Analysis of Tooth Surface Contact Stress of Involute Spur Gear

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Abstract. Through the establishment of a pair of spur gear contact models, based on Hertz contact theory, the tooth surface contact stress is calculated; then the Ansys finite element analysis software is used to simulate and analyse the stress distribution. Through the analysis and comparison of the two results, it is proved that the contact stress calculated by Hertz theory is relatively small, which is close to the results of the finite element simulation analysis. Theoretical calculation can verify the accuracy of the finite element simulation analysis model, and the finite element simulation analysis provides an effective way to accurately calculate the contact stress of the tooth surface.

1. Introduction
Gears are the most common and important transmission method in mechanical transmission. The research of tooth surface contact stress is the focus of strength design. Hertz first proposed a paper on gear contact problems in 1881 [1], which laid the theoretical foundation for gear contact research. The traditional method of calculating the contact strength of gears is based on the Hertz contact theory, but for gears with complex shapes and varying forces, numerical calculations cannot accurately calculate the stress distribution and deformation. Therefore, finite element analysis software can be used to accurately analyse the stress on the gear contact surface.

This paper takes a pair of spur gear meshing pair as an example, respectively applies Hertz contact theory and finite element analysis method to solve, compare and analyse the surface contact stress, and explore its laws.

2. Calculate tooth surface contact stress based on Hertz contact theory
The Hertz contact theory is based on the following four basic assumptions:

(1) The material in contact with the object is uniform and isotropic. (2) The load acting on the object produces only elastic deformation in the contact area, which obeys Hooke's law. (3) The size of the contact area is small compared to the radius of curvature of the surface of the contact object. (4) The load is perpendicular to the contact surface, that is, the contact surface is completely smooth, regardless of the friction with the contact object [1].

According to the Hertz contact strength theory, on the basis of satisfying the above four conditions, we simplified the tooth profile of the contact area into two cylinders with radii \( R_1 \) and \( R_2 \), and the contact area is shown in the figure. After analysis, it can be seen that the normal force \( F \) acts on the cylinder, so the contact surface undergoes local elastic deformation. The size of the contact area is a rectangular contact area with a width of \( 2a \) and a length of \( L \). The stress distribution in this contact
area is not uniform, and the pressure at the center of the contact surface is the largest, which is about
$4/\pi$ times the average pressure in value [3]. Therefore, the ratio of load distribution between teeth and
the comprehensive radius of curvature of the contact point together determine the contact stress at this
point.

![Figure 1. Schematic diagram of Hertz contact theory](image1)

![Figure 2. Schematic diagram of meshing curve](image2)

Table 1. Basic parameters of involute spur gear pair.

| Gear geometry parameters | Number of teeth | Module (m) | Pressure angle ($\alpha$) | Tooth width (b) (mm) | Modification coefficient ($x_1/x_2$) | Addendum height coefficient ($h^*_a$) | Clearance coefficient ($c^*$) |
|--------------------------|----------------|------------|--------------------------|---------------------|-------------------------------------|----------------------------------|-----------------------------|
| 12/27                    | 8              | 20         | 165                      | 0.5/0.25            | 1.0                                 | 0.4                              |

| Material parameters | Load parameters |
|---------------------|-----------------|
| Modulus of Elasticity ($E$ (Mpa)) | $E$ |
| Poisson's Ratio ($\nu$) | $\nu$ |
| Density ($\rho$ (kg/m$^3$)) | $\rho$ |
| Torque ($T$ (KN·m)) | $T$ |
| 2.06e11              | 0.273           | 7910       | 16.4                     |

According to the Hertz formula, the calculation formula of the contact half-width $a$ is as follows:

$$a = \sqrt{\frac{8F_L \rho_E}{\pi L E}}$$  (1)

The formula for calculating the maximum contact stress $\sigma_{H_{max}}$ is as follows:

$$\sigma_{H_{max}} = \sqrt{\frac{F_L E}{2\pi L \rho_E}}$$  (2)

It can be deduced that the average gear tooth surface contact pressure $\sigma_{H}$ can be calculated as follows:

$$\sigma_{H} = \frac{\pi}{4} \sigma_{H_{max}} = \frac{\pi}{4} \sqrt{\frac{F_L E}{2\pi L \rho_E}}$$  (3)

The equivalent elastic modulus $E$ of the tooth profile of the gear at the meshing point is:

$$\frac{1}{E} = \frac{1}{2} \left( 1 - \frac{v_1^2}{E_1} + 1 - \frac{v_2^2}{E_2} \right)$$  (4)
Where: $F_n$ contact line normal force; $L$ meshing zone actual contact line length, that is tooth width; $\rho_1, \rho_2, \rho_3$ are the radius of curvature and the comprehensive equivalent radius of curvature of gears $O_1$ and $O_2$ at the meshing point. $v_1, v_2$ they are the material Poisson’s ratio of gear $O_1$ and $O_2$.

The parameters in the theoretical formula of Hertzian contact strength need to be studied separately. The specific research process is as follows.

2.1. Calculate the normal force of the contact line
Assuming a pair of meshing transmission spur gears $O_1$ and $O_2$, the specific parameters are shown in the table. Establish a dimensionless function $\Gamma$ to describe the coordinates of the meshing line, and normalize the coordinates of the radius of curvature at any meshing point $K$, as shown in the figure. The torque applied to the driving wheel $O_1$ is $T_1$, the distance between the center of the driving wheel $O_1$ and any contact point $K$ on the tooth surface is $r_{k1}$, and the normal surface contact pressure angle is $\alpha_{k1}$. Point $A$ is the entry point, point $B$ is the entry point of the single tooth zone, point $C$ is the entry point of the single tooth zone, and point $D$ is the entry point. The $AB$ and $CD$ sections are the double-tooth meshing area, and the $BC$ section is the single-tooth meshing area.

The load distribution coefficient $K_H$ between the teeth in the double-tooth meshing area is related to the position of any meshing point and the meshing coincidence [4]. The degree of coincidence represents the average value of the tooth pairs engaged in meshing by colleagues in the double-tooth meshing zone. A higher degree of coincidence is of great significance for improving the stability and load-carrying capacity of gear transmission. The calculation is as follows:

$$e_a = \left[ z_1 \left( \tan \alpha_{a1} - \tan \alpha' \right) + z_2 \left( \tan \alpha_{a2} - \tan \alpha' \right) \right]/(2\pi)$$

The calculated actual coincidence degree is $e_a = 1.3170$. For gears without modification, the load distribution coefficient between teeth is calculated as follows [5] while ignoring the transmission error:

$$K_h = \begin{cases} 1 & \Gamma_B < \Gamma_K < \Gamma_C \\ \frac{1}{3} \left( 1 + \frac{\Gamma_K - \Gamma_A}{\Gamma_0} \right) & \Gamma_A < \Gamma_K < \Gamma_B \\ \frac{1}{3} \left( 1 + \frac{\Gamma_D - \Gamma_K}{\Gamma_0} \right) & \Gamma_C < \Gamma_K < \Gamma_D \\ \Gamma_0 = \pi m (e_a - 1) \cos \alpha' / r_{a1} \tan \alpha' \end{cases}$$

Using MATLAB to calculate the distribution of load distribution coefficient between teeth of a pair of meshing transmission gears along the meshing curve as shown in the figure:

![Figure 3. Load distribution coefficient between front teeth of spur gear](image1)

![Figure 4. Distribution diagram of radius of curvature of meshing gear](image2)
Analysing the above figure, it can be seen that as the meshing point moves on the meshing line and changes, the normal load can be expressed as follows:

\[ F_n = \frac{K_n \cdot T}{r_k \cdot \cos \alpha \cdot i} \tag{7} \]

### 2.2. Curvature radius of meshing point

The curvature radii of a pair of meshing transmission gear at the meshing point \( K \) are \( \rho_1 \) and \( \rho_2 \) respectively, and the comprehensive equivalent curvature radius is \( \rho_E \). The derivation process is as follows:

From the dimensionless function definition formulas \( \Gamma = \frac{PK}{N_iP} \) and \( \Gamma_{x_2} = \frac{N_iP}{N_iP} = \pm i \), we can get:

\[
\begin{align*}
\rho_1 &= \frac{(1 + \Gamma)}{(i \pm 1)} \cdot a' \sin \alpha' \\
\rho_2 &= \frac{(i - \Gamma)}{(i \pm 1)} \cdot a' \sin \alpha' \\
\rho_E &= \frac{\rho_1 \rho_2}{\rho_2 \pm \rho_1} \tag{8}
\end{align*}
\]

According to the above content, use MATLAB programming to calculate the curved surface curvature, \( \rho_1, \rho_2 \) of a pair of meshing transmission gears \( O_1 \) and \( O_2 \), the comprehensive equivalent curvature radius \( \rho_E \), and the distribution trend along the meshing curve, as shown in Figure 1.

From the analysis of the above figure, it can be seen from the radius of curvature of a single tooth that the radius of curvature \( \rho_1 \) of the driving wheel \( O_1 \) only has a numerical distribution on the actual meshing line \( AD \), and there is a minimum value at the actual meshing point \( A \), and at the actual meshing point \( D \) There is a maximum value at, and as the normalized coordinate value goes from \( N_1 \) to \( N_2 \), the value shows a linear increase trend. For the radius of curvature \( \rho_2 \) of the driven wheel \( O_2 \), its changing law is opposite to that of \( \rho_1 \). From this pair of meshing gears, the equivalent comprehensive radius of curvature \( \rho_E \) increases first and then decreases, if and only when \( \rho_1 = \rho_1, \rho_E = \rho_1 / 2 \).

### 2.3. Tooth surface contact pressure calculation

So far, the normal force calculation formula (7), the curvature radius formula of the meshing point (8), and the gear tooth surface contact average pressure \( \sigma_{hi} \) calculation formula (3) are combined with the table 1 For various parameters, the average pressure distribution diagram of the tooth surface of the pair of meshing spur gears can be obtained.
3. Calculation of tooth surface contact pressure based on finite element method
To use the finite element analysis method to conduct contact analysis on the meshing of planetary transmission gears, firstly complete the assembly model of the spur cylindrical transmission gear in Pro/E software. Define the material properties of the main and driven wheels according to Table 1. In the gear meshing process, the tooth surface comes into contact. It is necessary to integrate the motion state and tooth surface coincidence to set the contact surface and the target surface, and define the sliding friction coefficient of the tooth surface as 0.05. Therefore, the parts that do not affect the calculation results should be reasonably simplified to improve the rationality of meshing and the calculation accuracy. Use the hexahedral element type to divide the mesh, and set the initial mesh size to 10mm; select the fine method for local densification in the gear meshing area, and set the refined size to 1.5mm. According to the force, the boundary conditions are defined, the load is applied, and the static simulation analysis is performed to solve the tooth surface contact stress distribution. The results of tooth surface stress analysis are shown in the figure 6.

4. Comparative analysis
From the analysis of the stress cloud diagram, it can be seen that in Figure 1, based on the Hertz contact theory, through programming, the maximum contact stress of the tooth surface of the meshing gear pair appears at $\Gamma_{g}$, that is, the contact position of the tooth root of the driving wheel and the tooth top of the driven wheel. The maximum stress value is 978.66Mpa. The maximum contact stress obtained by the finite element simulation analysis appears on the contact tooth surface of the driving wheel and the driven wheel, and is closer to the tooth root area, with a value of 967.27 Mpa. Finite element simulation analysis and theoretical calculation analysis show that the maximum stress values obtained are basically the same at the alternating points of single and double teeth at the beginning of meshing, and the contact stress distribution is relatively uniform, without stress concentration. The deviation between the theoretical calculation and the finite element simulation analysis result is 1.2%, and the error is small, which proves that the modeling of the transmission gear model is reasonable and effective. On this basis, further research can be done.

5. Conclusion
In this paper, the Hertz contact theory and finite element simulation analysis are used to study the tooth surface stress of the straight tooth meshing gear pair. Through the analysis of the required results, it can be obtained that the finite element simulation and theoretical calculation errors are small, and the theoretical calculation can verify the accuracy of the finite element simulation analysis model. The finite element method can be used to solve this type of problem.
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