Brouwer’s Fan Theorem as an axiom and as a contrast to Kleene’s alternative

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Abstract  The paper is a contribution to intuitionistic reverse mathematics. We introduce a formal system called Basic Intuitionistic Mathematics BIM, and then search for statements that are, over BIM, equivalent to Brouwer’s Fan Theorem or to its positive denial, Kleene’s Alternative to the Fan Theorem. The Fan Theorem is true under the intended intuitionistic interpretation and Kleene’s Alternative is true in the model of BIM consisting of the Turing-computable functions. The task of finding equivalents of Kleene’s Alternative is, intuitionistically, a nontrivial extension of the task of finding equivalents of the Fan Theorem, although there is a certain symmetry in the arguments that we shall try to make transparent. We introduce closed-and-separable subsets of Baire space $\mathcal{N}$ and of the set $\mathcal{R}$ of the real numbers. Such sets may be compact and also positively noncompact. The Fan Theorem is the statement that Cantor space $C$, or, equivalently, the unit interval $[0, 1]$, is compact and Kleene’s Alternative is the statement that $C$, or, equivalently, $[0, 1]$, is positively noncompact. The class of the...
compact closed-and-separable sets and also the class of the closed-and-separable sets that are positively noncompact are characterized in many different ways and a host of equivalents of both the Fan Theorem and Kleene’s Alternative is found.

**Keywords**  Intuitionistic reverse mathematics · Fan Theorem · Weak König’s Lemma

**Mathematics Subject Classification**  03F55 · 03F60 · 03F35 · 03B30

1 Introduction

1.1 Brouwer’s program

L.E.J. Brouwer observed that, in mathematics, disjunctive and existential statements are sometimes interpreted constructively, and sometimes not. The so-called logical constants have no fixed meaning. As a consequence, mathematical propositions raise false expectations, like the principle of the excluded third one uses when proving them. Brouwer wanted to remove this ambiguity. He suggested to take the constructive reading of the logical constants as the first and more important one. The principle of the excluded middle \( P \lor \neg P \), and many theorems proven by it, then fail to be true.

The classical mathematician might feel he is wilfully misunderstood and retort that one should not try to change the *proper* meaning of, for instance, a disjunction \( P \lor Q \), which always is a weak one, perhaps, as the constructivists say: \( \neg (\neg P \land \neg Q) \): the assumption that both \( P \) and \( Q \) lead to a contradiction, leads to a contradiction.

This somewhat evasive reaction does not diminish the value of the undertaking to try to see if mathematical propositions make constructive sense. It is useful to attempt to approximate the somewhat elusive classical meaning of a statement by a circumscription in the constructive language. The intuitionistic mathematician, using a language that seems to be both more precise and more expressive than the language of the classical mathematician, sets himself the task to find out which of the various possible constructive explications of a classical statement make the most sense.

As is well-known, the logic that rules the behaviour of the constructively interpreted logical constants is a restriction of classical, that is: non-intuitionistic, logic: one obtains classical logic if one extends the rules of intuitionistic logic by the extra rule \( P \lor \neg P \), or, equivalently, the extra rule \( \neg \neg P \rightarrow P \). In a formal context, avoiding these extra rules, and starting from axioms permitted by the classical mathematician, one can not obtain conclusions that would be judged to be false by the classical mathematician.

However, Brouwer’s advice consisted of more than the ban on \( P \lor \neg P \): examining the concept of the continuum, he proposed new mathematical axioms. These axioms deserve to be studied seriously. The most important two of them are Brouwer’s Continuity Principle and Brouwer’s Thesis on Bars. The Continuity Principle makes no sense if one does not, like Brouwer, take the logical constants in their constructive sense, and neither does the Thesis on Bars, although, unlike in the case of the Continuity Principle, the formula expressing this Thesis is judged to be true by the classical reader. Brouwer used these axioms in order to prove intuitionistically that every function from the closed real interval \([0, 1]\) to the set \( \mathcal{R} \) of the real numbers is uniformly continuous.
Brouwer never put the truth of his axioms publicly into question. His axioms were not like scientific hypotheses to be confirmed or refuted by evidence later to be found. His axioms were the expression and explanation of his way of using and interpreting the logical constants and understanding the continuum. He assumed the reader to share his perception as the only natural one and did not want to invite him to doubt.

Of course, even Brouwer did not find absolute truth. His axioms should be considered as proposals for our common mathematical discourse. One should try them and reflect on them and, once one has done so, one could decide to accept, to revise or to reject them.

Brouwer abhorred a formal style of presenting his mathematics. He never used the word *axiom* for one of his assumptions. Only later, by the efforts of Kleene et al., see [19,22] and [34], intuitionistic analysis has been given a formal representation. It is possible now to formulate and discuss questions about the precise strength and import of Brouwer’s axioms.

In this paper we want to study the power of the Fan Theorem, the most famous consequence of Brouwer’s much further reaching Thesis on Bars, as an axiom in itself.

We also study the force of *Kleene’s Alternative to the Fan Theorem*, a positive denial of the Fan Theorem that becomes true if one assumes: every function from \( \mathbb{N} \) to \( \mathbb{N} \) is given by an algorithm.

### 1.2 The contents of the paper

We now describe the contents of the further sections of this paper.

In Sect. 2 we try to see how Brouwer convinced himself of the truth of his axioms, and, in particular, of the truth of the Fan Theorem.

In Sect. 3 we consider recursive counter-examples to the strict Fan Theorem found by Kleene and Specker.

In Sect. 4 we describe the classical argument proving D. König’s Infinity Lemma and we explain why both the proof and the Lemma itself fail from a constructive point of view.

In Sect. 5 we discuss the Reverse Mathematics Programme initiated by H. Friedman and S. Simpson, and we explain the possible meaning of such a programme for the intuitionistic mathematician.

In Sect. 6 we propose a formal system for basic intuitionistic mathematics, called \textbf{BIM}, that might serve as a starting point for Intuitionistic Reverse Mathematics.

In Sect. 7 we study closed-and-separable subsets of Baire space \( \mathcal{N} \). Such sets may be ‘compact’ and also may ‘positively fail to be compact’ and we see that both properties may be formulated in many ways.

In Sect. 8 we study closed-and-separable subsets of the set \( \mathcal{R} \) of the real numbers. Again, such sets may be ‘compact’ and also may ‘positively fail to be compact’ and we see that both properties may be formulated in many ways.

In Sect. 9 we prove that the Fan Theorem is equivalent to the statement that Cantor space \( \mathcal{C} \) is compact and to the statement that the real segment \([0, 1]\) is compact. We also prove that Kleene’s Alternative is equivalent to the statement that Cantor space \( \mathcal{C} \) positively fails to be compact, and to the statement that the real segment \([0, 1]\) positively fails to be compact. We finally prove a result, due to Frank Waaldijk, saying

\[ \square \text{ Springer} \]
that the Fan Theorem is equivalent to the statement that the composition of two functions continuous in the sense of E. Bishop is itself continuous in the sense of Bishop, and we see that Kleene’s Alternative is equivalent to the statement that there are two functions continuous in the sense of Bishop whose composition strongly fails to have this property.

In Sect. 10 we mention some equivalents of the Fan Theorem to be discussed elsewhere, and we briefly indicate some further questions and results in intuitionistic reverse mathematics.

I want to express my thanks to the referees of the paper and its earlier versions. Their suggestions and criticisms have led to various improvements.

2 Brouwer’s axioms

2.1 The continuity principle

We let \( \mathbb{N} \) denote the set of the natural numbers 0, 1, 2, \ldots and \( \mathcal{N} \) the set of all infinite sequences \( \alpha = (\alpha(0), \alpha(1), \alpha(2) \ldots) \) of natural numbers, that is, the set of all functions from \( \mathbb{N} \) to \( \mathbb{N} \). We let \( \mathbb{N}^* \) denote the set of all finite sequences \( s = (s(0), s(1), \ldots, s(n - 1)) \) of natural numbers. The number \( n \) is called the length of the finite sequence \( s = (s(0), s(1), \ldots, s(n - 1)) \).

For all \( \alpha \) in \( \mathcal{N} \), for all \( s \) in \( \mathbb{N}^* \), we say that the finite sequence \( s \) is an initial part of the infinite sequence \( \alpha \), or that \( \alpha \) passes through \( s \), or that \( s \) contains \( \alpha \), if and only if there exists \( n \) such that \( s = (\alpha(0), \alpha(1), \ldots, \alpha(n - 1)) \).

Brouwer’s Continuity Principle For every subset \( R \) of \( \mathcal{N} \times \mathbb{N} \), if, for every \( \alpha \) in \( \mathcal{N} \), we are able to find \( n \) in \( \mathbb{N} \) such that \( \alpha R n \), then for every \( \alpha \) in \( \mathcal{N} \) we are able to find \( m, n \) in \( \mathbb{N} \) such that for every \( \beta \) in \( \mathcal{N} \), if \( \beta \) passes through \( (\alpha(0), \alpha(1), \ldots, \alpha(m - 1)) \), then \( \beta R n \).

(We write “\( \alpha R n \)” while intending “\( (\alpha, n) \in R \)” and will do similarly in similar cases).

The Continuity Principle expresses the view that, if one is really able to find for every possible infinite sequence \( \alpha \) of natural numbers a suitable natural number \( n \), one must be able to find an appropriate number also if the sequence is given by a black box, step by step, without information on its evolution as a whole. The moment the appropriate number is given, only finitely many values of the infinite sequence will have been revealed.

The Continuity Principle is a truly revolutionary principle, making intuitionistic mathematics very different from its classical counterpart, see [39]. In this paper, its role is rather modest.

In Sect. 7.1.4, we want to refer to the following stronger version of the Continuity Principle:

Second Axiom of Continuous Choice For every subset \( R \) of \( \mathcal{N} \times \mathcal{N} \), if, for every \( \alpha \) in \( \mathcal{N} \) we are able to find \( \beta \) in \( \mathcal{N} \) such that \( \alpha R \beta \), then there exists \( \phi \) in \( \mathcal{N} \) enumerating a continuous function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( \alpha, \alpha R (\phi|\alpha) \).

In Sect. 7.5, the notion of an enumerable continuous function from \( \mathcal{N} \) to \( \mathcal{N} \) will be explained.
2.2 The thesis on bars in $\mathcal{N}$

For all $s = (s(0), s(1), \ldots, s(m - 1))$, $t = (t(0), t(1), \ldots, t(n - 1))$ in $\mathbb{N}^*$ we let $s \ast t$ be the finite sequence that we obtain by putting $t$ behind $s$, that is $s \ast t = (s(0), s(1), \ldots, s(m - 1), t(0), t(1), \ldots, t(n - 1))$.

For all $s, t$ in $\mathbb{N}^*$, we define: $s$ is an initial part of $t$, or: $t$ is an extension of $s$, notation: $s \subseteq t$, if and only if there exists $u$ in $\mathbb{N}^*$ such that $s = t \ast u$.

Let $B$ be a subset of $\mathbb{N}^*$. For every $s$ in $\mathbb{N}^*$, we define: $s$ meets $B$ if and only there exists $t$ in $B$ such that $t \subseteq s$. For every $\alpha$ in $\mathcal{N}$, we define: $\alpha$ meets $B$, or: $B$ bars $\alpha$, if and only if there exists $n$ such that $(\alpha(0), \alpha(1), \ldots, \alpha(n - 1))$ is in $B$. For every subset $X$ of $\mathcal{N}$, we define: $B$ is a bar in $X$ if and only if every $\alpha$ in $X$ meets $B$.

For every $s$ in $\mathbb{N}^*$, we let $\mathcal{N}_s$ be the set all $\alpha$ in $\mathcal{N}$ passing through $s$.

Brouwer asked the Kantian question how, if some subset $B$ of $\mathbb{N}^*$ is a bar in $\mathcal{N}$, it is possible that one has knowledge of this fact. He became convinced that every proof of the statement $B$ is a bar in $\mathcal{N}$ must be reducible, in some sense, to a canonical proof of that statement.

Such a canonical proof is an arrangement of statements. Each statement either has the form: $s$ belongs to $B$ or has the form: $B$ is a bar in $\mathcal{N}_s$.

The opening statements of a canonical proof are statements of the form: $s$ belongs to $B$. They are followed by the immediate conclusion: and therefore: $B$ is a bar in $\mathcal{N}_s$. There are two kinds of reasoning steps. A reasoning step of the first kind, or: a forward step, is a step with infinitely many premises. It has the form: $B$ is a bar in $\mathcal{N}_{s\ast(0)}$, $B$ is a bar in $\mathcal{N}_{s\ast(1)}$, $B$ is a bar in $\mathcal{N}_{s\ast(2)}$, $\ldots$, and therefore: $B$ is a bar in $\mathcal{N}_s$.

A reasoning step of the second kind, or: a backward step, has the form: $B$ is a bar in $\mathcal{N}_s$, and therefore: $B$ is a bar in $\mathcal{N}_{s\ast(n)}$. The conclusion of the proof is the statement: $B$ is a bar in $\mathcal{N} = \mathcal{N}_0$.

In a similar way, for every $t$ in $\mathbb{N}^*$, a canonical proof of the statement: $B$ is a bar in $\mathcal{N}_t$ is a proof with the same opening statements and reasoning steps, but now the conclusion of the proof is the statement: $B$ is a bar in $\mathcal{N}_t$. According to Brouwer, if $B$ is a bar in $\mathcal{N}_t$, there exists a canonical proof of the statement: $B$ is a bar in $\mathcal{N}_t$.

A canonical proof may be visualised as a tree. Because of the occurrence of reasoning steps of the first kind, the tree is, in general, an infinite tree.

Let $X$ be a subset of $\mathbb{N}^*$.

$X$ is an inductive subset of $X$ if and only if, for every $s$ in $\mathbb{N}^*$, if, for every $n, s \ast (n)$ belongs to $X$, then $s$ itself belongs to $X$.

$X$ is a monotone subset of $\mathbb{N}^*$ if and only if for every $s$ in $\mathbb{N}^*$, if $s$ belongs to $X$, then, for every $n, s \ast (n)$ belongs to $X$.

Let $B$ be a subset of $\mathbb{N}^*$. For each $s$ in $\mathbb{N}^*$, we define: $s$ is secured by $B$ if and only if there is a canonical proof of the statement: $B$ is a bar in $\mathcal{N}_s$. We let $\text{Sec}(B)$ be the set of all finite sequences of natural numbers that are secured by $B$.

Note that $\text{Sec}(B)$ is a monotone and inductive subset of $\mathbb{N}^*$, and that $B$ is a subset of $\text{Sec}(B)$.

Suppose that $C$ is another monotone and inductive subset of $\mathbb{N}^*$ including $B$, and suppose that $s$ belongs to $\text{Sec}(B)$. Now take a canonical proof of the statement: $B$ is a bar in $\mathcal{N}_s$, and replace, in this proof, every statement of the form: $B$ is a bar in $\mathcal{N}_u$ by the statement: $u$ belongs to $C$. The result is a proof of the statement: $s$ belongs to $C$.  

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We thus see that \( \text{Sec}(B) \) is a subset of \( C \).

Clearly, \( \text{Sec}(B) \) is the least monotone and inductive subset of \( \mathbb{N}^* \) including \( B \).

One might define \( \text{Sec}(B) \) as the least monotone and inductive subset of \( \mathbb{N}^* \) including \( B \), but the intuitionistic mathematician does not accept such impredicative definitions immediately.

We now may formulate Brouwer’s claim as follows:

**Brouwer’s Thesis on Bars in \( \mathcal{N} \)** For every subset \( B \) of \( \mathbb{N}^* \), if \( B \) is a bar in \( \mathcal{N} \), then the empty sequence \( ( \) \) belongs to the set \( \text{Sec}(B) \), that is, to the least monotone and inductive subset of \( \mathbb{N}^* \) including \( B \).

One may ask if the backward steps in a canonical proof may be avoided. Brouwer sometimes made his readers think this is possible.

I am grateful to one of the referees of this paper who called my attention to the fact that this question was treated also by Martino and Giaretta in [27]. Their conclusions are more or less the same as the conclusions in [46] and in the following discussion.

Let \( B \) be a subset of \( \mathbb{N}^* \). For every \( s \) in \( \mathbb{N}^* \), we define: \( s \) is forwardly secured by \( B \) if and only if there exists a canonical proof of the statement: \( B \) is a bar in \( \mathcal{N} \), that uses only the opening statements and the forward reasoning steps. We let \( S\text{sec}(B) \) be the set of all finite sequences of natural numbers that are forwardly secured by \( B \). Arguing as above, we find that \( S\text{sec}(B) \) is the least inductive set including \( B \). Note that \( S\text{sec}(B) \) is a subset of \( \text{Sec}(B) \).

Let \( X \) be a subset of \( \mathbb{N}^* \). \( X \) is a decidable subset of \( \mathbb{N}^* \), if for each \( s \) in \( \mathbb{N}^* \), either \( s \in X \) or \( s \notin X \).

Note that, if \( X \) is a decidable subset of \( \mathbb{N}^* \), then one may decide, for every \( s \) in \( \mathbb{N}^* \): either \( s \) meets \( X \) or \( s \) does not meet \( X \).

**First Observation** For every decidable subset \( B \) of \( \mathbb{N}^* \), for every \( s \) in \( \text{Sec}(B) \), either \( s \) meets \( B \), or \( s \) belongs to \( S\text{sec}(B) \).

(This observation was also made by Martino and Giaretta, see [27, p. 554], the proof of Proposition 3.3(b).)

**Proof** Let \( B \) be a decidable subset of \( \mathbb{N}^* \) and let \( C \) be the set of all \( s \) in \( \mathbb{N}^* \) with the property: if \( s \) does not meet \( B \), then \( s \) belongs to \( S\text{sec}(B) \). We now prove that the set \( C \) is monotone and inductive and then are sure that \( \text{Sec}(B) \) is a subset of \( C \).

Let us first show that \( C \) is monotone. Assume that \( s \) belongs to \( C \), and \( n \) to \( \mathbb{N} \) and that \( s \ast (n) \) does not meet \( B \). Then also \( s \) does not meet \( B \) and therefore: \( s \) belongs to \( S\text{sec}(B) \). Therefore: either \( s \) belongs to \( B \), or, for each \( i \), \( s \ast (i) \) belongs to \( S\text{sec}(B) \). As \( s \) does not meet \( B \), we may conclude: \( s \ast (n) \) belongs to \( S\text{sec}(B) \).

Clearly then, either \( s \ast (n) \) meets \( B \) or \( s \ast (n) \) belongs to \( S\text{sec}(B) \), that is: \( s \ast (n) \) belongs to \( C \).

Let us now prove that \( C \) is inductive. Assume that \( s \) belongs to \( \mathbb{N}^* \) and that, for each \( n \), \( s \ast (n) \) belongs to \( C \). Assume \( s \) does not meet \( B \). For each \( n \), either \( s \ast (n) \) belongs to \( B \), and therefore: \( s \ast (n) \) belongs to \( S\text{sec}(B) \) or \( s \ast (n) \) does not meet \( B \) and, therefore, as \( s \ast (n) \) belongs to \( C \), \( s \ast (n) \) belongs to \( S\text{sec}(B) \). We thus see: for each \( n \), \( s \ast (n) \) belongs to \( S\text{sec}(B) \). It follows that \( s \) itself belongs to \( S\text{sec}(B) \).

Clearly then, either \( s \) meets \( B \) or \( s \) belongs to \( S\text{sec}(B) \), that is: \( s \) belongs to \( C \).
Note that Brouwer’s Thesis, together with this First Observation, implies: *for every decidable subset $B$ of $\mathbb{N}^*$: if $B$ is a bar in $\mathcal{N}$, then $(\ )$ belongs to $S_{sec}(B)$.*

This First Observation is no longer true if we omit the condition: *$B$ is a decidable subset of $\mathbb{N}$: there exists a subset $B$ of $\mathbb{N}^*$ such $(\ )$ belongs to $Sec(B)$, but we can not prove: $(\ )$ belongs to $S_{sec}(B)$.*

The following example, due to Kleene, see Section 7.14 in [22], makes this clear.

Let $\alpha$ be an infinite sequence of natural numbers such that we are unable to decide: $\exists n[\alpha(n) = 0]$ or $\forall n[\alpha(n) \neq 0]$. Let $B$ be the subset of $\mathbb{N}^*$ consisting of all $s$ such that either: for some $n$, $s = (n)$ and $\alpha(n) \neq 0$ or: $s = (\ )$ and $\exists m[\alpha(m) = 0]$. Note that, for each $n$, either $\alpha(n) \neq 0$, and therefore: $(n)$ belongs to $Sec(B)$, or: $\alpha(n) = 0$, and therefore: $(\ )$ belongs to $Sec(B)$, and therefore: $(n)$ belongs to $Sec(B)$. We may conclude: $(\ )$ belongs to $S_{sec}(B)$.

On the other hand, if $(\ )$ belongs to $S_{sec}(B)$, then either: $(\ )$ belongs to $B$ and $\exists m[\alpha(m) = 0]$, or: for all $n$, $(n)$ belongs to $S_{sec}(B)$, and therefore: for all $n$, $(n)$ belongs to $B$, and $\alpha(n) \neq 0$. That is, if $(\ )$ belongs to $S_{sec}(B)$, then we may decide: $\exists n[\alpha(n) = 0]$ or $\forall n[\alpha(n) \neq 0]$.

Clearly then, $(\ )$ is secured by $B$, but we are unable to prove: $(\ )$ is forwardly secured by $B$.

The above formulation of Brouwer’s Thesis on Bars in $\mathcal{N}$: *for every subset $B$ of $\mathbb{N}^*$, if $B$ is a bar in $\mathcal{N}$, then $(\ ) \in Sec(B)$*, differs from Brouwer’s own formulation but seems to come close to his intentions, see [46].

Brouwer, in [13] and [10], claims: *for every subset $B$ of $\mathbb{N}^*$, if $B$ is a bar in $\mathcal{N}$, then $(\ ) \in S_{sec}(B)$.*

Kleene’s example shows this claim is wrong. Kleene himself conjectured that Brouwer, who used the expression *effective bar*, intended the bar $B$ to be a decidable subset of $\mathbb{N}^*$. As we just saw, this special case of Brouwer’s claim indeed follows from the Thesis on Bars in $\mathcal{N}$ as formulated above.

M. Dummett, in [15], rightly concluded from Kleene’s counterexample that Brouwer had made some mistake. Martino and Giaretta [27] discussed Dummett’s treatment of Brouwer’s argument. In [16], Dummett expresses his agreement with Martino and Giaretta.

It seems to me that Brouwer did not think of a restriction on the set $B$, certainly not in his 1954 publication [13], and probably also not in his 1927 publication [10], as he is quoting [10] in [13]. Brouwer’s claim, the *bar theorem*, seems to be an incorrect conclusion of the Thesis on Bars in $\mathcal{N}$ as formulated above. Brouwer probably at first had this Thesis in mind, but then deluded himself about the eliminability of the backward steps.

Kleene also noted that a strong form of Brouwer’s Continuity Principle may be used to conclude that Brouwer’s claim extends from the case that $B$ is a decidable subset of $\mathbb{N}$ to the case that $B$ is a monotone subset of $\mathbb{N}^*$. The following observation shows that this extension also follows from the Thesis on Bars in $\mathcal{N}$ as formulated above. One then may avoid the use of the Continuity Principle.

**Second Observation** *For every monotone subset $B$ of $\mathbb{N}^*$, for every $s$ in $\mathbb{N}^*$, if $s$ belongs to $Sec(B)$, then $s$ belongs to $S_{sec}(B)$.*

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Proof Let $B$ be a monotone subset of $\mathbb{N}^*$ and consider $Ssec(B)$. We now prove that the set $Ssec(B)$ is monotone and then, as $Ssec(B)$ is also inductive, are sure that $Sec(B)$ is a subset of $Ssec(B)$.

The proof that $Ssec(B)$ is monotone is easy. Assume that $s$ belongs to $Ssec(B)$, and $n$ to $\mathbb{N}$. Either $s$ belongs to $B$, and therefore: $s \ast (n)$ belongs to $B$ and to $Ssec(B)$ or, for each $i$, $s \ast (i)$ belongs to $Ssec(B)$. In both cases: $s \ast (n)$ belongs to $Ssec(B)$.

Clearly then, if $s$ belongs to $Ssec(B)$, then, for each $n, s \ast (n)$ belongs to $Ssec(B)$. □

2.3 The thesis on bars in $C$

Brouwer’s Thesis on Bars in $\mathcal{N}$ is a strong statement, too strong, actually, if one only wants to prove that every continuous function from $[0, 1]$ to $\mathcal{R}$ is uniformly continuous.

Let Cantor space $C$ be the set of all $\omega$ in $\mathcal{N}$ that assume no other values than 0, 1. Let $[0, 1]^*$ be the set of all finite sequences $s$ of natural numbers that assume no other values than 0, 1.

For every subset $B$ of $[0, 1]^*$, for every $n$, we define a subset $Sec_0(B)$ of $[0, 1]^*$, the set of all elements of $[0, 1]^*$ secured in at most $n$ stages by $B$ within $C$, as follows, by induction.

$Sec_0(B)$ coincides with $B$, and, for each $n$, for each $s$ in $[0, 1]^*$, $s$ belongs to $Sec_{n+1}(B)$ if and only if either: $s$ belongs to $B$ or: both $s \ast (0)$ and $s \ast (1)$ belong to $Sec_n(B)$, or: length({$s$}) $> 0$ and there exist $t$ in $[0, 1]^*$ and $i$ in $\{0, 1\}$ such that $s = t \ast (i)$ and $t$ belongs to $Sec_n(B)$.

One may prove, by induction: for each $n$, $Sec_n(B)$ is a subset of $Sec_{n+1}(B)$.

We let $Sec(B)$ be the set $\bigcup_{n \in \mathbb{N}} Sec_n(B)$.

Let $X$ be a subset of $[0, 1]^*$. Let $X$ be $01$-inductive if and only if, for each $t$ in $[0, 1]^*$, if $t$ belongs to $X$ then both $t \ast (0)$ and $t \ast (1)$ belong to $X$.

Let $B$ be a subset of $[0, 1]^*$. Note that $Sec(B)$ is a $01$-monotone and $01$-inductive subset of $[0, 1]^*$ including $B$.

Suppose that $C$ is another subset of $[0, 1]^*$ that is both $01$-monotone and $01$-inductive and includes $B$. Using induction, we may prove: for each $n$, $Sec_n(B)$ is a subset of $C$, and therefore: $Sec(B)$ is a subset of $C$. $Sec(B)$ thus is seen to be the least $01$-monotone and $01$-inductive subset of $[0, 1]^*$ including $B$.

Third Observation For every subset $B$ of $[0, 1]^*$, for every $s$ in $[0, 1]^*$, if $s$ belongs to $Sec(B)$, then there exists a finite subset $B'$ of $B$ such that $s$ belongs to $Sec(B')$.

Proof Let $B$ be a subset of $[0, 1]^*$.

Let $C$ be the set of all $s$ in $[0, 1]^*$ such that there exists a finite subset $B'$ of $B$ with the property: $s$ belongs to $Sec(B')$. □
Note that $B$ is a subset of $C$. Also note that $C$ is 01-monotone as, for each $s$ in \{0, 1\}*, for every finite subset $B'$ of $B$, if $s$ belongs to $Sec^{01}(B')$, then both $s * (0)$ and $s * (1)$ belong to $Sec^{01}(B')$. Finally, note that $C$ is 01-inductive as, for all finite subsets $B'$, $B''$ of $B$, if $s * (0)$ belongs to $Sec^{01}(B')$ and $s * (1)$ belongs to $Sec^{01}(B'')$, then $s$ belongs to $Sec^{01}(B' \cup B'')$.

As $C$ is both 01-monotone and 01-inductive and includes $B$, $Sec(B)$ is a subset of $C$, and we are done.

We let $Ssec^{01}(B)$ be the least 01-inductive subset of \{0, 1\}* including $B$. This set may be introduced in a way similar to the way we introduced the set $Sec^{01}(B)$.

For every $s$ in \{0, 1\}*, we say: $s$ is forwardly secured by $B$ if and only if $s$ belongs to $Ssec^{01}(B)$.

**Fourth Observation** For every subset $B$ of \{0, 1\}*, for every $s$ in \{0, 1\}*, if $s$ belongs to $Sec^{01}(B)$, then either $s$ meets $B$ or $s$ belongs to $Ssec^{01}(B)$.

**Proof** In view of the Third Observation it suffices to prove the statement for finite subsets $B$ of \{0, 1\}*. Let $B$ be a finite subset of \{0, 1\}* and let $C$ be the set of all $s$ in \{0, 1\}* such that either $s$ meets $B$ or $s$ belongs to $Sec^{01}(B)$.

Note that $B$ is a subset of $C$.

We now prove that $C$ is 01-monotone. Let $s$ belong to $C$ and consider $s * (0)$. As $B$ is finite, we may decide if $s * (0)$ meets $B$ or not. If $s * (0)$ does not meet $B$, then also $s$ does not meet $B$, and, therefore, $s$ belongs to $Sec(B)$, that is, either $s$ belongs to $B$, or both $s * (0)$ and $s * (1)$ belong to $Sec(B)$. As $s$ does not belong to $B$, we may conclude: $s * (0)$ belongs to $Ssec(B)$ and thus to $C$. In a similar way, one proves: $s * (1)$ belongs to $C$.

It is not difficult to see that $C$ is 01-inductive. Let $s$ belong to $C$ and assume both $s * (0)$ and $s * (1)$ belong to $C$. Either $s$ meets $B$ or both $s * (0)$ and $s * (1)$ belong to $B$ or to $Sec(B)$ and thus to $Ssec(B)$. In the latter case, it follows that $s$ itself belongs to $Sec(B)$ and thus to $C$.

Note that, if $B$ is finite, then we may decide, for every $s$ in \{0, 1\}* if $s$ meets $B$ or not. It now suffices to show, for every finite subset $B$ of \{0, 1\}*, for every $s$ in \{0, 1\}*, if $s$ belongs to $Ssec^{01}(B)$ and $s$ does not meet $B$, then both $s * (0)$ and $s * (1)$ belong to $Sec^{01}(B)$.

As $C$ is both 01-monotone and 01-inductive and includes $B$, $Sec(B)$ is a subset of $C$, and we are done.

We now present Brouwer’s Thesis on Bars in $C$ in two versions:

**Brouwer’s Thesis on Bars in $C$: the Fan Theorem**

(i) For every subset $B$ of \{0, 1\}*, if $B$ is a bar in $C$, then the empty sequence ( ) belongs to $Sec^{01}(B)$.

(ii) For every subset $B$ of \{0, 1\}*, if $B$ is a bar in $C$, then some finite subset of $B$ is a bar in $C$.

It is the second version that we intend to study in the remaining part of this paper.

Using both the Fan Theorem and Brouwer’s Continuity Principle one obtains the following conclusion.

\[ Springer \]
**Extended Fan Theorem** For every subset $R$ of $C \times \mathbb{N}$, if for every $\alpha$ in $C$ one may find $n$ in $\mathbb{N}$ such that $\alpha Rn$, then there exists $m$ in $\mathbb{N}$ with the property that for every $\alpha$ in $C$ one may find $n \leq m$ such that $\alpha Rn$.

The argument is as follows. Let $R$ be a subset of $C \times \mathbb{N}$ and assume that, for every $\alpha$ in $C$, there exists $n$ such that $\alpha Rn$. Let $B$ be the set of all finite sequences $s$ in $\{0, 1\}^*$ with the property that, for some $n$, for every $\alpha$ in $C$ passing through $s$, $\alpha Rn$. Brouwer’s Continuity Principle guarantees that $B$ is a bar in $C$. Using the Fan Theorem we find a finite subset $B'$ of $B$ that is a bar in $C$. We now determine, for every $s$ in $B'$, a natural number $n$ such that, for every $\alpha$ in $C$ passing through $s$, $\alpha Rn$. Let $m$ be the largest one among the finitely many natural numbers thus found.

The Fan Theorem can be formulated more generally. Cantor space $C$ is a so-called fan, but there are more fans than Cantor space.

A subset $F$ of $\mathbb{N}$ is called a fan or a finitary spread if and only if (i) $F$ is closed, that is, for every $\alpha$ in $\mathbb{N}$, if every initial part of $\alpha$ contains an element of $F$, then $\alpha$ itself belongs to $F$, and (ii) there exists a function $\beta$ from $\mathbb{N}^*$ to $\{0, 1\}$ such that, for all $s$ in $\mathbb{N}^*$, $\beta(s) = 0$ if and only if $s$ contains an element of $F$, and (iii) for every $s$ in $\mathbb{N}^*$ there are only finitely many numbers $n$ such that $s \ast (n)$ contains an element of $F$.

**Fan Theorem, general formulation** Let $F$ be a subset of $\mathbb{N}$ and a fan. For every subset $B$ of $\mathbb{N}^*$, if $B$ is a bar in $F$, then some finite subset of $B$ is a bar in $F$.

Using the Continuity Principle we obtain again a stronger statement.

**Extended Fan Theorem, general formulation** Let $F$ be a subset of $\mathbb{N}$ and a fan. For every subset $R$ of $F \times \mathbb{N}$, if for every $\alpha$ in $F$ we are able to find $n$ in $\mathbb{N}$ such that $\alpha Rn$, then there exists $m$ in $\mathbb{N}$ with the property that for every $\alpha$ in $F$ we are able to find $n \leq m$ such that $\alpha Rn$.

In [13,18] and [22], the name Fan Theorem is used for what we call the Extended Fan Theorem.

In the formal system to be introduced in Sect. 6 we introduce the strict Fan Theorem.

The strict Fan Theorem is obtained from the Fan Theorem by adding the condition that the set $B$ is a decidable subset of $\{0, 1\}^*$, in the strong sense that there exists a function $\alpha$ from $\{0, 1\}^*$ to $\{0, 1\}$ such that, for each $s$ in $\{0, 1\}^*$, $s \in B$ if and only if $\alpha(s) = 1$.

From then on we shall often use the name ‘Fan Theorem’ for the strict version.

Kleene and Vesley, put on their guard by Brouwer’s mistake with the bar theorem, also consider only the strict version of the Fan Theorem, see [22], *26.6a.

### 2.4 Countable choice

The following axiom would probably have been accepted by Brouwer and is mentioned in [22] and [19].

**First Axiom of Countable Choice** For every subset $R$ of $\mathbb{N} \times \mathbb{N}$, if, for each $m$, there exists $n$ such that $m Rn$, then there exists $\alpha$ such that, for every $m$, $m Ra(m)$.

Once one recognizes the possibility of constructing an infinite sequence of natural numbers step by step, choosing one value after another, without prescribing all future values at once by means of an algorithm, it seems legitimate to subscribe to this axiom. The complexity of the relation $R$ does not seem to play a rôle in this intuitive justifica-
tion. Nevertheless, we sometimes want to restrict this complexity and find out what the strength of the resulting axiom is. Such restrictions are also considered in [20] and [28].

Note that, if one assumes that every function in \( N \) is computable in Turing’s sense, it is not so obvious that the First Axiom of Countable Choice is true, although there are metamathematical results that seem to support this thought.

In Sect. 6 we shall consider some restricted versions of the First Axiom of Countable Choice.

In intuitionistic analysis, some other axioms of countable choice sometimes play a rôle, see [46] and [22], but we do not intend to use them in this paper.

2.5 Constructive mathematics without the Fan Theorem?

Bishop in his seminal works [6] and [7] endorsed Brouwer’s critical assessment of the language of mathematics but refused to adopt his axioms. He therefore is unable to prove that a pointwise continuous function from \([0, 1]\) to \( R \) is uniformly continuous on \([0, 1]\) and decides to call a function from \( R \) to \( R \) continuous only if it is uniformly continuous on every closed interval \([−N, N]\), where \( N \in \mathbb{N} \). This decision brings him into an awkward position, as we will see in Corollary 9.10.

3 Computable counterexamples

Kleene wanted to know if Brouwer’s axioms are compatible with the assumption that every infinite sequence of natural numbers is computable, that is, given by a standard algorithm in the sense of Church or Turing. He discovered that the strict Fan Theorem, and, a fortiori, Brouwer’s Thesis on Bars in \( N \), are not. A related discovery was made by Specker, see [33]. We now consider two examples showing this.

As Kleene himself had shown earlier there exist a computable subset \( T \) of \( \mathbb{N}^3 \) and a computable function \( U \) from \( \mathbb{N} \) to \( \mathbb{N} \) such that, for every computable \( \alpha \) in \( N \), one may find a natural number \( e \) with the property that, for all \( n \) in \( \mathbb{N} \), \( \alpha(n) = U(z_0) \) where \( z_0 \) is any number \( z \) such that \( T(e, n, z) \). The number \( e \) is called an index for the function \( \alpha \).

3.1 The first example

Let \( A \) be finite subset of \( \{0, 1\}^* \). We let \( \mu(\alpha) \), the measure of \( A \), be the number \( \sum_{s \in A} 2^{−\text{length}(s)} \).

**Lemma 3.1** Let \( A \) be a finite subset of \( \{0, 1\}^* \) such that \( \mu(A) < 1 \). Then there exists \( s \) in \( \{0, 1\}^* \) such that no initial part of \( s \) and no extension of \( s \) belongs to \( A \).

**Proof** By induction on max(\( \{\text{length}(s) | s \in A\} \)). We define: max(\( \emptyset \)) := 0.

If max(\( \{\text{length}(s) | s \in A\} \)) = 0, and \( \mu(A) < 1 \), then \( A = \emptyset \) and we choose \( s := ( ) \).

If max(\( \{\text{length}(s) | s \in A\} \)) > 0, we consider \( A_0 := \{s \in \{0, 1\}^* | (0) * s \in A\} \) and \( A_1 := \{s \in \{0, 1\}^* | (1) * s \in A\} \), and note: \( \mu(A) = \frac{1}{2}(\mu(A_0) + \mu(A_1)) \). Suppose \( \mu(A) < 1 \), then ( ) \( \notin A \) and either \( \mu(A_0) < 1 \) or \( \mu(A_1) < 1 \). If \( \mu(A_0) < 1 \), we find \( s \) such that \( s \) has no initial part or extension in \( A_0 \) and conclude: \( (0) * s \) has no initial part or extension in \( A \). If \( \mu(A_1) < 1 \), we find \( s \) such that \( s \) has no initial part or extension in \( A_1 \) and conclude: \( (1) * s \) has no initial part or extension in \( A \).
For each $s$ in $\mathbb{N}^*$ and each $\alpha$ in $\mathcal{N}$, we let $s \ast \alpha$ be the element of $\mathcal{N}$ that we obtain by putting $\alpha$ behind $s$, that is, for each $i < \text{length}(s)$, $(s \ast \alpha)(i) = s(i)$, and for each $i \geq \text{length}(s)$, $(s \ast \alpha)(i) = \alpha(i - \text{length}(s))$.

For each $n, \underline{n}$ is the element of $\mathcal{N}$ such that, for each $k, \underline{n}(k) = n$.

Let $C$ be the set of all $s$ in $\{0, 1\}^*$ such that, for some $e < \text{length}(s)$, there exists $z$ in $\mathbb{N}^{e+1}$ such that, for each $j \leq e, z(j) < \text{length}(s)$ and $T(e, j, z(j))$ and $U(z(j)) = s(j)$. Let $\alpha$ be a computable element of $\mathcal{C}$ and let $e$ be an index for $\alpha$. Find $z$ in $\mathbb{N}^{e+1}$ such that, for each $j \leq e, T(e, j, z(j))$ and $U(z(j)) = \alpha(j)$. Find $n$ such that $e < n$ and, for each $j \leq e, z(j) < n$. Note that $(\alpha(0), \alpha(1), \ldots, \alpha(n-1))$ belongs to $C$. We thus see that $C$ is a bar in the set of all computable elements of $\mathcal{C}$. Note that there is an algorithmic procedure to decide, for every $s$ in $\{0, 1\}^*$, if $s$ belongs to $C$ or not.

$C$ contains no finite sequence of length 0, at most one finite sequence of length 1, at most two finite sequences of length 2, and so on. Let $k > 0$ be given and let $s_0, s_1, \ldots, s_{k-1}$ be a list of $k$ elements of $\mathcal{C}$ such that, for each $i < k - 1, \text{length}(s_i) \leq \text{length}(s_{i+1})$. Note that, for each $i < k, \text{length}(s_i) \geq i + 1$. Therefore, for every $k$, for every subset $A$ of $\mathcal{C}$ that has $k$ elements, $\mu(A) \leq \sum_{i<k} 2^{-i-1} = 1 - 2^{-k}$.

We may conclude: for every finite subset $A$ of $\mathcal{C}$, $\mu(A) < 1$, and, by Lemma 3.1, there exists $s$ in $\{0, 1\}^*$ such that $s$ has no initial part or extension in $A$, so: $s \ast \underline{0}$ is a computable element of $\mathcal{C}$ not having an initial part in $A$.

It follows that $C$ is a bar in the set of all computable elements of $\mathcal{C}$, while every finite subset of $C$ positively fails to be so.

3.2 The second example

Let $B$ be the set of all finite sequences $s$ in $\{0, 1\}^*$ such that for some $e < \text{length}(s)$, for some $z < \text{length}(s)$, $T(e, e, z)$ and $s(e) = U(z)$.

Let $\alpha$ be a computable element of $\mathcal{C}$ and let $e$ be an index for $\alpha$. Find $z$ such that $T(e, e, z)$ and let $k = \max(e, z)$. The finite sequence $(\alpha(0), \alpha(1), \ldots, \alpha(k))$ belongs to $B$. We thus see that $B$ is a bar in the set of all computable elements of $\mathcal{C}$.

Note that there is an algorithmic procedure to decide, for every $s$ in $\{0, 1\}^*$, if $s$ belongs to $B$ or not.

For every $n$, one may define $s$ in $\{0, 1\}^*$ such that $s$ has length $n$ and, for all $e < n$, for all $z < n$, if $T(e, e, z)$, then $s(e) \neq U(z)$. Note that $s$ does not meet $B$. Given any finite subset $B'$ of $B$, one may calculate $n := \max(\{|\text{length}(s)|s \in B'\})$ and find $s$ such that $\text{length}(s) = n$ and $s$ does not meet $B$. The infinite sequence $s \ast \underline{0}$ then is a computable element of $\mathcal{C}$ that does not meet $B'$.

It follows that $B$ is a bar in the set of all computable elements of $\mathcal{C}$, while every finite subset of $B$ positively fails to be so.

Let us define subsets $K_0, K_1$ of $\mathbb{N}$ by: for each $i < 2$, for each $e, e \in K_i$ if and only if, for some $z, T(e, e, z)$ and $U(z) = i$. Note that $K_0, K_1$ are disjoint recursively enumerable subsets of $\mathbb{N}$.

Let $X$ be a subset of $\mathbb{N}$. $X$ separates $K_0$ from $K_1$ if and only if $K_0 \subseteq X$ and $K_1 \subseteq \mathbb{N} \setminus X$.

For each $\alpha$ in $\mathcal{C}$, we define $D_\alpha := \{n|\alpha(n) = 1\}$. $D_\alpha$ is called: the subset of $\mathbb{N}$ decided by $\alpha$. 

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\usepackage{amsmath} 
\usepackage{amssymb} 
\usepackage{hyperref} 
\begin{document} 
\section*{W. Veldman} 
\end{document}
Suppose \( \alpha \) is an element of \( C \) not meeting \( B \). Then, for each \( e \) in \( K_0 \), \( \alpha(e) = 1 \), and, for each \( e \) in \( K_1 \), \( \alpha(e) = 0 \), so \( D_\alpha \) separates \( K_0 \) from \( K_1 \).

Let \( \alpha \) be a computable element of \( C \), and let \( e \) be an index \( e \) for \( \alpha \). Then: either \( e \in K_0 \) and \( \alpha(e) = 0 \) or \( e \in K_1 \) and \( \alpha(e) = 1 \), that is: \( D_\alpha \) positively fails to separate \( K_0 \) from \( K_1 \).

4 König’s infinity lemma

König’s discovery of the Infinity Lemma almost coincides with Brouwer’s formulation of the Fan Theorem. We find the Lemma in the appendix of [23]. Its first appearance is in [24].

Let \( T \) be a subset of \( \mathbb{N}^* \). \( T \) is called a tree if and only if for each \( s \) in \( \mathbb{N}^* \), for each \( n \) in \( \mathbb{N} \), if \( s \ast (n) \) belongs to \( T \), then \( s \) belongs to \( T \). If \( T \) is a tree and \( \alpha \) belongs to \( \mathcal{N} \) then \( \alpha \) is called an infinite path of \( T \) if and only if every initial part of \( \alpha \) belongs to \( T \).

**Lemma 4.1** (Weak König’s lemma/weak infinity lemma) Let \( T \) be a subset of \( \{0, 1\}^\ast \) and a tree. If \( T \) is infinite, then \( T \) has an infinite path.

The (classical) proof of the Weak Infinity Lemma is as follows.

Suppose \( T \) is an infinite subtree of \( \{0, 1\}^\ast \). We build \( \alpha \) in \( C \) step by step, taking care that, for each \( n \) in \( \mathbb{N} \), there are infinitely many \( s \) in \( T \) such that \((\alpha(0), \alpha(1), \ldots, \alpha(n-1))\) is an initial part of \( s \). It then follows that \( \alpha \) is an infinite path of \( T \). At every step of the construction we have to apply the following **Pigeonhole Principle:**

For all subsets \( A, B \) of \( \mathbb{N} \), if \( \mathbb{N} = A \cup B \), then either \( A \) is infinite or \( B \) is infinite.

The construction of \( \alpha \) is by induction, as follows.

Let \( n \) be a natural number and assume we defined the first \( n \) values of \( \alpha \) and secured that \((\alpha(0), \alpha(1), \ldots, \alpha(n-1))\) is an initial part of infinitely many elements of \( T \). We define \( \alpha(n) := 0 \) if \((\alpha(0), \alpha(1), \ldots, \alpha(n-1), 0)\) is an initial part of infinitely many elements of \( T \) and \( \alpha(n) := 1 \) if not.

One may prove by induction that, for each \( n \), \((\alpha(0), \alpha(1), \ldots, \alpha(n-1))\) is an initial part of infinitely many elements of \( T \).

The pigeonhole principle applied in this proof is not valid constructively. In many cases where we are given subsets \( A, B \) of \( \mathbb{N} \) such that we are able to prove, for every natural number \( n \), either that \( n \) belongs to \( A \) or that \( n \) belongs to \( B \), we are unable to prove that \( A \) is infinite and also unable to prove that \( B \) is infinite.

For instance, let \( A \) be the set of all natural numbers \( n \) such that in the first \( n \) digits of the decimal expansion of \( \pi \) there occurs no uninterrupted sequence of 99 9’s and let \( B \) be the set all natural numbers that do not belong to \( A \).

(Note that the functions \( \alpha, \beta \) from \( \mathbb{N} \) to \( \{0, 1\} \) defined by: for each \( n \), \( \alpha(n) = 1 \) if and only if \( n \in A \) and \( \beta(n) = 1 \) if and only if \( n \in B \), are computable.)

The Infinity Lemma itself also fails constructively as it implies the pigeonhole principle we just refuted. We see this in the following way. Let \( A, B \) be subsets of \( \mathbb{N} \) and suppose that every natural number belongs either to \( A \) or to \( B \). We let \( T \) be the set of all finite sequences \( s = (s(0), s(1), \ldots, s(n-1)) \) such that either, for each \( j < n \), \( s(j) = 0 \) and there exists \( k \geq n \) such that \( k \) belongs to \( A \) or, for each
\( j < n, s(j) = 1 \) and there exists \( k \geq n \) such that \( k \) belongs to \( B \). If \( \alpha \) is a path of \( T \) and \( \alpha(0) = 0 \), then \( A \) is infinite and if \( \alpha \) is a path of \( T \) and \( \alpha(0) = 1 \), then \( B \) is infinite.

Observe that, if we assume, in the above argument, that \( A \) and \( B \) are decidable subsets of \( \mathbb{N} \), in the strong sense that there exist \( \alpha, \beta \) in \( C \) such that, for all \( n \in \mathbb{N}, n \) belongs to \( A \) if and only if \( \alpha(n) = 1 \) and \( n \) belongs to \( B \) if and only if \( \beta(n) = 1 \), then \( T \) will be an enumerable subset of \( \mathbb{N}^* \). One may construct a function \( \gamma \) from \( \mathbb{N} \) to \( \mathbb{N}^* \) enumerating \( T \) in the following way.

For all \( n, k \), \( \gamma(2^n(4k+1) - 1) = \overline{0n} \) if \( n < k \) and \( \alpha(k) = 1 \), and \( \gamma(2^n(4k+1) - 1) = ( ) \) if either \( n \geq k \) or \( \alpha(k) = 0 \), and \( \gamma(2^n(4k + 3) - 1) = \overline{0n} \) if \( n < k \) and \( \beta(k) = 1 \), and \( \gamma(2^n(4k + 3) - 1) = ( ) \) if either \( n \geq k \) or \( \beta(k) = 0 \).

In general, however, \( T \) will not be a decidable subset of \( \{0, 1\}^* \).

We thus see that, intuitionistically, the Infinity Lemma does not hold for enumerable subtrees of \( \{0, 1\}^* \). It also fails to be true in the case that \( T \) is a decidable subset of \( \{0, 1\}^* \), as will be clear from the following example.

Let \( A \) be the set of all natural numbers \( n \) such that either in the first \( n \) digits of the decimal expansion of \( \pi \) there occurs no uninterrupted sequence of 99 9’s or there occurs one and the first such sequence is completed at an even-numbered place in the decimal expansion of \( \pi \), and let \( B \) be the set all natural numbers \( n \) such that either in the first \( n \) digits of the decimal expansion of \( \pi \) there occurs no uninterrupted sequence of 99 9’s or there occurs one and the first such sequence is completed at an odd-numbered place in the decimal expansion of \( \pi \). Then every natural number will belong either to \( A \) or to \( B \), but we have no proof that \( A \) is infinite and no proof that \( B \) is infinite. For each \( n \), one may decide if there exists \( k \geq n \) such that \( k \) belongs to \( A \) and also if there exists \( k \geq n \) such that \( k \) belongs to \( B \). If we construct \( T \) as above, \( T \) will be a decidable subset of \( \{0, 1\}^* \) and an infinite tree. If \( T \) has an infinite path, either \( A \) is infinite or \( B \) is infinite.

Let \( T \) be a subset of \( \mathbb{N}^* \) and a tree. \( T \) is called finitely-branching if every element of \( T \) has only finitely many immediate prolongations in \( T \).

Lemma 4.2 [König’s (infinity) lemma] Let \( T \) be a subset of \( \mathbb{N}^* \) and a tree. If \( T \) is infinite and finitely-branching, then \( T \) has an infinite path.

Lemma 4.2 is a (non-constructive) statement slightly more general than Lemma 4.1 and it is proven similarly.

5 Classical and intuitionistic reverse mathematics

In the programme of Reverse Mathematics, started by H. Friedman and S. Simpson, see [32], one studies structures of the form \((\mathbb{N}, S, +, \cdot, 0, 1)\), where \( S \) is a subcollection of the collection \( \wp(\mathbb{N}) \) of all subsets of \( \mathbb{N} \). Formulas from Second Order Arithmetic may be interpreted in such structures. The collection \( S \) may be the collection of all subsets of \( \mathbb{N} \) whose characteristic function is computable, or the collection of all subsets of \( \mathbb{N} \) that may be defined by a formula from First Order Arithmetic, or “simply” \( \wp(\mathbb{N}) \) itself. The leading question of the project is:

\textit{How should we choose the collection} \( S \) \textit{of subsets of} \( \mathbb{N} \) \textit{if we want that this-or-that theorem becomes true in the model?}
From Kleene’s example, explained in Sect. 3, the classical mathematician may learn that Weak König’s Lemma is not true in the minimal model given by the class of the recursive/computable subsets of $\mathbb{N}$. For this reason, Weak König’s Lemma is seen as an implicit set existence axiom by Friedman and Simpson: it forces the existence of non-computable subsets of $\mathbb{N}$.

The Bolzano–Weierstrass-theorem turns out to be an even stronger set existence axiom than Weak König’s Lemma: there exist subcollections $S$ of $\wp(\mathbb{N})$ such that Weak König’s Lemma is true in the corresponding model for Second Order Arithmetic and the Bolzano–Weierstrass-Theorem is not. (The Bolzano–Weierstrass Theorem turns out to be equivalent to König’s Lemma for finitely-branching trees).

The purpose of (classical) Reverse Mathematics is to find out, for every theorem of classical analysis, which strength it has as a set existence axiom. The results of the project are impressive, see [32].

The classical Reverse Mathematics project seems to be driven at least partially by a certain distrust of the in many cases uselessly big totality of all subsets of $\mathbb{N}$.

The intuitionistic mathematician does not know at all what to do with the notion of the class $\wp(\mathbb{N})$ of all subsets of $\mathbb{N}$. In intuitionistic analysis, one prefers the notion of an infinite sequence of natural numbers as a primitive notion above the notion of a set of natural numbers. One also considers certain classes of subsets of $\mathbb{N}$ but the sets are then defined in terms of infinite sequences of natural numbers. One introduces, for example, given some $\alpha$ in $\mathcal{N}$, the set $D_\alpha$ of all natural numbers $n$ such that $\alpha(n) = 1$, that is: the subset of $\mathbb{N}$ decided by $\alpha$, and the set $E_\alpha$, consisting of all natural numbers $n$ such that, for some $m$, $\alpha(m) = n + 1$, that is: the subset of $\mathbb{N}$ enumerated by $\alpha$, see Sect. 7.1.4.

The intuitionistic mathematician may also learn from Kleene’s example. If one accepts the Fan Theorem one can not hold the view that every element of $C$ is given by a standard algorithm. On the other hand, given an element $\alpha$ of $C$ and a natural number, it is implicit in the notion of a constructive function that one must be able to find the value $\alpha(n)$ in an effective way. One thus has to admit the possibility of infinite sequences of natural numbers that may be effectively calculated but are not given by an algorithm. It is not so easy to give an example of such a sequence. As we saw, when considering the Continuity Principle, Brouwer was thinking of infinite sequences that are the result of free choices of the successive values. In such a case, we may come to know as many values as we want, but we do not know a rule that governs the process.

The Fan Theorem thus becomes an implicit function existence axiom.

In [32, pp. 136–137], Simpson declares that, in his programme, one takes the theorems of classical analysis as they stand without studying or judging their constructive content. For an intuitionistic mathematician, such an attitude is impossible. As it stands, a classical theorem has no intuitionistic meaning. The meaning ascribed to the statement by the classical mathematician evaporates if the statement is read constructively. The theorem is brought to fall, so to say, and there is a gamut of possible ways to make it constructively stand up again. Considering the various constructive statements that seem to come close to a given classical result, one has to find out first which of them are constructively true, and then which of the true ones might be useful.
Recall: we saw Brouwer’s defense of the Fan Theorem in Sect. 2, and we saw in Sect. 4 the constructive failure of Weak König’s Lemma, while, from a classical point of view, the two statements are equivalent.

An intuitionistic mathematician, if he wants to do Reverse Mathematics, must start from the theorems of intuitionistic analysis as they stand.

Simpson’s statement seems to be debatable on an even deeper level. Even classical Reverse Mathematics can not in truth be said to leave the meaning of the classical theorems unchanged. By establishing the precise connection of a given theorem with other results one comes to a different and better understanding of the meaning of the theorem.

The aim and effect of the intuitionistic critique of a classical result may be described in similar terms. A theorem is no longer the same once one has seen which sense it could or could not make from a constructive point of view. It seems very natural to do Reverse Mathematics intuitionistically and not to ignore questions about constructive validity when we are engaged upon finding out what might be the meaning of a given classical result.

6 The formal system BIM

The formal system BIM (for Basic Intuitionistic Mathematics) that we now want to introduce is similar to the system H introduced in [19].

There are two kinds of variables, numerical variables \( m, n, p, \ldots \), whose intended range is the set \( \mathbb{N} \) of the natural numbers, and function variables \( \alpha, \beta, \gamma, \ldots \), whose intended range is the set \( \mathcal{N} \) of all infinite sequences of natural numbers, that is, all functions from \( \mathbb{N} \) to \( \mathbb{N} \). There is a numerical constant 0. There are unary function constants \( 0 \) and \( S \), a name for the successor function, and \( K, L \), names for the projection functions. There is one binary function constant \( J \), a name for the pairing function. From these symbols numerical terms are formed in the usual way. The basic terms are the numerical variables and the numerical constant 0 and other terms are obtained from earlier constructed terms by the use of a function symbol. The function constants 0 and \( S \) and the function variables are at this stage the only function terms. As the theory develops, names for operations on infinite sequences will be introduced and more complicated function terms will appear.

There are two equality symbols, \( =_0 \) and \( =_1 \). The first symbol may be placed between numerical terms only and the second one between function terms only. When confusion seems improbable we simply write \( = \) and not \( =_0 \) or \( =_1 \). A basic formula is an equality between numerical terms or an equality between function terms. A basic formula in the strict sense is an equality between numerical terms. We obtain the formulas of the theory from the basic formulas by using the connectives, the numerical quantifiers and the function quantifiers. The logic of the theory is of course intuitionistic logic.

We first introduce

**Axiom 1** (Axiom of extensionality) \( \forall \alpha \beta [\alpha =_1 \beta \leftrightarrow \forall n [\alpha(n) =_0 \beta(n)]] \)

The Axiom of extensionality guarantees that every formula will be provably equivalent to a formula built up by means of connectives and quantifiers from basic formulas in the strict sense. The following axiom defines the function constants.
Axiom 2 \( \forall n[\neg(S(n) = 0)] \land \forall m \forall n[S(m) = S(n) \rightarrow m = n] \land \\
\forall n[0(n) = 0] \land \forall m \forall n[K(J(m, n)) = m \land L(J(m, n)) = n] \)

Thanks to the presence of the pairing function we may treat binary, ternary and other non-unary operations on \( \mathbb{N} \) as unary functions. “\( \alpha(m, n, p) \)” for instance will be an abbreviation of “\( \alpha(J(m, n), p) \)”.

The next axiom asks for the closure of the universe of functions under composition, primitive recursion and unbounded search

Axiom 3 \( \forall \alpha \forall \beta \exists \gamma \forall n[\gamma(n) = \alpha(\beta(n))] \land \\
\forall \alpha \forall \beta \exists \gamma \forall m \forall n[\gamma(m, 0) = \alpha(m) \land \gamma(m, S(n)) = \beta(m, n, \gamma(m, n))] \land \\
\forall \alpha \forall m \exists n[\alpha(m, n) = 0] \rightarrow \exists \gamma \forall m[\alpha(m, \gamma(m)) = 0]] \).

We sometimes call the formula \( \forall \alpha[\forall m \exists n[\alpha(m, n) = 0] \rightarrow \exists \gamma \forall m[\alpha(m, \gamma(m)) = 0]] \) the Minimal Axiom of Countable Choice. This formula expresses the restriction of the First Axiom of Countable Choice to decidable subsets of \( \mathbb{N} \times \mathbb{N} \).

We also need the Unrestricted Axiom Scheme of Induction:

Axiom 4 For every formula \( \phi = \phi(n) \) the universal closure of the following formula is an axiom: \( (\phi(0) \land \forall n[\phi(n) \rightarrow \phi(S(n))]) \rightarrow \forall n[\phi(n)] \)

The system consisting of the Axioms 1, 2, 3 and 4 will be called BIM. The system BIM is not very different from the system EL occurring in [34].

A formula \( \phi \) is called a \( \Sigma^0_1 \)-formula if it is of the form \( \exists \psi \forall \psi_0 \ldots \psi_{q-1}[\psi] \) where \( \psi \) is a formula built up from equalities between numerical terms only by means of connectives and restricted numerical quantifiers. If we should decide to restrict the formula \( \phi \) occurring in the Axiom Scheme of Induction to be a \( \Sigma^0_1 \)-formula, we might call the resulting system \( \text{BIM}_0 \). We would thereby follow Simpson and Friedman’s nomenclature, who distinguish the systems \( \text{WKL} \), containing Weak König’s Lemma and Unrestricted Induction as an axiom, and \( \text{WKL}_0 \), containing Weak König’s Lemma but restricting Induction to \( \Sigma^0_1 \)-formulas. In this paper we will not study the effect of restricting Induction.

We may add constants for the primitive recursive functions and relations with their defining equations to BIM and thus obtain conservative extensions of these systems. We want to assume that these constants and their defining equations form part already of the system BIM itself. In particular we assume that addition, multiplication and exponentiation have obtained their names and definitions and that there is a constant \( p \) denoting the function enumerating the prime numbers. We also have a notation for the function(s) from \( \mathbb{N}^k \) to \( \mathbb{N} \) coding finite sequences of natural numbers by natural numbers:

\[
\langle m_0, \ldots, m_{k-1} \rangle = 2^{m_0} \cdots (p(k - 2))^{m_{k-2}} \cdot (p(k - 1))^{m_{k-1}+1} - 1
\]

For each \( a \) we let \( \text{length}(a) \) be the least number \( i \) such that, for each \( j > i \), \( p(j) \) does not divide \( a + 1 \).

For each \( a \), for each \( i < \text{length}(a) - 1 \), we let \( a(i) \) be the greatest number \( q \) such that \( (p(i))^q \) divides \( a+1 \), and, if \( i = \text{length}(a) - 1 \) we let \( a(i) \) be the greatest number \( q \) such that \( (p(i))^{q+1} \) divides \( a + 1 \). Observe that for each \( a, a = \langle a(0), a(1), \ldots, a(i - 1) \rangle \), where \( i = \text{length}(a) \).
We let \( Bin \) denote the set of (code numbers of) binary sequences, so, for each \( a, a \in Bin \) if and only if \( \forall i < \text{length}(a), a(i) \leq 1 \).

We let \( * \) denote the binary function corresponding to concatenation of finite sequences, so for each \( a, b, a * b \) is the number coding the finite sequence that we obtain by putting the finite sequence coded by \( b \) behind the finite sequence coded by \( a \).

For each \( a \), for each \( n \leq \text{length}(a) \) we define: \( \overline{a}(n) = \langle a(0), \ldots, a(n - 1) \rangle \). If confusion seems unlikely, we sometimes write: “\( \overline{a}n \)” and not: “\( \overline{a}(n) \)”.

For all \( a, b \), we define: \( a \) is an initial segment of \( b \), notation: \( a \subseteq b \) if and only if there exists \( n \leq \text{length}(b) \) such that \( a = \overline{b}n \).

For all \( a, b \), we define: \( a \) is a proper initial segment of \( b \), notation: \( a \subset b \), if and only if \( a \subseteq b \) and \( a \neq b \).

For all \( a, b, a, b \) go different ways, notation: \( a \perp b \), if and only if neither \( a \subseteq b \) nor \( b \subseteq a \).

For each \( \alpha \), for each \( n \), we define \( \overline{\alpha}(n) = \langle \alpha(0), \ldots \alpha(n - 1) \rangle \). If confusion seems unlikely, we sometimes write: “\( \overline{\alpha}n \)” and not: “\( \overline{\alpha}(n) \)”.

For each \( a \), for each \( \alpha \), we define: \( \alpha \) passes through \( a \), or \( \alpha \) contains \( a \), if and only if there exists \( n \) such that \( \overline{\alpha}n = a \).

For all \( \alpha, \beta \), we define: \( \alpha \neq \beta \), \( \alpha \) is apart from \( \beta \) if and only if there exists \( n \) such that \( \alpha(n) \neq \beta(n) \).

We use the letter \( C \) in order to denote Cantor space, so “\( \alpha \in C \)” is an abbreviation for “\( \forall n[\alpha(n) = 0 \lor \alpha(n) = 1] \)”.

We now are able to formulate the (strict) Fan Theorem:

\[
\forall \beta[\forall \alpha \in C \exists n[\beta(\overline{\alpha}n) = 1] \rightarrow \exists m \forall \alpha \in C \exists n[\beta(\overline{\alpha}n) = 1 \land \overline{\alpha}n \leq m]]
\]

Observe that the Fan Theorem, as formulated here, is weaker than the axiom defended in the informal discussion in Sect. 2. We restrict ourselves to the case of a bar in \( C \) that is a decidable subset of \( \mathbb{N} \) in the strong sense that there exists a function \( \beta \) deciding which natural numbers code a finite sequence belonging to the bar.

We now introduce Kleene’s Alternative (to the Fan Theorem):

\[
\exists \beta[\forall \alpha \in C \exists n[\beta(\overline{\alpha}n) = 1] \land \forall m \exists \alpha \in C \forall n[\overline{\alpha}n \leq m \rightarrow \beta(\overline{\alpha}n) \neq 1]]
\]

We have seen, in Sect. 3, that, under the assumption that every element of \( \mathcal{N} \) is given by an algorithm, Kleene’s Alternative becomes a true statement.

Observe that BIM does not contain an axiom of countable choice, except for the Axiom of Unbounded Search, that we mentioned earlier in this section. We want to call the Axiom of Unbounded Search the Minimal Axiom of Countable Choice.

The following special case of the unrestricted First Axiom of Countable Choice would help us to prove, for instance, that every Cauchy sequence of rationals converges to a real number, see Sect. 8. We do want to make it an axiom of BIM, as it is not clear that this axiom holds in the universe of the computable functions from \( \mathbb{N} \) to \( \mathbb{N} \). In [28], several weak versions of the axiom of countable choice are studied.

\( \Pi^0_1 \)-First Axiom of Countable Choice

\[
\forall \alpha[\forall m \exists n p[\alpha(m, n, p) = 1] \rightarrow \exists \gamma \forall m \forall p[\alpha(m, \gamma(m), p) = 1]]
\]
In some cases, where we need to apply this special case of the First Axiom of Countable Choice, an even more special case of the axiom seems to be sufficient:

**Weak $\Pi^0_1$-First Axiom of Countable Choice**

\[
\forall \alpha [\forall m \exists n \forall p \geq n[\alpha(m, p) = 1] \rightarrow \exists \gamma \forall m \forall p \geq \gamma(m)[\alpha(m, p) = 1]]
\]

The Weak $\Pi^0_1$-First Axiom of Countable Choice enables us to show that every closed-and-separable subset of $\mathbb{R}$ that is totally bounded is also explicitly totally bounded, see Lemma 8.4, and that every subset of $\mathcal{N}$, if a fan, is an explicit fan, see Lemma 9.1. Joan Moschovakis made me see, against my earlier opinion, that it is probably not possible to prove, using only this Weak $\Pi^0_1$-First Axiom of Countable Choice, that every Cauchy sequence of rationals is an explicit Cauchy sequence of rationals, see Sect. 8.1.5.

In the next sections, we use variables $X, Y, F, \ldots$ on the class of subsets of $\mathbb{N}$ and variables $\mathcal{X}, \mathcal{Y}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \ldots$ on the class of subsets of $\mathcal{N}$. We use these variables in an informal sense only, in the same way one uses variables on classes of sets in the development of formal Zermelo–Fraenkel set theory.

**7 Compact and positively noncompact closed-and-separable subsets of $\mathcal{N}$**

7.1 Closed-and-separable or: closed-and-semilocated subsets of $\mathcal{N}$

In constructive mathematics, the notions of an open subset of $\mathcal{N}$ and a closed subset of $\mathcal{N}$ have to be handled with care. We hope the reader will not be confused by the many distinctions one is forced to make.

7.1.1 Weakly open and weakly closed subsets of $\mathcal{N}$

Let $\mathcal{G}$ be a subset of $\mathcal{N}$. $\mathcal{G}$ is a weakly open subset of $\mathcal{N}$ if and only if, for each $\alpha$ in $\mathcal{G}$, there exists $n$ such that every $\beta$ in $\mathcal{N}$ passing through $\overline{\alpha n}$ belongs to $\mathcal{G}$.

Let $\mathcal{X}$ be a subset of $\mathcal{N}$. The complement $\mathcal{X}^\sim$ of $\mathcal{X}$ is the set of all $\alpha$ in $\mathcal{N}$ such that the assumption $\alpha$ belongs to $\mathcal{X}$ leads to a contradiction. Note that, for every $\alpha$, if $\neg\neg(\alpha \in \mathcal{X})$, then $\alpha \in \mathcal{X}^\sim$.

Let $\mathcal{F}$ be a subset of $\mathcal{N}$. $\mathcal{F}$ is called weakly closed if and only if there exists a weakly open subset $\mathcal{G}$ of $\mathcal{N}$ such that $\mathcal{F}$ coincides with the complement $\mathcal{G}^\sim$ of $\mathcal{G}$.

Every weakly closed subset $\mathcal{F}$ of $\mathcal{N}$ is a stable subset of $\mathcal{N}$, that is: $\mathcal{F}^{\neg\neg}$ coincides with $\mathcal{F}$.

7.1.2 Sequentially closed subsets of $\mathcal{N}$

Let $\mathcal{F}$ be a subset of $\mathcal{N}$. $\mathcal{F}$ is a sequentially closed subset of $\mathcal{N}$ if and only if, for every $\gamma$ in $\mathcal{N}$, if, for each $n$, there exists $\alpha$ in $\mathcal{F}$ passing through $\overline{\gamma(n)}$, then $\gamma$ itself belongs to $\mathcal{F}$.

Every weakly closed subset of $\mathcal{N}$ is a sequentially closed subset of $\mathcal{N}$ but the converse is not true constructively. A sequentially closed subset of $\mathcal{N}$ may fail to be a stable subset of $\mathcal{N}$. Consider the following example. Let $P$ be an unsolved mathematical statement and let $\mathcal{F}$ be the set of all $\alpha$ in $\mathcal{N}$ such that $P \lor \neg P$. Clearly, $\mathcal{F}$ is
sequentially closed. The statement $F$ is a stable subset of $N$ is reckless, as we have a proof of $\neg(\neg P \lor \neg P)$ but not one of $P \lor \neg P$.

### 7.1.3 Closed-and-separable subsets of $N$

Let $\alpha$ belong to $N$ and $n$ to $\mathbb{N}$. We let $\alpha^n$ be the sequence $\beta$ such that, for all $m$ in $\mathbb{N}$, $\beta(m) = \alpha(J(n, m))$.

For later purposes, we extend this notation to finite sequences, as follows.

For all $s, n$ in $\mathbb{N}$ we let $s^n$ be the greatest number $t$ such that, for all $m < \text{length}(t)$, $J(n, m) < \text{length}(s)$ and $t(m) = s(J(n, m))$.

Let $\alpha$ belong to $N$. We let $CS_\alpha$ be the set of all $\beta$ in $N$ such that for every $n$ there exists $m$ with the property that $\alpha^m$ passes through $\overline{\beta}(n)$. We call a subset $F$ of $N$ a closed-and-separable subset of $N$ if and only if there exists $\alpha$ such that $F$ coincides with $CS_\alpha$.

Note that every closed-and-separable subset of $N$ is sequentially closed.

### 7.1.4 An intermezzo on decidable and enumerable subsets of $\mathbb{N}$

Let $\beta$ belong to $N$. We let $D_\beta$ be the set of all natural numbers $n$ such that $\beta(n) = 1$. We let $E_\beta$ be the set of all natural numbers $n$ such that, for some $m$, $\beta(m) = n + 1$.

We extend this notation to finite sequences, as follows. For every $b$ in $\mathbb{N}$, we let $D_b$ be the set of all natural numbers $n < \text{length}(b)$ such that $b(n) = 1$. We let $E_b$ be the set of all natural numbers $n$ such that, for some $m < \text{length}(b)$, $b(m) = n + 1$.

Let $X$ be a subset of $\mathbb{N}$ and let $\beta$ belong to $N$. $\beta$ decides $X$ if and only if $X$ coincides with $D_\beta$ and $\beta$ enumerates $X$ if and only if $X$ coincides with $E_\beta$. $X$ is a decidable subset of $\mathbb{N}$ if and only if some $\beta$ decides $X$ and $X$ is an enumerable subset of $\mathbb{N}$ if and only if some $\beta$ enumerates $X$.

Note that, for each $\beta$, $D_\beta = \bigcup_{n \in \mathbb{N}} D_{\beta n}$, and $E_\beta = \bigcup_{n \in \mathbb{N}} E_{\beta n}$.

For all $m, n$ in $\mathbb{N}$, we define: $(m, n) := J(m, n)$, $m' := K(m)$ and $m'' := L(m)$.

**Lemma 7.1** (i) Every decidable subset of $\mathbb{N}$ is an enumerable subset of $\mathbb{N}$.

(ii) The class of the decidable subsets of $\mathbb{N}$ is closed under the operations of union, intersection and taking the complement.

(iii) The class of the enumerable subsets of $\mathbb{N}$ is closed under the operations of union and intersection.

**Proof** (i) Let $\alpha$ in $N$ be given. Define $\gamma$ in $N$ such that, for each $n$, if $\alpha(n) = 1$, then $\gamma(n) = n + 1$, and if $\alpha(n) \neq 1$, then $\gamma(n) = 0$. Clearly, $D_\alpha = E_\gamma$.

(ii) Let $\alpha, \beta$ in $N$ be given. Define $\gamma, \delta, \varepsilon$ in $N$ such that, for each $n$, $\gamma(n) = \max(\alpha(n), \beta(n))$, $\delta(n) = \min(\alpha(n), \beta(n))$ and $\varepsilon(n) = 1 - \alpha(n)$ and note that $D_\gamma = D_\alpha \cup D_\beta$ and $D_\delta = D_\alpha \cap D_\beta$ and $D_\delta = \mathbb{N} \setminus D_\alpha$.

(iii) Let $\alpha, \beta$ in $N$ be given. Define $\gamma$ in $N$ such that, for each $n$, $\gamma(2n) = \alpha(n)$ and $\gamma(2n + 1) = \beta(n)$ and note: $E_\gamma = E_\alpha \cup E_\beta$. Define $\delta$ in $N$ such that, for each $n$, if $\alpha(n') = \alpha(K(n)) = \beta(L(n)) = \beta(n'')$, then $\delta(n) = \alpha(n')$ and, if not, then $\delta(n) = 0$ and note: $E_\delta = E_\alpha \cap E_\beta$. 

\[\square\]
The following lemma explains that, in both intended models of BIM, not every enumerable subset of \( \mathbb{N} \) is a decidable subset of \( \mathbb{N} \).

**Lemma 7.2**  
(i) Using Church’s Thesis one may prove:  
\[ \exists \gamma \forall \alpha \exists n(\lnot (n \in D_\alpha \leftrightarrow n \in E_\gamma)) \], that is: there exists an enumerable subset of \( \mathbb{N} \) that fails to be a decidable subset of \( \mathbb{N} \).

(ii) Using Brouwer’s Continuity Principle one may prove:  
\[ \forall \gamma \exists n(0 \in E_\gamma \lor 0 \notin E_\gamma) \], and therefore: \[ \forall \gamma \exists n(\lnot n \in E_\gamma \leftrightarrow n \in D_\alpha) \], that is, the assumption that every enumerable subset of \( \mathbb{N} \) is a decidable subset of \( \mathbb{N} \) leads to a contradiction.

(iii) Using Church’s Thesis one may prove:  
\[ \exists \gamma \exists n(E_\delta \subseteq \mathbb{N} \setminus E_\gamma \rightarrow (n \notin E_\delta \wedge n \notin E_\gamma)) \], that is, there exists an enumerable subset of \( \mathbb{N} \) such that its complement fails to be an enumerable subset of \( \mathbb{N} \).

(iv) Using the Second Axiom of Continuous Choice one may prove:  
\[ \forall \gamma \exists n(0 \notin E_\gamma \leftrightarrow 0 \in E_\delta) \], and therefore: \[ \forall \gamma \exists n(\lnot n \in E_\gamma \leftrightarrow n \in E_\delta) \], that is: the assumption that the complement of every enumerable subset of \( \mathbb{N} \) is an enumerable subset of \( \mathbb{N} \) leads to a contradiction.

**Proof**  
(i) Consider \( K_0 = \{ n \in \mathbb{N} | \exists z[T(n, n, z) \cup U(z) = 0] \} \). \( K_0 \) is a computably enumerable subset of \( \mathbb{N} \). Let \( D \) be a computably decidable subset of \( \mathbb{N} \). Find \( n \) such that, for all \( m, m \in D \) if and only if \( \exists z[T(n, m, m) \cup U(z) = 1] \). Note: if \( n \in D \), then \( n \notin K_0 \), and if \( n \notin D \), then \( n \in K_0 \).

(ii) Assume \( \forall \gamma \exists n(0 \notin E_\gamma \leftrightarrow 0 \in E_\delta) \). Using Brouwer’s Continuity Principle, find \( m \) such that *either* for every \( \gamma \), if \( \forall m = \tilde{0}m \), then \( 0 \in E_\gamma \), or, for every \( \gamma \), if \( \forall m = \tilde{0}m \), then \( 0 \notin E_\gamma \). Note that both alternatives are false, as \( 0 \notin K_0 \) and \( 0 \in E_{\tilde{0}m+1} \).

(iii) Consider \( K = \{ n \in \mathbb{N} | \exists z[T(n, n, z) \cup U(z) = 0] \} \). \( K \) is a computably enumerable subset of \( \mathbb{N} \). As is well-known, \( \mathbb{N} \setminus K \) is a productive set, that is, given any computably enumerable subset of \( \mathbb{N} \setminus K \), one may effectively find a number \( n \) such that \( n \in \mathbb{N} \setminus (K \cup E) \).

One proves this as follows. Let \( E \) be a computably enumerable subset of \( \mathbb{N} \). Find \( n \) such that, for every \( m, m \in E \) if and only if \( \exists z[T(n, m, m) \cup U(z) = 1] \). Assume \( E \subseteq \mathbb{N} \setminus K \). Note: \( n \in K \) if and only if \( \exists z[T(n, m, m) \cup U(z) = 1] \). Note: if \( n \in K \) and \( n \notin K \), \( K \) is a productive set.

Conclude: \( n \notin K \) and \( n \notin E \).

(iv) Assume \( \forall \gamma \exists n(0 \notin E_\gamma \leftrightarrow 0 \in E_\delta) \). Using the Second Axiom of Continuous Choice, see Sect. 2.1, we find \( \phi \in \mathcal{N} \) enumerating a continuous function from \( \mathcal{N} \) to \( \mathcal{N} \), (for this notion, see Sect. 7.5), such that \( \forall \gamma 0 \notin E_\gamma \leftrightarrow 0 \in E_{\phi(\gamma)} \). Note: \( 0 \notin E_0 \), therefore: \( 0 \in E_{\phi(0)} \). Find \( m \) such that \( (\phi(0))(m) = 1 \), and find \( n \) such that, for every \( \gamma \), if \( \forall n = \tilde{0}n \), then \( \forall \gamma (m) = (\phi(0))(m) = 1 \). Consider \( \delta := \tilde{0}n \) and note: \( 0 \in E_\delta \cap E_{\phi(\delta)} \). Contradiction.

**\( 7.1.5 \) Effectively open and effectively closed subsets of \( \mathcal{N} \)**

Let \( \mathcal{G} \) be a subset of \( \mathcal{N} \). \( \mathcal{G} \) is called *effectively open* if and only if there exists a decidable subset \( D \) of \( \mathbb{N} \) such that for all \( \alpha \in \mathcal{N} \), \( \alpha \) belongs to \( \mathcal{G} \) if and only if, for some \( n, \alpha n \) belongs to \( D \).

Let \( \beta \) belong to \( \mathcal{N} \) and let \( \mathcal{H} \) be the set of all \( \alpha \) such that, for some \( n, \alpha n \) belongs to \( E_\beta \). We claim that \( \mathcal{H} \) is effectively open. In order to see this, let \( \gamma \) be an element of
\( \mathcal{N} \) such that, for each \( s, \gamma(s) = 1 \) if and only if there exist \( i, j < \text{length}(s) \) such that \( \beta(i) = sj + 1 \). \( \mathcal{H} \) coincides with the set of all \( \alpha \) such that, for some \( n, \alpha n \) belongs to \( D_\gamma \).

Let \( \mathcal{F} \) be a subset of \( \mathcal{N} \). \( \mathcal{F} \) is called \textit{effectively closed} if and only if there exists an effectively open subset \( \mathcal{G} \) of \( \mathcal{N} \) such that \( \mathcal{F} \) coincides with the complement \( \mathcal{G}^\sim \) of \( \mathcal{G} \).

Note that \( \mathcal{F} \) is effectively closed if and only if there exists \( \beta \) in \( \mathcal{N} \) such that, for all \( \alpha \) in \( \mathcal{N} \), \( \alpha \) belongs to \( \mathcal{F} \) if and only if, for all \( n, \alpha n \) belongs to \( D_\beta \), that is \( \beta(\alpha n) = 1 \).

7.1.6 Sequentially closed sets with a frame that is either decidable or enumerable

Let \( X \) be a subset of \( \mathbb{N} \). \( X \) is a \textit{frame} if and only if, for each \( s \), \( s \) belongs to \( X \) if and only if, for some \( n, s * \langle n \rangle \) belongs to \( X \).

Let \( X \) be a frame and \( \alpha \) an element of \( \mathcal{N} \). \( \alpha \) is a \textit{member of} \( X \) if and only if, for each \( n, \alpha(n) \) belongs to \( X \). Observe that the set of all members of \( X \) is a sequentially closed subset of \( \mathcal{N} \).

Let \( X \) be a sequentially closed subset of \( \mathcal{N} \). The set \( X \) consisting of all \( s \) in \( \mathbb{N} \) that contain an element of \( \mathcal{F} \) is called the \textit{frame of} \( \mathcal{F} \). Note that the frame of \( \mathcal{F} \) is a frame and that \( \mathcal{F} \) coincides with the set of all members of its frame.

Let \( \mathcal{F} \) be a sequentially closed subset of \( \mathcal{N} \). \( \mathcal{F} \) is \textit{located} if and only if the frame of \( \mathcal{F} \) is a decidable subset of \( \mathbb{N} \). Subsets of \( \mathcal{N} \) that are both sequentially closed and located may be identified with the sets traditionally called \textit{spreads} in intuitionistic mathematics.

Note that every spread is an effectively closed subset of \( \mathbb{N} \) and that every effectively closed subset of \( \mathcal{N} \) is a weakly closed subset of \( \mathcal{N} \). The converse implications do not hold, as is explained in [43], Theorem 9.5. In [47] effectively closed subsets of \( \mathcal{N} \) are simply called \textit{closed} subsets of \( \mathcal{N} \).

Spreads, that is, sequentially closed sets with a decidable frame have always played an important rôle in intuitionistic mathematics. We sometimes call these subsets of \( \mathcal{N} \) \textit{closed-and-located} subsets of \( \mathcal{N} \).

In this paper, we propose to study the more extended class of sequentially closed sets with an enumerable frame. It turns out that these sets are the closed-and-separable subsets of \( \mathcal{N} \).

A subset \( X \) of \( \mathbb{N} \) is \textit{inhabited} or \textit{positively nonempty} if there exists \( n \) such that \( n \in X \). A subset \( X \) of \( \mathcal{N} \) is \textit{inhabited} or \textit{positively nonempty} if there exists \( \alpha \) such that \( \alpha \in X \).

**Theorem 7.3**

(i) Let \( \mathcal{F} \) be a closed-and-separable subset of \( \mathcal{N} \). The frame of \( \mathcal{F} \) is an enumerable subset of \( \mathbb{N} \).

(ii) Let \( \mathcal{F} \) be an inhabited sequentially closed subset of \( \mathcal{N} \) with an enumerable frame. \( \mathcal{F} \) is a closed-and-separable subset of \( \mathcal{N} \).

**Proof**

(i) Let \( \alpha \) belong to \( \mathcal{N} \). Observe that \( s \) belongs to the frame of \( \text{CS}_\alpha \) if and only if for some \( m, n, s = \alpha^m(n) \). Define \( \beta \) such that for all \( m, n, \beta(\text{J}(m, n)) = \alpha^m(n) + 1 \). Then \( \beta \) enumerates the frame of \( \text{CS}_\alpha \).

(ii) Let \( \mathcal{F} \) be a sequentially closed subset of \( \mathcal{N} \) with an enumerable frame and at least one element and let \( \beta \) enumerate the frame of \( \mathcal{F} \). Note that, for each \( s \), if \( s \) belongs to the frame of \( \mathcal{F} \), then there exists \( n \) such that \( s * \langle n \rangle \) belongs to the frame.
of \( \mathcal{F} \), that is, there is an immediate prolongation of \( s \) that also belongs to the frame of \( \mathcal{F} \). As \( \mathcal{F} \) is inhabited, there exists \( n \) such that \( \beta(n) \neq 0 \) and it does no harm to assume that \( \beta(0) \neq 0 \). We want to define \( \alpha \) in \( \mathcal{N} \) such that, for each \( n \), if \( \beta(n) \neq 0 \), then \( \alpha^n \) passes through \( \beta(n) - 1 \) and, if \( \beta(n) = 0 \), then \( \alpha^n \) passes through \( \beta(0) - 1 \). We further require that, for each \( n \), for each \( s \), if \( \alpha^n \) passes through \( s \), then \( \alpha^n \) also passes through \( \beta(j) - 1 \) where \( j \) is the least \( k \) such that \( \beta(k) > 0 \) and \( \beta(k) - 1 \) is an immediate prolongation of \( s \). It is not very difficult to verify that the sequentially closed set \( \mathcal{F} \) coincides with the closed-and-separable set \( \mathcal{CS}_\alpha \).

Because of Theorem 7.3, we sometimes call closed-and-separable subsets of \( \mathcal{N} \) closed-and-semilocated subsets of \( \mathcal{N} \).

7.2 Partial continuous functions from \( \mathcal{N} \) to \( \mathbb{N} \)

For all \( m, n \) in \( \mathbb{N} \), we defined: \( (m, n) := J(m, n) \), \( m' := K(m) \) and \( m'' := L(m) \). So, for each \( m, m = (m', m'') \).

Let \( X \) be a subset of \( \mathbb{N} \). \( X \) is a partial continuous function from \( \mathcal{N} \) to \( \mathbb{N} \) if and only if for all \( a, b \) in \( \mathbb{N} \), if both \( a \) and \( b \) belong to \( X \), and \( a' \) is an initial part of \( b' \), then \( a'' = b'' \).

Let \( X \) be a subset of \( \mathbb{N} \) that is a partial continuous function from \( \mathcal{N} \) to \( \mathbb{N} \).

Let \( \alpha \) belong to \( \mathcal{N} \) and let \( p \) be a natural number. We define: \( X \) maps \( \alpha \) onto \( p \), notation \( X : \alpha \mapsto p \), if and only if, for some \( n \), \((\alpha(n), p)\) belongs to \( X \).

We let \( \text{Dom}(X) \), the Domain of \( X \), be the set of all \( \alpha \) in \( \mathcal{N} \) such that, for some \( p \), \( X \) maps \( \alpha \) onto \( p \). Note that the Domain of \( X \) is a weakly open subset of \( \mathcal{N} \).

Suppose that \( \alpha \) belongs to \( \text{Dom}(X) \). Observe that there is exactly one natural number \( p \) with the property that \( X \) maps \( \alpha \) onto \( p \). We denote this number by \( X(\alpha) \).

Let \( \mathcal{F} \) be a subset of \( \text{Dom}(X) \). We then say: \( X \) is a continuous function from \( \mathcal{F} \) to \( \mathbb{N} \). We let \( \text{Ran}(X, \mathcal{F}) \), the Range of \( X \) on \( \mathcal{F} \) be the set of all numbers \( p \) such that, for some \( \alpha \) in \( \mathcal{F} \), \( X \) maps \( \alpha \) onto \( p \). We also define: \( \text{Ran}(X) := \text{Ran}(X, \text{Dom}(X)) \).

Suppose that \( \mathcal{F} \) is a closed-and-separable subset of \( \mathbb{N} \). Note that, by Theorem 7.3, the frame of \( \mathcal{F} \) is an enumerable subset of \( \mathbb{N} \). Note that, for each \( n, n \) belongs to \( \text{Ran}(X, \mathcal{F}) \) if and only if, for some \( s \) in the frame of \( \mathcal{F} \), \((s, n)\) is in \( X \). It follows that, if \( X \) is an enumerable subset of \( \mathbb{N} \), then also \( \text{Ran}(X, \mathcal{F}) \) is an enumerable subset of \( \mathbb{N} \).

Let \( \mathcal{F} \) be a subset of \( \text{Dom}(X) \). We say that \( X \) is uniformly continuous on \( \mathcal{F} \) if there exists \( n \) in \( \mathbb{N} \) such that for all \( \alpha, \beta \) in \( \mathcal{F} \), if \( \overline{\alpha}(n) = \overline{\beta}(n) \), then \( X(\alpha) = X(\beta) \).

7.3 Compact closed-and-separable subsets of \( \mathcal{N} \)

Let \( B \) be a subset of \( \mathbb{N} \) and let \( \gamma \) be an element of \( \mathcal{N} \). \( B \) bars \( \gamma \), or: \( \gamma \) meets \( B \), if and only if there exists \( n \) such that \( \overline{\gamma}n \) is in \( B \).

**Lemma 7.4** Let \( B \) be an enumerable subset of \( \mathbb{N} \). There exists a decidable subset \( C \) of \( \mathbb{N} \) such that every \( s \) in \( C \) has an initial part in \( B \) and, for each \( \gamma \), \( B \) bars \( \gamma \) if and only if \( C \) bars \( \gamma \).
Lemma 7.6 Let \( B \) be an enumerable subset of \( \mathbb{N} \). Find \( \beta \) in \( \mathcal{N} \) such that \( B \) coincides with \( E_\beta \). Now define \( \alpha \) in \( \mathcal{C} \) as follows. For each \( s \), \( \alpha(s) = 1 \) if and only if there exist \( t < s \) and \( n < s \) such that \( \gamma(n) = t + 1 \) and \( t \) is an initial part of \( s \). Note: every \( s \) in \( D_\alpha \) has an initial part in \( E_\beta \) and, for each \( \gamma \), if \( \gamma \) has an initial part in \( E_\beta \), then \( \gamma \) has an initial part in \( D_\alpha \).  

\[ \square \]

For every subset \( X \) of \( \mathbb{N} \), we define: \( X' := \{m' | m \in X \} \) and \( X'' := \{m'' | m \in X \} \).

Lemma 7.5  
(i) Let \( X \) be an enumerable subset of \( \mathbb{N} \). The sets \( X' \) and \( X'' \) are also enumerable subsets of \( \mathbb{N} \).

(ii) Let \( X, Y \) be enumerable subsets of \( \mathbb{N} \). The set \( A, \) consisting of all \( m \) in \( X \) such that \( m' \) is in \( Y \) is also an enumerable subset of \( \mathbb{N} \).

Proof (i) Find \( \gamma \) in \( \mathcal{N} \) such that \( X = E_\gamma \). Define \( \delta \) in \( \mathcal{N} \) such that, for each \( n \), if \( \gamma(n) = 0 \), then \( \delta(n) = 0 \), and if \( \gamma(n) > 0 \) then \( \delta(n) = (\gamma(n) - 1)' + 1 \). Clearly, \( X' = E_\delta \). The proof that \( X'' \) is enumerable is similar.

(ii) Find \( \gamma, \delta \) in \( \mathcal{N} \) such that \( X = E_\gamma \) and \( Y = E_\delta \). Define \( \varepsilon \) in \( \mathcal{N} \) such that, for each \( m, \) if, for some \( s, \gamma(m') = s + 1 \) and \( \delta(m'') = s' + 1 \), then \( \varepsilon(m) = s + 1 \), and if not, then \( \varepsilon(m) = 0 \). Clearly, \( E_\varepsilon \) is the set of all \( s \) in \( X \) such that \( s' \) is in \( Y \).  

\[ \square \]

Let \( B \) a subset of \( \mathbb{N} \) and let \( \mathcal{F} \) be a subset of \( \mathcal{N} \). \( B \) is a bar in \( \mathcal{F} \), or: \( B \) bars \( \mathcal{F} \), or: \( B \) is a covering of \( \mathcal{F} \) or: \( B \) covers \( \mathcal{F} \), if and only \( B \) bars every \( \gamma \) in \( \mathcal{F} \).

Let \( B \) be a subset of \( \mathbb{N} \). \( B \) is bounded-in-length if and only if there exists \( n \) in \( \mathbb{N} \) such that for all \( s \) in \( B \), \( \text{length}(s) \leq n \).

Let \( X \) be a subset of \( \mathbb{N} \). \( X \) is bounded if and only if there exist \( n \) in \( \mathbb{N} \) such that, for all \( m \) in \( X \), \( m \leq n \). \( X \) is finite if and only if, for some \( b \) in \( \mathbb{N} \), \( X \) coincides with \( D_b \).

Note that, in constructive mathematics, it is not true that every bounded subset of \( \mathbb{N} \) is a finite subset of \( \mathbb{N} \). It is also not true that every bounded and enumerable subset of \( \mathbb{N} \) is finite, a ‘fact’ that is used in classical reverse mathematics, see [32], Theorem II.3.9. A counter-example in Brouwer’s style is the set \( X \) consisting of all natural numbers \( n \) such that \( n = 0 \) and there exists an uninterrupted sequence of 99 9’s in the decimal expansion of \( \pi \). \( X \) is an enumerable and bounded subset of \( \{0\} \), but we are unable to decide if \( X \) is empty or has one element.

Let \( \mathcal{F} \) be a subset of \( \mathcal{N} \).

We define: \( \mathcal{F} \) is totally bounded if, for each \( n \), the set of all \( s \) in \( \mathbb{N} \) such that \( \text{length}(s) = n \) and \( s \) contains an element of \( \mathcal{F} \) is a finite subset of \( \mathbb{N} \).

A closed-and-separable subset of \( \mathbb{N} \) that is totally bounded also is called a finitary spread or a fan in intuitionistic mathematics.

We define: \( \mathcal{F} \) is compact if and only if \( \mathcal{F} \) is closed-and-separable and \( \mathcal{F} \) satisfies the condition mentioned in item (ii) of the next theorem: Every enumerable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) has a finite subset that is a bar in \( \mathcal{F} \).

Lemma 7.6 Let \( \mathcal{F} \) be a closed-and-separable subset of \( \mathbb{N} \) and let \( X \) be a decidable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \). There exists a continuous function \( Z \) from \( \mathcal{F} \) to \( \{0, 1\} \) such that, for all \( s, t, u \) in the frame of \( \mathcal{F} \), if \( s \sqsubseteq t \) and \( s \sqsubseteq u \) and \( t \perp u \) and \( s \) does not meet \( X \), then there exist \( \beta, \gamma \) in \( \mathcal{F} \) passing through \( s \) such that \( Z(\beta) = 1 - Z(\gamma) \).
Proof Let $\mathcal{F}$ be a closed-and-separable subset of $\mathbb{N}$ and let $X$ be a decidable subset of $\mathbb{N}$ that is a bar in $\mathcal{F}$. Let $Y$ be the set of all $s$ in $X$ that have no proper initial part in $X$. Note that $Y$ is a decidable subset of $\mathbb{N}$. Let $U$ be the set of all $s$ in $Y$ that belong to the frame of $\mathcal{F}$. Note that $U$ is a bar in $\mathcal{F}$ and an enumerable subset of $\mathbb{N}$. Find $\delta$ such that $U = E_\delta$.

We now define $\alpha$ in $C$, as follows, by induction. For each $n$, we distinguish three cases.

(1) $\delta(n) = 0$, or, for all $i < n$, $\delta(i) = 0$. Then $\alpha(n) = 0$.
(2) $\delta(n) > 0$ and there exists $i < n$ such that $\delta(i) = \delta(n)$. We then let $i_0$ be the least such $i$ and we define: $\alpha(n) = \alpha(i_0)$.
(3) $\delta(n) > 0$ and, for all $i < n$, $\delta(i) \neq \delta(n)$, and, for some $i < n$, $\delta(i) > 0$. Define $s := \delta(n) - 1$. Note that there exists $j < n$ such that $\delta(j) > 0$ and, for all $i < n$, if $\delta(i) > 0$, if we consider $t := \delta(j) - 1$ and $u := \delta(i) - 1$, then for each $k < \text{length}(u)$, if $u(k) = s(k)$, then $k < \text{length}(t)$, and $t(k) = s(k)$. Let $j_0$ be the least such $j$ and define: $\alpha(n) = 1 - \alpha(j_0)$. Note that $t := \delta(j_0) - 1$ is an element of the set $A := \{\delta(i) - 1 | i < n\}$ that has the longest possible initial segment in common with $s$ among all the elements of $A$.

We let $Z$ be the set of all $m$ such that, for some $n$, $m' = \delta(n) - 1$ and $m'' = \alpha(n)$. Note that $Z$ is an enumerable continuous function from $\mathcal{F}$ to $\{0, 1\}$.

Assume that $s$ belongs to the frame of $\mathcal{F}$ and does not meet $X$ and that there are $t, u$ in the frame of $\mathcal{F}$ such that $t \perp u$ and $s \sqsubseteq t$ and $s \sqsubseteq u$. Note that there are $t, u$ in $Y$ such that $t \perp u$ and $s \sqsubseteq t$ and $s \sqsubseteq u$. Let $j_0$ be the least $j$ such that $\delta(j) > 0$ and $s \sqsubseteq \delta(j) - 1$ and let $n_0$ be the least $n$ such that $n > j_0$ and $\delta(n) > 0$ and $\delta(n) \neq \delta(j_0)$ and $s \sqsubseteq \delta(n) - 1$. Note that $\alpha(n_0) = 1 - \alpha(j_0)$ and that for all $\beta, \gamma$ in $\mathcal{F}$, if $\beta$ passes through $\delta(j_0) - 1$ and $\gamma$ passes through $\delta(n_0) - 1$, then $Z(\beta) = 1 - Z(\gamma)$.

The next theorem characterizes compact closed-and-separable subsets of $\mathcal{N}$.

Theorem 7.7 Let $\mathcal{F}$ be a closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent:

(i) Every decidable subset of $\mathbb{N}$ that is a bar in $\mathcal{F}$ has a finite subset that is a bar in $\mathcal{F}$.
(ii) $\mathcal{F}$ is compact, that is: every enumerable subset of $\mathbb{N}$ that is a bar in $\mathcal{F}$ has a finite subset that is a bar in $\mathcal{F}$.
(iii) For every enumerable continuous function $X$ from $\mathcal{F}$ to $\mathbb{N}$, $\text{Ran}(X, F)$ is a finite subset of $\mathbb{N}$.
(iv) For every enumerable continuous function $X$ from $\mathcal{F}$ to $\mathbb{N}$, $\text{Ran}(X, F)$ is a bounded subset of $\mathbb{N}$.
(v) Every decidable subset of $\mathbb{N}$ that is a bar in $\mathcal{F}$ has a subset that is bounded-in-length and a bar in $\mathcal{F}$.
(vi) Every enumerable subset of $\mathbb{N}$ that is a bar in $\mathcal{F}$ has a subset that is bounded-in-length and a bar in $\mathcal{F}$.
(vii) $\mathcal{F}$ is totally bounded and every enumerable continuous function from $\mathcal{F}$ to $\mathbb{N}$ is uniformly continuous on $\mathcal{F}$.
(viii) $\mathcal{F}$ is totally bounded and every bounded enumerable continuous function from $\mathcal{F}$ to $\mathbb{N}$ is uniformly continuous on $\mathcal{F}$.
Proof (i) ⇒ (ii): Suppose that every decidable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) has a finite subset that is a bar in \( \mathcal{F} \).

Let \( X \) be an enumerable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \). Using Lemma 7.4 we find a decidable subset \( Y \) of \( \mathbb{N} \) that bars the same elements of \( \mathcal{N} \) as \( X \) and has the property that every element of \( Y \) has an initial part in \( X \). It follows that also \( Y \) is a bar in \( \mathcal{F} \). Let \( A \) be a finite subset of \( Y \) that is a bar in \( \mathcal{F} \). Find \( \gamma \) such that \( X = E_{\gamma} \). For each \( s \) in \( A \) there exists \( n \) such that \( \gamma(n) > 0 \) and \( t := \gamma(n) - 1 \) is an initial part of \( s \). We let \( B \) be the set of all \( t \) such that, for some \( s \) in \( A \), for some \( n \), \( \gamma(n) > 0 \) and \( t := \gamma(n) - 1 \) is an initial part of \( s \), while, for each \( i < n \), either \( \gamma(i) = 0 \) or \( \gamma(i) > 0 \) and \( \gamma(i) - 1 \) is not an initial part of \( s \).

\( B \) is a finite subset of \( X \) and a bar in \( \mathcal{F} \).

We thus see that every enumerable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) has a finite subset that is a bar in \( \mathcal{F} \).

(ii) ⇒ (iii): Suppose that every enumerable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) has a finite subset that is a bar in \( \mathcal{F} \).

Let \( X \) be an enumerable subset of \( \mathbb{N} \) that is a continuous function from \( \mathcal{F} \) to \( \mathbb{N} \). Note that the set \( Y \) consisting of all \( s \) in \( X' \) that belong to the frame of \( \mathcal{F} \) is a bar in \( \mathcal{F} \), and, by Lemma 7.5, an enumerable subset of \( \mathbb{N} \). Let \( A \) be a finite subset of \( Y \) that is a bar in \( \mathcal{F} \). Let \( B \) be the set of all natural numbers \( p \) such that, for some \( a \) in \( A \), \((a, p)\) is in \( A \). Note that \( B \) is a finite set and that \( B = Range(X, F) \).

We thus see that every enumerable continuous function from \( \mathcal{F} \) to \( \mathbb{N} \) has finite Range on \( \mathcal{F} \).

(iii) ⇒ (iv): Obvious.

(iv) ⇒ (v): Suppose that every enumerable continuous function from \( \mathcal{F} \) to \( \mathbb{N} \) has bounded Range on \( \mathcal{F} \).

Let \( X \) be a decidable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \). We let \( Y \) be the set of all \( s \) in \( \mathbb{N} \) such that \( s' \in X \) and no proper initial part of \( s' \) is in \( X \) and \( s'' = length(s) \). Note that \( Y \) is an enumerable continuous function from \( \mathcal{F} \) to \( \mathbb{N} \). Find \( m \) in \( \mathbb{N} \) such that \( m \) is an upper bound for the Range of \( Y \) on \( \mathcal{F} \). Note that the set of all \( a \) in \( X \) such that \( length(a) \leq m \) is a bar in \( \mathcal{F} \).

We thus see that every decidable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) has a subset that is bounded-in-length and a bar in \( \mathcal{F} \).

(v) ⇒ (vi): Suppose that every decidable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) has a subset that is bounded-in-length and a bar in \( \mathcal{F} \).

Let \( X \) be an enumerable subset of \( \mathbb{N} \) and a bar in \( \mathcal{F} \). Using Lemma 7.4 we let \( Y \) be a decidable subset \( \mathbb{N} \) barring the same elements of \( \mathcal{N} \) as \( X \) such that every \( s \) in \( Y \) has an initial part \( t \) in \( X \). Find \( m \) such that the set of all \( s \) in \( Y \) such that \( length(s) \leq m \) is a bar in \( \mathcal{F} \). Note that also the set of all numbers \( t \) in \( X \) such that \( length(t) \leq m \) is a bar in \( \mathcal{F} \).

We thus see that every enumerable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) has a subset that is bounded-in-length and a bar in \( \mathcal{F} \).

(vi) ⇒ (vii): Suppose that every enumerable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) has a subset that is bounded-in-length and a bar in \( \mathcal{F} \).

We use Theorem 7.3. Find \( \beta \) in \( \mathcal{N} \) such that \( \beta \) enumerates the frame of \( \mathcal{F} \), that is, for all \( s \), \( s \) contains an element of \( \mathcal{F} \) if and only if, for some \( n \), \( \beta(n) = s + 1 \). Let \( n \) be a natural number and let \( X \) be the set of all \( s \) in the frame of \( \mathcal{F} \) such that \( length(s) = 3n \),
and, for some $i < \text{length}(s)$, $\beta(i) = \bar{s}n + 1$. Note that $X$ is an enumerable subset of $\mathbb{N}$ and a bar in $\mathcal{F}$. Find $m$ such that the set of all $s$ in $X$ such that $\text{length}(s) \leq m$ is a bar in $\mathcal{F}$. Let $A$ be the set of all $s$ such that $\text{length}(s) = n$, and for some $i < m$, $\beta(i) = s + 1$. Note that $A$ is finite and that $A$ is the set of all $s$ in $\mathbb{N}$ such that $\text{length}(s) = n$ and $s$ contains an element of $\mathcal{F}$.

We thus see that $\mathcal{F}$ is a totally bounded subset of $\mathcal{N}$.

Let $Y$ be an enumerable continuous function from $\mathcal{F}$ to $\mathbb{N}$. Note that $Y'$ is an enumerable subset of $\mathbb{N}$ and a bar in $\mathcal{F}$. Find $n$ such that the set of all $s$ in $Y'$ such that $\text{length}(s) \leq n$ is a bar in $\mathcal{F}$. Note that, for all $\alpha, \beta$ in $\mathcal{F}$, if $\alpha n = \beta n$, then $Y(\alpha) = Y(\beta)$. We thus see that every enumerable continuous function from $\mathcal{F}$ to $\mathbb{N}$ is uniformly continuous on $\mathcal{F}$.

(vii) $\Rightarrow$ (viii): Obvious.

(viii) $\Rightarrow$ (i): Assume that $\mathcal{F}$ is totally bounded and that every bounded enumerable continuous function from $\mathcal{F}$ to $\mathbb{N}$ is uniformly continuous on $\mathcal{F}$.

Let $X$ be a decidable subset of $\mathbb{N}$ that is a bar in $\mathcal{F}$. Using Lemma 7.6, let $Z$ be an enumerable continuous function from $\mathcal{F}$ to $\{0, 1\}$ such that, for every $s$ in the frame of $\mathcal{F}$, if $s$ does not meet $X$ and there are $t, u$ in the frame of $\mathcal{F}$ such that $s \sqsubset t$ and $s \sqsubseteq u$ and $t \perp u$, then there are $\beta, \gamma$ in $\mathcal{F}$ passing through $s$ such that $Z(\beta) = 1 - Z(\gamma)$.

Find $n$ such that for all $\beta, \gamma$ in $\mathcal{F}$, if $\beta n = \gamma n$, then $Z(\beta) = Z(\gamma)$. Let $A$ be the set of all $s$ in $\mathbb{N}$ such that $\text{length}(s) = n$ and $s$ contains an element of $\mathcal{F}$. Note that $A$ is a finite subset of $\mathbb{N}$.

Let $s$ be an element of $A$. Note that, for all $t, u$ in the frame of $\mathcal{F}$, if $s \sqsubset t$ and $s \sqsubseteq u$, then $t \sqsubseteq u$ or $u \sqsubseteq t$.

Let $Y$ be the set of all elements $v$ of $X$ that do not have a proper initial part in $X$.

Note that, for every $s$ in $A$ there exists exactly one $t$ in $Y$ such that either $t \sqsubseteq s$ or $s \sqsubseteq t$. Let $B$ be the set of all $t$ in $Y$ such that, for some $s$ in $A$, $t \sqsubseteq s$ or $s \sqsubseteq t$. $B$ is a finite subset of $X$ and a bar in $\mathcal{F}$.

We thus see: every decidable subset of $\mathbb{N}$ that is a bar in $\mathcal{F}$ has a finite subset that is a bar in $\mathcal{F}$. □

7.3.1 A note on perfect closed-and-separable subsets of $\mathcal{N}$

Let $\mathcal{F}$ be a closed-and-separable subset of $\mathcal{N}$. We just saw, in Theorem 7.7(vii), that, if $\mathcal{F}$ is compact, then every enumerable continuous function from $\mathcal{F}$ to $\mathbb{N}$ is uniformly continuous on $\mathcal{F}$. From the fact that every enumerable continuous function from $\mathcal{F}$ to $\mathbb{N}$ is uniformly continuous on $\mathcal{F}$, we can not conclude that $\mathcal{F}$ is compact, as appears from the following simple example.

Let $\mathcal{F}$ be the set of all $\alpha$ in $\mathcal{N}$ such that, for each $n$, $\alpha(n + 1) = 0$. Note that $\mathcal{F}$ is a closed-and-separable subset of $\mathcal{N}$ and every continuous function is uniformly continuous on $\mathcal{F}$, as, for every such function $X$, for all $\alpha, \beta$ in $\mathcal{F}$, if $\alpha(0) = \beta(0)$, then $\alpha = \beta$ and $X(\alpha) = X(\beta)$. Note that the set $Y$ consisting of all $s$ such that $\text{length}(s) = 1$ is decidable bar in $\mathcal{F}$ and that every finite subset of $Y$ fails to be a bar in $\mathcal{F}$.
We can draw this conclusion, however, if we require \( \mathcal{F} \) to be totally bounded, see Theorem 7.7(vii), and also if we require \( \mathcal{F} \) to be perfect, according to the following definition.

A closed-and-separable subset \( \mathcal{F} \) of \( \mathcal{N} \) is called **perfect** if and only if, for each \( \alpha \) in \( \mathcal{F} \), for each \( n \), there exists \( \beta \) in \( \mathcal{F} \) such that \( \overline{\alpha n} = \overline{\beta n} \) and \( \alpha \neq \beta \).

**Theorem 7.8** Let \( \mathcal{F} \) be a perfect closed-and-separable subset of \( \mathcal{N} \). The following statements are equivalent:

(i) \( \mathcal{F} \) is compact.

(ii) Every enumerable continuous function from \( \mathcal{F} \) to \( \mathbb{N} \) is uniformly continuous on \( \mathcal{F} \).

(iii) Every bounded enumerable continuous function from \( \mathcal{F} \) to \( \mathbb{N} \) is uniformly continuous on \( \mathcal{F} \).

**Proof** (i) \( \Rightarrow \) (ii): See Theorem 7.7(vii).

(ii) \( \Rightarrow \) (iii): Obvious.

(iii) \( \Rightarrow \) (i): Assume that every bounded enumerable continuous function from \( \mathcal{F} \) to \( \mathbb{N} \) is uniformly continuous on \( \mathcal{F} \).

Let \( X \) be a decidable subset of \( \mathbb{N} \) and a bar in \( \mathcal{F} \). Use Lemma 7.6 and find an enumerable continuous function \( Z \) from \( \mathcal{F} \) to \( \{0, 1\} \) such that, for every \( s, t, u \) in the frame of \( \mathcal{F} \), if \( s \) does not meet \( X \) and \( s \sqsubseteq t \) and \( s \sqsubseteq u \) and \( t \perp u \), then there are \( \beta, \gamma \) in \( \mathcal{F} \) passing through \( s \) such that \( Z(\beta) = 1 - Z(\gamma) \).

Find \( n \) such that for all \( \beta, \gamma \) in \( \mathcal{F} \), if \( \overline{\beta n} = \overline{\gamma n} \), then \( Z(\beta) = Z(\gamma) \). Let \( A \) be the set of all \( s \) in \( \mathbb{N} \) such that \( \text{length}(s) = n \) and \( s \) contains an element of \( \mathcal{F} \). Note that every \( s \) in \( A \) meets \( X \). We thus see that every decidable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) has a subset that is bounded-in-length and a bar in \( \mathcal{F} \). According to Theorem 7.7(v), \( \mathcal{F} \) is compact.

\( \square \)

### 7.3.2 Dini’s Theorem

The following result from classical real analysis is due to U. Dini:

*Let \( f_0, f_1, f_2, \ldots \) be a sequence of continuous functions from the closed interval \([0, 1]\) to \( \mathbb{R} \) that is monotone nonincreasing and converges pointwise to 0, that is: \( \forall x \in [0, 1] \forall n [f_n(x) \geq f_{n+1}(x) \geq 0] \) and \( \forall x \in [0, 1] \forall \varepsilon > 0 \exists n [f_n(x) < \varepsilon] \). Then the sequence \( f_0, f_1, f_2, \ldots \) converges to 0 uniformly on \([0, 1]\), that is: \( \forall \varepsilon > 0 \exists n \forall x \in [0, 1] [f_n(x) < \varepsilon] \).*

We now prove a theorem that is closely connected with this result.

Let \( X \) be a subset of \( \mathbb{N} \). For each \( m \), we let \( X \upharpoonright m \) be the set of all \( n \) in \( \mathbb{N} \) such that \( (m, n) \) belongs to \( X \). Note that, if \( X \) is a decidable or an enumerable subset of \( \mathbb{N} \), then, for each \( m \), \( X \upharpoonright m \) is a decidable or an enumerable subset of \( \mathbb{N} \), respectively.

**Theorem 7.9** Let \( \mathcal{F} \) be a closed-and-separable subset of \( \mathcal{N} \). The following statements are equivalent:

(i) \( \mathcal{F} \) is compact.

(ii) For every enumerable subset \( X \) of \( \mathcal{N} \), if (1) for each \( n \), \( X \upharpoonright n \) is a continuous function from \( \mathcal{F} \) to \( \mathbb{N} \), and, for all \( \alpha \) in \( \mathcal{F} \), \( (X \upharpoonright n)(\alpha) \geq (X \upharpoonright (n + 1))(\alpha) \), and
(2) for all \( \alpha \) in \( \mathcal{F} \) there exists \( n \) such that \( (X \upharpoonright n)(\alpha) = 0 \), then (3) there exists \( n \) such that for all \( \alpha \) in \( \mathcal{F} \), \( (X \upharpoonright n)(\alpha) = 0 \).

Proof (i) \( \Rightarrow \) (ii): Suppose \( \mathcal{F} \) is compact. Now let \( X \) be a subset of \( \mathbb{N} \) satisfying the requirements in (ii). We let \( Y \) be the set of all \( s \) such that, for some \( n, (s, 0) \) is in \( X \upharpoonright n \). Note that \( Y \) is an enumerable subset of \( \mathbb{N} \) and a bar in \( \mathcal{F} \). Let \( Z \) be a finite subset of \( Y \) that is a bar in \( \mathcal{F} \). Find \( n \) such that \( Z \subseteq \bigcup_{m \leq n} X \upharpoonright m \) and note: for all \( \alpha \) in \( \mathcal{F} \), \( (X \upharpoonright n)(\alpha) = 0 \).

(ii) \( \Rightarrow \) (i): Let \( Y \) be a decidable subset of \( \mathbb{N} \) and a bar in \( \mathcal{F} \). We let \( X \) be the set of all numbers of the form \( (n, (s, i)) \) such that \( s \) belongs to \( Y \) and no proper initial part of \( s \) belongs to \( Y \) and either \( n + i = s \) or \( n \geq s \) and \( i = 0 \). Note that \( X \) is a decidable subset of \( \mathbb{N} \) and that, for each \( n \), \( X \upharpoonright n \) is a continuous function from \( \mathcal{F} \) to \( \mathbb{N} \). Note that, for each \( \alpha \) in \( \mathcal{F} \), \( (X \upharpoonright 0)(\alpha) = \alpha q \), where \( q \) is the least \( i \) such that \( \alpha i \) belongs to \( Y \), and, for each \( n \) in \( \mathbb{N} \), if \( (X \upharpoonright n)(\alpha) > 0 \), then \( (X \upharpoonright (n + 1))(\alpha) = (X \upharpoonright n)(\alpha) - 1 \), and, if \( (X \upharpoonright n)(\alpha) = 0 \), then \( (X \upharpoonright (n + 1))(\alpha) = 0 \). Note that, for every \( \alpha \) in \( \mathcal{F} \), for every \( n \), if \( (X \upharpoonright n)(\alpha) = 0 \), then there exists \( q \leq n \) such that \( \alpha q \) belongs to \( X \). Also note that, for all \( \alpha \) in \( \mathcal{F} \), there exists \( n \) in \( \mathbb{N} \) such that \( (X \upharpoonright n)(\alpha) = 0 \). Using (ii), we find \( n \) such that for all \( \alpha \) in \( \mathcal{F} \), \( (X \upharpoonright n)(\alpha) = 0 \). It follows that, for each \( \alpha \) in \( \mathcal{F} \), there exists \( q \leq n \) such that \( \alpha q \) belongs to \( X \), so the set of all \( s \) in \( X \) with the property: \( \text{length}(s) \leq n \) is a bar in \( \mathcal{F} \).

We thus see that every decidable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) has a subset that is bounded-in-length and a bar in \( \mathcal{F} \). Using Theorem 7.7 we conclude that \( \mathcal{F} \) is compact. \( \square \)

7.4 Closed-and-separable subsets of \( \mathcal{N} \) that are positively noncompact

A subset \( X \) of \( \mathbb{N} \) is positively infinite if and only if, for each \( n \), there exists \( m \) such that \( m > n \) and \( m \) belongs to \( X \).

A subset \( X \) of \( \mathbb{N} \) is positively unbounded-in-length if, for each \( n \), there exists \( s \) in \( X \) such that \( \text{length}(s) > n \).

Let \( \mathcal{F} \) be a subset of \( \mathcal{N} \), and let \( B \) be a subset of \( \mathbb{N} \). \( B \) positively fails to cover \( \mathcal{F} \) or: positively fails to be a bar in \( \mathcal{F} \) if and only if there exists \( \alpha \) in \( \mathcal{F} \) not contained in any element of \( B \).

Let \( \mathcal{F} \) be a subset of \( \mathcal{N} \) and let \( X \) be a subset of \( \mathbb{N} \) that is a continuous function from \( \mathcal{F} \) to \( \mathbb{N} \). \( X \) (positively) fails to be uniformly continuous on \( \mathcal{F} \) if and only if, for each \( m \), there exist \( \alpha, \beta \) in \( \mathcal{F} \) such that \( \alpha m = \beta m \) and \( X(\alpha) \neq X(\beta) \).

For each \( n \) in \( \mathbb{N} \) we let \( n \) denote the element of \( \mathcal{N} \) with the constant value \( n \).

Let \( \mathcal{F} \) be a closed-and-separable subset of \( \mathcal{N} \).

We define: \( \mathcal{F} \) is positively noncompact if and only if \( \mathcal{F} \) satisfies the condition mentioned in item (ii) of the next theorem: There exists an enumerable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) while every one of its finite subsets positively fails to be a bar in \( \mathcal{F} \).

The next theorem is a counterpart to Theorem 7.7 and characterizes closed-and-separable subsets of \( \mathcal{N} \) that are positively noncompact.

**Theorem 7.10** Let \( \mathcal{F} \) be a closed-and-separable subset of \( \mathcal{N} \). The following statements are equivalent:

\[ \text{Springer} \]
(i) There exists a decidable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) while every one of its finite subsets positively fails to be a bar in \( \mathcal{F} \).

(ii) \( \mathcal{F} \) is positively noncompact, that is: there exists an enumerable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) while every one of its finite subsets positively fails to be a bar in \( \mathcal{F} \).

(iii) There exists an enumerable continuous function \( X \) from \( \mathcal{F} \) to \( \mathbb{N} \) such that \( \text{Ran}(X, F) \) is positively infinite.

(iv) There exists an enumerable continuous function \( X \) from \( \mathcal{F} \) to \( \mathbb{N} \) such that \( \text{Ran}(X, F) \) is positively unbounded-in-length.

(v) There exists a decidable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) while every one of its subsets that is bounded-in-length positively fails to be a bar in \( \mathcal{F} \).

(vi) There exists an enumerable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) while every one of its subsets that is bounded-in-length positively fails to be a bar in \( \mathcal{F} \).

Proof (ii) \( \Rightarrow \) (i): Let \( X \) be an enumerable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) such that every finite subset of \( X \) positively fails to be a bar in \( \mathcal{F} \). Using Lemma 7.4 we find a decidable subset \( Y \) of \( \mathbb{N} \) that bars the same elements of \( \mathcal{N} \) as \( X \) and has the property that every element of \( Y \) has an initial part in \( X \). Let \( A \) be a finite subset of \( Y \). Find \( \gamma \) such that \( X = E_\gamma \). Let \( B \) be the set of all \( t \) such that, for some \( s \) in \( A \), for some \( n, \gamma(n) > 0 \) and \( t := \gamma(n) - 1 \) is an initial part of \( s \) and there is no \( i < n \) such that \( \gamma(i) > 0 \) and \( \gamma(i) - 1 \) is an initial part of \( s \). \( B \) is a finite subset of \( X \) and positively fails to be a bar in \( \mathcal{F} \). Find \( \alpha \) in \( \mathcal{F} \) such that no initial segment of \( \alpha \) belongs to \( B \). Conclude that no initial segment of \( \alpha \) belongs to \( A \), so \( A \) positively fails to be a bar in \( \mathcal{F} \).

We thus see that there exists a decidable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) while everyone of its finite subsets positively fails to be a bar in \( \mathcal{F} \).

(iii) \( \Rightarrow \) (ii): Let \( X \) be an enumerable continuous function from \( \mathcal{F} \) to \( \mathbb{N} \) such that \( \text{Ran}(X, F) \) is positively infinite. Let \( Y \) be the set of all \( s \) in \( X' \) that belong to the frame of \( \mathcal{F} \). Note that, by Theorem 7.3 and Lemma 7.5, \( Y \) is an enumerable subset of \( \mathbb{N} \) and a bar in \( \mathcal{F} \). Let \( A \) be a finite subset of \( Y \). Let \( B \) be the set of all \( n \) such that, for some \( s \) in \( A \), \((s, n)\) belongs to \( A \). Note that \( B \) is a finite subset of \( \text{Ran}(X, F) \). Find \( m \) such that \( m \) belongs to \( \text{Ran}(X, F) \) and not to \( B \). Find \( \alpha \) in \( \mathcal{F} \) such that \( X(\alpha) = m \). Note that no initial part of \( \alpha \) belongs to \( A \).

We thus see that there exists an enumerable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) while every one of its finite subsets positively fails to be a bar in \( \mathcal{F} \).

(iv) \( \Rightarrow \) (iii). Obvious.

(v) \( \Rightarrow \) (iv): Assume that \( X \) is a decidable subset of \( \mathbb{N} \) and a bar in \( \mathcal{F} \) while every subset of \( X \) that is bounded-in-length positively fails to be a bar in \( \mathcal{F} \). We let \( Y \) be the set of all \( s \) such that \( s' = s'' \) and \( s' \) belongs to \( X \) and no proper initial part of \( s' \) belongs to \( X \). Note that \( Y \) is an enumerable continuous function from \( \mathcal{F} \) to \( \mathbb{N} \), and, for every \( \alpha \) in \( \mathcal{F} \), \( Y(\alpha) \) is an initial part of \( \alpha \) that belongs to \( X \). For every \( n \), there exists \( \alpha \) such that no initial part of \( \alpha \) belongs to \( X \), and, therefore, \( \text{length}(Y(\alpha)) > n \).

We thus see that there exists an enumerable continuous function \( Y \) from \( \mathcal{F} \) to \( \mathbb{N} \) such that \( \text{Ran}(Y, F) \) is positively unbounded-in-length.

(vi) \( \Rightarrow \) (v): Assume that \( X \) is an enumerable subset of \( \mathbb{N} \) and a bar in \( \mathcal{F} \) while every subset of \( X \) that is bounded-in-length positively fails to be a bar in \( \mathcal{F} \). Using Lemma
7.4, find a decidable subset $Y$ of $\mathbb{N}$ that bars the same elements of $\mathcal{N}$ as $X$ such that every element of $Y$ has an initial part in $X$. Note that $Y$ is a bar in $\mathcal{F}$. For each $n$, find $\alpha$ in $\mathcal{F}$ such that no initial part of $\alpha n$ belongs to $X$ and observe that also no initial part of $\alpha n$ belongs to $Y$.

We thus see that there exists a decidable subset $Y$ of $\mathbb{N}$ that is a bar in $\mathcal{F}$ while every subset of $Y$ that is bounded-in-length positively fails to be a bar in $\mathcal{F}$.

(i) $\Rightarrow$ (vi): Let $X$ be a decidable subset of $\mathbb{N}$ that is a bar in $\mathcal{F}$ while every one of its finite subsets positively fails to be a bar in $\mathcal{F}$. Let $Y$ be the set of all $s$ in $\mathbb{N}$ such that $\text{length}(s)$ is an initial part of $s$ and an element of $X$. For each $\alpha$ in $\mathcal{F}$, for each $p$, $\alpha p$ is in $Y$ if and only if there exists $q$ such that $p = \alpha q$ and $p$ is in $X$. For every $n$, the set of all $s$ in $X$ such that $s \leq n$ is a finite subset of $X$ that positively fails to be bar in $\mathcal{F}$, so one may find $\alpha$ such that there is no $p$ such that $\alpha p$ is in $X$ and $\alpha p \leq n$, and, therefore, there is no $q \leq n$ such that $\alpha q$ is in $Y$.

We thus see that there exists an enumerable subset $Y$ of $\mathbb{N}$ that is a bar in $\mathcal{F}$, while every subset of $Y$ that is bounded-in-length positively fails to be a bar in $\mathcal{F}$. □

Theorem 7.11 Let $\mathcal{F}$ be a totally bounded closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent:

(i) $\mathcal{F}$ is positively noncompact.
(ii) There exists a bounded enumerable continuous function from $\mathcal{F}$ that positively fails to be uniformly continuous.
(iii) There exists an enumerable continuous function from $\mathcal{F}$ to $\mathbb{N}$ that positively fails to be uniformly continuous on $\mathcal{F}$.

Proof (i) $\Rightarrow$ (ii): Assume that $\mathcal{F}$ is positively noncompact. Using Theorem 7.10(v), we find a decidable subset $X$ of $\mathbb{N}$ that is a bar in $\mathcal{F}$ while every subset of $X$ that is bounded-in-length positively fails to be a bar in $\mathcal{F}$. Using Lemma 7.6 we find an enumerable continuous function $Z$ from $\mathcal{F}$ to $\{0, 1\}$ such that, for all $s$, $t$, $u$ in the frame of $\mathcal{F}$, if $s$ does not meet $X$ and $s \sqsubseteq t$ and $s \sqsubseteq u$ and $t \perp u$, then there are $\beta$, $\gamma$ in $\mathcal{F}$ passing through $s$ such that $Z(\beta) = 1 - Z(\gamma)$.

Let $n$ be a natural number. Let $A$ be the set of all $s$ in the frame $\mathcal{F}$ such that $\text{length}(s) = n$. Note that $A$ is a finite subset of $\mathbb{N}$. Find $s$ in $A$ such that no initial part of $s$ belongs to $X$. Note that there exist $\beta$, $\gamma$ in $\mathcal{F}$ passing through $s$ such that $Z(\beta) = 1 - Z(\gamma)$.

We thus see that $Z$ positively refuses to be uniformly continuous on $\mathcal{F}$.

(ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (i): Let $X$ be an enumerable continuous function from $\mathcal{F}$ to $\mathbb{N}$ that positively fails to be uniformly continuous on $\mathcal{F}$. Note that $X'$ is an enumerable subset of $\mathbb{N}$ and a bar in $\mathcal{F}$. For each $n$, find $\beta$, $\gamma$ in $\mathcal{F}$ such that $\beta n = \gamma n$ and $X(\beta) \neq X(\gamma)$. Note that no initial part of $\beta n$ is in $X'$.

We thus see that there exists an enumerable subset $Z = X'$ of $\mathbb{N}$ that is a bar in $\mathcal{F}$ while every subset of $Z$ that is bounded-in-length positively fails to be a bar in $\mathcal{F}$. Using Theorem 7.10, we conclude that $\mathcal{F}$ is positively noncompact. □
7.4.1 A second note on perfect closed-and-separable subsets of $\mathbb{N}$

Let $F$ be a closed-and-separable subset of $\mathbb{N}$ and suppose that $X$ is an enumerable continuous function from $F$ to $\mathbb{N}$ that positively fails to be uniformly continuous on $F$. $X'$ then is an enumerable subset of $\mathbb{N}$ that is a bar in $F$ while every subset of $X'$ that is bounded-in-length positively fails to be a bar in $F$. Using Theorem 7.10(vi), one may conclude that $F$ is positively noncompact. Conversely, the statement: if $F$ is positively noncompact, then there exists an enumerable continuous function from $F$ to $\mathbb{N}$ that positively fails to be uniformly continuous on $F$ is not true in general, as appears from the example we have given in Sect. 7.3.1.

The converse holds, however, if we add the condition: $F$ is totally bounded, as we saw in Theorem 7.11. The next Theorem states that the converse also holds if we add the condition: $F$ is perfect.

**Theorem 7.12** Let $F$ be a perfect closed-and-separable subset of $\mathbb{N}$. The following statements are equivalent:

(i) $F$ is positively noncompact.

(ii) There exists a bounded enumerable continuous function from $F$ to $\mathbb{N}$ that positively fails to be uniformly continuous on $F$.

(iii) There exists an enumerable continuous function from $F$ to $\mathbb{N}$ that positively fails to be uniformly continuous on $F$.

**Proof** (i) $\Rightarrow$ (ii): Suppose that $F$ is positively noncompact. Using Theorem 7.10(i) $\Rightarrow$ (v), we find a decidable subset $X$ of $\mathbb{N}$ that is a bar in $F$, while, for every $m$, the set of all $s$ in $X$ satisfying $\text{length}(s) \leq m$ positively fails to be a bar in $F$. Using Lemma 7.6 we find an enumerable continuous function $Z$ from $F$ to $\{0,1\}$ such that, for all $s, t, u$ in the frame of $F$, if $s$ does not meet $X$ and $s \sqsubseteq t$ and $s \sqsubseteq u$ and $t \perp u$, then there are $\beta, \gamma$ in $F$ passing through $s$ such that $Z(\beta) = 1 - Z(\gamma)$.

Let $n$ be a natural number. Find $\beta$ in $F$ such that $s := \beta n$ does not meet $X$. Note that, as $F$ is perfect, there exist $t, u$ in the frame of $F$ such that $s \sqsubseteq t$ and $s \sqsubseteq u$ and $t \perp u$. Conclude that there exist $\beta, \gamma$ in $F$ passing through $s$ such that $Z(\beta) = 1 - Z(\gamma)$.

We thus see that $Z$ positively fails to be uniformly continuous on $F$.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i): Let $X$ be an enumerable continuous function from $F$ to $\mathbb{N}$ that positively fails to be uniformly continuous on $F$. Note that $X'$ is an enumerable bar in $F$ such that, for each $n$, the set of all $s$ in $X'$ with the property $\text{length}(s) \leq n$ positively fails to be a bar in $F$. Using Theorem 7.10(v), we conclude that $F$ is positively noncompact.

$\square$

7.4.2 Dini’s Theorem again

Theorem 7.9 has a counterpart.

**Theorem 7.13** Let $F$ be a closed-and-separable subset of $\mathbb{N}$. The following statements are equivalent:
(i) \( \mathcal{F} \) is positively noncompact.

(ii) There exists an enumerable subset \( X \) of \( \mathbb{N} \) such that (1) for each \( n \), \( X \upharpoonright n \) is a continuous function from \( \mathcal{F} \) to \( \mathbb{N} \), and, for all \( \alpha \) in \( \mathcal{F} \), \( (X \upharpoonright n)(\alpha) \geq (X \upharpoonright (n+1))(\alpha) \), and (2) for all \( \alpha \) in \( \mathcal{F} \), there exists \( n \) in \( \mathbb{N} \) such that \( (X \upharpoonright n)(\alpha) = 0 \), and (3) for all \( n \), there exists \( \alpha \) in \( \mathcal{F} \) such that \( (X \upharpoonright n)(\alpha) \neq 0 \).

Proof (i) \( \Rightarrow \) (ii): Assume that \( \mathcal{F} \) is positively noncompact. Using Theorem 7.10, we let \( Y \) be a decidable subset of \( \mathbb{N} \) that is a bar in \( \mathcal{F} \) while, for each \( n \), the set of all \( s \) in \( Y \) with the property: \( s \leq n \) positively fails to be a bar in \( \mathcal{F} \). As in the proof of Theorem 7.9, we let \( X \) be the set of all numbers of the form \( (n, (s, i)) \) such that \( s \) belongs to \( Y \) and no proper initial part of \( s \) belongs to \( Y \) and either \( n + i = s \) or \( n \geq s \) and \( i = 0 \). Note that \( X \) is a decidable subset of \( \mathbb{N} \) and that, for each \( n \), \( X \upharpoonright n \) is a continuous function from \( \mathcal{F} \) to \( \mathbb{N} \). Note that, for each \( \alpha \) in \( \mathcal{F} \), \( (X \upharpoonright 0)(\alpha) = \overline{\alpha} q \), where \( q \) is the least \( i \) such that \( \overline{\alpha} i \) belongs to \( Y \), and, for each \( n \) in \( \mathbb{N} \), if \( (X \upharpoonright n)(\alpha) > 0 \), then \( (X \upharpoonright (n+1))(\alpha) = (X \upharpoonright n)(\alpha) - 1 \), and, if \( (X \upharpoonright n)(\alpha) = 0 \), then \( (X \upharpoonright (n+1))(\alpha) = 0 \). Note that, for every \( \alpha \) in \( \mathcal{F} \), for every \( s \), \( s \) is the least \( n \) such that \( (X \upharpoonright n)(\alpha) = 0 \) if and only if \( s \) is the initial part of \( \alpha \) that is in \( Y \). Clearly then, for all \( \alpha \) in \( \mathcal{F} \), there exists \( n \) in \( \mathbb{N} \) such that \( (X \upharpoonright n)(\alpha) = 0 \), and for each \( n \), there exists \( \alpha \) in \( \mathcal{F} \) such that \( (X \upharpoonright n)(\alpha) \neq 0 \).

(ii) \( \Rightarrow \) (i): Let \( X \) be a subset of \( \mathcal{N} \) satisfying the requirements. As in the proof of Theorem 7.9, we let \( Y \) be the set of all \( s \) such that, for some \( n \), \((s, 0)\) is in \( X \upharpoonright n \). Note that \( Y \) is an enumerable subset of \( \mathbb{N} \) and a bar in \( \mathcal{F} \). Let \( Z \) be a finite subset of \( Y \). Find \( n \) such that \( Z \subseteq \bigcup_{m \leq n} X \upharpoonright m \). Note that, for every \( \alpha \) in \( \mathcal{F} \), if \( \alpha \) has an initial part in \( Z \), then \( (X \upharpoonright n)(\alpha) = 0 \). Now find \( \beta \) in \( \mathcal{F} \) such that \( (X \upharpoonright n)(\beta) \neq 0 \) and note: \( \alpha \) does not have an initial part in \( Z \). We thus see that every finite subset of \( Y \) positively fails to be a bar in \( \mathcal{F} \).

Clearly then, \( \mathcal{F} \) is positively noncompact. \( \square \)

7.5 Partial continuous functions from \( \mathcal{N} \) to \( \mathcal{N} \)

Let \( X \) be a subset of \( \mathbb{N} \). \( X \) is a partial continuous function from \( \mathcal{N} \) to \( \mathcal{N} \) if and only if for all \( a \), \( b \), \( c \), \( d \) in \( \mathbb{N} \), if both \( (a, c) \) and \( (b, d) \) belong to \( X \) and \( a \) is initial part of \( b \), then either \( c \) is an initial part of \( d \) or \( d \) is an initial part of \( c \).

Let \( X \) be a subset of \( \mathbb{N} \) that is a partial continuous function from \( \mathcal{N} \) to \( \mathcal{N} \), and let \( \alpha \), \( \beta \) belong to \( \mathcal{N} \).

\( X \) maps \( \alpha \) onto \( \beta \), notation: \( X : \alpha \mapsto \beta \), if and only if for each \( n \) there exist \( m \), \( p \) such that \( n \leq p \) and \((\overline{\alpha} m, \overline{\beta} p)\) belongs to \( X \).

We let \( \text{dom}(X) \), the domain of \( X \), be the set of all \( \alpha \) in \( \mathcal{N} \) such that, for some \( \beta \) in \( \mathcal{N} \), \( X \) maps \( \alpha \) onto \( \beta \).

Suppose that \( \alpha \) belongs to \( \text{dom}(X) \). Observe that there is exactly one \( \beta \) in \( \mathcal{N} \) such that \( X \) maps \( \alpha \) onto \( \beta \). We denote this element of \( \mathcal{N} \) by \( X|\alpha \).

Let \( \mathcal{F} \) be a subset of \( \text{dom}(X) \). We then say: \( X \) is a continuous function from \( \mathcal{F} \) to \( \mathcal{N} \). We let \( \text{ran}(X, \mathcal{F}) \), the range of \( X \) on \( \mathcal{F} \), be the set of all \( \beta \) in \( \mathcal{N} \) such that, for some \( \alpha \) in \( \mathcal{F} \), \( X \) maps \( \alpha \) onto \( \beta \). We also define: \( \text{ran}(X) := \text{ran}(X, \text{dom}(X)) \).
7.5.1 A further note

Recall, from Sect. 7.1 that a subset $X$ of $\mathbb{N}$ is finite if and only if, for some $b$ in $\mathbb{N}$, $X$ coincides with $D_b$.

Recall that a subset $F$ of $\mathbb{N}$ is totally bounded if and only if, for each $n$, the set of all $s$ in $\mathbb{N}$ such that $\text{length}(s) = n$ and $s$ contains an element of $F$ is a finite subset of $\mathbb{N}$.

**Theorem 7.14** Let $F$ be a closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent:

(i) $F$ is compact.

(ii) For every enumerable continuous function $X$ from $F$ to $\mathcal{N}$, $\text{range}(X, F)$ is a totally bounded subset of $\mathcal{N}$.

**Proof** (i) $\Rightarrow$ (ii): Assume $F$ is compact. We prove that every enumerable continuous function from $F$ to $\mathcal{N}$ has totally bounded range on $F$.

Let $X$ be an enumerable continuous function from $F$ to $\mathcal{N}$. Let $n$ be a natural number. Let $Y$ be the set of all pairs $(s, t)$ such that $\text{length}(t) = n$ and, for some $u$, the pair $(s, t \ast u)$ belongs to $X$. Observe that $Y$ is an enumerable partial continuous function from $F$ to $\mathbb{N}$ and that, for each $\alpha$ in $F$, $Y(\alpha) = \langle X(\alpha) \rangle_n$. According to Theorem 7.7(ii) $\Rightarrow$ (iii), $\text{Range}(Y, F)$ is a finite subset of $\mathbb{N}$. It follows that the set of all $s$ in $\mathbb{N}$ such that $\text{length}(s) = n$ and $s$ contains an element of the range of $X$ on $F$ is a finite subset of $\mathbb{N}$.

We may conclude that $\text{range}(X, F)$ is a totally bounded subset of $\mathcal{N}$.

(ii) $\Rightarrow$ (i): Suppose that $F$ satisfies the requirement. Let $X$ be an enumerable continuous function from $F$ to $\mathcal{N}$. Let $n$ be a natural number. Let $Y$ be the set of all pairs $(s, (n) \ast 0^p)$ with the property that the pair $(s, n)$ belongs to $X$. Observe that $Y$ is an enumerable continuous function from $F$ to $\mathcal{N}$ and that, for all $\alpha$ in $F$, $Y(\alpha) = \langle X(\alpha) \rangle_0$. As $Y$ has totally bounded range, the set of all $n$ such that, for some $\alpha$ in $F$, $(Y|\alpha)(0) = n$ is a finite subset of $\mathbb{N}$, and, therefore, $\text{Range}(X, F)$ is a finite subset of $\mathbb{N}$.

We may conclude: every enumerable continuous function from $F$ to $\mathbb{N}$ has finite range on $F$.

According to Theorem 7.7(iii) $\Rightarrow$ (ii), $F$ is compact. ☐

Let $F$ be a subset of $\mathcal{N}$. $F$ is positively unbounded if and only if, for each $m$ there exists $\alpha$ in $F$ such that $\alpha(0) > m$.

The next theorem is a counterpart to Theorem 7.14.

**Theorem 7.15** Let $F$ be a closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent:

(i) $F$ is positively noncompact.

(ii) There exists an enumerable continuous function from $F$ to $\mathcal{N}$ with positively unbounded range.

**Proof** The theorem is an easy consequence of Theorem 7.10. ☐
8 The set $\mathcal{R}$ of the real numbers and its closed-and-separable subsets

8.1 Introducing $\mathcal{R}$

8.1.1 Introducing $\mathbb{Z}$

We assume that the operations $+,-,\cdot$ have been defined on $\mathbb{N}$.

We define a binary relation $<_{\mathbb{Z}}$ on $\mathbb{N}$ such that for all $m,n,m <_{\mathbb{Z}} n$ if and only if $m' + n'' < m' + n''$.

We define a binary relation $=_{\mathbb{Z}}$ on $\mathbb{N}$ such that for all $m,n,m =_{\mathbb{Z}} n$ if and only if $m' + n'' = m' + n''$.

We define binary operations $+_{\mathbb{Z}},-_{\mathbb{Z}}$ and $\cdot_{\mathbb{Z}}$ on $\mathbb{N}$ such that, for all $m,n,$

$m +_{\mathbb{Z}} n = (m' + n',m'' + n'')$,$$
m -_{\mathbb{Z}} n = (m' + n'',m'' + n')$$ and

$m \cdot_{\mathbb{Z}} n = (m' \cdot n' + m'' \cdot n'',m' \cdot n'' + m'' \cdot n')$.

We define $0_{\mathbb{Z}} = (0,0)$ and $1_{\mathbb{Z}} = (1,0)$.

8.1.2 Introducing $\mathbb{Q}$

We let $\mathbb{Z}^+$ be the set of all $z$ in $\mathbb{N}$ such that $0_{\mathbb{Z}} <_{\mathbb{Z}} z$.

We let $\mathbb{Q}$ be the set of all $n$ such that $n'' \in \mathbb{Z}^+$.

We define a binary relation $<_{\mathbb{Q}}$ on $\mathbb{N}$ such that for all $m,n$ in $\mathbb{Q},m <_{\mathbb{Q}} n$ if and only if $m' \cdot n'' <_{\mathbb{Z}} m' \cdot n''$.

We define a binary relation $=_{\mathbb{Q}}$ on $\mathbb{N}$ such that for all $m,n$ in $\mathbb{Q},m =_{\mathbb{Q}} n$ if and only if $m' \cdot n'' =_{\mathbb{Z}} m' \cdot n''$.

We define operations $+_{\mathbb{Q}},-_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$ on $\mathbb{N}$, such that for all $m,n$ in $\mathbb{Q},$

$m +_{\mathbb{Q}} n = (m' \cdot n'' +_{\mathbb{Z}} m'' \cdot n',m'' \cdot n'')$,$$
m -_{\mathbb{Q}} n = (m' \cdot n'' -_{\mathbb{Z}} m'' \cdot n',m'' \cdot n'')$$ and

$m \cdot_{\mathbb{Q}} n = (m' \cdot n',m'' \cdot n'')$.

We define $0_{\mathbb{Q}} = (0_{\mathbb{Z}},1_{\mathbb{Z}})$ and $1_{\mathbb{Q}} = (1_{\mathbb{Z}},1_{\mathbb{Z}})$.

We will allow ourselves to denote rational numbers like $\frac{2}{3}$ and $\frac{1}{\pi}$ in the usual way, leaving it to the reader to translate the resulting statements into our narrow formal framework.

8.1.3 Introducing $\mathbb{S}$

We let the set $\mathbb{S}$ of rational segments be the set of all $s$ in $\mathbb{N}$ such that both $s'$ and $s''$ belong to $\mathbb{Q}$ and $s' <_{\mathbb{Q}} s''$.

For each $s$ in $\mathbb{S}$, we define: midpoint($s$) = $\frac{1}{2}(s' +_{\mathbb{Q}} s'')$. We call midpoint($s$) the midpoint of the rational segment $s$.

We define operations $+_{\mathbb{S}},-_{\mathbb{S}},\cdot_{\mathbb{S}},\min_{\mathbb{S}},\max_{\mathbb{S}}$ and $\sqcup_{\mathbb{S}}$ on $\mathbb{N}$ such that for all $s,t$ in $\mathbb{S},$

$s +_{\mathbb{S}} s = (s' +_{\mathbb{Q}} t',s'' +_{\mathbb{Q}} t'')$,$$
s -_{\mathbb{S}} t = (s' -_{\mathbb{Q}} t',s'' -_{\mathbb{Q}} t'')$$
$s \cdot t = (u, v)$, where $u, v$ are the smallest and the greatest, respectively, among the four rational numbers $s' \cdot t', s' \cdot t''$, $s'' \cdot t'$ and $s'' \cdot t''$, respectively,

$$\min_S(s, t) = s \text{ if } s'' \leq_Q t',$$

$$\min_S(s, t) = t \text{ if } t'' \leq_Q s',$$

$$\min_S(s, t) = (\min(s', t'), \max(s'', t'')) \text{ otherwise,}$$

$$\max_S(s, t) = t \text{ if } s'' \leq_Q t',$$

$$\max_S(s, t) = s \text{ if } t'' \leq_Q s',$$

$$\max_S(s, t) = (\min(s', t'), \max(s'', t'')) \text{ otherwise, and}$$

$$\sqcup_S(s, t) = (\min(s', t'), \max(s'', t'')).$$

We also define binary relations $<_S, \leq_S, \#_S, \sim_S, \sqsubseteq_S$ and $\sqsubseteq_S$ on $\mathbb{N}$ such that for all $s, t \in S$,

- $s <_S t$ (s lies to the left of t) if and only if $s'' <_Q t'$,
- $s \leq_S t$ (s lies not to the right of t) if and only if $s' <_Q t''$,
- $s \#_S t$ (s lies apart from t) if and only if either $s <_S t$ or $t <_S s$,
- $s \sim_S t$ (s does not lie apart from, or: touches or: partially covers t) if and only if both $s \leq_S t$ and $t \leq_S s$,
- $s \sqsubseteq_S t$ (s is strictly included in t) if and only if $t' <_Q s'$ and $s'' <_Q t''$, and
- $s \sqsubseteq_S t$ (s is included in t) if and only if $t' \leq_Q s'$ and $s'' \leq_Q t''$.

Note that, for all $s, t \in S$, $s \sim_S t$ if and only if there exists $u \in S$ such that $u \sqsubseteq_S s$ and $u \sqsubseteq_S t$.

For each $s$ in $S$ we define: $\text{length}_S(s) := s'' -_Q s'$.

Note that each $s$ in $S$ has positive length.

### 8.1.4 Introducing $\mathcal{R}$

We now are ready to introduce real numbers.

Let $\alpha$ belong to $\mathcal{N}$. $\alpha$ is a real number if and only if (i) for each $n$, $\alpha(n)$ belongs to $S$, (ii) for each $n$, $\alpha(n + 1) \sqsubseteq_S \alpha(n)$ and (iii) for each $n$ there exists $p$ such that $\text{length}_S(\alpha(p)) \leq_Q \frac{1}{2^n}$. We let $\mathcal{R}$ be the set of the real numbers.

We often use $x, y, \ldots$ as variables over the set $\mathcal{R}$.

We introduce binary relations $<_\mathcal{R}, \leq_\mathcal{R}, \#_\mathcal{R}$ and $=_\mathcal{R}$ on $\mathcal{N}$ such that, for all $x, y$ in $\mathcal{R}$,

- $x < _\mathcal{R} y$ (x is really-smaller than y) if and only if, for some $n$, $x(n) <_S y(n)$,
- $x \leq_\mathcal{R} y$ (x is really-not-greater than y) if and only if, for all $n$, $x(n) \leq_S y(n)$,
- $x \#_\mathcal{R} y$ (x is really-apart from y) if and only if, for some $n$, $x(n) \leq_S y(n)$ and $x = _\mathcal{R} y$ (x really-coincides with y) if and only if both $x \leq_\mathcal{R} y$ and $y \leq_\mathcal{R} x$.

We introduce binary operations $+_\mathcal{R}, -_\mathcal{R}, \cdot_\mathcal{R}, \sup_\mathcal{R}$ and $\inf_\mathcal{R}$ on $\mathcal{N}$ such that for all real numbers $x, y$, for all $n$,

$$\begin{align*}
(x + _\mathcal{R} y)(n) &= x(n) +_S y(n), \\
(x - _\mathcal{R} y)(n) &= x(n) -_S y(n), \\
(x \cdot _\mathcal{R} y)(n) &= x(n) \cdot_S y(n), \\
(\sup_\mathcal{R}(x, y))(n) &= \max_S(x(n), y(n)), \\
(\inf_\mathcal{R}(x, y))(n) &= \min_S(x(n), y(n)).
\end{align*}$$

One proves easily that these operations, when applied to real numbers, produce real numbers.
We sometimes, if confusion seems unlikely, omit the subscripts from the function symbols and the relation symbols.

Observe that, for all real numbers \( x, y, z, x \leq_R z \) and \( y \leq_R z \) if and only if \( \sup_R(x, y) \leq_R z \). In general, we are unable to decide: \( \sup_R(x, y) =_R x \) or \( \sup_R(x, y) =_R y \).

For each \( \alpha \), for each \( n \), we let \( \alpha^n \) be the sequence \( \beta \) such that for all \( m, \beta(m) = \alpha(n, m) \).

Let \( \alpha \) belong to \( \mathcal{N} \). \( \alpha \) is called a \textit{sequence of real numbers} if and only if, for each \( n, \alpha^n \) is a real number.

Let \( \alpha \) be an element of \( \mathcal{N} \) such that, for each \( n, \alpha(n) \in \mathbb{S} \). We let \( \alpha', \alpha'' \) be the elements of \( \mathcal{N} \) satisfying, for each \( n, \alpha'(n) = (\alpha(n))^1 \) and \( \alpha''(n) = (\alpha(n))^'' \).

**Theorem 8.1** (Cantor intersection theorem) Let \( \alpha, \beta \) be sequences of real numbers such that \((i)\) for all \( n, \alpha^n \leq_R \alpha^{n+1} \leq_R \beta^{n+1} \leq_R \beta^n \) and \((ii)\) for all \( n \), there exists \( p \) such that \( \beta^p -_R \alpha^p \leq_R \frac{1}{2^n} \). Then there exists a real number \( x \) such that, for all \( n, \alpha^n \leq_R x \leq_R \beta^n \). Moreover, for all real numbers \( x, y \), if both for all \( n, \alpha^n \leq_R x \leq_R \beta^n \) and for all \( n, \alpha^n \leq_R y \leq_R \beta^n \), then \( x =_R y \).

**Proof** We define \( \delta \) in \( \mathcal{N} \) as follows. For each \( n \), we let \( \delta(n) \) be the least \( k \) such that \( \text{length}(\alpha^k(k'' \cup \beta^k(k'')) < \frac{1}{2^n} \). We then define the real number \( x \) as follows: for each \( n, x'(n) = \max_i \alpha^i(n) + \beta^i(n) - \frac{1}{2^n} \) and \( x''(n) = \min_i \alpha^i(n) + \beta^i(n) + \frac{1}{2^n} \). Note that \( x \) is well-defined and satisfies the requirements and that we only have to use the Minimal Axiom of Countable Choice in order to define \( x \).

\[ \Box \]

**8.1.5 A less effective definition**

Our definition of a real number resembles Brouwer’s own definition. In [18] and [32] a real number is defined as a Cauchy sequence of rationals.

Let \( \alpha \) be a sequence of rationals, that is, an element of \( \mathcal{N} \) such that, for each \( n, \alpha(n) \in \mathbb{Q} \).

\( \alpha \) is a \textit{Cauchy sequence of rationals} if and only if, for every \( m \), there exists \( n \) such that, for all \( p \), if \( p > n \), then \( |\alpha(n) - \alpha(p)| \leq \frac{1}{2^n} \).

\( \alpha \) is an \textit{explicit} Cauchy sequence of rationals if and only if there exists \( \gamma \) in \( \mathcal{N} \) such that for every \( m \), for all \( p \), if \( p > \gamma(m) \), then \( |\alpha(\gamma(m)) - \alpha(p)| \leq \frac{1}{2^n} \).

The \( \Pi^0_1 \)-First Axiom of Countable Choice, an axiom that we did not include in BIM, see Sect. 6, one easily proves that every Cauchy sequence of reals is an explicit Cauchy sequence of reals.

Suppose we are given an infinite sequence of reals \( \alpha \) and \( \gamma \) in \( \mathcal{N} \) such that, for every \( m \), for all \( p \), if \( p > \gamma(m) \), then \( |\alpha(\gamma(m)) - \alpha(p)| \leq \frac{1}{2^n} \). Let \( \beta \) in \( \mathcal{N} \) be such that, for all \( m, \beta(m) = (\alpha(\gamma(m)) - \frac{1}{2m-1}, \alpha(\gamma(m)) + \frac{1}{2m-1}) \). Note that \( \beta \) is a real number and that the sequence \( \alpha \) converges to \( \beta \), as, for every \( m > 2 \), for all \( p > \gamma(m), |\beta - \alpha(p)| < \frac{1}{2m-2} \).

As we observed in Sect. 6, the Weak \( \Pi^0_1 \)-First Axiom of Countable Choice does not seem sufficient for proving that every Cauchy sequence of rationals converges.
8.2 Closed-and-separable subsets of $\mathcal{R}$

Let $\mathcal{X}, \mathcal{Y}$ be subsets of $\mathcal{R}$. We say that $\mathcal{X}$ is a real subset of $\mathcal{Y}$, notation: $\mathcal{X} \subseteq_\mathcal{R} \mathcal{Y}$, if and only if every element of $\mathcal{X}$ really-coincides with an element of $\mathcal{Y}$. We say that $\mathcal{X}$ really-coincides with $\mathcal{Y}$, notation $\mathcal{X} =_\mathcal{R} \mathcal{Y}$ if and only if both $\mathcal{X} \subseteq_\mathcal{R} \mathcal{Y}$ and $\mathcal{Y} \subseteq_\mathcal{R} \mathcal{X}$.

Let $s$ belong to $\mathcal{S}$ and let $x$ be a real number. We say that $x$ belongs to $s$, or that $s$ contains $x$ if and only if, for some $n$, $x(n) \subseteq_\mathcal{S} s$.

Let $\mathcal{X}$ be a subset of $\mathcal{R}$. The closure $\overline{\mathcal{X}}$ of $\mathcal{X}$ is the set of all real numbers $x$ such that, for every $n$, $x(n)$ contains an element of $\mathcal{X}$. $\mathcal{X}$ is a closed subset of $\mathcal{R}$ if and only if $\overline{\mathcal{X}}$ really-coincides with $\mathcal{X}$.

Let $\alpha$ be a sequence of real numbers. We let $\mathcal{H}_\alpha$ be the set of all real numbers $x$ such that, for every $n$, there exists $m$ such that $\alpha^m$ belongs to $x(n)$. Observe that, for each sequence $\alpha$ of real numbers, $\mathcal{H}_\alpha$ is a closed subset of $\mathcal{R}$. Let $\mathcal{X}$ be a subset of $\mathcal{R}$. We call $\mathcal{X}$ a closed-and-separable subset of $\mathcal{R}$ if and only if there exists $\alpha$ such that $\mathcal{X}$ really-coincides with $\mathcal{H}_\alpha$.

Let $Y$ be a subset of $\mathcal{S}$. $Y$ is a (real) frame if and only if (i) for every $s$, $t$ in $\mathcal{S}$, if $s$ belongs to $Y$ and $s \subseteq_\mathcal{S} t$, then $t$ belongs to $Y$ and (ii) for every $s$ in $\mathcal{S}$, if $s$ belongs to $Y$ and $\text{length}_\mathcal{S}(s) >_Q 0_Q$, then there exists $t$ in $Y$ such that $t \subseteq_\mathcal{S} s$ and $\text{length}_\mathcal{S}(t) <_Q \frac{1}{2} \text{length}_\mathcal{S}(s)$.

Let $Y$ be a real frame and $x$ a real number. $x$ is a real member of $Y$ if and only if, for each $n$, $x(n)$ belongs to $Y$. Observe that the set of the real members of $Y$ is a closed subset of $\mathcal{S}$.

Let $\mathcal{X}$ be a closed subset of $\mathcal{R}$. The (real) frame of $\mathcal{X}$ is the set of all $s$ in $\mathcal{S}$ that contain an element of $\mathcal{X}$. Observe that the frame of $\mathcal{X}$ is a frame indeed and that $\mathcal{X}$ really-coincides with the set of the real members of its frame.

**Theorem 8.2** (i) Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$. The frame of $\mathcal{H}$ is an enumerable subset of $\mathcal{S}$.

(ii) Let $\mathcal{H}$ be a closed subset of $\mathcal{R}$ with an enumerable frame and at least one element. $\mathcal{H}$ is a closed-and-separable subset of $\mathcal{R}$.

**Proof** The proof is similar to the proof of Theorem 7.3 and is left to the reader. □

For all real numbers $x$, $y$ with the property $x \leq y$ we let $[x, y]$ be the set of all real numbers $z$ such that $x \leq z \leq y$.

**Lemma 8.3** Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$ and let $x$, $y$ be elements of $\mathcal{H}$ such that $x \leq_\mathcal{R} y$. The set $\mathcal{H} \cap [x, y]$ is also a closed-and-separable subset of $\mathcal{R}$.

**Proof** Let $\alpha$ be sequence of real numbers such that $\mathcal{H} = \mathcal{H}_\alpha$. Let $\gamma$ be a sequence of real numbers such that, for each $n$, $\gamma^n = \inf_\mathcal{R}(y, \sup_\mathcal{R}(x, \alpha^n))$ and note that $\mathcal{H} \cap [x, y]$ coincides with $\mathcal{H}_\gamma$. □

8.3 Bounded and totally bounded subsets of $\mathcal{R}$

Let $\mathcal{H}$ be a subset of $\mathcal{R}$ and let $x$ belong to $\mathcal{R}$. $x$ is an upper bound for $\mathcal{H}$, or $x$ bounds $\mathcal{H}$ from above if and only if, for each $y$ in $\mathcal{H}$, $y \leq_\mathcal{R} x$. $x$ is a least upper bound for
\[ \mathcal{H} \text{ if and only if } x \text{ is an upper bound for } \mathcal{H} \text{ and for each real number } z, \text{ if } z < \mathcal{R} x, \text{ then there exists } y \in \mathcal{H} \text{ such that } z < \mathcal{R} y. \] Note that, if both \( x \) and \( x' \) are a least upper bound for \( \mathcal{H} \), then \( x \) really-coincides with \( x' \).

As is well-known, it is not true constructively that every inhabited subset of \( \mathcal{R} \) that has an upper bound, also has a least upper bound. Consider for instance the set \( \mathcal{H} \) consisting of all real numbers such that \( \text{either: } x = 0 \text{ or: } (x = 1 \text{ and Riemann’s hypothesis is true).} \]

Let \( \mathcal{H} \) be a subset of \( \mathcal{R} \) and let \( x \) belong to \( \mathcal{R} \). It will be clear how we want to define: \( x \) is a lower bound of \( \mathcal{H} \) and: \( x \) is a greatest lower bound of \( \mathcal{H} \).

Let \( \mathcal{H} \) be a subset of \( \mathcal{R} \). \( \mathcal{H} \) is bounded from above if and only if \( \mathcal{H} \) has an upper bound and \( \text{bounded from below if and only if } \mathcal{H} \) has a lower bound. \( \mathcal{H} \) is a bounded subset of \( \mathcal{R} \) if and only if \( \mathcal{H} \) has both an upper bound and a lower bound. \( \mathcal{H} \) is a totally bounded subset of \( \mathcal{R} \) if and only if for each \( n \) there exists a finite subset \( B \) of \( \mathbb{S} \) covering \( \mathcal{H} \) such that, for each \( s \) in \( B \), \( \text{length}_{\mathbb{S}}(s) \leq \frac{1}{2^n} \) and \( s \) contains at least one element of \( \mathcal{H} \). Constructively, it is not true that every bounded subset of \( \mathcal{R} \) is totally bounded. \( \mathcal{H} \) is explicitly totally bounded if and only if there exists \( a \) such that, for each \( n \), \( D_{\alpha(n)} \) is a subset of \( \mathbb{S} \) covering \( \mathcal{H} \) such that, for each \( s \) in \( D_{\alpha(n)} \), \( \text{length}_{\mathbb{S}}(s) \leq \frac{1}{2^n} \) and \( s \) contains at least one element of \( \mathcal{H} \).

(Note that, for each \( n \), \( D_n \) is a finite subset of \( \mathbb{N} \), and \( D_0, D_1, \ldots \) is an enumeration of the finite subsets of \( \mathbb{N} \). Note that \( E_0, E_1, \ldots \) is another enumeration of the finite subsets of \( \mathbb{N} \). It does no harm to replace, in the above definition, “\( D_{\alpha(n)} \)” by “\( E_{\alpha(n)} \)”.

For each \( n \) we let \( \mathbb{R}S_n \) be the set of all \( s \) in \( \mathbb{S} \) such that \( s'' - Q s' = \frac{1}{2^n} \) and \( 2^{n+1} - Q s' \) is an integer.

We call the elements of \( \mathbb{R}S_n \) the regular rational segments of level \( n \).

These special rational segments were already used by Brouwer himself.

Note that, for each \( n \), for each \( s \) in \( \mathbb{S} \) such that \( \text{length}_{\mathbb{S}}(s) < \frac{1}{2^{n+1}} \), there exists \( t \) in \( \mathbb{R}S_n \) such that \( s \subseteq S \).

The Weak \( \Pi^0_1 \)-First Axiom of Countable Choice has been mentioned in Sect. 6.

**Lemma 8.4** (Using the weak \( \Pi^0_1 \)-first axiom of countable choice) Let \( \mathcal{H} \) be a closed-and-separable subset of \( \mathcal{R} \). If \( \mathcal{H} \) is totally bounded, then \( \mathcal{H} \) is explicitly totally bounded.

**Proof** Let \( \mathcal{H} \) be a closed-and-separable subset of \( \mathcal{R} \) that is totally bounded. Find \( \gamma \) such that \( E_\gamma \) is the frame of \( \mathcal{F} \). We define \( \delta \), by induction, as follows. For each \( n, p \), if \( p'' \in \mathbb{R}S_n \) and \( \gamma(p') > 0 \) and \( \text{length}_{\mathbb{S}}(\gamma(p') - 1) < \frac{1}{2^{n+1}} \) and \( \gamma(p') - 1 \subseteq S \) \( p'' \) and there is no \( q < p \) such that \( \delta((n, q)) > 0 \) and \( \gamma(p') - 1 \subseteq \delta((n, q)) - 1 \), then \( \delta((n, p)) = p'' + 1 \), and, if not, then \( \delta((n, p)) = 0 \).

Note that, for each \( n \), \( E_{\delta^n} \) is the set of all \( s \) in \( \mathbb{R}S_n \) such that, for some \( x \) in \( \mathcal{H} \), for some \( m, x(m) \subseteq S \).

We claim: for each \( n \), there exists \( m \) such that, for every \( p > m \), \( \delta^n(p) = \delta((n, p)) = 0 \).

We prove this claim as follows. Let \( n \) be a natural number. Let \( B \) be a finite subset of \( \mathbb{S} \) covering \( \mathcal{H} \) such that, for each \( s \) in \( B \), \( \text{length}_{\mathbb{S}}(s) < \frac{1}{2^{n+1}} \) and \( s \) contains an element of \( \mathcal{F} \). Let \( C \) be the set of all \( t \) in \( \mathbb{R}S_n \) such that, for some \( s \) in \( B \), \( s \subseteq S \). Note that \( C \) is a finite subset of \( \mathbb{R}S_n \) covering \( \mathcal{H} \) and that each \( t \) in \( C \) contains an element of \( \mathcal{H} \). Now find \( m \) such that \( C = E_{\delta^m} \). Note that, for every \( p \geq m, \delta^n(p) = 0 \).
Using the Weak $\Pi^0_1$-First Axiom of Countable Choice, find $\xi$ such that for each $n$, for every $p \geq \xi(n)$, $\delta^n(p) = 0$.

Note that, for each $n$, $E_{\delta^n}(\xi(n))$ is a finite subset of $\mathbb{S}$ covering $\mathcal{H}$ such that, for each $s$ in $E_{\delta^n}(\xi(n))$, $s$ contains an element of $\mathcal{H}$ and $\text{length}_{\mathbb{S}}(s) = \frac{1}{2^n}$.

We thus see that $\mathcal{H}$ is explicitly totally bounded. 

The following lemma gives a characterization of totally bounded subsets of $\mathcal{R}$ due to Freudenthal, see [17].

**Lemma 8.5** Let $\mathcal{H}$ be a subset of $\mathcal{R}$.

(i) $\mathcal{H}$ is totally bounded if and only if $\mathcal{H}$ is bounded and, for every $s, t$ in $\mathbb{S}$, if $s \subseteq t$, then either $t$ contains an element of $\mathcal{H}$ or $s$ does not contain an element of $\mathcal{H}$.

(ii) $\mathcal{H}$ is explicitly totally bounded if and only if $\mathcal{H}$ is bounded and there exists $\alpha$ in $\mathbb{C}$ such that, for all $s, t$ in $\mathbb{S}$, if $s \subseteq t$, then either $\alpha(s, t) = 1$ and $t$ contains an element of $\mathcal{H}$ or $\alpha((s, t)) = 0$ and $s$ does not contain an element of $\mathcal{H}$.

**Proof** (i) Let $\mathcal{H}$ be a totally bounded subset of $\mathcal{R}$. Let $s, t$ be elements of $\mathbb{S}$ such that $s \subseteq t$. Find $n$ such that, for all $u$ in $\mathbb{S}$, if $\text{length}_{\mathbb{S}}(u) < \frac{1}{2^n}$ and $s \approx_{\mathbb{S}} u$, then $u \not\subseteq t$. Now find a finite subset $B$ of $\mathbb{S}$ covering $\mathcal{H}$ such that, for each $s$ in $B$, $\text{length}_{\mathbb{S}}(s) \leq_{\mathbb{Q}} \frac{1}{2^n}$ and $s$ contains at least one element of $\mathcal{H}$. Either there exists $u$ in $B$ such that $s \approx_{\mathbb{S}} u$ and $t$ contains an element of $\mathcal{H}$, or there is no $u$ in $B$ such that $s \approx_{\mathbb{S}} u$ and $s$ does not contain an element of $\mathcal{H}$.

Conversely, let $\mathcal{H}$ be a bounded subset of $\mathcal{R}$ such that for every $s, t$ in $\mathbb{S}$, if $s \subseteq t$, then either $t$ contains an element of $\mathcal{H}$ or $s$ does not contain an element of $\mathcal{H}$. Find $M$ in $\mathbb{N}$ such that, for all $x$ in $\mathcal{H}$, $-M < x < M$. Let $n$ belong to $\mathbb{N}$. Find $p$ and two finite sequences $s_0, s_1, \ldots, s_p$ and $t_0, t_1, \ldots, t_p$ of elements of $\mathbb{S}$ such that the set $\{s_0, s_1, \ldots, s_p\}$ covers $[-M, M]$, and, for each $i \leq p$, $s_i \subseteq t_i$ and $\text{length}_{\mathbb{S}}(t_i) \leq_{\mathbb{Q}} \frac{1}{2^n}$.

Determine a finite subset $B$ of $t_0, t_1, \ldots, t_p$ such that, for every $i \leq p$, either: $t_i$ belongs to $B$ and $t_i$ contains an element of $\mathcal{H}$, or: $t_i$ does not belong to $B$ and $s_i$ does not contain an element of $\mathcal{H}$. Note that for every $x$ in $\mathcal{H}$, for every $i \leq p$, if $x$ is contained in $s_i$, then $t_i$ belongs to $B$. It follows that $B$ covers $\mathcal{H}$.

(ii) The proof is a straightforward effectivization of the proof of (i) and left to the reader. 

**Lemma 8.6** Let $\gamma$ be an infinite sequence of real numbers.

(i) The set $\mathcal{H}_\gamma$ is totally bounded if and only if, for each $m$, there exists $n$ such that, for each $i > n$, there exists $j < n$ such that $|\gamma^i - \gamma^j| < \frac{1}{2^m}$.

(ii) The set $\mathcal{H}_\gamma$ is explicitly totally bounded if and only if there exists $\delta$ in $\mathcal{N}$ such that, for each $m$, for each $i > \delta(m)$, there exists $j < \delta(m)$ such that $|\gamma^i - \gamma^j| < \frac{1}{2^m}$.

**Proof** (i) Let $\gamma$ be an infinite sequence of reals.

First assume that $\mathcal{H}_\gamma$ is totally bounded.

Let $m$ be a natural number. Find a finite subset $B$ of $s$ such that $B$ covers $\mathcal{H}_\gamma$ and, for every $s$ in $B$, $\text{length}_{\mathbb{S}}(s) < \frac{1}{2^m}$, and there exists $i$ such that $\gamma^i$ is contained in $s$. Find $k$ such that, for each $s$ in $B$, there exists $j < k$ such that $\gamma^j(j^m) \subseteq s$. Define $n := \max\{j^m|j < k\}$. Clearly, $n$ satisfies the requirements.
Now assume that for each \( m \), there exists \( n \) such that, for each \( i > n \), there exists \( j < n \) such that \( |y^i - y^j| < \frac{1}{2^n} \).

Let \( m \) be a natural number. Find \( n \) such that, for each \( i > n \), there exists \( j < n \) such that \( |y^i - y^j| < \frac{1}{2^n+1} \). Note that, for each \( x \in \mathcal{H}_y \), there exists \( j < n \) such that \( |y^j - x| < \frac{1}{2^n+1} \). For each \( j < n \), let \( k_j \) be the least \( k \) such that \( \text{length}_S(y^j(k)) < \frac{1}{2^n+1} \) and let \( s \) be the element of \( S \) such that \( s \) has the same midpoint as \( y^j(k_j) \) and \( \text{length}(s) = \frac{1}{2^n} \). Let \( B \) be the set of all \( s \) we thus obtain. Note that \( B \) is a finite subset of \( S \) covering \( \mathcal{H}_y \).

(ii) The proof is a straightforward effectivization of the proof of (i) and left to the reader. \( \square \)

The characterization given in Lemma 8.6(i) is reminiscent of Brouwer’s definition of located-compact sets, for instance in [9].

Recall that, for each \( s \) in \( S \), \( \text{midpoint}(s) = \frac{1}{2}(s' + s'') \).

For each element of \( S \), we let \( \text{double}_S(s) \) be the element of \( S \) satisfying:

\[
\text{midpoint}(\text{double}_S(s)) = \text{midpoint}(s) \quad \text{and} \quad \text{length}_S(\text{double}_S(s)) = 2 \cdot \text{length}_S(s).
\]

Note that, for all \( s, t \) in \( S \), if \( \text{length}_S(t) < \frac{1}{4} \text{length}_S(s) \) and \( t \approx_S s \), then \( \text{double}_S(t) \sqsubseteq_S \text{double}_S(s) \).

**Lemma 8.7** (Using the weak \( \Pi^0_1 \)-first axiom of countable choice) Every closed-and-separable and totally bounded subset of \( R \) has a largest element and a smallest element.

**Proof** Let \( \mathcal{H} \) be a closed-and-separable and totally bounded subset of \( R \). Using Lemma 8.4, we determine \( \alpha \) such that, for each \( n \), \( D_\alpha(n) \) is a subset of \( S \) covering \( \mathcal{H} \) such that, for each \( s \) in \( D_\alpha(n) \), \( \text{length}_S(s) \leq \frac{1}{2^n} \) and \( s \) contains at least one element of \( \mathcal{H} \). We make the innocent extra assumption that, for each \( n \), for each \( s \) in \( D_\alpha(n) \), \( \text{length}_S(s) = \frac{1}{2^n} \). We define \( \beta \) such that, for each \( n \), \( \beta(n) \) is the least \( s \) in \( D_\alpha(n) \) such that, for all \( t \) in \( D_\alpha(n) \), \( s \leq_S t \). Note that, for each \( n \), \( \text{length}_S(\beta(3n+1)) = \frac{1}{2^n} \text{length}_S(\beta(3n)) \) and \( \beta(3n) \approx_S \beta(3n+1) \). We define \( x \) such that, for each \( n \), \( x(n) = \text{double}_S(\beta(3n)) \). Note that, for each \( n \), \( x(n+1) \sqsubseteq_S x(n) \) and that \( x \) is a real number belonging to \( \mathcal{H} \), and that, for all \( y \) in \( \mathcal{H} \), \( x \leq_R y \). Clearly, \( x \) is the least element of \( \mathcal{H} \).

The proof that \( \mathcal{H} \) has a largest element is similar. \( \square \)

If one requires, in Lemma 8.7, the set \( \mathcal{H} \) to be explicitly totally bounded, the use of the \( \Pi^0_1 \)-First Axiom of Countable Choice may be avoided.

8.4 Partial continuous functions from \( R \) to \( R \)

A subset \( X \) of \( \mathbb{N} \) is a partial continuous function from \( R \) to \( R \) if and only if (i) for each \( a \) in \( X \), both \( a' \) and \( a'' \) are in \( S \), and (ii) for all \( a, b \) in \( X \), if \( a' \approx_S b' \), then \( a'' \approx_S b'' \).

Let \( X \) be a partial continuous function from \( R \) to \( R \) and let \( x, y \) be real numbers. \( X \) maps \( x \) onto \( y \), notation: \( X : x \mapsto y \), if and only if for each \( n \) there exist \( m, a \) such that \( a \) in \( X \) and \( x(m) \sqsubseteq_Q a' \) and \( a'' \sqsubseteq_Q y(n) \).
Lemma 8.8 Let $X$ be a partial continuous function from $\mathcal{R}$ to $\mathcal{R}$.

(i) Let $x$, $y$ be real numbers such that $X$ maps $x$ onto $y$. Then: $X$ is continuous at $x$, that is: for each $m$ there exists $n$ such that, for all real numbers $u$, $v$, if $X$ maps $u$ onto $v$ and $|x - u| < \frac{1}{2^m}$, then $|y - v| < \frac{1}{2^n}$.

(ii) For all real numbers $x$, $u$, $y$, $v$, if $X$ maps $x$ onto $y$ and $u$ onto $v$ and $y \not=_{\mathcal{R}} v$, then $x \not=_{\mathcal{R}} u$.

(iii) For all real numbers $x$, $u$, $y$, $v$, if $X$ maps $x$ onto $y$ and $u$ onto $v$ and $x =_{\mathcal{R}} u$, then $y =_{\mathcal{R}} v$.

Proof (i) Suppose that $X$ maps $x$ onto $y$. Let $m$ be a natural number. Find $q$ such that $\text{length}_{\mathcal{S}}(y(q)) <_{\mathbb{Q}} \frac{1}{2^{m+1}}$. Find $p$, $a$ such that $a \in X$ and $x(p) \sqsubseteq_{\mathcal{S}} a'$ and $a'' \sqsubseteq_{\mathcal{S}} y(q)$.

Consider $x(p)$ and $x(p+1)$ and note: $x'(p) <_{\mathbb{Q}} x'(p+1) <_{\mathbb{Q}} x''(p+1) <_{\mathbb{Q}} x''(p)$. Find $n$ such that $\frac{1}{2^n} <_{\mathbb{Q}} \min(x'(p+1) - x'(p), x''(p) - x''(p+1))$.

Note: for each real number $u$, if $|x - u| < \frac{1}{2^m}$, then, for each $s$, $x(p) \approx_{\mathcal{S}} u(s)$.

Let $u$ be a real number such that $|x - u| < \frac{1}{2^m}$ and let $v$ be a real number such that $X$ maps $u$ onto $v$. Find $t$ such that $\text{length}_{\mathcal{S}}(v(t)) <_{\mathbb{Q}} \frac{1}{2^{m+1}}$. Find $s$, $b$ such that $b \in X$ and $u(s) \sqsubseteq_{\mathcal{S}} b'$ and $b'' \sqsubseteq_{\mathcal{S}} v(t)$.

Note: $x(p) \approx_{\mathcal{S}} u(s)$ and, therefore: $a' \approx_{\mathcal{S}} b'$. Conclude: $a'' \approx_{\mathcal{S}} b''$ and $y(q) \approx_{\mathcal{S}} v(t)$ and $|y - v| < \frac{1}{2^n}$.

(ii) is an easy consequence of (i).

(iii) is an easy consequence of (ii).\hfill\Box

Let $X$ be a partial continuous function from $\mathcal{R}$ to $\mathcal{R}$. We let the domain of $X$, notation: $\text{dom}_{\mathcal{R}}(X)$, be the set of all real numbers $x$ such that for some $y$, $X$ maps $x$ onto $y$. We let the range of $X$, notation: $\text{ran}_{\mathcal{R}}(X)$, be the set of all real numbers $y$, such that, for some $x$ in $\text{dom}_{\mathcal{R}}(X)$, $X$ maps $x$ onto $y$.

Let $X$, $Y$ be partial continuous functions from $\mathcal{R}$ to $\mathcal{R}$. We say that $X$ restricts $Y$, or: $X$ is a restriction of $Y$, or: $Y$ extends $X$, if and only if $\text{dom}_{\mathcal{R}}(X)$ is a real subset of $\text{dom}_{\mathcal{R}}(Y)$ and for every $x$ in $\text{dom}_{\mathcal{R}}(X)$, for every $x'$ in $\text{dom}_{\mathcal{R}}(Y)$, for all real numbers $y$, $y'$, if $X$ maps $x$ onto $y$ and $Y$ maps $x'$ onto $y'$ and $x =_{\mathcal{R}} x'$, then $y =_{\mathcal{R}} y'$.

Let $\phi$ belong to $\mathcal{N}$ and suppose that $\phi$ enumerates a partial continuous function from $\mathcal{R}$ to $\mathcal{R}$. We define: $\text{dom}_{\mathcal{R}}(\phi) := \text{dom}_{\mathcal{R}}(E_\phi)$, and $\text{ran}_{\mathcal{R}}(\phi) := \text{ran}_{\mathcal{R}}(E_\phi)$. For every $x$ in $\text{dom}_{\mathcal{R}}(\phi)$ we let $\phi(x)$ be the real number $y$ such that

1. for each $m_0$, if $m_0$ is the least $m$ such that $\text{length}(m) = 3$ and $m(2) \in \mathbb{S}$ and $\text{length}_{\mathcal{S}}(m(2)) <_{\mathbb{Q}} 1_{\mathbb{Q}}$ and $\phi(m(0)) > 0$ and $a := \phi((m(0)) - 1$ satisfies: $x(m(1)) \sqsubseteq_{\mathcal{S}} a'$ and $a'' \sqsubseteq_{\mathcal{S}} m(2)$, then $y(0) = m_0(2)$, and,

2. for each $m_0$, for each $n$, if $m_0$ is the least $m$ such that $\text{length}(m) = 3$ and $m(2) \in \mathbb{S}$ and $\text{length}_{\mathcal{S}}(m(2)) <_{\mathbb{Q}} 1_{\mathbb{Q}}$ and $\phi(m(0)) > 0$ and $a := \phi((m(0)) - 1$ satisfies: $x(m(1)) \sqsubseteq_{\mathcal{S}} a'$ and $a'' \sqsubseteq_{\mathcal{S}} m(2) \sqsubseteq_{\mathcal{S}} y(n)$, then $y(n+1) = m_0(2)$.

Let $X$ be a partial continuous function from $\mathcal{R}$ to $\mathcal{R}$ and let $\mathcal{H}$ be a subset of $\mathcal{R}$. $X$ is called a continuous function from $\mathcal{H}$ to $\mathcal{R}$ if $\mathcal{H}$ is a real subset of $\text{dom}_{\mathcal{R}}(X)$.

8.4.1 Partial continuous functions from $\mathcal{R}$ to $\mathbb{Z}$

A subset $X$ of $\mathbb{N}$ is a partial continuous function from $\mathcal{R}$ to $\mathbb{Z}$ if and only if (i) for each $a$ in $X$, $a' \in \mathbb{S}$ and $a'' \in \mathbb{Z}$ and (ii) for all $a$, $b$ in $X$, if $a' \approx_{\mathcal{S}} b'$, then $a'' =_{\mathbb{Z}} b''$.\hfill\square
For each subset $X$ of $\mathbb{N}$, if $X$ is a partial continuous function from $\mathcal{R}$ to $\mathbb{Z}$, we let $\text{dom}_{\mathcal{R},\mathbb{Z}}(X)$ be the set of all real numbers $x$ such that, for some $a$ in $X$, for some $m, x(m) \sqsubseteq_S a'$.

Let $X$ be a subset of $\mathbb{N}$ and a partial continuous function from $\mathcal{R}$ to $\mathbb{Z}$. For each $x$ in $\text{dom}_{\mathcal{R},\mathbb{Z}}(X)$, for each $n$ in $\mathbb{Z}$, we define: $X$ maps onto $n$, notation: $X : x \mapsto n$ if and only if there exist $a$ in $X, m$ in $\mathbb{N}$ such that $x(m) \sqsubseteq_S a'$ and $a'' = n$. Note that, for all $x, y$ in $\text{dom}_{\mathcal{R},\mathbb{Z}}(X)$, for all $n, p$ in $\mathbb{Z}$, if $X : x \mapsto n$ and $X : y \mapsto p$ and $x \equiv_R y$, then $n = p$.

Assume that $\phi$ enumerates a partial continuous function from $\mathcal{R}$ to $\mathbb{Z}$ and let $x$ belong to $\text{dom}_{\mathcal{R},\mathbb{Z}}(\phi) := \text{dom}_{\mathcal{R},\mathbb{Z}}(E_{\phi})$. For each $x$ in $\text{dom}_{\mathcal{R},\mathbb{Z}}(X)$, we define $\phi(x)$ in $\mathbb{Z}$ as follows. We let $m$ be the least $n$ such that there exist $s$ in $S$, $z$ in $\mathbb{Z}$ such that $\phi(n') = (s, z) + 1$ and $x(n'') \sqsubseteq_S s$. We then find $s, z$ such that $\phi(m') = (s, z) + 1$ and define: $\phi(x) := z$.

8.5 Compact closed-and-separable-subsets of $\mathcal{R}$

Let $X$ be a subset of $\mathcal{S}$. $X$ is a connected subset of $\mathcal{S}$ if and only if, for all $s, t$ in $X$, there exist $n > 0$ in $\mathbb{N}$ and elements $s_0, s_1, \ldots, s_{n-1}$ of $X$ such that $s_0 = s$ and $s_{n-1} = t$ and, for each $i < n - 1, s_i \approx_S s_{i+1}$.

Let $X$ be a finite and connected subset of $\mathcal{S}$. We let $\bigcup X$ be the element $s$ of $\mathcal{S}$ satisfying: $s' = \min_{t \in X} t'$ and $s'' = \max_{t \in X} t''$.

Let $X, Y$ be subsets of $\mathcal{S}$. We define: $Y$ is a refinement of $X$ if and only if

1. for each $s$ in $X$ there exists a finite and connected subset $Z$ of $Y$ such that $s \sqsubseteq_S \bigcup Z$, and
2. for each $s$ in $Y$ there exists $t$ in $X$ such that $s \sqsubseteq_S t$.

Let $Y$ be a subset of $\mathcal{S}$. We define: $Y$ is shrinking in length if and only if, for each $m$, there exists $n$ such that, for all $s$ in $Y$, if $s > n$, then $\text{length}_S < \frac{1}{2m}$.

**Lemma 8.9** Let $X$ be an enumerable subset of $\mathcal{S}$.

There exists a decidable subset $Y$ of $\mathcal{S}$ that is shrinking in length and a refinement of $X$.

**Proof** Let $X$ be an enumerable subset of $\mathcal{S}$ and let $\gamma$ be an element of $\mathcal{N}$ enumerating $X$. For each $n$, if $\gamma(n) = 0$, we let $X_n$ be the empty set, and, if $\gamma(n) > 0$, we let $X_n$ be a finite and connected subset of $\mathcal{S}$ such that $\gamma(n) - 1 = \bigcup X_n$, and, for each $s$ in $X_n, \text{length}_S(s) < \frac{1}{2^n}$. We define $Y := \bigcup_{n \in \mathbb{N}} X_n$. $Y$ will be a decidable subset of $\mathcal{S}$, as, for each $m$, there exists $n$ such that, for all $s$ in $\mathcal{S}$, if $\text{length}_S < \frac{1}{2^n}$, then $s > m$, so, if $m$ belongs to $Y$, it must be an element of one of the finite sets $X_i$, where $i \leq n$, and therefore, we may decide: $m$ belongs to $Y$ or $m$ does not belong to $Y$. It will be clear that $Y$ is shrinking in length and a refinement of $X$. □

Let $\mathcal{H}$ be a subset of $\mathcal{R}$ and let $B$ be a subset of $\mathcal{S}$. $B$ is a bar in $\mathcal{H}$, or $B$ is a cover of $\mathcal{H}$, or: $B$ bars $\mathcal{H}$, or: $B$ covers $\mathcal{H}$, if and only if, for every $x$ in $\mathcal{H}$, there exists $s$ in $B$ such that, for some $n, x(n) \sqsubseteq_S s$.

Note that, for all subsets $X, Y$ of $\mathcal{S}$, if $Y$ is a refinement of $X$, then, for every $x$ in $\mathcal{R}$, there exist $s, m$ such that $s \in X$ and $s(m) \sqsubseteq_S s$ if and only if there exist $s, m$ such that $s \in Y$ and $x(m) \sqsubseteq_S s$. 

\( \text{Springer} \)
Now let $X$ be an enumerable subset of $\mathbb{N}$ and a partial continuous function from $\mathcal{R}$ to $\mathbb{Z}$. Note that $X'$ is an enumerable subset of $\mathbb{S}$. Let $Y$ be a decidable subset of $\mathbb{S}$ that is shrinking in length and a refinement of $X$. Let $Z$ be the set of all $a$ such that $a' \in Y$ and, for some $b$ in $X$, $a' \subseteq_b b'$ and $a'' =_Z b''$. Note that $Z$ is a decidable subset of $\mathbb{N}$ and a partial continuous function from $\mathcal{R}$ to $\mathbb{Z}$, and that $\text{dom}_{\mathcal{R}, \mathbb{Z}}(X) = \text{dom}_{\mathcal{R}, \mathbb{Z}}(Z)$, and, for every $x$ in $\text{dom}_{\mathcal{R}, \mathbb{Z}}(X)$, for each $n$ in $\mathbb{Z}$, $X : x \mapsto n$ if and only if $Z : x \mapsto n$.

Let $\mathcal{H}$ be a subset of $\mathcal{R}$ and let $F$ be a subset of $\mathbb{N}$ that is a partial continuous function from $\mathcal{H}$ to $\mathcal{R}$. $F$ is uniformly continuous on $\mathcal{H}$ if and only if, for each $m$, there exists an enumerable subset of $\mathbb{S}$ such that, for all $x, x' \in \mathcal{H}$, if $F$ maps $x$ onto $y$ and $x'$ onto $y'$ and $|x - x'| < \frac{1}{2^m}$, then $|y - y'| < \frac{1}{2^m}$.

Let $F$ be a partial continuous function from $\mathcal{R}$ to $\mathcal{R}$ and let $\mathcal{H}$ be a subset of $\text{dom}_{\mathcal{R}}(F)$. $F$ is bounded on $\mathcal{H}$ or: totally bounded on $\mathcal{H}$, respectively, if and only if the set of all $y$ such that $F$ maps an element of $\mathcal{H}$ onto $y$ is a bounded subset of $\mathcal{R}$, or: a totally bounded subset of $\mathcal{R}$, respectively.

**Lemma 8.10** Let $\mathcal{H}$ be a subset of $\mathcal{R}$ such that every enumerable covering of $\mathcal{H}$ has a finite subcovering. Then, for every finite covering $B$ of $\mathcal{H}$, one may determine $p$ such that, for all $x, y \in \mathcal{H}$, if $|x - y| < \frac{1}{2^p}$, then there exists $t$ in $B$ such that both $x$ and $y$ are contained in $t$.

**Proof** Let $B$ be a finite subcovering of $\mathcal{H}$. Let $C$ be the set of all $s$ in $\mathbb{S}$ such that, for some $t$ in $B$, $s \subseteq_t s'$ (that is: $t' \lessdot_Q s' \leq_\mathbb{S} s'' <_\mathbb{S} t''$), and note that $C$ is an enumerable covering of $\mathcal{H}$. Let $C'$ be a finite subset of $C$ covering $\mathcal{H}$. Let $s_0, s_1, \ldots s_k$ be a list of the elements of $C'$. For every $i \leq k$, find $t_i$ in $B$ such that $s_i \subset t_i$. Now find $p$ such that, for every $u$ in $\mathbb{S}$, for every $i \leq k$, if $\text{length}_{\mathbb{S}}(u) < \frac{1}{2^p}$ and $u \approx_\mathbb{S} s_i$, then $u \subset t_i$. Note that, for every $x$ in $\mathcal{H}$, for every $i \leq k$, if $x$ is contained in $s_i$ and $|x - y| < \frac{1}{2^p}$, then $y$ is contained in $t_i$. It follows that, for all $x, y \in \mathcal{H}$, if $|x - y| < \frac{1}{2^p}$, then there exists $t$ in $B$ such that both $x$ and $y$ are contained in $t$. \(\square\)

Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$.

We define: $\mathcal{H}$ is compact if and only if $\mathcal{H}$ satisfies the condition mentioned in item (ii) of the next theorem: Every enumerable subset of $\mathbb{N}$ that is a bar in $\mathcal{H}$ has a finite subset that is a bar in $\mathcal{H}$.

For every subset $\mathcal{X}$ of $\mathcal{R}$, for every $s$ in $\mathcal{S}$ we let $\mathcal{X} \cap s$ be the set of all elements $x$ of $\mathcal{X}$ such that $s' \leq_{\mathcal{R}} x \leq_{\mathcal{R}} s''$. Note that, if $\mathcal{X}$ is a closed-and-separable subset of $\mathcal{R}$, then also $\mathcal{X} \cap s$ is a closed-and-separable subset of $\mathcal{R}$, see Lemma 8.3.

**Theorem 8.11** Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$. The following statements are equivalent:

(i) Every decidable subset of $\mathcal{S}$ that is a bar in $\mathcal{H}$ has a finite subset that is a bar in $\mathcal{H}$.

(ii) $\mathcal{H}$ is compact, that is: every enumerable subset of $\mathcal{S}$ that is a bar in $\mathcal{H}$ has a finite subset that is a bar in $\mathcal{H}$.

(iii) $\mathcal{H}$ is totally bounded and every enumerable continuous function from $\mathcal{H}$ to $\mathcal{R}$ is uniformly continuous on $\mathcal{H}$.

(iv) $\mathcal{H}$ is totally bounded and every bounded enumerable continuous function from $\mathcal{H}$ to $\mathcal{R}$ is uniformly continuous on $\mathcal{H}$.
(v) Every enumerable continuous function from $\mathcal{H}$ to $\mathcal{R}$ has bounded range on $\mathcal{H}$.

(vi) Every enumerable continuous function from $\mathcal{H}$ to $\mathcal{R}$ that is positive everywhere on $\mathcal{H}$ has a positive lower bound on $\mathcal{H}$.

(vii) Every enumerable continuous function from $\mathcal{H}$ to $\mathcal{R}$ that is uniformly continuous on $\mathcal{H}$ and positive everywhere on $\mathcal{H}$ has a positive lower bound on $\mathcal{H}$.

Proof (i) $\Rightarrow$ (ii). Suppose that every decidable subset of $\mathbb{S}$ covering $\mathcal{H}$ has a finite subset covering $\mathcal{H}$. Let $X$ be an enumerable subset of $\mathbb{S}$ covering $\mathcal{H}$. Using Lemma 8.9, find a decidable subset $Y$ of $\mathbb{S}$ that is a refinement of $X$. Determine a finite subset $W$ of $Y$ that covers $\mathcal{H}$ and then determine a finite subset $U$ of $X$ such that, for each $s$ in $W$, there exists a finite and connected subset $Z$ of $U$ such that $s = \mathbb{S} \bigcap Z$. Clearly, $Z$ is a finite subset of $X$ covering $\mathcal{H}$.

(ii) $\Rightarrow$ (iii). Suppose that every enumerable subset of $\mathbb{S}$ covering $\mathcal{H}$ has a finite subset covering $\mathcal{H}$.

We first prove that $\mathcal{H}$ is totally bounded. Let $X$ be the frame of $\mathcal{H}$, that is, the set of all $s$ in $\mathbb{S}$ that contain an element of $\mathcal{H}$. Recall that $X$ is an enumerable subset of $\mathbb{S}$, see Theorem 8.2. Now let $n$ belong to $\mathbb{N}$. Note that the set $Y$ consisting of all $s$ in $X$ such that $\text{length}_X(s) < \frac{1}{2^n}$ is enumerable and a bar in $\mathcal{H}$. Let $Z$ be a finite subset of $Y$ that is a bar in $\mathcal{H}$. Note that $Z$ is a finite subset of $\mathbb{S}$ covering $\mathcal{H}$ such that, for every $s$ in $Z$, $\text{length}_\mathbb{S}(s) < \frac{1}{2^n}$, and $s$ contains an element of $\mathcal{H}$.

We thus see that $\mathcal{H}$ is totally bounded.

Note that the frame of $\mathcal{H}$ is an enumerable subset of $\mathbb{N}$. Find $\gamma$ such that $E_{\gamma}$ is the frame of $\mathcal{H}$.

Now let $\phi$ enumerate a continuous function from $\mathcal{H}$ to $\mathcal{R}$. We define $\beta$ in $\mathcal{N}$ as follows. For each $m$, if there exist $s$, $t$ in $\mathbb{S}$ such that $\gamma(m') = s + 1$ and $\phi(m'') = (s, t) + 1$ and $\text{length}_\mathbb{S}(t) \leq \frac{1}{2^n}$, then $\beta(m) = \gamma(m') = s + 1$, and if there are no such $s$, $t$, then $\beta(m) = 0$. Observe that $E_{\beta}$ covers $\mathcal{H}$ and find $k$ such that $E_{\beta(k)}$ covers $\mathcal{H}$. Now apply Lemma 8.10 and find $p$ such that, for all $x$, $y$ in $\mathcal{H}$, if $|x - y| < \frac{1}{2^n}$, then there exists $s$ in $E_{\beta(k)}$ such that $x$, $y$ both belong to $s$. It follows that, for all $x$, $y$ in $\mathcal{H}$, if $|x - y| < \frac{1}{2^n}$, then $|\phi(x) - \phi(y)| < \frac{1}{2^n}$.

We thus see that the function enumerated by $\phi$ is uniformly continuous on $\mathcal{H}$.

(iii) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (v). Suppose that $\mathcal{H}$ is totally bounded and every bounded enumerable continuous function from $\mathcal{H}$ to $\mathcal{R}$ is uniformly continuous on $\mathcal{H}$. We prove that every enumerable continuous function from $\mathcal{H}$ to $\mathcal{R}$ is bounded on $\mathcal{H}$.

Let $\phi$ enumerate a continuous function from $\mathcal{H}$ to $\mathcal{R}$. We let $\psi$ be an element of $\mathcal{N}$ enumerating a continuous function from $\mathcal{H}$ to $\mathcal{R}$ such that for every $x$ in $\mathcal{H}$, for every $n$ in $\mathbb{Z}$, if $2n \leq \mathcal{R} \phi(x) \leq \mathcal{R} 2n + 1$, then $\psi(x) = \mathcal{R} \phi(x) - 2n$ and, if $2n + 1 \leq \mathcal{R} \phi(x) \leq \mathcal{R} 2n + 2$, then $\psi(x) = \mathcal{R} 2n + 2 - \phi(x)$. Observe that $\psi$ enumerates a bounded function from $\mathcal{H}$ to $\mathcal{R}$. We may assume that this function is uniformly continuous on $\mathcal{H}$ and we determine $m$ such that for all $x$, $x'$ in $\mathcal{H}$, if $|x - x'| < \mathcal{R} \frac{1}{2^n}$ then $|\psi(x) - \mathcal{R} \psi(x')| < \mathcal{R} \frac{1}{2^n}$. Using the fact that $\mathcal{H}$ is totally bounded, we find a finite subset $C$ of $\mathbb{S}$ such that $C$ covers $\mathcal{H}$ and, for every $s$ in $C$, $\text{length}_\mathbb{S}(s) < \frac{1}{2^n}$ and $s$ contains an element of $\mathcal{H}$.
Let $s$ be an element of $C$ and $y$ an element of $\mathcal{H}$ contained in $s$. Observe that, either $\psi(y) <_R \frac{2}{3}$ and, therefore, for every $x$ in $\mathcal{H} \cap \overline{s}$, $\psi(x) <_R 1$, or $\psi(y) >_R \frac{1}{3}$ and, therefore, for every $x$ in $\mathcal{H} \cap \overline{s}$, $\psi(x) >_R 0$. We now prove that in both cases the function enumerated by $\phi$ is bounded on $H \cap \overline{s}$.

Let us first assume that for all $x$ in $\mathcal{H} \cap \overline{s}$, $\psi(x) <_R 1$. Observe that, for all $x$ in $\mathcal{H} \cap \overline{s}$, for all $n \in \mathbb{Z}$, $\phi(x) \#_R 2n + 1$.

We define $\rho$ as follows. For each $p$, for each $a$, for each $n \in \mathbb{Z}$, if $\phi(p) = a + 1$ and $a' \sqsubseteq s$ and $n \in a''$, then $\rho(p) = (a, n)$, and, if not, then $\rho(n) = 0$. Note that $\rho$ enumerates a continuous function from $\mathcal{H} \cap \overline{s}$ to $\mathbb{Z}$ with the property that, for all $x$ in $\mathcal{H} \cap \overline{s}$, for all $n \in \mathbb{Z}$, if $2n - 1 <_R \phi(x) <_R 2n + 1$, then $\rho(x) = \mathbb{Z} n$.

Note that $E_{\rho'}$ is an enumerable subset of $\mathbb{S}$ and a bar in $\mathcal{H} \cap \overline{s}$. We may assume that $E_{\rho'}$ is shrinking in length, and that $E_{\rho'}$ is a subset of the frame of $\mathcal{H} \cap \overline{s}$.

Note that, in case $\mathcal{H} = [0, 1]$, we may complete the proof quickly: the continuous function $E_{\rho'}$ is constant on $[0, 1] \cap \overline{s}$ and, therefore, the continuous function $E_{\rho}$ is bounded on $[0, 1] \cap \overline{s}$.

The following argument covers the general case.

Note that, for each $n$, $\rho(n) = (\rho'(n), \rho''(n))$.

We intend to define $\tau$ in $\mathcal{N}$ enumerating a continuous function from $\mathcal{H} \cap \overline{s}$ to $[0, 1] \cup \mathbb{Z}$ such that $\tau' = \rho'$. We define $\tau''$ as follows.

First, we define, for all $t, u \in \mathbb{S}$, a rational number $d(t, u)$ that we want to call the distance between $t$ and $u$. Let $t, u$ be elements of $\mathbb{S}$. If $t \approx \mathbb{S} u$, then $d(t, u) = 0$, and, if $t' <_Q u'$, then $d(t, u) = u'' - Q t'$, and, if $u'' <_Q t'$, then $d(t, u) = t'' - Q u'$. Note that, for all real numbers $x, y$, if there exist $m, n$ such that $x(m) \sqsubseteq t$ and $y(n) \sqsubseteq u$, then $|x - y| <_R d(t, u)$.

We let $\gamma$ be an element of $\mathcal{N}$ such that $E_{\gamma}$ is the frame of $\mathcal{H} \cap \overline{s}$. We define $\delta$ as follows. For each $n$, $\delta(n)$ is the least $k$ such that, for each $m < n$, there exists $j < k$ such that $\gamma(j) > 0$ and $\gamma(j) - 1 \sqsubseteq \rho'(m)$.

We define: $\tau''(0) = 0_{\mathbb{Z}}$. Now let $n > 0$ be given. We have to define $\tau''(n)$ and we distinguish two cases.

Case (i). There exists $m < n$ such that $\rho''(m) = \rho''(n)$. We let $m_0$ be the least such $m$ and we define: $\tau''(n) = \tau''(m_0)$.

Case (ii). For all $m < n$, $\rho''(m) \neq \rho''(n)$. Let $A$ be the finite set consisting of all $u$ such that, for some $j < \delta(n)$, $\gamma(j) = u + 1$, and, for some $m < n$, $u \sqsubseteq \rho'(m)$. Let $t$ be the least element of $A$ such that, for all $v$ in $A$, $d(u, \rho'(n)) \leq Q d(v, \rho'(n))$.

Find $m < n$ such that $t \sqsubseteq \rho'(m)$ and define: $\tau''(n) = 1_{\mathbb{Z}} - Q \tau'(m)$. Observe that, for all $v$ in $A$, for all $m < n$, if $v \sqsubseteq \rho'(m)$, then there will exist $x, y$ in $\mathcal{H}$ such that $|x - y| <_R d(v, \rho'(n))$ and $\tau(x) = \mathbb{Z} 1 - \tau(y)$.

Note that $E_{\tau}$ is a continuous function from $\mathcal{H} \cap \overline{s}$ to $[0, 1] \cup \mathbb{Z}$. As $E_{\tau}$ is easily extended to a partial continuous function from $\mathcal{H}$ to $\mathcal{R}$, we may conclude that $E_{\tau}$ is uniformly continuous on $H \cap \overline{s}$. Find $m$ such that, for all $x, y$ in $H \cap \overline{s}$, if $|x - y| <_R \frac{1}{2m + 1}$, then $\tau(x) = \mathbb{Z} \tau(y)$.

Note that $\mathcal{H} \cap \overline{s}$ is totally bounded. Let $B$ be a finite subset of $\mathbb{S}$ covering $\mathcal{H} \cap \overline{s}$ such that, for every $u$ in $B$, length$_{\mathbb{S}}(u) = \frac{1}{2m + 1}$ and: there exists $t$ in the frame of $\mathcal{H} \cap \overline{s}$ such that $t \sqsubseteq u$.

Find $p$ such that
(1) for each \( u \) in \( B \), there exist \( j, n \) such that such that \( j < p \) and \( n < p \) and \( \gamma(j) > 0 \) and \( \gamma(j) - 1 \subseteq u \) and \( \gamma(j) - 1 \subseteq \rho'(n) \), and,

(2) for each \( n > p \), \( \text{length}_S(\rho'(n)) \leq Q \frac{1}{2^m+1} \).

We now prove: for every \( n \geq 0 \), there exists \( j < p \) such that \( \rho''(n) = \mathbb{Z} \rho''(j) \).

Suppose there exists \( n \geq p \) such that, for all \( j < p \), \( \rho''(n) \neq \mathbb{Z} \rho''(j) \). Let \( n_0 \) be the least such \( n \). Find \( u \) in \( B \) and \( t \) in the frame of \( \mathcal{H} \) such that \( t \subseteq \rho'(n_0) \) and \( t \subseteq u \). Note: \( u \approx \rho'(n_0) \) and \( d(u, \rho'(n_0)) < Q \text{length}_S(u) + Q \text{length}_S(\rho'(n_0)) \leq Q \frac{1}{2^m} \). Find \( j, n \) such that \( j < \min(p, \delta(n)) \) and \( n < p \) and \( \gamma(j) > 0 \) and \( \gamma(j) - 1 \subseteq u \) and \( \gamma(j) - 1 \subseteq \rho'(n) \). Note that \( d(\gamma(j) - 1, \rho'(n_0)) < Q \frac{1}{2^m} \). It follows, as we observed above, that there exist \( x, y \) in \( \mathcal{H} \) such that \( |x - y| < \mathcal{R} \frac{1}{2^m} \) and \( \tau(x) = \mathbb{Z} 1 - \tau(y) \). Contradiction.

We have to conclude: for all \( n \geq 0 \), there exists \( j < p \) such that \( \rho''(n) = \mathbb{Z} \rho''(j) \).

Find \( m, n \) in \( \mathbb{Z} \) such that, for all \( j < p \), \( m \leq \mathbb{Z} \rho''(j) \leq \mathbb{Z} \). Conclude: for all \( x \) in \( \mathcal{H} \cap \mathcal{S} \), \( m \leq \mathbb{Z} \rho(x) \leq \mathbb{Z} n \), and, therefore, for all \( x \) in \( \mathcal{H} \cap \mathcal{S} \), \( 2m - 1 \leq \mathbb{R} \phi(x) \leq \mathbb{R} 2n + 1 \). We thus see that the function enumerated by \( \phi \) is bounded on \( \mathcal{H} \cap \mathcal{S} \).

By a similar argument one proves that, in the case that for every \( x \) in \( \mathcal{H} \cap \mathcal{S} \), \( \psi(x) > \mathcal{R} 0 \), the function enumerated by \( \phi \) is bounded on \( \mathcal{H} \cap \mathcal{S} \).

We may conclude that, for each \( s \) in \( C \), \( \phi \) bounded on \( \mathcal{H} \cap \mathcal{S} \). As \( C \) is a finite subset of \( \mathcal{S} \) covering \( \mathcal{H} \), \( \phi \) is bounded on \( \mathcal{H} \).

(\( v \)) \( \Rightarrow \) (vi). Suppose that every enumerable continuous function from \( \mathcal{H} \) to \( \mathcal{R} \) is bounded on \( \mathcal{H} \). Assume that \( \phi \) enumerates a continuous function from \( \mathcal{H} \) to \( \mathcal{R} \) such that, for all \( x \) in \( \mathcal{H} \), \( \phi(x) > \mathcal{R} 0 \). Determine \( \psi \) such that \( \psi \) enumerates a continuous function from \( \mathcal{H} \) to \( \mathcal{R} \) such that, for all \( x \) in \( \mathcal{H} \), \( \psi(x) = \mathcal{R} \frac{1}{\phi(x)} \). Find \( m \) in \( \mathbb{N} \) such that, for all \( x \) in \( \mathcal{H} \), \( \psi(x) < \mathcal{R} 2m \) and observe that, for all \( x \) in \( \mathcal{H} \), \( \phi(x) > \mathcal{R} \frac{1}{2^m} \).

(vi) \( \Rightarrow \) (vii). Obvious.

(vii) \( \Rightarrow \) (i). Suppose that for every \( \phi \), if \( \phi \) enumerates a continuous function from \( \mathcal{H} \) to \( \mathcal{R} \) that is uniformly continuous on \( \mathcal{H} \) and, for all \( x \) in \( \mathcal{H} \), \( \phi(x) > \mathcal{R} 0 \), then there exists \( m \) such that for all \( x \) in \( \mathcal{H} \), \( \phi(x) > \mathcal{R} \frac{1}{2^m} \).

Let \( X \) be a decidable subset of \( \mathcal{S} \) that is a covering of \( \mathcal{H} \). Using Lemma 8.9 we find an decidable subset \( Y \) of \( \mathcal{S} \) such that \( Y \) is shrinking in length and a refinement of \( X \).

Let \( s \) belong to \( \mathcal{S} \). We let \( f_s \) be an element of \( \mathcal{N} \) enumerating a continuous function from \( \mathcal{R} \) to \( \mathcal{R} \) with the property that, for every real number \( x \), \( f_s(x) = \sup_{\mathcal{R}}(0, \inf_{\mathcal{R}}(x - s', x'' - x)) \). Observe that \( f_s \) has the number \( \frac{1}{2} \text{length}_S(s) \) as its highest value. \( f_s \) is uniformly continuous as, for all real numbers \( x, x' \), \( |f_s(x) - f_s(x')| \leq \mathcal{R} |x - x'| \).

Observe that, for each \( n \), there exists \( s \) in \( Y \) such that, for every \( t \) in \( Y \), if \( t > s \), then, for every real number \( x \), \( f_s(x) < \mathcal{R} \frac{1}{2^m} \). We let \( \phi \) be an element of \( \mathcal{N} \) enumerating a continuous function from \( \mathcal{R} \) to \( \mathcal{R} \) with the property that, for all \( x \) in \( \mathcal{R} \), \( \phi(x) \) is the least upper bound of the set \( \{ f_t(x) | t \in X \} \), that is: (i) for every \( s \) in \( Y \), \( f_s(x) \leq \mathcal{R} \phi(x) \), and (ii) for every \( n \) there exists \( s \) in \( Y \) such that \( f_s(x) > \mathcal{R} \phi(x) - \frac{1}{2^m} \). The function enumerated by \( \phi \) is uniformly continuous, as, for all real numbers \( x, x' \), \( |\phi(x) - \phi(x')| \leq \mathcal{R} |x - x'| \).

Observe that, for every \( x \) in \( \mathcal{H} \), \( \phi(x) > \mathcal{R} 0 \). Find \( m \) such that for every \( x \) in \( \mathcal{H} \), \( \phi(x) > \mathcal{R} \frac{1}{2^m+1} \). Conclude that every element of \( \mathcal{H} \) belongs to some element \( s \) in
Y such that \( \text{length}_S(s) > Q \frac{1}{2^n} \). Now use the fact that Y is shrinking in length. Let Z be a finite subset of X that, for every s in Y such that \( \text{length}_S(s) > Q \frac{1}{2^n} \), there exists \( t \) in Z such that \( s \sqsubseteq_S t \). Note that Z covers \( \mathcal{H} \).

We thus see that every decidable subset of \( S \) covering \( \mathcal{H} \) has a finite subset covering \( \mathcal{H} \). \( \square \)

**Corollary 8.12** Let \( \mathcal{H} \) be a closed-and-separable subset of \( \mathcal{R} \). The following statements are equivalent:

(i) \( \mathcal{H} \) is compact.

(ii) Every enumerable continuous function from \( \mathcal{H} \) to \( \mathcal{R} \) has totally bounded range on \( \mathcal{H} \).

**Proof** (i) \( \Rightarrow \) (ii). Assume that every enumerable subset of \( S \) covering \( \mathcal{H} \) has a finite subset covering \( \mathcal{H} \). Let \( X \) be an enumerable subset of \( \mathbb{N} \) that is a continuous function from \( \mathcal{H} \) to \( \mathcal{R} \).

Let \( n \) belong to \( \mathbb{N} \). We let \( Y \) be the set of all \( s \) in \( S \) such that \( s \) is in the frame of \( \mathcal{H} \) for some \( t \) in \( S \), \( (s, t) \) is in \( X \) and \( \text{length}_S(t) < Q \frac{1}{2^n} \). As the frame of \( \mathcal{H} \) is enumerable, also \( Y \) is an enumerable subset of \( \mathbb{N} \), see Lemma 7.5. Note that \( Y \) is also a covering of \( \mathcal{H} \). Find a finite subset \( Z \) of \( Y \) covering \( \mathcal{H} \). Let \( W \) be the set of all \( t \) in \( S \) such that, for some \( s \) in \( S \), \( (s, t) \) is in \( Z \). \( W \) is a finite subset of \( S \) covering the range of \( X \) on \( \mathcal{H} \) and, for each \( t \) in \( W \), \( \text{length}_S(t) < Q \frac{1}{2^n} \), and \( t \) contains an element of the range of \( X \) on \( \mathcal{H} \).

We thus see that the range of \( X \) on \( \mathcal{H} \) is totally bounded.

(ii) \( \Rightarrow \) (i). Assume (ii) and conclude: every continuous function has bounded range on \( \mathcal{H} \). According to Theorem 8.11(v) \( \Rightarrow \) (ii), \( \mathcal{H} \) is compact. \( \square \)

### 8.5.1 A note on perfect closed-and-separable subsets of \( \mathcal{R} \)

A closed-and-separable subset \( \mathcal{H} \) of \( \mathcal{R} \) is **perfect** if and only if, for each \( x \) in \( \mathcal{H} \), for each \( n \), there exists \( y \) in \( \mathcal{H} \) such that \( x \#_\mathcal{R} y \) and \( |x - y| <_\mathcal{R} \frac{1}{2^n} \).

**Theorem 8.13** Let \( \mathcal{H} \) be a perfect closed-and-separable subset of \( \mathcal{R} \). The following statements are equivalent:

(i) \( \mathcal{H} \) is compact.

(ii) Every enumerable continuous function from \( \mathcal{H} \) to \( \mathcal{R} \) is uniformly continuous on \( \mathcal{H} \).

**Proof** (i) \( \Rightarrow \) (ii). See Theorem 8.11(ii) \( \Rightarrow \) (iv).

(ii) \( \Rightarrow \) (i). Let \( \mathcal{H} \) be a perfect closed-and-separable subset of \( \mathcal{R} \) such that every enumerable continuous function from \( \mathcal{H} \) to \( \mathcal{R} \) is uniformly continuous on \( \mathcal{H} \). We prove that \( \mathcal{H} \) is bounded, and then conclude that every enumerable continuous function has bounded range on \( \mathcal{H} \). Theorem 8.11(v) \( \Rightarrow \) (ii) then makes us see: \( \mathcal{H} \) is compact.

We first consider the function \( x \mapsto x^2 \). Find \( m \) such that, for all \( x, y \) in \( \mathcal{H} \), if \( |x - y| <_\mathcal{R} \frac{1}{2^m} \), then \( |x^2 - y^2| <_\mathcal{R} 1 \). Note that, for all \( x \) in \( \mathcal{H} \), if \( |x| >_\mathcal{R} 2^m + 1 \), then, for all \( y \) in \( \mathcal{H} \), if \( |x - y| <_\mathcal{R} \frac{1}{2^m} \), then \( |x^2 - y^2| =_\mathcal{R} |x + y| \cdot |x - y| > _\mathcal{R} 2^{m+1} \cdot |x - y| \), so \( |x - y| <_\mathcal{R} \frac{1}{2^{m+1}} \). We thus see that there are no \( x, y \) in \( \mathcal{H} \) such that \( |x| >_\mathcal{R} 2^m + 1 \) and \( \frac{1}{2^m+1} <_\mathcal{R} |x - y| <_\mathcal{R} \frac{1}{2^n} \).

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Let \( \alpha \) be an infinite sequence of real numbers such that \( H = H_\alpha \). We define infinite sequences \( \beta, \delta \) of real numbers as follows. For each \( n, \beta^n = \sup \{ \alpha^i | i \leq n \} - \frac{1}{2^{n+1}} \) and \( \delta^n = \inf \{ \alpha^i | i \leq n \} + \frac{1}{2^{n+1}}. \)

Note that, for each \( n \), if \( \beta^n < \delta^n - 2^m \), then: if \( \alpha^{n+1} < \beta^n \), then \( \alpha^{n+1} < \beta^n - \frac{1}{2^{n+1}} \); and, for each \( k \), if the length of both \( \alpha^{n+1}(k) \) and \( \beta^n(k) \) is smaller than \( \frac{1}{2^{n+2}} \), then \( \alpha^{n+1}(k) \leq \beta^n(k) \). Similarly, for each \( n \), if \( \beta^n > \delta^n + 2^m \), then: if \( \alpha^{n+1} > \delta^n \), then \( \alpha^{n+1} > \delta^n + \frac{1}{2^{n+1}} \); and, for each \( k \), if the length of both \( \alpha^{n+1}(k) \) and \( \delta^n(k) \) is smaller than \( \frac{1}{2^{n+2}} \), then \( \alpha^{n+1}(k) \geq \delta^n(k) \).

We now define \( \phi \) such that for each \( n \),

1. \( \phi^n \) enumerates a partial continuous function from \( R \) to \( R \), and
2. \( \text{dom}(\phi^n) = [\beta^n, \delta^n] \) and \( \phi^n(\beta^n) = R. \phi^n(\delta^n) = R. \) and,
3. if \( \alpha^{n+1} < R. \beta^n < R. -2^m - 1 \), then there exist \( x, y \) in \( H \cap \text{dom}(\phi^{n+1}) \) such that \( 0 < R. x - y < R. \frac{1}{2^{n+1}} \) and \( |\phi^{n+1}(x) - \phi^{n+1}(y)| = R. \) 1, and \( \beta^{n+1} < R. x < R. y < R. \beta^n \), and
4. if \( \alpha^{n+1} > R. \delta^n > R. +2^m + 1 \), then there exist \( x, y \) in \( H \cap \text{dom}(\phi^{n+1}) \) such that \( 0 < R. y - x < R. \frac{1}{2^{n+1}} \) and \( |\phi^{n+1}(x) - \phi^{n+1}(y)| = R. \) 1, and either \( \beta^{n+1} < R. x < R. y < R. \beta^n \) or \( \delta^n < R. x < R. y < R. \delta^n + 1 \).

Find \( \gamma \) such that \( E_\gamma \) is the frame of \( H \).

We let \( \phi^0 \) enumerate a continuous function with domain \([\beta^0, \delta^0]\) and the constant value 0.

Suppose \( n \) is a natural number and we defined \( \phi^n \). We then define \( \phi^{n+1} \) as follows. Consider \( \beta^{n+1} \) and \( \delta^{n+1} \) and let \( k_0 \) be the least \( k \) such that the length of each of the five segments \( \alpha^{n+1}(k), \beta^n(k), \beta^{n+1}(k), \delta^n(k) \) and \( \delta^{n+1}(k) \) is smaller than \( \frac{1}{2^{n+2}} \).

We now distinguish four cases.

Case (i). \( -2^m - 1 < Q. (\beta^{n+1}(k_0))' \) and \( (\delta^{n+1}(k_0))'' < Q. 2^m + 1 \). We let \( \phi^{n+1} \) be a continuous function with domain \([\beta^{n+1}, \delta^{n+1}]\) and the constant value 0.

Case (ii). \( (\beta^{n+1}(k_0))' \leq Q. -2^m - 1 \) and \( \alpha^{n+1}(k_0) < S. \beta^n(k_0) \). Define: \( s := \alpha^{n+1}(k_0) \) and note: \( s \cap [\beta^n, \delta^n] = \emptyset \) and \( s \subseteq [\beta^{n+1}, \delta^n] \). Note that \( s \) is in the frame of \( H \). We now use the fact that \( H \) is perfect. Let \( q \) be the least \( p \) such that \( \gamma(p') > 0 \) and \( \gamma(p'') > 0 \) and \( t := \gamma(p') - 1 \) and \( u := \gamma(p'') - 1 \) satisfy the conditions: \( t \subseteq S. s \) and \( u \subseteq Q. (s) \) and \( t \# S. u \) and \( d(t, u) < Q. \frac{1}{2^{n+1}} \). Define \( t := \gamma(q') - 1 \) and \( u := \gamma(q'') - 1 \). We let \( \phi^{n+1} \) enumerate a continuous function with domain \([\beta^{n+1}, \delta^{n+1}]\) such that, for all \( x \in L, \phi(x) = R. 0 \) and, for all \( y \in \bar{u}, \phi(y) = R. 1 \) and \( \phi^{n+1}(\beta^{n+1}) = R. \phi^{n+1}(\delta^{n+1}) = R. 0 \).

Case (iii). Case (ii) does not apply and \( (\delta^{n+1}(k_0))' \geq Q. 2^m + 1 \) and \( \alpha^{n+1}(k_0) > S. \delta^n(k_0) \). Define: \( s := \alpha^{n+1}(k_0) \) and note: \( s \cap [\beta^n, \delta^n] = \emptyset \) and \( s \subseteq [\beta^n, \delta^{n+1}] \). Note that \( s \) is in the frame of \( H \). Use the fact that \( H \) is perfect. Let \( q \) be the least \( p \) such that \( \gamma(p') > 0 \) and \( \gamma(p'') > 0 \) and \( t := \gamma(p') - 1 \) and \( u := \gamma(p'') - 1 \) satisfy the conditions: \( t \subseteq S. s \) and \( u \subseteq Q. (s) \) and \( t \# S. u \) and \( d(t, u) < Q. \frac{1}{2^{n+1}} \). Define \( t := \gamma(q') - 1 \) and \( u := \gamma(q'') - 1 \). We let \( \phi^{n+1} \) enumerate a continuous function with domain \([\beta^{n+1}, \delta^{n+1}]\) such that, for all \( x \in L, \phi(x) = R. 0 \) and, for all \( y \in \bar{u}, \phi(y) = R. 1 \) and \( \phi^{n+1}(\beta^{n+1}) = R. \phi^{n+1}(\delta^{n+1}) = R. 0 \).

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Case (iv). None of the previous cases applies. We let \( \phi^{n+1} \) enumerate a continuous function with domain \([\beta^{n+1}, \delta^{n+1}]\) extending the function enumerated by \( \phi^n \) such that, for all \( x \) in \([\beta^{n+1}, \beta^n] \cup [\delta^n, \delta^{n+1}]\), \( \phi^{n+1}(x) = R 0 \).

Now recall that, for all \( n, k \), \( \phi^n(k) = \phi((n, k)) \). Conclude that \( \phi \) enumerates a continuous function from \( \bigcup_{n \in \mathbb{N}} [\beta^n, \delta^n] \) to \( R \).

Note that, for each \( x \) in \( H \), there exists \( n \) such that \( |\alpha^n - x| < R \frac{1}{2m+1} \), and, therefore: \( \beta^n \leq R x \leq R \delta^n \). It follows that \( E_\phi \) is a continuous function from \( H \) to \( R \) and that \( E_\phi \) is uniformly continuous on \( H \). Find \( n \) such that, for all \( x, y \) in \( H \), if \( |x - y| < R \frac{1}{2m} \), then \( |\phi(x) - \phi(y)| < R 1 \). Conclude: there is no \( p \geq n \) such that \( \alpha^{p+1} < R -2^{m} - 1 \) and \( \alpha^{p+1} < R \inf \{\alpha^i | i \leq p\} \). Therefore, for all \( x \) in \( H \), \( x \geq R \inf \{\alpha^i | i \leq n\} \). A similar argument proves: for all \( x \) in \( H \), \( x \leq R \sup \{2^{m} + 1, \sup \{\alpha^i | i \leq n\}\} \).

\[ \square \]

8.5.2 Dini’s Theorem a third time

**Theorem 8.14** Let \( H \) be a closed-and-separable subset of \( R \). The following statements are equivalent:

(i) \( H \) is compact.

(ii) \( H \) has Dini’s property, that is, for every \( \phi \) in \( N \), if for each \( n \), \( \phi^n \) enumerates a continuous function from \( H \) to \( R \), and for each \( n \), for each \( x \) in \( H \), \( \phi^n(x) \geq \phi^{n+1}(x) \geq 0 \) and for each \( m \), for each \( x \) in \( H \), there exists \( n \) such that \( \phi^n(x) \leq R \frac{1}{2m} \), then for each \( m \) there exists \( n \) such that, for each \( x \) in \( H \), \( \phi^n(x) \leq R \frac{1}{2m} \).

**Proof** (i) \( \Rightarrow \) (ii). Suppose that every enumerable covering of \( H \) has a finite subcovering. Assume that \( \phi \) belongs to \( N \) and that, for each \( n \), \( \phi^n \) enumerates a continuous function from \( H \) to \( R \) such that for each \( n \), for each \( x \) in \( H \), \( \phi^n(x) \geq R \phi^{n+1}(x) \geq R 0 \) and for each \( m \), for each \( x \) in \( H \) there exists \( n \) such that \( \phi^n(x) \leq R \frac{1}{2m} \). Let \( m \) be a natural number. Let \( X \) be the set of all \( s \) in \( S \) such that, for some \( t \) in \( S \), for some \( n, (s, t) \) is in \( E_\phi^n \) and \( t'' < Q \frac{1}{2m} \). Observe that \( Y \) is an enumerable subset of \( S \) covering \( H \). Let \( Z \) be a finite subset of \( Y \) covering \( H \). Find \( n \) such that, for every \( s \) in \( Z \), there exists \( j \leq n \) such that, for some \( t, (s, t) \) is in \( E_\phi^n \). Observe that, for all \( x \) in \( H \), \( \phi^n(x) \leq R \frac{1}{2m} \).

(ii) \( \Rightarrow \) (i). Suppose that every sequence of continuous functions from \( H \) to \( R \) that pointwise decreases and converges to 0 converges to 0 uniformly on \( H \).

We intend to use Theorem 8.11(vi). We shall prove that the range of every continuous function from \( H \) to \( R \) that is positive at every point in \( H \) has a positive lower bound and then may conclude that \( H \) is compact.

Let \( \phi \) enumerate a continuous function from \( H \) to \( R \) such that for each \( x \) in \( H \), \( \phi(x) > R 0 \). We let \( \psi \) be an element of \( N \) such that, for each \( n \), \( \psi^n \) enumerates a continuous function from \( H \) to \( R \) with the property that, for all \( x \) in \( H \), \( \psi^n(x) = R (1 - \inf \{\phi(x), \frac{1}{2}\}) \). Observe that \( \psi \) is a sequence of continuous functions from \( H \) to \( R \) that pointwise decreases and converges to 0. Find \( n \) such that, for all \( x \) in \( H \), \( \psi^n(x) < R \frac{1}{2} \). Find \( p \) such that \( 1 - \sqrt{\frac{1}{2}} > R \frac{1}{2m} \) and observe that, for all \( x \) in \( H \), \( \phi(x) > R \frac{1}{2m} \). [Q.E.D.]
8.6 Closed-and-separable subsets of $\mathcal{R}$ that are positively noncompact

We want to prove a counterpart to Theorem 8.11. To this end we introduce the following notions.

Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$ and let $X$ be a subset of $\mathbb{S}$. $X$ positively fails to cover $\mathcal{H}$ if and only if there exists $x$ in $\mathcal{H}$ with the property that, for all $u$ in $X$, there is no $k$ such that $x(k) \sqsubseteq_S u$.

Let $\mathcal{H}$ be a subset of $\mathcal{R}$. $\mathcal{H}$ is positively unbounded from above if and only if for each $n$ in $\mathbb{N}$ there exists $x$ in $\mathcal{H}$ such that $x >_\mathcal{R} n$.

Let $Z$ be a finite subset of $\mathbb{S}$. We let the real closure $\mathbb{Z}$ of $Z$ be the set all real numbers $\alpha$ such that, for each $n$, there exists $u$ in $Z$ such that $u \approx_S \alpha(n)$.

Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$ and let $X$ be a continuous function from $\mathcal{H}$ to $\mathcal{R}$. $X$ positively fails to be uniformly continuous if and only if there exists $n$ such that, for each $m$, there exist $x, y$ in $\mathcal{H}$ such that $|x - y| < \frac{1}{2^n}$ and $|X(x) - X(y)| >_\mathcal{R} \frac{1}{2^n}$.

Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$.

We define: $\mathcal{H}$ is positively noncompact if and only if $\mathcal{H}$ satisfies the condition mentioned in item (ii) of the next theorem: There exists an enumerable subset of $\mathbb{S}$ covering $\mathcal{H}$ such that every finite subset of $\mathcal{H}$ positively fails to cover $\mathcal{H}$.

Theorem 8.15 Let $\mathcal{H}$ be a bounded closed-and-separable subset of $\mathcal{R}$. The following statements are equivalent:

(i) There exists a decidable subset of $\mathbb{S}$ covering $\mathcal{H}$ such that every finite subset of $\mathcal{H}$ positively fails to cover $\mathcal{H}$.

(ii) $\mathcal{H}$ is positively noncompact, that is: there exists an enumerable subset of $\mathbb{S}$ covering $\mathcal{H}$ such that every finite subset of $\mathcal{H}$ positively fails to cover $\mathcal{H}$.

(iii) There exists an enumerable continuous function from $\mathcal{H}$ to $\mathcal{R}$ with positively unbounded range.

(iv) There exists an enumerable continuous function from $\mathcal{H}$ to $\mathcal{R}$ that positively fails to be uniformly continuous.

Proof (i) $\Rightarrow$ (ii): obvious.

(ii) $\Rightarrow$ (iii): Let $X$ be an enumerable subset of $\mathbb{S}$ covering $\mathcal{H}$ such that every finite subset of $\mathcal{H}$ positively fails to cover $\mathcal{H}$. We let $Y$ be the set of all $s$ in $\mathbb{S}$ such that, for some $t$ in $X$, $s \sqsubseteq_S t$. Note that, for each $x$ in $\mathcal{R}$, $x$ is covered by $X$ if and only if $x$ is covered by $Y$. Also note that $Y$, like $X$, is an enumerable subset of $\mathbb{S}$.

We claim: For every finite subset $Z$ of $Y$ there exist $s, t$ in $\mathbb{S}$ such that $t$ belongs to $Y$ and $s \sqsubseteq_S t$ and $s$ contains an element of $\mathcal{H}$ and $s$ does not touch any element of $Z$, that is, for all $u$ in $Z$, not: $u \approx_S s$.

We prove this claim as follows. Let $Z$ be a finite subset of $Y$. Let $W$ be a finite subset of $X$ such that, for every $u$ in $Z$ there exists $t$ in $W$ such that $u \sqsubseteq_S t$. Find $\alpha$ in $\mathcal{H}$ not contained in any element of $W$. For each $u$ in $Z$, we first determine $t$ in $W$ such that $u \sqsubseteq_S t$ and then $n$ in $\mathbb{N}$ such that $\text{length}_S(\alpha(n)) < \min(t' - u', u'' - t'')$, and we conclude: not: $\alpha(n) \sqsubseteq_S t$ and, therefore, $\alpha(n) \#_S u$. Doing so for each $u$ in $Z$, we find...
$m$ such that, for all $u$ in $Z$, there exists $n \leq m$ such that $u \#_S \alpha(n)$, and, therefore, $u \#_S \alpha(m)$. Find $t$ in $Y$ such that $\alpha$ belongs to $t$. Determine $p$ such that $p > m$ and $\alpha(p) \subseteq I$. Define $s := \alpha(p)$ and note that $s, t$ satisfy the requirements.

Find $\gamma$ in $N$ such that $Y$ coincides with $E_\gamma$. Find $\delta$ such that the frame of $H$ coincides with $E_\delta$. Using the Minimal Axiom of Choice, we find $\alpha$ and $\beta$ in $N$ such that, for each $n$, $\alpha(n)$ and $\beta(n)$ belong to $S$ and there exists $p$ such that $\alpha(n) = \gamma(p') - 1$ and $\beta(n) = \delta(p'') - 1$, where $p$ is the least $q$ such that $\gamma(q') > 0$ and $\delta(q'') > 0$ and $\delta(q'') - 1 \subseteq \gamma(q') - 1$, and for each $i < n$, $\delta(q''_i) - 1 \#_S \alpha(i)$, and, if $\gamma(i) > 0$, then also $\delta(q''_i) - 1 \#_S \gamma(i) - 1$.

We now let $\psi$ be an element of $N$ such that, for each $n$, $\psi^n$ enumerates a continuous function from $[\alpha(0), \alpha(1), \ldots \alpha(n)] \cup E_{\gamma^n}$ to $R$ and $\psi^{n+1}$ extends $\psi^n$ and for all $x$ in $[\beta(n)]$, $\psi^n(x) > n$. Note that $\psi$ enumerates a partial continuous function from $R$ to $H$ and that this function is defined in every element of $H$. Also note that, for every $n$, there exists $x$ in $H$ such that $x$ is contained in $\beta(n)$ and therefore $\psi^n(x) = \psi(x) > n$. Clearly, $\psi$ enumerates a continuous function from $H$ to $R$ with unbounded range on $H$.

(iii) $\Rightarrow$ (iv): Let $\phi$ be an element of $N$ enumerating a function from $H$ to $R$ with positively unbounded range. Recall that $H$ is bounded and let $M$ be an element of $R$ such that, for every $x$ in $H$, $-M <_R x <_R M$. Let $m$ belong to $N$. Let $k$ be a natural number such that $\frac{k}{2^m + 1} > R 2M$. Find $x_0, x_1, \ldots x_{k-1}$ in $H$ such that, for each $j < k - 1$, $\phi(x_j) + 1 <_R \phi(x_{j+1})$. Find $i, j$ such $i < j < k$ and $|x_i - x_j| <_R \frac{1}{2^m}$ and note that $\phi(x_j) >_R \phi(x_i) + 1$. Clearly, the function enumerated by $\phi$ positively fails to be uniformly continuous.

(iv) $\Rightarrow$ (i): Let $\phi$ be an element of $N$ enumerating a function from $H$ to $R$ that positively fails to be uniformly continuous. Let $m$ be a natural number such that, for each $n$, there exist $x, y$ in $H$ such that $|x - y| <_R \frac{1}{2^m}$ and $|\phi(x) - \phi(y)| >_R \frac{1}{2^m}$. Let $X$ be the set of all $s$ such that, for some $t, p, \phi(p) = (s, t) + 1$ and length$_S(t) <_Q \frac{1}{2^m}$. Let $Y$ be the set of all $u$ such that, for some $p \leq u, \phi(p) > 0$ and $u \subseteq S (\phi(p) - 1)^\prime$. Observe that $Y$ is a decidable subset of $S$ covering $H$. We claim that every finite subset of $Y$ positively fails to cover $H$ and we prove this claim as follows.

Let $U$ be a finite subset of $Y$. Determine a finite subset $Z$ of $X$ such that, for each $u$ in $U$, there exists $s$ in $Z$ such that $u \subseteq S s$. Find $p$ such that, for all $t$ in $S$, if length$_S(t) <_Q \frac{1}{2^m}$ and $t$ contains elements $x, y$ of $H$ such that $|\phi(x) - \phi(y)| >_R \frac{1}{2^m}$. Note that there is no $s$ in $Z$ such that $t \subseteq S s$, and, therefore, for every $u$ in $U, u \#_S t$. Clearly, $U$ positively fails to cover $H$. \hfill $\square$

**Theorem 8.16** Let $H$ be a bounded closed-and-separable subset of $R$. The following statements are equivalent:

(i) $H$ is positively noncompact.

(ii) There exists a bounded enumerable continuous function from $H$ to $R$ that positively fails to be uniformly continuous.

**Proof** (i) $\Rightarrow$ (ii). Suppose $H$ is positively noncompact. As in the proof of the previous Theorem 8.15(ii) $\Rightarrow$ (iii), we may assume that we are given an enumerable subset $Y$ of $S$ covering $H$ such that for every finite subset $Z$ of $Y$ there exist $s, t$ in $S$ such that $t$ belongs to $Y$ and $s \subseteq S t$ and $s$ contains an element of $H$ and $s$ does not touch any element of $Z$, that is, for all $u$ in $Z, u \#_S s$.
Find $\gamma$ in $\mathcal{N}$ such that $Y$ coincides with $E_{\gamma}$. As in the proof of the previous Theorem 8.15(ii) $\Rightarrow$ (iii), we find $\alpha$ and $\beta$ in $\mathcal{N}$ such that, for each $n$, $\alpha(n)$ belongs to $Y$ and $\beta(n)$ belongs to the frame of $\mathcal{H}$ and $\beta(n) \in_S \alpha(n)$ and, for each $i < n$, $\beta(n) \in_S \alpha(i)$ and, if $i > 0$, then also $\beta(n) \in_S \gamma(i) - 1$.

We now let $\psi$ be an element of $\mathcal{N}$ such that, for each $n$, $\psi^n$ enumerates a continuous function from $[\alpha(0), \alpha(1), \ldots, \alpha(n)] \cup \mathcal{E}_{\psi^n}$ to $[-1, 1]$ and $\psi^{n+1}$ extends $\psi^n$ and either: for all $x$ in $[\beta(n)]$, $\psi^n(x) >_R \mathcal{R}$ 0 or: for all $x$ in $[\beta(n)]$, $\psi^n(x) <_R \mathcal{R}$ 0.

In order to decide if we make $\psi$ positive or negative on $\beta(n)$ we use a notion that we found useful already in the proof of Theorem 8.11.

For all $s, t$ in $S$, the rational number $d(s, t)$, that we want to call the distance between $s$ and $t$, is defined as follows. If $s \approx_S t$, then $d(s, t) = 0$, if $s^n <_Q t^n$, then $d(s, t) = t^n - Q s^n$, and if $t^n <_Q s^n$, then $d(s, t) = s^n - Q t^n$.

We now complete our definition of $\psi$. We first determine: for all $x$ in $\overline{\beta(0)}$, $\psi^0(x) >_R \mathcal{R}$ 0. For each $n > 0$, we let $i_0$ be the least $i < n$ such that, for all $j < n$, $d(\beta(i), \beta(n)) \leq_Q d(\beta(j), \beta(n))$. We determine: if, for all $x$ in $\overline{\beta(i_0)}$, $\psi^{i_0}(x) >_R \mathcal{R}$ 0, then, for all $x$ in $\overline{\beta(n)}$, $\psi^{i_0}(x) <_R \mathcal{R}$ 0, and, if, for all $x$ in $\overline{\beta(i_0)}$, $\psi^{i_0}(x) <_R \mathcal{R}$ 0, then, for all $x$ in $\overline{\beta(n)}$, $\psi^{i_0}(x) >_R \mathcal{R}$ 0.

Note that $\psi$ enumerates a partial continuous function from $\mathcal{R}$ to $[-1, 1]$ and that this function is defined in every element of $\mathcal{H}$. Also note that, for every $n$, $\beta(n)$ contains an element of $\mathcal{H}$.

We claim that the function enumerated by $\psi$ positively fails to be uniformly continuous on $\mathcal{H}$, as, for each $n$, one may find $x, y$ in $\mathcal{H}$ such that $|x - y| <_R \frac{1}{2^n}$ and $|\psi(x) - \psi(y)| >_R 1$.

We prove this claim as follows. Let $n$ be a natural number. As there exists $M$ in $\mathcal{R}$ such that, for each $x$ in $\mathcal{H}$, $-M <_R x <_R M$, and, for all $i, j$, if $i < j$, then $\beta(i) \in_S \beta(j)$, we may determine $p, q$ such that $p < q$ and $0 < d(\beta(p), \beta(q)) <_Q \frac{1}{2^n}$.

If the function enumerated by $\psi$ is positive on $\overline{\beta(p)}$ and negative on $\overline{\beta(q)}$, or conversely, negative on $\overline{\beta(p)}$ and positive on $\overline{\beta(q)}$, we choose $x, y$ in $\mathcal{H}$ belonging to $\overline{\beta(p)}$ and $\overline{\beta(q)}$, respectively, and we conclude: $|x - y| <_R \frac{1}{2^n}$ and $|\psi(x) - \psi(y)| >_R 1$.

If the function enumerated by $\psi$ is positive on both $\overline{\beta(p)}$ and $\overline{\beta(q)}$, then we let $i_0$ be the least $i < q$ such that, for all $j < q$, $d(\beta(i_0), \beta(q)) \leq_Q d(\beta(j), \beta(q))$. We then know: $\psi$ is negative on $\overline{\beta(i_0)}$ and positive on $\overline{\beta(q)}$ and $d(\beta(i_0), \beta(q)) <_Q \frac{1}{2^n}$, so we may finish the proof as before.

If the function enumerated by $\psi$ is negative on both $\overline{\beta(p)}$ and $\overline{\beta(q)}$, we argue similarly.

(ii) $\Rightarrow$ (i). This is a consequence of Theorem 8.15(iv) $\Rightarrow$ (ii).

8.6.1 Dini’s Theorem again

Theorem 8.17 Let $\mathcal{H}$ be a bounded closed-and-separable subset of $\mathcal{R}$. The following statements are equivalent:

(i) $\mathcal{H}$ is positively noncompact.

(ii) There exists $\phi$ in $\mathcal{N}$ enumerating a continuous function from $\mathcal{H}$ to $\mathcal{R}$ that is uniformly continuous on $\mathcal{H}$ and such that, for every $x$ in $\mathcal{H}$, $\phi(x) >_R 0$, and, for each $m$, there exists $x$ in $\mathcal{H}$ such that $\phi(x) <_R \frac{1}{2^m}$.
(iii) There exists $\phi$ in $\mathcal{N}$ enumerating a continuous function from $\mathcal{H}$ to $\mathcal{R}$ such that, for every $x$ in $\mathcal{H}$, $\phi(x) >_{\mathcal{R}} 0$, and, for each $m$, there exists $x$ in $\mathcal{H}$ such that $\phi(x) <_{\mathcal{R}} \frac{1}{2m}$.

(iv) There exists $\phi$ in $\mathcal{N}$ such that, for each $n$, $\phi^n$ enumerates a continuous function from $\mathcal{H}$ to $\mathcal{R}$, and for each $n$, for each $x$ in $\mathcal{H}$, $\phi^n(x) \geq_{\mathcal{R}} \phi^{n+1}(x) \geq_{\mathcal{R}} 0$, and, for each $m$, for each $x$ in $\mathcal{H}$, there exists $n$ such that $\phi^n(x) <_{\mathcal{R}} \frac{1}{2m}$, while also, there exists $m$ such that, for every $n$, there exists $x$ in $\mathcal{H}$ such that $\phi^n(x) >_{\mathcal{R}} \frac{1}{2m}$.

**Proof** (i) $\Rightarrow$ (ii): Let $X$ be an enumerable subset of $\mathbb{S}$ covering $\mathcal{H}$ such that every finite subset of $\mathcal{H}$ positively fails to cover $\mathcal{H}$. We now use a construction that we used earlier in the proof of Theorem 8.11(vii) $\Rightarrow$ (i). Using Lemma 8.9 we find a decidable subset $Y$ of $\mathbb{S}$ such that $Y$ is as a refinement of $X$ and shrinking in length.

For each $s$ in $\mathbb{S}$, we let $f_s$ be an element of $\mathcal{N}$ enumerating a continuous function from $\mathcal{R}$ to $\mathcal{R}$ with the property that, for every real number $x$, $f_s(x) =_{\mathcal{R}} \sup_{\mathcal{R}} (0, \inf_{\mathcal{R}} (x - s', s'' - x))$. Observe that $f_s$ has the number $\frac{1}{2} \text{length}_{\mathbb{S}}(s)$ as its highest value. $f_s$ is uniformly continuous as, for all real numbers $x$, $x'$, $|f_s(x) - f_s(x')| \leq_{\mathcal{R}} |x - x'|$.

Observe that, for each $n$, there exists $s$ in $Y$ such that, for every $t$ in $Y$, if $t > s$, then, for every real number $x$, $f_s(x) <_{\mathcal{R}} \frac{1}{2m}$. We let $\phi$ be an element of $\mathcal{N}$ enumerating a continuous function from $\mathcal{R}$ to $\mathcal{R}$ with the property that, for all $x$ in $\mathcal{R}$, $\phi(x)$ is the least upper bound of the set $\{|f_t(x)| : t \in Y\}$, that is: (i) for every $s$ in $Y$, $f_s(x) <_{\mathcal{R}} \phi(x)$, and (ii) for every $n$ there exists $s$ in $Y$ such that $f_s(x) >_{\mathcal{R}} \phi(x) - \frac{1}{2m}$.

Note: (1) for every $x$ in $\mathcal{H}$, $\phi(x) >_{\mathcal{R}} 0$, and (2) for every $m$, there exists $s$ such that, for all $t$ in $\mathcal{H}$, if $t > s$, then, for every $x$, $f_s(x) <_{\mathcal{R}} \frac{1}{2m}$, and there exists $x$ in $\mathcal{H}$ such that, for all $t$ in $Y$, if $t \leq s$, then $x$ is not contained in $t$, and, therefore, $\phi(x) <_{\mathcal{R}} \frac{1}{2m}$.

(ii) $\Rightarrow$ (iii): obvious.

(iii) $\Rightarrow$ (iv): Let $\phi$ be an element of $\mathcal{N}$ enumerating a continuous function from $\mathcal{H}$ to $\mathcal{R}$ such that, for every $x$ in $\mathcal{H}$, $\phi(x) >_{\mathcal{R}} 0$, and, for each $m$, there exists $x$ in $\mathcal{H}$ such that $\phi(x) <_{\mathcal{R}} \frac{1}{2m}$. We determine $\psi$ in $\mathcal{N}$ such that, for each $n$, $\psi^n$ enumerates a continuous function from $\mathcal{H}$ to $\mathcal{R}$ such that, for each $n$, for each $x$ in $\mathcal{H}$, $\psi^n(x) =_{\mathcal{R}} \left(1 - \inf_{\mathcal{R}} \left(\frac{1}{2}, \phi(x)\right)\right)^n$.

Note that, for each $p$, there exists $x$ in $\mathcal{H}$ such that $\phi(x) <_{\mathcal{R}} \frac{1}{2} - \sqrt[p]{\frac{1}{2}}$.

Also note that, for each $n$, there exists $p$ such that $\frac{1}{2} <_{\mathcal{R}} 1 - \sqrt[p]{\frac{1}{2}}$.

Conclude that, for each $n$, there exists $x$ in $\mathcal{H}$ such that $\phi(x) <_{\mathcal{R}} 1 - \sqrt[p]{\frac{1}{2}}$, and, therefore, $\sqrt[p]{\frac{1}{2}} <_{\mathcal{R}} 1 - \phi(x)$, so $\phi(x) <_{\mathcal{R}} \frac{1}{2}$ and $\sqrt[p]{\frac{1}{2}} <_{\mathcal{R}} 1 - \inf_{\mathcal{R}} \left(\frac{1}{2}, \phi(x)\right)$ and $\frac{1}{2} <_{\mathcal{R}} \psi^n(x)$.

Clearly, $\psi$ satisfies the requirements.

(iv) $\Rightarrow$ (i): Let $\phi$ be an element of $\mathcal{N}$, such that, for each $n$, $\phi^n$ enumerates a continuous function from $\mathcal{H}$ to $\mathcal{R}$, and for each $n$, for each $x$ in $\mathcal{H}$, $\phi^n(x) \geq_{\mathcal{R}} \phi^{n+1}(x) \geq_{\mathcal{R}} 0$, and, for each $m$, for each $x$ in $\mathcal{H}$, there exists $n$ such that $\phi^n(x) <_{\mathcal{R}} \frac{1}{2m}$, while also, there exists $m$ such that, for every $n$, there exists $x$ in $\mathcal{H}$ such that $\phi^n(x) >_{\mathcal{R}} \frac{1}{2m}$.

Let $m$ be a natural number with the last-mentioned property, that is, such that, for every $n$, there exists $x$ in $\mathcal{H}$ such that $\phi^n(x) >_{\mathcal{R}} \frac{1}{2m}$. 

\[ \text{ Springer} \]
Let $Y$ be the set of all $s$ in $S$ such that there exist $n, t$ such that $(s, t)$ belongs to $E_{\phi^s}$ and $t'' \leq \frac{1}{2^n}$. Note that $Y$ is an enumerable subset of $S$ covering $\mathcal{H}$ and that every finite subset of $Y$ positively fails to cover $\mathcal{H}$.  

\[ \square \]

9 Bringing in the Fan Theorem and its alternative

Let $X$, $Y$ be subsets of $\mathbb{N}$. We let $\text{Comp}[X, Y]$ be the set of all numbers $a$ such that, for some $b$, $(a', b)$ belongs to $Y$ and $(b, a'')$ belongs to $X$. One verifies easily that, if $X$ and $Y$ are enumerable subsets of $\mathbb{N}$, then also $\text{Comp}[X, Y]$ is an enumerable subset of $\mathbb{N}$, as follows.

For all $\alpha, \beta$, we let $C(\alpha, \beta)$ be the sequence $\gamma$ such that, for all $m$, if there exists $a < m$ such that, for some $i, j, b < m$, $\beta(i) = (a', b) + 1$ and $\alpha(j) = (b, a'') + 1$ and for all $k < m$, $\gamma(k) \neq a + 1$, then $\gamma(m) = a_0 + 1$, where $a_0$ is the least such $a$, and, if not, then $\gamma(m) = 0$.

Observe that, for all $\alpha, \beta, C(\alpha, \beta)$ enumerates $\text{Comp}[E_\alpha, E_\beta]$.

Suppose that $X$ is a partial continuous function from $\mathcal{N}$ to $\mathbb{N}$ and that $Y$ is a partial continuous function from $\mathcal{N}$ to $\mathbb{N}$. Observe that $\text{Comp}[X, Y]$ is a partial continuous function from $\mathcal{N}$ to $\mathbb{N}$ and that, for all $\alpha$, if $\alpha$ belongs to $\text{dom}(Y)$ and $Y|\alpha$ belongs to $\text{Dom}(X)$, then $\alpha$ belongs to $\text{Dom}(\text{Comp}[X, Y])$ and $(\text{Comp}[X, Y])(\alpha) = X(Y|\alpha)$.

Let $X, Y$ be subsets of $\mathbb{N}$ and suppose that both $X$ and $Y$ are partial continuous functions from $\mathcal{N}$ to $\mathcal{N}$. Observe that $\text{Comp}[X, Y]$ is a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$ and that, for all $\alpha$, if $\alpha$ belongs to $\text{dom}(Y)$ and $Y|\alpha$ belongs to $\text{Dom}(X)$, then $\alpha$ belongs to $\text{dom}(\text{Comp}[X, Y])$ and $(\text{Comp}[X, Y])(\alpha) = X(Y|\alpha)$.

Let $F$ be a closed-and-separable subset of $\mathcal{N}$. $F$ is compact if and only if every enumerable covering of $F$ has a finite subcovering, see Sect. 7.3. We know from Theorems 7.7, 7.8, 7.9 and 7.14 that closed-and-separable subsets of $\mathcal{N}$ with this property may be characterized in many ways.

$F$ is positively noncompact if and only if there exists an enumerable covering of $F$ such that every finite subset of the covering positively fails to cover $F$, see Sect. 7.4. We know from Theorems 7.10, 7.11, 7.12, 7.13 and 7.15 that closed-and-separable subsets of $\mathcal{N}$ that are positively noncompact may be characterized in many ways.

Let $F, G$ be closed-and-separable subsets of $\mathcal{N}$. We say that $F$ covers $G$ if and only if there exists an enumerable partial continuous function $X$ from $F$ to $\mathcal{N}$ such that $\text{ran}(X, F) = G$.

Let $b$ belong to $\mathbb{N}$. We call $b$ a binary sequence number if, for all $i < \text{length}(b)$, $b(i) < 2$. As in Sect. 6, $\text{Bin}$ denotes the set of the binary sequence numbers.

Preparing for the proof of Lemma 9.2, we define a function $D$ from $\mathbb{N}$ to $\text{Bin}$, as follows, by induction on the length of the argument:

(i) $D(\langle \rangle) = \langle \rangle$

(ii) for each $a, n$, $D(a * \langle n \rangle) = D(a) * \overline{D(n)} * \langle 1 \rangle$.

Note that, for every $c$ in $\text{Bin}$, there exists exactly one $a$ such that $D(a) = c * \langle 1 \rangle$.

A closed-and-separable subset of $\mathcal{N}$ with a decidable frame is called a spread, see Sect. 7.1.6. Let $F$ be a spread. We want to define a decidable subset $R$ of $\mathbb{N}$ that is a retraction of $\mathcal{N}$ onto $F$, that is: a continuous function from $\mathcal{N}$ to $F$ such that, for every $\gamma$ in $F$, $R|\gamma = \gamma$. Find $\alpha$ in $C$ such that $D_\alpha$ is the frame of $F$.  

\[ \triangle \]
We first define a function \( r \) from \( \mathbb{N} \) to \( D_\alpha \). The definition is by induction on the length of the argument, as follows.

(i) \( r(\{\}) = \{\} \).

(ii) For each \( b \), for each \( i \), if \( \alpha(r(b) \cdot \langle i \rangle) = 1 \), then \( r(b \cdot \langle i \rangle) = r(b) \cdot \langle i \rangle \), and if not, then \( r(b \cdot \langle i \rangle) = r(b) \cdot \langle i_0 \rangle \), where \( i_0 \) is the least \( i \) such that \( \alpha(r(b) \cdot \langle i \rangle) = 1 \).

We let \( R \) be the set of all numbers of the form \( (b \cdot c, r(b)) \).

It will be clear that \( R \) is an enumerable continuous function from \( \mathcal{N} \) onto \( \mathcal{F} \) such that for all \( \gamma \) in \( \mathcal{F} \), \( R[\gamma] = \gamma \).

For all \( s, t \) in \( \mathbb{N} \) we define: \( s \) and \( t \) go different ways. notation: \( s \bot t \), if and only if neither \( s \subseteq t \) nor \( t \subseteq s \). Let \( \mathcal{F} \) be a closed-and-separable subset of \( \mathcal{N} \). As we defined in Sect. 7.3.1, \( \mathcal{F} \) is a perfect closed-and-separable subset of \( \mathcal{N} \) if and only if, for each \( \alpha \) in \( \mathcal{F} \), for each \( n \), there exists \( \beta \) in \( \mathcal{F} \) such that \( \overline{\beta} n = \overline{\alpha} n \) and \( \beta \neq \alpha \). Note that \( \mathcal{F} \) is a perfect closed-and-separable subset of \( \mathcal{N} \) if and only if, for every \( s \), if \( s \) contains an element of \( \mathcal{F} \), then there exist \( t, u \) such that \( s \subseteq t \) and \( s \subseteq u \) and \( t \cap u \) and both \( t \) and \( u \) contain an element of \( \mathcal{F} \).

Let \( \mathcal{F} \) be a closed-and-separable subset of \( \mathcal{N} \) with a decidable frame. \( \mathcal{F} \) is a finitary spread or a fan if and only if, for each \( n \), the set of all \( s \) such that \( length(s) = n \) and \( s \) contains an element of \( \mathcal{F} \) is a finite subset of \( \mathbb{N} \).

\( \mathcal{F} \) is an explicit fan if and only if there exists \( \gamma \) such that, for each \( n \), \( D_{\gamma(n)} \) is the set all \( s \) such that \( length(s) = n \) and \( s \) contains an element of \( \mathcal{F} \).

The Weak \( \Pi^0_1 \)-First Axiom of Countable Choice has been mentioned in Sect. 6.

**Lemma 9.1** (Using the weak \( \Pi^0_1 \)-first axiom of countable choice) Let \( \mathcal{F} \) be a subset of \( \mathcal{N} \) that is a fan. \( \mathcal{F} \) is an explicit fan.

**Proof** Let \( \mathcal{F} \) be a subset of \( \mathcal{N} \) that is a fan.

Find \( \beta \) such that \( D_\beta \) is the frame of \( \mathcal{F} \).

Note that, for each \( n \), there exists \( p \) such that, for all \( s \geq p \), if \( length(s) = n \), then \( \beta(s) = 0 \). Using the Weak \( \Pi^0_1 \)-First Axiom of Countable Choice one finds \( \delta \) such that, for each \( n \), for all \( s \geq \delta(n) \), if \( length(s) = n \), then \( \beta(s) = 0 \).

Note that, for each \( n \), there exists \( p \) such that \( D_p \) is the set of all \( s < \delta(n) \) such that \( length(s) = n \) and \( \beta(s) = 1 \).

Using the Minimal Axiom of Countable Choice, one then may define \( \gamma \), such that, for each \( n \), \( D_{\gamma(n)} \) is the set of all \( s < \delta(n) \) such that \( length(s) = n \) and \( \beta(s) = 1 \), and, therefore, the set of all \( s \) such that \( length(s) = n \) and \( \beta(s) = 1 \).

**Lemma 9.2** (i) For all closed-and-separable subsets \( \mathcal{F}, \mathcal{G} \) of \( \mathcal{N} \), if \( \mathcal{F} \) is compact and \( \mathcal{F} \) covers \( \mathcal{G} \), then \( \mathcal{G} \) is compact.

(ii) For all closed-and-separable subsets \( \mathcal{F}, \mathcal{G} \) of \( \mathcal{N} \), if \( \mathcal{G} \) is positively noncompact and \( \mathcal{F} \) covers \( \mathcal{G} \), then \( \mathcal{F} \) is positively noncompact.

(iii) Let \( \mathcal{F} \) be a subset of \( \mathcal{N} \) that is a explicit fan. Then Cantor space \( \mathcal{C} \) covers \( \mathcal{F} \).

(iv) Every inhabited and perfect closed-and-separable subset of \( \mathcal{N} \) covers Cantor space \( \mathcal{C} \).

**Proof** (i) Suppose that \( \mathcal{F} \) and \( \mathcal{G} \) are closed-and-separable subsets of \( \mathcal{N} \). Assume also that \( \mathcal{F} \) covers \( \mathcal{G} \) and that \( \mathcal{F} \) is compact. Let \( X \) be an enumerable continuous function
from $\mathcal{F}$ onto $\mathcal{G}$. Let $Y$ be an enumerable continuous function from $\mathcal{G}$ to $\mathbb{N}$. Observe that $C(Y, X)$ has finite Range, as $\mathcal{F}$ is compact, see Theorem 7.7. Note that the Range of $Y$ coincides with the Range of $C(Y, X)$. We conclude that every continuous function from $\mathcal{G}$ to $\mathbb{N}$ has finite Range. By Theorem 7.7(iii), $\mathcal{G}$ is compact.

(ii) Suppose that $\mathcal{F}$ and $\mathcal{G}$ are closed-and-separable subsets of $\mathcal{N}$. Assume also that $\mathcal{F}$ covers $\mathcal{G}$ and that $\mathcal{G}$ is positively noncompact. Let $X$ be an enumerable continuous function from $\mathcal{F}$ onto $\mathcal{G}$. Using Theorem 7.10, find an enumerable continuous function $Y$ from $\mathcal{G}$ to $\mathbb{N}$ such that the Range of $Y$ is positively infinite. Note that the Range of $Y$ coincides with the Range of $C(Y, X)$. We conclude that there exists a continuous function from $\mathcal{F}$ to $\mathbb{N}$ with positively infinite Range. By Theorem 7.10(iii), $\mathcal{F}$ positively is noncompact.

(iii) Let $\mathcal{F}$ be a subset of $\mathcal{N}$ that is an explicit fan. Find $\gamma$ such that, for each $n$, $D_{\gamma(n)}$ is the set all $s$ such that $\text{length}(s) = n$ and $s$ contains an element of $\mathcal{F}$. Define $\delta$ such that, for each $n$, $\delta(n) = \max\{s(n) | s \in D_{\gamma(n+1)}\}$.

We let $X$ be the set of all numbers $(D(a) \ast c, a)$, where $a$ belongs to the frame of $\mathcal{F}$ and $c$ is a binary sequence number. Observe that $X$ is a decidable continuous function from a subset of $\mathcal{C}$ onto $\mathcal{F}$. We claim that the domain of $X$ is a fan. In order to verify this claim we show, how to decide, for each binary sequence number $b$, if $b$ contains an element of the domain of $X$ or not. Let $b$ be a binary sequence number. We distinguish two cases.

Case (i): For each $i < \text{length}(b)$, $b(i) = 0$. Find $k$ such that $b = \overline{0}k$. Note that $b$ contains an element of the domain of $X$ if and only if $k \leq \delta(0)$.

Case (ii): For some $i < \text{length}(b)$, $b(i) = 1$. Find $c, k$ such that $b = c \ast (1) \ast \overline{0}(k)$. Find $a$ such that $D(a) = c \ast (1)$. Observe that $b$ contains an element of the domain of $X$ if and only if there exists $i \leq \delta(\text{length}(a))$ such that $a \ast (i)$ is in the frame of $\mathcal{F}$ and $k \leq i$.

We let $Y$ be an enumerable retraction of $\mathcal{C}$ onto the domain of $X$.

Note that $\text{Comp}[X, Y]$ is an enumerable continuous function from $\mathcal{C}$ onto $\mathcal{F}$.

(iv) Let $\mathcal{F}$ be a perfect closed-and-separable subset of $\mathcal{N}$. Using Theorem 7.3, find $\gamma$ in $\mathcal{N}$ such that $E_{\gamma}$ is the frame of $\mathcal{F}$. Let $\delta$ be an element of $\mathcal{N}$ such that, for each $n, s$, if $\gamma(n) = s + 1$, then $\delta(n)$ is the least $p$ such that, for some $t, t' \gamma(p') = t' + 1$ and $\gamma(p'') = t'' + 1$ and $t' \perp t''$ and $s \subseteq t'$ and $s \subseteq t''$. Let $\epsilon$ be an element of $\mathcal{N}$ such that $\epsilon(()) = ()$ and, for each binary sequence number $s$, for each $n$, if $\gamma(n) = \epsilon(s) + 1$, then $\epsilon(s \ast (0)) = (\gamma(\delta(n)))^{t'} - 1$ and $\epsilon(s \ast (1)) = (\gamma(\delta(n)))^{t''} - 1$. Note that, for each $t$, one may decide if there exists a binary sequence number $s$ such that $t \subseteq \epsilon(s)$ or not. Let $\mathcal{G}$ be the set of all $\alpha$ in $\mathcal{F}$ such that, for each $n$, there exists a binary sequence number $s$ such that $\overline{a}n \subseteq \epsilon(s)$.

Note that $\mathcal{G}$ is a subset of $\mathcal{F}$ and a spread.

Let $X$ be the set of all numbers of the form $(\epsilon(s) \ast t, s)$ such that $s$ is a binary sequence number and $\beta(\epsilon(s) \ast t) = 1$.

Note that $X$ is a decidable continuous function from $\mathcal{G}$ onto $\mathcal{C}$.

Let $Y$ be a retraction from $\mathcal{N}$ onto $\mathcal{G}$.

Note that $\text{Comp}[X, Y]$ is an enumerable continuous function from $\mathcal{F}$ onto $\mathcal{C}$. $\square$

Let $X$ be a subset of $\mathbb{N}$. $X$ is a partial continuous function from $\mathcal{N}$ to $\mathcal{R}$ if and only if $X$ is a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$ and $\text{ran}(X)$ is a subset of $\mathcal{R}$.  

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Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$.

$\mathcal{H}$ is compact if and only if every enumerable covering of $\mathcal{H}$ has a finite subcovering, see Sect. 8.5. We know from Theorem 8.11, Corollary 8.12, and Theorem 8.14 that compact closed-and-separable subsets of $\mathcal{R}$ may be characterized in many ways.

$\mathcal{H}$ is positively noncompact if and only if there exists an enumerable covering of $\mathcal{H}$ such that every finite subset of the covering positively fails to cover $\mathcal{H}$, see Sect. 8.6. We know from Theorems 8.15, 8.16 and 8.17 that closed-and-separable subsets of $\mathcal{R}$ that are positively noncompact may be characterized in many ways.

Let $\mathcal{F}$ be a subset of $\mathcal{N}$ and let $\mathcal{H}$ be a subset of $\mathcal{R}$.

$\mathcal{F}$ covers $\mathcal{H}$ if and only if there exists an enumerable partial continuous function $X$ from $\mathcal{F}$ to $\mathcal{R}$ such that $\text{ran}(X)$ really-coincides with $\mathcal{H}$.

For each $s$ in $\mathcal{S}$, we let double$_{\mathcal{S}}(s)$ be the element $t$ of $\mathcal{S}$ such that length$_{\mathcal{S}}(t) = 2 \cdot$ length$_{\mathcal{S}}(s)$ and $t(0) + \mathcal{Q} t(1) = s(0) + \mathcal{Q} t(1)$, that is, the length of double$_{\mathcal{S}}(s)$ is twice the length of $s$, and double$_{\mathcal{S}}(s)$ and $s$ have the same midpoint.

Note that, for all $s, t$ in $\mathcal{S}$, if length$_{\mathcal{S}}(t) < \frac{1}{3} \text{length}_{\mathcal{S}}(s)$ and $s \approx_{\mathcal{S}} t$, then double$_{\mathcal{S}}(t) \sqsubseteq_{\mathcal{S}}$ double$_{\mathcal{S}}(s)$.

Let $\mathcal{H}, \mathcal{K}$ be subsets of $\mathcal{R}$.

$\mathcal{H}$ covers $\mathcal{K}$ if and only if there exists an enumerable partial continuous function $X$ from $\mathcal{H}$ to $\mathcal{R}$ such that $\text{ran}(X)$ really-coincides with $\mathcal{K}$.

A subset $X$ of $\mathcal{R}$ is inhabited if and only if there exists $x$ in $\mathcal{R}$ such that $x \in X$.

**Lemma 9.3**

(i) For all closed-and-separable subsets $\mathcal{F}$ of $\mathcal{N}$ and $\mathcal{H}$ of $\mathcal{R}$, if $\mathcal{F}$ is compact and $\mathcal{F}$ covers $\mathcal{H}$, then $\mathcal{H}$ is compact.

(ii) For all closed-and-separable subsets $\mathcal{F}$ of $\mathcal{N}$ and $\mathcal{H}$ of $\mathcal{R}$, if $\mathcal{F}$ covers $\mathcal{H}$ and $\mathcal{H}$ is positively noncompact, then $\mathcal{F}$ is positively noncompact.

(iii) Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$ that is explicitly totally bounded. Then Cantor space $\mathcal{C}$ covers $\mathcal{H}$.

(iv) For all closed-and-separable subsets $\mathcal{H}$, $\mathcal{K}$ of $\mathcal{R}$, if $\mathcal{H}$ is compact and $\mathcal{H}$ covers $\mathcal{K}$, then $\mathcal{K}$ is compact.

(v) For all closed-and-separable subsets $\mathcal{H}$, $\mathcal{K}$ of $\mathcal{R}$, if $\mathcal{H}$ covers $\mathcal{K}$ and $\mathcal{K}$ is positively noncompact, then $\mathcal{H}$ is positively noncompact.

(vi) Every inhabited perfect closed-and-separable subset of $\mathcal{R}$ covers $[0, 1]$.

**Proof** (i) Assume that $\mathcal{F}$ is a compact closed-and-separable subset of $\mathcal{N}$. Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$ and let $X$ be an enumerable continuous function from $\mathcal{F}$ to $\mathcal{R}$ such that $\text{ran}(X)$ really-coincides with $\mathcal{H}$. We prove that $\mathcal{H}$ is compact.

Let $Y$ be an enumerable subset of $\mathcal{S}$ covering $\mathcal{H}$. Let $Z$ be the set of all numbers $r$ such that for some $s, t$, $(r, s)$ belongs to $Y$ and $s \sqsubseteq_{\mathcal{S}} t$ and $t$ belongs to $X$. Observe that $Z$ is an enumerable subset of $\mathcal{N}$ covering $\mathcal{F}$. Let $Z'$ be a finite subset $Z$ covering $\mathcal{X}$. For every $r$ in $Z'$ we determine $s$ in $\mathcal{S}$ and $t$ in $Y$ such that $(r, s)$ belongs to $X$ and $s \sqsubseteq_{\mathcal{S}} t$. Let $Y'$ be the set of all elements $t$ of $Y$ we obtain in this way. $Y'$ is a finite subset of $Y$ covering $\mathcal{H}$.

We thus see that $\mathcal{H}$ is compact.

(ii) Assume that $\mathcal{F}$ is a closed-and-separable subset of $\mathcal{N}$. Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$ that is positively noncompact and let $X$ be enumerate a continuous function from $\mathcal{F}$ to $\mathcal{R}$ such that $\text{ran}(X)$ really-coincides with $\mathcal{H}$. We prove that $\mathcal{F}$ is positively noncompact.
Let $Y$ be an enumerable subset of $\mathbb{S}$ covering $\mathcal{H}$ such that every finite subset $Y'$ of $Y$ positively fails to cover $\mathcal{H}$. Let $Z$ be the set of all numbers $r$ such that for some $s, t, (r, s)$ belongs to $X$ and $s \subseteq_\mathbb{S} t$ and $t$ belongs to $Y$. Observe that $Z$ is an enumerable set covering $\mathcal{F}$. Let $Z'$ be a finite subset of $Z$. For every $r$ in $Z'$ we determine $s$ in $\mathbb{S}$ and $t$ in $Y$ such that $(r, s)$ belongs to $X$ and $s \subseteq_\mathbb{S} t$. Let $Y'$ be the finite set of all elements $t$ of $Y$ we obtain in this way. Find $y$ in $\mathcal{H}$ such that, for each $r$ in $Y'$, $y$ is not contained in $t$. Find $\alpha$ in $\mathcal{F}$ such that $\phi|\alpha = y$. Note that, for each $r$ in $Z'$, $\alpha$ does not pass through $r$.

We thus see that $\mathcal{F}$ is positively noncompact.

(iii) Let $\mathcal{H}$ be a closed-and-separable subset of $\mathcal{R}$ that is explicitly totally bounded. Find $\alpha$ such that, for each $n$, $D_\alpha(n)$ is a subset of $\mathbb{S}$ covering $\mathcal{H}$ such that, for each $s$ in $D_\alpha(n)$, $s$ contains at least one element of $\mathcal{H}$ and $\text{length}_\mathbb{S}(s) \leq \frac{1}{2^n}$. We may assume, without loss of generality, that, for each $n$, for each $s$ in $D_\alpha(n)$, $\text{length}_\mathbb{S}(s) = \frac{1}{2^n}$. We let $\mathcal{F}$ be the subset of $\mathcal{N}$ consisting of all $\beta$ such that, for each $n$, for some $s$ in $D_\alpha(n)$, $\beta(n) = \text{double}_\mathbb{S}(s)$ and $\beta(n + 1) \subseteq_\mathbb{S} \beta(n)$.

Clearly, $\mathcal{F}$ is a fan and every element of $\mathcal{F}$ really-coincides with an element of $\mathcal{H}$.

We now prove that every element of $\mathcal{H}$ really-coincides with an element of $\mathcal{F}$.

First note that, for each $n$, for each $s$ in $D_\alpha(n)$, there exists $t$ in $D_\alpha(n+1)$ such that $t \preccurlyeq_\mathbb{S} s$, and, therefore, $\text{double}_\mathbb{S}(t) \subseteq_\mathbb{S} \text{double}_\mathbb{S}(s)$.

Suppose that $x$ belongs to $\mathcal{H}$. Using the Minimal Axiom of Choice we find $\gamma$ such that, for each $n$, $\gamma(n)$ belongs to $D_\alpha(n)$ and $x$ is contained in $\gamma(n)$. It follows that, for each $n$, $\gamma(n) \preccurlyeq_\mathbb{S} \gamma(n+1)$. Let $\beta$ be the element of $\mathcal{N}$ such that, for each $n$, $\beta(n) = \text{double}_\mathbb{S}(\gamma(n))$. Observe that $\beta$ belongs to $\mathcal{F}$ and that $x$ really-coincides with $\beta$.

Using Lemma 9.2(iii) we find an enumerable continuous function from $\mathcal{C}$ onto $\mathcal{F}$. It follows that $\mathcal{C}$ covers both $\mathcal{F}$ and $\mathcal{H}$.

The proofs of (iv) and (v) are left to the reader.

(vi) Let $\mathcal{H}$ be an inhabited and perfect closed-and-separable subset of $\mathcal{R}$. Find $\gamma$ such that $E_\gamma$ is the frame of $\mathcal{H}$. We define $\delta$ in such a way that, for each $n$, if $\gamma(n) > 0$, then $\delta(n)$ is the least $p$ such that $\gamma(p') > 0$ and $\gamma(p'') > 0$ and, if we define $s := \gamma(n) - 1, t := \gamma(p') - 1$ and $u := \gamma(p'') - 1$, the following conditions are satisfied: $t \preccurlyeq_\mathbb{S} s, u \preccurlyeq_\mathbb{S} t \preccurlyeq_\mathbb{S} u'$ and $\text{length}_\mathbb{S}(t) \preccurlyeq \frac{1}{2}\text{length}_\mathbb{S}(s)$ and $\text{length}_\mathbb{S}(u) \preccurlyeq \frac{1}{2}\text{length}_\mathbb{S}(s)$. As $\mathcal{H}$ is perfect, $\delta$ is well-defined.

We now define a mapping $B_0$ from the set $\text{Bin}$ of binary sequence numbers to the frame of $\mathcal{H}$, as follows, by induction:

(i) $B_0(\langle \rangle) = \gamma(n) - 1$, where $n$ is the least $p$ such that $\gamma(p) > 0$.

(ii) For each $a$ in $\text{Bin}$, if $n$ is the least $p$ such that $B_0(s) = \gamma(p) - 1$, then $B_0(a * \langle 0 \rangle) = \gamma((\delta(n))') - 1$ and $B_0(a * \langle 1 \rangle) = \gamma((\delta(n))'') - 1$.

Note that, as $\mathcal{H}$ is inhabited, $B_0(\langle \rangle)$ is well-defined, and, therefore, the whole function $B_0$ is well-defined.

The function $B_0$ gives rise to a function from Cantor space $\mathcal{C}$ to $\mathcal{H}$. In order to describe this function correctly, we first define a function $C_0$ from $\text{Bin}$ to $\mathbb{N}$ by: $C_0(\langle \rangle) = \langle \rangle$, and, for each $a$ in $\text{Bin}$, for each $i$ in $\{0, 1\}$, $C_0(a * \langle i \rangle) = C_0(a) * \langle B_0(a * \langle i \rangle) \rangle$.

We now let $X_0$ be the set of all numbers $n$ such that $n'$ is in $\text{Bin}$ and $n'' = C_0(n')$. 

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$X_0$ now is a continuous function from $C$ into $\mathcal{H}$. The function is strongly injective in the following sense: for all $\alpha, \beta$ in $C$, if $\alpha \neq \beta$, then $X_0|\alpha \#_{\mathcal{R}} X_0|\beta$. In fact, the function $X_0$ is strictly order-preserving. Define, for each $\alpha, \beta$ in $\mathcal{N}, \alpha <_{\mathcal{N}} \beta$ if and only if, for some $n$, $\alpha n = \beta n$ and $\alpha(n) < \beta(n)$ and note: for all $\alpha, \beta$ in $C$, if $\alpha <_{\mathcal{N}} \beta$, then $X_0|\alpha <_{\mathcal{R}} X_0|\beta$.

We also define a mapping $B_1$ from the set $Bin$ to the set $S$ of rational segments, as follows, by induction:

(i) $B_1(()) = (0, 1)$

(ii) For each $a$ in $Bin$, for each $s$ in $S$, if $B_1(a) = s$, then $B_1(a\ast(0)) = (s', \frac{1}{2}s' + \frac{1}{2}s'')$ and $B_1(a\ast(1)) = (\frac{1}{2}s', +\frac{1}{2}s'', s'').$

The mapping $B_1$ gives rise to a function from $C$ to $[0, 1]$. Recall that, for all $s$ in $S$, double$_S(s)$ is the element $t$ of $S$ satisfying $t' + Q t'' = s' + Q s''$ and $t'' - Q t' = Q 2(s'' - Q s')$.

We define a function $C_1$ from $Bin$ to $\mathbb{N}$ by: $C_1(()) = ()$, and, for each $a$ in $Bin$, for each $i$ in $[0, 1]$, $C_1(a\ast(i)) = C_1(a)\ast\langle$double$_S(B_1(a\ast(i)))\rangle$.

We now let $X_1$ be the set of all numbers $n$ such that $n'$ is in $Bin$ and $n'' = C_0(n')$. $X_1$ now is a continuous function from $C$ to $[0, 1]$. $X_1$ is a surjective function: for each $x$ in $[0, 1]$ one may define $\alpha$ in $C$ such that $X_1|\alpha$ really-coincides with $x$.

We let $X$ be the set of all $n$ in $\mathbb{N}$ such that both $n'$ and $n''$ are in $S$ and there exists $a$ in $Bin$ such that $n' \subseteq S B_0(a)$ and $B_1(a) \subseteq n''$.

Note that $X$ is an enumerable subset of $\mathbb{N}$ and a partial continuous function from $\mathcal{R}$ to $\mathcal{R}$. For each $\alpha$ in $C$, $X$ maps $X_0|\alpha$ onto $X_1|\alpha$. Clearly, ran$_{\mathcal{R}}(X)$ really-coincides with $[0, 1]$, that is, with ran$_{\mathcal{R}}(X_1)$ and dom$_{\mathcal{R}}(X)$ really-coincides with ran$_{\mathcal{R}}(X_0)$.

We want to extend $X$ to a continuous function from $\mathcal{H}$ onto $[0, 1]$. In fact, we shall extend $X$ to a continuous function from $\mathcal{R}$ onto $[0, 1]$. The extended function should map every $x$ in $\mathcal{R}$ such that $s \leq_{\mathcal{R}} X_0|0$ onto $0$, and every $x$ in $\mathcal{R}$ such that $x \geq_{\mathcal{R}} X_0|1$ onto $1$. Furthermore, on every interval $[X_0|a\ast(0)\ast 1, X_0|a\ast(1)\ast 0]$, where $a$ is in $Bin$, the extended function should be a linear, and, in fact, decreasing function, as $X_1|a\ast(0)\ast 1 >_{\mathcal{R}} X_1|a\ast(1)\ast 0$, for instance: $X_1|((0)\ast 1) = \mathcal{R} \frac{1}{3}$ and $X_1|(1)\ast 0 = \mathcal{R} \frac{1}{3}$.

We first let $Y$ be the set of all $n$ in $\mathbb{N}$ such that, for some $a$ in $Bin$, for some $k$, for some $p$ in $\mathbb{Q}$ such that $0 \leq p \leq 1$, if we define $s_0 := B_0(a\ast(1)\ast \overline{0}k), s_1 := B_0(a\ast(0)\ast \overline{1}k), t_0 := B_1(a\ast(1)\ast \overline{0}k)$ and $t_1 := B_1(a\ast(0)\ast \overline{1}k)$, then $n' = (p \cdot Q \cdot s_0' + Q(1-p)\cdot Q s_1')$ and $n'' = (p \cdot Q t_0' + Q(1-p)\cdot Q t_1')$. We then let $U$ be the set of all $n$ in $\mathbb{N}$ such that both $n'$ and $n''$ are in $S$ and either, for some $k, n' \leq S B_0(\overline{0}k)$ and $B_1(\overline{0}k) \subseteq S n''$, or for some $k, n' \geq S B_1(\overline{1}k)$ and $B_1(\overline{1}k) \subseteq S n''$.

Finally, we let $Z$ be the set of all $n$ in $\mathbb{N}$ such that both $n'$ and $n''$ are in $S$ and there exists $u$ in $Y \cup U$ such that $n' \subseteq u'$ and $u'' \subseteq n''$.

Clearly, $Y$ is an enumerable continuous function from $\mathcal{R}$ to $[0, 1]$ mapping $\mathcal{H}$ onto $[0, 1]$.

We consider the closed real segment $[0, 1]$ consisting of all real numbers $x$ with the property $0 \leq x \leq 1$. $[0, 1]$ is a closed-and-separable subset of $\mathcal{R}$ that is explicitly totally bounded. We want to introduce the notion of a special covering of $[0, 1]$. A
Theorem 9.4  

The following statements are equivalent (in \( BIM \)):

(i)  \( C \) is compact.

(ii) \([0, 1]\) is compact.

(iii) Every enumerable special subcovering of \([0, 1]\) has a finite subset that is a special covering of \([0, 1]\).

**Proof**  

(i) \( \Rightarrow \) (ii). See Lemma 9.3(i) and (iii).

(ii) \( \Rightarrow \) (iii). Let \( X \) be an enumerable special subcovering of \([0, 1]\). We let \( X^+ \) be the set of all dyadic rational segments \( s \) such that either \( s \) belongs to \( X \), or there exist \( t, u \) in \( X \) such that \( t'' = u' \) and \( s = t \sqcup u \), that is, \( s' = t' \) and \( s'' = u'' \). Note that, for every dyadic rational \( p \), if \( 0 <_Q p <_Q 1 \), then there exists \( s \) in \( X^+ \) such that \( s' <_Q p <_Q s'' \). We let \( e \) be a function from \( \mathbb{N} \) to \( \mathbb{N} \) such that for every dyadic rational \( p \) such that \( 0 <_Q p <_Q 1 \), \( \varepsilon(p) \) is a positive rational, and, if \( 0 <_Q p <_Q 1 \), then, for some \( s \) in \( X^+ \), \( (q - \varepsilon(q), q + \varepsilon(q)) \sqsubseteq s \) and, for some \( s \) in \( X^+ \), \( \langle 0_Q, \varepsilon(0_Q) \rangle \sqsubseteq s \) and, for some \( s \) in \( X^+ \), \( \langle 1_Q, 1_Q \rangle \sqsubseteq s \).

We now prove that \( X^+ \) is a covering of \([0, 1]\).

Let \( x \) be an element of \([0, 1]\). We define an element \( \alpha \) of \( C \) in the following way. For each \( n \) we first consider if \( n \) is a dyadic rational such that \( 0 <_Q n <_Q 1 \). If not, then we define \( \alpha(n) = 0 \). If so, we calculate the least \( j \) such that \( \text{length}(x(j)) <_Q \varepsilon(n) \), calling this number \( j_0 \). If either \( n <_Q (x(j_0))' \) or \( (x(j_0))'' <_Q n \), we define \( \alpha(n) = 0 \), and if \( (x(j_0))' \leq_0 n \leq (x(j_0))'' \), we define: \( \alpha(n) = 1 \).
Note that, for each \( n \), if \( n \) is a dyadic rational such that \( 0 \leq R_n \leq 1 \), then, if \( \alpha(n) = 0 \), \( x \) is apart from \( n \), and, if \( \alpha(n) = 1 \), then \( x \) belongs to \( (n - Q \varepsilon(n), n + Q \varepsilon(n)) \).

We then define a real number \( x^* \) in \([0, 1]\), as follows. For each \( n \), if \( \alpha(n) = 0(n) \), then \( x^*(n) = x(n) \), and if \( \alpha(n) \neq 0(n) \), then \( n \) will be positive, and \( x^*(n) \) will be the first element \( s \) of \( S \) such that \( s \subseteq 0(0, 1) \) and \( s \subseteq x(n - 1) \) and \( \text{length}_S(s) \leq \frac{1}{2^n} \) and for every \( r < n \), if \( r \) is a dyadic rational from \([0, 1]\), then either \( r < Q s' \) or \( s'' < Q r \). Observe that \( x^* \) is an element of \([0, 1]\) apart from every dyadic rational and find \( s \) in \( X \) such that \( x^* \) belongs to \( s \). Determine a positive rational number \( q \) such that for every \( y \) in \([0, 1]\), if \( |y - x^*| < R q \), then \( y \) belongs to \( s \). Find \( n \) such that \( \text{length}_S(x(n)) < Q \frac{1}{3}q \) and distinguish two cases.

**Case (i).** \( |x^* - x| < R q \). Then \( x \) will belong to \( s \).

**Case (ii).** \( |x^* - x| > R \frac{1}{3}q \). Now there will exist \( n \) such that \( \alpha(n) = 1 \). Let \( n_0 \) be the first such \( n \). \( n_0 \) will be a dyadic rational number from \([0, 1]\) and \( |x - n_0| < R e(n_0) \), so \( x \) will belong to some element of \( X^+ \).

We may conclude that \( X^+ \) is indeed an enumerable covering of \([0, 1]\).

Let \( Y \) be a finite subset of \( X^+ \) that is a covering of \([0, 1]\). Let \( Y' \) be the set of all elements \( s \) of \( X \) touching at least one element of \( Y \), that is, such that, for some \( t \) in \( Y \), \( s \approx_S t \). A moment’s reflection shows that the finite set \( Y' \) is a special covering of \([0, 1]\).

We thus see that every enumerable special covering of \([0, 1]\) has a finite subset that is a special covering of \([0, 1]\).

(iii) \( \Rightarrow \) (i). Suppose that every enumerable special covering of \([0, 1]\) has a finite special subcovering. We show that \( \mathcal{C} \) is compact: every enumerable covering of \( \mathcal{C} \) has a finite subcovering.

Let \( X \) be an enumerable bar in \( \mathcal{C} \). Consider the set \( Y \) consisting of all numbers \( B(s) \) where \( s \) is a binary sequence number belonging to \( X \) and observe that \( Y \) is an enumerable special covering of \([0, 1]\). Determine a finite subset \( Y' \) of \( Y \) that is a special covering of \([0, 1]\). Let \( X' \) be the set of binary sequence numbers \( s \) such that \( B(s) \) belongs to \( Y' \). \( X' \) is a finite subset of \( X \) and a bar in \( \mathcal{C} \).

The equivalence of items (ii) and (iii) in Theorem 9.4 is also proven in [25], upon which [26] is based.

**Theorem 9.5** The following statements are equivalent (in BIM):

(i) \( \mathcal{C} \) is positively noncompact.

(ii) \([0, 1]\) is positively noncompact.

(iii) There exists an enumerable subset \( X \) of \( \mathbb{N} \) that is a special covering of \([0, 1]\) while every finite subset of \( X \) positively fails to cover \([0, 1]\).

**Proof** (ii) \( \Rightarrow \) (i). See Lemma 9.3(ii) and (iii).

(iii) \( \Rightarrow \) (ii). Let \( X \) be a special covering of \([0, 1]\) such that every finite subset of \( X \) positively fails to be a special covering of \([0, 1]\). We construct \( X^+ \) exactly as we did in the proof of Theorem 9.4(ii) \( \Rightarrow \) (iii). We again conclude that \( X^+ \) is an enumerable covering of \([0, 1]\). Let \( Y \) be finite subset of \( X^+ \). Let \( Y' \) be the set of all elements \( s \) of \( X \) touching at least one element of \( Y \). Note that \( Y' \) is a finite subset of \( X \). Find \( x \) in \([0, 1]\) such that, for each \( s \) in \( Y' \), \( x \) is not contained in \( s \). Conclude that also for each \( s \) in \( Y \), \( x \) is not contained in \( s \). We thus see that \( Y \) positively fails to cover \([0, 1]\).

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(i) ⇒ (iii). Suppose that \( C \) is positively noncompact. Let \( X \) be an enumerable bar in \( C \), such that every finite subset of \( X \) positively fails to cover \( C \). Consider the set \( Y \) consisting of all numbers \( B(s) \) where \( s \) is a binary sequence number belonging to \( X \) and observe that \( Y \) is an enumerable special covering of \([0, 1]\). Let \( Y' \) be a finite subset of \( Y \). Let \( X' \) be the set of binary sequence numbers \( s \) such that \( B(s) \) belongs to \( Y' \). Find \( \alpha \) in \( C \) such that no initial part of \( \alpha \) belongs to \( X' \). Find \( n \) such that no initial part of \( \overline{\alpha}n \) belongs to \( X' \). Observe that the midpoint of \( B(\overline{\alpha}n) \) does not belong to any element of \( Y' \). Thus we see that every finite subset of \( Y \) fails to be a special covering of \([0, 1]\).

Theorem 9.6 The following statements are equivalent (in BIM):

(i) the Fan Theorem: Cantor space \( C \) is compact.
(ii) Every explicitly totally bounded closed-and-separable subset of \( \mathbb{N} \), that is, every explicit fan, is compact.
(iii) The real closed interval \([0, 1]\) is compact.
(iv) Every explicitly totally bounded closed-and-separable subset of \( \mathbb{R} \) is compact.

Proof Immediate from Lemmas 9.2 and 9.3 and Theorem 9.4.

Theorem 9.7 The following statements are equivalent (in BIM):

(i) Kleene’s Alternative to the Fan Theorem: Cantor space \( C \) is positively noncompact.
(ii) Every inhabited and perfect closed-and-separable subset of \( \mathbb{N} \) is positively noncompact.
(iii) The real closed interval \([0, 1]\) is positively noncompact.
(iv) Every inhabited and perfect closed-and-separable subset of \( \mathbb{R} \) is positively noncompact.

Proof Immediate from Lemmas 9.2 and 9.3 and Theorem 9.5.

Corollary 9.8 The following statements are equivalent (in BIM):

(i) Every decidable bar in \( C \) has a bounded subbar.
(ii) Every decidable bar in \( C \) has a finite subbar.
(iii) the Fan Theorem: Every enumerable bar in \( C \) has a finite subbar.
(iv) Every enumerable continuous function from \( C \) to \( \mathbb{N} \) has finite Range.
(v) Every enumerable continuous function from \( C \) to \( \mathbb{N} \) has bounded Range.
(vi) Every enumerable bar in \( C \) has a bounded subbar.
(vii) Every enumerable continuous function from \( C \) to \( \mathbb{N} \) is uniformly continuous.
(viii) For every enumerable subset \( X \) of \( \mathbb{N} \), if (1) for each \( n \), \( X \upharpoonright n \) is a continuous function from \( C \) to \( \mathbb{N} \), and, for all \( \alpha \) in \( C \), \( (X \upharpoonright n)(\alpha) \geq (X \upharpoonright (n+1))(\alpha) \), and (2) for all \( \alpha \) in \( C \) there exists \( n \) such that \( (X \upharpoonright n)(\alpha) = 0 \), then (3) there exists \( n \) such that for all \( \alpha \) in \( C \), \( (X \upharpoonright n)(\alpha) = 0 \).
(ix) Every enumerable continuous function from \( C \) to \( \mathbb{N} \) has bounded range.
(x) Every enumerable continuous function from \( C \) to \( \mathbb{N} \) has totally bounded range.
(xi) Every enumerable continuous function from \( C \) to \( \mathbb{N} \) is uniformly continuous on \( C \).
(xii) Every enumerable covering of \([0, 1]\) has a finite subcovering.
(xiii) Every enumerable continuous function from \([0, 1]\) to \(\mathcal{R}\) has totally bounded range.
(xiv) Every enumerable continuous function from \([0, 1]\) to \(\mathcal{R}\) is uniformly continuous on \([0, 1]\).
(xv) Every bounded enumerable continuous function from \([0, 1]\) to \(\mathcal{R}\) is uniformly continuous on \([0, 1]\).
(xvi) Every enumerable continuous function from \([0, 1]\) to \(\mathcal{R}\) has a least upper bound.
(xvii) Every enumerable continuous function from \([0, 1]\) to \(\mathcal{R}\) is bounded from above.
(xviii) For every \(\phi\) in \(\mathcal{N}\), if \(\phi\) enumerates a continuous function from \([0, 1]\) to \(\mathcal{R}\), and for every \(x\) in \(\mathcal{H}\), \(\phi(x) > \mathcal{R} 0\), then there exists \(m\) such that for every \(x\) in \([0, 1]\), \(\phi(x) > \mathcal{R} \frac{1}{2^m}\).
(xx) (Dini’s Theorem:) For every \(\phi\) in \(\mathcal{N}\), if for each \(n\), \(\phi^n\) enumerates a continuous function from \([0, 1]\) to \(\mathcal{R}\), and for each \(n\), for each \(x\) in \([0, 1]\), \(\phi^n(x) \geq \mathcal{R} 0\) and for each \(m\), for each \(x\) in \([0, 1]\), there exists \(n\) such that \(\phi^n(x) \leq \mathcal{R} \frac{1}{2^n}\), then for each \(m\) there exists \(n\) such that, for each \(x\) in \(H[0, 1]\), \(\phi^n(x) \leq \mathcal{R} \frac{1}{2^n}\).

**Proof** Use Theorem 9.6 and Theorems 7.7, 7.8, 7.9, 7.14, 8.11, Corollary 8.12 and Theorem 8.14. \(\square\)

The equivalence of Dini’s Theorem and the Fan Theorem is a result occurring in [2]. The same fact was established, independently, in [4] and [5].

In his book [6], Bishop had posed the question if the statement Corollary 9.8(xix) is a true statement in constructive mathematics, see also [1], Section IV.8. The fact that, in Corollary 9.8, statement (xix) is an equivalent of the Fan Theorem, is due to Julian and Richman, see [21] and also [8], Chapter 6, Section 2. A nice and short argument proving the same fact may be found in [3]. In [3], the Fan Theorem is also shown to have some other intriguing equivalents.

The fact that, in Corollary 9.8, statement (xiv) is an equivalent of the Fan Theorem, is mentioned in Chapter 5 of [8].

Proofs of some of the equivalences following from Corollary 9.8 may be found in [26].

**Corollary 9.9** The following statements are equivalent (in BIM):

(i) There exists a decidable subset \(X\) of \(\mathbb{N}\) that is a bar in \(\mathcal{C}\) while every bounded subset of \(X\) positively fails to be a bar in \(\mathcal{C}\).

(ii) There exists a decidable subset \(X\) of \(\mathbb{N}\) that is a bar in \(\mathcal{C}\) while every finite subset of \(X\) positively fails to be a bar in \(\mathcal{C}\).

(iii) Kleene’s Alternative to the Fan Theorem: There exists an enumerable subset \(X\) of \(\mathbb{N}\) that is a bar in \(\mathcal{C}\) while every finite subset of \(X\) positively fails to be a bar in \(\mathcal{C}\).
(iv) There exists an enumerable subset X of \( \mathbb{N} \) that is a bar in \( C \) while every bounded subset of X positively fails to be a bar in \( C \).

(v) There exists an enumerable continuous function from \( C \) to \( \mathbb{N} \) with positively infinite Range.

(vi) There exists an enumerable continuous function from \( C \) to \( \mathbb{N} \) with Range \( \mathbb{N} \).

(vii) There is an enumerable continuous function from \( C \) to \( \mathbb{N} \) that positively fails to be uniformly continuous on \( C \).

(viii) There exists an enumerable subset X of \( \mathbb{N} \) such that (1) for each \( n, X \upharpoonright n \) is a continuous function from \( C \) to \( \mathbb{N} \), and, for all \( \alpha \in C \), \((X \upharpoonright n)(\alpha) \geq (X \upharpoonright (n + 1))(\alpha)\), and (2) for all \( \alpha \in C \), there exists \( n \in \mathbb{N} \) such that \((X \upharpoonright n)(\alpha) = 0\), and (3) for all \( n \), there exists \( \alpha \in C \) such that \((X \upharpoonright n)(\alpha) \neq 0\).

(ix) There is an enumerable continuous function from \( C \) to \( \mathbb{N} \) with positively unbounded range.

(x) There is an enumerable continuous function from \( C \) to \( \mathbb{N} \) with positively unbounded range.

(xi) There exists a bounded enumerable continuous function from \( [0, 1] \) to \( R \) such that, for every \( x \in H \), \( \phi(x) > R 0 \), and, for every \( m \in \mathbb{N} \), there exists \( x \) in \( [0, 1] \) such that \( \phi(x) < R \frac{1}{2^m} \).

(xii) There exists a bounded enumerable continuous function from \( [0, 1] \) to \( R \) such that, for every \( x \) in \( [0, 1] \), \( \phi(x) > R 0 \), and, for every \( m \) in \( \mathbb{N} \), there exists \( x \) in \( [0, 1] \) such that \( \phi(x) < R \frac{1}{2^m} \).

(xiii) There exists a bounded enumerable continuous function from \( [0, 1] \) to \( R \) such that, for every \( n \), \( \phi^n \) enumerates a continuous function from \( [0, 1] \) to \( R \), and, for every \( m \) in \( \mathbb{N} \), there exists \( x \) in \( [0, 1] \) such that \( \phi(x) < R \frac{1}{2^m} \).

(xiv) There exists a bounded enumerable continuous function from \( [0, 1] \) to \( R \) such that, for every \( n \), \( \phi^n \) enumerates a continuous function from \( [0, 1] \) to \( R \), and, for every \( m \) in \( \mathbb{N} \), there exists \( x \) in \( [0, 1] \) such that \( \phi(x) < R \frac{1}{2^m} \).

Proof Use Theorem 9.7 and Theorems 7.10, 7.11, 7.13, 8.15, 8.16 and 8.17.

Bishop did not accept Brouwer’s proof of the Fan Theorem. This refusal led him to propose another definition of the notion of continuity for (partial) functions from \( R \) to \( R \). We want to mention an important observation on Bishop’s proposal by Frank Waaldijk, see [52].

Let \( X \) be a subset of \( \mathbb{N} \) and a partial continuous function from \( R \) to \( R \). \( X \) is continuous in the sense of Bishop if and only if, for all real numbers \( x, y \) such that \( x \leq y \), if \( [x, y] \) is a real subset of the domain of \( X \), then \( X \) is uniformly continuous on \( [x, y] \).

The Fan Theorem implies that every enumerable partial continuous function from \( R \) to \( R \) is continuous in the sense of Bishop, see Theorem 7.8 and Theorem 9.6.

**Corollary 9.10** (F. Waaldijk) The following statements are equivalent (in BIM):

\[ \square \text{ Springer} \]
the Fan Theorem
(ii) For every \( \phi \in \mathcal{N} \) if \( \phi \) enumerates a uniformly continuous function from \([0, 1]\) to \( \mathbb{R} \) and, for every \( x \) in \([0, 1]\), \( \phi(x) > 0 \), then the function \( x \mapsto \frac{1}{\phi(x)} \) is uniformly continuous on \([0, 1]\).

(iii) For every \( \phi, \psi \in \mathcal{N} \), if both \( \phi \) and \( \psi \) enumerate a partial continuous function from \( \mathbb{R} \) to \( \mathbb{R} \) that is continuous in the sense of Bishop, then also \( C[\phi, \psi] \) enumerates a partial continuous function from \( \mathbb{R} \) to \( \mathbb{R} \) that is continuous in the sense of Bishop.

Proof (i) \( \Rightarrow \) (iii). See the remark preceding this Theorem.

(iii) \( \Rightarrow \) (ii). Observe that the function \( x \mapsto \frac{1}{x} \) is an enumerable continuous function from the set of all real numbers that are apart from 0 to \( \mathbb{R} \) and that this function is continuous in the sense of Bishop.

(ii) \( \Rightarrow \) (i). Using (ii), we may prove, for every \( \phi \), if \( \phi \) enumerates a continuous function from \([0, 1]\) to \( \mathbb{R} \) that is uniformly continuous on \([0, 1]\) and for every \( x \) in \([0, 1]\), \( \phi(x) > 0 \), then the function \( x \mapsto \frac{1}{\phi(x)} \) is uniformly continuous on \([0, 1]\) and, therefore, bounded on \([0, 1]\), so there exist \( m \) such that, for every \( x \) in \([0, 1]\), \( \phi(x) \geq \frac{1}{m} \). Using Theorem 7.11(x), we conclude that \([0, 1]\) is compact. The result now follows from Theorem 9.6.

As Frank Waaldijk pointed out, Corollary 9.10 makes one hesitate to adopt Bishop’s proposal.

Corollary 9.10 has a counterpart.

Let \( X \) be a subset of \( \mathbb{N} \) and a partial continuous function from \( \mathbb{R} \) to \( \mathbb{R} \). \( X \) positively fails to be continuous in the sense of Bishop if and only if there exist real numbers \( x, y \) such that \( x \leq y \) and \([x, y]\) is a real subset of the domain of \( X \) and \( X \) positively fails to be uniformly continuous on \([x, y]\).

Corollary 9.11 The following statements are equivalent (in BIM):

(i) Kleene’s alternative to the Fan Theorem.

(ii) There exists \( \phi \in \mathcal{N} \) enumerating a uniformly continuous function from \([0, 1]\) to \( \mathbb{R} \) such that, for every \( x \) in \([0, 1]\), \( \phi(x) > 0 \), and the function \( x \mapsto \frac{1}{\phi(x)} \) positively fails to be uniformly continuous on \([0, 1]\).

(iii) There exist \( \phi, \psi \in \mathcal{N} \), both of them enumerating a partial continuous function from \( \mathbb{R} \) to \( \mathbb{R} \) that is continuous in the sense of Bishop, while \( C[\phi, \psi] \) enumerates a partial continuous function from \( \mathbb{R} \) to \( \mathbb{R} \) that positively fails to be continuous in the sense of Bishop.

Proof The proof is left to the reader.

10 Concluding remarks

10.1 More equivalents of the Fan Theorem and of Kleene’s Alternative to the Fan Theorem

In [45, 49] and [50], some further equivalents of the Fan Theorem and also of Kleene’s Alternative are given. We mention some of them.
First, there are contrapositions of various axioms of countable choice. The first such results occur in [35], chapter 15, and [36]. Some contrapositions of an axiom of countable choice turn out to be equivalent to the Fan Theorem, and some positive failures of such contrapositions turn out to be equivalent to Kleene’s Alternative.

Secondly, the Fan Theorem is equivalent to certain statements concerning the intuitionistic determinacy of finite and infinite games, and Kleene’s Alternative is equivalent to positive failure of such determinacy in certain cases. The Intuitionistic Determinacy Theorem involved here is discussed in [48]. An early version of the Theorem occurs in [35], chapter 16.

Thirdly, the Fan Theorem turns out to be equivalent to a Uniform Contrapositive Intermediate Value Theorem, and Kleene’s Alternative proves to be equivalent to the positive failure of this theorem. Note that the Intermediate Value Theorem, as it is usually formulated, is not constructively valid.

Fourthly, the Fan Theorem turns out to be equivalent to the (contrapositive) Compactness Theorem for Classical Propositional Logic, and Kleene’s Alternative is equivalent to the positive failure of this theorem.

Fifthly, using the $\Pi^0_1$-First Axiom of Countable Choice as an extra axiom, one may prove that the Fan Theorem is equivalent to the following statement:

For all open subsets $G_0, G_1$ of $C$, if, for every $a$ in $C$, either $a^0$ belongs to $G_0$ or $a^1$ belongs to $G_1$, then either $G_0 = C$ or $G_1 = C$.

Finally, the Fan Theorem is equivalent to the approximate version of Brouwer’s own Fixed-Point-Theorem, see [49]. The approximate version of the Fixed-Point-Theorem was formulated and proved already by Brouwer himself, see [11] and [12]. The equivalence proof is an intuitionistic adaptation of an argument due to V.P. Orevkov, see [30]. This argument is also used by N. Shioji and K. Tanaka, see [31], and by Simpson, see [32], Theorem IV.7.7. Kleene’s Alternative proves to be equivalent to the positive failure of an approximate version of Brouwer’s Fixed-Point-Theorem, see also [49].

10.2 The Fan Theorem and the almost-Fan Theorem

Theorem 9.6 states that, in BIM, the Fan Theorem is equivalent to the statement that every explicitly totally bounded closed-and-separable subset of $N$ is compact. It follows from the Weak $\Pi^0_1$-First Axiom of Countable Choice that every totally bounded closed-and-separable subset of $N$ is explicitly totally bounded. Therefore, if we extend BIM with this axiom, the Fan Theorem becomes equivalent to the statement that every totally bounded closed-and-separable subset of $N$ is compact.

One may feel some surprise at this fact, as we learn from [32] that in the context of classical second order arithmetic König’s Lemma is definitely stronger than Weak König’s Lemma. König’s Lemma turns out to be equivalent to the so-called Arithmetical Comprehension Scheme.

In some sense, the rôle of König’s Lemma in the classical context may turn out to be more on a par with the rôle of the Almost-Fan Theorem, introduced in [40] and [41], in the intuitionistic context.
A decidable subset \( A \) of \( \mathbb{N} \) is called *almost-finite* if and only if every strictly increasing \( \gamma \) in \( \mathcal{N} \) contains an element of the complement of \( A \). Almost-finite subsets of \( \mathbb{N} \) are introduced and studied in \([37,38]\) and \([44]\).

Let \( \mathcal{F} \) be a spread. \( \mathcal{F} \) is called an *almost-fan* if and only if, for each \( n \), the set all \( s \) in \( \mathbb{N} \) such that length \( (s) = n \) and \( s \) contains an element of \( \mathcal{F} \) is almost-finite.

The (unrestricted) Almost-Fan Theorem, introduced in \([40]\) and \([41]\), is the following statement:

*Let \( \mathcal{F} \) be an almost-fan. Let \( B \) be subset of \( \mathcal{N} \) that is a bar in \( \mathcal{F} \). There exists a subset \( B' \) of \( B \) that is an almost-finite subset of \( \mathbb{N} \) and a bar in \( \mathcal{F} \).*

We often consider the restriction of this statement to the case that \( B \) is a decidable subset of \( \mathbb{N} \).

Like the Fan Theorem itself, the Almost-Fan Theorem is a consequence of Brouwer’s Thesis on Bars. The Fan Theorem may be derived from the Almost-Fan Theorem, as follows:

*Let \( B \) be a bar in \( \mathcal{C} \).*

Applying the Almost-Fan Theorem we find an almost-finite set \( B' \) that is a subset of \( B \) and a bar in \( \mathcal{C} \).

Let us define the statement \( QED \), \(`quod est demonstrandum, what has to be proved` rather than `quod erat demonstrandum, what we had to prove`) as follows:

\( QED := \) there exists \( n \) such that every binary sequence of length \( n \) has an initial part in \( B' \).

Note that, for each \( n \), there are \( 2^n \) binary sequence numbers of length \( n \) and we may decide if there exists a binary sequence number of length \( n \) that does not have an initial part in \( B' \), or not. Using the Minimal Axiom of Choice, we find \( \gamma \) in \( \mathcal{N} \) such that, for each \( n \), \( \gamma(n) \) is a binary sequence number of length \( n \), and, if there exists a binary sequence number of length \( n \) that does not have an initial part belonging to \( B' \), then \( \gamma(n) \) is such a binary sequence number. Using again the Minimal Axiom of Choice, we determine a sequence \( \delta \) in \( \mathcal{N} \) such that, for each \( n \), \( \delta(n) \) is the least \( s \) in \( B' \) such that \( s \) is an initial part of \( \gamma(n) \).

Note that, if, for some \( n \), length \( (\delta(n)) \leq n \), then \( \delta(n) \subseteq \gamma(n) \) and every binary sequence number of length \( n \) has an initial part belonging to \( B' \) and: \( QED \).

So, for each \( n \), either length \( (\delta(n)) > n \) or \( QED \).

We now let \( \varepsilon \) be an element of \( \mathcal{N} \) such that \( \varepsilon(0) = \langle \rangle \) and, for each \( n \), if length \( (\delta(\text{length}(\varepsilon(n)))) \) > length \( (\varepsilon(n)) \), then \( \varepsilon(n + 1) = \delta(\text{length}(\varepsilon(n))) \), (note that, in this case, \( \varepsilon(n + 1) \) belongs to \( B' \)), and, if not, then \( \varepsilon(n + 1) = \varepsilon(n) \ast \langle 0 \rangle \), (note that, in this case, \( QED \)).

So, for each \( n \), either \( \varepsilon(n + 1) \) belongs to \( B' \) or \( QED \).

Also note that, for each \( n \), length \( (\varepsilon(n + 1)) > \text{length}(\varepsilon(n)) \). It follows that, for each \( i, j \), if \( i < j \), then \( \varepsilon(i) \neq \varepsilon(j) \).

Using the fact that \( B' \) is almost-finite, find \( n \) such that \( \varepsilon(n + 1) \) does not belong to \( B' \).

Conclude: \( QED \).

Recall that a subset \( X \) of \( \mathcal{C} \) is an open subset of \( \mathcal{C} \) if and only there is an enumerable subset \( Y \) of \( \mathbb{N} \) such that for every \( \alpha \) in \( \mathcal{C} \), \( \alpha \) belongs to \( X \) if and only if, for some \( n, \alpha n \) belongs to \( Y \).
Let $\mathcal{X}$ be a subset $[0, 1]$. $\mathcal{X}$ is an open subset of $[0, 1]$ if and only if there is an enumerable subset $Y$ of $\mathbb{S}$ such that for every real number $\alpha$, $\alpha$ belongs to $\mathcal{X}$ if and only if, for some $s$ in $Y$, $\alpha$ belongs to $s$.

Let $\mathcal{X}$ be a subset $[0, 1]$. $\mathcal{X}$ is called a progressive subset of $[0, 1]$ if and only if, for every $x$ in $[0, 1]$, if every $y$ in $[0, 1]$ with the property $y \prec x$ belongs to $\mathcal{X}$, then $x$ itself belongs to $\mathcal{X}$.

For all $\alpha, \beta$ in $\mathbb{N}$, we define: $\alpha <_{\mathbb{N}} \beta$ if and only if there exists $n$ such that $\alpha n = \beta n$ and $\alpha(n) < \beta(n)$, and: $\alpha \leq_{\mathbb{N}} \beta$ if and only if, for all $n$, if $n$ is the least $i$ such that $\alpha(i) \neq \beta(i)$, then $\alpha(n) < \beta(n)$.

Let $\mathcal{X}$ be a subset $C$. $\mathcal{X}$ is called a progressive subset of $C$ if and only if, for every $\alpha$ in $C$, if every $\beta$ in $C$ with the property $\beta <_{\mathbb{N}} \alpha$ belongs to $\mathcal{X}$, then $\alpha$ itself belongs to $\mathcal{X}$.

Thierry Coquand has shown that the following two statements may be proved from Brouwer’s principle of induction on monotone bars in Baire space $\mathcal{N}$, see [40,41] and [42].

(i) Every progressive open subset of $C$ coincides with $C$.

(ii) Every progressive open subset of $[0, 1]$ coincides with $[0, 1]$.

We call the statements: the Principle of Open Induction on Cantor space $C$ on the unit interval $[0, 1]$.

In BIM, the two statements are equivalent.

It turns out, that, in BIM, the Almost-Fan Theorem implies the Principle of Open Induction on Cantor space, see [51]. In fact, the Almost-Fan Theorem might be a statement somewhat stronger than the Principle of Open Induction on Cantor space. In [51] we introduce the Strong Fan Theorem.

A decidable subset $A$ of $\mathbb{N}$ is called bounded-in-number if and only if there exists $n$ such that, for each $m$, the number of elements of the set $\{i \in A | i \leq m\}$ is at most $n$.

If a decidable subset $A$ of $\mathbb{N}$ is bounded-in-number, then $A$ is almost-finite, but the converse fails.

Let $\mathcal{F}$ be a spread. $\mathcal{F}$ is called an approximate fan if and only if, for each $n$, the set all $s$ in $\mathbb{N}$ such that $\text{length}(s) = n$ and $s$ contains an element of $\mathcal{F}$ is bounded-in-number.

The (unrestricted) Approximate-Fan Theorem or (unrestricted) Strong Fan Theorem is the following statement:

Let $\mathcal{F}$ be an approximate fan. Let $B$ be subset of $\mathcal{N}$ that is a bar in $\mathcal{F}$. There exists a subset $B'$ of $B$ that is an almost-finite subset of $\mathbb{N}$ and a bar in $\mathcal{F}$.

We often consider the restriction of this statement to the case that $B$ is a decidable subset of $\mathbb{N}$.

The Approximate-Fan Theorem stands to the Fan Theorem in intuitionistic reverse mathematics as König’s Lemma stands to Weak König’s Lemma in classical reverse mathematics.

The argument just given that proved the Fan Theorem from the Almost-Fan Theorem, also shows that the Fan Theorem is a consequence of the Strong Fan Theorem.

It is not so difficult to see that the Fan Theorem follows from the Principle of Open Induction on Cantor space $C$: 

Let $\beta$ be an element of $C$ such that $D_\beta$ is a bar in $C$. Let $X$ be the set of all $\alpha$ in $C$ such that, for some $n$, every $\gamma$ in $C$ with the property $\gamma \leq_N \alpha$ has an initial part in $D_{\beta n}$, $X$ is progressive and therefore coincides with $C$. In particular, the sequence $1$ belongs to $X$ and there exists $n$ such that every $\gamma$ in $C$ has an initial part in $D_{\beta n}$.

### 10.3 An axiom weaker than the Fan Theorem

The following statement is probably weaker than the Fan Theorem, but not provable in BIM.

\begin{itemize}
  \item (\ast) \textit{Every bounded enumerable continuous real function from $[0, 1]$ to $\mathbb{R}$ is Riemann-integrable.}
\end{itemize}

One should compare this statement to the principle \textit{Weak Weak König’s Lemma} in [32], page 397, X.1.7. For every finite subset $X$ of $\mathbb{N}$ we let $X'$ be the set of all binary sequence numbers that belong to $X$ but have no proper initial part than belongs to $X$, and we define: $\mu(X) := \sum_{s \in X'} 2^{-\text{length}(s)}$. The Weak Weak Fan Theorem should be formulated as follows:

\begin{itemize}
  \item For every $\beta$ in $C$, if $D_\beta$ is a bar in $C$, then, for all $m$, there exists $n$ such that $\mu(D_{\beta n}) > 1 - \frac{1}{2^m}$.
\end{itemize}

(The double adjective ‘weak’ in the name of the Weak Weak Fan Theorem is due to the fact that the Fan Theorem (not for fans in general, but for $C$ only) used to be called the Weak Fan Theorem, in analogy to the name ‘Weak König’s Lemma’ in classical reverse mathematics. The name ‘Weak Fan Theorem’ does not seem to be appropriate, after all, but we kept the name ‘Weak Weak Fan Theorem’, in order to avoid misunderstandings.)

The Weak Weak Fan Theorem is studied in [29].

One may prove that, in BIM, the statement (\ast) is equivalent to the Weak Weak Fan Theorem.

The statement:

\begin{itemize}
  \item (\diamond) \textit{Every enumerable continuous real function from $[0, 1]$ to $\mathbb{R}$ is Riemann-integrable}
\end{itemize}

is equivalent to the Fan Theorem, as the reader will understand after having had a look at Corollary 9.8. The statement (\diamond) follows from Corollary 9.8(xiv) and implies Corollary 9.8(xvii).

### 10.4 Intermediate reverse mathematics

In this paper, we studied Intuitionistic Reverse Mathematics. We did not think about non-intuitionistic principles, like Markov’s Principle, Weak König’s Lemma, or some of the omniscience principles introduced by Bishop, although some current research concerns such extensions of intuitionistic mathematics, see [20]. One might call this field \textit{intermediate} reverse mathematics, as it explores the many kind of things in between classical and intuitionistic reverse mathematics. If one pursues this subject, it seems that one, unlike the intuitionistic mathematician, does not put classical analysis
into question. One just wants to know how much unconstructivity one needs in order to establish its ‘results’.

10.5 Continuing intuitionistic reverse mathematics

The Fan Theorem is just one of the axioms of intuitionistic analysis. Further research in Intuitionistic Reverse mathematics should concern Brouwer’s Continuity Principle and Brouwer’s Thesis on Bars in \( \mathcal{N} \). Some results proven from these principles may be found in [39,48] and [44].

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