Symmetric implication zroupoids and weak associative laws

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Abstract
An algebra $A = (A, \rightarrow, 0)$, where $\rightarrow$ is binary and 0 is a constant, is called an implication zroupoid ($I$-zroupoid, for short) if $A$ satisfies the identities: $(x \rightarrow y) \rightarrow z \approx ((z' \rightarrow x) \rightarrow (y \rightarrow z))'$ and $0'' \approx 0$, where $x' := x \rightarrow 0$. An implication zroupoid is symmetric if it satisfies: $x'' \approx x$ and $(x \rightarrow y)' \approx (y \rightarrow x)'$. The variety of symmetric $I$-zroupoids is denoted by $S$. We began a systematic analysis of weak associative laws (or identities) of length $\leq 4$ in Cornejo and Sankappanavar (Soft Comput 22(13):4319–4333, 2018a. https://doi.org/10.1007/s00500-017-2869-z), by examining the identities of Bol–Moufang type, in the context of the variety $S$. In this paper, we complete the analysis by investigating the rest of the weak associative laws of length $\leq 4$ relative to $S$. We show that, of the (possible) 155 subvarieties of $S$ defined by the weak associative laws of length $\leq 4$, there are exactly 6 distinct ones. We also give an explicit description of the poset of the (distinct) subvarieties of $S$ defined by weak associative laws of length $\leq 4$.

Keywords Symmetric implication zroupoid · Weak associative law · Identity of Bol–Moufang type · Semilattice with least element 0

1 Introduction

Bernstein (1934), gave a system of axioms for Boolean algebras using implication as the only connective. His system, while not equational, could easily be modified into an equational one by using an additional constant. In 2012, the second author of this paper extended this “modified Bernstein’s theorem” to De Morgan algebras (see Sankappanavar 2012). Indeed, he showed in Sankappanavar (2012) that the varieties of De Morgan algebras, Kleene algebras and Boolean algebras are term-equivalent, respectively, to the varieties, $DM$, $KL$ and $BA$ (defined below) whose defining axioms use only an implication $\rightarrow$ and a constant 0.

The essential role played by the identity (I): $(x \rightarrow y) \rightarrow z \approx ((z' \rightarrow x) \rightarrow (y \rightarrow z))'$, where $x' := x \rightarrow 0$, in the axiomatization of $DM$, $KL$ and $BA$ motivated the second author to introduce a new equational class of algebras called “implication zroupoids” in Sankappanavar (2012).

Definition 1.1 An algebra $A = (A, \rightarrow, 0)$, where $\rightarrow$ is binary and 0 is a constant, is called a zroupoid. A zroupoid $A = (A, \rightarrow, 0)$ is an implication zroupoid ($I$-zroupoid, for short) if $A$ satisfies:

(I) $(x \rightarrow y) \rightarrow z \approx ((z' \rightarrow x) \rightarrow (y \rightarrow z))'$, where $x' := x \rightarrow 0$, and
(I0) $0'' \approx 0$.

$I$ denotes the variety of implication zroupoids. It is also known that this new variety $I$ contains the variety $SL$ (defined below) which is term-equivalent to the variety of $\lor$-semilattices with the least element 0 [see Cornejo and Sankappanavar (2016a), wherein the name “implicator groupoid” is used instead of “implication zroupoid”].

The varieties $DM$ and $SL$ are defined relative to $I$, respectively, by the following identities:

(DM) $(x \rightarrow y) \rightarrow x \approx x$ (De Morgan Algebras);
(SL) $x' \approx x$ and $x \rightarrow y \approx y \rightarrow x$. 

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The varieties $K\mathcal{L}$ and $B\mathcal{A}$ are defined relative to $D\mathcal{M}$, respectively, by the following identities:

$$(KL) \ x \mapsto (y \mapsto y) \approx y \mapsto y \ (Kleene algebras);$$

$$(BA) \ x \mapsto x \approx 0' \ (Boolean algebras).$$

According to Sankappanavar (2012), the variety $\mathcal{I}$ generalizes the variety of De Morgan algebras and also exhibits some interesting properties; for example, for the identity $x'' \approx y \approx x' \approx y$ holds in $\mathcal{I}$. Several new subvarieties of $\mathcal{I}$ are also introduced and investigated in Sankappanavar (2012). The (largely unexplored) lattice of subvarieties of $\mathcal{I}$ seems to be fairly complex. In fact, Problem 6 of Sankappanavar (2012) reveals that the variety $\mathcal{I}$ is not modular, which answers Problem 5 of Sankappanavar (2012) in the negative. The size of the poset of subvarieties of $\mathcal{I}$ is known to be at least 40 (as of October 24, 2018), but it is still unknown whether the lattice of subvarieties is finite or infinite.

The papers (Cornejo and Sankappanavar 2016a, b, c, 2017, 2018a, 2018b; Gusev et al. 2018) have addressed further, but still partially, the above-mentioned problem by introducing new subvarieties of $\mathcal{I}$ and investigating relationships among them. It follows from Gusev et al. (2018) that the lattice of subvarieties of $\mathcal{I}$ is not modular, which answers Problem 5 of Sankappanavar (2012) in the negative. The size of the set of subvarieties of $\mathcal{I}$ seems to be quite complex. In fact, Problem 6 of Sankappanavar (2012) reveals that the variety $\mathcal{I}$ is not modular, which answers Problem 5 of Sankappanavar (2012) in the negative. The size of the poset of subvarieties of $\mathcal{I}$ is known to be at least 40 (as of October 24, 2018), but it is still unknown whether the lattice of subvarieties is finite or infinite.

Given an $\mathcal{I}$-zroupoid $\mathbf{A}$, there are two operations $\land$ and $\lor$ on $\mathbf{A}$ defined as follows: $x \land y := (x \rightarrow y)'$ and $x \lor y := (x' \land y)'$. These operations give rise to the (derived) algebra $\langle \mathbf{A}, \land, \lor, 0 \rangle$ which has been investigated in Cornejo and Sankappanavar (2016a, 2017). Among the important subvarieties of $\mathcal{I}$, the two that are the most relevant to this paper are: $\mathcal{I}_{2,0}$ and $\mathcal{M}C$ which are defined relative to $\mathcal{I}$, respectively, by the following identities:

$$(I_{2,0}) \ x'' \approx x;$$

$$(MC) \ x \land y \approx y \land x, \text{ where } x \land y := (x \rightarrow y').$$

We are now ready to make the following definition which plays a fundamental role in the rest of this paper.

**Definition 1.2** Let $\mathbf{A} \in \mathcal{I}$. $\mathbf{A}$ is involutive if $\mathbf{A} \in \mathcal{I}_{2,0}$. $\mathbf{A}$ is meet-commutative if $\mathbf{A} \in \mathcal{M}C$. $\mathbf{A}$ is symmetric if $\mathbf{A}$ is both involutive and meet-commutative. Let $\mathcal{S}$ denote the variety of symmetric $\mathcal{I}$-zroupoids. In other words, $\mathcal{S} = \mathcal{I}_{2,0} \cap \mathcal{M}C$.

The investigations in Cornejo and Sankappanavar (2016a, 2017) reveal that the variety $\mathcal{S}$ has some interesting properties; for example, for $\mathbf{A} \in \mathcal{S}$, the (derived) algebra $\langle \mathbf{A}, \land, \lor, 0 \rangle$, where $x \lor y := (x' \land y)'$, is both a distributive bisemilattice and a Birkhoff system (i.e., satisfies $x \land (x \lor y) \approx x \lor (x \land y)$). The name “symmetric $\mathcal{I}$-zroupoids” for the members of $\mathcal{I}_{2,0} \cap \mathcal{M}C$ is new.

In the present paper, we continue our investigations into $\mathcal{S}$. More precisely, we are interested in the subvarieties of $\mathcal{S}$ defined by weak associative laws. A precise definition of a weak associative law appears in Kunen (1996), which is essentially restated below in our terminology.

**Definition 1.3** Let $n \in \mathbb{N}$ and let $\mathcal{L} := \langle \times \rangle$, where $\times$ is a binary operation symbol. Let $p$ be a (groupoid) term in the language $\mathcal{L}$. Then, $\text{Var}(p)$ denotes the set of distinct variables occurring in $p$. $p$ is of length $n$ if the number of (not necessarily distinct) occurrences of variables (in $p$) is $n$. An identity $p \approx q$ is said to be a weak associative law of length $n$ in $\mathcal{L}$ if the following conditions hold:

1. $p$ and $q$ are terms of length $n$;
2. $\text{Var}(p) = \text{Var}(q)$;
3. the variables in $p$ and $q$ occur in the same order (only the bracketings are possibly different). In other words, $p$ and $q$ are built from the same word using (possibly) different bracketings.

The following (general) problem was raised in Cornejo and Sankappanavar (2018a). (In the sequel, we use the words “law” and “identity” interchangeably.)

**Problem** Let $\mathcal{V}$ be a given variety of algebras (whose language includes a binary operation symbol, say, $\times$). Investigate the subvarieties of $\mathcal{V}$ defined by weak associative laws (with respect to $\times$) and their mutual relationships.

Special cases of the above problem have already been considered in the literature, wherein the weak associative laws chosen are the identities of Bol–Moufang type (i.e., weak associative laws of length 4 with 3 distinct variables), and the variety $\mathcal{V}$ is chosen to be the variety of quasigroups or the variety of loops (for more information about these identities in the context of quasigroups and loops, see Fenyes (1969), Kunen (1996), Phillips and Vojtechovsky, (2005a), Phillips and Vojtechovsky, (2005b)).

Let $\mathcal{W}$ denote the set of weak associative laws of size $\leq 4$. The systematic notation given in the next definition for the identities in $\mathcal{W}$ is influenced by the notation developed in Phillips and Vojtechovsky, (2005a) for Bol–Moufang identities.

Without loss of generality, we will assume that the variables in the terms $t_1$ and $t_2$ occur alphabetically in any weak associative identity $t_1 \approx t_2$. Given a word $X$ of variables, we refer to each possible way of bracketing $X$ that will transform $X$ into a term, as a “bracketing,” and we assign a number to each such bracketing and call it the “bracketing number” of that term.

We will now develop a notation for weak associative laws that helps us to investigate them.

**Definition 1.4** Let $n, m, p, q \in \mathbb{N}$ and let $X$ denote a word of length $n$ in which there are $m$ distinct variables occurring alphabetically (with some variables possibly repeated). We
denote by \((nmXpq)\) the weak associative identity \(t_1 \approx t_2\) of length \(n\), with \(m\) distinct variables, whose terms \(t_1\) and \(t_2\) are obtained from \(X\) and have \(p\) and \(q\) as their respective bracketing numbers. We denote by \(nmXpq\) the variety defined, relative to \(S\), by the weak associative identity \((nmXpq)\).

**Example 1.5** Let \(X\) be the word \(A:= (x, x, x)\) where \(n = 3\) and \(m = 1\). Then, there are two possible bracketings, numbered 1 and 2, with \(a\) as a place holder:

\[
1 : a \rightarrow (a \rightarrow a), \quad \text{and} \quad 2 : (a \rightarrow a) \rightarrow a.
\]

Thus, \((31A12)\) denotes the identity \((x \rightarrow (x \rightarrow x)) \approx (x \rightarrow x) \rightarrow x\) of length 3 with one variable, and with bracketing numbers 1 and 2, and \((31A12)\) denotes the subvariety of \(\mathcal{I}\) defined by \((31A12)\), relative to \(S\).

Here is a second example: Let \(X\) be the word \(B:= (x, y, x, z)\) where \(n = 4\) and \(m = 3\). Then, \((43B23)\) denotes the identity \((x \rightarrow ((y \rightarrow x) \rightarrow z)) \approx (x \rightarrow y) \rightarrow (x \rightarrow z)\) of length 4 with 3 distinct variables and with bracketing numbers 2 and 3 in the listing of possible bracketings given in Sect. 6.

Observe that Bol–Moufang identities are precisely the weak associative laws of the form \((43Xpq)\) of length 4 that have 3 distinct variables. It was shown in Cornejo and Sankappanavar (2018a) that there are 4 nontrivial subvarieties of \(\mathcal{I}\) of length 1 and 2, respectively, which are trivial. So, we will use it to find examples and to check some conjectures. We would like to acknowledge that the software “Prover 9/Mace 4” developed by McCune (2005–2010) has been useful to us in some of our findings presented in this paper. We have used it to find examples and to check some conjectures.

### 2 Preliminaries and properties of \(S\)

We refer the reader to the standard references (Balbes and Dwinger 1974; Burris and Sankappanavar 1981; Rasiowa 1974) for concepts and results used, but not explained, in this paper.

Recall from Sankappanavar (2012) that \(\mathcal{S L}\) is the variety of semilattices with a least element 0. It was shown in Cornejo and Sankappanavar (2016a) that \(\mathcal{S L} = C \cap \mathcal{I}_{1,0}\) where the subvarieties \(C\) and \(\mathcal{I}_{1,0}\) of \(\mathcal{I}\) are defined, respectively, by the identities \(x \rightarrow y \approx y \rightarrow x\) and \(x' \approx x\).

The two-element algebras \(2_s, 2_b\) were introduced in Sankappanavar (2012). Their operations \(\rightarrow\) are, respectively, as follows:

\[
\begin{array}{c|cc}
\rightarrow: & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

Recall that \(\mathcal{V}(2_b) = B.A\) and \(\mathcal{V}(2_s) = \mathcal{S L}\) (Cornejo and Sankappanavar 2016a, Corollary 10.4).

**Lemma 2.1** (Cornejo and Sankappanavar 2016a) \(\mathcal{MC} \cap \mathcal{I}_{1,0} \subseteq C \cap \mathcal{I}_{1,0} = \mathcal{S L}\).

**Lemma 2.2** (Sankappanavar 2012, Theorem 8.15) Let \(A\) be an \(\mathcal{I}\)-groupoid. Then, the following are equivalent:

1. \(0' \rightarrow x \approx x\),
2. \(x'' \approx x\),
3. \((x \rightarrow x)' \approx x\),
4. \(x' \rightarrow x \approx x\).

Recall that \(\mathcal{I}_{2,0}\) and \(\mathcal{MC}\) are the subvarieties defined, respectively, relative to \(\mathcal{I}\) by the equations

\[
x'' \approx x, \quad (I_{2,0})
\]
\[
x \land y \approx y \land x, \quad (MC)
\]

**Lemma 2.3** (Sankappanavar 2012) Let \(A \in \mathcal{I}_{2,0}\). Then,

1. \(x' \rightarrow 0' \approx 0 \rightarrow x\),
2. \(0 \rightarrow x' \approx x \rightarrow 0'\).

**Lemma 2.4** Let \(A \in \mathcal{I}_{2,0}\). Then, \(A\) satisfies:

1. \((x' \rightarrow 0') \rightarrow y \approx (x \rightarrow y') \rightarrow y\),
2. \(((0 \rightarrow x) \rightarrow y) \rightarrow x \approx y \rightarrow x\),
3. \((x \rightarrow (y \rightarrow x))' \approx (x \rightarrow y) \rightarrow x\),
4. \((y \rightarrow x) \rightarrow y \approx (0 \rightarrow x) \rightarrow y\),
5. \((0 \rightarrow x) \rightarrow (x \rightarrow y) \approx x \rightarrow (x \rightarrow y)\),
6. \((0 \rightarrow x) \rightarrow (0 \rightarrow y) \approx x \rightarrow (0 \rightarrow y)\),
7. \(x \rightarrow y \approx x \rightarrow (x \rightarrow y)\),
8. \(0 \rightarrow (0 \rightarrow x') \approx 0 \rightarrow x'\),
9. \(0 \rightarrow (x \rightarrow y) \approx x \rightarrow (0 \rightarrow y)\),
10. \(0 \rightarrow (x \rightarrow y') \approx 0 \rightarrow (x' \rightarrow y)\),
11. \(x \rightarrow (y \rightarrow x') \approx y \rightarrow x'\),
12. \((x \rightarrow y) \rightarrow (y \rightarrow x) \approx y \rightarrow x\),
13. \((x \rightarrow y) \rightarrow (y \rightarrow z) \approx (0 \rightarrow x') \rightarrow (y \rightarrow z)\),
14. \((x \rightarrow y)' \rightarrow y \approx x \rightarrow y\),
15. \((x \to y) \to ((0 \to y) \to z) \approx (x \to y) \to z\).

16. \((x \to y) \to ((z \to y) \to (u \to z)) \approx (x \to y) \to (u \to z)\).

**Proof** For the proofs of items (1), (3), (4), (9), (10), (11), we refer the reader to Cornejo and Sankappanavar (2016a), and for the proofs of items (2), (6), (7), (8) to Cornejo and Sankappanavar (2016b). Items (5), (12) are proved in Cornejo and Sankappanavar (2016c). For the proofs of items (13) and (14), we refer the reader to Cornejo and Sankappanavar (2017). Finally, for the proof of (15), the reader is referred to the proof of the equation (3.4) in the proof of Lemma 3.1 of Cornejo and Sankappanavar (2017).

Proof of (16): Let \(a, b, c, d \in A\). Hence, \((a \to b) \to ((c \to b) \to (d \to c)) \equiv (a \to b) \to ((d \to c) \to (b \to (d \to c)))\)'s theorem \((14)\) \((a \to b) \to ((d \to c) \to (b \to (d \to c))) \equiv (a \to b) \to ((d \to c) \to b) \to (d \to c)) \equiv (a \to b) \to ((0 \to b) \to (d \to c)) \equiv (a \to b) \to (d \to c)\), completing the proof.

Lemma 2.5 Let \(A \in T_{2,0}\) such that \(A \models 0 \approx 0', \) then \(A \models 0 \to x \approx x\).

**Proof** Let \(a \in A\). Then, \(a = 0' \to a = (0 \to 0) \to a = (0' \to 0) \to a \equiv 0 \to 0 \to a\).

Lemma 2.6 (Cornejo and Sankappanavar 2018a) Let \(A \in T_{2,0}\) such that \(A \models 0 \to x \approx x, \) then \(A \models (x \to y) \approx x' \to y'\).

Throughout the rest of this paper, \(A \in S\).

**Lemma 2.7** (Cornejo and Sankappanavar 2018a) \(A\) satisfies:

1. \(x \to (y \to z) \approx y \to (x \to z)\),
2. \(x' \to y \approx y' \to x\).

**Lemma 2.8** Let \(A \in S\) such that \(A \models 0 \to x \approx x \to x\), then

1. \(A \models 0 \to (x \to x) \approx x \to x\),
2. \(A \models 0 \to x' \approx 0 \to x\).

**Proof** Let \(a \in A\).

1. Observe that \(0 \to (a \to a) \equiv (a \to a) \to (a \to a)\) \(\overset{2.4(12)}{=} \)
   \(a \to a\).
2. \(0 \to a \equiv a \to a \overset{2.7(2)}{=} a' \to a' \overset{hyp}{=} 0 \to a'\).

This proves the lemma.

**Lemma 2.9** Let \(A \in S\) such that \(A \models 0 \to (x \to x) \approx x \to x\), then \(A\) satisfies the following identities:

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Let $E$ be the set consisting of the terms:

$$
t_1(x, y, z, t) = ((x \rightarrow y) \rightarrow z) \rightarrow t,
$$

$$
t_2(x, y, z, t) = z \rightarrow ((y \rightarrow x) \rightarrow t),
$$

$$
t_3(x, y, z, t) = (y \rightarrow x) \rightarrow (z \rightarrow t),
$$

$$
t_4(x, y, z, t) = ((z \rightarrow y) \rightarrow x) \rightarrow t,
$$

and

$$
t_5(x, y, z, t) = (y \rightarrow z) \rightarrow (x \rightarrow t).
$$

**Lemma 2.12** If $A \models 0 \rightarrow x \approx x$, then $A \models e_1 \approx e_2$ where $e_1, e_2 \in E$.

**Proof** Since $A \models 0 \rightarrow x \approx x$, by Lemma 2.6,

$$
A \models (x \rightarrow y) \rightarrow x' \rightarrow y'.
$$

Let $a, b, c, d \in A$. Then,

$$(a \rightarrow b) \rightarrow c \rightarrow d \overset{2.7(2)}{=} d' \rightarrow ((a \rightarrow b) \rightarrow c)' \overset{(2.3)}{=} (d \rightarrow ((a \rightarrow b) \rightarrow c))' \overset{2.7(4)}{=} (d \rightarrow (c' \rightarrow (a \rightarrow b)))' \overset{(2.3)}{=} c' \rightarrow (d \rightarrow (a \rightarrow b))' \overset{\overset{\text{r}}{\text{r}}}{=} c' \rightarrow (d \rightarrow (b' \rightarrow a))' \overset{(2.3)}{=} c' \rightarrow (d \rightarrow (b' \rightarrow a))' \overset{3.1(2)}{=} c \rightarrow (d \rightarrow (b' \rightarrow a))' \overset{(2.3)}{=} c \rightarrow (d \rightarrow (b' \rightarrow a))' \overset{2.7(1)}{=} (b \rightarrow a) \rightarrow (c \rightarrow d),
$$
proving $t_1 \approx t_2$ and $t_1 \approx t_3$.

Next, $((a \rightarrow b) \rightarrow c) \rightarrow d \overset{2.7(2)}{=} (c' \rightarrow (a \rightarrow b))' \rightarrow d \overset{(2.3)}{=} (c' \rightarrow (a' \rightarrow b))' \rightarrow d \overset{2.7(1)}{=} (a' \rightarrow (c' \rightarrow b)) \rightarrow d \overset{2.7(2)}{=} (c' \rightarrow b') \rightarrow a \rightarrow d \overset{(2.3)}{=} (c \rightarrow b) \rightarrow a \rightarrow d,
$$
proving $t_1 \approx t_4$.

Also, we have that

$$
c \rightarrow ((b \rightarrow a) \rightarrow d) \overset{2.7(2)}{=} c \rightarrow (d' \rightarrow (b \rightarrow a))' \overset{2.7(2)}{=} c \rightarrow (d \rightarrow (a \rightarrow b)) \overset{2.7(1)}{=} c \rightarrow (a \rightarrow (d' \rightarrow b)) \overset{2.7(1)}{=} a \rightarrow (c \rightarrow (d' \rightarrow b)) \overset{2.7(2)}{=} a \rightarrow ((c \rightarrow b)' \rightarrow d) \overset{2.7(2)}{=} a \rightarrow ((b' \rightarrow c') \rightarrow d) \overset{(2.3)}{=} a \rightarrow ((b \rightarrow c) \rightarrow d) \overset{2.7(1)}{=} (b \rightarrow c) \rightarrow (a \rightarrow d),
$$
proving $t_2 \approx t_5$. □

**Lemma 2.13** If $A \models 0 \rightarrow x \approx x$, then $A \models x \rightarrow ((x \rightarrow x) \rightarrow y) \approx (x \rightarrow (x \rightarrow x)) \rightarrow y$.

**Proof** By Lemma 2.12, $A \models e_1 \approx e_2$ for all $e_1, e_2 \in E$. Consider $a, b \in A$. We have that $a \rightarrow ((a \rightarrow a) \rightarrow b) \overset{2.11}{=} t_2(a, a, a, b) = t_4(a, a, a, b) = ((a \rightarrow a) \rightarrow a) \rightarrow b \overset{2.11}{=} (a \rightarrow (a \rightarrow a)) \rightarrow b$, proving the lemma. □

**Lemma 2.14** $A$ satisfies:

1. $(x \rightarrow x) \rightarrow (x \rightarrow x) \approx x \rightarrow (x \rightarrow x))$,
2. $((x \rightarrow x) \rightarrow x \approx x \rightarrow (x \rightarrow x)))$,
3. $(x \rightarrow y) \rightarrow (y \rightarrow z) \approx ((y \rightarrow x) \rightarrow y) \rightarrow z$,
4. $y \rightarrow ((x \rightarrow y) \rightarrow z) \approx ((y \rightarrow x) \rightarrow y) \rightarrow z$,
5. $(x \rightarrow x) \rightarrow (x \rightarrow y) \approx x \rightarrow ((x \rightarrow x) \rightarrow y)$,
6. $x \rightarrow ((x \rightarrow x) \rightarrow y) \approx (x \rightarrow x) \rightarrow x \rightarrow y$.

**Proof** Let $a, b, c \in A$.

1. Observe that $((a \rightarrow a) \rightarrow (a \rightarrow a) \overset{2.7(1)}{=} a \rightarrow ((a \rightarrow a) \rightarrow a) \overset{2.4(4)}{=} a \rightarrow ((0 \rightarrow a) \rightarrow a) \overset{2.7(1)}{=} (0 \rightarrow a) \rightarrow (a \rightarrow a) \overset{2.4(5)}{=} a \rightarrow (a \rightarrow a)$.
2. $((a \rightarrow a) \rightarrow a \overset{2.4(4)}{=} ((0 \rightarrow a) \rightarrow a) \overset{2.7(1)}{=} (0 \rightarrow a) \rightarrow (a \rightarrow a) \overset{2.4(5)}{=} a \rightarrow (a \rightarrow a)$.
3. $(a \rightarrow b) \rightarrow (b \rightarrow c) \overset{2.4(3)}{=} (0 \rightarrow a') \rightarrow (b \rightarrow c) \overset{1(2)}{=} (((b \rightarrow c)' \rightarrow 0) \rightarrow (a' \rightarrow (b \rightarrow c)))' \overset{1(2)}{=} ((b \rightarrow c) \rightarrow (a' \rightarrow (b \rightarrow c)))' \overset{2.7(2)}{=} ((b \rightarrow (c' \rightarrow a)) \rightarrow (b \rightarrow c))' \overset{2.7(2)}{=} (b \rightarrow (c' \rightarrow a)) \rightarrow (b \rightarrow c) \overset{1(2)}{=} ((b \rightarrow (c' \rightarrow a)) \rightarrow (b \rightarrow c))' \overset{1(2)}{=} ((b \rightarrow (c' \rightarrow a)) \rightarrow (b \rightarrow c))' \overset{1(2)}{=} (b \rightarrow (c' \rightarrow a)) \rightarrow (b \rightarrow c)$.
4. $b \rightarrow ((a \rightarrow b) \rightarrow c) \overset{2.7(1)}{=} (a \rightarrow b) \rightarrow (b \rightarrow c) \overset{(3)}{=} (b \rightarrow a) \rightarrow b \rightarrow c$.

For (5), use Lemma 2.7 (1), and for (6) use Lemma 2.7 (1) and item (3). (7) is a special case of (4). Finally, one can use Lemma 2.7 (1) and item (3) to prove (8). □

### 3 Weak associative laws of length 3

In this section, we examine all the weak associative laws of length 3.

#### 3.1 With one variable

The only word of length 3 with 1 variable is:

$A: \{x, x, x\}$.

Ways in which the word $A$ can be bracketed (where $a$ is just a place holder) are:

$1: a \rightarrow (a \rightarrow a), \quad 2: (a \rightarrow a) \rightarrow a$.

The only weak associative identity in this category is:

$1: (31A12) x \rightarrow (x \rightarrow x) \approx (x \rightarrow x) \rightarrow x$.

#### 3.2 With 2 variables

Possible words of length 3 with 2 variables are:

$A: \{x, x, y\}, \quad B: \{x, y, x\}, \quad C: \{x, y, y\}$.

Ways in which a word of length 3 can be bracketed:
The only word of length 3 with 3 variables is: 3.

1: (LALT) \( x \to (x \to y) \approx (x \to x) \to y \) (the left-
alternative law),
2: (FLEX) \( x \to (y \to x) \approx (x \to y) \to x \) (the flexible-
law),
3: (RALT) \( x \to (y \to y) \approx (x \to y) \to y \) (the right-
alternative law).

We should note that we did not follow our convention in this case, since these identities are well known by the above names. We let \( LALT \) in this case, since these identities are well known by the above names. We let \( LALT \), \( FLEX \) and \( RALT \) denote, respectively, the subvarieties \( S \) defined by (LALT), (FLEX) and (RALT).

**Theorem 3.1** \( FLEX = RALT = LALT = SL \).

**Proof** Let \( A \in LALT \cup FLEX \cup RALT \) and \( a \in A \).

First, let \( A \in LALT \). Then, \( a \overset{x}{\Rightarrow} a'' \overset{2.2(4)}{\Rightarrow} (a'' \to a')' \overset{x}{\Rightarrow} (a \to a')' = (a \to (a \to 0))' \overset{(LALT)}{\Rightarrow} (a \to a) \to 0' = (a \to a)'' \Rightarrow a \to a.

Next, let \( A \in FLEX \). Then, \( a \overset{2.2(4)}{\Rightarrow} a' \overset{2.2(4)}{\Rightarrow} (a'' \to a') \overset{FLEX}{\Rightarrow} x \Rightarrow (a \to a') \to a \overset{2.2(4)}{\Rightarrow} a \to (a' \to a) \overset{x}{\Rightarrow} a \to a.

Finally, let \( A \in RALT \). Then, \( a \overset{x}{\Rightarrow} a'' \overset{2.2(4)}{\Rightarrow} (a \to a)'' \overset{RALT}{\Rightarrow} (a \to (0 \to 0))' \overset{2.2(4)}{\Rightarrow} a \to (a \to a)'' \overset{2.2(4)}{\Rightarrow} a \overset{2.2(4)}{\Rightarrow} a' \overset{2.2(4)}{\Rightarrow} a \). Therefore, \( A \models x \approx x' \), and consequently, \( A \in SL \).

For the converse, it is easy to verify that \( 2_\lambda \) satisfies (FLEX), (RALT) and (LALT). The theorem follows since \( V(2_\lambda) = SL \).

**3.3 With 3 variables**

The only word of length 3 with 3 variables is:

\( A: \langle x, y, z \rangle \).

Ways in which a word of length 3 can be bracketed:

1: \( a \to (a \to a) \), \( 2: (a \to a) \to a \).

The only weak associative identity in this category is:

\( 33 \cdot A12 \) \( x \to (y \to z) \approx (x \to y) \to z \) (associative law).

**Lemma 3.2** \( 33 \cdot A12 = SL \).

**Proof** By Cornejo and Sankappanavar (2016a), \( SL \subseteq 33 \cdot A12 \). Hence, let us consider \( A \in 33 \cdot A12 \) and \( a \in A \).

Observe that \( 0' \overset{2.8(2)}{\Rightarrow} 0 \to 0' = 0 \to (0 \to 0) = (0 \to 0) \to 0 = 0' \to 0' \overset{2.8(1)}{\Rightarrow} 0 \). Then,

\( A \models 0' \approx 0. \)

Therefore, \( a \overset{2.2(4)}{\Rightarrow} a' \to a = (a \to 0) \to a \overset{33 \cdot A12}{\Rightarrow} a \to (0 \to a) \overset{(3.1)}{\Rightarrow} a \to (0' \to a) \overset{2.2(1)}{\Rightarrow} a \to a \). Consequently, \( A \in SL \) by Lemmas (2.1) and (2.10).

**4 Weak associative laws with length 4 and with 1 variable**

The only word of length 4 with 1 variable is:

\( A: \langle x, x, x, x \rangle \).

Ways in which a word of length 4 can be bracketed are:

1: \( a \to (a \to (a \to a)) \),
2: \( a \to ((a \to a) \to a) \),
3: \( (a \to a) \to (a \to a) \),
4: \( (a \to (a \to a)) \to a \),
5: \( ((a \to a) \to a) \to a \).

By now, we believe that the reader is well acquainted with our notation for identities. So, we will, no longer, present the list of the identities in this and the remaining categories.

**Lemma 4.1** The following hold:

1. \( 41 \cdot A12 = 41 \cdot A13 \),\n2. \( 41 \cdot A24 = 41 \cdot A34 \),\n3. \( 41 \cdot A25 = 41 \cdot A35 \),\n4. \( 41 \cdot A23 = S \),\n5. \( 41 \cdot A13 = 41 \cdot A15 = 41 \cdot A35 = S \),\n6. \( 41 \cdot A14 = 41 \cdot A34 = 41 \cdot A45 \).

**Proof** Items (1), (2), (3) and (4) follow from Lemma 2.7 (1). For (5), observe that \( S \subseteq 41 \cdot A13 \) by Lemma 2.14 (1), \( S \subseteq 41 \cdot A15 \) by Lemma 2.14 (2) and \( S \subseteq 41 \cdot A35 \) by Lemma 2.14 (6) and (5).

To prove (6), \( 41 \cdot A14 \subseteq 41 \cdot A34 \) follows from (41A14) and Lemma 2.14 (1). From (41A34), Lemma 2.7 (1) and Lemma 2.14 (6), we have that \( 41 \cdot A34 \subseteq 41 \cdot A45 \). The inclusion \( 41 \cdot A45 \subseteq 41 \cdot A14 \) follows from (41A45) and Lemma 2.14 (2).

**Lemma 4.2** \( 41 \cdot A14 = 31 \cdot A12 \).

**Proof** In view of Lemma 2.4 (7), we get \( 31 \cdot A12 \subseteq 41 \cdot A14 \). So, we will prove the converse. Let \( A \in 41 \cdot A14 \). Hence,
Lemma 5.2 We have

\[ \implies_2 (a \implies (a \implies a)) \implies (a \implies (a \implies a)) \]

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\[ \implies (a \implies (a \implies a)) \implies (a \implies (a \implies a)) \]

\[ \implies (a \implies (a \implies a)) \implies (a \implies
1. $42A14/a, 0$.
2. $42B14/a, a'$.
3. $42B24/a, a'$.
4. $42C24/a, 0$.
5. $42C45/a, 0$.
6. $42E12/a', a$.
7. $42E23/a', 0$.
8. $42E35/a', 0$.
9. $42F24/a, 0$.
10. $42F45/a, 0$.

We prove (7) as an illustration.

\[
\begin{align*}
&\overset{2.4(4)}{a \approx a'} \rightarrow a \overset{2.4(4)}{\approx} (a'' \rightarrow a') \rightarrow a \overset{x=1''}{\approx} (a \rightarrow a') \\
&\overset{2.4(4)}{= (0 \rightarrow a')} \rightarrow a \overset{2.7(2)}{\approx} (0 \rightarrow a')' \approx (0 \rightarrow a') \rightarrow a = (0 \rightarrow a')' \rightarrow a' \overset{2.4(23)}{\approx} (0 \rightarrow a') \rightarrow (a' \rightarrow 0) = a'' \rightarrow a''
\end{align*}
\]

Hence, the statement (7) is true. Similarly, one can verify the rest of the above statements is true, from which it follows that, in each of the above cases, $\mathcal{X} \models x \approx x$.

Then, applying Lemma 2.10, we get that $\mathcal{X} \models x' \approx x$.

The notation, where $x_0, y_0, x_1, y_1 \in \mathbf{A}$,

\[\mathcal{X}/x_0, y_0/x_1, y_1 / p \approx q\]

is an abbreviation for the following statement:

In the identity (X) that defines the variety $\mathcal{X}$, relative to $\mathcal{S}$, if we assign $x : = x_0, y : = y_0,$ (and simplify (X) using the appropriate lemmas from the list (s)), we obtain that $\mathbf{A} \models 0' \approx 0$, and then, we assign $x : = x_1, y := y_1$ in the identity (X) (and simplify it using $0' = 0$ and the list (s)), then $\mathbf{A} \models p \approx q$.

Secondly, consider the varieties associated with the following statements:

1. $42B45/0, 0/a, a'/x \rightarrow x \approx x$.
2. $42C34/0, 0/a, 0 / x \rightarrow x \approx x$.
3. $42E14/0, 0/a, 0 / x \rightarrow x \approx x$.
4. $42E24/0, 0/a, 0 / x \rightarrow x \approx x$.
5. $42E34/0, 0/a, 0 / x \rightarrow x \approx x$.
6. $42E45/0, 0/a, 0 / x \rightarrow x \approx x$.
7. $42F23/0, 0'/a, 0 / x \rightarrow x \approx x$.

As a sample, we prove (2) below:

\[
\begin{align*}
0 &\overset{2.4(4)}{\approx} 0' \rightarrow 0 \overset{2.4(4)}{\approx} (0'' \rightarrow 0') \rightarrow 0 \overset{0=0'}{\approx} (0 \rightarrow 0') \rightarrow 0 \\
&\overset{2.7(1)}{= (0 \rightarrow (0 \rightarrow 0)) \rightarrow 0 \overset{42C34}{(0 \rightarrow 0) \rightarrow (0 \rightarrow 0')} \\
&\overset{0'=0'}{= 0' \overset{2.4(1)}{\approx} 0'} \rightarrow a \overset{2.4(1)}{\approx} 0' \rightarrow (a \rightarrow a) \overset{0=0'}{= 0 \rightarrow (a \rightarrow a)} \overset{2.7(2)}{0 \rightarrow (a' \rightarrow a')} \overset{2.4(10)}{= 0 \rightarrow (a \rightarrow a')} \\
&\overset{0=0'}{=} 0 \rightarrow ((a \rightarrow a) \rightarrow 0) \overset{2.7(1)}{= (a \rightarrow a) \rightarrow (0 \rightarrow 0)} \overset{42C34}{(a \rightarrow (a \rightarrow 0)) \rightarrow 0 = (a \rightarrow a')} \overset{2.4(4)}{= a'' \overset{x=1''}{\approx} a''}
\end{align*}
\]

a. Hence, (2) holds. Similarly, one can verify that each of the above statements is true. So, it follows that in each case $\mathcal{X} \models x \rightarrow x \approx x$. Then, applying Lemma 2.10, we get that $\mathcal{X} \models x' \approx x$.

Thirdly, consider the varieties associated with the following statements:

1. $42C14/0, 0/a, 0 / x' \approx x$.
2. $42D14/0, 0/a, 0 / x' \approx x$.
3. $42D34/0, 0/a, 0 / x' \approx x$.
4. $42G23/0', 0/a, 0 / x' \approx x$.
5. $42G24/0, 0/a, 0 / x' \approx x$.
6. $42G35/0', 0/a, 0 / x' \approx x$.
7. $42G45/0, 0/a, 0 / x' \approx x$.

It is easy to verify that the above statements are true. Hence, it follows in each of the above cases that $\mathcal{X} \models x' \approx x$.

Thus, the varieties still left to consider are $42B34$ and $42F_{15}$.

Let $\mathbf{A} \in 42B34$ and let $a \in A$. Since $a \overset{x=1''}{\approx} a'' \overset{2.4(1)}{\approx} ((0 \rightarrow a')' \rightarrow a \overset{2.4(7)}{\approx} ((0 \rightarrow a) \rightarrow 0' \overset{x=1''}{\approx} 0 \rightarrow a)$, we have that

\[
\mathbf{A} \models x \approx 0 \rightarrow x.
\]

Therefore, $a \rightarrow a \overset{5.1}{\approx} (0 \rightarrow a) \rightarrow a \overset{2.4(4)}{\approx} (a \rightarrow a) \rightarrow a \overset{2.4(1)}{\approx} (a \rightarrow a) \rightarrow (0' \rightarrow a) \overset{42B34}{\approx} (a \rightarrow (a \rightarrow 0')) \rightarrow a \overset{2.4(7)}{\approx} (a \rightarrow 0') \rightarrow a \overset{2.4(4)}{\approx} (a \rightarrow a') \rightarrow a \overset{x=1''}{\approx} (a'' \rightarrow a') \rightarrow a \overset{2.4(4)}{\approx} a' \overset{2.4(4)}{\approx} a$. Hence,

\[
\mathbf{A} \models x \approx x \rightarrow x.
\]

Then, applying Lemma 2.10, we get that $\mathbf{A} \models x' \approx x$.

Let $\mathbf{A} \in 42F_{15}$ and let $a \in A$. If we replace $x := 0$ and $y := 0'$, we obtain that

\[
\mathbf{A} \models 0 \approx 0'.
\]

Since $a \overset{2.4(1)}{\approx} 0' \rightarrow a \overset{5.2}{\approx} 0 \rightarrow a \overset{2.4(4)}{\approx} (a' \rightarrow a) \overset{x=1''}{\approx} 0 \rightarrow (a' \rightarrow a) \overset{42F_{15}}{\approx} ((0 \rightarrow a') \rightarrow a) \overset{5.2}{\approx} (0 \rightarrow a') \rightarrow a \overset{2.4(1)}{\approx} (a' \rightarrow a) \rightarrow 0 \overset{2.7(2)}{\approx} (a \rightarrow a) \rightarrow 0$, the identity

\[
\mathbf{A} \models x \approx (x \rightarrow x)' \overset{5.3}{\approx} (a' \rightarrow a') \overset{2.7(2)}{\approx} (a \rightarrow a)' \overset{5.3}{\approx} a,
\]

proving the theorem. \(\square\)

Let $I$ be the set consisting of the following identities:

\[
42A24, 42A45, 42B13, 42B15, 42B23, 42B25, 42D24, 42D45, 42F_{25} \text{ and } 42G15.
\]
Lemma 5.5 Let \( x \in I \). If \( A \models (x) \), then \( A \models 0 \to x \approx x \).

**Proof** Let \( a \in A \). We consider the following cases:

**Case 1:** \( A \models (42A24) \). Then, \( a \equiv^{2.12(1)} 0' \to a \equiv^{2.2(4)} (0'' \to 0') \to a = (0 \to 0') \to a = (0 \to 0) \to a (42A24) \equiv 0 \to (0' \to a) = 0 \to (0' \to a) \equiv 0 \to a \).

**Case 2:** \( A \models (42A45) \). We have \( a \equiv^{2.12(1)} 0' \to a \equiv^{2.2(4)} (0'' \to 0') \to a = (0 \to 0') \to a = (0 \to 0) \to a (42A45) \equiv ((0 \to 0) \to a) = (0' \to a) \equiv 0 \to a \).

**Case 3:** \( A \models (42B13) \). In this case, we get \( x \equiv^y a'' = a' \to 0 \equiv^{2.12(1)} 0' \to (a' \to 0) \equiv (0 \to 0) \to (a' \to 0) (42B13) \equiv 0 \to (a' \to 0) = 0 \to (a' \to 0) \equiv 0 \to a'' \equiv x \).

**Case 4:** \( A \models (42B15) \). Then, \( a \equiv a'' = a' \equiv 0 \equiv^{2.2(4)} (0' \to a') \to 0 = (0 \to 0) \to a' \equiv (0 \to 0) \to a' (42B15) \equiv 0 \to (0 \to a') \equiv 0 \to a' \equiv 0 \to a'' \equiv x \).

**Case 5:** If \( A \models (42B23) \), then \( a \equiv^{2.12(1)} a'' = a' \equiv 0 \equiv^{2.2(4)} 0' \to (a' \to 0) \equiv (0 \to 0) \to (a' \to 0) (42B23) \equiv (0 \to a') \to 0 \equiv^{2.8(4)} 0 \to a'' \equiv 0 \to a \).

**Case 6:** \( A \models (42B25) \). One has \( a \equiv^{2.12(1)} (0' \to a') \to 0 = (0 \to 0) \to a' \equiv 0 (42B25) \equiv 0 \to (0 \to a') \equiv 0 \to a'' \equiv x \).

**Case 7:** \( A \models (42D24) \). Then, setting \( x := 0 \) and \( y := 0 \) in the identity \( (42D24) \), we obtain that \( 0' = 0 \). Then, apply Lemma 2.5.

**Case 8:** \( A \models (42D45) \). Then, set \( x := 0 \) and \( y := 0 \) in the identity \( (42D45) \) to obtain that \( 0' = 0 \). Now, apply Lemma 2.5.

**Case 9:** \( A \models (42F25) \). And consider \( x := 0 \) and \( y := 0' \) in the identity \( (42F25) \); we obtain that \( 0' = 0 \). Then, apply Lemma 2.5.

**Case 10:** \( A \models (42G15) \). Then, setting \( x := 0' \) and \( y := 0 \) in the identity \( (42G15) \), it is easy to obtain that \( 0' = 0 \). Then, apply Lemma 2.5.

**Theorem 5.6** If \( A \models 0 \to x \approx x \), then \( A \models (y) \) for all \( (y) \in I \).

**Proof** Observe that, by Lemma 2.12,

\[ A \models t_1 \approx t_2 \approx t_4 \approx t_5. \tag{5.4} \]

Also, by Lemma 2.13, \( A \models (42A24) \). Hence, by Lemma 2.14 (6), \( A \models (42A45) \). Now, \( A \models (42B13) \) is true, since \( (a \to a) \to (b \to a) \equiv^{2.7.1(1)} b \to ((a \to a) \to a) \equiv^{2.14(1)} b \to (a \to (a \to a)) \equiv^{2.7.1(1)} a \to (b \to (a \to a)) \).

Using Lemma 5.5 and Theorem 5.6, one can easily verify the following Theorem.

**Theorem 5.7** \( 42A24 = 42A45 = 42B13 = 42B15 = 42B23 = 42B25 = 42D24 = 42D45 = 42F25 = 42G15. \)

**Lemma 5.8** \( 42A24 = 31A12. \)
**Case 4:**

**Proof** Let \( A \in 31,412. \) Since, replacing \( x := 0 \) in the identity (31A12), we obtain \( 0 = 0' \), then \( a = a'' = a' \rightarrow 0 = a' \rightarrow 0' = 0 \rightarrow a \) by Lemma 2.3 (2). Then, by Lemma 2.13, \( A \in 42,424. \) Therefore, \( 31,412 \subseteq 42,424. \)

For the converse, let consider \( A \in 42,424. \) Let \( a \in A. \) By Lemma 5.5 \( A \models 0 \rightarrow x \approx x. \) (5.6)

Hence, \( (a \rightarrow a) \rightarrow a \overset{2.4(4)}{=} (0 \rightarrow a) \rightarrow a \overset{(5.6)}{=} a \rightarrow a \overset{2.4(7)}{=} a \rightarrow (a \rightarrow a). \)

Let \( J \) be the set consisting of the following identities: (42B12), (42B35), (42C23), (42C25), (42C35) and (42G25).

**Lemma 5.9** Let \( (x) \in J. \) If \( A \models (x), \) then \( A \models 0 \rightarrow (x \rightarrow x) \approx x \rightarrow x. \)

**Proof** Let \( a \in A. \) Consider the following cases:

Case 1: \( A \models (42B12). \) Then, \( a \rightarrow a \overset{2.4(4)}{=} a \rightarrow (a' \rightarrow a) \approx a \rightarrow ((a \rightarrow 0) \rightarrow a) \overset{(42B12)}{=} a \rightarrow (a \rightarrow (0 \rightarrow a)) \overset{2.7(1)}{=} a \rightarrow ((0 \rightarrow (a \rightarrow a)) \overset{2.7(1)}{=} 0 \rightarrow (a \rightarrow (a \rightarrow a)) \overset{2.4(7)}{=} 0 \rightarrow (a \rightarrow a). \)

Case 2: \( (x) \in (42B35). \) Hence, \( 0 \rightarrow (a \rightarrow a) \overset{2.4(6)\text{and}(9)}{=} (0 \rightarrow a) \rightarrow (0 \rightarrow a) \overset{2.7(1)}{=} 0 \rightarrow ((0 \rightarrow a) \rightarrow a) \overset{2.4(4)}{=} 0 \rightarrow ((a \rightarrow a) \rightarrow a) \overset{2.7(1)}{=} (a \rightarrow a) \rightarrow (0 \rightarrow a) \overset{(42B35)}{=} (a \rightarrow a) \rightarrow 0 \rightarrow a \overset{a \rightarrow (a \rightarrow a) \overset{2.4(4)}{=} a \rightarrow a.}{} \)

Case 3: \( (x) \in (42C23). \) We get \( a \rightarrow a \overset{2.2(1)}{=} 0' \rightarrow (a \rightarrow a) \approx (0 \rightarrow 0) \rightarrow ((a \rightarrow a) \rightarrow a) \overset{2.7(1)}{=} 0 \rightarrow ((0 \rightarrow a) \rightarrow a) \overset{2.4(6)\text{and}(9)}{=} 0 \rightarrow (a \rightarrow a). \)

Case 4: \( (x) \in (42C25). \) One has \( a \rightarrow a \overset{2.2(1)}{=} 0' \rightarrow (a \rightarrow a) \approx (0 \rightarrow 0) \rightarrow (a \rightarrow a) \overset{42C25}{=} 0 \rightarrow ((0 \rightarrow a) \rightarrow a) \overset{2.7(1)}{=} (a \rightarrow a) \rightarrow (0 \rightarrow a) \overset{2.4(6)\text{and}(9)}{=} 0 \rightarrow (a \rightarrow a). \)

Case 5: \( (x) \in (42C35). \) Then, \( 0 \rightarrow (a \rightarrow a)' \overset{2.7(2)}{=} (a \rightarrow a) \rightarrow 0' \approx (a \rightarrow a) \rightarrow (0 \rightarrow 0) \overset{(42C35)}{=} ((a \rightarrow a) \rightarrow 0 \rightarrow 0 \overset{x \approx x' \approx x}{=} a \rightarrow a. \) Therefore,

\[ A \models 0 \rightarrow (x \rightarrow x) \approx x \rightarrow x. \] (5.7)

Hence, \( a \rightarrow a \overset{2.5(7)}{=} 0 \rightarrow (a \rightarrow a) \overset{2.7(2)}{=} 0 \rightarrow (0 \rightarrow (a \rightarrow a)) \overset{2.4(8)}{=} 0 \rightarrow (a \rightarrow a)' \overset{2.4(9)}{=} 0 \rightarrow (0 \rightarrow a). \)

Case 6: \( (x) \in (42G25). \) We have \( 0 \rightarrow (a \rightarrow a) \overset{2.4(6)\text{and}(9)}{=} (0 \rightarrow a) \rightarrow (0 \rightarrow a) \overset{2.7(1)}{=} 0 \rightarrow ((0 \rightarrow a) \rightarrow a) \overset{2.4(4)}{=} 0 \rightarrow ((a \rightarrow a) \rightarrow a) \overset{(42G25)}{=} ((0 \rightarrow a) \rightarrow a) \overset{2.4(2)}{=} a \rightarrow a, \) which proves the lemma. \( \Box \)

**Theorem 5.10** If \( A \models 0 \rightarrow (x \rightarrow x) \approx x \rightarrow x, \) then \( A \models (y), \) for all \( (y) \in J. \)

**Proof** Let \( a, b \in A. \) By Lemma 2.9 (1) and (2), \( A \models (x \rightarrow y \rightarrow y) \approx ((x \rightarrow x) \rightarrow y)' \overset{2.4(9)}{=} ((x \rightarrow y \rightarrow y) \rightarrow (x \rightarrow y) \rightarrow y)' \overset{2.4(2)}{=} a \rightarrow a \rightarrow a, \) and \( A \models (x \rightarrow y \rightarrow z) \approx ((x \rightarrow x) \rightarrow y) \rightarrow z \overset{(5.9)}{=} ( (x \rightarrow y \rightarrow y) \rightarrow (x \rightarrow y) \rightarrow y)' \overset{(5.9)}{=} ((a \rightarrow a) \rightarrow (a \rightarrow a)) \rightarrow (b \rightarrow (a \rightarrow a)) \overset{2.4(6)\text{and}(9)}{=} (((0 \rightarrow (a \rightarrow a)) \rightarrow (b \rightarrow (a \rightarrow a))) \rightarrow (c \rightarrow (a \rightarrow a))) \rightarrow (d \rightarrow (a \rightarrow a)) \overset{2.4(6)\text{and}(9)}{=} (0 \rightarrow (a \rightarrow a)) \rightarrow (b \rightarrow (a \rightarrow a)) \overset{2.4(12)}{=} b \rightarrow (a \rightarrow a) \overset{2.7(1)}{=} b \rightarrow ((a \rightarrow a) \rightarrow (a \rightarrow a)) \overset{2.4(12)}{=} b \rightarrow (a \rightarrow a) \overset{2.7(1)}{=} a \rightarrow (a \rightarrow a) \overset{2.7(1)}{=} a \rightarrow (a \rightarrow a), \) the algebra \( A \) satisfies (42B12). By Lemmas 2.9 (3) and 2.7 (1), \( A \) satisfies (42C23). In view of (5.9) and (42C23), we have \( A \models (x \rightarrow x) \rightarrow (y \rightarrow y) \approx x \rightarrow ((x \rightarrow y) \rightarrow y) \) proving that the identity (42C25) holds in \( A. \)

It remains to verify that the algebra \( A \) satisfies identity (42G25). To finish off the proof, \( ((a \rightarrow a) \rightarrow b) \rightarrow b \overset{2.2(1)}{=} ((b' \rightarrow a) \rightarrow (b \rightarrow b)) \overset{x \approx x'}{=} (b' \rightarrow a) \rightarrow (b' \rightarrow a) \overset{2.4(7)}{=} b' \rightarrow (b' \rightarrow a) \overset{2.7(2)}{=} (b \rightarrow b) \rightarrow ((b' \rightarrow a) \rightarrow (b \rightarrow b))) \overset{2.7(2)}{=} (b \rightarrow b) \rightarrow ((b' \rightarrow a) \rightarrow (b \rightarrow b)) \overset{2.4(7)}{=} (b \rightarrow b) \rightarrow ((b' \rightarrow a) \rightarrow (b \rightarrow b)) \overset{2.7(2)}{=} a \rightarrow ((b \rightarrow b) \rightarrow (b \rightarrow b)) \overset{2.7(1)}{=} a \rightarrow (b \rightarrow b), \)

Consequently, \( A \models (42G25). \) \( \Box \)

Recall that the varieties 42B12, 42B35, 42C23, 42C25, 42C35 and 42G25 are defined, respectively, by the following:

\[ 42B12 \ x \rightarrow ((x \rightarrow y) \rightarrow x), \]
\[ 42B35 \ (x \rightarrow x) \approx (x \rightarrow y) \rightarrow x, \]
\[ 42C23 \ x \rightarrow ((x \rightarrow y) \rightarrow y) \approx (x \rightarrow x) \rightarrow (y \rightarrow y), \]
Lemma 5.12 Symmetric implication groups and weak associative laws

In view of Lemma 5.9 and Theorem 5.10, the following theorem is immediate.

Theorem 5.11 $42B12 = 42B35 = 42C23 = 42C25 = 42C35 = 42G25$.

Lemma 5.12 $42A12 = 42D23 = 42D35$.

Proof In view of Lemma 2.14 (7), we have

$\mathbf{A} \models x \rightarrow ((y \rightarrow x) \rightarrow x) \equiv ((x \rightarrow y) \rightarrow y)$.

Hence,

$\mathbf{A} \models (42D23)$ if and only if $\mathbf{A} \models (42D35)$,

proving that $42D23 = 42D35$.

Assume $\mathbf{A} \in 42A12$ and $a, b \in A$. Observe that $((a \rightarrow b) \rightarrow a) \equiv (a \rightarrow (b \rightarrow a))^{\gamma} \rightarrow a$ and $a \rightarrow (a \rightarrow (b \rightarrow a))^{\gamma} \rightarrow a \equiv (a \rightarrow (a \rightarrow b)) \rightarrow ((a \rightarrow b) \rightarrow a) \equiv ((a \rightarrow b) \rightarrow a) \rightarrow a$. Therefore, $\mathbf{A} \models (42A12)$.

Now assume that $\mathbf{A} \in 42D35$, and $a, b \in A$. Since $0 \rightarrow a \equiv (0 \rightarrow a)^{\gamma} \equiv ((0 \rightarrow a) \rightarrow 0) \equiv 0 \equiv (42D35) (0 \rightarrow a)$, $0 \rightarrow a$.

Therefore, $a \rightarrow ((a \rightarrow a) \rightarrow b) \equiv (a \rightarrow (b \rightarrow a)) \rightarrow (a \rightarrow b)$ and $(b \rightarrow a) 
\rightarrow ((a \rightarrow b) \rightarrow a) \equiv (a \rightarrow (b \rightarrow a)) \rightarrow (a \rightarrow b)$.

Hence, $\mathbf{A} \in 42A12$.

Let $K$ be the set consisting of the following identities:

(42C12), (42C13), (42C15), (42D12) and (42F35).

Lemma 5.13 Let $x \in K$. If $\mathbf{A} \models x$, then $\mathbf{A} \models 0 \rightarrow x \equiv x \rightarrow x$.

Proof Let $a \in A$.

If $\mathbf{A} \models (42C12), a \rightarrow a \equiv a' \rightarrow a' \rightarrow a'' \equiv a' \rightarrow a'' \equiv a'$.

If $\mathbf{A} \models (42C13), 0 \rightarrow a \equiv a' \rightarrow 0 \equiv a' \rightarrow a' \rightarrow 0 \equiv a' \rightarrow a' \rightarrow a'$.

Therefore, $a \rightarrow ((a \rightarrow a) \rightarrow b) \equiv (a \rightarrow (b \rightarrow a)) \rightarrow (a \rightarrow b)$.

Hence, by Lemmas 2.9 (3) and 2.7 (1), the algebra $\mathbf{A}$ satisfies

$0 \rightarrow (x \rightarrow x) \equiv x \rightarrow x$.

Hence, by Lemma 2.9 (3) and 2.7 (1), the algebra $\mathbf{A}$ satisfies (42C12).

In view of Theorem 5.11, the algebra $\mathbf{A}$ satisfies the identity (42B12) too.

Since $(a \rightarrow a) \rightarrow (b \rightarrow b) \equiv (0 \rightarrow a) \rightarrow (0 \rightarrow b)$, $0 \rightarrow (a \rightarrow b) \equiv a \rightarrow (b \rightarrow b) \equiv a \rightarrow (a \rightarrow (b \rightarrow b))$.

Therefore, $a \rightarrow ((a \rightarrow b) \rightarrow b)$.

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(6) Weak associative laws with length 4 and length 3 variables

Possible words of length 4 with 3 variables are:

\[
\begin{align*}
A: & \quad (x, x, y, z), \\
B: & \quad (x, y, x, z), \\
C: & \quad (x, y, y, z), \\
D: & \quad (x, y, z, x), \\
E: & \quad (x, y, z, y), \\
F: & \quad (x, y, z, z).
\end{align*}
\]

Ways in which a word of length 4 can be bracketed:

1. \(a \rightarrow ((a \rightarrow (a \rightarrow a)))\),
2. \(a \rightarrow ((a \rightarrow a) \rightarrow a)\),
3. \((a \rightarrow a) \rightarrow (a \rightarrow a)\),
4. \((a \rightarrow (a \rightarrow a)) \rightarrow a\),
5. \((a \rightarrow a) \rightarrow a\).

Note that these identities are precisely the identities of Bol–Moufang type which were, as mentioned in Introduction, analyzed in Cornejo and Sankappanavar (2018) with a slightly different notation wherein the first two digits were not used; for example, 43A23 was denoted by \(A_{23}\), etc.

Recall that the varieties 43A12, 43A23 and 43A23 are defined, respectively, by the following:

43A12: \(x \rightarrow (x \rightarrow (y \rightarrow z)) \approx x \rightarrow ((x \rightarrow y) \rightarrow z)\),
43A23: \(x \rightarrow ((x \rightarrow y) \rightarrow z) \approx (x \rightarrow x) \rightarrow (y \rightarrow z)\),
43F25: \(x \rightarrow ((y \rightarrow z) \rightarrow z) \approx ((x \rightarrow y) \rightarrow z)\).

Theorem 6.1 (Cornejo and Sankappanavar 2018) There are 4 nontrivial varieties of Bol–Moufang type that are distinct from each other: \(SL, 43A12, 43A23 \text{ and } 43F25\), and they satisfy the following inclusions:

1. \(SL \subset 43A23 \subset 43F25\),
2. \(SL \subset 43A12\),
3. \(B.A \subset 43A12 \subset 43F25\),
4. \(43F25 \subset S\),
5. \(SL = 43A23 \cap 43A12\).

Lemma 6.2 43F25 = 42B35.

Proof Let \(A \in 42B35\). By Lemma 5.9,

\[A \models 0 \rightarrow (x \rightarrow x) \approx x \rightarrow x.\]

Hence, using Lemmas 2.8 and 2.9, we have

\[A \models (x \rightarrow x) \rightarrow y' \approx ((x \rightarrow x) \rightarrow y)'.\]
\[
2.7(2) \Rightarrow (b \rightarrow c) \rightarrow ((c' \rightarrow a) \rightarrow c) \overset{(I)}{=} (b \rightarrow c) \rightarrow ((c' \rightarrow c') \rightarrow (a \rightarrow c')) \rightarrow (a \rightarrow c')) \overset{6.2(1)}{=} (b \rightarrow c) \rightarrow ((c' \rightarrow a) \rightarrow (a \rightarrow c')) \overset{6.2(1)}{=} ((c' \rightarrow (c' \rightarrow a)) \rightarrow (b \rightarrow c)) \rightarrow (c' \rightarrow c') \rightarrow (c' \rightarrow (c' \rightarrow (c' \rightarrow a))) \overset{2.4(16)}{=} (b \rightarrow c) \rightarrow (a \rightarrow c') \overset{2.7(1)}{=} a \rightarrow ((b \rightarrow c) \rightarrow c). \text{ Hence, } A \models (43F25), \text{ and consequently, } 42B35 \subseteq 43F25.
\]

Assume now that \( A \in 43F25 \) and \( a \in A \). Then, \( 0 \rightarrow (a \rightarrow a) \overset{2.3(1)}{=} 0 \rightarrow ((0' \rightarrow a) \rightarrow a) \overset{43F25}{=} ((0 \rightarrow 0') \rightarrow a) \rightarrow a \overset{2.2(4)}{=} (0' \rightarrow a) \rightarrow a \overset{2.2(1)}{=} a \rightarrow a \). Hence, using Lemma 2.9, we get
\[
A \models (x \rightarrow x) \rightarrow (y \rightarrow z) \Rightarrow ((x \rightarrow x) \rightarrow y) \Rightarrow (x \rightarrow x) \rightarrow z \quad (6.3)
\]

Using (6.3), we have that \( A \models (42B35) \). This completes the proof.

Lemma 6.3 \( 43A12 = 42C12 \).

**Proof** Let \( A \in 43A12 \) and \( a \in A \). Observe that \( a \rightarrow a \overset{2.7(2)}{=} a' \rightarrow a' \overset{6.2(7)}{=} a'' \rightarrow a'' \Rightarrow a' \rightarrow (a'' \rightarrow 0) \overset{43A12}{=} a' \rightarrow (a'' \rightarrow 0) \Rightarrow a' \rightarrow (0 \rightarrow 0) \overset{2.7(2)}{=} a' \rightarrow 0 \overset{7(2)}{=} \). Hence, by Theorem 5.14, \( A \in 42C12 \), implying \( 43A12 \subseteq 42C12 \).

Conversely, take \( A \in 42C12 \) and \( a, b, c \in A \). By Lemma 5.13,
\[
A \models (x \rightarrow x) \rightarrow y' \Rightarrow ((x \rightarrow x) \rightarrow y)' \Rightarrow (x \rightarrow x) \rightarrow y. \quad (6.4)
\]

Hence, by Lemmas 2.8 and 2.9,
\[
A \models (x \rightarrow x) \rightarrow (y \rightarrow z) \Rightarrow ((x \rightarrow x) \rightarrow y)' \Rightarrow (x \rightarrow x) \rightarrow z. \quad (6.5)
\]

Therefore,
\[
A \models (x \rightarrow x) \rightarrow (y \rightarrow z) \Rightarrow ((x \rightarrow x) \rightarrow y)' \Rightarrow (x \rightarrow x) \rightarrow z \quad (6.6)
\]

7 Weak associative laws with length 4 and with 4 variables

The only word of length 4 with 4 variables is:
\[
A: \{x, y, z \}.
\]

Ways in which a word of length 4 can be bracketed:

1. \( a \rightarrow ((a \rightarrow a) \rightarrow a) \),
2. \( a \rightarrow ((a \rightarrow a) \rightarrow a) \),
3. \( (a \rightarrow a) \rightarrow (a \rightarrow a) \),
4. \( (a \rightarrow (a \rightarrow a)) \rightarrow a \),
5. \( ((a \rightarrow a) \rightarrow a) \rightarrow a \).

Let \( L \) be the set consisting of the following varieties:
44A12, 44A13, 44A14, 44A15, 44A23, 44A24, 44A34, 44A35 and 44A45.

**Theorem 7.1** If \( \mathcal{K} \in L \), then \( \mathcal{K} = 33A12 \).

**Proof** It is easy to see that \( 33A12 \subseteq \mathcal{K} \). Let \( A \in \mathcal{K} \) and let \( a, b, c \in A \).
To facilitate a uniform presentation (and to make the proof shorter), we introduce the following notation, where \( u_0, x_0, y_0, z_0 \in A \):

The notation

\[ \mathcal{X}/u_0, x_0, y_0, z_0 \]

denotes the following statement:

In the identity (X) that defines the variety \( \mathcal{X} \), relative to \( S \), if we assign \( u := u_0, x := x_0, y := y_0, z := z_0 \) (and simplify it using Lemma 2.2 (1) and \( x \approx x' \)), then \( A \models x \rightarrow (y \rightarrow z) \approx (x \rightarrow y) \rightarrow z \).

Hence, we consider the varieties associated with the following statements:

1. \( 44,A12/0', a, b, c; \)
2. \( 44,A13/a, b, 0', c; \)
3. \( 44,A14/0', a, b, c; \)
4. \( 44,A15/0', a, b, c; \)
5. \( 44,A23/0', a, b, c; \)
6. \( 44,A24/a, 0', b, c; \)
7. \( 34,A12/0', a, b, c; \)
8. \( 44,A35/0', a, b, c; \)
9. \( 44,A45/a, b, c, 0. \)

It is routine to verify that each of the above statements is true, from which we conclude that, for \( \mathcal{X} \in L, \mathcal{X} \leq 33,A12 \).

\[ \square \]

Lemma 7.2 If \( A \in 31,A12 \cup 43,A23 \cup 44,A25 \), then \( A \models 0 \rightarrow x \approx x \).

**Proof** First, we will show that if \( A \in 31,A12 \cup 43,A23 \cup 44,A25 \), then \( A \models 0' \approx 0 \).

Let \( A \in 31,A12 \). Then, by Lemma 2.2 (4) and (31A12), 0' = 0 → 0' = 0 → (0 → 0) = (0 → 0) → 0 = 0' → 0 = 0.

Let \( A \in 43,A23 \). Then, 0 \( 2^{2}(1) \) 0' = 0 \( 2^{2}(1) \) 0' = 0' → 0 = 0

Let \( A \in 44,A24 \). Then, 0 \( 2^{2}(1) \) 0 → 0 \( 2^{2}(1) \) 0' = 0' → 0 = 0' → 0 = 0.

Therefore, for \( a \in A \), 0 → a = 0' → a = a by Lemma 2.2 (1).

\[ \square \]

Lemma 7.3 If \( A \in 0 \rightarrow x \approx x \), then \( A \in 31,A12 \cap 43,A23 \cap 44,A25 \).

\[ \square \]

**8 Main theorem**

The purpose of this section is to prove our main theorem. The following proposition will be needed to complete the proof of the main theorem.

In the following proposition, we combine all the results obtained so far both in this paper and in Cornejo and Sankappanavar (2018a). In the latter, each of the 60 varieties defined by the weak associative laws of length 4 with 3 variables was shown to be equal to one of the three varieties: 43,A12, 43,A23 and 43,F25. Therefore, we will need only consider these three varieties together with the remaining 95 varieties in the following result.

We recall here that the varieties 42,A12, 43,A12, 43,A23 and 43,A24 are defined, respectively, by the following:

\( (42,A12) \ x \rightarrow (x \rightarrow (x \rightarrow y)) \approx x \rightarrow ((x \rightarrow x) \rightarrow y), \)

\( (43,A12) \ x \rightarrow (x \rightarrow (y \rightarrow z)) \approx x \rightarrow ((x \rightarrow y) \rightarrow z), \)

\( (43,A23) \ x \rightarrow ((x \rightarrow y) \rightarrow z) \approx (x \rightarrow x) \rightarrow (y \rightarrow z), \)

\( (43,F25) \ x \rightarrow ((y \rightarrow z) \rightarrow z) \approx ((x \rightarrow y) \rightarrow z). \)

**Proposition 8.1** Each of the possible 155 weak associative subvarieties of \( S \) is equal to one of the following varieties:

\( SL, 43,A12, 43,A23, 42,A12, 43,F25 \) and \( S \).

**Proof** We have

1. \( SL \approx \mathcal{LAC}, \mathcal{L}E \mathcal{X} \approx \mathcal{RAC} \approx 33,A12 \approx 42,A14 \approx 42,B14 \approx 42,B24 \approx 42,B24 \approx 42,B34 \approx 42,B45 \approx 42,C14 \approx 42,C24 \approx 42,C34 \approx 42,C45 \approx 42,D14 \approx 42,D34 \approx 42,E12 \approx 42,E14 \approx 42,E23 \approx 42,E24 \approx 42,E34 \approx 42,E35 \approx 42,E45 \approx 42,F15 \approx 42,F23 \approx 42,F24 \approx 42,F45 \approx 42,G23 \approx 42,G24 \approx 42,G35 \approx 42,G45 \approx 42,G15 \approx 42,G14 \approx 42,F13 \approx 42,F14 \approx 42,F14. \)
We are now ready to present the main theorem of this paper.

**Theorem 8.2** (a) The following are the 6 varieties defined, relative to $S$, by the 155 weak associative laws of length $m \leq 4$ that are distinct from each other:

$SL$, $43A12$, $43A23$, $42A12$, $43F25$ and $S$.

(b) They satisfy the following relationships:

1. $SL \subseteq 43A23 \subseteq 43F25 \subseteq S$,
2. $SL \subseteq 43A12 \subseteq 42A12 \subseteq S$,
3. $BA \subseteq 43A12 \subseteq 43F25$,
4. $43A12 \not\subseteq 43A23$ and $43A23 \not\subseteq 43A12$,
5. $42A12 \not\subseteq 43F25$ and $43F25 \not\subseteq 42A12$,
6. $43A23 \not\subseteq 42A12$ and $42A12 \not\subseteq 43A23$.

**Proof** We first prove (b):

1. Follows from Theorem 6.1.
2. The statement $SL \subseteq 43A12$ follows from Theorem 6.1. Then, it remains to check that $43A12 \subseteq 42A12 \subseteq S$. Let $A \in 43A12$ and $a, b \in A$. By Proposition 8.1 (2), $A \in 42C12$. Observe that

42C12 $\models 0 \rightarrow x \approx x \rightarrow x$ (8.1)

by Lemma 5.13. Then, using Lemma 2.8 (1), we have 42C12 $\models 0 \rightarrow (x \rightarrow x) \approx x \rightarrow x$. Hence, by Lemma 2.9 (1),

$A \models (x \rightarrow x) \rightarrow y' \approx ((x \rightarrow x) \rightarrow y)'$. (8.2)

Hence, $a \rightarrow ((a \rightarrow a) \rightarrow b) \approx (a \rightarrow a) \rightarrow (a \rightarrow b)$.

We start with (b):

They satisfy the following relationships:

2. The statement $SL \subseteq 43A23 \subseteq 43F25 \subseteq S$, by Lemma 5.13. Then, using Lemma 2.8 (1), we have $SL \subseteq 43A12 \subseteq 42A12 \subseteq S$.

5. The following algebras show that $42A12 \not\subseteq 43F25$ and $43F25 \not\subseteq 42A12$, respectively.

6. The following algebras show that $43A23 \not\subseteq 42A12$ and $42A12 \not\subseteq 43A23$, respectively.

The proof of the theorem is now complete since (a) is an immediate consequence of Proposition 8.1 and (b).
We conclude the paper with the remark that it would be of interest to investigate the weak associative identities and, in particular, the identities of Bol–Moufang type, relative to \( I \) and other (important) subvarieties of \( I \).

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**Conflict of interest** Both authors declare that they have no conflict of interests.

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