MULTISET ESTIMATES AND COMBINATORIAL SYNTHESIS

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ABSTRACT. The paper addresses an approach to ordinal assessment of alternatives based on assignment of elements into an ordinal scale. Basic versions of the assessment problems are formulated while taking into account the number of levels at a basic ordinal scale [1,2,...,l] and the number of assigned elements (e.g., 1,2,3). The obtained estimates are multisets (or bags) (cardinality of the multiset equals a constant). Scale-posets for the examined assessment problems are presented. “Interval multiset estimates” are suggested. Further, operations over multiset estimates are examined: (a) integration of multiset estimates, (b) proximity for multiset estimates, (c) comparison of multiset estimates, (d) aggregation of multiset estimates, and (e) alignment of multiset estimates. Combinatorial synthesis based on morphological approach is examined including the modified version of the approach with multiset estimates of design alternatives. Knapsack-like problems with multiset estimates are briefly described as well. The assessment approach, multiset-estimates, and corresponding combinatorial problems are illustrated by numerical examples.

1. INTRODUCTION

In this article, a combinatorial approach to ordinal assessment of alternatives is suggested. The approach consists in assignment of elements (1, 2, 3,...) into an ordinal scale [1, 2, ..., l]. As a result, a multi-set based estimate is obtained, where a basis set involves all levels of the ordinal scale: \( \Omega = \{1,2,\ldots,l\} \) (the levels are linear ordered: \( 1 \succ 2 \succ 3 \succ \ldots \)) and the assessment problem (for each alternative) consists in selection of a multiset over set \( A \) while taking into account two conditions:

1. cardinality of the selected multiset equals a specified number of elements \( \eta = 1,2,3,\ldots \) (i.e., multisets of cardinality \( \eta \) are considered);
2. “configuration” of the multiset is the following: the selected elements of \( \Omega \) cover an interval over scale [1, l] (i.e., “interval multiset estimate”).

Note, fundamentals of multisets can be found in \([1,7,32,36]\). Evidently, the assessment case \( \eta = 1 \) corresponds to traditional ordinal assessment. Thus, an estimate \( e \) for an alternative \( A \) is (scale \([1, l]\), position-based form or position form): \( e(A) = (\eta_1, \ldots, \eta_\iota, \ldots, \eta_l) \), where \( \eta_\iota \) corresponds to the number of elements at the level \( \iota \) (\( \iota = 1, l \)). Here, the conditions above are:

Condition 1: \( \sum_{\iota=1}^{l} \eta_\iota = \eta \) (or \( |e(A)| = \eta \)).

Condition 2: if \( \eta_\iota > 0 \) and \( \eta_{\iota+2} > 0 \) then \( \eta_{\iota+1} > 0 \) (\( \iota = 1, l-2 \)).

On the other hand, the multiset estimate is:

\[
e(A) = \{ \underbrace{1,\ldots,1}_{\eta_1}, \underbrace{2,\ldots,2}_{\eta_2}, \underbrace{3,\ldots,3}_{\eta_3}, \ldots, \underbrace{l,\ldots,l}_{\eta_l} \}.
\]

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The number of multisets of cardinality $\eta$, with elements taken from a finite set of cardinality $l$, is called the “multiset coefficient” or “multiset number” \cite{7,32,36}:

$$\mu^{l,\eta} = \binom{l}{\eta} = \frac{l(l+1)(l+2)...(l+\eta-1)}{\eta!} = \binom{l+\eta-1}{\eta}.$$  

This number corresponds to possible estimates (without taking into account interval conditions 2). In the case of condition 2, the number of estimates is decreased.

**Example 1.** The ordinal assessment is the following: (a) the basic ordinal scale (basic set) is: $\Omega = \{1, 2, 3, ..., l\}$, (b) the number of elements or cardinality of multiset (estimate) is: $\eta = 1$. Estimates are:

- $\hat{e}_1^o = \{1\}$ and position-based form $\hat{e}_1^o = (1, 0, 0, 0, 0, ...)$,
- $\hat{e}_2^o = \{2\}$ and position-based form $\hat{e}_2^o = (0, 1, 0, 0, 0, ...)$,
- $\hat{e}_3^o = \{3\}$ and position-based form $\hat{e}_3^o = (0, 0, 1, 0, 0, ...)$, ...
- $\hat{e}_l^o = \{l\}$ and position-based form $\hat{e}_l^o = (0, 0, 0, 0, 0, 1)$.

**Example 2.** The basic ordinal scale (basic set) is: $\Omega = \{1, 2, 3, 4\}$, the number of elements or cardinality of multiset (estimate) is: $\eta = 3$.

**Case 1.** Estimates corresponding to basic ordinal assessment are: $e_1^o = \{1, 1, 1\}$ or in position-based form $e_1^o = (3, 0, 0, 0)$; $e_2^o = \{2, 2, 2\}$ or in position-based form $e_2^o = (0, 3, 0, 0)$; $e_3^o = \{3, 3, 3\}$ or in position-based form $e_3^o = (0, 0, 3, 0)$; $e_4^o = \{4, 4, 4\}$ or in position-based form $e_4^o = (0, 0, 0, 3)$.

**Case 2.** Examples of correct estimates are: $e_1' = \{1, 2, 3\}$ or in position-based form $e_1' = (1, 1, 0, 0, 0)$; $e_2' = \{2, 2, 3\}$ or in position-based form $e_2' = (0, 2, 1, 0)$; $e_3' = \{4, 4, 4\}$ or in position-based form $e_3' = (0, 0, 0, 3)$.

**Case 3.** Examples of incorrect estimates are: $e_1'' = \{1, 1, 3\}$ or in position-based form $e_1'' = (2, 0, 1, 0)$; $e_2'' = \{1, 3, 4\}$ or in position-based form $e_2'' = (1, 0, 1, 1)$; $e_3'' = \{2, 4, 4\}$ or in position-based form $e_3'' = (0, 1, 0, 2)$.

Basic versions of the assessment problems are formulated as $P_{l,\eta}^{l,\eta}$: (i) traditional assessment based on ordinal scale $[1, 2, 3]$; $P_{3}^{3,1}$; (ii) traditional assessment based on ordinal scale $[1, 2, 3, 4]$; $P_{4}^{4,1}$; (iii) assessment over ordinal scale $[1, 2, 3]$ based on assignment of two elements: $P_{3}^{2}$; (iv) assessment over ordinal scale $[1, 2, 3]$ based on assignment of three elements: $P_{3}^{3,2}$; and (v) assessment over ordinal scale $[1, 2, 3]$ based on assignment of four elements: $P_{4}^{4,3}$. In the article, the obtained scale-posets are presented and corresponding alternative evaluation and composition problems are described. In the case $\eta \geq 2$, the suggested assessment approach can be considered as a very simplified discrete version of fuzzy set based assessment \cite{37,39}. Fig. 1 depicts a framework of our assessment approach.

Further, operations over multiset estimates are examined: (a) integration of multiset estimates, (b) proximity for multiset estimates, (c) comparison of multiset estimates, (d) aggregation of multiset estimates (e.g., searching for a median estimate), (e) alignment of multiset estimates (and corresponding assessment problems). Combinatorial synthesis based on morphological approach is examined including the suggested modified version of the approach with multiset estimates of design alternatives. In addition, some knapsack-like problems with multiset estimates are briefly described as well. The assessment approach, multiset-estimates and the problems above are illustrated by numerical examples.
2. Basic Assessment Problems

In this section, several basic assessment problems are considered ($P^{3,1}$, $P^{4,1}$, $P^{3,2}$, $P^{3,3}$, $P^{3,4}$) (Table 1): assessment scale, order over the scale components.

| Assessment problem | Number of elements (cardinality of multiset) $\eta$ | Number of levels of basic ordinal scale $l$ | Type of scale | Type of estimate | Multiset coefficient (or $l$) | Number of multiset estimates (under condition 2) |
|-------------------|-----------------|-----------------|---------------|-----------------|-----------------|--------------------------------|
| 1                 | $P^{31}$        | 1               | 3             | linear order    | ordinal         | 3                                            |
| 2                 | $P^{41}$        | 1               | 4             | linear order    | ordinal         | 4                                            |
| 3                 | $P^{32}$        | 2               | 3             | linear order    | multiset        | 6                                            |
| 4                 | $P^{33}$        | 3               | 3             | poset           | multiset        | 10                                           |
| 5                 | $P^{34}$        | 4               | 3             | poset           | multiset        | 15                                           |

Fig. 2 illustrates the scale and estimates for problem $P^{3,1}$ (ordinal assessment, scale $[1, 3]$). In the case of scale $[1, 2, 3]$, the following semantic levels are often considered: excellent (1), good (2), and sufficient (3). Analogically, Fig. 3 illustrates the scale and estimates for problem $P^{4,1}$ (ordinal assessment, scale $[1, 4]$).
Fig. 4 illustrates the scale-poset and estimates for problem $P^{3,2}$ (assessment over scale [1, 3] with two elements; estimate (1, 0, 1) is not used).

Fig. 5 illustrates the scale-poset and estimates for problem $P^{3,3}$ (assessment over scale [1, 3] with three elements; estimates (2, 0, 1) and (1, 0, 2) are not used).

Fig. 6 illustrates the scale-poset and estimates for problem $P^{3,4}$ (assessment over scale [1, 3] with four elements; estimates (2, 0, 2), (3, 0, 1), and (1, 0, 3) are not used).
3. Operations over Multiset Estimates

The following operations are considered for the multiset estimates (or corresponding alternatives): (a) integration of several estimates (e.g., for composite systems), (b) proximity between estimates and comparison of the estimates, (c) comparison, ordering, selection of Pareto-efficient estimates (alternatives), and (d) aggregation (e.g., searching for a median estimate for the specified set of initial estimates).

3.1. Integrated Estimates. Integration of estimates (mainly, for composite systems) is based on summarization of the estimates by components (i.e., positions). Let us consider $n$ estimates (position form): estimate $e^1 = (\eta_1^1, ..., \eta_{i_1}^1, ..., \eta_{l_1}^1)$, ..., estimate $e^k = (\eta_1^k, ..., \eta_{i_k}^k, ..., \eta_{l_k}^k)$, ..., estimate $e^n = (\eta_1^n, ..., \eta_{i_n}^n, ..., \eta_{l_n}^n)$. Then, the integrated estimate is: estimate $e^I = (\eta_1^I, ..., \eta_{i_I}^I, ..., \eta_{l_I}^I)$, where $\eta_{i_I}^I = \sum_{k=1}^{n} \eta_{i_k}^k \forall i = 1, \ldots, l_I$. In fact, the operation $\bigcup$ is used for multiset estimates: $e^I = e^1 \bigcup \ldots \bigcup e^n \bigcup \ldots \bigcup e^n$. 

Fig. 6. Scale, estimates ($P^{3,4}$)
Further, some examples for integration of multiset-estimates are presented.

Example 3. $S = X \star Y \star Z$. Assessment problem $P^{3,1}$. Estimates of system parts are: (a) $X$: $e(X) = \{1\}$ or $e(X) = (1,0,0)$, (b) $Y$: $e(Y) = \{2\}$ or $e(Y) = (0,1,0)$, (c) $Z$: $e(Z) = \{1\}$ or $e(Z) = (1,1,0)$. As a result, the integrated estimate of composite system is (component-based summarization):

$S = X \star Y \star Z$: $e(S) = \{1,1,2\}$ or $e(S) = (2,1,0)$.

Example 4. $S = X \star Y \star Z \star V$. Assessment problem $P^{3,1}$. Estimates of system parts are: (a) $X$: $e(X) = \{1\}$ or $e(X) = (1,0,0)$, (b) $Y$: $e(Y) = \{2\}$ or $e(Y) = (0,1,0)$, (c) $Z$: $e(Z) = \{1\}$ or $e(Z) = (0,1,0)$, (d) $V$: $e(V) = \{3\}$ or $e(V) = (0,1,0)$. As a result, the integrated estimate of composite system is (component-based summarization):

$S = X \star Y \star Z \star V$: $e(S) = \{1,2,2,3\}$ or $e(S) = (1,2,1)$.

Example 5. $S = X \star Y \star Z$. Assessment problem $P^{3,2}$. Estimates of system parts are: (a) $X$: $e(X) = \{1\}$ or $e(X) = (1,1,0)$, (b) $Y$: $e(Y) = \{2\}$ or $e(Y) = (0,1,0)$, (c) $Z$: $e(Z) = \{1\}$ or $e(Z) = (0,2,0)$. As a result, the integrated estimate of composite system is (component-based summarization):

$S = X \star Y \star Z$: $e(S) = \{1,2,2,2\}$ or $e(S) = (1,4,0)$.

Example 6. $S = X \star Y \star Z$. Assessment problem $P^{3,3}$. Estimates of system parts are: (a) $X$: $e(X) = \{1\}$ or $e(X) = (1,1,1)$, (b) $Y$: $e(Y) = \{2\}$ or $e(Y) = (1,2,0)$, (c) $Z$: $e(Z) = \{1\}$ or $e(Z) = (0,2,1)$. As a result, the integrated estimate of composite system is (component-based summarization):

$S = X \star Y \star Z$: $e(S) = \{1,1,2,2,2,3,3\}$ or $e(S) = (2,5,2)$.

Example 7. $S = X \star Y \star Z$. Assessment problem $P^{3,4}$. Estimates of system parts are: (a) $X$: $e(X) = \{1\}$ or $e(X) = (1,2,1)$, (b) $Y$: $e(Y) = \{2\}$ or $e(Y) = (2,2,0)$, (c) $Z$: $e(Z) = \{1\}$ or $e(Z) = (1,1,2)$. As a result, the integrated estimate of composite system is (component-based summarization):

$S = X \star Y \star Z$: $e(S) = \{1,1,1,2,2,2,3,3\}$ or $e(S) = (4,5,3)$.

Generally, the integrated multiset estimate for multi-part system is based on assessment problem $P^{k,\eta \times m}$ (case of complete poset, i.e., without condition 2 on “interval”), where $P^{k,\eta}$ is assessment problem for system parts, $m$ is the number of subsystems (parts).

Example 8. $S = X \star Y \star Z$. Assessment problem $P^{3,4}$. The integrated estimate for $S$ is based on the assessment problem $P^{3,12}$ (case of complete poset).

Example 9. $S = X \star Y \star Z$. Assessment problem $P^{4,3}$. The integrated estimate for $S$ is based on the assessment problem $P^{4,9}$ (case of complete poset). Estimates of system parts are: (a) $X$: $e(X) = \{1\}$ or $e(X) = (3,0,0,0)$, (b) $Y$: $e(Y) = \{2\}$ or $e(Y) = (0,3,0,0)$, (c) $Z$: $e(Z) = \{1\}$ or $e(Z) = (0,0,0,3)$. As a result, the integrated estimate of composite system is (component-based summarization):

$S = X \star Y \star Z$: $e(S) = \{1,1,1,2,2,2,4,4,4\}$ or $e(S) = (3,3,0,3)$. 


This example is the reason to use the complete poset for the integrated multiset estimate.

3.2. Vector-like Proximity. Consider estimates of two alternatives $e(A_1), e(A_2)$ and the following vector-like proximity:

$$\delta(e(A_1), e(A_2)) = (\delta^-(A_1, A_2), \delta^+(A_1, A_2)),$$

where vector components are: (i) $\delta^-$ is the number of one-step changes: element of quality $i + 1$ into element of quality $i$ ($i = 1, l - 1$) (this corresponds to “improvement”); (ii) $\delta^+$ is the number of one-step changes: element of quality $i$ into element of quality $i + 1$ ($i = 1, l - 1$) (this corresponds to “degradation”).

This definition corresponds to change (edition) of $A_1$ into $A_2$. Vector-like proximity $\delta(e(A_1), e(A_2))$ is similar to vector-like proximity for rankings that was suggested in ([10], [11], [18]). Evidently, the following axioms are satisfied:

1. $\forall A_1, A_2: \delta(e(A_1), e(A_2)) \geq (0, 0)$ (nonnegativity).
2. $\delta(e(A_1), e(A_1)) = (0, 0)$ (identity).
3. $\forall A_1, A_2: \delta(e(A_1), e(A_2)) = (\delta^-(e(A_1), e(A_2)), \delta^+(e(A_1), e(A_2)))$, $\delta(e(A_2), e(A_1)) = (\delta^-(e(A_2), e(A_1)), \delta^+(e(A_2), e(A_1))) = (\delta^+(e(A_1), e(A_2)), \delta^-(e(A_1), e(A_2)))$.
4. $\forall A_1, A_2, A_3: \delta(e(A_1), e(A_2)) + \delta(e(A_2), e(A_3)) \geq \delta(e(A_1), e(A_3))$, here operation ‘+’ corresponds to summarization by components (triangle inequality).

In addition, the following is defined: $|\delta(e_1, e_2)| = |\delta^-(e_1, e_2)| + |\delta^+(e_1, e_2)|$. Evidently, $|\delta(e_1, e_2)| \leq \eta \times (l - 1)$. Fig. 7 depicts the domain of the proximity.

Fig. 7. Discrete domain of proximity

Now, let us consider numerical examples of the proximity.

**Example 10.** Assessment problem $P^{3,1}$.
Case 1. $e_1 = (0, 0, 1), e_2 = (0, 1, 0), \delta(e_1, e_2) = (1, 0), \delta(e_2, e_1) = (0, 1)$.
Case 2. $e_1 = (0, 0, 1), e_2 = (1, 0, 0), \delta(e_1, e_2) = (2, 0), \delta(e_2, e_1) = (0, 2)$.

**Example 11.** Assessment problem $P^{4,1}$.
Case 1. $e_1 = (0, 0, 0, 1), e_2 = (0, 1, 0, 0), \delta(e_1, e_2) = (2, 0), \delta(e_2, e_1) = (0, 2)$.
Case 2. $e_1 = (0, 0, 0, 1), e_2 = (1, 0, 0, 0), \delta(e_1, e_2) = (3, 0), \delta(e_2, e_1) = (0, 3)$.

**Example 12.** Assessment problem $P^{3,2}$.
Case 1. $e_1 = (0, 0, 2), e_2 = (0, 1, 1), \delta(e_1, e_2) = (1, 0), \delta(e_2, e_1) = (0, 1)$.
Case 2. $e_1 = (0, 0, 2), e_2 = (0, 2, 0), \delta(e_1, e_2) = (2, 0), \delta(e_2, e_1) = (0, 2)$. 

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Case 3. \( e_1 = (0, 0, 2), e_2 = (2, 0, 0), \delta(e_1, e_2) = (4, 0), \delta(e_2, e_1) = (0, 4) \).

Example 13. Assessment problem \( P^{3,3} \).

Case 1. \( e_1 = (0, 0, 3), e_2 = (0, 1, 2), \delta(e_1, e_2) = (1, 0), \delta(e_2, e_1) = (0, 1) \).

Case 2. \( e_1 = (0, 0, 3), e_2 = (0, 2, 1), \delta(e_1, e_2) = (2, 0), \delta(e_2, e_1) = (0, 2) \).

Case 3. \( e_1 = (0, 0, 3), e_2 = (3, 0, 0), \delta(e_1, e_2) = (6, 0), \delta(e_2, e_1) = (0, 6) \).

Case 4. \( e_1 = (0, 3, 0), e_2 = (1, 1, 1), \delta(e_1, e_2) = (1, 1), \delta(e_2, e_1) = (1, 1) \).

Example 14. Assessment problem \( P^{3,4} \).

Case 1. \( e_1 = (0, 0, 4), e_2 = (0, 1, 3), \delta(e_1, e_2) = (1, 0), \delta(e_2, e_1) = (0, 1) \).

Case 2. \( e_1 = (0, 0, 4), e_2 = (0, 3, 1), \delta(e_1, e_2) = (3, 0), \delta(e_2, e_1) = (0, 3) \).

Case 3. \( e_1 = (0, 0, 4), e_2 = (4, 0, 0), \delta(e_1, e_2) = (8, 0), \delta(e_2, e_1) = (0, 8) \).

Case 4. \( e_1 = (0, 4, 0), e_2 = (1, 2, 1), \delta(e_1, e_2) = (1, 1), \delta(e_2, e_1) = (1, 1) \).

Case 5. \( e_1 = (1, 1, 2), e_2 = (0, 4, 0), \delta(e_1, e_2) = (2, 1), \delta(e_2, e_1) = (1, 2) \).

Further, for two alternative \( A_1 \) and \( A_2 \) their proximity is: \( \delta(e_1(A_1), e_2(A_2)) = (\delta^-, \delta^+) \). Generally, the following is satisfied:

1. \( \delta^- = 0 \) and \( \delta^+ > 0 \iff A_1 > A_2 \).
2. \( \delta^- > 0 \) and \( \delta^+ = 0 \iff A_1 < A_2 \).
3. \( \delta^- > 0 \) and \( \delta^+ > 0 \iff A_1 \) and \( A_2 \) are incomparable.

3.3. Comparison of Estimates. In our study, the following comparison problems over estimates are considered:

1. Comparison of two estimates. This operation is based on estimate proximity.

2. Ordering of estimates from a specified set of estimates. This operation corresponds to a well-known linear ordering algorithm while taking into account in-comparable estimates (as “equivalent” ones). The computing complexity of the operation is \( O(n \log n) \) (\( n \) is the number of estimates).

3. Selection of Pareto-efficient estimates for a specified set of estimates. Here the simple algorithm is the following. The proximity matrix can be computed and a Pareto-efficient estimate (alternative) has its line without matrix element \( < \) (or \( > \)). The computing complexity of the algorithm is \( O(n^2) \) (\( n \) is the number of estimates). The basic linear ordering algorithm leads to the same result (complexity: \( O(n \log n) \)), but the described algorithm is more friendly for interactive procedure.

Evidently, in special cases the similar algorithms can be used to search for the “maximal” or the “minimal” estimate.

Example 15. Assessment problem \( P^{3,3} \).

The set of estimates is (Fig. 5): \( e_1^{3,3} = (0, 3, 0), e_2^{3,3} = (1, 1, 1), e_3^{3,3} = (0, 2, 1), e_4^{3,3} = (0, 1, 2), e_5^{3,3} = (0, 0, 4) \). Proximities for the estimates are presented as proximity matrix in Table 2. Pareto-efficient estimates are: \( e_4^{3,3}, e_5^{3,3} \). Here the “minimal” estimate exists: \( e_8^{3,3} \).

Example 16. Assessment problem \( P^{3,4} \).

The set of estimates is (Fig. 6): \( e_1^{3,4} = (2, 1, 1), e_2^{3,4} = (0, 4, 0), e_3^{3,4} = (1, 2, 1), e_4^{3,4} = (0, 2, 2), e_5^{3,4} = (0, 1, 3), e_6^{3,4} = (0, 0, 4) \). Proximities for the estimates are presented as proximity matrix in Table 3. Pareto-efficient estimates are: \( e_9^{3,4}, e_5^{3,4} \). Here the “minimal” estimate exists: \( e_8^{3,4} \).
3.4. Aggregation of Estimates. Here searching for a median estimate for the specified set of initial estimates is considered. Let \( E = \{ e_1, ..., e_n \} \) be the set of all possible estimates \((\text{or a corresponding set of possible alternatives})\), let \( D \) be the set of all possible estimates \((\text{or a corresponding set of possible alternatives})\) \((E \subseteq D)\). Thus, the median estimate is \((\text{e.g., [18, 31]})): \( M^g = \arg\min_{M \in D} \sum_{\kappa=1}^{n} \delta(M, e_\kappa); \)

\[(\text{b) simplified case of the median (approximation)} \text{ as } \text{“set median” over set } E:\]

\[M^s = \arg\min_{M \in E} \sum_{\kappa=1}^{n} \delta(M, e_\kappa).\]

The problem of searching for the “generalized median” is usually NP-hard, complexity of searching for “set median” is a polynomial problem \((O(n^2))\). In our study, simple problems are considered where the set of all multiset estimates is very limited \((\text{i.e., “multiset number”})\) and simple enumerative solving schemes can be used for “generalized median”.

**Example 17.** Assessment problem \(P^{3,4}\).

The initial set of estimates \( E \) is (Fig. 6): \( e^{3,4}_2 = (3, 1, 0), e^{3,4}_4 = (1, 3, 0), e^{3,4}_5 = (0, 4, 0), e^{3,4}_9 = (2, 1, 1), e^{3,4}_x = (1, 2, 1), e^{3,4}_6 = (0, 2, 2), e^{3,4}_7 = (0, 1, 3), e^{3,4}_{12} = (0, 0, 4). \) Table 4 contains proximities and the integrated estimate: \( \Upsilon_\kappa = \bigcup_{j \in E} \delta(e_\kappa, e_j). \) The median estimates are: \( \text{(a) “generalized median” } M^g = e^{3,4}_6 = (0, 3, 1) (e^{3,4}_6 \in E); \text{ (b) “set median” } M^s = e^{3,4}_{10} = (1, 2, 1) (e^{3,4}_{10} \in E). \)

| \( e_\kappa \) | \( e_j \) | \( e^{3,4}_2 \) | \( e^{3,4}_4 \) | \( e^{3,4}_5 \) | \( e^{3,4}_6 \) | \( e^{3,4}_7 \) | \( e^{3,4}_8 \) |
|---|---|---|---|---|---|---|---|
| \( e^{3,4}_2 \) = (3, 1, 0) | (0, 0) | (1, 1) | (0, 1) | (0, 2) | (0, 3) |
| \( e^{3,4}_4 \) = (1, 1, 1) | (1, 1) | (0, 0) | (0, 1) | (0, 2) | (0, 3) |
| \( e^{3,4}_5 \) = (0, 2, 1) | (1, 0) | (1, 0) | (0, 0) | (0, 1) | (0, 2) |
| \( e^{3,4}_7 \) = (0, 1, 2) | (2, 0) | (2, 0) | (1, 0) | (0, 0) | (0, 1) |
| \( e^{3,4}_8 \) = (0, 0, 3) | (3, 0) | (3, 0) | (2, 0) | (1, 0) | (0, 0) |

| \( e_\kappa \) | \( e_j \) | \( e^{3,4}_2 \) | \( e^{3,4}_4 \) | \( e^{3,4}_5 \) | \( e^{3,4}_6 \) | \( e^{3,4}_7 \) | \( e^{3,4}_8 \) |
|---|---|---|---|---|---|---|---|
| \( e^{3,4}_2 \) = (2, 1, 1) | (0, 0) | (1, 2) | (0, 1) | (0, 2) | (0, 3) |
| \( e^{3,4}_4 \) = (0, 4, 0) | (2, 1) | (0, 0) | (1, 1) | (0, 1) | (0, 2) | (0, 3) |
| \( e^{3,4}_5 \) = (1, 2, 1) | (1, 0) | (1, 1) | (0, 0) | (0, 1) | (0, 2) |
| \( e^{3,4}_6 \) = (0, 3, 1) | (2, 0) | (1, 0) | (1, 0) | (0, 0) | (0, 1) | (0, 2) |
| \( e^{3,4}_7 \) = (0, 2, 2) | (3, 0) | (2, 0) | (2, 0) | (1, 0) | (0, 0) | (0, 1) |
| \( e^{3,4}_8 \) = (0, 1, 3) | (4, 0) | (3, 0) | (3, 0) | (2, 0) | (1, 0) | (0, 0) |
Now, let us define the deviation of the median estimate \( M^g \) or \( M^s \):
\[
\Delta(M) = (\Delta^-(M), \Delta^+(M)), \quad \Delta^-(M) = \delta(M, \min_{\kappa=1,n} e_\kappa), \quad \Delta^+(M) = \delta(\max_{\kappa=1,n} e_\kappa, M). \]
In addition, \( |\Delta(M)| = \max\{|\Delta^-(M)|, |\Delta^+(M)|\} \).

**Example 18.** (Assessment problem \( P_{3,4} \), from the previous section). The initial set of estimates \( E \) is (Fig. 6): \( e_2^4, e_4^4, e_5^4, e_9^4, e_{10}^4, e_7^4, e_8^4, e_{12}^4 \). The median estimates are: (a) “generalized median” \( M^g = e_6^3 = (0, 3, 1) \); (b) “set median” \( M^s = e_{10}^3 = (1, 2, 1) \). The dispersions are:
\[
\Delta^-(M^g) = \delta((0, 3, 1), (0, 0, 4)) = (0, 3), \quad \Delta^+(M^g) = \delta((3, 1, 0), (0, 3, 1)) = (0, 4); \quad |\Delta(M^g)| = 4.
\]
\[
\Delta^-(M^s) = \delta((1, 2, 1), (0, 0, 4)) = (0, 4), \quad \Delta^+(M^s) = \delta((3, 1, 0), (1, 2, 1)) = (0, 3); \quad |\Delta(M^s)| = 4.
\]

3.5. Alignment of Estimates. Let \( \{e_1^{l_1}, ..., e_i^{l_i}, ..., e_n^{l_n}\} \) be an initial set of \( n \) multiset estimates. The alignment problem is:
\[
\{e_1^{l_1}, ..., e_i^{l_i}, ..., e_n^{l_n}\} \Rightarrow e_i^{l_i}, \quad \{P_1^{l_1}, ..., P_i^{l_i}, ..., P_n^{l_n}\} \Rightarrow P_i^{l_i}.
\]
In general, solving approach to this problem has to be based on special applied analysis by the domain expert(s). Let us consider a simplified approach to alignment of multiset alternatives as follows (Fig. 8):
\[
P_i^{l_i} \Rightarrow P_i^{l_i}, \quad \forall i = 1, n, \quad l = \max_{i=1,n} \{l_i\}, \quad \eta = \max_{i=1,n} \{\eta_i\}.
\]

\[
\begin{array}{c}
\eta_1^i \ldots \eta_i^i \Rightarrow \eta_1^i \ldots \eta_i^i \\
1 \ldots t' \quad 1 \ldots t
\end{array}
\]
\[
\begin{array}{c}
\eta_1^i \ldots \eta_i^i \Rightarrow \eta_1^i \ldots \eta_i^i \\
1 \ldots t' \quad 1 \ldots t
\end{array}
\]

(a) \( P_i^{l_i} \Rightarrow P_i^{l_i}, \quad \eta_i^i < \eta_i^0 \)

\[
\begin{array}{c}
\eta_1^i \ldots \eta_i^i \Rightarrow \eta_1^i \ldots \eta_i^i \\
1 \ldots t' \quad 1 \ldots t' \quad \ldots \quad 1 \ldots t'
\end{array}
\]

(b) \( P_i^{l_i} \Rightarrow P_i^{l_i}, \quad l' < t' \)

Fig. 8. Alignment of multiset estimates

**Example 19.** The initial set of multiset estimates is: \( e_1^{3,2} = (1, 1, 0), \quad e_2^{3,3} = (1, 1, 1), \quad e_3^{2,3} = (2, 1), \quad e_4^{4,4} = (0, 2, 1, 1), \quad e_5^{3,4} = (1, 2, 1) \), Fig. 9 depicts alignment.
4. Combinatorial Synthesis (morphological approach)

Here system composition as combinatorial synthesis of design alternatives (for system components) which are evaluated via the suggested assessment approach is examined. An additional problem consist in integration of design alternatives estimates into the total estimate of the composed system. This estimate integration is based on summarization of component estimates by estimate elements (i.e., position). As a result, system estimates are based on some analogical poset-like scales. This case corresponds to combinatorial synthesis in Hierarchical Morphological Multicriteria Design (HMMD) method. Descriptions of HMMD are presented in accessible literature: (i) two books ([11], [15]); (ii) journal articles ([9], [12], [13], [14], [17]); (iii) electronic preprint [19]. The method is based on morphological clique problem. In basic HMMD method, ordinal assessment for design alternatives is used (e.g., problems $P^{3,1}$, $P^{4,1}$). The section contains the following:

(1) a brief description of basic HMMD method (combinatorial synthesis with ordinal assessment of design alternatives);

(2) two numerical examples for basic HMMD method (three-component system and four-component system); and

(3) modified version of HMMD method that is based on the suggested multiset estimates for evaluation of design alternatives (numerical examples: (i) three-component system and assessment problem $P^{3,3}$, (ii) four-component system and assessment problem $P^{3,4}$, and (iii) three-layer hierarchical system.

Table 5 contains a list of the described combinatorial synthesis problems.

Table 5. Problem of combinatorial synthesis

| Number of system layers | Number of system parts | Assessment problem | Type of estimate for DAs | Version of HMMD |
|------------------------|------------------------|--------------------|--------------------------|-----------------|
| 1                      | 2                      | 3                  | $P^{3,1}$                 | basic           |
| 2                      | 2                      | 4                  | $P^{4,1}$                 | basic           |
| 3                      | 2                      | 3                  | $P^{3,3}$                 | modified        |
| 4                      | 2                      | 4                  | $P^{3,4}$                 | modified        |
| 5                      | 3                      | 3                  | $P^{3,3}$                 | modified        |
4.1. Basic Combinatorial Synthesis (ordinal assessment of alternatives).
An examined composite (modular, decomposable) system consists of components and their interconnection or compatibility (IC). Basic assumptions of HMMD are the following: (a) a tree-like structure of the system; (b) a composite estimate for system quality that integrates components (subsystems, parts) qualities and qualities of IC (compatibility) across subsystems; (c) monotonic criteria for the system and its components; (d) quality of system components and IC are evaluated on the basis of coordinated ordinal scales. The designations are: (1) design alternatives (DAs) for leaf nodes of the model; (2) priorities of DAs ($i = 1, l; 1$ corresponds to the best one); (3) ordinal compatibility for each pair of DAs ($w = 1, v; \nu$ corresponds to the best one). The basic phases of HMMD are:

Phase 1. design of the tree-like system model;
Phase 2. generation of DAs for leaf nodes of the model;
Phase 3. hierarchical selection and composing of DAs into composite DAs for the corresponding higher level of the system hierarchy;
Phase 4. analysis and improvement of composite DAs (decisions).

Let $S$ be a system consisting of $m$ parts (components): $R(1), ..., R(i), ..., R(m)$. A set of design alternatives is generated for each system part above. The problem is:

Find a composite design alternative $S = S(1) \ast ... \ast S(i) \ast ... \ast S(m)$ of DAs (one representative design alternative $S(i)$ for each system component/part $R(i)$, $i = 1, m$) with non-zero compatibility between design alternatives.

A discrete space of the system excellence on the basis of the following vector is used: $N(S) = (w(S); e(S))$, where $w(S)$ is the minimum of pairwise compatibility between DAs which correspond to different system components (i.e., $\forall R_{j_1}$ and $R_{j_2}, 1 \leq j_1 \neq j_2 \leq m$) in $S$, $e(S) = (\eta_1, ..., \eta_i, ..., \eta_l)$, where $\eta_i$ is the number of DAs of the $i$th quality in $S$. Thus, the modified problem is (two objectives, one constraint):

$$\max \ e(S),$$
$$\max \ w(S),$$
$$s.t. \ w(S) \geq 1.$$  

As a result, we search for composite decisions which are nondominated by $N(S)$ (i.e., Pareto-efficient solutions). “Maximization” of $e(S)$ is based on the corresponding poset. The considered combinatorial problem is NP-hard and an enumerative solving scheme is used.

4.2. Example: basic HMMD, three-component system. Here three component system and assessment problem $P^{3,1}$ are examined. Fig. 10 illustrate the composition problem by a numerical example, Table 6 contains compatibility estimates. In the example, Pareto-efficient composite DAs are:

(a) $S_1 = X_1 \ast Y_1 \ast Z_2$, $N(S_1) = (3; 1, 1, 1)$;
(b) $S_2 = X_2 \ast Y_1 \ast Z_2$, $N(S_2) = (2; 2, 0, 1)$; and
(c) $S_3 = X_3 \ast Y_2 \ast Z_1$, $N(S_3) = (1; 3, 0, 0)$.

Fig. 11 depicts the poset for integrated system estimate by components, Fig. 12 depicts the general poset of system quality.
4.3. Example: basic HMMD, four-component system. Here the following composition problem is considered:

(i) the system consists of 4 components,
(ii) assessment problem $P^{3,1}$ is used for the evaluation of DAs for components (Fig. 2), and
(iii) integrated system quality is based on poset (Fig. 13).

Let us describe the corresponding numerical example (a corrected version of the example from [9]). The system structure and DAs are presented in Fig. 14 (ordinal estimates of DAs are shown in parentheses). In addition, estimates of compatibility between DAs (scale $[0, 1, 2, 3, 4, 5]$) are presented in Table 7. Resultant Pareto-efficient composite DAs are:

(a) $S_1 = X_3 \ast Y_3 \ast Z_3 \ast V_4$, $N(S_1) = (4; 1, 3, 0)$;
(b) $S_2 = X_3 \ast Y_3 \ast Z_2 \ast V_3$, $N(S_2) = (3; 2, 2, 0)$; and
(c) $S_3 = X_3 \ast Y_3 \ast Z_2 \ast V_4$, $N(S_3) = (2; 3, 1, 0)$.
Fig. 13 illustrates the poset of quality for obtained composite DAs by components (i.e., \(e(S) = (\eta_1, \eta_2, \eta_3)\)). The general poset of quality (by \(N(S)\)) is depicted in Fig. 15.

![Diagram of poset](image)

Fig. 13. Poset \(e(S) = (\eta_1, \eta_2, \eta_3)\)

![Diagram of composition](image)

Fig. 14. Example of composition
Table 7. Compatibility estimates

|   | Y₃ | Y₂ | Y₁ | Z₂ | Z₁ | V₄ | V₂ | V₃ | V₁ |
|---|----|----|----|----|----|----|----|----|----|
| X₃ | 5  | 0  | 5  | 0  | 5  | 3  | 1  | 4  | 4  |
| X₁ | 0  | 5  | 1  | 3  | 0  | 2  | 5  | 3  | 1  |
| X₂ | 0  | 3  | 1  | 5  | 0  | 2  | 4  | 3  | 1  |
| Y₃ | 2  | 5  | 1  | 4  | 2  | 1  | 1  | 1  | 5  |
| Y₂ | 2  | 3  | 2  | 1  | 1  | 1  | 1  | 5  | 1  |
| Y₅ | 5  | 1  | 5  | 5  | 5  | 5  | 5  | 1  | 1  |
| Y₁ | 2  | 3  | 2  | 1  | 1  | 1  | 1  | 5  | 1  |
| Y₄ | 5  | 3  | 5  | 1  | 1  | 1  | 1  | 5  | 1  |
| Z₂ | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  |
| Z₁ | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  |
| Z₃ | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  |

4.4. Modified version of HMMD method. Here combinatorial synthesis is based on usage of multiset estimates of design alternatives for system parts. For the resultant system \( S = S(1) \ast ... \ast S(i) \ast ... \ast S(m) \) the same multiset estimate is examined as an aggregated estimate (“generalized median”) of corresponding multiset estimates of its components (i.e., selected DAs). In addition, a condition quality for the selected DAs is used. Thus, \( N(S) = (w(S); e(S)) \), where \( e(S) \) is the “generalized median” of estimates of the solution components. Evidently, the constraint for the resultant compatibility estimate in the obtained solution is used as well. Finally, the modified problem is (two objectives, two constraints):

\[
\begin{align*}
\max \, e(S) &= M^g = \arg \min_{M \in D} \bigcup_{i=1}^{m} \delta(M, e(S_i)), \\
\max \, w(S), \\
\text{s.t.} \, e(S_i) &\geq e_o, \, \forall i = 1, m; \\
w(S) &\geq 1.
\end{align*}
\]
Note, the usage of the aggregated multiset estimate for composite system $S$ provides the opportunity to use the same type of multiset estimate (e.g., $P_{3,2}^3$, $P_{3,3}^3$, $P_{3,4}^3$) computing of the system quality (i.e., $N(S)$) at various hierarchical layers of a multi-layer hierarchical system (during the ‘Bottom-Up’ system design framework). The constraint $e(s_i) \geq e_o, \forall i = 1, m$ provides selection of only “good” DAs (level $e_o$ can be changed, e.g., $e_6^{3,3} = (0, 2, 1)$). Here estimate $e_o$ is used as a “reference point”. Evidently, it is possible to use several similar constraints based on several “reference points” (e.g., $e_4^{3,3} = (0, 3, 0)$, $e_5^{3,3} = (1, 1, 1)$).

It is necessary to point out the basic modified version of HMMD method has been described. It is possible to consider two other versions:

**Version 1:** Integrated estimate for the composed system (objective 1):

$$\max e(S) = \bigoplus_{i=1}^{m} e(s_i).$$

This case leads to changing the assessment problem for system: $P_{l,\eta} \Rightarrow P_{l,\eta \times m}$.

**Version 2:** Aggregated estimate for the composed system (objective 1) as the “set median”:

$$\max e(S) = M^* = \arg \min_{M \in \{e(s_1), \ldots, e(s_m)\}} \left| \bigoplus_{i=1}^{m} \delta(M, e(S_i)) \right|.$$

Evidently, here an approximation is used and the computational complexity is decreased (to polynomial case, i.e., selection of a system component estimate).

### 4.5. Example: three-component system, three-element assessment.

Here three component system with assessment problem $P_{3,3}^3$ is considered. Thus, priorities of DAs are evaluated by assessment problem $P_{3,3}^3$ (see Fig. 5). The illustrative example is presented in Fig. 16 (3-position priorities are shown in parentheses), compatibility estimates from Table 6 are used. This example is close to example from Fig. 10. The poset for integrated estimates of quality for 3-component system and assessment problem $P_{3,3}^3$ is presented in Fig. 17. Note, the following points of the basic lattice are impossible: $< 8, 0, 1 >$, $< 7, 0, 2 >$, $< 5, 0, 4 >$, $< 4, 0, 5 >$, $< 2, 0, 7 >$, and $< 1, 0, 8 >$. 

![Fig. 16. Example of composition](image)
Fig. 17. Poset $e(S) = (\eta_1, \eta_2, \eta_3) (P^3,3)$
Now, let us consider three modified versions of HMMD: (i) integrated estimate for system quality (version 1), (ii) aggregated estimate for system quality (version 2, “set median”), and (ii) aggregated estimate for system quality (basic version, “generalized median”). The illustration is targeted to system estimates. The considered solutions correspond to the solutions in Fig. 10.

In the case 1 (integrated system estimate), the following solutions and their estimates are examined:

(a) $S_1^1 = X_1 \star Y_1 \star Z_2$, $N(S_1^1) = (3; 2, 5, 2)$,
(b) $S_1^2 = X_2 \star Y_1 \star Z_2$, $N(S_1^2) = (2; 4, 3, 2)$,
(c) $S_1^3 = X_3 \star Y_2 \star Z_1$, $N(S_1^3) = (1; 8, 1, 0)$.

Fig. 18 illustrates the general poset of system quality (each local poset for $w = 1, 2, 3$ corresponds to the poset from Fig. 17).

In the case 2 (aggregated system estimate as “set median”), the following solutions and their estimates are considered (the poset from Fig. 11 is used):

(a) $S_2^1 = X_1 \star Y_1 \star Z_2$, $N(S_2^1) = (3; 0, 3, 0)$,
(b) $S_2^2 = X_2 \star Y_1 \star Z_2$, $N(S_2^2) = (2; 2, 1, 0)$,
(c) $S_2^3 = X_3 \star Y_2 \star Z_1$, $N(S_2^3) = (1; 3, 0, 0)$.

In the case 3 (aggregated system estimate as “generalized median”), the following solutions and their estimates are considered (the poset from Fig. 11 is used):

(a) $S_3^1 = X_1 \star Y_1 \star Z_2$, $N(S_3^1) = (3; 0, 3, 0)$,
(b) $S_3^2 = X_2 \star Y_1 \star Z_2$, $N(S_3^2) = (2; 2, 0, 1)$ or $N(S_3^2) = (2; 1, 2, 0)$ (incomparable estimates, i.e., two “generalized medians”),
(c) $S_3^3 = X_3 \star Y_2 \star Z_1$, $N(S_3^3) = (1; 3, 0, 0)$.

Note the cases 2 and 3 are more easy from the viewpoint of the future usage in multi-layer systems at the higher hierarchical layer (i.e., assessment problem for composite system and the corresponding multiset system estimate are more easy).

4.6. Example: four-component system, four elements assessment. Here three component system with assessment problem $P^{3,4}$ is considered. Thus, priorities of DAs are evaluated by assessment problem $P^{3,4}$ (see Fig. 6). The illustrative example is presented in Fig. 19 (4-position priorities are shown in parentheses), compatibility estimates from Table 7 are used.
Three modified versions of HMMD are considered: (i) integrated estimate for system quality (version 1), (ii) aggregated estimate for system quality (version 2, “set median”), and (ii) aggregated estimate for system quality (basic version, “generalized median”). The illustration is targeted to system estimates. The considered solutions correspond to the solutions in Fig. 14.

\[
S = X \ast Y \ast Z \ast V \\
S_1^1 = X_3 \ast Y_5 \ast Z_3 \ast V_4 (4; 6, 9, 1) \\
S_2^1 = X_3 \ast Y_3 \ast Z_2 \ast V_4 (2; 12, 4, 0)
\]

**Fig. 19. Example of composition**

In the case 1 (integrated system estimate), the following solutions and their estimates are examined (Fig. 20 depicts the general poset of system quality):
(a) \( S_1^1 = X_3 \ast Y_5 \ast Z_3 \ast V_4 \), \( N(S_1^1) = (4; 6, 9, 1) \),
(b) \( S_2^1 = X_3 \ast Y_3 \ast Z_2 \ast V_4 \), \( N(S_2^1) = (2; 12, 4, 0) \).

**Fig. 20. Poset of system quality \( N(S) \)**

In the case 2 (aggregated system estimate as “set median”), the following solutions and their estimates are considered:
(a) \( S_1^2 = X_3 \ast Y_5 \ast Z_3 \ast V_4 \), \( N(S_1^2) = (4; 1, 3, 0) \),
(b) \( S_2^2 = X_3 \ast Y_3 \ast Z_2 \ast V_4 \), here two “set medians” exist: \( (3, 1, 0) \) or \( (4, 0, 0) \), the best “set median” is selected: \( N(S_2^2) = (2; 4, 0, 0) \).

In the case 3 (aggregated system estimate as “generalized median”), the following solutions and their estimates are considered (the poset from Fig. 11 is used):
(a) \( S_1^3 = X_3 \ast Y_5 \ast Z_3 \ast V_4 \), \( N(S_1^3) = (4; 1, 3, 0) \),
(b) \( S_2^3 = X_3 \ast Y_3 \ast Z_2 \ast V_4 \), here two “generalized medians” exist: \( (3, 1, 0) \) or \( (4, 0, 0) \), the best “generalized median” is selected: \( N(S_2^3) = (2; 4, 0, 0) \).
4.7. **Example: Three-layer Hierarchical System.** Here three-layer hierarchical system with assessment problem $P^{3,3}$ is examined (Fig. 21).

![Diagram of three-layer hierarchical system with assessment problem](image)

**Table 8. Compatibility**

|   | $J_1$ | $J_2$ | $J_3$ | $U_1$ | $U_2$ | $U_3$ |
|---|-------|-------|-------|-------|-------|-------|
| $H_1$ | 3 | 1 | 0 | 1 | 3 | 1 |
| $H_2$ | 2 | 1 | 2 | 1 | 2 | 2 |
| $H_3$ | 0 | 1 | 1 | 1 | 1 | 1 |
| $J_1$ | 2 | 3 | 2 |   |   |   |
| $J_2$ | 1 | 1 | 0 |   |   |   |
| $J_3$ | 3 | 2 | 1 |   |   |   |

The composition problem for subsystem $A$ corresponds to the example from the previous sections (Fig. 16, Table 6). Thus, the following composite solutions are examined (case of “generalized median”):

(a) $A_1 = X_1 \ast Y_1 \ast Z_2$, $N(A_1) = (3; 0, 3, 0)$,
(b) $A_2 = X_2 \ast Y_1 \ast Z_2$, $N(A_2) = (2; 1, 2, 0)$,
(c) $A_3 = X_1 \ast Y_1 \ast Z_2$, $N(A_3) = (1; 3, 0, 0)$.

The composition problem for subsystem $B$ corresponds to the example from the previous sections (Fig. 19, Table 7). Thus, the following composite solutions are examined (case of “generalized median”):
The following composite solutions are examined for subsystem C (compatibility estimates are presented in Table 8):

(a) \( C_1 = H_1 * J_1 * U_2, \) \( N(B_1) = (3; 0, 2, 1), \)
(b) \( C_2 = H_2 * J_3 * U_2, \) \( N(B_2) = (2; 1, 2, 0), \)
(c) \( C_3 = H_3 * J_3 * U_1, \) \( N(B_3) = (1; 3, 0, 0), \)

The synthesis problem at the higher hierarchical layer is depicted in Fig. 22.

Now, it is reasonable to consider several possible solving scheme for the composition:

Scheme 1. Combination of all possible composite solutions and an additional analysis of the solutions.

Scheme 2. Combination of all possible composite solutions and the selection of the best solution(s) while taking into account their integrated multiset estimates.

Scheme 3. Combination of all possible composite solutions and the selection of the best solution(s) while taking into account their aggregated multiset estimates.

The resultant composite solutions are (including their integrated estimates, assessment problem \( P^{3,10}, \) alignment of estimates is not used; compatibility estimates are considered concurrently: \( w' = \min \{w(A), w(B), w(C)\} \)):

\[
\begin{align*}
S_1 &= A_1 * B_1 * C_1, \quad N(S_1) = (3; 1, 8, 1); \\
S_2 &= A_1 * B_1 * C_2, \quad N(S_2) = (2; 2, 8, 0); \\
S_3 &= A_1 * B_1 * C_3, \quad N(S_3) = (1; 4, 6, 0); \\
S_4 &= A_1 * B_2 * C_1, \quad N(S_4) = (2; 4, 5, 1); \\
S_5 &= A_1 * B_2 * C_2, \quad N(S_5) = (2; 5, 5, 0); \\
S_6 &= A_1 * B_2 * C_3, \quad N(S_6) = (1; 7, 3, 0); \\
S_7 &= A_2 * B_1 * C_1, \quad N(S_7) = (2; 3, 6, 1); \\
S_8 &= A_2 * B_1 * C_2, \quad N(S_8) = (2; 4, 6, 0); \\
S_9 &= A_2 * B_1 * C_3, \quad N(S_9) = (1; 6, 4, 0); \\
S_{10} &= A_2 * B_2 * C_1, \quad N(S_{10}) = (2; 6, 3, 1); \\
S_{11} &= A_2 * B_2 * C_2, \quad N(S_{11}) = (2; 7, 3, 0); \\
S_{12} &= A_2 * B_2 * C_3, \quad N(S_{12}) = (1; 9, 1, 0); \\
S_{13} &= A_3 * B_1 * C_1, \quad N(S_{13}) = (1; 7, 2, 1); \\
S_{14} &= A_3 * B_1 * C_2, \quad N(S_{14}) = (1; 5, 5, 0); \\
S_{15} &= A_3 * B_1 * C_3, \quad N(S_{15}) = (1; 7, 3, 0); \\
S_{16} &= A_3 * B_2 * C_1, \quad N(S_{16}) = (1; 7, 2, 1); \\
S_{17} &= A_3 * B_2 * C_2, \quad N(S_{17}) = (2; 8, 2, 0); \\
S_{18} &= A_3 * B_2 * C_3, \quad N(S_{18}) = (1; 10, 0, 0).
\end{align*}
\]

The resultant composite solutions are (including their integrated estimates, assessment problem \( P^{3,10}, \) alignment of estimates is not used; compatibility estimates are considered concurrently: \( w' = \min \{w(A), w(B), w(C)\} \)):

\[
\begin{align*}
S_1 &= A_1 * B_1 * C_1(3; 1, 8, 1) \\
S_7 &= A_2 * B_1 * C_2(2; 8, 2, 0) \\
S_{15} &= A_3 * B_2 * C_3(1; 10, 0, 0)
\end{align*}
\]

Evidently, three Pareto-efficient solutions are obtained (Fig. 22):

(a) \( S_1 = A_1 * B_1 * C_1 = (X_1 * Y_1 * Z_2) * (T_3 * Q_3 * G_3 * V_4) * (H_1 * J_1 * U_2); \)
(b) \( S_{17} = A_3 * B_2 * C_2 = (X_3 * Y_2 * Z_1) * (T_3 * Q_3 * G_2 * V_4) * (H_2 * J_3 * U_2); \)
(c) \( S_{18} = A_3 * B_2 * C_3 = (X_3 * Y_2 * Z_1) * (T_3 * Q_3 * G_2 * V_4) * (H_3 * J_3 * U_1). \)

Fig. 23 illustrates the general poset of system quality (each local poset for \( w = 1, 2, 3 \) corresponds to assessment problem \( P^{3,10} \)).

Note, the usage of aggregated multiset estimates does not increase the dimension of assessment problem at the higher hierarchical layer. On the other hand, it may be
necessary to examine additional compatibility estimates at the higher hierarchical layer as well (i.e., compatibility between design alternatives for $A$, $B$, and $C$).

![Diagram of system quality $N(S)$](image)

**Fig. 23. Poset of system quality $N(S)$**

5. **Multiset Estimates in Knapsack-like Problems**

Let us consider basic knapsack-like problems and their modification in the case of multiset estimates. The basic knapsack problem is (e.g., [2], [6]):

$$\max \sum_{i=1}^{m} c_i x_i$$

$$\text{s.t. } \sum_{i=1}^{m} a_i x_i \leq b, \ x_i \in \{0, 1\}, \ i = 1, m$$

where $x_i = 1$ if item $i$ is selected, for $i$th item $c_i$ is a value ('utility'), and $a_i$ is a weight (i.e., resource requirement). Often nonnegative coefficients are assumed.

In the case of multiple choice problem, the items are divided into groups and it is necessary to select elements (items) or the only one element from each group while taking into account a total resource constraint (or constraints) (e.g., [2], [6]):

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n_i} c_{ij} x_{ij}$$

$$\text{s.t. } \sum_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij} x_{ij} \leq b; \ \sum_{j=1}^{n_i} x_{ij} \leq 1, \ i = 1, m; \ x_{ij} \in \{0, 1\}.$$ 

The knapsack-like problems above are NP-hard and can be solved by the following approaches ([2], [6]): (i) enumerative methods (e.g., Branch-and-Bound, dynamic programming), (ii) fully polynomial approximate schemes, and (iii) heuristics (e.g., greedy algorithms).

In the case of multiset estimates of item "utility" $e_i$, $i \in \{1, ..., i, ..., n\}$ (instead of $c_i$), the following aggregated multiset estimate can be used for the objective function ("maximization"): (a) an aggregated multiset estimate as the "generalized median", (b) an aggregated multiset estimate as the "set median", and (c) an integrated multiset estimate.
First, let us consider a special case of multiple choice problem as follows:

1. multiset estimates of item “utility” \( e_{i,j} \), \( i \in \{1, \ldots, i, \ldots, n\}, j = 1, q_i \) (instead of \( c_{ij} \)).

2. an aggregated multiset estimate as the “generalized median” (or “set median”) is used for the objective function (“maximization”).

The initial item set is:

\[
\{(1, 1), (1, 2), (1, q_1), \ldots, (i, 1), (i, 2), (i, q_i), \ldots, (n, 1), (n, 2), (n, q_n)\}.
\]

Boolean variable \( x_{i,j} \) corresponds to selection of the item \((i, j)\). The solution is a subset of the initial item set: \( S = \{(i, j) | x_{i,j} = 1\}\). The problem is:

\[
\text{max } e(S) = \max M = \arg\min_{M \in \mathcal{D}} \left| \bigcup_{(i,j) \in S = \{(i,j) | x_{i,j} = 1\}} \delta(M, e_{i,j}) \right|,
\]

s.t. \( \sum_{i=1}^{m} \sum_{j=1}^{q_i} a_{ij} x_{i,j} \leq b; \sum_{j=1}^{q_i} x_{ij} = 1; \ x_{ij} \in \{0,1\}. \)

Here \( \sum_{i=1}^{m} \sum_{j=1}^{q_i} = \sum_{(i,j) \in S} \). Evidently, this problem is similar to the above-mentioned combinatorial synthesis problem without compatibility of the selected items. Now let us consider a numerical example (Table 9). Some solutions for basic multiple choice problem (constraint: \( \sum_{j=1}^{q_i} x_{ij} = 1\)) are:

(a) \( S(b = 7) = \{(1, 2), (2, 1), (3, 2), (4, 2)\}, c(S(b = 7)) = 16.1; \)

(b) \( S(b = 8) = \{(1, 2), (2, 1), (3, 3), (4, 2)\}, c(S(b = 8)) = 17.1. \)

Here an estimate of computational complexity for a dynamic programming method is as follows:

\[
O(q_1 \mu^{l_{1\eta}}) + O(q_2 \mu^{l_{2\eta}}) + \ldots + O(q_m \mu^{l_{(m-1)\eta}}) \leq O(n \max_{i=1,m} \{q_i\} \mu^{l_{(m-1)\eta}}).
\]

| \((i, j)\) | Basic problem | Assessment \( P^{3,3}\) |
|-----------|---------------|----------------|
| \(a_{ij}\) | \(c_{ij}\) | \(e_{ij}^{3,3}\) |
| (1, 1)    | 1.3           | 3.4            | (1, 2, 0) |
| (1, 2)    | 3.1           | 8.1            | (3, 0, 0) |
| (1, 3)    | 0.7           | 1.3            | (0, 1, 2) |
| (2, 1)    | 2.0           | 4.1            | (2, 1, 0) |
| (2, 2)    | 1.3           | 2.3            | (0, 2, 1) |
| (3, 1)    | 3.0           | 2.6            | (0, 3, 0) |
| (3, 2)    | 0.6           | 1.3            | (0, 1, 2) |
| (3, 3)    | 1.6           | 2.7            | (0, 3, 0) |
| (4, 1)    | 2.0           | 2.7            | (0, 3, 0) |
| (4, 2)    | 1.3           | 2.6            | (0, 3, 0) |
| (4, 3)    | 3.3           | 4.2            | (2, 1, 0) |
| (4, 4)    | 2.3           | 3.4            | (1, 2, 0) |

Some solutions for multiple choice problem with multiset estimates and integrated multiset estimate for the solution (constraint: \( \sum_{j=1}^{q_i} x_{ij} = 1\)) are:

(a) \( S'(b = 7) = \{(1, 2), (2, 1), (3, 2), (4, 2)\}, e^{3,12}(S'(b = 7)) = (5, 5, 2); \)
\[(b) \quad S^I(b = 8) = \{(1, 2), (2, 1), (3, 3), (4, 2)\}, \quad e^{3.12}(S^I(b = 8)) = (5, 7, 0)\; \text{and}
\[(c) \quad S^I(b = 11.4) = \{(1, 2), (2, 1), (3, 1), (4, 3)\}, \quad e^{3.12}(S^I(b = 11.4)) = (7, 5, 0)\).

Some solutions for multiple choice problem with multiset estimates and “generalized median” estimate for the solution (constraint: \(\sum_{l=1}^{q_i} x_{ij} = 1\)) are:
\[(a) \quad S^g(b = 7) = \{(1, 2), (2, 1), (3, 2), (4, 2)\}, \quad e^{3.13}(S^g(b = 7)) = (1, 2, 0)\; \text{(two “generalized median” estimates exist: (1, 2, 0) and (0, 3, 0))};
\[(b) \quad S^g(b = 8) = \{(1, 2), (2, 1), (3, 3), (4, 2)\}, \quad e^{3.13}(S^g(b = 8)) = (2, 1, 0)\; \text{(two “generalized median” estimates exist: (2, 1, 0) and (1, 2, 0))};
\[(c) \quad S^g(b = 11.4) = \{(1, 2), (2, 1), (3, 1), (4, 3)\}, \quad e^{3.13}(S^g(b = 11.4)) = (2, 1, 0)\).

Some solutions for multiple choice problem with multiset estimates and “set median” estimate for the solution (constraint: \(\sum_{l=1}^{q_i} x_{ij} = 1\)) are:
\[(a) \quad S^A(b = 7) = \{(1, 2), (2, 1), (3, 2), (4, 2)\}, \quad e^{3.13}(S^A(b = 7)) = (2, 1, 0)\; \text{(two “median set” estimates exist: (0, 3, 0) and (2, 1, 0))};
\[(b) \quad S^A(b = 8) = \{(1, 2), (2, 1), (3, 3), (4, 2)\}, \quad e^{3.13}(S^A(b = 8)) = (2, 1, 0)\; \text{(two “median set” estimates exist: (0, 3, 0) and (2, 1, 0))};
\[(c) \quad S^A(b = 11.4) = \{(1, 2), (2, 1), (3, 1), (4, 3)\}, \quad e^{3.13}(S^A(b = 11.4)) = (2, 1, 0)\).

In other cases when the condition in knapsack-like problem is an inequality (e.g., \(\sum_{l=1}^{q_i} x_{ij} \leq 1\), it is necessary to consider an integrated multiset estimate for the solution. Here it is reasonable to describe a special approach to integration of different assessment multiset problems:
\[\{P^{l,\eta_1}, \ldots, P^{l,\eta_k}, \ldots, P^{l,\eta_m}\} \implies P^{l,\eta}\]
where \(\eta = \max_{\eta_i} \{\eta_i\}\). First, poset for assessment problem \(P^{l,\eta}\) is extended by addition of posets for problems \(P^{l,\eta_i}, \forall \eta_i < \eta\). Secondly, the preference rules of the following type are added (for comparison of multiset estimates from different posets above):
\[\text{(0, ..., 0, 1, 0, ..., 0)} \succ (0, ..., 0, \beta, \zeta_1, \ldots, \zeta_q),\]
where \(p_1 + p_2 \leq l, p_1 + 2 + p_3 \leq l, \beta \geq 1, \zeta_k \geq 0 \forall k = 1, p_3\). The described approach is targeted to comparison of multiset estimates with different numbers of elements: \(\eta_1 \neq \eta_2\). This is significant for basic knapsack problem and multiple choice problem because the number of elements in different solutions may be different.

**Example 20.**
\[(a) \quad e^{3.2}_1 = (1, 1, 0), \quad e^{3.3}_2 = (0, 0, 3), \quad e^{3.2}_1 > e^{3.3}_2.
\[(b) \quad e^{3.1}_1 = (1, 0, 0), \quad e^{3.3}_1 = (0, 3, 0), \quad e^{3.1}_1 > e^{3.3}_1.
\[(c) \quad e^{3.1}_5 = (0, 1, 0), \quad e^{3.5}_6 = (0, 0, 5), \quad e^{3.1}_5 > e^{3.5}_6.
\[(d) \quad e^{3.1}_5 = (1, 0, 0), \quad e^{3.5}_2 = (0, 3, 2), \quad e^{3.1}_5 > e^{3.5}_2.
\[(e) \quad e^{3.2}_2 = (0, 2, 0), \quad e^{3.5}_10 = (0, 1, 5), \quad e^{3.2}_2 > e^{3.5}_10.

Further, basic multiple choice problem with multiset estimates is:
\[
\max e(S) = \bigoplus_{(i,j) \in S = \{(i,j)\mid x_{i,j} = 1\}} e_{i,j},
\]
\[s.t. \quad \sum_{i=1}^{m} \sum_{j=1}^{q_i} a_{ij} x_{i,j} \leq b; \quad \sum_{j=1}^{q_i} x_{ij} \leq 1; \quad x_{ij} \in \{0, 1\}.\]
Here solutions for multiple choice problem with multiset estimates and the integrated estimate for the solution (constraint: $\sum_{j=1}^{q_i} x_{ij} \leq 1$) are (Table 9):

(a) $S(b = 5.1) = \{(1, 2), (2, 1)\}$, $e^f(S(b = 5.1)) = (5, 1, 0)$;
(b) $S(b = 8.4) = \{(1, 2), (2, 1), (4, 3)\}$, $e^f(S(b = 8.4)) = (7, 2, 0)$.

Note, $e^f(S(b = 8.4)) \succ e^f(S(b = 5.1))$. Here an estimate of computational complexity for a dynamic programming method is as follows:

$$O(q_1 \mu^{I, \eta}) + O(q_2 (\mu^{I,2\eta} + \mu^{I,\eta})) + \ldots + O(q_m (\mu^{I,(m-1)\eta} + \mu^{I,m-2\eta} + \ldots + \mu^{I,\eta})) \leq O(m^2 \max_{i=1,m} \{q_i\} \mu^{I,(m-1)\eta}).$$

Knapsack problem with multiset estimates and the integrated estimate for the solution is (solution $S = \{i|x_i = 1\}$):

$$\max \ e(S) = \biguplus_{i \in S = \{i|x_i = 1\}} e_i,$$

s.t. $\sum_{i=1}^{m} a_i x_i \leq b; \ x_i \in \{0, 1\}$.

A numerical example for knapsack problem is presented in Table 10.

| $i$ | Basic problem | Assessment $P^{3,3}$ |
|-----|---------------|---------------------|
|    | $a_i$ | $c_i$ | $e_i^{3,3}$ |
| 1   | 1.3  | 3.4  | (1, 1, 1) |
| 2   | 3.1  | 7.9  | (3, 0, 0) |
| 3   | 0.7  | 1.3  | (0, 1, 2) |
| 4   | 2.0  | 4.1  | (1, 2, 0) |
| 5   | 1.3  | 2.3  | (0, 2, 1) |
| 6   | 3.0  | 5.6  | (2, 1, 0) |
| 7   | 1.3  | 2.8  | (0, 3, 0) |

Some solutions for basic knapsack problem (constraint: $\sum_{j=1}^{q_i} x_{ij} \leq 1$) are:

(a) $S(b = 5.1) = \{1, 2, 5\}$, $c(S(b = 5.1)) = 13.2$;
(b) $S(b = 5.7) = \{1, 2, 7\}$, $c(S(b = 5.7)) = 14.1$;
(c) $S(b = 7.7) = \{1, 2, 4, 7\}$, $c(S(b = 7.7)) = 18.2$; and
(d) $S(b = 8.4) = \{1, 2, 3, 4, 7\}$, $c(S(b = 8.4)) = 19.5$.

Here $e^f(S(b = 8.4)) \succ e^f(S(b = 7.7)) \succ e^f(S(b = 5.7)) \succ e^f(S(b = 5.1))$.

Some solutions for knapsack problem with multiset estimates and the integrated estimate for the solution (constraint: $\sum_{j=1}^{q_i} x_{ij} \leq 1$) are:

(a) $S(b = 5.1) = \{1, 2, 5\}$, $e^f(S(b = 5.1)) = (4, 3, 2)$;
(b) $S(b = 5.7) = \{1, 2, 7\}$, $e^f(S(b = 5.7)) = (4, 4, 1)$;
(c) $S(b = 7.7) = \{1, 2, 4, 7\}$, $e^f(S(b = 7.7)) = (5, 6, 1)$; and
(c) $S(b = 8.4) = \{1, 2, 3, 4, 7\}$, $e^f(S(b = 8.4)) = (5, 7, 3)$.

Note, $e^f(S(b = 8.4)) \succ e^f(S(b = 7.7)) \succ e^f(S(b = 5.7)) \succ e^f(S(b = 5.1))$. Here an estimate of computational complexity for a dynamic programming method is as
follows:
$$O(\mu^1, \eta) + O((\mu^{1,2\eta} + \mu^1, \eta)) + \ldots + O((\mu^{1,(m-1)\eta} + \mu^1,(m-2)\eta) + \ldots + \mu^1, \eta) \leq O(m^2 \mu^1,(m-1)\eta).$$
Clearly, multiset estimates can be used for resource constraint(s) as well.

6. MULTISET ESTIMATES AND MULTI-ATTRIBUTE ALTERNATIVES

Let $A = \{A_1, \ldots, A_i, \ldots, A_n\}$ be the set of alternatives, $C = \{C_1, \ldots, C_j, \ldots, C_m\}$ be the set of attributes (criteria), $e^l, \eta_{i,j}$ be the multiset estimate of alternative $A_i$ upon criterion $C_j$ (i.e., assessment problem $P^{1, \eta}$ is used at all processing stages). Thus, the estimate vector for alternative $A_i$ is:

$$(e^l, \eta_{i,1}, \ldots, e^l, \eta_{i,j}, \ldots, e^l, \eta_{i,m}).$$

**Definition 1.** The aggregated multiset estimate for alternative $A_i$ is ("median" $M$ is considered as "generalized median" or "set median"):

$$e^l, \eta_M(A_i) = M = \arg\min_{M \in D} | \bigcup_{j=1}^{m} \delta(M, e^l, \eta_{i,j}) |.$$

**Definition 2.** The vector estimate for median alternative $A_M$ is defined as follows ("median" $M$ is considered as "generalized median" or "set median"):

$$(e^l, \eta_{M,1}, \ldots, e^l, \eta_{M,j}, \ldots, e^l, \eta_{M,m}),$$

where

$$e^l, \eta_{M,j} = \arg\min_{M \in D} | \bigcup_{i=1}^{m} \delta(M, e^l, \eta_{i,j}) |.$$

The aggregated multiset estimate for alternative $A_i$ ($e^l, \eta_M(A_i)$) and the median alternative $A_M$ can be used at preliminary stages in combinatorial synthesis.

Application of multiset estimates in real-world problems requires taking into account the following basic requirements:
1. correspondence to applied problem (and evaluation processes),
2. easy to use for domain experts,
3. limited computing complexity of the used computer procedures.

In many applications, the suggested multiset estimates can be used as a simple approximation of traditional fuzzy-set based estimates (or histogram-like estimates). Let us consider assessment problem $P^{6,4}$ as an example. Fig. 24 illustrates several estimates for problem $P^{6,4}$ (assessment over scale $[1, 2, 3, 4, 5, 6]$ with four elements.

Here "multiset coefficient" or "multiset number" is (i.e., the number of estimates):

$$\mu^{4,6} = \binom{6}{4} = \binom{6 + 4 - 1}{4} = 126.$$  

Evidently, the number (126) is the parameter for computational complexity and corresponding processing procedures will be sufficiently simple ("effective").
7. Conclusion

In the article, an approach to assessment of alternatives based on assignment of elements into an ordinal scale has been suggested. This kind of assessment corresponds to usage of special multiset based estimates (cardinality of multiset-estimate is a constant). Concurrently, the assessment approach provides evaluation from the viewpoint of uncertainty. Our evaluation process is based on an assignment procedure: for each alternative several elements (e.g., 1, 2, 3) are assigned to levels of an ordinal scale [1, 2, ..., l]. Here, the case of one element corresponds to well-known ordinal assessment. Several basic assessment problems are described: (i) traditional assessment on ordinal scale [1, 2, 3] \((P_{3,1})\); (ii) traditional assessment on ordinal scale [1, 2, 3, 4] \((P_{4,1})\); (iii) assessment on ordinal scale [1, 2, 3] based on assignment of two elements \((P_{3,2})\); (iv) assessment on ordinal scale [1, 2, 3] based on assignment of three elements \((P_{3,3})\); and (v) assessment on ordinal scale [1, 2, 3] based on assignment of four elements \((P_{3,4})\). For each assessment problem above, the following issues are examined: assessment scale, order over the scale components (poset). In addition, “interval multiset estimates” are suggested.

For the suggested multiset estimates, operations over the estimates are examined: (a) integration, (b) proximity, (c) comparison, (d) aggregation (e.g., by computing a median estimate), and (e) alignment. The suggested assessment approach is used for combinatorial synthesis (morphological approach, knapsack-like problems). The assessment approach, multiset-estimates and corresponding problems are illustrated by numerical examples.

In general, it is necessary to point out the following:
Note 1. The suggested assessment approach is based on expert judgment and/or computation procedures (including interactive mode and information visualization).

Note 2. The considered simplified versions of the assessment problem based on small numbers of elements and levels of used ordinal scale are very useful for data presentation/visualization and are often sufficient for many applications.

Note 3. The described assessment methods are very understandable and useful for domain experts. A similar assessment scale is widely used in financial engineering for evaluation of financial institutions (a version of the scale): 'AAA', 'AAB', 'ABB', 'BBB', 'BBC', 'BCD', etc. This corresponds to assessment problem $P_{3,4}$ with scale $[A, B, C, D]$ ($A > B > C > D$).

Note 4. The usage of the suggested assessment approach in composition problem leads to the extended version of HMMD method [11]

Note 5. The suggested approach can be considered as an approximation of traditional fuzzy set based assessment procedures. As a result, combinatorial estimates processing procedures based on posets (or lattices) can be used (presentation, computation).

In the future, the following research directions can be considered:

1. Usage of the suggested multiset assessment approach in decision support systems (including special visualization support subsystem and special human-computer interface).

2. Comparison of the suggested approach and approaches, based on fuzzy sets (including aspects of computation and human-computer interaction).

3. Usage of the suggested multiset approach for assessment of elements/alternatives in decision making problems: (i) sorting problems (e.g., [20], [25], [27], [38]); (ii) clustering, classification (e.g., [1], [10], [24], [25], [33], [21], [22]).

4. Analyzing the usage of the suggested multiset assessment in design and decision making procedures: (a) quality of the obtained solutions, (b) complexity of the solving schemes.

5. Special studies of algorithms and their computational complexity for considered combinatorial synthesis (e.g., combinatorial synthesis based on morphological approach and combinatorial synthesis based on knapsack-like problems) (e.g., [11], [15], [17], [19]) based on the suggested multiset assessment. Here the attention can be targeted to dynamic programming procedures. Note, analogues of FPTAS for knapsack-like problems (including intervals-based methods) (e.g., [4], [5], [8], [20], [28], [29], [30], [32], [35]) can be successfully used for combinatorial synthesis (knapsack-like problems) in case of integrated multiset estimates for solutions. On the other hand, various hybrid heuristics and man-machine procedures have to be widely used for the suggested versions of combinatorial synthesis problems.

6. Usage of the suggested approach for assessment of solution components in combinatorial optimization (e.g., scheduling problems, travelling salesman problem, routing problems, tree spanning problems, assignment problems, graph coloring problems) (e.g., [2], [11], [24]).

7. Conducting some special psychological studies of the suggested multiset assessment approach.

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