PT-symmetry breaking in complex nonlinear wave equations and their deformations

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Abstract
We investigate complex versions of the Korteweg–deVries equations and an Ito-type nonlinear system with two coupled nonlinear fields. We systematically construct rational, trigonometric/hyperbolic and elliptic solutions for these models including those which are physically feasible in an obvious sense, that is those with real energies, but also those with complex energy spectra. The reality of the energy is usually attributed to different realizations of an antilinear symmetry, as for instance PT-symmetry. It is shown that the symmetry can be spontaneously broken in two alternative ways either by specific choices of the domain or by manipulating the parameters in the solutions of the model, thus leading to complex energies. Surprisingly, the reality of the energies can be regained in some cases by a further breaking of the symmetry on the level of the Hamiltonian. In many examples, some of the fixed points in the complex solution for the field undergo a Hopf bifurcation in the PT-symmetry breaking process. By employing several different variants of the symmetries we propose many classes of new invariant extensions of these models and study their properties. The reduction of some of these models yields complex quantum mechanical models previously studied.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

One can adopt various points of view with regard to the usefulness of the study of complex classical and quantum mechanical systems. Being very orthodox one may just view the
complex systems as providing a larger setting which allows a better insight from a broader framework when restricted to the real physical system. A very successful example for this viewpoint is the more than 70 year old proposal of the analytical $S$-matrix \cite{1, 2}, which is still pursued nowadays, especially in 1+1 dimensions. Genuinely non-Hermitian systems such as dissipative ones are also well studied, but they are usually regarded as open and are therefore not self-consistent \cite{3}. In contrast, a more recent perspective allows us to regard certain complex quantum mechanical Hamiltonians also as perfectly acceptable self-consistent descriptions of physical systems \cite{4}, in the sense that they possess real energy eigenvalue spectra and well-defined unitary time evolution (see \cite{5–7} for recent reviews). Considering pseudo-/quasi-Hermitian systems provides a very clear conceptual view in this respect as it makes use of a similarity transformation towards a Hermitian Hamiltonian system for which everything is well defined in a standard conventional sense, although such transformations do not exist for all types of such systems. A more radical view is to give a direct meaning to the non-Hermitian Hamiltonians without any reference to the Hermitian system. This latter point of view has to be taken in order to explain recent experiments in which non-Hermitian systems are studied on optical lattices, see e.g. \cite{8}, as the observed gain and loss cannot be explained in a purely Hermitian setting. A further experimental realization has recently been proposed for graphene nanoribbons, where non-Hermitian Hamiltonians arise as effective Hamiltonians \cite{9}.

Largely inspired by the study of the quantum systems, classical complex systems have also been investigated recently. Naturally, one may also view them in various ways, either as directly meaningful or only sensible when transformed to a real system. Many classical models have already been investigated from these various perspectives, such as complex extensions of standard one particle real quantum mechanical potentials \cite{10–13}, non-Hamiltonian dynamical systems \cite{14}, chaotic systems \cite{15} and deformations of many-particle systems such as Calogero–Moser–Sutherland models \cite{16–19}. These investigations led to new insights into the quantum theories based on the features found in these classical models, such as tunnelling behaviour \cite{13, 20}, band structures \cite{21} and even a complex generalization of Bohr’s correspondence principle has been formulated \cite{22}.

Extensions of field theories of nonlinear wave type, such as Korteweg–deVries (KdV) equations \cite{23–25} and closely related models \cite{26–29} have also been investigated. These types of models will be the main subject of our investigations in this manuscript.

Pseudo- or quasi-Hermiticity is often equivalent to a simultaneous parity and time reversal, so-called $\mathcal{PT}$-symmetry. Remarkably, this property, or more general antilinear symmetry \cite{30}, of quantum mechanical systems is already visible on the classical level. It has been known for more than 50 years that when this symmetry is unbroken, the reality of the spectrum of the theory is guaranteed. *Unbroken* refers here to the property that

$$[\mathcal{PT}, H] = 0 \quad \text{and} \quad \mathcal{PT} \Phi = \Phi,$$  \hspace{1cm} (1.1)

i.e. the Hamiltonian $H$ is $\mathcal{PT}$-symmetric and its eigenfunctions $\Phi$ are also eigenstates of the $\mathcal{PT}$ or any other antilinear operator. In case the latter property does not hold, one speaks of *spontaneously broken $\mathcal{PT}$-symmetry* and the eigenenergies become complex conjugate pairs. The $\mathcal{PT}$-symmetry can be broken spontaneously in two different ways, either by tuning some of the parameters of the models appropriately or by manipulating the domain on which the model is defined, see e.g. \cite{31} for the latter possibility. One speaks of *broken $\mathcal{PT}$-symmetry* when neither of the relations in (1.1) hold. In this case, one usually expects complex energies, but we will demonstrate in this manuscript that in some cases real energy spectra can be produced by breaking the spontaneously broken theory further in a controlled way.
Here, we will trace these properties on the classical level and use the reality of the classical energy as a criterium to select physically meaningful complex models including their boundary and initial conditions. The role of $\Phi_1$ in (1.1) is in this case played by the solution of the classical field equation of motion, say $u(x, t)$. For a given Hamiltonian density $\mathcal{H}$ depending on $u(x, t)$, the energy on the interval $x \in [-a, a]$ is computed by

$$E = \int_{-a}^{a} \mathcal{H}[u(x)] \, dx = \oint_{\Gamma_1} \mathcal{H}[u(x)] \, du_{x}.$$  

Here and throughout the manuscript, we use the standard convention $u_x \equiv du/dx$. When the system possesses periodic solutions, that is when $u(a) = u(-a)$ along a path $\Gamma$ in the complex $u$-plane, we can employ the alternative contour integral version in (1.2). A simple argument [24] shows that the energy in the interval $x \in [-a, a]$ is guaranteed to be real if the symmetry property $\mathcal{H}^* [u(x)] = \mathcal{H}[u(-x)]$ holds for the Hamiltonian density. Here, we will present some unexpected examples of real energy solutions for which neither of the relations in (1.1) hold, i.e. the $\mathcal{P}\mathcal{T}$-symmetry is broken for the Hamiltonian and the solutions.

Here, we will investigate two different types of complexified wave equations. First of all, in section 2 we study the most immediate way to complexify wave equations by introducing complex boundary values and initial conditions for the KdV system. We investigate systematically the travelling wave solutions. We study complex trajectories in the complex plane of the KdV field especially with regard to their properties under $\mathcal{P}\mathcal{T}$-symmetry breaking. In section 3, we employ the different types of $\mathcal{P}\mathcal{T}$-symmetry as a construction principle to propose new extended versions of the KdV system. We investigate the new models in a similar fashion as their undeformed counterparts. Particular attention is paid to the question of what kind of conditions will lead to physical models, in the sense that they possess real energies. In section 4, we study the effect of complex boundary conditions and initial values on a nonlinear system with two fundamental fields coupled to each other, which is referred to as the Ito system for some specific parameter choice. We also investigate the $\mathcal{P}\mathcal{T}$-symmetry properties of these models. Due to the presence of an additional field when compared with the KdV system we can identify four different versions of $\mathcal{P}\mathcal{T}$-symmetry being realized in these systems, which we exploit in section 5 to construct new models. We study them from similar points of view as the previous ones. We draw our conclusions in section 6.

2. Complex KdV equation

Complex extensions of the KdV equation have been investigated already some time ago, for instance in [32–35]. However, $\mathcal{P}\mathcal{T}$-symmetry has only been utilized recently in [23–25] in order to understand some of their properties and in particular to construct new models. The standard KdV system is known to be a Hamiltonian system with density

$$\mathcal{H}_{\text{KdV}} = -\frac{\beta}{6} u^3 + \frac{\gamma}{2} u^2 x, \quad \beta, \gamma \in \mathbb{C},$$  

leading to the KdV equation in the form

$$u_t + \beta u u_x + \gamma u_{xxx} = 0.$$  

(2.1)

Usually, the constants $\beta$ and $\gamma$ are chosen to be real, but here we allow them to take complex values, thus including the possibility for $u(x, t)$ to be complex. $\mathcal{P}\mathcal{T}$-symmetry may then be realized in two alternative ways as

$$\mathcal{P}\mathcal{T}_+ : \quad x \mapsto -x, \quad t \mapsto -t, \quad i \mapsto -i, \quad u \mapsto u \quad \text{for } \beta, \gamma \in \mathbb{R},$$

(2.3)

$$\mathcal{P}\mathcal{T}_- : \quad x \mapsto -x, \quad t \mapsto -t, \quad i \mapsto -i, \quad u \mapsto -u \quad \text{for } i \beta, \gamma \in \mathbb{R},$$

(2.4)
both possibilities guaranteeing that $\mathcal{P}T_{\pm} : \mathcal{H}_{\text{KdV}} \mapsto \mathcal{H}_{\text{KdV}}$ holds. The underlying models respect one of the two symmetries and are therefore different as they correspond to two distinct choices of the coupling constant which may, however, be related by a simple rotation in $u$. Nonetheless, with regard to possible deformations to be discussed below, this second symmetry allows us to construct different types of new models.

A crucial feature of the model to be acceptable as physically consistent is the reality of the energy (1.2). In the case of the Hamiltonian density $\mathcal{H}_{\text{KdV}}$, the reality of the energy (1.2) is guaranteed for the two possibilities $u^\ast(x) = u(-x)$ when $\beta, \gamma \in \mathbb{R}$ or $u^\ast(x) = -u(-x)$ when $i\beta, \gamma \in \mathbb{R}$, resulting from $\mathcal{P}T_+$ or $\mathcal{P}T_-$, respectively.

2.1. $\mathcal{P}T$-symmetric, spontaneously broken and broken solutions

Let us now see which complex boundary conditions and initial values are physically permissible. In order to establish this, we first briefly recall how the travelling wave solutions of the KdV equations of motion may be constructed systematically. Integrating (2.2) twice leads to the equation

$$u_2^2 = \frac{2}{\gamma} \left( \kappa_2 + \kappa_1 u + \frac{c}{2} u^2 - \frac{\beta}{6} u^3 \right) = \lambda P(u)$$

with integration constants $\kappa_1, \kappa_2 \in \mathbb{C}$ and $P(u)$ denoting a third-order polynomial in $u$ multiplied by an overall constant $\lambda$. A further integration yields

$$\pm \sqrt{\lambda}(\zeta - \zeta_0) = \int du \frac{1}{\sqrt{P(u)}},$$

where we made the usual assumption that the field $u(x, t)$ acquires the form of a travelling wave $u(x, t) = u(\zeta)$ with $\zeta = x - ct$ and $c$ denoting the wave speed. In complete generality this is an elliptic integral, but it is instructive to generate simpler solutions by systematically making some specific assumptions on the factorization of the polynomial $P(u)$. This solution method may then be extended to the deformed cases.

Demanding specific boundary conditions will impose further restrictions or might be entirely incompatible with certain factorizations of $P(u)$. For instance, in case we wish to implement vanishing asymptotic boundary conditions for $u$ and its derivatives, the once integrated version of equations (2.2) and (2.5) implies

$$\lim_{\zeta \to \pm \infty} u, u_\zeta = 0 \Rightarrow \kappa_1 = \kappa_2 = 0.$$  

In the following we will study the solutions in the complex $u$-plane. For this purpose it is useful to separate $u$ into its real and imaginary parts $u^R$ and $u^I$, respectively, and decouple (2.5) into two first-order differential equations in these variables:

$$u^R_\zeta = \pm \text{Re}\left[\sqrt{\lambda} \sqrt{P(u^R + i u^I)}\right] \quad \text{and} \quad u^I_\zeta = \pm \text{Im}\left[\sqrt{\lambda} \sqrt{P(u^R + i u^I)}\right].$$

In this setup, we may then apply many of the techniques which have been developed for two-dimensional dynamical systems, see for instance [36]. Most immediate is the application of the linearization theorem at some fixed point $u_0$, converting the nonlinear system into

$$\begin{pmatrix} u^R \cr u^I \end{pmatrix} = J(u^R, u^I)|_{u = u_0} \begin{pmatrix} u^R_\zeta \cr u^I_\zeta \end{pmatrix},$$

with the Jacobian matrix

$$J(u^R, u^I)|_{u = u_0} = \begin{pmatrix} \pm \frac{\partial \text{Re}\left[\sqrt{\lambda} \sqrt{P(u)}\right]}{\partial u^R} & \pm \frac{\partial \text{Re}\left[\sqrt{\lambda} \sqrt{P(u)}\right]}{\partial u^I} \\
\pm \frac{\partial \text{Im}\left[\sqrt{\lambda} \sqrt{P(u)}\right]}{\partial u^R} & \pm \frac{\partial \text{Im}\left[\sqrt{\lambda} \sqrt{P(u)}\right]}{\partial u^I} \end{pmatrix}|_{u = u_0}.$$
Figure 1. Complex rational solutions of the KdV equation for different values of purely complex initial conditions $\zeta_0$: (a) $\mathcal{PT}$-symmetric solutions for $c=1$, $\beta=2$, $\gamma=3$ and $A=1/2$; (b) broken $\mathcal{PT}$-symmetric solutions for $c=1$, $\beta=2+i2$, $\gamma=3$ and $A=(1-i)/4$. Different trajectories are characterized by different initial conditions $\zeta_0$. Some values for the imaginary part of $\text{Im}\,\zeta_0<1$ are indicated on the trajectories. The unresolved white region corresponds to values for $\text{Im}\,\zeta_0>1$.

We denote the eigenvalues of $J(u=uf)$ by $j_1$, $j_2$. Provided the system (2.9) is simple and $\text{Re}\,j\neq0$ the linearization theorem applies, stating that the phase portraits of the systems (2.8) and (2.9) are qualitatively the same in some neighbourhood of the fixed point $uf$. The ten similarity classes for $2 \times 2$ matrices fully characterizing all possible behaviours for the fixed points of (2.8) are reported for reference in the appendix in table A1.

2.1.1. Rational solutions. Factorizing $P(u)$ at first in the simplest way as $P(u)=(u-A)^3$ with one constant $A$ leaves no freedom when solving (2.5), as all constants are fixed:

$$
\lambda = -\frac{\beta}{3\gamma}, \quad \kappa_1 = -\frac{c^2}{2\beta}, \quad \kappa_2 = \frac{c^3}{6\beta^2} \quad \text{and} \quad A = \frac{c}{\beta}.
$$

Clearly, asymptotically vanishing boundary conditions (2.7) are only possible for a static solution with $c=0$. The evaluation of expression (2.6) produces for this factorization the rational solution

$$
u(\zeta) = \frac{c}{\beta} - \frac{12\gamma}{\beta(\zeta - \zeta_0)^2},
$$

with additional integration constant $\zeta_0 \in \mathbb{C}$. Taking $\zeta_0$ as purely imaginary maintains the $\mathcal{PT}$-symmetry of the solution, whereas any real part may be compensated by a shift in $\zeta$, which is kept to be real. Independently of the parameter choice, asymptotically all rational solutions of the type (2.12) end up at $\lim_{\zeta \to \pm\infty} u(\zeta) = c/\beta$, as we can also observe in figure 1.

Focussing at first on the $\mathcal{PT}$-symmetric scenario, we note that while keeping the speed of the wave $c$ real there are two possible choices for the conjugation leading to physical solutions, namely $u^*(\zeta) = u(-\zeta)$ when $i\zeta_0, \beta, \gamma \in \mathbb{R}$ or $u^*(\zeta) = -u(-\zeta)$ when $i\zeta_0, i\beta, \gamma \in \mathbb{R}$. We depict some complex trajectories in the $u$-plane in figure 1(a) for several different initial
conditions \( \zeta_0 \). We have taken the plus and minus signs in (2.6) for the upper- and lower-half planes, respectively. We observe that for a specific branch the point \( A \) in the \( u \)-plane appears to be either a stable or an unstable improper asymptotic fixed point. Here, we adopt the notion for the characterizations of fixed points from the linearization (see the appendix for a classification), despite the fact that \( u_* \) is not a meromorphic function and the system is not easily linearized. This implies that only the choice with different branches for the upper and lower halves will give rise to closed orbits as depicted. For increasing \( \xi \), they run either out of the fixed point in the upper-half plane and into it in the lower-half or vice versa. The crossing of the trajectories with the real line is easily computed to be at \( u(0) = c/\beta + 12\gamma/\beta(\text{Im } \zeta_0)^2 \).

This makes it evident that only complex trajectories may be closed, whereas the real solution is the only trajectory drifting off to minus infinity.

Since the rational solution (2.12) does not have any free parameter left, as specified in (2.11), there is no possibility of breaking the \( \mathcal{PT} \)-symmetry spontaneously for this type of solution. We may, however, break the \( \mathcal{PT} \)-symmetry completely directly on the level of the Hamiltonian by fully complexifying \( \beta \) or \( \gamma \), an example of which is depicted in figure 1(b).

As expected, we observe that the symmetry \( u^*(\xi) = u(-\xi) \) has been lost, but instead the trajectories are almost symmetric about the line passing the two points \( A \) and \( u(0) \), which are now both located away from the real axis. However, the nature of the fixed point, being either an unstable or stable improper node, has not changed.

Considering the expression for the energy (1.2) it is evident that it will be real when we have the symmetry \( u^*(x) = u(-x) \). We also compute the expression explicitly for solution (2.12) by substituting it into (1.2). Then, the energy in the interval \([-a, a]\) is computed to

\[
E_a = -\frac{ac^2}{3\beta^2} \left( c + \frac{36\gamma}{a^2 - \zeta_0^2} \right) + \frac{72\gamma^2}{5\beta^2} \left[ \frac{10c (a^3 + 3a\zeta_0^2)}{(a^2 - \zeta_0^2)^3} - \frac{48\gamma (a^5 + 10a^3\zeta_0^2 + 5a\zeta_0^4)}{(a^2 - \zeta_0^2)^5} \right].
\]

(2.13)

Evidently, \( E_a \) is real even when \( i\zeta_0, \beta, \gamma \in \mathbb{R} \) or \( i\zeta_0, i\beta, \gamma \in \mathbb{R} \) and complex otherwise, that is for the \( \mathcal{PT} \)-symmetric and broken \( \mathcal{PT} \)-symmetric case, respectively.

2.1.2. Trigonometric/hyperbolic solutions. As the next possibility for the factorization we specify \( P(u) = (u-A)^2(u-B) \) involving now two constants \( A \) and \( B \), thus leaving one of them at our disposal when solving (2.5):

\[
\lambda = -\frac{\beta}{3\gamma}, \quad \kappa_1 = \frac{A}{2}(\beta A - 2c), \quad \kappa_2 = \frac{A^2}{6}(3c - 2\beta A) \quad \text{and} \quad B = \frac{3c}{\beta} - 2A.
\]

(2.14)

Having now some freedom in the choice of the constants one may ask which ones are the most natural to use for the symmetry breaking. As we will see, the constants \( A \) and \( B \) have a direct physical meaning and it appears therefore natural to view them as the free parameters to tune for a concrete model with fixed coupling constants \( \beta, \gamma \), rather than the integration constants \( \kappa_1 \) or \( \kappa_2 \) emerging more indirectly without immediate interpretation. By (2.7), vanishing asymptotic boundary conditions require the choice \( A = 0 \).

In general, the solution to (2.6) produces in this case the trigonometric/hyperbolic solution

\[
u(\xi) = B + (A - B) \tanh^2 \left[ \frac{1}{2} \sqrt{A - B} (\sqrt{\lambda} \xi - \zeta_0) \right].
\]

(2.15)

Let us first discuss the \( \mathcal{PT} \)-symmetric scenario for which all constants are taken to be real except for \( \zeta_0 \), which we still allow to be complex. When in that case either \( A < B, \lambda > 0 \) or \( A > B, \lambda < 0 \), we obtain a periodic solution with period \( T = 2\pi/(\sqrt{|A - B|} \sqrt{|\lambda|}) \) as
depicted in figure 2(a). The closed trajectories surround the point $A$, whereas the point $B$ is situated on its outside, following from the fact that on the real axis we always have $u(\zeta) < B$ or $u(\zeta) > B$, in the respective cases. This behaviour is also confirmed by the linearization (2.9) at the fixed point $A$. Parameterizing $A - B = r_{AB}e^{i\theta_{AB}}$ and $\lambda = r_{\lambda}e^{i\theta_{\lambda}}$ the Jacobian (2.10) is easily computed to

$$J(u)|_{u = A} = \left( \begin{pmatrix} \pm \sqrt{r_{AB}r_{\lambda}} \cos \left[ \frac{1}{2}(\theta_{AB} + \theta_{\lambda}) \right] & \mp \sqrt{r_{AB}r_{\lambda}} \sin \left[ \frac{1}{2}(\theta_{AB} + \theta_{\lambda}) \right] \\ \pm \sqrt{r_{AB}r_{\lambda}} \sin \left[ \frac{1}{2}(\theta_{AB} + \theta_{\lambda}) \right] & \pm \sqrt{r_{AB}r_{\lambda}} \cos \left[ \frac{1}{2}(\theta_{AB} + \theta_{\lambda}) \right] \end{pmatrix} \right), \quad (2.16)$$

with eigenvalues

$$j_1 = \pm \sqrt{r_{AB}r_{\lambda}} \exp \left[ \frac{1}{2}(\theta_{AB} + \theta_{\lambda}) \right] \quad \text{and} \quad j_2 = \pm \sqrt{r_{AB}r_{\lambda}} \exp \left[ -\frac{i}{2}(\theta_{AB} + \theta_{\lambda}) \right]. \quad (2.17)$$

Clearly for the two cases here, that is $A < B$, $\lambda > 0$ or $A > B$, $\lambda < 0$, the eigenvalues of the Jacobian are purely complex, i.e. $j_{1/2} = \pm i\sqrt{|A - B|\sqrt{|\lambda|}}$, indicating that the point $A$ is a centre. The result is independent of the sign in (2.8). The linearization at the fixed point $B$ does not exist, due to the occurrence of the square root.

For this periodic case, we may also compute the energy $E_T$ for one period $T$ from $-\pi/\sqrt{|A - B|\sqrt{|\lambda|}}$ to $\pi/\sqrt{|A - B|\sqrt{|\lambda|}}$ analytically to

$$E_T = \oint_{\Gamma} \mathcal{H}[u(\zeta)] \frac{du}{u_{\zeta}} = \oint_{\Gamma} \sqrt{\lambda}\sqrt{u - B(u - A)} \frac{\mathcal{H}[u]}{u_{\zeta}} du = -\pi \sqrt{B\gamma} \frac{A^3}{3\sqrt{A - B}}. \quad (2.18)$$

Here, the contour $\Gamma$ is taken to be any complex trajectory resulting for $i\zeta_0 \in \mathbb{R}$. Then, (2.18) follows from Cauchy’s residue theorem and the fact that $B$ will always be outside the contour $\Gamma$. We find that the energy is real and takes the same value for all trajectories independently of the concrete choice for the initial condition $\zeta_0$. Note that the energy for the real solution, i.e. $\zeta_0 \in \mathbb{R}$, cannot be computed in such an easy way as in that case $\Gamma$ does not form a
closed contour. Thus, demanding $PT$-symmetry leads inevitably to purely complex initial value conditions and constitutes a natural ‘$\epsilon$-prescription’ to deform the real solution with the purpose to compute the energy for one period.

For the remaining possibilities $A > B$, $\lambda > 0$ or $A < B$, $\lambda < 0$, all solutions tend asymptotically to $A$, which we depict for some examples in figure 2(b). The linearization (2.9) at the fixed point $A$ yields in these cases the two degenerate real eigenvalues $j_1 = j_2 = \sqrt{|A - B|}$ for $J$ depending on the plus or minus sign in (2.8), respectively. The Jacobian (2.16) is diagonalizable in these cases and therefore $A$ is an unstable or stable star node. As for the rational solutions, this implies that only the choice with different branches for the upper and lower halves will give rise to closed orbits as seen in figure 2(b). Now the energies can not be computed in a simple manner as for the periodic case since the singularity at $A$ is situated on the contour.

Next, let us embark on the case in which the $PT$-symmetry is spontaneously broken, which, unlike as for the rational solutions, is possible for (2.15) since we have additional parameters at our disposal. From (2.14) and (2.15) it is clear that when $2\beta K + \epsilon^2 < 0$ the constant $A$ becomes complex and we no longer have the property $u^*(\xi) = u(-\xi)$ ensuring the reality of the spectrum. The nature of the fixed point is now changed again to an unstable or stable focus depending on whether $\text{Re} j_i > 0$ or $\text{Re} j_i < 0$, respectively. Thus, once again closed orbits are obtained with the choice of different branches for the upper- and lower-half planes. This means of course that the periodic solution ceases to be periodic.

The energies $E_T$ for one period in the periodic solution corresponding to the two choices $A$, $B$, $\zeta_0$ and $A^*$, $B^*$, $\zeta_0^*$ occur in complex conjugate pairs. This is the typical scenario for spontaneously broken $PT$-symmetry, i.e. the Hamiltonian is still $PT$-symmetric but the solutions to the Schrödinger equation in the quantum case and in the classical case to the equation of motion are not. Formula (2.18) may still be used in this case for the computation of the energy even though the singularities are moved away from the real axis as they still lie within the contour. Once again the result does not depend on the choice of $\zeta_0$. We depict some trajectories for this type of spontaneous symmetry breaking in figure 3. The conjugate solution is simply obtained by a reflection about the real axis, as shown explicitly for the asymptotically constant solution in panel (b).

Finally, there is of course also the possibility of breaking $PT$-symmetry directly for the Hamiltonian itself. For instance, when $\beta \not\in \mathbb{R}$ and/or $\gamma \not\in \mathbb{R}$ we expect the energies to be complex. In that case, the complex conjugate energy would be obtained by considering a new type of Hamiltonian with $\beta^*$, $\gamma^*$ and thus it does not arise from within one specific model. We depict an example trajectory for this scenario in figure 4.

We observe from figure 4(a) that the periodic broken solution near the fixed point is qualitatively the same as the one for the spontaneously broken case, namely a stable or unstable focus. This behaviour follows from eigenvalues (2.17), which indicate that there is no distinction at this point in whether the complexification of the $j_i$ results from a spontaneous or a complete breaking of the $PT$-symmetry. Note that the trajectories still close even though this is not shown in figure 4(a), but this may be seen on a larger scaled plot. We find a similar behaviour for the fixed point of the broken asymptotic solution as depicted in figure 4(b).

So far we have simply broken the symmetry by choosing some more or less random complex value for $\gamma$. However, we can also carry out this process in a more controlled fashion producing periodic motion and even some non-Hermitian non-$PT$-symmetry cases with real energies. First of all, we observe from eigenvalues of the Jacobian (2.10) that the fixed point $A$ becomes a centre when $\theta_{AB} + \theta_j = \pi$, which is also compatible with solution (2.15) from which we note that a periodic motion occurs when $\lambda(A - B) < 0$. This allows for complex
Figure 3. Complex trigonometric/hyperbolic solutions of the KdV equation: (a) spontaneously broken $\mathcal{PT}$-symmetry of the periodic solution with $c = 1$, $\beta = 3/10$, $\gamma = 3$, $A = 4 + i/2$ and $B = 2 - i$ for $\text{Im} \zeta_0 = 0.3$ solid (black), $\text{Im} \zeta_0 = 0.3$ dotted (green) and $\text{Im} \zeta_0 = 0.1$ dashed (blue); (b) spontaneously broken $\mathcal{PT}$-symmetry of the asymptotically constant solution with $c = 1$, $\beta = 3/10$, $\gamma = -3$ for $A = 4 - i/2$, and $B = 2 + i$ for $\text{Im} \zeta_0 = -0.5$ solid (black) and $A = 4 + i/2$, $B = 2 - i$, for $\text{Im} \zeta_0 = 0.5$ dashed (blue).

Figure 4. Complex trigonometric/hyperbolic solutions of the KdV equation: (a) broken $\mathcal{PT}$-symmetry of the periodic solution with $A = 4$, $B = 2$, $\epsilon = 1$, $\beta = 3/10$, $\gamma = 3 + i/2$ and $\text{Im} \zeta_0 = 6$; (b) broken $\mathcal{PT}$-symmetry of the asymptotically constant solution with $A = 4$, $B = 2$, $\epsilon = 1$, $\beta = 3/10$, $\gamma = -3 + i/2$ and $\text{Im} \zeta_0 = 1/2$.

values of the parameters $\beta$, $\gamma$, $A$ and $B$. Combining this constraint with the last equation in (2.14) leads to

$$\begin{align}
A &= \frac{\sin \theta_\gamma}{|\beta| \sin(\theta_\gamma - \theta_\beta - \theta_A)} \exp(i \theta_A).
\end{align}$$

(2.19)
Thus, for a given model, that is for any fixed values of $\beta$ and $\gamma$, we obtain a periodic motion around the point $A$, given by the expression in (2.19) for any value of $\theta_A$. Indeed, this is the case as we observe for instance in figure 5(a). When using the constraint $\theta_{AB} + \theta_\lambda = \pi$ for the periodic motion in the expression for the energy (2.18) we find

$$E_T = -\frac{\pi}{3} \frac{\beta A^3}{\sqrt{|\lambda| |A - B|}}. \quad (2.20)$$

Hence, demanding that the energy is to be real leads to the further constraint $3\theta_A + \theta_\beta = 0, \pi$. Implementing this in (2.19) yields

$$E_T \in \mathbb{R} \quad \text{for} \quad A = \frac{\sin \theta_\gamma}{|\beta| \sin (\theta_\gamma - 2\theta_\beta/3)} \exp \left( \frac{-\theta_\beta}{3} \right). \quad (2.21)$$

This means for a given model, that is for any fixed values of $\beta$ and $\gamma$, we can find a point $A$, given by the expression in (2.21), around which the trajectory is periodic and the corresponding energy is real. This holds irrespective of whether $\beta$ and $\gamma$ are real or complex, or in other words whether the $\mathcal{PT}$-symmetry is intact or completely broken. In figure 5(b), we depict an example solution of (2.21) corresponding to a periodic motion with completely broken $\mathcal{PT}$-symmetry but real energy. An obvious question to ask at this point is whether by breaking this symmetry the system has acquired a new kind of anti-linear symmetry, which could serve to explain the reality of the energy. A systematic study of this more general issue will be presented elsewhere.

In general, we observe from the various cases studied in this subsection that when we let the parameters vary one of the fixed points for the periodic solution undergoes a Hopf bifurcation, i.e. it changes its behaviour from a stable focus to a centre and then to an unstable focus. As we can see from (2.17) that this behaviour is governed by the sign of $\pm \cos \left[ \frac{\pi}{2} (\theta_{AB} + \theta_\lambda) \right]$, this bifurcation can be achieved either by a spontaneous symmetry breaking, that is by varying the...
free parameter $A$, or by a complete breaking of the symmetry, by varying $\beta$ or $\gamma$. Both cases pass through the $\mathcal{P}\mathcal{T}$-symmetric case as realized by the centre.

2.1.3. Elliptic solutions. Next we specify $P(u) = (u - A)(u - B)(u - C)$ with three constants $A$, $B$ and $C$ leaving two constants at our disposal when solving (2.5):

$$\lambda = -\frac{\beta}{3\gamma}, \quad \kappa_1 = \frac{1}{6} [\beta(A^2 + AC + C^2) - 3c(A - C)],$$

(2.22)

$$\kappa_2 = \frac{AC}{6} [3c - \beta(A + C)] \quad \text{and} \quad B = \frac{3c}{\beta} - (A + C).$$

(2.23)

Vanishing asymptotic boundary conditions reduce this case to the previous one as by (2.7) they require either $A = B = 0$, $A = C = 0$ or $B = C = 0$. The evaluation of (2.6) yields in this case the elliptic solution

$$u(\zeta) = A + (B - A) \nu_s^2 \left[ \frac{1}{2} \sqrt{B - A} \sqrt{\lambda} (\zeta - \zeta_0) \right] \frac{A - C}{A - B}.$$  

(2.24)

with ns($\zeta$/m) being one of the Jacobi elliptic functions. Therefore, $u(\zeta)$ is a double periodic function

$$u(\zeta) = u(\zeta + \omega_1 + \omega_2)$$

(2.25)

with periods

$$\omega_1 = \frac{8}{\sqrt{B - A} \sqrt{\lambda}} K \left( \frac{A - C}{A - B} \right) \quad \text{and} \quad \omega_2 = i \frac{16}{\sqrt{B - A} \sqrt{\lambda}} K \left( \frac{C - B}{A - B} \right).$$

(2.26)

Here, $K(m)$ denotes a complete elliptic integral of the first kind. As expected, we recover the previous case when two of the constants $A$, $B$, $C$ coincide as in these cases the periods become dependent on each other, i.e. $\lim_{A \rightarrow B} \omega_1/\omega_2 = 1/2 \text{sign}(C - A)$, $\lim_{A \rightarrow C} \omega_1/\omega_2 = -i \infty$ and $\lim_{B \rightarrow C} \omega_1/\omega_2 = 0$.

Studying at first the $\mathcal{P}\mathcal{T}$-symmetric solutions, we find periodic solutions encircling some of the fixed points. For instance, in figure 6 we depict an example in which the fixed points $A$ and $B$ are encircled, whereas $C$ is situated outside of the trajectories. Different types of scenarios are also expected, changing for instance the sign in $\gamma$ will lead to a periodic orbit surrounding the points $B$ and $C$.

Next we break the $\mathcal{P}\mathcal{T}$-symmetry spontaneously by complexifying the free parameters of the solutions. We find that the periodic trajectories open up and cease to be periodic. In figure 7(a) we trace part of the trajectory to illustrate the behaviour. Starting at the point $u(-64) \approx 7.19 + i0.74$ the trajectory passes down between the points $A$ and $B$ and moves then up again surrounding once the points $C$ and $A$ in a clockwise sense. Thereafter, it encircles $C$ once more but instead of moving around $A$ it passes in between $C$ and $A$ surrounding $A$ in an anti-clockwise sense. It keeps progressing in an anti-clockwise manner encircling $C$ before passing from below between the points $A$ and $B$ reaching the point $u(18) \approx 6.36 + i3.45$. It appears that this type of movement is repeated indefinitely. In figure 7(b), we depict a wider range for $\zeta$ indicating that the region of the phase space depicted will eventually be filled densely by the trajectory, hence suggesting a chaotic behaviour. However, we do not observe a sensitivity towards the initial condition, which would be typical for a proper chaotic system.

Finally, we may also break the $\mathcal{P}\mathcal{T}$-symmetry completely by complexifying the parameters of the model $\beta$ on and $\gamma$. Examples for such a scenario are depicted in figure 8. The behaviour is similar to the one of the spontaneously broken case, i.e. the periodic motion has turned
Figure 6. $\mathcal{PT}$-symmetric complex elliptic solutions of the KdV equation with $A = 1$, $B = 3$, $C = 6$, $c = 1$, $\beta = 3/10$, $\gamma = -3$ for different values of $\text{Im}\,\zeta_0$.

Figure 7. Spontaneously broken $\mathcal{PT}$-symmetric complex elliptic solutions of the KdV equation for $\text{Im}\,\zeta_0 = 6$ with $A = 4$, $B = 5 - i/2$, $C = 1 + i/2$, $c = 1$, $\beta = 3/10$ and $\gamma = 3$: (a) $-64 \leq \zeta \leq 18$ solid (red) and $18 < \zeta \leq 200$ dashed (black); (b) $-200 < \zeta < 1400$.

into open trajectories with a non-compact limit set. Increasing the range for $\zeta$ will fill the depicted part of the phase space similarly as in the spontaneously broken $\mathcal{PT}$-symmetry case, thus suggesting a chaotic behaviour, albeit once again we do not observe the typical sensitivity towards the initial conditions.

Figures 7 and 8 are very reminiscent of the plots which may be found in section 5 of [12]. This is not surprising as formally the differential equations solved in there for the quantum mechanical setting for the potential $V \sim x^3$ are special cases of our more general treatment.
when making the identification $u \to x$ and $\zeta \to t$ for our travelling wave equation. With the further identifications
\begin{align*}
\kappa_1 &= 0, & \kappa_2 &= \gamma E, & \beta &= 6c g & \text{and} & \gamma &= -c \quad (2.27)
\end{align*}
equation (2.5) converts precisely into the quantum mechanical Hamiltonian
\begin{equation}
H = E = \frac{1}{2}p^2 + \frac{1}{2}x^2 - g x^3, 
\end{equation}
treated in [12]. The identification (2.27) explains why no analogue to our rational solution was found in [12], since $\kappa_1 = 0$ implies $c = 0$ and therefore the vanishing of $\kappa_2$, i.e. the confining potential, and $A$. However, as can be seen from (2.14), there should be an analogue to our trigonometric solution with energy $E = -4c^3/\gamma \beta^2$ when setting $A = 2c/\beta$. This energy depends on the coupling constant and is of course not freely choosable as in the elliptic case presented in this section where $\kappa_2$ is a free parameter. More potentials of the type treated in [12] can be obtained systematically from the deformations discussed below.

The new elliptic solutions recently found for generalized shallow wave equations [37] are expected to exhibit a similar behaviour as the solution discussed in this subsection. Complex soliton solutions will be presented elsewhere.

3. $\mathcal{PT}$-symmetric deformations of the KdV equation

Employing the standard arguments used in the study of non-Hermitian Hamiltonian systems with real eigenvalues, we may now $\mathcal{PT}$-symmetrically deform the Hamiltonian and maintain the possibility of having well-defined physical systems, e.g. we still obtain real energies resulting from the new models despite their non-Hermitian nature. The general principle is simply to deform $\mathcal{PT}$-anti-symmetric quantities, that means if we have the property $\mathcal{PT} : \phi(x,t) \mapsto -\phi(x,t)$ for some field $\phi(x,t)$, we define a deformation map $\delta_\varepsilon : \phi(x,t) \mapsto -i[\phi(x,t),\varepsilon]$. The undeformed case is recovered in the limit $\varepsilon = 1$. The new quantity will remain anti-$\mathcal{PT}$-symmetric with the crucial difference that the overall minus
sign is generated from the antilinear nature of the $PT$-operator, i.e. $i \mapsto -i$, rather than from $\phi(x,t) \mapsto -\phi(x,t)$. This means that for the Hamiltonian resulting from the density (2.1) we can make use of either of the two possibilities:

$$
\delta^+_\epsilon : u_x \mapsto u_{x,\epsilon} := -i(iu_x)^\epsilon \quad \text{or} \quad \delta^-_\epsilon : u \mapsto u_{\epsilon} := -i(iu)^\epsilon,
$$

(3.1)
depending on whether we choose $u(x,t)$ to be $PT$-symmetric or $PT$-anti-symmetric, respectively. Accordingly we define the deformed models with some suitable normalization by the following Hamiltonian densities:

$$
H^+_\epsilon = -\frac{\beta}{6} u^3 - \frac{\gamma}{1 + \epsilon} (iu_x)^{\epsilon+1}, \quad \text{and} \quad H^-_\epsilon = \frac{\beta}{(1 + \epsilon)(2 + \epsilon)} (iu)^{\epsilon+2} + \frac{\gamma}{2} u_x^2,
$$

(3.2)
with corresponding equations of motion

$$
\frac{d}{dt} u + \beta uu_x + \gamma u_{xxx,\epsilon} = 0 \quad \text{and} \quad \frac{d}{dt} u + i \beta uu_x + \gamma u_{xxx} = 0,
$$

(3.3)
respectively. The Hamiltonian $H^+_\epsilon$ was proposed in [24], whereas $H^-_\epsilon$ corresponds to a complex version of the generalized KdV equation. For the higher deformed derivatives we use here the notation $u_{x,\epsilon} := \partial u_{x,\epsilon}, u_{xxx,\epsilon} := \partial^2 u_{x,\epsilon}, \ldots, u_{xx,\epsilon} := \partial^{n-1} u_{x,\epsilon}$, which means we only deform the first derivative and keep acting on it with $\partial_x$ to define the higher order derivatives. On the level of the equation of motion we could also deform the dispersion term proportional to $\beta$, as investigated in [23] for the KdV equation. However, such deformations do not lead to Hamiltonian systems and the question of how $PT$-symmetry may be utilized in this scenario remains an open issue.

Let us now present some solutions to these equations.

### 3.1. The $H^+_\epsilon$-models

Proceeding similarly as in the previous section, we integrate the first equation in (3.3) twice and obtain the deformed version of equation (2.5):

$$
u^{(n)} = e^{\int \frac{i\pi}{2(1 + \epsilon)} (1 - \epsilon + 4n)} \left[ \frac{1}{\lambda_x} P(u) \right]^{i\epsilon},
$$

(3.4)
where we abbreviated $\lambda_x = -\beta(1 + \epsilon)/(6\gamma \epsilon)$ and denote different branches by $n$. The polynomial $P(u)$ is identical to the one introduced in (2.5), which means we can employ the same factorization as in the previous section. Integrating once more yields

$$
\zeta^{(n)} - \zeta_0 = e^{\int \frac{i\pi}{2(1 + \epsilon)} (\epsilon - 1 + 4n)} \int du \frac{1}{\left[ \lambda_x P(u) \right]^{i\epsilon}}.
$$

(3.5)
We may now proceed as before and specify further the form of $P(u)$. As for the case $\epsilon = 1$, in some specific cases we succeed to compute the remaining integral and subsequently solve the resulting equation for $u$, thus obtaining $u(\zeta)$. However, even when this is not possible analytically we can still investigate (3.5) numerically for all cases by viewing $\zeta$ as a function of the complex variable $u$ and plotting the contours of $\text{Im}[\zeta(u)] = \zeta_0$ in the $u$-plane, while taking special care about the different Riemann sheets labelled by $n$. One should distinguish here these types of Riemann sheets from those arising due to the technique employed in our solution procedure. Considering $\zeta$ as a function of $u$, as we do in some intermediate steps, will introduce new branch cuts which are sometimes seen in our figures, e.g. figure 22(b). However, we do not attribute any meaning to them in the $u$-plane. Genuine branch cuts can always be distinguished from the ‘technical’ ones by the fact that they have to be connected to the fixed points which are branch points.
3.1.1. Rational solutions. We start with the same assumption as in the previous section, namely $P(u) = (u - A)^3$, which will impose the same constraints (2.11) for the factorization.

The case $\varepsilon = 2$ is special since we may take the root $1/3$ in this case. Integrating (3.5) and solving the result for $u$ we find

$$u^{(n)}(\zeta) = A + \exp \left[ -\frac{i e^{\frac{2i\pi}{2\varepsilon^{3/2}}} \left( \frac{\beta}{\gamma} \right)^{1/3} (\zeta - \zeta_0) \right].$$

(3.6)

For the remaining integer values $\varepsilon \in \mathbb{Z}\backslash\{2\}$ we can also compute a particular solution

$$u(\zeta) = \frac{c}{\beta} + \exp \left( \frac{i \pi}{2} \frac{5 + \varepsilon}{2 - \varepsilon} \right) \left( \frac{6\varepsilon}{\beta} \right)^{\varepsilon-2} (1 + \varepsilon)^{\frac{\pi}{\varepsilon}} (\varepsilon - 2)^{\frac{\pi}{\varepsilon}} (\zeta - \zeta_0)^{\frac{\pi}{\varepsilon}}.$$  

(3.7)

However, in general we have to take care about the different branches present in (3.5). Our numerical findings for some specific cases are depicted in the figures below.

We supplement our numerical analysis with the prediction of some analytical features. Of special interest are the lines approaching or leaving the fixed points radially. When the symmetry is unbroken they can be defined equivalently as the lines for which the point $\zeta = \zeta_0$. We compute them by noting first that on one hand any point $\tilde{u}$ on the line radially crossing the point $u = A$ is characterized by a constant value for $\arg(\tilde{u} - A) =: \theta_0$. On the other hand, the change of $u$ with respect to $\zeta$ has to point into the same direction, i.e. $\arg(\pm \tilde{u} \zeta) = \theta_0$.

Hence, the lines of real initial conditions are determined by solving

$$\arg \left( \pm \tilde{u}^{(n)} \right) = \arg(\tilde{u}^{(n)} - A) + 2\pi m.$$

(3.8)

Employing (3.4) and the factorized form of $P(u)$ we can solve (3.8), obtaining

$$\theta_0^{(n,m,\pm)} := \arg(\tilde{u}^{(n)} - A) = \frac{\pi}{2(2 - \varepsilon)} \left[ \varepsilon - 1 + 4(m - n + m\varepsilon) - (1 + \varepsilon)(1 \pm 1) \right] + \frac{1}{\varepsilon - 2} \arg \lambda_\varepsilon.$$  

(3.9)

for the angle in which the trajectories with $\zeta_0 = 0$ enter the point $A$.

In figure 9, we show the first two members of the sequel parameterizing the deformation parameter as $\varepsilon = -n/(n+1)$ with $n = 1, 2, \ldots$. We observe that the pictures resemble flowers with $4 + 6n$ petals. We may predict the number of petals analytically as they are equal to the number of $\zeta_0 = 0$ solutions computable by (3.9). For instance, for the case $n = 3$ formula (3.9) predicts the 16 values $\theta_0 = (2\ell - 1)\pi/16$ for the parameter choice $c = 1, \beta = 2$ and $\gamma = 3$ with $\ell = 1, \ldots, 16$. We recognize these values in figure 9(b). Note that the wedge regions separate different solutions from each other as the point $A$ is always an asymptotic fixed point.

The trajectories appear to be qualitatively quite different for negative integer values of $\varepsilon$. For instance, in figures 10(a) and (b) we depict some trajectories for the models with $\varepsilon = -2$ and $\varepsilon = -3$, respectively. It appears that the fixed point $A$ is more like a saddle point, i.e. we exhibit trajectories more of a hyperbolic nature running away to infinity rather than converging asymptotically to $A$ as in figure 9. The $\zeta_0 = 0$ solutions are predicted once again correctly by (3.9) to be at $\theta_0 = (2\ell - 1)\pi/8$ and $\theta_0 = (2\ell - 1)\pi/5$.

For positive rational values of $\varepsilon$ a complete trajectory extends over several different Riemann sheets. Figures 11(a) and (b) show the solutions $\xi^{(1)}$ and $\xi^{(2)}$, respectively. The angles for the $\zeta_0 = 0$ solutions are predicted correctly by (3.9) to be $\theta_0 = 4\ell\pi/5$ for $\ell = 1, \ldots, 5$. A closed trajectory is obtained when passing the branch cut at $-\infty$ to $1/2$ from the upper-half plane in panel (a) to the lower-half plane in panel (b). In figure 13(a) a single trajectory is depicted for $\text{Im} \, \zeta_0 = 1$.

For the broken $\mathcal{PT}$-symmetry we present the solution in figure 12. The branch cut at $-\infty - i/4$ to $(1 - i)/4$ is passed from above in panel (a) to below in panel (b). This is illustrated
Figure 9. Complex $\mathcal{PT}$-symmetric rational solutions of the deformed KdV equation with $\lambda = 1/2$, $\epsilon = 1$, $\beta = 2$ and $\gamma = 3$ for (a) $\mathcal{H}^e_{1/2}$ and (b) $\mathcal{H}^e_{-2/3}$.

Figure 10. Complex $\mathcal{PT}$-symmetric rational solutions of the deformed KdV equation with $\lambda = 1/2$, $\epsilon = 1$, $\beta = 2$ and $\gamma = 3$ for (a) $\mathcal{H}^e_{-2}$ and (b) $\mathcal{H}^e_{-3}$.

in figure 13(b), where we depict just one single trajectory for Im $\zeta_0 = 1$. The trajectories for the $\mathcal{PT}$-symmetric and broken $\mathcal{PT}$-symmetric cases look qualitatively very similar, the major difference being that the fixed point has moved away from the real axis, thus leading to a loss of the symmetry.

Potentially there are of course many more possible values for $\epsilon$ to be considered. We conclude here just by presenting two more examples with $\epsilon$ being an integer, as these are the most common deformations usually considered. For these values we obtain yet another type of characteristics as more and more branches have to be taken into account for increasing $\epsilon$. In figures 14 and 15, we depict all branches for the trajectory with Im $\zeta_0 = 1$ for $\epsilon = 3$ and $\epsilon = 6$, respectively.
We observe some intricate winding behaviour near the fixed point, which, however, is not asymptotic in these cases. As is apparent from solution (3.7), the trajectories diverge to infinity into various directions depending on the chosen Riemann sheet. Evidently, the number of these asymptotes grows with increasing $\varepsilon$. Breaking the $PT$-symmetry will only distort the trajectories, giving rise to new directions of the asymptotes, but not changing their numbers.

3.1.2. Trigonometric/hyperbolic solutions. As in the non-deformed case, we assume next the factorization $P(u) = (u - A)^2(u - B)$, which will impose once more the constraints (2.14). Also in this case we can compute (3.5) numerically.
Figure 13. Single trajectory for the rational solution of $\mathcal{H}^{1}_{\alpha,\beta}$ with $\mathfrak{Im} \zeta_0 = 1$: (a) $\mathcal{P}\mathcal{T}$-symmetric solutions for the values as specified in figure 11; (b) broken $\mathcal{P}\mathcal{T}$-symmetric solutions for the values as specified in figure 12.

Figure 14. Single trajectory for the rational solution for of $\mathcal{H}^{3}_{\alpha,\beta}$ with $\mathfrak{Im} \zeta_0 = 1$: (a) $\mathcal{P}\mathcal{T}$-symmetric solutions for $A = 1/2$, $\epsilon = 1$, $\beta = 2$ and $\gamma = 3$; (b) broken $\mathcal{P}\mathcal{T}$-symmetric solutions for $A = 1/4(1 - i)$, $\epsilon = 1$, $\beta = 2 + 2i$ and $\gamma = 3$.

Again we supplement our numerical findings by some analytical results. We can predict once more the lines for which $\zeta_0 = 0$ explicitly following the arguments of the previous subsection, with the difference that we have to take the limit to either the point $A$ or $B$. For the angle of lines near the point $A$ we find

$$
\theta_{A}^{(n,m,\pm)} = \frac{\pi}{2(\epsilon - 1)} \left[ 1 - \epsilon + 4(n - m) - 4\epsilon m \right] + \frac{1}{\epsilon - 1} \arg(A - B) + \frac{1}{\epsilon - 2} \arg(\pm \lambda_n),
$$

(3.10)
Figure 15. Single trajectory for the rational solution for of \( H_0^+ \) with \( \text{Im} \zeta_0 = 1 \): (a) \( \mathcal{PT} \)-symmetric solutions for \( A = 1/2, c = 1, \beta = 2 \) and \( \gamma = 3 \); (b) broken \( \mathcal{PT} \)-symmetric solutions for \( A = 1/4(1 - i), c = 1, \beta = 2 + 2i \) and \( \gamma = 3 \).

Figure 16. Complex \( \mathcal{PT} \)-symmetric trigonometric/hyperbolic solutions of the deformed KdV equation with \( A = 4, B = 2, c = 1, \beta = 2 \) and \( \gamma = 3 \) for (a) \( H_{1/2}^- \) and (b) \( H_{2/3}^- \).

We depict some specific examples with different characteristics in figures 16 to 19.

In figure 16, we recognize a similar characteristic behaviour as for the rational solution of the same model depicted in figure 9. Roughly, the portrait in figure 16 corresponds to that in figure 9 with the difference that the single flower centre in the rational case situated as \( A \) has been pulled apart to the two points \( A \) and \( B \) in the trigonometric one. The overall effect is that some of the trajectories separated in the rational case in different wedges are

\[
\theta_{B}^{(n,m,\pm)} = \frac{\pi}{2} (\varepsilon - 1 - 4n) + (1 + \varepsilon)2\pi m + (\varepsilon - 1) \arg(B - A) - \arg(\pm\lambda_{\varepsilon}).
\]
Figure 17. Single trajectory for the complex $\mathcal{PT}$-symmetric trigonometric/hyperbolic solutions of the deformed KdV equation for $\mathcal{H}_{1/2}^\pm$ with the same values as specified in figure 16 for (a) $\text{Im} \zeta_0 = 1$ and (b) $\text{Im} \zeta_0 = -1$.

Figure 18. Broken $\mathcal{PT}$-symmetric trigonometric/hyperbolic solutions of the deformed KdV equation $\mathcal{H}_{1/2}^\pm$: (a) spontaneously broken $\mathcal{PT}$-symmetry with $A = 4 + i$, $B = 2 - 2i$, $c = 1$, $\beta = 3/10$ and $\gamma = 3$; (b) broken $\mathcal{PT}$-symmetry with $A = 4$, $B = 2$, $c = 1$, $\beta = 3/10$ and $\gamma = 3 + i$.

Joint in the trigonometric/hyperbolic case. To illustrate this we have extracted one single trajectory in figure 17. We observe a similar behaviour for the entire sequel parameterized by $\varepsilon = -n/(n + 1)$ with $n = 1, 2, \ldots$. Note also that unlike as in the case $\varepsilon = 1$, there is no distinction between a periodic and an asymptotically constant solution.

Having enough free parameters available we can now break the symmetry for this solution also spontaneously or completely as illustrated in figure 18. In both cases, we observe that the amount of wedges remains unchanged, the symmetry about the real axis is lost and the winding around the fixed points becomes more intricate. In addition, trajectories from
Figure 18. We note that instead trajectories from the light shaded region in the upper-half plane shaded region in the unbroken case.

Two constants free. We present here only the results for the cases certain wedge regions connect in different ways, e.g. in figure 16 we observe a connected figure 17. Such type of trajectories do not exist in the broken case as we can observe in figure 18. We note that instead trajectories from the light shaded region in the upper-half plane connect to a dark shaded region in the lower-half plane to the left of the corresponding light shaded region in the unbroken case.

Comparing next the solutions for \( \varepsilon \in \mathbb{R}^- \) we observe in figure 19(a) a similar transformation from the rational to the trigonometric/hyperbolic case as for the previous example. Again, we find that the rational solution resembles this solution for the same values of the deformation parameters, with the difference that the characteristic behaviour around the point \( A \) in the rational case has been distorted to the points \( A \) and \( B \).

The broken \( \mathcal{PT} \)-symmetric case for this model is presented in figure 19. In panel (b) we depict the spontaneously broken \( \mathcal{PT} \)-symmetric scenario for \( \mathcal{H}_{\omega,2} \), observing that the symmetry is lost since the trajectories rotate together with the two fixed points. The \( \mathcal{PT} \)-symmetry is completely broken in the presentation in (c) for the same model. Having kept the values for the fixed points, we note that in this case the orbits rotate around them.

We have also investigated other models observing similar patterns, which we will however not present here.

3.1.3. Elliptic solutions. Proceeding just as in the undeformed case we specify next
\[
P(u) = (u - A)(u - B)(u - C),
\]
which imposes constraints (2.22) and (2.23), thus leaving two constants free. We present here only the results for the cases \( \varepsilon = -2 \) and \( \varepsilon = -1/2 \) in figures 20 and 21, respectively.

In all three possible scenarios we observe a similar qualitative behaviour as for the trigonometric/hyperbolic solutions of the previous subsection with the difference that we have three instead of two fixed points.

3.2. The \( \mathcal{H}_\omega \)-models

Let us now turn to the second type of deformation. In order to construct the solutions we proceed in a similar manner as in the previous case. Integrating now twice the second equation in equation (3.3) we obtain
\[
\frac{u_x^2}{\gamma} = \kappa_2 + \kappa_1 u + \frac{c}{2} u^2 - \frac{\beta}{(1 + \varepsilon)(2 + \varepsilon)} \left( \frac{u^{2\gamma}}{u^{2\gamma - 1}} \right) = : \lambda Q(u), \tag{3.12}
\]
where

$$\lambda = -\frac{2\beta i^{\varepsilon}}{\gamma (1 + \varepsilon)(2 + \varepsilon)}.$$  \hspace{1cm} (3.13)

A crucial difference between $\mathcal{H}^+_{\varepsilon}$ and $\mathcal{H}^-_{\varepsilon}$ is that unlike as the polynomial $P(u)$, which was of fixed order 3, the order of $Q(u)$ depends on $\varepsilon$, meaning that we have more and more possibilities of factorizing it for growing $\varepsilon$. For instance, for a given integer value $n \in \mathbb{N}_0$, the factorization of $Q(u)$ as

$$Q(u) = (u - A_{1})^{\varepsilon + 2 - n} \prod_{i=1}^{n} (u - A_{i+1})$$  \hspace{1cm} (3.14)

admits solutions provided $n - 2 \leq \varepsilon \leq n + 1$ and $\varepsilon \in \mathbb{N}$. This allows of course for yet another infinity of possibilities.
When $\kappa_1 = \kappa_2 = 0$, we can find a closed solution valid for all $\varepsilon$ by integrating (3.12) and solving it for $u$:

$$u(\zeta) = \left( \frac{c(\varepsilon + 1)(\varepsilon + 2)}{i\varepsilon \beta \cosh \left( \frac{\sqrt{2}c(\varepsilon + 2)}{\varepsilon \beta} \right) + 1} \right)^{1/\varepsilon}.$$  \hspace{1cm} (3.15)

The generic scenario does not yield such a simple answer.

### 3.2.1. $H_{-2}$

This case is especially interesting as it corresponds to a complex version of the modified KdV equation. Specifying (3.14) for instance as $Q(u) = (u - A)^3(u - B)$ we can factorize the polynomial in (3.12) with the choice

$$\kappa_1 = -\frac{2c^{3/2}}{3\sqrt{-\beta}}, \quad \kappa_2 = -\frac{c^2}{4\beta}, \quad A = -\frac{B}{3} \quad \text{and} \quad B = -\frac{3\sqrt{c}}{\sqrt{-\beta}}.$$  \hspace{1cm} (3.16)

Note that this fixes all the boundary conditions for a given model. Solving (3.12) then yields a rational solution for the second equation in (3.3):

$$u(\zeta) = \sqrt{-\frac{c}{\beta}} \frac{2c\zeta^2 - 9\gamma}{3\gamma + 2c\zeta^2}.$$  \hspace{1cm} (3.17)

As is well known, for the real case one may construct solutions for the KdV equation from those of the modified KdV equation by means of a Miura transformation. We also expect this to hold for their complex versions. Indeed, using the transformation of the form

$$u_{KdV}(\zeta) = \frac{6\gamma}{\beta} u_{\xi} - u^2$$  \hspace{1cm} (3.18)

yields the rational solution of the KdV equations (2.12) from (3.17) when we identify $\zeta_0 = i\sqrt{3\gamma/2\beta}$ therein.

Assuming in (3.14) instead $Q(u) = (u - A)^2(u - B)(u - C)$ we can factorize the polynomial in (3.12) with the constraints

$$\kappa_1 = \frac{\beta C^2(\theta - 5C\beta) + 9c(\theta - 3C\beta)}{81\beta}, \quad \kappa_2 = \frac{C(2\theta - C\beta)(C\beta + \theta)^2}{324\beta^2},$$

$$A = -\frac{1}{2}(B + C) \quad \text{and} \quad B = \frac{2\theta - C\beta}{3\beta},$$  \hspace{1cm} (3.19, 3.20)

where we abbreviated $\theta := \sqrt{2}\sqrt{-\beta(\beta C^2 + 9c)}$. Note that in this case one constant remains free. The integration of (3.12) yields in this case a trigonometric solution, which, using the same Miura transformation (3.18), may also be converted into a solution of the KdV equations.

### 3.2.2. $H_{-4}$

In this case, the polynomial on the right-hand side of (3.12) is of sixth order. We present here just one very symmetric solution by assuming a factorization of the form $Q(u) = u^6(u^2 - B^2)(u^2 - C^2)$, which is possible with the simple choice

$$\kappa_1 = \kappa_2 = 0, \quad B = iC \quad \text{and} \quad C^4 = \frac{15c}{\beta}.$$  \hspace{1cm} (3.21)

Thus, we have made contact with solution (3.15). Parameterizing $B = r_B e^{i\theta_B}$ and $\lambda = r_\lambda e^{i\theta_\lambda}$ the eigenvalues of the Jacobian when linearized about $u = 0$ are easily computed to

$$j_1 = \pm i\sqrt{r_B} r^2 \exp \left[ \frac{i}{2}(4\theta_B + \theta_\lambda) \right] \quad \text{and} \quad j_2 = \mp i\sqrt{r_B} r^2 \exp \left[ -\frac{i}{2}(4\theta_B + \theta_\lambda) \right].$$  \hspace{1cm} (3.22)
This means that for the \( \mathcal{PT} \)-symmetric solution we always obtain two real degenerate eigenvalues and therefore a star node at \( u = 0 \). For the positive square root in (3.12) with \( B = \pm (15c/\beta)^{1/4} \) or with \( B = \pm i(15c/\beta)^{1/4} \) the node is stable or unstable, respectively. The stability property is reversed for the negative square root. Taking the branch cuts at \( \text{Im } u = 0, \text{Re } u = 0, \text{Im } u = \text{Re } u \) into account we obtain closed curves. The four zeros of \( Q(u) \), i.e. \( \pm (15c/\beta)^{1/4}, \pm i(15c/\beta)^{1/4} \), are surrounded by the trajectories. All these features can be seen in figure 22(a). We may also tune the coupling constants in such a way that \( 4\theta_B + \theta_\lambda = 0 \), which by (3.22) implies that the fixed point at the origin should become a centre. Indeed, for a specific choice we observe this in figure 22(b). Apart from the origin, the remaining four fixed points are now situated outside of the trajectories.

Since we have no free parameter left in our factorization this solution cannot be broken spontaneously. A complete breaking is carried out by complexifying \( \beta \) or \( \gamma \). When choosing \( \text{Im } \beta \neq 0 \) we can still achieve in (3.22) that \( 4\theta_B + \theta_\lambda = \pm \pi \), such that the star node nature of the fixed point is preserved despite the fact that the \( \mathcal{PT} \)-symmetry is lost. For a particular choice this behaviour is depicted in figure 23(a). More surprising is the fact that we can also achieve that \( 4\theta_B + \theta_\lambda = 0 \) in the broken case. This means the eigenvalues in (3.22) are purely imaginary and the fixed point at the origin is a centre. We depict this possibility in figure 23(b). This means we have closed trajectories even in the \( \mathcal{PT} \)-symmetrically broken case.

In contrast, for \( \text{Im } \gamma \neq 0 \) the eigenvalues \( j \) will become complex and the fixed point at the origin turns into an unstable or stable focus. Taking the branch cut structure into account we observed this behaviour in figure 24(a) for \( \beta \in \mathbb{R}, \gamma \in \mathbb{C} \) and in figure 24(b) for \( \beta \in \mathbb{C}, \gamma \in \mathbb{C} \).

Once again we observed that when identifying a coupling constant as a bifurcation parameter, the origin undergoes a Hopf bifurcation.

Making once more the identification \( u \to x \) and \( \zeta \to t \) for our travelling wave equation together with the identification

\[
\kappa_1 = 0, \quad \kappa_2 = \gamma E, \quad \text{and} \quad \beta = \gamma g(1 + \varepsilon)(2 + \varepsilon), \quad (3.23)
\]
equation (3.12) converts into the classical deformation of the harmonic oscillator

\[ H = E = \frac{1}{2} p^2 - \frac{c}{2} x^2 + g x^2 (i \epsilon)^2. \]  

(3.24)

Setting furthermore \( c = 0 \), which in our setting corresponds to a static solution, we obtain precisely the potential treated in the seminal paper by Bender and Boettcher [4].
4. Complex coupled nonlinear wave equation of Ito type

It has been known for some time that the KdV field \( u(x, t) \) may be coupled to a nonlinear field \( v(x, t) \) without destroying the integrability of the newly constructed system. This coupling is carried out by the introduction of an additional dispersion term involving the second field \( v(x, t) \) and the assumption of a specific form of the evolution equation for this field. The coupled system describing this type of scenario acquires the general form

\[
\begin{align*}
  u_t + \alpha vv_x + \beta uu_x + \gamma u_{xxx} &= 0, & \alpha, \beta, \gamma \in \mathbb{C}, \\
  v_t + \delta (uv)_x + \phi v_{xxx} &= 0, & \delta, \phi \in \mathbb{C}.
\end{align*}
\]

(4.1) (4.2)

Imposing the constraints \( \beta = 3\gamma, \phi = 1, \delta = 3 \) on the constants and leaving \( \alpha, \gamma \) free, the Hirota method can be applied to establish that the system (4.1) and (4.2) possesses \( N \)-soliton solutions [38]. When selecting in addition \( \gamma = -\frac{1}{2} \) or \( \alpha = -2, \beta = -6, \gamma = -1, \delta = -2 \) and \( \phi = 0 \) as in [40], it was established in both cases that the system possesses infinitely many charges. For \( \phi = 0 \), the system (4.1) and (4.2) was shown to possess soliton solutions of cusp type [41]. Note also that when complexifying the KdV field in (2.2) as \( u_{KdV} \to u + iv \) we obtain the system (4.1) and (4.2) for the special choice \( \alpha = -\beta_{KdV}, \beta = \beta_{KdV}, \gamma = \gamma_{KdV}, \delta = \beta_{KdV} \) and \( \phi = \gamma_{KdV} \). In the following we will mainly discuss the case \( \phi = 0 \), such that even the real case of (4.1) and (4.2) is distinct from the complexified version of the KdV equation.

It is straightforward to verify that for the choice \( \delta = \alpha \) the system of equations (4.1) and (4.2) results from a variation of a Hamiltonian whose density is given by

\[
H_I = -\frac{\alpha}{2} u v^2 - \frac{\beta}{6} u^3 + \frac{\gamma}{2} u^2 x + \frac{\phi}{2} v^2,
\]

(4.3)

when using the variational derivative and time evolution in the standard way

\[
w_t = \frac{\partial}{\partial x} \left( \sum_{n=0}^{\infty} (-1)^n \frac{\partial^6 H_I}{\partial u x^n} \right) x \quad \text{for} \quad w = u, v.
\]

(4.4)

The Hamiltonian resulting from the density (4.3) is manifestly \( PT \)-symmetric as it remains invariant under a simultaneous parity transformation and time reversal which may be realized in four alternative ways as

\[
PT_{++}: \ x \mapsto -x, \ t \mapsto -t, \ i \mapsto -i, \ u \mapsto u, \ v \mapsto v \quad \text{for} \quad \alpha, \beta, \gamma, \phi \in \mathbb{R},
\]

(4.5)

\[
PT_{+-}: \ x \mapsto -x, \ t \mapsto -t, \ i \mapsto -i, \ u \mapsto u, \ v \mapsto -v \quad \text{for} \quad \alpha, \beta, \gamma, \phi \in \mathbb{R},
\]

(4.6)

\[
PT_{-+}: \ x \mapsto -x, \ t \mapsto -t, \ i \mapsto -i, \ u \mapsto -u, \ v \mapsto v \quad \text{for} \quad i\alpha, i\beta, \gamma, \phi \in \mathbb{R},
\]

(4.7)

\[
PT_{--}: \ x \mapsto -x, \ t \mapsto -t, \ i \mapsto -i, \ u \mapsto -u, \ v \mapsto -v \quad \text{for} \quad i\alpha, i\beta, \gamma, \phi \in \mathbb{R},
\]

(4.8)

depending on whether we choose the fields \( u, v \) to be \( PT \) symmetric or anti-symmetric. All possibilities ensure that \( PT_{ij} : \mathcal{H}_I \to \mathcal{H}_I \) holds with \( i, j \in \{+, -\} \). First we will exploit these possibilities to explain the reality of the energies and in section 5 we use them to define new physically feasible models.
4.1. $\mathcal{PT}$-symmetric, spontaneously broken and broken solutions

As usual, in this context we assume that the fields acquire the form of a travelling wave $u(x, t) = u(\zeta)$ and $v(x, t) = v(\zeta)$ with $\zeta = x - ct$. In [42], it was shown for the case $\phi = 0$ that the possibility of having only one field to be a travelling wave and not the other is inconsistent. We extrapolate here without rigorous proof that this assumption can be made without loss of generality even in other cases. Let us briefly recall how these equations may be solved in a systematic way. To begin with we integrate (4.1) and (4.2) to

$$-cu + \frac{\alpha}{2} v^2 + \frac{\beta}{2} u^2 + \gamma u\zeta\zeta = \kappa_1,$$

$$-cv + a u v = \kappa_2,$$

with integration constants $\kappa_1, \kappa_2$. In the following we always exclude the case $\kappa_2 = 0$ as this implies the vanishing of the new field $v = 0$, which means a reduction to the deformed KdV equation, or a constant KdV field $u = \alpha/c$. Multiplying (4.9) by $u\zeta$, using (4.10) to replace $v$ by $u$ in (4.9), we can integrate once more and obtain

$$u_\zeta^2 = 2\frac{\gamma}{\gamma} \left( \kappa_3 + \kappa_1 u + \frac{c}{2} u^2 - \frac{\beta}{6} u^3 + \frac{\kappa_2^2}{2} \frac{1}{au - c} \right).$$

This equation is difficult to solve directly, but following [41] we trade the $u$ field for the $v$ field with

$$u = \frac{c}{\alpha} + \frac{\kappa_2}{\alpha v}$$

and

$$u_\zeta = -\frac{\kappa_2}{\alpha v^2} v_\zeta$$

and obtain

$$v_\zeta^2 = -\frac{v}{3\alpha'\kappa_2} \sum_{k=0}^4 a_k v^k,$$

where

$$a_0 = \beta \kappa_2, \quad a_1 = 3c\kappa_2^2(\beta - \alpha), \quad a_2 = 3\kappa_2(\beta c^2 - 2\alpha c^2 - 2\alpha^2\kappa_1),$$

$$a_3 = c^3(\beta - 3\alpha) - 6\alpha^2(\kappa_1 + \alpha\kappa_3), \quad a_4 = -3\alpha^3\kappa_2.$$

At first sight this looks even less encouraging than (4.11). However, now the right-hand side is a polynomial and in case we can factorize the sum $\sum_{k=0}^4 a_k v^k$ into some convenient form we may be able to integrate (4.13) similarly as in the previous section. Up to one integration the solution is therefore

$$\pm \sqrt{\lambda} (\zeta - \zeta_0) = \int dv \frac{1}{\sqrt{R(v)}},$$

with $\lambda = \alpha'^2/\gamma \kappa_2$ and $\lambda R(v)$ corresponding to the right-hand side of (4.13). We note that not all conceivable assumptions for the sum will lead to meaningful or non-trivial solutions. Taking for instance $R(v) = v^4(A + v)$ or $R(v) = v^3(A + v)(B + v)$ with some unknown constants $A$ and $B$ leads in both cases to $\kappa_2 = 0$ or $\beta = 0$, which we exclude for the above-mentioned reasons.

When demanding vanishing asymptotic boundary conditions for the $u$-field and its derivatives, relations (4.9) and (4.11) imply that we satisfy the additional constraints

$$\lim_{\zeta \to \pm \infty} u, u_\zeta, u_{\zeta\zeta} = 0 \quad \Rightarrow \quad \kappa_1 = \frac{\alpha \kappa_2^2}{2c^2}, \quad \kappa_3 = \frac{\kappa_2^2}{2c}, \quad \lim_{\zeta \to \pm \infty} v = -\frac{\kappa_2}{c}. \quad (4.16)$$
4.1.1. Type I solutions. The simplest possible factorization for the sum in (4.13) is $R(v) = v(v - A)^2$, which holds up to the constraints

$$
k_1 = \frac{c^2(2\beta \alpha - 9\alpha^2 - \beta^2)}{16\alpha^2\beta}, \quad k_2 = \frac{3\sqrt{3}c^2(\alpha - \beta)^2}{16\alpha^{3/2}(-\beta)^{3/2}}, \quad k_3 = \frac{c^3(3\alpha - \beta)(9\alpha^2 - 6\beta\alpha + 5\beta^2)}{96\alpha^3\beta^2}, \quad A = \frac{\sqrt{3}c(\beta - \alpha)}{4\alpha^{3/2}\sqrt{-\beta}}, \quad (4.17)
$$

For a given specific model, i.e. fixed $\alpha$, $\beta$, $\gamma$, $\phi$, this means that all remaining free parameters are fixed. Vanishing asymptotic boundary conditions for $u$ require by the first two equations in (4.16) only one further relation $\beta = -3\alpha$, despite the fact that one has to solve two constraining equations. Solving (4.15) for the given factorization yields

$$
\zeta - \zeta_0 = \pm \frac{1}{A\sqrt{\lambda}} \left[ \text{arctanh} \left( \frac{\sqrt{\lambda}}{\sqrt{A}} \right) + \frac{\sqrt{v}}{A - v} \right]. \quad (4.18)
$$

First we focus on the real solution, which is obtained from (4.18) for $\lambda > 0$ and $0 < v < A$. A further solution is obtained when replacing in (4.18) the arctanh by arcoth, which produces a real solution for $0 < A < v$. Again we are faced with the problem that we are not able to extract from (4.18) the function $v(\xi)$. Nonetheless, we can rewrite (4.18) as

$$
v(\xi) = A + \frac{\sqrt{v(\xi)}\sqrt{A}}{\text{arctanh} \left( \frac{\sqrt{\lambda}}{\sqrt{A}} \right) \pm A^{3/2}\sqrt{\lambda}(\xi - \zeta_0)} \quad (4.19)
$$

similarly as done in [41]. This is of course still not $v(\xi)$, but this form is very useful to extract various types of information in an analytical manner. Simply by considering the functions on the right-hand side of (4.19), we observe the asymptotic behaviour $\lim_{\xi \to \pm\infty} v(\xi) = A$. In principle, this is already sufficient to extract the qualitative behaviour of $v(\xi)$, since we have also assumed a simple factorization for the derivative of $v$ in (4.13). We can, however, be more precise because the formulation (4.19) is also ideally suited to be solved numerically. In principle, this possibility will become more powerful when considering more complicated scenarios. However, in these cases one also encounters more occurrences of the field $v$ and it is not always obvious for which one the equation should be solved.

For the purpose of a numerical study we discretize the equations as $v_{n+1} = F(v_n)$ and subsequently solve them iteratively for given values of $\zeta$. The fixed points $v_f(\xi)$ of such discretized equations are known to be stable if and only if $|F'[v_f(\xi)]| < 1$. We use this criterium to facilitate the numerical investigations. Concretely, we solve the recursive equation

$$
v_{n+1}(\xi) = A + \frac{\sqrt{v_n(\xi)}\sqrt{A}}{\text{arctanh} \left( \frac{\sqrt{\lambda}}{\sqrt{A}} \right) \pm A^{3/2}\sqrt{\lambda}(\xi - \zeta_0)} \quad (4.20)
$$

which converges very rapidly to a precision of $\sim 10^{-5}$ typically already for less than 150 iterations. We proceed similarly for the solution when $0 < A < v$. In figure 25, we depict two types of cusp solutions obtained in this manner.

In principle, we could also proceed in this manner when taking complex initial conditions, but it is simpler to produce the contour plot in the way outlined at the beginning of section 3.1. For the same values of the parameters as in figure 25 we depict our results in figure 26, indicating as before the imaginary parts of $\zeta_0$ on some particular trajectories.

As expected from (4.19) we observe in figures 26 that the complex solutions tend to the same asymptotic value as the real ones. The point $A$ is an unstable focus in the $\mathcal{PT}$-symmetric...
Figure 25. Cusp solutions with asymptotically vanishing boundary conditions for the $u$-field of the Ito-type equation with $A = 3$, $c = 1$, $\alpha = -1/3$, $\beta = 1$, $\gamma = -1/27^2$ and $\zeta_0 = 0$: (a) $v$-field; (b) $u$-field.

Figure 26. Complex type I solutions with asymptotically vanishing boundary conditions for the $u$-field and corresponding $v$-field of the Ito-type equation: (a) $\mathcal{PT}$-symmetric case with $A = 3$, $c = 1$, $\alpha = -1/3$, $\beta = 1$ and $\gamma = -1/27^2$; (b) same values as in panel (a) for a single trajectory with $\zeta_0 = 0.04$; (c) broken $\mathcal{PT}$-symmetric case with $A \approx 3.054 - 0.783i$, $c = 1$, $\alpha = -1/3$, $\beta = 1 - i$ and $\gamma = -1/27^2 - i/2$.

and its broken version. In panel (b) more Riemann sheets are taken into account, revealing more substructure compared to panel (a). In the $\mathcal{PT}$-symmetric case we also observe the crucial feature that $v^*(\zeta) = v(-\zeta)$ and $u^*(\zeta) = u(-\zeta)$, which guarantees the reality of the energy as defined by the expression in (1.2).
4.1.2. Type II solutions. Next we introduce an additional parameter $B$ and assume the factorization of the form $R(v) = v(v - A)^3(v - B)^2$, which imposes the four constraining equations

$$
\kappa_1 = \frac{(\beta - 2\alpha)c^2}{2a^2} + \frac{(A^2 + 4AB + B^2)\alpha}{2},
$$

(4.21)

$$
\kappa_2 = \frac{2AB(A + B)\alpha^3}{c(\beta - \alpha)},
$$

(4.22)

$$
\kappa_3 = \frac{2AB(A + B)^3\alpha^3}{c(\alpha - \beta)} + \frac{\sqrt{3}c(3\alpha - 2\beta)}{6a^3} - \frac{c(A^2 + 4AB + B^2)}{2}.
$$

(4.23)

$$
A + B = \pm \frac{3c(\alpha - \beta)}{2a^2} \sqrt{\alpha}.
$$

(4.24)

This means our five constants $\kappa_1, \kappa_2, \kappa_3, A, B$ are constrained by four equations, such that one of them remains free and thus allows us to adjust for some desired boundary conditions. We exclude the trivial solutions $A = B = A + B = 0$ as they all lead to $\kappa_2 = 0$. Solving (4.15) for this factorization gives

$$
\zeta - \zeta_0 = \pm \frac{2}{(A - B)\sqrt{\lambda}} \left[ \arctanh \left( \frac{\sqrt{v}}{\sqrt{A}} \right) - \arctanh \left( \frac{\sqrt{v}}{\sqrt{B}} \right) \right],
$$

(4.25)

which reduces to (4.18) in the limit $B \to A$. Arguing as in the previous case, the real solutions are obtained from (4.18) for $\lambda > 0$ and $0 < v < A < B$. For other configurations of the ordering we replace in (4.25) the arctanh by arcoth when the argument becomes greater than 1. For the same reasons as in the previous section we isolate $v$ from this equation and obtain

$$
v(\zeta) = A \tanh^2 \left[ \frac{1}{2} \sqrt{A} \sqrt{A(B - A)(\zeta - \zeta_0)} + \sqrt{\frac{A}{B}} \arctan \left( \frac{\sqrt{v(\zeta)}}{\sqrt{B}} \right) \right].
$$

(4.26)

Simply by considering the functions on the right-hand side of (4.26), we observe the asymptotic behaviour $\lim_{\zeta \to \pm \infty} v(\zeta) = A$. In this case the vanishing asymptotic boundary conditions for $u$ can be implemented without additional constraints on the model defining parameters $\alpha, \beta, \gamma, \phi$. For other orderings of $v, A, B$ we may also obtain $\lim_{\zeta \to \pm \infty} v(\zeta) = B$. When $\lim_{\zeta \to \pm \infty} v(\zeta) = A$ all constraints in (4.16) and (4.21)–(4.24) are satisfied with the choice

$$
A = \frac{c^2(\beta - 3\alpha)}{2\sqrt{3\alpha}^{3/2} \sqrt{\beta}} \quad \text{and} \quad B = -\frac{c\sqrt{\beta}}{\sqrt{3\alpha}^{3/2}},
$$

(4.27)

whereas when $\lim_{\zeta \to \pm \infty} v(\zeta) = B$ we need to exchange $A$ and $B$ in (4.27). For the solution reported in [41] the possible values for $\alpha, \beta, \gamma$ were restricted because the value of $B$ was pre-selected, but as we have shown here this is not necessary.

Choosing vanishing boundary conditions we depict the real solutions in figure 27 for the different regimes and real initial values $\zeta_0$.

Our findings disagree slightly with those in [41], where the case $0 < v < |A| < |B|$ was reported to be of cusp type, whereas we observe that it is of a standard soliton nature. Our numerical findings are in agreement with the factorization of the right-hand side of equation (4.13), which implies that $v(0) = 0$ and not infinity as needed for a cusp solution. We also find a new kink-type solution in the region $0 < |A| < v < |B|$ not reported by Kawamoto. In the regions $0 < v < A < B$ and $0 < A < B < v$, we observe explicitly the $PT$-symmetry $\zeta \to -\zeta + 2\zeta_0, v \to v$ and $u \to u$. 

30
Next we investigate some complex solutions by taking the initial values $\zeta_0$ to be purely imaginary. For this type of the factorization the linearization around the points $A$ and $B$ is straightforward as the square root in (4.15) can be taken. Parameterizing $A = r_A e^{i\theta_A}$, $B = r_B e^{i\theta_B}$ and $\lambda = r_\lambda e^{i\theta_\lambda}$ the eigenvalues of the Jacobian when linearized about $v = A$ are computed to

$$
\begin{align*}
j_k &= \pm \sqrt{r_A r_\lambda} \left[ \cos \left( \frac{3\theta_A}{2} + \frac{\theta_\lambda}{2} \right) r_A - \cos \left( \frac{\theta_A}{2} + \frac{\theta_\lambda}{2} \right) r_B \right] \\
&\quad + i (-1)^k \sqrt{r_A r_\lambda} \left[ \sin \left( \frac{3\theta_A}{2} + \frac{\theta_\lambda}{2} \right) r_A - \sin \left( \frac{\theta_A}{2} + \frac{\theta_\lambda}{2} \right) r_B \right]
\end{align*}
$$

for $k = 1, 2$. For the linearization around $v = B$ we obtain (4.28) with $A \leftrightarrow B$.

By tuning our free parameters we can produce any desired characteristic behaviour for the fixed points at $A$ and $B$. For instance, in the $PT$-symmetric setting for $A$, $B \in \mathbb{R}^+$ we always obtain two real degenerate eigenvalues when $\lambda \in \mathbb{R}^+$ and therefore star nodes at $v = A$ and $v = B$. In figure 28, we report an example of this type with $j_1 = j_2 = \mp 1/\sqrt{6}$.

We also observe in figure 28(a) that the complex trajectories surround the real solution with the asymptotic point or points in common. For instance, the trajectory with $\text{Im} \zeta_0 = 1$ corresponds to a complexified version of a real soliton solution in the regime $0 < v < A < B$ with asymptotic behaviour $\lim_{\zeta \to \pm \infty} v(\zeta) = A$. In the $u$-plane the real solution becomes a cusp solution running off to infinity, whereas the complex solutions close. We may also identify complexified versions of the kink solutions in the regime $0 < A < v < B$, such as for instance the trajectory with $\text{Im} \zeta_0 = 5$ in the $v$-plane with asymptotic behaviour $\lim_{\zeta \to \pm \infty} v(\zeta) = A$ and $\lim_{\zeta \to \pm \infty} u(\zeta) = B$. In the $u$-plane the qualitative behaviour remains the same with asymptotic behaviour $\lim_{\zeta \to \pm \infty} u(\zeta) = 0$ and $\lim_{\zeta \to \pm \infty} u(\zeta) = -c/\alpha$. These features may also be derived analytically from (4.26). The reality of the energy is once more guaranteed to be the symmetry $v^*(\zeta) = v(-\zeta)$ and $u^*(\zeta) = u(-\zeta)$.

Similarly as for the trigonometric solution of the KdV equation, by changing the parameters we can predict some periodic solutions. In the $PT$-symmetric setting this is achieved for $A, B \in \mathbb{R}^+$ when taking $\lambda \in \mathbb{R}^-$, as in this case the two eigenvalues become purely complex and therefore the fixed point becomes a centre. For this type of solution this can be achieved either for the point $v = A$ or $v = B$. We depict an example of such solutions in figure 29 with eigenvalues $j_{1/2} = \pm 1/\sqrt{6}$. In the $v$-plane, we observe that the complex trajectories encircle the points $A$ or $B$, whereas in the $u$-plane the trajectories surround the points $-c/\alpha$ and $0$ for $\text{Im} \zeta_0 > 0$ or $\text{Im} \zeta_0 < 0$, respectively.
Again we observe the symmetry relations $v^*(\zeta) = v(-\zeta)$ and $u^*(\zeta) = u(-\zeta)$ ensuring the reality of the energy, but as for the trigonometric solution of the KdV equation we may compute the energy in this case explicitly. Taking the trajectories surrounding $A$ in the $v$-plane to be the contour $\Gamma$ we compute

\begin{equation}
E_{Ta} = \oint_{\Gamma} \frac{\mathcal{H}[v(\zeta)]}{v_\zeta} \, dv = \oint_{\Gamma} \frac{\mathcal{H}[v]}{\sqrt{A\sqrt{v}(v - A)(v - B)}} \, dv
\end{equation}

\begin{equation}
= -\pi \sqrt{-\gamma \kappa_2} \left[ cA^2 + \kappa_2 A + \frac{B}{3} \left( \frac{c}{\alpha} + \frac{\kappa_2}{A} \right)^3 \right].
\end{equation}
For the trajectories surrounding $B$ in the $v$-plane we obtain in a similar way $E_{T_B} = E_{T_A}$ ($A \leftrightarrow B$).

Considering the amount of free parameters we have at our disposal, the expression for (4.28) also suggests that we will be able to generate any type of fixed point even for the broken $\mathcal{PT}$-symmetric scenario. Most unexpected is probably that we may even generate periodic orbits in that case. For this to happen we require $3\theta_A + \theta_\lambda = \pi$ and $\theta_A + 2\theta_B + \theta_\lambda = \pi$ to hold. A solution to these equations, leading to the eigenvalues $j_{1/2} = \pm i 2^{1/2}$, is presented in figure 30.

As already seen for the complex KdV system in section 2.1.2, we can also break the $\mathcal{PT}$-symmetry in a more controlled way and restore the reality of the energy given by (4.30). Making for instance the parameter choice

$$\alpha = r_u e^{-i \frac{\pi}{4} \arctan \left( \frac{7}{4} \sqrt{2} \right)}, \quad \beta = 3 r_u e^{i \frac{\pi}{4}}, \quad \gamma = \mu \frac{\beta - \alpha}{\alpha}, \quad c = -1, \quad A = \frac{c (\beta - \alpha)}{2 \sqrt{3 \alpha^3 / 2 \sqrt{-\beta}}}, \quad B = 2 A,$$

with $\mu \in \mathbb{R}^+$ and $r_u \in \mathbb{R}$ being unconstrained constants, we obtain for the energy of a periodic trajectory around the point $A$ the real expression

$$E_{T_A} = -\frac{\pi}{3 r_u^2} \sqrt{\frac{5 \mu}{3}}.$$  \hspace{1cm} (4.33)

Note that in this case we have two free parameters available allowing us to tune the real energy together with the model, despite the fact that any of these Hamiltonians is neither Hermitian nor $\mathcal{PT}$-symmetric. We depict an example for such type of trajectory in figure 31. We note that the trajectories are qualitatively the same as those with complex energies depicted for instance in figure 30.

---

**Figure 30.** Complex periodic trajectories for a type II broken $\mathcal{PT}$-symmetric Ito-type system for purely complex initial values $z_0$ with $\alpha = 1/(2 \sqrt{3})$, $\beta = (1 - 2 i \sqrt{1 - i})$, $\gamma = \frac{1}{4} (-1 + \sqrt{-1 - i})$, $c = -1$, $A = (1 + i) \sqrt{2}$, and $B = (2 + 2 i) \sqrt{2}$. (a) $v$-field; (b) $u$-field.
4.1.3. Type III solutions. Next we assume that \( R(v) = v(v - A)^2(B + v)(C + v) \), which is achievable upon the constraints

\[
\begin{align*}
\kappa_1 &= \frac{(\beta - 2\alpha)c^2 + [A^2 + 2A(B + C) + BC]\alpha^3}{2\alpha^2}, \\
\kappa_2 &= \frac{A[2BC + A(B + C)]\alpha^3}{c(\beta - \alpha)}, \\
\kappa_3 &= \frac{[c^2 - [A^2 + 2(B + C)A + BC]\alpha^2]c}{2\alpha^2} - \frac{A(2A + B + C)[2BC + A(B + C)]\alpha^3}{2(\beta - \alpha)c} - \frac{\beta e^3}{3\alpha^3}, \\
A &= \frac{-\sqrt{3}\sqrt{-\beta\alpha^3(\beta - B)(\beta - C)}\sqrt{\lambda}c}{\beta\alpha^3(B + C)^2}.
\end{align*}
\] (4.34)

Now we have two free parameters at our disposal. In this case, the solution of (4.15) leads to

\[
\zeta - \zeta_0 = \pm \frac{2}{A\sqrt{\beta B(\alpha - B)} \sqrt{\lambda} v} \left\{ F \left[ \arcsin \left( \frac{\sqrt{B}}{\sqrt{v}} \right) \frac{c}{B} \right] + \Pi \left[ \frac{\alpha A}{B}; \arcsin \left( \frac{\sqrt{B}}{\sqrt{v}} \right) \frac{c}{B} \right] \right\},
\] (4.35)

with \( F[\phi|m] \) denoting an elliptic function of the first and \( \Pi[n; \phi|m] \) an incomplete elliptic function.

Choosing \( B \) and \( C \) conveniently we compute some trajectories similarly as in the previous subsection and depict our results in figure 32.

We identify some trajectories surrounding the point \( A \) in the \( v \)-plane for which we compute the energy as

\[
E_{R_\alpha} = \int_\Gamma \mathcal{H}\{v(\zeta)\} \frac{dv}{\sqrt{\lambda}} = \frac{\mathcal{H}\{v\}}{\sqrt{\lambda \sqrt{(v - B)(v - C)v(v - A)}}} \frac{dv}{\sqrt{\lambda}}
\] (4.36)

\[
= \pi \frac{\sqrt{-\gamma \kappa_2}}{\alpha \sqrt{(A - B)(A - C)} A} \left[ cA^2 + \kappa_2 A + \frac{\beta}{3} \left( \frac{c}{\alpha A} + \frac{\kappa_2}{\alpha A} \right)^3 \right].
\] (4.37)
Figure 32. Complex periodic trajectories for the type III $\mathcal{PT}$-symmetric Ito-type system for purely complex initial values $\zeta_0$ with $\alpha = -\beta = 2\sqrt{6}/19$, $\gamma = -1$, $c = -1$, $A = 1$, $B = 4$ and $C = 8$. 

As seen for the type II solutions, in this case we may also break the $\mathcal{PT}$-symmetry and still obtain periodic solutions. Moreover, we can break the symmetry further, that is also on the level of the Hamiltonian, and still render the expression for $E_{\mathcal{T}_a}$ real. Since in this case we even have an additional parameter $C$ at our disposal, it is conceivable that we might even have a free variable left in the expression for the real energy. We leave this question for a future investigation.

An example for broken $\mathcal{PT}$-symmetry is depicted in figure 33.
5. $\mathcal{PT}$-symmetric deformations of the Ito-type equations

Since the Hamiltonian $\mathcal{H}_I$ admits the four different types of antilinear symmetries (4.5)–(4.8), we have now the options to apply the deformation maps $\delta^x_\epsilon$ and $\delta^-_\mu$ introduced in (3.1) to the fields $u$ and $v$ in different combinations. Accordingly, we define four different $\mathcal{PT}$-symmetric models with two deformation parameters $\epsilon$ and $\mu$ suitable normalized by the following Hamiltonian densities:

$$\mathcal{H}^+_{\epsilon,\mu} = -\frac{\alpha}{2} uv^2 - \frac{\beta}{6} u^3 - \frac{\gamma}{1 + \epsilon} (iu_x)^{\epsilon+1} - \frac{\phi}{1 + \mu} (iv_x)^{\mu+1},$$

$$\mathcal{H}^-_{\epsilon,\mu} = \frac{\alpha}{1 + \mu} u (iv)^{\mu+1} - \frac{\beta}{6} u^3 - \frac{\gamma}{1 + \epsilon} (iu_x)^{\epsilon+1} + \frac{\phi}{2} v^2,$$

$$\mathcal{H}^+_{\epsilon,\mu} = -\frac{\alpha}{2} u v^2 - \frac{i \beta}{(1 + \epsilon)(2 + \epsilon)} (iu)^{2\epsilon} + \frac{\gamma}{2} u^2 - \frac{\phi}{1 + \mu} (iv_x)^{\mu+1},$$

$$\mathcal{H}^-_{\epsilon,\mu} = \frac{\alpha}{1 + \mu} u (iv)^{\mu+1} - \frac{i \beta}{(1 + \epsilon)(2 + \epsilon)} (iu)^{2\epsilon} + \frac{\gamma}{2} u^2 + \frac{\phi}{2} v^2.$$

By construction we have $\mathcal{PT}_{\epsilon,\mu} : \mathcal{H}^l_{\epsilon,\mu} \mapsto \mathcal{H}^l_{\epsilon,\mu}$ with $i, j \in \{+, -\}$ and $\lim_{\epsilon,\mu \to 0} \mathcal{H}^l_{\epsilon,\mu} = \mathcal{H}_I$. The corresponding equations of motion resulting from (4.4) are

$$\mathcal{H}^+_{\epsilon,\mu} : u_t + \alpha (u v)_x + \beta u u_x + \gamma u_{xxx,x} = 0, \quad v_t + \alpha (u v)_x + \beta u u_x + \gamma v_{xxx,x} = 0,$$

$$\mathcal{H}^-_{\epsilon,\mu} : u_t + \alpha (u v)_x + \beta u u_x + \gamma u_{xxx,x} = 0, \quad v_t + \alpha (u v)_x + \beta u u_x + \gamma v_{xxx,x} = 0,$$

$$\mathcal{H}^+_{\epsilon,\mu} : u_t + \alpha (u v)_x + \beta u u_x + \gamma u_{xxx,x} = 0, \quad v_t + \alpha (u v)_x + \beta u u_x + \gamma v_{xxx,x} = 0,$$

$$\mathcal{H}^-_{\epsilon,\mu} : u_t + \alpha (u v)_x + \beta u u_x + \gamma u_{xxx,x} = 0, \quad v_t + \alpha (u v)_x + \beta u u_x + \gamma v_{xxx,x} = 0.$$

Naturally, there exist also possibilities of constructing non-Hamiltonian deformations. Noting for instance that the first equation related to $\mathcal{H}^+_{\epsilon,\mu}$ is also invariant under $\mathcal{PT}_{\epsilon,\mu}$ we may combine it with the second equation resulting from $\mathcal{H}^-_{\epsilon,\mu}$ and define the $\mathcal{PT}_{\epsilon,\mu}$-invariant system

$$u_t + \alpha (u v)_x + \beta u u_x + \gamma u_{xxx,x} = 0,$$

$$v_t + \alpha (u v)_x + \beta u u_x + \gamma v_{xxx,x} = 0.$$

We have now numerous new physically feasible theories to be investigated. Here, we will only present a few examples. Having two parameters at our disposal allows us to obtain more analytic expressions for the solutions. Technically we have two problems to overcome. First of all, we have to decouple the equations and subsequently carry out all the integrations. In order to achieve the first aim we focus mainly on the case $\phi = 0$ as this will allow us to express $u$ in terms of $v$ in a simple manner. This choice will eliminate the deformation term in $\mathcal{H}^+_{\epsilon,\mu}$ and we therefore concentrate here on $\mathcal{H}^+_{\epsilon,\mu}$ to study the interplay between the two parameters.

5.1. The model $\mathcal{H}^+_{\epsilon,\mu}(\alpha, \beta, \gamma, u, v)$

As a further simplification we set the constant $\kappa_2$ to zero, which results in the first integration of the equation for $v_t$. We may then integrate again and obtain a decoupled equation solely involving the field $u$:

$$u^{\epsilon+1} = \left[1 - \frac{\epsilon + 1}{\gamma} \left[\kappa_3 + \kappa_1 u + \frac{c}{2} u^2 - \frac{\beta}{6} u^3 + \frac{\alpha}{1 + \mu} \left(\frac{\alpha}{c} \right)^{\frac{1}{\gamma}} \frac{1}{2} u^{\gamma+1}\right]\right],$$

$$u^{\epsilon+1} = \left[1 - \frac{\epsilon + 1}{\gamma} \left[\kappa_3 + \kappa_1 u + \frac{c}{2} u^2 - \frac{\beta}{6} u^3 + \frac{\alpha}{1 + \mu} \left(\frac{\alpha}{c} \right)^{\frac{1}{\gamma}} \frac{1}{2} u^{\gamma+1}\right]\right].$$
with the two fields related as

$$u = \frac{c}{\alpha} (iv)^{1-\mu}.$$  \hfill (5.9)

To be able to carry out the final integration in (5.8) we would like the right-hand side to acquire the form of a factorizable polynomial in $u$. This is possible for several specific choices and with the additional choice of $\varepsilon$ we may even construct analytic solutions.

$\mathcal{H}^+_\varepsilon, 1/3(\alpha, \beta, \gamma, u, v)$. This case is entirely reducible to the deformation of the KdV $\mathcal{H}^+ \beta, \gamma, u)$ when noting that

$$\mathcal{H}^+ \beta, \gamma, u)(\varepsilon, u,\gamma-1, u) = \mathcal{H}^+_{\varepsilon, 1/3} \left[ \alpha, \beta, \gamma, u, \left( \frac{\alpha}{\beta} u \right)^{1/2} \right],$$  \hfill (5.10)

where we used the relation between $u$ and $v$ as specified in (5.9) for $\mu = 1/3$.

$\mathcal{H}^+_{\varepsilon, 1/2, 1/2}$. The choice $\mu = 1/2$ converts the right-hand side of (5.8) into a fourth-order polynomial. Let us next assume that the factorization is of the form $\lambda(u-A)^2(u-B)^2$, which is indeed possible if the following constraints hold:

$$\lambda = \frac{e^{i\pi/3} a^4}{6c^3}, \quad \kappa_1 = \frac{c^6 \beta^3 - 12c^3 \alpha^4 \beta}{48a^8}, \quad \kappa_3 = \frac{c^5(c^2 \beta^2 - 12c^4 \alpha^2)^2}{384a^{12}}.$$  \hfill (5.11)

$$A = \frac{c^3 \beta^3 + \sqrt{3} \sqrt{c^6 \beta^3 - 8c^4 \beta^2}}{4a^8}, \quad B = \frac{c^3 \beta^3 + \sqrt{3} \sqrt{c^6 \beta^3 - 8c^4 \beta^2}}{4a^8}.$$  \hfill (5.12)

Note that all the free parameters are fixed in this case, as a consequence of having already pre-selected $\kappa_2 = 0$. We analyse this model in figure 34.

We observe that the $PT_-$-symmetry manifests itself now through $v^+(\zeta) = -v(-\zeta)$ and $u^+(\zeta) = u(-\zeta)$. Furthermore, we recognize that the asymptotic limits of the $u$-field are $A$ and $B$.

Of course there are plenty more models one may explore.
5.2. The non-Hamiltonian deformation (5.7)

Let us briefly comment on one of the possibilities of constructing deformations of non-Hamiltonian systems (5.7). We consider again the case $\phi = 0$ and $\kappa_2 = 0$ and integrate the first equation in (5.7) for $\varepsilon = 1$ and $\mu \neq 3$:

$$u_\xi = 2 \frac{\mu - 1}{\gamma} \left[ \kappa_3 + \kappa_1 u + \frac{\epsilon}{2} u^2 - \frac{\beta}{6} u^3 + \frac{\alpha}{2} \left( \frac{C}{\alpha} \right)^{\gamma} \frac{1-\mu}{3-\mu} u \right].$$

(5.13)

The $v$-field is related to the $u$-field via (5.9). The boundary conditions have to be treated by distinguishing different cases. For instance, when $(3-\mu)/(1-\mu) \geq 0$ the vanishing asymptotic boundary conditions for $u$ and its derivatives demand that $\kappa_3 = \kappa_2 = 0$.

The simplest case to consider is $\mu = -1$, which is just the KdV system with a re-defined speed of the wave $c \rightarrow c + \alpha^2/c$.

A further simple example is to take $\mu = 1/3$ for which we can find an explicit solution $u(\xi)$. The right-hand side of (5.13) can be factorized into $\lambda(u - A)^2(u - B)^2$ with

$$\lambda = \frac{c^3}{4\epsilon^2}, \quad \kappa_1 = \frac{a^2 \beta^3 - 9c^4 \alpha^2 \beta}{27c^6}, \quad \kappa_3 = \frac{\alpha^2 \beta^2 - 9c^4 \alpha^2}{162c^6}, \quad \alpha = \frac{3^3 \epsilon c^6}{\beta \alpha^4 \beta^2 - 6c^10 \alpha^2}, \quad B = \frac{3^3 \epsilon c^6}{\beta \alpha^4 \beta^2 - 6c^10 \alpha^2}.$$

(5.14)

(5.15)

for which (5.13) can be integrated further and solved for $u$. We find

$$u(\xi) = \frac{B e^{\pm A(\xi - \xi_0)\sqrt{\gamma}} - A e^{\pm B(\xi - \xi_0)\sqrt{\gamma}}} {e^{\pm A(\xi - \xi_0)\sqrt{\gamma}} - e^{\pm B(\xi - \xi_0)\sqrt{\gamma}}} \quad \text{and} \quad v(\xi) = -i \left( \frac{\alpha u(\xi)}{c} \right)^{3/2}.$$

(5.16)

The system is easily linearized and qualitatively we find a similar behaviour as for the Hamiltonian systems, which we will, however, not present here in more detail. Thus, even though it is less clear how the $PT$-symmetry can be utilized, we find that they are not fundamentally different from the Hamiltonian systems.

6. Conclusions

The main focus of this paper was to investigate the effects of $PT$-symmetry and its breaking in complex nonlinear wave equations. In general, we find that unlike as claimed for orbits in quantum mechanical one-particle models [10–13], trajectories in the complex plane of the field of a $PT$-symmetric nonlinear system are not fundamentally distinct from those associated with systems spontaneously or completely broken $PT$-symmetry. Just from the general type of trajectory one cannot conclude which type of setting one is considering. Of course, invoking the information on the symmetry one can identify on this basis the $PT$-symmetric case over the broken ones. However, based on this criterium the spontaneously broken cases are indistinguishable from the completely broken ones. This behaviour extends to the type of fixed points one may find. We observed that essentially all types of fixed points, except saddle points, may occur, irrespective of the $PT$-symmetry properties of the model. When identifying and varying certain constants of the model as bifurcation parameter the fixed point may undergo a Hopf bifurcation.

It appears that the entire phase space is filled out in the broken case, thus suggesting a chaotic behaviour. However, from the Poincaré–Bendixson theorem we know that for a closed bounded and connected region in two dimensions this is impossible to occur.

3 Poincaré–Bendixson theorem: let $\phi_t$ be a flow for a two-dimensional dynamical system and let $D$ be a closed, bounded and connected set $D \in \mathbb{R}^2$, such that $\phi_t(D) \subset D$ for all time. Furthermore, $D$ does not contain any fixed point. Then, there exists at least one limit cycle in $D$. 

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The nature of the fixed points also indicates that the models are not Hamiltonian in the real and imaginary parts of $u$, as only saddle points and centres would emerge in that case, whereas we found different types of fixed points and the absence of saddle points.

With regard to the energies we confirmed that fully $\mathcal{PT}$-symmetric systems have real energies, which could be calculated explicitly in many cases. When breaking this symmetry spontaneously by complexifying some free parameters in the solutions we obtained complex energies. As expected, the models related to the complex conjugates of these parameters have complex conjugate energies. More surprising are the findings obtained in various models that one may regain the reality of the energy by breaking the symmetry further. For the type III solutions of the Ito-type systems it is conceivable that one might have free parameters in the expressions for the energy even when fixing the model. We conjecture that these models possess a different kind of antilinear symmetry yet to be identified.

Besides guaranteeing the reality of the energy, $\mathcal{PT}$-symmetry was noted to be useful for various other reasons. We found that the symmetry allows for a natural '$\varepsilon$-prescription' to facilitate the computation of the energies in the complex plane by means of the Cauchy theorem. Identifying the different types of $\mathcal{PT}$-symmetries also allows us to formulate new physically feasible models with real energies. This procedure constitutes a natural and more general framework for some models which have already been known before, such as the generalized KdV equations with the modified one as a special case.

Since many of the models discussed here are integrable, it is worth mentioning that most of the analysis carried out here for the Hamiltonian may also be performed for other conserved quantities of the same order in the fields having the same type of symmetry property. For some of the charges this is in fact not the case. Although in many deformed cases the question of integrability has not yet been answered decisively, the statement also holds for the lowest charges which are certain to exist.

The main difference between the deformed models and their undeformed counterparts is in general the occurrence of more and more Riemann sheets with increasing integer value for $\varepsilon$. We showed that in many cases non-integer values and even negative values for $\varepsilon$ give rise to interesting solvable models. The overall structure of the trajectories and the nature of the fixed points is not fundamentally different, except that they usual extend over several Riemann sheets. Clearly, we only presented here a limited number of solutions and many cases still need to be explored, especially for the second type of deformation for which even the factorization of the $P(u)$-function remains an open issue.

When comparing the Ito-type system with the KdV system we note that the former is not simply an add on to the latter, but gives rise to more complex structures. We found even for the undeformed case some new solutions such as those of kink-type hitherto not reported. We also pointed out some minor discrepancies when compared to the literature. With regard to the deformations, which are all entirely new proposals, the conclusions are similar as for the deformed KdV case. The major difference is that the interplay between the two deformation parameters allows for even more possibilities. Also in this case many possibilities remain unexplored.

As indicated at the end of sections 2.1.3 and 3.1.3 we can obtain simple quantum mechanical systems as special cases from our analysis. For instance, the archetype deformation of the harmonic oscillator Hamiltonian $H = p^2 + x^2(i\varepsilon)^\varepsilon$ can be obtained in various ways via the identification for the travelling wave $u \rightarrow x$ and $\zeta \rightarrow t$. As discussed, the deformed models $\mathcal{H}_\varepsilon$ yield precisely the quantum mechanical model with potential $V(x) = x^2(i\varepsilon)^\varepsilon$, of which some cases were studied in [43]. From the models presented here there are more possibilities of arriving at such potential, as for instance also the non-Hamiltonian models (5.7) give rise to such type of potentials as may be seen from (5.13). Exploiting these observations
allows us to obtain many solutions and properties of these systems easily from our analysis overlooked up to now. Obviously, one may also construct new interesting quantum mechanical models in this way which have not been investigated so far.

In addition, this opens up immediately the more general question of studying properties of these systems as continuous functions of $\varepsilon$, rather than selecting just certain specific values as in this paper. Building for instance on the analogy with the potential systems should certainly reveal fundamentally different kinds of behaviour in some regions, such that for instance $\mathcal{H}_\varepsilon$ will probably have a qualitatively different behaviour for negative values of $\varepsilon$. We leave these types of questions for future investigations.

Clearly, it would be very interesting to extend these types of analyses to other nonlinear field equations such as Burgers, Bussinesque, KP, generalized shallow water equations, extended KdV equations with compacton solutions, etc.

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Appendix. The ten similarity classes for $J$

For convenience, we recall here the ten different similarity classes characterizing the fixed points of a two-dimensional linear system (2.9) by the eigenvalues $j_1$ and $j_2$ of the Jacobian matrix $J$ at the fixed point.

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