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Left-invariant almost para-Kähler structures on six-dimensional nilpotent Lie groups

Abstract

In this paper, we consider left-invariant para-complex structures on six-dimensional nilpotent Lie groups. A complete list of six-dimensional nilpotent Lie groups that admit para-Kähler structures is obtained, explicit expressions for para-complex structures are found, and curvature properties of associated para-Kähler metrics are investigated. It is shown that paracomplex structures are nilpotent and the corresponding para-Kähler metrics are Ricci-flat.

1 Introduction

As is known [2], nilpotent Lie groups, with the exception of the Abelian case, do not admit left-invariant positive-definite Kähler metrics. However, pseudo-Kählerian (i.e. with a pseudo-Riemannian metric) structures may exist. In [4], a complete list of 13 classes of non-commutative six-dimensional nilpotent Lie groups was obtained that admits pseudo-Kähler structures. In [11], a more complete study of the above classes of six-dimensional pseudo-Kählerian nilpotent Lie groups was carried out. Recently, para-complex and para-Kähler structures have been of great interest. Therefore, the question of invariant para-Kähler structures on six-dimensional nilpotent Lie groups is natural. In this article, we will show that 15 classes of non-commutative six-dimensional nilpotent Lie groups admit para-Kähler structures.

We recall the main concepts and facts that will be used in the work. An almost para-complex structure on a $2n$-dimensional manifold $M$ is a field $J$ of endomorphisms $J : TM \to TM$ such that $J^2 = Id$, and the ranks of the eigen-distributions $T^\pm := \ker(Id \mp J)$ are equal. An almost paracomplex structure $J$ is said to be integrable if the distributions $T^\pm$ are involutive. In this case, the $J$ is called a para-complex structure. The Nijenhuis tensor $N$ of an almost paracomplex structure $J$ is defined by the equality $N_J(X,Y) = [X,Y] + [JX,JY] - J[JX,Y] - J[X,JY]$, for any vector fields $X, Y$ on $M$. As in the complex case, a para-complex structure $J$ is integrable if and only if $N_J = 0$. A para-Kähler manifold can be defined as a symplectic manifold $(M, \omega)$ with a compatible para-complex structure $J$, i.e. such that $\omega(JX, JY) = -\omega(X, Y)$. In this case, the metric $g(X, Y) = \omega(X, JY)$ is defined on $M$, which is a pseudo-Riemannian neutral signature. Note that $g(JX, JY) = -g(X, Y)$. The paper [11] presents a survey of the theory of para-complex structures and considers invariant para-complex and para-Kähler structures on homogeneous spaces.

Note that the phrase "para-Kählerian manifold" is also used in another sense. A. Gray in [3] noted that the remarkable geometric and topological properties of Kähler manifolds are largely due to the fact that the curvature tensor $R$ satisfies the special Kähler identity: $g(R(X, Y)Z, W) = g(R(X, Y)JZ, JW)$ for any vector fields $X, Y, Z, W$, where $J$ is a complex structure tensor compatible with the Riemannian metric $g$. However, the class of manifolds with this property is somewhat wider. In Rizza’s 1974 [9] paper, almost Hermitian manifolds satisfying the above Kähler identity are called para-Kähler manifolds. There are many papers devoted to the study of such para-Kähler manifolds, see, for example, [8] and [10], where one can also find references to classical and more recent papers.
In this paper, the para-Kähler manifolds are considered from the point of view of paracomplex geometry. For more details on such a para-Kählerian geometry, see in review \[\text{[1]}\]. Note for comparison that such para-Kähler manifolds satisfy \[\text{[1]}\] the following identity: 
\[g(R(X, Y)Z, W) = -g(R(X, Y)JZ, JW),\]
where \(J\) is the tensor of the para-complex structure, compatible with the pseudo-Riemannian metric \(g\).

We will consider left-invariant (almost) para-complex structures on the Lie group \(G\), which are given by the left-invariant endomorphism field \(J : TG \to TG\) of the tangent bundle \(TG\). Since such tensor \(J\) is defined by a linear operator on the Lie algebra \(\mathfrak{g} = T_eG\), we will say that \(J\) is an invariant almost para-complex structure on the Lie algebra \(\mathfrak{g}\). In this case, the integrability condition for \(J\) is also formulated at the Lie algebra level: \(N_J(X, Y) = [X, Y] + [JX, JY] - J[JX, Y] - J[X, JY] = 0\), for any \(X, Y \in \mathfrak{g}\). It follows from the integrability condition for \(J\) that the eigenspaces \(\mathfrak{g}^+\) and \(\mathfrak{g}^-\) of the operator \(J\) are subalgebras. Therefore, the para-complex Lie algebra \(\mathfrak{g}\) can be represented as a direct sum of two subalgebras:

\[\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-\].

We also note that for an element \(Z\) of the center \(\mathcal{Z}(\mathfrak{g})\) of the Lie algebra, the vector \(JZ\) may not be central, but it immediately follows from the integrability condition that \(ad_{JZ}\) commutes with \(J\):

\[\begin{align*}
[JZ, JX] &= J[JZ, X], & ad_{JZ} \cdot J &= J \cdot ad_{JZ}.
\end{align*}\]

A left-invariant symplectic structure \(\omega\) on a Lie group \(G\) is given by a 2-form \(\omega\) of maximum rank on the Lie algebra \(\mathfrak{g}\) and satisfying the condition \(\omega([X, Y], Z) - \omega([X, Z], Y) + \omega([Y, Z], X) = 0\), \(\forall X, Y, Z \in \mathfrak{g}\). In this case, the Lie algebra \(\mathfrak{g}\) will be called symplectic.

Recall that a subspace \(W \subset \mathfrak{g}\) is called \(\omega\)-isotropic if and only if \(\omega(W, W) = 0\) and \(W\) is called \(\omega\)-Lagrangian if it is \(\omega\)-isotropic and \(\omega(W, u) = 0\) implies that \(u \in W\). Subspaces \(U, V \subset W\) of a symplectic space \((W, \omega)\) is called \(\omega\)-dual if for any vector \(u \in U\) there exists a vector \(v \in V\) such that \(\omega(u, v) \neq 0\) and vice versa, \(\forall v \in V, \exists u \in U, \omega(u, v) \neq 0\).

A left-invariant para-Kählerian structure on a Lie group is given by a pair \((\omega, J)\) consisting of a symplectic form \(\omega\) on \(\mathfrak{g}\) and a para-complex structure \(J\) on \(\mathfrak{g}\) compatible with \(\omega\), i.e., such that \(\omega(JX, JY) = -\omega(X, Y)\). A compatible pair \((\omega, J)\) defines a para-Kählerian pseudo-Riemannian metric \(g(X, Y) = \omega(X, JY)\) on \(\mathfrak{g}\). Moreover, the subalgebras \(\mathfrak{g}^+\) and \(\mathfrak{g}^-\) are isotropic for the metric \(g\) and \(\omega\)-Lagrangian.

The lower central series of the Lie algebra \(\mathfrak{g}\) is defined inductively: \(C^0\mathfrak{g} = \mathfrak{g}, C^1\mathfrak{g}, \ldots\) \(C^k\mathfrak{g}\) is the center of the Lie algebra \(\mathfrak{g}\) for some \(k\). A Lie algebra \(\mathfrak{g}\) is called to be nilpotent if \(C^k\mathfrak{g} = 0\) for some \(k\). For a nilpotent Lie algebra, there is also an increasing central sequence of ideals \(\{\mathfrak{g}_k\}\): \(\mathfrak{g}_0 = \{0\} \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_{s-1} \subset \mathfrak{g}_p = \mathfrak{g}\), where the ideals \(\mathfrak{g}_k\) are defined inductively by the rule:

\[\mathfrak{g}_k = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset \mathfrak{g}_{k-1}\}, k \geq 1.\]

In particular, \(\mathfrak{g}_1 = \mathcal{Z}(\mathfrak{g})\) is the center of the Lie algebra. Moreover, the first derived ideal \(C^1\mathfrak{g}\) is included in the ideal \(\mathfrak{g}_{p-1}\). For a given almost para-complex structure \(J\), the ideals \(\{\mathfrak{g}_k\}\) defined above are generally not \(J\)-invariant. One can define an increasing sequence of \(J\)-invariant ideals \(\mathfrak{a}_k(J)\) as follows:

\[\mathfrak{a}_0(J) = 0, \mathfrak{a}_1(J) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset \mathfrak{a}_{k-1}(J)\} = [JX, \mathfrak{g}] \subset \mathfrak{a}_{k-1}(J), k \geq 1.\]

It is clear that \(\mathfrak{a}_k(J) \subset \mathfrak{g}_k\) for \(k \geq 1\). Obviously, the ideal \(\mathfrak{a}_1(J)\) lies at the center \(\mathcal{Z}(\mathfrak{g}) = \mathfrak{g}_1\) of the Lie algebra \(\mathfrak{g}\).  

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Definition 1.1. A left-invariant almost para-complex structure $J$ is called nilpotent if for a sequence of ideals $\{a_k(J)\}$ there exists a number $s$ such that $a_s(J) = \mathfrak{g}$.

Let $\nabla$ be the Levi-Civita connection corresponding to the left-invariant pseudo-Riemannian metric $g$. It is determined from the six-term formula [7], which for left-invariant vector fields $X, Y, Z$ on a Lie group takes the form: $2g(\nabla_X Y, Z) = g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X)$.

Recall that the curvature tensor $R(X, Y)$ is defined by the formula $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$, and the Ricci tensor $\text{Ric}$ is the convolution of the curvature tensor with respect to the first lower and upper indices, $\text{Ric}_{jk} = R^i_{ijk}$. The metric $g$ is called Ricci-flat if $\text{Ric} = 0$.

2 Left-invariant symplectic and para-Kähler structures on Lie algebras

Let $\mathfrak{g}$ be a Lie algebra with a symplectic form $\omega$, an almost paracomplex structure $J$ compatible with $\omega$, and a pseudo-Riemannian metric $g(X, Y) = \omega(X, JY)$. Let us present some simple facts about the first derived ideal $C^1(\mathfrak{g})$, the center $Z(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$, and the ideals $\{\mathfrak{g}_k\}$ and $\{a_k(J)\}$.

Proposition 2.1. $\omega(C^1(\mathfrak{g}), Z(\mathfrak{g})) = 0$ for any symplectic form $\omega$ on $\mathfrak{g}$.

Proof. Immediately follows from the formula $d\omega(X, Y, Z) = \omega([X, Y], Z) - \omega([X, Z], Y) + \omega([Y, Z], X) = \omega([X, Y], Z) = 0, \forall X, Y, Z \in \mathfrak{g}$. \hfill $\square$

Proposition 2.2. For any symplectic form $\omega$ on $\mathfrak{g}$ and an almost paracomplex structure $J$ compatible with $\omega$, we have $\omega(C^1(\mathfrak{g}) \oplus J(C^1(\mathfrak{g})), a_1(J)) = 0$.

Proof. Since $a_1(J)$ belongs to center $Z(\mathfrak{g})$, then $\omega(C^1(\mathfrak{g}), a_1(J)) = 0$. The equality $\omega(J(C^1(\mathfrak{g})), a_1(J)) = 0$ follows from the $J$-invariance of $a_1(J)$ and formulas $\omega(JX, JY) = -\omega(X, Y)$. \hfill $\square$

Corollary 2.3. For any almost para-Kähler structure $(\mathfrak{g}, \omega, g, J)$, the ideal $a_1(J) \subset \mathfrak{g}_1$ is orthogonal to the subspace $C^1(\mathfrak{g}) \oplus J(C^1(\mathfrak{g})): g(C^1(\mathfrak{g}) \oplus J(C^1(\mathfrak{g})), a_1(J)) = 0$.

Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $\mathfrak{g}_0 = \{0\} \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_{s-1} \subset \mathfrak{g}_p = \mathfrak{g}$ be the increasing central sequence of ideals.

Proposition 2.4. For any nilpotent almost para-Kähler structure $J$, the ideal $a_{p-1}(J)$ contains $C^1(\mathfrak{g}) \oplus J(C^1(\mathfrak{g}))$.

Proof. Since $\mathfrak{g}_p = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset a_{p-1}(J)$ and $[JX, \mathfrak{g}] \subset a_{p-1}(J)\} = \mathfrak{g}$, then $C^1(\mathfrak{g}) \subset a_{p-1}(J)$. Therefore $J(C^1(\mathfrak{g}) \subset J(a_{p-1}(J)) = a_{p-1}(J)$. \hfill $\square$

Proposition 2.5. For any left-invariant (pseudo) Riemannian structure $g$ on the nilpotent Lie algebra $\mathfrak{g}$, the following properties hold:

1. If the vector $X$ belongs to center $Z(\mathfrak{g})$ of the Lie algebra, then $\nabla_X Y = \nabla_Y X, \forall Y \in \mathfrak{g}$.

2. If the vectors $X$ and $Y$ belong to center of the Lie algebra, then $\nabla_X Y = 0$.

Proof. It follows from the formula $2g(\nabla_X Y, Z) = g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X)$ for the covariant derivative $\nabla$ on the Lie group. Indeed, if the vector $X$ lies at the center of the Lie algebra, then $2g(\nabla_X Y, Z) = g(X, [Z, Y])$ and $2g(\nabla_Y X, Z) = g([Y, X], Z) + g([Z, Y], X) + g(Y, [Z, X]) = g([Z, Y], X), \forall Z \in \mathfrak{g}$.
Corollary 2.6. If the vectors $X, Y, Z$ belong to the center of the Lie algebra, then $R(X, Y)Z = 0$.

Let $J$ be a nilpotent almost para-complex structure on a nilpotent Lie algebra $\mathfrak{g}$ and let $a_0 = \{0\} \subset a_1(J) \subset \cdots \subset a_n(J) \subset a_s(J)$ be the corresponding sequence of ideals. Note that although the ideals $a_k(J)$ are $J$-invariant, they need not be even-dimensional.

Proposition 2.7. If the vector $X$ belongs to the ideal $a_1(J) \subset Z(\mathfrak{g})$ of the Lie algebra, then $\nabla_X Y = \nabla_Y X = 0$, $\forall Y \in \mathfrak{g}$.

Proof. Let $X \in a_1(J) \subset \mathfrak{g}_1 = Z(\mathfrak{g})$ and $Z, Y \in \mathfrak{g}$. Then the covariant derivative formula and Corollary 2.3 imply that $2g(\nabla_X Y, Z) = g(X, [Z, Y]) = 0$. $\square$

Corollary 2.8. If the vector $X$ belongs to the ideal $a_1(J) \subset \mathfrak{g}_1 = Z(\mathfrak{g})$ of the Lie algebra, then $R(X, Y)Z = R(Z, Y)X = 0$, $\forall Y, Z \in \mathfrak{g}$.

Proof. It follows from $\nabla_X Y = \nabla_Y X = 0$, $\forall Y \in \mathfrak{g}$ and the formulas $R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - [X, Y]Z$ and $R(Z, Y)X = \nabla_Z(\nabla_Y X) - \nabla_Y(\nabla_Z X) - [Z, Y]X$. $\square$

3 Left-invariant para-Kähler structures on six-dimensional nilpotent Lie algebras

A classification list of six-dimensional symplectic nilpotent Lie algebras is presented in the article Goze M., Khakimdjanov Y., Medina A. [3]. Many Lie algebras in this list have an increasing sequence of ideals $Z(\mathfrak{g}) = \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathfrak{g}_3 = \mathfrak{g}$ of dimensions 2, 4, and 6 (we say that such a Lie algebra is of type (2,4,6)). We choose the complement $A$ to $\mathfrak{g}_2$ and the complement $B$ to $Z(\mathfrak{g})$ in $\mathfrak{g}_2$. Then such a Lie algebra can be represented as a direct sum of two-dimensional subspaces

$$\mathfrak{g} = A \oplus B \oplus Z(\mathfrak{g}).$$

From the definition of ideals $\mathfrak{g}_3 = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset \mathfrak{g}_2\} = \mathfrak{g}$ and $\mathfrak{g}_2 = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset Z(\mathfrak{g})\}$ it immediately follows that the additional subspaces $A$ and $B$ have the following properties:

$$[A, A] \subset \mathfrak{g}_2 = B \oplus Z(\mathfrak{g}), \quad [A, B] \subset Z(\mathfrak{g}).$$

For example, for the algebra $\mathfrak{g}_2$ of the list in [3] with Lie brackets $[e_1, e_2] = e_4, [e_1, e_4] = e_6, [e_2, e_3] = e_6$ we have: $Z(\mathfrak{g}) = \mathbb{R}\{e_5, e_6\}$, $\mathfrak{g}_2 = \mathbb{R}\{e_2, e_4, e_5, e_6\}$, $\mathfrak{g}_3 = \mathfrak{g}$. Then one can choose as subspaces $A$ and $B$: $B = \mathbb{R}\{e_3, e_4\}$ and $A = \mathbb{R}\{e_1, e_2\}$.

The symplectic structures of the classification list in [3] show that for algebras of type (2,4,6) additional subspaces $A$ and $B$ can be chosen so that the symplectic form $\omega$ on the subspace $B$ is non-degenerate, and the subspaces $A$ and $Z(\mathfrak{g})$ are $\omega$-isotropic and $\omega$-dual. For Lie algebras of other types, instead of $Z(\mathfrak{g})$ it is necessary to choose a two-dimensional subspace $C \subset Z(\mathfrak{g})$.

Theorem 3.1. Let a six-dimensional symplectic Lie algebra $(\mathfrak{g}, \omega)$ have a decomposition in the form of a direct sum of two-dimensional subspaces

$$\mathfrak{g} = A \oplus B \oplus C,$$

where $C \subset Z(\mathfrak{g})$, $[A, A] \subset B \oplus C$ and $[A, B] \subset C$. Assume that $B \oplus C$ is an Abelian subalgebra, the subspaces $A$ and $C$ are $\omega$-isotropic and $\omega$-dual, on subspace $B$ the form $\omega$ is non-degenerate and $\omega(B \oplus C, C) = 0$. Then for any nilpotent almost para-complex structure $J$ compatible with $\omega$ and for which the subspaces $B$ and $C$ are $J$-invariant, the Levi-Civita connection $\nabla$ of the corresponding pseudo-Riemannian metric $g_J$ has property location:
1. $\nabla_X Y \in B \oplus C$, $\forall X,Y \in \mathfrak{g}$.

2. $\nabla_X Y, \nabla_Y X \in C$, $\forall X \in \mathfrak{g}$, $\forall Y \in B \oplus C$.

3. $\nabla_X Y = 0$, $\forall X, Y \in B \oplus C$.

Proof. Property 1. Let $X,Y \in \mathfrak{g}$. Then $[X,Y] \subset B \oplus C$. Suppose that $\nabla_X Y$ does not belong to space $B \oplus C$, i.e., has a nonzero component from $A$. Then there exists a vector $Z \in C \subset Z(\mathfrak{g})$ such that $\omega(\nabla_X Y, Z) \neq 0$. At the same time, $2\omega(\nabla_X Y, Z) = 2g(\nabla_X Y, Z) = g([X,Y], Z) + g([Z,X], Y) + g([Z,Y], X) = g([X,Y], Z)$. The last equality follows from the property $\omega(B \oplus C, C) = 0$.

Property 2. Let now $X \in \mathfrak{g}$ and $Y \in B \oplus C$. Then $\nabla_X Y$ belongs to space $B \oplus C$. Assume that $\nabla_X Y$ does not belong to space $C$, i.e., has a nonzero component from $B$. Then the condition that the form $\omega$ is non-degenerate on $B$ and the equality $\omega(B \oplus C, C) = 0$ implies that there exists a vector $Z \in B$ such that $\omega(\nabla_X Y, Z) \neq 0$. At the same time, taking into account the commutativity of $B \oplus C$ and the inclusion $[A,B] \subset C \subset \mathfrak{a}_1(J) \subset Z(\mathfrak{g})$, we have: $2\omega(\nabla_X Y, Z) = 2g(\nabla_X Y, Z) = g([X,Y], Z) + g([Z,X], Y) + g([Z,Y], X) = g([X,Y], Z) + g([Z,X], Y) + \omega([X,Y], Z) = 0$. The last equality follows from the fact that $\omega(B \oplus C, C) = 0$. Thus, $\nabla_X Y \in C$. The inclusion $\nabla_Y X \in C$ follows from the formula $\nabla_X Y - \nabla_Y X = [X,Y]$.

Property 3. Let $X,Y \in B \oplus C$. Then, for any $Z \in \mathfrak{g}$, the following holds in exactly the same way: $2g(\nabla_X Y, Z) = g([X,Y], Z) + g([Z,X], Y) + g([Z,Y], X) = \omega([X,Y], Z) + \omega([Z,Y], X) = 0$.

Corollary 3.2. Under the assumptions of Theorem 1, if the vector $X$ belongs to space $B \oplus C$, then the following equalities hold: $R(X,Y)Z = R(Z,Y)X = 0$ for any $Y, Z \in \mathfrak{g}$.

Proof. Let $X \in B \oplus C$ and let $Y, Z \in \mathfrak{g}$. In the formula $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ we have $[X,Y] \in C \subset \mathfrak{a}_1(J)$, so by Proposition 2.7, $\nabla_{[X,Y]} Z = 0$. Further, $\nabla_Y Z \in B \oplus C$, so $\nabla_X \nabla_Y Z = 0$ by property 3 of the theorem. By property 2 of the theorem, we have $\nabla_X Z \in C \subset \mathfrak{a}_1(J) \subset Z(\mathfrak{g})$. Then $\nabla_Y \nabla_X Z = 0$ by Proposition 2.7.

Let $X \in B \oplus C$ and $R(Z,Y)X = \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X - \nabla_{[Z,Y]} X$. We have $[Z,Y] \in B \oplus C$, so $\nabla_{[Z,Y]} X = 0$. Since $\nabla_Y X \in C \subset \mathfrak{a}_1(J) \subset Z(\mathfrak{g})$, then $\nabla_Z \nabla_Y X = 0$ by Proposition 2.7. Similarly, $\nabla_Y \nabla_Z X = 0$. Therefore, $R(Z,Y)X = 0$.

We choose a basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ of the Lie algebra $\mathfrak{g}$ adapted to the decomposition $\mathfrak{g} = A \oplus B \oplus C$, i.e., such that $A = \mathbb{R}\{e_1, e_2\}$, $B = \mathbb{R}\{e_3, e_4\}$ and $C = \mathbb{R}\{e_5, e_6\}$.

Corollary 3.3. Under the assumptions of Theorem 1, for any $X,Y,Z \in \mathfrak{g}$, the inclusion $R(X,Y)Z \in C \subset Z(\mathfrak{g})$ holds. In the basis adapted to the expansion $\mathfrak{g} = A \oplus B \oplus C$, the curvature tensor can have, up to symmetries, only four non-zero components $R^1_{1,2,1}$, $R^6_{1,2,1}$, $R^5_{1,2,2}$, $R^6_{1,2,2}$. In particular, the Ricci tensor is zero.

Proof. Let $X,Y,Z \in \mathfrak{g}$. In the formula $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ we have $[X,Y] \in B \oplus C$, so $\nabla_{[X,Y]} Z \in C$. Further, $\nabla_Y Z \in B \oplus C$, so $\nabla_X \nabla_Y Z \in C$. Similarly, $\nabla_Y \nabla_Z X \in C$. Thus, $R(X,Y)Z \in C$. The assertion about nonzero components follows from Corollary 3.2.

Consider the question of which of the Lie algebras in the classification list in [6] admit compatible para-complex structures $(\omega, J)$. The results are presented in the table of Theorem
2 below. Each Lie algebra in the table has its number from the list of symplectic Lie algebras in [2]. For each symplectic Lie algebra \((\mathfrak{g}, \omega)\) of this table there exist multiparameter families of para-complex structures \(J\) compatible with \(\omega\). The table of Theorem 2 shows one of them, the simplest \(J\), which is presented in block form in the basis \(\{e_1, e_2, e_3, e_4, e_5, e_6\}\) of the Lie algebra and, in accordance with the expansion \(\mathfrak{g} = \mathbb{R}\{e_1, e_2\} \oplus \mathbb{R}\{e_3, e_4\} \oplus \mathbb{R}\{e_5, e_6\}\). The symbol \(R\) denotes the Riemann tensor. The dual basis is denoted by the symbols \(\{e^1, \ldots, e^6\}\).

**Theorem 3.4.** A six-dimensional nilpotent noncommutative Lie algebra admits a para-Kähler structure \((J, \omega)\) if and only if it is symplecto-isomorphic to one of the algebras in the table below. The admissible para-complex structures \(J\) are nilpotent, and the corresponding pseudo-Riemannian metrics are Ricci-flat.

| \(\mathfrak{g}\) | Lie brackets | Para-Kähler structure |
|-----------------|--------------|----------------------|
| \(\mathfrak{g}_6\) | \(\{e_1, e_2\} = e_3,\) \(\{e_1, e_3\} = e_4,\) \(\{e_1, e_4\} = e_5,\) \(\{e_2, e_3\} = e_6,\) | \(\omega = e^1 \wedge e^6 + e^2 \wedge e^4 + e^2 \wedge e^5 - e^3 \wedge e^4,\) \(J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\) \(R \neq 0\) |
| \(\mathfrak{g}_{10}\) | \(\{e_1, e_2\} = e_4,\) \(\{e_1, e_3\} = e_5,\) \(\{e_1, e_4\} = e_6,\) \(\{e_2, e_3\} = e_6,\) | \(\omega = e^1 \wedge e^6 + e^2 \wedge e^5 - e^3 \wedge e^4 - e^4 \wedge e^6,\) \(J = \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} -1 & 0 \\ -a(a+2) & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -a - 2 \\ 0 & 1 \end{pmatrix},\) \(R \neq 0\) when \(a \neq 0\) |
| \(\mathfrak{g}_{11}\) | \(\{e_1, e_2\} = e_4,\) \(\{e_1, e_3\} = e_5,\) \(\{e_2, e_3\} = e_6,\) \(\{e_1, e_4\} = e_6,\) | \(\omega = e^1 \wedge e^6 + e^2 \wedge e^5 - e^3 \wedge e^4 + \lambda e^2 \wedge e^6,\) \(J = \begin{pmatrix} -1 & -2\lambda \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\) \(R \neq 0\) when \(a \neq 1\) |
| \(\mathfrak{g}_{12}\) | \(\{e_1, e_2\} = e_4,\) \(\{e_1, e_3\} = e_5,\) \(\{e_2, e_3\} = e_6,\) \(\{e_1, e_4\} = e_6,\) | \(\omega = -e^1 \wedge e^5 + \lambda e^2 \wedge e^6 + (\lambda - 1)e^3 \wedge e^4,\) \(J = \begin{pmatrix} 0 & \lambda \alpha \\ \lambda \alpha & 0 \end{pmatrix} \times \begin{pmatrix} 0 & \lambda \alpha \\ \lambda \alpha & 0 \end{pmatrix} \times \begin{pmatrix} \lambda \alpha^2 - 1 \\ \lambda \alpha^2 - 1 \end{pmatrix} \times \begin{pmatrix} 0 & \lambda \alpha \\ \lambda \alpha & 0 \end{pmatrix},\) \(R \neq 0\) |
| \(\mathfrak{g}_{13}\) | \(\{e_1, e_2\} = e_4,\) \(\{e_1, e_3\} = e_5,\) \(\{e_1, e_4\} = e_6,\) \(\{e_2, e_3\} = e_6,\) | \(\omega_1 = e^1 \wedge e^6 + \lambda e^2 \wedge e^3 + (\lambda - 1)e^4 \wedge e^5,\) \(J_1 = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -a(\lambda + 1) & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -a \lambda & -1 \end{pmatrix},\) \(R = 0\) |
| \(\mathfrak{g}_{14}\) | \(\{e_1, e_2\} = e_4,\) \(\{e_1, e_3\} = e_5,\) \(\{e_1, e_4\} = e_6,\) | \(\omega = e^1 \wedge e^6 + b^2 \wedge e^3 + c^2 \wedge e^4,\) \(J = \begin{pmatrix} \frac{a \alpha^2 - 1}{1-a^2} & -b \\ -b & \frac{a \alpha^2 - 1}{1-a^2} \end{pmatrix},\) \(R = 0\) |
| \(\mathfrak{g}_{15}\) | \(\{e_1, e_2\} = e_4,\) \(\{e_2, e_3\} = e_5,\) \(\{e_1, e_4\} = e_6,\) | \(\omega = -e^1 \wedge e^6 + e^2 \wedge e^3 + e^2 \wedge e^6,\) \(J = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ a(\alpha - 2) \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 2 - a & -1 \end{pmatrix},\) \(R \neq 0\) |
| \(\mathfrak{g}_{16}\) | \(\{e_1, e_2\} = e_5,\) \(\{e_1, e_4\} = e_6,\) \(\{e_1, e_3\} = e_6,\) \(\{e_2, e_4\} = e_6,\) | \(\omega = e^1 \wedge e^6 + e^2 \wedge e^3 - e^4 \wedge e^5,\) \(J = e_1 \otimes e_6 - e_2 \otimes e^2 + e_3 \otimes e_3 - e_4 \otimes e^4 + e_5 \otimes e^5 - e_6 \otimes e^6 + \frac{1}{2a^2}e_2 \otimes e^3 + a e_4 \otimes e^1 + a e_6 \otimes e^5,\) \(R \neq 0\) |
### \( \mathfrak{g}_{17} \)

\[ \{e_1, e_3\} = e_5, \quad [e_1, e_4] = e_6, \quad [e_2, e_3] = e_6 \]

\[ \omega = e^i \wedge e^b + e^j \wedge e^a + e^k \wedge e^4 \]

\[ J = \begin{pmatrix} 1 & 0 & 0 \\ a^{-1} & b & -a \\ -a & 1 & 0 \end{pmatrix}, \quad R = 0 \]

### \( \mathfrak{g}_{18} \)

\[ \{e_1, e_2\} = e_4, \quad [e_1, e_3] = e_5, \quad [e_2, e_3] = e_6 \]

\[ \omega_1 = e^i \wedge e^b + \lambda e^a \wedge e^c + (\lambda - 1) e^a \wedge e^4, \quad \lambda \neq 0 \text{ and } \neq 1 \]

\[ J_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 0 \]

\[ \omega_2 = e^i \wedge e^3 + \lambda e^4 \wedge e^b - \lambda e^2 \wedge e^a \wedge e^b \wedge e^c - 2 \lambda e^3 \wedge e^4, \quad \lambda \neq 0 \]

\[ J_2 = J_1, \quad R = 0, \]

\[ \omega_3 = -e^i \wedge e^b + e^j \wedge e^a + 2 e^3 \wedge e^1 + e^4 \wedge e^3 \wedge e^5, \]

\[ J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad R = 0 \]

### \( \mathfrak{g}_{19} \)

\[ \{e_1, e_2\} = e_4, \quad [e_1, e_3] = e_5, \quad [e_1, e_4] = e_6, \]

\[ J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \quad R \neq 0 \]

### \( \mathfrak{g}_{21} \)

\[ \{e_1, e_2\} = e_4, \quad [e_1, e_3] = e_5, \quad [e_2, e_3] = e_6 \]

\[ J = \begin{pmatrix} a & b \\ 1-a^2 & -a \end{pmatrix}, \quad R \neq 0 \]

### \( \mathfrak{g}_{23} \)

\[ \{e_1, e_2\} = e_5, \quad [e_1, e_3] = e_6 \]

\[ \omega_1 = e^i \wedge e^b + e^j \wedge e^a + e^k \wedge e^4 \]

\[ J_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 0 \]

\[ \omega_2 = e^i \wedge e^4 + e^2 \wedge e^a + e^3 \wedge e^5 \]

\[ J_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 0 \]

\[ \omega_3 = e^j \wedge e^a + e^3 \wedge e^4 - e^3 \wedge e^6, \]

\[ J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R = 0 \]

### \( \mathfrak{g}_{24} \)

\[ \{e_2, e_3\} = e_5, \quad [e_1, e_4] = e_6 \]

\[ J = \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix}, \quad R \neq 0 \]

### \( \mathfrak{g}_{25} \)

\[ \{e_1, e_2\} = e_6 \]

\[ J = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}, \quad R = 0 \]

Table 1: Para-Kähler six-dimensional nilpotent Lie algebras

**Proof.** A left-invariant para-Kähler structure is a pair \((\omega, J)\) consisting of a symplectic form \(\omega\) and an integrable almost para-complex structure \(J\) compatible with \(\omega\) on the Lie algebra \(\mathfrak{g}\). Since the symplectic form is given in \(\mathfrak{g}\), the operator \(J\) must have the following properties: \(J^2 = Id, \quad \omega(JX, JY) = -\omega(X, Y)\) and \([X, Y] + [JX, JY] - J[JX, Y] - J[X, JY] = 0\). We write the compatible condition \(\omega(JX, JY) = -\omega(X, Y)\) as \(\omega(JX, Y) + \omega(X, JY) = 0\). Let the symplectic form \(\omega\) and operator \(J\) have matrices \(\omega_{ij}\) and \(J_{ij}\), \(J = J^a_i e_i \otimes e^a\) in the basis \(\{e_1, \ldots, e_6\}\) of the Lie algebra. Then the system of equations for finding the para-Kähler structure \((\omega, J)\) consists of the following equations for the variables:

\[
\begin{align*}
J^i_k J^k_j &= \delta^i_j, \\
\omega_{ij} J^k_j + \omega_{ik} J^j_k &= 0, \\
J^l_j J^m_i C^k_{lm} - J^i_l J^m_j C^k_{lj} - J^i_j J^m_k C^m_{il} + C^k_{ij} &= 0,
\end{align*}
\]

where \(\delta^i_j\) is the identity matrix, \(C^k_{ij}\) are the structure constants of the Lie algebra, and the
indices vary from 1 to 6. For the Lie algebras included in the table of Theorem 2, it is easy
to see that the reduced almost para-complex structures $J$ are compatible with the symplectic
forms $\omega$. The integrability of $J$ immediately follows from the fact that the proper subspaces
$\mathfrak{g}^+$ and $\mathfrak{g}^-$ are subalgebras. For all Lie algebras in this list, with the exception of $\mathfrak{g}_6$, the
conditions of Theorem 1 are satisfied. Therefore, the corresponding para-Kähler metrics for
them are Ricci flat.

Let us consider in more detail the Lie algebra $\mathfrak{g}_6$ with symplectic form $\omega = e^1 \wedge e^6 + e^2 \wedge e^4 + e^2 \wedge e^5 - e^3 \wedge e^4$. For the almost para-complex structure $J = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) \times \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right)$ we have the following sequence of ideals: $a_1(J) = Z(\mathfrak{g}) = \mathbb{R}\{e_5, e_6\}$, $a_2(J) = \mathbb{R}\{e_4, e_5, e_6\}$, $a_3(J) = \mathbb{R}\{e_3, e_4, e_5, e_6\}$, $a_4(J) = \mathfrak{g}$. Therefore the $J$ is nilpotent. The eigenspaces
$\mathfrak{g}^+$ and $\mathfrak{g}^-$ are formed by the vectors $\{e_1, e_4, e_5\}$ and $\{e_2, e_3, e_6\}$. It is easy to see that they are subalgebras. Therefore, the almost para-complex structure $J$ is integrable. The compatible condition $\omega(JX, JY) = -\omega(X, Y)$ is obviously satisfied. Using Maple system, the curvature
tensor is easily calculated. It has the following non-zero components: $R_{12,1}^3 = 1$, $R_{12,2}^3 = 1$, $R_{12,3}^6 = -1$, $R_{31,1}^3 = -1$, $R_{13,2}^6 = -1$. Therefore, the para-Kähler metric is also Ricci-flat.

Note that the most general para-Kähler structure for a given Lie algebra $\mathfrak{g}_6$ depends on five
parameters and has the form:

$$J = \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ J_1^3 & 0 & -1 & 0 & 0 & 0 \\ -J_2^3 J_4^3 & J_2^4 & J_3^4 & 1 & 0 & 0 \\ J_1^6 J_2^3 J_4^3 & J_2^5 & -J_2^4 - J_3^4 & 0 & 1 & 0 \\ J_1^6 J_2^3 J_4^3 & J_2^5 & -J_2^4 - J_3^4 & 0 & 1 & 0 \end{array}\right), \quad g_J = \left(\begin{array}{cccccc} J_1^6 & J_2^3 J_4^3 & J_3^3 & 0 & 0 \\ J_2^3 J_4^3 & J_4^4 + J_5^4 & -J_2^4 & 1 & 0 & 0 \\ J_3^3 & -J_2^4 & J_4^4 - J_3^4 & -1 & 0 & 0 \\ J_1^6 J_2^3 J_4^3 & J_2^5 & -J_2^4 - J_3^4 & 0 & 1 & 0 \\ J_1^6 J_2^3 J_4^3 & J_2^5 & -J_2^4 - J_3^4 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

For all Lie algebras in the list of work [3], which are not included in the table of Theorem 2, the system of equations for the para-Kähler structure has no real solutions. Consider, for example, the Lie algebra $\mathfrak{g}_1$ with Lie brackets $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$, $[e_1, e_5] = e_6$, $[e_2, e_3] = e_5$, $[e_2, e_4] = e_6$, and structure constants $C^3_{12} = 1$, $C^3_{21} = -1$, $C^4_{13} = 1$, $C^4_{31} = -1$, $C^5_{14} = 1$, $C^5_{41} = -1$, $C^6_{15} = 1$, $C^6_{51} = -1$, $C^6_{23} = 1$, $C^6_{52} = -1$, $C^6_{24} = 1$, $C^6_{45} = -1$. The symplectic form is $\omega = e_1 \wedge e_6 + (1 - \lambda) e_2 \wedge e_5 + \lambda e_3 \wedge e_4$, where $\lambda \neq 0$ and $\lambda \neq 1$. Consider a matrix of almost paracomplex structure $J = (J^i_j)$, whose elements are denoted by symbols $\psi_{ij}$, $J^i_j = \psi_{ij}$. From the compatibility condition $\omega_{kj} J^k_i + \omega_{ik} J^k_i = 0$ we obtain the following form of the matrix

$$J = \begin{bmatrix}
\psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} & \psi_{16} \\
\psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} & \psi_{25} & \frac{\psi_{15}}{1-\lambda} \\
\psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} & \frac{(1-\lambda) \psi_{24}}{\lambda} & \frac{\psi_{14}}{\lambda} \\
\psi_{41} & \psi_{42} & \psi_{43} & -\psi_{33} & \frac{(-1-\lambda) \psi_{25}}{\lambda} & -\frac{\psi_{13}}{\lambda} \\
\psi_{51} & \psi_{52} & \frac{\lambda \psi_{12}}{1-\lambda} & \frac{\lambda \psi_{13}}{1-\lambda} & -\psi_{22} & \psi_{12} \\
\psi_{61} & (1-\lambda) \psi_{51} & \lambda \psi_{41} & -\lambda \psi_{31} & (\lambda - 1) \psi_{21} & -\psi_{11}
\end{bmatrix}.$$

Now consider the system of integrability equations $N^k_{ij} = J^l_i J^m_j C^k_{lm} - J^l_i J^k_j C^m_{lj} - J^l_j J^k_i C^m_{il} + \ldots$
We obtain three simple equations:

$$N^1_{56} = J_5^1 J_m^m C_{lm} - J_5^1 J_m^m C_{16} - J_6^1 J_m^m C_{56} + C_{56}^1 = -J_6^1 J_m^m C_{56} = -J_6^1 J_6^1 C_{51}^6 = (J_6^1)^2 = (\psi_{16})^2.$$ 

We obtain that $N^1_{56} = 0$ when $\psi_{16} = 0$. Under this condition, we obtain

$$N^1_{45} = J_4^1 J_m^m C_{lm} - J_4^1 J_m^m C_{15} - J_5^1 J_m^m C_{45} + C_{45}^1 = -J_5^1 J_m^m C_{15} - J_5^1 J_m^m C_{45} =$$

$$= -J_5^1 J_5^6 C_{15}^5 - J_5^1 J_5^1 C_{41}^6 = -J_5^1 J_5^6 C_{41}^5 = J_5^1 J_5^6 = (\psi_{15})^2.$$ 

We obtain that $N^1_{45} = 0$ when $\psi_{15} = 0$. Under this condition, we obtain analogically $N^2_{35} = (\psi_{14})^2$ and $N^3_{26} = (\psi_{25})^2$. If $\psi_{14} = 0$ and $\psi_{25} = 0$, then $N^1_{23} = (\psi_{13})^2$. If $\psi_{13} = 0$, then $N^6_{26} = (\psi_{12})^2/(\lambda - 1), N^6_{51} = ((\psi_{24})^2(1 - \lambda))/\lambda, N^6_{35} = ((\psi_{23})^2(1 - \lambda))/\lambda$. If $\psi_{12} = 0, \psi_{24} = 0, \psi_{23} = 0$, then $N^1_{11} = (\psi_{34})^2 = 0$ with $\psi_{34} = 0$. Under this conditions, we obtain the following three simple equations: $N^1_{13} = 2\psi_{11} \psi_{33} + (\psi_{33})^2 + 1 = 0, N^2_{23} = (\psi_{22})^2 + 2\psi_{22} \psi_{33} + 1 = 0, N^6_{15} = (\psi_{11})^2 - 2\psi_{11} \psi_{22} + 1 = 0$. Their solutions are easy to find: $\psi_{11} = 1, \psi_{22} = 1, \psi_{33} = -1$, or $\psi_{11} = -1, \psi_{22} = -1, \psi_{33} = 1$. If they are taken into account, then some elements of $N^6_{ij}$ are non-zero constants. For example, in the first case, when $\psi_{11} = 1, \psi_{22} = 1, \psi_{33} = -1$, we get $N^6_{24} = 4$. Indeed, in respect that $J_2^1 = J_4^1 = J_4^2 = J_2^4 = 0$ and $J_2^2 = \psi_{22} = 1, J_4^4 = -\psi_{33} = 1, J_2^6 = -\psi_{11} = -1$, we obtain:

$$N^6_{24} = J_2^1 J_4^1 C_{6m}^6 - J_2^1 J_4^1 C_{6l}^6 - J_4^2 J_4^2 C_{6m}^6 + C_{6l}^6 = J_2^1 J_2^6 C_{15}^6 + J_2^6 J_1^1 C_{51}^6 + J_2^2 J_4^4 C_{66}^6 + J_2^4 J_4^2 C_{42}^6 -$$

$$- J_2^1 J_5^5 C_{14}^6 - J_2^5 J_6^6 C_{24}^6 - J_4^1 J_3^6 C_{21}^6 - J_4^6 J_3^6 C_{23}^6 = J_2^2 J_6^6 + 1 = 1 + 1 + 1 + 4 \neq 0.$$ 

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