MODULI SPACE OF GENERAL CONNECTIONS

STANISLAV DUBROVSKY

ABSTRACT. We consider local invariants of general connections (with torsion). The group of origin-preserving diffeomorphisms acts on a space of jets of general connections. Dimensions of moduli spaces of generic connections are calculated. Poincaré series of the geometric structure of connection is constructed, and shown to be a rational function, confirming the finiteness assertion of Tresse.

1. INTRODUCTION

A problem of finiteness of functional moduli in various local differential-geometric settings was discussed by Arnol’d in [A]. We consider general (not necessarily torsion-free) connections, under the action of smooth coordinate changes.

The structure of the resulting moduli space is reflected in the corresponding Poincaré series, explicitly calculated for the generic case (5.17). This series turns out to be a rational function (5.19), indicating a finite number of invariants. This confirms the finiteness assertion of Tresse [T] formulated for any “natural” differential-geometric structure.

Rationality was earlier confirmed for a symmetric (torsion-free) connection in [D1], and via a different approach (used also to obtain the result presented herein) in [D2]; and for Fedosov structure (a.k.a. symplectic connection) in [D3], by the author.

Similar results for Riemannian, Kähler and hyper-Kähler structures were obtained in [Sh1], and an explicit normal form for Riemannian structure - in [Sh2], by Shmelev. Earlier Vershik and Gershkovich investigated jet asymptotic dimension of moduli spaces of jets of generic distributions at origin in $\mathbb{R}^n$ in [VG], and their normal form in [G].

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2. PRELIMINARIES AND MAIN RESULT

Let $\mathcal{F}$ and $\mathcal{F}_k$ be spaces of germs and $k$-jets respectively of smooth connections at the origin in $\mathbb{R}^n$. Two smooth functions are said to have the same $k$-jet at the origin in $\mathbb{R}^n$ if their first $k$ derivatives are equal in some (hence any) local coordinates.

Two connections $\nabla$ and $\tilde{\nabla}$ are said to have the same $k$-jet at $0$ if for any smooth vector fields $X$, $Y$, and any smooth function $f$, the functions $\nabla_X Y(f)$ and $\tilde{\nabla}_X Y(f)$ have the same $k$-jet at $0$. (This is equivalent to Christoffel symbols of $\nabla$ and $\tilde{\nabla}$ having the same $k$-jet.)

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Lemma 2.2. \[ (2.2) \]

This is well-defined, since in the coordinate version of (2.1):

\[ j^k \Gamma \]

where \( \Gamma \) on the right is an arbitrary representative of the elements of \( k \)-jet. This defines the action on germs of connections.

We will frequently denote the connection and its Christoffel symbol with the same letter, e.g. \( \Gamma \), \( j^k \Gamma \) would stand for its \( k \)-jet.

There is an action of the group of germs of origin-preserving diffeomorphisms \( G := \text{Diff}(\mathbb{R}^n, 0) \) on \( \mathcal{F} \) and \( \mathcal{F}_k \). For \( \varphi \in G \), \( \nabla \) (or \( \Gamma \)) \( \in \mathcal{F} \) and \( j^k \Gamma \in \mathcal{F}_k \):

\[ \Gamma \mapsto \varphi^* \Gamma, \quad j^k \Gamma \mapsto j^k (\varphi^* \Gamma) , \]

where

\[ (\varphi^* \nabla)_X Y = \varphi_*^{-1} (\nabla_{\varphi_* X} \varphi_* Y) \]

Let us introduce a filtration of \( G \) by normal subgroups:

\[ G = G_1 \triangleright G_2 \triangleright G_3 \triangleright \ldots , \]

where

\[ G_k = \{ \varphi \in G \mid \varphi(x) = x + (\varphi_1(x), \ldots, \varphi_n(x)), \ \varphi_i = O(|x|^k), i = 1, \ldots, n \} . \]

The subgroup \( G_k \) acts trivially on \( k \)-jet.

For \( G \) the subgroup \( F \) \( \text{Diff}(\mathbb{R}) \) \( \in \mathcal{F}_p \) for \( k \geq p + 3 \). It means that the action of \( G \) coincides with that of \( G/G_{p+3} \) on each \( \mathcal{F}_p \). Now \( G/G_{p+3} \) is a finite-dimensional Lie group, which we will call \( K_p \). Denote by \( \text{Vect}_0(\mathbb{R}^n) \) the Lie algebra of \( \mathbb{C}^\infty \)-vector fields, vanishing at the origin. It acts on \( \mathcal{F} \) as follows:

**Definition 2.1.** For \( V \in \text{Vect}_0(\mathbb{R}^n) \) generating a local 1-parameter subgroup \( g^t \) of \( \text{Diff}(\mathbb{R}^n, 0) \), the Lie derivative of a connection \( \nabla \) in the direction \( V \) is a \((1,2)\)-tensor:

\[ L_V \nabla = \frac{d}{dt} \bigg|_{t=0} g^t \nabla \]

**Lemma 2.2.**

\[ (L_V \nabla)(X, Y) = [V, \nabla_X Y] - \nabla_{[V,X]} Y - \nabla_X [V, Y] \]

**Proof** Below the composition \( \circ \) is understood as that of differential operators acting on functions.

\[ (L_V \nabla)(X, Y) = \frac{d}{dt} \bigg|_{t=0} g_*^{-t} \nabla_{g_*^t X} g_*^t Y = \frac{d}{dt} \bigg|_{t=0} (g^t)^* \circ \nabla_{g_*^t X} g_*^t Y = (g^t)^* \circ (L_{g_*^t X} g_*^t Y) = L_V (\nabla_X Y) - \nabla_{L_V X} Y - \nabla_X (L_V Y) \]

This defines the action on germs of connections.

Now we can define the action of \( \text{Vect}_0(\mathbb{R}^n) \) on the space of jets \( \mathcal{F}_k \). For \( V \in \text{Vect}_0(\mathbb{R}^n) \):

\[ L_V (j^k \Gamma) = j^k (L_V \Gamma) , \]

where \( \Gamma \) on the right is an arbitrary representative of the \( j^k \Gamma \) on the left.

This is well-defined, since in the coordinate version of (2.1):

\[ (L_V \Gamma)_{ij}^l = V^k \frac{\partial \Gamma_{ij}^l}{\partial x^k} - \Gamma_{ij}^l \frac{\partial V^l}{\partial x^k} + \Gamma_{ij}^k \frac{\partial V^l}{\partial x^k} + \Gamma_{ij}^l \frac{\partial V^k}{\partial x^l} + \partial^2 V^l_{ij} \]

elements of \( k \)-th order and less are only coming from \( j^k \Gamma \), because \( V(0) = 0 \). Einstein summation convention in (2.2) above and further on is assumed.
Consequently, the action is invariantly defined. This can also be expressed as commutativity of the following diagram:

\[
\begin{array}{cccccc}
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \\
\pi_k & \pi_k & \pi_k & \pi_k & \pi_k & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \\
\end{array}
\]

where \( \pi_k \) is the projection from \( k \)-jets onto \((k - 1)\)-jets, \( F \) and \( \Pi \) denote the spaces of germs of connections and that of (1,2)-tensors, respectively, at 0.

Poincaré series will encode information about these actions for all \( k \).

The space

\[ M = \mathcal{F}/\text{Diff}(\mathbb{R}^n, 0) \]

of \( \text{Diff}(\mathbb{R}^n, 0) \)-orbits on \( \mathcal{F} \) is called the moduli space of connections at 0 on \( \mathbb{R}^n \).

We do not introduce any topology on \( M \). Similarly, the orbit space

\[ M_k = \mathcal{F}_k/\text{Diff}(\mathbb{R}^n, 0) = \mathcal{F}_k/K_k \]

is called the moduli space of connection \( k \)-jets.

The action of \( K_k \) is algebraic, a subspace \( \mathcal{F}_k^0 \subset \mathcal{F}_k \) of points on generic orbits (those of the largest dimension) is a smooth manifold, open and dense in \( \mathcal{F}_k \).

A subspace of points on orbits of any other given dimension is a manifold as well, albeit of a lesser dimension. We could consider the \( G \)-quotient for each of those subspaces, and have a moduli space of its own for each of the orbit types.

Let \( O_k \) denote a generic orbit. Denote by \( M_k^0 \) the moduli space of generic connections:

\[ M_k^0 = \mathcal{F}_k^0/\text{Diff}(\mathbb{R}^n, 0) = \mathcal{F}_k^0/K_k, \]

or the generic subspace of the moduli space \( M_k \). Its dimension is found as:

\[ \dim M_k^0 = \dim \mathcal{F}_k^0 - \dim O_k \]

However it is no longer true that the generic moduli (sub)space retains maximal dimension after the quotient is taken, even if we restrict our attention to algebraic Lie group actions. For an explicit counterexample see [MWZ].

Thus we define

\[ \dim M_k = \dim M_k^0 \]

for mere simplicity of notation. One more piece of notation:

\[ a_k = \begin{cases} 
\dim M_k, & k = 0 \\
\dim M_k - \dim M_{k-1}, & k \geq 1 
\end{cases} \]

and we can introduce our main object of interest.

**Definition 2.3.** The formal power series

\[ p_M(t) = \sum_{k=0}^{\infty} a_k t^k \]

is called the Poincaré series for the moduli space \( M \).

Our main result is the following theorem.
Theorem 2.4. Poncaré series coefficients $a_k = a(k)$ are polynomial in $k$, and the series has the form:

$$p_\Gamma(t) = n \sum_{k=1}^{\infty} \left[ n^2 \binom{n+k-1}{n-1} - \binom{n+k+1}{n-1} \right] t^k + \frac{n^2(n-3)}{2} + \delta_1^n + 2\delta_2^n (1-t).$$

($\delta$ is the Kronecker symbol).

It represents a rational function.

Remark 2.5. This complies with Tresse’ assertion that algebras of “natural” differential-geometric structures are finitely-generated.

Proof of this theorem is relegated to section 5.

To explain significance of rationality of Poincaré series we make the following

Remark 2.6. If a geometric structure is described by a finite number of functional moduli, then its Poncaré series is rational. In particular, if there are $m$ functional invariants in $n$ variables, then

$$p(t) = \frac{m}{(1-t)^n}$$

Indeed, dimension of the moduli spaces of $k$-jets is just the number of monomials up to the order $k$ in the formal power series of the $m$ given invariants:

$$\dim M_k = m \binom{n+k}{n}$$

For more details and slightly more general formulation see Theorem 2.1 in [Sh2].

3. Stabilizer of a generic $k$-jet

The main challenge in calculating a Poincaré series is finding the stabilizer of a generic $k$-jet. Here we do it in fixed normal coordinates. The suggestion to use special coordinates is due to Vlassov, [V].

Let $\nabla$ be a connection given in local coordinates around origin by its Christoffel symbols $\Gamma^i_{jk}$.

Definition 3.1. A geodesic curve is the solution of

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad x(0) = 0, \quad x'(0) = v.$$

Definition 3.2. A coordinate system $x$ is called an affine normal coordinate system when solutions of (3.4) are linear in the canonical parameter $t$:

$$x^i = a^i t.$$

These are exactly the coordinates induced from the tangent space by the exponential map of $\Gamma$

$$\exp_0(v) = x(1),$$

where $v \in U$, a small neighborhood of 0 in $T_0\mathbb{R}^n$, and $x(t)$ is the geodesic defined by (3.4).
Lemma 3.3. Coordinates $x$ on $\mathbb{R}^n$ are normal if and only if:

$$\Gamma^i_{jk}(x) x^j x^k \equiv 0.$$  

Corollary 3.4. Let $x$ be a normal coordinate system. We have:

$$\Gamma^i_{jk}(0) + \Gamma^i_{kj}(0) = 0$$

for distinct $j, k, l$ (i.e. terms with repeating lower indexes, as $\partial \Gamma^i_{jj} \partial x^k$, appear only once).

In general for $r = 1, 2, \ldots$,

$$S_{j,k\to j,k,\alpha_1\ldots\alpha_r} \left( \frac{\partial^r \Gamma^i_{jk}}{\partial x^{\alpha_1} \ldots \partial x^{\alpha_r}} \bigg|_{x=0} \right) = 0,$$

where $S_{j,k\to j,k,\alpha_1\ldots\alpha_r}$ is the sum of all terms obtained from the term inside the parentheses by replacing the pair $(j, k)$ by any ordered pair from the set $\{j, k, \alpha_1 \ldots \alpha_r\}$.

In fact, the converse holds as well, cf. \cite[VI.41]{Th} and \cite[Prop.4.2]{GRS}.

Namely, let $\{\Gamma^i_{jk\alpha_1\ldots\alpha_r} | r = 1, 2, \ldots\}$ be a set of numbers (or tensors given at a point), symmetric in $\{\alpha_1 \ldots \alpha_r\}$, satisfying (analogue of) (3.7) and $S_{j,k\to j,k,\alpha_1\ldots\alpha_r} \left( \Gamma^i_{jk\alpha_1\ldots\alpha_r} \right) = 0$

as in (3.9), and such that the power series

$$\sum_{r=1}^{\infty} \frac{1}{r!} \Gamma^i_{jka_1\ldots a_r} x^{a_1} \ldots x^{a_r}$$

converges near 0.

Then this series defines an object $\Gamma^i_{jk}(x)$ satisfying (3.6) in given coordinates $x$. Hence $\Gamma^i_{jk}(x)$ is a set of Christoffel symbols of a general connection in given local coordinates $x$, such that the system of coordinates $x$ is normal for it. We say that Cor 3.4 gives a complete set of identities for Christoffel symbols of a connection in normal coordinates.

For more details on normal coordinates, including proofs of the Lemma 3.3 and its corollary, consult \cite[VI.41]{Th}, \cite{Veb} or \cite[Sec.4]{GRS}.

Let us now turn to the problem of finding the stabilizer of a given connection $\Gamma$. From this point on through the rest of the section we assume that local coordinates $x$ are normal coordinates associated to $\Gamma$.

In these coordinates let us introduce grading in homogeneous components on $\Gamma$:

$$\Gamma = \Gamma_0 + \Gamma_1 + \ldots$$

and on diffeomorphism generating $V \in \text{Vect}_0(\mathbb{R}^n)$:

$$V = V_1 + V_2 + \ldots$$

($V_0 = 0$, since $V$ preserves the origin).
A diffeomorphism preserving \( \Gamma \) must preserve its set of geodesics, which in this coordinate system are lines \( (3.5) \), all parametrized by the canonical parameter \( t \). Such diffeomorphisms are linear maps, e.g.:

\[
V = V_1, \quad V_2 = V_3 = \ldots = 0.
\]

Then the stabilizer condition \( \mathcal{L}_V \Gamma = 0 \) considered as an equation on \( V \) becomes:

\[
\mathcal{L}_V \Gamma_0 = 0 \\
\mathcal{L}_V \Gamma_1 = 0 \\
\vdots \\
\mathcal{L}_V \Gamma_k = 0.
\]

(3.10)

We need to find all \( V \) solving this system for a generic \( \Gamma \). Write

\[
\gamma^l_{ij} := \Gamma^l_{ij}(0), \quad \gamma^k = \sum_{s=1}^n b^k_s x^s.
\]

(3.11)

In this notation the first equation of (3.11) is a linear system on \( b^k_i \):

\[
-\gamma^k_{ij} b^k_i + \gamma^k_{kj} b^k_j + \gamma^k_{ik} b^k_j = 0,
\]

(3.12)

indexed by \{\( (ijl), i < j \)\}, it is sufficient to consider \( i < j \) since exchanging them leads to the same equation. The coefficients \( \gamma \) are subject only to:

\[
\gamma^l_{ij} = -\gamma^l_{ji}
\]

(3.13)
due to (3.7).

We will presently show that for \( n \geq 3 \) this system is non-degenerate in general position. Thus the stabilizer is trivial (for any order jet). Exceptional dimensions 1 and 2 are considered in the next section.

It suffices to present one such connection (or just its constant part \{\( \gamma^l_{ij} \)\}) which makes the system non-degenerate. Namely, set the coefficients to:

\[
\gamma^\alpha_{\alpha \beta} = \begin{cases} 1, & \alpha < \beta \\ -1, & \alpha > \beta \end{cases}
\]

(3.14)

and zero otherwise (when the upper index fails to match either of the lower ones). Then (3.12) splits into the following non-trivial cases according to the index \( (ijl) \):

\[
l < i < j \\
i < j < l \\
implying b^\alpha_\beta = 0 \quad \text{for } \alpha \neq \beta,
\]

finishing the proof in dimensions 3 and higher (availability of three distinct indexes is essential).
4. Exceptions: the stabilizer in low dimensions

We start with the case \( n = 1 \). In this case we have only one function in one variable:

\[ \Gamma_{11}^1(x) \equiv 0, \]

because of (3.0). This means the stabilizer conditions (3.10) are all empty, and the stabilizer has the maximal possible dimension 1.

In the case \( n = 2 \), the stabilizer of the 0-jet is determined by (3.12). With only two distinct indexes available it makes for a system of two equations in four variables:

\[
\begin{align*}
\frac{\partial}{\partial x^l} \Gamma^i_j (0) &= \Gamma^i_j (0) \\
(122) &
\begin{pmatrix}
\gamma^2_{12} & \gamma^1_{12} & -\gamma^1_{12} & \gamma^2_{12} \\
-\gamma^2_{12} & \gamma^2_{12} & \gamma^1_{12} & \gamma^1_{12}
\end{pmatrix}
\end{align*}
\]

It is of full rank in general position, since the coefficients are only subject to (3.13), making the stabilizer of the 0-jet 2-dimensional.

For the 1-jet, in addition to the above, further restrictions on \( V \) are imposed by the second equation in (3.10). Denoting \( \frac{\partial}{\partial x^l} \Gamma^i_j (0) \) by \( \Gamma^i_j, l \), this equation produces the following system:

\[
\Gamma^i_{ij,k} b^k_s - \Gamma^i_{ij,s} b^k_k + \Gamma^i_{jk,s} b^k_i + \Gamma^i_{ik,s} b^k_j = 0.
\]

There are 16 equations indexed by arbitrary 4-tuples \((ijls)\). The coefficients satisfy:

\[
\Gamma^i_{11,1} = \Gamma^i_{22,2} = 0
\]

(4.16)

\[
\Gamma^i_{12,1} + \Gamma^i_{21,1} + \Gamma^i_{11,2} = 0, \quad i = 1, 2
\]

\[
\Gamma^i_{12,2} + \Gamma^i_{21,2} + \Gamma^i_{22,1} = 0,
\]

all consequences of (3.8). The twelve nontrivial equations have the matrix below:

\[
\begin{pmatrix}
\Gamma^1_{11,2} & -\Gamma^2_{11,2} & -\Gamma^1_{22,1} & \Gamma^2_{11,2} \\
2\Gamma^1_{11,2} & 0 & -(\Gamma^2_{11,2} + \Gamma^1_{22,1}) & 0 \\
\Gamma^1_{22,1} & -\Gamma^2_{22,1} & -\Gamma^1_{11,2} & \Gamma^2_{22,1} \\
0 & -(\Gamma^1_{11,2} + \Gamma^2_{22,1}) & 0 & 2\Gamma^1_{11,2} \\
\Gamma^2_{12,1} & 0 & -(\Gamma^2_{21,2} + \Gamma^1_{12,1}) & 0 \\
2\Gamma^2_{12,1} & 0 & -(\Gamma^2_{21,2} + \Gamma^1_{12,1}) & 0 \\
\Gamma^2_{21,1} & -\Gamma^1_{21,1} & -\Gamma^2_{12,2} & \Gamma^1_{21,1} \\
\Gamma^2_{21,1} & 0 & -(\Gamma^1_{12,1} + \Gamma^2_{21,1}) & 0 \\
2\Gamma^2_{21,1} & 0 & -(\Gamma^1_{12,1} + \Gamma^2_{21,1}) & 0 \\
\Gamma^2_{21,2} & -\Gamma^1_{21,2} & -\Gamma^2_{12,2} & \Gamma^1_{21,2} \\
0 & -(\Gamma^1_{12,1} + \Gamma^2_{22,1}) & 0 & 2\Gamma^1_{21,2} \\
\Gamma^2_{22,1} & -\Gamma^1_{22,1} & -\Gamma^2_{12,2} & \Gamma^1_{22,1} \\
\Gamma^2_{22,1} & 0 & -(\Gamma^1_{12,1} + \Gamma^2_{22,1}) & 0
\end{pmatrix}
\]

Setting the coefficients \( \Gamma^i_{12,j} = 2, \Gamma^i_{21,j} = \Gamma^i_{kk,j} = -1 \) \((i, j \text{ arbitrary}, k \neq j)\) produces a non-degenerate system (e.g. the top four equation minor is non-trivial).
Thus the only solution is zero and the stabilizer is trivial for jets of order one (and higher) in dimension two. To summarize:

**Proposition 4.1.** The stabilizer of a $k$-jet of a generic connection:
- for $n = 1$ is 1-dimensional for any $k$;
- for $n = 2$ is 2-dimensional for $k = 0$ and trivial for $k \geq 1$;
- for $n \geq 3$ is trivial for any $k$.

5. Poincaré series

We use the Proposition 4.1 above to find the dimension of a generic orbit:

$$\dim \mathcal{O}_k(\Gamma) = \dim(T_{id}(K_k/G_{\Gamma}))$$

$$= \dim(\{V|V = V_1 + V_2 + \ldots + V_{k+1} + V_{k+2}\}) - \delta^n_1 - 2\delta^n_2 \delta^k_0,$$

where $\delta$ is a Kronecker symbol, taking care of non-trivial stabilizers for various $k$ and $n$; $V_i$ is an $n$-component vector, each component a homogeneous polynomial of degree $i$ in $(x^1, \ldots, x^n)$. So we have:

$$\dim \mathcal{O}_k = n \sum_{m=1}^{k+2} \binom{n+m-1}{n-1} - \delta^n_1 \quad \text{for } k \geq 1,$$

$$\dim \mathcal{O}_0 = n \sum_{m=1}^{2} \binom{n+m-1}{n-1} - \delta^n_1 - 2\delta^n_2 = \frac{n^2(n+3)}{2} - \delta^n_1 - 2\delta^n_2.$$

Dimension of the moduli space of connection $k$-jets $\mathcal{M}_k$ is:

$$\dim \mathcal{M}_k = \dim \mathcal{F}_k - \dim \mathcal{O}_k,$$

where $\mathcal{F}_k$ is the space of connection $k$-jets.

For $k = 0$:

$$\dim \mathcal{M}_0 = \dim \mathcal{F}_0 - \dim \mathcal{O}_0 = \frac{n^2(n-3)}{2} + \delta^n_1 + 2\delta^n_2.$$

For $k \geq 1$:

$$\dim \mathcal{M}_k = \dim \mathcal{F}_k - \dim \mathcal{O}_k$$

$$= n^3 \sum_{m=0}^{k} \binom{n+m-1}{n-1} - n \sum_{m=1}^{k+2} \binom{n+m-1}{n-1} + \delta^n_1.$$

The Poincaré series is:

$$p_\Gamma(t) = \dim \mathcal{M}_0 + \sum_{k=1}^{\infty} (\dim \mathcal{M}_k - \dim \mathcal{M}_{k-1}) t^k$$

$$= n \sum_{k=1}^{\infty} \left[ n^2 \binom{n+k-1}{n-1} - \binom{n+k+1}{n-1} \right] t^k + \frac{n^2(n-3)}{2} + \delta^n_1 + 2\delta^n_2 (1 - t).$$

Note that since $\dim \mathcal{M}_0$ is exceptional, the linear term in the series had to be corrected by $-2\delta^n_2 t$. In dimension one $p_\Gamma(t) \equiv 0$. Simplifying the above, we obtain the following.
Fact. The Poincaré series $p_\Gamma(t)$ is a rational function. Namely,

$$p_\Gamma(t) = \delta_1^n + 2\delta_2^n(1-t) - n^2 + nD_\Gamma \left( \frac{1}{1-t} \right),$$

where $D_\Gamma$ is a differential operator of order $n - 1$:

$$D_\Gamma = n^2 \left( \frac{n + t \frac{d}{dt} - 1}{n - 1} \right) - \left( \frac{n + t \frac{d}{dt} + 1}{n - 1} \right),$$

with

$$\left( \frac{n + t \frac{d}{dt} - 1}{n - 1} \right) = \frac{1}{(n-1)!} \left( \frac{d}{dt} + 1 \right) \ldots \left( \frac{d}{dt} + n - 1 \right),$$

$$\left( \frac{n + t \frac{d}{dt} + 1}{n - 1} \right) = \frac{1}{(n-1)!} \left( \frac{d}{dt} + 3 \right) \ldots \left( \frac{d}{dt} + n + 1 \right).$$

Indeed, denote

$$\varphi_m(t) = \sum_{k=0}^\infty k^m t^k, \quad m \in \mathbb{Z}_+,$$

then

$$\varphi_m(t) = \sum_{k=0}^\infty k^{m-1} k t^{k-1} t = t \left( \sum_{k=0}^\infty k^{m-1} t^k \right)' = \left( t \frac{d}{dt} \right) \varphi_{m-1}(t) \quad \text{for} \quad m \in \mathbb{N}.$$ 

Thus

$$\varphi_m(t) = \left( \frac{d}{dt} \right)^m \varphi_0(t) = \left( \frac{d}{dt} \right)^m \left( \frac{1}{1-t} \right).$$

Hence,

$$\sum_{k=0}^\infty \left[ n^2 \left( \frac{n + k - 1}{n - 1} \right) - \left( \frac{n + k + 1}{n - 1} \right) \right] t^k = \left[ n^2 \left( \frac{n + t \frac{d}{dt} - 1}{n - 1} \right) - \left( \frac{n + t \frac{d}{dt} + 1}{n - 1} \right) \right] \left( \frac{1}{1-t} \right).$$

We have to account for the “extra” $0^{th}$ term, since the Poincaré series does not have it:

$$a_0 = n^2 - \left( \frac{n + 1}{n - 1} \right) = \frac{n(n - 1)}{2}.$$ 

So

$$p_\Gamma(t) = \frac{n^2(n - 3)}{2} + \delta_1^n + 2\delta_2^n(1-t) + n \sum_{k=0}^\infty \left[ n^2 \left( \frac{n + k - 1}{n - 1} \right) - \left( \frac{n + k + 1}{n - 1} \right) \right] t^k - na_0$$

$$= \delta_1^n + 2\delta_2^n(1-t) - n^2 + nD_\Gamma \left( \frac{1}{1-t} \right).$$

Now we shall derive the explicit formula for this rational function.

Lemma 5.1. For $N \geq 2$:

$$\left( \frac{N + t \frac{d}{dt} - 1}{N - 1} \right) \frac{1}{1-t} = \frac{1}{(1-t)^N},$$

$$\left( \frac{N + t \frac{d}{dt} + 1}{N - 1} \right) \frac{1}{1-t} = \frac{1}{(1-t)^N} + \frac{2}{(1-t)^{N-1}} + \frac{3}{(1-t)^{N-2}} + \ldots.$$
\[ \frac{k}{(1 - t)^{N-k+1}} + \ldots + \frac{N}{(1 - t)} \]

**Proof**  is by induction on \( N \). \( \Box \)

This together with the fact that \( p_T(t) \equiv 0 \) for \( n = 1 \) implies the following in all dimensions.

**Corollary 5.2.**

\[
(5.19) \quad p_T(t) = \delta_1^n + 2\delta_2^n (1 - t) - n^2 + n \left( \frac{n^2 - 1}{(1 - t)^n} - \frac{2}{(1 - t)^{n-1}} - \ldots - \frac{n}{(1 - t)} \right).
\]

This formula shows that \( p_T(t) \) is not an arbitrary rational function, but one of the form required by the Tresse finiteness claim, with poles exclusively at \( t = 1 \), cf. Remark 270.

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[V] A.T.Vlassov, private communication.

**Department of Mathematics, Northeastern University, Boston Mass. 02115**

**E-mail address:** dubr@neu.edu