Compressing Permutation Groups into Grammars and Polytopes.  
A Graph Embedding Approach*

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January 17, 2020

Abstract
It can be shown that each permutation group $G \subseteq S_n$ can be embedded, in a well defined sense, in a connected graph with $O(n + |G|)$ vertices. Some groups, however, require much fewer vertices. For instance, $S_n$ itself can be embedded in the $n$-clique $K_n$, a connected graph with $n$ vertices.

In this work, we show that the minimum size of a context-free grammar generating a finite permutation group $G \subseteq S_n$ can be upper bounded by three structural parameters of connected graphs embedding $G$: the number of vertices, the treewidth, and the maximum degree. More precisely, we show that any permutation group $G \subseteq S_n$ that can be embedded into a connected graph with $m$ vertices, treewidth $k$, and maximum degree $\Delta$, can also be generated by a context-free grammar of size $2^{O(k\Delta \log \Delta)} \cdot m^{O(k)}$. By combining our upper bound with a connection established by Pesant, Quimper, Rousseau and Sellmann [34] between the extension complexity of a permutation group and the grammar complexity of a formal language, we also get that these permutation groups can be represented by polytopes of extension complexity $2^{O(k\Delta \log \Delta)} \cdot m^{O(k)}$.

The above upper bounds can be used to provide trade-offs between the index of permutation groups, and the number of vertices, treewidth and maximum degree of connected graphs embedding these groups. In particular, by combining our main result with a celebrated $2^\Omega(n)$ lower bound on the grammar complexity of the symmetric group $S_n$ due to Glaister and Shallit [23] we have that connected graphs of treewidth $o(n/\log n)$ and maximum degree $o(n/\log n)$ embedding subgroups of $S_n$ of index $2^{cn}$ for some small constant $c$ must have $n^{\omega(1)}$ vertices. This lower bound can be improved to exponential on graphs of treewidth $n^{e}$ for $e < 1$ and maximum degree $o(n/\log n)$.

Keywords.  Permutation Groups, Context Free Grammars, Extension Complexity, Graph Embedding Complexity

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* Lars Jaffke acknowledges support from the Bergen Research Foundation. Mateus de Oliveira Oliveira acknowledges support from the Bergen Research Foundation and from the Research Council of Norway (Project Number 288761). Hans Raj Tiwary is partially supported by the grant GAČR 17-09142S.
1 Introduction

Let \( S_n \) be the set of permutations of the set \( \{1, \ldots, n\} \) and \( \text{str}(S_n) \) be the set of strings in \( \{1, \ldots, n\}^n \) encoding permutations in \( S_n \). The search for minimum size grammars generating the language \( \text{str}(S_n) \) has sparked a lot of interest in the automata theory and in the complexity theory communities, both in the study of lower bounds [19, 32, 21], and in the study of upper bounds [26, 3, 2]. In particular, a celebrated result due to Ellul, Krawetz and Shallit [19] states that any context-free grammar generating the language \( \text{str}(S_n) \) must have size \( 2^{\Omega(n)} \). In this work we complement this line of research by establishing upper bounds for the size of context-free grammars representing a given subgroup \( G \subseteq S_n \). These upper bounds are stated in terms of three structural parameters of connected graphs embedding \( G \): the number of vertices, the treewidth and the maximum degree.

We say that a permutation group \( G \subseteq S_n \) can be embedded in a graph \( X \) with vertex set \( [m] = \{1, \ldots, m\} \), if \( m \geq n \) and \( G \) is equal to the restriction of the automorphism group of \( X \) to its first \( n \) vertices \([n] = \{1, \ldots, n\}\). A more precise definition of the notion of graph embedding is given in Section 3. For a given class of connected graphs \( \mathcal{X} \), the \( \mathcal{X} \)-embedding complexity of \( G \), denoted by \( \text{gec}_\mathcal{X}(G) \), is defined as the minimum \( m \) such that \( G \) can be embedded in an \( m \)-vertex graph \( X \in \mathcal{X} \).

Given an alphabet \( \Sigma \), the symmetric grammar complexity (SGC) of a formal language \( L \subseteq \Sigma^n \) measures the minimum size of a context-free grammar accepting a permuted version of \( L \). As a matter of comparison, we note that online Turing machines working in space \( s \) and with access to a stack have symmetric grammar complexity \( 2^{O(s)} \) [25]. In this setting, the machine reads the input string \( w \in \Sigma^n \) from left to right, one symbol at a time. While reading this string, symbols can be pushed into or popped from the stack. The transitions relation depends on the current state, on the symbol being read at the input, and on the symbol being read at the top of the stack. The caveat is that the number of symbols used in the stack (which can be up to \( n \)) is not counted in the space bound \( s \), which can be much smaller than \( n \) (say \( s = O(\log n) \)). The SGC of a language \( L \subseteq \Sigma^n \) is also polynomially related to the minimum size of a read-once branching program with a stack accepting \( L \) (see for instance [33]).

1.1 Our Results

We show that the automorphism group of any graph with \( n \) vertices, maximum degree \( \Delta \) and treewidth \( k \) has symmetric grammar complexity at most \( 2^{O(k \Delta \log \Delta)} \cdot n^{O(k)} \) (Theorem 3). More generally, we show that the SGC of groups that can be embedded in \( m \)-vertex graphs of maximum degree \( \Delta \) and treewidth \( k \) is at most \( 2^{O(k \Delta \log \Delta)} \cdot m^{O(k)} \) (Theorem 5).

In linear programming theory, it can be shown that there are interesting polytopes \( P \subseteq \mathbb{R}^n \), which can only be defined with an exponential (in \( n \)) number of inequalities, but which can be cast as a linear projection of a higher dimensional polytope \( Q \) that can be defined with polynomially many variables and constraints. Such a polytope \( Q \) is called an extended formulation of \( P \). Extended formulations of polynomial size play a crucial role in combinatorial optimization because they provide an unified framework to obtain polynomial time algorithms for a large variety of combinatorial problems. For this reason, extended formulations of polytopes associated with formal languages and with groups have been studied intensively during the past decades, both from the perspective of lower bounds [38, 22, 40, 35, 4, 14, 30], and from the perspective of upper bounds [10, 15, 9, 16, 36, 40, 10, 15].

By combining our main theorem 5 with a connection established by Pesant, Quimper, Rousseau and Sellmann [34] between the extension complexity of a permutation group and the grammar complexity of a formal language, we show that any permutation group that can...
be embedded in a connected graph with \( m \) vertices, treewidth \( k \), and maximum degree \( \Delta \) can be represented by polytopes of extension complexity \( 2^{O(k \Delta \log \Delta)} \cdot m^{O(k)} \) (Theorem 16).

By combining our upper bound from Theorem 5 with the \( 2^{\Omega(n)} \) lower bound from [19], we obtain an interesting complexity theoretic trade-off relating the index of a permutation group with the size, treewidth and maximum degree of a graph embedding this group (Theorem 21).

As a corollary of this trade-off, we show that subgroups of \( S_n \) with index up to \( 2^c n \) for some small constant \( c \) have superpolynomial graph embedding complexity on classes of graphs with treewidth \( o(n/\log n) \) and maximum degree \( o(n/\log n) \) (Corollary 22). Additionally, this lower bound can be improved from super-polynomial to exponential on classes of graphs of treewidth \( n^\varepsilon \) (for \( \varepsilon < 1 \)) and maximum degree \( o(n/\log n) \) (Corollary 23). In particular, Corollary 23 implies exponential lower bounds for minor-closed families of connected graphs (which have treewidth \( \sqrt{n} \)).

1.2 Related Work

Proving lower bounds for the size of graphs embedding a given permutation group is a challenging and still not well understood endeavour. It is worth noting that it is still not known whether the alternating group \( A_n \) can be embedded in a graph with \( n^{O(1)} \) vertices. We note that by solving an open problem stated by Babai in [7], Liebeck has shown that any graph whose automorphism group is isomorphic to the alternating group (as an abstract group) must have at least \( 2^{\Omega(n)} \) vertices [31]. Nevertheless, a similar result has not yet been obtained in the setting of graph embedding of groups, and indeed, constructing an explicit sequence of groups that have superpolynomial graph embedding complexity is a long-standing open problem [8]. Our results in Corollary 22 and Corollary 23 provide unconditional lower bounds for interesting classes of graphs for any group of relatively small index (index at most \( 2^c n \) for some small enough constant \( c \)).

The crucial difference between the abstract isomorphism setting considered in [31] and our setting is in the way in which graphs are used to represent groups. In the setting of [31], given a group \( G \), the goal is to construct a graph \( X \) whose automorphism group is isomorphic to \( G \). On the other hand, in the graph embedding setting, we want the group \( G \) to be equal to the action of the automorphism group \( \text{Aut}(X) \) on its first \( [n] \) vertices. In the abstract isomorphism setting it has been shown by Babai that any class of graphs \( \mathcal{X} \) excluding a fixed graph \( H \) as a minor, there exists some finite group which is not isomorphic to the automorphism group of any graph in \( \mathcal{X} \) [6]. Our Corollary 23 can be regarded as a result in this spirit in the context of graph embedding. While the lower bound stated in Corollary 23 also applies to graphs that are not minor closed, this lower bound is only meaningful for graphs of maximum degree at most \( o(n/\log n) \).

We observe that in Theorem 5 an exponential dependence on the maximum degree parameter \( \Delta \) is unavoidable. Indeed, as stated above, the symmetric grammar complexity of the language \( \text{str}(S_n) \) is \( 2^{\Theta(n)} \). On the other hand, for each \( n \in \mathbb{N}_+ \), the symmetric group \( S_n \) can be embedded in the star graph \( K_{n,1} \) with vertex set \( V(K_{n,1}) = \{1, \ldots, n + 1\} \), and edge set \( E(K_{n,1}) = \{\{i, n + 1\} : i \in \{1, \ldots, n\}\} \), which is a connected graph of treewidth 1. Nevertheless, it is not clear to us whether the logarithmic factor \( \log \Delta \) can be shaved from the exponent of the upper bound \( 2^{O(k \Delta \log \Delta)} \cdot m^{O(k)} \). We also note that the connectedness requirement is also crucial for our upper bounds since \( S_n \) can be embedded in the discrete graph \( D_n \) with vertex set \( D_n = \{1, \ldots, n\} \), and edge set \( E(D_n) = \emptyset \).
2 Preliminaries

We let \( \mathbb{N} \) denote the set of non-negative integers and \( \mathbb{N}_+ = \mathbb{N} \setminus \{0\} \) denote the set of positive integers. For each \( n \in \mathbb{N}_+ \), we let \( [n] = \{1, \ldots, n\} \). For each finite set \( S \) we let \( \mathcal{P}(S) = \{ S' : S' \subseteq S \} \) denote the set of all subsets of \( S \). For each set \( S \) and each \( k \in \mathbb{N} \), we let \( \binom{S}{k} = \{ S' \subseteq S : |S'| = k \} \) be the set of subsets of \( S \) of size \( k \) and \( \binom{S^*}{k} \) the set of subsets of size at most \( k \). For a function \( f : X \to Y \) and a set \( X' \subseteq X \), we denote by \( f|_{X'} \) the restriction of \( f \) to \( X' \), i.e., the function \( f|_{X'} : X' \to Y \) with \( f|_{X'}(x) = f(x) \) for each \( x \in X' \).

Prefix Closed Sets. For each \( r \in \mathbb{N}_+ \), we let \( [r]^* \) be the set of all strings over \( [r] \), including the empty string \( \lambda \). Let \( p \) and \( u \) be strings in \( [r]^* \). We say that \( p \) is a prefix of \( u \) if there exists \( q \in [r]^* \) such that \( u = pq \). Note that \( u \) is a prefix of itself, and that the empty string \( \lambda \) is a prefix of each string in \( [r]^* \). A non-empty subset \( U \subseteq [r]^* \) is prefix closed if for each \( u \in U \), each prefix of \( u \) is also in \( U \). We note that the empty string \( \lambda \) is an element of any prefix closed subset of \( [r]^* \). We say that \( U \subseteq [r]^* \) is well numbered if for each \( p \in [r]^* \) and each \( j \in [r] \), the presence of \( pj \) in \( U \) implies that \( p1, \ldots, p(j - 1) \) also belong to \( U \).

Tree-Like Sets. We say that a subset \( U \subseteq [r]^* \) is tree-like if \( U \) is both prefix-closed and well-numbered. Let \( U \) be a tree-like subset of \( [r]^* \). If \( p \in U \), then we say that \( p \) is a child of \( p \), or interchangeably, that \( p \) is the parent of \( p \). If \( pu \in U \) for \( u \in [r]^* \), then we say that \( pu \) is a descendant of \( p \). For a node \( p \in U \) we let \( U|_p = \{ pu \in U : u \in [r]^* \} \) denote the set of all descendants of \( p \). Note that \( p \) is a descendant of itself and therefore, \( p \in U|_p \). A leaf of \( U \) is a node \( p \in U \) without children. We let \( \text{leaves}(U) \) be the set of leaves of \( U \), and \( \text{leaves}(U, p) \) be the set of leaves that are descendants of \( p \).

Terms. Let \( \Sigma \) be a finite set of symbols. An \( r \)-ary term over \( \Sigma \) is a function \( t : \text{Pos}(t) \to \Sigma \) whose domain \( \text{Pos}(t) \) is a tree-like subset of \( [r]^* \). We denote by \( \text{Ter}(\Sigma) \) the set of all terms over \( \Sigma \). If \( t_1, \ldots, t_r \) are terms in \( \text{Ter}(\Sigma) \), and \( a \in \Sigma \), then we let \( t = a(t_1, \ldots, t_r) \) be the term in \( \text{Ter}(\Sigma) \) which is defined by setting \( t(\lambda) = a \) and \( t(jp) = t_j(p) \) for each \( j \in [r] \) and each \( p \in \text{Pos}(t_j) \).

3 Embedding Permutation Groups in Graphs

For each finite set \( \Gamma \), we let \( \mathbb{S}(\Gamma) \) be the group of permutations of \( \Gamma \). If \( \Omega \subseteq \Gamma \) and \( \alpha \in \mathbb{S}(\Gamma) \), then we say that \( \alpha \) stabilizes \( \Omega \) setwise if \( \alpha(\Omega) = \Omega \). Alternatively, we say that \( \Omega \) is invariant under \( \alpha \). We let \( \alpha_\Omega \) be the permutation in \( \mathbb{S}(\Omega) \) which is defined by setting \( \alpha_\Omega(i) = \alpha(i) \) for each \( i \in \Omega \). In other words, \( \alpha_\Omega \) is the restriction of \( \alpha \) to \( \Omega \). If \( G \) is a subgroup of \( \mathbb{S}(\Gamma) \), then we let \( \text{stab}(G, \Omega) \) be the set of permutations in \( G \) that stabilize \( \Omega \) setwise. We say that a group \( G \) stabilizes \( \Omega \) if \( \text{stab}(G, \Omega) = G \). Alternatively, we say that \( \Omega \) is invariant under \( G \). We let \( G|_\Omega = \{ \alpha|_\Omega : \alpha \in G \} \) be the set of restrictions of permutations in \( G \) to \( \Omega \). In what follows, for each \( n \in \mathbb{N}_+ \) we write \( \mathbb{S}_n \) to denote \( \mathbb{S}([n]) \).

Graphs. Let \( m \in \mathbb{N}_+ \). An \( m \)-vertex graph is a pair \( X = ([m], E(X)) \), where \( E(X) \subseteq \binom{[m]}{2} \).

Isomorphisms and Automorphisms. If \( X \) and \( Y \) are two \( m \)-vertex graphs, then an isomorphism between \( X \) and \( Y \) is a permutation \( \alpha \in \mathbb{S}_m \) such that for each \( \{i, j\} \subseteq \binom{[m]}{2} \), \( \{i, j\} \in X \) if and only if \( \{\alpha(i), \alpha(j)\} \in Y \). An automorphism of \( X \) is an isomorphism between \( X \) and itself. We let \( \text{Iso}(X, Y) \) denote the set of all isomorphisms between \( X \) and \( Y \), and let \( \text{Aut}(X, X) = \text{Iso}(X, X) \) be the set of automorphisms of \( X \). If \( \Omega \subseteq [m] \) is invariant under \( \text{Aut}(X) \) then we define \( \text{Aut}(X, \Omega) = \text{Aut}(X)|_\Omega = \{ \alpha|_\Omega : \alpha \in \text{Aut}(X) \} \).

Definition 1. Let \( G \) be a subgroup of \( \mathbb{S}_n \) and \( X \) be a connected \( m \)-vertex graph where \( m \geq n \). We say that \( G \) is embeddable in \( X \) if \( \text{Aut}(X, [n]) = G \).
In other words, $G$ is embeddable in $X$ if the image of action of the automorphism group of $X$ on its first $n$ vertices is equal to $G$. We note that the requirement that the graph $X$ of Definition 1 is connected is crucial for our applications.

Let $\mathcal{X}$ be a class of connected graphs and $G$ be a subgroup of $S_n$. We say that $G$ is $\mathcal{X}$-embeddable if there exists some graph $X \in \mathcal{X}$ such that $G$ is embeddable in $X$. The $\mathcal{X}$-embedding complexity of $G$, denoted by $\text{gec}_\mathcal{X}(G)$ is the minimum $m$ such that $G$ is embeddable in a graph $X \in \mathcal{X}$ with at most $m$ vertices. If no such a graph $X \in \mathcal{X}$ exists, then we set $\text{gec}_\mathcal{X}(G) = \infty$.

4 Using Grammars to Represent Finite Permutation Groups

A context-free grammar is a 4-tuple $G = (\Sigma, B, R, B_1)$ where $\Sigma$ is a finite set of symbols, $B$ is a finite set of variables, $R \subseteq B \times (\Sigma \cup B)^*$ is a finite set of production rules, and $B_1 \in B$ is the initial variable of $G$. The notion of a string $w$ generated by $G$ can be defined with basis on the notions of $G$-parse-tree and yield of a $G$-parse-tree, which are inductively defined as follows.

1. For each $a \in \Sigma$ the term $t : \{\lambda\} \rightarrow \Sigma$ which sets $t(\lambda) = a$ is a $G$-parse-tree. Additionally, $\text{yield}(t) = a$.
2. If $\varepsilon$ is the empty symbol then the term $t : \{\lambda\} \rightarrow \Sigma$ which sets $t(\lambda) = \varepsilon$ is a $G$-parse-tree. Additionally, $\text{yield}(t) = \varepsilon$.
3. If $t_1, ..., t_r$ are $G$-parse-trees and $B \rightarrow t_1(\lambda)t_2(\lambda)...t_r(\lambda)$ is a production rule in $R$, then the term $t = B(t_1, ..., t_r)$ is a $G$-parse-tree. Additionally,
   $$\text{yield}(t) = \text{yield}(t_1) \cdot \text{yield}(t_2) \cdot ... \cdot \text{yield}(t_r).$$

In other words, the yield of $t$ is the concatenation of the yields of the subterms $t_1, ..., t_r$.

We say that a $G$-parse-tree $t$ is accepting if $t(\lambda) = B_1$. We say that a string $w \in \Sigma^*$ is generated by $G$ if there is an accepting $G$-parse-tree with yield($t$) = $w$. The language generated by $G$ is the set $L(G) = \{w \in \Sigma^* : w$ is generated by $G\}$ of strings generated by $G$. The size of $G$ is defined as
   $$|G| = \sum_{(B, u) \in R} (1 + |u|) \log(|\Sigma| + |B|),$$
where $|u|$ is the number of symbols/variables in $u$, $|\Sigma|$ is the number of elements in $\Sigma$ and $|B|$ is the number of elements in $B$. We denote by $G(\Sigma)$ the set of context-free grammars over the alphabet $\Sigma$.

A context-free grammar $G$ is said to be regular if each production rule is either of the form $(B, a)$ for some $B \in B$ and $a \in \Sigma$, or of the form $(B, aB')$ for some $B, B' \in B$ and some $a \in \Sigma$. We denote by $\mathcal{R}G(\Sigma)$ the set of regular context-free grammars over the alphabet $\Sigma$.

Complexity Measures. If $\alpha \in S_n$ and $w \in \Sigma^n$ then we let $\text{Perm}(w, \alpha) \overset{\text{def}}{=} w_{\alpha(1)}w_{\alpha(2)}...w_{\alpha(n)}$ be the string obtained by permuting the positions of $w$ according to $\alpha$. If $L \subseteq \Sigma^n$ then we let $\text{Perm}(L, \alpha) \overset{\text{def}}{=} \{\text{Perm}(w, \alpha) : w \in L\}$. In other words, $\text{Perm}(L, \alpha)$ is the language obtained by permuting the positions of each string $w \in L$ according to $\alpha$. The symmetric grammar complexity of a language $L \subseteq \Sigma^n$ is defined as the minimum size of a context-free grammar generating $\text{Perm}(L, \alpha)$ for some $\alpha \in S_n$. More precisely,
   $$\text{sgc}(L) = \min\{|G| : \alpha \in S, G \in \mathcal{G}(\Sigma), L(G) = \text{Perm}(L, \alpha)\}.$$
Then \( \beta \): \( \beta \cup B \rightarrow \beta \cup B \) follows from the fact that context-free languages are closed under \( \beta \).

Proof. Let \( \beta \) be a context-free grammar generating \( \text{Perm}(L, \alpha) \). Therefore, \( \beta \cup B \) is a left coset of \( H \) in \( G \). The following proposition, which will be used in the proofs of Lemma 10 and Theorem 5, follows from the fact that context-free languages are closed under homomorphisms.

Proposition 2. Let \( H \subseteq S_n \), and \( \alpha \) be a permutation in \( S_n \). Let \( \mathcal{G} \) be a context-free grammar such that \( L(\mathcal{G}) = \text{Perm}(\text{str}(H), \alpha) \). Then for each permutation \( \beta \in S_n \), there is a context-free grammar \( \mathcal{G}_\beta \) of size \( |\mathcal{G}_\beta| = |\mathcal{G}| \) generating \( \text{Perm}(\text{str}(\beta \circ H), \alpha) \).

Proof. Let \( \mathcal{G} = ([n], B, R, B_1) \) be a context free grammar generating \( \text{Perm}(\text{str}(H), \alpha) \). Let \( \beta : [n] \cup B \rightarrow [n] \cup B \) be the extension of \( \beta \) to the set \( [n] \cup B \) which sets \( \beta(a) = \beta(a) \) if \( a \in [n] \) and \( \beta(a) = a \) if \( a \in B \). For each string \( u = u_1 \ldots u_m \in ([n] \cup B)^* \) let \( \beta(u) = \beta(u_1) \ldots \beta(u_m) \). Finally, let \( \mathcal{G}' \) be the context-free grammar obtained from \( \mathcal{G} \) by replacing each production rule \( (B, u) \in R \) with the production rule \( (B, \beta(u)) \). Clearly, we have that \( |\mathcal{G}'| = |\mathcal{G}| \). Additionally, it is straightforward to verify that \( \mathcal{G}' \) generates a string \( a_1 a_2 \ldots a_n \in [n]^n \) if and only if \( \mathcal{G} \) generates the string \( \beta(a_1) \beta(a_2) \ldots \beta(a_n) \). Therefore, \( L(\mathcal{G}') = \text{Perm}(\text{str}(\beta \circ H), \alpha) \).

The following theorem, which will be crucial to the proof of our main result (Theorem 5), upper bounds the symmetric grammar complexity of the automorphism group of a graph in terms of the number of its vertices, its maximum degree, and its treewidth. If the latter two quantities are bounded, then this upper bound is polynomial in the number of its vertices.

Theorem 3. Let \( X \) be a connected graph with \( n \) vertices, treewidth \( k \) and maximum degree \( \Delta \). Then

\[
\text{sgc}(\text{Aut}(X)) \leq 2^{O(k \Delta \log \Delta)} \cdot n^{O(k)}.
\]

Additionally, one can construct in time \( 2^{O(k \Delta \log \Delta)} \cdot n^{O(k)} \) a permutation \( \alpha \in S_n \) and a context-free grammar \( \mathcal{G}(X) \) generating the language \( \text{Perm}(\text{str}(\text{Aut}(X)), \alpha) \).

Remark 4. If the graph \( X \) of Theorem 5 has pathwidth \( k \), then one may assume that \( \mathcal{G}(X) \) is a regular grammar. In other words, in this case, \( \text{reg-sgc}(\text{Aut}(X)) \leq 2^{O(k \Delta \log \Delta)} \cdot n^{O(k)} \).
Theorem can be simultaneously generalized in two ways. First, by allowing grammars to represent not only the automorphism group of a graph, but also groups that can be embedded in the graph. Second, not only the groups themselves but also left cosets of such groups can be represented in the same way. The result of these generalizations is stated in the next theorem.

**Theorem 5.** Let $G \subseteq S_n$, and suppose that $G$ is embeddable on a graph $X$ with $m$ vertices ($m \geq n$), maximum degree $\Delta$, and treewidth $k$. Then, for each $\beta \in S_n$,

$$\text{sgc}(\beta \circ G) \leq 2^{O(k\Delta \log \Delta)} \cdot m^{O(k)}.$$ 

Additionally, given $X$ and $\beta$, one can construct in time $2^{O(k\Delta \log \Delta)} \cdot m^{O(k)}$ a permutation $\alpha \in S_n$ (depending only on $X$) and a grammar $G_\beta$ generating the language $\text{Perm}(\text{str}(\beta \circ G), \alpha)$.

**Remark 6.** If the graph $X$ of Theorem has pathwidth $k$, then one may assume that $G_\beta(X)$ is a regular grammar. In other words, in this case, $\text{reg-sgc}(\beta \circ G) \leq 2^{O(k\Delta \log \Delta)} \cdot m^{O(k)}$.

### 4.1 Proof of Theorem 5

In this section, we will prove Theorem which establishes an upper bound for the symmetric grammar complexity of a permutation group $G$ in function of the size, treewidth and maximum degree of a graph embedding $G$. On the way to prove Theorem we will first prove Theorem The proofs of Remarks and follow by small adaptations of the proofs of Theorems and respectively.

**Subtree-Like Sets and Subterms.** Let $r \in \mathbb{N}_+$, $U \subseteq [r]^*$ be a tree-like set, $p, q \in U$ and $u$ be the longest common prefix of $p$ and $q$. Let $p = up'$ and $q = uq'$. The distance between $p$ and $q$ is defined as $|p'| + |q'|$. We call a set $M \subseteq U$ subtree-like if there exists a $p \in M$, such that $p$ is a prefix of every $q \in M$, and if the set $M' = \{u \mid pu \in M\}$ is prefix-closed. In particular, for each $p \in U$, the set $U|_p$ is subtree-like. One can obtain from $M'$ a tree-like set $M''$ by making $M'$ well-numbered in the obvious way. We call $M''$ the tree-like set induced by $M$. For a set $U' \subseteq U$, we call the smallest subtree-like set containing $U'$ the closest ancestral closure of $U'$. For any subtree-like set $P \subseteq \text{Pos}(t)$, we call $t|_P$ a subterm of $t$. If $P'$ is the induced tree-like set of $P$, then we call the corresponding term $t'$ with $\text{Pos}(t') = P'$ the t-term induced by $P$. For a position $p \in \text{Pos}(t)$, we denote by $t|_p$ the subterm of $t$ rooted at $p$, i.e. we let $t|_p \overset{\text{def}}{=} t|_{U|_p}$.

**Neighborhood of a Vertex, and Induced Subgraphs.** Let $X$ be a $n$-vertex graph. For a vertex $v \in [n]$, we let $N(v) \overset{\text{def}}{=} \{u \in [n] : \{v, u\} \in E(X)\}$ be the neighborhood of $v$. If $S \subseteq [n]$ then we let $N(S) = \bigcup_{v \in S} N(v)$ be the neighborhood of $S$. Finally, we let $\overline{N}(S) = N(S) \cup S$ be the closed neighborhood of $S$. The subgraph of $X$ induced by $S$ is defined as $X[S] = (S, E(X[S]))$ where $E(X[S]) = E(X) \cap \binom{S}{2}$.

**Tree decomposition as Terms.** If we regard the set $\binom{V(X)}{\leq k+1}$ as an alphabet, then each width-$k$ tree decomposition of a graph $X$ may be regarded as a term over $\binom{V(X)}{\leq k+1}$. More precisely, let $X$ be a $n$-vertex graph and $k \in \{0, 1, \ldots, n-1\}$. A width-$k$ tree decomposition (or simply tree decomposition, if $k$ is clear from the context) of $X$ is a term $t \in \text{Ter}(\binom{V(X)}{\leq k+1})$ satisfying the following axioms.

- **(T1)** $\bigcup_{p \in \text{Pos}(t)} t(p) = V(X)$

- **(T2)** For each vertex $v \in V(X)$ and each of its neighbors $u \in N(v)$, there is a position $p \in \text{Pos}(t)$ such that $\{v, u\} \subseteq t(p)$.

- **(T3)** For each vertex $v \in V(X)$, the set $\{p \in \text{Pos}(t) \mid v \in t(p)\}$ induces a subterm of $t$. 

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The treewidth of $X$, is defined as the smallest non-negative integer $k \in \mathbb{N}$ such that $X$ admits a width-$k$ tree decomposition.

**Annotated Tree Decompositions.** Let $X$ be an $n$-vertex graph, $S$ and $S'$ be subsets of $[n]$ such that $|S| = |S'|$, and $\nu : S \to S'$ be a bijection. We say that $\nu$ is a partial automorphism of $X$ if $\nu$ is an isomorphism from the subgraph $X[S]$ of $X$ induced by $S$ to the subgraph $X[S']$ of $X$ induced by $S'$. Now, we define the notion of annotated tree decomposition of a graph $X$. These are tree-decompositions whose bags are annotated with partial automorphisms.

**Definition 7 (Annotated Bags).** Let $X$ be an $n$-vertex graph and $k \in \{0, \ldots, n-1\}$. A $k$-annotated bag is a pair $(S, \nu)$, where $S \in (V(X) \leq k+1)$, and $\nu : \overline{N}(S) \to V(X)$ is a function satisfying the following two properties.

1. $\nu(\overline{N}(S)) = \overline{N}(\nu(S))$. In other words, the image of $\overline{N}(S)$ under $\nu$ is equal to the closed neighborhood of the image of $S$ under $\nu$.

2. $\nu$ is a partial automorphism of $X$.

We let $\mathcal{B}(X, k)$ be the set of all $k$-annotated bags of $X$. If $b$ is a $k$-annotated bag in $\mathcal{B}(X, k)$, then we denote the first coordinate of $b$ by $b.S$ and the second coordinate of $b$ by $b.\nu$. In other words, $b = (b.S, b.\nu)$. We let $\rho : \mathcal{B}(X, k) \to (V(X) \leq w+1)$ be the map that takes an annotated bag $b \in \mathcal{B}(X, k)$ and sends it to the bag $\rho(b) = b.S \in (V(X) \leq k+1)$. In other words, the map $\rho$ erases the second coordinate of the annotated bag $b$. We extend $\rho$ to terms in $\text{Ter}(\mathcal{B}(X, k))$ positionwise. More precisely, for each term $t \in \text{Ter}(\mathcal{B}(X, k))$, we let $\rho(t)$ be the term in $\text{Ter}(\mathcal{B}(X, k))$ where $\text{Pos}(\rho(t)) \stackrel{\text{def}}{=} \text{Pos}(t)$ and $\rho(t)(p) \stackrel{\text{def}}{=} \rho(t)(p)$ for each $p \in \text{Pos}(t)$. We say that a term $t \in \text{Ter}(\mathcal{B}(X, k))$ is an annotation of a term $t \in \text{Ter}(\mathcal{B}(X, k))$ if $\rho(t) = t$. Note that a term $t \in \text{Ter}(\mathcal{B}(X, k))$ may have many annotations.

In Definition 7, once a subset $S \subseteq V(X)$ is fixed, there are at most $O(n^{k+1})$ choices for the image of $S$ under the partial isomorphism $\nu$. Once such an image is fixed, for each vertex $x \in S$ there are at most $\Delta!$ ways of mapping the neighbors of $x$ to the neighbors of $\nu(x)$. Hence there are at most $(\Delta!)^{k+1}$ choices for obtaining a partial automorphism for a fixed image of $S$. Therefore, by noting that $\Delta! = 2^{O(\Delta \log \Delta)}$, we have the following observation.

**Observation 8.** Let $X$ be a graph of maximum degree $\Delta$ and let $k \in \{0, \ldots, n-1\}$. Then, $|\mathcal{B}(X, k)| \leq 2^{O(\Delta \log \Delta)}.n^{O(k)}$.

**Definition 9 (Annotated Tree Decomposition).** Let $\hat{t}$ be a term in $\text{Ter}(\mathcal{B}(X, k))$. We say that $\hat{t}$ is an annotated width-$k$ tree decomposition if the following conditions are satisfied.

1. $\rho(\hat{t})$ is a tree decomposition.

2. for each $p \in \text{Pos}(\hat{t})$ with children $p_1, \ldots, p_d$, and for each $j \in [d]$, the restriction of $\hat{t}(p)_{\nu}$ to $N[\hat{t}(p).S] \cap N[\hat{t}(p_j).S]$ is equal to the restriction of $\hat{t}(p_{\nu})$ to $N[\hat{t}(p).S] \cap N[\hat{t}(p_j).S]$.

Intuitively, the first condition states that if we take an annotated tree decomposition $\hat{t}$ and forget annotation then the result is a tree-decomposition of $X$. The second condition guarantees that the annotation is consistent along the whole tree decomposition, in the sense that for each vertex $x \in V(X)$, if the partial automorphism of one bag sends $x$ to vertex $x'$, then the partial automorphism of each bag sends $x$ to $x'$. Each annotated tree decomposition $\hat{t}$ gives rise to a map $\mu(\hat{t}) : V(X) \to V(X)$ which sets $\mu(\hat{t})|_{N[\hat{t}(p).S]} = \hat{t}(p)._{\nu}$ for each $p \in \text{Pos}(\hat{t})$. We call the map $\mu$ the annotation morphism of $\hat{t}$. The following lemma is the main technical tool of this section.
Lemma 10. Let $X$ be an $n$-vertex graph of treewidth $k$ and $\alpha \in \mathbb{S}_n$. Then, $\alpha$ is an automorphism of $X$ if and only if there exists an annotated tree decomposition $\mathbf{t}$ of $X$ such that $\alpha = \mu(\mathbf{t})$.

Proof. (Only if direction.) First, we show that if $\alpha$ is an automorphism of $X$ then there is an annotated tree decomposition $\mathbf{t}$ such that $\alpha = \mu(\mathbf{t})$. Let $\mathbf{t}$ be a width-$k$ tree decomposition of $X$ and suppose $\alpha$ is an automorphism. Then, we construct an annotated tree decomposition $\mathbf{t}$ with $\rho(\mathbf{t}) = \mathbf{t}$ by letting for each $p \in \text{Pos}(\mathbf{t})$, $\hat{\mathbf{t}}(p), S \overset{\text{def}}{=} \mathbf{t}(p)$ and $\hat{\mathbf{t}}(p), \nu \overset{\text{def}}{=} \alpha[\mathbf{N}[\mathbf{t}(p), S]]$. In other words, the annotation of each bag is simply the restriction of $\alpha$ to the closed neighborhood of $\hat{\mathbf{t}}(p), S$. Clearly, by construction we have that $\alpha = \mu(\mathbf{t})$. Therefore, it is enough to show that $\mathbf{t}$ is an annotated tree decomposition. Since $\alpha$ is an automorphism, one immediately verifies that for each $p \in \text{Pos}(\mathbf{t})$, $\nu_p \overset{\text{def}}{=} \hat{\mathbf{t}}(p), \nu$ is an isomorphism from $X[\mathbf{N}[S]]$ to $X[\mathbf{N}[\nu_p(S)]]$, i.e. condition (2) of Definition 7 holds. Condition (2) of Definition 9 is satisfied as well, since by construction, there exists a vertex $X$ such that $\alpha(\mathbf{t}) = \mathbf{t}(\mathbf{t})$, $\mathbf{t}$ is an annotated tree decomposition.

Proof. (If direction) Suppose that there is an annotated width-$k$ tree decomposition $\mathbf{t}$ of $X$ such that $\alpha = \mu(\mathbf{t})$. We show that $\alpha$ is an automorphism of $X$. First, we argue that $\alpha$ is well-defined. Since $\rho(\mathbf{t})$ is a tree decomposition, we have by (T1) that for each $x \in V(X)$, there is at least one position $p \in \text{Pos}(\mathbf{t})$ such that $x \in \mathbf{t}(p), S$. By property (T3) of tree decompositions and condition (2) of Definition 4 we can conclude that $\alpha(x)$ is assigned a unique value, and therefore that $\alpha$ is a well defined function. It remains to argue that $\alpha$ is surjective, injective, and indeed an automorphism.

Claim 11. Let $x, x', y' \in V(X)$. If $\alpha(x) = x'$ and $\{x', y'\} \in E(X)$, then there exists a $y \in V(X)$ such that $\{x, y\} \in E(X)$ and $\alpha(y) = y'$.

Proof. If $\alpha(x) = x'$, then there exists some position $p \in \text{Pos}(\mathbf{t})$ such that $\hat{\mathbf{t}}(p), \nu(x) = x'$. Now, Condition (2) of Definition 4 implies that $\hat{\mathbf{t}}(p), \nu|_{\mathbf{N}[x]}$ is a bijection from $\mathbf{N}[x]$ to $\mathbf{N}[x']$. Since, by assumption, $y' \in \mathbf{N}[x']$, there is a vertex $y \in \mathbf{N}[x]$ such that $\hat{\mathbf{t}}(p), \nu(y) = y'$, and hence $\alpha(y) = y'$.

From Claim 11, the following claim follows straightforwardly by induction on the length of paths.

Claim 12. Let $x, x', y' \in V(X)$. If $\alpha(x) = x'$ and there exists a path from $x'$ to $y'$, then there exists a vertex $y \in V(X)$ such that $\alpha(y) = y'$.

Proof. The proof is by induction on the length of paths. In the base case, the path has length 0. In this case, $x' = y'$ and the claim follows trivially by setting $y = x$. Now, let $r \geq 0$, and assume that the claim is true for every path of length at most $r$. Let $x'x'_1x'_2...x'_ry'$ be a path of length $r + 1$ from $x'$ to $y'$. Then, by the induction hypothesis, there exists $x_r \in V(X)$ such that $\alpha(x_r) = x'_r$. Now since $\{x'_r, y'\}$ is an edge in $E(X)$, by Claim 11 we have that there exists $y \in V(X)$ such that $\alpha(y) = y'$.

Since $X$ has at least one vertex, we have that there exist vertices $x_0, x'_0 \in V(X)$ such that $\alpha(x_0) = x'_0$. Now, since $X$ is connected, there is a path from $x'_0$ to any other vertex $y$ in $X$. Therefore, from Claim 12 and from the fact that $\alpha$ is well-defined, we can conclude that $\alpha$ is surjective.

Since $\alpha$ is a surjective map whose domain and codomain have the same size, we can infer that $\alpha$ is injective as well.

Now, suppose $\{x, y\} \in E(X)$. By property (T2) of tree decompositions, there is a position $p \in \text{Pos}(\mathbf{t})$ such that $\{x, y\} \subseteq \mathbf{t}(p), S$. Let $S \overset{\text{def}}{=} \hat{\mathbf{t}}(p), S$ and $\nu \overset{\text{def}}{=} \hat{\mathbf{t}}(p), \nu$. By condition (2) of Definition 4, $\nu$ is an isomorphism from $X[\mathbf{N}[S]]$ to $X[\mathbf{N}[\nu(S)]]$, so we know that $\{\nu(x), \nu(y)\} \in$.
Since $\nu(x) = \alpha(x)$ and $\nu(y) = \alpha(y)$, we have that $\{\alpha(x), \alpha(y)\} \in E(X)$. On the other hand, if $\{\alpha(x), \alpha(y)\} \in E(X)$, then by Claim [14] $\{x, y\} \in E(X)$.

This concludes the proof that $\alpha$ is an automorphism. \hfill \qed

**Definition 13.** A tree decomposition $t$ is called permutation yielding, if there is a bijection $\pi : \text{leaves}(\text{Pos}(t)) \rightarrow V(X)$ such that for each leaf $p \in \text{leaves}(\text{Pos}(t))$, $t(p) = \{\pi(p)\}$.

In other words, a tree decomposition $t$ is permutation yielding if each vertex occurs in precisely one leaf bag. The next lemma shows that any tree decomposition $t$ can be transformed in polynomial time into a permutation yielding tree decomposition of same width. We note that a statement analogous to Lemma [14] can also be obtained by observing that tree-decompositions can be converted in polynomial time into branch decompositions of roughly the same width [37]. We include a proof of Lemma [14] for completeness.

**Lemma 14.** Let $X$ be an $n$-vertex graph, $k \in \{0, \ldots, n-1\}$, and $t$ a width-$k$ tree decomposition of $X$. Then, one can construct from $t$ in polynomial time a permutation yielding width-$k$ tree decomposition.

**Proof.** For each position $p \in \text{Pos}(t)$, let $a(p)$ be the number of children of $p$ in $\text{Pos}(t)$. We can assume that for each $v \in V(X)$, there is at least one position $p \in \text{leaves}(\text{Pos}(t))$ such that $t(p) = \{v\}$; otherwise we could pick an arbitrary position $p \in \text{Pos}(t)$ with $v \in t(p)$ and add a leaf $p' = p \cdot (a(p) + 1)$ and set $t(p') = \{v\}$. Hence, we can assume that there is a set $S \subseteq \text{leaves}(\text{Pos}(t))$ containing precisely one leaf $p_v \in S$ with $t(p_v) = \{v\}$ for each $v \in V(X)$.

Let $S^+$ be the closest ancestral closure of $S$ in $\text{Pos}(t)$. We let $t'$ be the $t$-term induced by the closest ancestral closure of $S$. Note that $S = \text{leaves}(\text{Pos}(t|_{S^+}))$.

By construction, there is a bijection $\pi : \text{leaves}(\text{Pos}(t|_{S^+})) \rightarrow V(X)$ with $t(p) = \{\pi(p)\}$ for each $p \in \text{leaves}(\text{Pos}(t|_{S^+}))$. Let $p^* \in \text{Pos}(t) \setminus \text{Pos}(t|_{S^+})$ and consider $x, y \in t(p^*)$ (where possibly $x = y$). Let $q^* \in S^+$ denote the position that is closest to $p^*$ among all positions in $S^+$. Since there are leaves $p_x, p_y \in S$ such that $t(p_x) = \{x\}$ and $t(p_y) = \{y\}$ we can conclude by Property [T3] of tree decompositions that $\{x, y\} \subseteq t(q^*)$. We have argued that $t|_{S^+}$ satisfies Properties [T1] and [T2] of tree decompositions, and we observe that property [T3] remains intact on $t|_{S^+}$ as well.

Hence, the $t$-term induced by $S^+$ is a permutation yielding width-$k$ tree decomposition of $X$ and it is clear that the above construction can be implemented in polynomial time in the size of $t$.

Let $\hat{t}$ be an annotated tree decomposition with $r$ leaves, and let yield($\hat{t}$) = $(S_1, \nu_1) \ldots (S_r, \nu_r)$ be the yield of $\hat{t}$. In other words, yield($\hat{t}$) is the sequence of annotated bags obtained by reading the leaves of $\hat{t}$ from left to right. Then we define the annotation yield of $\hat{t}$ as the sequence yield$_\nu(\hat{t}) \overset{def}{=} \nu_1(S_1) \ldots \nu_m(S_r)$. Note that if $t$ is a permutation yielding tree decomposition of $X$, and $\hat{t}$ is an annotation of $t$, then $r = |V(X)|$, and yield$_\nu(\hat{t})$ is a string of singletons of the form $\{v_1\} \{v_2\} \ldots \{v_r\}$ where $v_i \in V(X)$ for each $i \in [r]$, and $v_i \neq v_j$ for $i \neq j$.

**Restatement of Theorem 3.** Let $X$ be a graph with $n$ vertices, treewidth $k$ and maximum degree $\Delta$. Then $\text{sgc}(\text{Aut}(X)) \leq 2^{O(k^{\Delta} \log \Delta)} \cdot n^{O(k)}$. Additionally, one can construct in time $2^{O(k^{\Delta} \log \Delta)} \cdot n^{O(k)}$ a permutation $\gamma$ and a context-free grammar $\mathcal{G}(X)$ generating $\text{Perm}(\text{str}(\text{Aut}(X)), \alpha)$.

**Proof.** Since the graph $X$ has treewidth $k$, one can construct in time $2^{O(k)} \cdot n^{O(1)}$ a width $O(k)$ tree decomposition $t$ of $X$. Additionally, from Lemma [14] one can assume that $t$ is yielding. Let yield($t$) = $\{v_1\} \{v_2\} \ldots \{v_n\}$. Then we let $\alpha_k$ be the permutation in $S_n$ with $\text{str}(\alpha_k) = v_1 v_2 \ldots v_n$. We set $\alpha = \alpha_k^{-1}$. Since $t$ can be constructed in time $2^{O(k)} \cdot n^{O(1)}$, so can the permutation $\alpha$.  

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We show that from \( t \) one can construct a context-free grammar \( \mathcal{G} \) accepting the language \( L(\mathcal{G}) = \text{Perm}(\text{str}(\text{Aut}(X)), \alpha) \). Intuitively, the parse trees accepted by the grammar \( \mathcal{G} \) correspond to annotations of \( t \), and by Lemma 10 these annotations correspond to automorphisms of \( X \). Formally, the grammar \( \mathcal{G} = (\Sigma, B, R, B_1) \) is defined as follows. We let \( \Sigma = V(X) = [n] \) and \( B = \text{Pos}(t) \times (\mathcal{B}(X,k) \cup \{B_1\}) \) where \( B_1 \) is the initial variable of \( \mathcal{G} \). Recall that \( \rho : \mathcal{B}(X,k) \rightarrow \{V(X)\}_{\leq k+1}^{n} \) is the map that erases the second coordinate from each annotated bag \( b \in \mathcal{B} \). The set \( R \) contains the following rules.

1. A rule \( B_1 \rightarrow (\lambda, b) \) for each annotated bag \( b \in \mathcal{B}(X,k) \) such that \( \rho(b) = t(\lambda) \). Intuitively, each such \( b \) is an annotated bag corresponding to the bag at the root of \( t \).

2. For each non-leaf position \( p \in \text{Pos}(t) \setminus \text{leaves}(\text{Pos}(t)) \), with children \( p_1, \ldots, p_d \), we have a rule
\[
(p, b) \rightarrow (p_1, b_1)(p_2, b_2) \cdots (p_d, b_d),
\]
for each sequence \( b, b_1, \ldots, b_d \) of annotated bags in \( \mathcal{B}(X,k) \) satisfying the following conditions:

(i) \( \rho(b) = t(p) \) and for \( j \in [d], \rho(b_j) = t(p_j) \).

(ii) for each \( j \in [d], b.\nu|_{S^*} = b_j.\nu|_{S^*} \) where \( S^* = b.S \cap b_j.S \).

3. A rule \( (p, b) \rightarrow j \) for each leaf position \( p \in \text{Pos}(t) \) with \( b.S = \{i\} \) and \( b.\nu(i) = j \).

These rules defined above ensure that if we take an accepting parse tree \( t \) of \( \mathcal{G} \) and remove its root (i.e the variable \( B_1 \)) and its leaves (which are labeled with numbers in \([n]\)) then we are left with an annotated version \( \hat{t} \) of the tree decomposition \( t \). By Lemma 10, \( \hat{t} \) is an annotation of \( t \) if and only if the map \( \mu(\hat{t}) : V(X) \rightarrow V(X) \) is an automorphism of \( X \). Therefore, since \( \text{str}(\mu(\hat{t})) = \text{yield}(t) \), we have that \( L(\mathcal{G}) = \text{Perm}(\text{str}(\text{Aut}(X)), \alpha) \).

Since we can assume that \( |\text{Pos}(t)| = O(kn) \) (see e.g. [17, Lemma 7.4]), and for each bag, there are at most \( 2^{O(k\Delta \log \Delta)} \cdot n^{O(k)} \) annotations, we have that \( |\mathcal{G}| = 2^{O(k\Delta \log \Delta)} \cdot n^{O(k)} \), as claimed.

**Restatement of Remark 4.** If the graph \( X \) of Theorem 2 has pathwidth \( k \), then one may assume that \( \mathcal{G}(X) \) is a regular grammar. In other words, in this case, \( \text{reg-sgc(Aut}(X)) \leq 2^{O(k\Delta \log \Delta)} \cdot n^{O(k)} \).

**Proof.** Any \( n \)-vertex graph of pathwidth \( k \) admits a path decomposition \( \mathcal{D} \) in which each vertex is introduced exactly once. Therefore, if we apply the construction that transforms tree decompositions in yielding tree decompositions stated in Lemma 14 then we actually obtain a path decomposition where each node has a child of arity 1 and a child of arity 0, except for the internal node farthest away from the root, which has two children of arity 0 (one of the children of arity 0 of each bag is a singleton containing the introduced vertex). Therefore, when applying the construction in Theorem 3 the resulting grammar is regular.

**Restatement of Theorem 5.** Let \( G \subseteq S_n \), and suppose that \( G \) is embeddable on a graph \( X \) with \( m \) vertices \( (m \geq n) \), maximum degree \( \Delta \), and treewidth \( k \). Then, for each \( \beta \in S_n \),
\[
\text{sgc}(\beta \circ G) \leq 2^{O(k\Delta \log \Delta)} \cdot m^{O(k)}.
\]
Additionally, given \( X \) and \( \beta \), one can construct in time \( 2^{O(k\Delta \log \Delta)} \cdot m^{O(k)} \) a permutation \( \alpha \in S_n \) (depending only on \( X \)) and a grammar \( \mathcal{G}_\beta \) generating the language \( \text{Perm}(\text{str}(\beta \circ G), \alpha) \).
Proof. This is a consequence of Theorem 4 together with the fact that context free grammars are closed under homomorphisms. More precisely, we first construct in time $2O(k \Delta \log \Delta) \cdot m^{O(k)}$ a permutation $\alpha' \in \mathbb{S}_m$, and a context-free grammar $G'$ such that $L(G) = \text{Perm}(\text{str}(\text{Aut}(X)), \alpha)$. Now let $h : [m] \setminus [n] \to \{\varepsilon\}$ be the map that sends each number in $[m] \setminus [n]$ to the empty symbol $\varepsilon$. Then using $G$, one can construct in time polynomial in $|G'|$ a context-free grammar $G''$ whose language $L(G'')$ is the homomorphic image of $L(G')$ under $h$. Additionally, one may assume that the grammar $G'$ has no production rule containing the empty-symbol $\varepsilon$. Let $\alpha = \alpha'|[n]$ be the permutation in $\mathbb{S}_n$ obtained by restricting $\alpha'$ to $[n]$. Note that $\alpha$ is well defined, since the fact that $G \subseteq \mathbb{S}_n$ is embeddable in $X$ implies that $\alpha'([n]) = [n]$, and therefore that $\alpha([n]) = [n]$. Then we have that the language accepted by $G''$ is $L(G'') = \text{Perm}(\text{str}(\text{Aut}(X)), \alpha)$.

Finally, let $\beta : [n] \to [n]$ be a permutation in $\mathbb{S}_n$. Then we can regard $\beta$ as a usual map from $[n]$ to $[n]$, and using again the fact that context-free languages are closed under homomorphism, we can construct in time $O(|G'')|$ a context-free grammar $G$ accepting the homomorphic image of $L(G'')$ under $\beta$. This homomorphic image is simply the language $\text{Perm}(\text{str}(\beta \circ G), \alpha)$.

Restatement of Remark 6. If the graph $X$ of Theorem 4 has pathwidth $k$, then one may assume that $G_\beta(X)$ is a regular grammar. In other words, in this case, $\text{reg-sgc}(\beta \circ \text{Aut}(X)) \leq 2^{O(k \Delta \log \Delta)} \cdot m^{O(k)}$.

Proof. The proof of this remark follows from Theorem 5 by using an argument analogous to the one used in the proof of Remark 4.

5 Polytopes for Permutation Groups

In linear-programming theory, the $n$-permutahedron is the polytope $P(\mathbb{S}_n)$ formed by the convex-hull of the set of permutations of the set $\{1, \ldots, n\}$. It can be shown that to define the permutahedron on the $n$-dimensional space, $2^{\Omega(n)}$ constraints are required. On the other hand, a celebrated result from Goemans states that the $n$-permutahedron has extended formulations with $O(n \log n)$ variables and constraints [24].

More generally, given a subgroup $G \subseteq \mathbb{S}_n$, one can define the $G$-hedron as the convex-hull of the permutations in $G$. The technique used in [24] to upper bound the extension complexity of polytope $P(\mathbb{S}_n)$, which is based on the existence of sorting networks of size $O(n \log n)$ [1], has been used to show that polytopes corresponding to certain families of groups have small extension complexity. This includes polytopes corresponding to the alternating group [39], and to finite reflection groups [28] [29] [27] [11]. Nevertheless, techniques to prove non-trivial upper bounds on the extension complexity of polytopes associated with general permutation groups based on structural properties of these groups are still lacking. We note that a trivial upper bound of $|G|$ can be obtained from the fact that the extension complexity of a polytope is upper bounded by its number of vertices. Nevertheless, $|G|$ may have up to $n! = 2^{\Omega(n \log n)}$ elements.

In this section, by combining our main theorem (Theorem 5) with a connection established in [34] between the grammar complexity of a given formal language $L \subseteq [n]^r$ (for $n, r \in \mathbb{N}_+$) and the extension complexity of the polytope $P(L)$ associated with $L$, we obtain a new approach for proving upper bounds on the extension complexity of a general permutation group $G \subseteq \mathbb{S}_n$ based on structural parameters of graphs embedding $G$ (Theorem 10). We note that Theorem 10 is more general in the sense that it also can be used to upper bound the extension complexity of polytopes associated with cosets of $G$.

Let $\mathcal{X}$ be a set of real variables. A real vector over $\mathcal{X}$ is a function $v : \mathcal{X} \to \mathbb{R}$. We let $\mathbb{R}^\mathcal{X}$ be the set of all real vectors over $\mathcal{X}$. Given a set $W = \{v_1, \ldots, v_r\}$ of real vectors, the convex-hull
of $W$ is the set
\[ \text{conv}(W) = \{ \sum_{i=1}^{k} \alpha_i v_i : \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1 \} \]
of all convex linear-combinations of vectors in $W$. A subset $P \subseteq \mathbb{R}^X$ is a polytope over $X$ if $P = \text{conv}(W)$ for some finite set $W$ of real vectors over $X$. For each such a polytope $P$, there is a finite set $\mathcal{E}$ of linear inequalities over $X$ such that $P$ is the set of vectors in $\mathbb{R}^X$ which satisfy each inequality in $\mathcal{E}$.

Let $X$ and $Y$ be sets of real variables with $X \cap Y = \emptyset$. We say that a $(X \cup Y)$-polytope $Q$ is an extended formulation of $P$ if there exists a linear projection $\rho : \mathbb{R}^{X \cup Y} \to \mathbb{R}^X$ such that $P = \rho(Q)$. The extension complexity of $P$, denoted by $xc(P)$, is defined as the least number of inequalities necessary to define an extended formulation of $P$.

For each $n \in \mathbb{N}_+$, we let $[n]^X$ be the set of all real vectors over $X$ whose coordinates are chosen from the set $[n]$. For $r \in \mathbb{N}_+$, let $w = w_1 \ldots w_r$ be a string in $[n]^r$, and let $X_r = \{x_1, \ldots, x_r\}$ be an ordered set of real variables. We let $\hat{w} : X_r \to [n]$ be the real vector over $X_r$ which sets $\hat{w}_i = w_i$ for each $i \in [r]$. Given a subset $L \subseteq [n]^r$, the $X_r$-polytope associated with $L$ is defined as $P(L) = \text{conv}(\{\hat{w} : w \in L\})$.

The following theorem, proved in [34], relates the grammar complexity of a subset $L \subseteq [n]$ with the extension complexity of the polytope $P(L)$.

**Theorem 15** ([34]). Let $\mathcal{G}$ be a context-free grammar such that $L(\mathcal{G}) \subseteq [n]^r$ for some $n, r \in \mathbb{N}_+$. Then the extension complexity of the polytope $P(L(\mathcal{G}))$ is upper bounded by $|\mathcal{G}|^{O(1)}$. Additionally, a system of inequalities defining $P(L(\mathcal{G}))$ can be constructed in time $|\mathcal{G}|^{O(1)}$.

If $G$ is a subgroup of $S_n$, and $\beta \in G$, then we let
\[ P(\beta \circ G) := P(\text{str}(\beta \circ G)) \]
be the polytope associated with the coset $\beta \circ G$. The following theorem, which is the main result of this section, follows by a direct combination of Theorem 5 with Theorems 13.

**Theorem 16.** Let $G \subseteq S_n$, and suppose that $G$ is embeddable on a graph $X$ with $m$ vertices ($m \geq n$), maximum degree $\Delta$, and treewidth $k$. Then, for each $\beta \in S_n$, the extension complexity of the polytope $P(\beta \circ G)$ is at most $2^{O(k\Delta \log \Delta)} \cdot m^{O(k)}$. Additionally, given $X$ and $\beta$, a system of inequalities defining $P(\beta \circ G)$ can be constructed in time $2^{O(k\Delta \log \Delta)} \cdot m^{O(k)}$.

### 6 Complexity Theoretic Tradeoffs

In 1969 Babai and Bouwer showed independently that any subgroup $G$ of $S_n$ can be embedded in a connected graph with $O(n + |G|)$ vertices [5, 13]. Note that $|G|$ can be as large as $n!$. Classifying which groups can, or cannot, be embedded on connected graphs with a much smaller number of vertices is an important problem in algebraic graph theory [5]. Indeed, constructing an explicit class of graphs with superpolynomial graph embedding complexity is still an open problem, although a conjecture of Babai states that the alternating group $A_n$ has graph embedding complexity $2^{O(n)}$ [7]. We note that Liebeck has shown that any graph whose automorphism group is isomorphic to the alternating group $A_n$ (as an abstract group) has an exponential number of vertices [31]. Nevertheless this result does not extend to the graph embedding setting.

In this section we use our main theorem to establish a trade-off between the index of a subgroup $G$ of $S_n$, and structural parameters of graphs embedding $G$. In particular, for several classes of graphs $X$, this trade-off can be used to prove lower bounds on the $X$-embedding...
complexity of subgroups of $\mathbb{S}_n$ of small index (i.e. index up to $2^c n$ for some small constant $c$). We start by stating the following immediate observation.

**Observation 17.** Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be context-free grammars. Then there is a context-free grammar $\mathfrak{G}_1 \cup \mathfrak{G}_2$ of size $O(|\mathfrak{G}_1| + |\mathfrak{G}_2|)$ such that $L(\mathfrak{G}_1 \cup \mathfrak{G}_2) = L(\mathfrak{G}_1) \cup L(\mathfrak{G}_2)$.

The next lemma states that the symmetric grammar complexity of a group $G$ is at most the index of a subgroup $H$ in $G$ times the symmetric grammar complexity of $H$. Recall that if $G$ is a group and $H$ is a subgroup of $G$, then the index of $H$ in $G$ is defined as $I_G(H) = \frac{|G|}{|H|}$.

**Lemma 18.** Let $H \subseteq G \subseteq S_n$. Then $sgc(G) \leq I_G(H) \cdot sgc(H)$.

**Proof.** Let $T$ be a left transversal of $H$ in $G$. In other words, $T$ contains exactly one element for each coset of $H$ in $G$. Then we have that $|T| = I_G(H)$ and that $G = \bigcup_{\beta \in T} \beta \circ H$. This implies that

$$str(G) = \bigcup_{\beta \in T} str(\beta \circ H).$$

(1)

From Equation (1), it is immediate that for each permutation $\alpha \in S_n$,

$$\text{Perm}(str(G), \alpha) = \bigcup_{\beta \in T} \text{Perm}(str(\beta \circ H), \alpha).$$

(2)

Now, let $sgc(H)$ be the symmetric context-free complexity of $H$. Then, for some permutation $\alpha \in S_n$, there exists a context-free grammar $\mathfrak{G}$ of size $sgc(H)$ such that $L(\mathfrak{G}) = \text{Perm}(str(H), \alpha)$. Therefore, by Proposition 2, for each $\beta \in S_n$, there is a context-free grammar $\mathfrak{G}_\beta$ of size $sgc(H)$ accepting the language $\text{Perm}(str(\beta \circ H), \alpha)$. By combining Equation (2) with Observation 17, we can infer that there is a context-free grammar $\mathfrak{G}'$ of size at most $\sum_{\beta \in T} |\mathfrak{G}_\beta| = I_G(H) \cdot sgc(H)$ accepting the language $\text{Perm}(str(G), \alpha)$.

It has been shown in [19] (Theorem 30) that the language $str(S_n)$ cannot be represented by context-free grammars of polynomial size. Since $str(S_n)$ is invariant under permutation of coordinates, i.e., $str(S_n) = \text{Perm}(str(S_n), \alpha)$ for any permutation $\alpha \in S_n$, we have that the symmetric context-free complexity of $str(S_n)$ is exponential.

**Theorem 19** (Theorem 30 of [19]). $sgc(S_n) \geq 2^{\Omega(n)}$.

Now, by combining Theorem 19 with Lemma 18 (for $G = S_n$), we have the following immediate corollary.

**Corollary 20.** Let $H$ be a subgroup of $S_n$. Then

$$sgc(H) \geq \frac{2^{\Omega(n)}}{I_{S_n}(H)}.$$  

By combining Theorem 5 with Corollary 20, we have a trade-off between the size of a group $H$, and the number of vertices, the treewidth and the maximum degree of a graph embedding $H$. Below, we write $\exp_2(x)$ to denote $2^x$.

**Theorem 21.** There exist positive real constants $c_1, c_2$, and $c_3$ such that for large enough $n$, and each subgroup $H$ of $S_n$, if $H$ is embeddable in a graph with $m$ vertices, maximum degree $\Delta$ and treewidth $k$, then

$$m \geq \exp_2 \left( \frac{c_1 n - c_2 k \Delta \log \Delta - c_3 \log I_{S_n}(H)}{k} \right).$$
Proof. By Corollary 20, there is a constant $c'_1$ such that for each sufficiently large $n$, each subgroup $H$ of $\mathbb{S}_n$,

$$\text{sgc}(H) \geq \frac{2^{c'_1 n}}{I_{\mathbb{S}_n}(H)}.$$ 

By Theorem 5 there exists constants $c'_2$ and $c'_3$ such that for each large enough $n$, if a subgroup $H$ of $\mathbb{S}_n$ can be embedded in a graph with $m \geq n$ vertices, maximum degree $\Delta$ and treewidth $k$ then

$$\text{sgc}(H) \leq 2^{c'_2 k \Delta \log \Delta} \cdot m^{c'_3 k}.$$ 

Therefore, we have that for sufficiently large $n$, if a subgroup of $\mathbb{S}_n$ can be embedded in a graph with $m$ vertices, maximum degree $\Delta$ and treewidth $k$, then

$$m \geq \left( \frac{2^{c'_1 n}}{I_{\mathbb{S}_n}(H) \cdot 2^{c'_2 k \Delta \log \Delta}} \right)^{1/c'_3 k}.$$ 

Therefore, by setting $c_1 = c'_1/c'_3$, $c_2 = c'_2/c'_3$ and $c_3 = 1/c'_3$, we have that

$$m \geq \exp_2 \left( \frac{c_1 n - c_2 k \Delta \log \Delta - c_3 \log I_{\mathbb{S}_n}(H)}{k} \right).$$

As a corollary of Theorem 21 we get the following lower bound stating that subgroups of $\mathbb{S}_n$ with small index (i.e. index at most $2^{c n}$ for some small constant $c$) cannot be embedded on graphs of treewidth $o(n/\log n)$, maximum degree $o(n/\log n)$ and a polynomial number of vertices.

**Corollary 22.** Let $\mathcal{X}$ be a class of connected graphs of treewidth $o(n/\log n)$ and maximum-degree $o(n/\log n)$. Then there is a function $f \in \omega(1)$, and a constant $c \in \mathbb{R}$, such that for each sufficiently large $n$, each subgroup $G$ of $\mathbb{S}_n$ of index $I_{\mathbb{S}_n}(G) \leq 2^{cn}$ has $\mathcal{X}$-embedding complexity at least $n^{f(n)}$.

For classes of graphs of treewidth $n^\epsilon$ (for $\epsilon < 1$), and maximum degree $o(n/\log n)$, Theorem 5 implies exponential lower bounds on the embedding complexity of groups of small index (i.e. index at most $2^{c n}$ for some small constant $c$).

**Corollary 23.** Let $\mathcal{X}$ be a class of connected graphs of treewidth $n^\epsilon$ (for $\epsilon < 1$) and maximum-degree $o(n/\log n)$. Then there exist constants $c, c' \in \mathbb{R}$, such that for each sufficiently large $n$, each subgroup $G$ of $\mathbb{S}_n$ of index $I_{\mathbb{S}_n}(G) \leq 2^{cn}$ has $\mathcal{X}$-embedding complexity at least $2^{c'n^{1-\epsilon}}$.

In particular, for some small $c, c' \in \mathbb{R}$, the graph embedding complexity of subgroups of $\mathbb{S}_n$ of index at most $2^{c n}$ is lower bounded by $2^{c'n \sqrt{n}}$ for any minor closed class of graphs of maximum degree $o(n \log n)$. Note that these classes of graphs have treewidth at most $\sqrt{n}$.

7 Conclusion and Open Problems

In this work, we have established new connections between three complexity measures for permutation groups: embedding complexity parameterized by treewidth and maximum-degree, symmetric grammar complexity and extension complexity. In particular, we have shown that groups that can be embedded in graphs of small treewidth and degree have small symmetric grammar complexity and small extension complexity. These results can also be used to translate
strong lower bounds on the symmetric grammar complexity or on the extension complexity of a group $G \subseteq S_n$ into lower bounds on the embedding complexity of $G$. In particular, using this approach, we have shown that subgroups $G \subseteq S_n$ of sufficiently small index have superpolynomial embedding complexity on classes of graphs of treewidth $o(n/\log n)$ and maximum degree $o(n/\log n)$.

Below, we state some interesting open problems related to our work.

**Problem 24.** Construct an explicit family of groups $\{G_n\}_{n \in \mathbb{N}^+}$ with superpolynomial graph embedding complexity, that is, such that $gec(G_n) = n^{\Omega(1)}$.

In particular, it is not known if the graph embedding complexity of the alternating group $A_n$ is superpolynomial. Note that the graph embedding complexity of the symmetric group $S_n$ is $n$, which is witnessed by $K_n$, the complete graph with vertex set $\{1, \ldots, n\}$.

**Problem 25.** Does the alternating group $A_n$ have superpolynomial graph embedding complexity?

The $n$-alternahedron polytope $P(A_n)$ is the polytope associated with the alternating group $A_n$. The technique used in [24] to prove an $O(n \log n)$ upper bound on the extension complexity of the $n$-permutahedron $P(S_n)$ was generalized in [39] to show that the extension complexity of the $n$-alternahedron is $O(n \log n)$. Therefore, if the answer to Problem 25 is positive, the alternating group $A_n$ is also a solution to the following problem.

**Problem 26.** Construct a family of groups $\{G_n\}_{n \in \mathbb{N}^+}$ of polynomial extension complexity and superpolynomial graph embedding complexity.

It is also worth noting that no superpolynomial lower bound for the extension complexity of permutation groups has been shown yet.

**Problem 27.** Construct an explicit family of groups $\{G_n\}_{n \in \mathbb{N}^+}$ with superpolynomial extension complexity.

We note that the upper bound of $O(n \log n)$ proved in [24] and in [39] on the extension complexity of $S_n$ and $A_n$ respectively are with respect the representation of permutations as strings of length $n$ over the alphabet $[n]$. In the realm of linear programming theory, another useful way of representing permutations of the set $1, \ldots, n$ is as 0/1-permutation matrices of dimension $n$. In this case, the polytope associated with a permutation group $G \subseteq S_n$ is the polytope $PM(G)$ formed by the convex hull of permutation matrices corresponding to the elements of $G$. In the case of the symmetric group $S_n$, the polytope $PM(S_n)$ is the well known Birkhoff polytope [12] that has extension complexity $\Theta(n^2)$. On the other hand, determining whether the polytope $PM(A_n)$ has polynomial extension complexity is an important open problem in linear-programming theory.

**Problem 28.** Does the polytope $PM(A_n)$ have polynomial extension complexity?

Acknowledgements. We thank Manuel Aprile, Laszlo Babai, Peter Cameron, Michael Fellows and Samuel Fiorini for valuable comments and suggestions. We thank Michel Goemans, Kanstantsin Pashkovich and Stefan Weltge for answering some of our questions by email.

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