ERGODIC THEOREMS IN SYMMETRIC SPACES OF \(\tau\)-MEASURABLE OPERATORS

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Abstract. In [11], a maximal ergodic theorem in noncommutative \(L_p\)-spaces was established and, among other things, corresponding individual ergodic theorems were obtained. In this article we derive maximal ergodic inequalities in noncommutative \(L_p\)-spaces and apply them to prove individual and Besicovitch weighted ergodic theorems in noncommutative \(L_p\)-spaces. Then we extend these results to noncommutative fully symmetric Banach spaces with Fatou property and non-trivial Boyd indices, in particular, to noncommutative Lorentz spaces \(L_{p,q}\). Norm convergence of ergodic averages in noncommutative fully symmetric Banach spaces is also studied.

1. Introduction and preliminaries

Let \(\mathcal{H}\) be a Hilbert space over \(\mathbb{C}\), \(B(\mathcal{H})\) the algebra of all bounded linear operators in \(\mathcal{H}\), \(\|\cdot\|_\infty\) the uniform norm in \(B(\mathcal{H})\), \(I\) the identity in \(B(\mathcal{H})\). If \(\mathcal{M} \subseteq B(\mathcal{H})\) is a von Neumann algebra, denote by \(P(\mathcal{M}) = \{e \in \mathcal{M} : e = e^2 = e^*\}\) the complete lattice of all projections in \(\mathcal{M}\). For every \(e \in P(\mathcal{M})\) we write \(e \perp = I - e\). If \(\{e_i\}_{i \in I} \subseteq P(\mathcal{M})\), the projection on the subspace \(\bigcap_{i \in I} e_i(\mathcal{H})\) is denoted by \(\bigwedge_{i \in I} e_i\).

A linear operator \(x : D_x \to \mathcal{H}\), where the domain \(D_x\) of \(x\) is a linear subspace of \(\mathcal{H}\), is said to be affiliated with the algebra \(\mathcal{M}\) if \(yx \subseteq xy\) for every \(y\) from the commutant of \(\mathcal{M}\).

Assume now that \(\mathcal{M}\) is a semifinite von Neumann algebra equipped with a faithful normal semifinite trace \(\tau\). A densely-defined closed linear operator \(x\) affiliated with \(\mathcal{M}\) is called \(\tau\)-measurable if for each \(\epsilon > 0\) there exists such \(e \in P(\mathcal{M})\) with \(\tau(e \perp) \leq \epsilon\) that \(e(\mathcal{H}) \subseteq D_x\). Let us denote by \(L_0(\mathcal{M}, \tau)\) the set of all \(\tau\)-measurable operators.

It is well-known [21] that if \(x, y \in L_0(\mathcal{M}, \tau)\), then the operators \(x + y\) and \(xy\) are densely-defined and preclosed. Moreover, the closures \(\overline{x + y}\) (the strong sum) and \(\overline{xy}\) (the strong product) and \(x^*\) are also \(\tau\)-measurable and, equipped with these operations, \(L_0(\mathcal{M}, \tau)\) is a unital \(*\)-algebra over \(\mathbb{C}\).

For every subset \(X \subseteq L_0(\mathcal{M}, \tau)\), the set of all self-adjoint operators in \(X\) is denoted by \(X^h\), whereas the set of all positive operators in \(X\) is denoted by \(X^+\). The partial order \(\leq\) in \(L_0^0(\mathcal{M}, \tau)\) is defined by the cone \(L_0^0(\mathcal{M}, \tau)\).

The topology defined in \(L_0(\mathcal{M}, \tau)\) by the family

\[ V(\epsilon, \delta) = \{x \in L_0(\mathcal{M}, \tau) : \|xe\|_\infty \leq \delta \text{ for some } e \in P(\mathcal{M}) \text{ with } \tau(e \perp) \leq \epsilon\} \]
It is known [25, Proposition 1] that such extensions $T$ admit unique positive continuous linear extensions $T : L_p \to L_p$, $1 \leq p < \infty$, and a unique positive ultraweakly continuous linear extension $T : \mathcal{M} \to \mathcal{M}$. In addition, $\|T(x)\| \leq \|x\|_p$ whenever $x \in L^*_p$, $1 \leq p < \infty$, and $\|T(x)\|_\infty \leq \|x\|_\infty$ if $x \in L^*_1 \cap \mathcal{M}^b$. In fact, $T$ is norm-continuous on all of $\mathcal{M}$:

**Proposition 1.1.** $\|T(x)\|_\infty \leq \|x\|_\infty$ for every $x \in \mathcal{M}^b$.

**Proof.** Assume first that $x = \sum_{k=1}^n \lambda_k e_k$, where $\lambda_k \in \mathbb{R}$, $0 \neq e_k \in \mathcal{P}(\mathcal{M})$, $1 \leq k \leq n$, and $e_i e_j = 0$ for all $1 \leq i, j \leq n$ with $i \neq j$. Since the trace $\tau$ is semifinite, for any given $k$ there exists a net $\{p^{(k)}_\alpha\}_{\alpha \in A} \subset \mathcal{P}(\mathcal{M})$, where $A$ is a base of neighborhoods of zero ultraweak topology ordered by inclusion, such that $p^{(k)}_\alpha \leq e_k$, $0 < \tau(p^{(k)}_\alpha) < \infty$ for all $k$ and $\alpha$, and $p^{(k)}_\alpha \to e_k$ ultraweakly for every $k$.

If we denote $x_\alpha = \sum_{k=1}^n \lambda_k p^{(k)}_\alpha$, then $\{x_\alpha\}_{\alpha \in A} \subset L^*_1 \cap \mathcal{M}^b$ and $x_\alpha \to x$ ultraweakly, thus $T(x_\alpha) \to T(x)$ ultraweakly. Since $\|T(x_\alpha)\|_\infty \leq \|x_\alpha\|_\infty = \|x\|_\infty$ and the unit ball of $\mathcal{M}$ is closed in ultraweak topology, we conclude that $\|T(x)\|_\infty \leq \|x\|_\infty$.

If $x \in \mathcal{M}^b$, due to spectral theorem, $x$ is the uniform limit of elements of the form $\sum_{k=1}^n \lambda_k e_k$, which implies that $\|T(x)\|_\infty \leq \|x\|_\infty$. \hfill \square
If \( x \in L_p, 1 \leq p \leq \infty \), is not necessarily self-adjoint, then \( x = \text{Re}(x) + i \cdot \text{Im}(x) \), where \( \text{Re}(x) = 2^{-1}(x + x^*) \in L_p^h \) and \( \text{Im}(x) = (2i)^{-1}(x - x^*) \in L_p^h \), and the inequality \( \|T(x)\|_p \leq 2\|x\|_p \) follows.

We will write \( T \in AC^+ = AC^+ (\mathcal{M}, \tau) \), where \( AC \) stands for absolute contraction, to indicate that \( T: L_1 + \mathcal{M} \to L_1 + \mathcal{M} \) is the unique positive linear extension of a positive linear map \( T: L_1 \cap \mathcal{M} \to L_1 \cap \mathcal{M} \) satisfying condition \( Y \). As we know, \( T \) is continuous on each \( L_p, 1 \leq p \leq \infty \). In the commutative case, similar transformations were considered in \([16]\) where they were called \( L_1 - L_\infty - \)contractions. Namely, it was assumed that a \( T: L_1 \to L_1 \) is a contraction that also contracts the \( L_\infty \)-norm in \( L_\infty \cap L_1 \).

The class of positive linear maps \( T: \mathcal{M} \to \mathcal{M} \) that was considered in \([11]\) is given by the condition
\[
(JX) \quad \|T(x)\|_\infty \leq \|x\|_\infty \quad \forall x \in \mathcal{M} \quad \text{and} \quad \tau(T(x)) \leq \tau(x) \quad \forall x \in L_1 \cap \mathcal{M}^+.
\]
It is clear that \((JX) \implies (Y)\). Besides, by Lemma 1.1 in \([11]\), a positive linear map \( T: \mathcal{M} \to \mathcal{M} \) that satisfies \((JX)\) uniquely extends to a positive linear contraction \( T \) in \( L_p, 1 \leq p < \infty \).

We shall write \( T \in DS^+ = DS^+ (\mathcal{M}, \tau) \) to indicate that \( T: L_1 + \mathcal{M} \to L_1 + \mathcal{M} \) is the unique positive linear extension of a positive linear map \( T: \mathcal{M} \to \mathcal{M} \) satisfying condition \((JX)\). Such \( T \) is often called positive Dunford-Schwartz transformation (see, for example, \([20]\)). Clearly, \( 2^{-1} T \in DS^+ \) whenever \( T \in AC^+ \).

Assume that \( T \in AC^+ \) and form its ergodic averages:
\[
(1) \quad M_n = M_n(T) = \frac{1}{n + 1} \sum_{k=0}^{n} T^k, \quad n = 1, 2, \ldots
\]

The following theorem provides a maximal ergodic inequality in \( L_1 \) for the ergodic averages given by \([1]\).

**Theorem 1.1.** \([25]\) If \( T \in AC^+ \), then for every \( x \in L_1^+ \) and \( \epsilon > 0 \), there is such \( e \in \mathcal{P}(\mathcal{M}) \) that
\[
\tau(e^+) \leq \frac{\|x\|_1}{\epsilon} \quad \text{and} \quad \sup_n \|eM_n(x)e\|_\infty \leq \epsilon.
\]

Here is a corollary of Theorem \([11] \) a noncommutative individual ergodic theorem of Yeadon:

**Theorem 1.2.** \([25]\) If \( T \in AC^+ \), then for every \( x \in L_1 \) the averages \( M_n(x) \) converge b.a.u. to some \( \hat{x} \in L_1 \).

The following noncommutative individual ergodic theorem was established in \([14]\) as an application of their maximal ergodic theorem \([11] \) Theorem 4.1].

**Theorem 1.3.** Let \( 1 < p < \infty \). If \( T \in DS^+ \), then
\begin{itemize}
  \item[(i)] for every \( x \in L_p \) the averages \( M_n(x) \) converge b.a.u. to some \( \hat{x} \in L_p \);
  \item[(ii)] if \( p \geq 2 \), these averages converge also a.u.
\end{itemize}

The next result was obtained in \([13]\) by utilizing the notion of uniform equicontinuity at zero of a family of additive maps into \( L_0(\mathcal{M}, \tau) \).

**Theorem 1.4.** Let \( 1 < p < \infty \). If \( T \in AC^+ \), then
\begin{itemize}
  \item[(i)] for every \( x \in L_p \) the averages \( M_n(x) \) converge b.a.u. to some \( \hat{x} \in L_p \);
  \item[(ii)] if \( p \geq 2 \), these averages converge also a.u.
In what follows, assuming that \( T \in \text{AC}^+(\mathcal{M}, \tau) \), we derive maximal ergodic inequalities for the averages \( (1) \) in \( L_p \) with \( 1 < p < \infty \) (see Theorem 2.1 below). As an application, a simplified proof of Theorem 1.4 is given.

Next, we use these maximal inequalities to prove Besicovitch weighted noncommutative ergodic theorem in \( L_p \), \( 1 < p < \infty \), (see Theorem 3.3 below). This theorem is an extension of the corresponding result of [3] for \( L_1 \).

Having available an individual ergodic theorem for noncommutative \( L^p \)-spaces with \( 1 \leq p < \infty \), allows us to establish its validity for noncommutative fully symmetric spaces with Fatou property and non-trivial Boyd indices. As a consequence, we obtain an individual ergodic theorem in noncommutative Lorentz spaces \( L_{p,q} \).

The last section of the article is devoted to a study of the mean ergodic ergodic theorem in noncommutative fully symmetric spaces in the case where \( T \in \text{DS}^+(\mathcal{M}, \tau) \).

2. Maximal ergodic inequalities in noncommutative \( L_p \)-spaces

Here is an extension of Theorem 1.1 to \( L_p \), \( 1 < p < \infty \):

**Theorem 2.1.** Let \( 1 \leq p < \infty \), and let \( T \in \text{AC}^+ \). Then for every \( x \in L_p \) and \( \epsilon > 0 \) there is \( e \in \mathcal{P}(\mathcal{M}) \) such that

\[
\tau(e^+) \leq 4 \left( \frac{\|x\|_p}{\epsilon} \right)^p \quad \text{and} \quad \sup_n \|eM_n(x)e\|_{\infty} \leq 8\epsilon.
\]

**Proof.** Fix \( \epsilon > 0 \). Assume that \( x \in L^+_p \), and let \( x = \int_0^\infty \lambda de_\lambda \) be its spectral decomposition. Since \( \lambda \geq \epsilon \) implies \( \lambda \leq \epsilon^{1-p} \lambda^p \), we have

\[
\int_{\epsilon}^\infty \lambda de_\lambda \leq \epsilon^{1-p} \int_{\epsilon}^\infty \lambda^p de_\lambda \leq \epsilon^{1-p} x^p.
\]

Then we can write

\[
x = \int_{0}^\epsilon \lambda de_\lambda + \int_{\epsilon}^\infty \lambda de_\lambda \leq x_\epsilon + \epsilon^{1-p} x^p,
\]

where \( x_\epsilon = \int_0^\epsilon \lambda de_\lambda \).

As \( x_\epsilon \in L_1 \), Theorem 1.1 entails that there exists \( e \in \mathcal{P}(\mathcal{M}) \) satisfying

\[
\tau(e^+) \leq \frac{\|x^p\|_1}{\epsilon^p} = \left( \frac{\|x\|_p}{\epsilon} \right)^p \quad \text{and} \quad \sup_n \|eM_n(x^p)e\|_{\infty} \leq \epsilon^p.
\]

It follows from (3) that

\[
0 \leq M_n(x) \leq M_n(x_\epsilon) + \epsilon^{1-p} M_n(x^p) \quad \text{and}
\]

\[
0 \leq eM_n(x)e \leq eM_n(x_\epsilon)e + \epsilon^{1-p} eM_n(x^p)e
\]

for every \( n \).

Since \( x_\epsilon \in \mathcal{M}^+ \), by Proposition 1.1, the inequality

\[
\|T(x_\epsilon)\|_{\infty} \leq \|x_\epsilon\|_{\infty} \leq \epsilon
\]

holds, and we conclude that

\[
\sup_n \|eM_n(x)e\|_{\infty} \leq \epsilon + \epsilon = 2\epsilon.
\]
If \( x \in L_p \), then \( x = (x_1 - x_2) + i(x_3 - x_4) \), where \( x_j \in L_p^+ \) and \( \|x_j\|_p \leq \|x\|_p \) for every \( j = 1, \ldots, 4 \). As we have shown, there exist \( e_j \in P(M) \) such that
\[
\tau(e_j) \leq \left( \frac{\|x_j\|_p}{\epsilon} \right)^p \quad \text{and} \quad \sup_n \|e_j M_n(x_j)e_j\|_\infty \leq 2\epsilon,
\]
j = 1, \ldots, 4. Taking into consideration Theorem 1.1 for \( p = 1 \), we see that the projector \( e = \sum_{j=1}^4 e_j \) satisfies (2) for every \( 1 \leq p < \infty \).

Remark 2.1. In the terminology of [11, Section 3], Theorem 2.1 asserts that the sequence \( \{M_n\} \) is of weak type \((p, p)\).

To refine Theorem 2.1 when \( p \geq 2 \) we turn to the following fundamental result of Kadison [13].

Theorem 2.2 (Kadison’s inequality). Let \( S : \mathcal{M} \to \mathcal{M} \) be a positive linear map such that \( S(\mathbb{I}) \leq \mathbb{I} \). Then \( S(x)^2 \leq S(x^2) \) for every \( x \in \mathcal{M} \).

We will need the following lemma (for a proof, see the proof of [3, Theorem 2.7] or [18] Theorem 3.1).

Lemma 2.1. Let \( \{a_{mn}\}_{m,n=1}^\infty \subset L_0(\mathcal{M}, \tau) \) be such that for any \( n \) the sequence \( \{a_{mn}\}_{m=1}^\infty \) converges in measure to some \( a_n \in L_0(\mathcal{M}, \tau) \). Then there exists \( \{a_{mkn}\}_{k,n=1}^\infty \) such that for any \( n \) we have \( a_{mkn} \to a_n \) a.u. as \( k \to \infty \).

Proposition 2.1 (cf. [11, proof of Remark 6.5]). Assume that \( 2 \leq p < \infty \) and \( T \in AC^+ \). Then for every \( x \in L_p^+ \) and \( \epsilon > 0 \), there exists \( e \in P(M) \) such that \( \tau(e^{-1}) \leq \epsilon \) and
\[
\|eM_n(x)^2 e\|_\infty \leq \|eM_n(x^2)e\|_\infty,
\]
n = 1, 2, \ldots

Proof. Let \( x = \int_{-\infty}^\infty \lambda d\nu_\lambda \) be the spectral decomposition of \( x \in L_p^+ \), and let \( x_m = \int_{-\infty}^m \lambda d\nu_\lambda \). Then, since \( x \in L_p^+ \), we clearly have \( \|x - x_m\|_p \to 0 \). Besides \( \|x^2 - x_m^2\|_p/2 \to 0 \), so \( \|M_n(x^2) - M_n(x_m^2)\|_p/2 \to 0 \) for every \( n \), which implies that
\[
M_n(x_m^2) \to M_n(x^2) \quad \text{in measure, } n = 1, 2, \ldots
\]
Also \( \|M_n(x) - M_n(x_m)\|_p \to 0 \) for every \( n \), hence \( M_n(x_m) \to M_n(x) \) in measure and
\[
M_n(x_m^2) \to M_n(x^2) \quad \text{in measure, } n = 1, 2, \ldots
\]
In view of Lemma 2.1 it is possible to find a subsequence \( \{x_{mk}\} \subset \{x_m\} \) such that
\[
M_n(x_{mk}^2) \to M_n(x^2) \quad \text{and} \quad M_n(x_{mk})^2 \to M_n(x^2) \quad \text{a.u., } n = 1, 2, \ldots
\]
Then one can construct such \( e \in P(M) \) that \( \tau(e^{-1}) \leq \epsilon \) and
\[
\|eM_n(x_{mk}^2)e\|_\infty \to \|eM_n(x^2)e\|_\infty \quad \text{and} \quad \|eM_n(x_{mk})^2e\|_\infty \to \|eM_n(x^2)e\|_\infty
\]
for every \( n \).

Since, by Kadison’s inequality,
\[
\|eM_n(x_{mk}^2)e\|_\infty \leq \|eM_n(x_{mk})e\|_\infty,
\]
k, \( n = 1, 2, \ldots \), the result follows.

□
Theorem 2.3. Let $2 \leq p < \infty$ and $T \in AC^+$. Then for every $x \in L_p$ and $\epsilon > 0$ there is such $e \in \mathcal{P}(\mathcal{M})$ that

$$\tau(e^\perp) \leq 4 \left( \frac{\|x\|_p}{\epsilon} \right)^p \quad \text{and} \quad \sup_n \|M_n(x)e\|_{\infty} \leq 2\sqrt{2}\epsilon.$$  

Proof. Assume first that $x \in L^h_p$. Since $x^2 \in L^+_p/2$, by the proof of Theorem 2.1 there exists $e_1 \in \mathcal{P}(\mathcal{M})$ such that

$$\tau(e_1^\perp) \leq \left( \frac{\|x^2\|_{p/2}}{\epsilon^2} \right)^{p/2} = \left( \frac{\|x\|_p}{\epsilon} \right)^p \quad \text{and} \quad \sup_n \|e_1 M_n(x^2)e_1\|_{\infty} \leq 2\epsilon^2.$$  

By Proposition 2.1 there is $e_2 \in \mathcal{P}(\mathcal{M})$ such that

$$\tau(e_2^\perp) \leq \left( \frac{\|x\|_p}{\epsilon} \right)^p \quad \text{and} \quad \sup_n \|e_2 M_n(x)^2 e_2\|_{\infty} \leq \sup_n \|e_2 M_n(x^2)e_2\|_{\infty}.$$  

Then, letting $e = e_1 \wedge e_2$, we obtain $\tau(e^\perp) \leq 2 \left( \frac{\|x\|_p}{\epsilon} \right)^p$ and

$$\sup_n \|M_n(x)e\|_{\infty} = \left( \sup_n \|M_n(x)e\|_{2,\infty}^2 \right)^{1/2} = \left( \sup_n \|e M_n(x^2)e\|_{\infty}^{\frac{1}{2}} \right)^{1/2} \leq \frac{\sqrt{2}}{\epsilon}.$$  

If $x \in L_p$, then $x = x_1 + ix_2$, where $x_j \in L_p^h$ and $\|x_j\|_p \leq \|x\|_p$, $j = 1, 2$, and we arrive at (6). \hfill \Box  

Let us now derive Theorem 1.4 using maximal ergodic inequalities given in Theorems 2.1 and 2.3.

Lemma 2.2 (see [2], Lemma 1.6). Let $X$ be a linear space, and let $S_n : X \to L$ be a sequence of additive maps. Assume that $x \in X$ is such that for every $\epsilon > 0$ there exists a sequence $\{x_k\} \subset X$ and a projection $e \in \mathcal{P}(\mathcal{M})$ satisfying the following conditions:

(i) the sequence $\{S_n(x + x_k)\}$ converges a.u. (b.a.u.) as $n \to \infty$ for each $k$;
(ii) $\tau(e^\perp) \leq \epsilon$;
(iii) $\sup_n \|S_n(x_k)e\|_{\infty} \to 0$ (resp., $\sup_n \|e M_n(x_k)e\|_{\infty} \to 0$) as $k \to \infty$.

Then the sequence $\{S_n(x)\}$ also converges a.u. (resp., b.a.u.)

Corollary 2.1. Let $1 \leq p < \infty$ (2 $\leq p < \infty$) and $T \in AC^+$. Then the set

$$\{x \in L_p : \{M_n(x)\} \text{ converges b.a.u.}\}$$

(resp., $\{x \in L_p : \{M_n(x)\} \text{ converges a.u.}\}$)

is closed in $L_p$.  

Proof. Denote $C = \{x \in L_p : \{a_n(x)\} \text{ converges b.a.u.}\}$. Fix $\epsilon > 0$. Theorem 2.1 implies that for every given $k \in \mathbb{N}$ there is such $\gamma_k > 0$ that for every $x \in L_p$ with $\|x\|_p < \gamma_k$ it is possible to find $e_{k,x} \in \mathcal{P}(\mathcal{M})$ for which

$$\tau(e_{k,x}^\perp) \leq \frac{\epsilon}{2^k} \quad \text{and} \quad \sup_n \|e_{k,x} a_n(x)e_{k,x}\|_{\infty} \leq \frac{1}{k}.$$  


Pick $x \in C$, the closure of $C$ in $L_p$. Given $k$, let $y_k \in C$ be such that 
\[\|y_k - x\|_p < \gamma_k.\] Denoting $y_k - x = x_k$, choose a sequence $\{e_k\} \subset \mathcal{P}(\mathcal{M})$ to be such that 
\[\tau(e_k) \leq \frac{c}{2^k} \quad \text{and} \quad \sup_n \|e_k a_n(x_k) e_k\|_\infty \leq \frac{1}{k}, \quad k = 1, 2, \ldots.\] Then we have $x + x_k = y_k \in C$ for every $k$. Also, letting $e = \bigwedge e_k$, we have 
\[\tau(e) \leq \epsilon \quad \text{and} \quad \sup_{k \geq 1} \|e M_n(x_k) e\|_\infty \leq \frac{1}{k}.\] Consequently, Lemma 2.2 yields $x \in C$.

Analogously, applying Theorem 2.3 instead of Theorem 2.1, we obtain the remaining part of the statement. \[\square\]

Proof. (ii) Since the map $T$ generates a contraction in the real Hilbert space $(L^2, \langle \cdot, \cdot \rangle)$ [25, Proposition 1], where $(x, y)_\tau = \tau(xy)$, $x, y \in L^2$, it is easy to verify that the set 
\[\mathcal{H}_0 = \{x \in L^2 : T(x) = x\} + \{x - T(x) : x \in L^2\}\] is dense in $(L^2, \| \cdot \|_2)$ (see, for example [10, Ch.VIII, §5]). Therefore, because the set $L^2 \cap \mathcal{M}$ is dense in $L^p$ and $T$ contracts $L^p$, we conclude that the set 
\[\mathcal{H}_1 = \{x \in L^2 : T(x) = x\} + \{x - T(x) : x \in L^2 \cap \mathcal{M}\}\] is also dense in $(L^2, \| \cdot \|_2)$. Besides, if $y = x - T(x)$, $x \in L^2 \cap \mathcal{M}$, then the sequence $M_n(y) = (n + 1)^{-1}(x - T^{n+1}(x))$ converges to zero with respect to the norm $\| \cdot \|_\infty$, hence a.u. Therefore $\mathcal{H}_1 + i\mathcal{H}_1$ is a dense subset on which the averages $M_n$ converge a.u. This, by Corollary 2.1, implies that $\{M_n(x)\}$ converges a.u. for all $x \in L^2$. Further, since the set $L^p \cap L^2$ is dense in $L^p$, Corollary 2.1 implies that the sequence $\{M_n(x)\}$ converges a.u. for each $x \in L_p$ (to some $\hat{x} \in L_0(\mathcal{M}, \tau)$). Then $\{M_n(x)\}$ converges to $\hat{x}$ in measure. Since $M_n(x) \in L_p$ and $\|M_n(x)\|_p \leq 2$, $n = 1, 2, \ldots$, by Theorem 1.2 in [3], $\hat{x} \in L_p$.

(i) By part (ii), we have b.a.u. convergence of the sequence $\{M_n(x)\}$ for all $x \in L^2$. But $L^p \cap L^2$ is dense in $L^p$, and Corollary 2.1 entails b.a.u. convergence of the averages $M_n(x)$ for all $x \in L^p$. Remembering that b.a.u. convergence implies convergence in measure (see Section 1), we conclude, as in part (ii), that the b.a.u. limit of the sequence $\{M_n(x)\}$ belongs to $L_p$. \[\square\]

3. Besicovitch Weighted Ergodic Theorem in Noncommutative $L_p$-Spaces

Everywhere in this section $T \in AC^+$. Assume that a sequence of complex numbers $\{\beta_k\}_{k=0}^\infty$ is such that $|\beta_k| \leq C$ for every $k$. Let us denote

\[M_{\beta,n} = \frac{1}{n + 1} \sum_{k=0}^n \beta_k T^k.\]

**Theorem 3.1.** If $1 \leq p < \infty$, then for every $x \in L_p$ and $\epsilon > 0$ there is $e \in \mathcal{P}(\mathcal{M})$ such that

\[\tau(e) \leq \epsilon \quad \text{and} \quad \sup_n \|e M_{\beta,n}(x) e\|_\infty \leq 192C \epsilon.\]
Proof. We have \( x = (x_1 - x_2) + i(x_3 - x_4) \), where \( x_j \in L^p \) and \( \|x_j\|_p \leq \|x\|_p \) for each \( j = 1, 2, 3, 4 \).

For a given \( j \), by Theorem 2.1, there is \( e_j \in \mathcal{P}(\mathcal{M}) \) such that
\[
\tau(e_j^+) \leq 4 \left( \frac{\|x_j\|_p}{\epsilon} \right)^p \quad \text{and} \quad \sup_n \|e_j M_n(x_j)e_j\|_{\infty} \leq 8\epsilon.
\]
Since \( 0 \leq Re\beta_k + C \leq 2C \), \( 0 \leq Im\beta_k + C \leq 2C \), and \( T^k(x_j) \geq 0 \) for every \( k \) and \( j \), it follows from the decomposition
\[
M_{\beta,n} = \frac{1}{n+1} \sum_{k=0}^n (Re\beta_k + C)T^k + \frac{i}{n+1} \sum_{k=0}^n (Im\beta_k + C)T^k - C(1+i)M_n
\]
that
\[
\sup_n \|e_j M_{\beta,n}(x_j)e_j\|_{\infty} \leq 6C \sup_n \|e_j M_n(x_j)e_j\|_{\infty} \leq 48C\epsilon, \quad j = 1, 2, 3, 4.
\]
Now, letting \( e = \frac{1}{4} \sum_{j=1}^4 e_j \), we obtain (8). \( \square \)

**Theorem 3.2.** If \( 2 \leq p < \infty \), then for every \( x \in L_p \) and \( \epsilon > 0 \) there is \( e \in \mathcal{P}(\mathcal{M}) \) such that
\[
\tau(e^+) \leq 8 \left( \frac{\|x\|_p}{\epsilon} \right)^p \quad \text{and} \quad \sup_n \|M_{\beta,n}(x)e\|_{\infty} \leq 4\sqrt{2C(\sqrt{2} + \sqrt{C})}\epsilon.
\]

**Proof.** Referring to decomposition (7), let us denote
\[
M_{\beta,n}^{(R)} = \frac{1}{n+1} \sum_{k=0}^n (Re\beta_k + C)T^k, \quad M_{\beta,n}^{(I)} = \frac{1}{n+1} \sum_{k=0}^n (Im\beta_k + C)T^k.
\]
We have \( x = x_1 + ix_2 \), where \( x_j \in L^p \) and \( \|x_j\|_p \leq \|x\|_p \), \( j = 1, 2 \). Since \( x_1^2 \in L^p \), it follows from the proof of Theorem 2.1 that there is \( f_1 \in \mathcal{P}(\mathcal{M}) \) such that
\[
\tau(f_1^+) \leq \left( \frac{\|x_1^2\|_{p/2}}{\epsilon^2} \right)^{p/2} = \left( \frac{\|x_1\|_p}{\epsilon} \right)^p \quad \text{and} \quad \sup_n \|f_1 M_n(x_1^2)f_1\|_{\infty} \leq 2\epsilon^2.
\]
Therefore we have
\[
\sup_n \|f_1 M_{\beta,n}^{(R)}(x_1^2)f_1\|_{\infty} \leq 4C\epsilon^2 \quad \text{and} \quad \sup_n \|f_1 M_{\beta,n}^{(I)}(x_1^2)f_1\|_{\infty} \leq 4C\epsilon^2.
\]
Since \( M_{\beta,n}^{(R)} : \mathcal{M} \to \mathcal{M} \) and \( M_{\beta,n}^{(I)} : \mathcal{M} \to \mathcal{M} \) are positive linear maps that are continuous in \( (L_p, \|\cdot\|_p) \) and such that \( (2C)^{-1}M_{\beta,n}^{(R)}(1) \leq I \) and \( (2C)^{-1}M_{\beta,n}^{(I)}(1) \leq I \) for each \( n \), we can find, as in Proposition 2.1, such \( g_{11}, g_{12} \in \mathcal{P}(\mathcal{M}) \) with \( \tau(g_{1j}^+) \leq (\|x_1\|_p/\epsilon)^p, \; j = 1, 2, \) that
\[
\sup_n \|g_{11} M_{\beta,n}^{(R)}(x_1^2)g_{11}\|_{\infty} \leq \sup_n \|g_{11} M_{\beta,n}^{(R)}(x_1^2)g_{11}\|_{\infty}
\]
and
\[
\sup_n \|g_{12} M_{\beta,n}^{(I)}(x_1^2)g_{12}\|_{\infty} \leq \sup_n \|g_{12} M_{\beta,n}^{(I)}(x_1^2)g_{12}\|_{\infty}.
\]
Furthermore, Proposition 2.1 implies that there exists \( h_1 \in \mathcal{P}(\mathcal{M}) \) with \( \tau(h_1^+) \leq (\|x_1\|_p/\epsilon)^p \) such that
\[
\sup_n \|h_1 M_n(x_1^2)h_1\|_{\infty} \leq \sup_n \|h_1 M_n(x_1^2)h_1\|_{\infty}.
\]
Setting $e_1 = f_1 \wedge g_{11} \wedge g_{12} \wedge h_1$, we see that
\[ \tau(e_1^+) \leq 4(||x_1||_p/\epsilon)^p, \quad \sup_n \|M_{n}(x_1)e_1\|_\infty \leq \sqrt{2}\epsilon, \]
\[ \sup_n \|M_{\beta,n}(x_1)e_1\|_\infty \leq 2\sqrt{C}\epsilon, \quad \text{and} \quad \sup_n \|M_{\beta,n}^{(l)}(x_1)e_1\|_\infty \leq 2\sqrt{C}\epsilon. \]

Now it follows from the decomposition (9) that
\[ \sup_n \|M_{\beta,n}(x_1)e_1\|_\infty \leq 2\sqrt{2C} (\sqrt{2} + \sqrt{C})\epsilon. \]

Similarly, we can find $e_2 \in \mathcal{P}(\mathcal{M})$ with $\tau(e_2^+) \leq 4(||x_2||_p/\epsilon)^p$ such that
\[ \sup_n \|M_{\beta,n}(x_2)e_2\|_\infty \leq 2\sqrt{2C} (\sqrt{2} + \sqrt{C})\epsilon. \]

Finally, $e = e_1 \wedge e_2 \in \mathcal{P}(\mathcal{M})$ satisfies (10). \( \square \)

Using maximal ergodic inequalities given in Theorems 3.1 and 3.2 as in the proof of Corollary 2.1, we obtain the following.

**Corollary 3.1.** Let $1 \leq p < \infty$ ($2 \leq p < \infty$). Then the set
\[ \{ x \in L_p : \{M_{\beta,n}(x)\} \text{ converges b.a.u.} \} \]
(resp., \( \{ x \in L_p : \{M_{\beta,n}(x)\} \text{ converges a.u.} \} \))
is closed in $L_p$.

Let $\mathbb{C}_1 = \{ z \in \mathbb{C} : |z| = 1 \}$ be the unit circle in $\mathbb{C}$. A function $P : \mathbb{Z} \to \mathbb{C}$ is said to be a *trigonometric polynomial* if $P(k) = \sum_{j=1}^{s} z_j \lambda_j^k$, $k \in \mathbb{Z}$, for some $s \in \mathbb{N}$, \( \{z_j\}_j \subset \mathbb{C} \), and \( \{\lambda_j\}_j \subset \mathbb{C}_1 \). A sequence \( \{\beta_k\}_{k=0}^{\infty} \subset \mathbb{C} \) is called a *bounded Besicovitch sequence* if

1. \( |\beta_k| \leq C < \infty \) for all $k$;
2. for every $\epsilon > 0$ there exists a trigonometric polynomial $P$ such that
\[ \limsup_n \frac{1}{n+1} \sum_{k=0}^{n} |\beta_k - P(k)| < \epsilon. \]

Assume now that $\mathcal{M}$ has a separable predual. The reason for this assumption is that our argument essentially relies on Theorem 1.22.13 in [20].

Since $L_1 \cap \mathcal{M} \subset L_2$, using Theorem 1.3 for $p = 2$ (or [5, Theorem 3.1]) and repeating steps of the proof of [3, Lemma 4.2], we arrive at the following.

**Proposition 3.1.** For any trigonometric polynomial $P$ and $x \in L_1 \cap \mathcal{M}$, the averages
\[ \frac{1}{n+1} \sum_{k=0}^{n} P(k)T^k(x) \]
converge a.u.

Then it is easy to verify the following (see the proof of [3, Theorem 4.4]).

**Proposition 3.2.** If \( \{\beta_k\} \) is a bounded Besicovitch sequence, then the averages \( \{\beta_k\} \) converge a.u. for every $x \in L_1 \cap \mathcal{M}$.

Here is an extension in [3, Theorem 4.6] to $L_p$-spaces.

**Theorem 3.3.** Assume that $\mathcal{M}$ has a separable predual. Let $1 < p < \infty$, and let \( \{\beta_k\} \) be a bounded Besicovitch sequence. If $T \in AC^+$, then for every $x \in L_p$ the averages \( \{\beta_k\} \) converge b.a.u. to some $\hat{x} \in L_p$. If $p \geq 2$, these averages converge a.u.
Proof. In view of Proposition 3.2 and Corollary 3.1 we only need to recall that the set $L_1 \cap \mathcal{M}$ is dense in $L_p$. The inclusion $\widehat{x} \in L_p$ follows as in the proof of Theorem 4.4. 

4. INDIVIDUAL ERODIC THEOREMS IN NONCOMMUTATIVE FULLY SYMMETRIC SPACES

Let $x \in L_0(\mathcal{M}, \tau)$, and let $\{e_\lambda\}_{\lambda \geq 0}$ be the spectral family of projections for the absolute value $|x|$ of $x$. If $t > 0$, then the $t$-th generalized singular number of $x$ (see [12]) is defined as

$$
\mu_t(x) = \inf\{\lambda > 0 : \tau(e_\lambda^x) \leq t\}.
$$

A Banach space $(E, \| \cdot \|_E) \subset L_0(\mathcal{M}, \tau)$ is called fully symmetric if the conditions

$$
x \in E, \ y \in L_0(\mathcal{M}, \tau), \ \int_0^s \mu_t(y) dt \leq \int_0^s \mu_t(x) dt \text{ for all } s > 0
$$

imply that $y \in E$ and $\|y\|_E \leq \|x\|_E$. It is known [9] that if $(E, \| \cdot \|_E)$ is a fully symmetric space, $x_n, x \in E$, and $\|x - x_n\|_E \to 0$, then $x_n \to x$ in measure. A fully symmetric space $(E, \| \cdot \|_E)$ is said to possess Fatou property if the conditions

$$
x_\alpha \in E^+, \ x_\alpha \leq x_\beta \text{ for } \alpha \leq \beta, \text{ and } \sup_\alpha \|x_\alpha\|_E < \infty
$$

imply that there exists $x = \sup_\alpha x_\alpha \in E$ and $\|x\|_E = \sup_\alpha \|x_\alpha\|_E$. The space $(E, \| \cdot \|_E)$ is said to have order continuous norm if $\|x_\alpha\|_E \downarrow 0$ whenever $x_\alpha \in E$ and $x_\alpha \downarrow 0$.

Let $L_0(0, \infty)$ be the linear space of all (equivalence classes of) almost everywhere finite complex-valued Lebesgue measurable functions on the interval $(0, \infty)$. We identify $L_\infty(0, \infty)$ with the commutative von Neumann algebra acting on the Hilbert space $L_2(0, \infty)$ via multiplication by the elements from $L_\infty(0, \infty)$ with the trace given by the integration with respect to Lebesgue measure. A Banach space $E \subset L_0(0, \infty)$ is called fully symmetric Banach space on $(0, \infty)$ if the condition above holds with respect to the von Neumann algebra $L_\infty(0, \infty)$.

Let $E = (E(0, \infty), \| \cdot \|_E)$ be a fully symmetric function space. For each $s > 0$ let $D_s : E(0, \infty) \to E(0, \infty)$ be the bounded linear operator given by $D_s(f)(t) = f(t/s)$, $t > 0$. The Boyd indices $p_E$ and $q_E$ are defined as

$$
p_E = \lim_{s \to \infty} \frac{\log s}{\log \|D_s\|_E}, \ \ q_E = \lim_{s \to +0} \frac{\log s}{\log \|D_s\|_E}.
$$

It is known [17] Ch.2, Proposition 2.b.2, Theorem 2.b.3] that $1 \leq p_E \leq q_E \leq \infty$ and if $1 \leq p < p_E \leq q_E < q \leq \infty$, then

$$
(11) \quad L_p(0, \infty) \cap L_q(0, \infty) \subset E(0, \infty) \subset L_p(0, \infty) + L_q(0, \infty),
$$

with the inclusion maps being continuous relative to $\|f\|_{L_p \cap L_q} = \max\{\|f\|_p, \|f\|_q\}$ and $\|f\|_{L_p + L_q} = \inf\{|g|_p + |h|_q : f = g + h, \ g \in L_p(0, \infty), \ h \in L_q(0, \infty)\}$.

A fully symmetric function space is said to have non-trivial Boyd indices

if $1 < p_E \leq q_E < \infty$. Note that the space $L_p(0, \infty)$, $1 < p < \infty$, has non-trivial Boyd indices:

$$
P_Lp(0, \infty) = q_Lp(0, \infty) = p
$$

[1] Ch.4, §4, Theorem 4.3].
If $E(0, \infty)$ is a fully symmetric function space, define

$$E(M) = E(M, \tau) = \{ x \in L_0(M, \tau) : \mu_t(x) \in E \}$$

and set

$$\|x\|_{E(M)} = \| \mu_t(x) \|_E, \ x \in E(M).$$

It is shown in [6] that $(E(M), \| \cdot \|_{E(M)})$ is a fully symmetric space. If $1 \leq p < \infty$ and $E = L_p(0, \infty)$, the space $(E(M), \| \cdot \|_{E(M)})$ coincides with the noncommutative $L_p$–space $(L_p(M, \tau), \| \cdot \|_p)$ because

$$\|x\|_p = \left( \int_0^\infty \mu_t^p(x) dt \right)^{1/p} = \|x\|_{E(M)}$$

[24] Proposition 2.4.

It was shown in [1] Proposition 2.2] that if $M$ is non-atomic, then every noncommutative fully symmetric $(E, \| \cdot \|_E) \subset L_0(M, \tau)$ is of the form $(E(M), \| \cdot \|_{E(M)})$ for a suitable fully symmetric function space $E(0, \infty)$.

Let $L_{p,q}(0, \infty)$, $1 \leq p, q < \infty$, be the classical function Lorentz space, that is, the space of all such functions $f \in L_0(0, \infty)$ that

$$\|f\|_{p,q} = \left( \int_0^\infty (t^{1/p} \mu_t(f))^q dt \right)^{1/q} < \infty.$$

It is known that for $q \leq p$ the space $(L_{p,q}(0, \infty), \| \cdot \|_{p,q})$ is a fully symmetric function space with Fatou property and order continuous norm. In addition, $L_{p,p} = L_p$. In the case $1 \leq p < q$, the function $\| \cdot \|_{p,q}$ is a quasi-norm on $L_{p,q}(0, \infty)$, but there exists a norm $\| \cdot \|_{(p,q)}$ on $L_{p,q}(0, \infty)$ that is equivalent to the norm $\| \cdot \|_{p,q}$ and such that $(L_{p,q}(0, \infty), \| \cdot \|_{(p,q)})$ is a fully symmetric function space with Fatou property and order continuous norm [1] Ch.4, §4]. In addition, if $1 \leq q \leq p < \infty$ ($1 < p < \infty, 1 \leq q < \infty$), then

$$P(L_{p,q}(0, \infty), \| \cdot \|_{p,q}) = q(L_{p,q}(0, \infty), \| \cdot \|_{p,q}) = p$$

[1] Ch.4, §4, Theorem 4.3] (resp.,

$$P(L_{p,q}(0, \infty), \| \cdot \|_{p,q}) = q(L_{p,q}(0, \infty), \| \cdot \|_{p,q}) = p$$

[1] Ch.4, §4, Theorem 4.5]).

Using function Lorentz space $(L_{p,q}(0, \infty), \| \cdot \|_{p,q}) ((L_{p,q}(0, \infty), \| \cdot \|_{(p,q)}))$, one can define noncommutative Lorentz space

$$L_{p,q}(M, \tau) = \left\{ x \in L_0(M, \tau) : \|x\|_{p,q} = \left( \int_0^\infty (t^{1/p} \mu_t(x))^q dt \right)^{1/q} < \infty \right\}$$

that is fully symmetric with respect to the norm $\| \cdot \|_{p,q}$ for $1 \leq q \leq p$ (resp., with respect to the norm $\| \cdot \|_{(p,q)}$ for $q > p > 1$). In addition, the norm $\| \cdot \|_{p,q}$ (resp., $\| \cdot \|_{(p,q)}$) is order continuous [7] Proposition 3.6] and satisfies Fatou property [8] Theorem 4.1]. These spaces were first introduced in the paper [14].

Following [15], a Banach couple $(X, Y)$ is a pair of Banach spaces, $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$, which are algebraically and topologically embedded in a Hausdorff
topological space. With any Banach couple \((X, Y)\) the following Banach spaces are associated:

(i) the space \(X \cap Y\) equipped with the norm
\[
\|x\|_{X \cap Y} = \max\{\|x\|_X, \|x\|_Y\}, \; x \in X \cap Y;
\]
(ii) the space \(X + Y\) equipped with the norm
\[
\|x\|_{X + Y} = \inf\{\|y\|_X + \|z\|_Y : x = y + z, y \in X, z \in Y\}, \; x \in X + Y.
\]

Let \((X, Y)\) be a Banach couple. A linear map \(T : X + Y \to X + Y\) is called a bounded operator for the couple \((X, Y)\) if both \(T : X \to X\) and \(T : Y \to Y\) are bounded.

Denote by \(B(X, Y)\) the linear space of all bounded linear operators for the couple \((X, Y)\). Equipped with the norm
\[
\|T\|_{B(X, Y)} = \max\{\|T\|_{X \to X}, \|T\|_{Y \to Y}\},
\]
this space is a Banach space. A Banach space \(Z\) is said to be intermediate for a Banach couple \((X, Y)\) if \(X \cap Y \subset Z \subset X + Y\) with continuous inclusions. If \(Z\) is intermediate for a Banach couple \((X, Y)\), then it is called an interpolation space for \((X, Y)\) if every bounded linear operator for the couple \((X, Y)\) acts boundedly from \(Z\) to \(Z\).

If \(Z\) is an interpolation space for a Banach couple \((X, Y)\), then there exists a constant \(C > 0\) such that \(\|T\|_{Z \to Z} \leq C\|T\|_{B(X, Y)}\) for all \(T \in B(X, Y)\). An interpolation space \(Z\) for a Banach couple \((X, Y)\) is called an exact interpolation space if \(\|T\|_{Z \to Z} = \|T\|_{B(X, Y)}\) for all \(T \in B(X, Y)\).

Every fully symmetric function space \(E = E(0, \infty)\) is an exact interpolation space for the Banach couple \((L_1(0, \infty), L_\infty(0, \infty))\) [15, Ch.II, §4, Theorem 4.3].

We need the following noncommutative interpolation result for the spaces \(E(M)\).

**Theorem 4.1.** [6, Theorem 3.4] Let \(E, E_1, E_2\) be fully symmetric function spaces on \((0, \infty)\). Let \(M\) be a von Neumann algebra with a faithful semifinite normal trace. If \((E_1, E_2)\) is a Banach couple and \(E\) is an exact interpolation space for \((E_1, E_2)\), then \(E(M)\) is an exact interpolation space for the Banach couple \((E_1(M), E_2(M))\).

It follows now from [15, Ch.II, Theorem 4.3] and Theorem 4.1 that every noncommutative fully symmetric space \(E(M)\), where \(E = E(0, \infty)\) is a fully symmetric function space, is an exact interpolation space for the Banach couple \((L_1(M), M)\).

Let \(T \in AC^\tau(M, \tau)\). Let \(E(0, \infty)\) be a fully symmetric function space. Since the noncommutative fully symmetric space \(E(M)\) is an exact interpolation space for the Banach couple \((L_1(M), \tau, M)\), we conclude that \(T(E(M)) \subset E(M)\) and \(T\) is a positive continuous linear map on \((E(M), \| \cdot \|_{E(M)})\). Thus
\[
M_n(x) = \frac{1}{n + 1} \sum_{k=0}^{n} T^k(x) \in E(M)
\]
for each \(x \in E(M)\) and all \(n\). Besides, the inequalities
\[
\|T(x)\|_1 \leq \|x\|_1, \; x \in L_1^h, \quad \|T(x)\|_\infty \leq \|x\|_\infty, \; x \in M^h
\]
implicate that
\[
\sup_{n \geq 1} \|M_n\|_{L_1^h \to L_1^h} \leq 1 \quad \text{and} \quad \sup_{n \geq 1} \|M_n\|_{M^h \to M^h} \leq 1.
\]
Therefore, due to the decomposition \( x = \text{Re}(x) + i \cdot \text{Im}(x) \), we have
\[
\sup_{n \geq 1} \| M_n \|_{L_p \rightarrow L_q} \leq 2 \quad \text{and} \quad \sup_{n \geq 1} \| M_n \|_{L_q \text{-symmetric } \rightarrow L_p} \leq 1.
\]
Since the noncommutative fully symmetric space \( E(\mathcal{M}) \) is an exact interpolation space for the Banach couple \((L_1(\mathcal{M}, \tau), \mathcal{M})\), we have
\[
(12) \quad \sup_{n \geq 1} \| M_n \|_{E(\mathcal{M}) \rightarrow E(\mathcal{M})} < 1.
\]
If \( T \in DS^+(\mathcal{M}, \tau) \), then
\[
(13) \quad \sup_{n \geq 1} \| M_n \|_{E(\mathcal{M}) \rightarrow E(\mathcal{M})} \leq 1.
\]

The following theorem is a version of Theorem 1.4 for noncommutative fully symmetric Banach spaces with non-trivial Boyd indices.

**Theorem 4.2.** Let \( E(0, \infty) \) be a fully symmetric function space with Fatou property and non-trivial Boyd indices. If \( T \in AC^+(\mathcal{M}, \tau) \), then for any given \( x \in E(\mathcal{M}, \tau) \),

(i) the averages \( M_n(x) \) converge b.a.u. to some \( \hat{x} \in E(\mathcal{M}, \tau) \);

(ii) if \( p_{E(0, \infty)} > 2 \), these averages converge a.u.

**Proof.** (i) According to [17, Theorem 2.b.3] (see (11)) and Theorem 1.4(i), there exist such \( 1 < p, q < \infty \) that
\[
E(\mathcal{M}, \tau) \subset L_p(\mathcal{M}, \tau) + L_q(\mathcal{M}, \tau).
\]
Then \( x = x_1 + x_2 \), where \( x_1 \in L_p(\mathcal{M}, \tau), x_2 \in L_q(\mathcal{M}, \tau) \), and, by Theorem 1.4(i), there exist such \( \hat{x}_1 \in L_p(\mathcal{M}, \tau) \) and \( \hat{x}_1 \in L_q(\mathcal{M}, \tau) \) that \( M_n(x_j) \) converge b.a.u. to \( \hat{x}_j, j = 1, 2 \).

Therefore
\[
M_n(x) \rightarrow \hat{x} = \hat{x}_1 + \hat{x}_2 \in L_p(\mathcal{M}, \tau) + L_q(\mathcal{M}, \tau) \subset L_0(\mathcal{M}, \tau)
\]
b.a.u., hence \( M_n(x) \rightarrow \hat{x} \) in measure. Since \( E(\mathcal{M}) \) satisfies Fatou property, the unit ball of \( E(\mathcal{M}) \) is closed in the measure topology [8, Theorem 4.1], and (12) implies that \( \hat{x} \in E(\mathcal{M}) \).

(ii) If \( p_{E(0, \infty)} > 2 \), then the numbers \( p \) and \( q \) in part (i) can be chosen such that \( 2 < p, q < \infty \). Utilizing Theorem 1.4(ii) and repeating the argument above, we conclude that the averages \( M_n(x) \) converge to \( \hat{x} \) a.u. \( \square \)

Since any function Lorentz space \( E = L_{p,q}(0, \infty) \) with \( 1 < p < \infty \) and \( 1 < q < \infty \) has non-trivial Boyd indices \( p_E = q_E = p \), we have the following corollary of Theorem 4.2.

**Theorem 4.3.** Let \( 1 < p < \infty \) and \( 1 \leq q < \infty \). Then, given \( x \in L_{p,q}(\mathcal{M}, \tau) \),

(i) the averages \( M_n(x) \) converge b.a.u. to some \( \hat{x} \in L_{p,q}(\mathcal{M}, \tau) \);

(ii) if \( p > 2 \), these averages converge a.u.

**Remark 4.1.** If \( 1 \leq q \leq p \), then \( L_{p,q}(\mathcal{M}, \tau) \subset L_{p,p}(\mathcal{M}, \tau) = L_p(\mathcal{M}, \tau) \) (see [14] and [23, Lemma 1.6]). Then it follows directly from Theorem 1.4 along with the ending of the proof of part (i) of Theorem 6.1 that for every \( x \in L_{p,q}(\mathcal{M}, \tau) \) the averages \( M_n(x) \) converge to some \( \hat{x} \in L_{p,q}(\mathcal{M}, \tau) \) b.a.u. (a.u. for \( p \geq 2 \)).

Utilizing Theorem 8.3 and following the proof of Theorem 8.1 we arrive at the following version of Theorem 8.3 for noncommutative fully symmetric Banach spaces with non-trivial Boyd indices.
**Theorem 4.4.** Let \( E = E(0,\infty) \) be a fully symmetric function space with Fatou property and non-trivial Boyd indices. Then for every \( x \in E(M) \) the averages \( \{ \hat{x} \} \) converge b.a.u. to some \( \hat{x} \in E(M) \). In addition, if \( p_{E(0,\infty)} > 2 \), then averages \( \{ \hat{x} \} \) converge a.u. to some \( \hat{x} \). As a corollary of Theorem 4.4, we obtain Besicovitch weighted ergodic theorem in noncommutative Lorentz spaces.

**Theorem 4.5.** Let \( 1 < p < \infty \) and \( 1 \leq q < \infty \). Then for any \( x \in L_{p,q}(M,\tau) \) the averages \( \{ \hat{x} \} \) converge b.a.u. to some \( \hat{x} \in L_{p,q}(M,\tau) \). If \( p > 2 \), these averages converge a.u.

**Remark 4.2.** If \( 1 \leq q \leq p \), then \( L_{p,q}(M,\tau) \subset L_{p}(M,\tau) \), and it follows directly from Theorem 3.3 along with the ending of the proof of part (i) of Theorem 3.1 that for every \( x \in L_{p,q}(M,\tau) \) the averages \( \{ \hat{x} \} \) converge to some \( \hat{x} \in L_{p,q}(M,\tau) \) b.a.u. (a.u. for \( p \geq 2 \)).

5. **Mean ergodic theorems in noncommutative fully symmetric spaces**

Let \( M \) be a von Neumann algebra with a faithful normal semifinite trace \( \tau \). In [24] the following mean ergodic theorem for noncommutative fully symmetric spaces was proven.

**Theorem 5.1.** Let \( E(M) \) be a noncommutative fully symmetric space such that
(i) \( L_1 \cap M \) is dense in \( E(M) \);
(ii) \( \| e_n \|_{E(M)} \to 0 \) for any sequence of projections \( \{ e_n \} \subset L_1 \cap M \) with \( e_n \downarrow 0 \);
(iii) \( \| e_n \|_{E(M)}/\tau(e_n) \to 0 \) for any increasing sequence of projections \( \{ e_n \} \subset L_1 \cap M \) with \( \tau(e_n) \to \infty \).

Then, given \( x \in E(M) \) and \( T \in DS^+(M,\tau) \), there exists \( \hat{x} \in E(M) \) such that \( \| \hat{x} - M_n(x) \|_{E(M)} \to 0 \).

It is clear that any noncommutative fully symmetric space \( (E(M), \| \cdot \|_{E(M)}) \) with order continuous norm satisfies conditions (i) and (ii) of Theorem 5.1. Besides, in the case of noncommutative Lorentz space \( L_{p,q}(M,\tau) \), the inequality \( p > 1 \) together with
\[
\| e \|_{p,q} = \left( \frac{p}{q} \right)^{1/q} \tau(e)^{1/p}, \quad e \in L_1 \cap P(M)
\]
imply that condition (iii) is also satisfied. Therefore Theorem 5.1 entails the following.

**Corollary 5.1.** Let \( 1 < p < \infty \), \( 1 \leq q < \infty \), \( T \in DS^+ \), and \( x \in L_{p,q}(M,\tau) \). Then there exists \( \hat{x} \in L_{p,q}(M,\tau) \) such that \( \| \hat{x} - M_n(x) \|_{p,q} \to 0 \).

The next theorem asserts convergence in the norm \( \| \cdot \|_{E(M)} \) of the averages \( M_n(x) \) for any noncommutative fully symmetric space \( (E(M), \| \cdot \|_{E(M)}) \) with order continuous norm, under the assumption that \( \tau(\mathbb{1}) < \infty \).

**Theorem 5.2.** Let \( \tau \) be finite, and let \( E(M,\tau) \) be a noncommutative fully symmetric space with order continuous norm. Then for any \( x \in E(M) \) and \( T \in DS^+ \) there exists \( \hat{x} \in E(M) \) such that \( \| \hat{x} - M_n(x) \|_{E(M)} \to 0 \).

**Proof.** Since the trace \( \tau \) is finite, we have \( M \subset E(M,\tau) \). As the norm \( \| \cdot \|_{E(M)} \) is order continuous, applying spectral theorem for selfadjoint operators in \( E(M,\tau) \),
we conclude that $\mathcal{M}$ is dense in $(E(\mathcal{M}, \tau), \| \cdot \|_{E(\mathcal{M})})$. Therefore $\mathcal{M}^+$ is a fundamental subset of $(E(\mathcal{M}, \tau), \| \cdot \|_{E(\mathcal{M})})$, that is, the linear span of $\mathcal{M}^+$ is dense in $(E(\mathcal{M}, \tau), \| \cdot \|_{E(\mathcal{M})})$.

Show that the sequence $\{M_n(x)\}$ is weakly sequentially compact for every $x \in \mathcal{M}^+$. Without loss of generality, assume that $0 \leq x \leq 1$. Since $T \in DS^+$, we have $0 \leq M_n(x) \leq M_n(1) \leq 1$ for any $n$. By [9, Proposition 4.3], given $y \in E^+(\mathcal{M}, \tau)$, the set $\{a \in E(\mathcal{M}, \tau) : 0 \leq a \leq y\}$ is weakly compact in $(E(\mathcal{M}, \tau), \| \cdot \|_{E(\mathcal{M})})$, which implies that the sequence $\{M_n(x)\}$ is weakly sequentially compact in $(E(\mathcal{M}, \tau), \| \cdot \|_{E(\mathcal{M})})$.

Since $\sup_{n \geq 1} \|M_n\|_{E(\mathcal{M})} \leq E(\mathcal{M}) \leq 1$ (see (13)) and

$$0 \leq \left\| \frac{T^n(x)}{n} \right\|_{E(\mathcal{M})} \leq \frac{\|x\|_{E(\mathcal{M})}}{n} \to 0$$

whenever $x \in \mathcal{M}^+$, the result follows by Corollary 3 in [10, Ch.VIII, §5].

Remark 5.1. In the commutative case, Theorem 5.2 was established in [22]. It was also shown that if $\mathcal{M} = L_{\infty}(0, 1)$, then for every fully symmetric Banach function space $E(0, 1)$ with the norm that is not order continuous there exists such $T \in DS^+$ and $x \in E(\mathcal{M})$ that the averages $M_n(x)$ do not converge in $(E(\mathcal{M}), \| \cdot \|_{E(\mathcal{M})})$.

The following proposition is a version of Theorem 5.1 for noncommutatively symmetric space with order continuous norm with condition (iii) being replaced by non-triviality of the Boyd indices of $E(0, \infty)$. Note that we do not require $T$ to be positive.

Proposition 5.1. Let $E(0, \infty)$ be a fully symmetric function space with non-trivial Boyd indices and order continuous norm. Then for any $x \in E(\mathcal{M}, \tau)$ and $T \in DS(\mathcal{M}, \tau)$ there exists such $\hat{x} \in E(\mathcal{M}, \tau)$ that $\|\hat{x} - M_n(x)\|_{E(\mathcal{M})} \to 0$.

Proof. By [17, Theorem 2. b. 3], it is possible to find such $1 < p, q < \infty$ that

$$\L_p(0, \infty) \cap L_q(0, \infty) \subset E(0, \infty) \subset \L_p(0, \infty) + L_q(0, \infty)$$

with continuous inclusion maps. Therefore Theorem 5.1 implies that the space $\mathcal{L} = \L_p(\mathcal{M}, \tau) \cap L_q(\mathcal{M}, \tau)$ is continuously embedded in $E(\mathcal{M}, \tau)$. Besides, it follows as in Theorem 5.2 that $\mathcal{L}^+$ is a fundamental subset of $(E(\mathcal{M}, \tau), \| \cdot \|_{E(\mathcal{M})})$.

Show that for every $x \in \mathcal{L}^+$ the sequence $\{M_n(x)\}$ is weakly sequentially compact in $(E(\mathcal{M}, \tau), \| \cdot \|_{E(\mathcal{M})})$. Since $p, q > 1$, the spaces $\L_p(\mathcal{M}, \tau)$ and $L_q(\mathcal{M}, \tau)$ are reflexive. Taking into account that $T \in DS$ and $x \in \L_p(\mathcal{M}, \tau) \cap L_q(\mathcal{M}, \tau)$, we conclude that the averages $\{M_n(x)\}$ converge in $(\L_p(\mathcal{M}, \tau), \| \cdot \|_p)$ and in $(L_q(\mathcal{M}, \tau), \| \cdot \|_q)$ to $\hat{x}_1 \in \L_p(\mathcal{M}, \tau)$ and to $\hat{x}_2 \in L_q(\mathcal{M}, \tau)$, respectively [10, Ch.VIII, §5, Corollary 4]. This implies that the sequence $\{M_n(x)\}$ converges to $\hat{x}_1$ and to $\hat{x}_2$ in measure, hence $\hat{x}_1 = \hat{x}_2 := \hat{x}$. Since $\mathcal{L}$ is continuously embedded in $E(\mathcal{M}, \tau)$, the sequence $\{M_n(x)\}$ converges to $\hat{x}$ with respect to the norm $\| \cdot \|_{E(\mathcal{M})}$, thus, it is weakly sequentially compact in $(E(\mathcal{M}, \tau), \| \cdot \|_{E(\mathcal{M})})$.

Now we can proceed as in the ending of the proof of Theorem 5.2. □
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