Hydrodynamic limit for interacting neurons

A. De Masi, A. Galves, E. Löcherbach and E. Presutti

Università degli Studi dell’Aquila, Universidade de São Paulo,
Université de Cergy-Pontoise and GSSI, L’Aquila

January 16, 2014

Abstract

This paper studies the hydrodynamic limit of a stochastic process describing the
time evolution of a system with $N$ neurons with mean-field interactions produced both
by chemical and by electrical synapses. This system can be informally described as
follows. Each neuron spikes randomly following a point process with rate depending
on its membrane potential. At its spiking time, the membrane potential of the spiking
neuron is reset to the value 0 and, simultaneously, the membrane potentials of the
other neurons are increased by an amount of energy $\frac{1}{N}$. This mimics the effect of
chemical synapses. Additionally, the effect of electrical synapses is represented by a
deterministic drift of all the membrane potentials towards the average value of the
system.

We show that, as the system size $N$ diverges, the distribution of membrane po-
tentials becomes deterministic and is described by a limit density which obeys a non
linear PDE which is a conservation law of hyperbolic type.

Key words: Hydrodynamic limit, Piecewise deterministic Markov process, Biological neu-
ral nets.

AMS Classification: 60F17; 60K35; 60J25

1 Introduction

This paper studies the hydrodynamic limit of a stochastic system describing interacting
neurons. The model we consider is inspired by a recent work of Galves and Löcherbach
(2013) where a new class of biological neuronal systems having a huge number of compo-
nents has been introduced.

The system we consider is made of $N$ neurons whose state is specified by $U^N(t) =
(U^N_1(t),\ldots,U^N_N(t))$, $t \geq 0$, $U^N(t) \in \mathbb{R}_+^N$, for some fixed integer $N \geq 1$. Each $U^N_i(t)$
models the membrane potential of neuron $i$ at time $t$, for $i = 1,\ldots,N$. Neurons interact
either by chemical or by electrical synapses. Our model does not consider external stimuli.

*e-mail addresses: anna.demasi@gmail.com, galves@usp.br, eva.loecherbach@u-cery.fr and
errico.presutti@gmail.com
Chemical synapses can be described as follows. Each neuron spikes randomly following a point process with rate depending on the membrane potential of the neuron. At its spiking time, the membrane potential of the spiking neuron is reset to an equilibrium potential 0. At the same time, simultaneously, the other neurons, which do not spike, receive an additional energy $\frac{1}{N}$ which is added to their membrane potential.

Electrical synapses occur through gap-junctions which allow neurons in the brain to communicate with one another. This induces an attraction between the values of the membrane potentials of each other and, as a consequence, a drift of the system towards its center of mass.

While the interaction among neurons are very complex and of different type (excitatory or inhibitory) our simple model only takes into account their average effect. This is a mean field type assumption. For the chemical synapses this is translated into the fact that when a neuron spikes the membrane potential of any other neuron increases by $1/N$. For the electrical synapses also the mean field type assumption implies that the drift felt by each neuron potential is described by a linear attraction towards the center of mass of the system.

Our model is an example of the class of processes introduced by Davis (1984) under the name of piecewise deterministic Markov processes. Processes in this class combine a deterministic continuous motion (in our case, due to the electrical synapses) with discontinuous, random jump events (in our case, the spike events). This is not the first time that piecewise deterministic Markov processes are used in the modelization of neuronal systems, see for instance the paper by Riedler, Thieullen and Wainrib (2012) in which processes of this type appear, however in a different context.

We regard the state of the neurons $U^N(t) = (U^N_1(t), \ldots, U^N_N(t))$ as a distribution of $1/N$ valued Dirac masses placed at the positions $U^N_1(t), \ldots, U^N_N(t)$. The main result of the present paper, presented in Theorem 2, is that in the limit as $N \to \infty$ this mass distribution becomes deterministic and it is described by a density $\rho_t(r)$. More precisely, in the limit, for any interval $I \subset \mathbb{R}_+$, $\int_I \rho_t(r) dr$ is the limit fraction of neurons whose membrane potentials are in $I$ at time $t$. The limit density $\rho_t(r)$ is proved to obey a nonlinear PDE which is a conservation law of hyperbolic type.

The usual approach to prove hydrodynamic limits in mean field systems is to show that propagation of chaos holds. In our case this amounts to prove that the membrane potentials $U^N_i(t)$ and $U^N_j(t)$ of any pair $i$ and $j$ of neurons get uncorrelated as $N \to \infty$. However, at each time that another neuron fires, it instantaneously affects both $U^N_i$ and $U^N_j$ by changing them with an additional amount $1/N$. Thus $U^N_i$ and $U^N_j$ are correlated, and propagation of chaos comes only by proving first that the firing activity of the other neurons – by propagation of chaos – is essentially deterministic. We are thus caught in a circular argument and it is not clear a priori that propagation of chaos holds. It is for this reason that in this paper we introduce an auxiliary process $Y^{(\delta)}$ which is a good approximation of the true process in the $N \to \infty$ limit, and for which it is easy to prove the hydrodynamic limit. Once the convergence for $Y^{(\delta)}$ is proved, we can then conclude by letting $\delta \to 0$.

Our paper is organized as follows. In Section 2 we introduce the process and state the main results, Theorem 1 and Theorem 2. Theorem 1 guarantees the existence of the process and gives upper bounds on the values of the potentials $U^N$ which are uniform in $N$. Theorem 2 gives existence and properties of the hydrodynamic limit.
Proofs are organized as follows: we first study the system under very restrictive assumptions on the firing rate \( f \), in such a case the proof of Theorem 1 becomes trivial and is given in Section 3. Even with such an assumption on \( f \) the proof of Theorem 2 remains rather complex. In Section 4 tightness of the sequence of processes indexed by \( N \) is proved. Section 5 introduces the sequence of auxiliary processes, and Section 6 proves the hydrodynamic limit theorem for this sequence. Section 7 concludes the proof of Theorem 2. In Section 8 we compare our model with others present in the literature. In the Appendix we extend the result to general firing rates \( f \). The main point is the proof of Theorem 1 which is given in Section A.3, together with some upper bounds for the maximal membrane potential of the process in the case of unbounded firing rate functions.

2 Model definition and main results

We consider a Markov process

\[ U^N(t) = (U_1^N(t), \ldots, U_N^N(t)), \ t \geq 0, \]

taking values in \( \mathbb{R}_+^N \), for some fixed integer \( N \geq 1 \), whose generator (see Theorem 1) is given for any smooth test function \( \varphi : \mathbb{R}_+^N \to \mathbb{R} \) by

\[
L\varphi(x) = \sum_{i=1}^{N} f(x_i) [\varphi(x + \Delta_i(x)) - \varphi(x)] - \lambda \sum_{i=1}^{N} \left( \frac{\partial \varphi}{\partial x_i}(x_i - \bar{x}) \right),
\]

where

\[
(\Delta_i(x))_j = \begin{cases} \frac{1}{N} & j \neq i \\ -x_i & j = i, \end{cases}
\]

\( \lambda \geq 0 \) a positive parameter modeling the strength of the gap junctions, \( \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \) and

**Assumption 1** \( f \in C^1(\mathbb{R}_+, \mathbb{R}_+) \) is strictly positive for \( x > 0 \) and non-decreasing.

In (2.1), the first term describes random jumps at rate \( f(x_i) \) due to spiking of neurons having potential \( x_i \). The function \( f \) is therefore called firing rate or spiking rate of the system. The second term, due to electrical synapses (gap junctions) describes the deterministic time evolution tending to attract the neurons to the common center of mass.

Our first theorem proves the existence of the process and gives some a priori estimates on the maximal membrane potential. In order to state these results, we have to introduce the following notation. Let \( N_i(t), t \geq 0, \) be the simple point process on \( \mathbb{R}_+ \) which counts the jump events of neuron \( i \) up to time \( t \) and let

\[
N(t) = \sum_{i=1}^{N} N_i(t)
\]

be the total number of jumps seen before time \( t \). For any \( x \in \mathbb{R}^N \), we define \( \| x \| = \max_{i=1,\ldots,N} x_i \). In this way,

\[
\| U^N(t) \| = \max_{i=1,\ldots,N} U_i^N(t) \quad \text{and} \quad \| N(t) \| = \max_{i=1,\ldots,N} N_i(t).
\]
Theorem 1 Let $f$ be a firing rate function satisfying Assumption 1.

1. For any $N \geq 1$ and any $x \in \mathbb{R}_+^N$ there exists a unique strong Markov process $U_N(t)$ taking values in $\mathbb{R}_+^N$ starting from $x$ whose generator is given by (2.1).

2. Denote by $P_{x}(N,\lambda)$ the probability law under which the process $U_N(t)$ starts from the initial configuration $U_N(0) = x = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$. Then for any $A > 0$ and $T > 0$ there exists $B$ such that
   \[
   \sup_{x: \|x\| \leq A} P_{x}(N,\lambda)\left[ \sup_{t \leq T} \|U_N(t)\| < B \right] \geq 1 - ce^{-CN},
   \]  
   where $c$ and $C$ are suitable constants.

The proof of Theorem 1 is given in the Appendix A.

We now give the main result of this paper. It shows that the process converges in the hydrodynamic limit, as $N \to \infty$, to a specified evolution which will be defined below.

Since the space where $U_N(t)$ takes values changes with $N$ it is convenient to identify configurations $U_N(t)$ with the associated empirical measure in the following way. Let $M$ be the space of all probability measures on $\mathbb{R}_+$. To any $x = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$ we associate the element of $M$ given by
   \[
   \mu_x := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i},
   \]  
   $\mu_x$ has the nice physical-biological interpretation of being the distribution of energies of the neurons.

We suppose that for all $N$, $U_N(0) = x^{(N)}$ and that the corresponding measure $\mu_x^{(N)}$ satisfies the following assumption.

Assumption 2
   \[
   \mu_x^{(N)} \rightharpoonup \psi_0(x)dx \quad \text{as } N \to \infty.
   \]  
   Here, $\psi_0$ is a smooth probability density on $\mathbb{R}_+$ with compact support $[0, R_0]$ such that the following properties are verified.

1. $\psi_0(0) = \frac{p_0}{p_0 + \bar{\lambda} \psi_0}$, where $p_0 = \int_0^\infty f(x) \psi_0(x)dx$ and $\bar{\psi}_0 = \int_0^\infty x \psi_0(x)dx$.

2. $\psi_0 > 0$ on $[0, R_0[$.

3. $\psi \equiv 0$ on $[R_0, \infty[$.

4. $\psi_0(x) \geq c(x - R_0)^2$, $c > 0$, in a left neighborhood of $R_0$.

We will eventually extend the definition of $\psi_0$ to the whole line by putting $\psi_0(x) = \psi_0(0)$ for all $x < 0$.

Remark 1 The above assumption is satisfied if we choose $x_1^N, \ldots, x_N^N \in \mathbb{R}_+$ i.i.d. according to $\psi_0(x)dx$. Condition 3. above is essential to our approach, since we will make repetitive use of the a priori energy estimate (2.4). Condition 4. could be relaxed to other rates of decay to 0 near $R_0$. Condition 1. implies conservation of mass of the system, in the sense that no particle dies, see also Remark 3 below.
Identifying $U^N(t)$ with the associated probability measure $\mu_{U^N(t)}$, we may identify the process with the element $\mathbb{R}_+ \ni t \to \mu_{U^N(t)}$ of the Skorokhod space $D(\mathbb{R}_+, S')$, where $S$ is the Schwartz space of all smooth functions $\psi: \mathbb{R} \to \mathbb{R}$. We write $\mu_{U^N_{[0,T]}}$ for the restriction of this process to $[0,T]$ which is an element of $D([0,T], S')$.

The following theorem is our main result.

**Theorem 2** Suppose that for all $N$ large enough, the initial configuration $x^{(N)}$ satisfies Assumption 2 above and Assumption 4 stated in Section 6 below. Then there exists a function $\rho_t(x) \geq 0$, $t \geq 0$, which is differentiable in $t \in \mathbb{R}_+$ and in $x \in \mathbb{R}_+$ such that for any fixed $T > 0$,

$$\mathcal{L}(\mu_{U^N_{[0,T]}}) \overset{w}{\to} \mathcal{P}_{[0,T]},$$

(weak convergence in $D([0,T], S')$) as $N \to \infty$, where $\mathcal{P}_{[0,T]}$ is the law on $D([0,T], S')$ supported by the distribution valued trajectory $\omega_t$ given by

$$\omega_t(\phi) = \int_0^\infty \phi(x) \rho_t(x) dx, \quad t \in [0,T].$$

Here, $\rho_t(x)$ is the unique solution of

$$\frac{\partial \rho_t(x)}{\partial t} = [\lambda x - \lambda \bar{\rho}_t - p_t] \frac{\partial \rho_t(x)}{\partial x} - [f(x) - \lambda] \rho_t(x), \quad x > 0, \quad (2.7)$$

$$\rho_0(x) = \psi_0(x), \quad \text{for all } x \geq 0, \quad \rho_t(0) = \frac{p_t}{p_t + \lambda \bar{\rho}_t} \quad \text{for all } t \geq 0 \quad (2.8)$$

which is of compact support and which satisfies

$$\int_0^\infty f(x) \rho_t(x) dx = p_t \quad \text{and} \quad \int_0^\infty x \rho_t(x) = \bar{\rho}_t \quad \text{for all } t \geq 0, \quad (2.9)$$

such that for all $x$ and $t$ in $\mathbb{R}_+$, $\rho_t(x) \geq 0$ and $\int_0^\infty \rho_t(x) dx = 1$. We have an explicit expression for $\rho_t(x)$. For all $x \geq 0$

$$\rho_t(x) = \psi_0 \left( e^{\lambda t} x - \lambda \int_0^t e^{\lambda s} \bar{\rho}_s ds - \int_0^t e^{\lambda s} p_s ds \right) \exp \left\{ - \int_0^t \left[ f \left( e^{\lambda (t-s)} x - \lambda \int_s^t e^{\lambda r} \bar{\rho}_r dr - \int_s^t e^{\lambda r} p_r dr \right) - \lambda \right] ds \right\}, \quad (2.10)$$

where we define $f(\cdot) = 0$ on $\mathbb{R}_-$.

To separate the difficulties we shall first prove Theorem 1 and Theorem 2 under a very restrictive assumption on $f$:

**Assumption 3** $f$ is a positive $C^1$–function which is non-decreasing, Lipschitz continuous, bounded and constant for all $x \geq x^{**}$ for some $x^{**} > 0$. We shall denote by $f^* = \|f\|_\infty$ the sup norm of $f$. 

---

1. Theorem
2. Theorem
3. Assumption
The proof of Theorem 1 under Assumption 3 is easy, it is given in the next section. In the successive sections we shall prove Theorem 2 under Assumption 3. In the Appendix we shall prove Theorem 1 in its original formulation (i.e. dropping Assumption 3) and then Theorem 2. However this last step is trivial because the estimate (2.4) implies that with probability going to 1 as \( N \to \infty \) all the energies are uniformly bounded in the time interval \([0, T]\) that we are considering. It is then possible to replace the true \( f \) with one satisfying Assumption 3 and which differs only for energies larger than those reached by the true process, so that we can use what was already proved under Assumption 3. The precise argument is given at the end of the Appendix.

### 3 Energy bounds under Assumption 3

Exploiting Assumption 3 we shall prove a statement stronger than in Theorem 1.

**Proposition 1** Let \( f \) satisfy Assumption 3 and call \( f^* = \|f\|_\infty \).

1. For any \( N \geq 1 \) and any \( x \in \mathbb{R}^N_+ \) there exists a unique strong Markov process \( U^N(t) \) starting from \( x \) taking values in \( \mathbb{R}^N_+ \) whose generator is given by (2.1).

2. Calling \( N(t) \) the total number of fires in the time interval \([0, t]\) we have

   \[
   N(t) \leq N^*(t) \quad \text{stochastically} \tag{3.11}
   \]

   where \( N^*(t) \) is a Poisson process with intensity \( Nf^* \), \( f^* = \|f\|_\infty \).

3. \( \sup_{t \leq T} \|U^N(t)\| \leq \|U^N(0)\| + \frac{N(t)}{N} \) and for any \( T > 0 \) there exist positive constants \( c \) and \( C \) such that for any \( N \) and any \( U^N(0) \):

   \[
   P_{U^N(0)}^{(N, \lambda)} \left[ \sup_{t \leq T} \|U^N(t)\| \leq \|U^N(0)\| + 2f^*T \right] \geq 1 - ce^{-CTN}. \tag{3.12}
   \]

**Proof** The existence of the process for each fixed \( N \) is now trivial as the firing rates are bounded. The variable \( N(t) \) is stochastically upper bounded by \( N^*(t) := \sum_{i=1}^N n_i(t) \), where \( (n_i(t)) \) are i.i.d. Poisson processes of intensity \( f^* \). \( N^*(t) \) is therefore a Poisson process with intensity \( Nf^* \). We have

\[
\sup_{t \leq T} \|U^N(t)\| \leq \|U^N(0)\| + \frac{N(t)}{N},
\]

because each firing event increases the rightmost particle by \( \frac{1}{N} \), while, in between firing events, the rightmost particle is attracted to the center of mass of the process and thus decreases. (3.12) then follows from item 2, because \( \{N(T) \geq B\} \) is an increasing event and thus the bound is reduced to large deviations for a Poisson variable, details are omitted.
4 Tightness

With this section we begin the proof of Theorem 2 (under Assumption 3). We start by proving tightness of the sequence of laws of \( \mu_{U_{[0,T]}} \).

**Proposition 2** Suppose that \( U^N(0) = x^{(N)} \) is such that Assumption 2 is verified. Then the sequence of laws of \( \mu_{U_{[0,T]}} \) is tight in \( D(\mathbb{R}_+, \mathcal{S}) \).

**Proof** For any test function \( \psi \in \mathcal{S} \) and all \( t \in [0, T] \), we write, by an abuse of notation,

\[
\langle U^N(t), \psi \rangle := \frac{1}{N} \sum_{i} \psi(U^N_i(t)) = \int \psi(x) \mu_{U^N(t)}(dx). \tag{4.13}
\]

By Mitoma 1983 it is sufficient to prove the tightness of \( \langle U^N(t), \psi \rangle, t \in [0, T] \in D([0, T], \mathbb{R}) \) for any fixed \( \psi \in \mathcal{S} \). In order to do so, we shall use a well known tightness criterion, see for instance Theorem 2.6.2 of De Masi and Presutti 1991, which requires that the \( L^2 \) norms of the “compensators” of \( \langle U^N(t), \psi \rangle \) are finite. The compensators are

\[
\gamma_1^N(t) = L\langle U^N(t), \psi \rangle, \quad \gamma_2^N(t) = L\langle U^N(t), \psi \rangle^2 - 2\langle U^N(t), \psi \rangle L\langle U^N(t), \psi \rangle; \tag{4.14}
\]

\( L \) the generator given by (2.11). The criterion requires that there exists a constant \( c \) so that

\[
\sup_{t \leq T} E[\gamma_1^N(t)]^2 \leq c, \quad \sup_{t \leq T} E[\gamma_2^N(t)]^2 \leq c. \tag{4.15}
\]

The proof of the criterion is based on the fact that

\[
\langle U^N(t), \psi \rangle - \int_{0}^{t} \gamma_1^N(s)ds =: M^N_t \quad \text{and} \quad (M^N_t)^2 - \int_{0}^{t} \gamma_2^N(s)ds =: N^N_t
\]

are martingales.

To prove (4.15) we start by calculating \( \gamma_1^N(t) = \frac{1}{N} \sum_{i} L\psi(U^N_i(t)) \).

\[
\gamma_1^N(t) = \frac{1}{N} \sum_{i} \left[ \sum_{j \neq i} f(U^N_j(t))[\psi(U^N_i(t) + \frac{1}{N}) - \psi(U^N_i(t))] + f(U^N_i(t))[\psi(0) - \psi(U^N_i(t))]) \right] + \frac{\lambda}{N} \sum_{i} \psi'(U^N_i(t))[\bar{U}(t) - U^N_i(t)],
\]

where \( \bar{U}(t) = \langle U^N(t), id \rangle \) is the average of the \( U^N_i(t) \). Expanding the discrete derivative, we get

\[
\gamma_1^N(t) = \langle U^N(t), f \rangle \langle U^N(t), \psi' \rangle - \langle U^N(t), \psi \rangle + \psi(0)\langle U^N(t), f \rangle
\]

\[
+ \lambda \left[ \langle U^N(t), \psi' \rangle \langle U^N(t), id \rangle - \langle U^N(t), \phi \rangle \right] + O(\frac{1}{N}),
\]

where \( \phi(x) = x\psi'(x) \) and

\[
O(\frac{1}{N}) := \frac{1}{N} \sum_{i} \left[ \sum_{j \neq i} f(U^N_j(t))[\psi(U^N_i(t) + \frac{1}{N}) - \psi(U^N_i(t))] - \frac{1}{N}\psi'(U^N_i(t)) \right].
\]
Since \( \psi, \psi' \) and \( \psi'' \) are bounded as well as \( f \) (thanks to Assumption 3) there is a constant \( c \) so that
\[
|\gamma^N_1(t)| \leq c \left( 1 + \langle U^N(t), \text{id} \rangle + |\langle U^N(t), \phi \rangle| \right) \leq c' \left( 1 + \frac{1}{N} \sum_i U^N_i(t)^2 \right).
\]

By Proposition 1, \( \sup_{t \leq T} E[|\gamma^N_1(t)|^2] \leq c \) for a constant \( c \) not depending on \( N \).

The proof of (4.15) for \( \gamma^N_2(t) \) is simpler. We write \( L = L_{\text{fire}} + L_\lambda \), where \( L_{\text{fire}} \phi \) and \( L_\lambda \phi \) are given by the first respectively second term on the right hand side of (2.1). Since \( L_\lambda \) acts as a derivative we have
\[
L_\lambda \langle U^N(t), \psi \rangle^2 - 2 \langle U^N(t), \psi \rangle L_\lambda \langle U^N(t), \psi \rangle = 0
\]
as it can be easily checked. We have
\[
\frac{1}{N^2} \sum_{i,j} L_{\text{fire}}(\psi(U^N_i(t))\psi(U^N_j(t)) =
\]
\[
= \frac{1}{N^2} \sum_{i \neq j} \left[ \sum_{k \neq i,j} f(U^N_k(t))\psi(U^N_i(t) + \frac{1}{N})\psi(U^N_j(t) + \frac{1}{N}) - \psi(U^N_i(t))\psi(U^N_j(t)) \right]
\]
\[
+ f(U^N_i(t))\psi(0)\psi(U^N_j(t) + \frac{1}{N}) - \psi(U^N_i(t))\psi(U^N_j(t))
\]
\[
+ f(U^N_i(t))\psi(0)\psi(U^N_j(t) + \frac{1}{N}) - \psi(U^N_i(t))\psi(U^N_j(t))
\]
\[
+ \frac{2}{N^2} \sum_i \left[ \sum_{k \neq i} f(U^N_k(t))\psi^2(U^N_i(t) + \frac{1}{N}) - \psi^2(U^N_i(t)) \right] + f(U^N_i(t))\psi^2(0) - \psi^2(U^N_i(t)) \right].
\]

The same arguments used earlier show that the \( L^2 \)-norm of this term is bounded uniformly in \( t \in [0,T] \) and in \( N \). The \( L^2 \)-norm of \(-2(U^N(t), \psi)L_{\text{fire}}\langle U^N(t), \psi \rangle \) is also bounded uniformly because \( |\langle U^N(t), \psi \rangle| \leq c \) and we have already proved the bound for \( L_{\text{fire}}\langle U^N(t), \psi \rangle \).

We have thus proved (4.15) and finished the proof. Observe that taking into account the signs we could prove that \( \gamma^N_2(t) \to 0 \) as \( N \to \infty \). 

\[\blacksquare\]

5 Coupling the true with an auxiliary process

The natural step after tightness is to prove propagation of chaos. This is however not so simple in our model because the firing of a neuron (i.e. when its energy jumps) affects simultaneously the state of the other neurons and not just their jumping rates, as usual in mean field models. For this reason the traditional approach to mean field systems fails here (as far as we can see) and we need a different strategy. We thus introduce an auxiliary process which is from one side a good approximation of the true one in the \( N \to \infty \) limit, and which, from the other side, is easy to handle in the same limit. The auxiliary process is defined in the present section where we prove that it is close to the true process uniformly in \( N \). In Section 6 we study the hydrodynamic limit for the approximating process. Section 7 will then conclude the proof of Theorem 2.
5.1 The auxiliary process

Since $N$ is fixed we shall drop the superscript $N$ from $U^N(t)$ unless ambiguities may arise. The auxiliary processes are labeled by $\delta > 0$, $\delta$ represents a time mesh. In fact we discretize the time axis $\mathbb{R}_+$ and the auxiliary process $Y^{(\delta)}(n\delta) = y = (y_1, \ldots, y_N)$ is defined at the discrete times $n\delta$, $n \in \mathbb{N}$. $(Y^{(\delta)}(n\delta))_{n \in \mathbb{N}}$ is a Markov chain. Its transition probability describes a process where neurons fire with constant firing rate $f(y_i)$ in the time interval $[n\delta, (n+1)\delta[$, all firing events after the first one are however suppressed; the final position at the end of the time interval is obtained by first letting the neurons evolve (for a time $\delta$) under the action of the gap-junction interaction and then taking into account the effect of the firings, the precise definition is given next.

We proceed by induction on $n$. Conditionally on $Y^{(\delta)}(n\delta) = y = (y_1, \ldots, y_N)$, we choose $N$ independent exponential random variables $\tau_1, \ldots, \tau_N$, which are independent of anything else, having intensities $f(y_i), i = 1, \ldots, N$, respectively. We put

$$\Phi_i(n) = 1\{\tau_i \leq \delta\}, 1 \leq i \leq N, \quad q = \frac{1}{N} \sum_{i=1}^{N} \Phi_i(n). \quad (5.16)$$

We write

$$\varphi_{\bar{y}, \delta}(y_i) = e^{-\lambda t} y_i + (1 - e^{-\lambda t}) \bar{y}, \quad 0 \leq t \leq \delta, \quad \bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i \quad (5.17)$$

for the deterministic flow attracting position $y_i$ to the center of mass $\bar{y}$ and set

$$Y^\delta_i((n+1)\delta) = \varphi_{\bar{y}, \delta}(y_i) + q, \quad \text{for all } i \text{ such that } \Phi_i(n) = 0. \quad (5.18)$$

Thus we keep constant the firing intensity of the particles. Neurons which do not fire follow the deterministic flow. Moreover, we suppose that they feel the additional factor $q$, generated by spiking of other neurons, only at the end of the interval $[n\delta, (n+1)\delta[$.

Let us now describe the evolution of the $Nq$ neurons that fire. Let $i_1, \ldots, i_{Nq}$ be the particle labels such that $\Phi_{i_j}(n) = 1, j = 1, \ldots, Nq$, ordered in such a way that $\tau_{i_j} > \tau_{i_{j+1}}$.

We then assign the position

$$Y^\delta_{i_{Nq}}((n+1)\delta) = \varphi_{\bar{y}, \delta}(0) + (q - \frac{1}{N}) = (1 - e^{-\lambda \delta}) \bar{y} + (q - \frac{1}{N}) \quad (5.19)$$

to the first particle which has fired. It is the position of a particle starting from 0 at time $n\delta$, evolving according to the flow and receiving an additional factor $q - \frac{1}{N}$ at time $(n+1)\delta$, due to the influence of the other spiking neurons (whose number is $Nq - 1$).

The remaining $Nq - 1$ particles that spike are distributed uniformly in the following manner. We put

$$d_n = \frac{\varphi_{\bar{y}, \delta}(0) + (q - \frac{1}{N})}{Nq - 1}, \quad \text{if } Nq - 1 > 0, \quad d_n = \varphi_{\bar{y}, \delta}(0), \quad \text{if } Nq - 1 = 0, \quad (5.20)$$

and

$$Y^\delta_{i_j}((n+1)\delta) := (j-1)d_n, j = 1, \ldots, Nq - 1. \quad (5.21)$$

The analogue of Proposition 1 holds for the auxiliary process as well. The basic point is:
Proposition 3 The variables \( \sum_{i=1}^{N} \Phi_i(n) \) are stochastically bounded by Poisson variables of intensity \( N f^* \delta, \quad f^* := \| f \|_{\infty}. \)

Proof \( \sum_{i=1}^{N} \Phi_i(n) \) is stochastically upper bounded by \( \sum_{i=1}^{N} Y_{i,n} \), where the \( (Y_{i,n})_{i,n} \) are i.i.d. Bernoulli random variables, \( P(Y_{i,n} = 1) = 1 - e^{-\delta f^*} \leq \delta f^* \).

Proceeding as in the proof of Proposition 1 for any \( T \) there is \( C \) so that for any initial datum \( x \), \( Y(\delta)(0) = x \),

\[
P_x \left[ \sup_{n: n \delta \leq T} \| Y(\delta)(n\delta) \| \leq \| x \| + 2f^*T \right] \leq e^{-CNT}. \tag{5.22}
\]

5.2 Coupling the auxiliary and the true process

In order to show that \( Y^\delta \) is close to the original process, we couple the two Markov chains \( (U(n\delta))_{n\geq 0} \) and \( (Y^\delta(n\delta))_{n\geq 0} \) in an appropriate way.

The coupling

We will couple the two processes in such a way that the pair \( (U(n\delta), Y^\delta(n\delta)) \), \( n \in \mathbb{N} \), will be a Markov chain taking values in \( \mathbb{R}^{N \times N} \). For any \( n = 0, 1, \ldots \), given \( (U(n\delta), Y^\delta(n\delta)) \), the values of \( (U((n+1)\delta), Y^\delta((n+1)\delta)) \) will be chosen with the simulation algorithm given below.

The algorithm uses the following variables.

\begin{itemize}
  \item \((x, y) \in \mathbb{R}_+^N \times \mathbb{R}_+^N \) and \( \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \). The strings \( x \) and \( y \) represent the state of the neurons in the two processes and \( \bar{x} \) gives the center of mass in \( x \).
  \item Independent random times \( \tau_i^1 \in (0, +\infty), \tau_i^2 \in (0, +\infty) \) and \( \tau_i \in (0, +\infty) \), for all \( i = 1, \ldots, N \). These variables will determine the times of possible updates.
  \item \( m = (m_1, \ldots, m_N) \in \{0, 1\}^N \). The variable \( m_i \) indicates the occurrence of a spike for neuron \( i \) in the auxiliary process.
  \item \( K \in \{0, \ldots, N\} \). The variable \( K \) counts the number of spikes in the auxiliary process.
  \item \( j = (j_1, \ldots, j_N) \in \{0, 1, \ldots, N\}^N \). The variable \( j_i \) is the label of the neuron associated with the \( i \)-th occurrence of a spike in the auxiliary process.
  \item \( L \in [0, \delta] \). The variable \( L \) indicates the remaining time after every update of the variables. The simulation algorithm stops when \( L = 0 \).
\end{itemize}

The deterministic flow attracting position \( x_i \) to the center of mass \( \bar{x} \), given in (5.17), will appear in the algorithm. For convenience of the reader we recall its definition here

\[
\varphi_{\bar{x},t}(x_i) = e^{-\lambda t} x_i + (1 - e^{-\lambda t}) \bar{x}, \quad 0 \leq t \leq \delta, \quad \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i.
\]

One can check that the following algorithm is indeed a coupling of the two processes and in the remaining of this section we shall prove that this is indeed a good coupling.
Algorithm 1 Coupling algorithm

1: Input: \((U(n\delta), Y^\delta(n\delta)) \in \mathbb{R}_+^N \times \mathbb{R}_+^N\)
2: Output: \((U((n+1)\delta), Y^\delta((n+1)\delta)) \in \mathbb{R}_+^N \times \mathbb{R}_+^N\)

3: Initial values: \((x, y) \leftarrow (U(n\delta), Y^\delta(n\delta)), K \leftarrow 0, L \leftarrow \delta, m_i \leftarrow 0, \text{ for all } i = 1, \ldots, N, j_i \leftarrow 0, \text{ for all } i = 1, \ldots, N\)

4: \textbf{while } L > 0 \textbf{ do}

5: \quad For \(i = 1, \ldots, N\), choose independent random times

6: \quad \bullet \quad \tau_i^1 \in (0, +\infty) \text{ with intensities } f(\varphi_{x,L}(x_i)) \wedge f(y_i)

7: \quad \bullet \quad \tau_i^2 \in (0, +\infty) \text{ with intensities } |f(\varphi_{x,L}(x_i)) - f(y_i)|

8: \quad \bullet \quad \tau_i \in (0, +\infty) \text{ with intensities } f(\varphi_{x,L}(x_i))

9: \quad \bullet \quad R = \inf_{1 \leq i \leq N; m_i \neq 0} (\tau_i^1 \wedge \tau_i^2) \wedge \inf_{1 \leq i \leq N; m_i = 1} \tau_i.

10: \quad \textbf{if } R \geq L \textbf{ then}

11: \quad \quad Stop situation:

12: \quad \quad \(x_i \leftarrow \varphi_{x,L}(x_i)\) for all \(i = 1, \ldots, N\)

13: \quad \quad L \leftarrow 0

14: \quad \quad \(y_i \leftarrow \varphi_{y,L}(y_i) + \frac{K}{N}\) for all \(i = 1, \ldots, N\) such that \(m_i = 0\)

15: \quad \quad \(y_{j_k} \leftarrow \varphi_{y,L}(0) + \frac{K-1}{N}\), if \(K \geq 1\)

16: \quad \quad \(y_{j_k} \leftarrow (K-k)\left[\frac{\varphi_{0,L}(0)+(K-1)/N}{K-1}1_{\{K>1\}}\right]\) for all \(k = 2, \ldots, K\)

17: \quad \textbf{else if } R = \tau_i^1 < L \textbf{ then}

18: \quad \quad \(m_i \leftarrow 1, K \leftarrow K + 1, j_K \leftarrow i, L \leftarrow (L - R)\)

19: \quad \quad \(x_i \leftarrow 0 \text{ and } x_j \leftarrow \varphi_{x,R}(x_j) + \frac{1}{N}, \text{ for all } j \neq i\)

20: \quad \textbf{else if } R = \tau_i^2 < L \textbf{ then}

21: \quad \quad \textbf{if } f(y_i) > f(\varphi_{x,R}(x_i)) \textbf{ then}

22: \quad \quad \quad \(m_i \leftarrow 1, K \leftarrow K + 1, j_K \leftarrow i, L \leftarrow (L - R)\), \(x_j \leftarrow \varphi_{x,R}(x_j)\) for all \(j\)

23: \quad \quad \quad \textbf{else if } f(y_i) \leq f(\varphi_{x,R}(x_i)) \textbf{ then}

24: \quad \quad \quad \quad L \leftarrow (L - R), x_i \leftarrow 0 \text{ and } x_j \leftarrow \varphi_{x,R}(x_j) + \frac{1}{N}, \text{ for all } j \neq i\)

25: \quad \textbf{end if}

26: \quad \textbf{else if } R = \tau_i < L \textbf{ then}

27: \quad \quad \(L \leftarrow (L - R)\)

28: \quad \quad \(x_i \leftarrow 0 \text{ and } x_j \leftarrow \varphi_{x,R}(x_j) + \frac{1}{N}, \text{ for all } j \neq i\)

29: \quad \textbf{end if}

30: \textbf{end while}

31: \((U((n+1)\delta), Y^\delta((n+1)\delta)) \leftarrow (x, y)\).

32: \textbf{return } (U((n+1)\delta), Y^\delta((n+1)\delta)).
5.3 Closeness between the auxiliary and the true process

The main result in this section, Theorem 3 below, states that the auxiliary and the true processes are close to each other. This means that for most neurons the potentials in the two processes are close to each other (proportionally to $\delta$), while the remaining ones constitute a small fraction of the totality (also proportional to $\delta$).

**Definition 1** A label $i \in \{1, \ldots, N\}$ is called “good at time $k\delta$” if for all $n = 1, \ldots, k$ the following is true.

Either $\Phi_{i}(n-1) = 0$ and $U_i$ has not fired during the whole time interval $[(n-1)\delta, n\delta]$.  

Or $\Phi_{i}(n-1) = 1$ and $U_i$ has fired exactly once in the time interval $[(n-1)\delta, n\delta]$.

We call $\mathcal{G}_n$ the set of good labels at time $n\delta$ and $M_n = N - |\mathcal{G}_n|$ the cardinality of its complement. If $i \in \mathcal{G}_k$ we call $D_i(k) := |U_i(k\delta) - Y_i^\delta(k\delta)|$. Finally, we set

$$\theta_n = \max\{D_i(k), i \in \mathcal{G}_n, k \leq n\}.$$  

Then the following holds.

**Theorem 3** Under Assumption 2, for any fixed $T > 0$, there exist $\delta_0 > 0$ and a constant $C$ depending on $\|f\|_\infty$ and on $T$ such that for all $\delta \leq \delta_0$,

$$\theta_n \leq C\delta \quad \text{and} \quad \frac{M_n}{N} \leq C\delta \quad \text{for all } n \text{ such that } n\delta \leq T,$$

with probability $\geq 1 - e^{-CN\delta^2}$.

**Strategy of proof.** It is clear that $M_{n-1} \leq M_n \leq \ldots$, because there is no recovery from not being a good label. We shall first prove that till when $\theta_n \leq c\delta$ the increments $M_n - M_{n-1} \leq c'\delta^2N$. In fact a label $i$ becomes bad at $n\delta$ if in the time interval $((n - 1)\delta, n\delta)$ there are either two or more fires of $U_i(\cdot)$ (which cost $0(\delta^2)$) or else the clock $\tau_i^\delta$ (recall the algorithm given in Subsection 5.2) rings, which also costs $0(\delta^2)$. Since $n\delta \leq T$ the sum of the increments is then bounded by $c\delta$ as desired. Thus there may be order of $c\delta N$ neurons which fire quite differently in the two processes but this produces a change for the potential of the good labels of the order of $\frac{1}{N}(c\delta N)T$ which is also what claimed in the theorem. The above heuristic argument can be made rigorous as we are going to show in what follows.

5.3.1 The stopped process

A technical difficulty in the proof of Theorem 3 comes from the possible occurrence of an anomalously large number of fires in one of the time steps $[(n - 1)\delta, n\delta]$. To avoid the problem we stop the process as soon as this happens and prove the theorem for such a stopped process. We then conclude by a large deviation estimate for the probability that the process is stopped before reaching the final time $T$. 
Recalling from Proposition 1 and Proposition 3 that the number of fires in an interval 
\([(n - 1)\delta, n\delta]\) in either one of the two processes is stochastically bounded by a Poisson 
variable of intensity \(f^*\delta N\) we stop the algorithm defining the coupled process as soon as 
the number of firings in either one of the two processes exceeds \(2f^*\delta N\) in one of the time 
steps \([(n - 1)\delta, n\delta]\). We call \(E\) the event when the process is stopped before reaching the 
final time. Then uniformly in the initial datum \(Y^{(\delta)}(0) = U(0) = x,\)
\[
P_x(E) \leq 2\frac{T}{\delta}e^{-CN\delta}. \tag{5.23}
\]
In what follows, \(C\) is a constant which may change from one appearance to another.

By an abuse of notation we denote by the same symbol the stopped processes and in 
the sequel, unless otherwise stated, we refer to the stopped process. We fix arbitrarily 
\(A > 0\) and consider the process starting from \(Y^{(\delta)}(0) = U(0) = x\) with \(\|x\| \leq A\). Writing 
\(B^* := B + A + 2f^*T\), we have by (3.12) for all \(t \leq T\) and all \(n\delta \leq T\)
\[
\|U(t)\| \leq B^*, \quad \|Y^{(\delta)}(n\delta)\| \leq B^* \quad \text{for the stopped processes}. \tag{5.24}
\]
It follows that the same bounds hold for the unstopped process with probability \(\geq 1 - e^{-CN\delta}\).

Thus by restricting to the stopped process we have:

- the firing rate of each neuron is \(\leq f^*\) and the number of fires of all neurons in any 
of the steps \([(n - 1)\delta, n\delta]\) is \(\leq 2f^*\delta N\).
- The bounds (5.24) are verified and as a consequence the center of mass in the \(U\) and 
  \(Y^{(\delta)}\) processes are \(\leq B^*\) so that the gap-junction drift on each neuron is \(\leq \lambda B^*\).

### 5.3.2 Bounds on the increments of \(M_n\)

We write \(M_n = M_{n-1} + |A_n \cap G_{n-1}| + |B_n \cap G_{n-1}| \leq M_{n-1} + |A_n| + |B_n \cap G_{n-1}|\) where 
recalling the algorithm given in Subsection 5.2 and Definition 4

- \(A_n\) is the set of all labels \(i\) for which a clock associated to label \(i\) rings at least twice 
during \([(n - 1)\delta, n\delta]\).
- \(B_n\) is the set of all labels \(i\) for which a clock associated to label \(i\) rings only once 
during \([(n - 1)\delta, n\delta]\), and it is the clock \(\tau^2_i\).

We shall prove that (for the stopped processes)
\[
P\left[|A_n| > N(\delta f^*)^2\right] \leq e^{-CN\delta^2}, \tag{5.25}
\]
\[
P\left[|B_n \cap G_{n-1}| > 2CN\delta[\theta_{n-1} + \delta]\right] \leq e^{-CN\delta^2} \tag{5.26}
\]
(recall \(C\) is a constant whose value may change at each appearance).

It will then follow that with probability \(\geq 1 - e^{-CN\delta^2}\)
\[
M_n \leq M_{n-1} + N(\delta f^*)^2 + 2CN\delta[\theta_{n-1} + \delta] \leq M_{n-1} + CN\delta[\theta_{n-1} + \delta]. \tag{5.27}
\]
Iterating the upper bound and using that $n \delta \leq T$, we will then conclude that with probability $\geq 1 - 2ne^{-CN\delta^2} \geq 1 - e^{-CN\delta^2}$, where $C$ depends on $T$,

$$\frac{M_n}{N} \leq C\delta \sum_{k=1}^{n-1} \theta_k + C\delta \leq C(\theta_{n-1} + \delta)$$

for all $n \leq \frac{T}{\delta}$, (5.28) having used that, by definition, $\theta_k \leq \theta_{n-1}$.

Proof of (5.25).

$|A_n|$ is stochastically upper bounded by $S^* := \sum_{i=1}^{N} 1_{\{N^*_i \geq 2\}}$, where $N^*_1, \ldots, N^*_N$ are independent Poisson variables of parameter $f^*\delta$, $f^* = \|f\|_{\infty}$. Then, writing $p^* = P(N^*_i \geq 2)$, we have

$$e^{-\delta f^*} \frac{1}{2} \delta^2 (f^*)^2 \leq p^* \leq \frac{1}{2} (\delta f^*)^2, \quad p^* \approx \frac{1}{2} (\delta f^*)^2 \text{ as } \delta \to 0.$$ 

Thus $S^*$ is the sum of $N$ Bernoulli variables (taking values in $\{0, 1\}$), each with average $p^*$. Then by the Hoeffding’s inequality, we get (5.25).

Proof of (5.26).

We shall prove that the random variable $|B_n \cap \mathcal{G}_{n-1}|$ (for the stopped process) is stochastically upper bounded by $\sum_{i=1}^{N} 1_{\{N_i \geq 1\}}$, where $N_i, i = 1, \ldots, N$, are independent Poisson variables of parameter $C(\theta_{n-1} + \delta)$. (5.26) will then follow straightly. We shorthand $y := Y^{(\delta)}((n-1)\delta), \ x := U(((n-1)\delta), \ y(\delta)) := Y^{(\delta)}(n\delta), \ x(t) := U(((n-1)\delta + t), \ t \in [0, \delta]$, and introduce independent random times $\sigma_i, i = 1, \ldots, N$, of intensity $|f(y_i) - f(x_i(t))|$, $t \in [0, \delta]$. Then $|B_n|$ is stochastically bounded by $\sum_{i=1}^{N} 1_{\sigma_i < \delta}$ because we are neglecting some of the conditions for being in $B_n$. We also obviously have

$$|B_n \cap \mathcal{G}_{n-1}| \leq \sum_{i=1}^{N} 1_{\sigma_i < \delta, i \in \mathcal{G}_{n-1}} \text{ stochastically.}$$

To control the right hand side we bound

$$|f(x_i(t)) - f(y_i)| \leq \|f\|_{\text{Lip}}|x_i(t) - y_i|,$$

$\|f\|_{\text{Lip}}$ the Lipschitz constant of the function $f$. Denote by $N_j(s, t)$ the number of firing of $U_j(\cdot)$ in the time interval $[s, t]$, then analogously to (5.14),

$$|x_i(t) - y_i| \leq |x_i - y_i| + \int_0^t \lambda e^{-\lambda(t-s)}|\bar{x}(s) - x_i|ds + \frac{1}{N} \sum_{j \neq i} N_j([((n-1)\delta, (n-1)\delta + t)]).$$

$|\bar{x}(s) - x_i| \leq B^*$ and $\sum_{j \neq i} N_j([((n-1)\delta, (n-1)\delta + t)]) \leq 2f^*\delta N$ because we are considering the stopped process. Then if $i \in \mathcal{G}_{n-1}$,

$$|x_i(t) - y_i| \leq \theta_{n-1} + B^*\delta + 2f^*\delta$$

and therefore

$$|f(x_i(t)) - f(y_i)| \leq \|f\|_{\text{Lip}} (\theta_{n-1} + B^*\delta + 2f^*\delta) \leq C(\theta_{n-1} + \delta).$$

so that

$$\sum_{i=1}^{N} 1_{\sigma_i < \delta, i \in \mathcal{G}_{n-1}} \leq \sum_{i=1}^{N} 1_{\sigma^*_i < \delta} \text{ stochastically,}$$

where the $\sigma^*_i$ are independent times of intensity $C(\theta_{n-1} + \delta)$. 

14
5.3.3 Bounds on $\theta_n$

The final bound on $\theta_n$ is reported in \((5.38)\) at the end of this subsection. We start by characterizing the elements $i \in G_n$ as $i \in G_{n-1} \cap (C_n \cup F_n)$ where:

1. $C_n$ is the set of all labels $i$ for which a clock associated to label $i$ rings only once during $\left[(n-1)\delta, n\delta]\right]$, and it is a clock $\tau_i^1$.
2. $F_n$ is the set of all labels $i$ which do not have any jump during $\left[(n-1)\delta, n\delta]\right]$.

In other words, we study labels $i$ which are good at time $(n-1)\delta$ and which stay good at time $n\delta$ as well. We shall use in the proofs the following formula for the potential $U_i(t)$ of a neuron which does not fire in the interval $[t_0, t]$:

$$U_i(t) = e^{-\lambda(t-t_0)}U_i(t_0) + \int_{t_0}^{t} e^{-\lambda(t-s)}\{\bar{U}(s)ds + \frac{1}{N}dN(s)\},$$  \hspace{1cm} (5.29)

$N(t)$ denoting the total number of fires till time $t$. For the $Y^{(\delta)}$ process we shall instead use \((5.17)\) and the expressions thereafter.

- Labels $i \in C_n \cap G_{n-1}$.

For such labels $i$ there is a random time $t \in [(n-1)\delta, n\delta]$ at which a $\tau_i^1$ event happens. Denoting by $N(t)$ the total number of fires till time $t$, by \((5.29)\)

$$U_i(n\delta) = \int_{t}^{\delta} \lambda e^{-\lambda(s-t)}\bar{U}(s)ds + e^{-\lambda\delta}\frac{1}{N}\int_{t}^{\delta} e^{\lambda s}dN(s),$$

because $U_i(t^+) = 0$. Since we are considering the stopped process, $\bar{U}(\cdot) \leq B^*$ and $N(n\delta) - N((n-1)\delta) \leq 2f^*\delta N$ so that $U_i(n\delta) \leq C\delta$. In the same way, $Y_{i}^{(\delta)}(n\delta) \leq C\delta$, and therefore

$$D_i(n) \leq C\delta, \hspace{1cm} \text{for the stopped process.} \hspace{1cm} (5.30)$$

Notice that the bound does not depend on $D_i(n-1)$.

- Labels $i \in F_n \cap G_{n-1}$.

This means that $i$ is good at time $(n-1)\delta$ and does not jump, neither in the $U$ nor in the $Y^{(\delta)}$ process. Let $U((n-1)\delta) = x$ and $Y^{(\delta)}((n-1)\delta) = y$. By \((5.29)\) and \((5.18)\)

$$|U_i(n\delta) - Y_{i}^{(\delta)}(n\delta)| = D_i(n)$$

is bounded by

$$D_i(n) \leq e^{-\lambda\delta}|x_i - y_i| + (1 - e^{-\lambda\delta})|\bar{x} - \bar{y}| + \int_{(n-1)\delta}^{n\delta} \lambda e^{-\lambda(n\delta-t)}|U_N(t) - U_N(0)|dt$$

$$+ \frac{1}{N} \int_{(n-1)\delta}^{n\delta} e^{-\lambda(n\delta-t)}dN(t) - Nq| \hspace{1cm} (5.31)$$

where $Nq$ is the total number of fires in the process $Y^{(\delta)}$ in the step from $(n-1)\delta$ to $n\delta$.

We bound the right hand side of \((5.31)\) by setting: $e^{-\lambda\delta}|x_i - y_i| \leq \theta_{n-1}$;

$$(1 - e^{-\lambda\delta})|\bar{x} - \bar{y}| \leq \lambda\delta\left(\theta_{n-1} + B^*\frac{M_{n-1}}{N}\right), \hspace{1cm} B^* \text{ as in } (5.24),$$
\[
\int_{(n-1)\delta}^{n\delta} \lambda e^{-\lambda(n\delta-t)}|\tilde{U}_N(t) - \tilde{U}_N(0)|dt \leq \lambda \delta \frac{1}{N} N((n-1)\delta, n\delta),
\]
where \(N((n-1)\delta, n\delta) := N(n\delta) - N((n-1)\delta).\) Writing

\[
\int_{(n-1)\delta}^{n\delta} e^{-\lambda(n\delta-t)} dN(t) = N((n-1)\delta, n\delta) + \int_{(n-1)\delta}^{n\delta} \{e^{-\lambda(n\delta-t)} - 1\} dN(t),
\]
we bound the last term on the right hand side of (5.31) as

\[
\frac{1}{N} \int_{(n-1)\delta}^{n\delta} e^{-\lambda(n\delta-t)} dN(t) - Nq| \leq \frac{1}{N} \left| N((n-1)\delta, n\delta) - Nq + \lambda \delta N((n-1)\delta, n\delta) \right|.
\]
Collecting all these bounds we then get

\[
D_i(n) \leq \theta_{n-1}(1 + \lambda \delta) + \lambda \delta B^* \frac{M_{n-1}}{N} + 2\lambda \delta \frac{1}{N} N((n-1)\delta, n\delta)
\]
\[
+ \frac{1}{N} \left| N((n-1)\delta, n\delta) - Nq \right|,
\]
and since we are considering the stopped process

\[
D_i(n) \leq \theta_{n-1}(1 + \lambda \delta) + \lambda \delta B^* \frac{M_{n-1}}{N} + 2\lambda \delta 2f^* \delta + \frac{1}{N} \left| N((n-1)\delta, n\delta) - Nq \right|. \tag{5.32}
\]
By the definition of the sets \(A_n, \ldots, F_n\) we have

\[
|N((n-1)\delta, n\delta) - Nq| \leq \sum_{j \in A_n} N_j((n-1)\delta, n\delta) + |B_n|. \tag{5.33}
\]
With probability \(\geq 1 - e^{-C\delta^2 N}\)

\[
|B_n| \leq |B_n \cap G_{n-1}| + |B_n \cap M_{n-1}| \leq 2CN\delta^2 + 2CN\delta \theta_{n-1} + |M_{n-1}| 2f^* \delta \quad \tag{5.34}
\]
having used (5.26) and that the number of neurons among those in \(M_{n-1}\) which fire in a time \(\delta\) is bounded by a Poisson variable of intensity \(f^* \delta |M_{n-1}|.\)

\[
P\left[ \sum_{j \in A_n} N_j((n-1)\delta, n\delta) \geq 4(f^* \delta)^2 N \right] \leq P\left[ \sum_{j \in A_n} N_j((n-1)\delta, n\delta) \right. \geq 4(f^* \delta)^2 N; |A_n| \leq (f^* \delta)^2 N \] \left. + P\left[ |A_n| > (f^* \delta)^2 N \right]. \tag{5.35}
\]
The last term is bounded using (5.25). Let \(A \subset \{1, \ldots, N\}, \ |A| \leq (f^* \delta)^2 N, \) then

\[
P\left[ \sum_{j \in A_n} N_j((n-1)\delta, n\delta) \geq 4(f^* \delta)^2 N \mid A_n = A \right] \leq P^* \left[ \sum_{j \in A} (N_j^* - 2) \geq 2(f^* \delta)^2 N \right],
\]
where \(P^*\) is the law of independent Poisson variables \(N_j^*, \ j \in A, \) each one of parameter \(f^* \delta\) and conditioned on being \(N_j^* \geq 2.\) Thus the probability that \(N^* - 2 = k\) is

\[
P^*[N_j^* - 2 = k] = Z_k^{-1} \frac{\xi^k}{(k+2)!}, \quad Z_k = \xi^{-2} \left( e^\xi - 1 - \xi \right), \quad \xi = f^* \delta.
\]
Denoting by $X_j$ independent Poisson variables of parameter $\xi$ we have that $N_j^* - 2 \leq X_j$ stochastically for $\xi$ small enough, hence for $\delta$ small enough. Indeed for any integer $k$ we have
\[ P^*[N_j^* - 2 \geq k] \leq P[X_j \geq k] \] (5.36)
because for $k \geq 1$,
\[ P^*[N_j^* - 2 \geq k] \leq e^{\xi \frac{2k^2}{(k + 2)!}}, \quad P[X_j \geq k] \geq e^{-\xi \frac{k^2}{k!},} \]
hence (5.36) when $3e^{-2\xi} \geq 2$.

Since $X = \sum_{j \in A} X_j$ is a Poisson variable of parameter $|A|\xi \leq (f^*\delta)^2Nf^*\delta$ we have
\[ P^*[\sum_{j \in A} (N_j^* - 2) \geq 2(f^*\delta)^2N] \leq P^*[X \geq 2(f^*\delta)^2N], \]
where the expectation $E^*(X)$ of $X$ is smaller (for $\delta$ small) than $\frac{1}{2}[2(f^*\delta)^2N]$. As a consequence, $P^*[X \geq 2(f^*\delta)^2N]$ is bounded as $1/n!$, where $n \approx 2(f^*\delta)^2N$, so that
\[ P^*[\sum_{j \in A} (N_j^* - 2) \geq 2(f^*\delta)^2N] \leq e^{-CN\delta^2}. \]

In conclusion for $i \in F_n \cap \mathcal{G}_{n-1}$:
\[ D_i(n) \leq \theta_{n-1}(1 + C\delta) + C\delta \frac{M_{n-1}}{N} + C\delta^2 \] (5.37)
with probability $\geq 1 - e^{-CN\delta^2N}$. Together with (5.30) this proves that with probability $\geq 1 - e^{-CN\delta^2N}$
\[ \theta_n \leq \max\{C\delta; \theta_{n-1}(1 + C\delta) + C\delta \frac{M_{n-1}}{N} + C\delta^2\}. \] (5.38)

### 5.3.4 Iteration of the inequalities

By (5.28), $\frac{M_n}{N} \leq C(\theta_{n-1} + \delta)$ for all $n\delta \leq T$ with probability $\geq 1 - \frac{T}{\delta}e^{-CN\delta^2}$. By (5.38), with probability $\geq 1 - \frac{T}{\delta}e^{-CN\delta^2N}$ we have
\[ \theta_n \leq \max\{C\delta; \theta_{n-1}(1 + C\delta) + C\delta \frac{M_{n-1}}{N} + C\delta^2\}. \]
Thus
\[ \theta_n \leq \max\left(C\delta, [1 + C\delta] \theta_{n-1} + C\delta^2\right). \]
Iterating this inequality we obtain
\[ \theta_n \leq C \sum_{k=0}^{n-1} [1 + C\delta]^k \delta^2 + (1 + C\delta)^n C\delta = C \frac{[1 + C\delta]^n - 1}{C\delta} \delta^2 + (1 + C\delta)^n C\delta \leq Ce^{CT}\delta, \]
where we have used once more that $n\delta \leq T$. Hence
\[ \theta_n \leq C\delta \]
for all $\delta \leq \delta_0$, with probability $\geq 1 - \frac{C}{\delta^2}e^{-CN\delta^2}$. This finishes the proof of Theorem 3.
5.4 Corollaries

We conclude the section with a corollary of the above results which will be used in the analysis of the hydrodynamic limit $N \to \infty$. Recall that by considering the associated empirical measures \((\ref{eq:empirical})\), we interpret $U$ and $Y^{(\delta)}$ as elements of the space $\mathcal{M}$ of all Borel probability measures on $\mathbb{R}_+$.

**Definition 2** We introduce the space $\mathcal{F}$ of smooth functions $\phi(m)$, $m \in \mathcal{M}$, which have the form

$$\phi(m) = h(m[a_1], ..., m[a_k]), \quad k \text{ a positive integer},$$

where $h(r_1, ..., r_k)$ is a smooth function on $\mathbb{R}^k$, uniformly Lipschitz continuous with Lipschitz constant $c_h$, i.e.

$$|h(r_1, ..., r_k) - h(r'_1, ..., r'_k)| \leq c_h \left( \sum_{i=1}^k |r_i - r'_i| \right).$$

The functions $a_i$, $i = 1, ..., k$, in \((\ref{eq:smooth})\) are $C^\infty$ functions on $\mathbb{R}$ each one with compact support contained in $\{|x| \leq c\}$, $c > 0$. Finally $m[a_i] = \int a_i \, dm$ denotes the integral of $a_i$ with respect to the measure $m(dx)$.

Let $c'$ be an upper bound for the derivatives $|a'_i(r)|$, $i = 1, ..., k$. We also introduce

$$\mathcal{T} = \left\{ t \in [0, T] : t = n2^{-k}T, k, n \in \mathbb{N} \right\}.$$  \hspace{1cm} (5.41)

Recall that $P^{(N,\lambda)}_x$ denotes the law under which $U(\cdot)$ starts from $U(0) = x$. Denote by $S^{(\delta,N,\lambda)}_x$ the law under which its approximation $Y^{(\delta)}(\cdot)$ starts from $x$ at time 0, and write $Q^{(\delta,N,\lambda)}_x$ for the probability law governing the coupled process defined above. By abuse of notation, we shall also denote the associated expectations by $P^{(N,\lambda)}_x, S^{(\delta,N,\lambda)}_x$ and $Q^{(\delta,N,\lambda)}_x$.

Then the following holds.

**Proposition 4** Let $t \in \mathcal{T}$, $\delta \in \{2^{-l}T, l \in \mathbb{N}\}$ such that $t = \delta n$ for some positive integer $n$. Let $\phi$ as in \((\ref{eq:smooth})\) with constants $c_h, c$ and $c'$. Then, with $C$ as in Theorem 3 ($C$ is independent of $\delta$)

$$|P^{(N,\lambda)}_x[\phi(\mu_{U(t)})] - S^{(\delta,N,\lambda)}_x[\phi(\mu_{Y^{(\delta)}(t)})]| \leq kc_h c' \frac{C}{\delta^2} e^{-C\delta^2 N} c + \delta(2kc_h c' C).$$  \hspace{1cm} (5.42)

**Proof** The left hand side of \((\ref{eq:4.1})\) is not changed if we replace $U(t)$ and $Y^{(\delta)}(t)$ by $U^*(t)$ and $Y^{\delta,*}(t)$ which are defined by setting

$U^*_i(t) = \min\{U_i(t), c\}, \quad Y^{\delta,*}_i(t) = \min\{Y^{(\delta)}_i(t), c\},$

$c$ as in Definition 2. Let $\phi$ be as in \((\ref{eq:smooth})\), then by \((\ref{eq:77})\)

$$|\phi(\mu_{U(t)}) - \phi(\mu_{Y^{(\delta)}(t)})| \leq kc_h c' \frac{1}{N} \sum_{i=1}^N |U^*_i(t) - Y^{\delta,*}_i(t)|,$$  \hspace{1cm} (5.43)

because $|U_i(t) - Y^{(\delta)}_i(t)| \leq |U^*_i(t) - Y^{\delta,*}_i(t)|$. Hence

$$|P^{(N,\lambda)}_x[\phi(\mu_{U(t)})] - S^{(\delta,N,\lambda)}_x[\phi(\mu_{Y^{(\delta)}(t)})]| \leq kc_h c' Q^{(\delta,N,\lambda)}_x \left[ \frac{1}{N} \sum_{i=1}^N |U^*_i(t) - Y^{\delta,*}_i(t)| \right].$$  \hspace{1cm} (5.44)
Let \( e^{-C\delta^2 N} \) be the bound on the bad events in the estimates of the coupled process, obtained in Theorem 3. Then

\[
Q_x^{(\delta,N,\lambda)} \left[ \frac{1}{N} \sum_{i=1}^N |U^*_i(t) - Y^*_i(t)| \right] \leq \frac{C}{\delta^2} e^{-C\delta^2 N} + 2C \delta
\]

(5.45)

where we used that \(|U^*_i(t) - Y^*_i(t)| \leq c\).

6 Hydrodynamic limit for the auxiliary process

The main result of this section is in Theorem 4 below. It states that the auxiliary process converges in the hydrodynamic limit to the evolution defined in Subsection 6.1. The convergence is in the sense of weak convergence of measures. Any configuration \( x \in \mathbb{R}^N_\lambda \) is in fact interpreted via (2.5) as an element \( \mu_x \) of \( \mathcal{M} \), the space of Borel probability measures on \( \mathbb{R}_+ \). When necessary we shall make explicit the dependence on \( N \) writing \( Y^{(\delta)} = Y^{(\delta,N)} \). We then suppose that for all \( \delta \in \{2^{-l}T, l \in \mathbb{N}\} \), \( Y^{(\delta,N)}(0) = x^{(N)} \) and that the corresponding measure \( \mu_{x^{(N)}} \) satisfies Assumption 2 of Section 2 and Assumption 4 below. We will then show that the law of \( \mu_{Y^{(\delta,N)}} \) converges weakly to a process supported by a single trajectory.

6.1 The limit trajectory

In this subsection we describe the limit law of \( \mu_{Y^{(\delta,N)}} \) denoted by \( \rho_0^{(\delta)}(r) \). We start with some heuristic considerations which will motivate the expression which defines \( \rho_0^{(\delta)}(r) \) and which foresee the way we shall prove convergence to \( \rho_0^{(\delta)}(r) \).

Consider an interval \( I = [a,b] \subset \mathbb{R}_+ \) of length \( \ell \) and center \( r \). We choose \( \ell = N^{-\alpha}, \alpha > 0 \) and properly small. The particles density of the initial configuration \( x^{(N)} \) in \( I \) is the average \( \mu_{x^{(N)}}[I] \); our Assumption 4 will ensure that \( \mu_{x^{(N)}}[I] = \frac{|x^{(N)} \cap I|}{N} \approx \rho_0(r)|I| \). At time \( \delta \) the particles initially in \( I \) and which do not fire will be in the interval \( J = [a',b'] \) (its center denoted by \( r' \)) where, recalling (5.10) and (5.17) for notation,

\[
a' = \varphi_{2^N,\delta}(a) + q^N, \quad b' = \varphi_{2^N,\delta}(b) + q^N,
\]

where \( q^N = q \) is the proportion of particles that have fired, see (5.16). By the definition of \( \varphi_{2^N,\delta} \), \( |J| = b' - a' = e^{-\lambda\delta}|I| \). The only particles in \( J \) at time \( \delta \) are those initially in \( I \) which do not fire, hence their number is approximately \( |x^{(N)} \cap I| e^{-f(r')\delta} \). Thus

\[
\rho_0^{(\delta)}(r')|I| \approx \mu_{x^{(N)}}(J) \approx e^{-f(r')\delta} \rho_0(r)|I|, \quad \rho_0^{(\delta)}(r') \approx e^{\lambda\delta} e^{-f(r')\delta} \rho_0(r),
\]

which gives \( \rho_0^{(\delta)}(r') \) in terms of \( \rho_0(r) \) once we consider \( r = r(r') \):

\[
r = \varphi_{2^N,\delta}^{-1}(r' - q^N) \approx \varphi_{\rho_0,\delta}^{-1}(r' - \rho_0^{(\delta)}(\delta)) = \varphi_{\rho_0,\delta}^{-1}(r') - e^{\lambda\delta} \rho_0^{(\delta)}(\delta),
\]

\[
\rho_0^{(\delta)}(r' - q^N) \approx \rho_0^{(\delta)}(r') - e^{\lambda\delta} \rho_0^{(\delta)}(\delta)
\]
where \( \bar{\rho}_0 = \int x \rho_0(x) dx \), \( \rho^{(\delta)}_0 = \int \rho_0(x) \frac{e^{-\delta f(x)}}{\delta} dx \) are obtained by letting \( N \to \infty \) under suitable assumptions on the convergence of \( \mu_{x,N} \to \rho_0 \). The inverse of \( \varphi_{x,\delta}(\cdot) \), see (5.17), is
\[
\varphi^{-1}_{x,\delta}(x) = e^{\lambda \delta} \left( x - (1 - e^{-\lambda \delta}) \bar{x} \right). 
\] (6.46)
The above gives a formula for \( \rho^{(\delta)}_{x,\delta}(r') \) for all
\[
r' \geq r'_0 = \varphi_{x,\delta}(0) + q^N = (1 - e^{-\lambda \delta}) \bar{x}^N + q^N; 
\] \( r'_0 \) is the same as in (5.19). The definition of \( Y^{(\delta,N)} \) is such that all the particles which have fired are put uniformly in \([0, r'_0]\), thus
\[
\rho^{(\delta)}_{x,\delta}(r') \approx \frac{\rho^{(\delta)}_{x,\delta}(0)}{(1 - e^{-\lambda \delta}) \bar{\rho}_0 + \rho^{(\delta)}_{x,\delta}}, 
\] \( r' \leq r'_0 \).
The definition of \( \rho^{(\delta)}_{n\delta}(r) \) will extend and formalize the above one to all \( n\delta \leq T \). We put \( \rho^{(\delta)}_{0}(x) = \psi_0(x) \), where \( \psi_0 \) is a smooth probability density on \( \mathbb{R}_+ \) satisfying Assumption 2. We then proceed inductively in \( n \) such that \( n\delta \leq T \).

Suppose that \( \rho^{(\delta)}_{n\delta} \) has already been defined. Then we put
\[
\rho^{(\delta)}_{n\delta}(x) = e^{\lambda \delta} \rho^{(\delta)}_{n\delta}(\varphi^{-1}_{n\delta,\delta}(x) - e^{\lambda \delta} \rho^{(\delta)}_{n\delta} \bar{x}) 
\exp \left( -f \left( \varphi^{-1}_{n\delta,\delta}(x) - e^{\lambda \delta} \rho^{(\delta)}_{n\delta} \bar{x} \right) \right), 
\] \( x > 0, \) (6.49)
where we define \( f(x) \equiv 0 \) for \( x < 0 \) and
\[
\rho^{(\delta)}_{n\delta}(x) \equiv \frac{\rho^{(\delta)}_{n\delta}(x)}{e^{\lambda \delta} \rho^{(\delta)}_{n\delta} \bar{x} - \varphi^{-1}_{n\delta,\delta}(0)} = \frac{e^{-\lambda \delta} \rho^{(\delta)}_{n\delta}}{\rho^{(\delta)}_{n\delta} \delta + (1 - e^{-\lambda \delta}) \rho^{(\delta)}_{n\delta}} \quad \text{for } x \leq 0. 
\] (6.50)
In this way, \( \rho^{(\delta)}_{n\delta} \) are probability densities on \( \mathbb{R}_+ \) for all \( n \), i.e.
\[
1 = \int_0^\infty \rho^{(\delta)}_{n\delta}(x) dx. 
\] (6.51)

**Remark 2** As \( \delta \to 0 \), the initial condition (6.50) reads as
\[
\rho^{(\delta)}_{n\delta}(0) \sim \frac{\rho^{(\delta)}_{n\delta}}{\rho^{(\delta)}_{n\delta} + \lambda \rho^{(\delta)}_{n\delta}} 
\] which corresponds to (2.8).
Notice that if $\rho_{n\delta}^{(\delta)}$ has support $]-\infty, R_0]$, then the support of $\rho_{(n+1)\delta}^{(\delta)}$ is included in $]-\infty, R_{n+1}]$, where
\[ R_{n+1} = e^{-\lambda\delta} R_n + p_{n\delta}^{(\delta)} \delta + (1-e^{-\lambda\delta}) \bar{\rho}_{n\delta}^{(\delta)}. \]
This leads to the following definition.

**Definition 3 (Edge)** We call $R_0$ the edge of the profile $\rho_0$ and
\[ R_n = e^{-\lambda\delta} R_{n-1} + p_{(n-1)\delta}^{(\delta)} \delta + (1-e^{-\lambda\delta}) \bar{\rho}_{(n-1)\delta}^{(\delta)} \tag{6.52} \]
the edge of $\rho_{n\delta}^{(\delta)}$.

Noticing that
\[ p_{n\delta}^{(\delta)} \leq \int p_{n\delta}^{(\delta)}(x) \frac{1-e^{-\delta f^*}}{\delta} dx = \frac{1-e^{-\delta f^*}}{\delta} \leq f^*, \tag{6.53} \]
it then follows from (6.49) that
\[ R_n \leq R_{n-1} + f^* \delta \leq R_0 + f^* n \delta \leq R_0 + f^* T, \tag{6.54} \]
since $n \delta \leq T$. Hence the supports of $\rho_{n\delta}^{(\delta)}$ are uniformly bounded. By iterating (6.49) and by using the explicit form of the inverse flow $\varphi_{x,\delta}^{-1}(x)$, we get
\[
\rho_{(n+1)\delta}^{(\delta)}(x) = e^{\lambda(n+1)\delta} \psi_0 \left( e^{\lambda(n+1)\delta} x - (1-e^{-\lambda\delta}) \sum_{k=0}^{n} e^{\lambda(k+1)\delta} \rho_{h\delta}^{(\delta)} - \sum_{k=0}^{n} e^{\lambda(k+1)\delta} p_{h\delta}^{(\delta)} \right) \\
\exp \left\{ -\sum_{k=0}^{n} \delta f \left( e^{\lambda(n+1-k)\delta} x - (1-e^{-\lambda\delta}) \sum_{h=k}^{n} e^{\lambda(k+1)\delta} \rho_{h\delta}^{(\delta)} - \sum_{h=k}^{n} e^{\lambda(k+1)\delta} p_{h\delta}^{(\delta)} \right) \right\}, \tag{6.55} \]
where $\psi_0$ is the initial density. Before proving the convergence of the approximating process to this limit trajectory, we observe the following two facts which will be useful in the sequel. By the smoothness of $\psi_0$ and of $f$, there exists a constant $c$ such that for any $t \in \mathcal{T}$ and any $\delta = 2^{-k}T$ with $k$ large enough
\[ |\rho_t^{(\delta)}(x) - \rho_t^{(\delta)}(y)| \leq c|x - y|. \tag{6.56} \]
Moreover, for any $t, s \in \mathcal{T}$ and any $\delta = 2^{-k}T$ with $k$ large enough,
\[ |\rho_t^{(\delta)}(x) - \rho_s^{(\delta)}(x)| \leq c|t - s|. \tag{6.57} \]

### 6.2 Energy discretization

Let $(Y^{(\delta)}(n\delta))_{n \leq T/\delta}$ be the auxiliary process defined in Subsection 5.1 starting from $x = x^{(N)}$. We suppose that $||x|| \leq R_0$. Recalling the definition of $\Phi_t(n)$ in (5.16) we put
\[
q(n\delta) = \frac{\sum_{i=1}^{N} \Phi_t(n)}{N}, \quad V(n\delta) = \frac{q(n\delta)}{\delta} \quad \text{and} \quad \bar{Y}^{(\delta)}(n\delta) = \frac{\sum_{i=1}^{N} Y_t^{(\delta)}(n\delta)}{N}. \tag{6.58} \]
and then define the sequence of random edges $R'_0 = R_0$,
\[
R'_n := e^{-\lambda \delta} R'_{n-1} + V((n-1)\delta) + \left(1 - e^{-\lambda \delta}\right) \bar{Y}((n-1)\delta). \tag{6.59}
\]

We introduce an energy mesh which depends on $N$ and on time, times having the form $t = n\delta, t \leq T$. The mesh at different times will be related as in the heuristic considerations in the beginning of this section.

**Definition 4 (Energy mesh)** Let $0 < \alpha \ll \frac{1}{6}$. Given $N$, let $r \in \left[\frac{1}{2}, 1\right]$ be such that $R_0$ is an integer multiple of $rN^{-\alpha}$. We then partition $(-\infty, R_0]$ into intervals
\[
\mathcal{I}_0 = \{I_{i,0}, i \geq 1\}, \quad I_{i,0} = [R_0 - i\ell, R_0 - (i - 1)\ell], \quad \ell = rN^{-\alpha}, \tag{6.60}
\]
and define $\mathcal{I}_0' = \{I'_{i,0}, i \geq 1\}$ by setting $I'_{i,0} = I_{i,0}$ so that at time $0, \mathcal{I}_0' = \mathcal{I}_0$. At times $n\delta$ we define $\mathcal{I}_{n\delta} = \{I_{i,n}, i \geq 1\}$ and $\mathcal{I}_{n\delta}' = \{I'_{i,n}, i \geq 1\}$ as the sequences of intervals
\[
I_{i,n} := [R_n - e^{-\lambda \delta n} i\ell, R_n - e^{-\lambda \delta n} (i - 1)\ell], \quad I'_{i,n} := [R'_n - e^{-\lambda \delta n} i\ell, R'_n - e^{-\lambda \delta n} (i - 1)\ell]. \tag{6.61}
\]

The strategy is to compare the “mass” of $\mu_{Y^{(\delta)}(n\delta)}$ in $I'_{i,n}$ and the mass in the corresponding interval $I_{i,n}$ (with same $i$) for the limit $\rho_{n\delta}^{(\delta)}$ (we still need to define the mass in intervals which are not contained in $\mathbb{R}_+$). We shall prove that for “most” intervals the corresponding masses are “very close” to each other. We thus need to specify the mass distributions in $\{x < 0\}$ and to define the bad intervals where the masses may differ. We start with the former. We have already extended the density $\rho_{n\delta}^{(\delta)}(x)$ to $x < 0$, see (6.50). For the particles we proceed analogously and extend $\mu_{Y^{(\delta)}(n\delta)}$ to the negative axis by adding an infinite mass:
\[
\left(\mu_{Y^{(\delta)}(n\delta)}\right)_{|_{-\infty,0}} := \frac{1}{N} \sum_{i=1}^{\infty} \delta_{-i\delta n}, \quad n\delta \leq T \tag{6.62}
\]
where in agreement with (5.20)
\[
d_n = \frac{(1 - e^{-\lambda \delta}) \bar{Y}((n\delta)) + (\delta V(n\delta) - \frac{1}{N})}{N \delta V(n\delta) - 1}. \tag{6.63}
\]
Notice that the choice (6.62) corresponds exactly to the initial configuration given in (5.21).

We introduce the following quantities for all $i, n$.
\[
N'_{i,n} = N_{i} \mu_{Y^{(\delta)}(n\delta)}(I'_{i,n}), \quad N_{i,n} = N \int_{I_{i,n}} \rho_{n\delta}^{(\delta)}(x) \, dx, \quad w_i = \int_{I_{i,0}} \psi_0(x) \, dx. \tag{6.64}
\]

### 6.3 Hydrodynamic limit

Good labels $i$ are when the intervals $I_{i,n}$ and $I'_{i,n}$ are at all times either both in $\{x \geq 0\}$ or both in $\{x \leq 0\}$. We thus set:

**Definition 5 (Bad intervals)** $I_{i,n}$ is bad, if there is $n_0 \leq n$ such that (at least) one of the following four properties holds.
1. $I_{i,n_0} \cap \{ x < 0 \} \neq \emptyset$ and $I_{i,n_0} \cap \{ x > 0 \} \neq \emptyset$.
2. $I'_{i,n_0} \cap \{ x < 0 \} \neq \emptyset$ and $I'_{i,n_0} \cap \{ x > 0 \} \neq \emptyset$.
3. $I_{i,n} \subset \{ x < 0 \}$ and $I'_{i,n} \subset \{ x > 0 \}$.
4. $I'_{i,n} \subset \{ x < 0 \}$ and $I_{i,n} \subset \{ x > 0 \}$.

$I'_{i,n}$ is bad if $I_{i,n}$ is bad. An interval is good if it is not bad.

**Assumption 4 (The initial configuration)** We suppose that the initial configuration $x = x^{(N)}$ is such that there is a constant $\kappa_0$ so that for all $N$ large enough,

$$|N'_{i,0} - N_{i,0}| \leq \kappa_0 w_i N \ell,$$

for all intervals $I_{i,0} \in \mathcal{I}_0$, (6.65)

(recall that by the definition of $\mathcal{I}_0$ all its intervals are good).

(6.65) is satisfied with probability going to 1 as $N \to \infty$ if $x_i, i = 1, \ldots, N$, are chosen independently with law $\psi_0(x)dx$; we shall prove analogous statements in the course of the proof of Theorem 4.

Since $\psi_0 \geq c(x - R_0)^2$, $c > 0$, in a left neighborhood of $R_0$,

$$w_i \geq c \ell^3,$$ (6.66)

while, “away” from $R_0$, $w_i \geq c \ell$, for some $c > 0$.

In order to compare $Y^{(\delta)}(n \delta), n \leq \delta^{-1}T$, and $\rho^{(\delta)} := (\rho^{(\delta)}_{n \delta}, n \leq \delta^{-1}T)$, we introduce the following distance.

**Definition 6 (Distances)** We define for any $n \leq \delta^{-1}T$,

$$B_n := \text{number of bad intervals in } \mathcal{I}_n$$ (6.67)

and set

$$d_n(Y^{(\delta)}, \rho^{(\delta)}) := B_n + \max_{I_{i,n}, \text{good}} \frac{|N'_{i,n} - N_{i,n}|}{w_i N \ell}$$

$$+ \frac{\delta}{\ell} |V((n - 1)\delta) - p^{(\delta)}_{(n-1)\delta}| + \frac{|\bar{Y}^{(\delta)}(n \delta) - \rho^{(\delta)}_{n \delta}|}{\ell}.$$ (6.68)

$d_n(Y^{(\delta)}, \rho^{(\delta)})$ depends on the times $\tau_j(k), j = 1, \ldots, N, k \leq n - 1$, where the $\tau_j(k)$ are the times which enter in the definition of $\Phi_j(k)$, see (5.19). We call $\mathcal{F}_n$ the $\sigma$-algebra generated by these variables. Observe that $Y_{d_n}^{(\delta)}, \bar{Y}^{(\delta)}(n \delta)$ and $V((n - 1)\delta)$ are $\mathcal{F}_n$-measurable.

We prove in Theorem 4 below that with large probability (going to 1 as $N \to \infty$) the distances $d_n(Y^{(\delta)}, \rho^{(\delta)})$ are bounded for all $n$ such that $n \delta \leq T$. The bounds are given by coefficients $\kappa_n$ which do not depend on $N$ but have a very bad dependence on $\delta$ for small $\delta$. $\delta$ however is a fixed parameter in this section and by the way $d_n$ is defined the bounds imply that $Y^{(\delta)}$ and $\rho^{(\delta)}$ become very close in most of the space as $N \to \infty$ (and keeping $\delta$ fixed). This will yield weak convergence as stated in Corollary 4 at the end of this section.
**Theorem 4** There exist \( \kappa_n > 0, \gamma \in ]0, 1[, c_1(n) \), increasing in \( n \), and a constant \( c_2 > 0 \) such that

\[
S_x^{(\delta,N,\lambda)}\left[ d_n(Y^{(\delta)}, \rho^{(\delta)}) \leq \kappa_n \right] \geq 1 - c_1(n)e^{-c_2 N^\gamma},
\]

(6.69)

for all \( n \) such that \( n\delta \leq T \).

**Proof.** The proof is by induction on \( N \) and we thus suppose that (6.69) holds for all \( j \leq n \). We finally have bounds on \( \mathcal{F}_n \), see Definition \( \mathcal{F}_n \) and introduce \( G_n = \bigcap_{j \leq n} \{ d_j(Y^{(\delta)}, \rho^{(\delta)}) \leq \kappa_j \} \),

\[
S_x^{(\delta,N,\lambda)}\left[ d_{n+1}(Y^{(\delta)}, \rho^{(\delta)}) > \kappa_{n+1} \right] \leq S_x^{(\delta,N,\lambda)}\left( 1_{G_n} S_x^{(\delta,N,\lambda)}\left[ d_{n+1}(Y^{(\delta)}, \rho^{(\delta)}) > \kappa_{n+1} \mid \mathcal{F}_n \right] \right) + nc_1(n)e^{-c_2 N^\gamma},
\]

and we thus need to prove that for some constant \( c \)

\[
S_x^{(\delta,N,\lambda)}\left[ d_{n+1}(Y^{(\delta)}, \rho^{(\delta)}) > \kappa_{n+1} \mid \mathcal{F}_n \right] \leq ce^{-c_2 N^\gamma}, \quad \text{on } G_n.
\]

(6.70)

From the conditioning we know that \( d_j(Y^{(\delta)}, \rho^{(\delta)}) \leq \kappa_j \) for all \( j \leq n \); we know also the value of \( Y^{(\delta)}(n\delta) \), say \( Y^{(\delta)}(n\delta) = y \), we know the location of the edges \( R'_j, j \leq n \), and we know which are the good and the bad intervals at time \( n\delta \).

**Consequences of being in \( G_n \)**

The condition to be in \( G_n \) does not only allow to control the quantities directly involved in the definition of \( d_n \) but also several other quantities. The first one is the difference \( |R'_n - R_n| \). Indeed, recalling (6.52) and (6.59),

\[
|R'_n - R_n| \leq e^{-\lambda \delta}|R'_{n-1} - R_{n-1}| + |V((n-1)\delta) - p^{(\delta)}(n-1)\delta| + (1 - e^{-\lambda \delta})|\bar{Y}(\delta)((n-1)\delta) - \bar{\rho}(\delta)(n-1)\delta|
\]

\[
\leq e^{-\lambda \delta}|R'_{n-1} - R_{n-1}| + \kappa_{n} \ell + \lambda \delta \kappa_{n-1} \ell
\]

\[
\leq \left( \sum_{j=1}^{n} \kappa_j \right)(1 + \lambda \delta) \ell,
\]

(6.71)

which in particular yields:

\[
|a_{i,n} - a'_{i,n}| = |b'_{i,n} - b_{i,n}| \leq \left( \sum_{j=1}^{n} \kappa_j \right)(1 + \lambda \delta) \ell,
\]

(6.72)

where \([a_{i,n}, b_{i,n}] := I_{i,n} \) and \([a'_{i,n}, b'_{i,n}] := I'_{i,n} \). We also get a bound on the increments of the number of bad intervals. Recalling (6.67) for notation we have in fact

\[
B_n \leq B_{n-1} + 1 + \frac{|R_n - R'_n|}{\ell}.
\]

(6.73)

We finally have bounds on \( N'_{i,n} \). Firstly we suppose that \( I_{i,n} \) is a good interval. Then \( N'_{i,n} \leq N'_{i,0} \), whence for \( N \) large enough

\[
N'_{i,n} \leq N_{i,0} + \kappa_0 w_i N \ell \leq (1 + \kappa_0 \ell) N w_i \leq 2 N w_i.
\]

(6.74)
We also have a lower bound. By (6.55), \( N_{i,n} \geq N_{i,0} e^{-f^* \delta n} \), hence
\[
N'_{i,n} \geq N_{i,n} - \kappa_n w_i N \ell \geq w_i N \left( e^{-f^* T} - \kappa_n \ell \right) \geq \frac{1}{2} w_i N \geq \frac{c}{2} \ell^3 N = \frac{c}{2} r^3 N^{1-3\alpha},
\]
for \( N \) large enough.

We shall only need an upper bound on \( N'_{i,n} \) when \( I_{i,n} \) is a bad interval. Recalling the definition of \( d_n \) in (5.20) and the definition (5.21) we notice that \( d_n \geq 1/N \), if \( N \delta V(n \delta) \geq 2 \), i.e. in case that at least two particles spike. At the first time a bad interval arises, the number of particles falling into it is upper bounded by \( \ell N \), i.e.
\[
N'_{i,n} \leq 1 \text{ for the number of particles falling into a bad interval. As a consequence,}
\]
\[
N'_{i,n} \leq N \ell + 1 \text{ for all bad intervals such that } I_{i,n} \cap \mathbb{R}_+ \neq \emptyset.
\]

**Expected fires in good intervals**

Recalling (5.16) and (6.58), we write
\[
V(n \delta) = \frac{1}{\delta N} \sum_i \Delta_i, \text{ where } \Delta_i = \sum_{j : y_j \in I'_{i,n}} \Phi_j(n),
\]
and call \( \langle \Delta_i \rangle \) its conditional expectation (given \( \mathcal{F}_n \), and hence given that \( Y(\delta)(n \delta) = y \), then
\[
\langle \Delta_i \rangle = \sum_{j : y_j \in I'_{i,n}} \left( 1 - e^{-\delta f(y_j)} \right).
\]

Call \( I'_{i,n} = [a'_{i,n}, b'_{i,n}] \). Since \( f \) is non decreasing, we have
\[
\langle \Delta_i \rangle \leq N'_{i,n} \left( 1 - e^{-\delta f(b'_{i,n})} \right) \leq N'_{i,n} \left( 1 - e^{-\delta f(a'_{i,n})} \right) + N'_{i,n} \left| e^{-\delta f(b'_{i,n})} - e^{-\delta f(a'_{i,n})} \right|.
\]
Moreover, \( f(b'_{i,n}) \leq f(a'_{i,n}) + \|f\|_{\text{Lip}} \ell \), which implies that
\[
\left| e^{-\delta f(b'_{i,n})} - e^{-\delta f(a'_{i,n})} \right| \leq \delta C \ell.
\]
Suppose first that \( I'_{i,n} \) is good so that \( |N'_{i,n} - N_{i,n}| \leq \kappa_n w_i N \ell \). Then by (6.74)
\[
\langle \Delta_i \rangle \leq \{ N_{i,n} + \kappa_n w_i N \ell \} (1 - e^{-\delta f(a'_{i,n})}) + \delta C \ell 2 N w_i
\]
\[
\leq N_{i,n} (1 - e^{-\delta f(a'_{i,n})}) + C(\kappa_n + 1) \delta w_i N \ell.
\]
Write \( I_{i,n} = [a_{i,n}, b_{i,n}] \), so that by (6.72), \( |a'_{i,n} - a_{i,n}| \leq K_n \ell \), where \( K_n = (\sum_{j=1}^n \kappa_n)(1 + \lambda \delta) \). Then
\[
\langle \Delta_i \rangle \leq N_{i,n} (1 - e^{-\delta f(a_{i,n})}) + (K_n + C(1 + \kappa_n)) \delta w_i N \ell.
\]
Since \( f \) is non decreasing and \( N_{i,n} = N \int_{I_{i,n}} \rho^{(\delta)}_{\kappa_n}(x) \, dx \),
\[
\langle \Delta_i \rangle \leq N \int_{I_{i,n}} \rho^{(\delta)}_{\kappa_n}(x) (1 - e^{-\delta f(x)}) \, dx + (K_n + C(1 + \kappa_n)) \delta w_i N \ell.
\]
An analogous argument gives

\[ \langle \Delta_i \rangle \geq N \int_{I_{i,n}} \rho_{\nu_\delta}^{(\nu)}(x)(1 - e^{-\delta f(x)})dx - (K_n + C(1 + \kappa_n))\delta w_i N \ell. \]  

(6.79)

**Fires fluctuations in good intervals**

Hoeffding’s inequality implies that for any \( b > 0 \),

\[ P \left[ |\Delta_i - \langle \Delta_i \rangle| \geq (N'_{i,n})^{b+\frac{1}{2}} \right] \leq 2e^{-2(N'_{i,n})^{2b}}. \]  

(6.80)

The contribution of \( I_{i,n} \) to \( p_{\nu_\delta}^{(\nu)} \) is called:

\[ p_{\nu_\delta}^{(\nu)} := N \int_{I_{i,n}} \rho_{\nu_\delta}^{(\nu)}(x)(1 - e^{-\delta f(x)})dx. \]

We then use (6.74) and (6.75) together with (6.78) and (6.80) to get

\[ P \left[ \bigcap_{I_{i,n} \text{ good}} \left\{ \Delta_i \leq \delta p_{\nu_\delta}^{(\nu)} + (K_n + C(1 + \kappa_n))\delta w_i N \ell + \left(2Nw_i\right)^{\frac{1}{2} + b} \right\} \right] \geq 1 - 2m_n e^{-2(\xi N^{1-3\alpha})^{2b}}, \]  

(6.81)

where \( m_n \) is an upper bound on the number of good intervals:

\[ m_n \leq \left( \frac{R_n}{e^{-\lambda n \delta \ell}} \vee \frac{R_n'}{e^{-\lambda n \delta \ell}} \right) + 1 \leq C \left( \frac{R_n}{\ell} + \kappa_n \right) + 1 \leq CN^\alpha, \]

because \( R_n \leq R_0 + c^*T \) and \( n\delta \leq T \).

As a consequence, the right hand side of (6.81) can be lower bounded by

\[ 1 - CN^\alpha e^{-2(\xi N^{1-3\alpha})^{2b}}. \]

By an analogous argument

\[ P \left[ \bigcap_{I_{i,n} \text{ good}} \left\{ \Delta_i \geq \delta p_{\nu_\delta}^{(\nu)} - (K_n + C(1 + \kappa_n))\delta w_i N \ell - \left(2Nw_i\right)^{\frac{1}{2} + b} \right\} \right] \geq 1 - CN^\alpha e^{-2(\xi N^{1-3\alpha})^{2b}}. \]  

(6.82)

Now we choose \( b \ll 1 \) and \( \alpha \ll \frac{1}{6} \) sufficiently small such that for \( N \) large enough, \( (2Nw_i)^{\frac{1}{2} + b} \leq C\delta(1 + \kappa_n)w_i N \ell \). Then

\[ P \left[ \bigcap_{I_{i,n} \text{ good}} \left\{ |\Delta_i - \delta p_{\nu_\delta}^{(\nu)}| \leq (K_n + 2C(1 + \kappa_n))\delta w_i N \ell \right\} \right] \geq 1 - CN^\alpha e^{-C'\xi N^{1-3\alpha})^{2b}} \geq 1 - Ce^{-CN^\gamma}, \]  

where \( \gamma = (1 - 3\alpha)2b \) and \( C \) a suitable constant.
The bounds on $V(n\delta)$ and on $B_{n+1}$

By (6.76)

$$|\delta V(n\delta) - \delta p_{n\delta}^{(\delta)}| \leq \left[ \sum_{I_{i,n} \text{ good}} (K_n + 2C(1 + \kappa_n)) \delta w_i \ell \right] + \frac{N\ell + 1}{N} B_n \leq \kappa'_{n+1} \ell. \quad (6.84)$$

Hence, we have proven the desired assertion for $V_n$ at time $(n+1)\delta$.

To bound $B_{n+1}$ (i.e. the number of bad intervals at time $(n+1)\delta$) we use the first inequality in (6.71) with $n \to n+1$ together with (6.84), so that by (6.73)

$$B_{n+1} \leq B_n + 1 + \left| \frac{R_{n+1} - R'_{n+1}}{\ell} \right| \leq B_n + 1 + \left( \sum_{j=1}^{n} \kappa_j + \kappa'_{n+1} \right) (1 + \lambda \delta)$$

whence the assertion concerning $B_{n+1}$.

Bounds on $|N'_{i,n+1} - N_{i,n+1}|$

Let $I_{i,n}$ be a good interval at time $n\delta$ which is contained in $\mathbb{R}_+$. Then it is good also at time $(n+1)\delta$ and we have

$$N'_{i,n+1} = \sum_{j : y_j \in I'_{i,n}} (1 - \Phi_j(n)) = N'_{i,n} - \Delta_i$$

and

$$N_{i,n+1} = N_{i,n} - \delta p_{i,n\delta}^{(\delta)}.$$

Thus

$$\frac{|N'_{i,n+1} - N_{i,n+1}|}{w_i N \ell} \leq \kappa_n + \frac{|\Delta_i - \delta p_{i,n\delta}^{(\delta)}|}{w_i N \ell},$$

and the desired bound follows from (6.83).

It remains to consider a good interval $I'_{i,n+1}$ such that $I'_{i,n} \subset \mathbb{R}_-$ (and hence also $I_{i,n} \subset \mathbb{R}_-$). Thus $I'_{i,n+1}$ consists entirely of “new born” particles which arise due to firing events where the energies are reset to 0. For such an interval,

$$\frac{N'_{i,n+1}}{N \ell} \in \left[ \frac{1}{N d_n} e^{-\lambda \delta n}, \frac{1}{N d_n} e^{-\lambda \delta n} + 1 \right].$$

But, recalling the definition of $d_n$ in (5.20) and of $\rho_{n\delta}^{(\delta)}(0)$ in (6.50), by continuity of $(u, p) \to \frac{e^{-\lambda \delta p}}{p \delta + (1 - e^{-\lambda \delta}) u}$, we have

$$\left| \frac{1}{d_n} - \rho_{n\delta}^{(\delta)}(0) \right| \leq C \kappa_n \ell. \quad (6.85)$$

Since $\rho_{(n+1)\delta}^{(\delta)}(x) 1_{I_{i,n+1}}(x) \equiv \rho_{n\delta}^{(\delta)}(0)$ on this interval, this implies that also for such intervals,

$$\frac{1}{N \ell} |N'_{i,n+1} - N_{i,n+1}| \leq C \kappa_n \ell = C \kappa_n w_i,$$

by definition of $w_i$. This concludes the bound of $|N'_{i,n+1} - N_{i,n+1}|$.

The bound on $|\bar{Y}^{(\delta)}((n+1)\delta) - \bar{\rho}^{(\delta)}_{(n+1)\delta}|$ follows from the bounds on $|N'_{i,n+1} - N_{i,n+1}|$ and $B_{n+1}$; details are omitted. This concludes the proof of Theorem [4].
Remark 3. Theorem 4 proves strong convergence of $\mu_{Y(t)}$ to $\rho_t^{(\delta)}(x)dx$. Indeed, it implies the convergence of the “densities” $\frac{N_i}{N}$ (notice that $w_i \leq \ell \to 0$ as $N \to \infty$).

In particular, we obtain the following convergence result.

Corollary 1 (Hydrodynamic limit for the approximating process) Let $t \in T$, $\delta \in \{2^{-lT}, l \in \mathbb{N}\}$ such that $t = \delta n$ for some positive integer $n$. Then almost surely, as $N \to \infty$,

$$
\mu_{Y(t)} \xrightarrow{w} \rho_t^{(\delta)}(x)dx.
$$

7 Hydrodynamic limit for the true process

We can conclude the proof of Theorem 2. The convergence in the hydrodynamic limit will be proved as a consequence of Corollary 1, which proves the convergence for the approximating process, of (6.56), (6.57), and of Proposition 2.

We start by extending $\rho_t^{(\delta)}(x)$ which is defined for all $t \in \{n\delta, n \leq T/\delta\}$ to a function defined for all $t \in [0, T]$, by linear interpolation. (6.56) and (6.57) remain true. We can thus consider $\rho^{(\delta)}$ as a continuous function of $(t, x) \in [0, T] \times \mathbb{R}_+$. (6.56) and (6.57) imply that the family of functions $\{\rho^{(\delta)}, \delta \in \{2^{-kT}\}\}$ is uniformly equicontinuous. Therefore there exists a subsequence $(k_n)_n$ such that $\rho_t^{(2^{-k_n}T)}(x)$ converges, as $n \to \infty$, to a continuous bounded function $\rho_t(x)$, uniformly in $t \in [0, T]$ and in $x$ for $0 \leq x \leq \max(R_0 + f^*T, B)$, where we recall that by (6.54), $R_0 + f^*T$ bounds the support of all $\rho^{(\delta)}$ and where $B$ is chosen according to Theorem 1.

Since we are working on compact support, we obtain, using dominated convergence, the following equalities for the limit function $\rho_t(x)$ from (6.51), (6.47) and (6.48).

$$
1 = \int_0^\infty \rho_t(x)dx, \quad p_t = \lim_{n \to \infty} \int_0^\infty \rho_t^{(2^{-k_n}T)}(x) = \int_0^\infty \rho_t(x)f(x)dx
$$

and

$$
\bar{\rho}_t = \lim_{n \to \infty} \bar{\rho}_t^{(2^{-k_n}T)} = \int_0^\infty x\rho_t(x)dx.
$$

Moreover, (6.55) implies, still by dominated convergence, that

$$
\rho_t(x) = \psi_0 \left( e^{\lambda x} - \lambda \int_0^t e^{\lambda s}\bar{\rho}_s ds - \int_0^t e^{\lambda s}p_s ds \right) \\
\exp \left\{ - \int_0^t \left[ f \left( e^{\lambda(t-s)}x - \lambda \int_s^t e^{\lambda r}\bar{\rho}_r dr \right) - \lambda \right] ds \right\}. \quad (7.86)
$$

Proposition 5. $\rho_t(x)$ is the unique solution of

$$
\frac{\partial \rho}{\partial t} = \left( \lambda x - \lambda \bar{\rho}_t - p_t \right) \frac{\partial \rho}{\partial x} - [f(x) - \lambda] \rho_t(x),
$$

solving the initial conditions $\rho_0(x) = \psi_0(x)$ for all $x$ and $\rho_t(0) = \frac{p_t}{\lambda + \lambda \bar{\rho}_t}$, for all $t > 0$, such that

$$
1 = \int_0^\infty \rho_t(x)dx, \quad p_t = \int_0^\infty f(x)\rho_t(x)dx, \quad \bar{\rho}_t = \int x\rho_t(x)dx.
$$
Remark 4 It is immediate to check that the function \( p_t(x) \) given in (7.86) indeed solves (7.87). Integrating equation (7.87) over \( x \in [0, \infty[ \) implies \( p_t(0) = \frac{p_t}{p_t+\lambda p_t} \).

Proof We address the uniqueness of the solution. The method of characteristics implies that any solution of (7.87) on any fixed time interval \([0, T]\) is given by the function \( g_t(x) = g(T-t, x) \), \( 0 \leq t \leq T \), where
\[
\begin{align*}
g(t, x) &= \psi_0(\varphi_{t,T}(x))e^{-\int_t^T (f-\lambda) (\varphi_{t,s}(x))ds}, \\
\varphi_{t,x}(x) &= x + \int_t^s [\lambda \varphi_{t,r}(x) - \lambda \bar{p}_r - p_r]dr, \\
p_r &= \int f(x)g_r(x)dx, \quad \bar{p}_r = \int xg_r(x)dx.
\end{align*}
\]

We use this explicit representation in order to prove uniqueness of the solution on the fixed time interval \([0, T]\). Let \((\varphi, g)\) associated to \((p, \bar{p})\) and \((\tilde{\varphi}, \tilde{g})\) associated to \((\tilde{p}, \bar{\rho})\) be two solutions on \([0, T]\), starting from the same initial condition \(\psi_0\), where \(\psi_0\) is of compact support \([0, R_0]\). Then there exists a constant \(d\) such that
\[
supp \varphi_t \cup supp \tilde{\varphi}_t \subset [0, e^{dT}R_0], \text{ for all } t \leq T.
\]

Let
\[
\Gamma := e^{\lambda T} (\|\psi_0\|_{Lip} + T\|\psi_0\|_\infty \|f\|_{Lip}) \int_0^{e^{dT}R_0} (x + f(x))dx
\]
and fix
\[
0 < t_0 < \frac{0,5}{\lambda(1+\Gamma)+\Gamma} \quad (7.88)
\]
We will use this choice of \(t_0\) in (7.89) below. It is easy to see that for \(0 \leq s \leq t_0\),
\[
|g_s(x) - \tilde{g}_s(x)| \leq e^{\lambda t_0} \left(\|\psi_0\|_{Lip} + t_0\|\psi_0\|_\infty \|f\|_{Lip}\right) \|\varphi - \tilde{\varphi}\|_{T-t_0,T},
\]
where
\[
\|\varphi - \tilde{\varphi}\|_{T-t_0,T} := \sup_{T-t_0 \leq r \leq s \leq T} |\varphi_{r,s}(x) - \tilde{\varphi}_{r,s}(x)|.
\]
Taking into account that \(supp(\psi_0) \subset [0, C]\) and writing \(C_s = \{x : \varphi_{T-s,T}(x) \in [0, R_0]\} \cup \{x : \varphi_{T-s,T}(x) \in [0, R_0]\}\), we obtain for all \(T - t_0 \leq s \leq T\),
\[
|p_s - \tilde{p}_s| \leq \int_{C_s} f(x)|g_s(x) - \tilde{g}_s(x)|dx
\]
\[
\leq e^{\lambda t_0} \left(\|\psi_0\|_{Lip} + t_0\|\psi_0\|_\infty \|f\|_{Lip}\right) \|\varphi - \tilde{\varphi}\|_{T-t_0,T} \int_{C_s} f(x)dx
\]
\[
\leq e^{\lambda t_0} \left(\|\psi_0\|_{Lip} + t_0\|\psi_0\|_\infty \|f\|_{Lip}\right) \|\varphi - \tilde{\varphi}\|_{T-t_0,T} \int_0^{e^{dT}R_0} f(x)dx
\]
\[
\leq \Gamma \|\varphi - \tilde{\varphi}\|_{T-t_0,T}.
\]

The same argument yields
\[
|\bar{p}_s - \bar{\rho}_s| \leq \Gamma \|\varphi - \tilde{\varphi}\|_{T-t_0,T}.
\]
Putting things together and using the equation of \( \varphi_{t,s}(x) \), we have for all \( T - t_0 \leq T - r \leq s \leq T \),
\[
|\varphi_{T-r,s}(x) - \tilde{\varphi}_{T-r,s}(x)| \leq t_0 (\lambda + \lambda \Gamma + \Gamma) \|\varphi - \tilde{\varphi}\|_{T-t_0,T}.
\]
therefore, by choice of \( t_0 \),
\[
\|\varphi - \tilde{\varphi}\|_{T-t_0,T} \leq 0.5 \|\varphi - \tilde{\varphi}\|_{T-t_0,T},
\]
which shows that \( \varphi_{T-r,s}(x) = \tilde{\varphi}_{T-r,s}(x) \) for all \( T - t_0 \leq T - r \leq s \leq T \).
Iterating the above argument over intervals \([T - kt_0, T - (k + 1)t_0]\), we obtain the desired uniqueness result.

We shall now prove that the true process converges to \( \rho_t(x)dx \) in the hydrodynamic limit. Call \( \mathcal{P}^N \) the law of the measure valued process \( \mu_{U,(t)}(x) \), \( t \in [0,T] \). By the tightness proved in Proposition 2, we have convergence by subsequences \( \mathcal{P}^N_i \) to a measure valued process \( \mathcal{P} \). We will show that any such limit measure \( \mathcal{P} \) is given by the Dirac measure supported by the single deterministic trajectory \( \rho_t(x)dx, t \in [0,T] \), where \( \rho_t(x) \) is the limit of \( \rho_t^{(0)}(x) \) found above.

First of all we state the following support property.

**Proposition 6** Any weak limit \( \mathcal{P} \) of \( \mathcal{P}^N \) satisfies
\[
\mathcal{P}(C([0, T], S')) = 1,
\]
where \( C([0, T], S') \) is the space of all continuous trajectories \([0, T] \to S'\).

**Proof** The proof is analogous to the proof of Theorem 2.7.8 in De Masi and Presutti 1991.

Let us denote the elements of \( C([0, T], S') \) by \( \omega = (\omega_t, t \in [0, T]) \) and let \( t \in [0, T] \). Suppose \( \mathcal{P} \) is the weak limit of \( \mathcal{P}^N \). We shall prove that \( \mathcal{P} \) is supported by \( \{\omega : \omega_t = \rho_t(x)dx\} \). Thus \( \mathcal{P} \) coincides with \( \rho_t(x)dx \) on the rationals of \([0, T]\) and by continuity on all \( t \in [0, T] \) and therefore any weak limit of \( \mathcal{P}^N \) is supported by \( \rho_t dx \).

The marginal of \( \mathcal{P} \) at time \( t \) is determined by the expectations
\[
\int h(\omega_t(a_1),\ldots,\omega_t(a_k))d\mathcal{P} =: \mathcal{P}_t(h)
\]
where, as in Definition 2, \( h \) is a smooth function on \( \mathbb{R}^k \), \( k \geq 1 \), and \( a_i \) are smooth functions on \( \mathbb{R}_+ \) with compact support. We need to show that
\[
\mathcal{P}_t(h) = h\left(\int a_1(x)\rho_t(x)dx, \ldots, \int a_k(x)\rho_t(x)dx\right).
\]

In the sequel, \( t \in \mathcal{T} \) and \( \delta \in \{2^{-n}T, n \geq 1\} \). For any \( \varepsilon > 0 \) there exists \( n_0 \) such that for all \( n \geq n_0 \),
\[
|h\left(\int a_1 \rho_t dx, \ldots, \int a_k \rho_t dx\right) - h\left(\int a_1 \rho_t^{(2^{-n}T)} dx, \ldots, \int a_k \rho_t^{(2^{-n}T)} dx\right)| \leq \varepsilon.
\]
Moreover there exists $N^\ast$ so that for all $N_i \geq N^\ast$,
\[
|\mathcal{P}_t^{N_i}(h) - \mathcal{P}_t(h)| \leq \varepsilon,
\]
where $\mathcal{P}_t^{N_i}(h) := P_x^{(N_i,\lambda)}(h(\mu_U(t)))$, see (5.42). By (5.42) for $\delta$ small enough and $N_i$ large enough
\[
|P_x^{(N_i,\lambda)}(h(\mu_U(t))) - S_x^{(\delta,N_i,\lambda)}(h(\mu_Y(\delta)(t)))| \leq \varepsilon.
\]
Applying Corollary 1 for $N_i$ large enough,
\[
|S_x^{(\delta,N_i,\lambda)}(h(\mu_Y(\delta)(t))) - h\left(\int \rho^{(\delta)}_1 a_1, \ldots, \int \rho^{(\delta)}_k a_k\right)| \leq \varepsilon.
\]
Collecting the above estimates and by the arbitrariness of $\varepsilon$ we then get (7.90). This finishes the proof of Theorem 2.

8 Discussion

Let us first compare our model to the Kuramoto rotators model which has been considered, for instance, in Kuramoto (1984), Tass (1997), Bertini, Giacomin and Pakdaman (2010). There are two main differences with our model:

- In the rotators model firing occurs when an angle crosses the north pole (i.e. it goes from a value smaller than $2\pi$ to a value larger than 0). However, unlike in our model, firing has no consequences as it does not affect the state of the other rotators.

- The other main difference with our model comes when we consider the gap junction type interactions. Two angles $\theta_i$ and $\theta_j$ with $\theta_i < 2\pi$ and $\theta_j > 0$ attract each other, in the sense that $\theta_i$ increases and $\theta_j$ decreases. Thus the gap junction interaction drives rotator $i$ towards firing. This reinforcement effect is an important ingredient for synchronization in Bertini, Giacomin and Pakdaman (2010). In our model the opposite happens. If neuron $i$ has a large energy $U_i$ and neuron $j$ has just fired so that its energy $U_j$ is small, the gap junction interaction increases $U_j$, while it lowers down energy $U_i$, making more improbable that neuron $i$ fires.

Delarue, Inglis, Rubenthaler and Tanrê (2012) consider a model which is close in spirit to ours, but where spiking appears according to an “integrate and fire” scheme once the membrane potential reaches a given excitation level. This corresponds to a firing rate given by a Heaviside function. In their work, the membrane potential of each single neuron is modelized by Brownian motion or by an Ornstein-Uhlenbeck process, and randomness comes in from this additional noise added to the continuous time evolution of each neuron.

A Proof of Theorem 1

A.1 The case $\lambda = 0$

In the case $\lambda = 0$, the proof of the above theorem is reduced to the following bound.
There exists a constant $K$ depending on $A$ and $T$ such that

$$\sup_{x: ||x|| \leq A} P^{(N,0)}_x [ |N(T)| < KN ] \geq 1 - ce^{-CN}. \quad (A.91)$$

Indeed (2.4) follows from (A.91) with $B = A + K$. We start by proving (A.91) bounding $|N(T)|$ in terms of the number of jumps of an auxiliary process which we define next.

**Definition 7** Fix $x^* > 2$ and let $N^*_i, i = 1, \ldots, N$, be independent Poisson processes of intensity $f(x^*)$ each, having jump times $T^*_{in}, n \geq 1, i = 1, \ldots, N$. Define auxiliary processes

$$U^+(t) = (U^+_1(t), \ldots, U^+_N(t)) \quad \text{and} \quad N^+(t) = (N^+_1(t), \ldots, N^+_N(t)), t \geq 0,$$

in the following way.

1. We start with $U^+_i(0) = x^* - 2 + \frac{i-1}{N}$, $N^+_i(0) = 0$, for any $i = 1, \ldots, N$.

2. Suppose that we have already defined $(U^+(t), N^+(t))$ up to time $t \geq 0$. The process is then updated at the time

$$\inf \{ T^*_{in} > t \} =: s.$$

Suppose that $s$ is the jump time of $N^*_i$. Then two cases are possible.

(a) $||U^+(s-)|| \leq x^* - \frac{2}{N}$. Then we put $U^+_i(s) = x^* - 2$, $U^+_i(s) = U^+_i(s- + \frac{1}{N}$ for all $j \neq i$, $N^+_i(s) = N^+_i(s- + 1$ and $N^+_j(s) = N^+_j(s)$ for all $j \neq i$.

(b) $||U^+(s-)|| = x^* - \frac{1}{N}$. Then we reset the whole system to $U^+(s) = U^+(0)$ and update $N^+_i(s) = N^+_i(s- + 1$ for all $i$.

The following properties hold:

1. For all $i = 1, \ldots, N$, we have $U^+_i(0) = x^* - 2 + \frac{i-1}{N}$ and $N^+_i(0) = 0$.

2. For all $i \neq j$ and all $t \geq 0$, $U^+_i(t) \neq U^+_j(t)$.

3. For all $i = 1, \ldots, N$ and all $t \geq 0$, $U^+_i(t) \in \{ x^* - 2, x^* - 2 + \frac{1}{N}, \ldots, x^* - \frac{1}{N} \}$.

4. For all $i = 1, \ldots, N$, $N^+_i(t)$ is a non decreasing jump process.

**Proposition 7** Let $\lambda = 0$ and $U(0) = x$ with $||x|| \leq A$. Set $x^* = A + 3$ and let $U^+(t)$ and $N^+(t)$ be defined as above. We couple $U(t)$ and $U^+(t)$ such that any time that $U_i(t) \leq x^*$ and $U_i(t)$ fires, then $U^+_i(t)$ fires as well. Then we have $N^+_i(t) \geq N_i(t)$ for all $i$ and all $t \geq 0$.

**Proof** We shall define a process $M(t)$ which depends on both $U(t)$ and $U^+(t)$ with the property that $N_i(t) \leq M_i(t) \leq N^+_i(t)$. The existence of $M(t)$ will thus prove the proposition. More precisely we will define a process $M(t)$ such that $M_i(0) = 0$ for all $i = 1, \ldots, N$, and such that the following holds.

$$M_i(t) \in \{ N_i(t), N_i(t) + 1 \}; \quad M_i(t) \leq N^+_i(t); \quad (A.92)$$

if $M_i(t) = N_i(t)$, then $U^+_i(t) \geq U_i(t) + \frac{d(t)}{N}$, where $d(t) = \sum_{j=1}^{N} 1_{M_j(t) > N_j(t)}$. 

32
It is clear that (A.92) is satisfied at time $t = 0$. Then the definition of $M(t)$, $t > 0$, is by induction on time. So suppose that $M(\cdot)$ has already been defined on $[0, T]$ and that on $[0, T]$, (A.92) is satisfied. Let $s > T$ be the first time after $T$ when there is a fire for $U(\cdot)$, or $U^+(\cdot)$, or for both, and suppose that the fire refers to particle $i$. Then the following cases appear.

1. If $U_i^+$ does not fire at time $s$, then we put $M(s) = M(T)$. 

2. Suppose $\|U^+(T)\| \leq x^* - \frac{2}{N}$. If $U_i^+$ fires, then we put $M_j(s) = M_j(T)$ for all $j \neq i$ and $M_i(s) = (M_i(T) + 1) \land (N_i(s) + 1)$. 

3. Suppose $\|U^+(T)\| = x^* - \frac{2}{N}$. If $U_i^+$ fires, then $M_j(s) = N_j(T) + 1$ for all $j \neq i$ and $M_i(s) = (M_i(T) + 1) \land (N_i(s) + 1)$. 

From the above definition it follows that $M(t)$ jumps as $N^+ (t)$ except for some of the jumps which are suppressed and which are exactly those which violate the condition $M_i(t) \leq N_i(t) + 1$. Hence the condition $M_i(s) \leq N_i(s)$ in (A.92) is satisfied. Let us then check that also the other conditions of (A.92) hold at time $s$. We start considering the case $\|U^+(T)\| \leq x^* - \frac{2}{N}$.

1. $U_i^+$ fires and $U_i$ does not. Then $M_i(s) = N_i(T) + 1$ and (A.92) holds for $i$ because $N_i(s) = N_i(T)$. For $j \neq i$, $U_j^+(s) = U_j^+(T) + \frac{1}{N}$ while $U(s) = U(T)$ and (A.92) is again satisfied even if $d(s) = d(T) + 1$.

2. Both $U_i$ and $U_i^+$ fire. In this case, $U_i^+(s) = x^* - 2$, $U_i(s) = 0$ and (A.92) holds for $i$, because $d(s) \leq N$ and $x^* - 2 = A + 1 \geq 1$. For $j \neq i$, (A.92) holds because $U_j^+(s) = U_j^+(T) + \frac{1}{N}$, $U_j(s) = U_j(T) + \frac{1}{N}$ and $d(s) = d(T)$.

3. $U_i$ fires and $U_i^+$ does not, hence $U_i(T) > x^*$. Since $U_i^+(T) < x^*$, we deduce from the second line of (A.92) that $M_i(T) = N_i(T) + 1$. Then $U_i(s) = 0$, $N_i(s) = N_i(T) + 1$ while $U^+(s) = U^+(T)$ and $M(s) = M(T)$, so that $d(s) = d(T) - 1$. (A.92) holds for the index $i$ (by the same argument used above) because $U_i(s) = 0$. Finally, (A.92) is again satisfied for $j \neq i$ because $d(s) = d(T) - 1$ and 

$$U_j^+(s) = U_j^+(T) \geq U_j(T) + \frac{d(T)}{N} = U_j(s) + \frac{d(s)}{N},$$

since $U_j(s) = U_j(T) + \frac{1}{N}$.

The case $\|U^+(T)\| = x^* - \frac{1}{N}$ is analogous. If $U_i$ fires and $U_i^+$ does not fire then we are exactly in the same situation as in the last item. If instead $U_i^+$ fires and $U_i$ does not fire then $M_i(s) = N_i(T) + 1$ and (A.92) holds for $i$ because $N_i(s) = N_i(T)$. For $j \neq i$ (A.92) holds because $M_j(s) = N_j(T) + 1 = N_j(s) + 1$. If both $U_i^+$ and $U_i$ fire, then $U_i^+(s) = x^* - 2 + \frac{1}{N}$, $U_i(s) = 0$ and (A.92) holds for $i$, because $d(s) \leq N$ and $x^* - 2 \geq 1$. For $j \neq i$ (A.92) holds because $M_j(s) = N_j(T) + 1 = N_j(s) + 1$.

(A.91) is now a corollary of the following proposition:

**Proposition 8** Given any $x^* > 3$ and $T > 0$ there exists a constant $K$ such that 

$$P \left[ |N^+(T)| < KN \right] \geq 1 - ce^{-C(N-1)},$$

where $c$ and $C$ are suitable constants depending on $f(x^*)$ and on $T$. 

33
Proof There are two contributions to $N^+(t)$. We denote $N^{+,1}(t)$ the contributions coming from fires when $\|U^+(t)\| < x^* - \frac{1}{N}$. The contributions coming from fires when $\|U^+(t)\| = x^* - \frac{1}{N}$ shall be denoted by $N^{+,2}(t)$. We bound their probabilities separately. The law of $|N^{+,1}(t)|$ is the law of a Poisson process in $[0,t]$ of intensity $Nf(x^*)$. Thus there exists a constant $C$ depending on $f(x^*)$, such that

$$P\big[|N^{+,1}(T)| \leq 2f(x^*)TN\big] \geq 1 - e^{-CNT},$$  \hfill (A.94)

To bound $|N^{+,2}(t)|$, we call $\tau_1,\ldots,\tau_k,\ldots$ the return times of $U^+$ to the initial configuration $x^+(0)$. If $\tau_{k+1} > T$ then $|N^{+,2}(T)| \leq kN$. To bound the probability that $\tau_{k+1} > T$, we observe that $\tau_1$ can be lower-bounded by the time of occurrence of the $N-1$-th fire in $N^+(t)$. Thus

$$P\big[\tau_1 > f(x^*)\big] \geq 1 - e^{-C(N-1)},$$  \hfill (A.95)

and therefore

$$P\big[\tau_{k+1} > (k + 1)f(x^*)\big] \geq 1 - (k + 1)e^{-C(N-1)}. \hfill (A.96)$$

By choosing $(k + 1)f(x^*) > T$, we obtain

$$P\big[|N^{+,2}(T)| < kN\big] \geq 1 - ce^{-C(N-1)}. \hfill (A.97)$$

The above Proposition achieves the proof of Theorem in the case $\lambda = 0$.

A.2 The case $\lambda \neq 0$

We study now the case $\lambda > 0$. Recall that the center of mass of configuration $U(t)$ is given by

$$\bar{U}_N(t) = \frac{1}{N} \sum_{i=1}^{N} U_i(t)$$ \hfill (A.98)

and let

$$K_1(t) = \sum_{i=1}^{N} \int_0^t 1_{\{U_i(s) \leq 1\}} dN_i(s)$$ \hfill (A.99)

be the total number of fires in $[0,t]$ when $U_i \leq 1$. Then we obtain the following.

Lemma 1 We have

$$\bar{U}_N(t) \leq \bar{U}_N(0) + \frac{K_1(t)}{N}$$ \hfill (A.100)

and

$$\|U(t)\| \leq \|U(0)\| + \frac{|N(t)|}{N}. \hfill (A.101)$$

Proof Suppose $U_i$ fires at time $t$, then

$$\bar{U}_N(t) = \frac{1}{N} \sum_{j \neq i} (U_j(t-)) + \frac{1}{N} = \bar{U}_N(t-) + \frac{N - 1}{N^2} - \frac{U_i(t-)}{N}.$$
Thus the center of mass decreases if \( U_i(t-) \geq 1 \), which implies (A.100).

Since between successive jumps the largest \( U_i(t) \) is attracted towards the center of mass we can upper bound its position by neglecting the action of the gap junction, whence (A.101).

For a fixed time horizon \( T \), we shall now bound the total number of fires as in Proposition 7 in restriction to the set \( \{ K_1(T) \leq KN \} \), for some fixed constant \( K > 0 \).

**Proposition 9** Let \( \lambda \neq 0 \), \( \| U(0) \| = \| x \| \leq A \) and set \( x^* = A + K + 3 \). We couple \( U(t) \) and \( U^+(t) \) in such a way that when \( U_i(t) \leq x^* \) and \( U_i(t) \) fires, then \( U_i^+(t) \) fires as well. Then, \( \{ K_1(T) \leq KN \} \), \( N_i^+(t) \geq N_i(t) \) for all \( i \) and for all \( t \in [0,T] \).

**Proof** Since we are working on the set \( \{ K_1(T) \leq KN \} \), we have \( \bar{U}_N(t) \leq \bar{U}_N(0) + K \leq A + K = x^* - 3 \) for all \( t \leq T \). This implies that

\[
\bar{U}_N(t) \leq U_i^+(t) - 1 \quad \text{for all} \quad t \leq T.
\]

(A.102)

As a consequence, if \( U_i(t) \geq U_i^+(t) - 1 \), then \( U_i(t) \geq \bar{U}_N(t) \), and the \( \lambda \)-potential induces a negative drift.

In order to prove the proposition, we define the process \( M(t) \) exactly as in the proof of Proposition 7 and check that the following properties still hold for all \( t \leq T \).

\[
M_i(t) \in \{ N_i(t), N_i(t) + 1 \}; \quad M_i(t) \leq N_i^+(t);
\]

(A.103)

if \( M_i(t) = N_i(t) \), then \( U_i^+(t) \geq U_i(t) + \frac{d(t)}{N} \), where \( d(t) = \sum_{j=1}^{N} 1_{M_j(t) > N_i(t)} \).

It is clear that (A.103) is satisfied at time \( t = 0 \). Suppose we have checked that the conditions of (A.103) hold up to time \( t \) and let \( s > t \) be the first jump time after time \( t \). We suppose that it is \( U_i \) or \( U^+_i \) which has a jump at time \( s \) and start considering the case \( \| U^+(t) \| \leq x^* - \frac{2}{N} \).

1. Suppose that \( U_i^+ \) has a jump at time \( s \), but not \( U_i \). Then by definition, \( M_i(s) = N_i(t) + 1 = N_i(s) + 1 \), and the condition (A.103) does not need to be checked for index \( i \). Let us check that condition (A.103) holds for all \( j \neq i \). We fix \( j \neq i \) such that \( M_j(t) = N_j(t) \) which implies that \( M_j(s) = M_j(t) = N_j(t) = N_j(s) \). Suppose that the condition of the second line of (A.103) does not hold at time \( s \), for \( j \). Then

\[
U_j^+(s-) < U_j^+(s) < U_j(s) + \frac{d(s)}{N} \leq U_j(s) + 1,
\]

(A.104)

which implies, using (A.102),

\[
U_j(s) > \bar{U}_N(s).
\]

Since \( U_j \) does not jump during \( [t,s] \) and since \( \bar{U}_N \) is constant on \( [t,s] \), equation (1) implies that \( U_j(s) \leq U_j(t) \). On the other hand, since (A.103) holds at time \( t \),

\[
U_j(t) \leq U_j^+(t) - \frac{d(t)}{N} = U_j^+(s) - \frac{1}{N} - \frac{d(t)}{N} \leq U_j^+(s) - \frac{d(s)}{N},
\]

even if \( d(s) = d(t) + 1 \). Hence

\[
U_j(s) \leq U_j(t) \leq U_j^+(s) - \frac{d(s)}{N},
\]

which is a contradiction with (A.104).
2. Suppose that both $U_i^+$ and $U_i$ have a jump at time $s$. Then $d(s) = d(t)$.

Holds for $i$ since $U_i(s) = 0$ and $U_i^+(s) \geq x^* - 2 \geq 1 \geq \frac{d(s)}{N}$. We show that (A.103) holds also for all $j \neq i$. Fix $j \neq i$ such that $M_j(t) = N_j(t)$ which implies that $M_j(s) = M_j(t) = N_j(t) = N_j(s)$. Suppose (A.103) does not hold at time $s$, for $j$. Then

$$U_j^+(s) < U_j(s) + \frac{d(s)}{N} \quad \text{and} \quad U_j^+(s-) < U_j(s-) + \frac{d(s-)}{N}.$$  \hspace{1cm} (A.105)

since $U_j^+(s-) = U_j^+(s) - \frac{1}{N}$, $U_j(s-) = U_j(s) - \frac{1}{N}$ and $d(s) = d(t)$. As a consequence, using the same argument as in the preceding step,

$$U_j(s-) = \bar{U}_N(s-), \quad \text{thus} \quad U_j(s-) \leq U_j(t).$$

Since (A.103) holds at time $t$,

$$U_j(t) \leq U_j^+(t) - \frac{d(t)}{N} = U_j^+(s-) - \frac{d(s)}{N}.$$  \hspace{1cm} (A.103)

Hence

$$U_j(s-) \leq U_j(t) \leq U_j^+(s-) - \frac{d(s)}{N} = U_j^+(s-) - \frac{d(s-)}{N},$$

which is a contradiction with (A.103).

3. Finally, suppose that $U_i$ fires at time $s$, and $U_i^+$ does not. Then (A.103) is satisfied for $i$ at time $s$. Consider $j \neq i$ such that $M_j(t) = N_j(t)$, hence $M_j(s) = N_j(s)$. Then two cases have to be considered. Either $U_j(t) > \bar{U}_N(t)$. Then $U_j(s-) \leq U_j(t)$, thus $U_j(s) \leq U_j(t) + \frac{1}{N} \leq U_j^+(t) - \frac{d(t)}{N} + \frac{1}{N}$, since (A.103) is satisfied at time $t$. Now, we use that $d(s) = d(t) - 1$, hence

$$U_j(s) \leq U_j^+(t) - \frac{d(s)}{N} + \frac{1}{N} = U_j^+(t) - \frac{d(s)}{N} = U_j^+(s) - \frac{d(s)}{N},$$

since $U_j^+(s) = U_j^+(t)$. Therefore, (A.103) is satisfied at time $s$ as well.

The second case we have to consider is when $U_j(t) < \bar{U}_N(t)$. Now suppose that (A.103) is not satisfied at time $s$. Then, since $d(s) = d(t) - 1$,

$$U_j^+(s) < U_j(s) + \frac{d(s)}{N} = U_j(s-) + \frac{d(t)}{N}$$

which implies

$$U_j(s-) > U_j^+(s) - 1, \quad \text{thus} \quad U_j(s-) > \bar{U}_N(s-)$$

which is a contradiction to $U_j(t) < \bar{U}_N(t)$.

The case $\|U^+(t)\| = x^* - \frac{1}{N}$ is analogous.

We are now able to finish the proof of Theorem 1.\hspace{1cm} \bullet
A.3 Proof of Theorem 1

The process $K_1(t)$ is stochastically bounded by a Poisson process with intensity $Nf(1)$. Therefore, there exists a constant $K$ such that

$$P\left[|K_1(T)| \leq KN \right] \geq 1 - e^{-CNT}.$$

We fix such a constant $K$ and work on the set $\{K_1(T) \leq KN\}$, where

$$\|U(t)\| \leq \|U(0)\| + \frac{|N(t)|}{N} \leq \|U(0)\| + \frac{|N^+(t)|}{N},$$

for all $t \leq T$. Then the claim follows from Proposition 8.

Finally, notice that the above construction gives implicitly the proof of the existence of the process $U(t)$, since the process can be constructed explicitly, by piecing together trajectories of the deterministic flow between successive jump times, once we know that the number of jumps of the process is finite almost surely on any finite time interval.

A.4 Proof of Theorem 2 for general firing rates.

Let $f$, $T$, $A$, $B$ as in Theorem 1 $x^{(N)}$ the initial state of the neurons as in Theorem 2 and such that $\|x^{(N)}\| \leq A$. Let $\psi$ be a bounded continuous function on $D([0, T], S')$. We need to prove that

$$\lim_{N \to \infty} \mathcal{P}^N_{[0, T]}(\psi) = \psi(\rho)$$

where $\mathcal{P}^N_{[0, T]}(\psi)$ is the expected value of $\psi$ under the law of $(\rho_{U^N})_{[0, T]}$ when the process $U^N$ starts from $x^{(N)}$ and $\psi(\rho)$ is the value of $\psi$ on the element $\rho := (\rho_t dt)_{t \in [0, T]}$ of $D([0, T], S')$.

Let $1_U$ be the characteristic function of the event $\{|\|U^N(t)\| \leq B, t \in [0, T]\}$. Then by Theorem 1

$$\lim_{N \to \infty} \left| \mathcal{P}^N_{[0, T]}(\psi) - \mathcal{P}^N_{[0, T]}(\psi 1_U) \right| = 0. \quad (A.106)$$

By an abuse of notation we call $\mathcal{P}^N_{[0, T]}(\psi 1_U)$ the law of the process with a firing rate $f^*(\cdot)$ which satisfies Assumption 4 and coincides with $f$ for $x \leq B$. Then

$$\mathcal{P}^N_{[0, T]}(\psi 1_U) = \mathcal{P}^N_{[0, T]}(\psi 1_U). \quad (A.107)$$

Since we have proved Theorem 2 under Assumption 4, we know convergence for the process with rate $f^*(\cdot)$ to a limit density that we call $\rho^* = (\rho^*_t)_{t \in [0, T]}$, so that

$$\lim_{N \to \infty} \mathcal{P}^N_{[0, T]}(\psi 1_U) = \psi(\rho^* 1_U). \quad (A.108)$$

As a consequence of (A.106) and (A.107),

$$\lim_{N \to \infty} \mathcal{P}^N_{[0, T]}(\psi) = \psi(\rho^* 1_U).$$

By the arbitrariness of $\psi$, $\rho^* = \rho^* 1_U$. Indeed, taking $\psi(\omega) = \sup\{\omega(t), t \leq T\} \wedge 1$, we have $\lim_{N \to \infty} \mathcal{P}^N_{[0, T]}(\psi) \equiv 1$, which implies that $\rho^*$ must have support in $[0, B]$. As a consequence,

$$\lim_{N \to \infty} \mathcal{P}^N_{[0, T]}(\psi) = \psi(\rho^* 1_U) = \psi(\rho^*),$$

and the limit $\rho^*$ is equal to the solution of the equation with the true firing rate $f$. This concludes the proof of the theorem.
Acknowledgments

We thank M. Cassandro and D. Gabrielli for their collaborative participation to the first stage of this work. We also thank B. Cessac, P. Dai Pra and C. Vargas for many illuminating discussions.

This work is part of USP project “Mathematics, computation, language and the brain”, FAPESP project “NeuroMat” (grant 2011/51350-6), USP/COFECUB project “Stochastic systems with interactions of variable range” and CNPq project “Stochastic modeling of the brain activity” (grant 480108/2012-9). ADM is partially supported by PRIN 2009 (prot. 2009TA2595-002). AG is partially supported by a CNPq fellowship (grant 309501/2011-3). AG and EL thank GSSI for hospitality and support.

References

[1] L. Bertini, G. Giacomin, and K. Pakdaman. Dynamical aspects of mean field plane rotators and the Kuramoto model. J. Stat. Phys., 138(1-3):270–290, 2010.

[2] M.H.A. Davis. Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. J. R. Stat. Soc., Ser. B, 46:353–388, 1984.

[3] F. Delarue, J. Inlis, S. Rubenthaler, and E. Tanré. Global solvability of a networked integrate-and-fire model of McKean-Vlasov type, 2012.

[4] A. Galves and E. Löcherbach. Infinite systems of interacting chains with memory of variable length - a stochastic model for biological neural nets. J. Stat. Phys., 5:896–921, 2013.

[5] Y. Kuramoto. Chemical oscillations, waves, and turbulence. Springer Series in Synergetics, Vol. 19, Berlin, New York, Heidelberg: Springer-Verlag. VIII, 1984.

[6] A. De Masi and E. Presutti. Mathematical methods for hydrodynamic limits. Lecture Notes in Mathematics. 1501. Berlin: Springer-Verlag, 1991.

[7] I. Mitoma. Tightness of probabilities on C([0,1];S_p) and D([0,1];S_p). Ann. Probab., 11:989–999, 1983.

[8] M. G. Riedler, M. Thieullen, and G. Wainrib. Limit theorems for infinite-dimensional piecewise deterministic markov processes. Applications to stochastic excitable membrane models, 2012.

[9] P. Tass. Phase and frequency shifts in a population of phase oscillators. Phys. Rev. E, 56:2043–2060, Aug 1997.