Statistical origin and properties of kappa distributions

George Livadiotis
Southwest Research Institute, USA
glivadiotis@swri.edu

Abstract. Classical particle systems reside at thermal equilibrium with their velocity distribution function stabilized into a Maxwell distribution. On the contrary, collisionless and correlated particle systems, such as space and astrophysical plasmas, are characterized by a non-Maxwellian behavior, typically described by kappa distributions, or combinations thereof. Empirical kappa distributions have become increasingly widespread across space and plasma physics. A breakthrough in the field came with the connection of kappa distributions to non-extensive statistical mechanics. Understanding the statistical origin of kappa distributions was the cornerstone of further theoretical developments and applications, namely, (i) the concept of temperature; (ii) the physical meaning of the kappa index; (iii) the $N$-particle description of kappa distributions; and the (iv) the generalization to phase-space kappa distribution of a Hamiltonian with non-zero potential.

1. Introduction

Classical particle systems are de facto characterized by the absence of correlations among the individual particle energies. The statistical behavior of these systems has been successfully described by the Boltzmann-Gibbs (BG) statistical mechanics and the Maxwellian distribution of velocities. Space plasmas are collisionless and correlated particle systems, characterized by stationary states out of thermal equilibrium that exhibit a non-Maxwellian behavior; this is typically described by kappa distributions and combinations thereof.

The kappa distributions have been used to describe numerous space plasma populations. Single types of kappa distributions are usually sufficient to satisfactorily model these plasma populations (e.g., [1-3]). Occasionally, more complicated models of kappa distributions are employed (e.g., [4-6]).

Among many others, some examples of space plasma applications are the following: (i) the inner heliosphere, including solar wind (e.g., [4, 7-18], solar spectra (e.g., [19,20]), solar corona (e.g., [21-24]), solar energetic particles (e.g., [25,26]), corotating interaction regions (e.g., [27]), and solar flares related (e.g., [28-31]); (ii) the planetary magnetospheres, including magnetosheath (e.g., [32,33]), near magnetopause (e.g., [34]), magnetotail (e.g., [35]), ring current (e.g., [36]), plasma sheet (e.g., [37-39]), ionosphere (e.g., [40]), magnetospheric substorms (e.g., [41]), magnetospheres of giant planets, such as Jovian (e.g., [42,43]), Saturnian (e.g., [2,44,45]), Uranian (e.g., [46]), Neptunian (e.g., [47]), magnetospheres of planetary moons, such as Io (e.g., [48]) and Enceladus (e.g., [49]), or cometary magnetospheres (e.g., [50,51]); (iii) the outer heliosphere and the inner heliosheath (e.g., [12-15,52-66]); (iv) beyond the heliosphere, including HII regions (e.g., [67]), planetary nebula (e.g., [68,69]), and supernova magnetospheres (e.g., [70]); or other space plasma-related analyses (e.g., [57,58,71-93]). (See also: [6,94,95], and references therein.)
Empirical kappa distributions have been introduced in mid-60’s by Binsack (1966) [96], Olbert (1968) [97], and Vasyliūnas (1968) [1], while the statistical origin of kappa distributions is determined within the framework of non-extensive statistical mechanics [94]. Their connection with statistical mechanics was shown and studied in detail in about half century later (see [98], and references therein). Non-extensive statistical mechanics is a consistent generalization of the classical statistical mechanics, which is based on a mono-parametric (q-index) entropic formula [99]. The theoretical q-exponential distribution, which results from the maximization of entropy in the canonical ensemble, and the empirical kappa distribution, have identical formulation under the transformation of their indices, $q = 1 + 1/\kappa$.

Having understood the physical underpinnings of the statistical mechanics connected with kappa distributions, it is straightforward to employ all the statistical well-grounded equations and capabilities for analyses that seek to apply kappa distributions in datasets, simulations, modeling, and theory of space plasmas. In Section 2 we describe the statistical origin of kappa distributions in detail. In Section 3 we show: (i) the concept of temperature; (ii) the physical meaning of the kappa index; (iii) the N-particle description of kappa distributions; and the (iv) the generalization to phase-space kappa distribution of a Hamiltonian with non-zero potential. Finally, Section 4 briefly summarizes the conclusions.

2. Statistical origin

2.1. Motivation

Classical particle systems reside in a special stationary state, the thermal equilibrium - the concept that any flow of heat (thermal conduction, thermal radiation) is in balance. At thermal equilibrium particles were described by the Boltzmann distribution of energies or the Maxwell distribution of velocities. This was the only known stationary state by the time BG statistical mechanics was built and applied. However, space plasmas have been observed in other stationary states. Since these are described by non-Maxwellian behavior and the Maxwell distribution was known as the only distribution interwoven with thermal equilibrium, the space plasma stationary states were called “out of thermal equilibrium”. They were also called “non-thermal”, but this is fairly unjustified, because these new types of stationary states are quite thermalized and surely described by a temperature.

Once we realized that non-equilibrium stationary states exist, the challenge is to understand their statistics and the involved statistical mechanics. They are characterized by a non-Maxwellian behavior, but the Maxwell distribution is still showing the right path to describe and express the new distributions and their statistics. Indeed, one way to understand this is via the variation of the temperature: If the particles are characterized by a Maxwellian distribution, where its temperature is not fixed but varies under certain statistics, it can lead to other stationary states (for details, see: [95], Chapter 6). Moreover, here we show some other way to understand how the Maxwell distribution and BG statistical mechanics can lead to the non-equilibrium stationary states observed in space plasmas.

Two basic functions are involved in the formulation of BG statistical mechanics: The exponential and logarithmic functions; the former describes the canonical distribution and the latter the entropic function. Both functions can be written using the Euler’s limit,

$$\exp(x) = \lim_{\kappa \to \infty} \left(1 - \frac{x}{\kappa}\right)^{-\kappa} \quad \Leftrightarrow \quad \ln(x) = \lim_{\kappa \to \infty} \left[\kappa \cdot \left(1 - x^{-1/\kappa}\right)\right], \quad (1)$$

Setting the definitions

$$\exp_\kappa(x) \equiv \left(1 - \frac{x}{\kappa}\right)^{-\kappa}, \quad \ln_\kappa(x) \equiv \kappa \cdot \left(1 - x^{-1/\kappa}\right), \quad (2)$$

known as deformed exponential and logarithmic functions, respectively, we have
Note that these functions are inverse to each other for any value of the kappa index.

The exponential function is used in the Boltzmann distribution, that is, the (canonical) distribution maximizes the BG entropy under the constraints of the canonical ensemble. This can be written as

\[ p_i \propto \exp \left( - \frac{\varepsilon_i}{k_B T} \right) = \lim_{\kappa \to \infty} \exp \left( - \frac{1}{\kappa} \cdot \frac{\varepsilon_i}{k_B T} \right) = \lim_{\kappa \to \infty} \left( 1 + \frac{1}{\kappa} \cdot \frac{\varepsilon_i}{k_B T} \right)^{-\kappa} . \]

On the other hand, the logarithmic function is used in the entropy formulation. This is expressed in terms of a discrete probability distribution \( \{ p_i \} \), which can be written as

\[ S = \sum_i p_i \cdot \ln(1/ p_i) = \lim_{\kappa \to \infty} \sum_i p_i \cdot \ln_\kappa(1/ p_i) = \lim_{\kappa \to \infty} \sum_i \left\{ \kappa \cdot \left[ p_i - p_i^{1/\kappa} \right] \right\} . \]

At this point, we note that while space physics community is used in the notation of the kappa index, the statistical mechanics community is using the \( q \)-index; both the indices are simply related:

\[ \kappa = \frac{1}{q-1} \iff q = 1 + \frac{1}{\kappa} . \]

Then, the canonical distribution and entropy are written in a limit form as follows:

\[ p_i \propto \exp \left( - \frac{\varepsilon_i}{k_B T} \right) = \lim_{\kappa \to \infty} \left( 1 + \frac{1}{\kappa} \cdot \frac{\varepsilon_i}{k_B T} \right)^{-\kappa} \]

\[ S = \sum_i p_i \cdot \ln(1/ p_i) = \lim_{\kappa \to \infty} \sum_i \left\{ \kappa \cdot \left[ p_i - p_i^{1/\kappa} \right] \right\} = \lim_{q \to 1} \sum_i \left( \frac{p_i - p_i^q}{q-1} \right) . \]

We define the kappa dependent distribution and entropy,

\[ p_{\kappa i} = \frac{1}{Z} \left( 1 + \frac{1}{\kappa} \cdot \frac{\varepsilon_i}{k_B T} \right)^{-\kappa} , \quad S_\kappa = \sum_i \left\{ \kappa \cdot \left[ p_i - p_i^{1/\kappa} \right] \right\} . \]

\( Z \) is an auxiliary partition function, e.g., see: [100]). Hence, we have that

\[ p_i = \lim_{\kappa \to \infty} p_{\kappa i} , \quad S = \lim_{\kappa \to \infty} S_\kappa . \]

Therefore, the special stationary state at thermal equilibrium can be described by a specific value of the kappa index of the distribution \( p_{\kappa i} \) and entropy \( S_\kappa \), that is, \( \kappa \to \infty \). However, what is the physical meaning of the states that correspond to finite values of the kappa index? As we will see, these finite kappa indices describe the non-equilibrium stationary states observed in space plasmas. The formed \( p_{\kappa i} \) and \( S_\kappa \) constitute the kappa distribution and its associated Tsallis entropy.

2.2. Entropy maximization

Consider a system described by a discrete energy distribution, that is, energy \( \varepsilon_1 \) with probability \( p_1 \), energy \( \varepsilon_2 \) with probability \( p_2 \), \ldots, energy \( \varepsilon_W \) with Probability \( p_W \). The classical BG entropy is given by

\[ S = S(p_1, p_2, \ldots, p_W) = - k_B \cdot \sum_{i=1}^{W} p_i \ln(p_i) , \]

while the Tsallis entropy is
\[ S = S(p_1, p_2, ..., p_W) = k_B \cdot \frac{1}{q-1} \left( 1 - \sum_{k=1}^{W} p_k^q \right) = k_B \cdot \frac{1}{q-1} \sum_{k=1}^{W} (p_k - p_k^q) . \]  

We expand the entropic form in terms of \((q-1)\):

\[ p_k^q = p_k \cdot e^{(q-1) \ln p_k} = p_k \cdot \left[ 1 + (q-1) \cdot \ln p_k + \frac{1}{2} (q-1)^2 \cdot \ln^2 p_k + O((q-1)^3) \right] \]

\[ \Rightarrow \sum_{k=1}^{W} p_k^q \approx 1 + (q-1) \sum_{k=1}^{W} p_k \ln p_k + \frac{1}{2} (q-1)^2 \sum_{k=1}^{W} p_k \ln^2 p_k + O((q-1)^3) \]

\[ \Rightarrow \frac{1}{q-1} \cdot \left( 1 - \sum_{k=1}^{W} p_k^q \right) \approx -\sum_{k=1}^{W} p_k \ln p_k - \frac{1}{2} (q-1) \sum_{k=1}^{W} p_k \ln^2 p_k + O((q-1)^2) . \]

Hence, the Tsallis entropy can be approximated by the BG entropy and expanding \((q-1)\) terms:

\[ S = k_B \cdot \frac{1}{q-1} \left( 1 - \sum_{k=1}^{W} p_k^q \right) \approx -k_B \cdot \sum_{k=1}^{W} p_k \ln p_k - k_B \cdot \frac{1}{2} (q-1) \sum_{k=1}^{W} p_k \ln^2 p_k + O((q-1)^2) . \]  

The expansion to infinite \((q-1)\) terms is

\[ S = k_B \cdot \frac{1}{q-1} \left( 1 - \sum_{k=1}^{W} p_k^q \right) = -k_B \cdot \sum_{k=1}^{W} p_k \ln p_k - k_B \cdot \sum_{n=2}^{\infty} \frac{1}{n!} (q-1)^n \cdot \sum_{k=1}^{W} p_k \ln^n p_k \]  

Therefore, Tsallis entropy recovers to BG entropy for \(q \to 1\).

The Gibbs’ path (1902) [101] for the maximization of the entropy \(S(p_1, p_2, ..., p_W)\) under the constraints of canonical ensemble, i.e., (i) normalization \(1 = \sum_{k=1}^{W} p_k\), and (ii) fixed internal energy \(U = \sum_{k=1}^{W} p_k \varepsilon_k\), involves maximizing the functional

\[ G(p_1, p_2, ..., p_W) = S(p_1, p_2, ..., p_W) + \lambda_1 \sum_{k=1}^{W} p_k - \lambda_2 \sum_{k=1}^{W} p_k \varepsilon_k . \]  

that is, setting

\[ \frac{\partial}{\partial p_j} G(p_1, p_2, ..., p_W) = 0 , \; \forall \; j : 1, ..., W . \]

Next, we examine the BG and Tsallis entropic formulations. For the BG entropy we find

\[ p_j = \exp(\lambda_1 - 1) \cdot \exp(-\lambda_2 \varepsilon_j) , \]

or using the mean-less energies, \(\varepsilon_j - U\),

\[ p_j = \exp(\lambda_1 - 1 - \lambda_2 U) \cdot \exp[-\lambda_2 (\varepsilon_j - U)] . \]

The case of Tsallis entropy is rather more complicated. The non-extensive statistical mechanics is interwoven with the concept of escort probabilities [102]. The escort probability distribution \(\{p_k\}_{k=1}^{W}\) is constructed from the ordinary probability distribution, \(\{p_k\}_{k=1}^{W}\), as \(p_k \propto p_k^q\), \(\forall \; k = 1, ..., W\). The interpretation for the internal energy \(U\) is given by the escort expectation value of energy, \(U = \sum_{k=1}^{W} p_k \varepsilon_k\), which complicates the functional \(G\) that needs to be maximized.
Another, fairly simpler way of approaching the entropy maximization within the framework of non-extensive statistical mechanics is given by avoiding the notion of escort probabilities \[100\]. Then, a different \(Q\)-index characterizes the canonical distribution, e.g., \(Q = q^{-1}\),

\[
p_j = \left[1 + (1-Q) \cdot (\lambda_1 - 1)\right]^{-\frac{Q}{q^{-1}}} \cdot \left[1 + (Q-1) \cdot \frac{\lambda_2 e_j}{1 + (1-Q) \cdot (\lambda_1 - 1)}\right]^{-\frac{Q}{q^{-1}}}.
\]

(18)

We enter the mean-less energies, \(e_j - U\),

\[
p_j = \left[1 + (1-Q) \cdot (\lambda_1 - 1 + \lambda_2 U)\right]^{-\frac{Q}{q^{-1}}} \cdot \left[1 + (Q-1) \cdot \frac{\lambda_2 (e_j - U)}{1 + (1-Q) \cdot (\lambda_1 - 1 + \lambda_2 U)}\right]^{-\frac{Q}{q^{-1}}}.
\]

(19)

Then, using the typical notation of the kappa index, \(\kappa \equiv (Q-1)^{-1}\), we construct the kappa distribution,

\[
p_j \propto \left[1 + \frac{1}{\kappa} \cdot \beta(e_j - U)\right]^{-\kappa^{-1}},
\]

(20)

where we set the (inverse) temperature as \(\beta \equiv \lambda_2 /[1 + (1-Q) \cdot (\lambda_1 - 1 + \lambda_2 U)]\).

2.3. General entropy

Here we assume that the entropic function is given in terms of an unknown function \(f(x)\):

\[
S = S(p_1, p_2, ..., p_W) = k_B \sum_{k=1}^W f(p_k).
\]

(21)

We claim that the exact form of \(f(x)\) can be determined by considering basic physical properties like the macroscopic additivity among energies and entropies.

First, the maximization of entropy gives

\[
f'(p_j) + \lambda_1 - \lambda_2 e_j = 0,
\]

(22)

and given the energy additivity among two parts of the system, A and B, i.e., \(e_{\gamma}^{A+B} = e_{\gamma}^A + e_{\gamma}^B\), we obtain (for constant \(\lambda_1\) and \(\lambda_2\)):

\[
f'(p_{\gamma}^{A+B}) - \lambda_1 = f'(p_{\gamma}^A) + f'(p_{\gamma}^B).
\]

(23)

Then, applying \(\sum_{\gamma=1}^W \sum_{\gamma=1}^W p_{\gamma}^{A+B}\times\) on both sides of Eq.(23), we derive

\[
\sum_{\gamma=1}^W \left[\pm f'(p_{\gamma}^{A+B}) - 1\right] p_{\gamma}^{A+B} = \sum_{\gamma=1}^W \left[\pm f'(p_{\gamma}^A) - 1\right] p_{\gamma}^A + \sum_{\gamma=1}^W \left[\pm f'(p_{\gamma}^B) - 1\right] p_{\gamma}^B.
\]

(24)

Second, the entropy additivity among the same two parts of the system, \(S^{A+B} = S^A + S^B\), gives

\[
\sum_{\gamma=1}^W f(p_{\gamma}^{A+B}) = \sum_{\gamma=1}^W f(p_{\gamma}^A) + \sum_{\gamma=1}^W f(p_{\gamma}^B).
\]

(25)

The only solution for having both relations (24) and (25) is to have the proportionality

\[
f(x) \propto \left[\frac{1}{\kappa} \cdot f'(x) - 1\right] \cdot x.
\]

(26)
Solving this differential equation we find
\[ f(x) = \lambda_1 \frac{x-x^q}{\lambda_1 c - 1}, \quad (27) \]
where \( c \) is the proportionality constant. Setting \( \lambda_1 \equiv k_B = 1 \) and \( q \equiv \frac{\lambda_1}{c} \), we end up with the Tsallis entropy, as given by Eq.(11),
\[ S(p_1, p_2, ..., p_w) = k_B \sum_{k=1}^W f(p_k) \text{ with } f(x) = \frac{1}{q-1} \cdot (x-x^q). \quad (28) \]
(For more details, see: \cite{103}.)

3. Statistical properties

3.1. The concept of temperature

The standard formulation of the kinetic energy is
\[ P(\varepsilon_K) \propto \left(1 + \frac{1}{\kappa - \frac{\varepsilon_K}{k_B T}} \cdot \varepsilon_K^{\frac{1}{\kappa - 1}} \right)^{-\kappa - 1} \cdot \varepsilon_K^\frac{1}{\kappa - 1}, \quad (29) \]
while sometimes a different formulation of a kappa distribution is also used,
\[ P(\varepsilon_K) \propto \left(1 + \frac{1}{\kappa \cdot k_B T} \cdot \varepsilon_K^{\frac{\kappa - 1}{\kappa}} \right)^{-\kappa - 1} \cdot \varepsilon_K^{\frac{1}{\kappa - 1}}, \quad (30) \]
where the parameter \( T_\kappa \) is not the actual temperature of the particle system, but a quantity dependent on both the actual temperature \( T \) and the kappa index.

The temperature can kinetically defined by the mean kinetic energy,
\[ \langle \varepsilon_K \rangle = \frac{1}{2} d_\kappa k_B T, \quad (31) \]
where \( d_\kappa \) is the kinetic degrees of freedom. If a thermal parameter other than the temperature is used for characterizing the kappa distribution, it must be expressed in terms of the temperature (and possibly, the kappa index). For example, in Eq.(30), the parameter \( T_\kappa \) can be derived from the mean kinetic energy,
\[ \langle \varepsilon_K \rangle = \frac{1}{2} d_\kappa k_B T_\kappa / (\kappa - \frac{1}{2} d_\kappa). \quad (32) \]
Comparing Eqs.(31,32) we derive the relation \( (\kappa - \frac{1}{2}) \cdot k_B T = \kappa \cdot k_B T_\kappa \); hence, we find that \( T_\kappa \) (and not \( T \)) depends on the kappa index, i.e., \( T_\kappa = T_\kappa(T, \kappa) \) (see: \cite{6}; \cite{95}, Chapter 1).

The correct definition and use of temperature in the formulation of the kappa distribution can be helpful for understanding the properties of this distribution. For example, we may ask, are there types of kappa distributions with different exponents? No, the generated types would be equivalent with the standard kappa distribution: Let’s assume that the kappa distribution can be generalized to have an arbitrary exponent,
\[ P(\varepsilon) \propto \left(1 + \frac{1}{\kappa - \frac{\varepsilon}{k_B T}} \cdot \varepsilon^{\frac{\kappa - a}{2}} \right)^{-\kappa - a} \cdot \varepsilon^{\frac{5}{2} - a} - a < \kappa \leq \infty. \quad (33) \]
How can this be equivalent to the standard kappa distribution, i.e.,
The parameter $T$ in Eq.(33) is not the actual temperature but a quantity depended on the actual temperature (and the kappa index). By substituting the parameter $T$ in Eq.(33) with the actual temperature $T_{\text{real}}$, derived from the mean kinetic energy,

$$<e> = \frac{3}{2}k_B T_{\text{real}} = \cdots = \frac{\kappa - \frac{3}{2}}{\kappa + a - \frac{5}{2}} \frac{1}{k_B} T_{\text{real}},$$

we find

$$P(e) \propto \left(1 + \frac{1}{\kappa - \frac{3}{2}} \frac{1}{k_B T_{\text{real}}}ight)^{-(\kappa + a - 1)}<e>, \quad \frac{3}{2} < \kappa + a - 1 \leq \infty .$$

Then, we note that the kappa index $\kappa$ appears always in the quantity $\kappa + a - 1$. Hence, we redefine the kappa index, so that $\kappa + a - 1 \rightarrow \kappa$, thus, the distribution (33) becomes

$$P(e) \propto \left(1 + \frac{1}{\kappa - \frac{3}{2}} \frac{1}{k_B T_{\text{real}}}ight)^{-(\kappa + a - 1)}<e>, \quad \frac{3}{2} < \kappa \leq \infty ,$$

which is the standard kappa distribution formulation.

3.2. The physical meaning of the kappa index

The kappa index depends on the kinetic degrees of freedom, $d_K$:

$$\kappa = \kappa(d_K) = \text{constant} + \frac{1}{2} d_K, \quad \text{with} \quad \kappa_0 = \kappa(d_K), -\frac{1}{2} d_K = \text{invariant kappa index}.$$

Therefore, the $d_K$-dimensional kappa distribution of the velocities of 1-particle can be written using the dependent kappa index, $\kappa$, or the invariant kappa index, $\kappa_0$,

$$P(\vec{u}; T; \kappa) \propto \left(1 + \frac{1}{\kappa - \frac{d_K}{2}} \frac{u^2}{\Theta^2}ight)^{-\kappa}, \quad \frac{3}{2} < \kappa - \frac{d_K}{2} \leq \infty .$$

The physical meaning of the kappa index is interwoven with the statistical correlation between the energy of the particles. It has been shown that a simple relation exists between the correlation and the kappa index, $R = \frac{1}{2} d_k / \kappa = \frac{1}{2} d_k / (\kappa_0 + \frac{1}{2} d_k)$ [59,78] (see also: [95], Chapter 5). The largest value of the kappa index is infinity, corresponding to the system residing at thermal equilibrium. This is a special stationary state for which the particles are characterized by zero correlation. The whole structure of classical statistical mechanics is based on this ideal property that the energies of any two particles are uncorrelated (i.e., zero covariance). On the other hand, the smallest possible value of the kappa index is $\kappa_0 \rightarrow 0$ (or $\kappa \rightarrow -\frac{1}{2} d_k$) and corresponds to the furthest state from thermal equilibrium, a state called anti-equilibrium [14].

The classical, single stationary state at equilibrium is generalized into a whole set of different non-equilibrium stationary states, each labeled by a separate kappa index, an arrangement called the $\kappa$-spectrum (Figure 1). Hence, the parameter kappa provides a suitable measure of the “thermodynamic distance” of stationary states from equilibrium.
The $\kappa$-spectrum. For $\kappa \to \infty$, the system resides at thermal equilibrium; for $\kappa \to \frac{3}{2}$, the furthest possible stationary state from thermal equilibrium is attained, the anti-equilibrium. This is shown comparing with the correlation coefficient $R$. The classical case of systems residing in thermal equilibrium ($\kappa \to \infty$) corresponds to zero correlation ($R \to 0$), while the other extreme state of “anti-equilibrium” ($\kappa \to \frac{3}{2}$) indicates a maximum correlation ($R \to 1$). (Adopted from [95], Chapter 5)

Furthermore, observations showed an empirical separation of the $\kappa$-spectrum that characterizes space plasmas [12,15,59,95]: While the inner heliosphere plasmas reside in near equilibrium states with large kappa indices, generally $\kappa>2.5$, analysis of the IBEX ENA-spectra showed that inner heliosheath resides in states far from equilibrium with small kappa indices $\kappa<2.5$ (Figure 2). The distinction in near-/far-equilibrium regions can be explained by the entropy associated with kappa distributions, which exhibits different functional behavior within the two regions (for details, see: [57] and [95], Chapters 2 and 10).

Density, temperature and kappa index are independent parameters. Various correlations and trends appear in several space plasmas do not constitute universal relations, as these are subject to the location, time, and plasma parameters. This behavior is expected, as there are many factors causing correlations between those three thermodynamic parameters (e.g., potential energy, e.g., [77] and [95], Chapters 3 and 5). This is similar to the classical case at thermal equilibrium, where density and temperature are independent parameters, but polytropes apply correlations and trends among their values.

Figure 3 illustrates the parameter values of space plasmas. In particular, it plots the representative values of density $n$, temperature $T$, and kappa index $\kappa$ of ~40 different space plasmas. The kappa indices were collected from the results of one or more published analyses. We also calculate the measure $M=1/(\kappa-0.5)$, an alternative of the kappa index $\kappa$. This is a measure of how far the system
resides from thermal equilibrium [12, 59, 80]. We observe a general trend characterizing space plasmas: The higher the temperature, the closer to thermal equilibrium, that is, the higher the kappa index.

3.3. The N-particle description of kappa distributions

The advantage of using the invariant kappa index, $\kappa_0$, instead of the dimensionality dependent kappa index, $\kappa$, is that we can express the $N$-particle kappa distribution, that is, the ($d_K N$)-dimensional kappa distribution of the velocities of $N$-particles

$$P(\vec{u}_1, \ldots, \vec{u}_N) = \left( \frac{4}{\pi \kappa_0^3} \right) \frac{1}{\kappa_0} \frac{\Gamma(\kappa_0 + 1 + \frac{d_K}{2} N)}{\Gamma(\kappa_0 + 1)} \left[ 1 + \frac{1}{\kappa_0^2} \sum_{n=1}^{N} \left( \vec{u}_{(n)} - \vec{u}_0 \right)^2 \right]^{-\kappa_0^{-1} \frac{1}{2} d_K N} \cdot (40)$$

The $N$-particle kappa distribution in terms of the kinetic energies, $\epsilon_{K(a)} = \frac{1}{2} m (\vec{u}_{(a)} - \vec{u}_0)^2$, is

$$P(\epsilon_{K(1)}, \ldots, \epsilon_{K(N)}) = \left( \frac{\kappa_0 k_B T}{2} \right)^{\frac{d_K}{2} N} \frac{\Gamma(\kappa_0 + 1 + \frac{d_K}{2} N)}{\Gamma(\kappa_0 + 1) \cdot \Gamma\left( \frac{d_K}{2} N \right)} \left( 1 + \frac{1}{\kappa_0^2} \sum_{n=1}^{N} \epsilon_{K(a)} \right)^{-\kappa_0^{-1} \frac{1}{2} d_K N} \prod_{n=1}^{N} \epsilon_{K(a)}^{-\frac{1}{2} d_K N} \cdot (41)$$

For example, the two-particle velocity distribution, with $d_K=3$ degrees of freedom per particle, is

$$P(\vec{u}_1, \vec{u}_2) = \frac{4}{\pi} \frac{1}{\kappa_0^3} (\kappa_0 + 1)(\kappa_0 + 2)(\kappa_0 + 3) \left[ 1 + \frac{1}{\kappa_0^2} \left( \frac{\vec{u}_{(1)} - \vec{u}_0}{\kappa_0^2} \right)^2 \right]^{-\kappa_0^{-4}} \cdot (42a)$$

or, in terms of the kinetic energy,

$$P(\epsilon_{K(1)}, \epsilon_{K(2)}) = \frac{4}{\pi} \frac{1}{k_B T} \left( \kappa_0 + 1 \right) \left( \kappa_0 + 2 \right) \left( \kappa_0 + 3 \right) \left[ 1 + \frac{1}{\kappa_0} \frac{\epsilon_{K(1)} + \epsilon_{K(2)}}{k_B T} \right]^{-\kappa_0^{-4}} \cdot (42b)$$

Therefore, the variance, co-variance, and (Pearson’s) correlation $R$, are given by
\[ \sigma_{(1)}^2 = \left( \mathbb{E}_K(1)^2 \right) - \left( \mathbb{E}_K(1) \right)^2 = \frac{3}{2} \frac{\kappa_0 + \frac{3}{2}}{\kappa_0 - 1}, \quad (43a) \]

\[ \sigma_{(2)}^2 = \left( \mathbb{E}_K(1)^2 \mathbb{E}_K(2)^2 \right) - \left( \mathbb{E}_K(1) \mathbb{E}_K(2) \right)^2 = \frac{1}{4} \frac{1}{\kappa_0 - 1}, \quad (43b) \]

\[ R = \frac{\sigma_{(2)}^2}{\sigma_{(1)}^2} = \frac{\frac{3}{2} \frac{3}{2}}{\kappa_0} = \frac{3}{\kappa_0}. \quad (43c) \]

### 3.4. Non-zero potential energy

So far we have presented and discussed kappa distributions of the velocities or kinetic energies, considering the absence of any potential energy. In the presence of a potential energy \( \Phi \), however, the kappa distribution depends on both the velocities (kinetic energies) and the positions, thus it describes the particle phase-space. Indeed, the phase-space kappa distribution is given by

\[
P(\vec{r}, \vec{u}) \propto \left[ 1 + \frac{1}{\kappa_0} \cdot \mathbb{E}_K(\vec{u}) \cdot \Phi(\vec{r}) \right]^{-\kappa_0 - \frac{1}{2} \frac{1}{d_k} + \frac{1}{2} \frac{d_k}{d_\Phi}}, \quad \Phi(\vec{r}) > 0 , \quad (44a)\]

\[
P(\vec{r}, \vec{u}) \propto \left[ \frac{\Phi(\vec{r}) \mathbb{E}_K(\vec{u})}{\kappa_0 k_B T} - 1 \right]^{-\kappa_0 - \frac{1}{2} \frac{1}{d_k} + \frac{1}{2} \frac{d_k}{d_\Phi}}, \quad \Phi(\vec{r}) < 0 , \quad (44b)\]

(the operator "[x]_+" means \([x]_+ = 0 \) when \(x \leq 0\)), where the kinetic and potential degrees of freedom are

\[
\frac{1}{2} d_k = \frac{\mathbb{E}_K}{k_B T}, \quad \frac{1}{2} d_\Phi = \frac{\Phi}{k_B T}. \quad (45)\]

If the phase-space kappa distribution is integrated over the positions, we arrive at the standard kappa distribution of velocities

\[
\int d\vec{r} \rightarrow P(\vec{u}) \propto \left[ 1 + \frac{1}{\kappa_0} \cdot \mathbb{E}_K(\vec{u}) \right]^{-\kappa_0 - \frac{1}{2} \frac{1}{d_k}}. \quad (46)\]

If the phase-space kappa distribution is integrated over the velocities, we derive the new type of positional kappa distribution

\[
\int d\vec{u} \rightarrow P(\vec{r}) \propto \left[ 1 + \frac{\Phi(\vec{r})}{\kappa_0 k_B T} \right]^{-\kappa_0 - \frac{1}{2} \frac{1}{d_\Phi}}, \quad \Phi(\vec{r}) > 0 , \quad (47a)\]

\[
\int d\vec{u} \rightarrow P(\vec{r}) \propto \left[ \frac{\Phi(\vec{r})}{\kappa_0 k_B T} - 1 \right]^{-\kappa_0 - \frac{1}{2} \frac{1}{d_\Phi}}, \quad \Phi(\vec{r}) < 0 . \quad (47b)\]

We proceed with central attractive potentials, \( \Phi(\vec{r}) = \Phi(r) \). Two characteristic examples are the attractive power-laws with positive or negative exponents, corresponding to oscillation and gravitation types, respectively:

(i) Oscillation type, \( \Phi(r) = \frac{1}{r^b} k \cdot r^{-b} \):

- Phase-Space distribution:
\[ P(\vec{r}, \vec{u}) \propto \left[ 1 + \frac{1}{\kappa_0} \cdot \left( \frac{E_K(\vec{u})}{k_B T} + \frac{d_B}{r_0} \left( \frac{\vec{p}}{r_0} \right) \right) \right]^{-1 - \frac{1}{d} + \frac{1}{d_k} - \frac{1}{d},} \]  

(48a)

- Positional distribution:

\[ P(\vec{r}) \propto \left[ 1 + \frac{d_B}{r_0} \left( \frac{\vec{p}}{r_0} \right) \right]^{-1 - \frac{1}{d} + \frac{1}{d_k},} \]  

(48b)

(ii) Gravitation type, \( \Phi(r) = -\frac{1}{2} k \cdot r^{-b} \):

- Phase-Space distribution:

\[ P(\vec{r}, \vec{u}) \propto \left[ \frac{1}{\kappa_0} \cdot \left( \frac{r_0}{\vec{p}} \right)^b - \frac{E_K}{k_B T} \right]^{-1 - \frac{1}{d} + \frac{1}{d_k},} \]  

(49a)

- Positional distribution:

\[ P(\vec{r}) \propto \left[ \frac{r_0}{\vec{p}} \right]^b - 1 \]  

(49b)

- Involved parameters for \( \Phi(r) = \pm \frac{1}{2} k r^{-b} \):

\[ r_0 = \left( \frac{1}{\kappa_0} \right)^{\frac{1}{b}} = \left( \frac{d_B k_B T}{\frac{1}{b} k} \right)^{\frac{1}{b}} , \quad \frac{1}{b} d_B = \pm \frac{\langle \Phi \rangle}{k_B T} = \frac{\frac{1}{b} k}{k_B T} r_0^{-b} = \frac{1}{b} d_t. \]  

(49c)

Below, we present two examples.

### 3.4.1. Generalized barometric formula

The potential energy is \( \Phi(z) \propto z \), e.g., \( \Phi(z) = mg \cdot z \) or \( \Phi(z) = qE \cdot z \). The Hamiltonian is \( H(z, \vec{u}) = m (\vec{u} - \vec{u}_0)^2 + mg \cdot z \). We find:

- Phase-Space distribution:

\[ P(z, E_K; \kappa_0, T) \propto \left[ 1 + \frac{1}{\kappa_0} \cdot \left( \frac{E_K}{k_B T} + \frac{z}{z_0} \right) \right]^{-1 - \frac{2}{d_k} + \frac{1}{d},} \]  

(50a)

- Positional distribution:

\[ P(z; \kappa_0, T) \propto \left( 1 + \frac{1}{\kappa_0} \cdot \frac{z}{z_0} \right)^{-1 - \frac{2}{d_k}} \]  

with \( z_0 = \frac{k_B T}{mg} \).  

(50b)

- Density, \( n \):

\[ n(z) \propto P(z) , \quad n(z) = n(0) \left( 1 + \frac{1}{\kappa_0} \cdot \frac{z}{z_0} \right)^{-1 - \frac{2}{d_k},} \]  

(50c)

- Local Temperature \( T(z) \):
\[ \frac{d}{dt} k_B T(z) = \int_0^\infty \frac{P(z, \varepsilon_K) \varepsilon_K d\varepsilon_K}{P(z, \varepsilon_K) d\varepsilon_K} = \frac{d}{dt} k_B T(z) \left( 1 + \frac{1}{\kappa_0} \frac{z}{z_0} \right) \] 

\[ T(z) = T(0) \left( 1 + \frac{1}{\kappa_0} \frac{z}{z_0} \right). \]  

(50d)

Note that the spatial average of the local temperature, \( T(z) \), is the global temperature, \( T \).

- Thermal Pressure, \( p(z) = n(z) \cdot k_B T(z) \):

The generalized barometric formula is given by

\[ p(z; \kappa_0) = p(0) \left( 1 + \frac{1}{\kappa_0} \frac{mg z}{k_B T} \right)^{-\kappa_0^{-1}} \]

with \( p(z; \kappa_0 \to \infty) = p(0) \exp \left( - \frac{mg z}{k_B T} \right). \)  

(50e)

3.4.2. Oscillation

Let the 1-dimensional oscillation described by the Hamiltonian: \( H(z, u_z) = \frac{1}{2} m (u_z - u_{z0})^2 + \frac{1}{2} k \cdot z^2 \).

- Phase-Space distribution:

\[ P(z, u_z) \propto \left[ 1 + \frac{1}{\kappa_0} \left( \frac{\varepsilon_k (u_z) + \frac{1}{2} \left( \frac{z}{z_0} \right)^2 }{k_B T} \right) \right]^{-\kappa_0^{-2}} \]

or \( P(z, u_z) \propto \left[ 1 + \frac{1}{\kappa_0} \frac{u_z^2 + (k/m) \cdot z^2}{\theta^2} \right]^{-\kappa_0^{-2}}. \)  

(51a)

- Positional distribution:

\[ P(z) \propto \left[ 1 + \frac{1}{2\kappa_0} \left( \frac{z}{z_0} \right)^2 \right]^{-\kappa_0^{-\frac{1}{2}}} \]

or \( P(z) \propto \left[ 1 + \frac{1}{\kappa_0} \frac{(k/m) \cdot z^2}{\theta^2} \right]^{-\kappa_0^{-\frac{1}{2}}}, \) with \( z_0 = \sqrt{\frac{k_B T}{k}}. \)  

(51b)

These kappa distributions were used by \([49]\) to describe the 1-dimensional speed and altitude of OH in the plasma near Enceladus. Interestingly, the kappa distribution used for describing the OH velocities and the kappa distribution used for describing the OH positions had the same kappa index and temperature, as it is predicted by the theory (Figure 4).

Figure 4. Density and velocity distribution of the OH cloud at Saturn. The vertical density profile is well fit to a positional kappa distribution (see: Eq.(51b); [95], Chapter 3). The velocity distribution of individual particles is non-Maxwellian, and also well fit to a kappa distribution. (Taken from [49])

4. Conclusions

Empirical kappa distributions have become increasingly widespread across space and plasma physics. Space plasmas from the solar wind to planetary magnetospheres and the outer heliosphere are systems out of thermal equilibrium, described by kappa distributions. A breakthrough in the field came with the connection of kappa distributions to non-extensive statistical mechanics. This paper presented important aspects of the statistical origin of kappa distributions, as well as theoretical developments and applications, such as (i) the concept of temperature; (ii) the physical meaning of the kappa index;
(iii) the $N$-particle description of kappa distributions; and the (iv) the generalization to phase-space kappa distribution of a Hamiltonian with non-zero potential.

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