BOUNDS ON VOLUME GROWTH OF GEODESIC BALLS FOR EINSTEIN WARPED PRODUCTS

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Abstract. The purpose of this note is to provide some volume estimates for Einstein warped products similar to a classical result due to Calabi and Yau for complete Riemannian manifolds with nonnegative Ricci curvature. To do so, we make use of the approach of quasi-Einstein manifolds which is directly related to Einstein warped products. In particular, we present an obstruction for the existence of such a class of manifolds.

1. Introduction

It has long been a goal of mathematicians to understand the geometry of Einstein manifolds as well as Einstein-type manifolds, for instance, Ricci solitons and \( m \)-quasi-Einstein manifolds. Surely, this is a fruitful problem in Riemannian Geometry. Ricci solitons model the formation of singularities in the Ricci flow and they correspond to self-similar solutions for this flow; for more details on this subject we recommend the survey by Cao [7]. On the other hand, one of the motivations to study \( m \)-quasi-Einstein metrics on a Riemannian manifold is its direct relation to Einstein warped products. For comprehensive references on such a theory, see [2], [3], [5], [9], [10], [11] and [20].

One fundamental ingredient to understand the behavior of Einstein warped products is the \( m \)-Bakry-Emery Ricci tensor which appeared previously in [1] and [17] as a modification of the classical Bakry-Emery tensor \( \text{Ric}_f = \text{Ric} + \nabla^2 f \). More exactly, the \( m \)-Bakry-Emery Ricci tensor is given by

\[
\text{Ric}_f^m = \text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df,
\]

where \( f \) is a smooth function on \( M^n \) and \( \nabla^2 f \) stands for the Hessian form.

A Riemannian manifold \( (M^n, g), n \geq 2 \), will be called \( m \)-quasi-Einstein manifold, or simply quasi-Einstein manifold, if there exist a smooth potential function \( f \) on \( M^n \) and a constant \( \lambda \) satisfying the following fundamental equation:

\[
\text{Ric}_f^m = \text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g.
\]
When $m$ goes to infinity, equation (1.2) reduces to the one associated with a Ricci soliton. Furthermore, when $m$ is a positive integer it corresponds to a warped product Einstein metric; for more details see, for instance, [5]. Following the terminology of Ricci solitons, an $m$-quasi-Einstein metric $g$ on a manifold $M^n$ will be called expanding, steady or shrinking, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Moreover, an $m$-quasi-Einstein manifold will be called trivial if $f$ is constant, otherwise it will be nontrivial. We notice that the triviality implies that $M^n$ is an Einstein manifold.

In order to proceed we recall a classical result which gives a characterization for Einstein warped products.

**Theorem 1** ([3], [11]). Let $M^n \times_u F^m$ be an Einstein warped product with Einstein constant $\lambda$, warping function $u = e^{-\frac{f}{m}}$ and Einstein fiber $F^m$. Then the weighted manifold $(M^n, g_M, e^{-\frac{f}{m}}dM)$ satisfies the $m$-quasi-Einstein equation (1.2). Furthermore the Einstein constant $\mu$ of the fiber satisfies

\[ \Delta f - |\nabla f|^2 = m\lambda - m\mu e^{\frac{2f}{m}}. \]

Conversely, if the weighted manifold $(M^n, g_M, e^{-\frac{f}{m}}dM)$ satisfies (1.2), then $f$ satisfies (1.3) for some constant $\mu \in \mathbb{R}$. Considering the warped product $N^{n+m} = M^n \times_u F^m$, with $u = e^{-\frac{f}{m}}$, and Einstein fiber $F$ with $\text{Ric}_F = \mu g_F$, then $N$ is also Einstein with $\text{Ric}_N = \lambda g_N$, where $g_N = g_M + u^2 g_F$.

Clearly, Theorem 1 shows that an $m$-quasi-Einstein structure provides a structure of Einstein warped product; for more details we recommend [11]. Therefore, classifying $m$-quasi-Einstein manifolds or understanding their geometry is definitely an important issue.

One should point out that some examples of expanding $m$-quasi-Einstein manifolds with arbitrary $\mu$ as well as steady quasi-Einstein manifolds with $\mu > 0$ were constructed in [3]. While Case [6] showed that steady $m$-quasi-Einstein manifolds with $\mu \leq 0$ are trivial. In [17], Qian proved that shrinking $m$-quasi-Einstein manifolds must be compact. Moreover, as already noticed in [11] the converse result is also true. From that it follows that an $m$-quasi-Einstein manifold is compact if and only if $\lambda > 0$. See also [20] for further discussion.

Based on the above results and inspired by works by Calabi and Yau, we shall investigate bounds on volume growth of geodesic balls for noncompact $m$-quasi-Einstein manifolds, in particular $\lambda$ must be nonpositive. For our purposes we recall that Calabi [4] and Yau [21] proved that every metric with nonnegative Ricci tensor on a noncompact smooth manifold satisfies

\[ \text{Vol}(B_p(r)) \geq cr \]

for any $r > r_0$ where $r_0$ is a positive constant and $B_p(r)$ is the geodesic ball of radius $r$ centered at $p \in M^n$, and $c$ is a constant that does not depend on $r$. In a similar way Munteanu and Sesum [14] obtained the same type of growth for a steady gradient Ricci soliton.

Now we state our first result concerning the growth of volume of geodesic balls for noncompact $m$-quasi-Einstein manifolds which is similar to the Calabi-Yau estimate.
Theorem 2. Let \((M^n, g, f)\) be a noncompact steady \(m\)-quasi-Einstein manifold with \(m \in (1, \infty]\). Then there exist constants \(c\) and \(r_0 > 0\) such that for any \(r > r_0\)

\[
\text{Vol}(B_p(r)) \geq cr.
\]

This immediately yields the following corollary.

Corollary 1. Let \(N = M^n \times_u F^m\) be a Ricci flat warped product. Then there exist constants \(c\) and \(r_0 > 0\) such that for any \(r > r_0\)

\[
\text{Vol}(B_p(r)) \geq cr,
\]

for geodesic balls of the base.

One question that naturally arises from the above results is to know what occurs on expanding \(m\)-quasi-Einstein manifolds. In this case, we obtain the following volume growth of geodesic balls.

Theorem 3. Let \((M^n, g, f, \lambda)\) be a noncompact expanding \(m\)-quasi-Einstein manifold with \(m \in (1, \infty)\) and \(\mu \leq 0\). Supposing that \(f \geq -k\) for some positive constant \(k\), then there exist constants \(c\) and \(r_0 > 0\) such that for any \(r > r_0\)

\[
\text{Vol}(B_p(r)) \geq cr.
\]

In the sequel, inspired by ideas developed in [8], [12] and [13] we shall prove an \(f\)-volume estimate of geodesic balls on expanding \(m\)-quasi-Einstein manifolds. More precisely, we have the following result.

Theorem 4. Let \((M^n, g, f, \lambda)\) be a noncompact expanding \(m\)-quasi-Einstein manifold with \(m \in [1, \infty)\) and \(\mu = 0\). Then there exists a constant \(c\) such that for any \(r > 1\)

\[
\text{Vol}_f(B_p(r)) \geq ce^{\sqrt{-\lambda mr}}.
\]

It should be emphasized that there are several further interesting obstructions to the existence of Einstein metrics. Based on this, in [3] (cf. page 265) the following question was posed:

“Does there exist a compact Einstein warped product with nonconstant warping function?”

In 2003 Kim and Kim [11] gave a partial answer for this question by means of the quasi-Einstein approach, while Case [6] studied this problem without compactness assumption for \(m\)-quasi-Einstein manifolds with \(\lambda = 0\) and \(\mu \leq 0\). Here, we use the weak Maximum Principle at infinity for the \(f\)-Laplacian to obtain a triviality result for Einstein warped products with negative scalar curvature. More precisely, we have the following result.

Theorem 5. Let \(N = M^n \times_u F^m\) be a complete Einstein warped product with Einstein constant \(\lambda < 0\), warping function \(u\) and Einstein constant of the fiber \(F\) satisfying \(\mu < 0\). If the warping function satisfies

\[
u \leq \sqrt{\frac{2\mu}{\lambda}},
\]

then \(u\) is a constant function and \(N\) is a Riemannian product.
2. Proof of the results

In order to set the stage for the proofs to follow, let us recall some classic equations. First, considering the function \( u = e^{-\frac{f}{m}} \) on \( M^n \), we immediately have \( \nabla u = -\frac{u}{m} \nabla f \) as well as

\[
\nabla^2 f - \frac{1}{m} df \otimes df = -\frac{m}{u} \nabla^2 u.
\]

Next, a straightforward computation involving (1.2) and (1.3) gives

\[
\frac{u^2}{m} (R - \lambda n) + (m - 1)|\nabla u|^2 = -\lambda u^2 + \mu.
\]

We also recall that Wang [19] proved that if \( \lambda \leq 0 \), then \( R \geq \lambda n \), from which it follows that

\[
(m - 1)|\nabla u|^2 \leq -\lambda u^2 + \mu.
\]

Now we are ready to prove the results.

2.1. Proof of Theorem 2.

Proof. To begin with, we notice that when \( m = \infty \) we have a gradient Ricci soliton and in this case the result follows from [14]. From now on we can assume that \( m \in (1, \infty) \). From this, since \( \lambda = 0 \) we get

\[
|\nabla u|^2 \leq \frac{\mu}{m - 1}.
\]

Taking into account that \( R \geq 0 \) and \( u > 0 \), we deduce

\[
\int_{B_p(r)} uR d\sigma \geq 0
\]

for each \( r > 0 \), where \( d\sigma \) denotes the Riemannian volume form. Consequently, if for all \( r > 0 \) we have

\[
\int_{B_p(r)} uR d\sigma = 0,
\]

then \( R = 0 \) on \( M^n \).

On the other hand, Wang [19] (see also [5]) proved that every \( m \)-quasi-Einstein manifold satisfies

\[
\frac{1}{2} \Delta R - \frac{m + 2}{2m} (\nabla f, \nabla R) = -\frac{m - 1}{m} |Ric - \frac{R}{n} g|^2 - \frac{n + m - 1}{mn} (R - n\lambda)(R - \frac{n(n - 1)}{n + m - 1} \lambda).
\]

In particular, in the steady case we have \( Ric = 0 \), provided that \( R = 0 \), whence relation (1.5) remounts to Calabi [4] and Yau [21].

It is well-known that such a metric is real analytic (cf. Proposition 2.8 in [10]), hence the zeroes of the scalar curvature \( R \) are isolated. Therefore, if \( R \geq 0 \), but \( R \neq 0 \), we choose \( p \in M^n \) such that \( R(p) > 0 \) and a ball \( B_p(r_0) \) with radius \( r_0 > 0 \) such that

\[
\int_{B_p(r_0)} uR d\sigma = mC_0
\]
is a positive constant. Then we use the trace of (2.1) to conclude that for all \( r \geq r_0 \)

\[
mC_0 = \int_{B_p(r_0)} uRd\sigma \leq \int_{B_p(r)} uRd\sigma = m \int_{B_p(r)} \Delta u d\sigma.
\]

Next we invoke Stokes formula and (2.4) to deduce

\[
mC_0 \leq m \int_{\partial B_p(r)} \frac{\partial u}{\partial \eta} ds \leq m \int_{\partial B_p(r)} |\nabla u| ds
\]

\[
\leq m \sqrt{\frac{\mu}{m-1} \cdot \text{Area}(\partial B_p(r))}.
\]

This implies that for \( r \geq r_0 \) we have

\[
(2.6) \quad \text{Area}(\partial B_p(r)) \geq c > 0
\]

for a uniform constant \( c \).

Finally, on integrating (2.6) from \( r_0 \) to \( r \) we arrive at

\[
\text{Vol}(B_p(r)) \geq c(r - r_0) \geq c_0 \cdot r
\]

for all \( r \geq 2r_0 \). This finishes the proof of the theorem. \( \square \)

2.2. Proof of Theorem 3

Proof. Firstly we notice that using the hypotheses on \( f \) and \( \mu \) in (1.8) we deduce

\[
(2.7) \quad |\nabla u|^2 \leq -\frac{\lambda e^{2k/m}}{m-1}.
\]

Next, taking into account that \( R \geq \lambda n \) and \( u > 0 \) we infer

\[
\int_{B_p(r)} u(R - \lambda n) d\sigma \geq 0
\]

for each \( r > 0 \). Moreover, if for all \( r > 0 \) we have

\[
\int_{B_p(r)} u(R - \lambda n) d\sigma = 0,
\]

then \( R = \lambda n \) on \( M^n \). Hence from (2.5) we deduce that \( M^n \) is Einstein, but this gives a contradiction. In fact, according to [6] an Einstein manifold with nontrivial expanding quasi-Einstein structure and potential function bounded from below does not exist.

From now on the proof looks like the one from the previous theorem. In particular, there is \( p \in M^n \) such that \( R(p) > \lambda n \). Since \( u \) is positive there exists \( r_0 > 0 \) such that

\[
\int_{B_p(r_0)} u(R - \lambda n) d\sigma = mC_0
\]

is a positive constant. Moreover, from the analyticity of \( R \) it follows that for all \( r \geq r_0 \)

\[
mC_0 \leq \int_{B_p(r)} u(R - \lambda n) d\sigma = m \int_{B_p(r)} \Delta u d\sigma
\]

\[
= m \int_{\partial B_p(r)} \frac{\partial u}{\partial \eta} ds \leq m \int_{\partial B_p(r)} |\nabla u| ds
\]

\[
\leq m \sqrt{\frac{-\lambda e^{2k/m}}{m-1} \cdot \text{Area}(\partial B_p(r))},
\]

(2.8)
where we have used Stokes formula and (2.7). Therefore, (2.8) allows us to deduce that for \( r \geq r_0 \)
\[
\text{Area}(\partial B_p(r)) \geq c > 0
\]
for an uniform constant \( c \).

In order to conclude it suffices to integrate (2.9) from \( r_0 \) to \( r \) to arrive at
\[
\text{Vol}(B_p(r)) \geq c(r - r_0) \geq c_0 \cdot r
\]
for all \( r \geq 2r_0 \), which gives the requested result. \( \square \)

2.3. Proof of Theorem 4.

Proof. First, since \( \mu = 0 \) we use (1.3) to infer
\[
\Delta e^{-f} = (-\Delta f + |\nabla f|^2)e^{-f} = -\lambda me^{-f}. 
\]
Now, upon integrating of (2.10) over \( B_p(r) \) we deduce
\[
-\lambda m \int_{B_p(r)} e^{-f} d\sigma = \int_{B_p(r)} \Delta e^{-f} d\sigma = \int_{\partial B_p(r)} \frac{\partial}{\partial \eta} (e^{-f}) ds. 
\]
On the other hand, from [19] we have \( |\frac{\partial f}{\partial \eta}| \leq |\nabla f| \leq \sqrt{-\lambda m} \). This enables us to use (2.11) to arrive at
\[
-\lambda m \int_{B_p(r)} e^{-f} d\sigma \leq \sqrt{-\lambda m} \int_{\partial B_p(r)} e^{-f} ds. 
\]
Next, denoting
\[
\xi(r) := \text{Vol}_f(B_p(r)) = \int_{B_p(r)} e^{-f} d\sigma,
\]
we can use (2.12) to get
\[
\xi'(r) \geq \sqrt{-\lambda m} \xi(r).
\]
Whence, on integrating this inequality from 1 to \( r \) we conclude that
\[
\xi(r) = \int_{B_p(r)} e^{-f} d\sigma \geq ce^{\sqrt{-\lambda m} r}
\]
for any \( c > 0 \). From here we conclude the proof of the theorem. \( \square \)

2.4. Proof of Theorem 5.

Proof. We start invoking Theorem 1 to deduce that \( (M^n, g, \nabla f, \lambda) \) is an expanding \( m \)-quasi-Einstein manifold with potential function \( f = -m \ln u \) and \( \mu < 0 \).

Next, we combine \( f \)-volume estimates obtained by Qian [17] and Theorem 9 in [16] to conclude that the weak Maximum Principle at infinity is valid for the \( f \)-Laplacian on \( (M^n, g, \nabla f, \lambda) \). We also highlight that the potential function \( f \) satisfies \( |\nabla f|^2 \leq -\lambda m \) (cf. [19], see also [18]). From this setting, we apply the weak Maximum Principle at infinity for \( |\nabla f|^2 \) to conclude that there exists a sequence \( \{p_k\} \subset M^n \), such that
\[
|\nabla f|^2(p_k) \geq |\nabla f|^2 - \frac{1}{k} \quad \text{and} \quad \Delta f |\nabla f|^2(p_k) \leq \frac{1}{k},
\]
where \( |\nabla f|^2 = \sup_M |\nabla f|^2 \).
We now recall the weighted Bochner formula:

\[(2.13) \quad \frac{1}{2} \Delta f |\nabla f|^2 = |\text{Hess} f|^2 + \text{Ric}(\nabla f, \nabla f) + \text{Hess} f(\nabla f, \nabla f) + \langle \nabla f, \nabla \Delta f \rangle.\]

Therefore, (1.2) and (1.3) substituted in (2.13) gives

\[(2.14) \quad \frac{1}{2} \Delta f |\nabla f|^2 \geq \lambda |\nabla f|^2 + \frac{1}{m} |\nabla f|^4 - 2\mu e^{2f/m} |\nabla f|^2\]

Now, since \(u \leq \sqrt{\frac{2\mu}{\lambda}}\) we immediately deduce

\[(2.15) \quad \frac{1}{2} \Delta f |\nabla f|^2 \geq \frac{1}{m} |\nabla f|^4.\]

From this, over \(\{p_k\} \) we get

\[\frac{1}{2k} \geq \frac{1}{m} \left( |\nabla f|^2 - \frac{1}{k} \right)^2.\]

So, when \(k\) goes to infinity we conclude that \( |\nabla f|^2 = 0 \) and this forces \( f \) to be constant which concludes the proof of the theorem. \( \square \)

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