A CLASS OF HEMIVARIATIONAL INEQUALITIES FOR ELECTROELASTIC CONTACT PROBLEMS WITH SLIP DEPENDENT FRICTION

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Dedicated to Professor Zdzislaw Denkowski on the occasion of his 65th birthday with deep admiration

Abstract. In this paper we deal with a class of inequality problems for static frictional contact between a piezoelectric body and a foundation. The constitutive law is assumed to be electrostatic and involves a nonlinear elasticity operator. The friction condition is described by the Clarke subdifferential relations of nonmonotone and multivalued character in the tangential directions on the boundary. We derive a variational formulation which is a coupled system of a hemivariational inequality and an elliptic equation. The existence of solutions to the model is a consequence of a more general result obtained from the theory of pseudomonotone mappings.

1. Introduction. In this paper we consider a mathematical model which describes the stationary contact problem with friction between a piezoelectric body and a rigid foundation. The body is assumed to be elasto-piezoelectric with a nonlinear elasticity operator. We first provide an abstract formulation in the form of operator inclusion of subdifferential type. For it we obtain the existence of solutions by using an approach based on a surjectivity result for a suitable operator in Banach spaces. Then we apply the result to static contact problem with friction for an elasto-piezoelectric body and we prove the existence of weak solutions. As far as the mechanical problem is concerned, the mathematical results have been delivered in [2, 12, 9, 22]. To our knowledge, except [15], there is no result in the literature dealing with hemivariational inequality for piezoelectric frictional contact problems. In contrast to [22], we do not assume that the elasticity operator is strongly monotone and lipschitzean. We also relax the assumption on the friction coefficient.

2000 Mathematics Subject Classification. Primary: 74M15, 47J20, 35J85, 74G25, 74G30; Secondary: 35R70, 74F20, 49J40.

Key words and phrases. hemivariational inequality, friction, piezoelectric, slip, subdifferential, contact problem, nonconvex, inclusion.

This research was partially supported by the State Committee for Scientific Research of the Republic of Poland (KBN) under the Grants 2 P03A 003 25 and 4 T07A 027 26. The author would like to thank Professors Shouchuan Hu, Xin Lu and Alain Miranville for very nice organization of the Sixth AIMS International Conference on Dynamical Systems, Differential Equations and Applications held in University of Poitiers, Poitiers, France, June 25–28, 2006. The author is grateful to Professor Zdzislaw Naniewicz for co-chaired the special session on “Hemivariational Inequalities, Nonsmooth and Nonconvex Variational Problems with Applications”.

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In this paper we propose an approach based on a general result for pseudomonotone operators. The main feature of the mechanical problem is a nonmonotone multidimensional and multivalued friction boundary condition. It is expressed as the Clarke subdifferential of a locally Lipschitz potential. Such formulation leads in a natural way to the study of a class of hemivariational inequalities. The novelty of this paper is that the friction subdifferential boundary condition has a nonmonotone character since it comes from a nonconvex and nondifferentiable potential. The friction boundary condition is supposed to depend on the displacement field, so in particular, the friction coefficient is allowed to be slip-dependent (see examples in Section 4). We mention that the result on a hemivariational inequality for viscoelastic problems with slip-dependent friction were considered in [16]. In the framework of variational inequalities the contact phenomena for piezoelectric bodies has been considered in [22]. On the other hand, the hemivariational inequalities modelling static and dynamic frictional contact problems without piezoelectric effects have been widely studied in recent years, cf. e.g. [19, 20, 7, 13, 14, 16, 15]. Finally, we remark that an extension of our result to a coupled dynamic system of piezoelectrics with nonmonotone friction boundary conditions seems to be, in a general case, an open problem. For a result on dynamic bilateral contact problem for viscoelastic piezoelectric materials with adhesion, we refer to [17]. We hope to report more on our efforts in this direction in a forthcoming paper.

The paper is organized as follows. In Section 2 we recall some notation and present some auxiliary material. The result on an abstract inclusion is given in Section 3. In Section 4 we state the mechanical problem, describe the classical model for the process, derive its variational formulation and prove existence of the weak solution to the system.

2. Preliminaries. In this section we introduce the notation and recall some definitions needed in the sequel.

We denote by $S_d$ the linear space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), or equivalently, the space $\mathbb{R}_{s}^{d \times d}$ of symmetric matrices of order $d$. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $S_d$ by

\[ u \cdot v = u_i v_i, \quad \|v\| = (v \cdot v)^{1/2} \quad \text{for all} \ u, v \in \mathbb{R}^d, \]

\[ \sigma : \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\|_{S_d} = (\tau : \tau)^{1/2} \quad \text{for all} \ \sigma, \tau \in S_d. \]

The summation convention over repeated indices is used, all indices take values in $\{1, \ldots, d\}$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary $\Gamma$ and let $n$ denote the outward unit normal vector to $\Gamma$. We use the following spaces

\[ H = L^2(\Omega; \mathbb{R}^d), \quad \mathcal{H} = \{ \tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega) \} = L^2(\Omega; S_d), \]

\[ H_1 = \{ u \in H : \varepsilon(u) \in \mathcal{H} \} = H^1(\Omega; \mathbb{R}^d), \quad \mathcal{H}_1 = \{ \tau \in \mathcal{H} : \Div \tau \in H_1 \}, \]

where $\varepsilon : H^1(\Omega; \mathbb{R}^d) \to L^2(\Omega; S_d)$ and $\Div : \mathcal{H}_1 \to L^2(\Omega; \mathbb{R}^d)$ denote the deformation and the divergence operators, respectively, given by $\varepsilon(u) = (\varepsilon_{ij}(u))$, $\varepsilon_{ij}(u) = \frac{1}{2}(u_{ij} + u_{ji})$, $\Div \sigma = (\sigma_{ij,j})$ and the index following a comma indicates a partial derivative. The spaces $H$, $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$ are Hilbert spaces equipped with the inner products

\[ \langle u, v \rangle_H = \int_{\Omega} u_i v_i \, dx, \quad \langle \sigma, \tau \rangle_{\mathcal{H}} = \int_{\Omega} \sigma : \tau \, dx, \]

\[ \langle u, v \rangle_{H_1} = \langle u, v \rangle_H + \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}}, \quad \langle \sigma, \tau \rangle_{\mathcal{H}_1} = \langle \sigma, \tau \rangle_{\mathcal{H}} + \langle \Div \sigma, \Div \tau \rangle_H. \]
The associated norms in $H, \mathcal{H}, H_1$ and $H_1$ are denoted by $\| \cdot \|_H, \| \cdot \|_{\mathcal{H}}, \| \cdot \|_{H_1}$ and $\| \cdot \|_{H_1}$, respectively.

For every $v \in H_1$ we denote by $v$ its trace $\gamma v$ on $\Gamma$, where $\gamma : H^{1/2}(\Omega; \mathbb{R}^d) \to H^{1/2}(\Gamma; \mathbb{R}^d) \subset L^2(\Gamma; \mathbb{R}^d)$ is the trace map. Given $v \in H^{1/2}(\Gamma; \mathbb{R}^d)$ we denote by $v_N$ and $v_T$ the usual normal and the tangential components of $v$ on the boundary $\Gamma$ defined by $v_N = v \cdot n$ and $v_T = v - v_N n$.

For a normed space $(X, \| \cdot \|_X)$, if $U \subset X$, then we write $\| U \|_X = \sup \{ \| x \|_X : x \in U \}$. Given a reflexive Banach space $Y$, we denote by $\langle \cdot, \cdot \rangle_{Y \times Y}$ (or simply $\langle \cdot, \cdot \rangle$) the pairing between $Y$ and its dual $Y^*$. Following [23, 18, 4] we recall some definitions.

**Definition 1.** An operator $T : Y \to Y^*$ is said to be pseudomonotone if
(i) it is bounded (i.e. it maps bounded subsets of $Y$ into bounded subsets of $Y^*$); 
(ii) $\langle Tu, u - v \rangle \leq \liminf \langle Tu_n, u_n - v \rangle$ for all $v \in Y$ whenever the sequence $\{u_n\}$ converges weakly in $Y$ to $u$ with lim sup $\langle Tu_n, u_n - u \rangle \leq 0$.

**Remark 1.** The condition (ii) of Definition 1 is equivalent (still under condition (i)) to the following one
(ii') if $u_n \to u$ weakly in $Y$ and $\limsup \langle Tu_n, u_n - u \rangle \leq 0$, then $Tu_n \to Tu$ weakly in $Y^*$ and $\lim \langle Tu_n, u_n - u \rangle = 0$.

**Definition 2.** A multivalued operator $T : Y \to 2^{Y^*}$ is said to be pseudomonotone if the following conditions hold:
(i) the set $Tv$ is nonempty, bounded, closed and convex for all $v \in Y$;
(ii) $T$ is usc from each finite dimensional subspace of $Y$ into $Y^*$ endowed with the weak topology;
(iii) if $v_n \in Y, v_n \to v$ weakly in $Y$ and $v_n^* \in Tv_n$ is such that $\limsup \langle v_n^*, v_n - v \rangle \leq 0$, then to each $y \in Y$, there exists $v^*(y) \in Tv$ such that $\langle v^*(y), v - y \rangle \leq \liminf \langle v_n^*, v_n - y \rangle$.

**Definition 3.** An operator $T : Y \to 2^{Y^*}$ is said to be generalized pseudomonotone if for every sequences $v_n \to v$ weakly in $Y$, $v_n^* \to v^*$ weakly in $Y^*$, $v_n^* \in Tv_n$ and $\limsup \langle v_n^*, v_n - v \rangle \leq 0$, we have $v^* \in Tv$ and $\langle v_n^*, v_n \rangle \to \langle v^*, v \rangle$.

The following result is well known, cf. e.g. [23, 5].

**Proposition 1.** If $T : Y \to 2^{Y^*}$ is a generalized pseudomonotone operator which is bounded and has nonempty, closed and convex values, then $T$ is pseudomonotone.

**Definition 4.** (cf. [3]) Let $h : E \to \mathbb{R}$ be a locally Lipschitz function, where $E$ is a Banach space. The generalized directional derivative of $h$ at $x \in E$ in the direction $v \in E$, denoted by $h^0(x; v)$, is defined by
$$h^0(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{h(y + tv) - h(y)}{t}.$$ 

The generalized gradient of $h$ at $x$, denoted by $\partial h(x)$, is a subset of a dual space $E^*$ given by $\partial h(x) = \{ \zeta \in E^* : h^0(x; v) \geq \langle \zeta, v \rangle \}_{E^* \times E}$ for all $v \in E$. A locally Lipschitz function $h$ is called regular (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $h^0(x; v)$ exists and satisfies $h^0(x; v) = h'(x; v)$ for all $v \in E$.

3. **Abstract result.** In this section we establish a result on the existence of solutions to an abstract inclusion.
Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^d, d \geq 1 \), with Lipschitz continuous boundary \( \Gamma \). Let \( X \) be a closed subspace of \( H^s(\Omega; \mathbb{R}^n) \), \( s \geq 1 \) and let \( Z = H^1(\Omega; \mathbb{R}^n) \) with a fixed \( \delta \in (1/2, 1) \). Let \( A: X \to X^* \) be an operator, \( N: Z \to 2^Z \) be a multivalued map and \( g \in X^* \). We consider the following problem

\[
\text{find } u \in X \text{ such that } g = Au + Nu. \tag{1}
\]

We say that an element \( u \in X \) is a solution to (1) if and only if there exists \( z \in Z^* \) such that \( g = Au + z \) and \( z \in Nu \).

The following hypotheses are needed in the sequel.

\[ H(A) : \quad A: X \to X^* \text{ is an operator which is pseudomonotone and coercive, i.e. } \langle Av, v \rangle_{X^* \times X} \geq c_A \|v\|^2_X \text{ for all } v \in X \text{ with } c_A > 0; \]

\[ H(N) : \quad N: Z \to 2^Z \text{ is a multivalued map such that} \]

(i) \( N \) has nonempty convex and weakly compact values;

(ii) \( N \) has a graph closed in \( Z \times (w - Z^*) \) topology;

(iii) \( \|Nz\|_{Z^*} \leq c_N(1 + \|z\|_Z) \) for all \( z \in Z \) with \( c_N > 0 \);

(iv) \( \langle Nz, z \rangle_{Z^* \times Z} \geq -c_1\|z\|_Z - c_2 \) for all \( z \in Z \) with \( c_1, c_2 > 0 \);

**Theorem 1.** If \( H(A), H(N) \) hold and \( g \in X^* \), then problem (1) has a solution.

**Proof.** Let \( g \in X^* \). We claim that the following operator \( F: X \to 2X^* \) defined by \( Fv = Av + Nv \) for \( v \in X \) is pseudomonotone and coercive. Then the result follows from the main surjectivity theorem of nonlinear analysis, cf. Section 32.4 of [23] or Theorem 1.3.70 of [5].

To establish that \( F \) is pseudomonotone, it is enough (see Proposition 1) to show that \( F \) is generalized pseudomonotone operator (since \( F \) is bounded with nonempty closed convex values). Let \( v_n \to v \) weakly in \( X \), \( v_n^* \to v^* \) weakly in \( X^* \), \( v_n^* \in Fv_n \) and \( \limsup \langle v_n^*, v_n - v \rangle_{X^* \times X} \leq 0 \). We need to show that \( v^* \in Fv \) and \( \langle v_n^*, v_n \rangle_{X^* \times X} \to \langle v^*, v \rangle_{X^* \times X} \). We have \( v_n^* = Av_n + w_n \) with \( w_n \in Nu_n \). By the boundedness of \( N \) (cf. \( H(N)(iii) \)), by passing to a subsequence if necessary, we obtain \( w_n \to w \) weakly in \( Z^* \) with \( w \in Z^* \) while the compactness of the embedding \( X \subset Z \) gives \( v_n \to v \) in \( Z \). Using \( H(N)(ii) \), we have \( w \in Nu \). Furthermore, from the equality

\[ \langle v_n^*, v_n - v \rangle_{X^* \times X} = \langle Av_n, v_n - v \rangle_{X^* \times X} + \langle w_n, v_n - v \rangle_{Z^* \times Z}, \]

we obtain \( \limsup \langle Av_n, v_n - v \rangle_{X^* \times X} = \limsup \langle v_n^*, v_n - v \rangle_{X^* \times X} \leq 0 \). Exploiting the pseudomonotonicity of \( A \) (cf. Remark 1), we get

\[ Av_n \to Av \text{ weakly in } X^* \quad \text{and} \quad \langle Av_n, v_n - v \rangle_{X^* \times X} \to \]

Passing to the limit in the equation \( v_n^* = Av_n + w_n \), we have \( v^* = Av + w \) which together with \( w \in Nu \) implies \( v^* \in Fv \). Next, we deduce

\[ \langle v_n^*, v_n \rangle_{X^* \times X} = \langle Av_n, v_n - v \rangle_{X^* \times X} + \langle Av_n, v \rangle_{X^* \times X} + \langle w_n, v_n \rangle_{Z^* \times Z} \to \]

\[ \to \langle Av, v \rangle_{X^* \times X} + \langle w, v \rangle_{Z^* \times Z} = \langle v^*, v \rangle_{X^* \times X} \]

which proves that \( F \) is pseudomonotone.

For the coercivity of \( F \), it is enough to observe that

\[ \langle Fv, v \rangle_{X^* \times X} = \langle Av, v \rangle_{X^* \times X} + \langle Nv, v \rangle_{Z^* \times Z} \geq cA \|v\|_X^2 - c_1 \|v\|_Z - c_2 \geq cA \|v\|_X^2 - c_0 c_1 \|v\|_X - c_2 \]

for all \( v \in X \), where \( c_0 > 0 \) is such that \( \|z\|_Z \leq c_0 \|\cdot\|_X \). Thus \( F \) is surjective which means that the problem (1) admits a solution. \( \square \)
4. Formulation of a class of piezoelectric contact problems with friction.

In this section we deal with a class of frictional problems for piezoelectric bodies that can be studied by employing Theorem 1. We now state the contact problem under consideration and give its variational formulation.

Consider an elastic piezoelectric body which initially occupies an open bounded subset $\Omega$ in $\mathbb{R}^d$, $d = 2, 3$. The boundary $\Gamma = \partial \Omega$ is assumed to be Lipschitz continuous. The body may come in frictional contact with an obstacle, the fixed foundation. We consider two partitions of $\Gamma$. A first partition is given by three mutually disjoint open parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$ such that $m(\Gamma_D) > 0$. The second one consists of two disjoint open parts $\Gamma_a$ and $\Gamma_b$ such that $m(\Gamma_a) > 0$. The body is subjected to volume forces of density $f_1$ and volume electric charges of density $q_1$. The body is clamped on $\Gamma_D$ and a surface tractions of density $f_2$ act on $\Gamma_N$. Moreover, the electric potential vanishes on $\Gamma_a$ and the surface electric charge of density $q_2$ is applied on $\Gamma_b$. On $\Gamma_C$ the body may come into contact with a foundation.

We denote by $u : \Omega \to \mathbb{R}^d$ the displacement field, by $\varepsilon(u) = (\varepsilon_{ij}(u))$, $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $i, j = 1, \ldots, d$ the strain tensor, by $\sigma : \mathbb{R} \to \mathbb{S}_d$, $\sigma = (\sigma_{ij})$ the stress tensor and by $D : \mathbb{R}^d \to \mathbb{R}^d$, $D = (D_i)$ the electric displacement field. We also denote $E(\varphi) = (E_i(\varphi))$ the electric vector field, where $\varphi : \Omega \to \mathbb{R}$ is an electric potential such that $E_i(\varphi) = -\frac{\partial \varphi}{\partial x_i}$.

We begin with the strong formulation of the problem of static deformation of an elastic piezoelectric body. The governing equations consist (cf. [10, 2, 12, 9, 1]) of the equilibrium equations given by

$$-\text{Div} \sigma = f_1 \quad \text{in } \Omega,$$
$$\text{div} D = q_1 \quad \text{in } \Omega,$$

and the stress-charge form of piezoelectric constitutive relations which have the form

$$\sigma = F \varepsilon(u) - P^\top E(\varphi) \quad \text{in } \Omega,$$
$$D = P \varepsilon(u) + \beta E(\varphi) \quad \text{in } \Omega.$$

We assume that $F : \mathbb{R} \times \mathbb{S}_d \to \mathbb{S}_d$ is a nonlinear elasticity operator, $P : \mathbb{R} \times \mathbb{S}_d \to \mathbb{R}^d$ and $P^\top : \mathbb{R} \times \mathbb{R}^d \to \mathbb{S}_d$ is a linear piezoelectric operator and its transpose, respectively and $\beta : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a linear electric permittivity operator. The operators are represented by

$$P(x, \varepsilon) = p(x) \varepsilon = \{p_{ijk}(x) \varepsilon_{jk}\}_{i=1}^d \quad \text{for } (x, \varepsilon) \in \Omega \times \mathbb{S}_d,$$
$$P^\top (x, E) = p^\top (x) E = \{p_{kij}(x) E_k\}_{i,j=1}^d \quad \text{for } (x, E) \in \Omega \times \mathbb{R}_d,$$
$$\beta(x, E) = \beta(x) E = \{\beta_{ij}(x) E_j\}_{i=1}^d \quad \text{for } (x, E) \in \Omega \times \mathbb{R}_d,$$

where $p(x) = \{p_{ijk}(x)\}$ are piezoelectric coefficients, $i, j, k \in \{1, \ldots, d\}$ (third order tensor field), $p^\top (x) = \{p_{ijk}(x)\}^\top = \{p_{kij}(x)\}$ are the transpose to $p(x)$ and $\beta(x) = \{\beta_{ij}(x)\}$ are dielectric coefficients, $i, j \in \{1, \ldots, d\}$ (second order tensor field). We use here the notation $p^\top$ to denote the transpose of the tensor $p$ given $p \varepsilon \cdot \xi = \varepsilon : p^\top \xi$ for $\varepsilon \in \mathbb{S}_d$ and $\xi \in \mathbb{R}^d$.

When the elasticity operator $F(x, \cdot)$ is linear, then $F(x, \varepsilon) = C(x) \varepsilon$ with the elasticity coefficients $C(x) = \{c_{ijkl}(x)\}$, $i, j, k, l = 1, \ldots, d$ (fourth order tensor field) which may be functions of position in a nonhomogeneous material. We also remark that the decoupled state (purely elastic and purely electric deformations) can be obtained by setting $p_{ijk} = 0$. 

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To complete the mechanical model, according to the description of the physical setting, we have

\[
\begin{align*}
  u &= 0 \quad \text{on } \Gamma_D, \\
  \sigma_n &= f_2 \quad \text{on } \Gamma_N, \\
  \varphi &= 0 \quad \text{on } \Gamma_a, \\
  D \cdot n &= g_2 \quad \text{on } \Gamma_b.
\end{align*}
\]

(6)

On the contact surface \( \Gamma_C \), we consider the subdifferential boundary condition in a tangential direction

\[
-\sigma_N = S \quad \text{on } \Gamma_C,
\]

(7)

\[
-\sigma_T = h(u) \partial j(u_T) \quad \text{on } \Gamma_C,
\]

(8)

where \( S \) is the normal load imposed on \( \Gamma_C \), \( h: \Gamma_C \times \mathbb{R}^d \to \mathbb{R} \) is prescribed and \( \partial j \) represents the Clarke subdifferential of the function \( j: \Gamma_C \times \mathbb{R}^d \to \mathbb{R} \) which is locally Lipschitz in its second variable.

The strong formulation of the problem consists in finding the displacement \( u: \Omega \to \mathbb{R}^d \) and the electric potential \( \varphi: \Omega \to \mathbb{R} \) such that (2)–(8) hold. To give a variational formulation of this problem we need the following hypotheses.

\begin{itemize}
  \item[(i)] \( h(\cdot, \xi) \) is measurable for all \( \xi \in \mathbb{R}^d \), \( h(x, \cdot) \) is continuous for a.e. \( x \in \Gamma_C \);
  \item[(ii)] \( 0 \leq h(x, \xi) \leq h_0 \) for a.e. \( x \in \Gamma_C \), for all \( \xi \in \mathbb{R}^d \) with \( h_0 > 0 \).
  \item[(iii)] \( j(\cdot, \xi) \) is measurable for all \( \xi \in \mathbb{R}^d \) and \( j(\cdot, 0) \in L^1(\Gamma_C) \);
  \item[(iv)] \( j(x, \cdot) \) is locally Lipschitz for a.e. \( x \in \Gamma_C \);
  \item[(v)] \( \| \partial j(x, \xi) \| \leq c_1 (1 + \| \xi \|) \) for all \( \xi \in \mathbb{R}^d \) with \( c_1 > 0 \);
  \item[(vi)] \( j^0(x, \xi; -\xi) \leq d_1 (1 + \| \xi \|) \) for all \( \xi \in \mathbb{R}^d \) with \( d_1 > 0 \).
\end{itemize}

Remark 2. The regularity hypothesis in \( H(j)(v) \) is satisfied, for instance, for a function which is represented as the difference of convex functions. More precisely, let us consider the following condition (for simplicity we omit the x-dependence).

\( H(d.c.) \): The function \( j: \mathbb{R}^d \to \mathbb{R} \) is locally Lipschitz and of d.c.-type, i.e. \( j(\xi) = j_1(\xi) - j_2(\xi) \) for \( \xi \in \mathbb{R}^d \), where \( j_k: \mathbb{R}^d \to \mathbb{R} \), \( k = 1, 2 \) are convex functions and one of the convex subdifferentials \( \partial j_k \) is assumed to be a singleton for every \( \xi \in \mathbb{R}^d \) and the growth conditions hold \( \| \eta \| \leq c_0 (1 + \| \xi \|) \) for \( \eta \in \partial j_k(\xi) \) for all \( \xi \in \mathbb{R}^d \), \( k = 1, 2 \) with \( c_0 > 0 \).

Under the hypothesis \( H(d.c.) \) either \( j \) or \( -j \) is regular in the sense of Clarke and \( \partial j(\xi) = \partial j_1(\xi) - \partial j_2(\xi) \) with \( \| \eta \| \leq c_0 (1 + \| \xi \|) \) for \( \eta \in \partial j(\xi) \) for \( \xi \in \mathbb{R}^d \), \( c_0 > 0 \).

We shortly comment on the friction condition (8). For a detailed description of the examples, we refer to [11, 14, 13, 16, 8, 21] and the references therein.

Example 1. (Contact with nonmonotone friction laws) Consider first, the simple case, when \( h = 1 \). This is a case of nonmonotone friction laws which are independent of the slip displacement. The friction law (8) takes the form \( -\sigma_T \in \partial j(x, u_T) \) on \( \Gamma_C \).

This law appears in the tangential direction of an adhesive interface and describes the partial cracking and crushing of the adhesive bonding material. We refer to Section 2.4 of [20] for several examples of the zig-zag friction laws which can be formulated in this form. As a model example we can consider a nonconvex function
j: \mathbb{R} \to \mathbb{R} given by j(r) = \min\{j_1(r), j_2(r)\}, where j_1(r) = ar^2, j_2(r) = \frac{a}{2}(r^2 + 1), a > 0. For more details, see [20, 14, 13].

**Example 2.** (Contact with slip dependent nonmonotone friction laws) Consider the friction law with slip dependent coefficient of the form \(-\sigma_T \in \mu(x, \|u_T\|) \partial_j(x, u_T)\) on \(\Gamma_C\), where the friction coefficient satisfies \(H(\mu) : \mu: \Gamma_C \times \mathbb{R} \to \mathbb{R}\) is a function such that

1. \(\mu(\cdot, r)\) is measurable for all \(r \in \mathbb{R}\), \(\mu(x, \cdot)\) is continuous for a.e. \(x \in \Gamma_C\);
2. \(0 \leq \mu(x, r) \leq \mu_0\) for a.e. \(x \in \Gamma_C\), for all \(r \in \mathbb{R}\) with \(\mu_0 > 0\).

By making a suitable choice of function \(h\) and the convex (hence Clarke’s regular) function \(j(x, \xi) = \|\xi\|\) in the contact boundary conditions (7), (8), we obtain a number of well known monotone friction laws. Two of them we recall below.

**Example 3.** (A version of Coulomb’s friction law) We consider a contact problem modeled with a version of Coulomb’s law of dry friction on \(\Gamma_C\):

\[
\begin{align*}
-\sigma_N &= S \\
\|\sigma_T\| &\leq \mu(x, \|u_T\|)|\sigma_N| \quad \text{with} \\
\|\sigma_T\| < \mu(x, \|u_T\|)|\sigma_N| &\implies u_T = 0 \\
\|\sigma_T\| = \mu(x, \|u_T\|)|\sigma_N| &\implies \sigma_T = -\lambda u_T \quad \text{with some } \lambda \geq 0.
\end{align*}
\]

Here \(S \in L^\infty(\Gamma_C)\), \(S \geq 0\) is a given normal stress and the coefficient of friction \(\mu\) satisfies \(H(\mu)\) of Example 2. This law was used in [22] for a piezolectric contact problem and in [6, 19, 8] for elastic and viscoelastic contact problems. If we take \(h(x, \xi) = S(x)\mu(x, \|\xi_T\|)\) and \(j(x, \xi) = \|\xi\|\), then the above Coulomb friction law has the form of (7) and (8). Since \(\partial \|\xi\|\) equals to \(\overline{B}(0, 1)\) if \(\xi = 0\) and \(\frac{\xi}{\|\xi\|}\) if \(\xi \neq 0\), the condition (8) is equivalent to

\[
\begin{align*}
\|\sigma_T\| &\leq S(x)\mu(x, \|u_T\|) \quad \text{if } u_T = 0 \\
-\sigma_T &= S(x)\mu(x, \|u_T\|) \frac{u_T}{\|u_T\|} \quad \text{if } u_T \neq 0.
\end{align*}
\]

**Example 4.** (Contact with the Tresca law) Consider the static version of the Tresca law (cf. e.g. [6, 8]) on \(\Gamma_C\):

\[
\begin{align*}
\|\sigma_T\| &\leq g \quad \text{with} \\
\|\sigma_T\| < g &\implies u_T = 0 \\
\|\sigma_T\| = g &\implies \sigma_T = -\lambda u_T \quad \text{with some } \lambda \geq 0,
\end{align*}
\]

where \(g \geq 0\) represents the friction bound, i.e. the magnitude of force at which slipping begins. This law can be put in the form (8) with \(h(x, \xi) = g\) and \(j(x, \xi) = \|\xi\|\).

We now pass to the variational formulation of the problem (2)–(8). We introduce the spaces for the displacement and the electric potential:

\[V = \{v \in H^1(\Omega; \mathbb{R}^d) : v = 0 \text{ on } \Gamma_D\}, \quad \Phi = \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_a\}\]

which are closed subspaces of \(H_1\) and \(H^1(\Omega)\). On \(V\) we consider the inner product and the corresponding norm given by \((u, v)_V = \langle \varepsilon(u), \varepsilon(v) \rangle_H\) and \(||v||_V = ||\varepsilon(v)||_H\) for \(u, v \in V\). Then \((V, ||\cdot||_V)\) is a Hilbert space. On \(\Phi\) we consider the inner product \((\varphi, \psi)_\Phi = (\varphi, \psi)_H^{1,1}(\Omega)\) for \(\varphi, \psi \in \Phi\). Then \((\Phi, ||\cdot||_\Phi)\) is also a Hilbert space.

Assuming sufficient regularity of the functions involved in the problem, using the Green formula, the constitutive relations and the equality \(\int_{\Gamma_C} \sigma n \cdot v d\Gamma =\)
\( \int_{\Gamma_C} (\sigma_{NVN} + \sigma_T \cdot v_T) \, d\Gamma \), we obtain the following variational formulation of the problem (2)–(8): find \( u \in V \) and \( \varphi \in \Phi \) such that
\[
\begin{align*}
\langle F(\varepsilon(u), \varepsilon(v)) \rangle_H &+ \langle P^T \nabla \varphi, \varepsilon(v) \rangle_H \\
+ \int_{\Gamma_C} h(x, u) j^0(x, u_T; v_T) \, d\Gamma &\geq \langle f, v \rangle_{V^* \times V} \quad \text{for all } v \in V \\
\langle \beta \nabla \varphi, \nabla \psi \rangle_H - \langle P \varepsilon(u), \nabla \psi \rangle_H &= \langle q, \psi \rangle_{\Phi^* \times \Phi} \quad \text{for all } \psi \in \Phi,
\end{align*}
\]
where
\[
\langle f, v \rangle_{V^* \times V} = \int_{\Omega} f_1 \cdot v \, dx + \int_{\Gamma_N} f_2 \cdot v \, d\Gamma + \int_{\Gamma_C} S v_N \, d\Gamma,
\]
\[
\langle q, \psi \rangle_{\Phi^* \times \Phi} = \int_{\Gamma_h} q_1 \psi \, dx + \int_{\Gamma_b} q_2 \psi \, d\Gamma.
\]

This weak formulation represents a coupled system of a hemivariational inequality and an elliptic equation. The existence theorem for (9) will be a consequence of a more general result which we prove below.

In the study of the problem (9), we will use the following hypotheses.

\( H(F) \) : The elasticity operator \( F : \Omega \times S_0 \to S_d \) satisfies the Carathéodory condition (i.e. \( F(\varepsilon, \cdot) \) is measurable on \( \Omega \) for all \( \varepsilon \in S_d \) and \( F(\cdot, \cdot) \) is continuous on \( S_d \) for a.e. \( x \in \Omega \)) and
\[
\begin{align*}
(\text{i}) & \quad \|F(x, \varepsilon)\|_{S_0} \leq \alpha_1 (b(x) + \|\varepsilon\|_{S_0}) \quad \text{for all } x \in S_d, \text{ a.e. } x \in \Omega \text{ with } b \in L^2(\Omega), \alpha_1 > 0; \\
(\text{ii}) & \quad (F(x, \varepsilon_1) - F(x, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geq 0 \quad \text{for all } \varepsilon_1, \varepsilon_2 \in S_d \text{ and a.e. } x \in \Omega; \\
(\text{iii}) & \quad F(x, \varepsilon) \geq \alpha_2 \|\varepsilon\|^2_{S_d} \quad \text{for all } x \in \Omega \text{ and a.e. } \varepsilon \in S_d \text{ and } \alpha_2 > 0.
\end{align*}
\]

\( H(\beta) \) : The dielectric coefficients \( \beta_{ij} \) satisfy \( \beta_{ij} = \beta_{ji} \in L^\infty(\Omega) \) and \( \beta_{ij}(x) \xi_i \xi_j \geq m_\beta \|\xi\|^2_{\mathbb{R}^d} \) for a.e. \( x \in \Omega \) and all \( \xi \in \mathbb{R}^d \) with \( m_\beta > 0 \).

\( H_0 \) : \( f_1 \in H, f_2 \in L^2(\Gamma_N; \mathbb{R}^d), q_1 \in L^2(\Omega), q_2 \in L^2(\Gamma_h), S \in L^\infty(\Gamma_C), S \geq 0 \).

The main existence result is as follows

**Theorem 2.** Under the hypotheses \( H(F), H(\beta), (H_0), H(h) \) and \( H(j) \), the problem (9) has at least one solution.

**Proof.** The idea is to apply Theorem 1 and show the existence of a solution to the inclusion (1) with suitable data \( X, A, N \) and \( g \). Then we establish that every solution to this inclusion is also a solution to (9).

First let \( X = V \times \Phi \subset H^1(\Omega; \mathbb{R}^{d+1}) \) be a Hilbert space endowed with the inner product \( (y, z)_X = (u, v)_V + (\varphi, \psi)_\Phi \) for \( y, z \in X \), \( y = (u, \varphi) \), \( z = (v, \psi) \). The corresponding norm is denoted by \( \| \cdot \|_X \). We observe that under the assumption \( (H_0) \) the elements \( f \) and \( g \) satisfy \( f \in V^* \) and \( g \in \Phi^* \), i.e. \( g = (f, q) \in X^* \).

Next, we introduce the operator \( A : X \to X^* \) defined by
\[
(Ay, z')_{X^* \times X} = \langle F(\varepsilon(u), \varepsilon(v)) \rangle_H + \langle P^T \nabla \varphi, \varepsilon(v) \rangle_H + \langle \beta \nabla \varphi, \nabla \psi \rangle_H - \langle P \varepsilon(u), \nabla \psi \rangle_H
\]
for all \( y, z \in X \), \( y = (u, \varphi) \) and \( z = (v, \psi) \). Subsequently, let \( N : Z = H^\delta(\Omega; \mathbb{R}^{d+1}) \to 2^{\mathbb{Z}^+} \) with \( \delta \in (1/2, 1) \) be the operator given by
\[
Nz = \gamma^+ \partial J(\gamma z, \gamma z) \quad \text{for } z \in Z,
\]
where \( J : L^2(\Gamma_C; \mathbb{R}^{d+1}) \times L^2(\Gamma_C; \mathbb{R}^{d+1}) \to \mathbb{R} \) is a functional defined by
\[
J((\Re, \Im), (w, \varphi)) = \int_{\Gamma_C} h(x, \Re(x)) j(x, w_T(x)) \, d\Gamma
\]
for all \( \Re, \Im \in L^2(\Gamma_C; \mathbb{R}^{d+1}) \) and \( w \in L^2(\Gamma_C; \mathbb{R}^{d+1}) \).
for \((\overline{w},\overline{\varphi}), (w, \varphi) \in L^2(\Gamma_C; \mathbb{R}^d) \times L^2(\Gamma_C)\), \(\partial J\) is the Clarke subdifferential of the functional \(J\) with respect to \((w, \varphi)\) and \(\gamma^*: L^2(\Gamma_C; \mathbb{R}^{d+1}) \to Z^*\) denotes the adjoint to \(\gamma\). Although \(J\) does not depend on \(\overline{\varphi}\) and \(\varphi\), for convenience, we keep the above notation.

The following two lemmata show the properties of the operators \(A\) and \(N\), respectively.

**Lemma 1.** If the hypotheses \(H(F), H(P)\) and \(H(\beta)\) hold, then the operator \(A: X \to X^*\) defined by \((10)\) satisfies \(H(A)\).

**Lemma 2.** Under hypotheses \(H(h)\) and \(H(j)\), the operator \(N: Z \to 2^{Z^*}\) defined by \((11)\) satisfies \(H(N)\).

For the proof of Lemma 1 we refer to Lemma 1 of [15]. The proof of Lemma 2 is based on the following result (cf. Lemma 11 in [16]).

**Proposition 2.** Suppose \(H(h)\) and \(H(j)(i)-(iv)\). Then the functional \(J\) defined by \((12)\) satisfies

\[
\begin{align*}
&\text{(1) } J(z_1, \cdot) \text{ is locally Lipschitz (in fact, Lipschitz on bounded subsets of } L^2(\Gamma_C; \mathbb{R}^{d+1})) \text{ for all } z_1 \in L^2(\Gamma_C; \mathbb{R}^{d+1}), \\
&\text{(2) } \|\partial J(z_1, z_2)\|_{L^2(\Gamma_C; \mathbb{R}^{d+1})} \leq \bar{c} (1 + \|z_2\|_{L^2(\Gamma_C; \mathbb{R}^{d+1})}) \text{ for all } z_1, z_2 \in L^2(\Gamma_C; \mathbb{R}^{d+1}) \\
&\text{with } \bar{c} > 0, \text{ where the subdifferential is taken with respect to } z_2; \\
&\text{(3) } J^0(z_1, z_2) = \bar{d} (1 + \|z_2\|_{L^2(\Gamma_C; \mathbb{R}^{d+1})}) \text{ for all } z_1, z_2 \in L^2(\Gamma_C; \mathbb{R}^{d+1}) \text{ with } \bar{d} > 0; \\
&\text{(4) For all } ((\overline{w},\overline{\varphi}), (w, \varphi); (v, \psi)) \in L^2(\Gamma_C; \mathbb{R}^{d+1}), \text{ we have}
\end{align*}
\]

\[
J^0((\overline{w},\overline{\varphi}), (w, \varphi); (v, \psi)) \leq \int_{\Gamma_C} h(x, \overline{w}(x)) j^0(x, w_T(x); v_T(x)) \, d\Gamma(x). \tag{13}
\]

If \(H(h)\) and \(H(j)(i)-(v)\) hold, then additionally, we have

\[
\text{(5) either } J(z_1, \cdot) \text{ or } -J(z_1, \cdot) \text{ is regular, respectively and in } (13) \text{ we have equality;}
\]

\[
\text{(6) the graph of } \partial J(\cdot, \cdot) \text{ is closed in } L^2(\Gamma_C; \mathbb{R}^{d+1})^2 \times (w - L^2(\Gamma_C; \mathbb{R}^{d+1})) \text{ topology.}
\]

From Lemmata 1, 2 and Theorem 1, we deduce that for \(g = (f, q) \in X^*\), there exists \(y \in X\), \(y = (u, \varphi) \in V \times \Phi\) such that

\[
A(u, \varphi) + \gamma^* \partial J(\gamma(u, \varphi), \gamma(u, \varphi)) \ni (f, q). \tag{14}
\]

We will show that under \(H(j)(i)-(iv)\), every element \((u, \varphi) \in X\) satisfying \((14)\) is a solution to \((9)\). If additionally either \(j(x, \cdot)\) or \(-j(x, \cdot)\) is regular, then the problems \((14)\) and \((9)\) are equivalent. Let \((u, \varphi) \in Y\) satisfies \((14)\). Then

\[
A(u, \varphi) + (\eta_1, \eta_2) = (f, q)
\]

with \((\eta_1, \eta_2) = \gamma^*(\xi_1, \xi_2)\) and \((\xi_1, \xi_2) \in \partial J(\gamma(u, \varphi), \gamma(u, \varphi))\). By the definition of Clarke’s subdifferential and Proposition 2, for all \((v, \psi) \in V \times \Phi\), we have

\[
\langle (\eta_1, \eta_2), (v, \psi) \rangle_{Z^* \times Z} = \langle \gamma^*(\xi_1, \xi_2), (v, \psi) \rangle_{Z^* \times Z} = \langle (\xi_1, \xi_2), \gamma(v, \psi) \rangle_{L^2(\Gamma_C; \mathbb{R}^{d+1})} \leq J^0(\gamma(u, \varphi), \gamma(u, \varphi); \gamma(v, \psi)) \leq \int_{\Gamma_C} h(x, u(x)) j^0(x, u_T(x); v_T(x)) \, d\Gamma(x).
\]

Hence

\[
\langle A(u, \varphi), (v, \psi) \rangle_{X^* \times X} + \int_{\Gamma_C} h(x, u) j^0(x, u_T; v_T) \, d\Gamma \geq \langle (f, q), (v, \psi) \rangle_{X^* \times X} \tag{15}
\]
for all \((v, \psi) \in X\). Now choosing \((v, 0) \in V \times \Phi\), we easily get the first inequality in (9). Next, we take successively \((0, \psi), (0, -\psi) \in V \times \Phi\) in (15), where \(\psi\) is arbitrary. We obtain the second equation in (9). The regularity of either \(j(x, \cdot)\) or \(-j(x, \cdot)\) implies, by Proposition 2, the regularity of \(J(w, v, \cdot, \cdot)\) or \(-J(w, v, \cdot, \cdot)\), respectively. In this case every solution to (9) also solves (14). This completes the proof of the theorem. 

\[
\begin{align*}
\Phi &\rightarrow V, \\
J &\colon V \times X \times X \rightarrow \mathbb{R}, \\
J &\colon V \times X \times X \rightarrow \mathbb{R},
\end{align*}
\]

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Received July 2006; 1st revision February 2007; 2nd revision September 2007.

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