Adelic Lefschetz formula for the action of a one-dimensional torus

S. O. Gorchinskiy, A. N. Parshin

1 Introduction

There exists a well-known Lefschetz formula for the number of fixed points in algebraic topology: if \( f : X \to X \) is an endomorphism of a compact oriented \( n \)-dimensional manifold \( X \) with a finite number \( N_f \) of fixed points and the determinant \( \det(1 - df) \) is positive at all the fixed points then

\[
N_f = \sum_{i=0}^{n} (-1)^i \text{Tr}(f^*|_{H^i(X,\mathbb{Q})}).
\]

In algebraic geometry, there exist cohomologies of coherent sheaves on algebraic varieties. So the following natural question arises: what is the Lefschetz number \( \text{Lef}(X, F, f) = \sum_{i=0}^{n} (-1)^i \text{Tr}(f^*|_{H^i(X,F)}) \) of the cohomologies of a coherent sheaf \( F \) on a projective variety \( X \) equal to, if we are given the lift of the endomorphism \( f \) up to the sheaf \( F \)? The answer on this question is given by a coherent (or holomorphic) Lefschetz formula \([1, 7]\).

Suppose that \( F \) is locally free, i.e. corresponds to a vector bundle \( F \), that the set \( Z \) of fixed points is finite, and that the intersection of the graph of the endomorphism with the diagonal is transversal at all the fixed points. Then the coherent Lefschetz formula has the form

\[
\text{Lef}(X, F, f) = \sum_{x \in Z} \frac{\text{Tr}(f|_{F_x})}{\det(1 - df|_{T_x})},
\]

where \( F_x \) is the fiber of the bundle \( F \) over point \( x \) and \( T_x \) is the tangent space to \( X \) at point \( x \).

In the present paper, we consider the case of the action of a one-dimensional torus \( \mathbb{G}_m \) on \( X \) instead of one endomorphism of the variety \( X \). In this case we can consider every summand in the Lefschetz formula, corresponding to a fixed point \( x \in Z \), as the trace of the action of the element \( f \in \mathbb{G}_m \) on the infinite-dimensional space \( \hat{O}_{X,x} \otimes F_x \), where \( \hat{O}_{X,x} \) is the local completed ring of the point \( x \). This remark was done first in \([11]\).

On the other hand, there exists a theory of adelic complexes \( \mathbb{A}_X(F)^* \) \([2, 9]\) for cohomologies of coherent sheaves \( F \). It allows to construct in a canonical way flasque resolutions of coherent sheaves and, hence, to compute their cohomologies. Because of its functoriality, the group \( \mathbb{G}_m \) acts on the adelic complex. For example, if \( X \) is a curve
and the sheaf $\mathcal{F}$ corresponds to the vector bundle $F$ then the complex $\mathbb{A}_X(\mathcal{F})^\bullet$ is equal to

$$
\mathcal{F}_\eta \otimes \prod_x \mathcal{O}_{X,x} \otimes F_x \to \prod_x K_x \otimes F_x
$$

Its cohomologies coincide with $H^\bullet(X, \mathcal{F})$. Here $K_x$ denotes the local field of the point $x$.

Thus, for the computation of the Lefschetz number it should be natural to define the traces of the action of $\mathbb{G}_m$ on each component of the adelic complex. This problem is rather non-trivial because of the fact that the considered linear spaces are infinite-dimensional over the field $k$. If it were solved then since the adelic complex contains $\prod_{x \in \hat{Z}} \mathcal{O}_{X,x} \otimes F_x$ as a summand, the proof of the coherent Lefschetz formula would be immediately reduced to the explanation of why the trace on the rest of the adelic complex is equal to zero. There are two heuristic considerations explaining why this should be true:

- The trace of the action of the group on any field is equal to zero
- The trace of the action of the group on the ”permutational” representation is equal to zero

The second statement is well known for finite groups in the finite-dimensional case while the first one can be reduced to the second one by the normal base theorem from Galois theory.

If we suppose that we have defined the trace of the action of the group on the components of the adelic complex that verifies these properties then it will be easy to see that it should be equal to zero on all the components except those that enter the right hand side part of the Lefschetz formula. The analogous reasoning should also take part in the case of an arbitrary dimension.

A direct realization of this plan seems to be not so simple. In this paper a somewhat roundabout way is proposed and is realized for the case of a locally free sheaf on a non-singular projective variety. Besides, the Lefschetz formula acquires the form

$$
\text{Tr}(\mathbb{G}_m, H^\bullet(X, \mathcal{F})) = \text{Tr}(\mathbb{G}_m, \mathbb{A}_X^{\text{fix}}(\mathcal{F}))
$$

where $\mathbb{A}_X^{\text{fix}}(\mathcal{F})$ is the ”fixed” part of the adelic complex, related to the set of fixed points (see definition 1). It coincides with the product $\prod_{x \in \hat{Z}} \mathcal{O}_{X,x} \otimes F_x$ that was considered above in the case of a curve.

The definition of the complex $\mathbb{A}_X^{\text{fix}}(\mathcal{F})$ is given for any scheme $X$ so it is possible to suppose that the adelic Lefschetz formula is valid for any proper scheme over the field $k$.

## 2 Endomorphisms and actions of algebraic groups

Let $X$ be a nonsingular projective algebraic variety $X$ defined over the field $k$. Let us consider an endomorphism $f : X \to X$ and a coherent sheaf $\mathcal{F}$ on $X$ with a chosen morphism (in the category of coherent sheaves) $\alpha : f^* \mathcal{F} \to \mathcal{F}$. We will call this morphism a lift of $f$ up to $\mathcal{F}$. We obtain an action of $f$ on the cohomologies of the sheaf $\mathcal{F}$ by
taking the composition of the canonical map \( H^i(X, \mathcal{F}) \rightarrow H^i(X, f^*\mathcal{F}) \) with the induced map \( \alpha_* : H^i(X, f^*\mathcal{F}) \rightarrow H^i(X, \mathcal{F}) \).

**Example 1.** If \( \mathcal{F} \) is the cotangent bundle \( \mathcal{T}_X^* \) on \( X \) then we will have the canonical lift \( \alpha : f^*\mathcal{T}_X^* \rightarrow \mathcal{T}_X^* \) that is dual to the differential \( df : \mathcal{T}_X \rightarrow f^*\mathcal{T}_X \).

**Example 2.** Let \( f : \mathbb{P}^N \rightarrow \mathbb{P}^N \) be a projective automorphism. There is a canonical lift on the sheaf \( \mathcal{O}_{\mathbb{P}^N}(N+1) \) since \( \mathcal{O}_{\mathbb{P}^N}(N+1) \cong \wedge^N \mathcal{T}_X \). Thus, for all \( n \in \mathbb{Z} \) there exist lifts \( \alpha_n \) on \( \mathcal{O}_{\mathbb{P}^N}(n(N+1)) \) such that \( \alpha_n \otimes \alpha_m = \alpha_{n+m} \). In particular, \( \alpha_1 \otimes \alpha_{-1} = \text{Id} \), where \( \text{Id} \) is the identity lift on the trivial bundle.

For the situation described above we define the **Lefschetz number** by

\[
\text{Lef}(X, \mathcal{F}, \alpha) = \sum_{i \geq 0} (-1)^i \text{Tr}(f|_{H^i(X, \mathcal{F})}).
\]

If \( f \) acts trivially on the variety \( X \) and on the sheaf \( \mathcal{F} \) as well then \( \text{Lef}(X, \mathcal{F}, f) = \chi(X, \mathcal{F}) \), and the Lefschetz number (as the element of the ground field \( k \)) is given by the Riemann-Roch theorem [5],[3].

**Remark 1.** If we summarize alternatively the Lefschetz numbers of the external powers of the cotangent bundle for a non-singular projective complex variety, then by Hodge decomposition we will get the "classical" Lefschetz number \( \sum_{i \geq 0} (-1)^i \text{Tr}(f|_{H^i(X, \mathcal{C})}) \).

In 1969 Donovan (see [7]) proposed to consider for a given variety \( X \) and its endomorphism \( f \) the category \( \text{Coh}(X, f) \) whose objects are coherent sheaves \( \mathcal{F} \) on \( X \) together with lifts \( \alpha \). The morphisms in this category are defined in a natural way: \( \text{Hom}((\mathcal{F}_1, \alpha_1), (\mathcal{F}_2, \alpha_2)) \) consists of such morphisms of coherent sheaves \( \varphi : \mathcal{F}_1 \rightarrow \mathcal{F}_2 \) that the following diagram is commutative:

\[
\begin{array}{ccc}
  f^*\mathcal{F}_1 & \xrightarrow{f^*\varphi} & f^*\mathcal{F}_2 \\
  \downarrow \alpha_1 & & \downarrow \alpha_2 \\
  \mathcal{F}_1 & \xrightarrow{\varphi} & \mathcal{F}_2
\end{array}
\]

It is possible to consider an analogous category \( \text{LocFr}(X, f) \) where coherent sheaves are replaced by locally free ones.

**Proposition 1.** Let \( f \) be an automorphism and suppose that there exists such a closed embedding of our variety into the projective space \( X \hookrightarrow \mathbb{P}^N \) that \( f \) can be extended to the projective automorphism of the whole \( \mathbb{P}^N \). Then

\[
K_0(\text{Coh}(X, f)) = K_0(\text{LocFr}(X, f)).
\]

**Proof.** The idea of the proof is absolutely the same as for the usual \( K_0(X) \). It follows from example 2 that there exist canonical lifts up to the sheaves \( \mathcal{O}_X(n(N+1)) \). It is enough to show that for any object \( (\mathcal{F}, \alpha) \) of the category \( \text{Coh}(X, f) \) one can construct its finite resolution using the objects of the category \( \text{LocFr}(X, f) \). To do this we twist \( \mathcal{F} \) by \( \mathcal{O}_X(n(N+1)) \) (in the category \( \text{Coh}(X, f) \)) such that \( \mathcal{F}(n(N+1)) \) would be generated by its global sections. Then we get a covering \( p : \bigoplus_i \mathcal{O}_X \cdot e_i \rightarrow \mathcal{F}(n(N+1)) \)
where $e_i$ denote the elements of some base of $H^0(X, \mathcal{F}(n(N + 1)))$. The action of $\alpha$ on $H^0(X, \mathcal{F}(n(N + 1)))$ defines the lift of $f$ up to a trivial bundle $\oplus_i \mathcal{O}_X \cdot e_i$. To get the resolution of the initial sheaf $\mathcal{F}$ we just twist it back and then repeat the described procedure several times for the kernels that will appear. We will stop in a finite number of steps since $X$ is non-singular.

Remark 2. The condition of proposition \[\square\] is equivalent to the existence of an isomorphic lift up to an ample sheaf, i.e. to the existence of an embedding $X \hookrightarrow \mathbb{P}^N = \mathbb{P}(V)$ with the extension of the action of $\mathbb{G}_m$ on $X$ to a linear action of $\mathbb{G}_m$ on $V$ (it is the consequence of example \[\square\]).

If $f$ acts on $X$ identically then the condition of proposition \[\square\] is obviously satisfied. If the sheaf $\mathcal{F}$ is locally free then $\alpha$ defines linear operators on each fiber of the corresponding bundle $\mathcal{F} \to X$. Since the coefficients of the characteristic polynomials of the action on fibers are regular functions on all of $X$, then they are constant and the characteristic polynomials are the same for all fibers. After we have passed to the algebraic closure of the field $k$ (we denote it by the same letter) we can consider the decomposition of a given linear space together with a given operator into Jordan blocks. The decomposition of fibers will also be ”constant over the base”, i.e. the bundle $\mathcal{F}$ decomposes into the sum over eigenvalues:

$$F = \bigoplus_{\lambda} F_{\lambda},$$

where $\lambda \in k$ are the roots of the characteristic polynomial of an arbitrary fiber and $F_{\lambda}$ are maximal subbundles such that the corresponding operator $\lambda \cdot Id - \alpha$ is nilpotent on them.

After we have applied the functor $K_0$, the Jordan blocks decompose in their order into the sum of one-dimensional spaces and we get the isomorphism of rings:

$$K_0(Coh(X, id_X)) = K_0(LocFr(X, id_X)) \cong \bigoplus \chi_{\lambda} F_{\lambda} \otimes \lambda,
\quad (F, \alpha) \mapsto \sum_{\lambda} F_{\lambda} \otimes \lambda,$$

where $Z[k]$ denotes the group ring of the multiplicative semi-group of the field $k$. We need to consider it for two reasons. First, the direct sum of two isomorphic bundles with different lifts is not equal in the $K_0$-group to the bundle itself with the sum of lifts, and, secondly, the tensor product of bundles together with lifts results in pairwise multiplication of their eigenvalues.

There exists a natural homomorphism from the ring $Z[k]$ to the field $k$ defined by the following formula:

$$w : Z[k] \to k
\sum_{\lambda} n_{\lambda} \cdot [\lambda] \mapsto \sum_{\lambda} n_{\lambda} \lambda.$$

Taking the composition of the constructed above isomorphism with the map $\chi \otimes w$, where $\chi : K_0(X) \to Z$ is the Euler characteristic of a sheaf, we obtain the map $\chi_L$:

$$\chi_L : K_0(Coh(X, id_X)) \to Z \otimes_Z k = k
\quad (F, \alpha) \mapsto \sum_{\lambda} \chi(F_{\lambda}) \cdot \lambda.$$
Obviously, 

\[ \text{Lef}(X, \mathcal{F}, id_X) = \chi_L(\mathcal{F}). \]

Thus, we see that for the computation of the Lefschetz number instead of knowing the element of the ring \( K_0(X, id_X) \) itself, it is enough to know its image in \( K_0(X) \otimes Z \) since the Lefschetz number is well defined for the elements of the latter ring.

Now we will formulate a very useful criterion of the invertibility of elements of the ring \( K_0(X) \otimes Z \). Recall that there exists a canonical decomposition \( K_0(X) \cong Z \oplus \widetilde{K}_0(X) \), where the \( Z \)-component corresponds to the virtual dimension of elements of \( K_0(X) \).

Moreover, the ideal \( \widetilde{K}_0(X) \) is nilpotent (see [3]). It follows that the element of \( K_0(X) \) is invertible if and only if its virtual dimension is an invertible integer, i.e. is equal to \( \pm 1 \). Then it is clear that there is an analogous criterion for the ring \( K_0(X) \otimes Z \) where the \( k \)-component will be equal to the trace of the action on an arbitrary fiber of a vector bundle \( \mathcal{F} \) with a lift \( \alpha \). Thus, the element \( (\mathcal{F}, \alpha) \) is invertible if and only if its trace on any fiber is not equal to zero.

Now let us return to the case where \( f \) is not identity. If \( f \) has a finite order then the set of fixed points \( Z \) is the disjoint union of non-singular subvarieties \( Z_\alpha \). Example [1] shows that there is a canonical action of \( f \) on the conormal bundle \( N^*_{Z_\alpha/X} \). Moreover, \( df \) does not act identically on any vector of any fiber \( N^*_{Z_\alpha/X} \). An elementary fact from linear algebra says that for a vector space \( V \) and an operator \( A \) acting on \( V \) the following identity is true:

\[ \det(1_V - A) = \sum_{i=0}^{\dim V} (-1)^i \text{Tr}(A|_{\wedge^i V}). \]

Consequently, our criterion of invertibility implies that the element \( \sum_{i \geq 0} (-1)^i \wedge^i N^*_{Z_\alpha/X} \)

is invertible in the ring \( K_0(Z_\alpha) \otimes W(k) \).

It is possible to show that the element \( \sum_{i \geq 0} (-1)^i \wedge^i N^*_{Z_\alpha/X} \) is invertible in the ring \( K_0(Z_\alpha) \otimes Z \). Donovan proved the following

**Theorem 1.** For an automorphism \( f \) of a finite order and its lift \( \alpha \) up to a coherent sheaf \( \mathcal{F} \), the following identity holds true:

\[ \text{Lef}(X, \mathcal{F}, \alpha) = \sum_\alpha \chi_L(\mathcal{F}|_{Z_\alpha} \otimes (\sum_{i \geq 0} (-1)^i \wedge^i N^*_{Z_\alpha/X})^{-1}), \]

where by the restriction \( \mathcal{F}|_{Z_\alpha} \) we mean the inverse image in \( K_0 \) groups with the induced action, in particular it coincides with the usual restriction \( \mathcal{F}|_{Z_\alpha} \) if the sheaf \( \mathcal{F} \) is locally free.

**Remark 3.** This theorem may be reformulated as follows:

\[ \text{Lef}(X, \mathcal{F}, \alpha) = \text{Lef}(Z, \mathcal{F}|_Z \otimes NL_{Z/X}, \alpha|_Z \otimes df_Z), \]

where \( NL_{Z/X} = (\sum_{i \geq 0} (-1)^i \wedge^i N^*_{Z_\alpha/X})^{-1} \), and \( df_Z \) denotes the canonical lift of \( f \) up to this “bundle”. 

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It is easy to see that if $F$ is locally free, i.e. corresponds to a vector bundle $\mathcal{F}$, and if $Z$ is a finite set then this formula will coincide with the Lefshetz formula that was mentioned in the introduction and was first proved by Atiah and Bott who used analytic methods \[1\]. The proof of Donovan is completely algebraic and does not depend on the properties of the field at all.

Now we consider an algebraic action of an algebraic group $G$ on a projective non-singular variety $X$ over the field $k$. Analogously to the case of the unique morphism, we consider a family of lifts corresponding to each element of the group $G$. The condition saying that it depends "algebraically" on the group may be stated as follows: there exists a morphism of sheaves $\alpha : \mathcal{m}^*F \to \mathcal{p}^*F$, where $\mathcal{m} : X \times G \to X$ is the map that defines the action of the group, and $\mathcal{p} : X \times G \to X$ is the projection on the second component. Thus, the case of the action of the group on the variety may be reduced, in some sense, to the case of the unique morphism. In particular, using it, we define the category $\text{Coh}(X, G)$ of the lifts of the action of the group up to $F$. Morphisms in this category are such morphisms of coherent sheaves $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{m}^*\mathcal{F}_1 & \xrightarrow{\mathcal{m}^*\varphi} & \mathcal{m}^*\mathcal{F}_2 \\
\downarrow \alpha_1 & & \downarrow \alpha_2 \\
\mathcal{p}^*\mathcal{F}_1 & \xrightarrow{\mathcal{p}^*\varphi} & \mathcal{p}^*\mathcal{F}_2
\end{array}
\]

We will also consider the category of lifts on only locally free sheaves $\text{LocFr}(X, G)$. If the variety $X$ is embedded in $\mathbb{P}^N$ and the action of $G$ on $X$ may be extended to the action on $\mathbb{P}^N$, then, like in proposition 2.1, we will get the isomorphisms of rings

$$K_0(\text{Coh}(X, G)) \cong K_0(\text{LocFr}(X, G)).$$

Suppose that $G$ acts on the sheaf $\mathcal{F}$, i.e. that for any two elements $f, g \in G$ we have $\alpha_{fg} = \alpha_f \circ \alpha_g$. Such a sheaf is called $G$-linearized. This may be reformulated in terms of the morphism $\alpha : \mathcal{m}^*\mathcal{F} \to \mathcal{p}^*\mathcal{F}$, giving a 1-cocycle condition on it (for details see \[10\]). For a locally free sheaf this means in terms of the bundle that the group $G$ acts on its space $\mathcal{F}$ (compatible with the action on the base $X$). For every $i$ we get a representation $\rho : G \to \text{End}_k(\mathcal{H}^i(X, \mathcal{F}))$.

Suppose that, like in the case of the unique morphism, the action of $G$ on $X$ is trivial and that the sheaf is locally free. Then we obtain a representation of the group in each fiber of the corresponding bundle. In fact, it is just a formalization of the notion of an "algebraic family of representations of the group $G". It is possible to consider the decomposition of the vector bundle into the sum $\bigoplus_{(\lambda(g))} F_{\lambda(g)} \otimes \lambda(g) \in K_0(X) \otimes_{\mathbb{Z}} k$ over eigenvalues as in the previous section for every element $g \in G$, by passing to the algebraic closure of the field $k$. It would be very nice if this decomposition depended "algebraically" on $g$, i.e. if we could get some element of the ring $K_0(X) \otimes_{\mathbb{Z}} k[G]$, that would reduce to $\bigoplus_{(\lambda(g))} F_{\lambda(g)} \otimes \lambda(g)$ for every $g \in G$. Unfortunately, in the case of an arbitrary group $G$ this decomposition doesn’t exist. However, if $G$ is an algebraic torus $T$, then $F$ may be decomposed into the sum over eigenvalues for all elements of the torus at the same time. Namely,

$$\bigoplus_{\chi} F_{\chi} \otimes \chi \in K_0(X) \otimes k[T],$$

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where $\chi$ runs over characters of the torus $T$ and $F_\chi$ is the biggest subbundle inside $F$ such that the action of any element $t \in T$ on it is the multiplication by $\chi(t)$. This follows from the fact that for any finite-dimensional representation of the torus there exists such an element $t_0 \in T$ that all the characters of this representation are uniquely defined by their values in $t_0$ — this is the algebraic analogue of the dense path on the real torus (cf. [6]). Hence, the decomposition of the bundle for the element $t_0$ will define the decomposition for the whole group. For the bundle $F$ together with the action of the torus $T$ on it we will denote this decomposition by $\psi(F)$.

**Remark 4.** It follows from the formulated criterion for the inversibility of elements of the ring $K_0(X)$ that the class $\psi(F)$ is invertible if and only if its trace on an arbitrary fiber is not identically zero as a function on the torus $T$.

Now let us return to the case of a nontrivial action of the torus on the variety. As in the case of a finite group the following result holds:

**Proposition 2.** The set of fixed points $Z = X^T$ is the disjoint union of non-singular subvarieties $Z_\alpha$.

The proof is given in [4]. Informally speaking, it comes from the fact that the tangent space to a point of $X^T$ coincides with the fiber of the component of $TX|_{X^T}$, on which the torus acts trivially. Its dimension, of course, is constant on each connected component of $X^T$.

**Remark 5.** From what was said before and from remark 4 it follows that the element $\psi(\sum_{i \geq 0} (-1)^i \Lambda^i N_+^*_{Z_\alpha/Z_\alpha^-})$ is invertible in the ring $K_0(X^T) \otimes \mathbb{Z} k(T)$.

The action of the torus also has another property that is very useful for our purposes.

**Proposition 3 (see [10]).** If the torus $T$ acts on the projective variety $X$ then there exists an embedding $X \hookrightarrow \mathbb{P}^N$ and the extension of the action up to $\mathbb{P}^N$ that is compatible with the embedding.

**Corollary 1.** There is an isomorphism of rings for the action of the torus $T$ on a non-singular projective variety $X$:

$$K_0(\text{Coh}(X,T)) \cong K_0(\text{LocFr}(X,T)).$$

From now on we suppose that the torus $T$ is a one-dimensional torus $\mathbb{G}_m$. Since every orbit is a regular map from $\mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$ to the projective variety $X$, it is possible to define the ”limits” $0(x), \infty(x) \in X$ for every point $x \in X$. This allows us to define for every connected component $Z_\alpha$ the set of ”incoming” points $Z_\alpha^+$ consisting of points $x \in X$ such that $0(x) \in Z_\alpha$. In a similar way we define the set of ”outgoing” points $Z_\alpha^-$ as the set of points $x \in X$ such that $\infty(x) \in Z_\alpha$.

**Example 3.** Let $X = \mathbb{P}^1$ and the element $\lambda \in \mathbb{G}_m$ act by the formula $\lambda(x_0 : x_1) = (\lambda x_0 : x_1)$. Then $Z = \{0\} \cup \{\infty\}$, $Z_0^+ = \mathbb{A}^1 = \{x_0 \neq 0\}$, and $Z_\infty^+ = \{\infty\}$.

This example illustrates the general situation.
Proposition 4 (see [4]). The subsets $Z_\alpha^+$ and $Z_\alpha^-$ are locally closed inside $X$ and have the structure of a non-singular algebraic variety. Besides, $X = \cup_\alpha Z_\alpha^+ = \cup_\alpha Z_\alpha^-$ and $Z_\alpha^+ \cap Z_\alpha^- = Z_\alpha$ for every $\alpha$.

Corollary 2. For every $\alpha$ there exists a neighbourhood $U$, containing $Z_\alpha$, such that $Z_\alpha^+$ and $Z_\alpha^-$ are closed inside $U$. For the ideal sheaves of corresponding subvarieties in $U$ the identity $\mathcal{I}_{Z_\alpha} = \mathcal{I}_{Z_\alpha^+} + \mathcal{I}_{Z_\alpha^-}$ holds true.

Consider the action of $\mathbb{G}_m$ on the restriction to $Z_\alpha$ of the tangent bundle $T_X$ of $X$. The non-zero characters of the torus are divided into positive and negative ones, i.e. into the ones that may be continuously extended to 0, and the ones that may be continuously extended to $\infty$, respectively. Then using a natural notation the decomposition of $T_X|_{Z_\alpha}$ over the characters may be written in the following way:

$$T_X|_{Z_\alpha} = (T_X|_{Z_\alpha})_0 \oplus (T_X|_{Z_\alpha})_+ \oplus (T_X|_{Z_\alpha})_-.$$ 

The following result easily comes from our definitions.

Proposition 5. The following equality of bundles holds: $(T_X|_{Z_\alpha})_0 = T_{Z_\alpha}$, $(T_X|_{Z_\alpha})_+ = N_{Z/Z_\alpha^+}$, $(T_X|_{Z_\alpha})_- = N_{Z/Z_\alpha^-}$, where $N_{Z/Z_\alpha^\pm}$ denote the normal bundles on $Z$ to $Z_\alpha^\pm$.

Let us also remark that $N_{Z/X} = N_{Z/Z_\alpha^+} \oplus N_{Z/Z_\alpha^-}$.

3 Adelic construction

For an algebraic variety $X$ over an arbitrary field $k$ one can define the adelic complex $\mathbb{A}_X^\bullet$ [2, 9, 8]. We recall that $\mathbb{A}_X^i$ is the complex of abelian groups and its $i$-th component is equal to

$$\mathbb{A}_X^i = \prod_{\{\eta_0, \ldots, \eta_i\}} '\hat{O}_{\eta_0, \ldots, \eta_i},$$

where the product $\prod'$ is taken over all flags of length $i$, i.e. over such sets of schematic points $\{\eta_0, \ldots, \eta_i\}$ that $\eta_{j+1} \in \eta_j$ and $\eta_{j+1} \neq \eta_j$ for every index $j$. The ring $\hat{O}_{\eta_0, \ldots, \eta_i}$ is defined as follows. Take the completed local ring $\hat{O}_{\eta_0^i, \ldots, \eta_i}$ and complete it on the ideal corresponding to $\eta_{i-1}$ (this ideal may not remain prime). Using this construction we obtain the ring $\hat{O}_{\eta_{i-1}, \eta_i}$. Then we repeat this procedure $i-1$ times for other schematic points in the flag and obtain the ring $\hat{O}_{\eta_0, \ldots, \eta_i}$.

The sign $'$ means that instead of considering not the whole direct product over flags we consider a subset inside it satisfying some special adelic condition [2, 9, 8]. The differential $d : \mathbb{A}_X^i \to \mathbb{A}_X^{i+1}$ is defined by the usual formula:

$$(da)_{\eta_0, \ldots, \eta_i} = \sum_{j=0}^{i} (-1)^j a_{\eta_0, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_i}.$$ 

Moreover, it is possible to define the adelic complex $\mathbb{A}_X(F)^\bullet$ of a (quasi)coherent sheaf $F$ on $X$ in a similar way. To do this we use modules $F_{\eta_0, \ldots, \eta_i}$ over rings $\hat{O}_{\eta_0, \ldots, \eta_i}$ that are
sequential completions of the module $\mathcal{F}_{\eta_i}$. Many notions of algebraic geometry can be reformulated in terms of higher-dimensional adeles. For instance, Serre duality, Chern classes, Chow groups and intersection numbers \cite{8}. In particular, the following general statement is true.

**Theorem 2.** There exists a functorial isomorphism $H^*(X, \mathcal{F}) \cong H^*(\mathbb{A}_X(\mathcal{F})^\bullet)$.

Consider the action of a one-dimensional torus $\mathbb{G}_m$ on a projective non-singular variety $X$ and a $\mathbb{G}_m$-linearized sheaf $\mathcal{F}$ on $X$.

**Definition 1.** We denote by $\mathbb{A}^\text{fix}_X(\mathcal{F})^\bullet$ the quotient of the adelic complex whose $i$-th component is equal to

$$\mathbb{A}^\text{fix}_X(\mathcal{F})^i = \prod_{\{\eta_0, \ldots, \eta_i\}^\prime} \hat{\mathcal{O}}_{\eta_0, \ldots, \eta_i},$$

where the product is taken over all flags that lie inside the set of fixed points $X^{\mathbb{G}_m}$.

**Example 4.** Suppose that we have finitely many fixed points. Then the complex $\mathbb{A}^\text{fix}_X(\mathcal{F})^\bullet$ consists of only one component of degree zero. It will be equal to the sum of completed local rings of fixed points:

$$\mathbb{A}^\text{fix}_X(\mathcal{F})^\bullet = \mathbb{A}^\text{fix}_X(\mathcal{F})^0 = \bigoplus_{x \in \mathbb{Z}} \hat{\mathcal{O}}_x.$$

Since the ideal sheaves $\mathcal{I}_{Z^+}$ and $\mathcal{I}_{Z^-}$ are well-defined in some open subset $U$ containing $Z$, we get the bifiltration on the complex $\mathbb{A}^\text{fix}_X(\mathcal{F})^\bullet$:

$$\mathbb{A}_{p,q}^\bullet = \mathbb{A}^\text{fix}_X(\mathcal{F}(\mathcal{I}_{Z^+})^p(\mathcal{I}_{Z^-})^q)^\bullet.$$

The quotients of this bifiltration may be found explicitly as shown below.

**Proposition 6.** Suppose that $\mathcal{F}$ is locally free. Then

$$\mathbb{A}_{p,q}/(\mathbb{A}_{p+1,q} + \mathbb{A}_{p,q+1}) \cong \mathbb{A}_Z((\mathcal{F}|_Z \otimes \text{Sym}^p(N^*_Z/Z^+_Z) \otimes \text{Sym}^q(N^*_Z/Z^-_Z))^\bullet).$$

**Proof.** In fact, the bifiltration is defined not only for adelic complex but also for the corresponding sheaf itself (on the open subset $U$). On the other hand the taking of the quotient of a finitely generated module $M$ over the ideal $I$ commutes with the completion:

$$\left(\lim_{\leftarrow} M/m^i M\right)/(I \hat{R}_m) = \lim_{\leftarrow} M/((m^i + I)M).$$

Consequently, it is enough to prove the required identity for the sheaf.

Since $\mathcal{F}$ is locally free we have

$$\mathcal{T}^p_{Z^+_Z} \mathcal{T}^q_{Z^-_Z} = \mathcal{F} \otimes \mathcal{T}^p_{Z^+_Z} \mathcal{T}^q_{Z^-_Z}.$$

Because $Z^+_Z, Z^-_Z$, and their "infinitesimal neighbourhoods" intersect transversely, we get $\text{Tor}_i^{\mathcal{O}_Z}(\mathcal{O}_Z/\mathcal{T}^p_{Z^+_Z}, \mathcal{O}_Z/\mathcal{T}^q_{Z^-_Z}) = 0$ for $i > 0$. Using the long exact sequence for $\text{Tor}$-groups, it is easy to see that $\text{Tor}_i^{\mathcal{O}_Z}(\mathcal{T}^p_{Z^+_Z}, \mathcal{T}^q_{Z^-_Z}) = 0$ for $i > 0$. From this it follows that

$$\mathcal{T}^p_{Z^+_Z} \mathcal{T}^q_{Z^-_Z} \cong \mathcal{T}^p_{Z^+_Z} \otimes \mathcal{T}^q_{Z^-_Z}.$$
Finally,
\[ \mathcal{F} \mathcal{T}_{Z_a}^p = \mathcal{T}_Z^p / \left( \mathcal{F} \mathcal{T}_{Z_a}^{p+1} \mathcal{T}_Z^p + \mathcal{F} \mathcal{T}_{Z_a}^p \mathcal{T}_Z^{p+1} \right) = \mathcal{F} \mathcal{T}_{Z_a}^p \mathcal{T}_Z^p \mathcal{O}_U / (\mathcal{I}_Z^p + \mathcal{I}_Z^{p+1}) \cong \mathcal{F}|_{Z_a} \otimes \text{Sym}^p(N_{Z_a/Z_a}^p |_{Z_a}) \otimes \text{Sym}^q(N_{Z_a/Z_a}^q |_{Z_a}). \]

Here we use the fact that if the quotient \( R/I \) of the local regular ring \( R \) remains regular, then \( I^p/I^{p+1} \cong \text{Sym}^p_{R/I}(I/I^2) \). The proof is finished. \( \square \)

Let us remark that the torus \( \mathbb{G}_m \) acts on all the complexes defined above: \( \mathbb{A}_X(\mathcal{F})^*, \mathbb{A}_X^{fix}(\mathcal{F})^*, \mathbb{A}_p^*, \mathbb{A}_q^*, \mathbb{A}_Z((\mathcal{F}|_Z \otimes \text{Sym}^p(N_{Z_a/Z_a}^p |_{Z_a}) \otimes \text{Sym}^q(N_{Z_a/Z_a}^q |_{Z_a})))^* \).

Let \( V^* \) be a bounded complex of vector spaces over \( k \) with finite-dimensional cohomologies. Let us define the trace \( \text{Tr}(\mathbb{G}_m, V^*) \) of the action of \( \mathbb{G}_m \) on \( V^* \) to be a regular function on \( \mathbb{G}_m \) that is equal to the sum
\[ \text{Tr}(\mathbb{G}_m, V^*) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(\mathbb{G}_m|_{H^i(V^*)}). \]

The problem is that this number is not well-defined by this formula if the cohomologies of the complex \( V^* \) are infinite-dimensional. To define the trace in this situation we embed the field of rational functions on \( \mathbb{G}_m \) in the completed local field of some point lying on the compactification of the torus \( \mathbb{P}^1 = \mathbb{G}_m \cup \{0\} \cup \{\infty\} \). Now we can sum series of rational functions on the torus and define the trace as the sum of the traces on the quotients of some suitable filtration. In this manner one can extend the notion of the trace to the case of complexes with infinite-dimensional cohomologies. However, in this case the trace may become a rational function on the torus.

**Example 5.** Let the complex consist of only one component — a complete one-dimensional local ring \( \mathcal{O} \) with a local parameter \( t \), and with the action of the torus \( \lambda : t \mapsto \lambda t \) where \( \lambda \in \mathbb{G}_m \). We have the filtration on \( \mathcal{O} \) by the degrees of the maximal ideal \( \mathcal{O} \supset t\mathcal{O} \supset t^2\mathcal{O} \supset \ldots \) while the torus acts on the \( i \)-th quotient of the filtration \( t^i\mathcal{O}/t^{i+1}\mathcal{O} \) by multiplying it by \( \lambda^i \). Therefore the trace of the action of \( \mathbb{G}_m \) on \( \mathcal{O} \) should be equal to the series \( \sum_{i \geq 0} \lambda^i = \frac{1}{1-\lambda} \) that converges in the completed local ring \( \hat{\mathcal{O}}_{\mathbb{P}^1,0} \supset k[\mathbb{G}_m] \).

**Example 6.** We join example 4 with example 5 and see that for the action from example 4 the trace of the action of \( \mathbb{G}_m \) on \( \mathbb{A}_p^{fix}(\mathcal{O}_{\mathbb{P}^1})^* \) is equal to
\[ \frac{1}{1-\lambda} + \frac{1}{1-\lambda^{-1}} = 1. \]

On the other hand the Lefschetz number \( \text{Lef}(\mathbb{P}^1, \mathcal{O}_X, \mathbb{G}_m) \) is also equal to 1: the sheaf \( \mathcal{O}_X \) has only one nontrivial cohomology \( H^0 \), which consists of constants on which the action of \( \mathbb{G}_m \) is the identity.

This example leads us to the following theorem.

**Theorem 3.** Let the sheaf \( \mathcal{F} \) be locally free. Then

1. For every natural \( p \) the series \( \sum_{q \geq 0} \text{Tr}(\mathbb{G}_m, \mathbb{A}_{p,q}^*/(\mathbb{A}_{p+1,q}^* + \mathbb{A}_{p+q+1}^*)) \) converges in the complete ring \( \hat{\mathcal{O}}_{\mathbb{P}^1,\infty} \) to the rational function \( \text{Tr}_p \in k(\mathbb{G}_m) \);
2. The series \( \sum_{p \geq 0} \text{Tr}_p \) converges in \( \hat{O}_{\mathbb{P}^1,0} \) to the regular function \( \text{Tr} \in k[\mathbb{G}_m] \);

3. The following equality is true:

\[
\text{Tr} = \text{Lef}(Z, \mathcal{F}|_Z \otimes NL_{Z/X}, \mathbb{G}_m).
\]

**Proof.** From the very beginning we pass to the algebraic closure of the field \( k \). By proposition 6 and theorem 2 we have the identity

\[
\text{Tr}_{p,q} = \text{Tr}(\mathbb{G}_m, A^p_{p,q}/(A^p_{p+1,q} + A^p_{p,q+1})) = \text{Lef}(Z, \mathcal{F}|_Z \otimes \text{Sym}^p(N^*_Z/Z^+_a) \otimes \text{Sym}^q(N^*_Z/Z^-_a), \mathbb{G}_m).
\]

Thus,

\[
\text{Tr}_{p,q} = \sum_{\alpha} \chi_L(\mathcal{F}|_{Z_{a}} \otimes \text{Sym}^p(N^*_Z/Z^+_a) \otimes \text{Sym}^q(N^*_Z/Z^-_a)).
\]

Consider the ring \( K_0(X) \otimes \hat{O}_{\mathbb{P}^1,\infty} \). We define the discrete valuation \( \| \cdot \| \) on this ring by the formula

\[
\| \sum_i F_i \otimes \lambda^{n_i} \| = \rho^{-\min_i \{n_i\}},
\]

where \( \rho \) is an arbitrary real number bigger than 1. It is easy to check that \( \| \cdot \| \) satisfies all the conditions of a discrete valuation and defines the structure of a metric space on the ring \( K_0(X) \otimes \hat{O}_{\mathbb{P}^1,\infty} \). Let us denote its completion by \( \hat{R} \). There exists a continuous map \( f : \hat{R} \to \hat{O}_{\mathbb{P}^1,\infty} \), that sends every element \( a \in K_0(X) \otimes \hat{O}_{\mathbb{P}^1,\infty} \) to a function in \( \hat{O}_{\mathbb{P}^1,\infty} \) by the formula

\[
f : a \mapsto \sum_{\alpha} \chi_L(\mathcal{F}|_{Z_{a}} \otimes \text{Sym}^p(N^*_Z/Z^+_a) \otimes a).
\]

Now let us prove the following statement.

**Lemma 1.** The following identity is true in the topological ring \( \hat{R} \):

\[
\sum_{q \geq 0} \text{Sym}^q F \otimes t^q = (\sum_{i \geq 0} (-1)^i \wedge^i F \otimes t^i)^{-1}
\]

for any vector bundle \( F \), where \( t \in \hat{O}_{\mathbb{P}^1,\infty} \) is any element that has a positive valuation at \( \infty \).

**Proof.** First, this series converges in \( \hat{R} \) by the definition of a discrete valuation on \( K_0(X) \otimes \hat{O}_{\mathbb{P}^1,\infty} \) and by the Cauchy criterion. Secondly, for any natural \( k \geq 1 \) there exists a well-known exact sequence

\[
0 \to \wedge^k F \to \wedge^{k-1} F \otimes F \to \wedge^{k-2} F \otimes \text{Sym}^2 F \to \ldots \to F \otimes \text{Sym}^{k-1} F \to \text{Sym}^k F \to 0
\]

This implies that

\[
(\sum_{q \geq 0} \text{Sym}^q F \otimes t^q) \cdot (\sum_{i \geq 0} (-1)^i \wedge^i F \otimes t^i) = 1.
\]
Now let us recall that $G_m$ acts on $N_{Z_\alpha/Z_{\alpha}}^*$ only with negative characters. Therefore we may decompose $N_{Z_\alpha/Z_{\alpha}}^*$ into a sum over characters

$$N_{Z_\alpha/Z_{\alpha}}^* = \bigoplus_{\chi-} (N_{Z_\alpha/Z_{\alpha}}^*)^\chi,$$

and apply lemma \[\text{Lemma 1}\] to every summand. Besides, since

$$\sum_{q \geq 0} \psi(Sym^q N_{Z_\alpha/Z_{\alpha}}^*) = \prod_{\chi-} \sum_{q \geq 0} Sym^q (N_{Z_\alpha/Z_{\alpha}}^*)^\chi \otimes \chi^q$$

and

$$\sum_{i \geq 0} (-1)^i \psi(\wedge^i N_{Z_\alpha/Z_{\alpha}}^*) = \prod_{\chi-} \sum_{i \geq 0} (-1)^i \wedge^i (N_{Z_\alpha/Z_{\alpha}}^*)^\chi \otimes \chi^i,$$

we obtain

$$Tr_p = \sum_{q \geq 0} f(\psi(Sym^q N_{Z_\alpha/Z_{\alpha}}^*)) = f(\sum_{q \geq 0} \psi(Sym^q N_{Z_\alpha/Z_{\alpha}}^*)) = f(\sum_{i \geq 0} (-1)^i \wedge^i (N_{Z_\alpha/Z_{\alpha}}^*)^{-1}) = \sum_{\alpha} \chi_L(\mathcal{F}|Z_\alpha \otimes Sym^p(N_{Z/Z_\alpha}^*) \otimes \sum_{i \geq 0} (-1)^i \wedge^i (N_{Z_\alpha/Z_{\alpha}}^*)^{-1}),$$

In particular, by remark \[\text{Remark 5}\] $Tr_p$ belongs to the field $k(G_m)$.

The same reasoning shows that

$$\sum_{p \geq 0} Tr_p = \sum_{\alpha} \chi_L(\mathcal{F}|Z_\alpha \otimes \sum_{i \geq 0} (-1)^i \wedge^i (N_{Z_\alpha/Z_{\alpha}}^*)^{-1} \otimes \sum_{i \geq 0} (-1)^i \wedge^i (N_{Z_\alpha/Z_{\alpha}}^*)^{-1}) =$$

$$= \sum_{\alpha} \chi_L(\mathcal{F}|Z_\alpha \otimes \sum_{i \geq 0} (-1)^i \wedge^i (N_{Z_\alpha/Z_{\alpha}}^*)^{-1}) = \text{Lef}(Z, \mathcal{F}|Z \otimes NL_{Z/X}, G_m).$$

Thus, the required indentity is proved. \[\square\]

Now it is natural to give

**Definition 2.**

$$Tr(\mathcal{G}_m, A_X^{fix}(\mathcal{F})) = \text{Lef}(Z, \mathcal{F}|Z \otimes NL_{Z/X}, G_m).$$

**Remark 6.** The Lefschetz formula, written for the action of $G_m$, has in this notation a very compact form:

$$Tr(\mathcal{G}_m, A_X(\mathcal{F})) = Tr(\mathcal{G}_m, A_X^{fix}(\mathcal{F})).$$

In what follows we will prove this formula.

**Remark 7.** We could consider an arbitrary coherent sheaf $\mathcal{F}$ on $X$ instead of a locally free one, construct the corresponding complex $A_{X}^{fix}(\mathcal{F})^\bullet$ and the same bifiltration on it $A_{p,q} = A_{X}^{fix}(\mathcal{F}^{T^p_{Z}, T^q_{Z}})^\bullet$. It is easy to see that in this case as well the quotient of complexes $A_{p,q}/(A_{p,q+1} + A_{p+1,q})$ is equal to $A_{Z}(\mathcal{F}^{T^p_{Z}, T^q_{Z}}/(\mathcal{F}^{T^{p+1}_{Z}, T^q_{Z}} + \mathcal{F}^{T^p_{Z}, T^{q+1}_{Z}}))^\bullet$. So we could consider a biseries

$$\sum_p (\sum_q Tr(\mathcal{G}_m, A_{p,q}/(A_{p,q+1} + A_{p+1,q})))$$

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and suppose that it converges to the function \( \text{Lef}(Z, \mathcal{F}|_Z \otimes NL_{Z/X}, \mathbb{G}_m) \) where by the restriction \( \mathcal{F}|_Z \) we mean the inverse image \( K_0 \) groups, i.e. this is the “usual” restriction of some flat resolution of \( \mathcal{F} \) on \( X \). The convergence is supposed for one index after another in the rings \( \mathcal{O}_{\mathbb{P}^1, 0} \) and \( \mathcal{O}_{\mathbb{P}^1, \infty} \) respectively (as in theorem \( 3 \)). In this case the formula from remark \( 8 \) would make sense for a coherent sheaf \( \mathcal{F} \) as well.

4 The exactness of the trace on bifiltrated complexes

The main statement of this section is the following

**Theorem 4.** Consider a \( \mathbb{G}_m \)-equivariant embedding \( i : Y \hookrightarrow X \) of non-singular projective varieties and a locally free \( \mathbb{G}_m \)-equivariant sheaf \( \mathcal{F} \) on \( Y \). Let us construct a locally free resolution consisting of \( \mathbb{G}_m \)-equivariant sheaves to the coherent sheaf \( i_* \mathcal{F} \) on \( X \):

\[
0 \to \mathcal{P}_n \to \ldots \to \mathcal{P}_0 \to i_* \mathcal{F} \to 0.
\]

Then

\[
\sum_{i=0}^{n} (-1)^i \text{Tr}(\mathbb{G}_m, A^{fix}_X(\mathcal{P}_i)^\bullet) = \text{Tr}(\mathbb{G}_m, A^{fix}_Y(\mathcal{F})^\bullet).
\]

**Remark 8.** In the natural notation the identities \( Z^{(\pm)}_Y = Y \cap Z^{(\pm)}_X \) are true. These identities are also true for tangent spaces (this follows from the explicit description of tangent spaces to the varieties \( Z^{(\pm)} \) in terms of the characters of the torus). Hence, the images of the ideal sheaves \( \mathcal{I}_{Z^{(\pm)}_X} \) under the natural map \( \mathcal{O}_X \to \mathcal{O}_Y \) are equal to \( \mathcal{I}_{Z^{(\pm)}_Y} \).

**Proof of theorem 4.** Consider an exact sequence

\[
0 \to A^{fix}_X(\mathcal{P}_n)^\bullet \to \ldots \to A^{fix}_X(\mathcal{P}_0)^\bullet \to A^{fix}_X(\mathcal{F})^\bullet \to 0.
\]

We have to prove the exactness of the trace, defined as the double limit over bifiltrations.

For this purpose we consider the quotient of complexes

\[
\mathcal{C} \colon 0 \to A^{fix}_X(\mathcal{P}_n/\mathcal{I}_{Z^+_X}^{p} + \mathcal{I}_{Z^-_X}^{q})^\bullet \to \ldots \to A^{fix}_X(\mathcal{P}_0/\mathcal{I}_{Z^+_X}^{p} + \mathcal{I}_{Z^-_X}^{q})^\bullet \to A^{fix}_X(\mathcal{F}/\mathcal{I}_{Z^+_X}^{p} + \mathcal{I}_{Z^-_X}^{q})^\bullet \to 0,
\]

where \( Z = Z_X = X^{\mathbb{G}_m} \) and \( \mathcal{I}_{Z^{\pm}} \) denote the ideal sheaves of incoming and outgoing components inside \( X \) that are defined in some neighbourhood of \( Z \). This sequence of complexes is not exact but has finite-dimensional cohomologies. Let \( \text{Tr}^{p,q}_{p,q} \) denote the alternated trace of the action of the torus \( \mathbb{G}_m \) on this cohomologies. Remark \( 8 \) implies that it is enough to prove that for every natural \( p \geq 0 \) there exists a limit \( \text{Tr}^{q}_{p,q} = \lim_{q \to \infty} \text{Tr}^{q}_{p,q} \) in the local field \( K_{\mathbb{P}^1, \infty} \) which takes a value in \( k(\mathbb{G}_m) \), and that there exists a limit \( \lim_{p \to \infty} \text{Tr}^{q}_{p} \) in the local field \( K_{\mathbb{P}^1, 0} \) that is equal to zero.

Since the trace of the action of \( \mathbb{G}_m \) on the bicomplex \( \mathcal{C} \) is just the "ordinary" trace of the action on finite-dimensional cohomologies, we may consider the "horizontal" cohomologies \( \mathcal{C} \) and compute the trace of the action of \( \mathbb{G}_m \) on its cohomologies. Explicitly, "horizontal" cohomologies are equal to

\[
A^{fix}_X(\text{Tor}_i^{\mathcal{O}_X}(i_* \mathcal{F}, \mathcal{O}_X/(\mathcal{I}_{Z^+_X}^{p} + \mathcal{I}_{Z^-_X}^{q}))^\bullet,
\]

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where $i \geq 1$. From the properties of the functor $\mathcal{T}or$ it immediately follows that $\mathcal{T}or_{i}^{O,X}(i_{*}\mathcal{F}, \mathcal{O}_{X}/(\mathcal{I}_{Z+}^{p} + \mathcal{I}_{Z-}^{q}))$ is supported inside $Z_{Y} = Y^{G_{m}}$. Thus, we see that

$$
\text{Tr}'_{p,q} = \sum_{i=1}^{n}(-1)^{i+1} \chi_{L}(\mathcal{L}_{i}^{p,q}),
$$

where $\mathcal{L}_{i}^{p,q}$ denote the sheaves $\mathcal{T}or_{i}^{O,X}(i_{*}\mathcal{F}, \mathcal{O}_{X}/(\mathcal{I}_{Z+}^{p} + \mathcal{I}_{Z-}^{q}))$. Therefore, analogously to the proof of theorem 3, we need to prove that for every $i \geq 1$ there exists a limit $L_{i}^{p} = \lim_{q \to \infty} \text{ch}_{L}(\mathcal{L}_{i}^{p,q})$ in the topological ring $K_{0}(X) \otimes \mathcal{O}_{P^{1},\infty}$ which takes a value in $K_{0}(X) \otimes k(\mathbb{G}_{m})$ and also that there exists a limit $\lim_{q \to \infty} L_{i}^{q}$ in the topological ring $K_{0}(X) \otimes \hat{\mathcal{O}}_{P^{1},0}$, which is equal to 0.

To begin with, we consider the fibers of the sheaves $\mathcal{L}_{i}^{p,q}$ over an arbitrary point $y \in Z_{Y}$. Since $\mathcal{F}$ is locally free, we have an equality of $\mathbb{G}_{m}$-modules

$$(\mathcal{L}_{i}^{p,q})_{y} = \text{Tor}_{i}^{O,X,y}(\mathcal{F}_{y}, \mathcal{O}_{X,y}/(\mathcal{I}_{Z+}^{p} + \mathcal{I}_{Z-}^{q})) = \mathcal{F}|_{y} \otimes \text{Tor}_{i}^{O,X,y}(\mathcal{O}_{Y,y}, \mathcal{O}_{X,y}/(\mathcal{I}_{Z+}^{p} + \mathcal{I}_{Z-}^{q})),
$$

where $\mathcal{F}|_{y}$ denotes the fiber over $y$ of the vector bundle on $Y$ that corresponds to the locally free sheaf $\mathcal{F}$.

Let us make the explicit calculations of the second factor. Consider the following local rings: $R_{+} = \hat{\mathcal{O}}_{Z+,y}$, $R_{-} = \hat{\mathcal{O}}_{Z-,y}$, $R_{0} = \hat{\mathcal{O}}_{Z,y}$, $R = R_{+} \otimes_{k} R_{-} \otimes_{k} R_{0}$, the maximal ideals $I_{+} \subset R_{+}$, $I_{-} \subset R_{-}$, and also the ideals of the subvariety $Y$ in the corresponding local rings $J_{+} \subset R_{+}$, $J_{-} \subset R_{-}$ and $J_{0} \subset R_{0}$. It is clear that $\hat{R} = \hat{\mathcal{O}}_{X,y}$, and that the completion of the ring $R_{+}/J_{+} \otimes_{k} R_{-}/J_{-} \otimes_{k} R_{0}/J_{0}$ coincides with $\hat{\mathcal{O}}_{X,y}$. Since the completion is an exact functor it commutes with the application of Tor. Let us also remark that the completion of a finite-dimensional vector space is the space itself. These two facts imply that

$$
\text{Tor}_{i}^{O,X,y}(\mathcal{O}_{Y,y}, \mathcal{O}_{X,y}/(\mathcal{I}_{Z+}^{p} + \mathcal{I}_{Z-}^{q})) = \text{Tor}_{i}^{\hat{\mathcal{O}},y}(\hat{\mathcal{O}}_{Y,y}, \hat{\mathcal{O}}_{X,y}/(\mathcal{I}_{Z+}^{p} + \mathcal{I}_{Z-}^{q})) =
$$

$$
= \text{Tor}_{R}(R_{+}/J_{+} \otimes_{k} R_{-}/J_{-} \otimes_{k} R_{0}/J_{0}, R_{+}/I_{Z+}^{p} \otimes_{k} R_{-}/I_{Z-}^{q} \otimes_{k} R_{0}).
$$

All the algebras over the field are flat over it (even infinite-dimensional). So we have the identity

$$
\text{Tor}_{R}(R_{+}/J_{+} \otimes_{k} R_{-}/J_{-} \otimes_{k} R_{0}/J_{0}, R_{+}/I_{Z+}^{p} \otimes_{k} R_{-}/I_{Z-}^{q} \otimes_{k} R_{0}) =
$$

$$
= \text{Tor}_{R}^{(p,q)}(R_{+}/J_{+}, R_{+}/I_{Z+}^{p}) \otimes_{k} \text{Tor}_{R}^{(p,q)}(R_{-}/J_{-}, R_{-}/I_{Z-}^{q}) \otimes_{k} R_{0}/J_{0}.
$$

In particular, we obtain that $\mathcal{L}_{i}^{p,q}$ are locally free sheaves on $Z_{Y}$ (a module over a local ring is free if and only if its completion is free because the completion is an exact functor).

Having considered minimal resolutions of modules $R_{\pm}/I_{\pm}^{(q)}$ over local rings $R_{\pm}$, we see that the valuations at 0 and $\infty$ of the characters of the representation of $\mathbb{G}_{m}$ in $\text{Tor}_{i}^{R_{\pm}}(R_{\pm}/J_{\pm}, R_{\pm}/I_{\pm}^{q})$ are not less than $p$ and $q$ respectively. Thus, for every fixed $p$ and for sufficiently large $q$ the locally free sheaves $\mathcal{L}_{i}^{p,q}$ can be decomposed into a sum over the characters of a "rather large" order at $\infty$ and a sum over all the remaining characters:

$$
\mathcal{L}_{i}^{p,q} = (\mathcal{L}_{i}^{p,q})_{1} \oplus (\mathcal{L}_{i}^{p,q})_{2}.
$$
At the level of completed fibers it corresponds to the decomposition

\[(\hat{\mathcal{L}}_{\mathbf{g}})_{y} = \oplus_{k>0}(\mathcal{F}|_{y} \otimes_{k} \text{Tor}^{R_{+}}_{-k}(R_{+}/J_{+}, R_{+}/I_{+}^{n}) \otimes_{k} \text{Tor}^{R_{-}}_{k}(R_{-}/J_{-}, R_{-}/I_{-}^{n}) \otimes_{k} R_{0}/J_{0} \oplus \mathcal{F}|_{y} \otimes_{k} \text{Tor}^{R_{+}}_{-k}(R_{+}/J_{+}, R_{+}/I_{+}^{n}) \otimes_{k} R_{-}/J_{-} \otimes_{k} I_{-}^{n} \otimes_{k} R_{0}/J_{0}).\]

Thus, similarly to the proof of theorem, it follows from lemma that the limit \(\lim_{q \to \infty} \text{ch}(\mathcal{L}_{\mathbf{g}}^{p,q})_{1} = 0\) is equal to zero in the ring \(K_{0}(X) \otimes \mathcal{O}_{\mathbb{P}^{1},0}\), and that there exists a limit \(\lim_{q \to \infty} \text{ch}(\mathcal{L}_{\mathbf{g}}^{p,q})_{2}\) that belongs to \(K_{0}(X) \otimes k(G_{m})\). Moreover, it tends to zero when \(p \to \infty\) in the ring \(K_{0}(X) \otimes \mathcal{O}_{\mathbb{P}^{1},0}\). This finishes the proof of theorem. \(\square\)

5 Main theorem and its proof

Now we have developed our machinery sufficiently to prove the main statement of this paper.

**Theorem 5.** For every locally free \(G_{m}\)-linearized sheaf \(\mathcal{F}\) on \(X\) the following equality is true:

\[\text{Tr}(G_{m}, \mathbb{A}_{X}(\mathcal{F})^{\bullet}) = \text{Tr}(G_{m}, \mathbb{A}_{X}^{fix}(\mathcal{F})^{\bullet}).\]

**Proof.** By proposition we may assume that we have a \(G_{m}\)-equivariant embedding \(i : X \to \mathbb{P}^{N}\). Besides, we may assume that the action of \(G_{m}\) on \(\mathbb{P}^{N}\) is linear, i.e. that \(G_{m}\) acts on the vector space \(V\) so that \(\mathbb{P}^{N} = \mathbb{P}(V)\), or, equivalently, that the sheaf \(\mathcal{O}_{\mathbb{P}^{N}}(1)\) is \(G_{m}\)-linearized.

Consider the sheaf \(i_{*}\mathcal{F}\) on \(\mathbb{P}^{N}\), together with the lift of the action of \(G_{m}\) from \(X\). The complexes \(\mathbb{A}_{X}(\mathcal{F})\) and \(\mathbb{A}_{X}^{fix}(\mathcal{F})\) do coincide with the complexes \(\mathbb{A}_{\mathbb{P}^{N}}(i_{*}\mathcal{F})\) and \(\mathbb{A}_{\mathbb{P}^{N}}^{fix}(i_{*}\mathcal{F})\) respectively, because, by definition, the fibers of the sheaf \(i_{*}\mathcal{F}\) outside \(X\) are equal to zero. Also, the bifiltration on the complex \(\mathbb{A}_{X}^{fix}(\mathcal{F})\) induced from \(\mathbb{P}^{N}\) coincides with the bifiltration, induced from \(X\): this comes from remark.

We construct a locally free resolution of the sheaf \(i_{*}\mathcal{F}\) on \(\mathbb{P}^{N}\):

\[0 \to \mathcal{P}_{n} \to \ldots \to \mathcal{P}_{0} \to i_{*}\mathcal{F} \to 0\]

As it was shown in theorem, the following equality holds true:

\[\sum_{i=0}^{n}(-1)^{i}\text{Tr}(G_{m}, \mathbb{A}_{X}^{fix}(\mathcal{P}_{i})) = \text{Tr}(G_{m}, \mathbb{A}_{X}^{fix}(\mathcal{F})).\]

Thus, it is enough to prove the statement of the theorem only for locally free sheaves on \(\mathbb{P}^{N}\). Moreover, the construction of the resolution in the proof of proposition implies that it is enough to consider locally free sheaves of the form \(\mathcal{E} = \oplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^{N}}(l)\). Here the action of the torus is just a tensor product of some \(n\)-dimensional representaion \(\rho\) of the torus \(G_{m}\) with the canonical action of \(G_{m}\) on \(\mathcal{O}_{\mathbb{P}^{N}}(l)\). It is clear that

\[\text{Tr}(G_{m}, \mathbb{A}_{\mathbb{P}^{N}}(\mathcal{E})) = \text{Lef}(\mathbb{P}^{N}, \mathcal{E}, G_{m}) = \text{Lef}(\mathbb{P}^{N}, \mathcal{O}_{\mathbb{P}^{N}}(l), G_{m}) \cdot \text{Tr}(\rho),\]

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and
\[ \text{Tr}(\mathbb{G}_m, A^{\text{fix}}_{\mathbb{P}N}(\mathcal{E})) = \text{Lef}((Z, \mathcal{E}|_Z \otimes NL_{\mathbb{P}N}/\mathbb{G}_m, \mathbb{G}_m) \cdot \text{Tr}(\rho), \]
where \( Z = (\mathbb{P}N)_{\mathbb{G}_m}. \) Hence we have reduced the general case to the case of the sheaf \( \mathcal{O}_{\mathbb{P}N}(l). \) Since all the linear representations of the torus are diagonalizable, the variety \( Z_{\mathbb{P}N} \) is just a disjoint union of projective subspaces \( \bigsqcup \mathbb{P}N_\alpha \) corresponding to the characters of the representation of \( \mathbb{G}_m \) in \( V. \) After all these remarks the case of the sheaf \( \mathcal{O}_{\mathbb{P}N}(l) \) is reduced to a straightforward but rather dull calculations that are similar to those in [5] on page 597. We omit further details.

**Corollary 3.** For the action of \( \mathbb{G}_m \) on the smooth projective variety \( X \) and an arbitrary \( \mathbb{G}_m \)-linearized coherent sheaf \( \mathcal{F} \) the following form of the Lefschetz trace formula in the above notation is true:
\[ \text{Lef}(X, \mathcal{F}, \mathbb{G}_m) = \text{Lef}(Z, \mathcal{F}|_Z \otimes NL_{\mathbb{P}N}/\mathbb{G}_m). \]

**Proof.** It is just a reformulation of theorem 5 for locally free sheaves. To prove it for an arbitrary coherent sheaf we remark that both sides of the needed equality are exact on sheaves, and use a locally free resolution of a coherent sheaf.

**Remark 9.** Despite the fact that the Lefschetz formula is true for coherent sheaves it remains impossible to prove that coherent sheaves satisfy the condition of remark 7.

## 6 Appendix: purely non-hyperbolic case

In one special case it is possible to prove that coherent sheaves satisfy the condition of remark 7. In this case the bifiltration on the quotients of adelic complexes turns out to be just an ordinary linear filtration, and it becomes possible to prove the exactness of the trace using a more general fact about filtrated complexes.

**Definition 3.** We say that the action of a one-dimensional torus is purely non-hyperbolic if for any connected component \( Z_\alpha \) of the set of fixed points \( Z \) either \( Z^+_{\alpha} \) or \( Z^-_{\alpha} \) is empty.

In this case the bifiltration on the complex \( A^{\text{fix}}_{\mathbb{P}N}(\mathcal{F})^\bullet \) is just a filtration by powers of either the ideal \( I_{Z^+_{\alpha}} \) or the ideal \( I_{Z^-_{\alpha}}. \)

**Definition 4.** Consider the representation \( \rho \) of the torus \( \mathbb{G}_m \) in an infinite-dimensional linear space \( V \) over the ground field \( k. \) We say that the order of \( \rho \) is at least \( k \) at point \( x \in \mathbb{P}^1 = \mathbb{C}_m, \) if for any \( \mathbb{G}_m \)-invariant subspace \( U \subset V \) and for any \( \mathbb{G}_m \)-equivariant finite-dimensional quotient \( U \rightarrow W \) the orders of all its characters as functions on \( \mathbb{G}_m \) are at least \( k \) at \( x. \)

**Remark 10.** Suppose that we have the action of the torus \( \mathbb{G}_m \) on the complex \( V^\bullet \) of linear spaces over \( k \) and a \( \mathbb{G}_m \)-equivariant filtration \( F^pV^\bullet \) on \( V^\bullet \) so that the orders of the representations \( F^pV^\bullet \) are at least \( p \) at the point \( x \in \mathbb{P}^1, \) and such that the cohomologies of the quotients \( F^pV^\bullet/F^{p+1}V^\bullet \) are finite-dimensional. It comes from our definitions that
the orders of the traces of $G_m$ on the cohomologies of $F^pV^\bullet/F^{p+1}V^\bullet$ are at least $p$. Then the series $\sum_p Tr(G_m, F^pV^\bullet/F^{p+1}V^\bullet)$ converges in the completed local ring $O_{\mathbb{P}^1,x}$. The sum of this series will be denoted by $Tr(G_m, F^\bullet V^\bullet)$.

**Example 7.** Let $X$ be a non-singular projective variety with the action of $G_m$ and $F$ be a locally free $G_m$-linearized sheaf on it. Then, up to an additive constant, the order of the action of $G_m$ on each component of $A_{fix}(F\mathcal{I}_{\mathbb{P}^1,0} + \mathcal{I}_{\mathbb{P}^1,\infty})$ is at least $p$ at $0$ and at least $q$ at $\infty$. This comes from the fact that on each component the bifiltration by the degrees of ideals $\mathcal{I}_{\mathbb{P}^1,\pm}$ induces the bifiltration on any finite-dimensional quotient of any subspace inside this component. Proposition 6 implies that the representation of the torus on the quotients of the initial bifiltration decomposes into the finite direct sum of ”scalar” representations, that is representations such that the torus acts on them by one character only. Besides, the orders of these characters are not less than $p$ at $0$ and not less than $q$ at $\infty$ up to an additive constant (this constant arises because of the characters of the action of $G_m$ on $F|_Z$).

Since every coherent sheaf on $X$ may be $G_m$-equivariantly covered by a locally free sheaf, the present statement remains valid for arbitrary coherent sheaves.

**Corollary 4.** Suppose that the action of the torus on $X$ is purely non-hyperbolic. Then for any coherent sheaf $F$ on $X$ the series

$\sum_p Tr(G_m, A_p^\bullet/A_{p+1}^\bullet)$,

corresponding to the filtration by the powers $\mathcal{I}_{\mathbb{P}^1,\pm}$, converges in the local ring $O_{\mathbb{P}^1,0}$ or $O_{\mathbb{P}^1,\infty}$.

Now we prove the following rather abstract statement.

**Proposition 7.** Let

$0 \to V_0^\bullet \to V_1^\bullet \to \ldots \to V_n^\bullet \to 0$

be an exact sequence of $G_m$-complexes. Suppose we have a $G_m$-equivariant filtration $F^pV_i^\bullet$ such that all together $F^pV_i^\bullet$ form a subcomplex in the initial sequence and satisfy the condition from remark 10 for some point $x \in \mathbb{P}^1$. Then

$\sum_{i=0}^n (-1)^i Tr(G_m, F^\bullet V_i^\bullet) = 0$.

**Proof.** Consider the quotients of the complexes $V_i^\bullet/F^pV_i^\bullet$. Together they form a bicomplex. The spectral sequence, whose first term consists of finite-dimensional cohomologies of these quotients $V_i^\bullet/F^pV_i^\bullet$ with respect to a ”vertical” differential, i.e. the differential from the adelic complex, tends to the cohomologies of the whole bicomplex. Let us remark that it does not matter on what finite-dimensional level of the spectral sequence we compute the trace of $G_m$. Hence, the trace of the torus on the cohomologies of the bicomplex is equal to

$\sum_{i=0}^n (-1)^i \sum_{j=0}^{p-1} Tr(G_m, F^j V_i^\bullet/F^{j+1}V_i^\bullet)$. 

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On the other hand we could compute the cohomologies of the bicomplex using the spectral sequence associated to the "horizontal" differential. The "horizontal" cohomologies of $V_j/F^pV_j$ turn out to be isomorphic to "horizontal" cohomologies of the subcomplex $F^pV_j$ because the initial sequence of complexes was exact. Consequently, the orders of the action of the torus on them, i.e. the orders at $x$ of the action of the torus on the cohomologies of the bicomplex, are at least $p$ up to an additive constant. So, the order at $x$ of the (alternated) sum

$$\sum_{i=0}^{n} (-1)^i \sum_{j=0}^{p-1} \text{Tr}(\mathbb{G}_m, F^jV_i^*/F^{j+1}V_i^*)$$

tends to infinity when $p \to \infty$, and the required statement is proved.

**Corollary 5.** Let the action of the torus on $X$ be purely non-hyperbolic and let

$$0 \to \mathcal{F}_1 \to \ldots \to \mathcal{F}_n \to 0$$

be an exact $\mathbb{G}_m$-equivariant sequence of coherent sheaves on $X$. Then

$$\sum_{i=0}^{n} (-1)^i \text{Tr}(\mathbb{G}_m, A^{fix}_X (\mathcal{F}_i)) = 0.$$ 

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