Silverman-Tate Height Inequality For Positive Characteristic

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Abstract

In the paper "Uniformity of Mordell-Lang" by Vesselin Dimitrov, Philipp Habegger and Ziyang Gao ([DGH20]), they use Silverman’s Height Inequality and they give a proof of the same in the appendix which makes use of Cartier divisors and hence drops the flatness assumption of structure morphisms of abelian schemes. Their proof makes use of Hironaka’s theorem on resolution of singularities which is unfortunately unknown for fields of positive characteristic. We try to slightly modify their ideas and use blow-ups in place of Hironaka to make the proof effective for any fields with product formula where heights can be defined. The main idea suggested by Prof. Gubler was the use of blow up in ideal of denominators to split divisors into effective parts.

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1 Introduction

We will repeat what the authors have done in the appendix of the paper just for the sake of clarity. All our varieties (i.e. reduced schemes of finite type and separated over fields) are defined over some field $k$ on which the product formula holds and hence we can define the notions of height. Note this is the main development from the version proved in the paper. We consider a regular, irreducible, quasi-projective variety $S$ as our base space. Over that we have the abelian scheme $\mathcal{A}$ which is our main object of interest with the structure morphism

$$\pi: \mathcal{A} \to S$$

On our abelian scheme, we have the ”multiplication by 2” morphism

$$[2]: \mathcal{A} \to \mathcal{A}$$

over $S$. Now we want to ”compactify” base scheme. Let $S$ be considered to be immersed in some projective space $\mathbb{P}^n_k$ and let us denote by $\overline{S}$ the projective closure of $S$. Note that we do not have necessarily that $\overline{S}$ is regular but just a projective, irreducible variety. Like in [DGH20], Appendix A(p 31) we will also assume that we are presented with a closed immersion

$$\mathcal{A} \hookrightarrow \mathbb{P}^m_k \times S$$

Then note that as $S$ is immersed inside $\mathbb{P}^n_k$, altogether we have an immersion

$$\mathcal{A} \hookrightarrow \mathbb{P}^m_k \times \mathbb{P}^n_k$$
and let us call by $\overline{A}$ the Zariski closure of $A$ under this immersion in the multi-projective space.

Next we come to the consideration of the Weil Heights that is our main object of interest. We have the canonical very ample line-bundle $O(1,1)$ on the multiprojective space $\mathbb{P}^{m,n}$. We denote as

$$L = O(1,1) |_{\overline{A}}$$

and we chose any representative of the height class $h_L$ of the line bundle $L$. Less abstractly if we denote a point $\overline{A}(k) \subseteq \mathbb{P}^{m,n}_k$ as $(P, s)$ where $P \in \mathbb{P}^{m,n}_k$ and $s \in \overline{S}(k)$, then we have

$$h_L(P, s) = h(P) + h_{\overline{S}}(s)$$

where $h$ is the canonical Weil Height of $\mathbb{P}^{m,n}_k$ and $h_{\overline{S}}$ is the Weil Height on $\overline{S}$ coming from its embedding in $\mathbb{P}^{m,n}_k$. Additionally if $\eta$ is the generic point of $S$ and $A_\eta$ is the generic fiber of $A$, then we can restrict the line bundle $L$ to $A_\eta$ (call it $L_\eta$). We assume that $L_\eta$ is an even ample line bundle on the abelian variety $A_\eta$ defined over $\kappa(S)$. Then note that in particular from the theorem of cubes, we have that

$$[2]^*L_\eta = L_\eta^\otimes 4.$$

With this language we are in a position to state the theorem that we wish to prove:

**Theorem 1.1.** There exists a constant $c > 0$ such that for all $P \in A(\overline{k})$ we have

$$|h_L([2]P) - 4 \cdot h_L(P)| \leq c \cdot \max\{1, h_{\overline{S}}(\pi(P))\}.$$

This theorem easily gives an interesting result concerning fiberwise Néron-Tate height and the global height $h_{\overline{S}}$. We recall the construction here.

**Construction.** Let $\eta \in S$ be the generic point of $S$ and consider the line bundle $L_\eta$ on the generic fiber. This is an ample line bundle, because it gives rise to a closed immersion $A_\eta \hookrightarrow \mathbb{P}^{m,n}_{\overline{k}(\eta)}$. We can define the Néron-Tate height on the generic fiber $A_\eta$ (recall $I_\eta$ is even):

$$\hat{h}_{A_\eta, L_\eta}(P) = \lim_{N \to \infty} \frac{h_\eta([N]P)}{N^2}, \quad P \in A_\eta(\overline{k}),$$

where $h_\eta$ is a Weil height obtained by the standard height on $\mathbb{P}^{m,n}_{\overline{k}(\eta)}$.

On the other fibers $A_s$, $s \in S(\overline{k})$, we can define the Néron-Tate heights induced by

$$L_s := L|_{A_s}, \quad s \in S(\overline{k}).$$

Note that $L_s$ is ample for every closed point $s \in S$. Moreover, they are also symmetric. Indeed, the symmetry of $L_\eta$ implies that

$$([-1]^*L \otimes L^{\otimes (-1)})|_{A_\eta} = [-1]^*L_\eta \otimes (L_\eta)^{\otimes (-1)}$$

is the trivial line bundle on $A_\eta$. Then, by a result from EGA (see Section 2) below, we can find some line bundle $M$ on $S$ such that $\pi^*M \cong L$. But then

$$[-1]^*L|_{A_s} \cong L|_{A_s}, \quad s \in S(\overline{k}),$$

because $\pi([N]P) = P$ for all $P \in A(\overline{k})$. Therefore, we can define the Néron-Tate height *fiberwise*:

$$\hat{h}_A(P) = \lim_{N \to \infty} \frac{\hat{h}([N]P)}{N^2},$$

where $\hat{h} = h$ is the induced Weil height from $\mathbb{P}^n(\overline{k})$ in the previous construction, if $P$ is a closed point of $A$; and $\hat{h} = h_\eta$ otherwise (we have $A(\overline{k}) = \bigcup_{s \in S} A_s(\overline{k})$).

Note that in any case $\hat{A}$ is non-negative by ampleness. We are ready to state our main result.
**Theorem 1.2.** Keeping the notations so far. Then there exists a constant $c > 0$ such that for all $P \in \mathcal{A}(\overline{k})$ we have

\[
|\hat{h}_A(P) - h(P)| \leq c \max(1, h_S(\pi(P))).
\]

**Proof of Theorem 1.2.** This is similar to the proof of the convergence of the Néron-Tate height (cf. [BG06, 9.2.4]). Pick any $P \in \mathcal{A}(\bar{k})$. Let $\ell \geq k \geq 0$ be integers. Using telescope-sum and triangle inequality, we find

\[
\left|\frac{h([2^\ell]P)}{4^{\ell}} - \frac{h([2^k]P)}{4^{k}}\right| \leq \frac{1}{4^k} \sum_{m=k}^{\ell-1} \left|\frac{h([2^{m+1}]P) - 4h([2^m]P)}{4^{m+1}}\right|.
\]

Apply Theorem 1.1 to the summands, we find a constant $c_1 > 0$ such that the right hand side is bounded by

\[
c_1 x \sum_{m=k}^{\ell-1} \frac{1}{4^{m+1}} \leq c_1 x \frac{1}{4^k}, \quad x := \max(1, h_S(\pi(P))).
\]

This means that $(\frac{h([2^\ell]P)}{4^{\ell}})_{\ell \geq 1}$ is a Cauchy sequence with limit $\hat{h}_A(P)$ – by definition of the Néron-Tate height. By taking $k = 0$ and considering $\ell \to \infty$ we obtain the inequality in the Theorem.

**2 Extension of [2]**

In the previous section we have considered the duplication morphism. However for using the height machines, we need that they are defined over the compactifications $\overline{\mathcal{X}}$. It is not always possible to extend it directly but we by-pass this problem by going to the products via the graph morphism and then identifying the image sub-scheme with $\mathcal{A}$ using the separatedness of $\mathcal{A}$. We consider the graph morphism corresponding to [2], namely:

$$
\mathcal{A} \xrightarrow{\Gamma[2]} \mathcal{A} \times_S \mathcal{A} \hookrightarrow \overline{\mathcal{X}} \times_{\overline{\mathcal{X}}} \overline{\mathcal{X}}.
$$

Note that as $\mathcal{A}$ is separated, the first morphism is a closed immersion while the second one is an open immersion. We denote by $\overline{\mathcal{A}}$ the Zariski closure of $\mathcal{A}$ under this immersion. Then note that due to $\Gamma[2]$ being a closed immersion, we can identify $\mathcal{A}$ as an open dense subscheme $\overline{\mathcal{A}}$. Moreover we have the two canonical projections $p_1$ and $p_2$ on the product $\overline{\mathcal{X}} \times_{\overline{\mathcal{X}}} \overline{\mathcal{X}}$ and note that by construction

$$
p_1 |_{\mathcal{A}} = \text{id}_{\mathcal{A}} \text{ and } p_2 |_{\mathcal{A}} = [2].
$$

To summarise, we have the following commutative diagram:

$$
\begin{array}{ccc}
\overline{\mathcal{X}} & \xrightarrow{p_1} & \overline{\mathcal{X}} \\
\downarrow & & \downarrow \\
\mathcal{A} & \xrightarrow{\text{id}} & \mathcal{A}
\end{array}
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{p_2} & \overline{\mathcal{X}} \\
\downarrow & & \downarrow \\
\mathcal{A} & \xrightarrow{[2]} & \mathcal{A}
\end{array}
$$

At this point we introduce the line bundle on $\overline{\mathcal{A}}$ which we are going to work with. We define

$$
\mathcal{F} = p_2^* \mathcal{L} \otimes (p_1^* \mathcal{L})^{-4}
$$

which is an element of $\text{Pic}(\overline{\mathcal{A}})$. Then note that we assume that the generic fiber $\mathcal{L}_\eta$ is even and hence by the theorem of cubes we have that the restriction $\mathcal{F} |_{\mathcal{A}_\eta}$ is trivial. Hence by [Gro67] Corollaire 21.4.13 , we get that there is a line bundle $\mathcal{M}$ on $S$ such that

$$
\mathcal{F} |_{\mathcal{A}} = \pi^* \mathcal{M}.
$$

We wish to extend this line bundle $\mathcal{M}$ to $\overline{S}$. We work towards this goal in the next section.
3 Extension of line bundle on the base

In the previous section, we came upon the existence of a line bundle \( \mathcal{M} \) on \( S \). We wish that \( \mathcal{M} \) comes from a line bundle \( \mathcal{M}' \) from above on \( S \). We can not wish to have this a-priori because we do not know that \( S \) is regular. However our idea here is to blow up \( S \) at certain closed sub-schemes lying outside \( S \) so that we do not disturb \( S \) and pass to a new \( S' \) where this extension is possible.

We begin with a Lemma that we learned last year from Klaus Kunnemer’s lecture notes on Algebraic Geometry course.

Lemma 3.1. *Let us have a coherent sheaf \( F \) on an open sub-scheme \( U \) of a noetherian scheme \( X \). Also suppose \( G \) is a quasi-coherent sheaf on \( X \) such that \( F \subseteq G \mid_U \). Then there exists a coherent sheaf \( F' \subseteq G \) on \( X \) such that \( F' \mid_U = F \).*

*Proof.* For an exact reference, we refer to Exercise 5.15(part (d)) in [Har13]. The proof is sketched step by step in Exercise 5.15.

Let us go back to the setting of our problem. With that we have the following:

Lemma 3.2. *If we have that \( \mathcal{M} \) is an invertible ideal sheaf on \( S \), then we can find a coherent ideal sheaf \( \mathcal{M}' \) on \( S \) such that \( \mathcal{M}' \mid_S = \mathcal{M} \).*

*Proof.* The proof is a direct application of Lemma 3.1 with \( \mathcal{M} = F, X = S, U = S \) and \( G = \mathcal{O}_S \) (the structure sheaf of \( S \)). Note that the condition that \( \mathcal{M} \) is an ideal sheaf is necessary so that we can choose \( G \) to be the structure sheaf itself.

Next we make a remark which is trivial but useful for our considerations later.

**Remark.** Let us consider the case in our hand. We have a line bundle \( \mathcal{M} \) on the regular irreducible variety \( S \). On a regular variety the notions of Weil and Cartier divisors (up to rational equivalence) and isomorphism classes of line bundles coincide. We take advantage of this regularity to assert that for any Cartier divisor \( D \), we can write \( D = C - E \) for effective Cartier divisors \( C \) and \( E \). It is very easy to see for Weil divisors and then using the equivalence between Cartier and Weil divisors on regular schemes.

Now let us consider a Cartier divisor \( D \) on \( S \) such that \( \mathcal{O}(D) \cong \mathcal{M} \). Then following the remark above, we can write \( D = C - E \) where \( C \) and \( E \) are effective divisors on \( S \). Then we see from the construction of \( \mathcal{O}(-) \) that both \( \mathcal{O}(-C) \) and \( \mathcal{O}(-E) \) are invertible ideal sheaves on \( S \). Our idea is to extend these ideal sheaves with the help of Lemma 3.2 and blow up in their supports one by one. More formally, we have the following lemma and its proof explains the above procedure:

Lemma 3.3. *There is a projective irreducible variety \( S' \) and a birational morphism \( \gamma: S' \to \overline{S} \) such that \( \gamma \mid_{\gamma^{-1}(S)} \) is an isomorphism. Moreover if we identify \( S \) as an open sub-scheme of \( S' \) via \( \gamma \mid_{\gamma^{-1}(S)} \) then we have a line bundle \( \mathcal{M}' \) on \( S' \) such that \( \mathcal{M}' \mid_S = \mathcal{M} \).*

*Proof.* We consider the notations from the paragraph above this theorem. As explained in the remark above, using 3.2 we find a coherent ideal sheaf \( \mathcal{C} \) on \( \overline{S} \) such that \( \mathcal{C} \mid_S = \mathcal{O}(-C) \). Then note that as \( \mathcal{C} \mid_S \) is invertible we get that \( \text{Supp}(\mathcal{C}) \) lies outside \( S \). Now we consider the blow-up of \( \overline{S} \)

\[ b_1: S_1 \to \overline{S} \]

corresponding to the closed sub-scheme of the ideal sheaf \( \mathcal{C} \). Clearly \( b_1 \) is an isomorphism when restricted to \( S \) because \( \text{Supp}(\mathcal{C}) \) lies outside \( S \) and hence we can consider \( S \) as an open sub-scheme of \( S_1 \) by identification via \( b_1 \). But now we have that the inverse image ideal sheaf \( \overline{\mathcal{C}} = b_1^{-1}\mathcal{C} \) (note that this
is the exceptional divisor or the inverse image ideal sheaf of \( C \) and hence in particular is an effective Cartier divisor by property of blow-ups. Note also that we have
\[
O(\mathcal{T}) |_S = O(-C)
\] (1)

Now via identification we view \( S \) as an open sub-scheme of \( S_1 \) and \( E \) is an effective Cartier divisor on \( S \). Thus we once again have that \( O(-E) \) is an invertible ideal sheaf and we can once again apply Lemma 3.2 to obtain a coherent ideal sheaf extension \( \mathcal{E} \) on \( S_1 \) such that \( O(\mathcal{E}) |_S = O(-E) \). Using a similar argument as before we obtain that \( \text{Supp}(\mathcal{E}) \) lies outside \( S \) and hence we can safely blow-up in the associated closed sub-scheme without disturbing \( S \). More formally we consider the blow-up
\[
b_2: S' \to S_1
\]
along the closed sub-scheme corresponding to the coherent ideal sheaf \( \mathcal{E} \). At first note that as pullbacks of Cartier divisors are again Cartier under dominant maps of integral schemes (that is pullback is well defined), we get that \( \mathcal{C} = b_2^{-1}(\mathcal{T}) \) is an invertible sheaf on \( S' \) with \( \mathcal{C} |_S = O(-C) \). Moreover, let us consider the effective Cartier divisor \( \mathcal{E} \) which is the exceptional divisor corresponding to the blow up \( b_2 \). Then note that arguing like above and using identifications of \( S \) via \( b_2 \), we get that
\[
O(\mathcal{E}) |_S = O(-E)
\] (2)

Now we are done because we take \( \mathcal{M}' = O(\mathcal{E}) - O(\mathcal{C}) \in \text{Pic}(S') \). Then from (1) and (2), we have
\[
\mathcal{M}' |_S = O(\mathcal{E}) |_S - O(\mathcal{C}) |_S = O(-E) - O(-C) = O(C - E) = O(D) = \mathcal{M}
\]

We are done with the Lemma by taking \( \gamma = b_1 \circ b_2 \).

We make two remarks before moving on to the next section.

- Firstly we remark about the point of this section. As we mentioned before, our goal is to pass to blow-ups where \( \mathcal{M} \) can be extended from \( S \) to the compactification of \( S \). Using 3.3 we can find such blow-ups (for example \( S' \)) and WLOG we will replace the old \( S \) by \( S' \). Now we can assume that \( \mathcal{M} \) has an extension to \( S \). Note that this change of base space does not affect the height of the base-space precisely because the blow-ups do not touch \( S \) (see 5.3 in Section 5).

The above procedure could be easily done by application of Hironaka on \( S \) and passing to a regular compactification. However Hironaka is unknown for positive characteristic but blow-ups work just fine. This is one of the modifications that we make from the proof of 1.1 in Appendix A of [DGH20].

- Secondly we comment on the regularity assumption on \( S \). We have used the regularity only prior to 3.2 throughout the paper. Using a result from [Liu02], we can get rid of this assumption. We have added a short appendix on this (see Appendix A). Note also that to apply Hironaka and changing the compactification of our base space (as in [DGH20]) it is essential that \( S \) is regular, otherwise the Hironaka morphism isn’t an isomorphism on \( S \).

### 4 Height Inequality Using Blow Ups

In this section we begin by talking about the ideal sheaf of denominators of a Cartier divisor. Let us have a Cartier divisor given by the local equations \( D = (U_i, f_i) \) where \( U_i = \text{Spec}(A_i) \) are affine and \( f_i \in \text{Frac}(A_i)^\times \) are the local equations of the Cartier divisor. Then we define the the ideal sheaf of denominators \( \mathcal{I}(D) \) to be
\[
\mathcal{I}(U_i) = \{ b \in A_i \mid b \cdot f_i \in A_i \}
\]
Note that clearly each $\mathcal{I}(U_i)$ is an ideal of $A_i$ and they glue together because $f_i$ and $f_j$ differ by an unit in the intersection $U_i \cap U_j$. Hence gluing these piece-wise sections we get an ideal sheaf $\mathcal{I}$ which we call the ideal sheaf of denominators.

Now let us digress and prove a lemma which will be necessary for us later:

**Lemma 4.1.** Let $X$ be an integral scheme, $U$ is an open sub-scheme of $X$ and $\mathcal{L} \in \text{Pic}(X)$ such that $\mathcal{L}|_U$ is trivial. Then we can take a Cartier divisor $D$ on $X$ such that $O(D) \cong \mathcal{L}$ and $\mathcal{I}(D)|_U$ is trivial.

**Proof.** From the construction of $\mathcal{I}(D)$ it is clear to see that $\mathcal{I}(D)|_U = \mathcal{I}(D|_U)$. Hence to show the claim it is enough to choose Cartier divisor corresponding to $\mathcal{L}$ such that its restriction to $U$ is effective. We begin by choosing any Cartier representative $D$ of $\mathcal{L}$. Then as we have $\mathcal{L}|_U$ is trivial, we get that $D|_U$ is principal i.e., there is an $f \in \kappa(X)$ such that $D|_U = (U, f)$. Then by replacing $D$ with $D' = D - (X, f)$ we get that $D'|_U = (U, 1)$ which is an effective Cartier divisor. However as we only change $D$ by a principal Cartier divisor, we stay in the same isomorphism class of $\mathcal{L}$. Hence we get that $\mathcal{L} \cong O(D')$ and $\mathcal{I}(D')|_U$ is trivial as wanted. \qed

We get a canonical structure map $\tilde{\mathcal{F}} \to \mathcal{S}$ which we also denote by $\pi$ for the sake of clarity. Now as explained in the introduction (but we repeat it for clarity and to re-iterate the importance of section 3), we know that $\mathcal{F}|_{\mathcal{A}_0}$ is trivial and hence according to [Gro67], Corollaire 21.4.13, we get that

$$\mathcal{F}|_{\mathcal{A}} \cong \pi^* \mathcal{M}$$

for some line bundle $\mathcal{M}$ on $S$. This where we will use the results of section 3. We can find a line-bundle $\mathcal{M}'$ on $\mathcal{S}$ such that $\mathcal{M}'|_S = \mathcal{M}$. In that case let us denote by

$$\mathcal{F}' = \mathcal{F} \otimes (\pi^* \mathcal{M}')^{-1}.$$ 

In that case we can see that $\mathcal{F}'|_{\mathcal{A}}$ is trivial. Hence by Lemma 1 above we can choose a Cartier divisor $D$ on $X = \mathcal{S}$ such that $\mathcal{I}(D)|_{\mathcal{A}}$ is trivial and $O(D) \cong \mathcal{F}'$. Note also that by the proof of Lemma 1, we can have $D|_{\mathcal{A}} = (\mathcal{A}, 1)$, the trivial cartier divisor. Let us denote by $\mathcal{S}$ the closed sub-scheme defined by the ideal sheaf $\mathcal{I}(D)$. Then as $\mathcal{I}(D)$ restricted to $\mathcal{A}$ is trivial, we get that $\mathcal{S}$ lies outside $\mathcal{A}$. Let us blow up $\tilde{\mathcal{F}}$ along $\mathcal{S}$. Note that as $\mathcal{S}$ lies outside $\mathcal{A}$, this blow up

$$\gamma_1 : \tilde{\mathcal{A}} \to \tilde{\mathcal{F}}$$

is an isomorphism over $\mathcal{A}$ and hence we aren’t changing our main object of interest (which was main point of the lemma above). Now as explained in [Full13], Chapter 2, Case 3 in the proof of the Commutativity Theorem 2.4 that

$$\gamma_1^* D = C - E$$

for some effective divisor $C$ in $\tilde{\mathcal{A}}$ and the exceptional divisor $E$ and we also have $C \cup E \subseteq \gamma_1^{-1}(\text{Supp}(D))$. For the sake of clarity, we re-iterate that we have the following diagram:

$$\begin{array}{ccc}
\tilde{\mathcal{A}} & \xrightarrow{\gamma} & \tilde{\mathcal{F}} \\
& \pi \downarrow & \\
& \mathcal{S} & 
\end{array}$$

Note now that as $D|_{\mathcal{A}}$ is trivial, we have that $\text{Supp}(D) \subseteq \tilde{\mathcal{F}} - \tilde{\mathcal{A}}$. Hence we get that $C \cup E$ lies outside $\mathcal{A}$ if we identify $\mathcal{A}$ with its inverse image via $\gamma_1$. Then we consider the closed subset $\mathcal{Y} = \pi \circ \gamma_1(C \cup E)$ which is closed as both $\gamma_1$ and $\pi$ are proper and equip $\mathcal{Y}$ with the reduced induced sub-scheme structure. We also note here that $\mathcal{Y} \subseteq \mathcal{S} - \mathcal{S}$ (set theoretic difference) because $\pi^{-1}(\mathcal{S}) = \mathcal{A}$. Next we consider the scheme-theoretic inverse image $(\pi \circ \gamma_1)^{-1}(\mathcal{Y})$ in $\mathcal{A}$ and call it $\mathcal{Y}$ and finally consider the blowup of $\tilde{\mathcal{A}}$ along $\mathcal{Y}$

$$\gamma_2 : \mathcal{A}' \to \tilde{\mathcal{A}}$$
Note here that as $\mathcal{Y}$ lies outside $S$, $\tilde{\mathcal{Y}}$ lies outside $\mathcal{A}$. Then note that the scheme theoretic inverse image of $\mathcal{Y}$ under $\tilde{\pi} = \pi \circ (\gamma_1 \circ \gamma_2)$ is a Cartier closed sub-scheme and let us call it $\mathcal{Y}_*$. Now we consider the change of the base space i.e. we consider the blow up

$$\gamma_* : S' \to \overline{S}$$

along the closed sub-scheme $\mathcal{Y}$ of $S$. Then note that as the scheme theoretic inverse image $\gamma_* \mathcal{Y}$ of $\mathcal{Y}$ under the morphism $\tilde{\pi}$ is a Cartier closed sub-scheme, we get the existence of a dotted row making the following commutative diagram by the universal property of blow-ups:

$$
\begin{array}{ccc}
\mathcal{A}' & \xrightarrow{\gamma_2} & \mathcal{A} \\
\downarrow{\gamma_1} & & \downarrow{\pi} \\
S' & \xrightarrow{\gamma_*} & \overline{S}
\end{array}
$$

Now we make a comment here that in the above diagram, each of the horizontal morphisms are isomorphisms over $\mathcal{A}$. For $\gamma_1$ it is because the sub-scheme of denominators lies outside $\mathcal{A}$ by Lemma 1 and for $\gamma_2$ it is because $\tilde{\mathcal{Y}}$ is contained in $\mathcal{A} - \mathcal{A}$ (set theoretic difference), being the inverse image of $\overline{S} - S$. Hence we can view $\mathcal{A}$ as an open dense subset of each of them.

We use the same letter for a Cartier closed sub-scheme and its associated effective divisor. By the property of blow-ups, we get that the divisor $\mathcal{Y}' = \gamma_*^{-1}(\mathcal{Y})$ is effective. It is the exceptional divisor corresponding to the blow-up $\gamma_*$. Let us call

$$\Omega = 2 \cdot \mathcal{Y}'$$

which is an effective divisor in $S'$. We claim that

$$\pi_*^* (\Omega) \pm \gamma_*^* (C - E)$$

is effective. Note it is enough to show that both

$$\gamma_*^* (C) \leq \pi_*^* (\mathcal{Y}')$$

The two conditions can be reformulated in the following way by [GW10] Cor.11.49):

$$\gamma_*^* (C) \subseteq \pi_*^* (\mathcal{Y}')$$

But this following conditions easily follow from diagram chasing and the construction of $\mathcal{Y}$. More precisely if $t \in \gamma_2^{-1}(C) \subseteq \mathcal{A}'$, which means that $\tilde{\pi} = \pi \circ (\gamma_1 \circ \gamma_2)$ sends $t$ to $\pi \circ \gamma_1 \circ \gamma_2 \subseteq \mathcal{Y}$. Hence we get that

$$\gamma_*(\pi_* (t)) \in \mathcal{Y} \Rightarrow \pi_* (t) \in \gamma_*^{-1}(\mathcal{Y}) = \mathcal{Y}'$$

and hence the inclusions follow. A similar argument also holds for $E$ by construction of $\mathcal{Y}$. Now passing to $O(\cdot)$, we get that

$$\Lambda_{\pm} = \Theta \pm (\gamma_1 \circ \gamma_2)^* O(D)$$

both have non-zero global sections where $\Theta$ is the line bundle associated to the effective Cartier divisor $\pi_*^* \Omega$. The effectively of the above line bundles proves that outside their supports, the height is bounded by a constant. Rigorously speaking, this means that there is a constant $B > 0$ that

$$h_{\Omega}(\pi_* (P)) \pm h_{\mathcal{F}}(\gamma_1 (\gamma_2 (P))) \geq B$$

for all $P \in \mathcal{A}'(k)$ outside the support. Upon further expansion and use of functoriality, we obtain

$$h_{\Omega}(\pi_* (P)) \pm h_{\mathcal{F}}(\gamma_1 (\gamma_2 (P))) \mp h_{O}\tilde{\pi}(P)) \geq B$$

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for all \( P \) outside the support. This in particular means that

\[
|h_{\mathcal{F}}(\gamma_1(\gamma_2(P)))| \leq |h_{\Omega}(\pi_*(P))| + |h_{\mathcal{M}}(\tilde{\pi}(P))| + B. \tag{1}
\]

Next we want to use inequality (1) above on all points of \( \mathcal{A}(k) \). Hence we claim that none of the points in \( \mathcal{A}(k) \) lies in the support of \( \Lambda_{\pm} \). Note that the support of \( \Lambda_{\pm} \) is given by \( \pi_*^{-1}(\mathcal{Y}') = \pi_*^{-1}(\gamma_*^{-1}(\mathcal{Y})) = \tilde{\pi}^{-1}(\mathcal{Y}) \). Hence if \( P \in \mathcal{A}(k) \) belongs to the supports, this implies that \( \pi_*(P) = \pi(P) \in \mathcal{Y} \). But clearly \( \mathcal{Y} \) lies outside \( S \) due to our choice of choosing blow ups so as to be isomorphic over \( \mathcal{A} \) in the category of \( S \)-schemes and \( \pi \) lands on \( S \) when restricted to \( \mathcal{A} \). Thus we get that none of the points of \( \mathcal{A}(k) \) lie in the support.

Hence we can use the inequality (1) and identifying points using that \( \gamma_1 \) and \( \gamma_2 \) are isomorphisms over \( \mathcal{A} \), we can write simply

\[
|h_{\mathcal{L}}([2]P) - 4 \cdot h_{\mathcal{L}}(P)| \leq |h_{\Omega}(\pi(P))| + |h_{\mathcal{M}}(\pi(P))| + B \text{ for all } P \in \mathcal{A}(k). \tag{2}
\]

Note that inequality (2) is almost ready to be the inequality in Theorem 1.1 except that in the right hand side we have the terms \( |h_{\mathcal{M}}| \text{ and } |h_{\Omega}| \text{ coming from line bundles } \mathcal{M} \text{ on the projective varieties } S' \text{ and } \overline{S} \). Each of the two heights \( h_{\mathcal{M}} \) and \( h_{\Omega} \) can be bounded by some constant times the canonical Weil heights \( h_{S'} \) and \( h_{\overline{S}} \). The reason is that the standard Weil heights are given by very ample line bundles into their respective projective embeddings. This will also be made more clear in the next section. Hence to get the final height inequality we have to show that the height \( h_{S'} \) is bounded by a constant times the height \( h_{\overline{S}} \) when we restrict it to \( \mathcal{A}(k) \).

We achieve this in the next section with the help of a lemma of Lang in his book which was pointed out by Zhelun.

## 5 Last Inequality

This section serves a bridging explanation for some assumptions we have made throughout the article. We have a recurring theme throughout the article where we pass via blow-up to a different compactification but the blow up does not touch certain open sub-schemes(that is an isomorphism over these open sub-schemes)which are typically \( \mathcal{A} \) or \( S \) for us. With help of Lemma 5.1 we show that the height functions only differ up to constant multiple factors when restricted to open sub-schemes over which these blow-ups are isomorphisms. On the light of the inequality in Theorem 1.1, this then justifies our change of base by application of blow-ups in section 3 and also shows us how to pass from inequality (2) in the previous section to the final inequality in Theorem 1.1.

So let

\[
\pi : \mathbb{P}^m' \supset S' \to \overline{S} \subset \mathbb{P}^m
\]

be a blow up morphism, where \( \overline{S} \) is a projective irreducible variety so that \( S' \) is also projective and irreducible variety. Moreover, we have an open subset \( S \subset \overline{S} \) which is not touched by the blow up, i.e. \( \pi^{-1}(S) \cong S \). Consider two line bundles \( L_1 \) and \( L_2 \) on \( S \); let \( h_1, h_2 \) be the Weil heights attached to them respectively, while \( h_{\overline{S}} \) denotes the Weil height on \( \overline{S} \) induced by the standard height on \( \mathbb{P}^m \) (resp. \( \mathbb{P}^m' \)) and the given closed immersion \( \overline{S} \subset \mathbb{P}^m \) (resp. \( S' \subset \mathbb{P}^m' \)). We want to bound \( h_1 \) and \( h_2 \) in terms of \( h_{\overline{S}} \) (at least on a sub-domain).

We are going to use some results from [Lan13].

For the next two assertions we adapt the convention that \( h_n \) denotes the standard height on \( \mathbb{P}^n \), \( n \in \mathbb{N} \).
Lemma 5.1. Let $V$ be a (locally) closed irreducible subvariety of $\mathbb{P}^n$ and suppose there is a morphism $f : V \to \mathbb{P}^\ell$ of finite type, defined over $k$ for some $\ell \in \mathbb{N}$. Then there exist constants $c_1, c_2 > 0$ such that
\[ \forall P \in V(k) : h_L(f(P)) \leq c_1 h_n(P) + c_2. \]

Proof. See [[Lan13], Prop.1.7 of Ch.IV]. □

Corollary 5.2. Let $V$ be an irreducible variety over $k$. Let $c$ be an ample line bundle and $L$ any line bundle on $V$. Pick a height $h_c$ associated to $c$ such that $h_c \geq 0$ (possible because $c$ has a non-zero global section). Then there exist constants $\gamma_1, \gamma_2 > 0$ such that
\[ \forall P \in V(k) : |h_L(P)| \leq \gamma_1 h_c(P) + \gamma_2. \]

Proof. (Cf. [Lang, Prop.5.4 of Ch.IV]) By a theorem from Serre, we can decompose $L = L_+ \otimes L_-(1)$ where $L_+$ and $L_-$ are ample line bundles on $V$ inducing closed immersions $V \hookrightarrow \mathbb{P}^{m_\pm}$ for some $m_\pm \in \mathbb{N}$. By the Height Machine we have
\[ h_L = h_{L_+} - h_{L_-} = h_{m_+} - h_{m_-} \]
as real-valued functions on $V(k)$ up to bounded functions.

On the other hand, since we choose $h_c$ to be non-negative, by the Height Machine we may assume without loss of generality that $c$ is very ample, inducing a closed immersion $V \hookrightarrow \mathbb{P}^n$ for some $n \in \mathbb{N}$. Note that in this case $h_c = h_n$ as real-valued functions on $V(k)$ up to bounded functions.

Applying Lemma 5.1 to the immersions $V \hookrightarrow \mathbb{P}^{m_\pm}$, using the additivity of the Height Machine and rearranging the terms resulting from Lemma 5.1 (the right hand side are both bounded by $\gamma h_c + O(1)$ for some $\gamma > 0$), we are done. □

Let us come back to our beginning situation. We will take $L_1 = \Omega$ and $L_2 = \gamma_* M'$ which are line bundles on $S'$ (following our notations from previous sections) Applying Corollary 5.2 with $V = S'$, $c = \mathcal{O}_{\nu'}(1)$, we are able to bound $h_1 = h_{\Omega}$ and $h_2 = h_{\gamma_* M'}$ with respect to $h_{S'}$. More precisely, using Lemma 5.2 we get positive constants $r_1, r_2, s_1, s_2$ such that
\[ |h_\Omega(P)| \leq r_1 \cdot h_{S'}(P) + s_1 \text{ and } |h_{M'}(P)| \leq r_2 \cdot h_{S'}(P) + s_2 \text{ for all } P \in S(k) \quad (3) \]
where we have used the identification of $\gamma^{-1}(S)$ and $S$ via $\gamma_*$ and the functoriality of height machine. Now note that $h_{S'}$ is non-negative. Since the closed points $P \in A(k)$ we concern are all mapped into $S(k) \subset \mathcal{S}(k)$ (recall our abelian scheme was $A \to S$), we are done by simply restricting the function $h_{S'}$ to $\pi^{-1}(S)(k) \subset S'(k)$. (We can use Lemma 5.1 and the isomorphism $\pi^{-1}(S) \cong S'$ to bound $h_{S'}$ by $h_{\mathcal{S}}$ on $\pi^{-1}(S)(k) = S(k)$.) More formally we have the following corollary of Lemma 5.1:

Corollary 5.3. Let us have two projective varieties $S'$ and $\mathcal{S}$ embedded in $\mathbb{P}^n_k$ and $\mathbb{P}^n_\nu$ respectively and let $h_m$ and $h_n$ be the standard Weil heights of these projective spaces respectively. Furthermore, suppose we have an open sub-scheme $S \subseteq \mathcal{S}$ and a morphism
\[ \pi : S' \to \mathcal{S} \]
such that $\pi |_{\pi^{-1}(S)}$ is an isomorphism onto $S$. Then we have positive constants $c_1, c_2, d_1, d_2$ such that
\[ c_1 \cdot h_n(P) - d_1 \leq h_m(P) \leq c_2 \cdot h_n(P) + d_2 \]
for all $P \in S(k)$ (where we identify $\pi^{-1}(S)$ and $S$ via $\pi$).

Proof. The proof consists of application of Lemma 5.1 once with $V = \pi^{-1}(S)$ and
\[ f = \pi |_{\pi^{-1}(S)} : \pi^{-1}(S) \to \mathbb{P}^n_k \]
and once in the reverse direction with \( V = S \) and \( f = (\pi \mid_{s^{-1}(S)})^{-1} : S \rightarrow \mathbb{P}^n_k \).

Note that for the application of the second case, it is vital that \( \pi \) is an isomorphism over \( S \). Once we write down the two inequalities obtained this way and after some algebraic manipulations, we clearly get the inequalities of the Lemma. 

Remark. Note that this lemma essentially shows that if we pass to a blow up of \( S \) which does not touch \( S \), the height functions (coming from the projective embeddings of \( S \) and \( S' \)) are equivalent to each other up to some universal constant multiples. In the view of the role of the height on the base space in the equality of Theorem 1.1, this clearly justifies the change of base by blow-up that we did in Section 3 because in the notation of the previous sections, we have \( h_m = h_{S'} \) and \( h_n = h_S \).

Now finally from the above we have

\[
|h_L([2]P) - 4 \cdot h_L(P)| \leq (r_1 + r_2) \cdot h_{S'}(\pi(P)) + (s_1 + s_2) + B \text{ for all } P \in A(k)
\]

Next we apply the inequality of Corollary 5.3 noting that \( h_m = h_{S'} \) and \( h_n = h_S \) to get that

\[
|h_L([2]P) - 4 \cdot h_L(P)| \leq c_2(r_1 + r_2)h_S(\pi(P)) + d_2(r_1 + r_2) + (s_1 + s_2) \text{ for all } P \in A(k)
\]

Now finally from the above we have

\[
|h_L([2]P) - 4 \cdot h_L(P)| \leq c \cdot \max\{1, h_S(\pi(P))\} \text{ for all } P \in A(k)
\]

where we take \( c = 2 \cdot \max\{c_2(r_1 + r_2), d_2(r_1 + r_2) + (s_1 + s_2)\} \) which finally proves Theorem 1.1 as \( c \) does not depend on \( P \) clearly.

Remark. The properties of heights we used so far are valid for the base field \( k \) from Section 1 (without restriction on characteristic) satisfying the Product Formula. For details see Chapter IV of Lang’s book mentioned above.

Appendix A  Regularity of \( S \)

We add a short appendix where we comment on the assumption that our base space \( S \) is regular. A careful analysis of the article will show immediately that the only place where we have used the regularity assumption on \( S \) is in section 3. We have chosen a Cartier divisor \( D \) on \( S \) such that \( O(D) \cong M \) prior to the proof of 3.3. After that we argued using the regularity of \( S \) that \( D \) can be written as \( D = C - E \) where \( C \) and \( E \) are effective Cartier divisors. However by [Liu02], Lemma 7.1.31 and Prop 7.1.32 we have that we can always write \( D \) as the difference of two effective Cartier divisors up to linear equivalence as long as \( S \) is a quasi-projective scheme over a noetherian affine scheme. This last hypothesis is always true when \( S \) is a variety over \( k \).

This means that without assuming that \( S \) is regular, we can change our \( D \) by rational functions to write them as the difference of two effective Cartier divisors. As rational equivalence does not change the isomorphism class of a line-bundle, we can choose \( D \) before 3.3 such that \( O(D) \cong M \) and \( D = C - E \) for effective Cartier divisors \( C, E \) on \( S \). After that we can follow the exact same procedure as we did for the proof of 3.2. This clearly shows that the regularity hypothesis on \( S \) is unnecessary.

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