Centroidal-Polygon: Solving First Order Ordinary Differential Equation using Centroidal mean to improve Euler Method

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Abstract. Euler method is one of the oldest and simplest numerical techniques to solve mathematical problems. However, the Euler method lacks accuracy compared to other methods such as the Runge-Kutta and the Adam Bashforth. This research proposed a new scheme by enhancing the Euler method using centroidal mean, i.e. combining the Euler method and the mean concept. In this research, the proposed scheme would be used to improve the accuracy of the first order Ordinary Differential Equation (ODE). The enhanced Euler method was used based on the improved Euler method, which is a midpoint method. To analyse the accuracy of the proposed scheme and the existing schemes, SCILAB 6.0 software was used to compare the results of different step sizes. This research compared the proposed scheme with the Polygon scheme by Zul Zamri and the Harmonic-Polygon scheme by Nurhafizah. By using Centroidal-Polygon as the proposed scheme, more accurate results were yielded. However, the scheme was limited to some of the linear and non-linear equations only.

1.0 INTRODUCTION

The numerical method is the basis of Engineering and Science. Most correct solution techniques utilise the numerical solution for solving problems in areas of science, such as semiconductor, population, weather forecast, physics, and others.

Ordinary Differential Equation (ODE) is one of the simplest techniques to solve numerical methods in Initial Value Problems (IVP’s), especially in cases where obtaining the exact solution is challenging or where there is an absence of closed-form analysis. This study used the Euler method as an effective technique to solve numerical methods [1].

The Euler method, also called a tangent line method or a one-step method, is the simplest numerical method for solving IVP in ODE [2]. The Euler method was developed by Leonhard Euler in 1768 [3], and it is suitable to be used in simple implementations, programming, and low-cost computations [2]. However, accuracy is one of the factors which necessitates the selection of another method to replace the Euler method [4].
This paper focuses on proposing a new scheme which would use an Improved Euler Method known as the Polygon scheme [5]. This paper aims to yield solutions which are more accurate in comparison to the exact solution. The new scheme in this paper was named Centroidal-Polygon. The idea to develop a new algorithm by enhancing other schemes had been proposed by Zul Zamri [5] and Nurhafizah [2 & 4]. Zul Zamri used an average of arithmetic mean for two points of coordinate.

\[ y_n = \frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{2} \]  

The construct of the method utilised the combination of the arithmetic polygon method and Centroidal mean in the equation. This paper proposed to compare other schemes and the centroidal mean with the exact solution. The proposed difference step sizes of \( h \) are 0.1, 0.01, and 0.001.

2.0 LITERATURE REVIEW

Before the Euler method was improved by combining of the Euler method and mean concept, there had been two simple modifications in the attempt to improve the Euler method. The methods are Heun’s method and the Midpoint method. Referring to the journal from Zul Zamri, the midpoint method is more stable compared to Heun’s method [5]. Moreover, the midpoint method, which is also called Improved Polygon, is simpler. This method is superior to the Euler method because it utilises the gradient estimate at the midpoint of the prediction interval [6].

Using the Midpoint method, Zul Zamri enhanced the Euler method by combining its equation with the mean concept. By combining these two methods, the result showed that the improved method is more accurate compared to the original Euler method.

Differential Equation (DE) contains one or several derivatives of an unknown function [6]. When the function in the equation is involved with one independent variable, the equation is called ODE. DE is classified in orders. As such, the first order would refer to the high derivative and second order would include the second derivative. The paper contains two conditions, which are linear and non-linear equations [7].

The proposed scheme was developed using the mean concept within two coordinate points of function to improve the method. Centroid mean refers to an object at the geometric centre. Other than that, Centroid is a simple concept used for any positive real number [8].

There are numerous techniques to solve ODE value problems. One such method is the Polygon scheme. However, the Euler method is less accurate and will yield a more significant error. Therefore, the researchers are compelled to choose more complicated methods, although the Euler method is easier. Thus, to improve the method, a study (Nurhafizah, 2015) suggested a new scheme to solve the Euler method by using another mean. Nurhafizah (2017) found a new solution for the Euler scheme to obtain more accurate values. The method is used to obtain an accurate solution by using the mean concept. Centroid mean provides an excellent solution [2 & 4] to solve the Euler solution, since an accurate result is possible.

The scheme was developed based on the original Euler method. Then, as improvement, the method employed two coordinating points, \( x \) and \( y \), as the primary points in the Euler equation. The modification of the scheme was done within the two coordinates by using the average concept. The modifications to the slope of the function at estimated midpoints would improve the accuracy of the Euler method [1].
3.0 RESEARCH METHODOLOGY

The scheme clearly describes the elements involved and then converts them into the programme code. Most applications solved the easy programme code which was developed to help understand its application [5]. Construction processes in mathematical software are as follows [9]:

a) Compare the proposed scheme with others
b) Develop the Algorithm
c) Document the details

The effectiveness of the developed scheme was then tested using computer implementation. The code programme used SCILAB 6.0 Programming [3]. Finally, all schemes were compared to the exact solutions by tabulating the error between the proposed scheme and the exact solution.

The technique to improve Euler is a Polygon. In this research [1], arithmetic mean had two coordinate points, which were used to improve the Euler Method [10]. The proposed scheme used Euler Method in equation (1),

\[ y_{n+1} = y_n + \Delta t f(x_0 y_0) \]

and the equation was then modified by using the concept of average. Mathematically, the Centroid or geometric centre is the arithmetic mean position of all the points. The Centroid is the mean position of all points in all coordinate direction, which is written as equation (2) below;

\[ y_{n+1} = y_0 + hf \left( \frac{2[f(x_0 y_0)^2 + f(x_0 y_0) f(x_1 y_1)] + f(x_1 y_1)^2}{3[(f(x_0 y_0) + f(x_1 y_1))]} \right) \]

Euler (E) \hspace{2cm} Mean (M)

\[ y_{n+1} = y_n + hf(x_n y_n) \]

\[ M = \frac{2((x_n)^2 + x_n y_n + (y_n)^2)}{3(x_n + y_n)} \]

E+M = CP

\[ CP = y_n + hf \left( \frac{x_n + (x_n + h)}{2}, \frac{y_n + y_n + f(x_n y_n)}{2} \right) + \frac{2((x_n)^2 + x_n y_n + (y_n)^2)}{3(x_n + y_n)} \]

\[ y_{n+1} = y_n + hf \frac{2((x_n)^2 + x_n (x_n + h) + (x_n h)^2)}{3(x_n + x_n + h)}, \frac{2(y_n)^2 + y_n + y_n + f(x_n y_n) + (y_n + h f(x_n y_n))^2}{3(y_n + y_n + h f(x_n y_n))} \]

For the comparison, this research compared the proposed scheme with the Polygon scheme by Zul Zamri and the Harmonic-Polygon by Nurhafizah.

This section compares three schemes of the modified Euler with the exact solution. The purpose of this research is to solve the Ordinary Differential Equation (ODE) over the interval from \( x = 0 \) to 20 for Problems 1, 2, and 3 using step sizes of 0.1, 0.01, and 0.001. Table 1 illustrates the exact solution for the First Order ODE. Table 2 shows the comparison results between Polygon, Harmonic-Polygon,
and Centroid-Polygon scheme. Relative error is based on two solutions. The solution then is calculated as

\[ \text{Error} = |\text{Ex} - \text{Ev}| \]

\( \text{Ex} = \) Exact value and \( \text{Ev} = \) Euler’s improved value

### Table 1: Set of problem First Order ODE

| Equation      | Exact Solution | Initial Value | Interval of Integration | Sources |
|---------------|----------------|---------------|-------------------------|---------|
| \( y' = -0.5y^3 \) | \( \frac{1}{\sqrt{x} + 1} \) | \( y(0)=1 \) | 0 ≤ \( x \) ≤ 20 | Gucolu |
| \( y' = -0.2y \) | \( e^{-0.2x} \) | \( y(0)=1 \) | 0 ≤ \( x \) ≤ 20 | Noraida |
| \( y' = -y \) | \( e^{-x} \) | \( y(0)=1 \) | 0 ≤ \( x \) ≤ 20 | Noraida |

### Table 2: Result for ODE problems

| Scheme  | Polygon | Harmonic-Polygon | Centroid-Polygon |
|---------|---------|------------------|------------------|
|         | 0.1     | 0.01             | 0.001            | 0.1     | 0.01   | 0.001 |
| Problem 1 | 0.105332 | 0.106684         | 0.106825         | 0.072182 | 0.073239 | 0.073343 |
|         | 0.059488 | 0.060530         | 0.060632         |         |        |       |
| Problem 2 | 0.223674 | 0.225536         | 0.225722         | 0.154729 | 0.156396 | 0.156563 |
|         | 0.136683 | 0.138240         | 0.138396         |         |        |       |
| Problem 3 | 0.356768 | 0.364596         | 0.365360         | 0.193406 | 0.202613 | 0.203530 |
|         | 0.19406  | 0.202613         | 0.203530         |         |        |       |

### 4.0 DISCUSSION

Observation in Table 2 shows the result of proposed scheme, Polygon (P) scheme, and Harmonic-Polygon (HP). The Centroidal-Polygon (CP) scheme is closer to the exact solution compared with other schemes. The CP scheme was suitable for solving Problem 1 and gave more accurate values than HP and P. Problem 1 equation is a non-linear equation. It would be used to ensure that the CP can be solved into the linear and non-linear first order ODE.

At step size of 0.1, the CP scheme gave the maximum error of 0.059488, while the HP scheme gave the maximum error of 0.072182, and the P scheme gave the maximum error of 0.105332. At the step size of 0.01, the CP gave the maximum error of 0.060530, while the HP scheme gave the maximum error of 0.073239, and the P scheme gave the maximum error of 0.106684. At the step size of 0.001, Table 2 shows that the P scheme gave the maximum error of 0.106825 compared to the CP scheme, which gave the maximum error of 0.060632. The
CP also obtained accuracy with the maximum error being less than that of the HP, i.e. 0.073343.

In Problem 2, the result shows that the CP scheme provided more accuracy than other schemes. At the 0.1 step size, the CP scheme gave the maximum error of 0.136683, while the HP scheme gave the maximum error of 0.154729, and the P scheme gave the maximum error of 0.223674. At step size of 0.01, the CP scheme gave the maximum error of 0.138240, while the HP scheme gave the maximum error of 0.156396, and the P scheme gave the maximum error of 0.225536. Meanwhile, at the large step size of 0.001, the CP gave the maximum error of 0.138396, whereas the HP scheme gave the maximum error of 0.156563 and the P scheme gave the maximum error of 0.2255722.

In problem 3, the result shows the CP scheme and HP scheme gave the same maximum error from an earlier loop. At a small step size, the CP and HP schemes gave the maximum error of 0.19406 while the P scheme gave the maximum error of 0.35678. At the 0.01 step size, the CP and HP schemes gave the maximum error of 0.202613. The P scheme gave the maximum error of 0.364596. At the large step size, the CP and HP schemes gave the maximum error of 0.203530, while the P scheme gave the maximum error of 0.365360.

Figure 1: Comparison of accuracy between CP, HP and P for non-linear equation (Problem 1)
Figure 2: Comparison of accuracy between CP, HP and P for linear equation (Problem 2)
The Figure 1 to 3 shows the value of maximum error in the difference between the value and Euler’s improved value. By referring to Figures 1 to 3, the CP scheme is more accurate compared with other schemes. However, Figures 1 and 2 show that the CP scheme produces the highest maximum error compared to the others scheme. Yet, after zero maximum error is reached, the CP schemes shows the lowest error compared to HP and P schemes.

Problem 3 shows that the value of CP and HP are the same. Even the values are the same, which shows that the CP and HP schemes are more accurate compared to the P scheme, with the difference of 0.16 between CP and P. The increase and decrease depend on the value of errors between the exact solution and the Euler improved value.

Although CP is generally more accurate, several disadvantages have been identified. CP method only can be used in several linear and non-linear equations. CP scheme is prone to develop stiff value in a specific equation after several looping. Other than that, if the CP method is applied in a multiplication operation, the value of the equation in y or x must not be zero.

Before using SCILAB as the software to solve Euler problems, researchers would manually compare schemes. The researchers used other equations to ascertain whether or not the CP scheme and other schemes could be used, before experimenting using SCILAB.

5.0 CONCLUSION

This study shows that the Centroid-Polygon scheme is advantageous as one of the solutions for First Order ODE. Using SCILAB 6.0 Programming, each scheme was tested and compared with the exact solution. Before this, the Euler Method could improve in small step sizes only and the Improved Euler method can solve in small and large step sizes, but the computational cost in terms of computational time is still high. Nevertheless, Centroid-Polygon can also be used in a big step size (h=0.1). The advantage of using a big step size is in terms of reducing the complexity of the solution and looping time. In conclusion, Centroid-Polygon is useful as one of the alternative schemes to solve the ODE equation. To improve the method, the Centroidal mean can use another method, such as to consider it with other equations, using non-standard EO or non-standard finite differentiation [11-12].

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