Calculation of the $K^0$-$\bar{K}^0$ mixing parameter via the QCD sum rules at finite energies

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ABSTRACT

The QCD finite energy sum rules method is used to show that the parameter of the $K^0$-$\bar{K}^0$ mixing $\hat{B}$ is mainly determined by the value of $g m_s \langle \overline{d}G_{\mu\nu}\sigma_{\mu\nu}d \rangle$ and the vacuum expectation values of four-quark operators. Assuming the hypothesis of vacuum dominance and/or unitarity symmetry to estimate the latter, it is found that $\hat{B} = 1.2 \pm 0.1$.

\[\text{\footnote{\textsuperscript{1}Copy of the paper published in \textit{Phys. Lett. B174} (1986) 104. Received 8 July 1985; revised manuscript received 11 April 1986. A couple of minor misprints is corrected. Comments, related to a (relatively small) change of the result due to changes in the values of incorporated phenomenological parameters, are added.}}\]
The $K_L$-$K_S$ mass difference is rather sensitive to the mixing of the t-quark and u- and c-quarks, so the analysis of the $K^0$-$\bar{K}^0$ system provides us with useful information about the values of mixing angles in the Kobayashi-Maskawa model and about the phase parameter $\delta$ of $CP$-violation. The $K_L$-$K_S$ mass difference $\Delta m$ can be presented as a sum of the long-distance dispersive contributions $\Delta m^L$ and the short-distance contributions $\Delta m^{SH}$:

$$\Delta m = \Delta m^L + \Delta m^{SH}. \quad (1)$$

$\Delta m^{SH}$ is related to the matrix element of the effective $\Delta S = 2$ hamiltonian (see e.g. Ref. [1]):

$$\Delta m^{SH} = 2\text{Re}(\bar{K}^0|H^\text{eff}|^2|K^0) = (G_F^2/16\pi^2)F(x_j, \theta_j)(M_W^2/m_K)(\bar{K}^0|\hat{O}|K^0), \quad (2)$$

where $\hat{O} = (s_L\gamma_\alpha d_L)^2\alpha_s(\mu)^{-2/9}$ is the renormalization-invariant operator of the hamiltonian, which arises in the calculation of the well-known box diagram [2], $G_F$ is the Fermi constant, and $M_W$ is the mass of the $W$-boson. The function $F(x_j, \theta_j)$ has the following form [3]:

$$F(x_j, \theta_j) = \text{Re}[\lambda_1^2 S(x_c)\eta_1 + \lambda_2^2 S(x_t)\eta_2 + 2\lambda_t\lambda_s S(x_c, x_t)\eta_3]. \quad (3)$$

Here $x_j = m_j^2/M_W^2$, $\lambda_i = V_{td}^*V_{ts}$ ($V_{ij}$ is the Kobayashi–Maskawa matrix), and the functions $S$ are defined as

$$S(x) = x\left[\frac{1}{4} + \frac{9}{4}(1-x)^{-1} - \frac{3}{2}(1-x)^{-2}\right] + \frac{3}{2}[x^3/(1-x)^3]\ln x, \quad (4)$$

$$S(x_i, x_j) = x_i x_j \left\{\left[\frac{1}{4} + \frac{3}{2}(1-x_i)^{-1} - \frac{3}{4}(1-x_i)^{-2}\right] \times \ln x_i/(x_i-x_j) + (i \leftrightarrow j) - \frac{3}{4}(1-x_i)^{-1}(1-x_j)^{-1}\right\}. \quad (5)$$

The coefficients $\eta_i$ make allowance for the strong-interaction corrections in the leading logarithmic approximation. For $\Lambda_{\text{QCD}} = 100$ MeV they have the following numerical values: $\eta_1 = 0.7$, $\eta_2 = 0.6$, and $\eta_3 = 0.4$ [1]. The coefficients $\lambda_i$ are connected with the mixing angles in the following way:

$$\text{Re}\lambda_t \simeq s_1^2 c_1^2, \quad \text{Re}\lambda_s \simeq s_1^2 c_1^2 s_2^4, \quad \text{Re}\lambda_t\lambda_s \simeq s_1^2 c_1^2 s_2^3 \quad (5)$$

where $s_j = \sin \theta_j$ and $c_j = \cos \theta_j$, and the experimentally acceptable hypothesis that $\sin \theta_3 \simeq 0$ and $\sin \theta_2 \ll \sin \theta_1$ is used. The mixing angles have the following numerical values: $s_1 = 0.229 \pm 0.006$, $c_1 = 0.9735 \pm 0.0015$, and $0.016 < s_2 < 0.095$ [4].

As follows from eq. (2), in order to calculate the short-distance contribution to the $K_L$-$K_S$ mass difference, it is necessary to find the value of the matrix element

$$\langle \bar{K}^0|\hat{O}|K^0\rangle = \frac{2}{3}f_K^2 m_K^2 \hat{B} \quad (6)$$

which is usually expressed through the dimensionless parameter $B$.

As for the long-distance contributions, they have been estimated using different phenomenological approaches. For instance, the authors of refs. [1, 2] propose to estimate these contributions by inserting the low-lying states between two $\Delta S = 1$ weak non-leptonic hamiltonians.
More precise estimates have been obtained in refs. \[8\]-\[10\]; the corresponding values of the parameter $D = \Delta m_L / \Delta m$ are $D = 0.10 \pm 0.41 \ [8]$, $D = 0.33 \pm 0.37 \ [9]$, and $D = 0.46 \pm 0.13 \ [10]$.

A lot of attention has been paid recently to estimating the parameter $B$. The first such estimate -- $B = 1$ -- has been obtained in ref. \[2\] using the vacuum dominance approximation. Substituting this value into formula (2) and taking into account only the contribution of the $c$-quark with the mass $m_c = 1.3 - 1.5 \text{ GeV}$, we find that the short-distance contribution is about 40-45% of the total value of $\Delta m$. The $t$-quark contribution is small: for $m_t = 40 \text{ GeV}$ and $\tau_B \geq 10^{-12} \text{ s}$ it does not exceed 2% of $\Delta m$. Thus at $B = 1$ the short-distance contribution does not saturate the experimental $K_L - K_S$ mass difference, which indicates that the long-distance effects may be also important. Unfortunately, the current estimates of $B$ depend considerably on both the particular model used in the calculation and the values of various parameters involved in the model \[2\],\[11\]-\[16\]. Thus at present it is not clear what is the real value of this important parameter.

This paper presents a new estimation of the value of $B$ within the QCD finite energy sum rules (FESR) method, which has demonstrated its efficiency in studying the properties of low-lying hadronic resonances \[17\],\[18\].

Within this approach, the problem to be solved is reduced to computing the following three-point correlator \[19\]:

$$T_{\mu\nu}(p, q) = i^2 \int \! dx dy \exp(ipx - iqy) \times \langle T j_5^\mu(x) \bar{O}(y) j_5^\nu(0) \rangle_0$$

(7)

$p_{\mu} q_{\nu} T(p^2, (p - q)^2, q^2) + \text{other structures}$

at small $q$ and large (euclidean) $p$: $|p^2| \geq 1 \text{ GeV}^2$. Here $j_5^\mu = \bar{d}\gamma_\mu\gamma_5 s$ is the interpolating field of the $K^0$ meson

$$\langle 0| j_5^\mu(0)|K^0(p)\rangle = if_K p_\mu \ , \ f_K = 1.17 f_\pi = 156 \text{ MeV}$$

The matrix element (6) is related to the amplitude $T(t) = T(-t, -t, 0)$ by means of the dispersion relation in the variable $t$:

$$T(t) = \int \! ds \frac{\rho(s)}{s + t} - \text{subtractions}$$

(8)

$$= f_K^2 \frac{(K^0|\bar{O}|K^0)}{(t + m_K^2)^2} + \frac{A}{t + m_K^2} + \ldots ,$$

where the single-pole contribution corresponds to the transition form factor of the $K^0$ meson, and the dots stand for the contributions of higher states.

In the vacuum dominance approximation $T_{\mu\nu}$ assumes the form

$$T_{\mu\nu}^{VD} = \frac{2}{3} \Pi_{\mu\alpha}(p) \Pi_{\nu\alpha}(q - p) ,$$

$$\Pi_{\mu\alpha}(p) = i \int \! dx \exp(ipx)\langle T(j_5^\mu(x)\bar{s}L(0)\gamma_\alpha d_L(0))\rangle_0 ,$$

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and the resulting value of $B$ proves to be $B^{VD} = 1$. Thus, there remains to be computed only the function $\Delta_{\mu\nu} = T_{\mu\nu} - T^{VD}_{\mu\nu}$, which is responsible for all the departures from the vacuum dominance prediction for $B$. In other words, within the sum rule approach we will calculate the value $B - 1$.

In the kinematical region we are interested in, there are three distinct contributions to $T_{\mu\nu}$, viz: the perturbation theory (PT) contribution; the power corrections due to the non-zero vacuum expectation values (VEV’s) of local operators [20]; and finally, the power corrections proportional to two-point correlators depending on the momentum $q$ (or, in other words, the VEV’s of bilocal operators [21]). These bilocal power corrections are due to the fact that one of the external momenta ($q_1$) is small, and thus, the contribution from the region of large $y \sim 1/|q|$ in the integral (7) cannot, generally speaking, be disregarded.

It is easy to check that the leading PT contribution to $T_{\mu\nu}$ is to be completely assigned to $T^{VD}_{\mu\nu}$. Thus, neglecting all the non-leading PT corrections we find that $\Delta_{\mu\nu}|_{p^2 \to -\infty} = \Delta^B_{\mu\nu} + \Delta^L_{\mu\nu}$, with $\Delta^L_{\mu\nu}$ ($\Delta^B_{\mu\nu}$) standing for the terms proportional to the VEV’s of (bi)-local operators.

The account of local operators with dimension $\leq 6$ leads to the following result for the function $\Delta^L_{\mu\nu}$:

$$\Delta^L_{\mu\nu} = p^\mu q^\nu \left(-5(pq)\langle \alpha_s G^2 \rangle/192\pi^3 - 4\langle \bar{d}s\bar{s}\rangleight) - 4\langle \bar{d}s\bar{s}\rangle + 2\langle \bar{s}s\bar{s}\bar{s}\rangle + 2\langle \bar{d}d\bar{d}d\rangle + m_s\langle \bar{g}dG_{\mu\nu}\sigma_{\mu\nu}\bar{d}\rangle/24\pi^2 (p - q)^{-2} + \text{other structures}$$

where the designation $\langle \bar{d}s\bar{s}\rangle$ stands for $\langle \bar{d}_L\gamma_\alpha s_L\bar{s}_L\gamma_\alpha d_L \rangle$ and so on; $G^2 = G^a_{\mu\nu}G^a_{\mu\nu}$; $G_{\mu\nu} = G_{\mu\nu}^a t^a$; $\text{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$; and $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]_\gamma$.

To estimate the contributions due to bilocal operators one needs to construct the Wilson expansion for the $T$-product $i \int dx \exp(ipx)T(j^5_\mu(x)j^5_\nu(0))$ at large euclidean $p$ (or, equivalently, at small $x$). A straightforward calculation shows that in the leading order in $\alpha_s$, the bilocal contribution to the tensor structure $p_\mu q_\nu$ is suppressed by the factor $p^{-6}$ and, thus, can be neglected within our approximation.

To proceed, the combination of four-quark operators appearing in (9) proves to transform as a component of an $SU^f(3)$ 27-plet. This means that the corresponding VEV is at least of second order in the unitary-symmetry breaking parameter. Moreover, the VEV of each four-quark operator in (9) vanishes if the vacuum saturation procedure [20] is to work. Thus, the corresponding contribution to $\Delta^L_{\mu\nu}$ is “doubly” forbidden and can be safely neglected. On the other hand, a straightforward estimation shows that should both of these suppressions be absent in the next-to-leading approximation, the corresponding contribution to $\Delta B$ might be uncomfortably large despite the loop suppression factor $\sim \alpha_s/\pi$. To clarify the situation we have calculated the $\alpha_s$-corrections to the coefficient functions of the four-quark terms in expansion (9) and have found that the appearing extra terms are die out after applying the vacuum saturation.

In order to extract information on the value of $B$ we employ the FESR technique to the func-
tion $T(t)$ multiplied by $(m_K^2 + t)$ to nullify the effect of the (unknown) single-pole contribution to the RHS of (8). The final sum rule has the form

$$B - 1 = \left( \frac{2}{3} f_K^4 m_K^2 \right)^{-1} \int_{s_0}^{s_0} \rho^{th}(s) (s + m_K^2) ds$$

$$= m_s \langle g d \bar{g} \sigma_{\mu\nu} G_{\mu\nu} d \rangle / 16\pi^2 m_K^2 f_K^4$$

where

$$\rho^{th}(s) = (2\pi i)^{-1} \lim_{\epsilon \to 0} \left[ T(-s - i\epsilon, -s - i\epsilon, 0) - T(-s + i\epsilon, -s + i\epsilon, 0) \right],$$

and $s_0 = 1.2$ GeV$^2$ is the duality interval of the $K^0$ meson [18].

Up to now we have neglected all the effects due to the non-zero anomalous dimensions of the operators under consideration. In the leading log approximation the account of these effects is carried out without any difficulty. We define the renormalization-group invariant quantity

$$\hat{B} = B(\mu)[\alpha_s(\mu)]^{-2/9}$$

and make use of the renormalization group technique [22] to rewrite eq. (10) as

$$\hat{B} = (1 + m_s \langle g d \bar{g} \sigma_{\mu\nu} G_{\mu\nu} d \rangle / 16\pi^2 m_K^2 f_K^4 |_{s_0})[\alpha_s(s_0)]^{-2/9}.$$
to $T_{\mu\nu}$. Indeed, we have shown that in the leading order in $\alpha_s$, the deviation of the actual value of $B$ from its vacuum dominance estimate $B^{VD} = 1$ is within the following limits: $0.1 \leq (\hat{B} - 1) \leq 0.3$, where the main uncertainty is due to our poor knowledge of the matrix element $\langle g\bar{d}G_{\mu\nu}\sigma_{\mu\nu}d \rangle$. However, the overall magnitude of $\hat{B} - 1$ proves to be small, which gives the above margins for $\hat{B}$.

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*Note added.* After this work has been issued as a preprint, we have learnt of another calculation of $B$ with the result $\hat{B} = (0.33 \pm 0.09)[\alpha_s(\mu^2)]^{2/9}$, which is quite different from ours. In fact, the authors of ref. [25] have combined the effective chiral Lagrangian approach with the finite energy sum rules method and calculated the parameter $B(t = 4m_K^2) = \langle 0|\hat{O}|K^0K^0\rangle/(2f_K^2m_K^2/3)$ rather than the parameter $B$ defined as in the formula (6).

To our mind, one of the possible reasons for this disagreement could be the noticeable variation of the function $B(t)$ between $t = 4m_K^2$ and $t = 0$ (within the chiral perturbation theory the quantity $4m_K^2$ can hardly be considered small in any way). It should be stressed that our approach meets no such problem since from the very beginning we are dealing with $B(0) \equiv B$.

Another possible explanation to the discrepancy could be the bad convergence of the power correction series for the correlator examined in ref. [25]. Let us recall that in this work the account of the two first power corrections have led to the decrease of the result by five times, while in our analysis the power corrections to the (known beforehand) factorizable contribution does not exceed 20% of the latter.

Leaving aside these technical subtleties, we feel that the main advantage of our approach is the exact account of the factorizable contributions and that the only chance to vary the result (11), say, by a factor of 2 or 3 is to have a strong violation of the vacuum saturation hypothesis for four-quark operators.

At present the problem of estimating the accuracy of the latter hypothesis and the possibility of the exact account of factorizable terms within our approach are under consideration.

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Comments

Since the time the paper was published (1986) the need in an accurate numerical value for the $\hat{B}_K$ has considerably raced up. Precise experimental data on $K^0 - \bar{K}^0$ system have appeared that allowed to reliably determine $\epsilon'/\epsilon$. The program of computing the next-to-leading correction to weak Hamiltonian in the effective theory approach has been completed \[1\]. However, no significant improvement in calculation of the matrix element has been achieved. The reason is clear: it is a non-PT problem. As such it is now actively discussed within lattice approach but the corresponding results are still far from being perfect.

Our calculation was made within the standard continuous QCD and the QCD Sum Rule Approach. The method we used factors out completely (and thereby takes into account) the factorizable contribution to the matrix element in all orders of PT and OPE expansion. The remaining non-factorizable part happens to be small in comparison to factorizable term, the latter being accounted exactly in our set-up.

The method is open to improvement: both high order PT corrections and next order power corrections in addition to those computed in the paper can be added. Note that PT corrections are really mandatory to compute to match the available NLO coefficient functions (see, e.g. \[2\]).

Having in mind a solid theoretical basis of our calculation we have just updated the prediction for $\hat{B}_K$ by taking into account a significantly changed value for $\Lambda_{\overline{MS}}^{(f=3)}$ from 100 MeV to the value of over 400 MeV, which corresponds to the world average value $\alpha_s(M_Z) \approx 0.118$. The updated version of eq. (11) with $\alpha_s(1.2 \text{ GeV}^2) = 0.69$, estimated in the leading order, is:

$$\hat{B} = 1.0 \pm 0.1$$

Note that the bulk of the change of the parameter $\hat{B}$ is due to the normalization (the change in the factor $\alpha_s(1.2 \text{ GeV}^2)^{-2/9}$) while other parameters of the calculations entering eq. (10) did not change much during last 15 years.

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