COMBINATORIAL QUANTUM FIELD THEORY AND GLUING FORMULA FOR DETERMINANTS

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Abstract. We define the combinatorial Dirichlet-to-Neumann operator and establish a gluing formula for determinants of discrete Laplacians using a combinatorial Gaussian quantum field theory. In case of a diagonal inner product on cochains we provide an explicit local expression for the discrete Dirichlet-to-Neumann operator. We relate the gluing formula to the corresponding Mayer-Vietoris formula by Burghelea, Friedlander and Kappeler for zeta-determinants of analytic Laplacians, using the approximation theory of Dodziuk. Our argument motivates existence of gluing formulas as a consequence of a gluing principle on the discrete level.

1. Introduction

1.1. Gluing formula for zeta-determinants. Investigation of the cut and paste behaviour of zeta-regularized determinants has been initiated by Forman [For92] and Burghelea-Friedlander-Kappeler in [BFK92]. Their studies have triggered further analysis by various authors including Park-Wojciechowski [PaWo05] and Lee [Lee03].

The approach of [For92] and [BFK92] to the gluing behavior of zeta-regularized determinants is purely analytic. In the present paper we use ideas from quantum field theoretic arguments to establish a gluing formula for determinants of combinatorial Laplacians and relate it to the gluing formulas for zeta-determinants of analytic Laplacians using the approximation theory of Dodziuk [Dod76], and its extension to manifolds with boundary by Müller [Müller78].

Our arguments naturally motivate existence of gluing formulas for the non-local spectral invariants such as zeta-regularized determinants as a consequence of the corresponding gluing principle in the discrete case.

Two main results are gluing formula for combinatorial Laplacians, and the continuum limit for the ratio of determinants of Laplace operators. An important part of the paper is the formulation of Gaussian quantum field theory for Whitney product on cochains.

1.2. Gluing formula for combinatorial determinants. Consider an $n$-dimensional cell complex $K$, possibly with an $(n-1)$-dimensional boundary $L$. Assume that the boundary subcomplex $L \subset K$ has three connected components $L_1, L_2$ and $L_3$. We write

$$L = L_1 \sqcup L_2 \sqcup L_3.$$

Assume that $K$ is equipped with a Riemannian structure (see §3) and that $L_1$ and $L_2$ are isomorphic via an isometry $f$ of cell complexes with Riemannian structure. Gluing the boundary components $L_1$ and $L_2$ via the isometric identification $f$ defines a new Riemannian cell complex $K_f$ with a single boundary subcomplex $L_3$. 

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Consider a combinatorial Hermitian vector bundle $E$ over $K$ with a flat connection $\alpha$, which gives rise to combinatorial coboundary operators on the cochains. $E$ pulls back to a vector bundle over $K$. This defines a scalar product on cochains with values in the corresponding vector bundle and gives rise to the combinatorial Laplacians $\Delta_K$ and $\Delta_{\tilde{\mathcal{M}}}$ on cochains of degree zero with Dirichlet boundary conditions at the corresponding boundary complexes.

Let $Q_K$ be the linear operator connecting the scalar products on $C^0(K, E)$, corresponding to the given Riemannian structure on $K$ and the diagonal (in the cell basis) Euclidean inner product. More explicitly, let $\langle \cdot, \cdot \rangle_0$ denote the Euclidean inner product on cochains. Then, for any $\phi, \psi \in C^0(K, E)$ we have $\langle \phi, \psi \rangle_K = \langle Q_K \phi, \psi \rangle_0$. We write $\Delta_K^c := Q_K \iota L \circ \Delta_K$ and $\Delta_{\tilde{\mathcal{M}}}^c := Q_{\tilde{\mathcal{M}}} \iota L_3 \circ \Delta_\tilde{\mathcal{M}}$ for the local Laplacians with respect to the Euclidean inner product on cochains with Dirichlet boundary conditions. In §6 we prove the following gluing formula for determinants of combinatorial Laplacians.

**Theorem 1.1.** Assume the Riemannian structure on $K$ is local. Then the determinants of combinatorial Laplacians satisfy the following identity

$$\frac{\det'(\Delta_{\Delta_K}^c)}{\det'(\Delta_{\Delta_{\tilde{\mathcal{M}}}^c}^c)} = \det' \mathcal{R}^c(K, \tilde{\mathcal{M}}),$$

where $\det'$ is the product of non-zero eigenvalues and $\mathcal{R}^c(K, \tilde{\mathcal{M}})$ is the composition of $Q_{L_2}$ with the combinatorial analog of the Dirichlet-to-Neumann map defined in (6.5).

The operators $Q_K$ satisfy their own intricate gluing law which we make explicit at the end of §6.

**1.3. Relating both gluing formulae in the discretization limit.** In §7 we establish the relationship between this gluing formula and the gluing formula for the analytic zeta-regularized determinants.

Let $(M, g)$ be a compact Riemannian manifold with boundary $\partial M$ that consists of three disjoint boundary components $N_1, N_2$ and $N_3$. Consider a flat Hermitian vector bundle $(E, h, \alpha)$, i.e. Hermitian bundle with a Hermitian flat connection $\alpha$. Flatness implies product structure over a collar neighborhood of $\partial M$. We denote by $\Delta_M$ the Laplace Beltrami operator acting of functions on $M$ with values in $E$ and with Dirichlet boundary conditions at the boundary.

Assume that $g$ is product near $N_1$ and $N_2$ and define a smooth Riemannian manifold $\tilde{M}$ by gluing a second copy of $M$ along $N_1 \sqcup N_2$. Note that $\partial \tilde{M}$ consists of two copies of $N_3$.

Let $N_1$ and $N_2$ be identified by an isometry $f$, and denote by $M_f$ the Riemannian manifold obtained from $M$ by gluing $N_1$ onto $N_2$. The flat Hermitian vector bundle $(E, h, \alpha)$ induces smooth flat vector bundles over $M_f$ and $\tilde{M}$. We write $\Delta_{\partial \tilde{M}}$ and $\Delta_M$ for the twisted Laplacians on $\tilde{M}$ and $M_f$, respectively, with Dirichlet boundary conditions at the respective boundaries.

Consider, as in §1.2, a simplicial complex $K$ which triangulates $M$ with subcomplexes $L_1, L_2$ and $L_3$ triangulating $N_1, N_2$ and $N_3$, respectively. Its double $\tilde{K}$ along $L_1 \sqcup L_2$, with the boundary subcomplex $L_3 \sqcup L_3$ is a simplicial decomposition of $\tilde{M}$. The simplicial complex $K_f$, obtained by gluing $K$ along the two identified boundary components $L_1$ and $L_2$, decomposes $M_f$. 

The pullbacks of the combinatorial analog of $E$ over $K$ define combinatorial vector bundles over $K_f$ and $\tilde{K}$. The combinatorial Riemannian structure on $K$ defines natural combinatorial Riemannian structures on $K_f$ and $\tilde{K}$. The metric structure on $M$ and the Whitney map an define a combinatorial Riemannian structure on $K$ together with the inner product on corresponding cochains with values in combinatorial vector bundle. Denote by $\Delta_{K_f}$ and $\Delta_{\tilde{K}}$ the combinatorial Laplace operators on cochains of degree zero, defined with respect to this inner product, with Dirichlet boundary conditions at the boundary. In §7 we prove the following

**Theorem 1.2.** As the mesh $\delta > 0$ of the triangulation $K$ goes to zero under standard subdivisions\(^1\)

$$
\lim_{\delta \to 0} \frac{\det' \Delta_{K_f}}{\det' \Delta_{\tilde{K}}} = \frac{\det_\zeta \Delta_{M_f}}{\det_\zeta \Delta_{\tilde{M}}}. \tag{1.1}
$$

By an application of Theorem 1.1, 1.2 and the gluing formula of Burghelea, Friedlander and Kappeler, which we recap in §2, we arrive at the following

**Corollary 1.3.** Let $\mathcal{R}_a(M_f, N_2)$ be the analytic Dirichlet-to-Neumann map and $\mathcal{C}_{M_f, N_2}$ the corresponding constants\(^2\) in the analytic Mayer-Vietoris formula. Then, as the mesh $\delta > 0$ of the triangulation $K$ goes to zero under standard subdivisions

$$
\lim_{\delta \to 0} \frac{(\det' \mathcal{R}_c(K_f, L_2))^2}{\det' \mathcal{R}_c(K_f, L_1 \cup L_2)} = \frac{\mathcal{C}_{M_f, N_2}^2}{\mathcal{C}_{M, N_1 \sqcup N_2}^2} \cdot \frac{(\det_\zeta \mathcal{R}_a(M_f, N_2))^2}{\det_\zeta \mathcal{R}_a(M, N_1 \sqcup N_2)}. \tag{1.2}
$$

The presented statements hold without the assumption of orientability for $K$ and $M$. Still, orientability is necessary for the discussion of combinatorial covariant derivatives in §3 as well as in the general setup of the quantum field theoretic framework in §6.1.2.

The paper is structured as follows. We first provide in §2 an overview over the fundamental elements of spectral geometry and the Mayer-Vietoris formula for zeta-determinants by Burghelea, Friedlander and Kappeler in the special case of scalar Laplace-Beltrami operators. With continue in §3 with a detailed construction of the combinatorial vector bundles, connections and discrete Laplacians, for the moment independent from the possibly underlying Riemannian geometry. In §4 we discuss the approximation theory of Dodziuk. §5 introduces the classical scalar free theory. Its critical point defines a Poisson and subsequently the Dirichlet-to-Neumann operators. We define the combinatorial Gaussian quantum field theory and prove the combinatorial gluing formula in §6. The last §7 establishes a link between the determinant gluing identities for discrete Laplacians and the Mayer-Vietoris formula by Burghelea, Friedlander and Kappeler. We conclude with an outlook of various research directions which are interesting in relation to the present analysis.

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\(^1\)Standard subdivisions were introduced by Whitney in [Whi57].

\(^2\)This constant has been explicitly identified in [Lee03].
the University of Amsterdam where important part of the work was done. The work was completed when both authors attended the thematic program "Modern Trends in TQFT" at the Erwin Schrödinger Institute in Vienna.

**Contents**

1. Introduction 1  
2. Analytic Laplacians and MV formula for their $\zeta$-regularized determinants 4  
3. Combinatorial vector bundles and Laplacians 8  
4. The approximation theory 13  
5. Classical free scalar field 15  
6. Discrete Quantum Gaussian field theory 18  
7. From identities for discrete Laplacians to BFK-identities 22  
8. Conclusion 24  
References 25

**2. Analytic Laplacians and MV formula for their $\zeta$-regularized determinants**

Here we will recall basic facts about differential forms on Riemannian manifolds, Laplace operators, and Mayer-Vietoris type formulae for $\zeta$-regularized determinants of Laplace operators. We assume orientability in some subsections for simplicity. For non-orientable spaces the arguments carry over after twisting with the orientation density bundle.

**2.1. Differential forms on Riemannian manifolds.**

**2.1.1. The twisted de Rham complex.** Let $M$ be a compact smooth manifold with boundary $\partial M$. Let $E$ be a Hermitian vector bundle over $M$ with finite dimensional fibers, with Hermitian metric $h$ and a flat connection $A$ on it. Define $\Omega^\bullet(M, E)$ as the space of $E$-valued differential forms and $\Omega^\bullet_c(M, E)$ as the subspace of smooth differential forms with compact support away from $\partial M$. Let $(\Omega^\bullet_c(M, E), d)$ be the twisted de Rham complex, where $d$ is the twisted (by the flat connection $A$) de Rham differential

$$d_q : \Omega^q_c(M, E) \to \Omega^{q+1}_c(M, E).$$

Locally, over a neighborhood $U$, the connection $A$ is a 1-form with coefficients in $End(V)$ and the twisted de Rham differential acts on forms as

$$d\omega = d_{dR}\omega + A \wedge \omega,$$

where $d_{dR}$ is the de Rham differential acting of $\Omega(U, V)$. The flatness of $A$ means $d^2 = 0$. Locally, this is equivalent to $d_{dR}A + A \wedge A = 0$. Each summand in the right hand side of this formula, in general, is defined only locally. The twisted differential however, is defined globally and this is why we need a flat connection on $E$ if we want to have globally defined differential on $\Omega^\bullet(M, E)$.

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Recall that, for a Hermitian vector bundle with fiber $V$, locally, a connection is an $End(V)$-valued 1-form. The structural group for such bundle is $G = U(V)$, the group of all unitary transformations of $V$. Similarly one can consider real Euclidean vector bundles with the structure group $O(V)$, and more generally, Hermitian $G$-bundles where $G$ is a compact real Lie group.
2.1.2. The scalar product. Now assume \((M, g)\) is an oriented Riemannian manifold of dimension \(n\). Let \(* : \Omega^q(M, E) \to \Omega^{n-q}(M, E)\) be the Hodge star operator, which yields a natural scalar product on the space \(\Omega^*(M, E)\)

\[
\langle \omega, \omega' \rangle_M = \int_M (\omega, \wedge * \omega'),
\]

where \((\cdot, \cdot)\) is the fiberwise Hermitian product.

With respect to the Riemannian metric on \(M\), each \(q\)-form \(\omega \in \Omega^*(M, E)\) has a natural decomposition into its normal and tangential components \(\omega = \omega_{\text{norm}} + \omega_{\text{tan}}\) at any point of the boundary \(\partial M\).

The Stokes formula applied to \(d(\omega \wedge * \omega')\) gives the integration by parts formula

\[
\langle d\omega, \omega' \rangle_M = \langle \omega, d^\prime \omega' \rangle_M + \langle \omega_{\text{tan}}', \omega_{\text{norm}} \rangle_{\partial M}. \tag{2.2}
\]

Here \(d^\prime\) is the formal adjoint of \(d\) with respect to the scalar product (2.1). In terms of the Hodge * operation \(d^\prime \omega = (-1)^{nq+n+1} * d * \omega\) when \(\omega \in \Omega^q(M, E)\).

2.1.3. Laplace operators. Here we recall the basic notions related to Laplace operators. Laplacians are defined by

\[
\Delta^q = d_q d_{q+1}^\prime + d_{q-1}^\prime d_q : \Omega^q(M, E) \to \Omega^q(M, E).
\]

We regard \(d_q\) and \(\Delta^q\) as unbounded operators in \(L^2 \Omega^*(M, E)\) with domain \(\Omega^*_c(M, E)\). Recall that the maximal extension \(d_{q,\text{max}}\) of \(d_q\) is a linear operator on \(L^2 \Omega^q(M, E)\) with domain

\[
\mathcal{D}(d_{q,\text{max}}) = \{ \omega \in L^2 \Omega^q(M, E) \mid d_q \omega \in L^2 \Omega^{q+1}(M, E) \},
\]

This is the space of forms \(\omega\) from \(L^2 \Omega^q(M, E)\) such that the differential \(d_q \omega\) is not just a distribution but actually a form in \(L^2 \Omega^{q+1}(M, E)\).

The minimal extension \(d_{q,\text{min}}\) of \(d_q\) with domain \(\mathcal{D}(d_{q,\text{min}}) \subset \mathcal{D}(d_{q,\text{max}})\) is the graph closure of \(d_q\) on \(\Omega^q_c(M, E)\). Ideal boundary conditions for the de Rham complex \((\Omega^*(M, E), d)\) is a choice of closed extensions \(D_q\) of \(d_q\) for each \(q = 0, \ldots, \dim M\) with

\[
\mathcal{D}(d_{q,\text{min}}) \subseteq \mathcal{D}(D_q) \subseteq \mathcal{D}(d_{q,\text{max}}),
\]

such that \(D_q : \mathcal{D}(D_q) \to \mathcal{D}(D_{q+1})\) and \(D_{q+1} \circ D_q = 0\). Such boundary conditions combine into a Hilbert complex in the sense of [BrLe92]. Ideal boundary conditions for the de Rham complex induce a self-adjoint extension \((D_q^* D_q + D_{q-1} D_{q-1}^*)\) for each \(\Delta^q\).

Two special cases of relative and absolute boundary conditions correspond to minimal and maximal extensions of \(d\)

\[
\begin{align*}
\Delta^q_{\text{rel}} &= d_{q,\text{min}}^* d_{q,\text{min}} + d_{q-1,\text{min}} d_{q-1,\text{min}}, \\
\Delta^q_{\text{abs}} &= d_{q,\text{max}}^* d_{q,\text{max}} + d_{q-1,\text{max}} d_{q-1,\text{max}}.
\end{align*} \tag{2.3}
\]

\footnote{If \(\iota : \partial M \to M\) is the natural inclusion of the boundary mapping, we have \(\omega_{\text{tan}} = \iota^* (\omega)\) and \(\omega_{\text{norm}} = *_{\partial} *_{\iota} (\omega)\) where * is the Hodge star operation on \(M\) and *_{\partial} is the Hodge star operation on \(\partial M\) corresponding to the metric induced from \(M\). Let \(\nu\) be a unit normal vector field on \(\partial M\) which is positive with respect to the orientation on \(M\). Then \(\omega_{\text{norm}} = \iota^* (\iota_\nu \omega)\) where \(\iota_\nu \omega\) is the contraction of \(\nu\) and \(\omega\).

Recall that the graph closure in our case is the closure with respect to the metric \(||\omega||^2 = \langle \omega, \omega \rangle + \langle d\omega, d\omega \rangle\).}
Explicitly, these boundary conditions are given as follows. The relative self-adjoint extension $\Delta^q_{\text{rel}}$ can be defined as the closure in $L^2\Omega^q(M, E)$ of the action of $\Delta^q$ on forms satisfying relative (or Dirichlet) boundary conditions $\omega_{\text{tan}} = 0$ and $(d_{q-1}\omega)_{\text{tan}} = 0$ at $\partial M$.

The absolute self-adjoint extension $\Delta^q_{\text{abs}}$ is given by the closure in $L^2\Omega^q(M, E)$ of the action of $\Delta^q$ on forms satisfying absolute (or Neumann) boundary conditions $\omega_{\text{norm}} = 0$ and $(d_q\omega)_{\text{norm}} = 0$ at $\partial M$.

In this paper we will focus on Laplace Beltrami operators with Dirichlet (relative) boundary conditions in degree $q = 0$.

### 2.2. Dirichlet-to-Neumann operator

Let $M$ be a Riemannian, smooth, oriented manifold with boundary $\partial M$. Denote by $\Delta_M$ the Hodge Laplace operator on $\Omega^0(M, E) = C^\infty(M, E)$. It is well known that for each $\eta \in C^\infty(\partial M, E)$ the Dirichlet boundary problem

$$\Delta_M \phi = 0, \quad \phi|_N = \eta. \quad (2.4)$$

has unique solution. Denote by $P_M : C^\infty(\partial M; E) \to C^\infty(M; E)$ the corresponding Poisson operator. In terms of $P_M$ the unique solution to the Dirichlet problem is $\phi = P_M \eta$.

The Dirichlet-to-Neumann operator $R^M_M$ may be defined implicitly as a linear map such that for any $\eta, \eta' \in C^\infty(\partial M, E)$

$$\langle dP_M \eta, dP_M \eta' \rangle = \langle R^M_M \eta, \eta' \rangle_{L^2(\partial M, E)}. \quad (2.5)$$

Explicitly, this leads by (2.2) to the following expression

$$R^M = \partial_\nu P_M : C^\infty(\partial M, E) \to C^\infty(\partial M, E),$$

where for any $\psi \in C^\infty(M, E)$ we write $\partial_\nu \psi := \iota^*(\iota_\nu d\psi) \in C^\infty(\partial M, E)$, with $\iota : \partial M \hookrightarrow M$ denoting the natural inclusion, and $\iota_\nu$ the contraction with the unit normal vector field on $\partial M$.

Recall that it is called the Dirichlet-to-Neumann map because the solution to the Neumann problem $\Delta_M \psi = 0$, $\partial_\nu \psi = \xi$ can be written as $\psi = P_M \eta$ where $\xi = R^M_M \eta$. In other words, $R^M_M$ maps Dirichlet boundary data to the Neumann one.

### 2.3. $\zeta$-regularized determinants

Fix Dirichlet boundary conditions for $\Delta_M$, and denote the resulting self-adjoint extension again by the same letter for the moment. It is known that $\Delta_M$ has non-negative pure point spectrum and that for every $t > 0$, $\exp(-t\Delta_M)$ is a trace class operator with an asymptotic expansion

$$\text{Tr} \left(e^{-t\Delta_M}\right) \sim \sum_{j=0}^{\infty} A_k(\sqrt{t})^{-\dim M+j}, \quad t \to 0^+. \quad (2.6)$$

Let $\{\lambda_j\}_{j \in \mathbb{N}}$ denote the eigenvalues of $\Delta_M$. It follows from (2.6) that Weyl’s law holds for the counting function of the eigenvalues. This implies that the zeta function

$$\zeta(s; \Delta_M) := \sum_{\lambda_j \neq 0} \lambda_j^{-s}$$
converges in the half-plane $\text{Re}(s) > \dim M/2$ and it can be expressed in terms of the trace of the heat operator by

$$
\zeta(s, \Delta_M) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-t\Delta} - P_q) \, dt, \quad \text{Re}(s) > \dim M/2, \tag{2.7}
$$

where $P_q$ denotes the orthogonal projection of $L^2(M, E)$ onto $\ker \Delta_M$. Then the asymptotic expansion (2.6) yields the meromorphic extension of the right hand side and hence, of the zeta function, to the whole complex plane $\mathbb{C}$. Furthermore it also follows from (2.6) that $\zeta(s, \Delta_M)$ is regular at $s = 0$. Hence we can define the $\zeta$-regularized determinant of $\Delta_M$ by the following expression

$$
\det_\zeta \Delta_M := \exp \left( - \frac{d}{ds} \bigg|_{s=0} \zeta(s, \Delta_M) \right).
$$

This discussion extends to other discrete operators with spectrum bounded from below, trace class heat operator and an asymptotic expansion of the form (2.6). In the finite dimensional case this defines exactly the determinant of a given self-adjoint linear operator.

### 2.4. Mayer-Vietoris type formula for $\zeta$-regularized determinants.

Let $(M, g)$ be a compact Riemannian manifold with boundary $\partial M$ that consists of three disjoint boundary components $N_1, N_2$ and $N_3$. Consider a flat Hermitian vector bundle $(E, h, \alpha)$, i.e. Hermitian bundle with a Hermitian flat connection $\alpha$. Flatness implies product structure over a collar neighborhood of $\partial M$. We denote by $\Delta_M$ the Laplace Beltrami operator acting of functions on $M$ with values in $E$ and with Dirichlet boundary conditions at the boundary.

Assume that $g$ is product near $N_1$ and $N_2$. Let $N_1$ and $N_2$ be identified by an isometry $f$, and denote by $M_f$ the Riemannian manifold obtained from $M$ by gluing $N_1$ onto $N_2$. The flat Hermitian vector bundle $(E, h, \alpha)$ induces a smooth flat vector bundle over $M_f$. We write $\Delta_{M_f}$ for the twisted Laplacian on $M_f$, with Dirichlet boundary conditions at $N_3$.

Let $P_M$ be the Poisson operator solving the Dirichlet problem on $M$. The Dirichlet-to-Neumann operator is this setting differs from (2.5) and is defined as

$$
\mathcal{R}_a(M_f, N_2) = (\partial_{\nu_1} + \partial_{\nu_2}) P_M.
$$

Here $\partial_{\nu_i}, i = 1, 2$, denotes the normal derivative at $N_i$ in the inward direction. The operators $\mathcal{R}_a(M_f, N_2)$ and $R^M$ are related by the following identity

$$
\langle \mathcal{R}_a(M_f, N_2) \eta, \eta \rangle_{L^2} = \langle R^M(\eta, \eta), (\eta, \eta) \rangle_{L^1 \cup L^2}.
$$

**Theorem 2.1.** [BFK92] There exists an explicitly determined constant $\mathcal{E}_{M_f, N_2} \in \mathbb{R}$ such that

$$
\frac{\det_\zeta \Delta_{M_f}}{\det_\zeta \Delta_M} = \mathcal{E}_{M_f, N_2} \det_\zeta \mathcal{R}_a(M_f, N_2).
$$

This is an example of Mayer-Vietoris type formulae obtained by Burgheldea, Friedlander and Kappeler for elliptic differential operators of possibly higher order. An explicit formula for $\mathcal{E}_{M_f, N_2}$ has been obtained by Yoonweon Lee [Lee03].

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6Dirichlet boundary conditions at $N_3$ may be replaced be any elliptic boundary conditions for $\Delta_M$ and $\Delta_{M_f}$.
3. Combinatorial vector bundles and Laplacians

3.1. Vector bundles and connections on a cell complex.

3.1.1. Vector bundles. Throughout this section we assume that $K$ an oriented $n$-dimensional CW complex. We denote the set of $q$-cells by $K_q$ and for each $\sigma \in K_q$ we will write $\partial \sigma$ for $(q-1)$-cells forming its boundary.

Definition 3.1. A vector bundle $E$ over a cell complex $K$ with the fibre $V$ is a triple $(E, \pi, K)$ where $E$ is the total space of the bundle, $\pi : E \to K$ is a projection such that for each cell $\sigma \in K$

$$E_\sigma = \pi^{-1}(\sigma) \approx V.$$  

as a vector space. The space $E_\sigma$ is called the fiber of $\pi$ over $\sigma$.

Any vector bundle over a cell complex $K$ is trivializable: a choice of a linear isomorphism $\xi_\sigma : E_\sigma \approx V$ for each $\sigma \in K$ induces an isomorphism $E \approx V \times K$ and brings $\pi$ to the natural projection $V \times K \to K$.

As usual, a section of $E$ is a map $s : K \to E$, such that $s \cdot \pi = id_K$. Denote the space of sections by $C^\ast(K, E)$. A trivialization of $E$ identifies the space of sections $C^\ast(K, E)$ with the space of maps $K \to V$. The space $C^\ast(K, E)$ is a vector space with $(\phi_1 + \phi_2)(\sigma) = \phi_1(\sigma) + \phi_2(\sigma) \in E_\sigma$. It has a natural $\mathbb{Z}$-grading with $C^q(K, E)$ being the space of sections $\phi : K_q \to E$ and co-chain complex structure with the differential

$$(d \phi)(\tau) = \sum_{\sigma \in \partial \tau} (-1)^{(|\tau|, |\sigma|)} \phi(\sigma),$$  \hspace{1cm} (3.1)

Here $\tau \in K_{q+1}$ and $(-1)^{(|\tau|, |\sigma|)}$ is plus when orientations of $\sigma$ and $\tau$ agree and it is minus when they are opposite. The identity $d^2 = 0$ is proven below.

This cochain complex should be regarded as a discrete version of the de Rham complex on a trivial vector bundle with the trivial flat connection.

3.1.2. The double of a cell complex. Let $K^\vee$ be the complex dual to $K$. By definition, there is a bijective correspondence $j : K_q \to K^\vee_{n-q}$. Moreover, the boundary operator $\partial$ on $K$ defines a coboundary mapping $\partial\gamma$ on $K^\vee$, such that for each $\sigma \in K_q$ and the cochain $(j\sigma)^*$ dual to $j\sigma \in K^\vee_{n-q}$, we find

$$\partial\gamma(j\sigma)^* = (j\partial\sigma)^*.$$  

Geometrically, 0-cells of $K^\vee$ can be identified with a choice of a point in the interior of each $n$-cell of $K$ (we will call such points centers of $n$-cells), 1-cells of $K^\vee$ can be identified with a choice of a point on each $(n-1)$-cell of $K$ (centers of $(n-1)$-cells) and connecting it with the center of each adjacent $n$-face by a segment, etc. We will denote a center of $\sigma$ by $p_\sigma$.

Clearly, a particular choice of the centers $p_\sigma$ for each $\sigma$ is irrelevant in the combinatorial picture and becomes of interest only when the cell complex is used to approximate smooth structures.

Define the cell complex $D(K)$ which we will call the double of $K$ as follows. For each $\sigma \in K$ choose a center $p_\sigma \in \text{int}(\sigma)$ of $\sigma$. Add new edges to $K$ which connect the center of $\sigma$ with centers of $\tau \in \partial \sigma$, i.e. $p_\sigma \in K^\vee$. Subdivide each 2-cell by new edges connecting its center to centers of boundary edges. For each 3-cell add 2-cells whose boundary consists of new edges connecting the center of the 3-cell to centers of its
boundary 2-cells and new edges connecting centers of boundary 2-cells to centers of their boundary etc. Proceeding iteratively in higher degrees defines the double cell complex \( D(K) \).

Note for example that by construction, each such 2-cell is a quadrilateral with one vertex \( p_\eta \) being the center of some \( \eta \in K_3 \), two others \( p_{\tau\pm} \) being centers of \( \tau\pm \in \partial \eta \), and the other \( p_\sigma \) where \( \sigma \in \partial \tau\pm \). Moreover all cells of \( D(K) \) are cubic and thus, \( D(K) \) is an example of a cubic cell complex.

Note that vertices (0-cells) of \( D(K) \) are in natural bijection with cells of \( K \). An example of the double of a 2-dimensional cell complex is shown on Fig. 1, where the cells of the double complex are bold.

![Figure 1. The double cell complex \( D(K) \) of a triangle.](image)

### 3.1.3. Connections on vector bundles.

Fix a bijection between cells of \( K \) and vertices of \( D(K) \). Then a vector bundle over \( K \) becomes a vector bundle over the set of vertices of \( D(K) \). Naturally any such vector bundle is trivial and isomorphic to \( V(D(K)) \times V \) where \( V(D(K)) \) is the set of vertices of \( D(K) \) and \( V \) is the vector space isomorphic to a fiber.

**Definition 3.2.** A connection \( \alpha \) on a vector bundle \( E \) over \( K \) is the collection of parallel transports, i.e. it assigns an linear isomorphism of fibers \( \alpha(\gamma) : E_\tau \rightarrow E_\sigma \) to each edge path \( \gamma \subset D(K) \) connecting cells \( \tau, \sigma \in K \).

**Definition 3.3.** The vector bundle \( E \) is Hermitian if each fiber is equipped with a Hermitian scalar product. A Hermitian connection is the collection of parallel transports which are unitary isomorphisms of fibers.

**Definition 3.4.** Given a connection \( \alpha \), its combinatorial covariant derivative \( d_\alpha : C^q(K, E) \rightarrow C^{q+1}(K, E) \) is:

\[
(d_\alpha \phi)(\tau) = \sum_{\sigma \in \partial \tau} (-1)^{\langle \tau, \sigma \rangle} \alpha(\tau, \sigma) \phi(\sigma).
\]

Here \( \phi \in C^q(K, E) \), \( \tau \in K^{q+1} \), \( (\tau, \sigma) = 0 \), if the orientation of \( \sigma \) coincides with the orientation induced by \( \tau \) on its boundary and \( (\tau, \sigma) = 1 \) if these orientations are opposite.

Consider four cells \( \eta \in K_{q+2} \), \( \tau\pm \in K_{q+1} \) and \( \sigma \in K_q \) such that \( \tau\pm \in \partial \eta \) and \( \sigma \in \partial \tau\pm \). For given \( \eta \) we will call such such quadrupels \((\eta, \tau\pm, \sigma)\) leaves of \( \eta \) and will denote the set of leaves of \( \eta \) by \( L(\eta) \). We identify each leaf with the corresponding \( \sigma \in \partial \partial \eta \) and write \( \sigma \in L(\eta) \). A choice of such \((\sigma, \eta)\) determines the associated cells.
Let us compute $d^2_\alpha$. For $\phi \in C^q(K, E)$ and $\eta \in K_{q+2}$ we have

\[
(d_\alpha d_\alpha \phi)(\eta) = \sum_{\tau \in \partial \eta} \sum_{\sigma \in \partial \tau} (-1)^{(\tau, \eta) + (\sigma, \tau)} \alpha(\eta, \tau) \alpha(\tau, \sigma) \phi(\sigma)
\]

\[
= \sum_{\sigma \in L(\eta)} \left( (-1)^{(\tau^+, \eta) + (\sigma, \tau^+)} \alpha(\eta, \tau^+) \alpha(\tau^+, \sigma)
+ (-1)^{(\tau^-, \eta) + (\sigma, \tau^-)} \alpha(\eta, \tau^-) \alpha(\tau^-, \sigma) \right) \phi(\sigma)
\]

\[
= \sum_{\sigma \in L(\eta)} (-1)^{(\sigma, \eta)} F(\sigma, \eta) \phi(\sigma),
\]

where we introduce the curvature of a connection as the collection of elements

\[
F_\alpha(\sigma, \eta) = \alpha(\eta, \tau^+) \alpha(\tau^+, \sigma) - \alpha(\eta, \tau^-) \alpha(\tau^-, \sigma)
\]

assigned to each quadrilateral of $D(k)$. In the last step of the computation we used the identity

\[
(-1)^{(\tau^+, \eta) + (\sigma, \tau^+)} + (-1)^{(\tau^-, \eta) + (\sigma, \tau^-)} = 0.
\]

**Definition 3.5.** The curvature of connection $\alpha$ on the leaf of $\eta$ containing $\sigma$ is defined by $F_\alpha(\sigma, \eta) \in \text{Hom}(E_\sigma, E_\eta)$.

**Definition 3.6.** The combinatorial connection $\alpha$ is flat if its curvature is zero for each pair $(\sigma \in L(\eta), \eta)$, i.e. if

\[
\alpha(\eta, \tau^+) \alpha(\tau^+, \sigma) = \alpha(\eta, \tau^-) \alpha(\tau^-, \sigma)
\]

for each such pair $(\sigma, \eta)$.

The following is now clear:

**Proposition 3.7.** The combinatorial connection is flat, if and only if

\[
d^2_\alpha = 0.
\]

One of the corollaries is the identity $d^2 = 0$ for the non-twisted differential. A flat combinatorial connection $\alpha$ defines a cochain complex $(C^*(K, E), d_\alpha)$. This complex should be regarded as a discrete analog of the de Rham complex with coefficients in a local system on $E$.

### 3.2. Combinatorial Laplacians.

#### 3.2.1. Riemannian structure on cell complexes.

A choice of metric on an oriented Riemannian manifold $M$ defines a scalar product on forms, i.e. an $L_2$-norm on the space of forms. It also defines a scalar product and a norm on forms on each submanifold of $M$. We will call such norm Riemannian norms.

**Definition 3.8.** A Riemannian norm on an $n$-dimensional cell complex $K$ as a mapping $| \cdot | : K_q \times K_q \to \mathbb{R}$ for each $q = 0, 1, \ldots, n$, such that the corresponding bilinear form on cochains is positive definite. The number $|\sigma, \tau|$ will be called the weight of the pair. A cell complex with a Riemannian norm is called a metrized cell complex.
Remark 3.9. An embedding of the cell complex $K$ to a Riemannian manifold $M$, gives a natural choice of a Riemannian norm on $K$. It is induced by the metric on $M$ and by the Whitney forms (see section 4.2.1). It assigns to each pair $(\sigma, \tau) \in K_q \times K_q$ the weight
\[ |\sigma, \tau| = \int_M W(\sigma) \wedge *W(\tau). \]

Remark 3.10. Note that if we want to define in a similar way a discrete analog of metric on $M$, we should define a similar systems of weights $|\sigma, \tau|_L$ for every subcomplex $L \subset K$. It is natural to require that the scalar products of cochains on sub-complexes should satisfy the natural compatibility condition. If two sub-complexes $L_1$ and $L_2$ of the same dimension contain a subcomplex $L_3$ of the same dimension then scalar products of $L_1$ and $L_2$ agree on $L_3$ and coincide with $L_3$ on $K$. If we want to discretize the Riemannian geometry we also need an analog of differential forms, which will bring in an $A_\infty$-structure on the cochains.

Define a Hermitian metric on a vector bundle $E$ over $K$ as a scalar product $h_\sigma : E_\sigma \times E_\sigma \to \mathbb{R}^+$ on fibres $E_\sigma$. Such a combinatorial vector bundle, equipped with a Hermitian metric is called the Hermitian vector bundle.

For a Hermitian vector bundle $E$ over a metrized cell complex $K$ define the bilinear form $\langle \cdot, \cdot \rangle_K$ on $C^q(K, E)$ as
\[ \langle \phi, \psi \rangle_K = \sum_{\sigma \in K_q} \sum_{\tau \in K_q} h(\phi(\sigma), \alpha(\sigma, \tau)\psi(\tau))|\sigma, \tau|. \tag{3.4} \]
We also require that this form is positive definite, i.e. that it is a scalar product on $C^q(K, E)^7$.

Recall that a connection $\alpha$ is Hermitian if for any $v \in E_\sigma$ and $w \in E_\tau$, the parallel transport is given by unitary transformations$^8$:
\[ h(\alpha(\tau, \sigma)v, w) = h(v, \alpha(\tau, \sigma)^{-1}w). \]

By definition, a parallel transport along the edge oriented from $\sigma$ to $\tau$ is the inverse to the one oriented from $\tau$ and $\sigma$: $\alpha(\sigma, \tau) = \alpha(\tau, \sigma)^{-1}$. For a Hermitian connection $\alpha$ the inner product on $C^q(K, E)$ defines the adjoint covariant derivative $d_\alpha^\vee : C^q(K, E) \to C^{q-1}(K, E)$ for any $\phi \in C^q(K, E)$ and $\psi \in C^{q-1}(K, E)$, as
\[ \langle d_\alpha^\vee \phi, \psi \rangle_K := \langle \phi, d_\alpha \psi \rangle_K. \]

The associated combinatorial Laplacian is defined as
\[ \Delta^K_\alpha := d_\alpha d_\alpha^\vee + d_\alpha^\vee d_\alpha : C^q(K, E) \to C^q(K, E). \tag{3.5} \]

Note that twisted combinatorial Laplacian are defined for all connections $\alpha$.

$^7$ The scalar product with $|\sigma, \tau| = |\sigma, \sigma|_L$ is called diagonal. However, scalar products need not be diagonal in general. For example, one of the geometrically most natural scalar products, the Whitney product is not diagonal. In case of a diagonal inner product, $d_\alpha$ can be computed explicitly
\[ (d_\alpha^\vee \phi)(\sigma) = \sum_{\tau, \sigma \in K_q} (-1)^{(|\tau, \sigma|)} \alpha(\sigma, \tau)\phi(\tau)|\tau, \sigma|_L. \]

$^8$Here we will focus on real scalar product, i.e. $h(\phi(\sigma), \psi(\sigma)) = h(\psi(\sigma), \phi(\sigma))$. In this case a connection is given by a collection of $h$-orthogonal parallel transports. In the complex Hermitian case when parallel transport is given by $h$-unitary matrices $h(\phi(\sigma), \psi(\sigma)) = h(\psi(\sigma), \phi(\sigma))$. We will use the term Hermitian for both complex Hermitian and real orthogonal cases.
If \( \alpha \) is flat, i.e. \( d_\alpha^2 = 0 \), then \((d_\alpha')^2 = 0\). Such two operators together with the scalar product give the discrete version of the Hodge structure. We will also use the notation \( \Delta_K \), whenever the connection \( \alpha \) is fixed.

3.2.2. Locality. We will say that the Riemannian norm on an \( n \)-dimensional cell complex \( K \) is local if the weight \( |\sigma, \tau| \) in the scalar product of \( q \)-cochains, \( q = 0, 1, \ldots, n \) can be non-zero only if

- there exists \( \eta \in K_{q+1} \) such that \( \sigma, \tau \subset \partial \eta \),
- the subcomplex \( \overline{\sigma \cup \tau} \) is connected.

The locality of the scalar product is equivalent to the following Mayer-Vietoris property. Assume that \( K \) admits a subcomplex \( L \) that consists of three connected components \( L = L_1 \sqcup L_2 \sqcup L_3 \). Let \( L_1 \) and \( L_2 \) be isometrically identified via \( f \). This defines a new chain complex \( K_f \) with a Riemannian norm induced from \( K \). Let \( \phi, \psi \) be cochains on \( K \), such that \( \phi|_{L_1} = f^*\phi|_{L_2} \) and \( \psi|_{L_1} = f^*\psi|_{L_2} \). Then the cochains lift to well-defined cochains \( \phi_f, \psi_f \) on \( K_f \) and the Mayer-Vietoris property is given by

\[
\langle \phi, \psi \rangle_K = \langle \phi_f, \psi_f \rangle_{K_f} + \langle \phi|_{L_1}, \psi|_{L_1} \rangle_{L_1}.
\]

For vertices this means \( |\sigma, \tau| \neq 0 \) only when \( \tau \) is connected to \( \sigma \) by an edge. For edges this means that \( \sigma \) and \( \tau \) belong to the boundary of some 2-cell and that they share a vertex.

The Whitney scalar product has minimal possible locality, the diagonal scalar product has maximal locality.

One of the reasons we define this notion of locality is that it is consistent with the locality of the classical action for scalar field theory.

3.2.3. Green formula. Assume that an \( n \)-dimensional cell complex \( K \) contains a subcomplex \( L \) with \( \dim(L) = n - 1 \) such that to each \( n - 1 \) cell in \( L \) belongs to the boundary of only one \( n \)-dimensional cell in \( K \). We will call \( L \) the boundary of \( K \).

We have a natural projection \( p : C^* (K, E) \to C^* (L, E) \) which is the restriction of a cochain on \( K \) to \( L \).

Define co-chains \( C^* (K, L, E) \) as the \( \ker(p) \subset C^* (K, E) \). These cochains are discrete analog of differential forms on a manifold with vanishing pullback to the boundary, i.e. of differential forms producing relative cohomologies. Given a flat connection \( \alpha \) on \( E \), it is clear that the subspace \( C^* (K, L; E) = \ker(p) \) is also a subcomplex with the differential \( d_{\alpha} \). We have an exact sequence:

\[
0 \to C^*(K, L; E) \to C^*(K; E) \xrightarrow{p} C^*(L; E) \to 0
\]

This exact sequence naturally splits because the spaces come with bases enumerated by cells. Denote this splitting by \( j : C^*(L; E) \to C^*(K; E) \).

The scalar product \( \langle \cdot, \cdot \rangle_K \) on \( C^*(K; E) \) defines the natural scalar product on the subcomplex \( C^*(K, L; E) \). The splitting defines the scalar product on \( C^*(L; E) \).

\[
\langle \phi, \psi \rangle_L = \langle j(\phi), j(\psi) \rangle_K
\]

Define the bilinear form \( \langle \phi, \psi \rangle_{K,L} \) on \( \phi, \psi \in C^*(K; E) \) as

\[
\langle \phi, \psi \rangle_{K,L} =: \langle \phi, \psi \rangle_K - \langle p(\phi), p(\psi) \rangle_L.
\]

when restricted to \( C^*(K, L; E) \) it coincides with the natural scalar product on this space induced from scalar product on cochains on \( K \).
There is a natural analog of the Green formula for $\phi \in C^q(K, E)$ and $\psi \in C^{q+1}(K, E)$:

$$\langle d_\alpha \phi, \psi \rangle_{K,L} = \langle \phi, d_\alpha^* \psi \rangle_{K,L} + \langle p(\phi), \psi_{\text{norm}} \rangle_L.$$  

Here

$$\langle p(\phi), \psi_{\text{norm}} \rangle_L = \langle p(\phi), p(d_\alpha^* \psi) \rangle_L - \langle p(d_\alpha \phi), p(\psi) \rangle_L$$

We will call $\psi_{\text{norm}}$ the normal component of $\psi$ at the boundary $L$. This defines the combinatorial Laplacian $\Delta_{K,L}$ on cochains $C^*(K, L; E)$, as above in (3.5). The operator $\Delta_{K,L}$ corresponds to the analytic Hodge Laplacian with relative boundary conditions. In degree zero, $\Delta_{K,L}$ is simply the restriction of $\Delta_K$ to $C^0(K, L; E) \subset C^0(K; E)$.

4. The approximation theory

Here we will outline the relation between combinatorial constructions and the smooth theory.

4.1. Cell approximation of vector bundles. Here we will focus the question how cell complexes with vector bundles and connections on them can be induced by a smooth triangulation on a manifold and a vector bundle on it with a connection. Consider an oriented compact smooth $M$ and a vector bundle $E$, with a connection $\alpha$.

From now on let $K$ is a simplicial complex of a smooth triangulation of $M$, identified with its embedding in $M$.

Denote by $D(K)$ the double of the cell complex $K$. Fix an embedding $D(K)$ in $M$ which makes it a cell decomposition of $M$. For each $\sigma \in K$ denote by $p_\sigma$ the point in the interior of $\sigma$ at which $\sigma$ intersects with its dual, recall Figure 1. Define the combinatorial vector bundle $E^c$, with a connection on it as

$$E^c_\sigma = E_{p_\sigma}, \quad \alpha^c(\sigma, \tau) = \alpha(p_\sigma, p_\tau), \tau \in \partial \sigma,$$

where $\alpha(p_\sigma, p_\tau)$ is the holonomy along the edge of $D(K)$ connecting $p_\sigma$ and $p_\tau$. This is the only place where the actual choice of centers is relevant for the construction.

When the connection is flat denote by $(C^*(K, E^c), d_c)$ the associated cochain complex.

Recall the definition of the complex $(C^*(K, F_\alpha), d_F)$ which has been considered in [Dod76] and [MÜl78] in the context of Dodziuk’s approximation theory. Here, $F_\alpha$ denotes the local system of flat sections.

For a simplex $\sigma \in K$ the open star of $\sigma \in K$, $\text{St}(\sigma)$, is defined as the union of $\sigma$ and of interiors of all simplices $\tau \in K$, which contain $\sigma$ as part of their boundary: $\sigma \in \partial \tau$. Every vector $v \in E_\sigma$ in the fiber $E_\sigma$ over $p_\sigma$ extends by the parallel transport $\alpha$ uniquely to a flat section $\Phi_\alpha v$ of $E$ over the open star of $\sigma \in K$.

Consequently we can identify each $\phi \in C^q(K, E^c)$ with a mapping $\Phi_\alpha \phi$ that assigns to any $\sigma \in K_\alpha$ a flat section $\Phi_\alpha \phi(\sigma) \in F_\alpha$. Under such an identification we find for any $v \otimes \sigma^* \in C^q(K, E^c)$, where $\sigma \in K_\alpha$, $v \in E_\sigma$ and $\sigma^*$ is the the dual to $\sigma$, that

$$d^c_\phi(\Phi_\alpha(v \otimes \sigma^*)) := \Phi_\alpha \circ d_c \circ \Phi_\alpha^{-1}(\Phi_\alpha(v \otimes \sigma^*)) = (\Phi_\alpha v) \otimes d\sigma^*.$$  

Thus the identification $\Phi_\alpha : (C^*(K, E^c), d_c) \to (C^*(K, F_\alpha), d^c_\phi)$ is an isomorphism of complexes. We have studied $(C^*(K, E^c), d_c)$ in the previous section only as it bears the advantage of being defined without using analytic data.
4.2. Cell approximation for Laplacians.

4.2.1. The Whitney map. Let $K$ be a smooth CW complex on $M$. Recall the definition of the Whitney map

$$W : C^\bullet(K, F_\alpha) \to \Omega^\bullet(M, E),$$

where $C^\bullet(K, F_\alpha)$ is defined above. The Whitney map was introduced by Whitney [Whi57], see also its use [Dod76, Müü78] in the spectral approximation theory.

Let $K_0 = \{\sigma_0, \ldots, \sigma_d\}$ be the vertices of $K$. Every $\sigma_i$ defines a barycentric coordinate function $\mu_i$ on $M$. Each $\mu_i$ is a continuous function on $M$ with support in the open star of $\sigma_i$. Moreover, since the triangulation is smooth, the restriction of $\mu_i$ to any simplex is smooth. Hence, the differential $d\mu_i \in L^2\Omega^1(M)$ exists in the distributional sense. Given an $q$-simplex $\tau = [\sigma_{i_0}, \ldots, \sigma_{i_q}] \in K_q$, with ascending sequence $\{i_k\}_k$ of indices, and some flat section $v \in F_\alpha$ supported over the open star of $\tau$, we define the Whitney map by $^9$

$$W(v \otimes \tau^*) = v \otimes \left(q! \sum_{k=0}^q (-1)^k \mu_{i_k} \wedge \cdots \wedge \hat{\mu}_{i_k} \cdots \wedge d\mu_{i_q}\right), \quad q > 0,$$

$$W(v \otimes \sigma_{i_0}^*) = v \otimes \mu_{i_0}, \quad q = 0.$$

4.2.2. The metric structures. The Riemannian structure on $M$ and the hermitian structure on $E$ define an $L^2$-structure on $E$-valued differential forms $\Omega^\bullet(M, E)$. The corresponding completion is usually denoted by $L^2\Omega^\bullet(M, E)$. Define the Hermitian metric on the combinatorial vector bundle $E^\sigma$ over $K$ by taking the Hermitian product of sections over a center $p_\sigma$ of each simplex $\sigma \in K$. We define the scalar product of any $\phi, \psi \in C^\bullet(K, E)$ in terms of the Whitney map as

$$\langle \phi, \psi \rangle_W^W := \langle W\phi, W\psi \rangle_{L^2}. \quad (4.1)$$

Note that for the Whitney scalar product $|\langle \sigma, \tau \rangle| \neq 0$ only if there exists a cell $\eta \in K$ such that $\sigma, \tau \subset \partial \eta$. This is an example of a local scalar product (not to confuse with a diagonal scalar products).

This definition provides an explicit example of our general construction in (3.4). Indeed, consider $\phi = v \otimes \sigma^*$ and $\psi = w \otimes \tau^*$, where $\sigma, \tau \in K_q$ and $v \in E_\sigma, w \in E_\tau$. Then we have

$$\langle \phi, \psi \rangle^W_K = \langle W(v \otimes \sigma^*), W(w \otimes \tau^*) \rangle_{L^2} = h(\Phi_\alpha v, \Phi_\alpha w) \langle W\sigma^*, W\tau^* \rangle = h_{p_\sigma}(v, \alpha(p_\sigma, p_\tau)) w) \langle W\sigma^*, W\tau^* \rangle = \langle h_{p_\sigma}(v, \alpha(p_\sigma, p_\tau)) w) \rangle_{(p_\sigma, p_\tau)}.$$

4.2.3. The approximation theory. Let $E$ be a flat Hermitian vector bundle. The de Rham map $R$ associates to any $f \in \Omega^p(M, E)$ a cochain $Rf \in C^0(K, E^c)$ as follows. In a local neighborhood of any simplex $\sigma \in K_q$ we fix a basis $\{v_1, \ldots, v_p\}$, $p = \text{rank } E$, of flat sections of $E$ and write $f$ locally as $f = \sum_{i=1}^p v_i \otimes f_i$, where each $f_i$ is an ordinary differential form. Then

$$Rf(\sigma) := \sum_{i=1}^p v_i(p_\sigma) \int_{\sigma} f_i. \quad (4.2)$$

where $p_\sigma$ is the center of $\sigma$.

---

$^9$Whitney map is a particular choice of a linear spline.
The approximation theory of forms by cochains is based on the fact that the composition of the de Rham and Whitney maps converge to identity when the mesh of the triangulation $K$ goes to zero under standard subdivisions, see [Dod76, Theorem 3.7] and also [Mül78] for twisted setup. The crucial results is the following

**Theorem 4.1.** [Dod76, Th. 5.30], [Mül78, Th. 4.6] The combinatorial zeta functions $\zeta(s, \Delta^K_{\eta}(\alpha))$ converge under standard subdivisions uniformly on compact subsets of the complex half plane $\text{Re}(s) > \dim M/2$ to the analytic zeta function $\zeta(s, \Delta_{\text{abs}})$, as the mesh $\eta$ of the triangulation $K(\eta)$ goes to zero.

Similar result holds for combinatorial zeta functions of $\Delta_{K,L}$ which converge to analytic zeta functions of the Laplacians with relative boundary conditions.

Finally, note that defining the combinatorial Laplacian $\Delta_K^q$ from an inner product on $C^*(K,E^c)$ depends on the choice of centers in $K^\vee$. However, $\Delta_K^q$ on $C^q(K,E^c)$ is equivalent to the corresponding Laplacian on $C^q(K,F^\alpha)$ under the isometric identification $\Phi_\alpha$, in particular combinatorial Laplacians defined with respect to different choices of centers, are unitarily or orthogonally equivalent and have the same spectral properties.

5. **Classical free scalar field**

Here we will outline classical field theory for free scalar field on a metrized cell complex with boundary. Having in mind corresponding Gaussian quantum field theory we will call such classical field theory Gaussian.

5.1. Classical scalar free theory on Riemannian manifolds.

5.1.1. **The action and its minimizers.** Let $(M, g)$ be a Riemannian compact oriented manifold, possibly with boundary and with a Hermitian vector bundle $(E, h)$ over $M$ with a Hermitian connection $\alpha$ on it. For classical scalar field theory on $M$ with values in $E$, the space of fields is the space of sections of $E$. We assume fields are smooth sections of $E$. The action functional is

$$S_M(\phi) := \frac{1}{2} \langle d^\ast_{\alpha} \phi, d_{\alpha} \phi \rangle_M + \frac{m^2}{2} \langle \phi, \phi \rangle_M. \quad (5.1)$$

The constant $m^2$ has the meaning of the square of the mass of a particle.

For the variation of the action we have

$$\delta S_M(\phi) = \langle (d^\ast_{\alpha} d_{\alpha} + m^2) \phi, \delta \phi \rangle_M + \langle \partial^\ast \phi, \delta \phi \rangle_{\partial M}.$$

Here, $\partial^\ast = \nu^\ast (\iota_{\nu^\ast} d_{\alpha} \psi)$ is the covariant normal derivative, with $\iota : \partial M \hookrightarrow M$ denoting the natural inclusion, and $\iota_{\nu}$ the contraction with the unit normal vector field on $\partial M$. The boundary term vanishes for Dirichlet boundary conditions, i.e. if we assume $\phi |_{\partial M} = \eta$. Euler Lagrange equations for this action are

$$\left( d^\ast_{\alpha} d_{\alpha} + m^2 \right) \phi = 0, \quad (5.2)$$

which for fixed boundary conditions $\phi |_{\partial M} = \eta$ admit a unique solution given in terms of the corresponding Poisson map $\phi_c = P_M \eta$. The value of $S_M$ at the critical point $\phi_\eta = P_M \eta$ is then

$$S_M(\phi_\eta) = \frac{1}{2} \langle R^M \eta, \eta \rangle_{\partial M}, \quad (5.3)$$
where \( R^M = \partial^a_v \circ P_M \) is the Dirichlet to Neumann operator for \( \Delta_M + m^2 \) given in terms of the Poisson operator \( P_M \).

The action 5.1 is local, i.e. for any submanifold \( N \subset M \) it satisfies the following property:

\[
S_M(\phi) = S_{M\setminus N}(\phi|_{M\setminus N}) + S_N(\phi|_N).
\]

When \( N \) is of codimension \( n - 1 \) disjoint from \( \partial M \), the second term on the right side is absent and the critical value of the action has the following gluing property:

\[
S_M(\phi_\eta) = \min_\eta (S_{cl(M\setminus N)}(\phi_\eta, \eta, \eta')),
\]

where \( cl(M\setminus N) \) denotes the closure of \( M\setminus N \) with boundary comprised of three components \( N \sqcup N \sqcup \partial M \) and \( \eta \) is the fixed boundary value of fields on \( \partial M \).

5.2. Classical scalar free theory on metrized cell complexes. In this section and in the rest of the paper we will focus on the scalar Bose field when fields are elements of \( C^0(K; E) \). Theories with fields from \( C^1(K; E) \) and from higher degree cochains usually involve gauge symmetry and we will not consider them here. In this section we assume that Riemannian norms on cell complexes are local.

5.2.1. The action. Let \((K, | \cdot |)\) be a metrized cell complex of dimension \( n \) with \((n - 1)\)-dimensional boundary subcomplex \( L \). Let \((E, h)\) a combinatorial Hermitian vector bundle over \( K \), with a Hermitian connection \( \alpha \).

Define the action\(^{10}\) of the free Bose scalar field theory as the following function on \( C^0(K, E) \) as

\[
S_K(\phi) := \frac{1}{2} \langle d_\alpha \phi, d_\alpha \phi \rangle_K + \frac{m^2}{2} \langle \phi, \phi \rangle_K = \frac{1}{2} \sum_{\tau, \tau' \in K} h(d_\alpha \phi(\tau), \alpha(\tau, \tau')d_\alpha \phi(\tau'))| (\tau, \tau') | (5.4)
+ \frac{m^2}{2} \sum_{\sigma, \sigma' \in K} h(\phi(\sigma), \alpha(\sigma, \sigma')m^2(\sigma')\phi(\sigma'))| (\sigma, \sigma') |.
\]

When \( L \subset K \) is a boundary subcomplex, define the boundary action as

\[
S_L(\phi) := \frac{1}{2} \langle d_\alpha \phi, d_\alpha \phi \rangle_L + \frac{1}{2} \langle \phi, m^2 \phi \rangle_L = \frac{1}{2} \sum_{\tau, \tau' \in L} h(d_\alpha \phi(\tau), \alpha(\tau, \tau')d_\alpha \phi(\tau'))| (\tau, \tau') | (5.5)
+ \frac{m^2}{2} \sum_{\sigma, \sigma' \in L} h(\phi(\sigma), \alpha(\sigma, \sigma')\phi(\sigma'))| (\sigma, \sigma') |.
\]

Here \( \phi \in C^0(L, E) \) and we emphasize that \( S_L \) is defined with respect to the extrinsic\(^{11}\) metrization of \( L \), obtained as the restriction of the Riemannian norm on \( K \) to \( L \). In other words, weights in \( S_L \) are the same as in \( S_K \). Geometrically, in case of

\(^{10}\)Note that though we used orientation on \( K \) for definition of \( d_\alpha \), orientability is not required for the definition of \( \langle d_\alpha \phi, d_\alpha \phi \rangle_K \) and the action makes sense on non-orientable complexes as well.

\(^{11}\)An intrinsic metrization of the boundary \( L \) never appears in our discussion.
the Whitney scalar product, the action depends on the Riemannian norm in a star neighborhood of $L$. Define

$$S_{K,L}(\phi) := S_K(\phi) - S_L(\phi|_L). \quad (5.6)$$

The action $S_{K,L}(\phi)$ by definition comprises the self-interaction between interior chains in $K \setminus L$ as well as the interaction between interior and the boundary chains, however does not contain interactions between the boundary chains themselves, i.e., it does not have terms $h(\phi(\sigma), \alpha(\sigma, \tau)\phi(\tau))$ where both $\sigma$ and $\tau$ are in $L$.

For local scalar products on cochains the action is also local in the following sense. Assume that $K$ admits a subcomplex $L$ that consists of three connected components $L = L_1 \sqcup L_2 \sqcup L_3$. Let $L_1$ and $L_2$ be isometrically identified via $f$. This defines a new chain complex $K_f$ with a single boundary subcomplex $L_3$ and a metrization induced from $K$. Let $\phi$ be a cochain on $K$, such that $\phi|_{L_1} = f^* \phi|_{L_2}$. Then $\phi$ lifts to well-defined cochain $\phi_f$ on $K_f$ and the locality of the action is expressed as follows

$$S_K(\phi) = S_{K_f}(\phi_f) + S_{L_2}(\phi|_{L_2}).$$

### 5.2.2. Minimizers of the Gaussian action with Dirichlet boundary conditions.

Let us describe the minimizer of the Gaussian action $S_{K,L}$ over the space of cochains $\phi \in C^0(K, E)$ with given Dirichlet boundary condition $\phi|_L = \eta \in C^0(L, E)$. Because for Dirichlet boundary conditions the variation $\delta \phi$ vanishes at $L$, the variations of $S_K, S_{K,L}$ are identical and

$$\delta S_{K,L}(\phi) = \delta S_K(\phi) = \langle (\Delta_K + m^2)\phi, \delta \phi \rangle_K = 0. \quad (5.7)$$

Explicitly, we have

$$\delta S_K(\phi) = \sum_{\tau, \tau' \in K} h(d_\alpha \phi(\tau), \alpha(\tau, \tau')d_\alpha \delta \phi(\tau')) |(\tau, \tau')|$$

$$+ m^2 \sum_{\sigma, \sigma' \in K} h(\phi(\sigma), \alpha(\sigma, \sigma')\delta \phi(\sigma')) |(\sigma, \sigma')|.$$  

For each vertex $\sigma \in K_0$, consider sets

$$\mathcal{V}(\sigma) := \{(\tau, \tau', \sigma') \mid \sigma \in \partial \tau, |(\tau, \tau')| \neq 0, \sigma' \in \partial \tau'\},$$

$$\mathcal{U}(\sigma) := \{\sigma' \mid \exists (\tau, \tau') : (\tau, \tau', \sigma') \in \mathcal{V}(\sigma)\}. \quad (5.8)$$

In case of a local (Whitney) inner product on a simplicial complex, $\mathcal{U}(\sigma) = \text{St}(\sigma) \cap K_0$ consists of all vertices in the closure of the open star of $\sigma$. We refer to $\mathcal{U}(\sigma)$ as the local neighborhood of $\sigma \in K_0$. Then

$$\delta S_K(\phi) = \sum_{\sigma \in K} \sum_{\mathcal{V}(\sigma)} (-1)^{|(\tau, \sigma) + (\tau', \sigma')} h(\phi(\sigma), \alpha(\sigma, \sigma')\delta \phi(\sigma')) |(\tau, \tau')|$$

$$+ m^2 \sum_{\sigma \in K} \sum_{\mathcal{U}(\sigma)} h(\phi(\sigma), \alpha(\sigma, \sigma')\delta \phi(\sigma')) |(\sigma, \sigma')|.$$

Consequently, the Euler-Lagrange equations for (5.7) can be written as

$$\sum_{\mathcal{V}(\sigma)} (-1)^{|(\tau, \sigma) + (\tau', \sigma')} h(\phi(\sigma'), \alpha(\sigma', \sigma)\phi(\sigma)) |(\tau, \tau')|$$

$$+ m^2 \sum_{\mathcal{U}(\sigma)} h(\phi(\sigma'), \alpha(\sigma', \sigma)\phi(\sigma')) |(\sigma, \sigma')| = 0.$$
We have such equation for each $\sigma \in (K \setminus L)_0$. Each equation is a linear equation involving vertices in the local neighborhood $U(\sigma)$.

Because of convexity\(^{12}\) of $S_{K,L}$ on fibers of $p$, the solution to this difference equation with Dirichlet boundary conditions $\phi|_L = \eta$, exists and is unique. We denote it by $\phi_\eta$. The solution $\phi_\eta$ is linear in $\eta$ and hence we can define the discrete version of the Poisson kernel as

$$\phi_\eta(\sigma) = \sum_{\sigma' \in L} P_{K,L}(\sigma, \sigma') \eta(\sigma'),$$

or $\phi_\eta = P_{K,L} \eta$. The value of $S_{K,L}(\phi_\eta)$ at the critical point $\phi_\eta$ is quadratic in $\eta$ and we can write

$$S_{K,L}(\phi_\eta) = \frac{1}{2} \langle \eta, R^K_L \eta \rangle_L,$$

(5.9)

where $R^K_L$ is the discrete version of the Dirichlet-to-Neumann operator.

**Remark 5.1.** For a diagonal inner product, the Poisson map and hence also the Dirichlet-to-Neumann operator are explicit (see [ClMa04, Theorem 2.1]). In this case the boundary value problem

$$(\Delta_K + m^2) \phi = 0, \text{ on vertices of } K \setminus L,$$

$$\phi(\sigma) = \eta(\sigma), \text{ for all } \sigma \in L.$$

(5.10)

has a unique solution $\phi = P_K \eta \in C^0(K, E)$, and assuming the scalar product on $C^*(K, E)$ is diagonal, the Poisson operator $P_K$ can be described explicitly by

$$P(P_K \eta) = - (P \Delta_K P + m^2)^{-1} P \Delta_K \eta, \quad (\text{Id} - P) R_K = \eta.$$

(5.11)

Here $P$ is the natural projection $P : C^0(K, E) \to C^0(K, L; E) \subset C^0(K, E)$ acting trivially on fibres of $E$.

Values of the action functional on critical points have the following gluing property in case of local scalar products. Assume as before that $K$ admits a subcomplex $L$ that consists of three connected components $L = L_1 \sqcup L_2 \sqcup L_3$. Let $L_1$ and $L_2$ be isometrically identified via $f$. This defines a new chain complex $K_f$ with a single boundary subcomplex $L_3$ and a metrization induced from $K$. Then

$$S_{K_f,L_3}(\phi_\eta) = \min_{\eta_{12}} (S_{K,L}(\phi_{12,\eta_{12}}) + S_L(\eta_{12})).$$

(5.12)

6. Discrete Quantum Gaussian field theory

6.1. The partition function.

6.1.1. Category of cobordisms. Recall that a quantum field theory on $n$-dimensional Riemannian manifolds can be regarded as functor from the category of Riemannian $n$-dimensional cobordisms to the category of vector spaces. An object in the category of Riemannian $n$-dimensional cobordisms is a smooth, oriented, compact $(n-1)$-dimensional Riemannian manifold $N$ with an $n$-dimensional smooth, oriented Riemannian collar $\mathcal{U}(N) \cong (-\epsilon, \epsilon) \times N$ equipped with a product Riemannian metric.

\(^{12}\)I.e. for all $0 < \theta < 1$ and $\phi, \psi \in C^0(K, E)$ with $\phi|_L, \psi|_L = \eta$

$$S_{K,L}(\theta \phi + (1 - \theta)\psi) \leq \theta S_{K,L}(\phi) + (1 - \theta)S_{K,L}(\psi).$$
A morphism between two objects $N_1$ and $N_2$ is a smooth, oriented, compact $n$-dimensional manifold $M$ such that $\partial M = \overline{N_1} \sqcup N_2$ with collars at each connected component of the boundary. The composition of morphism is gluing such that collars on the common boundary agree with both morphisms.

The combinatorial analog of this category is the category of metrized\textsuperscript{13} local $n$-dimensional cell complexes. An object in this category is an $(n-1)$-dimensional cell complex $L$ with a metrized $n$-dimensional collar complex $\mathcal{U}(L)$ which is homotopy equivalent to $L$. The metrization on $L$ is required to arise from the metric structures on $\mathcal{U}(L)$ by restriction.

For local metric structures, $\mathcal{U}(L)$ is the star neighborhood of $L$. For example, this is the case for Whitney scalar product.

6.1.2. Framework of QFT. The framework of an $n$-dimensional local quantum field theory, cf. [Ati90] and [Seg04], applied to metrized cell complexes consists of the following two assignments:

- To each $(n-1)$-dimensional metrized oriented cell complex $L$ with an $n$-dimensional collar $\mathcal{U}(L)$ we assign a $\mathbb{C}$-vector space $H(L)$ with a non-degenerate linear pairing
  $$( , )_L : H(L') \otimes H(L) \to \mathbb{C},$$
  where the cell complexes $L'$ and $L$ differ only by orientation.

- An orientation reversing automorphism $\sigma_L$ of $L$ lifts to an isomorphism of vector spaces $\hat{\sigma}_L : H(L) \to H(L')$, inducing the structure of a Hilbert space on $H(L)$\textsuperscript{14}. An orientation preserving isometry $f : L_1 \to L_2$ of metrized cell complexes lifts to an isometry $\hat{f} : H(L_1) \to H(L_2)$.

- The collar $\mathcal{U}(L)$ is separated by $L$ into subcomplexes $L_{\pm}$, such that $\mathcal{U}(L) = L^+ \cup_L L^-$. We will call $L^+$ and $L^-$ right and left neighborhoods of $L$ respectively. To each metrized $n$-dimensional cell complex $K$ such that $L^+ \subset K$ is a metrized cell subcomplex we assign the vector
  $$Z_{K,L} \in H(L).$$

These data should satisfy certain axioms. One of the most important axioms is the locality property of the partition function, also known as the gluing axiom. Assume that the boundary of a cell complex $K$ has three connected components $L_1$, $L_2$ and $L_3$. Assume that the corresponding right neighborhoods $L_1^+, L_2^+, L_3^+$ are disjoint. We write $L := L_1 \sqcup L_2 \sqcup L_3$. Assume that metrized cell complexes $L_1$ and $L_2$ together with their collars are isometric via an isometry $f$ of cell complexes. Let $K_f$ the result of the gluing the boundary component $L_2$ to $L_1$ via the orientation reversing isometry $\sigma_{L_2} \circ f$, and that the scalar product on $K$ is the pullback of the scalar product on $K_f$.

The partition function $Z_{K,L}$ is a vector in $H(L_1) \otimes H(L_2) \otimes H(L_3)$. Then we require the following

$$(( , )_{L_2} \otimes \text{id})(\hat{\sigma}_{L_2} \circ \hat{f} \otimes \text{id} \otimes \text{id})(Z_{K,L}) = Z_{K_f,L_3} \in H(L_3). \quad (6.1)$$

\textsuperscript{13}Recall that a metrized cell complex is a cell complex $L$ with scalar product on the corresponding cochain complex $C^*(L)$ (e.g. over $\mathbb{R}$). An isometry of cell complexes is an isomorphism of cell complexes which yields an isometry between the corresponding scalar product cochain spaces.

\textsuperscript{14}In case of non-orientable cell complexes, we only assign a Hilbert space structure on $H(L)$.\textsuperscript{15}
6.1.3. Gaussian QFT. Now let us construct a Gaussian quantum field theory which satisfies all these properties.

- To an \((n-1)\) dimensional metrized cell complex \(L\) with a local scalar product\(^{15}\) and an \(n\)-dimensional collar \(\mathcal{U}(L)\) and a Hermitian vector bundle \((E, h)\) we assign the Hilbert space
  \[
  H(L) = L^2(C^0(L; E)),
  \]
  with the scalar product
  \[
  (f, g)_{H(L)} := \int_{C^0(L; E)} \overline{f(\eta)} g(\eta) e^{-S_L(\eta)} d\eta.
  \]
  Here \(S_L\) is the classical Gaussian action defined as in (5.5) defined with respect to the scalar product on cochains given by restriction of the metric structure on the collar \(\mathcal{U}(L)\). We integrate with respect to the Euclidean measure \(d\eta\) on \(C^0(L; E)\). Note that \(S_L(\eta)\) is positive definite.

- If we assume that \(K\) is oriented, then the pairing \((\ , \ )_L : H(L') \otimes H(L) \to \mathbb{C}\) is given by the composition of the natural \(L^2\)-scalar product on \(H(L)\) and the lift \(\tilde{\sigma}_L\) of the orientation reversing automorphism \(\sigma_L : L \to L'\). Gaussian action is invariant under change of orientation, i.e. \(S_{\sigma_L(L)} = S_L\) and the Hermitian axiom of Atiyah-Segal is trivially satisfied. Moreover, the pullback \(\sigma_L^*\) to cochains is trivial in degree zero, so that \(H(L) = H(L')\) and the pairing \([\ , \ ]_L\) coincides with the scalar product on \(H(L)\).

- To the metrized cell complex \(K\) with boundary \(L \subset K\), equipped with its right neighborhood \(\mathcal{U}(L)^+ \subset K\) fitting \(K\) and with the weights on \(\mathcal{U}(L)^+\) given by weights on \(\mathcal{U}(L)\), we will assign the partition function \(Z_{K,L} \in H(L)\) as follows. Consider the natural projection \(p : C^0(K; E) \to C^0(L; E)\) which is the restriction to the boundary. For each \(\eta \in C^0(L; E)\) we set (recall (5.6))
  \[
  Z_{K,L}(\eta) := \int_{p^{-1}(\eta)} \exp (-S_K(\phi)) \, d\phi,
  \]
  Here the integration measure is the Euclidean measure on the vector space corresponding to the affine space \(p^{-1}(\{\eta\}) \subset C^0(K; E)\). The integral is convergent, so partition function \(Z_{K,L}(\eta)\) is defined because the classical action \(S_K\) is positive and strictly convex on each fiber \(p^{-1}(\eta)\). We have \(Z_{K,L} \in H(L)\) with \(\|Z_{K,L}\|_{H(L)} = Z_{K,\emptyset} \in \mathbb{C}\), \(^{16}\) where \(\tilde{K} := K \cup L, \sigma_L^{-1}(K)\) is the closed double of \(K\).

The gluing property is an exercise on Fubini’s theorem.

**Theorem 6.1.** The partition function \(Z_{K,L}\) satisfies the gluing axiom.

**Proof.** Recall that in (6.1) \(K\) is a cell complex with the boundary \(L = L_1 \sqcup L_2 \sqcup L_3\) and \(L_1\) and \(L_2'\) are isometric via orientation preserving isometry \(f_\sigma = \sigma_{L_2} \circ f : L_1 \to L_2'\) of cell complexes. Let us compute the left side of the identity (6.1). The pullback of \(f_\sigma\) to cochains defines \(\tilde{f}_\sigma : C^0(L_1; E) \to C^0(L_2'; E)\). The mapping \(\tilde{f}_\sigma : H(L_1) \to H(L_2')\) is the pull-back of \(f_\sigma^*\). For \((\eta_2', \eta_2, \eta_3) \in C^0(L_2'; E) \otimes C^0(L_2; E) \otimes C^0(L_3; E)\) we have
  \[
  (\tilde{f}_\sigma \otimes \text{id} \otimes \text{id})Z_{K,L}((\eta_2', \eta_2, \eta_3)) = Z_{K,L}(f_\sigma^*(\eta_2', \eta_2, \eta_3)).
  \]

\(^{15}\)in the sense of §3.2.2.

\(^{16}\)We identify \(H(\emptyset) \cong \mathbb{C}\).
Let $K_f$ be the cell complex obtained from $K$ by gluing the boundary component $L_2$ to $L_1$ via the orientation reversing isomorphism $f_\sigma$. The scalar product on $K_f$ is the pullback of the scalar product on $K$. Then any $\phi \in C^0(K_f; E)$ with $\phi|_{L_2} = \eta_2$ and $\phi|_{L_3} = \eta_3$ can be regarded as $\phi \in C^0(K; E)$ with $\phi|_L = (f^*\eta_2, \eta_2, \eta_3)$. Moreover, by construction

$$S_K(\phi) = S_{K_f}(\phi) + S_{L_2}(\eta_2).$$

Consequently we obtain

$$((\ , \ )_{L_2} \otimes \text{id})(\hat{\phi} \otimes \text{id} \otimes \text{id})(Z_{K,L})(\eta_3) = \int_{C^0(L_2; E)} Z_{K,L}(f^*\eta_2, \eta_2, \eta_3) e^{-S_{L_2}(\eta_2)} d\eta_2 = Z_{K_f,L_2}(\eta_3). \quad (6.3)$$

Write $\langle \cdot, \cdot \rangle_0$ for the Euclidean inner product on cochains (in the cell basis). Then there exists an endomorphism $Q_K$ on $C^0(K, E)$, Hermitian with respect to the Euclidean inner product, such that $\langle \phi, \psi \rangle_K = \langle Q_K \phi, \psi \rangle_0$ for any $\phi, \psi \in C^0(K, E)$. We write $\Delta_{K}^{\text{loc}} = Q_K \mid_{L} \circ \Delta_K$\footnote{The operator $\Delta_{K}^{\text{loc}}$ is local in a sense that it acts between nearest neighboring vertices.}, where $\Delta_K$ is the combinatorial Laplacian on $K$ with Dirichlet boundary conditions at the boundary subcomplex $L$. The Gaussian integral (6.2) is then easy to compute.

$$Z_{K,L}(\eta) = (2\pi)^{\frac{|K|L|}{2}} \frac{e^{-S_{K,L}(\phi)}}{\sqrt{\det(\Delta_{K}^{\text{loc}} + m^2 Q_K \mid_{L})}}.$$  

If we substitute this formula into the gluing identity for the partition function we will get the gluing identity for critical values of the classical action (5.12) in the exponent and the following identity of determinants in the pre-exponent:

$$\frac{\det(\Delta_{K_f}^{\text{loc}} + m^2 Q_{K_f} \mid_{L_2})}{\det(\Delta_{K}^{\text{loc}} + m^2 Q_K \mid_{L})} = \det \mathcal{R}_c^{\text{loc}}(K_f, L_2), \quad (6.4)$$

where $\mathcal{R}_c^{\text{loc}}(K_f, L_2) = Q_{L_2} \circ \mathcal{R}_c(K_f, L_2)$ and $\mathcal{R}_c(K_f, L_2)$ is defined by

$$\langle \mathcal{R}_c(K_f, L_2) \eta_2, \eta_2 \rangle_{L_2} := \langle R^K_2(\eta_2, \eta_2, 0), (\eta_2, \eta_2, 0) \rangle_{L_1 \cup L_2}. \quad (6.5)$$

This proves Theorem 1.1. It remains to identify a gluing relation for the endomorphism $Q_K$, associated to the quadratic form of the scalar product on $C^0(K, E)$. With respect to the direct sum decomposition $C^0(K_f \mid_{L_3}, E) = C^0(K \mid_{L}, E) \oplus C^0(L_2, E)$ we may write in the basis defined by the duals of the vertex elements

$$Q_{K_f \mid L_3} = \begin{pmatrix} Q_K \mid_{L} & A \\ A^t & Q_{L_2} \end{pmatrix}, \quad (6.6)$$

where $A : C^0(L_2, E) \to C^0(K \mid_{L}, E)$ describes the interaction in the inner product of $K$ between vertices at the boundary subcomplex $L_2$ with the interior vertices. From the block representation (6.6) and from (6.4) we find

$$\frac{\det(\Delta_{K_f} + m^2)}{\det(\Delta_K + m^2)} = \frac{\det Q_{K_f} \mid_{L_3} \det Q_{L_2}}{\det Q_{K_f} \mid_{L_3} \det \mathcal{R}(K_f, L_2)} \frac{\det Q_{L_2}}{\det(\bar{Q}_{L_2} - A^t \circ Q_{K_f} \mid_{L} \circ A)} \det \mathcal{R}(K_f, L_2). \quad (6.7)$$
7. FROM IDENTITIES FOR DISCRETE LAPLACIANS TO BFK-IDENTITIES

This section is devoted to a proof of Theorem 1.2. Here we basically apply the techniques by Dodziuk [Dod76] and Müller [Müll78].

We recall the setup and notation laid out in §1.3. Let \((M, g)\) be a compact Riemannian manifold with boundary \(\partial M\) that consists of two disjoint boundary components \(N_1, N_2\).\(^{18}\) Consider a flat Hermitian vector bundle \((E, h, \alpha)\). Flatness implies product structure over a collar neighborhood of \(\partial M\). We denote by \(\Delta_M\) the Laplace Beltrami operator acting of functions on \(M\) with values in \(E\) and with Dirichlet boundary conditions at the boundary.

Assume that \(g\) is product near \(N_1\) and \(N_2\) and define the closed double \(\tilde{M}\) by gluing a second copy of \(M\) along the boundary. Let \(N_1\) and \(N_2\) be identified by an isometry \(f\), and denote by \(M_f\) the closed Riemannian manifold obtained from \(M\) by gluing \(N_1\) onto \(N_2\). The flat Hermitian vector bundle \((E, h, \alpha)\) induces smooth flat vector bundles over \(M_f\) and \(\tilde{M}\). We write \(\Delta_{\tilde{M}}\) and \(\Delta_M\) for the twisted Laplacians on \(\tilde{M}\) and \(M_f\), respectively, with Dirichlet boundary conditions at the respective boundaries.

Consider a simplicial complex \(K\) which triangulates \(M\) with subcomplexes \(L_1, L_2\) triangulating \(N_1, N_2\), respectively. Its double \(\tilde{K}\) along \(L_1 \sqcup L_2\) is a simplicial decomposition of \(\tilde{M}\). The simplicial complex \(K_f\), obtained by gluing \(K\) along the two identified boundary components \(L_1\) and \(L_2\), decomposes \(M_f\).

The pullbacks of the combinatorial analog of \(E\) over \(K\) define combinatorial vector bundles over \(K_f\) and \(\tilde{K}\). The combinatorial Riemannian structure on \(K\) defines natural combinatorial Riemannian structures on \(K_f\) and \(\tilde{K}\). The metric structure on \(M\) and the Whitney map \(\phi\) define a combinatorial Riemannian structure on \(K\) together with the inner product on corresponding cochains with values in combinatorial vector bundle. Denote by \(\Delta_{K_f}\) and \(\Delta_{\tilde{K}}\) the combinatorial Laplace operators on cochains of degree zero, defined with respect to this inner product, with Dirichlet boundary conditions at the boundary.

Consider any covering \(\{U_\alpha\}_{\alpha \in A}\) of \(M_f\) by open subsets, such that the closure \(\overline{U_\alpha}\) of each is a submanifold of \(M_f\) with smooth boundary \(\partial \overline{U_\alpha}\). Choose a subordinate partition of unity \(\{\phi_\alpha\}\). Let \(\{\psi_\alpha\}\) be a family of functions \(\psi_\alpha \in C^\infty(U_\alpha)\) with compact support in \(U_\alpha\) such that \(\psi_\alpha \mid \text{supp} \phi_\alpha \equiv 1\).

Let \(\Delta_\alpha\) denote closure of the Hodge Laplace operator on \(C^\infty(U_\alpha, E)\) with absolute boundary conditions\(^{19}\). Denote by \(H_\alpha : L^2(U_\alpha, E) \to \ker \Delta_\alpha\) the corresponding harmonic projection and set \(D_\alpha := \Delta_\alpha + H_\alpha\). Then [Müll78, Definition 8.10] defines a pseudo-differential operator

\[
E_{M_f}(s) := \sum_{\alpha \in A} \phi_\alpha D_\alpha^{-s} \psi_\alpha, \tag{7.1}
\]

such that the difference between \((\Delta_{M_f} + H)^{-s}\) and \(E_{M_f}(s)\) is smoothing for every \(s \in \mathbb{C}\), cf. [Müll78, Theorem 8.11]. Here, obviously \(\Delta_{M_f}\) is the Hodge Laplacian on \(\Omega^0(M_f, E)\) and \(H\) the corresponding harmonic projection.

Given a smooth triangulation \(K_f\) of \(M_f\), we may choose an admissible covering \(\{U_\alpha\}_{\alpha \in A}\) of \(M_f\), such that \(K_f\) induces a smooth triangulation \(K_\alpha\) of each submanifold

\(^{18}\)By locality of the argument, we may assume without loss of generality that \(N_3, L_3 = \emptyset\).

\(^{19}\)The arguments of this section hold similarly for Hodge Laplacians on differential forms.
$U_\alpha \subset M$ and its boundary $\partial U_\alpha$. Let $\Delta_0^\alpha$ be the discrete Laplacian on $C^0(K_\alpha, E)$, $H_\alpha^c$ the corresponding harmonic projection. Write $D_\alpha^c := \Delta_0^\alpha + H_\alpha^c$. We denote by $\Delta_{K_f}$ the discrete Laplacian on $C^0(K_f, E)$. Recall the de Rham map from (4.2). Then [Mül78, Definition 8.12] defines a combinatorial parametrix

\[ E_{K_f}^c(s) := \sum_{\alpha \in A} \phi_\alpha W(D_\alpha^c)^{-s} A\psi_\alpha. \]  

(7.2)

Its trace is defined as follows. Let $\{a_i\}_{i=1}^N$ be an orthonormal basis of $C^0(K_f, E)$ with the scalar product induced by the Whitney map $W$ respective to the given triangulation $K_f$. Then

\[ \text{Tr} E_{K_f}^c(s) := \sum_{l=1}^N \langle E_{K_f}^c(s) W a_l, W a_l \rangle_{L^2(M_f, E)}. \]  

(7.3)

The crucial property of the presented construction is the following

**Theorem 7.1.** [Mül78, Theorem 8.43, 8.44] The family $\zeta(s, \Delta_{K_f}) - \text{Tr} E_{K_f}^c(s)$ is holomorphic in $s \in \mathbb{C}$ and locally uniformly (in $K_f$) bounded. Moreover, as the mesh $\delta > 0$ of the triangulation $K_f$ goes to zero under standard subdivisions, $\zeta(s, \Delta_{K_f}) - \text{Tr} E_{K_f}^c(s)$ converge uniformly on every compact subset of $\mathbb{C}$ to the holomorphic function $\zeta(s, \Delta_{M_f}) - \text{Tr} E_{M_f}(s)$.

Consider an admissible covering $\{U_\alpha\}_{\alpha \in A}$ of $M$, with $A = A_0 \cup A_1$ such that $\{U_\alpha\}_{\alpha \in A_0}$ covers a collar of the hypersurface $N_2$. $U_\alpha \subset M_\delta$ for each $\alpha \in A_1$. Then by locality of the individual summands in (7.1) we find

\[ E_{M_f}(s) = \sum_{\alpha \in A} \phi_\alpha D_\alpha^{-s} \psi_\alpha = \sum_{\alpha \in A_0} + \sum_{\alpha \in A_1} = \frac{1}{2} E_{\tilde{M}}(s). \]  

(7.4)

In order to establish a similar relation on the combinatorial level, write $P : L^2(M_f, E) \to WC^0(K_f, E)$ and $P_0 : L^2(U_\alpha, E) \to WC^0(K_\alpha, E)$ for the global and local orthogonal projections onto the image of the Whitney map, respectively. Then by definition $\text{Tr} E_{K_f}^c(s) = \text{Tr} E_{K_f}(s) P$ and moreover, the computations in the third displayed equation of [Mül78, p. 296] assert that for each $\alpha \in A$

\[ |\text{Tr}(\phi_\alpha W(D_\alpha^c)^{-s} A\psi_\alpha P) - \text{Tr}(\phi_\alpha W(D_\alpha^c)^{-s} A\psi_\alpha P_\alpha)| = \epsilon(\delta, s), \]  

(7.5)

converges uniformly to zero on compact subsets of $s \in \mathbb{C}$ as the mesh $\delta > 0$ of the triangulation $K_f$ goes to zero under standard subdivisions. The terms $\phi_\alpha W(D_\alpha^c)^{-s} A\psi_\alpha P_\alpha$ are local and the argument of (7.4) applies. This proves

\[ E_{K_f}(s) = \frac{1}{2} E_K(s) + \epsilon'(\delta, s), \]  

(7.6)

where as above $\epsilon'(\delta, s)$ converges uniformly to zero on compact subsets of $s \in \mathbb{C}$ as the mesh $\delta > 0$ of the triangulation $K_f$ goes to zero under standard subdivisions. Combining Theorem 7.1 with the relations (7.4) and (7.6) proves

**Theorem 7.2.** As the mesh $\delta > 0$ of the triangulation $K$ goes to zero under standard subdivisions

\[ \lim_{\delta \to 0} \frac{(\det' \Delta_{K_f})^2}{\det' \Delta_K} = \frac{(\det \zeta \Delta_{M_f})^2}{\det \zeta \Delta_M}. \]  

(7.7)
As an obvious consequence of Theorem 7.2, (6.7) and Theorem 2.1 we arrive at the following relation between the analytic and combinatorial DN operators

\[
\lim_{\delta \to 0} \frac{(\det' \mathcal{R}_a(K_f, L_2))^2}{\det' \mathcal{R}_a(K, L_1 \sqcup L_2)} \cdot \frac{\det(Q_{L_3 \sqcup L_3} - \tilde{A}' \circ Q_{(K \setminus L) \sqcup (K \setminus L)} \circ \tilde{A})}{\det(Q_{L_2} - A' \circ Q_{K \setminus L} \circ A)^2} = \frac{Q_{M, N_2}}{Q_{M, N_1 \sqcup N_2}} \cdot \frac{(\det \mathcal{R}_a(M_f, N_2))^2}{\det \mathcal{R}_a(M, N_1 \sqcup N_2)}
\]

where \( \tilde{A} = A \oplus A \). Corollary 1.3 now follows from the next

**Proposition 7.3.** For \( \delta > 0 \) sufficiently large

\[
\frac{\det(Q_{L_2 \sqcup L_3} - \tilde{A}' \circ Q_{(K \setminus L) \sqcup (K \setminus L)} \circ \tilde{A})}{\det(Q_{L_2} - A' \circ Q_{K \setminus L} \circ A)^2} = 1.
\]

**Proof.** Denote by \( \mathcal{U}_L \subset K_f \) the metrized collar complex of \( L_2 \subset K_f \). Let \( v \in C^0(K \setminus L) \) be an element in the image of \( A \). By locality of the Whitney inner product, \( v \) is supported inside the star neighborhood \( St(L_2) \) of \( L_2 \subset K_f \), which consists of all those vertices \( \tau \in K_f \), which are connected to \( L_2 \) by an edge. For any \( \omega \in C^0(K \setminus L) \)

\[
\langle \omega, Q_{K \setminus L}^{-1} v \rangle_K = \langle \omega, v \rangle_0
\]

and equals zero, if and only if \( \omega \) is supported in \( K_f \setminus St(L_2) \). By locality of the Whitney inner product, this in turn implies that \( Q_{K \setminus L}^{-1} v \) is supported inside \( St(St(L_2)) \). Assume \( \delta > 0 \) is sufficiently large, so that \( Q_{K_f}(St(St(L_2))) \subset \mathcal{U}_L \). Then we find

\[
Q_{\mathcal{U}_L \setminus L_2}(Q_{K \setminus L}^{-1} v) = Q_{K \setminus L}[Q_{K \setminus L}^{-1} v] = v,
\]

and hence \( Q_{K \setminus L}^{-1} v = Q_{\mathcal{U}_L \setminus L_2}^{-1} v \). Consequently \( Q_{K \setminus L}^{-1} \circ A \) depends only on the metrization over \( \mathcal{U}_L \) and the statement follows immediately from the product metric structure assumption. \(\square\)

**8. Conclusion**

This paper is a step towards constructing the quantum field theory of free scalar Bose field on a Riemannian manifold \( M \) as a limit of Gaussian quantum field theories on finite metrized cell approximations of \( M \) when the mesh of the approximation goes to zero. In quantum field theory and statistical mechanics such limit is known as a scaling limit (near a point of phase transition, which is mesh equals to zero in our case). The main step towards completion of such program is the characterization of the zero mesh limit of determinants of combinatorial Laplacians. One should expect that as mesh \( \epsilon \) goes to zero

\[
\log \det(\Delta_{K_\epsilon} + m^2) = 2N_\epsilon + \sum_{j=1}^{n-1} c_j \epsilon^j + \text{log. terms} + c_0 + \log \det(\Delta + m^2) + o(1).
\]

Here \( N_\epsilon \) is the number of vertices in \( K_\epsilon \), \( n \) is the dimension of the cell complex, constants \( c_0, \ldots, c_{n-1} \) are not universal, i.e. they depend on \( K_\epsilon \). This problem is largely open. For some results in this direction see [Ken00], [CJK10], [Sri14].

Another problem, closely related to this paper is the construction of topological quantum field theories based on an approximation of space times by cell complexes.
An example of such TQFT was constructed in [Mne09]. In this case we will have combinatorial torsions instead of determinants of Laplacians. Discrete version of the De Rham differential and of the exterior product are given by corresponding $A_{\infty}$ algebras. A very important but entirely understood question in this direction is what is the most natural discrete counterpart of Riemannian geometry (in particular of Hodge star operation).

We did not discuss here first order formulation of the classical field theory on cell complexes (see for example [CMR11] for classical field theories on manifolds). We will do it in a separate publication.

The approximation of space time by a complex for special box complexes is well known in constructive field theory. In this sense the present paper can be regarded as a development in constructive quantum field theory with the aim to construct an Atiyah-Segal style quantum field theory. In dimension 2 this was done in [Pic08] for quantum $P(\phi)^2$ theory directly in the continuum case. It would be interesting to derive these results from cell approximations.

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