1. Introduction

The aim of this paper is to formulate a unified programme for the linearisation of certain types of nonlinear systems. This has its roots in our original method presented in [4] and [5], and for the sake of completion, it would be instructive to briefly review it here.

We start by assuming that the integral operators $P$ and $Q$ satisfy the linear system

$$\begin{align*}
\partial_t Q &= A Q + B P \\
\partial_t P &= C Q + D P,
\end{align*}$$

where $A$ and $C$ are known bounded operators, while $B$ and $D$ are known possibly unbounded operators. Assuming further that there is a solution to this system in $P$ and $Q$, where $P$ and $Q - \text{id}$ are Hilbert-Schmidt operators at least for some time $T > 0$, we introduce a third integral operator $G$ via the relation $P = G Q$, which we call the Riccati relation. Then, formally, $G$ evolves according to the Riccati-type flow

$$\partial_t G = C + DG - G(A + BG).$$

In fact, in [4] and [5], we rigorously show that there exists a unique, well-behaved solution $G$ to the Riccati relation at least for some time $T > 0$, which indeed evolves according to such a flow. Of course, these being integral operators, all of the above can be translated to corresponding equations for
the kernels of $P, Q$ and $G$. At the kernel level, our Riccati relation reads
\[ p(x, y) = \int_{-\infty}^{0} g(x, z)q(z, y) \, dz, \]
which is reminiscent of the Marchenko equation and the role it plays in the classical theory of integrable systems, for example in the Zakharov–Shabat scheme (see [28]), as well as the work of Ablowitz et al. (see [3]). In our case however, this is a Fredholm integral equation, instead of a Volterra one, and results in a PDE for $g$ with a nonlocal nonlinearity.

The purpose of all this of course is to work in the opposite direction. That is, given a PDE with a nonlocal nonlinearity which we wish to solve, we investigate whether we can fit it into the above form so that we can generate a solution by solving the corresponding linear system. This is achieved in several cases in papers [4] and [5].

Recently, we turned our attention to classically integrable systems and in [10] we extended our method in a way that allowed us to deal with the NLS and KdV equations. This was motivated by the work of Ablowitz et al. [3], Dyson [13], McKean [21] and by a series of papers by Pöppe [24, 25, 26] and Pöppe and Sattinger [27]. Of particular importance for us was the realisation by Pöppe that the solution to a soliton equation is given by some function of the Fredholm determinant of the solution to the linearised soliton equation. Also see Grudsky and Rybkin [18, 19] and Blower and Newsham [6]. Non-commutative integrable systems, see Fordy and Kulisch [16], Nijhoff et al. [23], Nijhoff et al. [22], Fokas and Ablowitz [15], Ablowitz, Prinari and Trubatch [2], and latterly nonlocal integrable systems have recently received a lot of attention, see Ablowitz and Musslimani [1], Fokas [14] and Grahovski, Mohammed and Susanto [17].

In the current paper we extend the ideas presented in [10] even further by introducing a unified way of tackling integrable systems. We emphasize that we are actually able to linearise the more general nonlocal versions of these nonlinear systems. One key step in these decompositions is the use of a product rule that we shall introduce below. It is also worth noting that the proofs of these decompositions are minimal in the sense that, but for the aforementioned product rule, we only exploit the immediate consequences of the linear system. Moreover, we do not assume any commutativity between the operators involved, thus allowing us for example to consider matrix versions of the systems at hand. Having accomplished the linearisation of the nonlocal system, we can then deal with classically integrable systems by way of making a certain projection in the spirit of Pöppe.

These considerations will enable us to show how the same derivation yields essentially both the NLS and a coupled diffusion/anti-diffusion system with a cubic nonlinearity; these two cases being distinguished by a choice we are free to make retrospectively. Similarly, the cases of the KdV and the mKdV equations are treated in a unified manner.
In Section 2 we introduce some preliminary notions and results that we will be using throughout this paper, such as the observation functional and the product rule. We then describe the unification scheme in its most general form.

Then, in Section 3 we present the main results of this paper. We start by establishing existence and uniqueness properties and proceed to prove Theorems 3.4 and 3.5, where the linearisation of both nonlocal and local versions of different integrable systems is achieved without assuming commutativity.

Finally, in Section 4 we discuss some aspects of this work, as well as possible extensions we wish to explore.

## 2. Unification

### 2.1. Preliminaries

Let us first describe the general framework by introducing the types of operators we will be using and providing some necessary definitions.

As in [10], we shall be working with time-dependent Hilbert–Schmidt integral operators with parameter $x \in \mathbb{R}$; in general an operator $F \in \mathcal{J}_2$ with corresponding square-integrable kernel $f$ will be of the form

$$(F \phi)(y; x, t) = \int_{-\infty}^{0} f(y, z; x, t) \phi(z) \, dz,$$

for any square-integrable function $\phi$.

Moreover, we shall need another type of integral operator, namely an additive time-dependent Hilbert–Schmidt integral operator with parameter $x$.

**Definition 2.1 (Additive operator with parameter).** We say a given time-dependent Hilbert–Schmidt operator $H \in \mathcal{J}_2$ with corresponding square-integrable kernel $h$ is Hankel or additive with parameter $x \in \mathbb{R}$ if its action, for any square-integrable function $\phi$, is given by

$$(H \phi)(y; x, t) := \int_{-\infty}^{0} h(y + z + x; t) \phi(z) \, dz.$$

As in Pöppe [25], we introduce the following observation functional for any Hilbert–Schmidt operator.

**Definition 2.2 (Observation functional).** Given a Hilbert–Schmidt operator $F$ with corresponding square-integrable kernel $f = f(y, z)$, the observation functional $\langle \cdot \rangle$ is defined to be $\langle F \rangle := f(0, 0)$.

**Remark 2.3.** We note of course that in the case of a time-dependent Hilbert–Schmidt operator $F$ with parameter $x \in \mathbb{R}$, we have $\langle F \rangle = f(0, 0; x, t)$.

As mentioned previously, there is one key ‘product rule’ property. This is the following.
Lemma 2.4 (Product rule). Assume $H, H'$ are additive Hilbert–Schmidt operators with parameter $x$ and $F, F'$ are Hilbert–Schmidt operators. Then, the following product rule holds

$$(F \partial_x (HH')F')(y; z; x) = (FH)(y; 0; x)(H'F')(0; z; x).$$

As a special case, we have

$$\langle F \partial_x (HH')F' \rangle = \langle FH \rangle \langle H'F' \rangle.$$

Proof. This is essentially a consequence of the Fundamental Theorem of Calculus and the additivity of $H, H'$. For details see [10]. The second part of the statement is obtained by setting $y = z = 0$. □

2.2. Unification scheme. Assume $P = P(x, t)$ and $\tilde{P} = \tilde{P}(x, t)$ are additive Hilbert–Schmidt operators with corresponding integral kernels $p = p(y + z + x; t)$ and $\tilde{p} = \tilde{p}(y + z + x; t)$ and that $Q = Q(x, t)$ and $G = G(x, t)$ are Hilbert–Schmidt operators with corresponding integral kernels $q = q(y, z; x, t)$ and $g = g(y, z; x, t)$. Assume further that the operators $P, \tilde{P}, Q$ and $G$ satisfy the following system of linear equations:

$$\begin{align*}
\partial_t P &= d(\partial_x)P \\
\partial_t \tilde{P} &= \tilde{d}(\partial_x)\tilde{P} \\
Q &= \tilde{P}P \\
P &= G(id + Q),
\end{align*}$$

where $d, \tilde{d}$ are polynomials of $\partial_x$ with constant coefficients.

As described earlier, our goal will be later to extract a nonlinear flow for $G$ from this linear system and then apply the observation functional followed by the product rule whenever necessary, so as to show that the function $[G](x, t)$ satisfies the target PDE in each case. The different cases will be distinguished by the order of $d, \tilde{d}$ and, crucially, by the choice of $\tilde{P}$.

Before we move on to that, a few more preliminaries are in order. It is convenient to define

$$U := (id + Q)^{-1},$$

that is,

$$U = (id + \tilde{P}P)^{-1},$$

and

$$V := (id + \tilde{P}P)^{-1}.$$

Note that $PU^{-1} = V^{-1}P$, so we have $VP = PU = G$. Similarly, $U\tilde{P} = \tilde{P}V$ and, for completeness, we set $\tilde{G} := U\tilde{P}$.

Let us now mention a few identities for $U$ (and analogously for $V$) that will be useful later on. We list these in the next lemma.
Lemma 2.5 (U-identities). Let $U$ be defined as above. Then, the following identities hold:

(i) $\partial U = -U\partial QU$;
(ii) $\text{id} - U = UQ = QU$;
(iii) $U_x = -U_xQ - UQ_x = -Q_xU - QU_x$;
(iv) $U_{xx} = -2U_xQ_xU - UQ_{xx}U = -2UQ_xU_x - UQ_{xx}U$;
(v) $UQ_xU_x = U_xQ_xU$;
(vi) $U_{xxx} = -6U_xQ_xU_x - 3UQ_{xx}U_x - 3U_xQ_{xx}U - UQ_{xxx}U$.

Proof. (i) We have $UU^{-1} = \text{id}$, so differentiating $(\partial U)U^{-1} = -U\partial Q$ and hence $\partial U = -U(\partial Q)U$.

(ii) We have $\text{id} - U = UU^{-1} - U = U(U^{-1} - \text{id}) = UQ$, and similarly, $\text{id} - U = QU$.

(iii) These follow directly from differentiating (ii).

(iv) From the first part of (iii) we have $U_{xx} = -U_{xx}Q - 2U_xQ_x - UQ_{xx}$, so $U_{xx}(\text{id} + Q) = -2U_xQ_x - UQ_{xx}$ and hence $U_{xx} = -2U_xQ_xU - UQ_{xx}U$.

Similarly, $U_{xx} = -2UQ_xU_x - UQ_{xx}U$.

(v) Using (i) we have, $UQ_xU_x = -UQ_xUQ_xU = U_xQ_xU$.

(vi) Using the above we have

\[
U_{xxx} = -(UQ_xU)_x
= -U_{xx}Q_xU - UQ_xU_{xx} - 2U_xQ_xU_{xx}U - 2U_xQ_xU_x - 2UQ_{xx}U_x - UQ_{xxx}U,
\]

but

\[
-U_{xx}Q_xU - UQ_xU_{xx} = (2U_xQ_xU + UQ_{xx}U)Q_xU
+ UQ_x(2UQ_xU_x + UQ_{xx}U)
= -4U_xQ_xU_x - UQ_{xx}U_x - U_xQ_{xx}U,
\]

so

\[
U_{xxx} = -6U_xQ_xU_x - 3UQ_{xx}U_x - 3U_xQ_{xx}U - UQ_{xxx}U.
\]

\[\square\]

Remark 2.6. Note that the corresponding identities involving $V$ are obtained by replacing in the above every instance of $U$ with $V$ and every instance of $Q = \tilde{P}\tilde{P}$ with $\tilde{P}\tilde{P}$.

3. Application to integrable PDEs

3.1. Existence and Uniqueness results. Before we can proceed to the derivation of the nonlinear PDEs from the linear system, we need to establish some existence and uniqueness results. Firstly, let us mention our most abstract such result which applies to our general framework. For a proof of this see [10].
Lemma 3.3 (Existence and Uniqueness). Assume, for some $T > 0$, we know $Q \in C^\infty([0, T]; \mathcal{J}_2(V; V))$ and $P \in C^\infty([0, T]; \mathcal{J}_4(V; V^\perp))$, where $\mathcal{J}_4(V; V^\perp)$ is a closed subspace of $\mathcal{J}_2(V; V^\perp)$. Further, assume $\det_2(id + Q_0) \neq 0$. Then, there exists a $T' > 0$, with $T' \leq T$, such that $\det_2(id + Q(t)) \neq 0$, for $t \in [0, T']$. In particular, there exists a unique solution $G \in C^\infty([0, T']; \mathcal{J}_4(V; V^\perp))$ to the linear Fredholm equation $P = G(id + Q)$.

Now we also need to establish regularity results for the linear PDEs satisfied by the integral kernels $p, \tilde{p}$ of $P, \tilde{P}$ respectively, i.e.

$$\partial_t p = d(\partial_x p)$$
$$\partial_t \tilde{p} = d(\partial_x \tilde{p}),$$

in order to ensure that these generate corresponding Hilbert–Schmidt operators $P, \tilde{P}$. Again, we only state here the lemma for completeness; for further details and the proof see [10].

Lemma 3.2 (Dispersive linear PDE properties). Assume $p = p(x; t)$ is a solution to the general dispersive linear partial differential equation above. Let $w : \mathbb{R} \to \mathbb{R}$ denote an arbitrary polynomial function with constant non-negative coefficients, whose Fourier transform we denote by $\hat{w} = \hat{w}(k)$, while $W : \mathbb{R} \to \mathbb{R}_+$ denotes the specific function $W : x \mapsto 1 + x^2$. Then, $p$ and its Fourier transform $\hat{p} = \hat{p}(k; t)$ satisfy the following properties for all $k \in \mathbb{R}$ and $t \geq 0$:

1. $\|\partial p\|_{L^2}^2 \leq (2\pi)^2 \|W^{1/2}p\|_{L^2}^2$;
2. $p(0) \in H(\mathbb{R}; \mathbb{C}) \Rightarrow p(0) \in L_w^2(\mathbb{R}; \mathbb{C})$;
3. $p(0) \in L_w^2(\mathbb{R}; \mathbb{C}) \Rightarrow p(0) \in H^1(\mathbb{R}; \mathbb{C})$;
4. $p(k; t) = e^{id(2\pi k)}p(k; 0)$;
5. $\hat{p}^*(k; t)p(k; t) = \hat{p}^*(k; 0)p(k; 0)$;
6. $\|w(\partial)p(t)\|_{L^2}^2 = \|\hat{w}p(t)\|_{L^2}^2 = \|\hat{w}p(0)\|_{L^2}^2 = \|w(\partial)p(0)\|_{L^2}^2$;
7. $p(0) \in H^2(\mathbb{R}; \mathbb{C}) \Rightarrow p(t) \in H^2(\mathbb{R}; \mathbb{C})$;
8. $\|\partial p\|_{L^2}^2 \leq (2\pi)^2(2t^2\|d^p(0)\|_{L^2}^2 + \|\partial p(0)\|_{L^2}^2)$;
9. $p(0) \in H^1(\mathbb{R}; \mathbb{C}) \cap L_w^2(\mathbb{R}; \mathbb{C}) \Rightarrow p(t) \in L_w^2(\mathbb{R}; \mathbb{C})$;
10. $p(t) \in L^2(\mathbb{R}; \mathbb{C}) \Rightarrow \text{additive P}(t) \in \mathcal{J}_2$.

Finally, we establish the sense in which a solution generated by the linear system exists.

Lemma 3.3 (Existence and Uniqueness: unified PDE prescription). Assume that $p_0, \tilde{p}_0 \in H(\mathbb{R}; \mathbb{C}) \cap L_w^2(\mathbb{R}; \mathbb{C})$ and $\det(id + Q(x; 0)) \neq 0$. Then, there exists a $T > 0$ such that, for each $t \in [0, T]$ and $x \in \mathbb{R}$, we have:

1. The solutions $p = p(y + x; t)$ and $\tilde{p} = \tilde{p}(y + x; t)$ to $\partial_t p = d(\partial_x p)$ and $\partial_t \tilde{p} = d(\partial_x \tilde{p})$, respectively, are such that $p(\cdot + x; t), \tilde{p}(\cdot + x; t) \in H^1(\mathbb{R}; \mathbb{C}) \cap L_w^2(\mathbb{R}; \mathbb{C})$ with $p(x; 0) = p_0(x)$ and $\tilde{p}(x; 0) = \tilde{p}_0(x)$, and thus $P(x; t), \tilde{P}(x; t) \in \mathcal{J}_2$ and are smooth functions of $x$ and $t$. 
(ii) The function given by
\[ q(y, z; x, t) = \int_{-\infty}^{0} \tilde{p}(y + \xi + x; t)p(\xi + z + x; t) \, d\xi, \]
i.e. the kernel corresponding to \( Q \), is such that \( Q(x, t) \in J_1 \) and is a smooth function of \( x \) and \( t \).

(iii) \( \det(\text{id} + Q(x; t)) \neq 0 \).

(iv) There exists a unique \( g \in C^\infty([0, T]; C^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{C})) \) which satisfies the linear Fredholm equation
\[ p(y + z + x; t) = g(y, z; x, t) + \int_{-\infty}^{0} g(y, \xi; x, t)q(\xi, z; x, t) \, d\xi. \]

Proof. (i) The time regularity of \( p \) follows from the spatial regularity assumed on the initial data. The regularity of \( p = p(x + t) \) with respect to \( x \) follows from the dispersive linear PDE Lemma, we deduce \( P(x, t) \in J_2 \) and is a smooth function of \( x \) and \( t \). The same arguments apply for \( \tilde{P} \).

(ii) Since \( P(x, t) \), \( \tilde{P}(x; t) \) are smooth in \( x, t \) so is \( Q(x; t) \). Hence, since \( \det(\text{id} + Q(x; 0)) \neq 0 \), there exists \( T' > 0 \) such that \( \det(\text{id} + Q(x; t)) \neq 0 \) for \( t \in [0, T] \); if \( T' < T \) we reset \( T \) to be \( T' \).

(iii) This is established by a slight modification of the argument used to obtain the corresponding abstract result in the Existence and Uniqueness Lemma, due to the fact that here \( Q = \tilde{P}P \in J_1 \).

3.2. Nonlinear Schrödinger and coupled diffusion/anti-diffusion system. We now restrict our choice of operators \( d \) and \( \tilde{d} \) by considering the following linear system:

\[
\begin{align*}
\partial_t P &= \mu_1 \partial_x^2 P + \mu_2 \partial_x^3 P \\
\partial_t \tilde{P} &= \tilde{\mu}_1 \partial_x^2 \tilde{P} + \tilde{\mu}_2 \partial_x^3 \tilde{P} \\
Q &= \tilde{P}P \\
P &= G(\text{id} + Q).
\end{align*}
\]

(3.1)

With regard to the parameters \( \mu_j \) and \( \tilde{\mu}_j \), a priori these can be regarded as arbitrary complex numbers. However, as it turns out, the structure of the computations needed to prove the following two theorems is such that we will eventually require \( \tilde{\mu}_j = \pm \mu_j \). Moreover, even though in principle the two cases can be worked out simultaneously, we shall distinguish between the second-order and the third order cases, that is, we impose \( \mu_1 \mu_2 = 0 \). This is because the derivations in each case are not naturally linked to each other, and hence they are better showcased by being treated separately. Lastly, as
we show below, some special values of \( \mu_j \) and \( \tilde{\mu}_j \) yield specific well-known integrable systems.

Let us first focus on the case where \( \mu_2 = \tilde{\mu}_2 = 0 \). We state and prove our first result which leads to the NLS and a coupled diffusion/anti-diffusion system with a cubic nonlinearity.

**Theorem 3.4 (Second-order decomposition).** Assume the Hilbert–Schmidt operators \( P, Q \) and \( G \) satisfy the linear system (3.1) and their corresponding kernels satisfy all the assumptions of Lemma 3.3. Then, for some \( T > 0 \), the integral kernel \( g = g(y, z; x, t) \) corresponding to \( G \) satisfies, for every \( t \in [0, \infty) \),

(i) the nonlocal equation

\[
i\partial_t g(y, z; x, t) = \partial_x^2 g(y, z; x, t) + 2g(y, 0; x, t)g^*(0, 0; x, t)g(0, z; x, t),
\]

when choosing \( \tilde{P} = P^\dagger \) and \( \mu_1 = -i \).

In particular, the functions \( \langle G \rangle(x, t) = g(0, 0; x, t) \) satisfies the nonlinear Schrödinger equation

\[
i\partial_t \langle G \rangle = \partial_x^2 \langle G \rangle + 2|\langle G \rangle|^2 \langle G \rangle.
\]

(ii) the nonlocal system

\[
\begin{aligned}
\partial_t g(y, z; x, t) &= \partial_x^2 g(y, z; x, t) + 2g(y, 0; x, t)\tilde{g}(0, 0; x, t)g(0, z; x, t) \\
\partial_t \tilde{g}(y, z; x, t) &= -\partial_x^2 \tilde{g}(y, z; x, t) - 2\tilde{g}(y, 0; x, t)g(0, 0; x, t)\tilde{g}(0, z; x, t),
\end{aligned}
\]

when choosing \( \tilde{P} = P(x, -t) \) and \( \mu_1 = 1 \).

In particular, the functions \( \langle G \rangle(x, t) = g(0, 0; x, t) \) and \( \langle \tilde{G} \rangle(x, t) = \tilde{g}(0, 0; x, t) \) satisfy the coupled diffusion/anti-diffusion system with cubic nonlinearity

\[
\begin{aligned}
\partial_t \langle G \rangle &= \partial_x^2 \langle G \rangle + 2\langle G \rangle \langle \tilde{G} \rangle \langle G \rangle \\
\partial_t \langle \tilde{G} \rangle &= -\partial_x^2 \langle \tilde{G} \rangle - 2\langle \tilde{G} \rangle \langle G \rangle \langle \tilde{G} \rangle.
\end{aligned}
\]

**Proof.** Using \( G = PU \), we compute

\[
\begin{aligned}
\partial_t G - \mu_1 \partial_x^2 G &= -PU\partial_t(\tilde{P} P)U - 2\mu_1 P_x U_x - \mu_1 P U_{xx} \\
&= -PU(\tilde{\mu}_1 \tilde{P}_x P + \mu_1 \tilde{P} P_x)U + 2\mu_1 P_x U(\tilde{P} P)_x U \\
&\quad + \mu_1 P U_x (\tilde{P} P)_x U + \mu_1 P U (\tilde{P} P)_{xx} U_x \\
&= (-\tilde{\mu}_1 + \mu_1) P U \tilde{P}_x P_x U + 2\mu_1 P_x U (\tilde{P} P)_x U + \mu_1 P U_x (\tilde{P} P)_x U \\
&\quad + 2\mu_1 P U \tilde{P}_x P_x U + \mu_1 P U (\tilde{P} P)_{xx} U_x \\
&= (-\tilde{\mu}_1 + \mu_1) P U \tilde{P}_x P_x U + 2\mu_1 (P_x U (\tilde{P} P)_x U + P U_x (\tilde{P} P)_{xx} U_x \\
&\quad + PU \tilde{P}_x P_x U).
\end{aligned}
\]

Making the choice \( \tilde{\mu}_1 = -\mu_1 \), we get

\[
\partial_t G - \mu_1 \partial_x^2 G = 2\mu_1 (PU(\tilde{P}_x P)_x U + P_x U (\tilde{P} P)_x U + P U_x (\tilde{P} P)_{xx} U_x),
\]
and so, for the corresponding kernels, denoted by $[\cdot]$, after applying the product rule, we have

$$
\partial_t [G](y, z) - \mu_1 \partial_x^2 [G](y, z)
= 2\mu_1 ([PU\tilde{P}_x](y, 0)[PU](0, z) + [P_xU\tilde{P}](y, 0)[PU](0, z)
+ [PU_x\tilde{P}](y, 0)[PU](0, z))
= 2\mu_1 [(PU\tilde{P})_x(y, 0)[PU](0, z).
$$

Now, the identity $id - V = VP\tilde{P} = PU\tilde{P}$, gives $V_x = - (PU\tilde{P})_x$. But also, by the definition of $V$, we have $V_x = -V(P\tilde{P})_xV$. Hence, applying the product rule once more we get

$$
\partial_t [G](y, z) - \mu_1 \partial_x^2 [G](y, z) = 2\mu_1 [V(P\tilde{P})_xV](y, 0)[PU](0, z)
= 2\mu_1 [VP(y, 0)[\tilde{P}V](0, 0)[PU](0, z)
= 2\mu_1 [G](y, 0)[\tilde{G}](0, 0)[G](0, z).
$$

We can now retrospectively restrict the choices of $\tilde{P}$ and $\mu_1$, provided these, together with our earlier choice of $\tilde{\mu}_1 = -\mu_1$, are consistent.

One such set of choices is $\tilde{P} = P^\dagger$ and $\mu_1 = -i$ (so $\tilde{\mu}_1 = i$), where $P^\dagger$ denotes the adjoint of the complex-valued Hilbert–Schmidt operator $P$. In this case, $V^\dagger = V$ and hence, $G = \tilde{P}^\dagger V = (VP)^\dagger = G^\dagger$. Also, since $G^\dagger$ has kernel $g^*(z, y; x, t)$, we conclude that the kernel $g$ satisfies the nonlocal equation

$$
i\partial_t g(y, z; x, t) = \partial_x^2 g(y, z; x, t) + 2g(y, 0; x, t)g^*(0, 0; x, t)g(0, z; x, t).
$$

In particular, the function $\langle G(x, t) = g(0, 0; x, t)$ satisfies the nonlinear Schrödinger equation

$$
i\partial_t \langle G \rangle = \partial_x^2 \langle G \rangle + 2|\langle G \rangle|^2 \langle G \rangle.
$$

Another consistent set of choices we can make is $\tilde{P} = P(x, -t)$ and $\mu_1 = 1$ (so $\tilde{\mu}_1 = -1$). In this case, $V(x, t) = U(x, -t)$ and hence, $\tilde{G} = \tilde{P}V = (PU)(x, -t) = G(x, -t)$. So we conclude that the kernels $g$ and $\tilde{g}$ satisfy the nonlocal system

$$
\partial_t g(y, z; x, t) = \partial_x^2 g(y, z; x, t) + 2g(y, 0; x, t)\tilde{g}(0, 0; x, t)g(0, z; x, t)
\partial_t \tilde{g}(y, z; x, t) = -\partial_x^2 \tilde{g}(y, z; x, t) - 2\tilde{g}(y, 0; x, t)g(0, 0; x, t)\tilde{g}(0, z; x, t),
$$

In particular, the functions $\langle G(x, t) = g(0, 0; x, t)$ and $\langle \tilde{G}(x, t) = \tilde{g}(0, 0; x, t)$ satisfy the coupled diffusion/anti-diffusion system with cubic nonlinearity

$$
\partial_t \langle G \rangle = \partial_x^2 \langle G \rangle + 2\langle G \rangle \langle \tilde{G} \rangle \langle G \rangle
\partial_t \langle \tilde{G} \rangle = -\partial_x^2 \langle \tilde{G} \rangle - 2\langle \tilde{G} \rangle \langle G \rangle \langle \tilde{G} \rangle.
$$

□
3.3. **KdV and mKdV equations.** Finally, turning to the case where $\mu_1 = \tilde{\mu}_1 = 0$, we formulate and derive the corresponding result for the KdV/mKdV cases.

**Theorem 3.5 (Third-order decomposition).** Assume the Hilbert–Schmidt operators $P, Q$ and $G$ satisfy the linear system (3.1) and their corresponding kernels satisfy all the assumptions of Lemma 3.3. Then, for some $T > 0$, the integral kernel $g = g(y, z; x, t)$ corresponding to $G$ satisfies, for every $t \in [0, \infty]$,

(i) the nonlocal equation

$$
\partial_t g(y, z; x, t) + \partial_x^3 g(y, z; x, t) + 3g(y, 0; x, t)g(0, 0; x, t)\partial_x g(0, z; x, t) + 3(\partial_x g(y, 0; x, t))g(0, 0; x, t)g(0, z; x, t) = 0,
$$

when choosing $\tilde{P} = P$ and $\mu_2 = -1$.

In particular, the function $\langle G \rangle(x, t) = g(0, 0; x, t)$ satisfies the matrix-mKdV equation

$$
\partial_t \langle G \rangle + \partial_x^3 \langle G \rangle + 3\langle G \rangle^2 \partial_x \langle G \rangle + 3(\partial_x \langle G \rangle)^2 = 0.
$$

(ii) the nonlocal equation

$$
\partial_t g(y, z; x, t) + \partial_x^3 g(y, z; x, t) = 3\partial_x g(y, 0; x, t)g(0, z; x, t),
$$

when choosing $\tilde{P} = \text{id}$, $\mu_2 = -1$.

In particular, the function $\langle G \rangle(x, t) = g(0, 0; x, t)$ satisfies the (primitive) form of the KdV equation

$$
\partial_t \langle G \rangle + \partial_x^3 \langle G \rangle = 3(\partial_x \langle G \rangle)^2.
$$

**Proof.** As before, we treat $\tilde{P}$ and $\mu_2, \tilde{\mu}_2$ arbitrarily for the moment. Recall $G = PU = VP$, and eventually we will consider $G = \frac{1}{2}(PU + VP)$. This is because, in the absence of commutativity, we shall require this symmetric representation of $G$ in order to group terms together appropriately later on. For the sake of presentation, we will split the proof into separate steps.

**Step 1:** Considering $G = PU$.

Let us first take $G = PU$ and compute

$$
\partial_t G - \mu_2 \partial_x^3 G = P_t U - PU Q_t U - \mu_2 (P_{xxx} U + 3P_{xx} U_x + 3P_x U_{xx} + PU_{xxx})
$$

$$
= -PU(Q_t - \mu_2 Q_{xxx})U + \mu_2 (3P_{xx} U Q_x U + 6P_x U_x Q_x U + 3P U_{xx} U_x + 3P U Q_{xx} U_x + 3P U_x Q_{xx} U_x).
$$

We have

$$
Q_t - \mu_2 Q_{xxx} = \tilde{\mu}_2 \tilde{P}_{xxx} P + \mu_2 \tilde{P} P_{xxx} - \mu_2 (\tilde{P}_{xxx} P + 3\tilde{P} x P_x + 3\tilde{P} P_{xx} + \tilde{P} P_{xxx}),
$$
so we let \( \tilde{\mu}_2 = \mu_2 \) and thus

\[
\frac{1}{\mu_2} \partial_t G - \partial_x^2 G
\]

\[
= 3PU \tilde{P}_x P_x U + 3PU \tilde{P}_x P_x U + \{3P_{xx} U \tilde{P}_x PU\} + \{3P_{xx} U \tilde{P}_x PU\}
+ 6P_x U_x \tilde{P}_x PU + 6P_x U_x \tilde{P}_x PU + \{3P_{xx} U \tilde{P}_x PU\} + 6P_x \tilde{P}_x PU
+ 3P_x \tilde{P}_x PU + 6P_x \tilde{P}_x PU + 6P_x \tilde{P}_x PU + 3P_{xx} \tilde{P}_x PU
+ 6P_x \tilde{P}_x P_x U + \{3P \tilde{P}_x U \tilde{P}_x U\} + \{3P U \tilde{P}_x PU\} + 6P_x \tilde{P}_x P_x U
+ 3P_x \tilde{P}_x PU.
\]

(3.2)

Step 2: Considering \( G = VP \).

Similarly, for \( G = VP \), the same calculation gives

\[
\frac{1}{\mu_2} \partial_t G - \partial_x^2 G
\]

\[
= 3VP_{xx} \tilde{P}_x VP + 3VP_{xx} \tilde{P}_x VP + \{3VP_{xx} \tilde{P}_x VP\} + \{3VP \tilde{P}_x VP\}
+ 6V_x \tilde{P}_x VP VP + 6V_x \tilde{P}_x VP VP + 3VP_{xx} \tilde{P}_x VP + 6V_x \tilde{P}_x VP
+ \{3VP_{xx} \tilde{P}_x VP\} + 6V_x \tilde{P}_x VP VP + 3VP_{xx} \tilde{P}_x VP + 6V_x \tilde{P}_x VP
+ 3V_x \tilde{P}_x VP.
\]

(3.3)

In each case above, our aim is to first manipulate the terms in curly brackets appropriately and then factorise the combined r.h.s., so as to eventually extract a nonlinear term in \([G], \tilde{G}\) and their derivatives.

Step 3: Cancellations from \((3.2)\).

Let us first focus on \((3.2)\) and the first three terms in curly brackets, namely \(3P_{xx} U \tilde{P}_x PU, 3P_{xx} U \tilde{P}_x PU \& 3P_{xx} U \tilde{P}_x PU\). In what follows, we will be using the fact that \(PU = VP\), whereby

\[
P_x U + PU_x = V_x P + VP_x,
\]

and

\[
P_{xx} U + 2P_x U_x + PU_{xx} = V_{xx} P + 2V_x P_x + VP_{xx}.
\]

With these in mind, we have

\[
3P_{xx} U \tilde{P}_x PU = \{3VP_{xx} \tilde{P}_x VP\} + 6V_x \tilde{P}_x PU + 3V_{xx} P \tilde{P}_x PU
- 6P_x U_x \tilde{P}_x PU - 3PU_{xx} \tilde{P}_x PU,
\]

\[
3P_{xx} U \tilde{P}_x PU = 3VP_{xx} \tilde{P}_x PU + 6V_x \tilde{P}_x PU + 3V_{xx} P \tilde{P}_x PU - 6P_x U_x \tilde{P}_x PU
- 3PU_{xx} \tilde{P}_x PU
= \{3VP_{xx} \tilde{P}_x VP\} + \{3VP_{xx} \tilde{P}_x VP\} - 3VP_{xx} \tilde{P}_x PU
+ 6V_x \tilde{P}_x PU + 3VP_{xx} \tilde{P}_x PU - 6P_x U_x \tilde{P}_x PU - 3PU_{xx} \tilde{P}_x PU,
\]

\[
3P_{xx} U \tilde{P}_x PU
\]
and

\[ 3P_x U \tilde{P}_{xx} PU = \{\{3V_x \tilde{P}_{xx} VP\}\} + \{\{3V_x P \tilde{P}_{xx} VP\}\} - 3PU_x \tilde{P}_{xx} PU. \]

Let \( R \) be the sum of the terms in curly brackets in (3.2) excluding the ones in double curly brackets in the expressions just above. Then, \( R = 0 \). Indeed,

\[ R = 6V_x P_x \tilde{P}_x PU + 6V_x P_x \tilde{P}_x P_x U - 6P_x U_x \tilde{P}_x PU - 6P_x U_x \tilde{P}_x P_x U \]
\[ + 3V_{xx} P \tilde{P}_x PU + 3V_{xx} P \tilde{P}_x P_x U - 3PU_{xx} \tilde{P}_x PU - 3PU_{xx} \tilde{P}_x P_x U \]
\[ - 3V_{xx} \tilde{P}_x PU_x + 3P_U \tilde{P}_P_{xx} U_x \]
\[ = 6V_x P_x Q_x U - 6P_x U_x Q_x U + 3V_{xx} P Q_x U - 3P_{xx} U_x Q_x U - 3V_{xx} P Q_x U_x \]
\[ + 3V \tilde{P}_x P_{xx} U_x \]
\[ = 3(2V_x P_x - P_x U_x) + V_{xx} P - PU_{xx} Q_x U - 3V_{xx} P Q_x U_x + 3V \tilde{P}_x P_{xx} U_x \]
\[ = 3(P_{xx} U - VP_{xx})Q_x U - 3V P_{xx} Q_x U_x + 3V \tilde{P}_x P_{xx} U_x \]
\[ = -3V P_{xx} (QU)_x - 3P_{xx} U_x + 3V \tilde{P}_x P_{xx} U_x \]
\[ = -3V P_{xx} (QU)_x - 3P_{xx} U_x + 3V \tilde{P}_x P_{xx} U_x \]
\[ = 0. \]

Step 4: Cancellations from (3.3).

Similarly, for the first three terms in curly brackets of (3.3), we have

\[ 3V P_x \tilde{P} V P_{xx} = 3V P_x \tilde{P} P_{xx} U + 6V P_x \tilde{P} P_x U_x + 3V P_x \tilde{P} PU_{xx} - 6V P_x \tilde{P} V_x P_x \]
\[ - 3V P_x \tilde{P} V_{xx} P \]
\[ = \{\{3P_x U \tilde{P} P_{xx} U\}\} + \{\{3PU_x \tilde{P} P_{xx} U\}\} - 3V_x P \tilde{P} P_{xx} U \]
\[ + 6V P_x \tilde{P} P_x U_x + 3V P_x \tilde{P} PU_{xx} - 6V P_x \tilde{P} V_x P_x - 3V P_x \tilde{P} V_{xx} P, \]

\[ 3V P \tilde{P}_x V_{xx} = \{\{3PU \tilde{P}_x P_{xx} U\}\} + 6V P \tilde{P}_x P_x U_x + 3V P \tilde{P}_x PU_{xx} \]
\[ - 6V P \tilde{P}_x V_x P_x - 3V P \tilde{P}_x V_{xx} P, \]

and

\[ 3V P \tilde{P}_{xx} V P_x = \{\{3PU \tilde{P}_{xx} P_x U\}\} + \{\{3PU \tilde{P}_{xx} PU_x\}\} - 3V P \tilde{P}_{xx} V_x P. \]

Let \( R' \) be the sum of the terms in curly brackets in (3.3) excluding the ones in double curly brackets in the expressions just above. Then, \( R' = 0 \).
Indeed,

\[ R' = 6VP_x\tilde{P}P_xU_x + 6VP\tilde{P}_xP_xU_x - 6VP_x\tilde{P}\tilde{V}_xP_x - 6VP\tilde{P}_xV_xP_x \\
+ 3VP_x\tilde{P}PU_{xx} + 3VP\tilde{P}_xPU_{xx} - 3VP_x\tilde{P}\tilde{V}xP - 3VP\tilde{P}_xV_{xx}P \\
- 3V_xP\tilde{P}P_xU + 3V_xP_{xx}\tilde{P}V \]

\[ = 6V(P\tilde{P})_xP_xU_x - 6V(P\tilde{P})_xV_xP_x + 3V(P\tilde{P})_xPU_{xx} - 3V(P\tilde{P})_xV_{xx}P \\
- 3V_xP\tilde{P}P_xU + 3V_xP_{xx}\tilde{P}U \]

\[ = 3V(P\tilde{P})_x(P_xU_x - V_xP_x) + PU_{xx} - V_{xx}P - 3V_xP\tilde{P}P_xU \\
+ 3V_xP_{xx}\tilde{P}U \]

\[ = 3V(P\tilde{P})_x(VP_{xx} - P_{xx}U) - 3V_xP\tilde{P}P_{xx}U + 3V_xP_{xx}\tilde{P}U \]

\[ = -3(VP\tilde{P})_xP_{xx}U - 3V_xP_{xx} + 3V_xP_{xx}\tilde{P}U \]

\[ = -3(VP\tilde{P})_xP_{xx}U - 3V_xP_{xx}(id - QU) \]

\[ = -3(VP\tilde{P})_xP_{xx}U - 3V_xP_{xx}U \]

\[ = 0. \]

We now combine equations \((3.2)\) and \((3.3)\), utilising the results above and then rearranging and grouping terms appropriately.

**Step 5:** Combine \((3.2)\) and \((3.3)\).

\[
\frac{1}{\mu_2} \partial_t G - \partial_x^3 G = \frac{1}{2} \left( \frac{1}{\mu_2} \partial_t - \partial_x^3 \right) (PU + VP) \\
= \frac{1}{2} \left( 3PU\tilde{P}_{xx}P_xU + 3PU\tilde{P}_xP_{xx}U + 6P_xU_x\tilde{P}_xPU + 6P_xU_x\tilde{P}_xP_xU \\
+ 6P_x\tilde{P}_xP_xU + 3P_xU\tilde{P}P_{xx}U + 6PU_x\tilde{P}_xPU + 6PU_x\tilde{P}_xP_xU \\
+ 3PU\tilde{P}_{xx}PU_x + 6PU_x\tilde{P}_xP_xU + 6PU_x\tilde{P}_xP_xU + 3PU_x\tilde{P}P_{xx}U \\
+ 3V_P_{xx}\tilde{P}V P + 3V_P_{xx}\tilde{P}V P + 3VP_{xx}\tilde{P}V P + 3VP_{xx}\tilde{P}V P + 3VP_{xx}\tilde{P}V P \\
+ 3V_P_{xx}\tilde{P}V P + 3V_P_{xx}\tilde{P}V P + 6V_P_{xx}\tilde{P}V P + 6V_P_{xx}\tilde{P}V P \\
+ 3V_P_{xx}\tilde{P}V P + 3V_P_{xx}\tilde{P}V P + 6V_P_{xx}\tilde{P}V P + 3V_xP_P_{xx}V P \\
+ 3P_xU\tilde{P}P_{xx}U + 3PU_x\tilde{P}P_{xx}U + 3PU_x\tilde{P}P_{xx}U + 3PU_x\tilde{P}P_{xx}U \\
+ 3PU_x\tilde{P}P_{xx}U + 3PU_x\tilde{P}P_{xx}U + 3PU_x\tilde{P}P_{xx}U + 3PU_x\tilde{P}P_{xx}U \right). 
\]
Step 6: Rearrange by grouping $U$-terms and $V$-terms.

\[
\frac{1}{\mu_2} \partial_t G - \partial_x^3 G
\]

\[
= 3 \left( (PU \tilde{P}_x P_x U + PU \tilde{P}_x P_x U) + (P_x U \tilde{P}_x P_x U + P_x U \tilde{P}_x P_x U) + (P_x U \tilde{P}_x P_x U) \right)
\]

\[
+ 3 \left( V (P_x \tilde{P}_x P_x V + P_x \tilde{P}_x P_x V) + (V_x \tilde{P}_x P_x V + V_x \tilde{P}_x P_x V) + (V_x \tilde{P}_x P_x V + V_x \tilde{P}_x P_x V) \right)
\]

Step 7: Introduce total derivatives to prepare for product rule.

\[
\frac{1}{\mu_2} \partial_t G - \partial_x^3 G
\]

\[
= 3 \left( PU (\tilde{P}_x P_x)_x U + P_x U (\tilde{P}_x P_x)_x U + P_x U (\tilde{P}_x P_x)_x U + PU (\tilde{P}_x P_x)_x U_x \right)
\]

\[
+ 3 \left( V (P_x \tilde{P}_x P_x V + V_x \tilde{P}_x P_x V + V_x \tilde{P}_x P_x V + V_x \tilde{P}_x P_x V) \right)
\]

Step 8: Consider kernels and apply the product rule.

\[
\frac{1}{\mu_2} \partial_t [G] - \partial_x^3 [G]
\]

\[
= 3 \left( [PU \tilde{P}_x][P_x U] + [P_x U \tilde{P}][PU_x] + [P_x U \tilde{P}][P_x U] + [PU_x \tilde{P}][PU_x] \right)
\]

\[
+ 3 \left( [V P_x][\tilde{P}_x V] + [V P_x][\tilde{P}_x V] + [V P_x][\tilde{P}_x V] + [V P_x][\tilde{P}_x V] \right)
\]

Step 9: mKdV derivation.
Choosing $\tilde{P} = P$ and $\mu_2 = -1$ (so $\tilde{\mu}_2 = -1$), we see that $[G]$ satisfies a matrix-mKdV equation

$$\partial_t [G] + \partial_x^3 [G] + 3[G]^2 \partial_x [G] + 3(\partial_x[G])[G]^2 = 0.$$  

**Step 10: KdV derivation.**

Moreover, if we were to go back just before Step 8 and choose $\tilde{P} = \text{id}, \mu_2 = -1$, we would get

$$\partial_t G + \partial_x^3 G = 3 \left( P_x U P_y U_x + P_x U P_{xy} U_x + PU_x P_x U + PU_x P_{xx} U \right)$$

$$+ 3 \left( V_x P_x V P + V P_{xx} V P + V_x P_x V_x P + V P_{xx} V_x P \right).$$

In this case though we have $Q = P$ and $V = U$, so

$$\partial_t G + \partial_x^3 G$$

$$= 3 \left( Q_x U Q_x U_x + Q_x U Q_{xx} U + QU_x Q_x U_x + QU_x Q_{xx} U \right)$$

$$+ 3 \left( U_x Q_x U Q_x + U Q_{xx} U Q_x + U Q_x U Q_x + U Q_{xx} U Q_x \right)$$

$$= 3 \left( -U_x Q_x U_x - QU_x Q_x U_x - U_x Q_{xx} U - QU_x Q_{xx} U + QU_x Q_x U_x \right)$$

$$+ 3 \left( -U_x Q_x U_x - U_x Q_x U_x - U Q_{xx} U_x - U Q_x U_x Q + U_x Q_x U_x Q \right)$$

$$= 3 \left( U_x Q_x U_x + U_x (Q^2)x U_x + U (Q_x Q)_x U_x + U (Q^2)_x U_x \right).$$

Applying now the functional $[\cdot]$ and the product rule, we obtain

$$\partial_t [G] + \partial_x^3 [G] = 3 \left( [U_x Q][Q_x U] + [U_x Q][Q U_x] + [U Q_x][Q U_x] + [U Q_x][Q_x U] \right)$$

$$= 3[U_x][U_x],$$

so, since in this case $U_x = -G_x$, we see that $[G]$ satisfies the KdV equation

$$\partial_t [G] + \partial_x^3 [G] = 3(\partial_x[G])^2.$$

□

4. Discussion

We now discuss a few of the extensions we wish to look upon in the future. Firstly, we note that the unified process for the mKdV and the KdV equations had to be split before applying the observation functional. This is
because the product rule is no longer valid in the KdV case when \( \tilde{P} = \text{id} \) as this does not have an additive kernel. Perhaps strangely though, if we were to set \( \tilde{P} = \text{id} \) anyway after taking the functional in the main calculation, we would indeed get the KdV nonlinearity \( [G_{x}]^{2} \) but with a factor of 6 instead of 3. This leads to the conclusion that there may be a unified derivation, matching more closely the one for the second-order case, at the very end of which we arrive at either the KdV or the mKdV equations depending on the choice we make. In principle, this could be resolved in two ways. One would require to have at hand a more general product rule, that is one that does not assume an additive kernel, but for which the additive-kernel result is a special case. We could then proceed to Step 8 above by carrying \( \tilde{P} \) as either \( P \) or \( \text{id} \) without issues. Another way would be to aim for the usual KdV equation rather than the primitive form which we actually derive above. That is to say, by assuming \( \tilde{P} = \text{id} \) as we have done here, we are bound to aim for the primitive form of the KdV equation (see also [25, 3]). However, it is possible that there exists an operator \( \tilde{P} \) with additive kernel that leads to the usual KdV, in which case we could proceed as above in a straightforward manner.

Secondly, it would be justified to say that the derivation of the third-order case could be made shorter by identifying the underlying patterns in the calculations involved. Indeed, by identifying certain algebraic relations, it should be possible derive the whole hierarchy of integrable systems to which the ones discussed herein belong. For example, the induction argument used by Pöppe in [25] to derive the KdV hierarchy points towards this direction.

Thirdly, we would like to provide numerical simulations showcasing the validity and the computational benefit of our approach as the ones we have performed in our previous papers [4, 5, 10]. The case of the mKdV should be of interest.

And finally, we are interested in investigating in particular the applicability of the nonlocal versions of integrable systems described in this paper, as well as exploring other types of such systems and the extent to which these can be accommodated by our method.

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