Research Article

Inertial Forces and Torques Acting on a Spinning Annulus

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The many publications related to the gyroscope theory consider the action of the inertial torques on the spinning disc. All of them have simplified expressions for the inertial torques and mathematical models of the gyroscope motions do not validate by practice. Recent research in the theory of the gyroscopic effects for rotating objects solved problems with mathematical models for interrelated inertial torques generated by the spinning disc, bar, and ring and their motions. Practitioners of engineering designed gyroscopic devices with spinning rotors whose geometry can be an annulus or similar designs. Such spinning annulus generates inertial torques whose expressions differ from the disc bar and ring and hence another mathematical model describes the motions of the gyroscopic devices. The value of gyroscopic effects of the devices with spinning annulus is bigger than for the disc-type rotor. This manuscript presents mathematical models for the inertial torques generated by the spinning annulus and the interrelated angular velocities of the gyroscopic devices about axes of rotation.

1. Introduction

The gyroscopic effects in engineering mechanics are the most complex problems that are being unsolved for a long time [1–5]. Beginning from the Industrial Revolution, scientists yield only partial analytical solutions but did not solve the entire gyroscopic effects. For practical applications were worked out numerical models with the expensive software for gyroscopic effects. The physics of the gyroscopic effects of the simple spinning disc remained unexplained and mysterious for a couple of centuries [6–10]. Scientists and researchers of our time continue to describe gyroscopic effects without success which can be seen in many publications each year [11–19]. Recent investigations of the gyroscopic effects showed their nature is more complex and based on several physical principles that were discovered at different times. Scientists of 18-19 centuries could not solve gyroscopic effects in principle because the concept of mechanical energy conservation was formulated at the beginning of twenty century. From this time, researchers have all the analytical tools for formulation of the gyroscope theory but did not do it.

The latest studies of the physics of the gyroscopic effects yield mathematical models for the inertial torques and the interrelated dependency of the angular velocities of the spinning disc motions about axes of rotation. The solution to gyroscopic effects is based on the principle of mechanical energy conservation [20–22]. The method of deriving mathematical models for inertial torques shows the action of the system of the centrifugal, Coriolis forces generated by the rotating mass, and the change in the angular momentum of the spinning disc. The dependency of the angular velocities of the spinning disc around axes of motions interrelates the inertial torques. Table 1 presents the expressions of inertial torques acting on the spinning disc and the dependency of the angular velocities of it’s motions.

The practice tests validate the action of the system of the interrelated inertial torques on the spinning disc. Practitioners should use the derived method for the modeling of the gyroscopic effects for any designs of the spinning objects [22]. The mathematical models for the inertial torques generated by the centrifugal and Coriolis forces and the change in the angular momentum acting on the spinning flat annulus and...
its interrelated motions around two axes are present as the contribution of the manuscript.

2. Centrifugal Forces and Torques Acting on a Spinning Annulus

The method for the analytical solution for all inertial torques acting on the spinning disc is described in several publications [21, 22]. The inertial torques are generated by the distributed mass elements of the spinning annulus disposed on the circle of 2/3 of its radius. This method can be applied to rotating objects of different forms. For the annulus, the radius of the disposition of the distributed mass elements on the circle should be defined. The rotating mass elements produce the plane of centrifugal forces of the spinning annulus around axis \( ox \) with an angular velocity of \( \omega \) in a counter-clockwise direction considered in Figure 1. The mass element \( m \) is disposed on the circle of radius \( r \) which is perpendicular to the axis of the spinning annulus.

The radius \( r \) of disposing of the mass elements of the annulus is defined from the truncated sector with the small angle \( \Delta \delta \) and the arcs of the external and internal radii \( R_e \) and \( R_i \), respectively. The rotating mass elements generate centrifugal forces. The action of the external torque to the spinning annulus has manifested the inclination of its plane with the rotating centrifugal forces and the turns of the annulus around axes. These motions are presented at the Cartesian 3D coordinate system \( \Sigma oxyz \) in Figure 1. The external torque applied to the spinning annulus generates several inertial torques (Figure 2).

The plane of the rotating centrifugal forces turns around axis \( ox \) along the diameter line and changes in the directions of centrifugal force vectors \( f_{ct} \) of the mass elements. The inclined plane of rotating centrifugal forces on angle \( \Delta \gamma \) around axis \( ox \) is presented in the plane \( xoy \). The action of the centrifugal force vectors \( f_{ct} \) is radial and their components \( f_{ct,z} \) produce the torques around axes \( ox \) and \( oy \). The values of torques acting around axis \( ox \) are maximal at 90° and 270° and zero at 0° and 180° (Figure 2(a)). The values of torques acting around axis \( oy \) are maximal at 0° and 180° and zero at 90° and 270° (Figure 2(b)). The integrated product of the components’ vector of the centrifugal forces \( f_{ct,z} \) and their radii relative to axes \( ox \) and \( oy \) give the torques \( T_{ct} \) acting around two axes. The torque acting around axis \( ox \) resists the action of the external torque \( T \). The torque acting around axis \( oy \) turns the spinning annulus in the counter-clockwise direction.

The mathematical models for the inertial torques generated by the centrifugal forces of the mass elements are expressed as follows:

\[
\Delta T_{ct} = -f_{ct,z}y_m = -ma_y y_m, \quad (1)
\]

\[
\Delta T_{ct} = f_{ct,z}x_m = ma_x x_m, \quad (2)
\]

where \( \Delta T_{ct} \) is the centrifugal torque generated by the mass element; \( f_{ct,z} \) is the component of the centrifugal force; \( y_m \) and \( x_m \) are the distance from the mass element to axes \( oy \) and \( ox \), respectively; \( m \) is the mass of the annulus disposed on the circle of the radius \( r \); \( \alpha = \omega r^2 \) is the radial acceleration; \( \omega \) is the angular velocity of the spinning annulus; the signs (+) and (-) mean the counter-clockwise and clockwise direction, respectively.

The following equations express the component of the centrifugal force acting along axis \( ox \):

(i) Around axis \( ox \)

\[
f_{ct,z} = f_{ct} \sin \alpha \sin \Delta \gamma = m r \omega^2 \sin \alpha \sin \Delta \gamma = \frac{Mr \omega^2}{2\pi} \Delta \delta \Delta \gamma \sin \alpha. \quad (3)
\]

(ii) Around axis \( oy \)

\[
f_{ct,z} = f_{ct} \cos \alpha \sin \Delta \gamma = m r \omega^2 \cos \alpha \sin \Delta \gamma = \frac{Mr \omega^2}{2\pi} \Delta \delta \Delta \gamma \cos \alpha, \quad (4)
\]

where \( f_{ct} = m r \omega^2 \) is the centrifugal force of the mass element \( m \); \( r = (M/2\pi) \Delta \delta \) in which \( M \) is the mass of the annulus; \( \Delta \delta \) is the sector angle of the mass element’s disposition; \( \alpha \) is the angle of the mass element’s disposition; \( \Delta \gamma \) is the angle of turn for the annulus plane, \( \sin \Delta \gamma = \Delta \gamma \) is the trigonometric identity for the small values of the angle.

The radius \( r \) of the circle of disposing of the mass elements \( m \) of the annulus is defined from the truncated sector with the small angle \( \Delta \delta \) and the arcs of the external and internal radii \( R_e \) and \( R_i \), respectively (Figure 1). The radius \( r \) is expressed

| Type of the torque generated by | Action | Equation |
|--------------------------------|--------|---------|
| Centrifugal forces            | Resistance | \( T_{ct} = (4/9)\pi^2 f \omega x \) |
| Coriolis forces               | Precession | \( T_{ct} = (8/9)\pi^2 f \omega x \) |
| Change in angular momentum    | Precession | \( T_{am} = f \omega x \) |
| Dependency of angular velocities of spinning disc rotations about axes | | \( \omega_y = (8\pi^2 + 17)\omega x \) |
by the equation of the centroid of the truncated sector

$$r = \frac{\Sigma A_i r_i}{A_i} = \frac{\left(\frac{2\pi}{3} (R_i^3 - R_e^3) / 2\pi - \left(\frac{\pi}{3} (R_e^2 - R_i^2) / 2\pi\right)\right)}{\left(\frac{\pi}{3} (R_e^2 - R_i^2) / 2\pi\right)} \times \Delta \delta / 2\pi$$

$$= \frac{2(R_i^3 - R_e^3)}{3(R_e^2 - R_i^2)},$$

(5)

where $A_e = (\pi R_e^2 / 3) \pi$ and $A_i = (\pi R_i^2 / 3) \pi$ are the area of the sectors of radii $R_e$ and $R_i$ and $(2/3)R_e$ and $(2/3)R_i$ are the radii of the mass elements disposition, respectively; $A_{y_\alpha} = (\pi (R_e^2 - R_i^2) / 3) \pi$ is the area of the truncated sector.

Substituting the expressions of $f_{ct,z}$ of Equations (3)–(5) into Equations (1) and (2) and transformation yield the expressions of the inertial centrifugal torques produced by the mass element:

$$\Delta T_{ct} = \frac{M (R_i^3 - R_e^3) \alpha^2}{3n(R_e^2 - R_i^2)} \Delta \delta \Delta y \times \sin \alpha \times y_m, \quad (6)$$

$$\Delta T_{ct} = \frac{M (R_i^3 - R_e^3) \alpha^2}{3n(R_e^2 - R_i^2)} \Delta \delta \Delta y \times \cos \alpha \times x_m, \quad (7)$$

where $y_m = r \sin \alpha$ and $x_m = r \cos \alpha$ are the distance from the mass element of the annulus to the axis $x$ to the centroid relative to axes $x$ and $y$. The action of the torque is defined by a concentrated load at the centroid points at the semi-circles along axes $x$, $y$, and $z$. The known integrated equation calculates the disposition of the centroid (point $A$ of Figure 2(a) and point $B$ of Figure 2(b)) [6–9].

$$l_z = \int_0^\alpha f_{ct,z} l_m d\alpha,$$

(8)

where $f_{ct,z} = (M (R_e^3 - R_i^3) / 3\pi(R_e^2 - R_i^2)) \Delta \delta \Delta y \times \sin \alpha$, $l_z = \int_0^\alpha l_m d\alpha$.

Substituting Equations (4) and (5) and other components into Equation (8) and transformation yield the following expression.

(i) About axis $x$

$$y_A = \frac{\int_0^\alpha f_{ct,z} y_m d\alpha}{\int_0^\alpha f_{ct,z} d\alpha} = \frac{\int_0^\alpha (M (R_i^3 - R_e^3) / 3\pi(R_e^2 - R_i^2)) \Delta \delta \Delta y \times (2/3) ((R_e^2 - R_i^2) / (R_i^2 - R_e^2)) \sin \alpha \sin \alpha d\alpha}{\int_0^\alpha (M (R_i^3 - R_e^3) / 3\pi(R_e^2 - R_i^2)) \Delta \delta \Delta y \sin \alpha d\alpha}$$

$$= \frac{(M (R_i^3 - R_e^3) / 3\pi(R_e^2 - R_i^2)) \Delta \delta \Delta y (2/3) ((R_e^2 - R_i^2) / (R_i^2 - R_e^2)) \int_0^\alpha (1 - \cos 2\alpha) d\alpha}{\int_0^\alpha \sin \alpha d\alpha}$$

$$= \frac{(M (R_i^3 - R_e^3) / 3(R_e^2 - R_i^2)) \int_0^\alpha (1 - \cos 2\alpha) d\alpha}{\int_0^\alpha \sin \alpha d\alpha}.$$
(ii) About axis oy

\[ x_B = \frac{\int_{\alpha=0}^{\alpha} \int_{r=0}^{R} f(r, \alpha) x(r, \alpha) \, dr \, d\alpha}{\int_{\alpha=0}^{\alpha} \int_{r=0}^{R} f(r, \alpha) x(r, \alpha) \, dr \, d\alpha} = \frac{\int_{\alpha=0}^{\alpha} (M(R^2 - R_r^2) \omega^2/3\pi(R^2 - R_r^2)) \Delta \delta \times \Delta y \times (2/3)((R^2 - R_r^2)/(R^2 - R_f^2)) \cos \alpha \cos \alpha \, d\alpha}{\int_{\alpha=0}^{\alpha} (M(R^2 - R_r^2) \omega^2/3\pi(R^2 - R_r^2)) \Delta \delta \times \Delta y \cos \alpha \, d\alpha} \]

\[ = \frac{(2/(3 \times 2))((R^2 - R_f^2)/(R^2 - R_r^2)) \int_{\alpha=0}^{\alpha} (1 + \cos 2\alpha) \, d\alpha}{\int_{\alpha=0}^{\alpha} (1 + \cos 2\alpha) \, d\alpha} \]

\[ = \frac{((R^2 - R_f^2)/(3(R^2 - R_r^2)) \int_{\alpha=0}^{\alpha} (1 + \cos 2\alpha) \, d\alpha}{\int_{\alpha=0}^{\alpha} (1 + \cos 2\alpha) \, d\alpha} , \]

**Figure 2**: Schematic of acting centrifugal forces, torques, and motions around axis ox (a) and oy (b) of the spinning annulus.
where the expression \((M(R_2^3 - R_1^3)\omega^2/3\pi(R_2^3 - R_1^3))\Delta\delta \times \Delta\gamma\) is constant of Equations (9) and (10); the expressions \(\sin^2\alpha = (1 - \cos 2\alpha)/2\) and \(\cos^2\alpha = (1 + \cos 2\alpha)/2\) are a trigonometric identity, and other parameters are as specified above.

Equations (6) and (7) are presented in differential and integral forms. Substituting expressions of Equations (9) and (10) into Equations (6) and (7), replacing \(\sin\alpha = \int_0^\pi \cos \alpha \, d\alpha\) and \(\cos\alpha = -\int_0^\pi \sin \alpha \, d\alpha\) by the integral expressions with defined limits, respectively, the following equations emerge:

(i) About axis \(ox\)

\[
\int_0^{T_o} dT_{ca} = \frac{M(R_2^3 - R_1^3)\omega^2}{3\pi(R_2^3 - R_1^3)} \times \int_0^\pi d\delta \times \int_0^\pi dy \times \int_0^\pi \cos \alpha \, d\alpha \\
\times \left(\frac{R_2^3 - R_1^3}{3(R_2^3 - R_1^3)}\right) \times \frac{1}{\int_0^\pi \sin \alpha \, d\alpha}.
\]

(11)

(ii) About axis \(oy\)

\[
\int_0^{T_o} dT_{ca} = \frac{M(R_2^3 - R_1^3)\omega^2}{3\pi(R_2^3 - R_1^3)} \times \int_0^\pi d\delta \times \int_0^\pi dy \times (-) \int_0^\pi \sin \alpha \, d\alpha \\
\times \left(\frac{R_2^3 - R_1^3}{3(R_2^3 - R_1^3)}\right) \times \frac{1}{\int_0^\pi \cos \alpha \, d\alpha}.
\]

(12)

Solutions of integral Equations (11) and (12) yield the following:

(i) About axis \(ox\)

\[
T_{ca}|_{T_o} = \frac{M}{9} \times \left(\frac{R_2^3 - R_1^3}{R_2^3 - R_1^3}\right)^2 \omega^2 \times (\delta|_0^\pi) \times (y|_0^\pi) \\
\times 2(2\sin \alpha|_{|0}^{\pi/2}) \times \left(1 - (1/2) \sin 2\alpha|_0^\pi\right) \\
\times \cos \alpha|_0^{\pi/2} \times \left(1 + (1/2) \sin 2\alpha|_0^\pi\right) \\
\times 2 \sin \alpha|_0^{\pi/2}.
\]

(13)

(ii) About axis \(oy\)

\[
T_{ca}|_{T_o} = \frac{M}{9} \times \left(\frac{R_2^3 - R_1^3}{R_2^3 - R_1^3}\right)^2 \omega^2 \times (\delta|_0^\pi) \times (y|_0^\pi) \\
\times (\cos \alpha|_0^\pi) \times \left(1 + (1/2) \sin 2\alpha|_0^\pi\right) \\
\times 2 \sin \alpha|_0^{\pi/2}.
\]

(14)

that gave rise to the following:

(i) About axis \(ox\)

\[
T_{ca} = \frac{M}{9} \left(\frac{R_2^3 - R_1^3}{R_2^3 - R_1^3}\right)^2 \omega^2 \times (\pi - 0) \times (y - 0) \times 2(1 - 0) \\
\times \frac{[(\pi - 0) - (1/2)(0 - 0)]}{2(1 - 0)} = \frac{M}{9} \left(\frac{R_2^3 - R_1^3}{R_2^3 - R_1^3}\right)^2 \omega^2 \pi y.
\]

(15)

(ii) About axis \(oy\)

\[
T_{ca} = \frac{M}{9} \left(\frac{R_2^3 - R_1^3}{R_2^3 - R_1^3}\right)^2 \omega^2 \times (\pi - 0) \times (y - 0) \times (-1 - 1) \\
\times \frac{[(\pi - 0) + (1/2)(0 - 0)]}{2(1 - 0)} = \frac{M}{9} \left(\frac{R_2^3 - R_1^3}{R_2^3 - R_1^3}\right)^2 \omega^2 \pi y,
\]

(16)

where all components are as specified above.

Equations (15) and (16) are almost identical except for the signs (±) of counter and clockwise action around axes \(ox\) and \(oy\). The inertial torque \(T_{ca}\) depends on the variable angle \(\gamma\) that expresses the angular velocity \(\omega\) of the annulus rotation about axis \(ox\) per time \(t\). The differential equation expresses the change in torque \(T_{ca}\) per time

\[
\frac{dT_{ca}}{dt} = \pm \frac{M}{9} \left(\frac{R_2^3 - R_1^3}{R_2^3 - R_1^3}\right)^2 \omega^2 \pi dy/dt,
\]

(17)

where \(t = a/\omega\) is the time taken relative to the angular velocity of the annulus.

The differential of time \(t\) is \(dt = da/\omega\); the expression \(dy/dt = \omega\) is the angular velocity of the spinning annulus around axis \(ox\). Substituting the defined components into Equation (17) and transforming yield:

\[
\frac{\omega dT_{ca}}{da} = \pm \frac{M}{9} \left(\frac{R_2^3 - R_1^3}{R_2^3 - R_1^3}\right)^2 \omega^2 \omega \pi y.
\]

(18)

Separation of the variables of Equation (18), transformation, and presentation by the integral form with defined limits yield:

\[
\int_0^{T_o} dT_{ca} = \pm \frac{M}{9} \left(\frac{R_2^3 - R_1^3}{R_2^3 - R_1^3}\right)^2 \omega \omega \pi da.
\]

(19)

Solution of Equation (19) yields

\[
T_{ca}|_{T_o} = \pm \frac{M}{9} \left(\frac{R_2^3 - R_1^3}{R_2^3 - R_1^3}\right)^2 \omega \omega \pi a|_0^\pi.
\]

(20)

The centrifugal torques act on the upper and lower sides of the annulus about axis \(ox\), and its left and right sides about axis \(oy\). Then, the total torque \(T_{ca}\) acting about axes \(ox\) and \(oy\) is increased double.
\[ T_{cr} = \pm \frac{2M}{9} \left( \frac{R^3_e - R^3_i}{R^2_e - R^2_i} \right)^2 \omega \omega_x n^2 \]
\[ = \pm \frac{4}{9} (R^2_e + R^2_i) \left( \frac{R^3_e - R^3_i}{R^2_e - R^2_i} \right)^2 \pi^2 j \omega_x, \]  
where \( J = (1/2)M(R^2_e + R^2_i) \) is the annulus moment of inertia, and other components are as specified above.

3. Coriolis Forces and Torques Acting on a Spinning Annulus

The Coriolis acceleration and force generated by mass elements are revealed when the spinning annulus turns around an axis perpendicular to its axle. The integrated Coriolis force produced by the rotating mass elements of the annulus generates the torque counteracting the external torque [10]. Figure 3 shows the rotation of the mass element \( m \) of the annulus that turns on the angle \( \Delta \gamma \) around axis \( ox \). The turning of the annulus around axis \( ox \) changes the direction of the tangential velocity vectors of mass elements. The change in the tangential velocity of mass elements produces the acceleration in which a product with a mass yields the Coriolis forces of the mass elements, where \( \Delta V = V \sin \Delta \gamma \), \( V = r \cos \alpha \times \omega \), and \( \sin \Delta y = \Delta y \) for the small angle (Figure 3). The maximal changes of the velocity vectors \( V \) are on the line of axis \( ox \). The tangential velocity \( V \) whose vector is parallel to axis \( o\alpha \), i.e., on 90° and 270° does not change. The changes in tangential velocity vectors are presented by the components \( V_z \) that are parallel to the annulus axle \( oz \).

The torque generated by the Coriolis force of the rotating mass element is expressed by
\[ \Delta T_{cr} = -f_{cr}y_m = -m \alpha \omega x y_m, \]  
where \( \Delta T_{cr} \) is the torque generated by Coriolis force \( f_{cr} \) of the annulus mass element \( m \); \( \alpha_z \) is the Coriolis acceleration along axis \( oz \); \( y_m = r \sin \alpha \) is the distance from the mass element to axis \( ox \); the sign (-) means the action in the clockwise direction around axis \( ox \), and the other components are represented in Equation (2).

The differential form of changes in tangential velocity vectors is \( dV/dt = a_z \) and change in the angle of rotation around axis \( ox \) is \( dy/dt = \omega_z \).

Then, the Coriolis acceleration is:
\[ \alpha_z = r \omega x \cos \alpha. \]  
The Coriolis force of the mass element is presented:
\[ f_{cr} = \frac{M}{2\pi} \times \Delta \delta \omega_x \sin \alpha = \frac{M}{3\pi} \left( \frac{R^3_e - R^3_i}{R^2_e - R^2_i} \right) \times \Delta \delta \times \omega_x \sin \alpha. \]  
(24)

Then, the Coriolis torque is:
\[ \Delta T_{cr} = -f_{cr}y_m = -\frac{M}{3\pi} \left( \frac{R^3_e - R^3_i}{R^2_e - R^2_i} \right) \omega \omega_x \times \Delta \delta \times \cos \alpha \times y_m, \]  
(25)

where all parameters are as specified above.

The centroid for the torque \( \Delta T_{cr} \) is point \( C \) of Figure 3, which is defined by Equation (8).
Substituting Equation (26) into Equation (25), replacing \( \cos \alpha = \int_0^\pi \sin \alpha \, d\alpha \) by the integral expression with defined limits, and presenting other components by the integral forms, the following equation emerges:

\[
y_c = \frac{\int_0^{\pi} \omega_x \, d\alpha}{\int_0^{\pi} \sin \alpha \, d\alpha} = \frac{M/3\pi((R_3^2 - R_2^2)/R_2^2)\omega_x \times \Delta\delta \int_0^{\pi} \sin \alpha \, d\alpha}{M/3\pi((R_4^2 - R_2^2)/R_2^2)\omega_x \times \Delta\delta \int_0^{\pi} \sin \alpha \, d\alpha} = \frac{2/3((R_4^2 - R_2^2)/R_2^2)\int_0^{\pi} \sin \alpha \, d\alpha}{\int_0^{\pi} \sin \alpha \, d\alpha}.
\]

Solution of integral Equation (27) yields:

\[
T_{cr}|_{0}^{\pi} = \frac{2M}{9\pi} \left( \frac{R_3^3 - R_4^3}{R_2^3 - R_4^3} \right)^2 \omega_x \times (\theta|_0^\pi) \times (\cos \alpha|_0^\pi) \times \frac{2(\sin^2 \alpha/2)|_0^{\pi/2}}{2 \sin \alpha|_0^{\pi/2}},
\]

that gave rise to the following:

\[
T_{cr} = -\frac{2M}{9\pi} \left( \frac{R_3^3 - R_4^3}{R_2^3 - R_4^3} \right)^2 \omega_x \times (\pi - 0) \times (-1 - 1) \times (1 - 0) \times \frac{1}{-2(1 - 0)} = -\frac{2M}{9} \left( \frac{R_3^3 - R_4^3}{R_2^3 - R_4^3} \right)^2 \omega_x,
\]

where all components are as specified above.

The inertial torque \( T_{cr} \) acts on the upper and lower sides of the annulus. Multiplying Equation (29) by two yields the full expression of \( T_{cr} \):

\[
T_{cr} = \frac{2 \times 2M}{9} \left( \frac{R_3^3 - R_4^3}{R_2^3 - R_4^3} \right)^2 \omega_x = -\frac{8}{9(R_2^3 + R_4^3)} \left( \frac{R_3^3 - R_4^3}{R_2^3 - R_4^3} \right)^2 \omega_x,
\]

where \( J = (1/2)M(R_2^3 + R_4^3) \) is the annulus moment of inertia; the sign (-) expresses the clockwise direction.

4. Attributes of Inertial Torques Acting on a Spinning Annulus

The load torque applied to the spinning annulus produces the system of the inertial torques generated by the rotating mass [22]. Among them is the change in the angular momentum of a spinning disc, which is Euler’s fundamental principle of gyroscope theory [1–5]. The motion of the spinning annulus around axis \( oy \) (Figure 3) manifests the change in the annulus angular momentum in the counter-clockwise direction which is called precession. The expression of the change in the angular momentum is \( T_{am} = J\omega_x \) where all components are presented above. The system of the inertial torques produces the resistance and precession torques and motions of the spinning annulus around axes of the rotation. Table 2 represents expressions of the inertial torques generated by pseudo forces of the spinning annulus.

The external torque \( T \) generates all inertial torques that depend on the mass moment of the annulus’s inertia \( J \), its angular velocity \( \omega \), and the angular velocity \( \omega_x \) of the spinning annulus about axis \( ox \).

The action of all torques around axes \( ox \) and \( oy \) on the spinning object is shown in Figure 4 for the given symmetrical disposing of its supports [22]. The interrelated action of the inertial torques is considered for the horizontal disposition of the spinning object.

The action of the inertial torques of the spinning annulus around axes \( ox \) and \( oy \) expresses analytically the equality of their mechanical energies that enables for deriving of the dependency of its interrelated angular velocities around axes of motions (Figure 4) [22].

\[
-T_{ct,x} - T_{ct,y} - T_{am,y} = T_{ct,x} + T_{am,x} - T_{ct,y} - T_{cr,y},
\]

where expressions of the inertial torques are presented by Equations (21) and (30), and \( T_{am} = J\omega_x \).
Substituting expressions of the defined torques into Equation (31) yields the following:

\[
-\frac{4}{9(R_i^4 + R_f^4)} \left( \frac{R_i^3 - R_f^3}{R_i^2 - R_f^2} \right)^2 \pi^2 i \omega_x \\
-\frac{8}{9(R_i^4 + R_f^4)} \left( \frac{R_i^3 - R_f^3}{R_i^2 - R_f^2} \right)^2 f \omega_x \\
-\frac{4}{9(R_i^4 + R_f^4)} \left( \frac{R_i^3 - R_f^3}{R_i^2 - R_f^2} \right)^2 \pi^2 f \omega_y - f \omega_y \\
= \frac{4}{9(R_i^4 + R_f^4)} \left( \frac{R_i^3 - R_f^3}{R_i^2 - R_f^2} \right)^2 \pi^2 i \omega_x + f \omega_x \\
-\frac{4}{9(R_i^4 + R_f^4)} \left( \frac{R_i^3 - R_f^3}{R_i^2 - R_f^2} \right)^2 \pi^2 f \omega_y \\
-\frac{8}{9(R_i^4 + R_f^4)} \left( \frac{R_i^3 - R_f^3}{R_i^2 - R_f^2} \right)^2 f \omega_y,
\]

where the signs (-) and (+) mean the clockwise and counterclockwise directions of the action of the inertial torques around two axes, respectively.

Simplification and transformation of Equation (32) yield the following:

\[
\omega_y = -\left[ \frac{(8/(9(R_i^4 + R_f^4)))\left((R_i^3 - R_f^3)/(R_i^2 - R_f^2)\right)^2 (\pi^2 + 1) + 1}{1 - (8/(9(R_i^4 + R_f^4)))\left((R_i^3 - R_f^3)/(R_i^2 - R_f^2)\right)^2} \right] \omega_x.
\]

The expressions of the inertial torques and their action, and the dependency of interrelated angular velocities of the spinning annulus around axes are presented in Table 2.

The expressions of torques and the dependency angular velocities (Table 2) formulate the mathematical models for the motions of the gyroscopic devices with the annulus rotor. Comparative analysis of formulas for the spinning annulus shows when its internal radius \( R_i = 0 \), all of them are converted to the formulas for the disc-type rotor presented in Table 1. This fact validates the mathematical correctness of the formulas. The values of formulas in Table 2 give results bigger than the formulas in Table 1.

### 5. Working Example

The annulus has a mass of 1.0 kg and the external and internal radii of 0.1 m and 0.06 m, respectively. The annulus rotates at
3000 rpm around the axle and precesses with an angular velocity of 0.06 rpm. Determine the values of the torques generated by the centrifugal and Coriolis force, the change in the angular momentum, and the angular velocity of precession. Substituting the data into equations of Table 2 and transformation yields the following results.

(i) The torque $T_{c1}$ generated by the centrifugal forces

$$T_{c1} = \frac{4}{9} \times \left(0.1^2 + 0.06^2 \right) \times \frac{3000 \times 2\pi}{2} \times \frac{0.06 \times 2\pi}{60} \times \frac{0.06^2}{2} \times 0.068 \times 3 \times 2 = 0.06496634Nm^2,$$

$$J = \frac{1}{2} M (R_1^2 + R_2^2) = \frac{1}{2} \times 1.0 \times (0.1^2 + 0.06^2) = 0.0068Nm^2.$$

(ii) The torque generated by Coriolis $T_{cr}$ forces

$$T_{cr} = \frac{8}{9} \times \left(0.1^2 - 0.06^2 \right) \times \frac{3000 \times 2\pi}{2} \times \frac{0.06 \times 2\pi}{60} = 0.013164Nm^2.$$

(iii) The torque $T_{am}$ of the change in the angular momentum

$$T_{am} = J \omega_x = 0.0068 \times \frac{3000 \times 2\pi}{2} \times \frac{0.06 \times 2\pi}{60} = 0.013422Nm^2.$$

(iv) The angular velocity of precession

$$\omega_x = \left( \frac{8}{9} \times (0.1^2 + 0.06^2) \right) \times \left( \frac{0.1^2 - 0.06^2}{0.1^2 - 0.06^2} \right) \times \left( \frac{0.06^2}{2} \times 1 + \frac{2}{1} \right) \times \frac{3.816023rad/s}{60} = 36.440337rpm.$$

For the disc-type rotor (Table 1), the dependency of the angular velocities is obtained if the $R_1 = 0$ in Table 2. Substituting the working example and computing yield:

$$\omega_y = \left(8 \pi^2 + 17 \right) \times \frac{0.06 \times 2\pi}{60} = 0.602914rad/s = 5.682336rpm.$$

6. Results and Discussion

Gyroscopic effects are remaining as sophisticated mathematical problems because the inertial torques generated by the rotating mass depend on the geometry of the spinning objects. The practice of engineering designs of the gyroscopic devices shows different geometry of their main components which are the rotors. The value of the inertial torques generated by the spinning rotors depends on their technical data. The working example shows the spinning annulus generates bigger values of the inertial torques and the dependency of the angular velocities than the disc-type rotor. This result gives the tip for the practitioners to design gyroscopic devices with a high value of the inertial torques and the angular velocity of precession. Practitioners can design gyroscopic devices with a necessary output of technical data that depends on the geometry of the spinning rotor. This result opens a new analytical approach to finding optimal designs of the spinning rotor that respond to the quality of gyroscopic work.

7. Conclusion

Aerospace and engineering industries try to solve gyroscopic problems and find optimal designs for gyroscopic devices that give the necessary technical data. The new theory of gyroscopic effects for rotating objects opens new methods for computing the technical data for gyroscopic devices. The method for deriving mathematical models for the gyroscopic effects enables finding the output data of gyroscopic devices that depend on the geometry of the main unit which is the spinning rotor. Optimization of the rotor’s design by criteria of the aerospace industry is a new direction for investigations in engineering. The fundamental principles of the theory of gyroscopic effects for rotating objects and methods enable solving them in engineering. The known numerical modeling for gyroscopic effects with expensive software is going past and engineering science receives new analytical methods.

Data Availability

The authors declare that the data supporting the findings of this study are available within the article. R. Usubamatov wrote the mathematical models of inertial torques. S. Kapayea corrected the text and references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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