About solution of multipoint boundary problem of static analysis of deep beam with the use of combined application of finite element method and discrete-continual finite element method. Part 1: formulation of the problem and general principles of approximation

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Abstract. This paper is devoted to formulation and general principles of approximation of multipoint boundary problem of static analysis of deep beam with the use of combined application of finite element method (FEM) discrete-continual finite element method (DCFEM). The field of application of DCFEM comprises structures with regular physical and geometrical parameters in some dimension (“basic” dimension). DCFEM presupposes finite element approximation for non-basic dimension while in the basic dimension problem remains continual. DCFEM is based on analytical solutions of resulting multipoint boundary problems for systems of ordinary differential equations with piecewise-constant coefficients.

1 Formulation of the problem

Let’s consider multipoint boundary problem of static analysis of deep beam (Fig. 1) [7, 10-12,14,15,17-23,25,27-32]. Some elements of notation system is presented at Fig. 1. Corresponding design model is two-dimensional theory of elasticity [9].

Let’s Ω be domain occupied by structure, Ω = \{(x_1,x_2): 0 < x_1 < l_1, 0 < x_2 < l_2 \} or

\[ Ω = \bigcup_{k=1}^{n_0-1} Ω_k, Ω_k = \{(x_1,x_2): 0 < x_1 < l_{1,k}, x_{2,k}^b < x_2 < x_{2,k+1}^b \}; \] (1)

where \( x_1, x_2 \) are coordinates (\( x_2 \) corresponds to basic dimension); \( x_{2,k}^b, k = 1, 2, ..., n_b \) are coordinates of corresponding boundary points (cross-sections) along basic dimension (for multipoint boundary problems \( n_b > 2 \)); \( Ω_k, k = 1, 2, ..., n_b - 1 \) are subdomains of \( Ω \).

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Let $l_{i,k} = \text{const}$ for $x_{2,k}^b < x_2 < x_{2,k+1}^b$. However in general case $l_i = l_i(x_2) \neq \text{const}$ (parameter $l_i$ can be piecewise-constant).

Let $\omega_k$, $k = 1, 2, \ldots, n_b - 1$ are extended subdomains, bordering subdomains $\Omega_i \subset \omega_i$, $k = 1, 2, \ldots, n_b - 1$,

$$\omega = \bigcup_{k=1}^{n_b-1} \omega_k, \quad \omega_k = \{(x_1, x_2) : 0 < x_1 < l_i, \ x_{2,k}^b < x_2 < x_{2,k+1}^b\} \supset \Omega_i;$$

(2)

2 General principles of domain approximation

Let physical and geometrical parameters within one group of subdomains from (2) is constant in some dimension (“basic” dimension). Thus, it is recommended to use DCFEM [1-5,30,35] for approximation of these subdomains (discrete-continual design model is introduced). Let physical and geometrical parameters within other group of subdomains from (2) is arbitrary varying. FEM [6,8,9,13,24,26,33,34] can be effectively used for approximation here. Thus, combined application of DCFEM and FEM is advisable.

So-called approximation parameter $\rho_i$ can be introduced in accordance with the following rule: $\rho_i = 1$ for approximation with the use of FEM; $\rho_i = 2$ for approximation with the use of DCFEM.

3 About numbering of subdomains

Various approaches can be used for numbering of subdomains.

The first approach provides separate numbering of subdomains with different types of approximation:

$$k_i = k_i(k) = \sum_{\omega_i} |\rho_\omega - 2|; \quad k_2 = k_2(k) = \sum_{\omega_i} |\rho_\omega - 1|,$$

(3)
where $k$ is initial number of subdomain $\omega_k$; $k_i = k_i(k)$ is the corresponding number of subdomain with approximation with the use of FEM; $k_i = k_i(k)$ is the corresponding number of subdomain with approximation with the use of DCFEM.

Various approaches can be used for numbering of subdomains. We can certainly construct inverse relationships by tabulating the results of calculations using formulas (3).

Thus we can rewrite (2) in the following form:

$$\omega = \bigcup_{k=1}^{N_{fe}} \omega_{fe}^{k} + \bigcup_{k=2}^{N_{fe}} \omega_{dc}^{k}.$$  \hspace{1cm} (5)

Besides, the following relations are valid:

$$N_{fe} = \sum_{x=1}^{w-1} |\rho_x - 2|; \hspace{0.5cm} N_{dc} = \sum_{x=1}^{w-1} |\rho_x - 1|; \hspace{0.5cm} N_{fe} + N_{dc} = n - 1,$$

where $\omega_{fe}^{k}$, $k_i = 1, 2, \ldots, N_{fe}$ are subdomain with approximation with the use of FEM; $\omega_{dc}^{k}$, $k_i = 1, 2, \ldots, N_{dc}$ are subdomain with approximation with the use of DCFEM.

The second approach, on the contrary, is based on a linked numbering of subdomains with different types of approximation. Formula (2) can be used, where

$$\omega_i = \begin{cases} \omega_{fe}^{k}, & \text{if} \quad \rho_i = 1 \\
\omega_{dc}^{k}, & \text{if} \quad \rho_i = 2 \end{cases}$$  \hspace{1cm} (7)

while formulas (3) and (4) are not required. The second approach is used in this paper.

4 About numbering of finite elements and discrete-continual finite elements

Let’s consider arbitrary subdomain $\omega_{fe}^{k}$. We can introduce notation

$$l_{2,k}^{fe} = x_{2,k+1}^{fe} - x_{2,k}^{fe} \quad \text{if} \quad \rho_i = 1.$$  \hspace{1cm} (8)

Let’s $x_{1,i}^{fe}$, $i = 1, 2, \ldots, N_{1,i}^{fe}$ and $x_{2,j}^{fe}$, $j = 1, 2, \ldots, N_{2,j}^{fe}$ are coordinates coordinates (along $x_1$ and $x_2$) of nodes of finite elements, which are used for approximation of domain $\omega_{fe}^{k}$; $(N_{1,i}^{fe} - 1)$ and $(N_{2,j}^{fe} - 1)$ are numbers of finite elements along coordinates $x_1$ and $x_2$, which are used for approximation of $\omega_{fe}^{k}$. Three-index system is used for numbering of finite elements, which are used for approximation of $\omega_{fe}^{k}$. Typical number of has the form $(k,i,j)$, where $k$ is the number of subdomain with approximation with the use of FEM, $i$ and $j$ are numbers of elements (along $x_1$ and $x_2$); $\omega_{fe}^{k,i,j}$ is corresponding finite element.

Let’s consider arbitrary subdomain $\omega_{dc}^{k}$. We can introduce notation
\[ l^\text{de}_{2,k} = x^b_{2,k+1} - x^b_{2,k} \quad \text{if} \quad \rho_k = 2 . \] (8)

Let’s \( x^{de}_{i,j} \), \( i = 1, 2, \ldots, N^{de}_{1,k} \) are coordinates (along \( x_i \)) of nodes (nodal lines) of discrete-continual finite elements, which are used for approximation of domain \( \Omega^e_k \); \( (N^{de}_{1,k} - 1) \) is the number of discrete-continual finite elements, which are used for approximation of \( \Omega^e_k \). Two-index notation system is used for numbering of discrete-continual finite elements, which are used for approximation of \( \Omega^e_k \). Typical number of has the form \((k,i)\), where \( k \) is the number of subdomain, \( i \) is the number of element (along \( x_i \)).

We can introduce notation

\[
N_{1,k} = \begin{cases} 
N^{\text{de}}_{1,k} & \text{if } \rho_k = 1; \\
N_{1,k} & \text{if } \rho_k = 2; \\
N_{2,k} = N^{\text{de}}_{2,k} & \text{if } \rho_k = 1; \\
N_{2,k} & \text{if } \rho_k = 2. 
\end{cases} \] (9)

\[
x_{1,k,j} = \begin{cases} 
x^{\text{de}}_{1,k,j} & \text{if } \rho_k = 1; \\
x^{\text{de}}_{1,k,j} & \text{if } \rho_k = 2; \\
x_{2,k,j} = x^{\text{de}}_{2,k,j} & \text{if } \rho_k = 1; \\
x_{2,k,j} & \text{if } \rho_k = 2. 
\end{cases} \] (10)

\[
l^e_{2,k} = \begin{cases} 
l^e_{2,k} & \text{if } \rho_k = 1; \\
l^\text{de}_{2,k} & \text{if } \rho_k = 2. 
\end{cases} \] (11)

It should be noted that in the simplest cases (such case in considered in the distinctive paper) discretization of structure is constant along \( x_1 \) throughout the domain (otherwise the mathematical constructions given below are substantially more complicated). We have

\[ N_{1,k} = N_i, \quad k = 1, 2, \ldots, n_k - 1; \quad x_{1,k,j} = x_{1,j}, \quad k = 1, 2, \ldots, n_k - 1, \quad i = 1, 2, \ldots, N_i. \] (12)

In should be noted that notation from [3-5,30] is also used in this paper.

5 Discrete (finite element) approximation model for subdomain

Let’s consider arbitrary subdomain \( \Omega^e_k \). Discrete (finite element) approximation model for the considering two-dimensional problems presupposes finite element approximation along \( x_1 \) and \( x_2 \). Thus extended subdomain \( \Omega^e_k \) is divided into finite elements,

\[
\Omega^e_k = \bigcup_{i=1}^{N_i-1} \bigcup_{j=1}^{N_{2,k}-1} \Omega^e_{i,j}, \quad \Omega^e_{i,j} = \{(x_1, x_2) : x_{1,j} < x_1 < x_{1,j+1}, \quad x_{2,j} < x_2 < x_{2,j+1}\}. \] (13)

Lame constants for finite element are defined by formulas:

\[
\overline{\lambda}_{i,j} = \theta_{i,j} \lambda; \quad \overline{\mu}_{i,j} = \theta_{i,j} \mu, \quad \text{where} \quad \theta_{i,j} = \begin{cases} 
1, & \Omega^e_{i,j} \subset \Omega_k; \\
0, & \Omega^e_{i,j} \not\subset \Omega_k; 
\end{cases} \] (14)

\( \theta_{i,j} \) is the characteristic function of element \( \Omega^e_{i,j} \).
Basic nodal unknowns are displacement components $u_1^{(k)}$, $u_2^{(k)}$ (superscript “(k)”) hereinafter corresponds to the number of considered subdomain i.e. $\omega_k = \omega_k^k$). Thus for node $(k, i, j)$ we have the following unknowns: $u_1^{(k,i,j)}$, $u_2^{(k,i,j)}$.

Bilinear approximation of unknowns is used within finite element (conventional plane rectangular 4-node finite element of two-dimensional problem of elasticity theory (Fig. 2)).

![Fig. 2. Finite element and its local coordinate system.](image)

Computing of partial derivatives of displacements, deformations and stresses within the finite element, nodal stresses and nodal deformations with allowance for averaging is described in [3-5,30].

As known, FEM is reduced to the solution of systems of $2N_iN_{2,\delta}$ linear algebraic equations:

$$K_k\bar{U}_k = \bar{R}_k,$$

where $\bar{U}_k$ is global vector of nodal unknowns (subscript “(k)" corresponds to the number of subdomain $\omega_k = \omega_k^k$),

$$\bar{U}_k = [(\bar{u}_n^{(k,1,1)})^T \ldots (\bar{u}_n^{(k,i,1)})^T \ldots (\bar{u}_n^{(k,K,1)})^T \ldots (\bar{u}_n^{(k,1,2)})^T \ldots (\bar{u}_n^{(k,i,2)})^T \ldots (\bar{u}_n^{(k,K,2)})^T \ldots (\bar{u}_n^{(k,1,3)})^T \ldots (\bar{u}_n^{(k,i,3)})^T \ldots (\bar{u}_n^{(k,K,3)})^T]^T$$

$$\bar{u}_n^{(k,i,j)} = [u_1^{(k,i,j)} \ldots u_2^{(k,i,j)}]^T, \quad i = 1, 2, \ldots, N_i, \quad j = 1, 2, \ldots, N_{2,\delta}.$$ (16)

$K_k$ is global stiffness matrix of order $2N_iN_{2,\delta}$; $\bar{R}_k$ is global right-side vector of order $2N_iN_{2,\delta}$ (global load vector).

6 Discrete-continual approximation model for subdomain

Let’s consider arbitrary subdomain $\omega_k^{dc}$. Discrete-continual approximation model is used for two-dimensional problems. It presupposes mesh approximation for non-basic dimension of extended domain (along $x_1$) while in the basic dimension (along $x_2$) problem remains continual. Thus subdomain $\omega_k$ is divided into discrete-continual finite elements

$$\omega_k^{dc} = \bigcup_{j=1}^{N_{2,\delta}} \omega_k^{dc,j}, \quad \omega_k^{dc,j} = \{(x_1, x_2) : \ x_{1,j} < x_1 < x_{1,j+1}, \ x_{2,j} < x_2 < x_{2,j+1}\}.$$ (18)
Lame constants for finite element are defined by formulas:

\[ \theta_{k,j} = \theta_{k,j}^{(k)}, \quad \overline{\theta}_{k,j} = \theta_{k,j}^{(k)}, \mathbf{\mu}, \quad \text{where} \quad \theta_{k,j} = \begin{cases} 1, & \omega_{k,j} \in \Omega_i; \\ 0, & \omega_{k,j} \not\in \Omega_i. \end{cases} \]  

(19)

\( \theta_{k,j} \) is the characteristic function of element \( \omega_{k,j}^{(k)} \).

Basic nodal unknown functions are displacement components \( u_i^{(k)}, u_2^{(k)} \) and their derivatives \( v_i^{(k)}, v_2^{(k)} \) with respect to \( x_2 \) (superscript “(k)” hereinafter corresponds to the number of considered subdomain i.e. \( \omega_k = \omega_i^{(k)} \)). Thus for node \((k,i)\) we have the following unknown functions: \( u_1^{(k,i)}, u_2^{(k,i)} \) and \( v_1^{(k,i)}, v_2^{(k,i)} \).

Linear approximation is used for unknown functions within discrete-continual finite element.

DCFEM is reduced at some stage to the solution of systems of \( 4N_i \) first-order ordinary differential equations:

\[ \overline{U}_k''(x_2) = A_k \overline{U}_k(x_2) + \overline{R}_k(x_2), \]  

(20)

where \( \overline{U}_k \) is global vector of nodal unknowns (subscript “(k)” corresponds to the number of subdomain \( \omega_k = \omega_i^{(k)} \)),

\[ \overline{U}_k = \overline{U}_k(x_2) = \begin{bmatrix} (\overline{u}_k) \ (\overline{v}_k) \end{bmatrix}^T; \]  

(21)

\[ \overline{u}_k = \overline{u}_k(x_2) = \begin{bmatrix} (\overline{u}_1^{(k,i)}) \ (\overline{u}_2^{(k,i)}) \ \ldots \ (\overline{u}_n^{(k,i)}) \end{bmatrix}^T; \]  

(22)

\[ \overline{v}_k = \overline{v}_k(x_2) = \begin{bmatrix} (\overline{v}_1^{(k,i)}) \ (\overline{v}_2^{(k,i)}) \ \ldots \ (\overline{v}_n^{(k,i)}) \end{bmatrix}^T; \]  

(23)

\[ \overline{u}_n^{(k,i)}(x_2) = \begin{bmatrix} u_1^{(k,i)} \ u_2^{(k,i)} \end{bmatrix}^T, \overline{v}_n^{(k,i)}(x_2) = \begin{bmatrix} v_1^{(k,i)} \ v_2^{(k,i)} \end{bmatrix}^T; \]  

(24)

\( A_k \) is global matrix of coefficients of order \( 4N_i \); \( \overline{R}_k(x_2) \) is the right-side vector of order \( 4N_i \). Correct analytical solution of (21) is defined by formula

\[ \overline{U}_k(x_2) = E_k(x_2) \overline{C}_k + \overline{S}_k(x_2); \]  

(25)

\[ E_k(x_2) = \varepsilon_k(x_2 - x_{2,k}^i) - \varepsilon_k(x_2 - x_{2,2,i}^i); \overline{S}_k(x_2) = \varepsilon_k(x_2) \ast \overline{R}_k(x_2); \]  

(26)

\( \varepsilon_k(x_2) \) is the fundamental matrix-function of system (20), which is constructed in the special form convenient for problems of structural mechanics \([1-5,35]; * \) is convolution notation; \( \overline{C}_k \) is the vector of constants of order \( 4N_i \).

7 Multilevel approximation model for domain

Passing to multilevel approximation for domain is described in \([3-5]\) for the simplest case of two-point boundary problem (Fig. 3). In a more general case, this transition is carried out in a similar way, although, of course, also with the use of more complicated formulas.
8 Software and verification samples

We should stress that all methods and algorithms considered in this paper have been realized in software. The main purpose of Analysis system CSASA2Dm (DCFEM + FEM) is semianalytical structural analysis (static structural analysis of deep beam within two-dimensional theory of elasticity), based on combined application of FEM and DCFEM. Programming environment is Microsoft Visual Studio 2013 Community and Intel Parallel Studio 2017XE (Fortran programming language [16]) with Intel MKL Library [8]. Software is designed for Microsoft Windows 8.1/10.

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