Analytic solutions for the Λ-FRW Model

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The high precision attained by cosmological data in the last few years has increased the interest in exact solutions. Analytic expressions for solutions in the Standard Model are presented here for all combinations of Λ = 0 , Λ ≠ 0, κ = 0 and κ ≠ 0, in the presence and absence of radiation and nonrelativistic matter. The most complete case (here called the ΛγCDM Model) has Λ ≠ 0, κ ≠ 0, and supposes the presence of radiation and dust. It exhibits clearly the recent onset of acceleration. The treatment includes particular models of interest such as the ΛCDM Model (which includes the cosmological constant plus cold dark matter as source constituents).

**Key words**: Cosmology, standard model, exact solutions

**Running head**: Λ-FRW Model
1 INTRODUCTION

Recent cosmological data [1, 2, 3] suggest a present-day Universe dominated by dark energy, though with a significant contribution of matter (visible and dark). This brings to the fore a Friedmann-Robertson-Walker model with a strong de Sitter component, that is, with a cosmological constant. In the usual approach to standard cosmology the history of the Universe is divided into eras, each one dominated by one of the source-constituents. Separate solutions are used for each era. The intermediate periods were described by approximations which were quite acceptable until recently. The precision of present-day data, however, makes it desirable to have solutions as complete and exact as possible. An exact solution for the Standard Model can in principle be obtained provided an equation of state (giving the pressure in terms of the energy density) is given for each constituent. For all times except the first few seconds (the equation of state for a very-very-high-temperature black-body is unknown), the main difficulty concerns dark matter. It is our aim here to provide a short, compact presentation, in terms of present-day parameters, of the solutions [4, 5, 6] for a dust dark matter, which can be called "ΛγCDM Model". This complete, unified version is a necessity whenever applications and comparisons are in view. We shall, however, proceed step by step, the reason for that being that the general, complete solution is actually implicit (gives time as a function of the scale parameter) and involves elliptic functions of the first and the third kinds. Such a progressive presentation seems advisable, describing solutions for particular values of the cosmological constant and curvature parameters, presence and absence of matter and radiation, with increasing complexity.

Section 2 is a short introduction to the Friedmann equations [7, 8, 9, 10, 11], the main objective being the introduction of convenient parameters and notation. Maybe too much detail on well-known results is given in section 3 concerned with de Sitter solutions. The assumptions there made concerning inflation will, however, propagate to the ensuing solutions. In effect, we use the standard exponential inflation there found as a kind of "correspondence principle": all other solutions will generalize that case, and reduce to it for the corresponding choice of parameters. Section 4 treats the model with radiation, cosmological constant and curvature. The limits to particular cases, as the pure radiation model, are readily obtained. In section 5 we analyze the case for matter plus another contribution, either the cosmological constant or curvature. Adding any other contribution leads to a solution as involved as the general case, which is given in Section 6.
2 GENERAL OUTLOOK

Let us start fixing notation by recalling Einstein’s equations with a cosmological–
constant and source energy content modeled by a perfect fluid:

\[ R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8 \pi G}{c^4} \, T_{\mu\nu}, \]

\[ T_{\mu\nu} = (p + \rho c^2) \, u_\mu u_\nu - p \, g_{\mu\nu}. \]  

(1)

Here \( \rho = \epsilon / c^2 \) is the mass equivalent of the energy density. We use metric sig-
nature (+, −, −, −), so that \( u_\mu u^\mu = 1 \) for timelike flux lines. Spacetimes with
homogeneous and isotropic space sections are described by the Robertson–
Walker line element

\[ ds^2 = c^2 dt^2 - a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \]  

(2)

with \( \kappa = 0, \pm 1 \), which reduces Einstein’s equations to the two Friedmann
equations for the scale parameter \( a(t) \):

\[ \dot{a}^2 = \left[ 2 \left( \frac{4 \pi G}{3} \right) \rho + \frac{\Lambda c^2}{3} \right] a^2 - \kappa c^2, \]  

(3)

\[ \ddot{a} = \left[ \frac{\Lambda c^2}{3} - \frac{4 \pi G}{3} \left( \rho + \frac{3p}{c^2} \right) \right] a. \]  

(4)

Combining the last two equations, one finds the expression for energy
conservation,

\[ \frac{d\rho}{da} = - \frac{3}{a} \left( \rho + \frac{p}{c^2} \right). \]  

(5)

The total matter energy density \( \epsilon_m = \rho_m c^2 = \rho_b c^2 + \rho_r c^2 \) includes both non-
relativistic matter (“baryons”), dark or not, and radiation. It is convenient
to relate to \( \Lambda \) the “dark energy” density

\[ \epsilon_\Lambda = \frac{\Lambda c^4}{8 \pi G}. \]  

(6)

In terms of the Hubble function

\[ H(t) = \frac{\dot{a}(t)}{a(t)}, \]  

(7)

Eqs. (3,4) assume the forms

\[ H^2 = \frac{8 \pi G}{3c^2} [\epsilon_m + \epsilon_\Lambda] - \frac{\kappa c^2}{a^2}, \]  

(8)

\[ \dot{H} = \frac{\kappa c^2}{a^2} - \frac{3}{2} \frac{8 \pi G}{3c^2} [\epsilon_m + p_m]. \]  

(9)
Actually, these equations acquire their most convenient form in terms of dimensionless variables, obtained by dividing by the present-day value $H^2(t = t_0) = H_0^2$ and introducing the parameters

$$\Omega_b(t) = \frac{\rho_b(t)}{\rho_{\text{crit}}} ; \quad \Omega_m(t) = \Omega_b + \Omega_\gamma = \frac{\rho_b}{\rho_{\text{crit}}} + \frac{\rho_\gamma}{\rho_{\text{crit}}} ,$$

(10)

$$\Omega_\kappa(t) = -\frac{\kappa c^2}{a^2 H_0^2} ; \quad \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} = \frac{\rho_\Lambda}{\rho_{\text{crit}}} ,$$

(11)

with $\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G}$. The first equation, in particular, becomes

$$\frac{H^2}{H_0^2} = \frac{\rho_m}{\rho_{\text{crit}}} - \frac{\kappa c^2}{a^2 H_0^2} + \frac{\Lambda c^2}{3H_0^2} \equiv \Omega_m(t) + \Omega_\gamma(t) + \Omega_\Lambda ,$$

(12)

which leads to the usual normalization for present-day values,

$$\Omega_{m0} + \Omega_{\kappa0} + \Omega_\Lambda = 1 ,$$

(13)

or

$$(\Omega_{m0} + \Omega_\Lambda) - 1 = \Omega_{\text{total}} - 1 = \frac{\kappa c^2}{a_0^2 H_0^2} .$$

(14)

Observation will fix the value of $\kappa$: we are forced to choose $\kappa = +1$ if $\Omega_{\text{total}} > 1$ and $\kappa = -1$ if $\Omega_{\text{total}} < 1$. In both cases the present-day value of the scale factor is determined by $a_0 = \sqrt{\frac{\kappa}{\Omega_{\text{total}} - 1}} \frac{c}{H_0}$. If $\Omega_{\text{total}} = 1$, then $\kappa = 0$ and $a_0$ remains undetermined.

Pressure in equation (5) is the sum $p = p_m = p_b + p_\gamma$, the necessary expressions being given by thermodynamic considerations. Non-relativistic matter is represented as an ideal gas, i.e., $p_b = n_b kT$. This pressure, however, is negligible against its related density (dust approximation): $(\rho_b + p_b/c^2) = \frac{\rho_b}{\rho_{\text{crit}}} (mc^2 + kT) \simeq \rho_b$, since $mc^2 \gg kT$. The radiation equation of state is that of a black body [12], $p_\gamma = \frac{\rho_\gamma c^2}{3} = \frac{\epsilon}{3}$.

The conservation equation (5), which is

$$\left[ \frac{d\rho_b}{da} + \frac{3\rho_b}{a} \right] + \left[ \frac{d\rho_\gamma}{da} + 4\frac{\rho_\gamma}{a} \right] = 0 ,$$

(15)

can be solved immediately under the hypothesis of independence of the components: matter and radiation have their own separated dynamics with respect to the scale factor $a$. The result is the pair

$$\rho_b = \rho_{0b} \left( \frac{a_0}{a} \right)^3 = \rho_{0b} (1 + z)^3 ; \quad \rho_\gamma = \rho_{0\gamma} \left( \frac{a_0}{a} \right)^4 = \rho_{0\gamma} (1 + z)^4$$

(16)
where $z$ is the red-shift and $\rho_0 = \rho(a_0)$. These results for the relationship between density and scale factor for matter and radiation is a natural consequence of the baryonic number conservation, which is strongly supported by the very high value of the proton lifetime [13]. In fact, if the number of baryons $N_b = n_b a^3$ is set constant, $\frac{dN_b}{dt} = 0$, it follows that $n_b a^3 = n_{b0} a^3$, $n_b$ being the numerical baryonic density. The first of equations (16) then follows because $\rho_b = n_b (m_b c^2)$, and leads to the second once inserted into (15).

Finally, we introduce the notations

$$\gamma c^2 = \frac{8\pi G}{3} \rho_{\gamma 0} a_0^4 = \Omega_{\gamma 0} H_0^2 a_0^4,$$

(17)

$$M c^2 = \frac{8\pi G}{3} \rho_{b0} a_0^3 = \Omega_{b0} H_0^2 a_0^3,$$

(18)

$$\frac{c^2}{L^2} = \frac{\Lambda c^2}{3} = \Omega_{\Lambda} H_0^2,$$

(19)

in terms of which equations (3,4) take on the simple forms we shall be using:

$$\frac{\dot{a}^2}{a^2} = \frac{c^2}{L^2} + \frac{\gamma c^2}{a^4} + \frac{M c^2}{a^3} - \frac{\kappa c^2}{a^2};$$

(20)

$$\frac{\ddot{a}}{a} = \frac{c^2}{L^2} - \frac{\gamma c^2}{a^4} - \frac{M c^2}{2 a^3}.$$

(21)

In proceeding to our step-by-step approach, it will be convenient to start from the simplest and dominant solution, de Sitter spacetimes. They will help fixing some aspects to be retained in the other cases. Notice that (20, 21) impose severe conditions on parameters for the particular solutions we shall be examining. An example which we shall bypass is the case $\Lambda = \gamma = M = 0$, which would only be admissible for $\kappa = 0, -1$.

3 DE SITTER SOLUTIONS

We recall that there are two different kinds of such spaces:

1. de Sitter spacetime proper: indicated dS; a one-sheeted hyperbolic 4-space with topology $R^1 \times S^3$, an inclusion in the pseudo–Euclidean space $E^{4,1}$ satisfying $(A, B = 0, 1, 2, 3, 4)$

$$\eta_{AB} \xi^A \xi^B = \eta_{\alpha \beta} \xi^\alpha \xi^\beta - (\xi^4)^2 = - L^2$$

(22)

Its scalar curvature $R$ is — within our conventions — negative.
2. anti–de Sitter spacetime: indicated AdS: a two-sheeted hyperbolic 4-space with topology $S^1 \times \mathbb{R}^3$, an inclusion in the pseudo–Euclidean space $E^{3,2}$ satisfying

$$\eta_{AB} \xi^A \xi^B = \eta_{\alpha\beta} \xi^\alpha \xi^\beta + (\xi^4)^2 = L^2.$$  \hspace{1cm} (23)

It has $R > 0$. With the notation $\eta_{44} = s$, dS and AdS can be put together in

$$\eta_{AB} \xi^A \xi^B = \eta_{\alpha\beta} \xi^\alpha \xi^\beta + s (\xi^4)^2 = sL^2.$$  \hspace{1cm} (24)

Both dS ($s = -1, \Lambda > 0$) and AdS ($s = +1, \Lambda < 0$) are solutions of the sourceless Einstein’s equations with a cosmological constant related to the pseudo-radius $L$ and the scalar curvature by

$$\Lambda = -s \frac{3}{L^2} = -\frac{R}{4}. \hspace{1cm} (25)$$

As they are sourceless solutions and have homogeneous and isotropic space sections, these spacetimes satisfy equations (20) and (21) with vanishing $\gamma$ and $M$:

$$\dot{a}^2 = -s \left(\frac{c}{L}\right)^2 a^2 - \kappa c^2; \hspace{1cm} (26)$$

$$\ddot{a} = -s \left(\frac{c}{L}\right)^2 a. \hspace{1cm} (27)$$

Let us start with case $\kappa = 0$. The second equation has solutions of the general form $a(t) = A_0 e^{\alpha t} + B_0 e^{\beta t}$, actually with $\alpha = \beta$. Thus, the general expression reduces to the form $a(t) = A e^{\alpha t}$. But then $\dot{a}/a = H = \alpha = \pm \sqrt{-s} \frac{c}{L}$. The only way to have a real positive constant $H$ is to choose $s < 0$ and the upper sign. These choices lead to

$$a(t) = A e^{\frac{c}{L} t}, \hspace{1cm} (28)$$

the standard inflationary solution of exponential type. The initial condition is set for the beginning of the time measure, i.e., $A = a(t = 0)$. We could equivalently choose as “initial” value the present time $t_0$, such that $A = a_0 e^{\frac{-c}{L} t_0}$.

Notice that $\Lambda = -s \frac{3}{L^2} > 0$ is essential for inflation — only the dS space (and not the AdS) can lead to inflation. Another point: the usual treatment considers sometimes solutions with initial value $A = a(0) = 0$. Here, however, there will be no inflation if the initial value of $a(t)$ is zero. We shall from now on consider only $A > 0, \Lambda > 0$, and look for solutions generalizing this
result, that is, which reduce to (28) when only a cosmological term is present and \( \kappa = 0 \).

For \( \kappa \neq 0 \), the solution allowing for inflation is

\[
a(t) = A \cosh \frac{ct}{L} + \sqrt{A^2 - \kappa L^2} \sinh \frac{ct}{L} = \frac{1}{2} \left[ \left( A + \sqrt{A^2 - \kappa L^2} \right) e^{\frac{ct}{L}} + \left( A - \sqrt{A^2 - \kappa L^2} \right) e^{-\frac{ct}{L}} \right]. \tag{29}
\]

If we introduce the function

\[
f(t) = \left( A + \sqrt{A^2 - \kappa L^2} \right) e^{\frac{ct}{L}} \tag{30}
\]

and use the relation

\[
\left( A - \sqrt{A^2 - \kappa L^2} \right) = \frac{\kappa L^2}{\left( A + \sqrt{A^2 - \kappa L^2} \right)} \quad (\kappa \neq 0), \tag{31}
\]

solution (29) takes on the form

\[
a(t) = \frac{1}{2} \left[ f(t) + \frac{\kappa L^2}{f(t)} \right]. \tag{32}
\]

Summing up, the solution is

\[
\Lambda = \frac{3}{L^2} \neq 0: \begin{cases} a(t) = \frac{1}{2} \left[ f(t) + \frac{\kappa L^2}{f(t)} \right] \quad \text{with} \\ f(t) = \left( A + \sqrt{A^2 - \kappa L^2} \right) e^{\frac{ct}{L}} \end{cases} \tag{33}
\]

and the Friedmann-Robertson-Walker form of the inflationary de Sitter line element will consequently be

\[
ds^2 = c^2 dt^2 - \frac{1}{4} \left[ f(t) + \frac{\kappa L^2}{f(t)} \right]^2 \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \tag{34}
\]

4 RADIATION WITH \( \kappa \neq 0, \Lambda \neq 0 \)

From all the evidence we have today, the Universe has passed an initial stage during which, besides the cosmological term, ultrarelativistic matter or radiation provided a significant contribution as a source in Einstein’s equations. This era is described by equations (20) and (21) with \( M = 0 \),

\[
a^2 = \left[ \frac{\gamma c^4}{a^4} + \frac{c^2}{L^2} \right] a^2 - \kappa c^2; \tag{35}
\]

\[
\ddot{a} = \left[ \frac{c^2}{L^2} - \frac{\gamma c^2}{a^4} \right] a. \tag{36}
\]
We solve the first of these equations by direct integration (completing squares and changing variables) with our conventional initial condition \(a(0) = A\). The result reads:

\[
a(t) = \sqrt{\frac{\kappa L^2}{2} - \frac{1}{2} \left[ g(t) + \frac{(\kappa L^2)^2 - \gamma L^2}{g(t)} \right]}
\]

with the definition

\[
g(t) \equiv \left[ \left( \frac{\kappa L^2}{2} - A^2 \right) - \sqrt{\gamma L^2 + A^2 (A^2 - \kappa L^2)} \right] e^{2 \frac{t}{L}}.
\]

The solution above involves, besides the cosmological constant and \(\kappa\), other quantities hidden in parameter \(\gamma\): the present-day value \(a_0\) of the expansion parameter and the actual value of the radiation density \(\rho_{r0}\). At \(t = 0\) we recover the initial condition \(a(0) = A\), as it should be.

It is a good test of consistency to analyze some limits. (i) In the absence of radiation, \(\gamma = 0\), solution (37) reduces dutifully to the de Sitter solution for \(\kappa \neq 0\), equation (29). (ii) In the limit of no cosmological constant, \(L \to +\infty\), the scale factor (37) is simplified to

\[
a(t) = \sqrt{A^2 + 2\sqrt{\gamma - \kappa A^2} ct - \kappa (ct)^2}.
\]

(iii) Taking only \(\kappa = 0\) in (37), we are left with:

\[
a(t) = \sqrt{\frac{1}{2} \left\{ \left[ A^2 + \sqrt{\gamma L^2 + A^4} \right] e^{2\frac{t}{L}} + \left[ A^2 - \sqrt{\gamma L^2 + A^4} \right] e^{-2\frac{t}{L}} \right\}}.
\]

(iv) If we, in addition the restriction \(\kappa = 0\), perform the limit \(\Lambda \to 0\) (i.e. \(L \to \infty\)), it follows

\[
a(t) = \sqrt{A^2 + 2\sqrt{\gamma} ct},
\]

the conventional solution for a radiation dominated universe, in which \(a \propto t^{1/2}\).

5 NON-RELATIVISTIC MATTER

This section deals with the possible cosmological models in absence of radiation: \(\gamma = 0\). It is divided in two special cases: first, besides \(M \neq 0\), one assumes \(\Lambda \neq 0\) but \(\kappa = 0\); and, then, one considers no cosmological constant, \(\Lambda = 0\), but \(\kappa \neq 0\).
5.1 Matter with $\Lambda \neq 0, \kappa = 0$ (LCDM Model)

Equation (8) can be used to eliminate the term $\epsilon_m$ from (9). The dust approximation $p = 0$ leads then, for $\kappa = 0$, to

$$\dot{H} = \frac{3}{2} \left( \frac{c^2}{L^2} - H^2 \right),$$  \hspace{1cm} (42)

whose integration is straightforward:

$$H(t) = \frac{c}{L} \left[ \frac{c}{L} + H(0) \right] e^{\frac{c}{3} \frac{\dot{H}}{L}} + \left[ \frac{c}{L} - H(0) \right].$$  \hspace{1cm} (43)

The integration constant $H(0)$ is obtained from (20) in terms of previously defined measurable quantities:

$$H^2(0) = \frac{M c^2}{A^3} + \frac{c^2}{L^2}.$$  \hspace{1cm} (44)

The expansion parameter follows from expression (20) for $H^2$:

$$a(t) = \left( M c^2 \right)^{\frac{2}{3}} \left[ H^2(t) - \frac{c^2}{L^2} \right]^{-\frac{1}{3}},$$

or, substituting (43) with the choice of the upper sign,

$$a(t) = \frac{A}{2^2} \left[ 1 + \sqrt{1 + \frac{M L^2}{A^3}} \right] e^{\frac{3}{2} \frac{\dot{H}}{L}} + \left[ 1 - \sqrt{1 + \frac{M L^2}{A^3}} \right] e^{-\frac{3}{2} \frac{\dot{H}}{L}}.$$  \hspace{1cm} (45)

The choice of sign is adequate, as it leads to the appropriate limits: (i) $M \rightarrow 0$ reduces (45) to (28); and (ii) with $L \rightarrow \infty$ (i.e. $\Lambda \rightarrow 0$) we find

$$a(t) = \left( A^{3/2} + \frac{3}{2} \sqrt{M c t} \right)^{\frac{2}{3}},$$

the solution for a matter dominated universe. In terms of still another convenient function,

$$j(t) = \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{M L^2}{A^3}} \right] e^{\frac{3}{2} \frac{\dot{H}}{L}},$$  \hspace{1cm} (46)

solution (45) becomes

$$a(t) = A \left( j(t) - \frac{M L^2}{2 A^3} \frac{1}{j(t)} \right)^{\frac{2}{3}}.$$  \hspace{1cm} (47)
5.2 Matter with $\Lambda = 0$, $\kappa \neq 0$

Equations (20) and (21) are, in this case,

$$\dot{a}^2 + \kappa c^2 - \frac{Mc^2}{a} = 0$$  (48)
$$\ddot{a} + \frac{Mc^2}{2a^2} = 0.$$  (49)

The first equation,

$$\int \frac{ada}{\sqrt{Ma - \kappa a^2}} = c \int dt,$$  (50)

is better solved separately for each value of $\kappa$:

(I) Case $\kappa = 0$ can be used as a test: integration gives $a^{3/2} - A^{3/2} = \frac{3}{2}\sqrt{Mc} t$, just the result of the previous section.

(II) Case $\kappa > 0$, actually $\kappa = 1$: a change of variable puts the integrals into the form

$$\int_{A}^{a} \frac{\sqrt{y} dy}{\sqrt{1 - \frac{y}{M}}} = \sqrt{Mc} \int_{0}^{t} dt.$$  (51)

Integration gives $a(t)$ in an implicit way:

$$ct = \sqrt{MA - A^2} - \sqrt{Ma - a^2} + M \left[ \arcsin \left( \frac{a}{M} \right) - \arcsin \left( \frac{A}{M} \right) \right].$$  (52)

(III) Case $\kappa < 0$, actually $\kappa = -1$: integration now gives

$$ct = \sqrt{Ma + a^2} - \sqrt{MA + A^2} + M \ln \left( \frac{\sqrt{A} + \sqrt{A + M}}{\sqrt{a} + \sqrt{a + M}} \right).$$  (53)

When $M > 0$, a function of $\frac{a(t)}{M}$ turns up:

$$\frac{e^{\sqrt{\frac{a}{M}} + (\frac{\sqrt{a}}{M})^2}}{\sqrt{\frac{a}{M}} + \sqrt{1 + \frac{a}{M}}} = \frac{e^{\sqrt{\frac{A}{M}} + (\frac{\sqrt{A}}{M})^2}}{\sqrt{\frac{A}{M}} + \sqrt{1 + \frac{A}{M}}} e^{\frac{t}{\sqrt{Mc}}}.$$  (54)
This can be rewritten as

\[
\begin{align*}
F(x) &= \frac{e^{\sqrt{x}+1}}{\sqrt{x+1}+x} \\
F\left[\frac{a(t)}{M}\right] &= F\left[\frac{A}{M}\right] e^{\frac{ct}{M}}.
\end{align*}
\]  

(55)

Function \(F(x)\) has actually a very simple, monotonic behavior, and so has \(a(t)\).

We have in this section solved Friedmann’s equations with a dust source \((M \neq 0)\), choosing solutions of inflationary type. Actually, only the cases with \(\text{either } \Lambda \neq 0 \text{ or } \kappa \neq 0\) have been presented. The case with both \(\Lambda \neq 0\) and \(\kappa \neq 0\) is much more involved. In fact, all the remaining classes of solutions are practically as involved as the general case including all the four components \(\Lambda, \kappa, M\) and \(\gamma\). Let us then proceed directly to that case.

6 GENERAL CASE: \(\gamma \neq 0, M \neq 0, \Lambda \neq 0, \kappa \neq 0\)

This \(\Lambda\gamma CDM\) case is described by the complete equations (20,21). We repeat one of them:

\[
\dot{a}(t)^2 + \kappa c^2 - \frac{\gamma c^2}{a(t)^2} - \frac{Mc^2}{a(t)} - \frac{a(t)^2c^2}{L^2} = 0.
\]  

(56)

We shall integrate this first-order equation, which is equivalent to

\[
a(t) \dot{a}(t) = \sqrt{\frac{c^2 a(t)^4}{L^2} + \gamma c^2 + Mc^2a(t) - \kappa c^2a(t)^2}
\]  

(57)

or

\[
\frac{c}{L} \int_0^t dt = \int_A^{a(t)} \frac{a da}{\sqrt{a^4 + \gamma L^2 + ML^2a - \kappa L^2a^2}}
\]  

(58)

The procedure to arrive at the solution is long and rather cumbersome. Let us proceed step by step:

1. Rewrite the integral above in terms of the roots \(\{r_i\}\) of the denominator:

\[
\int \frac{a da}{\sqrt{a^4 + \gamma L^2 + ML^2a - \kappa L^2a^2}} = \int \frac{a da}{\sqrt{(a-r_1)(a-r_2)(a-r_3)(a-r_4)}}
\]
2. Introduce some constants, in terms of which the roots will be expressed later on:

\[ W = \left[ 27(ML^2)^2 - 2(\kappa L^2)^3 + 72\gamma\kappa L^4 \right]^2, \]

\[ U = W + \sqrt{W^2 - 4[(\kappa L^2)^2 + 12\gamma L^2]^3} \]

\[ V = \frac{1}{3} \left\{ 2\kappa L^2 - \left( \frac{U}{2} \right)^{\frac{1}{3}} - \left( \frac{2}{U} \right)^{\frac{1}{3}} [(\kappa L^2)^2 + 12\gamma L^2] \right\}, \]

\[ X = (2\kappa L^2 - V). \] (59)

3. The roots will then be:

\[ r_1 = \frac{1}{2} \left( \sqrt{V} - \sqrt{X - \frac{2ML^2}{\sqrt{V}}} \right), \]

\[ r_2 = \frac{1}{2} \left( \sqrt{V} + \sqrt{X - \frac{2ML^2}{\sqrt{V}}} \right), \]

\[ r_3 = -\frac{1}{2} \left( \sqrt{V} + \sqrt{X + \frac{2ML^2}{\sqrt{V}}} \right), \]

\[ r_4 = -\frac{1}{2} \left( \sqrt{V} - \sqrt{X + \frac{2ML^2}{\sqrt{V}}} \right). \] (60)

4. Solutions will be implicit, and involve elliptic functions of two kinds [14, 15, 16]:

- the elliptic integral of first kind with parameter \( m \) and amplitude \( \phi \):

  \[ F[\phi, m] = \int_0^{\phi} \frac{1}{\sqrt{1 - m \sin^2 \theta}} d\theta. \] (61)

- the elliptic integral of third kind with parameter \( m \), characteristic \( n \) and amplitude \( \phi \):

  \[ \Pi[\phi, n, m] = \int_0^{\phi} \frac{1}{(1 - n \sin^2 \theta)\sqrt{1 - m \sin^2 \theta}} d\theta. \] (62)

5. The characteristic and the parameter turning up will be:

\[ n = \left( \frac{r_2 - r_4}{r_1 - r_4} \right) ; \quad m = \left( \frac{r_1 - r_3}{r_2 - r_3} \right) n. \] (63)

6. The amplitude turning up will be

\[ \phi(a) = \arcsin \sqrt{\frac{a - r_2}{n(a - r_1)}}. \] (64)
7. The implicit solution will then be

\[
\frac{ct}{2L} = \frac{(r_1 - r_2) \left\{ \Pi[\phi(a), n, m] - \Pi[\phi(A), n, m] \right\}}{\sqrt{(r_2 - r_3)(r_1 - r_4)}} + \frac{r_1 \left\{ F[\phi(a), m] - F[\phi(A), m] \right\}}{\sqrt{(r_2 - r_3)(r_1 - r_4)}}.
\]  

(65)

We see from their definitions that both elliptic functions vanish when \( \phi = 0 \). The intricacy of this solution justifies its use only in the general case. It does reduce to the expected particular cases: this can be more easily verified by confronting the plots of each particular case with the corresponding graphics constructed from the general scale factor obtained from (65).

![Figure 1: The scale factor \( a(t)/a_0 \) in terms of \( t \) (in Gyear). The two higher curves are reference models (pure radiation and matter, respectively) exhibiting an overall deceleration. The full curve comes from the \( \Lambda \gamma \)CDM universe, which coincides with the \( \Lambda \)CDM model in the scale used. It presents an inflection point, at which \( \Lambda \) starts dominating and acceleration sets on. The graphs shown in Figure 1 include the general solution (full line, the \( \Lambda \gamma \)CDM Model), and two other reference cases: the radiation dominated Universe (dashed line) and the matter dominated Universe (dotted curve). Following the observational data, the curvature parameter is set to zero for all functions \( a(t), \kappa = 0 \). The values of \( L, M \) and \( \gamma \) are chosen in accordance with the present day values [3, 13]: \( \Omega_{\text{total}} = 1.02 \pm 0.02, \Omega_{b0} = 0.27 \pm 0.04 \) (contribution of baryonic and dark matter) and \( \Omega_{\gamma0} = (4.9 \pm 0.5) \times 10^{-5} \).
and \( H_0 = (71^{+0.004}_{-0.003}) \text{ km sec}^{-1} \text{ Mpc}^{-1} \). Time \( T = ct \) \((c = 1)\) is given in Gyear and the scale factor is measured in units of its present day value \( a_0 \), an arbitrary constant. When considering pure radiation (pure matter), we take \( \Omega_{\gamma 0} = \Omega_{\text{total}} \) \((\Omega_{b0} = \Omega_{\text{total}})\).

The \( \Lambda \gamma CDM \) model gives 13.8 Gyear for the age of the universe. Unlike the two other models, it exhibits an inflection point for not too large a \( z \), which starts the present day acceleration. As expected, the effect of matter and radiation surpasses largely that of the cosmological constant in the early Universe, and a pure radiation model evolves much faster than a matter dominated universe. In the present and future, however, the cosmological constant becomes more important than radiation and matter and dominates the cosmic dynamics.

Another interesting plot is that of matter and \( \Lambda \) \((\text{with } \gamma = \kappa = 0)\), the \( \Lambda CDM \) model. It is in fact also plotted in Figure 1, but coincides with the complete solution in the scale used. At small scales, however, it is clear that the general \( \Lambda \gamma CDM \) model \((\Omega_{b0} = 0.27, \Omega_{\Lambda} = 0.73, \gamma = \kappa = 0)\) differs from the \( \Lambda CDM \) model, which corresponds to the lower curve in Figure 2, which is an enlargement of Figure 1 for early times. The radiation is the dominant component in this primeval period: see the dashed curve close to the vertical axis, or the interval \((t \lesssim 10 \text{ kyear})\) where complete model \(\text{(full line)}\) overcomes the matter one \(\text{(dotted line)}\).

7 FINAL COMMENTS

The solutions given above hold for all times, as long as dark matter is a dust formed by baryons or, at least, when its energy density behaves according to the pure volumetric law \( \epsilon_m = \epsilon_{m0} (1 + z)^3 \). Notice that Eq.(21) is, in time units of \( H_0^{-1} \),

\[
\frac{\ddot{a}}{a} = \Omega_{\Lambda} - \Omega_{\gamma 0} (1 + z)^3 - \frac{1}{2} \Omega_{b0} (1 + z)^3, \tag{66}
\]

so that acceleration will change sign when

\[
\Omega_{\Lambda} = \Omega_{\gamma 0} (1 + z)^4 + \frac{1}{2} \Omega_{b0} (1 + z)^3. \tag{67}
\]

This condition is \( \kappa \)-independent. For the present-day favored values \( \Omega_{\Lambda} = 0.73, \Omega_{b0} = 0.27, \Omega_{\gamma 0} = 4.88 \times 10^{-5} \), this take-over happens rather recently, at \( z \approx 0.75 \), or \( a(t) \approx 0.57 a_0 \). The limitations of the \( \Lambda \gamma CDM \) model can be seen in the estimate value coming up for the cross-over, which seems too high: \((1 + z) = \Omega_{b0}/\Omega_{\gamma 0} \approx 5530\), to be compared with the preferred WMAP value \( z \approx 3234 \).
We have seen that the $\Lambda CDM$ model, described by equation (45), has a simpler form than the general $\Lambda\gamma CDM$ solution, equation (65): the latter gives the scale factor as an implicit function of time, unlike the former. Figure 1 shows also that the $\Lambda CDM$ model coincides with the complete solution at recent times. Nevertheless, the curve of $\Lambda\gamma CDM$ model differs from the one of $\Lambda CDM$ in the early universe (Figure 2), when radiation has a dominant role. Whenever precision is needed, the general solution (65) is to be used. This general solution will be valid as long as we take the combination of ultra-relativistic and non-relativistic ideal particles in the presence of a vacuum energy with equation of state $p = -\epsilon$.

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