The \( P_1 - P_{1NC} \) Finite Element Method for 1D wave simulation using Shallow Water Equations

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Abstract. We study a simple numerical scheme based on a new type of Finite Element Method (FEM) to solve the 1D Shallow Water Equations. In the new scheme, the surface elevation variable is approximated by a linear continuous basis function (\( P_1 \)) and the velocity potential variable is approximated by the one-dimensional discontinuous linear non-conforming basis function (\( P_{1NC} \)). Here, we implement the \( P_1 - P_{1NC} \) finite element pair to solve the 1D Shallow Water Equations on a structured grid, whereas the Runge Kutta method is adopted for time integration. We verified the resulting scheme by conducting several simulations such as a standing wave simulation, and propagation of an initial hump over sloping bathymetry. The resulting scheme free from numerical damping error, conservative and both standing wave and shoaling phenomena are well simulated.

Keywords: finite element method, non-conformal basis function, structured grid, Shallow Water Equations.

1. Introduction

One of the numerical methods is very popular for determining solutions for Differential Equations (DE) is the Finite Element Method (FEM). The first principle of FEM is to approach the solution of DE by using variational formulation. Types of FEM are distinguished based on basis function. In general, there are two types of basis functions, i.e. continuous and discontinuous basis function, with their elements can be a triangular, quadrilateral, etc. The first use of FEM related to the discontinuous basis function is proposed by Crouzeix-Raviart in 1973, known as the Crouzeix-Raviart element (CR Element). Application of CR element to the hyperbolic system (such as Shallow Water Equations) known as \( P_1 - P_{0} \) finite element pair. It means that the velocity variable is approximated by \( P_{0} \) basis functions, where its continuously only across the triangle boundaries at midpoint nodes. Meanwhile, the surface elevation variable is approximated by \( P_0 \) basis function which is piecewise constant.

Analogous to CR Element method, Hua and Thomasset [1] propose a new finite element method that modifies \( P_0 \) to be \( P_1 \) so the surface elevation variable is approximated by a continuous piecewise linear basis function, called "hat function". Their new finite element pair was successfully solved a two-dimensional Shallow Water Equations. They proposed a staggered scheme in space with a set of continuous basis function (\( P_1 \)) for elevation variables and discontinuous basis function (\( P_{1NC} \)) for velocity variables. Instead of using classical FEM as

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in [2] and [3], some authors have been succeeded in using this two-dimensional $P_1 - P_{1NC}^1$ finite element pair: Greenberg et al [5], Hanert et al [6], Harig et al [7], Cui et al [8].

Following this successful, in this paper, we apply a simple numerical scheme based on $P_1 - P_{1NC}^1$ finite element pair to solve 1D SWE as governed in section 2. As far as the author’s knowledge, there is no $P_1 - P_{1NC}^1$ element method pairs that are used to solve the 1D Shallow Water Equations. This is due to the discontinuous basis function for 1D which is related to the two-dimensional discontinuous basis function $P_{1NC}^1$, which has not been proposed. In the case of Hua and Thomasset [1], the discontinuous basis function is constructed directly from a two-dimensional Shallow Water Equations problem. Here, a new approach using 1D $P_{1NC}^1$ basis function is discussed in section 3 together with $P_1$ basis function. In section 4, we discussed a variational numerical scheme to seek the solution of the Shallow Water Equations. Result and discussion are given in section 5, while the conclusion and remarks will be given in the last section.

2. Governing Equation

Let $x, z, t$ denote horizontal, vertical coordinates and time respectively. We consider a layer of ideal fluid bounded above by free surface $z = \eta(x,t)$ and below by an impermeable topography $z = -d(x)$ with horizontal velocity of the fluid particles $u(x,t)$ and total depth $h(x,t) = d(x) + \eta(x,t)$ as sketches in figure 1. Under the gravitational acceleration $g = 9.8 m/s^2$ as the restoring force, the surface motion in shallow area is governed by Shallow Water Equations as follows.

\[
\begin{align*}
\partial_t h &= -\partial_x (hu) \\
\partial_t u + g \partial_x \eta + u \partial_x u &= -C_f u |u| / h
\end{align*}
\]

where $C_f$ is friction coefficient. Here we consider linearized SWE with non-moving bottom, i.e. $\partial_t d = 0$ as follows.

\[
\begin{align*}
\partial_t \eta + \partial_x (du) &= 0 \\
\partial_t u + g \partial_x \eta &= 0
\end{align*}
\]  

Under the assumption that the flow of the fluid is irrotational, there exist a scalar $\phi(x,t)$ such that $u(x,t) = \phi_x(x,t)$. So that equation (1) can be written in another form as follows.

\[
\begin{align*}
\partial_t \eta &= -\partial_x (d \partial_x \phi) \\
\partial_t \phi &= -g \eta
\end{align*}
\]  

Figure 1: Sketch of fluid domain.
3. \( P_1 - P_{NC}^1 \) basis function

One of the essential parts of the finite element method is the characteristic of the basis function. In this paper two basis functions are used, i.e. \( P_1 \) and \( P_{NC}^1 \) basis function. The term “\( P_1 \)” means that the physical variables are represented by a set of piecewise linear basis functions which is continuous over the whole domain, in finite element literature its called conformal basis function. Here we use hat function denoted as \( T \) as illustrated in figure 2a and defined as follows.

\[
T_k(x) = \begin{cases} 
  \frac{x - x_{k-1}}{\Delta x}, & x_{k-1} \leq x < x_k, \\
  \frac{x_{k+1} - x}{\Delta x}, & x_k \leq x < x_{k+1}, \quad k = 1, 2, \ldots, N + 1, \\
  0, & x < x_{k-1}, x \geq x_{k+1}.
\end{cases}
\]

Meanwhile, the term “\( P_{NC}^1 \)” means that the physical variables are represented by a set of piecewise linear basis function which is discontinuous in some part of the domain, in finite element literature its called non-conformal basis function. Here we used \( P_{NC}^1 \) basis function denoted as \( \psi \) as illustrated in figure 2b and defined as follows.

\[
\psi_k(x) = \begin{cases} 
  1 - 2 \left( \frac{x - x_{k-1}}{\Delta x} \right), & x_{k-1} \leq x < x_k \\
  2 \left( \frac{x - x_k}{\Delta x} \right) - 1, & x_k \leq x < x_{k+1}, \quad k = 1, 2, \ldots, N, \\
  0, & x < x_{k-1}, x \geq x_{k+1}.
\end{cases}
\]

where \( N, \Delta x \) denoted numbers of sub-intervals and spatial grid size respectively which is discussed later in the next section. Note that this \( P_{NC}^1 \) is a 1D version of the two-dimensional non-conformal basis function \( P_{NC}^1 \) as discussed in [1].

4. 1-D Finite Element Discretization

In this section, we seek the solution of (2-3) by derived its weak formulation with constant depth \( d(x) = d_0 \). To derive the weak form, let \( \vec{V}(x), \vec{W}(x) \) be a test functions of equation (2-3) in test space \( E \) and \( \Phi \) respectively and satisfying hard wall boundary condition such as

\[
E = \{ \eta(x,t) : \eta_a(a,t) = 0, \eta_b(b,t) = 0 \}, \quad \Phi = \{ \phi(x,t) : \phi_x(a,t) = 0, \phi_x(b,t) = 0 \}
\]

then by multiplying (2-3) by test function \( \vec{V}(x), \vec{W}(x) \) respectively to yield

\[
\int_a^b [\partial_t \eta(x,t) \vec{V}(x)]dx = -\int_a^b \partial_x [d_0 \partial_x \phi(x,t)] \vec{V}(x)dx
\]

\[
\int_a^b \partial_t \phi(x,t) \vec{W}(x)dx = -g \int_a^b \eta(x,t) \vec{W}(x)dx
\]
Applying integration by part for the right side of equation (5) we have
\[ \int_a^b \eta \tilde{V}(x) dx = - \left[ d_0 \phi_x \tilde{V}(x) \right]_a^b - d_0 \int_a^b \phi_x \tilde{V}_x(x) dx \]
The first term vanishes because \( \phi_x(a,t), \phi_x(b,t) \in \Phi \). So we have
\[ \int_a^b \eta \tilde{V}(x) dx = d_0 \int_a^b \phi_x \tilde{V}_x(x) dx \] (7)
is the weak formulation of equation (2-3).

Next, we assume that interval \([a,b]\) is divided into \(N\) numbers sub-intervals with \(N+1\) grid points such that \(a = x_0 < x_1 < ... < x_{N+1} = b\) where \(a, b \in \mathbb{R}\). Suppose that those \(N\) sub-intervals has spatial grid size with equal length \(\Delta x = \frac{(b-a)}{N}\) and nodal points \(x_k = k\Delta x\), for \(k = 0, 1, 2, ..., N + 1\). A finite element approximation to the exact solution of equation (2-3) are found by replacing \(\eta(x,t)\) and \(\phi(x,t)\) by finite element approximation solution \(\eta^E(x,t)\) and \(\phi^E(x,t)\) respectively where \(\eta^E(x,t) \in E\) and \(\phi^E(x,t) \in \Phi\). Hence, \(\eta^E(x,t)\) and \(\phi^E(x,t)\) can be stated as a linear combination of conformal basis function \((P_1)\),"or simply hat function" and non-conformal basis function \((P_1^{NC})\) defined in section 3 such that,
\[ \eta(x,t) \approx \eta^E(x,t) = \sum_{k=1}^{N+1} \eta_k(t) T_k(x), \] (8)
\[ \phi(x,t) \approx \phi^E(x,t) = \sum_{k=1}^{N+1} \phi_k(t) \psi_k(x), \] (9)
where \(\eta_k, \phi_k\) denote surface elevation and velocity potential nodal values. Because the test function \(\tilde{V}(x), \tilde{W}(x)\) belong to test space \(E\) and \(\Phi\) respectively then we can take \(\tilde{V}(x), \tilde{W}(x)\) as \(T_k(x), \psi_k(x), k = 1, 2, ..., N + 1\). After rearrangement between integral, differential and summation and using (8-9) the discrete weak formulation of (6-7) can be written as follows.
\[ \partial_t \sum_{k=1}^{N+1} \eta_k(t) \int_a^b T_k(x) T_j(x) \, dx = d_0 \sum_{k=1}^{N+1} \phi_k(t) \int_a^b \partial_x \psi_k(x) \partial_x T_j(x) \, dx, \] (10)
\[ \partial_x \sum_{k=1}^{N+1} \phi_k(t) \int_a^b \psi_k(x) \psi_j(x) \, dx = -g \sum_{k=1}^{N+1} \eta_k(t) \int_a^b T_k(x) \psi_j(x) \, dx. \] (11)

In matrix form it reads as follows.
\[ \begin{pmatrix} M & 0 \\ 0 & M_c \end{pmatrix} \partial_t \vec{Y}(t) = \begin{pmatrix} 0 & d_0 S_1 \\ -g S_2 & 0 \end{pmatrix} \vec{Y}(t), \quad \vec{Y}(t) = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{N+1} \\ \phi_1 \\ \vdots \\ \phi_{N+1} \end{pmatrix} \] (12)
where: for \(k, j = 0, 1, ..., N + 1\),
\[ M = [m_{kj}], \quad m_{kj} = \int_a^b T_k(x) T_j(x) \, dx, \quad M_c = [m^*_{kj}], \quad m^*_{kj} = \int_a^b \psi_k(x) \psi_j(x) \, dx \]
\[ S_1 = [s_{kj}], \quad s_{kj} = \int_a^b \partial_x \psi_k(x) \partial_x T_j(x) \, dx, \quad S_2 = [s^*_{kj}], \quad s^*_{kj} = \int_a^b T_k(x) \psi_j(x) \, dx \]
After some direct calculation, the entry of the mass matrix $M$ and $M^c$ are

$$[m_{kj}] = \Delta x \times \begin{cases} 
1/3, & k = j = 1, N + 1 \\
2/3, & k = j \\
1/6, & k = j \pm 1 \\
0, & \text{elsewhere}
\end{cases}$$

and the entry for two other matrices are:

$$[s_{kj}] = \left(\Delta x\right)^2 \times \begin{cases} 
-2, & k = j = 1, N + 1 \\
-2, & k = j \\
1, & k = j \pm 1 \\
0, & \text{elsewhere}
\end{cases}, 
[s^*_kj] = \Delta x \times \begin{cases} 
-2/6, & k = j = 1, N + 1 \\
-2/6, & k = j \\
1/6, & k = j \pm 1 \\
0, & \text{elsewhere}
\end{cases}.$$

All four resulting matrices were tridiagonal, which is a low computational cost. Furthermore, for the general case in which the depth is varying, we should modify the entry of $S_1$ as follows.

$$S_1 = [s_{kj}], 
 s_{kj} = \int_a^b d(x)\partial_x\psi_k(x)\partial_xT_j(x)dx, \ k, j = 1, ..., N + 1 \quad (13)$$

Numerical quadrature formula such as trapezoidal, Simpson’s rule, Romberg, Gauss-Legendre can be adopted to compute (13). Meanwhile, equation (12) is an ordinary differential equation system of the first order. This system can be solved by using time integration schemes such as Euler, Crank-Nicholson, Runge-Kutta, etc. In this paper, we use the ode solver from MATLAB toolbox, i.e. ODE45.

5. Result and Discussion

5.1. Standing Wave

Take a computational domain $0 \leq x \leq 900$, constant depth $d(x) = d_0 = 80m$, $g = 9.8m/s^2$, observation time $T = 500$ s, $\Delta x = 1$ m, initial value $\vec{Y}(0) = [\eta_0, \phi_0]$, where $\eta_0 = \cos\left(\frac{\pi x}{900}\right)$, $\phi_0 = 0$ denoted initial surface profile and initial velocity potential respectively. The result of the simulation using the proposed scheme is shown in figure 3. The free surface evolution continuously oscillates for a long period of simulation as shown in figure 3a. We also recorded
the wave signal during the simulation at several locations, \( x = 0, 450, 900 \) m as plotted in figure 3b. It is clear from the plot of surface elevation \( \eta \) at \( x = 0 \) and \( x = 900 \) m, the wave oscillates with constant amplitude i.e. 1 m tall. Meanwhile, the free surface at \( x = 450 \) m stays zero. Furthermore, we can observe that the wave has a period \( P \approx 64 \) s. Hence, the wave frequency is \( f = \frac{1}{P} \approx 0.015625 \) Hz. We can conclude that our proposed scheme is free from numerical damping error.

5.2. Wave Shoaling

Take a computational domain \( 0 \leq x \leq 4000, T = 120 \) s, initial value as multiple Gaussian humps with amplitude = 2m tall, the width of the first and second hump’s are 300 m long, and we settle down the center of the first and second hump’s at \( x = 300 \) and 1500 respectively. Bottom topography is flat depth \( d_1 = 100 \) m (upstream) and \( d_2 = 2 \) m (downstream) connected with plane beach with slope 100 : 2000. The result from our second simulation is presented in figure 4. Suppose from steady-state water condition, a multiple Gaussian initial condition breaks to be four solitary waves propagates with half of its amplitude \( \approx 1 \) m tall. At \( x = 2000 \), the wave starts to climb in the sloping area and the wave amplitude starts to increase compared to the waves that are still in the deeper area. In this sloping area, the shoaling phenomena happen, i.e. wave amplitude increases as depth decreases (amplified). To capture shoaling phenomena we use Maximum Temporal Amplitude (MTA) \( \eta_{\text{max}} = \max_t \eta(x,t) \), tools to capture the propagation history as introduced in [9]. It is clear from the plot of \( \eta \) at several location \( x = 1901 \) and \( x = 3567 \) that the maximum amplitude of the wave is \( \eta_{\text{max}} = 1 \) and \( \eta_{\text{max}} = 2.615 \). When then sloping area is ended at \( x = 3500 \), the waves enter the shallow area and the shoaling phenomena are over. The maximum amplitude achieves = 2.635. Furthermore, the shoaling phenomena must satisfy WKB formula \( A(x) \) defined as

\[
A(x) = A_1 \left( \frac{d_1}{d(x)} \right)^{1/4}
\]

where \( A_1, d_1 \) denoted maximum wave amplitude before shoaling and maximum water depth respectively. The second simulation result as in figure 4 shows that the MTA relatively close to WKB formulation. It means shoaling effects can be simulated by our scheme properly.

![Figure 4: Snapshots of free surfaces at subsequent times.](image-url)
It is well known that the numerical conservative scheme is better than non-conservative ones. Here we also conduct numerical experiments using several spatial grid size $\Delta x = 0.1, 0.5, 1$ m and temporal grid size $\Delta t = 0.1, 0.5, 1$ s for several observation time $T = 0, 10, 100, 500$ s. Note that the shallow water equation posses mass conservative law as follows.

$$M(t) = \int_a^b \eta(x,t)dx = \int_a^b \eta_0(x,t)dx.$$  \hspace{1cm} (15)

Table 1: Numerical results of $M(t)$.

| time  | $\Delta x = \Delta t = 0.5$ | $\Delta x = 1, \Delta t = 0.5$ | $\Delta x = 1, \Delta t = 0.1$ | $\Delta x = 0.1, \Delta t = 0.5$ |
|-------|-----------------|-----------------|-----------------|-----------------|
| 0     | 531.7361532216762 | 531.7361532201761 | 531.7361532201761 | 531.736153221564 |
| 10    | 531.7361532201761 | 531.7361532199730 | 531.7361532199744 | 531.736153221561 |
| 100   | 531.7361532189391 | 531.7361532189430 | 531.7361531739579 | 531.7361532221561 |
| 500   | 531.7361532363818 | 531.736153237358  | 531.736153237358  | 531.7361532221561 |

We calculate the discrete law of conservative mass by applying numerical quadrature $\text{trapz}$ in Matlab ToolBox to the (15). Table I shows that the discrete law of conservative mass for several combinations of spatial and temporal grid size in subsequent observation time is relatively constant. It is indicated that our proposed finite element scheme is conservative.

6. Conclusion and Future Works

The non-conformal basis function ($P_{NC}^1$) has been used in the sense of the 1D case. Together with conformal basis function (hat function) a simple numerical finite element pair $P_1 - P_{NC}^1$ scheme to study Shallow Water Equations has been presented. The resulting scheme free from numerical damping error, conservative, and both standing wave and shoaling phenomena were well simulated.

From a numerical point of view, a flexible scheme can be obtained and extended to a two-dimensional problem. Flexibility comes from the fact that computation can use various boundary conditions including complex geometry. Future research will be focused on the enhancement of the numerical model to incorporate absorbing boundary conditions as well as wave influx. Importantly, this $P_1 - P_{NC}^1$ finite element pair will be applied to the fully nonlinear SWE model.

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References

[1] Hua, B., Thomasset, F 1984. *Tellus*. 36A 157
[2] Lawrence, C., Aditya, D., & Van Groesen, E. 2018. *Wave motion*. 76 78-102.
[3] Kumar, P. 2018. *Ocean Engineering*. 165 386-398.
[4] Daniel Y. Le Roux., Andrew Staniforth., Charles A. Lin 1998. *Monthly Weather Review*. 126 1931
[5] Greenberg, D. A., Murty, T. S., Ruffman, A 1993. *Marine Geodesy*. 16 153
[6] Hanert, E., Le Roux, D. Y., Legat, V., Deleersnijder, E 2005. *Ocean Modelling*. 10 115
[7] Harig, S., Chaeroni, Pranowo, W. S., Behrens, J 2008. *Ocean Dynamics*. 58 429
[8] Cui, H., Pietrzak, J.D., Stelling, G.S 2010. *Ocean Modelling*. 35 16
[9] Andonowati A, Marwan U, van Groesen EWC 2003. *Proc. Int. Conf. on Port and Maritime R&D and Technology (Singapore)*. p 111-116