Hadrons of Arbitrary Spin and Heavy Quark Symmetry

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Abstract

We present a general construction of the spin content of the Bethe-Salpeter amplitudes (covariant wave functions) for heavy hadrons with arbitrary orbital excitations, using representations of $L \otimes O(3,1)$. These wave functions incorporate the symmetries manifest in the heavy quark limit. In the baryonic sector we clearly differentiate between the $\Lambda$ and $\Sigma$-type excited baryons. We then use the trace formalism to

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evaluate the weak transitions of ground state heavy hadrons to arbitrary excited heavy hadrons. The contributions of excited states to the Bjorken sum rule are also worked out in detail.
1 Introduction

Recent advances in heavy quark physics have not only led to the development of the heavy quark effective theory [HQET] [1]-[3] but also have contributed to a better understanding of relativistic wave functions (Bethe-Salpeter amplitudes) of both heavy and light hadrons [4]-[6]. In our previous work, in the mesonic sector, we have used the Bethe-Salpeter formulation to derive the consequences of the heavy quark symmetry for weak transitions involving both s-wave and p-wave mesons [3]. In [3], we developed the B-S approach to heavy quark symmetry for s-wave baryons. In this paper we extend the B-S approach to present covariant wavefunctions for heavy hadrons of arbitrary spin. We then derive explicit expressions for the current induced transitions of s-wave heavy hadrons to arbitrary spin heavy hadrons, in both the mesonic and baryonic sectors. In [7] wavefunctions for arbitrary spin states have also been proposed. The approach followed in [7] is different from the present work as the wave functions are developed there explicitly from the heavy quark spinor with subsequent projections to particular spin states, whereas we construct the wave functions directly in terms of the polarisation tensors (generalised Rarita-Schwinger spinors in the case of baryons) of the hadrons using representations of the Lorentz group. Although in the mesonic sector the wave functions look different one can recast the forms proposed in [7] into the simpler forms given in the present work. During the course of this work we became aware that heavy meson Bethe-Salpeter wave functions have been also constructed in [8] albeit following a different approach. Their wave functions are equivalent to the ones developed here from considering representations of the Lorentz group.

As far as the heavy baryon wave functions are concerned, in [7] only a certain subset of the possible baryonic states are constructed. This is an artifact of the manner of construction which involves taking the product of a totally symmetric tensor of rank \( j \) with the heavy quark spinor. Such states do not exhaust the possible baryonic excited states. As we shall see in section 3 on baryons other symmetry types are also possible. These are discussed in detail in the present work. We present a systematic approach towards constructing any \( \Lambda \)-type or \( \Sigma \)-type excited resonance.

In this work we consider hadrons as bound states of quarks and antiquarks and we concentrate on mesons and baryons containing one heavy quark \( Q \).
The natural object to describe a bound state is the Bethe-Salpeter amplitude which we will define for mesons as

$$M_{\alpha \beta}(x_1, x_2) = \langle 0 | T \psi_{Q\alpha}(x_1) \bar{\psi}_{Q\beta}(x_2) | M, P \rangle,$$  \hspace{1cm} (1.1)

where $| M, P \rangle$ is a mesonic state with a certain momentum $P$, spin and other quantum numbers.

Similarly, for heavy baryons the corresponding BS amplitude is defined as

$$B_{\alpha\beta\gamma}(x_1, x_2, x_3) = \langle 0 | T \psi_{q\alpha}(x_1) \psi_{q\beta}(x_2) \psi_{Q\gamma}(x_3) | B, P \rangle.$$  \hspace{1cm} (1.2)

Here $| B \rangle$ represents a particular baryon state. In eqs. (1.1) and (1.2), $\psi_Q$ represents the heavy quark field while the $\psi_q$'s represent the light quark fields.

As is by now very well known, in the heavy quark mass limit, $m_Q \to \infty$, in the leading order of the HQET, the heavy quark spin is decoupled from the light degrees of freedom and as a consequence the B-S amplitudes satisfy the Bargmann-Wigner(Dirac) equation on the heavy label (in momentum space):

$$(\gamma - 1)_{\alpha}^\delta M_{\delta \beta}(p_1, p_2) = 0$$  \hspace{1cm} (1.3)

and

$$(\gamma - 1)_{\gamma}^\delta B_{\alpha\beta\gamma}(p_1, p_2, p_3) = 0,$$  \hspace{1cm} (1.4)

where $\nu$ is the four velocity of the hadron. These equations imply that the heavy meson and heavy baryon B-S amplitudes can be written in momentum space as

$$M_{\alpha \beta}(p_1, p_2) = \chi_{\alpha}^\delta(p_1, p_2) A_{\delta \beta}(p_1, p_2)$$  \hspace{1cm} (1.5)

and

$$B_{\alpha\beta\gamma}(p_1, p_2, p_3) = \chi_{\rho\delta\gamma}(p_1, p_2, p_3) A_{\alpha\beta\gamma}(p_1, p_2, p_3),$$  \hspace{1cm} (1.6)

where the $\chi$'s are projection operators satisfying

$$(\gamma - 1)_{\alpha}^{\alpha'} \chi_{\alpha'}^\delta = (\gamma - 1)_{\gamma}^{\gamma'} \chi_{\rho\delta\gamma'} = 0.$$  \hspace{1cm} (1.7)

They project out particular spin and parity states from the orbital wave functions $A$. The Dirac indices on these operators indicate that these transform as product representations of the Lorentz group, whereas the momentum arguments indicate that they also have specific transformation properties under the 4-dimensional rotation group. A particular spin parity projection
operator corresponds to reducing such a product representation down to its required spin content. This can be done in an elegant manner through the use of the Bargmann-Wigner wave functions and by a careful consideration of the Lorentz group. We will do this in detail in the next two sections for mesons and baryons. Although the projection operators developed here are used in the context of heavy hadrons, they have nothing to do, per se, with heavy quark physics. In fact, one can use them also in studying light hadrons . After all, we are just constructing representations of the Lorentz group, albeit representations which are particularly suited to the heavy quark limit.

The consequences of eqs. (1.5) and (1.6) have been worked out for s-wave to s-wave and s-wave to p-wave heavy meson semileptonic decays in and for heavy baryon s-wave to s-wave semileptonic decays in and . In sections 4 and 5 we will consider semileptonic decays of ground state heavy hadrons to arbitrary heavy orbital excitations. We also calculate the contributions of these excitations to the Bjorken sum rules. We mention that our covariant description is also well suited to calculate one-pion and photon transitions between heavy hadrons of the same flavour .

2 Wave Functions of Heavy Mesons of Arbitrary Spin

First let us consider the mesonic case. Essentially we would like to discover how reduces under the Lorentz group, . Now the infinitesimal generators of Lorentz transformations acting on are

\[ J_{\mu\nu} = S_{\mu\nu} + i \sum_{j=1}^{2} (p_{j\mu} \frac{\partial}{\partial p_{j\nu}} - p_{j\nu} \frac{\partial}{\partial p_{j\mu}}) , \]

(2.1)

where are the generators of the Lorentz group acting on the product space \([(\frac{1}{2},0) \oplus (0, \frac{1}{2})] \otimes [(\frac{1}{2},0) \oplus (0, \frac{1}{2})]\) or

\[ S_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} \otimes 1 + 1 \otimes \frac{1}{2} \sigma_{\mu\nu} , \]

(2.2)

with the acting on the Dirac labels and of . Therefore, has to be reduced in terms of representations of . It is easy to
show that the $O(3, 1)$ generators can be written as

$$\sum_{j=1}^{2} (p_{j\mu} \frac{\partial}{\partial p_{j}^{\nu}} - p_{j\nu} \frac{\partial}{\partial p_{j}^{\mu}}) = (P_{\mu} \frac{\partial}{\partial P_{\nu}} - P_{\nu} \frac{\partial}{\partial P_{\mu}}) + (k_{\mu} \frac{\partial}{\partial k_{\nu}} - k_{\nu} \frac{\partial}{\partial k_{\mu}}),$$

(2.3)

where $P = p_1 + p_2$ and $k = p_1 - p_2$.

Hence

$$J_{\mu\nu} = S_{\mu\nu} + i(k_{\mu} \frac{\partial}{\partial k_{\nu}} - k_{\nu} \frac{\partial}{\partial k_{\mu}}) + i(P_{\mu} \frac{\partial}{\partial P_{\nu}} - P_{\nu} \frac{\partial}{\partial P_{\mu}}).$$

(2.4)

The last term in eq. (2.4) describes the overall orbital angular momentum of the centre of mass of the system with respect to an external origin. The first two terms give the genuine total internal angular momentum operator of the $Q\bar{q}$ system,

$$M_{\mu\nu} = S_{\mu\nu} + L_{\mu\nu},$$

(2.5)

with

$$L_{\mu\nu} = i(k_{\mu} \frac{\partial}{\partial k_{\nu}} - k_{\nu} \frac{\partial}{\partial k_{\mu}}).$$

(2.6)

The “spin”, i.e. the total angular momentum of the $Q\bar{q}$ pair is described by the Pauli-Lubanski operator

$$W_{\mu} = \frac{1}{2} \epsilon_{\mu\nu\kappa\lambda} P_{\nu} J_{\kappa\lambda} = \frac{1}{2} \epsilon_{\mu\nu\kappa\lambda} P_{\nu} M_{\kappa\lambda}.$$

(2.7)

Because of the $\epsilon_{\mu\nu\kappa\lambda}$ the overall orbital angular momentum operator drops out, as expected, and we are left with the relative internal orbital momentum $L_{\mu\nu}$ plus $S_{\mu\nu}$. The square of the Pauli-Lubansky operator gives the magnitude of the “total spin” of the system

$$W^2 = -M^2 J(J + 1),$$

(2.8)

where $M^2 = P^2$ is the invariant mass of the system.

It is easy now to look at the $S_{\mu\nu}$ and orbital parts separately. In the rest frame, $P = (M, \vec{Q})$,

$$W^\mu = M(0, M_{23}, M_{31}, M_{12}) = M(0, J_1, J_2, J_3),$$

(2.9)

with

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

(2.10)
and
\[ J_i = \epsilon_{ijk} (S_{jk} + L_{jk}) , = S_i + L_i , \] (2.11)
where the \( S_i \) are the usual spin matrices and the \( L_i \) are the orbital angular momenta. We have thus reduced the group from \( \mathcal{L} \otimes O(3,1) \) to \( SU(2) \otimes O(3) \). As expected, the total angular momentum is given by the usual vector addition of the \( Q\bar{q} \) spin and the relative orbital angular momentum. The operators \( S_{\mu\nu} \) and \( L_{\mu\nu} \) (or \( S_i, L_i \)) act on different spaces. So we have to separately find the appropriate irreducible representations of \( \mathcal{L} \) and \( O(3,1) \).

The reduction of multispinor representations of the Lorentz group to their irreducible \( SU(2) \) components was first done by Bargmann and Wigner \[11\] and was later worked out in detail in the 60’s by Delbourgo, Salam and Strathdee \[14\],\[15\], and other workers \[16\] in the context of relativistic \( SU(6) \). Of course such reductions have nothing to do, per se, with relativistic \( SU(6) \).

This is a point emphasized in a modern review of the procedure as given in \[17\].

The basic approach is very simple. The four component Dirac indices of a multispinor \( \chi_{\alpha_1...\alpha_n}^{\beta_1...\beta_m}(p_i) \) are reduced to two components by imposing the Bargmann-Wigner equations on each of the upper and lower indices
\[ (\gamma - 1)_{\alpha}^{\alpha_1} \chi_{\alpha_1...\alpha_n}^{\beta_1...\beta_m} = 0 \]
\[ \vdots \]
\[ (\gamma - 1)_{\alpha}^{\alpha_n} \chi_{\alpha_1...\alpha_n}^{\beta_1...\beta_m} = 0 \] (2.12)
and
\[ \chi_{\alpha_1...\alpha_n}^{\beta_1...\beta_m}(\gamma + 1)_{\beta_1}^{\beta} = 0 \]
\[ \vdots \]
\[ \chi_{\alpha_1...\alpha_n}^{\beta_1...\beta_m}(\gamma + 1)_{\beta_m}^{\beta} = 0. \] (2.13)

The \( \chi \)'s are spinor functions of the total momentum, \( P = \sum p_i \), or in modern usage the total velocity, \( v = P/M \), and the relative momenta \( k_i \). Also following current usage, we have written the Bargmann-Wigner equations in terms of the operators \( (\gamma \pm 1) \) rather than \( (P \pm M) \).

Imposing the equations (2.12) and (2.13) reduces the multispinor to a product representation of \( SU(2) \). We then use the charge conjugation matrix
$C$ to lower the upper indices. Then choosing a given symmetry of the indices we pick out a particular irreducible representation of $SU(2)$.

Following such a procedure for the mesonic projector in eq. (1.3) leads to the following general form

$$\chi_{\alpha \beta}(v, k) = \sqrt{M}[1 + \frac{v^2}{2}\Gamma(k)\frac{1 - \gamma_5}{2}]_{\alpha \beta},$$

(2.14)

where $\Gamma(k)$ is a positive parity matrix. The $\sqrt{M}$ factor is a normalization factor which we have introduced here for later convenience when we go to heavy hadrons, where one wants to factor out the heavy mass scale. The overall parity of $\chi$ is negative and is fixed by the Bargmann-Wigner equations [18].

Now to take the direct product of $L \otimes O(3, 1)$ we require $\Gamma(k)$ to transform appropriately under $L_{\mu \nu}$, eq. (2.6). To do this we go to the rest frame ($v = (1, 0)$), and look for the appropriate eigenstates of $\vec{L}^2$ with $\vec{L}$ defined as in eq. (2.11) or more specifically

$$L_1 = i(k_2 \frac{\partial}{\partial k_3} - k_3 \frac{\partial}{\partial k_2}),$$

(2.15)

Thus for s-waves, we require $\vec{L}^2 \Gamma(k) = 0$ which means that $\Gamma(k)$ can at most depend on $k$ in the form $k^2_\perp$, where

$$k^\mu_\perp = k^\mu - v \cdot k v^\mu,$$

(2.16)

such that

$$v \cdot k_\perp = 0.$$  

(2.17)

Now expanding $\Gamma$ in terms of the 16 gamma matrices we find that they all reduce to just two non-vanishing terms (as $v^2 = 1$),

$$\Gamma(k) = \gamma_5 \epsilon_5$$

(2.18)

or

$$\Gamma(k) = \gamma^\mu \epsilon_\mu,$$

(2.19)

with $v^\mu \epsilon_\mu = 0$. Eq. (2.18) describes the 0$^-$ bound state of a $Q\bar{q}$ pair. $\epsilon_5$ is the “polarisation” pseudoscalar. (For a point particle this would just be

\[2\text{For pseudoscalars and scalars we shall omit these } \epsilon \text{'s later. In general we could have an unknown function of } k^2_\perp \text{ in eqs. (2.18) and (2.19) but these can be absorbed in the unknown functions } A^\delta_\alpha \text{ in eq. (1.3)}.\]
the pseudoscalar field e.g. the pion field.) Eq. (2.19) describes the $1^{-}$ bound state with $\epsilon_{\mu}$ identified as the polarisation vector.

For the p-waves we require

$$\vec{L}^2 \Gamma(k) = 2\Gamma(k). \quad (2.20)$$

This is easily solved by

$$\Gamma(k) = k_\perp \cdot \vec{\Gamma}(k_\perp^2), \quad (2.21)$$

where $\vec{\Gamma}_\mu$ can at most be a vector function of the Lorentz scalar $k_\perp^2$. $\vec{\Gamma}_\mu$ is also a bispinor in Dirac space. Observe that the vector $k_\perp^\mu$ satisfying $v \cdot k_\perp = 0$ has the correct degrees of freedom to describe an angular momentum one object. Constructing all possible forms we get the p-wave reduced spin wave functions $\vec{\Gamma}_\mu$ listed in Table 1. This table is reproduced from [6]. Note that we cannot use $v_\mu$ in the construction as this annihilates on $k_\perp$ in eq. (2.21).

| state | JPC    | $\vec{\Gamma}_\mu$           |
|-------|--------|-------------------------------|
| $^3P_0$ | 0++   | $\gamma_\mu/\sqrt{3}$       |
| $^3P_1$ | 1++   | $-\gamma_5 \sigma^{\nu}_{\mu} \epsilon_\nu$ |
| $^3P_2$ | 2++   | $\gamma_\nu \epsilon_\nu^\mu$ |
| $^1P_1$ | 1+-   | $\gamma_5 \epsilon_\mu$      |

Table 1: Reduced spin wave functions $\vec{\Gamma}$ for mesonic p-wave states.

$\epsilon_\mu$ and $\epsilon_{\mu\nu}$ are the spin 1 and spin 2 polarisation tensors satisfying $v^\mu \epsilon_\mu = v^\mu \epsilon_{\mu\nu} = 0$ with $\epsilon_{\mu\nu}$ symmetric and traceless. The Bargmann-Wigner equations ensure that

$$\frac{1 + \not{v}}{2} \cdot \vec{\Gamma} \frac{1 - \not{v}}{2} = 0. \quad (2.22)$$

The above states correspond to an LS coupling scheme. In [6] we have shown how this can be transformed to another basis, the jj basis, more suitable for heavy quark physics, i.e. where the spin of the light quark and the orbital angular momentum are added first to get the total light quark angular momentum with respect to the heavy quark. The heavy quark spin is added

$^3$Again these $\vec{\Gamma}_\mu$'s can, in general, be multiplied by unknown functions of $k_\perp^2$ which will be absorbed in the $A_{\alpha}^\delta$ in eq. (1.5). This remark applies to all subsequent constructions of higher spin meson and baryon projection operators in the rest of this paper and will not be repeated again.
later. We will come back to the jj basis when we consider the general L-wave case.

The covariant wave functions, eqs. (2.14) and (2.21), with $\bar{\Gamma}$ as in Table 1, for p-waves were first constructed for heavy $\mathcal{Q}\bar{\mathcal{Q}}$ quarkonium systems [19]. However, we now see that these have nothing intrinsically to do with heavy quarkonium systems but are rather consequences of the reduction of $\mathcal{L} \otimes O(3,1)$.

We can now proceed to construct projection operators for d-waves by looking for solutions of

$$\vec{L}^2 \Gamma(k) = 6 \Gamma(k). \quad (2.23)$$

This is solved immediately by

$$\Gamma(k) = (k_\perp^\mu k_\perp^\nu - \frac{1}{3} k_\perp^2 g_\perp^{\mu\nu}) \bar{\Gamma}_\mu\nu, \quad (2.24)$$

where

$$g_\perp^{\mu\nu} = g^{\mu\nu} - v^\mu v^\nu. \quad (2.25)$$

$\bar{\Gamma}_{\mu\nu}$ is a tensor spinor such that

$$\frac{1 + y^\nu v^\mu \bar{\Gamma}_{\mu\nu}}{2} - \frac{1 - y^\nu}{2} = 0. \quad (2.26)$$

Thus, all the states $2^{-+}, 3^{--}, 2^{--}, 1^{--}$ arising from $(0,1) \otimes 2$ of $0(3)_{spin} \otimes 0(3)_{orbital}$ can be immediately listed as in Table 2.

| state | JPC | $\bar{\Gamma}_{\mu\nu}$ |
|-------|-----|-------------------------|
| $^3D_1$ | 1$^{--}$ | $\sqrt{\frac{2}{3}} \gamma_\mu \epsilon_\nu$ |
| $^3D_2$ | 2$^{--}$ | $-\sqrt{\frac{2}{3}} i \gamma_5 \sigma^\lambda \gamma_\mu \epsilon_{\lambda\nu}$ |
| $^3D_3$ | 3$^{--}$ | $\gamma^\lambda \epsilon_{\lambda\mu}$ |
| $^1D_2$ | 2$^{++}$ | $\gamma_5 \epsilon_{\mu\nu}$ |

Table 2: Reduced spin wave functions $\bar{\Gamma}$ for mesonic d-wave states.
The $\epsilon$'s are the usual symmetric traceless polarisation tensors satisfying $v^\mu \epsilon_\mu = v^\nu \epsilon_\nu = v^\rho \epsilon_\rho = 0$. These are normalised to $2M$ with respect to the traceless tensor $\frac{1}{2} (g_1^{\mu \nu} g_1^{\nu \rho} + g_1^{\mu \rho} g_1^{\nu \nu} - \frac{2}{3} g_1^{\mu \nu} g_1^{\nu \rho})$ (See Appendix A).

One can now immediately generalise to any higher orbital angular momentum. For the general orbital angular momentum $L$ case we need the solution to

$$\vec{L}^2 \Gamma(k) = L(L+1) \Gamma(k). \quad (2.27)$$

This is solved by

$$\Gamma(k) = N_{\mu_1 \mu_2 \ldots \mu_L} (k) \bar{\Gamma}_{\mu_1 \mu_2 \ldots \mu_L}. \quad (2.28)$$

$N_{\mu_1 \mu_2 \ldots \mu_L} (k)$ is the symmetric, traceless tensor product of $L k_\perp$'s satisfying

$$v_\mu N_{\mu_1 \ldots \mu_L} (k) = 0 \quad (2.29)$$

for all $i$. Specifically

$$N_{\mu_1 \mu_2 \ldots \mu_L} (k) = k_{\perp 1} k_{\perp 2} \ldots k_{\perp L} - \frac{k_{\perp 1}^2}{2L-1} \sum_{i<j} g_{\perp i j} \prod_{n \neq i, j} k_{\perp n}^4 \rightleftharpoons \frac{k_{\perp 1}^4}{(2L-1)(2L-3)} \sum_{i<j, i<k<l} g_{\perp i j} g_{\perp k l} \prod_{n \neq i, j, k, l} k_{\perp n}^4. \quad (2.30)$$

We also have

$$\frac{1 + \sqrt{2} v_\mu \bar{\Gamma}_{\mu_1 \ldots \mu_L} \Gamma_{\mu_1 \ldots \mu_L}^\dagger \frac{1 - \sqrt{2}}{2} = 0. \quad (2.31)$$

for all $i$.

In table 3 we list the $\bar{\Gamma}_{\mu_1 \mu_2 \ldots \mu_L}$ for the states $L = 1, L, L = 1; L$ arising from $(1, 0) \otimes L$ of $O(3)$ spin $\otimes O(3)$ orbital.

Here the $\epsilon$'s are the usual fully symmetric, traceless, transverse polarisation tensors. See Appendix A for the normalisation of these wave functions.

Having constructed the projection operators we now recall that the B-S amplitudes for the heavy meson are obtained by substituting these operators in eq. (1.5). Using Tables 2 and 3, we can now easily consider heavy to heavy semi-leptonic decays involving d- and higher waves. This will be done in the next section.

The wave functions as developed up to now are quite general and can be used in the context of both heavy and light mesons. However as pointed out in [8], in the case of heavy mesons it is convenient to use a different basis
Table 3: Reduced spin wave functions $\bar{\Gamma}$ for mesonic L-wave states.

| state   | $\mathbf{JPC}$ | $\Gamma_{\mu_1...\mu_L}$                        |
|---------|----------------|-------------------------------------------------|
| $^3L_{L-1}$ | $(L - 1)(-)^{L+1}(-)^{L+1}$ | $\frac{\sqrt{2L-1}}{2L+1}\gamma_{\mu_1} \epsilon_{\mu_2...\mu_L}$ |
| $^3L_L$    | $L(-)^{L+1}(-)^{L+1}$ | $-\sqrt{\frac{L}{L+1}}i\gamma_5 \sigma_{\lambda\mu_1} \epsilon_{\lambda\mu_2...\mu_L}$ |
| $^3L_{L+1}$ | $(L + 1)(-)^{L+1}(-)^{L+1}$ | $\gamma_{\lambda\mu_1...\mu_L}$ |
| $^1L_L$    | $L(-)^{L+1}$ | $\gamma_5 \epsilon_{\mu_1...\mu_L}$ |

Table 3: Reduced spin wave functions $\bar{\Gamma}$ for mesonic L-wave states.

than that used for the states in Tables 2 and 3. These states are in the LS coupling scheme and are eigenstates of $C$-parity. We need to move to the $jj$ scheme. In this scheme, for heavy mesons, the degenerate states in the d-wave case are the pairs $(1\frac{3}{2}, 2\frac{3}{2})$ and $(2\frac{5}{2}, 3\frac{3}{2})$ where the subscripts $3/2$ and $5/2$ indicate the total angular momentum of the light quark system $(2 \otimes 1/2 = 5/2 \oplus 3/2)$. The two states in each pair are related to each other by flipping the heavy quark spin i.e. they are degenerate because of the heavy quark spin symmetry. This can be demonstrated by applying the spin flip operator $\gamma_5 \epsilon_{\lambda\mu_1...\mu_L}$ to the appropriate $\bar{\Gamma}$. The $1^-$ and $3^-$ states are unchanged whereas the $2^-$ states in this new basis are linear combinations of the LS states

\[
|2_{-\frac{3}{2}}\rangle = \sqrt{\frac{3}{5}}|2_{-}\rangle + \sqrt{\frac{2}{5}}|2_{+}\rangle
\]
\[
|2_{-\frac{5}{2}}\rangle = -\sqrt{\frac{2}{5}}|2_{-}\rangle + \sqrt{\frac{3}{5}}|2_{+}\rangle.
\]  

(2.32)

In Table 4, we list the $\bar{\Gamma}$'s for the appropriate degenerate states of the d-wave heavy meson.

Similarly in the general case one has to go to the heavy quark basis states $|L_{L-1/2}^{(-)L+1}\rangle$ and $|L_{L+1/2}^{(-)L+1}\rangle$. Recall that total light quark angular momentum is
Table 4: The d-wave heavy meson reduced spin wave functions

| state      | $\Gamma_{\mu\nu}$                      |
|------------|----------------------------------------|
| $1_{-3/2}$ | $\sqrt{\frac{2}{3}} \gamma_\mu \epsilon_\nu$ |
| $2_{-3/2}$ | $\sqrt{\frac{2}{5}} \gamma_5 \gamma^\lambda \gamma_\mu \epsilon_{\lambda\nu}$ |
| $2_{-5/2}$ | $\frac{1}{\sqrt{15}} \gamma_5 (5 g_\mu^\lambda - 2 \gamma^\lambda \gamma_\mu) \epsilon_{\lambda\nu}$ |
| $3_{-5/2}$ | $\gamma_\lambda \epsilon_{\lambda\mu\nu}$ |

either $L - 1/2$ or $L + 1/2$. Thus

\[
| L^{(-)}_{L+1/2} \rangle = \sqrt{\frac{L+1}{2L+1}} | L^{(-)}_{L+1} \rangle + \sqrt{\frac{L}{2L+1}} | L^{(-)}_{L} \rangle \\
| L^{(-)}_{L+1} \rangle = -\sqrt{\frac{L}{2L+1}} | L^{(-)}_{L+1} \rangle + \sqrt{\frac{L+1}{2L+1}} | L^{(-)}_{L} \rangle.
\]

In table 5, we list the $\bar{\Gamma}$'s for the appropriate pairs of degenerate states of the L-wave heavy meson.

Tables 2-5 are some of the main results of this paper. Notice the simplicity and elegance of the wave functions. Heavy meson wave functions have also been constructed by Falk [7] following a different route, which always explicitly involves the heavy quark spinor and always requires projection operators to go to particular states. Notice that in our construction there is no sign of the heavy quark spinor. Our construction is directly in terms of the appropriate polarisation tensors of the heavy meson state itself. The forms of the wave functions for heavy mesons given in [7] can be transformed into the forms given above.

In the above we have listed just the “pure” spin states characterized by particular orbital angular momentum. Of course physical states could in general be mixtures of these states preserving the heavy quark spin symmetry.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
state & $\Gamma_{\mu_1 \ldots \mu_L}$ \\
\hline
$(L - 1)_{L-1/2}^{(-)_{L+1}}$ & $\sqrt{\frac{2L-1}{2L+1}} \gamma_{\mu_1} \epsilon_{\mu_2 \ldots \mu_L}$ \\
$L_{L-1/2}^{(-)_{L+1}}$ & $\sqrt{\frac{L}{2L+1}} \gamma_5 \gamma_{\mu_1} \epsilon_{\lambda \mu_2 \ldots \mu_L}$ \\
$L_{L+1/2}^{(-)_{L+1}}$ & $\frac{1}{\sqrt{(L+1)(2L+1)}} \gamma_5 ((2L + 1) g_{\mu_1}^\lambda - L \gamma_{\lambda \gamma_{\mu_1}}) \epsilon_{\lambda \mu_2 \ldots \mu_L}$ \\
$(L + 1)_{L+1/2}^{(-)_{L+1}}$ & $\gamma_{\lambda \epsilon_{\lambda \mu_1 \ldots \mu_L}}$ \\
\hline
\end{tabular}
\caption{The L-wave heavy meson reduced spin wave functions}
\end{table}

3 Heavy Baryon Wave Functions of Arbitrary Orbital Angular Momentum

We repeat the procedure of sec. 2 for the baryon spin projector $\chi_{\alpha \beta \gamma}(p_1, p_2, p_3)$ in eq. (1.6). Here $p_1, p_2, p_3$ are the quark momenta with $P = M v = p_1 + p_2 + p_3$. We construct the s-, p- and d-wave baryon projection operators in detail and then indicate how to generalise to an arbitrary orbital excitation. Recall that the $\chi_{\alpha \beta \gamma}$ satisfy the Bargmann-Wigner equations and that the heavy baryon B-S amplitude is obtained from eq. (1.6) by substituting with the appropriate $\chi$.

Here the infinitesimal generators of the Lorentz transformations acting on $\chi_{\alpha \beta \gamma}(p_1, p_2, p_3)$ are

$$J_{\mu \nu} = S_{\mu \nu} + i \sum_{j=1}^{3} (p_{j \mu} \frac{\partial}{\partial p_{j \nu}} - p_{j \nu} \frac{\partial}{\partial p_{j \mu}}),$$

where $S_{\mu \nu}$ are the generators of the Lorentz group acting on the space $[(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})]^3$ or

$$S_{\mu \nu} = \frac{1}{2} \sigma_{\mu \nu} \otimes 1 \otimes 1 + 1 \otimes \frac{1}{2} \sigma_{\mu \nu} \otimes 1 + 1 \otimes 1 \otimes \frac{1}{2} \sigma_{\mu \nu}.$$

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with the \( S_{\mu\nu} \) acting on the Dirac labels \( \alpha, \beta \) and \( \gamma \). As before we reduce \( \chi_{\alpha\beta\gamma} \) in terms of representations of \( \mathcal{L} \otimes O(3,1) \).

Defining two independent relative four momenta as

\[
k_3 = \frac{1}{2}(p_1 - p_2)
\]

and

\[
K_3 = \frac{1}{3}(p_1 + p_2 - 2p_3)
\]

we can write

\[
J_{\mu\nu} = S_{\mu\nu} + L_{\mu\nu}^{(k_3)} + L_{\mu\nu}^{(K_3)} + i(P_{\nu} \frac{\partial}{\partial P_{\nu}} - P_{\mu} \frac{\partial}{\partial P_{\mu}}).
\]

Here

\[
L_{\mu\nu}^{(k)} = i(k_{\mu} \frac{\partial}{\partial k_{\nu}} - k_{\nu} \frac{\partial}{\partial k_{\mu}}).
\]

\( L_{\mu\nu}^{(k)} \) is the angular momentum operator for the relative orbital angular momentum of the quark pair, \( q_1q_2 \), while \( L_{\mu\nu}^{(K)} \) is the angular momentum operator for the relative orbital angular momentum between the centre of mass (c.m.) of the pair \( q_1q_2 \) and the third quark \( q_3 \). Of course, one could also decompose \( J_{\mu\nu} \) in terms of \( k_1, k_2 \) and \( K_1, K_2 \) where

\[
k_1 = \frac{1}{2}(p_2 - p_3), \quad k_2 = \frac{1}{2}(p_3 - p_1)
\]

and

\[
K_1 = \frac{1}{3}(p_2 + p_3 - 2p_1), \quad K_2 = \frac{1}{3}(p_3 + p_1 - 2p_2).
\]

However out of the six momenta, \( k_i \) and \( K_i \), only two, plus the total momentum \( P \), are independent and we choose \( k_3 \) and \( K_3 \). Recall that the third quark momenta \( p_3 \) is associated with the Dirac label \( \gamma \).

As in the mesonic case the last term in eq. (3.3) describes the overall orbital angular momentum of the c.m. of the system with respect to an external origin and drops out of the Pauli-Lubansky operator. Thus

\[
W_\mu = \frac{1}{2} \epsilon_{\mu\nu\kappa\lambda} P^\nu J^{\kappa\lambda} = \frac{1}{2} \epsilon_{\mu
u\kappa\lambda} P^\nu M^{\kappa\lambda},
\]

with

\[
M_{\mu\nu} = S_{\mu\nu} + L_{\mu\nu}^{(k_3)} + L_{\mu\nu}^{(K_3)}.
\]
### 3.1 S-wave Baryon Projection Operators

The s-wave baryon projection operators are disposed of quite easily. The \( \chi_{\alpha\beta\gamma} \) for s-waves can only be functions of \( k_{3\perp}^2 \) or \( K_{3\perp}^2 \) and must satisfy the Bargmann-Wigner equations on each label. Thus one can in general have two possibilities \([14, 15, 17]\), either antisymmetric or symmetric in the \( \alpha\beta \) indices:

\[
\chi^\Lambda_{(\alpha\beta)}\gamma = \chi^0_{\alpha\beta}\Psi_{\gamma} \tag{3.11}
\]

or

\[
\chi^\Sigma_{(\alpha\beta)}\gamma = \chi^{1,\mu}_{\alpha\beta}\Psi_{\mu,\gamma} \tag{3.12}
\]

where \( \chi^0_{\alpha\beta} \equiv [(\not_\gamma + 1)\gamma_5 C]_{\alpha\beta} \) and \( \chi^{1,\mu}_{\alpha\beta} = [(\not_\gamma + 1)\gamma^\mu C]_{\alpha\beta} \) with

\[
v^\mu\Psi_\mu = 0 \tag{3.13}
\]

and

\[
(\not_\gamma - 1)\Psi = (\not_\gamma - 1)\Psi_\mu = 0. \tag{3.14}
\]

We have put the superscripts \( \Lambda \) and \( \Sigma \) because we will soon show that these correspond to the projection operators for the \( \Lambda \) and \( \Sigma \)-type baryons respectively. These are the only two independent possibilities because any other like \( [(\not_\gamma + 1)\gamma_5 C]_{\beta,\gamma}\Psi_\alpha \) can always be recast in the forms given in eqs. (3.11) and (3.12) by splitting into \( \alpha\beta \) symmetric and antisymmetric pieces. Specifically

\[
[(\not_\gamma + 1)\gamma_5 C]_{\beta,\gamma}\Psi_\alpha = -\frac{1}{2}[(\not_\gamma + 1)\gamma_5 C]_{\alpha\beta}\Psi_{\gamma} + \frac{1}{2}[(\not_\gamma + 1)\gamma^\mu C]_{\alpha\beta}[(\gamma_\mu + v_\mu)\gamma_5 \Psi]_\gamma
\]

and

\[
[(\not_\gamma + 1)\gamma^\mu C]_{\beta,\gamma}\Psi_{\mu,\alpha} = -\frac{1}{2}[(\not_\gamma + 1)\gamma^\mu C]_{\alpha\beta}\Psi_{\mu,\gamma} + \frac{1}{2}[(\not_\gamma + 1)\gamma_5 C]_{\alpha\beta}[(\gamma_\mu \gamma_5 \Psi_\mu)_{\gamma}]. \tag{3.15}
\]

The meaning of \( \chi^0_{\alpha\beta} \) and \( \chi^{1,\mu}_{\alpha\beta} \) is obvious. \( \chi^0_{\alpha\beta} \) represents the antisymmetric spin zero, \( S_{q_1q_2} = 0 \) state of the quarks 1 and 2, whereas the transverse part of \( \chi^{1,\mu}_{\alpha\beta}, \chi^{1,\mu}_{\perp\alpha\beta}, (v_\mu \chi^{1,\mu}_{\perp\alpha\beta} = 0) \), represents the symmetric spin 1, \( S_{q_1q_2} = 1 \), state of the first two quarks. Because of the relations (3.13) and (3.16) one can

---

\(^4\)From now on we shall often use the usual bracket notation to indicate symmetry properties of indices. Square brackets,\([\],\), around indices will represent antisymmetrisation of the indices enclosed whereas curly brackets,\(\{\},\), around indices indicates symmetrisation.
always express the spin of any pair of quarks in terms of the 1,2 pair. One
notes also that \( \chi^1_{\mu,\perp} = [(\not \! v + 1)(\gamma^\mu - v^\mu)C] \). But since \( v^\mu\Psi_\mu \) in eq. (3.13)
and in subsequent wave functions, one can use interchangeably \( \chi^1_{\mu} \) or \( \chi^1_{\perp} \). However it is important to remember the transversality. In summary, \( \chi^0_{\alpha\beta} \)
and \( \chi^1_{\alpha\beta,\perp} \) represent the 0+ and 1+ spin states of the \( q_1q_2 \) pair.

We can now differentiate the \( \Lambda_Q \) and \( \Sigma_Q \)-type of baryons by considering
flavour symmetry. The \( \Lambda_Q \)-type baryons are antisymmetric in flavour for the
first two quarks. Hence to preserve overall symmetry we see that the \( \chi^\Lambda_{\alpha\beta,\gamma} \)
in eq. (3.11) is the correct, \( \frac{1}{2}^+ \), \( \Lambda \) wave function with
\[
\Psi_\gamma = u_\gamma, \quad (3.17)
\]
the usual Dirac spinor.

Similarly, \( \chi^{\Sigma}_{\{\alpha\beta\},\gamma} \) describes the symmetric \( \Sigma_Q \)-type baryons. Here it is
easy to see that \( \Psi_{\mu,\gamma} \) decomposes uniquely as
\[
\begin{align*}
\frac{3}{2}^+ & : \quad \Psi_{\mu,\gamma} = u_{\mu,\gamma} \\
\frac{1}{2}^+ & : \quad \Psi_{\mu,\gamma} = \frac{1}{\sqrt{3}}(\gamma_{\perp\mu}\gamma_5 u)_\gamma \\
& \quad = \frac{1}{\sqrt{3}}[(v_\mu + \gamma_\mu)\gamma_5 u]_\gamma. \quad (3.18)
\end{align*}
\]
where \( \gamma_{\perp\mu} = \gamma_\mu - \not \! v v_\mu \). The particular form of the spin 1/2 in eq. (3.18) is
required by the conditions (3.14).

### 3.2 P-wave Baryon Projection Operators

We now proceed to construct the projection operators for p-wave baryons.
In contrast to the mesonic case, there are now two ways to get p-waves. Let
us denote by \( L_{k_3} \) the orbital angular momentum of the quark pair \( q_1q_2 \) and
by \( L_{K_3} \) the orbital angular momentum between the centre of mass of the pair
\( q_1q_2 \) and the third quark \( q_3 \). P-wave baryons are obtained by taking either
\( (L_{k_3} = 1, L_{K_3} = 0) \) or \( (L_{k_3} = 0, L_{K_3} = 1) \). Hence we require that \( \chi_{\alpha\beta,\gamma} \) should
be of the form
\[
\chi_{\alpha\beta,\gamma} = k^\mu_{3\perp} \bar{\chi}_{\mu,\alpha\beta,\gamma} \quad \text{or} \quad K^\mu_{3\perp} \bar{\chi}_{\mu,\alpha\beta,\gamma}, \quad (3.19)
\]
with \( \bar{\chi}_{\mu,\alpha\beta,\gamma} \) satisfying the Bargmann-Wigner equations on all the indices
\( \alpha, \beta, \gamma \). Note also that although both \( k_3 \) and \( K_3 \) are of mixed symmetry
(under the interchange of $p_1, p_2, p_3$), $k_3$ is antisymmetric under $p_1 \leftrightarrow p_2$ whereas $K_3$ is symmetric. We construct $\bar{\chi}_{\mu,\alpha\beta\gamma}$ using $\chi_{0,\alpha\beta}$ and $\chi_{1,\mu,\alpha\beta}$ and ensuring overall symmetry with respect to flavour $\otimes$ spin $\otimes$ space for the quarks 1 and 2.

Without worrying about the choice of $k_{3\perp}^\mu$ or $K_{3\perp}^\mu$, it is easy to construct $\bar{\chi}_{\mu,\alpha\beta\gamma}$. With $\chi_{0,\alpha\beta}$ we have simply

$$\bar{\chi}_{\mu,\alpha\beta\gamma} = \chi_{0,\alpha\beta} \Psi_{\mu,\gamma}, \quad (3.20)$$

with $\Psi_{\mu,\gamma}$ satisfying

$$(\gamma' - 1)\Psi_{\mu} = 0 \text{ and } v^\mu \Psi_{\mu} = 0. \quad (3.21)$$

Thus, $\Psi_{\mu}$ decomposes uniquely into the $\frac{3}{2}^-$ and $\frac{1}{2}^-$ states as in eq. (3.18). These two states arise from $(S_{q_1q_2} = 0 \otimes L = 1) \otimes S_{q_3} = \frac{3}{2}, \frac{1}{2}$ where $S_{q_3} = 1/2$ is the spin of the third quark (later we will identify the third quark with the heavy quark). Since $\chi_{0,\alpha\beta}$ is antisymmetric under $\alpha \leftrightarrow \beta$, to ensure overall symmetry for the $\Lambda$-type excitation (flavour antisymmetric) we would have $K_{3\perp}^\mu \bar{\chi}_{\mu,\alpha\beta\gamma}$, i.e. the $L_{K_3} = 1$ orbital angular momentum. On the other hand for the $\Sigma$-type we would have $k_{3\perp}^\mu \bar{\chi}_{\mu,\alpha\beta\gamma}$, i.e. the $L_{K_3} = 1$ orbital angular momentum.

With $\chi_{1,\mu,\alpha\beta}$ we have three possibilities. Here we combine the $S_{q_1q_2} = 1$ with $L = 1$ to give $J_{q_1q_2} = 2, 1, 0$ where $J_{q_1q_2}$ is the total angular momenta of quarks 1 and 2 with respect to the third quark. These three different sets of states can be written as ($k$ is either $k_3$ or $K_3$):

1. $J_{q_1q_2} = 2$

$$\frac{1}{2} (k^\mu_{\perp} \chi_{1,\perp}^{\mu} + k^\nu_{\perp} \chi_{1,\perp}^{\mu} - \frac{2}{3} k_{\perp} \cdot \chi_{\perp} g^{\mu\nu}_{\perp})_{\alpha\beta} \Psi_{\mu,\gamma}, \quad (3.22)$$

with $\Psi_{\mu,\nu,\gamma}$ a symmetric, traceless tensor-spinor satisfying

$$(\gamma' - 1)\Psi_{\mu,\nu} = 0 \text{ and } v^\mu \Psi_{\mu,\nu} = 0. \quad (3.23)$$

5The negative parity is obvious from physical reasons. Also it can be seen from eqs. (3.19) that since $\chi_{\alpha\beta,\gamma}$ is always of positive parity, $\bar{\chi}_{\mu,\alpha\beta,\gamma}$ must necessarily be odd as $k^\mu_{\perp}(K^\mu_{\perp})$ is odd. Thus from eq. (3.20) we see that $\Psi_{\mu,\gamma}$ is also odd since $\chi_{1,\alpha\beta}^{\mu}$ transforms like $0^+$. The parity of the subsequent states eqs. (3.22)-(3.24) is odd for a similar reason since $\chi_{1,\alpha\beta}^{\mu}$ transforms like $1^+$. We have continued to use the usual spinors $u, (u_{\mu}, u_{\mu\nu})$, satisfying the Dirac equation $(\gamma' - 1)u = 0$ but always remembering that they describe negative parity particles. One could as well use $\gamma_5 v$. 

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\(\Psi_{\mu\nu}\) can easily be decomposed as (for normalisations see Appendix A)

\[
\begin{align*}
\frac{5^-}{2} : & \quad \Psi_{\mu\nu} = u_{\mu\nu} \\
\frac{3^-}{2} : & \quad \Psi_{\mu\nu} = \frac{1}{\sqrt{10}}(\gamma_{\perp\mu}\gamma_5 u_{\nu} + \gamma_{\perp\nu}\gamma_5 u_{\mu}).
\end{align*}
\]

(3.24)

(3.25)

Here one could also have a term like \(g_{\mu\nu}^\perp u\). However this would annihilate when folded into the expression (3.22) because the angular momentum tensor of the \(q_1 q_2\) pair is traceless. Such a term will be called a non-propagating mode.

Thus the two nonvanishing structures correspond exactly to the two possibilities expected from \((J_{q_1 q_2} = 2) \otimes (S_{q_3} = 1/2) = \frac{5^-}{2}, \frac{3^-}{2}\). \(u_{\mu\nu}\) is the usual 5/2 generalised, symmetric Rarita-Schwinger tensor-spinor and \(u_{\mu}\) is the 3/2 Rarita-Schwinger spinor.

2. \(J_{q_1 q_2} = 1\)

\[
\frac{1}{2}(k_{\perp}^{\mu}x_{\perp}^{1,\nu} - k_{\perp}^{\nu}x_{\perp}^{1,\mu})\alpha\beta\Psi_{\mu\nu,\gamma}. \quad (3.26)
\]

Here \(\Psi_{\mu\nu,\gamma}\) is an antisymmetric tensor-spinor satisfying eqs. (3.23). \(\Psi_{\mu\nu}\) is easily decomposed as

\[
\begin{align*}
\frac{3^-}{2} : & \quad \Psi_{\mu\nu} = \frac{1}{\sqrt{2}}(\gamma_{\perp\mu}\gamma_5 u_{\nu} - \gamma_{\perp\nu}\gamma_5 u_{\mu}) \\
\frac{1^-}{2} : & \quad \Psi_{\mu\nu} = \frac{1}{2\sqrt{6}}[\gamma_{\perp\mu}, \gamma_{\perp\nu}]u. \quad (3.27)
\end{align*}
\]

These correspond exactly to the two possibilities expected from \((J_{q_1 q_2} = 1) \otimes (S_{q_3} = \frac{3}{2}) = \frac{3^-}{2}, \frac{1^-}{2}\).

3. \(J_{q_1 q_2} = 0\)

\[
\frac{1}{\sqrt{3}}(k_{\perp}^{\mu}x_{\perp}^{1,\nu})\alpha\beta g^\mu_{\perp} u_{\nu}\Psi_{\mu\nu,\gamma}. \quad (3.28)
\]

Again \(\Psi_{\mu\nu}\) is a traceless, symmetric tensor satisfying eqs. (3.23). Here \(\Psi_{\mu\nu}\) decomposes uniquely as

\[
\frac{1^-}{2} : \quad \Psi_{\mu\nu} = \frac{1}{\sqrt{3}}g_{\perp\mu\nu} u. \quad (3.29)
\]

This state corresponds to the one state expected from \((J_{q_1 q_2} = 0) \otimes (S_{q_3} = \frac{1}{2}) = \frac{1^-}{2}\).
The Lorentz structure of the states given by the eqs. (3.22)-(3.29) is valid for both \( \Lambda_q \) and \( \Sigma_q \) type baryons. Since these wave functions (3.22)-(3.29) are symmetric under \( \alpha \leftrightarrow \beta \), to get \( \Lambda \)-type baryons we take \( k = k_3 \) and for \( \Sigma \)-type we take \( k = K_3 \) in these equations to ensure overall symmetry.

### 3.3 Baryon Projection Operators for the First Positive Parity Excitations

There are three possibilities in constructing the first positive parity excitations. In the subsequent equations of this subsection, \( L \) refers to the total orbital angular momentum of the quark pair, \( q_1 \) and \( q_2 \). By first positive parity excitations we mean all those excitations with \( L_{\mathbf{K}3} + L_{\mathbf{k}3} = 2 \).

1. \[ L_{\mathbf{k}3} = 2, L_{\mathbf{K}3} = 0, L = 2 \]
   Such an orbital state is represented by the symmetric traceless tensor \( N^{\mu_1 \mu_2}(k_3) \) as defined in eq. (2.30).

2. \[ L_{\mathbf{k}3} = 0, L_{\mathbf{K}3} = 2, L = 2 \]
   Such an orbital state is represented by the symmetric traceless tensor \( N^{\mu_1 \mu_2}(K_3) \).

3. \[ L_{\mathbf{k}3} = 1, L_{\mathbf{K}3} = 1 \]
   \[ (a) L = 2 \]
   Here the orbital state is represented by the tensor
   \[
   N^{\{\mu_1, \mu_2\}}(K_3, k_3) = \frac{1}{2}(K_3^{\mu_1}k_3^{\mu_2} + K_3^{\mu_2}k_3^{\mu_1} - \frac{2}{3}K_3^{\mu_1} \cdot k_3^{\mu_2}).
   \]

   \[ (b) L = 1 \]
   Here the orbital state is represented by the tensor
   \[
   N^{[\mu_1, \mu_2]}(K_3, k_3) = \frac{1}{2}((K_3^{\mu_1}k_3^{\mu_2} - K_3^{\mu_2}k_3^{\mu_1}) + K_3^{\mu_1} \cdot k_3^{\mu_2}).
   \]

   \[ (c) L = 0 \]
   Here the orbital state is represented by the scalar
   \[
   N(K_3, k_3) = \frac{1}{\sqrt{3}}K_3 \cdot k_3 g_3^{\mu_1 \mu_2}.
   \]

The orbital states (1) and (2) are symmetric under interchange of \( p_1 \) and \( p_2 \) while the orbital states in (3) are antisymmetric under the same interchange. Using these symmetry properties we now construct the projection operators for the \( \Lambda \)-type and \( \Sigma \)-type baryons.
3.3.1 Λ-type first positive parity excitation projection operators

The Λ-type baryon is in a flavour antisymmetric state for the quarks labelled 1 and 2. Because of the requirement of overall symmetry, the orbital states (1) and (2) above can only combine with the antisymmetric spin zero, \( S_{q_1q_2} = 0 \), state of the quarks 1 and 2, i.e. with \( \chi^0_{\alpha\beta} \), leading to the total light quark angular momentum of \( J_{q_1q_2} = 2 \). This in turn leads to the total baryon angular momentum states \( \frac{5}{2}^+ \) and \( \frac{3}{2}^+ \). The physical states will in general be mixtures of the states arising from the two orbital states (1) and (2). We list these projection operators below:

(1) \( L_{K_3} = 2, L_{k_3} = 0, L = 2 \) or (2) \( L_{K_3} = 0, L_{k_3} = 2, L = 2 \)

In both cases we have

\[
N^{\mu_1\mu_2} \chi^0_{\alpha\beta} \Psi_{\mu_1\mu_2,\gamma},
\]

where \( N^{\mu_1\mu_2} \) is either \( N^{\mu_1\mu_2}(K_3) \) or \( N^{\mu_1\mu_2}(k_3) \). \( \Psi_{\mu_1\mu_2} \) decomposes into its \( \frac{5}{2}^+ \) and \( \frac{3}{2}^+ \) components just as in eqs. (3.23) and (3.24), (3.25),

\[
\begin{align*}
\frac{5}{2}^+ & : \quad \Psi_{\mu_1\mu_2} = u_{\mu_1\mu_2} \\
\frac{3}{2}^+ & : \quad \Psi_{\mu_1\mu_2} = \frac{1}{\sqrt{10}} (\gamma_{\perp\mu_1} \gamma_5 u_{\mu_2} + \gamma_{\perp\mu_2} \gamma_5 u_{\mu_1}) .
\end{align*}
\]

In contrast, the orbital states (3), being antisymmetric with respect to the interchange of \( p_1 \) and \( p_2 \), can only combine with the symmetric, spin one, \( S_{q_1q_2} = 1 \), state of the quarks 1 and 2, i.e. with \( \chi^1_{\mu\nu} \). These will give rise to the states whose projection operators are discussed in the rest of this subsection. All the \( \Psi \)s which are about to appear are transverse with respect to \( v_\mu \) and satisfy the Dirac equation. Furthermore they are traceless in the vector labels. The symmetry of these labels will be specified as we proceed.

(3) (a) \( L_{K_3} = 1, L_{k_3} = 1, L = 2 \)

The total orbital angular momentum, \( L = 2 \), here combines with the spin, \( S_{q_1q_2} = 1 \), to give total light quark angular momenta \( J_{q_1q_2} = 3, 2, 1 \), for which the projection operators are listed below.

(i) \( J_{q_1q_2} = 3 \)

\[
\frac{1}{3} \left[ (\gamma_{\perp\mu} N^{\mu_1\mu_2}) - \frac{2}{5} g_{\mu\nu} N^{\mu_1\nu} \right] + \text{symmetrised in } \mu, \mu_1, \mu_2 |_{\alpha\beta} \Psi_{\mu_1\mu_2,\gamma}
\]

(3.36)
with $\Psi_{\mu_1\mu_2\gamma}$ fully symmetric. $\Psi_{\mu_1\mu_2}$ can easily be decomposed as

\[
\begin{align*}
\frac{7}{2}^+: \quad \Psi_{\mu_1\mu_2} &= u_{\mu_1\mu_2} \tag{3.37} \\
\frac{5}{2}^+: \quad \Psi_{\mu_1\mu_2} &= \frac{1}{\sqrt{21}} (\gamma_{\perp\mu} \gamma_5 u_{\mu_1\mu_2} + \gamma_{\perp\mu_1} \gamma_5 u_{\mu_2\mu} + \gamma_{\perp\mu_2} \gamma_5 u_{\mu_1\mu_2}) \\
&= \frac{1}{\sqrt{3}} \gamma_{\perp\mu} \gamma_5 u_{\mu_1\mu_2} , \tag{3.38}
\end{align*}
\]

where in the last step we have used the fact that $\Psi_{\mu_1\mu_2}$ is folded into a symmetric tensor in eq. (3.36). (There are, of course, as usual non-propagating modes just as in the p-wave case. From now we shall just ignore them.) The two non-vanishing states correspond exactly to the two expected from $(J_{q_1q_2} = 3) \otimes (S_{q_2} = \frac{1}{2}) = \frac{7}{2}^+, \frac{5}{2}^+$. $u_{\mu_1\mu_2}$ and $u_{\mu_1\mu_2}$ are the generalised $\frac{7}{2}$ and $\frac{5}{2}$ Rarita-Schwinger spinors.

\begin{align*}
(ii) \quad J_{q_1q_2} &= 2 \\
\frac{1}{3} [x_{\perp\mu}^1 N^{\{\mu_1\mu_2\}} - x_{\perp\mu_1}^1 N^{\{\mu_2\}} &- \frac{1}{2} g_{\perp\mu_2}^\mu \chi_{\perp\nu} N^{\{\nu\mu_1\}} + \frac{1}{2} g_{\perp\mu_1}^\mu \chi_{\perp\nu} N^{\{\nu\mu_2\}}]_{\alpha\beta} \Psi_{[\mu_1\mu_2],\gamma} , \tag{3.39}
\end{align*}

with $\Psi_{[\mu_1\mu_2],\gamma}$ mixed symmetric. $\Psi_{[\mu_1\mu_2]}$ can easily be decomposed as (see Appendix B for the construction and Appendix A for the normalisation)

\[
\begin{align*}
\frac{5}{2}^+: \quad \Psi_{[\mu_1\mu_2]} &= \frac{1}{\sqrt{2}} (\gamma_{\perp\mu} \gamma_5 u_{\mu_1\mu_2} - \gamma_{\perp\mu_1} \gamma_5 u_{\mu_2\mu_2}) \tag{3.40} \\
\frac{3}{2}^+: \quad \Psi_{[\mu_1\mu_2]} &= \frac{1}{3 \sqrt{5}} \left\{ (\gamma_{\perp\mu_1} \gamma_{\perp\mu_2}) u_{\mu_2} + (\gamma_{\perp\mu} \gamma_{\perp\mu_2}) u_{\mu_1} - \gamma_{\perp\mu_1} \gamma_{\perp\mu_2} u_{\mu} \right\} . \tag{3.41}
\end{align*}
\]

\begin{align*}
(iii) \quad J_{q_1q_2} &= 1 \\
[x_{\perp\mu}^1 N^{\{\nu\mu_2\}}]_{\alpha\beta} \Psi_{\mu_2,\gamma} . \tag{3.42}
\end{align*}

$\Psi_{\mu_2}$ can easily be decomposed as

\[
\begin{align*}
\frac{3}{2}^+: \quad \Psi_{\mu_2} &= u_\mu_2 \tag{3.43} \\
\frac{1}{2}^+: \quad \Psi_{\mu_2} &= \frac{1}{\sqrt{3}} \gamma_{\perp\mu_2} \gamma_5 u . \tag{3.44}
\end{align*}
\]
Here the total angular momentum of quarks 1 and 2 is either 2, 1 or 0, i.e. $(L = 1) \otimes (S_{q_1 q_2} = 1) = (J_{q_1 q_2} = 2, 1, 0)$. The appropriate projectors are listed below. (See Appendix B for details of the construction)

(i) $J_{q_1 q_2} = 2$

\[
\begin{align*}
[x_{\perp}^{\mu_1} N^{\mu_2}] - \frac{1}{2} g_{\perp}^{\mu_1} x_{\perp}^{\nu_2} N^{\nu_1} + \frac{1}{2} g_{\perp}^{\mu_2} x_{\perp}^{\nu_1} N^{\nu_2}]_{\alpha \beta} \Psi_{[\mu_1 \mu_2], \gamma},
\end{align*}
\]

(3.45)

with $\Psi_{[\mu_1 \mu_2], \gamma}$ mixed symmetric. It can easily be decomposed as

\[
\begin{align*}
\frac{5^+}{2} & : \Psi_{\mu_1 \mu_2} = u_{\mu_2} \\
\frac{3^+}{2} & : \Psi_{\mu_1 \mu_2} = \frac{1}{3\sqrt{6}} \{[\gamma_{\perp \mu_1}, \gamma_{\perp \mu_2}] u_{\mu_1} + (\gamma_{\perp \mu_1}, \gamma_{\perp \mu_2} u_{\mu_2} - \gamma_{\perp \mu_2} \gamma_{\perp \mu_1} u_{\mu_1}) \}.
\end{align*}
\]

(3.46)

(3.47)

Note that these $\frac{5^+}{2}$ and $\frac{3^+}{2}$ are slightly different from the corresponding states in (3)(a)(ii), eqs. (3.40 and 3.41). For the details of this construction, please see the Appendix B.

(ii) $J_{q_1 q_2} = 1$

\[
(\chi_{\perp}^{\mu_1} N^{[\mu_2]}_{\nu_1 2})_{\alpha \beta} \Psi_{\mu_2, \gamma}
\]

(3.48)

$\Psi_{\mu_2}$ can easily be decomposed as

\[
\begin{align*}
\frac{3^+}{2} & : \Psi_{\mu_2} = u_{\mu_2} \\
\frac{1^+}{2} & : \Psi_{\mu_2} = \frac{1}{\sqrt{3}} \gamma_{\perp \mu_2} \gamma_5 u.
\end{align*}
\]

(3.49)

(3.50)

(iii) $J_{q_1 q_2} = 0$

\[
\begin{align*}
\frac{1^+}{2} : \Psi_{[\mu_1 \mu_2]} = \frac{1}{6\sqrt{6}} \{[\gamma_{\perp \mu_1}, \gamma_{\perp \mu_1}] \gamma_{\perp \mu_2} + [\gamma_{\perp \mu_1}, \gamma_{\perp \mu_2}] \gamma_{\perp \mu_1} + [\gamma_{\perp \mu_2}, \gamma_{\perp \mu_1}] \gamma_{\perp \mu_1} \} \gamma_5 u \\
&= \frac{1}{\sqrt{6}} \gamma_{\perp \mu_1} \gamma_{\perp \mu_2} \gamma_5 u. 
\end{align*}
\]

(3.51)

(3.52)

(3.53)
In the last step we have used the fact that $\Psi_{[\mu_1\mu_2]}$ is folded into a fully antisymmetric tensor in eq. (3.51).

(3) (c) $L_{K_3} = 1, L_{k_3} = 1, L = 0$

Here the $L = 0$ orbital angular momentum of the quarks 1 and 2 combines with the $S_{q_1q_2} = 1$ spin state to give a total angular momentum $J_{q_1q_2} = 1$ for the quark pair. The projection operator is

$$ (K_{3\perp} \cdot k_{3\perp} \chi_{\perp}^{1,\mu})_{\alpha\beta} \Psi_{\mu,\gamma}. $$  \hspace{1cm} (3.54)

$\Psi_\mu$ decomposes in the usual manner as

$$ \begin{align*}
\frac{3}{2}^+ : & \quad \Psi_\mu = u_\mu \\
\frac{1}{2}^+ : & \quad \Psi_\mu = \frac{1}{\sqrt{3}} \gamma_\mu \gamma_5 u. \hspace{1cm} (3.55) \\
\frac{1}{2}^- : & \quad \Psi_\mu = \frac{1}{\sqrt{3}} \gamma_\mu \gamma_5 u. \hspace{1cm} (3.56)
\end{align*} $$

We have now come to the end of enumerating the projection operators for the first positive parity excited $\Lambda$-type baryons.

### 3.3.2 $\Sigma$-type first positive parity excitation projection operators

These are in some sense easier to write down. Here because the flavour state of quarks 1 and 2 is symmetric, the symmetric orbital angular momentum $L = 2$ states (1) and (2) above combine with the symmetric spin one state, $S_{q_1q_2} = 1, \chi_{\alpha\beta}^{1,\mu}$, to preserve overall symmetry. This gives rise to the total angular momenta, $J_{q_1q_2} = 3, 2, 1$, of the quark pair $q_1$ and $q_2$ with respect to the third quark. We treat cases (1) and (2) together. In contrast the antisymmetric orbital states (3), with total orbital angular momentum $L = 2, 1$ or 0, arising from $L_{K_3} = 1, L_{k_3} = 1$ necessarily combine with the antisymmetric spin zero state to give total $J_{q_1q_2} = 2, 1$ or 0.

\[ \begin{array}{ccc}
(1) & L_{K_3} = 2, L_{k_3} = 0, L = 2 & (2) L_{K_3} = 0, L_{k_3} = 2, L = 2 \\
(1) & (a) J_{q_1q_2} = 3 & \\
\end{array} \]

$$ \frac{1}{3}[(\chi_{\perp}^{1,\mu} N^{\mu_1\mu_2} - 2/5 g_{\perp}^{\mu_1} \chi_{\perp}^{1,\mu} N^{\nu\mu_2}) + \text{symmetrised in } \mu, \mu_1, \mu_2]_{\alpha\beta} \Psi_{\mu_1\mu_2,\gamma}, $$  \hspace{1cm} (3.57)

with $\Psi_{\mu_1\mu_2,\gamma}$ fully symmetric. It can easily be decomposed as

$$ \begin{align*}
\frac{7}{2}^+ : & \quad \Psi_{\mu_1\mu_2} = u_{\mu_1\mu_2} \\
\frac{3}{2}^- : & \quad \Psi_{\mu_1\mu_2} = u_{\mu_1\mu_2} \hspace{1cm} (3.58)
\end{align*} $$

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\[ \frac{5^+}{2}: \quad \Psi_{\mu_1\mu_2} = \frac{1}{\sqrt{21}} \gamma_{\mu_1\gamma_5} u_{\mu_1\mu_2} \]
\[ = \sqrt{\frac{3}{7}} \gamma_{\mu_1\mu_2} \cdot \]  (3.59)

Again in the last step we have used the fact that in eq. (3.57) \( \Psi_{\mu_1\mu_2} \) is traced into a fully symmetric tensor.

(1) \( J_{q_1 q_2} = 2 \)

\[ \frac{5^+}{3} \left[ \gamma^{1_{\mu_1}} N^{\mu_1\mu_2_1} - \gamma^{1_{\mu_1}} N^{\mu_1\mu_2} - \frac{1}{2} g^{\mu_1\mu_2_1} N^{\nu_{\mu_1}} + \frac{1}{2} g^{\mu_1\mu_2_1} N^{\nu_{\mu_2}} \right]_{\alpha\beta} \Psi_{[\mu_1\nu_1],\nu_2,\gamma} \]  (3.60)
with \( \Psi_{[\mu_1\nu_1],\nu_2,\gamma} \) mixed symmetric. As usual it decomposes as (see Appendix B for construction)

\[ \frac{5^+}{2}: \quad \Psi_{[\mu_1\nu_1],\nu_2} = \frac{1}{\sqrt{2}} (\gamma_{\mu_1\gamma_5} u_{\mu_1\nu_2} - \gamma_{\nu_1\gamma_5} u_{\mu_1\nu_2}) \]  (3.61)

\[ \frac{3^+}{2}: \quad \Psi_{[\mu_1\nu_1],\nu_2} = \frac{1}{3\sqrt{5}} \left\{ (\gamma_{\mu_1,\gamma_5} u_{\mu_2} + (\gamma_{\mu_1\gamma_5} u_{\mu_1} - \gamma_{\mu_1\gamma_5} u_{\mu_2}) \right\} . \]  (3.62)

(1) \( J_{q_1 q_2} = 1 \)

\[ [\chi_{\nu_{\mu_1}} N^{\nu_{\mu_2}}]_{\alpha\beta} \Psi_{\mu,\gamma} , \]  (3.63)
with \( \Psi_{\mu,\gamma} \) being written as usual as

\[ \frac{5^+}{2}: \quad \Psi_{\mu} = u_{\mu} \]  (3.64)

\[ \frac{1^+}{2}: \quad \Psi_{\mu} = \frac{1}{\sqrt{3}} \gamma_{\mu_5} u . \]  (3.65)

(3) \( L_{K_3} = 1, L_{K_3} = 1 \)

(3) \( (a) L = 2, J_{q_1 q_2} = 2 \)

\[ [\chi^{0_{\mu_1\nu_1}} N^{\mu_1\mu_2}]_{\alpha\beta} \Psi_{\mu_1\nu_2,\gamma} \]  (3.66)
with \( N^{(\mu_1\mu_2)} \) as defined in eq. (3.30). \( \Psi_{\mu_1\nu_2} \) is symmetric and is as usual given by

\[ \frac{5^+}{2}: \quad \Psi_{\mu_1\nu_2} = u_{\mu_1\nu_2} \]  (3.67)

\[ \frac{3^+}{2}: \quad \Psi_{\mu_1\nu_2} = \frac{1}{\sqrt{10}} (\gamma_{\mu_1\gamma_5} u_{\nu_2} + \gamma_{\mu_2\gamma_5} u_{\mu_1}) . \]  (3.68)
\( L = 1, J_{q_1q_2} = 1 \)

\[
(\chi^0 N^{[\mu_1\mu_2]})_{\alpha\beta} \Psi_{[\mu_1\mu_2], \gamma}, \tag{3.69}
\]

with \( N^{[\mu_1\mu_2]} \) as defined in eq. (3.31). \( \Psi_{[\mu_1\mu_2]} \) is antisymmetric and is given by

\[
\frac{3}{2}^+: \quad \Psi_{[\mu_1\mu_2]} = \frac{1}{\sqrt{2}}(\gamma_{\perp\mu_1}\gamma_5 u_{\mu_2} - \gamma_{\perp\mu_2}\gamma_5 u_{\mu_1}) \tag{3.70}
\]

\[
\frac{1}{2}^+: \quad \Psi_{[\mu_1\mu_2]} = \frac{1}{2\sqrt{6}}[\gamma_{\perp\mu_1}, \gamma_{\perp\mu_2}] u. \tag{3.71}
\]

\( L = 0, J_{q_1q_2} = 0 \)

\[
(\chi^0 K_{3\perp} \cdot k_{3\perp})_{\alpha\beta} \Psi_{\gamma}, \tag{3.72}
\]

with \( \Psi \) given by

\[
\frac{1}{2}^+: \quad \Psi = u. \tag{3.73}
\]

### 3.4 \((-1)^L\) Parity Excitation Baryon Projection Operators

By \((-1)^L\) parity excitations we mean all those states with \( L_{K_3} + L_{k_3} = L \). These contain total orbital angular momenta starting from \( L \) down to 1 if \( L \) is odd and 0 if \( L \) is even. The situation for the general \( L \)-wave resonances becomes rather complicated because of the large number of possible states due to the the \( L + 1 \) partitions of \( L \), arising from the choice of the angular momenta \( L_{K_3} \) and \( L_{k_3} \). We will not do the full construction here but just indicate how one goes about it.

Let us take \( L_{k_3} = n \). Then for the \((-1)^L\) parity excitations we must have \( L_{K_3} = L - n \). A particular partition of \( L \), i.e. a particular choice of \( n \), \( 0 < n < L \), signifies taking the direct product of the symmetric tensor \( N^{\mu_1...\mu_n}(k_3) \) with the other symmetric tensor \( N^{\mu_1...\mu_{L-n}}(K_3) \). This direct product can be decomposed into the direct sum of orbital angular momenta ranging from \( L \) to \( |L - 2n| \) corresponding to the various Young tableaux present in the reduction of the product of two fully symmetric Young tableaux. All these different irreducible, traceless product tensors will have symmetry \((-)^n\) under the interchange \( p_1 \leftrightarrow p_2 \), i.e. they will be symmetric if \( n \) is even and...
antisymmetric if \( n \) is odd. Thus one can differentiate the states arising from \( n \) even or odd.

(i)(a) \( n \) even. \( \Lambda \)-type baryon

In this case to preserve overall symmetry of the wave function the irreducible orbital angular momentum tensors would have to be combined with the antisymmetric spin zero, \( S_{q_1q_2} = 0 \), state of the \( q_1q_2 \) pair represented by \( \chi^{0}_{\alpha\beta} \). This gives rise to the following total angular momenta, \( J_{q_1q_2} \), of the quark pair, \( q_1q_2 \), with respect to the third quark,

\[
J_{q_1q_2} = L, L - 1, \ldots |L - 2n| .
\] (3.74)

Then combining these with the spin of the third quark \( S_{q_3} = \frac{1}{2} \) gives the total angular momentum (spin) of the baryon.

(i)(b) \( n \) even. \( \Sigma \)-type baryon

In this case to preserve overall symmetry the orbital angular momentum tensors would have to be combined with the symmetric spin one, \( S_{q_1q_2} = 1 \), state of the quark pair \( q_1q_2 \) represented by \( \chi^{1,\mu}_{\alpha\beta} \). Such an operation will give rise to the following groupings of the total angular momentum of the pair \( q_1q_2 \) with respect to the third quark,

\[
J_{q_1q_2} = (L+1, L, L-1); (L, L-1, L-2); \ldots (|L-2n| +1, |L-2n|, |L-2n|-1) .
\] (3.75)

Again the total angular momentum (spin) of the baryon is obtained by combining finally with \( S_{q_3} = \frac{1}{2} \).

(ii)(a) \( n \) odd. \( \Lambda \)-type baryon

Now the situation is reversed. To preserve overall symmetry the orbital tensors have to be combined with \( \chi^{1,\mu}_{\alpha\beta} \) in the usual manner as in (i)(b) above.

(ii)(b) \( n \) odd. \( \Sigma \)-type baryon

Here the orbital tensors are combined with \( \chi^{0}_{\alpha\beta} \) to preserve symmetry as in (i)(a) above.

All these tensors representing the different values of \( J_{q_1q_2} \) will have \( L \) or \( L + 1 \) indices of various symmetry types. Now to construct the projection operators one has to multiply them into tensor spinors \( \Psi_{\mu_1 \cdots \mu_N \gamma} \) (where \( N \) is either \( L \) or \( L + 1 \)) of the corresponding symmetry. Because of the Bargmann-Wigner equations these must satisfy the Dirac equation and the transversality condition

\[
u^{\mu_1} \Psi_{\mu_1 \cdots \mu_N} = 0
\] (3.76)
on each index.

These are easy to construct, in any particular case, by noting that if $\phi$ satisfies the Dirac equation then so do $\gamma_{\perp\mu}\gamma_5\phi$ and $\gamma_{\perp\mu_1}\gamma_{\perp\mu_2}\phi$ plus obviously being transverse. One then has to use appropriately symmetrised chains of $\gamma_{\perp\mu}\gamma_5$ and/or $\gamma_{\perp\mu_1}\gamma_{\perp\mu_2}$ along with generalised traceless, symmetric Rarita-Schwinger spinors $u_{\mu_1...\mu_n}$ to construct the projection operators.

We have illustrated now how to construct any arbitrary orbital excited baryonic projection operator. As usual they will come in pairs which correspond to the heavy quark basis. Note that in the case $L=2$, treated above, the two partitions $L_{K_3}=2$, $L_{k_3}=0$ and $L_{k_3}=2$ give rise to the same Lorentz structure and to the same kind of states. Similarly from the $L+1$ partitions in the L-wave case only a certain number can give rise to possible different kinds of states. When $L$ is odd there are $(L+1)/2$ partitions giving rise to possible different Lorentz structures, whereas when $L$ is even there are $(L+2)/2$ possibilities.

Writing down all the projection operators in the general case is thus rather cumbersome so we shall not pursue it further. However the highest weight L-wave states are relatively easier to write down. We will not list them here but will show them in the next subsection when we put everything together in the final form of the Bethe-Salpeter amplitudes.

### 3.5 Heavy Baryon Bethe-Salpeter Amplitudes

The heavy baryon B-S amplitudes are now obtained from eq. (1.6) by substituting with the appropriate projection operators developed in subsections 3.1 to 3.3 above. The $B_{\alpha\beta\gamma}$ now satisfy the Bargmann-Wigner equation on one label $\gamma$. We identify the third quark with the heavy quark. Of course, we are listing just the “pure” spin states characterized by definite values of the orbital angular momentum. In general the physical mass eigenstates may be mixtures of states preserving the heavy quark spin symmetry.

#### 3.5.1 S-wave Heavy Baryons

We define the following quantities

$$S^0_{\alpha\beta} = \chi^0_{\delta\rho} A^{\delta\rho}_{\alpha\beta},$$

$$S^\mu_{\alpha\beta} = \chi^1_{\perp\delta\rho} A^{\delta\rho}_{\alpha\beta}. \tag{3.77}$$

27
$S^0$ and $S^\mu$ are the projections of the $0^+$ and $1^+$ total ($J_{q_1,q_2}$) angular momenta of the light quark pair, $q_1,q_2$ with respect to the heavy quark. These are functions of $K_3$ and $k_3$. We can now write the s-wave heavy B-S amplitude in compact notation, dropping the Dirac indices $\alpha, \beta$ and $\gamma$. However remember that the light quark indices are on the $S$’s whereas the heavy quark index $\gamma$ is always on the explicit spinor structure. Thus

$$\Lambda - \text{type} :$$
\[
\frac{1}{2}^+: \quad S^0 u
\]

$$\Sigma - \text{type} :$$
\[
\frac{3}{2}^+: \quad S^\mu u_\mu
\]
\[
\frac{1}{2}^+: \quad S^\mu \frac{1}{\sqrt{3}} \gamma_{\perp} \gamma_5 u.
\]

(3.78)

### 3.5.2 P-wave Heavy Baryons

For the p-wave case, we define the following quantities:

\[
P_{k,\alpha\beta}^\mu = k_{\perp}^\mu \delta_\rho A_{\alpha\beta}^{\delta_\rho},
\]
\[
P_{k,\alpha\beta}^{\{\mu\nu\}} = \frac{1}{2} \left( k_{\perp}^{\mu} \chi_{\perp}^{1,\nu} + k_{\perp}^{\nu} \chi_{\perp}^{1,\mu} - \frac{2}{3} k_{\perp} \cdot \chi_{\perp}^{\mu\nu} \right) \delta_\rho A_{\alpha\beta}^{\delta_\rho},
\]
\[
P_{k,\alpha\beta}^{[\mu\nu]} = \frac{1}{2} \left( k_{\perp}^{\mu} \chi_{\perp}^{1,\nu} - k_{\perp}^{\nu} \chi_{\perp}^{1,\mu} \right) \delta_\rho A_{\alpha\beta}^{\delta_\rho},
\]
\[
P_{k,\alpha\beta} = (k_{\perp} \cdot \chi_{\perp}^{1}) \delta_\rho A_{\alpha\beta}^{\delta_\rho}.
\]

(3.79)

$P_k^\mu, P_k^{\{\mu\nu\}}, P_k^{[\mu\nu]}$ and $P_k$ are the projections of the $1^-, 2^-, 1^-$ and $0^-$ total angular momenta respectively of the light quark pair. Of course these $P$’s are functions of both $k_3$ and $K_3$. The subscript $k$ in the $P$’s is to be understood as a label indicating which $k$ ($k_3$ or $K_3$) we use in the definitions (3.79).

The pairs of states listed below are separately degenerate because of the heavy quark spin symmetry.

**$\Lambda - \text{type}$:**

(i) $L_{K_3} = 1, S_{q_1,q_2} = 0, J_{q_1,q_2} = 1.$

\[
\frac{3}{2}^- : \quad P_{K_3}^\mu u_\mu
\]
\[
\frac{1}{2}^- : \quad P_{K_3}^\mu \frac{1}{\sqrt{3}} \gamma_{\perp} \gamma_5 u.
\]

(3.80)
\[ (ii) \quad L_{k_3} = 1, \quad S_{q_1 q_2} = 1, \quad J_{q_1 q_2} = 2, 1, 0. \]

(a) \[ J_{q_1 q_2} = 2 \]

\[ \begin{align*}
5^- & : \quad P_{k_3}^{(\mu \nu)} u_{\mu \nu} \\
3^- & : \quad P_{k_3}^{(\mu \nu)} \frac{1}{\sqrt{10}} (\gamma_{\perp \mu} \gamma_5 u_\nu + \gamma_{\perp \nu} \gamma_5 u_\mu) \\
\end{align*} \]

(b) \[ J_{q_1 q_2} = 1 \]

\[ \begin{align*}
3^- & : \quad P_{k_3}^{[\mu \nu]} \frac{1}{\sqrt{2}} (\gamma_{\perp \mu} \gamma_5 u_\nu - \gamma_{\perp \nu} \gamma_5 u_\mu) \\
1^- & : \quad P_{k_3}^{[\mu \nu]} \frac{1}{2 \sqrt{6}} [\gamma_{\perp \mu}, \gamma_{\perp \nu}] u \\
\end{align*} \]

(c) \[ J_{q_1 q_2} = 0 \]

\[ \begin{align*}
\frac{1}{2}^- & : \quad P_{k_3} u. \quad (3.83) \\
\end{align*} \]

For the Σ-type heavy baryons one has to simply interchange \( k_3 \) and \( K_3 \) in eqs. (3.80) - (3.83).

### 3.5.3 First Positive Parity Excited Heavy Baryons

**A. First Positive Parity Excited Heavy Λ**

For the first positive parity excited heavy Λ the projections of the possible total angular momenta of the light quark pair are represented by the following matrices:

\[ \begin{align*}
2^+ & \quad D_{k \alpha \beta}^{[\mu_1 \mu_2]} = N^{\mu_1 \mu_2} (k) \chi_\alpha^{(0)} A_\beta^{[\delta \rho]} A_{\alpha \beta}^{0}, \\
3^+ & \quad D_{[\alpha \beta]}^{[\mu_1 \mu_2]} = \frac{1}{2} \left( \chi^{(1)}_{\perp \mu} N^{(\mu_1 \mu_2)} - \frac{1}{2} g^{[\mu_1}_{\perp \mu} \chi^{(1)}_{\perp \nu} N^{(\nu \mu)} + 2 g^{[\mu_1}_{\perp \mu} \chi^{(1)}_{\perp \nu} N^{(\nu \mu)} \right) \delta_{\alpha \beta} A_{\alpha \beta}^{0}, \\
2^+ & \quad D_{[\alpha \beta]}^{[\mu_1 \mu_2]} = \frac{1}{3} \left( \chi^{(1)}_{\perp \mu} N^{(\mu_1 \mu_2)} - \frac{1}{2} g^{[\mu_1}_{\perp \mu} \chi^{(1)}_{\perp \nu} N^{(\nu \mu)} + \frac{1}{2} g^{[\mu_1}_{\perp \mu} \chi^{(1)}_{\perp \nu} N^{(\nu \mu)} \right) \delta_{\alpha \beta} A_{\alpha \beta}^{0}, \\
1^+ & \quad D_{[\alpha \beta]}^{[\mu_1 \mu_2]} = \left( \chi^{(1)}_{\perp \mu} N^{(\mu_1 \mu_2)} + \frac{1}{2} g^{[\mu_1}_{\perp \mu} \chi^{(1)}_{\perp \nu} N^{(\nu \mu)} \right) \delta_{\alpha \beta} A_{\alpha \beta}^{0}, \\
0^+ & \quad D_{[\alpha \beta]}^{[\mu_1 \mu_2]} = \frac{1}{3} \left( \chi^{(1)}_{\perp \mu} N^{(\mu_1 \mu_2)} + \chi^{(1)}_{\perp \mu} N^{(\mu_1 \mu_2)} + \chi^{(1)}_{\perp \mu} N^{(\mu_1 \mu_2)} \right) \delta_{\alpha \beta} A_{\alpha \beta}^{0}, \\
1^+ & \quad D_{T[\alpha \beta]}^{[\mu_1 \mu_2]} = K_{3 \perp} \delta_{\alpha \beta} A_{\alpha \beta}^{0}. \\
\end{align*} \]

(3.84)
With the help of these matrices, we can now write down the B-S amplitudes for the pairs of degenerate first positive parity excited heavy Λ states.

(1) \( L_k^3 = 2, L_k^3 = 0, L = 2, S_{q_1 q_2} = 0, J_{q_1 q_2} = 2 \)

\[
\begin{align*}
\frac{5}{2}^+ : & \quad D^{\mu_1 \mu_2}_{K_3} u_{\mu_1 \mu_2} \\
\frac{3}{2}^+ : & \quad D^{\mu_1 \mu_2}_{K_3} \frac{1}{\sqrt{10}} (\gamma_{\perp \mu_1} \gamma_5 u_{\mu_2} + \gamma_{\perp \mu_2} \gamma_5 u_{\mu_1}) 
\end{align*}
\]

(3.85)

(2) \( L_k^3 = 0, L_k^3 = 2, L = 2, S_{q_1 q_2} = 0, J_{q_1 q_2} = 2 \)

\[
\begin{align*}
\frac{5}{2}^+ : & \quad D^{\mu_1 \mu_2}_{K_3} u_{\mu_1 \mu_2} \\
\frac{3}{2}^+ : & \quad D^{\mu_1 \mu_2}_{K_3} \frac{1}{\sqrt{10}} (\gamma_{\perp \mu_1} \gamma_5 u_{\mu_2} + \gamma_{\perp \mu_2} \gamma_5 u_{\mu_1})
\end{align*}
\]

(3.86)

(3) (a) (i) \( L_k^3 = 1, L_k^3 = 1, L = 2, S_{q_1 q_2} = 1, J_{q_1 q_2} = 3 \)

\[
\begin{align*}
\frac{7}{2}^+ : & \quad D^{(\mu_1 \mu_2)}_{\Lambda} u_{\mu_1 \mu_2} \\
\frac{5}{2}^+ : & \quad D^{(\mu_1 \mu_2)}_{\Lambda} \frac{3}{7} \gamma_{\perp \mu} \gamma_5 u_{\mu_1 \mu_2}
\end{align*}
\]

(3.87)

(3) (a) (ii) \( L_k^3 = 1, L_k^3 = 1, L = 2, S_{q_1 q_2} = 1, J_{q_1 q_2} = 2 \)

\[
\begin{align*}
\frac{5}{2}^+ : & \quad D^{[\mu_1 \mu_2]}_{\Lambda} \frac{1}{\sqrt{2}} (\gamma_{\perp \mu_1} \gamma_5 u_{\mu_1 \mu_2} - \gamma_{\perp \mu_2} \gamma_5 u_{\mu_2} u_{\mu_1}) \\
\frac{3}{2}^+ : & \quad D^{[\mu_1 \mu_2]}_{\Lambda} \frac{1}{3 \sqrt{5}} \{[\gamma_{\perp \mu_1}, \gamma_{\perp \mu_2}] u_{\mu_2} + (\gamma_{\perp \mu_2} \gamma_5 u_{\mu_1} - \gamma_{\perp \mu_1} \gamma_{\perp \mu_2} u_{\mu_1})\}
\end{align*}
\]

(3.88)

(3) (a) (iii) \( L_k^3 = 1, L_k^3 = 1, L = 2, S_{q_1 q_2} = 1, J_{q_1 q_2} = 1 \)

\[
\begin{align*}
\frac{3}{2}^+ : & \quad D^{\mu}_{S} u_{\mu} \\
\frac{1}{2}^+ : & \quad D^{\mu}_{S} \frac{1}{\sqrt{3}} \gamma_{\perp \mu} \gamma_5 u
\end{align*}
\]

(3.89)

(3) (b) (i) \( L_k^3 = 1, L_k^3 = 1, L = 1, S_{q_1 q_2} = 1, J_{q_1 q_2} = 2 \)

\[
\begin{align*}
\frac{5}{2}^+ : & \quad D^{[\mu_1 \mu_2]}_{\Lambda} \frac{1}{\sqrt{2}} (\gamma_{\perp \mu_1} \gamma_5 u_{\mu_1 \mu_2} - \gamma_{\perp \mu_2} \gamma_5 u_{\mu_2} u_{\mu_1}) \\
\frac{3}{2}^+ : & \quad D^{[\mu_1 \mu_2]}_{\Lambda} \frac{1}{3 \sqrt{5}} \{[\gamma_{\perp \mu_1}, \gamma_{\perp \mu_2}] u_{\mu} + (\gamma_{\perp \mu_2} \gamma_5 u_{\mu_1} - \gamma_{\perp \mu_1} \gamma_{\perp \mu_2} u_{\mu_1})\}
\end{align*}
\]

(3.90)
(3) (b) (ii) \( L_{k_3} = 1, L_{k_3} = 1, L = 1, S_{q_1q_2} = 1, J_{q_1q_2} = 1 \)

\[
\begin{align*}
\frac{3^+}{2} : & \quad D^\mu_A u_\mu \\
\frac{1^+}{2} : & \quad D^\mu_A \frac{1}{\sqrt{3}} \gamma_\mu \gamma_5 u
\end{align*}
\] (3.91)

(3) (b) (iii) \( L_{K_3} = 1, L_{K_3} = 1, L = 1, S_{q_1q_2} = 1, J_{q_1q_2} = 0 \)

\[
\begin{align*}
\frac{1^+}{2} : & \quad D^{[\mu_1\mu_2]}_{\Sigma;k_0,\alpha} \frac{1}{6\sqrt{6}} \{ [\gamma_{\mu_1}, \gamma_{\mu_2}] [\gamma_{\mu_1}, \gamma_{\mu_2}] [\gamma_{\mu_1}, \gamma_{\mu_2}] \} \gamma_5 u \\
& = D^{[\mu_1\mu_2]}_{\Sigma;k_0,\alpha} \frac{1}{\sqrt{6}} \gamma_{\mu_1} \gamma_{\mu_2} \gamma_5 u
\end{align*}
\] (3.92)

(3) (c) \( L_{K_3} = 1, L_{K_3} = 1, L = 0, S_{q_1q_2} = 1, J_{q_1q_2} = 1 \)

\[
\begin{align*}
\frac{3^+}{2} : & \quad D^\mu_T u_\mu \\
\frac{1^+}{2} : & \quad D^\mu_T \frac{1}{\sqrt{3}} \gamma_\mu \gamma_5 u
\end{align*}
\] (3.93)

B. First Positive Parity Excited Heavy \( \Sigma \)

For the first positive parity heavy \( \Sigma \) the projections of the possible total angular momenta of the light quark pair are represented by the following matrices:

\[
\begin{align*}
3^+ D^{[\mu_1\mu_2]}_{\Sigma;k_0,\alpha} &= \frac{1}{3} \left( \chi_{\mu_1} N^{\mu_1\mu_2}(k) - \frac{2}{3} g^{\mu_1\mu_2} \chi_{\mu_1} N^{\mu_1\mu_2}(k) \right) + \text{symmetrised } (\mu_1, \mu_2)]\delta_\rho A^{\rho}_{\alpha,\beta}, \\
2^+ D^{[\mu_1\mu_2]}_{\Sigma;k_0,\alpha} &= \frac{2}{3} \left( \chi_{\mu_1} N^{\mu_1\mu_2}(k) - \frac{1}{3} g^{\mu_1\mu_2} \chi_{\mu_1} N^{\mu_1\mu_2}(k) \right) \\
1^+ D^{[\mu_1\mu_2]}_{\Sigma;k_0,\alpha} &= \left( \chi_{\mu_1} N^{\mu_1\mu_2}(k) \right)\delta_\rho A^{\rho}_{\alpha,\beta}, \\
2^+ D^{[\mu_1\mu_2]}_{\Sigma;0,\alpha} &= N^{[\mu_1\mu_2]} \gamma_0 A^{\alpha,\beta}_{\rho}, \\
1^+ D^{[\mu_1\mu_2]}_{\Sigma;0,\alpha} &= N^{[\mu_1\mu_2]} \gamma_0 A^{\alpha,\beta}_{\rho}, \\
0^+ D^{[\mu_1\mu_2]}_{\Sigma;0,\alpha} &= K_{3;\parallel} \gamma_0 \chi_{\delta,\beta} A^{\alpha,\beta}_{\rho}.
\end{align*}
\] (3.94)

One can now write down the B-S amplitudes for the pairs of degenerate first positive parity excited heavy \( \Sigma \) baryons.

(1) (a) \( L_{K_3} = 2, L_{K_3} = 0, L = 2, S_{q_1q_2} = 1, J_{q_1q_2} = 3 \)

\[
\begin{align*}
\frac{7^+}{2} : & \quad D^{[\mu_1\mu_2]}_{\Sigma;K_3} u_{\mu_1\mu_2} \\
\frac{5^+}{2} : & \quad D^{[\mu_1\mu_2]}_{\Sigma;K_3} \sqrt{\frac{3}{7}} \gamma_\mu \gamma_5 u_{\mu_1\mu_2}
\end{align*}
\] (3.95)
(1) (b) \( L_{K_3} = 2, L_{k_3} = 0, L = 2, S_{q_1q_2} = 1, J_{q_1q_2} = 2 \)

\[ \begin{align*}
\frac{5}{2}^+ & : D_{\Sigma;K_3}^{[\mu_1\mu_2]} \frac{1}{\sqrt{2}} (\gamma_{\perp \mu_1} \gamma_5 u_{\mu_1\mu_2} - \gamma_{\perp \mu_1} \gamma_5 u_{\mu_2}) \\
\frac{3}{2}^+ & : D_{\Sigma;K_3}^{[\mu_1\mu_2]} \frac{1}{3\sqrt{5}} \{ [\gamma_{\perp \mu_1} \gamma_{\perp \mu_2}] u_{\mu_2} + (\gamma_{\perp \mu_1} \gamma_{\perp \mu_2} u_{\mu_1} - \gamma_{\perp \mu_1} \gamma_{\perp \mu_2} u_{\mu_2}) \}
\end{align*} \tag{3.96} \]

(1) (c) \( L_{K_3} = 2, L_{k_3} = 0, L = 2, S_{q_1q_2} = 1, J_{q_1q_2} = 1 \)

\[ \begin{align*}
\frac{3}{2}^+ & : D_{\Sigma;K_3}^{[\mu_1\mu_2]} u_{\mu} \\
\frac{1}{2}^+ & : D_{\Sigma;K_3}^{[\mu_1\mu_2]} \frac{1}{\sqrt{10}} (\gamma_{\perp \mu_1} \gamma_5 u_{\mu_2} - \gamma_{\perp \mu_2} \gamma_5 u_{\mu_1}) \\
\end{align*} \tag{3.97} \]

The states arising from the orbital angular momentum case (2), \( L_{K_3} = 0, L_{k_3} = 2 \) are identical to the above except for the replacement of \( K_3 \) by \( k_3 \) in the D’s.

(3) (a) \( L_{K_3} = 1, L_{k_3} = 1, L = 2, S_{q_1q_2} = 0, J_{q_1q_2} = 2 \)

\[ \begin{align*}
\frac{5}{2}^+ & : D_{\Sigma}^{[\mu_1\mu_2]} u_{\mu_1\mu_2} \\
\frac{3}{2}^+ & : D_{\Sigma}^{[\mu_1\mu_2]} \frac{1}{\sqrt{10}} (\gamma_{\perp \mu_1} \gamma_5 u_{\mu_2} + \gamma_{\perp \mu_2} \gamma_5 u_{\mu_1}) \\
\end{align*} \tag{3.98} \]

(3) (b) \( L_{K_3} = 1, L_{k_3} = 1, L = 1, S_{q_1q_2} = 0, J_{q_1q_2} = 1 \)

\[ \begin{align*}
\frac{3}{2}^+ & : D_{\Sigma}^{[\mu_1\mu_2]} \frac{1}{\sqrt{2}} (\gamma_{\perp \mu_1} \gamma_5 u_{\mu_2} - \gamma_{\perp \mu_2} \gamma_5 u_{\mu_1}) \\
\frac{1}{2}^+ & : D_{\Sigma}^{[\mu_1\mu_2]} \frac{1}{2\sqrt{6}} [\gamma_{\perp \mu_1}, \gamma_{\perp \mu_2}] u \\
\end{align*} \tag{3.99} \]

(3) (c) \( L_{K_3} = 1, L_{k_3} = 1, L = 0, S_{q_1q_2} = 0, J_{q_1q_2} = 0 \)

\[ \frac{1}{2}^+ : D_{\Sigma} u. \tag{3.100} \]

### 3.5.4 L-wave Heavy Baryons

Although, as we have pointed out above, the general case for any arbitrary higher excitation is rather involved, we can easily generalise these results to the “highest weight” states of any orbital excitation. In such a case, the total orbital angular momentum \( L \) will be represented by the transverse,
symmetric, traceless product of $L$ momenta (both $k_{3\perp}$ or $K_{3\perp}$ can appear). This will be an obvious generalisation of eq. (2.30). It does not matter which of the different, $L + 1$, possibilities we choose as long as we are only concerned with the Lorentz structure and not with the flavour. Let us continue to call such a product of $L$ momenta, $k$, as $N^{\mu_1 \mu_2 \ldots \mu_L}$. This orbital angular momenta has then to be combined, on the light side, with the total spin of the light quarks, which is either 0 or 1 and which is represented as before by $\chi^0_{\alpha\beta}$ or $\chi^1_{\perp, \alpha\beta}$.

Thus one will get the following states, all of parity $(-1)^L$:

\[
((S_{q_1 q_2} = 0 \otimes L) = (J_{q_1 q_2} = L)) \otimes (S_Q = 1/2) = L + 1/2, L - 1/2
\]

\[
((S_{q_1 q_2} = 1 \otimes L) = (J_{q_1 q_2} = L + 1, L, L - 1)) \otimes (S_Q = 1/2) = (L + 3/2, L + 1/2) \oplus (L + 1/2, L - 1/2) \oplus (L - 1/2, L - 3/2).
\]

(3.101)

The total light angular momentum projections ($J_{q_1 q_2} = L + 1, L, L - 1$) arising from the orbital angular momentum, $L$, and the light quark spin 1, is obtained in the usual way by taking the symmetric, antisymmetric product and trace of $N^{\mu_1 \mu_2 \ldots \mu_L}$ with $\chi^1_{\perp, \mu}$. For details of the mixed symmetric tensor, see Appendix B. Define

\[
\phi^{(\mu_1 \mu_2 \ldots \mu_L)}_{\alpha\beta} = N^{\mu_1 \mu_2 \ldots \mu_L} \chi^0_{\delta\rho, \alpha\beta} A^{\delta\rho}
\]

(3.102)

\[
\phi^{\{\mu_1 \mu_2 \ldots \mu_L\}}_{\alpha\beta} = \left[\chi^1_{\perp} N^{\mu_1 \mu_2 \ldots \mu_L} + \sum_{i=1}^{L} \chi^1_{\perp} N^{\mu_1 \ldots \mu_i-1 \mu_i \mu_{i+1} \ldots \mu_L} \right. \\
- \frac{2}{2L + 1} \sum_{i=1}^{L} g^{\mu_1}_{\mu_2} \chi^1_{\perp, \nu} N^{\nu \mu_1 \ldots \mu_i-1 \mu_i \mu_{i+1} \ldots \mu_L} \\
- \frac{2}{2L + 1} \sum_{i,j} g^{\mu_1}_{\mu_2} \chi^1_{\perp, \nu} N^{\nu \mu_1 \ldots \mu_i-1 \mu_i \mu_{i+1} \ldots \mu_L} \delta_{\rho} A^{\delta\rho}_{\alpha\beta}
\]

(3.103)

\[
\phi^{[\mu_1 \mu_2 \ldots \mu_L]}_{\alpha\beta} = \frac{L}{2(L + 1)} \left[\chi^1_{\perp} N^{\mu_1 \ldots \mu_L} - \chi^1_{\perp} N^{\mu_1 \ldots \mu_L} \right. \\
- \frac{2}{L} \sum_{i=2}^{L} g^{\mu_1}_{\mu_2} \chi^1_{\perp, \nu} N^{\nu \mu_1 \ldots \mu_i-1 \mu_i \mu_{i+1} \ldots \mu_L}
\]

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+ \frac{2}{L} \sum_{i=2}^{L} g_{\mu_1}^{\mu_{1_1}} \chi_{\mu_{L_1}}^{1} N^\mu_{\mu_{2_1}} \cdots \mu_{L_{i-1}} \mu_{L_{i+1}} \cdots \mu_{L_{L}} \delta_{\rho} A_{\alpha \beta}^{\delta_{\rho}}

(3.104)

\phi_{\alpha \beta}^{\mu_2 \cdots \mu_L} = (\chi_{\mu_1}^1 N^{\mu_1 \mu_2 \cdots \mu_L}) \delta_{\rho} A_{\alpha \beta}^{\delta_{\rho}}.

(3.105)

These $\phi$’s contain the “brown muck” information, or, better, lack of information. They represent respectively the following total angular momentum and parity of the light quarks:

\begin{align*}
\phi^{\mu_1 \mu_2 \cdots \mu_L} : & \quad L^{(-L)} \\
\phi^{\mu_1 \mu_2 \cdots \mu_L} : & \quad (L + 1)^{(-L)} \\
\phi^{\mu_1 \mu_2 \cdots \mu_L} : & \quad L^{(-L)} \\
\phi^{\mu_2 \cdots \mu_L} : & \quad (L - 1)^{(-L)}.
\end{align*}

(3.106)

We can now write down the B-S amplitudes for the L-wave heavy baryons of parity $(-1)^L$. Each pair of states listed is separately degenerate because of the heavy quark spin symmetry.

(i) $L = L$, $S_{q1q2} = 0$, $J_{q1q2} = L$.

\begin{align*}
L + \frac{1}{2} : & \quad \phi^{\mu_3 \cdots \mu_L} u_{\mu_1 \cdots \mu_L} \\
L - \frac{1}{2} : & \quad \phi^{\mu_3 \cdots \mu_L} \frac{1}{\sqrt{L(2L + 1)}} [\gamma_{\perp \mu_1} \gamma_5 u_{\mu_2 \cdots \mu_L} \\
& \quad + \gamma_{\perp \mu_2} \gamma_5 u_{\mu_1 \mu_3 \cdots \mu_L} + \cdots \gamma_{\perp \mu_L} \gamma_5 u_{\mu_1 \cdots \mu_{L-1}}] \\
& \quad = \phi^{\mu_3 \cdots \mu_L} \sqrt{\frac{L}{2L + 1}} \gamma_{\perp \mu_1} \gamma_5 u_{\mu_2 \cdots \mu_L}.
\end{align*}

(3.107)

(ii) $L = L$, $S_{q1q2} = 1$, $J_{q1q2} = L + 1, L, L - 1$.

(a) $J_{q1q2} = L + 1$.

\begin{align*}
L + \frac{3}{2} : & \quad \phi^{\mu_1 \mu_2 \cdots \mu_L} u_{\mu_1 \cdots \mu_L} \\
L + \frac{1}{2} : & \quad \phi^{\mu_1 \cdots \mu_L} \sqrt{\frac{L + 1}{2L + 3}} \gamma_{\perp \mu} \gamma_5 u_{\mu_1 \cdots \mu_L}.
\end{align*}

(3.108)

This last can also be written in fully symmetrised form as in (3.107).

(b) $J_{q1q2} = L$.

\begin{align*}
L + \frac{1}{2} : & \quad \phi^{\mu_1 \mu_2 \cdots \mu_L} \frac{1}{\sqrt{2}} \left( \gamma_{\perp \mu} \gamma_5 u_{\mu_1 \cdots \mu_L} - \gamma_{\perp \mu} \gamma_5 u_{\mu_2 \cdots \mu_L} \right);
\end{align*}
\[ L - \frac{1}{2} : \phi^{[\mu_1]_1\ldots\mu_L} \frac{1}{L+1} \sqrt{\frac{L}{2(2L+1)}} \left\{ [\gamma_{\perp \mu}^1, \gamma_{\perp \mu_1}] u_{\mu_2\ldots\mu_L} \right. \\
+ (L - 1)(\gamma_{\perp \mu}^1 \gamma_{\perp \mu_2} u_{\mu_3\mu_5\ldots\mu_L} - \gamma_{\perp \mu_1} \gamma_{\perp \mu_2} u_{\mu_3\mu_5\ldots\mu_L}) \right\}. \] (3.109)

\( (c) \ J_{q_1 q_2} = L - 1. \)

\[ L - \frac{1}{2} : \phi^{\mu_2\ldots\mu_L} u_{\mu_2\ldots\mu_L} , \]

\[ L - \frac{3}{2} : \phi^{\mu_2\ldots\mu_L} \sqrt{\frac{L - 1}{2L - 1}} \gamma_{\mu_2} \gamma_{\perp 5} u_{\mu_3\ldots\mu_L} . \] (3.110)

The \( u_{\mu_1\ldots\mu_L} \) in eqs. (3.107) - (3.110) are the appropriate generalised symmetric, traceless Rarita-Schwinger spinors satisfying

\[ v^{\mu_1} u_{\mu_1\ldots\mu_L} = 0 , \]

\[ \gamma_{\mu_1} u_{\mu_1\ldots\mu_L} = 0 , \]

\[ (\not{\gamma} - 1) u_{\mu_1\ldots\mu_L} = 0 . \] (3.111)

These spinors represent bound states with parity \((-)^L\). (See footnote on page 13).

As already remarked, Falk [7] has followed a different route to construct wave functions of heavy baryons with arbitrary spin. However, in his construction he only has the symmetric wave functions (3.107), (3.108) and (3.110). We see that, in general, there will always be mixed symmetry type wave functions like (3.109). In fact, this observation has important consequences for the decay matrix elements as we shall see in a subsequent section.

Flavour is easily accounted for as we have already discussed. If one requires \( \Lambda \)-type heavy baryons, antisymmetric under flavour, overall symmetry symmetry is assured by taking the appropriate number of \( k_{3L} \)'s and \( K_{3L} \)'s to ensure that \( L^{\mu_1\ldots\mu_L} \) in eq. (3.102) is symmetric under \( p_1 \leftrightarrow p_2 \) and antisymmetric under \( p_1 \leftrightarrow p_2 \) in eqs. (3.103) - (3.105). For \( \Sigma \)-type heavy baryons we have the opposite situation.

For the construction of the \( p \)- and higher wave baryons we were inspired by the seminal work, [15], of Salam, Delbourgo, Rashid and Strathdee. In that work they had constructed the mixed symmetry tensor which we have used in describing the \( p \)-wave baryons. However, they had the wrong particle (spin and parity) content because they did not include internal momenta. Once one
does that it is easy to show that the mixed symmetry, $\bar{U}(12)$ representation, contains nothing but the set of p-wave baryons which we have constructed above.

4 Heavy Meson Transitions

In this section we calculate the current induced flavour changing transitions between ground state heavy mesons and ground and excited state mesons of different flavour. Exemplary processes are the $b \to c$ transitions $B \to D, D^*, D^{**} \ldots$.

We define the transition matrix elements as

$$M_\lambda = \langle M_2(v_2)|J^V_{\lambda}M_1(v_1)\rangle.$$  

(4.1)

Here $M_1$ and $M_2$ are the initial and final mesons with velocities $v_1$ and $v_2$, respectively. Closely following the presentation in [3] the transition matrix element can be written in the form of a trace.

First we introduce tensor-valued reduced meson projectors $\chi_{\mu_1 \ldots \mu_L}$ in analogy to the traceless, symmetric tensors $\bar{\Gamma}_{\mu_1 \ldots \mu_L}$ in eq. (2.28). Thus using eq. (2.28), one can write the $\chi$ of eq. (2.14) as

$$\chi = N^{\mu_1 \ldots \mu_L} \chi_{\mu_1 \ldots \mu_L}.$$  

(4.2)

Specifically,

$$\chi_{\mu_1 \ldots \mu_L} = \frac{1 + \gamma^\mu}{2} \bar{\Gamma}_{\mu_1 \ldots \mu_L} \frac{1 - \gamma^\nu}{2}.$$  

(4.3)

The transition matrix element (4.1) can then be written as

$$M_\lambda = Tr\{\bar{\chi}_2(L\pm)\bar{\mu}_3 \ldots \mu_L \gamma_\lambda(1 - \gamma^5)\chi_1\} N(v_1)^{\mu_1 \ldots \mu_L} \xi_{L\pm}(\omega),$$  

(4.4)

where $N^{\mu_1 \ldots \mu_L}(v_1)$ is identical to the symmetric traceless tensor $N^{\mu_1 \ldots \mu_L}$ defined in (2.30) except for the replacement $k_\perp \to v_{1\perp} = v_1 - \omega v_2$ where $\omega = v_1.v_2$. $\chi_1$ is the incoming reduced spin wave function for ground state mesons, eqs. (2.14, 2.18, and 2.19). The subscript $L\pm$ refers to the two pairs of degenerate states in table (5) labelled by the subscripts $L + \frac{1}{2}$ and $L - \frac{1}{2}$, respectively. $\xi_{L\pm}(\omega)$ are two independent form factor functions of the variable $\omega$ defined above.
The last two factors in eq. (4.4) characterize the covariance structure of the light side transition which is parametrised according to [6]

\[ \int d^4k_1 d^4k_2 A_1(k_1) T(k_1, k_2; v_1, v_2) A_2(k_2) N^{\mu_1...\mu_L}(k_2) = \xi_{L\pm}(\omega) N^{\mu_1...\mu_L}(v_1). \] (4.5)

The single covariant \( N^{\mu_1...\mu_L}(v_1) \) in eq. (4.4) is the most general covariant that can be written down for the light-side transition given by the overlap integral on the l.h.s. of eq. (4.5) when taken “between the projectors” in the trace eq. (4.4). At first glance one would also write down a second covariant

\[ N^{\mu_1...\mu_L}(v_1) = \sum_{i=1}^L \gamma^{\mu_i}_{\perp 12} N^{\mu_1...\mu_{i-1}\mu_{i+1}...\mu_L}(v_1) - \frac{2}{2L-1} \sum_{i<j} g^{\mu_i\mu_j}_{\perp 12} \gamma^{\mu_j}_{\perp 2v} N^{\mu_1...\mu_{i-1}\mu_{j+1}...\mu_L}(v_1). \] (4.6)

To avoid confusion from now on we add a subscript to the symbol \( \perp \) to indicate transversality with respect either to the incoming velocity \( v_1 \) or the final velocity \( v_2 \). The symbol \( \perp_i \) hence means perpendicular to the velocity \( v_i \). However, the covariant \( N^{\mu_1...\mu_L}(v_1) \) is not linearly independent of \( N^{\mu_1...\mu_L}(v_1) \) when taken in the trace eq. (4.4). In fact one can easily check that \( N^{\mu_1...\mu_L} \) is linearly related to \( N^{\mu_1...\mu_L} \) for the transitions to the \((L-\frac{1}{2})\) states and identical to zero for transitions to the \((L+\frac{1}{2})\) states. We are now in a position to write down the trace expressions. Using the expressions for \( \bar{\Gamma} \) from Table 5 in eq. (4.4) one can write the matrix elements for the transitions from the ground state s-wave mesons to the L-wave mesons as (all the matrix elements are multiplied by a common factor \( \sqrt{M_1 M_2} \)):

(a) \((0^-, 1^-) \to (L-1)L-\frac{1}{2}\)

\[ -\sqrt{\frac{2L-1}{2L+1}}(1+\omega) Tr \left\{ \left[ v^\mu_1 + (1-\omega) \frac{\gamma^{\mu_2}}{2L-1} \right] \frac{1+\gamma^5}{2} \gamma_{\lambda(1-\gamma_5)} \xi_1 \right\} v^\mu_2 \ldots v^\mu_L \epsilon_{\mu_2...\mu_L}^* \xi_{L-}(\omega). \] (4.7)

(b) \((0^-, 1^-) \to L_{L-\frac{1}{2}}\)

\[ \sqrt{\frac{L}{2L+1}}(1+\omega) Tr \left\{ \gamma^{\mu_1 \gamma_5} \frac{1+\gamma^2}{2} \gamma_{\lambda(1-\gamma_5)} \xi_1 \right\} v^\mu_2 \ldots v^\mu_L \epsilon_{\mu_1...\mu_L}^* \xi_{L-}(\omega). \] (4.8)
\[ (c) \ (0^-, 1^-) \rightarrow L_{L+\frac{1}{2}} \]
\[ = \frac{1}{\sqrt{(L+1)(2L+1)}} Tr \{ [L(1+\omega)\gamma^{\mu_1} + (2L+1)v_1^{\mu_1}] \]
\[ \times \gamma_5^{1+\frac{\omega}{2}} \gamma_\lambda (1-\gamma_5) \chi_1 \} v_1^{\mu_2} \ldots v_1^{\mu_L} \epsilon_{\mu_1 \ldots \mu_L}^* \xi_{L+}(\omega) \] (4.9)

\[ (d) \ (0^-, 1^-) \rightarrow (L+1)_{L+\frac{1}{2}} \]
\[ Tr \{ \gamma^\nu \frac{1+\omega}{2} \gamma_\lambda (1-\gamma_5) \chi_1 \} v_1^{\mu_1} \ldots v_1^{\mu_L} \epsilon_{\nu \mu_1 \ldots \mu_L}^* \xi_{L+}(\omega) \] (4.10)

Notice that all these amplitudes vanish at \( \omega = 1 \) where \( v_1 = v_2 \). To summarize we see that to describe all the heavy s- to L-wave transitions we need just two form factors. This is a tremendous simplification.

As an obvious first application of our formalism we consider the contributions of the current induced transitions \( 0^- \rightarrow (L-1)_{L-\frac{1}{2}}, 0^- \rightarrow L_{L-\frac{1}{2}} \) and \( 0^- \rightarrow L_{L+\frac{1}{2}}, 0^- \rightarrow (L+1)_{L+\frac{1}{2}} \) to the Bjorken sum rule \([10]\). Technically the easiest route to do this is to first calculate the longitudinal helicity amplitude, as done in \([3]\), and then to square it in order to obtain the contribution of a given excitation to the Bjorken sum rule. This is an elegant device that avoids the tedium of having to do spin sums in squared covariant matrix elements. To this end one needs an explicit representation of the helicity 0 component of the spin \( j \) polarisation tensor in terms of products of the spin 1 polarisation vector. One has
\[ \epsilon_{\mu_1 \ldots \mu_j}(0) = \sqrt{\frac{j!}{(2j-1)!!}} \epsilon_{\mu_1}(0) \ldots \epsilon_{\mu_j}(0) + \ldots, \] (4.11)

where the ellipsis stand for terms involving transverse spin 1 polarisation vectors. A glance at the structure of the transition matrix elements eqs. \([4.7-4.10]\) shows that these extra transverse terms do not contribute to the longitudinal helicity amplitudes and thus their explicit form is not needed here. It is then rather straightforward to calculate the longitudinal helicity amplitudes. Upon squaring the longitudinal amplitudes and dividing out the longitudinal structure function, \([3]\), \( K_L = 4M_1M_2[\omega(M_1^2 + M_2^2) - 2M_1M_2]/q^2 \), one obtains
\[ 1 = \frac{\omega + 1}{2} \left\{ \xi(\omega)^2 + \sum_{L \geq 1} \frac{L!(\omega^2 - 1)^L}{(2L + 1)!!} (L|\xi_{L-}(\omega)|^2 + (L+1)|\xi_{L+}(\omega)|^2) \right\}, \] (4.12)
where $\xi(\omega)$ denotes the reduced form factor function of the ground-state to ground-state transition. Compared to [6] we no longer absorb a factor $(\omega + 1)$ in the reduced form factor $\xi_{L-}(\omega)$, i.e. one has $\xi_{L+}^*(\omega) = (\omega + 1)\xi_{L-}(\omega)$ where $\xi_{L+}(\omega)$ is the s-wave to p-wave reduced form factor introduced in [6]. If one identifies $\xi(\omega) = \xi_{0+}(\omega)$ one can rewrite eq. (4.12) as a single sum, i.e. one has

$$
1 = \frac{\omega + 1}{2} \sum_{L \geq 0} \frac{L!}{(2L + 1)!!} (\omega^2 - 1)L|\xi_{L-}(\omega)|^2 + (L + 1)|\xi_{L+}(\omega)|^2 . \quad (4.13)
$$

5 Heavy Baryon Transitions

Using the heavy baryon wave functions developed in section 3, we now calculate the current induced flavour changing transitions between ground state heavy baryons and heavy baryon orbital excitations of different flavours. The procedure has already been outlined in the previous section in the case of mesons. The transitions between s-wave baryons has already been treated in [4] and [5]. We shall not repeat them here. As before the transition matrix element can be written in terms of a trace [4], [5].

5.1 Decay of heavy s-wave baryons to heavy p-wave baryons

We first consider the decays of s- to p-wave heavy baryons using the wave functions eqs. (3.78-3.83). We present separately the results for decays of $\Lambda_{Q}$ type and $\Sigma(\Omega)$ type baryons.

5.1.1 Heavy $\Lambda_{Q}$ to heavy p-wave $\Lambda_{Q'}$.

We shall first demonstrate the general structure of the matrix element for the heavy s-wave $\Lambda_{Q}$ to the heavy p-wave $\Lambda_{Q'}$'s. We will then see that we need only two independent form factors to describe the decays of the $\Lambda_{Q}(\frac{1}{2})$ to all the p-wave $\Lambda_{Q'}$'s $(\frac{5}{2}^{-}, \frac{3}{2}^{-}; \frac{3}{2}^{-}, \frac{1}{2}^{-}; \frac{3}{2}^{-}, \frac{1}{2}^{-}; \frac{1}{2}^{-})$. We list the matrix elements according to the various final p-wave states developed in section 3.

(i) $\Lambda_{Q}, \frac{1}{2}^{+} \to$ p-wave $\Lambda_{Q'}$ with $S_{q_1q_2} = 0$.

$$
\frac{1}{2}^{+} \to \frac{3}{2}^{-} : \bar{u}_{\mu} \gamma_{\lambda}(1 - \gamma_{5})u M^{\mu}
$$

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Here $M^\mu$ is the overlap integral

$$M^\mu = Tr \int d\Gamma \bar{P}_{K_3}^{\mu}(k_3^{(2)}, K_3^{(2)}) T(k_3^{(1)}, K_3^{(1)}, k_3^{(2)}, K_3^{(2)}; v_1, v_2) S(k_3^{(1)}, K_3^{(1)}),$$

(5.2)

where $d\Gamma = d^4k_3^{(1)} d^4k_3^{(2)} d^4k_3^{(2)} d^4K_3^{(2)}$ and $k_3^{(1)}, K_3^{(1)} (k_3^{(2)}, K_3^{(2)})$ are the two pairs of initial and final relative momenta. $v_1, v_2$ are the velocities of the initial and final baryons, respectively. In eq. (5.2), we have explicitly displayed the arguments of $\bar{P}$ and $S$. We will not display these in subsequent equations. Clearly $M^\mu$ is a vector function of the velocities, $v_1$ and $v_2$, satisfying $v_2^\mu M^\mu = 0$. Hence the most general nonvanishing covariant is $v_{\mu\perp}^1$ leading to just one form factor $f_{1}^{(1)}$, so that eqs. (5.1) can be written as:

$$\frac{1}{2}^+ \rightarrow \frac{1}{2}^- : \frac{1}{\sqrt{3}} \bar{u} \gamma_5 \gamma_{\mu\perp} \gamma(1 - \gamma_5) u M_{\mu}, $$

(5.1)

(ii) $\Lambda'_{Q} \frac{1}{2}^+ \rightarrow$ to p-wave $\Lambda_Q'$ with $S_{q_1 q_2} = 1$.

(a) $S_{q_1 q_2} = 1, J_{q_1 q_2} = 2$

$$\begin{align*}
\frac{1}{2}^+ \rightarrow \frac{5}{2}^- : & \quad \bar{u}^\mu \gamma(1 - \gamma_5) u M_{\mu
u}^\nu \\
\frac{1}{2}^+ \rightarrow \frac{3}{2}^- : & \quad \frac{\sqrt{2}}{3} \bar{u} \gamma_5 \gamma_\perp^{\mu\nu} \gamma(1 - \gamma_5) u M_{\mu
u
u}^\nu,
\end{align*}$$

with

$$M_{\mu\nu}^\nu = Tr \int d\Gamma \bar{P}_{k_3^{(2)}}^{\mu\nu} T S,$$

(5.5)

a negative parity symmetric, traceless tensor satisfying $v_{2\mu} M^{\mu\nu} = 0$. Such a tensor does not exist. Hence $\Lambda_Q$ does not decay to this pair of $\frac{5}{2}^-$ and $\frac{3}{2}^-$ states.

(b) $S_{q_1 q_2} = 1, J_{q_1 q_2} = 1$

$$\begin{align*}
\frac{1}{2}^+ \rightarrow \frac{3}{2}^- : & \quad \sqrt{2} \bar{u}_\nu \gamma_5 \gamma_\perp^{\mu\nu} \gamma(1 - \gamma_5) u M^{\mu\nu} \\
\frac{1}{2}^+ \rightarrow \frac{1}{2}^- : & \quad \frac{1}{2} \sqrt{2} \bar{u} \gamma_\perp^{\mu\nu} \gamma_\perp^{\mu\nu} \gamma(1 - \gamma_5) u M^{\mu\nu},
\end{align*}$$

(5.6)
with

$$M^{[\mu \nu]} = Tr \int d\Gamma \bar{P}_{k_3}^{[\mu \nu]} T S$$

(5.7)

a negative parity antisymmetric tensor satisfying $v_{2\mu} M^{[\mu \nu]} = 0$. Such a tensor does exist uniquely, namely $i \epsilon^{\mu \nu \rho \kappa} v_{1\rho} v_{2\kappa}$, leading to one form factor, $f_2^{(1)}$, for this pair of transitions. Thus the matrix elements eq. (5.6) can be written as:

$$\frac{1}{2}^+ \rightarrow \frac{3}{2}^- : \sqrt{2} i f_2^{(1)} \epsilon^{\mu \nu \rho \kappa} v_{1\rho} v_{2\kappa} \bar{u}_\nu \gamma_5 \gamma_\mu \gamma_\lambda (1 - \gamma_5) u$$

$$\frac{1}{2}^+ \rightarrow \frac{1}{2}^- : \frac{1}{2} \sqrt{3} f_2^{(1)} \epsilon^{\mu \nu \rho \kappa} v_{1\rho} v_{2\kappa} \bar{u}_\nu \gamma_\mu \gamma_\lambda (1 - \gamma_5) u.$$  

(5.8)

On using standard $\epsilon^{\mu \nu \rho \lambda}$ identities, these matrix elements can also be written as:

$$\frac{1}{2}^+ \rightarrow \frac{3}{2}^- : \sqrt{2} f_2^{(1)} v_1^\mu \bar{u}_\mu \gamma_\lambda (1 - \gamma_5) u$$

$$\frac{1}{2}^+ \rightarrow \frac{1}{2}^- : \sqrt{2} f_2^{(1)} \bar{u} \gamma_5 (\omega + \gamma_1) \gamma_\lambda (1 - \gamma_5) u.$$  

(5.9)

(c)$S_{q_1q_2} = 1, J_{q_1q_2} = 0$

$$\frac{1}{2}^+ \rightarrow \frac{1}{2}^- : \bar{u} \gamma_5 (1 - \gamma_5) u M,$$  

(5.10)

with

$$M = Tr \int d\Gamma \bar{P}_{k_3} T S$$

(5.11)

a pseudoscalar. It is not possible to construct such a pseudoscalar with the two vectors available to us, $v_1$ and $v_2$. Hence the $\Lambda_{Q', \frac{3}{2}^-}$ does not decay to this $\frac{1}{2}^-$. To summarise we see that all the decays of the ground state heavy $\Lambda$ to p-wave heavy $\Lambda$‘s are controlled by just two form factors which we have called $f_1^{(1)}$ and $f_2^{(1)}$.

5.1.2 Heavy $\Sigma_Q(\Omega_Q)$ to heavy p-wave $\Sigma_{Q'}(\Omega_{Q'})$.

Using the B-S amplitudes developed in earlier sections we can also immediately write down the matrix elements for the semi-leptonic decays of the $\Sigma(\Omega)$ type ground state heavy baryons to the corresponding p-wave heavy
baryons. The initial ground state is either $\frac{3}{2}^+$ or $\frac{1}{2}^+$. We list below, in obvious matrix form, the weak current matrix elements for the transition to the various p-wave states.

(i) $\Sigma_Q(\Omega_Q)(\frac{3}{2}^+,\frac{1}{2}^+) \rightarrow \text{p-wave } \Sigma_Q'(\Omega_Q')$ with $(S_{q_1q_2} = 0; \frac{3}{2}^-\frac{1}{2}^-)$.

\[
\left( \frac{1}{\sqrt{3}} \bar{u}_{\gamma 5} \gamma_{\perp 2\mu} \right) \gamma_{\lambda}(1 - \gamma_5) \left( \frac{1}{\sqrt{3}} \bar{u}_{\gamma 5} \gamma_{\perp 1\nu} u^{\nu} \right) M^{\mu\nu}, \tag{5.12}
\]

with

\[
M^{\mu\nu} = Tr \int d\Gamma \tilde{P}_{k_3}^{\mu\nu} T S^{\nu} \tag{5.13}
\]

a negative parity tensor satisfying $v_2^{\mu} M^{\mu\nu} = v_1^{\nu} M^{\mu\nu} = 0$. The general unique form of this tensor is $i \epsilon^{\mu\nu\rho\kappa} v_1^{\rho} v_2^{\kappa}$ leading to just one form factor, $g_1^{(1)}$. The matrix elements eq. (5.12) can then be written as

\[
ig_1^{(1)} \epsilon^{\mu\nu\rho\kappa} v_1^{\rho} v_2^{\kappa} \left( \frac{1}{\sqrt{3}} \bar{u}_{\gamma 5} \gamma_{\perp 2\mu} \right) \gamma_{\lambda}(1 - \gamma_5) \left( \frac{1}{\sqrt{3}} \bar{u}_{\gamma 5} \gamma_{\perp 1\nu} u^{\nu} \right). \tag{5.14}
\]

(ii) $\Sigma_Q(\Omega_Q)(\frac{3}{2}^+,\frac{1}{2}^+) \rightarrow \text{p-wave } \Sigma_Q'(\Omega_Q')$ with $S_{q_1q_2} = 1$.

(a) $(\frac{3}{2}^+,\frac{1}{2}^+) \rightarrow (J_{q_1q_2} = 2; \frac{5}{2}^-\frac{3}{2}^-)$

\[
\left( \frac{1}{\sqrt{5}} \bar{u}_{\gamma 5} \gamma_{\perp 2\nu} \right) \gamma_{\lambda}(1 - \gamma_5) \left( \frac{1}{\sqrt{3}} \bar{u}_{\gamma 5} \gamma_{\perp 1\nu} u^{\nu} \right) M^{(\mu\nu)}_{\kappa}, \tag{5.15}
\]

with

\[
M^{(\mu\nu)}_{\kappa} = Tr \int d\Gamma \tilde{P}_{k_3}^{(\mu\nu)} T S^\kappa \tag{5.16}
\]

a negative parity, traceless and symmetric in $\{\mu, \nu\}$, tensor satisfying $v_2^{\mu} M^{(\mu\nu)}_{\kappa} = v_1^{\nu} M^{(\mu\nu)}_{\kappa} = 0$. Thus there are two form factors as $M^{(\mu\nu)}_{\kappa}$ can in general be written as

\[
M^{(\mu\nu)}_{\kappa} = g_2^{(1)} (v_{1\perp}^{\mu} v_{1\perp}^{\nu} - \frac{1}{3} v_{1\perp}^{\nu} g^{\mu\nu}_{1\perp}) v_{2\perp} + g_3^{(1)} (v_{1\perp}^{\mu} g^{\mu\kappa}_{1\perp 2\perp} + v_{1\perp}^{\nu} g^{\nu\kappa}_{1\perp 2\perp} - \frac{2}{3} g^{\nu\kappa}_{1\perp 2\perp} v_{1\perp} v_{1\perp} g^{\nu\kappa}_{1\perp 2\perp}), \tag{5.17}
\]

where

\[
v_{1\perp} = v_1 - \omega v_2, \quad v_{2\perp} = v_2 - \omega v_1 \tag{5.18}
\]
and

\[ g_{\perp_{2} \perp_{1}}^{\mu \kappa} = g^{\mu \kappa} - (v_{1}^{\nu} v_{1}^{\kappa} + v_{2}^{\nu} v_{2}^{\kappa}) + \omega v_{1}^{\nu} v_{2}^{\kappa}, \quad (5.20) \]

so that \( v_{2 \nu} g_{\perp_{2} \perp_{1}}^{\mu \kappa} = v_{1 \kappa} g_{\perp_{2} \perp_{1}}^{\mu \kappa} = 0 \). Thus the matrix element eq. (5.15) becomes

\[ \{ g_{2}^{(1)} v_{1}^{\mu} v_{1}^{\nu} v_{2}^{\kappa} + g_{3}^{(1)} (v_{2}^{\mu} g^{\nu \kappa} + v_{1}^{\nu} g^{\mu \kappa}) \} \left( \frac{\bar{u}_{\mu \nu}}{\sqrt{2}} \gamma_{5} \gamma_{\perp_{2} \perp_{1}} \right) \gamma_{\lambda}(1 - \gamma_{5}) \left( \frac{u_{\kappa}}{\sqrt{3}} \gamma_{\perp_{1} \perp_{5}} \right) M_{\mu \nu \kappa}^{[\mu \nu \kappa]}, \quad (5.21) \]

with

\[ M_{\mu \nu \kappa}^{[\mu \nu \kappa]} = Tr \int d\Gamma \bar{\mathcal{P}}_{K_{3}^{(2)}}^{[\mu \nu \kappa]} T S^{\kappa}, \quad (5.23) \]

a \([\mu \nu] \) antisymmetric, negative parity tensor such that \( v_{2 \mu} M_{\mu \nu \kappa}^{[\mu \nu \kappa]} = v_{1 \kappa} M_{\mu \nu \kappa}^{[\mu \nu \kappa]} = 0 \). (5.24)

The unique form for such a tensor is

\[ M_{\mu \nu \kappa}^{[\mu \nu \kappa]} = g_{4}^{(1)} (g_{\perp_{2} \perp_{1}}^{\mu \kappa} v_{1 \perp} - g_{\perp_{1} \perp_{2}}^{\mu \kappa} v_{1 \perp}) = g_{4}^{(1)} (g_{\perp_{2} \perp_{1}}^{\mu \nu} v_{1 \perp} - g_{\perp_{1} \perp_{2}}^{\mu \nu} v_{1 \perp}) \quad (5.25) \]

Hence the matrix elements eq. (5.22) become

\[ g_{4}^{(1)} (g^{\mu \kappa} v_{1}^{\nu} - g^{\nu \kappa} v_{1}^{\mu}) \left( \frac{\bar{u}_{\mu \nu}}{\sqrt{2}} \gamma_{5} \gamma_{\perp_{2} \perp_{1}} \right) \gamma_{\lambda}(1 - \gamma_{5}) \left( \frac{u_{\kappa}}{\sqrt{3}} \gamma_{\perp_{1} \perp_{5}} \right) M_{\mu \nu \kappa}^{[\mu \nu \kappa]} \quad (5.26) \]

(c) \( (\frac{3}{2}^{+} , \frac{1}{2}^{+}) \rightarrow (J_{q_{1} q_{2}} = 0; \frac{1}{2}^{-}) \)

\[ \bar{u} \gamma_{\lambda}(1 - \gamma_{5}) \left( \frac{u_{\mu}}{\sqrt{3}} \gamma_{\perp_{1} \perp_{5}} \right) M_{\mu}, \quad (5.27) \]

with

\[ M_{\mu} = Tr \int d\Gamma \bar{\mathcal{P}}_{K_{3}^{(2)}} T S^{\mu} \quad (5.28) \]

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a vector satisfying \( v_{1\mu}M^\mu = 0 \). Such a vector is uniquely \( v_{2\perp} \), giving rise to just one form factor \( g_5^{(1)} \). The matrix element can be written as

\[
g_5^{(1)} \bar{u} \gamma_\lambda (1 - \gamma_5) \left( \frac{1}{\sqrt{3}} (\not{p}_2 + \omega) \gamma_5 u \right). \tag{5.29}
\]

Thus all the ground state heavy \( \Sigma(\Omega) \) transitions to all the heavy p-wave \( \Sigma(\Omega) \) states are described by just five form factors.

### 5.2 Decay of s-wave heavy baryons to first positive parity excited heavy baryons

In this subsection, we present the counting of form factors for the weak decays of s-wave heavy baryons to the first positive parity excited heavy baryons.

#### 5.2.1 Heavy \( \Lambda_Q \) to first positive parity excited \( \Lambda_{Q'} \)

We list the matrix elements in terms of the final states.

1. \( L_{K_3} = 2, L_{k_3} = 0, L = 2, S_{q_1q_2} = 0, J_{q_1q_2} = 2 \)
2. \( L_{K_3} = 2, L_{k_3} = 2, L = 2, S_{q_1q_2} = 0, J_{q_1q_2} = 2 \)

\[
\frac{1^+}{2} \rightarrow \frac{3^+}{2} : \frac{2}{\sqrt{2}} \bar{u}_{\mu_1} \gamma_5 \gamma_{\perp 2\mu_2} \gamma_\lambda (1 - \gamma_5) u M^{\mu_1\mu_2}.
\tag{5.30}
\]

Here \( M^{\mu_1\mu_2} \) is a positive parity, symmetric, traceless tensor transverse to \( v_{2\perp} \). The only non-vanishing form for this tensor is \( M^{\mu_1\mu_2} = f_1^{(2)} N^{\mu_1\mu_2}(v_1) \) with \( N^{\mu_1\mu_2} \) defined as in eq. (2.30), but with argument \( v_{1\perp} \). Although the Lorentz structure is the same, we will in general have two independent form factors.

\(^6\)From now on we will not write down the full expressions for the matrices \( M \) which appear in the transition amplitudes. In the case of the transitions to the next excited positive parity baryons they all have the general form

\[
M = Tr \int d\Gamma \bar{D}TS
\tag{5.31}
\]

where the \( D \)'s and the \( S \)'s are as defined in subsections 3.5.3 and 3.5.1 respectively.
arising from the cases (1) and (2). Let us call these form factors \( f_1^{(2)} \) and \( f_1^{(2)} \). Hence we can write the matrix elements (5.30) in terms of one of these form factors as

\[
\begin{align*}
\frac{1}{2}^+ &\rightarrow \frac{5}{2}^- : \quad f_1^{(2)} v_1^{\mu_1} v_1^{\mu_2} \bar{u}_{\mu_1 \mu_2} \gamma_\lambda (1 - \gamma_5) u \\
\frac{1}{2}^- &\rightarrow \frac{3}{2}^- : \quad \sqrt{\frac{2}{5}} f_1^{(2)} v_1^{\mu_1} \bar{u}_{\mu_1} \gamma_5 (\not\gamma_1 + \omega) \gamma_\lambda (1 - \gamma_5) u.
\end{align*}
\] (5.32)

The same structure holds with the other form factor.

(3) (a) (i) \( L_{K_3} = 1, L_{K_3} = 1, L = 2, S_{q_1 q_2} = 1, J_{q_1 q_2} = 3 \)

\[
\begin{align*}
\frac{1}{2}^+ &\rightarrow \frac{7}{2}^- : \quad \bar{u}_{\mu_\perp \mu_2} \gamma_\lambda (1 - \gamma_5) u M^{\mu_\perp \mu_2} \\
\frac{1}{2}^+ &\rightarrow \frac{5}{2}^- : \quad \sqrt{\frac{3}{7}} \bar{u}_{\mu_\perp \mu_2} \gamma_5 \gamma_\perp \gamma_\mu \gamma_\lambda (1 - \gamma_5) u M^{\mu_\perp \mu_2}.
\end{align*}
\] (5.33)

Here \( M^{\mu_1 \mu_2} \) is a positive parity, symmetric, traceless tensor transverse with respect to \( v_2 \). It is impossible to construct such a tensor with the two vectors, \( v_1 \) and \( v_2 \) available to us. Hence this pair of decays are not allowed.

(3) (a) (ii) \( L_{K_3} = 1, L_{K_3} = 1, L = 2, S_{q_1 q_2} = 1, J_{q_1 q_2} = 2 \)

\[
\begin{align*}
\frac{1}{2}^+ &\rightarrow \frac{5}{2}^- : \quad \sqrt{2} \bar{u}_{\mu_1 \mu_2} \gamma_5 \gamma_\perp \gamma_\mu \gamma_\lambda (1 - \gamma_5) u M^{[\mu_1 | \mu_2} \\
\frac{1}{2}^+ &\rightarrow \frac{3}{2}^- : \quad \frac{2}{3 \sqrt{5}} (\bar{u}_{\mu_2} \gamma_\perp \mu_1 + \bar{u}_{\mu_1} \gamma_\perp \mu_2) \gamma_\perp \gamma_\mu \gamma_\lambda (1 - \gamma_5) u M^{\mu_1 | \mu_2}. \tag{5.34}
\end{align*}
\]

\( M^{[\mu_1 | \mu_2} \) is a positive parity, mixed symmetry tensor transverse to \( v_2 \). There is a unique nonvanishing form for this tensor giving rise to just one new form factor, \( f_2^{(2)} \), i.e.

\[
M^{[\mu_1 | \mu_2} = i f_2^{(2)} \varepsilon^{\mu_1 \rho \kappa} v_1^\rho v_2^\kappa v_1^{\mu_2} v_1^{\mu_2}. \tag{5.35}
\]

Thus the matrix elements (5.34) can be written as

\[
\begin{align*}
\frac{1}{2}^+ &\rightarrow \frac{5}{2}^- : \quad \sqrt{2} i f_2^{(2)} \varepsilon^{\mu_1 \rho \kappa} v_1^\rho v_2^\kappa v_1^{\mu_2} \bar{u}_{\mu_1 \mu_2} \gamma_5 \gamma_\mu \gamma_\lambda (1 - \gamma_5) u \\
\frac{1}{2}^+ &\rightarrow \frac{3}{2}^- : \quad \frac{2}{3 \sqrt{5}} i f_2^{(2)} \varepsilon^{\mu_1 \rho \kappa} v_1^\rho v_2^\kappa \{v_1^{\mu_2} \bar{u}_{\mu_2} \gamma_\mu_1 + \bar{u}_{\mu_1} (\not\gamma_1 - \omega)\} \gamma_\mu \gamma_\lambda (1 - \gamma_5) u.
\end{align*}
\] (5.36)

Using \( \epsilon \) identities one can write these matrix elements in simpler form as

\[
\begin{align*}
\frac{1}{2}^+ &\rightarrow \frac{5}{2}^- : \quad \sqrt{2} f_2^{(2)} v_1^{\mu_1} v_1^{\mu_2} \bar{u}_{\mu_1 \mu_2} \gamma_\lambda (1 - \gamma_5) u
\end{align*}
\]
\[
\frac{1}{2}^+ \rightarrow \frac{3}{2}^+ : \quad \frac{2}{\sqrt{3}} f_{(2)}^{(2)} v_1^\mu \bar{u}_\mu \gamma_5 (\not{\gamma}_1 + \omega) \gamma_\lambda (1 - \gamma_5) u.
\] (5.37)

(3) (a) (iii) \( L_{K_3} = 1, L_{k_3} = 1, L = 2, S_{q_1q_2} = 1, J_{q_1q_2} = 1 \)

\[
\frac{1}{2}^+ \rightarrow \frac{5}{2}^+ : \quad \bar{u}_{\mu_2} \gamma_\lambda (1 - \gamma_5) u M^{\mu_2}
\]

\[
\frac{1}{2}^+ \rightarrow \frac{3}{2}^+ : \quad \frac{1}{\sqrt{3}} \bar{u} \gamma_5 \gamma_\perp \gamma_\perp (1 - \gamma_5) u M^{\mu_2}.
\] (5.38)

\( M^{\mu_2} \) is an axial vector transverse with respect to \( v_2 \). It is not possible to construct such a vector from the two vectors at our disposal. Hence these decays are forbidden.

(3) (b) (i) \( L_{K_3} = 1, L_{k_3} = 1, L = 1, S_{q_1q_2} = 1, J_{q_1q_2} = 2 \)

\[
\frac{1}{2}^+ \rightarrow \frac{5}{2}^+ : \quad \sqrt{2} \bar{u}_{\mu_2} \gamma_5 \gamma_\perp \gamma_\perp \gamma_\perp \gamma_\perp \gamma_\perp (1 - \gamma_5) u M^{\mu_1\mu_2}
\]

\[
\frac{1}{2}^+ \rightarrow \frac{3}{2}^+ : \quad \frac{2}{3\sqrt{5}} (\bar{u}_\mu \gamma_\perp \gamma_\perp \gamma_\perp \gamma_\perp \gamma_\perp + \bar{u}_{\mu_2} \gamma_\perp \gamma_\perp \gamma_\perp \gamma_\perp \gamma_\perp ) \gamma_\lambda (1 - \gamma_5) u M^{\mu_1\mu_2}.
\] (5.39)

Here \( M^{\mu_1\mu_2} \) is a positive parity, mixed symmetry tensor, transverse with respect to \( v_2 \). There is a unique non vanishing tensor, of this type, giving rise to one form factor, \( f_3^{(2)} \):

\[
M^{\mu_1\mu_2} = i f_3^{(2)} \epsilon_{\mu_1\mu_2\rho\kappa} v_1^\rho v_2^\kappa v_1^\mu.
\] (5.40)

Hence the matrix elements (5.39) can be written as

\[
\frac{1}{2}^+ \rightarrow \frac{5}{2}^+ : \quad \sqrt{2} i f_3^{(2)} \epsilon_{\mu_1\mu_2\rho\kappa} v_1^\rho v_2^\kappa v_1^\mu \bar{u}_{\mu_2} \gamma_5 \gamma_\mu_1 \gamma_\lambda (1 - \gamma_5) u
\]

\[
\frac{1}{2}^+ \rightarrow \frac{3}{2}^+ : \quad \frac{2}{3\sqrt{5}} i f_3^{(2)} \epsilon_{\mu_1\mu_2\rho\kappa} v_1^\rho v_2^\kappa \{ v_1^\mu \bar{u}_\mu \gamma_5 \gamma_\mu_1 + \bar{u}_{\mu_2} (\not{\gamma}_1 - \omega) \gamma_\mu_1 \} \gamma_\lambda (1 - \gamma_5) u.
\] (5.41)

Again using \( \epsilon \) identities these can further be simplified to

\[
\frac{1}{2}^+ \rightarrow \frac{5}{2}^+ : \quad \sqrt{2} f_3^{(2)} v_1^\mu \bar{u}_\mu \gamma_5 (\not{\gamma}_1 + \omega) \gamma_\lambda (1 - \gamma_5) u
\]

\[
\frac{1}{2}^+ \rightarrow \frac{3}{2}^+ : \quad \frac{2}{\sqrt{5}} f_3^{(2)} v_1^\mu \bar{u}_\mu \gamma_5 (\not{\gamma}_1 + \omega) \gamma_\lambda (1 - \gamma_5) u.
\] (5.42)
(3) (b) (ii) \( L_{K_3} = 1, L_{k_3} = 1, L = 1, S_{q_1q_2} = 1, J_{q_1q_2} = 1 \)
\[
\begin{align*}
\frac{1}{2} &\rightarrow \frac{3}{2} : \bar{u}_{\mu_2} \gamma_\lambda (1 - \gamma_5) u M^{\mu_2} \\
\frac{1}{2} &\rightarrow \frac{3}{2} : \frac{1}{\sqrt{3}} \bar{u} \gamma_5 \gamma_{\perp 2\mu_2} \gamma_\lambda (1 - \gamma_5) u M^{\mu_2}. 
\end{align*}
\] (5.43)

We see immediately that these transition amplitudes vanish as it is not possible to construct an axial vector \( M^{\mu_2} \), transverse to \( v_2 \), from the two velocity vectors available to us.

(3) (b) (iii) \( L_{K_3} = 1, L_{k_3} = 1, L = 1, S_{q_1q_2} = 1, J_{q_1q_2} = 0 \)
\[
\begin{align*}
\frac{1}{2} &\rightarrow \frac{1}{2} : \frac{1}{\sqrt{6}} \bar{u} \gamma_\perp 2\mu_1 \gamma_{\perp 2\mu_2} \gamma_\lambda (1 - \gamma_5) u M^{[\mu_1 \mu_2]}. 
\end{align*}
\] (5.44)

Here \( M^{[\mu_1 \mu_2]} \) is a totally antisymmetric, positive parity tensor, transverse with respect to \( v_2 \). There is a unique nonvanishing form for this tensor and hence we get one form factor, \( f^{(2)}_1 \):
\[
M^{[\mu_1 \mu_2]} = i f^{(2)}_4 (\omega^2 - 1) \epsilon^{\mu_1 \mu_2 \rho \rho} v_{2\rho}. 
\] (5.45)

In writing the coupling (5.43) we have exhibited an explicit dynamical threshold factor which would arise, for example, in a dynamical calculation from the different mode structures of excited and ground state baryons in overlap integrals. Hence the matrix element (5.44) becomes
\[
\begin{align*}
\frac{1}{2} &\rightarrow \frac{1}{2} : \frac{1}{\sqrt{6}} i f^{(2)}_4 (\omega^2 - 1) \epsilon^{\mu_1 \mu_2 \rho \rho} v_{2\rho} \bar{u} \gamma_\mu \gamma_{\mu_1} \gamma_{\mu_2} \gamma_\lambda (1 - \gamma_5) u. 
\end{align*}
\] (5.46)

Once more this can be further simplified to
\[
\begin{align*}
\frac{1}{2} &\rightarrow \frac{1}{2} : - \sqrt{6} f^{(2)}_4 (\omega^2 - 1) \bar{u} \gamma_\lambda (1 - \gamma_5) u. 
\end{align*}
\] (5.47)

(3) (c) \( L_{K_3} = 1, L_{k_3} = 1, L = 0, S_{q_1q_2} = 1, J_{q_1q_2} = 1 \)
\[
\begin{align*}
\frac{1}{2} &\rightarrow \frac{3}{2} : \bar{u}_\mu \gamma_\lambda (1 - \gamma_5) u M^\mu \\
\frac{1}{2} &\rightarrow \frac{1}{2} : \frac{1}{\sqrt{3}} \bar{u} \gamma_5 \gamma_{\perp 2\mu} \gamma_\lambda (1 - \gamma_5) u M^\mu. 
\end{align*}
\] (5.48)

Again these transition amplitudes vanish because we cannot construct an axial vector transverse to \( v_2 \).

In conclusion, we see that the transition amplitudes for the ground state heavy \( \Lambda \) to the next positive parity excited \( \Lambda \)'s are described in terms of just five form factors.
5.2.2 Heavy $\Sigma_Q(\Omega_Q)$ to first positive parity excited $\Sigma_Q'(\Omega_Q')$.

We list the matrix elements, in terms of the final states in the usual matrix form as we have done above in the case of decays to p-wave states.

(1) $L_{K_3} = 1, L_{k_3} = 1, L = 2, S_{q_1q_2} = 0, J_{q_1q_2} = 2$

$\left( \frac{3}{2}^+, \frac{1}{2}^+ \right) \rightarrow \left( \frac{3}{2}^+, \frac{1}{2}^+ \right)$

$$
\left( \sqrt{\frac{2}{5}} \bar{u}_{\mu_1\mu_2} \gamma_5 \gamma_{\perp 2\mu_2} \right) \gamma_\lambda (1 - \gamma_5) \left( \frac{1}{\sqrt{3}} u_\nu \gamma_{\perp \mu_1} \gamma_5 u \right) M^{(\mu_1\mu_2)\nu}. 
$$

(5.49)

Here $M^{(\mu_1\mu_2)\nu}$ is a positive parity tensor, symmetric and traceless in $\mu_1, \mu_2$, satisfying

$$
v_{2\mu_1} M^{(\mu_1\mu_2)\nu} = v_{1\nu} M^{(\mu_1\mu_2)\nu} = 0. 
$$

(5.50)

There is just one such tensor giving rise to one form factor, $g^{(2)}_1$:

$$
M^{(\mu_1\mu_2)\nu} = g^{(2)}_1 i (\epsilon^{\mu_1\nu\rho\kappa} v_{1\rho} v_{2\kappa} v_{1\mu_2}^{\mu_1} + \epsilon^{\mu_2\nu\rho\kappa} v_{1\rho} v_{2\kappa} v_{1\mu_1}^{\mu_1}). 
$$

(5.51)

Hence the matrix elements (5.49) can be written as

$$
g^{(2)}_1 i (\epsilon^{\mu_1\nu\rho\kappa} v_{1\rho} v_{2\kappa} v_{1\mu_2}^{\mu_1} + \epsilon^{\mu_2\nu\rho\kappa} v_{1\rho} v_{2\kappa} v_{1\mu_1}^{\mu_1}) \left( \sqrt{\frac{2}{5}} \bar{u}_{\mu_1\mu_2} \gamma_5 \gamma_{\perp 2\mu_2} \right) \gamma_\lambda (1 - \gamma_5) \left( \frac{1}{\sqrt{3}} u_\nu \gamma_{\perp \mu_1} \gamma_5 u \right).
$$

(5.52)

(1) $L_{K_3} = 1, L_{k_3} = 1, L = 1, S_{q_1q_2} = 0, J_{q_1q_2} = 1$

$\left( \frac{3}{2}^+, \frac{1}{2}^+ \right) \rightarrow \left( \frac{3}{2}^+, \frac{1}{2}^+ \right)$

$$
\left( \frac{1}{\sqrt{6}} \bar{u}_{\mu_2} \gamma_5 \gamma_{\perp 2\mu_1} \right) \gamma_\lambda (1 - \gamma_5) \left( \frac{1}{\sqrt{3}} u_\nu \gamma_{\perp \mu_1} \gamma_5 u \right) M^{(\mu_1\mu_2)\nu},
$$

(5.53)

where $M^{(\mu_1\mu_2)\nu}$ is a positive parity tensor antisymmetric in $\mu_1, \mu_2$ and satisfies

$$
v_{2\mu_1} M^{(\mu_1\mu_2)\nu} = v_{1\nu} M^{(\mu_1\mu_2)\nu} = 0. 
$$

(5.54)

We can construct just one such tensor leading to one form factor, $g^{(2)}_7$:

$$
M^{(\mu_1\mu_2)\nu} = g^{(2)}_7 i \epsilon^{\mu_1\mu_2\rho\kappa} v_{1\rho} v_{2\kappa} v_{2\nu}^{\nu}.
$$

(5.55)

Hence, using $\epsilon$ identities, we can write the matrix elements (5.53) as

$$
g^{(2)}_7 \left( \frac{\sqrt{2} v_1^\mu \bar{u}_\mu}{\sqrt{3} u_5 (\omega + \nu_1)} \right) \gamma_\lambda (1 - \gamma_5) \left( \frac{1}{\sqrt{3}} (\nu_2^\mu u_\nu) \gamma_5 u \right).
$$

(5.56)

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\[ L_{K_3} = 1, L_{k_3} = 1, L = 0, S_{q_1q_2} = 0, J_{q_1q_2} = 0 \)
\[ (\frac{3}{2}, \frac{1}{2}) \rightarrow (\frac{1}{2}) \]

\[ \bar{u}\gamma(1 - \gamma_5) \begin{pmatrix} u_\nu \\ \frac{1}{\sqrt{3}}\gamma_{\perp\nu}\gamma_5u \end{pmatrix} M^\nu, \quad (5.57) \]

where \( M^\nu \) is an axial vector, transverse to \( v_1 \). It is not possible to construct such an axial vector. Hence this transition is forbidden.

\[ (2) \quad (a) \quad L_{K_3} = 2, L_{k_3} = 0, L = 2, S_{q_1q_2} = 1, J_{q_1q_2} = 3 \]
\[ (\frac{3}{2}, \frac{1}{2}) \rightarrow (\frac{7}{2}, \frac{5}{2}) \]

\[ \left( \sqrt{\frac{2}{3}} \bar{u}_{\mu\nu} g_\mu g_\nu \right) \gamma(1 - \gamma_5) \begin{pmatrix} u_\nu \\ \frac{1}{\sqrt{3}}\gamma_{\perp\nu}\gamma_5u \end{pmatrix} M^{(\mu\mu\nu)} \quad (5.58) \]

Here \( M^{(\mu\mu\nu)} \) is a positive parity tensor, symmetric and traceless in the labels \( \mu, \mu_1, \mu_2 \) and satisfying the following transversality conditions:

\[ v_{2\mu} M^{(\mu\mu\nu)} = v_{1\nu} M^{(\mu\mu\nu)} = 0. \quad (5.59) \]

The general form for such a tensor is

\[ M^{(\mu\mu\nu)} = g_2^{(2)} N^{(\mu\mu\nu)}(v_1) v_2 \]
\[ + g_3^{(2)} \{ v_1^{(\mu\mu\nu)} v_2 \} - \frac{2}{5} v_1^{(\mu\mu\nu)} g_1^{(\mu\mu\nu)} g_2^{(\mu\mu\nu)} - \frac{1}{5} g_1^{(\mu\mu\nu)} g_2^{(\mu\mu\nu)} g_3^{(\mu\mu\nu)} \} \quad (5.60) \]

Here \( N^{(\mu\mu\nu)} \) is the traceless, transverse tensor defined in eq. (2.30). \( v_{1\perp}, v_{2\perp} \) and \( g_{\perp\perp} \) are as defined in eqs. (5.19-5.20).

Hence the matrix elements, eq. (5.58), for this set of transitions can be written as

\[ \{ g_2^{(2)} v_1^{(\mu\mu\nu)} v_2^{(\mu\mu\nu)} v_2^{(\mu\mu\nu)} + g_3^{(2)} v_1^{(\mu\mu\nu)} v_1^{(\mu\mu\nu)} g_2^{(\mu\mu\nu)} \}
\times \left( \sqrt{\frac{2}{3}} \bar{u}_{\mu\nu} g_\mu g_\nu \right) \gamma(1 - \gamma_5) \begin{pmatrix} u_\nu \\ \frac{1}{\sqrt{3}}\gamma_{\perp\nu}\gamma_5u \end{pmatrix} \] \( (5.61) \)

\[ (2) \quad (b) \quad L_{K_3} = 2, L_{k_3} = 0, L = 2, S_{q_1q_2} = 1, J_{q_1q_2} = 2 \]
the form factors as above except that the five form factors, labelled $i = 2$ transitions of the heavy $\Sigma$ to the first positive parity excited heavy $\Sigma$'s.

Clearly there are two such tensors giving rise to two form factors:

$$M^{[\mu_1\mu_2]} = g_{4}^{(2)}[v_{11}^{\mu_1}(g_{12}^{\mu_2}v_{11}^{\mu_2} - g_{12}^{\mu_2}v_{11}^{\mu_2}) - \frac{1}{2}(v_{11}^{\mu_2}v_{11}^{\mu_2} - g_{12}^{\mu_2}v_{11}^{\mu_2} + v_{11}^{\mu_2}g_{12}^{\mu_2} - g_{12}^{\mu_2}g_{12}^{\mu_2})].$$

(5.64)

In this case we will not simplify the matrix element (5.62) further.

$$(3^+, 1^+) \rightarrow (\frac{3}{2}, \frac{3}{2})$$

where $M^{[\mu_1\mu_2]}$ is a positive parity tensor, antisymmetric with respect to the indices $\mu, \mu_1$ and satisfying the conditions

$$v_{2\mu}M^{[\mu_1\mu_2]} = v_{2\mu}M^{[\mu_1\mu_2]} = v_{1\nu}M^{[\mu_1\mu_2]} = 0.$$  

(5.63)

It is also traceless with respect to the indices $\mu, \mu_2$. There is a unique tensor satisfying these conditions, giving rise to one form factor, $g_{4}^{(2)}$:

$$M^{[\mu_1\mu_2]} = g_{4}^{(2)}[v_{11}^{\mu_1}(g_{12}^{\mu_2}v_{11}^{\mu_2} - g_{12}^{\mu_2}v_{11}^{\mu_2}) - \frac{1}{2}(v_{11}^{\mu_2}v_{11}^{\mu_2} - g_{12}^{\mu_2}v_{11}^{\mu_2} + v_{11}^{\mu_2}g_{12}^{\mu_2} - g_{12}^{\mu_2}g_{12}^{\mu_2})].$$

(5.64)

In this case we will not simplify the matrix element (5.62) further.

$$(3^+, 1^+) \rightarrow (\frac{3}{2}, \frac{3}{2})$$

Here $M^{[\mu]}$ is a positive parity tensor satisfying

$$v_{2\mu}M^{[\mu]} = v_{1\nu}M^{[\mu]} = 0.$$  

(5.66)

Clearly there are two such tensors giving rise to two form factors:

$$M^{[\mu]} = g_{5}^{(2)}v_{11}^{\mu}v_{21}^{\mu} + g_{6}^{(2)}g_{12}^{\mu}.$$  

(5.67)

Hence the matrix elements can be written as

$$(g_{5}^{(2)}v_{11}^{\mu}v_{21}^{\mu} + g_{6}^{(2)}g_{12}^{\mu})\left(\frac{1}{\sqrt{3}}\bar{u}_5\gamma_5\gamma_{2\mu}\right)\gamma_{\lambda}(1 - \gamma_5)\left(\frac{1}{\sqrt{3}}u_\lambda\gamma_{1\nu}\gamma_5\right).$$

(5.68)

Case (3) with $L_{K_3} = 0, L_{k_3} = 2$ gives rise to the same Lorentz structure for the form factors as above except that the five form factors, labelled $g_{i}^{(2)}$ with $i = 2 \ldots 6$, will in general be different from the above five, $g_{i}^{(2)}$, $i = 2 \ldots 6$, because they will involve different overlap integrals.

To summarise we have a total of twelve form factors to describe the transitions of the heavy $\Sigma$ to the first positive parity excited heavy $\Sigma$'s.
5.3 Decay of s-wave baryons to L-wave baryons

We now generalise the results of the previous subsections to the decays of ground state baryons to the “highest weight” L-wave excitations using the wave functions developed in subsection 3.5.4.

5.3.1 Heavy $\Lambda_Q$ to L-wave $\Lambda_{Q'}$.

As before we shall list the matrix elements according to the various final L-wave states with parity $(-1)^L$.

(i) $\Lambda_Q, \frac{1}{2}^+ \rightarrow$ L-wave $\Lambda_{Q'}$ with $S_{q_1 q_2} = 0$.

\[
\begin{align*}
\frac{1}{2}^+ & \rightarrow L + \frac{1}{2}: \quad \bar{u}_{\mu_1 \ldots \mu_L} \gamma_\lambda (1 - \gamma_5) u M^{\mu_1 \ldots \mu_L} \\
\frac{1}{2}^+ & \rightarrow L - \frac{1}{2}: \quad \sqrt{\frac{L}{2L + 1}} \bar{u}_{\mu_2 \ldots \mu_L} \gamma_5 \gamma_{\perp 2} \gamma_\lambda (1 - \gamma_5) u M^{\mu_1 \ldots \mu_L}, \quad (5.69)
\end{align*}
\]

with $M^{\mu_1 \ldots \mu_L}$ an L-th rank, symmetric, traceless tensor, transverse with respect to $v_2$ and with parity $(-1)^L$. Such a tensor is uniquely given by the tensor $N^{\mu_1 \ldots \mu_L} (v_1)$ which is defined as in eq. (2.30) with $k_\perp$ replaced by $v_1 \perp$. Hence there is just one form factor, here called $f_1^{(L)}$, and the transition matrix elements, eq. (5.69), can be written as

\[
\begin{align*}
\frac{1}{2}^+ & \rightarrow L + \frac{1}{2}: \quad f_1^{(L)} v_1^{\mu_1} \ldots v_1^{\mu_L} \bar{u}_{\mu_1 \ldots \mu_L} \gamma_\lambda (1 - \gamma_5) u \\
\frac{1}{2}^+ & \rightarrow L - \frac{1}{2}: \quad f_1^{(L)} \sqrt{\frac{L}{2L + 1}} v_1^{\mu_2} \ldots v_1^{\mu_L} \bar{u}_{\mu_2 \ldots \mu_L} \gamma_5 (\omega + \gamma_1) \gamma_\lambda (1 - \gamma_5) u.
\end{align*}
\]

(5.70)

(ii) $\Lambda_Q, \frac{1}{2}^+ \rightarrow$ to L-wave $\Lambda_{Q'}$ with $S_{q_1 q_2} = 1$.

(a) $S_{q_1 q_2} = 1, J_{q_1 q_2} = L + 1$

\[
\begin{align*}
\frac{1}{2}^+ & \rightarrow L + \frac{3}{2}: \quad \bar{u}_{\mu_1 \ldots \mu_L} \gamma_\lambda (1 - \gamma_5) u M^{\mu_1 \ldots \mu_L} \\
\frac{1}{2}^+ & \rightarrow L + \frac{1}{2}: \quad \sqrt{\frac{L + 1}{2L + 1}} \bar{u}_{\mu_1 \ldots \mu_L} \gamma_5 \gamma_{\perp 2} \gamma_\lambda (1 - \gamma_5) u M^{\mu_1 \ldots \mu_L}, \quad (5.71)
\end{align*}
\]

with $M^{\mu_1 \ldots \mu_L}$ a $(-1)^L$ parity, symmetric tensor satisfying $v_2 \mu M^{\mu_1 \ldots \mu_L} = 0$. It is impossible to construct such an unnatural parity tensor with the two
vectors available to us. Hence \( \Lambda Q \) does not decay to this pair of \( L + \frac{3}{2} \) and \( L + \frac{1}{2} \) states.

(b) \( S_{q_1 q_2} = 1, J_{q_1 q_2} = L \)

\[
\begin{align*}
\frac{1}{2}^+ & \rightarrow L + \frac{1}{2} : \sqrt{2} \bar{u}_{\mu_1 \ldots \mu_L} \gamma_5 \gamma_{\perp 2\mu} \gamma_\lambda (1 - \gamma_5) u M[^{\mu_\mu_1 \mu_\mu_2 \ldots \mu_L}] \\
\frac{1}{2}^+ & \rightarrow L - \frac{1}{2} : \frac{1}{L + 1} \sqrt{\frac{2L}{2L + 1}} (\bar{u}_{\mu_2 \ldots \mu_L} \gamma_{\perp 2\mu_1} \gamma_{\perp 2\mu}) \\
& \quad + (L - 1) \bar{u}_{\mu_1 \mu_3 \ldots \mu_L} \gamma_{\perp 2\mu_2} \gamma_{\perp 2\mu} \gamma_\lambda (1 - \gamma_5) u M[^{\mu_\mu_1 \mu_\mu_2 \ldots \mu_L}].
\end{align*}
\]

with \( M[^{\mu_\mu_1 \mu_\mu_2 \ldots \mu_L}] \) a \((-1)^L\) parity, \( L + 1 \) index mixed symmetry, traceless tensor transverse with respect to \( v_2 \). Such a tensor does exist uniquely, namely

\[
M[^{\mu_\mu_1 \mu_\mu_2 \ldots \mu_L}] = i f_2^{(L)} [\epsilon^{\mu_\mu_1 \rho \kappa} v_{1\rho} v_{2\kappa}, N^{\mu_\mu_2 \ldots \mu_L}] (v_1)
\]

leading to one form factor, \( f_2^{(L)} \) for this pair of transitions. The extra terms indicated by dots in eq. (5.73) are needed to make the tensor traceless but they do not contribute to the matrix elements. Using \( \epsilon \) identities one can then write the matrix elements eq. (5.72) as

\[
\begin{align*}
\frac{1}{2}^+ & \rightarrow L + \frac{1}{2} : \sqrt{2} f_2^{(L)} v_{1\mu_1} \cdots v_{1\mu_L} \bar{u}_{\mu_1 \ldots \mu_L} \gamma_\lambda (1 - \gamma_5) u \\
\frac{1}{2}^+ & \rightarrow L - \frac{1}{2} : f_2^{(L)} \sqrt{\frac{2L}{2L + 1}} v_{1\mu_1} \cdots v_{1\mu_L} \bar{u}_{\mu_1 \ldots \mu_L - 1} \gamma_5 (\nu_1 + \omega) \gamma_\lambda (1 - \gamma_5) u.
\end{align*}
\]

(5.74)

(c) \( S_{q_1 q_2} = 1, J_{q_1 q_2} = L - 1 \)

\[
\begin{align*}
\frac{1}{2}^+ & \rightarrow L - \frac{1}{2} : \bar{u}_{\mu_2 \ldots \mu_L} \gamma_\lambda (1 - \gamma_5) u M^{\mu_\mu_2 \ldots \mu_L} \\
\frac{1}{2}^+ & \rightarrow L - \frac{3}{2} : \sqrt{\frac{L - 1}{2L - 1}} \bar{u}_{\mu_3 \ldots \mu_L} \gamma_{\perp 2\mu_2} \gamma_\lambda (1 - \gamma_5) u M^{\mu_\mu_2 \ldots \mu_L},
\end{align*}
\]

(5.75)

with \( M^{\mu_\mu_2 \ldots \mu_L} \) a \((L - 1)\) index symmetric tensor with parity \((-1)^L\). It is not possible to construct such a tensor with the two vectors available to us, \( v_1 \) and \( v_2 \). Hence the \( \Lambda Q, \frac{1}{2}^+ \) does not decay to these \((L - \frac{1}{2})\) and \((L - \frac{3}{2})\) states.
From the analysis given in this subsection it would seem that there are only two form factors governing the transition of ground state $\Lambda$'s to L-wave excited $\Lambda$'s. However this is misleading as in the general case there are many partitions of the total orbital angular momentum $L$ giving rise to the same Lorentz structure and the same symmetry properties under $p_1 \leftrightarrow p_2$. For each such set of excited $\Lambda$ states there will be two form factors.

5.3.2 Heavy $\Sigma_Q(\Omega_Q)$ to L-wave $\Sigma_{Q'}(\Omega_{Q'})$.

We now list all the matrix elements for the transitions of the ground state $\frac{3}{2}^+, \frac{1}{2}^+$ heavy $\Sigma(\Omega)$ states to the corresponding highest weight L-wave excitations.

\[ (i) \Sigma_Q(\Omega_Q)(\frac{3}{2}^+, \frac{1}{2}^+) \rightarrow \text{L-wave } \Sigma_{Q'}(\Omega_{Q'}) \text{ with } S_{q_1q_2} = 0, J_{q_1q_2} = L; L + \frac{1}{2}, L - \frac{1}{2}. \]

\[
\left( \frac{-i\mu_1...\mu_L}{\sqrt{2L+1}} \gamma_{\mu_2...\mu_L} \gamma_{\mu_1} \right) \gamma_\lambda (1 - \gamma_5) \left( \frac{u_\nu}{\sqrt{3}} \gamma_{\nu} \gamma_5 u \right) M^{\mu_1...\nu L \lambda}, \quad (5.76)
\]

with $M^{\mu_1...\nu L \lambda}$ a $(L + 1)$ rank, parity $(-1)^L$ tensor, symmetric and traceless in the indices $\mu_1 \ldots \mu_L$ and satisfying

\[ v_{2\mu} M^{\mu_1...\nu L \lambda} = v_{1\nu} M^{\mu_1...\mu L \lambda} = 0. \quad (5.77) \]

The general unique form of this tensor is

\[
M^{\mu_1...\nu L \lambda} = g_1^{(L)} i v_{1\nu} v_{2\lambda} \left\{ \sum_{i=1}^L \epsilon^{\mu_1...\nu \rho_1...\mu_L} N^{\mu_1...\nu L \lambda} \right\} (v_1)
\]

\[
- \frac{2}{L + 1} \sum_{i<j} g_2^{\mu_1...\mu_L} \epsilon^{\mu_\nu \rho \kappa} N^{\mu_1...\nu L \lambda} (v_1)(v_2) \right\}, \quad (5.78)
\]

leading to just one form factor, $g_1^{(L)}$. When one substitutes this in eq. (5.77) we see that the second term does not contribute, and in fact from the first term only the terms without the traces in the tensor $N(v_1)$ survive giving the matrix elements as

\[
\begin{align*}
&\left( \frac{-i\mu_1...\mu_L}{\sqrt{2L+1}} \gamma_{\mu_2...\mu_L} \gamma_{\mu_1} \right) \gamma_\lambda (1 - \gamma_5) \left( \frac{u_\nu}{\sqrt{3}} \gamma_{\nu} \gamma_5 u \right) \times \\
&\times \left( \frac{-i\mu_1...\mu_L}{\sqrt{2L+1}} \gamma_{\mu_2...\mu_L} \gamma_{\mu_1} \right) \gamma_\lambda (1 - \gamma_5) \left( \frac{u_\nu}{\sqrt{3}} \gamma_{\nu} \gamma_5 u \right). \quad (5.79)
\end{align*}
\]
(ii) $\Sigma_Q(\Omega_Q), \frac{1^+}{2} \rightarrow$ to L-wave $\Sigma_{Q'}(\Omega_{Q'})$ with $S_{q_1q_2} = 1.$

(a) $(\frac{3^+}{2}, \frac{1^+}{2}) \rightarrow J_{q_1q_2} = L + 1; L + \frac{3}{2}, L + \frac{1}{2}$

\[
\left( \frac{\bar{u}_{\mu_1\ldots\mu_L}}{\sqrt{\frac{L+1}{2L+3}}} \bar{u}_{\mu_1\ldots\mu_L} \gamma_5 \gamma_{\perp 2\mu} \right) \gamma_\lambda (1 - \gamma_5) \left( \frac{u_\nu}{\sqrt{3}} \gamma_{\perp 1\nu} \gamma_5 u \right) M^{\{\mu_1\ldots\mu_L\}\nu}, \tag{5.80}
\]

with $M^{\{\mu_1\ldots\mu_L\}\nu}$ a parity $(-1)^L$ tensor, symmetric and traceless in $\{\mu_1 \ldots \mu_L\}$, satisfying

\[
v_{2\mu} M^{\{\mu_1\ldots\mu_L\}\nu} = v_{1\nu} M^{\{\mu_1\ldots\mu_L\}\nu} = 0. \tag{5.81}
\]

Thus there are two form factors, as $M^{\{\mu_1\ldots\mu_L\}\nu}$ can in general be written as

\[
M^{\{\mu_1\ldots\mu_L\}\nu} = g_2^{(L)} v_{1\nu} N^{\mu_1\ldots\mu_L} (v_1)
+ g_3^{(L)} \left[ \bar{v}_{\mu_2+1} N^{\mu_1\ldots\mu_L} (v_1) + \sum_{i=1}^L g_{\mu_2+1}^{\nu} N^{\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_L} (v_1) + \ldots \right]. \tag{5.82}
\]

The dots in the above equation stand for terms required to make the tensor traceless but these do not contribute to the matrix elements. Thus the matrix elements, eq. (5.80), become

\[
\left( \frac{\bar{u}_{\mu_1\ldots\mu_L}}{\sqrt{\frac{L+1}{2L+3}}} \bar{u}_{\mu_1\ldots\mu_L} \gamma_5 \gamma_{\perp 2\mu} \right) \gamma_\lambda (1 - \gamma_5) \left( \frac{u_\nu}{\sqrt{3}} \gamma_{\perp 1\nu} \gamma_5 u \right) \times \left\{ g_2^{(L)} v_1^{\mu_1} \ldots v_1^{\mu_L} v_2^{\nu} + g_3^{(L)} v_1^{\mu_1} \ldots v_1^{\mu_L} g^{\nu} \right\} \tag{5.83}
\]

(b) $(\frac{3^+}{2}, \frac{1^+}{2}) \rightarrow J_{q_1q_2} = L; L + \frac{1}{2}, L - \frac{1}{2}$

\[
\left( \frac{\bar{u}_{\mu_1\ldots\mu_L}}{\sqrt{\frac{L+1}{2L+1}}} \bar{u}_{\mu_1\ldots\mu_L} \gamma_5 \gamma_{\perp 2\mu} \right) \gamma_\lambda (1 - \gamma_5) \left( \frac{u_\nu}{\sqrt{3}} \gamma_{\perp 1\nu} \gamma_5 u \right) M^{[\mu_1]\ldots[\mu_L\nu]} \tag{5.84}
\]

with $M^{[\mu_1]\ldots[\mu_L\nu]}$ a parity $(-1)^L$ tensor, mixed symmetric and traceless in $\mu, \mu_1 \ldots \mu_L$, such that

\[
v_{2\mu} M^{[\mu_1]\ldots[\mu_L\nu]} = v_{1\nu} M^{[\mu_1]\ldots[\mu_L\nu]} = v_{1\nu} M^{[\mu_1]\ldots[\mu_L\nu]} = 0. \tag{5.85}
\]

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The unique form for such a tensor is

\[ M^{[\mu_1\mu_2\ldots\mu_L\nu]} = g_4^{(L)} [(g_{12}^{\mu_2} v_{12}^{\mu_1} - g_{12}^{\mu_1} v_{12}^{\mu_2}) N^{\mu_2\ldots\mu_L}(v_1) \]

\[ - \frac{1}{L} \sum_{i=2}^{L} g_{i2}^{\mu_1} (v_{i1}^{\mu_1} N^{\mu_2\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_L}(v_1) - g_{i2}^{\mu_1} v_{i1}^{\nu} N^{\mu_2\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_L}(v_1)) \]

\[ + \frac{1}{L} \sum_{i=2}^{L} (\mu, \mu_1 \text{interchanged})]. \]

(5.86)

Thus we have one form factor \( g_4^{(L)} \) to describe these transitions.

(c) \( \left( \frac{3}{2}, \frac{1}{2}^+ \right) \rightarrow J_{q_1 q_2} = L - 1; L - \frac{1}{2}, L - \frac{3}{2} \)

\[ \left( \frac{-1}{2L-1} \bar{u}_{\mu_2\ldots\mu_L} \gamma_5 \gamma_{\perp 2} \mu L \right) \gamma_\lambda (1 - \gamma_5) \left( \frac{1}{\sqrt{3}} \gamma_{\perp 1} \gamma_{\perp 2} \mu L \right) M^{\mu_2\ldots\mu_L \nu}, \]

(5.87)

with \( M^{\mu_2\ldots\mu_L \nu} \) a parity \((-1)^L\) tensor, symmetric and traceless in \( \mu_2\ldots\mu_L \), such that

\[ v_{2\mu_2} M^{\mu_2\ldots\mu_L \nu} = v_{1\nu} M^{\mu_2\ldots\mu_L \nu} = 0. \]

(5.88)

One can construct two such tensors giving rise to two form factors, \( g_5^{(L)} \) and \( g_6^{(L)} \), i.e.

\[ M^{\mu_2\ldots\mu_L \nu} = g_5^{(L)} N^{\mu_2\ldots\mu_L}(v_1) v_{2\perp}^{\nu} \]

\[ + g_6^{(L)} \left\{ \sum_{i=2}^{L} [g_{i2}^{\mu_2} v_{i2}^{\mu_1} + v_{i2}^{\mu_1} v_{i2}^{\nu} + \omega v_{i2}^{\nu}] N^{\mu_2\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_L}(v_1) \right\} \]

\[ - \frac{1}{L+1} \sum_{i<j} g_{i2}^{\mu_2} g_{j2}^{\nu} N^{\rho \mu_2\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_L}(v_1) \}. \]

(5.89)

Here too the total number of form factors is not just six. As in the \( \Lambda_Q \) case there are many partitions of the total orbital angular momentum \( L \) giving rise to the same Lorentz structure and the same symmetry properties under \( p_1 \leftrightarrow p_2 \). For each such partition there will be five form factors describing the transitions.

### 5.4 Bjorken sum rule for heavy baryon decays

In this subsection we give the complete contributions of the s-wave, p-wave and the first positive parity excitations, as well as of the highest weight \( L \)-wave states, to the Bjorken sum rule for the decays of the heavy s-wave
baryons. Other contributions can be calculated easily given the wave functions.

### 5.4.1 Bjorken sum rule for $\Lambda_Q$ decays

Let us now consider the contributions of the various $\Lambda_Q \rightarrow \Lambda_{Q'}^*$ transitions to the Bjorken sum rule. Again we use the same trick as in the mesonic case in that we first calculate the longitudinal transition amplitudes. In the baryonic case one needs the Clebsch-Gordon decomposition

$$u_{\mu_1 \ldots \mu_j}(\pm \frac{1}{2}) = \sqrt{\frac{j + 1}{2j + 1}} \frac{j!}{(2j - 1)!!} u(\pm \frac{1}{2}) \epsilon_{\mu_1}(0) \cdots \epsilon_{\mu_j}(0) + \cdots \quad (5.90)$$

where the ellipsis at the end of (5.90) stand for terms involving transverse spin 1 polarisation vectors. These, however, are not needed in explicit form since they do not contribute to the longitudinal helicity projection because of the HQET structure of the transitions. Upon squaring the longitudinal amplitudes, summing over the respective degenerate partners and dividing out the $K_L$-factor [3] one obtains

$$1 = |f(\omega)|^2 + \sum_{L \geq 1} (\omega^2 - 1)^L \frac{L!}{(2L - 1)!!} (|f_1^{(L)}(\omega)|^2 + 2|f_2^{(L)}(\omega)|^2 + 2|f_3^{(2)}(\omega)|^2 + 9|f_4^{(2)}(\omega)|^2) + \ldots \quad (5.91)$$

Here $f(\omega)$ is the single form factor appearing in the ground state to ground state heavy $\Lambda_Q$ decay treated in [4] and [5], with $f(1) = 1$. In this sum rule we have included all the contributions up to the first positive parity excitations. However, for the general $L$ case we have included only one of the many highest weight contributions. The other possible highest weight states will give exactly the same form of contributions as in the sum above. We have also not shown the contributions from the other types of higher excitations i.e. other than highest weight. However our method of construction is general enough to allow us to calculate the contribution of the decay to any arbitrary excitation given its angular momentum structure.
5.4.2 Bjorken sum rule for $\Sigma_{Q}(\Omega_{Q})$ decays

Following the same general procedure one can calculate the contribution of any excitation to the Bjorken sum rule for the decay of $\Sigma_{Q}$ or $\Omega_{Q}$. We write the sum rule as:

$$1 = |g_{1}^{(0)}|^2 \frac{(2 + \omega^2)}{3} + |g_{2}^{(0)}|^2 \frac{(\omega^2 - 1)^2}{3} - (g_{1}^{(0)} g_{2}^{(0)*} + g_{1}^{(0)} g_{2}^{(0)*}) \frac{\omega(\omega^2 - 1)}{3}$$

$$+ \sum_{L \geq 1} \left\{ \frac{1}{3} |g_{1}^{(L)}|^2 \frac{(L + 1)!}{(2L - 1)!!} L(\omega^2 - 1)^L \right. $$

$$+ \frac{1}{3} \frac{(L + 1)!}{(2L + 1)!!} (\omega^2 - 1)^L [ |g_{2}^{(L)}|^2 (\omega^2 - 1)^2 + |g_{3}^{(L)}|^2 (L + 1) ((L + 1) \omega^2 + (L + 2))$$

$$+ (g_{2}^{(L)} g_{3}^{(L)*} + g_{3}^{(L)} g_{2}^{(L)*})(L + 1) \omega(\omega^2 - 1)]$$

$$+ \frac{2}{3} |g_{1}^{(L)}|^2 \frac{(L + 1)!}{L (2L - 1)!!} (\omega^2 - 1)^L$$

$$+ \frac{1}{3} \frac{(L - 1)!}{(2L - 1)!!} (2L - 1) (\omega^2 - 1)^{L - 2} [ |g_{5}^{(L)}|^2 (\omega^2 - 1)^2 + |g_{6}^{(L)}|^2 ((L - 1) \omega^2 + L (L - 1))$$

$$+ (g_{5}^{(L)} g_{6}^{(L)*} + g_{6}^{(L)} g_{5}^{(L)*})(L - 1) (\omega^2 - 1) \omega]$$

$$+ \frac{2}{3} |g_{2}^{(2)}|^2 (\omega^2 - 1)^2 + \ldots. \right\} (5.92)$$

where the form factors $g_{1}^{(0)}$ and $g_{2}^{(0)}$ parametrize the $\Sigma$ s-wave to s-wave decays, [4], [5].

$$(-g_{1}^{(0)} g_{\mu} + g_{2}^{(0)} u_{\mu} v_{\nu}) \left( \frac{\bar{u}_{\mu}}{\sqrt{3}} \gamma_{1} \gamma_{2} \right) \gamma_{\lambda} (1 - \gamma_{5}) \left( \frac{u_{\nu}}{\sqrt{3}} \gamma_{1} \gamma_{2} \gamma_{5} \right). (5.93)$$

As one can see in subsection 5.1.2, there is no form factor $g_{6}^{(1)}$, whereas it seems to be present in the sum above. However, this is misleading as the factor $(L - 1)$ always occurring with $g_{6}^{(L)}$ kills $g_{6}^{(1)}$.

As in the $\Lambda_{Q}$ case, the above sum rule contains all the contributions up to the first positive parity excitations if one also includes the contributions from the form factors $g_{i}^{(2)}, i = 2 \ldots 6$, mentioned after eq. (5.68). These give the same formal structure as the $g_{i}^{(2)}, i = 2 \ldots 6$. Again, for the general $L$ case we have included only one of the many highest weight contributions and have not shown the contributions from the other types of (non-highest weight) excitations, which can of course be calculated easily, if so desired.
The Bjorken sum rule for $\Sigma_Q(\Omega_Q)$ decays has also been studied in [21]. However, in that work the contribution of $g_1^{(1)}$ is missing and further the factor in front of the $g_1^{(1)}$ contribution is $\frac{12}{9}$ rather than $\frac{20}{9}$.

6 Conclusions

In this paper we have presented a general method to construct wavefunctions for arbitrary orbital excitations of heavy hadrons. In doing this we have utilised the representations of the group $L \otimes O(3,1)$. We then used these wave functions to calculate the form factors for the weak transitions of s-wave heavy hadrons to arbitrarily excited heavy hadrons. Of course, given these wave functions we can also study the form factor structure for the transitions between arbitrary heavy excited states as the need arises. The contributions of excited states to the Bjorken sum rule have also been worked out in some detail. One can also use these same projection operators for studying transitions from heavy to light hadrons. This is currently under investigation and the results will be presented elsewhere.

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A Normalisation

In this appendix we present a general approach to the normalisation of the projection operators for mesons and baryons. The polarisation vectors for a meson of spin $J$ are normalised as

$$\epsilon^{*\mu_1...\mu_J}\epsilon_{\mu_1...\mu_J} = (-1)^J,$$

and the generalised Rarita-Schwinger spinors for baryons with spin $(J + \frac{1}{2})$ are normalised such that

$$\bar{u}^{\mu_1...\mu_J}u_{\mu_1...\mu_J} = (-1)^J2M.$$  \hspace{1cm} (A.2)

Recall that both the meson and baryon polarisation tensors are symmetric and traceless in all the indices.

The meson projection operators are relatively easy to normalise because they are all of the form

$$N^{\mu_1...\mu_L}(k)(1 + \frac{\not\epsilon}{2})\Gamma_{\mu_1...\mu_L}(1 - \frac{\not\epsilon}{2}).$$  \hspace{1cm} (A.3)

Thus to normalise $\Gamma_{\mu_1...\mu_L}$ we consider

$$Tr\left(\frac{1 - \not\epsilon}{2}\Gamma_{\mu_1...\mu_L}^*(\frac{1 + \not\epsilon}{2})\Gamma_{\nu_1...\nu_L}\mu_1...\mu_L = 2M,$$

where

$$G^{\mu_1...\mu_L;\nu_1...\nu_L} = \int d^4k d^4k' N^{\mu_1...\mu_L}(k)N^{\nu_1...\nu_L}(k)f(k,k')$$  \hspace{1cm} (A.5)
and \( f(k, k') \) is an appropriate weight function. This \( G^{\mu_1 \ldots \mu_L, \nu_1 \ldots \nu_L} \) is the required “metric” tensor to take the trace over the Lorentz indices in eq. \( (A.4) \). This tensor \( G \) must reflect the symmetry and trace properties of \( N^{\mu_1 \ldots \mu_L} \). Hence it must be separately symmetric and traceless in the sets of indices \((\mu_1, \ldots, \mu_L)\) and \((\nu_1, \ldots, \nu_L)\). Also

\[
v_{\mu_1} G^{\mu_1 \ldots \mu_L, \nu_1 \ldots \nu_L} = v_{\nu_1} G^{\mu_1 \ldots \mu_L, \nu_1 \ldots \nu_L} = 0. \tag{A.6}
\]

Of course, in general, such a tensor consists of many terms but fortunately all our polarisation tensors have at least \((L - 1)\) indices and are traceless so that we need only the first few terms in \( G \). These are

\[
G^{\mu_1 \ldots \mu_L, \nu_1 \ldots \nu_L} = \frac{1}{L!} [g_{\perp}^{\mu_1 \ldots \mu_L, \nu_1 \ldots \nu_L} \\
- \frac{2}{2L - 1} \sum_{i=2}^{L} g_{\perp}^{\mu_1 \nu_i} g_{\perp}^{\nu_1 \mu_2 \ldots \mu_{i-1} \mu_1 \ldots \mu_L, \nu_{i-1} \nu_{i+1} \ldots \nu_L}] + \ldots. \tag{A.7}
\]

The dots indicate terms which necessarily contain \( g_{\perp}^{\mu_2 \nu_3} \) plus other such terms which annihilate on the \( \epsilon \)'s. In the above equation we have used the following definitions

\[
g_{\perp}^{\mu_1 \ldots \mu_L, \nu_1 \ldots \nu_L} = \sum_{i=1}^{L} g_{\perp}^{\mu_1 \nu_i} g_{\perp}^{\mu_2 \ldots \mu_L, \nu_1 \ldots \nu_{i-1} \nu_{i+1} \ldots \nu_L} \\
\vdots \\
g_{\perp}^{\mu_1 \mu_2 \nu_3, \nu_1 \nu_2 \nu_3} = g_{\perp}^{\mu_1 \nu_1} (g_{\perp}^{\mu_2 \nu_2} g_{\perp}^{\nu_3} + g_{\perp}^{\mu_2 \nu_3} g_{\perp}^{\nu_2}) \\
+ g_{\perp}^{\mu_2 \nu_2} (g_{\perp}^{\mu_1 \nu_1} g_{\perp}^{\nu_3} + g_{\perp}^{\mu_1 \nu_3} g_{\perp}^{\nu_2}) \\
+ g_{\perp}^{\mu_1 \nu_3} (g_{\perp}^{\mu_2 \nu_2} g_{\perp}^{\nu_1} + g_{\perp}^{\mu_2 \nu_1} g_{\perp}^{\nu_3}). \tag{A.8}
\]

We have a further simplification as in the normalisation equation \( (A.4) \) we can drop the subscript \( \perp \) from the \( g^{\mu \nu} \)'s appearing in the definition of the tensor \( G \) as the velocity vector \( v_\mu \) will always annihilate on the \( \Gamma \)'s. All our normalisations of the mesonic wavefunctions have been done with the “metric” tensor in eq. \( (A.7) \).

For the general baryonic case, the situation is very complicated. However, in the present paper, we have only considered baryonic projection operators which are either fully symmetric in the Lorentz indices or at most have the form

\[
A[^{\mu_1 \mu_2} \mu_3 \ldots \mu_L] \Psi[^{\mu_1 \mu_2} \mu_3 \ldots \mu_L], \tag{A.9}
\]
where $A[^\mu_1^\mu_2]\ldots[^\mu_L]$ is a tensor antisymmetric in $\mu_1\mu_2$ and symmetric under interchange of the other labels and traceless with respect to any pair of indices. For the symmetric $\Psi$'s we use the same “metric” as in eq. (A.7), whereas for the mixed symmetric $\Psi$'s we need the following:

$$G[^\mu_1^\mu_2]\ldots[^\mu_L;[^\nu_1^\nu_2]\ldots^\nu_L] = \frac{1}{2(L-1)!}[(g_\perp^\mu^\nu g_\perp^\mu_1^\nu_1 - g_\perp^\mu_1^\nu_1 g_\perp^\mu^\nu)g_\perp^\mu_2\ldots^\nu_L]
+ \frac{1}{L} \sum_{i=2}^{L} g_\perp^{\mu_i^\nu_i} (g_\perp^\mu_1^\nu g_\perp^\nu_1^\mu_2\ldots^\nu_{i-1}^\mu_{i+1}\ldots^\mu_L^\nu_2\ldots^\nu_L
- g_\perp^\mu_1^\nu g_\perp^\mu_2\ldots^\nu_{i-1}^\mu_{i+1}\ldots^\mu_L^\nu_2\ldots^\nu_L)
+ \frac{1}{L} \sum_{i=2}^{L} g_\perp^\mu_1^\mu_i^\mu_j^\nu_i (g_\perp^\mu_1^\nu g_\perp^\nu_1^\mu_2\ldots^\nu_{i-1}^\mu_{i+1}\ldots^\mu_L^\nu_2\ldots^\nu_L
- g_\perp^\mu_1^\nu g_\perp^\mu_2\ldots^\nu_{i-1}^\mu_{i+1}\ldots^\mu_L^\nu_2\ldots^\nu_L)]$$

(A.10)

The extra terms not shown necessarily involve $g_\perp^\mu_1^\mu_2$ and generally $g_\perp^\mu_1^\mu_j$ with $j > i \geq 2$. These extra terms annihilate on the heavy side spinor functions $\Psi_{\mu_1\ldots^\mu_L}$.

## B Mixed Symmetry Baryon Projection Operators

The baryon projection operators, which we have considered in this paper, contain tensor spinors $\Psi_{\mu_1\ldots^\mu_L}$ which are either fully symmetric in the indices or are of mixed symmetry of the form $[^\mu_1^\mu_2]\ldots^\nu_L$. These spinors satisfy the Dirac equation and the transversality condition

$$\gamma^\mu_i \Psi_{\mu_1\ldots^\mu_L} = 0,$$

(B.1)
on each label. To construct these $\Psi$'s we note once again that if $\phi$ satisfies the Dirac equation then $\gamma^\mu_1^\gamma^\mu_3^5\phi$ and $\gamma^\mu_1^\gamma^\mu_3^\gamma^\mu_2^5\phi$ also satisfy the Dirac equation plus, obviously, the transversality condition (B.1). Thus the construction of the fully symmetric and fully antisymmetric tensors is obvious.

We now demonstrate how we have constructed the mixed tensors e.g. eqs. (3.39)-(3.41), (3.47) and the generalisation to eq. (3.109). To construct the
projection operator \( [3.35] \) we need to construct a traceless, mixed symmetry tensor from \( \chi_{\perp}^{1,\mu} \) and \( N^{\{\mu_1\mu_2\}} \) to obtain total angular momentum 2. We decompose

\[
\begin{array}{ccc}
\phantom{X} & \otimes & \phantom{X} \\
\end{array}
\begin{array}{ccc}
\phantom{X} & \oplus & \phantom{X}
\end{array}.
\]

(B.2)

Thus to construct the mixed symmetric tensor we simply subtract the fully symmetric tensor from the direct product. The fully symmetric tensor is

\[
\phi^{\{\mu_1\mu_2\}} = \frac{1}{3}[\chi_{\perp}^{1,\mu} N^{\{\mu_1\mu_2\}} + \chi_{\perp}^{1,\mu_1} N^{\{\mu_2\}} + \chi_{\perp}^{1,\mu_2} N^{\{\mu_1\}}]
\]

(B.3)

and hence the mixed symmetry tensor is

\[
\chi_{\perp}^{1,\mu} N^{\{\mu_1,\mu_2\}} - \phi^{\{\mu_1\mu_2\}}.
\]

(B.4)

We make this traceless and write the mixed symmetric, traceless tensor as

\[
\frac{2}{3}[2 \chi_{\perp}^{1,\mu} N^{\{\mu_1,\mu_2\}} - \chi_{\perp}^{1,\mu_1} N^{\{\mu_2\}} - \chi_{\perp}^{1,\mu_2} N^{\{\mu_1\}} + g_{1,\mu_1}^{\mu_2} N^{\{\nu\}} - \frac{1}{2} g_{1,\mu_1}^{\mu_2} N^{\{\nu\}} - \frac{1}{2} g_{1,\mu_2}^{\mu_1} N^{\{\nu\}}] \times \frac{2}{3}[2 \gamma_{\perp,\mu_1} \gamma_5 u_{\mu_1,\mu_2} - \gamma_{\perp,\mu_1} \gamma_5 u_{\mu_2} - \gamma_{\perp,\mu_2} \gamma_5 u_{\mu_1}].
\]

(B.5)

Now for the \( \frac{5}{2}^+ \) case, eq. (3.40), we need to multiply the above tensor by a similar mixed symmetry, traceless tensor constructed with \( \gamma_{\perp,\mu} \gamma_5 u_{\mu_1,\mu_2} \) i.e.

\[
\frac{2}{3}[2 \chi_{\perp}^{1,\mu} N^{\{\mu_1,\mu_2\}} - \chi_{\perp}^{1,\mu_1} N^{\{\mu_2\}} - \chi_{\perp}^{1,\mu_2} N^{\{\mu_1\}} + g_{1,\mu_1}^{\mu_2} N^{\{\nu\}} - \frac{1}{2} g_{1,\mu_1}^{\mu_2} N^{\{\nu\}} - \frac{1}{2} g_{1,\mu_2}^{\mu_1} N^{\{\nu\}}] \times \frac{2}{3}[2 \gamma_{\perp,\mu_1} \gamma_5 u_{\mu_1,\mu_2} - \gamma_{\perp,\mu_1} \gamma_5 u_{\mu_2} - \gamma_{\perp,\mu_2} \gamma_5 u_{\mu_1}].
\]

(B.6)

Hence we obtain

\[
\frac{1}{3}[2 \chi_{\perp}^{1,\mu} N^{\{\mu_1,\mu_2\}} - \chi_{\perp}^{1,\mu_1} N^{\{\mu_2\}} - \chi_{\perp}^{1,\mu_2} N^{\{\mu_1\}} + g_{1,\mu_1}^{\mu_2} N^{\{\nu\}} - \frac{1}{2} g_{1,\mu_1}^{\mu_2} N^{\{\nu\}} - \frac{1}{2} g_{1,\mu_2}^{\mu_1} N^{\{\nu\}}] \times \frac{2}{3}[2 \gamma_{\perp,\mu_1} \gamma_5 u_{\mu_1,\mu_2} - \gamma_{\perp,\mu_1} \gamma_5 u_{\mu_2} - \gamma_{\perp,\mu_2} \gamma_5 u_{\mu_1}].
\]

(B.7)

We simplify this by noting that both the factors in the product are symmetric under \( \mu_1 \leftrightarrow \mu_2 \), so that we can write the product as

\[
\frac{1}{3}[2 \chi_{\perp}^{1,\mu} N^{\{\mu_1,\mu_2\}} - \chi_{\perp}^{1,\mu_1} N^{\{\mu_2\}} - \chi_{\perp}^{1,\mu_2} N^{\{\mu_1\}} + g_{1,\mu_1}^{\mu_2} N^{\{\nu\}} - \frac{1}{2} g_{1,\mu_1}^{\mu_2} N^{\{\nu\}} - \frac{1}{2} g_{1,\mu_2}^{\mu_1} N^{\{\nu\}}] \times \frac{2}{3}[2 \gamma_{\perp,\mu_1} \gamma_5 u_{\mu_1,\mu_2} - \gamma_{\perp,\mu_1} \gamma_5 u_{\mu_2} - \gamma_{\perp,\mu_2} \gamma_5 u_{\mu_1}].
\]

(B.8)
But now the second factor is antisymmetric under $\mu \leftrightarrow \mu_1$. Therefore it projects out the antisymmetric piece from the first term leading to
\[
\frac{1}{3} \left[ 1^{1.\mu} N^{\{\mu_1 \mu_2\}} - \chi^{1.\mu_1} N^{\{\mu_2\}} - \frac{1}{2} g^{\mu_1} \chi^{1.\nu} N^{\{\nu\mu_2\}} + \frac{1}{2} g^{\mu_1} \chi^{1.\nu} N^{\{\nu\mu_2\}} \right] \times \left( \gamma_{\perp \mu} \gamma_5 u_{\mu_1 \mu_2} - \gamma_{\perp \mu_1} \gamma_5 u_{\mu_2} \right). \tag{B.9}
\]
This is precisely eqn. (3.39) combined with eq. (3.40) upto a normalisation factor.

To construct the other $\binom{3^+}{2}$, eq. (3.41), member of the degenerate multiplet $\binom{5^+}{2}, \binom{3^+}{2}$ we proceed as follows. Here we need to construct a mixed symmetry tensor from $\gamma_{\perp \mu}, \gamma_{\perp \mu_1}$ and $u_{\mu_2}$. We can take either
\[
\begin{array}{c}
\begin{array}{c}
\bigotimes \bigotimes \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bigoplus \\
\end{array}
\end{array}
\]
and then
\[
\begin{array}{c}
\begin{array}{c}
\bigotimes \bigotimes \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bigoplus \bigoplus \\
\end{array}
\end{array}
\]

or
\[
\begin{array}{c}
\begin{array}{c}
\bigotimes \bigotimes \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bigoplus \bigoplus \\
\end{array}
\end{array}
\]

To make contact with the other member of the multiplet and also for later generalisation to the $L > 2$ case we follow the first route. We first construct the symmetric tensor
\[
F_{\mu_1 \mu_2} = \frac{1}{2} (\gamma_{\perp \mu_1} u_{\mu_2} + \gamma_{\perp \mu_2} u_{\mu_1}) \tag{B.13}
\]
Hence the traceless, mixed symmetry tensor is
\[
\frac{2}{3} \gamma_{\perp \mu} F_{\mu_1 \mu_2} - \frac{1}{3} \gamma_{\perp \mu_1} F_{\mu_2 \mu_2} - \frac{1}{3} \gamma_{\perp \mu_2} F_{\mu_1 \mu_1} \tag{B.14}
\]
Following the same procedure as in the $\binom{5^+}{2}$ case above the product
\[
\begin{array}{c}
\begin{array}{c}
\frac{1}{3} \left[ 2 \chi^{1.\mu} N^{\{\mu_1 \mu_2\}} - \chi^{1.\mu_1} N^{\{\mu_2\}} - \chi^{1.\mu_2} N^{\{\mu_1\}} \right] \\
\end{array}
\end{array}
\]
\[
+ g^{\mu_1} \chi^{1.\nu} N^{\{\nu\mu_2\}} - \frac{1}{2} g^{\mu_1} \chi^{1.\nu} N^{\{\nu\mu_2\}} - \frac{1}{2} g^{\mu_1} \chi^{1.\nu} N^{\{\nu\mu_1\}} \right] \times \left( \gamma_{\perp \mu} \gamma_5 u_{\mu_1 \mu_2} - \gamma_{\perp \mu_1} \gamma_5 u_{\mu_2} \right). \tag{B.15}
\]

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reduces to
\[
\frac{1}{3} \chi_{\perp}^{1,\mu} N^{\{\mu_1\mu_2\}} - \chi_{\perp}^{1,\mu_1} N^{\{\mu_2\}} - \frac{1}{2} g_{\perp}^{\mu_1\mu_2} \chi_{\perp}^{1,\nu} N^{\{\nu_1\}} + \frac{1}{2} g_{\perp}^{\mu_1\mu_2} \chi_{\perp}^{1,\nu} N^{\{\nu_2\}} \times \frac{1}{2} \left\{ \gamma_{\perp \mu} \gamma_{\perp \mu_1} u_{\mu_2} + \left( \gamma_{\perp \mu} \gamma_{\perp \mu_2} u_{\mu_1} - \gamma_{\perp \mu_1} \gamma_{\perp \mu_2} u_{\mu} \right) \right\}
\] (B.16)

which is the same as eq. (3.41) up to normalisation. The mixed symmetry projection operators, eqs. (3.60)-(3.62), for the \(\Sigma\)-type first positive parity excitations are constructed in the same way.

We now come to the construction of the projection operators (3.45)-(3.47). Here the \(J_{q_1q_2} = 2\) is represented by the mixed symmetry tensor arising from the product of the antisymmetric tensor, \(N^{[\mu_1\mu_2]}\), defined in eq. (3.31) and \(\chi_{\perp}^{1,\mu}\). The decomposition is
\[
\bigoplus = \bigotimes \bigoplus \bigotimes \bigotimes .
\] (B.17)
The fully antisymmetric tensor is
\[
T^{[\mu_1\mu_2]} = \frac{1}{3} \left[ \chi_{\perp}^{1,\mu} N^{[\mu_1\mu_2]} + \chi_{\perp}^{1,\mu_1} N^{[\mu_2\mu]} + \chi_{\perp}^{1,\mu_2} N^{[\mu_1\mu]} \right].
\] (B.18)
The mixed symmetry tensor is then
\[
\chi_{\perp}^{1,\mu} N^{[\mu_1\mu_2]} - T^{[\mu_1\mu_2]} = \frac{1}{3} \left[ 2 \chi_{\perp}^{1,\mu} N^{[\mu_1\mu_2]} - \chi_{\perp}^{1,\mu_1} N^{[\mu_2\mu]} - \chi_{\perp}^{1,\mu_2} N^{[\mu_1\mu]} \right].
\] (B.19)
Note that this is antisymmetric under \(\mu_1 \leftrightarrow \mu_2\) and annihilates on the mixed symmetry tensor constructed earlier for the \(5^+\), eq. (B.6), which was symmetric under \(\mu_1 \leftrightarrow \mu_2\). However, instead of eq. (B.6) we are free to choose another mixed symmetry tensor, (change of labels only!),
\[
\frac{2}{3} \gamma_{\perp \mu_1} \gamma_5 u_{\mu_2\mu} - \frac{1}{3} \gamma_{\perp \mu_2} \gamma_5 u_{\mu_1\mu} - \frac{1}{3} \gamma_{\perp \mu} \gamma_5 u_{\mu_1\mu_2} .
\] (B.20)
This is symmetric under \(\mu_2 \leftrightarrow \mu\) and does not annihilate on the mixed symmetry tensor eq. (B.19). Actually to proceed further we have to make eq. (B.19) traceless. Doing that we can write the projection operator for the \(5^+\) as
\[
\left( \frac{2}{3} \chi_{\perp}^{1,\mu} N^{[\mu_1\mu_2]} - \frac{1}{3} \chi_{\perp}^{1,\mu_1} N^{[\mu_2\mu]} \right) \times \frac{2}{3} \left( 2 \gamma_{\perp \mu_1} \gamma_5 u_{\mu_2\mu} - \gamma_{\perp \mu_2} \gamma_5 u_{\mu_1\mu} - \gamma_{\perp \mu} \gamma_5 u_{\mu_1\mu_2} \right).
\] (B.21)
Then using the fact that the second factor is symmetric under $\mu \leftrightarrow \mu_2$ we can write the above as

$$
\left( \chi^{1,\mu}_1 N^{[\mu_1\mu_2]} - \frac{1}{2} g^{\mu_1}_1 \chi^{1,\mu}_1 N^{[\nu_2]} + \frac{1}{2} g^{\mu_2}_1 \chi^{1,\mu}_1 N^{[\nu_1]} \right)
\times \frac{2}{3} (2 \gamma_{\mu_1} \gamma_5 u_{\mu_2 \mu} - \gamma_{\mu_2} \gamma_5 u_{\mu_1 \mu} - \gamma_{\mu} \gamma_5 u_{\mu_1 \mu_2} ).
$$

(B.22)

Now the first factor is antisymmetric under $\mu_1 \leftrightarrow \mu_2$ and will thus pick out the corresponding antisymmetric piece from the second bracket finally leading to

$$
\left( \chi^{1,\mu}_1 N^{[\mu_1\mu_2]} - \frac{1}{2} g^{\mu_2}_1 \chi^{1,\mu}_1 N^{[\nu_2]} + \frac{1}{2} g^{\mu_1}_1 \chi^{1,\mu}_1 N^{[\nu_1]} \right)
\times \frac{1}{2} \left( \gamma_{\mu_1} \gamma_5 u_{\mu_2 \mu} - \gamma_{\mu_2} \gamma_5 u_{\mu_1 \mu} \right),
$$

(B.23)

which is precisely eqs. (3.43) and (3.44) up to a normalisation factor.

For the $\frac{3}{2}^+$ partner, eq. (3.47), of this $\frac{5}{2}^+$ we need to construct, as earlier, a mixed symmetry tensor from two $\gamma_\perp$'s and from the Rarita-Schwinger spinor, $u_\mu$. Following by now familiar steps such a mixed symmetry tensor is

$$
\frac{2}{3} \gamma_{\mu_1} G_{\mu_1 \mu_2} - \frac{1}{3} \gamma_{\mu_1} G_{\mu_2 \mu} - \frac{1}{3} \gamma_{\mu_2} G_{\mu_1 \mu} ,
$$

(B.24)

where

$$
G_{\mu_1 \mu_2} = \frac{1}{2} (\gamma_{\mu_1} u_{\mu_2} - \gamma_{\mu_2} u_{\mu_1} ).
$$

(B.25)

Multiplying the tensor in eq. (B.24) into the mixed symmetry tensor in eq. (B.19), appropriately made traceless, and following by now a familiar procedure we arrive at the projection operator eq. (3.47), for the $\frac{3}{2}^+$, up to a normalisation factor.

We now come to the mixed symmetric tensors arising in the highest weight $L$-wave states, i.e. we want to construct eq. (3.104) combined with eq. (3.109). The fully symmetric tensor constructed from $\chi^{1,\mu}_1$ and the fully symmetric tensor $N^{[\mu_1 \cdots \mu_L]}$ is

$$
T^{\{\mu_1 \cdots \mu_L\}} = \frac{1}{L+1} \left( \chi^{1,\mu}_1 N^{[\mu_1 \cdots \mu_L]} + \sum_{i=1}^L \chi^{1,\mu_i} N^{[\mu_1 \cdots [\mu_{i-1} \mu_{i+1} \cdots \mu_L]} \right).
$$

(B.26)

Hence the mixed symmetry tensor is just

$$
T^{\mu_1 \cdots \mu_L}_{\mu_1 \cdots \mu_L} = \chi^{1,\mu}_1 N^{[\mu_1 \cdots \mu_L]} - T^{\{\mu_1 \cdots \mu_L\}}
$$

$$
= \frac{L}{L+1} \chi^{1,\mu}_1 N^{[\mu_1 \cdots \mu_L]} - \frac{1}{L+1} \sum_{i=1}^L \chi^{1,\mu_i} N^{[\mu_1 \cdots [\mu_{i-1} \mu_{i+1} \cdots \mu_L]}.
$$

(B.27)
For the $L + \frac{1}{2}$ spinor part of the projector, we follow a similar construction of the mixed symmetry tensor from $\gamma_{\perp \mu}$ and the fully symmetric tensor $\gamma_5 u_{\mu_1 \ldots \mu_L}$ to obtain

$$\frac{L}{L+1} \gamma_{\perp \mu} \gamma_5 u_{\mu_1 \ldots \mu_L} - \frac{1}{L+1} \sum_{i=1}^{L} \gamma_{\perp \mu_i} \gamma_5 u_{\mu_1 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_L}.$$  \hspace{1cm} \text{(B.28)}

Now making the tensor in eq. (B.27) traceless and multiplying into expression (B.28) we can write the projection operator for $L + \frac{1}{2}$ as

$$\frac{1}{L+1} \left[ L \chi_{\perp}^{1 \mu} N_{\mu_1 \ldots \mu_L} - \sum_{i=1}^{L} \chi_{\perp}^{1 \mu_i} N_{\mu_1 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_L} \right.$$

$$\left. - \frac{L-1}{L} \sum_{i=1}^{L} g_{\mu \nu}^{\mu_i} \chi_{\perp}^{1 \nu} N_{\mu_1 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_L} + \frac{2}{L} \sum_{i<j} g_{\mu \nu}^{\mu_i \nu_j} \chi_{\perp}^{1 \nu} N_{\mu_1 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_{j-1} \mu_{j+1} \ldots \mu_L} \right].$$

$$\times \frac{1}{L+1} \left( L \gamma_{\perp \mu} \gamma_5 u_{\mu_1 \ldots \mu_L} - \sum_{i=1}^{L} \gamma_{\perp \mu_i} \gamma_5 u_{\mu_1 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_L} \right).$$ \hspace{1cm} \text{(B.29)}

Both these factors are symmetric under any interchange $\mu_i \leftrightarrow \mu_j$ and thus using the fact that these are all dummy indices we can write

$$\frac{1}{L+1} \left[ L \chi_{\perp}^{1 \mu} N_{\mu_1 \ldots \mu_L} - \sum_{i=1}^{L} \chi_{\perp}^{1 \mu_i} N_{\mu_1 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_L} \right.$$

$$\left. - \frac{L-1}{L} \sum_{i=1}^{L} g_{\mu \nu}^{\mu_i} \chi_{\perp}^{1 \nu} N_{\mu_1 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_L} + \frac{2}{L} \sum_{i<j} g_{\mu \nu}^{\mu_i \nu_j} \chi_{\perp}^{1 \nu} N_{\mu_1 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_{j-1} \mu_{j+1} \ldots \mu_L} \right].$$

$$\times \frac{L}{L+1} \left( \gamma_{\perp \mu} \gamma_5 u_{\mu_1 \ldots \mu_L} - \gamma_{\perp \mu_1} \gamma_5 u_{\mu_2 \ldots \mu_L} \right).$$ \hspace{1cm} \text{(B.30)}

But now the second factor is antisymmetric under $\mu \leftrightarrow \mu_1$ and hence projects out the corresponding antisymmetric part from the first factor leading to

$$\frac{L}{2(L+1)} \left( \chi_{\perp}^{1 \mu} N_{\mu_1 \ldots \mu_L} - \chi_{\perp}^{1 \mu_1} N_{\mu_2 \ldots \mu_L} \right.$$

$$\left. - \frac{2}{L} \sum_{i=2}^{L} g_{\mu \nu}^{\mu_i} \chi_{\perp}^{1 \nu} N_{\mu_1 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_L} + \frac{2}{L} \sum_{i=2}^{L} g_{\mu \nu}^{\mu_1 \nu_i} \chi_{\perp}^{1 \nu} N_{\mu_2 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_L} \right].$$

$$\times \left( \gamma_{\perp \mu} \gamma_5 u_{\mu_1 \ldots \mu_L} - \gamma_{\perp \mu_1} \gamma_5 u_{\mu_2 \ldots \mu_L} \right).$$ \hspace{1cm} \text{(B.31)}

Upto a normalisation factor this is just the first of the eqs. (B.109). For the second $L - \frac{1}{2}$ member of the degenerate pair we proceed by first constructing the fully symmetric $L$ index tensor

$$F_{\mu_1 \ldots \mu_L} = \frac{1}{L} \sum_{i=1}^{L} \gamma_{\perp \mu_i} u_{\mu_1 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_L}.$$ \hspace{1cm} \text{(B.32)}

Now make the fully symmetric $(L + 1)$ tensor from combining the above with $\gamma_{\perp \mu}$

$$\frac{1}{L+1} \left( \gamma_{\perp \mu} F_{\mu_1 \ldots \mu_L} + \sum_{i=1}^{L} \gamma_{\perp \mu_i} F_{\mu_1 \ldots \mu_i-1 \mu_{i+1} \ldots \mu_L} \right).$$ \hspace{1cm} \text{(B.33)}
Thus the mixed symmetric \((L+1)\) index tensor is

\[
U_{\mu_1...\mu_L}^{\mu_1...\mu_L} = \frac{1}{L+1} (\gamma_{\mu_1...\mu_L} F_{\mu_1...\mu_L} + \sum_{i=1}^{L} \gamma_{\mu_1...\mu_{i-1}\mu_{i+1}...\mu_L} )
\]

\[
= \frac{L}{L+1} (L\gamma_{\mu_1...\mu_L} F_{\mu_1...\mu_L} - \sum_{i=1}^{L} \gamma_{\mu_1...\mu_{i-1}\mu_{i+1}...\mu_L} ) . \tag{B.34}
\]

Now we multiply this into the mixed symmetry tensor of eq. (B.27) and using the familiar arguments about symmetry and dummy indices we finally arrive at

\[
\frac{L}{2(L+1)} (\chi_{\mu_1...\mu_L}^1 N^{\mu_1...\mu_L} - \chi_{\mu_1...\mu_L}^1 N^{\mu_2...\mu_L})
\]

\[
- \frac{2}{L} \sum_{i=2}^{L} g_{\mu_1...\mu_L}^{\mu_1...\mu_L} N^{\mu_1...\mu_{i-1}\mu_{i+1}...\mu_L} + \frac{2}{L} \sum_{i=2}^{L} g_{\mu_1...\mu_L}^{\mu_1...\mu_L} N^{\mu_2...\mu_{i-1}\mu_{i+1}...\mu_L} \cdot \tag{B.35}
\]

\[
\times \left\{ [\gamma_{\mu_1...\mu_L}] u_{\mu_2...\mu_L} + (L-1)(\gamma_{\mu_1...\mu_L} u_{\mu_1\mu_2...\mu_L} - \gamma_{\mu_1...\mu_{i-1}u_{i+1}...\mu_L} ) \right\} .
\]

Upto a normalisation factor this is the same as the second of the eqs. (3.109).