Long-term memory contribution as applied to the motion of
discrete dynamical systems

A.A. Stanislavsky

Institute of Radio Astronomy, Ukrainian National Academy of Sciences
4 Chervonopraporna St., Kharkov 61002, Ukraine

Abstract

We consider the evolution of logistic maps under long-term memory. The memory effects are characterized by one parameter $\alpha$. If it equals to zero, any memory is absent. This leads to the ordinary discrete dynamical systems. For $\alpha = 1$ the memory becomes full, and each subsequent state of the corresponding discrete system accumulates all past states with the same weight just as the ordinary integral of first order does in the continuous space. The case with $0 < \alpha < 1$ has the long-term memory effects. The characteristic features are also observed for the fractional integral depending on time, and the parameter $\alpha$ is equivalent to the order index of fractional integral. We study the evolution of the bifurcation diagram among $\alpha = 0$ and $\alpha = 0.15$. The main result of this work is that the long-term memory effects make difficulties for developing the chaos motion in such logistic maps. The parameter $\alpha$ resembles a governing parameter for the bifurcation diagram. For $\alpha > 0.15$ the memory effects win over chaos.

PACS numbers: 05.40.-a, 05.45.-a, 05.60.-k, 82.40.Bj
The treatment of nonlinear dynamics in terms of discrete maps (difference equations produced by numerical methods) is a very important step in studying the qualitative behaviour of continuous systems described by differential equations. The logistic map represents one of the most important examples of an one-dimensional discrete nonlinear map with the bifurcation scenario well known. The complicated behaviour exhibited by the logistic map is typical for a whole class of dynamical systems. Fractional calculus occupies an appreciable place in order to describe various kinds of wave propagation in complex media, fractional kinetics of Hamiltonian systems, anomalous diffusion and relaxation, etc. The fractional operator is a natural generalization of the ordinary differentiation and integration. When the operator depends on time, it is characterized by long-term memory effects. The effects correspond to intrinsic dissipative processes in the physical systems. In the application to discrete maps this means that their present state evolution depends on all past states with a power weight. Appearance of long-term memory effects in the logistic map makes the coupling among states stronger. This feature is plainly directed against the development of the chaotic dynamics.

I. INTRODUCTION

The evolution of discrete dynamical systems is described by the variables measured in discrete time steps. The behavior of such systems is governed by return maps. They connect the \((n+1)\)th value of variables with the preceding \(n\)th value of the variables through a func-
tional dependence. A discrete dynamical system can demonstrate a complex development even in simple systems with one variable [1]. If the system falls into a series of bifurcations, it exhibits a transition from a variety of periodic cycles to chaos.

A statistical treatment of the macroscopic equation of motion leads to memory effects in dynamical Onsager coefficients, if Markovian-like approximations are not admissible [2]. The application of memory effects to the discrete systems signifies that their dynamics at time \( t + 1 \) will depend only not on time \( t \), but on former times \( t - 1, t - 2, t - 3 \) and so on. Here time is equivalent to the number of step. Therefore, \( n + 1 \) can be substituted for \( t + 1 \) as well as \( n \) can be \( t \), and \( t - 1, t - 2, \ldots \) can be replaced by \( n - 1, n - 2, \ldots \) respectively. The introduction of a retardation of the linear term in the logistic equation modifies the Feigenbaum scenario [3]. In this case the periodic orbits are shifted. Several forms of nonlinear maps with memory were also considered by a formal way in [4].

Last time the interest to the study of long-term memory effects and fractional kinetics in physical systems increased too much [5–8]. The fractional operator with respect to time is characterized by long-term memory effects, whereas one with respect to coordinates possesses non-local (long-range) effects [9, 10]. The direct relationship among the long-term memory effects, fractional calculus and the stable distributions from the theory of probability has been established in [11, 12]. Then it was shown [13] that perturbed by a periodic force, the nonlinear oscillator with fractional derivative exhibits a chaotic motion called the fractional chaotic attractor. Although the fractional nonlinear oscillator behaves like the stochastic attractor in phase space, being periodically perturbed, the role of the polynomial dissipation leads to a degradation of the attractor structure. The aim of this work is to present the analysis of the influence of long-term memory effects on the behavior of discrete systems.

The paper starts with a discrete conception of long-term memory effects (Section II).
They are characterized by the parameter similar to the order index of fractional integral. By means of the computer simulation of bifurcation diagrams for the quadratic and triangular maps (Section III) with the memory contribution, we demonstrate how the memory effects strangle the chaotic motion of discrete dynamical systems.

II. MATHEMATICAL DESCRIPTION OF LONG-TERM MEMORY

The mapping \( x_{n+1} = f(x_n) \) does not have any memory, as the value \( x_{n+1} \) only depends on \( x_n \). The introduction of memory means that the discrete value \( x_{n+1} \) is connected with the previous values \( x_n, x_{n-1}, x_{n-2}, \ldots, x_1 \). Particularly, any discrete system will have a full memory, if each state of the system is a simple sum of all previous states:

\[
x_{n+1} = \sum_{i=1}^{n} f(x_i),
\]

where \( f(x) \) is an arbitrary function suitable for the discrete map. However, the above expression may tend to infinity because of the sum accumulating all the values. In order to have a fixed point in this mapping, the expression should be normalized as

\[
x_{n+1} = \frac{1}{n} \sum_{i=1}^{n} f(x_i). \quad (1)
\]

When \( p \) is a fixed point, the map (1) gives

\[
x_{n+1} = \frac{1}{n} \sum_{i=1}^{n} p = p.
\]

The starting conjection is that the map under long-term memory is expressed in terms of

\[
x_{n+1} = \frac{1}{n^\alpha} \sum_{i=1}^{n} b_i f(x_i),
\]

where the weights \( b_i \) and the parameter \( \alpha \) characterize the non-ideal memory effects.
Let the weights take the form

\[
c_i^{(n)} = \begin{cases} 
(1 + \alpha)n^\alpha - n^{\alpha+1} + (n - 1)^{\alpha+1}, & \text{if } i = 0; \\
1, & \text{if } i = n; \\
(n - i + 1)^{\alpha+1} - 2(n - i)^{\alpha+1} + (n - i - 1)^{\alpha+1}, & \text{if } 0 < i < n - 1.
\end{cases}
\]

In this connection it should be pointed out that the similar representation is used for the robust algorithm of numerical fractional integration [14]. It has found an interesting application for the analysis of the nonlinear reaction with fractional dynamics [15] as well as for the study of fractional oscillations [16]. Then the normalized mapping is written as

\[
x_{n+1} = \frac{\sum_{i=0}^{n} c_i^{(n)} f(x_i)}{\sum_{i=0}^{n} c_i^{(n)}}.
\]

Here the sum starts with \(i = 0\) (though it could be realized from \(i = 1\)) by reason of the simplicity in writing the weights \(c_i^{(n)}\). To calculate the normalization factor presents no difficulty, namely

\[
\sum_{i=0}^{n} c_i^{(n)} = (1 + \alpha)n^\alpha.
\]

It turns out that the factor is a power function just as one in our foregoing conjection.

Now consider the following limit cases. If \(\alpha = 0\), the memory is absent. Really, in this case all the weights are equal to zero with the exception of \(c_n^{(n)} = 1\). Thus, we come to an ordinary map \(x_{n+1} = f(x_n)\). For \(\alpha = 1\) the memory effects will be full, namely

\[
c_i^{(n)} = \begin{cases} 
1, & \text{if } i = 0; \\
1, & \text{if } i = n; \\
2, & \text{if } 0 < i < n - 1.
\end{cases}
\]

Then this map becomes \(x_{n+1} = \frac{1}{n}[f(x_0) + f(x_n)]/2 + \sum_{i=1}^{n-1} f(x_i)]\). The full memory is ideal because each new state of the discrete system sticks to the system’s memory and has the same action upon next states as all the others in memory. When \(0 < \alpha < 1\), the memory
effects have a long-term dependence so that
\[ x_{n+1} = \frac{1}{(1 + \alpha)n^\alpha} \sum_{i=0}^{n} c_i^{(n)} f(x_i). \] (3)

Certainly the memory is not ideal, as the contribution of more early states is noticeably less than the contribution of following ones on the present state of such systems. This is something like a part of system states being lost. Recall that if a system does not remember any previous state except for the present, it has no memory effects.

In the next section we will show the results of numerical simulations for the triangular and quadratic maps under the long-term memory effects. This allows us to estimate their influence on the evolution of the discrete systems. The memory effects are governed by means of the parameter \( \alpha \), whereas the parameter of type \( r \) serves as an order parameter for the onset of chaos.

III. NUMERICAL RESULTS

The triangular map is expressed in terms of the function
\[ \Delta(x) = r\left(1 - 2\mid \frac{1}{2} - x \mid \right), \]
and the quadratic map is written as
\[ x_{n+1} = qx_n(1 - x_n), \]
where \( r, q \) are constants. These logistic maps are very popular in the theory of deterministic chaos. They help easily to understand main features of discrete dynamical systems. In particular, the quadratic map is the simplest nonlinear difference equation, appears in many contexts, for example, a strongly damped kicked rotator or the growth model of a population in a closed area. For the triangular map with \( r < 1/2 \) the origin \( x = 0 \) is the only stable
The bifurcation diagrams for the triangular logistic map with long-term memory effects.

The parameter $\alpha$ accounts for the memory contribution, the constant $r$ relates to the onset of chaos for this map (all details in the text).

fixed point to which all points in the interval $[0 \ 1]$ are attracted. The value $r$ plays a role of “order parameter”, which indicates the onset of chaos. For $r > 1/2$ two unstable fixed points emerge. If the magnitude of the parameter $r$ increases still more, then the information about the position of a point in $[0 \ 1]$ is lost. This may result in chaos. The function $\Delta(x)$ is a simple model which for $r > 1/2$ generates chaotic sequences $x_0, \Delta(x_0), \Delta[\Delta(x_0)], \ldots$, and due to its simple form, all quantities in this chaotic state can be calculated explicitly. We go the same way for the maps taking into account the long-term memory effects.

The long-term memory distinguishes itself by a slow decay contribution of former values for discrete maps. The procedure of calculating the string of system states requires to evaluate in the coefficients $c_i^{(n)}$. Therefore, the duration of calculations notably increases
FIG. 2: The bifurcation diagrams for the quadratic logistic map with long-term memory effects. The parameter $\alpha$ characterizes the memory influence, the value $q$ is the constant of the map (see details in the text).

with the growth of $n$. To remember “the simple dynamical systems do not necessarily lead to simple dynamical behavior” [17], we guess that a weak memory effects with $\alpha \to 0$ almost will not influence on the complicated behavior of such discrete systems. By means of numerical simulation we establish the evolution of the simple logistic maps under long-term memory, when the parameter $\alpha$ tends to one. For this purpose the bifurcation diagram is constructed. Figure 1 shows a set of bifurcation diagrams to the triangular map with various values $\alpha$. A number of diagrams, represented in Fig. 2 corresponds to the quadratic case. The number of iterations $n$ equals to 3500. In these figures it is seen that the memory effects step-by-step in the growth of $\alpha$ block up the development of chaotic behavior in the logistic maps. After $\alpha > 0.155$ the evolution of the given maps tends to a regular stable fixed point.
What can we say about a sequence of iterates if there are unstable fixed points? The \( n \)th iterate \( \Delta^n(x) \) is piecewise linear and has the slope \( \frac{d}{dx} \Delta^n(x) = 2^n \), except for the countable set of points \( j \cdot 2^{-n} \) with \( j = 0, 1, \ldots, 2^n \). The Liapunov exponent measures the average loss of information about the position of a point in \([0, 1]\) after one iteration. For the general triangular map under long-term memory effects the Liapunov exponent becomes

\[
\lambda(x) = \lim_{n \to \infty} \frac{1}{n} \ln \left| \sum_{i=0}^{n} c_i^{(n)} \frac{d}{dx} \Delta^i(x) \right| \leq \lim_{n \to \infty} \frac{1}{n} \ln \left\{ \sum_{i=0}^{n} c_i^{(n)} \left| \frac{d}{dx} \Delta^i(x) \right| \right\}
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \ln \left\{ \sum_{i=0}^{n} c_i^{(n)} (2r)^i \right\} = \lim_{n \to \infty} \frac{1}{n} \ln \left\{ (2r)^n \sum_{i=0}^{n} c_i^{(n)} (2r)^{i-n} \right\}
\]

\[
= \ln(2r).
\]

The equality \( \lambda(x) = \ln(2r) \) holds true for \( \alpha = 0 \), when any memory effects are missing. If \( 0 < \alpha < 1 \), then the Liapunov exponent decreases. Therefore, the chaotic states typical for the ordinary triangular map die out with the growth of memory effects, i.e. when the parameter \( \alpha \) tends to one.

IV. CONCLUSIONS

We have shown that if a discrete dynamical system is exposed to the long-term memory effects, then the chaotic sequences which are presented in the system without any memory are squeezed out. As the memory mounts, chaos and its traces disappear. The magnitude \( \alpha_{\text{crit}} \approx 0.155 \) looks like a “critical point” that indicates a transition from chaos to order. It should be noticed that the fine structure in the iterates for the maps is just washed out because of the memory effects. Nevertheless, for sufficiently small values of \( \alpha \) there is still a remarkable transition to chaos.
Acknowledgements

The author acknowledges G.M. Zaslavsky for fruitful discussions, M. Edelman for his help as well as the referees for their useful remarks.

[1] H. G. Schuster, Deterministic Chaos: An Introduction, Physik-Verlag, Wrihelm, 1984.
[2] H. Risken, The Fokker-Planck Equation, Berlin-Heidelberg, Springer-Verlag, 1989.
[3] E. Fick, M. Fick, and G. Hausmann, Phys. Rev. A44 (1991) 2469.
[4] A. Fulinski, A. S. Kleszkowski, Phys. Scripta 35 (1987) 119.
[5] R. Metzler, J. Klafter, J. Phys. A: Math. Gen. 37 (2004) R161.
[6] A. Piryatinska, A. I. Saichev, W. A. Woyczynski, Physica A349 (2005) 375.
[7] G. M. Zaslavsky, Hamiltonian Chaos and Fractional Dynamics, Oxford University Press, 2005.
[8] G. M. Zaslavsky, Phys. Rep. 371 (2002) 461.
[9] V. E. Tarasov, G. M. Zaslavsky, Chaos 16 (2006) 023110.
[10] V. E. Laskin, G. M. Zaslavsky, Physica A 368 (2006) 38.
[11] A. A. Stanislavsky, Theor. and Math. Phys. 138 (2004) 418.
[12] M. M. Meerschaert, H.-P. Scheffler, J. Appl. Probab. 41 (2004) 623.
[13] G. M. Zaslavsky, A. A. Stanislavsky and M. Edelman, Chaos 16 (2006) 013102.
[14] K. Diethelm, N. J. Ford, A. D. Freed, Yu. Luchko, Comp. Methods Appl. Mech. Eng. 194 (2005) 743.
[15] A. A. Stanislavsky, Appl. Math. and Comp. 174 (2006) 1122.
[16] A. A. Stanislavsky, Physica A 354 (2005) 101.
[17] R. M. May, Nature 261 (1976) 459.