A second alternative approach for the study of the Muckenhoupt class $A_1(\mathbb{R})$

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Abstract
We find the exact best possible range of those $p > 1$ for which any $\varphi \in A_1(\mathbb{R})$, with $A_1$-constant equal to $c$, must also belong to $L^p$. In this way we provide alternative proofs of the results in [2] and [10].

1 Introduction
The study of Muckenhoupt weights has been proved to be important in analysis. One of the most important facts about these is their self improving property. A way to express this is through the so-called reverse Hölder inequalities (see [3], [4] and [6]).

For an interval $J$ on $\mathbb{R}$, we define the class $A_1(J)$ to be the set of all those $\varphi : J \to \mathbb{R}^+$ for which there exists a constant $c \geq 1$, such that the following inequality is satisfied

$$\frac{1}{|J|} \int_J \varphi(x)dx \leq c \cdot \text{essinf}_{J}(\varphi),$$

for every subinterval $I$ of $J$, where $|\cdot|$ is the Lebesque measure on $\mathbb{R}$. The least constant $c$ for which (1.1) holds, is called the $A_1$-constant of $\varphi$ and is denoted by $[\varphi]_1$. We will say then that $\varphi$ belongs to the class $A_1(J)$ with constant $c$, and we will write $\varphi \in A_1(J, c)$.

The study of weights in the class $A_1(J, c)$ has been seen for the first time in [2]. In that paper the study of such weights has been given through the notion of the non-increasing rearrangement of $\varphi$, denoted by $\varphi^*$, which is a non-negative and non-increasing function defined on $(0, |J|)$. It is characterized by the following two additional properties. It is equimeasurable to $\varphi$ (in the sense that $|\{\varphi > \lambda\}| = |\{\varphi^* > \lambda\}|$, for every $\lambda > 0$) and is also left continuous. All these properties define uniquely $\varphi^*$ as can be seen in [1], [5] or [8]. Nevertheless an equivalent definition of $\varphi^*$ can be given by the following formula:

$$\varphi^*(t) = \sup_{E \subseteq J} \left[ \inf_{x \in E} \varphi(x) \right], \text{ for } t \in (0, |J|),$$

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as can be seen in [8].

In [2] now the following result has been proved:

**Theorem 1.** Let \( \varphi \in A_1(J,c) \). The \( \varphi^* \) satisfies:

\[
\frac{1}{t} \int_0^t \varphi^*(y)dy \leq c\varphi^*(t), \text{ for } t \in (0,|J|].
\] (1.2)

That is \( \varphi^* \) belongs to the class \( A_1(J) \), with \( A_1 \)-constant not more than \( c \).

The above theorem describes the \( A_1 \)-properties of \( \varphi^* \), in terms of those of \( \varphi \). It was used effectively by the authors in [2] in order to prove the following:

**Theorem 2.** Let \( \varphi \in A_1(J,c) \). Then \( \varphi \in L^p \) for every \( p \in [1,\frac{c}{c-1}] \). Moreover, the following inequality must hold for every subinterval \( I \) of \( J \), and every \( p \) in the above range,

\[
\frac{1}{|I|} \int_I \varphi^p(x)dx \leq \frac{1}{c^{p-1}(c - p - pc)} \left( \frac{1}{|I|} \int_I \varphi(x)dx \right)^p.
\] (1.3)

Additionally, the above inequality is sharp, that is the constant appearing in the right side cannot be decreased.

Our aim in this paper is to give an alternative proof of Theorem 2, by using Theorem 1 and certain techniques involving the well known Hardy operator on the line. Additionally, we need to mention that in [7] and [9] related problems for estimates for the range of \( p \) in higher dimensions have been treated.

The paper is organized as follows: In Section 2 we give a brief discussion of the proof of the Theorem 1, as is presented in [2] and in Section 3 we provide the proof of Theorem 2.

## 2 \( \varphi^* \) as an \( A_1 \) weight on \( \mathbb{R} \).

Before we present the proof of Theorem 1 we give the following covering Lemma as can be seen in [2].

**Lemma 2.1.** Let \( E \) be a measurable bounded subset of \( \mathbb{R} \) and \( \epsilon > 0 \). More precisely suppose that \( E \subseteq I \), for a certain bounded interval \( I \) of \( \mathbb{R} \). Then there exists a sequence \( \{I_\nu\}_{\nu=1}^\infty \) of subintervals of \( I \) with disjoint interiors and a subset \( E_1 \) of \( E \) with the properties that \( |E_1| = |E| \) and

i) \( E_1 \subseteq \bigcup_{\nu=1}^\infty I_\nu \)

ii) \( (1 - \epsilon)|I_\nu| \leq |I_\nu \cap E| < |I_\nu| \), for every \( \nu \).

We now proceed to the **Proof of Theorem 1**: Suppose without loss of generality that \( J = (0,1) \) and that \( \varphi \) satisfies \((1.1)\), for every subinterval \( I \) on the above interval. Fix \( t \in (0,1] \) and \( \epsilon > 0 \). Let \( E_t \) be a subset of \( (0,1) \) such that \( |E_t| = t \) and \( \varphi(x) \leq \varphi^*(t) \),
for any $x \notin E_t$. Using Lemma 2.1 we produce a subset $E_{t,1}$ of $E_t$, such that $|E_{t,1}| = t$ and $E_{t,1} \subseteq \bigcup_{\nu=1}^{\infty} I_{\nu}$, where for every $\nu = 1, 2, ...$ the following holds:

\[(1 - \epsilon)|I_{\nu}| \leq |I_{\nu} \cap E_t| < |I_{\nu}|,\]  

(2.1) for a suitable family $(I_{\nu})_{\nu=1}^{\infty}$ of subintervals of $(0, 1)$. By the strict inequality in (2.1), we conclude that $I_{\nu}$ contains a set of positive measure in the complement of $E_t$, therefore we must have that

\[\text{essinf}_{x \in I_{\nu}} \varphi(x) \leq \varphi^*(t),\]

so by using (1.1) and (2.1) we have a consequence that

\[\begin{align*}
\int_0^t \varphi^*(y)dy &= \int_{E_t} \varphi(x)dx = \int_{E_{t,1}} \varphi(x)dx \leq \sum_{\nu=1}^{\infty} \int_{I_{\nu}} \varphi(x)dx \leq \epsilon \sum_{\nu=1}^{\infty} |I_{\nu}| \cdot \varphi^*(t) \\
&\leq \frac{\epsilon}{1 - \epsilon} \left(\sum_{\nu=1}^{\infty} |I_{\nu} \cap E_t|\right) \cdot \varphi^*(t) = \frac{\epsilon}{1 - \epsilon} \cdot t \cdot \varphi^*(t) \\
\Rightarrow \frac{1}{t} \int_0^t \varphi^*(y)dy &\leq \frac{c}{1 - \epsilon} \varphi^*(t),
\end{align*}\]

for every $\epsilon > 0$. Letting $\epsilon \to 0^+$, we conclude (1.2) for any $t \in (0, 1]$.

3 $L^p$ integrability of $A_1$ weights on $\mathbb{R}$.

We will now prove the following:

**Lemma 3.1.** Let $g : (0, 1) \to \mathbb{R}^+$ be a non-increasing, left continuous function which satisfies the following inequality:

\[\frac{1}{t} \int_0^t g(y)dy \leq c \cdot g(t), \forall t \in (0, 1)\]  

(3.1) for a fixed $c > 1$. Then for any $p \in [1, \frac{c}{c-1})$ the following is the:

\[\int_0^1 g^p(y)dy \leq \frac{1}{c^{p-1}(c + p - pc)} \left( \int_0^1 g(y)dy \right)^p.\]  

(3.2)

Moreover, inequality (3.2) is best possible.

**Proof:** Fix a $p$ such that $1 \leq p < \frac{c}{c-1}$ and let $F = \int_0^1 g^p(y)dy$ and $f = \int_0^1 g(y)dy$. Then by Hölder’s inequality $f^p \leq F$. We need to prove that

\[F \leq \frac{1}{c^{p-1}(c + p - pc)} \cdot f^p.\]  

(3.3)
We define the following function:

\[ H_p : \left[ 1, \frac{p}{p-1} \right] \rightarrow [0, 1] \]

by \( H_p(z) = pz^{p-1} - (p-1)z^p \). Then we easily see that \( H_p \) is one to one and onto. We denote its inverse function by \( \omega_p \), defined on \([0, 1]\), which is decreasing as \( H_p \) also is. We shall prove that (3.3) holds, equivalently, \( H_p(c) \leq \frac{p}{p-1} \) \( \Leftrightarrow c \geq \omega_p \left( \frac{p}{p-1} \right) =: \tau \).

Suppose on the contrary that \( c < \tau \). We are going to reach to a contradiction.

Define the following function on \((0, 1)\), by \( g_1(t) = \frac{t}{t^{1+\frac{1}{p}}} \). This is obviously non-increasing and continuous \((0, 1)\). Additionally, it satisfies for any \( t \in (0, 1) \) the following equality.

\[ \frac{1}{t} \int_0^t g_1(y)dy = \tau \cdot g_1(t) \]  

(3.4)

Indeed: \( \frac{1}{t} \int_0^t g_1(y)dy = \frac{1}{t} \int_0^t y^{-1+\frac{1}{p}}dy = \frac{1}{t} \left[ \frac{y^{1+\frac{1}{p}}}{1+\frac{1}{p}} \right]_{y=0}^{t} = \frac{1}{t} \cdot t^{1+\frac{1}{p}} = \tau \cdot \left( \frac{t}{t^{1+\frac{1}{p}}} \right) = \tau g_1(t) \).

Moreover, it satisfies \( \int_0^1 g(y)dy = f \) and \( \int_0^1 g^p(y)dy = F \). The first equation is obvious, in view of (3.4). As for the second it is equivalent to \( \frac{p}{p-1} \int_0^1 y^{-p+\frac{1}{p}}dy = F \Leftrightarrow \int_0^1 g^p(y)dy = F \Leftrightarrow \int_0^1 (1+\frac{1}{p})y^{-p+\frac{1}{p}}dy = F \Leftrightarrow \int_0^1 (1+\frac{1}{p})y^{-p+\frac{1}{p}}dy = F \Leftrightarrow \tau = \omega_p \left( \frac{p}{p-1} \right) \), which is true by the definition of \( \tau \).

We are now aiming to prove that the following inequality is satisfied:

\[ \int_0^t g(y)dy \leq \int_0^t g_1(y)dy, \text{ for any } t \in (0, 1). \]  

(3.5)

For this reason we define the following subset of \((0, 1)\):

\[ G = \left\{ t \in (0, 1) : \int_0^t g(y)dy > \int_0^t g_1(y)dy \right\}, \text{ and we suppose that } G \text{ is non empty.} \]

By the continuity of the involving integral functions on \( t \) we have as a consequence that \( G \) is an open subset of \((0, 1)\). Since \( G \neq \emptyset \Rightarrow G = \bigcup_{\nu} I_{\nu} \), where \((I_{\nu})_\nu\) is a (possibly finite) sequence of pairwise disjoint open integrals on \((0, 1)\).

Let us choose one of them, \( I_{\nu} = (\alpha_{\nu}, b_{\nu}) \). Since \( \alpha_{\nu} \notin G \)

\[ \Rightarrow \int_{\alpha_{\nu}}^{b_{\nu}} g(y)dy \leq \int_{\alpha_{\nu}}^{b_{\nu}} g_1(y)dy. \]  

(3.6)

Let now \( (x_n)_{n} \subseteq I_{\nu} \) be a sequence such that \( x_n \rightarrow \alpha_{\nu}, \text{ as } n \rightarrow \infty \). Since \( x_n \in G, \forall n = 1, 2, ... \) we must have that \( \int_{0}^{x_n} g(y)dy > \int_{0}^{x_n} g_1(y)dy \), so letting \( n \rightarrow \infty \) we conclude that

\[ \int_{0}^{b_{\nu}} g(y)dy \geq \int_{0}^{b_{\nu}} g_1(y)dy. \]  

(3.7)

By (3.6) and (3.7) we see that \( \int_{\alpha_{\nu}}^{b_{\nu}} g(y)dy = \int_{\alpha_{\nu}}^{b_{\nu}} g_1(y)dy \). In the same way we prove that \( \int_{\alpha_{\nu}}^{b_{\nu}} g(y)dy = \int_{\alpha_{\nu}}^{b_{\nu}} g_1(y)dy \). As a consequence, we must have that

\[ \int_{\alpha_{\nu}}^{b_{\nu}} g(y)dy = \int_{\alpha_{\nu}}^{b_{\nu}} g_1(y)dy. \]  

(3.8)
Let now $t \in I_\nu = (\alpha_\nu, b_\nu)$. Since $t \in G$ and because of (3.1) and (3.4) and the assumption on $\tau$, we must have the following: $c g(t) \geq \frac{1}{\tau} \int_0^t g(y)dy > \frac{1}{\tau} \int_0^t g_1(y)dy = \tau \cdot g_1(t)$ thus $g(t) > g_1(t)$, for every $t \in I_\nu$. This is impossible in view of (3.8). We note the following (which can be seen in [5], p.88).

**Lemma 3.2.** Let $\varphi_1, \varphi_2 : (0, 1] \to \mathbb{R}^+$ be integrable functions. Then the following are equivalent

i) $\int_0^t \varphi_1^*(y)dy \leq \int_0^t \varphi_2^*(y)dy$, for every $t \in (0, 1]$.

ii) $\int_0^1 G(\varphi_1(x))dx \leq \int_0^1 G(\varphi_2(x))dx$

for any $G$ convex, non-negative, increasing and left continuous function on $[0, +\infty)$.

We consider now two cases:

A) We have equality in (3.5) for every $t \in (0, 1]$. That is

$$\int_0^t g(y)dy = \int_0^t g_1(y)dy, \forall t \in (0, 1].$$

This gives immediately as a consequence that $g(t) = g_1(t)$ almost everywhere on $(0, 1]$, and since $g_1$ is continuous on $(0, 1]$, we must have that $g(t) = g_1(t), \forall t \in (0, 1] \Rightarrow g(t) = \frac{t}{1+t^{1+\frac{1}{p}}}, \forall t \in (0, 1] \Rightarrow \frac{1}{\tau} \int_0^t g(y)dy = \tau g(t), \forall t \in (0, 1].$ Then in view of (3.1) we conclude that $c \geq \tau$ which is a contradiction since we have supposed the opposite inequality.

B) There exists a $t_0 \in (0, 1)$ such that:

$$\int_0^{t_0} g(y)dy < \int_0^{t_0} g_1(y)dy.$$  

Then, by continuity reasons, we have as a consequence that there exists a $\delta > 0$ such that

$$\int_0^{t_0} g(y)dy < \int_0^{t_0} g_1(y)dy, \forall t \in (t_0 - \delta, t_0 + \delta) = I_\delta. \quad (3.9)$$

We define now the quantities $d_1, d_2$ by the following equations:

$$\frac{1}{\delta} \int_{t_0-\delta}^{t_0} g_1(y)dy = d_1 \quad \text{and} \quad \frac{1}{\delta} \int_{t_0-\delta}^{t_0+\delta} g_1(y)dy = d_2. \quad (3.10)$$

Then by Hölder’s inequality on the interval $(t_0 - \delta, t_0)$ for $g_1$, we conclude that

$$\frac{1}{\delta} \int_{t_0-\delta}^{t_0} g_1^p(y)dy > d_1^p, \quad (3.11)$$

which is a strict inequality since $g_1$ is strictly decreasing (therefore not constant) on the interval $(t_0 - \delta, t_0)$. In the same way we have

$$\frac{1}{\delta} \int_{t_0}^{t_0+\delta} g_1^p(y)dy > d_2^p. \quad (3.12)$$

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Then since \( g_1 \) is decreasing we have that \( d_2 < d_1 \). We define now the following nonincreasing (as can be easily seen) function on \((0,1] \)

\[
g_2(t) = \begin{cases} 
g_1(t), & t \in (0,1] \setminus (t_0 - \delta, t_0 + \delta) \\
d_1 & t \in [t_0 - \delta, t_0) \\
d_2 & t \in [t_0, t_0 + \delta].
\end{cases}
\]  

(3.13)

By (3.9) and since \( g_1 \) is decreasing we easily see that we can choose \( \delta > 0 \) small enough, so that

\[
\int_0^t g(y)dy \leq \int_0^t g_2(y)dy, \text{ for any } t \in (0,1].
\]  

(3.14)

Additionally, because of (3.11) and (3.12) we must have that

\[
\int_0^1 g_2^p(y)dy < \int_0^1 g_1^p(y)dy = F.
\]

Since (3.14) holds for any \( t \in (0,1] \) and because of Lemma 3.2 we conclude that \( \int_0^1 g^p(y)dy \leq \int_0^1 g_2^p(y)dy < F \), by considering the function \( G(t) = t^p \). This is obviously a contradiction according to the way that \( F \) is defined. In this way we derive the proof of our Lemma.

We now proceed to the:

**Proof of Theorem 2**: Without loss of generality we suppose that \( \mathcal{J} = (0,1) \).

Let \( p \in [1, \frac{2}{c-1} \) and \( \mathcal{I} \subseteq (0,1) \) and let also \( \varphi_{\mathcal{I}} = \varphi|_{\mathcal{I}} \) the restriction of \( \varphi \) to \( \mathcal{I} \).

Consider now the function \( g : (0,|\mathcal{I}|) \to \mathbb{R}^+ \), defined by \( g = (\varphi_{\mathcal{I}})^+ \). Then since \( \varphi_{\mathcal{I}} \in A_1(\mathcal{I}) \) with \( A_1 \) constant not more than \( c \), we must have by using Theorem 1 that \( \frac{1}{t} \int_0^t g(y)dy \leq cg(t) \), for any \( t \in (0,|\mathcal{I}|] \). Thus by Lemma (3.1) we have as a consequence that

\[
\frac{1}{|\mathcal{I}|} \int_0^{|\mathcal{I}|} g^p(y)dy \leq \frac{1}{c^{p-1}(c + p - pc)} \left( \frac{1}{|\mathcal{I}|} \int_0^{|\mathcal{I}|} g(y)dy \right)^p,
\]

which is:

\[
\frac{1}{|\mathcal{I}|} \int_{\mathcal{I}} \varphi^p(x)dx \leq \frac{1}{c^{p-1}(c + p - pc)} \left( \frac{1}{|\mathcal{I}|} \int_{\mathcal{I}} \varphi(x)dx \right)^p.
\]

The relation (1.3) is proved.

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