MASUR’S CRITERION DOES NOT HOLD IN THE THURSTON METRIC

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Abstract. We construct a counterexample for an analogue of Masur’s criterion in the setting of Teichmüller space equipped with the Thurston metric. For that, we find a minimal, filling, non-uniquely ergodic lamination \( \lambda \) on the seven-times punctured sphere with uniformly bounded annular projection distances. Then we show that a geodesic in the corresponding Teichmüller space that converges to \( \lambda \), stays in the thick part for the whole time.

1. Introduction

The Thurston metric is an asymmetric Finsler metric on Teichmüller space that was first introduced by Thurston in [Thu86]. The distance between marked hyperbolic surfaces \( X \) and \( Y \) is defined as the log of the infimum over the Lipschitz constants of maps from \( X \) to \( Y \), homotopic to the identity. Thurston showed that when \( S \) has no boundary, the distance can be computed by taking the ratios of the hyperbolic lengths of the geodesic representatives of simple closed curves (s.c.c.):

\[
\text{d}_{\text{Th}}(X, Y) = \sup_{\alpha - \text{s.c.c.}} \log \frac{\ell_\alpha(Y)}{\ell_\alpha(X)}.
\]

A class of oriented geodesics for this metric called stretch paths was introduced in [Thu86]. Given a maximal geodesic lamination \( \nu \) on a hyperbolic surface \( X \), a stretch path starting from \( X \) is obtained by stretching the leaves of \( \nu \) and extending this deformation to the whole surface. The stretch path is controlled by the horocyclic foliation, obtained by foliating the ideal triangles in the complement of \( \nu \) by horocyclic arcs and endowed with the transverse measure that agrees with the hyperbolic length along the leaves of \( \nu \). That is, the projective class of the horocyclic foliation is invariant along the stretch path.

Thurston showed that there exists a geodesic between any two points in Teichmüller space equipped with this metric that is a finite concatenation of stretch path segments. In general, geodesics are not unique: the length ratio in Equation (1) extends continuously to the compact space of projective measured laminations \( \overline{\mathcal{ML}}(S) \) and the supremum is usually (in a sense of the word) realized on a single point which is a simple closed curve, thus leaving freedom for a geodesic.

The following is our main theorem:

**Theorem 1.1.** There are Thurston stretch paths in a Teichmüller space with minimal, filling, but not uniquely ergodic horocyclic foliation, that stay in the thick part for the whole time.

The theorem contributes to the study of the geometry of the Thurston metric in comparison to the better studied Teichmüller metric. Namely, our result is in contrast with a criterion for the divergence of Teichmüller geodesics in the moduli space, given by Masur:

**Theorem 1.2** (Masur’s criterion, [Mas92]). Let \( q \) be a unit area quadratic differential on a Riemann surface \( X \) in the moduli space \( \mathcal{M}(S) \). Suppose that the vertical foliation of \( q \) is minimal but not uniquely ergodic. Then the projection of the corresponding Teichmüller geodesic \( X_t \) to the moduli space \( \mathcal{M}(S) \) eventually leaves every compact set as \( t \to \infty \).

**Remark 1.3.** The horocyclic foliation is a natural analogue of the vertical foliation in the setting of the Thurston metric, see [Mir08], [CF21].

**Remark 1.4.** Compare Theorem 1.1 to a result of Brock and Modami in the case of the Weil-Petersson metric on Teichmüller space [BM15]: they show that there exist Weil-Petersson geodesics with minimal, filling, non-uniquely ergodic ending lamination, that are recurrent in the moduli space, but not contained in any compact subset. Hence our counterexample disobeys Masur’s criterion even more than in their setting of the Weil-Petersson metric.

Despite being asymmetric, and in general admitting more than one geodesic between two points, the Thurston metric exhibits some similarities to the Teichmüller metric. For example, it differs from the Teichmüller metric by at most a constant in the thick part\(^1\) and there is an analog of Minsky’s product region theorem [CR07]; every Thurston

\(^1\)here the constant \( C(\varepsilon) \) depends on the thick part \( T_\varepsilon(S) \).
geodesic between any two points in the thick part with bounded combinatorics is cobounded\footnote{for every $x, y \in \mathcal{T}(S)$ with $K$–bounded combinatorics (Definition 2.2 in \cite{LRT12}), every $G(x, y)$ is in the $\varepsilon(K, S)$–thick part.} \cite{LRT12}; the shadow of a Thurston geodesic to the curve graph is a reparameterized quasi-geodesic \cite{LRT15}.

Nevertheless, the Thurston metric is quite different from the Teichmüller metric. For one, the identity map between them is neither bi-Lipschitz \cite{Li03}, nor a quasi-isometry \cite{CR07}. In the Teichmüller metric, whenever the vertical and the horizontal foliations of a geodesic have a large projection distance in some subsurface, the boundary of that subsurface gets short along the geodesic\footnote{for every $\varepsilon > 0$ there exists $K$ such that $d_W(\mu_+, \mu_-) > K$ implies inf $\ell_{\partial W}(G(t)) < \varepsilon$.} \cite{Raf05}. However, it follows from our construction that the endpoints of a cobounded Thurston geodesic do not necessarily have bounded combinatorics. The reason behind it is that a condition equivalent to a curve getting short along a stretch path that is expressed in terms of the subsurface projections of the endpoints is more restrictive than in the case of the Teichmüller metric \cite{Raf14}, and involves only the annular subsurface of $\alpha$ (see Theorem 2.10 for a precise and more general statement). This allows us to produce our counterexample by constructing a minimal, filling, non-uniquely ergodic lamination with uniformly bounded annular subsurface projections.

The construction will be done on the seven-times punctured sphere. First, in Section 3 we construct a minimal, filling, non-uniquely ergodic lamination $\lambda$ using a modification of the machinery developed in \cite{LLR18}. Namely, we choose a partial pseudo-Anosov map $\tau$ supported on a subsurface $Y$ homeomorphic to the three-times punctured sphere with one boundary component. We pick a finite-order homeomorphism $\rho$, such that the subsurface $\rho(Y)$ is disjoint from $Y$, and the orbit of the subsurface $Y$ under $\rho$ fills the surface. Then we set $\varphi = \tau \circ \rho$ and provided with a sequence of natural numbers $\{r_i\}_{i=1}^\infty$ and a curve $\gamma_0$, define

$$\Phi_i = \varphi_{r_i} \circ \ldots \circ \varphi_{r_1}, \quad \gamma_i = \Phi_i(\gamma_0).$$

We show that under a mild growth condition on the coefficients $r_i$, the sequence of curves $\gamma_i$ forms a quasi-geodesic in the curve graph and converges to an ending lamination $\lambda$ in the Gromov boundary. In Section 4, we introduce a $\Phi$-invariant bigon track and provide matrix representations of the maps $\tau$ and $\tau \circ \rho$. In Section 5, we let $\gamma_0$ be a multicurve and produce coarse estimates for the intersection numbers between the pairs of multicurves in the sequence $\gamma_i$. In Section 6, we show that $\lambda$ is non-uniquely ergodic and we find all ergodic transverse measures on $\lambda$. In Section 7, we prove that $\lambda$ has uniformly bounded annular subsurface projections. Finally, in Section 8 we show that there are Thurston stretch paths whose horocyclic foliation is $\lambda$, that stay in the thick part of the Teichmüller space for the whole time.

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\section{Background}

\subsection{Notation.}
We adopt the following notation. Given two quantities (or functions) $A$ and $B$, we write $A \asymp_{K, C} B$ if $\frac{1}{K} B - C \leq A \leq KB + C$. Further, unless explicitly stated, by the following notation we will mean that there are universal constants $K \geq 1, C \geq 0$ such that

- $A \asymp B$ means $A \leq B + C$.
- $A \asymp B$ means $A \leq KB$.
- $A \asymp B$ means $A - C \leq B \leq A + C$.
- $A \asymp B$ means $\frac{1}{K} B \leq A \leq KB$.
- $A \asymp B$ means $A \leq KB + C$.
- $A \asymp B$ means $\frac{1}{K} B - C \leq A \leq KB + C$.

\subsection{Curves and markings.}
Let $S = S_{g, n}$ be the oriented surface of genus $g \geq 0$ with $n \geq 0$ punctures and with negative Euler characteristic. A simple closed curve on $S$ is called essential if it does not bound a disk or a punctured disk. We will call a curve on $S$ the free homotopy class of an essential simple closed curve on $S$. Given two curves $\alpha$ and $\beta$ on $S$, we will denote the minimal geometric intersection number between their representatives by $i(\alpha, \beta)$. A multicurve is a collection of pairwise disjoint curves on $S$. A pants decomposition $P$ on $S$ is a maximal multicurve on $S$, i.e. whose complement in $S$ is a disjoint union of three-times punctured spheres. A collection of curves $\Gamma$ is
called filling if for any curve $\beta$ on $S$: $i(\alpha, \beta) > 0$ for some $\alpha \in \Gamma$. A marking $\mu$ on $S$ is a filling collection of curves. The intersection number between two collections of curves $\Gamma$ and $\Gamma'$ is defined as

$$i(\Gamma, \Gamma') = \sum_{\gamma \in \Gamma, \gamma' \in \Gamma'} i(\gamma, \gamma').$$

### 2.3. Curve graph

The curve graph $C(S)$ of a surface $S$ is a graph whose vertex set $C_0(S)$ is the set of all curves on $S$. Two vertices $\alpha$ and $\beta$ are connected by an edge if the underlying curves realize the minimal possible geometric intersection number for two curves on $S$. This means that $i(\alpha, \beta) = 0$, i.e. the curves are disjoint, unless $S$ is one of the exceptional surfaces: if $S$ is the punctured torus, then $i(\alpha, \beta) = 1$, and if $S$ is the four-times punctured sphere, then $i(\alpha, \beta) = 2$. The curve graph is the 1-skeleton of the curve complex, introduced by Harvey in [Har81]. The metric $d_S$ on the curve graph is induced by letting each edge have unit length. Masur and Minsky showed in [MM99] that the curve graph is Gromov hyperbolic using Teichmüller theory.

**Theorem 2.1.** [MM99] The curve graph $C(S)$ is Gromov hyperbolic.

Later, Bowditch gave another proof of this result and showed that the hyperbolicity constant of $C(S_{g,n})$ is bounded above by a function that is logarithmic in $g + n$ [Bow06]. It was then shown that the hyperbolicity constant is uniformly bounded independently by Bowditch [Bow14], Aougab [Aoug13], Hensel, Przytycki, Webb [HPW15], Clay, Rafi, Schleimer [CRS14].

Although the compact annulus $A$ is not a surface of negative Euler characteristic, it is crucial for us to consider it and we separately define its curve graph. Let the vertices of $C(A)$ be the arcs connecting two boundary components of $A$, up to homotopies that fix the endpoints. Two vertices are connected by an edge of length 1 if the underlying arcs have representatives with disjoint interiors. It is easy to check that $C(A)$ is quasi-isometric to $\mathbb{Z}$ with the standard metric, hence also Gromov hyperbolic (see Section 2.4 in [MM00] for more details).

### 2.4. Measured laminations and measured foliations

We denote the space of geodesic laminations on $S$ equipped with the Hausdorff topology by $G\mathcal{L}(S)$. For the background on geodesic laminations we refer to Chapter 4 in [CEG86]. We fix some definitions. A geodesic lamination is minimal if it does not contain any proper sublamination. A geodesic lamination is maximal if it is not contained in any lamination as a proper subset. A geodesic lamination is filling if the connected components of its complement are open disks or once punctured open disks. A geodesic lamination is chain-recurrent if it is in the closure of the set of multicurves in $G\mathcal{L}(S)$.

We denote the space of measured laminations on $S$ equipped with the weak* topology by $M\mathcal{L}(S)$. For the background on measured laminations we refer to Chapter 8 in [Mar16]. The stump of a geodesic lamination is its maximal sublamination that admits a transverse measure of full support. We note that a minimal, filling geodesic lamination admits a transverse measure of full support. A geodesic lamination is uniquely ergodic if it supports a unique transverse measure up to scaling. Otherwise it is non-uniquely ergodic.

We denote the space of projective measured laminations on $S$ equipped with the quotient topology of $M\mathcal{L}(S) \setminus \{0\}$ by $P,M\mathcal{L}(S)$. For a non-zero measured lamination $\eta \in M\mathcal{L}(S)$, we denote its projective class by $[\eta] \in P,M\mathcal{L}(S)$. The intersection number $i(\cdot, \cdot)$ extends continuously to the space of measured laminations (for a further extension to the space of geodesic currents see Chapter 8 in [Mar16]). We say that the intersection number between two projective measured laminations equals zero if it holds for every pair of their representatives in $M\mathcal{L}(S)$.

Consider the subspace of $P,M\mathcal{L}(S)$ consisting of projective measured laminations with minimal and filling support. Consider the quotient of this subspace by identifying the laminations that have the same support. The resulting space equipped with the quotient subspace topology is the space of ending laminations $E\mathcal{L}(S)$. Alternatively, the topology of $E\mathcal{L}(S)$ can be described as follows: a sequence $\{\nu_i\}$ of minimal, filling geodesic laminations converges to $\nu \in E\mathcal{L}(S)$ if every limit point of $\{\nu_i\}$ in $G\mathcal{L}(S)$ contains $\nu$ as a sublamination. We refer to [Ham06] for more details. Klarreich proved the following:

**Theorem 2.2.** [Klar99] The Gromov boundary of the curve graph $C(S)$ is homeomorphic to the space of ending laminations $E\mathcal{L}(S)$. If a sequence of curves $\{\nu_i\}$ is a quasi-geodesic in $C(S)$ that converges to $\nu \in E\mathcal{L}(S)$, then any limit point of $\{\nu_i\}$ in $P,M\mathcal{L}(S)$ projects to $\nu$ under the forgetful map.

We denote the space of measured foliations on $S$ equipped with the weak* topology by $M\mathcal{F}(S)$. For the background on measured foliations we refer to [FLP12]. The spaces $M\mathcal{F}(S)$ and $M\mathcal{L}(S)$ are canonically identified, and we will sometimes not distinguish between measured laminations and measured foliations; similarly for their projectivizations $P,M\mathcal{L}(S)$ and $P,M\mathcal{F}(S)$.

### 2.5. Teichmüller space and Thurston boundary

A marked hyperbolic surface is a complete finite-area Riemannian surface of constant curvature $-1$ with a fixed homeomorphism from the underlying topological surface $S$. Two marked hyperbolic surfaces $X$ and $Y$ are called equivalent if there is an isometry between $X$ and $Y$ in the
2.6. Mapping class group. The mapping class group of a surface $S$ is the group of the isotopy classes of orientation-preserving self-homeomorphisms of $S$. The mapping class group acts continuously on the space of projective measured laminations $PMC(S)$. A non-periodic element of the mapping class group that has no invariant multicurves is called pseudo-Anosov. A pseudo-Anosov mapping class has exactly two fixed points in $PMC(S)$ that represent a pair of transverse measured laminations that are minimal, filling and uniquely ergodic. Moreover, given a pseudo-Anosov mapping class $\Psi$, there is a number $\lambda_\Psi > 1$ such that

$$\Psi(\nu^n) = \lambda_\Psi \nu^n, \quad \Psi(\nu^s) = \lambda_\Psi^{-1} \nu^s.$$  

The (classes of the) laminations $\nu^{n,s}$ in Equation (2) are called the unstable and stable laminations of $\Psi$, respectively. We refer to [FLP12], [FM12] for more background on pseudo-Anosov homeomorphisms.

2.7. Subsurface projections. By a subsurface $Y \subset S$ we mean the isotopy class of a proper, closed, connected, embedded subsurface, such that its boundary consists of curves on $S$ and its punctures agree with those of $S$. Whenever we talk about curves or laminations on $Y$, we think of the boundary components of $Y$ as punctures. We allow $Y$ to be an annular subsurface, whose core curve is a curve on $S$. We assume $Y$ is not a three-times punctured sphere.

The subsurface projection is a map $\pi_Y : GL(S) \to 2^\mathcal{C}(Y)$ from the space of geodesic laminations on $S$ to the power set of the vertex set of the curve graph of $Y$. Equip $S$ with a hyperbolic metric. Let $\bar{Y}$ be the Gromov compactification of the cover of $S$ corresponding to the subgroup $\pi_1(Y) = \pi_1(S)$ with the hyperbolic metric pulled back from $S$. There is a natural homeomorphism of $\mathcal{C}^\prime(Y)$ to $\bar{Y}$, allowing to identify the curve graphs $\mathcal{C}(Y)$ and $\mathcal{C}(\bar{Y})$. For any geodesic lamination $\nu$ on $S$, let $\bar{\nu}$ be the closure of the complete preimage of $\nu$ in $Y$. Suppose that $Y \subset S$ is a nonannular subsurface. An arc $\beta \subset Y$ is essential if no component of $\bar{Y} \setminus \beta$ has closure which is a disk. For each essential arc $\beta \subset Y$, let $N_\beta$ be a regular neighborhood of $\beta \cup \partial Y$. Define $\pi_Y(\nu)$ to be the union of all curves which are either curve components of $\bar{\nu}$ or curve components of $\partial N_\beta$, where $\beta$ is an essential arc in $\bar{\nu}$. If $Y \subset S$ is an annular subsurface, define $\pi_Y(\nu)$ to be the union of all arcs $\beta$ in $\bar{\nu}$ that connect two boundary components of $Y$.

We say that a lamination $\nu$ intersects the subsurface $Y$ essentially if $\pi_Y(\nu)$ is non-empty. The projection distance between two laminations $\nu, \nu' \in GL(S)$ that intersect $Y$ essentially is

$$d_Y(\nu, \nu') = \text{diam}_{\mathcal{C}(Y)}(\pi_Y(\nu) \cup \pi_Y(\nu')).$$

If $Y$ is an annular subsurface with the core curve $\alpha$, we will write $d_\alpha(\nu, \nu')$ instead of $d_Y(\nu, \nu')$ for convenience (when the quantity makes sense). More generally, if $\Gamma$ is a collection of laminations, we define $\pi_Y(\Gamma) = \cup_{\nu \in \Gamma} \pi_Y(\nu)$ and denote by $d_Y(\Gamma)$ the quantity $\text{diam}_{\mathcal{C}(Y)}(\pi_Y(\Gamma))$. We say that a collection of laminations $\Gamma$ intersects the subsurface $Y$ essentially if $\pi_Y(\Gamma)$ is non-empty. Similarly, if $\Gamma, \Gamma'$ are collections of laminations that intersect $Y$ essentially, we define $d_Y(\Gamma, \Gamma') = \text{diam}_{\mathcal{C}(Y)}(\pi_Y(\Gamma) \cup \pi_Y(\Gamma'))$. A collection of subsurfaces $\Gamma$ is called filling if for any $\nu \in GL(S)$ there is $Y \in \Gamma$ such that $\pi_Y(\nu)$ is non-empty.

The following lemma provides an upper bound for a subsurface projection distance in terms of intersection numbers.

**Lemma 2.3** ([Hem01], Lemma 2.1; [MM00], Section 2.4). If $Y \subset S$ is a subsurface or $Y = S$, and $\alpha, \beta$ are curves on $S$ that intersect $Y$ essentially, then

$$d_Y(\alpha, \beta) \leq 2i(\alpha, \beta) + 2.$$  

If $Y$ is an annular subsurface the above bound holds with multiplicative and additive factors $1$.

We state the Bounded geodesic image theorem proved by Masur and Minsky in [MM00].

**Theorem 2.4.** [MM00] Given a surface $S$ there is a constant $M = M(S)$ such that whenever $Y$ is a subsurface and $g = \{\gamma_i\}_{i \in I}$ is a geodesic in $C(S)$ such that $\gamma_i$ intersects $Y$ essentially for all $i \in I$, then $d_Y(g) \leq M$.

Later, Webb proved that the value of $M$ can be chosen to be independent of the surface [Webb15]. We state a corollary of Theorem 2.4, which follows from the stability of quasi-geodesics in Gromov hyperbolic spaces (Theorem 1.7, Chapter III.H in [BH99]):
Lemma 2.7. Given $k \geq 1, c \geq 0$ and a surface $S$, there exists a constant $A = A(k, c, S)$ such that the following holds. Let $\{\gamma_i\}_{i \in I}$ be a $(k, c)$-quasi-geodesic in $C(S)$ which is also 1-Lipschitz and let $Y$ be a subsurface of $S$. If every $\gamma_i$ intersects $Y$ essentially, then for every $i, j \in I$:
$$d_Y(\gamma_i, \gamma_j) \leq A.$$ We say that two subsurfaces $Y, Z$ are overlapping if the multicurve $\partial Y$ intersects $Z$ essentially and the multicurve $\partial Z$ intersects $Y$ essentially. The following relationship between subsurface projection distances was found in [Behr06] and an elementary proof with explicit constants was later obtained in [Man13]:

Theorem 2.6 (Behrstock inequality). If $Y, Z \subset S$ are overlapping subsurfaces and $\alpha$ is a lamination that intersects both of them essentially, then
$$d_Y(\alpha, \partial Z) \geq 10 \implies d_Z(\alpha, \partial Y) \leq 4.$$

We also state a useful lemma on the convergence of the projection distances (we note that the definition of the projection distance in [BM15] is slightly different from ours, but this only results in a bounded change of the additive error compared to their statement).

Lemma 2.7 ([BM15], Lemma 2.7). Suppose that a sequence of curves $\{\nu_i\}$ converges to a lamination $\nu$ in the Hausdorff topology on $G\mathcal{L}(S)$. Let $Y$ be a subsurface, so that $\nu$ intersects $Y$ essentially. Then for any geodesic lamination $\nu'$ that intersects $Y$ essentially we have
$$d_Y(\nu, \nu') \leq d_Y(\nu_i, \nu')$$
for all $i$ sufficiently large.

Finally, we state the following proposition:

Proposition 2.8 ([Min00], p. 121-122). Let $\nu$ be the unstable or stable lamination of a pseudo-Anosov map $\Psi$ on a surface $S$ and let $\Gamma$ be a collection of curves on $S$. Then there is a constant $C_{\Psi, \Gamma} > 0$ such that if $Y \subset S$ is a subsurface such that $\Gamma$ intersects $Y$ essentially, then
$$d_Y(\nu, \Gamma) \leq C_{\Psi, \Gamma}.$$ 2.8. Relative twisting. In Section 2.7, the projection distances between laminations for the annular subsurfaces were defined. Here we extend the definition to allow us to compute projection distances between a lamination and a point in Teichmüller space, and between two points in Teichmüller space. We will refer to any of these quantities as the relative twisting about a curve $\alpha$.

Suppose $\alpha$ is a curve, $X$ is a point in Teichmüller space and $\nu$ is a geodesic lamination on $S$. Suppose that $\nu$ intersects $\alpha$ essentially. Consider the Gromov compactification of the annular cover $X_\alpha$ that corresponds to the cyclic subgroup $\langle \alpha \rangle$ in the fundamental group $\pi_1(S)$, with the hyperbolic metric pulled back from $X$. Consider the complete preimage $\tilde{\nu}$ of $\nu$ in $X_\alpha$. Let $\alpha^\perp$ be a geodesic arc in $X_\alpha$ that is perpendicular to the geodesic in the homotopy class of the core curve. Define $d_\alpha(X, \nu)$ to be the maximal distance between $\tilde{\omega}$ and $\alpha^\perp$ in $\mathcal{C}(X_\alpha)$, where $\tilde{\omega}$ is any arc of $\tilde{\nu}$ that connects two boundary components of $X_\alpha$ and $\alpha^\perp$ is any perpendicular. We refer to Section 3 in [Min06] for another way to measure the amount that a lamination twists around a curve in a hyperbolic surface using the projection of lifts in the universal cover. We note that the quantity in their definition differs from ours by at most a bounded change.

Lastly, we define $d_\alpha(X, Y)$, where $X, Y$ are two points in Teichmüller space. Let $S_\alpha$ be the compactification of the annular cover that corresponds to $\alpha$. Let $X_\alpha, Y_\alpha$ be the compactified covers with the hyperbolic metrics defined as before. Using the first metric, construct a geodesic arc $\alpha^\perp_X$ perpendicular to the geodesic in the homotopy class of the core curve. Similarly, construct a geodesic arc $\alpha^\perp_Y$. Define $d_\alpha(X, Y)$ to be the maximal distance between $\alpha^\perp_X$ and $\alpha^\perp_Y$ in $\mathcal{C}(S_\alpha)$, over all possible choices of the perpendiculars.

2.9. Thurston metric on Teichmüller space. We assume that $S$ has no boundary. For a background on the Thurston metric we refer to [Thu86] and [PT07], while here we mention the necessary notions and state the results that we will use.

In [Thu86], Thurston showed that the unique largest chain-recurrent lamination $\Lambda(X, Y)$, called the maximally stretched lamination, such that any map from $X$ to $Y$ realizing the infimum in Equation (1), multiplies the arc length along the lamination by the factor of $\exp(\tau_{p, q}(X, Y))$. Generically, $\Lambda(X, Y)$ is a curve ([Thu86]; Section 10).

For a maximal lamination $\nu$, Thurston constructed a homeomorphism $\mathcal{F}_\nu : \mathcal{T}(S) \to \mathcal{MF}($, where $\mathcal{MF}($ is the subspace of measured foliations transverse to $\nu$ and standard near the cusps (the latter means that every puncture has a neighborhood in which the leaves are homeotopic to that puncture and the transverse measure of a (non-compact) arc going out to a cusp is infinite). The image of a point $X$ in the Teichmüller space under $\mathcal{F}_\nu$ is the horocyclic foliation of the pair $(X, \nu)$. The space $\mathcal{MF}(\nu)$ has a natural cone structure given by the shearing coordinates which
produce an embedding \( s_\nu : T(S) \to \mathbb{R}^{\dim T(S)} \) such that the image is an open convex cone. We refer to \[\text{Bon96}, \text{Thb14}\] for the details of the construction. We assume that \( \nu \) is not an ideal triangulation of \( S \). The stretch paths form open rays from the origin in the image of \( s_\nu \). Namely, given any \( X \) in Teichmüller space \( T (S) \), a maximal lamination \( \nu \), and \( t \in \mathbb{R} \), we let \( \text{stretch}(X, \nu, t) \) be a unique point in \( T(S) \), such that
\[ s_\nu(\text{stretch}(X, \nu, t)) = e^t s_\nu(X). \]
Every stretch path converges to the projective class of the horocyclic foliation in the Thurston boundary as \( t \to \infty \) ([Pap91], Theorem 5.1). Every stretch path such that the stump of \( \nu \) is uniquely ergodic converges to the projective class of the stump of \( \nu \) as \( t \to - \infty \) [Thb07]. We summarize these results in one theorem.

**Theorem 2.9** ([Pap91], [Thb07]). Suppose that \( \nu \) is a maximal lamination on \( S \) that is not an ideal triangulation. The stretch path \( \text{stretch}(X, \nu, t) \) converges to the projective class of the horocyclic foliation \([F_\nu(X)]\) in the Thurston boundary as \( t \to \infty \). Every stretch path \( \text{stretch}(X, \nu, t) \) such that stump(\( \nu \)) is uniquely ergodic converges to the projective class of the stump \( \text{stump}(\nu) \) in the Thurston boundary as \( t \to - \infty \).

### 2.10. Twisting parameter along a Thurston geodesic.

We introduce the notions necessary to state Theorem 2.10. We say that a curve \( \alpha \) interacts with a lamination \( \nu \) if \( \alpha \) is a leaf of \( \nu \) or if \( \nu \) intersects \( \alpha \) essentially. We call \([a, b]\) the \( \varepsilon \)-active interval for \( \alpha \) along a Thurston geodesic \( G(t) \) if \([a, b]\) is the maximal interval such that \( \ell_{\alpha}(a) = \ell_{\alpha}(b) = \varepsilon \). We use the notation \( \text{Log}(x) = \max(1, \log(x)) \). Denote \( X_t = G(t) \).

**Theorem 2.10** ([DLRT20], Theorem 3.1). There exists a constant \( \varepsilon_0 > 0 \) such that the following statement holds. Let \( X, Y \in T_\varepsilon(S) \) and \( \alpha \) be a curve that interacts with \( \Lambda(X, Y) \). Let \( G \) be any geodesic from \( X \) to \( Y \) and \( \ell_\alpha = \min_{t} \ell_\alpha(t) \). Then
\[ d_\alpha(X, Y) \geq \frac{1}{\ell_\alpha} \text{Log} \frac{1}{\ell_\alpha}. \]
If \( \ell_\alpha < \varepsilon_0 \), then \( d_\alpha(X, Y) \geq d_\alpha(X_a, X_b) \), where \([a, b]\) is the \( \varepsilon_0 \)-active interval for \( \alpha \). Further, for all sufficiently small \( \ell_\alpha \), the relative twisting \( d_\alpha(X_t, \Lambda(X, Y)) \) is uniformly bounded for all \( t \leq a \) and \( \ell_\alpha(t) \geq e^{t-b} \ell_\alpha(b) \) for all \( t \geq b \). All errors in this statement depend only on \( \varepsilon_0 \).

**Remark 2.11.** We note that the statement of Theorem 2.10 remains true if the condition \( X, Y \in T_\varepsilon(S) \) is replaced with the weaker condition \( \ell_\alpha(X), \ell_\alpha(Y) \geq \varepsilon_0 \). The proof is identical. This will be crucial for us to make Corollary 8.3.

### 3. Construction of the lamination

In this section we construct a quasi-geodesic \( \{\alpha_i\} \) in the curve graph of the seven-times punctured sphere \( S_{0,7} \) converging to the ending lamination \( \lambda \) in the Gromov boundary. We thank the referee for suggesting simpler proofs.

#### 3.1. Alpha sequence.

Denote by \( S = S_{0,7} \) the seven-times punctured sphere, obtained by doubling a regular heptagon on the plane along its boundary. Consider four curves \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) on \( S \) as shown in Figure 1.

![Figure 1](image_url)

**Figure 1.** The curves \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) and \( \delta_0, \delta_1 \) on \( S \).

Let \( \rho \) be the finite order homeomorphism of \( S \) which is realized by the counterclockwise rotation along the angle of \( \frac{2 \pi}{7} \). In other words, the map \( \rho \) rotates \( S \) by 3 ‘clicks’ counterclockwise. Let \( Y_0, Y_1, Y_2, Y_3 \) be the subsurfaces of \( S \) with the boundary curves \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \), respectively, and with 3 punctures each. Denote by \( \tau \) the partial pseudo-Anosov map on \( S \) supported on the subsurface \( Y_2 \) and obtained as the composition of two half-twists \( \tau = H_3^1 \circ H_0 \) (the core curves are shown in Figure 1).

For any \( n \in \mathbb{N} \), let \( \varphi_n = \tau^n \circ \rho \). Let \( \{r_n\}_{n=1}^{\infty} \) be a strictly increasing sequence of natural numbers. We will impose further conditions on \( \{r_n\} \) in Section 5. Set
\[ \Phi_i = \varphi_{r_1} \varphi_{r_2} \cdots \varphi_{r_{i-1}} \varphi_{r_i}. \]

Define the curves \( \alpha_i = \Phi_i(\alpha_0) \) for every \( i \in \mathbb{N} \). Denote by \( Y_i \) the subsurface with the boundary curve \( \alpha_i \) and 3 punctures.
Observe that for any $a, b, c \in \mathbb{N}$:

\[
\begin{align*}
\alpha_1 &= \varphi_c(a_0), \alpha_2 = \varphi_b \varphi_c(a_0), \alpha_3 = \varphi_a \varphi_b \varphi_c(a_0); \\
Y_1 &= \varphi_c(Y_0), Y_2 = \varphi_b \varphi_c(Y_0), Y_3 = \varphi_a \varphi_b \varphi_c(Y_0).
\end{align*}
\]

(4)

In particular, for $i = 1, 2, 3$ we have that $\Phi_i(a_0) = \alpha_i, \Phi_i(Y_0) = Y_i$.

We begin with the observations on the sizes of the subsurface projections between the curves in the sequence $\{\alpha_i\}$.

**Claim 3.1.** There is a constant $c > 0$, so that for every $i \geq 2$

\[
d_Y(\alpha_{i-2}, \alpha_{i+2}) \geq cr_{i-1} - 1.
\]

**Proof.** First we expand the expression using Equation (3), then simplify it by applying Equation (4) and using the fact that the mapping class group acts on the curve graph by isometries, and then apply the triangle inequality:

\[
d_Y(\alpha_{i-2}, \alpha_{i+2}) = d_Y(\Phi_{i-2}(\alpha_0), \Phi_{i-2}(\alpha_0)) \\
= d_Y(\alpha_0, \tau r_{i-1}(\rho a_3)) \\
\geq d_Y(\alpha_0, \tau r_{i-1}(\rho a_3)) - d_Y(\tau r_{i-1}(\rho a_3), \tau r_{i-1}(\rho a_3)) \\
= d_Y(\alpha_0, \tau r_{i-1}(\rho a_3)) - 1.
\]

(5)

Since the mapping class $\tau$ restricts to a pseudo-Anosov map on the surface $Y_2$, by Proposition 3.6 in [MM99] we have $d_Y(\alpha_0, \tau^n(\alpha_0)) \geq cn$ for some $c > 0$, so the result follows.

**Lemma 3.2.** There is a constant $i_0 \in \mathbb{N}$ such that for every $i_0 \leq i < j < k$ with $j - i \geq 2, k - j \geq 2$, the curves $\alpha_i, \alpha_k$ intersect $Y_j$ essentially and

\[
d_Y(\alpha_i, \alpha_k) \geq cr_{j-1} - 9.
\]

**Proof.** Since the sequence $\{r_n\}$ is strictly increasing, we can choose $i_0$ such that $cr_{i_1} - 9 \geq 10$ for all $i \geq i_0$. The proof is by induction on $n = k - i$.

**Base:** $n = 4$. It follows from Claim 3.1.

**Step.** Suppose that $k - i = n + 1$. We show that the curve $\alpha_i$ intersects the subsurface $Y_j$ essentially, the case of the curve $\alpha_k$ is similar. If $j - i < 4$, it follows from Equation (4) together with $i(\alpha_0, \alpha_2) > 0, i(\alpha_0, \alpha_3) > 0$. If $j - i \geq 4$, then applying the induction hypothesis to the triple $i < i + 2 < j$, we obtain $d_Y(\alpha_i, \alpha_k) \geq cr_{i+1} - 9$. If $i(\alpha_i, \alpha_j) = 0$, then since the subsurface projection distance for the disjoint curves is at most 2 ([MM00], lemma 2.2), we have $d_Y(\alpha_i, \alpha_j) \leq 2$, which contradicts the choice of $i_0$. Therefore, $i(\alpha_i, \alpha_j) \neq 0$ and hence the curve $\alpha_i$ intersects $Y_j$ essentially.

Now we prove that $d_Y(\alpha_i, \alpha_k) \geq cr_{j-1} - 9$. By the triangle inequality, we have

\[
d_Y(\alpha_i, \alpha_k) \geq d_Y(\alpha_i, \alpha_j) + d_Y(\alpha_j, \alpha_k).
\]

Hence $d_Y(\alpha_i, \alpha_k) \geq d_Y(\alpha_{i-2}, \alpha_{i+2}) - d_Y(\alpha_{i-2}, \alpha_{i+2}) - d_Y(\alpha_{i-2}, \alpha_{i+2})$. If $j - 2 - i < 2$, then $\alpha_{i-2}, \alpha_{i+2}$ are disjoint and $d_Y(\alpha_{i-2}, \alpha_{i+2}) \leq 2$. If $j - 2 - i \geq 2$, then by the induction hypothesis we have $d_Y(\alpha_{i-2}, \alpha_{i+2}) \geq cr_{j-3} - 9 \geq cr_{i+1} - 9 \geq 10$.

Since $\partial Y_j = \alpha_j$, by Theorem 2.6 we have $d_Y(\alpha_i, \alpha_{i-2}) \leq 4$. Similarly, $d_Y(\alpha_{i+2}, \alpha_k) \leq 4$. Together with Claim 3.1, we obtain

\[
d_Y(\alpha_i, \alpha_k) \geq d_Y(\alpha_{i-2}, \alpha_{i+2}) - d_Y(\alpha_{i-2}, \alpha_{i+2}) - d_Y(\alpha_{i-2}, \alpha_{i+2}) \geq cr_{j-1} - 1 - 4 - 4 = cr_{j-1} - 9.
\]

Next, we prove the main result of the section.

**Proposition 3.3.** The path $\{\alpha_i\}$ is a quasi-geodesic in the curve graph $C(S)$.

**Proof.** Let $i_1 \in \mathbb{N}$ be such that $i_1 \geq i_0 + 1$ and $cr_{i_1} - 9 \geq M + 1$ for all $i \geq i_1$, where $i_0$ is the constant from Lemma 3.2 and $M$ is the constant from Theorem 2.4. We prove that if $i_1 \leq i < k$ with $k - i \geq 7d - 4$ for $d \in \mathbb{N}$, then $d_Y(\alpha_i, \alpha_k) \geq d$.

Let $G$ be a geodesic between $\alpha_i$ and $\alpha_k$ in the curve graph. By Lemma 3.2 and Theorem 2.4, for each $j \in \{i + 2, \ldots, k - 2\}$ there is a curve $v$ in $G$ such that $v$ does not intersect the subsurface $Y_j$ essentially. We show that if a curve $v$ does not intersect $Y_j$ and $Y_{j'}$ essentially for $j, j' \in \{i + 2, \ldots, k - 2\}$, then $|j - j'| < 7$. Assume on the contrary that $|j - j'| \geq 7$. Observe that for every $k \in \mathbb{N}$, the subsurfaces $\{Y_k, Y_{k+1}, Y_{k+2}, Y_{k+3}\}$ fill $S$. Indeed, by Equation (4) it is sufficient to consider the case $k = 0$, which easily follows from Figure 1. This observation allows us to find $m \in \mathbb{N}$ with $j + 2 \leq m \leq j' - 2$, such that the curve $v$ intersects $Y_m$ essentially. From Lemma 3.2 we know
that $d_{Y_m}(\alpha_j, \alpha_{j'}) \geq cr_{m-1} - 9 \geq 10$. On the other hand, since $i(v, \alpha_j) = i(v, \alpha_{j'}) = 0$, by the triangle inequality we have
\[ d_{Y_m}(\alpha_j, \alpha_{j'}) \leq d_{Y_m}(\alpha_j, v) + d_{Y_m}(v, \alpha_{j'}) \leq 2 + 2 = 4, \]
contradiction.

For each $j \in \{i+2, \ldots, k-2\}$ map the curve $\alpha_j$ to some vertex in $G$ that does not intersect $Y_j$ essentially. We have shown that this map is at most 7-to-1. Also by Lemma 3.2 it omits the endpoints of $G$, therefore if $k - i \geq 7d - 4$, then $|\{i + 2, \ldots, k - 2\}| \geq 7d - 7$ and $d_S(\alpha_i, \alpha_k) \geq d$. It follows that path $\{\alpha_i\}$ is a quasi-geodesic. \qed

We obtain an immediate corollary from Theorem 2.2:

**Corollary 3.4.** There is an ending lamination $\lambda$ on $S$ representing a point in the Gromov boundary of $C(S)$, such that
\[ \lim_{i \to \infty} \alpha_i = \lambda. \]
Furthermore, every limit point of $\{\alpha_i\}$ in $\mathbb{P}ML(S)$ defines a projective class of transverse measure on $\lambda$.

In the remainder of the section we prove more claims about the sequence $\{\alpha_i\}$ that will be used in Section 7.

Let $i_1 \in \mathbb{N}$ be the constant from the proof of Proposition 3.3. We show:

**Lemma 3.5.** For every $i_1 \leq i < j$ with $j - i \geq 5$, the curves $\alpha_i, \alpha_j$ fill $S$.

**Proof.** The triples $i < i + 2 < j$, $i < i + 3 < j$ satisfy the conditions of Lemma 3.2. Hence
\[ d_{Y_{i+2}}(\alpha_i, \alpha_j), d_{Y_{i+3}}(\alpha_i, \alpha_j) \geq cr_{i+1} - 9 \geq \max\{10, M + 1\}. \]
If $\alpha_i, \alpha_j$ are disjoint, then $d_{Y_{i+2}}(\alpha_i, \alpha_j) \leq 2$, contradiction. If $d_S(\alpha_i, \alpha_{i+2}) = 2$, let $\{\alpha_i, \alpha', \alpha_j\}$ be a geodesic in the curve graph between $\alpha_i$ and $\alpha_j$. By Theorem 2.4, the curve $\alpha'$ does not intersect $Y_{i+2}$ and $Y_{i+3}$ essentially. A curve that does not intersect $Y_{i+2}$ and $Y_{i+3}$ essentially is either $\alpha_{i+2}$ or $\alpha_{i+3}$: indeed, by Equation (4) it is enough to consider the case $i = 0$, which follows from Figure 1. Equation (4) also gives $d_S(\alpha_i, \alpha_{i+2}) = d_S(\alpha_i, \alpha_{i+3}) > 1$, contradiction. Therefore the curves $\alpha_i, \alpha_j$ fill $S$. \qed

**Remark 3.6.** The sequence of subsurfaces $\{\gamma_i\}_{i \geq j}$ satisfies the conditions of Theorem 4.1 in [BLMR20] for sufficiently large $j \in \mathbb{N}$ with $m = 2, n = 3$. This gives another proof of Proposition 3.3.

Let $i_0 \in \mathbb{N}$ be the constant from Lemma 3.2. We show:

**Claim 3.7.** For each $i \geq i_0$, there is a unique curve $\beta_i$ on $S$ such that $i(\beta_i, \alpha_i) = i(\beta_i, \alpha_{i+4}) = 0$. Further, $\beta_i$ is disjoint from $\alpha_{i+1}, \alpha_i, \alpha_{i+2}$ and $\alpha_{i+3}$.

**Proof.** Let $\beta_i$ be a curve such that $i(\beta_i, \alpha_i) = i(\beta_i, \alpha_{i+4}) = 0$. By Claim 3.1, we have $d_{Y_{i+2}}(\alpha_i, \alpha_{i+4}) \geq cr_{i+1} - 1 \geq 10$. If the curve $\beta_i$ intersects $Y_{i+2}$ essentially, we have $d_{Y_{i+2}}(\alpha_i, \alpha_{i+4}) \leq d_{Y_{i+2}}(\alpha_i, \beta_i) + d_{Y_{i+2}}(\beta_i, \alpha_{i+4}) \leq 2 + 2 = 4$, contradiction.

By applying the homeomorphism $\Phi_i^{-1}$, replace the triple $\{\beta_i, \alpha_i, \alpha_{i+4}\}$ with the triple $\{\Phi_i^{-1}(\beta_i), \alpha_0, \varphi_{r_{i+1}}(\alpha_3)\}$ using Equation (4). Denote the curve $\Phi_i^{-1}(\beta_i)$ by $\beta_0$. We proved that $\beta_0$ does not intersect $Y_2$ essentially. Together with $i(\beta_0, \alpha_0) = 0$, from Figure 1 we have that either $\beta_0 = \alpha_1$ or $\beta_0 \subset Y_1$. Put the curves $\alpha_1, \alpha_2, \rho_3$ in minimal position and apply the homeomorphism $\tau_{r_{i+1}}$. This gives representatives of the curves $\alpha_1$ and $\varphi_{r_{i+1}}(\alpha_3)$ that are in minimal position, which shows that $i(\alpha_1, \varphi_{r_{i+1}}(\alpha_3)) > 0$, therefore $\beta_0 \neq \alpha_1$. This also shows that there is a unique curve $\beta_0$ in $Y_1$ as in Figure 2 such that $i(\beta_0, \varphi_{r_{i+1}}(\alpha_3)) = 0$. Hence the curve $\beta_i = \Phi_i(\beta_0)$ is unique. Further, the curve $\beta_0$ is disjoint from the curves $\alpha_1, \alpha_2, \alpha_3$, hence by Equation (4) $\beta_i$ is disjoint from $\alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}$. \qed

![Figure 2](https://via.placeholder.com/150)

**Figure 2.** The curve $\beta_0$ on $S$. 
Claim 3.8. For each $i \in \mathbb{N}$ there are exactly three curves on $S$ that are disjoint from $\alpha_i$ and $\alpha_{i+3}$. For each $i \geq i_0 + 1$, only one of them intersects both $\alpha_{i-1}$ and $\alpha_{i+4}$ essentially. Further, this curve intersects $\alpha_{i+1}$ and $\alpha_{i+2}$ essentially.

Proof. For the first statement, by Equation (4) it is sufficient to consider the case $i = 0$, and it follows from Figure 1 that these curves are $\rho^{-1}(\beta_0), \beta_0$ and $\rho^3(\beta_0)$. By Claim 3.7, the curves $\Phi_i(\rho^{-1}(\beta_0)) = \beta_{i-1}$, $\Phi_i(\beta_0) = \beta_i$ do not intersect essentially either $\alpha_{i-1}$ or $\alpha_{i+4}$ for $i \geq i_0 + 1$. By Claim 3.7, the curve $\Phi_i(\rho^3(\beta_0))$ intersects $\alpha_{i-1}$ and $\alpha_{i+4}$ essentially for $i \geq i_0 + 1$. Further, the curve $\rho^3(\beta_0)$ intersects $\alpha_1$ and $\alpha_2$ essentially, hence $\Phi_i(\rho^3(\beta_0))$ intersects $\alpha_{i+1}$ and $\alpha_{i+2}$ essentially.

Claim 3.9. If a curve on $S$ is disjoint from $\alpha_i$ and $\alpha_{i+2}$ for some $i \in \mathbb{N}$, then it is also disjoint from $\alpha_{i+1}$.

Proof. By Equation (4) it is sufficient to consider the case $i = 0$. Notice that the curve $\alpha_1$ is a boundary component of a unique subsurface which is filled by the curves $\alpha_0$ and $\alpha_2$. Therefore a curve on $S$ that is disjoint from $\alpha_0$ and $\alpha_2$, is also disjoint from $\alpha_1$, which shows the claim.

Claim 3.10. For each $i \in \mathbb{N}$ there is no curve on $S$ that is disjoint from $\alpha_{i+1}, \alpha_{i+2}$ and intersects $\alpha_i, \alpha_{i+3}$ essentially.

Proof. By Equation (4) it is sufficient to consider the case $i = 0$. If a curve $\gamma$ on $S$ is disjoint from $\alpha_1$ and $\alpha_2$, then one of the following holds: $\gamma = \alpha_1, \gamma = \alpha_2, \gamma \subset Y_1, \gamma \subset Y_2$. If $\gamma = \alpha_1$ and $\gamma \subset Y_1$, then $\gamma$ is disjoint from $\alpha_0$, if $\gamma = \alpha_2$ or $\gamma \subset Y_2$, then $\gamma$ is disjoint from $\alpha_3$, so the result follows.

We have the following corollary:

Corollary 3.11. If a curve $\gamma$ on $S$ is disjoint from some curves in the sequence $\{\alpha_i\}_{i \geq i_1}$, then one of the following holds: $\gamma$ is disjoint from 5 consecutive curves, $\gamma$ is disjoint from two curves $\alpha_i, \alpha_j$ with $j - i = 3$, $\gamma$ is disjoint from 3 consecutive curves or $\gamma$ is disjoint from 1 curve.

Proof. Let $\ell \geq i_1$ be the smallest index so that $\gamma$ is disjoint from $\alpha_\ell$ and $r \geq \ell$ be the largest index so that $\gamma$ is disjoint from $\alpha_r$. By Lemma 3.5, we have $r - \ell \leq 4$. If $r - \ell = 4$, then by Claim 3.7, $\gamma$ is disjoint from 5 consecutive curves. If $r - \ell = 3$, then by Claim 3.8, $\gamma$ is disjoint only from $\alpha_\ell$ and $\alpha_r$. If $r - \ell = 2$, then by Claim 3.9, $\gamma$ is disjoint from 3 consecutive curves. The case $r - \ell = 1$ is impossible by Claim 3.10. If $r - \ell = 0$, then $\gamma$ is disjoint from 1 curve in $\{\alpha_i\}_{i \geq i_1}$.

4. INVARIANT BIGON TRACK

In this section, we introduce a maximal birecurrent bigon track on $S$ that is invariant under the homeomorphisms $\Phi_i$ defined in Equation (3). We refer the reader to [HP92] for more details on train tracks and specifically to §3.4 in [HP92] for more details on bigon tracks. The bigon track $T$ is shown in Figure 3:

![Figure 3](image)

Figure 3. The bigon track $T$ with a numbering of some of its branches.

The complement to $T$ in $S$ consists of 7 punctured monogons, 3 trigons and one bigon. The shaded region in Figure 4 shows a part of the bigon in the complement of $T$.

Let $V(T)$ be the convex cone consisting of all non-negative real assignments of weights to the branches of $T$ that satisfy the switch conditions. Pick the ordered subset of 9 branches of $T$ as in Figure 3. Notice that every non-negative assignment of weights to the chosen branches can be uniquely promoted to a vector in $V(T)$. Denote by $e_i$ the vector in $V(T)$ that assigns the weight 1 to the $i$-th branch ($i = 1, \ldots, 9$) and the weight 0 to all other branches in the chosen set. It follows that $V(T)$ is the non-negative orthant in the vector space $W(T)$ of all real assignments of weights to the branches of $T$ (that satisfy the switch conditions) with basis $e_1, \ldots, e_9$.

The dimension of the space of measured laminations on $S$ is equal to 8, and the natural map from $V(T)$ to $\mathcal{ML}(S)$ is not injective because $T$ has a bigon. Namely, we can show:
Claim 4.1. The space of measured laminations carried by $T$ is naturally identified with the linear quotient cone $V'(T) = V(T) / \sim$, where for $\mu_1, \mu_2 \in V(T)$ we let $\mu_1 \sim \mu_2$ when $\mu_1 - \mu_2 \in \text{span}(2e_2 - 2e_4 + e_6 - e_8 + e_9) \subset W(T)$.

Proof. According to Proposition 3.4.1 in [HP92] and since $\dim V(T) - \dim \mathcal{ML}(S) = 1$, it is sufficient to find two distinct vectors $v_1, v_2 \in V(T)$ that correspond to the same measured lamination. Indeed, it then follows that vectors $\mu_1, \mu_2 \in V(T)$ correspond to the same measured lamination if and only if $\text{span}(\mu_1 - \mu_2) = \text{span}(v_1 - v_2) \subset W(T)$. Consider $v_1 = 4e_2 + 2e_6 + 2e_9$ and $v_2 = 4e_4 + 2e_8$. We leave it for the reader to verify that both of them correspond to the curve in Figure 5. \qed

Proposition 4.2. The bigon track $T$ is $\Phi_i$-invariant.

Proof. It is enough to check that $T$ is invariant under the mapping classes $\tau$ and $\tau \circ \rho$. We refer to Figure 6 and Figure 7 for the verification. \qed

Figure 4. The bigon in the complement of $T$.

Figure 5. A curve on $S$ that can be represented as a vector in $V(T)$ in two different ways.

Figure 6. The action of $\tau$ on $T$. 
Denote by $A$ the matrix of the induced action of $\tau$ on the cone $V(T)$ in the basis $\{\epsilon_1, \ldots, \epsilon_n\}$. Similarly, denote by $B$ the matrix of the induced action of $\tau \circ \rho$ on the cone $V(T)$ in the same basis. We show:

**Proposition 4.3.** The matrices $A$ and $B$ are as follows:

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Further, the vector $v = (\phi, 1, 0, 0, 0, 0, 0, 0, 0)^T$ is an eigenvector of $A$ with the eigenvalue $\phi^2$, where $\phi = \frac{1 + \sqrt{5}}{2}$.

**Proof.** Let $w_i = 2e_i$. The matrices $A$ and $B$ do not change if expressed in the basis $\{w_1, \ldots, w_9\}$. It is sufficient to find the images of the vectors $w_i$, $i = 1, \ldots, 9$. We refer to Figure 8, Figure 9, Figure 10, Figure 11, Figure 12, Figure 13, Figure 14, Figure 15, Figure 16 and leave the verification for the reader. Finally, the vector $v = (\phi, 1, 0, 0, 0, 0, 0, 0, 0)^T$ corresponds to the unstable lamination of $\tau$ on $S$.

**Figure 7.** The action of $\rho$ followed by $\tau$ on $T$.

**Figure 8.** The curve corresponding to the vector $w_1$ and its images under $\tau$ and $\tau \circ \rho$, respectively.
Figure 9. The curve corresponding to the vector $w_2$ and its images under $\tau$ and $\tau \circ \rho$, respectively.

Figure 10. The curve corresponding to the vector $w_3$ and its images under $\tau$ and $\tau \circ \rho$, respectively.

Figure 11. The curve corresponding to the vector $w_4$ and its images under $\tau$ and $\tau \circ \rho$, respectively.

Figure 12. The curve corresponding to the vector $w_5$ and its images under $\tau$ and $\tau \circ \rho$, respectively.
5. Estimating the intersection numbers

Let $\gamma_0$ be the multicurve on $S$ that corresponds to the vector $w_1 + w_3 \in V(T)$ as in Figure 17. Define the multicurves $\gamma_i = \Phi_i(\gamma_0)$. In this subsection we will coarsely estimate the intersection numbers between pairs of multicurves in the sequence $\{\gamma_i\}$. To state the result we introduce some notation.
Let \( f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2} \) for \( n \geq 2 \) be the Fibonacci sequence. Define the numbers \( c_i = 2f_{2r_i-2} \) for \( i \geq 1 \). We assume that the sequence \( \{r_n\} \) satisfies

\[
c_1 \neq 0, \quad \sum_{i=1}^{\infty} \frac{c_i}{c_{i+1}} < \infty.
\]

We prove:

**Proposition 5.1.** There is a constant \( i_2 \in \mathbb{N} \) such that for \( i_2 \leq i < j \) with odd \( j - i \), the following holds:

\[
i(\gamma_{i-1}, \gamma_j) \geq i(\gamma_i, \gamma_j) \approx c_{i+1}c_{i+3} \ldots c_j.
\]

The multiplicative constants are independent of \( i \) and \( j \).

To prove this proposition we will study the asymptotic behavior of the matrix products involving matrices \( A \) and \( B \) from Proposition 4.3. We start with elementary observations about Fibonacci sequence.

**Claim 5.2.** For \( m \geq 0 \), the following holds: \( 2f_{m+1} + f_m = f_{m+3}, 2f_{m+1} + f_m + f_{m+2} = f_{m+4} \).

**Proof.** We have:

\[
2f_{m+1} + f_m = f_{m+1} + f_{m+1} + f_m = f_{m+1} + f_{m+1} + f_{m+1} = f_{m+3}.
\]

\[
2f_{m+1} + f_m + f_{m+2} = f_{m+1} + f_{m+1} + f_{m+2} = f_{m+4}.
\]

Let \( \phi = \frac{1 + \sqrt{5}}{2} \) be the golden ratio.

**Claim 5.3.** For \( m \geq 1 \), the following holds:

\[
\phi^{-2}f_{2m} - \phi^{-2m} = f_{2m-2}, \quad \phi^{-1}f_{2m} - \phi^{-2m} = f_{2m-1}, \quad \phi f_{2m} + \phi^{-2m} = f_{2m+1}, \quad \phi^2 f_{2m} + \phi^{-2m} = f_{2m+2}.
\]

**Proof.** By Binet’s formula, we have \( f_{2m} = \frac{\psi^m - \psi^{-m}}{\sqrt{5}} \), where \( \psi = \frac{1 - \sqrt{5}}{2} \). Since \( \psi^2 = \phi^{-2} \), we have \( f_{2m} = \frac{\phi^m - \phi^{-2m}}{\sqrt{5}} \).

Since \( \phi^2 - \phi^{-2} = \phi - \phi^{-1} = \sqrt{5} \), we have

\[
\phi^{-2}f_{2m} - \phi^{-2m} = \frac{\phi^{2m} - \phi^{-2m} - \phi^{-2m}\sqrt{5}}{\sqrt{5}} = \frac{\phi^{2m} - \phi^{-2m}(\phi^{-2} + \sqrt{5})}{\sqrt{5}} = \frac{\phi^{2m} - \phi^{-2m} + \phi^{-2m}\sqrt{5}}{\sqrt{5}} = f_{2m-2}.
\]

\[
\phi^{-1}f_{2m} - \phi^{-2m} = \frac{\phi^{2m} - \phi^{-2m}}{\sqrt{5}} = \frac{\phi^{2m} - \phi^{-2m}(\phi^{-1} + \sqrt{5})}{\sqrt{5}} = \frac{\phi^{2m} - \phi^{-2m} + \phi^{-2m}\sqrt{5}}{\sqrt{5}} = f_{2m-1}.
\]

\[
\phi f_{2m} + \phi^{-2m} = \frac{\phi^{2m+1} - \phi^{-2m+1} + \phi^{-2m}\sqrt{5}}{\sqrt{5}} = \frac{\phi^{2m+1} - \phi^{-2m+1}(\phi^{-2} - \sqrt{5})}{\sqrt{5}} = \frac{\phi^{2m+1} - \phi^{-2m+1} - \phi^{-2m}\sqrt{5}}{\sqrt{5}} = f_{2m+1}.
\]

\[
\phi^2 f_{2m} + \phi^{-2m} = \frac{\phi^{2m+2} - \phi^{-2m+2} + \phi^{-2m}\sqrt{5}}{\sqrt{5}} = \frac{\phi^{2m+2} - \phi^{-2m+2}(\phi^2 - \sqrt{5})}{\sqrt{5}} = \frac{\phi^{2m+2} - \phi^{-2m+2} - \phi^{-2m}\sqrt{5}}{\sqrt{5}} = f_{2m+2}.
\]

Next, we show:
Claim 5.4. For \( n \geq 1 \), the matrix \( A^n \) is as follows:

\[
A^n = \begin{pmatrix}
    f_{2n+1} & f_{2n} & 0 & 0 & f_{2n} & f_{2n-1} - 1 & f_{2n} & f_{2n+2} - 1 & 0 \\
    f_{2n} & f_{2n-1} & 0 & 0 & f_{2n-1} - 1 & f_{2n-2} + 1 & f_{2n-1} - 1 & f_{2n+1} - 1 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Proof. The proof is by induction.

Base: \( n = 1 \). It holds since \( f_2 = 1, f_3 = 2, f_4 = 3 \).

Step. Using Claim 5.2, we calculate:

\[
A^{n+1} = A^n \cdot A =
\]

\[
\begin{pmatrix}
    2f_{2n+1} + f_{2n} & f_{2n+1} + f_{2n} & 0 & 0 & f_{2n+1} + f_{2n} & f_{2n} + f_{2n-1} - 1 & f_{2n+1} + f_{2n} & 2f_{2n+1} + f_{2n} + f_{2n+2} - 1 & 0 \\
    2f_{2n} + f_{2n-1} & f_{2n} + f_{2n-1} & 0 & 0 & f_{2n} + f_{2n-1} - 1 & f_{2n} + f_{2n-2} + 1 & f_{2n} + f_{2n-1} - 1 & 2f_{2n} + f_{2n-1} + f_{2n+1} - 1 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    f_{2n+3} & f_{2n+2} & 0 & 0 & f_{2n+2} & f_{2n+1} - 1 & f_{2n+2} & f_{2n} + 2 & f_{2n+4} - 1 & 0 \\
    f_{2n+2} & f_{2n+1} & 0 & 0 & f_{2n+1} - 1 & f_{2n+2} & f_{2n} & f_{2n+4} - 1 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

Corollary 5.5. For \( n \geq 1 \), the matrix \( A^n B \) is as follows:

\[
A^n B = \begin{pmatrix}
    0 & 0 & f_{2n} & f_{2n+1} - 1 & f_{2n+2} & f_{2n+2} - 1 & f_{2n} & 0 & f_{2n-1} - 1 \\
    0 & 0 & f_{2n-1} - 1 & f_{2n} + 1 & f_{2n+1} - 1 & f_{2n+1} - 1 & f_{2n} + 1 & 0 & f_{2n-2} - 1 \\
    1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Proof. Direct check.

Claim 5.6. For \( n \geq 1 \), the matrix \( A^n B \) can be expressed as follows:

\[
(7) \quad A^n B = 2f_{2n} N + M + \phi^{-2n} L,
\]
where
\[
N = \begin{pmatrix}
0 & 0 & 1/2 & \phi/2 & \phi^2/2 & \phi^3/2 & 1/2 & 0 & 1/2\phi \\
0 & 0 & 1/2\phi & 1/2 & \phi/2 & \phi^2/2 & 1/2\phi & 0 & 1/2\phi^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \\
M = \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & -1 & 1 & -1 & -1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}, \\
L = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\
0 & -1 & 0 & 1 & 1 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Further, the following holds:

1. $N^2 = 0$ and $\text{rk}(N) = 1$.
2. $(MN)^2 = MN$ and $MN(v) = v$ for $v = (0, 0, \phi, 1, 0, 0, 0, 0, 0)^T$.
3. $(NM)^2 = NM$.
4. $NL = LN = 0$.
5. $L^2 = 0$.

**Proof.** Equation (7) holds by Corollary 5.5 together with Claim 5.3. The rest is a direct check. □

Let $\| \cdot \|$ denote the operator norm induced by the standard norm on $V(T)$ with basis $\{e_1, \ldots, e_9\}$.

**Claim 5.7.** There is a constant $C > 0$ such that for $m, n \in \mathbb{N}$, the following holds:

\[
\left\| \frac{A^n B}{2f_{2n}} - N \right\| \leq C \cdot \frac{1}{f_{2n}}, \quad \left\| \frac{A^m B A^n B}{2f_{2n}} - MN \right\| \leq C \cdot \frac{f_{2m}}{f_{2n}}.
\]

**Proof.** By Claim 5.6, we have

\[
\frac{A^n B}{2f_{2n}} - N = \frac{1}{2f_{2n}}(M + \phi^{-2n}L).
\]

Hence

\[
\left\| \frac{A^n B}{2f_{2n}} - N \right\| \leq \frac{1}{2f_{2n}} \left\| M + \phi^{-2n}L \right\| \leq \frac{1}{2f_{2n}} \left( \left\| M \right\| + \left\| L \right\| \right).
\]

By Claim 5.6, we have

\[
\frac{A^m B A^n B}{2f_{2n}} = \frac{f_{2m}}{f_{2n}} \left( MN + \frac{1}{2f_{2m}}M^2 + \frac{\phi^{-2m}}{2f_{2m}}LM + \frac{\phi^{-2n}}{2f_{2m}}ML \right).
\]

Hence

\[
\frac{A^m B A^n B}{2f_{2n}} - MN = \frac{f_{2m}}{f_{2n}} \left( MN + \frac{1}{2f_{2m}}M^2 + \frac{\phi^{-2m}}{2f_{2m}}LM + \frac{\phi^{-2n}}{2f_{2m}}ML \right).
\]

Therefore

\[
\left\| \frac{A^m B A^n B}{2f_{2n}} - MN \right\| \leq \frac{f_{2m}}{f_{2n}} \left( \left\| MN \right\| + \frac{1}{2f_{2m}} \left\| M \right\|^2 + \frac{\phi^{-2m}}{2f_{2m}} \left\| LM \right\| + \frac{\phi^{-2n}}{2f_{2m}} \left\| ML \right\| \right)
\]

\[
\leq \frac{f_{2m}}{f_{2n}} \left( \left\| MN \right\| + \left\| M \right\|^2 + \left\| LM \right\| + \left\| ML \right\| \right).
\]

Letting $C = \max\{\left\| M \right\| + \left\| L \right\|/2, \left\| MN \right\| + \left\| M \right\|^2 + \left\| LM \right\| + \left\| ML \right\|\}$ concludes the proof. □

Observe that the matrix $A^n B$ is the induced matrix of the homeomorphism $\varphi_{n+1}$ since $\varphi_{n+1} = \tau^{n+1} \circ \rho = \tau^n \circ (\tau \circ \rho)$. Then the matrix $P_i$ defined as $P_i = A^{r_i-1} B A^{r_{i+1}-1}$ for $i \geq 1$ corresponds to $\varphi_r \varphi_{r_{i+1}}$. We show:
Claim 5.8. There are constants $C' > 0$ and $j_0 \in \mathbb{N}$ such that for $j_0 \leq j < k$ with odd $k - j$, the following holds:

$$\left\| \frac{P_j}{c_{j+1}} \cdot \frac{P_{j+2}}{c_{j+3}} \cdot \ldots \cdot \frac{P_{k-1}}{c_k} - MN \right\| \leq C' \cdot \sum_{i=j}^{\infty} \frac{c_i}{c_{i+1}}.$$  

$$\left\| \frac{A^{r_j+1-1}B}{c_{j+1}} \cdot \frac{P_{j+2}}{c_{j+3}} \cdot \ldots \cdot \frac{P_{k-1}}{c_k} - MN \right\| \leq C' \cdot \sum_{i=j}^{\infty} \frac{c_i}{c_{i+1}}.$$  

Proof. By the definition of $P_i$ and $c_i$ we have

$$\frac{P_i}{c_{i+1}} = \frac{A^{r_i-1}BA^{r_i+1-1}B}{2f_{2^{r_{i+1}+2}}},$$  

therefore by Claim 5.7, we get

$$\left\| \frac{P_i}{c_{i+1}} - MN \right\| \leq C \cdot \frac{c_i}{c_{i+1}}.$$  

It follows from Equation (6) that $\sum_{i=1}^{\infty} \frac{\left\| P_i/c_{i+1} - MN \right\|}{c_{i+1}} < \infty$. Since the matrix $MN$ is idempotent by Claim 5.6, we can invoke Lemma 9.1 (see Equation (23)) to conclude that there is a constant $j_0 \in \mathbb{N}$ such that for $j_0 \leq j < k$:

$$\left\| \frac{P_j}{c_{j+1}} \cdot \frac{P_{j+2}}{c_{j+3}} \cdot \ldots \cdot \frac{P_{k-1}}{c_k} - MN \right\| \leq 2 \cdot \left( \frac{1}{C} \cdot \sum_{i=j}^{\infty} \frac{c_i}{c_{i+1}} \right) \cdot \|MN\|^2.$$  

Together with the triangle inequality and the first inequality in Claim 5.7, we obtain:

$$\left\| \frac{A^{r_j+1-1}B}{c_{j+1}} \cdot \frac{P_{j+2}}{c_{j+3}} \cdot \ldots \cdot \frac{P_{k-1}}{c_k} - MN \right\| \leq \left\| \frac{A^{r_j+1-1}B}{c_{j+1}} \cdot \frac{P_{j+2}}{c_{j+3}} \cdot \ldots \cdot \frac{P_{k-1}}{c_k} - \frac{A^{r_j+1-1}B}{c_{j+1}} MN \right\| +$$

$$+ \left\| \frac{A^{r_j+1-1}B}{c_{j+1}} MN - MN \right\| \leq \left( \|N\| + \frac{2C}{c_{j+1}} \right) \cdot 2 \cdot \left( \frac{1}{C} \cdot \sum_{i=j}^{\infty} \frac{c_i}{c_{i+1}} \right) \cdot \|MN\|^2 + \frac{2C}{c_{j+1}} \cdot \|MN\|.$$  

Letting $C' = (\|N\| + 2C) \cdot 2C\|MN\|^2 + 2C\|MN\|$ concludes the proof. \qed

We prove the main result of the section:

Proof of Proposition 5.1. Using Equation (3), we can write

$$i(\gamma_{i-1}, \gamma_j) = i(\Phi_{i-1}(\gamma_0), \Phi_{i-1}(\varphi_{r_1} \ldots \varphi_{r_j}(\gamma_0))) = i(\gamma_0, \varphi_{r_1} \ldots \varphi_{r_j}(\gamma_0)).$$  

We can express the multicurve $\varphi_{r_1} \ldots \varphi_{r_j}(\gamma_0)$ as a vector in $V(T)$ as follows: $P_iP_{i+2} \ldots P_{i-1}(w_1 + w_3)$. Notice that the measured lamination that corresponds to the vector $MN(w_1 + w_3) = \frac{1}{2}w_1 + \frac{1}{2}w_2$ is the unstable lamination of the homeomorphism $\rho \tau \rho^{-1}$, which has a positive intersection number with the curve that corresponds to $w_3$, hence also with the multicurve $\gamma_0$. Since the natural map $V(T) \to ML(S)$ and the intersection number $i(\cdot, \cdot)$ are continuous, by Claim 5.8 we can choose $i_2 \in \mathbb{N}$ so that for $i_2 \leq i < j$, the intersection number of the measured lamination $P_{i+2} \ldots P_{i-1}(w_1 + w_3)$ and $\gamma_0$ is bounded above and below from zero, where the bound is independent of $i$ and $j$. Hence for $i_2 \leq i < j$, the intersection number $i(\gamma_{i-1}, \gamma_j)$ is equal to $c_{i+1}c_{i+3} \ldots c_j$ up to a fixed multiplicative constant.

Similarly, we can write

$$i(\gamma_j, \gamma_j) = i(\Phi_j(\gamma_0), \Phi_j(\varphi_{r_{i+1}} \ldots \varphi_{r_j}(\gamma_0))) = i(\gamma_0, \varphi_{r_{i+1}} \ldots \varphi_{r_j}(\gamma_0)).$$  

We can express the multicurve $\varphi_{r_{i+1}} \ldots \varphi_{r_j}(\gamma_0)$ as a vector in $V(T)$ as follows: $A^{r_j+1-1}BP_{i+2}P_{i+4} \ldots P_{i-1}(w_1 + w_3)$. Notice that the measured lamination that corresponds to the vector $MN(w_1 + w_3) = \frac{1}{2}w_1 + \frac{1}{2}w_2$ is the unstable lamination of the homeomorphism $\tau$, which has a positive intersection number with the curve that corresponds to $w_1$, hence also with the multicurve $\gamma_0$. By Claim 5.8, for $i_2 \leq i < j$ the intersection number of the measured lamination $A^{r_j+1-1}BP_{i+2} \ldots P_{i-1}(w_1 + w_3)$ and $\gamma_0$ is bounded above and below from zero, where the bound is independent of $i$ and $j$. Hence for $i_2 \leq i < j$, the intersection number $i(\gamma_i, \gamma_j)$ is equal to $c_{i+1}c_{i+3} \ldots c_j$ up to a fixed multiplicative constant. \qed
6. Non-unique Ergodicity

In this section we show that the ending lamina \( \lambda \) constructed in Section 3 is not uniquely ergodic. Namely, we prove that the appropriately scaled subsequences of multicurves \( \{ \gamma_i \} \) with even and odd indices converge to non-zero measured laminations that are not multiples of each other. Further, we show that the limiting measured laminations are ergodic and are the only ergodic transverse measures on \( \lambda \).

Claim 6.1. There are \( \lambda_e, \lambda_o \in \mathcal{ML}(S) \) such that the following holds as \( n \to \infty \):

\[
\frac{\gamma_{2n}}{c_2c_4 \ldots c_{2n}} \to \lambda_e, \quad \frac{\gamma_{2n+1}}{c_1c_3 \ldots c_{2n+1}} \to \lambda_o.
\]

Proof. Notice that the vector \( \left( \prod_{i=1}^n \frac{P_{2i-1}}{c_{2i}} \right)(w_1 + w_3) \) corresponds to \( \frac{\gamma_{2n}}{c_{2i}c_4 \ldots c_{2n}} \) and the vector \( \frac{A_{i-1}^{-1}B}{c_1} \left( \prod_{i=1}^n \frac{P_{2i-1}}{c_{2i+1}} \right)(w_1 + w_3) \) corresponds to \( \frac{\gamma_{2n+1}}{c_1c_3 \ldots c_{2n+1}} \). By Equation (6), Claim 5.7 and Lemma 9.1, the infinite products \( \prod_{i=1}^\infty \frac{P_{2i-1}}{c_{2i}} \) and \( \prod_{i=1}^\infty \frac{P_{2i}}{c_{2i+1}} \) converge. Hence the vectors \( \left( \prod_{i=1}^n \frac{P_{2i-1}}{c_{2i}} \right)(w_1 + w_3) \) and \( \frac{A_{i-1}^{-1}B}{c_1} \left( \prod_{i=1}^n \frac{P_{2i-1}}{c_{2i+1}} \right)(w_1 + w_3) \) converge as \( n \to \infty \), and the result follows.

Claim 6.2. As \( n \to \infty \),

\[
\frac{i(\gamma_{2n}, \lambda_e)}{i(\gamma_{2n}, \lambda_o)} \to 0, \quad \frac{i(\gamma_{2n+1}, \lambda_e)}{i(\gamma_{2n+1}, \lambda_o)} \to \infty.
\]

Proof. Suppose that \( 2n \geq i_2 \) where \( i_2 \in \mathbb{N} \) is the constant from Proposition 5.1. Then by Proposition 5.1 for \( m > n \) we have:

\[
i \left( \gamma_{2n}, \frac{\gamma_{2m}}{c_2c_4 \ldots c_{2m}} \right) \leq \frac{i\left( \gamma_{2n}, \gamma_{2m} \right)}{c_{2c_4} \ldots c_{2m}} = \frac{1}{c_2c_4 \ldots c_{2m}}.
\]

Since it holds for every \( m > n \), by passing to the limit as \( m \to \infty \), we have \( i(\gamma_{2n}, \lambda_e) \leq \frac{1}{c_2c_4 \ldots c_{2n}} \). In particular, \( i(\gamma_{2n}, \lambda_e) \neq 0 \).

Similarly, for \( m > n \) we have

\[
i \left( \gamma_{2n+1}, \frac{\gamma_{2m+1}}{c_1c_3 \ldots c_{2m+1}} \right) \leq \frac{i\left( \gamma_{2n+1}, \gamma_{2m+1} \right)}{c_1c_3 \ldots c_{2m+1}} = \frac{1}{c_1c_3 \ldots c_{2m+1}}.
\]

Since it holds for every \( m > n \), by passing to the limit as \( m \to \infty \), we have \( i(\gamma_{2n+1}, \lambda_e) \leq \frac{1}{c_1c_3 \ldots c_{2n+1}} \). In particular, \( i(\gamma_{2n+1}, \lambda_e) \neq 0 \).

Putting this together, we obtain

\[
i(\gamma_{2n}, \lambda_e) \leq \frac{1}{c_2c_4 \ldots c_{2n}}, \quad i(\gamma_{2n+1}, \lambda_e) \leq \frac{1}{c_1c_3 \ldots c_{2n+1}}.
\]

It follows from Equation (6) that \( \frac{c_{2n-1}}{c_{2n}} \to 0 \) as \( n \to \infty \). Hence \( \frac{c_1c_3 \ldots c_{2n-1}}{c_2c_4 \ldots c_{2n}} \to 0 \), and therefore \( \frac{i(\gamma_{2n}, \lambda_e)}{i(\gamma_{2n}, \lambda_o)} \to 0 \) as \( n \to \infty \). Similarly, for \( m > n \) we have:

\[
i \left( \gamma_{2n+1}, \frac{\gamma_{2m}}{c_2c_4 \ldots c_{2m}} \right) \leq \frac{i\left( \gamma_{2n+1}, \gamma_{2m} \right)}{c_2c_4 \ldots c_{2m}} = \frac{1}{c_2c_4 \ldots c_{2m}},
\]

Since it holds for every \( m > n \), by passing to the limit as \( m \to \infty \), we have \( i(\gamma_{2n+1}, \lambda_e) \leq \frac{1}{c_2c_4 \ldots c_{2n}} \). It follows from Equation (6) that \( \frac{c_{2n+1}}{c_2c_4 \ldots c_{2n}} \to \infty \), hence \( \frac{c_1c_3 \ldots c_{2n+1}}{c_2c_4 \ldots c_{2n}} \to \infty \) and therefore \( \frac{i(\gamma_{2n+1}, \lambda_e)}{i(\gamma_{2n+1}, \lambda_o)} \to \infty \) as \( n \to \infty \).

Corollary 6.3. The measured laminations \( \lambda_e, \lambda_o \) are non-zero and are not the multiples of each other.

Proof. It was shown in Claim 6.2 that \( i(\gamma_{2n}, \lambda_e) \neq 0 \) and \( i(\gamma_{2n}, \lambda_o) \neq 0 \) for \( 2n \geq i_2 \), hence \( \lambda_e \neq 0 \) and \( \lambda_o \neq 0 \). If \( \lambda_e, \lambda_o \) are multiples of each other, then the sequence \( \frac{i(\gamma_{2n}, \lambda_e)}{i(\gamma_{2n}, \lambda_o)} \) is constant, which contradicts Claim 6.2.

Proposition 6.4. The ending lamina \( \lambda \) is not uniquely ergodic.

Proof. The measured lamina \( \lambda_e \) can be expressed as \( \lambda_e = \lambda'_e + \lambda''_e \), where \( \lambda'_e \) is the measured lamina that corresponds to the vector \( \left( \prod_{i=1}^\infty \frac{P_{2i-1}}{c_{2i}} \right)(w_1) \) and \( \lambda''_e \) is the measured lamina that corresponds to the vector \( \left( \prod_{i=1}^\infty \frac{P_{2i-1}}{c_{2i}} \right)(w_3) \). The simple closed curve that corresponds to the vector \( \prod_{i=1}^n \frac{P_{2i-1}}{c_{2i}}(w_1) \) is at distance 2 from the curve \( c_{2n} \) in the curve graph for each \( n \geq 1 \), hence the sequence of curves \( \prod_{i=1}^n \frac{P_{2i-1}}{c_{2i}}(w_1), n \geq 1 \) converges to \( \lambda \) in the Gromov boundary as \( n \to \infty \). Then by Theorem 2.2, the measured lamina \( \lambda'_e \) is either supported on \( \lambda \) or zero. Repeating the same argument for \( \lambda''_e \) and since \( \lambda \neq 0 \) by Corollary 6.3, we obtain that \( \lambda \) is supported on \( \lambda \). By a similar argument, the measured lamina \( \lambda_o \) is supported on \( \lambda \). By Corollary 6.3, \( \lambda \) is not uniquely ergodic.
Let $C(\lambda)$ denote the convex cone of transverse measures supported on $\lambda$. Since the measured lamination $\lambda_n$ is carried by $T$, the ending lamination $\lambda$, being the support of $\lambda_n$, is carried by $T$. Hence every measured lamination in $C(\lambda)$ is carried by $T$. In fact, we can show more:

Claim 6.5. For every $n \in \mathbb{N}$, the image of the convex cone $P_1 P_3 \ldots P_{2n-1}(V(T))$ under the natural map to $\mathcal{ML}(S)$ contains $C(\lambda)$.

Proof. Notice that $P_1 P_3 \ldots P_{2n-1}(V(T))$ is isomorphic to the convex cone of the non-negative real assignments of weights satisfying the switch conditions to the branches of the train track $\varphi_{r_1} \ldots \varphi_{r_n}(T)$. It is then sufficient to show that the measured lamination $\lambda_n$ is carried by the train track $\varphi_{r_1} \ldots \varphi_{r_n}(T)$. Indeed, in this case every measured lamination in $C(\lambda)$ is carried by $\varphi_{r_1} \ldots \varphi_{r_n}(T)$. Since the measured lamination corresponding to the vector $(\prod_{i=n+1}^{\infty} \frac{P_{2i-1}}{c_{2i}})(w_1 + w_3)$ is carried by $T$ by Proposition 4.2, the measured lamination corresponding to the vector $\frac{P_1}{c_2} \frac{P_3}{c_4} \ldots \frac{P_{2n-1}}{c_{2n}}(\prod_{i=n+1}^{\infty} \frac{P_{2i-1}}{c_{2i}})(w_1 + w_3)$ is carried by $\varphi_{r_1} \ldots \varphi_{r_n}(T)$. Since the latter measured lamination is $\lambda$, the result follows.

To find all ergodic transverse measures on $\lambda$, we study the shapes of the convex cones $P_1 P_3 \ldots P_{2n-1}(V(T))$ as $n \to \infty$. Roughly speaking, we will show that for each $n \in \mathbb{N}$, the set of the generators of the cone $P_1 P_3 \ldots P_{2n-1}(V(T))$ can be divided into two subsets such that the angles between pairs of generators within each of the subsets converge to zero as $n \to \infty$ (Lemma 6.10). From this the upper bound on the number of ergodic transverse measures will follow.

Endow $V(T)$ with the standard inner product with respect to the basis $\{e_1, \ldots, e_9\}$. We start with the following helpful observation:

Claim 6.6. For $i \in \mathbb{N}$,
$$\langle P_i(e_1), e_1 \rangle = \frac{c_i}{2}, \quad \langle P_i(e_3), e_3 \rangle = \frac{c_i+1}{2}.$$  

Proof. By Equation (8), we have
$$c_i = c_{i+1} MN + c_i N M + M^2 + \phi^{-2(r_{i-1})} LM + \phi^{-2(r_{i+1})} ML.$$  

Notice that $MN(e_1) = 0$ since $e_1 \notin \ker N$. We have $NM(e_1) = \frac{1}{2} e_1 + \frac{1}{2 \phi} e_2$. It is also a direct check that $c(e_1) = (M N(e_1), e_1) = 0$, hence $\langle P_1(e_1), e_1 \rangle = \frac{c_1}{2}$. Similarly, we have $MN(e_3) = \frac{1}{2} e_3 + \frac{1}{2 \phi} e_4$ and $NM(e_3, e_3) = \langle M^2(e_3), e_3 \rangle = \langle LM(e_3), e_3 \rangle = \langle ML(e_3), e_3 \rangle = 0$. Hence $\langle P_i(e_3), e_3 \rangle = \frac{c_i+1}{2}$.

Next, we prove:

Claim 6.7. For every $n \in \mathbb{N}$ the following holds. If $e_i \notin \ker N$, then
$$\|P_1 P_3 \ldots P_{2n-1} MN(e_i)\| \geq \frac{c_{i+1} c_i}{\phi^{2n+1}}.$$

If $e_i \in \ker N$, then
$$\|P_1 P_3 \ldots P_{2n-1} N M(e_i)\| \geq \frac{c_{i+1} c_i}{\phi^{2n+1}}.$$

Proof. Notice that if $e_i \notin \ker N$, then $\langle MN(e_i), e_3 \rangle \geq \frac{c_i}{2 \phi}$. Since the matrices $P_i$ are non-negative for $i \in \mathbb{N}$, we have $\langle P_i(v), e_3 \rangle \geq \langle P_i(v, e_3), e_3 \rangle$ for all $v \in V(T)$. Applying Claim 6.6 $n$ times, it follows that $\langle P_1 P_3 \ldots P_{2n-1} MN(e_i), e_3 \rangle \geq \frac{c_{i+1} c_i}{\phi^{2n+1}}$, therefore $\|P_1 P_3 \ldots P_{2n-1} MN(e_i)\| \geq \frac{c_{i+1} c_i}{\phi^{2n+1}}$.

Similarly, if $e_i \in \ker N$, then $\langle NM(e_i), e_3 \rangle \geq \frac{c_i}{2 \phi}$. Since the matrices $P_i$ are non-negative for $i \in \mathbb{N}$, together with Claim 6.6 it follows that $\langle P_1 P_3 \ldots P_{2n-1} N M(e_i), e_1 \rangle \geq \frac{c_{i+1} c_i}{\phi^{2n+1}}$, hence $\|P_1 P_3 \ldots P_{2n-1} N M(e_i)\| \geq \frac{c_{i+1} c_i}{\phi^{2n+1}}$.

Let $K_i$ be the matrix defined as $K_i = M^2 + \phi^{-2(r_{i-1})} LM + \phi^{-2(r_{i+1})} ML$ for $i \in \mathbb{N}$. Then
$$P_i = c_{i+1} MN + c_i N M + K_i.$$  

Notice that $\|K_i\| \leq \|M\|^2 + \|LM\| + \|ML\|$ for all $i \in \mathbb{N}$.

Claim 6.8. There is a constant $D > 0$ such that for every $n \in \mathbb{N}$ and $1 \leq i \leq 9$ the following holds:
$$\|P_1 P_3 \ldots P_{2n-1} MN(e_i)\| \leq D^{n+1} c_{i+1} c_i \ldots c_{2n-1},$$
$$\|P_1 P_3 \ldots P_{2n-1} N M(e_i)\| \leq D^{n+1} c_{i+1} c_i \ldots c_{2n},$$
$$\|P_1 P_3 \ldots P_{2n-1} K_{2n+1}(e_i)\| \leq D^{n+1} c_{i+1} c_i \ldots c_{2n}.$$
Proof. Consider the first inequality. Expressing each matrix $P_i$ in the product $P_1 P_2 \ldots P_{n-1} N M$ as in Equation (12) and opening up the brackets, we obtain a sum of $3^n$ matrices with coefficients. It follows from the identity $N^2 = 0$ that $c_1 c_3 \ldots c_{2n-1}$ is the largest coefficient in the sum which is multiplied by a non-zero matrix. Since the norm of each matrix in the sum is at most $(\sup_i \{\|MN_i\|, \|NM_i\|, \|K_i\|\})^{n+1}$, by the triangle inequality we have

$$\|P_1 P_2 \ldots P_{n-1} MN(e_i)\| \leq 3^n \cdot c_2 c_4 \ldots c_{2n} \cdot \left( \sup_i \{\|MN_i\|, \|NM_i\|, \|K_i\|\} \right)^{n+1}.$$ 

Letting $D = 3 \cdot (\sup_i \{\|MN_i\|, \|NM_i\|, \|K_i\|\})$ concludes the first inequality. The second and the third inequalities follow by a similar argument and noticing that $c_2 c_4 \ldots c_{2n}$ is the largest coefficient in the corresponding sum. □

Remark 6.9. It is possible to obtain better upper bounds for $\|P_1 P_2 \ldots P_{n-1} MN(e_i)\|$ and $\|P_1 P_2 \ldots P_{n-1} K_{2n+1}(e_i)\|$ using Claim 5.8, but weaker bounds will suffice for our purposes.

Lemma 6.10. There is a constant $D' > 0$ such that for every $n \in \mathbb{N}$ the following holds. If $e_i, e_j \notin \ker N$, then

$$1 - \cos \angle(P_1 P_2 \ldots P_{n+1}(e_i), P_1 P_2 \ldots P_{n+1}(e_j)) \leq (D')^{n+1} \cdot \frac{c_1 c_3 \ldots c_{2n+1}}{c_2 c_4 \ldots c_{2n+2}}.$$ 

If $e_i, e_j \in \ker N$, then

$$1 - \cos \angle(P_1 P_2 \ldots P_{n+1}(e_i), P_1 P_2 \ldots P_{n+1}(e_j)) \leq (D')^{n+1} \cdot \frac{c_2 c_4 \ldots c_{2n}}{c_1 c_3 \ldots c_{2n+1}}.$$ 

Further, in either case $\angle(P_1 P_2 \ldots P_{n+1}(e_i), P_1 P_2 \ldots P_{n+1}(e_j)) \to 0$ as $n \to \infty$.

Proof. By Equation (12), we can write

$$P_1 P_2 \ldots P_{n+1} = P_1 P_2 \ldots P_{n-1}(c_2 c_4 \ldots c_{2n+1} + c_{2n+1} N M + c_{2n+1} N M + K_{2n+1}).$$

Consider the first inequality. Let

$$w_1 = c_{2n+2} P_1 P_2 \ldots P_{n-1} MN(e_i), \quad v_2 = c_{2n+2} P_1 P_2 \ldots P_{n-1} MN(e_j),$$

$$w_1 = P_1 P_2 \ldots P_{n-1} (c_{2n+1} N M + K_{2n+1})(e_i), \quad w_2 = P_1 P_2 \ldots P_{n-1} (c_{2n+1} N M + K_{2n+1})(e_j).$$

Notice that $w_1 + w_1 = P_1 P_2 \ldots P_{n+1}(e_i)$ and $v_2 + v_2 = P_1 P_2 \ldots P_{n+1}(e_j)$. We also have $\angle(v_1, v_2) = 0$ since the vectors $MN(e_i)$ and $MN(e_j)$ are collinear. By Claim 6.7 and Claim 6.8, we have

$$\frac{c_2 c_4 \ldots c_{2n+2}}{2^{n+1} \phi} \leq ||v_1||, ||v_2|| \leq D^{n+1} \cdot c_2 c_4 \ldots c_{2n+2},$$

so

$$||w_1||, ||w_2|| \leq D^{n+1} \cdot (c_1 c_3 \ldots c_{2n+1} + c_2 c_4 \ldots c_{2n}) \leq 2 \cdot D^{n+1} \cdot c_1 c_3 \ldots c_{2n+1}.$$ 

Then by Lemma 9.2, we have

$$1 - \cos \angle(P_1 P_2 \ldots P_{n+1}(e_i), P_1 P_2 \ldots P_{n+1}(e_j)) \leq 2 \cdot \left( 4 \cdot D^{n+2} \cdot c_2 c_4 \ldots c_{2n+2} \cdot c_1 c_3 \ldots c_{2n+1} + 4 \cdot D^{2n+2} \cdot (c_1 c_3 \ldots c_{2n+1})^2 \right)$$

$$\leq 2^{2n+2} \phi^2 \cdot 8 \cdot D^{2n+2} \cdot \frac{c_2 c_4 \ldots c_{2n+2} \cdot c_1 c_3 \ldots c_{2n+1}}{(c_2 c_4 \ldots c_{2n+2})^2}$$

$$\leq 2^{2n+6} \phi^2 \cdot D^{2n+2} \cdot \frac{c_1 c_3 \ldots c_{2n+1}}{c_2 c_4 \ldots c_{2n+2}}.$$ 

For the second inequality, notice that $c_2 c_4 \ldots P_1 P_3 \ldots P_{n-1} (MN(e_i)) = c_2 c_4 \ldots P_1 P_3 \ldots P_{n-1} (MN(e_j)) = 0$ since $N(e_i) = N(e_j) = 0$. We let

$$v_1' = c_{2n+1} P_1 P_2 \ldots P_{n-1} N M(e_i), \quad v_2' = c_{2n+1} P_1 P_2 \ldots P_{n-1} N M(e_j),$$

$$w_1' = P_1 P_2 \ldots P_{n-1} K_{2n+1}(e_i), \quad w_2' = P_1 P_2 \ldots P_{n-1} K_{2n+1}(e_j).$$

Notice that $v_1' + v_1' = P_1 P_2 \ldots P_{n+1}(e_i)$ and $v_2' + v_2' = P_1 P_2 \ldots P_{n+1}(e_j)$. We also have $\angle(v_1', v_2') = 0$ since the vectors $MN(e_i)$ and $MN(e_j)$ are collinear. Then by Lemma 9.2, Claim 6.7 and Claim 6.8, we have

$$1 - \cos \angle(P_1 P_2 \ldots P_{n+1}(e_i), P_1 P_2 \ldots P_{n+1}(e_j)) \leq 2 \cdot \left( 2 \cdot D^{2n+2} \cdot c_1 c_3 \ldots c_{2n+1} \cdot c_2 c_4 \ldots c_{2n} + D^{2n+2} \cdot (c_2 c_4 \ldots c_{2n})^2 \right)$$

$$\leq 2^{2n+2} \phi^2 \cdot D^{2n+2} \cdot \frac{3 c_1 c_3 \ldots c_{2n+1} \cdot c_2 c_4 \ldots c_{2n}}{(c_1 c_3 \ldots c_{2n+1})^2}$$

$$\leq 2^{2n+3} \phi^2 \cdot 3 \cdot D^{2n+2} \cdot \frac{c_2 c_4 \ldots c_{2n}}{c_1 c_3 \ldots c_{2n+1}}.$$


Letting $D' = 2^4 \cdot 3 \cdot D^2$ concludes the desired inequalities. By Equation (6), we have $\frac{c_n}{c_{n+1}} \to 0$ as $n \to \infty$. Hence for sufficiently large $n$ the upper bounds for $1 - \cos \langle P_1P_3 \ldots P_{2n+1}(e_i), P_1P_3 \ldots P_{2n+1}(e_j) \rangle$ decrease at least exponentially with $n$, therefore $\langle P_1P_3 \ldots P_{2n+1}(e_i), P_1P_3 \ldots P_{2n+1}(e_j) \rangle \to 0$ as $n \to \infty$. 

\begin{proposition}
\textbf{Proposition 6.11.} The measured laminations $\lambda_c, \lambda_o$ are ergodic. Further, any transverse measure on $\lambda$ is a linear combination of $\lambda_c$ and $\lambda_o$.
\end{proposition}

\begin{proof}
Let $\Delta = \{x_1 e_1 + \ldots + x_9 e_9 \mid \sum_{i=1}^9 x_i = 1\} \subset V(T)$ be the standard unit simplex. Notice that the set $P_1P_3 \ldots P_{2n-1}(V(T)) \cap \Delta$ is the convex hull of the points span$\langle P_1P_3 \ldots P_{2n-1}(e_i) \rangle \cap \Delta$ for $i = 1, \ldots, 9$. Since the infinite product $\prod_{i=1}^{\infty} P_{2n-1}(e_i)$ converges (see the proof of Claim 6.1), the sequence of compact sets $P_1P_3 \ldots P_{2n-1}(V(T)) \cap \Delta$ converges in the Hausdorff metric on $V(T)$ as $n \to \infty$. It follows from Lemma 6.10 that the limiting set is either an interval or a point. If $\lambda$ admits at least three ergodic transverse measures up to scalar, then the set of points in $\Delta$ that correspond to measured laminations in $C(\lambda)$ contains a convex triangle. Let $R > 0$ be the radius of the circumscribed circle of such triangle. For sufficiently large $n$ so that the Hausdorff distance from $P_1P_3 \ldots P_{2n-1}(V(T)) \cap \Delta$ to the limiting set is less than $R/8$, the set $P_1P_3 \ldots P_{2n-1}(V(T)) \cap \Delta$ cannot contain a 2-dimensional disk of radius $r$: it is immediate if the limiting set is a point and if the limiting set is an interval it follows from Lemma 9.3 since $R > 2\sqrt{2(8-1)}R/8$. Since the projective class of every non-zero measure in $C(\lambda)$ is represented in $P_1P_3 \ldots P_{2n-1}(V(T)) \cap \Delta$ by Claim 6.5, we arrive at a contradiction. Hence $\lambda$ admits at most two ergodic transverse measures up to scalar. Together with Proposition 6.4 we obtain that $\lambda$ admits exactly two ergodic transverse measured up to scalar.

Let $\lambda_1, \lambda_2$ be ergodic transverse measures on $\lambda$ that are not multiples of each other. Then we can write

$$\lambda_c = \alpha_1 \lambda_1 + \alpha_2 \lambda_2, \quad \lambda_o = \beta_1 \lambda_1 + \beta_2 \lambda_2$$

for $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$. Since $\lambda_o \neq 0$, at least one of the numbers $\beta_1, \beta_2$ is non-zero. Without loss of generality, assume that $\beta_1 \neq 0$. Since $\lambda_1, \lambda_2$ are filling, we can write

$$\frac{i(\gamma_{2n+1}, \lambda_c)}{i(\gamma_{2n+1}, \lambda_o)} = \frac{\alpha_1 \cdot i(\gamma_{2n+1}, \lambda_1) + \alpha_2 \cdot i(\gamma_{2n+1}, \lambda_2)}{\beta_1 \cdot i(\gamma_{2n+1}, \lambda_1) + \beta_2 \cdot i(\gamma_{2n+1}, \lambda_2)} \leq \frac{\alpha_1 \cdot i(\gamma_{2n+1}, \lambda_1) + \alpha_2 \cdot i(\gamma_{2n+1}, \lambda_2)}{\beta_1 \cdot i(\gamma_{2n+1}, \lambda_1)} = \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_1} \frac{i(\gamma_{2n+1}, \lambda_2)}{i(\gamma_{2n+1}, \lambda_1)}$$

Since by Claim 6.2, $\frac{i(\gamma_{2n+1}, \lambda_2)}{i(\gamma_{2n+1}, \lambda_1)} \to \infty$ as $n \to \infty$, it follows that $\frac{i(\gamma_{2n+1}, \lambda_c)}{i(\gamma_{2n+1}, \lambda_o)} \to \infty$ as $n \to \infty$. If $\beta_2 \neq 0$ we have

$$\frac{i(\gamma_{2n+1}, \lambda_c)}{i(\gamma_{2n+1}, \lambda_o)} = \frac{\alpha_1 \cdot i(\gamma_{2n+1}, \lambda_1) + \alpha_2 \cdot i(\gamma_{2n+1}, \lambda_2)}{\beta_1 \cdot i(\gamma_{2n+1}, \lambda_1) + \beta_2 \cdot i(\gamma_{2n+1}, \lambda_2)} \leq \frac{\alpha_1 \cdot i(\gamma_{2n+1}, \lambda_1) + \alpha_2 \cdot i(\gamma_{2n+1}, \lambda_2)}{\beta_2 \cdot i(\gamma_{2n+1}, \lambda_2)} = \frac{\alpha_1}{\beta_2} \frac{i(\gamma_{2n+1}, \lambda_1)}{i(\gamma_{2n+1}, \lambda_2)} + \frac{\alpha_2}{\beta_2} \frac{i(\gamma_{2n+1}, \lambda_2)}{i(\gamma_{2n+1}, \lambda_1)}$$

Letting $n \to \infty$, we get $\limsup_{n \to \infty} \frac{i(\gamma_{2n+1}, \lambda_c)}{i(\gamma_{2n+1}, \lambda_o)} \leq \frac{\alpha_1}{\beta_2}$, contradiction. Hence $\beta_2 = 0$. A similar argument using that $\frac{i(\gamma_{2n+1}, \lambda_c)}{i(\gamma_{2n+1}, \lambda_o)} \to \infty$ as $n \to \infty$ by Claim 6.2 shows that one of the numbers $\alpha_1, \alpha_2$ is zero. Therefore $\lambda_c, \lambda_o$ are ergodic transverse measures themselves, and the result follows.

\end{proof}

\section{7. Relative twisting bounds}

In this section we prove that the lamination $\lambda$ constructed in Section 3 has uniformly bounded annular projection distances. To show this, we return to the sequence of curves $\{\alpha_n\}$, defined in Section 3.1. First we show:

\begin{lemma}
\textbf{Lemma 7.1.} For every $i \geq 2$

$$d_{\alpha_i}(\alpha_{i-2}, \alpha_{i+2}) \leq 7.$$

\end{lemma}

\begin{proof}
By Equation (4), we have

$$d_{\alpha_i}(\alpha_{i-2}, \alpha_{i+2}) = d_{\alpha_2}(\alpha_0, \varphi_{r-1} \alpha_3) = d_{\alpha_2}(\alpha_0, \tau_{r-1} \rho_{\alpha_3}).$$

Choose a marked complete hyperbolic metric $X$ on $S$ of finite volume. Let $\tilde{\alpha}_2$ and $\tilde{\rho}_{\alpha_3}$ be geodesic lifts of $\alpha_2$ and $\rho_{\alpha_3}$ in the universal cover $\tilde{X} \cong \mathbb{H}^2$ intersecting at the point $O \in \mathbb{H}^2$. Let $\delta_0, \delta_1$ be the curves on $S$ shown in Figure 1. Let $\delta_0$ and $\delta_1$ be the geodesic lifts of $\delta_0$ in $\mathbb{H}^2$ that intersect $\tilde{\rho}_{\alpha_3}$ at the points $A, B \in \mathbb{H}^2$, respectively, such that the geodesic segment $[AB] \subset \mathbb{H}^2$ contains the point $O$ and does not contain any other intersection points of lifts of $\delta_0$ with $\tilde{\rho}_{\alpha_3}$. Let $p, q \in \partial \mathbb{H}^2$ be the endpoints of $\overrightarrow{\delta_0}$ and let $r, s \in \partial \mathbb{H}^2$ be the endpoints of $\overrightarrow{\delta_1}$. Let $(pq) \subset \partial \mathbb{H}^2$ be the open interval such that $r, s \notin (pq)$ and let $(rs) \subset \partial \mathbb{H}^2$ be the open interval such that $p, q \notin (rs)$. See Figure 18 (left).

Let $H_{\delta_0} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of the half-twist $H_{\delta_0}$ such that $\tilde{H}_{\delta_0}(O) = O$. Similarly, let $H_{\delta_1}^{-1} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of the half-twist $H_{\delta_1}^{-1}$ such that $\tilde{H}_{\delta_1}(O) = O$. Then the map $\overline{\tau} : \mathbb{H}^2 \to \mathbb{H}^2$ defined as $\overline{\tau} = H_{\delta_1}^{-1} \circ \tilde{H}_{\delta_0}$ is a lift of $\tau$. We prove that for every $n \in \mathbb{N}$, the endpoints of the curve $\overline{\tau}^n(\rho_{\alpha_3})$ in $\mathbb{H}^2$ belong to $(pq) \cup (rs)$, from which lemma will follow as we now show. Denote by $g_n$ the geodesic joining the endpoints of $\overline{\tau}^n(\rho_{\alpha_3})$. Then since $i(\alpha_0, \delta_0) = 0$, the set of geodesic lifts of $\alpha_0$ in $\mathbb{H}^2$ that intersect both $\overline{g}_2$ and $g_n$ coincides with the set of geodesic lifts of $\alpha_0$.
in $\mathbb{H}^2$ that intersect both $\tilde{\alpha}_2$ and $\tilde{\rho}\tilde{\alpha}_3$. Then the projections of $g_n$ and $\tilde{\rho}\tilde{\alpha}_3$ to the annular cover that corresponds to the hyperbolic isometry with the axis $\tilde{\alpha}_2$ and translation length $\ell_{\alpha_2}(X)$ intersect at most once. It follows that $|d_{\alpha_2}(\alpha_0, \tau^n(\rho\alpha_3)) - d_{\alpha_2}(\alpha_0, \rho\alpha_3)| \leq 4$ for every $n \in \mathbb{N}$. Then by Lemma 2.3 we have
\[
d_{\alpha_2}(\alpha_0, \tau^{r'-1}(\rho\alpha_3)) \leq d_{\alpha_2}(\alpha_0, \rho\alpha_3) + 4 \leq (i(\alpha_0, \rho\alpha_3) + 1) + 4 = 7.
\]

Now we prove that for every $n \in \mathbb{N}$, the endpoints of the curve $\tilde{\tau}^n(\rho\tilde{\alpha}_3)$ in $\partial\mathbb{H}^2$ belong to $(pq) \cup (rs)$. Let $\tilde{\delta}_1$ be the geodesic lift of $\delta_1$ in $\mathbb{H}^2$ that intersects $\tilde{\delta}_0'$ at the point $C \in \mathbb{H}^2$ such that the geodesic segment $[BC] \subset \mathbb{H}^2$ does not contain any other intersection points of lifts of $\delta_1$ with $\tilde{\delta}_0'$. Let $t, v \in \partial\mathbb{H}^2$ be the endpoints of $\tilde{\delta}_1$ and let $(tv) \subset \partial\mathbb{H}^2$ be the open interval such that $r \in (tv)$. Let $(rv) \subset \partial\mathbb{H}^2$ be the open interval such that $t \notin (rv)$. Let $w \in \partial\mathbb{H}^2$ be the endpoint of $\tilde{\rho}\tilde{\alpha}_3$ such that $w \in (rs)$. Let $(rw) \subset \partial\mathbb{H}^2$ be the half-open interval such that $v \in (rw)$. See Figure 18 (right).

It follows from Proposition 2.1 in [Sm01] that the boundary extension of $H_{\tilde{\delta}_0}$ fixes $r, s \in \partial\mathbb{H}^2$ and moves all points in $(rs)$ counterclockwise (Proposition 2.1 is about Dehn twists, but the argument applies to half-twists as well). Similarly, the boundary extension of $H_{\tilde{\delta}_1}^{-1}$ fixes $t, v \in \partial\mathbb{H}^2$ and moves all points in $(tv)$ clockwise. Further, every point in $(rw)$ is either fixed under the boundary extension of $H_{\tilde{\delta}_1}^{-1}$ (such as the point $v$) or moves clockwise (such as any point in $(rv)$). Since $i(\rho\alpha_3, \delta_1) = 0$, no geodesic lift of $\delta_1$ intersects $\tilde{\rho}\tilde{\alpha}_3$, therefore no point in $(rw)$ moves past $w$ under the boundary extension of $H_{\tilde{\delta}_1}^{-1}$. It follows that the boundary extension of $\tilde{\tau} = H_{\tilde{\delta}_1}^{-1} \circ H_{\tilde{\delta}_0}$ maps $(rw)$ to itself. Hence for every $n \in \mathbb{N}$, the boundary extension of $\tilde{\tau}^n$ maps the point $w \in \partial\mathbb{H}^2$ to a point in $(rs)$. By a similar argument, the other endpoint of $\tilde{\rho}\tilde{\alpha}_3$ is mapped to a point in $(pq)$ under the boundary extension of $\tilde{\tau}^n$ for every $n \in \mathbb{N}$, which concludes the proof.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure18.png}
\caption{Left: lifts of the curves $\alpha_2, \rho\alpha_3, \delta_0$ in the universal cover. Right: lifts of the curves $\alpha_2, \rho\alpha_3, \delta_0, \delta_1$ in the universal cover.}
\end{figure}

Let $\mu_1 = \{\alpha_{i_1}, \alpha_{i_1+5}\}$ be a collection of curves on $S$, where $i_1$ is the constant from Lemma 3.5. By Lemma 3.5 the collection of curves $\mu_1$ is a marking on $S$. We prove:

**Proposition 7.2.** There is a constant $E \in \mathbb{N}$ such that the following holds. For every curve $\gamma$ on $S$ there is $j_\gamma \in \mathbb{N}$ such that for all $j \geq j_\gamma$ the curve $\alpha_j$ intersects $\gamma$ essentially and
\[
d_\gamma(\mu_1, \alpha_j) \leq E.
\]

**Proof.** If the curve $\gamma$ intersects every curve $\alpha_j$ essentially for $j \geq i_1$, then by Corollary 2.5 we have
\[
d_\gamma(\mu_1, \alpha_j) = \max\{d_\gamma(\alpha_{i_1}, \alpha_j), d_\gamma(\alpha_{i_1+5}, \alpha_j)\} \leq A
\]
for every $j \geq i_1$. Otherwise, the curve $\gamma$ is disjoint from some curves in the sequence $\{\alpha_i\}_{i \geq i_1}$. If $\gamma$ is disjoint from $\alpha_{i_1}$, then by Lemma 3.5, $\gamma$ intersects every $\alpha_j$ essentially for $j \geq i_1 + 5$. Then by Corollary 2.5
\[
d_\gamma(\mu_1, \alpha_j) = d_\gamma(\alpha_{i_1+5}, \alpha_j) \leq A
\]
for every $j \geq i_1 + 5$. If $\gamma$ intersects $\alpha_{i_1}$ essentially, let $\ell > i_1$ be the smallest index so that $\gamma$ is disjoint from $\alpha_\ell$ and $r \geq \ell$ be the largest index so that $\gamma$ is disjoint from $\alpha_r$. By Lemma 3.5, we have $r - \ell \leq 4$. Let $j_\gamma = r + 1$. Then for $j \geq j_\gamma = r + 1$ by the triangle inequality we have

$$d_\gamma(\mu_1, \alpha_\ell) \leq d_\gamma(\mu_1, \alpha_{\ell-1}) + d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+1}) + d_\gamma(\alpha_{\ell+1}, \alpha_j).$$

By Corollary 2.5, $d_\gamma(\alpha_{\ell+1}, \alpha_j) \leq A$. Next, we show that $d_\gamma(\mu_1, \alpha_{\ell-1}) \leq \max\{A, 4\} + d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+1})$, thus it will remain to find an upper bound for $d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+1})$. If $\gamma$ intersects $\alpha_{i_1}$ essentially and is disjoint from $\alpha_{i_1+5}$, then by Corollary 2.5

$$d_\gamma(\mu_1, \alpha_{\ell-1}) = d_\gamma(\alpha_{i_1}, \alpha_{\ell-1}) \leq A.$$

Suppose that $\gamma$ intersects both $\alpha_{i_1}$ and $\alpha_{i_1+5}$ essentially. If $\ell - 1 \geq i_1 + 5$, then by Corollary 2.5

$$d_\gamma(\mu_1, \alpha_{\ell-1}) = \max\{d_\gamma(\alpha_{i_1}, \alpha_{\ell-1}), d_\gamma(\alpha_{i_1+5}, \alpha_{\ell-1})\} \leq A,$$

If $\ell - 1 < i_1 + 5$ and $r + 1 \leq i_1 + 5$, then by the triangle inequality and Corollary 2.5

$$d_\gamma(\alpha_{i_1+5}, \alpha_{\ell-1}) \leq d_\gamma(\alpha_{i_1+5}, \alpha_{r+1}) + d_\gamma(\alpha_{r+1}, \alpha_{\ell-1}) \leq A + d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}).$$

Therefore we have

$$d_\gamma(\mu_1, \alpha_{\ell-1}) = \max\{d_\gamma(\alpha_{i_1}, \alpha_{\ell-1}), d_\gamma(\alpha_{i_1+5}, \alpha_{\ell-1})\} \leq \max\{A, A + d_\gamma(\alpha_{\ell-1}, \alpha_{r+1})\} = A + d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}).$$

If $\ell - 1 < i_1 + 5$ and $r + 1 > i_1 + 5$, then since $\gamma$ intersects $\alpha_{i_1+5}$ essentially, we have $\ell < i_1 + 5$ and $r > i_1 + 5$. Then the curves in $\{\alpha_\ell\}_{\ell \geq i_1}$ that are disjoint from $\gamma$ are not consecutive. It follows from Corollary 3.11 and Claim 3.8 that either $i_1 + 5 = r - 1$ or $i_1 + 5 = r - 2$ and $\gamma$ intersects $\alpha_{r-1}$ essentially. In the first case, by Lemma 2.3 and Equation (4) we have

$$d_\gamma(\alpha_{i_1+5}, \alpha_{r+1}) = d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) \leq i(\alpha_{\ell-1}, \alpha_{r+1}) + 1 = i(\alpha_0, \alpha_2) + 1 = 3.$$

In the second case, by the triangle inequality

$$d_\gamma(\alpha_{i_1+5}, \alpha_{r+1}) = d_\gamma(\alpha_{r-2}, \alpha_{r+1}) \leq d_\gamma(\alpha_{r-2}, \alpha_{r-1}) + d_\gamma(\alpha_{r-1}, \alpha_{r+1}).$$

Notice that $d_\gamma(\alpha_{r-2}, \alpha_{r-1}) = 1$ since $\alpha_{r-2}$ and $\alpha_{r-1}$ are disjoint. Since $d_\gamma(\alpha_{r-1}, \alpha_{r+1}) \leq 3$, we have $d_\gamma(\alpha_{i_1+5}, \alpha_{r+1}) \leq 4$. We obtain that if $\ell - 1 < i_1 + 5$ and $r + 1 > i_1 + 5$, then

$$d_\gamma(\mu_1, \alpha_{\ell-1}) = \max\{d_\gamma(\alpha_{i_1}, \alpha_{\ell-1}), d_\gamma(\alpha_{i_1+5}, \alpha_{\ell-1})\}
\leq \max\{A, d_\gamma(\alpha_{i_1+5}, \alpha_{r+1}) + d_\gamma(\alpha_{r+1}, \alpha_{\ell-1})\}
\leq \max\{A, 4 + d_\gamma(\alpha_{r+1}, \alpha_{\ell-1})\}
\leq \max\{A, 4\} + d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}).

Now we find an upper bound for $d_\gamma(\alpha_{\ell-1}, \alpha_{r+1})$. Depending on the value of $r - \ell$, we consider the following cases:

Case: $r - \ell = 4$. By Claim 3.7, we have $\gamma = \beta_\ell$. By Equation (4), we have

$$d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) = d_{\beta_\ell}(\alpha_{\ell-1}, \alpha_{r+1}) = d_{\beta_\ell}(\phi_{\ell-1}^{-1}(\alpha_0), \gamma_{r+1}, \gamma_{r+2}(\alpha_3)).$$

Let $\nu'$, $\nu''$ denote the limits of the laminations $\phi_{\ell-1}^{-1}(\alpha_0)$, $\gamma_{r+1}$, $\gamma_{r+2}(\alpha_3)$ in the Hausdorff topology as $\ell \to \infty$, respectively. Notice that $\nu'$ and $\nu''$ intersect $\beta_\ell$ essentially. Then by Lemma 2.7, there is $\ell_0 \in \mathbb{N}$ such that for every $\ell \geq \ell_0$, we have

$$d_{\beta_\ell}(\phi_{\ell-1}^{-1}(\alpha_0), \gamma_{r+1}, \gamma_{r+2}(\alpha_3)) \leq d_{\beta_\ell}(\nu', \nu'').$$

Then we have

$$d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) \leq \max\left\{\sum_{\ell_0}^{\ell} \left\{d_{\beta_\ell}(\phi_{\ell-1}^{-1}(\alpha_0), \gamma_{r+1}, \gamma_{r+2}(\alpha_3))\right\}, d_{\beta_\ell}(\nu', \nu'') + 16\right\}.

Case: $r - \ell = 3$. By Claim 3.8 and the triangle inequality we can write

$$d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) = d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+4}) \leq d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+1}) + d_\gamma(\alpha_{\ell+1}, \alpha_{\ell+2}) + d_\gamma(\alpha_{\ell+2}, \alpha_{\ell+4}).$$

Notice that $d_\gamma(\alpha_{\ell+1}, \alpha_{\ell+2}) = 1$ since $\alpha_{\ell+1}$ and $\alpha_{\ell+2}$ are disjoint. By Lemma 2.3 we also have

$$d_\gamma(\alpha_{\ell+1}, \alpha_{\ell+1}) \leq i(\alpha_{\ell+1}, \alpha_{\ell+1}) + 1 = i(\alpha_0, \alpha_2) + 1 = 3.$$

Similarly, $d_\gamma(\alpha_{\ell+2}, \alpha_{\ell+4}) \leq 3$. Hence $d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) \leq 7$.

Case: $r - \ell = 2$. If $\gamma$ is disjoint from $\alpha_\ell$ and $\alpha_{\ell+2}$, then one of the following holds: $\gamma = \beta_{\ell-1}$, $\gamma = \alpha_{\ell+1}$, $\gamma \subset Y_{\ell+1}$. Indeed, by applying the homeomorphism $\phi_{\ell-1}^{-1}$ and by Equation (4) it is enough to consider the case $\ell = 1$, which follows from Figure 1. If $\gamma = \beta_{\ell-1}$, then by Claim 3.7 $\gamma$ is disjoint from $\alpha_{\ell+3}$, which is impossible. If $\gamma = \alpha_{\ell+1}$, then by Lemma 7.1 we have

$$d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) = d_{\alpha_{\ell+1}}(\alpha_{\ell-1}, \alpha_{\ell+3}) \leq 7.$$
If \( \gamma \subset Y_{\ell+1} \), we have
\[
d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+1}) = d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+3}) = d_{\Phi^{-1}_\gamma}(\alpha_0, \tau^\ell(\rho_3)),
\]
where \( \Phi^{-1}_\gamma \gamma \) is a curve in \( Y_2 \). Denote the curve \( \Phi^{-1}_\gamma \gamma \) by \( \gamma' \). Let \( \nu_\tau \) denote the unstable lamination of \( \tau \). Notice that \( \nu_\tau \) intersects every curve in \( Y_3 \) essentially and that the curves \( \tau^n(\rho_3) \) converge in the Hausdorff topology as 
\( n \to \infty \) to a lamination that contains \( \nu_\tau \). Then for sufficiently large \( n \in \mathbb{N} \) so that Lemma 2.7 applies and the curve \( \tau^n(\rho_3) \) intersects \( \gamma' \) essentially, we have
\[
d_{\gamma'}(\alpha_0, \tau^n(\rho_3)) \lesssim d_{\gamma'}(\alpha_0, \nu_\tau).
\]
By Proposition 2.8, we have \( d_{\gamma'}(\alpha_0, \nu_\tau) \leq C_{\tau, \alpha_0}. \) By the triangle inequality, we have
\[
d_{\gamma'}(\alpha_0, \tau^\ell(\rho_3)) \leq d_{\gamma'}(\alpha_0, \tau^n(\rho_3)) + d_{\gamma'}(\tau^n(\rho_3), \tau^\ell(\rho_3)) \leq (C_{\tau, \alpha_0} + 9) d_{\tau^\ell(\gamma)}(\tau^n(\rho_3), \rho_3),
\]
where \( \tau^\ell(\gamma) \) is a curve in \( Y_2 \). Denote the curve \( \tau^\ell(\gamma) \) by \( \gamma'' \). Since \( r_\ell \in \mathbb{N} \) is fixed, for sufficiently large \( n \in \mathbb{N} \) so that Lemma 2.7 applies, together with Proposition 2.8 we have
\[
d_{\gamma''}(\tau^n(\rho_3), \rho_3) \lesssim d_{\gamma''}(\nu_\tau, \rho_3) \leq C_{\tau, \rho_3}.
\]
Therefore we have
\[
d_{\gamma}(\alpha_{\ell-1}, \alpha_{\tau+1}) \leq C_{\tau, \alpha_0} + C_{\tau, \rho_3} + 18.
\]
\textbf{Case:} \( r - \ell = 1 \). This case is impossible by Claim 3.10.
\textbf{Case:} \( r - \ell = 0 \). By Lemma 2.3 we have
\[
d_{\gamma}(\alpha_{\ell-1}, \alpha_{\ell+1}) = d_{\gamma}(\alpha_{\ell-1}, \alpha_{\ell+3}) \leq i(\alpha_{\ell-1}, \alpha_{\ell+3}) + 1 = i(\alpha_0, \alpha_2) + 1 = 3.
\]
Finally, according to Equation (16), Equation (17), Equation (18), Equation (19), if we let
\[
E = \max\{A, 4\} + 2 \cdot \max \left\{ \max_{\ell \leq \ell_0} \{d_{\beta_\ell}(\alpha_{\ell+1}, \alpha_{\ell+3})\}, d_{\beta_\ell}(\alpha_0, \alpha_2) + 16\} \right\}, C_{\tau, \alpha_0} + C_{\tau, \rho_3} + 18
\]
then \( d_{\gamma}(\mu_1, \alpha_j) \leq E \) for every \( j \geq j_\gamma = r + 1 \), which concludes the proof.

In the following corollary, \( \lambda \) is the non-uniquely ergodic ending lamination on \( S \) constructed in Section 3.

\textbf{Corollary 7.3.} \textit{There is a constant \( E' \in \mathbb{N} \) such that \( d_{\gamma}(\mu_1, \lambda) \leq E' \) for all curves \( \gamma \) on \( S \).}

\textbf{Proof.} By Corollary 3.4, there is a subsequence of \( \{\alpha_i\} \) that converges in the Hausdorff topology on \( GL(S) \) to a geodesic lamination \( \lambda' \) that contains \( \lambda \). Taking an index \( i \in \mathbb{N} \) in the subsequence sufficiently large so that Lemma 2.7 applies for the annular subsurface of a curve \( \gamma \) on \( S \) and so that \( \alpha_i \) intersects \( \gamma \) essentially, we obtain
\[
d_{\gamma}(\mu_1, \alpha_i) \lesssim d_{\gamma}(\mu_1, \lambda').
\]
Since \( \lambda \subset \lambda' \), we have \( d_{\gamma}(\mu_1, \lambda) \leq d_{\gamma}(\mu_1, \lambda') \). Taking \( i \in \mathbb{N} \) sufficiently large so that Proposition 7.2 applies as well, we have
\[
d_{\gamma}(\mu_1, \lambda) \leq d_{\gamma}(\mu_1, \lambda') \leq d_{\gamma}(\mu_1, \alpha_i) + 8 \leq E + 8.
\]
Letting \( E' = E + 8 \) concludes the proof.

We remark that not all projection distances for \( \lambda \) are uniformly bounded. We prove the following:

\textbf{Claim 7.4.} Let \( \nu \) be a minimal, filling geodesic lamination on \( S \) such that \( d_Y(\mu_1, \nu) \leq G \) for some constant \( G > 0 \) and all subsurfaces \( Y \subset S \). Then
\[
d_Y(\nu, \lambda) \geq c r_{i-1} - G - 18
\]
for all \( i \geq i + 2 \).

\textbf{Proof.} By Lemma 3.2, we have
\[
d_Y(\mu_1, \alpha_i) \geq c r_{i-1} - 9
\]
for all \( i \geq i + 2 \) and \( j \geq i + 2 \). By an argument, similar to the one in Corollary 7.3, we have
\[
d_Y(\mu_1, \lambda) \geq (c r_{i-1} - 9) - 9
\]
for all \( i \geq i + 2 \). Then by the triangle inequality we have
\[
d_Y(\nu, \lambda) \geq d_Y(\mu_1, \lambda) - d_Y(\mu_1, \nu) \geq c r_{i-1} - 18 - G
\]
for all \( i \geq i + 2 \).

We obtain the following corollary which is in contrast with Theorem 1.1.
Corollary 7.5. Suppose $X_i$ is a Teichmüller geodesic such that the support of the lamination that corresponds to its vertical foliation contains the support of $\lambda$ constructed in Section 3, and such that the support of the lamination that corresponds to its horizontal foliation contains the support of $\nu$ as in Claim 7.4. Then for all sufficiently large $i \in \mathbb{N}$, the minimal length $\ell_{\alpha_i}$ of the curve $\alpha_i$ along $X_i$ satisfies:
\[ \ell_{\alpha_i} \ll \frac{1}{r_{i-1}}. \]

Proof. Since the sequence $\{r_i\}$ is strictly increasing, we can choose $i_0 \geq i_1 + 2$ such that $e r_{i_1} - G - 18 \geq \frac{e}{2} r_{i_1}$ for all $i \geq i_0$. Then the statement follows from Claim 7.4 and Theorem 6.1 in [Ra05]. In particular, $X_i$ does not stay in the thick part of the Teichmüller space. Moreover, it follows from Theorem 1.2 that $X_i$ diverges in the moduli space as $t \to \infty$. \qed

8. Geodesics in the thick part

In this section we prove Theorem 1.1. First, we prove some technical lemmas.

Lemma 8.1. Let $X_n \in T(S)$ be a sequence in Teichmüller space converging to $[\xi]$ in the Thurston boundary, and let $\eta_n$ be a curve on $S$ such that $\ell_{\eta_n}(X_n) \leq C$ for some $C > 0$. If $[\eta]$ is a limit point of the sequence $[\eta_n]$ in the Thurston boundary, then $i(\xi, \eta) = 0$.

Proof. By definition, there is a sequence $\{a_n\}$ of positive numbers, such that $a_n X_n \to \xi$ as geodesic currents. We have (Prop. 15 in [Bon88]):
\[ i(a_n X_n, a_n X_n) = a_n^2 i(X_n, X_n) = a_n^2 \pi^2 |\chi(S)|. \]
By the continuity of the intersection number, $i(a_n X_n, a_n X_n) \to i(\xi, \xi) = 0$, since $\xi \in \mathcal{ML}(S)$. Hence $a_n^2 \to 0$, and in particular, $a_n \to 0$. By definition, there is a sequence $\{b_n\}$ of non-negative numbers, such that $b_n \eta_n \to \eta$ as geodesic currents. Let $\gamma$ be a filling collection of curves on $S$, then $i(\gamma, b_n \eta_n) = b_n i(\gamma, \eta_n) \to b_n$. We also have $i(\gamma, b_n \eta_n) \to i(\gamma, \eta) < \infty$. Hence the sequence $\{b_n\}$ is bounded from above, so suppose $b_n \leq B$ for some $B > 0$. Then
\[ i(a_n X_n, b_n \eta_n) = a_n b_n \ell_{\eta_n}(X_n) \leq a_n B C. \]
Since $i(a_n X_n, b_n \eta_n) \to i(\xi, \eta)$, we obtain $i(\xi, \eta) = 0$. \qed

Let $B(S)$ be a Bers constant of $S$. We prove:

Lemma 8.2. Let $X_n, Y_n \in T(S)$ be sequences in Teichmüller space converging to $[\xi]$ and $[\zeta]$ in the Thurston boundary, respectively. Suppose that the supports of $\xi$ and $\zeta$ are minimal and filling. If $\alpha$ is a curve on $S$ such that $\ell_\alpha(X_n), \ell_\alpha(Y_n) > B(S)$ for all $n \in \mathbb{N}$, then
\[ d_\alpha(X_n, Y_n) \preceq d_\alpha(\xi, \zeta) \]
for infinitely many $n \in \mathbb{N}$.

Proof. It follows from the definition of a Bers constant for that for every $n \in \mathbb{N}$ there are curves $\eta_n$ and $\nu_n$ on $S$ that intersect $\alpha$ essentially such that $\ell_{\eta_n}(X_n), \ell_{\nu_n}(Y_n) \leq B(S)$. By the triangle inequality we have
\[ d_\alpha(X_n, Y_n) \leq d_\alpha(X_n, \eta_n) + d_\alpha(\eta_n, Y_n) + d_\alpha(\nu_n, Y_n). \]
Similarly,
\[ d_\alpha(\eta_n, \nu_n) \leq d_\alpha(\eta_n, X_n) + d_\alpha(X_n, Y_n) + d_\alpha(Y_n, \nu_n). \]
Hence
\[ d_\alpha(X_n, Y_n) - d_\alpha(\eta_n, \nu_n) \leq d_\alpha(X_n, \eta_n) + d_\alpha(Y_n, \nu_n). \]
It is sufficient to show that $d_\alpha(X_n, \eta_n), d_\alpha(Y_n, \nu_n) \preceq 0$, and that $d_\alpha(\eta_n, \nu_n) \preceq d_\alpha(\xi, \zeta)$ for infinitely many $n \in \mathbb{N}$.

We show that the relative twisting coefficients $d_\alpha(X_n, \eta_n)$ are uniformly bounded, the case of $d_\alpha(Y_n, \nu_n)$ is identical. Let $\ell_n = \ell_\alpha(X_n)$. By the Collar Lemma ([FM12], Section 13.5), the $\omega_n$-neighborhood (collar) of the geodesic representative of $\alpha$ in $X_n$ for $\omega_n = \arcsinh \left( \frac{1}{\sinh(\ell_n/2)} \right)$ is embedded in $X_n$. Consider an arc $\tilde{\eta}_n$ of the geodesic representative of $\eta_n$ inside the collar of $\alpha$ in $X_n$ with one endpoint on $\alpha$ and the other endpoint on the boundary of the neighborhood. Since the collar is embedded, the length of $\tilde{\eta}_n$ is at most $B(S)$. From the trigonometry of right triangles, we find a lower bound on the angle $\delta_n$ that $\tilde{\eta}_n$ makes with $\alpha$ in $X_n$:
\[ \sin \delta_n \geq \frac{\sinh \omega_n}{\sinh B(S)}. \]
Denote by $L_n$ the length of the orthogonal projection of a lift of $\eta_n$ on a lift of $\alpha$ in the universal cover of $X_n$ that intersect at the angle $\delta_n$. Then from the angle of parallelism formula, we have $\cosh \frac{L_n}{2} = 1$. Since $\sinh x \leq e^{x^2/2}$ and $\arccosh x \leq \ln 2x$ for $x > 0$, we find:
\[ L_n \leq 2 \arccosh \frac{\sinh B(S)}{\sinh \omega_n} = 2 \arccosh(\sinh B(S) \sin(\ell_n/2)) \leq 2 \ln(\sinh B(S)e^{\ell_n/2}) \leq \ell_n + 2B(S) - 2 \ln 2 < 3 \ell_n. \]
We estimate the relative twisting coefficients (see [Min96], Section 3):
\[
    d_\alpha(X_n, \eta_n) \leq_2 \frac{L_\alpha}{\ell_n} \leq_3 0.
\]
We show that \(d_\alpha(\eta_n, \nu_n) \leq d_\alpha(\xi, \zeta)\) for infinitely many \(n \in \mathbb{N}\). Let \([\eta] \in \mathbb{P}ML(S)\) be the limit of a subsequence of \(\{\eta_n\}\). By Lemma 8.1, \(i(\xi, \eta) = 0\). Since \(\xi\) is minimal and filling, we have \(\text{supp}(\xi) = \text{supp}(\eta)\), in particular \(\eta\) intersects \(\alpha\) essentially and \(d_\alpha(\xi, \zeta) = d_\alpha(\eta, \zeta)\). Let \(\eta'\) be the limit of a further subsequence of \(\{\eta_n\}\) in \(\mathbb{G}L(S)\). Then \(\text{supp}(\eta) \subseteq \text{supp}(\eta')\), hence \(d_\alpha(\eta, \zeta) \leq_1 d_\alpha(\eta', \zeta)\). By Lemma 2.7, \(d_\alpha(\eta', \zeta) \leq d_\alpha(\eta_n, \zeta)\) for infinitely many \(n \in \mathbb{N}\). By a similar argument for \(\{\nu_n\}\), we have \(d_\alpha(\eta_n, \zeta) \leq d_\alpha(\eta_n, \nu_n)\), hence \(d_\alpha(\xi, \zeta) \leq d_\alpha(\eta_n, \nu_n)\) for infinitely many \(n \in \mathbb{N}\), which proves the lemma.

Together with Theorem 2.10, we obtain the following corollary:

**Corollary 8.3** (Bounded annular combinatorics implies cobounded). Let \(G(t), t \in \mathbb{R}\) be a stretch path in \(T(S)\) with the horocyclic foliation \(\mathcal{F}\) such that \(G(t) \to [\xi] \in \mathbb{P}ML(S)\) as \(t \to -\infty\). Suppose that the supports of \(\xi\) and \(\zeta\) are minimal and filling. If there exists a number \(K \in \mathbb{N}\) such that \(d_\alpha(\xi, \zeta) \leq K\) for all curves \(\alpha\) on \(S\), then there exists \(\varepsilon(K) > 0\) such that \(G(t)\) lies in the thick part \(T_\varepsilon(S)\) for all \(t \in \mathbb{R}\).

**Proof.** Suppose that there is a curve \(\alpha\) on \(S\) that gets shorter than \(\varepsilon_0\) along the geodesic \(G(t)\), where \(\varepsilon_0 > 0\) is the constant in the statement of Theorem 2.10 — otherwise there is nothing to prove. Since \(G(t)\) is a stretch path, Theorem 2.10 is applicable. Let \([a, b]\) be the \(\varepsilon_0\)-active interval for \(\alpha\). Indeed, this interval in bounded: for example, if there is a sequence \(t_i \to \infty\) such that \(\ell_i(G(t_i)) \leq \varepsilon_0\), then by Lemma 8.1 we have \(i(\alpha, \xi) = 0\), which is impossible since \(\xi\) is minimal and filling. By a similar argument it can be shown that there are infinitely many numbers \(m \in \mathbb{N}\) such that \(\ell_0(G(-m)) > B(S)\). By choosing large enough \(n\), so that the interval \([-n, n]\) contains the interval \([a, b]\) and Lemma 8.2 applies for \(X_n = G(-n), Y_n = G(n)\), we conclude by combining Theorem 2.9, Theorem 2.10 with the condition \(d_\alpha(\xi, \zeta) \leq K\) that there is a lower bound on the minimal length of \(\alpha\) along \(G(t)\) that depends only on \(K\).

Finally, we prove our main result.

**Proof of Theorem 1.1.** Let \([\lambda]\) be the projective class of some non-zero transverse measure on the non-uniquely ergodic ending lamination \(\lambda\) constructed in Section 3. Let \(\nu\) be the unstable or stable lamination of a pseudo-Anosov \(\Psi\) map on \(S\), and let \(\tilde{\nu}\) be a maximal lamination on \(S\) obtained from \(\nu\) by adding finitely many leaves. Consider the projective measured foliation on \(S\) that corresponds to \([\lambda]\) and that is standard near the cusps; we also denote it by \([\lambda]\). Since \(\nu\) is minimal, filling and uniquely ergodic, the set of projective measured foliations transverse to \(\tilde{\nu}\) contains \([\lambda]\). Thus there is a point \(X \in T(S)\) such that \([\mathcal{F}_{\tilde{\nu}}(X)] = [\lambda]\) (see Section 2.9). Since stump(\(\tilde{\nu}\) = \(\nu\), by Theorem 2.9 the stretch path stretch(\(X, \tilde{\nu}, t\)) converges to \([\lambda]\) as \(t \to \infty\) and to \([\nu]\) as \(t \to -\infty\).

By Corollary 8.3, to prove that stretch(\(X, \tilde{\nu}, t\)) stays in the thick part, it is sufficient to show that the relative twisting coefficients \(d_\alpha(\nu, \lambda)\) are uniformly bounded for all curves \(\alpha\) on \(S\). Let \(\mu_1\) be the marking on \(S\) from Proposition 7.2. By the triangle inequality, we have
\[
    d_\alpha(\nu, \lambda) \leq d_\alpha(\nu, \mu_1) + d_\alpha(\mu_1, \lambda).
\]
By Proposition 2.8 and Corollary 7.3, we have
\[
    d_\alpha(\nu, \lambda) \leq C_{\Psi, \mu_1} + E',
\]
which completes the proof. □

9. Appendix

9.1. Convergence Lemma. Let \(\| \cdot \|\) denote the operator norm. Then \(\|Y\| \geq 1\) for any nontrivial idempotent matrix \(Y\). The following lemma is a slight improvement over Lemma 11.1 in [BGT22].

**Lemma 9.1.** Let \(Y\) be an idempotent matrix and let \(\{\Delta_i\}_{i=1}^{\infty}\) be a sequence of matrices such that \(\sum_{i=1}^{\infty} \|\Delta_i\| < \infty\). Let \(\varepsilon_j = \sum_{i=j}^{\infty} \|\Delta_i\|\) for \(j \in \mathbb{N}\). Then there is \(j_0 \in \mathbb{N}\) such that for every \(j \geq j_0\), the infinite product
\[
    \prod_{i=j}^{\infty} (Y + \Delta_i)
\]
converges to a matrix \(X_j\) with \(\|X_j - Y\| \leq 2\varepsilon_j\|Y\|^2\). Moreover, the kernel of \(Y\) is contained in the kernel of \(X_j\).
Proof. Let \( j_0 \in \mathbb{N} \) be such that \( \varepsilon_{j_0} \leq \frac{1}{2\|Y\|} \). Now fix some \( j \geq j_0 \). For \( k \geq j \), write

\[
Y + \Sigma_k = \prod_{i=j}^{k} (Y + \Delta_i).
\]

Then \((Y + \Sigma_k)(Y + \Delta_{k+1}) = Y + \Sigma_{k+1}\) and since \( Y^2 = Y \) it follows that

\[
\Sigma_{k+1} = \Sigma_k Y + Y \Delta_{k+1} + \Sigma_k \Delta_{k+1}.
\]

Multiplying on the right by \( Y \) and using \( Y^2 = Y \) we get

\[
\Sigma_{k+1} Y = \Sigma_k Y + Y \Delta_{k+1} Y + \Sigma_k \Delta_{k+1} Y
\]

and applying the norm

\[
\|\Sigma_{k+1} Y\| \leq \|\Sigma_k Y\| + \|\Delta_{k+1}\| \cdot \|Y\|^2 + \|\Sigma_k\| \cdot \|\Delta_{k+1}\| \cdot \|Y\|.
\]

For \( m > j \), by applying these inequalities for \( k = j, \ldots, m - 1 \) we get

\[
\|\Sigma_m Y\| \leq \left( \|\Sigma_j Y\| + \left( \|\Delta_{j+1}\| + \ldots + \|\Delta_m\| \right) \cdot \|Y\|^2 + \left( \|\Sigma_j\| \cdot \|\Delta_{j+1}\| + \ldots + \|\Sigma_{m-1}\| \cdot \|\Delta_m\| \right) \cdot \|Y\| \right) \cdot \|Y\|.
\]

Putting this together with Equation (21) and using \( \|Y\|^2 \geq \|Y\|, \|Y\| \geq 1 \), we get

\[
\|\Sigma_{k+1}\| \leq \|\Sigma_k Y\| + \|\Delta_{k+1}\| \cdot \|Y\| + \|\Sigma_k\| \cdot \|\Delta_{k+1}\| \leq \left( \|\Delta_j\| + \ldots + \|\Delta_{k+1}\| \right) \cdot \|Y\|^2 + \left( \|\Sigma_j\| \cdot \|\Delta_{j+1}\| + \ldots + \|\Sigma_k\| \cdot \|\Delta_{k+1}\| \right) \cdot \|Y\| \leq \frac{\varepsilon_j \|Y\|^2}{1 - \varepsilon_{j+1} \|Y\|} \cdot \|Y\|.
\]

Now we show by induction that \( \|\Sigma_k\| \leq \frac{\varepsilon_j \|Y\|^2}{1 - \varepsilon_{j+1} \|Y\|} \) for all \( k \geq j \).

**Base:** \( k = j \). Since \( \Sigma_j = \Delta_j \), we have \( \|\Sigma_j\| = \|\Delta_j\| = \varepsilon_j - \varepsilon_{j+1} \). Next, using \( \|Y\|^2 \geq 1 \) we trivially have

\[
(e_j - \varepsilon_{j+1})(1 - \varepsilon_{j+1} \|Y\|) \leq e_j - \varepsilon_{j+1} \leq \varepsilon_j \leq \varepsilon_j \|Y\|^2.
\]

By the choice of \( j_0 \), we have \( 1 - \varepsilon_{j+1} \|Y\| > 0 \), hence by dividing both sides by \( (1 - \varepsilon_{j+1} \|Y\|) \), we obtain

\[
\|\Sigma_j\| \leq \frac{\varepsilon_j \|Y\|^2}{1 - \varepsilon_{j+1} \|Y\|}
\]

as desired.

**Step.** By Equation (22), we have

\[
\|\Sigma_{k+1}\| \leq \varepsilon_j \|Y\|^2 + \frac{\varepsilon_j \|Y\|^2}{1 - \varepsilon_{j+1} \|Y\|} \left( \|\Delta_{j+1}\| + \ldots + \|\Delta_{k+1}\| \right) \cdot \|Y\| \leq \varepsilon_j \|Y\|^2 + \frac{\varepsilon_j \|Y\|^2}{1 - \varepsilon_{j+1} \|Y\|} \cdot \varepsilon_{j+1} \|Y\| = \frac{\varepsilon_j \|Y\|^2}{1 - \varepsilon_{j+1} \|Y\|},
\]

By the choice of \( j_0 \), we also have \( \frac{\varepsilon_j \|Y\|^2}{1 - \varepsilon_{j+1} \|Y\|} \leq 2\varepsilon_j \|Y\|^2 \). This shows that

\[
\|\Sigma_k\| \leq \|Y\|^2.
\]

It also follows that \( \|X_j - Y\| \leq 2\varepsilon_j \|Y\|^2 \) if we assume the convergence.

To prove the convergence, we show that the partial products form a Cauchy sequence. For \( j < k < m \),

\[
\prod_{i=j}^{m} (Y + \Delta_i) - \prod_{i=j}^{k} (Y + \Delta_i) = \prod_{i=k}^{m} (Y + \Delta_i) \left( \prod_{i=j}^{m} (Y + \Delta_i) - (Y + \Delta_k) \right)
\]

and applying the norm

\[
\left\| \prod_{i=j}^{m} (Y + \Delta_i) - \prod_{i=j}^{k} (Y + \Delta_i) \right\| \leq \left\| \prod_{i=k}^{m} (Y + \Delta_i) \right\| \left\| (\prod_{i=k}^{m} (Y + \Delta_i) - Y) + \|\Delta_k\| \right\| \leq (\|Y\|^2 + 2\varepsilon_j \|Y\|^2)(2\varepsilon_k \|Y\|^2 + \|\Delta_k\|)
\]

which proves the sequence is Cauchy.
For the last statement, let $v$ be a unit vector with $Yv = 0$. Then for $k \geq j$, \[
\|X_kv\| = \|(X_k - Y)v\| \leq \|X_k - Y\| \cdot \|v\| \leq 2\varepsilon_k \|Y\|^2.\]
Since $X_j = (Y + \Sigma_{k-1})X_k$, we have \[
\|X_jv\| \leq \|(Y + \Sigma_{k-1})\| \|X_kv\| \leq (\|Y\| + 2\varepsilon_j \|Y\|^2)2\varepsilon_k \|Y\|^2).
\]
Since this is true for all $k \geq j$, letting $k \to \infty$ yields $X_jv = 0$. 

9.2. Angle Estimate Lemma. Let $V$ be an inner product space.

Lemma 9.2. Let $v_1, v_2, w_1, w_2 \in V$ be such that $\angle(v_1, v_2) = 0$. Then
\[
1 - \cos \angle(v_1 + w_1, v_2 + w_2) \leq \frac{2(\|v_1\| \cdot \|w_2\| + \|v_2\| \cdot \|w_1\| + \|w_1\| \cdot \|w_2\|)}{\|v_1\| \cdot \|v_2\|}.
\]

Proof. Writing the definition of the cosine of the angle, using the triangle inequality, the fact that $\langle v_1, v_2 \rangle = \|v_1\| \cdot \|v_2\|$ and that $\langle v, w \rangle = -\|v\| \cdot \|w\|$ for $v, w \in V$, we get
\[
\cos \angle(v_1 + w_1, v_2 + w_2) = \frac{(v_1 + w_1, v_2 + w_2)}{\|v_1 + w_1\| \cdot \|v_2 + w_2\|} \geq \frac{(v_1 + w_1, v_2 + w_2)}{(\|v_1\| + \|w_1\|)(\|v_2\| + \|w_2\|)}
\]
(24)
\[
\geq \frac{\|v_1\| \cdot \|w_2\| - \|v_2\| \cdot \|w_1\| - \|v_2\| \cdot \|w_2\| - \|v_1\| \cdot \|w_1\|}{(\|v_1\| + \|w_1\|)(\|v_2\| + \|w_2\|)}.
\]
Then by Equation (24) and since $\|w_1\|, \|w_2\| \geq 0$,
\[
1 - \cos \angle(v_1 + w_1, v_2 + w_2) \leq \frac{2(\|v_1\| \cdot \|w_2\| + \|v_2\| \cdot \|w_1\| + \|w_1\| \cdot \|w_2\|)}{(\|v_1\| + \|w_1\|)(\|v_2\| + \|w_2\|)}.
\]
(25)

9.3. Interval Neighborhood Lemma.

Lemma 9.3. Let $I \subset \mathbb{R}^n$, $n \geq 2$ be a closed line segment. Let $I_r \subset \mathbb{R}^n$ be the $r$-neighborhood of $I$ for $r > 0$. Then for every 2-dimensional closed disk $D_r \subset \mathbb{R}^n$ of radius $r > 2\sqrt{(n-1)}$, $D_r \not\subset I_r$.

Proof. Without loss of generality assume that $I = \{(x_1, 0, \ldots, 0) | -t < x_1 < t\}$ for some $t > 0$. Let $B_r = \{(x_1, x_2, \ldots, x_n) | -t - r < x_i < t + r, -r < x_i < r, 2 < i < n\}$. Notice that $I_r \subset B_r$. We prove that $D_r \not\subset B_r$, hence $D_r \not\subset I_r$.

Assume on the contrary that $D_r \subset B_r$. Let $c \in D_r$ be the center of $D_r$ and $a, b \in \partial D_r$ be such that the vector $a - c$ is perpendicular to the vector $b - c$. Thus there are points $a, b, c \in B_r$ such that $\|a - c\| = R, \|b - c\| = R$ and $\langle a - c, b - c \rangle = 0$. Let $v_i$ denote the $i$-th coordinate of a vector $v \in \mathbb{R}^n$. Since $\|(a - c)_i, (b - c)_i\| < 2r$ for $2 < i < n$, we have \[
R^2 = \|(a - c)^2\| \leq (a - c)_1^2 + (n - 1) \cdot 4r^2, R^2 = \|(b - c)^2\| \leq (b - c)_1^2 + (n - 1) \cdot 4r^2.
\]
Hence $(a - c)_1^2 \geq R^2 - 4(n - 1)r^2, (b - c)_1^2 \geq R^2 - 4(n - 1)r^2$. Then by the triangle inequality we have \[
\|(a - c, b - c)\| \geq \|(a - c)_1, (b - c)_1\| - \sum_{i=2}^n \|(a - c)_i, (b - c)_i\| \geq R^2 - 4(n - 1)r^2 - 4(n - 1)r^2 = R^2 - 8(n - 1)r^2.
\]
Since $R > 2\sqrt{(n-1)}r$, we have $\|(a - c, b - c)\| > 0$, contradiction. 

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