Equivalent Binary Quadratic Form and the Extended Modular Group

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Abstract

Extended modular group \( \Pi = \langle R, T, U : R^2 = T^2 = U^3 = (RT)^2 = (RU)^2 = 1 \rangle \), where \( R : z \rightarrow -z \), \( T : z \rightarrow -\frac{1}{z} \), \( U : z \rightarrow -\frac{1}{z+1} \), has been used to study some properties of the binary quadratic forms whose base points lie in the point set fundamental region \( F_\Pi \) (See [1, 5]). In this paper we look at how base points have been used in the study of equivalent binary quadratic forms, and we prove that two positive definite forms are equivalent if and only if the base point of one form is mapped onto the base point of the other form under the action of the extended modular group and any positive definite integral form can be transformed into the reduced form of the same discriminant under the action of the extended modular group and extend these results for the subset \( \mathbb{Q}^*(\sqrt{-n}) \) of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-m}) \).

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1 Introduction

The modular group \( PSL(2, \mathbb{Z}) \) has finite presentation \( \Pi = \langle T, U : T^2 = U^3 = 1 \rangle \) where \( T : z \to -\frac{1}{z} \) and \( U : z \to -\frac{1}{z+1} \) are elliptic transformations and their fixed points in the upper half plane are \( i \) and \( e^{2\pi i/3} \). The Modular group \( PSL(2, \mathbb{Z}) \) is the free product of \( C_2 \) and \( C_3 \). If \( n \) is finite, then the cyclic group of order \( n \) is denoted by \( C_n \) and an infinite cyclic group is denoted by \( C_\infty \). The modular group \( PSL(2, \mathbb{Z}) \) is the group of linear fractional transformations of the upper half of the complex plane which have the form \( z \to \frac{az+b}{cz+d} \) where \( a, b, c \) and \( d \) are integers, and \( ad - bc = 1 \). The group operation is given by the composition of mappings. This group of transformations is isomorphic to the projective special linear group \( PSL(2, \mathbb{Z}) \), which is the quotient of the 2-dimensional special linear group over the integers by its center. In other words, \( PSL(2, \mathbb{Z}) \) consists of all matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( a, b, c \) and \( d \) are integers, and \( ad - bc = 1 \).

We assume the transformation \( R(z) = -\overline{z} \) which represent the symmetry with respect to imaginary axis. Then the group \( \overline{\Pi} = \Pi \cup R\Pi \) is generated by the transformations \( R, T \) and \( U \) and has representation

\[
\overline{\Pi} = \langle R, T, U : R^2 = T^2 = U^3 = (RT)^2 = (RU)^2 = 1 \rangle.
\]

\( \Pi \) is called the extended modular group and \( \Pi \) is a subgroup of index 2 in \( \overline{\Pi} \). Therefore \( \Pi \) is a normal subgroup of \( \overline{\Pi} \) (See [1, 15].

A binary quadratic form is a polynomial in two variable of the form

\[ F = F(X, Y) = aX^2 + bXY + cY^2 \]

with real coefficients \( a, b, c \). We denote \( F \) simply by \([a, b, c] \). The discriminant of \( F \) is denoted by \( \Delta(F) = b^2 - 4ac \). \( F \) is an integral form if and only if \( a, b, c \in \mathbb{Z} \), and \( F \) is positive definite if and only if \( \Delta(F) < 0 \) and \( a, c > 0 \). The form \( F \) represent an integer \( n \) if there are integers \( x, y \) such that \( F(x, y) = n \) and the representation of \( n \) is primitive if \( (x, y) = 1 \).

Lagrange was the first to introduce the theory of quadratic forms, later Legendre expended the theory of quadratic forms, and greatly magnified even later by Gauss. It is proved by Gauss that the set \( C_\Delta \) of primitive reduced forms of discriminant \( \Delta \) is an abelian group in a natural way.

Tekcan and Bizim in [1] discussed various properties of the binary quadratic form in connection with the extended modular group. Tekcan in [2] derived
cycles of the indefinite quadratic forms and cycles of the ideals. Continuing
his work on the quadratic forms, Tekcan used base points of the quadratic
forms and derived cycle and proper cycle of an indefinite quadratic form [3].
In [18] Dani and Nogueira consider the actions of $SL(2, \mathbb{Z})$ and $SL(2, \mathbb{Z})_+$
on the projective space $\mathbb{P}$ and on $\mathbb{P} \times \mathbb{P}$ and the results are obtained on orbits-
closures to derive a class of binary quadratic forms.
The form $F = [a, b, c]$ is said to be almost reduced if $a \leq c$ and $|b| \leq a$, with
$a, c > 0$ and $\Delta(F) < 0$. This condition is equivalent to $|Re(z)| \leq \frac{1}{2}$ and
$|z| \geq 1$. Thus $F$ is almost reduced if $\tau = z(F)$ lies in the fundamental region
$F_{\Pi}$ described by $|Re(z)| \leq \frac{1}{2}$ and $|z| \geq 1$ as shown in figure 1.

![Figure 1](image)

The form $F = [a, b, c]$ is said to be reduced if $|b| \leq a \leq c$, and in case
$a = |b|$, then $a = b$ and in case $a = c$ then $b \geq 0$ [5], then the corresponding
fundamental region is denoted by $F_{\Pi}$ as shown in figure 2.
Flath [5] proved that the number of almost reduced forms with discriminant $\Delta(F) = b^2 - 4ac$ is finite. In [8] it has been proved that $\alpha$ is mapped onto $\alpha$ under the action of $PSL(2, \mathbb{Z})$ if and only if the quadratic form $f$ is equivalent to $-f$. It is well known that the set

$$Q^*(\sqrt{n}) := \left\{ \frac{a + \sqrt{n}}{c} : a, c, b = \frac{a^2 - n}{c} \in \mathbb{Z} \text{ and } (a, b, c) = 1 \right\}$$

is a $G$-subset of the real quadratic field $Q(\sqrt{m})$ under the action of $PSL(2, \mathbb{Z})$. In [10] it was proved that there exist two proper $G$-subsets of $Q^*(\sqrt{n})$ when $n \equiv 0 \pmod{p}$ and four $G$-subsets of $Q^*(\sqrt{n})$ when $n \equiv 0 \pmod{pq}$.

Malik and Riaz in [11] extended this idea to determine four proper $G$-subsets of $Q^*(\sqrt{n})$ with $n \equiv 0 \pmod{2pq}$, We generalized this result for $n \equiv 0 \pmod{p_1p_2...p_r}$ and proved that there are exactly $2^r$ number of $G$-subsets of $Q^*(\sqrt{n})$. We also proved for $h = 2k + 1 \geq 3$ there are exactly two $G$-orbits of $Q^*(\sqrt{2^h})$ namely $\langle 2^k \sqrt{2} \rangle^G$ and $\langle \frac{2^k \sqrt{2}}{1} \rangle^G$. In the same paper we used subgroup $G^* = \langle yx \rangle$ and $G^{**} = \langle yx, y^2x \rangle$ to determine the $G$-subsets and $G$-orbits of $Q^*(\sqrt{n})$.

If $p$ is an odd prime, then $\lambda \not\equiv 0 \pmod{p}$ is said to be a quadratic residue of $p$ if there exists an integer $r$ such that $r^2 \equiv \lambda \pmod{p}$.

The quadratic residues of $p$ form a subgroup $R$ of the group of nonzero integers modulo $p$ under multiplication and $|R| = (p-1)/2$. [4]
The norm of an element $\alpha$ in $Q^*(\sqrt{n})$ is defined by $N(\alpha) = \alpha\overline{\alpha} = \frac{a^2 - n}{c}$. The Legendre symbol $(a/p)$ is defined as 1 if $a$ is a quadratic residue of $p$ otherwise it is defined by $-1$.

**Theorem 1.1**\footnote{[1]}

(i) Let $p$ be any odd prime such $p \equiv 1 \pmod{4}$ and $a$ be quadratic residue of $p$ then $p - a$ is a quadratic residue of $p$.

(ii) Let $p$ be any odd prime such $p \equiv 3 \pmod{4}$ and $a$ be quadratic residue of $p$ then $p - a$ is a quadratic non-residue of $p$.

**Theorem 1.2**\footnote{[5]}

The number of almost reduced forms $h(\Delta)$, with given discriminant $\Delta < 0$, is finite.

## 2 The relation between quadratic forms and the extended modular group.

It is interesting to determine equivalent binary quadratic forms, the interest has risen in the last decade as the possibilities to study binary quadratic forms in combination with mathematical software has improved. There is a strong relation between equivalent binary quadratic forms and their base points (See [1, 3]).

Two forms $F = [a, b, c]$ and $G = [A, B, C]$ are said to be equivalent if there exists an element $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \bar{\Pi}$, where $r, s, t, u$ are integers and $ru - st = 1$, such that $gF(X, Y) = G(X, Y)$ or $F(rX + sy, tX + uY) = G(X, Y)$

the coefficients of $G$ in terms of the coefficients of $F$ are as follows

$A = ar^2 + btr + ct^2$

$B = 2ars + bru + bst + 2ctu$

$C = as^2 + bsu + cu^2$.

Then it is easy to see that $\Delta(F) = (ru - st)^2\Delta(gF)$ but $ru - st = 1$ implies that $\Delta(F) = \Delta(gF)$\footnote{[17]}, that is any two equivalent forms have the same discriminant, but the converse is not true in general. The definition of $gF$ is a group action on the set of forms because it satisfies the two axioms

(i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} F = F$  
(ii) $g(hF) = (gh)F$.  

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Gauss (1777-1855) introduced the group action of extended modular group \( \mathbb{\Pi} \) on the set of forms. We use the standard notations as used in \([1]\). Let 
\[ U = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \]
be the upper half of the complex plane. For a positive definite form 
\( F = [a, b, c] \) with \( \Delta(F) < 0 \) there is a unique 
\[ z = x + iy = z(F) \in U \] such that 
\[ F = a(X + zY)(X + \bar{z}Y) \]
then \( z = x + iy \) is called base point of \( F \) in \( U \).

By comparison we get \( x = \frac{b}{2a} \) and \( y = \frac{1}{2a} \sqrt{-\Delta(F)} > 0 \) so 
\[ z = x + iy = \frac{b + i \sqrt{-\Delta(F)}}{2a} \in U. \]
Conversely for any \( w = x + iy \in U \) there exists a positive definite quadratic 
form \( F = \left[ \frac{1}{|w|^2}, \frac{2x}{|w|^2}, 1 \right] \) with base point \( w \) and discriminant \( \Delta(F) < 0 \).

There is a one-one correspondence between the set of positive definite forms 
with fixed discriminant and the set of base points of these forms \([1]\).

In the next theorem we show that there is a strong relation between equivalent binary quadratic forms and their base points. We proved that two 
positive definite forms are equivalent if and only if the base point of one form 
is mapped onto the base point of the other form under the action of the 
extended modular group.

**Theorem 2.1**

Two positive definite forms \( F = [a, b, c] \) and \( G = [A, B, C] \) are equivalent if and only if \( \alpha = g(\beta), g \in \mathbb{\Pi} \), where \( \alpha \) and \( \beta \) are the base points of \( F \) and \( G \) respectively.

**Proof.**

Let \( F = [a, b, c] = aX^2 + bXY + cY^2 \) and \( G = [A, B, C] = Au^2 + Buv + Cv^2 \) 
are equivalent and \( \alpha = z(F) \) and \( \beta = z(G) \) are the base points of \( F \) and \( G \) respectively, then we can assume that

\[ F = a(X + \alpha Y)(X + \bar{\alpha} Y) \]
\[ G = A(u + \beta v)(u + \bar{\beta} v) \]

Since \( F \) and \( G \) are equivalent then exist an element \( g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathbb{\Pi} \), where \( r, s, t, u \) are integers and \( ru - st = 1 \), such that \( u = rX + sY \) and \( v = tX + uY \).
Thus by \[16\] we have
\[
a(X + \alpha Y)(X + \overline{\alpha} Y) = A(u + \beta v)(u + \overline{\beta} v)
\]
\[
= A((rX + sY + \beta(tX + uY))(rX + sY + \overline{\beta}(tX + uY))
\]
\[
= A((r + \beta t)X + (s + \beta u)Y)((r + \overline{\beta} t)X + (s + \overline{\beta} u)Y)
\]
\[
= A'(X + \left(\frac{s + \beta u}{r + \beta t}\right) Y)(X + \left(\frac{s + \overline{\beta} u}{r + \overline{\beta} t}\right) Y)
\]
where \[A' = A(r + \beta t)(r + \overline{\beta} t)\].

Thus, if \[\alpha = \frac{s + \beta u}{r + \beta t},\] then \[\overline{\alpha} = \frac{s + \overline{\beta} u}{r + \overline{\beta} t}\]. It is clear that \(\alpha = g'(\beta)\) and \(\overline{\alpha} = g'(\overline{\beta})\) under the linear fractional transformation

\[g'(z) = \frac{s + uu}{r + zt},\] where \(st - ru = 1\).

Conversely suppose that there exists a linear fractional transformation \(g(z) = \frac{s + uu}{r + zt} \in \Pi\), where \(st - ru = 1\) such that \(\alpha = g(\beta)\), then \(\alpha\) and \(\overline{\alpha}\) are images of \(\beta\) and \(\overline{\beta}\) under \(g\), where \(\alpha\) and \(\beta\) are the base points of \(F\) and \(G\) respectively, then clearly \(gF = G\). This proves the result. \(\Box\)

The following results are the immediate consequences of the above theorem.

**Corollary 2.2**
Two forms \(F\) and \(G\) are equivalent if and only if their base points lie in the same fundamental region \(F\Pi\).

**Corollary 2.3**
If \(F\) and \(G\) are two equivalent binary quadratic forms then both forms are either almost reduced quadratic forms or reduced quadratic forms.

**Theorem 2.4**
The binary quadratic form \(F = x^2 + py^2\) is equivalent to \(\lambda F = \lambda x^2 + \lambda py^2\) if and only if \((\lambda/p) = 1\), For any \(\lambda \in \mathbb{Z}\) and \(p\) be any prime.

**Proof.**
Let the binary quadratic form \(F\) be equivalent to \(\lambda F\), then there exists an element \(g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \Pi\), where \(r, s, t, u\) are integers and \(ru - st = 1\), such that \(gF(x, y) = \lambda F(x, y)\)

\[\Leftrightarrow F(rx + sy, tx + uy) = \lambda F(x, y)\]

\[\Leftrightarrow (rx + sy)^2 + p(tx + uy)^2 = \lambda x^2 + \lambda py^2\] (Comparing the coefficients of \(x^2\))

\[\Leftrightarrow r^2 + pt^2 = \lambda\]

\[\Leftrightarrow r^2 \equiv \lambda (mod\ p)\].

Thus \(\lambda\) is a quadratic residue modulo \(p\). \(\Box\)
Example 2.5
The form $x^2 + 37y^2$ is equivalent to the form $-x^2 - 37y^2$ because $(\lambda/37) = 1$ but the form $x^2 + 79y^2$ is not equivalent to the form $-x^2 - 79y^2$ because $(\lambda/79) = -1$. □

Theorem 2.6
Let $F$ be a positive definite integral form with $\alpha = z(F)$ as the base points of $F$, and let $g(\alpha) = \beta$ under the action of the extended modular group. Then $F$ is equivalent to its reduced form $F_R$ whose base point is $\beta = z(F_R)$.

Proof.
Let $F = ax^2 + bxy + cy^2$ be primitive positive definite form and let $n$ be the smallest integer represented by $F$, put $F_R = Au^2 + Buv + Cv^2$ and $\alpha = z(F)$ and $\beta = z(F_R)$ be the base points of $F$ and $F_R$ respectively, If $g(\alpha) = \beta$ then

$$
\begin{pmatrix}
    r & s \\
    t & u
\end{pmatrix}
\begin{pmatrix}
    \alpha \\
    \beta
\end{pmatrix} = 1
$$

$$
\Rightarrow r\alpha + s = \beta u + t\alpha \beta \Rightarrow r\alpha + s - \beta u - t\alpha \beta = 0
$$

$$
\Rightarrow (r\alpha + s) - (u + t\alpha) \beta = 0
$$

$$
\Rightarrow r\alpha + s = 0 \text{ and } (u + t\alpha) \beta = 0, \beta \neq 0
$$

$$
\Rightarrow r\alpha + s = 0 \text{ and } u + t\alpha = 0
$$

$$
\begin{pmatrix}
    \alpha \\
    \beta
\end{pmatrix}
\begin{pmatrix}
    r & s \\
    t & u
\end{pmatrix}
\begin{pmatrix}
    \alpha \\
    \beta
\end{pmatrix} = 1
$$

$$
\Rightarrow \begin{pmatrix}
    \alpha \\
    \beta
\end{pmatrix}
\begin{pmatrix}
    r & s \\
    t & u
\end{pmatrix}
\begin{pmatrix}
    \alpha \\
    \beta
\end{pmatrix} = 1
$$

$$
\Rightarrow s = -\alpha r \text{ and } u = -\alpha t
$$

$$
ru = st \Rightarrow ru - st = 1 \text{ this shows that } g \in \bar{\Pi}, \text{ put } u = rx + sy \text{ and } v = tx + uy
$$

then $F$ is transformed into the form $F_R$ as follows:

$$
F_R(x, y) = (ar^2 + brt + ct^2)x^2 + (2ars + bru + bst + 2ctu)xy + (as^2 + bsu + cu^2)y^2
$$

or $F_R(x, y) = nx^2 + Bxy + Cy^2$, where

$$
n = ar^2 + brt + ct^2
$$

$$
B = 2ars + bru + bst + 2ctu
$$

$$
C = as^2 + bsu + cu^2.
$$

Then $\Delta(F) = (ru - st)^2\Delta(F_R)$ this implies that $\Delta(F) = \Delta(F_R)$. Now by definition $a \leq c \Rightarrow 4a^2 \leq 4ac = b^2 - \Delta(F) \leq a^2 - \Delta(F)$, where $\Delta(F) < 0$ implies that $3a^2 \leq (-\Delta(F))$ and finally $a \leq \sqrt{(-\Delta(F))}/3$, Thus by [17], we see that $|B| < n$ then $F$ is properly equivalent to $F_R$, since $F$ is positive definite, and $F_R(0,1) = C$ which implies that $C \in \mathbb{N}$, and $C \geq n$ by the minimality of $n$. This proves the result. □
The relation between quadratic forms and orbits of imaginary quadratic fields.

The imaginary quadratic fields are defined by the set $\mathbb{Q}(\sqrt{-m}) = \{a + b\sqrt{-m} : a, b \in \mathbb{Q}\}$, where $m$ is a square free positive integer. We shall denote the subset

$$\left\{\frac{a + \sqrt{-n}}{c} : a, c, b = \frac{a^2 + n}{c} \in \mathbb{Z}, c \neq 0\right\}$$

by $\mathbb{Q}^*(\sqrt{-n})$. The imaginary quadratic fields are very useful in different branches of Mathematics. The integers in $\mathbb{Q}(\sqrt{-1})$ are called Gaussian integers, and the integers in $\mathbb{Q}(\sqrt{-3})$ are called Eisenstein integers.

We denote the modular group $\Pi$ by $G$ for our convenience and use coset diagrams to investigate $G$-subsets and $G$-orbits of $\mathbb{Q}^*(\sqrt{n})$ under the action of the modular group $\Pi$ (See [10, 11]).

In [8] it has been proved that for any quadratic form $F = Ax^2 + Bxy + Cy^2$ if $\alpha$ and $\overline{\alpha}$ belong to the subset $\mathbb{Q}^*(\sqrt{n})$ of the real quadratic field then there exist a rational number $\lambda$ such that $F(x, y) = \lambda(cx^2 - 2axy + by^2)$. We extend this result for the subset $\mathbb{Q}^*(\sqrt{-n})$ of the imaginary quadratic field in the next Lemma.

**Lemma 3.1**

For any positive definite binary quadratic form $F = [A, B, C] = Ax^2 + Bxy + Cy^2$ if $\alpha$ is base point of $F$ then there exists a rational number $A'$ such that $F(x, y) = A'(cx^2 - 2axy + by^2)$.

**Proof.**

Let $\alpha = \frac{a + \sqrt{-n}}{c} \in \mathbb{Q}^*(\sqrt{-n})$ with $b = \frac{a^2 + n}{c}$ such that $\alpha$ be the base point of $F$. Then there exists a rational number $A'$ such that $F(x, y) = A'(cx^2 - 2axy + by^2)$.

Further it has been proved in [8] that $\alpha$ is mapped onto $\overline{\alpha}$ under the action of $PSL(2, \mathbb{Z})$ on $\mathbb{Q}^*(\sqrt{p})$, where $\alpha = \frac{a + \sqrt{p}}{c}$, $b = \frac{a^2 - p}{c}$ and $(a, b, c) = 1$ if and only if the quadratic form $f(x, y) = cx^2 - 2axy + by^2$ is equivalent to $-f$. We generalized this result for the subset $\mathbb{Q}^*(\sqrt{-n})$ of the imaginary quadratic field.
Theorem 3.2
Under the action of $PSL(2, \mathbb{Z})$ on $\mathbb{Q}^*(\sqrt{-n})$, $\alpha$ is mapped onto $\beta$ if and only if the binary quadratic forms $F$ and $G$ are equivalent, where $\alpha = z(F)$ and $\beta = z(G)$ are the base points of $F$ and $G$ respectively.

Proof.
We know that the modular group $PSL(2, \mathbb{Z})$ is a subgroup of index 2 in $\Pi$, also $\Pi$ is a normal subgroup of $\overline{\Pi}$. Let $\alpha = \frac{a+\sqrt{-n}}{c} \in \mathbb{Q}^*(\sqrt{-n})$ with $b = \frac{a^2+n}{c}$ such that $\alpha$ be the base point of $F$ and $\beta$ be the base points of $G$ respectively. Since $\alpha = g(\beta)$ under the action of $PSL(2, \mathbb{Z})$ on $\mathbb{Q}^*(\sqrt{-n})$, then by Theorem 2.1, $F$ is equivalent to $G$. This proves the result. $\square$

Conclusion
There is a one-one correspondence between the set of positive definite forms with fixed discriminant and the set of base points of these forms. We prove that there is a strong relation between equivalent binary quadratic forms and their base points in a sense that two positive definite forms are equivalent if and only if the base point of one form is mapped onto the base point of the other form under the action of the extended modular group and two forms are equivalent if and only if their base points lie in the same fundamental region. These results can be extended to determine the quadratic ideals for a given quadratic irrationals, and for determining proper cycles of an indefinite quadratic forms as done in [2, 3].

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