Repulsive Dark Matter
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ABSTRACT

It seems necessary to suppress, at least partially, the formation of structure on subgalactic scales. As an alternative to warm or collisional dark matter, I postulate a condensate of massive bosons interacting via a repulsive interparticle potential, plus gravity. This leads to a minimum lengthscale for bound objects, and to superfluidity. Galactic dynamics may differ significantly from that of more generic dark matter in not unwelcome ways, especially in the core. Such particles can be realized as quanta of a relativistic massive scalar field with a quartic self-interaction. At high densities, the equation of state has the same form as that of an ideal relativistic gas despite the interactions. If the nonrelativistic lengthscale is of order a kiloparsec, then the energy density in these particles was comparable to that of photons at early times, but small enough to avoid conflict with primordial nucleosynthesis.

1. Introduction

At the time of writing, the combination ("ΛCDM") of cold dark matter, a Harrison-Zeldovich spectrum of initial fluctuations, and a small cosmological constant appears to be in good agreement with many observational constraints: the angular power spectrum of the temperature of the cosmic microwave background, the large-scale distribution of galaxies and clusters, and the growth of present-day structures by gravitational instability (Bahcall et al. 1999, and references therein). Recently, however, it has become clear that ΛCDM overpredicts structure on small scales. N-body and hydrodynamic simulations of galaxy formation within the ΛCDM model predict that the dark halos of galaxies should have singular cores (Navarro, Frenk, & White 1996; Navarro & Steinmetz 2000), contrary to observation (Carignan 1985; Swaters, Madore, & Trewhella 1999). A separate though related problem is that late accretion of dark matter clumps or satellite galaxies is likely to shred the disks of spirals (Toth & Ostriker 1992; Klypin et al. 1999).

Clustering on small scales could be suppressed by an upper limit to the phase-space density of the dark-matter particles due either to thermal entropy or, if they are fermions, degeneracy pressure; the particle mass would have to lie in the range $10^2 \text{ eV} < m < 10^3 \text{ eV}$ (Hogan & Dalcanton 2000).

Another proposal is that the dark-matter particles are heavy ($m \gg \text{GeV}$) but self-interacting (Spergel & Steinhardt 1999). Since the simulations tend to produce a dark-matter velocity disper-
sion that decreases towards the center of the cusp, occasional collisions “heat” the central regions and produce a finite-density core, at least temporarily. Once the inner regions become roughly isothermal, the core will recollapse; the collision cross section must be chosen so that this does not happen within the age of the galaxy. This proposal seems unlikely to alleviate the second problem, viz. the damage to galactic disks.

This paper proposes a third solution to the problem of small-scale power: a Bose-Einstein condensate of dark matter particles, hence similar to the axion, but interacting via a repulsive potential of finite range. I show in §2 that cores would have a minimum size independent of their density or mass, and that the dark matter would behave as a superfluid. Such particles arise fairly naturally as quanta of a self-interacting scalar field (§3). For plausible choices of the minimum core size, the interaction energy per particle would have been comparable to the rest mass somewhat before the universe became matter-dominated, and at earlier times it would have a relativistic equation of state \( p \approx \rho/3 \), hence slightly increasing the effective number of degrees of freedom in the radiation field, but not enough to violate the constraints of primordial nucleosynthesis. With respect to the standard cold-dark matter spectrum, density fluctuations in the linear regime would be suppressed on comoving scales less than a few megaparsecs (§4).

While this work was being written up, I became aware that my colleague P. J. E. Peebles has been working along similar lines (Peebles & Vilenkin 1999; Peebles 1999, 2000). There is also closely related earlier work (Tkachëv 1985, 1991). This seems to be a genuine case of convergent evolution, and the fact that independent groups arrived at similar results is perhaps reassuring in such a wildly speculative domain. I started with a nonrelativistic many-body view of these particles (§2) and then sought a relativistic framework for them (§3), whereas Peebles & Vilenkin (1999) seem to have proceeded in the opposite direction. With further development, the nonrelativistic but fully quantum-mechanical viewpoint may prove useful in studying the galactic dynamics of this form of dark matter.

2. Minimum dark-matter core radius

Suppose that nonrelativistic bosons interact via a two-particle potential \( U(r_1 - r_2) \) of finite range. The potential energy of \( N \) such bosons in a common single-particle momentum state \( \psi(r) = V^{-1/2} e^{i\mathbf{p} \cdot \mathbf{r}} \) in volume \( V \) is

\[
W_N = \frac{N(N-1)}{2V} \tilde{U}(0) \equiv \frac{N(N-1)}{2V} \int d\mathbf{r}' U(\mathbf{r'}),
\]

(1)

where \( \tilde{U}(\mathbf{p'}) \) is the fourier transform of \( U(\mathbf{r'}) \), whose range is assumed to be small compared to the linear dimensions of \( V \). If \( N, V \to \infty \) at fixed number density \( n = N/V \), then the potential energy per unit volume is \( w(n) = n^2 \tilde{U}(0)/2 \). For the moment, \( w(n)/n \ll mc^2 \), the boson rest-mass energy.

Macroscopically, one has a polytropic gas of adiabatic index \( \gamma = 2 \); in other words, the pressure
is related to the mass density $\rho \approx mn$ by

$$p = K \rho^2, \quad K = \tilde{U}(0)/2m^2.$$  \hfill (2)

If the gas is self-gravitating, there is a well-known spherical equilibrium with the density profile (Chandrasekhar 1939):

$$\rho(r) = \rho(0) \frac{\sin(r/a)}{r/a}, \quad a = \sqrt{\frac{K}{2\pi G}}.$$  \hfill (3)

The radius of the sphere, $\pi a$, is independent of the central density $\rho(0)$, which determines the total mass, $4\pi a^3 \rho(0)$. Dark-matter halos do not have the profile (3), but this does not rule out the model. If the particles are not all in the same momentum state, then their relative motions make an additional contribution to the pressure, which allows the halo to have a power-law density profile outside the core. Indeed, axionic dark matter is usually assumed to be a Bose-Einstein condensate like the one considered here but without the repulsive interaction, so the pressure support of dark halos in that model is due entirely to relative motions. Still, since (2) gives the minimum pressure, (3) is the most compact possible equilibrium. The conventional definition of the core radius in terms of the central density and its second derivative is $r_c \equiv [-3\rho(0)/\rho''(0)]^{1/2}$, thus $r_{c,\text{min}} = 3a$.

The nonrelativistic approximation breaks down when the pressure (2) is comparable to the rest mass density. This happens when $\rho \approx c^2/Ga^2 \equiv \rho_{\text{rel}}$. If this bosonic dark matter dominates the total present-day mass density, then the mean mass density was equal to $\rho_{\text{rel}}$ at redshift

$$1 + z_{\text{rel}} = \left(\frac{8\pi G \rho_{\text{rel}}}{3\Omega H_0^2}\right)^{1/3} \approx 2.1 \times 10^5 \Omega_{0.3}^{-1/3} h_{50}^{-2/3} r_{c,\text{kpc}}^{-2/3}.$$  \hfill (4)

where $\Omega \equiv \Omega/0.3$, $h_{50} \equiv H_0/(50 \text{ km s}^{-1} \text{ Mpc}^{-1})$, and $r_{c,\text{kpc}} = r_{c,\text{min}}/\text{kpc}$. The question how the dark matter behaves at higher redshifts is deferred to §§3-4. It suffices for now that $1 + z_{\text{rel}}$ is comfortably larger than the redshift of matter-radiation equality (Peebles 1993): $1 + z_{\text{eq}} = 1.8 \times 10^3 \Omega_{0.3} h_{50}^2$.

The interaction makes the gas a superfluid. If one particle is removed from the Bose-Einstein condensate and put into a single-particle state with momentum $p + q \neq p$, then the potential energy (1) is replaced by

$$W_{N-1} + \frac{N-1}{N} \left[ \tilde{U}(0) + \tilde{U}(q) \right] = W_N + \frac{N-1}{N} \tilde{U}(q).$$

The first term on the left is the interaction of the $N-1$ particles in the condensate with one another, and the second is the interaction of the condensate with the extracted particle; the piece involving $\tilde{U}(q)$ is the exchange energy resulting from symmetrization of the $N$-particle wavefunction. I assume that the range of $U(r)$ is sufficiently short so that $\tilde{U}(q) \approx \tilde{U}(0)$. The energetic penalty for removing a particle from the condensate is then approximately equal to the potential energy per particle pair, $n\tilde{U}(0)$. Thus if the condensate streams past an obstacle (an external potential) at
speed $v$, scattering out of the condensate is impossible if the kinetic energy per particle is less than this energy penalty, \textit{i.e.} if the relative velocity
\[ v < \sqrt{2nU(0)/m} \equiv v_{\text{crit}}(n). \] 
Similarly, when two condensates of density $n_{1,2}$ and momenta $p_{1,2}$ stream through one another, dissipation occurs only if $|p_1 - p_2| \geq \sqrt{2mv_{\text{crit}}(n_1 + n_2)}$. This is not to say that the two streams do not interact, but rather that they interact only through the mean-field energy $n_1n_2U(0)$ per unit volume. Inside a core supported mainly by the repulsive interaction [eq. (3)], $v_{\text{crit}}$ is comparable to the virial velocity.

It will be interesting to study whether superfluid dark matter would have any distinctive consequences for galactic dynamics other than the minimum core size. Attention naturally focuses on dissipative processes, such as dynamical friction: \textit{i.e.}, irreversible transfer of energy and momentum between the dark and baryonic matter \textit{via} their gravitational interaction (cf. Binney & Tremaine 1987). In collisionless systems, dynamical friction involves upon single-particle resonances (e.g. Tremaine & Weinberg 1984), much like Landau damping in plasmas. As long as the condition (5) is satisfied, however, a perturbing gravitational potential interacts coherently with the condensate, and all particles have the same resonant frequencies because they share a common macroscopic wavefunction. Thus for example, a rotating galactic bar may experience little drag against the dark matter; this may circumvent an important argument against dense dark halos in barred spirals (Debattista & Sellwood 1998). The question will require a quantitative analysis, however, because even in the innermost parts parts of the galaxy, not all of the dark matter will be in the condensate.

3. Relativistic era

Dark matter with the properties described in §2 arise as quanta of a self-interacting relativistic scalar field $\phi$ with lagrangian density
\[ \mathcal{L} = -\sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + V(\phi) \right). \] 
Without loss of generality, the minimum of $V(\phi)$ occurs at $\phi = 0$, and $V(\phi) = \frac{m^2}{2}\phi^2/2 + (\text{higher powers})$. Potentials of the form
\[ V(\phi) = \frac{1}{2}m^2\phi^2 + \kappa\phi^4 \] 
are of particular interest, though one might want to add a constant $V(0) = \Lambda/8\pi G$ to produce a present-day cosmological constant. In lowest-order perturbation theory, the interaction energy of a state $|\Psi_N(0)\rangle$ consisting of $N$ quanta at rest in volume $V$ is, in Minkowski space,
\[ \int_{V} d^3 r \langle \Psi_N(0) | : \kappa\phi^4(r, t) : | \Psi_N(0) \rangle = \frac{6\kappa}{(2m)^2} \frac{N(N - 1)}{V}, \]
and therefore \( \tilde{U}(0) = 3\kappa/m^2 \).

Semiclassical methods give the same result, which is important because they are not restricted to perturbation theory. Thus if \( \phi \) were a spatially uniform classical field, then \( \mathcal{L} \) could be regarded as the lagrangian of a one-dimensional oscillator with an explicit time dependence via the metric \( g_{\mu\nu} \rightarrow \text{diag}(-1, a^2, a^2, a^2) \) in an Einstein-de Sitter universe. The momentum conjugate to \( \phi \) is \( \varpi = a^3 \dot{\phi} \), the hamiltonian is

\[
\mathcal{H} = \frac{\varpi^2}{2a^3} + a^3V(\phi),
\]

and the action in the oscillator is given by an integral over one complete cycle:

\[
\mathcal{I} = \frac{1}{2\pi} \oint \varpi d\phi = \frac{a^3}{\pi \sqrt{2}} \oint \sqrt{a^{-3}\mathcal{H} - V(\phi)} \, d\phi.
\]

(Semiclassically, \( \mathcal{I} \) becomes the number of quanta per comoving volume, \( na^3 \), while \( \mathcal{H} \) becomes the energy per comoving volume, \( \rho a^3 \). (In this section, \( \rho \) will be the total energy density, not the rest-mass density \( m \) of \( \S \).) The definition (9) makes sense only when the oscillation frequency \( \omega = (\partial \mathcal{H}/\partial \mathcal{I})_a \) is much larger than the current Hubble expansion rate \( \dot{a}/a \), in which case \( \mathcal{I} \) is an adiabatic invariant and hence \( na^3 \) is conserved. By direct expansion of the quadrature (9) to first order in \( \kappa \) and inversion of series, one has

\[
\mathcal{H} = m\mathcal{I} + \frac{3\kappa}{2m^2a^3} \mathcal{I}^2 + O(\kappa^2\mathcal{I}^3)
\]

hence

\[
\rho = mn + \frac{3\kappa}{2m^2} n^2 + O(\kappa^2 n^3),
\]

in agreement with previous results for the nonrelativistic (small-\( n \)) regime. In the opposite limit of large \( n \), the quadrature (9) is dominated by \( \phi \gg m/\sqrt{\kappa} \); neglecting the mass term in \( V(\phi) \), one has

\[
\rho \approx 3^{4/3}\pi^2 \Gamma^{-8/3}(1/4) \kappa^{1/3} n^{4/3} \approx 1.377 \kappa^{1/3} n^{4/3},
\]

as if this were a noninteracting relativistic gas: \( p = -\partial(\rho V)/\partial V = \rho/3 \). Eq. (9) can be evaluated to an exact expression for \( n(\rho) \) in terms of complete elliptic integrals.

We are now in a position to estimate the mass \( m \) and average number density \( \bar{n}(z) \) of these quanta. From eqs. (3) & (10), it follows that the minimum core radius \( r_{c,\text{min}} = 3a \) depends only upon \( m^4/\kappa \) and fundamental constants, so

\[
m\text{c}^2 = \left( \frac{27\hbar^3 c^3 \kappa}{4\pi G r_{c,\text{min}}^2} \right)^{1/4} \approx 10.7 \, \kappa^{1/4} r_{c,\text{pc}}^{-1/2} \text{eV}.
\]

Apart from the dimensionless coupling \( \kappa \), this is the geometric mean of the Planck mass and the mass whose Compton wavelength is \( 2\pi r_{c,\text{min}} \). Furthermore, if this form of dark matter dominates the mass density today, then

\[
\bar{n}(z) = \frac{\Omega_{\text{crit}}}{m}(1 + z) \approx 74. \, \kappa^{-1/4} r_{c,\text{pc}}^{-1/2} \Omega_{0.3} h_{50}^2 (1 + z)^3 \text{cm}^{-3}.
\]
Prior to the redshift (4) when particles followed the relativistic equation of state (11), they would have contributed a constant fraction of the total energy density, equivalent to an increase

$$\Delta N_\nu \approx 0.14 \frac{r_{c,\text{kpc}}^2}{\mathcal{H}_0^2} (\Omega_0 h_0^2)^{4/3}$$

in the number of effectively massless neutrinos (assuming $N_\nu \approx 3$), which is compatible with the constraints from primordial nucleosynthesis (Olive & Thomas 1999).

### 4. DISCUSSION

We have seen that small-scale structure can be suppressed even if the dark matter is completely cold and bosonic, provided that it has a repulsive interaction. At first blush, the idea seems less natural than the alternatives—warm or degenerate fermionic dark matter—which have been much more widely discussed. I am not aware of a strong particle-physics motivation for matter with these properties.

Nevertheless, in working through the consequences of the basic idea, one is intrigued by some satisfying coincidences.

(i) From a nonrelativistic viewpoint ($\S 2$), the equation of state (2) results from a generic two-body interaction of finite range among massive bosons; relativistically, it emerges from the simplest nonlinear field theory (6)-(7).

(ii) The nonrelativistic equation of state implies a characteristic lengthscale and a minimum dark-halo core radius. If this lengthscale is of order a kiloparsec, as the observations suggest (Hogan & Dalcanton 2000), then the dark matter began to be nonrelativistic at the lowest possible redshift that growth of structure would permit, viz. $z_{\text{eq}} \sim z_{\text{rel}}$. The result (14) that the energy density in these hypothetical quanta would have been comparable to the energy density in photons at early times is really the same coincidence. Both are independent of $\kappa$ and $m$, because the relevant combination of these quantities is already fixed by $r_{c,\text{min}}$ & $\Omega$.

With regard to the second point, Peebles (2000) has estimated that the model may be a little too successful at suppressing small-scale power during the linear regime. Density fluctuations that come within the horizon before $z_{\text{eq}}$ not only do not grow, but actually decay, until their physical size is larger than the “Jeans length” $a$. He finds that this constraint is marginally inconsistent with the quartic model unless $r_{c,\text{min}} \leq 0.5$ kpc. Pending more precise observations, however, one may be impressed that the model marginally survives this test without appeal to an adjustable parameter.¹

¹Peebles notes that slightly sub-quartic potentials $V(\phi) = m^2 \phi^2/2 + \kappa |\phi|^q$ with $q \approx 3.7$ would fit this constraint more comfortably. But one would have to sacrifice (i).
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