A problem involving the $p$-Laplacian operator

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Abstract

Using a variational technique we guarantee the existence of a solution to the resonant Lane-Emden problem $-\Delta_p u = \lambda |u|^{q-2}u, u|_{\partial \Omega} = 0$ if and only if a solution to $-\Delta_p u = \lambda |u|^{q-2}u + f, u|_{\partial \Omega} = 0, f \in L^p(\Omega)$ ($p'$ being the conjugate of $p$) , exists for $q \in (1,p) \cup (p,p^*)$ under a certain condition for both the cases, i.e., $1 < q < p < p^*$ and $1 < p < q < p^*$ - the sub-linear and the super-linear cases.

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1 Introduction

The study of partial differential equations involving a $p$-laplacian differential operator has become a major case of study in the recent times although it is still far from being completely understood, especially when $p = 1$ or $\infty$. A few evidences of the limiting case can be found in [1], [2]. When $p = 2$, the usual Laplacian is obtained for which a vast literature exists ([4], [5] and the references therein). For $p \neq 2$ the p-Laplace operator has physical applications in the study of non-Newtonian fluids (dilatant fluids when $p > 2$) [6]. In practical life most of the problems are non-linear by nature for which a numerical solution is sought for, however, unearthing the existence of solution leads to a rich theory hidden behind the partial differential equation. The problems we are going to address in this article are the following.

Let $\Omega$ be a bounded subset of $\mathbb{R}^n$, $n \geq 3$ with a Lipschitz boundary $\partial \Omega$. Given $1 < p < \infty$ and $q \in (1,p) \cup (p,p^*)$, where $p^* = \frac{np}{n-p}$ if $1 < p < n$ and $p^* = \infty$ if $p \geq n$, we consider the following problems.

1. $-\Delta_p u = \lambda |u|^{q-2}u, u|_{\partial \Omega} = 0$. This problem is also known as the resonant Lane-Emden problem.

2. $-\Delta_p u = \lambda |u|^{q-2}u + f, f \in L^p(\Omega), u|_{\partial \Omega} = 0$.

where $\lambda$ is a real number, $\Delta_p = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$. Throughout this paper we shall refer the problems in 1 and 2 as the first and the second problem respectively.

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We call the first problem to be of sub-critical type if $1 < q < p < p^*$ and of super-critical type when $p^* > q > p > 1$. It is found in [7] that a unique solution exists to the first problem for the sub-critical case whereas uniqueness is lost for the super-critical case. Readers interested in knowing more about the first problem can refer to examples found in [8], [9], where the domain is ring shaped for $q \sim p^*$ and the solution is non-unique. Kawohl [10] showed the same but the domain which was considered is of annulus type with the annulus being sufficiently small in size. Dancer [11] showed that if $p = 2$ and $\Omega$ is a general domain then a unique solution exists to the first problem. Uniqueness is also guaranteed in [12] for the sub-linear case whereas a subdifferential method has been used to prove existence in [13] for both sub and super critical cases.

In this paper we will use a well known variational technique to show the existence of a solution in $W^{1,p}_0(\Omega) = \{ v \in L^p(\Omega) : \nabla v \in L^p(\Omega), v|_{\partial \Omega} = 0 \}$. A Fredholm type alternative is also proposed thus showing a connection between the first and the second problem. We organize the paper into two sections. In Section 2 we give the Mathematical formulation. In Section 3 we discuss a few preliminary results and the main result.

2 Mathematical formulation

The following definitions and theorems will be used in the main result we prove.

2.1 Definition: Let $X$ be a Banach space and $H : X \to \mathbb{R}$ a $C^1$ functional. It is said to satisfy the Palais-Smale condition (PS) if the following holds:

Whenever $\{u_n\}$ is a sequence in $X$ such that $\{H(u_n)\}$ is bounded and $H'(u_n) \to 0$ strongly in $X'$ (the dual space), then $\{u_n\}$ has a strongly convergent subsequence.

The (PS) condition is a strong condition as very “well-behaved” function do not satisfy it (Example: $f(x) = c$, $x \in \mathbb{R}$, $c$ a real constant).

We now state the following important theorem due to Ambrosetti and Rabinowitz [14] which is a common tool used in the theory of modern PDEs.

Mountain-pass theorem: Let $H : X \to \mathbb{R}$ be a $C^1$ functional satisfying (PS). Let $u_0, u_1 \in X$, $c_0 \in \mathbb{R}$ and $r > 0$ such that

1. $||u_1 - u_0|| > r$

2. $H(u_0), H(u_1) < c_0 \leq H(v), \forall v$ such that $||v - u_0|| = r$. Then $H$ has a critical value $c \geq c_0$ defined by

$$c = \inf_{\Gamma \in \varphi} \max_{t \in [0,1]} H(\Gamma(t))$$  (1)

where $\varphi$ is the collection of all continuous paths $\Gamma : [0,1] \to X$ such that $\Gamma(0) = u_0$, $\Gamma(1) = u_1$.  

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2.2 Weak formulation of the problem: We now give the weak formulation of the first problem. We say that $u \in W^{1,p}_{0}(\Omega)$ is a weak solution of the first problem if

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} uv dx = 0$$

for every $v \in W^{1,p}_{0}(\Omega)$.

The weak solutions of the Lane-Emden problem are the critical points of the energy function defined by

$$J_{q}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^{p}dx - \frac{\lambda}{q} \int_{\Omega} |u|^{q}dx.$$  

(3)

The following compact embedding theorems, due to Rellich-Kondrasov have been used in our work.

- if $p < n$, $W^{1,p}_{0}(\Omega) \hookrightarrow L^{q}(\Omega)$, $1 \leq q < p^{*}$,
- if $p = n$, $W^{1,n}_{0}(\Omega) \hookrightarrow L^{q}(\Omega)$, $1 \leq q < \infty$,
- if $p > n$, $W^{1,p}_{0}(\Omega) \hookrightarrow C(\bar{\Omega})$.

We consider the non-homogeneous counterpart of the first problem - which is the second problem - and is as follows.

$$-\Delta_{p} u = \lambda|u|^{q-2} u + f,$$

$$u|_{\partial \Omega} = 0,$$

(4)

where $f \in L^{p'}(\Omega)$, $p'$ being the conjugate of $p$ and is equal to $\frac{p}{p-1}$. Let the corresponding functional be denoted by $J$ which is defined as follows.

$$J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^{p}dx - \frac{\lambda}{q} \int_{\Omega} |u|^{q}dx - \int_{\Omega} f u dx.$$  

(5)

The Fréchet derivative of $J$, which is in $W^{-1,p'}_{0}(\Omega)$ where $p' = \frac{p}{p-1}$, is

$$< J'(u), v > = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} uv dx - \int_{\Omega} f v dx,$$

(6)

$\forall v \in W^{1,p}_{0}(\Omega)$. Thus $u \in W^{1,p}_{0}(\Omega)$ is a weak solution of the second problem if

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} uv dx - \int_{\Omega} f v dx = 0.$$  

For the sake of further analysis we redefine the functional as follows.

$$J_{q}(u) = -\chi_{(1,p)}(q) J(u) + \chi_{(p,p')}(q) J(u),$$

(7)

where $\chi$ is the indicator function. From the sections which follow we shall use the functional in (7).
3 Few preliminary results and the main theorem

The main result of this paper is as follows. The problem \(-\Delta_p u = \lambda |u|^{q-2}u, u|_{\partial \Omega} = 0\) has a weak solution if and only if the problem \(-\Delta_p u = \lambda |u|^{q-2}u + f, u|_{\partial \Omega} = 0\), where \(f \in L^{p/p-1}(\Omega)\), has a weak solution. We prove the result for \(p < n\). The case of \(p \geq n\) follows the same proof as in the case \(p < n\) which is based on the results on compact embedding stated after equation (3). But first we present a few technical lemmas on which the proof of this result relies upon.

We first assume that a solution exists to the problem

\[
- \Delta_p u = \lambda |u|^{q-2}u, \\
u|_{\partial \Omega} = 0.
\]

(8)

**Theorem 3.1.** The mapping \(J_q\) defined in (7) is a \(C^1\)-functional over \(W^1_{0,p}(\Omega)\).

**Proof.** We first prove that the functional \(J'\) is continuous which will imply that \(J'_q\) is continuous and hence the theorem will follow. Consider

\[
|J'(u), v| \leq \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx + |\lambda| \int_\Omega |u|^{q-1}v dx + \int_\Omega |f||v| dx
\]

\[
\leq ||\nabla u||_{\frac{p}{p-1}}||\nabla v||_p + |\lambda||u||_{\frac{q}{q-1}}||v||_q + ||f||_{\frac{p}{p-1}}||v||_p
\]

\[
\leq \left[||\nabla u||_{\frac{p}{p-1}} + C_1|\lambda||u||_{\frac{q}{q-1}} + C_2||f||_{\frac{p}{p-1}}\right]||\nabla v||_p, \forall v \in W^1_{0,p}(\Omega), (9)
\]

where \(C_1, C_2\) are the constants due to the embedding of \(W^1_{0,p}(\Omega)\) in \(L^q(\Omega)\) for \(q \in [1,p^*]\). From (8)\&(9) one can see that \(J\) is a \(C^1\) functional over \(W^1_{0,p}(\Omega)\).

**Theorem 3.2.** There exists \(u_0, u_1 \in W^1_{0,p}(\Omega)\) and a positive real number \(c_0\) such that \(J_q(u_0), J_q(u_1) < c_0\) and \(J_q(v) \geq c_0\), for every \(v\) satisfying \(||v - u_0||_{1,p} = r\).

**Proof.** Let \(u_0 = 0\). Clearly \(u_0\) is a solution of (8) and \(J_q(0) = 0\). Now let \(v \in B(0,1)\) in \(W^1_{0,p}(\Omega)\) and consider \(v = u_0 + r w\) for \(r > 0\) and hence \(||v - u_0||_{1,p} = r\). We first show the existence of \(r\) such that \(||v - u_0||_{1,p} = r_0\) and for which \(J(v) \geq c_0\) for each \(v \in B(0,r_0)\).

Let \(p < q < p^*\). Now

\[
J_q(u_0 + r w) - J_q(u_0) = \frac{r^p}{p} \int_\Omega |\nabla w|^p dx - \frac{r^q\lambda}{q} \int_\Omega |w|^q dx - r \int_\Omega fwdx,
\]

\[
= \frac{r^p}{p} - \frac{r^q\lambda}{q} \int_\Omega |w|^q dx - r \int_\Omega fwdx.
\]

(10)

Further, \(|w|_{1,p} = 1\) and hence \(\int_\Omega w^p dx \leq \int_\Omega |w|^p dx \leq c||w||_p^p \leq c_1||w||_{1,p} = c_1\). Similarly, \(\int_\Omega w^q dx \leq c_2\). Using these arguments leads to

\[
J_q(u_0 + r w) - J_q(u_0) \geq r \left[ \frac{r^{p-1}}{p} - \frac{r^{q-1}\lambda}{q} c_2 - c_1^{1/p}||f||_{p'} \right],
\]

\[
= c'.
\]

(11)
We first analyze the term \[ \frac{r^{p-1}}{p} - \frac{r^{q-1}}{q} c_2 - c_1^{1/p} \|f\|_{p'} = F(r) \] (say). Clearly \( F(0) < 0 \) and for \( r_0 = \left( \frac{q(p-1)}{p(q-1)} \right)^{1/p} \) we see that \( F'(r_0) = 0 \). A bit of calculus guarantees that \( F''(r_0) < 0 \) and hence \( r_0 \) is a maximizer of \( F \). If \( 0 < \lambda < \lambda_1 = \frac{q(p-1)}{p(q-1)} \left( \frac{p(q-1)}{q-p} \right)^{\frac{1}{q-p}} c_1^{1/p} \|f\|_{p'} \), then \( F(r_0) > 0 \). As \( r \to \infty \) we have \( F(r) \to -\infty \). Hence there exists \( r_1, r_2 > 0 \) and \( r_1 < r_0 < r_2 \) such that \( F(r) > 0 \) for each \( r \in (r_1, r_2) \). We choose \( r = r_0 \) such that \( \|v - u_0\|_{1,p} = r_0 \) and for which \( J_q(v) \geq c' \) for each \( v \in B(0, r_0) \). Similarly, if \( 1 < q < p \) then according to the definition of \( J_q \) we now have

\[
J_q(u_0 + rw) - J_q(u_0) = -J(u_0 + rw) + J(u_0) \\
\geq r \left[ -\frac{r^{p-1}}{p} + \frac{r^{q-1}}{q} \lambda_0 c_2 + \frac{c_1^{1/p}}{p} \|f\|_{p'} \right], \\
= c''.
\] (12)

Using the same argument as for the case of \( p < q < p^* \) we find \( r \) and \( 0 < \lambda < \lambda_2 \) such that \( J_q(v) \geq c'' \) for all \( \|v - u_0\| = r \). We choose \( \lambda' = \min\{\lambda_1, \lambda_2\} \) such that \( 0 < \lambda < \lambda' \) and \( c_0 = \min\{c', c''\} \).

**Choice of \( u_1 \):** Let \( w_p \) be the first eigen vector of \(-\Delta_p \), i.e., \(-\Delta_p w_p = \lambda_p|w_p|^{p-1}w_p \), where \( \lambda_p \) is the first eigen value of \(-\Delta_p \). The first eigen value of the \( p \)-laplacian operator is strictly positive [3]. Consider the function \( g = kw_p, k \in \mathbb{R}, \|w_p\|_{1,p} = 1 \) and \( p < q < p^* \). Note that,

\[
J_q(g) = \left( \frac{k^p}{p} - \frac{\lambda k^q}{q} \int_{\Omega} |w_p|^q dx \right) - kC,
\]

where \( C = \int_{\Omega} f w_p dx \). Since \( p < q < p^* \), we observe \( k \) can be chosen arbitrarily large so that \( \frac{k^p}{p} - \frac{\lambda k^q}{q} \int_{\Omega} |w_p|^q dx - k_0C < 0 \). Then \( J_q(kw_p) < 0 \) and hence \( J_q(kw_p) < J_q(u_0) \). Thus we can choose \( u_1 = kw_p \), where \( k_0 > r_0 \). Then \( \|u_1 - u_0\|_{1,p} > r_0 \). Similarly for \( 1 < q < p \) we have

\[
J_q(g) = \left( -\frac{k^p}{p} + \frac{\lambda k^q}{q} \int_{\Omega} |w_p|^q dx \right) + kC,
\]

and \( k \) can be chosen large enough to make \( J_q(g) < 0 \). Hence the result. \( \square \)

**Theorem 3.3.** \( J_q \) satisfies the Palais-Smale condition.

**Proof.** Let us consider the case for which \( p < q < p^* \). The other case for \( 1 < q < p \) follows similarly. Let \( u_n \) be a sequence in \( W_0^{1,p}(\Omega) \) such that \( |J_q(u_n)| \leq M \) and \( J_q'(u_n) \to 0 \) as \( n \to \infty \) in \( W_0^{1,p'}(\Omega) \), \( p' \) being the conjugate of \( p \). Now

\[
J_q(u_n) = \frac{1}{p} \int_{\Omega} |\nabla u_n|^p dx - \frac{\lambda}{q} \int_{\Omega} |u_n|^q dx - \int_{\Omega} fu_n dx, \quad (13)
\]

\[
\langle J_q'(u_n), v \rangle = \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx - \lambda \int_{\Omega} |u_n|^{q-2} u_n v dx - \int_{\Omega} fv dx, \forall v \in W_0^{1,p}(\Omega).
\] (14)
Consider the following.

\[
\langle J'_q(u_m), u_m \rangle = \int_\Omega |\nabla u_m|^p dx - \lambda \int_\Omega |u_m|^q dx - \int_\Omega fu_m dx, \tag{15}
\]

\[
J_q(u_m) = \frac{1}{p} \int_\Omega |\nabla u_m|^p dx - \frac{\lambda}{q} \int_\Omega |u_m|^q dx - \int_\Omega fu_m dx,
\]

\[
= \frac{1}{p} |u_m|_{1,p}^p - \frac{\lambda}{q} \int_\Omega |u_m|^q dx - \int_\Omega fu_m dx,
\]

\[
\lambda \int_\Omega |u_m|^q dx = \frac{q}{p} |u_m|_{1,p}^p - qJ_q(u_m) - q \int_\Omega fu_m dx,
\]

\[
\frac{p-q}{p} |u_m|_{1,p}^p = \langle J'_q(u_m), u_m \rangle - qJ_q(u_m) - q \int_\Omega fu_m dx. \tag{16}
\]

This implies that \( |u_m|_{1,p} \) is bounded. The above inequality in (14) clearly shows that \( u_n \) is bounded in \( W_0^{1,p}(\Omega) \) and hence by Eberlein-Šmulian’s theorem (refer Dunford-Schwartz [1; p. 430] [15]) it has a weakly convergent subsequence, say \( u_{n_k} \), in \( W_0^{1,p}(\Omega) \).

**Claim.** The sequence \( \{u_{n_k}\} \) is strongly convergent in \( W_0^{1,p}(\Omega) \).

**Proof.** Applying limit \( k \to \infty \) to (14) (refer Appendix) and using the strong convergence of \( (u_{n_k}) \) in \( L^q(\Omega) \) due to compact embedding we obtain

\[
\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \lambda \int_\Omega |u|^{q-2} u v dx + \int_\Omega f v dx, \tag{17}
\]

and we pass on the limit to (15) we get

\[
\lim_{n \to \infty} |u_{n_k}|_{1,p}^p = \lambda \int_\Omega |u|^q dx + \int_\Omega fu dx = |u|_{1,p}^p. \tag{18}
\]

Thus a weakly convergent sequence which is convergent in norm is strongly convergent. Hence \( u_{n_k} \to u \) in \( W_0^{1,p}(\Omega) \) as \( k \to \infty \).

Thus by the Mountain-pass theorem an extreme point for \( H \) exists in \( W_0^{1,p}(\Omega) \)

We summarize the results proved in Theorems 3.1, 3.2 and 3.3 in the form of a theorem as follows.

**Theorem 3.4.** Suppose \(-\Delta_p u = \lambda |u|^{q-2} u, u|_{\partial \Omega} = 0 \) has a solution. Then

1. the functional \( J_q = -\chi_{(1,p)}(q)J(u) + \chi_{(p,p')} (q)J(u) \) where \( J(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{\lambda}{q} \int_\Omega |u|^q dx - \int_\Omega fu dx \) is \( C^1 \) and satisfies the Palais-Smale condition,

2. \( J_q \) satisfies the hypothesis of the Mountain-Pass theorem.

Therefore \( J_q \) has an extreme point in \( W_0^{1,p}(\Omega) \). In other words \(-\Delta_p u = \lambda |u|^{q-2} u + f, f \in L^p(\Omega), u|_{\partial \Omega} = 0 \) has a solution whenever \( \lambda \in (0, \lambda') \) where \( \lambda' = \min\{\lambda_1, \lambda_2\} \) as found in Theorem 3.2.
Conversely, suppose a solution to the problem
\[
- \Delta_p u = \lambda |u|^{q-2}u + f, \quad f \in L^p(\Omega), \quad |u|_{\partial \Omega} = 0.
\]
(19)

We subdivide this situation into two different cases - namely, \(1 < q < p\) (the sub-linear case) and \(1 < p < q < p^*\) (the super-linear cases). Let \((f_n) \subset L^p(\Omega)\) be a sequence such that \(f_n \to 0\) in \(L^p(\Omega)\). By the assumption, to each \(f_n\) there exists a solution, say \(u_n\).

We have \(q \in (1, p) \cup (p, p^*)\) and
\[
B[u, v] = \int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla vdx - \lambda \int_{\Omega} |u|^{q-2}uvdx,
\]
(20)

where \(B\) is a ‘non linear form’ in two variables \(u\) and \(v\). It is easy to check that \(B(., .)\) is the Fréchet derivative of the \(C^1\) functional \(\frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{1}{q} \int_{\Omega} |u|^q\) and hence is continuous. Clearly, for each \(v \in W_0^{1,p}(\Omega)\) we have
\[
B[u_n, v] = \int_{\Omega} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla vdx - \lambda \int_{\Omega} |u_n|^{q-2}u_nvdx,
\]
\[
= \int_{\Omega} f_nvdx,
\]
\[
\leq ||f_n||_{p'} ||v||_p \to 0 \quad \text{as} \quad n \to \infty.
\]
(21)

Hence \(\int_{\Omega} f_nvdx \to 0\) as \(n \to \infty\). Consider \(T_n(v) = \int_{\Omega} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla vdx\). Then \(T_n\) is bounded linear over \(W_0^{1,p}(\Omega)\) and \(||T_n|| = |||\nabla u_n|^{p-1}||_{p'}\). From the above definition of \(T_n\), for a fixed \(v \in W_0^{1,p}(\Omega)\) we have the sequence \((T_n(v))\) to be bounded which implies that \((T_n(v))\) is pointwise bounded. Thus by the uniform boundedness principle \((||T_n||)\) is bounded. Thus \(||\nabla u_n||_p\) is bounded. Hence, there exists a subsequence \((u_{n_k})\) which weakly converges to \(u_\infty\) with respect to the \(||\cdot||_{1,p}\) in \(W_0^{1,p}(\Omega)\). Hence we have
\[
\lim_{k \to \infty} \int_{\Omega} |\nabla v|^{p-2}\nabla v \cdot \nabla u_{n_k} dx = \int_{\Omega} |\nabla v|^{p-2}\nabla v \cdot \nabla u_\infty dx, \forall v \in W_0^{1,p}(\Omega).
\]
(22)

for a fixed \(l\). Therefore, since \(u_{n_k} \rightharpoonup u_\infty\) in \(W_0^{1,p}(\Omega)\) implies that \(|\nabla u_{n_k}|^{p-1} \rightharpoonup |\nabla u_\infty|^{p-1}\) (for a subsequence) in \(L^p(\Omega)\) (Refer Appendix). But \(W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow W^{-\frac{1}{p}, p'}(\Omega)\) and hence
\[
\lim_{l \to \infty} \int_{\Omega} |\nabla u_{n_l}|^{p-2}\nabla u_{n_l} \cdot \nabla vdx = \int_{\Omega} |\nabla u_\infty|^{p-2}\nabla u_\infty \cdot \nabla vdx, \forall v \in W_0^{1,p}(\Omega),
\]
(23)
Hence, \( \lim_{k \to \infty} \int_{\Omega} |\nabla u_{n_k}|^p dx = \int_{\Omega} |\nabla u_\infty|^p dx \). It immediately can be concluded that there exists a \( u_\infty \) such that \( u_n \to u_\infty \) in \( W_0^{1,p}(\Omega) \). Hence using the continuity of \( B[.,.] \) in (20) we have

\[
\lim_{n \to \infty} B[u_n, v] = \lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v - \lim_{n \to \infty} \lambda \int_{\Omega} |u_n|^{q-2} u_n v dx,
\]

\[
= \lim_{n \to \infty} \int_{\Omega} f_n v dx,
\]

\[
\Rightarrow B[u_\infty, v] = 0, \forall v \in W_0^{1,p}(\Omega).
\]

In other words

\[
\int_{\Omega} |\nabla u_\infty|^{p-2} \nabla u_\infty \cdot \nabla v dx - \lambda \int_{\Omega} |u_\infty|^{q-2} u_\infty v dx = 0, \forall v \in W_0^{1,p}(\Omega).
\]

We summarize the result proved as follows.

**Theorem 3.5.** Suppose \(-\Delta_{p} u = \lambda |u|^{q-2} u + f, f \in L^p(\Omega), u|_{\partial \Omega} = 0 \) has a solution. If \( u_n \) is a solution of the PDE corresponds to \( f_n \), where \( f_n \subset L^p(\Omega) \) such that \( f_n \to 0 \) in \( L^p(\Omega) \), then we have \( B[u_\infty, v] = 0 \) for each \( v \in W_0^{1,p}(\Omega) \) and thus \( u_\infty \) is a solution to \(-\Delta_{p} u = \lambda |u|^{q-2} u, u|_{\partial \Omega} = 0 \).

4 Appendix

We show that

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx, \forall v \in W_0^{1,p}(\Omega).
\]

We divide the explanation into two cases:

**Case 1:** When \( p > 2 \).

This implies that \( p' \), the conjugate of \( p \), should be less than 2, i.e., \( 1 < p' < 2 < p \). Thus we have \( W_0^{1,p}(\Omega) \rightarrow_{\text{compact}} L^{p'}(\Omega) \) (since \( W_0^{1,q}(\Omega) \rightarrow_{\text{compact}} L^{q}(\Omega) \) for \( q \in [1,p^*) \)).

Since \( \nabla u_n \) converges weakly to, say \( \nabla u \), in \( L^{p}(\Omega) \), hence \( |\nabla u_n| \rightarrow |\nabla u|, v \to 0 \) for each \( v \in L^{p}(\Omega) \). Thus \( \nabla u_n \to \nabla u \) in \( L^{p}(\Omega) \). Hence \( ||\nabla u_n||_{p'} \to ||\nabla u||_{p'} \) because \( p' < 2 < p \). By the Riesz-Fischer theorem [10], there exists a subsequence of \( \nabla u_n \) which converges pointwise a.e., i.e., \( \nabla u_n(x) \to \nabla u(x) \). So \( \nabla u_n(x)^{p-1} \to |\nabla u(x)|^{p-1} \) and hence \( |\nabla u_n|^{p-1} \to |\nabla u|^{p-1} \) in \( L^{p}(\Omega) \). Thus we have \( \lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx, \forall v \in W_0^{1,p}(\Omega) \).

**Case 2:** When \( p < 2 \).

This implies that \( p' \), the conjugate of \( p \), should be greater than 2, i.e., \( p < 2 < p' \).

Look at the map \( F : W_0^{1,p}(\Omega) \to L^{p'}(\Omega) \) defined by \( u \mapsto |\nabla u|^{p-1} \). Consider the range of \( F \), i.e., \( R(F) = \{|\nabla u|^{p-1} : u \in W_0^{1,p}(\Omega)\} \).

Observe that the map \( F \) is bounded in the sense that bounded sets are mapped to bounded
sets. Hence if \( u_n \rightharpoonup u \) in \( W_0^{1,p}(\Omega) \) implies that \( (u_n) \) is bounded in \( W_0^{1,p}(\Omega) \). Hence \((F(u_n)) = (|\nabla u_n|^{p-1})\) is bounded in \( L^{p'}(\Omega) \). Since \( L^{p'}(\Omega) \) is reflexive, hence there exists a subsequence of \( |\nabla u_n|^{p-1} \) which weakly converges to, say, \( w \) in \( L^{p'}(\Omega) \).

We have the following.
\[ u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ so } |\nabla u_n|^{p-1} \rightharpoonup w \text{ in } L^{p'}(\Omega). \] This implies that
\[ \langle |\nabla u_n|^{p-1} - w, v \rangle \to 0, \forall v \in L^{p}(\Omega) \]
Since \( p < 2 < p' \) hence \( |\nabla u_n|^{p-1} - w \in L^{p}(\Omega) \). Thus \( \| |\nabla u_n|^{p-1} - w|_2 \to 0 \) and hence \( \| |\nabla u_n|^{p-1} - w|_p \to 0 \). Therefore we have a subsequence of \( (|\nabla u_n|^{p-1}) \) such that \( |\nabla u_n|^{p-1} \to w \) pointwise a.e. (implying \( |\nabla u_n| \to w^{\frac{1}{p-1}} \) pointwise a.e.) and so \( |\nabla u_n| \to w^{\frac{1}{p-1}} \) in \( L^{p}(\Omega) \). Hence \( w = |\nabla u|^{p-1} \).

Thus in all the above cases we found the following.
\[
\lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{26}
\]
Hence by the compact embedding due to Rellich-Kondrachov it can be concluded \( u_n \to u \) in \( L^{q}(\Omega) \). Thus we also have
\[
\lim_{n \to \infty} \int_{\Omega} |u_n|^{q-2} u_n \cdot \nabla v dx = \int_{\Omega} |u|^{q-2} u \cdot \nabla v dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{27}
\]

5 Conclusions

The resonant Lane-Emden problem has been studied. An existence result has been established to the non-homogeneous Lane-Emden problem for the sub-linear - \( 1 < q < p < p^* \) and the super-linear case - \( 1 < p < q < p^* \) for \( \lambda \in (0, \lambda') \) - \( \lambda' \) being sufficiently large - if it is assumed that a non-trivial solution exists to the homogeneous Lane-Emden problem for the sub-linear - \( 1 < q < p < p^* \) and the super-linear case - \( 1 < p < q < p^* \), which is basically an eigen value problem. We further proved the ‘converse’ that if the non-homogeneous problem has a solution then a solution to the homogeneous problem exists for both the sub and the super critical cases.

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