A Maximization Algorithm of Pseudo-convex Quadratic Functions on Closed Convex Sets in Euclidean Spaces

Youssef Jabri, Abdessamad Jaddar and Mourad Taj

ABSTRACT: We give an algorithm to find maxima of pseudo-convex quadratic functions on closed convex sets and show its convergence. Some computational results are given at the end.

Key Words: Pseudo-convexity, quadratic function, maxima, algorithm, computation.

Contents

1 Introduction 1

2 Properties of the function under study 2

3 Optimality Conditions and Algorithm 3

4 Numerical Simulations 5

1. Introduction

Our aim in this paper is the search for the maxima of vectorial pseudo-convex quadratic functions. The motivation behind the choice of these functions is mainly computation. We shall give an algorithm to find such maxima and give some computational results at the end. Many papers were devoted to the numerical search of minima and maxima of convex functions, we cite for example [1,8]. In this paper, we show that some results of Enkhbat and Ibaraki [1,8] given in a context of convex functions can be carried on to pseudo-convex functions.

We will give some necessary and sufficient conditions of optimality in the third section. We will also derive an algorithm to apply this program. We deal with the convergence of the algorithm in the fourth section. Some numerical results from problems that were treated in [1,2,7,8,9,10] are given at the end for illustration.

Consider a quadratic and pseudo-convex \( f: \mathbb{R}^n \to \mathbb{R} \). The problem we are interested in is:

\[
(P) \quad \left\{ \begin{array}{l}
\text{maximize } f(x), \\
\text{for } x \in C, \text{ a closed convex set}.
\end{array} \right.
\]

Recall that a differentiable function \( f \) is pseudo-convex (cf. [6]) if:

\[
\langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow f(y) \geq f(x), \quad \forall x, y \in C.
\]

Since \( f \) is quadratic, there is a square real matrix \( Q \) of order \( n \) and \( x, p \in \mathbb{R}^n \) such that:

\[
f(x) = \frac{1}{2} \langle Qx, x \rangle + \langle p, x \rangle.
\]

The derivative of \( f \) at \( x \) is \( \nabla f(x) = Qx + p \).

In this work, we do not suppose that \( Q \) is symmetric, nevertheless we can always transform our problem via a change of basis and a change of variable to the maximization of a function of the form \( f(x) = f(x_0) + \frac{1}{2} \langle Dy, y \rangle \) where \( D \) is a diagonal matrix and \( y \) that will be clarified below.

2010 Mathematics Subject Classification: 90C46, 90C25, 90C20, 90C26.
Submitted June 12, 2019. Published January 13, 2020.
2. Properties of the function under study

We begin by the following result.

**Proposition 2.1.** Let \( f(x) = \frac{1}{2}(Qx, x) + \langle p, x \rangle \), where \( Q \) is not necessarily a symmetric matrix. We can make a change of variable to obtain

\[
f(x) = f(x_0) + \frac{1}{2}(Dy, y),
\]

where \( D \) is a diagonal matrix and \( x_0 \) is a vector such that \( \nabla f(x_0) = 0 \).

**Proof.** Claim: Consider two matrices defined as follows:

\[
A = [a_{ij}]_{1 \leq i, j \leq n} \quad \text{and} \quad Q = [q_{ij}]_{1 \leq i, j \leq n},
\]

with \( a_{ii} = q_{ii} \) and \( \forall i \neq j, a_{ij} = \frac{1}{2}(q_{ij} + q_{ji}) \). Then, \( A \) is symmetric and \( \langle Qx, x \rangle = \langle Ax, x \rangle \).

Indeed, by construction of the matrix \( A \), \( a_{ij} = a_{ji} \). So \( A \) is symmetric. On the other hand,

\[
\langle Qx, x \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} x_i x_j
\]

\[
= \sum_{i=1}^{n} q_{ii} x_i^2 + \sum_{j=1}^{n} \sum_{i=1; j \neq i}^{n} (q_{ij} + q_{ji}) x_i x_j
\]

\[
= \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{j=1}^{n} \sum_{i=1; j \neq i}^{n} (a_{ji} + a_{ij}) x_i x_j
\]

\[
= \langle A, x \rangle
\]

For the second part, we refer to Best [4] who shows that if \( f(x) = \frac{1}{2} \langle Ax, x \rangle + \langle p, x \rangle \), where \( A \) is a symmetric matrix. We can make a change of variable and obtain

\[
f(x) = f(x_0) + \frac{1}{2}(Dy, y),
\]

where \( D \) is a diagonal matrix and \( x_0 \) is a vector such that \( \nabla f(x_0) = 0 \).

Indeed, \( A \) is real symmetric, so there is an orthogonal matrix \( S \) and a diagonal matrix \( D \) such that \( A = S D S^T \) and \( S^T = S^{-1} \). Also, \( Ax_0 + p = 0 \) implies that \( x_0 \) is a solution of a linear system. Consider the vector \( y \) such that \( x = Sy + x_0 \).

Therefore, whatever the real matrix \( Q \) that defines the function \( f(x) \), we can always find a symmetric matrix \( D \) such that

\[
f(x) = \frac{1}{2} \langle Qx, x \rangle + \langle p, x \rangle = f(x_0) + \frac{1}{2} \langle Dy, y \rangle
\]

That way we have an equivalent formulation of \( f \) with terms of the form \( y_i^2 \) only and no \( y_i y_j \).

**Characterization of pseudoconvex quadratic functions**

Denote by \( \nu(D) \) the number of negative eigenvalues in \( D \) and \( \pi(D) \) the number of positive eigenvalues.

Suppose in the sequel that \( D = (\lambda_i)_i \) is a diagonal matrix and \( \nu(D) = 1, \pi(D) = k - 1 \) and \( k \leq n \).

We sort the eigenvalues of the matrix \( D \) so that \( \lambda_1 \leq 0 \). Denote the sets:

\[
T^+_k = \left\{ x \in \mathbb{R}^k; \sum_{i=1}^{k} \lambda_i x_i^2 < 0; \ x_1 > 0 \right\} \quad \text{and} \quad T^-_k = \left\{ x \in \mathbb{R}^k; \sum_{i=1}^{k} \lambda_i x_i^2 < 0; \ x_1 < 0 \right\}
\]

\[
T^+ = \left\{ x \in \mathbb{R}^n; \ f(x) < 0; \ x_1 > 0 \right\} \quad \text{and} \quad T^- = \left\{ x \in \mathbb{R}^n; \ f(x) < 0; \ x_1 > 0 \right\}
\]
According to Greub [3], the following sets represent the solid cones:
\[
T^+ = \{ x \in \mathbb{R}^n; f(x) \leq 0; \ x_1 \geq 0 \} \text{ and } T^- = \{ x \in \mathbb{R}^n; f(x) \leq 0; \ x_1 \geq 0 \}
\]
And according to Ferland, [2], if the real diagonal matrix \( D \) satisfies \( \nu(D) = 1 \), then the quadratic form \( f(x) = \langle Dx, x \rangle \) is pseudoconvex on either of the sets \( T^+ \setminus N \) and \( T^- \setminus N \) defined by:
\[
T^+ \setminus N = \{ x \in \mathbb{R}^n; x \in T^+ \text{ and } Dx \neq 0 \} \text{ and } T^- \setminus N = \{ x \in \mathbb{R}^n; x \in T^- \text{ and } Dx \neq 0 \}
\]

**Proposition 2.2.** If a function \( f \) is quadratic pseudoconvex, then we can easily check that:

(i) \( \forall x, y \in C, \text{ if } \langle Ax, y - x \rangle \geq 0, \text{ then } f(y) \geq f(x) \)

(ii) \( \forall x, y \in C, \text{ if } f(y) < f(x), \text{ then } \langle Ax, y - x \rangle < 0. \)

(iii) When \( f \) is pseudoconvex, it is quasiconvex.

(iv) \( \forall x, y \in C, \text{ if } f(y) = f(x), \text{ then } \langle Ax, y - x \rangle \leq 0 \)

### 3. Optimality Conditions and Algorithm

We define a level set as follows:

\[
C_x = \{ y \in \mathbb{R}^n; f(y) = f(x) \}
\]

**Theorem 3.1** (Hassouni and Jaddar [5]). A vector \( x^* \) is a solution of (P), if and only if

\[
\forall y \in C_{x^*} \text{ and } \forall x \in C, \langle Ay, x - y \rangle \leq 0.
\]

For the construction of the algorithm to solve our problem, we define the following functions:

\[
Y(y) = \max_{y \in C_x} \langle Ay, x - y \rangle \quad \text{and} \quad X(x) = \max_{y \in C} Y(y).
\]

**Theorem 3.2.** If \( X(x^*) \leq 0 \), then \( x^* \) is a solution of (P).

**Proof.** By definition of \( X(x) \) and \( Y(y) \), \( \forall y \in C_x \):

\[
X(x) = \max_{y \in C} Y(y) \geq Y(y) = \max_{y \in C_x} \langle Ay, x - y \rangle \geq \langle Ay, x - y \rangle
\]

Hence, \( \forall y \in C_x, X(x) \geq \langle Ay, x - y \rangle \). So, for \( x^* \), we have

\[
\forall y \in C_{x^*}, 0 \geq X(x^*) \geq \langle Ay, x^* - y \rangle.
\]

By Theorem 3, we have that \( x^* \) is a solution of (P). \( \square \)

**Algorithm**

See Figure 1 below for our algorithm to find the maxima of \( f \).

**Lemma 3.3.** The sequence \( (f(x^{(k)}))_k \) is strictly increasing.

**Proof.** For the general case, we suppose that \( x^{(k)} \) is not a solution of the problem.

First, let’s prove that the sequence is increasing. Since \( x^{(k)} \) is not a solution, \( X(x^{(k)}) > 0 \). Hence

\[
\langle Ay^{(k)}, x^{(k+1)} - y^{(k)} \rangle > 0 \implies \langle Ay^{(k)}, x^{(k+1)} - y^{(k)} \rangle \geq 0
\]

By (i), \( f(x^{(k+1)}) \geq f(y^{(k)}) = f(x^{(k)}) \). So the sequence \( (f(x^{(k)}))_k \) is increasing.

Let’s prove now that the sequence is strictly increasing.
By absurdum, suppose that \( \exists k \) such that \( f(x^{(k+1)}) = f(x^{(k)}) \). Then, \( f(y^{(k)}) = f(x^{(k+1)}) \), and by (iv), \( \langle Ay^{(k)}, x^{(k+1)} \rangle \leq \langle Ay^{(k)}, y^{(k)} \rangle \), so

\[
\langle Ay^{(k)}, x^{(k+1)} - y^{(k)} \rangle \leq 0
\]

But, since \( x^{(k)} \) is not a solution of the problem and satisfies \( \langle Ay^{(k)}, x^{(k+1)} - y^{(k)} \rangle > 0 \). A contradiction.

\[\square\]

**Lemma 3.4.**

\[\exists L \in \mathbb{R} \text{ such that } \lim_{k \to \infty} f(x^{(k)}) = L\]

**Proof.** The sequence \( (f(x^{(k)}))_k \) is strictly increasing and bounded from above by \( f(x^*) \), where \( x^* \) is a solution of (P). So for all \( k = 1, 2, \ldots \), we have \( f(x^{(0)}) < f(x^{(k)}) \leq f(x^*) \), and the function \( f(x) \) is bounded. Since it’s also continuous, we deduce that:

There exists \( L \in \mathbb{R} \) such that \( f(x^{(k)}) = L \).

\[\square\]

**Lemma 3.5.**

\[\lim_{k \to \infty} X(x^{(k)}) = 0\]
**Proof.** We have

\[
X(x^{(k)}) = \langle Ay^{(k)}, x^{(k+1)} - y^{(k)} \rangle \\
= \langle Ay^{(k)}, x^{(k+1)} \rangle - 2f(y^{(k)}) \\
= \langle A(y^{(k)} - x^{(k+1)} + x^{(k+1)}), x^{(k+1)} \rangle - 2f(x^{(k)}) \\
= \langle A(y^{(k)} - x^{(k+1)}), x^{(k+1)} \rangle + \langle Ax^{(k+1)} - x^{(k+1)} \rangle - 2f(x^{(k)}) \\
= 2(f(x^{(k+1)}) - f(x^{(k)})) + \langle A(y^{(k)} - x^{(k+1)}), x^{(k+1)} \rangle
\]

By Lemma 3.3, \( f(x^{(k+1)}) > f(x^{(k)}) \). By (ii), \( \langle Ax^{(k+1)}, y^{(k)} - x^{(k+1)} \rangle < 0 \).
Hence, \( X(x^{(k)}) < 2(f(x^{(k+1)}) - f(x^{(k)})) \). Since \( X(x^{(k)}) = \max_{x \in C} Y(y^{(k)}) \geq \max_{y \in C_v(k)} \langle A, y, x^{(k)} - y \rangle \geq 0 \).

We get \( 0 \leq X(x^{(k)}) < 2(f(x^{(k+1)}) - f(x^{(k)})) \), which gives us \( \lim_{k \to \infty} X(x^{(k)}) = 0 \).

### 4. Numerical Simulations

Now, that we have an algorithm to find solutions of \((P)\), we will use it on the following set of problems:

\[
P_1 \quad f(x) = x_1^2 + x_2^2 + x_3^2 + (x_3 - x_4)^2 \rightarrow \max \\
\quad -2.3 \leq x_i \leq 2.7, \quad i = 1, 2, 3, 4 \tag{4.1}
\]

\[
f(x) = 4(x_1 - 1)^2 + 25(x_2 - 2)^2 \rightarrow \max \\
\quad 8.3x_1 + 20.5x_2 \leq 170.15 \\
\quad -7.5x_1 + 18x_2 \leq 135 \\
\quad -10.5x_1 + 7.7x_2 \leq 80.85 \\
\quad -3.7x_1 - 10.2x_2 \leq 37.74 \\
\quad -2.7x_1 - 13x_2 \leq 35.1 \\
\quad 4.5x_1 - 7x_2 \leq 31.5 \\
\quad -20 \leq x_1 \leq 20, \quad -20 \leq x_2 \leq 20 \tag{4.2}
\]

\[
P_3 \quad -(n - i + 1) \leq x_i \leq n + 0.5i \quad i = 1, 2, \ldots, n \tag{4.3}
\]

\[
f(x) = -x_1^2 + x_2^2 \\
\quad -x_1 - x_2 \leq -6 \\
\quad 0.4x_1 - x_2 \leq 1 \\
\quad -x_1 + x_2 \leq -2 \\
\quad x_1 + x_2 \leq 13 \\
\quad 0.5x_1 + x_2 \leq 8.5 	ag{4.4}
\]

\[
P_5 \quad f(x) = -0.5x_1^2 - 2x_1x_2 - 7x_1x_3 - 5x_1 \\
\quad i \leq x_i \leq n + 3i, \quad i = 1, 2, 3 \tag{4.5}
\]

\[
f(x) = -x_1^2 + \sum_{i > 1} \|x\|^2 \\
\quad i + 0.5 \leq x_i \leq n + 0.5i, \quad i = 1, 2, \ldots, n \tag{4.6}
\]

The results of the numerical simulations are as follows \((n)\) is the number of variables:

| Problem | n | \(x^{(0)}\) | \(f(x^{(0)})\) | \(x^{(*)}\) | \(f(x^{(*)})\) |
|---------|---|-------------|-----------------|-------------|-----------------|
| \(P_1\) | 4 | \(-2.29\) | 15.7323 | \((2.7; 2.7; 2.7; -2.3)^T\) | 46.87 |
| \(P_2\) | 2 | \((-7.21907; 0.65376\) | 315.5215 | \((0.97; 7.91)^T\) | 871.946 |
| \(P_3\) | 10 | \(0.01\) | 0.001 | \(x_1^* = 10.5, x_{i+1}^* = x_{i-1}^* + 0.5\) | 1646.25 |
| \( P_4 \) | 30 | \( x_1^{(0)} = 0.01 \) | 0.003 | \( x_1^* = 30.5 \), \( x_{i>1}^* = x_{i-1}^* + 0.5 \) | 43313.75 |
| \( P_3 \) | 70 | \( x_1^{(0)} = 0.01 \) | 0.007 | \( x_1^* = 70.5 \), \( x_{i>1}^* = x_{i-1}^* + 0.5 \) | 546148.75 |
| \( P_3 \) | 100 | \( x_1^{(0)} = 0.01 \) | 0.01 | \( x_1^* = 100.5 \), \( x_{i>1}^* = x_{i-1}^* + 0.5 \) | 1589587.5 |
| \( P_4 \) | 2 | \( x_1 = 4.01 \), \( x_2 = 1.01 \) | -15.6 | \( x_1 = 4 \), \( x_2 = 2 \) | -12.00 |
| \( P_5 \) | 3 | \( x_1^{(0)} = 6.01 \), \( x_{i>1}^{(0)} = x_{i-1}^{(0)} + 3 \) | -661.67 | \( x^* = (1, 2, 3)^T \) | -30.50 |
| \( P_6 \) | 5 | \( x_1^{(0)} = 1.51 \), \( x_{i>1}^{(0)} = x_{i-1}^{(0)} + 1 \) | 67.04 | \( x_1^* = 1.5 \), \( x_2^* = 6.0 \), \( x_{i>2}^* = x_{i-2}^* + 0.5 \) | 181.250 |
| \( P_6 \) | 20 | \( x_1^{(0)} = 1.51 \), \( x_{i>1}^{(0)} = x_{i-1}^{(0)} + 1 \) | 3084.84 | \( x_1^* = 1.5 \), \( x_2^* = 21.0 \), \( x_{i>2}^* = x_{i-2}^* + 0.5 \) | 12495.0 |

**Acknowledgments**

We thank the referee for the valuable comments and suggestions.

**References**

1. Enkhbat R. and Ibaraki T., On the maximization and minimization of a quasiconvex function. *J. Nonlinear Convex Anal.*, 4, no. 1, 43–76, (2003).
2. Ferland J.A., Maximal domains of quasi-convexity and pseudo-convexity for quadratic functions. *Math. Programming*, 3, 178-192, (1972).
3. Greub W. H., *Linear Algebra. Fourth edition*. Graduate Texts in Mathematics, No. 23., Springer-Verlag, New York-Berlin, (1975).
4. Best, Michael J., *Quadratic programming with computer programs*. Advances in Applied Mathematics. CRC Press, Boca Raton, FL, (2017).
5. Hassouni A. and Jaddar A., On generalized monotone multifunctions with applications to optimality conditions in generalized convex programming. *J. Inequal. Pure Appl. Math.*, 4, no. 4, Article 67, 11 pp., (2003).
6. Mangasarian O.L., Pseudo-convex functions. *J. Soc. Induct. Appl. Math. Ser. A Control*, 3, 281-290, (1965).
7. Martos B., Quadratic programming with a quasiconvex objective function. *Operations Res.*, 19, 87-97, (1971).
8. Rentsen E., An algorithm for maximizing a convex function over a simple set. *J. Global Optim.*, 8, no. 4, 379-391, (1996).
9. Schaible S., Second-order characterizations of pseudoconvex quadratic functions. *J. Optim. Theory Appl.*, 21, no. 1, 15-26, (1977).
10. Schaible S., Quasiconvex, pseudoconvex, and strictly pseudoconvex quadratic functions. *J. Optim. Theory Appl.*, 35, no. 3, 303-338, (1981).
11. Press W.H.; Teukolsky S.A.; Vetterling W.T. and Flannery B.P., *Numerical recipes in Fortran 77. The art of scientific computing, second edition*. Cambridge University Press, Cambridge, (1996).