Operator covariant transform and local principle

Vladimir V Kisil

School of Mathematics, University of Leeds, Leeds LS2 9JT, UK

E-mail: kisilv@maths.leeds.ac.uk

Received 31 October 2011, in final form 10 January 2012
Published 30 May 2012
Online at stacks.iop.org/JPhysA/45/244022

Abstract

We describe connections between the localization technique introduced by I B Simonenko and operator covariant transform produced by nilpotent Lie groups.

This article is part of a special issue of Journal of Physics A: Mathematical and Theoretical devoted to ‘Coherent states: mathematical and physical aspects’.

PACS numbers: 02.20.Sv, 02.30.Tb

1. Introduction

In 1965, Simonenko [22, 23] pioneered the localization technique in the theory of operators. It still remains an important tool in this area, see for example [2, 4, 8, 9, 21, 24]. Many questions addressed by this technique, e.g. boundary value problems, are rooted in mathematical physics. We also discuss connections with quantum mechanics in the final section of this paper.

The localization method was developed in various directions and there is no possibility to mention all works based on numerous existing variants and modifications of the localization technique. Several generalizations, e.g. within $C^*$-algebras setup [3, proposition 4.5], capture the abstract skeleton of the localization technique. However, the idea of ‘localization’ has an explicit geometrical meaning, which often escapes those general schemes.

We present here a different point of view on the original works of Simonenko, which highlights the role of groups in the constructions. Thus, it is not a generalization but rather an attempt to link certain geometrical meaning of locality with a homogeneous structure of nilpotent Lie groups. This paper originated from our earlier works [10–16] revised in the light of recent research [17, 18].

The paper is organized as follows. Section 2 collects preliminary information from other works, which will be used here. In section 3, we use the homogeneous structure of nilpotent Lie groups to define basic elements of localization. Operators which are invariant under certain group action are the main building blocks for localization, we demonstrate this in section 4. Section 5 gives the summary of our observations which lead to new directions for further research.

1 On leave from Odessa University.
2. Preliminaries

2.1. Classic localization technique

We present here the fundamental definitions from the work of I B Simonenko [22, 23] formulated for operators on $L^p(\mathbb{R}^n)$. The essential norm of an operator is defined by

$$||A|| = \inf_{K} ||A - K||,$$

where the infimum is taken over all compact operators $K$. For a measurable set $F \subset \mathbb{R}^n$, we define the projection operator $P_F : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ by

$$[P_F f](x) = \begin{cases} f(x), & \text{if } x \in F; \\ 0, & \text{otherwise}. \end{cases} \quad (1)$$

The operators, most suitable for the localization method, are defined as follows.

**Definition 1** [22, section 1.1]. An operator $A$ is of local type if for any two closed disjoint sets, $F_1$ and $F_2$, the operator $P_{F_1} A P_{F_2}$ is compact.

The cornerstone definition for the whole theory is as follows.

**Definition 2** [22, section 1.2]. Operators $A, B : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ are called equivalent at a point $x_0$, if for any $\varepsilon > 0$ there is a neighborhood $u$ of $x_0$ such that $||A P_u - B P_u|| < \varepsilon$ and $||P_u A - P_u B|| < \varepsilon$. This is denoted $A \sim_x B$.

As usual there are two stages in this method: analysis and synthesis. Local equivalence decomposes operators into families of local representatives. Now we define the opposite process of a reconstruction.

**Definition 3** [22, section 1.5]. Let $A_x$ be a family of operators $L^p(X) \to L^p(X)$ depending on $x \in X$. An operator $A : L^p(X) \to L^p(X)$ is an envelope of $A_x$ if for every $x$ we have $A \sim_x A_x$.

An envelope can be built [22, section 1.5] as the limit $A$ of a sequence $A_n$ which is defined by the expression

$$A_n = \sum_{j=1}^{n} P_{u_j} A_{x_j} P_{u_j}, \quad (2)$$

where sets $u_n$ make a decomposition of $X$ and $x_n \in u_n$.

2.2. Covariant transform

The following concept is a natural development of the coherent states (wavelets) based on group representations.

**Definition 4** [17, 18]. Let $\rho$ be a representation of a group $G$ in a space $V$, and $F$ be an operator from $V$ to a space $U$. We define a covariant transform $W$ from $V$ to the space $L(G, U)$ of $U$-valued functions on $G$ by the formula

$$W : v \mapsto \hat{v}(g) = F(\rho(g^{-1})v), \quad v \in V, \ g \in G. \quad (3)$$

Operator $F$ will be called the fiducial operator in this context.
We borrow the name for operator $F$ from fiducial vectors of Klauder and Skagerstam [20]. The wavelet transform, which is a particular case of the covariant transform, corresponds to the fiducial operator which is a linear functional. Thus, its image consists of scalar-valued functions. It seems to be the most favourable situation, cf [18, remark 3], and was believed to be the only possible one for a long time. The moral of this work is that the covariant transform can be useful even in the other extreme limit, if the range of the fiducial operator is the entire space $V$.

By the way, we do not require that the fiducial operator $F$ be linear in general; however, it will be always linear in this work. Sometimes the positive homogeneity, i.e. $F(tv) = tF(v)$ for $t > 0$, alone can be already sufficient, see [18, 19].

The following property is inherited by the coherent transform from the wavelet one.

**Theorem 5** [17, 18], The covariant transform (3) intertwines $\rho$ and the left regular representation $\Lambda_1$ on $L(G, U)$:

$$W_{\rho}(g) = \Lambda_1(g)W.$$  

Here, $\Lambda_1$ is defined as usual by

$$\Lambda_1(g) : f(h) \mapsto f(g^{-1}h).$$  

(4)

The next result follows immediately.

**Corollary 6.** The image space $W(V)$ is invariant under the left shifts on $G$.

2.3. Inverse covariant transform

An object invariant under the left action $\Lambda_1$ (4) is called left invariant. For example, let $L$ and $L'$ be two left invariant spaces of functions on $G$. We say that a pairing $\langle \cdot, \cdot \rangle : L \times L' \to \mathbb{C}$ is left invariant if

$$\langle \Lambda_1(g)f, \Lambda_1(g)f' \rangle = \langle f, f' \rangle,$$

for all $f \in L$, $f' \in L'$.  

(5)

**Remark 7.** (1) We do not require the pairing to be linear in general.

(2) If the pairing is invariant on space $L \times L'$, then it is not necessarily invariant (or even defined) on the whole $C(G) \times \mathbb{C}$.

(3) An invariant pairing on $G$ can be obtained from an invariant functional $l$ by the formula $\langle f_1, f_2 \rangle = l(f_1f_2)$. Such a functional is often associated with the (quasi-) invariant measures.

**Example 8.** Let $G$ be the $ax + b$ group, cf example 12 below. There are essentially two non-trivial invariant pairings for it. The first one is based on the left Haar measure $\frac{da \, db}{a^2}$ and integration over the entire group:

$$\langle f_1, f_2 \rangle = \int_{\infty}^{\infty} \int_{0}^{\infty} f_1(a, b) \, f_2(a, b) \, \frac{da \, db}{a^2}.  $$  

(6)

Another invariant pairing on $G$, which is not generated by the Haar measure, is

$$\langle f_1, f_2 \rangle = \lim_{a \to 0} \int_{-\infty}^{\infty} f_1(a, b) \, f_2(a, b) \, db.$$  

(7)

This pairing participates in the definition of the inner product on the Hardy space; thus, we call it Hardy-type pairing [18].
For a representation \( \rho \) of \( G \) in \( V \) and \( v_0 \in V \), we fix a function \( w(g) = \rho(g)v_0 \). We assume that the pairing can be extended in its second component to this \( V \)-valued function, say, in the weak sense.

**Definition 9.** Let \( \langle \cdot, \cdot \rangle \) be a left invariant pairing on \( L \times L' \) as above, let \( \rho \) be a representation of \( G \) in a space \( V \), we define the function \( w(g) = \rho(g)v_0 \) for \( v_0 \in V \). The inverse covariant transform \( \mathcal{M} \) is a map \( L \rightarrow V \) defined by the pairing

\[
\mathcal{M} : f \mapsto \langle f, w \rangle, \quad \text{where} \quad f \in L.
\]  

There is an easy consequence of this definition.

**Proposition 10.** The inverse wavelet transform intertwines the left regular representation and \( \rho(g) \).

### 3. Semidirect products and localization

Let \( G \) be an \( m \)-dimensional exponential nilpotent Lie group of length \( k \). That means that

- we can identify \( G \) with its Lie algebra \( g \sim \mathbb{R}^m \) through the exponential map,
- there is a linear space decomposition
  \[
g = \bigoplus_{j=1}^k V_j, \quad \text{such that} \quad [V_i, V_j] \in V_{i+j},\]  

where \([V_i, V_j]\) denotes the space of all commutators \([x, y] = xy - yx \) with \( x \in V_i, y \in V_j \) and \( V_i = \{0\} \) for all \( i > k \).

**Example 11.** Here are two most fundamental examples.

1. The group of Euclidean shifts in \( \mathbb{R}^n \)—a nilpotent group of length 1.
2. The Heisenberg group \( \mathbb{H}^n \) \([6, 7]\)—a nilpotent group of dimensionality \( m = 2n + 1 \) and length 2. Its element is \((s, x, y)\), where \( x, y \in \mathbb{R}^n \) and \( s \in \mathbb{R} \). The group law on \( \mathbb{H}^n \) is given as follows:
   \[
   (s, x, y) \cdot (s', x', y') = \left(s + s', x + x', y + y' + \frac{1}{2}(xy' - x'y)\right).\]  

For a generic group \( G \) described above there is a one-parameter group of automorphisms of \( g \) defined in terms of decomposition (9):

\[
\tau_t(v_j) = t^j v_j, \quad \text{for} \quad v_j \in V_j, \quad t \in \mathbb{R}_+.\]  

The exponential map sends \( \tau_t \) to automorphisms of the group \( G \) by the Baker–Campbell–Hausdorff formula. Thus, we consider the semidirect product \( \tilde{G} = G \times \mathbb{R}_+ \) of the group \( G \) and positive reals with the group law:

\[
(t, g) \cdot (t', g') = (tt', g \cdot \tau_t(g')), \quad \text{where} \quad t, t' \in \mathbb{R}_+, \quad g, g' \in G.
\]

**Example 12.** Returning to groups introduced in example 11.

1. If \( G \) is the group of shifts on the real line \( \mathbb{R} \), then the above semidirect \( \tilde{G} \) product is the \( ax + b \) group (or affine group). The group \( \tilde{G} \) is isomorphic to \( \mathbb{R}_+ \times \mathbb{R} \) with the group law
   \[
   (a, b) \cdot (a', b') = (aa', ab + b), \quad \text{where} \quad a, a' \in \mathbb{R}_+, \quad b, b' \in \mathbb{R}.
   \]
(2) For the Heisenberg group $\mathbb{H}^n$ the above automorphism is $\tau_r(s, x, y) = (t^2 s, tx, ty)$ [5]; thus, the respective group law on $\mathbb{R}^n$ is

$$(t, s, x, y) \cdot (t', s', x', y') = \left( t' t + t^2 s', s + t^2 s', x + t x', y + t y' \right).$$

There is a linear action of $\hat{G}$ on functions over $G$ cooked by the ‘$ax + b$-recipe’:

$$[\rho(t, g) f](g') = t^k f(\tau_{t^{-1}}(g^{-1} \cdot g')),$$

where $k = \sum_j j \cdot \dim V_j$. This action is an isometry of $L_p(G) = L_p(G, d\mu)$, where $d\mu$ is the Haar measure on $G$ (recall that it is unimodular as a nilpotent one). Then we can define the respective representation $\rho_d$ of $\hat{G} \times \hat{G}$ on operators [16, 18, 19]:

$$\rho_d(t, g; t', g') : A \mapsto \rho(t^{-1}, \tau_{t^{-1}}(g^{-1})) A \rho(t', g'),$$

for a linear operator $A : L_p(G) \to L_p(G)$.

Let $F_e \subset G$ be a bounded closed subset, which contains a neighbourhood of the unit $e \in G$. We will denote by $F_{(e, g)} = (t, g) \cdot F_e$ for $(t, g) \in \hat{G}$, its image under the left action of $\hat{G}$ on $G$. Define the associated projection $P_e = P_{F_e}$ by (1). It is a straightforward verification that

$$\rho_d(t, g; t, g) P_e = P_{F_{(e, g)}}, \quad \text{where} \quad F_{(e, g)} = (t, g) \cdot F_e.$$  \hspace{1cm} (14)

We shall use a simpler notation $P_{(e, g)} = P_{F_{(e, g)}}$ again. The exact form of $F_e$ is not crucial for the following construction, but the following property simplifies technical issues.

**Definition 13.** We say that $F_e$ is $r$-self-covering if for any two intersecting sets $F_{(1, g_1)}$ and $F_{(1, g_2)}$ there is such $g \in G$ that $F_{(e, g)}$ covers the union of $F_{(1, g_1)}$ and $F_{(1, g_2)}$.

For example, the closed unit ball in $\mathbb{R}^n$ is 2-self-covering with no other $F_e$ having a smaller value of $r$ for the self-covering property.

For a Banach space $V$, we denote by $B(V)$ the collection of all bounded linear operators $V \to V$.

**Definition 14.** We select a fiducial operator $F : B(L_p(G)) \to B(L_p(G))$ by the identity

$$F(A) = P_e A P_e, \quad \text{where} \quad A \in B(L_p(G)).$$  \hspace{1cm} (15)

Then Simonenko presymbol $\hat{S}_A(t, g; t', g')$ of an operator $A$ is the covariant transform (3) generated by the representation $\rho_d$ (13) and the fiducial operator $F$ (15):

$$\hat{S}_A(t, g; t', g') = F(\rho_d(t, g; t', g') A) = P_e \rho(t^{-1}, \tau_{t^{-1}}(g^{-1})) A \rho(t', g') P_e.$$  \hspace{1cm} (16)

Thus, the Simonenko presymbol is $B(L_p(G))$-valued function on $\hat{G} \times \hat{G}$. We can consider a definition of the alternative presymbol:

$$\tilde{S}_A(t, g; t', g') = P_{(t, g)} A P_{(t', g')}.$$  \hspace{1cm} (16)

which is closer to the original geometrical spirit of Simonenko’s works [22, 23]. However, there is an easy explicit connection between them:

$$\tilde{S}_A((t, g)^{-1}; (t', g')^{-1}) = \rho_d(t, g; t', g') \hat{S}_A(t, g; t', g'),$$

which is a local transformation of the function value at every point. Thus, both symbols shall bring equivalent theories, although each of them seems to be more suitable for particular purposes.

For operators of local type the whole presymbol is excessive due to the following result.
Proposition 15. Let \( F_r \) be \( r \)-self-similar and \( A \) be an operator of local type. Then, for any reals \( t > t' > 0 \) and \( g \in G \) the operator \( \hat{S}_A(t, g; t_1, g_2) \) with \( t_i > t, i = 1, 2 \) can be expressed as a finite sum

\[
\hat{S}_A(t_1, g_1; t_2, g_2) = \sum_{k=1}^{n} B_k \hat{S}_A(t', h_k; t', h_k) C_k,
\]

(17)

for some \( h_k \in F_{(t_1, g_1)} \cup F_{(t_2, g_2)} \) and constant operator coefficients \( B_k \) and \( C_k \), which do not depend on \( A \).

Proof. We will proceed in terms of the equivalent presymbol \( \tilde{S}_A \) (16), since it better reflects geometrical aspects. We also note that if we obtain the decomposition

\[
\hat{S}_A(t_1, g_1; t_2, g_2) = \sum_{k=1}^{n} B_k \hat{S}_A(t_k, h_k; t_k, h_k) C_k,
\]

with all \( t_k \leq t' \), then we will be able to replace \( t_k \) by \( t' \) with the simultaneous change of coefficients \( B_k \) and \( C_k \) in order to get the required identity (17).

Now we put \( t'' = t'/r \) and find a finite covering of the compact sets \( F_{(r', h_k)} \) by the interiors of sets \( F_{(r, h_k)} \) with \( h_k \in F_{(t_1, g_1)} \cup F_{(t_2, g_2)} \). Using the inclusion–exclusion principle we can write

\[
P_{(r, g)} = \sum_k P_{(r', h_k)} - \sum_{k,l} P_{(r', h_k)} P_{(r', h_l)} + \cdots
\]

\[
- \sum_{k,l} P_{(r', h_k)} P_{(r', h_l)} - \sum_{k,l} P_{(r', h_k)} P_{(r', h_l)} - \cdots,
\]

where all sums are finite and the number of sums is finite as well. Moreover, each term in the summation contains at least one projection \( P_{(r', h_k)} \). We use this decomposition for the presymbol \( P_{(r, g)} A P_{(r, g)} \) of an operator \( A \) of local type. Then we need to take care only of the terms \( P_{(r', h_k)} A P_{(r', h_l)} \), where \( F_{(r', h_k)} \) and \( F_{(r', h_l)} \) intersect. Due to the \( r \)-self-covering property, each such term can be represented as \( B_m P_{(r', h_m)} A P_{(r', h_m)} C_m \) for some \( h_m \in F_{(t_1, g_1)} \cup F_{(t_2, g_2)} \) with \( B_m \) and \( C_m \) depending on the geometry of sets only.

Thus, for the operators of local type we give the following definition.

Definition 16. For an operator \( A \) of local type, we define Simonenko symbol \( S_A(t, g) = \hat{S}_A(t, g; t, g) \), that is,

\[
S_A(t, g) = P_r \rho(t^{-1}, (g^{-1}) A \rho(t, g) P_r.
\]

Corollary 17. For an operator \( A \) of local type, the value of the presymbol \( \tilde{S}_A(t', g'; t'', g'') \) at a point \( (t', g'; t'', g'') \in G \times G \) is completely determined by the values of symbol \( S_A(t, g) \), \( g \in G \) for an arbitrary fixed \( t \) such that \( t \leq \min(t', t''). \)

Corollary 18. The operators \( A \) and \( B \) of local type are equal if and only if for any \( \varepsilon > 0 \) there is a positive \( t < \varepsilon \) such that \( S_A(t, g) = S_B(t, g) \) for all \( g \in G \).

In other words, even the symbol \( S_A(t, g) \) contains excessive information: in a sense we shall look for values of \( \lim_{t \to 0} S_A(t, g) \) only. We conclude this section by the restatement of definition 2.

Definition 19. Two operators \( A \) and \( B \) of local type are equivalent at a point \( g \in G \), denoted by \( A \overset{g}{\sim} B \), if

\[
\lim_{t \to 0} \|S_{A-B}(t, g)\| = 0.
\]
4. Localization and invariance

The paper of Simonenko [23] already contains results which can be easily adopted to covariant transform setup. This was already used in our previous work [10–15] to study singular integral operators on the Heisenberg group. In this section, we provide such restatements of results in terms of the representation from (12). Proofs will be omitted since they are easy modifications of the original ones [23].

**Definition 20.** An operator is called homogeneous if it commutes with all transformations \( \rho(t, e), t \in \mathbb{R}^+ \) (12). If an operator commutes with \( \rho(1, g), g \in G \) (12), then it is called shift-invariant.

There is an immediate consequence of theorem 5.

**Corollary 21.** The symbols of a homogeneous (or shift-invariant) operator is a function on \( \tilde{G} \), which is invariant under the action of the subgroup \( \mathbb{R}^+ \subset \tilde{G} \) (or \( G \subset \tilde{G} \), respectively).

Thus, homogeneous shift-invariant operators have constant symbols. Tame behaviour of operators from those classes is described by the following statements, cf [23, section 2.2].

**Lemma 22.** For two homogeneous operators \( A \) and \( B \) the following are equivalent:

1. \( A \sim B \), where \( e \in G \) is the unit;
2. \( S_A(t, e) = S_B(t, e) \) for certain \( t \in \mathbb{R}^+ \);
3. \( A = B \).

**Lemma 23.** [23, section 2.2] For two homogeneous shift-invariant operators \( A \) and \( B \) the following are equivalent:

1. \( A \sim B \) for certain \( g \in G \);
2. \( S_A(t, g) = S_B(t, g) \) for certain \( (t, g) \in \tilde{G} \);
3. \( A = B \).

A shift-invariant operator on \( G \) can be associated with a convolution. A convolution, which is also a homogeneous operator, shall have singular kernels. A study of such convolutions can be carried out by means of (non-commutative) harmonic analysis on \( G \). For the (commutative) Euclidean group this was illustrated in [22, 23]. A non-commutative example of the Heisenberg group can be found in [10–15]. It is also possible to study these operators through further versions of wavelet (coherent) transform, e.g. the Berezin-type symbols [16]. In the common case, boundedness of the Berezin symbols corresponds to the boundedness of the operator, and if the symbol vanishes at infinity then the operator is compact.

Once a good description of singular convolutions is obtained (through covariant transform or several such transforms applied in a sequence), we can consider the class of operators which can be reduced to them.

**Definition 24** [23, section 3.1]. A linear operator \( A \) of local type is called a generalized singular integral, if \( A \) is equivalent at every point of \( G \) to some homogeneous shift-invariant operator.

The final step of the construction is synthesis of an operator from the field of local representatives using the inverse covariant transform from subsection 2.3. To this end, we need to chose an invariant pairing on the group \( \tilde{G} \), keeping the \( ax + b \) group as an archetypal example. For operators of local type all the information is concentrated in the arbitrary small
neighborhood of the subgroup $G \subset \bar{G}$, cf corollary 18. Thus, we select the Hardy-type functional (7) instead of the Haar one (6). Let $d\mu$ be the Haar measure on the group $G$. Then the following integral
\[ \langle f_1, f_2 \rangle = \lim_{t \to 0} \int_G f_1(t, g)f_2(t, g)\,d\mu(g), \tag{18} \]
defines an invariant pairing on the group $\bar{G}$.

We again make use of the fiducial operator $F(A) = PAP$ (15). In the language of wavelet theory, we may say that analysing and reconstructing vectors are the same. The respective transformation $\rho_f(t, g)F$ by an element of the group $\bar{G}$ is defined through the identity $[\rho_f(t, g)F](A) = P(t, g)AP(t, g)$ for an arbitrary $A$. Consequently, the inverse covariant transform (8) sends an operator valued function $A(t, g)$ to an operator through the invariant pairing:
\[ \mathcal{M} : A(t, g) \mapsto A = \lim_{t \to 0} \int_G P(t, g)A(t, g)P(t, g)\,d\mu(g). \]

The last integral may be realized through Riemann-type sums which lead to the approximation (2) of an envelope of $A(t, g)$.

5. Concluding remarks

In this work, we outlined an interpretation of the classical Simonenko’s localization method [22, 23] in the context of recently formulated covariant transform [17, 18]. The original localization was used to study singular integral operators, which are convolutions on the Euclidean group. Our interpretation allows us to make a straightforward modification of the localization technique for non-commutative nilpotent Lie groups. The crucial role is played by the one-parameter group of automorphism realized as dilations.

Once local representatives are obtained, they can be studied further by other forms of wavelet (covariant) transform. The Berezin symbol seems to be very suitable for this task. Such a chain (Simonenko–Berezin–...) of covariant transforms shall lead to the full dissection of the initial operator into a very detailed symbol, which may even be scalar valued. The opposite process, reconstruction of an operator from its symbol or local representatives, can be done by the inverse covariant transform, which uses the same group structure.

The original coherent states in quantum mechanics are obtained from the ground state of the harmonic oscillator by a unitary action of the Weyl–Heisenberg group [1, chapter 1]. The next standard move is a decomposition of an arbitrary state into a linear superposition of coherent states, which form an overcomplete set. Consequently, observables can be investigated through such decompositions of states.

However, observables are primary notions of the quantum theory; thus, direct techniques, which circumvent decomposition of states, look preferable. Classical coherent states have the best possible (within the Heisenberg uncertainty relations) localization in the phase space. Thus, our localization on nilpotent Lie groups, in particular the Heisenberg group, has particular significance for quantum theory. Any observable corresponding to an operator of local type can be represented as a compact operator and a continuous field of local representatives. Compact operators have a discrete spectrum with a complete set of eigenvectors, each having at most a finite degeneracy. Local representatives correspond to observables which are highly localized on the phase space. Thus, operators of local type are a large set of quantum observables admitting efficient calculations of their spectrum.

It would be interesting to look for a similar construction in other classes of Lie groups. For example, Toeplitz operators on the Bergman space [24] may be treated through the group
$SL_2(\mathbb{R})$ [19], which is semi-simple. Such groups do not admit a group of dilation-type global automorphisms; thus, some adjustments to the scheme are required at this point.

Another interesting direction of development is operators of non-local type. They may look very different from the viewpoint of geometrical localization; however, in terms of covariant transform, the distinction is not so huge. For operators of local type, their Simonenko presymbol over $\tilde{G} \times \tilde{G}$ is excessive, and we can consider only the symbol in a small vicinity of the boundary $G$ of the diagonal in $\tilde{G} \times \tilde{G}$. For operators of non-local type the presymbol on the whole group $\tilde{G} \times \tilde{G}$ shall be used. This topic deserves further consideration.

Acknowledgment

I am grateful to anonymous referees for useful comments and suggestions, which helped to improve the paper.

References

[1] Ali S T, Antoine J-P and Gazeau J-P 2000 Coherent states, wavelets and their generalizations Graduate Texts in Contemporary Physics (New York: Springer)
[2] Bottcher A, Karlovich Y I and Spitkovsky I M 2002 Convolution operators and factorization of almost periodic matrix functions Operator Theory: Advances and Applications vol 131 (Basel: Birkhäuser)
[3] Douglas R G 1972 Banach Algebra Techniques in the Theory of Toeplitz Operators (Providence, RI: American Mathematical Society)
[4] Duduchava R, Saginashvili A and Shargorodsky E 1997 On two-dimensional singular integral operators with conformal Carleman shift J. Oper. Theor. 37 263–79
[5] Dynin A S 1975 Pseudodifferential operators on the Heisenberg group Dokl. Akad. Nauk SSSR 225 1245–8
[6] Folland G B 1989 Harmonic analysis in phase space Annals of Mathematics Studies vol 122 (Princeton, NJ: Princeton University Press)
[7] Howe R 1980 Quantum mechanics and partial differential equations J. Funct. Anal. 38 188–254
[8] Karlovich Y and Silbermann B 2004 Fredholmness of singular integral operators with discrete subexponential groups of shifts on Lebesgue spaces Math. Nachr. 272 55–94
[9] Karlovich Y and Spitkovsky I 1995 Factorization of almost periodic matrix functions J. Math. Anal. Appl. 193 209–32
[10] Kisil V V 1992 Algebra of two-sided convolutions on the Heisenberg group Dokl. Akad. Nauk SSSR 325 20–3 Translated in Kisil V V 1994 Russ. Acad. Sci. Dokl.—Math. 46 12–6
[11] Kisil V V 1993 On the algebra of pseudodifferential operators that is generated by convolutions on the Heisenberg group Sibirsk. Mat. Zh. 34 75–85 (in Russian)
[12] Kisil V V 1994 Local behavior of two-sided convolution operators with singular kernels on the Heisenberg group Mat. Zametki 56 41–55, 158 (in Russian)
[13] Kisil V V 1994 The spectrum of the algebra generated by two-sided convolutions on the Heisenberg group and by operators of multiplication by continuous functions Dokl. Akad. Nauk SSSR 337 439–41 Translated in Kisil V V 1995 Russ. Acad. Sci. Dokl.—Math. 50 92–7
[14] Kisil V V 1995 Connection between two-sided and one-sided convolution type operators on a non-commutative group Integral Equps Oper. Theor. 22 317–32
[15] Kisil V V 1996 Local algebras of two-sided convolutions on the Heisenberg group Mat. Zametki 59 370–381, 479
[16] Kisil V V 1999 Wavelets in Banach spaces Acta Appl. Math. 59 79–109 (arXiv:math/9807141)
[17] Kisil V V 2010 Wavelets beyond admissibility Progress in Analysis and its Applications, Proc. 10th ISAAC Congress (London, 13–18 July 2009) ed M Ruzhansky and J Wirth (Singapore: World Scientific) pp 219–25
[18] Kisil V V 2011 Covariant transform J. Phys.: Conf. Ser. 284 012038 (arXiv:1011.3947)
[19] Kisil V V 2012 Erlangen programme at large: an overview Advances in Applied Analysis ed S V Rogosin and A A Koroleva (Basel: Birkhäuser) pp 1–78 (arXiv:1106.1686)
[20] Klauder J R and Skagerstam B-S 1985 Coherent States: Applications in Physics and Mathematical Physics (Singapore: World Scientific)
[21] Rabinovich V and Samko S 2011 Pseudodifferential operators approach to singular integral operators in weighted variable exponent Lebesgue spaces on Carleson curves Integral Equps Oper. Theor. 69 405–44
[22] Simonenko I B 1965 A new general method of investigating linear operator equations of singular integral equation type. I Izv. Akad. Nauk SSSR Ser. Mat. 29 567–86
[23] Simonenko I B 1965 A new general method of investigating linear operator equations of singular integral equation type. II Izv. Akad. Nauk SSSR Ser. Mat. 29 757–82
[24] Vasilevski N L 2008 Commutative algebras of Toeplitz operators on the Bergman space Operator Theory: Advances and Applications vol 185 (Basel: Birkhäuser)