EXISTENCE OF STATIONARY STOCHASTIC BURGERS EVOLUTIONS ON $\mathbb{R}^2$ AND $\mathbb{R}^3$

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Abstract. We prove that the stochastic Burgers equation on $\mathbb{R}^d$, $d < 4$, forced by gradient noise that is white in time and smooth in space, admits spacetime-stationary solutions. These solutions are thus the gradients of solutions to the KPZ equation on $\mathbb{R}^d$ with stationary gradients. The proof works by proving tightness of the time-averaged laws of the solutions in an appropriate weighted space.

1. Introduction

Consider the stochastic Burgers equation

$$du = \frac{1}{2} [\Delta u - \nabla (|u|^2)] dt + d(W)$$

(1.1)

on $\mathbb{R} \times \mathbb{R}^d$, where $W$ is a Wiener process in time with smooth values in space and $d$ is the Itô time differential (so $d(W)$ is white in time and smooth and of gradient type in space). To be precise, let $dW$ be a space-time white noise on $\mathbb{R} \times \mathbb{R}^d$ and let $V = \rho \ast W$, where $\rho \in C^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is a spatial mollifier and $\ast$ denotes convolution in space. We will always consider in strong solutions to (1.1), as defined in [15]. Our goal is to prove the following.

Theorem 1.1. If $d < 4$, then there exist statistically spacetime-stationary solutions to (1.1).

Equation (1.1) is related via the Cole–Hopf transform[26, 11] to the KPZ equation[28] and multiplicative stochastic heat equation (SHE). Explicitly, if $\phi$ solves the SHE

$$d\phi = \frac{1}{2} \Delta \phi dt - \phi dW,$$

then $h = -\log \phi$ solves the KPZ equation

$$dh = \frac{1}{2} [\Delta h - |\nabla h|^2] dt + dV,$$

and $u = \nabla h$ solves (1.1). It is easy to see by a variance computation, using the Feynman–Kac formula, that when $d = 2$, or when $d \geq 3$ and $\|\rho\|_{L^2(\mathbb{R}^d)}$ is sufficiently large, the SHE started at $\phi(\cdot, \cdot) \equiv 1$ has diverging pointwise statistics at $t \to \infty$. This implies the same for the KPZ solutions. Thus it is expected that these equations, unlike (1.1), do not admit space-time stationary solutions on the whole space. So informally, Theorem 1.1 indicates that the obstruction to stationary KPZ solutions is the growth of fluctuations of the zero-frequency mode, which is destroyed by the gradient. The present work thus extends the theory of long-time KPZ solution behavior. Long-time/large-space scaling limits for the multidimensional SHE and KPZ equations, especially phrased in terms of solving the (renormalized) equations with driving noise $V$ that is white in space as well as time, have been of substantial recent interest. See [4, 8, 10, 9, 22, 23] for some results in $d = 2$ and [16, 34, 31, 24, 18, 19] for some results in $d \geq 3$.

Similar problems to ours have been considered in one spatial dimension in [2, 1, 3, 17]. We obtain the existence results of these papers in our two- and three-dimensional settings. The functional-analytic framework that we use is the same as that of [17]. (We completely avoid the consideration of Lagrangian minimizers and directed polymers used in [2, 1, 3] and most other earlier works in this area.) To show the existence of stationary solutions, we prove that the time-averaged laws of the solutions are tight in a space in which the equation is well-posed. We use a simple but fundamental idea of [17, Section 3] to bound the pointwise variance of the solution at a random time using the Cole–Hopf transform and Jensen’s inequality. (See Section 3 below.) However, the tools used in [17] to bound the derivatives of the solutions (similar to those used in the periodic setting in [5]) are not available in dimension $d \geq 2$. Moreover, the growth at infinity that is allowed by a variance bound in $d \geq 2$ is too much for the equation (1.1) to be well-posed. These issues pose serious challenges for proving compactness.

Stationary solutions for (1.1) are known to exist in $d \geq 3$ when $V$ is multiplied by a sufficiently small constant—the so-called weak noise regime, which does not exist in $d = 1, 2$. This can be seen as a consequence of the fact that
the stationary solutions exist for the stochastic heat equation in this regime \([16, 34, 32, 18]\), and was studied directly in \([29]\). When the noise is stronger, stationary solutions are thought to exist for the Burgers equation but not for the SHE. This paper thus extends the existence of stationary solutions for Burgers in \(d = 3\) to the strong-noise setting.

In another direction, the present work can be seen as an extension of some of the results of \([27, 21, 6]\) about stationary solutions for the multidimensional stochastic Burgers equation on a periodic domain. The problem of tightness on the whole space is very different from on a periodic domain, since on the whole space Poincaré-type inequalities are not available to control the solution in terms of its derivatives, and controlling the growth of the solutions at infinity becomes crucial. However, a key part of our argument is the Kruzhkov maximum principle\([30]\), which was introduced in the context of the periodic stochastic Burgers equation in \([6]\) to give bounds on the slopes of the rarefactions of the solutions. Mathematically, this means bounds on \((\partial_t u_i)^+\) for \(i = 1, \ldots, d\).

The literature on stationary solutions for the stochastic Burgers equation on a one-dimensional compact domain is by now quite extensive. We mention for example the seminal papers \([33, 20]\). Also relevant are the more recent results \([5, 7]\), which use PDE-style techniques closer to those in the present paper.

We do not address questions of uniqueness of the stationary measures, nor do we prove anything about convergence to them. In \([5]\) on the one-dimensional torus, and \([17]\) on the line, this was studied using the \(L^1\) contraction properties of the one-dimensional Burgers equation. The Burgers equation in \(d \geq 2\) does not have an \(L^1\) contraction property. On the multidimensional torus, \([6]\) replaced this property with an \(L^\infty\) contraction property for the mean-zero integral of \(u\), which solves the KPZ equation \((3.2)\) up to a zero-frequency term. When solutions are unbounded as they are when the equation is considered on the whole space, it is not clear if or how this argument can be adapted. Using an approach based on directed polymers, \([3]\) proved convergence to the stationary solution and a one-force-one-solution principle for the stochastic Burgers equation on the real line. However, the proof relied on the ordering of the real line and on a relaxation property for the Burgers equation coming from the fact that the forcing was discrete-time “kicks.”

**Strategy of the proof and outline of the paper.** The technical heart of our work is the use of a Kruzhkov maximum principle in a nonperiodic setting. We want to show that, as in \([6]\), \((\partial_t u_i)^+\) can be controlled by the solution to a Riccati-type equation that brings any initial data down to size of order 1 in time of order 1. Physically, this corresponds to the fact that rarefaction waves are destroyed by the nonlinearity in Burgers evolution, in contrast to shocks, which are actually encouraged by the nonlinearity and are only prevented by the viscosity term. The situation is immediately much more complicated than that considered in \([6]\) because we do not expect global-in-space bounds, and so we must work with the equation for the weighted solution. We do this Section 4, obtaining the derivative bound \((4.4)\), below.

Since the equation for the maximum of \((\partial_t u_i)^+\) also involves \(u\) itself, it is necessary to control \(u\) in terms of \((\partial_t u_i)^+\) and \textit{a priori} controlled quantities to close the argument. Unlike in the mean-0 periodic case, a function on the whole space cannot be controlled by its derivatives. The only available \textit{a priori} controlled quantity is the pointwise \(L^2\) bound on \(u\) discussed above and proved in Section 3. Thus we need to pass from \(L^2\) control in space to \(C^0\) control in space (which is necessary to close the maximum principle argument for \((\partial_t u_i)^+\), using only a bound on the positive part of the diagonal terms of the Jacobian of \(u\). Our tool for this is Proposition 5.1, which is stated for general differentiable functions \(R^d \to R^d\) of gradient type.

The other key challenge to overcome is that for the Burgers equation to be well-posed, the initial conditions must grow sublinearly at infinity. Otherwise, mass from infinity could reach the origin in finite time. The proof that the Burgers equation is well-posed under this assumption is the same as that of \([17,\text{Theorem 2.3}]\) in one spatial dimension. We summarize it in Section 2 below. A variance bound on a space-stationary function \(g : Z^d \to R\) restricts the growth of \(g\) to be less than order \(|x|^{d/2+\delta}\) for any \(\delta > 0\). This is insufficient when \(d \geq 2\). Thus, we need to “stretch out” the mesh on which we sample the values of \(u(t,\cdot)\), by using the bound on \((\partial_t u_i)^+\) to interpolate. (See the proof of Proposition 5.1.) This makes \(d < 4\) tractable.

In Section 6, we put the bounds together to establish tightness of the time-averaged laws of the Burgers solutions. Our bounds are in terms of solutions to Riccati-type equations. Since our \textit{a priori} control of the solutions established in Section 3 is only \(L^2\) in time, we need to use the well-posedness of the equation to upgrade this to \(C^0\) control of the solutions on very short time scales. It turns out, however, that these time scales are long enough to find a time
at which a spatial $L^2$ norm is not too large, using the $L^2$ in time control. Then we can iterate the argument. This is done in the proofs of Lemma 6.1 and Proposition 6.3. Once the time-averaged laws of the solutions are controlled uniformly in time, in a space in which the equation (1.1) is well-posed, showing the existence of stationary solutions and proving Theorem 1.1 is a routine application of parabolic regularity and a Krylov–Bogoliubov argument.

**Notation.** We will often write $x^+ = \max\{x, 0\}$. If $x \in \mathbb{R}^d$, we define $|x|$ to be the Euclidean norm of $x$. For a metric space $\mathcal{X}$ and a normed space $\mathcal{Y}$, we let $C(\mathcal{X}; \mathcal{Y})$ be the space of bounded $\mathcal{Y}$-valued continuous functions on $\mathcal{X}$, equipped with the supremum norm. We define more function spaces in Section 2 below.

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2. WELL-POSEDNESS

In order to discuss the well-posedness theory for the equation (1.1), we first need to introduce weighted function spaces. By a weight we simply mean a positive function $w : \mathbb{R}^d \to \mathbb{R}$. We will use weights of the form

$$p_\ell.K(x) = (\langle x \rangle + K)^\ell, \quad p_\ell = p_{\ell, 1}, \quad \xi(x) = (\log((\langle x \rangle + 1))^{3/4}, \quad \langle x \rangle = \sqrt{1 + |x|^2}. \quad (2.1)$$

If $w$ is a weight, $k \in \mathbb{Z}_{\geq 0}$, and $\alpha \in (0, 1)$, define the norms

$$\|f\|_{C_w^k} = \sup_{x \in \mathbb{R}^d} \max_{j \in \{0, \ldots, k\}} |f^{(j)}(x)| w(x), \quad \|f\|_{C_w^\alpha} = \max \left\{ \|f\|_{C_w^k}, \sup_{|x-y| \leq 1} \frac{|f(x) - f(y)|}{w(x)|x-y|^\alpha} \right\},$$

and define $C_w^\alpha$ to be the space of functions for which the $C_w^\alpha$ norm is finite. Define $\mathcal{A}_w^\ell$ to be the subspace of $\mathbb{R}^d$-valued elements of $C_w^\ell$ consisting of functions which are gradients, equipped with the $C_w^\alpha$ norm. We will abbreviate $C_w = C_w^0$ and $\mathcal{A}_w = \mathcal{A}_w^0$. The following compactness lemma is analogous to [17, Proposition 2.2].

**Lemma 2.1.** Let $\alpha > 0$ and $\ell, \ell_1, \ell_2$ be such that $\ell_1 < \ell$. Then the embedding $C_{p_{\ell_1}} \cap C_{p_{\ell_2}}^{\alpha} \hookrightarrow C_{p_{\ell}}$ is compact.

We now establish that the equation (1.1) is well-posed in the space $C_{p_\ell}$ as long as $\ell < 1$. As discussed in the introduction, the restriction to $\ell < 1$ is essential. The proof given for the one-dimensional case in [17, Section 2] goes through for general $d$ without essential modification, but we will need some lemmas from the construction in this section, so we sketch the proof here.

We will define solutions to periodic problems and then obtain whole-space solutions as almost-sure limits. Thus we need a sequence of periodic driving noises that converge to the full-space noise almost surely in a weighted space...

**Lemma 2.2.** We have a sequence of space-stationary Wiener processes $V^{[L]}$, $L \in [1, \infty)$, so that $V^{[L]}$ agrees in law with $\rho * W^{[L]}$, where $W^{[L]}$ is an $\ell \mathbb{Z}^d$-periodic space-time white noise, and for any $T, \ell > 0$ we almost surely have

$$\lim_{L \to \infty} \|V^{[L]} - V\|_{C([0,T]; C_{p_\ell})} = 0. \quad (2.2)$$

**Proof.** For each $L \in [1, \infty)$, let $\chi^{[L]} \in C_c^\infty$ be a function so that $\text{supp} \chi^{[L]} \subset [-L, L]$ and $\sum_{j \in \mathbb{Z}^d} \chi^{[L]}(x + Lj)^2 = 1$ for all $x \in \mathbb{R}^d$. Define $W^{[L]} = \sum_{j \in \mathbb{Z}^d} \tau_{Lj}(\chi^{[L]}W)$, where $\tau_{Lj}$ denotes spatial translation by $Lj$, and define $V^{[L]} = \rho * W^{[L]}$. A simple covariance computation shows that $V^{[L]}$ has the desired law, and the convergence (2.2) is also clear from the smoothness and Gaussianity of the fields. □

For notational convenience, put $V^{[\infty]} = V$. We note that if $\psi$ solves the linear problem

$$d\psi = \frac{1}{2} \Delta \psi \, dt + d(\nabla V^{[L]}), \quad (2.3_L)$$

and $v$ (classically) solves the problem

$$\partial_t v = \frac{1}{2} \Delta v - \frac{1}{2} \nabla (|v|^2), \quad (2.4\psi)$$

then $u = v + \psi$ solves the problem

$$du = \frac{1}{2} (|u|^2) \, dt + d(\nabla V^{[L]}), \quad (2.5_L)$$

which is (1.1) with the periodized noise $V^{[L]}$ in place of the whole-space noise $V$. The stochastic PDE (2.3_L) has Gaussian solutions, while (2.4\psi) is a PDE with classical solutions. Thus we will generally work with (2.3_L) and (2.4\psi), and add the solutions together to obtain solutions to (2.5_L).
The fact that, for any $LZ^d$-periodic $u \in C$, there exist global-in-time solutions to (2.5) with initial condition $u(0, \cdot) = u$ can be proved by a fixed-point argument, as was done in [6]. (Similar arguments were carried out in [12, 13, 5, 17].) The global-in-time well-posedness theory on the whole space is somewhat more involved, since appropriate weights must be used: certainly if the initial conditions are to live in $C_p$, then $\ell$ must be required to be strictly less than 1, or else mass from infinity could accumulate at the origin in finite time. The following theorem is analogous to [17, Theorem 2.3].

**Theorem 2.3.** Fix $0 < m < m^* < 1$ and $M, R, T < \infty$. There is a random constant $X = X(m, m^*, M, R, T)$, finite almost surely, and an $\alpha > 0$ so that the following holds. If $L \in (0, \infty)$ and $u \in \mathcal{A}_{p_m}$ is $LZ^d$-periodic, and $\|u\|_{C_p} \leq M$, then there is a unique $u \in C([0,T];\mathcal{A}_{p_m})$ with

$$\|u\|_{C([0,T];\mathcal{A}_{p_m})} \leq X$$

and so that $u(0, \cdot) = u$ and $u$ is a strong solution to (2.5). Moreover, if $L' \in (0, \infty)$, $\tilde{u} \in \mathcal{A}_{p_m}$ is $L'Z^d$-periodic and such that $\|\tilde{u}\|_{C_p} \leq M$, and $\tilde{u}$ is the strong solution to (2.5) with initial condition $\tilde{u}(0, \cdot) = \tilde{u}$, then

$$\|u - \tilde{u}\|_{C([0,T];C([-R,R]^d))} \leq X[\|u - \tilde{u}\|_{C_p} + \|\psi[L] - \psi[L']\|_{C([0,T];C_p)}].$$

(2.7)

Since the proof of Theorem 2.3 is a straightforward generalization of that of [17, Theorem 2.3], we only sketch it. The existence theory for the periodic problem is a standard fixed-point argument along the lines of those discussed in [12, 15, 6]. The fact that $\|u(t, \cdot)\|_{C_{p_m,K}}$ is bounded uniformly in $L$ (the first part of (2.6)) follows from a maximum principle argument along the lines of [17, Proposition 2.10], which we establish below as Proposition 2.4, both because we will use it in the sequel and because its proof statement requires attention to the fact that the solution $u$ is assumed to be of gradient type, which is a vacuous assumption in $d = 1$. Finally, the existence theory for the nonperiodic problem, the bound on $\|u\|_{C([0,T];C_p)}$ in (2.6), and (2.7) follow from parabolic regularity estimates essentially identical to those given in [17].

We now establish uniform-in-$L$ bounds on $\|u\|_{C([0,T];C_{p_m})}$, i.e. bound our in the following proposition, which is very similar to [17, Proposition 2.10], shows that $\|u(t, \cdot)\|_{C_{p_m,K}}$ grows in $t$ at most like the solution to a Riccati equation of the form $f' = cf^2$, where the constant $c$ can be made small by choosing $K$ large relative to the size of the noise. Increasing $K$ reduces the maximum gradient of $p_m,K$, attenuating the effect on $\|u(t, \cdot)\|_{C_{p_m,K}}$ of movement of mass in the solution $u(t, \cdot)$. This movement of mass is all that we have to be concerned about, as the addition of mass coming from the noise is finite in the weighted norm, and the nonlinearity in (1.1) is a gradient so does not add mass.

**Proposition 2.4.** For any $m \in (0, 1)$, $T \in (0, \infty)$, and $\varepsilon \in (0, (1 - m)/T)$, there is a constant $C = C(m, T, \varepsilon) < \infty$ so that the following holds. Let $L \in [0, \infty)$, $\psi \in C([0,1];A_{\varepsilon})$, $\nu \in C([0,1];\mathcal{A}_{p_m,K})$ be a solution to (2.4), and $u = \nu + \psi$. (Recall the definition (2.1) of $\xi$.) There is a constant $K_0 = K_0(d, e, \|\psi\|_{C([0,T];C_{\varepsilon'})}) < \infty$ so that if $K \geq K_0$, then for all

$$t \in [0, K^{-1} - \varepsilon],$$

we have

$$\|u(t, \cdot)\|_{C_{p_m,K}} \leq \min\{\|u(0, \cdot)\|_{C_{p_m,K}}^{-1}, 1\} - K^{-1/2}t^{-1} + \|\psi\|_{C([0,T];C_{p_m,K})},$$

(2.8)

The proof of Proposition 2.4 will be based on the maximum principle for the weighted solution $v$, so first we show what happens when we differentiate a weighted version of $v$.

**Lemma 2.5.** Suppose $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is differentiable and a gradient, $\nu : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a gradient and solves (2.4) for some $L \in (0, \infty)$, $\alpha : \mathbb{R} \to \mathbb{R}^d$ is differentiable in time and twice-differentiable in space, and $\gamma = av$. Then we have, for each $i \in\{1, \ldots, d\}$, that

$$\partial_t \gamma_i = \gamma_i \partial_t (\log a) + \frac{1}{2} \Delta \gamma_i - \nabla (\log a) \cdot \nabla \gamma_i + \frac{1}{2} \gamma_i \Delta (\log a)^2 - \Delta (\log a)$$

$$- (a^{-1} \gamma + \psi) \cdot (\nabla \gamma_i - \gamma_i \nabla (\log a)) - (\gamma + a \psi) \cdot \partial_\gamma.$$

(2.9)

**Proof.** By the product rule we have

$$\partial_t \gamma = v \partial_\gamma + a \partial_\gamma v = \gamma \partial_t (\log a) + \frac{a}{2} [\Delta v - \nabla (|v + \psi|^2)].$$
It is a general fact about functions \( f, g \), and \( h = fg \) that
\[
\begin{align*}
\frac{\partial}{\partial t} (f g) &= (\nabla g - g \nabla f) \cdot \nabla f + h \nabla (f g) \cdot \nabla f = (\Delta g - 2\nabla (f g) \cdot \nabla f) + h (f g - \Delta g - 2\nabla (f g) \cdot \nabla f - \Delta (f g)).
\end{align*}
\]
Therefore, we have
\[
\frac{\partial}{\partial t} (f g) = \Delta f g - 2\nabla (f g) \cdot \nabla f + h (f g - \Delta f g - 2\nabla (f g) \cdot \nabla f - \Delta (f g)).
\]
(2.10)

On the other hand, using the fact that \( v + \psi \) is a gradient and also (2.10) again, we have
\[
\frac{1}{2} a \partial_t (|v + \psi|^2) = a (v + \psi) \cdot \partial_t (v + \psi) = (a - 1) \gamma \cdot (\nabla (v + \psi)) - (a - 1) \gamma \cdot (\nabla (v + \psi)) + (a + \psi) \cdot \partial_t \psi.
\]
Plugging this into (2.11), we obtain (2.9).
\[
\Box
\]

Now we are ready to prove Proposition 2.4.

**Proof of Proposition 2.4.** Let \( a(t, x) = 1/p_{m+1, K}(x) \) and put \( \gamma = av \). Multiplying (2.9) (with this choice of \( a \)) by \( \gamma_i \) and summing over \( i \), we obtain
\[
\begin{align*}
\frac{1}{2} \partial_t |\gamma|^2 &= |\gamma|^2 \partial_t (\log a) + \frac{1}{2} \Delta |\gamma|^2 - |\nabla \gamma|^2 - \frac{1}{2} \nabla (\log a) \cdot \nabla |\gamma|^2 + \frac{1}{2} |\nabla (\log a)|^2 - \Delta (\log a) \\
&= (a - 1) \gamma \cdot (\nabla (\log a)) - (a - 1) \gamma \cdot (\nabla (\log a)) + (1 + a \psi) \cdot \partial_t \psi.
\end{align*}
\]
(2.11)

At a local maximum of \( |\gamma|^2(t, \cdot) \), we have \( \Delta |\gamma|^2 \leq 0 \) and \( \nabla |\gamma|^2 = 0 \), so at such a local maximum, using the Cauchy–Schwarz inequality we obtain
\[
\begin{align*}
\frac{1}{2} \partial_t |\gamma|^2 &\leq |\gamma|^2 \partial_t (\log a) + \frac{1}{2} |\gamma|^2 (|\nabla (\log a)|^2 - \Delta (\log a)) + (a - 1) |\gamma| |\psi| |\nabla (\log a)| + (a - 1) |\gamma| |\nabla (\log a)| + (1 + a \psi) |\nabla (\log a)| + (a - 1) |\nabla (\log a)| + (1 + a \psi) |\nabla (\log a)|.
\end{align*}
\]
(2.12)

Using the simple bounds
\[
\begin{align*}
|\nabla (\log a)(t, x)| &\leq 1, \quad \Delta (\log a) \leq d + 2, \quad |a - 1| |\nabla (\log a)| \leq K^{(-1)}(x + K), \quad \partial_t (\log a)(t, x) = -a \log((x + K),
\end{align*}
\]
(2.13)

we can derive from (2.12) that, at a local maximum \( x \) of \( |\gamma|^2(t, \cdot) \) such that \( |\gamma(t, x)|^2 \geq 1 \),
\[
\frac{1}{2} \partial_t |\gamma|^2 \leq K^{(-1)} |\gamma|^3 + |\gamma|^2 - a \log((x + K),
\]
(2.14)

Now we assume \( K \) is so large (depending on \( d, \epsilon \) and \( \|\psi\|_{C([0,T], C^1)} \)) that the term in brackets is guaranteed to be negative regardless of \( t \) and \( x \). Then, at a local maximum \( x \) of \( |\gamma|^2(t, \cdot) \) such that \( |\gamma(t, x)|^2 \geq 1 \), we have \( \partial_t |\gamma|^2 \leq 2K^{(-1)} |\gamma|^3 \). By [25, Lemma 3.5], this implies that if we define
\[
\Gamma(t) = \max_{x \in \mathbb{R}^d} |\gamma(t, x)|^2 = \|v(t, \cdot)\|_{C([0,T], C^1)}^2,
\]
(2.14)

then \( \Gamma : [0, T] \rightarrow \mathbb{R} \) is Lipschitz and we have \( \Gamma(t) \leq 2K^{(-1)} \Gamma(t)^{3/2} \) whenever \( \Gamma(t) \geq 1 \), which means that
\[
\Gamma(t)^{1/2} \leq \min\{\Gamma(0)^{-1/2}, 1\} - K^{(-1)} t
\]
whenever \( 0 < K^{(-1)} \min\{\Gamma(0)^{-1/2}, 1\} \). Recalling (2.14), this implies (2.8).
\[
\Box
\]

3. **Pointwise Variance Bound**

In the one-dimensional setting, sufficient compactness of solutions to (2.5L) to obtain existence of stationary solutions was obtained in [17, Section 3] from \( L^2 \) bounds on the solution and its derivative at a point. In dimension \( d > 1 \), such an \( L^2 \) bound on the derivative is not available due to contributions from terms of the form \( \partial_i u_j \) for \( i \neq j \). However, we do have the same \( L^2 \) bound on the solution at a point. The below proposition and its proof are the same as [17, (3.1)], although we need to obtain bounds in the \( L^2 \)-periodic setting, uniformly in \( L \).

**Proposition 3.1.** There is a constant \( C = C(\rho) < \infty \), depending only on \( \rho \), so that if \( L \in [1, \infty) \) and \( u \) solves (2.5L) with initial condition \( u(0, \cdot) \equiv 0 \), then for all \( t \geq 0 \) and \( x \in \mathbb{R}^d \), we have the bound
\[
\frac{1}{T} \int_0^T \mathbf{E} u(t, x)^2 \, dt \leq C.
\]
(3.1)
Suppose that $a$ will be necessary to set up the argument. The next several sections will develop the necessary bounds on the terms. Introduce weights complicates the situation enormously. In this section, we perform the derivative computations that

\begin{equation}
E h(t, x) = \log E \phi(t, x) = 0 \text{ for all } t \geq 0, x \in \mathbb{R}.
\end{equation}

Therefore,

\begin{equation}
0 \leq E h(T, x) = \int_0^T E -|\nabla h(t, x)|^2 + \sum_{k \in \mathbb{Z}^d} \rho^2(Lk) dt = T \sum_{k \in \mathbb{Z}^d} \rho(Lk) - \int_0^T E |u(t, x)|^2 dt
\end{equation}

by (3.2) and the space-stationarity of $h$, and $\sum_{k \in \mathbb{Z}^d} \rho^2(Lk)$ is bounded above independently of $L$, so we get (3.1).

\section{The Kruzhkov Maximum Principle}

Now we deploy a version of the Kruzhkov maximum principle [30]. This type of argument was used for the stochastic Burgers equation on the $d$-dimensional torus in [6, Theorem 4.2]. In the whole-space setting, the need to introduce weights complicates the situation enormously. In this section, we perform the derivative computations that will be necessary to set up the argument. The next several sections will develop the necessary bounds on the terms.

\begin{lemma}
Suppose that $a : \mathbb{R}^d \to \mathbb{R}$ is twice-differentiable, $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is twice-differentiable, and $v$ solves (2.4). Define $u = v + \psi$. Fix $i \in \{1, \ldots, d\}$ and define $z = a \partial_i v$. Then

\begin{equation}
\partial_i z \leq \frac{1}{2} \partial_a \Delta z - \nabla \log a \cdot \nabla z - \frac{1}{z}[a^2 \nabla \log a \cdot \nabla z - \nabla \log a \cdot \nabla z] - a^{-1} |z + a \partial_i \psi|^2 - u \cdot \nabla z.
\end{equation}

\end{lemma}

\begin{proof}
Without loss of generality, we can assume that $i = 1$. Define $q = \partial_1 v$. Taking derivatives in (2.4), we have

\begin{equation}
\partial_i q = \frac{1}{2} \Delta q - \frac{1}{2} \partial_i (v + \psi)^2 = \frac{1}{2} \Delta q - \frac{1}{2} \partial_i (v + \psi)^2 - u \cdot \partial_i (v + \psi) \leq \frac{1}{2} \Delta q - |q + \partial_1 \psi|^2 - u \cdot \nabla q.
\end{equation}

where in the last inequality we used that $u \cdot \partial_1 v = u \cdot \nabla q$ since $v$ is a gradient. Now $z = aq$, so

\begin{equation}
\partial_i z = a \partial_i q \leq \frac{1}{2} \partial_a \Delta z - a |q + \partial_1 \psi|^2 - u \cdot (a \nabla q + a \partial_1 \psi).
\end{equation}

Applying the two calculus identities (2.10) to (2.2) yields (4.1) in the case $i = 1$:

\begin{equation}
\partial_i z \leq \frac{1}{2} \Delta z - \nabla \log a \cdot \nabla z - \frac{1}{z}[\nabla \log a \cdot \nabla z - \nabla \log a \cdot \nabla z] - a^{-1} |z + a \partial_i \psi|^2 - u \cdot \nabla z.
\end{equation}

\end{proof}

Now we specialize to the case when the partial derivative is evaluated at a sufficiently large local maximum. The arguments in Sections 5 below will be motivated by the need to understand solutions to the key differential inequality (4.4), proved in the following proposition.

\begin{proposition}
Let $t, \varepsilon \in (0, 1)$ and $i \in \{1, \ldots, d\}$, and fix $v, u, z$ as in the statement of Lemma 4.1. If $t > 0$, $x_*$ is a local maximum of $z(t, \cdot)$, and

\begin{equation}
z(t, x_*) \geq 4(d + 3) + \|\psi(t, \cdot)\|_{C^{d+1}_{\bar{t}, \nu}}.
\end{equation}

then

\begin{equation}
\partial_i z(t, x_*) \leq -\frac{1}{8} \langle x_* \rangle ^2 + |u(t, x_*)| \left( \frac{z(t, x_*)}{\langle x_* \rangle ^2 + |u(t, x_*)|} + \|\psi(t, \cdot)\|_{C_{\bar{t}, \nu}} \right).
\end{equation}

\end{proposition}

\begin{proof}
Define $a(x) = (x + K)^{-t}$. As in (2.13), it is simple to estimate that

\begin{equation}
|\nabla \log a(x)| \leq (x + K)^{-1} \leq 1; \quad |\Delta \log a(x)| \leq d + 2.
\end{equation}

Note also that

\begin{equation}
a(x)^{1/\varepsilon} |\partial_i \psi(t, x)| \leq \sup_{x \in \mathbb{R}^d} \frac{|\partial_i \psi(t, x)|}{(x + K)^{\varepsilon}} \leq \sup_{x \in \mathbb{R}^d} \frac{|\partial_i \psi(t, x)|}{(x)^{\varepsilon}} \leq \|\psi(t, \cdot)\|_{C_{\bar{t}, \nu}}.
\end{equation}

Considering (4.1), noting that $\nabla z(t, x_*) = 0$ and $\Delta z(t, x_*) \leq 0$ since $x_*$ is a local maximum of $z$, and applying (4.5) and (4.6), we obtain

\begin{equation}
\partial_i z(t, x_*) \leq \frac{1}{2} (d + 3)|z(t, x_*)| - a(x_*)^{-1} |(z(t-x_*) - \|\psi\|_{C_{\bar{t}, \nu}})^{\varepsilon}|^2 + |u(t, x_*)| \left( \frac{\ell|z(t, x_*)|}{\langle x_* \rangle ^2 + a(x_*)^{1-\ell} \|\psi(t, \cdot)\|_{C_{\bar{t}, \nu}}} \right).
\end{equation}

The assumption (4.3) yields the inequalities \( (z(t,x)) - \|b(t)\|_{C^2_y}^2 \geq t \frac{1}{2} \|z(t,x)\| \) and \( \frac{1}{2} (d + 1) \|z(t,x)\| \leq \frac{1}{k} \|z(t,x)\|^2 \), and plugging these and the assumption that \( \ell < 1 \) into (4.7) implies (4.4).

\[ \square \]

5. The Sobolev-type inequality

The leading \(-z^2\) term in (4.4), intuitively, should bring the positive part of \( z \) down to a quantity of order-1 in time of order-1, independently of the initial conditions. In order to make this rigorous, of course, we need to control the second term of (4.4), and in particular control \( u(t,x) \). Besides \( z(t,x) \), the only quantities available to control \( u(t,x) \) are \( L^2 \) norms of \( u(t,\cdot) \) with respect to finite measures, whose second moments can be \emph{a priori} controlled by Proposition 3.1 and Fubini's theorem. In this section, we prove a bound for general functions \( u \) of gradient type which will be sufficient for our purposes. If \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is twice-differentiable, define \( \Delta^+ f = \sum_{i=1}^d (\partial_{i,i} f)^+ \). The goal of this section is to prove the following proposition, which we will apply later on with \( \nabla f = u(t,\cdot) \).

**Proposition 5.1.** Let \( m > 0 \) and \( \alpha > 1 \) satisfy \( m > d/(2\alpha) \), and put \( \ell = m - 1 + 1/\alpha \). There is a constant \( C < \infty \) and a finite measure \( \nu \) on \( \mathbb{R}^d \) so that if \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is twice-differentiable and \( K \in [0,\infty) \), then

\[ \|\nabla f\|_{C_{\nu}(\mathbb{R}^d)} \leq C (\|\Delta^+ f\|_{C_{\nu}(\mathbb{R}^d)} + \|\nabla f\|_{L^2(\mathbb{R}^d,d\nu)}) \]

in which we use the notation \( \|f\|_{L^2(\mathbb{R}^d,d\nu)} = \left( \int_{\mathbb{R}^d} |f(x)|^2 \nu(dx) \right)^{1/2} \).

Proposition 5.1 follows from the following local version. For \( x \in \mathbb{R}^d \) and \( r > 0 \), define the cube \( Q(x,r) = \{ y \in \mathbb{R}^d : |x-y|_{\infty} \leq r \} \), where \( |x|_{\infty} = \max_{i \in \{1,\ldots,d\}} |x_i| \), and the grid \( Q(x,r) = \{ x^k \cap [x + (\frac{1}{4} r + \frac{1}{2} r \mathbb{Z})^d] \} \) of \( 4^d \) points in a cubic lattice in \( Q(x,r) \).

**Proposition 5.2.** If \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is twice-differentiable, \( x \in \mathbb{R}^d \), and \( r \in (0,\infty) \), then

\[ \|\nabla f\|_{C(Q(x,r))} \leq 84d (6r \|\Delta^+ f\|_{C(Q(x,9r))} + \|\nabla f\|_{C(Q(x,9r))}) \]

Before proving Proposition 5.2, we show how it implies Proposition 5.1.

**Proof of Proposition 5.1.** For \( k = 0,1,2,\ldots \), let \( k^* = m \), and let \( x_{k,1}, \ldots, x_{k,m_k} \in \mathbb{R}^d \) be chosen so that the cubes \( Q_{k,j} = Q(x_{k,j}, k^{a-1}) \), \( j = 1,\ldots,m_k \), cover \( A_k \) and \( m_k \leq Ck^{d-1} \) for a constant \( C \) independent of \( k \). Define \( Q_{k,j} = Q(x_{k,j}, 9k^{a-1}) \), \( Q_{k,j} = Q(x_{k,j}, 9k^{a-1}) \), and

\[ \nu = \sum_{k=0}^\infty \frac{1}{p_m(k^\alpha)^2} \sum_{j=0}^{m_k} \sum_{y \in Q_{k,j}} \delta_y, \]

where \( \delta_y \) denotes a Dirac measure at \( y \). We note that

\[ \nu(Q_{k,j}) = \sum_{k=0}^\infty \frac{1}{p_m(k^\alpha)^2} \sum_{j=0}^{m_k} \sum_{y \in Q_{k,j}} 1 \leq \sum_{k=0}^\infty \frac{1}{p_m(k^\alpha)^2} \sum_{j=0}^{m_k} 4^d \leq C \sum_{k=0}^\infty \langle k \rangle^{-2a(\alpha+d-1)} < \infty, \]

for some constant \( C \) (depending on the dimension \( d \)), where the penultimate inequality comes from the bound \( m_k \leq Ck^{d-1} \), and the fact that the final sum is finite comes from the assumption \( m > d/(2\alpha) \).

By Proposition 5.2, we have for each \( k,j \) that

\[ \|\nabla f\|_{C(Q_{k,j})} \leq 84d \left( 6k^{a-1} \|\Delta^+ f\|_{C(Q_{k,j})} + \|\nabla f\|_{C(Q_{k,j})} \right), \]

so

\[ \frac{\|\nabla f\|_{C(Q(x_k,k^{a-1}))}}{p_m(k^\alpha)} \leq 84d \left( 6k^{a-1} \frac{\|\Delta^+ f\|_{C(Q_{k,j})}}{p_m(k^\alpha)} + \frac{\|\nabla f\|_{C(Q_{k,j})}}{p_m(k^\alpha)} \right) \leq C \left( \|\Delta^+ f\|_{C_{\nu}(\mathbb{R}^d)} + \|\nabla f\|_{L^2(\mathbb{R}^d,d\nu)} \right) \]

for some \( C < \infty \) depending only on \( d \). Taking a supremum over all \( k,j \), we obtain (5.1).
We will spend the rest of this section proving Proposition 5.2. Define, for $f : \mathbb{R}^d \to \mathbb{R}$ and $S \subset \mathbb{R}^d$, the oscillation \( \text{osc}(f; S) = \sup_S f - \inf_S f \). Our proof relies on the observation that if the oscillation of $f$ is controlled on the set of four points lying on a line, and we have a one-sided bound on the second derivative of $f$ in the direction of the line, then we can obtain a bound on the oscillation of $f$ on the segment connecting the middle two points, and also on the directional derivative of $f$ in the direction of the line on this segment. We make this precise in Lemmas 5.5 and 5.6 below. Thus we will seek to control the oscillation of a multidimensional function $f$ on a cube, which we do by induction on the dimension, successively extruding a controlled set in each coordinate direction. First we establish the following elementary lemma about oscillation.

**Lemma 5.3.** If $S_0 \subset \mathbb{R}^d$ and $(S_i)_{i \in I}$ is a family of subsets of $\mathbb{R}^d$, indexed by some index set $I \neq \emptyset$, and $S_0 \cap S_i \neq \emptyset$ for each $i \in I$, and we put $S = S_0 \cup \bigcup_{i \in I} S_i$, then we have

$$\text{osc}(f; S) \leq \text{osc}(f; S_0) + 2 \sup_{i \in I} \text{osc}(f; S_i).$$

**Proof.** Let $x_1, x_2 \in S$. Choose $i_1, i_2 \in \{0\} \cup I$ so that $x_j \in S_{i_j}, \ i = 1,2$, and select $y_j \in S_0 \cap S_{i_j}$ arbitrarily. Then we have

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(y_1)| + |f(y_1) - f(y_2)| + |f(y_2) - f(x_2)| \leq \text{osc}(f; S_{i_1}) + \text{osc}(f; S_0) + \text{osc}(f; S_{i_2}).$$

This implies (5.2). \(\square\)

### 5.1. One-dimensional inequalities

Before we carry out our induction procedure in Subsection 5.2, we prove the necessary statements for functions of one variable. The following inequalities are all simple consequences of the fundamental theorem of calculus.

**Lemma 5.4.** Suppose that $f : [0,1] \to \mathbb{R}$ is differentiable. Then we have

$$\|f\|_{C([0,1])} \leq \max\{|f(0)|, |f(1)|\} + \|(f')^+\|_{C([0,1])}.$$  

**Proof.** We have, for each $x \in [0,1]$, that

$$f(1) - \int_x^1 (f'(y))^+ \, dy \leq f(1) - \int_x^1 f'(y) \, dy = f(x) = f(0) + \int_0^x f'(y) \, dy \leq f(0) + \int_0^x (f'(y))^+ \, dy,$$

so (5.3) follows from the resulting estimate

$$|f(x)| \leq \max\{|f(0)|, |f(1)|\} + \int_0^x (f'(y))^+ \, dy \leq \max\{|f(0)|, |f(1)|\} + \|(f'(y))^+\|_{C([0,1])}.$$ \(\square\)

**Lemma 5.5.** Suppose that $f : [-1,2] \to \mathbb{R}$ is twice-differentiable. Then we have

$$\|f''\|_{C([0,1])} \leq \text{osc}(f; [-1,2] \cap \mathbb{Z}) + \frac{1}{2} \|(f''')^+\|_{C([-1,2])}.$$  

**Proof.** Let $x_0 \in [0,1]$ be such that $|f'(x_0)| = \|f'\|_{C([0,1])}$. Assume without loss of generality that $f''(x_0) \leq 0$; otherwise, consider $\tilde{f}(x) = f(1/2 - x)$. Then we have, for all $x \in [x_2, 2]$, that

$$f'(x) \leq f'(x_0) + \int_{x_0}^x (f''(t))^+ \, dt \leq -\|f''\|_{C([0,1])} + (x - x_0) \|(f''')^+\|_{C([-1,2])}.$$  

This implies that

$$\text{osc}(f; [-1,2] \cap \mathbb{Z}) \geq -(f(2) - f(1)) = -\int_1^2 f'(x) \, dx \geq \|f''\|_{C([0,1])} - \frac{1}{2} \|(f''')^+\|_{C([-1,2])},$$

and (5.4) follows. \(\square\)

**Lemma 5.6.** Suppose that $f : [-1,2] \to \mathbb{R}$ is twice-differentiable. Then we have

$$\text{osc}(f; [0,1]) \leq \text{osc}(f; [-1,2] \cap \mathbb{Z}) + \frac{1}{2} \|(f''')^+\|_{C([-1,2])}.$$  

**Proof.** Fix $0 \leq x_1 < x_2 \leq 1$ so that $|f(x_2) - f(x_1)| = \text{osc}(f; [0,1])$. The mean value theorem gives an $x_0 \in (x_1, x_2)$ with

$$|f'(x_0)| = \frac{|f(x_2) - f(x_1)|}{x_2 - x_1} \geq |f(x_2) - f(x_1)| = \text{osc}(f; [0,1]),$$

and then Lemma 5.5 yields (5.5). \(\square\)
5.2. **Induction on the dimension.** In order to carry out the inductive procedure, we first prove a lemma controlling the oscillation of the extrusion of a set in a coordinate direction. Let \( e_1, \ldots, e_d \) be the standard basis of \( \mathbb{R}^d \). In this section, we will add sets \( A, B \subset \mathbb{R}^d \) in the Minkowski sense, meaning \( A + B = \{a + b \mid a \in A, b \in B \} \), and also define the product of a set \( A \subset \mathbb{R}^d \) with a vector \( v \) as \( Av = \{av \mid a \in A \} \).

**Lemma 5.7.** Suppose that \( f : \mathbb{R}^d \to \mathbb{R} \) is twice-differentiable, \( i \in \{1, \ldots, d\} \), and \( S \subset \mathbb{R}^d \). Then, for any \( x \in S \),

\[
osc(f; S + [0, 1]e_i) \leq 28\| (\partial_{ii} f)^+ \|_{C(S+[1,2]e_i)} + 3\max_{\ell=-1}^{2} \| \partial f(x + \ell e_i) \| + 2osc(f; S + \ell e_i)).
\]

(5.6)

**Proof.** By Lemma 5.6, we have, for each \( x \in S \), that

\[
osc(f; x + [0, 1]e_i) \leq osc(f; x + ([-1, 2] \cap Z)e_i) + \frac{1}{2}\| (\partial_{ii} f)^+ \|_{C(x+[1,2]e_i)}
\]

\[
\leq osc(f; S + ([-1, 2] \cap Z)e_i) + \frac{1}{2}\| (\partial_{ii} f)^+ \|_{C(S+[1,2]e_i)}.
\]

(5.7)

By Lemma 5.3, we have

\[
osc(f; S + [0, 1]e_i) \leq osc(f; S) + 2\sup_{x \in S} osc(f; x + [0, 1]e_i)
\]

\[
\leq osc(f; S) + 2 osc(f; S + ([-1, 2] \cap Z)e_i) + \| (\partial_{ii} f)^+ \|_{C(x+[1,2]e_i)}
\]

\[
\leq 3 osc(f; S + ([-1, 2] \cap Z)e_i) + \| (\partial_{ii} f)^+ \|_{C(S+[1,2]e_i)}.
\]

(5.8)

where in the second inequality we used (5.7). Also by Lemma 5.3, we have, for any \( x \in S \), that

\[
osc(f; S + ([-1, 2] \cap Z)e_i) \leq \| \partial f \|_{L^1([x+[-1,2]e_i]; \mathcal{H}^d)} + 2\max_{\ell=-1}^{2} osc(f; S + \ell e_i),
\]

(5.9)

where \( \mathcal{H}^1 \) denotes the one-dimensional Hausdorff measure on \( x + [-1, 2]e_i \). Using (a rescaled version of) Lemma 5.4, we have

\[
\| \partial f \|_{L^1([x+[-1,2]e_i]; \mathcal{H}^d)} \leq 3\| \partial f \|_{C(x+[1,2]e_i)} \leq 3(\max\{ | \partial f(x - e_i)|, | \partial f(x + 2e_i)| \}) + 3\| (\partial_{ii} f)^+ \|_{C(x+[1,2]e_i)}.
\]

(5.10)

Plugging (5.10) into (5.9), and then (5.9) into (5.8), we obtain (5.6) from the bound

\[
osc(f; S + [0, 1]e_i) \leq 9\max\{ | \partial f(x - e_i)|, | \partial f(x + 2e_i)| \} + 28\| (\partial_{ii} f)^+ \|_{C(S+[1,2]e_i)} + 6\max_{\ell=-1}^{2} osc(f; S + \ell e_i).
\]

\( \square \)

Lemma 5.7 then forms the heart of the inductive step in the following proposition.

**Proposition 5.8.** For any twice-differentiable function \( f : \mathbb{R}^d \to \mathbb{R} \), we have

\[
osc(f; [0, 1]^d) \leq 28d \left( \| \Delta f \|_{C([-1,2]^d)} + \| \nabla f \|_{C((-1,2]^d)} \right).
\]

Proof. We will prove by induction the stronger statement that for each \( i \in \{0, \ldots, d\} \) and \( y \in ([-1, 2] \cap Z)^{d-i} \),

\[
\max_{y \in ([-1, 2] \cap Z)^{d-i}} osc(f; [0, 1]^{d-i} \times \{y\}) \leq 28i \left( \| \Delta f \|_{C([-1,2]^d)} + \| \nabla f \|_{C((-1,2]^d)} \right).
\]

(5.11)

The base case \( i = 0 \) requires no explanation. For the inductive step, we assume that

\[
\max_{y \in ([-1, 2] \cap Z)^{d-i+1}} osc(f; [0, 1]^{d-i} \times \{y\}) \leq 28(i-1) \left( \| \Delta f \|_{C([-1,2]^d)} + \max_{w \in ([-1, 2] \cap Z)^d} | \nabla f(w) | \right).
\]

(5.12)

Now for all \( z \in ([-1, 2] \cap Z)^d \), we have by Lemma 5.7 that

\[
osc(f; [0, 1]^{d-i} \times \{z\}) \leq 28\| (\partial_{ii} f)^+ \|_{C([0,1]^{i-1} \times [-1,2] \times \{z\})} + 3\max_{\ell=-1}^{2} \| \partial f(0_{i-1}, \ell, z) \| + 2osc(f; [0, 1]^{d-i} \times \{\ell, z\}) - \| \nabla f \|_{C((-1,2]^d)}\]

(5.13)

Here and henceforth, \( 0_{i-1} \) denotes the vector of length \( i-1 \) all of whose coordinates are 0. We note that

\[
\max_{\ell=-1}^{2} \| \partial f(0_{i-1}, \ell, z) \| \leq \| \nabla f \|_{C([-1,2]^d)}
\]

(5.14)

and, by the inductive hypothesis (5.12), we have

\[
\max_{\ell=-1}^{2} osc(f; [0, 1]^{d-i} \times \{\ell, z\}) \leq 28(i-1) \left( \| \Delta f \|_{C([-1,2]^d)} + \| \nabla f \|_{C((-1,2]^d)} \right).
\]

(5.15)
Plugging (5.14) and (5.15) into (5.13), we obtain
\[
\text{osc}(f;[0,1]^d \times \{z\}) \leq 28\varepsilon \| \Delta^+ f \|_{C([-1,2]^d)} + (9 + 28(i - 1))\| \nabla f \|_{C([-1,2]^d)}(\{z\})^d,
\]
which implies (5.11).

It only remains to extract from our oscillation bound a bound on the gradient of \( f \).

**Proposition 5.9.** If \( f : \mathbb{R}^d \to \mathbb{R} \) is twice-differentiable, then
\[
\| \nabla f \|_{C([1/3,2/3]^d)} \leq 84d(\| \Delta^+ f \|_{C([-1,2]^d)} + \| \nabla f \|_{C([-1,2]^d)}(\{z\})^d).
\]  
(5.16)

**Proof.** Applying a rescaled version of Lemma 5.5 in each coordinate direction, we obtain
\[
\sup_{d \in \mathbb{N}} \| \nabla f \|_{C([1/3,2/3]^d)} \leq \text{osc}(f;[0,1]^d) + \frac{1}{18}\| \Delta^+ f \|_{C([0,1]^d)}.
\]
Then we obtain (5.16) by estimating, using Proposition 5.8,
\[
\| \nabla f \|_{C([1/3,2/3]^d)} \leq 84d \left( \| \Delta^+ f \|_{C([-1,2]^d)} + \| \nabla f \|_{C([-1,2]^d)}(\{z\})^d \right) + \frac{1}{3}\| \Delta^+ f \|_{C([0,1]^d)}.
\]

Now Proposition 5.2 follows from translating and rescaling (5.16).

6. PROOF OF TIGHTNESS

6.1. The periodic case. To complete our tightness proof, we will use the maximum principle described in Section 4, so it is simplest to first work in the periodic setting and then let the periodicity go to infinity. Throughout this subsection, we fix exponents
\[
m \in (2d/(d+4), 1), \quad \ell \in (m/2, (1 + 2/d)m - 1), \quad \varepsilon \in (0, \varepsilon - m/2).
\]  
(6.1)

The requirement \( d < 4 \) is necessary so that the interval \((2d/(d+4), 1)\) is nonempty. We note that if \( m \) is chosen arbitrarily in \((2d/(d+4), 1)\), then \( \ell \) and \( \varepsilon \) can be chosen to satisfy (6.1). Let \( \nu \) be the finite measure on \( \mathbb{R}^d \) given by the statement of Proposition 5.1 with \( \alpha = 1/(1 + \ell - m) \), noting that \( m > d/(2\alpha) \).

**Lemma 6.1.** For every \( M \in [1, \infty) \), there is a \( K_0 = K_0(d, M) \in [1, \infty) \) so that if \( K \geq K_0 \), then there is an \( A = A(d, M, K) < \infty \) so that the following holds. Suppose that \( L \in [1, \infty) \) and \( \psi \in C([0,1]; \mathcal{A}^2_\nu) \) is \( LZ^d \)-periodic. Let \( \nu \in C([0,1]; \mathcal{A}^{\phi}_m) \) be an \( LZ^d \)-periodic solution to (2.4) and let \( u = \nu + \psi \). If
\[
\int_0^1 \| u(s, \cdot) \|^2_{L^2([\mathbb{R}^d, \nu])} \, ds, \| \psi \|_{C([0,1]; \mathcal{A}^2_\nu)} \leq M
\]
then there exists a \( t_* \in [0, \min\{K^{1-\ell} \| u(0, \cdot) \|_{C_{\phi_m,K}}^{-1}, 1\}] \) so that \( \| u(t_*, \cdot) \|_{C_{\phi_m,K}} \leq \max\{A, \frac{1}{2} \| u(0, \cdot) \|_{C_{\phi_m,K}}\} \).

**Proof.** Let us first agree to take \( A \geq K^{1-\ell} \), so we can assume that
\[
\| u(0, \cdot) \|_{C_{\phi_m,K}} > K^{1-\ell},
\]
(6.3)
as otherwise there is nothing to show. Define
\[
Z(t) = \max_{i=1} \| (\partial_i v_i) \|_{C_{\phi,K}}, \quad z_i(t, x) = \frac{\partial_i v_i(t, x)}{p_{\phi,K}(x)}; \quad (i, s(t), x(s(t))) \in \arg\max_{(i, s, x) \in \{1, \ldots, d\} \times \mathbb{R}^d} (z_i(t, x), \partial_i z_i(t, x)),
\]
where in the argmax we use the lexicographic ordering on \( \mathbb{R} \times \mathbb{R} \). In other words, \((i, s(t), x(s(t)))\) is chosen first to maximize \( z_i(t, x) \), and then, among the maximizers of \( z_i(t, x) \), to maximize \( \partial_i z_i(t, x) \). The argmax is guaranteed to exist since \( v \) is assumed to be \( LZ^d \)-periodic. Note that \( Z(t) = z_i(t, x(s(t))) \).

By Proposition 4.2, (6.2) and [25, Lemma 3.5], for almost every \( t \) such that \( Z(t) \geq 4(d + 3) + M \), we have
\[
Z'(t) \leq -\frac{1}{8} p_{\phi,K}(x(s(t)) \frac{Z(t)}{Z(t) + |u(t, x(t))|} + \frac{M}{p_{\phi,K}(x(s(t)) + M})
\]
(6.4)
Now Proposition 5.1 and (6.2) imply that there is a constant C, depending only on the dimension \( d \), so that
\[
\| u(t, x(t)) \|_{L^2([\mathbb{R}^d, \nu])} \leq C(Z(t) + \| u(t, \cdot) \|_{L^2([\mathbb{R}^d, \nu])} + M).
\]
(6.5)
Also, Proposition 2.4 and (6.3) imply that there is a $K_1 = K_1(M)$ so that if $K \geq K_0$ and $t < K^{1-\ell} \| u(0, \cdot) \|_{C_{pm,K}}^\frac{1}{2}$ then
\[ |u(t, x(t))| \leq \left( \| u(0, \cdot) \|_{C_{pm,K}}^\frac{1}{2} - K^{(1-\ell)t} \right)^{-1} + M \right] p_{m+\ell,t,K}(x_0(t)). \]
In particular, if $t \leq t_0 := \frac{1}{K} K^{1-\ell} \| u(0, \cdot) \|_{C_{pm,K}}^\frac{1}{2}$, then we have
\[ |u(t, x(t))| \leq \left( 2 \| u(0, \cdot) \|_{C_{pm,K}} + M \right) p_{m+\ell,t,K}(x_0(t)). \tag{6.6} \]
Using (6.5) and (6.6) in (6.4), if $Z(t)$ is the solution of the Riccati equation, then we have
\[ Z'(t) \leq -\frac{1}{8} p_{\ell,K}(x_0(t)) Z(t)^2 + C[Z(t) + \| u(t, \cdot) \|_{L^2(R^d, \nu)} + M] Z(t) p_{m+\ell-1,K}(x_0(t)) \]
for almost every $t$ such that $Z(t) \geq \max \{ 4(d+3) + M, \| u(0, \cdot) \|_{C_{pm,K}}^{1/2} \}$. So by Lemma 6.2 below we have, for all $t \leq t_0$,
\[ Z(t) \leq \exp \left( \int_0^t \| u(t, \cdot) \|_{L^2(R^d, \nu)} \, dt \right) \max \left\{ 8(d+3) + 2M, 2 \| u(0, \cdot) \|_{C_{pm,K}}^{1/2}, 64(M^{1/2}) \right\} \]
\[ \leq e^{M} \max \left\{ 8(d+3) + 2M, 2 \| u(0, \cdot) \|_{C_{pm,K}}^{1/2}, 64(K^{1/2}) \right\}. \]
Now if $t \in [t_0/2, t_0]$, then by the definition of $t_0$, we have $t \geq \frac{1}{4} K^{1-\ell} \| u(0, \cdot) \|_{C_{pm,K}}^{-1}$, so
\[ Z(t) \leq e^{M} \max \left\{ 8(d+3) + 2M, 2 \| u(0, \cdot) \|_{C_{pm,K}}^{1/2}, 256K^{-1} \| u(0, \cdot) \|_{C_{pm,K}} \right\}. \tag{6.7} \]

Next we observe, using (6.2) and the definition of $t_0$, that
\[ \frac{1}{t_0/2} \int_{t_0/2}^{t_0} \| u(t, \cdot) \|_{L^2(R^d, \nu)} \, dt \leq \frac{2}{t_0} \int_{t_0/2}^{t_0} \| u(t, \cdot) \|_{L^2(R^d, \nu)}^2 \, dt \leq \sqrt{2M/t_0} \leq 2 \sqrt{MK^{-\frac{1}{2}}} \| u(0, \cdot) \|_{C_{pm,K}}^{1/2}. \]
Thus we have a $t_* \in [t_0/2, t_0]$ so that
\[ \| u(t_*, \cdot) \|_{L^2(R^d, \nu)} \leq 2 \sqrt{MK^{-\frac{1}{2}}} \| u(0, \cdot) \|_{C_{pm,K}}^{1/2} \leq 2 \sqrt{M} \| u(0, \cdot) \|_{C_{pm,K}}^{1/2} \]
Then we have, again using Proposition 5.1,
\[ \| u(t_*, \cdot) \|_{C_{pm,K}} \leq C \| u(t_*, \cdot) \|_{L^2(R^d, \nu)} + Z(t_*) + M \]
\[ \leq C \left( \sqrt{2M} \| u(0, \cdot) \|_{C_{pm,K}}^{1/2} + \right. \left. e^{M} \max \left\{ 8(d+3) + 2M, 2 \| u(0, \cdot) \|_{C_{pm,K}}^{1/2}, 256K^{-1} \| u(0, \cdot) \|_{C_{pm,K}} \right\} + M \right). \tag{6.8} \]
Let $K \geq 2048e^M$, so we have an $A_1 = A_1(d, K, M)$ so that if $\| u(0, \cdot) \|_{C_{pm,K}} \geq A_1$, then (6.8) implies $\| u(t_*, \cdot) \|_{C_{pm,K}} \leq \frac{1}{2} \| u(0, \cdot) \|_{C_{pm,K}}^{-1}$. This completes the proof, since we can take $K_0 = \max \{ K_1, K_2, 2048e^M \}$ and $A = \max \{ K^{1-\ell}, A_1 \}$ (recalling the original agreement leading to (6.3)).

It remains to prove the Riccati equation estimate we used in the proof of Lemma 6.1.

**Lemma 6.2.** Suppose that $T > 0$, $h : [0, T] \to [\mathbb{R}]$ is Lipschitz, $a > 0$, $f \in L^1([0, T])$, $b > 0$, and for almost every $t \in [0, T]$ such that $h(t) \geq b$, we have $h'(t) \leq -ah(t)^2 + f(t)h(t)$. Then we have
\[ h(t) \leq \exp \left\{ \| f \|_{L^1([0, t])} \right\} \max \left\{ 2b, \left[ h(0)^{-1} + at/2 \right]^{-1} \right\}. \tag{6.9} \]
Proof. Assume without loss of generality that \( f(t) \geq 0 \) for all \( t \). Let \( F(t) = \exp \left\{ - \int_0^t f(s) \, ds \right\} \) and let \( H(t) = F(t) h(t) \). Note in particular that \( F(t) \leq 1 \) for all \( t \). Then, for almost every \( t \in [0, T] \) such that \( H(t) \geq b \), we have \( h(t) \geq b \) and thus

\[
H'(t) \leq -\omega(t)F(t)h(t)^2 \leq -aH(t)^2.
\]

Then (6.9) follows from standard bounds on solutions to the Riccati equation:

\[
\exp \left\{ -\|f\|_{L^1([0,T])} \right\} h(t) = H(T) \leq \max \left\{ 2b[H(0)^{-1} + aT/2]^{-1} \right\} = \max \left\{ 2b[H(0)^{-1} + aT/2]^{-1} \right\}. \quad \Box
\]

We now iterate Lemma 6.1 to show that our dynamics brings, in time 1, any initial condition down to a size depending only on the forcing on \([0, 2]\) and the integrated \( L^2 \) norm \( \int_0^T \|u(s, \cdot)\|_{L^2(\mathbb{R}^d, \nu^s)}^2 \, ds \). The last quantity, when \( u \) is taken to solve (2.5L) with initial condition 0, will be a priori bounded using Proposition 3.1.

**Proposition 6.3.** For every \( M \in [1, \infty) \), there is a \( B = B(d, M) < \infty \) so that the following holds. Suppose that \( L \in [1, \infty) \) and \( \psi \in C([0,2]; \mathcal{A}_c^2) \) is \( L \)-periodic. Let \( v \) be an \( L \)-periodic solution to (2.4) and let \( u = v + \psi \). If

\[
\int_0^2 \|u(s, \cdot)\|^2_{L^2(\mathbb{R}^d, \nu^s)} \, ds \|\psi\|^2_{C([0,2]; C_c^0)} \|\psi\|^2_{C([0,2]; C_{p, \nu}^2)} \leq M
\]

then there exists a \( t \in [0, 2] \) so that \( \|u(t, \cdot)\|_{C_{p, \nu}^m} \leq B \).

**Proof.** Choose \( K_0 \) depending on \( M \) as in Lemma 6.1 and fix \( K \geq K_0 \). (The choice \( K = K_0 \) is fine.) We inductively define a finite sequence of times \( t_0, t_1, \ldots, t_N \) as follows. Let \( t_0 = 0 \). If \( t_k \) has been defined and either \( t_k \geq 1 \) or \( \|u(t_k, \cdot)\|_{C_{p, \nu}^m, K} \leq A \), then let \( N = k \) and stop. Otherwise, by Lemma 6.1, since (6.10) implies (6.2) for the equation considered as starting at time \( t_k < 1 \), there is an \( A = A(d, K, M) < \infty \) so that we can find a

\[
t_{k+1} \in [t_k, t_k + K^{-1-\ell} \|u(t_k, \cdot)\|_{C_{p, \nu}^m, K}^{-1}]
\]

such that \( \|u(t_{k+1}, \cdot)\|_{C_{p, \nu}^m, K} \leq \max \left\{ A, \frac{1}{2} \|u(t_k, \cdot)\|_{C_{p, \nu}^m, K} \right\} \). Thus, for each \( 0 \leq k \leq N - 1 \), we have

\[
\|u(t_k, \cdot)\|_{C_{p, \nu}^m, K} \geq 2^{N-k} \|u(t_{N-1}, \cdot)\|_{C_{p, \nu}^m, K}.
\]

In the case when \( \|u(t_{N-1}, \cdot)\|_{C_{p, \nu}^m, K} > A \) and thus \( t_{N-1} \geq 1 \), this means that by (6.11) we get

\[
1 \leq t_N = \sum_{k=1}^{N} (t_k - t_{k-1}) \leq K^{1-\ell} \sum_{k=0}^{N-1} \|u(t_k, \cdot)\|_{C_{p, \nu}^m, K}^{-1} \leq K^{-1-\ell} \|u(t_k, \cdot)\|_{C_{p, \nu}^m, K}^{-1} \sum_{k=0}^{N-1} 2^{(N-k)} \leq K^{1-\ell} \|u(t_{N-1}, \cdot)\|_{C_{p, \nu}^m, K}^{-1},
\]

so \( \|u(t_{N-1}, \cdot)\|_{C_{p, \nu}^m, K} \leq L^{1-\ell} \). Since the alternative is that \( \|u(t_{N-1}, \cdot)\|_{C_{p, \nu}^m, K} \leq A \), in either case we have

\[
\|u(t_{N-1}, \cdot)\|_{C_{p, \nu}^m} \leq 2^m \|u(t_{N-1}, \cdot)\|_{C_{p, \nu}^m, K} \leq (1 + K)^m \max \left\{ A, K^{1-\ell} \right\}.
\]

\( \Box \)

Up until now all of our arguments have been agnostic to the initial conditions. The value of this, of course, is that they can be applied to the equation started at any time. The next proposition shows how we carry this out.

**Proposition 6.4.** Let \( d \leq 4 \). For each \( \delta > 0 \), we have a constant \( Q(\delta) \) so that the following holds. Let \( L \in [1, \infty) \) and let \( u \) be the solution to (2.5L) with \( u(0, \cdot) \equiv 0 \). Also, let \( T \geq 0 \) and let \( S_T \sim \text{Uniform}([0, T]) \) be independent of everything else. Then \( P[\|u(3 + S_T, \cdot)\|_{C_{p, \nu}^m} > Q(\delta)] < \delta \).

**Proof.** Let \( \psi \) solve (2.3L) with \( \psi(0, \cdot) \equiv 0 \). Standard Gaussian process estimates, using the smoothness of the mollifier \( \rho \), show that there is a constant \( C < \infty \), independent of \( L \), so that \( E[\|\psi\|_{C([1, T+3]; C^2_{\nu})}^2 E[\|\psi\|_{C([1, T+3]; C^2_{p, \nu})}^2] \leq C \) for each \( T \geq 0 \). Also, by Proposition 3.1 and Fubini’s theorem, using the fact that \( \psi \) is a finite measure, we have a (different) constant \( C < \infty \), also independent of \( L \), so that \( \frac{1}{T} \int_0^{T} \|u(s, \cdot)\|_{L^2(\mathbb{R}^d, \nu^s)}^2 \, ds \leq C \) for each \( s \geq 0 \). Markov’s inequality thus implies that there is a \( Q(\delta) < \infty \), independent of \( L \), so that

\[
P \left( \max \left\{ \|\psi\|_{C([1, T+3]; C^2_{\nu})}^2, \|\psi\|_{C([1, T+3]; C^2_{p, \nu})}^2 \right\} \frac{1}{T} \int_0^{T} \|u(s, \cdot)\|_{L^2(\mathbb{R}^d, \nu^s)}^2 \, ds > Q(\delta) \right) < \delta.
\]
By Proposition 6.3, since \( u - \psi \) is \( LZ^d \)-periodic and solves (2.4\( \phi \)), this means that there is a \( Q_1(\delta) < \infty \), still independent of \( L \), so that

\[
P( \text{there is a } t \in [[S_T], [S_T] + 2] \text{ so that } \|u(t, \cdot)\|_{C^m_p} \leq Q_1(\delta) ) \geq 1 - \delta.
\]

Thus by Theorem 2.3, there is a \( Q_2(\delta) < \infty \), not depending on \( L \), so that

\[
P( \|u(S_T + \cdot, \cdot)\|_{C^m_p} \leq Q_2(\delta) ) \geq 1 - \delta. \quad \square
\]

6.2. The non-periodic case. All of the estimates above have been uniform in the periodicity length \( L \), so it should not be surprising that they hold for \( L = \infty \) as well. For each \( L \in [1, \infty) \) let \( u^{[L]} \) be a solution to (2.5\( L \)) with initial condition \( u^{[L]}(0, \cdot) \equiv 0 \). Our first lemma passes to the limit \( L \to \infty \) in Proposition 6.4, showing that the family \( (u^{[\omega]}(S_T, \cdot))_{T \geq 0} \) is bounded in probability with respect to the \( C_{p_\omega} \) norm whenever \( \omega \in (2d/(d + 4), 1) \). Since the equation (1.1) is well-posed in this space, we can then upgrade this boundedness in probability to tightness by using the parabolic regularity statement in Theorem 2.3, which we do in Proposition 6.6 below.

**Lemma 6.5.** If \( d < 4 \), then for any \( \omega \in (2d/(d + 4), 1) \) and any \( \delta > 0 \), we have a constant \( Q(\delta) < \infty \) so that, for all \( T > 0 \), if \( S_T \sim \text{Uniform}([0, T]) \) is independent of everything else, we have

\[
P( \|u^{[\omega]}(S_T + \cdot, \cdot)\|_{C^m_p} \geq Q(\delta) ) \leq \delta. \quad (6.12)
\]

**Proof.** By (2.7) of Theorem 2.3 and (2.2) of Lemma 2.2, for fixed \( t > 0 \) and \( M > 0 \), \( \|u^{[L]}(\cdot) - u^{[\omega]}(\cdot)\|_{C([0, T]; C([-M, M]^d))} \) approaches 0 as \( L \) goes to infinity, almost surely. This means that for fixed \( T > 0 \) and \( M > 0 \), we almost surely have

\[
\lim_{L \to \infty} \|u^{[L]}(S_T + \cdot, \cdot) - u^{[\omega]}(S_T + \cdot, \cdot)\|_{C([\omega, M]^d)} = 0. \quad (6.13)
\]

Choose \( m \in (2d/(d + 4), \omega) \) and choose \( \ell, \varepsilon \) so that \( m, \ell, \varepsilon \) satisfy (6.1). Proposition 6.4 implies that for each \( \delta > 0 \) and \( n \in \mathbb{N} \), there is a \( Q_n(\delta) < \infty \) so that

\[
\sup_{L \in [1, \infty)} P( \|u^{[L]}(S_T + \cdot, \cdot)\|_{C^m_p} > Q_n(\delta) ) = \delta/2^{n+1}. \quad (6.14)
\]

By (6.13), for fixed choice of \( M_n \), we can choose \( L_n \) so large that

\[
P( \|u^{[L_n]}(S_T + \cdot, \cdot) - u(S_T + \cdot, \cdot)\|_{C([-M_n, M_n]^d)} \geq 1 ) \leq \delta/2^{n+1}. \quad (6.15)
\]

Combining (6.14) and (6.15) with a union bound, and then union bounding over all \( n \), we have

\[
P\left( \sup_{n \in \mathbb{N}} \frac{\|u^{[\omega]}(S_T + \cdot, \cdot)\|_{C^m_p} \mathbf{1}_{[-M_n, M_n]^d}}{1 + Q_n(\delta)} \geq 1 \right) \leq \delta. \quad (6.16)
\]

for any sequence \( \{M_n\} \). Now define \( M_0(\delta) = 0 \) and choose \( (M_n(\delta))_{n \in \mathbb{N}} \) so that \( M_n(\delta) \) is nondecreasing in \( n \) and there is a constant \( \bar{Q}(\delta) \) so that \( 1 + Q_n(\delta) \leq \bar{Q}(\delta) \) for each \( n \). If we define \( A_n(\delta) \) to be the square annulus \([0, \bar{Q}(\delta)] 

Combining this with (6.16), we obtain (6.12). \( \square \)

**Proposition 6.6.** For each \( T \geq 0 \), let \( S_T \sim \text{Uniform}([0, T]) \) be independent of everything else. If \( d < 4 \), for any \( \omega \in (2d/(d + 4), 1) \), the family \( (u^{[\omega]}(S_T + \cdot, \cdot))_{T \geq 0} \) is tight in the topology of \( \mathcal{A}_{p_\omega} \).

**Proof.** Choose \( \omega', \omega'' \) so that \( 2d/(d + 4) < \omega' < \omega'' < \omega \). By Lemma 6.5, for each \( \delta > 0 \) we have a constant \( Q(\delta) < \infty \) so that

\[
P( \|u^{[\omega]}(S_T + \cdot, \cdot)\|_{C^m_{p_{\omega'''}}} \geq Q(\delta) ) \leq \delta.
\]
for all $T \geq 0$. In light of Theorem 2.3, this means that there is an $\alpha > 0$ (independent of $\delta$) and a $Q' = Q'(\delta) < \infty$ so that

$$
P \left( \| u^{(\infty)} (S_T + 4, \cdot) \|_{C_{p_{\omega}}} + \| u^{(\infty)} (S_T + 4, \cdot) \|_{C_{p_{2\omega}}} \geq Q' \right) \leq \delta.
$$

This means that the random variable $u^{(\infty)} (S_T + 4, \cdot)$ is bounded in probability in the topology of $\mathcal{A}_{p_{\omega}} \cap \mathcal{A}_{p_{2\omega}}$. But this space embeds compactly into $\mathcal{A}_{p_{\omega}}$ by Lemma 2.1, so the proposition is proved. \hfill \Box

We can now show that stationary solutions exist for (1.1), using a Krylov–Bogoliubov argument (see e.g. [14]).

**Proof of Theorem 1.1.** Fix $\omega \in (2d/(d+4), 1)$. Let $u$ be a solution to (1.1) with initial condition $u(0, \cdot) \equiv 0$. For each $T \geq 0$, let $S_T \sim \text{Uniform}([0, T])$ be independent of everything else. By Proposition 6.6 and Prokhorov’s theorem, for any $\omega \in (2d/(d+4), 1)$ we have an increasing sequence $T_k \uparrow \infty$ and a random $u^* \in \mathcal{A}_{p_{\omega}}$ so that

$$
u(S_{T_k} + 4, \cdot) \xrightarrow{\text{law}} u^* \quad (6.17)
$$

with respect to the topology of $C_{p_{\omega}}$. It is not difficult to see that $u^*$ is space-stationary in law. Let $u^*$ be a solution to (1.1) with initial condition $u^*(0, \cdot) = u^*$. Fix $M > 0$ and let $f \in C([-M, M]^d; \mathbb{R}^d)$, the space of bounded continuous functions on $C([-M, M]^d; \mathbb{R}^d)$. By the Skorokhod representation theorem, we can find a family $(u^{(k)})_{k \in \mathbb{N}}$ of random variables on the same probability space as $u^*$, taking values in $\mathcal{A}_{p_{\omega}}$, so that $\text{Law}(u^{(k)}) = \text{Law}(u(S_{T_k} + 4, \cdot))$ and $\lim_{k \to \infty} \| u^{(k)} - u^* \|_{C_{p_{\omega}}} = 0$ almost surely. By the bounded convergence theorem and the fact that $f \in C([-M, M]^d; \mathbb{R}^d) \subset C(C_{p_{\omega}}(\mathbb{R}^d; \mathbb{R}^d))$, this means that

$$
\lim_{k \to \infty} \mathbf{E} | f(u^{(k)}) - f(u^*) | = 0.
$$

(6.18)

Now fix $t > 0$ and let $u^{[k]}$ be a solution to (1.1) with initial condition $u^{[k]}(0, \cdot) = u^{[k]}$. By (6.18), Theorem 2.3, and the bounded convergence theorem, we have

$$
\lim_{k \to \infty} \mathbf{E} | f(u^{[k]}(t, \cdot)) - f(u^*(t, \cdot)) | = 0.
$$

(6.19)

On the other hand, by construction we have $\text{Law}(u^{[k]}(t, \cdot)) = \text{Law}(u(S_{T_k} + 4 + t, \cdot))$. Thus, for fixed $t$, we have

$$
| \mathbf{E} f(u^*(0, \cdot)) - \mathbf{E} f(u^*(t, \cdot)) | \leq | \mathbf{E} f(u^*(0, \cdot)) - \mathbf{E} f(u(S_{T_k} + 4, \cdot)) | + | \mathbf{E} f(u(S_{T_k} + 4 + t, \cdot)) - \mathbf{E} f(u(S_{T_k} + 4 + t, \cdot)) |
$$

$$
+ | \mathbf{E} f(u^{[k]}(t, \cdot)) - \mathbf{E} f(u^*(t, \cdot)) |.
$$

The first term on the right-hand side goes to 0 as $k \to \infty$ by (6.17), the second by a simple coupling argument and the bounded convergence theorem (as in [17, (4.12)]), and the third by (6.19). This means that $\mathbf{E} f(u^*(0, \cdot)) = \mathbf{E} f(u^*(t, \cdot))$ for all $f \in C([-M, M]^d; \mathbb{R}^d)$ for any $M > 0$, hence for all $f \in C(C_{p_{\omega}}(\mathbb{R}^d; \mathbb{R}^d))$. (See [17, Proposition 4.2].) Thus, $u^*$ is statistically stationary in time. \hfill \Box

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