Abstract

We consider the construction of insurance premiums that are monotonically increasing with respect to a loading parameter. By introducing weight functions that are totally positive of higher order, we derive higher monotonicity properties of weighted transformed premiums. We deduce that the greater the degree of randomness of insured risks, the higher the order of total positivity that should be required for the chosen weight functions. We examine seven classes of weight functions that have appeared in the literature, and we ascertain the higher order total positivity properties of those functions.

1 Introduction

Consider the problem of estimating the premiums that an insurance operation is to charge its clients in order to underwrite their risks. On the one hand, the insurer is limited by competition as to how much it may charge to underestimate a given risk. On the other hand the insurer, so as to remain solvent, necessarily must charge premiums that are suitably large in order to cover its insured risks and its operating expenses.

To formulate this problem probabilistically, suppose that we have a probability triplet \((\Omega, \mathcal{A}, P)\), consisting of a sample space \(\Omega\), a sigma-algebra \(\mathcal{A}\) of subsets of \(\Omega\),
and a probability measure $P(\cdot)$ on $\mathcal{A}$. We suppose that there corresponds to $P(\cdot)$ a positive random variable $X : \Omega \to \mathbb{R}_+$, representing the net premium that the insurer charges to cover a randomly chosen risk, with finite mean $\mathbb{E}[X] = \int_{\Omega} X(\omega) P(\mathrm{d}\omega)$. As the mean premium $\mathbb{E}[X]$ will cover only the average insured risk, the insurer will charge an amount $H[X]$ that is loaded, meaning that $H[X] \geq \mathbb{E}[X]$.

Sendov, Wang, and Zitikis (2011) provided a comprehensive account of the construction of loaded premiums. One such method begins by choosing a nonnegative weight function $w(\lambda, x)$ that depends on a loading parameter $\lambda > 0$. We refer readers to Jones and Zitikis (2007) and Sendov, et al. (2011) for motivating accounts of actuarial and insurance problems that give rise to loading parameters.

We are particularly interested in the weighted premium,

$$H[\lambda, X] = \frac{\mathbb{E}[X w(\lambda, X)]}{\mathbb{E}[w(\lambda, X)]},$$

where it is assumed that, for each $\lambda > 0$, the function $x \mapsto w(\lambda, x)$ is Borel-measurable and monotonically increasing in $x$. Suppose that the weight function $(\lambda, x) \mapsto w(\lambda, x)$ is totally positive of order 2, i.e.,

$$w(\lambda_1, x_1) w(\lambda, x_2) \geq w(\lambda_1, x_2) w(\lambda_2, x_1),$$

whenever $\lambda_1 > \lambda_2$ and $x_1 \geq x_2$. Sendov, et al. (2011, Theorem 2.1) prove that $H[\lambda, X]$ is non-decreasing in $\lambda$, thereby relating the study of weighted premiums and the theory of total positivity. The insurance implication is that if $w(\lambda, x)$ is totally positive of order two then riskier ventures, with risk measured by the parameter $\lambda$, will not be assigned lower weighted net premiums.

Noting the general theory of total positivity (cf., Karlin, 1968), we wish to determine the behavior of the weighted premium $H[\lambda, X]$ for weights $w(\lambda, x)$ that are totally positive of order higher than two. In this paper, we study the behavior of $H[\lambda, X]$, and generalizations of it, when the weight $w(\lambda, x)$ is totally positive of any given order $k$.

Our results may be described as follows. In Section 2, we introduce the theory of total positivity, providing a self-contained introduction to results needed in the sequel.

In Section 3, we consider weighted transformed premiums that generalize $H[\lambda, X]$. We establish monotonicity properties of these transformed premiums, recovering as a special case the monotonocity result of Sendov, et al. (2011, Theorem 2.1). Also, we explain, with reference to the concept of index of dispersion, the necessity of choosing weight functions that are totally positive of order greater than two if the underlying risk is known to exhibit a greater degree of randomness.

In Section 4, we consider seven classes of weight functions treated previously by Sendov, et al. (2011). We ascertain the higher order total positivity properties of these weight functions, proving that five of them are strictly totally positive of order infinity, one is totally positive of order infinity, and one is not totally positive of order three.
Finally, in Section 5, we summarize with concluding remarks on the implications of working with weighted premiums that are totally positive of higher order.

2 Total positivity

For $k \in \mathbb{N}$, a weight function $w : \mathbb{R}^2 \to \mathbb{R}$ is **totally positive of order $k$**, denoted $\text{TP}_k$, if for all $\lambda_1 > \cdots > \lambda_k$, $x_1 > \cdots > x_k$, and for all $r = 1, \ldots, k$, the $r \times r$ determinant,

$$
\det \left( w(\lambda_i, x_j) \right) := \begin{vmatrix} w(\lambda_1, x_1) & \cdots & w(\lambda_1, x_r) \\ \vdots & \ddots & \vdots \\ w(\lambda_r, x_1) & \cdots & w(\lambda_r, x_r) \end{vmatrix} \geq 0.
$$

The function $w(\lambda, x)$ is **totally positive of order infinity**, denoted $\text{TP}_\infty$, if $w(x, \lambda)$ is $\text{TP}_k$ for all $k \geq 1$. Similarly, the function $w(\lambda, x)$ is **strictly totally positive of order $k$**, denoted $\text{STP}_k$ if the $r \times r$ determinant $\det \left( w(\lambda_i, x_j) \right)$ is strictly positive for all $\lambda_1 > \cdots > \lambda_k$, $x_1 > \cdots > x_k$, and all $r = 1, \ldots, k$. Further, $w(\lambda, x)$ is **strictly totally positive of order infinity**, denoted $\text{STP}_\infty$, if $w(\lambda, x)$ is $\text{STP}_k$ for all $k \geq 1$. The function $w(\lambda, x)$ is called **reverse-rule of order $k$**, denoted $\text{RR}_k$, if $(-1)^{r(r-1)/2} \det \left( w(\lambda_i, x_j) \right)$ is nonnegative for all $r = 1, \ldots, k$; if this holds for all $k \geq 1$ then we say that $w(\lambda, x)$ is **reverse-rule of order infinity**.

Throughout the remainder of the paper, we will assume that all integrals or sums converge absolutely. Whenever it is necessary to provide explicit conditions under which such convergence holds then we will provide the details.

The **Binet-Cauchy formula** often is stated in terms of calculating the minors of a matrix product, $AB$, from the minors of $A$ and $B$; see Karlin (1968, p. 1). We will need a continuous and a discrete generalization of this formula: Let $\nu$ be a Borel-finite measure on a totally ordered measure space $\mathcal{X}$. Also, for $r \in \mathbb{N}$, let $\phi_1, \ldots, \phi_r$ and $\psi_1, \ldots, \psi_r$ be complex-valued functions on $\mathcal{X}$. The Binet-Cauchy formula is that the $r \times r$ determinant with $(i,j)$th entry $\int_{\mathcal{X}} \phi_i(x) \psi_j(x) \, d\nu(x)$ satisfies the identity

$$
\det \left( \int_{\mathcal{X}} \phi_i(x) \psi_j(x) \, d\nu(x) \right) = \int_{x_1 > \cdots > x_r} \det \left( \phi_i(x_j) \right) \det \left( \psi_j(x_j) \right) \prod_{j=1}^r d\nu(x_j). \tag{2.1}
$$

For the case in which $\mathcal{X} = \mathbb{N}_0$, the set of nonnegative integers, and $\nu$ is a discrete measure on $\mathbb{N}_0$ with weights $\nu(m)$, $m = 0, 1, 2, \ldots$, the Binet-Cauchy formula is the statement that

$$
\det \left( \sum_{m=0}^\infty \phi_i(m) \psi_j(m) \nu(m) \right) = \sum_{m_1 > \cdots > m_r \geq 0} \det \left( \phi_i(m_j) \right) \det \left( \psi_j(m_j) \right) \prod_{j=1}^r \nu(m_j). \tag{2.2}
$$

The continuous version of the **Basic Composition Formula** is that if the weight functions $w_1(\lambda, x)$ and $w_2(\lambda, x)$ are $\text{TP}_k$ on $\mathbb{R}^2$, and if $\nu$ is a sigma-finite measure on $\mathbb{R}$,
then the weight function

\[ w(\lambda, x) = \int_{\mathbb{R}} w_1(\lambda, t)w_2(t, x) \, d\nu(t) \]  

(2.3)

also is TP\(_k\) on \(\mathbb{R}^2\).

The discrete version of the Basic Composition Formula, analogous to (2.2), is that if \(w_1(\lambda, x)\) and \(w_2(\lambda, x)\) are TP\(_k\) on \(\mathbb{N}_0 \times \mathbb{N}_0\), and \(\nu\) is a discrete measure on \(\mathbb{N}_0\) with nonnegative weights \(\nu(m), m = 0, 1, 2, \ldots\), then the function

\[ w(\lambda, x) = \sum_{m=0}^{\infty} w_1(\lambda, m)w_2(m, x)\nu(m) \]  

(2.4)

also is TP\(_k\) on \(\mathbb{N}_0 \times \mathbb{N}_0\).

We remark that a crucial difference between total positivity of order two and total positivity of higher orders is that if a positive function \(w(\lambda, x)\) is TP\(_2\) then the function \(1/w(\lambda, x)\) is RR\(_2\). However this result does not generally extend to TP\(_k\) functions for \(k > 2\). This explains why some of the weight functions considered in Section 4 have relatively straightforward TP\(_2\) or RR\(_2\) properties, while their higher-order total positivity properties are more difficult to establish. We refer to Carlson and Gustafson (1983, (1.5)) for further remarks on this point.

3 Monotonicity properties of weighted transformed premiums

Let \(X\) be a nonnegative random variable with probability density function \(g\). Sendov, et al. (2011) derived a monotonicity property of the weighted premium function,

\[ H[\lambda, X] = \frac{\mathbb{E}[w(\lambda, X) X]}{\mathbb{E}[w(\lambda, X)]}, \]

where the expectations are taken with respect to the distribution of \(X\). Sendov, et al. (loc. cit., Theorem 2.1) proved that if \(w(\lambda, x)\) is TP\(_2\) then the function \(\lambda \mapsto H[\lambda, X]\) is non-decreasing. We shall generalize this property in two ways. First, for a function \(f : \mathbb{R}_+ \to \mathbb{R}_+\) we consider the weighted transformed premium,

\[ H[\lambda, f(X)] = \frac{\mathbb{E}[w(\lambda, X) f(X)]}{\mathbb{E}[w(\lambda, X)]}, \]

whenever these expectations exist. Let \(Y\) be the random variable that has the weighted probability density function,

\[ \frac{w(\lambda, y)}{\mathbb{E}[w(\lambda, X)]} g(y) \]  

(3.1)
y ≥ 0 and, as defined earlier, g is the density function of X. Then

\[ H[\lambda, f(X)] \]

also be viewed as the expectation \( \mathbb{E}_Y f(Y) \), where the expectation is with respect to the distribution of Y. We will establish the monotonicity of \( H[\lambda, f(X)] \) for the case in which \( f \) is monotonically increasing.

Second, for the case in which the weight function \( w(\lambda, x) \) is TP \(_k\) or STP \(_k\), we obtain generalizations of the monotonicity property arising from the case in which \( k = 2 \).

**Theorem 3.1.** Suppose that the weight function \( w(\lambda, x) \) is TP \(_k\), \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is a non-decreasing function, and \( \lambda_1 > \cdots > \lambda_k \). Then, all minors of the \( k \times k \) determinant

\[
\det \left( H[\lambda_i, (f(X))^{k-j}] \right)
\]

(3.2)

are nonnegative. Further, if \( w(\lambda, x) \) is STP \(_k\) and the set of points of increase of \( f \) contains an open set then all minors of the matrix (3.2) are positive.

**Proof.** Consider the \( m \times m \) minor of (3.2) corresponding to rows \( r_1, \ldots, r_m \) and columns \( c_1, \ldots, c_m \), where \( r_1 < \cdots < r_m \) and \( c_1 < \cdots < c_m \). Applying the Binet-Cauchy formula (2.1) with \( \mathfrak{X} = \mathbb{R}, \phi_i(x) = w(\lambda_{c_i}, x) \) and \( \psi_i(x) = (f(x))^{k-r_i}, i = 1, \ldots, m \), and \( dv(x) = g(x) \, dx \), we obtain

\[
\det \left( \mathbb{E}[w(\lambda_{c_i}, X)(f(X))^{k-r_j}] \right)
\]

\[
= \int_{x_1 > \cdots > x_m} \cdots \int \det \left( w(\lambda_{c_i}, x_j) \right) \det \left( (f(x_j))^{k-r_i} \right) \prod_{j=1}^m g(x_j) \, dx_j.
\]

(3.3)

Since \( w(\lambda, x) \) is TP \(_r\) and \( \lambda_{c_1} > \cdots > \lambda_{c_m} \) then \( \det \left( w(\lambda_{c_i}, x_j) \right) \) is nonnegative on the orthant \( \{(x_1, \ldots, x_m) : x_1 > \cdots > x_m\} \).

As for the second determinant in the integrand in (3.3), let \( \theta_i = k - r_i - m + i, i = 1, \ldots, m \), and set \( \theta = (\theta_1, \ldots, \theta_m) \). Since \( 1 \leq r_1 < \cdots < r_m \leq k \) then \( k - m \geq \theta_1 \geq \cdots \geq \theta_m \geq 0 \). The determinant

\[
\det \left( t_j^{k-r_i} \right) \equiv \det \left( t_j^{\theta_i+m-i} \right)
\]

is well-known; see Macdonald (1995, p. 40). In particular, this determinant is divisible by the product \( \prod_{1 \leq i < j \leq m} (t_i - t_j) \), and the ratio of these two polynomials defines the Schur function,

\[
\chi_\theta(t_1, \ldots, t_m) = \frac{\det \left( t_j^{\theta_i+m-i} \right)}{\prod_{1 \leq i < j \leq m} (t_i - t_j)}.
\]

(3.4)

It is straightforward to verify that \( \chi_\theta(t_1, \ldots, t_m) \) is a homogeneous polynomial of degree \( \theta_1 + \cdots + \theta_m \). It is also well-known (Macdonald, 1995, p. 75) that the coefficients appearing in the monomial expansion of \( \chi_\theta(t_1, \ldots, t_m) \) are nonnegative integers. Therefore, \( \chi_\theta(t_1, \ldots, t_m) > 0 \) for \( t_1, \ldots, t_m > 0 \). Writing (3.4) in the form

\[
\det \left( t_j^{k-r_i} \right) = \prod_{1 \leq i < j \leq m} (t_i - t_j) \cdot \chi_\theta(t_1, \ldots, t_m),
\]
it follows that \( \det \left( t_j^{k-r_j} \right) > 0 \) for all \( t_1, \ldots, t_m > 0 \). Consequently, by substituting \( t_i = f(x_i) \), we obtain
\[
\det \left( (f(x_j))^{k-r_j} \right) = \prod_{1 \leq i < j \leq m} \left( f(x_i) - f(x_j) \right) \cdot \chi(f(x_1), \ldots, f(x_m)),
\]
and since \( f \) is increasing then it follows that the determinant in (3.5) is nonnegative for \( x_1 > \cdots > x_m \).

Therefore, the integrand in (3.3) is nonnegative for \( \lambda_1 > \cdots > \lambda_m \) and \( x_1 > \cdots > x_m \), so it follows that \( \det \left( \mathbb{E}[w(\lambda_i, X)(f(X))^{k-r_j}] \right) \geq 0 \). Since \( m, r_1, \ldots, r_m \) and \( c_1, \ldots, c_m \) were chosen arbitrarily then we deduce that all minors of the determinant \( \det \left( \mathbb{E}[w(\lambda_i, X)(f(X))^{k-r_j}] \right) \) are nonnegative.

If \( w(\lambda, x) \) is STP \(_k\), then \( \det \left( w(\lambda_i, x_j) \right) > 0 \) for all \( \lambda_1 > \cdots > \lambda_m \) and \( x_1 > \cdots > x_m \). If also the set of points of increase of \( f \) contains an open set then the determinant (3.5) is positive on an open set in the orthant \( \{(x_1, \ldots, x_m) : x_1 > \cdots > x_m\} \). Then, the integrand in (3.3) is positive on an open set, so it follows that \( \det \left( \mathbb{E}[w(\lambda_i, X)(f(X))^{k-r_j}] \right) > 0 \).

For \( j = 1, \ldots, r \), we divide by \( \mathbb{E}[w(\lambda_i, X)] \) the \( j \)th column of the determinant \( \det \left( \mathbb{E}[w(\lambda_i, X)(f(X))^{k-r_j}] \right) \). Since
\[
\frac{\mathbb{E}[w(\lambda_i, X)(f(X))^{k-r_j}]}{\mathbb{E}[w(\lambda_i, X)]} = H[\lambda_i, (f(X))^{k-r_j}]
\]
then we find that \( \det \left( H[\lambda_i, (f(X))^{k-r_j}] \right) \geq 0 \) for all \( \lambda_1 > \cdots > \lambda_m \). As before, it follows that all minors of \( \det \left( H[\lambda_i, (f(X))^{k-r_j}] \right) \) are nonnegative.

Finally, for the case in which \( w(\lambda, x) \) is STP \(_k\) and \( f \) is strictly increasing on an open set, we deduce analogously that \( \det \left( H[\lambda_i, (f(X))^{k-r_j}] \right) > 0 \) for all \( \lambda_1 > \cdots > \lambda_m \). Therefore, all minors of \( \det \left( H[\lambda_i, (f(X))^{k-r_j}] \right) \) are positive. \( \square \)

**Remark 3.2.** (1) Consider the case in which \( k = 2 \). As \( H[\lambda, 1] \equiv 1 \), Theorem 3.1 provides that if \( f \) is increasing, \( w(\lambda, x) \) is TP \(_2\), and if \( \lambda_1 > \lambda_2 \) then
\[
\begin{bmatrix}
H[\lambda_1, f(X)] & H[\lambda_1, 1] \\
H[\lambda_2, f(X)] & H[\lambda_2, 1]
\end{bmatrix} = H[\lambda_1, f(X)] - H[\lambda_2, f(X)] \geq 0;
\]
that is, the function \( \lambda \mapsto H[\lambda, f(X)] \) is non-decreasing. For the case in which \( f(x) = x \), we recover the result of Sendov, et al. (2011, Theorem 2.1).

Let \( \mu_\lambda \) denote \( H[\lambda, f(X)] \); equivalently, \( \mu_\lambda \) is the mean of \( f(Y) \) with respect to the weighted distribution (3.1). Then the hypothesis that \( w(\lambda, x) \) is TP \(_2\) leads to the conclusion that \( \mu_\lambda \) is increasing in \( \lambda \).

(2) Suppose that \( k = 3 \); then Theorem 3.1 provides that if \( f \) is increasing, \( w(\lambda, x) \) is TP \(_3\), and if \( \lambda_1 > \lambda_2 > \lambda_3 \) then all minors of the determinant
\[
\begin{bmatrix}
H[\lambda_1, (f(X))^2] & H[\lambda_1, f(X)] & 1 \\
H[\lambda_2, (f(X))^2] & H[\lambda_2, f(X)] & 1 \\
H[\lambda_3, (f(X))^2] & H[\lambda_3, f(X)] & 1
\end{bmatrix}
\]
are nonnegative. In particular, the $2 \times 2$ minor,
\[
\begin{vmatrix}
H[\lambda_1, (f(X))^2] & H[\lambda_1, f(X)] \\
H[\lambda_2, (f(X))^2] & H[\lambda_2, f(X)]
\end{vmatrix}
\]
\[
= H[\lambda_1, (f(X))^2]H[\lambda_2, f(X)] - H[\lambda_1, f(X)]H[\lambda_2, (f(X))^2]
\]  \hspace{1cm} (3.6)
is nonnegative.

Suppose that the weight function $w(\lambda, x)$ is differentiable in $\lambda$. Also, suppose that its partial derivative, $\partial w(\lambda, x)/\partial \lambda$, is integrable and that
\[
H_1[\lambda, f(X)] := \frac{\partial}{\partial \lambda} H[\lambda, f(X)]
\]
exists. Dividing (3.6) by $\lambda_1 - \lambda_2$ and then letting $\lambda_1, \lambda_2 \to \lambda$, we obtain
\[
0 \leq \lim_{\lambda_1, \lambda_2 \to \lambda} \frac{H[\lambda, (f(X))^2]H[\lambda, f(X)] - H[\lambda_1, f(X)]H[\lambda_2, (f(X))^2]}{\lambda_1 - \lambda_2}
\]
\[
= H_1[\lambda, (f(X))^2]H[\lambda, f(X)] - H_1[\lambda, f(X)]H[\lambda, (f(X))^2]
\]
\[
= H[\lambda, f(X)]H[\lambda, (f(X))^2] \frac{\partial}{\partial \lambda} \log \frac{H[\lambda, (f(X))^2]}{H[\lambda, f(X)]};
\]
equivalently,
\[
\frac{\partial}{\partial \lambda} \log \frac{H[\lambda, (f(X))^2]}{H[\lambda, f(X)]} \geq 0.
\]
Hence, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and $w(\lambda, x)$ is TP$_3$ then the function $\lambda \mapsto H[\lambda, (f(X))^2]/H[\lambda, f(X)]$ is increasing.

To interpret this result within a statistical context, define
\[
\sigma^2_\lambda := H[\lambda, (f(X))^2] - (H[\lambda, f(X)])^2;
\]
representing the variance of $f(Y)$ with respect to the weighted distribution (3.1). Then,
\[
\frac{H[\lambda, (f(X))^2]}{H[\lambda, f(X)]} = \frac{\sigma^2_\lambda + \mu^2_\lambda}{\mu_\lambda} = \frac{\sigma^2_\lambda}{\mu_\lambda} + \mu_\lambda.
\]
The ratio $\sigma^2_\lambda/\mu_\lambda$ is known classically as the index of dispersion or the variance-to-mean ratio (Cox and Lewis, 1966, p. 72). The index of dispersion is a measure of the degree of randomness of the underlying random variable, and we denote it by $VMR_\lambda$. Therefore, the assumption that $w(\lambda, x)$ is TP$_3$ leads to the conclusions that $\mu_\lambda$ and $VMR_\lambda + \mu_\lambda$ are increasing functions of $\lambda$.

We can see from these remarks that as $k$, the order of total positivity of the weight function $w(\lambda, x)$, increases, we are able to deduce correspondingly more detailed aspects of the monotonicity properties of $H[\lambda, f(X)]$ as a function of $\lambda$. In particular, if an insurer expects greater randomness for increasing values of the loading parameter $\lambda$ then the insurer should calculate premiums using weight functions that are STP$_k$ for larger values of $k$. 

For \( k = 2 \), we note that a consequence of the proof of Theorem 3.1 is that it provides in (3.3) an explicit representation for the difference \( H[\lambda_1, f(X)] - H[\lambda_2, f(X)] \) as the integral of a nonnegative function, viz.,

\[
\mathbb{E}[w(\lambda_1, X)] \mathbb{E}[w(\lambda_2, X)] \left( H[\lambda_1, f(X)] - H[\lambda_2, f(X)] \right) \\
= \iint_{x_1 > x_2} (f(x_1) - f(x_2)) \begin{vmatrix} w(\lambda_1, x_1) & w(\lambda_1, x_2) \\ w(\lambda_2, x_1) & w(\lambda_2, x_2) \end{vmatrix} g(x_1)g(x_2) \, dx_1 \, dx_2. \tag{3.7}
\]

Hence, the nonnegativity of the difference, \( H[\lambda_1, f(X)] - H[\lambda_2, f(X)] \), is obtained immediately. Further, as the following result shows, the integral representation (3.7) combined with an estimate on the variation of \( f(x) \) leads to an upper bound on \( H[\lambda_1, f(X)] - H[\lambda_2, f(X)] \).

**Corollary 3.3.** Suppose that \( (\lambda, x) \mapsto w(\lambda, x) \) is \( TP_2 \) and \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is a non-decreasing function that satisfies a uniform Lipschitz condition of order 1, viz.,

\[
|f(x_1) - f(x_2)| \leq |x_1 - x_2|
\]

for all \( x_1 \) and \( x_2 \). Then, for all \( \lambda_1 > \lambda_2 \),

\[
H[\lambda_1, f(X)] - H[\lambda_2, f(X)] \leq H[\lambda_1, X] - H[\lambda_2, X]. \tag{3.8}
\]

**Proof.** Applying to (3.7) the Lipschitz condition on \( f \), we obtain

\[
\mathbb{E}[w(\lambda_1, X)] \mathbb{E}[w(\lambda_2, X)] \left( H[\lambda_1, f(X)] - H[\lambda_2, f(X)] \right) \\
\leq \iint_{x_1 > x_2} (x_1 - x_2) \begin{vmatrix} w(\lambda_1, x_1) & w(\lambda_1, x_2) \\ w(\lambda_2, x_1) & w(\lambda_2, x_2) \end{vmatrix} g(x_1)g(x_2) \, dx_1 \, dx_2 \\
\equiv \iint_{x_1 > x_2} \det(x_1^{j-2}) \cdot \det(w(\lambda_i, x_j)) g(x_1)g(x_2) \, dx_1 \, dx_2. \tag{3.10}
\]

Applying the Binet-Cauchy formula (2.1), we deduce that (3.10) equals

\[
\det(\mathbb{E}[w(\lambda_i, X)X^{j-2}]) = \det(\mathbb{E}[w(\lambda_1, X)]H[\lambda_i, X^{j-2}]) \\
= \mathbb{E}[w(\lambda_1, X)] \mathbb{E}[w(\lambda_2, X)] \left( H[\lambda_1, X] - H[\lambda_2, X] \right). \tag{3.11}
\]

On comparing (3.9) and (3.11), and clearing the common terms on each side of that inequality, we obtain (3.8). \( \square \)

**Remark 3.4.** For specified degree-of-variation on \( f \), the bound in Corollary 3.3 provides an upper limit on the increase in the premium \( H[\lambda, f(X)] \) resulting from an increase in \( \lambda \), the loading parameter. This enables an insurer to assess the extent to which it is charging suitable additional amounts for perceived increases in risk as measured by higher values of the loading parameter.
Concepts of total positivity of higher order are germane to other insurance-related problems. For \( u, v > 0 \), define the upper incomplete gamma function,

\[
\Gamma(u, v) = \int_v^\infty x^{u-1} e^{-x} \, dx,
\]

and for \( c > 0 \), the ratio,

\[
R_c(u, v) = \frac{\Gamma(u + c, v)}{\Gamma(u, v)}.
\]

This function was shown by Furman and Zitikis (2008) to arise in the study of losses from collections of insurable risks, and their Proposition 2.1 proved that the function \( R_c(u, v) \) is strictly increasing in \( u \) for each fixed \( v \) and \( c \). Extending this observation, we have the following result for an \( r \times r \) determinant whose entries are based on the function \( R_c \).

**Proposition 3.5.** Suppose that \( c_1 > \cdots > c_r \geq 0 \) and \( u_1 \cdots > u_r > 0 \). Then,

\[
\det (R_{c_j}(u_i, v)) > 0. \tag{3.12}
\]

**Proof.** Consider the determinant,

\[
\det (\Gamma(u_i, v) R_{c_j}(u_i, v)) = \det (\Gamma(u_i + c_j, v)) = \det \left( \int_v^\infty x^{u_i+c_j-1} e^{-x} \, dx \right).
\]

Applying the continuous version of the Binet-Cauchy formula, (2.3), with \( \phi_i(x) = x^{u_i} \), \( \psi_j(x) = x^{c_j-1} \), and \( d\nu(x) = e^{-x} \, dx \), we find that

\[
\det (\Gamma(u_i, v) R_{c_j}(u_i, v)) = \int_{x_1, \ldots, x_r, v} \det (x_i^{u_j}) \det (x_i^{c_j-1}) \prod_{j=1}^{r} e^{-x_j} \, dx_j. \tag{3.13}
\]

As shown later in (4.1), the determinants \( \det (x_i^{u_j}) \) and \( \det (x_i^{c_j-1}) \) are positive for \( x_1 > \cdots > x_r, c_1 > \cdots > c_r \) and \( u_1 \cdots > u_r \). Therefore, the integrand in this multiple integral is positive on an open subset of \( \mathbb{R}^r \), so the integral is positive. Hence, the determinant on the left-hand side of (3.13) is positive; and by extracting the factors \( \Gamma(u_i, v) \) from that determinant, we obtain (3.12).

As a special case of (3.12), suppose that \( r = 2, c_1 = c, \) and \( c_2 = 0 \); since \( R_0(u, v) = 1 \) then (3.12) reduces to the monotonicity result of Furman and Zitikis (2008). More generally, Proposition 3.5 can be applied to obtain inequalities for the higher moments of sums of risks similar to the way in which higher moment inequalities are described in Remark 3.2.
4 Seven classes of weight functions

In this section, we determine the total positivity and strict total positivity properties of some classes of weight functions treated by Sendov, et al. (2011, Section 3).

Example 4.1. Let \( w_1(\lambda, x) = e^{\lambda x} \). The corresponding weighted premium \( H[\lambda, X] \) is called the Esscher premium; see Sendov, et al. (2011), and references given therein. It is well-known (Karlin, 1968, p. 15) that the weight function \( w_1 \) is STP\(_\infty\).

Indeed, by a result of Gross and Richards (1989, p. 233), for each \( r \geq 2 \), the \( r \times r \) determinant, \( \det (w_1(\lambda_i, x_j)) \) has an integral representation,

\[
\prod_{1 \leq i < j \leq r} (\lambda_i - \lambda_j)(x_i - x_j) = \int_U \Phi(\Lambda, X, u) \, d\nu(u),
\]

where \( U \) is a certain set of \( r \times r \) matrices, \( \Lambda = (\lambda_1, \ldots, \lambda_r) \), \( X = (x_1, \ldots, x_r) \), \( \Phi(\Lambda, X, u) \) is a strictly positive function, and \( \nu \) is a probability measure on \( U \). This integral formula yields immediately the positivity of the determinant \( \det (w_1(\lambda_i, x_j)) \) for \( \lambda_1 > \cdots > \lambda_r \) and \( x_1 > \cdots > x_r \). Hence \( w_1 \) is STP\(_r\) for all \( r \) and therefore also STP\(_\infty\).

More generally, we see that, if \( F : \mathbb{R} \to \mathbb{R} \) is strictly increasing then the weight function \( \tilde{w}_1(\lambda, x) = \exp(\lambda F(x)) \) is STP\(_\infty\). The corresponding premium is known as the Aumann-Shapley premium; cf. Furman and Zitikis (2012).

A consequence of the STP\(_\infty\) property of \( w_1 \) is that the weight function \( w_1(\log \lambda, x) = \lambda^x \), \((\lambda, x) \in \mathbb{R}_+ \times \mathbb{R}\), is STP\(_\infty\). That is, for any \( r \in \mathbb{N} \), the \( r \times r \) determinant,

\[
\det (\lambda_i^{x_j}) > 0 \quad (4.1)
\]

for \( \lambda_1 > \cdots > \lambda_r > 0 \) and \( x_1 > \cdots > x_r \). This result holds because the transformation \( \lambda \to \log \lambda \) is strictly increasing and therefore preserves the total positivity properties of \( w_1(\lambda, x) \). In the actuarial literature, this weight function gives rise to a weighted premium known as the size-biased premium; cf. Furman and Zitikis (2012). We shall apply (4.1) repeatedly in the sequel.

Example 4.2. Let \( w_2(\lambda, x) = 1\{x > \lambda\} \), the weight function corresponding to the conditional tail expectation (CTE) premium, \( H[\lambda, X] = \mathbb{E}(X|X > \lambda) \). It is well-known from Karlin (1968, p. 16) that the weight function \( w_2 \) is TP\(_\infty\). Indeed, for any \( r = 1, 2, \ldots \), and for \( \lambda_1 > \cdots > \lambda_r \) and \( x_1 > \cdots > x_r \),

\[
\det (w_2(\lambda_i, x_j)) = \begin{cases} 
1, & \text{if } x_1 > \lambda_1 > x_2 > \lambda_2 > \cdots > x_r > \lambda_r \\
0, & \text{otherwise}
\end{cases} \quad (4.2)
\]

which proves that the determinant is nonnegative.
Example 4.3. Let \( w_3(\lambda, x) = 1 - e^{-x/\lambda} \). The corresponding weighted premium \( H[\lambda, X] \) is called the \textit{Kamps premium} (see Sendov, et al. (2011) and the references therein). The weight function \( w_3 \) is \( \text{STP}_\infty \), as we now prove.

It is straightforward to verify that

\[
    w_3(\lambda, x) = \int_0^x \lambda^{-1} e^{-t/\lambda} \, dt = \int_0^\infty w(\lambda, t) w_2(t, x) \, dt,
\]

where \( w(\lambda, t) = \lambda^{-1} e^{-t/\lambda} \), and \( w_2(t, x) = 1 \{ x > t \} \) is the weight function given in Example 4.2. Applying the Binet-Cauchy formula (2.1), we obtain for each \( r \geq 2 \),

\[
    \det (w_3(\lambda_i, x_j)) = \int t_1 > \cdots > t_r \det (w(\lambda_i, t_j)) \det (w_2(t_i, x_j)) \, dt_1 \cdots dt_r. \tag{4.3}
\]

Suppose that \( \lambda_1 > \cdots > \lambda_r \) and \( x_1 > \cdots > x_r \). By Example 4.1, the function \( w(\lambda, t) \) is \( \text{STP}_\infty \), so \( \det (w(t_i, x_j)) \) is positive on the orthant \( \{(t_1, \ldots, t_r) : t_1 > \cdots > t_r \} \). Also, by (4.2), the determinant \( \det (w_2(t_i, x_j)) \) is positive on an open neighborhood in the same orthant. Therefore, the integrand in (4.3) is positive on an open set, hence the integral is positive for any choice of \( (\lambda_1, \ldots, \lambda_r) \) and \( (x_1, \ldots, x_r) \). Therefore, \( w_3(\lambda, x) \) is \( \text{STP}_r \) for all \( r \), hence it is \( \text{STP}_\infty \).

These results for \( w_3(\lambda, x) \) also extend to a more general class of weight functions. For each nonnegative integer \( k \), define the weight function

\[
    w_{3,k}(\lambda, x) = k! \left[ 1 - e^{-x/\lambda} \sum_{j=0}^{k} \frac{(x/\lambda)^j}{j!} \right].
\]

For \( k = 0 \), \( w_{3,k}(\lambda, x) \) reduces to \( w_3(\lambda, x) \), the weight function corresponding to Kamps’ premium. By repeated integration-by-parts, we obtain

\[
    w_{3,k}(\lambda, x) = \lambda^{-(k+1)} \int_0^x t^k e^{-t/\lambda} \, dt = \int_0^\infty w(\lambda, t) w_2(t, x) \, dt,
\]

where \( w(\lambda, t) = \lambda^{-(k+1)} e^{-t/\lambda} \) and \( w_2(t, x) \) is the weight function in Example 4.2. Finally, we proceed using arguments similar to the case of \( w_3 \): Since \( w(\lambda, t) \) and \( w_2(\lambda, t) \) are \( \text{TP}_\infty \) then, by applying the Basic Composition Formula (2.3) and the Binet-Cauchy formula (2.1), we deduce that \( w_{3,k}(\lambda, x) \) is \( \text{STP}_\infty \).

Example 4.4. The fourth weight function considered by Sendov, et al. (2011, Section 3) is

\[
    \tilde{w}_4(\lambda, x) = \exp \left( \frac{(1 + x)^\lambda - 1}{\lambda} \right) - x,
\]
λ, x > 0. We replace x by \( e^x - 1 \), a transformation that is strictly increasing and therefore preserves any total positivity properties of \( \bar{w}_4(\lambda, x) \). Then we are to determine the total positivity properties of

\[
w_4(\lambda, x) := \bar{w}_4(\lambda, e^x - 1) = \exp \left( f(\lambda, x) \right) - e^x + 1,
\]

λ, x > 0, where

\[
f(\lambda, x) = \frac{e^{\lambda x} - 1}{\lambda}.
\]

We also define \( f(0, x) \) by right-continuity:

\[
f(0, x) := \lim_{\lambda \to 0^+} f(\lambda, x) = x.
\]

Then,

\[
w_4(\lambda, x) = \exp \left( f(\lambda, x) \right) - \exp \left( f(0, x) \right) + 1. \tag{4.4}
\]

For \( r \in \mathbb{N} \), let \( \lambda_1 > \cdots > \lambda_r > 0, x_1 > \cdots > x_r > 0 \), and consider the \( r \times r \) determinant,

\[
\det \left( w_4(\lambda_i, x_j) \right) = \begin{vmatrix}
    w_4(\lambda_1, x_1) & \cdots & w_4(\lambda_1, x_r) \\
    w_4(\lambda_2, x_1) & \cdots & w_4(\lambda_2, x_r) \\
    \vdots & \ddots & \vdots \\
    w_4(\lambda_{r-1}, x_1) & \cdots & w_4(\lambda_{r-1}, x_r) \\
    w_4(\lambda_r, x_1) & \cdots & w_4(\lambda_r, x_r)
\end{vmatrix}.
\]

For \( i = 1, \ldots, r - 1 \), we subtract row \( i + 1 \) from row \( i \), obtaining

\[
\det \left( w_4(\lambda_i, x_j) \right) = D_1 + D_2,
\]

where

\[
D_1 = \begin{vmatrix}
    w_4(\lambda_1, x_1) - w_4(\lambda_2, x_1) & \cdots & w_4(\lambda_1, x_r) - w_4(\lambda_2, x_r) \\
    w_4(\lambda_2, x_1) - w_4(\lambda_3, x_1) & \cdots & w_4(\lambda_2, x_r) - w_4(\lambda_3, x_1) \\
    \vdots & \ddots & \vdots \\
    w_4(\lambda_{r-1}, x_1) - w_4(\lambda_r, x_1) & \cdots & w_4(\lambda_{r-1}, x_r) - w_4(\lambda_r, x_1) \\
    w_4(\lambda_r, x_1) - w_4(0, x_1) & \cdots & w_4(\lambda_r, x_r) - w_4(0, x_1)
\end{vmatrix}
\]

and

\[
D_2 = \begin{vmatrix}
    w_4(\lambda_1, x_1) - w_4(\lambda_2, x_1) & \cdots & w_4(\lambda_1, x_r) - w_4(\lambda_2, x_r) \\
    w_4(\lambda_2, x_1) - w_4(\lambda_3, x_1) & \cdots & w_4(\lambda_2, x_r) - w_4(\lambda_3, x_1) \\
    \vdots & \ddots & \vdots \\
    w_4(\lambda_{r-1}, x_1) - w_4(\lambda_r, x_1) & \cdots & w_4(\lambda_{r-1}, x_r) - w_4(\lambda_r, x_1) \\
    w_4(0, x_1) & \cdots & w_4(0, x_1)
\end{vmatrix}. \tag{4.5}
\]

Define

\[
w_{41}(\lambda, x) = \frac{\partial}{\partial \lambda} \omega_4(\lambda, x) \equiv \frac{\partial}{\partial \lambda} \exp \left( f(\lambda, x) \right),
\]
and set $\lambda_{r+1} \equiv 0$. By Taylor’s theorem, there exists $\rho_i \in (\lambda_{i+1}, \lambda_i)$ such that

$$w_4(\lambda_i, x) - w_4(\lambda_{i+1}, x) = (\lambda_i - \lambda_{i+1}) w_{41}(\rho_i, x),$$

(4.6)

$i = 1, \ldots, r$. Therefore,

$$D_1 = \det \left( (\lambda_i - \lambda_{i+1}) w_{41}(\rho_i, x_j) \right)$$

$$\equiv \prod_{i=1}^{r} (\lambda_i - \lambda_{i+1}) \cdot \det \left( w_{41}(\rho_i, x_j) \right),$$

where $\lambda_1 > \rho_1 > \lambda_2 > \rho_2 > \cdots > \lambda_r > \rho_r > 0$. So, to prove that $D_1$ is positive, it suffices to show that $w_{41}$ is STP$_r$, and we begin by observing from (4.4) that

$$\det \left( w_{41}(\lambda_i, x_j) \right) = \det \left( \frac{\partial}{\partial \lambda_i} \exp \left( f(\lambda_i, x_j) \right) \right)$$

$$= \frac{\partial^r}{\partial \lambda_1 \cdots \partial \lambda_r} \det \left( \exp \left( f(\lambda_i, x_j) \right) \right).$$

(4.7)

We now recall the Bell (or exponential) polynomials $B_k$, $k = 0, 1, 2, \ldots$, defined through the generating function,

$$\exp \left( u(e^t - 1) \right) = \sum_{k=0}^{\infty} B_k(u) \frac{t^k}{k!}.$$ 

(4.8)

We refer to Comtet (1974, p. 133 ff.) and Roman (1984, pp. 63-67) for further details on these polynomials. For $k \geq 1$, $B_k(u)$ is monic and of degree $k$; moreover,

$$B_k(u) = \sum_{m=1}^{k} S(k, m) u^m,$$

(4.9)

where the coefficients $S(k, m)$ are the Stirling numbers of the second kind, viz., the number of partitions of a set of size $k$ into $m$ non-empty subsets (Comtet, 1974, p. 50). In particular, $S(k, 1) = S(k, k) = 1$, and $S(k, m) = 0$ if $m > k$.

An alternative representation for the Bell polynomials arises from the observation that the left-hand side of (4.8) is, for $u > 0$, the moment-generating function of $U$, a Poisson-distributed random variable with mean parameter $u$; therefore,

$$B_k(u) = E(U^k) = \sum_{m=0}^{\infty} \frac{e^{-u} u^m}{m!} m^k.$$  

(4.10)

We now apply to (4.10) the discrete Binet-Cauchy formula (2.2) with $\phi_i(m) = e^{-u_i} u_i^m$ and $\psi_k(m) = m^k$, each of which is STP$_\infty$ by (4.1), and weights $\nu(m) = 1/m!$. Written explicitly, we have, for $k_1 > \cdots > k_r \geq 0$ and $u_1 > \cdots > u_r > 0$,

$$\det \left( B_k(u_j) \right) = e^{-(u_1 + \cdots + u_r)} \sum_{m_1 > \cdots > m_r \geq 0} \frac{1}{m_1! \cdots m_r!} \det(u_i^{m_j}) \det(m_i^{k_j}).$$
Then the positivity of \( \det B_k(u_j) \) follows from the positivity of each determinant inside the summation. Since \( r \) was chosen arbitrarily then it follows that \( B_k(u) \) is STP\(_\infty \) in \((k,u)\).

Define

\[
\tilde{B}_k(\lambda) := \lambda^k B_k(\lambda^{-1}) = \sum_{m=0}^{k-1} S(k, k-m)\lambda^m.
\]  

(4.11)

Then, by (4.8) and (4.9),

\[
\exp(f(\lambda, x)) = \sum_{k=0}^{\infty} B_k(\lambda^{-1})(\lambda x)^k \frac{1}{k!} = \sum_{k=0}^{\infty} \tilde{B}_k(\lambda) x^k.
\]

(4.12)

Applying to (4.12) the Binet-Cauchy formula (2.1), we obtain for \( \lambda_1 > \cdots > \lambda_r > 0 \) and \( x_1 > \cdots > x_r > 0 \),

\[
det\left(\exp(f(\lambda_i, x_j))\right) = det\left(\sum_{k=0}^{\infty} \frac{1}{k!} \tilde{B}_k(\lambda_i) x_j^k\right)
\]

\[
= \sum_{k_1 > \cdots > k_r \geq 0} \frac{1}{k_1! \cdots k_r!} det(\tilde{B}_{k_1}(\lambda_i)) det(x_i^{k_j}).
\]

(4.13)

By (4.1), \( det(x_i^{k_j}) > 0 \) for \( x_1 > \cdots > x_r \) and \( k_1 > \cdots > k_r \).

We note two consequences of (4.11). First, since \( S(k,1) = 1 \) then the polynomial \( \tilde{B}_k \) is monic and of degree \( k - 1 \). Second, since \( S(k,m) = 0 \) for \( m > k \) and \( S(k,k) = 1 \) then the polynomials \( \{\tilde{B}_k(\lambda) : k = 0,1,2,\ldots\} \) satisfy a linear system of equations in terms of \( \{\lambda^m : m = 0,1,2,\ldots\} \), with a triangular matrix of coefficients having the \((k,m)\)th entry equal to \( S(k,m) \), \( 1 \leq m \leq k \). Writing out these equations for rows \( k_r, k_{r-1}, \ldots, k_1 \), in that order, and for columns \( 1,2,\ldots,r \) results in a matrix equation,

\[
\tilde{B} = S\Lambda,
\]

(4.14)

where the \( r \times r \) matrix \( \tilde{B} \) has \((i,j)\)th entry \( \tilde{B}_{k-r+1}(\lambda_i) \), \( i,j = 1,\ldots,r \); \( S \) is \( r \times k_1 \) with \((i,j)\)th entry \( S(k_{r-i+1}, k_{r-i+1} - j + 1) \), \( i = 1,\ldots,r \); \( j = 1,\ldots,k_1 \); and \( \Lambda \) is \( k_1 \times r \) with \((i,j)\)th entry \( \lambda_i^{j-1} \), \( i = 1,\ldots,k_1 \), \( j = 1,\ldots,r \).

Each \( r \times r \) minor of \( \Lambda \), being of the form \( det(\lambda_i^{j_l}) \) with \( \lambda_1 > \cdots > \lambda_r \) and \( l_1 < \cdots < l_r \), is non-zero and has sign \((-1)^{r(r-1)/2} \) as \( r(r-1)/2 \) row interchanges are needed to order the \( \lambda_i \) and \( l_i \) similarly.

To determine the sign of the minors of \( S \), we apply the results of Brenti (1995, Section 5); cf. Mongelli (2012). According to those results, the infinite matrix \( S(k,m) \) is totally positive, i.e., all minors of the matrix \( S(k,m) \), where \( k \) and \( m \) are similarly ordered, are nonnegative. In the \( i \)th row of \( S \), the columns are indexed by the decreasing sequence \( k_{r-i+1} - j + 1, j = 1,\ldots,k_i \); and in the \( j \)th column, the rows are indexed by the increasing sequence \( k_{r-i+1}, i = 1,\ldots,r \); therefore, it follows that each non-zero \( r \times r \) minor of \( S \) also has sign \((-1)^{r(r-1)/2} \). Further, if \( k_1,\ldots,k_r \) are consecutive
integers then the resulting matrix is lower triangular with non-zero diagonal entries, so the corresponding minor of $S$ is non-zero.

By the classical Binet-Cauchy formula, $\det(\tilde{B})$ equals a sum of products of $r \times r$ minors of $S$ and $\Lambda$ (Karlin, 1968, p. 1). By the preceding discussion, each such product is nonnegative; this establishes the positivity of each $r \times r$ minor of $\tilde{B}$, proving that it is at least $\text{TP}_r$. Further, the sum of all such products of minors is positive since some minors of $S$, and all minors of $\Lambda$, are non-zero. Therefore, $\tilde{B}$ is $\text{STP}_r$; and since $r$ is arbitrary then it is $\text{STP}_\infty$.

To complete the proof that $w_{41}(\lambda, x)$ is $\text{STP}_r$, we need to show that (4.7) is positive. By the same argument as at (4.14), infra, we find that

$$
\frac{\partial^r}{\partial \lambda_1 \cdots \partial \lambda_r} \det (\tilde{B}_{k_j}(\lambda_{i}))
$$

(4.15)

is a sum of products of minors of $S$ with derivatives of minors of $\Lambda$. However, the derivatives of the minors of $\Lambda$ are the form

$$
\frac{\partial^r}{\partial \lambda_1 \cdots \partial \lambda_r} \det (\lambda_{i}^{l_j}) \equiv \det \left( \frac{\partial}{\partial \lambda_i} \lambda_{i}^{l_j} \right) = \frac{l_1 \cdots l_r}{\lambda_1 \cdots \lambda_r} \det (\lambda_{i}^{l_j}),
$$

(4.16)

which is of the same sign, viz., $(-1)^{r(r-1)/2}$, as each $r \times r$ minor of $\Lambda$. Therefore, the derivatives (4.15) are nonnegative, and some are positive. By differentiating the series (4.13), it follows that $w_{41}(\lambda, x)$ is $\text{STP}_r$, and hence is $\text{STP}_\infty$.

For future reference, we note that in addition to (4.7) being positive for $\lambda_1 > \cdots > \lambda_r$ and $x_1 > \cdots > x_r$, there also holds

$$
\det \left( \frac{\partial^2}{\partial x_j \partial \lambda_i} \exp (f(\lambda_i, x_j)) \right) > 0 \quad (4.17)
$$

under the same conditions. To prove this, we observe that the above determinant equals

$$
\frac{\partial^r}{\partial x_1 \cdots \partial x_r} \frac{\partial^r}{\partial \lambda_1 \cdots \partial \lambda_r} \det \left( \exp (f(\lambda_i, x_j)) \right),
$$

so we can expand this determinant using (4.13). As shown before, the resulting terms in $(\lambda_1, \ldots, \lambda_r)$ are nonnegative. Also, the resulting terms in $(x_1, \ldots, x_r)$ are of the form (4.16) (with each $\lambda_i$ replaced by $x_i$), and hence also are nonnegative. Moreover, it is straightforward to see that some of these terms are non-zero. Therefore, (4.17) is positive for $\lambda_1 > \cdots > \lambda_r$ and $x_1 > \cdots > x_r$.

Turning to the determinant $D_2$ in (4.5), we note first that

$$
w_4(0, x) := \lim_{\lambda \to 0^+} w_4(0, x) = 1.
$$
Therefore,
\[
D_2 = \begin{vmatrix}
w_4(\lambda_1, x_1) - w_4(\lambda_2, x_1) & \cdots & w_4(\lambda_1, x_r) - w_4(\lambda_2, x_r) \\
w_4(\lambda_2, x_1) - w_4(\lambda_3, x_1) & \cdots & w_4(\lambda_2, x_r) - w_4(\lambda_3, x_r) \\
\vdots & \ddots & \vdots \\
w_4(\lambda_{r-1}, x_1) - w_4(\lambda_r, x_1) & \cdots & w_4(\lambda_{r-1}, x_r) - w_4(\lambda_r, x_r) \\
1 & \cdots & 1
\end{vmatrix}.
\]

We again apply Taylor’s theorem, as in \((4.6)\), to each entry in rows 1, \ldots, \(r-1\), obtaining
\[
D_2 = \prod_{i=1}^{r-1} (\lambda_i - \lambda_{i+1}) \cdot D_3
\]
where
\[
D_3 = \begin{vmatrix}
w_41(\rho_1, x_1) & \cdots & w_41(\rho_1, x_r) \\
w_41(\rho_2, x_1) & \cdots & w_41(\rho_2, x_r) \\
\vdots & \ddots & \vdots \\
w_41(\rho_{r-1}, x_1) & \cdots & w_41(\rho_{r-1}, x_r) \\
1 & \cdots & 1
\end{vmatrix}.
\]

Carrying out elementary column operations, subtracting column \(j+1\) from column \(j\), for \(j = 1, \ldots, r-1\), we obtain
\[
D_3 = \begin{vmatrix}
w_41(\rho_1, x_1) - w_41(\rho_1, x_2) & \cdots & w_41(\rho_1, x_{r-1}) - w_41(\rho_1, x_r) & w_41(\rho_1, x_r) \\
\vdots & \ddots & \vdots & \vdots \\
w_41(\rho_{r-1}, x_1) - w_41(\rho_{r-1}, x_2) & \cdots & w_41(\rho_{r-1}, x_{r-1}) - w_41(\rho_{r-1}, x_r) & w_41(\rho_{r-1}, x_r) \\
0 & \cdots & 0 & 1
\end{vmatrix}
\]
\[
= \begin{vmatrix}
w_41(\rho_1, x_1) - w_41(\rho_1, x_2) & \cdots & w_41(\rho_1, x_{r-1}) - w_41(\rho_1, x_r) \\
\vdots & \ddots & \vdots & \vdots \\
w_41(\rho_{r-1}, x_1) - w_41(\rho_{r-1}, x_2) & \cdots & w_41(\rho_{r-1}, x_{r-1}) - w_41(\rho_{r-1}, x_r)
\end{vmatrix}.
\]

Define
\[
w_{411}(\lambda, x) := \frac{\partial}{\partial x}w_{41}(\lambda, x) = \frac{\partial^2}{\partial \lambda \partial x} \exp\left(f(\lambda, x)\right).
\]
Applying Taylor’s theorem again, we find that there exists \(y_j \in (x_{j+1}, x_j)\) such that
\[
w_{41}(\rho_i, x_j) - w_{41}(\rho_i, x_{j+1}) = (x_j - x_{j+1})w_{411}(\rho_i, y_j),
\]
\(i, j = 1, \ldots, r-1\); therefore,
\[
D_3 = \det \left((x_j - x_{j+1})w_{411}(\rho_i, y_j)\right) = \prod_{j=1}^{r-1}(x_j - x_{j+1}) \cdot \det\left(w_{411}(\rho_i, y_j)\right).
\]
Noting that
\[
\det\left( w_{411}(\rho_i, y_j) \right) = \det \left( \frac{\partial^2}{\partial \rho_i \partial y_j} \exp \left( f(\rho_i, y_j) \right) \right)
\]
is of the form (4.17), it follows that \( D_3 \), and therefore \( D_2 \) is a product of terms, each of which is positive for \( x_1 > \cdots > x_r, y_1 > \cdots > y_{r-1} \), and \( \rho_1 > \cdots > \rho_{r-1} \); therefore \( D_2 > 0 \). Consequently, \( w_4(\lambda, x) \), and hence \( \tilde{w}_4(\lambda, x) \), are STP, and since \( r \) was chosen arbitrarily then they both are STP\(_\infty\).

**Example 4.5.** Let
\[
w_5(\lambda, x) = \frac{(1 + \lambda)^x - 1}{\lambda x},
\]
\( \lambda, x > 0 \). We show that \( w_5(\lambda, x) \) is STP\(_\infty\).

We observe that
\[
w_5(\lambda, x) = \lambda^{-1} \int_0^\lambda (1 + t)^{x-1} dt
\]
\[
= \lambda^{-1} \int_0^{\infty} 1\{\lambda > t\} (1 + t)^{x-1} dt.
\]
Recall that the function \( w_2(\lambda, t) = 1\{\lambda > t\} \) is TP\(_\infty\); moreover, the corresponding \( r \times r \) determinants are positive on an open set in \( \mathbb{R}_+^r \). Also, the function \( w(t, x) = (1 + t)^{x-1} \) is STP\(_\infty\). Therefore, by the Basic Composition Formula (2.3), \( w_5(\lambda, x) \) is STP\(_\infty\).

**Example 4.6.** Let
\[
w_6(\lambda, x) = \frac{\lambda x}{\log(1 + \lambda x)},
\]
\( \lambda, x > 0 \). We shall prove that \( w_6 \) is STP\(_\infty\) on the region \( \{(\lambda, x) : \lambda > 0, x > 0, \lambda x < 1\} \).

First, we note that for all \( \lambda, x > 0 \),
\[
w_6(\lambda, x) = \int_0^1 (1 + \lambda x)^t dt.
\]
(4.18)

For \( \lambda x < 1 \), we expand the integrand, obtaining
\[
(1 + \lambda x)^t = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (-t)_k \lambda^k x^k,
\]
where \( (a)_k = a(a+1)(a+2) \cdots (a+k-1) \) is the classical rising factorial. Substituting this series into the integral at (4.18) and integrating term-by-term, we obtain
\[
w_6(\lambda, x) = \sum_{k=0}^{\infty} \theta_k \lambda^k x^k,
\]
(4.19)
where
\[
\theta_k = \frac{(-1)^k}{k!} \int_0^1 (-t)_k \, dt = \frac{1}{k!} \int_0^1 t(1-t)(2-t) \cdots (k-1-t) \, dt.
\]

This representation shows immediately that \( \theta_k > 0 \) for all \( k \geq 0 \).

By applying to (4.19) the discrete version (2.4) of the Basic Composition Formula, with \( w_1(\lambda, k) = \lambda^k \), \( w_2(k, x) = x^k \), and \( \nu(k) = \theta_k \), we deduce that \( w_6 \) is STP\(_r\) for all \( r \); hence, \( w_6(\lambda, x) \) is STP\(_\infty\) on the region \( \{ (\lambda, x) : \lambda > 0, x > 0, \lambda x < 1 \} \). By the discrete Binet-Cauchy formula (2.2), we also obtain the representation,

\[
\det (w_6(\lambda, x)) = \sum_{k_1 > \cdots > k_r \geq 0} \theta_{k_1} \cdots \theta_{k_r} \det (\lambda^{k_i}) \det (x^{k_j}),
\]

As regards the total positivity properties of \( w_6(\lambda, x) \) on \( \mathbb{R}^2_+ \), a high-precision numerical evaluation, with accuracy to 120 significant digits, determined that for \( (x_1, x_2, x_3) = (20000, 0.3, 0.1) \) and \( (\lambda_1, \lambda_2, \lambda_3) = (3, 0.4, 0.1) \), the \( 3 \times 3 \) determinant,

\[
\det (w_6(\lambda_i, x_j)) = -5.1748811 \ldots < 0.
\]

Therefore, \( w_6(\lambda, x) \) is not TP\(_3\) on \( \mathbb{R}^2_+ \).

**Example 4.7.** Let

\[
w_7(\lambda, x) = \log(1 + \lambda + x) \frac{x}{\lambda + x} \frac{1}{\log(1 + x)},
\]

\( \lambda, x > 0 \). It is straightforward to verify that

\[
\frac{\log(1 + \lambda + x)}{\lambda + x} = \int_0^\infty (1+t)^{-1}(1+\lambda+x+t)^{-1} \, dt.
\]

We remark that this integral representation arose in work of Carlson and Gustafson (1983) on the total positivity properties of mean value kernels; in the notation of Carlson and Gustafson, the function in (4.20) is denoted by \( R_{-1}(1,1;1+\lambda+x,1) \). Writing

\[
(1+\lambda+x+t)^{-1} = \int_0^\infty e^{-(1+\lambda+x+t)u} \, du,
\]

substituting this formula into (4.20), and applying Fubini’s theorem to justify an interchange of the order of integration, we obtain

\[
\frac{\log(1 + \lambda + x)}{\lambda + x} = \int_0^\infty (1+t)^{-1} \int_0^\infty e^{-(1+\lambda+x+t)u} \, du \, dt
\]

\[
= \int_0^\infty e^{-\lambda u} e^{-\mu x} \, d\nu(u),
\]
where the positive measure \( \nu \) is given explicitly by

\[
d\nu(u) = \left[ \int_0^\infty e^{-tu}(1+t)^{-1} dt \right] e^{-u} du,
\]

\( u > 0 \). Consequently, we have obtained an integral representation,

\[
w_7(\lambda, x) = \frac{x}{\log(1+x)} \int_0^\infty e^{-\lambda u} e^{-ux} d\nu(u).
\]

Applying the Binet-Cauchy formula (2.1), we obtain

\[
\det \left( w_7(\lambda_i, x_j) \right) = \left[ \prod_{j=1}^r \frac{x_j}{\log(1+x_j)} \right] \int \cdots \int e^{-\lambda u_1} \cdots e^{-\lambda u_r} \, d\mu(u_1) \cdots d\nu(u_r)
\]

for \( \lambda_1 > \cdots > \lambda_r \) and \( x_1 > \cdots > x_r \). Since the sign of each determinant in the integrand equals \((-1)^{r(r-1)/2}\) then their product is positive everywhere on the range of integration. Therefore, \( w_7(\lambda, x) \) is STP_r for all \( r \geq 1 \), hence it is STP_\infty.

### 5 Conclusions

In this paper, we have explored the implications for the loading monotonicity problem of the use of higher-order totally positive weight functions for constructing weighted premiums. In doing this, we applied results from the areas of total positivity (Karlin, 1968) and symmetric functions (Macdonald, 1995). As a consequence, we obtained monotonicity properties of weighted transformed premiums, and an upper bound under Lipschitz hypotheses for the increase in the weighted premium in response to an increase in the loading parameter.

Further, we examined the higher order total positivity properties of a class of kernels that have appeared in the actuarial literature. We established the highest order of total positivity of each of these kernels, thereby adding to the collection of examples of strictly totally positive kernels.

We related the use of weight functions that are totally positive of higher order to the degree of randomness of insured risks. Thus, it appears that a broad list of TP_\infty, and even STP_\infty, weight functions is needed to develop weighted premiums to underwrite insurable risks of any degree of randomness, and our paper initiates such a list.

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