Number of generators of ideals in Jordan cells of the family of graded Artinian algebras of height two *

Nasrin Altafi
Department of Mathematics, KTH Royal Institute of Technology, S-100 44 Stockholm, Sweden.

Anthony Iarrobino
Department of Mathematics, Northeastern University, Boston, MA 02115, USA.

Leila Khatami
Union College, Schenectady, New York, 12308, USA.

Joachim Yaméogo
Université Côte d’Azur, CNRS, LJAD, FRANCE.

Abstract
We let $A = R/I$ be a standard graded Artinian algebra quotient of $R = k[x, y]$, the polynomial ring in two variables over a field $k$ by an ideal $I$, and let $n$ be its vector space dimension. The Jordan type $P_\ell$ of a linear form $\ell \in A_1$ is the partition of $n$ determining the Jordan block decomposition of the multiplication on $A$ by $\ell$ – which is nilpotent. The first three authors previously determined which partitions of $n = \dim_k A$ may occur as the Jordan type for some linear form $\ell$ on a graded complete intersection Artinian quotient $A = R/(f, g)$ of $R$, and they counted the number of such partitions for each complete intersection Hilbert function $T$ [AIK].

We here consider the family $G_T$ of graded Artinian quotients $A = R/I$ of $R = k[x, y]$, having arbitrary Hilbert function $H(A) = T$. The Jordan cell $V(E_P)$ corresponding to a partition $P$ having diagonal lengths $T$ is comprised of all ideals $I$ in $R$ whose initial ideal is the monomial ideal $E_P$ determined by $P$. These cells give a decomposition of the variety $G_T$ into affine spaces. We determine the generic number $\kappa(P)$ of generators for the ideals in each cell $V(E_P)$, generalizing a result of [AIK]. In particular, we determine those partitions for which $\kappa(P) = \kappa(T)$, the generic number of generators for an ideal defining an algebra $A$ in $G_T$.

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We also count the number of partitions $P$ of diagonal lengths $T$ having a given $\kappa(P)$. A main tool is a combinatorial and geometric result allowing us to split $T$ and any partition $P$ of diagonal lengths $T$ into simpler $T_i$ and partitions $P_i$, such that $V(E_P)$ is the product of the cells $V(E_{P_i})$, and $T_i$ is single-block: $G_{T_i}$ is a Grassmannian.

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1 Introduction.

Let $A$ be a standard-graded Artinian algebra $A = \bigoplus_{i=0}^{j} A_i$ over a field $k$, and let $\ell \in A_1$ be a linear form. The multiplication map $m_\ell : A \to A : a \to \ell \cdot a$ is nilpotent. The Jordan type $P_\ell = P_{\ell,A}$ is a partition of $n = \dim_k A$, giving the Jordan block decomposition of the multiplication map $m_\ell$. We consider standard graded Artinian algebra quotients $A = R/I$ where $I$ is an ideal of $R = k[x,y]$, the polynomial ring $R$ in two variables over $k$. We will assume that $I_1 = 0$, so the height (or codimension) $\dim_k A_1$ of $A$ is two. The order of the graded ideal $I$ is the lowest degree of a (non-zero) element.

The Hilbert function $T = H(A)$ of such a graded Artinian algebra $A$ in codimension two is a sequence of the following form

$$T = (1,2,\ldots,d,t_d,t_{d+1},\ldots,t_j,0)$$

where $d \geq t_d \geq t_{d+1} \geq \cdots \geq t_j > 0$, \hspace{0.5cm} (1.1)

where $t_i = \dim_k A_i$, $j$ is the socle degree of $T$, $d$ is the order of $T$ – the order of any ideal $I$ such that $H(R/I) = T$ – and $n = |T| = \sum t_i = \dim_k A$.

The first three authors in [AIK] determined all possible Jordan types $P_\ell$ of linear forms $\ell$ for complete intersection (CI) graded Artinian algebras of height two: they assumed that the field $k$ either has characteristic zero, or is infinite of
characteristic $p > j$, the socle degree of the algebra. In this paper we make the same assumption on characteristic because we use a standard-basis result due to J. Briançon and A. Galligo in showing Lemma 3.6 that requires the restriction. Our main results in this paper generalize those of [AIK] to all height two Hilbert functions.

Let $P$ be a partition and let $E_P$ be the monomial ideal determined by the Ferrers diagram of $P$. The diagonal lengths of the Ferrers diagram of $P$ is just the Hilbert function $T_P = H(R/E_P)$ (Definition 2.1). The set $\mathcal{P}(T)$ of partitions having diagonal lengths $T$ or, equivalently, the set of monomial ideals $E_P$ such that $H(R/E_P) = T$, has been studied by the second and last author in [IY1], as well as by others, including [Bu, Br, Br-Ga, MR, Con]. We denote by $G_T$ the family of graded Artinian quotients $A = R/I$ where $I$ is an ideal of $R = k[x,y]$, for which the Hilbert function $H(A) = T$. This is a smooth projective variety $G_T$, that is locally an affine space of known dimension (Proposition 2.7). Let $\mathcal{V}(E_P)$ be the affine cell of $G_T$ that parametrizes the ideals of $R$ having initial ideal $E_P$ (Definition 2.4).

We will term $\mathcal{V}(E_P)$ a Jordan cell. L. Göttsche showed that the cells $\mathcal{V}(E_P)$ for $P \in \mathcal{P}(T)$ form a cellular decomposition of $G_T$ [Gö1]; this followed the analogous results of G. Ellingsrud and S.A. Strømme for the punctual Hilbert scheme of projective space $\mathbb{P}^2$, using the A. Białynicki-Birula theorem [ES1, ES2, B-B]. This cellular decomposition was as well studied by G. Gotzmann [Gm], and the fourth author [Y2, Y3]. Generators and relations for height two graded ideals have been studied by many, as [Bu, Br, Br-Ga, MR, Con]; J.O. Kleppe studies a scheme analogue of $G_T$ [Kl]. L. Evain discusses an equivariant Hilbert scheme, involving weights of the variables [Ev]; some combinatorics of cells in [IY1] are seen in a larger context in [LW]. The cells $\mathcal{V}(E_P)$ and their connection with generators and relations of ideals have also been studied by A. Conca and G. Valla in [CoVa].

By a generic element of an irreducible algebraic variety $X$ we will mean an element belonging to a certain non-empty Zariski-dense open subset $U \subset X$.

The main results of this paper, Theorems 3.11 and 5.15 give $\kappa(P)$, the minimum number of generators for the ideal $I$ defining a generic element $A = R/I$ in the cell $\mathcal{V}(E_P)$, for each partition $P \in \mathcal{P}(T)$, where $T$ is a Hilbert function of a height two graded Artinian algebra, namely a sequence of the form given in Equation (1.1). In Theorem 6.1 we give the number of partitions $P \in \mathcal{P}(T)$ with $\kappa(P) = k$, for any positive integer $k$.

**Summary.** Decomposition of cells. We first prove in Section 2.3 a new combinatorial and geometric result allowing us to associate to an arbitrary Hilbert function $T$ and to any partition $P$ of diagonal lengths $T$ their components, a set of simpler, single-block sequences $T_i$ and partitions $P_i$ of diagonal lengths $T_i$. We show that the Jordan cell $\mathcal{V}(E_P)$ of $G_T$ is in a natural way the product of its single-block components, the cells $\mathcal{V}(E_{P_i})$ of $G_{T_i}$ (Theorem 2.27). This decomposition is closely related to the hook codes that had been studied in [IY1] and that we define in Section 2.2.

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1 The local analogue $Z_T$ of $G_T$ has been termed a vertical cell by J. Briançon [Br]; $G_T$ has been termed an Ellingsrud-Strømme-Göttsche cell [IY1], defined using a $\mathbb{C}^*$ action. The fourth author showed that these concepts are the same both for $G_T$ and the associated parameter space $Z_T$ for local algebras [Y2].
Single-block case. In the single-block case the Hilbert function sequence $T$ of Equation (1.1) satisfies $d = j$, so

$$T = (1, 2, \ldots, d, t, 0),$$

(1.2)

that is $t_i = i + 1$ for $0 \leq i \leq d - 1$ and $t_d = t$. An ideal $I \subset R$ defining an algebra $A = R/I$ of Hilbert function $T$ of Equation (1.2) satisfies

$$I = V \oplus m^{d+1},$$

(1.3)

where we let $V = I_d \subset R_d$ and $m$ is the maximal ideal of $R$. Thus, the projective variety $G_T$ in the single-block case is isomorphic to the Grassmannian Grass$(s, R_d)$, $s = d + 1 - t$, parametrizing $s$-dimensional subspaces $V \subset R_d$. In Theorem 3.11 we determine the integer $\kappa(P)$, the number of generators of a generic ideal in the cell $V(E_P)$, for single-block partitions, and in Theorem 4.2 we give the number of special single-block partitions $P$, namely partitions $P$ with $\kappa(P) > \kappa(T)$, the generic - and minimum - number of generators for an ideal defining an algebra $A$ in $G_T$. More generally, in Corollary 4.3 we determine the number of partitions $P \in \mathcal{P}(T)$ with $\kappa(P) = k$, for any positive integer $k$.

General case. In Theorem 5.15 we determine $\kappa(P)$ for an arbitrary partition $P \in \mathcal{P}(T)$ in terms of the $\kappa(P_i)$ of their single-block components. We show that $P$ is special ($\kappa(P) \neq \kappa(T)$) if and only if some component $P_i$ is special (Theorem 5.18). In Theorem 6.1 we determine $\kappa(P)$ for arbitrary Jordan types $P$, and we give the number of partitions $P \in \mathcal{P}(T)$ with $\kappa(P) = k$, for any positive integer $k$; this result also determines the number of special partitions in $\mathcal{P}(T)$ (Corollary 6.3), and we recover the number of CI partitions shown in [AIK] (Corollary 6.4).

Hook code. We explain in Section 2.2 the hook code for partitions $P$ of diagonal lengths $T$. The hook code of $P$ is a sequence $\Omega(P) = (\mathfrak{h}_d(P), \ldots, \mathfrak{h}_j(P))$ of partitions-in-a-box $\mathfrak{B}_i(T), d \leq i \leq j$, where the box $\mathfrak{B}_i(T) = (\delta_{i+1}) \times (1 + \delta_i)$; that is, the Ferrers diagram of each $\mathfrak{h}_i(P)$ has at most $\delta_{i+1}$ rows and $(1 + \delta_i)$ columns (Definition 2.9). The partitions of diagonal lengths $T$ are completely determined by their hook code $\Omega(P)$. Also, the component partition $P_i$ of diagonal lengths $T_i$ has as its hook code the degree-$i$ component $\mathfrak{h}_i(P)$ of $\Omega(P)$ (Proposition 2.22).

The proofs involve a careful study of standard generators and relations for the ideals $I$ defining algebras in the cell $V(E_P)$, using in particular the hook code of a partition $P$. We then compare these invariants to those for the partition $P : x$ corresponding to the ideals $I : x$. This allows us to compare $\kappa(P)$ with $\kappa(P : x)$, and we thus determine how to compute $\kappa(P)$ from the hook code $\Omega(P)$.

2 Cells of the variety $G_T$ and their hook codes.

2.1 The variety $G_T$ and the cells $V(E_P)$.

We need some basic notions from [LY1, LY2] (see also [AIK] §4.1).

Recall that we consider graded Artinian quotients $A = R/I$, where $I$ is an ideal of $R = k[x, y]$ the polynomial ring over an arbitrary field $k$. The Hilbert
function of $A$ is the sequence $H(A) = (1, t_1, \ldots, t_j)$ where $t_i = \dim_k A_i$ and $j$ is the socle degree of $A$ that is $A_j \neq 0$, $A_{j+1} = 0$. The family of all such quotients having Hilbert function $H(A) = T$ is denoted by $G_T$, which has a natural structure of subvariety $G_T \subset \Pi_{d \leq i \leq j} \text{Grass}(t_i, R_i)$, where $\text{Grass}(t_i, R_i)$ parametrizes quotients $A_i = R_i/I_i$ of vector space dimension $t_i$. Thus we have

$$\iota : G_T \to \Pi_{d \leq i \leq j} \text{Grass}(t_i, R_i) : A = R/I \to (R_d/I_d, R_{d+1}/I_{d+1}, \ldots, R_j/I_j).$$

We now explain the affine cell decomposition $G_T = \bigcup_{P \in \mathcal{P}(T)} V(E_P)$ where $P$ runs through the set $\mathcal{P}(T)$ of partitions having diagonal lengths $T$ (Theorem 2.8).

**Definition 2.1** (The monomial ideal $E_P$ and diagonal lengths of $P$). Given a partition $P = (p_1, p_2, \ldots, p_t)$ of $n = \sum p_i$ where $p_1 \geq p_2 \geq \cdots \geq p_t$, we let $C_P$ be the set of $n$ monomials that fill the Ferrers diagram $F_P$ of $P$ as follows: for $i \in [1, t]$ the, $i$-th row counting from the top of $F_P$ is filled by the monomials $y^i x^{p_1 - 1}, y^i x^{p_2 - 1}, \ldots, y^i x^{p_i - 1}$. We let $E_P$ be the complementary set of monomials to $C_P$ and denote by $(E_P)$ the monomial ideal generated by $E_P$. The diagonal lengths $T_P$ of $P$ is the Hilbert function $T_P = H(R/E_P)$.

In a Ferrers diagram of monomials associated to a partition $P$ of $n$, the $x$-degrees of monomials increase as we go from left to right and the $y$-degrees increase as we go from top to bottom. We count the columns from left to right and the rows from top to bottom. See Figure 1 for the Ferrers diagram of the partition $P = (5, 3, 1)$ of diagonal lengths $T = (1, 2, 3, 2, 1)$; and Example 5.23 and Figure 20 for that of $P = (10^2, 4, 3, 2^5)$.

![Figure 1: Ferrers diagram for $P = (5, 3, 1)$](image1)

**Definition 2.2.** A hook of a partition $P$ is a subset of the Ferrers diagram $F_P$ consisting of a hook-corner $c$, an arm $(c, xc, \ldots, \nu = x^{u-1}c)$ and a leg $(c, yc, \ldots, \mu = y^{v-1}c)$, such that $x\nu \in E_P$ and $y\mu \in E_P$ (Figure 2). The arm length is $u$ and the leg length is $v$; the hook has arm-leg difference $u - v$. We term the monomial $\nu$ the hand, and the monomial $\mu$ the foot of the hook.

![Figure 2: Difference-one hook with hand $h$, foot $f$, corner $c$, $P = (4,1,1)$](image2)
Figure 3: Difference-one hook with corner $x$ for $P = (4, 3, 1)$.

**Example 2.3.** Let $P = (4, 3, 1)$. $P$ has diagonal lengths $T_P = (1, 2, 3, 2)$. The hook with corner $x$ in the Ferrers diagram $C_P$ has arm length 3, foot length 2, hand $x^3$, foot $yx$, so has (arm − leg) difference one (Figure 3).

**Definition 2.4** (Initial ideal of $I$, and the Jordan cell $V(E_P)$). We order the monomials of degree $i$ by $x^i < x^{i−1}y < \cdots < y^i$ (lex order). The *initial monomial* $\mu(f) = \min(f)$ of a form $f = \sum_k a_k y^k x^{i−k}, a_k \in k$ is the monomial $\mu(f) = y^s x^{i−s}$ of highest $y$-degree $s$ among those with non-zero coefficients $a_k$. Given an ideal $I \subset \mathbb{R} = k[x, y]$, defining the Artinian quotient $A = R/I$ we denote by $\text{in}(I)$ the ideal

$$\text{in}(I) = (\{\text{in}(f), f \in I\})$$

generated by the initial monomials of all elements of $I$. We may identify $\text{in}(I)$ with an ideal $E_P$ for a partition $P = P(I)$ of diagonal lengths $T = H(A) = H(R/E_P)$.

We denote by $V(E_P)$ the affine variety parametrizing all ideals $I \subset \mathbb{R}$ having initial ideal $E_P$ (for the affine variety structure see Theorem 2.8 and [IY2, Prop. 2.6], or Theorem 2.27 below).

When counting the minimal number of generators of an ideal $I \in V(E_P)$ we will refer to the leading terms of these generators as corner-monomials of $E_P$.

**Definition 2.5** (corner-monomial of $E_P$). Let $P$ be a partition of an integer $n$. Denote by $E_P \subset \mathbb{R} = k[x, y]$ the monomial ideal associated to $P$ (Definition 2.1). An element of a minimal set of generators of $E_P$ is called a corner-monomial of $E_P$.

**Example 2.6.** Let $P = (4, 4, 2, 1, 1)$. Then the corner-monomials of $E_P$ are $x^4, x^2 y^2, xy^3$ and $y^5$ (see Figure 4).

For the first of the next two results see [Ia1, Theorems 2.9, 2.12], or [IY1, §3-B,Theorem 3.12, §3-F]; the cellular decomposition in the second was shown by L. Göttsche [Gal], and the hook count for the dimension is [IY1, Theorem 3.12]. Further results involving the intersections of closures of cells was shown by the last author in [Y2, Y3], relying in part on methods of J. Briançon [Br]. Recall that we denote by $P(T)$ the set of all partitions of $n = |T|$ having diagonal lengths $T$. We denote by $\delta_i(T)$ the difference $\delta_i(T) = t_{i−1} − t_i$, for $i \geq d$.

**Proposition 2.7** (The smooth projective variety $G_T$). [Ia1, Thm. 3.13]. The variety $G_T$ parametrizing all ideals $I$ of $R = k[x, y]$ satisfying $H(R/I) = T$ is a smooth irreducible projective variety, that is locally an affine space of dimension $\sum_{i\geq d}(\delta_i + 1)(\delta_i+1)$: it has a connected cover by opens in the same affine space.
Figure 4: Ferrers diagram of $P = (4, 4, 2, 1, 1)$: corner-monomials of $E_P$ (in blue).

**Theorem 2.8** (Cellular decomposition of $G_T$). [Y2] [Y3]. The Jordan cell $\mathcal{V}(E_P)$ is an affine space of dimension equal to the total number of difference-one hooks in $C_P$, viewed as the Ferrers diagram of the partition $P$ (Definition 2.1).

The variety $G_T$ has a finite decomposition into affine cells,

$$G_T = \bigcup_{P \in \mathcal{P}(T)} \mathcal{V}(E_P).$$

(2.1)

### 2.2 Hook code of $P$.

We review the hook code, using results from [IY1] [IY2]. First, given $T$ satisfying Equation (1.1) we define a sequence $\mathfrak{B}(T)$ of rectangular partitions or boxes. We let

$$\delta_i(T) = t_{i-1} - t_i$$

for $i \geq d(T)$. We set

$$\mathfrak{B}(T) = (\mathfrak{B}_d(T), \mathfrak{B}_{d+1}(T), \ldots, \mathfrak{B}_j(T)),$$

where

$$\mathfrak{B}_i(T) = (\delta_{i+1}) \times (1 + \delta_i),$$

a rectangular box,

(2.2)

with height $\delta_{i+1}$ and base $1 + \delta_i$: so $\mathfrak{B}_i(T)$ has $\delta_{i+1}$ parts, each $1 + \delta_i$. We order the monomials of degree $i$ by $x^i < x^{i-1} y < \cdots < y^i$ (lex order); certain of these monomials are hands of degree-$i$ hooks of $P$, that is end elements of rows of $C_P$ from Definition 2.1 and we order these correspondingly.

**Definition 2.9.** Suppose that the partition $P$ has diagonal lengths $T$. The (difference-one) hook code of $P$ is the sequence

$$\Omega(P) = (\mathfrak{h}_d(P), \ldots, \mathfrak{h}_j(P))$$

(2.3)

where $\mathfrak{h}_i(P)$ is a partition that enumerates the difference-one hooks of hand-degree $i$, according to their $\delta_{i+1}$ degree-$i$ hands. That is, the $k$-th part of $\mathfrak{h}_i(P)$ is the number of difference-one hooks having the $k$-th possible degree-$i$ hand.

**Remark 2.10.** It is not hard to see that the number of difference-one hooks per hand is in the interval $[0, \delta_{i+1}]$, and that is non-increasing: so $\mathfrak{h}_i(P)$ is a partition, and $\mathfrak{h}_i(P) \subset \mathfrak{B}_i(T)$: the degree-$i$ hook partition fits into the box $\mathfrak{B}_i(T)$. 

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Thus, the code is determined by arranging the difference-one hooks of $P$ first, according to their hand-degree $i$, then according to their “hand monomial,” determining for each degree $i \in [d,j]$ a partition $h_i(P)$.

We denote by $Q(T)$ the set of all $(j+1-d)$-tuples of partitions $(h_d, \ldots, h_j)$ satisfying, $h_i \subset \mathfrak{B}_i(T)$. Here $Q(T)$ is a lattice under the product structure given by inclusion for each component $h_i$; that is, $Q \leq Q'$ if each $h_i \subset h_i'$, in the sense that the Ferrers diagram for $h_i$ fits inside that of $h_i'$.

When $T$ is a sequence satisfying Equation (1.1) - of socle degree $j$ - we write $T^\vee$ for the partition obtained from $T$ as the conjugate (switch rows and columns in the Ferrers graph) of the partition having $j+1$ parts $\{t_0, t_1, \ldots, t_j\}$. Thus, $T^\vee$ gives the lengths of the rows of the bar-graph of $T$. The partition $T^\vee$ always has the maximum hook code $\mathfrak{B}(T)$ possible for a partition of diagonal lengths $T$.

**Example 2.11** (Hook code for $P = (6, 3, 3, 3)$). Let $T = (1, 2, 3, 4, 3, 2, 0)$ where $d = 4$; we have $\delta_4 = 4 - 3 = 1, \delta_5 = 3 - 2 = 1, \delta_6 = 2 - 0 = 2$. Then $\mathfrak{B}(T) = (\mathfrak{B}_4, \mathfrak{B}_5) = ((1 \times 2)_4, (2 \times 2)_5)$. The partition $T^\vee = (6, 5, 3, 1)$ has the maximum hook code $Q(T^\vee) = \mathfrak{B}(T)$. But $P = (6, 3^3)$ has the hook code $Q(P) = (1_4, (2, 1)_5)$: the degree four hand monomial is $y^2 x^2$ with a single difference-one hook, with corner $y^2$; the degree-5 hand monomials are $x^5$ with two hooks with corners $x, x^4$, and $y^2 x^5$ with one hook, corner $y^2 x$. See Figure 5 where we visualize the hooks by showing their corners, blue for degree 4 and red for degree 5.

![Figure 5: Hook code for $P = (6, 3, 3, 3): Q(P) = (1_4, (2, 1)_5)$.

The following is stated as part of [Y1] Theorem 3.27, and shown in [Y2] Theorem 1.17. Recall that $\mathcal{P}(T)$ is the set of partitions having diagonal lengths $T$. We denote by $q: \mathcal{P}(T) \to Q(T)$ the hook code map taking $P$ to $Q(P)$. For a partition $h_i \subset \mathfrak{B}_i$ we denote by $h_i^c$ the complement of $h_i$ in $\mathfrak{B}_i$. For an element $h = (h_d, \ldots, h_j) \in Q(T)$, we denote by $h^c = (h_d^c, \ldots, h_j^c)$ the complement in $\mathfrak{B}(T)$. Recall that, given $P \in \mathcal{P}(T)$ we denote by $P^\vee$ the conjugate partition (switch rows and columns in the Ferrers graph of $P$); evidently $P^\vee \in \mathcal{P}(T)$.

**Theorem 2.12.** Let $T$ satisfy Equation (1.1), and let $P \in \mathcal{P}(T)$. Then the map $q: \mathcal{P}(T) \to Q(T)$ is an isomorphism of sets satisfying $q(P^\vee) = (q(P))^c$.

We endow $\mathcal{P}(T)$ with the structure of a lattice via the isomorphism $q$ to $Q(T)$ (see Definition 2.12).

2In the study of Lefschetz properties and Jordan type of Artinian algebras, multiplication by $x$ having Jordan type $T^\vee$ on $A$ corresponds to the the strong Lefschetz property of the pair $(A, x)$ [IMMI § 2.3].

3There is an alternative poset structure $\mathcal{P}_{alt}(T)$ on $\mathcal{P}(T)$, related to the sequences of degree-$i$ monomials in $C_P$. The inverse $Q(T) \cong P(T) \to \mathcal{P}_{alt}(T)$ is an inclusion of posets, not an isomorphism of lattices as stated incorrectly in [Y1] Theorem 3.27. See the discussion in [Y2].
The second and last author showed that the dimension of the cell $\mathcal{V}(E)$ is the total number of difference-one hooks in the partition $P_E$ determined by $E$ (Theorem 2.8): this is just the height of $\Omega(P)$ in the lattice $\mathcal{Q}(T)$. It follows from the A. Bialynicki-Birula result \cite{B-B} that over the complexes, the Betti numbers of $G_T$ may be deduced from the cellular decomposition \cite{IY1} Theorem 3.28, Theorem 3.29.

2.3 The cell $\mathcal{V}(E_P)$, and its component decomposition.

Throughout this section $T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0)$ will be a Hilbert function satisfying Equation (1.1) and $P$ will be a partition of diagonal lengths $T$. We denote by $E_P$ the monomial ideal associated to $P$ and let $\mathcal{V}(E_P)$ be the Jordan cell of $G_T$ associated to $P$. Our purpose here is to give a description of $\mathcal{V}(E_P)$ as a product of cells of “small” Grassmannians (Theorem 2.27).

The component Hilbert functions of $T$.

Let $P$ be a partition with Hilbert function $T = (1, \ldots, d, t_d, \ldots, t_j, 0)$ and difference-one hook code $\Omega(P) = (\delta_d, \delta_{d+1}, \ldots, \delta_j)$ (Definition 2.3). For $i = d, \ldots, j$ we let $T_i$ be the single-block Hilbert function

\[ T_i = (1, \ldots, d_i - 1, d_i, t_{d_i}, 0), \quad (2.4) \]

where $d_i = \delta_i(T) + \delta_{i+1}(T)$, and $t_{d_i} = \delta_{i+1}$. We set $t_{d-1} := d$ and $t_{j+1} := 0$. (There is a shift in degrees, $T_i$ parametrizes ideals of order (initial degree) $d_i = t_{i-1} - t_{i+1} = \delta_i(T) + \delta_{i+1}(T)$).

Definition of the component partitions of $P$.

In the next pages we give a construction of the component partitions $P_i$ corresponding to the single-block Hilbert function $T_i$, from the partition $P$ having diagonal lengths $T$ (Definition 2.17).

We will first define the $i$-th block of $P$, denoted by $P_i$ (Definition 2.17). We will show that it is the partition with diagonal lengths $T_i$ and hook code the $i$-th component $\Omega(P_i) = (h_i) = (\delta_i(P))$ of the hook code of $P$ (Proposition 2.22). This depends on defining border (foot) and hand monomials, respectively, of $P$ in degree $i$, giving vector spaces $V_{i1}, V_{i2}$, respectively. We will show that the partition $P_i$ may be derived simply from the monomials $V_{i1} \cup V_{i2}$ (Proposition 2.19). A reader on a first look may just use this Proposition as a definition of $P_i$.

Recall that for a Hilbert function $T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0)$, we set $\delta_i = t_{i-1} - t_i \quad (d \leq i \leq j + 1)$.

Given $P = (p_1, \ldots, p_s)$ a partition of diagonal lengths $T$, for each $i \quad (d \leq i \leq j)$ we construct a vector space $V_i$ of dimension $n_i = \delta_i + \delta_{i+1} + 1$ such that to each element $I$ of the cell $\mathcal{V}(E_P)$ we can associate a subvector space $I_{V_i}$ of $V_i$ with $\dim_k (I_{V_i}) = \delta_i + 1$. So the vector space $I_{V_i}$ is an element of $\text{Grass}(\delta_i + 1, V_i)$ and belongs to a cell described by the partition $h_i(P)$, the degree-$i$ block of the difference-one hook code of $P$.

We first define (as in \cite{LY2}, but in a more strict way) the horizontal-border monomials and vertical-border monomials of $E_P$. 
**Definition 2.13.** Let $P$ be a partition of diagonal lengths $T$. Denote by $E_P$ the monomial ideal of $k[x,y]$ associated to $P$.

(i). We say a monomial $x^a y^b \in E_P$ is a horizontal-border monomial of $E_P$ if $(b > 0$ and $x^a y^{b-1} \notin E_P$).

(ii). We say a monomial $x^a y^b \in E_P$ is a vertical-border monomial of $E_P$ if $a > 0$ and $x^{a-1} y^b \notin E_P$.

Denote by $A(P)_i$ the set of degree-$i$ horizontal-border monomials of $E_P$, $B(P)_i$ the set of degree-$i$ vertical-border monomials of $E_P$ and $(C_P)_i$ the set of degree-$i$ monomials that are not in $E_P$.

**Example 2.14.** Let $P = (9, 7^2, 4^2, 2, 1^2) \in \mathcal{P}(T)$ where $T = (1, 2, 3, 4, 5, 6, 7, 5, 2, 0)$, see Figure 5. The sets of degree-7 and 8 horizontal-border monomials of $E_P$ are given by $A(P)_7 = \{x^4 y^3, x^2 y^5, xy^6\}$ and $A(P)_8 = \{x^7 y, x^5 y^3, x^3 y^5, y^8\}$. Also we observe that the vertical-border monomials of degree 7, $B(P)_7$, is equal to $A(P)_7$.

In degree 8 the only vertical-border monomial in $E_P$ is $x^7 y$.

**Claim 2.15.** (i). $|A(P)_i| = \begin{cases} t_i - t_i + 1 = \delta_i + 1, & \text{if } x^i \in (C_P)_i \\ t_i - t_i = \delta_i, & \text{if } x^i \notin (C_P)_i \end{cases}$

(ii). $|B(P)_{i+1}| = \begin{cases} t_i - t_{i+1} + 1 = \delta_{i+1} + 1, & \text{if } y^{i+1} \in (C_P)_{i+1} \\ t_i - t_{i+1} = \delta_{i+1}, & \text{if } y^{i+1} \notin (C_P)_{i+1} \end{cases}$

**Proof of claim.** (Note: these formulas have been established in [Y2] to define difference-a hook partitions.)

One can consider the following maps

$$
\varphi_i : (C_P)_{i-1} \rightarrow (C_P)_i \cup A(P)_i ; \quad \psi_i : (C_P)_i \rightarrow (C_P)_{i+1} \cup B(P)_{i+1}.
$$

The maps $\varphi_i$ and $\psi_i$ are injective. Also note that if $x^i \in (C_P)_i$, then $x^i$ is the only element of $(C_P)_i \cup A(P)_i$ that is not in the image of $\varphi_i$, so we have $|(C_P)_{i-1}| = |(C_P)_i| + |A(P)_i| - 1$. If $x^i \notin (C_P)_i$ then $\varphi_i$ is a bijection, thus $|(C_P)_{i-1}| = |(C_P)_i| + |A(P)_i|$. The formula for $|A(P)_i|$ follows from the fact that for any integer $l$, $|(C_P)_l| = t_l$. Using the same arguments one can verify the formula for $|B(P)_{i+1}|$. \(\square\)

**Remark 2.16.** (i). Each monomial $x^a y^b$ in $A(P)_i$ is just below a degree-$(i-1)$ foot monomial $(x^a y^{b-1})$ of $P$, thus $|A(P)_i|$ counts the number of degree-$(i-1)$ foot monomials in the Ferrers diagram of $P$.

(ii). The elements of $B(P)_{i+1}$ are each just right to a degree-$i$ hand monomial, so $|B(P)_{i+1}|$ counts the number of degree-$i$ hand monomials in the Ferrers diagram of $P$. In Definition 2.17, we will consider the first (numbering from top to bottom–lex order) $\delta_i$ degree-$i$ hand monomials of $P$.

(iii). Recall that $P = (p_1, \ldots, p_s)$, with $p_1 \geq p_2 \geq \cdots \geq p_s > 0$. If $x^i \notin (C_P)_i$, then $p_1 \leq i$. 

10
With the following key definition we are able to construct the component $P_i$ from $P$, in a fashion that is convenient for our later algebraic proofs, that involve an induction from $I : x$ to $I$. For illustration see Example 2.20 and Figure 9, and as well Example 2.23 and Figure 10.

**Definition 2.17.** [Definition of $P_i$] For any positive integer $n \in \mathbb{N}$, denote by $\text{Mon}(R_n)$ the set of degree $n$ monomials of $R = k[x, y]$ and recall the lex order on $\text{Mon}(R_n)$: $x^n < x^{n-1}y < \cdots < x^1y^{n-1} < y^n$. Let $P \in \mathcal{P}(T)$. For every $i \in [d, j]$ we define the set $V_i$ as the following

\[
V_i = \begin{cases} 
A(P)_i, & \text{if } x^i \in (C P)_i, \\
A(P)_i \cup \{x^ay^b\}, & \text{if } x^i \notin (C P)_i,
\end{cases}
\]

where $x^ay^b$ is the last (lex order) degree-$i$ vertical-border monomial above $M_{i1}$ that is the first (lex order) monomial in $A(P)_i$. We note that $\dim V_i = \delta_i + 1$.

We now define the set $V_2$ to be the first (lex order) $\delta_{i+1}$ hand monomials in $(C P)_i$. We denote by $V_i$ the vector space spanned by $V_{i1} \cup V_{i2}$.

By definition, $V_i$ has dimension $d_i + 1$ where $d_i = \delta_i + \delta_{i+1}$. The set $V_{i1} \cup V_{i2}$ is lex ordered and we can consider the one-to-one correspondence

\[
s_i : V_{i1} \cup V_{i2} \rightarrow \text{Mon}(R_{d_i})
\]

that respects the lex ordering (the $k$-th element of $V_{i1} \cup V_{i2}$ is associated to the $k$-th element of $\text{Mon}(R_{d_i})$).

The vector space $s_i(V_{i1})$ has dimension $\delta_i + 1$, so $\langle s_i(V_{i1}) \rangle + \mathfrak{m}^{d_i+1}$ is the monomial ideal $(E_{P_i})$ for a unique partition $P_i$ of diagonal lengths the single-block Hilbert function $T_i = (1, \ldots, d_i - 1, d_i, t_{d_i}, 0)$ where $d_i = \delta_i + \delta_{i+1}$ and $t_{d_i} = \delta_{i+1}$.

**Example 2.18.** Let $P = (9, 7^2, 4^2, 2, 1^2)$ be the partition considered in Example 2.14 with diagonal lengths $T = (1, 2, 3, 4, 5, 6, 7, 5, 2, 0)$. (See Figure 6). The difference-one hook code of $P$ is $\Omega(P) = ((3, 2, 0)_7, (4, 3)_8)$. We have $(\delta_7, \delta_8, \delta_9) = (2, 3, 2), T_7 = (1, 2, 3, 4, 5, 3)$ and $T_8 = (1, 2, 3, 4, 5, 2)$.

(a) In degree 7 we have $V_{7,1} = A(P)_7 = \{x^4y^3, x^2y^5, x^6y\}, V_{7,2} = \{x^6y, x^3y^4, y^7\}$, so $V_7$ has basis $(x^6y, x^4y^3, x^3y^4, x^2y^5, x^6y, xy^6, y^7)$ in the lex order. We consider the bijection $s_7 : V_{7,1} \cup V_{7,2} \rightarrow \text{Mon}(R_3)$ given by:

\[
\begin{align*}
s_7(x^6y) &= x^5, & s_7(x^4y^3) &= x^4y, & s_7(x^3y^4) &= x^3y^2 \\
s_7(x^2y^5) &= x^2y^3, & s_7(xy^6) &= xy^4, & s_7(y^7) &= y^9.
\end{align*}
\]

We then get a one block partition $P_7$ (see Figure 7).

(b) In degree 8, $V_{8,1} = A(P)_8 = \{x^7y, x^5y^3, x^3y^5, y^8\}, V_{8,2} = \{x^8, x^6y^2\}$. $V_8$ has basis $(x^8, x^7y, x^6y^2, x^5y^3, x^3y^5, y^8)$ in the lex order.

The bijection $s_8 : V_{8,1} \cup V_{8,2} \rightarrow \text{Mon}(R_3)$ given by:

\[
\begin{align*}
s_8(x^8) &= x^5, & s_8(x^7y) &= x^4y, & s_8(x^6y^2) &= x^3y^2 \\
s_8(x^5y^3) &= x^2y^3, & s_8(x^3y^5) &= xy^4, & s_8(y^8) &= y^9.
\end{align*}
\]

This gives us a one block partition $P_8$ (see Figure 8).
Figure 6: Ferrers diagram of $P = (9, 7^2, 4^2, 2, 1^2)$: border monomials are marked in blue and hand monomials are marked in red (Example 2.18).

Figure 7: Ferrers diagram of $P_7 = (6, 4^2, 2, 1^2)$: horizontal-border monomials are marked in blue and hand monomials are marked in red (Example 2.18).

**Proposition 2.19.** Using the notation of Definition 2.17, let

$$V_{i1} \cup V_{i2} = \left\{ x^{\alpha_0} y^{\beta_0}, \ldots, x^{\alpha_{d_i}} y^{\beta_{d_i}} \right\},$$

where $\alpha_0 < \alpha_1 < \cdots < \alpha_{d_i}$ (so $\beta_0 > \beta_1 > \cdots > \beta_{d_i}$). Let $P'$ be the partition obtained from $P$ by removing any column of $P$ whose index does not belong to the set $\{\alpha_0, \alpha_1, \ldots, \alpha_{d_i}\}$ and any row of $P$ whose index does not belong to the set $\{\beta_0, \beta_1, \ldots, \beta_{d_i}\}$. Then $P' = P_i$.

**Proof.** In constructing the $i$-th component of $\Omega(P)$ (in the difference-one hook code of $P$), we only need the elements of $V_{i2}$ (degree $i$ hand monomials) and the elements of $V_{i1}$ (related to degree $i$ horizontal-border monomials). The purpose of the bijection $s_i : V_{i1} \cup V_{i2} \rightarrow \text{Mon}(R_{d_i})$ is to let us focus on these monomials. So
by definition of the bijection $s_i$, the partition $P_i$ is obtained from $P$ by ignoring any column of $P$ whose index does not belong to the set $\{\alpha_0, \alpha_1, \ldots, \alpha_d\}$ and any row of $P$ whose index does not belong to the set $\{\beta_0, \beta_1, \ldots, \beta_d\}$. Deleting these unnecessary rows and columns will result in showing only the relevant degree $i$ hands and degree $i - 1$ feet of $P$. □

In the following example we visualize the set $V_i = V_{i1} \cup V_{i2}$, from Definition 2.17 by looking at the Ferrers diagram of a partition $P$.

Example 2.20. Consider the two-block partition $P = (6^2, 3^2, 2^2)$ with diagonal lengths $T = (1, 2, 3, 4, 5, 3, 1)$, see Figure 8. Consider the diagonal corresponding to degree-5 monomials of $k[x, y]$, see the grey bubbles in Figure 9. Then the set of degree-5 horizontal-border monomials of $P$, $A(P)_5$ can correspond to the bubbles outside of the Ferrers diagram that are right below the horizontal edges of $P$. So $A(P)_5 = \{x^3 y^2, x^2 y^3, y^5\}$, see the blue monomials in the left side of Figure 9. Since the largest part of $P$ is greater than 5 (i.e., $x^5 \in (C_P)_5$), then $V_{51}$ is the same as $A(P)_5$. To obtain $A(P)_6$, notice that the largest part of $P$ is at most 6 (i.e., $x^6 \notin (C_P)_6$) then $V_{61}$ also includes the first degree-6 vertical-border monomial of $P$ that is above all monomials in $A(P)_6$. This monomial corresponds to a bubble outside of the Ferrers diagram that is immediately to the right of a horizontal edge of $P$, see the red monomial on the the right of Figure 9. So $A(P)_6 = \{x^4 y^2, xy^5\}$ and $V_{61} = \{x^6, x^4 y^2, xy^5\}$. Finally, monomials in $V_{i2}$ consists of the first $\delta_{i+1}$ hand monomials of $P$. These correspond to bubbles inside the Ferrers diagram that are at the end of a row of $P$. So $V_{52} = \{x^5, x^4 y, xy^4\}$ and $V_{62} = \{x^5 y, x^3 y^3, x^2 y^4, y^6\}$, see the black monomials in Figure 9.

To visualize Proposition 2.19, we fill out the degree-$i$ grey bubbles that correspond to $V_i$ by their monomials. We then remove all rows and columns of $P$ that do not include a filled degree-$i$ bubble, see Figure 9.

Remark 2.21. For $i \in [d, j]$, it may happen that $t_i = t_{i+1}$, so $\delta_{i+1} = 0$. In that case we have:

(i). The rectangular box $B_i(T) = (\delta_{i+1}) \times (1 + \delta_i)$ of Equation 2.2 is empty and so the degree-$i$ partition $h_i(P)$ in Equation 2.3 is empty.

(ii). $V_{i2}$ is empty and so $s_i(V_{i1}) = \text{Mon}(R_d)$. 

Figure 8: Ferrers diagram $P_8 = (6, 4^2, 2, 1)$: horizontal-border monomials are marked in blue and hand monomials are marked in red (Example 2.18b).
Figure 9: Illustration of Example 2.20. On the left, the monomials in blue represent $V_{5,1} = A(P)_5$. On the right, the monomials in blue represent $A(P)_6$ and the monomial in red represents the additional vertical-border monomial in $V_{6,1}$. 
(iii). The partition \( P_t \) associated to the monomial ideal \( \langle s_i(V_{t_i}) \rangle + m^{d_t+1} \) is just the basic triangle \( \Delta_{d_t} = \Delta_{\delta_t} = (\delta_t, \delta_t - 1, \ldots, 1) \). Of course, if \( \delta_t = 0 \), then \( \Delta_{d_t} = \emptyset \) and \( \langle s_i(V_{t_i}) \rangle + m^{d_t+1} = R \).

The following Proposition follows directly from Definition 2.9 and Definition 2.17.

**Proposition 2.22.** The difference-one hook code of \( P_t \) is exactly that of the \( i \)-th component of \( \Omega(P) \) (in the difference-one hook code of \( P \)).

Note, however, the shift in degree: the difference-one hook code of \( P_t \) occurs in the degree \( d_i = t_{i-1} - t_{i+1} = \delta_i(T) + \delta_{i+1}(T) \).

**Example 2.23.** Consider the three-block partition \( P = (15, 12^4, 11, 7, 6^2, 5, 3^4) \) with diagonal lengths \( T = (1, 2, \ldots, 13, 10_{13}, 6_{14}, 3_{15}, 0) \) and hook code

\[
\Omega(P) = ((3, 1^2, 0)_{13}, (5, 4, 1)_{14}, (2^2, 1)_{15}).
\]

In Figure 10, we illustrate the process of decomposing \( P \) into its single-block components, \( P_{13} = (7^2, 5, 4^2, 3, 1^2) \), \( P_{14} = (8, 6^2, 4, 3, 2^2) \), and \( P_{15} = (6, 5^2, 4, 2^2) \) of diagonal lengths, respectively, \( T_{13} = (1, \ldots, 7, 4, 0) \), \( T_{14} = (1, \ldots, 7, 3, 0) \), and \( T_{15} = (1, \ldots, 6, 3, 0) \).

In each part of Figure 10, the bubbles correspond to degree-\( i \) monomials. The hand monomials of \( V_{12} \) are black and the horizontal-border monomials, which all belong to \( V_{11} \) are blue. For \( i = 15 \), since \( x^{15} \notin (C_P)_{15} \), in addition of the horizontal-border monomials, the set \( V_{15,2} \) also includes a vertical-border monomial that is illustrated in red. The rest of the bubbles which are in light grey determine the rows and columns of \( P \) that need to be removed, according to Proposition 2.19 in order to obtain the corresponding single-block component.

---

**The projection map \( \pi \) from \( \mathbb{V}(E_P) \) to a product of cells of “small” Grassmannians.**

**Definition 2.24** (The map \( \pi \) of \( \mathbb{V}(E_P) \) to the product of small Grassmannians).

Suppose \( P \in \mathcal{P}(T) \) and let \( I \in \mathbb{V}(E_P) \). Denote by \( W_i \) the vector space generated by \( V_{i1} \cup (C_P)_i \) for \( i \in [d_j] \). Note that \( \dim_k W_i = \delta_i + 1 + t_i \). It is straightforward to see that the vector space \( I_{W_i} = I \cap W_i \) has dimension \( (\delta_i + 1) \). The leading monomial of any non-zero element of \( I_{W_i} \) belongs to \( V_{11} \) and conversely, given an element \( M \) of \( V_{11} \), there is an element of \( I_{W_i} \) whose leading monomial is \( M \).

Let \( K_i \) be the vector space generated by \( V_{i3} = (C_P)_i \setminus V_{12} \). We write \( W_i \) as a direct sum \( W_i = V_i \oplus K_i \) and consider the projection on the first factor \( pr_1 : W_i \rightarrow V_i \) and let \( I_{V_i} = pr_1(I_{W_i}) \). Then \( I_{V_i} \in \text{Grass}(\delta_i + 1, V_i) \).

Thus, we have constructed a morphism \( \mathbb{V}(E_P) \xrightarrow{\pi} \prod_{i=d}^{i=j} \text{Grass}(\delta_i + 1, V_i) \), and, taking the product \( \pi = (\pi_d, \ldots, \pi_j) \) we have a morphism

\[
\mathbb{V}(E_P) \xrightarrow{\pi} \prod_{i=d}^{i=j} \text{Grass}(\delta_i + 1, V_i).
\]  

\(^4 \text{We use the term “small” Grassmannian to distinguish these from the (large) Grassmannians determined by the projection } G_T \rightarrow \prod_{d \leq i \leq j} \text{Grass}(i + 1 - t_i, i + 1) \text{ given by } I \rightarrow (I_d, \ldots, I_j). \)
Obtaining $P_{13} = (7^2, 5, 4^2, 3, 1^2)$ from $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$

Obtaining $P_{14} = (8, 6^2, 4, 3, 2^2)$ from $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$

Obtaining $P_{15} = (6, 5^2, 4, 2^2)$ from $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$

Figure 10: Finding single-block components for partition $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$ of Example 2.23 using Proposition 2.19.
Remark 2.25. Note that by Definition 2.24 and the definition of the difference-one hook code, the image of \( \pi_i \) is a Schubert cell \( \mathcal{V}(E_i) \) in \( \text{Grass}(\delta_i + 1, V_i) \), whose dimension is \(|Q(P)_i|\), the length of the \( i \)-th block of the hook code of \( P \).

When \( \delta_{i+1} = 0 \) (that is, \( t_i = t_{i+1} \)) we have \( \dim(V_i) = \delta_i + 1 \) and \( \text{Grass}(\delta_i + 1, V_i) \) is just one point: the \( i \)-th block of the hook code of \( P \) in this case is empty, so has length zero (see Remark 2.21).

Lemma 2.26 (Morphism \( \pi \) to the component small Grassmannians). Let \( P \in \mathcal{P}(T) \). The morphism \( \pi \) of (2.7) determines a morphism

\[
\pi : \mathcal{V}(E_P) \rightarrow \prod_{i=1}^{i=j} \mathcal{V}(E_P_i).
\]

Proof. The bijection \( s_i : V_i \rightarrow \text{Mon}(R_{d_i}) \) of Equation (2.5) induces a linear isomorphism \( \hat{s}_i : V_i \rightarrow R_{d_i} \). We thus have an isomorphism \( \hat{s}_i : \text{Grass}(\delta_i + 1, V_i) \rightarrow \text{Grass}(\delta_i + 1, R_{d_i}) \), taking \( I_{V_i} = pr_1(I_{V_i}) \) to the subspace \( \hat{s}_i(I_{V_i}) \subset R_{d_i} \). Then by Definition 2.17, Equation 2.7 and Remark 2.25 determine the morphism \( \pi \) of the Lemma.

Theorem 2.27. The morphism \( \pi \) of Lemma 2.26 is an isomorphism from the Jordan cell \( \mathcal{V}(E_P) \) onto its image \( \prod_{i=1}^{i=j} \mathcal{V}(E_P_i) \).

Proof. Each \( \mathcal{V}(E_P) \) is an affine space: from Theorem 2.8 and Proposition 2.22 the dimension of \( \mathcal{V}(E_P) \) is the sum \( \sum \dim \mathcal{V}(E_{P_i}) \). We know that the difference-one hook code of \( P_i \) is the \( i \)-th component \( (b_i(P)) \).

Using the notation of Definition 2.17 consider the set \( V_{i1} \), which contains all the degree-\( i \) horizontal-border monomials of \( E_P \).

Let \( V_{i1} = \{ M_{i1}, \ldots, M_{i,\delta_i}, M_{i,\delta_i+1} \} \) (numbered from top to bottom). Denote by \( b_{i,l} \) the number of degree-\( i \) hand monomials above \( M_{i,l} \). Then we have \( b_{i,\delta_i} \geq \cdots \geq b_{i,1} \), and the affine space \( \mathcal{V}(E_P) \) has dimension \( \sum_{l=1}^{l=\delta_i+1} b_{i,l} = |\{ (b_i(P)) \}|. \)

We may think of \( \mathcal{V}(E_P) \) as \( \prod_{l=1}^{l=\delta_i+1} k^{b_{i,l}} \) (\( b_{i,l} \) free parameters for each \( M_{i,l} \)). Thus, if \( P \) is a single-block partition, then we are done. Suppose \( P \) is not a single-block partition and let \( h\text{Mon}(E_P) \) be the set of horizontal-border monomials of \( E_P \).

Suppose \( h\text{Mon}(E_P) = \{ y^{\beta_0}, xy^{\beta_1}, \ldots, x^{m}y^{\beta_m} \} \), with \( \beta_0 \geq \beta_1 \geq \cdots \geq \beta_m \).

For \( 0 \leq l \leq m \), let \( b_l \) be the number of degree-(\( l + \beta_l \)) hand monomials above \( x^l y^{\beta_l} \).

Let \( P' \) be the partition obtained by deleting the first column of \( P \). The morphism \( \mathcal{V}(E_P) \rightarrow \mathcal{V}(E_{P'}) : I \mapsto (I : x) \) is a trivial fibration whose fiber has dimension \( b_0 \) and we have \( \mathcal{V}(E_P) \cong k^{b_0} \times \mathcal{V}(E_{P'}) \) (see for example [Y1], Prop. 1.7). It is easy to see that \( E_{P'} \) has one less horizontal-border monomial than \( E_P \). The horizontal-border monomials of \( E_{P'} \) are deduced from that of \( E_P \) by dividing each of the monomials of the set \( \{ xy^{\beta_1}, \ldots, x^{m}y^{\beta_m} \} \) by \( x \). Also, for \( l > 0 \), the number of degree-(\( l + \beta_l \)) hand monomials above \( x^l y^{\beta_l} \) is the same as that of degree-(\( l-1 + \beta_l \)) hand monomials above \( x^{l-1}y^{\beta_l} \) (for the new partition \( P' \)).

Suppose \( P = (p_1^n, \ldots, p_s^n) \), with \( p_1 > \cdots > p_s > 0 \). Iterating the trivial fibration \( \mathcal{V}(E_P) \rightarrow \mathcal{V}(E_{P'}) : I \mapsto (I : x) \), \((p_1-1)\) times, we see that the partition associated
to \((I : x^{p_1 - 1})\) is the partition \(P_{p_1 - 1} = (1^{p_1})\). This partition \((1^{p_1})\) is that of a zero dimensional cell of Hilbert function \(T_{p_1 - 1} = (1, \ldots, 1)\). We then obtain the proof of the Theorem by induction.

\[\square\]

Examples of the projection map \(\pi\).

\textbf{Example 2.28.} Let \(T = (1, 2, 2, 1)\). We have \(\delta_2 = t_1 - t_2 = 0\), \(\delta_3 = t_2 - t_3 = 1\), \(\delta_4 = t_3 - t_4 = 1\), and \(\mathcal{B}(T) = (\mathcal{B}_2, \mathcal{B}_3) = ((1 \times 1)_2, (1 \times 2)_3)\). Here the product of “small” Grassmannians is

\[G = \text{Grass}(\delta_2 + 1, \delta_2 + 1 + \delta_3) \times \text{Grass}(\delta_3 + 1, \delta_3 + 1 + \delta_4) = \text{Grass}(1, 2) \times \text{Grass}(2, 3)\].

Consider the partition \(P = (3, 3)\).

\[
P : \begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ }
\end{array}
\]

By Definition \([2.24]\) we have \(V_2 = (y^2, x^2)\), \(W_2 = (y^2, xy, x^2)\), \(V_3 = (xy^2, x^2y, x^3)\) and \(W_3 = V_3\).

Any element \(I\) in the cell \(\mathbb{V}(E_P)\) is of the form \(I = (y^2 + a_1 xy + a_2 x^2, x^3)\), with \((a_1, a_2) \in k^2\). So \(\mathbb{V}(E_P)\) is a two dimensional affine space.

For \(I = (y^2 + a_1 xy + a_2 x^2, x^3) \in \mathbb{V}(E_P)\) we get \(I \cap W_2 = (y^2 + a_1 xy + a_2 x^2)\) and the projection of \(I \cap W_2 \) on \(V_2\) is \(I_{V_2} = (y^2 + a_2 x^2)\) \(\in \text{Grass}(1, V_2)\). Also \(I \cap W_3 = \langle xy^2 + a_1 x^2 y, x^3 \rangle = I_{V_3} \in \text{Grass}(2, V_3)\). We view \(\mathbb{V}(E_P)\) as a product of two cells, one in \(\text{Grass}(1, 2)\) corresponding to single-block \(T_2 = (1, 1)\) and another one in \(\text{Grass}(2, 3)\) corresponding to \(T_3 = (1, 2, 1)\).

The difference-one hook code of \(P = (3, 3)\) is

\[
\Omega(P) = ((1)_2, (1)_3) \subset \mathcal{B}(T) = ((1 \times 1)_2, (1 \times 2)_3).
\]

The code \((1)_2\) corresponds to the vector space \(V_2\) and the small cell \(\mathbb{V}(E_{P_2}) = \{ (y^2 + a_2 x^2), a_2 \in k \} \subset \text{Grass}(1, 2) \cong \text{Grass}(1, V_2)\):

\[
P_2 : \begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ }
\end{array}
\]

The code \((1)_3\) corresponds to the vector space \(V_3\) and the small cell \(\mathbb{V}(E_{P_3}) = \{ (xy^2 + a_1 x^2 y, x^3), a_1 \in k \} \subset \text{Grass}(2, 3) \cong \text{Grass}(2, V_3)\) (note, these are labelled by degree: \(V_i \subset R_i)\):

\[
P_3 : \begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ }
\end{array}
\]

So we get \(\mathbb{V}(E_P)\) as a product of two affine lines: \(\mathbb{V}(E_P) = \mathbb{V}(E_{P_2}) \times \mathbb{V}(E_{P_3})\).

\textbf{Example 2.29.} Let \(T = (1, 2, 3, 4, 5, 4, 2)\) and consider \(P = (5^3, 3, 1^2)\). (See Figure \([11]\) We have \(\langle \delta_5, \delta_6, \delta_7 \rangle = (1, 2, 2)\). The two single-block Hilbert functions associated to \(T\) are \(T_3 = (1, 2, 3, 2)\) and \(T_6 = (1, 2, 3, 4, 2)\) (see Definition \([2.17]\) ).
We will view the cell \(V(E_P)\) as a product of cells in \(\text{Grass}(2, 4) \times \text{Grass}(3, 5)\).

The difference-one hook code of \(P\) is \(\Omega(P) = ((1, 1), (2, 0))\).

The partition \(P_5 = (3^2, 2)\) of diagonal lengths \(T_5\) has hook code \(\Omega(P_5) = ((1, 1))\) (the degree-5 block of \(\Omega(P)\)). The partition \(P_6 = (4^2, 2, 1^2)\) of diagonal lengths \(T_6\) has hook code \(\Omega(P_6) = ((2, 0))\) (the degree-6 block of \(\Omega(P)\)).

![Ferrers diagram of \(P = (5^3, 3, 1^3)\): hand monomials are marked in red and border monomials are marked in blue (Example 2.29).](image)

We now display the isomorphism \(\pi : V(E_P) \rightarrow V(E_{P_5}) \times V(E_{P_6})\).

By Definition 2.24 we have

\[
V_5 = \langle xy^4, x^2y^3, x^4y, x^5 \rangle, W_5 = R_5, V_6 = \langle y^6, x^2y^4, x^3y^3, x^4y^2, x^5y \rangle = W_6.
\]

The projection of \(I_5\) onto \(V_5\) is a 2-dimension vector space

\[
I_{V_5} = \langle xy^4 + \alpha_1 x^2 y^3 + \alpha_2 x^4 y, x^5 \rangle, (\alpha_1, \alpha_2) \in \mathbb{k}^2.
\]

The projection of \(I_6\) onto \(V_6\) is a 3-dimension vector space

\[
I_{V_6} = \langle x^2 y^4 + a_1 x^4 y^2, x^3 y^3 + a_2 x^4 y^2, x^5 y \rangle, (a_1, a_2) \in \mathbb{k}^2.
\]

So we have \(I_{V_5} \in \text{Grass}(2, V_5) \cong \text{Grass}(2, 4)\), \(I_{V_5} \in \text{Grass}(3, V_6) \cong \text{Grass}(3, 5)\) and \(V(E_P)\) can be viewed as \(V(E_{P_5}) \times V(E_{P_6})\).

**Note.** Suppose we are given \((L_5, L_6) \in V(E_{P_5}) \times V(E_{P_6})\) with \(L_5 = \langle xy^4 + \alpha_1 x^2 y^3 + \alpha_2 x^4 y, x^5 \rangle, (\alpha_1, \alpha_2) \in \mathbb{k}^2\) and \(L_6 = \langle x^2 y^4 + a_1 x^4 y^2, x^3 y^3 + a_2 x^4 y^2, x^5 y \rangle, (a_1, a_2) \in \mathbb{k}^2\). Then, using standard basis techniques (Theorem 1.1.9 of [Br], Propositions 2 and 3 of [Br-Ga]) one can see that there is a unique ideal \(I \in V(E_P)\) such that \(I_{V_5} = L_5\) and \(I_{V_6} = L_6\):

\[
I = \langle x^5, x^3 y^3 + a_2 x^4 y^2, x(y^4 + a_1 x^2 y^2) + \alpha_1 (x y^3 + a_2 x^2 y^2) + \alpha_2 x^3 y, y^7 \rangle.
\]

In connection with Lemma 5.7 where we will be counting the degree \(i+1\) relations and corner-monomials (generators -Definition 2.25) of \(E_P\) it is interesting to note that,
Lemma 2.30. The bijection $s_i : V_{i1} \cup V_{i2} \longrightarrow \text{Mon}(R_{d_i})$ (Equation 2.5) induces

(i). a one to one correspondence between the degree $i + 1$ relations of $E_P$ and
the degree $d_i + 1 = \delta_i + \delta_{i+1} + 1$ relations of $E_{P_i}$

(ii). a one to one correspondence between the first (numbering from top to bottom-
lex order) $\delta_{i+1}$ degree $i + 1$ vertical-border monomials of the ideal $E_P$ and
the degree $d_i + 1$ vertical-border monomials of $E_{P_i}$

(iii). a one to one correspondence between the degree $i + 1$ corner-monomials of
$E_P$ and the degree $d_i + 1$ corner-monomials of $E_{P_i}$.

Proof. (i). Suppose that the monomial $x^\alpha y^{i+1-\alpha}$ $(0 < \alpha < i+1)$ corresponds to
a degree $i + 1$ relation. Then $x^\alpha y^{i+1-\alpha}$ is a horizontal-border monomial
of $E_P$ and $x^\alpha y^{i-\alpha}$ is a vertical-border monomial of $E_P$. Now, consider
the set $(E_P)_{i,\alpha}$ of degree $i$ horizontal-border monomials of $E_P$ that are
above $x^\alpha y^{i-\alpha}$. If this set is empty, then $x^i \notin (C_P)_i$, so $x^\alpha y^{i-\alpha} \in V_{i1}$
and the degree $i + 1$ relation of $E_P$ corresponding to $x^\alpha y^{i+1-\alpha}$ is sent to a
degree $d_i + 1$ relation of $E_{P_i}$. If the set $(E_P)_{i,\alpha}$ is not empty, let $x^\alpha y^{i-\alpha'}$
be the first element of $(E_P)_{i,\alpha}$ just above $x^\alpha y^{i-\alpha}$. By definition, $s_i$ sends
$x^\alpha y^{i+1-\alpha}$ and $x^\alpha y^{i-\alpha'}$ to two consecutive horizontal-border monomials
of $E_{P_i}$, resulting to a degree $d_i + 1$ relation of $E_{P_i}$.

(ii). By Definition 2.13 we know that any degree $i+1$ vertical-border monomial is
just to the right of a unique degree $i$ hand monomial. The second statement of
the Lemma is then just a remark based on the fact that any element of
$V_{i2}$ is a degree $i$ hand monomial of $P$ and $s_i$ sends the elements of $V_{i2}$ to
the degree $d_i$, hand monomials of $E_{P_i}$.

(iii). Suppose $x^\alpha y^{i+1-\alpha}$ is a degree $i + 1$ hook corner of $E_P$.
a) If $\alpha = 0$, then one can easily see that $y^{d_i} \notin E_{P_i}$, so $y^{d_i+1}$ is a corner-
monomial of $E_{P_i}$.
b) If $\alpha > 0$ then the corner-monomial $x^\alpha y^{i+1-\alpha}$ is also a vertical-border
monomial of $E_P$ that will correspond via $s_i$ to a degree $d_i + 1$ corner-
monomial of $E_{P_i}$.

\[\square\]

3 Number of generators for a single-block partition.

We first state the known bounds for the number of generators of a graded ideal $I$
of Hilbert function $H(R/I) = T$ for arbitrary $T$ (Lemma 3.3). In Theorem 3.11
we determine the number of generators $\kappa(P)$ for generic ideals in the cell $\mathbb{V}(E_P)$
where $P$ has diagonal lengths $T$ satisfying the single-block Equation (3.3).

3.1 Lower bound $\kappa(T)$ on the number of generators of
an ideal $I$ in $G_T$.

We recall Equation (1.1) for an arbitrary codimension two Hilbert function $T$:

\[T = (1, 2, \ldots, d - 1, d, t_d, \ldots, t_j, 0)\]

where $d \geq t_d \geq t_{d+1} \geq \cdots \geq t_j > 0$. 


Here \( j \) is the (highest) socle degree of \( A = R/I \). Recall from Section 2.1 that \( G_T \) is the irreducible projective variety parametrizing the graded ideals \( I \) in \( R = k[x, y] \) such that \( A = R/I \) has Hilbert function \( T \).

**Definition 3.1** (Order of a Hilbert function \( T \)). Let \( T \) be a sequence satisfying Equation (1.1). Set \( \nu(T) = d \), usually called the order of \( T \); that is \( \nu(T) \) is the order of graded ideals \( I \in G_T \) — that define an Artinian algebra \( A = R/I \) of Hilbert function \( T \).

**Definition 3.2.** We let \( \kappa(T) \) be the minimum number of generators for the ideal \( I \) corresponding to a generic element of \( G_T \). If \( P \) is a partition of diagonal lengths \( T \), we denote by \( E_P \) the monomial ideal associated to \( P \) and set \( \kappa(P) \) to be the minimum number of generators for a generic element \( I \) in the cell \( \mathcal{V}(E_P) \).

Given a sequence \( T \) satisfying Equation (1.1), recall that we denote by \( \delta_i \) the first difference function of \( T \):

\[
\delta_i = t_{i-1} - t_i \quad \text{for} \quad i \in [\nu(T), j+1].
\]

(iii) The ideal \( \kappa(T) \) is open dense in \( G_T \) — that is \( \kappa(T) \) is non-special. But for \( \kappa(T) = \kappa(P) \) we say \( P \) is special. If \( \kappa(P) = \kappa(T) \) we say \( P \) is non-special.

**Example 3.5.** Let \( T = (1, 2, 3, 2, 1) \). We have \( \kappa(T) = 2 \), as the generic ideal in \( G_T \) is a complete intersection of generator degrees \((3, 3)\). For \( P = (5, 3, 1) \) we also have \( \kappa(P) = 2 \); here \( \mathcal{V}(E_P) \) is open dense in \( G_T \), so \( P = (5, 3, 1) \) is non-special. But for \( P = (3, 3, 1^3) \) we have \( \kappa(P) = 3 \) since an \( R \)-relation between the generators \( y^3, y^2x + \cdots \) cannot yield the generator \( x^3 \); so \( P = (3, 3, 1^3) \) is special.
3.2 Single-block partitions \( P \), and \( \kappa(P) \).

Henceforth in this section we let \( T \) be a Hilbert function that satisfies

\[
T = (1, 2, \ldots, d - 1, d, t_d, 0).
\]

(3.3)

where \( d \geq t_d \) and we let \( s = d + 1 - t_d \). We term this a single-block Hilbert function.

In this case, \( G_T \) is isomorphic to the Grassmannian variety \( \text{Grass}(s, R_d) \) where \( R_d \) is the vector space of the degree \( d \) homogeneous forms of \( R = k[x,y] \):

\[
\Phi : G_T \rightarrow \text{Grass}(s, R_d)
I \mapsto I_d
\]

Also, by Equation 3.2 we have for a single-block Hilbert function

\[
\kappa(T) = s + \delta, \text{ where } \delta = \max\{t_d + 1 - s, 0\}.
\]

(3.4)

Let \( P \) be a partition of diagonal lengths \( T \). The corners of the Ferrers diagram of \( P \) correspond to monomials \( x^a y^b \) that belong to a minimal set of generators for the monomial ideal \( E_P \). We may call such monomials, corner-monomials of \( P \).

Let \( I \in G_T \) be an element of the Jordan cell \( V(E_P) \). Then the corner-monomials of \( P \) are leading terms of a system of generators \( B(I) \) of \( I \). The system \( B(I) \) may not be minimal. By definition of \( T \), a minimal set of generators of \( E_P \) must contain \( s \) degree \( d \) \( \nu(T) \) corner-monomials. These degree \( d \) corner-monomials are leading monomials for the degree \( d \) elements of the system of generators \( B(I) \).

Since we are looking for a minimal set of generators for \( I \), we want a criterion to decide that a degree \( d + 1 \) element of \( B(I) \) can be obtained using a relation involving degree \( d \) elements of \( B(I) \). That is where corner “kick-off” comes into play.

Let \( a \) be an integer such that \( 0 \leq a < d = \nu(T) \) and set \( d' = d - a \). Suppose \( m \) is an integer such that \( 1 \leq m \leq d' \). For any integer \( i \) such that \( 0 \leq i \leq m \), set \( K_i = x^a \cdot (x^{m-i} y^{d'-m+i}) \). The \( K_i \)'s form a set of \( m + 1 \) consecutive degree \( d \) monomials in two variables:

\[
x^{a+m} y^{d'-m}, x^{a+m-1} y^{d'-m+1}, \ldots, x^{a+1} y^{d'-1}, x^a y^d.
\]

From these \( m + 1 \) consecutive monomials we have \( m \) relations: \( yK_i - xK_{i+1} = 0 \), \( (0 \leq i < m) \). Suppose the \( K_i \)'s are leading monomials of some elements of \( B(I) \). Note that by definition, if \( f_0, \ldots, f_m \) are degree-\( i \) forms such that \( f_i \) has leading monomial \( K_i \), then \( x^a \) divides any element of the ideal generated by \( (f_0, \ldots, f_m) \). Thus, assuming that \( \dim_k (R_1 \cdot (f_0, \ldots, f_m)) = 2(m + 1) \) requires \( 2(m + 1) \leq d' + 2 = \dim_k (R_{d+1}) \), that is, \( 2m \leq d' \).

For simplicity we now assume \( a = 0 \), so \( K_i = x^{m-i} y^{d'-m+i} \), \( 2m \leq d \), and we let \( N_1, \ldots, N_m \) be the \( m \) degree \( d + 1 \) monomials given by

\[
\begin{cases}
N_i = x^{\alpha_i} y^{\beta_i}, & \alpha_i + \beta_i = d + 1, \\
0 \leq \beta_1 < \beta_2 < \cdots < \beta_m < d - m \\
m + 1 < \alpha_m < \alpha_{m-1} < \cdots < \alpha_1 \leq d + 1
\end{cases}
\]

Concerning the next Lemma, although J. Briançon and A. Galligo state their standard basis result that we use in characteristic zero, it is valid also for characteristic greater than the socle degree \( d \). This is the key step in the paper where we need to restrict the characteristic of \( k \).
Lemma 3.6 (How to kick off corners). Assume that the characteristic of $k$ is zero, or that $k$ is infinite of characteristic $p$ greater than the socle degree $d$. With the above notation, there exist $m+1$ degree $d$ forms $f_0, \ldots, f_m$ such that $f_i$ has leading monomial $K_i$ and $N_i$ is a leading monomial of a degree $d+1$ element of the ideal generated by $(f_0, \ldots, f_m)$.

Proof. Using a technique of standard basis calculations developed by J. Briançon and A. Galligo in [Br-Ga] (requiring the restriction on the characteristic of $k$), we can inductively construct $f_0, \ldots, f_m$ such that $N_i \in (f_0, \ldots, f_m)$. Let

$$f_0 = x^m y^{d-m}, \quad f_1 = x^{m-1} y^{d-m+1} + \lambda_1 x^{\alpha_1 - 1} y^{\beta_1},$$

where $\lambda_1 \in k$. One can see that $x f_1 - y f_0 = \lambda_1 x^{\alpha_1} y^{\beta_1}$, so if $\lambda_1 \neq 0$, we have

$$N_1 = x^{\alpha_1} y^{\beta_1} \in (f_0, \ldots, f_m).$$

In general, for $0 \leq i < m$, suppose that we have $f_i = x^{m-i} u_i + \lambda_i x^{\alpha_i} y^{\beta_i}$, where $u_i$ is a degree $d-m+i$ form such that $u_i(0,y) = y^{d-m+i}$.

Then we set

$$f_{i+1} = x^{m-i-1} y \left( u_i + \lambda_i x^{\alpha_i-1} y^{\beta_i} \right) + \lambda_{i+1} x^{\alpha_{i+1}-1} y^{\beta_{i+1}}.$$

So, $x f_{i+1} - y f_i = \lambda_{i+1} x^{\alpha_{i+1}+1} y^{\beta_{i+1}+1}$ and for $\lambda_{i+1} \neq 0$, we have $N_{i+1} \in (f_0, \ldots, f_m)$. Note that for $i = 0$, $u_0 = y^{d-m}, \lambda_0 = 0$; for $i = 1$, $u_1 = y^{d-m+1}$; thus, inductively, we have constructed $f_0, \ldots, f_m$ such that $N_i \in (f_0, \ldots, f_m)$.

Remark 3.7 (Choosing which corner should be kicked off). Given $r$ indices $i_1, \ldots, i_r$ such that $1 \leq i_1 < \cdots < i_r \leq m$, in the inductive construction of $(f_0, \ldots, f_m)$ of Lemma 3.6 if we let $\lambda_{i_l} = 0$ $(1 \leq l \leq r)$, then none of the monomials $N_i$ will be kicked off. So, if $\lambda_{i_l} = 0$ for $1 \leq l \leq r$, then $N_i \notin (f_0, \ldots, f_m)$.

We remind the reader of the Definition 2.2 and Figure 3 of a difference-one hook, and Definition 2.9 of the hook code. In the next Lemma and Theorem a hook code of $P$ has a single non-zero partition $b_d(P) = \Omega(P)$, which for short we term its hook code. Note that $n = \delta_{d+1}$ is the number of parts of $b_d(P)$ and some parts may be zero.

Lemma 3.8 (Counting the corner-monimials of $P$). Let $T = (1, 2, \ldots, d, t_d = t, 0), t > 0$ and set $s = d+1-t$. Suppose that $P$ is a partition of diagonal lengths $T$ and difference-one hook code $\Omega(P) = (h^1, \ldots, h^s)$ (where $s \geq h_1 > h_2 > \cdots > h_n \geq 0$). Then the minimum number of generators $b_1(E)$ of the monomial ideal $E$ is given by the following formula.

$$b_1(E) = s + t - n, \quad \text{if } h_1 < s \text{ and } h_n > 0$$

$$b_1(E) = s + t - n + 1, \quad \text{if } h_1 = s \text{ and } h_n > 0, \text{ or } h_1 < s \text{ and } h_n = 0$$

$$b_1(E) = s + t - n + 2, \quad \text{if } h_1 = s \text{ and } h_n = 0.$$

\[^{5}\text{See [Br] Theorem I.1.9], [Br-Ga] Props. 2,3], also [Pir] §1.\]
Proof. This is an easy count that we obtain by looking at the Ferrers diagram of $P$. □

Note that $b_1(E_P) - \kappa(P)$ counts the number of degree $d+1$ corner-monomials we have been able to kick-off.

**Example 3.9.** Suppose $T = (1, 2, 3, 4, 5, 6, 7, 8, 4)$. Then $d = 8$, $t_d = 4$ and $s = 5$. Let $P$ be the partition of diagonal lengths $T$ defined by $P = (9, 7^2, 6, 4^2, 2, 1, 2)$ (See Figure 12). We have $\Omega(P) = (5, 4^2, 3)$. The monomial ideal $E_P$ associated to $P$ is generated by $(y^8, xy^7, x^2y^6, x^4y^4, x^7y, x^6y^3, x^3)$. Using Lemma 3.6 we see that the degree 9 corners of $P$ associated to $x^6y^3$ and $x^9$ can be kicked-off using the degree 8 corners associated to the consecutive monomials $y^8, xy^7, x^2y^6$.

![Figure 12: Kicking off corners of the partition $P = (9, 7^2, 6, 4^2, 2, 1)$ (Example 3.9).](image)

Let $P$ be a partition of diagonal lengths $T$. Suppose $P = (p_1, \ldots, p_m)$, with $p_1 \geq p_2 \geq \cdots \geq p_m$. Let $P' = (p'_1, \ldots, p'_m)$ with $p'_i = p_i - 1$. Let $T'$ be the Hilbert function associated to $P'$. If $I$ is an element of the cell $\mathbb{V}(E_P)$ of $G_T$, then $(I : x)$ is an element of the cell $\mathbb{V}(E_{P'})$ of $G_{T'}$. In fact we have a morphism $\varphi : \mathbb{V}(E_P) \rightarrow \mathbb{V}(E_{P'})$ defined by $I \mapsto (I : x)$ whose fiber is an affine space of dimension the number of difference-one hooks having their feet at $y^{m-1}$ [Y2, Proposition 2.6].

**Lemma 3.10.** Assume that $T = (1, \ldots, d, t_d, 0)$, and that $P$ is a partition having diagonal lengths $T$ and difference-one hook code $\Omega(P) = (h^{l_1}_1, \ldots, h^{l_n}_n)$. Set $s = d + 1 - t_d$. Suppose that $I$ is a generic ideal in the cell $\mathbb{V}(E_P)$ and let $\bar{I} = (I : x)$.

(a) If $h_n = 0$ then $\bar{I} \in \mathbb{V}(E_{P'})$ where $P'$ is the partition of diagonal lengths $\bar{T} = (1, \ldots, d-1, t-1)$ and hook code $\Omega(P') = (h^{l_1}_1, \ldots, h^{l_n-1}_n)$. Furthermore, in this case $\kappa(P) = \kappa(P') + 1$.  

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(b) If $h_n > 0$ then $\bar{I} \in \mathbb{V}(E_P)$ where $P$ is the partition of diagonal lengths $T = (1, \ldots, d - 1, t)$ and hook code $\Omega(P) = ((h_1 - 1)^1, \ldots, (h_n - 1)^{h_n})$.

Furthermore, in this case

$$\kappa(P) = \begin{cases} 
\kappa(P), & \text{if } \kappa(P) \geq s \\
s, & \text{if } \kappa(P) = s - 1.
\end{cases}$$

\textbf{Proof.} We note that the Ferrers diagram of $\bar{P}$ is always obtained from the Ferrers diagram of $P$ by removing the first column. Let $\mathcal{B} = \{f_1, \ldots, f_\kappa\}$ be a minimal set of generators for $\bar{I}$ (the $f_i$’s are ordered according to their leading monomials, from top to bottom in the Ferrers diagram).

Part (a) If $h_n = 0$, then $\mathcal{B} = \{xf_1, \ldots, xf_\kappa, y^{d + 1}\}$ is a minimal set of generators for $I$. Thus the equality in part (a) holds.

Part (b). If $h_n > 0$ then by definition $s = s - 1$. This in particular implies that in this case $\kappa(P) \geq s - 1$.

Assume that $h_n = 1$. Then the leading term of $f_\kappa$ is $y^d$ (we can even set $f_\kappa = y^d$ here). Let $g$ be a generic polynomial with leading term $y^d$. Then $\mathcal{B} = \{xf_1, \ldots, xf_{\kappa - 1}, g\}$ is a minimal set of generators for $I$. Thus in this case $\kappa(P) = \kappa(\bar{P})$. We also note that since $\kappa(P) \geq s$ (Definition 3.1 and Lemma 3.3) the equality $\kappa(P) = \kappa(\bar{P})$ in particular implies that when $h_n = 1$, we have $\kappa(\bar{P}) \geq s$.

Next, assume that $h_n > 1$. Suppose $\kappa(\bar{P}) = s - 1$ (this is the minimum value possible for $\kappa(\bar{P})$). In this case, all the degree $d$ corner-monomials of $P'$ have been kicked-off. After multiplication by $x$, these degree $d$ corner-monomials of $P'$ become degree $d + 1$ corner-monomials of $P$, so are kicked-off by $(xf_1, \ldots, xf_\kappa)$ and therefore $\mathcal{B} = \{xf_1, \ldots, xf_\kappa, g\}$, where $g$ is a generic polynomial with leading term $y^d$, is a minimal set of generators of the generic element of $\mathbb{V}(E_P)$, so $\kappa(P) = \kappa(\bar{P}) + 1$.

Now, suppose $\kappa(\bar{P}) > s - 1$. This means that there is at least one degree-$d$ form in any minimal set of generators of a generic element of $\mathbb{V}(E_P)$. So we have $\bar{B} = \{f_1, \ldots, f_{\kappa - s + 1}, f_{\kappa - s + 2}, \ldots, f_\kappa\}$, $\deg(f_{\kappa - s + 1}) = d$, $\deg(f_{\kappa - s + 2}) = \cdots = \deg(f_\kappa) = d - 1$. Note that $f_\kappa$ has leading monomial $y^{d-1}$. Now, let $g = yf_\kappa + \lambda f_{\kappa - s + 1}$ ($\lambda \neq 0$). Then $xg - y(xf_\kappa) = \lambda x f_{\kappa - s + 1}$. Since $\lambda \neq 0$, this means that $xf_{\kappa - s + 1}$ can be kicked off. If $\mathcal{B} = \{f_1, \ldots, f_{\kappa - s + 1}, f_{\kappa - s + 2}, \ldots, f_\kappa\}$ is a minimal set of generators of $(I : x) = \bar{I}$, then $\mathcal{B} = \{xf_1, \ldots, xf_{\kappa - s}, xf_{\kappa - s + 2}, \ldots, xf_\kappa, g\}$ is a minimal set of generators of $I$. Thus $\kappa(P) = \kappa(\bar{P})$.

Recall that for a partition $P$ of diagonal lengths $T$, we denote the minimum number of generators for a generic element $I$ in the cell $\mathbb{V}(E_P)$ by $\kappa(P)$. Also note that, as discussed at the beginning of this section, if $T = (1, \ldots, d, t_d, 0)$ is a single-block Hilbert function, then a minimal system of generators for $I$ consists of $s = d + 1 - t_d$ generators of degree $d$ and $(\kappa(P) - s)$ generators of degree $d + 1$. The following theorem provides an explicit formula for $\kappa(P)$ in the single-block case.

\textbf{Theorem 3.11} (The invariant $\kappa(P)$ for a single-block $T$). Assume that $T = (1, \ldots, d, t_d = t, 0)$, set $s = d + 1 - t$ and let $P$ be a partition of diagonal lengths
$T$ and difference-one hook code $\Omega(P) = (h_1^k, \ldots, h_n^k)$. For $k = 1, \ldots, n$, let $\tau_k = \sum_{i=k}^n l_i - h_k$. Then

$$\kappa(P) = s + \max\{t + 1 - s, 0, \tau_k\}_{k=1,\ldots,n}. \quad (3.5)$$

**Proof.** We prove the theorem by induction on $d$.

First assume that $d = 2$.

If $t = 2$, then $s = 1$. In this case there are three partitions of diagonal lengths $T$, namely

(i) Partition $P = (3, 2)$ with hook code $\Omega(P) = (1^2)$, $\kappa(P) = 3$ and
$$s + \max\{t + 1 - s, 0, \tau_1\} = 1 + \max\{2, 0, 2 - 1\} = 3;$$

(ii) Partition $P = (3, 1^2)$ with hook code $\Omega(P) = (1, 0)$, $\kappa(P) = 3$ and
$$s + \max\{t + 1 - s, 0, \tau_1, \tau_2\} = 1 + \max\{2, 0, 2 - 1, 1 - 0\} = 3;$$

(iii) Partition $P = (2^2, 1)$ with hook code $\Omega(P) = (0^2)$, $\kappa(P) = 3$ and
$$s + \max\{t + 1 - s, 0, \tau_1\} = 1 + \max\{2, 0, 2 - 0\} = 3.$$

On the other hand, if $t = 1$, then $s = 2$. In this case, there are three partitions of diagonal lengths $T$, namely

(i) Partition $P = (3, 1)$ with hook code $\Omega(P) = (2)$, $\kappa(P) = 2$ and
$$s + \max\{t + 1 - s, s, s + \tau_1\} = 2 + \max\{0, 0, 1 - 2\} = 2;$$

(ii) Partition $P = (2^2)$ with hook code $\Omega(P) = (1)$, $\kappa(P) = 3$ and
$$s + \max\{t + 1 - s, 0, \tau_1\} = 2 + \max\{0, 0, 1 - 1\} = 2;$$

(iii) Partition $P = (2, 1^2)$ with hook code $\Omega(P) = (0)$, $\kappa(P) = 3$ and
$$s + \max\{t + 1 - s, 0, \tau_1\} = 2 + \max\{0, 0, 1 - 0\} = 3.$$

This shows that the desired equality holds when $d = 2$.

Now assume that $d > 2$ and that Equation (3.5) holds for any partition of diagonal lengths $(1, \ldots, d', t, 0)$ with $d' < d$.

Suppose that $P$ is a partition of diagonal lengths $T = (1, \ldots, d, t)$ and hook code $\Omega(P) = (h_1^k, \ldots, h_n^k)$. Let $\bar{P}$ be the partition associated to $P$ defined in Lemma 3.10. Then by the inductive hypothesis Equation (3.5) holds for $\kappa(\bar{P})$.

**Case 1.** Assume that $h_n = 0$. Then $\bar{t} = t - 1$, $\bar{s} = s$, and for $k = 1, \ldots, n$, we have $\bar{\tau}_k = \tau_k - 1$. Thus

$$\kappa(\bar{P}) = \bar{s} + \max\{\bar{t} + 1 - \bar{s}, 0, \tau_k\}_{k=1,\ldots,n} = s + \max\{t - s, 0, \tau_k - 1\}_{k=1,\ldots,n}.$$
Since \( \tau_n = l_n - h_n = l_n \geq 1 \), we have

\[
\max\{t - s, 0, \tau_k - 1\}_{k=1,...,n} = \max\{t - s, \tau_k - 1\}_{k=1,...,n} = \max\{t + 1 - s, \tau_k\}_{k=1,...,n} - 1.
\]

Thus using part (a) of Lemma 3.10 we have

\[
\kappa(P) = \kappa(\bar{P}) + 1
\]

\[
s + \max\{t + 1 - s, \tau_k\}_{k=1,...,n}
\]

\[
s + \max\{t + 1 - s, 0, \tau_k\}_{k=1,...,n}.
\]

**Case 2.** Assume that \( h_n > 0 \). Then \( \bar{t} = t, \bar{s} = s - 1 \), and for \( k = 1, \ldots, n \), we have \( \bar{\tau}_k = \tau_k + 1 \). By the inductive hypothesis

\[
\kappa(\bar{P}) = \bar{s} + \max\{\bar{t} + 1 - \bar{s}, 0, \bar{\tau}_k\}_{k=1,...,n}
\]

\[
= s - 1 + \max\{t + 1 - s + 1, 0, \tau_k + 1\}_{k=1,...,n}.
\]

If \( h_n = 1 \), then \( \tau_n = l_n - 1 \geq 0 \). Furthermore, if \( \kappa(\bar{P}) \geq s \) then \( t + 1 \geq s \) or \( \tau_k \geq 0 \) for some \( k \). In either of these cases, we have

\[
\max\{t + 1 - s + 1, 0, \tau_k + 1\}_{k=1,...,n} = \max\{t + 1 - s + 1, \tau_k + 1\}_{k=1,...,n}.
\]

Thus, using Lemma 3.10 we have

\[
\kappa(P) = \kappa(\bar{P}) + 1
\]

\[
= s - 1 + \max\{t + 1 - s + 1, \tau_k + 1\}_{k=1,...,n}
\]

\[
= s + \max\{t + 1 - s, \tau_k\}_{k=1,...,n}
\]

\[
= s + \max\{t + 1 - s, 0, \tau_k\}_{k=1,...,n}.
\]

Finally, if \( \kappa(\bar{P}) = s - 1 \), then \( t + 1 \leq s - 1 \) and \( \tau_k + 1 \leq 0 \), for all \( k = 1, \ldots, n \). This in particular implies that in this case \( s + \max\{t + 1 - s, 0, \tau_k\}_{k=1,...,n} = s \).

By Lemma 3.10 we also have

\[
\kappa(P) = \kappa(\bar{P}) + 1
\]

\[
= s - 1 + 1
\]

\[
= s.
\]

Recall that a partition in \( \mathcal{P}(T) \) is special if \( \kappa(P) > \kappa(T) \) from Equation (3.2).

**Corollary 3.12** (Special partitions). Assume that \( P \) is a single-block partition. Then \( P \) is special if and only if some \( \tau_k \) from Theorem 3.11 satisfies \( \tau_k > \delta \) where \( \delta = \max\{t_d + 1 - s, 0\} \).

**Proof.** This follows from Equation (3.2) and Theorem 3.11. \( \square \)

**Remark 3.13.** We note that if at least one entry in the hook code of \( P \) is zero, then \( \tau_n = l_n > 0 \). Thus, in this case \( \kappa(P) \geq s + 1 \). This in particular implies that in part (b) of Lemma 3.10 if \( h_n = 1 \) then the hook code of \( \bar{P} \) has a zero entry and therefore \( \kappa(\bar{P}) \geq s + 1 = s \).
3.3 Lattice path correspondence.

In this subsection we introduce a one-to-one correspondence between single-block partitions with a given Hilbert function $T = (1, 2, \ldots, d - t_d = t, 0)$ and the northeast lattice paths from $(0, 0)$ to $(s, t)$. This correspondence will in particular provide a straightforward geometric illustration of the statement of Theorem 3.11.

**Definition 3.14.** Let $P$ be a single-block partition of diagonal lengths $T = (1, \ldots, d, t_d = t, 0)$ and difference-one hook code $Q(P) = (h^n_1, \ldots, h^n_{l_P})$. Recall that $s = d + 1 - t$. Let $L(P)$ be the NE lattice path form $(0, 0)$ to $(s, t)$ represented by the word

$$L(P) = E^{h_0} N^{l_0} E^{h_{n-1} - h_0} N^{l_{n-1}} \cdots E^{h_1 - h_2} N^{l_1} E^{s - h_1}.$$ 

In other words, to obtain $L(P)$ from $Q(P)$, we start at the origin in $\mathbb{Z}^2$, move to the right by $h_n$ steps, then move up by $l_n$ steps, then move to the right by $h_{n-1} - h_n$ steps and up by $l_{n-1}$ steps, etc. Note that we may have $h_n = 0$ or $h_1 = s$, and therefore the path may start or end with northward steps.

**Lemma 3.15.** Let $T = (1, \ldots, d, t_d = t, 0)$ and $s = d + 1 - t$. The map sending a partition $P$ of diagonal lengths $T$ to the NE lattice path $L(P)$ defined in Definition 3.14 is a 1-1 correspondence between the set of partitions of diagonal lengths $T$ and the set of NE lattice paths from $(0, 0)$ to $(s, t)$.

**Proof.** By Theorem 2.12 the map sending each partition $P$ of diagonal lengths $T$ to its difference-one hook code $Q(P)$ is an isomorphism. Moreover, the following map from the set of NE lattice paths from $(0, 0)$ to $(s, t)$ to the set of difference-one hook codes for partitions of diagonal lengths $T$ is the inverse of the map defined in Definition 3.14. Consider a NE lattice path from $(0, 0)$ to $(s, t)$ given by a word $L = E^{e_r} N^{n_r} \cdots E^{e_1} N^{n_1} E^{e_0}$ where $e_r$ and $e_0$ are non-negative while the rest of $e_i$ and $n_i$'s are positive. Then the corresponding partition $P$ is the partition with diagonal lengths $T = (1, 2, \ldots, s + t - 1, t, 0)$ and difference-one hook code

$$Q(P) = \left( \left( \sum_{i=1}^{r} e_i \right)^{n_1}, \ldots, \left( \sum_{i=k}^{r} e_i \right)^{n_k}, \ldots, e_r^{n_r} \right).$$

□

**Example 3.16.** Consider the single-block partition $P = (5, 4, 2^2)$ of diagonal lengths $T = (1, 2, 3, 4, 3, 0)$. Then $Q(P) = (2^2, 1)$. Therefore $L(P)$ is the path corresponding to the word $ENENN$. Note that the first horizontal step corresponds to the entry 1 in the hook-code. It is then followed by one step up, corresponding to the multiplicity of the entry 1 in $Q(P)$. Then there is another step to the right which corresponds to the difference $2 - 1$ of the consecutive entries in $Q(P)$, followed by two steps up because of the multiplicity 2 of the entry 2 in $Q(P)$. See Figure 13 for a visualization of this, as well as similar path correspondences for two other partitions of the same diagonal lengths.

The key observation about the lattice path correspondence defined above is that $\max \{ \tau_k | k = 1, \ldots, n \}$ is in fact the maximum value $b$ for which the line $y = x + b$ intersects $L(P)$, excluding the end points.

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Figure 13: NE lattice path correspondence for three different partitions of diagonal lengths $T = (1, 2, 3, 4, 3, 0)$. See Example 3.16.
Therefore, in order to find $\kappa(P)$ for a partition $P$, we consider the corresponding lattice path $\mathcal{L}(P)$. If $\mathcal{L}(P)$ does not cross the line $y = x + \delta$, where $\delta = \max\{0, t + 1 - s\}$, then $\kappa(P) = s + \delta$. Otherwise, $\kappa(P) = s + b$, where $b$ is the largest integer such that $\mathcal{L}(P)$ intersects the line $y = x + b$. See Figure [13].

**Remark 3.17.** In general, the following steps lead to a visual and relatively straightforward way of finding $\kappa(P)$ for a single-block partition $P$ of diagonal lengths $T = (1, \ldots, d, t_d = t, 0)$ and difference-one hook code $\Omega(P) = (h^1_1, \ldots, h^n_n)$. See Figure [13].

- Consider the NE lattice path $\mathcal{L}(P)$ associated with the hook code $\Omega(P)$. Note that this is a path from $(0,0)$ to $(s,t)$ where the eastward movements are determined by entries of the hook code ($h_k$'s) while the northward steps are determined by the multiplicities of the entries of the hook code ($l_k$'s). The corners of the path are at $(h_k, \sum_{i=k}^n l_i)$ for $k = 1, \ldots, n$.

- Consider the set $L$ of all lines of slope 1 that intersect $\mathcal{L}(P)$. If $y = x + \delta$ is not in the set, add it to $L$. Intersect all lines in $L$ with the vertical line $x = s$ and consider the most northerly intersection point. The $y$-coordinate of this point is $\kappa(P)$.

## 4 Partitions in $\mathcal{P}(T)$ having a given number of generators, for single-block $T$.

We begin with a result counting the total number of partitions having diagonal lengths a single-block Hilbert function $T$. We then in Theorem [4.2] count those associated to a given generic number of generators $\kappa(P)$. Throughout the section $T$ will be a single-block Hilbert function $T = (1, 2, \ldots, d, t_d = t, 0)$ of Equation (3.3), we let $s = d + 1 - t$ and we set $\delta = \max\{t + 1 - s, 0\} = \max\{2t - d, 0\}$.

**Lemma 4.1.** The number of partitions having the single-block diagonal lengths $T = (1, 2, \ldots, d, t_d = t)$ satisfies

$$\# \mathcal{P}(T) = \binom{s + t}{t}. \quad (4.1)$$

**Proof.** By Theorem [2.12] $\# \mathcal{P}(T)$ counts the total number of partitions whose Ferrers diagram can be placed in a $t \times s$ box $\mathcal{B}_d(T)$ or, equivalently, the number of lattice paths from $(0,0)$ to $(s,t)$, which satisfies (4.1). □

By Theorem [3.11] the number of generators for a generic ideal in the cell $V(E_P)$ is $\kappa(P) = s + \max\{\delta, \tau_k\}_{k=1,\ldots,n}$. In particular, for all partitions $P$ of diagonal lengths $T$, we have

$$\kappa(T) = s + \delta \leq \kappa(P) \leq s + t.$$

**Theorem 4.2** (Number of special partitions of diagonal lengths $T$). Let $T = (1, \ldots, d, t, 0)$, $s = d + 1 - t$, and $\delta = \max\{t + 1 - s, 0\}$. Assume that $k$ is an
**Case 1.** Assume that $s \leq t$. Then $\delta = t + 1 - s > 0$.

\[
\kappa(P) = s + \delta
\]
(Here $s = 2$, $t = 3$ and $\delta = 2$.)

**Case 2.** Assume that $s > t$. Then $\delta = 0$.

\[
\kappa(P) = s + \delta = s
\]
(Here $s = 6$ and $t = 4$.)

Figure 14: Finding $\kappa(P)$ from $\mathcal{L}(P)$. See Remark 3.17.
integer such that \( s + \delta < k \leq s + t \). Then the number of partitions \( P \) of diagonal lengths \( T \) and \( \kappa(P) \geq k \) is
\[
\binom{s+t}{k}.
\] (4.2)

In particular the number of special partitions of diagonal lengths \( T \) is
\[
\binom{s+t}{s+\delta+1} = \binom{s+t}{\min\{s-1, t\}}.
\] (4.3)

And the number of non-special partitions of diagonal lengths \( T \) is
\[
\binom{s+t}{s} - \binom{s+t}{s+\delta+1}.
\] (4.4)

**Proof.** Consider an integer \( k \) such that \( s + \delta < k \leq s + t \). Let \( \gamma = k - s \). Then \( \delta < \gamma \leq t \). Using the correspondence established in Section 3.3, partitions \( P \) of diagonal lengths \( T \) and \( \kappa(P) \geq k \) correspond to paths from \((0,0)\) to \((s,t)\) intersecting the line \( y = x + \gamma \). Note that the inequality \( \delta < \gamma \) implies that the line \( y = x + \gamma \) is above the line \( y = x + \delta \).

To count the number of paths from \((0,0)\) to \((s,t)\) intersecting the line \( y = x + \gamma \), we first count the number of paths from \((0,0)\) to \((s,t)\) that stay weakly below the line \( y = x + \gamma - 1 \). We will then subtract this number from the total number of paths from \((0,0)\) to \((s,t)\), which is \( \binom{s+t}{s} \).

Since \( \delta < \gamma \) and \( \delta = \max\{0,t+1-s\} \), both endpoints \((0,0)\) and \((s,t)\) are weakly below \( y = x + \gamma - 1 \). Therefore the set of paths weakly below the line \( y = x + \gamma - 1 \) is non empty.

Using a simple vertical translation by \( \gamma - 1 \) units, it is clear that the number of paths from \((0,0)\) to \((s,t)\) that are weakly below the line \( y = x + \gamma - 1 \) is the same as the number of paths from \((0,-(\gamma - 1))\) to \((s,t-(\gamma - 1))\) that are weakly below the line \( y = x \). By Theorem 10.3.1 of [Kc], the number of such paths is
\[
\binom{s+t}{s} - \binom{s+t}{s+\gamma}.
\]
This completes the proof of the theorem. \( \square \)

**Corollary 4.3.** Let \( T = (1, \ldots, d, t_d = t, 0) \), \( s = d + 1 - t \), and \( \delta = \max\{t+1-s, 0\} \). For a positive integer \( k \), we define \( \mu(T,k) \) to be the number of partitions \( P \) with diagonal lengths \( T \) and \( \kappa(P) = k \). Then
\[
\mu(T,k) = \begin{cases} 
\binom{s+t}{s} - \binom{s+t}{s+\delta+1}, & \text{if } k = s + \delta \text{ (non-special } P), \\
\binom{s+t}{k} - \binom{s+t}{k+1}, & \text{if } s + \delta < k \leq s + t, \\
0, & \text{otherwise}.
\end{cases}
\]

**Remark 4.4.** We note that for \( k = s + \delta \), the number \( \mu(T,k) \) is the coefficient of the degree \( k \) term in \((1 + z)^{s+t} (z^\delta - \frac{1}{z}) \) while for \( s + \delta < k < s + t \), the number \( \mu(T,k) \) is the same as the coefficient of the degree \( k \) term in \((1 + z)^{s+t} (1 - \frac{1}{z}) \).

**Remark 4.5.** In [Y1] Section 2C it was shown that for any single-block Hilbert function \( T \), there will be a unique minimal finite set of special partitions of
diagonal lengths $T$, such that any special partition is in the closure of the minimal set. The following example shows that the special cells do not form an irreducible subfamily of $G_T$.

**Example 4.6** (Single-block table). Let $T = (1, 2, 3, 4, 2, 0)$, then $t = 2, s = 3$ and $\mathfrak{B}(T) = (\mathfrak{B}_4(T)) = (2 \times 3)$, and there are $\binom{5}{2} = 10$ partitions of diagonal lengths $T$. We give Figure 15 for these, specifying the hook code, and $\kappa(P)$ for each. Here $\delta(T) = \max\{0, s+1-t\} = 0$ and $\kappa(T) = 3$. We have placed conjugate partitions in symmetric positions from the center line; the two middle partitions of hook codes $(3, 0)$ and $(2, 1)$ are self-conjugate. Note also that the conjugate partition $P^\vee$ has the complementary hook code in $\mathfrak{B}_4(T) = (3, 3)$.

Figure 16 gives the specialization diagram for $\mathcal{P}(T)$, corresponding to inclusion of the Ferrers diagrams for the hook codes $h_4(P)$ (on the left). We see from the table that the cells in $\kappa(P) \geq 4$ are the union of the closures of cells having hook codes $(1, 1)$ and $(3, 0)$: $\kappa(P) = 4$ includes also the cells with hook codes $(2, 0)$ and $(1, 0)$, while the cell with hook code $(0, 0)$ is the unique with $\kappa(P) = 5$ (these cells are colored red/blue on the left of Figure 16). Thus, the subvariety of cells corresponding to special partitions is here the union of two irreducible components, of dimensions three (closure of $(3, 0)$) and two (closure of $(1,1)$), respectively.

### 5 Number of generators for multiblock partitions.

Throughout this section, $T = (1, \ldots, d, t_d, \ldots, t_j, 0)$, and $P$ is a partition lengths $T$ and difference-one hook code $\Omega(P) = (h_d, \ldots, h_j)$.

Recall from Equation (2.3) that for $i = d, \ldots, j$, we set $\delta_i = t_i - 1 - t_i, t_{d-1} := d$ and

$$T_i = (1, \ldots, \delta_i + \delta_{i+1}, \delta_{i+1}, 0).$$
Figure 16: Specialization diagram for $\mathcal{P}(T)$, $T = (1, 2, 3, 4, 2, 0)$. See Example 4.6.
As we saw in Definition 2.17 and Proposition 2.22, a partition \( P \in \mathcal{P}(T) \) can be decomposed into single-block “component” partitions \( P_i \). For \( i = d, \ldots, j \), the partition \( P_i \) has diagonal lengths \( T_i \) and difference-one hook code \( \mathcal{h}_i \). We note that by construction the hook code for \( P_i \) is \( \mathcal{h}_i \); however, the hand-degree for the hooks in \( P_i \) is \( \delta_i + \delta_{i+1} \), and they correspond to hooks in \( P \) of hand-degree \( i \). We showed in Theorem 2.24 that the cells \( \text{V}(E_P) \) are naturally the product of the corresponding cells \( \text{V}(E_{P_i}) \).

In this section we count the minimum number of generators for ideals in the cell associated to an arbitrary partition. The main results of this section are Theorem 5.10 and Theorem 5.15. Theorem 5.10 specifies the number \( \kappa(P_i) \) of degree- \( i \) generators of an ideal \( I \) defining a generic element of \( \text{V}(E_P) \). In Theorem 5.15 we provide a formula for \( \kappa(P_i) \) in terms of the \( \kappa(P_i) \) of single-block components of \( P \). The value \( \kappa(P_i) \) for a single-block partition was determined in Theorem 3.11.

The \( i \)-th block \( \mathcal{h}_i(P) \) of the hook code \( \mathcal{Q}(P) \) is a key element of our statements. So we first give a formula for \( \kappa(P_i) \) when \( \mathcal{h}_i(P) \) is empty. Recall that the case \( \mathcal{h}_i = \emptyset \) occurs when \( t_i = t_{i+1} \) and in this case, we can think of \( P_i \) as the basic triangle \( \Delta_{\delta_i} \) (Remark 2.21).

**Proposition 5.1.** If the block \( \mathcal{h}_i(P) \) is empty, then \( \kappa(P_i) = \delta_i + 1 \).

**Proof.** Recall that the case \( \mathcal{h}_i = \emptyset \) occurs when \( t_i = t_{i+1} \) and in this case, we can think of \( P_i \) as the basic triangle \( \Delta_{\delta_i} \) (see Remark 2.21). It is then clear that the monomial ideal whose cobasis is the basic triangle \( \Delta_{\delta_i} \) has exactly \( \delta_i + 1 \) generators, the degree \( \delta_i \) monomials. Note that the basic triangle \( \Delta_{\delta_i} \) is empty when \( \delta_i = 0 \).

**Example 5.2.** Consider the partition \( P = (8, 7, 4, 2, 1, 1) \) of diagonal lengths \( T = (1, 2, 3, 4, 5, 4, 2, 2, 0) \). The degree 6 block \( \mathcal{h}_6(P) \) is empty \((t_6 = t_7 = 2)\). \( \kappa(P_6) = \delta_6 + 1 = 3 \) and this number corresponds to the three border monomials \( (x^4y^2, x^3y^3, y^6) \) of \( E_P \).

From now on, we will assume that the \( i \)-th block \( \mathcal{h}_i(P) \) of the hook code \( \mathcal{Q}(P) \) is not empty. For \( i = d, \ldots, j \), the degree \( i \) block \( \mathcal{h}_i \) in the hook code of \( P \) can be written as

\[
\mathcal{h}_i = \left( h_{i,1}^{l_{i,1}}, \ldots, h_{i,k}^{l_{i,k}}, \ldots, h_{i,n_i}^{l_{i,n_i}} \right)
\]

(5.1)

where \( \sum_{k=1}^{n_i} l_{i,k} = \delta_{i+1} \) and \( \delta_i + 1 \geq h_{i,1} > h_{i,2} > \cdots > h_{i,n_i} \geq 0 \).

In the following, we define two sequences of integers \( R(P)_i \) and \( G(P)_i \) for single-block components \( P_i \) of \( P \) that allow us to count degree \( i+1 \) relations and degree \( i+1 \) generators of the monomial ideal \( E_P \) associated to \( P \).

**Definition 5.3** (Sequences of degree \( i+1 \) relations and generators). Denote by \( R_{i,k} \) the set of degree \( i+1 \) relations below the \((l_{i,1} + \cdots + l_{i,k})\)-th degree
Let \( G_{i,1} \) be the set of degree \( i + 1 \) corner-monomials of \( E_P \) and for \( 2 \leq k \leq n_i \), denote by \( G_{i,k} \) the set of degree \( i + 1 \) corner-monomials of \( E_P \) below the \((l_{i,1} + \cdots + l_{i,k-1} + 1)\)-th degree \( i \) hand of \( P \). Then \( G(P)_i = (g_{i,1}, \ldots, g_{i,n_i}) \) where \( g_{i,k} \) counts the number of elements of \( G_{i,k} \).

**Remark 5.4** (Chains of degree \( i + 1 \) relations and generators). Note that using the above notation, we have the following sequences of inclusions.

\[
R_{i,n_i} \subset R_{i,n_i-1} \subset \cdots \subset R_{i,k+1} \subset R_{i,k} \subset \cdots \subset R_{i,1}
\]

\[
G_{i,n_i} \subset G_{i,n_i-1} \subset \cdots \subset G_{i,k+1} \subset G_{i,k} \subset \cdots \subset G_{i,1}.
\]

So by definition, \( R(P)_i = (r_{i,1}, \ldots, r_{i,n_i}) \) and \( G(P)_i = (g_{i,1}, \ldots, g_{i,n_i}) \) are non-increasing sequences.

We now use the hook code \( \mathbf{h}_i = \left(h_{i,1}^{l_{i,1}}, \ldots, h_{i,k}^{l_{i,k}}, \ldots, h_{i,n_i}^{l_{i,n_i}}\right) \) to give formulas for the integers \( r_{i,k} \) and \( g_{i,k} \) of Definition 5.3.

**Observation 5.5.** By definition of the difference-one hook code, for any integer \( 1 \leq k \leq n_i \), there are \((h_{i,k} - h_{i,k+1})\) horizontal-border monomials below the \((l_{i,1} + \cdots + l_{i,k})\)-th degree \( i \) hand and above the \((l_{i,1} + \cdots + l_{i,k} + 1)\)-th degree \( i \) hand of \( P \). Let \((M_{i,1}, \ldots, M_{i,h_{i,k} - h_{i,k+1}})\) be the list of these monomials, ordered from top to bottom. Then there is exactly one linear (occurring in degree \((i + 1)\)) relation between any two consecutive monomials \( M_{i,m} \) and \( M_{i,m+1} \) \((1 \leq m \leq h_{i,k} - h_{i,k+1})\). So, we have exactly \((h_{i,k} - h_{i,k+1} - 1)\) linear relations below the \((l_{i,1} + \cdots + l_{i,k})\)-th degree-\(i\) hand and above the \((l_{i,1} + \cdots + l_{i,k} + 1)\)-th degree-\(i\) hand of \( P \).

Also we observe that each degree-\(i\) hand of \( P \) is just left of a degree \((i + 1)\) vertical-border monomial of \( E_P \). We use this property to count the degree \((i + 1)\) corner-monomials of \( E_P \).

**Example 5.6.** Consider the partition \( P = (12^2, 11, 8^2, 7, 6^2, 3^2, 2^2) \) (see Figure 17) whose Hilbert function is \( T = (1, 2, \cdots, 10, 11, 10, 4) \). Here we have \( h_{12} = (6^2, 3, 1) \). So \( h_{12,1} = 6, h_{12,2} = 3, h_{12,3} = 1, l_{12,1} = 2, l_{12,2} = 1 \) and \( l_{12,3} = 1 \). There are \((h_{12,1} - h_{12,2}) = 3\) horizontal-border monomials that are below the second hand monomial \( x^{10}y^2 \) and above the third \((l_{12,1} + 1 = 3)\) hand monomial \( x^5y^7 \). Also we have exactly two linear relations that are below \( x^{10}y^2 \) and above \( x^5y^7 \).

**Lemma 5.7** (Counting degree \( i + 1 \) relations and corner-monomials of \( E_P \)).

Let \( \mathbf{h}_i = \left(h_{i,1}^{l_{i,1}}, \ldots, h_{i,k}^{l_{i,k}}, \ldots, h_{i,n_i}^{l_{i,n_i}}\right) \) be the \(i\)-th block of the hook code \( \mathbf{Q}(P) \).

Then the sequences \( R(P)_i = (r_{i,1}, \ldots, r_{i,n_i}) \) and \( G(P)_i = (g_{i,1}, \ldots, g_{i,n_i}) \) of Definition 5.3 are given by the following numbers.

(a) If \( h_{i,n_i} > 0 \), then
Figure 17: Ferrers diagram of $P = (12, 11, 8^2, 7, 6^2, 3^2, 2^2)$: horizontal-border monomials are marked in blue and hand monomials are marked in red (Example 5.6).

- $g_{i,1} = \begin{cases} \delta_{i+1} - (n_i - 1) & \text{if } h_{i,1} = \delta_i + 1 \\ \delta_{i+1} - n_i & \text{if } h_{i,1} < \delta_i + 1 \end{cases}$

- $g_{i,k} = \sum_{j=1}^{n_i} l_{i,j} - (n_i + 1 - k)$, for $2 \leq k \leq n_i$

- $r_{i,k} = h_{i,k} - (n_i + 1 - k)$, for $1 \leq k \leq n_i$.

(b) If $h_{i,n_i} = 0$, then

- $g_{i,1} = \begin{cases} \delta_{i+1} - (n_i - 2) & \text{if } h_{i,1} = \delta_i + 1 \\ \delta_{i+1} - (n_i - 1) & \text{if } h_{i,1} < \delta_i + 1 \end{cases}$

- $g_{i,k} = \sum_{j=1}^{n_i} l_{i,j} - (n_i - k)$, for $2 \leq k \leq n_i$

- $r_{i,k} = h_{i,k} - (n_i - k)$ for $1 \leq k \leq n_i$.

Proof. Note that by definition of the difference-one hook code, we have

$\delta_i + 1 \geq h_{i,1} > \cdots > h_{i,n_i} \geq 0$ and $\sum_{k=1}^{n_i} l_{i,k} = \delta_{i+1}$.

The numbers given in the Lemma come directly from Observation 5.5.

(a) If $h_{i,n_i} > 0$, then we have
Remark 5.8. Using the formulas for \( r_{i,k} \) and \( g_{i,k} \), if we let \( \theta_{i,k} = g_{i,k} - r_{i,k} \) (\( 1 \leq k \leq n_i \)), we obtain

- \( \theta_{i,1} = \left\{ \begin{array}{ll} \delta_i + 1 - h_{i,1} & \text{if } h_{i,1} = \delta_i + 1 \\ \delta_i + 1 - h_{i,1} - 1 & \text{if } h_{i,1} < \delta_i + 1 \end{array} \right. \)

- \( \theta_{i,k} = \sum_{j=k}^{n_i} l_{i,j} - h_{i,k} \) for \( 2 \leq k \leq n_i \).

Remark 5.9. For \( 1 \leq k \leq n_i \), the formulas in Lemma 5.7 can be rewritten in a compact way using the invariants of the hook code Equation (5.2)

- \( g_{i,k} = \sum_{j=k}^{n_i} l_{i,j} - (n_i + 1 - k) + \max\{1 - h_{i,n_i}, 0\} + \max\{h_{i,k} - \delta_i, 0\} \);

- \( r_{i,k} = h_{i,k} - (n_i + 1 - k) + \max\{1 - h_{i,n_i}, 0\} \).

So, \( \theta_{i,k} = \sum_{j=k}^{n_i} l_{i,j} - h_{i,k} + \max\{h_{i,k} - \delta_i, 0\} \) for \( 1 \leq k \leq n_i \).

As in Theorem 5.11, let \( \tau_{i,k} = \sum_{j=k}^{n_i} l_{i,j} - h_{i,k} \). Then \( \tau_{i,k} = \theta_{i,k} \) for \( 2 \leq k \leq n_i \) and \( \tau_{i,1} = \theta_{i,1} - \max\{h_{i,1} - \delta_i, 0\} \). These are the ingredients for the formula Equation (5.2) for \( \beta_{i+1,0}(P) = N_{i+1} - N_i \) in Theorem 5.10, which gives the number of generators of degree-\( i \) for an ideal \( I \) defining a generic element \( A \) of \( \mathbb{V}(E_P) \).
**Theorem 5.10.** Let \( P \in \mathcal{P}(T), T = (1, \ldots, d, t_d, \ldots, t_j, 0) \) have hook code \( \Omega(P) = (h_d, \ldots, h_i, \ldots, h_j) \). Suppose \( h_i = (h_{i,1}^1, \ldots, h_{i,k}^i, \ldots, h_{i,n_i}^i) \) is the \( i \)-th block of the hook code \( \Omega(P) \) of \( P \). Let \( I \) be a generic element of the cell \( \mathfrak{V}(E_P) \) and \( \mathcal{G}(I) \) a minimal set of generators of \( I \), and \( \beta_{i,0}(P) \) the number of degree-\( i \) generators. We denote by \( \kappa(P) = \beta_0(P) = (\beta_{d,0}(P), \ldots, \beta_{j+1,0}(P)) \). For \( d \leq m \leq j + 1 \), let

\[
N_m = \# \{ f \in \mathcal{G}(I), \text{ such that } \deg(f) \leq m \}.
\]

Then \( N_d = d + 1 - t_d \), and using the previously defined numbers \( r_{i,k} \) and \( g_{i,k} \), for \( i \in [d, j] \), we have for \( \beta_{i+1,0}(P) = N_{i+1} - N_i \),

\[
N_{i+1} - N_i = \max \{ 0, r_{i,k} - g_{i,k} \}_{1 \leq k \leq n_i} = \max \{ 0, \theta_{i,k} \}_{1 \leq k \leq n_i} = \max \{ \delta_{i+1} - \delta_i, 0, \tau_{i,k} \}_{1 \leq k \leq n_i}.
\] (5.2)

**Proof.** By standard basis construction techniques (see Theorem I.1.9 of [Br]), Propositions 2 and 3 of [Br-Ga]), one can first see that if \( r_{i,n_i} \geq g_{i,n_i} \), then there are enough degree \( i + 1 \) relations to kick out all of the \( g_{i,n_i} \) generators that are just above these relations. Also, if \( r_{i,n_i} < g_{i,n_i} \), then we need \( g_{i,n_i} - r_{i,n_i} \) extra generators whose leading terms are corner- monomials of \( E_P \) below the \((l_{i,1} + \cdots + l_{i,n_i} + 1)\)-th degree \( i \) hand of \( P \).

It is clear that if for all \( k \) \((1 \leq k \leq n_i)\) we have \( r_{i,k} \geq g_{i,k} \), then we can kick out all degree \( i + 1 \) generators whose leading terms are degree \( i + 1 \) corner-monomials of \( E_P \).

Now, suppose there exists an integer \( k \) such that \( r_{i,k} < g_{i,k} \). Then we can inductively consider the following sets and numbers.

\[
S_0 = \{ k \in \mathbb{N}, \ r_{i,k} < g_{i,k} \}, \quad s_0 = \max (S_0);
S_1 = \{ k \in \mathbb{N}, \ k < s_0, \ r_{i,k} - r_{i,s_0} < g_{i,k} - g_{i,s_0} \}, \quad s_1 = \max (S_1);
\]

\[
\vdots
\]

\[
S_q = \{ k \in \mathbb{N}, \ k < s_{q-1}, \ r_{i,k} - r_{i,s_{q-1}} < g_{i,k} - g_{i,s_{q-1}} \}, \quad s_q = \max (S_q);
S_{q+1} = \emptyset.
\]

The meaning of the sets \( S_0, \ldots, S_q \) and the numbers \( s_0, \ldots, s_q \) is the following:

- First, we have \( r_{i,s_0+1} \geq g_{i,s_0+1}, \ldots, r_{i,n_i} \geq g_{i,n_i} \) and \( r_{i,s_0} < g_{i,s_0} \). So we need \( g_{i,s_0} - r_{i,s_0} \) generators whose leading terms are in \( G_{i,s_0} \), the \( s_0 \)-th part of the chain \( G_{i,n} \subset G_{i,n-1} \subset \cdots \subset G_{i,s_0} \subset G_{i,s_0-1} \subset \cdots \subset G_{i,1} \). This means we have used all the relations in the \( s_0 \)-th part of the chain \( R_{i,n} \subset R_{i,n-1} \subset \cdots \subset R_{i,s_0} \subset R_{i,s_0-1} \subset \cdots \subset R_{i,1} \).
- Since we have used all the relations in \( R_{i,s_0} \), if we are looking for more extra generators, the next step is to consider the chains of inclusions

\[
(R_{i,s_0-1} - R_{i,s_0}) \subset \cdots \subset (R_{i,1} - R_{i,s_0})
\]

\[
(G_{i,s_0-1} - G_{i,s_0}) \subset \cdots \subset (G_{i,1} - G_{i,s_0}).
\]
If the set \( S_1 = \{ k \in \mathbb{N}, \ k < s_0, \ r_{i,k} - r_{i,s_0} < g_{i,k} - g_{i,s_0} \} \) is not empty, then we set \( s_1 = \max(S_1) \) and continue looking for extra generators until \( S_{q+1} = \emptyset \) and \( S_q \neq \emptyset \) for some index \( q \).

By construction, the number of degree \( i + 1 \) extra generators needed is

\[
g_{i,s_0} - r_{i,s_0} + \sum_{j=1}^{j=q} ((g_{i,s_j} - g_{i,s_{j-1}}) - (r_{i,s_j} - r_{i,s_{j-1}})) = g_{i,s_q} - r_{i,s_q}.
\]

It is then clear that

\[
N_{i+1} - N_i = g_{i,s_q} - r_{i,s_q} = \max \{ 0, g_{i,k} - r_{i,k} \}_{1 \leq k \leq n_i}.
\]

From the formulas in Remark 5.8 we trivially have \( N_{i+1} - N_i = \max \{ 0, \theta_{i,k} \}_{1 \leq k \leq n_i} \).

Now to show the last equality in Equation 5.2 notice that if \( h_{i,1} = \delta_i + 1 \), then \( \theta_{i,1} = \delta_{i+1} - \delta_i = \tau_{i,1} + 1 \). We then have \( \{ 0, \theta_{i,k} \}_{1 \leq k \leq n_i} = \{ \delta_{i+1} - \delta_i, 0, \tau_{i,k} \}_{2 \leq k \leq n_i} \) because \( \theta_{i,k} = \tau_{i,k} \) for \( 2 \leq k \leq n_i \). Thus

\[
\max \{ 0, \theta_{i,k} \}_{1 \leq k \leq n_i} = \max \{ \delta_{i+1} - \delta_i, 0, \tau_{i,k} \}_{1 \leq k \leq n_i}.
\]

Otherwise, if \( h_{i,1} < \delta_i + 1 \), we have \( \theta_{i,k} = \tau_{i,k} \) for \( 1 \leq k \leq n_i \) and \( \delta_{i+1} - \delta_i \leq \tau_{i,1} \). So again \( \max \{ 0, \theta_{i,k} \}_{1 \leq k \leq n_i} = \max \{ \delta_{i+1} - \delta_i, 0, \tau_{i,k} \}_{1 \leq k \leq n_i} \). \( \square \)

**Corollary 5.11.** Let \( P \in \mathcal{P}(T), T = (1, \ldots, d, t_d, \ldots, t_j, 0) \) have hook code \( \mathcal{Q}(P) = (h_d, \ldots, h_l, \ldots, h_j) \). Suppose \( h_i = (h_{i,1}^{l_{i,1}}, \ldots, h_{i,k}^{l_{i,k}}, \ldots, h_{i,n_i}^{l_{i,n_i}}) \) is the \( i \)-th block of the hook code \( \mathcal{Q}(P) \) of \( P \) and let \( P_i \) be the single-block partition related to \( h_i \). Using the notation of Theorem 5.10, let \( \kappa(P) = \beta_0(P) = (\beta_{d,0}(P), \ldots, \beta_{j+1,0}(P)) \). Then for \( d \leq i \leq j, \beta_{i+1,0}(P) = \kappa(P_i) - (\delta_i + 1) \).

**Proof.** Recall that the single-block partition \( P_i \) related to

\[
h_i = (h_{i,1}^{l_{i,1}}, \ldots, h_{i,k}^{l_{i,k}}, \ldots, h_{i,n_i}^{l_{i,n_i}})
\]

is \( T_i = (1, \ldots, \delta_i + \delta_i + 1 - 1, \delta_i + \delta_i + 1, \delta_i + 1) \) \( \{2,3\} \). Applying Theorem 3.11 to the single-block case where \( d = \delta_i + \delta_i + 1, t = \delta_i + 1 \) and \( s = \delta_i + 1 \), we get \( \kappa(P_i) = \delta_i + 1 + \max \{ \delta_i + 1 - \delta_i, 0, \tau_{i,k} \}_{1 \leq k \leq n_i} \). That is \( \kappa(P_i) - (\delta_i + 1) = \max \{ \delta_i + 1 - \delta_i, 0, \tau_{i,k} \}_{1 \leq k \leq n_i} \). By Theorem 5.10 we have \( \beta_{i+1,0}(P) = N_{i+1} - N_i = \max \{ \delta_i + 1 - \delta_i, 0, \tau_{i,k} \}_{1 \leq k \leq n_i} \), so \( \beta_{i+1,0}(P) = \kappa(P_i) - (\delta_i + 1) \). \( \square \)

**Remark 5.12.** Let \( h_i = (h_{i,1}^{l_{i,1}}, \ldots, h_{i,k}^{l_{i,k}}, \ldots, h_{i,n_i}^{l_{i,n_i}}) \) be the \( i \)-th block of the hook code \( \mathcal{Q}(P) \). Moving along the \( i \)-th diagonal of \( P \) from top to bottom, we see that \( \sum_{j=n_i}^{j=k} l_{i,j} \) is, by definition, the number of degree-\( i \) hand monomials of \( P \) below the \( (l_{i,1} + \ldots + l_{i,k-1}) \)-th degree-\( i \) hand monomial of \( P \). Also, by definition, \( h_{i,k} \) is the number of degree \( i \) horizontal-border monomials of \( P \).
below the \((l_{i,1} + \ldots + l_{i,k-1} + 1)\)-th degree-\(i\) hand monomial of \(P\).

We may visualize the key integer \(\tau_{i,k}\) related to \(\beta_{i+1,0}(P)\) in the Ferrers diagram of \(P\) by coloring the corresponding \(\sum_{j=k}^{n_i} l_{i,j}\) hand monomials in red and the \(h_{i,k}\) horizontal-border monomials in blue, in the next example (Figure 18).

**Example 5.13.** Let \(T = (1, 2, \ldots, 12, 13, 12, 6)\) and consider the partition \(P\) of diagonal lengths \(T\) given by \(P = (14^2, 12, 11^2, 10, 7^2, 5^3, 4, 3, 1)\). \(T\) is a two-block Hilbert function. The hook code of \(P\) is \(\Omega(P) = (h_{13}, h_{14})\). We have \(\delta_{13} = 1, \delta_{14} = 6, h_{13}(P) = (2^3, 1^1, 0^2)\) and \(h_{14}(P) = (6^1, 4^2, 2^3)\).

Note that \(h_{13} = 2 = \delta_{13} + 1, h_{14,1} = 6 < \delta_{14} + 1\) and \(\delta_{14} - \delta_{13} = 5\).

- \(\tau_{13,1}, \tau_{13,2}\) and \(\tau_{13,3}\) are computed using the hook code block \(h_{13} = (2^3, 1^1, 0^2)\).

  - \(\tau_{13,1} = 3 + 1 + 2 - 2 = 4\). We may visualise \(\tau_{13,1}\) by coloring the sequence \((2^3, 1^1, 0^2)\): \((2^3, 1^1, 0^2) = (2^3, 1^1, 0^2)\);

  \[\tau_{13,1} = \text{sum of red integers minus the blue integer}\]

  \[\tau_{i,k} = \sum_{j=k}^{n_i} l_{i,j} - h_{i,k}\]

  - \(\tau_{13,2} = 1 + 2 - 1 = 2\). \(\tau_{13,2}\) can be visualised by coloring the subsequence \((1^1, 0^2)\) of \((2^3, 1^1, 0^2)\): \((2^3, 1^1, 0^2) = (2^3, 1^1, 0^2)\), so \(\tau_{13,2} = \text{sum of red integers minus the blue integer}\).

  - \(\tau_{13,3} = 2 - 0 = 2\), computed using \((2^3, 1^1, 0^2) = (2^3, 1^1, 0^2)\); \(\tau_{13,3} = \text{sum of red integers minus the blue integer}\).

We then find that \(\beta_{14,0}(P) = \max\{\delta_{14} - \delta_{13}, 0, \tau_{13,1}, \tau_{13,2}, \tau_{13,3}\} = 5\).

- to compute \(\tau_{14,1}, \tau_{14,2}\) and \(\tau_{14,3}\) we can use the same coloring method on the hook code block \(h_{14} = (6^1, 4^2, 2^3)\).

  - \(\tau_{14,1} = 1 + 2 + 3 - 6 = 0\); this is \(\tau_{14,1} = \text{sum of red integers minus the blue integer}\), using the coloring \((6^1, 4^2, 2^3)\).

  - \(\tau_{14,2} = 2 + 3 - 4 = 1\) and \(\tau_{14,3} = 3 - 2 = 1\) using the colorings \((6^1, 4^2, 2^3), (6^1, 4^2, 2^3)\).

We then find that \(\beta_{15,0}(P) = \max\{\delta_{15} - \delta_{14}, 0, \tau_{14,1}, \tau_{14,2}, \tau_{14,3}\} = 1\).

We illustrate \(\tau_{14,2}\) in Figure 18 by coloring the degree-\(i\) hand monomials and the degree-\(i\) horizontal-border monomials as suggested in Remark 5.12.
Figure 18: $P = (14^2, 12, 11^2, 10, 7^2, 5^3, 4, 3, 1): \tau_{14,2} = 2 + 3 - 4 = 1$

**Remark 5.14.** Using the notation of Lemma 3.3 and Theorem 5.10 one has

(i). $[\delta_{i+1} - \delta_i]^{+} \leq \max \{\delta_{i+1} - \delta_i, 0, \tau_{i,k}\}_{1 \leq k \leq n_i} = N_{i+1} - N_i$, so the number of degree $i + 1$ generators of $I \in V(E_P)$ is at least $[\delta_{i+1} - \delta_i]^{+}$ (this is statement i. of Lemma 3.3).

(ii). Suppose $P$ is the partition associated to the generic cell of $G_T$. Then for the $i$-th block $h_i = (h_{i,1}, \ldots, h_{i,k}, \ldots, h_{i,n_i})$ of the hook code $\Omega(P)$ of $P$, we have $n_i = 1$, $h_{i,1} = \delta_i + 1$ and $l_{i,1} = \delta_{i+1}$, that is $h_i = (\delta_i + 1)^{\delta_{i+1}}$. In this case, $g_{i,1} = \delta_{i+1}$, $r_{i,1} = \delta_i$ and formula 5.2 of Theorem 5.10 gives $N_{i+1} - N_i = [\delta_{i+1} - \delta_i]^{+}$ (this is statement ii. of Lemma 3.3).

(iii). An immediate corollary of Theorem 5.15 in the single-block case recovers Theorem 3.11. More precisely, for a single-block partition $P_i$ with Hilbert function $T_i = (1, \cdots, d - 1, d, t_d, 0)$ where $d = \delta_i + \delta_{i+1}$ and $t_d = \delta_{i+1}$. Denote the difference-one hook code of $P_i$, $\Omega(P_i)$, by $h_i$ as above which satisfies

$$\delta_i + 1 \geq h_{i,1} > \cdots > h_{i,n_i} \geq 0 \quad \text{and} \quad \sum_{k=1}^{k=n_i} l_{i,k} = \delta_{i+1}.$$  

Then $\kappa(P_i) = \delta_i + 1 + \max \{0, g_{i,k} - r_{i,k}\}_{1 \leq k \leq n_i}$.
We can now state and prove the main theorem of this section.

**Theorem 5.15** (Decomposition of \( \kappa(P) \) into components). Let \( P \) be a partition of lengths \( T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0) \). Then

\[
\kappa(P) = \sum_{i=d}^{j} \kappa(P_i) - (t_d - t_j) - (j - d).
\]

Moreover, a minimal set of generators for a generic ideal in the cell \( \mathbb{V}(E_P) \) includes \( d + 1 - t_d \) generators of degree \( d \) and \( \kappa(P_i) - (\delta_i + 1) \) generators of degree \( i + 1 \), for \( i \in [d, j] \).

**Proof.** Let \( I \) be a generic element of the cell \( \mathbb{V}(E_P) \), \( \mathcal{G}(I) \) a minimal set of generators of \( I \), and \( \beta_{i,0}(P) \) the number of degree-\( i \) generators of \( I \). From Theorem 5.10 and Corollary 5.11, we have \( \beta_{i+1,0}(P) = N_{i+1} - N_i = \kappa(P_i) - (\delta_i + 1) \) for all \( i \in [d, j] \). By definition, \( \kappa(P) = \# \mathcal{G}(I) = N_{j+1} = N_d + \sum_{i=d+1}^{j} \beta_{i,0}(P) \).

Since \( \beta_{i+1,0}(P) = \kappa(P_i) - (\delta_i + 1) \), we have \( \kappa(P) = N_d + \sum_{i=d}^{j} (\kappa(P_i) - (\delta_i + 1)) \).

Clearly, \( N_d = d + 1 - t_d \). Recall that by definition, \( \delta_i = t_{i-1} - t_i \). We then have

\[
\kappa(P) = N_d + \sum_{i=d}^{j} (\kappa(P_i) - (\delta_i + 1))
\]

\[
= d + 1 - t_d + \sum_{i=d}^{j} \kappa(P_i) - \sum_{i=d}^{j} (t_{i-1} - t_i) - \sum_{i=d}^{j} 1
\]

\[
= \sum_{i=d}^{j} \kappa(P_i) - (t_d - t_j) - (j - d).
\]

\( \square \)

**Remark 5.16.** Note that if \( t_i = t_{i+1} \), then it is clear that \( N_{i+1} - N_i = 0 \). Also, we found (Proposition 5.1) that when \( t_i = t_{i+1} \), we have \( \kappa(P_i) = \delta_i + 1 \). So the empty hook blocs contribute to zero in the sum \( \kappa(P) = N_d + \sum_{i=d}^{j} (\kappa(P_i) - (\delta_i + 1)) \) that computes \( \kappa(P) \) in Theorem 5.15.

**Proposition 5.17.** Let \( P \) be a partition of diagonal lengths \( T = (1, 2, \ldots, d-1, t_d, \ldots, t_j, 0) \) and suppose \( h_i = (h_{i,1}^{l_1}, \ldots, h_{i,k}^{l_k}, \ldots, h_{i,n_i}^{l_i}) \) is the \( i \)-th block of the hook code \( \Omega(P) \) of \( P \).

Denote by \( b_{i+1}(E_P) \) the number of degree \( i + 1 \) corner-monomials (generators) of \( E_P \). Then we have

\[
b_d = d + 1 - t_d,
\]

\[
b_{i+1}(E_P) = \delta_{i+1} - n_i + \max\{1 - h_{i,n_i}, 0\} + \max\{h_{i,1} - \delta_i, 0\}.
\]
Figure 19: Ferrers diagram of Example 5.19. Each labeled box represents a hook corner, it is labeled by a ◦ if its hand-degree is 13, with a • if the hand-degree is 14, and with a dark • when the hand-degree is 15.

Proof. The number of degree $d$ corner-monomials of $E_P$ is of course $d+1-t_d$. By Definition 5.3 we have $b_{i+1}(E_P) = g_{i,1}$. The proof of the Proposition then follows directly from Lemma 5.7 and Remark 5.9. □

Recall from Definition 3.4 that a partition $P$ of diagonal lengths $T$ is special if $\kappa(P) \neq \kappa(T)$. In other words, $P$ is special if $\kappa(P)$ does not have the minimum value $\kappa(T)$ possible for partitions of diagonal lengths $T$, from Equation 3.2. The following immediate corollary of Theorem 5.15 gives a necessary and sufficient condition for a partition $P$ to be special. Recall that Corollary 3.12 specifies when a single-block partition is special.

Theorem 5.18 (Component Theorem for $P$ special). Assume that $T$ is a Hilbert function of height $d$ and socle degree $j$, and that the partition $P$ of diagonal lengths $T$ decomposes into single-block partitions $P_d, \ldots, P_j$. Then $P$ is special if and only if $P_i$ is special, for some $i \in [d, j]$.

Proof. By Theorem 5.15 the value of $\kappa(P)$ is minimum - equal to $\kappa(T)$ - if and only if $\kappa(P_i)$ is minimum for all $i \in [d, j]$. Thus $P$ is non-special if and only if at least one component $P_i$ is non-special for an integer $i \in [d, j]$. □

Example 5.19. Consider the partition $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$ from Example 2.23. Then $P$ has diagonal lengths $T = (1, 2, \ldots, 13, 10_{13}, 6_{14}, 3_{15}, 0)$ and hook code (see Figure 19)

$$\Omega(P) = ((3, 1^2, 0)_{13}, (5, 4, 1)_{14}, (2^2, 1)_{15}).$$

Then, as we saw in Example 2.23 $P$ can be decomposed into the following three single-block partitions. Partition $P_{13} = (7^2, 5, 4^2, 3, 1^2)$ of diagonal lengths $T_{13} = (1, \ldots, 7, 4, 0)$ and hook code $\Omega(P_{13}) = (3, 1^2, 0)$, partition $P_{14} = (8, 6^2, 4, 3, 2^2)$ of diagonal lengths $T_{14} = (1, \ldots, 7, 3, 0)$ and hook code $\Omega(P_{14}) = (5, 4, 1)$, and partition $P_{15} = (6, 5^2, 4, 2^2)$ of diagonal lengths $T_{15} = (1, \ldots, 6, 3, 0)$ and hook code $\Omega(P_{15}) = (2^2, 1)$. 

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By Theorem 3.14 we have
\[ \kappa(P_{13}) = 6, \quad \kappa(P_{14}) = 5, \quad \text{and} \quad \kappa(P_{15}) = 5. \]
Thus, by Theorem 5.15 we have
\[ \kappa(P) = 6 + 5 + 5 - (10 - 3) - 2 = 7. \]

Let \( I \) be a generic ideal in the cell \( \mathbb{V}(E_P) \). Then a minimal set of

generators for \( I \) consists of seven generators. Of these seven generators,
four have degree 13, two have degree 14, and one has degree 15.

**Elementary and non-elementary Hilbert functions.**

A key aspect to understanding the family \( G_T \) is that, when the sequence
\( T \) of Equation 1.1 has a constant subsequence \( (s, s, \ldots) \) with \( s < d \), the
order of \( T \), then \( G_T \) splits into the direct product of simpler parameter
spaces. We explain here briefly consequences for our analysis of generators
for ideals in Jordan cells.

We say that a sequence \( T \) satisfying Equation 1.1 is **elementary** if there
is no integer \( i \in [d, j] \) such that \( t_i = t_{i+1} < d \) [Y1, §4A]; then we also say
that \( G_T \) is elementary. It is well known (see [Ia1, §4B], [Ia2, Lemma 2.2])
that when a Hilbert function \( T = H(R/I) \) satisfies \( t_i = t_{i+1} = s < d \) then
there is a form \( f \in R_s \) such that
\[ I_i = fR_{i-s} \quad \text{and} \quad I_{i+1} = f \cdot R_{i+1-s}. \]  \( \text{(5.4)} \)

It follows that \( f|I_u \) for \( u \leq i+1 \). This is usually shown using the properties
of \( \tau(V) = \dim_k R_1 V - \dim_k V \) for vector subspaces \( V \subset R_i \); this integer is
the number of generators of an “ancestor ideal” \( I = (V) \oplus_{u=1} V : R_u \); and
\( \tau(I_i) = 1 \) when \( t_i = t_{i+1} \) (ibid.).

We will define implicitly in the next Theorem “elementary factors” \( T(i) \)
of Hilbert sequences \( T \) which have constant subsequences of height \( s < d \).
These factors have no relation with the single-block components \( T_i \) for each
\( T \), defined in Equation 2.4, and a major topic for us. In fact if \( T \)
plits into elementary components \( T(i) \) they are not usually single-block.

**Lemma 5.20.** [Y1, Lemma 4.2] There is a decomposition of \( G_T \) as a
product
\[ G_T = \prod_k G_{T(k)} \quad \text{for} \quad T(k) \quad \text{elementary.} \]

**Proof.** Assume there is a single maximal consecutive subsequence \( t_i = t_{i+1} = \cdots = t_{i+k} = s \) with \( k \geq 1 \) and \( s < d \). Then consider \( T(1) = (1, 2, \ldots, s_{i-1}, s, s_{i+k}, t_{i+k+1}, \ldots, t_j), \) and \( T(2) \) defined by \( T(2)_u = t_{u+s} - s \) for \( u \leq i - s \). Let \( p_i \in G_T \) be a point parametrizing the graded ideal \( A \) such that \( A = R/I \) satisfies \( H(A) = T \). Then we let \( I(1) = (f_s, I) \). We have \( I_{u+s} = f_s V_u \) for \( 0 \leq u \leq i - s \); we define \( I(2)_u = V_u \) for \( u \in [0, i - s] \).
Then the pair \((I(1), I(2))\) determines \( I \) and conversely. This proves the
Lemma for \( k = 2 \), it is straightforward to extend it to \( k \geq 2 \). \( \square \)
Remark 5.21. Let $T = (1, 2, \cdots, d, t_d, \cdots, t_i, 0)$ be a Hilbert function as in Equation (1.1) and $P$ a partition of diagonal lengths $T$. Suppose there is an integer $i \in [d, j - 1]$ such that $t_i = t_{i+1} = \cdots = t_{i+k}$ with $k \geq 1$ and $s < d$. Let $I$ be a generic element of the cell $\mathbb{V}(E_P)$ and $G(I)$ a minimal set of generators of $I$. Let $G(I)_1 = \{ f \in G(I), \text{degree}(f) \geq i + 1 \}$ and $G(I)_2 = \{ f \in G(I), \text{degree}(f) \leq i \}$. Setting $m_1 = |G(I)_1|$ and $m_2 = |G(I)_2|$, we get $\kappa(P) = m_1 + m_2$. We know from Equation (5.4) that there is a degree $s$ form $f_s$ such that $f_s$ divides each of the elements $f_1, \ldots, f_{m_2}$ of $G(I)_2$. Let $I(1) = (f_s, I)$ and $I(2) = (I : f_s)$. Then $I(1)$ is a generic element of a cell $\mathbb{V}(E_{P(1)})$ and $I(2)$ is a generic element of a cell $\mathbb{V}(E_{P(2)})$. It is clear that $\kappa(P(1)) = m_1 + 1$ and $\kappa(P(2)) = m_2$, so $\kappa(P) = \kappa(P(1)) + \kappa(P(2)) - 1$.

Proposition 5.22. Suppose that the variety $G_T$ decomposes as $G_T = \prod_{k=1}^{k=r} G_{T(k)}$ with each $T(k)$ elementary. Then any cell $\mathbb{V}(E_P)$ of $G_T$ decomposes as
\[
\mathbb{V}(E_P) = \prod_{k=1}^{k=r} \mathbb{V}(E_{P(k)}) \text{ for } P(k) \text{ a partition of diagonal lengths } T(k).
\]
Also $\kappa(P) = \sum_{k=1}^{k=r} \kappa(P(k)) - r + 1$.

Proof. The Proposition follows from Remark 5.21.$\square$

Example 5.23. (See Figure 20) Let $P = (10^2, 4, 3^2, 2^5)$ be the partition of diagonal lengths
\[
T = (1, 2, \ldots, 6_5, 5_6, 4_7, 4_8, 4_9, 2_{10}, 0),
\]
and difference-one hook code
\[
\mathfrak{Q}(P) = ((0)_6, (1, 0)_9, (2, 1)_{10}).
\]
Let $I$ be a generic element of $\mathbb{V}(E_P)$. The elementary components of $T$, explained in Lemma 5.20 and Remark 5.21 are
\[
T(1) = (1, 2, 3, 4, 4_4, 5_5, \ldots, 4_9, 2_{10}, 0), \quad \text{and } T(2) = (1, 2, 1).
\]
As it is explained in Remark 5.21 we let $I(1) = (f_4, I)$ be a generic element in the cell $\mathbb{V}(E_{P(1)})$ with the Hilbert function $T(1)$. Also $I(2) = (I : f_4)$ is a generic element in the cell $\mathbb{V}(E_{P(2)})$ with the Hilbert function $T(2)$. We have $P(1) = (10^2, 2^8)$ and $P(2) = (2, 1^2)$ which are subpartitions of $P$ in different colors in Figure 20. We easily see that $\kappa(P(1)) = \kappa(P(2)) = 3$ and therefore by Proposition 5.22 we get
\[
\kappa(P) = \kappa(P(1)) + \kappa(P(2)) - 1 = 5.
\]
We could also compute \( \kappa(P) \) by decomposition of \( P \) and \( T \) into single-block components, see Equation 2.4. Single-block component partitions \( P_6, \ldots, P_{10} \) of diagonal lengths \( T_6, \ldots, T_{10} \) as follows,

\[
P_6 = (2,1,1), \quad P_7 = (1), \quad P_8 = (0), \quad P_9 = (3,1,1) \quad \text{and} \quad P_{10} = (4,4,2,2),
\]
\[
T_6 = (1,2,1), \quad T_7 = (1,0), \quad T_8 = (0), \quad T_9 = (1,2,2,0) \quad \text{and} \quad T_{10} = (1,2,3,4,2,0).
\]

The hook codes of \( P_6, P_9 \) and \( P_{10} \) are \( h_6 = (0) \), \( h_9 = (1,0) \) and \( h_{10} = (2,1) \) respectively. Using Theorem 3.11 we get that

\[
\kappa(P_6) = \kappa(P_9) = \kappa(P_{10}) = 3,
\]

and for \( P_7 = \Delta_1 \) and \( P_8 = \Delta_0 \) by Remark 2.21 we conclude that

\[
\kappa(P_7) = 2, \quad \kappa(P_8) = 1.
\]

Therefore, Theorem 5.18 implies that

\[
\kappa(P) = 3 + 2 + 1 + 3 + 3 - (4 - 2) - (10 - 6) = 5
\]

We also note that of these five generators, two generators have degree 6 and one generator has degree 7 (corresponding to generators of \( P_6 \)), and two have degree 10.

Note that \( \dim G_{T_{10}} = 2(3) = 6 \), \( \dim G_{T_9} = (2)(1) = 2 \), and \( \dim G_T = 8 \), since \( G_T \) is fibred over \( \mathbb{P}_4 \) parametrizing the generator \( f_4 \) of \( I_6 \) by a Grassmannian Grass(2,4) parametrizing \( I_{10}/f_4 R_6 \), a two-dimensional subspace of \( R_{10}/f_4 R_6 \), which has dimension four.

6 Number of cells of special multiblock partitions.

Using Corollary 4.3 and Theorem 5.15 we are able to count the number of multiblock partitions with a given number of generators.
Theorem 6.1. Assume that \( T = (1, \ldots, d, t_d, \ldots, t_j, 0) \) and for \( d \leq i \leq j \), let \( T_i = (1, \ldots, t_i - t_{i+1}, t_i - t_{i+1}, 0) \). Then for every positive integer \( k \), the number of partitions \( P \) of diagonal lengths \( T \) and \( \kappa(P) = k \), denoted by \( \mu(T, k) \), satisfies

\[
\mu(T, k) = \sum_{(k_d, \ldots, k_j) \in Q_k} \left( \prod_{i=d}^{j} \mu(T_i, k_i) \right),
\]

where \( Q_k = \{(k_d, \ldots, k_j) \in \mathbb{Z}^{j+1-d} \mid k_d + \cdots + k_j = k + (t_d - d) - (t_j - j)\} \).

Proof. This is an immediate consequence of Theorem 5.1. Also recall that Corollary 4.3 provides an explicit formula for \( \mu(T, k) \), for every \( d \leq i \leq j \).

Remark 6.2. For each \( i \in [d,j] \), by Corollary 4.3, \( \mu(T, k) \) is non-zero if and only if \( \max\{t_i - t_{i+1} + 1, t_{i-1} - t_i + 1\} + 1 \leq k_i \leq t_{i-1} - t_{i+1} + 1 \). Thus in Equation (6.1) we are effectively taking the sum over the points in the hyperplane defined by \( k_d + \cdots + k_j = k + (t_d - d) - (t_j - j) \) in the hyper cubes obtained by the product of line segments of the form \( [\max\{t_i - t_{i+1}, t_{i-1} - t_i\} + 1, t_{i-1} - t_{i+1} + 1] \) in \( \mathbb{Z}^{j+1-d} \).

Recall that \( \mathcal{P}(T) \) is the set of all partitions of diagonal lengths \( T \). Denote by \( A \) the cardinality of \( \mathcal{P}(T) \). We have from [Y, Theorem 3.30], or as a consequence of Equation 6.4 below, that

\[
A = \prod_{d \leq i \leq j} \left( \frac{t_i - t_{i+1} + 1}{t_i - t_{i+1}} \right).
\]

A refinement, grading by the dimension of the cells, gives the Betti numbers of \( \text{Gr}_T \) [Y, Equation 3.34]. Recall from Definition 3.4 that a partition \( P \) of diagonal lengths \( T \) is called special if \( \kappa(P) > \kappa(T) \), and denote by \( S \) the number of special partitions of diagonal lengths \( T \).

Using Definition 2.17, we decompose a partition \( P \) of diagonal lengths \( T = (1, \ldots, d, t_d, \ldots, t_j, 0) \) into \( j+1-d \) single-block partitions, \( P_d, \ldots, P_j \), where for each \( d \leq i \leq j \), the diagonal lengths of \( P_i \) is the sequence \( T_i = (1, \ldots, t_i - t_{i+1}, t_i - t_{i+1}, 0) \). For each \( d \leq i \leq j \), we denote the total number of partitions of diagonal lengths \( T_i \) by \( A_i \) (see Lemma 4.1 and Equation (6.4) below) and the number of special partitions of diagonal lengths \( T_i \) by \( S_i \). The number of special partitions is equal to \( \sum_{k > \kappa(T)} \mu(T, k) \), where \( \mu(T, k) \) is described in the above theorem.

In the following, we provide the number of special partitions of diagonal lengths \( T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0) \), using the inclusion-exclusion principal.

Corollary 6.3 (Number of special partitions). The number of special partitions of diagonal lengths \( T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0) \) is equal to

\[
S = \sum_{i=1}^{j-d+1} (-1)^{i+1} \left( \sum_{\lambda \subseteq \{d, \ldots, j\}, |\lambda| = i} S_{\lambda} A_{\{d, \ldots, j\} \setminus \lambda} \right),
\]

(6.3)
where \( S_\lambda = \prod_{i \in \lambda} S_i \) and \( A_{\{d,\ldots,j\} \setminus \lambda} = \prod_{i \in \{d,\ldots,j\} \setminus \lambda} A_i \).

**Proof.** Theorem 5.18 implies that \( P \) is special if and only if \( P_i \) is special for some \( i \in [d, j] \).

Note that for each \( i \in [d, j] \) the number of partitions of diagonal lengths \( T_i \) is equal to

\[
A_i = \left( t_{i-1} - t_{i+1} + 1 \right). \tag{6.4}
\]

On the other hand, Theorem 4.2 provides the number of special single-block partitions. Using Equation 4.3 for each \( d \leq i \leq j \) we obtain the number of special partitions of diagonal lengths \( T_i \) as the following

\[
S_i = \left( t_{i-1} - t_{i+1} + 1 \right). \tag{6.5}
\]

where \( \delta_i = \max\{2t_i - 2t_{i+1} - t_{i-1} + t_{i+1}, 0\} = \max\{2t_i - t_{i+1} - t_{i-1}, 0\} \).

Now using the inclusion-exclusion principal we get the equality of Equation (6.3). \( \square \)

As a consequence of the above Theorem, we recover a result of [AIK, Theorem 3.7] providing the number of complete intersection Jordan types \( P \in \mathcal{P}(T) \). Recall that a complete intersection Jordan type of diagonal lengths \( T \) is a partition \( P \) of diagonal lengths \( T \) such that \( \kappa(P) = 2 \).

**Corollary 6.4.** (a) The number of complete intersection Jordan types of diagonal lengths

\[
T = (1_0, 2_1, \ldots, (d - 1)_d, d_{d-1}, (d - 1)_d, \ldots, 2_{2d-3}, 1_{2d-2})
\]

is equal to \( 2^{d-1} \).

(b) The number of complete intersection Jordan types with diagonal lengths

\[
T = (1_0, 2_1, \ldots, (d - 1)_d, d_{d-1}, \ldots, d_{d+k-2}, (d - 1)_{d+k-1}, \ldots, 2_{2d-4+k}, 1_{2d-3+k})
\]

where \( k \geq 2 \) is equal to \( 2^d \).

**Proof.** (a) In this case we have that \( j = 2d - 2 \) and the number of blocks in this case is equal to \( d - 1 \), we also have \( t_{d-1} = d, t_d = d - 1, \ldots, t_j = 1 \).

For each \( d \leq i \leq j \) we have that \( T_i = (1, 2, 1) \), and clearly \( A_i = 3 \) and \( S_i = 1 \). So the total number of partitions of diagonal lengths \( T \) is \( A = 3^{d-1} \). On the other hand, using (6.3), we obtain the number of special partitions

\[
S = \sum_{i=1}^{d-1} (-1)^{i+1} \sum_{\lambda \subseteq \{d,\ldots,2d-2\}, |\lambda| = i} 1^i \cdot 3^{d-1-i}
\]

\[
= \sum_{i=1}^{d} (-1)^{i+1} \binom{d - 2}{i} 3^{d-1-i}
\]

\[
= 3^{d-1} - 2^{d-1}.
\]
Thus the number of complete intersection Jordan types with the Hilbert function in \((a)\) is equal to \(A - S = 2^{d-1}\).

\((b)\) In this case we have that \(j = 2d - 3 + k\) and \(t_{d-1} = \ldots = t_{k+d-2} = d, t_{k+d-1} = d - 1, \ldots, t_{2d+k-3} = 1\). For each \(i \in [d, d + k - 3]\) we have \(T_i = 0\) and clearly \(A_i = 1\) and \(S_i = 0\). We have \(T_{d+k-2} = (1,1)\), so \(A_{d+k-2} = 2\) and \(S_{d+k-2} = 0\). There are \(d - 1\) more components for each \(i \in [d + k - 1, 2d + k - 3]\) where \(T_i = (1,2,1)\), \(A_i = 3\) and \(S_i = 1\), similar to the previous case. So the total number of partitions in this case is \(A = 2 \cdot 3^{d-1}\)

Using Equation (6.3) we obtain the number of special partitions

\[
S = \sum_{i=1}^{d+k-2} (-1)^{i+1} \sum_{\lambda \subseteq \{d,\ldots,2d-2\}, |\lambda| = i} 1^i \cdot 3^{d-1-i} \cdot 2
\]

\[
= 2 \sum_{i=1}^{d-1} (-1)^{i+1} \binom{d+k-3}{i} 3^{d-1-i}
\]

\[
= 2 \cdot 3^{d-1} - 2^d.
\]

Therefore the number of complete intersection Jordan types in this case is equal to \(A - S = 2^d\).  \(\square\)
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