A Whittaker-Plancherel Inversion Formula for $\text{SL}_2(\mathbb{C})$

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Abstract. In this paper, we establish a Whittaker-Plancherel inversion formula for $\text{SL}_2(\mathbb{C})$ from the analytic perspective of the Bessel transform of Bruggeman and Motohashi. The formula gives a decomposition of the Whittaker-Fourier coefficient of a compactly supported function on $\text{SL}_2(\mathbb{C})$ in terms of its Bessel coefficients attached to irreducible unitary tempered representations of $\text{SL}_2(\mathbb{C})$.

1. Introduction

Let $G = \text{SL}_2(\mathbb{C})$. Define

$$N = \left\{ n(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathbb{C} \right\}, \quad A = \left\{ s(z) = \begin{pmatrix} z & \ast \\ \ast & z^{-1} \end{pmatrix} : z \in \mathbb{C}^\times \right\}.$$ 

Let $B = AN$ be the Borel subgroup of $G$. Let

$$w = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$ 

Let $C_c^\infty(G)$ be the space of smooth and compactly supported functions on $G$. For $\lambda \in \mathbb{C}^\times$, define the additive character $\psi = \psi_\lambda$ by

$$\psi_\lambda(u) = e(\lambda u + \bar{\lambda}u),$$

with the usual abbreviation $e(x) = e^{2\pi i x}$.

For a function $f \in C_c^\infty(G)$ define the associated Whittaker function by

$$W_\psi^f(g) = \int_{\mathbb{C}} f(n(u)g)\psi(u)du, \quad g \in G.$$ 

Next, we introduce the Bessel coefficients of a function $f \in C_c^\infty(G)$.

$$W_\psi(f) = W_\psi^f(1) = \int_{\mathbb{C}} f(n(u))\psi(u)du.$$ 

1In number theory, when $f$ is a function on $G$ (not compactly supported) that comes from a Maass cusp form, there is a similar definition of $W_\psi_f(f) = W_\psi^f(1)$ related to the $\lambda$-th Fourier coefficient of $f$. 

2010 Mathematics Subject Classification. 22E46, 33C10.

Key words and phrases. Whittaker-Plancherel inversion formula, Bessel functions, Bessel transform.

The second author is supported by the Construct Program of the Key Discipline in Hunan Province.
For $t \in \mathbb{R}$ and $m \in \mathbb{Z}$, let $\pi_{t,m}$ denote the principal series representation of $G$ unitarily induced from the following character of $B$,

$$s(z)n(u) \rightarrow |z|^{2it}(z/|z|)^m.$$ 

Precisely, $\pi_{t,m}$ is the representation of $G$ by right shifts on the space of smooth functions $\varphi : G \rightarrow \mathbb{C}$ satisfying

$$\varphi(s(z)n(u)g) = |z|^{2it+2}(z/|z|)^m\varphi(g), \quad g \in G.$$ 

It is well known that these $\pi_{t,m}$ are all the irreducible unitary tempered representations of $G$, up to the equivalence $\pi_{t,m} \cong \pi_{-t,-m}$.

Let $J_\nu(z)$ be the classical Bessel function of the first kind,

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n(z/2)^{\nu+2n}}{n!(\nu+n+1)}.$$ 

We first introduce

$$J_{t,m}(z) = J_{-it-\frac{1}{2}m}(z)J_{-it+\frac{1}{2}m}(\bar{z}).$$ 

The function $J_{t,m}(z)$ is well defined for $z \in \mathbb{C}^\times$ in the sense that the expression on the right of (1.5) is independent on the choice of the argument of $z$ modulo $2\pi$. Define

$$J_{t,m}(z) = \begin{cases} 
\frac{1}{i\sinh(\pi t)} (J_{t,m}(z) - J_{-it,-m}(z)), & \text{if } m \text{ is even,} \\
\frac{1}{i\cosh(\pi t)} (J_{t,m}(z) + J_{-it,-m}(z)), & \text{if } m \text{ is odd.}
\end{cases}$$ 

It is understood that in the non-generic case when $t = 0$ and $m$ is even the right hand side should be replaced by its limit.

For $\pi = \pi_{t,m}$ and $\psi = \psi_\nu$, we define the attached Bessel function $j_{\pi,\psi}$, supported on the open Bruhat cell $BuB = NAuN$, such that

$$j_{\pi,\psi}(s(z)w) = 2\pi^2 |\lambda|^2 J_{t,m}(4\pi \lambda z),$$ 

and that $j_{\pi,\psi}$ is left and right $(\psi, N)$-equivariant, namely,

$$j_{\pi,\psi}(n(u)s(z)wn(v)) = \psi(u)\psi(v)j_{\pi,\psi}(s(z)w).$$ 

Let $f \in C_c^\infty(G)$. We now define the Bessel coefficient of $f$ attached to $\pi$ by

$$J_{\pi,\psi}(f) = \int_G f(g)j_{\pi,\psi}(g)dg,$$

where $dg$ is the Haar measure on $G$ defined by $dg = |z|^{-4}du^cd^z$ for the coordinates $g = n(u)s(z)wn(v)$ on the Bruhat cell, with $d^z = |z|^{-2}dz$ as usual. The Bessel coefficient (Bessel distribution) $J_{\pi,\psi}(f)$ has been investigated in [CQ]. Some results of [CQ] will be recollected in Appendix [B] but at this moment one only needs to know that the integral in (1.9) is convergent for all $f \in C_c^\infty(G)$.

Our Whittaker-Plancherel inversion formula for $G = SL_2(\mathbb{C})$ is as follows.

**Theorem 1.1.** Let notation be as above. For $f \in C_c^\infty(G)$ we have

$$W_{\psi_\nu}(f) = \frac{1}{32\pi^4|\lambda|^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} J_{\pi_{t,m},\psi}(f)(t^2 + m^2/4)dt.$$ 

The definition of $J_{t,m}(z)$ is slightly different from that in [Q4] or [CQ]. The new normalization here is only for notational convenience.
Observe that $J_{\mu,m}(z)$ is an even or odd function according as $m$ is even or odd. So the formula \[(1.11)\] splits into a pair of similar formulae if parity conditions are imposed on $f$. Also note that $\pi_{\mu,m}$ is trivial on the center if and only if $m$ is even, in which case $\pi_{\mu,m}$ may be regarded as representation of $PSL_2(\mathbb{C})$. So the even case of \[(1.10)\] is the following formula for $G/\{\pm 1\} = PSL_2(\mathbb{C})$.

**Corollary 1.2.** Suppose that $f \in C_c^\infty(G)$ is even, namely, $f(g) = f(-g)$ for all $g \in G$. Then
\[
W_\phi(f) = \frac{1}{32\pi|\lambda|^2} \sum_{d=m-\infty}^{\infty} \int_{-\infty}^{\infty} J_{\pi,2\lambda}\phi(f)(r^2 + d^2) \, dt.
\]

**Remarks.** A similar Whittaker-Plancherel formula for $G = SL_2(\mathbb{R})$ was obtained by Baruch and Mao [BM3]. They employ an analytic Bessel inversion formula of Kuznetsov\(^3\) while developing his celebrated trace formula for $PSL_2(\mathbb{Z})$ in [Kuz]. Both our main results and methods are in parallel with those of [BM3] although the analysis here is quite different. Likewise, the Bessel inversion formula that we use here was discovered by Bruggeman and Motohashi in their work on the Kuznetsov trace formula for $PSL_2(\mathbb{Z}[i])$.

Consider the $\tilde{\psi}$-Whittaker space $C_c^\infty(N\backslash G; \tilde{\psi})$ of smooth functions $W : G \rightarrow \mathbb{C}$, compactly supported modulo $N$, satisfying
\[
W(n(u)g) = \tilde{\psi}(u)W(g), \quad g \in G.
\]
Then each Whittaker function in $C_c^\infty(N\backslash G; \tilde{\psi})$ is a $W_\phi^\psi$ associated to some $f \in C_c^\infty(G)$. Recall that $W_\phi(f)$ is $W_\psi(f)^1$. On the other hand, in view of \[(1.8)\], we have
\[
J_{\pi,\phi}(f) = \int_C \int_{C^\times} W_\phi^\psi(s(z)u(n(v))j_{\pi,\phi}(s(z)u)\psi(v)|z|^{-4}d^\times z d\nu.
\]
Hence, by simple comparison, the reader may find that our formula is an analogue to the Plancherel formula of Harish-Chandra for $SL_2(\mathbb{C})$. See for example [Kna] Theorem 11.2. In contrast, the formula for $SL_2(\mathbb{R})$ in [BM3] is an analogue of [Kna] Theorem 11.6.

A general decomposition formula for the space $L^2(N\backslash G; \psi)$ of square integrable $\psi$-Whittaker functions of a real reductive group $G$ was obtained in [Wal2] §15.9. Our formula, in contrast, is point-wise and does not follow from his decomposition formula as indicated in [BM3]. Moreover, our proof is completely different from the proof of Wallach, since we do not use Harish-Chandra’s Plancherel formula. In particular, our proof brings to the front objects and tools which we think are interesting by themselves: Bessel functions and Bessel distributions of representations, orbital integrals and the Bessel transform of Bruggeman and Motohashi.

Recently, it was pointed out in [vdBK] that a lemma in the last chapter of [Wal2] is not correct. In response, Wallach gave a fixed proof of his Whittaker-Plancherel theorem in [Wal3]. He also presented a point-wise Whittaker-Plancherel formula for a $K$-finite $f$ in Harish-Chandra’s Schwartz space on $G$ (see [Wal3] Theorem 48). Moreover, the

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\(^3\)According to an anonymous referee, this formula should indeed be attributed to Sears and Titchmarsh [ST] (4.4)-(4.7)] (see also [HF] (4.14.1), (4.14.2)). The formula of Kuznetsov in [Kna] Appendix] however is slightly more general.
Whittaker-Plancherel formula for a reductive $p$-adic group $G$ was developed by Sakellaridis, Venkatesh [SV] §6.3] and Delorme [Del] by different methods. The proof of Sakellaridis and Venkatesh is relatively short and can be readily adapted to real groups as in the works of Beuzart-Plessis [BP1, BP2] (see [BP2] Proposition 2.14.2).

Finally, some remarks on the technical details are in order. Our analysis is quite different from that in Baruch and Mao [BM3]. However, if we use the differential equation instead of the recurrence relations for Bessel functions, along with an observation of the referee, we would have a simpler proof in the SL$_2$($\mathbb{R}$) case. Moreover, their estimates for the Bessel transforms may be considerably improved. See §5.1

**Assumption.** Subsequently, we shall assume, as we may without essential loss of generality, that $\lambda = 1$ so that $\psi(u) = \psi_1(u) = e(u + \overline{u})$. It is only up to the conjugation by $s(\sqrt{\lambda})$. Also, the $\psi$ will be suppressed from the notation for simplicity.

In fact, for general $\lambda$, we fix a $\sqrt{\lambda}$ and set $f_\lambda(g) = f(s(\sqrt{\lambda})^{-1}g(s(\sqrt{\lambda})))$. Then we readily see that $W_{\psi_1}(f) = W_{\psi_1}(f_\lambda)/|\lambda|^2$. Also, we may verify that $J_{\pi,\psi_1}(f) = J_{\pi,\psi_1}(f_\lambda)$. Hence the formula (1.10) in the general case follows from the case $\lambda = 1$.

**Acknowledgements.** Thanks are due to the anonymous referee for many remarks and suggestions, and for a crucial observation (Lemma 5.4) that greatly simplifies our arguments in an earlier draft of the paper and that leads us to a considerably improved estimate for the Bessel transform.

## 2. Orbital integrals and the Bessel transform

In this section, we start our study with the Bessel coefficients $J_\pi(f)$ on the right-hand side of (1.10). It will be shown how one can express $J_\pi(f)$ as the Bessel transform of an orbital integral for $f$ in a simple way.

### 2.1. Orbital integrals.

Recall that $\psi(u) = e(u + \overline{u})$. For $f \in C_c^\infty(G)$, define

\begin{equation}
O_f(z) = \int f(n(u)s(z)wn(v))\psi(u)\psi(v)dudv, \quad z \in \mathbb{C}^\times.
\end{equation}

Since the integrand is compactly supported, the integral in (2.1) is absolutely convergent. Moreover, it is easy to check that $O_f(z)$ is a smooth function on $\mathbb{C}^\times$ which vanishes around $z = 0$. These orbital integrals were investigated by Jacquet [Jac] §7 in a different context.

By the definitions in (1.9) and (2.1), along with (1.8), we have

\begin{align*}
J_\pi(f) &= \int_{\mathbb{C}^\times} O_f(z) d\hat{z} \\
&= \int f(n(u)s(z)wn(v))j_\pi(n(u)s(z)wn(v)) \cdot |z|^{-4}duded^\times z \\
&= \int j_\pi(s(z)w) \left( \int f(n(u)s(z)wn(v))\psi(u)\psi(v)dudv \right) |z|^{-4}d^\times z \\
&= \int j_\pi(s(z)w)O_f(z)|z|^{-4}d^\times z.
\end{align*}

More explicitly, in view of (1.7), we have

\begin{equation}
J_{\pi,\psi_1}(f) = 2\pi^2 \int_{\mathbb{C}^\times \setminus \{0\}} J_{0,m}(4\pi z)|z|^{-2}d^\times z.
\end{equation}
Now we set
\[ G_f(z) = \frac{O_f(z/4\pi)}{|z|^2} \]
so that
\[ J_{\pi \sigma}(f) = 32\pi^4 \int_{\mathbb{C} \setminus \{0\}} J_{it,m}(z) G_f(z) d^\times z. \]
This immediately brings into mind the Bessel transform of Bruggeman and Motohashi.

2.2. The Bessel transform of Bruggeman and Motohashi. Let \( G(z) \) be a function in the space \( L^1(\mathbb{C} \setminus \{0\}, d^\times z) \). Following [BMS] [LG], we define its Bessel transform by
\[ \hat{G}(it,m) = \int_{\mathbb{C} \setminus \{0\}} J_{it,m}(z) G(z) d^\times z. \]
Since \( J_{it,m}(z) \) is of order \( O(1) \) at 0 and \( O(1/|z|) \) at \( \infty \) except for the non-generic case when \( t = 0 \) and \( m \) even (see (2.5) and (4.7)), it follows that these integrals are absolutely convergent if \( (it,m) \) is generic.

The Bessel inversion formula of Bruggeman and Motohashi in [BMS] was originally for \( \text{PSL}_2(\mathbb{C}) \). It was soon generalized to \( \text{SL}_2(\mathbb{C}) \) in the thesis of Lokvenec-Guleska [LG]. We reformulate her formula in the following theorem:

**Theorem 2.1** (Bruggeman, Motohashi and Lokvenec-Guleska). Assume that \( G(z) \in L^1(\mathbb{C} \setminus \{0\}, d^\times z) \cap L^2(\mathbb{C} \setminus \{0\}, d^\times z) \) is continuous and that
\[ \hat{G}(it,m) = O(1/(t^2 + m^2 + 1)^\theta(|m| + 1)) \]
for some \( \theta > 1 \). Then
\[ G(z) = \frac{1}{8} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} (-)^m J_{it,m}(z) \hat{G}(it,m)(t^2 + m^2/4) dt. \]

**Corollary 2.2.** Assume that \( G(z) \) is a smooth function on \( \mathbb{C} \setminus \{0\} \) such that \( G(z) \) vanishes in a neighborhood of 0 and \( G(z) = O(1/|z|^{\rho}) \) for some \( \rho > 0 \). Assume also that \( \hat{G}(it,m) = O(1/(t^2 + m^2 + 1)^\theta(|m| + 1)) \) for some \( \rho > 1 \). Then the Bessel inversion formula \( (2.5) \) holds for \( G(z) \).

**Remark 2.3.** The formula \( (2.5) \) may be found in [BMS, Theorem 11.1] and [LG, Theorem 12.2.1], but it is proven only for function \( G(z) \) compactly supported on \( \mathbb{C} \setminus \{0\} \). It is not sufficient for our purpose. They however also proved a Parseval-Plancherel formula for the Bessel transform, and it would enable us to prove \( (2.5) \) for \( G(z) \) as in Theorem 2.1. This will be done in Appendix A.

**Remark 2.4.** This Bessel inversion formula is the bridge to the “Kloosterman form” of the Kuznetsov trace formula for \( \text{SL}_2(\mathbb{C}) \) from its “spectral form” in [BMS, LG]. There is however a direct approach to the Kloosterman form by a representation theoretic method of Cogdell and Piatetski-Shapiro in [Q3]. Compare also [Kuz] and [CPS] for \( \text{SL}_2(\mathbb{R}) \).

**Remark 2.5.** In [Q1, Chapter 3], the Bessel inversion formula for \( \text{SL}_2(\mathbb{C}) \), along with the \( \text{GL}_1(\mathbb{C}) \times \text{GL}_2(\mathbb{C}) \) local functional equations, is applied in a formal way to derive a formula of Bessel functions for \( \text{GL}_3(\mathbb{C}) \). A similar formula is obtained for \( \text{GL}_3(\mathbb{R}) \) by the Bessel inversion formula of Kontorovich, Lebedev and Kuznetsov (Sears and Titchmarsh).

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4Note that Lokvenec-Guleska normalizes the Bessel functions in an unnatural way to be all even.
One of the main technical results of this work is to show that the function $G_f(z)$ satisfies the conditions in Corollary 2.2 for every $f \in C_c^\infty(G)$. For this, we shall first need the results of Jacquet [Jac] on the asymptotic of orbital integrals $O_f(z)$ as $|z| \to \infty$. In particular, Jacquet proves that the Whittaker coefficient $W(f)$ on the left-hand side of (1.10) arises in the leading terms of the asymptotic formula for $O_f(z)$ (if $f$ is even or odd).

3. Asymptotic of orbital integrals

We now recollect Jacquet’s results on orbital integrals [Jac §7] as follows. Since it is so elegant, we are prompted to include his proof here.

**Theorem 3.1 (Jacquet).** Let $f \in C_c^\infty(G)$. Then there are two compactly supported smooth functions $H_+$ and $H_-$ on $\mathbb{C}$ such that

\[ O_f(z) = \psi(2z)|z|H_+(1/z) + \psi(-2z)|z|H_-(1/z), \]

and

\[ H_+(0) = W_f(1)/2, \quad H_-(0) = W_f(-1)/2. \]

**Remark 3.2.** There is an alternative proof of the theorem by Baruch and Mao in [BM3] for $G = SL_2(\mathbb{R})$, directly using the method of stationary phase. It would however be quite technical to generalize their arguments to $\mathbb{C}$. Jacquet’s original proof applies the Parseval-Plancherel formula of Weil as a substitute for stationary phase. It works not only for $\mathbb{R}$ and $\mathbb{C}$ but also for non-Archimedean local fields.

**Proof of Theorem 3.1** Let

\[ N^- = \left\{ n^-(u) = \begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix} : u \in \mathbb{C} \right\}. \]

We have

\[ G = NAwN \cup N^-AN. \]

Thus we may assume that the support of $f$ is contained in one of these cells. Recall the definitions of $W_f(g)$ and $O_f(z)$ in (1.2) and (2.1). If $f$ is supported on $NAwN$, then $W_f(1) = W_f(-1) = 0$ and $O_f(z)$ is compactly supported on $\mathbb{C}^\times$. Now assume that the support of $f$ is contained in $N^-AN$. Then the function $T$ defined by

\[ T(u,z) = \int f(n(u)s(z)n(v))\psi(v)dv \]

is smooth and compactly supported on the product $\mathbb{C} \times \mathbb{C}^\times$. It is clear that

\[ T(0,1) = W_f(1), \quad T(0,-1) = W_f(-1). \]

Note that for $u \in \mathbb{C}^\times$ we have

\[ \begin{pmatrix} 1 & u \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z & v \\ z^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & u^{-1} \\ u^{-1} & 1 \end{pmatrix} = \begin{pmatrix} z^{-1}u & 1 \\ zu^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & v - z^2u^{-1} \\ 1 & 1 \end{pmatrix}. \]

After suitable changes of variables, $O_f(z)$ may be reformulated as follows

\[ O_f(z) = |z|^2 \int T((uz)^{-1},u)\psi(z(u+u^{-1}))du. \]

Since the phase function $u + u^{-1}$ has two stationary points 1 and -1, we may introduce a suitable partition of unity and write

\[ O_f = K_0 + K_+ + K_- \]
so that \( K_0 \) is a Schwartz function on \( \mathbb{C}^\times \) and that \( K_\pm \) is of the form
\[
K_\pm(z) = |z|^2 \int T_\pm((uz)^{-1}, u) \psi(z(u + u^{-1}))du;
\]
the function \( T_\pm \) is smooth of compact support on \( \mathbb{C} \times \mathbb{C}^\times \); furthermore, the projection of the support of \( T_\pm \) to the second factor is contained in a small neighborhood of \( \pm 1 \) and
\[
T_\pm(0, \pm 1) = T(0, \pm 1) = W_f(\pm 1).
\]
In the integral for \( K_\pm \), we first introduce a new variable given by
\[
u = w_2 \Rightarrow \nu = u_1 - \frac{u_1 - u_1^{-1}}{2} \Rightarrow \nu = u^{-1/2},
\]
where the square root is chosen to be the principal branch. We find then
\[
K_\pm(z) = \psi(2z)|z|^2 \int \Phi_\pm(z^{-1}, w) \psi(zw^2)dw,
\]
with
\[
\Phi_\pm(z, w) = T_\pm(2w^{-1}, u) |dw|/|du|^2.
\]
At this point, we invoke the Parseval-Plancherel identity of Weil,
\[
(3.3)
\]
in which \( \Phi \) is a Schwartz function on \( \mathbb{C} \) and \( \hat{\Phi} \) is its Fourier transform
\[
\hat{\Phi}(u) = \int \Phi(v)\psi(uv)dv.
\]
As a consequence of (3.3), we have
\[
K_\pm(z) = \psi(2z)|z|^2 H_\pm(1/z),
\]
where we have set
\[
H_\pm(z) = \frac{1}{2} \int \hat{\Phi}_\pm(z, w) \psi(- \frac{1}{2}w^2z)dw;
\]
\[
\hat{\Phi}_\pm(z, w) = \int \Phi_\pm(z, v)\psi(vw)dv.
\]
Now \( H_\pm \) is certainly smooth near 0 and
\[
2H_\pm(0) = \int \hat{\Phi}_\pm(0, w)dw = \Phi_\pm(0, 0) = T_\pm(0, \pm 1) = W_f(\pm 1).
\]
Q.E.D.

3.1. **Proof of Theorem 1.1.** We are now ready to prove Theorem 1.1. We shall leave two questions to § 5. The first point is that our function \( G_f(z) \) that was defined in (2.2) satisfies the conditions of Corollary 2.2 so that we can apply the Bessel inversion formula for \( G_f(z) \). The second is a change of order of a limit, summation and integration that will be pointed out in the course of the proof.

Let \( f \in C_\infty(G) \). We can write \( f \) as the sum of an even and an odd function. So, by linearity, we may well assume that \( f \) is itself even or odd. As the arguments for the two cases are similar, we shall only prove the even case, that is, Corollary 1.2.

Now let \( f \) be even. Then \( O_f(z) \) and hence \( G_f(z) \) are even functions on \( \mathbb{C} \setminus \{0\} \). Observe that
\[
W_f(1) = W_f(-1) = W(f).
\]
It follows from (2.2), (3.1) and (3.2) that
\[ W(f) = \lim_{z \to \infty} \frac{4\pi \Re \{G_f(z)\}}{\cos(z + \bar{z})}, \]
where the limit is taken on a set of \( z \) such that the denominator \( \cos(z + \bar{z}) \) is bounded away from 0. By (2.5), we have
\[ G_f(z) = \frac{1}{8} \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} J_{i,2d}(z) G_f(it, 2d)(t^2 + d^2) \, dt. \]
Hence
\[ W(f) = \lim_{z \to \infty} \frac{\pi |z|}{2 \cos(z + \bar{z})} \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} J_{i,2d}(z) G_f(it, 2d)(t^2 + d^2) \, dt. \]
Assume now that it is legitimate to interchange the order of the limit, summation and integration in (3.4). This will be justified using the dominated convergence theorem in \( \S \) 5.2. We then have
\[ W(f) = \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\pi |z| J_{i,2d}(z)}{2 \cos(z + \bar{z})} \bar{G}_f(it, 2d)(t^2 + d^2) \, dt. \]
It follows from the asymptotic formula for \( J_{i,2d}(z) \) (see (4.7)) that
\[ \lim_{z \to \infty} \frac{\pi |z| J_{i,2d}(z)}{2 \cos(z + \bar{z})} = 1. \]
Consequently,
\[ W(f) = \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{G}_f(it, 2d)(t^2 + d^2) \, dt. \]
In view of (2.3) and (2.4), we conclude that
\[ W(f) = \frac{1}{32\pi^4} \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} J_{\pi,2d}(f)(t^2 + d^2) \, dt. \]

4. Estimates for the Bessel function \( J_{i,m}(z) \)

4.1. Preliminaries on the Bessel function \( J_{\mu,m}(z) \). In this section, we recollect some results on the Bessel function \( J_{\mu,m}(z) \) for \( \mu \in \mathbb{C} \) and \( m \in \mathbb{Z} \). We do not feel it is necessary to restrict ourselves to Bessel functions \( J_{i,m}(z) \) of pure imaginary order.

Let \( J_{\nu}(z) \) be the Bessel function of the first kind of order \( \nu \). Let \( H_{\nu}^{(1)}(z) \) and \( H_{\nu}^{(2)}(z) \) be the Hankel functions of order \( \nu \). They all satisfy the Bessel differential equation
\[ z^2 \frac{d^2w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = 0. \]

4.1.1. We have the definition
\[ J_{\mu,m}(z) = \begin{cases} \frac{1}{\sin(\pi \mu)} (J_{\mu,m}(z) - J_{-\mu,-m}(z)), & \text{if } m \text{ is even}, \\ \frac{1}{i \cos(\pi \mu)} (J_{\mu,m}(z) + J_{-\mu,-m}(z)), & \text{if } m \text{ is odd}. \end{cases} \]
with
\[ J_{\mu,m}(z) = J_{-\mu-\frac{1}{2}m}(z) J_{\mu+\frac{1}{2}m}(\xi). \]
It is readily seen from the series expansion of \( J_r(z) \) (see (1.4)) that
\[
J_r(z) \ll_r |z^r|, \quad |z| \leq 1.
\]
It follows that if \( \mu \) is generic, that is \( 2\mu \not\in 2\mathbb{Z} + m \), then
\[
(4.4) \quad J_{\mu,m}(z) \ll_{\mu,m} |z|^{-2\mu} + |z|^{2\mu}, \quad |z| \leq 1.
\]
In particular,
\[
(4.5) \quad J_{0,m}(z) \ll_{t,m} 1, \quad |z| \leq 1,
\]
except for the non-generic case when \( t = 0 \) and \( m \) even. In the non-generic case, we have a slightly worse estimate (see (Q4) (2.24))
\[
(4.6) \quad J_{0,m}(z) \ll_m \log(2/|z|), \quad |z| \leq 1, \quad (m \text{ even}).
\]
We have a second expression of \( J_{\mu,m}(z) \) in terms of Hankel functions. Define
\[
H_{\mu,m}^{(1,2)}(z) = H_\mu^{(1,2)}(z) H_{\mu - m}^{(1,2)}(z).
\]
By the connection formulae (Wat) 3.6 (1),(2)], we have
\[
J_{\mu,m}(z) = \frac{i}{2} \left( (-)^m e^{\pi i \mu} H_{\mu,m}^{(1)}(z) - e^{-\pi i \mu} H_{\mu,m}^{(2)}(z) \right).
\]
From the asymptotic formula of Hankel functions (Wat) 7.2 (5),(6)], we deduce that
\[
(4.7) \quad J_{\mu,m}(z) \sim \frac{1}{\pi |z|} \left( (-)^m e \left( \frac{1}{2} \mu (z + \bar{z})/2\pi \right) + e \left( -\mu (z + \bar{z})/2\pi \right) \right), \quad |z| \to \infty.
\]
It is then clear that
\[
(4.8) \quad J_{\mu,m}(z) = O(1/|z|), \quad |z| \to \infty.
\]
Moreover, it follows from (Olv) §7.13.1 that
\[
(4.9) \quad J_{\mu,m}(z) \ll 1/|z|, \quad |z| > |\mu|^2 + m^2 + 1,
\]
where the implied constant is absolute.

4.1.2. Recurrence formulae. Recall the following recurrence formula (Wat) 3.2 (2)],
\[
(4.10) \quad 2J_r(z) = J_{r-1}(z) - J_{r+1}(z).
\]
By (4.10), it is straightforward to derive the corresponding recurrence formulae
\[
2i \partial J_{\mu,m}(z)/\partial \bar{z} = J_{\mu + 1,m+1}(z) + J_{\mu - 1,m-1}(z),
\]
\[
(4.11) \quad 2i \partial J_{\mu,m}(z)/\partial \bar{z} = J_{\mu + 1,m-1}(z) + J_{\mu - 1,m+1}(z).
\]
It follows from (4.8) and (4.11) that
\[
(4.12) \quad \partial J_{\mu,m}(z)/\partial \bar{z}, \quad \partial J_{\mu,m}(z)/\partial \bar{z} = O(1/|z|), \quad |z| \to \infty.
\]

4.1.3. Differential equations. Define
\[
\nabla = \bar{z} \frac{\partial^2}{\partial \bar{z}^2} + z \frac{\partial}{\partial z} + \bar{z}^2 = \left( \frac{\partial}{\partial \bar{z}} \right)^2 + \bar{z}.
\]
and its conjugation
\[
\nabla = \bar{z} \frac{\partial^2}{\partial \bar{z}^2} + z \frac{\partial}{\partial z} + \bar{z}^2 = \left( \frac{\partial}{\partial \bar{z}} \right)^2 + \bar{z}.
\]
It is clear from (4.11) (4.13) that \( J_{\mu,m}(z) \) is an eigenfunction of both \( \nabla \) and \( \nabla \):
\[
(4.15) \quad \nabla J_{\mu,m}(z) = (\mu + m/2)^2 J_{\mu,m}(z),
\]
\[
(4.16) \quad \nabla J_{\mu,m}(z) = (\mu - m/2)^2 J_{\mu,m}(z).
\]
4.1.4. An integral representation. In the polar coordinates, we have the following integral representation of \( J_{\mu,m}(xe^{i\phi}) \) (see [Qi2 Corollary 6.17] or [BMS Theorem 12.1]),

\[
J_{\mu,m}(xe^{i\phi}) = \frac{2(-i)^m}{\pi} \int_0^\infty y^{2\mu-1} E_{\mu}(ye^{i\phi})^{-m} J_m (xY(ye^{i\phi})) \, dy,
\]

with

\[
Y(z) = \left| z + z^{-1} \right|, \quad E(z) = \left( z + z^{-1} \right) / \left| z + z^{-1} \right|.
\]

The integral in (4.17) is absolutely convergent only when \( |\text{Re} \mu| < \frac{1}{4} \).

4.2. Uniform bounds for the Bessel function \( J_{\mu,m}(z) \). The following uniform bound for \( |z| J_{\mu,m}(z) \) will be needed in \( \S 5 \) to conclude the proof of Theorem 1.1.

**Lemma 4.1.** We have uniformly

\[
|z| J_{\mu,m}(z) \leq t^2 + m^2 + 1.
\]

**Proof.** It follows from [HM Proposition 8] that

\[
\sqrt{x} J_m(x) \leq |m| + 1, \quad x > 0.
\]

By the integral representation (4.17) (4.18), we have

\[
|J_{\mu,m}(xe^{i\phi})| \leq \frac{2}{\pi} \int_0^{\infty} |J_m(xY(ye^{i\phi}))| \, dy / y,
\]

with \( Y(ye^{i\phi}) \) defined as in (4.18). Thus (4.20) yields

\[
J_{\mu,m}(xe^{i\phi}) \leq \frac{|m| + 1}{\sqrt{x}} \int_0^{\infty} \frac{dy}{y \sqrt{Y(ye^{i\phi})}}.
\]

Note that we always have

\[
Y(ye^{i\phi}) \geq |y - 1/y|,
\]

and that the integral

\[
\int_0^{\infty} \frac{dy}{y |y - 1/y|^{1/2}}
\]

is bounded. Hence

\[
J_{\mu,m}(xe^{i\phi}) \leq \frac{|m| + 1}{\sqrt{x}}.
\]

When \( |z| \leq t^2 + m^2 + 1 \), it follows from (4.21) that

\[
J_{\mu,m}(z) \leq \frac{|m| + 1}{\sqrt{|z|}} \leq \frac{t^2 + m^2 + 1}{|z|}.
\]

When \( |z| > t^2 + m^2 + 1 \), the bound \( J_{\mu,m}(z) \leq 1/|z| \) in (4.9) is more than sufficient. Q.E.D.

Note that if we use (4.27) with \( r = 0 \) in Lemma 4.3 below, the uniform bound (4.19) may be improved into

\[
|z| J_{\mu,m}(z) \leq (t^2 + m^2 + 1)^{2/3}.
\]

The following two lemmas will only be used in Appendix A. So the reader is advised to directly jump to \( \S 5 \). In these lemmas, the estimates are focused on the aspect of \( t \) and \( m \) in order to optimize the condition on \( G(t,m) \) in Theorem 2.1 or Corollary 2.2. In view of Theorem 5.1, the reader may find that the optimized condition is far from necessary.
Lemma 4.2. We have
\[
\mathcal{J}_{\nu,m}(z) \leq \begin{cases} 
\frac{\exp\left(\frac{|z|^2}{4}/|t| + 1\right)}{\sqrt{t^2 + m^2}}, & \text{if } m \text{ is even}, \\
\frac{\exp\left(\frac{|z|^2}{2}\right)}{\sqrt{t^2 + m^2}}, & \text{if } m \text{ is odd}.
\end{cases}
\]
(4.23)

Proof. Let us assume that \((it, m)\) is generic, as otherwise (4.23) is trivial. Recall from [Wat. 3.13 (1)] that for \(\nu \neq -1, -2, -3, \ldots\)
\[
J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1)} (1 + \theta),
\]
(4.24)
where
\[
|\theta| \leq \exp\left(\frac{|z|^2}{4|\nu_0 + 1|}\right) - 1,
\]
(4.25)
and \(|\nu_0 + 1|\) is the smallest of the numbers \(|\nu + 1|, |\nu + 2|, |\nu + 3|, \ldots\). Note that the bound in (4.25) is awful when \(|\nu_0 + 1|\) is very close to 0. By modifying the arguments in [Wat. §2.11], we may also prove
\[
|\theta| \leq \frac{\exp\left(\frac{|z|^2}{4} - 1\right)}{|\nu_0 + 1|}.
\]
(4.26)
We now apply these to \(J_{\nu,m}(z)\) as defined in (1.5). For \(\nu = -it + \frac{1}{2}m\), we have
\[
|\nu_0 + 1| \geq \begin{cases} 
|t|, & \text{if } m \text{ is even}, \\
|t + \frac{1}{2}|, & \text{if } m \text{ is odd}.
\end{cases}
\]
Also note that
\[
\left|\frac{(z/2)^{-it + \frac{1}{2}m}}{\Gamma(-it - \frac{1}{2}m + 1) \Gamma(-it + \frac{1}{2}m + 1)}\right| = \begin{cases} 
\frac{|\sinh(\pi t)|}{\pi \sqrt{t^2 + m^2/4}}, & \text{if } m \text{ is even}, \\
\frac{\cosh(\pi t)}{\pi \sqrt{t^2 + m^2/4}}, & \text{if } m \text{ is odd}.
\end{cases}
\]
It then follows from (4.24)-(4.26) that
\[
\mathcal{J}_{\nu,m}(z) \ll \begin{cases} 
\min\{\exp\left(\frac{|z|^2}{4}|t|\right), \exp\left(\frac{|z|^2}{4}/|t| + 1\right)\}, & \text{if } m \text{ is even}, \\
\exp\left(\frac{|z|^2}{4|it + 2|}\right), & \text{if } m \text{ is odd}.
\end{cases}
\]
(4.23)
This clearly implies (4.23). Q.E.D.

Lemma 4.3. For any given \(0 < r < \frac{1}{2}\), we have
\[
\mathcal{J}_{\nu,m}(z) \ll \frac{1}{(m + 1)|z|^{1/3 - r}}.
\]
(4.27)
Proof. For this lemma, we need some uniform bounds for the classical Bessel function \(J_\nu(x)\) of real argument. First of all, it is well known that for all \(x\) real and \(\nu \geq 0\),
\[
|J_\nu(x)| \leq 1.
\]
See for example [AS. 9.1.60]. Moreover, the following bounds are obtained by L. J. Landau [Lan] in his study of monotonicity properties of Bessel functions,
\[
|J_\nu(x)| < b/\sqrt[3]{x}, \quad |J_\nu(x)| \leq c/|x|^{1/3},
\]
(4.27)
with $b = 0.674885...$ and $c = 0.785746....$ Combining the bounds above, we deduce easily that, for any $0 \leq r \leq \frac{1}{3}$,

\begin{equation}
J_{\nu}(x) \ll_{\epsilon} \frac{1}{(\nu + 1)^{r}|x|^{1/3-r}}, \quad x \text{ real, } \nu \geq 0.
\end{equation}

We now return to the proof. By the arguments that lead us to (4.21), if we apply (4.28) in place of (4.20) then

\begin{align*}
J_{it,m}(xe^{i\theta}) \ll_{\epsilon} & \frac{1}{(|m| + 1)^{r}x^{1/3-r}} \int_{0}^{\infty} dy \frac{dy}{yY(ye^{i\theta})y^{1/3-r}} \\
& \ll_{\epsilon} \frac{1}{(|m| + 1)^{r}x^{1/3-r}} \int_{0}^{\infty} dy \frac{dy}{y|y - 1/y|^{1/3-r}} \\
& \ll_{\epsilon} \frac{1}{(|m| + 1)^{r}x^{1/3-r}},
\end{align*}

provided that $0 \leq r < \frac{1}{3}$. Q.E.D.

5. Completion of the proof of Theorem 1.1

To conclude the proof of Theorem 1.1 we need to prove that $G_{f}(z)$ satisfies the requirements of Corollary 2.2 so that Bessel inversion holds and that the change of order of the limit, summation and integration which we employed in the proof is justified.

5.1. Bounds on the Bessel transform. First, we prove the following estimate for the Bessel transform $\tilde{G}_{f}(it,m)$ as defined in (2.4). Our proof can be easily adapted in the real case so that the estimates in [BM3] may be considerably improved. This is based on the idea of applying the differential operator $\nabla$ (instead of recurrence relations as in [BM3]) and an observation of the anonymous referee as in Lemma 5.4.

**Theorem 5.1.** We have $\tilde{G}_{f}(it,m) = O(1/(t^{2} + m^{2} + 1)^{4})$ for arbitrary $A \geq 0$, with the implied constant depending on $A$ and $f$.

First of all, we need a simple formula for partial integration. It will be very convenient to integrate on differential forms in terms of $z$ and $\bar{z}$. In particular, the volume form on \( \mathbb{C} \) is $idz \wedge d\bar{z}$ (there is a slight abuse of notation as $dz$ was the Haar measure on $\mathbb{C}$). By Stokes’ theorem, one can easily derive the following formula; the proof is left to the reader as an exercise.

**Lemma 5.2.** Let $F$, $G \in C^{\infty}(\mathbb{C} \setminus \{0\})$. Assume that $F(z)G(z) = o(1)$ for both $|z| \to 0$ and $|z| \to \infty$. We have

\begin{equation}
\int_{C \setminus \{0\}} \frac{\partial F(z)}{\partial z} \cdot G(z) \frac{idz \wedge d\bar{z}}{z^{2} \bar{z}} = - \int_{C \setminus \{0\}} F(z) \cdot \frac{\partial G(z)}{\partial z} \frac{idz \wedge d\bar{z}}{z^{2} \bar{z}},
\end{equation}

provided that one of the integrals is convergent.

**Corollary 5.3.** Let $\nabla$ be defined as in (4.13). Let $G(z)$ be a smooth function that vanishes around $0$ and such that $G(z) = o(1), \partial G(z)/\partial z = o(1)$ for $|z| \to \infty$. We have

\begin{equation}
\int_{C \setminus \{0\}} \tilde{J}_{it,m}(z) \cdot G(z) d^{	imes}z = \frac{1}{(it + m/2)^{2}} \int_{C \setminus \{0\}} \tilde{J}_{it,m}(z) \cdot \nabla G(z) d^{	imes}z;
\end{equation}

the integral on the left is convergent.
Proof. The identity ([5.1]) follows from two applications of Lemma 5.2 along with
\[ \nabla J_{\alpha,m}(z) = (it + m/2)^2 J_{\alpha,m}(z). \]

See (4.15). Note that the condition of Lemma 5.2 may be verified by the bounds for \( J_{\alpha,m}(z) \) and \( \partial J_{\alpha,m}(z)/\partial z \) in (4.8) and (4.12) together with the conditions on \( G(z) \). Q.E.D.

The following lemma is due to the anonymous referee.

Lemma 5.4. Define \( \mathcal{V} \) to be the space of functions on \( \mathbb{C} \setminus \{0\} \) of the form
\[ G(z) = \frac{e((z + \bar{z})/2\pi)}{|z|} H_+(1/z) + \frac{e(-z/2\pi)}{|z|} H_-(1/z), \]
where \( H_\pm \in C^\infty(\mathbb{C}) \). Then \( \mathcal{V} \) is stable by the differential operator \( \nabla \). Moreover, if \( G(z) \) is a function in \( \mathcal{V} \) as above then \( G(z) = O(1/|z|) \), \( \partial G/z = O(1/|z|) \) for \( |z| \to \infty \).

Proof. By direct calculations, we find that
\[ \frac{\partial}{\partial z} \left\{ \frac{e((z + \bar{z})/2\pi)}{|z|} H_\pm(1/z) \right\} = \frac{e((z + \bar{z})/2\pi)}{|z|} \cdot \left( \left( -\frac{1}{2z} \pm i \right) H_\pm(1/z) - \frac{1}{z^2} H_\pm^1(1/z) \right), \]
and
\[ \nabla \left\{ \frac{e((z + \bar{z})/2\pi)}{|z|} H_\pm(1/z) \right\} = \frac{e((z + \bar{z})/2\pi)}{|z|} \cdot \left( \frac{1}{z^2} H_\pm(1/z) + 2 \left( \frac{1}{z} + i \right) H_\pm^1(1/z) + \frac{1}{z^2} H_\pm^2(1/z) \right), \]
with
\[ H_\pm^1(z) = \partial H_\pm(z)/\partial z, \quad H_\pm^2(z) = \partial^2 H_\pm(z)/\partial z^2. \]

Then our assertions become obvious. Q.E.D.

In view of (2.2) and Theorem 3.1, we have \( G_f \in \mathcal{V} \). Hence Lemma 5.4 implies that \( \nabla^k G_f \in \mathcal{V} \) for any integer \( k \geq 0 \) and that Corollary 5.3 is applicable for each \( \nabla^k G_f \). Theorem 5.1 follows immediately from applying Corollary 5.3 repeatedly, along with a final estimation by Lemma 5.4.

From Theorem 5.1 we infer that the conditions in Corollary 2.2 are satisfied by \( G_f(z) \).

5.2. Change of order of limit, summation and integration. Next, we prove that the expression inside the limit in (3.4) is convergent absolutely and uniformly in \( z \). By (4.19), we have
\[ \frac{|z| J_{\alpha,2d}(z)}{\cos(z + \bar{z})} \leq t^2 + d^2 + 1, \]
which is uniform on the set of \( z \) such that \( |\cos(z + \bar{z})| > \frac{1}{2} \) say. By Theorem 5.1 with \( A = 4 \) say, we have
\[ G_f(it, 2d) \leq \frac{1}{(t^2 + d^2 + 1)^4}. \]

Consequently,
\[ \sum_{d=-\infty}^{\infty} \sum_{d=-\infty}^{\infty} \left| \frac{|z| J_{\alpha,2d}(z)}{\cos(z + \bar{z})} \right| G_f(it, 2d) dt \leq \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \left( t^2 + d^2 \right) dt \leq \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dt}{(t^2 + d^2 + 1)^2} < \infty. \]
Hence we can use the dominated convergence theorem to move the limit into the sum and integral in (3.4). Similar is the odd case.

Appendix A. Proof of Theorem 2.1

Recall the Bessel transform defined in (2.4). We have the following Parseval-Plancherel formula due to Bruggeman, Motohashi and Lokvenec-Guleska ([BM5 (11.3)] and [LG Proposition 12.2.2]),

\[
\int_{\mathbb{C}\setminus\{0\}} G(z) \overline{F(z)} dz = \frac{1}{8} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{G}(it, m) \hat{F}(it, m) (t^2 + m^2/4) dt,
\]

for functions \( G, F \in C_{c}^{\infty}(\mathbb{C}\setminus\{0\}) \). Observe that

\[
\overline{F(it, m)} = (-)^m \hat{F}(it, m).
\]

It follows from (A.1) that the Bessel transform establishes an (abstract) isometry between the Hilbert spaces

\[
L^2(\mathbb{C}\setminus\{0\}, d^x z) \longrightarrow L^2(\mathbb{Z} \times \mathbb{R}, (t^2 + m^2/4) dt/8).
\]

Now let \( G(z) \in L^1(\mathbb{C}\setminus\{0\}, d^x z) \cap L^2(\mathbb{C}\setminus\{0\}, d^x z) \) and \( F(z) \in C_{c}^{\infty}(\mathbb{C}\setminus\{0\}) \) so that their Bessel transform \( \hat{F}(it, m) \) and \( \hat{F}(it, m) \) given as in (2.4) are both well defined. We may then write (A.1) in the following way

\[
\int_{\mathbb{C}\setminus\{0\}} G(z) \overline{F(z)} dz = \frac{1}{8} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} (-)^m \hat{G}(it, m) \hat{F}(it, m) (t^2 + m^2/4) \overline{F(z)} dz dt.
\]

Thus the Bessel inversion formula (2.5) in Theorem 2.1 follows immediately if \( G(z) \) is continuous and the right hand side converges absolutely. This in turn follows from our assumption \( \hat{G}(it, m) = O(1/(t^2 + m^2 + 1)^q(|m| + 1)) \) \((q > 1)\) in Theorem 2.1 along with the uniform bounds for \( \hat{J}_{it, m}(z) \) in Lemma 4.2 and 4.3. In fact, by the estimates in (4.23) and (4.27), we have

\[
\hat{J}_{it, m}(z) \leq \begin{cases} 1/(|t| + |m|), & \text{if } |t| > 1, \\ 1, & \text{if } |t| \leq 1, \end{cases}
\]

which is uniform on any given compact subset of \( \mathbb{C}\setminus\{0\} \), say on the support of \( F \). Hence the sum of integrals on the right hand side of (2.5) is uniformly and absolutely bounded by

\[
\sum_{m=-\infty}^{\infty} \frac{1}{|m| + 1} \left( \int_{-\infty}^{-1} + \int_{1}^{\infty} \frac{dt}{(|t| + |m|)^{2q-1}} + \int_{-1}^{1} \frac{dt}{(|m| + 1)^{2q-2}} \right) < \infty,
\]

desired.

Appendix B. Representation theory of Bessel functions and distributions for \( SL_2(\mathbb{C}) \)

In this appendix, we give a brief review of Bessel distributions and Bessel functions for \( SL_2(\mathbb{C}) \) in [CQ]. For additional results on Bessel distributions and Bessel functions see [Bar1], [BM4], [CPS] and [Mot].

B.1. Bessel distributions for \( SL_2(\mathbb{C}) \). The Bessel distribution \( J_{\pi, \phi} \) given by (1.5)-(1.9) were defined in an ad hoc way. The definition of \( J_{\pi, \phi} \) in representation theory is another story.

Let \( G = SL_2(\mathbb{C}) \) and \( K = SU_2(\mathbb{C}) \). Let \( \pi \) be an infinite dimensional irreducible unitary representation of \( G \) on a Hilbert space \( H \) and \( \langle , \rangle \) be a \( G \)-invariant nonzero inner product on
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$H$. Let $H_\infty$ be the subspace of smooth vectors in $H$. Let $\psi = \psi_\lambda$ be an additive character as defined in (1.1). It is well known that there exists a nonzero continuous $\psi$-Whittaker functional $L$ on $H_\infty$, unique up to scalars, satisfying

$$L(\pi(n(u))v) = \psi(u)L(v), \quad n(u) \in N, \ v \in H_\infty.$$  

Let $$W_v(g) = L(\pi(g)v), \quad v \in H_\infty, \ g \in G,$$

be the Whittaker function corresponding to $v$. We normalize the Whittaker functional $L$ so that

$$\langle v_1, v_2 \rangle = \int_{\mathbb{C}^\times} W_{v_1}(s(z))\overline{W_{v_2}(s(z))}d^\times z.$$  

It is well known that for every $v_1, v_2 \in H_\infty$ the integral above is absolutely convergent and indeed gives a $G$-invariant inner product on $H_\infty$.

For every $f \in C_\infty^c(G)$ and $v \in H$ we define

$$\pi(f)v = \int_G f(g)\pi(g)v \, dg,$$

where $dg$ is the Haar measure on $G$ defined after (1.9).

Let $\{v_i\}$ be an orthonormal basis in $H_\infty$ such that each $v_i$ is contained in a $K$-isotypic component (see [WalI] Lemma 8.1.1 for this condition). We define the Bessel distribution $J_{\pi,\psi}$ by

$$J_{\pi,\psi}(f) = \sum_i L(\pi(f)v_i)\overline{L(v_i)}.$$  

(B.1)

The distribution is independent on the choice of such orthonormal basis.

The main result on Bessel distributions for $SL_2(\mathbb{C})$ is a regularity theorem in [CQ].

**Theorem B.1.** Let $\pi$ be an irreducible unitary representation of $G$. Then there exists a real analytic function $j_{\pi,\psi} : BuB \to \mathbb{C}$ which is locally integrable on $G$ such that

$$J_{\pi,\psi}(f) = \int_G f(g)j_{\pi,\psi}(g) \, dg,$$

for all $f \in C_\infty^c(G)$.

**B.2. Bessel functions for $SL_2(\mathbb{C})$.** The interpretation in representation theory of Bessel functions $j_{\pi,\psi}$ defined as in (1.5)-(1.8) is the following kernel formula.

**Theorem B.2.** Let $\pi$ be an irreducible unitary representation of $G$. We have

$$W_v(s(y)w) = \int_{\mathbb{C}^\times} j_{\pi,\psi}(s(yz))W_v(s(z))\overline{d^\times z}\text{ for all } v \in H_\infty.$$  

This kernel formula is proven in [Qi2] Chapter 4 in full generality; some of its special cases may be found in [BM4, Mot] and [BBA]. It is first introduced in [CPS] for $SL_2(\mathbb{R})$.

It is proven in [CQ] that the function $j_{\pi,\psi}$ in Theorem B.1 is exactly the Bessel function defined as in (1.3)-(1.8). Thus coincide the two Bessel distributions $J_{\pi,\psi}$ defined in the Introduction and §B.1.

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