Onset of spontaneous scalarization in spinning Gauss-Bonnet black holes

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It has recently been proved numerically that spinning black holes in Einstein-scalar theories which are characterized by a non-minimal negative coupling of the scalar field to the Gauss-Bonnet invariant of the curved spacetime may develop exponentially growing instabilities. Intriguingly, it has been demonstrated that this tachyonic instability, which marks the onset of the spontaneous scalarization phenomenon in the Einstein-Gauss-Bonnet-scalar theory, characterizes spinning black holes whose dimensionless angular momentum parameter \( \tilde{a} \equiv a/M \) is larger than some critical value \( \tilde{a}_{\text{crit}} \equiv 0.505 \). In the present paper we prove, using analytical techniques, that the critical rotation parameter which marks the boundary between bald Kerr black holes and hairy (scalarized) spinning black holes in the Einstein-Gauss-Bonnet-scalar theory is given by the exact dimensionless relation \( \tilde{a}_{\text{crit}} = \frac{1}{2a} \).

I. INTRODUCTION

Various no-hair theorems have established the physically interesting fact that, within the framework of classical general relativity, black holes with spatially regular horizons cannot support static scalar hairy configurations which are minimally coupled to gravity \( \eta > 0 \). This no-hair property also characterizes the black-hole solutions of the non-linearly coupled Einstein-scalar field equations in which the scalar fields are non-minimally coupled to the Ricci scalar \( R \) of the corresponding curved spacetimes \( S, B \).

Intriguingly, it has recently been revealed that spatially regular scalar fields which are non-minimally coupled to the Gauss-Bonnet curvature invariant \( \mathcal{G} \) can be supported in static \( S, B \) and stationary \( S, B \) black-hole spacetimes (see \( \text{[13,14]} \) for the closely related phenomenon of charged black holes that support spatially regular scalar fields which are non-minimally coupled to the electromagnetic tensor of the charged spacetime).

In particular, it has been explicitly proved in the physically important works \( \text{[5,12]} \) that, within the extended framework of a Scalar-Tensor-Gauss-Bonnet theory whose action contains a direct coupling term of the form \( f(\phi)\mathcal{G} \) between the scalar field \( \phi \) and the Gauss-Bonnet curvature invariant \( \mathcal{G} \), the canonical Schwarzschild and Kerr black-hole solutions of the Einstein field equations may become unstable to linearized perturbations of the non-minimally coupled scalar fields. This is a tachyonic instability which is characterized by the presence of an effective negative (squared) mass term of the form \( -\eta \phi\mathcal{G} \) in the wave equation of the linearized non-minimally coupled scalar field, where the physical parameter \( \eta \) controls the strength of the non-minimal coupling between the scalar field and the Gauss-Bonnet curvature invariant in the linearized regime [see Eq. (7) below].

The exponentially growing tachyonic instabilities of the linearized non-minimally coupled scalar fields in the Schwarzschild and Kerr \( \text{[12]} \) black-hole spacetimes mark the onset of the spontaneous scalarization phenomenon in the extended Einstein-Gauss-Bonnet-scalar theory \( \text{[5,12]} \). In particular, as explicitly demonstrated in \( \text{[5,12]} \), the boundary between bald (scalarless) black-hole solutions of the field equations and hairy (scalarized) composed black-hole-scalar-field configurations in the Scalar-Tensor-Gauss-Bonnet theory is marked by the presence of Schwarzschild (or Kerr) black holes that support linearized spatially regular configurations of the non-minimally coupled scalar fields (see also \( \text{[13,14]} \)).

The spontaneous scalarization phenomenon of spinning black holes in the Einstein-Gauss-Bonnet-scalar theory with positive values of the coupling parameter \( \eta \) has been studied in the physically important work \( \text{[11]} \). Most recently, Dima, Barausse, Franchini, and Sotiriou \( \text{[12]} \) have studied numerically the (in)stability properties of spinning Kerr black holes to linearized perturbations of the non-minimally coupled scalar fields with negative values of the physical coupling parameter \( \eta \). Intriguingly, it has been revealed in \( \text{[12]} \) that the non-minimally coupled Einstein-Gauss-Bonnet-scalar theory with \( \eta < 0 \) is characterized by the presence of a critical black-hole rotation parameter \( \tilde{a} \equiv a/M \approx 0.505 \),

\[
\left( \frac{a}{M} \right)_{\text{crit}} \approx 0.505 ,
\]

which separates stable Kerr black holes from rapidly spinning Kerr black holes that develop exponentially growing instabilities in response to linearized perturbations of the non-minimally coupled scalar fields.

The physically interesting numerical results presented in \( \text{[12]} \) implies, in particular, that black holes with sufficiently high spins \( a \geq a_{\text{crit}} \approx 0.505 M \) in the non-minimally coupled Einstein-Gauss-Bonnet-scalar theory with negative
values of the physical coupling parameter $\eta$ are expected to be characterized by the intriguing spontaneous scalarization phenomenon.

The main goal of the present paper is to explore, using analytical techniques, the onset of the spontaneous scalarization phenomenon (the onset of the tachyonic instabilities) in spinning black-hole spacetimes of the extended Scalar-Tensor-Gauss-Bonnet theory with negative values of the physical coupling parameter $\eta$. In particular, below we shall explicitly prove that the critical rotation parameter $(a/M)_{\text{crit}} \simeq 0.505$ that was first computed numerically in the physically interesting work [12], and which marks the boundary between bald (scalarless) Kerr black holes and hairy (scalarized) spinning black-hole spacetimes of the non-minimally coupled Einstein-Gauss-Bonnet-scalar theory, can be determined analytically.

II. DESCRIPTION OF THE SYSTEM

We shall analyze the onset of the spontaneous scalarization phenomenon in the composed Einstein-Gauss-Bonnet-scalar theory in which the scalar field is characterized by a non-minimal negative coupling to the Gauss-Bonnet curvature invariant of a spinning Kerr black hole. The curved black-hole spacetime is described by the line element

$$ds^2 = \frac{-\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [a dt - (r^2 + a^2) d\phi]^2 ,$$

where $M$ and $a$ are respectively the mass and angular momentum per unit mass of the black hole. The metric functions in (2) are given by the functional expressions $\Delta \equiv r^2 - 2Mr + a^2$ and $\rho^2 \equiv r^2 + a^2 \cos^2 \theta$. The outer and inner horizon radii of the Kerr black-hole spacetime are determined by the roots of the metric function $\Delta$:

$$r_{\pm} = M \pm (M^2 - a^2)^{1/2} .$$

The composed Einstein-Gauss-Bonnet-scalar theory is characterized by the action [9, 10]

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi + f(\phi) \mathcal{G} \right] .$$

(4)

Here

$$\mathcal{G} \equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$$

(5)

is the Gauss-Bonnet invariant of the curved spacetime. For a spinning Kerr black hole, the Gauss-Bonnet invariant is given by the ($r$ and $\theta$ dependent) functional expression [12]

$$\mathcal{G}_{\text{Kerr}}(r, \theta) = \frac{48M^2}{(r^2 + a^2 \cos^2 \theta)^6} \cdot \left( r^6 - 15a^2 r^4 \cos^2 \theta + 15a^4 r^2 \cos^4 \theta - a^6 \cos^6 \theta \right) .$$

(6)

The term $f(\phi)\mathcal{G}$ in the action (4) controls the non-minimal coupling between the scalar field and the Gauss-Bonnet invariant of the curved and spinning black-hole spacetime. In the linearized regime, the scalar coupling function has the universal quadratic behavior [9, 10, 13]

$$f(\phi) = \frac{1}{2} \eta \phi^2 ,$$

(7)

where the physical parameter $\eta$, which can be positive [11] or negative [12], controls the strength of the non-minimal coupling between the scalar field and the Gauss-Bonnet curvature invariant [25]. As explicitly shown in [9, 10, 13], the simple quadratic form (7) of the scalar coupling function guarantees that the bald (scalarless) Schwarzschild and Kerr black-hole spacetimes are valid solutions of the Einstein field equations in the trivial $\phi \to 0$ limit.

A variation of the action (4), which characterizes the non-minimally coupled Einstein-Gauss-Bonnet-scalar theory, with respect to the scalar field $\phi$ yields the differential equation [26]

$$\nabla^\mu \nabla_\mu \phi = - f'(\phi) \mathcal{G} ,$$

(8)

which determines the spatial and temporal behavior of the non-minimally coupled scalar field in the spinning black-hole spacetime.
As shown in [12], a projection of the differential equation [8] onto a basis of the angular-dependent spherical harmonic functions $Y_{lm}(\theta, \varphi)$ [27,28] yields, in the linearized regime, the coupled spatio-temporal differential equations [12]

$$
\left[(r^2+a^2)^2-a^2\Delta(1-c_{im}^2)\right]\ddot{\psi}_{lm} + a^2\Delta(c_{l+2,m}^n\ddot{\psi}_{l+2,m} + c_{l-2,m}^n\ddot{\psi}_{l-2,m})
+ i\lambda_0 M r \dot{\psi}_{lm} - (r^2+a^2)^2\dot{\psi}_{lm} - 2i\lambda_0 (r^2+a^2 - 2a^2\Delta/r)\dot{\psi}_{lm}
+ \Delta l(l+1) + 2M/r - a^2/2 + 2i\lambda_0/r\dot{\psi}_{lm} + \Delta \sum_{j} \langle lm|\mu^2_{\text{eff}}(r^2+a^2\cos^2\theta)|jm\rangle \psi_{jm} = 0, 
$$

where

$$
\psi_{lm}(t,r) \equiv \int r\phi Y_{lm} Y_{lm}^{*} d\Omega \equiv \langle lm|\phi|lm\rangle 
$$

and [12]

$$
c^{m}_{jl} \equiv \langle lm|\cos^2\theta|jm\rangle. 
$$

Interestingly, the differential equation [9] is characterized by the presence of an effective mass term $\mu^2_{\text{eff}}$, which is a direct consequence of the non-minimal coupling between the scalar field and the Gauss-Bonnet curvature invariant [9]. This spatially-dependent effective mass term is given by the mathematical expression [12]

$$
\mu^2_{\text{eff}} = -\eta \mathcal{G}. 
$$

In the next section we shall study analytically the onset of the spontaneous scalarization phenomenon in the composed Einstein-Gauss-Bonnet-scalar theory [3] with negative values [12] of the non-minimal coupling parameter $\eta$. As discussed in [7,12], this intriguing physical phenomenon is related to the presence of an effective negative (squared) mass term in the wave equation of the non-minimally coupled scalar field which, in the vicinity of the black-hole horizon, acts as an effective binding potential.

### III. ONSET OF THE SPONTANEOUS SCALARIZATION PHENOMENON IN SPINNING GAUSS-BONNET BLACK-HOLE SPACETIMES

In the present section we shall determine, using analytical techniques, the critical value $(a/M)_{\text{crit}}$ of the dimensionless black-hole rotation parameter which marks the onset of tachyonic instabilities in the non-minimally coupled Einstein-Gauss-Bonnet-scalar theory [3] with negative values [12] of the non-minimal coupling parameter $\eta$. As explicitly demonstrated in [7,12], these tachyonic instabilities are closely related to the intriguing phenomenon of black-hole spontaneous scalarization. In particular, the critical rotation parameter $(a/M)_{\text{crit}}$ marks the boundary between bald Kerr black holes and hairy (scalarized) spinning black holes in the Einstein-Gauss-Bonnet-scalar theory [3] with negative values of the coupling parameter $\eta$ [29].

As nicely demonstrated numerically in [12], for a given value of the Kerr black-hole rotation parameter $a$, the onset of the tachyonic instabilities in the composed black-hole-nonminimally-coupled-linearized-scalar-field system is marked by the presence of marginally-stable scalar modes with

$$
m^* = 0 
$$

which are characterized by an infinitely long instability timescale. In particular, at the onset of the tachyonic instability, the mixed term $\Delta \sum_j \langle lm^*|\mu^2_{\text{eff}}(r^2+a^2\cos^2\theta)|jm^*\rangle \psi_{jm^*}$ in Eq. (9) can be replaced by a single term of the form

$$
\Delta \langle l_1 m^*|0|\mu^2_{\text{eff}}(r^2+a^2\cos^2\theta)|l_2 m^* = 0\rangle \psi_{l_1 m^* = 0} = 0
$$

at asymptotically late times.

Interestingly, it has been revealed in [7,12] that the physically intriguing phenomenon of black-hole spontaneous scalarization is related to the presence of an effective binding potential well [an effective negative (squared) mass term, see Eqs. (15) and (14)] in the vicinity of the black-hole outer horizon whose two turning points $\{r_{\text{in}}, r_{\text{out}}\}$ are characterized by the relation $r_{\text{out}} \geq r_{\text{in}} = r_+$. In particular, the marginally-stable critical black hole (with $a = a_{\text{crit}}$),
which marks the boundary between bald Kerr black holes and hairy (scalarized) spinning black holes in the Einstein-
Gauss-Bonnet-scalar theory \(4\) with negative values of the coupling parameter \(\eta\), is characterized by the presence of a degenerate binding potential well whose two turning points merge at the outer horizon of the black hole \[30\].

\[
\langle l_1 m^* = 0 | \mu_2^2 (r^2 + a^2 \cos^2 \theta) | l_2 m^* = 0 \rangle_{r=r_{in}=r_{out}(a_{\text{crit}})} = 0 \quad \text{for} \quad a = a_{\text{crit}} .
\]  

(15)

Taking cognizance of Eqs. \(3\), \(6\), \(10\), \(12\), \(13\), and \(15\), one finds that the critical rotation parameter \(a_{\text{crit}}\) of the non-minimally coupled Einstein-Gauss-Bonnet-scalar theory \(4\) with \(\eta < 0\) can be determined by the (rather cumbersome) integral relation

\[
\int_0^{\pi} \frac{1 - 15a^2 r_+^4 \cos^2 \theta + 15a^4 r_+^2 \cos^2 \theta - a^6 \cos^6 \theta}{(r_+^2 + a^2 \cos^2 \theta)^5} \cdot Y_{l_1 m^* = 0}(\cos \theta) Y_{l_2 m^* = 0}(\cos \theta) \sin \theta d\theta = 0 \quad \text{for} \quad a = a_{\text{crit}} .
\]  

(16)

Interestingly, and most importantly for our analysis, the resonance equation \(16\) can be solved analytically.

To see this, it proves useful to define the dimensionless variables

\[
\hat{a} \equiv \frac{a_{\text{crit}}}{r_+}
\]  

(17)

and

\[
x \equiv \hat{a} \cdot \cos \theta ,
\]  

(18)

in terms of which the characteristic integral equation \(16\) can be written in the form

\[
\int_0^{\hat{a}} \frac{1 - 15x^2 + 15x^4 - x^6}{(1 + x^2)^5} \cdot Y_{l_1 m^* = 0}(x/\hat{a}) Y_{l_2 m^* = 0}(x/\hat{a}) dx = 0 .
\]  

(19)

The integral \(19\) can be evaluated analytically. To illustrate this, we shall now present some examples:

1. For \((l_1, l_2) = (0, 0)\) one obtains from the integral \(19\) the resonance equation

\[
\hat{a}^4 - 8\hat{a}^2 + 3 = 0 ,
\]  

whose solution is given by the compact dimensionless expression

\[
\hat{a} = \sqrt{4 - \sqrt{13}} .
\]  

(20)

(21)

Taking cognizance of Eqs. \(3\), \(17\), and \(21\), one finds

\[
\left( \frac{a}{M} \right)_{\text{crit}} = \sqrt{\frac{11 + \sqrt{13}}{18}} \simeq 0.9008 \quad \text{for} \quad (l_1, l_2) = (0, 0) .
\]  

(22)

It is worth emphasizing that the analytically derived value \(22\) agrees remarkably well with the numerical results presented in \([31]\) (see, in particular, Fig. 2 of \([31]\)).

(2) For \((l_1, l_2) = (1, 1)\) one obtains from the integral \(19\) the resonance equation

\[
3\hat{a}^4 - 8\hat{a}^2 + 1 = 0 ,
\]  

whose solution is given by the compact dimensionless expression

\[
\hat{a} = \sqrt{4 - \sqrt{13}} .
\]  

(23)

(24)

Taking cognizance of Eqs. \(3\), \(17\), and \(24\), one finds

\[
\left( \frac{a}{M} \right)_{\text{crit}} = \sqrt{\frac{11 - \sqrt{13}}{18}} \simeq 0.6409 \quad \text{for} \quad (l_1, l_2) = (1, 1) .
\]  

(25)

It is worth emphasizing again that the analytically derived value \(25\) agrees remarkably well with the numerical results presented in \([31]\) (see, in particular, Fig. 1 of \([31]\)).
Interestingly, in the asymptotic $l_1 = l_2 \to \infty$ limit the spherical harmonic function $Y_{l_1 m_1}(\cos \theta)$ approaches a delta-function which is peaked around the poles $\theta = 0, \pi$ (or, equivalently, around $x = \pm \hat{a}$) \[32\]. In this large-$l$ limit the integral equation \[19\] yields the remarkably simple resonance relation

$$1 - 15\hat{a}^2 + 15\hat{a}^4 - \hat{a}^6 = 0,$$

whose solution is given by

$$\hat{a} = 2 - \sqrt{3}.$$  \[27\]

Taking cognizance of Eqs. (3), (17), and (27), one finds the simple dimensionless expression

$$\left( \frac{a}{M} \right)_{\text{crit}} = \frac{1}{2} \quad \text{for} \quad l_1 = l_2 \to \infty$$

for the critical black-hole rotation parameter.

Remarkably, the analytically derived critical black-hole spin parameter \[28\] agrees extremely well with the critical value \[1\] that was first computed numerically in the physically important work \[12\]. As nicely shown numerically in \[12\], the critical black-hole rotation parameter $a_{\text{crit}}$ marks the onset of tachyonic instabilities (the onset of the spontaneous scalarization phenomenon) in the non-minimally coupled Einstein-Gauss-Bonnet-scalar theory \[4\]. In particular, slowly spinning Kerr black holes with rotation parameters in the regime $a < a_{\text{crit}}$ do not develop tachyonic instabilities and cannot carry spatially regular scalar hairy configurations in the field theory \[4\] with negative values of the physical coupling parameter $\eta$.

\section{IV. SUMMARY}

It has recently been proved \[7-12\] that black holes in Einstein-Gauss-Bonnet-scalar theories in which the scalar field is non-minimally coupled to the Gauss-Bonnet invariant of the curved spacetime can support spatially regular bound-state configurations of the scalar field.

In particular, the recently published important work \[12\] has revealed the physically intriguing fact that in Einstein-Gauss-Bonnet-scalar theories with negative values of the physical coupling parameter $\eta$, there exists a critical black-hole rotation parameter [see Eq. (1)],

$$\bar{a}_{\text{crit}} \equiv \left( \frac{a}{M} \right)_{\text{crit}} \simeq 0.505,$$

which separates stable Kerr black holes from rapidly spinning black holes that develop exponentially growing tachyonic instabilities in response to linearized perturbations of the non-minimally coupled scalar fields.

Motivated by the intriguing numerical observation \[29\] presented in \[12\], in the present paper we have used analytical techniques in order to study the onset of the spontaneous scalarization phenomenon of spinning black holes in the non-minimally coupled Einstein-Gauss-Bonnet-scalar theory \[4\] with negative values of the coupling parameter $\eta$.

In particular, we have proved that the critical black-hole rotation parameter $\bar{a}_{\text{crit}}$, which marks the boundary between bald Kerr black holes and hairy (scalarized) spinning black holes in the Einstein-Gauss-Bonnet-scalar theory, is given by the simple dimensionless relation [see Eq. (28)]

$$\bar{a}_{\text{crit}} = \frac{1}{2}.$$  \[30\]

It is interesting to note that the analytically derived critical black-hole rotation parameter \[30\] agrees remarkably well with the corresponding numerically computed critical rotation parameter \[29\] of \[12\].

Finally, it is worth emphasizing again that the physical significance of the critical black-hole rotation parameter \[30\] stems from the fact that it marks the boundary between bald (scalarless) Kerr black holes and hairy (scalarized) black-hole solutions of the non-minimally coupled Einstein-Gauss-Bonnet-scalar theory with negative values of the physical coupling parameter $\eta$ \[53\].
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[1] J. D. Bekenstein, Phys. Rev. D 5, 1239 (1972).
[2] T. P. Sotiriou, Class. Quant. Grav. 32, 214002 (2015).
[3] C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D 24, 1542014 (2015).
[4] T. P. Sotiriou and V. Faraoni, Phys. Rev. Lett. 108, 081103 (2012).
[5] A. E. Mayo and J. D. Bekenstein and, Phys. Rev. D 54, 5059 (1996).
[6] S. Hod, Phys. Lett. B 771, 521 (2017) arXiv:1911.08371, S. Hod, Phys. Rev. D 96, 124037 (2017) arXiv:2002.05093.
[7] T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. Lett. 112, 251102 (2014); T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. D 90, 124063 (2014).
[8] T. P. Sotiriou, Lect. Notes Phys. 892, 3 (2015) arXiv:1404.2955.
[9] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. 120, 131103 (2018).
[10] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou, and E. Berti, Phys. Rev. Lett. 120, 131104 (2018).
[11] P. V. P. Cunha, C. A. R. Herdeiro, and E. Radu, Phys. Rev. Lett. 123, 011101 (2019).
[12] A. Dima, E. Barausse, N. Franchini, and T. P. Sotiriou, arXiv:2006.03095v2.
[13] C. A. R. Herdeiro, E. Radu, N. Sanchis-Gual, and J. A. Font, Phys. Rev. Lett. 121, 101102 (2018).
[14] P. G. S. Fernandes, C. A. R. Herdeiro, A. M. Pombo, E. Radu, and N. Sanchis-Gual, Class. Quant. Grav. 36, 134002 (2019) arXiv:1909.05079.
[15] S. Hod, Phys. Lett. B 798, 135025 (2019) arXiv:2002.01948.
[16] S. Hod, Phys. Rev. D 101, 104025 (2020) arXiv:2005.10268.
[17] To the best of our knowledge, most studies in the physics literature of the tachyonic instabilities of black holes to scalar perturbations in extended Scalar-Tensor-Gauss-Bonnet theories have focused on the case of spherically symmetric spacetimes.

The physically interesting case of non-spherically symmetric spinning black holes in extended Scalar-Tensor-Gauss-Bonnet theories has been studied in [11,12].

[18] S. Hod, Phys. Rev. D 100, 064039 (2019) arXiv:1912.07680.
[19] S. Hod, The Euro. Phys. Jour. C 79, 966 (2019).
[20] Here $M$ is the black-hole mass and $a \equiv J/M$ is the angular momentum per unit mass of the black hole. We shall henceforth assume $a > 0$ without loss of generality.
[21] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation,* (W. H. Freeman, San Francisco, 1973).
[22] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, (Oxford University Press, New York, 1983).
[23] We shall use natural units in which $8\pi G = c = \hbar = 1$.
[24] Here $(t, r, \theta, \phi)$ are the Boyer-Lindquist spacetime coordinates.
[25] Note that $\eta$, the physical coupling parameter of the theory, has the dimensions of length$^2$.
[26] Here $f' \equiv df/d\phi$.
[27] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, 1988).

[28] The angular parameters $\{l, m\}$ are characterized by the relation $l \geq |m|$.
[29] Note that spinning black holes with non-minimally coupled scalar hair and a negative coupling parameter $\eta$ are expected to be characterized by the relation $a \geq a_{\text{crit}}$ [12].
[30] It is worth noting that, as explicitly demonstrated numerically in [12], the critical boundary (with $a = a_{\text{crit}}$) between stable and unstable Kerr black holes in the composed theory [13] with negative values of the physical coupling parameter $\eta$ is characterized by the relation $\eta \to -\infty$. Thus, in the $a \to a_{\text{crit}}$ ($\eta \to -\infty$) limit the last term in Eq. (5), which is proportional to the coupling parameter $\eta$ [see Eqs. (5) and (12)], dominates the spatial behavior of the effective potential well [the effective (squared) mass term] in the vicinity of the black-hole horizon.
[31] A. Dima, E. Barausse, N. Franchini, and T. P. Sotiriou, arXiv:2006.03095v1.
[32] In particular, one finds the functional behavior $Y_{l-\infty m=0}(\theta = 0) = \sqrt{l}/2\pi \to \infty$ for the spherical harmonic functions in the asymptotic $l \to \infty$ limit.
[33] In particular, as emphasized in [12], spinning Kerr black holes with rotation parameters in the regime $a < a_{\text{crit}}$ do not develop tachyonic instabilities to perturbations of the non-minimally coupled scalar fields with negative values of the physical coupling parameter $\eta$. Thus, these black holes (with $a < a_{\text{crit}}$) are not expected to support spatially regular hairy configurations of the non-minimally coupled scalar fields.