A NOTION OF ROBUSTNESS FOR CYBER-PHYSICAL SYSTEMS

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ABSTRACT. Robustness as a system property describes the degree to which a system is able to function correctly in the presence of disturbances, i.e., unforeseen or erroneous inputs. In this paper, we introduce a notion of robustness termed input-output dynamical stability for cyber-physical systems (CPS) which merges existing notions of robustness for continuous systems and discrete systems. The notion captures two intuitive aims of robustness: bounded disturbances have bounded effects and the consequences of a sporadic disturbance disappear over time. We present a design methodology for robust CPS which is based on an abstraction and refinement process. We suggest several novel notions of simulation relations to ensure the soundness of the approach. In addition, we show how such simulation relations can be constructed compositionally. The different concepts and results are illustrated throughout the paper with examples.

1. INTRODUCTION

Robustness describes the ability of a system to function correctly in the presence of disturbances, e.g., unmodeled dynamics or unforeseen events. Disturbances arise whenever certain assumptions imposed on the system or the environment at design-time are violated during run-time. Since a system and its environment are only partly known at design-time, disturbances are unavoidable and robustness of is natural requirement in every system design.

In this paper we present a methodology for the design of robust Cyber-Physical Systems (CPS). We establish robustness with respect to continuous disturbances, possibly arising from sensor noise or actuator imprecisions, as well as discrete disturbances to account for potential failures of the cyber components, like faulty communication channels, hardware or software errors.

Technically, we formalize robustness as input-output dynamical stability as a formal notion of robustness of CPS, combining well-known notions of robustness for control systems, such as input-to-state stability [30] and input-to-state dynamical stability [11], with a recently introduced notion of robustness for discrete systems [34, 33]. Input-output dynamical stability provides two guarantees that are intuitively related with a robust design: first, bounded disturbances have bounded consequences and second, the nominal system behavior is eventually resumed after the occurrence of a sporadic disturbance.

We provide a computational framework based on a three-step abstraction and refinement procedure. The first step, consists of computing a discrete abstraction or symbolic model, i.e., a finite-state substitute of a given CPS. In the second step,
we employ the algorithms developed in \cite{34,33} to synthesize a robust controller for the symbolic model. The last step, consist in the refinement of the controller obtained on the abstract domain to the concrete CPS.

We follow the usual approach, which is based on simulation relations and alternating simulation relations, to ensure the soundness of the abstraction and refinement scheme. Simulation relations provide a mathematical tool to compare the dynamical behavior of a concrete system and its symbolic model in terms of behavioral inclusion. In this paper, we enrich the well-known constructs of (alternating) simulation relations \cite{18,1,9,31} to facilitate the comparison of two systems in terms of robustness.

We recently introduced in \cite{26} contractive simulation relations to capture a certain stability or contraction property that is often observed in the concrete system \cite{31,23,24,10} with the goal to reduce the complexity of the symbolic models. The focus of \cite{26} was the verification of robustness using contractive simulation relations. In this paper we focus on the synthesis of robust controllers which naturally leads to the notion of contractive Alternating Simulation Relations (ASR). By using contractive ASR we are allowed to ignore continuous disturbances on the abstract domain, while still providing robustness of the CPS with respect to continuous as well as discrete disturbances. As we will illustrate with an example in Section 7 this might lead to a separation of concerns, where a continuous design caters to continuous disturbances and the discrete design on the abstract domain caters to discrete disturbances. Yet, the refined design provides robustness with respect to both continuous and discrete disturbances.

While it is straightforward to construct symbolic models together with contractive ASR for continuous control systems following the methods presented in \cite{31,23,24,10}, it is less clear how to construct such models and relations for CPS. In Section 6 we provide a compositional scheme. This approach is in particular useful for CPS, since the overall symbolic model of the CPS can be constructed from the individual symbolic models of the physical part and the cyber part of the CPS.

In summary, the contribution in this paper as follows: 1) we introduce input-output dynamical stability as formal notion of robustness of CPS and propose an abstraction/refinement scheme for the synthesis of robust controllers; 2) we show how to refine a robust design found on the abstract domain to a robust design on the concrete domain whenever the symbolic model is related to the concrete system by an ASR; 3) when using contractive ASR we tailor the design on the abstract domain to discrete disturbances, while ensuring the robustness of the refined design with respect to continuous and discrete disturbances; 4) we provide a compositional scheme to the construction of symbolic models of CPS.

1.1. Related work. Robustness has been studied in the control systems community for more than fifty years, see \cite{38}, and formalized in many different ways including operator finite gains, bounded-input bounded-output stability, input-to-state stability, input-output stability, and several others, see e.g. \cite{30}. Moreover, robustness investigations have been conducted for different system models such as continuous-time systems, sampled-data systems, networked control systems, and general hybrid systems \cite{19,20,4,28}. The notion of robustness described in this paper benefited from all this prior work and was directly inspired by input-to-state stability \cite{30} and its quantitative version: input-to-state dynamical stability \cite{11}.
Unlike the framework presented in this paper, most of the existing research on robustness of nonlinear control systems does not consider constructive procedures for the verification and controller synthesis enforcing robustness. The only exceptions known to the authors are [12, 39, 13]. Unfortunately, the finite-state models that are used in those approaches represent approximations of the concrete dynamics, rather than abstractions. Hence, the soundness of those methods is not ensured.

Robustness for discrete systems also has a long standing history. For example, Dijkstra’s notion of self-stabilizing algorithms in the context of distributed systems [7] requires the “nominal” behavior of the system to be resumed in finitely many steps after the occurrence of a disturbance. As explained in [34, 33], self-stabilizing systems are a special case of robust systems, as defined in this paper. In addition to self-stabilization, there exist several different notions of robustness for discrete systems. For example, in [29] a systematic literature review is presented, where the authors distill and categorize more than 9000 papers on software robustness. In the following, we focus on the few approaches that provide quantitative measures of robustness for discrete systems and thereby are close to the framework presented in this paper.

Let us first mention two notions of robustness for systems over finite alphabets [35] and reactive systems [3] that we think are the closest to the definition of robustness discussed in this paper. Similarly to our methodology, the deviation of the system behavior from its “nominal” behavior as well as the disturbances are quantified. A system is said to be robust if its deviation from the “nominal” behavior is proportional to the disturbance causing that deviation. Although, this requirement captures the first intuitive goal of robustness, those definitions do not require that the effect of a sporadic disturbance disappears over time. See [34, 33] for a more rigorous comparison of the robustness definitions.

Note that the work in [3] on reactive systems demonstrates how to quantify disturbances and their effects on the system behavior in order to characterize safety specifications in terms of robustness inequalities. However, it is unclear how to quantify disturbances and their effects in order to encode liveness specifications. Some possible notions are given in [2, 8, 36], where the robustness of a system is expressed as the ratio of the number of assumptions and guarantees the system meets. Those notions of robustness are incompatible with our definition of robustness, and further work is needed if we would like to express liveness specifications through the notion of robustness presented in this paper.

There exist different studies that characterize the robustness of discrete systems in terms of a Lyapunov function, as it is done in [6, 21] for discrete event systems, or in [15] for $\omega$-regular automata and in [25] for software programs. Note that Lyapunov functions represent a tool to establish robustness inequalities, but do not provide a direct quantification of the effect of disturbances on the system behavior. Hence, further work is needed to related Lyapunov functions, like those presented in [6, 21, 15, 25], to a robustness inequality that directly quantifies the consequences of disturbances on the system behavior.

Another interesting method to characterize robustness for programs is outlined in [16] and [5]. Programs are interpreted as function that map input data to output data. A program is said to be robust if the associated input-output function is continuous. In comparison to our approach, in [16] a program is assumed to
terminate on all inputs and is interpreted as a static function while we consider CPS whose executions are non-terminating.

A preliminary version of this contribution appears in [27] where we announce the main results presented in this paper. In comparison to [27], we provide detailed proofs of all statements. Moreover, the result on the compositional construction of contractive alternating simulation relations presented in Section 6 is new.

2. Preliminaries

We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural numbers and by $\mathbb{B}_n(r)$ the closed ball centered at $x \in \mathbb{R}^n$ with radius $r \in \mathbb{R}_{\geq 0}$. We identify $\mathbb{B}(r)$ with $\mathbb{B}_0(r)$. We use $|x|$ and $|x|_2$ to denote the $\infty$-norm and two-norm of $x \in \mathbb{R}^n$, respectively. Given $x \in \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$, we use $|x|_A := \inf_{x' \in A} |x-x'|_2$ to denote the Euclidean distance between $x$ and $A$. Given a set $A \subseteq \mathbb{R}^n$ we use $[A]_\eta := \{x \in A \mid \exists k \in \mathbb{Z}^n : x = 2k\eta\}$ to denote a uniform grid in $A$. For $a, b \in \mathbb{R}$ with $a \leq b$, we denote the closed, open, and half-open intervals in $\mathbb{R}$ by $[a, b]$, $]a, b[\text{, } [a, b]\text{ and }]a, b[$, respectively. For $a, b \in \mathbb{Z}$, $a \leq b$ we use $[a; b]$, $]a; b[\text{, } [a; b]\text{ and }]a; b[$, to denote the corresponding intervals in $\mathbb{Z}$.

Given a function $f : A \rightarrow B$ and $A' \subseteq A$ we use $f(A') := \{f(a) \in B \mid a \in A'\}$ to denote the image of $A'$ under $f$. A set-valued function or mapping $f$ from $X$ to $Y$ is denoted by $f : X \rightrightarrows Y$. Its domain is defined by $\text{dom} f := \{x \in X \mid f(x) \neq \emptyset\}$.

Given a sequence $a : \mathbb{N} \rightarrow A$ in some set $A$, we use $a_i$ to denote its $i$-th element and $a_{i:j}$ to denote its restriction to the interval $[0; t]$. The set of all finite sequences is denoted by $A^*$. The set of all infinite sequences is denoted by $A^\omega$ and we think of elements $a \in A^\omega$ as sequences $a : \mathbb{N} \rightarrow A$. Given a relation $R \subseteq A \times B$ we use $\pi_A(R)$ and $\pi_B(R)$ to denote its projection onto the set $A$ and $B$, respectively.

We use the following classes of comparison functions:

- $\mathcal{K} := \{\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \mid \alpha \text{ is continuous and strictly increasing with } \alpha(0) = 0\}$
- $\mathcal{L} := \{\alpha : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} \mid \alpha \text{ is strictly decreasing with } \lim_{t \rightarrow \infty} \alpha(t) = 0\}$
- $\mathcal{KL} := \{\beta : \mathbb{R}_{\geq 0} \times \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} \mid \forall t \in \mathbb{N} : \beta(\cdot, t) \in \mathcal{K} \text{ and } \forall c \in \mathbb{R}_{\geq 0} : \beta(c, \cdot) \in \mathcal{L}\}$
- $\mathcal{KL}D := \{\beta \in \mathcal{KL} \mid \forall c \in \mathbb{R}_{\geq 0}, s, t \in \mathbb{N} : \beta(c, 0) = c \land \beta(c, s+t) = \beta(c, s) + t\}$

Note that we work only with discrete-time systems and for this reason we have defined the domain of class $\mathcal{L}$ functions as $\mathbb{N}$.

3. Robustness for CPS

Since CPS exhibit a rich dynamical behavior through the interaction of discrete and continuous components we need an adequate mathematical description that is able to represent its complex dynamics. We use a general notion of transition system as the underlying model of CPS.

**Definition 1.** A system $S$ is a tuple $S = (X, X_0, U, r)$ consisting of

- a set of states $X$;
- a set of initial states $X_0 \subseteq X$;
- a set of inputs $U$ containing the distinguished symbol \(\perp\);
- a transition map $r : X \times U \rightrightarrows X$.

A behavior of $S$ is a pair of sequences $\langle \xi, \nu \rangle \in (X \times U)^\omega$, that satisfies $\xi_0 \in X_0$ and $\xi_{t+1} = r(\xi_t, \nu_t)$ for all times $t \in \mathbb{N}$.

A state $x \in X$ is called reachable if there exists $T \in \mathbb{N}$ and sequences $\xi \in X_T$, $\nu \in U_T^{-1}$ with $\xi_{t+1} = r(\xi_t, \nu_t)$ for all $t \in [0; T]$, $\xi_0 \in X_0$, and $\xi_T = x$. 


A system is called non-blocking if \( r(x,u) \neq \emptyset \) for any reachable state \( x \) and any \( u \in U \). It is called finite if \( X \) and \( U \) are finite sets and otherwise it is called infinite.

Behaviors are defined as infinite sequences since we have in mind reactive systems, such as control systems, that are required to interact with its environment for arbitrarily long periods of time. In particular, we are interested in understanding the effect of disturbances on the system behavior. Therefore, the inputs in \( U \) are to be interpreted as disturbance inputs. Nevertheless, in order to allow for the possibility of absence of disturbances, we assume that \( U \) contains a special symbol \( \perp \in U \) that indicates that no disturbance is present.

For simplicity of presentation, we assume throughout this section that the system is non-blocking, i.e., for every state and (disturbance) input there exists at least one successor state to which the system can transition.

In order to be able to talk about robustness properties, we endow our notion of system with cost functions \( I \) and \( O \) that we use to describe the desired behavior and to quantify disturbances.

**Definition 2.** A system with cost functions is a triple \((S, I, O)\) where \( S \) is a system and \( I : X \times U \rightarrow \mathbb{R}_{\geq 0} \) and \( O : X \times U \rightarrow \mathbb{R}_{\geq 0} \) are the input cost function and output cost function, respectively.

We now introduce a notion of robustness following well-known notions of robustness for control systems, see e.g. [30]. In particular, we follow the notion of input-to-state dynamical stability introduced in [11] and generalize it here to CPS using the cost functions \( I \) and \( O \).

**Definition 3.** Let \((S, I, O)\) be a system with cost functions, \( \gamma \in K, \mu \in KLD \) and \( \rho \in \mathbb{R}_{\geq 0} \). We say that \( S \) is \((\gamma, \mu, \rho)\)-practically input-output dynamically stable \((\gamma, \mu, \rho)\)-pIODS with respect to \((I, O)\) or that \((S, I, O)\) is \((\gamma, \mu, \rho)\)-pIODS if the following inequality holds for every behavior of \( S \):

\[
O(\xi_t, \nu_t) \leq \max_{t' \in [0; t]} \mu(\gamma(I(\xi_{t'}, \nu_{t'}))), t - t' + \rho, \quad \forall t \in \mathbb{N}.
\]  

We say that \((S, I, O)\) is pIODS if there exist \( \gamma \in K, \mu \in KLD \) and \( \rho \in \mathbb{R}_{\geq 0} \) such that \((S, I, O)\) is \((\gamma, \mu, \rho)\)-pIODS.

We say that \((S, I, O)\) is \((\gamma, \mu)\)-IODS if it is \((\gamma, \mu, 0)\)-pIODS, and IODS if there exist \( \gamma \in K, \mu \in KLD \) such that \((S, I, O)\) is \((\gamma, \mu)\)-IODS.

If the cost functions are clear from the context or are irrelevant to the discussion, we abuse the terminology and call a system \( S \) pIODS/IODS without referring to the cost functions.

In our previous work [33, 34] we used IODS as a notion of robustness for cyber systems. The underlying model were transducers, i.e., maps \( f : U^* \rightarrow Y^* \) that process input streams in \( U^* \) into output streams in \( Y^* \). In that framework, the cost functions were defined on sequences of input symbols and output symbols, i.e., \( I : U^* \rightarrow \mathbb{N} \) and \( O : Y^* \rightarrow \mathbb{N} \). In order to formulate such cost functions in the current framework we can compose the transducers computing the input and output costs with the system being modeled so that input and output costs are readily available as functions on the states and inputs of the composed system.

Let us describe how the IODS inequality \([11]\) realizes the intuitive notion of robustness described in the introduction. For the following discussion, suppose we
are given a system with cost functions \((S, I, O)\) that is \((\gamma, \mu)\)-IODS. We use the output cost to specify preferences on the system behaviors: less preferred behaviors have higher costs. In particular, the cost should be zero for the nominal behavior. Similarly, we use the input costs to quantify the disturbances. Hence, the input costs should be zero if no disturbances are present, i.e., \(I(\xi_t, \nu_t) = 0\) when \(\nu = \perp^\infty\). Since, \(\gamma(0) = 0\) and \(\mu(0, s) = 0\) for all \(s \in \mathbb{N}\), zero input cost implies zero output cost which, in turn, implies that the system follows the desired behavior. Moreover, inequality (1) implies that bounded disturbances lead to bounded deviations from the nominal behavior. Suppose \(I(\xi_t, \nu_t) \leq c\) holds for some \(c \in \mathbb{R}_{\geq 0}\) for all \(t \in \mathbb{N}\). Note that \(\gamma\) is monotonically increasing and \(\mu(c, t) \leq \mu(c, 0) = c\) holds for all \(t \in \mathbb{N}\). Therefore, (1) becomes

\[
O(\xi_t, \nu_t) \leq \gamma(c) \quad \forall t \in \mathbb{N}.
\]

In addition, inequality (1) ensures that the effect of a sporadic disturbance vanishes over time. Suppose there exists \(t' \in \mathbb{N}\) after which the input cost is zero, i.e., \(I(\xi_t, \nu_t) = 0\) for all \(t \geq t'\). Then it follows from the definition of \(\mu \in KLD\) that

\[
\mu(\gamma(I(\xi_{t'}, \nu_{t'})), t - t') \to 0, \quad t \to \infty.
\]

Hence, the output cost is forced to decrease to zero as time progresses.

We refer the reader to our previous work [33, 34] for a further demonstration of the usefulness of inequality (1) to express robustness of cyber systems. We showed in [33, 34] that verifying if a cyber system is robust can be algorithmically solved in polynomial time. Similarly, the problem of synthesizing a controller to enforce robustness of a cyber system is solvable in polynomial time. Moreover, we provided some examples of robust cyber systems in the sense of inequality (1).

4. Preservation of IODS by Simulation Relations

In this section we introduce simulation relations between two systems and answer the following question:

**Under what conditions is \(pIODS\) preserved by simulation relations?**

We consider three different types of relations: exact simulation relations (SR), approximate simulation relations (aSR) and approximate contractive simulation relations (acSR).

Bisimilarity and (bi)simulation relations were introduced in computer science by Milner and Park in the early 1980s, see e.g. [18], and have proven to be a valuable tool in verifying the correctness of programs. Approximate SR [9, 23, 32] have been introduced in the control community as a generalization of SR in order to enlarge the class of systems which admit discrete abstractions (or symbolic models). We refine the notion of aSR to acSR, with the aim of capturing a contraction property that is often observed in concrete systems, see e.g. [14, 22, 23]. Intuitively, the existence of a SR from system \(S\) to system \(\hat{S}\) implies that for every behavior of \(S\) there exists a behavior of \(\hat{S}\) satisfying certain properties. In the classical setting, one would ask that the output of the two related behaviors coincides, from which behavioral inclusion follows. For our purposes, as we want to preserve the IODS inequality, we require that the input costs and output costs satisfy \(\hat{I} \leq I\) and \(O \leq \hat{O}\) along those related behaviors. The satisfaction of these inequalities allows us to conclude that \((\hat{S}, \hat{I}, \hat{O})\) being \(pIODS\) implies that \((S, I, O)\) is \(pIODS\).
For notational convenience we use $R_X := \pi_{X \times \hat{X}}(R)$ to denote the projection of a relation $R \subseteq X \times \hat{X} \times U \times \hat{U}$ on $X \times \hat{X}$. Moreover, we use $U(x) := \{u \in U \mid r(x, u) \neq \emptyset\}$ to denote the set of inputs for which the right-hand-side is non-empty.

4.1. Exact simulation relations.

**Definition 4.** Let $S$ and $\hat{S}$ be two systems. A relation $R \subseteq X \times \hat{X} \times U \times \hat{U}$ is said to be a simulation relation (SR) from $S$ to $\hat{S}$ if:

1. for all $x_0 \in X_0$ exists $\hat{x}_0 \in \hat{X}_0$ such that $(x_0, \hat{x}_0) \in R_X$;
2. for all $(x, \hat{x}) \in R_X$ and $u \in U(x)$ there exists $\hat{u} \in \hat{U}(\hat{x})$ such that
   a. $(x, \hat{x}, u, \hat{u}) \in R$;
   b. for all $x' \in r(x, u)$ there exists $\hat{x}' \in \hat{r}(\hat{x}, \hat{u})$ such that $(x', \hat{x}') \in R_X$.

Let $(S, I, O)$ and $(\hat{S}, \hat{I}, \hat{O})$ be two systems with cost functions. We call a SR $R$ form $S$ to $\hat{S}$ an input-output simulation relation (IOSR) from $(S, I, O)$ to $(\hat{S}, \hat{I}, \hat{O})$ if

$$\hat{I}(\hat{x}, \hat{u}) \leq I(x, u) \quad \text{and} \quad O(x, u) \leq \hat{O}(\hat{x}, \hat{u})$$

(2)

holds for all $(x, \hat{x}, u, \hat{u}) \in R$.

Note that the notion of IOSR for systems with input and output costs is a straightforward extension of the well-known definition of SR for the usual definition of system, see [32].

**Lemma 1.** Let $S$ and $\hat{S}$ be two systems. Suppose there exists an SR $R$ from $S$ to $\hat{S}$, then for every behavior $(\xi, \nu)$ of $S$ there exists a behavior $(\hat{\xi}, \hat{\nu})$ of $\hat{S}$ such that

$$(\xi_t, \hat{\xi}_t, \nu_t, \hat{\nu}_t) \in R, \quad t \in \mathbb{N}.\tag{3}$$

**Proof.** The proof follows by similar arguments as the proof of [32] Proposition 4.9 and is omitted here.

Simulation relations preserve IODS in the following sense.

**Theorem 1.** Let $(S, I, O)$ and $(\hat{S}, \hat{I}, \hat{O})$ be two systems with cost functions and suppose there exists an IOSR $R$ from $(S, I, O)$ to $(\hat{S}, \hat{I}, \hat{O})$. If $(\hat{S}, \hat{I}, \hat{O})$ is $(\gamma, \mu, \rho)$-pIODS then $(S, I, O)$ is $(\gamma, \mu, \rho)$-pIODS.

**Proof.** Since we assume that $(\hat{S}, \hat{I}, \hat{O})$ is $(\gamma, \mu, \rho)$-pIODS, any behavior $(\hat{\xi}, \hat{\nu})$ of $\hat{S}$ satisfies

$$\hat{O}(\hat{\xi}_t, \hat{\nu}_t) \leq \max_{t' \in [0,t]} \mu(\gamma(\hat{I}(\hat{\xi}_{t'}, \hat{\nu}_{t'})), t - t') + \rho$$

for all times $t \in \mathbb{N}$. From Lemma 1 it follows that for every behavior $(\xi, \nu)$ of $S$ there exists a behavior $(\hat{\xi}, \hat{\nu})$ of $\hat{S}$ such that (3) holds. Now we derive the inequality

$$O(\xi_t, \nu_t) \leq \hat{O}(\hat{\xi}_t, \hat{\nu}_t) \leq \max_{t' \in [0,t]} \mu(\gamma(\hat{I}(\hat{\xi}_{t'}, \hat{\nu}_{t'})), t - t') + \rho \leq \max_{t' \in [0,t]} \mu(\gamma(I(\xi_{t'}, \nu_{t'})), t - t') + \rho$$

for all $t \in \mathbb{N}$. The last inequality follows from $\hat{I}(\hat{\xi}_t, \hat{\nu}_t) \leq I(\xi_t, \nu_t)$ and the monotonicity properties of the functions $\gamma$ and $\mu$. Since we can repeat this argument for any behavior $(\xi, \nu)$ of $S$ we see that $(S, I, O)$ is $(\gamma, \mu, \rho)$-pIODS.

\[\square\]
Note how preservation of pIODS is contra-variant, i.e., while the direction of the simulation relation is from system $S$ to system $\hat{S}$, the propagation of pIODS is from system $\hat{S}$ to system $S$. Moreover, by taking $\rho = 0$ it follows that $\hat{S}$ being IODS implies $S$ is IODS.

4.2. Approximate simulation relations. Exact simulation relations are often too restrictive when one seeks to relate a physical system to a finite-state abstraction or symbolic model. In this case, approximate simulation relations were shown to be adequate in the sense that they can be shown to exist for large classes of physical systems \[9, 32\].

Definition 5. Let $(S, I, O)$ and $(\hat{S}, \hat{I}, \hat{O})$ be two systems with cost functions. A SR $R$ from $S$ to $\hat{S}$ is called an $\varepsilon$-approximate input-output SR ($\varepsilon$-aIOSR) from $(S, I, O)$ to $(\hat{S}, \hat{I}, \hat{O})$ if every $(x, \hat{x}, u, \hat{u}) \in R$ satisfies:

$$\hat{I}(\hat{x}, \hat{u}) \leq I(x, u) + \varepsilon \quad \text{and} \quad O(x, u) \leq \hat{O}(\hat{x}, \hat{u}) + \varepsilon. \quad (4)$$

Note that the definition of aIOSR is again a straightforward extension of the well-known notion of approximate SR of systems, see \[32\]. For $\varepsilon = 0$ the notion of exact IOSR is recovered. However, the notion of aIOSR introduces some flexibility as it allows, for example, the inequality $O(x, u) - \varepsilon \leq \hat{O}(\hat{x}, \hat{u}) \leq O(x, u)$ to hold which is not possible for IOSR. This flexibility is important when we are dealing with infinite state systems where an abstract state in $\hat{X}$ corresponds to a set of states in $X$.

Theorem 2. Let $(S, I, O)$ and $(\hat{S}, \hat{I}, \hat{O})$ be two systems with cost functions and suppose there exists an $\varepsilon$-aIOSR $R$ from $(S, I, O)$ and $(\hat{S}, \hat{I}, \hat{O})$. If $(\hat{S}, \hat{I}, \hat{O})$ is $(\gamma, \mu, \rho)$-pIODS, then $(S, I, O)$ is $(\gamma', \mu', \rho')$-pIODS with $\gamma'(\varepsilon) = 2\gamma(2\varepsilon)$ and $\rho' = \mu(\gamma'(\varepsilon), 0) + \varepsilon + \rho$.

Proof of Theorem 2. Using the same arguments as in the proof of Theorem 1 we choose for any behavior $(\xi, \nu)$ of $S$ a behavior $(\hat{\xi}, \hat{\nu})$ of $\hat{S}$ satisfying \[\Xi\]. Then we obtain:

$$O(\xi_t, \nu_t) \leq \hat{O}(\hat{\xi}_t, \hat{\nu}_t) + \varepsilon \leq \max_{t' \in [0, t]} \mu(\gamma(\hat{I}(\hat{\xi}_{t'}, \hat{\nu}_{t'})), t - t') + \varepsilon + \rho \quad (5)$$

Now we can use Lemma 5 in the appendix with $\mu$, $\gamma$, and $\varepsilon$ to obtain $\gamma'(\varepsilon) = 2\gamma(2\varepsilon)$, $\sigma(\varepsilon) = \mu(2\gamma(2\varepsilon), 0)$ and conclude

$$O(\xi_t, \nu_t) \leq \max_{t' \in [0, t]} \mu(\gamma'(\hat{I}(\hat{\xi}_{t'}, \hat{\nu}_{t'})), t - t') + \sigma(\varepsilon) + \varepsilon + \rho$$

which completes the proof. \[\square\]

4.3. Contractive simulation relations. The construction of abstractions or symbolic models for physical systems described in \[23, 24, 32\] results in simulation relations that satisfy a certain contraction property. Here we introduce a notion of simulation that captures those contraction properties.

In the following definition of contractive simulation relation from $S$ to $\hat{S}$, we use a function $d : U \times \hat{U} \rightarrow \mathbb{R}_{\geq 0}$ to measure the “mismatch” between two inputs $u \in U$ and $\hat{u} \in \hat{U}$. In various examples, in which we show that two systems are related, the set of inputs $\hat{U}$ of system $\hat{S}$ is actually a subset $\hat{U} \subseteq U$ of the set of inputs of
system \(S\) and we simply use a norm \(| · |\) in \(U\) as distance function \(d(u, \hat{u}) = |u - \hat{u}|\), see Example 1 Example 2 and Section 7. However, in the following definition, we simply assume we are given a function \(d : U \times \hat{U} \to \mathbb{R}_{\geq 0}\) without referring to any underlying metric or norm.

**Definition 6.** Let \(S\) and \(\hat{S}\) be two systems, let \(\kappa, \lambda \in \mathbb{R}_{\geq 0}, \beta \in [0, 1]\) be some parameters and consider a map \(d : U \times \hat{U} \to \mathbb{R}_{\geq 0}\). We call a parameterized (by \(\varepsilon \in [\kappa, \infty[\)) relation \(R(\varepsilon) \subseteq X \times X \times U \times \hat{U}\) an-approximate \((\beta, \lambda)\)-contractive simulation relation \(((\kappa, \beta, \lambda)\)-acSR) from \(S\) to \(\hat{S}\) with distance function \(d\) if \(R(\varepsilon) \subseteq R(\varepsilon')\) holds for all \(\varepsilon \leq \varepsilon'\) and for all \(\varepsilon \in [\kappa, \infty[\) we have

\[
\begin{align*}
(1) \forall x_0 \in X_0, \exists \tilde{x}_0 \in \tilde{X}_0 : (x_0, \tilde{x}_0) & \in R_X(\kappa); \\
(2) \forall (x, \tilde{x}) \in R_X(\varepsilon), \forall u \in U(x), \exists \tilde{u} \in U(\tilde{x}) : \\
(a) \quad (x, \tilde{x}, u, \tilde{u}) & \in R(\varepsilon) \\
(b) \quad \forall x' \in r(x, u), \exists \tilde{x}' \in \hat{r}(\tilde{x}, \tilde{u}) : \\
& (x', \tilde{x}') \in X_{\varepsilon + \beta | \varepsilon | + \lambda | \varepsilon | + | d(u, \hat{u}) |).
\end{align*}
\]

Let \((S, I, O)\) and \((\hat{S}, \hat{I}, \hat{O})\) be two systems with cost functions. We call a \((\kappa, \beta, \lambda)\)-acSR \(R(\varepsilon)\) from \(S\) to \(\hat{S}\) with distance function \(d\) a \(\beta\)-approximate \((\beta, \lambda)\)-contractive input-output SR \(((\kappa, \beta, \lambda)\)-acIOSR) from \((S, I, O)\) to \((\hat{S}, \hat{I}, \hat{O})\) with distance function \(d\) if there exist \(\gamma_O, \gamma_I \in \mathbb{K}\) such that

\[
\begin{align*}
\hat{I}(\hat{x}, \hat{u}) & \leq I(x, u) + \gamma_I(\varepsilon') \\
O(x, u) & \leq \hat{O}(\hat{x}, \hat{u}) + \gamma_O(\varepsilon')
\end{align*}
\]

holds for all \((x, \hat{x}, u, \hat{u}) \in R(\varepsilon)\) and \(\varepsilon' = \max\{\varepsilon, d(u, \hat{u})\}\).

Recall that in generalizing IOSR to aIOSR we merely relaxed the inequalities on the costs functions by a constant parameter \(\varepsilon\), compare 2 and 1. Here, we even go one step further, and relax the inequalities using the generalized gain functions \(\gamma_I\) and \(\gamma_O\), where \(\varepsilon'\) in 1 depends on the parameter \(\varepsilon\) that appears in the definition of the acSR \(R(\varepsilon)\) and on the input mismatch measured in terms of \(d\). This change, in combination with the definition of acSR, allows us to quantify the relaxation in the cost function inequalities as a function of the difference of input histories, see Theorem 3 and the subsequent discussion. Before, we make those statements more precise, let us first introduce an example to illustrate the notion of acSR.

**Example 1.** We consider a scalar disturbed linear system

\[
x^+ = 0.6x + u.
\]

on the bounded set \(D := [-1, 1]\). We start our analysis by casting \(7\) as a system \(S\) with \(X := \mathbb{R}, X_0 := D, U := \mathbb{R}\) and \(r(x, u) := \{0.6x + u\}\).

Note that \(D\) is forward invariant with respect to \(7\) in the absence of disturbances, i.e., when \(u = 0\). Later on, we analyze the invariance property in the presence of disturbances. This motivates our choice of cost functions with \(O(x, u) := |x|_D\) and \(I(x, u) := |u|\).

We now introduce a symbolic model \(\hat{S}\) of \(S\) with \(\hat{X} := \{D\}_{0.2}, \hat{X}_0 := \hat{X}, \hat{U} := \{0\}\) and

\[
\hat{x}' \in \hat{r}(\hat{x}, \hat{u}) : \iff |\hat{x}' - 0.6\hat{x}| \leq 0.2.
\]

Note that since \(O(\hat{x}, \hat{u}) = I(\hat{x}, \hat{u}) = 0\) for all \(\hat{x} \in \hat{X}\) and \(\hat{u} \in \hat{U}\), we define the cost functions for \(\hat{S}\) to be \(\hat{O} := 0\) and \(\hat{I} := 0\). We also introduce the relation
$R(\varepsilon) := R_X(\varepsilon) \times \mathbb{R} \times \{0\}$ with

$$R_X(\varepsilon) := \{(x, \hat{x}) \in X \times \hat{X} \mid |x - \hat{x}| \leq \varepsilon\}$$

and show that $R(\varepsilon)$ is a $(0.2, 0.6, 1)$-acSR from $S$ to $\hat{S}$ with distance function $d(u, 0) := |u|$.

Point 1) in Definition 7 is easily verified. Now let $(x, \hat{x}) \in R_X(\varepsilon)$ and $u \in U$. We pick $0 \in \hat{U}$ and observe that $(x, \hat{x}, u, 0) \in R(\varepsilon)$ holds by definition of $R(\varepsilon)$. We proceed with 2.b) of Definition 6. For $x' \in r(x, u)$ there exists $\hat{x}' \in \hat{r}(\hat{x}, 0)$ with

$$|x' - \hat{x}'| \leq 0.2 + 0.6x + u - 0.6\hat{x} \leq 0.2 + 0.6\varepsilon + |u|$$

and it follows that $R(\varepsilon)$ is a $(0.2, 0.6, 1)$-acSR from $S$ to $\hat{S}$. Moreover, the inequalities (1) are satisfied with $\gamma_1 = 0$ and $\gamma_0(\varepsilon) = c$. Hence, $R(\varepsilon)$ is an acIOSR from $(S, I, O)$ to $(\hat{S}, \hat{I}, \hat{O})$.

Let us now emphasize that there exists no $\varepsilon$-aIOSR $\hat{R}$ from $(S, I, O)$ to $(\hat{S}, \hat{I}, \hat{O})$ for any finite symbolic model $\hat{S}$. For the sake of contradiction, suppose there exists an $\varepsilon$-aIOSR $\hat{R}$ from $(S, I, O)$ to $(\hat{S}, \hat{I}, \hat{O})$ and $\hat{S}$ is finite. Since $\hat{S}$ is finite, there necessarily exists a state $\hat{x} \in \hat{X}$ and input $\hat{u} \in \hat{U}$ such that the set of related states and inputs $\{(x, u) \in X \times U \mid (x, \hat{x}, u, \hat{u}) \in \hat{R}\}$ is unbounded. As a consequence, we find for any constant $c \in \mathbb{R}$, a pair $(x, u)$ with $(x, \hat{x}, u, \hat{u}) \in \hat{R}$ so that $O(x, u) = |x|_D > \hat{O}(\hat{x}, \hat{u}) + c$ and $\hat{R}$ cannot be an aIOSR since (4) is violated.

Conversely, if we bound the set of states and inputs of (7) but consider the modified dynamics $x^+ = x + u$, then it is easy to compute a relation $\hat{R}$ that is an $\varepsilon$-aIOSR from $(S, I, O)$ to $(\hat{S}, \hat{I}, \hat{O})$, but there is no acSR from $\hat{S}$ to $S$.

We resume the analysis of this example at the end of this section, where we continue the robustness analysis of the invariance property of $D$ with respect to $S$.

The previous example demonstrates that we can use acIOSR to relate an infinite system $S$ with an unbounded set of states and/or inputs, with a finite system $\hat{S}$, which is not possible using aIOSR.

We point out that any $(\kappa, \beta, \lambda)$-aIOSR $R(\varepsilon)$ from $(S, I, O)$ to $(\hat{S}, \hat{I}, \hat{O})$ is also an aIOSR, whenever the maximal distance between two related elements in $U$ and $\hat{U}$ is bounded. Let $\alpha \in \mathbb{R}_{\geq 0}$ be given such that $d(u, \hat{u}) \leq \alpha$ holds for all $(u, \hat{u}) \in \pi_{U \times \hat{U}}(R(\varepsilon))$ and $\varepsilon \in \mathbb{R}_{\geq 0}$. Now we fix $\varepsilon$ such that $\kappa + \beta \varepsilon + \lambda \alpha \leq \varepsilon$ holds. Note that we can always find such an $\varepsilon$ as we assume $\beta \in [0, 1]$. Then the relation $R' := R(\varepsilon)$ is an aSR from $S$ to $\hat{S}$. This observation follows immediately from the definition of $R(\varepsilon)$ since $\kappa + \beta \varepsilon + \lambda \alpha \leq \varepsilon$ implies that $R(\kappa + \beta \varepsilon + \lambda \alpha) \subseteq R'$ in turn implies that $R'$ is a SR from $S$ to $\hat{S}$. Moreover, if $R(\varepsilon)$ is an acIOSR then $R'$ is an $\varepsilon'$-aIOSR from $(S, I, O)$ to $(\hat{S}, \hat{I}, \hat{O})$ with $\varepsilon' := \max\{\varepsilon, \gamma_0(\max\{\varepsilon, \alpha\}), \gamma_1(\max\{\varepsilon, \alpha\})\}$.

Before we explain how the notions of acsr and acIOSR capture the contraction property of $S$, we provide a result that mimics Lemma 1.

**Theorem 3.** Let $S$ and $\hat{S}$ be systems and let $R(\varepsilon)$ be a $(\kappa, \beta, \lambda)$-aIOSR from $S$ to $\hat{S}$ with distance function $d$. Then there exist $\mu_{\Delta} \in KLD$ and $\gamma_{\Delta}, \kappa_{\Delta} \in \mathbb{R}_{\geq 0}$ such that for every behavior $(\xi, \nu)$ of $S$ there exists a behavior $(\hat{\xi}, \hat{\nu})$ of $\hat{S}$ so that the two behaviors satisfy

$$(\xi_t, \hat{\xi}_t, \nu_t, \hat{\nu}_t) \in R(\varepsilon_t), \quad t \in \mathbb{N},$$

with $\varepsilon_{t+1} \leq \max_{t' \in [0,t]} \mu_{\Delta}(\gamma_{\Delta}d(\nu_{t'}, \hat{\nu}_{t'}), t - t') + \kappa_{\Delta}$.


Proof. First, we show by construction that for every behavior \((\xi, \nu)\) of \(S\) there exists a behavior \((\tilde{\xi}, \tilde{\nu})\) of \(\tilde{S}\) such that \((\xi_t, \tilde{\xi}_t, \nu_t, \tilde{\nu}_t) \in R(\varepsilon_t)\) holds for all \(t \in \mathbb{N}\) where \(\varepsilon_t\) satisfies
\[
\varepsilon_{t+1} = \kappa + \beta \varepsilon_t + \lambda d(\nu_t, \tilde{\nu}_t), \quad \varepsilon_0 = \kappa.
\]
We define the sequences \(\tilde{\xi} : \mathbb{N} \to \tilde{X}\) and \(\tilde{\nu} : \mathbb{N} \to \tilde{U}\) inductively. For the base case \(t = 0\), we choose \(\tilde{x}_0 \in \tilde{X}_0\) such that \((\xi_0, \tilde{x}_0) \in R_X(\kappa)\) and \(\tilde{\nu}_0 \in \tilde{U}\) such that \((\xi_0, \tilde{x}_0, \nu_0, \tilde{\nu}_0)\) satisfies 2.a) with \(\varepsilon_0 = \kappa\) and 2.b) of Definition 8. Now suppose \((\xi_t, \xi_t', \nu_t, \tilde{\nu}_t')\) satisfies 2.a) with \(\varepsilon_t\) satisfying 9 and 2.b) of Definition 6 for all \(t' \in [0; t]\). We choose \(\xi_{t+1} \in \tilde{\nu}(\xi_t, \tilde{\nu}_t)\) such that \((\xi_{t+1}, \tilde{\xi}_{t+1}) \in R(\varepsilon_{t+1})\) which in turn implies that we can fix \(\tilde{\nu}_{t+1} \in \tilde{U}\) such that \((\xi_{t+1}, \tilde{\xi}_{t+1}, \nu_{t+1}, \tilde{\nu}_{t+1})\) satisfies 2.a) with \(\varepsilon_{t+1}\) that satisfies 9 and 2.b) of Definition 6. It follows that \((\tilde{\xi}, \tilde{\nu})\) is a behavior of \(\tilde{S}\) and satisfies the claim.

In the remainder of the proof we use a discrete-time version of [11] Lemma 15 to show that there exist \(\mu_{\Delta} \in \mathcal{KLD}\) and \(\gamma_{\Delta}, \kappa_{\Delta} \in \mathbb{R}_{\geq 0}\) such that we have
\[
\varepsilon_t \leq \max_{t' \in [0; t]} \mu_{\Delta}(\gamma_{\Delta} d(\nu_t, \tilde{\nu}_t), t - t') + \kappa_{\Delta}.
\]

We fix \(\kappa_{\Delta} := \kappa/(1 - \beta), \gamma_{\Delta} := \lambda/(\beta' - \beta)\) and \(g(c) := \beta' c\) for some \(\beta' \in \mathbb{R}_{[1]}\). Now it suffices to verify that \(V(\varepsilon) := \|\varepsilon\|_{\beta \kappa_{\Delta}}\) satisfies \(\gamma_{\Delta} d(\nu_t, \tilde{\nu}_t) \leq V(\varepsilon_t) \implies V(\varepsilon_{t+1}) \leq V(\varepsilon_t)\) holds. Then it follows from [11] Lemma 15 that 10 holds. □

Theorem 8 exposes one of the key features of an acIOSR. The membership \((\xi_t, \tilde{\xi}_t, \nu_t, \tilde{\nu}_t) \in R(\varepsilon_t)\) implies \(O(\xi_t, \nu_t) = O(\tilde{\xi}_t, \tilde{\nu}_t) + \gamma_O(\varepsilon_t)\). Hence, the bound on the output cost \(O\) of \(S\) in terms of the output cost \(O\) of \(\tilde{S}\) depends on the parameter \(\varepsilon_t\) which is time-varying. In comparison to the definition of aIOSR (see [11]) this parameter varies over time. We established with Theorem 8 a bound on \(\varepsilon_t\) in terms of the difference (measured by \(\lambda d\)) of the input histories \(d(\nu_t, \tilde{\nu}_t)\) with \(t' \in [0; t]\). If we are able to match a disturbance \(\nu_t\) of \(S\) closely (in terms of \(d\)) by a disturbance \(\tilde{\nu}_t\) of \(\tilde{S}\), we know that the output cost \(O\) of \(\tilde{S}\) provides a good estimate for the output cost \(O\) of \(S\). Moreover, if after a certain \(t' \in \mathbb{N}\) the difference in the input behaviors is zero, i.e., \(d(\nu_t, \tilde{\nu}_t) = 0\) for all \(t \geq t'\), then the bound on \(\varepsilon_t\) approaches \(\kappa_{\Delta}\) as \(t \to \infty\). Here, we clearly exploit the contraction parameter \(\beta \in [0, 1]\) together with the requirement 2.b) in the Definition 6 where the successor states satisfy \((\xi_{t+1}, \tilde{\xi}_{t+1}) \in R(\kappa + \beta \varepsilon)\) whenever \((\xi_t, \tilde{\xi}_t, \nu_t, \tilde{\nu}_t) \in R(\varepsilon)\) and \(d(\nu_t, \tilde{\nu}_t) = 0\).

With the following corollary, we provide a bound on \(\varepsilon_t\) that depends solely on the behavior \((\xi, \nu)\) of \(S\) and not on the choice of a related behavior \((\tilde{\xi}, \tilde{\nu})\) of \(\tilde{S}\).

**Corollary 1.** Given the premises of Theorem 8, let the function \(\Gamma : X \times U \to \mathbb{R}_{\geq 0} \cup \{\infty\}\) be given by
\[
\Gamma(x, u) := \sup \{d(u, \tilde{u}) \mid \exists \tilde{x}, \exists \tilde{x} : (x, \tilde{x}, u, \tilde{u}) \in R(\varepsilon)\}.
\]

For any two behaviors \((\xi, \nu)\) and \((\tilde{\xi}, \tilde{\nu})\) of \(S\) and \(\tilde{S}\), respectively, that satisfy 8, \(\varepsilon_t\) in 8 is bounded by
\[
\varepsilon_{t+1} \leq \max_{t' \in [0; t]} \mu_{\Delta}(\gamma_{\Delta} \Gamma(\xi_t, \nu_t), t - t') + \kappa_{\Delta}
\]
with \(\kappa_{\Delta} = \kappa/(1 - \beta), \gamma_{\Delta} = \lambda/(\beta' - \beta)\) and \(\mu_{\Delta}(r, t) = (\beta')^t r\) for any \(\beta' \in \mathbb{R}_{[1]}\).

We are now ready to state the main result of this section where we show that pIODS is preserved under acIOSR. As in the in case of SR and aSR, the proof
strategy is to establish a plIODS inequality for $S$ in terms of the pIODS inequality given for $\hat{S}$. For acIODS, the estimates of the cost functions $I$ and $O$ in terms of the cost functions $\hat{I}$ and $\hat{O}$ depend on the time varying parameter $\varepsilon_t$. That is reflected in the following theorem, by a modification of the input costs $I$ of $S$ to $I' = \max \{ I, \Gamma \}$. Here, $\Gamma$ is the function that we used in Corollary 1 to establish a bound on $\varepsilon_t$. It represents the mismatch of the inputs $U$ and $\hat{U}$ measured in terms of $d$.

**Theorem 4.** Let $(S, I, O)$ and $(\hat{S}, \hat{I}, \hat{O})$ be systems with costs functions and suppose there exists a $(\kappa, \beta, \lambda)$-acIOSR $R(\varepsilon)$ from $(S, I, O)$ to $(\hat{S}, \hat{I}, \hat{O})$ with distance function $d$. Then, $(\hat{S}, \hat{I}, \hat{O})$ being plIODS implies that $(S, I', O)$ is plIODS, with $I'(x, u) := \max \{ I(x, u), \Gamma(x, u) \}$ and $\Gamma$ given by (11).

In the proof of Theorem 4 we use two lemmas, Lemma 6 and Lemma 7, which are given in the appendix.

**Proof of Theorem 4.** Let $(\xi, \nu)$ and $(\hat{\xi}, \hat{\nu})$ be a behavior of $S$ of $\hat{S}$, respectively, that satisfy (5). Using the fact that $\hat{S}$ is $(\hat{\gamma}, \hat{\mu}, \hat{\rho})$-plIODS, (6), and Lemma 5 we obtain

$$O(\xi_t, \nu_t) \leq \hat{O}(\hat{\xi}_t, \hat{\nu}_t) + \gamma_O(\varepsilon_t)$$

$$\leq \max_{t' \in [0;t]} \hat{\mu}(\hat{\gamma}(I(\xi_{t'}, \nu_{t'}), t - t')) + \gamma_O(\varepsilon_t) + \hat{\rho}$$

$$\leq \max_{t' \in [0;t]} \hat{\mu}(\hat{\gamma}(I(\xi_{t'}, \nu_{t'}), t - t')) + \gamma_O(\varepsilon_t) + \hat{\rho}$$

with $\hat{\gamma}'(c) = 2\hat{\gamma}(2c)$ and $\gamma_O(c) = \hat{\mu}(\hat{\gamma}(\gamma_I(c), 0) + \gamma_O(c)$. We use the bound on $\varepsilon_t$ from Corollary 1 and obtain

$$O(\xi_t, \nu_t) \leq \max_{t' \in [0;t]} \hat{\mu}(\hat{\gamma}(I(\xi_{t'}, \nu_{t'}), t - t'))$$

$$+ \gamma'_I(\max_{t' \in [0;t]} \mu(\gamma(\gamma_I c, t), t - t')) + \gamma_O(\varepsilon_t) + \hat{\rho}$$

(12)

for $\gamma'_I := \max \{ \gamma_I, 1 \}$ and $\gamma'_I(c) := \gamma_I(2c)$. We use Lemma 6 to choose $\mu'_I \in \mathcal{K}LD$ such that $\gamma'_I(\mu(\gamma_I c, t)) = \mu'(\gamma_I c, t)$. Now we use Lemma 7 to choose $\mu \in \mathcal{K}LD$ such that

$$\max_{t' \in [0;t]} \mu(c, t') + \max_{t' \in [0;t]} \mu'(c, t') \leq \max_{t' \in [0;t]} \mu(2c, t')$$

holds. Then, by defining $\gamma(c) := 2 \max \{ \gamma'_I(c), \gamma'_I(\gamma_I c) \}$ the rhs of (12) is bounded by

$$O(\xi_t, \nu_t) \leq \max_{t' \in [0;t]} \mu(\gamma(\max \{ I(\xi_{t'}, \nu_{t'}), \Gamma(\xi_{t'}, \nu_{t'}) \}, t - t')) + \rho.$$

with $\rho := \gamma_O(\varepsilon_t) + \hat{\rho}$. 

If the inequality $\hat{I} \leq I$ holds, we can provide a plIODS type inequality for $S$ that can be easily described in terms of the parameters of the pIODS inequality of $\hat{S}$.

**Corollary 2.** Given the premises of Theorem 4 suppose $\gamma_O$ satisfies $\gamma_O(r + r') \leq \gamma_O(r) + \gamma_O(r')$ and that $\hat{I}(\hat{x}, \hat{u}) \leq I(x, u)$ holds for all $(x, \hat{x}, u, \hat{u}) \in R(\varepsilon)$ and $(\hat{S}, I, O)$ is $(\hat{\gamma}, \hat{\mu}, \hat{\rho})$-plIODS, then every behavior $(\xi, \nu)$ of $S$ satisfies

$$O(\xi_t, \nu_t) \leq \max_{t' \in [0;t]} \hat{\mu}(\hat{\gamma}(I(\xi_{t'}, \nu_{t'}), t - t')) + \gamma_O(\varepsilon_t) + \hat{\rho}$$

(13)
with \( \gamma'_\Delta(r) = \max\{r, \gamma_\Delta(r)\} \), \( \mu_\Delta \) and \( \kappa_\Delta \) from Corollary 7.

Even though in Theorem 4, contrary to the results in Theorem 1 and Theorem 2, we do not state the parameters \((\mu, \gamma, \rho)\) of the pIODS inequality for \( S \) in dependency of the parameters \((\hat{\mu}, \hat{\gamma}, \hat{\rho})\), inequality (13) provides us with some insights. The first term in the inequality (13) follows from the fact that we were able to successfully verify pIODS for \( \hat{S} \). The second term in (13) accounts for the “mismatch” between the inputs \( U \) and \( \hat{U} \). The last two terms, i.e., the constant offset \( \gamma_O(\kappa_\Delta) + \hat{\rho} \), is a result of the lower bound on the parameter \( \varepsilon \geq \kappa \) and \( \hat{\rho} \) from the pIODS inequality of \( \hat{S} \).

Let us conclude this section with an application of Theorem 4 to Example 1.

**Example 1** (continued). Recall that, every behavior \((\hat{\xi}, \hat{\nu})\) of \( \hat{S} \) satisfies \( \hat{O}(\hat{\xi}_t, \hat{\nu}_t) = 0 \) for all \( t \in \mathbb{N} \). Therefore \((\hat{S}, \hat{I}, \hat{O})\) is \((\hat{\gamma}, \hat{\mu})\)-iods with \( \hat{\gamma} = \hat{\mu} = 0 \). We obtain \( \Gamma \) for this example by \( \Gamma(x, u) = |u| \) and the input cost \( \Gamma' \) coincides with \( I = \max\{I, \Gamma\} = \Gamma' \). In addition, the inequality \( I \leq I \) holds and we can apply Corollary 2 to obtain the pIODS inequality for every behavior \((\xi, \nu)\) of \( S \) as

\[
|\xi|_D \leq \max_{r \in [0,1]} \mu_\Delta(\gamma_\Delta|\nu_r|; t - t') + \kappa_\Delta
\]  

(14)

with \( \kappa_\Delta = 0.2/0.4 \), \( \gamma_\Delta = 1/(\beta' - 0.6) \) and \( \mu_\Delta(r, t) = (\beta')^t r \) for any \( \beta' \in [0.6, 1] \).

Let us shortly describe how this inequality shows the robustness of the invariance of \( D \) with respect to \( S \) against the disturbances \( \nu \). First, let us ignore the constant \( \kappa_\Delta \) on the right-hand-side of (14). Then, the distance between the state \( \xi_t \) and \( D \) is proportional to the norm of the disturbance \( \nu_t \). Moreover, the effect of a disturbance at some time \( t' \) disappears over time since \( \beta^{t-t'} \frac{\gamma_\Delta}{\nu_r} \) approaches zeros as \( t \to \infty \). The constant \( \kappa_\Delta \) appears in (14) because we established the inequality through the use of the symbolic model \( \hat{S} \) and represents the effect of quantization.

5. Controller Design

So far we interpreted the set of inputs \( U \) of a system \( S \) as disturbance inputs over which we had no control. However, in this section, we assume that the input set \( U \) is composed of a set of control inputs \( U^c \) and a set of disturbance inputs \( U^d \), i.e., \( U = U^c \times U^d \). Moreover, we introduce a controller that is allowed to modify the system behavior by imposing restrictions on the control inputs \( U^c \). In our framework, a controller for \( S \) consists of a system \( S_C \) and a relation \( R_C \). The controlled system \( S_C \times R_C \) \( \hat{S} \) is given by the composition of \( S_C \) with \( S \) where \( R_C \) is used to restrict the control inputs \( U^c \) depending on the current state of \( S_C \) and \( S \).

In [34], a synthesis approach has been developed to construct a controller \((\hat{S}_C, \hat{R}_C)\) rendering a finite system \( \hat{S} \) IODS, i.e., the composed system \( \hat{S}_C \times \hat{R}_C \) \( \hat{S} \) is IODS. In order to apply those results to a (possibly infinite) CPS \( S \) we first compute a finite symbolic model \( \hat{S} \) of \( S \) and then provide a procedure to transfer (or refine) a controller \((\hat{S}_C, \hat{R}_C)\) that is designed for \( \hat{S} \) to a controller \((S_C, R_C)\) for \( S \). This brings us to the main question answered in this section:

Given \((S, I, O)\), what are the conditions that a symbolic model \((\hat{S}, \hat{I}, \hat{O})\) of \((S, I, O)\) needs to satisfy so that the existence of a controller \((\hat{S}_C, \hat{R}_C)\) for \( \hat{S} \) rendering

\footnote{Technically, the controller in [34] is defined in a slightly different manner from \((\hat{S}_C, \hat{R}_C)\). However, it is straightforward to obtain a controller \((\hat{S}_C, \hat{R}_C)\) from the controller given in [34].}
\( \dot{S}_C \times \hat{R}_C \hat{S} \) pIODS, implies the existence of a controller \((S_C, R_C)\) for \(S\) rendering \(S_C \times \hat{R}_C S\) pIODS?

A well-known approach for controller refinement in connection with symbolic models is based on alternating simulation relations (ASR), see [1] and [32, Chapter 4.3]. In this section, we extend this approach to approximate contractive alternating input-output SR (acAIOSR). An intuitive version of the main result proved in this section is:

Consider two systems \((S, I, O)\) and \((\hat{S}, \hat{I}, \hat{O})\), and let \(R\) be an acAIOSR from \((\hat{S}, \hat{I}, \hat{O})\) to \((S, I, O)\). Suppose there exists a controller \((\hat{S}_C, \hat{R}_C)\) for \(\hat{S}\) such that \((S_C \times \hat{R}_C, \hat{S}_C, \hat{I}, \hat{O})\) is pIODS. Then there exists a controller \((S_C, R_C)\) for \(S\) such that \((S_C \times \hat{R}_C, S, I, O)\) is pIODS.

We provide a precise formulation of this statement in Theorem 5 after we formalize the notions of acAIOSR, controller, and composition of a system with a controller. Moreover, we explain how \((S_C, R_C)\) can be constructed from \((\hat{S}_C, \hat{R}_C)\).

5.1. Alternating simulation relations. In the following definition of an ASR we use a refined notion of input sets associated to states given by:

\[
U^c(x) := \{u^c \in U^c \mid \forall u^d \in U^d : r(x, u^c, u^d) \neq \varnothing\}.
\]

**Definition 7.** Let \(S\) and \(\hat{S}\) be two systems, let \(\kappa, \lambda \in \mathbb{R}_{\geq 0}\) and \(\beta \in [0, 1]\) be some parameters and consider the map \(d : \hat{U} \times U \rightarrow \mathbb{R}_{\geq 0}\). We call a parameterized (by \(\varepsilon \in [\kappa, \infty]\)) relation \(R(\varepsilon) \subseteq \hat{X} \times X \times \hat{U} \times U\) a \(\kappa\)-approximate \((\beta, \lambda)\)-contractive alternating simulation relation \(((\kappa, \beta, \lambda)\text{-acASR})\) from \(\hat{S}\) to \(S\) with distance function \(d\) if \(R(\varepsilon) \subseteq R(\varepsilon')\) holds for all \(\varepsilon \leq \varepsilon'\) and we have for all \(\varepsilon \in [\kappa, \infty]\)

1. \(\forall \hat{x}_0 \in \hat{X}_0, \exists x_0 \in X_0 : (\hat{x}_0, x_0) \in R_X(\kappa);\)
2. \(\forall (x, \hat{x}) \in R_X(\varepsilon), \forall \hat{u}^c \in U^c(\hat{x}), \exists u^c \in U^c(x),\)
   \(\forall u^d \in U^d, \exists \hat{u}^d \in U^d: \)
   \(\hat{r}(\hat{x}, \hat{u}) : (\hat{x}', \hat{x}) \in R_X(\kappa + \beta \varepsilon + \lambda d(\hat{u}, u));\)
   with \(u := (u^c, u^d), \hat{u} := (\hat{u}^c, \hat{u}^d)\).

Let \((S, I, O)\) and \((\hat{S}, \hat{I}, \hat{O})\) be two systems with cost functions. We call a \((\kappa, \beta, \lambda)\text{-acASR} R(\varepsilon)\) from \(\hat{S}\) to \(S\) with distance function \(d\) a \(\kappa\)-approximate \((\beta, \lambda)\)-contractive alternating input-output SR \(((\kappa, \beta, \lambda)\text{-acAIOSR})\) from \((\hat{S}, \hat{I}, \hat{O})\) to \((S, I, O)\) with distance function \(d\) if there exist \(\gamma_0, \gamma_1 \in \mathcal{K}\) such that

\[
\hat{I}(\hat{x}, \hat{u}) \leq I(x, u) + \gamma_1(\varepsilon')
\]
\[
O(x, u) \leq \hat{O}(\hat{x}, \hat{u}) + \gamma_0(\varepsilon')
\]

with \(\varepsilon' := \max\{\varepsilon, d(\hat{u}, u)\}\) for all \((\hat{x}, x, \hat{u}, u) \in R(\varepsilon)\).

We call a relation \(R(\varepsilon)\) acASR (acAIOSR) if there exists \(\beta \in [0, 1], \kappa, \lambda \in \mathbb{R}_{\geq 0}\) such that \(R(\varepsilon)\) is a \((\kappa, \beta, \lambda)\text{-acASR (acAIOSR)}\) from \(\hat{S}\) to \(S\) \(((\hat{S}, \hat{I}, \hat{O})\) to \((S, I, O)\)).

We illustrate acAIOSR using an example from the literature.

**Example 2** (DC-DC boost converter). We consider a popular example from the literature, the boost DC-DC converter, see for example [10, 17]. The dynamics of the boost converter is given by a two-dimensional switched linear system \(\xi(t) = \)

\[ A_\alpha \xi(t) + B \text{ with } A_\alpha \in \mathbb{R}^{2 \times 2}, B \in \mathbb{R}^2 \text{ and } \alpha \in \{1, 2\}. \] In [10], a symbolic model \( \hat{S} \) of the sampled dynamics of the boost converter \( S \) is used to compute a controller rendering the set \( D = [1.3, 1.7] \times [5.7, 5.8] \) positively invariant. Similarly to the approach in this paper, a symbolic model \( \hat{S} \) together with an approximate ASR \( \hat{R} \) is first computed. In the second step, a controller \( (\hat{S}_C, \hat{R}_C) \) for \( \hat{S} \) is computed to render \( D \) positively invariant with respect to the symbolic model \( \hat{S}_C \times \hat{R}_C \hat{S} \). Afterwards, a controller for \( S \) is obtained by refining the controller \( (\hat{S}_C, \hat{R}_C) \).

Note, as the controller refinement in [10] is based on an \( \varepsilon \)-approximate ASR with constant \( \varepsilon \in \mathbb{R}_{\geq 0} \), a disturbance \( w \in \mathbb{R}^2 \) on the system dynamics \( \xi(t) = A_\alpha \xi(t) + B + w \) might lead to a state \( \xi(r) \) such that the composed system is blocking. Therefore, the resulting controller is prone to fail in the presence of disturbances. Contrary to that, we exploit the contractivity of the matrices \( A_\alpha \) and construct a robust controller using the introduced notion of acAIOSR.

We refer the reader to [17] for a detailed exposition of the boost converter. In this example, we simply use the same parameters as in [10], and obtain the sampled dynamics of the boost converter as \( \xi_{t+1} = A_\nu \xi_t + B_\nu + \omega_t \) with the system matrices given by

\[
A_1 = \begin{bmatrix} 0.9917 & 0 \\ 0 & 0.9964 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.1660 \\ 0 \end{bmatrix},
\]
\[
A_2 = \begin{bmatrix} 0.9903 & -0.0330 \\ 0.0354 & 0.9959 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1659 \\ 0.0030 \end{bmatrix}.
\]

Note that in contrast to [10] we add \( \omega_t \in \mathbb{R}^2 \) to model various disturbances. We introduce the system \( S = (X, X_0, U, r) \) associated with the boost converter by defining \( X := \mathbb{R}^2 \), \( X_0 := D \), \( U := U^c \times U^d \) with \( U^c := \{1, 2\} \) and \( U^d := \mathbb{R}^2 \). Note that the inputs \( (u^c, u^d) \in U \) of the system \( S \) correspond to the control input \( u^c = u \) and the disturbance \( u^d = w \). The transition function is given by \( r(x, (u^c, u^d)) := \{A_{u^c}x + B + u^d\} \). We use the cost functions \( I(x, (u^c, u^d)) := |u^d| \) and \( O(x, u) := |x| \) to quantify the disturbances and to encode the desired behavior.

The symbolic model \( S = (X, X_0, U, \hat{r}) \) that is used in [10] is based on a discretization of \( D \):

\[
\hat{X} := \hat{X}_0 := D \cap \{x \in \mathbb{R}^2 \mid x_i = k_i/2/\sqrt{2} \kappa, i \in \{1, 2\}, k_i \in \mathbb{Z}\}
\]

with \( \kappa = 0.25 \cdot 10^{-3}/\sqrt{2} \). The inputs are given by \( \hat{U} := \hat{U}^c \times \hat{U}^d \) with \( \hat{U}^c := \{1, 2\} \) and \( \hat{U}^d := \{0\} \). The transition function is implicitly given by \( \hat{x}' = \hat{r}(\hat{x}, (\hat{u}, 0)) \iff |\hat{x}' - A_{u^{c}}\hat{x} - B_{\hat{a}}|_2 \leq \kappa \).

We set the cost functions for \( \hat{S} \) simply to \( \hat{I}(\hat{x}, \hat{u}) := 0 \) and \( \hat{O}(\hat{x}, \hat{u}) := 0 \) since \( I(\hat{x}, \hat{u}) = O(\hat{x}, \hat{u}) = 0 \) holds for all \( \hat{x} \) and \( \hat{u} \). Let us introduce the relation \( R(\varepsilon) := R_X(\varepsilon) \times R_U \) with

\[
R_X(\varepsilon) := \{(\hat{x}, x) \in \hat{X} \times X \mid |\hat{x} - x|_2 \leq \varepsilon\}
\]
\[
R_U := \{(\hat{u}^c, 0), (u^c, u^d) \in \hat{U} \times U \mid |u^c - \hat{u}^c|_2 \leq \varepsilon\}.
\]

We now show that \( R(\varepsilon) \) is a \( (\kappa, \beta, \lambda) \)-acAIOSR from \( \hat{S} \) to \( S \) with \( d((u^c, u^d), (\hat{u}^c, 0)) := |u^d|_2 + \beta \cdot 0.997 \geq \max\{|A_{12}|, |A_{22}|\} \) and \( \lambda = 1 \). We first note that \( R(\varepsilon) \subseteq R(\varepsilon') \) holds whenever \( \varepsilon \leq \varepsilon' \). By definition of \( \hat{X}_0 \) we can see that for every \( \hat{x}_0 \in \hat{X}_0 \) there exists a \( x_0 \in X_0 \) such that \( (x_0, \hat{x}_0) \in R_X(\kappa) \). We proceed by checking 2) of Definition [2]. Let \((\hat{x}, x) \in R_X(\varepsilon) \) and \( u^c \in U^c \). We choose \( u^c = \hat{u}^c \) and observe that for every \( u^d \in U^d \) we have \((\hat{x}, x, (\hat{u}^c, 0), (u^c, u^d)) \in R(\varepsilon) \) and \((\hat{x}', x') \in R_X(\kappa + \beta \varepsilon + \lambda |u^d|_2) \).
with $x' \in r(x, (u^c, u^d))$, $\hat{x}' \in \hat{r}(\hat{x}, (\hat{u}^c, 0))$ since
\[
|x' - \hat{x}'|_2 \leq \kappa + |A_{u^c} x + u^d - A_{u^c} \hat{x}|_2 \leq \kappa + \beta \varepsilon + |u^d|_2
\]
which shows that $R(\varepsilon)$ is an $(\kappa, \beta, \lambda)$-acASR from $\hat{S}$ to $S$. As the inequalities (15) hold for $\gamma_\iota = 0$ and $\gamma_\iota(c) = c$ we conclude that $R(\varepsilon)$ is an acAIOSR from $(\hat{S}, \hat{I}, \hat{O})$ to $(S, I, O)$.

Similarly to previous examples, we exploited the contraction property of the control system to construct an acASR from the symbolic model $\hat{S}$ to $S$.

We resume the example after we presented the main theorem of this section, where we refine the controller for the symbolic model $\hat{S}$ to a controller for $S$.

5.2. System composition. In this subsection, we define a general notion of system composition between two systems $S_1$ and $S_2$ with respect to a relation $H \subseteq X_1 \times X_2 \times U_1 \times U_2$. Afterwards, we introduce the notion of system composition for the case when $H$ is an acASR $R(\varepsilon)$ from $S_1$ to $S_2$. In the next subsection, we use the definition of system composition to define the controlled system.

Definition 8. The composition of system $S_1$ and $S_2$ with respect to the relation $H \subseteq X_1 \times X_2 \times U_1 \times U_2$, is denoted by $S_{12} := S_1 \times_H S_2$ and defined by:

1. $X_{12} := X_1 \times X_2$;
2. $X_{120} := (X_{10} \times X_{20}) \cap H_X$;
3. $U_{12} := U_1 \times U_2$;
4. $(x_1', x_2') \in r_{12}((x_1, x_2), (u_1, u_2)) : \iff$
   (a) $x_2' \in r_1(x_1, u_1)$;
   (b) $x_1' \in r_2(x_2, u_2)$;
   (c) $(x_1, x_2, u_1, u_2) \in H$ and $(x_1', x_2') \in H_X$.

If $H$ is an $(\kappa, \beta, \lambda)$-acASR $R(\varepsilon)$ from $S_1$ to $S_2$ with distance function $d$, then we exchange 2) by $X_{120} := (X_{10} \times X_{20}) \cap R_X(\kappa)$ and 4.c) by
\[
(x_1, x_2, u_1, u_2) \in R(e(x_1, x_2)), \quad \text{and} \quad (x_1', x_2') \in R_X(\varepsilon')
\]
with $\varepsilon' := \kappa + e(x_1, x_2)\beta + \lambda d(u_1, u_2)$ and $e(x_1, x_2) := \inf\{\varepsilon \in \mathbb{R}_{\geq 0} \mid (x_1, x_2) \in R_X(\varepsilon)\}$.

Intuitively, our definition of system composition corresponds to the well-known definition of parallel composition of the systems $S_1$ and $S_2$ with synchronization defined by $H$, respectively $R(\varepsilon)$. The only transitions allowed on the composed system $S_1 \times_H S_2$ are those for which the corresponding states and inputs belong to $H$, i.e., $(x_1, x_2, u_1, u_2) \in H$. It is shown in [32] how this notion of composition can describe series, parallel, feedback and several other interconnections. For the case that $H$ is an acASR $R(\varepsilon)$, we require that $(x_1, x_2, u_1, u_2) \in R(\varepsilon)$ where we fix $\varepsilon = e(x_1, x_2)$. With our particular choice of $\varepsilon = e(x_1, x_2)$ we restrict the transitions of the composed system $S_1 \times_{R(\varepsilon)} S_2$ to those states and inputs that are related by the smallest $\varepsilon = e(x_1, x_2)$ possible. In general it is not ensured that the infimal $\varepsilon = e(x_1, x_2)$ is actually attained by the states $(x_1, x_2)$. Therefore, we assume in the following that
\[
e(x_1, x_2) < \infty \implies (x_1, x_2) \in R_X(e(x_1, x_2)). \quad (16)
\]
Note that this assumption is often satisfied in practice where $R_X(\varepsilon)$ is for example defined by $|x_1 - x_2| \leq \varepsilon$. 
5.3. The controlled system and controller refinement. In the following, we use the composition of two systems $S_C$ and $S$ with respect to a parameterized relation $R_C(\varepsilon)$ to define the controlled system $S_C \times_{R_C(\varepsilon)} S$, when the relation $R_C(\varepsilon)$ is an acASR from $S_C$ to $S$. From a control perspective, the controller $(S_C, R_C(\varepsilon))$ for $S$ can be implemented in a feedback loop as follows. Let us denote the set of initial states $x \in X_0$ for which there exists $x_C \in X_{C0}$ such that $(x_C, x) \in R_C(\varepsilon(\kappa))$ by $X'_0$. Then initially, i) the controller measures the system state $x \in X'_0$ and determines a related controller state $x_C \in X_{C0}$ such that $(x_C, x) \in R_C(\varepsilon(\kappa));$ ii) the controller picks the control inputs $u_C^e$ and $u^e$ according to 2) in Definition 4 and applies $u^e$ to $S;$ iii) the disturbance chooses $u^d \in U^d$ and $x' \in r(x, (u^e, u^d));$ iv) the controller measures the new state $x'$ and chooses $x'_C$, and $u^d_C \in U^d_S$ such that $x'_C \in r_C(x_C, (u^e_C, u^d_C))$ and $(x'_C, x') \in R_C(\varepsilon'(\kappa))$ for $\varepsilon' = \varepsilon(x'_C, x')$. Now the cycle continues with ii).

Note that in this scenario, the disturbance inputs $U^d_C$ of the controller $S_C$ are not considered as external inputs, but are allowed to be chosen by the controller. This leads us to the following the definition.

Definition 9. Given a system $S$, we call the pair $(S_C, R_C(\varepsilon))$ a controller for $S$ if $S_C$ is a system, $R_C(\varepsilon)$ is an acASR from $S_C$ to $S$ and the composed system $S_C \times_{R_C(\varepsilon)} S$ is non-blocking, in the sense that for all reachable states $(x_C, x)$ there exist $(u_C^e, u^e) \in U^e_C \times U^e$ such that for all $u^d \in U^d$ there exist $u^d_C \in U^d_S$ for which $r'(((x_C, x), ((u_C^e, u^e), (u^d_C, (u^d)))) \neq \emptyset$, where $r'$ is the transition map of the composed system.

The interested reader may wish to consult [32 Chapter 6.1] for detailed explanations of why the composition between a controller and a system is only well defined when the relation $R_C$ is alternating. Note that the assumption (17) is consistent with the use of extended alternating simulation relations in the definition of the feedback composition in [32 Definition 6.1].

Let us remark that the controller $(S_C, R_C)$ rendering the system $\hat{S}$ pIODS that we obtain from the approach in [34] is given in terms of a system $\hat{S}_C$ and an alternating simulation relation (ASR) from $S_C$ to $\hat{S}$ rather than an acASR. The definition of an ASR is given in [32 Definition 4.22]. Instead of repeating the definition here, we define it in terms of an acASR.

Definition 10. Let $S$ and $\hat{S}$ be two systems and let $R(\varepsilon)$ be a $(0,0,0)$-acASR from $\hat{S}$ to $S$. The relation $\hat{R} := R(0)$ is called an alternating simulation relation (ASR) from $\hat{S}$ to $S$.

The composition $S_1 \times_{R_1} S_2$ of $S_2$ and $S_1$ with respect to an ASR $R_{12}$ follows from Definition 5 with $H = R_{12}$. Similarly, the definition of a controller $(S_C, R_C)$ in terms of an ASR follows in a straightforward manner from Definition 9. No confusion between acASR and ASR should arise, since we always include the parameter $\varepsilon$ in the notation when we refer to an acASR (acAIOSR).

In the following, we assume that an ASR $R_{12}$ from $S_2$ to $S_1$ satisfies

$$(x_1, x_2, (u_1^e, u_1^d), (u_2^e, u_2^d)) \in R_{12} \implies (x_1, x_2, (u_1^e, u_2^e)) \text{ satisfies 2.a) of Def. 9 (17)}$$

This implication (17) results in no loss of generality since we can always construct an ASR $R'_1_{12}$ that satisfies (17) from an ASR $R_{12}$ by simply removing the elements that don’t satisfy (17).
Lemma 2. \[ I_C((x_C, x), (u_C, u)) := I(x, u) \] and \[ O_C((x_C, x), (u_C, u)) := O(x, u) \].

Like in Corollary 3, we define the function
\[
\Gamma(x, u) := \sup \{ d(\hat{u}, u) : \exists \xi, \exists \hat{x} : (\hat{x}, x, \hat{u}, u) \in R(\varepsilon) \} \tag{18}
\]
for an acAISOR \( R(\varepsilon) \) from \( \hat{S} \) to \( S \) with distance function \( d \) and refer to \( R(\varepsilon) \) as an acAISOR from \( \hat{S} \) to \( S \) with \( \Gamma \).

Now we are ready to state the main theorem.

Theorem 5. Given two systems with cost functions \((S, I, O)\) and \((\hat{S}, \hat{I}, \hat{O})\), let \( R(\varepsilon) \) be an acAISOR from \((\hat{S}, \hat{I}, \hat{O})\) to \((S, I, O)\) with \( \Gamma \) and let \( R(\varepsilon) \) satisfy (17). Suppose there exists an acAISOR \( R_{12}(\varepsilon) \) from \( \hat{S}_{12} := \hat{S}_1 \times \hat{S}_2 \) to \( S_3 \) with distance function \( d_{12} := d_{12}(u_1, u_2, u_3) \) satisfying (10).

Lemma 3. Consider the systems \( S_1 \) and \( S_2 \). Let \( R_{12}(\varepsilon) \) be an acAISOR from \( S_1 \) to \( S_2 \) such that \( (\hat{S}_1, \hat{R}_C) \) is a pIROS. Then \( (\hat{S}_1, \hat{R}_{C}(\varepsilon)) \) is a pIROS with \( \hat{R}_C \) and \( \hat{R}_{C}(\varepsilon) \) satisfies (10).

Lemma 4. Consider the systems \( S_1 \) and \( S_2 \). Let \( R_{12}(\varepsilon) \) be a \((\kappa, \beta, \lambda)\)-acAISOR from \( S_1 \) to \( S_2 \) with \( d_{12}(u_1, u_2) \) satisfying (10). Then there exists an acAISOR \( \hat{S}_2 := \hat{S} \times \hat{R}_{C}(\varepsilon) \) to \( S_2 \) with distance function \( d_{12} := d_{12}(u_1, u_2) \) that satisfies (10).

Proof of Theorem 5. We apply Lemma 2 for \( S_1 = \hat{S}_C, S_2 = \hat{S}, S_3 = S, R_{12} = \hat{R}_C \) and \( R_{23}(\varepsilon) = R(\varepsilon) \). It follows that there exists an acAISOR \( R_C(\varepsilon) \) from \( \hat{S}_C \times \hat{R}_{C}(\varepsilon) \) to \( S \) with distance function \( d_C((\hat{u}_C, \hat{u}), u) := d(\hat{u}, u) \) and \( R_C(\varepsilon) \) satisfies (10). We apply Lemma 3 to see that \( (\hat{S}_C, \hat{R}_C(\varepsilon)) \) with \( \hat{S}_C := \hat{S} \times \hat{R}_{C}(\varepsilon) \) is a controller for \( S \). Now it follows from Lemma 4 that there exists an acAISOR \( \hat{R}'(\varepsilon) \) from \( \hat{S}_C \times \hat{R}_C(\varepsilon) \) to \( S \) with distance function \( d'(\hat{u}_C, u) := d_C((\hat{u}_C, \hat{u}), u) \).

Note that the cost functions for the composed systems \( \hat{S}_C \times \hat{R}_C \hat{S} \) and \( S \times R_C(\varepsilon) \) are given by
\[
\hat{I}_C((\hat{x}_C, \hat{x}), (\hat{u}_C, u)) := \hat{I}(\hat{x}, u), \quad \hat{O}_C((\hat{x}_C, \hat{x}), (\hat{u}_C, u)) := \hat{O}(\hat{x}, \hat{u}),
\]
\[
I_C((u_C, x), (u_C, u)) := I(x, u), \quad O_C((u_C, x), (u_C, u)) := O(x, u).
\]

We proceed by showing that \( R'(\varepsilon) \) is actually a \((\kappa, \beta, \lambda)\)-acIORS form \( (S, I, O) \) to \( (\hat{S}_C \times \hat{R}_C(\varepsilon), \hat{I}_C, \hat{O}_C) \). By carefully checking the proof of the Lemmas 2 and 4, we see that \( (x_C, x, (\hat{x}_C, \hat{x}), (\hat{u}_C, \hat{u}), (u_C, u)) \in R'(\varepsilon) \) implies \( x_C = (\hat{x}_C, \hat{x}), u_C = (\hat{u}_C, \hat{u}) \) and \( (\hat{x}, \hat{x}, \hat{u}, u) \in R(\varepsilon) \). As \( R(\varepsilon) \) is an acAISOR from \( (\hat{S}, \hat{I}, \hat{O}) \) to \( (S, I, O) \) we obtain the inequalities
\[
\hat{I}_C((\hat{x}_C, \hat{x}), (\hat{u}_C, \hat{u})) := \hat{I}(\hat{x}, \hat{u}) \leq I(x, u) + \gamma_I(\varepsilon) \tag{19}
\]
\[
\hat{O}_C((\hat{x}_C, \hat{x}), (\hat{u}_C, \hat{u})) := \hat{O}(\hat{x}, \hat{u}) \leq O(x, u) + \gamma_O(\varepsilon) \tag{20}
\]
for all \((x_C, x), (\hat{x}_C, \hat{x}), (u_C, u), (\hat{u}_C, \hat{u})\) ∈ \(R'(\varepsilon)\) and \(\varepsilon' = \max\{\varepsilon, \delta'((u_C, u), (\hat{u}_C, \hat{u}))\} \).

We apply Theorem 4 to \((S_C × R_C(\varepsilon)) S, I_C, O_C)\) and \((\hat{S}_C × R_{C, \hat{S}}, \hat{I}_C, \hat{O}_C)\) with distance function \(d'\) and obtain that \((S_C × R_C(\varepsilon)) S, I_C, O_C)\) is pIODS with the modified input costs \(I'_C((x_C, x), (u_C, u)) = \max\{I(x, u), \Gamma(x, u)\}\).

**Remark 1.** Note that we use Theorem 4 to see that the controlled system \(S_C × R_C(\varepsilon)\) \(S\) is pIODS. If \(\gamma_0\) satisfies the triangle inequality and \(I(\hat{x}, \hat{u}) ≤ I(x, u)\) holds for every \((\hat{x}, x, \hat{u}, u)\) ∈ \(R(\varepsilon)\) and \(\varepsilon \in \mathbb{R}_{≥ 0}\), the premises of Corollary 2 are satisfied and it follows that every behavior \(((\hat{\xi}_C, \hat{\xi}), (\hat{\nu}_C, \hat{\nu}))\) of \((S_C × R_C(\varepsilon)) S\) satisfies \(13\).

**Remark 2.** Note that the controller \((S_C, R_C(\varepsilon))\) for \(S_C\) is given by \(S_C = \hat{S}_C × \hat{R}_C \hat{S}\) where \(R_C(\varepsilon)\) equals \\{((\hat{x}_C, \hat{x}), (\hat{u}_C, \hat{u}), u) | (\hat{x}_C, \hat{x}, \hat{u}, u) \in R(\varepsilon) \wedge (\hat{x}_C, \hat{x}) \in \hat{R}_C, x\\}, \text{ see } (20)\.

Moreover, the parameters \(\kappa, \beta\) and \(\lambda\) and distance function \(d'\) of the \((\kappa, \beta, \lambda)\)-acIOASR \(R'(\varepsilon)\) from \(S_C × R_C(\varepsilon) S\) to \(S_C\) coincide with the parameters and distance function \(d\) of the \((\kappa, \beta, \lambda)\)-IOASR from \(\hat{S}\) to \(S\) given in the premise of Theorem 3.

**Example 2 (DC-DC boost converter (continued)).** Let \((\hat{S}_C, \hat{R}_C)\) denote the controller from 10 that renders \(D\) positively invariant with respect to \(S_C := \hat{S}_C × \hat{R}_C \hat{S}\). Therefore, any behavior \(((\hat{\xi}_C, \hat{\xi}), (\hat{\nu}_C, \hat{\nu}))\) of \(\hat{S}_C × \hat{R}_C \hat{S}\) satisfies \(O(\hat{\xi}_t, \hat{\nu}_t) = I(\hat{\xi}_t, \hat{\nu}_t) = 0\) and it follows that \(\hat{S}_C × \hat{R}_C \hat{S}\) is \((\hat{\gamma}, \hat{\mu})\)-IODS with \(\hat{\gamma} = 0\) and \(\hat{\mu} = 0\).

We apply Theorem 4 and conclude that \(S_C × R_C(\varepsilon) S\) is pIODS with input costs \(\max\{|f, \Gamma\} = |w'\|_2\), since \(\Gamma\) induced by \(R(\varepsilon)\) and \(d\) is given by \(|w'\|_2\). Note that the assumptions of Corollary 2 hold and we can conclude that any behavior \(((\xi_C, \xi), (\nu_C, \nu))\) of \(S_C × R_C(\varepsilon) S\) satisfies

\[|\xi_t|_D ≤ \max_{t' \in [0, t]} \mu_D(\gamma_D(|\nu'_t|_2), t - t') + \kappa_D\]

where with \(\mu_D(r, t) := (\beta')^t r, \gamma_D = 1/(\beta' - \beta)\) and \(\kappa_D := \kappa/(1 - \beta)\) for some \(\beta' \in |\beta, 1|\).

The pIODS inequality implies that the system may leave the set \(D\) in the presence of disturbances, however in absence of disturbances the system either stays in \(D + \mathbb{B}(\kappa_D)\) or asymptotically approaches \(D + \mathbb{B}(\kappa_D)\). Moreover, contrary to the approach in 10 the closed-loop system \(S_C × R_C(\varepsilon) S\) is non-blocking even in the presents of unbounded disturbances.

Note that in this example, the contraction property of the system matrices enabled us to establish an acIOASR from the symbolic model to the concrete system. As a consequence, we could neglect the continuous disturbances on the symbolic model, but nevertheless establish the pIODS inequality. We demonstrate in Section 7 how this procedure leads to a separation of concerns in the robust controller design for CPS, where a continuous “low-level” controller and a discrete “high-level” controller provides robustness with respect to continuous and discrete disturbances, respectively. In particular, we use a low-level feedback controller to enforce the contraction property needed to establish an acIOASR from the symbolic model (without continuous disturbances) to the concrete CPS. Then we use the synthesis approach in 34 to design a discrete high-level controller that renders the symbolic model robust against discrete disturbances. Afterwards, we refine the discrete controller to the concrete CPS according to Remark 2 and obtain from Theorem 5 that the controlled CPS is robust against the continuous as well as discrete disturbances.
6. A Compositional Result

In this section, we show how aCASR are preserved under composition. We analyse four systems $S_1$, $\hat{S}_1$, $S_2$, and $\hat{S}_2$ and assume the existence of the relations $R_i(\varepsilon)$, $i \in \{1, 2\}$ with $R_i(\varepsilon)$ being an aCASR from $S_i$ to $\hat{S}_i$. Then we show how to construct a relation $\hat{H}$ such that there is an aCASR $R(\varepsilon)$ from $\hat{S}_1 \times_H \hat{S}_2$ to $S_1 \times_H S_2$.

Note that this result is useful to construct symbolic models that are alternatingly related with CPS $S_{12} := S_1 \times_H S_2$ that is given by the composition of a system $S_1$, representing the physical part and system $S_2$, representing the cyber part. The compositional result enables us to construct a symbolic model of the concrete CPS in two steps. In the first step, we compute symbolic models for the individual parts $S_1$ and $S_2$. In the second step, we combine those symbolic models to obtain a symbolic for the composed CPS. Usually, the cyber part of a CPS is already finite and an abstraction of $S_2$ may not be necessary. In that case, the construction of a symbolic model of $S_{12}$ is reduced to the computation of symbolic model for the physical part $S_1$ using, e.g., the methods presented in [23, 24, 10] and [32, Chapter 11]. We don’t provide further details on how to construct such models here, but refer the reader to Example 2 and Section 7 where we illustrate those approaches with concrete examples.

We begin with the derivation of the compositional result. Let $S_i$, $\hat{S}_i$, $i \in \{1, 2\}$ be four systems, and let the relations $R_i(\varepsilon)$ be aCASR from $\hat{S}_i$ to $S_i$. Suppose we are given $H \subseteq X_1 \times X_2 \times U_1 \times U_2$, then we define the relation $\hat{H} \subseteq \hat{X}_1 \times \hat{X}_2 \times \hat{U}_1 \times \hat{U}_2$ by

$$\{(\hat{x}_1, \hat{x}_2, \hat{u}_1, \hat{u}_2) \mid \exists x_i, u_i : (\hat{x}_i, x_i, \hat{u}_i, u_i) \in R_i(\varepsilon), i \in \{1, 2\} \land (x_1, x_2, u_1, u_2) \in (H)\}$$

and $R(\varepsilon) \subseteq \hat{X}_{12} \times \hat{X}_{12} \times \hat{U}_{12} \times \hat{U}_{12}$ by

$$\{(\hat{x}_{12}, x_{12}, \hat{u}_{12}, u_{12}) \mid (\hat{x}_1, x_1, \hat{u}_1, u_1) \in R_1(\varepsilon), (\hat{x}_2, x_2, \hat{u}_2, u_2) \in H\}$$

We use the following assumption

(21) $(\hat{x}_1, x_1) \in R_{1, X}(\varepsilon), i \in \{1, 2\} \land (x_1, x_2) \in H_X \implies \exists u_i, \hat{u}_i : (\hat{x}_i, x_i, \hat{u}_i, u_i) \in R_i(\varepsilon) \land (x_1, x_2, u_1, u_2) \in H$

Intuitively, we ensure with this assumption that if $(x_1, x_2) \in H_X$ and the states $\hat{x}_i$ are related to $x_i$ for $i \in \{1, 2\}$ then $(\hat{x}_1, \hat{x}_2) \in \hat{H}_X$.

Theorem 6. Let $S_i$, $\hat{S}_i$, $i \in \{1, 2\}$ be four systems, and let the relations $R_i(\varepsilon)$ be $(\kappa_i, \beta_i, \lambda_i)$-aCASR from $\hat{S}_i$ to $S_i$ with distance function $d_i$. Let $H \subseteq X_1 \times X_2 \times U_1 \times U_2$ be a relation and $\hat{H} \subseteq \hat{X}_1 \times \hat{X}_2 \times \hat{U}_1 \times \hat{U}_2$ be obtained from (20). If (21) holds, then $R(\varepsilon)$ as defined in (20) is an $(\kappa, \beta, \lambda)$-aCASR from $\hat{S}_1 \times_H \hat{S}_2$ to $S_1 \times_H S_2$ with $\kappa := \max_i \{\kappa_i\}$, $\beta := \max_i \{\beta_i\}$ and $\lambda := \max_i \{\lambda_i\}$ with distance function $d_{12}(u_{12}, \hat{u}_{12}) := \max_i \{d_i(u_i, \hat{u}_i)\}$.

Proof. The property $R(\varepsilon) \subseteq R(\varepsilon')$ whenever $\varepsilon \leq \varepsilon'$ is directly inherited from $R_i(\varepsilon)$. Let $\hat{x}_{12} \in \hat{X}_{12}$ which implies $\hat{x}_i \in X_{i0}, i \in \{1, 2\}$. Therefore, there exist $x_i \in X_{i0}$ with $(\hat{x}_i, x_i) \in R_i(\kappa_i)$ and thereby we have $(\hat{x}_{12}, x_{12}) \in R_X(\kappa)$. Consider $(\hat{x}_{12}, x_{12}) \in R_X(\varepsilon)$ and $\hat{u}_{12} \in \hat{U}_{12}(\hat{x}_{12})$. This implies $(\hat{x}_1, x_1) \in R_{1, X}(\varepsilon)$ and $\hat{u}_1 \in \hat{U}_1(\hat{x}_1)$. (21)

By (22), we can pick $u_1 \in U_1^c$ such that the tuple $(\hat{x}_1, x_1, \hat{u}_1, u_1)$ satisfies 2.a) in Definition 7. Let $u_1^d \in U_1^d$ and $x_{12}' \in r_{12}(x_{12}, u_{12})$ where $u_i = (u_i^c, u_i^d)$ and

(22)
The system \( u_{12} = (u_1, u_2) \). By our choice of \( \hat{u} \) there exist \( \hat{u}^i \) such that \( (\hat{x}, x, \hat{u}, u) \in R_i(\varepsilon) \) with \( \hat{u}_i = (\hat{u}^i, \hat{u}^c) \) and it follows that \( (\hat{x}_{12}, x_{12}, u_{12}, u_{12}) \in R(\varepsilon) \) where \( \hat{u}_{12} = (\hat{u}_1, \hat{u}_2) \).

For \( i \in \{1, 2\} \), we choose \( \hat{x}'_i \in \hat{r}_i(\hat{x}, \hat{u}) \) such that \( (\hat{x}'_i, x'_i) \in R_i(X(\varepsilon')) \) with \( \varepsilon'_i = \kappa_i + \beta_i \varepsilon + \lambda_i(\hat{u}_i, u_i) \). It remains to show that \( \hat{x}'_{12} \in \hat{r}_{12}(\hat{x}_{12}, \hat{u}_{12}) \) from which it follows that \( (\hat{x}'_{12}, x'_{12}) \in R(X(\varepsilon')) \) with \( \varepsilon' = \kappa + \beta \varepsilon + \lambda(\hat{u}_{12}, u_{12}) \). We need to check 4.(c) in Definition 8.

Since \( (\hat{x}_i, x_i, \hat{u}_i, u_i) \in R_i(\varepsilon) \) and \( (x_1, x_2, u_1, u_2) \in H \), we have \( (\hat{x}_1, \hat{x}_2, \hat{u}_1, \hat{u}_2) \in \hat{H} \) and it remains to show that \( \hat{x}'_{12} \in H_X \). That follows by 20, since we know that \( (\hat{x}'_i, x'_i) \in R_i(X(\varepsilon')) \) and \( (x'_1, x'_2) \in H_X \).

7. A Mobile Robot Example

In this section, we demonstrate our results in terms of a simple example with a robot moving in the plane equipped with an omnidirectional drive. We model the sampled dynamics of the robot by the difference equation

\[
\xi_{t+1} = \xi_t + \nu_t
\]

where \( \xi_t \in \mathbb{R}^2 \) is the position of the robot and \( \nu_t \in \mathbb{R}^2 \) is the control input. We assume that the control signal is sent to the mobile robot over a wireless communication channel with possible package dropouts. We apply the presented abstraction and refinement framework to design a robust controller for the robot over the lossy channel. As a first step, we construct a symbolic model that alternatesimulates the robot. Here we use Theorem 6 to construct symbolic models of the physical part and cyber part individually and then compose those models to obtain a symbolic model of the overall robot with communication channel. Afterwards, we apply the approach from 34 to synthesize a robust controller for the symbolic model. Finally, we apply Theorem 6 to refine the controller for the symbolic model to the robot.

The system description. We assume that the robot drive is equipped with low-level controllers that we use to enforce the sampled-data dynamics

\[
\xi_{t+1} = 0.8\xi_t + \nu_t + \omega_t. \tag{23}
\]

We use \( \omega_t \in \mathbb{R}^2 \) to model actuator errors and/or sensor noise. A real-world example of a robot that fits our assumptions is Robotino, see 37. We cast 23 as the system \( S_1 = (X_1, X_{10}, U_1, r_1) \) with \( X_1 = \mathbb{R}^2 \), \( X_{10} = \{x_{10}\} \), \( U_1 = U_1^c \times U_1^d \), \( U_1^c = U_1^d = \mathbb{R}^2 \) and \( r_1 \) is defined in the obvious way.

Moreover, we assume that the high-level control signal \( u \) is sent to the actuator via a wireless connection where package dropouts might occur. However, for simplicity of the presentation, we assume that two packages are never dropped consecutively. We use the system \( S_2 = (X_2, X_{20}, U_2, r_2) \) with \( X_2 = \{a_0, a_1\} \), \( X_{20} = X_2 \) and \( U_2 = U_2^d = D \) and \( D = \{\perp, \top\} \) to model that behavior. The dynamics \( r_2 \) of the system \( S_2 \) is illustrated in Figure 1. Our model of the wireless communication acts like a switch with respect to the control input \( \hat{u} \in \mathbb{R}^2 \). If a package dropout occurs, i.e., \( x_2 = a_1 \), we apply zero as control input \( u = 0 \). If no dropout occurs, i.e., \( x_2 = a_0 \), the control input is \( u = \hat{u} \) since the robot successfully received a control update. The transition between the nominal state \( x_2 = a_0 \) and the state when a package dropout occurs \( x_2 = a_1 \) is modelled by the perturbation signal \( \top \). The continuation of the nominal behavior, i.e., no package dropout occurs is modelled by the nominal input \( \perp \).
We define the composed system $S_{12} := S_1 \times_H S_2$ using the relation $H \subseteq X_1 \times X_2 \times U_1 \times U_2$ which is implicitly given by

$$(x_1, x_2, (u^c_1, u^d_1), u_2) \in H \iff (x_2 = a_1 \implies u^c_1 = 0).$$

In this way only the zero control input $u^c_1 = 0$ is allowed when the system $S_2$ is in state $x_2 = a_1$.

We would like to enforce a periodic behavior which we express as a cycle along the states displayed in Figure 2. In order to express our desired behavior in terms of the output costs, we introduce a system $S_3 = (X_3, \{r_i\}, U_3, r_3)$ with $X_3 = \{r_i\}$, $i \in \{0, \ldots, 7\}$, $X_{30} = \{r_0\}$, $U_3 = \{\epsilon\}$ and $r_3(x_3, u_3)$ given according to Figure 2.

The reference states $r_i \in \mathbb{R}^2$ are given by

- $r_0 = [0, 0]^\top$,
- $r_1 = [1, 0]^\top$,
- $r_2 = [2, 0]^\top$,
- $r_3 = [3, 0]^\top$,
- $r_4 = [3, 1]^\top$,
- $r_5 = [2, 1]^\top$,
- $r_6 = [1, 1]^\top$,
- $r_7 = [0, 1]^\top$.

The overall system is obtained as the composition of the three systems $S_{123} = S_{12} \times_G S_3$ with respect to $G := X_{12} \times X_3 \times U_{12} \times U_3$. We define the output costs $O : X_1 \times X_3 \to \mathbb{R}_{\geq 0}$ by

$$O(x_1, x_3) := |x_1 - x_3|$$

and choose the input costs $I : X_2 \times U^d_1 \to \mathbb{R}_{\geq 0}$ simply as

$$I(x_2, u^d_1) := I_d(x_2) + |u^d_1|,$$

with $I_d(a_0) := 0$ and $I_d(a_1) := 1$. Note that we omit the independent variables in $O$ and $I$. The value of the output costs indicates how well the robot is following the nominal behavior. The costs are zero, if the robot follows the system $S_3$ and non-zero otherwise. The input costs are used to quantify the possible disturbances.

The symbolic model. We continue with the construction of the symbolic model $\hat{S}_{123}$ for $S_{123}$, where we construct symbolic models $\hat{S}_i$, $i \in \{1, 2, 3\}$ for each subsystem $S_i$, respectively, and then use Theorem 6 to compose the individual models $\hat{S}_i$ to $\hat{S}_{123}$.

First we introduce the symbolic model $\hat{S}_1$ of $S_1$ based on a discretization of the state space and input space of $S_1$. We choose $\bar{X}_1 = [-1, 4]^2 \mathbb{K}$, $\bar{U}^c_1 = [-3, 3]^2 \mathbb{K}$ and...
is an \( (0.05, 0.8, 1) \)-acASR from \( \tilde{S}_1 \) to \( S_1 \) with distance function \( d_1((\tilde{u}_1^i, 0), (u_1^i, u_1^d)) = |u_1^d| \).

The symbolic models for \( S_2 \) and \( S_3 \) are directly given by \( \tilde{S}_2 = S_2 \) and \( \tilde{S}_3 = S_3 \) since \( S_2 \) and \( S_3 \) are finite. It is straightforward to see that the relations \( R_i := \{(\tilde{x}_i, x_i, \tilde{u}_i, u_i) | \tilde{x}_i = x_i \land \tilde{u}_i = u_i \}, i \in \{2, 3\} \) are \( (0, 0, 0) \)-acASR from \( \tilde{S}_i \) to \( S_i \) with distance functions \( d_i(\tilde{u}_i, u_i) = 0 \).

Now we apply Theorem \ref{thm:main} to see that \( R_{12}(\varepsilon) \subseteq \tilde{X}_{12} \times X_{12} \times \tilde{U}_{12} \times U_{12} \) given by

\[
\{(\tilde{x}_{12}, x_{12}, \tilde{u}_{12}, u_{12}) | (\tilde{x}_1, x_1, \tilde{u}_1, u_1) \in R_{1}(\varepsilon) \land x_2 = \tilde{x}_2 \land u_2 = \tilde{u}_2 \}
\]
is an \( (0.05, 0.8, 1) \)-acASR from \( \tilde{S}_{12} := \tilde{S}_1 \times \tilde{S}_2 \) to \( S_{12} \) with distance function \( d_{12}(\tilde{u}_{12}, u_{12}) = d_{1}(\tilde{u}_1, u_1) = |u_1^d| \).

We choose the cost functions \( \tilde{I} \) and \( \tilde{O} \) for \( \tilde{S}_{123} \) to be \( I_2(\tilde{x}_2) \) and \( \tilde{O}(\tilde{x}_1, \tilde{x}_3) := |\tilde{x}_1 - \tilde{x}_3|_\kappa \). We remark that the cost functions satisfy \( \tilde{I}_t = 0 \) and \( \gamma_0(c) = c + \max\{\varepsilon, \kappa\} \) and thereby follows that \( R_{123}(\varepsilon) \) is an acAIOSR from \( \tilde{S}_{123} \) to \( S_{123} \).

We use the synthesis approach in \[34\] to compute a controller \( (\hat{S}_C, \hat{R}_C) \) that renders the system \( \hat{S}_{123} \) IODS. As a result, we obtain the IODS inequality

\[
|\hat{\xi}_{1,t} - \hat{\xi}_{3,t}|_\kappa \leq \max_{\varepsilon \in [0,\varepsilon]} \{1.4 \tilde{I}_d(\hat{\xi}_t') - 1.4(t - t')\}
\]  

(24)

for every behavior \( (\hat{\xi}, \hat{v}) \) of the controlled system \( S_C := \hat{S}_C \times \hat{R}_C \tilde{S}_{123} \). Note that with \( \gamma = \eta = 1.4 \) the effect of the disturbance \( \hat{x}_2 = a_1 \) at time \( t \) disappears after one step.

**Controller refinement.** We now apply Theorem \ref{thm:refinement} to refine the controller for \( \tilde{S}_{123} \) to a controller for \( S_{123} \). First, note that \( R_{123}(\varepsilon) \) is a \( (0.05, 0.8, 1) \)-acAIOSR from \( \tilde{S}_{123} \) to \( S_{123} \) with \( \Gamma(\tilde{x}_{123}, (\tilde{u}_{123}^i, u_{23})) = |u_{123}^d| \) that satisfies \( \tilde{I}_t = 0 \) and \( R_{123}(\varepsilon) \) satisfies \( \tilde{I}_t \). Moreover, \( (\hat{S}_C \times \hat{R}_C \tilde{S}, \hat{I}, \hat{O}) \) is IODS with the inequality \( \tilde{I}_t \). As a consequence there exists a controller \( (S_C, R_C(\varepsilon)) \) for \( S_{123} \) and the controlled system is pIODS. Furthermore, since \( \hat{I}(\hat{x}, \hat{u}) \leq I(x, u) \) for all related tuples \( (\hat{x}, x, \hat{u}, u) \) we can apply Corollary \ref{cor:refinement} and the inequality

\[
O(\hat{x}_t) \leq \max_{t' \in [0,\varepsilon]} \{1.4 \tilde{I}_d(\hat{\xi}_t') - 1.4(t - t')\} + \max_{t' \in [0,\varepsilon]} \frac{1}{\beta'_{123}}(\beta'_{123}^t - t') \max_{t' \in [0,\varepsilon]} \{1.4 \tilde{I}_d(\hat{\xi}_t') - 1.4(t - t')\} + 0.25
\]

follows for any behavior \( (\hat{\xi}, \hat{v}) \) of \( S_C \times R_C(\varepsilon) \) \( S \) and any \( \beta'_{123} \in [0.8, 1] \).

This example demonstrates nicely how our results enable us to separate the design procedure to establish robustness with respect to continuous and discrete disturbances. We used the low-level controllers of the robot to enforce the contractive dynamics \[24\] so that \( S \) admits an acAIOSR. We used the discrete design procedure \[34\] to establish the IODS inequality \( \tilde{I}_t \) for the symbolic model with
respect to the discrete disturbances. As the previous pIODS inequality shows, the final controlled system is robust with respect to both discrete as well as continuous disturbances.

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Lemma 5. For every \( \mu \in \mathcal{KLD} \), \( \gamma \in \mathcal{K} \) and \( c' \in \mathbb{R}_{\geq 0} \) the following inequality holds

\[
\mu(\gamma(c + c'), t) \leq \mu(\gamma'(c), t) + \sigma(c')
\]

for all \( c, t \in \mathbb{R}_{\geq 0} \) with \( \gamma'(c) = 2\gamma(2c) \) and \( \sigma(c') = \mu(\gamma'(c'), 0) \).

Proof of Lemma 5. We apply the fact \( a + b \leq \max\{2a, 2b\} \) twice. First, for \( \gamma \) we get \( \gamma(c + c') \leq \gamma(\max\{2c, 2c'\}) \leq \gamma(2c) + \gamma(2c') \). Then for \( \mu \) we obtain

\[
\mu(\gamma(c + c'), t) \leq \mu(\gamma(2c) + \gamma(2c'), t) \\
\leq \mu(\max\{2\gamma(2c), 2\gamma(2c')\}, t) \\
\leq \mu(2\gamma(2c), t) + \mu(2\gamma(2c'), 0).
\]

Lemma 6. Suppose we are given \( \gamma \in \mathcal{K} \) and \( \mu \in \mathcal{KLD} \). Then there exists \( \mu' \in \mathcal{KLD} \) such that

\[
\gamma(\mu(c, t)) = \mu'(\gamma(c), t)
\]

holds for all \( c \in \mathbb{R}_{\geq 0} \) and \( t \in \mathbb{N} \).

Proof. We define \( \mu' : \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}_{\geq 0} \) iteratively by

\[
\mu'(c, 0) := r, \quad \mu'(c, t + 1) := g(\mu'(c, t))
\]

for all \( c \in \mathbb{R}_{\geq 0} \) and \( t \in \mathbb{N} \), where \( g(c) := \gamma(\mu(c, 1)) \). It is easy to see by induction over \( t \in \mathbb{N} \) that \( \mu' \) satisfies (25). Hence, \( \mu' \) is a \( \mathcal{K} \mathcal{L} \) function and by the iterative definition follows that \( \mu' \in \mathcal{KLD} \). □
Lemma 7. Suppose we are given $\mu_a, \mu_b \in \mathcal{KLD}$. Then there exists $\mu \in \mathcal{KLD}$ such that

$$\max_{t' \in [0,t]} \mu_a(c, t') + \max_{t' \in [0,t]} \mu_b(c, t') \leq \max_{t' \in [0,t]} \mu(2c, t')$$

holds for all $c \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{N}$.

Proof. First let us remark that

$$\max_{t' \in [0,t]} \mu_a(c, t') + \max_{t' \in [0,t]} \mu_b(c, t')$$

holds for all $c \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{N}$. Now we define $\mu : \mathbb{R}_{\geq 0} \times \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ recursively for all $c \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{N}$ by $\mu(c,0) := c$, $\mu(c,t+1) := g(\mu(c,t))$ with $g(c) := \max\{2\mu_a(c,1), 2\mu_b(c,1)\}$. To show (20), in view of (27), it suffices to show that $2\mu_j(c,t) \leq \mu(c,t)$ holds for all $j \in \{a, b\}$, $c \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{N}$. We fix $j \in \{a, b\}$ and $c \in \mathbb{R}_{\geq 0}$ and proceed by induction over $t \in \mathbb{N}$. The base case $t = 0$ is trivial. Suppose the induction hypothesis holds, then we derive $2\mu_j(c,t+1) = 2\mu_j(\mu_j(c,t),1) \leq 2\mu_j(\mu(2c,t),1) \leq g(\mu(2c,t)) = \mu(2c,t+1)$.

Proof of Lemma 3. We show that the relation $R_{123}(\varepsilon)$ defined by

$$\{(x_{12}, x_3, u_{12}, u_3) \in X_{12} \times X_3 \times U_{12} \times U_3) \mid (x_2, x_3, u_2, u_3) \in R_{23}(\varepsilon) \wedge (x_1, x_2) \in R_{12}, \forall x \}$$

fulfills the claim of the lemma.

First let us note that $R_{123}(\varepsilon') \subseteq R_{123}(\varepsilon)$ whenever $\varepsilon' \leq \varepsilon$ is directly inherited from the inclusion $R_{23}(\varepsilon') \subseteq R_{23}(\varepsilon)$ for $\varepsilon' \leq \varepsilon$. Moreover, $e(x_{12}, x_3) = e(x_2, x_3)$ whenever $(x_1, x_2) \in R_{12}, \forall x$ which implies that $(x_2, x_3) \in R_{23}(\varepsilon(x_2, x_3))$ and $(x_1, x_2) \in R_{12}, \forall x$ whenever $e(x_1, x_2) < \infty$. Hence, $R_{123}(\varepsilon)$ satisfies (16).

We proceed by checking 1) of Def. 4. Let $x_{12} \in X_{12} \subseteq R_{12}, \forall x$. Since for every $x_2 \in X_{20}$ there is $x_3 \in X_{30}$ with $(x_2, x_3) \in R_{23}(\varepsilon(k),\forall x)$, there exists $x_3 \in X_{30}$ with $(x_{12}, x_3) \in R_{123}(\varepsilon(k),\forall x)$.

Let us now check 2) of Def. 4. Let $(x_1, x_3) \in R_{123}(\varepsilon(k),\forall x)$ and $u_{12} \in U'_{12}(x_{12})$. This implies:

- a) $(x_2, x_3) \in R_{23}(\varepsilon(k),\forall x)$ and $(x_1, x_2) \in R_{12}(\varepsilon,k)$;
- b) $r_{12}(x_{12}, (u_{12}^c, u_{12}^d)) \not\in \emptyset$ for any $u_{12}^c \in U_{12}^c$.

Since $(x_2, x_3) \in R_{23}(\varepsilon(k),\forall x)$ and $u_{12}^c \in U_{12}^c(x_{12})$ we can choose $u_{12}^c \in U_3^c$ so that 2.a) of Def. 7 holds. Now for $u_{12}^d \in U_{12}^d$ and $x_3' \in r_3(x_3, u_3)$ we can pick $u_{12}^d \in U_2^d$ and $x_2' \in r_2(x_2, u_2)$ such that $(x_2, x_3, u_2, u_3) \in R_{23}(\varepsilon(k),\forall x)$ and $(x_2', x_3') \in R_{23}(\varepsilon')$ with $\varepsilon' = \kappa + \beta \varepsilon + \lambda d_{23}(u_2, u_3)$.

Moreover, from b) and (17) follows that $(x_1, x_2, u_1^c, u_2^c)$ satisfy 2.a) of Def. 7. Therefore, there exist $u_{12}^c \in U_{12}^c$ and $x_1' \in r_1(x_1, u_1)$ for our choice of $u_{12}^c$ and $x_2'$ so that $(x_1, x_2, u_1, u_2) \in R_{12}$ and $(x_1', x_2') \in R_{12}(\varepsilon,\forall x)$.

In the previous two paragraphs we showed $(x_{12}, x_{31}, u_{12}, u_3) \in R_{123}(\varepsilon(k),\forall x)$ and $(x_2', x_3') \in R_{23}(\varepsilon(k),\forall x)$ which implies that $R_{123}(\varepsilon(k),\forall x)$ is a $(\kappa, \beta, \lambda)$-acSR from $S_{12}$ to $S_3$ with the distance function given by $d_{123}(u_{12}, u_3) = d_{23}(u_2, u_3)$.

Proof of Lemma 5. We only need to show that $S_{12} = S_1 \times R_{123}(\varepsilon(k),\forall x)$ is non-blocking as defined in Def. 4. By definition of $S_{12}$ and (10) every reachable state $x_{12}$ of $S_{12}$ satisfies $(x_1, x_2) \in R_{12}(\varepsilon(x_1, x_2))$. Now it is easy to check with the help 2.a) in the Def. 7 that $S_{12}$ satisfies the non-blocking condition.
Proof of Lemma 4. We leave it to reader to check that the relation $R_{121}(\varepsilon) \subseteq X_{12} \times X_1 \times U_{12} \times U_1$ given by

$$\{(x_{12}, x_1', u_{12}, u_1') \mid (x_1, x_2, u_1, u_2) \in R_{12}(\varepsilon) \land x_1 = x_1' \land u_1 = u_1'\}$$

is an acSR from $S_{12}$ to $S_1$. □