ON THE GLOBAL MONODROMY OF A LEFSCHETZ FIBRATION ARISING FROM THE FERMAT SURFACE OF DEGREE 4

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Abstract

A complete description of the global monodromy of a Lefschetz fibration arising from the Fermat surface of degree 4 is given. As a by-product we get a positive relation among right hand Dehn twists in the mapping class group of a closed orientable surface of genus 3.

1. Introduction

The motivation of this work is an interest in the topological monodromy of surface bundles obtained by the following way. Let \( X \subseteq \mathbb{P}_N \) be a complex surface embedded in the complex projective space of dimension \( N \). We denote by \( \mathbb{P}_N^N \) the dual projective space of \( \mathbb{P}_N \), i.e., the space of all hyperplanes of \( \mathbb{P}_N \). The dual variety \( X^\vee \) of \( X \) is, by definition, the set of all hyperplanes of \( \mathbb{P}_N \) tangent to \( X \) at some point. Then we have a complex analytic family of compact Riemann surfaces over \( \mathbb{P}_N^N \setminus X^\vee \); the fiber over \( H \in \mathbb{P}_N^N \setminus X^\vee \) is the hyperplane section \( H \cap X \).

If we regard such a family as an oriented surface bundle, its bundle structure is totally encoded (at least when the genus of \( H \cap X \) is \( \geq 2 \)) in the associated topological monodromy \( \rho \) from the fundamental group \( \pi_1(\mathbb{P}_N^N \setminus X^\vee) \), which is non-trivial when \( X^\vee \) is a hypersurface, to the mapping class group \( \Gamma_g \) of a closed orientable surface of genus \( g \), where \( g \) is the genus of \( H \cap X \). If a finite presentation of \( \pi_1(\mathbb{P}_N^N \setminus X^\vee) \) and a description of \( \rho \) in terms of this presentation are obtained, we might say that the topological monodromy \( \rho \) is understood. However such a nice situation may not be expected in general. One reason for this is the difficulty of the computations of \( \pi_1(\mathbb{P}_N^N \setminus X^\vee) \), see [4].

Instead we consider to cut \( \mathbb{P}_N^N \setminus X^\vee \) by a generic line. Let \( L \) be a line (1-dimensional projective subspace) of \( \mathbb{P}_N^N \) and consider the restriction of the family over \( \mathbb{P}_N^N \setminus X^\vee \) to \( L \setminus (L \cap X^\vee) \). We focus on the associated topological monodromy \( \rho' \) from \( \pi_1(L \setminus (L \cap X^\vee)) \) to \( \Gamma_g \). If \( L \) meets \( X^\vee \) transversely, \( L \cap X^\vee \) consists of finitely many points and the inclusion \( L \setminus (L \cap X^\vee) \hookrightarrow \mathbb{P}_N^N \setminus X^\vee \) induces the surjection on the fundamental group level (the Zariski theorem of Lefschetz...
type, see [6]). Thus for instance to know the group $\text{Im}(\rho)$, called the universal monodromy group in [4], it suffices to consider $\rho'$ instead of $\rho$. Moreover, theory of Lefschetz pencils can be applied to the study of $\rho'$, as follows. There is a natural family of algebraic curves over $L$; the fiber over $H \in L$ is the (possibly singular) hyperplane section $H \cap X$. As in [8] or [9], this family turns out to be a Lefschetz fibration in the sense of [5], Definition 8.1.4. In particular all the singular fibers, which are over $L \cap X^\vee$, have one nodal singularity and the local monodromy around each point of $L \cap X^\vee$ is the right hand Dehn twist along a simple closed curve, called the vanishing cycle. Thus the determination of the positions of all the vanishing cycles on a fixed reference fiber will lead to a complete description of the global monodromy $\rho'$. Also, as a by-product we will get a positive relation among right hand Dehn twists in $\Gamma_g$, since $\pi_1(L\setminus L \cap X^\vee)$ admits a presentation by a standard generating system subject to one relation, see the paragraph before Theorem 1.1.

In this paper we investigate a particular example. Hereafter $X$ is the Fermat surface of degree 4, namely the smooth hypersurface in $\mathbb{P}_3$ defined by the equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0,$$

where $[x_0 : x_1 : x_2 : x_3]$ is a homogeneous coordinate system of $\mathbb{P}_3$. In this case $X^\vee$ is an irreducible hypersurface of $\mathbb{P}_3$, whose defining equation will be given in section 2. Let

$$\mathcal{F} := \{(x, H) \in \mathbb{P}_3 \times (\mathbb{P}_3 \setminus X^\vee) ; x \in H \cap X\}$$

and let

$$\rho : \pi_1(\mathbb{P}_3 \setminus X^\vee, b_0) \to \Gamma_3$$

be the associated topological monodromy of the second projection $\pi : \mathcal{F} \to \mathbb{P}_3 \setminus X^\vee$, where $b_0$ is a base point. Note that for each $H \in \mathbb{P}_3 \setminus X^\vee$, the hyperplane section $\pi^{-1}(H) = H \cap X \subset H \cong \mathbb{P}_2$ is a non-singular plane curve of degree 4.

To state the result we prepare a terminology. Let $\{v_i\}_i$ be a set of $n$ points in $\mathbb{P}_3$ and choose a base point $b_0$ of $\mathbb{P}_3 \setminus \{v_i\}_i$. We say a set of $n$ based loops $\{\lambda_i\}_i$ is a standard generating system for $\pi_1(\mathbb{P}_3 \setminus \{v_i\}_i, b_0)$ if each $\lambda_i$ is free homotopic to a loop nearby $v_i$ going once around $v_i$ by counter-clockwise manner, and their product $\lambda_1\lambda_2 \cdots \lambda_n$ is trivial as an element of $\pi_1(\mathbb{P}_3 \setminus \{v_i\}_i, b_0)$.

**Theorem 1.1.** Let $X$ be the Fermat surface of degree 4 and $L$ a line of $\mathbb{P}_3$ meeting $X^\vee$ transversely. Choose a base point $b_0$ of $L \setminus (L \cap X^\vee)$. Then there is a standard generating system $\lambda_1, \ldots, \lambda_{36}$ for $\pi_1(L \setminus (L \cap X^\vee), b_0)$ such that the monodromy $\rho'(\lambda_i)$ is given by the right hand Dehn twist along the simple closed curve $C_i$ on a genus 3 surface as shown in Figure 1.1. Here, $\rho'$ is the composition of $\pi_1(L \setminus (L \cap X^\vee), b_0) \to \pi_1(\mathbb{P}_3 \setminus X^\vee, b_0)$ induced by the inclusion and (1.1).

Since $\lambda_1\lambda_2 \cdots \lambda_{36} = 1$ and $\rho'$ is an anti-homomorphism (see the conventions below), we immediately have the following
Figure 1.1
Corollary 1.2. Let us denote by $t_i$ the right hand Dehn twist along $C_i$. Then the relation $t_{36}t_{35}\cdots t_2t_1 = 1$ holds in the mapping class group $\Gamma_3$.

Also we can show the following

**Corollary 1.3.** The topological monodromy (1.1) is surjective. In other words, the universal monodromy group $\text{Im}(\rho)$ coincides with $\Gamma_3$.

**Proof.** The set of the right hand Dehn twists along the seven simple closed curves $C_9$, $t_9(C_8)$, $C_{28}$, $t_{28}^{-1}(C_1)$, $C_{10}$, $t_{10}^{-1}(C_{11})$, and $t_2^{-1}t_{28}^{-1}(C_6)$ constitutes a Dehn-Lickorish-Humphries generating system of $\Gamma_3$ (see [7], Corollary 4.2.F). Thus $\rho'$ is surjective, so is $\rho$. \qed

The study of the global monodromy of a holomorphic fibration of Riemann surfaces over a Riemann surface via numerical analysis is initiated by Ahara [2] and Matsumoto [10]. They introduced a holomorphic fibration $f : V_n \to \mathbb{P}_1$, where $V_n$ is the Fermat surface of degree $n$. Their fibration is not a Lefschetz fibration and has more degenerated singular fibers. Their method was to express the general fibers as branched coverings of $\mathbb{P}_1$ and analyze the motions of the critical points of these branched coverings. The analysis is based on Newton approximation, see [2], section 3. Based on the result of [2], the global monodromy was described in terms of Dehn twists for the case $n = 5$ in [10] (in this case the genus of the general fibers is 3). Recently, Ahara and Awata [1] determined how general fibers of $f$ degenerate to the singular fibers for all $n$, without numerical analysis.

The rest of the paper is devoted to the proof of Theorem 1.1. Note that the total space of our Lefschetz fibration $\pi : \mathcal{F} \to L$ (see section 5) is the blow up of $X$ at 4 points. We adopt the same method as [2], [10]. In section 2 we give

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.1}
\caption{Figure 1.1 (continued)}
\end{figure}
the defining equation of \( X^\vee \). In section 3 we cut \( \mathbb{P}^3 \backslash X^\vee \) by a line \( L = L(c_1, c_2) \) whose defining equation has two parameters \( c_1 \) and \( c_2 \). For a suitable choice of \( c_1 \) and \( c_2 \), \( L \) will meet \( X^\vee \) transversely. We will introduce a homogeneous coordinate system \([u : v] \) to \( L \) and will denote by \( \mathcal{C}_L \) the set \( L \backslash \{[0 : 1]\} \). Then we proceed to express general fibers as 4-branched coverings of \( \mathbb{P}_1 \). In section 4 we introduce a projection \( p_v \) from \( X_v = v \cap X \) to \( \mathbb{P}_1 \) for \( v = [1 : v] \in \mathcal{C}_L \subset \mathbb{P}^3 \) and prove its “tameness” over \([1 : 0] \in \mathbb{P}_1 \) (see Lemma 4.1). Section 5 is a preparation for sections 6 and 7. We choose explicit values for \( c_1 \) and \( c_2 \). Most of the results in sections 6 and 7 depend on numerical analysis using a computer. In section 6 we describe the projection \( p_0 : X_0 \to \mathbb{P}_1 \) and in section 7 we analyze motions of the critical values of \( p_v \) caused by movements of \( v \) along suitable chosen paths in \( L \) and give a complete description of the topological monodromy \( \rho^v : \pi_1(L \backslash (L \cap X^\vee)) \to \Gamma_3 \). Theorem 1.1 will easily follow from Proposition 7.1.

### Conventions about topological monodromy

It is sometimes confusing that there are different kinds of conventions about product of paths or product of maps, so let us fix the conventions in this paper: 1) for any two mapping classes \( f_1 \) and \( f_2 \), the multiplication \( f_1 \circ f_2 \) means that \( f_2 \) is applied first, 2) for any two homotopy classes of based loops \( \ell_1 \) and \( \ell_2 \), their product \( \ell_1 \cdot \ell_2 \) means that \( \ell_1 \) is traversed first.

Let \( \Sigma \) be a closed oriented surface and \( \pi : E \to B \) an oriented \( \Sigma \)-bundle. Choose a base point \( b_0 \in B \) and fix an identification \( \phi : \Sigma \cong \pi^{-1}(b_0) \). For each based loop \( \ell : [0, 1] \to B \), consider the pull back \( \ell^*(E) \to [0, 1] \). Since \([0, 1] \) is contractile there exists a trivialization \( \Phi : \Sigma \times [0, 1] \to \ell^*(E) \) such that \( \Phi(x, 0) = \phi(x) \). By assigning the isotopy class of \( \phi^{-1} \circ \Phi(x, 1) \) to the homotopy class of \( \ell \), we obtain a map \( \rho \), called the topological monodromy of \( \pi : E \to B \), from \( \pi_1(B, b_0) \) to the mapping class group of \( \Sigma \). Under the conventions above, \( \rho \) is an anti-homomorphism, i.e., for \( \ell_1, \ell_2 \in \pi_1(B, b_0) \) we have \( \rho(\ell_1 \cdot \ell_2) = \rho(\ell_2) \rho(\ell_1) \).

### 2. The defining equation of \( X^\vee \)

Our first task is to describe the defining equation of \( X^\vee \). The result might be known, but we give it here since our numerical analysis by a computer performed in sections 6 and 7 will heavily use it. To begin with, we compute the degree of \( X^\vee \). By using the formula of Katz [8] (5.5.1), it is computed as

\[
\deg(X^\vee) = \int_X \frac{(1 + h)^2}{c([X])} = \int_X \frac{(1 + h)^2}{1 + 4h} = 36.
\]

Here, \( h \in H^2(\mathbb{P}_3; \mathbb{Z}) \) denotes the hyperplane class and \( c([X]) \in H^*(\mathbb{P}_3; \mathbb{Z}) \) denotes the total Chern class of the divisor \( X \). Let \([x_0 : x_1 : x_2 : x_3] \) be the homogeneous coordinate system of \( \mathbb{P}^3 \) dual to \([x_0 : x_1 : x_2 : x_3] \). Namely, \([x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 \) is the hyperplane of \( \mathbb{P}_3 \) defined by

\[
x_0x_0 + x_1x_1 + x_2x_2 + x_3x_3 = 0.
\]
Proposition 2.1. Let \( \omega = \exp(2\pi \sqrt{-1}/3) \) and let \( \beta_i \) be a formal indeterminate such that \( \beta_i^3 = x_i \) for \( i = 0, 1, 2, \) and \( 3. \) Then the defining equation of \( X^\omega \) is given by

\[
(2.1) \quad \prod_{0 \leq i, j, k \leq 2} (\beta_i \omega + \omega^j \beta_j + \omega^k \beta_k + \omega^{i+j+k}) = 0.
\]

Remark that the left hand side of (2.1) is invariant under the transformations \( \{ \phi_i \}_{i=0}^3 \) where \( \phi_i \) is defined by \( \phi_i(\beta_j) = \omega \beta_j \) for \( j \neq i \) and \( \phi_i(\beta_i) = \beta_i. \) Thus it is in fact a homogeneous polynomial in \( x_i \)'s and the degree is 36.

Proof of Proposition 2.1. Since we know the degree of \( X^\omega \) is also equal to 36, it suffices to show that \( x \in X^\omega \) if and only if \( x \) satisfies the equation (2.1).

Let \( x = [x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 \) and assume that \( x_0 = 1. \) Let \( P(x_0, x_1, x_2, x_3) = x_0^3 + x_1^3 + x_2^3 + x_3^3. \) By definition, \( x \in X^\omega \) if and only if there exists a point \( y = [y_0 : y_1 : y_2 : y_3] \in \mathbb{P}^3 \) such that

\[
(2.2) \quad \begin{cases}
P(y_0, y_1, y_2, y_3) = 0 \\
y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0 \\
[P_{x_0}(y) : P_{x_1}(y) : P_{x_2}(y) : P_{x_3}(y)] = [1 : x_1 : x_2 : x_3],
\end{cases}
\]

where \( P_{x_i} \) is the partial derivative of \( P \) with respect to \( x_0, \) etc. Since \( P_{x_i} = 4x_i^3 \) we see that \( y_0 \neq 0, \) by the third equation of (2.2). Thus we may assume \( y_0 = 1 \) and we have

\[
(2.3) \quad y_1^3 = x_1, \quad y_2^3 = x_2, \quad y_3^3 = x_3.
\]

Under (2.3), the first and the second equations of (2.2) are equivalent. Therefore \( x \in X^\omega \) if and only if there exists \( (y_1, y_2, y_3) \in \mathbb{C}^3 \) such that

\[
\begin{cases}
y_1^3 = x_1, \quad y_2^3 = x_2, \quad y_3^3 = x_3 \\
1 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0.
\end{cases}
\]

Let \( \beta_i \) be a complex number such that \( \beta_i^3 = x_i \) for \( i = 1, 2, \) and \( 3. \) Then, \( x \in X^\omega \) if and only if there exist \( i_1, i_2, i_3 \in \{0, 1, 2\} \) such that

\[
1 + \omega^{i_1} \beta_1^3 + \omega^{i_2} \beta_2^3 + \omega^{i_3} \beta_3^3 = 0,
\]

namely \( \beta = [1 : \beta_1 : \beta_2 : \beta_3] \) satisfies the equation (2.1). This completes the proof. \( \square \)

3. Cutting \( X^\omega \) by a line of special type

Let \( c_1 \) and \( c_2 \) be complex numbers and \( L = L(c_1, c_2) \) the line of \( \mathbb{P}^3 \) defined by

\[
c_1^3 x_0 - x_1 = c_2^3 x_0 - x_2 = 0.
\]
We introduce a homogeneous coordinate system \([u : v]\) of \(L\) by assigning \([x_0 : x_1 : x_2 : x_3] = [u : c_1^2 u : c_2^2 u : v]\) to \([u : v]\).

**Proposition 3.1.** The defining equation of \(L \cap X^\vee \subset L\) is given by

\[
\prod_{0 \leq i_1, i_2 \leq 2} ((1 + \omega^i c_1^4 + \omega^{i_2} c_2^4)u^4 + v^4) = 0.
\]

**Proof.** Let \(\beta_i, 0 \leq i \leq 3\) be the formal elements as in Proposition 2.1 and suppose \(c_1 \beta_0 - \beta_1 = c_2 \beta_0 - \beta_2 = 0\). Then \(\beta_0^4 + \omega^i c_1^4 + \omega^2 c_2^4 = (1 + \omega^i c_1^4 + \omega^{i_2} c_2^4)\beta_0^4 + \omega^i \beta_3^4\), and

\[
\prod_{0 \leq i_1, i_2 \leq 2} ((1 + \omega^i c_1^4 + \omega^{i_2} c_2^4)\beta_0^4 + \omega^i \beta_3^4) = ((1 + \omega^i c_1^4 + \omega^{i_2} c_2^4)\beta_0^4)^3 + (\beta_3^4)^3
\]

\[= (1 + \omega^i c_1^4 + \omega^{i_2} c_2^4)^3 x_0^4 + x_3^4.
\]

Note that \(\omega\) is a primitive third root of unity. Combining this computation with Proposition 2.1, we have the result. \(\square\)

Suppose \(c_1\) and \(c_2\) are chosen so that
1. for any pair \((i_1, i_2)\), we have \(1 + \omega^i c_1^4 + \omega^{i_2} c_2^4 \neq 0\),
2. for any two distinct pairs \(i = (i_1, i_2)\) and \(j = (j_1, j_2)\), the roots of \(f_i(v) = v^4 + (1 + \omega^i c_1^4 + \omega^{i_2} c_2^4)^3\) and those of \(f_j(v) = v^4 + (1 + \omega^j c_1^4 + \omega^{j_2} c_2^4)^3\) are all different.

Then by Proposition 3.1, \(L \cap X^\vee\) consists of \(36\) points therefore \(L\) meets \(X^\vee\) transversely. Moreover, \(L \cap X^\vee\) is contained in \(L \setminus \{[1 : 0], [0 : 1]\}\). For simplicity we write \(C_L\) instead of \(L \setminus \{[0 : 1]\}\), and we identify \(C_L\) with \(C\) by \(v \mapsto [1 : v], v \in C\). Choose \(0 \in C_L\) as a base point of \(L \setminus (L \cap X^\vee)\). By the Zariski theorem of Lefschetz type [6], (for our purpose, a weaker statement in [9], (7.4.1) is sufficient) the natural homomorphism

\[
(3.1) \quad \pi_1(L \setminus (L \cap X^\vee), 0) \to \pi_1(P^3 \setminus X^\vee, b_0)
\]

induced by the inclusion is surjective (we denote by \(b_0\) the image of \(0 \in C_L\) by the inclusion). From now on, we assume that \(c_1\) and \(c_2\) satisfy the two conditions above and will focus on the surface bundle

\[
\pi' : \mathcal{F}' \to L \setminus (L \cap X^\vee)
\]

where \(\mathcal{F}' = \pi^{-1}(L \setminus (L \cap X^\vee))\) and \(\pi = \pi|_{\mathcal{F}'}\). The associated topological monodromy

\[
\rho' : \pi_1(L \setminus (L \cap X^\vee), 0) \to \Gamma_3
\]

is the composition of (3.1) and (1.1).
4. A lemma on the hyperplane section by \( v \in \mathbf{C}_L \)

Let \( v \in \mathbf{C}_L \). We denote by \( X_v \) the hyperplane section \( v \cap X \), whose defining equation is

\[
\begin{aligned}
&x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \\
x_0 + c_1^3 x_1 + c_2^3 x_2 + v x_3 = 0.
\end{aligned}
\]

Eliminating the indeterminate \( x_0 \), we obtain

\[
(c_1^3 x_1 + c_2^3 x_2 + v x_3)^4 + x_1^4 + x_2^4 + x_3^4 = 0.
\]

Let \( E_v = E_v(x_1, x_2, x_3) \) be the left hand side of this equation. Then by regarding \([x_1 : x_2 : x_3]\) as a homogeneous coordinate system of \( \mathbf{P}_2 \), \( X_v \) is identified with the plane curve determined by \( E_v \). Under this identification, consider the projection

\[
p_v : X_v \to \mathbf{P}_1, \quad [x_1 : x_2 : x_3] \mapsto [x_1 : x_3].
\]

Lemma 4.1. If \( |c_1|^4 + |c_2|^4 < 1 \), the following holds: for any \( v \in \mathbf{C}_L \),

1. the plane curve \( X_v \) has no singularities on the line \( x_3 = 0 \), and
2. the projection \( p_v \) does not branch over \([1 : 0] \in \mathbf{P}^1\).

Proof. For simplicity, we write \( E \) instead of \( E_v \). Suppose \( E = E_{x_1} = E_{x_2} = E_{x_3} = 0 \) as has a solution \([x_1 : x_2 : 0]\) for some \( v \in \mathbf{C}_L \). If \( v \neq 0 \), we have \( c_1^3 x_1 + c_2^3 x_2 = 0 \) since \( E_{x_3} = 0 \). Substituting this into \( E_{x_1} = E_{x_2} = 0 \) we have \( x_1 = x_2 = 0 \), a contradiction. Thus it suffices to consider the case when \( v = 0 \). Suppose \( x_2 = 1 \). Then we have

\[
\begin{aligned}
E_{x_1} &= 4 c_1^3 (c_1^3 x_1 + c_2^3)^3 + 4 x_1^3 = 0 \\
E_{x_2} &= 4 c_2^3 (c_1^3 x_1 + c_2^3)^3 + 4 = 0.
\end{aligned}
\]

By the second equation of (4.1), we have

\[
(c_1^3 x_1 + c_2^3)^3 = -c_2^{-3}.
\]

Substituting this into the first equation of (4.1), we have \( x_1^3 = (c_1/c_2)^3 \) therefore we can write \( x_1 = \omega^j c_1/c_2 \) for some \( j \), \( 0 \leq j \leq 2 \). Substituting this into (4.2) we have a necessary condition \( (c_1^4 \omega^j + c_2^4)^3 = -1 \). But this is impossible by our assumption \( |c_1|^4 + |c_2|^4 < 1 \). If we assume \( x_1 = 1 \) a similar argument leads to a contradiction. This establishes the first part.

To show the second part, it suffices to show the following: for \((x_1, x_3) = (1, 0)\), the equation \( E = E_{x_2} = 0 \) does not have any solution in \( x_2 \). The argument is similar to the first part. Suppose \( x_2 \in \mathbf{C} \) satisfies

\[
\begin{aligned}
E &= (c_1^4 + c_2^4 x_2)^4 + 1 + x_2^4 = 0 \\
E_{x_2} &= 4 c_2^3 (c_1^3 + c_2^3 x_2)^3 + 4 x_2^3 = 0.
\end{aligned}
\]

By the second equation of (4.3), we have

\[
(c_1^3 + c_2^3 x_2)^3 = -x_2^2.
\]
Substituting this into the first equation of (4.3), we see that \( x_2 = \omega^j c_2 / c_1 \) for some \( j, 0 \leq j \leq 2 \). Substituting this into (4.4) we have \((e_1^4 + e_2^4 \omega) \)\(^3 = -1\), a contradiction. \( \square \)

5. A special choice of \( c_1 \) and \( c_2 \)

Henceforth, let \( c_1 = 7/8 \) and \( c_2 = 3/4 \). For this choice, the conditions for \( c_1 \) and \( c_2 \) given in section 3 and the assumption of Lemma 4.1 are satisfied.

To study \( p' \) (see section 3) we also consider \( \mathcal{F} = \{(x, H) \in \mathbb{P}^2 \times \mathcal{L}; x \in H \cap X\} \) and the second projection \( \pi : \mathcal{F} \to L \). By the transversality of \( L \) and \( X^\vee \), it follows that \( \mathcal{F} \) is non-singular and \( \pi : \mathcal{F} \to L \) is a Lefschetz fibration (see section 1). The set of critical values of \( \pi \) is \( L \cap X^\vee = \{v_1, \ldots, v_{36}\} \). For each \( v_i \), there is a unique critical point \( \tilde{v}_i \) in \( \pi^{-1}(v_i) \) and for a suitable choice of local holomorphic coordinates, the projection \( \pi \) looks like \( (z_1, z_2) \mapsto z_1^2 + z_2^2 \) near \( \tilde{v}_i \). In this local model, the singular fiber \( \pi^{-1}(v_i) \) looks like \( \Sigma_0 = \{z_1^2 + z_2^2 = 0\} \), which is obtained from the smooth fibers \( \Sigma_\varepsilon = \{z_1^2 + z_2^2 = \varepsilon\}, \varepsilon > 0 \) by collapsing the simple closed curves \( C_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2; x_1^2 + x_2^2 = \varepsilon\} \). The curve \( C_\varepsilon \) is called the vanishing cycle. By the Picard-Lefschetz formula ([5], p. 295), the local monodromy around each \( v_i \) is the right hand Dehn twist along the corresponding vanishing cycle.

Recall that the defining equation of \( X_v = x \cap X \) is
\[
(c_1^3 x_1 + c_2^3 x_2 + v x_3 )^4 + x_1^4 + x_2^4 + x_3^4 = 0.
\]
By Lemma 4.1, \( p_v \) is unramified over \( [1 : 0] \in \mathbb{P}_1 \). Thus we focus on \( p_v \) restricted to \( \mathbb{P}_1 \setminus \{[1 : 0]\} \), which is identified with \( C \) by \( x_1 \mapsto [x_1 : 1], x_1 \in C \). Let
\[
F^v(x_2) := (c_1^3 x_1 + c_2^3 x_2 + v x_3 )^4 + x_1^4 + x_2^4 + 1
\]
and \( G^v(x_1) \) the discriminant of \( F^v \), regarded as a polynomial in \( x_2 \) and \( Q(v) \) the discriminant of \( G^v(x_1) \) regarded as a polynomial in \( x_1 \). \( G^v(x_1) \) is a polynomial of degree 12 in \( x_1 \). By definition \( v \in C \) is a root of \( Q(v) \) if and only if there is a root of \( G^v \) with multiplicity \( \geq 2 \). As we will see in section 7, \( G^v \) has this property hence \( Q(v_i) = 0 \) for \( i = 1, \ldots, 36 \). Therefore if \( v \) is not a root of \( Q \) the curve \( X_v \) is non-singular and all the roots of \( G^v \), which correspond to the critical values of \( p_v \), are simple. By the Riemann-Hurwitz formula we see that the total branching order of each critical value of \( p_v \) is 1. This means that over each critical value there is an exactly one critical point of \( p_v \), near which \( p_v \) looks like \( z \mapsto z^2 \) for a suitable choice of local coordinates.

6. Description of the reference fiber

In this section, we describe the reference fiber \( X_0 = \pi^{-1}(0) \) as a 4-fold branched covering \( p_0 : X_0 \to \mathbb{P}_1 \). As in the last section we focus on \( p_0 \) restricted to \( \mathbb{P}_1 \setminus \{[1 : 0]\} \cong C \).

The roots of \( G^0(x_1) \) are numerically computed and we denote them by \( a_1, \ldots, a_{12} \) as shown in the following schematic figure:
Here, \( a_1 \approx 0.709187 + 0.642143 \sqrt{-1} \), \( a_2 \approx 0.692307 + 0.692307 \sqrt{-1} \), \( a_3 \approx 0.642143 + 0.709187 \sqrt{-1} \) and \( a_{i+3} = \sqrt{-1} a_i \) for \( 1 \leq i \leq 9 \).

For \( x_1 \in \mathbb{C} \), the points in the fiber \( p_0^{-1}(x_1) \) correspond to the roots of \( F_0^0(x_1) \) by \([x_1 : x_2 : 1] \mapsto x_2 \). Now we choose 0 as a base point of \( \mathbb{C} \setminus \{a_i\} \). The fiber \( p_0^{-1}(0) \) corresponds to the roots of

\[
F_0^0(x_2) = (c_2^{12} + 1)x_2^4 + 1,
\]

i.e., \( \{s_k\}_{k=1}^4 \) where \( s_k = (1 + c_2^{12})^{-1/4} \exp((2k-1)\pi \sqrt{-1}/4) \).

We will investigate the monodromy

\[
\chi : \pi_1(\mathbb{C} \setminus \{a_i\}, 0) \to \mathfrak{S}_4
\]

of the unramified 4-covering \( p_0^{-1}(\mathbb{C} \setminus \{a_i\}) \to \mathbb{C} \setminus \{a_i\} \). Here, \( \mathfrak{S}_4 \) is the symmetric group on the four letters \( s_1, s_2, s_3, \) and \( s_4 \).

For each \( j = 1, \ldots, 12 \), let \( m_j \) be the straight line segment from 0 to \( a_j \) and \( \ell_j \) be a based loop in \( \mathbb{C} \setminus \{a_i\} \) going from 0 to a point nearby \( a_j \) along \( m_j \), then going once around \( a_j \) by counter-clockwise manner and then coming back to 0 along \( m_j \), as shown in the following figure.
By numerical analysis using a computer, we see that $\chi(\ell_j)$ is given by the following table:

| $j$ | $\chi(\ell_j)$ | $j$ | $\chi(\ell_j)$ |
|-----|----------------|-----|----------------|
| 1   | (12)           | 7   | (34)           |
| 2   | (13)           | 8   | (13)           |
| 3   | (14)           | 9   | (23)           |
| 4   | (23)           | 10  | (14)           |
| 5   | (24)           | 11  | (24)           |
| 6   | (12)           | 12  | (34)           |

For example, $\chi(\ell_1) = (12)$ means $\chi(\ell_1)$ is the transposition of $s_1$ and $s_2$, etc. Let $S_k = \{0 : s_k : 1\}$ and $\tilde{a}_j$ the unique critical point of $p_0$ over $a_j$, and $\tilde{m}_j$ the connected component of $p_0^{-1}(m_j)$ containing $\tilde{a}_j$ as an interior point. Then $p_0^{-1}(0) = \{S_k\}_{k=1}^4$ and we can draw the picture of $S_k$, $\tilde{a}_j$, and $\tilde{m}_j$ on $X_0$ by using the table above, which determines the topological type of the branched covering $p_0 : X_0 \rightarrow \mathbb{P}_1$. See the figure below.

![Figure 6.3](image)

For example, $\tilde{m}_1$ is the unique path from $S_1$ through $\tilde{a}_1$ to $S_2$, corresponding to the data $\chi(\ell_1) = (12)$. In section 7 this figure will be a key to find the vanishing cycles.

### 7. Finding the vanishing cycles

In this section we give a complete description of

$$
p' : \pi_1(\Lambda(L \cap X^\vee), 0) \rightarrow \Gamma_3
$$

and finish the proof of Theorem 1.1. Our task is to determine the position of all the vanishing cycles in $X_0$. We will achieve this by investigating the motions of
the critical values of $p_e$ along a suitably chosen path from 0 to each point of $L \cap X^\nu = \{v_1, \ldots, v_{36}\}$.

Now we arrange indices of $v_i$'s and let $\mu_i : [0, 1] \to C_L$ be a simple path from 0 to $v_i$, satisfying $Q(\mu_i(t)) \neq 0$ for $t \in [0, 1)$, as shown in Figure 7.1.

Approximate values of $v_i$'s are: $v_1 \approx 0.600851 + 0.315483\sqrt{-1}$, $v_2 \approx 0.963952 + 0.064039\sqrt{-1}$, $v_3 \approx 0.999689 + 0.470655\sqrt{-1}$, $v_4 \approx 1.059535 + 0.794167\sqrt{-1}$, $v_5 \approx 1.145495 + 1.145495\sqrt{-1}$, $v_i = \text{Im}(v_{10-i}) + \text{Re}(v_{10-i})$ for $6 \leq i \leq 9$, and $v_{i+9} = \sqrt{-1}v_i$ for $1 \leq i \leq 27$. Each $\mu_i$ consists of 4 straight line segments, as shown in the figure. Here, $\zeta_i$ is a root of $(1 + \omega^i\zeta_1^4)^3 + \zeta_i^4$ such that $\text{Re}(\zeta_i) > 0$, $\text{Im}(\zeta_i) > 0$ for $i = 1, 2, 3$ and $\mu_{i+9} = \sqrt{-1}\mu_i$ for $1 \leq i \leq 27$.

Let $\lambda_i$ be a based loop in $C_L \setminus (L \cap X^\nu)$ going from 0 to a point nearby $v_i$ along $\mu_i$, then going once around $v_i$ by counter-clockwise manner and then coming back to 0 along $\mu_i$. Then $\{\lambda_1, \ldots, \lambda_{36}\}$ is a standard generating system for $\pi_1(L \setminus (L \cap X^\nu), 0)$ in the sense of section 1.
For a while we fix \( i, \ 1 \leq i \leq 36 \). For each \( t \in [0,1) \) the roots of \( G^i t(x_1) \) are all simple, therefore we can choose complex valued continuous functions \( a_1(t), \ldots, a_{12}(t) \) such that \( G^i t(a_j(t)) = 0 \) and \( a_j(0) = a_j \) for \( j = 1, \ldots, 12 \). We have \( a_j(t) \neq a_k(t) \) for \( t \in [0,1) \) and \( (j,k) \) with \( j \neq k \).

By continuity, \( a_j(t) \) is uniquely extended to a continuous function on the unit interval \([0,1] \). We would like to study what happens when \( t \) approaches 1. By numerical analysis using a computer, we can investigate the motions of \( a_j(t) \), \( 1 \leq j \leq 12 \).

**Observation 1.** There exist two indices \( \delta = \delta(i) \) and \( \varepsilon = \varepsilon(i) \), \( 1 \leq \delta < \varepsilon \leq 12 \) such that \( a_0(t) = a_1(t) \) and \( a_j(t) \neq a_k(t) \) for any pair \((j,k)\) with \( j < k \) other than \((\delta,\varepsilon)\), see the table below. In particular, the number of roots of \( G^i \) is 11.

| \( i \) | \( (\delta(i), \varepsilon(i)) \) |
|---|---|
| 1 | (3, 6) |
| 2 | (1, 4) |
| 3 | (2, 5) |
| 4 | (1, 7) |
| 5 | (2, 8) |
| 6 | (3, 9) |
| 7 | (2, 11) |
| 8 | (3, 12) |
| 9 | (1, 10) |

**Observation 2.** For any root \( a_j(1) \) of \( G^0 \), the number of roots of \( F_{a_j(1)}^i \) is 3.

Let \( \{ \gamma_i^t : [0,1] \to C \}_{0 \leq t \leq 1} \) be a continuous family of paths constructed by the following way. First choose a real number \( t_0 < 1 \) sufficiently near 1, and for \( t \in [t_0,1] \), let \( \gamma_i^t \) be the straight path joining \( a_0(t) \) and \( a_i(t) \). Next extending the motions of \( a_i(t) \)'s for \( t \in [0,t_0] \), we have an ambient isotopy \( \tau : C \times [0,t_0] \to C \) of \( C \) such that \( \tau(x_1,t_0) = x_1 \) and \( \tau(a_i(t_0),t) = a_i(t) \), \( 1 \leq i \leq 12 \). Finally we set \( \gamma_i(s) = \tau(\gamma_i^t(s),t) \) for \( t \in [0,t_0] \). Note that we may assume that \( \gamma_i^{t_0} = \sqrt{-1} \gamma_i^0 \). This follows from the fact that \( x_1 \in C \) is a root of \( G^i \) if and only if \( \sqrt{-1}x_1 \) is a root of \( G^{i+1} \). Then we have

**Observation 3.** \( \gamma_i^0 \), \( 1 \leq i \leq 9 \) look like Figure 7.2.

By construction the family \( \{ \gamma_i^t \}_{0 \leq t \leq 1} \) satisfies the following three conditions:
1. For each \( t \in [0,1] \), \( \gamma_i^t(0) = a_0(t) \) and \( \gamma_i^t(1) = a_i(t) \),
2. For each \( t \neq 1 \), \( \gamma_i^t \) is a simple path not meeting \( \{ a_j(t) \}_{j \neq \delta, \varepsilon} \).
3. \( \gamma_i^1(s) = a_0(1) = a_i(1) \), for \( s \in [0,1] \).

Let \( C_i(t) \) be the connected component of \( p_{\mu(t)}^{-1}(\gamma_i^t([0,1])) \) containing the critical points of \( p_{\mu(t)} \) over \( a_0(t) \) and \( a_i(t) \). We can draw the picture of \( C_i(0) \) on \( X_0 \) in Figure 6.3, then we see that it is a simple closed curve in \( X_0 \), and isotopic to \( C_i \) if we identify \( X_0 \) with the genus 3 surfaces in Figure 1.1 by an obvious
manner. The simplicity of the roots of $G_{\mu_i(t)}(x_1)$ for $t \in [0,1)$ implies that the
topological type of $p_{\mu_i(t)}$ is the same as $p_0$, therefore $C_i(t)$ is also a simple closed
curve in $X_{\mu_i(t)}$ for $t \in [0,1)$. On the other hand $C_i(1) = p_{\epsilon_i}^{-1}(a_i(1))$ consists of a
single point, which is a unique singular point of $X_{\epsilon_i}$.

\[\begin{array}{cccc}
\gamma_1^0 & \gamma_2^0 & \gamma_3^0 & \gamma_4^0 \\
& & & \\
\gamma_5^0 & \gamma_6^0 & \gamma_7^0 & \gamma_8^0 \\
& & & \\
\gamma_9^0 & & & \\
\end{array}\]

\text{Figure 7.2}

Let $D$ be the unit closed disk and choose a continuous family
$\{i^t_j : D \to C\}_{0 \leq t \leq 1}$ of embeddings of $D$ such that $i^t_j(D)$ contains $\gamma^t_j([0,1])$ and
does not meet $\{a_j(t)\}_{j \neq t, \epsilon}$. Let $A_i(t)$ be the connected component of $p_{\mu_i(t)}^{-1}(i^t_j(D))$
containing $C_i(t)$. For $t \in [0,1)$, $A_i(t)$ is homeomorphic to an annulus, and $A_i(1)$ is homeomorphic to the space obtained from an annulus by collapsing a non-null-
homologous simple closed curve in it.

Let $M_i$ be the quotient space of $X_0 \times [0,1]$ obtained by identifying all of
$\mathcal{C}(0) \times \{1\}$ to a single point. Using $\{i^t_j\}_{0 \leq t \leq 1}$, we have a diffeomorphism

$$\bigcup_{0 \leq t \leq 1} \partial A_i(t) \cong \partial A_i(0) \times [0,1]$$
\( \partial A_i(t) \) is the boundary of \( A_i(t) \) compatible with the natural projections onto \([0,1]\). By the observations, we can extended it to a diffeomorphism

\[
\bigcup_{0 \leq t \leq 1} X_{\mu_i(t)} \setminus \text{Int} A_i(t) \cong (X_0 \setminus \text{Int} A_i(0)) \times [0,1]
\]

\( \text{Int} A_i(t) \) is the interior of \( A_i(t) \). Moreover, using \( \{t'_i\}_{0 \leq t \leq 1} \) again we can extend (7.1) to a homeomorphism from \( \tilde{\pi}^{-1}(\mu_i) = \bigcup_{0 \leq t \leq 1} X_{\mu_i(t)} \) to \( M_i \) also compatible with the projections onto \([0,1]\). Here \( (X_0 \setminus \text{Int} A_i(0)) \times [0,1] \) is understood to be a subspace of \( M_i \) by an obvious manner.

The existence of the homeomorphism \( \tilde{\pi}^{-1}(\mu_i) \cong M_i \) implies that \( C_i(0) \) is the vanishing cycle along \( \mu_i \). In summary, we have proved the following.

**Proposition 7.1.** The monodromy \( \rho'(\lambda_i) \in \Gamma_3 \) is the right hand Dehn twist along \( C_i \).

Now we can complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** We write \( L_0 \) instead \( L = L(7/8, 3/4) \) and let \( L_1 \) be a line of \( P^3 \) meeting \( X^c \) transversely. Choose a base point \( b_1 \in L_1 \setminus (L_1 \cap X^c) \). Since the set of all lines of \( P^3 \) meeting \( X^c \) transversely is Zariski open hence path connected, there exist a continuous path \( \{L(t)\}_{t \in [0,1]} \) of lines of \( P^3 \) such that \( L(t) \) meets \( X^c \) transversely and \( L(0) = L_0 \), \( L(1) = L_1 \). Let \( \mathcal{F}' := \{(x,H) \in P_3 \times (\tilde{L}(t) \setminus (\tilde{L}(t) \cap X^c)) \cap H \} \). Then there exist continuous families of homeomorphisms \( \{\psi_t : L_0 \setminus (L_0 \cap X^c) \to L(t) \setminus (L(t) \cap X^c)\}_{0 \leq t \leq 1} \) and \( \{\Psi_t : \mathcal{F}' \to \mathcal{F}'\}_{0 \leq t \leq 1} \) such that \( \pi'_t \circ \Psi_t = \psi_t \circ \pi'_t \) where \( \pi'_t \) is the second projection. Now \( \{\psi_t(\lambda_i)\}_{t} \) is a standard generating system for \( \pi_1(L_1 \setminus (L_1 \cap X^c), \psi_1(0)) \) such that the image of \( \psi_1(\lambda_i) \) under the associated topological monodromy is the right hand Dehn twist along \( C_i \). The result follows by considering an isomorphism \( \pi_1(L_1 \setminus (L_1 \cap X^c), \psi_1(0)) \cong \pi_1(L_1 \setminus (L_1 \cap X^c), b_1) \) induced by a path from \( \psi_1(0) \) to \( b_1 \).

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**References**

[1] K. AharA and I. Awata, On the global monodromy of a fibration of the Fermat surface of degree \( n \) (full version), MIMS preprint, 2009.

[2] K. AharA, On the topology of Fermat type surface of degree 5 and the numerical analysis of algebraic curves, Tokyo J. Math. 16 (1993), 321–340.

[3] A. Beauville, Le groupe de monodromie des familles universelles d’hypersurfaces et d’intersections complètes, Lecture notes in math. 1194, Springer Berlin, 1986, 8–18.
I. Dolgachev and A. Libgober, On the fundamental group of the complement to a discriminant variety, Algebraic geometry, Chicago, 1980, Lecture notes in math. 862, Springer-Verlag New York, 1981, 1–25.

R. Gompf and A. Stipsicz, 4-Manifolds and Kirby Calculus, Graduate studies in mathematics 20, American Mathematical Society, 1999.

H. Hamm, Lefschetz theorems for singular varieties, Part I, Arcata Singularities Conference, Proc. symp. pure math. 40, American Mathematical Society, 1983, 547–557.

N. V. Ivanov, Mapping class groups, Handbook of geometric topology, Elsevier Science B. V. 2002, 523–633.

N. Katz, Pinceaux de Lefschetz: théorème d’existence, Groupe de monodromy en geometrie algebrique [SGA 7 II], Lecture notes in math. 340, Springer-Verlag Berlin, 1973, 212–253.

K. Lamotke, The topology of complex projective varieties after S. Lefschetz, Topology 20 (1981), 15–51.

Y. Matsumoto, On the topological structure of the Fermat surface of degree 5, Workshop on Geometry and Topology (Hanoi, 1993), Kodai Math. J. 17 (1994), 560–570.

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