Learnability with Indirect Supervision Signals

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Abstract

Learning from indirect supervision signals is important in real-world AI applications when, often, gold labels are missing or too costly. In this paper, we develop a unified theoretical framework for multi-class classification when the supervision is provided by a variable that contains nonzero mutual information with the gold label. The nature of this problem is determined by (i) the transition probability from the gold labels to the indirect supervision variables and (ii) the learner’s prior knowledge about the transition. Our framework relaxes assumptions made in the literature, and supports learning with unknown, non-invertible and instance-dependent transitions. Our theory introduces a novel concept called separation, which characterizes the learnability and generalization bounds. We also demonstrate the application of our framework via concrete novel results in a variety of learning scenarios such as learning with superset annotations and joint supervision signals.

1 Introduction

We are interested in the problem of multiclass classification where direct and gold annotations for the unlabeled instance are expensive or inaccessible, and instead the observation of a dependent variable of the true label is used as supervision signal. Examples include learning from noisy annotations [1, 16, 21], partial annotations [11, 17, 9] or feedback from an external world [10, 5].

To extract the information contained in a dependent variable, the learner should have certain prior knowledge about the relation between the true label and the supervision signal, which can be expressed in various forms. For example, in the noisy label problem, the noisy rate is assumed to be bounded by a constant (such as the Massart noise [18, 12]). In the superset problem, the true label is commonly assumed to be contained in (or consistent with) the superset annotation [11, 17].

As in [8, 24, 32], we model the aforementioned relation using a transition probability, which is the distribution of the observable variable conditioned on the label and instance. The transition enables the learner to induce a prediction of the observable via the prediction of the label, and construct loss functions based on the induced prediction and the observable.

In this paper, instead of assuming that the learner fully knows the transition, we formalize the concept of transition class, a set that contains all the candidate transitions, to describe more general forms of prior information. Also, we define the concept of separation to quantify whether the information is enough to distinguish different labels. With these concepts, we are able to study a variety of learning scenarios with unknown, non-invertible and instance-dependent transitions in a unified way. We show this under the realizability assumption (also called separable in linear classification), a commonly made assumption (such as [2, 14, 17]) that assumes that the true classifier is in the hypothesis space.

Our goal is to develop a unified theoretical framework that can (i) provide learnability conditions for general indirect supervision problems, (ii) describe what prior knowledge is needed about the transition, and (iii) characterize the difficulty of learning with indirect supervision.

Specifically, in this paper, our main contribution includes:

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1. We decompose the learnability condition of a general indirect supervision problem into three aspects: complexity, consistency and identifiability and provide a unified learning bound for the problem (Theorem 4.2).

2. We propose a simple yet powerful concept called separation, which encodes the prior knowledge about the transition using statistical distance between distributions over the annotation space and uses it to characterize consistency and identifiability (Theorem 5.2).

3. We formalize two ways to achieve separation: total variation and joint supervision, and use them to derive concrete novel results of practical learning problems of interest (Section 5.2 and 5.3).

All proofs of the theoretical results are presented in the supplementary material.

2 Related Work

Specific Indirect Supervision Problems. Our work is motivated by many previous studies on the problem of learning in the absence of gold labels. Specially, the problem of classification under label noise dates back to [1] and has been studied extensively over the past decades. Our work is mostly related to (i) Theoretical analysis of PAC guarantees and consistency of loss functions, including learning with bounded noise [18, 16, 2], and instance-dependent noise [25, 19, 7], (ii) Algorithms for learning from noisy labels, including using the inverse information of the transition [21, 32], and inducing predictions of noisy label (which is more similar to our formulation) [6, 30].

Superset (also called partial label) problems, where the annotation is given as a subset of the annotation space, arises in various forms in standard multiclass classification and structured prediction [11, 9, 15, 22]. While it is possible to extend some approaches in the theory of noisy problems to the superset case, the superset problem focuses on the case of a large and complex annotation space, and some of the assumptions (such as “known transition”) would be too strong in practice. On the theoretical side, [11] defines ambiguity degree to characterize the learning bound. [17] provides an insightful discussion of the PAC-learnability of the superset problem and proposes the concept of induced hypothesis. This two papers motivate the approach pursued in this paper.

Frameworks for Indirect Supervision. Our supervision scheme is similar to [29, 24], which model the label as a latent variable of the indirect supervision signal. [8, 9] study the problem of designing consistent loss functions for superset problems when the transition (they call it mixing) matrix is partially known. The discussion can be applied to a wider range of problems such as noisy and semi-supervised learning. Our goal is mostly similar to [32], which further develops the ideas from [27, 9, 21] and develops a general framework of learning from data with reconstructible corruption, using the inverse of a known, instance-independent transition matrix, to construct unbiased estimator of the classification loss and derive generalization bounds.

3 Preliminaries

We will use $\mathbb{P}(\cdot)$ to denote probability, $\mathbb{E}[\cdot]$ to denote expectation, $\mathbb{1}\{\cdot\}$ to denote the indicator function and $p(\cdot)$ to denote the density function or more generally, the Radon-Nikodym derivative.

We denote the source variable as $X$, which takes value in an input space $\mathcal{X}$ and denote the target label as $Y$, which takes value in a label space $\mathcal{Y}$. We assume $|\mathcal{Y}| = c$ is finite and identify the elements in $\mathcal{Y}$ as $\{y_1, y_2, \ldots, y_c\}$. The goal is to learn a mapping $h_0 : \mathcal{X} \to \mathcal{Y}$. The hypothesis class $\mathcal{H}$ contains candidate mappings $h : \mathcal{X} \to \mathcal{Y}$. The loss function for hypothesis $h$ and sample $(x, y)$ is denoted as $\ell(h(x), y)$. The risk of a hypothesis $h \in \mathcal{H}$ is defined as $R(h) := \mathbb{E}_{X,Y}[\ell(h(x), y)]$, where $x$ is sampled independently from a (unknown) distribution $D_X$. We will focus on the realizable case, i.e., there is a classifier $h_0 \in \mathcal{H}$ such that $R(h) = 0$. As in standard PAC-learning theory, we use the zero-one loss for the gold sample $(x, y)$ (although we may not observe $y$): $\ell(h(x), y) = \mathbb{1}\{h(x) \neq y\}$.

The annotation $O$ (also called supervision signal) is a random variable that is not independent with $Y$ (or equivalently, $O$ and $Y$ has positive mutual information). The dependence between $X$ and $O$ conditioned on $Y$ is allowed but not required. $O$ takes value in an annotation space denoted as $\mathcal{O}$. We also assume $|\mathcal{O}| = s < \infty$ and identify the elements in $\mathcal{O}$ as $\{o_1, o_2, \ldots, o_s\}$. For convenience, when using $y_i$ and $o_i$ as subscripts, we regard $y_i, o_i$ as its index $i$. For example, for
any indexed quantity \( a_i \), we denote \( a_{u_i} = a_i \). We denote the probability simplex of dimension \( s \) as:

\[
D_s = \{ w \in \mathbb{R}^s : \sum_{i=1}^s w_i = 1, \ w_i \geq 0 \},
\]

which represents the set of all distributions over \( O \).

Examples of annotation \( O \): (i) In the noisy problem, the true label is replaced (due to mislabeling or corruption) by another label with certain probabilities. Therefore, \( O = \mathcal{Y} \). (ii) In the superset annotation problem, the learner observes \( o \) which is a subset of \( \mathcal{Y} \) (hopefully but not necessarily \( o \) contains the true label \( y \)). In this case, \( O = 2^\mathcal{Y} \); the power set of \( \mathcal{Y} \).

In our framework, the learner predicts \( O \) using the graphical model shown in fig. 1. The conditional distribution of \( O \) given \( X = x \) and \( Y = y \) can be identified by a mapping from \( x \) to a transition matrix \( T_0(x):= \mathbb{P}(O = o_j | X = x, Y = y_j)_{ij} \). A key point of this paper is that in general we do not assume that the learner (or learning algorithm) has full information of \( T_0(x) \). Instead, we define a transition hypothesis to be a candidate transition \( T(x) \) (also denoted as \( T \) for convenience) that maps the instance \( x \) to a stochastic matrix of size \( c \times s \). For a fixed \( x \), the \( s \)-th row of a transition \( T(x) \) represents a distribution over \( O \), and is denoted as \( (T(x))_s \). The set of all candidate transition hypotheses is called the transition class, denoted as \( \mathcal{T} \). We assume \( T_0 \in \mathcal{T} \). When it is needed to distinguish transition hypothesis from classifiers in \( \mathcal{H} \), we will call the latter one a base hypothesis. With transition hypothesis \( T \), a base hypothesis \( \hat{y} = h(x) \) naturally induces a probability distribution \( (T(x))_{h(x)} \). We call it induced hypothesis, denoted as \( T \circ h \).

One may penalize \( T \circ h \) by evaluating its prediction of \( O \) on the dataset. More precisely, in our framework, the learner will be penalized by provided with an annotation loss \( \ell_O(\hat{y}, T, (x, o)) : \mathcal{Y} \times \mathcal{T} \times \mathcal{X} \times O \mapsto \mathbb{R} \). A natural example is the cross-entropy loss, which approximates the marginal probability of \( O \):

\[
\ell_O(h(x), T, (x, o)) := -\log \mathbb{P}(o | x, h(x), T)
\]

The annotation risk is defined as \( R_O(T \circ h) := \mathbb{E}_{x,o}[\ell_O(h(x), T, (x, o))] \). A training set \( S = \{(x^{(i)}, o^{(i)})\}_{i=1}^m \) contains independent samples of \( X \) and \( O \). The empirical annotation risk associated with the training sample \( S \) is then defined as \( \tilde{R}_O(h \circ T | S) := \frac{1}{m} \sum_{i=1}^m \ell_O(h(x^{(i)}), T, (x^{(i)}, o^{(i)})) \).

In summary, the learner’s input includes: the spaces \( \mathcal{X}, \mathcal{Y}, \mathcal{O} \), the hypothesis class \( \mathcal{H} \) and transition class \( \mathcal{T} \), the training set \( S = \{(x^{(i)}, o^{(i)})\}_{i=1}^m \), and the loss functions \( \ell, \ell_O \).

We call a hypothesis class \( \mathcal{H} \) learnable if there is a learning algorithm \( A : \cup_{m=1}^\infty (\mathcal{X} \times \mathcal{O})^m \mapsto \mathcal{H} \) such that: for any distribution \( D_X \) over \( \mathcal{X} \) and \( T_0 \in \mathcal{T} \), when running \( A \) on datasets \( S^{(m)} \) of \( m \) independent samples of \( (X, O) \), we have \( R(A(S^{(m)})) \) converges to 0 in probability as \( m \to \infty \). In particular, we define the Empirical Risk Minimizer to be a mapping \( \text{ERM} : \cup_{m=1}^\infty (\mathcal{X} \times \mathcal{O})^m \mapsto \mathcal{H} \) such that \( \text{ERM}(S) \in \text{argmin}_{h \in \mathcal{H}, T \in \mathcal{T}} \tilde{R}_O(h \circ T | S) \), where the argmin operator only returns the base hypothesis (although the empirical risk is minimized over both base and transition hypotheses).

### 4 General Learnability Conditions

In this section, we present theorem 4.2 that decomposes the learnability of a general indirect supervision problem into three aspects: complexity, consistency and identifiability. After that, we propose proposition 4.3 to help verifying the complexity condition. The other two conditions will be further studied in the next section.

We assume \( \ell_O \) takes value in an interval \([0, b]\) for some constant \( b > 0 \). To characterize the learnability, a key step is to describe the complexity of the function class

\[
\ell_O \circ \mathcal{T} \circ \mathcal{H} := \{(x, o) \mapsto \ell_O(T, (x, h(x), o)) : h \in \mathcal{H}, T \in \mathcal{T}\}
\]

To do so, we use the following generalized version of VC-dimension proposed in [3]. It enables us to bound the Rademacher complexity [3] of \( \ell_O \circ \mathcal{T} \circ \mathcal{H} \) (which provides the flexibility to study arbitrary
loss function) via the Natarajan dimension [20] of \( \mathcal{H} \) (proposition 4.3) (which is in general easier to compute than Rademacher complexity).

**Definition 4.1.** We adopt the following definitions from [3]:

1. (VC-class) A class \( \mathcal{C} \) of subsets of a set \( \mathcal{Z} \) is said to **shatter** a finite subset \( \mathcal{Z} \subseteq \mathcal{Z} \) if
   \[
   \{ C \cap \mathcal{Z} : C \in \mathcal{C} \} = 2^{\mathcal{Z}}
   \]
   Moreover, \( \mathcal{C} \) is called a **VC-class** with dimension no larger than \( k \) if there exists an integer \( k \) such that \( \mathcal{C} \) cannot shatter any subset of \( \mathcal{Z} \) with more than \( k \) elements.

2. (weak VC-major) The function class \( \ell_{\mathcal{O}} \circ \mathcal{T} \circ \mathcal{H} \) is said to be **weak VC-major** with dimension \( d \) if \( d \) is the smallest integer such that for all \( u \in \mathbb{R} \), the set family
   \[
   \mathcal{C}_u \overset{\text{def}}{=} \{ \{ (x, o) : \ell_{\mathcal{O}}(h(x), T, (x, o)) > u \} : h \in \mathcal{H}, T \in \mathcal{T} \}
   \]
   is a VC-class of \( \mathcal{X} \times \mathcal{O} \) with dimension no larger than \( d \).

Now we are able to state the main result in this section:

**Theorem 4.2.** If the following conditions are satisfied

\[\text{[C1] (Complexity)} \quad \ell_{\mathcal{O}} \circ \mathcal{T} \circ \mathcal{H} \text{ is weak VC-major with dimension } d < \infty.\]

\[\text{[C2] (Consistency)} \quad h_0 \subseteq \mathop{\text{argmin}}_{h \in \mathcal{H}, T \in \mathcal{T}} R_{\mathcal{O}}(T \circ h).\]

\[\text{[C3] (Identifiability)} \quad \eta = \mathop{\text{inf}}_{h \in \mathcal{H}, T \in \mathcal{T} : R(h) > 0} \frac{R_{\mathcal{O}}(T \circ h) - \mathop{\text{inf}}_{T \in \mathcal{T}} R_{\mathcal{O}}(T \circ h_0)}{R(h)} > 0.\]

Then, \( \mathcal{H} \) is learnable. That is, for any \( \delta \in (0, 1) \), with probability of at least \( 1 - \delta \), we have:

\[
R(\text{ERM}(S^{(m)})) \leq \frac{2b}{\eta} \left( \sqrt{\frac{2\Gamma_m(d)}{m}} + \frac{4\Gamma_m(d)}{m} + \frac{2\log(4/\delta)}{m} \right).
\]

(2)

where \( \Gamma_m(d) \) is defined in [3] by \( \Gamma_m(d) \overset{\text{def}}{=} \log \left[ \sum_{j=0}^{\min\{d, m\}} \binom{m}{j} \right] = d \log m(1 + o(1)) \) as \( m \to \infty \).

This implies \( R(\text{ERM}(S^{(m)})) \to 0 \) in probability as \( m \to \infty \).

Bound (2) suggests that the difficulty of the learning can be characterized by (i) the identifiability level \( \eta \), which mainly depends on the nature of the indirect supervision and how about: learner’s prior information of the transition hypothesis, and will be further studied in the next section. (ii) the weak VC-major \( d \) of \( \ell_{\mathcal{O}} \circ \mathcal{T} \circ \mathcal{H} \), which depends on the modeling choice. We present the following results that bound \( d \) by the Natarajan dimension of \( \mathcal{H} \) and the weak-VC major dimension of function class:

\[
\ell_{\mathcal{O}} \circ \mathcal{T} \overset{\text{def}}{=} \{ (\hat{x}, \hat{o}) \rightarrow \ell_{\mathcal{O}}(\hat{y}, T, (x, o)) : T \in \mathcal{T} \}
\]

**Proposition 4.3.** Suppose the Natarajan dimension of \( \mathcal{H} \) is \( d_\mathcal{H} < \infty \) and the weak-VC major dimension of \( \ell \circ \mathcal{T} \) is \( d_\mathcal{T} < \infty \). Then, the weak-VC major dimension of \( \ell_{\mathcal{O}} \circ \mathcal{H} \), \( d \), can be bounded:

\[
d \leq 2 \left( (d_\mathcal{H} + d_\mathcal{T}) (\log(6(d_\mathcal{H} + d_\mathcal{T}))) + 2d_\mathcal{H} \log c \right) \quad \text{where } c = |\mathcal{Y}|
\]

The reason that we do not study the complexity of \( \mathcal{T} \) separately is that the annotation loss may not depend on \( T \) (i.e., \( \ell_{\mathcal{O}}(\hat{y}, T_1, (x, o)) = \ell_{\mathcal{O}}(\hat{y}, T_2, (x, o)) \) for any \( T_1, T_2 \in \mathcal{T} \)). See proposition 5.5 for an example of such a loss. To show applications of proposition 4.3, we study the following cases:

**Example 4.4.** In the following cases, we first compute/bound \( d_\mathcal{T} \), then \( d \) can be bounded by \( d_\mathcal{H} \):

1. When the true transition is known or the annotation loss function only depends on \( (\hat{y}, o) \), we have \( d_\mathcal{T} = 0 \); hence \( d \leq 2d_\mathcal{H}(\log(6d_\mathcal{H}) + 2\log c) \). This is conceptually similar to the Lemma 3.4 in [17], which bounds the VC-dimension of the induced hypothesis class for the noise-free superset problem.

\[1\text{This argmin operator only returns the base hypothesis.}\]
2. When all transition hypotheses in T are instance-independent and the annotation loss only depends on \((T, \hat{y}, o)\) (e.g., the cross-entropy loss defined in (1)), then \(d_T\) can be trivially bounded by \(d_T \leq cs = |Y \times O|\); hence \(d \leq 2((d_H + cs) \log(6(d_H + cs))) + 2d_H \log c\).

3. Suppose the instance is embedded in a vector space \(X = \mathbb{R}^p\). Consider the problem (Example 5.1.3 in [19]) of binary classification with uniform noise rate which is modeled as a Logistic regression: \(P(O \neq y|x, y) = S(w^T x)\) where \(S\) is the sigmoid function and \(w\) is the parameter. Then the cross-entropy loss becomes: 
\[-\log(S(w^T x)) - \log(1 - S(w^T x)).\]
We have \(d_T \leq 2p + 2\). See supplementary material for a proof.

5 Separation

Throughout this section we assume [C1] of Theorem 4.2 holds. We will first propose a concept called separation, which provides an intuitive way to understand the learnability and helps to verify [C2] and [C3]; then we study two ways to ensure separation, and their application in real problems.

5.1 Learning by Separation

Without any prior knowledge, the transition class will contain all possible transitions. In this case, learnability cannot be ensured since a wrong label \(\hat{y}\) can also induce a good prediction of \(O\) via an incorrect transition hypothesis. Hence, certain kind of prior knowledge is needed to restrict the range of \(T\). To formalize this idea, we first introduce an extension of the KL-divergence.

\textbf{Definition 5.1} (KL-divergence between Two Sets of Distributions). Given two sets of distributions \(\mathcal{D}_1\) and \(\mathcal{D}_2\), we define the KL-divergence between them as:
\[
KL(\mathcal{D}_1 \parallel \mathcal{D}_2) \overset{\text{def}}{=} \inf_{D_1 \in \mathcal{D}_1, D_2 \in \mathcal{D}_2} KL(D_1 \parallel D_2)
\]

Now we are able to state the main result of this section:

\textbf{Theorem 5.2} (Separation). For all \(x \in X\), we denote the induced distribution families by label \(y_i\) as \(\mathcal{D}_i(x) \overset{\text{def}}{=} \{(T(x))_i : T \in T\} \subseteq \mathcal{D}_O\) (recall that \((T(x))_i\) is the \(i\)th row of \(T(x)\)), and the set of all possible predictions of the label as \(\mathcal{H}(x) \overset{\text{def}}{=} \{h(x) : h \in \mathcal{H}\} \subseteq Y\). Suppose
\[
\gamma \overset{\text{def}}{=} \inf_{(x,i,j):p(x,y_i)>0,j \neq i,y_j \in \mathcal{H}(x)} KL(D_i(x) \parallel D_j(x)) > 0
\]
Then \(\mathcal{H}\) is learnable from the observations of \((X, O)\) with \(\eta \geq \gamma > 0\) via the ERM of cross-entropy loss (1). We call \(\gamma\) the separation degree.

Moreover, if (3) is not satisfied, then there exists a sequence of transitions \(\{T^{(k)}\}_k\) \((T^{(k)} \in T)\) and distributions \(\{D_X^{(k)}\}_k\) over \(X\) such that \(\lim_k \eta^{(k)} = 0\), where \(\eta^{(k)}\) is defined the same as \(\eta\) in [C3], with the expectation (in the definition of the risk functions) being taken according to \(T^{(k)}\) and \(D_X^{(k)}\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{separation.png}
\caption{(a) Illustration of \textit{separation}. A (predicted) label \(y_i\) will induce a distribution family over \(O\) called \(D_i(x)\). Different families are separated by a minimal “distance” \(\gamma\). (b) Illustration of \textit{joint supervision}. By adding new supervision signals, separation of particular pairs of labels is preserved.}
\end{figure}
Theorem 5.2 is important in two ways: (i) It provides a way to characterize the prior knowledge of the learner about the transition using the KL-divergence and reveals its connection with the identifiability of labels. (ii) The “moreover” result shows if separation is not satisfied, then the induced distribution of $O$ by different labels can be arbitrarily close, and hence the learning of $Y$ from $O$ can be arbitrarily difficult. An illustration of separation is shown in fig. 2 (a). Yet, a drawback of the cross-entropy loss used in theorem 5.2 is that it can be unbounded when there is a zero element in the transition. This problem will be partly solved in proposition 5.5 by introducing a new annotation loss.

As the simplest application, we introduce the case where the transition is fully known to the learner:

**Example 5.3** (Full Information of the Transition). Suppose the transition $T(x)$ is known to the learner, i.e., $T = \{T_0\}$, by Theorem 5.2, we know $H$ is learnable if

$$\inf_{(x,i,j) : p(x) > 0, i \neq j, y_i \in H(x)} \text{KL}(T_0(x)_i \parallel T_0(x)_j) > 0$$

Notice that this is a weaker assumption than the invertibility assumption of $T_0(x)$, which is used in [32] (called reconstructible corruption). This is because we assume a deterministic rule for $X \to Y$ but a randomized process for $X, Y \to O$, hence the latter one is capable to encode the deterministic rule even when $\dim(\text{range}(T))$ is smaller than $\dim(Y)$.

### 5.2 Separation by Total Variation

In this subsection, we introduce a way to guarantee separation by controlling the KL-divergence using total variation distance, which is done via the well-known Pinsker’s inequality [31]:

**Lemma 5.4** (Pinsker’s inequality, proposed in [23], see [31] for an introduction). If $P$ and $Q$ are two probability distributions on the same measurable space $(\Omega, \mathcal{F})$, then

$$\|P - Q\|_{TV} \leq \sqrt{\text{KL}(P \parallel Q)}/2$$

where $\| \cdot \|_{TV}$ is the total variance distance: $\|P - Q\|_{TV} \overset{\text{def}}{=} \sup_{A \in \mathcal{F}} |P(A) - Q(A)|$. Moreover, if $\Omega$ is countable (in our case, $\Omega = O$ is finite and hence, then the total variance distance is equivalent to the $L^1$-distance: $\|P - Q\|_{TV} = \frac{1}{2} \|P - Q\|_1$.

This inequality implies we can ensure separation by controlling the $L^1$-distance. To show a concrete example, we introduce the concentration condition. The intuition behind concentration is that the information of different labels in $Y$ is concentrated in relatively different annotations $O$. Formally:

**Proposition 5.5** (Concentration). A sufficient condition for (3) is that for every $1 \leq i \leq c$, there exists a set $S_i \subset O$ (we call them concentration sets) such that

$$\gamma_C \overset{\text{def}}{=} \inf_{(i,j,x,T) : T \in T, p(x) > 0, j \neq i} \mathbb{P}_T(O \in S_i | x, y_i) - \mathbb{P}_T(O \in S_j | x, y_i) > 0$$

(4)

where $\mathbb{P}_T(\cdot)$ is the conditional probability defined by transition $T$. Under this condition, we can relate identifiability and separation degree by $\gamma \geq \gamma_C \geq 2 \gamma^2$. Since a condition imposed on all $T \in T$ can be regarded as an assumption imposed on the true transition $T_0$, condition (4) can be rewritten as:

$$\gamma_C = \inf_{(i,j,x) : p(x,y_i) > 0, i \neq j} \mathbb{P}(O \in S_i | x, y_i) - \mathbb{P}(O \in S_j | x, y_i) > 0$$

(5)

Also, in this case, one can also ensure learnability by the ERM which minimizes the following transition-independent annotation loss

$$\ell_C(h(x), T, (x, o)) \overset{\text{def}}{=} \mathbb{1}\{o \notin S_{h(x)}\}$$

(6)

For this annotation loss, we can bound the identifiability level by $\eta \geq \gamma_C$.

**Example 5.6** (Superset with Noise). For superset with noise problem where $O$ is a random subset of $Y$ and $O = 2^Y$, let $S_i = \{o : y_i \in o\} \subset O$, the conditional (5) becomes

$$\gamma_C = \inf_{p(x,y_i) > 0, i \neq j} \mathbb{P}(y_i \in O | x, y_i) - \mathbb{P}(y_j \in O | x, y_i) > 0$$

(7)

This generalizes the small ambiguity condition proposed in [11, 17], which assumes $\mathbb{P}(y_i \in O | x, y_i) = 1$ (i.e., the gold label always lies in the superset). [11, 17] also defines a superset loss, which is the special case of (6). We extend the discussion to allow the presence of noise.
The following example can be regarded as a special case of Example 5.6.

**Example 5.7 (Label Noise).** For noisy problem where \( O = Y \), let \( S_j = \{ y_i \} \), condition (4) becomes
\[
\gamma_c = \inf_{p(x,y_i) > 0, i \neq j} P(O = y_i | x, y_i) - P(O = y_j | x, y_i) > 0
\]
(8)

This generalizes the Massart noise condition [18] of binary classification, which assumes the noise rate is lower bounded by \( 1/2 \) minus a constant. We extend the discussion to multiclass case.

Also, notice that (6) is simply the zero-one loss for \( O \), which means learnability can still be guaranteed if one ignores the noisy process and learns \( O \) as clean label. This partly explains the empirical study in [26], which tests the robustness of neural networks (without additional denoising process) to noise in annotations. [26] proposes a parameter called \( \delta \)-degree which is similar to \( \gamma_c \) and observes that the performance of the network decreases as \( \delta \) decreases, as our learning bound (2) suggests.

We can further generalize proposition 5.5 by encoding functional prior information of the transition:

**Proposition 5.8 (Evidence).** A sufficient condition for (3) is that there exists Lipschitz (with respect to \( L^1 \)-norm) functions \( \Phi_{ij} : \mathbb{R}^c \to \mathbb{R}, 1 \leq i, j \leq c \) (we call them evidence) with Lipschitz constants \( L_{ij} \) such that
\[
\gamma_{ij} \defeq \inf_{T \in T, p(x,y_i) > 0, y_j \in H(x)} \Phi_{ij}(\langle T(x) \rangle_i) - \sup_{T \in T, p(x,y_i) > 0, y_j \in H(x)} \Phi_{ij}(\langle T(x) \rangle_j) > 0
\]
(9)

In this case, the separation degree can be bounded by \( \gamma \geq 1/2 \min_{i \neq j} (\gamma_{ij}/L_{ij})^2 \).

In particular, the dot product with a fixed vector \( \Phi_u(t) = \langle u, t \rangle \) (\( \langle \cdot, \cdot \rangle \) is the dot product) is Lipschitz with Lipschitz constant \( L_u \leq \|u\|_{\infty} \). As an example, given sets \( S_i \subseteq O \) \((1 \leq i \leq c)\), letting \( \Phi_{ij}(\cdot) = \langle \sum_{k \in S_i} e_k - \sum_{k \in S_j} e_k, \cdot \rangle \) recovers proposition 5.5, where \( e_k \) is the \( k \)th standard unit basis vector of \( \mathbb{R}^c \). Another example is given in Example 5.12.

5.3 Separation by Joint Supervision

When a weak supervision signal cannot ensure learnability individually, it needs to be used with other forms of annotations together to supervise the learning task. Our goal in this subsection is to provide a way to describe the effect of using multiple sources of annotations jointly. We will show that joint supervision can improve, preserve or even damage the separation.

First, we formulate the problem of joint supervision. For simplicity, we only consider the case that we have two sources of annotations \( O_1, O_2 \), and the general case can be discussed in a similar way. For each \( O_k, k \in \{ 1, 2 \} \), denote its annotation space as \( O_k \), its transition as \( T_k(x) \) and its transition classes as \( T_k \). We focus on the scenario that for each instance \( x \), there is only one type of annotation. Then the joint annotation space is \( O = O_1 \cup O_2 \). We model the annotation type \( 1 \{ O = O_k \} \) as a random variable that is independent with \( X \) and all the \( O_k \), and the probability \( P(O = O_1) = \lambda \) is known to the learner. Then the joint annotation is defined as: \( O = 1 \{ O = O_1 \} O_1 + 1 \{ O = O_2 \} O_2 \).

Next, we quantify the supervision power of an annotation if separation is not guaranteed via a local version of the separation (degree):

**Definition 5.9 (Pairwise Separation).** Define the separation degree of \( y_i \) to \( y_j \) as
\[
\gamma_{i \to j} \defeq \inf_{x : p(x,y_i) > 0, y_j \in H(x)} KL(D_i(x) \parallel D_j(x))
\]
(10)

We say the labels \( y_i \) is separated from \( y_j \) if \( \gamma_{i \to j} > 0 \). The separation degree \( \gamma = \min_{i \neq j} \gamma_{i \to j} \).

This definition gives a probabilistic formulation of the intuition that a (weak) supervision signal can help distinguish certain pairs of labels. For example, a noisy annotation for multiclass classification may break the condition (8) due to a large noise rate for certain labels, but it can still provide information to separate other labels if (8) is satisfied for any other pairs of \((i, j)\).

When there are no additional constraints on the joint transition, one can construct the joint transition simply by combining the candidate transitions in \( T_1, T_2 \). For example, the induced distribution family by \( y_i \) of joint supervision can be naturally constructed by
\[
D_i(x) = \{ \lambda D_1 + (1 - \lambda) D_2 : D_1 \in D_{i1}(x), D_2 \in D_{i2}(x) \}
\]
(11)
where \( D_{i1} \) and \( D_{i2} \) are the induced distribution family by \( y_i \) of \( O_1 \) and \( O_2 \). In this case, we present the following result to characterize the learnability under joint supervision \( O \):
Proposition 5.10 (No Free Separation). Suppose the separation degrees of \( y_i \) to \( y_j \) of \( O_1 \) and \( O_2 \) are \( \gamma_{i \rightarrow j_1} \) and \( \gamma_{i \rightarrow j_2} \) respectively. Then, if the joint transition class is constructed as (11), then the separation degrees of \( y_i \) to \( y_j \) for the joint supervision satisfies:

\[
\gamma_{i \rightarrow j} \leq \lambda \gamma_{i \rightarrow j_1} + (1 - \lambda) \gamma_{i \rightarrow j_2}
\]

Also, if \( O_1 \cap O_2 = \emptyset \), then the two sides are equal. As a consequence, a necessary condition of that \( y_i \) is separated from \( y_j \) by the joint signal \( O \) is that \( y_i \) must be separated from \( y_j \) by one of \( O_1, O_2 \).

The condition \( O_1 \cap O_2 = \emptyset \) means that the learner distinguishes different annotations. For example, in a crowdsourcing setting, we have two annotators and each provides a noisy annotation, then \( O_1, O_2 = \mathcal{Y} \). But as long as the learner distinguishes the annotations of the two annotators, we can still write \( O_1 \cap O_2 = \emptyset \). Without this condition, even if both \( \gamma_{i \rightarrow j_1}, \gamma_{i \rightarrow j_2} > 0 \), we can still have \( \gamma_{i \rightarrow j} = 0 \). See the supplementary material for an example. This explains the empirical study of [13], which observes that for crowdsourcing, the model performance improves if annotator identifiers are input as features. However, one should note that the tradeoff is the complexity: distinguishing different annotations will in general require more parameters to model the joint transition.

Remark 5.11. Proposition 5.10 shows that without constraints, the joint supervision does not create new separation, however, it can preserve the separation between labels by the original supervision signals. So in this view, the weak supervision signal can be regarded as a “building block” for the separation degrees of \( y_i \) to \( y_j \).

If there do exist constraint about the two transition classes, Proposition 5.10 no longer holds and joint supervision may create new separation. To illustrate, consider the following artificial example:

Example 5.12 (Learning from Difference). Given a binary classification problem where \( \mathcal{Y} = \{±1\} \). Suppose we have two annotators \( O_1 \) and \( O_2 \) and each provides a noisy annotation with an unknown, uniform, instance-independent noise, i.e., \( \eta_1 \overset{\text{def}}{=} \mathbb{P}(O_1 \neq y|x, y = -1) = \mathbb{P}(O_1 \neq y|x, y = +1) \), \( \eta_2 \overset{\text{def}}{=} \mathbb{P}(O_2 \neq y|x, y = -1) = \mathbb{P}(O_2 \neq y|x, y = +1) \), where \( \eta_1, \eta_2 \) do not depend on \( x \).

Now, suppose it is known that the first annotator provides a better quality of annotation, i.e., there is a \( \gamma \in \mathbb{R} \) (known to the learner) such that \( \eta_1 - \eta_2 \leq \gamma < 0 \). Then, the joint transition is modeled as:

\[
T = \begin{bmatrix}
\lambda(1 - \eta_1) & \lambda \eta_1 \\
\lambda \eta_1 & \lambda(1 - \eta_1)
\end{bmatrix}
\begin{bmatrix}
\lambda(1 - \eta_1)(1 - \eta_2) & (1 - \lambda)\eta_2 \\
(1 - \lambda)\eta_2 & \lambda(1 - \eta_1)(1 - \eta_2)
\end{bmatrix}
= \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}
\]

To apply proposition 5.8, define \( \Phi(\cdot) = \langle \hat{e}_1/\lambda - \hat{e}_2/(1 - \lambda), \cdot \rangle \), then \( \Phi(D_1) = \eta_2 - \eta_1 \geq \gamma \) and \( \Phi(D_2) = \eta_1 - \eta_2 \leq -\gamma \). So by proposition 5.8, the classification problem is learnable. Notice that without joint supervision, separation is not guaranteed since we do not restrict \( \eta_1 \) or \( \eta_2 \) individually.

This example it is necessary to model possible constraints between different supervision sources, which help to reduce the size of the joint transition class and may improve the separation degree.

6 Conclusion and Future Work

In this paper, we provide a unified framework for analyzing the learnability of multiclass classification with indirect supervision. Our theory builds upon two key components: (i) The construction of the induced hypothesis class and its complexity analysis, which allows us to indirectly supervise the learning by minimizing the annotation risk. (ii) A formal description of the prior knowledge about the transition and its encoding in the learning condition and bound, which allows us to bound the classification error by the annotation risk.

The notion of separation depends on the annotation loss being used. The KL-divergence may be replaced by other statistical distances, as long as the distance can induce a loss function. However, the idea behind separation is invariant: the prior knowledge needs to be strong enough to distinguish different labels via the observable. Moreover, theorem 5.2 shows that separation is a sufficient and almost necessary condition, and the later examples show separation is also practically useful and can easily produce learnability conditions. Therefore, we believe the the concepts introduced are general, and that our analysis tools can be applied in many other supervision scenarios.

One limitation of our work is that the definition of learnability requires us to handle every possible \( D_X \), and the consequence is that we need to ensure separation at every \( x \in X \). In future work, we may try to relax the learnability conditions by encoding prior knowledge of \( D_X \), which can be obtained from unlabeled data. Another thing to explore is to extend the discussion to the agnostic case as well as the case where \( T_0 \notin T \).
Broader Impact

Our work mostly focuses on theoretical aspects of learning, however, it provides better understanding and thus can suggest new machine learning scenarios and algorithms for learning from indirect observations; this addresses a key challenge to machine learning today, and will help machine learning researchers to reduce the cost of and need for labeled data. Our theory may have positive and negative impact on the privacy protection of sensitive data. On one hand, the theory suggests that one can alter the forms of data (via a probabilistic transition) to ensure privacy while keeping its usefulness (learnability). On the other hand, it might be possible for an attacker to recover sensitive information about the data indirectly through a related dataset.

7 Appendix: Proofs

7.1 Proof of Theorem 4.2

We need several intermediate results to prove this. First, we introduce the definition of the averaged Rademacher complexity.

Definition 7.1 (Averaged Rademacher Complexity [4]). The averaged Rademacher complexity [4] of \( \ell_{O} \circ T \circ H \) with respect to \( m \) samples is defined as

\[
\mathcal{R}_{m}(\ell_{O} \circ T \circ H) \overset{\text{def}}{=} \mathbb{E}_{\epsilon,x,o} \left[ \frac{1}{m} \sup_{h \in H, T} \left( \sum_{i=1}^{m} \epsilon(i) \ell_{O}(h(x(i)), T, (x(i), o(i))) \right) \right] \tag{12}
\]

where \( \epsilon \overset{\text{ind}}{\sim} \text{Uniform}\{-1, +1\} \) are the so-called Rademacher random variables and the expectation is taken over \( m \) i.i.d. samples of \( \epsilon, x, o \).

The first lemma bounds the empirical risk via the averaged Rademacher complexity.

Lemma 7.2 (Adapted from the proof of Theorem 26.5 in [28]). In this lemma and its proof, for convenience, we let the ERM algorithm return the induced hypothesis in \( H \circ T \) (rather than the base hypothesis only).

Given any \( \delta \in (0, 1) \), with probability of at least \( 1 - \delta \), we have

\[
R_{O}(\text{ERM}(S^{(m)})) - \inf_{h,T} R_{O}(T \circ h) \leq 2\mathcal{R}_{m}(\ell_{O} \circ T \circ H) + 2b \sqrt{\frac{2 \log(4/\delta)}{m}}
\]

Proof. Let \( T^* \circ h^* \) be any induced hypothesis in \( T \times H \). Given dataset \( S^{(m)} \), we have,

\[
\begin{align*}
R_{O}(\text{ERM}(S^{(m)})) &- R_{O}(T^* \circ h^*) \\
= & R_{O}(\text{ERM}(S^{(m)})) - \hat{R}_{O}(\text{ERM}(S^{(m)})) + \hat{R}_{O}(\text{ERM}(S^{(m)})) - \hat{R}_{O}(T^* \circ h^*) \\
\leq & R_{O}(\text{ERM}(S^{(m)})) - \hat{R}_{O}(\text{ERM}(S^{(m)})) + \hat{R}_{O}(T^* \circ h^*) - R_{O}(T^* \circ h^*) \\
\end{align*}
\]

By Theorem 26.5 (i) of [28], we have that with probability of at least \( 1 - \delta/2 \),

\[
R_{O}(\text{ERM}(S^{(m)})) - \hat{R}_{O}(\text{ERM}(S^{(m)})) \leq 2\mathcal{R}_{m}^{'}(\ell_{O} \circ T \circ H) + b \sqrt{\frac{2 \log(4/\delta)}{m}}
\]

where \( \mathcal{R}_{m}^{'}(\ell_{O} \circ T \circ H) \) is defined slightly differently in [28] as:

\[
\mathcal{R}_{m}^{'}(\ell_{O} \circ T \circ H) \overset{\text{def}}{=} \mathbb{E}_{\epsilon,x,o} \left[ \frac{1}{m} \sup_{h \in H, T} \left( \sum_{i=1}^{m} \epsilon(i) \ell_{O}(h(x(i)), T, (x(i), o(i))) \right) \right] \tag{13}
\]

It can be seen that \( \mathcal{R}_{m}^{'}(\ell_{O} \circ T \circ H) \leq \mathcal{R}_{m}(\ell_{O} \circ T \circ H) \) since the two quantities only differ by the absolute value. Hence

\[
R_{O}(\text{ERM}(S^{(m)})) - \hat{R}_{O}(\text{ERM}(S^{(m)})) \leq 2\mathcal{R}_{m}(\ell_{O} \circ T \circ H) + b \sqrt{\frac{2 \log(4/\delta)}{m}}
\]
By Hoeffding’s inequality, we have that with probability of at least $1 - \delta/2$,

$$\tilde{R}_\Omega(T^* \circ h^*) - R_\Omega(T^* \circ h^*) \leq b\sqrt{\frac{\log(4/\delta)}{2m}}$$

Combining the inequalities, we have that with probability of at least $1 - \delta$,

$$R_\Omega(\text{ERM}(S^{(m)})) - R_\Omega(T^* \circ h^*) \leq 2\mathbb{R}_m(\ell_\Omega \circ \mathcal{T} \circ \mathcal{H}) + 2b\sqrt{\frac{2\log(4/\delta)}{m}}$$

Since the above inequality holds for any $T^* \circ h^* \in \mathcal{T} \times \mathcal{H}$, so taking infimum for $T^* \circ h^*$ gives the desired result.

The second lemma bounds the averaged Rademacher complexity via the weak VC-major, which is provided in [3].

**Lemma 7.3** (Adapted from the Theorem 2.1 in [3]). Suppose the weak VC-major dimension of $\ell_\Omega \circ \mathcal{T} \circ \mathcal{H}$ is $d$, then,

$$m\mathbb{R}_m(\ell_\Omega \circ \mathcal{T} \circ \mathcal{H}) \leq \sigma \log \left(\frac{eb}{\sigma}\right) \sqrt{2m\Gamma_m(d)} + 4b\Gamma_n(d)$$

where $e$ is the base of the natural logarithm and

$$\sigma \overset{\text{def}}{=} \sup_{h \in \mathcal{H}, T \in \mathcal{T}} \sqrt{\mathbb{E}_{x,o}[\ell_\Omega^2(T, (x, h(x), o))]} \in (0, b]$$

**Proof.** The proof of the Theorem 2.1 in [3] is long and is presented in the section 3 of [3]. Here we only point out how to use Theorem 2.1 of [3] (equation (2.8) of the paper) to derive our lemma.

First, the Theorem 2.1 of [3] bounds an empirical process (denoted as $\mathbb{E}[\mathcal{Z}(\mathcal{F})]$) in the paper, where $\mathcal{F}$ is a function class and here we let $\mathcal{F} = \ell_\Omega \circ \mathcal{T} \circ \mathcal{H}$ rather than the averaged Rademacher complexity (denoted as $\mathbb{E}[\mathcal{Z}(\mathcal{F})]$ in the paper). However, the proof of Theorem 2.1 of [3] aims to bound the averaged Rademacher complexity $\mathbb{E}[\mathcal{Z}(\mathcal{F})]$ and then uses the relation $\mathbb{E}[\mathcal{Z}(\mathcal{F})] \leq 2\mathbb{E}[\mathcal{Z}(\mathcal{F})]$ (Lemma 2.1 of [3]) to obtain the bound for $\mathbb{E}[\mathcal{Z}(\mathcal{F})]$. Therefore, the proof of the Theorem 2.1 in [3] tells:

$$\mathbb{E}[\mathcal{Z}(\mathcal{F})] = m\mathbb{R}_m(\ell_\Omega \circ \mathcal{T} \circ \mathcal{H}) \leq \sigma \log \left(\frac{e}{\sigma}\right) \sqrt{2m\Gamma_m(d)} + 4\Gamma_n(d)$$

Second, in the Theorem 2.1 of [3], it is assumed that the functions in $\mathcal{F}$ is bounded in the interval $[0, 1]$. Hence, we need scale the annotation loss to $\ell_\Omega/b$ in order to use the theorem (i.e., let $f = \ell_\Omega/b$ in the definition of $\mathbb{E}[\mathcal{Z}(\mathcal{F})]$), i.e., equation (1.2) of [3]). Also, in this case, the supreme of variance (15) is scaled to $\sigma/b$. So, the inequality (16) is rewritten as:

$$\mathbb{E}[\mathcal{Z}(\mathcal{F})] = \frac{m}{b}\mathbb{R}_m(\ell_\Omega \circ \mathcal{T} \circ \mathcal{H}) \leq \frac{\sigma}{b} \log \left(\frac{e}{\sigma/b}\right) \sqrt{2m\Gamma_m(d)} + 4\Gamma_n(d)$$

Rearranging the inequality gives the desired result.

Now, we are able to give the proof of the original theorem:

**Proof.** By [C2], we have

$$\inf_{h,T} R_\Omega(T \circ h) = \inf_{T} R_\Omega(T \circ h_0)$$

Therefore, by lemma 7.2, we have that with probability of at least $1 - \delta$,

$$R_\Omega(\text{ERM}(S^{(m)})) - \inf_{T} R_\Omega(T \circ h_0) \leq 2\mathbb{R}_m(\ell_\Omega \circ \mathcal{T} \circ \mathcal{H}) + 2b\sqrt{\frac{2\log(4/\delta)}{m}}$$

By [C3], we have that with probability of at least $1 - \delta$,

$$R(\text{ERM}(S^{(m)})) \leq \frac{1}{\eta} \left(2\mathbb{R}(\ell_\Omega \circ \mathcal{T} \circ \mathcal{H}) + 2b\sqrt{\frac{2\log(4/\delta)}{m}}\right)$$

(18)
By [C1] and lemma 7.3, we bound the Rademacher Complexity by

$$R_m(\ell_\mathcal{O} \circ T \circ \mathcal{H}) \leq \sigma \log \left( \frac{eb}{\sigma} \right) \sqrt{\frac{2\Gamma_m(d)}{m}} + 4 \frac{b}{m} \Gamma_n(d)$$

(19)

Now the result follows by combining (18) and (19).

7.2 Proof of Proposition 4.3

Proof. First, we translate weak-VC major to the language of standard VC-dimension [33]: For a fixed $u \in \mathbb{R}$ and every $h \in \mathcal{H}, T \in \mathcal{T}$, we define an binary classifier: $f_{h,T,u}(x,o) = \mathbb{1} \{\ell_\mathcal{O}(h(x),T,(x,o)) > u\}$ and denote $F_u := \{f_{h,T,u} : h \in \mathcal{H}, T \in \mathcal{T}\}$ as the set of such classifiers. Then $C_u$ shatters a set in $\mathcal{X} \times \mathcal{O}$ if and only if $\ell_\mathcal{O} \circ T \circ \mathcal{H}$ is weak VC-major with dimension $d$ if $d = \max_{u \in \mathbb{R}} VC(F_u) < \infty$, where $VC(\cdot)$ is the VC dimension for hypothesis class of binary classifiers.

Let $M$ be the maximum number of distinct ways to classify $d$ points in $\mathcal{X}$ by $\mathcal{H}$. Then for $d$ points in $\mathcal{X} \times \mathcal{O}$, suppose there are at most $M$ ways to assign multi-class labels to each point. By Natarajan’s lemma [20] of multiclass classification, we have

$$M \leq d^{d_H} e^{2d_H}$$

(20)

For each way of assignment, it forms a set of $d$ points in $\mathcal{X} \times \mathcal{Y} \times \mathcal{O}$, and for these $d$ points, by Sauer-Shelah lemma, there are at most

$$\sum_{i=0}^{d_T} \binom{d}{i} \leq \left( \frac{ed}{d_T} \right)^{d_T}$$

ways to classify if $\ell_\mathcal{O}(\hat{y},T,(x,o)) > u$ by $\mathcal{T}$, so in total we have

$$2^d \leq M \sum_{i=0}^{d_T} \binom{d}{i} \leq M \left( \frac{ed}{d_T} \right)^{d_T}$$

where $e$ is the base of the natural logarithm. Therefore, $M \geq 2^d (d_T/ed)^{d_T}$. Then, by (20)

$$d^{d_H} e^{2d_H} \geq M \geq 2^d \left( \frac{d_T}{ed} \right)^{d_T}$$

Taking logarithm in both side, we have

$$d_H \log d + 2d_H \log e \geq d \log 2 + d_T (\log d_T - \log d - 1)$$

Rearrange the inequality,

$$d \log 2 + d_T (\log d_T - 1) \leq (d_H + d_T) \log d + 2d_H \log e$$

$$\leq (d_H + d_T) \left( \frac{d}{6(d_H + d_T)} + \log(6(d_H + d_T)) \right) + 2d_H \log e$$

$$\leq d/6 + (d_H + d_T) (\log(6(d_H + d_T)) - 1) + 2d_H \log e$$

where the second step follows from the first-order Taylor series expansion of logarithm function at the point $6(d_H + d_T)$. Therefore,

$$d \leq \frac{(d_H + d_T) (\log(6(d_H + d_T))) + 2d_H \log e - d_T (\log(d_T))}{\log 2 - 1/6}$$

$$\leq 2 \left( (d_H + d_T) (\log(6(d_H + d_T))) + 2d_H \log e \right)$$

where the last step follows from $\log 2 - 1/6 < 1/2$. 

\qed
7.3 Proof of Example 4.4

The first two conclusions of Example 4.4 are straightforward. We prove the last statement:

**Proof.** Given $2p + 3$ points in $\mathcal{X} \times \mathcal{Y} \times \mathcal{O}$, without loss of generality, suppose there are at least $p + 2$ points such that $o \neq y$. For these points, the value of annotation loss only depends on $x$, i.e., $\log(S(w^T x))$. For any $u \in \mathbb{R}$, the classifier

$$f_{h,T,u}(x,o) = 1\{\ell_\mathcal{O}(h(x), T, (x, o)) > u\} = 1\{\log(S(w^T x)) < -u\}$$

is a linear classifier with decision boundary $w^T x = e^{-u}$. Since the VC dimension of hyperplanes of dimension $p$ is $p + 1$, we know these linear classifiers cannot classify $p + 2$ points arbitrarily. Therefore, the original $2p + 3$ points cannot be classified arbitrarily, and we have $d_T \leq 2p + 2$. □

7.4 Proof of Theorem 5.2

**Proof.** Denote the cross-entropy of two distributions $D_1$ and $D_2$ as $H(D_1, D_2)$ and the entropy of a distribution $D$ as $H(D)$. With cross-entropy loss, for a fixed $x \in \mathcal{X}$ we have that

$$\mathbb{E}_o[\ell_\mathcal{O}(h(x), T, (x, o))] - \mathbb{E}_o[\ell_\mathcal{O}(h_0(x), T_0, (x, o))]$$

$$= H(T_0(x))_{h_0(x)} - H((T_0(x))_{h_0(x)}), (T_0(x))_{h_0(x)})$$

$$= KL((T_0(x))_{h_0(x)} \| (T(x))_{h_0(x)})$$

If $h(x) \neq h_0(x)$, then by the separation condition we have that

$$KL((T_0(x))_{h_0(x)} \| (T(x))_{h_0(x)}) \geq \gamma$$

Also, if $h(x) = h_0(x)$, we have

$$KL((T_0(x))_{h_0(x)} \| (T(x))_{h_0(x)}) = KL((T_0(x))_{h_0(x)} \| (T_0(x))_{h_0(x)}) = 0$$

Therefore, for a fixed $h \in \mathcal{H}$

$$R_\mathcal{O}(h \circ T) - \inf_{T \in \mathcal{T}} R_\mathcal{O}(h \circ T)$$

$$= \mathbb{P}(h(x) \neq h_0(x)) \inf_{T,h(x) \neq h_0(x)} \mathbb{E}_o[\ell_\mathcal{O}(h(x), T, (x, o)) - \mathbb{E}_o[\ell_\mathcal{O}(h_0(x), T, (x, o)))]$$

$$\geq \mathbb{P}(h(x) \neq h_0(x)) \inf_{T,h(x) \neq h_0(x)} \mathbb{E}_o[\ell_\mathcal{O}(h(x), T, (x, o)) - \mathbb{E}_o[\ell_\mathcal{O}(h_0(x), T, (x, o)))]$$

$$\geq \mathbb{P}(h(x) \neq h_0(x)) \inf_{T,h(x) \neq h_0(x)} KL((T_0(x))_{h_0(x)} \| (T(x))_{h_0(x)})$$

$$\geq \mathbb{P}(h(x) \neq h_0(x)) \gamma \geq 0$$

This shows the consistency condition [C2]. Also, if $\mathbb{P}(h(x) \neq h_0(x)) > 0$, notice that $\mathbb{P}(h(x) \neq h_0(x)) = R(h)$, we have

$$\eta = \inf_{R(h)>0} \frac{R_\mathcal{O}(h \circ T) - \inf_{T \in \mathcal{T}} R_\mathcal{O}(h \circ T)}{R(h)} \geq \gamma \frac{R(h)}{R(h)} = \gamma > 0$$

This shows the identifiability condition [C3].

Moreover, if the condition (3) is not satisfied, by definition we have

$$\gamma = \inf_{(x,i,j):p(x,y_i) > 0,j \neq i,y_j \in \mathcal{H}(x)} KL(D_i(x) \| D_j(x))$$

$$= \inf_{(x,i,j):p(x,y_i) > 0,j \neq i,y_j \in \mathcal{H}(x),D_i \in \mathcal{D}_i(x),D_j \in \mathcal{D}_j(x)} KL(D_i \| D_j)$$

$$= 0$$

Then, by the definition of infimum, we have for any $k \in \mathbb{N}^+$, there exists a 5-tuple

$$\left(\left.x(k), y_i(k), y_j(k), D_i(k)(x(k)), D_j(k)(x(k))\right) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{D}_i \times \mathcal{D}_j\right)$$

such that
\begin{itemize}
  \item $p(x, y^{(k)}_i) > 0$
  \item $y^{(k)}_i \neq y^{(k)}_j$
  \item There is a $h_\in \mathcal{H}$ with $h_\in(x^{(k)}) = y^{(k)}_j$
  \item $\text{KL}(D^{(k)}_i(x^{(k)}) \parallel D^{(k)}_j(x^{(k)})) < \frac{1}{k}$
\end{itemize}

Now, let $D^{(k)}_X$ be the point mass distribution with probability one to be $x^{(k)}$, i.e., $D^{(k)}_X(\{x^{(k)}\}) = 1$. Then, we have $h_0(x^{(k)}) = y^{(k)}_i$ since $h_0$ has zero classification error. Also, let $T_0^{(k)} \in \mathcal{T}$ be such that its $i^{th}$ row is $D^{(k)}_i$, and $T^{(k)}_{\in \mathcal{T}}$ be such that its $j^{th}$ row is $D^{(k)}_j$. We have
\[
\eta^{(k)} = \inf_{h \in \mathcal{H} : R(h) > 0} \frac{R_\mathcal{O}(h \circ T) - \inf_{T \in \mathcal{T}} R_\mathcal{O}(h_0 \circ T)}{R(h)}
= \inf_{h \in \mathcal{H} : R(h) > 0} \frac{R_\mathcal{O}(h \circ T) - R_\mathcal{O}(h_0 \circ T^{(k)})}{R(h)}
\leq R_\mathcal{O}(h_\in \circ T^{(k)}) - R_\mathcal{O}(h_0 \circ T^{(k)})
\leq \text{KL}(D^{(k)}_i(x^{(k)}) \parallel D^{(k)}_j(x^{(k)}))
\leq \frac{1}{k}
\]

Let $k \to \infty$ and the desired result follows.

\section*{7.5 Proof of Proposition 5.5}

Proof. First, for any $(x, y_i) \in \mathcal{X} \times \mathcal{Y}$ with $p(x, y_i) > 0$ and $D_i \in \mathcal{D}_i(x)$, $D_j \in \mathcal{D}_j(x)$, by Pinsker’s inequality, we have
\[
\text{KL}(D_i \parallel D_j) \geq 2\|D_i - D_j\|_1^2 = \frac{1}{2}\|D_i - D_j\|_1^2
= \frac{1}{2} \left( \sum_{o \in \mathcal{O}} |D_i(o) - D_j(o)| \right)^2
\geq \frac{1}{2} \left( |D_i(S_i - S_j) - D_j(S_i - S_j)| + |D_i(S_j - S_i) - D_j(S_j - S_i)| \right)^2
\geq \frac{1}{2} \left( |D_i(S_i - S_j) - D_j(S_i - S_j)| + |D_i(S_j - S_i) - D_j(S_j - S_i)| \right)^2
\geq \frac{1}{2} \left( |D_i(S_i - S_j) - D_j(S_i - S_j)| + |D_i(S_j - S_i) - D_j(S_j - S_i)| \right)^2
\geq \frac{1}{2} \left( 2\gamma C \right)^2 = 2\gamma C
\]

where $D_i(\cdot)$ is the probability measure over $\mathcal{O}$ defined by $D_i$, and $S_i - S_j$ is the set subtraction: $S_i - S_j \overset{\text{def}}{=} \{ o : o \in S_i \land o \notin S_j \} \subset \mathcal{O}$. Taking infimum on both sides of the inequality gives the first result. Another proof for this result can be found in the proof of Proposition 5.8.

Next, consider the annotation loss $\ell_\mathcal{O}(h(x), T, (x, o)) = 1 \{ o \notin S_{h(x)} \}$ and its ERM. Then we have
\[
\mathbb{E}_{x, o}[\ell_\mathcal{O}(h(x), T, (x, o))] - \mathbb{E}_{x, o}[\ell_\mathcal{O}(h_0(x), T, (x, o))]
= p(o \notin S_{h(x)}) - p(o \notin S_{h_0(x)})
\geq p(h(x) \neq h_0(x)) \inf_{x : h(x) \neq h_0(x)} (p(o \in S_{h_0(x)}) - p(o \in S_{h(x)}))
\geq p(h(x) \neq h_0(x)) \gamma C = R(h) \gamma C
\]
Therefore,
\[ \eta = \inf_{R(h) > 0} \frac{R_\mathcal{O}(h \circ T) - \inf_{T \in \mathcal{T}} R_\mathcal{O}(h_0 \circ T)}{R(h)} \geq \frac{\gamma_C R(h)}{R(h)} = \gamma_C > 0 \]
as claimed. \hfill \Box

### 7.6 Proof of Proposition 5.8

**Proof.** Since \( \Phi_{ij} \) is Lipschitz, then for any \( a, b \in \mathbb{R}^r \), we have
\[ |\Phi_{ij}(a) - \Phi_{ij}(b)| \leq L_{ij} \|a - b\|_1 \]
Hence, given \( (x, i, j) \) such that \( p(x, y_i) > 0, j \neq i \) and \( y_j \in \mathcal{H}(x) \), then for any \( D_i \in \mathcal{D}_i(x) \) and \( D_j \in \mathcal{D}_j(x) \), by Lipschitz property we have
\[ \|D_i - D_j\|_1 \geq \frac{1}{L_{ij}} |\Phi_{ij}(D_1) - \Phi_{ij}(D_2)| \geq \frac{\gamma_{ij}}{L_{ij}} \]
Therefore, by Pinsker’s inequality, we have
\[ \text{KL}(D_i \parallel D_j) \geq \frac{1}{2} \|D_i - D_j\|_2^2 \geq \frac{1}{2} \left( \frac{\gamma_{ij}}{L_{ij}} \right)^2 \geq \frac{1}{2} \min_{i \neq j} \left( \frac{\gamma_{ij}}{L_{ij}} \right)^2 \]
Taking infimum on the left hand side of the inequality gives the desired result.

In particular, if \( \Phi \) represents the inner product with a fixed vector \( u \), i.e., \( \Phi(a) = \langle u, a \rangle \), then \( \Phi \) is Lipschitz since for any \( a, b \in \mathbb{R}^r \), by the Hölder’s inequality, we have
\[ |\Phi(a) - \Phi(b)| = |\langle u, a - b \rangle| \leq \|u\|_\infty \|a - b\|_1 \]
Also, we can bound the Lipschitz constant of \( \Phi \) by \( L \leq \|u\|_\infty \).

To recover the concentration condition, given sets \( S_i \subset \mathcal{O}(1 \leq i \leq c) \), for any \( i \neq j \), let
\[ \Phi_{ij}(a) = \left\langle \sum_{k : a_k \in S_i} \hat{e}_k - \sum_{k : a_k \in S_j} \hat{e}_k, a \right\rangle \]
Then \( \Phi_{ij}((T(x))_i) = \mathbb{P}_T(O \in S_i | x, y_i) - \mathbb{P}_T(O \in S_j | x, y_j) \) and \( \Phi_{ij}((T(x))_j) = \mathbb{P}_T(O \in S_j | x, y_j) - \mathbb{P}_T(O \in S_i | x, y_i) \). Then, the concentration condition (5) will imply that
\[ \inf_{T \in \mathcal{T}, p(x, y_i) > 0, y_j \in \mathcal{H}(x)} \Phi_{ij}((T(x))_i) - \sup_{T \in \mathcal{T}, p(x, y_i) > 0, y_j \in \mathcal{H}(x)} \Phi_{ij}((T(x))_j) \geq 2 \gamma_C > 0 \]
Moreover, since \( \left\| \sum_{k : a_k \in S_i} \hat{e}_k - \sum_{k : a_k \in S_j} \hat{e}_k \right\|_\infty = 1 \), the separation degree can be bounded by \( \gamma \geq \frac{1}{2} \min_{i \neq j} (\gamma_{ij})^2 = 2 \gamma_C^2 \). \hfill \Box

### 7.7 Proof of Proposition 5.10

**Proof.** Given \( D_i \in \mathcal{D}_i(x) \) and \( D_j \in \mathcal{D}_j(x) \), write \( D_i = \lambda D_{i1} + (1 - \lambda) D_{i2} \) and \( D_j = \lambda D_{j1} + (1 - \lambda) D_{j2} \), where \( D_{i1} \in \mathcal{D}_{i1}(x), D_{j1} \in \mathcal{D}_{j1}(x), D_{i2} \in \mathcal{D}_{i2}(x), D_{j2} \in \mathcal{D}_{j2}(x) \). The summation \( D_i = \lambda D_{i1} + (1 - \lambda) D_{i2} \) means that we combine \( D_{i1} \) and \( D_{i2} \) as distributions over \( O \) such that \( D_i(o) = \lambda \mathbb{1}\{o \in O_1\} D_{i1}(o) + (1 - \lambda) \mathbb{1}\{o \in O_2\} D_{i2}(o) \) for any \( o \in O \).

The first result basically follows from the convexity of KL-divergence: we have
\[ \text{KL}(D_i \parallel D_j) = \text{KL}(\lambda D_{i1} + (1 - \lambda) D_{i2} \parallel \lambda D_{j1} + (1 - \lambda) D_{j2}) \leq \lambda \text{KL}(D_{i1} \parallel D_{j1}) + (1 - \lambda) \text{KL}(D_{i2} \parallel D_{j2}) \] (21)
Hence,
\[ \lambda \text{KL}(D_{i1} \parallel D_{j1}) + (1 - \lambda) \text{KL}(D_{i2} \parallel D_{j2}) \geq \inf_{x : p(x, y_i) > 0, y_j \in \mathcal{H}(x)} \text{KL}(D_i \parallel D_j) = \gamma_{i \rightarrow j} \]
Take infimum again on the left hand side of the inequality, we have
\[ \lambda \gamma_{i \rightarrow j1} + (1 - \lambda) \gamma_{i \rightarrow j2} \geq \gamma_{i \rightarrow j} \]
More over, if \( \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset \), then in (21), we have

\[
KL(\lambda D_{i1} + (1 - \lambda) D_{i2} \parallel \lambda D_{j1} + (1 - \lambda) D_{j2}) \\
\begin{align*}
&= \sum_{o \in \mathcal{O}_1} (\lambda D_{i1}(o) + (1 - \lambda) D_{i2}(o)) \log \left( \frac{\lambda D_{i1}(o) + (1 - \lambda) D_{i2}(o)}{\lambda D_{j1}(o) + (1 - \lambda) D_{j2}(o)} \right) \\
&= \sum_{o \in \mathcal{O}_1} \lambda D_{i1}(o) \log \left( \frac{\lambda D_{i1}(o)}{\lambda D_{j1}(o)} \right) + \sum_{o \in \mathcal{O}_2} (1 - \lambda) D_{i2}(o) \log \left( \frac{(1 - \lambda) D_{i2}(o)}{(1 - \lambda) D_{j2}(o)} \right) \\
&= \lambda KL(D_{i1} \parallel D_{j1}) + (1 - \lambda) KL(D_{i2} \parallel D_{j2})
\end{align*}
\]

Hence taking infimum on both sides gives \( \lambda \gamma_{i \rightarrow j1} + (1 - \lambda) \gamma_{i \rightarrow j2} = \gamma_{i \rightarrow j} \).

The above discussion shows that if \( \gamma_{i \rightarrow j} > 0 \), then one of \( \gamma_{i \rightarrow j1} \) and \( \gamma_{i \rightarrow j2} \) must be positive.

Moreover, we show by a simple example that if \( \mathcal{O}_1 \cap \mathcal{O}_2 \neq \emptyset \), then even if both \( \lambda \gamma_{i \rightarrow j1} \) and \( \gamma_{i \rightarrow j2} \) are positive, we can still have \( \lambda \gamma_{i \rightarrow j} = 0 \). Consider a binary classification \( \mathcal{Y} = \{\pm 1\} \) with two noisy annotations (crowdsourcing with two annotators) \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \). Suppose the transitions of the two annotations are known to the learner and are given by constant matrices

\[
T_1(x) = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix} \text{ and } T_2(x) = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}
\]

Then, individually, both the annotations can ensure separation. However, suppose \( \lambda = 1/2 \), then in this case, if the annotations are mixed (i.e., the learner do not distinguish the annotations of different annotators, and hence \( \mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 = \mathcal{Y} \)), then for any \( x, y \),

\[
\mathbb{P}(O = y|x, y) = \lambda \mathbb{P}(O_1 = y|x, y) + (1 - \lambda) \mathbb{P}(O_2 = y|x, y) = 1/2
\]

Here we used the condition that \( \mathbb{I}(O = O_1) \) is independent with \( X \). Now, it is not possible to learn \( Y \) from the observation of \( O \) since \( O \) is purely a random noise that is independent of \( Y \).

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