VOLUME FUNCTIONS OF LINEAR SERIES

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Abstract. The volume of a Cartier divisor is an asymptotic invariant, which measures the rate of growth of sections of powers of the divisor. It extends to a continuous, homogeneous, and log-concave function on the whole Néron–Severi space, thus giving rise to a basic invariant of the underlying projective variety. Analogously, one can also define the volume function of a possibly non-complete multigraded linear series. In this paper we will address the question of characterizing the class of functions arising on the one hand as volume functions of multigraded linear series and on the other hand as volume functions of projective varieties. In the multigraded setting, relying on the work of Lazarsfeld and Mustaţă [16] on Okounkov bodies, we show that any continuous, homogeneous, and log-concave function appears as the volume function of a multigraded linear series. By contrast we show that there exists countably many functions which arise as the volume functions of projective varieties. We end the paper with an example, where the volume function of a projective variety is given by a transcendental formula, emphasizing the complicated nature of the volume in the classical case.

INTRODUCTION

Let \( X \) be a smooth complex projective variety of dimension \( n \) over the complex numbers, let \( D \) be a Cartier divisor on \( X \). The volume of \( D \) is defined as

\[
\text{vol}_X(D) = \limsup_{k \to \infty} \frac{\dim_{\mathbb{C}}(H^0(X, \mathcal{O}_X(kD)))}{k^n/n!}.
\]

The volume and its various versions have recently played a crucial role in several important developments in higher dimensional geometry, e.g. [23], [10].

In the classical setting of ample divisors, the volume of \( D \) is simply its top self-intersection. Starting with the work of Fujita [7], Nakayama [18], and Tsuji [24], it became gradually clear that the volume of big divisors — that is, ones with \( \text{vol}_X(D) > 0 \) — displays a surprising number of properties analogous to that of ample ones. Notably, it depends only on the numerical class of \( D \), it is homogeneous of degree \( n \), and satisfies a Lipschitz-type property ([15, Section 2.2.C]). Consequently, one can extend the volume to a continuous function

\[
\text{vol}_X : N^1(X)_\mathbb{R} \to \mathbb{R}_+,
\]

where \( N^1(X) \) is the Néron–Severi group of numerical equivalence classes of Cartier divisors on \( X \), and by \( N^1(X)_\mathbb{R} \) we mean the finite-dimensional real vector space of numerical equivalence.

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classes of $\mathbb{R}$-divisors. Besides continuity and homogeneity, another important feature of the volume function is log-concavity of degree $n$, i.e. for any two classes $\xi, \xi' \in \text{Big}(X)_\mathbb{R}$ we have
\[
\text{vol}_X(\xi + \xi')^{1/n} \geq \text{vol}_X(\xi)^{1/n} + \text{vol}_X(\xi')^{1/n}.
\]
Given a sufficient amount of information, the volume function associated to a variety can be explicitly computed under special circumstances. Examples include all smooth surfaces, toric varieties, and homogeneous spaces.

However, beside what has been mentioned above, relatively little is known about its global behavior, and understanding it more clearly remains a very important quest.

In [16], Lazarsfeld and Mustaţă showed that in fact most of the properties of $\text{vol}_X$ are quite formal in nature, and their validity can be extended to the non-complete multigraded setting. Specifically, fix Cartier divisors $D_1, \ldots, D_\rho$ on $X$ (where $\rho$ is an arbitrary positive integer for the time being, but soon it will be $\text{dim}_\mathbb{R} N^1(X)_\mathbb{R}$), and write $mD = m_1D_1 + \ldots + m_\rho D_\rho$ for $m = (m_1, \ldots, m_\rho) \in \mathbb{N}^\rho$. A multigraded linear series $W_\star$ on $X$ associated to $D_1, \ldots, D_\rho$ consists of subspaces
\[
W_m \subseteq H^0(X, \mathcal{O}_X(mD)) ,
\]
such that $R(W_\star) = \oplus W_m$ is a subalgebra of the section ring
\[
R(D_1, \ldots, D_\rho) = \oplus_{m \in \mathbb{N}^\rho} H^0(X, \mathcal{O}_X(mD)) .
\]
The support of $W_\star$ is the closed convex cone in $\mathbb{R}^\rho_+$ spanned by all multi-indices $m \in \mathbb{N}^\rho$ such that $W_m \neq 0$. Given $a \in \mathbb{N}^\rho$, set
\[
\text{vol}_{W_\star}(a) \overset{\text{def}}{=} \limsup_{k \to \infty} \frac{\dim_{\mathbb{C}}(W_{k,a})}{k^n/n!}.
\]
Exactly as in the complete case, the above assignment defines the volume function of $W_\star$
\[
\text{vol}_{W_\star} : \mathbb{N}^\rho \to \mathbb{R}_+.
\]
Based on earlier work of Okounkov, [19] and [20], the authors of [16] associate a convex cone — the so-called Okounkov cone — to a multigraded linear series on a projective variety. With the help of convex geometry and semigroup theory they show that (under very mild hypotheses) the formal properties of the global volume function persist in the multigraded setting. Precisely as in the global case, the function $m \mapsto \text{vol}_{W_\star}(m)$ extends uniquely to a continuous function
\[
\text{vol}_{W_\star} : \text{int}(\text{supp}(W_\star)) \to \mathbb{R}_+ ,
\]
which is homogeneous, log-concave of degree $n$, and extends continuously to the entire $\text{supp}(W_\star)$, (cf. Remark 1.4). The construction generalises the classical case: whenever $X$ is an irreducible projective variety, the cone of big divisors $\text{Big}(X)_\mathbb{R}$ is pointed and $\text{vol}_X$ vanishes outside of it. Pick Cartier divisors $D_1, \ldots, D_\rho$ on $X$, whose classes in $N^1(X)_\mathbb{R}$ generate a cone containing $\text{Big}(X)_\mathbb{R}$. Then $\text{vol}_X = \text{vol}_{W_\star}$ on $\text{Big}(X)_\mathbb{R}$, where $W_\star = R(D_1, \ldots, D_\rho)$.

In this "in vitro" setting, we prove first that in fact any continuous, homogeneous, and log-concave function arises as the volume function of an appropriate multigraded linear series.

\[\text{In their interesting paper} \ [3], \text{Boucksom-Favre-Jonsson found a nice formula for the derivative of} \ \text{vol}_X \text{in any direction.}\]
Theorem A. Let $K \subseteq \mathbb{R}_+^\rho$ be a closed convex cone with non-empty interior, $f : K \to \mathbb{R}_+$ a continuous function, which is non-zero, homogeneous, and log-concave of degree $n$ in the interior of $K$. Then there exists a smooth, projective variety $X$ of dimension $n$ and Picard number $\rho$, a multigraded linear series $W_\bullet$ on $X$ and a positive constant $c > 0$ such that $\text{vol}_{W_\bullet} \equiv c \cdot f$ on the interior of $K$. Moreover we have $\text{supp}(W_\bullet) = K$.

As a consequence, we observe that the volume function $\text{vol}_{W_\bullet}$ of a multigraded linear series $W_\bullet$ can be pretty wild. To support this claim, let us point out the following connection. Alexandroff [1] showed that a function as in Theorem A is almost everywhere twice differentiable; at the same time, one can give examples of functions of this sort, which are nowhere three times differentiable (cf. Remark 1.2). This gives a positive answer to [16, Problem 7.2].

As for the proof of Theorem A, the linear series we construct lives on a smooth projective toric variety $X$ of dimension $n$ and Picard number $\rho$.

For the proof of Theorem A we first check that any pointed cone in $\mathbb{R}_+^\rho \times \mathbb{R}_+^\rho$, modulo shrinking, is the Okounkov cone of some multigraded linear series on $X$. This will show further that the volume of the linear series is the same as the Euclidean volume function of slices of its Okounkov cone. We finish by verifying that any function as in Theorem A is the Euclidean volume function of a cone.

It follows from Theorem A that there exist uncountably many volume functions in the non-complete case. In comparison, in the complete case we prove that in fact there are only countably many of them:

Theorem B. Let $V_\mathbb{Z} = \mathbb{Z}^\rho$ be a lattice inside the vector space $V_\mathbb{R} = V_\mathbb{Z} \otimes \mathbb{Z} \mathbb{R}$. Then there exist countably many functions $f_j : V_\mathbb{R} \to \mathbb{R}_+$ with $j \in \mathbb{N}$, so that for any irreducible projective variety $X$ of dimension $n$ and Picard number $\rho$, we can construct an integral linear isomorphism $\rho_X : V_\mathbb{R} \to N^1(X)_\mathbb{R}$ with the property that $\text{vol}_X \circ \rho_X = f_j$ for some $j \in \mathbb{N}$.

We prove Theorem B in the case of smooth varieties, from which the general case follows easily by appealing to resolution of singularities. The heart of the proof is a careful analysis of the variation of the volume function in families coming from multi-graded Hilbert schemes. This approach will also help us to establish the analogous statement for the ample, nef, big, and pseudoeffective cones. We would like to point out that the claim regarding the countability of ample or nef cones also follows from the work of Campana and Peternell [4] on the algebraicity of the ample cone.

An amusing application of Theorem B concerns the set of volumes $V \subseteq \mathbb{R}_+$, which is the set of all non-negative real numbers arising as the volume of a Cartier divisor on some irreducible projective variety. Using Theorem B, one can deduce that $V$ has the structure of a countable multiplicative semigroup (cf. Remark 2.3). By contrast, in the last section we give an example of a four-fold whose volume function is given by a transcendental function, deepening further the mystery surrounding the volume function in the classical case. In particular, the same example provides a Cartier divisor with transcendental volume, thus the set of volumes $V$ contains also transcendental numbers.
The transcendental nature of the volume function in a geometrically simple situation leads to a new relation linking complex geometry to diophantine questions. In their inspiring work [14] (see also [25]), Kontsevich and Zagier make a strong argument for the concept of a period. According to their definition, a complex number $\alpha$ is a period, if it can be written as the integral of a rational function with rational coefficients over an algebraic domain (a subset or Euclidean space determined by polynomial inequalities with rational coefficients).

By their very definition, periods are countable in number, and contain all algebraic numbers. On the other hand, various transcendental numbers manifestly belong to this circle, $\pi$ or the natural logarithms of positive integers among them. It is straightforward to verify that periods form a ring with respect to the usual operations on real numbers. Although it is obvious from cardinality considerations that most real numbers are not elements of this ring, so far there is not a single real number that has been proved not to be a period (although there are notable candidates such as $e$, the base of the natural logarithm).

Using results of Lazarsfeld and Mustață from [16], the volume of a Cartier divisor $D$ can be written as
\[ \text{vol}_X(D) = \int_{\Delta_{Y^*}(D)} 1, \]
where $\Delta_{Y^*}(D)$ is the Okounkov body of $D$ with respect to any admissible complete flag $Y^*$ of subvarieties in $X$. This means that $\text{vol}(D)$ — originally defined as the asymptotic rate of growth of the number of global sections of multiples of $D$ — is a period in a very natural way, whenever $\Delta_{Y^*}(D)$ is an algebraic domain for some suitably chosen admissible flag. This happens in all the cases that have been explicitly computed so far, leading to the following question.

**Question.** Is the volume of an integral Cartier divisor on an irreducible projective variety always a period?

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1. **Volume functions of non-complete linear series**

In this section we study the volume function of a multigraded linear series and verify that any non-zero continuous homogeneous log-concave function appears (modulo compressing) as the volume function of some multigraded linear series on a smooth projective toric variety of dimension $n$ and Picard number $\rho$.

First we introduce some notation. In course of this section $X$ will be a smooth projective toric variety of dimension $n$, corresponding to a fan $\Delta(X)$ in $N_\mathbb{R} \cong \mathbb{R}^n$, so that the torus $T = N \otimes \mathbb{Z} \mathbb{C}^*$ acts on $X$. Let $v_1, \ldots, v_{n+\rho}$ be the primitive ray generators of $\Delta(X)$, where the ordering is chosen in such a way that $v_1, \ldots, v_n$ form a $\mathbb{Z}$-basis of $N$ and generate a maximal
cone in \( \Delta(X) \). Let \( D_1, \ldots, D_{n+\rho} \) be the corresponding \( T \)-invariant prime divisors of \( X \). By general theory (see [5, Chapter 4] or [8, Section 3.3]), every effective Cartier divisor on \( X \) is a linear combination of the \( D_i \)'s with non-negative integral coefficients. Also, we can and will assume that \( D_{n+1}, \ldots, D_{n+\rho} \) form a \( \mathbb{Z} \)-basis of \( \text{Pic}(X) \).

Fix big Cartier divisors \( H_1, \ldots, H_{\rho} \in \text{Pic}(X) \), which form a \( \mathbb{Z} \)-basis of the Picard group of \( X \), and consider the integral linear isomorphism

\[
\rho_X : \mathbb{R}^\rho \to N^1(X)_{\mathbb{R}},
\]

where \( \rho_X(e_i) = H_i \) for all \( i = 1, \ldots, \rho \). Its image \( \rho_X(\mathbb{R}^\rho_+) \) is the positive closed convex cone spanned by the \( H_i \)'s and contained in the big cone of \( X \). Under these circumstances, Theorem A is implied by the following result.

**Theorem 1.1.** Let \( f : K \to \mathbb{R}_+ \) be a continuous function, which is non-zero, homogeneous and log-concave of degree \( n \) in the interior of \( K \). Then there exists a multigraded linear series \( W_\bullet \) on \( X \) satisfying

\[
W_m \subseteq H^0(X, \mathcal{O}_X(\rho_X(m)))
\]

for any \( m \in \mathbb{N}^\rho \) such that \( \text{supp}(W_\bullet) = K \) and \( \text{vol}_{W_\bullet} \equiv c \cdot f \) in the interior of \( K \) for some positive constant \( c \).

**Remark 1.2.** Functions as in Theorem A can be constructed from continuous concave functions \( g : B \to \mathbb{R}_+ \) defined on a bounded convex body \( B \subseteq \mathbb{R}^{\rho-1}_+ \). Let \( H \subseteq \mathbb{R}^\rho_+ \) be an affine hyperplane, not containing the origin, so that \( H \cap K = B \) is bounded. Then the function

\[
g \overset{\text{def}}{=} \sqrt{f} : B \to \mathbb{R}_+
\]

has all the properties we need. So to find irregular functions as in Theorem A, we can focus on continuous and concave functions. As an example, we can take a negative, bounded and nowhere differentiable continuous function defined on a closed interval. Upon integrating it twice, we obtain a continuous, concave and nowhere three times differentiable function.

The main idea of the proof of Theorem 1.1 is to use Okounkov cones, as introduced by Lazarsfeld and Mustaţă in [16]. We briefly recall their construction here. Let \( Z \) be a projective variety and fix a flag

\[
Y_\bullet : Y_0 = \overline{Z} \supseteq Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_{n-1} \supseteq Y_n = \{ \text{pt} \}
\]

of irreducible subvarieties of codimension \( i \) in \( Z \), such that each \( Y_i \) is nonsingular at the point \( Y_n \). We will call this an admissible flag. Let \( D \) be a Cartier divisor on \( Z \) and \( s \in H^0(Z, \mathcal{O}_X(D)) \) a non-zero section. Denote by \( D' \) the effective Cartier divisor associated to \( s \). By restricting to a suitably small open neighborhood of the point \( Y_n \) in which the divisor \( Y_i \subseteq Y_{i-1} \) is Cartier for each \( i = 1, \ldots, n \), we can recursively define an integral vector

\[
\nu(s) = (\nu_1(s), \ldots, \nu_n(s)) \in \mathbb{N}^n
\]

as follows. Let \( D_0 = D' \), and \( \nu_i(s) \) be the coefficient of \( Y_i \) in \( D_{i-1} \), where \( D_i = (D_{i-1} - \nu_i Y_i)|_{Y_i} \). Now, given a multigraded linear series \( V_\bullet \) on \( Z \), we define the multigraded semigroup of \( V_\bullet \) with respect to the admissible flag \( Y_\bullet \) to be

\[
\Gamma(V_\bullet) \overset{\text{def}}{=} \Gamma_{Y_\bullet}(V_\bullet) = \{ (\nu(s), m) \mid 0 \neq s \in V_m \} \subseteq \mathbb{N}^n \oplus \mathbb{N}^\rho.
\]
The Okounkov cone $\Delta_{Y \bullet}(V \bullet) \subseteq \mathbb{R}_+^n \times \mathbb{R}_+^p$ is defined to be the closed convex cone spanned by $\Gamma(V \bullet)$. The said cone is pointed, and the slice $\Delta_{Y \bullet}(V \bullet) \cap (\mathbb{R}_+^n \times \{m\})$ is a compact convex set by [16, Lemma 1.10]. At the same time if $p_2 : \mathbb{R}_+^n \times \mathbb{R}_+^p \to \mathbb{R}_+^p$ is the second projection, then it is easy to see that $\text{supp}(V \bullet) = p_2(\Delta_{Y \bullet}(V \bullet))$.

Returning to Theorem 1.1 we choose on $X$ the following flag of $T$-invariant subvarieties:

$$Y \bullet : Y_0 = X \supseteq Y_1 = D_1 \supseteq Y_2 = D_1 \cap D_2 \supseteq \ldots \supseteq Y_n = D_1 \cap \ldots \cap D_n.$$ 

The choice we made upon the arrangement of the $v_i$'s tells us that each $Y_i$ is a smooth $T$-invariant subvariety of codimension $i$ in $X$.

Next, take $V \bullet = (V_m)_{m \in \mathbb{N}^p}$ to be the complete multigraded linear series on $X$ associated to $H_1, \ldots, H_\rho$, i.e. set

$$V_m = H^0(X, \mathcal{O}_X(\rho_X(m))) = H^0(X, \mathcal{O}_X(m_1 H_1 + \ldots + m_\rho H_\rho))$$

for each $m = (m_1, \ldots, m_\rho) \in \mathbb{N}^p$, and let $\Delta_{Y \bullet}(V \bullet) \subseteq \mathbb{R}_+^n \times \mathbb{R}_+^p$ be the Okounkov cone of $V \bullet$ with respect to the flag $Y \bullet$.

With this in hand we can proceed to prove Theorem 1.1. We first show that any non-empty closed convex cone $\Delta \subseteq \Delta_{Y \bullet}(V \bullet)$ is the Okounkov cone of some multigraded linear subseries $W \bullet \subseteq V \bullet$.

**Proposition 1.3.** Let $\Delta \subseteq \Delta_{Y \bullet}(V \bullet)$ be a closed convex cone with non-empty interior. Then $\Delta$ is the Okounkov cone of an appropriately chosen multigraded linear subseries $W \bullet \subseteq V \bullet$ with respect to $Y \bullet$. Moreover, $p_2(\Delta) = \text{supp}(W \bullet)$ and we have

$$\text{vol}_{\mathbb{R}}(\Delta \cap (\mathbb{R}_+^n \times \{m\})) = \frac{1}{n!} \cdot \text{vol}_\bullet(m),$$

for any $m \in (\text{int}(p_2(\Delta)) \cap \mathbb{N}^p)$, where the volume on the left-hand side denotes the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$.

**Proof.** By the general theory of toric varieties lattice points $u \in M_\mathbb{Z} = N_\mathbb{Z}$ correspond to characters $\chi^u$ that are rational functions on $X$. For an effective $T$-invariant divisor $D = a_1 D_1 + \ldots + a_\rho D_\rho$ with $a_i \in \mathbb{N}$, there exists a basis of isotypical sections of $H^0(X, \mathcal{O}_X(D))$ of the form $\chi^u$. Since every effective integral divisor on $X$ is linearly equivalent to a $T$-invariant divisor and $V \bullet$ is a complete multigraded linear series, Lazarsfeld and Mustaţă show [16, Section 6.1] that the multigraded semigroup of $V \bullet$ is described as

$$\Gamma_{Y \bullet}(V \bullet) = \{ (\nu_{Y \bullet}(\chi^u), m) \mid m \in \mathbb{N}^p, \chi^u \in H^0(X, \mathcal{O}_X(\rho_X(m))) \}.$$

In particular, $V \bullet$ is a finitely generated semigroup. Consequently, $\Delta_{Y \bullet}(V \bullet) \cap (\mathbb{N}^n \times \mathbb{N}^p)$ is the saturation of $\Gamma_{Y \bullet}(V \bullet)$, hence for any vector $\sigma \in \Delta_{Y \bullet}(V \bullet) \cap (\mathbb{N}^n \times \mathbb{N}^p)$, there is a natural number $m = m_\sigma > 0$ such that $m_\sigma \cdot \sigma \in \Gamma_{Y \bullet}(V \bullet)$.

Set $\Gamma_m(V \bullet) = \Gamma(V \bullet) \cap (\mathbb{N}^n \times \{m\})$ for $m \in \mathbb{N}^p$, and let $\Gamma_m = \Delta \cap \Gamma_m(V \bullet)$. Denote by $W_m$ the linear span of the rational functions

$$\{ \chi^u \mid (\nu_{Y \bullet}(\chi^u), m) \in \Gamma_m \}.$$

Then $W \bullet = (W_m)_{m \in \mathbb{N}^p}$ will be the multigraded linear series we are interested in.
Because $\Delta$ is a cone, and $\Gamma(V_\bullet)$ is a semigroup, $\Gamma = \cup \Gamma_m \subseteq \mathbb{N}^n \times \mathbb{N}^\rho$ is also a semigroup. Thus we have
$$W_m \cdot W_n \subseteq W_{m+n}, \text{ for any } m, n \in \mathbb{N}^\rho.$$

It remains to show that $\Delta_{V_\bullet}(W_\bullet) = \Delta$. Since $\Gamma = \Gamma_{V_\bullet}(V_\bullet) \cap \Delta$, $\Delta \cap (\mathbb{N}^n \times \mathbb{N}^\rho)$ is the saturated semigroup of $\Gamma$. Therefore it suffices to check that $\Delta$ is generated by the set $\Delta \cap (\mathbb{N}^n \times \mathbb{N}^\rho)$. As $\Delta$ is pointed, one can choose an affine hyperplane $H$ with rational coefficients, not containing the origin, such that $\Delta \cap H$ is compact and generates $\Delta$. The set $\Delta$ has non-empty interior, hence the same can be said about $\Delta \cap H$. Therefore the set of its rational points is dense and generates $\Delta$ as a closed convex cone. The second claim follows from [16, Lemma 1.3].

Remark 1.4. The previous proposition implies that the function
$$\text{vol}_{W_\bullet} : \text{int}(\text{supp}(W_\bullet)) \to \mathbb{R}_+$$
equals the volume of slices of $\Delta$. Thus it is continuous, homogeneous and log-concave of degree $n$ in the interior of $\text{supp}(W_\bullet)$. Since $W_\bullet \subseteq V_\bullet$, the function $(\text{vol}_{W_\bullet})^{1/n}$ is bounded:
$$(\text{vol}_{W_\bullet}(v))^{1/n} \leq k_1||v||, \text{ for all } v \in \text{supp}(W_\bullet)$$
for some $k_1 > 0$. The concavity of $(\text{vol}_{W_\bullet})^{1/n}$ implies that the function vol$_{W_\bullet}$ satisfies a Hölder-type condition of degree $n$ (see [21, Theorem 1.5.1]):
$$|(\text{vol}_{W_\bullet}(v))^{1/n} - (\text{vol}_{W_\bullet}(w))^{1/n}| \leq k_2||v - w||$$
for all $v, w \in \text{int}(\text{supp}(W_\bullet))$. The boundedness and the Hölder-type condition above imply that the function vol$_{W_\bullet}$ can be extended continuously to the whole support of $W_\bullet$.

To finish off the proof of Theorem 1.1 our last task is to show that any function as in Theorem 1.1 is the volume function of some cone $\Delta \subseteq \mathbb{R}^n_+ \times \mathbb{R}^\rho_+$, defined as the Euclidean volume of the slices $\Delta \cap (\mathbb{R}^n_+ \times \{v\})$ for $v \in \mathbb{R}^n_+$. For lack of a suitable reference we will give a proof of this fact.

Proposition 1.5. If $f : K \rightarrow \mathbb{R}_+$ is a function as in Theorem A, then there exists a closed convex cone in $C \subseteq \mathbb{R}^n_+ \times K$ with nonempty interior such that
$$f(v) = \text{vol}_{\mathbb{R}^n}(\{w \in \mathbb{R}^n_+ \mid (w, v) \in C\})$$
for all $v \in \text{int}(K)$.

Proof. In order to lighten the presentation, we define
$$r(v) \overset{\text{def}}{=} \sqrt[n]{f(v)/C_n}$$
for every $v \in K$, where $C_n$ is a positive constant chosen such that the volume of a ball with radius $r(v)$ equals $f(v)$. The idea is to find first a linear map $g : \mathbb{R}^\rho \rightarrow \mathbb{R}^n$ with the property that the ball
$$B_{g(v)}(r(v)) = \{w \in \mathbb{R}^n \mid ||w - g(v)|| \leq r(v)\}$$
is contained in $\mathbb{R}^n_+$ for all $v \in K$. To find such a map we proceed as follows. Since the cone $K$ is pointed, it is possible to choose a linear form $l$, strictly positive on $K \setminus \{0\}$. The function $\sqrt[n]{f(v)/l(v)}$ is homogeneous of degree 0, and continuous on $K$, hence it is
bounded from above. Hence after choosing an appropriate positive constant \( k \), the linear map \( g(v) = k(l(v), \ldots, l(v)) \) has the required property. The cone \( C \) that we are looking for, will be the closure of the open set

\[
C' = \{ (w, v) \in \text{int}(\mathbb{R}_+^n) \times \text{int}(K) \mid ||w - g(v)|| < r(v) \}.
\]

It remains to show that \( C' \) is an open convex cone. As \( f \) is homogeneous of degree \( n \) in the interior of \( K \), \( C' \) is an open cone contained in \( \mathbb{R}_+^n \times \mathbb{R}_+^n \) such that

\[
\text{vol}_{\mathbb{R}_+^n} \{ w \in \mathbb{R}_+^n \mid (w, v) \in C \} = \text{vol}_{\mathbb{R}_+^n} (B_{g(v)}(r(v))) = f(v)
\]

for any \( v \in \text{int}(K) \). As for the convexity of \( C' \), choose two points \( (w_1, v_1), (w_2, v_2) \in C' \) and let \( (w_3, v_3) = (w_1 + w_2, v_1 + v_2) \). From the fact that \( g \) is linear, we obtain

\[
||w_3 - g(v_3)|| \leq ||w_1 - g(v_1)|| + ||w_2 - g(v_2)|| \leq \sqrt{\frac{f(v_1)}{C_n}} + \sqrt{\frac{f(v_2)}{C_n}} \leq \sqrt{\frac{f(v_3)}{C_p}}.
\]

Here the first inequality is the triangle inequality, the second is a consequence of \( \Delta_{V^*}(V') \) for \( i = 1, 2 \), and the last follows from the log-concavity of \( f \). This tells us that \( (w_3, v_3) \in C' \) and therefore \( C' \) is indeed convex.

\[ \square \]

Remark 1.6. To conclude the proof of Theorem 1.1 we need to verify that every cone \( C \subseteq \mathbb{R}_+^n \times \mathbb{R}_+^n \) can be shrunk in the direction \( \mathbb{R}_+^n \times \{ * \} \) so that it is contained in \( \Delta_{V^*}(V') \). If this is valid — because we shrink along \( \mathbb{R}_+^n \times \{ * \} \) — then the volume function compresses without changing its domain. This shrinking procedure is possible as the intersection of the boundary of the cone \( \Delta_{V^*}(V') \) with any face \( \mathbb{R}_+^n \times \mathbb{R}_+^{n-1} \) of the cone \( \mathbb{R}_+^n \times \mathbb{R}_+^n \) has a non-empty interior, since we chose the \( H_i \)'s to be big Cartier divisors. Thus the intersection \( \Delta_{V^*}(V') \cap (\mathbb{R}_+^n \times \{ * \}) \) has always non-empty interior.

2. COUNTABILITY OF VOLUME FUNCTIONS FOR COMPLETE LINEAR SERIES.

One of the consequences of the previous section is that for non-complete multigraded linear series there are uncountably many different volume functions. By contrast, we will prove that there are only countably many volume functions for all irreducible projective varieties.

Theorem 2.1. Let \( V = \mathbb{Z}^p \) be a lattice inside the vector space \( V_{\mathbb{R}} \). Then there exist countably many closed convex cones \( A_i \subseteq V_{\mathbb{R}} \) and functions \( f_j : V_{\mathbb{R}} \to \mathbb{R} \) with \( i, j \in \mathbb{N} \), so that for any smooth projective variety \( X \) of dimension \( n \) and Picard number \( \rho \), we can construct an integral linear isomorphism

\[
\rho_X : V_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}
\]

with the properties that

\[
\rho_X^{-1}(\text{Nef}(X)_{\mathbb{R}}) = A_i, \text{ and } \text{vol}_X \circ \rho_X = f_j
\]

for some \( i, j \in \mathbb{N} \).

Remark 2.2. (1) Theorem 2.1 quickly implies Theorem B: let \( X \) be an irreducible projective variety and let \( \mu : X' \to X \) be a resolution of singularities of \( X \). The pullback map

\[
\mu^* : N^1(X)_{\mathbb{R}} \to N^1(X')_{\mathbb{R}}
\]
is linear, injective, and \( \text{vol}_X = \text{vol}_{X'} \circ \mu^* \) by \cite{Kollar} Example 2.2.49. Since the map \( \mu^* \) is defined by choosing \( \dim(N^1(X)_\mathbb{R}) \) integral vectors, the countability of the volume functions in the smooth case implies that the same statement is valid for the collection of irreducible varieties. As \( \text{Nef}(X)_\mathbb{R} = (\mu^*)^{-1}(\text{Nef}(X')_\mathbb{R}) \), the same statement holds for nef cones.

(2) From the fact that \( \text{Amp}(X)_\mathbb{R} = \text{int}(\text{Nef}(X)_\mathbb{R}) \), we can see that Theorem B remains valid for ample cones as well. Much the same way, the cone of big divisors can be described as

\[
\text{Big}(X)_\mathbb{R} = \{ D \in N^1(X)_\mathbb{R} \mid \text{vol}_X(D) > 0 \};
\]

its closure is known to be equal to the pseudo-effective cone \( \text{Eff}(X)_\mathbb{R} \). Hence we conclude that Theorem B is also valid for the big and pseudoeffective cones.

**Remark 2.3** (The semigroup of volumes). Let

\[
\mathbb{V} = \{ a \in \mathbb{R}_+ \mid a = \text{vol}_X(D) \text{ for some pair } (X, D) \}\]

where \( X \) is some irreducible projective variety and \( D \) a Cartier divisor on \( X \). By Theorem 2.1 \( \mathbb{V} \) is countable. Moreover, using the Künneth formula \cite[Proposition 4.5]{Kollar}, one can show that the set \( \mathbb{V} \) has the structure of a multiplicative semigroup with respect to the product of real numbers. Beyond this fact very little is known about \( \mathbb{V} \). It is certainly true by \cite[Example 2.3.6]{Kollar} and the semigroup structure of \( \mathbb{V} \) that all non-negative rational numbers are contained in \( \mathbb{V} \). At the same time we do not know whether all algebraic numbers appear as volumes of Cartier divisors. Going in the other direction, we provide an example in Section 3 of a pair \( (X, D) \) such that the volume \( \text{vol}_X(D) \) is transcendental.

We will make some preparations. Let \( \phi : \mathcal{X} \to T \) be a smooth projective and surjective morphism of relative dimension \( n \) between two quasi-projective varieties. Suppose further that \( T \) and each fiber of \( \phi \) is irreducible and reduced. If we are given \( \rho \) Cartier divisors \( D_1, \ldots, D_\rho \) on \( \mathcal{X} \) then we say that a closed point \( t_0 \in T \) admits a good fiber if \( D_1|_{X_{t_0}}, \ldots, D_\rho|_{X_{t_0}} \) form a basis for the Néron–Severi space \( N^1(X_{t_0})_\mathbb{R} \). The main ingredient of the proof of Theorem 2.1 is the following statement.

**Proposition 2.4.** Let \( \phi : \mathcal{X} \to T \) be a family as above and suppose that there exists a closed point \( t_0 \in T \), admitting a good fiber. Then for all closed points \( t \in T \) the Cartier divisors \( D_1|_{X_t}, \ldots, D_\rho|_{X_t} \) are linear independent in \( N^1(X_t)_\mathbb{R} \).

**Proof.** First notice that \( D_1|_{X_t}, \ldots, D_\rho|_{X_t} \) are linearly dependent in \( N^1(X_t)_\mathbb{R} \) if and only if they are linear dependent over integers. Therefore we only need to show that given a Cartier divisor \( D \) on \( \mathcal{X} \) such that \( D|_{X_{t_0}} \not\equiv_{\text{num}} 0 \), one has \( D|_{X_t} \not\equiv_{\text{num}} 0 \) for any \( t \in T \).

We use induction on the dimension of the fibers. First assume that \( \dim(\mathcal{X}) = \dim(T) + 1 \). As \( X_{t_0} \) is a smooth irreducible curve, the condition \( D|_{X_{t_0}} \not\equiv_{\text{num}} 0 \) is equivalent to \( (D.X_{t_0}) \neq 0 \). The morphism \( \phi \) is smooth and \( T \) irreducible, therefore the function

\[
t \in T \longrightarrow (D.X_t)
\]

is globally constant. Consequently, \( D|_{X_t} \neq 0 \) for any \( t \in T \) as we wanted.

In the general case, when \( n \geq 2 \), let \( t_1 \in T \setminus \{ t_0 \} \) and choose a line bundle \( A \) on \( \mathcal{X} \) which is very ample relative to the map \( \phi \). Bertini’s Theorem and generic smoothness says
that for a general section $W$ of $A$, the fiber $W_t = W \cap X_t$ is smooth and irreducible for all $t$'s in some open neighborhood of $t_0$. The same statement holds for $t_1$, and using the fact that $T$ is irreducible, one can choose a general section $W$ and an open neighborhood $U \subseteq T$ containing both $t_0$ and $t_1$, such that $W_t$ is smooth and irreducible for all $t \in U$. Now, as $W$ is general, the map

$$\phi_W^U = \phi|_{W \cap \phi^{-1}(U)} : W \cap \phi^{-1}(U) \rightarrow U$$

is flat and of relative dimension $n - 1$. Because each fiber of $\phi_W^U$ is smooth, $\phi_W^U$ is smooth as well. With this in hand, suppose that $D|_{W_{t_0}} \neq \text{num} 0$. By applying induction to the family $\phi_W^U$, we obtain $D|_{W_{t_1}} \neq \text{num} 0$, hence $D|_{X_{t_1}} \neq \text{num} 0$.

Whenever $D|_{W_{t_0}} = \text{num} 0$, we have two cases. If $n = 2$, we can use the fact that $W_{t_0}$ is an ample section of $X_{t_0}$ and deduce from the Hodge Index Theorem that $(D|_{X_{t_0}})^2 < 0$. Hence by flatness one obtains that $(D|_{X_{t_1}})^2 < 0$ and therefore $D|_{X_{t_1}} \neq \text{num} 0$. When $n \geq 3$, one can use a higher-dimensional version of the Hodge Index Theorem [13, Corollary I.4.2] and deduce that the condition $D|_{W_{t_0}} = \text{num} 0$ implies $D|_{X_{t_0}} = \text{num} 0$, contradicting our assumptions. \hfill $\square$

**Proof of Theorem 2.1.** Our first step is to embed every smooth projective variety $X$ of dimension $n$ and Picard number $\rho$ into a product of projective spaces, i.e.

$$X \subseteq Y = \mathbb{P}^{2n+1} \times \ldots \times \mathbb{P}^{2n+1}$$

with the property that the restriction map

$$\rho_X : V_{\mathbb{R}} := N^1(Y)_{\mathbb{R}} = \mathbb{R}^\rho \rightarrow N^1(X)_{\mathbb{R}}$$

is an integral linear isomorphism.

To this end fix $\rho$ very ample Cartier divisors $D_{1,X}, \ldots, D_{\rho,X}$ on $X$, which form a $\mathbb{Q}$-basis of the Néron–Severi group $N^1(X)_{\mathbb{Q}}$. As $X$ is a smooth variety, [22, Theorem 5.4.9] implies that for each $D_{i,X}$ there exists an embedding $X \subseteq \mathbb{P}^{2n+1}$ with $O_X(D_{i,X}) = O_{\mathbb{P}^{2n+1}}(1)|_X$. With this in hand, we embed $X$ in $Y$ in the following manner

$$X \subseteq \underbrace{X \times \ldots \times X}_{\rho \text{ times}} \subseteq Y,$$

where the first embedding is given by the diagonal. The corresponding restriction map $\rho_X$ on the Néron–Severi groups is an integral linear isomorphism identifying the semigroup $\mathbb{N}^\rho \subseteq N^1(Y)_{\mathbb{Z}}$ with the one generated by $D_{1,X}, \ldots, D_{\rho,X}$ in $N^1(X)_{\mathbb{Z}}$.

Next, we construct countably many families such that each smooth variety $X$ embedded in $Y$ as in (1) appears as a fiber in at least one of them. We will use multigraded Hilbert schemes of subvarieties embedded in $Y$ for this purpose. Before introducing them, we note that each line bundle on $Y$ is of the form

$$O_Y(m) = p_1^*(O_{\mathbb{P}^{2n+1}}(m_1)) \otimes \ldots \otimes p_\rho^*(O_{\mathbb{P}^{2n+1}}(m_\rho))$$

with $m = (m_1, \ldots, m_\rho) \in \mathbb{Z}^\rho$ and $p_i : Y \rightarrow \mathbb{P}^{2n+1}$ being the $i$th projection. For a closed subscheme $X \subseteq Y$, one can define its multigraded Hilbert function as

$$P_{X,Y}(m) = \chi(X,(O_Y(m))|_X),$$

for all $m \in \mathbb{Z}^\rho$.
The Hilbert functor $H_{Y,P}(T)$ parameterizes families of closed subschemes $Z \subseteq Y \times T$ flat over $T$ such that for any $t \in T$ the multigraded Hilbert function of the scheme-theoretical fiber $Z_t \subseteq Y$ equals $P$. In [11, Corollary 1.2], Haiman and Sturmfels prove that this functor is representable, i.e. for any $\rho \geq 1$ and $P$ as above the multigraded Hilbert functor $H_{Y,P}$ is represented by a projective scheme $\text{Hilb}_{Y,P}$ and by an universal family

$$
U_P \subseteq Y \times \text{Hilb}_{Y,P} \xrightarrow{pr_2} \text{Hilb}_{Y,P}
$$

with the property that there is a bijection between the closed subschemes of $Y$ with the multigraded Hilbert function equal $P$ and the scheme theoretical fibers of $\phi$.

In the case when $X$ is a smooth projective variety of dimension $n$ and Picard number $\rho$ embedded in $Y$ as in (1), its multigraded Hilbert function $P_{X,Y}$ is a polynomial with rational coefficients and of total degree at most $n \cdot \rho$. Hence there are countably many polynomials of this form, and therefore countably many families such that any smooth projective variety of dimension $n$ and Picard number $\rho$ appears as a fiber in at least one of them.

By what we said above, it is enough to verify countability for one of these flat families. Fix one of them, and call it $\phi : \mathcal{X} \to T$. Without loss of generality we can assume that $T$ is irreducible and reduced. The fact that $\phi$ is flat implies by [9, Theorem 12.2.4] that the set of all $t \in T$ for which $X_t$ is smooth, irreducible, and reduced, is open.

Arguing inductively on $\dim T$, we can restrict our attention to a non-empty open subset of $T$ and assume that all the fibers of $\phi$ are smooth, irreducible, and reduced. This will imply furthermore that the map $\phi$ is smooth, so it is enough to prove countability under this additional condition.

The embedding $\mathcal{X} \subseteq Y \times T$ tells us that $\mathcal{X}$ comes equipped with $\rho$ Cartier divisors $D_1, \ldots, D_{\rho}$, the restriction of the canonical base of $\text{Pic}(Y)$. By setting $V_\mathbb{R} \overset{\text{def}}{=} \mathbb{R}D_1 \oplus \cdots \oplus \mathbb{R}D_{\rho}$, we can assume further that there exists a closed point $t_0 \in T$ such that $X_{t_0}$ is a smooth variety embedded in $Y$ as in (1). Hence the Cartier divisors $D_1|_{X_{t_0}}, \ldots, D_{\rho}|_{X_{t_0}}$ form an $\mathbb{R}$-basis for $N^1(X_{t_0})_\mathbb{R}$, and the family $\phi : \mathcal{X} \to T$ satisfies the conditions in Proposition 2.4. We conclude that the map

$$
\rho_{X_t} : V_\mathbb{R} = \mathbb{R}D_1 \oplus \cdots \oplus \mathbb{R}D_{\rho} \to N^1(X_t)_\mathbb{R}, \quad \text{where } \rho_{X_t}(D_i) := D_i|_{X_t}
$$

is an injective integral linear morphism for all $t \in T$.

With these preparations behind us we can move on to complete the proof. Let us write

$$
A_t \overset{\text{def}}{=} \rho_{X_t}^{-1}(\text{Nef}(X_t)_\mathbb{R}), \quad \text{and } f_t \overset{\text{def}}{=} \text{vol}_{X_t} \circ \rho_{X_t}
$$

for each $t \in T$. We need to show that both sets $(A_t)_{t \in T}$ and $(f_t)_{t \in T}$, are countable. Actually, it is enough to check that there exists a subset $F = \cup F_m \subseteq T \ (B = \cup B_m \subseteq T)$ consisting of a countable union of proper Zariski-closed subsets $F_m \nsubseteq T$ (resp. $B_m \nsubseteq T$), such that $A_t$ (resp. $f_t$) is independent of $t \in T \setminus F \ (t \in T \setminus B)$. This reduction immediately implies
Theorem 2.1, because one can argue inductively on \( \dim(T) \) and apply this reduction for each
family \( \phi : \phi^{-1}(F_m) \to F_m \) containing a good fiber.

We first prove the above reduction for nef cones. The set of all cones \( (A_t)_{t \in T} \) has the
following property: if \( t_0 \in T \), then there exists a subset \( \bigcup F_{t_0}^m \not\subseteq T \), which does not contain
\( t_0 \) and consists of a countable union of proper Zariski-closed sets such that
\[
A_{t_0} \subseteq A_t, \text{ for all } t \in T \setminus \bigcup F_{t_0}^m.
\]
To verify this claim choose an element \( D \in A_{t_0} \cap \mathbb{R}^\rho \). By [15, Theorem 1.2.17] on the behaviour
of nefness in families, there exists a countable union \( F_{t_0,D} \subseteq T \) of proper subvarieties of \( T \),
not containing \( t_0 \) such that \( D \in A_t \), for all \( t \)-s outside of \( F_{t_0,D} \). As \( A_{t_0} \) is a closed pointed
cone, the set \( A_{t_0} \cap \mathbb{R}^\rho \) is countable and generates \( A_{t_0} \) as a closed convex cone. Thus the cone
\( A_{t_0} \) is included in \( A_t \) for all \( t \)-s outside of the subsets \( F_{t_0,D} \) with \( D \in A_{t_0} \cap \mathbb{R}^\rho \). Our base field
is uncountable, therefore the union of all of the \( F_{t_0,D} \)-s still remains a proper subset of \( T \).

Denoting \( A \overset{\text{def}}{=} \bigcup_{t \in T} A_t \), it is enough to find a closed point \( t \in T \) with \( A_t = A \). Note that
\( A \subseteq V_{\mathbb{R}} = \mathbb{R}^\rho \) is second countable, so there exists a countable set
\( \{ t_i \in T \mid i \in \mathbb{N} \} \) such that \( A = \bigcup_{i \in \mathbb{N}} A_{t_i} \)
according to [17, Theorem 30.3].

By [2], for every \( i \in \mathbb{N} \) there exists a countable union of proper Zariski-closed subsets
\( F_i \not\subseteq T \) with the property that
\[
A_{t_i} \subseteq A_t, \text{ for all } t \in T \setminus F_i,
\]
and as before \( \bigcup F_i \) remains a proper subset. This proves Theorem 2.1 in the case of nef cones
because we have \( A_{t_i} \subseteq A_t \) and hence \( A_t = A \) for each \( t \in T \setminus \bigcup F_i \) and \( i \in \mathbb{N} \).

Next, we turn out attention to the case of volume functions. We assumed each fiber
\( X_t \) to be smooth and irreducible. As the volume function is continuous, and homogeneous
of degree \( n \), it is actually enough to prove that for any \( D \in V_{\mathbb{R}} \) the volume \( \text{vol}_{X_t}(D|_{X_t}) \) is
independent of \( t \in T \setminus B \).

Pick a Cartier divisor \( D \in V_{\mathbb{R}} \). It is a consequence of the Semicontinuity Theorem [12,
Theorem III.12.8], that for any \( d \in \mathbb{N} \) there exists a proper Zariski-closed subset \( B_{D,d} \not\subseteq T \),
such that
\[
h^0(t, \mathcal{O}_X(dD)) = \dim_C H^0(X_t, \mathcal{O}_X(dD)|_{X_t}) \text{ is independent of } t \in T \setminus B_{D,d}.
\]
The definition of the volume implies that \( \text{vol}_{X_t}(D|_{X_t}) \) is independent of \( t \in T \setminus \bigcup_{d \in \mathbb{N}} B_{D,d} \)
and, because \( V_{\mathbb{R}} \) is countable, the union of \( \bigcup B_{D,d} \) for all \( D \in V_{\mathbb{R}} \) and \( d \in \mathbb{N} \), is a countable
union of proper Zariski-closed subsets properly contained in \( T \).

3. An example of a transcendental volume function

The aim of this section is to give an example of a four-fold \( X \) such that the volume
function \( \text{vol}_X \) is given by a transcendental function over an open subset of \( N^1(X)_{\mathbb{R}} \). We
utilize a construction of Cutkosky (see [6] or [15, Chapter 2.3]).
Let $E$ be a general elliptic curve (that is, one without complex multiplication), and set $Y = E \times E$. [15] Lemma 1.5.4 gives a full description of all the cones on $Y$. Let $f_1, f_2$ denote the divisor classes of the fibers of the projections $Y \to E$, and $\Delta$ the class of the diagonal, then

$$\text{Nef}(Y)_R = \overline{\text{Eff}(Y)}_R = \{ x \cdot f_1 + y \cdot f_2 + z \cdot \Delta \mid xy + xz + yz \geq 0, x + y + z \geq 0 \}.$$ 

Setting $H_1 = f_1 + f_2 + \Delta$, $H_2 = -f_1$ and $H_3 = -f_2$, we define the vector bundle

$$V = \mathcal{O}_{E \times E}(H_1) \oplus \mathcal{O}_{E \times E}(H_2) \oplus \mathcal{O}_{E \times E}(H_3);$$

$\pi : X = \mathbb{P}(V) \to Y$ will be the four-fold of our interest.

**Proposition 3.1.** With notation as above, there exists a non-empty open set in $\text{Big}(X)_R$, where the volume is given by a transcendental formula.

**Proof.** The characterization of line bundles on projective space bundles, and the fact that the function $\text{vol}_X$ is continuous, and homogeneous on $\text{Big}(X)_R$, imply that it is enough to handle line bundles of the form

$$A = \mathcal{O}_{F(Y)}(1) \otimes \pi^*(\mathcal{O}_Y(L'))$$

with $L' = c_1 f_1 + c_2 f_2 + c_3 \Delta$ a Cartier divisor on $Y$ with $(c_1, c_2, c_3) \in \mathbb{N}^3$. Using the projection formula the volume of $A$ is given by

$$\text{vol}_X(A) = \sum_{a_1 + a_2 + a_3 = m} \frac{h^0(mL' + a_1 H_1 + a_2 H_2 + a_3 H_3)}{m^4/24},$$

where the sum is taken over all natural $a_i$'s. Riemann-Roch on the K3 surface $Y$ has the simple form

$$h^0(Y, \mathcal{L}) = \frac{1}{2}(\mathcal{L}^2)$$

for an ample line bundle $\mathcal{L}$. It is not hard to show that in the sum above only the contribution of ample divisors $L = L' + a_1 H_1 + a_2 H_2 + a_3 H_3$ counts, hence one can write the volume as

$$\text{vol}_X(A) = \lim_{m \to \infty} \frac{4!}{2m^4} \sum_{a_1 + a_2 + a_3 = m} ((mc_1 + a_1 - a_2)f_1 + (mc_2 + a_1 - a_3)f_2 + (mc_3 + a_1)\Delta)^2.$$ 

Upon writing $x_i = a_i/m$ and making a change of coordinates $T : \mathbb{R}^3 \to \mathbb{R}^3$,

$$T(c_1) = c_1 + x_1 - x_2, \quad T(c_2) = c_2 + x_1 - x_3, \quad T(c_3) = c_3 + x_1,$$

we can write

$$\text{vol}_X(A) = \int_{\Gamma(c_1, c_2, c_3)} (T(c_1)f_1 + T(c_2)f_2 + T(c_3)\Delta)^2,$$

where the set $\Gamma(c_1, c_2, c_3)$ is the intersection of the image of the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ by the map $T$ and the nef cone $\text{Nef}(Y)_R$. As a more concrete example, when $c_1 = c_2 = c_3 = 1/4$ it looks like Figure 1. Indeed, one can use Maple to find that $\text{vol}(A)$ is given by a transcendental formula in the $c_i$'s. 

**Remark 3.2.** In the example above, using the same idea it is not hard to see that $\text{vol}_X(\mathcal{O}_X(1))$ is a transcendental number.
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