Scalar Extension of Abelian and Tannakian Categories

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Abstract

We introduce and develop the notion of scalar extension for abelian categories. Given a field extension $F'/F$, to every $F$-linear abelian category $\mathcal{A}$ satisfying a suitable finiteness condition we associate an $F'$-linear abelian category $\mathcal{A} \otimes_F F'$ and an exact $F$-linear functor $t : \mathcal{A} \to \mathcal{A} \otimes_F F'$. This functor is universal among $F$-linear right exact functors with target an $F'$-linear abelian category.

We discuss various basic properties of this concept, among others compatibilities with multilinear endofunctors such as tensor products, and the permanence of favourable properties of the functors and categories involved. We obtain the notion of scalar extension for Tannakian categories, which allows us to deduce consequences for the algebraic monodromy groups of Tannakian categories.

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Introduction

This article initiates a study of “scalar extension” of abelian categories, in the case where the scalars are fields. We understand this to mean that to a field extension $F'/F$ and an $F$-linear abelian category $\mathcal{A}$ we wish to associate an $F'$-linear abelian category $\mathcal{A} \otimes_F F'$ and an $F$-linear exact functor $t: \mathcal{A} \rightarrow \mathcal{A} \otimes_F F'$

which is in a certain sense “universal” among certain $F$-linear functors with values in an $F'$-linear abelian category.

We construct $\mathcal{A} \otimes_F F'$ and $t$ in Subsections 1.2 and 1.3 under the assumption that $\mathcal{A}$ is $F$-finite: All objects have finite length, and all endomorphism algebras are finite $F$-dimensional (Definition 1.1.1). In the case of Tannakian categories, our construction has been used before, we know of the instances [DeM82] and [Mil92].

What is original in our approach is to characterise this construction by finding its universal property, applicable to all abelian categories. Namely, every right-exact $F$-linear functor $V: \mathcal{A} \rightarrow \mathcal{B}$ with target an $F'$-linear abelian category has a right-exact $F'$-linear “extension” $V': \mathcal{A} \otimes_F F' \rightarrow \mathcal{B}$ which is unique “up to unique isomorphism”:

This is the content of Subsection 1.4. We refer to Theorem 1.4.1 for the precise formulation of this universal property, where the underlying 2-categorical nonsense is formulated in precise, down-to-earth terms.

Examples of this process are plentiful, and show that our abstract notion of scalar extension coincides with what intuition suggests. For instance, if $E$ is a finite-dimensional $F$-algebra and $\mathcal{A}$ is the category of finite $F$-dimensional $E$-modules, then $\mathcal{A} \otimes_F F'$ is the category of finite $F'$-dimensional $(F' \otimes_F E)$-modules.

We hasten to add that an exact functor on $\mathcal{A}$ need not extend to an exact functor on $\mathcal{A} \otimes_F F'$. However, by categorical nonsense the category $(\mathcal{A}^{\text{op}} \otimes_F F')^{\text{op}}$ has the universal property that left exact functors do extend.
So a possible direction of further research might be the following question:
Under which conditions do the categories $(\mathcal{A}^{op} \otimes_F F')$ and $(\mathcal{A} \otimes_F F')^{op}$ coincide?

Throughout the article, we shall systematically disregard set-theoretic difficulties. For hints towards a solution of these, and general categorical background, we refer to [KaS06].

Motivation and Overview

The content of this article is not meant to be “l’art pour l’art”. My main motivation for its presentation are two applications to Tannakian duality, which I use in my article [Sta08]. For the first, recall that to a Tannakian category $\mathcal{T}$ over $F$ with fibre functor $\omega$ over $F'$ there is associated a linear algebraic group $G_\omega(\mathcal{T})$ over $F'$, the algebraic monodromy group of $\mathcal{T}$ with respect to $\omega$. It turns out – Theorem 3.1.7 – that the functor $\omega'$ induced by $\omega$ using the universal property identifies $\mathcal{T} \otimes_F F'$ with the category of finite-dimensional representations of $G_\omega(\mathcal{T})$ over $F'$. In this way, we obtain as our first application a weak form of non-neutral Tannakian duality, which uses only the input of neutral Tannakian duality.

To prove this fact, we must first show that $\mathcal{T} \otimes_F F'$ carries a tensor product, and that “everything is compatible” with tensor products. For this, in Subsection 2.1 we will consider more generally a multilinear functor $\mathcal{A}^\times \to \mathcal{A}$ and study the induced multilinear functor $(\mathcal{A} \otimes_F F')^\times \to \mathcal{A} \otimes_F F'$. Since in tensor categories with duals right exact functors are automatically exact – Lemma 2.2.3 – the proof of our first application ensues rather easily.

The second application is a partial answer to the following question: In diagram (⋆), under which conditions are favourable properties of $V'$ equivalent to corresponding “relatively” favourable properties of $V$? An example has been given above, the question of being exact. The two others we focus on are the following: When is $V'$ fully faithful? And when is the essential image of $V'$ closed under subquotients? Taken together, we ask: When, in terms of properties of $V$ and the field extension $F'/F$, is $V'$ an equivalence of categories?

The relative version of being fully faithful is to be $F'/F$-fully faithful, a categorical version of the Tate conjecture on homomorphisms in algebraic geometry, see Definition 1.1.2. We prove that $t$ is $F'/F$-fully faithful in Proposition 1.3.6. For tensor categories with duals, we prove that $V$ is $F'/F$-fully faithful if and only if $V'$ is fully faithful in Subsection 2.3.

I have not achieved a full clarification of what the relative version of the essential image being closed under subquotients is. As a kludge, in the special case of separable field extensions and the context of tensor categories with duals, we study functors which map semisimple objects to semisimple objects, we call this property semisimple on objects. If $F' = F$, this property
is equivalent to the essential image being closed under subquotients for exact fully faithful functors by Proposition 1.1.4. If $F'/F$ is separable, we prove that $t$ is semisimple on objects in Proposition 1.5.1 and that $V$ is semisimple on objects if and only if $V'$ is in Proposition 1.5.3.

Our second application is then a partial answer to the question of “recognising induced equivalences of categories”. It is developed in Subsection 2.4 and states the following. Let $F'/F$ be a separable field extension, $\mathcal{T}$ a Tannakian category over $F$, $\mathcal{T}'$ a Tannakian category over $F'$ and $V : \mathcal{T} \to \mathcal{T}'$ an $F$-linear exact tensor functor. If $V$ is $F'/F$-fully faithful and semisimple on objects, then the induced functor

$$V' : \mathcal{T} \otimes_F F' \to ((V\mathcal{T})_\otimes$$

is an equivalence of Tannakian categories, where $((V\mathcal{T})_\otimes$ denotes the strictly full Tannakian subcategory of $\mathcal{T}'$ generated by the image of $V$.

**Motivic Monodromy Groups**

Here is an example of how our second application may be put to use. Let $\mathcal{A}$ be the $\mathbb{Q}$-linear abelian subcategory of the category of pure Grothendieck motives – up to isogeny and numerical equivalence – generated by abelian varieties over a given number field $K$. By [Jan92], it is a Tannakian category. Choose a prime number $\ell$, and let $\mathcal{B}$ denote the category of finite-dimensional continuous representations of the absolute Galois group $\Gamma := \text{Gal}(K_{\text{sep}}/K)$ of $K$ over $\mathbb{Q}_\ell$. This is obviously a Tannakian category, and it is known – Proposition 3.3.3 – that the algebraic monodromy group of its strictly full Tannakian subcategory $((V)_\otimes$ generated by a given Galois representation $V$ with respect to the forgetful functor may be identified with the Zariski closure of the image of $\Gamma$ in $\text{GL}(V)(\mathbb{Q}_\ell)$. By [Fal83], the functor $V_\ell$ of rational Tate modules is indeed $\mathbb{Q}_\ell$-fully faithful (Tate’s conjecture!) and semisimple on objects. Since all objects of $\mathcal{A}$ are semisimple by Poincaré reducibility and [Jan92], this latter property means that all rational Tate modules of abelian varieties are semisimple, and is hence a special case of the Grothendieck-Serre conjecture on étale cohomology groups of pure motives. Therefore, our theorem allows to conclude that the algebraic monodromy group of an abelian variety over a number field – its “motivic” monodromy group – coincides with the Zariski-closure of the image of Galois. Our application of Theorem 2.4.1 in [Sta08] is an analogue of this example, with Anderson $A$-motives replacing abelian varieties, and the main result of [Sta08], as advertised in its title, replacing [Fal83].

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1 Abelian Categories

1.1 Properties of Algebras, Categories and Functors

In this subsection, we collect several algebraic and categorical notions for later reference. Let $F$ be a field. Recall that a category is $F$-linear if all Hom-sets are endowed with $F$-vector space structures in such a way that composition of homomorphisms is $F$-bilinear.

1.1.1 Definition. Let $\mathcal{A}$ be an abelian category.

(a) $\mathcal{A}$ is finite if all objects are of finite length.

(b) Assume that $\mathcal{A}$ is $F$-linear. Then $\mathcal{A}$ is $F$-finite if it is finite and the endomorphism algebra of each object is finite $F$-dimensional.

1.1.2 Definition. Let $F'/F$ be a field extension. Consider an $F$-linear category $\mathcal{C}$ and an $F'$-linear category $\mathcal{C}'$. An $F$-linear functor $V : \mathcal{C} \to \mathcal{C}'$ is $F'/F$-fully faithful if the induced homomorphism $F' \otimes F \text{Hom}_F(X,Y) \to \text{Hom}_{F'}(V(X),V(Y)), \ f' \otimes h \mapsto f' \cdot V(h)$ is an isomorphism for all objects $X,Y$ of $\mathcal{C}$.

More loosely speaking, we might say that an $F'/F$-fully faithful functor is relatively fully faithful if the field extension $F'/F$ is clear from the context.

1.1.3 Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. An exact functor $V : \mathcal{A} \to \mathcal{B}$ is semisimple on objects if it maps all semisimple objects of $\mathcal{A}$ to semisimple objects of $\mathcal{B}$.

Let us consider a consequence of the juxtaposition of the two properties “$F'/F$-fully faithful” and “semisimple on objects” in the special case $F' = F$.

1.1.4 Proposition. Let $\mathcal{A}$ be a finite $F$-linear abelian category, $\mathcal{B}$ an $F$-linear abelian category, and $V : \mathcal{A} \to \mathcal{B}$ an $F$-linear, exact, fully faithful functor semisimple on objects. Then the essential image of $V$ is closed under subquotients in $\mathcal{B}$.

Proof. By symmetry, it is enough to show that the essential image of $V$ is closed under subobjects in $\mathcal{B}$. So let $A$ be an object of $\mathcal{A}$ and $B$ an arbitrary subobject of $V(A)$ in $\mathcal{B}$. We must show that $B \cong V(A_0)$ for some object $A_0$ of $\mathcal{A}$. We proceed by induction on $\ell := \text{lg}(B)$, the length of a composition series of $B$. The case $\ell = 0$ is trivial. If $\ell = 1$, but $A$ is not simple, choose a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

with nonzero objects $A', A''$ of $\mathcal{A}$. Consider the composite homomorphism $h : B \to V(A) \to V(A'')$. There are two possibilities:
(a) “$h \neq 0$”: In this case, $B$ is a subobject of $V(A'')$ since $B$ is simple, and $\lg(A'') < \lg(A)$.

(b) “$h = 0$”: In this case, $B$ is a subobject of $V(A')$, and $\lg(A') < \lg(A)$.

Since $\lg(A) < \infty$, by repeating this process with $A'$ or $A''$ instead of $A$, depending on which case we arrive at, we find a simple object $A_1$ of $\mathcal{A}$ such that $B$ is a subobject of $V(A_1)$. Now $V(A_1)$ is semisimple since $V$ is semisimple on objects, so $B$ is a quotient object of $V(A_1)$. Since $V$ is fully faithful, the composite homomorphism $g : V(A_1) \to B \to V(A_1)$ is of the form $V(f)$ for some homomorphism $f \in \text{End}(A_1)$. Set $A_0 := \text{im}(f)$. Since $V$ is exact, we see that $B = \text{im}(g) \cong V(A_0)$, as required.

We turn to the case $\ell = \lg(B) > 1$. Choose a short exact sequence

$$0 \to B' \to B \to B'' \to 0$$

with nonzero objects $B', B''$ of $\mathcal{B}$. By induction hypothesis, $B' \cong V(A')$ and $B'' \cong V(A'')$ for objects $A', A''$ of $\mathcal{A}$. Consider the induced commutative diagram with exact rows

$$
\begin{array}{ccc}
0 & \to & V(A') & \to & B & \to & V(A'') & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & V(A') & \to & V(A) & \to & V(A/A') & \to & 0,
\end{array}
$$

using the fact that $V$ is exact. The dotted vertical arrow is of the form $g = V(f)$ since $V$ is fully faithful. Set $A''' := \text{coker}(f)$. Consider the composite homomorphism

$$h : V(A) \to V(A/A') \to V(A'')$$

where we again use the fact that $V$ is exact. The Snake Lemma implies that $B = \ker(h)$. Since $V$ is fully faithful, $h$ is of the form $V(f')$ for some homomorphism $f' : A \to A''$. Set $A_0 := \ker(f')$. Since $V$ is exact, we see that $B = \ker(h) \cong V(A_0)$, as required.

In the situation of Definition 1.1.3 if $F'/F$ is a field extension, $\mathcal{A}$ is $F$-linear and $\mathcal{B}$ is $F'$-linear, experience tells us not to expect an exact functor $V : \mathcal{A} \to \mathcal{B}$ to be semisimple on objects in the absence of separability assumptions. Hence we recall the definition of separability for $F$-algebras, which extends the usual definition of separability for finite field extensions.

1.1.5 Definition. A field extension $F'/F$ is separable if the ring $F' \otimes_F F'$ contains no nilpotent elements, where $\overline{F}$ denotes an algebraic closure of $F$.

1.1.6 Definition. A finite-dimensional semisimple $F$-algebra $E$ is separable if the center of each simple factor is a separable field extension of $F$. 
1.1.7 Proposition. Let $E$ be a finite-dimensional semisimple $F$-algebra, and consider a field extension $F'/F$. If either

(a) $F'/F$ is a separable field extension, or

(b) $E$ is a separable $F$-algebra,

then $F' \otimes_F E$ is a semisimple $F'$-algebra.

Proof. (a): [Bou58, §7, no. 3, Corollaire 1 to Proposition 3(b)].

(b): [Bou58, §7, no. 5, Proposition 6 and Corollary to Proposition 7]. $	herefore$

At the end of Subsection 1.3 we will need the property given in the following definition, which in contrast to semisimplicity is invariant under field extensions. For more information, see [Lam99].

1.1.8 Definition. A finite-dimensional $F$-algebra $E$ is Frobenius if there exists an isomorphism $E \cong \text{Hom}_F(E, F)$ of right $E$-modules.

1.1.9 Proposition. Let $E$ be a finite-dimensional $F$-algebra.

(a) If $E$ is a semisimple $F$-algebra, then $E$ is Frobenius.

Assume that $E$ is Frobenius.

(b) For every field extension $F'/F$ the $F'$-algebra $F' \otimes_F E$ is Frobenius.

(c) We have $\text{soc}(E) \cong E/\text{rad}(E)$ as right $E$-modules, where $\text{soc}(E)$ denotes the maximal semisimple right $E$-submodule of $E$, and $\text{rad}(E)$ denotes the maximal semisimple right $E$-module quotient of $E$.

Proof. Set $E^\vee := \text{Hom}_F(E, F)$, considered as a right $E$-module using the left $E$-module structure of $E$.

(a): By additivity, we may assume that $E$ is a simple $F$-algebra. Then, up to isomorphism, there exists only one simple right $E$-module, and so the isomorphism class of a right $E$-module is determined by its dimension over $F$. Since $\dim_F E = \dim_F E^\vee$, it follows that $E$ and $E^\vee$ are isomorphic.

(b): By assumption, $E \cong E^\vee$, and hence $F' \otimes_F E \cong F' \otimes_F \text{Hom}_F(E, F) \cong \text{Hom}_{F'}(F' \otimes_F E, F')$ as claimed.

(c): By duality, $\text{soc}(E^\vee) = (E/\text{rad}(E))^\vee$. Since $E$ is a Frobenius $F$-algebra, we obtain an induced isomorphism $\text{soc}(E) \cong \text{soc}(E^\vee)$. Now $E/\text{rad}(E)$ is a semisimple $F$-algebra, so item (a) implies that $(E/\text{rad}(E))^\vee \cong E/\text{rad}(E)$ as right $E/\text{rad}(E)$-modules, and thus as right $E$-modules. Taken together, we obtain a composite isomorphism

$$\text{soc}(E) \cong \text{soc}(E^\vee) = (E/\text{rad}(E))^\vee \cong E/\text{rad}(E)$$

of right $E$-modules, as claimed. $	herefore$
1.2 The Category $\mathcal{A} \otimes_F F'$

Let $F$ be a field, and consider an $F$-linear abelian category $\mathcal{A}$.

1.2.1 Definition. An ind-object of $\mathcal{A}$ is a filtered direct system $(X_i)_{i \in I}$ of objects of $\mathcal{A}$. A homomorphism of two given ind-objects $(X_i)_{i \in I}$ and $(Y_j)_{j \in J}$ of $\mathcal{A}$ is an element of $\varprojlim_i \varprojlim_j \text{Hom}_\mathcal{A}(X_i, Y_j)$. We obtain the $F$-linear abelian category ind $\mathcal{A}$ of ind-objects of $\mathcal{A}$.

We have a natural functor $\mathcal{A} \to \text{ind } \mathcal{A}$, mapping an object $X$ of $\mathcal{A}$ to the object $(X_i)_{i \in I_\emptyset}$ given by $I_\emptyset := \{\emptyset\}$ and $X_\emptyset := X$. It is $F$-linear, exact and fully faithful. Abusing notation, we identify $\mathcal{A}$ with its essential image in ind $\mathcal{A}$ under this natural functor.

Recall that $\mathcal{A}$ is finite if all of its objects have finite length.

1.2.2 Lemma. Let $\mathcal{A}$ be finite.

(a) $\mathcal{A}$ is closed under subquotients in ind $\mathcal{A}$.

(b) Every object of ind $\mathcal{A}$ is a union of subobjects in $\mathcal{A}$.

Proof. [Del87, §4.1 and Lemme 4.2.1].

Let $F'/F$ be a field extension.

1.2.3 Definition. An $F'$-module in ind $\mathcal{A}$ is an ind-object of $\mathcal{A}$ together with an $F$-linear ring homomorphism $\phi : F' \to \text{End}_{\text{ind } \mathcal{A}}(X)$. A homomorphism of $F'$-modules is a homomorphism of ind-objects which commutes with the respective actions of $F'$. We obtain the $F'$-linear abelian category (ind $\mathcal{A}$)$_{F'}$ of $F'$-modules in ind $\mathcal{A}$.

Recall that $\mathcal{A}$ is $F$-finite if it is finite and the endomorphism algebra of each object is finite $F$-dimensional.

1.2.4 Definition. Let $\mathcal{A}$ be $F$-finite. The scalar extension of $\mathcal{A}$ from $F$ to $F'$ is the full subcategory $\mathcal{A} \otimes_F F'$ of (ind $\mathcal{A}$)$_{F'}$ consisting of all $F'$-modules of finite length. It is $F'$-linear, abelian and finite.

1.2.5 Remark. In the next section, we will see that $\mathcal{A} \otimes_F F'$ is $F'$-finite.

1.2.6 Examples. (a) If $E$ is a finite-dimensional $F$-algebra and $\mathcal{A}$ is the category of finite $F$-dimensional left $E$-modules, then $\mathcal{A} \otimes_F F'$ is the category of finite $F'$-dimensional left $(F' \otimes_F E)$-modules.

(b) If $G$ is an affine group scheme over $F$ and $\mathcal{A}$ is the category $\text{Rep}_F(G)$ of finite-dimensional representations of $G$ over $F$, then $\mathcal{A} \otimes_F F'$ is the category $\text{Rep}_{F'}(G_{F'})$ of finite-dimensional representations of $G_{F'} := G \times_{\text{Spec}(F)} \text{Spec}(F')$ over $F'$. For a proof, we refer to [Del87].
The Functor \( t : \mathcal{A} \to \mathcal{A} \otimes_F F' \)

Let \( F \) be a field, and consider an \( F \)-finite \( F \)-linear abelian category \( \mathcal{A} \).

1.3.1 Definition. Consider an object \( X \) of \( \text{ind} \mathcal{A} \), an \( F \)-subalgebra \( E \subset \text{End}_{\text{ind} \mathcal{A}}(X) \), and a free right \( E \)-module \( M \). The external tensor product \( M \otimes_E X \) of \( M \) with \( X \) over \( E \) is the object of \( \text{ind} \mathcal{A} \) representing the functor from \( \text{ind} \mathcal{A} \) to left \( E \)-modules given by \( Y \mapsto \text{Hom}_E(M, \text{Hom}_{\text{ind} \mathcal{A}}(X,Y)) \). In other words, we require a natural isomorphism

\[
\text{Hom}_{\text{ind} \mathcal{A}}(M \otimes_E X, Y) \xrightarrow{\sim} \text{Hom}_E(M, \text{Hom}_{\text{ind} \mathcal{A}}(X,Y)).
\]

Note that \( M \otimes_E X \) exists, it is a direct sum of \( \text{rk}_E(M) \) copies of \( X \).

The external tensor product is an exact \( F \)-linear functor in its first variable if \( X \) and \( E \) are fixed, and in its second variable if \( E = F \) and \( M \) is fixed.

1.3.2 Remark. Consider the situation of Definition 1.3.1. If \( M \) is a free right \( E \)-module of finite rank, then \( M \otimes_E X \) also represents the functor \( Z \mapsto \text{Hom}_E(M, \text{Hom}_{\text{ind} \mathcal{A}}(X,Z)) \) on \( \text{ind} \mathcal{A} \), so one has a natural isomorphism

\[
M \otimes_E \text{Hom}_{\text{ind} \mathcal{A}}(Z,X) \xrightarrow{\sim} \text{Hom}_{\text{ind} \mathcal{A}}(Z, M \otimes_E X).
\]

Let \( F'/F \) be a field extension. For every object \( X \) of \( \text{ind} \mathcal{A} \), the external tensor product \( F' \otimes_F X \) has a natural \( F' \)-module structure, using the action of \( F' \) on itself by multiplication \( \mu \). We obtain an exact \( F \)-linear functor

\[
(1.3.3) \quad t = t_{F'/F} : \text{ind} \mathcal{A} \to (\text{ind} \mathcal{A})_{F'}, \quad X \mapsto (F' \otimes_F X, \mu \otimes \text{id}).
\]

We also let \( t \) denote its restriction to \( \mathcal{A} \).

1.3.4 Proposition. For every object \( X \) of \( \text{ind} \mathcal{A} \) and \( Y = (Y, \psi) \) of \( (\text{ind} \mathcal{A})_{F'} \), the restriction homomorphism

\[
\text{Hom}_{(\text{ind} \mathcal{A})_{F'}}(t(X), Y) \to \text{Hom}_{\text{ind} \mathcal{A}}(X, Y)
\]

is an isomorphism. In other words, \( t \) is left adjoint to the forgetful functor from \( F' \)-modules in \( \text{ind} \mathcal{A} \) to \( \text{ind} \mathcal{A} \) itself.

Proof. We construct an inverse \( e \) to the restriction homomorphism. Given a homomorphism \( h : X \to Y \), the induced homomorphism

\[
F' \to \text{Hom}(X,Y), \quad f' \mapsto \left( X \xrightarrow{h} Y \xrightarrow{\psi(f')} Y \right)
\]

corresponds to a unique homomorphism \( e(h) : F' \otimes_F X \to Y \) by the definition of \( F' \otimes_F X \). By construction, \( e(h) \) is a homomorphism of \( F' \)-modules.
1.3.5 Remark. Given an $F'$-module $X = (X, \phi)$ in $\text{ind} \mathcal{A}$, Proposition 1.3.4 implies that there exists a natural homomorphism $t(X) \rightarrow X$ corresponding to $\text{id}_X$ via $\phi$. Note that this homomorphism is surjective.

Recall the notion $F'/F$-fully faithful, introduced in Definition 1.1.2.

1.3.6 Proposition. The functor $t : \mathcal{A} \rightarrow (\text{ind} \mathcal{A})_{F'}$ is $F'/F$-fully faithful.

Proof. We must show that for all objects $X, Y$ of $\mathcal{A}$ the natural homomorphism
\[
F' \otimes_F \text{Hom}_{\text{ind} \mathcal{A}}(X, Y) \rightarrow \text{Hom}_{(\text{ind} \mathcal{A})_{F'}}(t(X), t(Y))
\]
is an isomorphism. By Proposition 1.3.4, the target of this homomorphism coincides with $\text{Hom}_{\text{ind} \mathcal{A}}(X, F' \otimes F Y)$, so we must show that the natural homomorphism
\[
F' \otimes_F \text{Hom}_{\text{ind} \mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\text{ind} \mathcal{A}}(X, F' \otimes_F Y)
\]
is an isomorphism.

Injectivity: Given a non-zero element $h'$ of $F' \otimes_F \text{Hom}(X, Y)$, there exists a finite $F$-dimensional subspace $V \subset F'$ such that $h'$ arises from an element $\tilde{h}'$ of $V \otimes_F \text{Hom}(X, Y)$. By Remark 1.3.2 we have a natural isomorphism $V \otimes_F \text{Hom}(X, Y) \cong \text{Hom}(X, V \otimes_F Y)$. Now the commutative diagram (disregard $h$ and $\tilde{h}$ for the moment)
\[
\begin{array}{c}
\tilde{h}' \in V \otimes_F \text{Hom}(X, Y) \xrightarrow{\cong} \text{Hom}(X, V \otimes_F Y) \ni \tilde{h} \\
\downarrow & \\
 h' \in F' \otimes_F \text{Hom}(X, Y) \rightarrow \text{Hom}(X, F' \otimes_F Y) \ni h
\end{array}
\]
implies that $h'$ is mapped to a non-zero element $h$ of $\text{Hom}(X, F' \otimes_F Y)$.

Surjectivity: Consider an element $h$ of $\text{Hom}(X, F' \otimes_F Y)$. Since $\mathcal{A}$ is finite the object $X$ has finite length, so the image $\text{im}(h)$ of $h$ is of finite length as well. The object $F' \otimes_F Y$ is the union over all finite $F$-dimensional subspaces $W \subset F'$ of its subobjects $W \otimes_F Y$. It follows that $\text{im}(h) \subset V \otimes_F Y$ for some finite $F$-dimensional vector subspace $V \subset F'$.

Therefore, $h$ arises from an element $\tilde{h}$ of $\text{Hom}(X, V \otimes_F Y)$. Now the commutative diagram (1.3.7) shows that $h$ is the image of an element $h'$ of $F' \otimes_F \text{Hom}(X, Y)$.

1.3.8 Remark. If $\mathcal{A}$ is not finite, then $t$ need not be $F'/F$-fully faithful. Here is a counter-example: Set $F := \mathbb{Q}$ and let $\mathcal{A}$ be the category of all $\mathbb{Q}$-vector spaces. Consider $X := \bigoplus_{j \in \mathbb{N}} \mathbb{Q}$, $Y := \mathbb{Q}$ and $F' := \overline{\mathbb{Q}}$, an algebraic closure of of $\mathbb{Q}$. As $\mathbb{Q}$-vector space $F'$ is isomorphic to $\bigoplus_{i \in \mathbb{N}} \mathbb{Q}$. Then the homomorphism
\[
F' \otimes_F \text{Hom}(X, Y) \rightarrow \text{Hom}(X, F' \otimes_F Y)
\]
is not surjective. Indeed, we have $F' \otimes_F \text{Hom}(X, Y) \cong \bigoplus_{i \in \mathbb{N}} \prod_{j \in \mathbb{N}} \mathbb{Q}$, whereas $\text{Hom}(X, F' \otimes_F Y) \cong \prod_{j \in \mathbb{N}} \bigoplus_{i \in \mathbb{N}} \mathbb{Q}$. The latter strictly contains the former.

Next, we wish to show that the image of $\mathcal{A}$ under $t$ lies in $\mathcal{A} \otimes_F F'$. For this, we study how simple objects of $\mathcal{A}$ “split up” under $t$.

1.3.9 Definition. Let $S$ be a simple object of $\text{ind} \mathcal{A}$. An object of $\text{ind} \mathcal{A}$ is $S$-isotypic if it is a direct sum of copies of $S$.

1.3.10 Lemma. Let $S$ be a simple object of $\text{ind} \mathcal{A}$ and set $E := \text{End}_{\text{ind} \mathcal{A}}(S)$. The functor $(-) \otimes_E S$ is an equivalence of $F$-linear abelian categories between the category of (free) right $E$-modules and the full subcategory of $S$-isotypic objects of $\text{ind} \mathcal{A}$.

Proof. A quasi-inverse functor is given by $\text{Hom}_{\text{ind} \mathcal{A}}(S, -)$.

We turn to a construction. Let $S$ be a simple object of $\mathcal{A}$ and set $E := \text{End}(S)$. Since $S$ is simple, $E$ is a skew field and all right $E$-modules are free. Let $\text{mod}-E$ denote the $F$-finite $F$-linear abelian category of finitely generated $E$-modules, and $\langle S \rangle$ the $F$-finite $F$-linear abelian category of $S$-isotypic objects in $\mathcal{A}$. Restricting the statement of Lemma 1.3.11 to objects of finite length, we obtain an equivalence of $F$-finite $F$-linear abelian categories $(-) \otimes_E S : \text{mod}-E \xrightarrow{\cong} \langle S \rangle$. Let $F'/F$ be a field extension. We obtain an induced equivalence of finite $F'$-linear abelian categories

$$(\text{mod}-E) \otimes_F F' \xrightarrow{\cong} \langle S \rangle \otimes_F F'.$$

As in Example 1.2.6(a), we have $\text{mod}-(E \otimes_F F') \cong (\text{mod}-E) \otimes_F F'$. On the other hand, the inclusion $\langle S \rangle \subset \mathcal{A}$ induces a natural fully faithful $F$-linear exact functor $\langle S \rangle \otimes_F F' \subset \mathcal{A} \otimes_F F'$. Setting $E' := E \otimes_F F'$ we obtain a fully faithful $F'$-linear exact functor

$$\text{mod}-E' \cong (\text{mod}-E) \otimes_F F' \xrightarrow{\cong} \langle S \rangle \otimes_F F' \subset \mathcal{A} \otimes_F F',$$

which we denote as $(-) \otimes_{E'} t(S)$.

1.3.11 Proposition. Let $S$ be a simple object of $\mathcal{A}$, set $E := \text{End}_{\mathcal{A}}(S)$ and $E' := F' \otimes_F E$. The functor $(-) \otimes_{E'} t(S)$ gives rise to an inclusion preserving bijection from the set of right ideals of $E'$ to the set of subobjects of $t(S)$ in $(\text{ind} \mathcal{A})_{E'}$.

Proof. Recall that $E$ is a skew field and all right $E$-modules are free. Set $S' := F' \otimes_F S$ and note that $S'$ is an $S$-isotypic object of $\text{ind} \mathcal{A}$.
Consider the following diagram of lattices:

\[
\begin{array}{ccc}
\{ \text{right } E\text{-submodules of } E' \} & \xrightarrow{(-) \otimes_E S} & \{ \text{S-isotypic subobjects of } S' \} \\
\{ \text{right ideals of } E' \} & \xrightarrow{(-) \otimes_{E'} t(S)} & \{ \text{subobjects of } t(S) \}
\end{array}
\]

The upper row is a bijection by Lemma 1.3.10 and it preserves inclusions by construction. The second row corresponds to the \(F'\)-stable objects in the upper row, using the action of \(F'\) on \(E'\) and \(S'\), respectively. Since the bijection in the first row is natural, it induces a bijection in the second row. Finally, we may clearly identify the objects of the second row with the vertically corresponding objects of the third row. Unraveling the definition of \((-) \otimes_{E'} t(S)\) we see that the resulting diagram commutes, so we have proven our claim.

\[\therefore\]

1.3.12 Corollary. The image of \(t : \mathcal{A} \to (\text{ind } \mathcal{A})_F\) is contained in \(\mathcal{A} \otimes_F F'\) and thus we obtain a well-defined functor \(t : \mathcal{A} \to \mathcal{A} \otimes_F F'\).

Proof. We must show that \(t(X)\) has finite length as \(F'\)-module for every object \(X\) of \(\mathcal{A}\). Since \(\mathcal{A}\) is finite and \(t\) is exact, we may assume that \(X =: S\) is simple. Since \(\mathcal{A}\) is \(F\)-finite, the endomorphism ring \(E := \text{End}_{\mathcal{A}}(S)\) is finite \(F\)-dimensional. It follows that \(F' \otimes_F E\) has finite length as a right module over itself, and so \(t(S)\) has finite length by Proposition 1.3.11. \[\therefore\]

Summing up what we have achieved so far, to our \(F\)-finite abelian category \(\mathcal{A}\) we have associated a finite \(F\)-linear abelian category \(\mathcal{A} \otimes_F F'\) and an \(F\)-linear exact \(F'/F\)-fully faithful functor \(t : \mathcal{A} \to \mathcal{A} \otimes_F F'\).

The remainder of this subsection is devoted to showing that \(\mathcal{A} \otimes_F F'\) is not only finite, but indeed \(F'\)-finite. The most natural approach would be to show that for every object \(X\) of \(\mathcal{A} \otimes_F F'\) there exist objects \(X_0\) and \(X^0\) of \(\mathcal{A}\) together with an epimorphism \(t(X_0) \to X\) and a monomorphism \(X \to t(X^0)\), since then by \(F'/F\)-full faithfulness of \(t\) and \(F\)-finiteness of \(\mathcal{A}\) it would follow that

\[\dim_{F'} \text{End}(X) \leq \dim_{F'} \text{Hom}(t(X_0), t(X^0)) = \dim_F \text{Hom}(X_0, X^0) < \infty.\]

1.3.13 Lemma. For every object \(X\) of \(\mathcal{A} \otimes_F F'\) there exists an object \(X_0\) of \(\mathcal{A}\) and an epimorphism \(t(X_0) \to X\).
1.3 The Functor $t : \mathcal{A} \to \mathcal{A} \otimes F F'$

**Proof.** Let $X = (X, \phi)$ be an object of $\mathcal{A} \otimes F F'$. By Lemma 1.2.2(b) the object $X$ is the union of its subobjects $X_i$ lying in $\mathcal{A}$. Consider the homomorphisms $h_i : t(X_i) \subset t(X) \to X$ given as in Remark 1.3.5. Since the image of $h_i$ contains $X_i$ and $X$ has finite length, there exists an index $i$ such that $t(X_i) \to X$ is an epimorphism, so we may choose $X_0 := X_i$ together with the epimorphism $h_i$. 

However, for a general object $X$ of $\mathcal{A} \otimes F F'$ it seems difficult to find a monomorphism $X \to t(X_0)$ with $X_0$ an object of $\mathcal{A}$. Hence we modify the natural approach sketched above, by first “reducing to the simple case” and then using the fact the endomorphism algebra of a simple object is Frobenius together with Proposition 1.3.11 to find a monomorphism as desired in the simple case.

1.3.14 Definition. Let $X$ be an object of a finite abelian category.

(a) The *socle* $soc(X)$ of $X$ is the sum of its simple subobjects. Note that $X$ is semisimple if and only if $X = soc(X)$.

(b) The *socle filtration* of $X$ is the ascending exhaustive filtration defined as follows: We set $soc^0(X) := 0$, $soc^1(X) := soc(X)$ and for $i \geq 1$ recursively

$$soc^{i+1}(X) := \pi_i^{-1}\left(soc(X/ soc^i X)\right)$$

where $\pi_i$ is the canonical projection of $X$ onto $X/ soc^i(X)$.

(c) The *semisimplification* $X^{ss} := \bigoplus_{i \geq 1} soc^i(X)/ soc^{i-1}(X)$ of $X$ is the object underlying the graded object associated to the socle filtration of $X$.

(d) The *socle length* of $X$ is the smallest integer $\ell$ such that $soc^\ell(X) = X$. We denote it by $slg(X)$.

The assignments (a-c) are functorial in $X$.

We may now “reduce to the simple case”.

1.3.15 Proposition. Given two objects $X$, $Y$ of a finite $F$-linear abelian category, we have $\dim_F Hom(X,Y) \leq \dim_F Hom(X^{ss}, Y^{ss})$.

**Proof.** We proceed by induction on $\ell := \max\{slg(X), slg(Y)\}$. If $\ell \leq 1$ we have $X = X^{ss}$ and $Y = Y^{ss}$, so the statement of this proposition is trivial.

Assume that $\ell \geq 2$. For every $f \in Hom(X,Y)$ we have $f(soc X) \subset soc Y$, so we obtain a diagram

$$
\begin{array}{cccccccc}
0 & \longrightarrow & soc X & \longrightarrow & X & \longrightarrow & X/ soc X & \longrightarrow & 0 \\
0 & \longrightarrow & soc Y & \longrightarrow & Y & \longrightarrow & Y/ soc Y & \longrightarrow & 0 \\
\end{array}
$$

with $f|_{soc X} : soc X \to soc Y$, $f : X \to Y$, $f|_{X/ soc X} : X/ soc X \to Y/ soc Y$. 

(1.3.16)
Let $K$ denote the kernel of the induced homomorphism

$$\text{Hom}(X, Y) \to \text{Hom}(\text{soc} X, \text{soc} Y) \oplus \text{Hom}(X/\text{soc} X, Y/\text{soc} Y),$$

$$f \mapsto (f|_{\text{soc} X}, [f]_{X/\text{soc} X}).$$

For $f \in K$ the Snake Lemma applied to (1.3.16) gives us an exact sequence

$$0 \to \text{soc} X \to \ker f \to X/\text{soc} X \xrightarrow{\delta(f)} \text{soc} Y \to \text{coker} f \to Y/\text{soc} Y \to 0.$$ 

Since $\delta(f)$ is natural in $f \in K$ we obtain an $F$-linear homomorphism

$$\delta : K \to \text{Hom}(X/\text{soc} X, \text{soc} Y).$$

If $\delta(f) = 0$, then $\text{lg}(\ker f) = \text{lg}(\text{soc} X) + \text{lg}(X/\text{soc} X) = \text{lg}(X)$, so $\ker f = X$ and $f = 0$. This shows that $\delta$ is injective.

Now the definition of $K$ and the injectivity of $\delta$ show that the dimension of $\text{Hom}(X, Y)$ is bounded above by

$$\dim_F \text{Hom}(\text{soc} X, \text{soc} Y) + \dim_F \text{Hom}(X/\text{soc} X, \text{soc} Y) + \dim_F \text{Hom}(X/\text{soc} X, Y/\text{soc} Y).$$

By construction, all objects involved have socle length $< \ell$, so by induction hypothesis the last displayed quantity is bounded above by

$$\dim_F \text{Hom}(\text{soc} X, \text{soc} Y) + \dim_F \text{Hom}(X/\text{soc} X, (Y/\text{soc} Y)^{ss}) + \dim_F \text{Hom}(\text{soc} X, (Y/\text{soc} Y)^{ss}) + \dim_F \text{Hom}(X/\text{soc} X, (Y/\text{soc} Y)^{ss}),$$

where we have added an extra term $\dim_F \text{Hom}(\text{soc} X, (Y/\text{soc} Y)^{ss}) \geq 0$.

However, this last displayed quantity is precisely the dimension of

$$\text{Hom}(X^{ss}, Y^{ss}) = \text{Hom}(\text{soc}(X) \oplus (X/\text{soc} X)^{ss}, \text{soc}(Y) \oplus (Y/\text{soc} Y)^{ss}),$$

so we have achieved our goal.

We may now exploit the fact that simple objects have Frobenius endomorphism algebras.

1.3.17 Proposition. Let $X$ be a simple object of $\mathcal{A} \otimes_F F'$. There exists a simple object $S$ of $\mathcal{A}$ together with both an epimorphism $t(S) \to X$ and a monomorphism $X \to t(S)$.

Proof. By Lemma 1.3.13 there exists an object $X_0$ of $\mathcal{A}$ and an epimorphism $t(X_0) \to X$. Let us first show that we may assume that $X_0$ is simple. If $X_0$ is not simple, then it has length $\geq 2$ and we may choose a simple subobject $Y_0 \subset X_0$. Consider the restriction $t(Y_0) \leftarrow t(X_0) \to X$. If it is non-zero, then it is an epimorphism because $X$ is simple and we may choose $S := Y_0$. If it is zero, we obtain a factor homomorphism $t(X_0/Y_0) = t(X_0)/t(Y_0) \to X$
which remains an epimorphism. Since $\mathscr{A}$ is finite, the claim follows by induction on the length of $X_0$.

It remains to show that $X$ embeds into $t(S)$. Set $E := \text{End}(S)$. Since $t$ is $F'/F$-fully faithful by Proposition [1.3.6] we may identify $E' := F' \otimes_F E$ and $\text{End}(t(S))$. The kernel of our epimorphism $t(S) \to X$ corresponds to a maximal right ideal $I'$ of $E'$ by Proposition [1.3.11]. Since $E'/I'$ is simple, it embeds into $E'/\text{rad}(E')$, the largest semisimple right $E'$-module quotient of $E'$. Since $E$ is a skew field, it is Frobenius by Proposition [1.1.9] (a). Therefore $E'$ is also Frobenius by Proposition [1.1.9] (b), and so Proposition [1.1.9] (c) shows that $E'/\text{rad}(E') \cong \text{soc}(E')$, the largest semisimple right $E'$-submodule of $E'$. Taken together, we obtain an injection

\[ E'/I' \hookrightarrow E'/\text{rad}(E') \cong \text{soc}(E') \subset E'. \]

Applying the exact functor $(-) \otimes_{E'} t(S)$, we obtain an induced monomorphism

\[ X = (E' \otimes_{E'} t(S))/\left(I' \otimes_{E'} t(S)\right) = (E'/I') \otimes_{E'} t(S) \hookrightarrow E' \otimes_{E'} t(S) = t(S), \]

as desired.

1.3.18 Theorem. (a) $\mathscr{A} \otimes_F F'$ is an $F'$-finite $F'$-linear abelian category.

(b) $t : \mathscr{A} \to \mathscr{A} \otimes_F F'$ is an $F$-linear exact $F'/F$-fully faithful functor.

Proof. (b): We have seen that $t$ is an $F$-linear exact functor. Proposition 1.3.6 shows that it is $F'/F$-fully faithful.

(a): By definition, $\mathscr{A} \otimes_F F'$ is a finite $F'$-linear abelian category. It remains to show that $\text{End}(X)$ is finite $F'$-dimensional for every object $X$ of $\mathscr{A} \otimes_F F'$. Since $\dim_F \text{End}(X) \leq \dim_F \text{End}((X)_{ss})$ by Proposition 1.3.15, it is sufficient to show that $\text{End}(X)$ is finite-dimensional for all simple $X$. By Proposition 1.3.17 we may choose an object $S$ of $\mathscr{A}$, an epimorphism $t(S) \to X$ and a monomorphism $X \to t(S)$. We obtain an $F'$-linear injection

\[ \text{End}(X) = \text{Hom}(X, X) \hookrightarrow \text{Hom}(t(S), t(S)) \overset{(b)}{\cong} F' \otimes_{F'} \text{End}\mathscr{A}(S). \]

The target is finite-dimensional since $\mathscr{A}$ is $F$-finite, thus so is the source.

1.4 Universal Property for Abelian Categories

Let $F'/F$ be a field extension, and consider an $F$-finite $F$-linear abelian category $\mathscr{A}$. By Theorem 1.3.18 we have an associated $F'$-finite $F'$-linear abelian category $\mathscr{A} \otimes_F F'$ and an $F$-linear exact functor $t : \mathscr{A} \to \mathscr{A} \otimes_F F'$.

The goal of this subsection is to show that this functor is “universal” among right exact $F'$-linear functors with target an $F'$-linear abelian category. By this we mean that every such functor $V : \mathscr{A} \to \mathcal{B}$ “factors"
through \( \mathcal{A} \otimes_F F' \) via a right exact \( F' \)-linear functor \( V : \mathcal{A} \otimes_F F' \to \mathcal{B} \), and does so “uniquely”:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{t} & \mathcal{A} \otimes_F F' \\
\downarrow{V} & & \downarrow{V'} \\
\mathcal{B} & & \\
\end{array}
\]

Since we are working with functors, we have to be more precise in stating this universal property.

1.4.1 Theorem. Let \( \mathcal{B} \) be an \( F' \)-linear abelian category, and consider a right exact \( F \)-linear functor \( V : \mathcal{A} \to \mathcal{B} \). Then:

(a) There exists a right exact \( F' \)-linear functor \( V' : \mathcal{A} \otimes_F F' \to \mathcal{B} \) and an isomorphism of functors \( \alpha : V \Rightarrow V' \circ t \).

(b) If \((V'_1, \alpha_1)\) and \((V'_2, \alpha_2)\) both have the property stated in (a), then there exists a unique isomorphism of functors \( \beta' : V'_1 \Rightarrow V'_2 \) such that \( \alpha_2, X = \beta'_t(X) \circ \alpha_{1, X} \) for every \( X \in \mathcal{A} \).

1.4.2 Remark. One might expect (since \( t \) is exact) that if the functor \( V \) in the statement of Theorem 1.4.1 is exact, then \( V' \) is also exact. This is false in general.

The idea behind the proof of Theorem 1.4.1 is to use the purported right exactness of \( V' \) for the proof of its existence. After all, by Lemma 1.3.13 and Theorem 1.3.18 every object \( X \) of \( \mathcal{A} \otimes_F F' \) possesses a presentation

\[
t(X_1) \xrightarrow{\sum_i \lambda_i \otimes f_i} t(X_0) \to X \to 0,
\]

with \( X_0, X_1 \in \mathcal{A} \), and finitely many \( \lambda_i \in F' \) and \( f_i \in \text{Hom}_\mathcal{A}(X_1, X_0) \). Therefore, by right exactness and \( F' \)-linearity of \( V' \), we should have

\[
V'(X) \cong \text{coker} \left( V(X_1) \xrightarrow{\sum_i \lambda_i V(f_i)} V(X_0) \right).
\]

However, since there is no canonical such presentation, it seems difficult to verify that this idea gives us a well-defined functor \( V' \) directly. Hence, we take a detour through the respective ind-categories, where canonical presentations exist. We begin by supplementing Lemma 1.2.2.

1.4.3 Definition. Let \( \mathcal{B} \) be an \( F \)-linear abelian category, and let

\[
V : \mathcal{A} \to \mathcal{B}
\]

be an \( F \)-linear functor. The ind-extension of \( V \) is the \( F \)-linear functor \( \text{ind} V : \text{ind} \mathcal{A} \to \text{ind} \mathcal{B} \) mapping an object \((X_i)_{i \in I}\) of \( \text{ind} \mathcal{A} \) to \( \text{ind} V((X_i)_{i \in I}) := (V X_i)_{i \in I} \) in \( \text{ind} \mathcal{B} \), and a homomorphism \( f = \lim_i \lim_j f_{ij} \) in

\[
\text{Hom}_{\text{ind} \mathcal{A}}((X_i)_{i \in I}, (Y_j)_{j \in J}) = \lim_i \lim_j \text{Hom}_\mathcal{A}(X_i, Y_j)
\]

to \( \text{ind} V(f) := \lim_i \lim_j V(f_{ij}) \).
1.4.4 Lemma. (a) \( \text{ind}(V) \) is a functor extending \( V \) and functorial in \( V \).

(b) If \( V \) is right exact, then \( \text{ind}(V) \) is right exact.

Proof. (a): [KaS06, Proposition 6.1.9], (b): [KaS06, Corollary 8.6.8]. \( \blacksquare \)

1.4.5 Lemma. Every \( F' \)-module \( X \) in \( \text{ind} \mathcal{A} \) has a functorial presentation

\[
\Pi(X) : t(X_1) \xrightarrow{d_1} t(X_0) \xrightarrow{d_0} X \to 0
\]

with \( X_0, X_1 \in \text{ind} \mathcal{A} \).

Proof. For every object \( X = (X, \phi) \) of \( (\text{ind} \mathcal{A})_{F'} \), let \( \tilde{\phi} \) denote the natural surjective homomorphism \( t(X) \to X \) as in Remark 1.3.5.

We may now define our presentation: Given \( X \) as above, we set \( X_0 := \tilde{X} \) and \( d_0 := \tilde{\phi} \). Then \( \ker(d_0) \) is an \( F' \)-module \( (X_1, \phi_1) \), and we set \( d_1 := \tilde{\phi}_1 \).

We obtain an exact sequence

\[
t(X_1) \xrightarrow{d_1} t(X_0) \xrightarrow{d_0} X \to 0
\]

of \( F' \)-modules, which we denote as \( \Pi(X) \).

Let us show that \( \Pi(X) \) is functorial in \( X \): Given another \( F' \)-module \( Y = (Y, \psi) \) and a homomorphism \( f : X \to Y \), we set \( f_0 := \text{id} \otimes f \) and \( f_1 := \text{id} \otimes (f_0|_{(X_1, \phi_1)}) \). We obtain a diagram

\[
\begin{array}{c}
\Pi(X) : & F' \otimes_F X_1 & \xrightarrow{f_1} & F' \otimes_F X_0 & \xrightarrow{f_0} & X & \to 0 \\
 & f & \downarrow & f_0 & \downarrow & f & \\
\Pi(Y) : & F' \otimes_F Y_1 & \xrightarrow{f_1} & F' \otimes_F Y_0 & \xrightarrow{f_0} & Y & \to 0
\end{array}
\]

This diagram commutes by definition, so we have constructed a canonical homomorphism \( \Pi(f) : \Pi(X) \to \Pi(Y) \).

\( \blacksquare \)

1.4.6 Lemma. Let \( \text{ind} V : \text{ind} \mathcal{A} \to \text{ind} \mathcal{B} \) be a right exact \( F \)-linear functor. Let \( \text{ind} \mathcal{A} \) denote the full subcategory of \( (\text{ind} \mathcal{A})_{F'} \) with the image of \( \text{ind} \mathcal{A} \) under \( t \) as objects. There exists a natural \( F' \)-linear functor

\[
\text{ind} \tilde{V} : \text{ind} \mathcal{A} \to \text{ind} \mathcal{B}
\]

such that \( \text{ind} V = \text{ind} \tilde{V} \circ t \).

Proof. Since \( \text{ind} \tilde{V} \) is to extend \( \text{ind} V \), on objects \( t(X) \) of \( \text{ind} \mathcal{A} \) we must and may set

\[\text{ind} \tilde{V}(t(X)) := \text{ind} V(X)\]

Given two objects \( t(X) \) and \( t(Y) \) of \( \text{ind} \mathcal{A} \), we have \( X = \lim_{i \in I} X_i \) and \( Y = \lim_{j \in J} Y_j \) for directed sets \( I, J \) and objects \( X_i, Y_j \) of \( \mathcal{A} \). Recall that both
$t$ and $\text{ind} V$ are right exact and note that both commute with direct sums; it follows that both commute with direct limits – they are “continuous”.

Considering first the special case that $X = X_i$ and $Y = Y_j$ for some $i$ and $j$, by Proposition 1.3.6 we have $\text{Hom}(tX_i, tY_j) = F' \otimes_F \text{Hom}(X_i, Y_j)$, so since $\text{ind} \tilde{V}$ is to be $F'$-linear and extend $\text{ind} V$ we see that $\text{ind} V : \text{Hom}(t(X_i), t(Y_j)) \to \text{Hom}(\text{ind} V(X_i), \text{ind} V(Y_j))$ must be the $F'$-linear extension of $\text{ind} V : \text{Hom}(X_i, Y_j) \to \text{Hom}(\text{ind} V(X_i), \text{ind} V(Y_j))$.

Returning to the general case, using the special case given above for all $i$ and $j$ we define $\text{ind} \tilde{V}$ on homomorphisms as

$$\text{Hom}(t(X), t(Y)) = \text{Hom}(\lim_{i} t(X_i), \lim_{j} t(Y_j)),$$

since $t$ is continuous.

$$\text{Hom}(\lim_{i} \text{ind} V(X_i), \lim_{j} \text{ind} V(Y_j)) = \text{Hom}(\text{ind} \tilde{V}(X), \text{ind} \tilde{V}(Y)),$$

by definition.

The conscientious reader will check that our definition of $\text{ind} \tilde{V}$ is well-defined. Obviously, it extends $\text{ind} V$ in the sense that $\text{ind} \tilde{V} \circ t = \text{ind} V$.

1.4.7 Lemma. Let $\text{ind} V : \mathcal{A} \to \mathcal{B}$ be a right exact $F'$-linear functor. There exists a right exact $F'$-linear functor

$$\text{ind} V' : (\text{ind} \mathcal{A})_{F'} \to \mathcal{B}$$

and an isomorphism of functors $\text{ind} \alpha : \text{ind} V \Rightarrow (\text{ind} V') \circ t$.

Proof. By Lemma 1.4.6 our given functor $\text{ind} V$ extends naturally to an $F'$-linear functor

$$\text{ind} \tilde{V} : t(\text{ind} \mathcal{A}) \to \text{ind} \mathcal{B},$$

where $t(\text{ind} \mathcal{A})$ denotes the full subcategory of $(\text{ind} V)_{F'}$ which has as objects the essential image of $\text{ind} \mathcal{A}$ under $t$.

In particular, given an $F'$-module $X = (X, \phi)$ in $\text{ind} \mathcal{A}$, we may apply $\text{ind} \tilde{V}$ to the portion $t(X_1) \overset{d_1}{\longrightarrow} t(X_0)$ of the presentation $\Pi(X)$ given by Lemma 1.4.5 and set

$$\text{ind} V'(X) := \text{coker} \left( \text{ind} V(X_1) \overset{\text{ind} \tilde{V}(d_1)}{\longrightarrow} \text{ind} V(X_0) \right).$$

Given a second object $Y$ and a homomorphism $f : X \to Y$ in of $F'$-modules, we may apply $\text{ind} \tilde{V}$ to the portion

$$\begin{array}{ccc}
t(X_1) & \longrightarrow & t(X_0) \\
\downarrow f_1 & & \downarrow f_0 \\
t(Y_1) & \longrightarrow & t(Y_0)
\end{array}$$
of the homomorphism \( \Pi(f) \) of presentations given in the proof of Lemma 1.4.5. Now the universal property of cokernels implies that there is exactly one homomorphism \( \text{ind} V'(f) : \text{ind} V'(X) \to \text{ind} V'(Y) \) completing the image of the above commutative square under \( \text{ind} \tilde{V} \) to a commutative diagram:

\[
\begin{array}{cccccc}
\text{ind} \tilde{V}(t(X_1)) & \to & \text{ind} \tilde{V}(t(X_0)) & \to & \text{ind} V'(X) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{ind} \tilde{V}(t(Y_1)) & \to & \text{ind} \tilde{V}(t(Y_0)) & \to & \text{ind} V'(Y) & \to & 0 \\
\end{array}
\]

The universal property of cokernels also shows that \( \text{ind} V'(\text{id}_X) = \text{id}_{\text{ind} V'(X)} \) for all \( X \), and that \( \text{ind} V'(gf) = \text{ind} V'(g) \text{ind} V'(f) \) for all pairs of composable homomorphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \), so \( \text{ind} V' \) is indeed a functor \( (\text{ind} \mathcal{A})_F' \to \text{ind} \mathcal{B} \).

Let us prove that \( \text{ind} V' \) is right exact, so let \( X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \) be a right exact sequence in \( (\text{ind} \mathcal{A})_F' \). We obtain the following commutative diagram:

\[
\begin{array}{cccccc}
\text{ind} V(X_1) & \to & \text{ind} V(X_0) & \to & \text{ind} V'(X) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{ind} V(Y_1) & \to & \text{ind} V(Y_0) & \to & \text{ind} V'(Y) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{ind} V(Z_1) & \to & \text{ind} V(Z_0) & \to & \text{ind} V'(Z) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & \\
\end{array}
\]

The rows are the sequences defining \( \text{ind} V' \) on objects, so they are exact by definition. Since \( \tilde{V} \) is right exact and the vertical homomorphisms arise from \( \text{ind} \mathcal{A} \), the first two columns are exact. Hence, by the 3 \( \times \) 3-Lemma, the remaining column is exact, which is what we had to prove.

Finally, let us construct an isomorphism \( \text{ind} \alpha : \text{ind} V \Rightarrow (\text{ind} V') \circ t \) of functors. We let \( K \) be the kernel of the multiplication \( \mu \) of \( F' \), so we have an exact sequence of \( F' \)-vector spaces

\[
K \to F' \otimes_{F'} F' \xrightarrow{\mu} F' \to 0.
\]

For every object \( X \) of \( \text{ind} \mathcal{A} \), this induces an exact sequence

\[
(1.4.8) \quad K \otimes_{\text{pr} \text{ind} V} V(X) \to (F' \otimes_{\text{pr} F'} F') \otimes_{\text{pr} \text{ind} V} V(X) \to F' \otimes_{\text{pr} \text{ind} V} V(X) \to 0
\]
in \( \text{ind} \mathcal{B} \). We use this observation to construct the following diagram:

\[
\begin{array}{c}
\text{ind} \mathcal{V}(t(X)) & \overset{\text{ind} V(d_1)}{\longrightarrow} & \text{ind} \mathcal{V}(t(X_0)) & \overset{\text{ind} V'(t(X))}{\longrightarrow} & 0 \\
\text{ind} \mathcal{V}(K \otimes_F X) & \overset{\cong}{\longrightarrow} & \text{ind} \mathcal{V}(F' \otimes_F X) & \overset{\cong}{\longrightarrow} & \text{ind} V'(t(X)) \\
K \otimes_F \text{ind} V(X) & \overset{\cong}{\longrightarrow} & F' \otimes_F \text{ind} V(X) & \overset{\cong}{\longrightarrow} & \text{ind} V'(t(X)) \\
K \otimes_F \text{ind} V(X) & \overset{\cong}{\longrightarrow} & (F' \otimes_F F') \otimes_F \text{ind} V(X) & \overset{\cong}{\longrightarrow} & F' \otimes_F \text{ind} V(X) & \longrightarrow & 0 \\
\end{array}
\]

The first row is the definition of \( \text{ind} \mathcal{V}'(t(X)) \), which we unravel in the second row. The isomorphisms connecting the second and third row are canonical, as are the epimorphism and the isomorphism connecting the third row with the fourth, which is the exact sequence (1.4.8). One can check that this diagram commutes, so by the Five Lemma we obtain a canonical isomorphism \( \text{ind} V'(t(X)) \rightarrow F' \otimes_F \text{ind} V(X) \). Precomposing the inverse of this isomorphism with the canonical isomorphism \( \text{ind} V(X) \rightarrow F' \otimes_F \text{ind} V(X) \), we obtain an isomorphism

\[
(\text{ind} \alpha)_X : \text{ind} V(X) \overset{\cong}{\longrightarrow} \text{ind} V'(t(X))
\]

as desired. By construction, \( (\text{ind} \alpha)_X \) is natural in \( X \), so \( \text{ind} \alpha \) is a homomorphism of functors. Therefore \( \text{ind} \alpha \) is an isomorphism of functors, since we have already seen that \( (\text{ind} \alpha)_X \) is an isomorphism for each \( X \in \mathcal{A} \).

1.4.9 Lemma. Let \( \mathcal{A} \) be an \( F \)-finite \( F \)-linear abelian category and \( V : \mathcal{A} \rightarrow \mathcal{B} \) be a right exact \( F \)-linear functor. Let \( \text{ind} V' \) be the right exact \( F' \)-linear functor associated to \( \text{ind} V \) via Lemma 1.4.7. Then there exists a functor

\[
V' : \mathcal{A} \otimes_F F' \longrightarrow \mathcal{B}
\]

such that \( V' \) fulfills the requirements of Theorem 1.4.7(a) and the following diagram commutes up to isomorphism of functors:

\[
\begin{array}{c}
\text{ind} \mathcal{A} \otimes_F F' \overset{\text{ind} V'}{\longrightarrow} \text{ind} \mathcal{B} \\
\mathcal{A} \otimes_F F' \overset{V'}{\longrightarrow} \mathcal{B}
\end{array}
\]

Proof. By Lemma 1.4.4, \( V \) induces a right exact \( F \)-linear functor \( \text{ind} V : \mathcal{A} \rightarrow \text{ind} \mathcal{B} \). By Lemma 1.4.7, \( \text{ind} V \) induces a right exact \( F' \)-linear
functor $\text{ind} V' : (\mathcal{A})_{F'} \to \text{ind} \mathcal{B}$. We obtain the following diagram, which commutes up to isomorphism of functors:

\[
\begin{array}{c}
\mathcal{A} \\ \downarrow V \\
\text{ind} \mathcal{A} \\
\text{ind} V' \\
\mathcal{B} \\
\end{array}
\]

We let $V'$ be the restriction of $\text{ind} V'$ to $\mathcal{A} \otimes F' \subset (\mathcal{A})_{F'}$. If we prove that the image of $V'$ lies in the essential image of $\mathcal{B}$ in $\text{ind} \mathcal{B}$, then we will have shown that the following diagram commutes up to isomorphism of functors:

\[
\begin{array}{c}
\mathcal{A} \\
\downarrow V \\
\mathcal{B} \\
\end{array} 
\quad \text{ind} \mathcal{B} 
\]

So let us do this: Given $X$ in $\mathcal{A} \otimes F'$, by Lemma 1.3.13 and Theorem 1.3.18 there exists a right exact sequence

\[ t(X_1) \xrightarrow{g} t(X_1) \to X \to 0 \]

in $(\mathcal{A})_{F'}$, with $X_0, X_1 \in \mathcal{A}$ and $g \in F' \otimes \text{Hom}_\mathcal{A}(X_0, Y_0)$. Since $\text{ind} V'$ is right exact, and its restriction to $\mathcal{A}$ is isomorphic to $V$, the induced sequence

\[ V(X_1) \xrightarrow{(\text{ind} V')(g)} V(X_0) \to V'(X) \to 0 \]

is exact in $\text{ind} \mathcal{B}$. Since $\mathcal{B} \to \text{ind} \mathcal{B}$ is exact, it follows that $V'(X)$ is isomorphic to the cokernel of the homomorphism $\text{ind} V'(g)$ as calculated in the full subcategory $\mathcal{B}$.

We turn to the unicity of our extensions $\text{ind} V'$ and $V'$.

1.4.10 Lemma. Let $\text{ind} V_1, \text{ind} V_2 : \mathcal{A} \to \text{ind} \mathcal{B}$ be two right exact $F$-linear functors, and let $(\text{ind} V_1', \text{ind} \alpha_1), (\text{ind} V_2', \text{ind} \alpha_2)$ be extensions as in Lemma 1.4.7 of $\text{ind} V_1, \text{ind} V_2$, respectively.

For every homomorphism of functors $\text{ind} \beta : \text{ind} V_1 \Rightarrow \text{ind} V_2$ there exists a unique homomorphism of functors $\text{ind} \beta' : \text{ind} V_1' \Rightarrow \text{ind} V_2'$ such that $\text{ind} \alpha_{2,X} \circ \text{ind} \beta X = \text{ind} \beta_{t(X)}' \circ \text{ind} \alpha_{1,X}$ for all $X \in \mathcal{A}$.

Moreover, $\text{ind} \beta$ is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if $\text{ind} \beta'$ is.

Proof. For $X \in (\mathcal{A})_{F'}$, the sequences $\text{ind} V_1'(\Pi(X))$ are both exact, since both $\text{ind} V'_1$ are right exact by assumption. They are connected by means of
the following commutative diagram with exact rows:

\[
\begin{array}{c}
\text{ind } V'(t(X_1)) \rightarrow \text{ind } V'(t(X_0)) \rightarrow \text{ind } V'(X) \rightarrow 0 \\
\downarrow \text{(ind } \alpha_{1,X_1})^{-1} \downarrow \text{(ind } \alpha_{1,X_0})^{-1} \\
\text{ind } V(X_1) \rightarrow \text{ind } V(X_0) \\
\downarrow \text{ind } \beta_{X_1} \downarrow \text{ind } \beta_{X_0} \\
\text{ind } V(X_1) \rightarrow \text{ind } V(X_0) \\
\downarrow \text{ind } \alpha_{2,X_1} \downarrow \text{ind } \alpha_{2,X_0} \\
\text{ind } V'_2(t(X_1)) \rightarrow \text{ind } V'_2(t(X_0)) \rightarrow \text{ind } V'_2(X) \rightarrow 0
\end{array}
\]

By the universal property of cokernels, we obtain a unique homomorphism \( \text{ind } \beta_X' : V'_1(X) \rightarrow V'_2(X) \) completing the diagram to a homomorphism of right exact sequences. By the Five Lemma, \( \text{ind } \beta_X' \) is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if \( \text{ind } \beta \) is. Now by construction \( \text{ind } \beta_X' \) is natural in \( X \), so \( \text{ind } \beta' : \text{ind } V'_1 \Rightarrow \text{ind } V'_2 \) is a homomorphism of functors, which is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if \( \text{ind } \beta \) is.

The same diagram shows that any homomorphism \( \text{ind } V'_1 \Rightarrow \text{ind } V'_2 \) which restricts to \( (\text{ind } \alpha_2) \circ (\text{ind } \beta) \circ (\text{ind } \alpha_1)^{-1} \) on \( t(\text{id } \mathcal{A}) \) must coincide with \( \text{ind } \beta' \).

It remains to show that \( \text{ind } \beta' \) restricts in such a way. But this again follows from the same diagram, since if \( X = t(\tilde{X}) \) for \( \tilde{X} \in \text{ind } \mathcal{A} \), then \( \text{ind } \alpha_{2,\tilde{X}} \circ \text{ind } \beta_{\tilde{X}} \circ (\text{ind } \alpha_{1,\tilde{X}})^{-1} \) fits in the same place as \( \text{ind } \beta'_{t(\tilde{X})} \), so the two homomorphisms must coincide by the universal property of cokernels.

\[ \therefore \]

**1.4.11 Lemma.** *Given two pairs \((V'_1, \alpha_i), (V'_2, \alpha_2)\) extending \( V \) as in Theorem 1.4.7(a), there exists a unique isomorphism of functors \( \beta' : V'_1 \Rightarrow V'_2 \) such that \( \beta'_{t(X)} \circ \alpha_{1,X} = \alpha_{2,X} \) for all \( X \in \mathcal{A} \).*

**Proof.** Given two such pairs of data \((V'_1, \alpha_i), (V'_2, \alpha_2)\), using Lemma 1.4.4 we obtain two pairs of data \((\text{ind } V'_1, \text{ind } \alpha_i)\) extending \( V \) as in Lemma 1.4.7. Lemma 1.4.10 applied to \( \text{ind } \beta := \text{id } \text{ind } V \) shows that there exists an isomorphism of functors \( \text{ind } \beta' : \text{ind } V'_1 \Rightarrow \text{ind } V'_2 \) such that \( \beta'_{t(X)} \circ \text{ind } \alpha_{1,X} = \text{ind } \alpha_{2,X} \) for all \( X \in \text{ind } \mathcal{A} \). The restriction \( \beta' \) of \( \text{ind } \beta' \) to \( \mathcal{A} \cap F \times F' \subset (\text{ind } \mathcal{A} \cap F') \) is then an isomorphism of functors \( V'_1 \Rightarrow V'_2 \) with the required properties.

Let us show that this \( \beta' \) is unique. Given two isomorphisms of functors \( \beta'_1, \beta'_2 : V'_1 \Rightarrow V'_2 \) with an identification of isomorphisms

\[ \beta'_1 |_{t(\mathcal{A})} = \alpha_2 \circ \alpha_1^{-1} = \beta'_2 |_{t(\mathcal{A})} : V'_1 \Rightarrow V'_2, \]

applying \( \text{ind } (\cdot) \) gives us an identification of isomorphisms

\[ \text{ind } \beta'_1 |_{t(\text{ind } \mathcal{A})} = \text{ind } (\alpha_2 \circ \alpha_1^{-1}) = \text{ind } \beta'_2 |_{t(\text{ind } \mathcal{A})} : \text{ind } V'_1 \Rightarrow \text{ind } V'_2. \]
by Lemma 1.4.4 where \( \text{ind}(\alpha_2 \circ \alpha_1^{-1}) = \text{ind} \alpha_2 \circ \text{ind} \alpha_1^{-1} \) and clearly \( t(\text{ind} \mathcal{A}) = \text{ind} t(\mathcal{A}) \). Lemma 1.4.10 shows that \( \text{ind} \beta_1' = \text{ind} \beta_2' \), so restricting to \( t(\mathcal{A}) \) we obtain \( \beta_1' = \beta_2' \) as desired.

Proof of Theorem 1.4.1. (a): Lemma 1.4.9, (b): Lemma 1.4.11.

1.4.12 Proposition. Let \( \mathcal{A} \) be an \( F \)-finite \( F \)-linear abelian category, \( \mathcal{B} \) an \( F' \)-linear abelian category, and \( V_1', V_2' : \mathcal{A} \otimes_F F' \to \mathcal{B} \) two right exact \( F' \)-linear functors. Then for every homomorphism of functors \( \beta : V_1' \circ t \Rightarrow V_2' \circ t \) there exists a unique homomorphism of functors \( \beta' : V_1' \Rightarrow V_2' \) extending \( \beta \).

Moreover, \( \beta \) is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if \( \beta' \) is such.

Proof. This may be deduced from Lemma 1.4.10 as in the proof of Lemma 1.4.11.

1.5 Permanence of Semisimplicity on Objects

Let \( F \) be a field. Recall that an exact functor between abelian categories is semisimple on objects if it maps semisimple objects to semisimple objects. Recall also the notion of separability as given in Definition 1.1.6.

1.5.1 Proposition. Let \( \mathcal{A} \) be an \( F \)-finite \( F \)-linear abelian category. Assume that \( F'/F \) is a separable field extension. Then \( t : \mathcal{A} \to \mathcal{A} \otimes_F F' \) is semisimple on objects.

Proof. Let \( X \) be a semisimple object of \( \mathcal{A} \), and set \( E := \text{End}_{\mathcal{A}}(X) \). It is a semisimple finite-dimensional \( F \)-algebra by assumption. We must show that \( t(X) \) is semisimple, and may assume that \( X \) is simple since \( t \) is additive. By Proposition 1.3.11, \( t(X) \) is semisimple if and only if the \( F' \)-algebra \( F' \otimes_F E \) is semisimple. And \( F' \otimes_F E \) is semisimple by Proposition 1.1.7(a) since \( E \) is semisimple and \( F'/F \) is separable.

Even if \( F'/F \) is not separable, \( t(X) \) may be semisimple:

1.5.2 Proposition. Let \( \mathcal{A} \) be an \( F \)-finite \( F \)-linear abelian category. Let \( X \) be a semisimple object of \( \mathcal{A} \) for which \( \text{End}(X) \) is a separable \( F \)-algebra. Then \( t(X) \) is semisimple.

Proof. The algebra \( F' \otimes_F E \) is semisimple by Proposition 1.1.7(b) since \( E \) is separable and semisimple. So we may repeat the proof of Proposition 1.5.1 mutatis mutandis.

1.5.3 Proposition. Let \( \mathcal{A} \) be an \( F \)-finite \( F \)-linear abelian category, \( \mathcal{B} \) an \( F' \)-linear abelian category, \( V : \mathcal{A} \to \mathcal{B} \) a right exact \( F \)-linear functor and \( V' : \mathcal{A} \otimes_F F' \to \mathcal{B} \) the induced right exact \( F' \)-linear functor. Assume that \( F'/F \) is a separable field extension. Then \( V \) is semisimple on objects if and only if \( V' \) is semisimple on objects.
Proof. If $V'$ is semisimple on objects, then so is $V = V' \circ t$, a composition of such functors by Proposition 1.5.1. Conversely, if $V$ is semisimple on objects, let $X$ be a semisimple object of $\mathcal{A} \otimes_F F'$. We must show that $V'(X)$ is semisimple, and may assume that $X$ is simple, since $V'$ is additive. By Proposition 1.3.17(a) there exists a simple object $S$ of $\mathcal{A}$ such that $X$ is a quotient of $t(S)$. Since $V'$ is right exact, this implies that $V'(X)$ is a quotient of $V'(t(S)) = V(S)$, which is semisimple by assumption. Therefore, $V'(X)$ itself is semisimple.

\section{Tensor Categories}

\subsection{Scalar Extension of Tensor Categories}

Let $F'/F$ be a field extension, and consider an $F$-finite $F$-linear abelian category $\mathcal{A}$ with associated scalar extension functor $t: \mathcal{A} \rightarrow \mathcal{A} \otimes_F F'$.

\subsection{Definition}

An $F$-multilinear endofunctor of $\mathcal{A}$ is a functor

$$M: \mathcal{A}^n \rightarrow \mathcal{A}$$

which is $F$-linear in each each argument, for some integer $n \geq 1$.

\subsection{Proposition}

Let $\mathcal{A}$ be an $F$-finite $F$-linear abelian category, and $n \geq 1$ an integer.

(a) Let $M: \mathcal{A}^n \rightarrow \mathcal{A}$ be a right exact $F$-multilinear functor. Then there exists a right exact $F'$-multilinear functor $M': (\mathcal{A} \otimes_F F')^n \rightarrow \mathcal{A} \otimes_F F'$ together with an isomorphism $\alpha: t \circ M \Rightarrow M' \circ (t \times_n)$ of functors.

(b) Let $M_1, M_2: \mathcal{A}^n \rightarrow \mathcal{A}$ be two right exact $F$-multilinear functors, and let $(M'_1, \alpha_1), (M'_2, \alpha_2)$ be extensions as in (a) of $M_1, M_2$ respectively. Then, for every homomorphism of functors $\beta: M_1 \Rightarrow M_2$ there exists a unique homomorphism of functors $\beta': M'_1 \Rightarrow M'_2$ such that $t \circ \beta = \alpha_2 \circ \beta'_t \times_n$ in the sense that for every $n$-tuple of objects $(X_1, \ldots, X_n) \in \mathcal{A}^n$ the following diagram commutes:

$$
\begin{array}{ccc}
M'_1(t(X_1), \ldots, t(X_n)) & \xrightarrow{\alpha_1(X_1, \ldots, X_n)} & t(M_1(X_1, \ldots, X_n)) \\
\beta'(tX_1, \ldots, tX_n) \downarrow & & \downarrow \text{id} \otimes \beta(X_1, \ldots, X_n) \\
M'_2(t(X_1), \ldots, t(X_n)) & \xrightarrow{\alpha_2(X_1, \ldots, X_n)} & t(M_1(X_1, \ldots, X_n))
\end{array}
$$

Proof. This is one of the proofs in mathematics which does not become much clearer by writing it down in detail. The case $n = 1$ follows from Theorem 1.4.1 and Proposition 1.4.12 applied to $V := t \circ M$. We settle for
a sketch of the construction of $M'$ in the case $n = 2$. We set $\otimes := M$ and will denote the desired extension $M'$ by $\otimes'$. Let us abbreviate notation by setting $\mathcal{A}' := \mathcal{A} \otimes F'$.

For every $Y \in \mathcal{A}$, let

$$- \otimes' t(Y) := (t \circ (- \otimes Y))' : \mathcal{A}' \to \mathcal{A}'$$

denote the scalar extension of $t \circ (- \otimes Y)$ as in Theorem 1.4.1(a). It is an $F'$-linear right exact functor. It is also functorial in $Y$, since a homomorphism $f : Y_1 \to Y_2$ induces a homomorphism of functors $t \circ (- \otimes Y_1) \Rightarrow t \circ (- \otimes Y_2)$ given for $X \in \mathcal{A}$ by $\text{id} \otimes f : X \otimes Y_1 \to X \otimes Y_2$. By Proposition 1.4.12 this induces a unique homomorphism of functors $- \otimes' t(Y_1) \Rightarrow - \otimes' t(Y_2)$. Therefore, we obtain a right exact functor

$$- \otimes' t(-) : \mathcal{A}' \times \mathcal{A} \to \mathcal{A}'$$

which is $F'$-linear in the first variable and $F$-linear in the second.

For every $X \in \mathcal{A}'$, let

$$X \otimes' (-) := (X \otimes' -)' : \mathcal{A}' \to \mathcal{A}'$$

denote the scalar extension of $X \otimes' -$ as in Theorem 1.4.1(a). It is an $F'$-linear right exact functor. By similar reasoning as before, it is functorial in $X$. Therefore, we obtain a right exact $F$-bilinear functor

$$(-) \otimes' (-) : \mathcal{A}' \times \mathcal{A}' \to \mathcal{A}'$$

It fulfills what is required in item (a).

For an introduction to the theory of tensor categories, we refer to [DeM82] and [Del90]. We will repeat only the definitions to fix notation.

2.1.3 Definition. (a) An abelian tensor category is an abelian category $\mathcal{A}$ together with a right exact biadditive functor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, its tensor product, which is assumed to be equipped with sufficiently many (associativity, commutativity and unity) constraints such that the tensor product of an unordered finite set of objects is well-defined. In particular, there exists a unit object $1$. One tends to suppress mention of the constraints.

(b) An abelian tensor category over $F$ is an abelian tensor category together with a ring isomorphism $F \to \text{End}(1)$. Using this isomorphism and the constraints, $\mathcal{T}$ becomes $F$-linear and $\otimes$ is $F$-bilinear.

(c) A tensor functor is a functor $V : \mathcal{T} \to \mathcal{T}$ between two abelian tensor categories, which is assumed to be equipped with tensor constraints, that is, canonical isomorphisms $V(X) \otimes V(Y) \cong V(X \otimes Y)$ compatible with the constraints of $\mathcal{T}$ and $\mathcal{T}$.
(d) A morphism of tensor functors $V, W : \mathcal{T} \to \mathcal{T}$ is a natural transformation $\eta : V \Rightarrow W$, which is assumed to be compatible with the tensor constraints of $V$ and $W$. We let $\text{Hom}^\otimes(V, W)$ denote the set of morphisms of tensor functors $V \Rightarrow W$ and let $\text{Aut}^\otimes(V)$ denote the group of automorphisms of $V$ as tensor functor.

Given an $F$-finite abelian tensor category $(\mathcal{T}, \otimes)$ over $F$, Proposition \[2.1.2\] provides a natural candidate for a tensor product $\otimes'$ on $\mathcal{T} \otimes_F F'$. The following proposition demonstrates that our instincts are correct.

2.1.4 Theorem. Let $(\mathcal{T}, \otimes)$ be an $F$-finite abelian tensor category over $F$. Let $F'/F$ be a field extension.

(a) $(\mathcal{T} \otimes_F F', \otimes')$ is an abelian tensor category over $F'$.

(b) $\tau : \mathcal{T} \to \mathcal{T} \otimes_F F'$ is a tensor functor.

Proof. By assumption, $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a right exact $F$-bilinear functor and comes equipped with an associativity constraint $\phi$, commutativity constraint $\psi$, unit object $1$ and isomorphism $F \to \text{End}(1)$. Set $\mathcal{T}' := \mathcal{T} \otimes_F F'$.

By Proposition \[2.1.2\], the induced functor $\otimes' : \mathcal{T}' \times \mathcal{T}' \to \mathcal{T}'$ is right exact and $F'$-bilinear. The associativity constraint $\phi' : \otimes' \circ (\text{id} \times \otimes') \Rightarrow \otimes' \circ (\otimes \times \text{id})$ has a unique extension to an isomorphism of functors $\phi' : \otimes' \circ (\text{id} \times \otimes') \Rightarrow \otimes' \circ (\otimes' \times \text{id})$ by Proposition \[2.1.2\] for $n = 3$, and the commutativity constraint $\psi' : \otimes \Rightarrow \otimes' \circ s$ has a unique extension to an isomorphism of functors $\psi' : \otimes' \Rightarrow \otimes' \circ s$ by Proposition \[2.1.2\] for $n = 2$, where $s$ denotes the “switch” functor $C \times C \to C \times C$, $(X, Y) \to (Y, X)$ for any category $C$.

It remains to check that three relations hold among $\phi'$ and $\psi'$ (namely, $\psi' \circ \psi' = \text{id}$, the Pentagon Axiom and the Hexagon Axiom), and that there exists a unit object $1' \in \mathcal{T}'$ for which $F' \to \text{End}_{\mathcal{T}'}(1')$ is an isomorphism.

Each of these three relations state that certain natural transformations (constructed using $\phi'$ and $\psi'$) of certain functors $\mathcal{T}'^n \to \mathcal{T}'$ (constructed using $\otimes'$) are equal. The first states that $\psi'_{Y, X} \circ \psi'_{X, Y} = \text{id}_{X \otimes' Y}$ for all $X, Y \in \mathcal{T}'$. The Pentagon Axiom states that $\phi' \circ \phi' = (\phi' \circ \text{id}) \circ \phi' \circ (\text{id} \otimes \phi')$ in the sense that for every quadruple $(X, Y, Z, T)$ of objects of $\mathcal{T}'$ the following diagram commutes:

$$\begin{array}{ccc}
X \otimes' (Y \otimes' (Z \otimes' T)) & \xrightarrow{} & (X \otimes' Y) \otimes' (Z \otimes' T) & \xrightarrow{} & ((X \otimes' Y) \otimes' Z) \otimes' T \\
\downarrow & & \downarrow & & \downarrow \\
X \otimes' ((Y \otimes' Z) \otimes' T) & \xrightarrow{} & (X \otimes' (Y \otimes' Z)) \otimes' T \\
\end{array}$$

The Hexagon Axiom states that $\phi' \circ \psi' \circ \phi' = (\psi' \otimes \text{id}) \circ \phi' \circ (\text{id} \otimes \phi')$ in the sense that for every triple $(X, Y, Z)$ of objects of $\mathcal{T}'$ the following diagram commutes:

$$\begin{array}{ccc}
X \otimes' (Y \otimes' Z) & \xrightarrow{} & (X \otimes' Y) \otimes' Z & \xrightarrow{} & Z \otimes' (X \otimes' Y) \\
\downarrow & & \downarrow & & \downarrow \\
X \otimes' (Z \otimes' Y) & \xrightarrow{} & (X \otimes' Z) \otimes' Y & \xrightarrow{} & (Z \otimes' X) \otimes' Y \\
\end{array}$$
In all cases, Proposition 2.1.2(b) and the assumption that \( T \) is a tensor category show that the stated relations hold. Let us prove the first relation \( \psi' \circ \psi' = \text{id} \) as an example. Now \( \psi' \circ \psi' \) is a homomorphism of functors \( \otimes' \rightarrow \otimes' \). Its restriction to \( \otimes \) is equal to \( \psi \circ \psi \) by definition, and is equal to the identity endomorphism of \( \otimes \), since \( \psi' \) extends \( \psi \) and \( T \) is a tensor category. So \( \psi' \circ \psi' \) is an extension of the identity endomorphism of \( \otimes \). Since the identity endomorphism of \( \otimes' \) is another extension of the identity endomorphism of \( \otimes \), Proposition 2.1.2(b) shows that \( \psi' \circ \psi' \) and the identity endomorphism of \( \otimes' \) coincide! The proofs that the Pentagon and Hexagon axioms hold are similar, if somewhat more involved notationally.

It remains to show that there exists a unit object of \((T', \otimes')\) with endomorphism ring \( F' \), and we claim that \( t(1) \) is one for every unit object \( 1 \) of \((T, \otimes)\). To say that \( 1 \) is a unit object means that there exists an isomorphism \( u : 1 \rightarrow 1 \otimes 1 \) and that \( 1 \otimes (-) \) is an equivalence of categories \( T \rightarrow T \).

Now \( t(u) : t(1) \rightarrow t(1 \otimes 1) \cong t(1) \otimes' t(1) \) is an isomorphism since \( t \) is a functor. Let \( V \) be a quasi-inverse of the restriction \( 1 \otimes (-) \) of the functor \( t(1) \otimes' (-) \). Then \( t \circ V \), the scalar extension of \( t \circ V \), is a quasi-inverse of the functor \( t(1) \otimes' (-) \), this may again be proved using Proposition 2.1.2(b). Finally, \( F' \rightarrow \text{End}_{T'}(t(1)) \) is an isomorphism since \( 1 \) has endomorphism ring \( F \) and \( t \) is \( F/F \)-fully faithful.

(b): This statement is true by construction, since we have given \( T' \) a structure of tensor category extending that of \( T \).

2.1.5 Proposition. Let \( T \) be an \( F \)-finite abelian tensor category over \( F \), \( T' \) an \( F' \)-linear abelian tensor category, \( V : T \rightarrow T' \) an \( F' \)-linear right exact tensor functor. Then the \( F' \)-linear functor \( V' : T \otimes_F F' \rightarrow T' \) induced by Theorem 1.4.1 is a tensor functor.

Proof. The proof is similar to the proof of Theorem 2.1.4, using Proposition 2.1.2 and the precise definition of tensor functors. We suppress it.

2.1.6 Remark. It follows that \( t : T \rightarrow T \otimes_F F' \) has a universal property with respect to tensor categories, with right exact tensor functors replacing the right exact functors of Theorem 1.4.1.

2.2 The Influence of Duals

2.2.1 Definition. (a) An object \( X \) of an abelian tensor category is dualisable if there exists an object \( X^\vee \) – its dual – together with homomorphisms \( \delta : 1 \rightarrow X \otimes X^\vee \) and \( \text{ev} : X \otimes X^\vee \rightarrow 1 \) such that the composite homomorphisms \( X \rightarrow X \otimes X^\vee \otimes X \rightarrow X \) and \( X^\vee \rightarrow X^\vee \otimes X \otimes X^\vee \rightarrow X^\vee \) are equal to the respective identities. If \( X \) is dualisable, then so is \( X^\vee \) and one has a canonical isomorphism \( X \cong X^{\vee \vee} \).
An abelian tensor category is **rigid** if all of its objects are dualisable.

The **dual** of a homomorphism \( f : X \to Y \) in a rigid abelian tensor category is the unique homomorphism \( f^\vee : Y^\vee \to X^\vee \) satisfying

\[
ev_Y \circ (\text{id}_Y \otimes f) = ev_X \circ (f^\vee \otimes \text{id}_X) : Y^\vee \otimes X \to 1.
\]

The **internal Hom** of two objects \( X, Y \) of a rigid abelian tensor category is the object \( \text{Hom}(X, Y) := X^\vee \otimes Y \).

A **pre-Tannakian category** over \( F \) is an \( F \)-finite rigid abelian tensor category over \( F \).

A subcategory \( \mathcal{S} \) of a pre-Tannakian category \( \mathcal{T} \) is a **strictly full pre-Tannakian subcategory** if it is full and closed under direct sums, tensor products, duals and subquotients in \( \mathcal{T} \).

Given a set \( S \) of objects of a pre-Tannakian category \( \mathcal{T} \), we let \( (\{(S)\}_\otimes \) denote the smallest strictly full pre-Tannakian subcategory of \( \mathcal{T} \) containing \( S \). We also set \( (\{X\}_\otimes := (\{X\})_\otimes \) for any object \( X \) of \( \mathcal{T} \).

A pre-Tannakian category \( \mathcal{T} \) is **finitely generated** if \( \mathcal{T} = (\{X\}_\otimes \) for some object \( X \in \mathcal{T} \).

**2.2.2 Proposition.** Let \( \mathcal{T} \) be a pre-Tannakian category over \( F \), and consider a field extension \( F'/F \). Then \( \mathcal{T} \otimes F' \) is a pre-Tannakian category over \( F' \).

**Proof.** \( \mathcal{T} \otimes F' \) carries the natural structure of abelian tensor category given by Theorem 2.1.4. We must show that it is rigid, so let \( X \) be an object of \( \mathcal{T} \otimes F' \). By Lemma 2.3.13 there exists a presentation

\[
t(X_1) \xrightarrow{t} t(X_0) \to X \to 0
\]

of \( X \) with objects \( X_0, X_1 \) of \( \mathcal{T} \). Since \( \mathcal{T} \) is rigid, the objects \( X_i \) are dualisable. Since \( t \) is a tensor functor, so are the objects \( t(X_i) \), with duals \( t(X^\vee_i) \). But every object of an abelian tensor category which possesses a presentation by dualisable objects is dualisable. Namely, \( X^\vee := \ker(f^\vee) \) is a dual of \( X = \ker(f) \).

For pre-Tannakian categories, we obtain yet another universal property of \( t \) with respect to exact tensor functors, due to the following fact.

**2.2.3 Lemma.** Let \( V : \mathcal{S} \to \mathcal{T} \) be a tensor functor. Assume that \( \mathcal{S} \) is rigid. Then \( V \) is exact if and only if it is right exact.

**Proof.** Every tensor functor commutes with duals. Dualisation is an exact functor. So if \( 0 \to X' \to X \to X'' \) is a left exact sequence in \( \mathcal{S} \), then its image under \( V \) may be identified with the dual of the image of its dual, which must therefore be left exact. \( \therefore \)
We end this subsection with the following two observations.

2.2.4 Lemma. Let \( \mathcal{I}, \mathcal{J} \) be abelian tensor categories, \( V : \mathcal{I} \to \mathcal{J} \) an exact tensor functor, and assume that \( \mathcal{I} \) is rigid. If \( \mathcal{J} \neq 0 \), then \( V \) is faithful.

**Proof.** An exact functor is faithful if and only if it maps all non-zero objects to non-zero objects. A dualisable object \( X \in \mathcal{I} \) is non-zero if and only if \( X \otimes X^\vee \to 1 \) is surjective, and this criterion is respected by right exact tensor functors. So if \( \mathcal{J} \neq 0 \), that is, if \( 1_\mathcal{J} \neq 0 \), then \( V \) is faithful. \( \Box \)

2.2.5 Lemma. Let \( \mathcal{T} \) be a pre-Tannakian category over \( F \), \( \mathcal{T}' \) an abelian tensor category over \( F' \), and consider two exact \( F \)-linear tensor functors \( V', W' : \mathcal{T} \otimes_F F' \to \mathcal{T}' \). Let \( \eta : V' \Rightarrow W' \) be a natural transformation. Then \( \eta \) is a morphism of tensor functors if and only if its restriction along \( t \) is such.

**Proof.** Again, as in Theorem 2.1.4, this is a matter of checking that certain natural transformations are equal, and we suppress it. \( \Box \)

### 2.3 Permanence of Relative Full Faithfulness

2.3.1 Proposition. Let \( \mathcal{A} \) be a pre-Tannakian category over \( F \), \( \mathcal{B} \) an \( F' \)-linear abelian tensor category, \( V : \mathcal{A} \to \mathcal{B} \) an exact \( F \)-linear tensor functor and \( V' : \mathcal{A} \otimes_F F' \to \mathcal{B} \) the induced exact \( F' \)-linear functor. Then \( V \) is \( F'/F \)-fully faithful if and only if \( V' \) is fully faithful.

**Proof.** If \( V' \) is fully faithful, then its restriction \( V = V' \circ t \) is \( F'/F \)-fully faithful since \( t \) is \( F'/F \)-fully faithful by Lemma 1.3.6.

Conversely, let us assume that \( V \) is \( F'/F \)-fully faithful. We first prove that for every \( X \in \mathcal{A} \otimes_F F' \) and every \( Y \in \mathcal{A} \), the homomorphism

\[
V' : \text{Hom}_{\mathcal{A}'}(X, t(Y)) \to \text{Hom}_{\mathcal{B}}(V'(X), V(Y))
\]

is an isomorphism. With the help of Lemma 1.3.13 we choose a presentation

(2.3.2) \( t(X_1) \to t(X_0) \to X \to 0 \)

of \( X \). Applying \( \text{Hom}(-, t(Y)) \) to this sequence, and applying \( \text{Hom}(-, V(Y)) \) to the right exact sequence which is the image of (2.3.2) under \( V' \), we obtain a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}(X, t(Y)) & \longrightarrow & \text{Hom}(t(X_0), t(Y)) & \longrightarrow & \text{Hom}(t(X_1), t(Y)) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Hom}(V'(X), V(Y)) & \longrightarrow & \text{Hom}(V(X_0), V(Y)) & \longrightarrow & \text{Hom}(V(X_1), V(Y)) & \longrightarrow & 0
\end{array}
\]
The two last vertical arrows are isomorphisms since both $t$ and $V$ are $F'/F$-fully faithful functors. By the Five Lemma, the first vertical arrow is an isomorphism, as claimed.

In general, consider $X$ and $Y$ in $\mathcal{A} \otimes_F F'$. The dual of a presentation of $Y^\vee$ gives us a copresentation

\[(2.3.3) \quad 0 \to Y \to t(Y^0) \to t(Y^1)\]

of $Y$. Applying $\text{Hom}(X, -)$ to this sequence, and applying $\text{Hom}(-, V'Y)$ to the left exact sequence which is the image of \[(2.3.3)\] under $V'$, we obtain a diagram

\[
\begin{array}{cccccc}
0 & \to & \text{Hom}(X, Y) & \to & \text{Hom}(X, t(Y^0)) & \to & \text{Hom}(X, t(Y^1)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Hom}(V'(X), V'(Y)) & \to & \text{Hom}(V'(X), V(Y^0)) & \to & \text{Hom}(V'(X), V(Y^1))
\end{array}
\]

By what we have already proven, the last two vertical arrows are isomorphisms, so by the Five Lemma so is the first, and we have shown that $V'$ is fully faithful.

\[\therefore\]

### 2.4 Induced Equivalences

#### 2.4.1 Theorem. Let $F'/F$ be a separable field extension, $\mathcal{F}$ a pre-Tannakian category over $F$, $\mathcal{F}'$ a pre-Tannakian category over $F'$ and consider an $F$-linear exact tensor functor $V : \mathcal{F} \to \mathcal{F}'$. Let $((V\mathcal{F})_\otimes)$ denote the strictly full pre-Tannakian subcategory of $\mathcal{F}'$ generated by the essential image of $V$.

If $V$ is $F'/F$-fully faithful and semisimple on objects, then the functor

\[V' : \mathcal{F} \otimes_F F' \to ((V\mathcal{F})_\otimes)\]

induced by $V$ is an equivalence of pre-Tannakian categories.

\[\therefore\]

### 3 Tannakian Categories

#### 3.1 Scalar Extension of Tannakian Categories

Let $F$ be a field.
### 3.1 Scalar Extension of Tannakian Categories

#### 3.1.1 Definition.

(a) Let $R$ be a commutative $F$-algebra. A fibre functor over $R$ of a pre-Tannakian category $\mathcal{T}$ over $F$ is a faithful $F$-linear exact tensor functor $\omega$ from $\mathcal{T}$ to the category of $R$-modules which has values in the rigid subcategory of finitely generated projective $R$-modules.

(b) A neutral fibre functor is a fibre functor over $F$ itself, it takes values in the category $\text{Vec}_F$ of finite-dimensional $F$-vector spaces.

(c) A Tannakian category over $F$ is a pre-Tannakian category for which there exists a fibre functor over some field extension $F'/F$. If there exists a neutral fibre functor, we say that $\mathcal{T}$ is neutral.

(d) A subcategory $S$ of a Tannakian category $\mathcal{T}$ is a strictly full Tannakian subcategory if it is a strictly full pre-Tannakian subcategory of $\mathcal{T}$, that is, if it is full and closed under direct sums, tensor products, duals and subquotients in $\mathcal{T}$.

We start by checking that our notion of scalar extension for abelian categories gives rise to a notion of scalar extension for Tannakian categories. In particular, Tannakian categories may be “neutralised”. Together with the following Theorem 3.1.7, we generalise [DeM82, Proposition 3.11] and substantiate [Mil92, Proposition A.12].

#### 3.1.2 Proposition.

Let $\mathcal{T}$ be a pre-Tannakian category over $F$, and consider a field extension $F'/F$. For every commutative $F'$-algebra $R'$ the restriction functor

$$
\left( \text{fibre functors on } \mathcal{T} \otimes_F F' \text{ over } R' \right) \xrightarrow{(-) \circ t} \left( \text{fibre functors on } \mathcal{T} \text{ over } R' \right)
$$

is an equivalence of categories.

**Proof.** The given functor $\text{res} := (-) \circ t$ maps fibre functors on $\mathcal{T} \otimes_F F'$ to fibre functors on $\mathcal{T}$ since $t$ is exact, $F$-linear, faithful by either Proposition 1.3.6 or Lemma 2.2.4 and a tensor functor by Theorem 2.1.4(b). Hence, $\text{res}$ is well-defined. It is fully faithful by Proposition 1.4.12 and Lemma 2.2.5.

To show that $\text{res}$ is essentially surjective, let $\omega$ be a fibre functor on $\mathcal{T}$ over a given $F'$-algebra $R'$. The $F$-linear right-exact functor $\omega'$ on $\mathcal{T} \otimes_F F'$ induced by Theorem 1.4.1 fulfills $\text{res}(\omega') \cong \omega$ by item (a) of that theorem. Now $\omega'$ is exact by Lemma 2.2.3, faithful by Lemma 2.2.4 and a tensor functor by Proposition 2.1.6. A priori, $\omega'$ has values in the category of $R'$-modules. However, since $\mathcal{T} \otimes_F F'$ is rigid by Proposition 2.2.2, the essential image of $\omega'$ must consist of dualisable $R'$-modules (cf. the proof of Proposition 2.2.2). It is well known that a dualisable $R'$-module is finitely generated and projective, see [Del87]. Hence, $\omega$ is a fibre functor on $\mathcal{T} \otimes_F F'$ over $R'$, and we are done.
3.1.3 Theorem. Let \( \mathcal{T} \) be a Tannakian category over \( F \), and consider a field extension \( F'/F \).

(a) \( \mathcal{T} \otimes_F F' \) is a Tannakian category over \( F' \).

(b) If \( \mathcal{T} \) has a fibre functor over \( F' \), then \( \mathcal{T} \otimes_F F' \) is neutral.

Proof. (a): By Proposition \[2.2.2\] we know that \( \mathcal{T} \otimes_F F' \) is a pre-Tannakian category over \( F' \). By assumption, there exists a fibre functor of \( \mathcal{T} \) over some field extension \( L/F \). Choose a field extension \( L'/F \) containing both \( F' \) and \( L \). Then \((L' \otimes_L -) \circ \omega\) is a fibre functor of \( \mathcal{T} \) over \( L' \). By Proposition \[3.1.2\] it extends to a fibre functor of \( \mathcal{T} \otimes_F F' \) over \( L' \).

(b): In this case, we may choose \( L' = L = F' \). \( \therefore \)

The starting point of Tannakian duality is the idea that the category of finite-dimensional representations of a linear algebraic group is in a certain sense dual to the group itself. Reversing this point of view, we wish to associate a group to a Tannakian category.

3.1.4 Definition. (a) The algebraic monodromy group of the Tannakian category \( \mathcal{T} \) over \( F \) with respect to a given fibre functor \( \omega \) over a field extension \( F'/F \) is the functor

\[
G_\omega(\mathcal{T}) : (\text{(Commutative } F'-\text{Algebras)}) \longrightarrow (\text{(Groups)})
\]

mapping a commutative \( F' \)-algebra \( R' \) to the group \( \text{Aut}^\otimes (R' \otimes_{F'} \omega(-)) \) of tensor automorphisms of the functor \( R' \otimes_{F'} \omega(-) \) which maps an object \( X \) of \( \mathcal{T} \) to the \( R' \)-module \( R' \otimes_{F'} \omega(X) \).

(b) The algebraic monodromy group \( G_\omega(X) \) of an object \( X \) of \( \mathcal{T} \) with respect to \( \omega \) is the algebraic monodromy group of the strictly full Tannakian subcategory \( (\{X\})_\otimes \) of \( \mathcal{T} \) that \( X \) generates, with respect to the restriction of \( \omega \).

The theory of Tannakian categories comes in two flavours, neutral and non-neutral. The former is relatively simple to understand, whereas the latter is more advanced and more closely connected to groupoids than groups. The non-neutral theory is developed in [Del90]. Nevertheless, the aim of this section is to understand part of the non-neutral theory, using only our results on scalar extension and the neutral theory which we recall in the next two theorems.

3.1.5 Theorem. Let \( G \) be an algebraic group over \( F \). Then \( G \) represents the monodromy group of \( \text{Rep}_F(G) \) with respect to the forgetful functor \( \text{Rep}_F(G) \to \text{Vec}_F \).

Proof. [DeM82, Theorem 2.8]. \( \therefore \)
3.1.6 Theorem. Let \( \mathcal{T} \) be a neutral Tannakian category over \( F \), and fix a neutral fibre functor \( \omega \).

(a) \( G_\omega(\mathcal{T}) \) is representable by an affine group scheme over \( F \).

(b) \( G_\omega(\mathcal{T}) \) is of finite type if and only if \( \mathcal{T} \) is finitely generated.

(c) If \( \mathcal{T} \) is finitely generated, then \( \omega(X) \) is a faithful representation of \( G_\omega(\mathcal{T}) \) for every \( X \in \mathcal{T} \) with \( \mathcal{T} = ((X)_\otimes) \).

(d) \( \omega \) induces an equivalence of categories \( \mathcal{T} \rightarrow \text{Rep}_F(G_\omega(\mathcal{T})) \).

Proof. \([\text{Saa72}] \) or \([\text{DeM82}, \text{Theorem 2.11}] \).

We end this subsection with a version of Theorem 3.1.6 for non-neutral Tannakian categories and a consequence of Theorem 2.4.1.

3.1.7 Theorem. Let \( \mathcal{T} \) be a Tannakian category over \( F \), and fix a fibre functor \( \omega \) over a field extension \( F'/F \).

(a) \( G_\omega(\mathcal{T}) \) is an affine group scheme over \( F' \).

(b) \( G_\omega(\mathcal{T}) \) is of finite type if and only if \( \mathcal{T} \) is finitely generated.

(c) If \( \mathcal{T} \) is finitely generated, then \( \omega(X) \) is a faithful representation of \( G_\omega(\mathcal{T}) \) for every \( X \in \mathcal{T} \) with \( \mathcal{T} = ((X)_\otimes) \).

(d) \( \omega \) induces an equivalence of categories \( \mathcal{T} \otimes_F F' \rightarrow \text{Rep}_{F'}(G_\omega(\mathcal{T})) \).

Proof. By Theorem 3.1.3 \( \mathcal{T} \otimes_F F' \) is a Tannakian category, and the functor \( \omega' \) induced by \( \omega \) is a neutral fibre functor. Therefore, Theorem 3.1.6 applies to the pair \( (\mathcal{T} \otimes_F F', \omega') \).

It remains to show that \( G_\omega(\mathcal{T}) \) and \( G_{\omega'}(\mathcal{T} \otimes_F F') \) coincide. But given an \( F'\)-algebra \( R' \), Proposition 3.1.2 shows that the natural homomorphism

\[
\text{Aut}^\otimes \left( (R' \otimes_{F'} -) \circ \omega' \right) \rightarrow \text{Aut}^\otimes \left( (R' \otimes_{F'} -) \circ \omega \right)
\]

is an isomorphism, so we are done.

3.1.8 Proposition. Let \( F'/F \) be a separable field extension, \( F''/F' \) any field extension, \( \mathcal{T} \) a Tannakian category over \( F \), \( \mathcal{T}' \) a Tannakian category over \( F' \) with fibre functor \( \omega \) over \( F'' \) and consider an \( F \)-linear exact tensor functor \( V : \mathcal{T} \rightarrow \mathcal{T}' \). Assume that \( V \) is \( F'/F \)-fully faithful and semisimple on objects.

For every object \( X \) of \( \mathcal{T} \), there exists a canonical isomorphism of algebraic monodromy groups

\[
G_{\omega \circ V}(X) \cong G_\omega(V(X)).
\]
Proof. By Theorems 3.1.7(d) and 3.1.5, the monodromy group $G_{\omega \circ V}(X)$ coincides with the monodromy group of $t(X)$ as calculated in $\mathcal{T} \otimes_{F} F'$ with respect to $(\omega \circ V)':$

$$G_{\omega \circ V}(X) \xrightarrow{\cong} G_{(\omega \circ V)'}(t(X)).$$

Applying Theorem 2.4.1 to the Tannakian categories $(\mathcal{X}) \otimes_{F} F'$ and $(\mathcal{V}(X)) \otimes_{F} F'$, we obtain an equivalence of categories

$$(\mathcal{X}) \otimes_{F} F' \xrightarrow{\cong} (\mathcal{V}(X)) \otimes_{F} F'.$$

Clearly, this implies the existence of an isomorphism

$$G_{(\omega \circ V)'}(t(X)) \xrightarrow{\cong} G_{\omega}(V(X)),$$

which is what was left to prove.

### 3.2 Reductivity of Monodromy Groups

Let $F$ be a field.

**3.2.1 Proposition.** Let $V$ be a finite-dimensional $F$-vector space, and consider a closed algebraic subgroup $G \subset \text{GL}(V)$. If $V$ is semisimple as a representation of $G$, and $\text{End}_G(V)$ is a separable $F$-algebra, then the identity component $G^0$ is a reductive group.

**Proof.** Let $\overline{F}$ be an algebraic closure of $F$. Since $\text{End}_G(V)$ is both semisimple and separable over $F$, the $\overline{F}$-algebra $\overline{F} \otimes_{F} E$ is semisimple by Proposition 1.1.7(b). By the same assumptions, $\overline{F} \otimes_{F} V$ is a semisimple representation of $G_{\overline{F}}$, the base change of $G$ to $\overline{F}$, using Proposition 1.5.2 applied to Example 1.2.6(b). Therefore we may assume that $F$ is algebraically closed.

Let $U$ be the unipotent radical of $G$, and let $V^U \subset V$ denote the subvector space consisting of those elements fixed (pointwise) by $U$. Since $U$ is normal in $G$, $V^U$ is a $G$-stable subspace of $V$. We claim that $V^U = V$. If not, since $V$ is semisimple, we may write $V = V^U \oplus V'$ for some $G$-stable complement $V'$ of $V^U$. Since $U$ operates unipotently on $V'$, it follows that $(V')^U \neq 0$, which is a contradiction to the definition of $V'$ as a complement of $V^U$. Therefore $V^U = V$. Since $G$ operates faithfully on $V$, it follows that $U = 1$, which means that $G^0$ is reductive.

**3.2.2 Corollary.** Let $\mathcal{T}$ be a Tannakian category over $F$, fix a fibre functor $\omega$ over some field extension $F'/F$, and choose an object $X$ of $\mathcal{T}$. If $X$ is semisimple and $\text{End}(X)$ is a separable $F$-algebra, then the identity component of $G_{\omega}(X)$ is a reductive group over $F'$.

**Proof.** The vector space $\omega(X)$ is a faithful representation of $G_{\omega}(X)$ by Proposition 3.1.7(c). Therefore, Proposition 3.2.1 applies to it, and we are done.

\qed
3.3 An Application: Representation-Valued Fibre Functors

We close this article with an application of our results to “representation-valued” fibre functors. Let $\Gamma$ be a profinite group. Let $F$ be a global field, $F' \supset F$ a local field arising by completing $F$ at some place. It is well-known that the field extension $F'/F$ is separable. Let $\mathcal{T}$ be any Tannakian category over $F$, and let $\text{Rep}_{F'} \Gamma$ denote the category of finite-dimensional continuous representations of $\Gamma$ over $F'$.

We assume that we are given a faithful exact $F$-linear tensor functor

$$V : \mathcal{T} \rightarrow \text{Rep}_{F'} \Gamma,$$

a “representation-valued fibre functor”, which is additionally both $F'/F$-fully faithful and semisimple on objects. Examples are given by the rational Tate module functors on either the Tannakian category of pure Grothendieck motives generated by abelian varieties up to isogeny or the Tannakian category of Anderson $A$-motives up to isogeny.

For every object $X$ of $\mathcal{T}$, let $\Gamma(X)$ denote the image of $\Gamma$ in $\text{Aut}_{F'}(V(X))$, and let $G(X)$ denote the algebraic monodromy group of $X$ with respect to the fibre functor on $\mathcal{T}$ arising by postcomposing $V$ with the forgetful functor $U : \text{Rep}_{F'} \Gamma \rightarrow \text{Vec}_{F'}$.

There exists a unique reduced algebraic subgroup of $\text{GL}(V(X))$ which has as set of $F'$-rational points the Zariski closure of $\Gamma$ in $\text{GL}(V(X))(F')$, and it is natural to hope that this group coincides with $G(X)$:

3.3.1 Theorem. (a) The natural homomorphism $\Gamma(X) \rightarrow G(X)(F')$ is injective and has Zariski-dense image.

(b) If $X$ is semisimple and $\text{End}_\mathcal{T}(X)$ is a separable $F$-algebra, then $G(X)^\circ$, the identity component of $G(X)$, is a reductive group.

We need some preparations.

3.3.2 Lemma. Let $V$ be a finite-dimensional $F'$-vector space, and consider an algebraic subgroup $G \subset \text{GL}(V)$ together with a Zariski-dense subgroup $\Gamma \subset G(F')$ of its $F'$-rational points. Then:

(a) A linear subspace $W \subset V$ is $G$-stable if and only if it is $\Gamma$-stable.

(b) We have $\text{End}_G(V) = \text{End}_\Gamma(V)$.

Proof. (a): Given a linear subspace $W \subset V$ the stabiliser $H := \text{Stab}_G(W)$ is an algebraic subgroup of $G$. If $W$ is $G$-stable, then the $F'$-valued points of $H = G$ contain $\Gamma$, so $W$ is $\Gamma$-stable.

Conversely, if $W$ is $\Gamma$-stable, then $H(F')$ contains $\Gamma$. Since $\Gamma$ is dense in $G(F')$, this implies that $H = G$, and so $W$ is $G$-stable.

(b): We note that $\text{End}_G(V)$ is the maximal $G$-stable subspace of $V^\vee \otimes V$ on which $G$ acts trivially, and similarly $\text{End}_\Gamma(V)$ is the maximal $\Gamma$-stable
subspace on which $\Gamma$ acts trivially. By a similar argument as in (a), these two spaces must coincide.

**3.3.3 Proposition.** Let $V$ be a finite-dimensional $F'$-vector space, consider a subgroup $\Gamma \subset \text{GL}(V)(F')$ with associated algebraic group $G := \Gamma_{\text{Zar}} \subset \text{GL}(V)$. Let $V_{\text{cont}}$ represent $V$ considered as a continuous representation of $\Gamma$ over $F'$, and let $V_{\text{alg}}$ represent $V$ considered as a representation of $G$ over $F'$.

(a) The natural functor

$$
(V_{\text{alg}}) \otimes - \to (V_{\text{cont}}) \otimes
$$

between the strictly full Tannakian subcategories of $\text{Rep}_{F'} G$ and of $\text{Rep}_{F'} \Gamma$ generated by $V_{\text{alg}}$ and $V_{\text{cont}}$, respectively, is an equivalence of Tannakian categories.

(b) In particular, $G$ is the algebraic monodromy group of $V_{\text{cont}}$.

**Proof.** (a): Any object of $(V_{\text{alg}}) \otimes$ yields a continuous representation of $\Gamma$, and this gives rise to the desired exact $F'$-linear tensor functor; let us denote it by $C$. We wish to employ Theorem 2.4.1 to conclude that $C$ is an equivalence of Tannakian categories, so we must show that $C$ is fully faithful and semisimple, let us do this.

Consider $W \in (V_{\text{alg}}) \otimes$, let $G_W$ denote the image of $G$ in $\text{GL}(W)$ and let $\Gamma_W$ denote the image of $\Gamma$ in $G_W(F')$. By continuity, $\Gamma_W$ is dense in $G_W(F')$, so Lemma 3.3.2(b) shows that $\text{End}_G(W) = \text{End}_{F'}(CW)$. Since this is true for all $W$, we conclude that $C$ is fully faithful. If $W$ is simple, Lemma 3.3.2(a) shows that $CW$ is simple. In particular, $C$ is semisimple on objects.

(b): It is well-known (cf. [Wat79, Theorem 3.5]) that $(V_{\text{alg}}) \otimes$ is equivalent to $\text{Rep}_{F'}(G)$. Thus, by Theorem 3.1.5 $G$ is the algebraic monodromy group of $V_{\text{alg}}$, and so by (a) $G$ is also the algebraic monodromy group of $V_{\text{cont}}$.

**Proof of Theorem 3.3.1** (a): By Corollary 3.1.8 we have

$$
G_U(V(X)) \cong G_{U \circ V}(X).
$$

By Proposition 3.3.3, $\Gamma(X) \subset G_{U \circ V}(X)(F')$ is Zariski dense.

(b): By our assumptions or Theorem 3.1.7(c), $G(X)$ is a closed algebraic subgroup of $\text{GL}(V(X))$, and $V(X)$ is semisimple as a representation of $G(X)$, since $X$ is semisimple and $V$ is semisimple on objects. Since $V$ is $F'/F$-fully faithful, $\text{End}(V(X)) = F' \otimes_F \text{End}(X)$, which is a separable $F'$-algebra since $\text{End}(X)$ is a separable $F$-algebra. Therefore, the assumptions of Theorem 3.2.1 hold true, and $G(X)^c$ is a reductive group. 

\[.\]
References

[Bou58] N. Bourbaki, Élements de mathématique, Algèbre, Chapitre 8: Modules et anneaux semi-semiples, Hermann, 1958.

[DeM82] P. Deligne, J. S. Milne: Tannakian categories, in: Hodge cycles, motives, and Shimura varieties, LNM 900, Springer-Verlag, 1982.

[Del87] P. Deligne: Le groupe fondamental de la droite projective moins trois points, in: Galois groups over $\mathbb{Q}$, 79–297, Springer-Verlag, 1987.

[Del90] P. Deligne: Catégories tannakiennes, in: The Grothendieck Festschrift, Vol. II, 111–195, Birkhäuser, 1990.

[Fal83] G. Faltings: Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, Invent. math. 73 (1983), 349-366.

[Jac90] N. Jacobson: Basic Algebra II, 2nd ed., W. H. Freeman and company, 1990.

[Jan92] U. Jannsen: Motives, numerical equivalence, and semi-simplicity, Invent. Math. 107, no. 3, (1992), 447–452.

[KaS06] M. Kashiwara, P. Shapira: Categories and sheaves, GDM 332, Springer-Verlag, 2006.

[Lam99] T. Y. Lam: Lectures on modules and rings, GTM 189, Springer-Verlag, 1999.

[Mil92] J. S. Milne: The points on a Shimura variety modulo a prime of good reduction, in: The Zeta Function of Picard Modular Surfaces, 151–253, Publ. Centre de Rech. Math., 1992.

[Saa72] N. Saavedra Rivano: Catégories tannakiennes, LNM 265, Springer-Verlag, 1972.

[Sta08] N. Stalder: The semisimplicity conjecture for $A$-motives, preprint, 2008.

[Wat79] W. C. Waterhouse: Introduction to affine group schemes, GTM 66, Springer-Verlag, 1979.