Precision Data-enabled Koopman-type Inverse Operators for Linear Systems

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Abstract: The advent of easy access to large amount of data has sparked interest in directly developing the relationships between input and output of dynamic systems. A challenge is that in addition to the applied input and the measured output, the dynamics can also depend on hidden states that are not directly measured. The main contribution of this work is to identify the information needed (in particular, the past history of the output) to remove the hidden state dependence in Koopman-type inverse operators for linear systems. Additionally, it is shown that the time history of the output should be augmented with the instantaneous time derivatives of the output to achieve precision of the inverse operator. This insight into the required output (history and instantaneous derivative) information, to remove the hidden-state dependence and improve the precision of data-enabled inverse operators, is illustrated with an example system.

Keywords: Dynamics, hidden state dependency, neural-network models, inverse models

1. INTRODUCTION

With increasing ease of collecting data and low cost storage, there is increasing interest to use data-enabled methods developing models for prediction and control, Abraham et al. (2017); Mamakoukas et al. (2021); Hewing et al. (2020); Asadi et al. (2021); Piche et al. (2000); Kabzan et al. (2019); Kocijan et al. (2004). Such models can be optimized to best fit the data and methods are also available to estimate the error bounds on the predictions, e.g., using predicted time derivatives of the observables Mamakoukas et al. (2021). However, the conditions under which such data-enabled models can achieve sufficient precision remains unclear. A major challenge is that the model (i.e., the relationship between the input and the measurable outputs) can be dependent on the system’s internal states, which are hidden in the sense that they are not directly measured, nor inferred using standard observer designs since they require prior knowledge of the system dynamics.

Several approaches are available to address the lack of direct access to the hidden states. One approach is to represent the dynamics through Markov models with a predefined number of hidden states, and then minimize the model prediction error Tarbouriech et al. (2020); Yoon et al. (2019); Pohle et al. (2017). A difficulty is that the optimal selection of the number of hidden states can be computationally expensive, and there is no guarantee that the resulting models will achieve the desired precision. A second class of approaches to handle the lack of direct access to the hidden states is to model the system dynamics (flow) in a lifted observable space (with generalized functions of the observables) using Koopman operator theory Schmid and Sesterhenn (2008); Mezić (2005). Recent techniques include sparse identification of nonlinear dynamical systems (SINDy) Brunton et al. (2016) and linearization Dynamic Mode Decomposition (DMD) Kutz et al. (2016). Nevertheless, with a finite number of states, there is uncertainty about how to select a sufficient set of generalized observable functions to achieve a specified level of prediction precision. A third class of approaches is to use time history of the input and output data to find forward models, e.g., with (i) transfer function models in the frequency domain Devasia (2017); Yan et al. (2021); (ii) autoregressive models with extra input (ARX) Ljung et al. (1987) as well as nonlinear ARX (NARX) Kocijan et al. (2004); Pham et al. (2010); (iii) time-delayed information in the Koopman operator framework Kamb et al. (2020); and fitting a relation between the time-delayed output data and the inverse input Butterworth et al. (2012); Blanken and Oomen (2020); Aarnoudse et al. (2021). Again, determining the type of data needed to capture the input-output relationship (with high precision) when models are not available a-priori remains uncertain.

When precision of the inverse is not sufficient, it can be improved using iterative techniques, with the inverse of the plant considered as the learning operator, Ghosh and Paden (2001); Fine et al. (2009); Teng and Tsao (2015); Spiegel et al. (2021). Nevertheless, increasing the precision of the inverse model can improve ILC convergence. The goal of this article is to identify the type of output data needed to develop inverse (output-to-input) operators, with a desired level of precision. Rather than the two step processes of first learning forward models and second using model-predictive control (MPC) to optimally select the control input, the proposed approach seeks to solve the inverse problem of directly finding the input for a given output, e.g., similar to Devasia et al. (1996): Willems et al.
and internal states $\eta$ system be (LTI) single-input-single-output (SISO) system. Let the inverse operator be developed for linear time-invariant (illustrated with a simulation example). Overall, the work need to explicitly capture the hidden state dynamics) by using time-delayed observations of the output, along with the output’s time derivatives.

The main contribution of this paper is to propose a Koopman-type time-delay and output-derivative-based data-enabled algorithm that minimizes the impact of the hidden state dependency and achieves precision (with the need to explicitly capture the hidden state dynamics) by using time-delayed observations of the output, along with the output’s time derivatives.

2. PROBLEM FORMULATION AND SOLUTION

The inverse operator is developed for linear time-invariant (LTI) single-input-single-output (SISO) system. Let the system be

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with states $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}$ and output $y(t) \in \mathbb{R}$ with matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in 1 \times \mathbb{R}^n$.

**Assumption 1.** (System properties). The system described in (1) and (2) is stable (i.e., $A$ is Hurwitz), hyperbolic (no zeros on the imaginary axis), and has relative degree $r \leq n$ (i.e., the difference between the number of poles and the number of zeros).

**Assumption 2.** The desired output $y_d$, specified in inverse operator problems, is sufficiently smooth, and has bounded time derivatives up to the relative degree $r$.

2.1 Hidden state dependency

The system state $x$ can split into state components $\xi$ that directly depend on the output and its time derivatives

$$\xi(t) = \left[ y(t), \dot{y}(t), \ldots, \frac{d^{r-1}}{dt^{r-1}} y(t) \right] \in \mathbb{R}^{r+1}$$

and internal states $\eta$,

$$\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} = Sx(t)$$

such that in the new coordinates, (1) can be written as, e.g., see Marino and Tomei (1995), Example 4.1.3,

$$\dot{\xi}(t) = A_1 \xi(t) + A_2 y(t) + B_1 u(t)$$
$$\dot{\eta}(t) = A_3 y(t) + A_4 \eta(t)$$

where

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{r+1 \times 1}, \quad A_3 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

and the eigenvalues of matrix $A_4$ are the zeros of the transfer function of system (1) and (2).

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s + \cdots + b_{n-r} s^{n-r}}{a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + s^n}. \quad (7)$$

Note that the internal state $\eta$ is only driven by the output $y = \xi_1$. Moreover, due to the relative degree $r$ assumption, the input $u$ is directly related to the $r^{th}$ derivative of the output, and therefore, the $r^{th}$ row of (5) can be written as

$$y^{(r)}(t) = \frac{d^r y(t)}{dt^r} = CA^r x + CA^{r-1} Bu(t)$$
$$= CA^r S^{-1} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} + b_{n-r} u(t)$$

and the matrices $A_1$ and $A_2$ in (5) are given by

$$A_1 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_\xi & 0 & \cdots & 0 \end{bmatrix}$$

where $A_\xi$ and $A_{\eta}$ are the last rows of matrices $A_1$ and $A_2$ respectively.

2.2 Research problem

The desired output and its derivatives, $(y^{(r)}_d, \xi_d)$ can be used to predict the inverse input $u_d$ from (8), as

$$u_d(t) = b_{n-r}^{-1} \left[ y^{(r)}_d(t) - A_\xi \xi_d(t) - A_{\eta} \eta_d(t) \right]. \quad (9)$$

which depends on the internal states $\eta$ that are hidden or not directly measured. The goal is to minimize the internal state effects on the inverse model, by addressing the following research problems.

(i) Finding the hidden state from output: Develop an operator that maps the time history of the output $y$ with length $T$ to an estimate of the hidden state $\eta$ at time $t$

$$\eta(t) = \mathbb{E}[y(t - T : t)]. \quad (10)$$

(ii) Koopman-type inverse operator: Using the operator in (10), develop a data-enabled Koopman-type inverse operator $\hat{G}^{-1}$ that uses the history of the desired output and its time derivatives to predict the inverse input as

$$\hat{u}_d(t) = \hat{G}^{-1} \left[ y_d(t - T : t), \xi_d(t), y^{(r)}_d(t) \right]. \quad (11)$$

(iii) Inverse operator precision: Quantify the error $\| \hat{u}_d(t) - u_d(t) \|_2$ dependence on each argument of $\hat{G}^{-1}$.  

2.3 Solution

**Finding the hidden state from output** If the system is minimum-phase ($A_4$ is Hurwitz), i.e., (7) has no zeros on
the right half plane, then \( \eta(t) \) can be obtained from the history of the output by solving (6)

\[
\eta(t) = \int_{-\infty}^{t} e^{A_{3}(t-\tau)} A_{3} y(\tau) d\tau \\
\triangleq \mathbb{H}[y(-\infty; t)].
\]

In practice, such an operator is hard to capture in a data-enabled way since it requires an infinite window. Therefore, an estimate \( \hat{\eta} \) is obtained with an approximate operator \( \hat{\mathbb{H}} \) with a finite time history length \( T \) is defined

\[
\hat{\eta}(t) \triangleq \int_{t-T}^{t} e^{A_{3}(t-\tau)} A_{3} y(\tau) d\tau \\
\triangleq \hat{\mathbb{H}}[y(t-T; t)].
\]

The approximate operator \( \hat{\mathbb{H}} \) approaches the exact operator \( \mathbb{H} \) exponentially as the time history \( T \) increases.

**Lemma 1.** If the output trajectory is bounded,

\[
M = \max_{\tau \in [-\infty, t-T]} \| y(\tau) \|_2 < \infty,
\]

then the error in computing the hidden state \( \eta(t) \) decays exponentially with the time history \( T \), i.e., there exists positive scalars \( \alpha > 0, \beta > 0 \) such that

\[
\| \Delta \eta(t) \|_2 \leq \beta \| \eta(t) - \hat{\eta}(t) \|_2 \leq \beta_1 e^{-\alpha_1 T}.
\]

**Proof.** Since the system is assumed to be minimum phase, the and the eigenvalues of matrix \( A_1 \), which are the zeros of the transfer function of system (1), lie in the open left-half of the complex plane, i.e., the matrix \( A_1 \) is Hurwitz. Then, there exists positive scalars \( \kappa_1 > 0, \alpha_1 > 0 \) such that, Desoer and Vidyasagar (1975)

\[
\| e^{A_1 t} \|_2 \leq \kappa_1 e^{-\alpha_1 t}.
\]

Then, from (12,13), the approximation error can be bounded as

\[
\| \eta(t) - \hat{\eta}(t) \|_2 \leq \| \int_{-\infty}^{t-T} e^{A_{3}(t-\tau)} A_{3} y(\tau) d\tau \|_2 \\
\leq M \| A_{3} \|_2 \int_{-\infty}^{t-T} \kappa_1 e^{-\alpha_1(t-\tau)} d\tau \\
\leq M \| A_{3} \|_2 \int_{-\infty}^{t-T} \kappa_1 e^{-\alpha_1 T} d\tau \\
= M \| A_{3} \|_2 \frac{\kappa_1}{\alpha_1} e^{-\alpha_1 T}.
\]

The result follows with

\[
\beta_1 = M \| A_{3} \|_2 \frac{\kappa_1}{\alpha_1}.
\]

**Koopman-type inverse operator.** Given an estimate \( \hat{\eta} \) of the internal state \( \eta \), the inverse operator prediction in (11) can be estimated as

\[
\hat{u}_d(t) = b_{n-r}^{-1} \left[ y_d^{(r)}(t) - A_2 \xi_d(t) - A_2 \hat{\eta}(t) \right] \\
= b_{n-r}^{-1} \left[ y_d^{(r)}(t) - A_2 \xi_d(t) - A_2 \hat{\mathbb{H}}[y_d(t-T; t)] \right] \\
\triangleq \hat{G}^{-1}[y_d^{(r)}(t), \xi_d(t), y_d(t-T; t)].
\]

**Remark 1.** In addition to sufficient time history (large \( T \)) of the output to accurately find the internal state (to let \( \Delta \eta \to 0 \)), information about the derivatives of the output (up to the relative degree \( r \) at time \( t \), i.e., \( y_d^{(r)}(t), \xi(t) \)) are also needed for precisely computing the inverse input \( u_d \) in (11) as illustrated in Fig. 1.

![Fig. 1. The inverse operator’s dependence on the hidden state is removed by use of past output history and current time derivatives of the output.](image)

**Koopman-type forward operators using output history**

The output \( y \) can be related to the input as

\[
y(t+T_f) = C \int_{-\infty}^{t+T_f} e^{A(t-\tau)} B u(\tau) d\tau
\]

and approximated by

\[
\hat{y}(t+T_f) = C \int_{t-T_f}^{t+T_f} e^{A(t-\tau)} B u(\tau) d\tau.
\]

Therefore, using arguments similar to the proof of Lemma 1, the error in computing the output using just the history of input \( u \) tends to zero as the time history of the input increases, i.e., as \( T \to \infty \). Thus, it is possible to find a map that only depends on the input and its past history,

\[
\hat{y}(t+T_f) = \hat{G}_u[u(t-T_f; t+T_f)],
\]

which justifies the use of ARX models to capture forward linear system models using past input history (and augmented by the output history). In contrast, with Koopman-type operators where past history of the observable output is used to predict future values, the forward model prediction can be written as

\[
\hat{y}(t+T_f) = C e^{A T_f} \hat{x}(t) + C \int_{t}^{t+T_f} e^{A(t+T_f-\tau)} B u(\tau) d\tau
\]

\[
= C e^{A T_f} S^{-1} \left[ \xi(t) \right] + C \int_{t}^{t+T_f} e^{A(t+T_f-\tau)} B u(\tau) d\tau
\]

using (4)

\[
= C e^{A T_f} S^{-1} [\hat{\mathbb{H}}[y_d(t-T; t)](t)] \\
+ C \int_{t}^{t+T_f} e^{A(t+T_f-\tau)} B u(\tau) d\tau
\]

\[
\triangleq \hat{G}[y(t-T; t), \xi(t), u(t+T_f)].
\]
Therefore, past history of the output can also be used to develop Koopman-type forward operators, provided access is available to current time derivatives of the output $\eta(t)$.

Inverse operator precision The inverse operator depends not only on the past history of the output (to remove the hidden state $\eta$ dependency) but also on the output and its time derivatives at the current time instant $t$. The impact of the time history $T$, output and its time derivatives on the precision of the operator is quantified in the next lemma.

Lemma 2. The prediction error of the inverse operator is bounded, i.e. there exist positive scalars $L_1 > 0$, $L_2 > 0$, $L_3 > 0$ such that the error between the predicted input $\hat{u}_d(t)$ and the true input $u_d(t)$ is

$$||\hat{u}_d(t) - u_d(t)||_2 \leq L_1||\Delta y^{(r)}_d(t)||_2 + L_2||\Delta \xi_d(t)||_2 + L_3 e^{-\alpha t T}.$$ (24)

Proof. From (9) and (19),

$$||\hat{u}_d(t) - u_d(t)||_2 \leq |b_n^{-1}| [||\Delta y^{(r)}_d(t)||_2 + ||A_k||_2||\Delta \xi_d(t)||_2 + ||A_k||_2||\Delta \eta_d(t)||_2],$$ (25)

where $\Delta y^{(r)}_d(t) = \hat{y}^{(r)}_d(t) - y^{(r)}_d(t)$, $\Delta \xi_d(t) = \hat{\xi}_d(t) - \xi_d(t)$ and $\Delta \eta_d(t) = \hat{\eta}_d(t) - \eta_d(t)$. The results follows from (15) with

$$L_1 = |b_n^{-1}|, \quad L_2 = L_1||A_k||_2, \quad L_3 = L_1||A_k||_2\beta_1. \quad (26)$$

Remark 2. (Data-enabled algorithm). Known values of the desired output and its derivatives, specified with a sampling period $\Delta t$ and time history $T$ can be used to estimate a discrete-time inverse operator from (19) as

$$\hat{u}_d[m] = G_d^{-1}[y_d[m - m_T : 1 : m], \xi_d[m], y^{(r)}_d[m]],$$ (27)

where $[m]$ indicates value at time $t_m = m \Delta t$, and $m_T = T / \Delta t$. Data-enabled algorithms can be used to learn the operator $G_d^{-1}$, since (27) maps a finite number of variables (desired output and its time derivatives) to the inverse input at time $t_m$.

3. SIMULATION RESULTS

In this section, an example system is introduced, followed by the data-enabled learning of the inverse operator.

3.1 Example system

Consider the following two-mass-spring-damper system, where the input $u$ is the force acting on mass $m_2$ and its displacement $x_2$ as shown in Fig. 2.

$$\frac{d}{dt}X = AX + Bu \quad (28)$$

$$y = x_2 = CX \quad (29)$$

where $X = [x_1 \ x_2 \ x_2']$, $C = [0 \ 0 \ 1]$, $A = \begin{bmatrix} 0 & k_2 & 0 \\ 0 & 0 & k_1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $m_1 = 10, m_2 = 5, k_1 = 110, c_1 = 68, a = k_1 / 2, k_2 = 75$ and $c_2 = 60$ in SI units. The relative degree of the system is $r = 2$ and the input-output relation is given by $\hat{y}(t) = -25y(t) - 12\hat{y}(t) + 25x_1(t) + 12\hat{x}_1(t) + 11u(t). \quad (31)$

3.2 Preliminary selections

Selection of the data-enabled model types to evaluate, the sampling time (which needs to be sufficiently small to reduce discretization error), the evaluation metric, and sufficiently smooth output trajectories for model evaluation are described below.

(i) A two-layer feedforward neural-net (created through MATLAB function feedforwardnet() with default activation function) is used to learn the inverse operator from data.

(ii) For the two-layer neural-net, each model pool consists of 5 candidates with different number $N \in \{5, 10, 20, 40, 80\}$ of neurons in the hidden layer.

(iii) The sampling frequency is varied from 5 Hz to 20 Hz, which is substantially higher than the system bandwidth of 1.7 Hz.

$$y_{\eta,k} = \frac{a}{s + a} y_{d,k} \quad (32)$$

where $a = 2\pi$ (cut-off frequency as 1 Hz), which is less than the system’s bandwidth of 1.7 Hz, and example trajectories are shown in Fig. 4.
For a given time history $T$ and sampling time $\Delta t$, as in Remark 2, the evaluation metrics for the data-enabled inverse operator with $N$ neurons in the hidden layer are selected as the mean $e_{u,N}$ and maximum $\tau_{u,N}$ normalized prediction error over the ten evaluation trajectories $\hat{y}_{d,k}(\cdot)$, i.e.,

$$e_{u,N} = \frac{1}{10} \sum_{k=1}^{10} \max_{m} \left| \frac{\hat{u}_d[m] - u_{d,k}[m]}{max_m |u_{d,k}[m]|} \right| \times 100\% \quad (33)$$

$$\tau_{u,N} = \max_{k=1,\ldots,10} \max_{m} \left| \frac{\hat{u}_d[m] - u_{d,k}[m]}{max_m |u_{d,k}[m]|} \right| \times 100\%, \quad (34)$$

where the ideal inverse $u_{d,k}$ was found using (9) where $\eta_u$ was obtained through (12). Moreover, the smallest normalized prediction error over different number of neurons in the hidden layer is defined as

$$e_u = e_{u,N^*}, \quad e_u = e_{u,N^*} \quad \text{where} \quad N^* = \arg\min_{N} e_{u,N} \quad (35)$$

to quantify the precision of the inverse operator.

For the noisy case, additive white gaussian noise with signal-to-noise ratio of 20 is separately added to each output and its time derivatives. Simulations were done in MATLAB with ode45() with sampling rate of 100 Hz (to be consistent with the evaluation metrics from (33) to (35)). Input, output and the output’s time derivatives (upto the fourth order) were collected. Second order derivative was obtained from (31). Third and fourth order derivatives for training purposes were estimated from the data, using finite difference as,

$$\left[ y^{(3)}[m] \right] = \frac{1}{12(\Delta t)} \left[ -1 \cdot \frac{8}{\Delta t^2} \cdot \frac{30}{\Delta t^3} \cdot \frac{1}{\Delta t^4} \right] \left[ \begin{array}{c} y[m+2] \\ y[m+1] \\ y[m] \\ y[m-1] \\ y[m-2] \end{array} \right]. \quad (36)$$

To investigate the reduction of the impact of hidden states on the prediction precision of data-enabled inverse operators, the performance of the data-enabled inverse operators was assessed for different time history $T$ of the output. In this part of the study, the number of time derivatives of the output used was the same as the relative degree of the example system. The relative degree $r = 2$ can be established by applying a step input — a corresponding discontinuity will appear in $y^{(r)}$, while the lower order derivatives ($y, \dot{y}$ in this example) remain continuous as seen in Fig. 5. Then, from (27),

$$\hat{u}_d[m] = G_d^{-1} \left[ y_d[m] - m_T : 1 : m \right], \hat{y}_d[m], \hat{y}_d[m]. \quad (37)$$

The inverse operator’s prediction error $e_u$ (35) was obtained for varying output time history $T$ ($0.1, 0.2, 0.4, 0.8, 1.6, 3.2, \ldots$ s), for different sampling time $\Delta t \in \{0.05s, 0.1s, 0.2s\}$, and for different number $N$ of neurons in the hidden layer, and plotted in Fig. 6 for the case without noise in the training data. The associated prediction errors are tabulated in Table 2 for the fastest sampling time $\Delta t = 0.05$ s.

The precision of the inverse operator improves with larger output time history $T$, as seen in Table 2, where the evaluation values of the two-layer neural net with different $N$ neurons in the hidden layer are listed. Note that typically $N^* \leq 20$ yields good precision for this application from Table 2. Over all selections of neuron numbers $N$, the variation of the smallest prediction error $e_u = e_{u,N^*}$ (35) with sampling time of $\Delta t = 0.05$ s (20 Hz) fits an exponential decay curve $e_u(T) \approx 1.88e^{-2.18T}$, shown in red in Fig. 6. This exponential improvement in precision is expected from Lemma 2, which predicts an exponential
The precision of the in-



backward error is


terms of prediction error $e_u$ (35) exponentially improves with respect to different window length $T$ of output history, for different sampling times, $\Delta t = 0.05s(20Hz, \text{ blue}), 0.1s(10Hz, \text{ cyan}), 0.2s(5Hz, \text{ red}).$

Similar results are seen over different $N^*$ neurons in the hidden layer: 5 triangle ($\triangle$), 10 (square $\Box$), 20 (diamond $\Diamond$), 40 (pentagram $\star$), and 80 (circle $\circ$). The fitted exponential decay (red line) is obtained with sampling time of $\Delta t = 0.05 s$ (20 Hz, blue).

Table 2. Inverse operator’s precision improvement in terms of prediction error $e_{u,N}$ (33) and $\tau_{u,N}$ (34) for varying output time history $T$ and number $N$ of neurons in the hidden layer, with sampling time $\Delta t = 0.05 s$.

| $T$ | 5  | 10 | 20 | 40 | 80 |
|-----|----|----|----|----|----|
|     | $e_{u,N}(\%)$ as in (33) |     |    |    |    |
| 0.1 | 1.78 | 2.23 | 1.64 | 2.05 | 2.43 |
| 0.2 | 0.79 | 0.88 | 0.87 | 0.88 | 0.98 |
| 0.4 | 0.95 | 0.85 | 0.88 | 0.92 | 0.91 |
| 0.8 | 0.46 | 0.51 | 0.49 | 0.48 | 0.52 |
| 1.6 | 0.14 | 0.12 | 0.12 | 0.14 | 0.16 |
| 3.2 | 0.05 | 0.04 | 0.01 | 0.01 | 0.05 |

| $\tau_{u,N}(\%)$ as in (34) |     |    |    |    |    |
| 0.1 | 3.17 | 3.72 | 4.73 | 5.61 | 6.28 |
| 0.2 | 1.22 | 1.59 | 1.56 | 1.48 | 1.76 |
| 0.4 | 1.17 | 1.30 | 1.33 | 1.75 | 1.69 |
| 0.8 | 0.54 | 0.65 | 0.61 | 0.67 | 1.01 |
| 1.6 | 0.20 | 0.16 | 0.18 | 0.33 | 0.44 |
| 3.2 | 0.08 | 0.02 | 0.02 | 0.02 | 0.14 |

Decay of error in the estimation of the hidden states, dependent on $\|e^{A_T}\|_2$ from (16), and shown in Fig. 6. Thus, the impact of hidden states on the prediction precision of data-enabled inverse operator can be reduced by using sufficient time history of the desired output.

Remark 3. (Reducing hidden state dependence). In the following simulations, the time history $T$ is chosen to be sufficiently large $T^* = 3.2 s$, which results in a normalized error $e_u = 0.01 \%$.

3.5 Need to include output time derivatives

From (24) in Lemma 2, even if the hidden state error is reduced by having sufficiently large time history $T$, (as shown in the previous subsection), current time derivatives of the output $\dot{y}_d(t), y_d^{(1)}(t)$ are needed to achieve precision prediction with the inverse operator. Therefore, the impact of adding time-derivative information is investigated through the following two steps, for different sampling periods $\Delta t \in \{0.05s, 0.1s, 0.2s\}$ and for different number $N$ of neurons in the hidden layer.

(i) Incrementally including higher-order time derivatives of the output when learning the inverse operator $G_{d,l}^{-1}$ that predicts the inverse input $\hat{u}_d$ similar to (37), where output time derivatives till order $l$ ($0 \leq l \leq 4$) are included in the data-enabled operator learning, e.g., with $l = i \geq 0$,

$$\hat{u}_d[m] = G_{d,l}^{-1}[y_d(m - m_T : 1 : m)],$$

$$y_d^{(i)}[m], y_d^{(i-1)}[m], \ldots, y_d^{(0)}[m]],$$

where $G_{d,2}^{-1} = G_{d,3}^{-1}$ in (37).

(ii) Adding the output’s time derivatives $\dot{y}_d(t), \ddot{y}_d(t)$ to NARX-type inverse operators where the inverse operator is learned using both input and output time history, i.e., to compare

$$\hat{u}_d[m] = \text{NARX}[y_d(m - m_T : 1 : m),$$

$$u_d(m - m_T : 1 : m - 1)],$$

$$\ddot{y}_d[m], \dot{y}_d[m], u_d(m - m_T : 1 : m - 1)].$$

The corresponding prediction performance, in terms of errors $e_u$ and $e_s$ in (35), for $T^* = 3.2 s$ and $\Delta t = 0.05 s$ are tabulated in Table 3, and plotted in Fig 7 for $T^* = 3.2 s$ and different sampling time $\Delta t \in \{0.05s, 0.1s, 0.2s\}$.

Table 3. Prediction error $e_u, e_s$ (35) for inverse operators from (38) to (40) with $\Delta t = 0.05 s$.

| $e_u(\%)$ | $e_s(\%)$ | $\bar{e}_u(\%)$ | $\bar{e}_s(\%)$ |
|----------|----------|----------------|----------------|
|          |          | Noise free training data |
| $G_{d,0}$ | 3.13 | 9.82 | $G_{d,4}$ | 0.01 | 0.02 |
| $G_{d,1}$ | 0.74 | 2.10 | NARX | 1.60 | 5.93 |
| $G_{d,2}^{-1} = G_{d,3}^{-1}$ | 0.01 | 0.02 | \text{NARX}$^*$ | 0.01 | 0.02 |
| $G_{d,3}^{-1}$ | 0.01 | 0.02 | \text{NARX}$^*$ | 0.01 | 0.02 |
|          |          | Noisy training data |
| $G_{d,0}$ | 53.91 | 114.68 | $G_{d,4}$ | 0.41 | 0.78 |
| $G_{d,1}$ | 11.53 | 37.82 | NARX | 3.89 | 17.95 |
| $G_{d,2}^{-1} = G_{d,3}^{-1}$ | 0.53 | 1.05 | \text{NARX}$^*$ | 0.21 | 0.45 |
| $G_{d,3}^{-1}$ | 0.65 | 1.32 | \text{NARX}$^*$ | 0.21 | 0.45 |

Impact of including derivatives. The precision of the inverse operator depends on inclusion of the output derivative up to order $r$ (the relative degree). When the number of derivatives $l$ (included in the training and evaluation) is increased from $l = 0$ to $l = 4$, the precision of the inverse operator improves significantly when all the required number ($l = 2 = r$) of time derivative features are included in the training and evaluation data. In particular, the maximum error $\bar{e}_u$ in (35) reduces from 9.82% to 0.02% for the case with noise free training data and from 114.68% to 1.05% for the case with noisy training data as seen in Table 3. Therefore, there is substantial improvement in the inverse operator’s precision (especially in the presence of noise) when time derivatives upto the required order of 2 are included.

Impact on NARX-type inverse operator. Inclusion of time derivatives is also important for NARX-type inverse operators where both input and output time history are used in the inverse operator. This can be seen by comparing NARX (39) without time derivatives and NARX$^*$.
Conceptually, information about the derivatives up to the relative degree \( r \) are available in the time history of the output and only the \( r^{th} \) time derivative \( y^{(r)}[m] \) is directly affected by the input \( u[m] \). In particular, output derivatives can be related to the output time history using finite difference techniques, especially in the noise free case, and hence direct computation of the derivatives might not appear to be critical if time history of the output is used during training. Nevertheless, including computed or measured values (even with some noise) of the time derivative \( \dot{y}[m] \) (which is not directly affected by the input \( u[m] \)) still can improve the precision of the inverse operator as seen in Fig. 7 and Table 3. In particular, the maximum error \( \overline{e}_u \) in (35) reduces from 9.82% to 2.10% for the case with noise free training data and from 114.68% to 37.82% for the case with noisy training data as seen in Table 3. Therefore, while the noise free case precision could be improved by smaller sampling time \( \Delta t \) without the inclusion of \( \dot{y} \), for the noisy case, direct measurements of the output time derivatives can substantially improve the inverse operator training, and lead to better precision in its predictions. Moreover, the precision of the inverse operator is further improved by including time derivatives up to the required order of \( r \) (relative degree).

4. CONCLUSION

This work showed that Koopman-type data-enabled inverse operators can have high precision if a sufficient large time history of the output is included to reduce the impact of hidden internal states. Additionally, measurements of the instantaneous output time derivatives (upto the relative degree) are required during training to improve the data-enabled inverse operator precision. Our ongoing work is aimed at extending these results to Koopman-type data-enabled inverse operators for nonlinear nonminimum-phase systems.

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Appendix A. EVALUATION TRAJECTORIES

Expressions of \(y_{0,k}(t)\) for \(k = 1, 2, \ldots, 10\) and \(0 \leq t \leq 10\).

Trapezoidal shape (\(k = 1\))

\[
y_{0,1}(t) = \begin{cases} 
0.4(t - 1) & 1 \leq t < 3 \\
0.8 & 3 \leq t < 6 \\
0.4(8 - t) & 6 \leq t < 8 \\
0 & \text{otherwise.}
\end{cases}
\]

Triangle wave (\(k = 2\))

\[
y_{0,2}(t) = \begin{cases} 
t - 2 & 2 \leq t < 3 \\
3.7 - 0.9t & 3 \leq t < 5 \\
1.2(8 - t) & 7 \leq t < 8 \\
0 & \text{otherwise.}
\end{cases}
\]

Square wave (\(k = 3\))

\[
y_{0,3}(t) = \begin{cases} 
1 & 2 \leq t < 4 \\
-1 & 4 \leq t < 6 \\
1 & 6 \leq t < 8 \\
0 & \text{otherwise.}
\end{cases}
\]

Serrated wave mixture (\(k = 4\))

\[
y_{0,4}(t) = \begin{cases} 
2(t - 1)/3 & 1 \leq t < 2.5 \\
2(4 - t)/3 & 2.5 \leq t < 4 \\
8(t - 4)/15 & 4 \leq t < 5 \\
8(6 - t)/15 & 5 \leq t < 6 \\
0.4(t - 6) & 6 \leq t < 7.5 \\
0.4(9 - t) & 7.5 \leq t < 9 \\
0 & \text{otherwise.}
\end{cases}
\]

Monotonic (\(k = 5\)): \(y_{0,5}(t) = 0.001(x^{3.2} - x^2)\) Sine wave

\#1 (\(k = 6\)) \(y_{0,6}(t) = \sin(0.4\pi t) - 0.9\sin(0.6\pi t) + 0.2\sin(\pi t)\)

Sine wave \#2 (\(k = 7\)) \(y_{0,7}(t) = 1.5\sin(0.7\pi t) - 0.5\sin(0.4\pi t)\)

Sine wave \#3 (\(k = 8\)) \(y_{0,8}(t) = -0.5\sin(0.3\pi t) - 0.6\sin(0.7\pi t) + 0.2\sin(1.2\pi t)\)

Sine wave \#4 (\(k = 9\)) \(y_{0,9}(t) = 0.7\sin(0.26\pi t) + 0.3\sin(1.3\pi t) - 0.2\sin(1.4\pi t)\)

Slow chirp wave (\(k = 10\)): \(y_{0,10}(t) = 0.35\sin(x^{1.5})\).