On boundary value problems for inhomogeneous Beltrami equations

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Abstract

We give here results on the existence of nonclassical solutions of the Hilbert boundary value problem in terms of the so-called angular limits (along nontangent curves to the boundary) for Beltrami equations with sources in Jordan domains satisfying the quasihyperbolic boundary condition by Gehring–Martio, generally speaking, without \((A)\)–condition by Ladyzhenskaya–Ural’tseva and, in particular, without the known outer cone condition that were standard for boundary value problems in the PDE theory. Assuming that the coefficients of the problem are functions of countable bounded variation and the boundary data are measurable with respect to the logarithmic capacity, we prove the existence of locally Hölder continuous solutions of the problem.

Moreover, we prove similar results on the Hilbert boundary value problem with its arbitrary measurable boundary data as well as coefficients in finitely connected Jordan domains along the so-called general Bagemihl–Seidel systems of Jordan arcs to their boundaries. In the same terms, we formulate theorems on the existence of regular solutions of the known Riemann boundary value problems, including nonlinear, with arbitrary measurable coefficients for the nonhomogeneous Beltrami equations. We give also the representation of obtained solutions through the so-called generalized analytic functions with sources.

Finally, we formulate similar results on Poicare and Neumann problems for the Poisson type equations that are main in hydromechanics (mechanics of incompressible fluids) in anisotropic and inhomogeneous media. We give here the representation of their solutions through the so-called generalized harmonic functions with sources that describe the corresponding physical processes in isotropic and homogeneous media.

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1 Introduction

The present paper is a natural continuation of articles [11], [16]–[24], [46] and [47] devoted to the Riemann, Hilbert, Dirichlet, Poincare and, in particular, Neumann boundary value problems in Jordan domains for quasiconformal, analytic, harmonic and the so-called $A$–harmonic functions with arbitrary boundary data that are measurable with respect to logarithmic capacity, as well as with respect to the natural parameter in domains with rectifiable boundaries, see [38]–[42]. Relevant definitions with history notes and necessary comments on the previous results can be found e.g. in [21] and [23]. We recall here only some of them.

The first part of article [21] was devoted to the proof of existence of nonclassical solutions of Riemann, Hilbert and Dirichlet boundary value problems with arbitrary measurable boundary data with respect to logarithmic capacity for the equations of the Vekua type

$$\partial z h(z) = g(z)$$  \hspace{1cm} (1.1)

with real valued functions $g$ in classes $L^p(D)$, $p > 2$, in the corresponding domains $D \subset \mathbb{C}$. We called continuous solutions $h$ of the equations (1.1) with first Sobolev partial derivatives generalized analytic functions with the sources $g$. Note that these results are valid for complex valued sources $g$, see also [22], and that equation (1.1) is the complex form of the Poisson equation, which describes physical processes in homogeneous and isotropic media.

As a consequence, the second part of article [21] contained the proof of existence of nonclassical solutions to the Poincare problem on the directional derivatives and, in particular, to the Neumann problem with arbitrary measurable boundary data with respect to logarithmic capacity for the Poisson equations

$$\Delta U(z) = G(z)$$  \hspace{1cm} (1.2)

with real valued functions $G$ of a class $L^p(D)$, $p > 2$. For short, we called continuous solutions to (1.2) in $W^{2,p}_{\text{loc}}(D)$ generalized harmonic functions with the sources $G$. Note that by the Sobolev embedding theorem, see Theorem I.10.2 in [43], such functions belong to the class $C^1$.

Moreover, in [23] we studied the Hilbert boundary value problem for the so-called Beltrami equation that is the complex form of the main equation of the hydromechanics (incompressible fluid mechanics) in anisotropic and inhomogeneous media, however, without any sources.

Recall that the Beltrami equation is the equation of the form

$$f_{\bar{z}} = \mu(z)f_z$$  \hspace{1cm} (1.3)

where $\mu : D \to \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e., $f_{\bar{z}} = \partial \bar{f} = (f_x - if_y)/2$, $f_z = \partial f = (f_x + if_y)/2$, $z = x + iy$, $f_x$ and $f_y$ are partial derivatives of the function $f$ in $x$ and $y$, respectively. Note that continuous functions with the generalized derivative $f_{\bar{z}} = 0$ are analytic functions, see e.g. Lemma 1 in [2].
Equation (1.3) is said to be nondegenerate if $||\mu||_\infty < 1$ that we assume later on. Homeomorphic solutions $f$ of nondegenerate (1.3) in $W^{1,2}_{loc}$ are called quasiconformal mappings or sometimes $\mu$–conformal mappings. Its continuous solutions in $W^{1,2}_{loc}$ are called $\mu$–conformal functions. Existence theorems see e.g. in monographs [1], [8] and [33].

In the present paper, we study the boundary value problems for the Beltrami equations with sources. Namely, here we will research the nonhomogeneous Beltrami equations in the complex plane $\mathbb{C}$ or in its domains $D$:

$$\omega_\bar{z} = \mu(z) \cdot \omega_z + \sigma(z) .$$  

(1.4)

Following [2], see also monograph [1], let us first assume that the source $\sigma : \mathbb{C} \to \mathbb{C}$ belongs to class $L_p(\mathbb{C})$ for some $p > 2$ with

$$kC_p < 1 , \quad k := ||\mu||_\infty < 1 ,$$  

(1.5)

where $C_p$ is the norm of the known operator $T : L_p(\mathbb{C}) \to L_p(\mathbb{C})$ defined through the Cauchy principal limit of the singular integral

$$(Tg)(\zeta) := \lim_{\varepsilon \to 0} \left\{ -\frac{1}{\pi} \int_{|z-\zeta|>\varepsilon} \frac{g(z)}{(z-\zeta)^2} \, dx dy \right\} , \quad z = x + iy .$$  

(1.6)

As known, $||Tg||_2 = ||g||_2$, i.e. $C_2 = 1$, and by the Riesz convexity theorem $C_p \to 1$ as $p \to 2$. Thus, there are such $p$, whatever the value of $k$ in (1.4).

Let us denote by $B_p$ the Banach space of functions $\omega$, defined on the whole plane $\mathbb{C}$, which satisfy a global H"older condition of order $1 - 2/p$, which vanish at the origin, and whose generalized derivatives $\omega_z$ and $\omega_\bar{z}$ exist and belong to $L_p(\mathbb{C})$. The norm in $B_p$ is defined by

$$\|\omega\|_{B_p} := \sup_{z_1, z_2 \in \mathbb{C}} \frac{|\omega(z_1) - \omega(z_2)|}{|z_1 - z_2|^{1-2/p}} + \|\omega_z\|_p + \|\omega_\bar{z}\|_p .$$  

(1.7)

The principal result in [2], Theorem 1, is the following statement:

**Theorem A.** Let condition (1.5) hold and $\sigma \in L_p(\mathbb{C})$ for $p > 2$. Then the equation (1.4) has a unique solution $\omega^{\mu, \sigma} \in B_p$. This is the only solution with $\omega(0) = 0$ and $\omega_z \in L_p(\mathbb{C})$.

Its following consequence holds, see Theorem 4 and Lemma 8 in [2].

**Theorem B.** Let $\mu : \mathbb{C} \to \mathbb{C}$ be in $L_\infty(\mathbb{C})$ with compact support and $k := ||\mu||_\infty < 1$. Then there exists a unique $\mu$–conformal mapping $f^\mu$ in $\mathbb{C}$ which vanishes at the origin and satisfies condition $f^\mu_\bar{z} - 1 \in L_p(\mathbb{C})$ for any $p > 2$ with (1.5). Moreover, $f^\mu(z) = z + \omega^{\mu, \mu}(z)$.
2 Factorization of nonhomogeneous equations

The following simple statement follows by point (vii) of Theorem 5 in [2].

Remark 1. Let \( D \) be a bounded domain in \( \mathbb{C} \) and let \( \mu : D \to \mathbb{C} \) be in class \( L_\infty(D) \) with \( k := \|\mu\|_\infty < 1 \). If \( \omega_1 \) and \( \omega_2 \) are continuous solutions of (1.4) in \( D \) of class \( W^{1,2}_{loc}(D) \), then \( \omega_2 - \omega_1 = A \circ f^\mu \), where \( \mu \) was extended onto \( \mathbb{C} \) by zero outside of \( D \) and \( A \) is an analytic function in the domain \( D^* := f^\mu(D) \).

Note that here we assumed nothing on \( \sigma \).

The next representation of solutions of (1.4) is much more important.

Lemma 1. Let \( D \) be a bounded domain in \( \mathbb{C} \), \( \mu : D \to \mathbb{C} \) be in class \( L_\infty(D) \) with \( k := \|\mu\|_\infty < 1 \) and let \( \sigma : D \to \mathbb{C} \) be in class \( L_p(D), p > 2 \), with condition (1.3). Then each continuous solution \( \omega \) of equation (1.4) in \( D \) of class \( W^{1,p}(D) \) has the representation as a composition \( h \circ f^\mu \mid_D \), where \( h \) is a generalized analytic function in the domain \( D^* := f^\mu(D) \) with the source \( g \in L_p(D^*), p_* := p^2/2(p-1) \in (2,p) \),

\[
g := \left(f^\mu \frac{\sigma}{J}\right) \circ (f^\mu)^{-1}, \tag{2.1}
\]

where \( J \) is the Jacobian of a quasiconformal mapping \( f^\mu : \mathbb{C} \to \mathbb{C} \) with some extension of \( \mu \) onto \( \mathbb{C} \). Inversely, if \( h \) is a generalized analytic function with source (2.1), then \( \omega := h \circ f^\mu \) is a solution of (1.4) of class \( C^\alpha_{loc} \cap W^1_{loc}(D) \), where \( \alpha = 1 - 2/q \) and \( q := p^2/2(p_* - 1) \in (2,p_*) \).

Remark 2. Note that if \( h \) is a generalized analytic function with the source \( g \) in the domain \( D^* \), then \( H = h + A \) is so for any analytic function \( A \) in \( D^* \) but \( |A|^p \) can be integrable only locally in \( D^* \). The source in (2.1) is always in class \( L_{p_*}(D^*) \), \( p_* := p^2/2(p-1) \in (2,p) \), in view of Theorem A with \( \sigma \) extended onto \( \mathbb{C} \) by zero outside of \( D \). Here we may assume that \( \mu \) is extended onto \( \mathbb{C} \) by zero outside of \( D \). However, any other extension of \( \mu \) keeping condition (1.3) seems suitable here, too.

Proof. To be short, let us apply here the notation \( f \) instead of \( f^\mu \). Let us consider the function \( h := \omega \circ f^{-1} \). First of all, note that by point (iii) of Theorem 5 in [2] \( f^* := f^{-1} \mid_{D^*} \), \( D^* := f(D) \), is of class \( W^{1,p}(D^*) \). Then, arguing as under the proof of Lemma 10 in [2], we obtain that \( h \in W^{1,p}(D^*) \), where \( p_* := p^2/2(p-1) \in (2,p) \). Since \( \omega = h \circ f \), we get also, see e.g. formulas (28) in [2] or formulas I.C(1) in [1], that

\[
\omega_z = (h_\zeta \circ f) \cdot f_z + (h_\zeta \circ f) \cdot \bar{f}_z ,
\]

and, thus,

\[
\sigma(z) = \omega_{\bar{z}} - \mu(z) \omega_z = (h_\zeta \circ f) \bar{f}_z (1-|\mu(z)|^2) = (h_\zeta \circ f) J(z)/f_z ,
\]

where \( J(z) = |f_z|^2 - |f_{\bar{z}}|^2 = |f_z|^2 (1-|\mu(z)|^2) \) is the Jacobian of \( f \), i.e.,

\[
h_{\zeta} = g(\zeta) := \left(f_z \frac{\sigma}{J}\right) \circ f^{-1}(\zeta) .
\]
Similarly, applying Lemma 10 in [2] and the Sobolev embedding theorem, see Theorem I.10.2 in [43], we come to the inverse conclusion.

\[ \square \]

3 Hilbert problem with angular limits

In this section, we prove the existence of nonclassical solutions of the Hilbert boundary value problem with arbitrary boundary data that are measurable with respect to logarithmic capacity for nonhomogeneous Beltrami equations. The result is formulated in terms of the angular limit that is a traditional tool of the geometric function theory, see e.g. monographs [10, 29, 34, 36] and [37].

Recall that the classic boundary value problem of Hilbert, see [26], was formulated as follows: To find an analytic function \( f(z) \) in a domain \( D \) bounded by a rectifiable Jordan contour \( C \) that satisfies the boundary condition

\[
\lim_{z \to \zeta, z \in D} \text{Re} \left\{ \lambda(\zeta) f(z) \right\} = \varphi(\zeta) \quad \forall \zeta \in C ,
\]  

(3.1)

where the coefficient \( \lambda \) and the boundary date \( \varphi \) of the problem are continuously differentiable with respect to the natural parameter \( s \) and \( \lambda \neq 0 \) everywhere on \( C \). The latter allows to consider that \( |\lambda| \equiv 1 \) on \( C \). Note that the quantity \( \text{Re} \left\{ \lambda f \right\} \) in (3.1) means a projection of \( f \) into the direction \( \lambda \) interpreted as vectors in \( \mathbb{R}^2 \).

The reader can find a comprehensive treatment of the theory in excellent books [5, 6, 25, 44]. We also recommend to make familiar with historic surveys in monographs [13, 35, 45] on the topic with an exhaustive bibliography and take a look at our recent papers, see Introduction.

Next, recall that a straight line \( L \) is tangent to a curve \( \Gamma \) in \( \mathbb{C} \) at a point \( z_0 \in \Gamma \) if

\[
\limsup_{z \to z_0, z \in \Gamma} \frac{\text{dist} (z, L)}{|z - z_0|} = 0 .
\]  

(3.2)

Let \( D \) be a Jordan domain in \( \mathbb{C} \) with a tangent at a point \( \zeta \in \partial D \). A path in \( D \) terminating at \( \zeta \) is called nontangential if its part in a neighborhood of \( \zeta \) lies inside of an angle with the vertex at \( \zeta \). The limit along all nontangential paths at \( \zeta \) is called angular at the point.

Following [23], we say that a Jordan curve \( \Gamma \) in \( \mathbb{C} \) is almost smooth if \( \Gamma \) has a tangent q.e. (quasi everywhere) with respect to logarithmic capacity, see e.g. [32] for the term. In particular, \( \Gamma \) is almost smooth if \( \Gamma \) has a tangent at all its points except its countable collection. The nature of such a Jordan curve \( \Gamma \) can be complicated enough because this countable collection can be everywhere dense in \( \Gamma \), see e.g. [9].

Recall that the quasihyperbolic distance between points \( z \) and \( z_0 \) in a domain \( D \subset \mathbb{C} \) is the quantity

\[
k_{D}(z, z_0) := \inf_{\gamma} \int_{\gamma} ds/d\zeta (\zeta, \partial D) ,
\]  

where \( \gamma \) is a curve in \( D \).
where \( d(\zeta, \partial D) \) denotes the Euclidean distance from the point \( \zeta \in D \) to \( \partial D \) and the infimum is taken over all rectifiable curves \( \gamma \) joining the points \( z \) and \( z_0 \) in \( D \), see [13].

Further, it is said that a domain \( D \) satisfies the **quasihyperbolic boundary condition** if there exist constants \( a \) and \( b \) and a point \( z_0 \in D \) such that

\[
kd(z, z_0) \leq a + b \ln \frac{d(z, \partial D)}{d(z_0, \partial D)} \quad \forall z \in D. 
\]

The latter notion was introduced in [14] but, before it, was first implicitly applied in [7]. By the discussion in [23], every smooth (or Lipschitz) domain satisfies the quasihyperbolic boundary condition but such boundaries can be nowhere locally rectifiable.

Note that it is well–known the so–called \((A)\)–condition by Ladyzhenskaya–Ural’tseva, which is standard in the theory of boundary value problems for PDE, see e.g. [31]. Recall that a domain \( D \) in \( \mathbb{R}^n \), \( n \geq 2 \), is called satisfying \((A)\)-condition if

\[
\text{mes } D \cap B(\zeta, \rho) \leq \Theta_0 \text{ mes } B(\zeta, \rho) \quad \forall \, \zeta \in \partial D , \, \rho \leq \rho_0
\]

for some \( \Theta_0 \) and \( \rho_0 \in (0, 1) \), where \( B(\zeta, \rho) \) denotes the ball with the center \( \zeta \in \mathbb{R}^n \) and the radius \( \rho \), see 1.1.3 in [31].

A domain \( D \) in \( \mathbb{R}^n \), \( n \geq 2 \), is said to be satisfying the **outer cone condition** if there is a cone that makes possible to be touched by its top to every point of \( \partial D \) from the completion of \( D \) after its suitable rotations and shifts. It is clear that the latter condition implies \((A)\)–condition.

Probably one of the simplest examples of an almost smooth domain \( D \) with the quasihyperbolic boundary condition and without \((A)\)–condition is the union of 3 open disks with the radius 1 centered at the points 0 and 1 ± i. It is clear that this domain has zero interior angle at its boundary point 1.

Given a Jordan domain \( D \) in \( \mathbb{C} \), we call \( \lambda : \partial D \to \mathbb{C} \) a function of bounded variation, write \( \lambda \in BV(\partial D) \), if

\[
V_\lambda(\partial D) := \sup \sum_{j=1}^k |\lambda(\zeta_{j+1}) - \lambda(\zeta_j)| < \infty
\]

where the supremum is taken over all finite collections of points \( \zeta_j \in \partial D, \, j = 1, \ldots, k, \) with the cyclic order meaning that \( \zeta_j \) lies between \( \zeta_{j+1} \) and \( \zeta_{j-1} \) for every \( j = 1, \ldots, k \). Here we assume that \( \zeta_{k+1} = \zeta_1 = \zeta_0 \). The quantity \( V_\lambda(\partial D) \) is called the **variation of the function** \( \lambda \).

Now, we call \( \lambda : \partial D \to \mathbb{C} \) a function of **countable bounded variation**, write \( \lambda \in CBV(\partial D) \), if there is a countable collection of mutually disjoint arcs \( \gamma_n \) of \( \partial D, \, n = 1, 2, \ldots \) on each of which the restriction of \( \lambda \) is of bounded variation and the set \( \partial D \setminus \bigcup \gamma_n \) has logarithmic capacity zero. In particular, the latter holds true if the set \( \partial D \setminus \bigcup \gamma_n \) is countable. It is clear that such functions can be singular enough.
Theorem 1. Let $D$ be a Jordan domain with the quasihyperbolic boundary condition, $\partial D$ have a tangent q.e., $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and let $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity. Suppose also that $\mu : D \to \mathbb{C}$ is of class $L^\infty(D)$ with $k := \|\mu\|_\infty < 1$, it is Hölder continuous in a neighborhood of $\partial D$, $\sigma \in L^p(D)$ and condition (1.2) hold for some $p > 2$.

Then equation (1.4) has solutions $\omega : D \to \mathbb{C}$ of class $C^0_{\text{loc}} \cap W^{1,q}(D)$, where $\alpha = 1 - 2/q$ and $q \in (2, p)$, smooth in the neighborhood of $\partial D$ with the angular limits
\[
\lim_{z \to \zeta, z \in D} \text{Re} \left\{ \frac{\lambda(\zeta)}{\omega(z)} \right\} = \varphi(\zeta) \quad \text{q.e. on } \partial D.
\] (3.6)

Furthermore, the space of all such solutions $\omega$ of the equation (1.4) has infinite dimension for each fixed $\lambda$, $\varphi$, $\mu$ and $\sigma$.

Remark 3. By the construction in the proof below, each such solution has the representation $\omega = h \circ f_D$, where $f = f^\mu : \mathbb{C} \to \mathbb{C}$ is a quasiuniform mapping with the corresponding extension of $\mu$ onto $\mathbb{C}$ and $h$ is a generalized analytic function with the source $g \in L^p(D)$, $p_* := p^2/(p - 1) \in (2, p)$, in (2.1), $D_* := f(D)$, and the angular limits
\[
\lim_{w \to \xi, w \in D_*} \text{Re} \left\{ \frac{\lambda(\xi)}{h(w)} \right\} = \Phi(\xi) \quad \text{q.e. on } \partial D_* ,
\] (3.7)
with $\Lambda := \lambda \circ f^{-1}$, $\Phi := \varphi \circ f^{-1}$. Also, $q = p^2/(p_* - 1) \in (2, p_*).

Proof. First of all, let us choose a suitable extension of $\mu$ onto $\mathbb{C}$ outside of $D$. By hypotheses of Theorem 1 $\mu$ belongs to a class $C^\alpha$, $\alpha \in (0, 1)$, for an open neighborhood $U$ of $\partial D$ inside of $D$. By Lemma 1 in [21] $\mu$ is extended to a Hölder continuous function $\mu : U \cup \mathbb{C} \setminus D \to \mathbb{C}$ of the class $C^\alpha$. Then, for every $k_* \in (k, 1)$, there is an open neighborhood $V$ of $\partial D$ in $\mathbb{C}$, where $\|\mu\|_\infty \leq k_*$ and $\mu$ in $C^\alpha(V)$. Let us choose $k_* \in (k, 1)$ so close to $k$ that $k_* C_\mu^p < 1$ and set $\mu \equiv 0$ outside of $U \cup V$.

By the Measurable Riemann Mapping Theorem, see e.g. [1], [8] and [33], there is a quasiconformal mapping $f = f^\mu : \mathbb{C} \to \mathbb{C}$ a.e. satisfying the Beltrami equation (1.3) with the given extended complex coefficient $\mu$ in $\mathbb{C}$. Note that the mapping $f$ has the Hölder continuous first partial derivatives in $V$ with the same order of the Hölder continuity as $\mu$, see e.g. [27] and also [28]. Moreover, its Jacobian
\[
J(z) \neq 0 \quad \forall \ z \in V,
\] (3.8)
see e.g. Theorem V.7.1 in [33]. Hence $f^{-1}$ is also smooth in $V_* := f(V)$, see e.g. formulas IC(3) in [1].

Now, the domain $D_* := f(D)$ satisfies the boundary quasihyperbolic condition because $D$ is so, see e.g. Lemma 3.20 in [14]. Moreover, $\partial D_*$ has q.e. tangents, furthermore, the points of $\partial D$ and $\partial D^*$ with tangents correspond each to other in one-to-one manner because the mappings $f$ and $f^{-1}$ are smooth there. In addition, the function $\Lambda := \lambda \circ f^{-1}$ belongs to the class $\mathcal{CBV}(\partial D_*)$ and $\Phi := \varphi \circ f^{-1}$ is measurable with respect to logarithmic capacity, see e.g.
Remark 2.1 in [23]. Next, by Remark 3 the source \( g : D_\ast \to \mathbb{C} \) in (2.1) belongs to class \( L_{p_\ast}(D_\ast) \), where \( p_\ast = p^2/(p - 1) \in (2, p) \). Thus, by Theorem 1 in [21] the space of all generalized analytic functions \( h : D_\ast \to \mathbb{C} \) with the source \( g \) and the angular limits (3.4) q.e. on \( \partial D_\ast \) has infinite dimension. Finally, by Lemma 1 we obtain the rest of conclusions.

In particular case \( \lambda \equiv 1 \), we obtain the consequence of Theorem 1 on the Dirichlet problem for the nonhomogeneous Beltrami equations.

4 Hilbert problem along special curve systems

Let \( D \) be a domain in \( \mathbb{C} \) whose boundary consists of a finite collection of mutually disjoint Jordan curves. A family of mutually disjoint Jordan arcs \( J_\zeta : [0, 1] \to \mathcal{D} \), \( \zeta \in \partial D \), with \( J_\zeta([0, 1]) \subset D \) and \( J_\zeta(1) = \zeta \) that is continuous in the parameter \( \zeta \) is called a Bagemihl–Seidel system or, in short, of class BS in \( D \).

Lemma 2. Let \( D \) be a bounded domain in \( \mathbb{C} \) whose boundary consists of a finite number of mutually disjoint Jordan curves, \( \{\gamma_\zeta\}_{\zeta \in \partial D} \) be a family of Jordan arcs of class BS and let \( \lambda : \partial D \to \mathbb{C} \), \( |\lambda(\zeta)| \equiv 1 \), \( \varphi : \partial D \to \mathbb{R} \) and \( \psi : \partial D \to \mathbb{R} \) be measurable with respect to logarithmic capacity.

Suppose also that \( \mu : D \to \mathbb{C} \) is of class \( L_\infty(D) \) with \( k := \|\mu\|_\infty < 1 \), \( \sigma \in L_p(D) \) and condition (1.7) hold for some \( p > 2 \). Then the equation (1.3) has solutions \( \omega : D \to \mathbb{C} \) of class \( C^\ast_{\text{loc}} \cap W^{1, \beta}_{\text{loc}} \) with \( \alpha = 1 - 2/q \) for some \( q \in (2, p) \) such that along the arcs \( \gamma_\zeta \)

\[
\lim_{z \to \zeta} \Re \left\{ \overline{\lambda(\zeta)} \cdot \omega(z) \right\} = \varphi(\zeta) \quad \text{q.e. on } \partial D, \tag{4.1}
\]
\[
\lim_{z \to \zeta} \Im \left\{ \overline{\lambda(\zeta)} \cdot \omega(z) \right\} = \psi(\zeta) \quad \text{q.e. on } \partial D. \tag{4.2}
\]

Remark 4. By the construction in the proof below, each such solution has the representation \( \omega = h \circ f_\xi \), where \( f = f^\mu : \mathbb{C} \to \mathbb{C} \) is a quasiconformal mapping with \( \mu \) extended by zero onto \( \mathbb{C} \) outside of \( D \) and \( h \) is a generalized analytic function with the source \( g \in L_{p_\ast}(D_\ast) \), \( p_\ast := p^2/(p - 1) \in (2, p) \), in (2.1), \( D_\ast := f(D) \), and with the limits along Jordan arcs \( \Gamma_\xi := f(\gamma_{f^{-1}(\xi)}) \), \( \xi \in \partial D_\ast \),

\[
\lim_{w \to \xi, w \in D_\ast} \Re \left\{ \overline{\lambda(\xi)} \cdot h(w) \right\} = \Phi(\xi) \quad \text{q.e. on } \partial D_\ast, \tag{4.3}
\]
\[
\lim_{w \to \xi, w \in D_\ast} \Im \left\{ \overline{\lambda(\xi)} \cdot h(w) \right\} = \Psi(\xi) \quad \text{q.e. on } \partial D_\ast, \tag{4.4}
\]

where \( \lambda := \lambda \circ f^{-1} \), \( \Phi := \varphi \circ f^{-1} \), \( \Psi := \psi \circ f^{-1} \).

Moreover, more precisely, \( q = p_\ast^2/(p_\ast - 1) \in (2, p_\ast) \).

Proof. First of all note that functions \( \lambda, \Phi \) and \( \Psi \) in Remark 4 are measurable with respect to logarithmic capacity on \( \partial D_\ast \) because the quasiconformal
mapping $f$ is Hölder continuous on the compact set $\partial D$, see e.g. Remark 2.1 in [23]. Consequently, by Theorem 2 in [21], see also Remark 2 above, there exists a generalized analytic function $h$ with the source $g$ in (2.1) satisfying conditions (4.3) and (4.4) q.e. on $\partial D_\ast$. Thus, the rest of conclusions follows by Lemma 1, see again Remark 2.1 in [23], because the inverse quasiconformal mapping $f^{-1}$ is Hölder continuous on the compact set $\partial D_\ast$, too.

**Remark 5.** The space of all generalized analytic functions $h$ with the source $g$ in (2.1) satisfying Hilbert boundary condition (4.3) q.e. on $\partial D_\ast$ along the Jordan arcs $\Gamma_\xi := f(\gamma f^{-1}(\xi))$, $\xi \in \partial D_\ast$, has infinite dimension for any prescribed $g$, $\Phi$, $\Lambda$ and $\{\Gamma_\xi\}_{\xi \in \partial D_\ast}$ of class $BS$ because the space of all functions $\Psi : \partial D_\ast \to \mathbb{R}$ which are measurable with respect to logarithmic capacity has infinite dimension.

The latter is valid even for its subspace of continuous functions $\Psi : \partial D_\ast \to \mathbb{R}$. Indeed, every Jordan component of $\partial D_\ast$ can be mapped with a homeomorphism onto the unit circle $\partial \mathbb{D}$. However, by the Fourier theory, the space of all continuous functions $\tilde{\Psi} : \partial \mathbb{D} \to \mathbb{R}$, equivalently, the space of all continuous $2\pi$-periodic functions $\Psi : \mathbb{R} \to \mathbb{R}$, just has infinite dimension.

Thus, by Lemma 2, Remarks 4 and 5 we obtain the following statement on solutions of the equation (1.4), satisfying the Hilbert boundary condition (4.1) q.e. along Behgemil-Seidel systems of Jordan arcs.

**Theorem 2.** Let $D$ be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, $\{\gamma_\zeta\}_{\zeta \in \partial D}$ be a family of Jordan arcs of class $BS$ in $D$, $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.

Suppose also that $\mu : D \to \mathbb{C}$ is of class $L_\infty(D)$ with $k := \|\mu\|_\infty < 1$, $\sigma \in L_p(D)$ and condition (1.3) hold for some $p > 2$. Then the equation (1.4) has solutions $\omega : D \to \mathbb{C}$ of class $C^\alpha_{loc} \cap W^{1,q}_{loc}$ with $\alpha = 1 - 2/q$ for some $q \in (2,p)$ that satisfy the Hilbert boundary condition (4.1) q.e. in the sense of the limits along $\gamma_\zeta$.

Furthermore, the space of all such solutions $\omega$ has infinite dimension for any fixed $\mu$, $\sigma$, $\varphi$, $\lambda$ and $\{\gamma_\zeta\}_{\zeta \in \partial D}$.

Moreover, each such solution has the representation $\omega = h \circ f|_D$, where $f = f^\mu : \mathbb{C} \to \mathbb{C}$ is a quasiconformal mapping with $\mu$ extended by zero onto $\mathbb{C}$ outside of $D$ and $h$ is a generalized analytic function with the source $g \in L_{p_\ast}(D_\ast)$, $p_\ast := p^2/(p-1) \in (2,p)$, in (2.7), $D_\ast := f(D)$, satisfying the Hilbert boundary condition (4.3) q.e. on $\partial D_\ast$ in the sense of the limits along Jordan arcs $\Gamma_\xi := f(\gamma f^{-1}(\xi))$, $\xi \in \partial D_\ast$.

In particular case $\lambda \equiv 1$, we obtain the corresponding consequence on the Dirichlet problem for the nonhomogeneous Beltrami equations (1.4) along any prescribed Bagemihl-Seidel systems of Jordan arcs.
5 Riemann problem and special curve systems

Recall that the classical setting of the Riemann problem in a smooth Jordan domain $D$ of the complex plane $\mathbb{C}$ is to find analytic functions $f^+: D \to \mathbb{C}$ and $f^-: \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ that admit continuous extensions to $\partial D$ and satisfy the boundary condition

$$f^+(\zeta) = A(\zeta) \cdot f^-(\zeta) + B(\zeta) \quad \forall \zeta \in \partial D$$

(5.1)

with its prescribed Hölder continuous coefficients $A: \partial D \to \mathbb{C}$ and $B: \partial D \to \mathbb{C}$. Recall also that the Riemann problem with shift in $D$ is to find analytic functions $f^+: D \to \mathbb{C}$ and $f^-: \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ satisfying the condition

$$f^+(\alpha(\zeta)) = A(\zeta) \cdot f^-(\zeta) + B(\zeta) \quad \forall \zeta \in \partial D$$

(5.2)

where $\alpha: \partial D \to \partial D$ was a one-to-one sense preserving correspondence having the non-vanishing Hölder continuous derivative with respect to the natural parameter on $\partial D$. The function $\alpha$ is called a shift function. The special case $A \equiv 1$ gives the so-called jump problem and then $B \equiv 0$ gives the problem on gluing of analytic functions.

Arguing similarly to the proof of Lemma 2, on the base of Lemma 1, we reduce the proof of the next theorem below to Theorem 3 in [21] on the Riemann problem for generalized analytic functions with sources.

**Theorem 3.** Let $D$ be a domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, functions $A: \partial D \to \mathbb{C}$ and $B: \partial D \to \mathbb{C}$ be measurable with respect to logarithmic capacity, \(\{\gamma_+^\zeta\}_{\zeta \in \partial D}\) and \(\{\gamma_-^\zeta\}_{\zeta \in \partial D}\) be families of Jordan arcs of class B$\infty$ in $D$ and $\mathbb{C} \setminus \overline{\Delta}$, correspondingly. Suppose also that $\mu: \mathbb{C} \to \mathbb{C}$ is of class $L_\infty(\mathbb{C})$ with $k := \|\mu\|_\infty < 1$, $\sigma \in L_p(\mathbb{C})$ has compact support and condition (1.5) hold for some $p > 2$.

Then the nonhomogeneous Beltrami equation (1.4) has solutions $\omega^+: D \to \mathbb{C}$ and $\omega^-: \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ of class $C^{\alpha}_{\text{loc}} \cap W^{1,q}_{\text{loc}}$ with $\alpha = 1 - 2/q$ for some $q \in (2, p)$, satisfying the Riemann boundary condition (5.1) q.e. on $\partial D$, where $\omega^+(\zeta)$ and $\omega^-(\zeta)$ are limits of $\omega^+(z)$ and $\omega^-(z)$ as $z \to \zeta$ along $\gamma_+^\zeta$ and $\gamma_-^\zeta$, correspondingly.

Furthermore, the space of all such couples of solutions $(\omega^+, \omega^-)$ of (1.4) has infinite dimension for any fixed $\mu$, $\sigma$, couples $(A, B)$ and collections $\gamma_+^\zeta$ and $\gamma_-^\zeta$, $\zeta \in \partial D$.

Moreover, each such couples of solutions $(\omega^+, \omega^-)$ of (1.4) has the representation in the form of the composition of the corresponding couples $(h^+, h^-)$ of generalized analytic functions with the source $g$ in (2.1) and $f^\mu: \mathbb{C} \to \mathbb{C}$.

Theorem 3 is a special case of the following lemma, whose proof is reduced to Lemma 1 in [21] on the Riemann boundary value problem with shifts for generalized analytic functions with source $g$ given by (2.1) on the base of Lemma 1 above, that may have of independent interest.
Lemma 3. Under the hypotheses of Theorem 3, let in addition that \( \alpha : \partial D \to \partial D \) be a homeomorphism keeping components of \( \partial D \) such that \( \alpha \) and \( \alpha^{-1} \) have \((N)\)-property with respect to logarithmic capacity.

Then the nonhomogeneous Beltrami equation \( (1.4) \) has solutions \( \omega^+ : D \to \mathbb{C} \) and \( \omega^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C} \) of class \( C^\infty_{\text{loc}} \cap W^{1,q}_{\text{loc}} \) with \( \alpha = 1 - 2/q \) for some \( q \in (2,p) \), satisfying the Riemann boundary condition with shift \( (6.1) \) q.e. on \( \partial D \), where \( \omega^+(\zeta) \) and \( \omega^-(\zeta) \) are limits of \( \omega^+(z) \) and \( \omega^-(z) \) as \( z \to \zeta \) along \( \gamma^+ \) and \( \gamma^- \), correspondingly.

Furthermore, the space of all such couples of solutions \( (\omega^+,\omega^-) \) has infinite dimension for any fixed \( \mu, \sigma \), couples \( (A,B) \) and collections \( \gamma^+ \) and \( \gamma^- \), \( \zeta \in \partial D \).

Again, each such couple of solutions \( (\omega^+,\omega^-) \) of \( (1.4) \) has the representation in the form of the composition of the corresponding couples \( (h^+,h^-) \) of generalized analytic functions with the source \( g \) in \( (2.1) \) and \( f^\mu : \mathbb{C} \to \mathbb{C} \).

6 Nonlinear Riemann boundary value problems

We are able by our scheme above to formulate also a series of results on nonlinear Riemann boundary value problems in terms of Bagemihl–Seidel systems for Beltrami equations with sources and representations.

For instance, special nonlinear boundary value problems of the form

\[
\omega^+(\zeta) = \varphi(\zeta, \omega^-(\zeta)) \quad \text{q.e. on } \zeta \in \partial D
\]  

(6.1)

are solved if \( \varphi : \partial D \times \mathbb{C} \to \mathbb{C} \) satisfies the Carathéodory conditions with respect to logarithmic capacity, i.e., if \( \varphi(\zeta, w) \) is continuous in the variable \( w \in \mathbb{C} \) for q.e. \( \zeta \in \partial D \) and it is measurable with respect to logarithmic capacity in the variable \( \zeta \in \partial D \) for all \( w \in \mathbb{C} \). Later on, we sometimes say in short "C-measurable" instead of the expression "measurable with respect to logarithmic capacity".

Theorem 4. Let \( D \) be a domain in \( \mathbb{C} \) whose boundary consists of a finite number of mutually disjoint Jordan curves, functions \( A : \partial D \to \mathbb{C} \) and \( B : \partial D \to \mathbb{C} \) be measurable and \( \varphi : \partial D \times \mathbb{C} \to \mathbb{C} \) satisfy the Carathéodory conditions with respect to logarithmic capacity, \( \{\gamma^+_\zeta\}_{\zeta \in \partial D} \) and \( \{\gamma^-_\zeta\}_{\zeta \in \partial D} \) be families of Jordan arcs of class BS in \( D \) and \( \mathbb{C} \setminus \overline{D} \), correspondingly.

Suppose also that \( \mu : \mathbb{C} \to \mathbb{C} \) is of class \( L^\infty_{\text{loc}}(\mathbb{C}) \) with \( k := \|\mu\|_{\infty} < 1, \sigma \in L^p_{\text{loc}}(\mathbb{C}) \) has compact support and condition \( (1.7) \) hold for some \( p > 2 \). Then the nonhomogeneous Beltrami equation \( (1.4) \) has solutions \( \omega^+ : D \to \mathbb{C} \) and \( \omega^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C} \) of class \( C^\infty_{\text{loc}} \cap W^{1,q}_{\text{loc}} \) with \( \alpha = 1 - 2/q \) for some \( q \in (2,p) \), satisfying the nonlinear Riemann boundary condition \( (6.1) \) q.e. on \( \partial D \), where \( \omega^+(\zeta) \) and \( \omega^-(\zeta) \) are limits of \( \omega^+(z) \) and \( \omega^-(z) \) as \( z \to \zeta \) along \( \gamma^+ \) and \( \gamma^- \), correspondingly.

Furthermore, the space of all such couples of solutions \( (\omega^+,\omega^-) \) has infinite dimension for any fixed \( \mu, \sigma \), couples \( (A,B) \) and collections \( \gamma^+ \) and \( \gamma^- \), \( \zeta \in \partial D \).

Proof. The spaces of solutions of such problems always have infinite dimension. Indeed, by the Egorov theorem, see e.g. Theorem 2.3.7 in [12], see also
Section 17.1 in [30], the function \( \varphi(\zeta, \psi(\zeta)) \) is \( C \)-measurable in \( \zeta \in \partial D \) for every \( C \)-measurable function \( \psi : \partial D \to \mathbb{C} \) if the function \( \varphi \) satisfies the Carathéodory conditions, and the space of all \( C \)-measurable functions \( \psi : \partial D \to \mathbb{C} \) has the infinite dimension, see e.g. arguments in Remark 5.

Finally, applying Theorem 3 firstly to each component of \( \mathbb{C} \setminus \overline{D} \) with \( A \equiv 0 \) and \( B = \psi \) on the corresponding component of \( \partial D \), we see that \( \omega^- \) can be just \( \psi \), and secondly to the domain \( D \) with \( A \equiv 0 \) and \( B(\zeta) := \varphi(\zeta, \psi(\zeta)), \zeta \in \partial D \), we come to the desired conclusion. \( \square \)

Similarly we also obtain the following consequence of Lemma 3 on the non-linear Riemann boundary value problems with shifts.

Lemma 4. Under the hypotheses of Theorem 4, let in addition that \( \alpha : \partial D \to \partial D \) be a homeomorphism keeping components of \( \partial D \) such that \( \alpha \) and \( \alpha^{-1} \) have \( (N) \)-property with respect to logarithmic capacity.

Then the nonhomogeneous Beltrami equation (1.4) has solutions \( \omega^+ : D \to \mathbb{C} \) and \( \omega^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C} \) of class \( C^\alpha_{\text{loc}} \cap W^{1,q}_{\text{loc}} \) with \( \alpha = 1 - 2/q \) for some \( q \in (2, p) \), satisfying the nonlinear Riemann boundary condition with shift

\[
\omega^+(\alpha(\zeta)) = \varphi(\zeta, \omega^-(\zeta)) \quad \text{q.e. on} \quad \zeta \in \partial D ,
\]

where \( \omega^+(\zeta) \) and \( \omega^-(\zeta) \) are limits of \( \omega^+(z) \) and \( \omega^-(z) \) as \( z \to \zeta \) along \( \gamma^+_\zeta \) and \( \gamma^-_\zeta \), correspondingly.

Furthermore, the space of all such couples of solutions \( (\omega^+, \omega^-) \) has infinite dimension for any fixed \( \mu, \sigma \), couples \( (A, B) \) and collections \( \gamma^+_\zeta \) and \( \gamma^-_\zeta, \zeta \in \partial D \).

Representations of solutions in Theorem 4 and Lemma 4 as compositions of generalized analytic functions \( h^+ \) and \( h^- \) with the source \( g \) in (2.1) and quasiconformal mappings \( f^\mu : \mathbb{C} \to \mathbb{C} \) remain also valid.

7 Mixed nonlinear boundary-value problems

In order to demonstrate the potentiality of our approach, we give here also a couple of results on mixed nonlinear boundary value problems in terms of Bagemihl–Seidel systems for Beltrami equations with sources.

Namely, arguing similarly to the last section and to the proof of Theorem 1, we are able to reduce the proof of the following theorem to Theorem 4 in [21] on the base of Lemma 1 and Remark 2.

Theorem 5. Let \( D \) be a domain in \( \mathbb{C} \) whose boundary consists of a finite number of mutually disjoint Jordan curves, \( \varphi : \partial D \times \mathbb{C} \to \mathbb{C} \) satisfy the Carathéodory conditions and \( \nu : \partial D \to \mathbb{C}, |\nu(\zeta)| \equiv 1 \), be measurable with respect to the logarithmic capacity, \( \{\gamma^+_\zeta\}_{\zeta \in \partial D} \) and \( \{\gamma^-_\zeta\}_{\zeta \in \partial D} \) be families of Jordan arcs of class \( \text{BS} \) in \( D \) and \( \mathbb{C} \setminus \overline{D} \), correspondingly.

Suppose also that \( \mu : \mathbb{C} \to \mathbb{C} \) is of class \( L_\infty(\mathbb{C}) \) with \( k := ||\mu||_\infty < 1 \), \( \sigma \in L_p(\mathbb{C}) \) has compact support and condition (1.5) hold for some \( p > 2 \), they are both Hölder continuous in an open neighborhood \( V \) of \( \partial D \).
Then nonhomogeneous Beltrami equation (1.4) has solutions \( \omega^+ : D \to \mathbb{C} \) and \( \omega^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C} \) of class \( C^1_{\text{loc}} \cap W^{1,q}_{\text{loc}} \) with \( \alpha = 1 - 2/q \) for some \( q \in (2, p) \), that both are smooth in \( V \) and that satisfy the mixed nonlinear boundary condition

\[
\omega^+(\zeta) = \varphi \left( \zeta, \left[ \frac{\partial \omega}{\partial \nu} \right]^- \right) \quad \text{q.e. on } \partial D ,
\]

where \( \omega^+(\zeta) \) and \( \left[ \frac{\partial \omega}{\partial \nu} \right]^-(\zeta) \) are limits of the functions \( \omega^+(z) \) and \( \frac{\partial \omega^-}{\partial \nu}(z) \) as \( z \to \zeta \) along \( \gamma^+_\zeta \) and \( \gamma^-_\zeta \), correspondingly.

Furthermore, the space of all such couples \( (\omega^+, \omega^-) \) has infinite dimension for any such prescribed functions \( \sigma, \varphi, \nu \) and collections \( \gamma^+_\zeta \) and \( \gamma^-_\zeta \), \( \zeta \in \partial D \).

Indeed, by the Measurable Riemann Mapping Theorem, see e.g. [1], [8] and [33], there is a quasiconformal mapping \( f = f^\mu : \mathbb{C} \to \mathbb{C} \) a.e. satisfying the Beltrami equation (1.3) with the given complex coefficient \( \mu \) in \( \mathbb{C} \). Note also that the mapping \( f \) has locally Hölder continuous first partial derivatives in \( V \) with the same order of the Hölder continuity as \( \mu \), see e.g. [27] and also [28]. Moreover, its Jacobian

\[
J(z) \neq 0 \quad \forall \ z \in V ,
\]

see e.g. Theorem V.7.1 in [33]. Hence \( f^{-1} \) is also smooth in \( V_* := f(V) \), see e.g. formulas I.C(3) in [1]. Moreover, by Lemma 10 in [2] for \( \omega := \omega^- \) and \( h = h^- \) we have the connection between derivatives of \( \omega \) and \( h \) in the corresponding directions

\[
\frac{\partial \omega}{\partial \nu} = \omega_* \cdot \nu + \omega_\nu \cdot \nu = \nu \cdot (h_w \circ f \cdot f_z + h_\bar{w} \circ f \cdot \overline{f_z}) + \nu \cdot (h_w \circ f \cdot f_z + h_\bar{w} \circ f \cdot \overline{f_z})
\]

\[
= h_w \circ f \cdot (\nu \cdot f_z + \overline{f_z}) + h_\bar{w} \circ f \cdot (\nu \cdot \overline{f_z} + \overline{f_z}) = h_w \circ f \cdot \frac{\partial f}{\partial \nu} + h_\bar{w} \circ f \cdot \frac{\overline{f}}{\partial \nu}
\]

\[
= (N_* \cdot h_w + \overline{N_*} \cdot h_\bar{w}) \circ f = \frac{\partial h}{\partial N_*} \circ f , \quad N_* := \frac{\partial f}{\partial \nu} \circ f^{-1} .
\]

Theorem 5 is a special case of the following lemma on the corresponding mixed boundary value problem with shift.

**Lemma 5.** Under the hypotheses of Theorem 4, let in addition \( \beta : \partial D \to \partial D \) be a homeomorphism keeping components of \( \partial D \) such that \( \beta \) and \( \beta^{-1} \) have \((N)\)-property with respect to logarithmic capacity.

Then nonhomogeneous Beltrami equation (1.4) has solutions \( \omega^+ : D \to \mathbb{C} \) and \( \omega^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C} \) of class \( C^1_{\text{loc}} \cap W^{1,q}_{\text{loc}} \) with \( \alpha = 1 - 2/q \) for some \( q \in (2, p) \), that both are smooth about \( \partial D \) and that satisfy the following mixed nonlinear boundary condition

\[
\omega^+(\beta(\zeta)) = \varphi \left( \zeta, \left[ \frac{\partial \omega}{\partial \nu} \right]^- \right) \quad \text{q.e. on } \partial D ,
\]
where \( \omega^+(\zeta) \) and \( \frac{\partial \omega^+}{\partial \nu}(\zeta) \) are limits of the functions \( \omega^+(z) \) and \( \frac{\partial \omega^+}{\partial \nu}(z) \) as \( z \to \zeta \) along \( \gamma^+_{\zeta} \) and \( \gamma^-_{\zeta} \), correspondingly.

Furthermore, the space of all such couples \( (\omega^+, \omega^-) \) has infinite dimension for any such prescribed functions \( \sigma, \varphi, \nu \) and collections \( \gamma^+_{\zeta} \) and \( \gamma^-_{\zeta} \), \( \zeta \in \partial D \).

**Remark 6.** On the base of Remark 6 in [21], under the hypotheses of Theorem 5, the similar statement can be derived for the boundary gluing conditions of the form

\[
\left[ \frac{\partial \omega^+}{\partial \nu} \right](\beta(\zeta)) = \varphi \left( \zeta, \left[ \frac{\partial \omega^-}{\partial \nu} \right](\zeta) \right) \quad \text{q.e. on } \partial D. \tag{7.4}
\]

Again, representations of solutions in Theorem 5, Lemma 5 and Remark 6 as compositions of generalized analytic functions \( h^+ \) and \( h^- \) with the source \( g \) in (2.1) and quasiconformal mappings \( f^\mu : \mathbb{C} \to \mathbb{C} \) remain valid.

## 8 Poincaré and Neumann problems in terms of angular limits

It is well-known, see Theorem 16.1.6 in [3], that nonhomogeneous Beltrami equations in a domain \( D \) of the complex plane \( \mathbb{C} \) are closely connected with the divergence type equations of the form

\[
\text{div} \ A(z) \nabla u(z) = g(z), \tag{8.1}
\]

where \( A(z) \) is the matrix function:

\[
A = \begin{pmatrix}
\frac{1-|\mu|^2}{1+|\mu|^2} & -2i\text{Im}\mu \\
-2i\text{Re}\mu & \frac{1-|\mu|^2}{1+|\mu|^2}
\end{pmatrix}. \tag{8.2}
\]

As we see, the matrix function \( A(z) \) in (8.2) is symmetric and its entries \( a_{ij} = a_{ij}(z) \) are dominated by the quantity

\[
K_\mu(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|},
\]

and, thus, they are bounded if the Beltrami equation is not degenerate.

Vice verse, uniformly elliptic equations (8.1) with symmetric \( A(z) \) whose entries are measurable and \( \det A(z) \equiv 1 \) just correspond to nondegenerate Beltrami equations with coefficient

\[
\mu = \frac{1}{\det (I + A)} (a_{22} - a_{11} - 2ia_{21}) = \frac{a_{22} - a_{11} - 2ia_{21}}{1 + \text{Tr} A + \det A}. \tag{8.3}
\]

\( M_{2 \times 2}(D) \) denotes the collection of all such matrix functions \( A(z) \) in \( D \).

Note that (8.1) are the main equation of hydromechanics (mechanics of incompressible fluids) in anisotropic and inhomogeneous media.
Given such a matrix function $A$ and a quasiconformal mapping $f^\mu : D \to \mathbb{C}$, we have already seen in Lemma 1 of [16], by direct computation, that if a function $T$ and the entries of $A$ are sufficiently smooth, then
\[
\text{div} [A(z) \cdot \nabla (T(f^\mu(z)))] = J(z) \cdot \Delta T(f^\mu(z)) .
\] (8.4)

In the case $T \in W^{1,2}_{\text{loc}}$, we understand the identity (8.4) in the distributional sense, see Proposition 3.1 in [17], i.e., for all $\psi \in C^1_c(D)$,
\[
\int_D \langle A(\nabla T), \nabla \psi \rangle \, dm_z = \int_D J \cdot \langle M^{-1}(\nabla T), \nabla \psi \rangle \, dm_z ,
\] (8.5)

where $M$ is the Jacobian matrix of the mapping $f^\mu$ and $J$ is its Jacobian.

**Theorem 6.** Let $D$ be a Jordan domain with the quasihyperbolic boundary condition, $\partial D$ have a tangent q.e., $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be in $CBV(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity. Suppose that $A \in M^{2\times 2}(D)$ has Hölder continuous entries and $g \in L^p(D)$ for some $p > 2$.

Then there exist smooth weak solutions $u : D \to \mathbb{R}$ of equation (8.1) that have the angular limits of its derivatives in directions $\nu$
\[
\lim_{z \to \xi, z \in D} \frac{\partial u}{\partial \nu} (z) = \varphi(\xi) \quad \text{q.e. on} \quad \partial D .
\] (8.6)

Furthermore, the space of all such solutions has infinite dimension for each fixed collection of functions $A$, $g$, $\nu$ and $\varphi$ of given classes.

Here $u$ is called a weak solution of equation (8.1) if
\[
\int_D \{ \langle A(z)\nabla u(z), \nabla \psi \rangle + g(z) \cdot \psi(z) \} \, dm(z) = 0 \quad \forall \psi \in C^1_c(D) .
\] (8.7)

**Remark 7.** By the construction in the proof below, each such solution $u$ has the representation $u = U \circ f|_D$, where $f = f^\mu : \mathbb{C} \to \mathbb{C}$ is a quasiconformal mapping with a suitable extension of $\mu$ in (8.3) and $U$ is a generalized harmonic function with the source ($J$ is Jacobian of $f$)
\[
G := (g/J) \circ f^{-1}
\] (8.8)
of class $L_p(D_*)$ in the domain $D_* := f(D)$ that has the angular limits
\[
\lim_{w \to \xi, w \in D_*} \frac{\partial U}{\partial N}(w) = \Phi(\xi) \quad \text{q.e. on} \quad \partial D_* ,
\] (8.9)

with
\[
N(\xi) := \left\{ \frac{\partial f}{\partial \nu} \cdot \left| \frac{\partial f}{\partial \nu} \right|^{-1} \right\} \circ f^{-1}(\xi) , \quad \xi \in \partial D_* ,
\] (8.10)
and
\[
\Phi(\xi) := \left\{ \varphi \cdot \left| \frac{\partial f}{\partial \nu} \right|^{-1} \right\} \circ f^{-1}(\xi) , \quad \xi \in \partial D_* .
\] (8.11)
Proof. By the hypotheses of the theorem μ given by (8.3) belongs to a class $C^\alpha(D)$, $\alpha \in (0,1)$, and by Lemma 1 in [21] $\mu$ is extended to a Hölder continuous function $\mu : \mathbb{C} \to \mathbb{C}$ of the class $C^\alpha$. Then, for every $k_* \in (k,1)$, there is an open neighborhood $V$ of $\partial D$, where $\|\mu\|_\infty \leq k_*$ and $\mu$ is of class $C^\alpha(V)$. We set $\mu \equiv 0$ in $\mathbb{C} \setminus V$.

By the Measurable Riemann Mapping Theorem, see e.g. [1], [8] and [32], there is a quasiconformal mapping $f = f^\mu : \mathbb{C} \to \mathbb{C}$ a.e. satisfying the Beltrami equation (8.3) with the given extended complex coefficient $\mu$ in $\mathbb{C}$. Note that the mapping $f$ has the Hölder continuous first partial derivatives in $V$ with the same order of the Hölder continuity as $\mu$, see e.g. [27] and also [28]. Moreover, its Jacobian

$$J(z) \neq 0 \quad \forall \, z \in V,$$

see e.g. Theorem V.7.1 in [32]. Hence $f^{-1}$ is also smooth in $V_* := f(V)$, see e.g. formulas I.C(3) in [1].

Now, the domain $D_* := f(D)$ satisfies the boundary quasihyperbolic condition because $D$ is so, see e.g. Lemma 3.20 in [14]. Moreover, $\partial D_*$ has q.e. tangents, furthermore, the points of $\partial D$ and $\partial D^*$ with tangents correspond each to other in a one-to-one manner because the mappings $f$ and $f^{-1}$ are smooth. In addition, the function $N$ in (8.10) belongs to the class $CBV(\partial D_*)$ and $\Phi$ in (8.11) is measurable with respect to logarithmic capacity, see e.g. Remark 2.1 in [29]. Next, the source $G : D_* \to \mathbb{R}$ in (8.8) belongs to class $L_p(D_*)$, see e.g. point (vi) of Theorem 5 in [2]. Thus, by Theorem 5 in [21] the space of all generalized analytic functions $U : D_* \to \mathbb{R}$ with the source $G$ and the angular limits (8.9) q.e. on $\partial D_*$ has infinite dimension. Finally, by Proposition 3.1 in [17] the functions $u := U \circ f$ give the desired solutions of equation (8.1) because by Lemma 10 in [2]

$$\frac{\partial u}{\partial \nu} = u_z \cdot \nu + u_z \cdot \overline{\nu} = \nu \cdot (U_w \circ f \cdot f_z + U_{\overline{w}} \circ f \cdot \overline{f_z}) + \overline{\nu} \cdot (U_w \circ f \cdot f_z + U_{\overline{w}} \circ f \cdot \overline{f_z})$$

$$= U_w \circ f \cdot (\nu \cdot f_z + \overline{\nu} \cdot \overline{f_z}) + U_{\overline{w}} \circ f \cdot (\nu \cdot \overline{f_z} + \overline{\nu} \cdot f_z) = U_w \circ f \cdot \frac{\partial f}{\partial \nu} + U_{\overline{w}} \circ f \cdot \frac{\overline{\partial f}}{\partial \nu}$$

$$= \left( N \cdot U_w + \overline{N} \cdot U_{\overline{w}} \right) \circ f \cdot \frac{\partial f}{\partial \nu} \circ f \cdot \frac{\overline{\partial f}}{\partial \nu} = \frac{\partial U}{\partial N} \circ f \cdot \frac{\partial f}{\partial \nu} \circ f \cdot \frac{\overline{\partial f}}{\partial \nu} = \frac{\partial \Phi}{\partial \nu} \circ f \cdot \frac{\partial f}{\partial \nu} \circ f \cdot \frac{\overline{\partial f}}{\partial \nu}$$

where the direction $N$ is given by (8.10). \hfill \Box

Remark 8. We are able to say more in the case of $\text{Re} \, n(\zeta)\nu(\zeta) > 0$, where $n(\zeta)$ is the inner normal to $\partial D$ at the point $\zeta$. Indeed, the latter magnitude is a scalar product of $n = n(\zeta)$ and $\nu = \nu(\zeta)$ interpreted as vectors in $\mathbb{R}^2$ and it has the geometric sense of projection of the vector $\nu$ into $n$. In view of (8.6), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $u(\zeta)$ of $u(z)$ as $z \to \zeta$ in $D$ along the straight line passing through the point $\zeta$ and being parallel to the vector $\nu$ because along this line

$$u(z) = u(z_0) - \int_0^1 \frac{\partial u}{\partial \nu}(z_0 + \tau(z - z_0)) \, d\tau. \quad (8.13)$$
Thus, at each point with condition (8.6), there is the directional derivative
\[
\frac{\partial u}{\partial \nu}(\zeta) := \lim_{t \to 0} \frac{u(\zeta + t \cdot \nu) - u(\zeta)}{t} = \varphi(\zeta) ,
\] (8.14)

In particular, in the case of the Neumann problem, \( \text{Re } n(\zeta) \nu(\zeta) \equiv 1 > 0 \), where \( n = n(\zeta) \) denotes the unit interior normal to \( \partial D \) at the point \( \zeta \), and we have by Theorem 6 and Remark 8 the following significant result.

**Corollary 1.** Let \( D \) be a Jordan domain in \( \mathbb{C} \) with the quasihyperbolic boundary condition, the unit inner normal \( n(\zeta), \zeta \in \partial D \), belong to the class \( \text{CBV}(\partial D) \) and \( \varphi : \partial D \to \mathbb{R} \) be measurable with respect to logarithmic capacity. Suppose that \( A \in M_{2 \times 2}(D) \) has H"older continuous entries and \( g \in L^p(D) \) for some \( p > 2 \).

Then there exist smooth weak solutions \( u : D \to \mathbb{R} \) of equation (8.1) such that q.e. on \( \partial D \) there exist:

1) the finite limit along the normal \( n(\zeta) \)
\[
u(\zeta) := \lim_{z \to \zeta} u(z) ,
\]
2) the normal derivative
\[
\frac{\partial u}{\partial n}(\zeta) := \lim_{t \to 0} \frac{u(\zeta + t \cdot n(\zeta)) - u(\zeta)}{t} = \varphi(\zeta) ,
\]
3) the angular limit
\[
\lim_{z \to \zeta} \frac{\partial u}{\partial n}(z) = \frac{\partial u}{\partial n}(\zeta) .
\]

Furthermore, the space of such solutions \( u \) of has infinite dimension.

**Remark 9.** Moreover, such solutions \( u \) have the representation \( u = U \circ f|_D \), where \( f = f^\mu : \mathbb{C} \to \mathbb{C} \) is a quasiconformal mapping with a suitable extension of \( \mu \) in (8.3) onto \( \mathbb{C} \) outside of \( D \) described in the proof of Theorem 6 and \( U \) is a generalized harmonic function with the source \( G \) in (8.8) of class \( L_p(D_*) \) in the domain \( D_* := f(D) \) that satisfies the corresponding Neumann condition (8.9).

## 9 Poincaré problem in Bagemihl–Seidel system

Arguing similarly to the last section, see also Section 4, we obtain by Theorem 6 in [21] the following statement.

**Theorem 7.** Let \( D \) be a Jordan domain in \( \mathbb{C} \), \( \nu : \partial D \to \mathbb{C} \), \( |\nu(\zeta)| \equiv 1 \), and \( \varphi : \partial D \to \mathbb{C} \) be measurable functions with respect to the logarithmic capacity
and let \{γ_ζ\}_{ζ \in \partial D} be a family of Jordan arcs of class BS in D. Suppose that \(A \in M^{2 \times 2}(D)\) has Hölder continuous entries and \(g : D \to \mathbb{R}\) is of class \(L_p(D)\) for some \(p > 2\).

Then there exist smooth weak solutions \(u : D \to \mathbb{R}\) of equation (8.1) that have the limits along \(γ_ζ\)
\[
\lim_{z \to ζ, z \in D} \frac{∂u}{∂ν}(z) = ϕ(ζ) \quad \text{q.e. on } \partial D .
\]  

Furthermore, the space of such solutions \(u\) has infinite dimension.

**Remark 10.** Moreover, each such solution \(u\) has the representation \(u = U \circ f|_D\), where \(f = f^μ : \mathbb{C} \to \mathbb{C}\) is a quasiconformal mapping with \(μ\) in (8.3) extended onto \(\mathbb{C}\) outside of \(D\) as in the proof of Theorem 6, and \(U\) is a generalized harmonic function with the source \(G\) in (8.8) of class \(L_p(D^*)\) in the domain \(D^* := f(D)\) that satisfies the corresponding Poincare condition (8.9) along \(Γ_ζ := f(γ_f^{-1}(ζ)), ζ \in \partial D^*\).

### 10 Riemann–Poincaré type problems

Arguing similarly to the last section and to Sections 5 and 6, we reduce the proof of existence of solutions \(u^±\) in the next statement to Corollary 5 in [21] on such problems for the corresponding generalized harmonic functions \(U^±\) through the representation \(u^± = U^± \circ f^µ\), where \(f = f^µ : \mathbb{C} \to \mathbb{C}\) is a quasiconformal mapping with \(μ\) in (8.3).

**Theorem 8.** Let \(D\) be a Jordan domain in \(\mathbb{C}\), \(ϕ : \partial D \times \mathbb{R} \to \mathbb{R}\) satisfy the Carathéodory conditions, \(ν\) and \(ν^* : \partial D \to \mathbb{C}\), \(|ν(ζ)| \equiv 1\), \(|ν^*(ζ)| \equiv 1\), are measurable with respect to the logarithmic capacity, and let \(\{γ^+_ζ\}_{ζ \in \partial D}\) and \(\{γ^-_ζ\}_{ζ \in \partial D}\) be families of Jordan arcs of class BS in D and \(\mathbb{C} \setminus \overline{D}\), correspondingly. Suppose that \(A \in M^{2 \times 2}(\mathbb{C})\) has Hölder continuous entries and \(g : \mathbb{C} \to \mathbb{R}\) is of class \(L_p(\mathbb{C})\) for some \(p > 2\) with compact support.

Then there exist smooth weak solutions \(u^+ : D \to \mathbb{R}\), \(u^- : \mathbb{C} \setminus \overline{D} \to \mathbb{R}\) of equation (8.1) such that
\[
\left[ \frac{∂u^+}{∂ν^*} \right]^+(ζ) = ϕ(ζ, \left[ \frac{∂u^-}{∂ν} \right]^-(ζ)) \quad \text{q.e. on } \partial D ,
\]
where \(\left[ \frac{∂u^+}{∂ν^*} \right]^+(ζ)\) and \(\left[ \frac{∂u^-}{∂ν} \right]^-(ζ)\) are limits of the directional derivatives \(\frac{∂u^+}{∂ν^*}(z)\) and \(\frac{∂u^-}{∂ν}(z)\) as \(z \to ζ\) along \(γ^+_ζ\) and \(γ^-_ζ\), correspondingly.

Furthermore, the function \(\left[ \frac{∂u^-}{∂ν} \right]^-(ζ)\) can be arbitrary measurable with respect to the logarithmic capacity and, correspondingly, the space of all such couples \((u^+, u^-)\) has the infinite dimension for any such prescribed functions \(A\), \(ϕ\), \(ν\), \(ν^*\) and collections \(γ^+_ζ\) and \(γ^-_ζ\), \(ζ \in \partial D\).
Finally note that the present paper creates the basis for articles on the corresponding results for semi-linear equations (with nonlinear sources) of mathematical physics that will be published elsewhere.

References

[1] Ahlfors L. (1966) Lectures on Quasiconformal Mappings. New York: Van Nostrand.

[2] Ahlfors, L.V., Bers, L. (1960) Riemann’s mapping theorem for variable metrics. Ann. Math. (2) 72, 385-404.

[3] Astala K., Iwaniec T., Martin G.J. Elliptic differential equations and quasiconformal mappings in the plane. – Princeton Math. Ser. 48. – Princeton: Princeton Univ. Press, 2009.

[4] Bagemihl, F., Seidel, W. (1955). Regular functions with prescribed measurable boundary values almost everywhere. Proc. Nat. Acad. Sci. U.S.A., 41, pp. 740–743.

[5] Begehr, H. (1994). Complex analytic methods for partial differential equations. An introductory text. River Edge, NJ: World Scientific Publishing Co., Inc.

[6] Begehr, H., Wen, G.Ch. (1996). Nonlinear elliptic boundary value problems and their applications. Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 80. Harlow: Longman.

[7] Becker, J., Pommerenke, Ch. (1982). Hölder continuity of conformal mappings and nonquasiconformal Jordan curves. Comment. Math. Helv., 57(2), 221–225.

[8] Bojarski B., Gutlyanskii V., Martio O., Ryazanov V. (2013) Infinitesimal geometry of quasiconformal and bi-lipschitz mappings in the plane. EMS Tracts in Mathematics, Vol. 19. Zürich: European Mathematical Society.

[9] Dovgoshey, O., Martio, O., Ryazanov, V., Vuorinen, M. (2006). The Cantor function. Expo. Math., 24(1), 1–37.

[10] Duren, P.L. (1970). Theory of Hp spaces. Pure and Applied Mathematics, Vol. 38. New York-London: Academic Press.

[11] Efimushkin, A.S., Ryazanov, V.I. (2015). On the Riemann–Hilbert problem for the Beltrami equations in quasidisks. Ukr. Mat. Bull., 12(2), 190–209; transl. in (2015). J. Math. Sci., 211(5), 646–659.

[12] Federer, H. (1969). Geometric Measure Theory. Springer-Verlag. Berlin.

[13] Gakhov, F.D. (1990). Boundary value problems. Dover Publications. Inc. New York.
[14] Gehring, F.W., Martio, O. (1985). Lipschitz classes and quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. A I Math.* 10, 203–219.

[15] Gehring, F.W., Palka, B.P. (1976). Quasiconformally homogeneous domains. *J. Analyse Math.*, 30, 172–199.

[16] Gutlyanskii, V., Nesmelova, O., Ryazanov, V. (2017). On a model semi-linear elliptic equation in the plane. *J. Math. Sci., New York* 220, No. 5, 603-614 (2017); transl. (2016) from *Ukr. Mat. Visn.* 13, No. 1, 91-105.

[17] Gutlyanskii, V., Nesmelova, O., Ryazanov, V. (2018). On quasiconformal maps and semi-linear equations in the plane. *J. Math. Sci., 229*(1), 7–29; transl (2017) from *Ukr. Mat. Visn.*, 14(2), 161–191.

[18] Gutlyanskii, V., Nesmelova, O., Ryazanov, V. (2019). To the theory of semi-linear equations in the plane. *J. Math. Sci., 242*(6), 833–859; transl. (2019) from *Ukr. Mat. Visn.*, 16(1), 105–140.

[19] Gutlyanskii, V., Nesmelova, O., Ryazanov, V. (2020). On a quasilinear Poisson equation in the plane. *Anal. Math. Phys., 10*(1), Paper No. 6, 1–14.

[20] Gutlyanskii, V., Nesmelova, O., Ryazanov, V. (2020). Semi-linear equations and quasiconformal mappings. *Complex Var. Elliptic Equ., 65*(5), 823–843.

[21] Gutlyanskii, V., Nesmelova, O., Ryazanov, V., Yefimushkin, A. (2021). Logarithmic potential and generalized analytic functions. *J. Math. Sci., New York* 256, No. 6, 735-752; transl. from *Ukr. Mat. Visn.* 18, No. 1, 12-36.

[22] Gutlyanskii, V., Nesmelova, O., Ryazanov, V., Yefimushkin, A. (2021). On boundary-value problems for semi-linear equations in the plane. *J. Math. Sci., New York* 259, No. 1, 53-74; transl. from *Ukr. Mat. Visn.* 18, No. 3, 359-388.

[23] Gutlyanskii, V.Ya., Ryazanov, V.I., Yakubov, E., Yefimushkin, A.S. (2020). On Hilbert boundary value problem for Beltrami equation. *Ann. Acad. Sci. Fenn. Math.*, 45(2), 957–973.

[24] Gutlyanskii, V., Ryazanov, V., Yefimushkin, A. (2015). On the boundary value problems for quasiconformal functions in the plane. *Ukr. Mat. Bull., 12*(3), 363–389; transl. in (2016). *J. Math. Sci., 214*(2), 200–219.

[25] Heinonen, J., Kilpeläinen, T., Martio, O. (1993). *Nonlinear potential theory of degenerate elliptic equations*. Oxford Mathematical Monographs. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York.
[26] Hilbert, D. (1904). Über eine Anwendung der Integralgleichungen auf eine Problem der Funktionentheorie. Verhandl. des III Int. Math. Kongr., Heidelberg.

[27] Iwaniec T. Regularity of solutions of certain degenerate elliptic systems of equations that realize quasiconformal mappings in n-dimensional space. Differential and integral equations. Boundary value problems. - 1979. - p. 97–111. - Tbilisi: Tbilis. Gos. Univ.

[28] Iwaniec T. Regularity theorems for solutions of partial differential equations for quasiconformal mappings in several dimensions. Dissertationes Math. (Rozprawy Mat.) - 1982. - 198. - 45 pp.

[29] Koosis, P. (1998). Introduction to $H^p$ spaces. Cambridge Tracts in Mathematics. 115. Cambridge Univ. Press, Cambridge.

[30] Krasnosel’ski, M.A., Zabreiko, P.P., Pustyl’nik, E.I., Sobolevskii, P.E. (1976). Integral operators in spaces of summable functions. Monographs and Textbooks on Mechanics of Solids and Fluids, Mechanics: Analysis. Noordhoff International Publishing. Leiden.

[31] Ladyzhenskaya, O.A., Ural’tseva, N.N. (1968). Linear and quasilinear elliptic equations. Academic Press, New York-London; transl. from (1964) Lineinye i kvazilineinye uravneniya ellipticheskogo tipa. Nauka, Moscow.

[32] Landkof, N.S. (1972). Foundations of modern potential theory. Die Grundlehren der mathematischen Wissenschaften. 180. Springer-Verlag: New York-Heidelberg.

[33] Lehto O., Virtanen K.J. (1973) Quasiconformal mappings in the plane. Springer-Verlag: Berlin, Heidelberg.

[34] Luzin, N.N. (1951). Integral and trigonometric series. Editing and commentary by N.K. Bari and D.E. Men’shov. Gosudarstv. Izdat. Tehn.-Teor. Lit. Moscow–Leningrad (in Russian).

[35] Muskhelishvili, N.I. (1992) Singular integral equations. Boundary problems of function theory and their application to mathematical physics. Dover Publications Inc. New York.

[36] Pommerenke, Ch. (1992). Boundary behaviour of conformal maps. Grundlehren der Mathematischen Wissenschaften. Fundamental Principles of Mathematical Sciences, 299. Springer-Verlag, Berlin.

[37] Priwalow, I.I. (1956). Randeigenschaften analytischer Funktionen. Hochschulbücher für Mathematik. 25. Deutscher Verlag der Wissenschaften. Berlin.

[38] Ryazanov, V. (2014). On the Riemann–Hilbert problem without index. Ann. Univ. Buchar. Math. Ser. 5(LXIII)(1), 169–178.
[39] Ryazanov, V. (2015). On Hilbert and Riemann problems. An alternative approach. *Ann. Univ. Buchar. Math. Ser. 6*(LXIV)(2), 237–244.

[40] Ryazanov, V. (2015). Infinite dimension of solutions of the Dirichlet problem. *Open Math. (the former Central European J. Math.)*, 13(1), 348–350.

[41] Ryazanov, V. (2017). On Neumann and Poincare problems for Laplace equation. *Anal. Math. Phys.* 7. no. 3. pp. 285-289.

[42] Ryazanov, V. (2021). On Hilbert and Riemann problems for generalized analytic functions and applications. *Anal. Math. Phys.* 11(5). Published online: 22 November 2020.

[43] Sobolev, S.L. (1963). *Applications of functional analysis in mathematical physics.* Transl. of Math. Mon. 7. AMS, Providence, R.I.

[44] Trogdon, Th., Olver, Sh. (2016). *Riemann–Hilbert problems, their numerical solution, and the computation of nonlinear special functions.* Society for Industrial and Applied Mathematics (SIAM). Philadelphia.

[45] Vekua, I.N. (1962). *Generalized analytic functions.* Pergamon Press. London-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass.

[46] Yefimushkin, A. (2016). On Neumann and Poincare problems in A–harmonic analysis. *Advances in Analysis, 1*(2), 114–120.

[47] Yefimushkin, A., Ryazanov, V. (2016). On the Riemann-Hilbert problem for the Beltrami equations. *Complex analysis and dynamical systems VI. Part 2,* 299–316. Contemp. Math. 667. Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI.

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