An efficient high-probability algorithm for Linear Bandits

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Abstract

For the linear bandit problem, we extend the analysis of algorithm ComBEXP from Combes et al. [2015] to the high-probability case against adaptive adversaries, allowing actions to come from an arbitrary polytope. We prove a high-probability regret of \(O(T^{2/3})\) for time horizon \(T\). While this bound is weaker than the optimal \(O(\sqrt{T})\) bound achieved by GeometricHedge in Bartlett et al. [2008], ComBEXP is computationally efficient, requiring only an efficient linear optimization oracle over the convex hull of the actions.

1 Introduction

We study sequential prediction problems with linear losses and bandit feedback against an adaptive adversary. At every round \(t\) the forecaster chooses an action \(x_t\), and the adversary chooses a loss function \(L_t\), and the forecaster suffers the loss \(L_t(x_t)\). The forecaster learns only the suffered loss after each round, while the adversary learns the forecaster’s action \(x_t\). The forecaster’s aim is to minimize regret, which is the difference between the incurred loss and the loss of the best single action in hindsight:

\[
\sum_{t \in T} L_t(x_t) - \min_{x \in A} \sum_{t \in T} L_t(x).
\]

In this work we focus on establishing regret bounds holding with high-probability with an efficient algorithm.

For algorithms with bandit feedback, exploration (occasionally playing random actions for learning) is a crucial feature, however it does not have to be explicit as recently shown in Neu [2015], where exploration is achieved via skewing loss estimators. One of the most studied regret minimization algorithm is EXP, which iteratively updates the probabilities of each action via multiplication with factors exponential in its (estimated) loss. The variant EXP/three.taboldstyle for multi-armed bandit problems first appeared in Auer et al. [2002], however optimal high-probability regret bounds were first achieved in Dani and Hayes [2006]. The linear bandit setting is a generalization of the multi-armed bandit setting where, utilizing the linearity of losses, the goal is to improve the dependence on the number of actions in the regret bound, which might be exponential in the dimension \(n\). At the same time linear losses come naturally into play when considering actions with a combinatorial structure, such as e.g., matchings, spanning trees, \(m\)-sets; see Cesa-Bianch and Lugosi [2012], Audibert et al. [2013] for an extensive discussion. For the linear bandit setting, the EXP-variant ComBand (Combinatorial Bandit) from Cesa-Bianch and Lugosi [2012] has optimal \(O(\sqrt{T})\) expected regret, and in Bartlett et al. [2008] the modified version GeometricHedge achieves \(O(\sqrt{T})\) regret with high probability. While these regret bounds practically do not depend on the number of actions, both maintain a distribution over the (possibly exponentially large) action set \(A\), which is infeasible in general due to the large data size, even though ComBand is still efficient for many specific problems. Recently, a modification of the ComBand algorithm called ComEXP (see Algorithm 1) was derived in Combes et al. [2015], which achieves general computational efficiency by not maintaining a distribution of \(x_t\) but only the desired expectation \(\hat{x}_t\) of the distribution, and generating a new sparse approximate distribution at every round.

In this work we provide a high-probability regret bound of \(O(T^{2/3})\) for ComEXP against adaptive adversaries, while generalizing it to general polytopes. The obtained bounds are any-time, i.e., the parameter choice is independent
of the time horizon $T$. Finally, our algorithm maintains computational efficiency given an efficient linear programming oracle over the underlying polytope (the convex hull of actions). For comparison, we also show an $O(T^{2/3})$ regret in the high-probability setting for the original COMBAND.

The maximal matching problem is a good example where the linear programming oracle approach is useful, as it has a polynomial time linear optimization algorithm [Edmonds 1965], but no polynomial-size polyhedral description [Rothvoss 2014].

**Related work**

Our work is most closely related to the line of works on combinatorial bandit problems. The algorithm COMBAND first appeared in [Cesa-Bianchi and Lugosi 2012], while GEOMETRICHEDGE comes from [Bartlett et al. 2008], and COMBEXP appeared in [Combes et al. 2015]. Using interior point methods, an efficient algorithm with $O(\sqrt{T})$ expected regret for linear bandit problems has been established in [Abernethy et al. 2008].

For multiarmed bandit problems, the original version of EXP3 has high-probability regret $\Omega(T^{2/3})$ against some adaptive adversaries [Dani and Hayes 2006, Theorem 1.2], however variants with optimal $O(\sqrt{T})$ regret exists, e.g., using accountants to control the exploration rate (see [Dani and Hayes 2006]), or via the recent EXP3-IX with implicit exploration (see [Neu 2015]).

For convex loss functions, optimal high-probability regret bounds have been obtained in [Hazan and Li 2016] with running time being poly-exponential in the dimension, and in [Bubeck et al. 2016, Theorem 1] with polynomial running time provided the number of constraints of the underlying polytope is polynomial in the dimension. Optimal regret bounds in expectation was first obtained in [Bubeck and Eldan 2015]. However the case of convex loss does not subsume the combinatorial/linear case, as with convex loss all inner points of the convex set are actions; with linear losses the actions are limited to the vertices of the underlying polytope in most cases.

We refer the interested reader to the excellent survey of [Bubeck and Cesa-Bianchi 2012] on bandit problems.

**Contribution**

Our main contribution is a high-probability regret bound for COMBEXP from [Combes et al. 2015] for adaptive adversaries over actions coming from arbitrary polytopes $P \subseteq \mathbb{R}^n$. Our algorithm, being a slight generalization of COMBEXP, maintains computational efficiency. In particular, our contribution can be summarized as follows:

(i) **High-probability bounds for an efficient algorithm.** For COMBEXP we establish a high-probability regret of

$$O\left(\frac{B^2 + nB}{\min\{\lambda, 1\}} \cdot \ln \frac{2n + 2}{\delta}\right) T^{2/3},$$

with probability $1 - \delta$, where $B$ is the $\ell_2$-diameter of $P$, and $\lambda$ is a lower bound on the smallest eigenvalue of the exploration covariance matrix, see Theorem 3.1 for the exact regret bound.

For comparison we show that the same method already provides a high-probability regret bound of $O(T^{2/3})$ for the original COMBAND, albeit a suboptimal one as GEOMETRICHEDGE achieves $O(\sqrt{T})$ regret.

(ii) **Generalization of COMBEXP and computational efficiency.** We generalize COMBEXP to actions arising from arbitrary polytopes contained in $\mathbb{R}^n$ and to the case of adaptive adversaries. We maintain computational efficiency of COMBEXP providing running times relative to a linear programming oracle over the underlying polytope $P$, separating the complexity for learning from the complexity of linear optimization over $P$.

All our bounds are any-time, i.e., holding uniformly for all times $T$. In particular, our parameter choices are independent of $T$.

**Outline**

After a brief summary of the regret minimization framework in Section 2 we reanalyze COMBEXP in Section 3. For completeness we present a similar analysis for COMBAND in Section 4.

We relegated various related materials to the the Appendix. In Section A we provide an any-time version of EXP with time-varying parameters maintaining generalized distributions, defined by an arbitrary convex set in the positive
orthant, instead of the probability simplex. We prove an $O(\sqrt{T})$ regret bound in the full information case by standard arguments, which forms the basis for our regret bounds for the bandit case. In Section B we recall concentration inequalities that we use to establish high-probability bounds. Finally, in Sections C and D we provide (already known) efficient algorithms for projection and distribution generation, which are key components in our algorithms. We include those for completeness of exposition and to make parameters explicit.

2 Preliminaries

We will briefly recall the regret minimization framework to define our notation. In the sequential prediction problem with linear losses, at every round $t$ the forecaster chooses an action $x_t$ from a finite set $A \subseteq \mathbb{R}^n$ and the adversary chooses a loss vector $L_t \in \mathbb{R}^n$. The forecaster suffers the loss $\ell_t := L_t^T x_t$. The goal of the forecaster is to minimize the regret

$$\sum_{t=1}^T L_t^T x_t - \min_{x \in A} \sum_{t=1}^T L_t^T x.$$ 

Against an oblivious adversary, who chooses the $L_t$ independently of the forecaster’s actions, this is the extra loss suffered by not playing the best single action in hindsight. However, this interpretation is clearly incorrect against an adaptive adversary (the notion of policy regret from Arora et al. [2012] matches this interpretation). Nevertheless the above notion of regret proved to be useful in many areas.

With bandit feedback the forecaster learns only the loss $\ell_t$ but not the actual loss vector $L_t$. An adaptive adversary learns the forecaster’s action $x_t$ after round $t$, and can use it in later rounds to choose his actions.

We make various standard assumptions to bound the regret. The most important one is that the per round loss is bounded, i.e., $|L_t^T x| \leq 1$ for all $x \in A$. Under reasonably assumptions, this also implies that the set $A$ of possible actions $A$ is bounded and we assume that $\|x\|_2 \leq B$ and $\|x\|_1 \leq B_1$, with suitable positive numbers $B, B_1$. Clearly, one can always choose $B_1 = nB$, however we obtain finer bounds by keeping them separate. The bounds $B_1$ and $B$ also serve as a proxy for the sparsity of the actions.

Following Cesa-Bianchi and Lugosi [2012] for ComBand, we shall use a fixed arbitrary distribution $\mu$ on $A$ for exploration, whose fitness for exploration is measured by a positive lower bound $\lambda$ on the smallest eigenvalue of its covariance matrix $J$:

$$J := \mathbb{E}_{y \sim \mu} [yy^T] \succeq \lambda I.$$ 

Here and below we denote by $M \preceq N$ that $N - M$ is a positive semi-definite matrix for symmetric matrices $M$ and $N$. When $A$ is small then $\mu$ is typically the uniform distribution over $A$. For large $A$, common choices are the uniform distribution on a barycentric spanner of $A$ (see Hazan et al. [2014]), or the distribution on contact points of the maximal volume ellipsoid contained in the convex hull $P$ of $A$ arising from John’s decomposition (John’s exploration; see Dani et al. [2007]), transferred to $A$. In the latter two cases, $J = I$ and $\lambda = 1/n$ using the scalar product on $\mathbb{R}^n$ induced by the additional structure. John’s ellipsoid can be approximately estimated with a worse lower bound $\lambda = 1/n^{3/2}$ by Grötschel et al. [1993], however a constant factor approximation is NP-hard by Nemirovski [2006].

Recall that a barycentric spanner is a linear basis $v_1, \ldots, v_n$ in $P$ (the convex hull of $A$), such that every element of $P$ is a linear combination of the $v_i$ with coefficients from $[-1, +1]$. The basis $v_1, \ldots, v_n$ is a $C$-approximate barycentric spanner for some $C > 1$ if every element of $P$ is a linear combination of the $v_i$ with coefficients from $[-C, +C]$. A $C$-approximate barycentric spanners can be efficiently computed by $O(n^2 \ln n / \ln C)$ calls to a linear optimization oracle over $P$ by Awerbuch and Kleinberg [2004], which actually computes a spanner consisting of vertices of $P$. In this paper we deliberately avoid using the scalar product induced by the structure to be able to directly use the bounds available in the original space of the problem. Fortunately, the uniform distribution on an approximate barycentric spanner has a close to optimal minimal eigenvalue even in the original space, see Lemma 5.1 which allows us to preserve sparsity of the original space. As such we assume that we have access to an exploration distribution over actions with sparse support of size $n$, where $n$ is the dimension of the vector space, from which we can efficiently sample. Note that for specific problems exploration distributions with better minimal eigenvalue can be explicitly given; we refer the interested reader to Cesa-Bianchi and Lugosi [2012] and follow-up work for a large set of such examples.

Let $u := \mathbb{E}_{y \sim \mu} [y]$ denote the expectation of $\mu$ and let $e$ denote the Euler constant. Instead of dealing directly with $A$, it will be more convenient to use the convex hull $P$ of $A$, then $A$ contains the vertex set of $P$ (and in many applications the two are equal). We shall use the Kullback–Leibler divergence as Bregman divergence of the function
fast linear programming oracles are available. A shifting vector maintaining a distribution with sparse support, to allow fast sampling and fast computation of the covariance matrix \( \text{C}_t \). The positive parameters \( \eta_t, \gamma_t \) control the learning rate and exploration rate of the algorithm. The role of the shifting vector \( a \in \mathbb{R}^n \) is to avoid singularity issues with Kullback–Leibler divergence. Except for the shifting vector \( a \), these ideas already appeared in Combes et al. [2015].

The algorithm contains four resource-consuming steps: (1) projection (Line 10), (2) distribution generation (Line 3), (3) sampling from the distribution, and (4) computing the covariance matrix. All the other steps are fast, depending only polynomially on the dimension.

The major factor for the running time of sampling from the distribution and computing the covariance matrix is the sparsity of the generated distribution, i.e., the number of possible outcomes. Sparse distributions (number of outcomes polynomial in the dimension) of sufficient accuracy can be efficiently generated by the decomposition algorithm from Mirrokni et al. [2015], which we summarize as Algorithm 5 in Section 1 for the reader’s convenience. Common choices of the exploration distribution \( \mu \) are sparse, as discussed above, notwithstanding non-sparse distributions for \( \mu \) are also acceptable which have an efficient sampling method and a precomputed covariance matrix. Therefore we will disregard the complexity of sampling and computation of the covariance matrix.

Finally, the projection step (Line 10) can be efficiently accomplished by the Frank–Wolfe algorithm (also called conditional gradient), which we recall in Algorithm 4 in Section 1. Note that if Algorithm 4 is used for the projection step, it already provides a sparse linear decomposition of the desired expectation \( \hat{x}_{t+1} \) with accuracy \( \varepsilon = 0 \), and therefore makes a separate linear decomposition step unnecessary. Nevertheless it might be advantageous for specific polytopes to use a specialized, more efficient projection algorithm and/or decomposition algorithm.

All in all, we measure complexity of only the most time-consuming tasks: projection and linear decomposition, requiring the linear decomposition to be sparse. We report complexity of Algorithms 4 and 5 mentioned above in the total number of linear optimization oracle calls over \( P \). This relative complexity is often useful in applications where fast linear programming oracles are available.

Now we are ready to state our main theorem on the regret and complexity of Combes.

**Theorem 3.1 (High-probability regret bound for Combes for adaptive adversaries).** For \( n \geq 1 \) and with the choice

\[
\gamma_t := \frac{t^{-1/3}}{2} \quad \text{and} \quad \eta_t := \min \{ \gamma_t^2, \gamma_t \lambda \}
\]

Algorithm 4 achieves for any time \( T \geq 1 \) the following regret: With probability at least \( 1 - \delta \), for any \( x \in P \) we have

\[
\sum_{t=1}^T (L_t^2 x_t - L_t^1 x) \leq 4 \frac{\text{KL}(a + x, a + \hat{x}_1)}{\min\{1, 2 \lambda T^{1/3}\}} + B_1 \sqrt{\frac{3 \ln n + 2}{\delta}} + \left( (e - 2) \frac{\|a\|_1 + B_1}{\lambda} + 2 + \frac{B (B + \varepsilon)}{\lambda} \right) \frac{3}{4} T^{2/3} + O \left( \max \left\{ 1, \frac{B^2 \max\{\|a\|_1 + B_1, \lambda \}, B_1 \max\{1, B \}, B + \varepsilon}{\sqrt{\lambda}} \right\} \sqrt{T} \right) \ln \frac{2n + 2}{\delta}. \tag{1}
\]

In particular, assuming \( a - a_i \leq z_i \leq \beta - a_i \) for some \( 0 < a < \beta \) for all \( z \in P \):
3.1 Proof of Theorem 3.1

In this section we will prove Theorem 3.1. We focus on the main regret bound, Equation (1), the other results easily follow from it. See Propositions C.1 and D.1 for the complexity of Algorithms 4 and 5. The inequality $\text{KL}(a + y, a + \hat{x}_1) \leq \frac{4B^2}{\alpha}$ is derived using $\ln z \leq z - 1$:

$$\text{KL}(a + y, a + \hat{x}_1) = \sum_{i=1}^{n} (a_i + y_i) \ln \frac{a_i + y_i}{a_i + \hat{x}_{1,i}} - \sum_{i=1}^{n} (a_i + y_i) + \sum_{i=1}^{n} (a_i + \hat{x}_{1,i})$$

$$\leq \sum_{i=1}^{n} (a_i + y_i) \left( \frac{a_i + y_i}{a_i + \hat{x}_{1,i}} - 1 \right) - \sum_{i=1}^{n} (a_i + y_i) + \sum_{i=1}^{n} (a_i + \hat{x}_{1,i}) = \sum_{i=1}^{n} \frac{(y_i - \hat{x}_{1,i})^2}{a_i + \hat{x}_{1,i}} \leq \frac{\|y - \hat{x}_1\|_2^2}{\alpha} \leq \frac{4B^2}{\alpha}.$$
Let $\mathbb{E}_t[-] := \mathbb{E}[- | x_1, L_{1, \ldots, 1, L_{t-1, L_t}]$ denote the conditional expectation operator given the history preceding round $t$ and also the adversary’s action in round $t$. In particular, $C_t = \mathbb{E}_t[x_t x_t^\top] = (1 - \gamma_t)P_t + \gamma_t I$, with $P_t := \mathbb{E}_{x \sim p_t}[xx^\top]$. We first establish some basic bounds on quantities occurring in Algorithm 3.

**Lemma 3.2 (Basic bounds).** Let $y \in P$ be arbitrary.

\[
\|C_t^{-1}\|_2 \leq \frac{1}{\gamma_t \lambda} \tag{3}
\]

\[
\|\hat{L}_t\|_2 \leq \frac{B}{\gamma_t \lambda} \tag{4}
\]

\[
\|L_t\|_2 \leq \frac{B}{\lambda} \tag{5}
\]

**Proof.** Equation (3) follows from $C_t \succeq \gamma_t I \succeq \gamma_t \lambda I$. Inequality (4) follows via

\[
|\hat{L}_t| = |\ell_t \cdot C_t^{-1} x_t| \leq |\ell_t| \cdot \|C_t^{-1}\|_2 \cdot \|x_t\|_2 \leq \frac{B}{\gamma_t \lambda}.
\]

Finally, (5) follows from the estimation

\[
\|L_t\|_2 = \|L_t^\top J J^{-1}\|_2 = \|\mathbb{E}_{y \sim \mu} \left[ L_t^\top y y^\top J^{-1} \right]\|_2 \leq \mathbb{E}_{y \sim \mu} \left[ \|L_t^\top y\|_2 \cdot \|y\|_2 \cdot \|J^{-1}\|_2 \right] \leq \frac{B}{\lambda}.
\]

We now estimate the pieces of Equation (2). The following series of upper bounds are independent of the concrete choice of the parameters $\gamma_t$, $\eta_t$. However, for the reader’s convenience in the last inequality of each estimation we make the bound explicit by substituting the values for $\gamma_t$, $\eta_t$ by the choices given in Theorem 3.1. We will tacitly use the following inequality to estimate sums like $\sum_{t=1}^T \gamma_t$:

\[
\sum_{t=1}^T \gamma_t^\alpha \leq \begin{cases} \frac{T^{\alpha+1}}{\alpha+1} & \text{if } \alpha + 1 \neq 0, \\ T^\alpha & \text{if } \alpha = 0, \\ T^{\alpha+1} & \text{if } \alpha + 1 > 0, \end{cases} \quad 0 < \alpha < 0
\]

We first estimate the regret when using the loss estimators $\hat{L}_t$. For this we use a generalized variant of EXP (see Lemma A.1), which works with arbitrary convex sets contained in the positive orthant.

**Lemma 3.3.**

\[
\sum_{t=1}^T (\hat{L}_t^\top x_t - \hat{L}_t^\top \hat{x}_t) \leq \frac{KL(a + x, a + \hat{x}_t)}{\eta_t} + \sum_{t=1}^{T-1} \gamma_t \eta_t \sum_{t=1}^n (a_i + \hat{x}_{t,i})^2 \tag{6}
\]

\[
\leq 4 \frac{KL(a + x, a + \hat{x}_t)T^2/3}{\min\{1, 2\lambda T^{1/3}\}} + \frac{3}{4} T^{2/3} + (e - 2) \sum_{t=1}^T \eta_t \sum_{t=1}^n (a_i + \hat{x}_{t,i})^2 \tag{6}
\]

**Proof.** This follows from Lemma A.1 with the $\hat{L}_t$ as loss vectors and the $a + \hat{x}_t$ as played actions. Note that $a$ cancels on the left hand side in $(\hat{L}_t^\top (a + \hat{x}_t) - \hat{L}_t^\top (a + x))$. \hfill \Box

In a next step we estimate the last term of Equation (6).

**Lemma 3.4.** With probability at least $1 - \delta$

\[
\sum_{t=1}^T \eta_t \sum_{t=1}^n (a_i + \hat{x}_{t,i})^2 \leq \frac{\|a\|_1 + B_1}{\lambda} \sum_{t=1}^T \frac{\eta_t}{\gamma_t} + \frac{(\|a\|_1 + B_1)B^2}{\lambda^2} \sqrt{\frac{1}{2} \sum_{t=1}^T \frac{\eta_t^2}{\gamma_t^2} \ln \frac{1}{\delta}} \tag{7}
\]

\[
\leq \frac{\|a\|_1 + B_1}{\lambda} \frac{3}{4} T^{2/3} + \frac{(\|a\|_1 + B_1)B^2}{\lambda^2} \sqrt{\frac{1}{2} T \ln \frac{1}{\delta}}.
\]

\[
\leq \frac{\|a\|_1 + B_1}{\lambda} \frac{3}{4} T^{2/3} + \frac{(\|a\|_1 + B_1)B^2}{\lambda^2} \sqrt{\frac{1}{2} T \ln \frac{1}{\delta}}.
\]
Proof. This is a special case of the Azuma–Hoeffding inequality (recalled in Theorem B.1) using the bounds

\[ 0 \leq \sum_{i=1}^{n}(a_i + \hat{x}_{t,i})\hat{L}_{i,j}^2 \leq \sum_{i=1}^{n}(a_i + \hat{x}_{t,i}) \left( \frac{B}{\gamma_t \lambda} \right)^2 \leq \frac{\|a\|_1 + B_t}{(\gamma_t \lambda)^2} \]

and

\[
\mathbb{E}_t \left[ \sum_{i=1}^{n}(a_i + \hat{x}_{t,i})\hat{L}_{i,j}^2 \right] = \mathbb{E}_t \left[ \sum_{i=1}^{n}(a_i + \hat{x}_{t,i})\ell_t^2 e_i^T C_t^{-1} x_i^T \right] = \sum_{i=1}^{n}(a_i + \hat{x}_{t,i})\ell_t^2 e_i^T C_t^{-1} C_t^{-1} e_i \\
\leq \sum_{i=1}^{n}(a_i + \hat{x}_{t,i})\ell_t^2 C_t^{-1} e_i \leq \frac{\|a\|_1 + B_t}{\gamma_t \lambda}.
\]

Next we bound the difference between the true loss \(L_t^T x_t\) and the expected estimated loss \(\hat{L}_t^T \hat{x}_t\).

Lemma 3.5. With probability at least \(1 - \delta\)

\[
\sum_{t=1}^{T} (L_t^T x_t - \hat{L}_t^T \hat{x}_t) \leq \left( 1 + \frac{B(B + \epsilon)}{\lambda} \right) \sum_{t=1}^{T} \gamma_t + \sqrt{2 \left( \frac{2}{\delta} \frac{B(B + \epsilon)(B + \epsilon)}{\lambda} \sum_{t=1}^{T} \gamma_t \right)} \ln \frac{1}{\delta} \\
+ \frac{3}{4} \left( \frac{B}{\sqrt{\gamma_T (1 - \gamma_T) \lambda}} + \epsilon \lambda \right) \ln \frac{1}{\delta} \\
\leq \left( 1 + \frac{B(B + \epsilon)}{\lambda} \right) \sum_{t=1}^{T} \gamma_t + \frac{B}{3 \sqrt{\gamma_T (1 - \gamma_T) \lambda}} \ln \frac{1}{\delta} + \left[ 3 + \frac{B^2 \epsilon}{\lambda} + O \left( \max \left\{ 1, \frac{B + \epsilon}{\sqrt{\lambda}} \right\} \sqrt{T} \right) \right] \max \left\{ 1, \ln \frac{1}{\delta} \right\} \\
\leq \left( 1 + \frac{B(B + \epsilon)}{\lambda} \right) \frac{3}{4} T^{2/3} + \frac{\sqrt{2B}}{3 \sqrt{\lambda}} (T^{1/6} + T^{-1/6}) \ln \frac{1}{\delta} + \left[ 3 + \frac{B^2 \epsilon}{\lambda} + O \left( \max \left\{ 1, \frac{B + \epsilon}{\sqrt{\lambda}} \right\} \sqrt{T} \right) \right] \max \left\{ 1, \ln \frac{1}{\delta} \right\}.
\]

(8)

Proof. Let \(\tilde{x}_t := \mathbb{E}_{x \sim p_t} [x]\) and \(\bar{x}_t := \mathbb{E}_t [x_t] = (1 - \gamma_t) \tilde{x}_t + \gamma_t u\). As also \(\|\hat{x}_t - \tilde{x}_t\|_2 \leq \gamma t \epsilon\), we have \(\mathbb{E}_t = (1 - \gamma_t) \tilde{x}_t + \gamma_t v\) for \(v := u + \frac{1-\gamma_t}{\gamma_t} (\tilde{x}_t - \hat{x}_t)\) with \(\|v\|_2 \leq B + \epsilon (1 - \gamma_t) \leq B + \epsilon\). We consider the martingale difference sequence

\[ X_t := L_t^T x_t - \hat{L}_t^T \hat{x}_t - \mathbb{E}_t [L_t^T x_t - \hat{L}_t^T \hat{x}_t] = \hat{L}_t^T \hat{x}_t - \hat{L}_t^T \hat{x}_t + L_t^T x_t - \hat{L}_t^T \hat{x}_t = L_t^T x_t - \hat{L}_t^T \hat{x}_t + \gamma_t L_t^T (\hat{x}_t - \bar{x}_t). \]

Note that as \(\hat{x}_t^T \hat{x}_t = \mathbb{E}_{x \sim p_t} [x] \mathbb{E}_{x \sim p_t} [x] \mathbb{T} \leq \mathbb{E}_{x \sim p_t} [x x^T] = P_t \leq C_t/(1 - \gamma_t)\)

\[ (\hat{L}_t^T \hat{x}_t)^2 = \hat{L}_t^T \hat{x}_t \hat{x}_t^T \hat{L}_t = \ell_t^2 x_t^T C_t^{-1} x_t \hat{x}_t^T C_t^{-1} x_t \leq x_t^T C_t^{-1} P_t C_t^{-1} x_t \leq \frac{x_t^T C_t^{-1} x_t}{1 - \gamma_t} \leq \frac{B^2}{\gamma_t \lambda (1 - \gamma_t)} \leq \frac{B^2}{\gamma_T \lambda (1 - \gamma_T)}, \]

and

\[ |\hat{L}_t^T \hat{x}_t - \hat{L}_t^T \hat{x}_t| \leq |\hat{L}_t|_2 \cdot \|\hat{x}_t - \tilde{x}_t\|_2 \leq \frac{B^2 \epsilon}{\gamma_t \lambda} \gamma t \epsilon \frac{B^2 \epsilon}{\lambda}, \]

hence

\[ |X_t| \leq |L_t^T x_t| + |L_t^T \bar{x}_t| + |\hat{L}_t^T \hat{x}_t| + |\hat{L}_t^T \hat{x}_t - \hat{L}_t^T \hat{x}_t| + |\hat{L}_t^T \hat{x}_t| \leq 3 + \frac{B^2 \epsilon}{\lambda} + \frac{B}{\sqrt{\gamma_T (1 - \gamma_T) \lambda}}, \]

and the variance of \(X_t\) is easily bounded by:

\[ \text{Var}_t [X_t] \leq \mathbb{E}_t \left[ (L_t^T x_t - \hat{L}_t^T \hat{x}_t)^2 \right] = \mathbb{E}_t \left[ (\ell_t \cdot (1 - x_t^T C_t^{-1} \hat{x}_t))^2 \right] \]

\[ \leq \mathbb{E}_t \left[ (1 - x_t^T C_t^{-1} \hat{x}_t)^2 \right] = \mathbb{E}_t \left[ 1 - 2x_t^T C_t^{-1} \hat{x}_t + x_t^T C_t^{-1} x_t \hat{x}_t^T C_t^{-1} \hat{x}_t \right] \]

\[ = 1 - 2\bar{x}_t^T C_t^{-1} \hat{x}_t + \hat{x}_t^T C_t^{-1} \hat{x}_t = 1 - \frac{1 - 2\gamma t}{(1 - \gamma_t)^2} x_t^T C_t^{-1} x_t + \frac{\gamma_t^2}{(1 - \gamma_t)^2} (v - 2\bar{x}_t^T C_t^{-1} v) \leq 1 + \frac{\gamma_t (3B + \epsilon)(B + \epsilon)}{(1 - \gamma_t)^2 \lambda}. \]
Hence Benett’s inequality (Theorem B.2, Fan et al., 2012 (18)) applied to the martingale difference sequence $X_t$ provides

$$
\sum_{t=1}^{T} \left( L_t^\top x_t - L_t^\top \hat{x}_t + \gamma_t L_t^\top (\hat{x}_t - v) \right) \leq \frac{1}{3} \left( \frac{B}{\sqrt{T(1 - \gamma T)\lambda}} + 3 + \frac{B^2}{\lambda} \right) \ln \frac{1}{\delta} + \sqrt{2 \left( T + \frac{(3B + \varepsilon)(B + \varepsilon)}{\lambda} \sum_{t=1}^{T} \frac{\gamma_t}{(1 - \gamma_t)^2} \right) \ln \frac{1}{\delta}}.
$$

The claim follows by using $|L_t^\top (\hat{x}_t - v)| \leq 1 + B(B + \varepsilon)/\lambda$.

Finally, we bound the difference between the true loss $L_t^\top x$ and the estimated loss $\hat{L}_t^\top x$ for any point $x \in P$.

**Lemma 3.6.** For all $0 < \delta < 1$ with probability at least $1 - \delta$ for every $x \in \mathbb{R}^n$ simultaneously

$$
\sum_{t=1}^{T} (\hat{L}_t^\top x - L_t^\top x) \leq \frac{\|x\|_1}{3} \left( \frac{B}{\lambda} + \frac{1}{T\lambda} \right) \ln \frac{2n}{\delta} + \|x\|_1 \sqrt{\frac{2}{\lambda} \sum_{t=1}^{T} \frac{1}{\gamma_t} \ln \frac{2n}{\delta}}.
$$

In particular, with probability at least $1 - \delta$, for all $x \in P$ simultaneously

$$
\sum_{t=1}^{T} (\hat{L}_t^\top x - L_t^\top x) \leq \frac{B_1}{3\lambda} \left( B + 2T^{1/3} \right) \ln \frac{2n}{\delta} + B_1 T^{2/3} \sqrt{1 + \frac{4}{3T} \sqrt{\frac{3}{\lambda} \ln \frac{2n}{\delta}}}.
$$

(9)

**Remark 3.7.** Restricting the statement for all $x \geq 0$, the $\ln(2n/\delta)$ can be replaced by $\ln(n/\delta)$.

**Proof.** Let $x = \pm e_i$ be a coordinate vector or its negation. Then

$$
\text{Var}_t [\hat{L}_t^\top x - L_t^\top x] \leq \mathbb{E}_t \left[ (\hat{L}_t^\top x)^2 \right] \leq \mathbb{E}_t \left[ x^\top C_t^{-1} x_t x_t^\top C_t^{-1} x \right] = x^\top C_t^{-1} x \leq \frac{1}{\gamma_t \lambda},
$$

and

$$
|\hat{L}_t^\top x - L_t^\top x| \leq \frac{B}{\lambda} + \frac{1}{\gamma_t \lambda}.
$$

Hence by Benett’s inequality (Theorem B.2, Fan et al., 2012 (18)) the claim follows for a fixed vector $x = \pm e_i$ with probability at least $1 - \delta/(2n)$. Hence by the union bound, it holds for all $x = \pm e_i$ simultaneously with probability at least $1 - \delta$. Finally, the inequality for a general $x$ follows by taking linear combinations with the absolute values of the coefficients of $x$.

Summing up (6), (7), (8) (substituting $\delta/(2n + 2)$ for $\delta$ in the latter two) and (9) (substituting $2n\delta/(2n + 2)$ for $\delta$), with probability at least $1 - \delta$ yields (1) of Theorem 3.1.

### 4 A high-probability regret bound for ComBand

In this section we will show that ComBand of Cesa-Bianchia and Lugosi, 2012 achieves a high-probability regret bound of $O(T^{2/3})$ without any modifications. While this is worse than the optimal regret of $O(\sqrt{T})$ obtained by GEOMETRICHEDGE in Bartlett et al. [2008], it shows that already Algorithm 2, the vanilla version of ComBand without any correction terms suffices to achieve a high-probability regret bound.

**Theorem 4.1.** With the choice

$$
\eta_t := \frac{\gamma_t \lambda}{B^2} \quad \text{and} \quad \gamma_t := \frac{t^{-1/3}}{2}.
$$
Algorithm 2 \textsc{ComBand}

\textbf{Require:} Losses $L_t$, action set $A \subseteq \mathbb{R}^d$, positive parameters $\eta_1 \geq \eta_2 \geq \ldots, 1/2 \geq \gamma_1 \geq \gamma_2 \geq \ldots$

\textbf{Ensure:} actions $x_t \in A$

1: \textbf{for} $t = 1$ to $T$ \textbf{do}
2: \hspace{1em} $w_t(x) \leftarrow \sum_{i=1}^{t-1} \hat{L}_i^T x$ \hspace{1em} for all $x$
3: \hspace{1em} $W_t \leftarrow \sum_x w_t(x)$
4: \hspace{1em} $p_t(x) \leftarrow w_t(x) / W_t$ \hspace{1em} for all $x$
5: \hspace{1em} $q_t \leftarrow (1 - \gamma_t) p_t + \gamma_t \mu$
6: \hspace{1em} Sample $x_t \sim q_t$.
7: \hspace{1em} Observe loss $\ell_t := L_t^T x_t$.
8: \hspace{1em} $C_t \leftarrow \mathbb{E}_{x \sim q_t} [xx^T]$
9: \hspace{1em} $\hat{L}_t \leftarrow \ell_t C_t^{-1} x_t$
10: \textbf{end for}

\textbf{Algorithm 2} achieves regret

\begin{equation}
\left( \frac{\sqrt{3B}}{\sqrt{\lambda}} \sqrt{\ln \frac{N + 2}{\delta} + n \frac{3(e - 2)\lambda}{4B^2} + \frac{3}{2}} \right) T^{2/3} + O \left( \frac{\ln n}{B^2} + \left( 1 + \frac{B^2}{\lambda} \right) \ln \frac{N + 2}{\delta} \right) \sqrt{T} \leq O \left( \frac{B}{\sqrt{\lambda}} \sqrt{\ln \frac{N + 2}{\delta} + n \frac{\lambda}{B^2}} \right) T^{2/3}
\end{equation}

with probability at least $1 - \delta$ for $0 < \delta < 1$.

\textbf{Remark 4.2.} Similar to Theorem 3.1 it is possible to change the $\ln((N + 2)/\delta)$ in the coefficient of $T^{2/3}$ to the possibly much smaller $\ln((n + 2)/\delta)$ with a suitable altering of the other constants. However, since an $T^{1/3} \ln N$ term will still remain in the regret bound, this does not seem to be a significant improvement.

We use the same notation as in Section 3.1 for \textsc{ComEXP}, which we recall here for the reader’s convenience. Let $\mathbb{E}_t[-] := \mathbb{E}[-|x_1, L_1, \ldots, x_{t-1}, L_{t-1}, L_t]$ denote the conditional expectation operator given the history preceding round $t$ and also the adversary’s action in round $t$. Let $\hat{x}_t := \mathbb{E}_{x \sim p_t} [x]$ and $P_t := \mathbb{E}_{x \sim p_t} [xx^T]$ denote the expectation and variance of distribution $p_t$, respectively. Note that $C_t = \mathbb{E}_t [x_t x_t^T] = (1 - \gamma_t) P_t + \gamma_t I$.

\textbf{Lemma 4.3 (Basic bounds).} Let $x$, $y_1$, and $y_2$ be arbitrary actions.

(i) \textbf{Bounds on size}

\begin{equation}
|y_1^T C_t^{-1} y_2| \leq \frac{B^2}{\gamma_t \lambda}
\end{equation}

\begin{equation}
|\hat{L}_t^T x| \leq \frac{B^2}{\gamma_t \lambda}
\end{equation}

(ii) \textbf{Bounds on expectation}

\begin{equation}
\mathbb{E}_t \left[ x_t^T C_t^{-1} x_t \right] = n
\end{equation}

\textbf{Proof.} Equation (11) follows from the bounds $\|y_1\|_2, \|y_2\|_2 \leq B$ and $\|C_t^{-1}\|_2 \leq 1/(\gamma_t \lambda)$, as $C_t \geq \gamma_t I \geq \gamma_t \lambda I$.

Inequality (12) follows via

\[ |\hat{L}_t^T x| = |\ell_t \cdot x_t^T C_t^{-1} x| \leq \frac{B^2}{\gamma_t \lambda}. \]

To prove (13), we use a trick using the trace function to compute the expectation:

\[ \mathbb{E}_t \left[ x_t^T C_t^{-1} x_t \right] = \mathbb{E}_t \left[ \text{Tr}(C_t^{-1} x_t x_t^T) \right] = \text{Tr}(C_t^{-1} C_t) = \text{Tr}(I) = n. \]
Remark 4.4. One can similarly prove \( \mathbb{E}_{y \sim p_t} \left[ y^T P_t^{-1} y \right] = n \), but it will not be used in the following.

As in the case of ConnEXP, the lemmas below are independent of the choice of the \( \gamma_t, \eta_t \) except for the last formula in each lemma, where we particularize the bounds by substituting parameters.

First instead of the real regret, we estimate the regret computed using the estimators \( \tilde{L}_t \).

**Lemma 4.5.** With probability at least \( 1 - \delta \)

\[
\sum_{t=1}^{T} (\tilde{L}_t^T \tilde{x}_t - \tilde{L}_t^T x) \leq \frac{\ln N}{\eta T} + (e - 2) \left( n \sum_{t=1}^{T} \frac{\eta_t}{1 - \gamma_t} + \frac{B^2}{\lambda} \sqrt{\frac{1}{2} \sum_{t=1}^{T} \frac{\eta_t^2}{(1 - \gamma_t)^2} \cdot \ln \frac{1}{\delta}} \right)
\]

\[
\leq \frac{2B^2 \ln N}{\lambda} T^{1/3} + (e - 2) \left( n \frac{3\lambda}{4B^2} (T^{2/3} + 2T^{1/3}) + \frac{B}{\sqrt{\lambda}} \sqrt{2T \ln \frac{1}{\delta}} \right).
\]

**Proof.** By Lemma [A.1]

\[
\sum_{t=1}^{T} (\tilde{L}_t^T \tilde{x}_t - \tilde{L}_t^T x) \leq \frac{\ln N}{\eta T} + (e - 2) \sum_{t=1}^{T} \eta_t \mathbb{E}_{y \sim p_t} \left[ (\tilde{L}_t^T y)^2 \right].
\]

To estimate the last term, first note that

\[
\mathbb{E}_{y \sim p_t} \left[ (\tilde{L}_t^T y)^2 \right] = \mathbb{E}_{y \sim p_t} \left[ \tilde{L}_t^T y y^T L_t \right] = \tilde{L}_t^T P_t L_t = \tilde{L}_t^T x_t C_t^{-1} P_t C_t^{-1} x_t \leq \frac{x_t^T C_t^{-1} x_t}{1 - \gamma_t}.
\]

So far combining our estimates provides

\[
\sum_{t=1}^{T} (\tilde{L}_t^T \tilde{x}_t - \tilde{L}_t^T x) \leq \frac{\ln N}{\eta T} + (e - 2) \sum_{t=1}^{T} \eta_t \frac{x_t^T C_t^{-1} x_t}{1 - \gamma_t}.
\]

(14)

To estimate the last term on the right-hand side, we apply the Azuma–Hoeffding inequality using (11) and (13) for bounding the summands and their expectation, which readily proves the lemma:

\[
\sum_{t=1}^{T} \eta_t \frac{x_t^T C_t^{-1} x_t}{1 - \gamma_t} \leq n \sum_{t=1}^{T} \frac{\eta_t}{1 - \gamma_t} + \frac{B^2}{\lambda} \sqrt{\frac{1}{2} \sum_{t=1}^{T} \frac{\eta_t^2}{(1 - \gamma_t)^2} \cdot \ln \frac{1}{\delta}}.
\]

We turn our attention to the difference between the real loss vectors \( L_t \) and their estimators \( \tilde{L}_t \). We start by comparing the loss of the played action.

**Lemma 4.6.** With probability at least \( 1 - \delta \)

\[
\sum_{t=1}^{T} (L_t^T x_t - \tilde{L}_t^T \tilde{x}_t) \leq 2 \sum_{t=1}^{T} \frac{\gamma_t + \frac{1}{3}}{\gamma T} \left( 2 + \frac{B}{\sqrt{\gamma T} \lambda (1 - \gamma_T)} \right) \ln \frac{1}{\delta} + \frac{1}{3} \left( 2 + \frac{\sqrt{2B}}{\sqrt{\lambda}} (T^{1/6} + T^{-1/6}) \right) \ln \frac{1}{\delta}
\]

\[
\leq \frac{2}{3} T^{2/3} + \frac{1}{3} \left( 2 + \sqrt{2B} \sqrt{T} \ln \frac{1}{\delta} + \sqrt{2} \left( T + \frac{9B^2}{\lambda} T^{2/3} \right) \ln \frac{1}{\delta} \right).
\]

**Proof.** Let \( x_t \) := \( E_t \left[ x_t \right] = (1 - \gamma_t) \bar{x}_t + \gamma_t \bar{x} \) denote the conditional expectation of \( x_t \) given the history before round \( t \) and loss \( L_t \). The statement is a special case of Benett’s inequality (see Theorem [B.3]) for the martingale

\[
X_t := L_t^T x_t - \tilde{L}_t^T \tilde{x}_t - E_t \left[ L_t^T x_t - \tilde{L}_t^T \tilde{x}_t \right] = L_t^T x_t - \tilde{L}_t^T \tilde{x}_t - \tilde{L}_t^T \tilde{x}_t + L_t^T x_t - \tilde{L}_t^T \tilde{x}_t + \gamma_t L_t^T (\tilde{x}_t - u).
\]

Note that \( \tilde{x}_t, \tilde{x}_t \leq P_t \) by Jensen’s inequality, therefore

\[
(L_t^T \tilde{x}_t)^2 = L_t^T \tilde{x}_t \tilde{x}_t^T L_t = \tilde{L}_t^T x_t C_t^{-1} \tilde{x}_t \tilde{x}_t^T C_t^{-1} x_t \leq \gamma_t C_t^{-1} P_t C_t^{-1} x_t \leq \frac{x_t^T C_t^{-1} x_t}{1 - \gamma_t} \leq \frac{B^2}{\gamma t \lambda (1 - \gamma_t)}.
\]
Proof. As customary for concentration inequalities, we start by a variance and size estimate:

\[ |X_t| \leq 1 + \frac{B}{\sqrt{\gamma_t \lambda (1 - \gamma_t)}} + 2\gamma_t \leq 2 + \frac{B}{\sqrt{\gamma T \lambda (1 - \gamma T)}}, \]

and the variance of \( X_t \) is easily bounded by:

\[
\text{Var}_t [X_t] \leq E_t \left[ (L_t^T x_t - \hat{L}_t^T \tilde{x}_t)^2 \right] = E_t \left[ (\ell_t \cdot (1 - x_t^T C_t^{-1} \tilde{x}_t))^2 \right] \\
\leq E_t \left[ (1 - x_t^T C_t^{-1} \tilde{x}_t)^2 \right] = E_t \left[ 1 - 2x_t^T C_t^{-1} \tilde{x}_t + x_t^T C_t^{-1} x_t x_t^T C_t^{-1} \tilde{x}_t \right] \\
= 1 - 2\tilde{x}_t^T C_t^{-1} \tilde{x}_t + \tilde{x}_t^T C_t^{-1} \tilde{x}_t = 1 - \frac{2\lambda}{(1 - \gamma_t)^2} \tilde{x}_t^T C_t^{-1} \tilde{x}_t + \frac{\gamma_t^2}{(1 - \gamma_t)^2} (u - 2\tilde{x}_t)^T C_t^{-1} u \leq 1 + \frac{3\gamma_t B^2}{(1 - \gamma_t)^2 \lambda}.
\]

Benett’s inequality provides

\[
\sum_{t=1}^T (L_t^T x_t - \hat{L}_t^T \tilde{x}_t + \gamma_t L_t^T (x_t - u)) \leq \frac{1}{3} \left( 2 + \frac{B}{\sqrt{\gamma T \lambda (1 - \gamma T)}} \right) \ln \frac{1}{\delta} + \sqrt{2 \left( T + \frac{3B^2}{\lambda} \sum_{t=1}^T \frac{\gamma_t}{(1 - \gamma_t)^2} \right) \ln \frac{1}{\delta}}.
\]

The claim follows by using \(|L_t^T (x_t - u)| \leq 2\).

Now we compare the losses \( L_t \) with their estimator \( \hat{L}_t \) for all fixed actions.

**Lemma 4.7.** For all \( 0 < \delta < 1 \) with probability at least \( 1 - \delta \) for every \( x \in A \) simultaneously

\[
\sum_{t=1}^T (\hat{L}_t^T x - L_t^T x) \leq \frac{1}{3} \left( 1 + \frac{B^2 T}{\gamma T \lambda} \right) \ln \frac{N}{\delta} + \sqrt{\frac{2B^2 T}{\lambda} \sum_{t=1}^T \frac{1}{\gamma_t} \ln \frac{N}{\delta}} \\
\leq \frac{1}{3} \left( 1 + \frac{B^2 T^{1/3}}{\lambda} \right) \ln \frac{N}{\delta} + \frac{\sqrt{\frac{3B}{\lambda}} T^{2/3}}{N^{1/3}} + \frac{4}{3T^{1/3}} \sqrt{\ln \frac{N}{\delta}}.
\]

**Proof.** As customary for concentration inequalities, we start by a variance and size estimate:

\[
\text{Var}_t [\hat{L}_t^T x - L_t^T x] \leq E_t \left[ (\hat{L}_t^T x)^2 \right] \leq E_t \left[ x^T C_t^{-1} x_t x_t^T C_t^{-1} x_t \right] = x^T C_t^{-1} x \leq \frac{B^2}{\gamma_t \lambda},
\]

and

\[
|\hat{L}_t^T x - L_t^T x| \leq 1 + \frac{B^2}{\gamma_t \lambda}.
\]

Also note that \( \hat{L}_t^T x - L_t^T x \) is a martingale difference sequence. Hence by Benett’s inequality (see Theorem B.2) the claim follows for a fixed action \( x \) with probability at least \( 1 - \delta/N \). Therefore by the union bound, it holds for all \( x \in A \) simultaneously with probability at least \( 1 - \delta \).

Summing up (14), (15) (substituting \( \delta/(N + 2) \) for \( \delta \)), and (16) (substituting \( N\delta/(N + 2) \) for \( \delta \)), we obtain (10) with probability at least \( 1 - \delta \).

## 5 Concluding remarks

We would like to mention that our method could be immediately strengthened to provide an optimal high-probability regret of \( O(\sqrt{T}) \) using the correction term of GeometricHedge (see Bartlett et al. [2008]) and the identity

\[
\mathbb{E}_n \left[ \sum_{i \in [d]} \hat{M}_i(n) \hat{X}_i^2(n) \right] = \mathbb{E}_n \left[ X(n)^T M(n) M(n)^T \Sigma_{i=1}^{d} \Sigma_{i=1}^{d} M(n)^T M(n) X(n) \right],
\]
used for establishing the $O(\sqrt{T})$ regret bound for the expected case under oblivious adversaries in [Combes et al., 2015, supplementary material, proof of Theorem 6]. However, we were unable to verify this identity, which is equivalent to

$$E_n \left[ \sum_{i \in [d]} \tilde{M}_i(n) \tilde{X}_i(n)^2 \right] = E_n \left[ \left( \sum_{i \in [d]} \tilde{M}_i(n) \tilde{X}_i(n) \right)^2 \right],$$

and as such we only claim the weaker bound of $O(T^{2/3})$. This is the only obstacle to combining ComEXP with GEOMETRIC-HEDGE to obtain an efficient algorithm with optimal high-probability regret $O(\sqrt{T})$ for the adaptive case using our method.

To put this into context, without the above identity also for the expected regret case under oblivious adversaries we were only able to establish an $O(T^{2/3})$ regret bound, matching our high-probability regret bound for adaptive adversaries.

References

J. Abernethy, E. Hazan, and A. Rakhlin. Competing in the dark: An efficient algorithm for bandit linear optimization. In Proceedings of the 21st Annual Conference on Learning Theory (COLT, pages 263–274, July 2008. URL http://colt2008.cs.helsinki.fi/papers/123-Abernethy.pdf.

R. Arora, O. Dekel, and A. Tewari. Online bandit learning against an adaptive adversary: from regret to policy regret. In J. Langford and J. Pineau, editors, In Proceedings of the 29th International Conference on Machine Learning, pages 1503–1510, New York, NY, USA, July 2012. Omnipress. ISBN 978-1-4503-1285-1.

J.-Y. Audibert, S. Bubeck, and G. Lugosi. Regret in online combinatorial optimization. Mathematics of Operations Research, 39(1):31–45, 2013.

P. Auer, N. Cesa-Bianchi, Y. Freund, and R. E. Schapire. The nonstochastic multiarmed bandit problem. Siam J. Comput., 32(1):48–77, 2002.

B. Awerbuch and R. D. Kleinberg. Adaptive routing with end-to-end feedback: Distributed learning and geometric approaches. In Proceedings of the thirty-sixth annual ACM symposium on Theory of computing, pages 45–53, New York, NY, USA, June 2004. ISBN 1-58113-852-0. doi:10.1145/1007352.1007367.

P. L. Bartlett, V. Dani, T. Hayes, S. Kakade, A. Rakhlin, and A. Tewari. High-probability regret bounds for bandit online linear optimization. In 21th Annual Conference on Learning Theory (COLT 2008), July 2008. URL http://eprints.qut.edu.au/45706/1/30-Bartlett.pdf.

S. Bubeck and N. Cesa-Bianchi. Regret analysis of stochastic and nonstochastic multi-armed bandit problems. Foundations and Trends in Machine Learning, 5(1):1–122, December 2012. doi:10.1561/2200000024.

S. Bubeck and R. Eldan. Multi-scale exploration of convex functions and bandit convex optimization. COLT 2016, arXiv:1507.06580v1, 2015.

S. Bubeck, R. Eldan, and Y. T. Lee. Kernel-based methods for bandit convex optimization. COLT 2016, arXiv preprint arxiv:1607.03084, July 2016.

N. Cesa-Bianchhia and G. Lugosi. Combinatorial bandits. Journal of Computer and System Sciences (Special Issue: Cloud Computing 2011), 78(5):1404–1422, September 2012. doi:10.1016/j.jcss.2012.01.001.

R. Combes, M. S. T Talebi Mazraeh Shahi, A. Proutiere, and M. Lelarge. Combinatorial bandits revisited. In C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama, and R. Garnett, editors, Advances in Neural Information Processing Systems 28, pages 2116–2124. Curran Associates, Inc., 2015. URL http://papers.nips.cc/paper/5831-combinatorial-bandits-revisited.pdf.

1As of October 2016, we are discussing the matter with the authors of Combes et al. [2019].
V. Dani and T. P. Hayes. How to beat the adaptive multi-armed bandit. *arXiv preprint, arXiv:cs/0602053*, February 2006.

V. Dani, T. P. Hayes, and S. Kakade. The price of bandit information for online optimization. In *Advances in Neural Information Processing Systems*, volume 20, pages 345–352, 2007. URL 
[http://machinelearning.wustl.edu/mlpapers/papers/NIPS2007_758](http://machinelearning.wustl.edu/mlpapers/papers/NIPS2007_758)

J. Edmonds. Maximum matching and a polyhedron with 0,1-vertices. *J. Res. Nat. Bur. Standards B*, 69:125–130, 1965.

X. Fan, I. Grama, and Q. Liu. Hoeffding’s inequality for supermartingales. *Stochastic Processes and their Applications*, 122:3545–3559, 2012. doi:10.1016/j.spa.2012.06.009.

M. Grötschel, L. Lovász, and A. Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, second edition, 1993. ISBN 3-540-56740-2.

E. Hazan and Y. Li. An optimal algorithm for bandit convex optimization. *arXiv preprint, arXiv:1603.04350*, March 2016.

E. Hazan, Z. Karnin, and R. Meka. Volumetric spanners: an efficient exploration basis for learning. In *JMLR: Workshop and Conference Proceedings*, volume 35, pages 1–15, 2014.

M. Jaggi. Revisiting Frank–Wolfe: Projection-free sparse convex optimization. In *Proceedings of the 30th International Conference on Machine Learning (ICML-13)*, pages 427–435, 2013.

S. Lacoste-Julien and M. Jaggi. On the global linear convergence of Frank–Wolfe optimization variants. In C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama, and R. Garnett, editors, *Advances in Neural Information Processing Systems*, volume 28, pages 496–504. Curran Associates, Inc., 2015. URL [http://papers.nips.cc/paper/5925-on-the-global-linear-convergence-of-frank-wolfe-optimization](http://papers.nips.cc/paper/5925-on-the-global-linear-convergence-of-frank-wolfe-optimization)

C. H. Lim and S. J. Wright. Efficient Bregman projections onto the permutahedron and related polytopes. In *Proceedings of the 19th International Conference on Artificial Intelligence and Statistics*, pages 1205–1213, 2016. URL [http://www.jmlr.org/proceedings/papers/v51/lim16.html](http://www.jmlr.org/proceedings/papers/v51/lim16.html)

V. S. Mirrokni, R. P. Leme, A. Vladu, and S. C. wai Wong. Tight bounds for approximate Carathéodory and beyond. *arXiv preprint, arXiv:1512.08602*, 2015.

A. Nemirovski. Efficient methods for large-scale convex optimization problems. *Ekonomika i Matematicheskie Metody*, 15, 1979.

A. Nemirovski. Advances in convex optimization: Conic programming. In *Proceedings of the International Congress of Mathematicians*. EMS-European Mathematical Society Publishing House, 2006.

G. Neu. Explore no more: Improved high-probability regret bounds for non-stochastic bandits. In C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama, and R. Garnett, editors, *Advances in Neural Information Processing Systems 28*, pages 3168–3176. Curran Associates, Inc., 2015. URL [http://papers.nips.cc/paper/5732-explore-no-more-improved-high-probability-regret-bounds](http://papers.nips.cc/paper/5732-explore-no-more-improved-high-probability-regret-bounds)

T. Rothvoß. The matching polytope has exponential extension complexity. *Proceedings of STOC*, pages 263–272, 2014.
A Time-varying EXP algorithm with projections

Let $\mathbb{R}^n_{>0} := \{ x \in \mathbb{R}^n : x_i > 0 \text{ for all } i \in [n] \}$ be the strictly positive orthant. In this section we provide a version of EXP (see Algorithm 1 that computes points in an arbitrary convex set $P \subseteq \mathbb{R}^n_{>0}$ (as compared to distributions in the probability simplex), and (2) that is any-time, i.e., the parameter choice is independent of $T$ and the regret bounds hold uniformly for any $t \leq T$. We explicitly allow arbitrary dependence between the parameters $\eta_t$, input $L_t$, and the points $x_t$ computed by the algorithm, to ease the use of the regret bound in applications.

Algorithm 3 EXP for convex sets contained in $\mathbb{R}^n_{>0}$ with time varying parameters

Require: convex set $P \subseteq \mathbb{R}^n_{>0}$, start point $x_1 \in P$, loss vectors $L_t \in \mathbb{R}^n$, positive parameters $\eta_1 \geq \eta_2 \geq \ldots$, satisfying $\eta_t L_{i,t} \geq -1$ for $i = 1, \ldots, n$

for $t = 1$ to $T - 1$ do
\begin{align*}
\tilde{x}_{t+1,i} &\leftarrow x_{1,t}^{1-\eta_{t+1}/\eta_t} x_{1,t}^{\eta_{t+1}/\eta_t} e^{-\eta_{t+1} L_{i,t}} \text{ for all } i \in [n].
\end{align*}
Find $x_{t+1} \in P$ such that $\text{KL}(z, x_{t+1}) \leq \text{KL}(z, \tilde{x}_{t+1}) + \varepsilon_t$ for all $z \in P$. (approximate Bregman projection)
end for

Lemma A.1. Let $P \subseteq \mathbb{R}^n_{>0}$ be a convex set and let $\eta_t L_{i,t} \geq -1$ for all $t$ and $i$, the vector $x_t$ computed by Algorithm 3 satisfy the following:
\begin{align*}
\sum_{t=1}^{T} L_t^T x_t - \sum_{t=1}^{T} L_t^T y &\leq \frac{\text{KL}(y, x_1)}{\eta_T} + \sum_{i=1}^{T-1} \frac{\varepsilon_t}{\eta_{t+1}} + (e - 2) \sum_{t=1}^{T} \eta_t \sum_{i=1}^{n} x_{t,i} L_{i,t}^2.
\end{align*}

Proof. The proof is an extension of the standard analysis of EXP, using the potential $(\text{KL}(y, x_t) - \text{KL}(y, x_1))/\eta_t$ to measure progress:
\begin{align*}
\text{KL}(y, x_t) - \text{KL}(y, x_1) &= \sum_{i=1}^{n} y_i \ln \frac{x_{1,i}}{x_{t,i}} - \sum_{i=1}^{n} x_{1,i} + \sum_{i=1}^{n} x_{t,i}. \quad (17)
\end{align*}

We compare this with the potential in the next round, first using $\tilde{x}_{t+1}$ instead of $x_{t+1}$:
\begin{align*}
\frac{\text{KL}(y, \tilde{x}_{t+1}) - \text{KL}(y, x_1)}{\eta_{t+1}} - \frac{\text{KL}(y, x_1) - \text{KL}(y, x_{t+1})}{\eta_t} - L_t^T y &\leq \frac{1}{\eta_t} \sum_{i=1}^{n} x_{t,i} \left( e^{-\eta_{t+1} L_{i,t}} - 1 \right) \\
&\leq \frac{1}{\eta_t} \sum_{i=1}^{n} x_{t,i} \left( -\eta_{t+1} L_{i,t} + (e - 2)(\eta_{t+1} L_{i,t})^2 \right) = -L_t^T x_t + (e - 2) \eta_t \sum_{i=1}^{n} x_{t,i} L_{i,t}^2,
\end{align*}

Summing up for $t = 1, \ldots, T$ and rearranging leads to (using the value $\eta_{T+1} = \eta_T$)
\begin{align*}
\sum_{t=1}^{T} L_t^T x_t - \sum_{t=1}^{T} L_t^T y - (e - 2) \sum_{t=1}^{T} \frac{\sum_{i=1}^{n} x_{t,i} L_{i,t}^2}{\eta_t} &\leq \sum_{t=1}^{T-1} \frac{\text{KL}(y, x_{t+1}) - \text{KL}(y, \tilde{x}_{t+1})}{\eta_{t+1}} + \frac{\text{KL}(y, x_1) - \text{KL}(y, x_{T+1})}{\eta_T} \\
&\leq \sum_{t=1}^{T-1} \frac{\varepsilon_t}{\eta_{t+1}} + \frac{\text{KL}(y, x_1)}{\eta_T},
\end{align*}

using $\text{KL}(y, \tilde{x}_{T+1}) \geq 0$ and $\text{KL}(y, x_{T+1}) - \text{KL}(y, \tilde{x}_{T+1}) \leq \varepsilon_t$. The claim follows by rearranging. \qed
B Concentration inequalities

We will use the following concentration inequalities.

**Theorem B.1** (Azuma–Hoeffding inequality). For a martingale difference sequence $X_t$ with $a_t \leq X_t \leq b_t$ almost surely for constants $a_t$, $b_t$, we have with probability at least $1 - \delta$

$$
\sum_{t=1}^{T} X_t \leq \sqrt{\frac{\sum_{t=1}^{T} (b_t - a_t)^2 \ln(1/\delta)}{2}}.
$$

While the following inequality is stated only for $b = 1$ in [Fan et al., 2012, (18)] it easily generalizes via scaling to arbitrary $b > 0$.

**Theorem B.2** (Benett’s inequality [Fan et al., 2012, (18)]). For a supermartingale difference sequence $X_t$ bounded above by a positive constant $X_t \leq b$, for any $v \geq 0$ with probability at least $1 - \delta$:

$$
\sum_{t=1}^{T} \text{Var}_t[X_t] \geq v \quad \text{or} \quad \sum_{t=1}^{T} X_t \leq \frac{b \ln(1/\delta)}{3} + \sqrt{2v \ln(1/\delta)}.
$$

C Projection for Kullback–Leibler divergence

We will now describe a generic, efficient, simple Frank–Wolfe algorithm for the projection step in Line 10 of Algorithm 1. We remark that there are many possibilities for improvements, such as, e.g., employing advanced variants of the Frank–Wolfe algorithm (see e.g., Lacoste-Julien and Jaggi 2015) or using customized algorithms for specific polytopes. For example, in the case of the simplex $P = \{ x \geq 0 | \sum_{i=1}^{n} x_i = 1 \}$, the projection of $x$ is simply $x / \sum_{i=1}^{n} x_i$ and for the the permutahedron there exist very fast, specialized projection methods (see e.g., Lim and Wright 2016).

**Algorithm 4** Projection for KL

**Require:** linear optimization oracle over a polytope $P \subseteq [\alpha, \beta]^n$, $\alpha > 0$, upper bound $B$ for the $\ell_2$-diameter of $P$, accuracy $\epsilon > 0$, point $x \in \mathbb{R}^n$

**Ensure:** $y_K \in P$ with KL$(z, y_K) \leq \text{KL}(z, x) + \epsilon$ for all $z \in P$

$y_0 \in P$ any point

$K \leftarrow \left\lceil \frac{4B^2\beta}{\alpha^2\epsilon^2} \right\rceil$

for $k = 1$ to $K$

$s \leftarrow \arg \min_{z \in P} \sum_{i=1}^{n} z_i \ln(y_{k-1,i}/x_i)$ \{Linear optimization oracle call\}

$y_k \leftarrow ((k-1)y_{k-1} + 2s)/(k+1)$

end for

return $y_K$

**Proposition C.1.** Given a polytope $P \subseteq [\alpha, \beta]^n$ with $\alpha > 0$, an upper bound $B$ for the $\ell_2$-diameter of $P$, as well as an accuracy $\epsilon > 0$, Algorithm 4 computes an approximate projection with $O\left(\frac{B^2 \beta}{\alpha \epsilon^2}\right)$ oracle calls.

**Proof.** As the algorithm calls the oracle once per iteration, the bound on the number of oracle calls is immediate. To prove the claimed accuracy of the returned point $y_K$, note that the algorithm is the Frank–Wolfe algorithm for the function $f(z) := \text{KL}(z, x)$. Recall that the gradient $\nabla f(z)$ of $f$ at $z$ is given by $(\nabla f(z))_i = \ln(z_i/x_i)$ and the Hessian is a diagonal matrix $\nabla^2 f(z) = \text{diag}(1/z_1, 1/z_2, \ldots, 1/z_n)$. As $1/\beta \leq 1/z_i \leq 1/\alpha$ for $z \in P$, the function $f$ is $1/\alpha$-smooth and $1/\beta$-strongly convex on $P$ in the $\ell_2$-norm, and has curvature $C_f \leq B^2/\alpha$. Let $x^* := \arg \min_{z \in P} f(z)$, i.e., the Bregman projection of $x$ to $P$. By [Jaggi, 2013, Theorem 1], $f(y_K) - f(x^*) \leq 2C_f/(K+2)$, therefore by strong convexity

$$
\frac{1}{2\beta} \|y_K - x^*\|^2_2 \leq \text{KL}(y_K, x) - \text{KL}(x^*, x) \leq \frac{2B^2}{\alpha(K+2)}.
$$
Let $z \in P$ be arbitrary. By the Pythagorean Theorem we have $\text{KL}(z, x^*) \leq \text{KL}(z, x)$ and thus

$$\text{KL}(z, y_K) - \text{KL}(z, x) \leq \text{KL}(z, y_K) - \text{KL}(z, x^*) = \sum_{i=1}^{n} z_i \ln \frac{x_i^*}{y_{K,i}} - \sum_{i=1}^{n} x_i^* + \sum_{i=1}^{n} y_{K,i}$$

$$\leq \sum_{i=1}^{n} z_i \left( \frac{x_i^*}{y_{K,i}} - 1 \right) - \sum_{i=1}^{n} x_i^* + \sum_{i=1}^{n} y_{K,i} = \sum_{i=1}^{n} \frac{z_i}{y_{K,i}} (x_i^* - y_{K,i}) \leq \frac{B}{\alpha} \|x^* - y_K\|_2$$

$$\leq \frac{B \sqrt{2p}}{\alpha} \sqrt{\text{KL}(y_K, x) - \text{KL}(x^*, x)} \leq \frac{2B^2 \sqrt{p}}{\alpha^{3/2} \sqrt{p + 2}} \leq \epsilon.$$  

Plugging in $K = \left[ \frac{4B^4 \alpha^2}{\epsilon^2} \right]$ as set by the algorithm provides the result. \hfill \Box

## D Linear decomposition

For the convenience of the reader, we briefly recall the decomposition algorithm (Algorithm 5) of Mirrokni et al. [2015] that for a polytope $P$ approximately decomposes any point $x \in P$ into a convex combination of vertices of $P$, using a linear optimization oracle over $P$. The algorithm uses Mirror Descent (see Nemirovski [1979]) to find a convex combination.

**Proposition D.1** (Mirrokni et al. [2015, Theorem 3.5]). Given a polytope $P$ with diameter at most $2D$ in $\ell_2$-norm, and a point $x \in P$, Algorithm 5 computes with $O(D^2/\epsilon^2)$ calls to a linear optimization oracle over $P$ a multiset $x_1, \ldots, x_k$ of vertices for $k = \left[ \frac{4D^2}{\epsilon^2} \right]$ such that $\|\sum_{i=1}^{k} x_i/k - x\|_2 \leq \epsilon$.

**Algorithm 5 Linear decomposition**

**Require:** linear optimization oracle over polytope $P$, an inner point $x \in P$, precision $\epsilon$

**Ensure:** vertices $x_1, \ldots, x_k \in P$ such that $\|x - \sum_{i} \lambda_i x_i/k\|_2 \leq \epsilon$

1. $k \leftarrow \left[ \frac{4D^2}{\epsilon^2} \right]$
2. $\eta \leftarrow 4\epsilon(p - 1)$
3. $y_1 \leftarrow 0; z_1 \leftarrow 0$
4. for $t = 1$ to $k$ do
   1. Choose vertex $x_t \in \arg\min_{y \in P} y_1^T y$ \{Linear optimization oracle call\}
   2. $z_{t+1} \leftarrow z_t - \eta (x - x_t)$
   3. if $\|z_{t+1}\|_2 > 1$ then
      1. $y_{t+1} \leftarrow z_{t+1}/\|z_{t+1}\|_2$
   4. else
      1. $y_{t+1} \leftarrow z_{t+1}$
   5. end if
5. end for
6. return $x_1, \ldots, x_k$

## E Fitness of barycentric spanners for exploration

Let $\lambda_{\min}(\mu)$ denote the minimal eigenvalue of the covariance matrix $\mathbb{E}_{x \sim \mu} [xx^T]$ of a distribution $\mu$. For exploration one wishes to find a $\mu$ with a high minimal eigenvalue $\lambda_{\min}(\mu)$. Here we show that a uniform distribution on any approximate barycentric spanner achieves within an $O(n^2)$ factor the best possible minimal eigenvalue using any scalar product on $\mathbb{R}^n$. The free choice of scalar product and hence orthonormal basis allows preserving sparse representation of a polytope $P$. 
Lemma E.1. Let \( v_1, \ldots, v_n \) be a \( C \)-approximate barycentric spanner of a polytope \( P \subseteq \mathbb{R}^n \). Then the uniform distribution \( \mu_{v_1, \ldots, v_n} \) on the spanner satisfies

\[
\lambda_{\min}(\mu_{v_1, \ldots, v_n}) \geq \frac{\lambda_{\min}(\mu)}{C^2 n^2}
\]

for any distribution \( \mu \) over \( P \).

Proof. Using that the \( v_i \) form a barycentric spanner, there are coefficients \( \lambda_{x,i} \) for all \( x \in P \) satisfying

\[
x = \sum_i \lambda_{x,i} v_i, \quad |\lambda_{x,i}| \leq C.
\]

In particular, with \( a_x := \sum_{i=1}^n |\lambda_{x,i}| \leq Cn \) by Jensen’s inequality

\[
xx^\top \leq \sum_{i=1}^n a_x |\lambda_{x,i}| v_i v_i^\top \leq C^2 n \sum_{i=1}^n v_i v_i^\top.
\]

Hence \( \mathbb{E}_{x \sim \mu} [xx^\top] \preceq C^2 n \sum_{i=1}^n v_i v_i^\top = C^2 n^2 \mathbb{E}_{x \sim \mu_{v_1, \ldots, v_n}} [xx^\top] \), from which the claim follows. \( \square \)