AN EFFICIENT HEURISTIC APPROACH COMBINING MAXIMAL ITEMSETS AND AREA MEASURE FOR COMPRESSING VOLUMINOUS TABLE CONSTRAINTS

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ABSTRACT

Constraint Programming is a powerful paradigm to model and solve combinatorial problems. While there are many kinds of constraints, the table constraint is perhaps the most significant-being the most well-studied and has the ability to encode any other constraints defined on finite variables. However, constraints can be very voluminous and their size can grow exponentially with their arity. To reduce space and the time complexity, researchers have focused on various forms of compression. In this paper we propose a new approach based on maximal frequent itemsets technique and area measure for enumerating the maximal frequent itemsets relevant for compressing table constraints. Our experimental results show the effectiveness and efficiency of this approach on compression and on solving compressed table constraints.

1 Introduction

Constraint Programming (CP) is a powerful paradigm to model and solve combinatorial problems. While there are many kinds of constraints, like automatas [28], and MDDs (Multivalued Decision Diagrams) [7], the table constraint is perhaps the most significant being the most well studied and has the ability to encode any other constraints defined on finite domain (FD) variables. A canonical way of defining a FD constraint is simply to define the allowed (or disallowed) tuples of values, thus the constraint is defined as a table hence the term table constraint. Table constraints are widely used for modelling applications of the real world, for instance, to encode user’s preferences, to model database configuration problems, etc. Sometimes, table constraints provide the unique natural or practical way for a non-expert user to express her constraints.

Over the last decade, research on table constraints has mainly focused on the development of fast algorithms to enforce generalized arc consistency (GAC), which is a property that corresponds to the maximum level of filtering when constraints are treated independently. There has been a large body of work on GAC algorithms for table constraints dating from the GAC4 [1] and GAC-Schema [30] (See [29] for a detailed survey on techniques and algorithms for (G)AC on table constraints). Most of these algorithms propose different techniques for improving the implementation of the seeksupports function, it searches supports for each domain value. Among successful techniques, we find [29]:
• Residue Supports. The idea of residue support is to record previously found supports, called residues, then seek supports from the residues to skip some checkings. Initially introduced for ensuring optimal complexity [31], newer GAC algorithms also use the residue idea, e.g. STRbit [33] and Compact-table (CT) [32].

• Simple Tabular Reduction (STR) [35] is one of the most successful techniques for filtering table constraints. The idea of STR is to remove invalid tuples from tables as search goes deeper, and restore them upon backtrack. STR reduces the number of tuples of a table as search goes deeper, saving unnecessary tuple checks. Different variants of Simple Tabular Reduction (STR) have been proposed and proved to be quite competitive like STR2 [35] and STR3 [34].

• Bitwise Representation uses bit vectors to represent the domain and supports. It has been exploited more recently to the enforcement of GAC. Wang et al. [33] propose a bitwise encoding together with the algorithm STRbit. Compact-table (CT) is another approach based on bitwise representation. Both approaches use bit vectors to record all valid tuples in a table (non-zero words in the bit vectors) during search.

As said before, table constraints are important for modeling parts of many problems, but they admit practical boundaries because the memory space required to represent them may grow exponentially with their arity which can slow down their solving. To reduce space and the time complexity researchers have focused on various forms of compression. The intuition behind employing compact representations is that significant compression of tables should reduce running time for enforcing GAC. Multi-valued Decision Diagrams (MDDs) [7] and bit-wise based algorithms are two examples of compact representations. More compact representations were also proposed to revise existing GAC algorithms, such as the c-tuples, short-supports, slice-tables, smart-tables [11] and segmented-tables [26]. The corresponding GAC algorithms of different compact representations includes: GAC-ctuple [5], STR2-C and STR3-C [37], and STRbit-C [33] for c-tuples; shortSTR2 [38] and shortCT [32] for short-supports; STR-slice [3] for slice-table; smartSTR [59] and smartCT [50] for smart-tables.

Other approaches propose to use data mining techniques for compressing table constraints, like the Microstructure Based Compression method [6], sliced-table [3] and FPTCM+ [2]. The sliced-table approach exploits an FP-Tree structure to enumerate the frequent itemsets from a table constraint and uses the notion of the savings that can be offered by an itemset to select the frequent itemsets that are relevant for compression. The FPTCM+ approach is an improvement of the sliced-table method, it uses the concept of compression rate to enumerate frequent itemsets that are more relevant for compression.

In this paper, we go one step further in exploiting data mining approaches to compress table constraints. We propose to use the maximum frequent itemset (MFI) to cover a maximum number of variables in the scope of the table constraint. This allows to reduce the size of the tuples in the resulting compressed tables. To achieve better compression, we select the MFI covering a maximum number of tuples (i.e., high frequency). However, the larger the MFI, the lower the frequencies. A better compromise between the length and the frequency of MFI is to exploit the area measure (the product of the length of an itemsets and its frequency value) such that we select the MFI with higher area values. To mine the set of MFI, the value of the minimum frequency threshold $S_{min}$ has to be fixed, for this we dynamically fix for each table constraint the value of $S_{min}$ by using the TopK approach. Finally, the relevance and the effectiveness of our approach is highlighted through a set of experiments on benchmarks downloaded from https://bitbucket.org/pschaus/xp-table/src/master/instances/. The obtained results are very promising. The remainder of this paper is organized as follows. In Section 2, we give some definitions related to Constraint Satisfaction Problems (CSPs) and frequent itemsets mining. Section 3 reviews some related works. Section 4 is devoted to our proposition called MFI-Compression. In section 5, we calculated the time complexity of our approach. Section 6 explains how solving the compressed constraints. Experiments, carried out in this work, are presented in Section 7. We conclude with some remarks and avenue for future works in Section 8.

2 Background

In this section some concepts related to Constraint Satisfaction Problems (CSPs) and Data Mining are formally defined [24, 27, 25].

2.1 Constraint Satisfaction Problem

Constraint Satisfaction Problem (CSP) was formally defined by U. Montanari [4] as a finite set of variables $\mathcal{X} = \{x_1, \cdots, x_n\}$ with finite domains $\mathcal{D} = \{D_1, \cdots, D_n\}$. Each $D_i$ is the set of possible values that can be assigned to $x_i$, and a finite set of constraints $\mathcal{C} = \{c_1, \cdots, c_m\}$. A constraint $c_i \in \mathcal{C}$ is a pair $(S(c_i), R(c_i))$, where:

- $S(c_i) \subseteq \mathcal{X}$ is the scope of the constraint $c_i$. It represents the set of variables involved in $c_i$;
• \( R(c_i) \subseteq \prod_{x_k \in S(c_i)} D_k \) is a relation that defines the set of tuples allowed for the variables of \( c_i \).

The size of the set \( S(c_i) \) is called the arity of the constraint \( c_i \). A unary constraint is a constraint of arity one, a binary constraint is a constraint of arity two, a non-binary constraint is a constraint of arity greater than two.

The size of a constraint relation \( R(c_i) \) is the product of the arity of \( c_i \) by the number of tuples in \( R(c_i) \). The relation of a constraint can be specified extensionally by explicitly listing its acceptable tuples, or intensionally by specifying an expression that tuples in the constraint must satisfy. Example Table 1 shows a CSP instance defined in extension.

Example 1 Consider the following CSP defined in extension: \( X = \{x_0, \ldots, x_4\} \), \( D = \{D_0, \ldots, D_4\} \) where, \( D_0 = \{0,1\} \), \( D_1 = \{0,1,2\} \), \( D_2 = \{0,1,2\} \), \( D_3 = \{0,1,2,3\} \), \( D_4 = D_5 \).

\[ C = \{c_0\} \text{ where } c_0 = ((x_0, x_1, x_2, x_3, x_4), R(c_0)) \text{ and } R(c_0) = \{(0 0 0 0 2), (0 0 0 1 2), (0 2 0 2 0), (0 0 1 1 2), (0 0 1 2 0), (0 0 1 3 2), (1 0 2 1 1), (1 0 2 3 0), (1 1 2 0 1), (1 1 2 2 2), (1 1 2 3 0)\} \]

An assignment is a pair \((x_i, a)\), which means that the variable \( x_i \in X \) is assigned the value \( a \in D_i \). A partial assignment (noted \( \overrightarrow{A} \)) is a set of assignments to distinct variables in \( X \). A complete assignment is an assignment to all variables in \( X \). We say that a partial assignment satisfies a constraint \( c_i \) if the restriction of the assignment to the scope \( S(c_i) \) is an acceptable (satisfying) tuple. A solution to a CSP instance \( P = \langle X, D, C \rangle \) is a complete assignment that satisfies all constraints of \( C \). Solving a CSP \( P \) consists in checking whether \( P \) admits at least one solution. It is a NP-hard problem. If no solution exists, the CSP is said to be inconsistent or unsatisfied.

There exists many complete and incomplete techniques for solving CSPs. Most “efficient” complete methods rely on a depth-first search with backtracking combined with Constraint propagation and variable/value ordering heuristics. In the worst case, their time complexity is in \( O(d^n) \) (with \( n \) is the number of variables and \( d \) is the size of the largest domain) while being generally linear in space.

Depth-First Search methods explore a search tree in a systematic way by recursively choosing the next unassigned variable to assign and by choosing a value in its domain for the assignment (the branch part) until a solution is found or it can be proved that the subtree rooted at the current search node has no solution. At each search node, constraint propagation is performed to filter the domains of variables so that values that cannot be part of a solution are removed from the domains of unassigned variables. When one domain of a variable becomes empty, this means that the last instantiated variable conducts some constraints to be violated. Hence, the algorithm needs to backtrack in order to consider another possible value for this variable. Most solvers maintain generalized arc consistency for the table constraint.

Definition 1 (Support) A support of a constraint \( c \in C \) is a set of assignments to exactly the variables in \( S(c) \) such that \( c \) is satisfied. A support of \( c \) that includes the assignment \((x_i, a)\) is called a support of \( x_i \) in \( c \).

Definition 2 (Generalized arc consistency (GAC)) A constraint \( c \) is GAC if there exists a support for all values in the current domains of the variables in \( S(c) \). A CSP is GAC if all of its constraints are GAC.

2.2 Frequent itemset mining

Let \( I \) be a set of \( n \) distinct literals called items, an itemset (or pattern) is a non-null subset of \( I \). The language of itemsets corresponds to \( \mathcal{L}_{I} = 2^{I} \setminus \emptyset \). A transaction data set is a multi-set of \( m \) itemsets of \( \mathcal{L}_{I} \). Each itemset, usually called a transaction or object, is a data set entry.

Let \( \mathcal{T}D \) be a transaction data set, \( u \in \mathcal{L}_{I} \) be an itemset, and \( \text{match}[\cdot] : \mathcal{L}_{I} \times \mathcal{L}_{I} \rightarrow \{\text{true}, \text{false}\} \) a matching operator. Table 1 presents an example of a transaction data set \( \mathcal{T}D \) where each tuple (transaction) \( t_i \) is described by items denoted \( A, \ldots, E \).

Definition 3 (Coverage and Frequency) Let \( \mathcal{T}D \) be a transaction database over a set of items \( I \), the set of identifiers of tuples in which an itemset \( u \) appears is called the coverage of \( u \):

\[
\text{cover}(u) = \{ t \in T | \forall i \in u, (i, t) \in \mathcal{T}D \}
\]

The frequency of an itemset \( u \) is the size of its coverage: \( \text{freq}(u) = | \text{cover}(u) | \).

1^For an itemset \( u \in \mathcal{L}_{I} \) and a transaction \( t \), \( \text{match}(p, t) = \text{true} \iff p \text{ covers the transaction } t \).
When a database is very dense or the value of the minimal support \( S \) has been expended.

Regarding the algorithmic approaches for mining closed itemsets, much effort on developing sophisticated algorithms has been expended. LCM (Linear time Closed frequent itemset Mining) is one of the most prominent and performer algorithm for this task. LCMmax is an extension of LCM dedicated to mine maximal frequent itemsets (MFI).

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**Example 2** Consider the transaction data set \( \mathcal{T}\mathcal{D} \) in Table 1. We have for \( u = EC \), \( \text{cover}(u) = \{t_0, t_1, t_2, t_3, t_4\} \) and \( \text{freq}(u) = 5 \).

**Example 3** By considering the transaction data set \( \mathcal{T}\mathcal{D} \) in Table 1 and \( S_{\text{min}} = 2 \), the itemset \( u = EC \) is a frequent itemset because \( \text{freq}(u) > S_{\text{min}} \).

**Definition 4** Let \( \mathcal{T}\mathcal{D} \) be a transaction database over a set of items \( \mathcal{I} \), and let \( S_{\text{min}} \) be a minimal support threshold. We note the collection of frequent itemsets in \( \mathcal{T}\mathcal{D} \) with respect to \( S_{\text{min}} \) by: \( \mathcal{F}(\mathcal{T}\mathcal{D}, S_{\text{min}}) = \{ u \in \mathcal{L}_\mathcal{I} \mid \text{freq}(u) \geq S_{\text{min}} \} \), or simply \( \mathcal{F} \) if \( \mathcal{T}\mathcal{D} \) and \( S_{\text{min}} \) are clear from the context.

**Definition 5 (Frequent Itemset Mining Problem)** Let \( S_{\text{min}} \) be a minimal support threshold. The frequent itemset mining problem is the computation of the set of all itemsets \( u \) having frequency in the data set exceeding \( S_{\text{min}} \): \( \text{freq}(u) \geq S_{\text{min}} \).

When a database is very dense or the value of the minimal support \( S_{\text{min}} \) is set too low, mining all the frequent itemsets can be impractical because of the huge number of possible frequent itemsets. To limit the number of output, several reduction techniques based condensed representations of patterns have been proposed in the context of the frequency measure \[13, 14, 15, 16\]. The most popular ones are closed and maximal itemsets.

**Definition 6 (Closed frequent itemset)** A frequent itemset \( u_i \in \mathcal{F}(\mathcal{T}\mathcal{D}, S_{\text{min}}) \) is closed iff \( \forall u_j \in \mathcal{L}_\mathcal{I}, u_j \subset u_i \Rightarrow \text{freq}(u_j) < \text{freq}(u_i) \).

**Example 4** Consider \( S_{\text{min}} = 2 \). From Table 1 we get four frequent closed itemsets which are: \( CE(5), CDE(4), AC\{2\}, ADE(2), BCE(3), BCDE(2), ABCE(3) \). The value between \( ⟨ \rangle \) indicates the frequency of an itemset.

Since the collection of all frequent itemsets is downward closed, meaning that any subset of a frequent itemset is frequent, it can be represented by its maximal elements, the so called maximal frequent itemsets.

**Definition 7 (Maximal frequent itemset)** A frequent itemset \( u_i \in \mathcal{F}(\mathcal{T}\mathcal{D}, S_{\text{min}}) \) is called maximal iff \( \forall u_j \in \mathcal{L}_\mathcal{I}, u_j \supseteq u_i \Rightarrow \text{freq}(u_j) < S_{\text{min}} \).

**Example 5** In Table 1 if we impose that \( S_{\text{min}} = 2 \), we obtain two maximal frequent itemsets: \( BCDE(2), ABCE(2) \) and \( A\{2\} \).

Other studies attempt to integrate user preferences into the mining task in order to limit the number of extracted patterns such as the TopK pattern mining approaches \[57, 58\]. By associating each pattern with a rank score, such as frequency, this approach returns an ordered list of the \( k \) patterns with the highest score to the user.

**Definition 8 (TopK frequent itemsets)** Let \( k \) be an integer. \( \text{TopK} \) w.r.t. the frequency measure is the set of \( k \) best frequent itemsets:

\[
\{ u \in \text{ang} \mathcal{I} \mid \text{freq}(u) \geq S_{\text{min}} \land \exists x_1, \ldots, x_k \in \mathcal{L}_\mathcal{I} : \forall 1 \leq j \leq k, \text{freq}(x_j) > \text{freq}(u) \}
\]

**Example 6** In our running example (with \( S_{\text{min}} = 4 \)), the top-7 frequent itemsets are: \( E\{5\}, C\{5\}, D\{4\}, EC\{5\}, ED\{4\}, CD\{4\}, ECD\{4\} \).

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**Table 1**: Transactional dataset \( \mathcal{T}\mathcal{D} \).

| tid | C | D | E | A | B |
|-----|---|---|---|---|---|
| t0  |  |   |   |   |   |
| t1  |   |   |  |   |   |
| t2  |   |   |  |   |   |
| t3  |   |   |  |   |   |
| t4  |   |   |  |   |   |

**Example 6** In our running example (with \( S_{\text{min}} = 4 \)), the top-7 frequent itemsets are: \( E\{5\}, C\{5\}, D\{4\}, EC\{5\}, ED\{4\}, CD\{4\}, ECD\{4\} \).
main feature is to have a linear complexity w.r.t the number of closed itemsets. LCMmax enumerates the set of all closed frequent itemsets (CFI) by backtracking and exploits pruning and maximality checking techniques to accelerate the computation time and to avoid storing the MFI previously found in memory.

In addition to the frequency, other interestingness measures, like the area, can be exploited.

**Definition 9 (area of itemset)** The area of an itemset $u$ is the size of the itemset $|u|$ multiplied by its frequency $\text{freq}(u)$:

$$\text{area}(u) = |u| \times \text{freq}(u)$$  \hspace{1cm} (3)

In the sequel, we show how to take advantage of maximal patterns to compress table constraints by selecting those that maximize the area measure.

### 2.3 Constraint based compression by itemset mining

In this section we show how a table constraint $R(c_i)$ associated with a constraint $c_i$ can be represented as a transactional dataset $\mathcal{T}D_{c_i}$. Then, we show how to compress $R(c_i)$ using itemset mining techniques.

Let $P = (X, D, C)$ be a CSP and $R(c_i)$ be a table constraint associated with a constraint $c_i \in C$. The transactional dataset $\mathcal{T}D_{c_i}$ is defined as follows:

(i) the union of the domains of the variables in the scope of $c_i$ represents the set of items of $\mathcal{I}$,

(ii) the set of values involved in the tuple $t \in R(c_i)$ forms a transaction in $\mathcal{T}$.

In this context, an itemset represents an assignment of some variables involved in the scope of $c_i$. Table 2 shows the transactional dataset $\mathcal{T}D_{c_0}$ associated with the table constraint $R(c_0)$ of Example 1. If we consider $S_{\text{min}} = 2$, the following assignments: $(x_0 = 1, x_2 = 2)$ represent an example of a frequent itemset of Table 2 that covers the tuples $t_7, t_8, t_9$ and $t_{10}$.

The main idea behind the use of pattern mining to derive a compact representation of the table constraint is to use frequent itemsets extracted from the transaction dataset as a summary of a set of transactions. These transactions are replaced by each frequent itemset that covers them. The resulting compressed constraint relation consists of a set of entries where each entry contains an itemset and its corresponding sub-table.

**Definition 10 (Sub-table)** The sub-table $St$ associated with an itemset $u$ of a constraint $c_i$ consists in the remaining parts of tuples in the coverage of $u$ after removing $u$ from each tuple.

**Definition 11 (Entry)** An entry for a table constraint $R(c)$ is a pair $(u, St)$ such that $u$ is a frequent itemset and $St$ its corresponding sub-table.
Example 7  Table \[2 \]b shows the entry corresponding to the itemset \( u = \{x_0 = 0, x_1 = 0, x_2 = 1\} \) and its resulting sub-table.

Definition 12 (Default table)  A default-table for a table constraint \( R(c) \) is a table that contains all tuples that can not be compressed with the mined frequent itemsets.

Definition 13 (Compressed table constraint)  A compressed table constraint \( R'(c) \) of \( R(c) \) is represented by a set of entries associated to the set of non-overlapping frequent itemsets and a default-table.

Let \( u \) be a frequent itemset, \( f \) its frequency and \( T \) the set of compressed transactions. Let \( \text{size}_a \) (resp. \( \text{size}_b \)) be the size of \( T \) after (resp. before) compression. To assess the quality of a summary \( u \) of a set of transactions \( T \), we define the following metric: The size \( \text{size}_a \) of the transactions after their compression is equal to the length of \( u \) plus the size of its corresponding sub-table. The size of the sub-table is obtained by multiplying the arity of the sub-table (\( \text{arity} - |u| \)) by the frequency of \( u : \text{size}_a = |u| + (\text{arity} - |u|) \times f \). The compression ratio of a \( T \) w.r.t. itemset \( u \) is obtained as follows: \( \text{Rate} = 1 - \frac{\text{size}_a}{\text{size}_b} \) where \( \text{size}_b = \text{arity} \times f \).

3 Related Works

In this section, we review some compact representations proposed in the literature for table constraints.

Katsirelos and Walsh [5] have proposed first a compact representation of constraint relations. They exploit a decision tree to represent the original constraint relation as a disjunction of tuples. Then, they extract from this decision tree a set of compact tuples called c-tuples that will be used to represent the constraint relation as a conjunction of c-tuples. Thus a compact representation can exponentially reduce the size of a constraint relation and the time complexity required to enforce GAC (Generalized Arc Consistency).

Cheng et al. [7] have proposed a new form of compression based on Multi-valued Decision Diagrams (MDD). The size of a tree is often smaller than the size of the constraint relation. That is why the authors have proposed such a structure to perform an effective support checking. They also proposed to merge the identical sub-tries in the decision tree to reduce the time required for support checking, thus obtaining a directed acyclic graph (DAG), called a multi-valued decision diagram (MDD). Two notable algorithms using MDDs as main data structure are \textit{mddc} [7] and \textit{MDD4R} [41]. The former does not modify the decision diagram and performs a depth-first search of the MDD during propagation to detect which parts of the MDD are consistent or not. \textit{MDD4R} dynamically maintains the MDD by deleting nodes and edges that do not belong to a solution.

Jabbour et al. [6] have proposed a SAT based approach for compressing table constraints. They introduced two new rewriting rules for reducing the size of the constraint network as well as the size of the constraint relations while preserving the original structure of the constraints. They used closed itemsets to compute a summary of tuples of each table constraint.

Some variants of STR algorithms work on compressed table representations. \textit{STR2-C} and \textit{STR3-C} [37] works on the Cartesian Product representation (c-tuple) of tuples to compress tables.

Wang et al. [33] proposes a bitwise encoding of the dual table representations together with the algorithms \textit{STRbit} and \textit{STRbit-C}. To get the bitwise representation, the original table is first partitioned so that each subtable have \( w \) tuples where \( w \) corresponds to the natural word size of processor with \( O(1) \) bit vector operations. Compact-table (CT) is another state-of-the-art algorithm, also based on bitwise representation. Both CT and \textit{STRbit(-c)} use bit vectors to record all valid tuples in a table (non-zero words in the bit vectors) during search.

Gharbi et al. [3] have introduced sliced-table [3], a new compression method based on FP-Tree structure to enumerate the frequent itemsets relevant for compressing constraint relations. The proposed approach takes as input a constraint relation to compress and returns a set of entries and a default table which contains tuples that are not compressed. To decide either an itemset \( u \) of the FP-Tree corresponding to a constraint relation is relevant for compression or not, the authors proposed to compute the savings that can be obtained by factoring \( u \). The saving of \( u \) is computed by the following formula: \( |u| \times (freq(u) - 1) \), where \( u \) is an itemset and \( freq(u) \) its frequency.

Audemard et al. [26] introduced the notion of segmented table that generalize compressed tables. A segmented constraint is represented with a set of segmented tuples. Where each segment of a segmented table constraint can be represented with universal values (*), ordinary values or sub-tables. Then authors proposed an algorithm for enforcing GAC on segmented tables.
4 A new heuristic approach based on maximal patterns for compressing table constraints

In this section, we detail our heuristic approach, called MFI-Compression, based on maximal itemsets for compressing table constraints.

4.1 Our approach in nutshell

To achieve a better compression, our approach first selects maximal itemsets (MFI). Indeed, considering itemsets involving more variables in the scope of the table constraint allows to reduce the size of the tuples in the resulting compressed tables. Moreover, we aim to select those covering a large number of tuples in the table, that is, those with high frequency. However, the larger the MFI, the lower the frequencies. It is thus necessary to ensure a better compromise between these two criteria, i.e. length and frequency. To that end, we propose to exploit the area measure to achieve such a compromise. Consider, for instance, the set of MFI in Table 3a extracted from our running example in Table 2a. The first MFI with size three covers 3 tuples, while the second MFI with size four covers only 2 tuples. According to the area measure, it would be more interesting to select the first MFI because its area (equal to 9) and its compression ratio (equal to 10.9%) are larger than the second MFI’s area (equal to 8) and compression ratio (equal to 7.6%), respectively.

Data: $TD$: table constraint $c$ to compress, $k$: number of CFI to fix $S_{min}$.

Result: $R^c$: compressed table constraint.

\begin{verbatim}
S ← Select(M); /* see section 4.2 */
F_k ← TopK(TD, k); /* see section 4.2 */
S_{min} ← \min_{F \in F_k} freq(F);
M ← LCM\_{max}(TD, S_{min});
create the compressed table constraint $R^c$;
Return $R^c$
\end{verbatim}

Algorithm 1: MFI-Compression

Consider again the MFI in Table 3a. We can see that the tuple $t_{10}$ appears in the coverage of the first and the sixth MFI, while it can be compressed using only one MFI. To prevent compressing tuples more than once, we remove all the compressed tuples from the coverage of the remaining non yet selected MFI. The main steps of our MFI-Compression heuristic are depicted in Algorithm 1:

(a) select the best value for the minimum frequency threshold $S_{min}$;
(b) extract the MFI using $LCM_{max}$ with the value of $S_{min}$ found in step (i);
(c) choose heuristically the most relevant MFI maximizing the area measure;
(d) create the compressed table constraint $R^c$.

4.2 Setting the parameter $S_{min}$ and mining candidate MFI

Finding the suitable minimum threshold value $S_{min}$ for each table constraint is challenging. If its value is maintained too low, too many MFI can be mined, and the relevant ones can hardly be found among the resulting massive set of MFI. Similarly, if the value of $S_{min}$ is too high, too few number of MFI can be generated, and some MFI relevant for compressing the table constraint can be missed. To generate a good set of MFI candidates regardless of the table constraints, MFI-Compression dynamically fix for each table constraint the value of $S_{min}$ by using the TopK approach (line 1, Algorithm 1). Let $k$ be a user-defined value. We first generate the top–$k$ closed most frequent itemsets, then we set $S_{min}$ to the lower frequency value among all the mined CFI (line 2, Algorithm 1). Finally, using the $LCM_{max}$ method, we extract from the constraint table all the MFI w.r.t. $S_{min}$ (line 3, Algorithm 1). For instance, if we consider Table 2 and $k = 10$, according to our TopK approach, $S_{min} = 2$.

4.3 Selecting heuristically non-overlapping MFI

To ensure a better compression, we have to select the MFI for which both the length (or size) and the frequency values are maximized. However, maximizing simultaneously these two conflicting objectives is challenging because the larger the MFI, the lower its frequency. Instead, we propose to maximize the area criterion since it represents a good compromise between these two criteria.
Several approaches in the literature use the concept of a tile and its area as an objective interestingness measure for itemsets. A tile consists of a block of ones in a binary database. For instance, the top-\(k\) tiles problem which asks for the \(k\) tiles that have the largest area is known to be NP-hard \cite{18} even for \(k = 1\). We propose in this paper a greedy algorithm which finds a sub-set of non-overlapping MFI maximizing the area.

As pointed out earlier, each tuple of a table constraint can be compressed using at most only one MFI. To select only non-overlapping MFI with the largest area values, our algorithm sorts the MFI in decreasing order of their area value, selects the first MFI for compression and removes those for whose coverage overlap with the coverage of the selected MFI. Algorithm 2 details the different steps for selecting heuristically the MFI relevant for compression. It exploits a data structure \(E\) with two elements, the MFI \(p\) and its area value \(area\). A function \(Comparator\) is defined to perform a pairwise comparison between itemsets (lines 1-10, Algorithm 2). First, we compute for each MFI \(u\) its area and insert the pair \((u, area)\) in the list \(L\) (lines 12-15, Algorithm 2). Second, we sort the elements of \(L\) in decreasing order of their area values using the function \(Comparator\) (line 16, Algorithm 2). Finally, we select the first MFI \(u\) from the ordered list \(L\), add it to the list \(S\) of MFI relevant for compression, remove \(u\) from \(L\) and remove from \(L\) all the MFI that overlap with the coverage of \(u\) (lines 18 - 28, Algorithm 2). This process is repeated until there are no more MFI to select.

Sorting the set of MFI of Table 3a leads to the ordering of Table 3b. From this ordering, we select the first MFI, add it to the set \(S\) and remove from Table 3b all those that overlap with this MFI, i.e. the MFI with \(idx = 6\). In the second iteration of the while loop of line 17, the MFI with \(idx = 2\) is selected, added to \(S\) = \{\((x_0 = 1, x_1 = 1, x_2 = 2)\)\} and the MFI with \(idx = 5\) is removed. Finally, when there are no more MFI to select, Algorithm 2 returns the set \(S = \{\langle x_0 = 1, x_1 = 1, x_2 = 2\rangle, \langle x_0 = 0, x_1 = 0, x_2 = 0, x_4 = 2\rangle, \langle x_0 = 0, x_1 = 0, x_2 = 1, x_4 = 2\rangle, \langle x_0 = 0, x_3 = 2, x_4 = 0\rangle\}\).
Table 3: Running MFI-Compression on Table 2.

(a) The MFI extracted from Table 2 with $S_{\text{min}} = 2$. Covers and area are also shown.

| Idx | Maximal frequent itemsets | Coverage | Area |
|-----|---------------------------|----------|------|
| 1   | $\langle x_0 = 1, x_1 = 1, x_2 = 2 \rangle$ | $\{t_8, t_9, t_{10}\}$ | 9 |
| 2   | $\langle x_0 = 0, x_1 = 0, x_2 = 0, x_4 = 2 \rangle$ | $\{t_0, t_1\}$ | 8 |
| 3   | $\langle x_0 = 0, x_1 = 0, x_2 = 1, x_4 = 2 \rangle$ | $\{t_3, t_5\}$ | 8 |
| 4   | $\langle x_0 = 0, x_3 = 2, x_4 = 0 \rangle$ | $\{t_2, t_4\}$ | 6 |
| 5   | $\langle x_0 = 1, x_1 = 0, x_3 = 1, x_4 = 2 \rangle$ | $\{t_1, t_3\}$ | 8 |
| 6   | $\langle x_0 = 0, x_2 = 1, x_3 = 3, x_4 = 0 \rangle$ | $\{t_7, t_{10}\}$ | 8 |

(b) Sorting the MFI w.r.t area.

| Idx | Area |
|-----|------|
| 1   | 9    |
| 2   | 8    |
| 3   | 8    |
| 5   | 8    |
| 6   | 8    |
| 4   | 6    |

4.4 Creating the compressed table constraint

The last step of MFI-Compression algorithm consists to create the compressed table constraint $R_c$. This is done by associating an entry for each each MFI in $S$. However, as tuples six and seven of Table 2 cannot be compressed using $S$, a default entry is then created for these two tuples. The final compressed table constraint is showed in Table 4.

Table 4: Compressed table constraint.

| $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|------|------|------|------|------|
| 0    | 0    | 1    | 2    | 1    |
| 1    | 1    | 2    | 0    | 1    |
| 2    | 0    | 0    | 2    | 0    |

Entry $e_1$.

| $x_0$ | $x_3$ | $x_4$ | $x_1$ | $x_2$ |
|------|------|------|------|------|
| 0    | 0    | 0    | 2    | 2    |
| 0    | 0    | 0    | 1    | 1    |

Entry $e_2$.

| $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|------|------|------|------|------|
| 1    | 0    | 2    | 1    | 2    |
| 1    | 0    | 2    | 3    | 0    |

Entry $e_3$.

| $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|------|------|------|------|------|
| 0    | 0    | 0    | 2    | 0    |
| 1    | 0    | 2    | 3    | 0    |

Entry $e_4$.

5 Complexity analysis

To analyse the time complexity of MFI-Compression, we analyse the time complexity of each step. Let $n$ be the number of MFI.

- The time complexity of the TopK CFI and MFI using LCM (lines 1, 2 and 3, Algorithm 2) is linear in the number of CFI.
- The sort of line 16 of Algorithm 2 can be done in $O(n \log n)$.
- The time complexity to select the MFI relevant for compression (lines 17 to 29, Algorithm 2) is $O(n((n+1)/2)) = O(n^2)$ in the worst case.

So the time complexity of MFI-Compression is $\theta(n^2)$ where $n$ is the number of CFI.

6 Enforcing GAC on CSP compressed with MFI-Compression

To enforce GAC on the CSP compressed using the MFI-Compression method, we used the STR-Slice algorithm which is an optimized variant of STR2 that works on compressed table constraints, i.e. a set of entries where each entry consists on an itemset and its corresponding sub-table.

To maintain GAC, STR-Slice checks the validity of entries, where an entry is said to be valid if both of its itemset and at least one tuple of its corresponding sub-table are valid. The method uses a limit pointers to save the index of the latest valid entry and the index of the latest valid sub-tuple of the sub-table corresponding to each valid entry. When restoring entries and sub-tuples, the method just has to modify the value of the limit pointers. We denoted by STR-MFIC the combination of MFI-Compression with structure of STR-Slice used to enforce GAC on table constraints.
Example 6.1 Consider the compressed constraint relation of Table 4. Let entriesLimit (resp. limit) be the index of the last current (valid) entry (resp. the index of the latest valid sub-tuple in the entry). Firstly, all the entries are valid so entriesLimit = 5. STR-Slice is called after an event is generated. In Table 5, considering that the new event is \( x_1 \neq 0 \) (i.e., the removal of the value 0 from \( \text{dom}(x_1) \)), STR-Slice starts checking the validity of the current entries (from 1 to entriesLimit). For the first entry, the itemset is not valid. We do not need to check the validity of its sub-table. We consider the entry as not valid. The second entry is valid because it does not contain \( x_1 = 0 \). Like the entry \( e_1 \) the entry \( e_3 \) is not valid. For the entry \( e_4 \), the validity of the itemset \( u = (x_0 = 0, x_3 = 2, x_4 = 0) \) is checked. Since \( u \) remains valid, the sub-table is scanned. Only the sub-tuple \((x_1 = 0, x_2 = 1)\) remains invalid, thus the value of limit = 1. For \( df \), the two tuples are not valid. So the value of entriesLimit is 2.

7 Experiments

The experimental evaluation is designed to determine how (in terms of CPU time) STR-MFIC compares to the state-of-the-art of GAC-based algorithms.

7.1 Experimental protocol

We performed experiments on the same benchmarks used in [3]. Table 6 summarizes the characteristics of each of them. For each benchmark, we give the number of its instances (\( Ins_{nbr} \)), the maximum number of variables (\( X \)) in an instance, the largest domain (\(|D|\)), the largest number of relations (\( R_{nbr} \)), the size of the largest relation (\( R_{max} \)), the largest arity of relations (\( arity \)), the greatest number of constraints (\( C_{nbr} \)).

Table 6: Characteristics of the used benchmarks.

| Benchmark         | \( Ins_{nbr} \) | \( X \) | \(|D|\) | \( R_{nbr} \) | \( R_{max} \) | \( arity \) | \( C_{nbr} \) |
|-------------------|-----------------|--------|--------|-------------|-------------|-------------|-------------|
| bddLarge          | 35              | 21     | 2      | 1           | 57971       | 18          | 133         |
| bddSmall          | 35              | 21     | 2      | 1           | 6945        | 15          | 2713        |
| randsJC2500       | 10              | 40     | 8      | 40          | 2500        | 7           | 40          |
| randsJC5000       | 10              | 40     | 8      | 40          | 5000        | 7           | 40          |
| randsJC7500       | 10              | 40     | 8      | 40          | 7500        | 7           | 40          |
| randsJC10000      | 10              | 40     | 8      | 40          | 10000       | 7           | 40          |
| Crossword-Lex-Vg  | 63              | 288    | 26     | 2           | 3607        | 18          | 34          |
| Crossword-Words-Vg| 65              | 320    | 25     | 2           | 68064       | 20          | 36          |
| Modified-Renault  | 50              | 111    | 42     | 142         | 48721       | 10          | 159         |

The implementation of STR-MFIC was carried out in the Oscar solver The implementation of algorithms of the stat-of-the-art selected for our comparison are also available in the Oscar solver. All experiments were conducted on Intel (R) Core(TM), i5 - 7200 CPU, 2.5 GHz with a RAM of 4 GB, running the Ubuntu 64 bits 20.04 LTS operating system.

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\(^2\)Data sets are available at [https://bitbucket.org/pschaus/xp-table/src/master/instances/](https://bitbucket.org/pschaus/xp-table/src/master/instances/)

\(^3\)Solver available at [https://bitbucket.org/oscarlib/oscar/src/dev/](https://bitbucket.org/oscarlib/oscar/src/dev/)
system. A time limit of 1,800 seconds has been used per instance. When the runtime exceeds this limit the resolution stops and the instance is considered as failed.

For our experiments, we fixed the initial value of $S_{\text{min}}$ to 2 for STR-Slice, contrary to MFI-Compression that exploits the TopK mining method to fix its value. To be relevant for compression an MFI must cover at least two tuples and a tuple can be compressed using one and only one MFI. For this, to fix the value of $S_{\text{min}}$ we varied the value of $k$ on the number of tuples in the table constraint to compress. After several experiments the value of $k$ was varied between 20% and 60% of the number of tuples of the table constraint to compress then the value of $S_{\text{min}}$ was set to the average of frequency values returned by the TopK algorithm.

7.2 Comparing STR-MFIC with STR-Slice and STR2

The STR-MFIC and STR-Slice [3] are both based on itemsets mining technique for compression and use the same structure (entries) of compressed table constraints. The main differences between the two methods are: (i) STR-MFIC dynamically fix, for each table constraint, the value of the minimum threshold $S_{\text{min}}$ while for STR-Slice the value of $S_{\text{min}}$ is fixed to 2. (ii) STR-MFIC compresses table constraints using MFI while STR-Slice compresses them using frequent itemsets. The two methods are based on STR-slice [3] to solve the compressed CSP. STR-slice [3] is an optimized version of STR2 for compressed CSP. Hence the interest of comparing them.

In Table 7, we reported for each method STR-MFIC, STR-Slice and STR2 and for each benchmark the number of solved instances ($\text{inst}_s$) within 1800s and the average CPU time of solving an instance of each benchmark.

- number of solved instances: the three methods solved the same number instances except for the benchmark randsJC1000 where STR-Slice did not solved any instance and for crossword-lexVg where STR-MFIC and STR2 solved more instances (7) compared then STR-Slice.
- average CPU time: STR-MFIC performed better compared to STR-Slice and STR2 on the average CPU time required to solve each instance of the different benchmarks except for crossword-lexVg where the average CPU time required by STR-Slice is less then the one required by STR-MFIC and STR2. Also for randsJC5000 and randsJC7500, STR2 solved each instance on average CPU time less than that of STR-MFIC and STR-Slice.

Figure 1 shows the cumulative curves of the solving CPU time(s) obtained for STR-MFIC, STR-Slice and STR2 for the selected benchmarks. The x-axis represents the solved instances while the y-axis the cumulative CPU time. Even if both STR-MFIC and STR-Slice compress the table constraints before there solving, STR-MFIC behaved better on most benchmarks. To clarify the obtained results, Table 8 gives some details about the compression process of some selected benchmarks. For each benchmark and for each method, we reported, in percentage ($\%$), the number of compressed tuples ($c$-tup), the compression rate ($c$-rate), the average number of itemsets ($|M|$), frequent itemsets for STR-Slice and MFI for STR-MFIC, mined from each instance of a benchmark, the
Table 7: Comparing the STR-MFIC, STR-Slice and STR2 on the number of solved instances ($inst_s$) and average solving CPU time ($CPU_t$).

| benchmark   | STR-MFIC $inst_s$ | STR-MFIC $CPU_t(s)$ | STR-Slice $inst_s$ | STR-Slice $CPU_t(s)$ | STR2 $inst_s$ | STR2 $CPU_t(s)$ |
|-------------|------------------|---------------------|-------------------|--------------------|---------------|------------------|
| bddLarge    | 35               | 60                  | 35                | 382                | 35            | 65               |
| bddSmall    | 35               | 28                  | 35                | 195                | 35            | 37               |
| crossword-lexVg | 35        | 338                 | 28                | 325                | 35            | 352              |
| crossword-words | 27          | 33                  | 23                | 157                | 23            | 46               |
| modifiedRenault | 39         | 72                  | 39                | 88                 | 39            | 150              |
| randsJC2500 | 10              | 10                  | 10                | 35                 | 10            | 12               |
| randsJC5000 | 10              | 154                 | 10                | 553                | 10            | 130              |
| randsJC7500 | 10              | 636                 | 10                | 1373               | 10            | 563              |
| randsJC10000| 10              | 703                 | 0                 | TO                 | 10            | 750              |

average length ($|u|$) and the average frequency value ($freq(u)$) of each itemset $u$. STR-Slice compressed more tuples and offered better compression rate compared to STR-MFIC. But when comparing the number and the frequency of itemsets used for compression, we can see that STR-Slice used a very large number of itemsets with low frequencies while STR-MFIC used less number of itemsets with high frequencies. For example for the benchmark randsJC2500, STR-MFIC compressed about 20% of tuples of each instance with only 86 MFI with an average frequency equals to 16. STR-Slice compressed about 20% more tuples compared to STR-MFIC, using 470 frequent itemsets with an average frequency equals to 3. So, STR-Slice compress table constraint with a very large number of frequent itemsets with smallest frequencies, therefore the resulting compressed table is composed of a large number of smallest entries.

Solving compressed table constraints with high number of smallest entries can slow down the solving process due to the number of entries to iterate.

Table 8: Comparing STR-MFIC and STR-Slice on number of compressed tuples ($c-tup$), compression rate ($c-rate$), number of itemsets ($|M|$), length of itemsets ($|u|$) and frequency of itemsets ($freq(u)$).

| benchmark   | method     | $c-tup$(%) | $c-rate$(%) | $|M|$ | $|u|$ | $freq(u)$ |
|-------------|------------|------------|-------------|------|------|----------|
| Crossword-LexVg | STR-MFIC   | 44.3       | 22.1        | 88   | 4    | 18       |
|             | STR-Slice  | 57.13      | 29.45       | 667  | 3    | 2        |
| Crossword-WordsVg | STR-MFIC   | 52.04      | 29.04       | 95   | 4    | 30       |
|             | STR-Slice  | 69.4       | 18.8        | 376  | 4    | 3        |
| randsJC2500 | STR-MFIC   | 39.2       | 17.9        | 86   | 3    | 16       |
|             | STR-Slice  | 59.27      | 28.04       | 470  | 3    | 3        |
| randsJC5000 | STR-MFIC   | 38.14      | 15.6        | 98   | 3    | 24       |
|             | STR-Slice  | 71.26      | 34.87       | 1039 | 3    | 3        |
| randsJC7500 | STR-MFIC   | 36.17      | 15.8        | 91.8 | 3    | 28       |
|             | STR-Slice  | 76.44      | 38.4        | 1604 | 3    | 3        |
| randsJC10000| STR-MFIC   | 35.5       | 16.1        | 91   | 3    | 32       |
|             | STR-Slice  | 79.54      | 40.91       | 2177 | 3    | 3        |
| bddLarge    | STR-MFIC   | 24.5       | 13.4        | 25   | 7    | 82       |
|             | STR-Slice  | 91         | 58          | 1655 | 6    | 3        |

7.3 Comparing STR-MFIC with state-of-the-art GAC-based algorithms

Our last experiment aims at comparing our approach STR-MFIC with state-of-the-art algorithms enforcing GAC on table constraints. The tested GAC algorithms are STR3 [34], shortSTR2 [38], STRBit [33], MDD4R [41], GAC4 [1].
GAC4R [48] and CT [32]. Figure 3 depicts the curves of cumulative CPU times obtained for the STR-MFIC method (the red curve) and the selected GAC algorithms of the state-of-the-art among all instances of the used benchmarks. Clearly, compression approaches based on Bitwise representation such as CT and STRbit are the best performer methods with a slight advantage to CT. Even if STRbit dominates our STR-MFIC method, we notice that their corresponding curves get closer and closer until they are almost identical after 230 solved instances. The shortSTR2 and STR-MFIC are competitive such that their cumulative curves are almost identical for the first 180 solved instances then we can clearly see that the cumulative curve of STR-MFIC dominates that of shortSTR2 and solved more instances. Comparing to the other selected GAC-based algorithms, the cumulative curve of STR-MFIC dominates all the others cumulative curves and solves more instances.

**Figure 2:** The cumulated CPU time (s) of selected GAC algorithms. The x-axis represents the solved instances and the y-axis the total solving time. There are 250 instances from 9 divers benchmarks.

**Figure 3:** The cumulated CPU time (s) of our STR-MFIC and some STR variants algorithms.

In Figure 4 depicts the curves of cumulative CPU times obtained for three benchmarks chosen arbitrarily from those selected to conduct our experiments.
We can see that for the three benchmarks, the curves of cumulative CPU times of STR-MFIC [3] dominate that of STR-Slice, STR3 [34], shortSTR2 [38], MDD4R [41], GAC4 [41] and GAC4R [48] and solve more instances. While Comparing to

- **STRbit [33]**: for the benchmark Crossword-words-vg, even if STRbit [33] solved more instances, the curve of cumulative CPU times of STR-MFIC dominates that of STRbit [33]. For the benchmark Crossword-lex-vg, the two methods STR-MFIC and STRbit [33] solve the same number of instances and their cumulative curves are competitive. For the 35 first instances, the cumulative curve of STRbit [33] dominates that of STR-MFIC then the one of STR-MFIC dominates it. Finally, even if STRbit [33] and STR-MFIC solved the same number of instance of the benchmark randsJC10000, the curve of cumulative CPU times of STRbit [33] dominates that of STR-MFIC.

- **CT [32]**: for the benchmarks Crossword-words-vg and Crossword-lex-vg, the two curves of cumulative CPU times of STR-MFIC and CT [32] are almost identical for the first 28 instances of each benchmark, then we remark that the ones of CT [32] dominate that of STR-MFIC.

Figure 4: Curves of cumulative CPU times obtained for STR-MFIC and the state-of-the-art GAC-based methods for the benchmarks crosswords-words-vg, crosswords-lex-vg and randsJC10000.

8 Conclusion

In this paper, we have proposed a new approach based on data mining techniques for compressing table constraints. Our approach, called MFI-Compression, enumerates from a table constraint the maximal frequent itemsets (MFI) relevant for compression. To cope with the problem of fixing the minimum support \( S_{min} \), we proposed to use the TopK approach. The coverages of MFI in a table constraint can overlap each other, a tuple of a table constraint can be compressed using one and only one MFI. To respect this condition and compress a table constraint more efficiently, we proposed to select from the set of MFI only those having the largest area and do not overlap each other. To solve the compressed CSP, we used the STR-slice [3] which is an optimized variant of STR2 for compressed CSP. We called the combination of the two methods MFI-Compression with STR-slice by STR-MFIC. We evaluated our contributions on different benchmarks, and compared it to some GAC-based methods of the state-of-the-art, namely the different variant of STR (STR2, STR3, STR-Slice, shortSTR2 and STRbit), GAC4, GAC4R, MDD4R and Compact-table. The results showed that compressing table constraints using STR-MFIC enables to solve the resulting CSP in less time compared to STR-Slice [3] also based on a data mining technique. The results obtained for our method are competitive.
with that obtained for the other selected GAC-based methods of the state-of-the-art except for CT [32] which obtains better results in most cases. As future work we will try find an efficient solution to fix the value $S_{min}$.

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