CHARACTERIZATION OF MIXED $n$-JORDAN HOMOMORPHISMS AND PSEUDO $n$-JORDAN HOMOMORPHISMS

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Abstract. In this article, a new notion of $n$-Jordan homomorphism namely the mixed $n$-Jordan homomorphism is introduced. It is proved that how a mixed $(n + 1)$-Jordan homomorphism can be a mixed $n$-Jordan homomorphism and vice versa. By means of some examples, it is shown that the mixed $n$-Jordan homomorphisms are different from the $n$-Jordan homomorphisms and the pseudo $n$-Jordan homomorphisms. As a consequence, it shown that every mixed Jordan homomorphism from Banach algebra $A$ into commutative semisimple Banach algebra $B$ is automatically continuous. Under some mild conditions, every unital pseudo 3-Jordan homomorphism can be a homomorphism.

1. Introduction and Preliminaries

Let $A$ and $B$ be complex Banach algebras and $\varphi : A \to B$ be a linear map. Then, $\varphi$ is called an $n$-homomorphism if for all $a_1, a_2, \cdots, a_n \in A$,
$$\varphi(a_1a_2\cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots \varphi(a_n).$$

The concept of an $n$–homomorphism was studied for complex algebras in [5] and [9]. A linear map $\varphi$ between Banach algebras $A$ and $B$ is called an $n$-Jordan homomorphism if
$$\varphi(a^n) = \varphi(a)^n, \quad a \in A.$$ 

This notion was introduced by Herstein in [10]. A 2-homomorphism (2-Jordan homomorphism) is called simply a homomorphism (Jordan homomorphism). It is clear that every $n$-homomorphism is an $n$-Jordan homomorphism, but in general the converse is false. Indeed, it was Ancochea [2] who firstly studied the connection of Jordan homomorphisms and homomorphisms. The results of Ancochea were generalized and extended in several ways in [11] and [12]. There are plenty of known examples of $n$-Jordan homomorphism which are not $n$-homomorphism. For $n = 2$, it is proved in [11] that some Jordan homomorphism on the polynomial rings can not be homomorphism. Also, each homomorphism is an $n$-homomorphism for every $n \geq 2$, but the converse does not hold in general. For instance, if $h : A \to B$ is a homomorphism, then $g := -h$ is a 3-homomorphism which is not a homomorphism [5]. Herstein [10] proved the following result.

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Theorem 1.1. If \( \varphi \) is a Jordan homomorphism of a ring \( R \) onto a prime ring \( R' \) of characteristic different from 2 and 3, then either \( \varphi \) is a homomorphism or an anti-homomorphism.

After that, Zelazko in [16] presented the upcoming result (see also [15]).

Theorem 1.2. Suppose that \( A \) is a Banach algebra, which need not be commutative, and suppose that \( B \) is a semisimple commutative Banach algebra. Then each Jordan homomorphism \( \varphi : A \to B \) is a homomorphism.

This result has been proved by the third author in [18] for 3-Jordan homomorphism with the extra condition that the Banach algebra \( A \) is unital. In other words, he presented the next theorem.

Theorem 1.3. Suppose that \( A \) is a unital Banach algebra, which need not be commutative, and suppose that \( B \) is a semisimple commutative Banach algebra. Then each 3-Jordan homomorphism \( \varphi : A \to B \) is a 3-homomorphism.

After that, An [1] extended the above theorem for all \( n \in \mathbb{N} \) in [1] and showed that for unital ring \( A \) and ring \( B \) with \( \text{char}(B) > n \), every \( n \)-Jordan homomorphism from \( A \) into \( B \) is an \( n \)-homomorphism (\( n \)-anti-homomorphism) provided that every Jordan homomorphism from \( A \) into \( B \) is a homomorphism (anti-homomorphism). Recently, the second author and İnceboz extended Theorem 1.2 for \( n \in \{3, 4\} \) in [4] without the Banach algebra \( A \) is that of being unital by considering an extra condition on the mapping \( \varphi \) as

\[
\varphi(a^2b) = \varphi(ba^2), \quad a, b \in A.
\]

Some significant results concerning Jordan homomorphisms and their automatic continuity on Banach algebras obtained by the third author in [17], [19] and [20]. For the commutative case, Lee in [13] and Gselmann in [8] every \( n \)-Jordan homomorphism between two commutative Banach algebras is an \( n \)-homomorphism where \( n \) is an arbitrary and fixed positive integer. Later, this problem solved in [3] based on the property of the Vandermonde matrix, which is different from the methods that are used in [8] and [13].

Let \( A \) and \( B \) be rings (algebras), and let \( B \) be a right [left] \( A \)-module. Then a linear map \( \varphi : A \to B \) is said to be pseudo \( n \)-Jordan homomorphism if there exist an element \( w \in A \) such that

\[
\varphi(a^nw) = \varphi(a)^n \cdot w, \quad [\varphi(a^nw) = w \cdot \varphi(a)^n] \quad (a \in A).
\]

The element \( w \) is called Jordan coefficient of \( \varphi \). The concept of pseudo \( n \)-Jordan homomorphism was introduced and studied by Ebadian et al., in [6]. They also investigated the automatic continuity such homomorphisms on commutative \( C^* \)-algebras and semisimple (non unital) Banach algebras.

In section 2, we introduce the notion of mixed \( n \)-Jordan homomorphism on algebras. We prove that every 3-Jordan homomorphism \( \varphi \) from algebra \( A \) into \( \varphi \)-commutative algebra \( B \) such that \( \varphi(ab - ba) = 0 \) is a mixed Jordan homomorphism. Also, we discuss
the automatic continuity of mixed Jordan homomorphisms. We show that under which conditions a mixed \((n+1)\)-Jordan homomorphism is a mixed \(n\)-Jordan homomorphism and vice versa. We prove that every \(n\)-Jordan homomorphism on non-commutative Banach algebras is a \(n\)-homomorphism with different conditions as in [1, Theorem 2.4] and [3, Theorem 2.2]. It is of interest to know whether the converse of [6, Theorem 2.3] holds. In the last section, we answer to this question. In fact, we show that every unital pseudo \((n+1)\)-Jordan homomorphism is pseudo \(n\)-Jordan homomorphism with the same Jordan coefficient.

2. Mixed \(n\)-Jordan homomorphisms

Let \(A\) and \(B\) be complex algebras and \(\varphi : A \rightarrow B\) be a linear map. Then \(\varphi\) is called a mixed \(n\)-Jordan homomorphism if for all \(a, b \in A\),

\[\varphi(a^n b) = \varphi(a)^n \varphi(b).\]

A mixed 2-Jordan homomorphism is said to be mixed Jordan homomorphism. Clearly, every \(n\)-homomorphism is an mixed \((n-1)\)-Jordan homomorphism for \(n \geq 3\), and every mixed \(n\)-Jordan homomorphism is \((n+1)\)-Jordan homomorphism but the converse is not true in general. The following example illustrates this fact.

Example 2.1. Let \(A = \left\{ \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} : X, Y \in M_2(\mathbb{C}) \right\}\), and define \(\varphi : A \rightarrow A\) by

\[
\varphi \left( \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \right) = \begin{bmatrix} X & 0 \\ 0 & Y^T \end{bmatrix},
\]

where \(Y^T\) is the transpose of matrix \(Y\). Then, for all \(U \in A\), we have

\[
\varphi(U^n) = \varphi(U)^n = \begin{bmatrix} X^n & 0 \\ 0 & (Y^n)^T \end{bmatrix}.
\]

Thus, \(\varphi\) is \(n\)-Jordan homomorphism, but \(\varphi\) is not mixed \((n-1)\)-Jordan homomorphism. In fact, for

\[
U = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}, \quad V = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.
\]

We have \((Y^{(n-1)}B)^T \neq (Y^T)^{(n-1)}B^T\). Therefore, \(\varphi(U^{(n-1)}V) \neq \varphi(U)^{(n-1)}\varphi(V)\).

A left ideal \(\mathcal{I}\) of an algebra \(A\) is a modular left ideal if there exists \(u \in A\) such that \(A(e_A - u) \subseteq \mathcal{I}\), where \(A(e_A - u) = \{ x - xu : x \in A \}\). The Jacobson radical \(\text{Rad } (A)\) of \(A\) is the intersection of all maximal modular left ideals of \(A\). An algebra \(A\) is called semisimple whenever its Jacobson radical \(\text{Rad } (A)\) is trivial. Also, every \(C^*\)-algebra is semisimple.

Let \(n\) be an integer \(n \geq 2\). Recall that an associative ring \(R\) is of characteristic not \(n\) if \(na = 0\) implies \(a = 0\) for every \(a \in R\), and \(R\) is of characteristic greater than \(n\) if \(n!a = 0\) implies \(a = 0\) for every \(a \in R\).

Here, we bring some trivial observations are as follows:
• It is known that every mixed $n$-Jordan homomorphism is $(n + 1)$-Jordan homomorphism, and so by [3, Theorem 2.2], every mixed $n$-Jordan homomorphism $\varphi$ between commutative algebras $A$ and $B$ is $(n + 1)$-homomorphism.

• Let $A$ be a unital Banach algebra and $B$ be a Banach algebra with $\text{char}(B) > n$. By [1, Theorem 2.4], every mixed $n$-Jordan homomorphism from $A$ into $B$ is an $(n + 1)$-homomorphism if every Jordan homomorphism from $A$ into $B$ is a homomorphism. So, under such assumptions and that $B$ is a semisimple commutative Banach algebra, every surjective mixed $n$-Jordan homomorphism $\varphi : A \to B$ is automatically continuous by [1, Corollary 2.5].

**Lemma 2.2.** Let $\varphi$ be a mixed Jordan homomorphism between algebras $A$ and $B$. Then

(i) $\varphi$ is mixed $(2n)$-Jordan homomorphism for all $n \in \mathbb{N}$.

(ii) if $A$ is unital, then $\varphi(x)\varphi(e) = \varphi(e)\varphi(x)$, where $e$ is the identity of $A$.

(iii) the mapping $\psi : x \mapsto \varphi(x)\varphi(e)$ is a homomorphism.

**Proof.** Suppose that $\varphi$ is mixed Jordan homomorphism. Then for all $a, b \in A$, 

$$\varphi(a^2b) = \varphi(a)^2\varphi(b).$$  \hfill (1)

Replacing $b$ by $a^2b$ in (1), gives 

$$\varphi(a^4b) = \varphi(a)^2\varphi(a^2b) = \varphi(a)^4\varphi(b).$$

Thus, $\varphi$ is mixed 4-Jordan homomorphism. This argument can be repeated to achieve the desired result of part (i).

For the part (ii), replacing $b$ by $b + e$ in (1), we have 

$$2\varphi(ab) = [\varphi(a)\varphi(e) + \varphi(e)\varphi(a)]\varphi(b).$$  \hfill (2)

for all $a, b \in A$. Switching $b$ by $e$ in (2), we find 

$$2\varphi(a) = \varphi(a)\varphi(e)^2 + \varphi(e)\varphi(a)\varphi(e).$$  \hfill (3)

for all $a \in A$. Multiplying $\varphi(e)$ from the left in (3), and using the equality $\varphi(e)^3 = \varphi(e)$, we get 

$$2\varphi(a)\varphi(e) = \varphi(a)\varphi(e) + \varphi(e)\varphi(a)\varphi(e)^2.$$  \hfill (4)

for all $a \in A$. Similarly, 

$$2\varphi(e)\varphi(a) = \varphi(e)\varphi(a)\varphi(e)^2 + \varphi(e)^2\varphi(a)\varphi(e).$$  \hfill (5)

The relations (4) and (5) necessitate that 

$$2[\varphi(e)\varphi(a) - \varphi(a)\varphi(e)] = \varphi(e)^2\varphi(a)\varphi(e) - \varphi(a)\varphi(e).$$  \hfill (6)

Multiplying $\varphi(e)$ from the right in (6), we arrive at 

$$2\varphi(e)[\varphi(e)\varphi(a) - \varphi(a)\varphi(e)] = \varphi(e)\varphi(a)\varphi(e) - \varphi(e)\varphi(a)\varphi(e) = 0.$$
Thus,
\[ \varphi(e)^2 \varphi(a) = \varphi(e) \varphi(a) \varphi(e). \] (7)
for all \( a \in \mathcal{A} \). Once more, multiplying \( \varphi(e) \) from the right in (7), we get
\[ \varphi(e) \varphi(a) = \varphi(e)^2 \varphi(a) \varphi(e). \] (8)
It now follows from (6) and (8) that \( \varphi(e) \varphi(a) = \varphi(a) \varphi(e) \), for all \( a \in \mathcal{A} \). This completes the proof of (ii).

By the part (ii) and (2) we see that \( \varphi(ab) = \varphi(a) \varphi(b) \varphi(e) \), for all \( a,b \in \mathcal{A} \). □

For certain calculations, we use the notation \([a,b] = ab - ba\) which is called the Lie product of \( a \) and \( b \). Let \( \varphi : \mathcal{A} \rightarrow \mathcal{B} \) be a map between Banach algebras \( \mathcal{A} \) and \( \mathcal{B} \). Then, we say that \( \mathcal{B} \) is \( \varphi \)-commutative if for all \( a,b \in \mathcal{A} \), \([\varphi(a),\varphi(b)] = 0\).

Note that every commutative Banach algebra is \( I \)-commutative, where \( I \) is the identity map.

**Example 2.3.** (i) Consider the Banach algebras \( \mathcal{A} = \left\{ \begin{bmatrix} u & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : u, a, b, c \in \mathbb{C} \right\} \), \( \mathcal{B} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\} \) with the usual sum and product. Define the linear map \( \varphi : \mathcal{A} \rightarrow \mathcal{B} \) by
\[ \varphi \left( \begin{bmatrix} u & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}. \]
Then, \( \mathcal{B} \) is non-commutative Banach algebra, but it is \( \varphi \)-commutative.

(ii) Let \( \mathcal{A} \) as the part (i). Consider \( \varphi : \mathcal{A} \rightarrow \mathbb{C} \) defined via \( \varphi \left( \begin{bmatrix} a & b & c \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix} \right) = u \). Then, \( \varphi(XY) = \varphi(YX) \) and \([\varphi(X),\varphi(Y)] = 0\) for all \( X,Y \in \mathcal{A} \). We see that \( \varphi \) is not 3-Jordan homomorphism and so it is neither mixed Jordan homomorphism nor 3-homomorphism.

**Lemma 2.4.** Let \( \varphi \) be an \( n \)-Jordan homomorphism from unital Banach algebra \( \mathcal{A} \) into \( \varphi \)-commutative Banach algebra \( \mathcal{B} \). Then, for all \( a \in \mathcal{A} \),
\[ \varphi(a) = \varphi(e)^{n-1} \varphi(a) = \varphi(a) \varphi(e)^{n-1}. \]

**Proof.** Let \( \varphi \) be a Jordan homomorphism. Then
\[ \varphi((a + 3)^2 - 2(a + 2)^2 + a^2) = \varphi(a + 3)^2 - 2\varphi(a + 2)^2 + \varphi(a)^2. \] (9)
for all \( a \in \mathcal{A} \). Let \( e \) be the identity of \( \mathcal{A} \). By assumption \( \varphi(e) = \varphi(e)^2 \). So, (9) gives
\[ 2\varphi(a) = \varphi(a)\varphi(e) + \varphi(e)\varphi(a). \] (10)
for all \( a \in \mathcal{A} \). On the other hand,

\[ [\varphi(a), \varphi(e)] = \varphi(a)\varphi(e) - \varphi(e)\varphi(a) = 0, \quad (a \in \mathcal{A}). \quad (11) \]

It follows from (10) and (11) that \( \varphi(a) = \varphi(e)\varphi(a) = \varphi(a)\varphi(e), \) for all \( a \in \mathcal{A} \). Now, assume that \( n = 3 \). Then

\[ \varphi((a + 2)^3 - 2(a + 1)^3 + a^3) = \varphi(a + 2)^3 - 2\varphi(a + 1)^3 + \varphi(a)^3. \quad (12) \]

for all \( a \in \mathcal{A} \). Since \( \varphi(e) = \varphi(e)^3 \), the relation (12) implies that

\[ 3\varphi(a) = \varphi(a)\varphi(e)^2 + \varphi(e)^2\varphi(a) + \varphi(e)\varphi(a)\varphi(e). \quad (13) \]

for all \( a \in \mathcal{A} \). By the \( \varphi \)-commutativity of \( \mathcal{B} \), we have

\[ \varphi(a)\varphi(e) = \varphi(e)\varphi(a), \quad (a \in \mathcal{A}). \quad (14) \]

for all \( a \in \mathcal{A} \). Plugging (13) into (14), we find \( \varphi(a) = \varphi(e)^2\varphi(a) = \varphi(a)\varphi(e)^2 \). Similarly, one can obtain the result for all \( n \geq 4 \). □

**Lemma 2.5.** Let \( \mathcal{A} \) be unital Banach algebra with unit \( e \), and \( \varphi : \mathcal{A} \rightarrow \mathcal{B} \) be a \( n \)-Jordan homomorphism. Then

\[ \varphi(a^2) = \varphi(a)^2\varphi(e)^{n-2} \quad (a \in \mathcal{A}). \]

**Proof.** See the proof of Theorem 2.4 from [1]. □

As mentioned that every \( n \)-Jordan homomorphism between commutative Banach algebras \( \mathcal{A} \) and \( \mathcal{B} \) is an \( n \)-homomorphism [3, Theorem 2.2]. In view of the proof of this theorem, we see the same result holds by the weaker condition on \( \mathcal{B} \), as \( \varphi \)-commutativity. However, in the next result, we show that under some conditions every \( n \)-Jordan homomorphism on non-commutative Banach algebras can be an \( n \)-homomorphism.

**Theorem 2.6.** Every \( n \)-Jordan homomorphism \( \varphi : \mathcal{A} \rightarrow \mathcal{B} \) from unital Banach algebra \( \mathcal{A} \) into \( \varphi \)-commutative Banach algebra \( \mathcal{B} \) satisfying the following condition is a \( n \)-homomorphism.

\[ \varphi(x^2) = 0 \implies \varphi(x) = 0, \quad (x \in \mathcal{A}). \quad (15) \]

**Proof.** Define a mapping \( \psi : \mathcal{A} \rightarrow \mathcal{B} \) through

\[ \psi(a) := \varphi(a)\varphi(e)^{n-2}, \quad (a \in \mathcal{A}). \]

By Lemma 2.5, \( \psi \) is Jordan homomorphism and \( \mathcal{B} \) is \( \psi \)-commutative. Also, Lemma 2.4 necessitates that \( \psi \) satisfies the condition (15). We wish to show that \( \psi \) is a homomorphism. For all \( a, b \in \mathcal{A} \), we have

\[ \psi(ab + ba) = \psi(a)\psi(b) + \psi(b)\psi(a). \quad (16) \]

The \( \psi \)-commutativity of \( \mathcal{B} \) implies that the equality (16) converts to

\[ \psi(ab + ba) = 2\psi(a)\psi(b). \quad (17) \]
The mapping $\psi$ is a Jordan homomorphism and hence
\[ \psi([a, b]^2) = [\psi(a), \psi(b)]^2 = 0. \tag{18} \]
Since $\psi$ satisfies the condition (15), it concludes from (18) that
\[ \psi(ab - ba) = 0. \tag{19} \]
Plugging (17) into (19), we have
\[ \psi(ab) = \psi(a)\psi(b). \tag{20} \]
By Lemma 2.4, we get
\[ \psi(a)\varphi(e) = \varphi(a)\varphi(e)^{n-1} = \varphi(a). \tag{21} \]
It follows from Lemma 2.4 and the relation (20) that
\[ \varphi(a_1a_2\cdots a_n) = \psi(a_1a_2\cdots a_n)\varphi(e) = \psi(a_1)\psi(a_2)\cdots \psi(a_n)\varphi(e) = \varphi(a_1)\varphi(a_2)\cdots \varphi(a_n)\varphi(e)^{(n-1)^2} = \varphi(a_1)\varphi(a_2)\cdots \varphi(a_n). \tag{22} \]
Thus, $\varphi$ is $n$–homomorphism. □

**Theorem 2.7.** Let $\varphi$ be a 3-Jordan homomorphism from algebra $A$ into $\varphi$-commutative algebra $B$ such that $\varphi([a, b]) = 0$ for any $a, b \in A$. Then, $\varphi$ is mixed Jordan homomorphism.

**Proof.** By assumption
\[ \varphi(x^3) = \varphi(x)^3, \tag{23} \]
for all $x \in A$. Replacing $x$ by $a + b$ in (23), we obtain
\[ \varphi(aba + ba^2 + a^2b + b^2a + ab^2 + bab) = 3[\varphi(a)^2\varphi(b) + \varphi(a)\varphi(b)^2]. \tag{24} \]
Switching $b$ by $-b$ in (22), we get
\[ \varphi(-aba - ba^2 - a^2b + b^2a + ab^2 + bab) = 3[-\varphi(a)^2\varphi(b) + \varphi(a)\varphi(b)^2], \tag{25} \]
for all $a, b \in A$. The equalities (22) and (23) show that
\[ \varphi(b^2a + ab^2 + bab) = 3\varphi(a)\varphi(b)^2, \quad (a, b \in A). \tag{26} \]
Since $\varphi([a, b]) = 0$, we have
\[ \varphi(b^2a) = \varphi(ab^2) = \varphi(bab), \tag{27} \]
for all $a, b \in A$. It follows from (26) and (27) that $\varphi(b^2a) = \varphi(b)^2\varphi(a)$, for all $a, b \in A$. Therefore, $\varphi$ is mixed Jordan homomorphism. □

As a consequence of Theorem 2.7, we have the next result.
Corollary 2.8. Let \( \varphi \) be a mapping from algebra \( A \) into \( \varphi \)-commutative Banach algebra \( B \), such that \( \varphi([a,b]) = 0 \) for all \( a, b \in A \). Suppose \( \delta > 0 \) and \( \varphi \) satisfy
\[
|\varphi(a^2) - \varphi(a)^2| \leq \delta,
\]
then \( \varphi \) is mixed Jordan homomorphism.

The following theorem is a well-known result, due to Šilov, concerning the automatic continuity of homomorphisms between Banach algebras.

Theorem 2.9. Let \( A \) and \( B \) be Banach algebras such that \( B \) is commutative and semisimple. Then, every homomorphism \( \varphi : A \rightarrow B \) is automatically continuous.

In 1967, B. E. Johnson proved that if \( \varphi : A \rightarrow B \) is a surjective homomorphism between a Banach algebra \( A \) and a semisimple Banach algebra \( B \), then \( \varphi \) is automatically continuous and then the Johnson’s result extended to \( n \)-homomorphism in \([7]\). One may refer to \([5]\) for automatic continuity of 3-homomorphism.

We say that a linear map \( \varphi : A \rightarrow B \) is a co-Jordan homomorphism if for all \( a \in A \), \( \varphi(a^2) = -\varphi(a)^2 \). For example, the function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( \varphi(a) = -a \) is a co-Jordan homomorphism. Here, we show that Theorem 2.9 holds for mixed Jordan homomorphisms.

Theorem 2.10. Let \( \varphi \) be a mixed Jordan homomorphism from Banach algebra \( A \) into \( \mathbb{C} \). Then, \( \varphi \) is automatically continuous.

Proof. Suppose that there exist \( x_0 \in A \) such that \( \|x_0\| < 1 \) and \( \varphi(x_0) = 1 \). Take \( y = \sum_{n=1}^{\infty} x_0^n \). Then, \( x_0 + x_0^2y = y - x_0^2 \), and so
\[
1 + \varphi(y) = \varphi(x_0) + \varphi(x_0)^2 \varphi(y) = \varphi(x_0 + x_0^2y) = \varphi(y) - \varphi(x_0^2).
\]
Thus, \( \varphi(x_0^2) = -1 \). Since \( \varphi \) is mixed Jordan homomorphism, we have
\[
\varphi(a^2b) = \varphi(a)^2\varphi(b).
\]
(27)
for all \( a, b \in A \). Replacing \( a \) by \( u + v \) in (27), we get
\[
\varphi((uv + vu)b) = 2\varphi(u)\varphi(v)\varphi(b).
\]
(28)
for all \( a, u, v \in A \). Interchanging \( b \) by \( x_0^2 \) in (28), we obtain
\[
\varphi((uv + vu)x_0^2) = 2\varphi(u)\varphi(v)\varphi(x_0^2) = -2\varphi(u)\varphi(v).
\]
(29)
for all \( a, u, v \in A \). Substituting \( (u,v) \) into \( (u^2, x_0^2) \) in (29), we arrive at
\[
\varphi(u^2x_0^4) + \varphi(x_0^2u^2x_0^2) = -2\varphi(u^2)\varphi(x_0^2) = 2\varphi(u^2).
\]
(30)
for all \( u \in A \). Also, \( \varphi(x_0^4) = \varphi(x_0)^2\varphi(x_0^2) = -1 \), and
\[
\varphi(x_0^2u^2x_0^3) = \varphi(x_0)^2\varphi(u^2x_0^3) = \varphi(x_0)^2\varphi(u^2)\varphi(x_0) = -\varphi(u)^2.
\]
for all $u \in \mathcal{A}$. So by (30), we have $\varphi(u)^2 = -\varphi(u^2)$ for all $u \in \mathcal{A}$. Hence, $\varphi$ is co-Jordan homomorphism and it is continuous by [20, Proposition 2.1]. If there is no $x_0 \in \mathcal{A}$ such that $\|x_0\| < 1$ and $\varphi(x_0) = 1$, then for all $x \in \mathcal{A}$ with $\|x\| < 1$ we have $|\varphi(x)| \leq 1$. Therefore, $\varphi$ is continuous. □

The upcoming corollary is a direct consequence of Theorem 2.10.

**Corollary 2.11.** Let $\varphi$ be a mixed Jordan homomorphism from Banach algebras $\mathcal{A}$ into commutative semisimple Banach algebra $\mathcal{B}$. Then $\varphi$ is automatically continuous.

**Proof.** Let $(a_n) \subseteq \mathcal{A}$, $a_n \rightarrow 0$ and $\varphi(a_n) \rightarrow b$. Suppose that $h \in \mathfrak{M}(\mathcal{B})$, where $\mathfrak{M}(\mathcal{B})$ is the maximal ideal space of $\mathcal{B}$. Then, $h \circ \varphi$ is a mixed Jordan homomorphism and so it is continuous by Theorem 2.10. Thus,

$$ h(b) = \lim_n h(\varphi(a_n)) = \lim_n h \circ \varphi(a_n) = 0. $$

Since $\mathcal{B}$ is semisimple, we have $b = 0$, and thus $\varphi$ is continuous by the close graph theorem. □

The next result is the same as Theorem 2.9 for 3-homomorphism.

**Corollary 2.12.** Let $\varphi$ be a 3-homomorphism from Banach algebras $\mathcal{A}$ into commutative semisimple Banach algebra $\mathcal{B}$. Then $\varphi$ is automatically continuous.

A linear map $\varphi$ between unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called **unital** if $\varphi(e) = e'$, where $e$ and $e'$ are the unit element of $\mathcal{A}$ and $\mathcal{B}$, respectively.

**Theorem 2.13.** Every unital mixed $(n + 1)$-Jordan homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism.

**Proof.** Let $\varphi$ be a mixed $(n + 1)$-Jordan homomorphism. Then we have

$$ \varphi(a^{n+1}b) = \varphi(a)^{n+1}\varphi(b), \quad (31) $$

for all $a, b \in \mathcal{A}$. Since $\varphi$ is unital, then with $b = e$ in (31) we get

$$ \varphi(a^{n+1}) = \varphi(a)^{n+1}, \quad (32) $$

for all $a \in \mathcal{A}$. Replacing $a$ by $a + \lambda e$ in (32), where $\lambda$ is a complex number, and compare powers of $\lambda$, we arrive at

$$ \varphi(a^2) = \varphi(a^2), \quad (33) $$

hence $\varphi$ is a Jordan homomorphism. Interchanging $a$ by $a + \lambda e$ in (31), we botain

$$ \varphi(a^{n+1}b) = (\varphi(a) + \lambda \varphi(e))^{n+1}\varphi(b). \quad (34) $$

Comparing powers of $\lambda^n$ in (34) and using $\varphi(e) = e'$, one deduce that $\varphi$ is a homomorphism. □

The next corollary follows immediately from Theorem 2.13.
Corollary 2.14. Every unital mixed \((n+1)\)-Jordan homomorphism \(\varphi: A \to B\) is a mixed \(n\)-Jordan homomorphism.

The following example shows the condition being unital for Banach algebras \(A\) and \(B\) in Theorem 2.13 is essential.

Example 2.15. Let
\[
A = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\},
\]

and define \(\varphi: A \to A\) via
\[
\varphi \left( \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.
\]

Then, \(\varphi(X^2) \neq \varphi(X)^2\), for all \(X \in A\). Hence, \(\varphi\) is not Jordan homomorphism, and so it is not homomorphism. But for all \(n \geq 3\) and for all \(X, Y \in A\), we have \(\varphi(X^nY) = \varphi(X)^n\varphi(Y)\). Therefore, \(\varphi\) is mixed \(n\)-Jordan homomorphism for all \(n \geq 3\).

Corollary 2.16. Let \(\varphi: A \to B\) be a unital mixed \(n\)-Jordan homomorphism. Then, \(\varphi\) is continuous under one of the following conditions.

(i) \(B\) is semisimple and commutative.
(ii) \(B\) is semisimple and \(\varphi\) is surjective.
(iii) \(B\) is \(C^*\)-algebra and \(\varphi\) is surjective.

Proof. By Corollary 2.14, \(\varphi\) is a homomorphism and so it is \(n\)-Jordan homomorphism. Thus, the result follows from Corollaries 2.9, 2.10 and 2.11 of [20].

In the upcoming result, we prove the converse of Theorem 2.13 under some conditions. The idea of the proof is taken from [6, Theorem 2.6]. We include the proof for the sake of completeness.

Theorem 2.17. Let \(A, B\) be Banach algebras such that \(A\) is unital and \(\varphi: A \to B\) be mixed \(n\)-Jordan homomorphism. Suppose that there exists an idempotent \(p\) in \(A\) such that \(\varphi(ab) = \varphi(a)\varphi(b)\) for all \(a, b, x \in A\) with \(ab = px\). Then, \(\varphi(a^{n+1}px) = \varphi(a)^{n+1}\varphi(px)\) for all \(a, x \in A\). In particular, \(\varphi\) is mixed \((n+1)\)-Jordan homomorphism.

Proof. Let \(e\) be a until element of \(A\) and \(a \in A\). For \(\lambda \in \mathbb{C}\), with \(|\lambda| < 1/\|a\|\), \(e - \lambda\) is invertible and \((e - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n\). Then
\[\varphi(px) = \varphi((e - \lambda a)(e - \lambda a)^{-1}px)\]
\[= \varphi(e - \lambda a)\varphi((e - \lambda a)^{-1}px)\]
\[= (\varphi(e) - \lambda \varphi(a))\varphi\left(\sum_{n=0}^{\infty} \lambda^n a^n px\right)\]
\[= \varphi(e)\varphi(px) + \varphi\left(\sum_{n=1}^{\infty} \lambda^n a^n px\right) - \lambda \varphi(a)\varphi\left(\sum_{n=0}^{\infty} \lambda^n a^n px\right)\]
\[= \varphi(px) + \sum_{n=1}^{\infty} \lambda^n \varphi(a^n px) - \lambda \varphi(a)\varphi\left(\sum_{n=0}^{\infty} \lambda^n a^n px\right)\]
\[= \varphi(px) + \sum_{n=0}^{\infty} \lambda^{n+1} \varphi(a^{n+1} px) - \varphi(a)\sum_{n=0}^{\infty} \lambda^{n+1} \varphi(a^n px)\]

Hence, \(\sum_{n=0}^{\infty} \lambda^{n+1}[\varphi(a^{n+1} px) - \varphi(a)\varphi(a^n px)] = 0\) for \(\lambda \in \mathbb{C}\), with \(|\lambda| < 1/||a||\). Thus, \(\varphi(a^{n+1} px) = \varphi(a)\varphi(a^n px)\) for \(n = 0, 1, 2, \cdots\). For \(n \geq 1\), we have
\[\varphi(a)\varphi(a^n px) = \varphi(a)\varphi(a)^n \varphi(px) = \varphi(a)^{n+1} \varphi(px).\]

So, \(\varphi(a^{n+1} px) = \varphi(a)^{n+1} \varphi(px)\) for all \(a, x \in A\). If we take \(p = e\), we get \(\varphi(a^{n+1} x) = \varphi(a)^{n+1} \varphi(x)\). This finishes the proof. \(\square\)

We should note that Theorem 2.17 is true if “mixed” is deleted from it.

In the following result, under certain conditions, we prove that each mixed Jordan homomorphism is a mixed \(n\)-Jordan homomorphism.

**Theorem 2.18.** Let \(\varphi : A \to B\) be a unital mixed Jordan homomorphism. Then, \(\varphi\) is mixed \(n\)-Jordan homomorphism.

**Proof.** The proof follows from Corollary 2.14. \(\square\)

**Proposition 2.19.** Let \(\varphi : A \to B\) be a linear map such that
\[\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| \leq \delta\|a\|^n\|b\|,\]  
for all \(a, b \in A\) and for some \(\delta > 0\). If \(B\) is commutative and semisimple, then \(\varphi\) is continuous.

**Proof.** Replacing \(b\) by \(a\) in (35), we get
\[\|\varphi(a^{n+1}) - \varphi(a)^{n+1}\| \leq \delta\|a\|^{n+1}.\]  
Thus, \(\varphi\) is almost \(n\)-Jordan homomorphism and so it is continuous by [20, Theorem 3.4]. \(\square\)
Theorem 2.20. Let \( \varphi : A \rightarrow B \) be a linear map such that
\[
\| \varphi(a^n b) - \varphi(a)^n \varphi(b) \| \leq \delta(\|a\| + \|b\|),
\]
for all \( a, b \in A \) and for some \( \delta > 0 \). Then, \( \varphi \) is \((n+1)\)-Jordan homomorphism.

Proof. At first, we consider the inequality
\[
\| \varphi(a^n b) - \varphi(a)^n \varphi(b) \| \leq \delta(\|a\| - \|b\|),
\]
for all \( a, b \in A \). Putting \( a = b \) in (38), we get \( \varphi(a^{n+1}) = \varphi(a)^{n+1} \) and so \( \varphi \) is \((n+1)\)-Jordan homomorphism. Now, assume that
\[
\| \varphi(a^n b) - \varphi(a)^n \varphi(b) \| \leq \delta(\|a\| + \|b\|).
\]
for all \( a, b \in A \). Replacing \( b \) by \( a \) in (39), we find
\[
\| \varphi(a^{n+1}) - \varphi(a)^{n+1} \| \leq 2\delta\|a\|.
\]
for all \( a \in A \). Setting \( a = 2^m x \), we obtain
\[
\| \varphi(a^{n+1}) - \varphi(a)^{n+1} \| \leq \frac{\delta 2^{m+1}}{2^{m(n+1)}} \|a\|.
\]
for all \( a \in A \). Letting \( m \to \infty \), we obtain \( \varphi(a^{n+1}) = \varphi(a)^{n+1} \) and hence \( \varphi \) is \((n+1)\)-Jordan homomorphism. \( \square \)

3. Pseudo \( n \)-Jordan Homomorphisms

We commence with a definition.

Definition 3.1. Let \( A \) and \( B \) be rings (algebras), and let \( B \) be a right [left] \( A \)-module. We say that a linear mapping \( \psi : A \rightarrow B \) is a pointwise pseudo \( n \)-Jordan homomorphism if for each \( a \in A \) there exists an element \( \omega_a \in A \) such that \( \psi(a^n \omega_a) = \psi(a)^n \cdot \omega_a \) \( [\psi(\omega_a a^n) = \omega_a \cdot \psi(a^n)] \). We say that \( \omega_a \) is a Jordan coefficient of \( \psi \) depended on \( a \).

It is obvious that every \( n \)-Jordan homomorphism from unital Banach algebra \( A \) into \( B \) which is unitary Banach \( A \)-module is a pseudo \( n \)-Jordan homomorphism. Also, as we see that for a pseudo \( n \)-Jordan homomorphism, there are infinitely many Jordan coefficient.

Every pseudo \( n \)-Jordan homomorphism is a pointwise pseudo \( n \)-Jordan homomorphism. Now, let \( \varphi \) be a mixed \( n \)-Jordan homomorphism such that has a fixed point, say \( \omega \). Then, \( \varphi \) is a pseudo \( n \)-Jordan homomorphism with a Jordan coefficient \( \omega \). The following example indicates this fact that the converse is false in general.

Example 3.2. Let
\[
U_3(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}
\]
be the algebra of $2 \times 2$ matrices with the usual sum and product. Let $\psi : U_2(\mathbb{R}) \rightarrow U_2(\mathbb{R})$ be a linear map defined by

$$\psi \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

For every $n \in \mathbb{N}$, we have

$$\psi \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^n \right) = \begin{bmatrix} a^n \sum_{k=0}^{n-1} a^{n-k-1}bc^k \\ 0 \end{bmatrix} = \begin{bmatrix} a^n \sum_{k=0}^{n-1} a^{n-k-1}bc^k \\ 0 \end{bmatrix}$$

and

$$\psi \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right)^n = \begin{bmatrix} a & a^{n-1}b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a^{n-1}b \\ 0 & 0 \end{bmatrix}$$

Thus, $\psi$ is not an $n$-Jordan homomorphism and so it is not mixed $(n-1)$-Jordan homomorphism. Assume that $t, s \in \mathbb{R}$. Put $\omega = \begin{bmatrix} s & t \\ 0 & 0 \end{bmatrix}$. Then,

$$\psi \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^n \omega \right) = \psi \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right)^n.$$

This means that $\psi$ is a pseudo $n$-Jordan homomorphism. On the other hand, for each $a, b, c \in \mathbb{R}$, take $\omega_{a,b,c} = \begin{bmatrix} s & 0 \\ -sc^{1-n} \sum_{k=2}^{n-2} a^{n-k-2}bc^k \\ 0 \end{bmatrix}$. Then,

$$\psi \left( \omega_{a,b,c} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^n \right) = \omega_{a,b,c} \psi \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right)^n.$$

This means that $\psi$ is a pointwise pseudo $n$-Jordan homomorphism. Note that

$$\psi \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^n \omega_{a,b,c} \right) \neq \psi \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right)^n \omega_{a,b,c} \text{ and } \psi \left( \omega \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^n \right) \neq \omega \psi \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right)^n.$$

Here, we remind that the part (3) of [6, Example 2.2] is not true. Indeed, it is corrected in Example 3.2.

It is known that every Jordan homomorphism is $n$-Jordan homomorphism [19]. The next example shows that the same result is false for mixed Jordan homomorphisms and pseudo $n$-Jordan homomorphisms.

**Example 3.3.** Let $\mathcal{A}$ be a Banach algebra and $f : \mathcal{A} \rightarrow \mathcal{A}$ be a homomorphism. Define $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ by $\varphi(x) = -f(x)$. Then $\varphi$ is a 3-homomorphism. Thus,

$$\varphi(x^2a) = \varphi(x)^2 \varphi(a),$$

So $\varphi$ is a mixed Jordan homomorphism, but $\varphi$ is not a mixed 3-Jordan homomorphism. Suppose that $\varphi$ has a fixed point, say $a$. Hence $\varphi(a) = a$. Thus,

$$\varphi(x^2a) = \varphi(x)^2 \varphi(a) = \varphi(x)^2a,$$
So \( \varphi \) is a pseudo Jordan homomorphism with a Jordan coefficient \( a \), but \( \varphi \) is not a pseudo 3-Jordan homomorphism. Note that for all \( n \in \mathbb{N} \), \( \varphi \) is a pseudo \((2n)\)-Jordan homomorphism with a Jordan coefficient \( a \), but \( \varphi \) is not a pseudo \((2n+1)\)-Jordan homomorphism for each \( n \in \mathbb{N} \).

In the sequel, \( \binom{n}{k} \) is the binomial coefficient defined for all \( n, k \in \mathbb{N} \) with \( n \geq k \) by \( n!/k!(n-k)! \).

Here and subsequently, let \( A \) and \( B \) be unital Banach algebras and \( B \) be a right \( A \)-module. Also, it is assumed that \( \varphi \) between unital Banach algebras \( A \) and \( B \) is unital. The following result is the converse of [6, Theorem 2.3].

**Theorem 3.4.** Every unital pseudo \((n+1)\)-Jordan homomorphism \( \varphi : A \to B \) with a Jordan coefficient \( w \) is a pseudo \( n \)-Jordan homomorphism.

**Proof.** We firstly have

\[
\varphi((a + le)^{n+1}w) = (\varphi(a + le))^{n+1} \cdot w, \tag{42}
\]

for all \( a, b \in A \), where \( l \) is an integer with \( 2 \leq l \leq n \). It follows from the equality \( (42) \) and assumption that

\[
\sum_{i=1}^{n} l^i \binom{n+i}{i} [\varphi(a^i w) - \varphi(a)^i \cdot w] = 0, \quad (2 \leq l \leq n), \tag{43}
\]

for all \( a \in A \). We can rewrite the equalities in \( (43) \) as follows

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
2 & 2^2 & \cdots & 2^n \\
3 & 3^2 & \cdots & 3^n \\
\vdots & \vdots & \ddots & \vdots \\
n & n^2 & \cdots & n^n
\end{bmatrix}
\begin{bmatrix}
\Gamma_1(a, w) \\
\Gamma_2(a, w) \\
\Gamma_3(a, w) \\
\vdots \\
\Gamma_n(a, w)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

for all \( a \in A \), where \( \Gamma_i(a, w) = \binom{n+i}{i} [\varphi(a^i w) - \varphi(a)^i \cdot w] \) for all \( 1 \leq i \leq n \). It is shown in [3, Lemma 2.1] that the above square matrix is invertible. This implies that \( \Gamma_i(a, w) = 0 \) for all \( 1 \leq i \leq n \) and all \( a \in A \). In particular, \( \Gamma_n(a, w) = 0 \). This means that \( \varphi \) is a pseudo \( n \)-Jordan homomorphism. \( \square \)

The next Corollary follows from Theorem 3.4.

**Corollary 3.5.** Every unital \((n+1)\)-Jordan homomorphism \( \varphi : A \to B \) is an \( n \)-Jordan homomorphism.

**Example 3.6.** Let

\[
A = \left\{ \begin{bmatrix}
0 & a & b \\
0 & 0 & c \\
0 & 0 & 0
\end{bmatrix} : \ a, b, c \in \mathbb{R} \right\},
\]
and define $\varphi : A \to A$ by

$$
\varphi \left( \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix},
$$

Then, for all $X \in A$, $\varphi(X^2) \neq \varphi(X)^2$. Hence, $\varphi$ is not Jordan homomorphism, but for all $n \geq 3$ and all $X \in A$, we have $\varphi(X^n) = \varphi(X)^n$. Therefore, $\varphi$ is $n$-Jordan homomorphism for all $n \geq 3$. Assume that $s, t, r \in \mathbb{R}$ is arbitrary. Put

$$
w = \begin{bmatrix} 0 & s & t \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix}.
$$

Hence, for all $n \in \mathbb{N}$ and $X \in A$, we get $\varphi(X^n w) = \varphi(X)^n w$. Therefore, $\varphi$ is a pseudo $n$-Jordan homomorphism. In other words, we showed the condition that being unital for Banach algebras $A$ and $B$ in Corollary 3.5 is essential.

Suppose that $A$ is a Banach algebra and $M$ is an $A$-module. Let $w \in A$. Then $w$ is called a left (right) separating point of $M$ if the condition $wx = 0$ ($xw = 0$) for $x \in M$ implies that $x = 0$ [14].

In the following result, under certain conditions, we prove that each pseudo Jordan homomorphism is pseudo $n$-Jordan homomorphism.

**Theorem 3.7.** Let $\varphi : A \to B$ be a unital pseudo Jordan homomorphism with a Jordan coefficient $w$ such that $w$ is a right separating point of $B$. Then

(i) $\varphi$ is $n$-Jordan homomorphism.

(ii) $\varphi$ is pseudo $n$-Jordan homomorphism.

**Proof.** Assume that $\varphi$ is a pseudo Jordan homomorphism, then

$$
\varphi(a^2 w) = \varphi(a)^2 \cdot w,
$$

for all $a \in A$. Replacing $a$ by $a + b$ in (44), we get

$$
\varphi[(ab + ba)w] = [\varphi(a)\varphi(b) + \varphi(b)\varphi(a)] \cdot w.
$$

Since $\varphi$ is unital, one can show that

$$
\varphi[(ab + ba)w] = \varphi(ab + ba) \cdot w.
$$

It follows from (45) and (46) that

$$
(\varphi(ab + ba) - [\varphi(a)\varphi(b) + \varphi(b)\varphi(a)]) \cdot w = 0
$$

Since $w$ is a right separating point of $B$, we get

$$
\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a).
$$
Thus, $\varphi$ is a Jordan homomorphism. Now by Lemma 2.6 of [19], $\varphi$ is $n$-Jordan homomorphism. The Part (2) follows from above Lemma and (1).

From Theorem 3.7, [3, Theorem 2.3] and [1, Corollary 2.5], we have the following trivial consequence.

**Corollary 3.8.** By hypotheses of Theorem 3.7, $\varphi$ is $n$-homomorphism if either

(i) $A$ and $B$ are commutative, or
(ii) $B$ is commutative and semisimple.

**Corollary 3.9.** Let $\varphi : A \to B$ be a unital pseudo $n$-Jordan homomorphism with a Jordan coefficient $w$ such that $w$ is a right separating point of $B$. Then $\varphi$ is continuous with each of the following conditions.

(i) $B$ is semisimple and commutative.
(ii) $B$ is semisimple and $\varphi$ is surjective.
(iii) $B$ is $C^\ast$-algebra and $\varphi$ is surjective.

**Proof.** By Theorem 3.4, $\varphi$ is pseudo Jordan homomorphism and by Theorem 3.7 it is $n$-Jordan homomorphism. Thus, the result follows from Corollaries 2.9, 2.10 and 2.11 of [20].

**Theorem 3.10.** Let $\varphi : A \to B$ be a unital pseudo $3$-Jordan homomorphism with a Jordan coefficient $w$ such that $w$ is a right separating point of $B$. Suppose that $B$ is commutative and

$$\varphi(abcw) = \varphi(acbw), \quad (a, b, c \in A).$$

Then, $\varphi$ is homomorphism.

**Proof.** Assume that $e$ is an unit element of $A$, letting $a = e$ in (48), we get

$$\varphi(bcw - cbw) = 0,$$

for all $b, c \in A$. Thus, $\varphi((ab)cw) = \varphi(c(ab)w) = \varphi(c(ba)w)$ and hence

$$\varphi(a(bc)w) = \varphi((bc)aw) = \varphi(b(ca)w) = \varphi(b(ac)w).$$

That is

$$\varphi(abcw) = \varphi(xyzw),$$

whenever $(x, y, z)$ is a permutation of $(a, b, c)$. Since $\varphi$ is a pseudo 3-Jordan homomorphism, $\varphi(a^3w) = \varphi(a)^3 \cdot w$, for all $a \in A$. Replacing $a$ by $a + b$, we get

$$\varphi[(ab^2 + b^2a + a^2b + b - ab + bab)w] = [3\varphi(a)\varphi(b)^2 + 3\varphi(a)^2\varphi(b)] \cdot w.$$  

(50)

Interchanging $b$ by $-b$ in (50), we obtain

$$\varphi[(ab^2 + b^2a - a^2b - b + bab)w] = [3\varphi(a)\varphi(b)^2 - 3\varphi(a)^2\varphi(b)] \cdot w.$$  

(51)

The relations (50), and (51) imply that

$$\varphi[(ab^2 + b^2a + bab)w] = [3\varphi(a)\varphi(b)^2] \cdot w.$$  

(52)
Replacing $b$ by $b - c$ in (52), we deduce
$$\varphi[(abc + acb + bac + bca + cab + cba)w] = [6\varphi(a)\varphi(b)\varphi(c)] \cdot w.$$  \hfill (53)

It follows from (49) and (53) that
$$\varphi(abcw) = [\varphi(a)\varphi(b)\varphi(c)] \cdot w,$$  \hfill (54)
for all $a, b, c \in A$. Put $c = e$ in (54), we arrive at
$$(\varphi(ab) - [\varphi(a)\varphi(b)]) \cdot w = 0.$$  

Since $w$ is a right separating point of $B$, we get
$$\varphi(ab) = \varphi(a)\varphi(b), \quad a, b \in A.$$  

Thus, $\varphi$ is a homomorphism. □

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