MATHEMATICAL MODELING OF THE STIRLING ENGINE

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Abstract

The paper presents mathematical models which have been developed by the authors, and the results of which may be used to design an experimental refrigeration unit operating in the Stirling cycle. The systems engineered and based on Stirling cycle may be considered as an alternative to commonly employed internal combustion engines and Linde circulation cooling systems. The article presents a time discretization model assuming the cylinders as adiabatic spaces. The model enables the size optimization of all particular elements of the Stirling device such as: heat exchangers, the regenerator, the cylinders, piston motion and phase displacement. The advantage of modelling is the calculation speed when compared to modelling based on full Navier-Stokes system of equations (CFD) therefore enabling the dimensioning of the device. The results obtained using the simplified model have been verified by the other modelling type – that is the full 3D CFD, in which the whole working space including the heat exchangers and the regenerator has been modelled. Additionally, a dynamic MESH option has been applied in order to simulate the movement of the pistons in the cylinders.

Keywords: Stirling engine; Stirling cooler; Stirling cycle numerical modelling; cogeneration; optimization of Stirling engine;

1. Introduction

The Stirling cycle was invented by a Scottish clergyman, Robert Stirling. He obtained a patent for the invention

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at the beginning of the 19th century in 1816 and two years later the first working version was used to pump out water from a quarry. Despite its theoretical efficiency being equal to the efficiency of the Carnot cycle, the development of Stirling engine was not as dynamic as the evolution and expansion of the steam engine or internal combustion engine. The main obstacle in the design was the shortcomings in materials necessary to build the working unit and a very complex thermodynamic description (difficult to depict at that time). The advancement in material technology especially in the second half of the 20th century and the development in computer technology employing numerical calculation methods are the reasons why technological barriers from more than 100 years ago are now history.

Engineers and scientists are presently rediscovering the concept of the Stirling engine because it possesses many positive attributes such as high theoretical efficiency, compact and relatively simple design, environmentally inoffensive working medium. All of these make it ideal for contemporary conditions. An almost forgotten 200 year-old idea has revived in our times and has the potential to stand up to currently used engines employing thermodynamic cycle in wide variety of field technology from cooling systems, industrial engines and cogeneration [3, 4, 7, 10, 14, 20].

Devices employing the Stirling cycle are still expensive to build simply because of limited production orders. In case of Stirling engine specific working conditions (high temperature and high pressure) have a detrimental effect on the price, especially the cost of the prototype unit. To offset the high cost, the designers try to lower it by implementing new materials and search for optimum work parameters lowering the average pressure in the working chamber. In case of cooling devices where there is no need for high work temperature the mass production cost can be reduced significantly. Initiating production of coolers of any kind employing the Stirling cycle is justified as the cooling medium widely used in contemporary cooling systems has a devastating effect on the natural environment contributing to the ozone layer depletion and the greenhouse effect.

| Nomenclature                                      |
|--------------------------------------------------|
| M       | total gas mass in the machine, kg               |
| m_C    | mass of gas in a warm cylinder, kg             |
| m_E    | mass of gas in a cold cylinder, kg             |
| m_HC   | mass of gas in a warm exchanger, kg            |
| m_HE   | mass of gas in a cold exchanger, kg            |
| m_R    | mass of gas in a regenerator exchanger, kg     |
| T_C    | temperature in the warm cylinder, °C           |
| T_HC   | temperature in the heat exchanger, °C          |
| T_R    | temperature in the regenerator, °C             |
| T_HE   | temperature in the cold exchanger, °C          |
| T_E    | temperature in the cold cylinder, °C           |
| c_p    | specific heat at constant pressure, J/kgK      |
| c_v    | specific heat at constant volume, J/kgK        |
| V_C    | V_C(θ) volume of hot cylinder, m³              |
| V_E    | V_E(θ) volume of cold cylinder, m³             |
| V_HC   | const volume of hot exchanger, m³              |
| V_HE   | const volume of cold exchanger, m³             |
| V_R    | const volume of regenerator, m³                |
| W      | work of thermal cycle, W                       |
| Q      | total amount of the heat (Fig. 2), W           |
| p      | pressure, Pa                                   |
| Φ      | actual instantaneous shaft angle position      |
| γ      | coefficient of compressibility                 |
2. Mathematical models

A Stirling machine is a device employing thermodynamic cycle which, in theory, is described as a group of thermodynamic processes consisting of two isotherms and two isochores. Theoretically, the efficiency of the Stirling cycle is equal to the Carnot cycle.

Stirling cycle devices are divided into three groups: alpha, beta and gamma as their geometrical configurations and design differ. Additional configuration represents the so-called thermoacoustic device with traveling wave [5, 11, 12, 15].

The best performers are alpha configured devices. It is the result of the least number of the irreversible processes and the smallest clearance volume of all configurations. The disadvantage of this configuration is the necessity of sealing both pistons and adaptation of the transfer drive with a phase shift. Illustration (Fig.1) depicts a diagram configuration of the device type alpha.

In the real Stirling device the gaseous working medium is enclosed in space consisting of the cylinders, heat exchangers and the regenerator. During operation, the gas moves in the working space and is subject to transformation of thermodynamic. The working medium, however, never fully vacates the elements of the device. During the cycle the working medium remains in the working space. The dead space effect of the working spaces on the device must be minimized. The simplest theoretical analyses of the cycle processes can be performed using Schmidt's analysis [2, 7, 12, 13, 15, 19, 21].

2.1. Simplified numerical adiabatic model (Fig.2)

The simplified Stirling device simulation method presented in this paper employs adiabatic analysis of the cylinder spaces. This approach gives more realistic results than a simple isothermal Schmidt analysis. Therefore, it can be used in the optimization procedure. This procedure requires multiple model calculations for decisive values of variables in one iterative step. The adiabatic model is described by equations (6 – 14). The adiabatic model needs the initial conditions from which calculation starts. Thus, the first step is calculated using a simple isothermal analysis described by equations (1 – 5).

It is assumed that thermodynamic process takes place in particular subsections of the device treated as separate control volumes. One cycle of the device (one rotation of the shaft) is divided into elementary angles, where the state of the gas is considered as established. As a result, states of the gas in individual components of the working space are determined in any given, discrete instants of the Stirling cycle (assuming \( \theta = 2\pi/\text{frequency} \)). In the modelling, the irreversibility of the processes occurring in the exchangers is assumed. Gas flow resistance, the regenerator and housing heat loss data must be considered.
Fig. 2. Diagram of the space discretization in the thermodynamic model

ISOTHERMAL MODEL

Mass balance

\[ M = m_C + m_{HC} + m_R + m_{HE} + m_E \]  \hspace{1cm} (1)

Equation of state for each gas volume

\[ M = \frac{pV}{RT} \]  \hspace{1cm} (2)

Pressure in the working space as function of the momentary shaft position

\[ P(\phi) = \left( MR \left( \frac{v_C(\phi)}{\tau_C} + \frac{v_{HC}}{\tau_{HC}} + \frac{v_R}{\tau_R} + \frac{v_{HE}}{\tau_{HE}} + \frac{v_E(\phi)}{\tau_E} \right)^{-1} \right) \]  \hspace{1cm} (3)

Average gas temperature of the regenerator calculated from the formula

\[ T_R = \left( \frac{T_{HC} - T_{HE}}{\ln \left( \frac{T_{HE}}{T_{HC}} \right)} \right) \]  \hspace{1cm} (4)

Work cycle calculations obtained by integration of the formulas

\[ W_C = \int pdV_C(\phi), \quad W_E = \int pdV_E(\phi), \quad W = W_C + W_E \]  \hspace{1cm} (5)

ADIABATIC MODEL

First law of thermodynamic for any volume space can be presented by the following equations

\[ dQ + c_pTdm = dW + c_vd(mT) \]  \hspace{1cm} (6)

For the adiabatic cylinder

\[ c_pTdm = dW + c_vd(mT) \]  \hspace{1cm} (7)
The law of conservation mass

\[ dm_c + dm_{HC} + dm_R + dm_{HE} + dm_E = 0 \]  \hspace{1cm} (8)

Equation of state

\[ Vdp + pdV = R(Tdm + mdT) \]  \hspace{1cm} (9)

For given moment of time \( T_{gas} = \text{const} \)

\[ \frac{dp}{p} + \frac{dV}{V} = \frac{dm}{m} \rightarrow m \left( \frac{dp}{p} + \frac{dV}{V} \right) = \frac{1}{RT} (dp + dVp) \]  \hspace{1cm} (10)

For individual sections of the device

\[ dm_E = \frac{1}{RT_E} (pdV_E + \frac{dpV_E}{γ}); \quad dm_{HC} = \frac{1}{RT_{HC}} dpV_{HC}; \quad dm_R = \frac{1}{RT_R} dpV_R; \quad dm_{HE} = \frac{1}{RT_{HE}} dpV_{HE} \]  \hspace{1cm} (11)

Where: \( γ = \frac{c_p}{c_v} \)

Substituting (11) and (8) after transformation obtain the differential equation for the pressure as a function of shaft position

\[ dp = -γp \left( \frac{dp}{p} + \frac{dV}{V} \right) \left( \frac{V_c}{T_C} + \frac{V_E}{T_E} + γ\frac{V_R}{T_R} + γ\frac{V_HC}{T_{HC}} + γ\frac{V_{HE}}{T_{HE}} \right)^{-1} \]  \hspace{1cm} (12)

From equation (9) we obtain for cylinders

\[ dT_C = T_C \left( \frac{dp}{p} + \frac{dV_C}{V_C} - \frac{dm_C}{m_C} \right), \quad dT_E = T_E \left( \frac{dp}{p} + \frac{dV_E}{V_E} - \frac{dm_E}{m_E} \right) \]  \hspace{1cm} (13)

Heat flux for individual exchangers: hot, cold and the regenerator

\[ dQ_{HC} = \frac{dpV_{HC}c_p}{R} - c_p(T_{C-HC}dm_{C-HC} - T_{HC-R}dm_{HC-R}) \]  \hspace{1cm} (14)

\[ dQ_{HE} = \frac{dpV_{HE}c_p}{R} - c_p(T_{E-HE}dm_{E-HE} - T_{HE-R}dm_{HE-R}) \]

\[ dQ_R = \frac{dpV_Rc_p}{R} - c_p(T_{HC-R}dm_{HC-R} - T_{R-HE}dm_{R-HE}) \]

The solution obtained from the system of equations together with proper time steps of discretization allow for the determination of the parameters of the device in any given conditions and the size of its elements (exchangers, regenerator, cylinders' diameter, piston stroke, phase shift) [6, 19].

2.2. CFD model

The above formulated mathematical model does not allow for the analysis of the cylinder shape factor or the driving gear factor (instantaneous piston speed), both factors influencing the device’s performance. The model is simple and computationally cheap but it gives non-realistic representation of the heat exchangers as the impact of oscillating character of the gas motion on heat transfer is not included accurately.
The results obtained and based on an adiabatic model are compared with the results obtained in the 3D CFD model which allows to model the gas flow in real geometry of working space. The model uses a moving mesh which allows for piston movement simulation in the cylinders.

CFD Model assumptions:

- Laminar flow - based on simplified model calculations specifying instantaneous Reynolds-number value,
- Using the symmetry of the device - modelling ½ of the working space,
- Applied half-structure hybrid mesh,
- Regenerator - porous deposit model programmed,
- Cylinder spaces modelled adopting “moving mesh”- piston movement simulation,
- Piston movement simulation in Ross-Yoke mechanism model, using numeric derivatives,
- Number elements of MESH – 650000.

3. Both models results comparison

Table 1 (Tab.1.) shows the thermal power comparison of the adiabatic and the CFD models.

| Model          | CFD (W) | Adiabatic model (W) |
|----------------|---------|---------------------|
| Cooling power - upper source [W] | 368.7 | 313.0 |
| Cooling power - lower source [W]  | 220.2 | 215.9 |
| Mechanical work – net (W)        | 218.5 | 214.8 |

The graph (Fig.3.) below portrays an instantaneous heat flux and the instantaneous temperature values in individual sections of the device during the full work cycle [6, 19].

where:
- hot cylinder (adiabatic model), 2- hot cylinder CFD, 3- hot exchanger CFD, 4- cold cylinder (adiabatic model), 5- cold cylinder CFD, 6- cold exchanger CFD, 7- gas boundary: hot exchanger/hot cylinder, 8- boundary gas: hot exchanger/regenerator, 9- boundary gas: cold exchanger/cold cylinder, 10- boundary gas: cold exchanger/regenerator.

![Fig. 3.](image-url)
Compatibility of the adiabatic model with the full 3D CFD model is attained in global heat flux, gas velocity in individual sections and both global and local pressures. The above comparison reveals phase shift in functional instantaneous heat flux and some temperature value differences (identification of the inaccuracies is the subject of the paper) [6, 19].

4. Model of the device

Based on the results obtained from both models, the prototype of a Stirling cycle cooling device has been designed (Fig.4). During the designing process particular emphasis was put on the future possibility of modification of the device by exchanging the subassemblies. It may be possible to modify the diameter of some subsections, the length of the heat exchanger or the whole regenerator which itself contains an interchangeable head. Particularly noteworthy is the kinematic pair piston-cylinder solution enabling dry friction work. A gas cycle in Stirling engines is a closed one where no gas exchange occurs, therefore, the device must be kept hermetic.

Fig. 4. View of the Stirling cooler elements with the Ross-Yoke kinetic mechanism

Fig. 5. The cross section of the oxide layer (magnification 300x) [18]

The devices where the piston-cylinder kinematic pair demand an oil lubrication, contact between the gas and the lubricant causes the translocation of the latter into the working space. It is the reason for which cylinder oil lubrication is troublesome or outright impossible in the hermetic Stirling devices. The most critical problem is the prospect of the lubricating oil entering the regenerator's ducts. To maintain high energy output of the regenerator, the ducts must be of very small hydraulic radius and the oil-free pistons shall work in dry friction working conditions.

This kind of design forces the adaptation of entirely different materials used for cylinder bearing surface or piston ring compared to traditionally built compressors [1, 9, 17, 18, 19]. It is suggested that the cylinder sleeves must be built with aluminium based alloy and the piston rings with polytetrafluoroethylene composite. The tests have shown that[1, 17, 18] tribological properties of such association promise preferable sliding cooperation as compared to tarflen-graphite made rings where the cylinder bearing surface is made out of chrome plated steel alloy.

The oxide layer of the cylinder bearing surface has a tubular fibrous structure. The miniscule tubes are set perpendicular to the cylinder wall (Fig.5), their pores filled with solid state like lubricant additionally decrease the friction diminishing the abrasive wear of the oxide layer and the piston rings.

5. Conclusions

The paper presents numerical models developed by the authors that can be used to design devices performing the Stirling cycle. The first one is a simplified numerical model with time discretization based on the ideal adiabatic analysis. The results from this model have been compared with the results from the other model – that is the 3D CFD model, in which the authors mapped the entire working space including the heat exchangers and the regenerator. In CFD model the net deformation used allows for piston movement simulation in the cylinders.

The authors have achieved the compliance of adiabatic model with the full 3D CFD model based on the full
system of Navier-Stokes equations, regarding global heat flux and some temperature values differences, global and local pressure. In the results a phase drift can be observed in momentary functions of heat flux and some differences in temperature values can be noticed (the identification of the inaccuracies is the subject of the current work).

The results obtained from both models have confirmed that the polished simplified numerical model with discretization of time, due to its very short calculation time, can find future applications in the design and optimization of such devices. On the basis of the numerical models results an experimental Stirling cooler has been designed. The research on the Stirling cooler are going to be used to verify the results obtained from numerical models and innovative design solutions applied.

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