Classical behaviour in quantum systems: the case of straight tracks in a cloud chamber

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Abstract
The aim of this paper is to discuss in a pedagogical way the problem of the emergence of classical behaviour in certain physical systems which, in principle, are correctly described by quantum mechanics. It is stressed that the limit $\hbar \to 0$ is not sufficient and the crucial role played by the environment must be taken into account. In particular, recalled is the old problem raised by Mott in 1929 (Proc. R. Soc. Lond. A 126 79) concerning the straight tracks observed in a cloud chamber, produced by an $\alpha$-particle emitted by a source in the form of a spherical wave. The conceptual relevance of the problem for a clearer understanding of the classical limit is discussed in a historical perspective. Moreover, a simple mathematical model is proposed, where the result of Mott is obtained in a rigorous mathematical way.

1. Introduction

Quantum mechanics (QM) is a well-established theory formulated in the first half of the last century to account for the behaviour of microscopic systems, i.e. systems in which a typical action is comparable in size with Planck’s constant $\hbar$. Nevertheless in the non-relativistic regime, QM is a universal theory and can also be applied to the macroscopic world or in general to situations where the classical description is valid to a high degree of accuracy. In such cases, a relevant question is to guarantee that the ‘correspondence principle’ holds, i.e. to show that the quantum mechanical description reduces to the classical one through a suitable limit procedure. It is well known that, under appropriate conditions on the initial state of the system, this reduction can be obtained taking, roughly speaking, the limit $\hbar \to 0$. As a matter of fact, in some relevant situations such a procedure turns out to be inadequate, and the role of the environment must be taken into account in order to show how a classical behaviour

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can appear in a quantum system. The corresponding dynamical mechanism is known in the physical literature under the name of decoherence (see e.g. [18]).

In this paper, a general review of the problem is given in a historical and pedagogical perspective, with some emphasis on the conceptual implications of the different proposed approaches. Furthermore, we shall describe the specific case considered by Mott [28] in an old paper of 1929, where a possible heuristic explanation of the straight tracks (i.e. classical trajectories) observed in a cloud chamber is given.

In our opinion, the approach proposed by Mott, which was rather neglected for a long time, is particularly relevant for a clearer understanding of the classical limit and it should be more carefully studied. In this perspective, we shall also mention a first mathematical result obtained in a simplified one-dimensional case, which has recently been proved in [12]. Here we shall limit ourselves to describing the result and giving some comments on it. In particular, our aim is to show a simple example of the mathematical model where, under precise assumptions on the physical parameters of the model, the range of validity of the result is clearly established. We believe that this kind of approach can be useful at a pedagogical level for a deeper comprehension of the problem.

We shall start in section 2 recalling the basic rules of QM with no pretence of full rigor and generality. The only aim is to underline the crucial points of the so-called standard formulation (see e.g. [31]), with a brief discussion about the difficulty arising from the measurement problem and in general from the existence of superposition states for macroscopic objects. Because of lack of space, we will not mention other possible formulations of the theory, such as Bohmian mechanics [13], GRW theory [20] or many worlds formulation [14]. For the reader interested in the conceptual foundation of QM we refer to the books [2, 17].

In section 3, we discuss the problem of the classical limit and the important role of decoherence induced by the environment which results in the dynamical suppression of the quantum correlations present in a superposition state.

In section 4, we shall briefly describe how a cloud chamber works and analyse the problem of the appearance of straight tracks produced by the passage of an α-particle. In particular, we shall describe the explanation proposed by Mott based on the quantum mechanical analysis of the α-particle plus the atoms of the chamber.

In section 5, following the same line of thought, we consider a simplified one-dimensional model of a cloud chamber and we describe a theorem, proved in [12], which can be considered as a mathematical proof, in a simple case, of the heuristic arguments given by Mott. Some final remarks will conclude the paper.

2. Basic rules of QM

As is well known, classical mechanics (CM) provides a description of the motion of a system of n point particles in \( \mathbb{R}^3 \) based at kinematical level on their positions \( x_1, \ldots, x_n \) and momenta \( p_1, \ldots, p_n \). This means that at each time the state of the system is identified by a point in a 6n-dimensional space, called phase space. Given the initial state and the forces acting on the system, in principle it is possible to solve Newton’s equation and determine the state of the system at any other time. Furthermore, a classical observable is represented by a smooth real function defined on the phase space of the system. In particular, the predicted value of an observable \( f \) when the system is in the state \( (x_1, \ldots, x_n, p_1, \ldots, p_n) \) is simply given by \( f(x_1, \ldots, x_n, p_1, \ldots, p_n) \). An important point is that in CM the algebra of observables is commutative and this fact allows a prediction of the value of all the observables relative to a given system.
At the beginning of the 20th century, it became clear that CM, and classical electromagnetism, fail when applied to atomic systems and it was realized that a radically new description was required. The new theory was elaborated in the years 1925–1926 due to the work of Heisenberg and Schrödinger and it was formulated in a rigorous mathematical language by von Neumann in 1932. The crucial point is that QM is based on a new and more abstract kinematics. More precisely, the commutative algebra of observables typical of CM must be replaced by a non-commutative algebra of self-adjoint operators in a suitable Hilbert space and, at each time, the state of the system is given by a vector in the same space. We summarize here the basic rules for a system of \( n \) quantum particles in \( \mathbb{R}^3 \), avoiding generality and technical difficulties and, in particular, neglecting the specific rules required for systems of identical particles.

1. All the information on the system is encoded in the wavefunction \( \psi_t(x_1, \ldots, x_n) \), \( x_j \in \mathbb{R}^3 \), which is an element of the Hilbert space \( L^2(\mathbb{R}^{3n}) \) with \( \|\psi_t\| = 1 \). The wavefunction \( \psi_t \) is referred to as the (pure) state of the system at time \( t \).

2. Given the initial state \( \psi_0 \), the state at time \( t \) is the solution of the Schrödinger equation

\[
\frac{i\hbar}{\partial t} \psi_t = \sum_{j=1}^{n} -\frac{\hbar^2}{2m_j} \Delta_j \psi_t + V(x_1, \ldots, x_n) \psi_t
\]  

(2.1)

with initial datum \( \psi_0 \), where \( m_j \) is the mass of the \( j \)th particle, \( \Delta_j \) denotes the Laplace operator relative to the coordinates of the \( j \)th particle and \( V \) is the interaction potential.

3. An observable relative to the system is represented by a self-adjoint operator in \( L^2(\mathbb{R}^{3n}) \). In a system made of a single quantum particle, simple examples of quantum observables are position and momentum. The position is represented by \( \hat{x}_k \), defined as the multiplication operator by \( x_k \), \( k = 1, 2, 3 \), where \( x_k \) denotes the \( k \)th component of the position of the particle. Analogously the momentum is represented by the differential operator \( \hat{p}_k = -i\hbar \frac{\partial}{\partial x_k} \), \( k = 1, 2, 3 \). One can easily check that the two observables do not commute and in fact, at least formally, they satisfy the Heisenberg commutation relations \( [\hat{x}_k, \hat{p}_l] = i\hbar \delta_{kl} I \), where \( I \) denotes the identity operator.

4. The predictions of the theory are given by Born’s rule and, in general, are of probabilistic type. In the special case of the position observable relative to a system made of one quantum particle, the Born’s rule reduces to

\[
P(x \in \Omega; \psi) = \int_\Omega dx |\psi(x)|^2,
\]  

(2.2)

where the lhs denotes the probability that the position of the particle described by the state \( \psi \) is found in a set \( \Omega \subset \mathbb{R}^3 \). The prescription can easily be extended to the case of observables different from the position making use of the spectral theorem for self-adjoint operators.

We list here some comments on the above rules.

(i) As already mentioned, QM is formulated as a universal theory (in the non-relativistic regime) and in principle it can be used to describe both micro and macro systems.

(ii) The predictions of the theory are in excellent agreement with experiments.

(iii) Except in some special cases, the Born’s rule gives only probabilistic predictions. In particular, formula (2.2) means that the theory can only predict the statistical distribution of the detected positions in a large number of experiments made in identical conditions. We note that a quantum particle, when detected in a single experiment, always appears localized in a well-defined position which, in general, cannot be predicted by the theory.
Since we have assumed that the state $\psi$ gives complete information on the system, the above probability cannot be considered epistemic, as it is common in classical physics. Such a completeness assumption is typical of the standard formulation of QM.

(iv) For a single observable $A$, it is always possible to find a state $\psi$ such that $\Delta_\psi A$ is arbitrarily small, where $\Delta_\psi A$ denotes the mean square deviation of the statistical distribution of the possible values of $A$ in the state $\psi$. This means that the values of the observable $A$ in the state $\psi$ can be predicted with arbitrary accuracy. On the other hand, the non-commutative character of the algebra of observables implies that one cannot predict, with arbitrary accuracy, the value of all the observables relative to a given system. In particular, for a quantum particle in the state $\psi(x)$ the Heisenberg uncertainty principle states

$$\Delta_\psi \hat{x}_k \Delta_\psi \hat{p}_l \geq \frac{\hbar}{2} \delta_{kl}. \quad (2.3)$$

(v) A crucial point of the theory is linearity. This means that if $\psi_1(x)$ and $\psi_2(x)$ are two states, then also the sum $\psi_1(x) + \psi_2(x)$, suitably normalized, is a possible state (superposition principle). This apparently trivial fact has strong physical consequences, due to the fact that the predictions (see (2.2)) are given by a quadratic expression with respect to the state. In particular, the probability density for the position is

$$|\psi_1(x) + \psi_2(x)|^2 = |\psi_1(x)|^2 + |\psi_2(x)|^2 + 2\Re(\psi_1(x)\psi_2(x)). \quad (2.4)$$

From formula (2.4) it is clear that the situation described by $\psi_1(x) + \psi_2(x)$ cannot be considered in any sense as the 'sum' of the situations described by $\psi_1(x)$ and $\psi_2(x)$ separately. In particular, the last term in (2.4) is responsible for the appearance of interference effects, typical of waves, in the statistical distribution of the detected positions in a large number of identical experiments. Such effects can be directly observed in the so-called two-slit experiment.

We note that a completely different situation occurs when we know that the system is in the state $\psi_1(x)$ with probability $p_1$ and in the state $\psi_2(x)$ with probability $p_2$, where $p_1 + p_2 = 1$. In such a case, the probability involved has an epistemic nature and the system is described by a so-called classical statistical mixture of the two pure states $\psi_1(x)$ and $\psi_2(x)$.

(vi) Another important issue is the occurrence of 'entanglement' for systems composed by more than one particle. An entangled state is a state that cannot be factorized in a product of one-particle states. At a kinematical level, this means that if a system is described by an entangled state $\psi(x_1, \ldots, x_n)$, it is not possible to associate a definite (pure) state with each subsystem. The situation is again radically different from the classical case and it is the origin of the 'non-local effects' which can be produced on a subsystem $A$ acting on another spatially separated subsystem $B$.

(vii) It should be stressed that superposition principle and entanglement are the most peculiar aspects of the quantum description. They are responsible for the strange behaviour of the quantum objects but also for the extraordinary success of the theory in a large variety of applications.

(viii) It is clear from the above remarks that the behaviour of a quantum object cannot be reduced to that of a classical particle or a classical wave. In the early days of QM, this fact was interpreted in terms of the so-called wave–particle duality, i.e. it was claimed that a quantum object behaves like a wave or like a particle depending on the given physical situation and the two aspects are mutually exclusive. This duality was considered a special case of a general philosophical view known as Bohr’s complementarity principle. At that time, such a view was a reasonable attempt to understand the new phenomenology in terms of a more traditional intuition based on concepts of classical physics. More than
80 years of the birth of QM, it seems reasonable to accept that the behaviour of a quantum object is something completely different from the behaviour of a classical object. At a psychological level, it would probably be helpful to use a new name, e.g. ‘quanton’, as discussed in [1].

Even in a short introduction to the standard formulation of QM, it is worth mentioning some conceptual difficulties arising in the description of the measurement process.

In order to exemplify the problem, let us consider the simple case of a quantum particle described by the state $\psi_j$, $j = 1, 2$, where $\psi_j$ is a wave packet well localized around the position $x_j$. The quantum particle is supposed to interact with a macroscopic measurement apparatus designed to measure the position of the particle. Before the interaction, the apparatus is described by the state $\Phi_0$, corresponding to the ‘index’ at rest.

Let us assume that after the interaction the system is still in $\psi_j$ and the apparatus is in $\Phi_j$, i.e. a state corresponding to the index in the position $j$ and showing that the particle position is $x_j$. In other words, the situation after the measurement for the whole system + apparatus is described by the product state $\psi_j \otimes \Phi_j$. It is intuitively clear that this is the typical response of an apparatus, and we remark that model interactions between system and apparatus producing situations of this type can be effectively constructed. The problem arises if the system is initially in the superposition state $\psi_1 + \psi_2$. As a pure consequence of the linearity of the evolution, after the interaction the state of system + apparatus is $\psi_1 \otimes \Phi_1 + \psi_2 \otimes \Phi_2$, which is a typical entangled state corresponding to the superposition of $\psi_1 \otimes \Phi_1$ and $\psi_2 \otimes \Phi_2$. In principle, this means that one cannot assign a definite position to the index of the apparatus and some interference effects could be observed for the position of the index, like in the case of the two-slit experiment. Obviously the situation is unsatisfactory since we always see the index of a macroscopic apparatus in a well-defined position.

In order to avoid the above difficulty, one possibility is to invoke an additional rule called wave packet reduction. Roughly speaking the rule simply states that the macroscopic apparatus must behave classically and it cannot be found in a superposition state. From a technical point of view, this means that after the measurement the superposition state $\psi_1 + \psi_2$ instantaneously reduces to the classical statistical mixture of the two possible states $\psi_1 \otimes \Phi_1$ and $\psi_2 \otimes \Phi_2$. The critical aspect is that such reduction cannot be described by a dynamics governed by the Schrödinger equation, and one has to admit that in some situations the Schrödinger dynamics fails and must be replaced by a different dynamical rule.

This is a rather strange fact since, after all, a macroscopic apparatus is made of atoms whose behaviour is described by the Schrödinger equation. Moreover, it is not precisely prescribed when and under which circumstances the wave packet reduction should occur and this introduces a further ambiguity in the application of the rule.

Another possible point of view is to insist that the only existing dynamical rule is the Schrödinger dynamics. In this case, one has to consider specific models of system + apparatus described by the Schrödinger equation. Then the problem becomes to prove that the replacement of the superposition state with the statistical mixture can be done at the cost of an error so small that it would be practically undetectable in a real experiment. Such models have been effectively constructed and some interesting results in this direction are available [23, 30].

This suggests that in practical circumstances the wave packet reduction is an innocuous procedure and the standard formulation of QM can be considered valid for all practical purposes [2].

Nevertheless also with this approach a delicate conceptual difficulty remains. Indeed QM would be a fundamental theory which acquires a precise meaning only through an approximation procedure, although very accurate, depending on the specific model employed.
In conclusion, one has to admit that the description of the measurement process within the standard formulation of QM is not completely satisfactory from a conceptual point of view, and it is not surprising that the question has stimulated an intense epistemological debate which is still active.

3. The classical limit

There are many examples in the history of physics of a theory $T_1$, describing very accurately a certain class of phenomena, which turns out inadequate when applied to an emerging new set of experimental data. In these cases, one tries to construct a new theory $T_2$ for the explanation of the experimental data with the requirement that $T_2$ must be ‘more general’ than $T_1$, i.e. it must reduce to $T_1$ in a suitable limit. The underlying idea is that a physical theory is not only a set of rules useful to predict the results of experiments, but it is also supposed to provide a unified description of a wider and wider portion of the physical reality. A typical example is the special relativity as a Lorentz-invariant generalization of CM. In this case, it is easy to see that the former reduces to the latter in the limit $c \to \infty$, where $c$ is the speed of light. From the technical and conceptual point of view, the limit procedure is greatly simplified by the fact that both theories deal with point particles characterized by position and momentum and evolving in time in ordinary space.

Another example is wave optics as a generalization of geometrical optics, required to account for interference and diffraction effects of light. In this case, the two theories are based on a different spacetime description of light propagation, i.e. as a wave and as a ray respectively. The technical difficulty here is to specify in which sense the propagation of a wave can reproduce the propagation of a ray. This can be done with a natural choice of the initial datum and exploiting the ‘short wavelength’ limit for the corresponding solution of Maxwell equations. After a non-trivial mathematical analysis of the problem, one can conclude that in the limit the laws of geometrical optics are recovered.

The problem of the classical limit of QM is surely more delicate than the previous examples.

Apparently the situation is similar to the case of optics, since we compare a theory based on point particles propagating along trajectories (CM) with a theory based on a kind of ‘wave’ solution of the Schrödinger equation (QM). Nevertheless one realizes that the situation is deeply different if one observes that the wave representing the quantum state does not describe a physical object distributed in ordinary space like in optics. Rather it is only a probability amplitude for the position distribution of the quantum object which, when detected, is well localized in a given point. Moreover, the superposition principle introduces a crucial difficulty since we cannot give any definite meaning to the position of the quantum object in a superposition state. In other words, the standard formulation of QM does not provide a spacetime description of the behaviour of a quantum object easily comparable with the classical one. For these reasons, the classical limit of QM is a very delicate problem both from the technical and the conceptual point of view (see e.g. [9] for a historical analysis of the problem).

The traditional approach is essentially based on the analysis of the solutions of the Schrödinger equation in the limit $\hbar \to 0$ for a suitable choice of the initial state. We recall that in this context the limit $\hbar \to 0$ simply means that the typical action of the system is large with respect to the Planck constant.

Usually one considers two possible initial states, chosen in analogy with the case of optics. The first one is the WKB state defined by an amplitude independent of $\hbar$ and a highly oscillating phase for $\hbar$ small. In this case one can show, for $\hbar$ small and for short times, that
the corresponding solution of the Schrödinger equation has the same form, with amplitude and phase governed by the classical transport and Hamilton–Jacobi equations, respectively. This means that in the limit the quantum state propagates like a classical fluid and in this sense the classical description is recovered.

A second reasonable choice of initial state is the coherent state, i.e. a wave packet well concentrated in position and momentum around a point \((x_0, p_0)\) in the classical phase space for \(\hbar\) small. One can prove that the time evolution, for \(\hbar\) small and a time interval not too long, is again described by a wave packet well concentrated in position and momentum around the classical trajectory starting from \((x_0, p_0)\).

A precise statement and a mathematical proof of the above qualitative pictures have required much technical work, and many refined and detailed results are now available in the literature [29].

Despite their mathematical elegance, such results cannot be considered satisfactory for a complete understanding of the classical limit of QM. The reason is that the approach is crucially based on the choice of specific, essentially classical initial states while, in many cases, the classical behaviour should emerge also starting from a genuine quantum state, like a superposition state. In these cases the usual procedure \(\hbar \to 0\) is not sufficient.

The difficulty was clearly recognized by Einstein in his correspondence with Born, in the years 1953–1954 [4]. The answer given by the founders of the standard formulation of QM (see e.g. the Born–Pauli correspondence in [4]) was based on the rule of the wave packet reduction and therefore it was outside the context of an evolution governed by the Schrödinger equation. In any case at that time the question was considered relevant only at foundational level without practical consequences. Starting around 1980, some remarkable experimental progresses have made possible a detailed exploration of the classical/quantum border. In particular, genuine superposition states of quantum systems have been prepared and their evolution analysed in real experiments, often in connection with problems arising in the new emerging field of quantum information. At a theoretical level, a new impulse was given to the construction of models in which one can analyse if and how much a superposition state can survive or eventually can be reduced to a classical statistical mixture. It is important to remark that here one is interested in a quantitative analysis of the problem. Therefore, any explanation based on the wave packet reduction cannot help and one is forced to consider the problem entirely in the context of the Schrödinger equation.

The idea behind this theoretical analysis is that the quantum coherence present in a genuine superposition state is very fragile. This implies that even a weak interaction of the system with the environment can destroy the superposition state and a classical behaviour of the system can emerge. The dynamical mechanism producing the reduction or the suppression of the quantum coherence for a system interacting with the environment is usually called decoherence. In the last few years, considerable work has been done to control and quantify the decoherence induced by the environment in many interesting situations. Here we want to mention the analysis of the reduction of the interference effects for a heavy particle due to the scattering of light particles, in a typical two-slit experiment [24, 26]. Another interesting case is the explanation of the molecular localization for pyramidal molecules, like \(\text{NH}_3\), due to the dipole–dipole interaction among the molecules [8, 19, 25]. In the following section, we shall concentrate on a further example, i.e. the emergence of classical trajectories in a cloud chamber, which seems particularly interesting for a better understanding of the classical limit of QM.
4. \( \alpha \)-particle in a cloud chamber

A cloud chamber is an experimental apparatus which was constructed and improved by Wilson in the years 1911–1912 (see e.g. [27] for a description of the original apparatus). The cloud chamber proved to be particularly useful for the investigation of the properties of various atomic and subatomic particles. In particular, it was used to observe the trajectory of an \( \alpha \)-particle emitted by a radioactive source of radium placed in the chamber. The way of working the apparatus can be schematically described as follows. The chamber is filled by a supersaturated vapour which can undergo local phase transitions induced by the exchange of even a small amount of energy. The \( \alpha \)-particle released by the source propagates and interacts with the atoms of the gas in the chamber inducing ionization. The ionized atoms produce a local phase transition of the gas and therefore the formation of small drops of cloud near their positions. The sequence of such drops is the visible track that one can directly observe in the chamber. The tracks usually have the form of straight lines (or of curved lines whenever magnetic and/or electric fields are applied) and they can be considered as the experimental manifestation of the ‘trajectory’ of the \( \alpha \)-particle.

The theoretical explanation of this phenomenology must be given in the framework of QM due to the microscopic character of the processes involved. In particular, the successful analysis of the radioactive decay of a nucleus with emission of an \( \alpha \)-particle was achieved by Gamow [16] and by Condon and Gurney [7]. It was shown that the emitted \( \alpha \)-particle is described by a wavefunction having the form of a spherical wave, centred in the source, and isotropically propagating in space.

It was soon realized that this fact is apparently in contrast with the experimentally observed straight tracks in the cloud chamber, which are typical of a particle behaviour. The problem was intensively discussed in the early days of QM since it was a serious and interesting test for the new theory. In particular, it was considered important to clarify the connection between the particle-like and the wave-like behaviour of a quantum system.

The problem was summarized by Born [15] in his communication at the Solvay Conference in 1927: ‘If a spherical wave is associated with every act of emission, how can it be understood that the trace of the \( \alpha \)-particle appears as an (almost) straight line?’

A further clear enunciation of the problem was given by Darwin [11]: ‘Consider for example one of the most striking manifestations of particle characters, the ray tracks of \( \alpha \)-particles in a Wilson cloud chamber. We have to connect this in some way with the theory of radioactive disintegration as presented by Gamow. In that theory the radium nucleus contains what we may call an \( \alpha \)-wave which is slowly and continuously leaking out as a spherical wave.’

We finally mention the words of Mott [28]: ‘It is a little difficult to picture how it is that an outgoing spherical wave can produce a straight line; we think intuitively that it should ionize atoms at random throughout space’.

It was clearly recognized by Heisenberg in his Chicago lectures in 1929 [22] that in the framework of the standard formulation of QM one has two possible points of view to reconcile theory with experimental data. The first one is based on the wave packet collapse and the second on the consideration of the Schrödinger equation for the \( \alpha \)-particle and the atoms of the gas.

According to the first point of view, the gas in the chamber must be considered as a (classical) measurement apparatus which repeatedly measures the position of the \( \alpha \)-particle. The result of the measurement operated by the first atom of the gas is the reduction of the original spherical wave to a wave packet localized near the position of the atom and propagating along the line joining the source with the atom. Such a wave packet is subject to an unavoidable
spreading which is again reduced by the position measurement of the next atom and so on. This mechanism would explain the appearance of an almost classical trajectory of the $\alpha$-particle. The explanation is surely simple and effective, but on the other hand it can be criticized. In fact, even if one accepts the rule of the collapse, it is not clear why a microscopic system, like an atom of the gas, should behave as a measurement apparatus. At least, it should be specified under which physical conditions this can happen.

If one takes seriously the second point of view, the atoms of the gas are considered as quantum objects together with the $\alpha$-particle and one has to derive the effect, i.e. the straight tracks, from the Schrödinger equation applied to the whole quantum system. In this context, a crucial role is played by the entanglement produced by the interaction between the $\alpha$-particle and each atom of the gas. The basic idea was expressed in [11]: ‘Before the very first collision (the wavefunction) can be represented as the product of a spherical wave for the $\alpha$-particle, by a set of more or less stationary waves for the atoms. But the first collision changes this product into a function in which the two types of coordinates are inextricably mixed, and every subsequent collision makes it worse.’ Such a complicated function contains a phase factor and ‘without in the least seeing the details, it looks quite natural to expect that this phase factor will have some special character, such as vanishing, when the various co-ordinates satisfy a condition of collinearity. So without pretending to have mastered the details, we can understand how it is possible that the $\psi$ function, so to speak, not to know in what direction the track is to be, but yet to insist that it should be a straight line.’

The program enunciated by Darwin was concretely carried out by Mott in [28]. He studied a simplified model with only two atoms of the gas placed at fixed positions $a_1$, $a_2$ and the $\alpha$-particle initially described by a spherical wave centred in the point $O$, with $|Oa_2| > |Oa_1|$. He computed the ionization probability of both atoms and he found that such a probability is essentially zero unless the second atom in $a_2$ lies on the straight line joining $O$ with $a_1$. The basic idea is the following. The ionization of the first atom in $a_1$ produces a well-localized wave packet starting from $a_1$ with momentum directed from $O$ to $a_1$. Therefore if we require that also the second atom be ionized, the only possibility is that its position is on the trajectory of the wave packet. The analysis exploited a perturbative expansion up to second order of the solution of the time-independent Schrödinger equation and made use of the stationary phase method, essentially along the line suggested in [11]. Although the result was obtained only through heuristic arguments, it was based on a beautiful physical intuition and the approach to the problem must be surely considered pioneering. As a matter of fact the work was rather neglected for a long time. This was probably a consequence of a too rigid acceptance of the standard formulation of QM, rather common in the scientific community during the first years of the theory. In particular, a special emphasis was given to the role of the act of measurement, and then to the rule of the wave packet collapse, in order to solve possible conceptual difficulties of the theory. In this context, it is clear that the approach proposed by Mott, entirely based on the study of the Schrödinger equation, could appear rather misleading.

In particular, it could obscure the role of the act of observation which should be necessarily invoked at a certain point. We note that in principle the objection is correct but, nevertheless, it is not irrelevant that this point can be postponed. In this sense Mott’s approach is more interesting from the physical point of view since it allows a quantitative analysis of the processes involved which is excluded with the wave packet collapse.

5. A one-dimensional model

In recent years, further elaborations on the subject of the straight tracks in a cloud chamber have been worked out [3, 5, 6, 10, 21, 32]. Here we want to describe a simplified
one-dimensional model consisting of a test particle and two harmonic oscillators, where a
rigorous mathematical analysis can be carried out [12]. A superposition state of two wave
packets centred in the origin with opposite momentum plays the role of the spherical wave of
the $a$-particle and the oscillators replace the atoms to be ionized. Under suitable assumptions
on the physical parameters of the model, a detailed analysis of the time evolution of the system
can be performed using time-dependent perturbation theory. The result is a quantitative
estimate of the joint excitation probability of the oscillators. In particular, it is shown that
such a probability is essentially zero if the two oscillators are placed on opposite sides of the
origin, while it has a finite, non-zero value in the other case. The work is entirely within the
line of reasoning of Mott and it develops some aspects of the model which are not considered
in [28]. In particular, the analysis is performed following the time evolution of the system and,
moreover, quantitative estimates are given showing precisely under which physical conditions
the effect can be seen.

Let us introduce the model. We denote by $\Psi(R, r_1, r_2, t)$ the wavefunction of the system,
where $R$ is the coordinate of the test particle and $r_1$, $r_2$ are the coordinates of the oscillators
placed in positions $a_1$, $a_2$ respectively, with $|a_2| > a_1 > 0$. The Schrödinger equation reads
\[ i\hbar \frac{\partial \Psi}{\partial t} = H \Psi \]  
\[ H = -\frac{\hbar^2}{2M} \Delta_R - \frac{\hbar^2}{2m} \Delta_{r_i} + \frac{1}{2} m \omega^2 (r_1 - a_1)^2 - \frac{\hbar^2}{2m} \Delta_{r_2} + \frac{1}{2} m \omega^2 (r_2 - a_2)^2 \]
\[ + \lambda V(\delta^{-1}(R - r_1)) + \lambda V(\delta^{-1}(R - r_2)), \]  
(5.1)
(5.2)
where $M$ is the mass of the test particle, $m$, $\omega$ are the mass and the frequency of the oscillators,
$V$ is an interaction potential and $\lambda$, $\delta$ are two positive parameters. As an initial state we choose
\[ \Psi_0(R, r_1, r_2) = \psi(R) \phi_{a_1}^{\sigma_1}(r_1) \phi_{a_2}^{\sigma_2}(r_2), \]  
where $\psi(R)$ is the one-dimensional spherical wave for the test particle
\[ \psi(R) = \psi^+(R) + \psi^-(R) = \frac{\mathcal{N}}{\sqrt{\sigma}} e^{-\frac{\sigma}{2\sigma^2}} e^{i\frac{\sigma}{2} R} + \frac{\mathcal{N}}{\sqrt{\sigma}} e^{-\frac{\sigma}{2\sigma^2}} e^{-i\frac{\sigma}{2} R} \]  
(5.3)
(5.4)
with $\sigma$, $P_0 > 0$ and $\mathcal{N}$ being the normalization factor. Moreover $\phi_{n_i}^{\sigma_i}(r_i)$, $i = 1, 2$, denotes the
eigenfunction of the oscillator in $a_i$ corresponding to the energy level indexed by $n_i \in \mathbb{N}$.

We remark that in the absence of interaction, the test particle would be described by
the free evolution of the superposition state (5.4), i.e. a coherent superposition of two wave
packets, one propagating on the right and the other on the left. Therefore, according to the
standard formulation of QM, the test particle would be completely delocalized and its position
would have no meaning.

The relevant object of the analysis is $P_{n_1n_2}(t)$, i.e. the probability that both oscillators are
excited in the states labelled by $n_1$, $n_2 \neq 0$ at time $t$. From a direct application of Born’s rule
one has
\[ P_{n_1n_2}(t) = \int_R dR |f_{n_1n_2}(R, t)|^2, \]  
(5.5)
where $f_{n_1n_2}(R, t)$ are the coefficients of the expansion
\[ \Psi(R, r_1, r_2, t) = \sum_{n_1, n_2} f_{n_1n_2}(R, t) \phi_{n_1}^{\sigma_1}(r_1) \phi_{n_2}^{\sigma_2}(r_2). \]  
(5.6)

The computation is done for $t > |a_1| v_0 = t_2$, where $v_0 = \frac{P_0}{M}$ and $t_2$ is the time required
to a classical particle to go from the origin to $a_2$. An important point is to specify the set of
assumptions on the physical parameters of the model. In particular, we assume
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(1) \( \lambda_0 \equiv \frac{\lambda}{Mv_0^2} \ll 1; \)

(2) the dimensionless quantities \( \frac{m}{M}, \frac{\hbar}{Mv_0}, \frac{\sigma_i}{|a_i|}, \frac{v_0}{|a_i|}, i = 1, 2, \) are all \( O(\varepsilon) \), where \( \lambda_0 \ll \varepsilon \ll 1. \)

Let us briefly discuss the physical meaning of (1), (2). Assumption (1), together with the condition \( \lambda_0 \ll \varepsilon \ll 1, \) guarantees that the problem can be analyzed in a perturbative regime. In fact, the first- and second-order corrections in perturbation theory turn out to be of order \( \lambda_0 \varepsilon^{-1} \) and \( \lambda_0^2 \varepsilon^{-2} \) respectively. In (2) we require that the mass and the kinetic energy of the test particle be much larger than the mass and the spacing of the energy levels of the oscillators. In particular this, together with (1), means that momentum and energy loss for the test particle are negligible. Moreover, the initial wave packets of the test particle are assumed to be well localized and the interaction potential is required to be short range. Finally, the characteristic time of the oscillators \( \omega^{-1} \) is assumed to be much smaller than the flight times \( \tau_i = \frac{|a_i|}{v_0} \) of the test particle.

The mathematical result can be formulated as follows.

**Theorem 1.** Let us assume (1), (2) and \( n_1, n_2 \neq 0 \). Then, up to second order in perturbation theory and for \( t > \tau_2 \), one has

(i) for \( a_2 < 0 < a_1 \)

\[
\mathcal{P}_{n_1n_2}(t) \leq \varepsilon^4 \left( \frac{\lambda_0}{\varepsilon} \right)^4 A^{(k)}_{n_1n_2}(t), \quad \text{for any } k \in \mathbb{N}, \quad (5.7)
\]

(ii) for \( 0 < a_1 < a_2 \)

\[
\mathcal{P}_{n_1n_2}(t) = \left( \frac{\lambda_0}{\varepsilon} \right)^4 B_{n_1n_2}(t), \quad (5.8)
\]

where \( A^{(k)}_{n_1n_2}(t), B_{n_1n_2}(t) \), for \( t \) of order of \( \tau_2 \), are quantities of order 1 which are explicitly estimated.

The theorem clearly expresses the fact that the probability \( \mathcal{P}_{n_1n_2}(t) \) computed in case (i) is negligible with respect to the same probability computed in case (ii). For the proof, we refer to [12] and here we only add some comments on the meaning of the result.

In general, QM can only predict the probabilities of the possible events during the evolution of a given system. In our model in case (i) we are concerned with four possibilities, which we may call possible histories: no oscillator is excited, only the oscillator on the right is excited, only the oscillator on the left is excited, both oscillators are excited. The theorem states that the last history has a negligible probability and then we can only observe just one excited oscillator. If we observe that the oscillator on the right (resp. on the left) is excited, we can affirm that the test particle is localized on the right (resp. on the left). This means that the interaction determines the localization of the test particle (on the right or on the left) while, before the interaction, the test particle was completely delocalized and no meaning could be given to its position. In this rather indirect sense, the test particle behaves like a classical particle propagating along a well-defined trajectory.

We emphasize that the result strongly depends on the assumptions (1), (2) and one should expect a completely different behaviour if other physical assumptions are introduced. This fact once again shows the difference with respect to the approach based on the wave packet collapse.
6. Concluding remarks

In sections 4 and 5, we have discussed a specific case of classical behaviour emerging in a quantum system. We have emphasized that the pioneering approach introduced by Mott was rather neglected for a long period and only recently rediscovered and applied to a number of other situations, where the environment plays a role in the suppression of quantum coherence (decoherence theory). We have also underlined that the birth and development of such an interesting field of research was prevented for a long time on the basis of pure ideological motivations, corresponding to a too-rigid interpretation of the theory. In any case, we have now a well-established set of results in the physical literature which allows a quantitative description of the emergence of classical behaviour in many concrete situations.

It is worth mentioning that such a description is surely valid for all practical purposes, in complete analogy with the problem of the measurement process mentioned in section 2. Therefore, once again one can ask whether it is satisfactory also from the conceptual point of view.

The answer depends on the idea of scientific explanation or scientific theory one has in mind. There are surely many different opinions on this question and we want just to recall two possible views which, roughly speaking, can be summarized as follows.

According to an instrumentalist view, a scientific theory should not pretend to describe the objective world. Rather, it is only a set of rules, formulated in mathematical language, useful to organize the empirical experience and to predict the results of experiments. Such a view influenced the founders (Bohr, Heisenberg, Pauli) of the standard formulation of QM and it is rather common in the physics community.

On the other side, there is a more realistic view which was strongly supported by Einstein. According to this view, a scientific theory is a conceptual construction created to capture elements of the reality, which is supposed to exist independently of the observer.

It is clear that if one accepts the instrumentalist point of view, then an explanation for all practical purposes can be considered satisfactory while, on the other hand, a realistic view would force one to a deeper analysis of the theory or, eventually, of its interpretation.

For a rather long period, in the scientific community the opinion was spread out that QM constrains to reject the realistic view. This idea was based on a consideration of the mathematical formalism and of its interpretation (according to the standard formulation) as an inextricable whole, obscuring the necessary distinction between the two levels.

It was later realized that such distinction is crucial and that many different and coherent interpretations of the mathematical formalism are possible. In particular, some of these interpretations are compatible with a realistic view. It should be remarked that the choice of appropriate interpretation cannot be decided on a purely scientific level. In fact, genuinely philosophical questions are involved in this choice, and one can only discuss positive or negative aspects of each position without pretending to close the debate.

In conclusion, we stress that the discussion on such foundational questions are of interest not only at an epistemological level but also because it is a source of new and possibly stimulating approaches to scientific problems. At least from this pragmatic point of view, it is relevant for the concrete development of the scientific knowledge.

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