_4^4 & x_5^4 & x_6^4 \\
x_3^8 & x_4^8 & x_5^8 & x_6^8 \\
x_4^{16} & x_5^{16} & x_6^{16} \\
x_5^{32} & x_6^{32} \\
\end{bmatrix}.
\]

Then for any three pairwise different and fixed elements \( x_1, x_2, x_3 \), the total number of different choices of \( (x_4, x_5, x_6) \) such that \( \overline{M}_{(1,2,4,5)} = 0 \) is equal to \( \frac{(q-5)(q-8)}{6} \) (regardless of the ordering of \( x_4, x_5, x_6 \)).

**Proof.** Similarly to the proof of Lemma 32, we can easily derive this lemma by Lemma 17.

**Theorem 42.** Let \( m \) be an odd integer with \( m > 3 \) and \( q = 2^m \). Then \( \overline{C}_{(1,2,4,5)} \) is a \([q, 6, q-6]\) NMDS code over \( \mathbb{F}_q \) with weight enumerator

\[
A(z) = 1 + \frac{q(q-1)^2(q-2)(q-5)(q-8)}{720}z^{q-6} + \frac{q(q-1)^2(q-2)(3q-14)}{60}z^{q-5} + \]

\[
\frac{(q-1)^2(q-2)(q-5)(q-7)(q-8)^2}{720}z^{q-1}.
\]
Lemma 45. The following lemma is useful for determining the minimum locality for nontrivial linear codes whose 
were proposed for the recovery of data by Gopalan, Huang, Simitci and Yikhanin [14]. Let 
recoverable codes (LRCs for short) are applied in distributed storage and cloud storage. LRCs 
\( (q, 2, 1) \)-LRC, we have

\[
\frac{q(q-1)^2(q-2)(q^2-3q+10)}{48} \cdot q^{q-4} + \frac{q(q-1)^2(q-2)(q^2+5q+10)}{18} \cdot q^{q-3} + \\
\frac{q(q-1)^2(9q^3+21q^2+22q+160)}{48} \cdot q^{q-2} + \frac{q(q-1)(22q^4+33q^3+57q^2-72q+260)}{60} \cdot q^{q-1} \\
+ \frac{(q-1)(265q^5+134q^4+21q^3+424q^2-844q+720)}{720} \cdot q^{q-1}.
\]

Moreover, the minimum weight codewords of \( \mathcal{C}_{(1,2,4,5)} \) support a \( 3-(q, q-6, \frac{(q-5)(q-6)(q-7)(q-8)^2}{120}) \) simple design and the minimum weight codewords of \( \mathcal{C}_{(1,2,4,5)} \) support a \( 3-(q, 6, \frac{(q-5)(q-8)}{6}) \) simple design.

Proof. Similarly to the proof of Theorem [21] we can prove this theorem by Lemma [41].

7. Optimal locally recoverable codes

In this section, we study the minimum locality of the codes constructed in this paper. Locally 
recoverable codes (LRCs for short) are applied in distributed storage and cloud storage. LRCs 
were proposed for the recovery of data by Gopalan, Huang, Simitci and Yikhanin [14]. Let \( [n] := \{1, 2, \ldots, n\} \) for a positive integer \( n \). Let \( \mathcal{C} \) be an \( [n, k, d] \) linear code over \( \mathbb{F}_q \). For every \( i \in [n] \), if there exist a subset \( R_i \subseteq [n] \setminus \{i\} \) of size \( r \) \( (r < k) \) and a function \( f_i(x_1, x_2, \ldots, x_r) \) on \( \mathbb{F}_q^r \) such that 
\( c_i = f_i(c_R) \) for each \( c = (c_1, c_2, \ldots, c_n) \) in \( \mathcal{C} \), then \( \mathcal{C} \) is called an \( (n, k, d, q; r) \)-LRC, where \( c_R \) is the 
projection of \( c \) at \( R_i \). \( R_i \) is said to be the recovering set of \( c_i \). The minimum \( r \) such that a linear code 
\( \mathcal{C} \) is an \( (n, k, d, q; r) \)-LRC is called the minimum locality of this code.

Lemma 43. [14, Singleton-like bound] For any \( (n, k, d, q; r) \)-LRC, we have

\[
d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2.
\]

A LRC is said to be distance-optimal \( (d\)-optimal for short) if it achieves the Singleton-like bound.

Lemma 44. [1] Cadambe-Mazumdar bound] For any \( (n, k, d, q; r) \)-LRC,

\[
k \leq \min_{r \in \mathbb{Z}^+} \left\lceil rt + k_{opt}^{(q)}(n - t(r + 1), d) \right\rceil,
\]

where \( \mathbb{Z}^+ \) denotes the set of all positive integers, \( k_{opt}^{(q)}(n, d) \) is the largest possible dimension of 
a linear code with alphabet size \( q \), length \( n \), and minimum distance \( d \).

A LRC is said to be dimension-optimal \( (k\)-optimal for short) if it achieves the Cadambe-
Mazumdar bound.

Let \( \mathcal{B}_i(C) \) denote the set of supports of all codewords with Hamming weight \( i \) in \( C \). The 
following lemma is useful for determining the minimum locality for nontrivial linear codes whose 
minimum distance is larger than 1.

Lemma 45. [28] Let \( C \) be a nontrivial linear code of length \( n \) and put \( d^\perp = d(C^\perp) \). If \( (\mathcal{P}, \mathcal{B}_{d^\perp}(C^\perp)) \) is a \( 1-(n, d^\perp, \lambda^1) \) design with \( \lambda^1 \geq 1 \), then \( C \) has minimum locality \( d^\perp - 1 \).
Besides, all NMDS codes constructed in this paper are both LRCs.

### 8. Summary and concluding remarks

In this paper, we presented several infinite families of near MDS codes holding $t$-designs over $\mathbb{F}_{2^m}$. Our main contributions are listed in the following:

1. In Section 3, two families of $[q-1,3,q-4]$ NMDS codes holding 2-designs were constructed;
2. In Section 4, two families of $[q-1,4,q-5]$ NMDS codes holding 2-designs were constructed;
3. In Section 5, four families $[q-1,5,q-6]$ NMDS codes holding 2-designs and a family of $[q,5,q-5]$ NMDS codes holding 3-designs were constructed;
4. In Section 6, three families $[q-1,6,q-7]$ NMDS codes holding 2-designs and two families of $[q,6,q-6]$ NMDS codes holding 3-designs were constructed.

We remark that only the first two families of NMDS codes satisfy the Assmus-Mattson Theorem. Though other families of NMDS codes do not satisfy the Assmus-Mattson Theorem, they still hold $t$-designs. Besides, all NMDS codes constructed in this paper are both $d$-optimal and $k$-optimal LRCs.

| Table 1: Optimal locally repairable codes |
|------------------------------------------|
| Optimal locally repairable codes         | parameters                                   |
| $C_D, C_H$                               | $(q - 1, 3, q - 4; 2)$                       |
| $C_D^\perp, C_H^\perp$                   | $(q - 1, q - 4, 3; q - 5)$                   |
| $C_{(1,3)}, C_{(2,3)}$                   | $(q - 1, 4, q - 5; q - 3)$                   |
| $C_{(1,3)}^\perp, C_{(2,3)}^\perp$      | $(q - 1, q - 5, 4; q - 6)$                   |
| $C_{(2,3,4)}, C_{(1,2,3)}, C_{(1,2,4)}$ | $(q - 1, 5, q - 6, 4; q - 4)$               |
| $C_{(2,3,4)}^\perp, C_{(1,2,3)}^\perp$ | $(q - 1, q - 6, 5, q - 7)$                   |
| $C_{(1,2,4)}$                            | $(q, 5, q - 5; q - 4)$                      |
| $C_{(1,2,4)}^\perp$                     | $(q, q - 5, 5; q - 6)$                      |
| $C_{(1,2,3,4)}, C_{(2,3,4,5)}, C_{(1,2,4,5)}$ | $(q - 1, 6, q - 7; q - 5)$               |
| $C_{(1,2,3,4)}^\perp, C_{(2,3,4,5)}^\perp$ | $(q - 1, q - 7, 6; q - 8)$               |
| $C_{(1,2,3,4)}^\perp, C_{(1,2,4,5)}^\perp$ | $(q, 6, q - 6; q - 5)$                      |
| $C_{(1,2,3,4)}^\perp, C_{(1,2,4,5)}$   | $(q, q - 6, 6; q - 7)$                      |

**Theorem 46.** All the locally repairable codes listed in table 7 are both $d$-optimal and $k$-optimal.

**Proof.** Note that all NMDS codes in Sections 3, 4 hold 2-designs or 3-designs. Then the desired conclusion follows from Lemmas 43, 44 and 45. 

In [13, 20, 21, 28], optimal locally repairable codes of distances 3 and 4 were constructed. We remark that the optimal locally repairable codes of distance 3 and 4 in this paper have different parameters from those in these papers.
Constructing NMDS codes holding $t$-designs for $t \geq 2$ is very challenging. All known infinite families of NMDS codes holding $t$-designs have small dimensions. It is open to construct NMDS codes holding $t$-designs with general dimensions. If Conjecture 36 in this paper is true, then this question can be tackled. The reader is invited to prove our conjecture.

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