Dynamical Friction and Resonance Trapping in Planetary Systems

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ABSTRACT
A restricted planar circular three-body system, consisting of the Sun and two planets, is studied as a simple model for a planetary system. The mass of the inner planet is considered to be larger and the system is assumed to be moving in a uniform interplanetary medium with constant density. Numerical integrations of this system indicate a resonance capture when the dynamical friction of the interplanetary medium is taken into account. As a result of this resonance trapping, the ratio of orbital periods of the two planets becomes nearly commensurate and the eccentricity and semimajor axis of the orbit of the outer planet and also its angular momentum and total energy become constant. It appears from the numerical work that the resulting commensurability and also the resonant values of the orbital elements of the outer planet are essentially independent of the initial relative positions of the two bodies. The results of numerical integrations of this system are presented and the first-order partially averaged equations are studied in order to elucidate the behavior of the system while captured in resonance.

Key words: planetary dynamics, resonance capture, averaging.

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1 INTRODUCTION

The dragging effect of the interplanetary medium and its role in formation and evolution of planetary systems are quite well known and have long been investigated by many authors such as Weidenschilling et al. (1985 & 1993), Patterson (1987), Peale (1993), Malhotra (1993), Beaugé et al. (1993 and 1994a&b), Murray (1994) and Gomes (1995). Recently, it has also been shown that the dynamical friction of this medium can account for the stability of planetary orbits in the early stages of their dynamical evolution (Melita and Woolfson 1996, here after MW). Using a formulation developed by Dodd and McCrea (1952, see also Binney and Tremaine 1987) and by integrating a general three-body system, MW showed that the system of Sun and two massive planets, subject to dynamical friction and accretion, will be captured into a near mean-motion resonance when the inner body is more massive. As a result of this resonance trapping, the eccentricities of the both planets become constant and their semimajor axes decrease monotonically.

A simple planetary model is studied here based upon the results of Melita and Woolfson (1996). The simplifications introduced in this study make it possible to analyze the averaged dynamics of the system, analytically. The model used in this paper is a restricted planar circular three-body system with an inner planet that is more massive. Since MW showed that the effect of accretion will not change the end results qualitatively, this effect is not included in the dynamical equations of the system here. However, the frictional effect of the interplanetary medium is the main source of non-gravitational dissipation and is therefore taken into account.

The model under investigation is presented in section 2. Section 3 is concerned with the results of the numerical integrations. In order to analyze the numerical results analytically, a newly developed averaging technique for dissipative systems is introduced in section 4. This averaging method has been developed by Chicone et al. (1996-7a&b) and was used in their analysis of dynamical behavior of binary systems subject to external gravitational
radiation as well as radiation reaction damping. This averaging technique provides a very helpful tool to study the dynamics of the system at resonance. In section 5, the application of this method to the dynamical equations of the system are discussed. The first-order averaged system at resonance is presented in section 6 and in section 7, conclusions of this study are presented by reviewing the results and discussing the possible extensions and applications.

2 THE MODEL

A restricted planar circular three-body system consisting of a central star (hereafter \( S \)) and two planets is considered with the inner planet more massive. It is assumed that the gravitational attraction of the planets on the central star is so small that its motion is negligible. This allows for the consideration of an inertial coordinate system with its origin on \( S \). It is also assumed that after taking all perturbative effects into account, the orbital motion of the inner planet (hereafter \( I \)) is uniformly circular with a given period. Dynamics of the outer planet (hereafter \( O \)) is thus the focus of attention and is determined by the gravitational attractions of the central star and the inner planet and also by dissipation due to dynamical friction caused by the interplanetary medium.

In the inertial coordinate system under consideration, the equation of motion for planet \( O \) can be written as

\[
 m_o \frac{d^2\vec{r}_o}{dt^2} = -G \frac{Mm_o}{r_o^3} \vec{r}_o - G \frac{m_im_o}{|\vec{r}_o - \vec{r}_I|^3} (\vec{r}_o - \vec{r}_I) + \vec{R}_f ,
\]

where the indices \( I \) and \( O \) stand for the inner and the outer planets respectively, and \( \vec{R}_f \) represents the dynamical friction force of the interplanetary medium and is given by (Dodd & McCrea 1952, Binney & Tremaine 1987)

\[
 \vec{R}_f = -2\pi \rho_o \frac{G^2m_o^2}{W^3} \ln \left( 1 + \frac{S^2W^4}{G^2m_o^2} \right) \vec{W} .
\]
In this equation, \( \overrightarrow{W} \) is the relative velocity of \( \mathcal{O} \) with respect to the medium, \( \rho_o \) is the uniform density of the medium and

\[
S = r_o \left( \frac{m_o}{2M} \right)^{1/3},
\]

(3)

where \( M \) represents the mass of the central star.

Equation (1) can be simplified by dividing both sides by \( m_o \) and also by letting \( \overrightarrow{R} = \overrightarrow{R}_f/m_o \) and \( \overrightarrow{r} = \overrightarrow{r}_o \). The main equation under consideration can therefore be written as

\[
\frac{d^2 \overrightarrow{r}}{dt^2} = -G \frac{M}{r^3} \overrightarrow{r} - G \frac{m_I}{|\overrightarrow{r} - \overrightarrow{r}_I|^3} (\overrightarrow{r} - \overrightarrow{r}_I) + \overrightarrow{R}.
\]

(4)

It is more convenient to write this equation in dimensionless form. Introducing \( t_o \) and \( r_o \) as the quantities that carry units of time and length respectively, \( t \) and \( r \) can be written as \( t = t_o \hat{t} \) and \( r = r_o \hat{r} \), where \( \hat{t} \) and \( \hat{r} \) are their corresponding dimensionless variables. Equation (4) has now the form

\[
\frac{d^2 \overrightarrow{r}}{d\hat{t}^2} = -K \frac{\overrightarrow{r}}{r^3} - \varepsilon K \frac{(\overrightarrow{r} - \overrightarrow{r}_I)}{|\overrightarrow{r} - \overrightarrow{r}_I|^3} + \overrightarrow{\hat{R}},
\]

(5)

where

\[
K = \frac{GMt_o^2}{r_o^3},
\]

(6)

\[
\varepsilon = \frac{m_I}{M},
\]

(7)

and

\[
\overrightarrow{\hat{R}} = \overrightarrow{\hat{R}} \left( \frac{t_o^2}{r_o} \right).
\]

(8)

In the rest of the calculations, I choose a set of units such that \( K = 1 \). Therefore from equation (6), \( t_o \) and \( r_o \) will be related as \( t_o^2 = r_o^3/GM \). This relation implies that \( r_o \) can be viewed as the orbital radius of a planet that orbits \( S \) on a circular path with a period \( 2\pi t_o \). In the model presented in this paper, planet \( \mathcal{I} \) has this type of motion. Therefore,
in the rest of this paper, I set \( r_0 = r_I \) and \( t_0 = T_I / 2\pi \) where \( T_I \) represents the (given) orbital period of planet \( I \). Dropping the hat signs, equation (5) can be simplified as

\[
\frac{d^2 \vec{r}}{dt^2} = -\frac{\vec{r}}{r^3} - \frac{\varepsilon}{|\vec{r} - \vec{r}_I|^3} \left[ \frac{\vec{r}}{|\vec{r} - \vec{r}_I|} \right] + \vec{R}, \tag{9}
\]

where \( \vec{r}_I \) is the unit vector along \( \vec{r}_I \).

Let us now assume that the motion is planar. Introducing a polar coordinate system on the plane of the orbits and with its origin at the location of the central star, the dynamical equations of planet \( O \) will be given by

\[
P_r = \dot{r} , \tag{10}
\]

\[
P_\theta = r^2 \dot{\theta} , \tag{11}
\]

\[
\dot{P}_r = \frac{P_\theta^2}{r^3} - \frac{1}{r^2} - \frac{\varepsilon}{|\vec{r} - \vec{r}_I|^3} \left[ r - \cos(\theta - \theta_I) \right] + (R_x \cos \theta + R_y \sin \theta) , \tag{12}
\]

and

\[
\dot{P}_\theta = -\varepsilon \frac{r}{|\vec{r} - \vec{r}_I|^3} \sin(\theta - \theta_I) + r(-R_x \sin \theta + R_y \cos \theta) , \tag{13}
\]

where \( R_x \) and \( R_y \) are the \( x \) and \( y \) components of \( \vec{R} \), respectively, and \( \theta_I = 2\pi t/T_I + \) constant. In dimensionless form and with a time origin such that this constant value becomes zero, we have \( \theta_I = t \), where \( t \) is the dimensionless time variable.

In order to calculate \( R_x \) and \( R_y \), we need to turn our attention to the interplanetary medium. It is assumed that this medium is freely rotating around \( S \) (Kiang 1962, Dormand
Therefore its velocity at any point is perpendicular to the position vector at that point with a dimensionless magnitude equal to \( v_\mu = r^{-1/2} \). The relative velocity \( \vec{W} = \vec{v} - v_\mu \), will therefore have a radial component equal to \( \dot{r} \) and a tangential component equal to \( r \dot{\theta} - r \omega_\mu \) where \( \omega_\mu = r^{-3/2} \) is the dimensionless angular frequency of the medium at distance \( r \). As a result, the magnitude of \( \vec{W} \) will be equal to

\[
W^2 = \dot{r}^2 + r^2 (\dot{\theta} - \omega_\mu)^2 .
\] (14)

The components of \( \vec{W} \) are easily obtained noting that at any distance \( r \), the angle between \( \vec{v}_\mu \) and the \( x \)-axis is equal to \( (\theta + \frac{\pi}{2}) \). Therefore

\[
W_x = \dot{r} \cos \theta - r (\dot{\theta} - \omega_\mu) \sin \theta ,
\] (15)

and

\[
W_y = \dot{r} \sin \theta + r (\dot{\theta} - \omega_\mu) \cos \theta .
\] (16)

From equation (2) and in dimensionless form, \( R_x \) and \( R_y \) will then be equal to

\[
R_x = -\frac{A}{W^3} \ln (1 + Br^2 W^4) \left[ \dot{r} \cos \theta - r (\dot{\theta} - \omega_\mu) \sin \theta \right] ,
\] (17)

and

\[
R_y = -\frac{A}{W^3} \ln (1 + Br^2 W^4) \left[ \dot{r} \sin \theta + r (\dot{\theta} - \omega_\mu) \cos \theta \right] ,
\] (18)

where \( A \) and \( B \) are positive parameters given by

\[
A = 2\pi \left( \frac{\rho_0 r^3}{M} \right) \left( \frac{m\omega}{M} \right) , \quad B = 2^{-2/3} \left( \frac{m\omega}{M} \right)^{-4/3},
\] (19)

such that \( A << 1 \) and \( AB << 1 \) for any realistic system.
3 NUMERICAL RESULTS

Equations (10) to (13) were numerically integrated for different values of the perturbation parameter \( \varepsilon \) and density \( \rho_0 \) using a stiff integration routine. Following MW, integrations were first performed on the Sun-Jupiter-Saturn system starting from the present near (5:2) commensurability. The density of the interplanetary medium was taken to be constant and equal to \( 10^{-11} \) Kg m\(^{-3}\) which is equivalent to 16 times the mass of Jupiter uniformly spread in a sphere of radius 50 au. In complete agreement with the results of MW, a near (2:1) commensurability is obtained for all initial relative positions of the two planets. A typical case is presented in Figure 1.

![Figure 1](image-url)

**Figure 1.** Graph of the ratio of orbital period of the outer planet (i.e., Saturn) to that of the inner one (i.e., Jupiter) against time. Integrations were performed with Jupiter on the x-axis with \( \theta_I = 0 \) and Saturn at \( \theta_O = 45^\circ \). The initial values of \( r \), \( P_\theta \) and \( P_r \) were calculated from \( r = a(1-e^2)/(1+e \cos \hat{\nu}) \), \( P_\theta^2 = a(1-e^2) \) and \( P_r = e \sin \hat{\nu}/P_\theta \), where \( a=1.838046 \) is the present dimensionless value of the mean semimajor axis of Saturn’s orbit, \( e = 0.0556 \) is its present mean orbital eccentricity and \( \hat{\nu} = 15^\circ \) is its initial true anomaly (see section 4 and also Figure 4). The timescale is \( 10^4 T_J/2\pi \) years.
As a result of this resonance trapping, the orbital eccentricity and semimajor axis of the outer planet (i.e., Saturn) and also its angular momentum and total energy become essentially constant (Figure 2).

Since the present density of the interplanetary medium is much smaller than $10^{-11} \text{Kg} \text{m}^{-3}$, integrations were also performed with lower values of density for the actual Sun-Jupiter-Saturn system. A typical case is shown in Figure 3, where the system is captured in a (3:1) resonance but leaves this state after a short time.
In another series of runs, the planetary masses in the Sun-Jupiter-Saturn system were changed while keeping their mass ratio constant and equal to the present mass ratio of Jupiter and Saturn. The density of the interplanetary medium was kept equal to the 16 times the mass of the inner planet uniformly distributed inside a sphere of radius 50 au and the mass of the central star was unchanged and equal to that of the Sun. Integrations were performed for different values of $\varepsilon$ and it was observed that only for the values of $\varepsilon$ in the range $10^{-2}$ to $10^{-4}$, the system was captured in resonance (Figure 4). The interesting result was that starting from present near (5:2) commensurability between Saturn and Jupiter, all resonances for this range occurred near (2:1). It is important to mention that for the values of $\varepsilon$ smaller than $10^{-4}$, the planetary model under consideration will no longer be physically valid. Numerical results indicate that in these cases, the amplitude of oscillations of $r$ grows rapidly with time which results in change of the configuration of the system. That is, for long amplitudes, planet $O$ approaches $I$ and crosses its orbit. This results in an exchange of positions between $O$ and $I$ and also causes the quantity $|\mathbf{r} - \mathbf{r}_I|$ in the dynamical equations of the system, reaches zero at certain times.
Figure 4. Graphs of the ratios of orbital periods (left column) and orbital eccentricities (right column) versus time, for $\varepsilon = 10^{-2}$ (top), and $\varepsilon = 10^{-4}$ (bottom). The initial values are given by $(a, e, \theta, \dot{v}) = (1.838046, 0.0556, 75^\circ, 30^\circ)$ and the timescale is $(10^4 T_i / 2\pi)$ years.

Numerical experiments on the actual Sun-Jupiter-Saturn system, were also performed with larger values of the mass of the central star $M$, but no resonance capture was obtained. A typical case is presented in Figure 5.

Figure 5. Ratio of orbital periods (left) and orbital eccentricity of the outer planet (right) against time for a central star with a mass equal to 1000 times the present mass of the Sun. The initial values and the timescale are the same as those in Figure 4.
The rest of this paper is devoted to the analysis of the results of the numerical integrations presented in this section. For this purpose, I will use a method of averaging called "Partial Averaging Near a Resonance" (Chicone et al. 1996) which is described in the next section.

4 AVERAGING

Consider the following perturbation problem in a $k$-dimensional Euclidean space $\mathbb{R}^k$,

\[
\dot{u} = f(u) + \epsilon b(u, t, \epsilon), \quad u \in \mathbb{R}^k.
\]  

(20)

In this system $\epsilon$ is the perturbation parameter, $b$ is a periodic function of time with a period $\eta > 0$ and the overdot represents the derivative with respect to time $t$. If the unperturbed system, i.e. $\dot{u} = f(u), u \in \mathbb{R}^k$ is integrable, one can always find a set of canonical coordinate transformations to write equation (20) in terms of a set of action-angle variables $(\mathcal{L}, \phi) \in \mathbb{R}^k \times \mathbb{R}^\ell$ such that

\[
\dot{\mathcal{L}} = \epsilon F(\mathcal{L}, \phi) + O(\epsilon^2),
\]  

(21)

and

\[
\dot{\phi} = \Omega(\mathcal{L}) + \epsilon B(\mathcal{L}, \phi) + O(\epsilon^2).
\]  

(22)

In these equations $\phi$ is a $2\pi$ modulo $\ell$-dimensional vector of angular variables and $F$ and $B$ are $2\pi$ periodic functions of $\phi$. The most important feature of equations (21) and (22) is that in the angular equation (22), there appears a term which is independent of the perturbation parameter $\epsilon$. This indicates that, in the first order of approximations (i.e., $\epsilon$), $\phi(t)$ can be considered as a fast-changing angular variable while $\mathcal{L}(t)$ moves slowly away from its unperturbed value. Therefore, when studying the dynamics of the system (20), the time evolution of the averaged value of $\mathcal{L}(t)$ over $\phi(t)$ can be taken as a reasonable approximation to the evolution of $\mathcal{L}(t)$. 
The above statement is often referred to as the "Averaging Principle." This principle states that if $\mathcal{L} = \mathcal{L}_0$ is the initial value for the action variable $\mathcal{L}$ in system (21) and (22), then for sufficiently small $\epsilon$, the solution to the initial value problem

$$\dot{J} = \epsilon \bar{F}(J) \quad ; \quad J(0) = \mathcal{L}_0 ,$$

will provide a useful approximation for the evolution of the action variable $\mathcal{L}$ with an error of order $\epsilon^2$ over a timescale of order $\epsilon^{-1}$. In this equation $\bar{F}$ is the averaged value of $F$ and is given by

$$\bar{F}(J) = \frac{1}{(2\pi) \epsilon} \int_{N\epsilon} F(J, \phi) \, d\phi .$$

In the system presented in this paper, the unperturbed system (i.e. Sun-Saturn) is Hamiltonian. Also, in the perturbation problem given by equations (10) to (13), $\epsilon$ and $A$ are small quantities. Therefore we can use the averaging principle to analyze the dynamics of this system while captured in resonance. To this end, one needs to write the equations of motion in terms of appropriate action-angle variables that will provide us with a fast-changing angular variable. This is accomplished by writing the equations of motion in terms of Delaunay’s variables which can be described as follows.

If at any time $t$ all perturbative forces are removed, the orbit of the outer planet will become an ellipse with its focus always at the origin of coordinates. This ellipse is tangent to the actual orbit at point $\vec{r}(t)$ and is therefore called an osculating ellipse (Brouwer & Clemence 1961, Kovalevsky 1967, Hagihara 1972 and Gutzwiller 1998). One can use the orbital elements of the osculating ellipse to introduce the Delaunay action-angle variables. In our planar problem, the relevant Delaunay variables are given by (Kovalevsky 1967, Sternberg 1969, Hagihara 1972)

$$L = a^{1/2} ,$$

$$l = \hat{u} - e \sin \hat{u} ,$$
\[ G = \left[ a(1-e^2) \right]^{1/2}, \]  

and

\[ g = \theta - \hat{v}, \]

where \( a \) and \( e \) are the semimajor axis and eccentricity of the osculating ellipse, respectively, and \( l \) is the mean anomaly. In these equations \( L \) and \( G \) are action variables, while \( l \) and \( g \) are their corresponding angular variables. The variables \( \hat{v} \) and \( \hat{u} \) are, respectively, the true anomaly and the eccentric anomaly of the outer planet as illustrated in Figure 6.

The polar coordinates and the momenta of the outer planet are related to the Delaunay variables by

\[ P_{\theta}^2 = G^2 = r(1 + e \cos \hat{v}) \quad \text{and} \quad P_r = e \hat{v}/G. \]

Using these equations along with replacing \( \theta \) from equation (28), the equations of motion in terms of the Delaunay variables can be written as

\[
\frac{dL}{dt} = a(1-e^2)^{-1/2} \left[ F_r e \sin \hat{v} + F_\theta (1 + e \cos \hat{v}) \right],
\]

\[
\frac{dG}{dt} = r F_\theta,
\]

\[
\frac{dl}{dt} = \omega + \frac{r}{e} a^{-1/2} \left[ F_r (-2e + \cos \hat{v} + e \cos^2 \hat{v}) - F_\theta (2 + e \cos \hat{v}) \sin \hat{v} \right],
\]

and

\[
\frac{dg}{dt} = \frac{1}{e} \left[ a(1-e^2) \right]^{1/2} \left[ -F_r \cos \hat{v} + F_\theta \left( 1 + \frac{1}{1+e \cos \hat{v}} \right) \sin \hat{v} \right],
\]

where

\[
F_r = \varepsilon \frac{\cos(\theta - \theta_I)}{|\vec{r} - \vec{r}_I|^3} - r (R_x \cos \theta + R_y \sin \theta),
\]

\[
F_\theta = -\varepsilon \frac{\sin(\theta - \theta_I)}{|\vec{r} - \vec{r}_I|^3} - (R_x \sin \theta - R_y \cos \theta),
\]

and the Keplerian frequency of the osculating ellipse is given by \( \omega = L^{-3} \).
Quantities $F_r$ and $F_\theta$ are, in fact, representing perturbations due to the gravitational attraction of the inner planet and also the dynamical friction of the interplanetary medium. It is evident that only the angular variable $l$, i.e. the mean anomaly, has an associated Keplerian frequency unaffected by perturbations. This immediately suggests that $l$ is the appropriate "fast" angular variable for averaging purposes. In the next section, the averaging principle is applied to the system of equations (29) to (32) and the averaged dynamics of the system while captured in resonance is discussed.
5 AVERAGED SYSTEM AT RESONANCE

As mentioned above, equations (29) to (32) represent the dynamics of the system in terms of action variables \((L, G)\) and angular variables \((l, g)\). The objective of this section is to apply the averaging principle to the equations (29) to (32) while the system is at resonance. In general, resonance occurs when the Keplerian frequency \(\omega\) becomes commensurate with the frequency of the external perturbation. In the system presented in this paper, the external perturbation is due to gravitational attraction of the inner planet and its frequency is given by \(\omega_I = 2\pi/T_I\). Therefore in this system, resonance capture means that relatively prime integers \(m\) and \(n\) exist such that \(m\omega = n\omega_I\). Since in this model the inner planet is orbiting the central star on a circular path with a fixed orbital period, the immediate result of resonance trapping is a constant value for the orbital period of the outer planet. A constant orbital period in turn, results in a constant value for the semimajor axis of the osculating ellipse and therefore a constant value for the action variable \(L\). At resonance, \(L = L_0 = (m/n\omega_I)^{1/3}\). However, as it is evident from Figure 2, the value of \(L = a^{1/2}\) during resonance is not entirely constant, but oscillates around its Keplerian value \(L_0\). Let \(D\) be a measure of these oscillations and also let the oscillations of the mean anomaly from its Keplerian value \(L_0^{-3}t\) be given by \(\varphi\). Requiring \(\dot{L}\) and \(\dot{l}\) to have the same power of \(\varepsilon\) in the lowest order of approximation, one can write

\[
L = L_0 + \varepsilon^{1/2} D ,
\]

(35)

and

\[
l = \frac{1}{L_0^3} t + \varphi ,
\]

(36)

where the first order of approximation here has perturbation parameter \(\varepsilon^{1/2}\).

Equations (35) and (36) are indeed the necessary transformations that one needs to employ in order to write equations (29) to (32) for the system at resonance. In order to study the averaged dynamics of the system at this state, the averaging method presented
in section 4 must then be applied to the transformed equations. However, it is important to mention that the averaging principle is valid only for systems with one angular variable (Chicone et al. 1997a). In the system presented in this paper, there are two angular variables $l$ and $g$. One can, however, show that the average rate of variation of $g$ is in fact negligible and therefore it can be considered as a slow-changing quantity. This becomes evident by writing equations (29) to (32) as

$$\frac{dL}{dt} = -\varepsilon \frac{\partial H_{\text{ext}}}{\partial l} + \varepsilon \Delta R_L , \quad (37)$$

$$\frac{dG}{dt} = -\varepsilon \frac{\partial H_{\text{ext}}}{\partial g} + \varepsilon \Delta R_G , \quad (38)$$

$$\frac{dl}{dt} = \frac{1}{L^3} + \varepsilon \frac{\partial H_{\text{ext}}}{\partial L} + \varepsilon \Delta R_l , \quad (39)$$

and

$$\frac{dg}{dt} = \varepsilon \frac{\partial H_{\text{ext}}}{\partial G} + \varepsilon \Delta R_g , \quad (40)$$

where $\Delta = AB/\varepsilon$,

$$H_{\text{ext}} = -\frac{1}{|\vec{r} - \vec{r}_l|} , \quad (41)$$

and $R_L$, $R_G$, $R_l$ and $R_g$ represent the contributions of dynamical friction in their corresponding equations. I will show in section 6 that from these contributions, only $R_L$ will appear in the first-order averaged system.

Equations (39) and (40) indicate that while $l$ has a perturbation-free frequency $L^{-3}$, the rate of change of $g$ is proportional to $\varepsilon$. This implies that, in comparison to $\dot{l}$, the rate of variation of $g$ can be neglected and therefore, for the purpose of this study, equations (29) to (32) can be considered to have only one fast-changing angular variable that is $l$. Equations (37) and (39) have now the general form of the system (21) and (22) and can therefore be used as the appropriate grounds for applying our averaging method. That means, after
replacing $L$ and $l$ in equations of motion (37) to (40) by their equivalent expression given by (35) and (36) and averaging the resulting equations over the fast-changing angular variable $l$, the averaged equations will represent a system whose dynamics, in the first order of approximation, will be the same as the dynamical behavior of the original system at resonance, for a time interval $\varepsilon^{-1/2}t_0$ where $t_0$ is the unit of time introduced in section 2. To do this, let us differentiate equations (35) and (36) with respect to time and expand $1/L^3$ in powers of $\varepsilon^{1/2}$ as

$$\frac{1}{L^3} = \frac{1}{L_0^3} \left[ 1 - \varepsilon^{1/2} \left( \frac{3D}{L_0} \right) + \varepsilon \left( \frac{6D^2}{L_0^2} \right) + O(\varepsilon^{3/2}) \right].$$

Equations (37) to (40) will now have the form

$$\dot{D} = -\varepsilon^{1/2} F_{11} - \varepsilon D F_{12} + O(\varepsilon^{3/2}),$$

$$\dot{G} = -\varepsilon F_{22} + O(\varepsilon^{3/2}),$$

$$\dot{\varphi} = -\varepsilon^{1/2} \left( \frac{3D}{L_0^4} \right) + \varepsilon \left( \frac{6D^2}{L_0^5} + F_{32} \right) + O(\varepsilon^{3/2}) ,$$

$$\dot{g} = \varepsilon F_{42} + O(\varepsilon^{3/2}),$$

where

$$F_{11} = \frac{\partial}{\partial l} H_{ext} - \Delta R_L ,$$

$$F_{22} = \frac{\partial}{\partial g} H_{ext} - \Delta R_G ,$$

$$F_{32} = \frac{\partial}{\partial l} H_{ext} + \Delta R_l ,$$

$$F_{42} = \frac{\partial}{\partial G} H_{ext} + \Delta R_g ,$$

and $F_{12} = \partial F_{11}/\partial L$ are evaluated at $(L_0, G, L_0^{-3} t + \varphi, g)$. In labeling these terms, I have followed the convention of Chicone et al. (1997b).
The averaged equations of the system at resonance are obtained by averaging equations (43) to (46) over the mean anomaly $l$. However, application of the averaging principle requires an appropriate averaging transformation that renders the system of (43) to (46) into a new system such that in the lowest order, it becomes exactly the first-order averaged system (Chicone et al. 1997b). To obtain this averaging transformation, one has to define

$$\lambda(G, \varphi, g, t) = F_{11}(G, \frac{1}{L_0^3} t + \varphi, g, t) - \bar{F}_{11}, \quad (51)$$

where

$$\bar{F}_{11}(G, \varphi, g) = \frac{\omega_f}{2\pi m} \int_0^{2\pi m/\omega_f} F_{11}(G, \frac{1}{L_0^3} t + \varphi, g, t) \, dt, \quad (52)$$

is the averaged value of $F_{11}$. Introducing $\Lambda(G, \varphi, g, t)$ by

$$\frac{\partial}{\partial t} \Lambda(G, \varphi, g, t) = \lambda(G, \varphi, g, t), \quad (53)$$

such that $\bar{\Lambda} = 0$, one can use the averaging transformations $D = \hat{D} - \varepsilon^{1/2} \Lambda(\hat{G}, \hat{\varphi}, \hat{g}, t)$, $G = \hat{G}$, $\varphi = \hat{\varphi}$, $g = \hat{g}$ to write the equations (43) to (46) as

$$\dot{\hat{D}} = -\varepsilon^{1/2} \hat{F}_{11} - \varepsilon \hat{D} \left( F_{12} + \frac{3}{L_0^3} \frac{\partial \Lambda}{\partial \varphi} \right) + O(\varepsilon^{3/2}), \quad (54)$$

$$\dot{\hat{G}} = -\varepsilon F_{22} + O(\varepsilon^{3/2}), \quad (55)$$

$$\dot{\hat{\varphi}} = -\varepsilon^{1/2} \left( \frac{3\hat{D}}{L_0^4} \right) + \varepsilon \left( \frac{6\hat{D}^2}{L_0^5} + F_{32} + \frac{3\Lambda}{L_0^3} \right) + O(\varepsilon^{3/2}), \quad (56)$$

$$\dot{\hat{g}} = \varepsilon F_{42} + O(\varepsilon^{3/2}). \quad (57)$$

Since equations (10) to (13) represent a perturbation problem of order $\varepsilon$, I will keep the terms proportional to $\varepsilon$ in equations above and neglect the $O(\varepsilon^{3/2})$ terms. The averaged dynamics of the system will then be given by
\[ \dot{D} = -\varepsilon^{1/2} \bar{F}_{11} - \varepsilon \tilde{D} \bar{F}_{12} , \]  
\[ \dot{G} = -\varepsilon \bar{F}_{22} , \]  
\[ \dot{\phi} = -\varepsilon^{1/2} \left( \frac{3\tilde{D}}{L_0^4} \right) + \varepsilon \left( \frac{6\tilde{D}^2}{L_0^5} + \bar{F}_{32} \right) , \]  
\[ \dot{g} = \varepsilon \bar{F}_{42} . \]  

where \( \bar{F}_{12} , \bar{F}_{22} , \bar{F}_{32} , \) and \( \bar{F}_{42} \) are calculated in the same fashion as \( \bar{F}_{11} \).

### 6 FIRST ORDER AVERAGED DYNAMICS

In the present work, only the dynamics of the first-order averaged system at resonance will be investigated. In this approximation, the first-order averaged equations have perturbation parameter \( \varepsilon^{1/2} \) and are given by

\[ \dot{D} = -\varepsilon^{1/2} \bar{F}_{11} , \quad \dot{\phi} = -\varepsilon^{1/2} \left( \frac{3\tilde{D}}{L_0^4} \right) , \quad \dot{G} = \dot{g} = 0 , \]  

where the terms with perturbation parameter \( \varepsilon \) and higher have been neglected.

Equation (62) is, in fact, a general representation of the first-order averaged system at resonance. In this equation, it is \( \bar{F}_{11} \) that contains information about the physical properties of the system. For the three-body system presented in this paper, \( \bar{F}_{11} \) is given by equation (47) where

\[ R_L = -\frac{a}{AB} (1 - e^2)^{-1/2} \left\{ R_x C - R_y D \right\} . \]  

In this equation

\[ C(L,G,l,g) = \sin(\hat{v} + g) + e \sin g , \]  

and

\[ D(L,G,l,g) = \cos(\hat{v} + g) + e \cos g . \]
Since for the actual system of Sun-Jupiter-Saturn, numerical integrations have indicated a resonance lock with a near (2:1) commensurability (see section 3), one has to set $m = 2, n = 1$ and $\omega_i = 1$, which is the angular frequency of Jupiter in dimensionless form, when one calculates $\bar{F}_{11}$ using integration (52). This integration requires $H_{\text{ext}}$ and $R_L$ to be written in terms of the mean anomaly $l$. Appendices A and B contain details of these calculations. There, it has been shown that in the lowest order in eccentricity, $R_L$ is proportional to $e^2$. That is,

$$R_L = -\frac{1}{2} a_0^2 e^2 \left( 1 - \frac{3}{4} \cos^2 l \right)^{1/2} \cos l ,$$

(66)

where $a_0 \simeq 1.625$ represents the resonant value of the semimajor axis of Saturn (see Figure 2). It is evident from this equation that the averaged value of $R_L$ over the mean anomaly $l$ will become zero. The first non-vanishing term in the averaged value of $R_L$ appears as a term proportional to $e^3$. Numerical value of this term is so small that its contribution in dynamics of the first-order averaged system becomes actually negligible (see Figure 2; the numerical value of the orbital eccentricity of Saturn at resonance is in average about 0.022 with an amplitude of oscillation equal to 0.01).

The appearance of the higher orders of eccentricity in equation (66) requires $H_{\text{ext}}$ to be expanded to higher orders in $e$. However, it turns out that even in expanding $H_{\text{ext}}$ to the second order in eccentricity, non-vanishing terms after averaging, are at most of order $10^{-4}$. A comparison between these values and the order of magnitude of the term proportional to $e$ in that expansion (i.e. $10^{-2}$), indicates that the contributions of $O(e^2)$ terms are entirely negligible. Therefore, in the rest of these calculations, I will neglect the contribution of dynamical friction and will consider $F_{11} \simeq \partial H_{\text{ext}}/\partial l$. The external Hamiltonian $H_{\text{ext}}$ will also be expanded only to the first order in eccentricity. Appendix A shows that in that order, the only contribution of $H_{\text{ext}}$ in $\bar{F}_{11}$ appears as the term proportional to $\cos(2l + g - \theta_i)$. Denoting this term by $\mathcal{H}_{\text{ext}}$,

$$\mathcal{H}_{\text{ext}} = -\frac{e \sigma}{2a_0} \cos(2l + g - \theta_i) ,$$

(67)
where
\[
\sigma = \sum_{h=0}^{\infty} \left[ \frac{\Gamma\left(\frac{3}{2} + h\right)}{a_0^h h! \Gamma\left(\frac{3}{2}\right)} \right]^2 \left\{ 1 + \left(\frac{2h + 3}{h + 1}\right) \left[ 1 - \frac{3}{4a_0^2} \left(\frac{2h + 5}{h + 2}\right) \right] \right\} .
\] (68)

From equation (52), \( \bar{F}_{11} \) will now be equal to
\[
\bar{F}_{11} = \frac{e \sigma}{a_0} \sin(2\varphi + g) ,
\] (69)
and the first order averaged system at resonance will simply be obtained by substituting for \( \bar{F}_{11} \) in equation (62). That is,
\[
\dot{D} = -\frac{e \sigma}{a_0} \varepsilon^{1/2} \sin(2\varphi + g) ,
\] (70)

and
\[
\dot{\varphi} = -\varepsilon^{1/2} \left( \frac{3D}{L_0^4} \right) ,
\] (71)

where \( G \) and \( g \) are constants, and the tildes have been dropped for the sake of simplicity.

From these two equations it follows that
\[
\ddot{D} - \frac{6e \sigma \varepsilon}{a_0^4} D \cos(2\varphi + g) = 0 ,
\] (72)
and
\[
\ddot{\varphi} - \frac{3e \sigma \varepsilon}{a_0^4} \sin(2\varphi + g) = 0 .
\] (73)

Equation (73) is the equation of a mathematical pendulum. This equation can be attributed to a Hamiltonian \( H_\varphi \) in the form
\[
H_\varphi(\varphi, p_\varphi) = \varepsilon^{1/2} \left[ \frac{1}{2} p_\varphi^2 + U(\varphi) \right] ,
\] (74)

with a potential given by
\[
U(\varphi) = \frac{3e \sigma}{2a_0} \cos(2\varphi + g) + \text{Const.}
\] (75)
Figure 7 shows the graph of $U(\varphi)$ against $\varphi$. The periodic characteristic of $U(\varphi)$ indicates that for all values of $\varphi$ except for those corresponding to the maxima of $U(\varphi)$, the averaged system will fall into one of the potential wells and oscillates around a stable point forever. The resonance lock is established in this manner. For those values of $\varphi$ where $U(\varphi)$ becomes maximum, the system will be in an unstable equilibrium. A slight deviation from this instability will result in a resonance capture.

![Figure 7](image_url)

**Figure 7.** Graph of $U(\varphi)$ against $\varphi$. The periodic nature of $U(\varphi)$ guarantees resonance capture for essentially all initial relative positions of the two planets.

Equation (72), too, shows a harmonic behavior (Figure 8). It is important to remember that $D$ represents the deviation of $L$ from its resonant value $L_0$. From the averaging principle presented in section 4, it is expected that $D$ and $L$ will have the same general behavior for time intervals of duration $\varepsilon^{-1/2}t_0$. This is illustrated in Figure 8 for the system presented in this paper.
In an attempt to study analytically, the resonance capture phenomenon reported by Melita and Woolfson (1996), a restricted planar circular model of the Sun-Jupiter-Saturn system subject to dynamical friction with a freely rotating homogeneous interplanetary medium was studied. Numerical integrations of this system indicated a resonance lock with the same commensurability as reported by MW. The method of partial averaging was employed to study the dynamical evolution of the system while captured in resonance. By averaging the equations of motion over a fast-changing angular variable, these equations were reduced to the dynamical equations of a Hamiltonian system whose dynamics would be partially equivalent to the dynamical behavior of the main system. The application of this averaging method to the Sun-Jupiter-Saturn system where ε = 0.001, resulted in the first-order averaged system (70) and (71). Although this system is Hamiltonian and it is not expected to fully illustrate the dynamical evolution of the main dissipative system, but in a time interval $\varepsilon^{-1/2}T_I/2\pi$, it can present a reasonable approximation to the dynamics of the original system. The harmonic characteristic of the potential function of this Hamiltonian system guarantees the occurrence of resonance lock for all initial relative positions of the two planets. Also, the time evolution of the action variable of the first-
order averaged system illustrates an oscillatory behavior for the semimajor axis as well as the angular momentum of the main system over time scales of order $\varepsilon^{-1/2}$.

As mentioned above, the first-order averaged system at resonance is Hamiltonian. In general at this order, the dissipative effects of the non-gravitational perturbations appear as external torques in the equation of the mathematical pendulum presented by the differential equation of the angular variable $\varphi$. In the system presented in this paper, the averaged value of this torque in the lowest order in eccentricity, is proportional to $e^3$ and is entirely negligible when compared to the contribution of the gravitational attraction of the inner planet. The damping effect of this perturbation on the dynamical behavior of the original system can be illustrated by extending the calculations to the second-order partially averaged system. This will also allow for investigating the effects of the perturbation parameter $\varepsilon$ and the density of the interplanetary medium $\rho_0$ on the time variation of the orbital elements of the outer planet and their resonant values (Haghighipour, in preparation).

The accreting force of the interplanetary medium was also neglected in this study. Although the density of the interplanetary medium was considered much higher than its present value, the masses of the planets were kept constant and equal to their present values. Numerical integrations performed by Melita and Woolfson (1996) have indicated that accretion will only change the results quantitatively. Therefore, one can apply the exact analysis presented in this paper, to the dynamical behavior of the system at resonance when accretion is also included.

Numerical integrations were also carried out for different values of the perturbation parameter $\varepsilon$ and the density of the interplanetary medium $\rho_0$. The results indicated that when $\varepsilon$ is in the range $10^{-2}$ to $10^{-4}$, a Sun-Jupiter-Saturn-like system will be captured in a near (2:1) resonance when the density of the interplanetary medium is taken to be equal to 16 times the mass of the inner planet spread uniformly in a sphere of radius 50 au. However, for a real Sun-Jupiter-Saturn system in a less dense medium, the resonance
capture occurs for a shorter time interval and also moves to (3:1) commensurability.

The ideas presented in this paper can be applied to any dynamical system at resonance. Of particular interest would be extension of calculations to include the gravitational tidal effects as another source of dissipation. This can be applicable to analyze the dynamical behavior of triple systems (Arzoumanian 1996) as well as newly discovered planetary systems (see for instance, Holland et al. 1998).

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APPENDIX A: SIMPLIFICATION OF $H_{ext}$

From equation (41) and in dimensionless form, $H_{ext}$ is given by

$$H_{ext} = -[r^2 - 2r \cos(\theta - \theta_i) + 1]^{-\frac{1}{2}}. \tag{A1}$$

It is evident from this equation that in order to express $H_{ext}$ in terms of the mean anomaly $l$, one needs to write $r$ and $\cos(\theta - \theta_i)$ in terms of $l$. This is possible by writing $r = G^2(1 + e \cos \hat{v})^{-1}$ and $\theta = g + \hat{v}$, and expanding $\cos \hat{v}$ and $\sin \hat{v}$ as (Kovalevsky 1967)

$$\cos \hat{v} = -e + 2 \left( \frac{1 - e^2}{e} \right) \sum_{j=1}^{\infty} \cos(jl) J_j(je), \tag{A2}$$

and

$$\sin \hat{v} = (1 - e^2)^{1/2} \sum_{j=1}^{\infty} \sin(jl) \left[ J_{j-1}(je) - J_{j+1}(je) \right]. \tag{A3}$$

In these equations, $J_j$ is Bessel function of order $j$ and is given by

$$J_j(s) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!(j+\nu)!} \left( \frac{s}{2} \right)^{j+2\nu}. \tag{A4}$$

As mentioned in section 6, $H_{ext}$ is expanded to the first order in eccentricity. Substituting for $\cos \hat{v}$ and $\sin \hat{v}$ in the expressions for $r$ and $\cos(\theta - \theta_i)$ and considering terms of order $e$, one can therefore write

$$r \simeq a \left( 1 - e \cos l \right), \tag{A5}$$

and

$$\cos(\theta - \theta_i) \simeq \cos(l + g - \theta_i) + e \left[ \cos(2l + g - \theta_i) - \cos(g - \theta_i) \right]. \tag{A6}$$

The external Hamiltonian $H_{ext}$ will therefore be given by

$$H_{ext} = -\left[1 + a^2 - 2a \cos(l + g - \theta_i) \right]^{-\frac{1}{2}}$$

$$\left[ 1 + \frac{1}{2} e \frac{2a^2 \cos l + a \cos(2l + g - \theta_i) - 3a \cos(g - \theta_i)}{1 + a^2 - 2a \cos(l + g - \theta_i)} \right]. \tag{A7}$$
At resonance, the semimajor axis of the outer planet is almost constant with an average value \( a_0 \approx 1.625 \). One can use this to simplify equation (A7) by expanding \( H_{ext} \) in terms of \( a_0^{-1} \). The first term of this equation can be expanded using the relation

\[
\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \frac{1}{r} \sum_{N=0}^{\infty} \left(\frac{r_1}{r_2}\right)^N P_N(\cos \Theta),
\]

where \( \Theta \) is the angle between \( \vec{r}_1 \) and \( \vec{r}_2 \) and \( P_N(\cos \Theta) \) is Legendre polynomial of order \( N \). In using equation (A8) in (A7), \( \Theta = l + g - \theta_I \). Therefore the Legendre polynomial \( P_N(\cos \Theta) \) will produce terms proportional to \( \cos N(l + g - \theta_I) \). Due to the harmonic nature of these terms, their averaged values obtained from equation (52) will all vanish. Therefore, the term in equation (A7) that is independent of the eccentricity \( e \) will have no contribution in \( \bar{F}_{11} \). The remaining term in equation (A7) can also be expanded using the relation

\[
(1 - 2\tau \cos \alpha + \tau^2)^{-\lambda} = \sum_{q=0}^{\infty} C_{q}^{\lambda}(\cos \alpha) \tau^q, \quad |\tau| < 1,
\]

where \( C_{q}^{\lambda}(\cos \alpha) \) are Gegenbauer polynomials given by

\[
C_{q}^{\lambda}(\cos \alpha) = \sum_{h=0}^{q} \frac{\Gamma(\lambda + h)\Gamma(\lambda + q - h)}{h!(q-h)!\Gamma(\lambda)^2} \cos[(q - 2h)\alpha].
\]

In expanding the second term in equation (A7), \( \lambda = 3/2, \tau = a_0^{-1} \) and \( \alpha = l + g - \theta_I \). From equation (36) where \( L_0^3 = 2 \), the only non-vanishing terms in integration (52) are the ones proportional to \( \cos(2l + g - \theta_I) \). Expansion (A10) produces terms proportional to \( \cos[(q - 2h)(l + g - \theta_I)] \). Therefore after multiplying the numerator of the relevant term in equation (A7) by expansion (A9), the contribution of \( \cos l \) appears in the terms where \( q = 2h + 1 \), that of \( \cos(2l + g - \theta_I) \) in the terms where \( q = 2h \) and for \( \cos(g - \theta_I) \) when \( q = 2h + 2 \). The contributing terms in \( H_{ext} \) (i.e., the terms with non-vanishing averaged values) can thus be written as in equation (67).
APPENDIX B : SIMPLIFICATION OF $R_L$

Substituting for $R_x$ and $R_y$ from equations (17) and (18) in (63), $R_L$ can be written as

$$R_L = -\frac{a}{B W^3} (1 - e^2)^{-1/2} \ln(1 + Br^2 W^4) \left[ a(1 - e^2)(\dot{\theta} - \omega_\mu) + e \dot{r} \sin \hat{v} \right] . \tag{B1}$$

The first step in simplifying $R_L$ is to calculate the expression inside the bracket of equation (B1). From equation (A5), the angular frequency of the medium $\omega_\mu = r^{-3/2}$, and the angular frequency of the outer planet $\dot{\theta} = Gr^{-2}$, are approximately equal to

$$\omega_\mu \simeq a^{-3/2} \left( 1 + \frac{3}{2} e \cos l \right) , \tag{B2}$$

and

$$\dot{\theta} \simeq a^{-3/2} \left( 1 + 2e \cos l \right) , \tag{B3}$$

where $G$ has been replaced by its equivalent expression from equation (27). Also from equation (A3) and to the first order in eccentricity, the radial momentum $\dot{r} = e \sin \hat{v}/G$ can be written as

$$\dot{r} \simeq e a^{-1/2} \sin l . \tag{B4}$$

Substituting these values in the expression inside the bracket of equation (B1) and neglecting the terms of the second order and higher in eccentricity,

$$\left[ a(1 - e^2)(\dot{\theta} - \omega_\mu) + e \dot{r} \sin \hat{v} \right] \simeq \frac{1}{2} e a^{-1/2} \cos l . \tag{B5}$$

It is now necessary to simplify the logarithmic term in equation (B1). Numerical computations indicate that, when the system is at resonance (i.e. $10^{-4} \leq \varepsilon \leq 10^{-2}$), $Br^2 W^4$
is very small in comparison to 1 (Figure B1). Therefore one can use the approximation \( \ln(1 + \delta) \simeq \delta \) for \( \delta \ll 1 \) to write this term as \( \ln(1 + Br^2W^4) \simeq Br^2W^4 \).

![Figure B1](image.png)

**Figure B1.** The graph of \( Br^2W^4 \) against time while the system is captured in resonance. \( a, e, \theta, \dot{v} \) and \( \varepsilon \) are respectively equal to 1.838046, 0.0556, 45°, 15° and 0.001. This figure shows that at resonance, \( Br^2W^4 \) is almost constant with an order of magnitude at most equal to \( 10^{-2} \). The timescale is \( (10^4 T_I/2\pi) \) years.

Replacing \( W^2 \) from equation (14) and using equations (A5) and (B2) to (B4), \( W \) can be written as

\[
W \simeq a^{-1/2} e \left( 1 - \frac{3}{4} \cos^2 l \right)^{1/2}.
\]  

(B6)

Finally from equations (B5) and (B6), \( R_L \) can be written as in equation (66) where in the lowest order, eccentricity appears as \( e^2 \).