COMBINATORIAL EXPRESSIONS FOR THE TAU FUNCTIONS OF \textit{q}-PAINLEVÉ V AND III EQUATIONS

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ABSTRACT. We derive series representations for the tau functions of the \textit{q}-Painlevé V, III\textsubscript{1}, III\textsubscript{2}, and III\textsubscript{3} equations, as degenerations of the tau functions of the \textit{q}-Painlevé VI equation in [JNS]. Our tau functions are expressed in terms of \textit{q}-Nekrasov functions. Thus, our series representations for the tau functions have explicit combinatorial structures. We show that general solutions to the \textit{q}-Painlevé V, III\textsubscript{1}, III\textsubscript{2}, and III\textsubscript{3} equations are written by our tau functions. We also prove that our tau functions for the \textit{q}-Painlevé III\textsubscript{1}, III\textsubscript{2}, and III\textsubscript{3} equations satisfy the three-term bilinear equations for them.

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1. INTRODUCTION

The \textit{q}-Painlevé equations [RGH], [KNY] are \textit{q}-difference analogs of the Painlevé equations, which were introduced as new special functions beyond elliptic functions and the hypergeometric functions more than one hundred years ago [P], [G], and are now considered as important special functions with many applications both in mathematics and physics.

Similarly as for other integrable systems, tau functions play a crucial role of the studies of the Painlevé equations. Recent discovery by [GIL] states that the tau function of the sixth Painlevé equation is a Fourier transform of conformal blocks at \(c = 1\), which admit explicit combinatorial formulas by AGT correspondence [AGT]. Series representations of the tau functions of other types are also studied in [GIL1], [N], [N1], [BLMST] for differential cases, [BSI], [INS] for \textit{q}-difference cases.

In [JNS], a general solution \((y, z)\) to the \textit{q}-Painlevé VI equation [JS] was expressed by the tau functions having \textit{q}-Nekrasov type expressions, and it was conjectured that the tau functions satisfy the bilinear equations for the \textit{q}-Painlevé VI equation. In this paper, we give explicit expressions for general solutions to the \textit{q}-Painlevé V, III\textsubscript{1}, III\textsubscript{2}, and III\textsubscript{3} equations using degenerations of the tau functions of the \textit{q}-Painlevé VI equation. We also give conjectures on the bilinear equations satisfied by the tau functions of the \textit{q}-Painlevé V equation and prove that the tau functions of the \textit{q}-Painlevé III\textsubscript{1}, III\textsubscript{2}, and III\textsubscript{3} equations satisfy the bilinear equations.

Our \textit{q}-difference equations are as follows. (i) the \textit{q}-Painlevé VI equation:

\[
\frac{y\bar{y}}{a_3a_4} = \frac{(\bar{z} - b_1 t)(\bar{z} - b_2 t)}{(\bar{z} - b_3)(\bar{z} - b_4)}, \quad \frac{z\bar{z}}{b_3b_4} = \frac{(y - a_1 t)(y - a_2 t)}{(y - a_3)(y - a_4)}.
\]

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(ii) the \(q\)-Painlevé V equation:
\[
\frac{y\overline{y}}{a_3a_4} = -\frac{(z - b_1t)(z - b_2t)}{z - b_3}, \quad \frac{z\overline{z}}{b_3} = -\frac{(y - a_1t)(y - a_2t)}{a_4(y - a_3)}.
\]

(iii) the \(q\)-Painlevé \(\text{III}_1\) equation:
\[
\frac{y\overline{y}}{a_3a_4} = -\frac{z(z - b_2t)}{z - b_3}, \quad \frac{z\overline{z}}{b_3} = -\frac{y(y - a_1t)}{a_4(y - a_3)}.
\]

(iv) the \(q\)-Painlevé \(\text{III}_2\) equation:
\[
\frac{y\overline{y}}{a_3a_4} = -\frac{z^2}{z - b_3}, \quad \frac{z\overline{z}}{b_3} = -\frac{y(y - a_2t)}{a_4(y - a_3)}
\]

(v) the \(q\)-Painlevé \(\text{III}_3\) equation:
\[
\frac{y\overline{y}}{a_3} = z^2, \quad \frac{z\overline{z}}{b_3} = -\frac{y(y - a_2t)}{y - a_3}
\]

Here, \(y, z\) are functions of \(t\), \(\overline{y} = y(qt), \overline{z} = z(qt)\), and \(a_i, b_i (i = 1, 2, 3, 4)\) are parameters.

From the point of view of Sakai’s classification for the discrete Painlevé equations [S1], the \(q\)-Painlevé VI, V, \(\text{III}_1\), \(\text{III}_2\) and \(\text{III}_3\) equations are derived from the symmetries/surfaces of type \(D^{(1)}_5/A^{(1)}_3\), \(A^{(1)}_4/A^{(1)}_4\), \(E^{(1)}_2/A^{(1)}_5\), \(E^{(1)}_2/A^{(1)}_6\) and \(A^{(1)}_1/A^{(1)}_7\), respectively.

The degeneration scheme of Painlevé equations is as follows.

\[
\begin{array}{cccccc}
\text{PVI} & \xrightarrow{y_1} & \text{PV} & \xrightarrow{y_2} & \text{P}_{\text{III}1} & \xrightarrow{y_3} \text{P}_{\text{III}2} & \xrightarrow{y_4} \text{P}_{\text{III}3} \\
& \xrightarrow{a_2} & \text{P}_{\text{IV}} & \xrightarrow{a_3} & \text{P}_{\text{II}} & \xrightarrow{a_4} & \text{P}_{\text{I}}
\end{array}
\]

The degeneration pattern of the \(q\)-Painlevé equations we use is similar to the one in [Mu] but not exactly same. Rather, our limiting procedure is a \(q\)-version for the one used in [GIL] in order to derive combinatorial expressions of tau functions of \(\text{PVI}, \text{P}_{\text{III}1}, \text{P}_{\text{III}2},\) and \(\text{P}_{\text{III}3}\) from the Nekrasov type expression of the tau function of \(\text{PVI} [GIL]\).

For the case of the \(q\)-Painlevé \(\text{III}_3\) equation, series representations for the tau functions was proposed in [BS1], which are expressed by \(q\)-Virasoro Whittaker conformal blocks which equal Nekrasov partition functions for pure \(SU(2)\) 5d theory [AY], [Y]. Our tau functions for the \(q\)-Painlevé \(\text{III}_3\) equation obtained by the degeneration are equivalent to them.

Our plan is as follows. In Section 2, we recall the result on \(q\)-Painlevé VI equation in [JNS]. In Section 3–6, we compute limits of tau functions and derive combinatorial expressions of general solutions and bilinear equations for \(q\)-Painlevé V, \(\text{III}_1\), \(\text{III}_2\) and \(\text{III}_3\) equations.

\textit{Notations.} Throughout the paper we fix \(q \in \mathbb{C}^\times\) such that \(|q| < 1\). We set
\[
[u] = (1 - q^u)/(1 - q), \quad (a; q)_N = \prod_{j=0}^{N-1} (1 - aq^j),
\]
\[
(a_1, \ldots, a_k; q)_\infty = \prod_{j=1}^{k} (a_j; q)_\infty, \quad (a; q, q)_\infty = \prod_{j,k=0}^{\infty} (1 - aq^{j+k}).
\]

We use the \(q\) Gamma function, \(q\) Barnes function and the theta function defined by
\[
\Gamma_q(u) = \frac{(q; q)_\infty}{(q^u; q)_\infty} (1 - q)^{-u}, \quad G_q(u) = \frac{(q^u; q, q)_\infty}{(q; q, q)_\infty} (q; q, q)_{\infty}^{-u-1}(1 - q)^{-(u-1)(u-2)/2},
\]
\[
\theta(u) = q^{u(u-1)/2} \Theta_q(q^u), \quad \Theta_q(x) = (x, q/x, q; q)_\infty,
\]
which satisfy $\Gamma_q(1) = G_q(1) = 1$ and

\[
\begin{align*}
\Gamma_q(u + 1) &= [u] \Gamma_q(u), \quad G_q(u + 1) = \Gamma_q(u) \Gamma_q(u), \\
\theta(u + 1) &= -\theta(u) = \theta(-u).
\end{align*}
\]

A partition is a finite sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_l)$ such that $\lambda_1 \geq \cdots \geq \lambda_l > 0$. Denote the length of the partition by $\ell(\lambda) = l$. The conjugate partition $\lambda' = (\lambda_1', \ldots, \lambda_l')$ is defined by $\lambda_j' = \# \{i | \lambda_i \geq j\}$, $\ell' = \lambda_1$. We regard a partition as a Young diagram. Namely, we regard a partition $\lambda$ also as the subset $\{(i, j) \in \mathbb{Z}^2 | 1 \leq j \leq \lambda_i, \ i \geq 1\}$ of $\mathbb{Z}^2$, and denote its cardinality by $|\lambda|$. We denote the set of all partitions by $\mathcal{P}$. For $\square = (i, j) \in \mathbb{Z}^2$ we set $a_1(\square) = \lambda_i - j$ (the arm length of $\square$) and $\ell_\lambda(\square) = \lambda_j' - i$ (the leg length of $\square$). In the last formulas we set $\lambda_i = 0$ if $i > \ell(\lambda)$ (resp. $\lambda_j' = 0$ if $j > \ell(\lambda')$). For a pair of partitions $(\lambda, \mu)$ and $u \in \mathbb{C}$ we set

\[
N_{\lambda, \mu}(u) = \prod_{\square \in \lambda} \left( 1 - q^{\ell_\lambda(\square) - a_\mu(\square)} - 1 u \right) \prod_{\square \in \mu} \left( 1 - q^{\ell_\mu(\square) + a_\lambda(\square)} + 1 u \right)
\]

which we call a Nekrasov factor.

2. Results on $q$-P$_{VI}$ from [JNS]

In this section, we recall the results of [JNS] on the $q$-Painlevé VI equation. Define the tau function by

\[
\tau_{VI}^{\lambda} \left[ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\theta_8 \\
\theta_9 \\
\theta_{10}
\end{array} \right] = \sum_{n \in \mathbb{Z}} s^{n} t^{(n+e^\lambda - \theta_2^\lambda - \theta_4^\lambda) / 2} C \left[ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\theta_8 \\
\theta_9 \\
\theta_{10}
\end{array} \right] \frac{\prod_{\ell, e' = \pm} G_q(1 + e \theta_\infty - \theta_1 - e' \theta_0) G_q(1 + e \sigma - \theta_1 + e' \theta_0)}{G_q(1 + 2 \sigma) G_q(1 - 2 \sigma)}
\]

with the definition

\[
C \left[ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\theta_8 \\
\theta_9 \\
\theta_{10}
\end{array} \right] = \frac{\prod_{e, e' = \pm} \left( q^\epsilon_\infty - \theta_1 - e' \theta_0 \right) \prod_{e, e' = \pm} \left( q^\epsilon_\infty - \theta_1 + e' \theta_0 \right) N_{\lambda, \mu}(q^\epsilon_\infty - \theta_1 - e' \theta_0)}{\prod_{e, e' = \pm} \left( q^\epsilon_\infty - \theta_1 + e' \theta_0 \right) N_{\lambda, \mu}(q^\epsilon_\infty - \theta_1 + e' \theta_0)}
\]

Put

\[
\begin{align*}
\tau_{VI}^{\lambda} &= \tau_{VI}^{\lambda} \left[ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\theta_8 \\
\theta_9 \\
\theta_{10}
\end{array} \right], \quad \tau_{VI}^{\lambda} = \tau_{VI}^{\lambda} \left[ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\theta_8 \\
\theta_9 \\
\theta_{10}
\end{array} \right], \\
\tau_{VI}^{\lambda} &= \tau_{VI}^{\lambda} \left[ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\theta_8 \\
\theta_9 \\
\theta_{10}
\end{array} \right], \quad \tau_{VI}^{\lambda} = \tau_{VI}^{\lambda} \left[ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\theta_8 \\
\theta_9 \\
\theta_{10}
\end{array} \right], \\
\tau_{VI}^{\lambda} &= \tau_{VI}^{\lambda} \left[ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\theta_8 \\
\theta_9 \\
\theta_{10}
\end{array} \right], \quad \tau_{VI}^{\lambda} = \tau_{VI}^{\lambda} \left[ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\theta_8 \\
\theta_9 \\
\theta_{10}
\end{array} \right], \\
\tau_{VI}^{\lambda} &= \tau_{VI}^{\lambda} \left[ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\theta_8 \\
\theta_9 \\
\theta_{10}
\end{array} \right], \quad \tau_{VI}^{\lambda} = \tau_{VI}^{\lambda} \left[ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\theta_8 \\
\theta_9 \\
\theta_{10}
\end{array} \right].
\]

Here and after we write $\tilde{f}(t) = f(qt), \ f(t) = f(t/q)$.

**Theorem 2.1. [JNS]** The functions $y$ and $z$ defined by

\[
y = q^{-\theta_1 - 1} t; \quad \tau_{VI}^{\lambda} \tau_{VI}^{\lambda} = \tau_{VI}^{\lambda} \tau_{VI}^{\lambda} - \tau_{VI}^{\lambda} \tau_{VI}^{\lambda}, \quad z = \frac{\tau_{VI}^{\lambda} \tau_{VI}^{\lambda} - \tau_{VI}^{\lambda} \tau_{VI}^{\lambda}}{\tau_{VI}^{\lambda} \tau_{VI}^{\lambda} - \tau_{VI}^{\lambda} \tau_{VI}^{\lambda}}
\]

are solutions to the $q$-Painlevé VI equation

\[
\begin{align*}
\frac{y}{a_3 a_4} &= \frac{(\overline{z} - b_1) (\overline{z} - b_2) t}{(\overline{z} - b_3) (\overline{z} - b_4)}, \quad \frac{z \overline{z}}{b_3 b_4} = \frac{(y - a_1) (y - a_2) t}{(y - a_3) (y - a_4)},
\end{align*}
\]
with the parameters
\[
\begin{align*}
a_1 &= q^{-2\theta_1-1}, \quad a_2 = q^{-2\theta_1-2\theta_1-1}, \quad a_3 = q^{-1}, \quad a_4 = q^{-2\theta_1-1}, \\
b_1 &= q^{-\theta_0-\theta_1-\theta_1}, \quad b_2 = q^{0_0-\theta_1-\theta_1}, \quad b_3 = q^{0_0-1/2}, \quad b_4 = q^{-\theta_0-1/2}.
\end{align*}
\]

Theorem 2.1 was obtained by constructing the fundamental solution of the Lax-pair for \(q\)-PVI in [JS], in terms of \(q\)-conformal blocks in [AFS]. The method of construction of the fundamental solution is a \(q\) analogue of the CFT approach used in [ILT]. In the derivation of Theorem 2.1 convergence of the fundamental solution was assumed and it has not been proved. Recently, analyticity of K-theoretic Nekrasov functions in certain domains was discussed in [FM].

**Conjecture 2.2.** [NS] The tau functions \(\tau_i^{VI}\) \((i = 1, \ldots, 8)\) satisfy the following bilinear equations.

\[
\begin{align*}
(2.3) & \quad \tau_1^{VI} \tau_2^{VI} - q^{-2\theta_1} t \tau_3^{VI} \tau_4^{VI} - (1 - q^{-2\theta_1}) t \tau_5^{VI} \tau_6^{VI} = 0, \\
(2.4) & \quad \tau_1^{VI} \tau_2^{VI} - t \tau_3^{VI} \tau_4^{VI} - (1 - q^{-2\theta_1}) t \tau_5^{VI} \tau_6^{VI} = 0, \\
(2.5) & \quad \tau_1^{VI} \tau_2^{VI} - \tau_3^{VI} \tau_4^{VI} + (1 - q^{-2\theta_1}) t \tau_5^{VI} \tau_7^{VI} r_8^{VI} = 0, \\
(2.6) & \quad \tau_1^{VI} \tau_2^{VI} - q^{-2\theta_1} \tau_3^{VI} \tau_4^{VI} + (1 - q^{-2\theta_1}) t \tau_5^{VI} \tau_7^{VI} \tau_8^{VI} = 0, \\
(2.7) & \quad \tau_5^{VI} \tau_6^{VI} + q^{-\theta_0-\theta_1-\theta_0} t \tau_7^{VI} \tau_8^{VI} - \tau_1^{VI} \tau_2^{VI} = 0, \\
(2.8) & \quad \tau_5^{VI} \tau_6^{VI} + q^{-\theta_0+\theta_0+\theta_1-\theta_0} t \tau_7^{VI} \tau_8^{VI} - \tau_1^{VI} \tau_2^{VI} = 0, \\
(2.9) & \quad \tau_5^{VI} \tau_6^{VI} + q^{\theta_0+2\theta_1} \tau_7^{VI} \tau_8^{VI} - \tau_1^{VI} \tau_2^{VI} = 0, \\
(2.10) & \quad \tau_5^{VI} \tau_6^{VI} + q^{\theta_0+2\theta_1} \tau_7^{VI} \tau_8^{VI} - \tau_1^{VI} \tau_2^{VI} = 0.
\end{align*}
\]

Then, the function \(y,z\)

\[
y = q^{-2\theta_1-1} t \tau_3^{VI} \tau_4^{VI} \tau_1^{VI} \tau_2^{VI}, \quad z = -q^{\theta_0-\theta_1-1} t \tau_5^{VI} \tau_6^{VI} \tau_7^{VI} \tau_8^{VI}.
\]

solves \(q\)-PVI 2.2.

The function \(y\) in Conjecture 2.2 is expressed as the same form in Theorem 2.1 while the function \(z\) in Conjecture 2.2 is not. By the bilinear equations (2.7) and (2.8), we obtain the expression of \(z\) in (2.11) from the expression of \(z\) in (2.1).

We note that in [NS] we have a Lax pair with respect to the shift \(t \to qt\), namely, a fundamental solution of the linear \(q\)-difference equations

\[
(2.12) \quad Y(qx, t) = A(x, t) Y(x, t), \quad Y(x, qt) = B(x, t) Y(x, t)
\]

for certain 2 by 2 matrices \(A(x, t)\) and \(B(x, t)\) was constructed in terms of \(q\)-Nekrasov functions. From (2.12) we obtain the four-term bilinear equation in Remark 3.5 of [NS]:

\[
(2.13) \quad \tau_1^{VI} \tau_2^{VI} - \tau_1^{VI} \tau_2^{VI} = \frac{q_1^{1/2+\theta_0} - q_1^{1/2-\theta_0}}{q^{-\theta_0} - q^{\theta_0}} q^{\theta_0-1} t \left( \tau_3^{VI} \tau_4^{VI} - \tau_3^{VI} \tau_4^{VI} \right).
\]

3. From \(q\)-PVI to \(q\)-PV

In this section, we take a limit of the tau functions of \(q\)-PVI to \(q\)-PV. For \(\tilde{\theta} = (\theta_*, \theta_0)\) and \(\tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)\) we set

\[
\tau(\tilde{\theta}, \tilde{\alpha}, s, \sigma, t) = \sum_{n \in \mathbb{Z}} s^n t^{(\sigma+\alpha_5+n)^2-(\theta_1+\alpha_3)^2-(\theta_0+\alpha_4)^2} C_V \left[ \tilde{\theta}, \tilde{\alpha}, \sigma, n \right]
\]
Proposition 3.1. 

\[ \text{Proof.} \]

Let \( Z = \prod_{\varepsilon = \pm} G_q(1 - \theta_1 - \alpha_1 - \varepsilon(\sigma + \alpha_5 + n)) \]

Then we have

\[ Z \left[ \tilde{\theta}, \tilde{\sigma}, \varepsilon \right] = \sum_{(\lambda_+, \lambda_-) \in \mathcal{Y}^2} t^{\lambda_+ + \lambda_-} \prod_{\varepsilon \in \Lambda} \frac{N_{\phi, \lambda}(q^{-1})^{\lambda_+ + \lambda_-}}{N_{\lambda_+ \lambda_-}(q^{\lambda_+ - \lambda_-})}, \]

where

\[ f_\lambda(u) = \prod_{\square \in \lambda} \left( -q^{\ell(\square) + a_\phi(\square) + 1} u^{-1} \right). \]

We set

\[ a_1 = 1/2, 0, 0, 0, 0, 0, \quad a_2 = -1/2, 0, 0, 0, 0, 0, 0, \quad a_3 = 0, 0, 0, 1/2, 1/2, 1/2, 1/2, 1/2, \quad a_4 = 0, 0, 0, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2. \]

We define tau functions for \( q\)-P\_V by

\[ \tau_i^V(\tilde{\theta}, s, \sigma, t) = \tau_i(\tilde{\theta}, a_1, s, \lambda, t) \quad (i = 1, \ldots, 8). \]

Let

\[ C(\tilde{\alpha}) = q^{-\Lambda(\alpha + \alpha_5)^2 - (\theta_0 + \alpha_3 - \alpha_5)^2} \prod_{\varepsilon = \pm} G_q(1 - \theta_1 - \alpha_1 - \alpha_2 + \varepsilon(\sigma + \alpha_5))^{-1}. \]

**Proposition 3.1.** Set

\[ \theta_1 + \theta_\infty = \Lambda, \quad \theta_1 - \theta_\infty = \theta_*, \quad t = q^\Lambda t_1, \quad s = s q^{-2\Lambda} \prod_{\varepsilon = \pm} \Gamma_q(-\theta_1 - \theta_1 + \varepsilon \sigma + \frac{1}{2})^{-\varepsilon}. \]

Then we have

\[ C(a_i) \tau_i^V(\tilde{\theta}, s, \sigma, t) \rightarrow \tau_i^V(\tilde{\theta}, s, \sigma, t_1) \quad (\Lambda \rightarrow \infty) \]

for \( i = 1, \ldots, 8 \). Here, we denote by \( \tau_i^V(\tilde{\theta}, s, \sigma, t) \) the tau functions of \( q\)-P\_V\_I presented in the previous section.

**Proof.** First, we verify the limit of the series part. For any partition \( \lambda \) we have

\[ N_{\phi, \lambda}(q^{-\Lambda} u) q^{\Lambda[\lambda]} = \prod_{\square \in \lambda} \left( q^{\Lambda} - q^{\ell(\square) + a_\phi(\square) + 1} u \right) \rightarrow f_\lambda(u^{-1}) \quad (\Lambda \rightarrow \infty). \]

Hence, the series \( Z \left[ \frac{\theta_1}{\theta_0}, \theta_i | \sigma, t \right] \) goes to \( Z_V \left[ (\theta_1, \theta_1, \theta_0), \sigma, t \right] \) as \( \Lambda \rightarrow \infty \).

Second, we examine the limits of the coefficients of \( Z \). By the identities \( \text{(1.1)} \) on \( q \) Gamma function and \( q \) Barnes function, for \( n \in \mathcal{Z} \) we have

\[ \prod_{\varepsilon = \pm} G_q(1 - x + \varepsilon(\sigma + n)) = \prod_{\varepsilon = \pm} G_q(1 - x + \varepsilon \sigma) \Gamma_q(-x + \varepsilon \sigma)^{\varepsilon n} \]

\[ \times \prod_{i=0}^{[n]-1} \prod_{j=0}^{[n]-1} [-x + \sigma + j] \prod_{i=0}^{[n]-1} (-x - \sigma - j). \]

Using the identity above, we compute the coefficient of \( Z \) in \( \tau_i^V \) multiplied by \( C(a_1) \) as follows.

\[ C(a_1) s^n C \left[ \frac{\theta_1}{\theta_\infty} + \frac{\theta_1}{\theta_0} | \sigma + n \right] t^{(\sigma + n)^2 - \theta_1^2 - \theta_0^2} \]
The functions

\[ y = q^{-\theta_*-1} t \frac{\tau_3 \tau_4}{\tau_1 \tau_2}, \quad z = -\frac{\tau_1 \tau_2 - \tau_1 \tau_2}{q^{\theta_*/2+1/2} \tau_1 \tau_2} \]

solves the \( q \)-Painlevé \( V \) equation

\[ \frac{y}{a_5 a_4} = -\frac{(z - b_1 t)(z - b_2 t)}{z - b_3}, \quad \frac{z}{a_4 (y - a_3)} = -\frac{(y - a_1 t)(y - a_2 t)}{a_4 (y - a_3)} \]

with the parameters

\[
\begin{align*}
  a_1 &= q^{-\theta_*-1}, \quad a_2 = q^{-\theta_*-1}, \quad a_3 = q^{-1}, \quad a_4 = q^{-3\theta_*/2-1/2}, \\
  b_1 &= q^{-\theta_*-1}, \quad b_2 = q^{-\theta_*-1}, \quad b_3 = q^{-\theta_*-1}.
\end{align*}
\]

\[ \lim_{\lambda \to \infty} y \rightarrow y_1 = q^{-\theta_*-1} t \frac{\tau_3 \tau_4}{\tau_1 \tau_2}, \quad q^{-\lambda/2} z \rightarrow z_1 = -\frac{\tau_1 \tau_2 - \tau_1 \tau_2}{q^{\theta_*/2+1/2} \tau_1 \tau_2} \]

It is easy to see that the \( q \)-Painlevé \( V \) equation \( (3.3) \) degenerates the \( q \)-Painlevé \( V \) equation \( (2.2) \) for \( y = y_1 \) and \( z = z_1 \) as \( \Lambda \rightarrow \infty \).

Since we also have

\[ C(a_5)C(a_6) = C(a_7)C(a_8), \quad C(a_1)C(a_2) = C(a_5)C(a_6), \]

we obtain the following conjecture.

**Conjecture 3.3.** The tau functions \( \tau_i \) (\( i = 1, \ldots, 8 \)) satisfy the following bilinear equations.

\[
\begin{align*}
  \tau_1 \tau_2 - q^{-\theta_*} t \tau_3 \tau_4 - (1 - q^{-\theta_*} t) \tau_5 \tau_6 &= 0, \\
  \tau_1 \tau_2 - \tau_3 \tau_4 + (1 - q^{-\theta_*} t) q^{2\theta_* \tau_7} \tau_8 &= 0, \\
  \tau_1 \tau_2 - q^{2\theta_* \tau_3} \tau_4 + q^{2\theta_* \tau_7} \tau_8 &= 0, \\
  \tau_5 \tau_6 + q^{-1/2 \tau_7} \tau_8 - \tau_1 \tau_2 &= 0, \\
  \tau_5 \tau_6 + q^{\theta_*+1/2 \tau_7} \tau_8 - q^{\theta_* \tau_3} \tau_4 &= 0.
\end{align*}
\]
Proposition 3.4. We have

(3.10) \[ \tau_1 \tau_2 - \tau_5 \tau_6 = 0, \]

(3.11) \[ \tau_5 \tau_6 - \tau_1 \tau_2 = 0, \]

(3.12) \[ \tau_1 \tau_2 - \tau_1 \tau_2 = -q^{-1/2} \frac{t}{q^{\theta_0} - q^{-\theta_0}} \left( \tau_3 \tau_4 - \tau_3 \tau_4 \right). \]

Proof. By definition we have

\[ \tau_5 = q^{-\sigma^2 + \theta^2_1 + \theta^2_2} t_1, \]
\[ \tau_6 = q^{\sigma^2 - \theta^2_1 - \theta^2_2} t_2. \]

Hence we obtain the identity (3.10) and (3.11).

The identity (3.12) is a direct consequence of (2.13) by the limit (3.1) as \( \Lambda \to \infty. \) \( \square \)

4. FROM \( q \)-P\( \text{V} \) TO \( q \)-P\( \text{III}_1 \)

In this section, we take a limit of the tau functions of \( q \)-P\( \text{V} \) to \( q \)-P\( \text{III}_1 \). For \( \tilde{\theta} = (\theta_*, \theta_*) \) and \( \tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \) we set

\[ \tau(\tilde{\theta}, \tilde{\alpha}, s, \sigma, t) = \sum_{n \in \mathbb{Z}} s^n t^{(\sigma + a_5 + n)^2 - a_5^2 - a_4^2} C_{\text{III}_1} \left[ \tilde{\theta}, \tilde{\alpha}, \sigma, n \right] \times \left[ \theta_* - \alpha_1 + \alpha_2, \theta_* + \alpha_3 - \alpha_4, \sigma + \alpha_5 + n, q^{-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4} t \right], \]

with

\[ C_{\text{III}_1} \left[ \tilde{\theta}, \tilde{\alpha}, \sigma, n \right] = (q - 1)^{-2n(n + 2a_5)} \frac{q^{-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} n(n + 2\sigma + \sigma n - 2\alpha_3 + \alpha_4 a_5)}{G_q(1 + 2\epsilon(\sigma + a_5 + n))}. \]

Let us define the tau functions for \( q \)-P\( \text{III}_1 \) by

\[ \tau_i^{\text{III}_1} (\tilde{\theta}, s, \sigma, t) = \tau(\tilde{\theta}, a_i, s, \sigma, t) \quad (i = 1, \ldots, 8). \]

Put

\[ C(\tilde{\alpha}) = q^{-\Lambda((\sigma + a_5)^2 - a_5^2 - a_4^2)} \left( t_1 \right)^{2\theta_3} t_0^2 + 2\alpha_3 t_1 + 2a_4 n \prod_{\epsilon = \pm} G_q(1 - \theta_0 - \theta_1 - \alpha_3 - \alpha_4 + \epsilon(\sigma + a_5))^{-1}. \]

Proposition 4.1. Set

(4.1) \[ \theta_t + \theta_0 = \Lambda, \quad \theta_t - \theta_0 = \theta_*, \quad t = q^\Lambda t_1, \quad s = \tilde{s} q^{-2\sigma \Lambda} \prod_{\epsilon = \pm} \Gamma_q \left( -\theta_0 - \theta_1 + \epsilon \sigma \right)^{-\epsilon}. \]

Then we have

\[ C(a_i) \tau_i^\text{V} \to \tau_i^{\text{III}_1} \quad (\Lambda \to \infty) \]

for \( j = 1, \ldots, 8. \) \( \square \)
Proposition 4.2. We have
\[
\begin{align*}
\tau_1 \tau_2 - \tau_5 \tau_6 &= 0, \\
q^{-\sigma} \tau_3 \tau_4 - q^\sigma \tau_7 \tau_8 &= 0, \\
\tau_5 \tau_6 - \tau_1 \tau_2 &= 0, \\
q^\sigma \tau_7 \tau_8 - q^{-\sigma} \tau_3 \tau_4 &= 0, \\
\tau_1 \tau_2 - \tau_1 \tau_2 &= \frac{q^{-\sigma-1/2}}{q-1} t \tau_3 \tau_4.
\end{align*}
\]

The limit of (3.12) by (4.1) as \( \Lambda \to \infty \) is the identity (4.4).

Theorem 4.3. The functions
\[
y = -\frac{q^{-\theta_* - \sigma - 1}}{1 - q} \tau_3 \tau_4, \quad z = \frac{q^{-\theta_* + 2 + \sigma - 1}}{1 - q} \tau_7 \tau_8
\]
solve the q-Painlevé III\(_1\) equation
\[
\frac{y\overline{y}}{a_3 a_4} = -\frac{z(z - b_2 t)}{z - b_3}, \quad \frac{z\overline{z}}{b_3} = -\frac{y(y - a_2 t)}{a_4(y - a_3)}
\]
with the parameters
\[
a_2 = q^{-\theta_* - \theta_* - 1}, \quad a_3 = q^{-1}, \quad a_4 = q^{-3\theta_* - 2 - 1/2}, \\
b_2 = q^{-\theta_* - 1/2}, \quad b_3 = q^{-\theta_* - 1/2 - 1/2}.
\]

Furthermore, the tau functions \( \tau_i \) (\( i = 1, \ldots, 8 \)) satisfy the following bilinear equations.
\[
\begin{align*}
(1 - q)\tau_1 \tau_2 + q^{\theta_* + \sigma} t \tau_3 \tau_4 - (1 - q) \tau_5 \tau_6 &= 0, \\
(1 - q)\tau_1 \tau_2 + q^{\theta_* - \sigma} \tau_3 \tau_4 - q^{\theta_* + \sigma} \tau_7 \tau_8 &= 0, \\
(1 - q)\tau_5 \tau_6 - q^{\sigma - 1/2} t \tau_7 \tau_8 - (1 - q) \tau_1 \tau_2 &= 0, \\
(1 - q)\tau_5 \tau_6 - q^{\sigma - 1/2} \tau_7 \tau_8 + q^{-\sigma} \tau_3 \tau_4 &= 0.
\end{align*}
\]

Proof. By definition we have
\[
C(a_1) = C(a_2), \quad C(a_1)C(a_2) = \frac{(1 - q^{-\theta_* - \theta_* - 1})}{1 - q} C(a_3)C(a_4).
\]
Hence, by (4.1) the solution \((y, z)\) of the \(q\)-Painlevé V equation degenerates as
\[
y \to y_1 = -\frac{q^{\theta_+ - \sigma - 1}}{1 - q} t \frac{\tau_3 \tau_4}{\tau_1 \tau_2}, \quad z \to z_1 = -\frac{\tau_1 \tau_2 - \tau_1 \tau_2}{q^{\theta_+ + 1/2} \tau_1 \tau_2} \quad (\Lambda \to \infty).
\]

Also, the \(q\)-Painlevé V equation (3.3) degenerates the \(q\)-Painlevé III_1 equation (4.6) for \(y = y_1\) and \(z = z_1\) as \(\Lambda \to \infty\). From (4.2), (4.3), and (4.4), we obtain the expression of \(z\) in (4.5).

Next we prove the bilinear equations (4.7)-(4.10). Substituting \(\tau_5 \tau_6\) and \(\tau_7 \tau_8\) by (4.2) and (4.3) into (4.4), we obtain the identity (4.9). The identity (4.10) is obtained by substituting the expression (4.5) of \((y, z)\) into the \(q\)-Painlevé III_1 equation:
\[
y \frac{\sqrt{y}}{a_3 a_4} = -\frac{\overline{\mathcal{Z}} - b_2 t}{\mathcal{Z} - b_3},
\]
and using the already proved bilinear equations (4.2), (4.3), and (4.9).

In order to prove (4.7) and (4.8), we use the following transformation
\[
(4.11) \quad (\tilde{\theta}_*, \tilde{\theta}_*, \tilde{\sigma}, \tilde{s}, \tilde{t}) = (-\theta_+, -\theta_+, \sigma - \frac{1}{2}, C_s, q^{-\theta_+ - 1} t)\]
where
\[
C = (q - 1)^2 q^{(\sigma - 1)(1 + 2\theta_+ + 2\theta_+) + 1/2},
\]

Then by straightforward calculations we have \(\tilde{\tau}_{2i} = K_{8-2i} \tau_{8-2i}, \tau_{2i-1} = K_{9-2i} \tau_{9-2i} (i = 1, 2, 3, 4)\), where we denote by \(\tilde{\tau}_i\) the tau functions with parameters \((\tilde{\theta}_*, \tilde{\theta}_*, \tilde{\sigma}, \tilde{s}, \tilde{t})\) and by \(\tau_i\) the tau functions with parameters \((\theta_*, \theta_*, \sigma, s, t)\), and

\[
K_1 = s t^{-1/4} q^{1/2(1 + 2\theta_+ + 2\theta_+)(1 - 2\sigma^2/2 + (1 + 2\theta_+ + 2\theta_+)(7 - 6\theta_+ - 6\theta_+))/8} (q - 1)^2
\]
\[
\times \frac{G_q(\theta_* + \sigma - \frac{1}{2}) G_q(\theta_* + \sigma + \frac{3}{2})}{G_q(\theta_* - \sigma + \frac{1}{2}) G_q(\theta_* + \sigma + \frac{1}{2}) G_q(-\theta_* - \sigma + 2) G_q(-\theta_* + \sigma + 2)},
\]

\[
K_2 = t^{1/4} q^{1/2(1 + 2\theta_+ + 2\theta_+ + 1 - 4\sigma^2)/8} \frac{G_q(\theta_* + \sigma + \frac{3}{2}) G_q(\theta_* - \sigma + \frac{3}{2})}{G_q(-\theta_* + \sigma + \frac{3}{2}) G_q(-\theta_* - \sigma + \frac{3}{2}) G_q(-\theta_* + \sigma + 1) G_q(-\theta_* - \sigma + 1)}
\]
\[
\times \frac{G_q(\theta_* + \sigma + \frac{1}{2}) G_q(\theta_* - \sigma + \frac{1}{2}) G_q(-\theta_* - \sigma + 2) G_q(-\theta_* + \sigma + 2)},
\]

\[
K_3 = s t^{1/4} q^{1/2(1 - 2\theta_+ - 2\theta_+)(1 - 2\sigma^2)/8} (q - 1)^2
\]
\[
\times \frac{G_q(\theta_* + \sigma + \frac{1}{2}) G_q(\theta_* - \sigma + \frac{1}{2})}{G_q(-\theta_* + \sigma + \frac{3}{2}) G_q(-\theta_* + \sigma + \frac{1}{2}) G_q(-\theta_* + \sigma + 2) G_q(-\theta_* - \sigma + 2)}
\]
\[
\times \frac{G_q(\theta_* + \sigma + \frac{3}{2}) G_q(\theta_* - \sigma + \frac{3}{2})}{G_q(-\theta_* + \sigma + \frac{3}{2}) G_q(-\theta_* + \sigma - \frac{1}{2}) G_q(-\theta_* + \sigma + 1) G_q(-\theta_* - \sigma + 1)}
\]
\[
\times \frac{G_q(\theta_* + \sigma - \frac{1}{2}) G_q(\theta_* - \sigma - \frac{1}{2}) G_q(-\theta_* - \sigma + 2) G_q(-\theta_* + \sigma + 2)},
\]

\[
K_4 = t^{1/4} q^{1/2(1 + 2\theta_+ + 2\theta_+ + 1 - 2\sigma^2)/8} \frac{G_q(\theta_* + \sigma + \frac{3}{2}) G_q(\theta_* - \sigma + \frac{3}{2})}{G_q(-\theta_* + \sigma + \frac{3}{2}) G_q(-\theta_* + \sigma - \frac{1}{2}) G_q(-\theta_* + \sigma + 1) G_q(-\theta_* - \sigma + 1)}
\]
\[
\times \frac{G_q(\theta_* + \sigma + \frac{1}{2}) G_q(\theta_* - \sigma + \frac{1}{2}) G_q(-\theta_* - \sigma + 2) G_q(-\theta_* + \sigma + 2)},
\]

\[
K_5 = s t^{-1/4} q^{1/2(1 + 2\theta_+ - 2\theta_+)(1 + 2\theta_+ + 2\theta_+)(1 - 6\theta_+ + 6\theta_+)(1 - 6\theta_+ + 6\theta_+)/8} (q - 1)^2
\]
\[
\times \frac{G_q(\theta_* + \sigma + \frac{3}{2}) G_q(\theta_* - \sigma + \frac{3}{2})}{G_q(-\theta_* + \sigma + \frac{3}{2}) G_q(-\theta_* + \sigma - \frac{1}{2}) G_q(-\theta_* + \sigma + 1) G_q(-\theta_* - \sigma + 1)}
\]
\[
\times \frac{G_q(\theta_* + \sigma - \frac{1}{2}) G_q(\theta_* - \sigma - \frac{1}{2}) G_q(-\theta_* - \sigma + 2) G_q(-\theta_* + \sigma + 2)},
\]

\[
K_6 = t^{1/4} q^{1/2(1 + 2\theta_+ + 2\theta_+)(1 - 2\sigma^2)/8} \frac{G_q(\theta_* + \sigma + \frac{3}{2}) G_q(\theta_* - \sigma + \frac{3}{2})}{G_q(-\theta_* + \sigma + \frac{3}{2}) G_q(-\theta_* + \sigma - \frac{1}{2}) G_q(-\theta_* + \sigma + 1) G_q(-\theta_* - \sigma + 1)}
\]
\[
\times \frac{G_q(\theta_* + \sigma + \frac{1}{2}) G_q(\theta_* - \sigma + \frac{1}{2}) G_q(-\theta_* - \sigma + 2) G_q(-\theta_* + \sigma + 2)},
\]

\[
K_7 = s t^{1/4} q^{1/2(1 + 2\theta_+ - 2\theta_+)(1 - 2\sigma^2)/8} \frac{G_q(\theta_* + \sigma + \frac{3}{2}) G_q(\theta_* - \sigma + \frac{3}{2})}{G_q(-\theta_* + \sigma + \frac{3}{2}) G_q(-\theta_* + \sigma - \frac{1}{2}) G_q(-\theta_* - 2 + 2) G_q(-\theta_* + 2)}
\]
\[
\times \frac{G_q(\theta_* + \sigma - \frac{1}{2}) G_q(\theta_* - \sigma - \frac{1}{2}) G_q(-\theta_* - 2 + 2) G_q(-\theta_* + 2)}
\]
we set

\[ \text{we obtain the following theorem by the degeneration.} \]

\[ K_8 = t^{1/4} q^{-(1+2\alpha_5+2\alpha_4)(1-2\alpha)^2/8} \frac{G_q(\theta_\ast + \sigma + \frac{1}{2}) G_q(\theta_\ast - \sigma + \frac{3}{2})}{G_q(-\theta_\ast - \sigma + \frac{3}{2}) G_q(-\theta_\ast + \sigma + \frac{1}{2})} \frac{G_q(\theta_\ast + \sigma + 1) G_q(\theta_\ast - \sigma + 2)}{G_q(-\theta_\ast + \sigma + 1) G_q(-\theta_\ast + \sigma + 2)}. \]

By definition we have

\[ (4.12) \quad \frac{K_3 K_4}{K_7 K_8} = q^{-2\sigma}, \quad \frac{K_1 K_2}{K_7 K_8} = -q^{\theta_\ast - 2\sigma + 1/2}, \]

\[ (4.13) \quad \frac{K_5 K_6}{K_3 K_4} = q^{\theta_\ast + 2\sigma - 1/2} t^{-1}, \quad \frac{K_1 K_2}{K_3 K_4} = -q^{\theta_\ast + 1/2} t^{-1}. \]

Applying the transformation (4.11) to the bilinear equations (4.9) and (4.10) and using the relations (4.12) and (4.13), we obtain the identities (4.7) and (4.8).

We note that the bilinear equations (3.4), (3.6), (3.7), and (3.8) for $q$-$P_V$ tau functions degenerate (4.7), (4.8), (4.9), and (4.10), respectively.

5. FROM $q$-$P_{III}$ TO $q$-$P_{III}$

In this section, we take a limit of the tau functions of $q$-$P_{III_1}$ to $q$-$P_{III_2}$. For $\theta_\ast$ and $\tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ we set

\[ \tau(\theta_\ast, \tilde{\alpha}, s, \sigma, t) = \sum_{n \in \mathbb{Z}} s^n t^{(\sigma + \alpha_5 + n)^2 - \alpha_3^2 - \alpha_4^2} C_{III_2}[\theta_\ast, \tilde{\alpha}, \sigma, n] Z_{III_2}[\theta_\ast - \alpha_1 + \alpha_2, \sigma + \alpha_5 + n, q^{-\alpha_1 - \alpha_2 - 2\alpha_3} t], \]

with

\[ C_{III_2}[\theta_\ast, \tilde{\alpha}, \sigma, n] = (q - 1)^{-3n(n + 2\alpha_5)} q^{-(\alpha_1 + \alpha_2 + 2\alpha_3)n(n + 2\sigma) + 2n - 4\alpha_3 n} \prod_{\varepsilon = \pm} G_q(1 - \theta_\ast + \alpha_1 - \alpha_2 + \varepsilon(\sigma + \alpha_5 + n)) G_q(1 + 2\varepsilon(\sigma + \alpha_5 + n))^2, \]

\[ Z_{III_2}[\theta_\ast, \sigma, t] = \sum_{(\lambda_+, \lambda_-) \in \mathbb{Z}^2} t^{(\lambda_+ + |\lambda_-|)} \prod_{\varepsilon = \pm} N_{\lambda_+, \lambda_-}(q^{-\theta_\ast + \varepsilon \sigma}) f_{\lambda_+}(q^{\varepsilon \sigma})^{-1} / \prod_{\varepsilon, \varepsilon' = \pm} N_{\lambda_+ + \lambda_-}(q^{(\varepsilon - \varepsilon') \sigma}). \]

Let us define the tau functions for $q$-$P_{III_2}$ by

\[ \tau_{III_2}^j(\theta_\ast, s, \sigma, t) = \tau(\theta_\ast, a_i, s, \sigma, t) \quad (i = 1, \ldots, 8). \]

Put

\[ C(\tilde{\alpha}) = q^{-\Lambda((\sigma + \alpha_5)^2 - \alpha_3^2 - \alpha_4^2)} \prod_{\varepsilon = \pm} G_q(1 - \theta_\ast - \alpha_3 + \alpha_4 + \varepsilon(\sigma + \alpha_5))^{-1}. \]

**Proposition 5.1.** Set

\[ \theta_\ast = \Lambda, \quad t = q^{\Lambda} t_1, \quad s = \bar{s} q^{-2\sigma \Lambda} \prod_{\varepsilon = \pm} \Gamma_q(-\theta_\ast + \frac{1}{2} + \varepsilon \sigma)^{-\varepsilon}. \]

Then we have

\[ C(a_j) \tau_{III_1}^j(\theta_\ast, \theta_\ast, s, \sigma, t) \to \tau_{III_2}^j(\theta_\ast, \bar{s}, \sigma, t_1) \quad (\Lambda \to \infty) \]

for $j = 1, \ldots, 8$. \:

In what follows, we abbreviate $\tau_{III_2}^j(\theta_\ast, s, \sigma, t)$ to $\tau_i$. Since we have the relation

\[ C(a_1)C(a_2) = \frac{1 - q^{-\Lambda + \sigma}}{1 - q} C(a_3)C(a_4), \]

we obtain the following theorem by the degeneration.
Theorem 5.2. The functions

\[
y = \frac{q^{-\theta_*-1}}{(1-q)^2} \frac{\tau_3 \tau_4}{\tau_1 \tau_2}, \quad z = -\frac{q^{-\theta_*/2+2\sigma-1}}{(1-q)^2} \frac{\tau_7 \tau_8}{\tau_5 \tau_6}
\]

solve the q Painlevé III\textsubscript{2} equation

\[
\frac{y \overline{y}}{a_3 a_4} = -\frac{\overline{z}^2}{z - b_3}, \quad \frac{z \overline{z}}{b_3} = -\frac{y(y - a_2 t)}{a_4 (y - a_3)}
\]

with the parameters

\[
a_2 = q^{-\theta_*-1}, \quad a_3 = q^{-1}, \quad a_4 = q^{-3 \theta_*/2-1/2}, \quad b_2 = q^{-\theta_*/2}, \quad b_3 = q^{-\theta_*/2-1/2}.
\]

Furthermore, the tau functions \(\tau_i\) (\(i = 1, \ldots, 8\)) satisfy the following bilinear equations.

\[
(1-q)^2 \tau_1 \tau_2 - t q^{-\theta_*} \tau_3 \tau_4 - (1-q)^2 \tau_5 \tau_6 = 0,
\]

\[
(1-q)^2 \tau_1 \tau_2 - \tau_3 \tau_4 + q^{2 \sigma} \tau_7 \tau_8 = 0,
\]

\[
\tau_5 \tau_6 + q^{2 \sigma-1/2} (1-q)^2 t \tau_7 \tau_8 - \tau_1 \tau_2 = 0,
\]

\[\Box\]

6. FROM q-P\textsubscript{III\textsubscript{2}} TO q-P\textsubscript{III\textsubscript{3}}

In this section, we take a limit of the tau functions of q-P\textsubscript{III\textsubscript{2}} to q-P\textsubscript{III\textsubscript{3}}. For \(\bar{a} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)\) we set

\[
\tau(\bar{a}, s, \sigma, t) = \sum_{n \in \mathbb{Z}} s^n t^{(\sigma + \alpha_5 + n)^2 - \alpha_3^2 - \alpha_2^2} C_{\text{III\textsubscript{3}}} [\bar{a}, \sigma, n] Z_{\text{III\textsubscript{3}}} [\sigma + \alpha_5 + n, q^{-2 \alpha_2 - 2 \alpha_3} t],
\]

with

\[
C_{\text{III\textsubscript{3}}} [\bar{a}, \sigma, n] = (q - 1)^{-4n(n+2\alpha_3)} q^{-(2 \alpha_2 + 2 \alpha_3)n(n+2\sigma)+2\sigma n-4\alpha_3 \alpha_5 n} \prod_{\epsilon = \pm} \frac{1}{G_q (1+2\epsilon (\sigma + \alpha_5 + n))},
\]

\[
Z_{\text{III\textsubscript{3}}} [\sigma, t] = \sum_{(\lambda_+, \lambda_-) \in \mathbb{Y}^2} \frac{1}{t^{(\lambda_+ + |\lambda_-|)} N_{\lambda_+ \lambda_-} (q^{\epsilon - \epsilon'})}.
\]

Let us define the tau functions for q-P\textsubscript{III\textsubscript{3}} by

\[
\tau_{\text{III\textsubscript{3}}}^j (s, \sigma, t) = \tau(a_i, s, \sigma, t) \quad (i = 1, \ldots, 8).
\]

Put

\[
C(\bar{a}) = q^{-2 \lambda (\sigma + \alpha_5)^2 - \alpha_3^2 - \alpha_2^2} \prod_{\epsilon = \pm} G_q (1-\theta_* + \alpha_1 - \alpha_2 + \epsilon (\sigma + \alpha_3))^{-1}.
\]

Proposition 6.1. Set

\[
\theta_* = \Lambda, \quad t = q^\Lambda t_1, \quad s = \bar{s} q^{-2 \sigma \Lambda} \prod_{\epsilon = \pm} \Gamma_q (\Lambda + \frac{1}{2} + \epsilon \sigma)^{-\epsilon}.
\]

Then we have

\[
C(a_i) \tau_{\text{III\textsubscript{2}}}^j (\theta_*, s, \sigma, t) \rightarrow \tau_{\text{III\textsubscript{3}}}^j (\bar{s}, \sigma, t_1) \quad (\Lambda \rightarrow \infty)
\]

for \(j = 1, \ldots, 8\).

\[\Box\]

In what follows, we abbreviate \(\tau_{\text{III\textsubscript{3}}}^j (s, \sigma, t)\) to \(\tau_j\).
**Theorem 6.2.** The functions

\[ y = \frac{q^{-1}}{(1-q)^2} \frac{\tau_3 \tau_4}{\tau_1 \tau_2}, \quad z = -\frac{q^{2\sigma-1}}{(1-q)^2} \frac{\tau_7 \tau_8}{\tau_5 \tau_6} \]

solves the $q$-Painlevé III equation

\[ \frac{\dot{y} \dot{y}}{a_3} = z^2, \quad z \ddot{z} = -\frac{y(y-a_2 t)}{y-a_3} \]

with the parameters

\[ a_2 = q^{-1}, \quad a_3 = q^{-1}, \quad b_2 = 1. \]

Furthermore, the tau functions $\tau_i$ ($i = 1, \ldots, 8$) satisfy the following bilinear equations.

\[ (1-q)^2 \tau_1 \tau_2 - t \tau_3 \tau_4 - (1-q)^2 \tau_5 \tau_6 = 0, \]

\[ (1-q)^2 \tau_1 \tau_2 - \tau_3 \tau_4 + q^{2\sigma} \tau_7 \tau_8 = 0. \]

As suggested in [BS1 (2.9)–(2.11)], the bilinear equation (6.3) is derived from (6.2) by the transformation $\sigma \rightarrow \sigma + 1/2$, $s \rightarrow q^{-1}(1-q)^{-4}s$.

**Remark 6.3.** The tau function $T_c(q^{2\sigma}, s; q|t)$ proposed in [BS1] for the $q$-Painlevé III equation is related to our tau functions by

\[ T_c(q^{2\sigma}, s; q|t) = \frac{(-1)^{2\sigma^2}}{(1-q)^{4\sigma^2}} \frac{1}{T_1 \left( \frac{(-1)^{4\sigma} s}{q^{2\sigma} (1-q)} \right)}, \]

\[ T_c(q^{2\sigma+1}, s; q|t) = \frac{(-1)^{2(\sigma+1/2)^2}}{(1-q)^{4(\sigma+1/2)^2}} t^{1/4} \frac{1}{T_3 \left( \frac{(-1)^{4\sigma} s}{q^{2\sigma} (1-q)} \right)}. \]

**Remark 6.4.** $q$-$P(A'_1)$ in [Mu] (or $q$-$P(A_1^{(1)}/A_7^{(1)})$ in [KNY] (8.14)) is

\[ \frac{\dot{y} \dot{y}}{a_4} = -\frac{z(z-b_2 t)}{z-b_3}, \quad \frac{z \ddot{z}}{b_3} = \frac{y^2}{a_4} \]

where $y = y(t)$, $z = z(t)$, and $a_4, b_1, b_2, b_3$ are complex parameters. Replacing $y, z$ in (6.1) by $y, z$, we obtain $q$-$P(A'_1)$ with $a_4 = 1, b_2 = 1$, and $b_3 = q^{-1}$.

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