Dynamics and symmetries in chiral $SU(N)$ gauge theories

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Abstract

Dynamics and symmetry realization in various chiral gauge theories in four dimensions are investigated, generalizing a recent work by M. Shifman and the present authors [1], by relying on the standard ’t Hooft anomaly matching conditions and on some other general ideas. These requirements are so strong that the dynamics of the systems are severely constrained. Color-flavor or color-flavor-flavor locking, dynamical Abelianization, and combinations of these, are powerful ideas which often leads to solutions of the anomaly matching conditions. Moreover, a conjecture is made on generation of a mass hierarchy associated with symmetry breaking in chiral gauge theories, which has no analogues in vector-like gauge theories such as QCD.
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1 Introduction

Our world has a nontrivial chiral structure. The macroscopic structures such as biological bodies often have approximately left-right symmetric forms, but not exactly. At the molecular levels, $O(10^{-6} \text{cm})$, the structure of DNA has a definite chiral spiral form. At the microscopic length scales of the fundamental interactions, $O(10^{-14} \text{cm})$, the left- and right-handed quarks and leptons have different couplings to the $SU(3) \times SU_L(2) \times U_Y(1)$ gauge bosons. In spite of the impressive success of the standard model, and after many years of theoretical studies of four dimensional gauge theories, our understanding of strongly-coupled chiral gauge theories is today surprisingly limited. An almost half-century of studies of vector-like gauge theories like $SU(3)$ quantum chromodynamics (QCD), based on lattice simulations with ever more powerful computers, and roughly $\sim 25$ years of beautiful theoretical developments in models with $\mathcal{N} = 2$ supersymmetries, both concern vector-like theories only. Perhaps it is not senseless to make some more efforts to understand this class of gauge theories, which Nature might be making use of, in an as yet unknown way to us.

Urged by such a motivation we have recently revisited the physics of some chiral gauge theories [1]. In the present paper we generalize the analysis done there to a wider class of models, and try to learn some general lessons from them. We use as guiding light the standard ‘t Hooft anomaly matching conditions [15]. To be concrete, we shall limit ourselves to $SU(N)$ gauge theories with a set of Weyl fermions in a complex representation of the gauge group. Also only asymptotically free type of models will be considered, as weakly coupled infrared-free theories can be reliably analyzed in perturbation theory, as

\footnote{See however [2]-[13] for partial list of earlier studies of these theories. See [14] for a recent work on the infrared fixed point in a class of chiral gauge theories.}
in the case of the standard electroweak model. For simplicity we shall restrict ourselves to various irreducibly chiral SU(N) theories, with \( N_\psi \) fermions \( \psi^{(ij)} \) in the symmetric representation, \( N_\chi \) fermions \( \chi_{[ij]} \) in the anti-antisymmetric representation, and a number of anti-fundamental (or fundamental) multiplets, \( \eta^c \) (or \( \tilde{\eta}^{a^i} \)). The number of the latter is fixed by the condition that the gauge group be anomaly free.

Figure 1 gives a schematic representation of the various irreducibly SU(N) chiral theories we shall be interested in. Both \( N_\psi \) and \( N_\chi \) can go up to 5 without loss of asymptotic freedom for large \( N \). The ones we will explicitly consider are summarized in Table 1 with their \( b_0 \) coefficient. The gauge interactions in these models become strongly coupled in the infrared. There are no gauge-invariant bifermion condensates, no mass terms or potential terms (of renormalizable type) can be added to deform the theories, no \( \theta \) parameter exists.

The main question we would like to address ourselves, given a model of this sort, is how to solve the ’t Hooft anomaly matching conditions in the IR and if there are more than one apparently possible dynamical scenarios, all consistent with the matching conditions.

The paper is organized as follows. In Section 2 we revisit the \((N_\psi, N_\chi) = (1,1)\) model previously considered in [1]. In Sections 3–10 we consider respectively the models \((N_\psi, N_\chi) = (1,0), (2,1), (3,0), (0,1), (0,2), (0,3), (2,1), (1,−1)\). In Section 11 we discuss the pion decay constant and a possible new hierarchy mechanism. We conclude in Sec-

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For example we do not consider addition of fundamental-antifundamental pairs of fermions. Models of this type, in the simplest cases \((N_\psi, N_\chi) = (1,0), (0,1)\), have been studied in [9].
2 Revisiting the \((N_\psi, N_\chi) = (1, 1)\) ("\(\psi\chi\eta\)"") model

We first review the analysis of the model with left-handed fermion matter fields

\[ \psi^{[ij]}, \quad \chi^{[ij]}, \quad \eta^A, \quad A = 1, 2, \ldots 8, \quad (2.1) \]

a symmetric tensor, an anti-antisymmetric tensor and eight anti-fundamental multiplets of \(SU(N)\), and add a few new comments with respect to \[1\] \[3\] It is asymptotically free, the first coefficient of the beta function being,

\[ b_0 = \frac{1}{3} [11N - (N + 2) - (N - 2) - 8] = \frac{9N - 8}{3}. \quad (2.2) \]

It is a very strongly coupled theory in the infrared and unlikely to flow into an infrared-fixed point CFT. A nonvanishing instanton amplitude

\[ \langle \psi \psi \ldots \psi \chi \chi \ldots \chi \eta \ldots \eta \rangle \neq 0 \quad (2.3) \]

involves \(N + 2\ \psi\)'s, \(N - 2\ \chi\)'s and \(8\ \eta\)'s.

The model has a global \(SU(8)\) symmetry. It has also three \(U(1)\) symmetries, \(U_\psi(1), U_\chi(1), U_\eta(1)\), of which two combinations are anomaly-free. They can be taken e.g., as

\[ U_1(1) : \quad \psi \rightarrow e^{i\frac{\alpha}{N+2}}\psi, \quad \eta \rightarrow e^{-i\frac{\alpha}{8}}\eta; \]
\[ U_2(1) : \quad \psi \rightarrow e^{i\frac{\alpha}{N+2}}\psi, \quad \chi \rightarrow e^{-i\frac{\alpha}{N-2}}\chi. \quad (2.4) \]

\[ ^3 \text{Earlier studies on this model can be found in \[6\ \[7\ \[11].} \]
There are also anomaly-free discrete subgroups $\mathbb{Z}_{N+2} \otimes \mathbb{Z}_{N-2} \otimes \mathbb{Z}_8$ of $U_\psi(1)$, $U_\chi(1)$, $U_\eta(1)$, which are not broken by the instantons. However, they are not independent of each other, in view of the nonanomalous symmetries (2.4) The global continuous symmetry of the $\psi - \chi - \eta$ model is

$$G_1 = SU(8) \times U_1(1) \times U_2(1) \, .$$

(2.5)

### 2.1 Partial color-flavor locking

Possible dynamical scenarios in this model have been analyzed and discussed in [1]. It was proposed that a possible phase (valid for $N \geq 12$) can be described by the nonvanishing bi-fermion condensates

$$\langle \phi^i A \rangle = \langle \psi^{ij} \eta^A_{i} \rangle \quad \text{and} \quad \langle \tilde{\phi}^j \rangle \equiv \langle \psi^{ik} \chi_{kj} \rangle .$$

(2.6)

More concretely, the proper realization of the global $SU(8)$ symmetry has led us to assume the following form for these condensates:

$$\langle \psi^{ij} \eta^A_{i} \rangle = \Lambda^3 \begin{pmatrix} c 1_8 \\ 0_{N-8,8} \end{pmatrix}^{i A} , \quad \langle \psi^{ik} \chi_{kj} \rangle = \Lambda^3 \begin{pmatrix} a 1_8 \\ d_1 \\ \vdots \\ d_{N-12} \\ b 1_4 \end{pmatrix}^{i j} ,$$

(2.7)

where

$$8a + \sum_{i=1}^{N-12} d_i + 4b = 0 , \quad a, d_i, b \sim O(1) .$$

(2.8)

The symmetry breaking pattern is, therefore,

$$SU(N)_c \times SU(8)_f \times U(1)^2 \rightarrow SU(8)_{cf} \times U(1)^{N-11} \times SU(4)_c .$$

(2.9)

The theory dynamically Abelianizes (in part). $SU(8) \subset SU(N)$ is completely Higgsed but due to color-flavor (partial) locking no NG bosons appear in this sector (the would-be NG bosons make the $SU(8) \subset SU(N)$ gauge bosons massive.) Only $SU(4) \subset SU(N)$ remains unbroken and confining. The remainder of the gauge group Abelianizes. The baryons

$$\tilde{B}_j^A = \psi^{ik} \chi_{kj} \eta^A_i \sim \eta^A_j \, , \quad (9 \leq j \leq N-4)$$

(2.10)

and

$$B^{(AB)} = \psi^{ij} \eta^A_i \eta^B_j ,$$

(2.11)
symmetric in the flavor indices ($A \leftrightarrow B$)\(^4\) remain massless and together saturate the ‘t Hooft anomaly matching condition for $SU(8)$:

\[ 8 + 4 + N - 12 = N . \]  

(2.12)

Note that both nonanomalous continuous $U_{1,2}(1)$’s are broken by the two condensates. Actually, for some $N$, a discrete symmetry survives the condensates of the form (2.6), and the discrete anomaly matching must be taken into account.

### 2.1.1 Discrete symmetries

Under the discrete symmetries the fields transform as

\[
\begin{align*}
\mathbb{Z}_{N+2} &\subset U_\psi(1) : \quad \psi \to e^{\frac{2\pi k}{N+2}} \psi , \quad k = 0, 1, \ldots N + 1 ; \\
\mathbb{Z}_{N-2} &\subset U_\chi(1) : \quad \chi \to e^{\frac{2\pi \ell}{N-2}} \chi , \quad \ell = 0, 1, \ldots N - 3 ; \\
\mathbb{Z}_8 &\subset U_\eta(1) : \quad \eta \to e^{\frac{2\pi m}{8}} \eta , \quad m = 0, 1, \ldots 7 .
\end{align*}
\]

(2.13)

A discrete subgroup survives the condensates (2.6) if

\[
\frac{k}{N+2} - \frac{\ell}{N-2} \in \mathbb{Z} , \quad \frac{k}{N+2} - \frac{m}{8} \in \mathbb{Z} .
\]

(2.14)

Clearly there are no discrete surviving symmetry for odd $N$. For $N$ even, the above shows that there remains a $\mathbb{Z}_2$ symmetry, for $N = 4n$, $n \in \mathbb{Z}$, or a $\mathbb{Z}_4$ symmetry, for $N = 4n + 2$.

To be concrete, consider $N = 14$. The conditions above read in this case

\[
\frac{k}{16} - \frac{\ell}{12} \in \mathbb{Z} , \quad \frac{k}{16} - \frac{m}{8} \in \mathbb{Z} .
\]

(2.15)

The transformation

\[
\begin{align*}
\psi &\to e^{\frac{\pi i}{2}} \psi , \quad \chi \to e^{-\frac{\pi i}{2}} \chi , \quad \eta \to e^{-\frac{\pi i}{2}} \eta ,
\end{align*}
\]

(2.16)

generates $\mathbb{Z}_4$, which is kept unbroken by $\langle \psi \chi \rangle$ and $\langle \psi \eta \rangle$. The $\mathbb{Z}_4$ charge of $(\psi, \chi, \eta)$ fields are $(1, -1, -1)$ Mod 4.

Consider the discrete anomaly $SU(8)^2 \mathbb{Z}_4$\([16]\). In the UV, the only contribution is from the $\eta$ fields, which gives

\[ N \cdot 1 \cdot (-1) = -N = -14 . \]  

(2.17)

\(^4\)If the massless $B^{(AB)}$ were antisymmetric in the flavor indices, they would contribute $8 - 4 = 4$ to the $SU(8)$ anomaly. We would then need $N - 4$ massless fermions of the form $B^A_j \sim \eta^A_j$, but this is impossible as the latter arises from the Abelianization of the rest of the color gauge group, $SU(N - 8)$. 

7
In the IR, $\eta_j^A (9 \leq j \leq N - 4)$ gives
\[(N - 12) \cdot 1 \cdot (-1) = -2 , \quad (2.18)\]
whereas $B^{(AB)} = \psi^{ij} \eta_i^A \eta_j^B$ contribute
\[1 \cdot (8 + 2) \cdot (-1) = -10 , \quad (2.19)\]
total of
\[-2 - 10 = -12 . \quad (2.20)\]
The difference between UV and IR is
\[-14 - (-12) = -2 \neq 0 \mod 4 . \quad (2.21)\]
Thus the discrete $SU(8)^2 \mathbb{Z}_4$ anomaly does not match for $N = 14$. A similar situation is found for all $N$ of the form $4n + 2$, $n = 3, 4, 5, \ldots$.

As for the discrete $Grav^2 \mathbb{Z}_4$ anomaly, we count only the $\mathbb{Z}_4$ charges and the multiplicities: in the UV, it is
\[N \cdot 1 \cdot (-1) = N = -14 , \quad (2.22)\]
whereas in the IR the value is
\[2 \cdot 1 \cdot (-1) + \frac{8 \cdot 9}{2} \cdot (-1) = -38 . \quad (2.23)\]
The difference is
\[38 - 14 = 24 = 0 \mod 4 , \quad (2.24)\]
so it is matched.

For $N = 4n$, the conditions
\[\frac{k}{4n + 2} - \frac{\ell}{4n - 2} \in \mathbb{Z} , \quad \frac{k}{4n + 2} - \frac{m}{8} \in \mathbb{Z} . \quad (2.25)\]
leaves a $\mathbb{Z}_2$ symmetry generated by the transformations with $k = 2n + 1$, $\ell = 2n - 1$ and $m = 4$. It is easy to verify that all the discrete anomalies involving $\mathbb{Z}_2$ are matched in the UV and in the IR.

The fact that the discrete anomaly matching does not work for $N = 4n + 2$ renders the scenario (2.6)-(2.9) not likely to be realized for any $N$. There are however other possibilities as discussed below.
2.2 Color-flavor locking and dynamical Abelianization: an alternative scenario

Another possible phase, for \( N \geq 8 \), which was not considered in [1], is described by the condensates (2.6), but this time of the form

\[
\langle \psi^{ij} \eta_j^A \rangle = \Lambda^3 \left( \begin{array}{c}
c_1 \\
0_{N-8,8}
\end{array} \right)^A_i, \quad \langle \psi^{ik} \chi_{kj} \rangle = \Lambda^3 \left( \begin{array}{c}
d_1 \\
0_{N-8,8}
\end{array} \right)^i_j.
\] (2.26)

The symmetry breaking pattern is:

\[
SU(N) \times SU(8) \times U(1)^2 \rightarrow SU(8)_{cf} \times U(1)^{N-8}.
\] (2.27)

As \( U(1)^{N-8} \) is an Abelian subgroup of the color \( SU(N) \), whereas both nonanomalous flavor \( U(1) \) are broken by the condensates, we shall consider only the \( SU(8)_{cf}^3 \) anomalies. Indicating the color indices up to 8 by \( i_1 \) or \( j_1 \) while those larger than 8 by \( i_2 \) or \( j_2 \), one has the decomposition of the fields in \( SU(8)_{cf} \) multiplets, see Table 2. The massless baryons

|         | fields                  | \( SU(8)_{cf} \) |
|---------|-------------------------|-----------------|
| UV      | \( \psi^{ij}_{i_1 j_1} \) | \[\square\]      |
|         | \( \psi^{ij}_{i_2 j_2} \) | \[\square\]      |
|         | \( \chi_{i_1 j_1} \)     | \[\square\]      |
|         | \( \chi_{i_2 j_2} \)     | \[\square\]      |
|         | \( \eta_{j_1}^A \)       | \[\square\]      |
|         | \( \eta_{j_2}^A \)       | \[\square\]      |
| IR      | \( \tilde{B}_{j_2}^A \sim \eta_{j_2}^A \) | \[\square\]      |
|         | \( B^{[A]j_1} \sim A(\eta_{j_1}^A) \) | \[\square\]      |
|         | \( \tilde{B}^{[i]j_1} \sim \chi_{i_1 j_1} \) | \[\square\]      |

Table 2: \( B^{[AB]} \sim \psi^{ij}_i^A \eta_j^B \) and \( \tilde{B}^{[AB]} \sim (\psi \eta)^A \chi_{ij}(\psi \eta)^B j \). \( \eta_{j_2}^A \) are weakly coupled due to the Abelianization of the \( SU(N-8) \times U(1) \subset SU(N) \). They can be interpreted as \( \tilde{B}_j^A \sim (\psi \chi)_j^A \). The color indices up to 8 are indicated by \( i_1 \) or \( j_1 \) while those larger than 8 by \( i_2 \) or \( j_2 \).
are shown in the lower part of the Table 2. The $SU(8)^3$ matching works, as in the infrared,

\[(N - 8) + (8 - 4) + (8 - 4) = N .\]  

(2.28)

As for the discrete symmetry, the surviving symmetry is either $\mathbb{Z}_2$, for $N = 4n, n \in \mathbb{Z}$, or $\mathbb{Z}_4$ symmetry, for $N = 4n + 2$ under which the fields $\psi, \chi, \eta$ are charged with $(1, -1, -1)$. An inspection of Table 2 shows that all discrete anomaly matching is also satisfied in this case, in contrast to the previous case.

### 2.3 Partial color-flavor locking for $N \leq 8$

For $N < 8$ the scenario above is not viable. It is possible however that the color-flavor locking still takes place in a different way (this possibility was not considered in [1] either). Let us assume that

\[\langle \psi^{ij} \eta_{ij}^A \rangle = \Lambda^3 \left( c_{1N} \left| 0_{N,8-N} \right. \right)^{iA}, \quad \langle \psi^{ik} \chi_{kj} \rangle = 0 .\]  

(2.29)

The symmetry breaking pattern is now

\[SU(N) \times SU(8) \times U(1)^2 \rightarrow SU(N)_{cf} \times SU(8 - N) \times \tilde{U}(1) .\]  

(2.30)

The fermions decompose as in Table 3. The massless baryons which saturate the anomalies

| fields | $SU(N)_c$ | $SU(8-N)$ | $\tilde{U}(1)$ |
|--------|----------|-----------|--------------|
| UV     | $\psi$   | $\chi$    | $\eta^{A_1}$ | $\eta^{A_2}$ |
|        | $\psi$   | $\chi$    | $\eta^{A_1}$ | $\eta^{A_2}$ |
| IR     | $B^{[A_1]}$ | $B^{[A_1]}$ | $B^{[A_1]}$ | $B^{[A_1]}$ |

Table 3: Partial color-flavor locking for $N \leq 8$ and the $SU(8)$ anomaly matching of Subsection 2.3. $A_1, B_1$ stand for the flavor indices up to $N(<8)$, $A_2, B_2$ for the rest.
are made of
\[ B^{[A_1 B_1]} = \psi^{ij} \eta^A_i \eta^B_j \sim \phi, \quad B^{[A_1 B_2]} = \psi^{ij} \eta^A_i \eta^B_j \sim \eta^B_2, \]
\[ \hat{B}^{[A B]} = \psi^{ik} \chi_{ij} \psi^{j\ell} \eta^B_\ell \sim \chi_{A B}. \]  

(2.31)

2.4 Full Abelianization and general \( N \)

The dynamical scenarios (2.9) assumes that \( N \geq 12 \), whereas the one in (2.27) requires \( N \geq 8 \) and (2.30) requires \( N \leq 8 \).

Still another option, consistent for any value of \( N \), considered in [1], is that the gauge group dynamically Abelianizes completely, by the adjoint condensates
\[ \langle \psi^{ij} \eta^A_j \rangle = 0, \quad \langle \psi^{ik} \chi_{kj} \rangle = \Lambda^3 \left( \begin{array}{c} d_1 \\ \vdots \\ d_N \end{array} \right)_i \] with \( \sum_j d_j = 0 \) and no other particular relations among \( d_j \)'s. No color-flavor locking takes place. The symmetry breaking occurs as:
\[ SU(N)_c \times SU(8)_f \times U(1)_c^2 \rightarrow \prod_{\ell=1}^{N-1} U_\ell(1) \times SU(8)_f \times \tilde{U}(1), \]  

(2.33)

where \( \tilde{U}(1) \) is an unbroken combination of the two nonanomalous \( U(1) \)'s, (2.4), with charges:
\[ \psi : 2, \quad \chi : -2, \quad \eta : -1. \]  

(2.34)

The fields \( \eta^A_i \) are all massless and weakly coupled (only to the gauge bosons from the Cartan subalgebra which we will refer to as the photons; they are infrared free) in the infrared. Also, some of the fermions \( \psi^{ij} \) do not participate in the condensates. Due to the fact that \( \psi^{(ij)} \) are symmetric whereas \( \chi_{[ij]} \) are antisymmetric, actually only nondiagonal elements of \( \psi^{(ij)} \) condense and get mass. The diagonal fields \( \psi^{(ii)} \), \( i = 1, 2, \ldots, N \) remain massless and weakly coupled. Also there is one NG boson. The anomaly matching works as shown in Table 4.

3 \( (N_\psi, N_\chi) = (1, 0) \)

Let us review the \( (N_\psi, N_\chi) = (1, 0) \) model studied in [3, 8, 9, 10, 13]. The matter fermions are
\[ \psi^{(ij)}, \quad \eta^B_\ell, \quad B = 1, 2, \ldots, N + 4, \]  

(3.1)
or

\[
\begin{array}{c}
\begin{array}{c}
\psi
\end{array}
\end{array} + (N+4) \begin{array}{c}
\eta
\end{array}.
\]

(3.2)

The first coefficient of the beta function is

\[
b_0 = \frac{1}{3} \left[ 11N - (N+2) - (N+4) \right] = \frac{9N - 6}{3}.
\]

(3.3)

The (continuous) symmetry of this model is

\[
SU(N)_c \times SU(N+4)_f \times U(1),
\]

(3.4)

where \( U(1) \) is an anomaly-free combination of \( U_\psi(1) \) and \( U_\eta(1) \), with

\[
Q_\psi : N+4, \quad Q_\eta : -(N+2).
\]

(3.5)

There are also discrete symmetries

\[
\mathbb{Z}_\psi = \mathbb{Z}_{N+2} \subset U_\psi(1), \quad \mathbb{Z}_\eta = \mathbb{Z}_{N+4} \subset U_\eta(1).
\]

(3.6)

### 3.1 Chirally symmetric phase in the \((1,0)\) model

Let us first examine the possibility that no condensates form, the system confines and the flavor symmetry is unbroken \[3\]. The candidate massless baryons are:

\[
B^{[AB]} = \psi_{ij}^A \eta_{ij}^B, \quad A, B = 1, 2, \ldots, N+4,
\]

(3.7)

antisymmetric in \( A \leftrightarrow B \). All the \( SU(N+4)_f \times U(1) \) anomalies are saturated by \( B^{[AB]} \) as can be seen by inspection of the Table 5. The discrete anomaly \( \mathbb{Z}_\psi SU(N)^2 \) is also matched, as can be easily checked.
\[
\langle \psi^{(ij)} \eta^B \rangle = c \Lambda^3 \delta^{jB}, \quad j, B = 1, 2, \ldots N, \quad (3.8)
\]

in which the symmetry is reduced to

\[
SU(N)_{cf} \times SU(4)_f \times U'(1). \quad (3.9)
\]

As this forms a subgroup of the full symmetry group, (3.4), it is quite easily seen, by making the decomposition of the fields in the direct sum of representations in the subgroup, that a subset of the same baryons saturate all of the triangles associated with the reduced symmetry group, see Table 6.

| fields         | $SU(N)_{cf}$ | $SU(4)_f$          | $U'(1)$ |
|----------------|--------------|-------------------|--------|
| $UV \psi$      |              | $\frac{N(N+1)}{2} \cdot$ | 1      |
| $\eta^A$       | $(N+4) \cdot$ | $N \cdot$         | -1     |
| $IR B^{[AB]}$  | $(N+4)(N+3) \cdot$ | $-N$             |        |

Table 6: Color-flavor locked phase in the (1,0) model, discussed in Subsection 3.2. $A_1$ or $B_1$ stand for $A, B = 1, 2, \ldots, N$. $A_2$ or $B_2$ the rest of the flavor indices.

The discrete anomaly $Z_\psi$ is broken by the condensate $\psi \eta$. There is (for generic $N$) no combination between $Z_\psi$ and $Z_\eta$ which survives, therefore there is no discrete anomaly matching condition.

It is not known which of the possibilities, 3.1 or 3.2, is realized in the (1,0) model. The low-energy degrees of freedom are $\frac{(N+4)(N+3)}{2}$ massless baryons in the former case, and $\frac{N^2+7N}{2}$ massless baryons together with $8N+1$ Nambu-Goldstone bosons, in the latter. Thus
the complementarity [18], as noted in [1], does not work here even though the (dynamical) Higgs scalars $\psi\eta$ are in the fundamental representation of color.

4 \quad (N_\psi, N_\chi) = (2, 0)

This is a straightforward generalization of the $\psi\eta$ model above. The matter fermions are

$$ \psi^{(ij, m)}, \quad \eta^B_i, \quad m = 1, 2, \quad B = 1, 2, \ldots, 2(N + 4), \quad (4.1) $$

or

$$ 2\square + 2(N + 4)\square. \quad (4.2) $$

The (continuous) symmetry of this model is

$$ SU(N)_c \times SU(2)_f \times SU(2N + 8)_f \times U(1), \quad (4.3) $$

where $U(1)$ is an anomaly-free combination of $U_\psi(1)$ and $U_\eta(1)$,

$$ U(1) : \quad \psi \rightarrow e^{i\alpha/2(N+2)}\psi, \quad \eta \rightarrow e^{-i\alpha/2(N+4)}\eta. \quad (4.4) $$

The first coefficient of the beta function is

$$ b_0 = \frac{1}{3} \left[ 11N - 2(N + 2) - 2(N + 4) \right] = \frac{7N - 12}{3}, \quad (4.5) $$

which is positive for $N \geq 2$.

4.1 No chiral symmetry breaking in the $(2, 0)$ model?

Let us first assume that no condensates form and no flavor symmetry breaking occurs. Assuming confinement, the possible massless baryons are

$$ B^{mAB} = \psi^{ij, m}_i \eta^A_i \eta^B_j. \quad (4.6) $$

They cannot however saturate the triangles associated with the flavor symmetry

$$ SU(2)_f \times SU(2N + 8)_f \times U(1). \quad (4.7) $$

For instance the $SU(2N + 8)^3$ anomaly, which is equal to $N$ in the UV, would be at least $\sim 2N$ for any baryon like (4.6) and thus it is not reproduced in any way in IR. We must conclude that confinement phase with unbroken flavor symmetries cannot be realized in this system. This is in contrast to the $(1, 0)$ model, reviewed in Subsection 3.1.
4.2 Partial color-flavor locking?

Let us consider next a partial color-flavor locking condensates

\[ \langle \psi^{ij} \eta_i^B \rangle = c A^3 \delta^{jB}, \quad j, B = 1, 2, \ldots N, \quad (4.8) \]

which breaks the symmetry to

\[ SU(N)_{\text{cf}} \times SU(N + 8) \times \tilde{U}(1); \quad (4.9) \]

\[ SU(2) \text{ is broken. } \tilde{U}(1) \text{ is a linear combination of } [1,4] \text{ and} \]

\[ U_1(1) = \left( \begin{array}{cc} \frac{1}{N} & 0 \\ 0 & \frac{1}{N+8} \end{array} \right) \subset SU(2(N + 4)). \quad (4.10) \]

So the unbroken \( \tilde{U}(1) \) acts on the fields as

\[ \psi \rightarrow e^{i{\alpha \over 2(N+2)}} \psi; \]

\[ \eta_i^A \rightarrow e^{-i{\alpha \over 2(N+2)}} \eta_i^A, \quad (A = 1, 2, \ldots N); \]

\[ \eta_i^A \rightarrow e^{-i{\alpha(N+4) \over 2(N+2)(N+8)}} \eta_i^A, \quad (A = N + 1, N + 2, \ldots 2N + 8). \quad (4.11) \]

The charges with respect to \( \tilde{U}(1) \) are:

\[ \psi : 1, \quad \eta^< : -1, \quad \eta^> : -{N + 4 \over N + 8}. \quad (4.12) \]

The massless baryons are assumed to be of the form,

\[ B^{AB} = \psi^{ij} \eta_i^A \eta_j^B, \quad A, B = 1, 2, \ldots N \quad (4.13) \]

and

\[ \tilde{B}^{AB} = \psi^{ij} \eta_i^A \eta_j^B, \quad A = 1, 2, \ldots N, \quad B = N + 1, \ldots 2N + 8. \quad (4.14) \]

Here we must choose \( B^{AB} \) in the symmetric or antisymmetric representation of the \( SU(N)_{\text{cf}} \) group while \( \tilde{B}^{AB} \) is in the \((N, N + 8)\) of \( SU(N)_{\text{cf}} \times SU(N + 8)_{\text{flavor}} \). \( \tilde{U}(1) \) charges of \( B^{AB} \) and \( \tilde{B}^{AB} \) are

\[ B^{AB} : -1; \quad \tilde{B}^{AB} : -{N + 4 \over N + 8}. \quad (4.15) \]

These assumptions are made such that the \( SU(N + 8)^3_f \) and \( \tilde{U}(1)SU(N + 8)^3_f \) anomalies are matched in the UV and IR; however, it is easy to verify that the triangles \( \tilde{U}(1)^3 \) and \( SU(N)^3_{\text{cf}} \) cannot be matched. Therefore the phase (4.8), (4.9), cannot be realized.
4.3 A possible phase: a double color-flavor locking

Another possibility is to assume a double $SU(N)$ color-flavor-flavor locking

$$
\langle \psi^{ij,1} \eta^B \rangle = c \Lambda^3 \delta^i, B, \quad j, B = 1, 2, \ldots N ,
\langle \psi^{ij,2} \eta^B \rangle = c' \Lambda^3 \delta^i, B-N, \quad j = 1, 2, \ldots N , \quad B = N + 1, \ldots 2N ,
$$

(4.16)

The symmetry is broken to

$$SU(N)_{cf} \times \tilde{U}(1) \times U'(1) \times SU(8) .
$$

(4.17)

where $\tilde{U}(1)$ acts as before:

$$
\psi : 1 ; \quad \eta^{B \leq 2N} : -1 ; \quad \eta^{B > 2N} : -\frac{1}{2} .
$$

(4.18)

There are

$$3N^2 + 32N + 3
$$

(4.19)

NG bosons. $U'(1)$ is a subgroup of $SU(2)_{ff}$ defined below, (4.26), (4.27) which survives the condensates (4.16).

In order to saturate all the anomalies, one assumes that somehow only

$$
\tilde{B}^{A,B} = \psi^{ij,1} \eta^A \eta^B , \quad A = 1, 2, \ldots , N , \quad B = 2N + 1, \ldots 2N + 8 .
$$

(4.20)

or

$$
\tilde{B}^{A,B} = \psi^{ij,1} \eta^A \eta^B , \quad A = N + 1, N + 2, \ldots , 2N , \quad B = 2N + 1, \ldots 2N + 8 .
$$

(4.21)

(but not both) remain massless. One could write these states as

$$
\tilde{B}^B_a = \sum_{A=1}^{2N} \sum_{m=1,2} c_{a,m,A} \psi^{ij,m} \eta^A \eta^B , \quad a = 1, 2, \ldots , N , \quad B = 2N + 1, \ldots 2N + 8 .
$$

(4.22)

Furthermore, we shall need also two types of baryons

$$
B^{[AB],1} = \psi^{ij,1} \eta^A \eta^B , \quad A, B = 1, 2, \ldots , N ,
B^{[AB],2} = \psi^{ij,1} \eta^A \eta^B , \quad A, B = N + 1, N + 2, \ldots , 2N ,
$$

(4.23)

both antisymmetric in $AB$, all of them remaining massless. It is a simple exercise to check that all anomalies, $SU(8)^3, SU(8)^2 \tilde{U}(1), \tilde{U}(1)^3, \tilde{U}(1), SU(N)^3, SU(N)^2 \tilde{U}(1)$ are matched.

In conclusion, the double c-f locking phase, with massless baryons $\tilde{B}^{A,B}$, or $\tilde{B}^{A,B}$, or analogous states with $1 \leftrightarrow 2$, together with $B^{AB,1}$ and $B^{AB,2}$ (both antisymmetric in $AB$), is consistent with anomaly matching. The asymmetric way $\psi^{ij,1}$ and $\psi^{ij,2}$ appears in the
IR baryons is consistent as the $SU(2)$ is broken.

4.4 Phase with unbroken $SU(2)$

Another phase is the one with an unbroken $SU(2)$ symmetry. Assume (4.16) with the same coefficients

$$c = c' .$$

The symmetry is broken to

$$SU(N)_{cf} \times \tilde{U}(1) \times SU(2)_{ff} \times SU(8) ,$$

where $SU(2)_{ff}$ is a linear combination of $SU(2)_{f}$ and

$$SU(2) \subset SU(2N) \subset SU(2N + 8)$$

which interchange the first and second $N$ flavors. The charges of the unbroken $SU(2)$ are:

$$\left( \psi^{ij,1} \right) \sim 2 , \quad \left( \eta^{A \leq N}_{i} \right) \sim 2^{*} .$$

The $\tilde{U}(1)$ charges are as before,

$$\psi : 1 , \quad \eta^{B \leq 2N} : -1 , \quad \eta^{B>2N} : -\frac{1}{2} .$$

The baryons are

$$B^{A,C} = \sum_{i,j} \left( \psi^{ij,1}_{i} \eta^{A \leq N}_{i} \eta^{C}_{j} + \psi^{ij,2}_{i} \eta^{N<A \leq 2N}_{i} \eta^{C}_{j} \right) , \quad C > 2N ,$$

which is a $SU(2)$ singlet; the others are

$$B^{[A_1B_1],1} = \psi^{ij,1}_{i} \eta^{A_1}_{i} \eta^{B_1}_{j} , \quad A_1, B_1 = 1, 2, \ldots, N ,$$

$$B^{[A_2B_2],2} = \psi^{ij,2}_{i} \eta^{A_2}_{i} \eta^{B_2}_{j} , \quad A_2, B_2 = N + 1, N + 2, \ldots, 2N ,$$

which form a doublet. Their $\tilde{U}(1)$ charges are:

$$B^{A,C} : -\frac{1}{2} ; \quad B^{[AB],m} : -1 .$$

The anomaly saturation can be again seen quickly by inspecting Table 7. The discrete symmetries $\mathbb{Z}_\psi = \mathbb{Z}_{2(N+2)}$ and $\mathbb{Z}_\eta = \mathbb{Z}_{2(N+4)}$ are both broken by the condensates. Also,
Table 7: An $SU(2)$ flavor-flavor locked symmetric phase in the $(2,0)$ model, discussed in Subsection 4.4. A$_i$ or B$_i$ ($i = 1, 2$) indicate the flavor indices up to $2N$, C the rest, $2N + 1, \ldots, 2N + 8$.

Witten’s $SU(2)$ anomaly matches: there are

$$\frac{N(N+1)}{2} + N^2 \quad (4.32)$$

left-handed $SU(2)$ doublets in the UV, whereas the corresponding number in the IR is

$$\frac{N(N-1)}{2} \quad : \quad (4.33)$$

the difference is

$$N(N+1) \quad , \quad (4.34)$$

which is always even.

4.4.1 Remarks on less symmetric phases

The less symmetric phases discussed in Subsection 4.3 can be derived from the most symmetric phase discussed here. Namely, when the bi-fermion condensates have no special relations, some of the global symmetries are broken, and a multiplet (irrep) with respect to such a subgroup (e.g., $SU(N)_{ct}$ or $SU(2)$) is replaced by a simple multiplicity of states of similar types, both for elementary fermions and for composite ones. Clearly the anomaly saturation valid in the most symmetric case imply similar results by subset of fermions / subgroups in the less symmetric phases.

5 \quad $(N_\psi, N_\chi) = (3, 0)$

Let us consider a further generalization. The matter fermions are

$$\psi^{(ij,m)}, \quad \eta^{i}_B, \quad m = 1, 2, 3, \quad B = 1, 2, \ldots, 3(N + 4), \quad (5.1)$$
or

\[ 3 \square + 3(N + 4) \square. \]  

(5.2)

The (continuous) symmetry of this model is

\[ SU(N)_c \times SU(3)_f \times SU(3N + 12)_f \times U(1), \]  

(5.3)

where \( U(1) \) is the anomaly-free combination of \( U_\psi(1) \) and \( U_\eta(1) \),

\[ U(1) : \quad \psi \rightarrow e^{i\alpha/3(N+2)} \psi, \quad \eta \rightarrow e^{-i\alpha/3(N+4)} \eta. \]  

(5.4)

The first coefficient of the beta function is

\[ b_0 = \frac{1}{3} [11N - 3(N + 2) - 3(N + 4)] = \frac{5N - 18}{3}, \]  

(5.5)

which is positive for \( N \geq 4 \). It can be seen that, as for the \((2,0)\) model, chiral symmetric phase and partial color-flavor locking do not provide solutions to the anomaly matching.

### 5.1 Triple color-flavor locking

Generalizing Subsection 4.3, one may assume a triple color-flavor locking here:

\[
\begin{align*}
\langle \psi^{(ij,1)} \eta^{B} \rangle &= c \Lambda^3 \delta^{ij} \delta^{B}, & j, B &= 1, 2, \ldots N, \\
\langle \psi^{(ij,2)} \eta^{B} \rangle &= c' \Lambda^3 \delta^{ij} \delta^{B-N}, & j, B &= N + 1, N + 2, \ldots 2N, \\
\langle \psi^{(ij,3)} \eta^{B} \rangle &= c'' \Lambda^3 \delta^{ij} \delta^{B-2N}, & j, B &= 2N + 1, 2N + 2, \ldots 3N.
\end{align*}
\]  

(5.6)

The symmetry realization is

\[ SU(N)_{cf} \times \tilde{U}(1) \times SU(12). \]  

(5.7)

where \( \tilde{U}(1) \) acts as

\[
\begin{align*}
\psi &\rightarrow e^{i\alpha/3(N+2)} \psi; \\
\eta^A &\rightarrow e^{-i\alpha/3(N+2)} \eta^A, & A &= 1, 2, \ldots, 3N; \\
\check{\eta}^A &\rightarrow e^{-i\alpha/6(N+2)} \check{\eta}^A, & A &= 3N + 1, 3N + 2, \ldots, 3N + 12.
\end{align*}
\]  

(5.8)

or by renormalizing the charges:

\[ \check{Q}_\psi : 1; \quad \check{Q}_\eta : -1; \quad \check{Q}_{\check{\eta}} : \frac{1}{2}. \]  

(5.9)
We now check the matching with massless baryons

\[ \hat{B}^{A, B} = \psi^{ij, 1} \eta_i^A \eta_j^B, \quad A = 1, 2, \ldots, N, \quad B = 3N + 1, \ldots 3N + 12, \]
\[ B^{[AB], 1} = \psi^{ij, 1} \eta_i^A \eta_j^B, \quad A, B = 1, 2, \ldots, N, \]
\[ B^{[AB], 2} = \psi^{ij, 1} \eta_i^A \eta_j^B, \quad A, B = N + 1, N + 2, \ldots, 2N, \]
\[ B^{[AB], 3} = \psi^{ij, 1} \eta_i^A \eta_j^B, \quad A, B = 2N + 1, 2N + 2, \ldots, 3N. \]  
(5.10)

The \( \hat{U}(1) \) charge of these baryons are:

\[ \hat{B}^{A, B} : \quad \frac{-1}{2}; \]  
(5.11)
\[ B^{[AB], 1}, \quad B^{[AB], 2}, \quad B^{[AB], 3} : \quad -1. \]  
(5.12)

It can be readily verified that the anomalies with respect to \( SU(12)^3, SU(12)^2 \hat{U}(1), \hat{U}(1)^3, \hat{U}(1), SU(N)^3, SU(N)^2 \hat{U}(1) \) agree in the UV and in the IR.

### 5.2 \( SU(3) \) symmetric phase

As in Subsection 4.4 one may assume a more symmetric form of the condensates (5.6) with equal coefficient

\[ c = c' = c''. \]  
(5.13)

In this case, a diagonal \( SU(3) \) between \( SU(3)_\psi \) and \( SU(3) \subset SU(3N) \) remains unbroken. The low-energy symmetry realization is then

\[ SU(N)_{\text{cf}} \times SU(3) \times \hat{U}(1) \times SU(12). \]  
(5.14)

There are two more triangles, \( SU(3)^3 \) and \( SU(3)^2 \hat{U}(1) \), in addition to six types of anomalies considered in the previous Subsection. The charges with respect to this \( SU(3) \) are

\[ \begin{pmatrix} \psi^{ij,1} \\ \psi^{ij,2} \\ \psi^{ij,3} \end{pmatrix} \sim \mathbf{2}, \quad \begin{pmatrix} \eta_i^{A \leq N} \\ \eta_i^{N < A \leq 2N} \\ \eta_i^{2N < A \leq 3N} \end{pmatrix} \sim \mathbf{2}^*. \]  
(5.15)

The massless baryons are

\[ B^{A, C} = \sum_{i,j} (\psi^{ij,1} \eta_i^{A \leq N} \eta_j^C + \psi^{ij,2} \eta_i^{N < A \leq 2N} \eta_j^C + \psi^{ij,3} \eta_i^{2N < A \leq 3N} \eta_j^C), \quad C > 3N, \]  
(5.16)
which is an $SU(3)$ singlet; the others are

\[
\begin{align*}
B^{[AB],1} &= \psi^{ij} \cdot \eta^{i} \cdot \eta^{j}, & A, B = 1, 2, \ldots, N , \\
B^{[AB],2} &= \psi^{ij} \cdot 2\eta^{i} \cdot \eta^{j}, & A, B = N + 1, N + 2, \ldots, 2N , \\
B^{[AB],3} &= \psi^{ij} \cdot 2\eta^{i} \cdot \eta^{j}, & A, B = 2N + 1, 2N + 2, \ldots, 3N ,
\end{align*}
\]

which form an anti-triplet, $\bar{3}$.

Again it is convenient to have the decomposition of the fields with respect to the unbroken groups. The saturation of the anomalies $SU(12)^3$, $SU(12)^2\tilde{U}(1)$, $\tilde{U}(1)^3$, $\tilde{U}(1)^3$, $SU(N)^3$, $SU(N)^2\tilde{U}(1)$, $SU(3)^3$, and $SU(3)^2\tilde{U}(1)$ is seen at once, by inspection of Table 8.

| fields | $SU(N)_{cf}$ | $SU(12)$ | $SU(3)$ | $\tilde{U}(1)$ |
|--------|-------------|-----------|---------|---------------|
| UV     | $\psi$      | $3 \cdot \mathbb{1}$ | $3N(N+1)/2 \cdot (\cdot)$ | $N(N+1)/2 \cdot \mathbb{1}$ | $1$ |
| $\eta^{A \leq 3N}$ | $3 \cdot (\mathbb{1} \oplus \mathbb{1})$ | $3N^2 \cdot (\cdot)$ | $N^2 \cdot \mathbb{1}$ | $-1$ |
| $\eta^{A > 3N}$ | $12 \cdot \mathbb{1}$ | $N \cdot \mathbb{1}$ | $12N \cdot (\cdot)$ | $-\frac{1}{2}$ |
| IR     | $B^{A,\bar{C}},$ | $12 \cdot \mathbb{1}$ | $N \cdot \mathbb{1}$ | $12N \cdot (\cdot)$ | $-\frac{1}{2}$ |
| $B^{[AB],m}$ | $3 \cdot \mathbb{1}$ | $3N(N-1)/2 \cdot (\cdot)$ | $N(N-1)/2 \cdot \mathbb{1}$ | $-1$ |

Table 8: Color-favor-flavor locked $SU(3)$ symmetric phase in the $(3,0)$ model, discussed in Subsection 5.2.

As seen in the $(2,0)$ model, Subsection 4.4.1, less symmetric phases are possible in the $(3,0)$ model as well. The condensates are of more general forms in those cases, with unequal values, and one or both of the symmetries $SU(N)_{cf}$ and $SU(3)$ can be broken spontaneously. The set of the baryons $B^{A,\bar{C}}$ and $B^{[AB],m}$ will continue to saturate the anomaly triangles of the remaining symmetries.

6 \quad $(N_{\psi}, N_{\chi}) = (0, 1)$

Let us review the $(N_{\psi}, N_{\chi}) = (0,1)$ model studied in [3, 8, 9, 10, 13, 1]. The matter fermions are

\[
\chi^{[ij]}, \quad \tilde{\eta}^{B,j}, \quad B = 1, 2, \ldots, (N - 4) .
\]

The first coefficient of the $\beta$ function is

\[
b_0 = \frac{1}{3} \left[ 11N - (N - 2) - (N - 4) \right] = \frac{9N + 6}{3} .
\]
The (continuous) symmetry is

\[ SU(N)_c \times SU(N-4)_f \times U(1), \quad (6.3) \]

where the anomaly free \( U(1) \) charge is

\[ \chi : \quad N - 4; \quad \tilde{\eta}^{Bj} : \quad -(N - 2). \quad (6.4) \]

There are also discrete symmetries

\[ \mathbb{Z}_\chi = \mathbb{Z}_{N-2} \subset U_\psi(1), \quad \mathbb{Z}_{\tilde{\eta}} = \mathbb{Z}_{N-4} \subset U_\eta(1). \quad (6.5) \]

### 6.1 Chirally symmetric phase in the (0, 1) model

Let us first examine the possibility that no condensates form, the system confines and the flavor symmetry is unbroken \([3]\). The massless baryons are

\[ B^{\{CD\}} = \chi_{[ij]} \tilde{\eta}^i C \tilde{\eta}^j D, \quad C, D = 1, 2, \ldots, (N - 4), \quad (6.6) \]

symmetric in \( C \leftrightarrow D \). They have the \( U(1) \) charge: \(-N\). The matching of the anomalies can be read off Table 9.

| fields | \( SU(N)_c \) | \( SU(N-4) \) | \( U(1) \) |
|--------|--------------|--------------|------------|
| UV \( \chi \) | \( (N-4) \cdot \) | \( \frac{N(N-1)}{2} \cdot (\cdot) \) | \( N - 4 \) |
| IR \( \tilde{\eta}^A \) | \( \frac{(N-4)(N-3)}{2} \cdot (\cdot) \) | \( (\cdot) \) | \(-N \) |

Table 9: Confinement and unbroken symmetry in the (0, 1) model

### 6.2 Color-flavor locked vacuum

It was pointed out \([9]\) that this system may instead develop a condensate of the form

\[ \langle \chi_{[ij]} \tilde{\eta}^{Bj} \rangle = \text{const.} \Lambda^3 \delta^B \quad i, B = 1, 2, \ldots, N - 4, \quad (6.7) \]

namely,

\[ \square \otimes \square \rightarrow \square \oplus \ldots \quad (6.8) \]
The symmetry is broken to
\[ SU(N - 4)_{\text{cf}} \times U'(1) \times SU(4)_{c} . \quad (6.9) \]

The massless baryons (6.6) saturate all the anomalies associated with \( SU(N - 4)_{\text{cf}} \times U'(1) \). There remains the \( \chi_{i_2 j_2} \) fermions which remain massless and strongly coupled to the \( SU(4)_{c} \). We may assume that \( SU(4)_{c} \) confines, and the condensate
\[ \langle \chi \chi \rangle \neq 0 , \quad (6.10) \]
in
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\end{array}
or

\[
2 \square + 2(N - 4) \square .
\]  

(7.2)

The first coefficient of the \( \beta \) function is

\[
b_0 = \frac{1}{3} |11N - 2(N - 2) - 2(N - 4)| = \frac{1}{3}(7N + 12) .
\]  

(7.3)

7.1 No chiral symmetry breaking in the \((0, 2)\) model?

The symmetry is

\[
SU(N)_c \times SU(2)_f \times SU(2N - 8)_f \times U(1) ,
\]  

(7.4)

where the anomaly free \( U(1) \) charge is

\[
\chi : \quad N - 4 ; \quad \tilde{\eta}^{Bj} : \quad -(N - 2) .
\]  

(7.5)

Let us assume that the massless baryons are

\[
B^{(CD),m} = \chi^{m}_{[ij]} \tilde{\eta}^C \tilde{\eta}^D , \quad C, D = 1, 2, \ldots 2(N - 4) , \quad m = 1, 2 .
\]  

(7.6)

symmetric in \( CD \). They have the \( U(1) \) charge \(-N\). There is no way \( B^{(CD)} \) can saturate the anomalies in \( SU(2N - 8)_f \times U(1) \).

One concludes that the confinement phase with unbroken chiral symmetry \( SU(2) \times SU(2N - 8)_f \times U(1) \) is not possible. This is, again, in contrast to the \((N_\psi, N_\chi) = (0, 1)\) model.

7.2 Color-flavor locking

Let us instead assume a color-flavor locked diagonal VEV,

\[
\langle \chi^{j}_{[ij]} \tilde{\eta}^B \rangle = c \Lambda^3 \delta_j^B , \quad j, B = 1, 2, \ldots, N - 4 ,
\]

\[
\langle \chi^{j}_{[ij]} \tilde{\eta}^B \rangle = c' \Lambda^3 \delta_j^{B-(N-4)} , \quad j = 1, \ldots, N - 4 ; \quad B = N - 3 \ldots 2N - 8 .
\]  

(7.7)

Then the symmetry is broken to

\[
SU(N - 4)_{cf} \times U'(1) \times SU(4)_c .
\]  

(7.8)

where \( U(1)' \) is the unbroken linear combination between the anomaly free \( U(1), \) \((7.5)\), and a subgroup of the color \( SU(N) \), \( \text{diag}(\frac{1}{N-4} \mathbb{1}_{N-4}, -\frac{1}{4} \mathbb{1}_4) \). We assume that the massless
baryons are
\[
B^{(CD),1} = \chi^{\frac{1}{i_j}}_{[i_j]} \tilde{\eta}^C \tilde{\eta}^D, \quad C, D = 1, 2, \ldots N - 4, \\
\tilde{B}^{(CD),1} = \chi^{\frac{1}{i_j}}_{[i_j]} \tilde{\eta}^C \tilde{\eta}^D, \quad C, D = N - 3, N - 2, \ldots 2(N - 4).
\] (7.9)

The charges under \(SU(N - 4)_{\text{cf}} \otimes U'(1)\) are given in Table 11 where the \(U(1)'\) charges are appropriately renormalized by a common factor. All anomalies \(SU(N - 4)_{\text{cf}}^3, U(1)'^3, U(1)'^3, U(1)'SU(N - 4)_{\text{cf}}^2\) work out fine.

| fields | \(SU(N - 4)_{\text{cf}}\) | \(U'(1)\) |
|--------|------------------|---------|
| UV     |                  |         |
| \(\chi^1_{[i_1j_1]}\) | \(4 \cdot \) | \(\frac{1}{2}\) |
| \(\chi^1_{[i_1j_2]}\) | ( )          | 0       |
| \(\chi^1_{[i_2j_2]}\) | \(4(N - 4)\) | -1      |
| \(\tilde{\eta}^{B,i_1}\) | \(8 \cdot \) | -\frac{1}{2} |
| IR     | \(B^{(CD1)}\)   | -1      |
|        | \(\tilde{B}^{(CD1)}\) | -1 |

Table 11: Color-flavor locking in the (0,2) model. The color index \(i_1\) or \(j_1\) runs up to \(N - 4\). The rest is indicated by \(i_2\) or \(j_2\).

### 7.3 Phase with unbroken \(SU(2)\)

Assume instead that the condensates \((7.7)\) occur with with the same coefficients
\[
c = c'.
\] (7.10)

Then the residual symmetry is bigger
\[
SU(N - 4)_{\text{cf}} \times SU(2) \times U'(1) \times SU(4)_{\text{cf}}.
\] (7.11)

The baryons are
\[
B^{(CD),1} = \chi^{\frac{1}{i_j}}_{[i_j]} \tilde{\eta}^C \tilde{\eta}^D, \quad C, D = 1, 2, \ldots N - 4, \\
B^{(CD),2} = \chi^{\frac{2}{i_j}}_{[i_j]} \tilde{\eta}^C \tilde{\eta}^D, \quad C, D = N - 3, N - 2, \ldots 2(N - 4),
\] (7.12)
symmetric in \(CD\). The charges with respect to this \(SU(2)\) are
\[
\left( \begin{array}{c} \chi^{\frac{1}{i_j}}_{[i_j]} \\ \chi^{\frac{2}{i_j}}_{[i_j]} \end{array} \right) \sim 2, \quad \left( \begin{array}{c} \tilde{\eta}^{A \leq N - 4}_{i_i} \\ \tilde{\eta}^{N - 4 \leq A \leq 2N - 8}_{i_i} \end{array} \right) \sim 2^*. \] (7.13)
The charges of the fields with respect to the unbroken symmetries are in Table 12. The saturation of all seven types of triangles can be seen by inspection.

| fields   | $SU(N - 4)_{cf}$ | $SU(2)$ | $U'(1)$ | $SU(4)_c$ |
|----------|------------------|---------|---------|------------|
| UV       |                  |         |         |            |
| $\chi_{[i_1 j_1]}^m$ | 2 · [ ] | $(N-4)(N-5) \cdot \square$ | $\frac{1}{2}$ | $(N - 4)(N - 5) \cdot (\cdot)$ |
| $\chi_{[i_1 j_2]}^m$ | 8 · [ ] | $4(N - 4) \cdot \square$ | $\frac{1}{2}$ | $2(N - 4) \cdot \square$ |
| $\chi_{[i_2 j_2]}^m$ | 12 · (·) | 6 · [ ] | 0       | 2 · [ ]   |
| $\tilde{\eta}^B_{i_1}$ | 2 · (· + ·) | $(N - 4)^2 \cdot \square$ | $-1$       | $2(N - 4)^2 \cdot (\cdot)$ |
| $\tilde{\eta}^B_{i_2}$ | 8 · [ ] | $4(N - 4) \cdot \square$ | $-\frac{1}{2}$ | $2(N - 4) \cdot \square$ |
| IR       |                  |         |         |            |
| $B^{CD,m}$ | 2 · [ ] | $(N-4)(N-3) \cdot \square$ | $-1$       | $(N - 4)(N - 3) \cdot (\cdot)$ |

Table 12: $SU(2)$ symmetric phase in the $(0, 2)$ model. $i_1, j_1$ stand for the color indices up to $N - 4$, $i_2, j_2$ the last four.

$SU(2)$ has no (perturbative) triangle anomaly but it does have a global anomaly (Witten). It can be readily checked that the difference of the number of the doublets in the UV and in the IR is even.

As in the $(0, 1)$ model, the fermions $\chi_{[i_2 j_2]}^m$ remain massless and coupled strongly by the unbroken color $SU(4)_c$. It is possible that they condense as

$$\langle \epsilon^{ijk\ell} \chi_{ij}^m \chi_{k\ell}^n \rangle \neq 0, \quad m, n = 1, 2.$$  \hspace{1cm} (7.14)

As they are symmetric in $m, n$, the symmetry is broken as

$$SU(2) \rightarrow SO(2) = U(1),$$  \hspace{1cm} (7.15)

in a scenario similar to tumbling.

So after all $SU(2)$ is dynamically broken. The fate of the unbroken, residual $SU(4)_c$ is similar to what happens in the second (XSB) scenario in Subsection 6.2.

**8** \hspace{1cm} $(N_\psi, N_\chi) = (0, 3)$

The model to be considered now is

$$\chi_{[ij]}^m, \quad \tilde{\eta}^B_{i}, \quad m = 1, 2, 3, \quad B = 1, 2, \ldots, 3(N - 4).$$  \hspace{1cm} (8.1)

or

$$3[ ] + 3(N - 4)\square.$$  \hspace{1cm} (8.2)
The first coefficient of the \( \beta \) function is

\[
b_0 = \frac{1}{3} [11N - 3(N - 2) - 3(N - 4)] = \frac{1}{3} (5N + 18) . \tag{8.3}
\]

The symmetry is

\[
SU(N)_c \times SU(3) \times SU(3N - 12)_f \times U(1) , \tag{8.4}
\]

where the anomaly free \( U(1) \) charge is

\[
\chi : \quad N - 4 , \quad \tilde{\eta}^{Bj} : \quad -(N - 2) . \tag{8.5}
\]

Again, the option of confinement with no flavor symmetry breaking is excluded.

### 8.1 Color-flavor locking

Let us try to generalize the color-flavor locking of the \((N_\psi, N_\chi) = (0, 2)\) case to our \((N_\psi, N_\chi) = (0, 3)\) model, by assuming

\[
\langle \chi_{[ij]}^1 \tilde{\eta}^B \rangle = e^{3 \delta_j^B} \neq 0 , \quad j, B = 1, 2, \ldots, N - 4 ,
\]

\[
\langle \chi_{[ij]}^2 \tilde{\eta}^B \rangle = e^{3 \delta_j^B - (N - 4)} \neq 0 , \quad j = 1, 2, \ldots, N - 4 , \quad N - 3 \leq B \leq 2N - 8 ,
\]

\[
\langle \chi_{[ij]}^3 \tilde{\eta}^B \rangle = e^{3 \delta_j^B - (N - 4)} \neq 0 , \quad j = 1, 2, \ldots, N - 4 , \quad 2N - 7 \leq B \leq 3N - 12 . \tag{8.6}
\]

Then the symmetry breaking pattern is

\[
SU(N - 4)_c \times U'(1) \times SU(3) \times SU(4)_c . \tag{8.7}
\]

where \(U'(1)'\) is the unbroken linear combination between the anomaly free \(U(1)\), \(\text{[8.3]}\), and a subgroup of the color \(SU(N)\), \(\text{diag}(4 \mathbb{1}_{N-4}, -(N - 4) \mathbb{1}_4)\). The would-be \(SU(3)\) multiplets are:

\[
\begin{pmatrix}
\chi_{[ij]}^1 \\
\chi_{[ij]}^2 \\
\chi_{[ij]}^3
\end{pmatrix}
\sim 3 , \\
\begin{pmatrix}
\tilde{\eta}_{i}^{A \leq N-4} \\
\tilde{\eta}_{i}^{N-4 < A \leq 2N-8} \\
\tilde{\eta}_{i}^{2N-8 < A \leq 3N-12}
\end{pmatrix}
\sim 3^* . \tag{8.8}
\]

We assume that the massless baryons are

\[
B^{(CD),1} = \chi_{[ij]}^1 \tilde{\eta}^C \tilde{\eta}^D , \quad C, D = 1, 2, \ldots, N - 4 ,
\]

\[
B^{(CD),2} = \chi_{[ij]}^2 \tilde{\eta}^C \tilde{\eta}^D , \quad C, D = N - 3, \ldots, 2(N - 4) ,
\]

\[
B^{(CD),3} = \chi_{[ij]}^3 \tilde{\eta}^C \tilde{\eta}^D , \quad C, D = 2N - 7, \ldots, 3(N - 4) . \tag{8.9}
\]

symmetric in \(CD\). These baryons transform as \(3^*\).
Table 13: The decomposition of the fields in the $(0, 3)$ model. The color indices are divided into two groups: $i_1, j_1$ run up to $N - 4$; $i_2, j_2$ the rest. Moreover, the color and flavor indices are combined as in Subsection 8.1.

Unlike what happens to the $(0, 2)$ model, or to the $(3, 0)$ model, however, here the unbroken $SU(3)$ symmetry cannot be realized manifestly in the infrared: $SU(3)^3$ triangles do not match in the UV and IR, see Table 13.

A possibility is that the condensates $(8.6)$ take unequal values. With $SU(3)$ broken, the baryons $B^{(CD),m}$ saturate the anomalies in $SU(N - 4)_{cf} \times U'(1) \times SU(4)c$.

Another possibility is suggested by the presence of massless fermions $\chi_{[ij]}^m$ $(i_\geq, j_\geq)$, which interact strongly with the remaining gauge group $SU(4)c$. It is possible that the condensates

$$\langle \epsilon^{ijk}\chi_{[ij]}^m\chi_{[kl]}^n \rangle \neq 0, \quad m, n = 1, 2, 3.$$  

(8.10)

form. As they are symmetric in $m, n$, the symmetry is broken as

$$SU(3) \to SO(3)$$  

(8.11)

which is free of anomalies.

9 \quad \left(N_{\psi}, N_{\chi}\right) = (2, 1)

Next consider the $SU(N)$ gauge model with the chiral fermion sector

$$\psi^{(ij),m}, \quad \chi_{[ij]}, \quad \eta^{B}_j, \quad m = 1, 2, \quad B = 1, 2, \ldots, N + 12,$$

(9.1)

or

$$2\begin{array}{c}
\hline
\hline
\end{array} + \begin{array}{c}
\hline
\hline
\end{array} + (N + 12)\begin{array}{c}
\hline
\hline
\end{array}.$$  

(9.2)
The symmetries of the theory are

\[ SU(N)_c \times SU(2)_f \times SU(12 + N)_f \times U(1)^2 . \]  

(9.3)

The two \( U(1) \)'s are anomaly-free combinations of \( U_\psi(1), U_\chi(1), U_\eta(1) \), which can be taken as

\[ U_1(1) : \quad \psi \rightarrow e^{i\frac{N+2}{N+12}} \psi , \quad \eta \rightarrow e^{-i\frac{N+12}{N+2}} \eta ; \]
\[ U_2(1) : \quad \psi \rightarrow e^{i\frac{N+2}{N+12}} \psi , \quad \chi \rightarrow e^{-i\frac{N+12}{N+2}} \chi . \]  

(9.4)

The first coefficient of the \( \beta \) function is

\[ b_0 = \frac{1}{3} \left[ 11N - 2(N+2) - (N-2) - (12 + N) \right] = \frac{1}{3} (7N - 14) . \]  

(9.5)

### 9.1 Color-flavor locking?

A possibility is that a (partial) color-flavor locking condensate

\[ \langle \psi^{\{ij\}, 1} \eta_i^B \rangle = c \Lambda^3 \delta^i B , \quad i, B = 1, 2, \ldots, N \]  

(9.6)

develops, where the direction of the \( SU_\psi(2) \) breaking is arbitrarily. Let us assume that there is no adjoint condensate \( \langle \psi \chi \rangle \). The unbroken symmetry is

\[ SU(N)_{cf} \times SU(12)_f \times \tilde{U}(1) , \]  

(9.7)

where \( \tilde{U}(1) \) charges are

\[ Q_\psi = 1 , \quad Q_\chi = -\frac{N-8}{N-2} , \quad Q_\eta = -1 . \]  

(9.8)

The candidate baryons are:

\[ B^{C,D,m} = \psi^{\{ij\},m} \eta_i^C \eta_j^D . \]  

(9.9)

An inspection shows that these baryons do not saturate the \( G_f \) anomalies, and one concludes that the phase (9.6) is not possible.

### 9.2 Color-flavor-flavor locking?

Let us assume, for \( N \leq 12 \), the condensates of the form,

\[ \langle \psi^{\{ij\}, 1} \eta_j^{B_1} \rangle = c \Lambda^3 \delta^i B_1 , \]
\[ \langle \psi^{\{ij\}, 2} \eta_j^{B_2} \rangle = c \Lambda^3 \delta^i B_2 - N , \]  

(9.10)

\[ ]
where the flavor indices $B_1$ runs up to $N$, $B_2$ from $N + 1$ to $2N$. The symmetry is broken to

$$SU(N)_{\text{cf}} \times SU(2)_{\text{ff}} \times SU(12 - N)_{\ell} \times U'(1).$$  \hspace{1cm} (9.11)

| fields | $SU(N)_{\text{cf}}$ | $SU(2)$ | $SU(12 - N)$ | $U'(1)$ |
|--------|------------------|----------|--------------|---------|
| UV     | $\psi^{(ij),m}$  | $2 \cdot \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{array}$ | $\frac{N(N+1)}{2} \cdot \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$ | $N(N+1) \cdot (\cdot)$ | $1$ |
|        | $\chi_{[ij]}$    | $\frac{N(N-1)}{2} \cdot (\cdot)$ | $\frac{N(N-1)}{2} \cdot (\cdot)$ | $-\frac{N-8}{N-2}$ |
|        | $\eta^{B_1,j}_i, \eta^{B_2,j}_i$ | $2 \cdot \overline{(\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \text{ } \\ \text{ } \end{array})}$ | $N^2 \cdot \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$ | $2N^2 \cdot (\cdot)$ | $-1$ |
|        | $\eta^{B_3,j}_i$ | $(12 - N) \cdot \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$ | $N(12 - N) \cdot (\cdot)$ | $N \cdot \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$ | $-1$ |
| IR     | $B^{[A_1,B_2],m}$ | $2 \cdot \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \text{ } \\ \text{ } \\ \text{ } \end{array}$ | $\frac{N(N-1)}{2} \cdot \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$ | $N(N-1) \cdot (\cdot)$ | $-1$ |
|        | $B^{[A_1,B_3],m}$ | $(12 - N) \cdot \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$ | $N(12 - N) \cdot (\cdot)$ | $N \cdot \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$ | $-1$ |

Table 14: $SU(2)$ symmetric phase in the $(2, 1)$ model. $A_1, B_1$ stand for the flavor indices up to $N$; $A_2, B_2$ from $N + 1$ to $2N$, $A_3, B_3$ the last $12 - N$. The anomaly matching fails in this case.

The candidate baryons have the form,

$$B^{AB,m} = \psi^{(ij),m} \eta^A_i \eta^B_j,$$

but it is not possible to achieve the anomaly matching.

### 9.3 Dynamical Abelianization

Assuming that the adjoint condensate forms

$$\langle \psi^{(ij),1} \chi_{[jk]} \rangle = c^j \Lambda^3 \delta^i_k, \quad j, k = 1, 2, \ldots, N,$$

with $c^j$’s all different the Cartan subgroup of $SU(N)_c$ survives in the infrared. $SU(2)_f$ is broken. There is a $U(1)$ symmetry which remains unbroken, $\tilde{U}(1)$, under which

$$\psi : N + 12; \quad \chi : -(N + 12); \quad \eta : -(N + 6).$$

The unbroken symmetry group is

$$SU(N + 12)_f \times \tilde{U}(1).$$

The low energy degrees of freedom are the fermion fields $\eta^B_j$ which are unconfined and are weakly coupled to the $U(1)^{N-1}$ photons, the diagonal $\psi^{(ij),1}$ and all of $\psi^{(ij),2}$. Also there are $3 + 1 = 4$ NG bosons.

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The anomaly equalities for $SU(12+N)_3^f, \tilde{U}(1)SU(12+N)_3^2, \tilde{U}(1)_3^3, \tilde{U}(1)$ can be straightforwardly checked, see Table 15.

| fields | $SU(N+12)$ | $\tilde{U}(1)$ |
|--------|-------------|----------------|
| UV     | $\psi$      | $2 \cdot \frac{N(N+1)}{N(N-1)} \cdot (\cdot)$ | $N+12$ |
|        | $\chi$      | $\frac{N}{2} \cdot (\cdot)$ | $-(N+12)$ |
|        | $\eta^A$    | $-(N+6)$ |
| IR     | $\psi^{i1,1}$ | $N \cdot (\cdot)$ | $N+12$ |
|        | $\psi^{i2,2}$ | $\frac{N(N+1)}{N} \cdot (\cdot)$ | $N+12$ |
|        | $\psi \chi \eta^A \sim \eta^A$ | $-\frac{1}{2}N$ |

Table 15: The decomposition of the fields in the $(2,1)$ model, assuming the complete dynamical Abelianization.

10. $(N_\psi, N_\chi) = (1, -1)$

Consider now a model with

$$
\psi^{ij}, \tilde{\chi}^{[ij]}, \eta^A_i, A = 1, 2, \ldots 2N,
$$

or

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} + 2N \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array},
\end{array}
$$

i.e., a symmetric tensor, an antisymmetric tensor and $2N$ anti-fundamental multiplets of $SU(N)$. The first coefficient of the beta function is

$$
b_0 = \frac{1}{3} [11N - (N + 2) - (N - 2) - 2N] = \frac{7N}{3}.
$$

The symmetry of the system is

$$
SU(N)_c \times SU(2N)_f \times U_1(1) \times U_2(1)
$$

times some discrete symmetry. The $U(1)$ charges are:

$$
U_1(1): \quad Q_\psi = \frac{1}{N+2}, \quad Q_{\tilde{\chi}} = -\frac{1}{N-2}, \quad Q_\eta = 0;
$$

$$
U_2(1): \quad Q_\psi = \frac{1}{N+2}, \quad Q_{\tilde{\chi}} = 0, \quad Q_\eta = -\frac{1}{2N},
$$

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Possible baryon states are

$$B^{AB} = \psi^{(ij)} \eta_i^A \eta_j^B, \quad \hat{B}^{AB} = \chi^{(ij)} \eta_i^A \eta_j^B,$$

both of which could form either symmetric or antisymmetric tensors in the flavor. Confinement without chiral symmetry breaking appears excluded: there is no way $B^{AB}$ or $\hat{B}^{AB}$ can match the UV $SU(2N)_f$ anomaly, $N$.

### 10.1 Color-flavor locking

Let us try a color-flavor locking

$$\langle \psi^{(ij)} \eta_j^A \rangle = c \Lambda^3 \delta^{iA}, \quad i, A = 1, 2, \ldots, N,$$
$$\langle \chi^{(ij)} \eta_j^A \rangle = c' \Lambda^3 \delta^{iA}, \quad i, A = 1, 2, \ldots, N.$$

The symmetry is broken to

$$SU(N)_{cf} \times SU(N)_f \times \hat{U}(1)$$

where $\hat{U}(1)$ is an unbroken combination of $U_{1,2}(1)$, with charges,

$$\hat{U}(1): \quad Q_\psi = -1, \quad Q_\chi = -1, \quad Q_\eta = 1.$$

Again we list the fields and their decomposition in the low-energy symmetry groups. Assuming that the only massless baryons are $B^{AB}$, with $A \leq N$, $B \geq N$, the anomaly matching is obvious, see Table 16.

| fields | $SU(N)_{cf}$ | $SU(N)_f$ | $\hat{U}(1)$ |
|--------|-------------|-------------|-------------|
| UV     | $\psi$      | $N(N+1)/2 \cdot \cdot \cdot$ | -1          |
|        | $\chi$      | $N(N-1)/2 \cdot \cdot \cdot$ | -1          |
| $\eta^A_1$ | $\oplus$ | $N^2 \cdot \cdot \cdot$ | 1           |
| $\eta^A_2$ | $N \cdot \cdot \cdot$ | $N \cdot \cdot \cdot$ | 1           |
| IR     | $B^{A_1B_2}$ | $N \cdot \cdot \cdot$ | 1           |

Table 16: The color-flavor locking scheme for the $(1, -1)$ model. The flavor indices $A_1, B_1$ stand for those up to $N$, $A_2, B_2$ for $N+1, \ldots, 2N$.  

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11 Pion decay constant in chiral theories

After these exercises with various \((N_\psi, N_\chi)\) models, it would be useful to try to draw some lessons. One concerns the nature of the Nambu-Goldstone bosons (called “pions” below symbolically) and the quantity analogous to the pion-decay constant in the chiral \(SU(2)_L \times SU(2)_R\) QCD. As we shall see, there is some qualitative difference between the wisdom about the chiral dynamics with light quarks in QCD which is a vector-like theory, and what is to be expected in general chiral theories.

Consider any global continuous symmetry \(G_f\) and the associated conserved current \(J_\mu\), the field \(\phi\) (elementary or composite) which condenses and break \(G_f\), and the field \(\tilde{\phi}\) which is transformed into \(\phi\) by the \(G_f\) charge

\[
Q \equiv \int d^3x J_0, \quad [Q, \tilde{\phi}] = \phi, \quad \langle \phi \rangle \neq 0. \tag{11.1}
\]

Thus

\[
\lim_{q_\mu \to 0} i q_\mu \int d^4x e^{-i q \cdot x} \langle 0| T\{J_\mu(x) \tilde{\phi}(0)\}|0\rangle = \\
\lim_{q_\mu \to 0} \int d^4x e^{-i q \cdot x} \partial_\mu \langle 0| T\{J_\mu(x) \tilde{\phi}(0)\}|0\rangle = \\
\int d^3x \langle 0|[J_0(x), \tilde{\phi}(0)]|0\rangle = \langle 0|[Q, \tilde{\phi}(0)]|0\rangle = \langle 0|\phi(0)|0\rangle \neq 0. \tag{11.2}
\]

This Ward-Takahashi like identity implies that the two-point function

\[
\int d^4x e^{-i q \cdot x} \langle 0| T\{J_\mu(x) \tilde{\phi}(0)\}|0\rangle \tag{11.3}
\]

is singular at \(q \to 0\). If the \(G_f\) symmetry is broken spontaneously such a singularity is due to the massless NG boson, \(\pi\), such that

\[
\langle 0|J_\mu(q)|\pi\rangle = i q_\mu F_\pi, \quad \langle \pi|\tilde{\phi}(0)|0\rangle \neq 0, \tag{11.4}
\]

such that the two point function behaves as

\[
q^\mu \cdot q_\mu \frac{F_\pi \langle \pi|\tilde{\phi}(0)\rangle}{q^2} \sim \text{const.} \tag{11.5}
\]

at \(q \to 0\).

In the standard \(SU(2)_L \times SU(2)_R \to SU(2)_V\) chiral symmetry breaking in QCD, the quarks are

\[
\psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \psi_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \tag{11.6}
\]
and by taking
\[ \phi = \bar{\psi}_R \psi_L + \text{h.c.}, \quad \tilde{\phi} = \bar{\psi}_R \tilde{t}^a \psi_L - \text{h.c.;} \quad J^{5,a}_\mu = i \bar{\psi}_L \sigma_\mu t^a \psi_L - (L \leftrightarrow R) \]  \hspace{1cm} (11.7)

\[ t^a = \frac{\tau^a}{2}, \quad a = 1, 2, 3. \]  \hspace{1cm} (11.8)

It is believed that the field
\[ \langle \phi \rangle = \langle \bar{u}_R u_L + \bar{d}_R d_L + \text{h.c.} \rangle \sim -\Lambda^3, \]  \hspace{1cm} (11.9)

condenses, leaving $SU(2)_V$ unbroken; the axial $SU(2)_A$ is broken. In the QCD $\Lambda$ is of the same order of the confinement mass scale, the dynamically generated mass scale of QCD,
\[ \Lambda \sim 200 \text{ MeV}. \]  \hspace{1cm} (11.10)

The pions are associated with the interpolating field
\[ \pi^a \sim \tilde{\phi}^a = \bar{\psi}_R t^a \psi_L - \text{h.c.} \sim \bar{\psi}_D \gamma^5 t^a \psi_D \]  \hspace{1cm} (11.11)

(where $\psi_D$ is the Dirac spinors for the quarks). It is natural to expect that the pion decay constant, the amplitude with which the current operator $J^{5,a}_\mu$ produces the pions from the vacuum, is of the same order of magnitude as $\Lambda$ itself,
\[ F_\pi \sim \Lambda. \]  \hspace{1cm} (11.12)

Indeed, the best experimental estimate for $F_\pi$ is
\[ F_\pi \sim 130 \text{ MeV}, \]  \hspace{1cm} (11.13)
cfr. with (11.10).

Now let us study the case of chiral gauge theories, as those considered in this paper. To be concrete, consider the dynamical scenarios, Subsection 4.3 in the $(2,0)$ model. The symmetry breaking pattern is
\[ SU(N)_c \times SU(2)_t \times SU(2N + 8)_t \times U(1) \rightarrow SU(N)_{\text{ef}} \times \tilde{U}(1) \times U'(1) \times SU(8). \]  \hspace{1cm} (11.14)

The Nambu-Goldstone modes are associated with the breaking
\[ SU(2)_t \times SU(2N + 8)_t \rightarrow SU(8) \times U'(1), \]  \hspace{1cm} (11.15)

There are
\[ 3N^2 + 32N + 3 \]  \hspace{1cm} (11.16)

NG bosons.
To simplify the discussion let us concentrate our attention to the two NG bosons associated with the $SU(2) \rightarrow SU'(1)$ breaking. The $SU(2)$ current is

$$J^a_\mu = i \bar{\psi}^{ij,m} \bar{\sigma}_\mu \left( \frac{\tau^a}{2} \right)_{mn} \psi^{ij,n},$$

and the charges are

$$Q^a = \int d^3 x J^a_0.$$ (11.17)

One can choose

$$\tilde{\phi}^b = \sum_{i,j,k,B} (\psi^{ij,m})_i^B \eta^B_i \left( \frac{\tau^b}{2} \right)_{mn} \psi^{kj,m} \eta^B_k,$$ (11.19)

in (11.2), so that

$$\langle [Q^a, \tilde{\phi}^b] \rangle = \delta^{ab} \langle \sum_{i,j,k,B} (\psi^{ij,m})_i^B \eta^B_i \left( \frac{\tau^b}{2} \right)_{mn} \psi^{kj,m} \eta^B_k \rangle \neq 0.$$ (11.20)

An important issue here is the fact that even though the dynamical gauge and flavor symmetry breaking are (by assumption) determined by the “dynamical Higgs scalar” condensates

$$\langle \psi^{ij,1} \eta^B_i \rangle = c \Lambda^3 \delta^{ij,B}, \quad j,B = 1,2,\ldots,N,$$ (11.21)

$$\langle \psi^{ij,2} \eta^B_i \rangle = c' \Lambda^3 \delta^{ij,B-N}, \quad j = 1,2,\ldots,N, \quad B = N+1,\ldots,2N,$$

at some mass scale, $\Lambda$, the pion interpolating fields appearing in the WT identity must be gauge invariants such as (11.19), which are necessarily four-fermion composites. On the other hand, the “pion decay constant” is defined as usual,

$$\langle 0 | J^a_\mu | \pi^a \rangle = i q_\mu F_\pi,$$ (11.22)

as the amplitude with which the current operator produces the NG bosons from the vacuum. It is quite possible that the pion decay constant in chiral theories is such that

$$F_\pi \ll \Lambda,$$ (11.23)

as the bifermion current operator must produce pions, which are four-fermion composite particles, from the vacuum.\[5\]

\[5\] Naturally the same discussion holds for other $3N^2 + 32N + 1$ NG bosons, but the expressions would become more clumsy.

\[6\] Large $N$ scaling would ruin this hierarchy so $N$ must be kept finite.
Another way of seeing the same question is to think of the pion effective action,

\[ \mathcal{L}(\tilde{\phi}^a, \partial_\mu \tilde{\phi}^a) = \frac{1}{2} \partial_\mu \tilde{\phi}^a \partial^\mu \tilde{\phi}^a + \ldots, \tag{11.24} \]

in which the interaction strength among the pions is given by \( F_\pi \). The effective action involves eight-fermion, sixteen-fermion, etc. amplitudes, and the result such as (11.23) could well be realized by the complicated strong interaction dynamics.

### 12 Discussion

Let us recapitulate the class of \((N_\psi, N_\chi)\) models, analyzed here. The gauge group is taken to be \(SU(N)\). By \(N_\psi, N_\chi\) are indicated the numbers of Weyl fermions \(\psi\) or \(\chi\) in the representations

[Diagram of representations]

\[ \text{or} \quad \text{[Diagram of representations]} \tag{12.1} \]

Let us take \(N_\psi \geq 0\). In the case \(N_\chi < 0\), \(-N_\chi\) indicates the number of the fields \(\tilde{\chi}\) in the representation

[Diagram of representations]

\[ \text{instead. The number of the fermions in the antifundamental (or fundamental) representations } \eta^a \text{ (or } \tilde{\eta}^a) \text{ is fixed by the condition that the gauge group } SU(N) \text{ be anomaly free. Also we restrict the numbers } N_\psi, N_\chi \text{ such that the model is asymptotically free.} \]

The systems considered here are rather rigid. No fermion mass terms can be added in the Lagrangian and this also means that no gauge-invariant bifermion condensates can form. They cannot be deformed by addition of any other renormalizable potential terms either, including the topological \( \theta F_{\mu\nu} \tilde{F}^{\mu\nu} \) term. The presence of massless chiral matter fermions means all values of \(\theta\) are equivalent to \(\theta = 0\). The vacuum, apart from possible symmetry breaking degeneration, is expected to be unique. The system is strongly coupled in the infrared. Our ignorance about these simple models, after more than a half century of studies of quantum field theories, certainly is severely hindering our capability of finding any application of them in a physical theory describing Nature.

In the absence of other theoretical tools, we have insisted in this paper upon trying to find possible useful indications following the standard \’t Hooft anomaly matching constraints (for application of some new ideas such as the generalized symmetries and higher-form gauging to these chiral gauge theories, see [19]). The main lesson to be learned is perhaps the fact that color-flavor (or color-flavor-flavor) locking and dynamical Abelianization, in various combinations, always provides natural ways to solve these consistency constraints and to find possible phases of the system.

The strategy we used in paper, for all the models, is summarized as follows. First we
chose a set of bi-fermions operators that may condense. Since we do not have a gauge invariant bi-fermion in our theories, we chose among the gauge-non-invariant ones, possibly guided by the maximal attractive channel (MAC) criterion. Condensations has two important effects: it breaks part or all of the color symmetry and it breaks part or all of the flavor symmetry. The broken part of the gauge group is dynamically Higgsed. The unbroken part confines or remains in the IR if it is in the Coulomb phase (as for the dynamical Abelianization). We then have to look at the anomaly matching conditions. The part of the flavor symmetry that is broken by the condensate is saturated by massless NG boson poles. For the unbroken part instead, we need to find a set of fermions in the IR to match the computation in the UV. We then decompose the UV fermion into direct sum of representations of the unbroken flavor subgroup that remains unbroken. Unlike the UV representation, which is chiral, the IR decomposed representations have in general vectorial subsets. All the vectorial parts can be removed since they presumably get massive and in any case do not contribute to the ’t Hooft anomaly of the unbroken group. Other fermions remain in the IR as massless baryons and saturate the ’t Hooft anomalies.

The fact that in models $(1, 0)$ and $(0, 1)$ one can find a set of candidate massless fermions saturating the anomalies of the full unbroken flavor symmetries, seems to be fortuitous, rather than being a rule. In fact, no analogous set of candidate massless baryons can be found in other $(2, 0)$, $(3, 0)$, or $(0, 2)$, $(0, 3)$ models. On the other hand, the color-flavor breaking (dynamical Higgs) phase of the $(1, 0)$ and $(0, 1)$ models finds natural generalizations in these more complicated systems.

In this sense, our proposal shares a common feature with the tumbling scheme, but does not follow literally the MAC criterion with the multi-scale chains of dynamical gauge symmetry breaking, as in the original proposal [2]. There are a few cases, however, in which the appearance of hierarchy of mass scales, for reasons entirely different from that in the tumbling mechanism, is rather natural.

The local gauge symmetries can never be “truly” spontaneously broken, and any dynamical or elementary Higgs mechanism (including the case of the standard Higgs scalar in the Weinberg-Salam electroweak theory) must be re-interpreted in a gauge-invariant fashion. What happens in the chiral gauge theories considered here is that the system produces a bifermion composite states such as

\[ \psi(x) \eta(x), \quad \psi(x) \chi(x), \]

which then act as an effective Higgs scalar field. As these “dynamical” Higgs fields are still strongly coupled in general, the way their condensates and consequent flavor symmetry breaking is reinterpreted in a gauge invariant fashion may be more complicated than in

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7As explained by ’t Hooft, the Higgs VEV of the form \( \langle \phi \rangle = v \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) found in all textbooks on the electroweak theory, is just a gauge dependent way of describing the gauge-invariant VEV \( \langle \phi^\dagger \phi \rangle \), so is the statement such as the left hand fermion being equal to \( \psi_L = \left( \begin{array}{c} e_L \\ \nu_L \end{array} \right) \).
the standard electroweak theory where the Higgs scalars are weakly coupled and described by perturbation theory. The proposed dynamical Higgs mechanism does however have a definite statement about the flavor symmetry breaking: the latter is described by the condensate of the composite (dynamical) Higgs fields such as above, at the mass scale associated with them.

This brings us to a possibly relevant observation made in Section[11]. A study of chiral Ward-Takahashi identities shows that, in contrast to what happens in vector-like gauge theory such as QCD, the system might generate a hierarchy of mass scales, between the mass scale of the condensates of the composite Higgs fields (12.3), “Λ”, and the quantity corresponding to the pion decay constant, “$F_\pi$". The latter is the amplitude that the (broken) symmetry current produces a NG boson (“pion”) from the vacuum. The fact that in chiral gauge theories the current is a two-fermion operator, while the pions are in general four-fermion composites, in contrast to what happens in the case of axial symmetry breaking in vector-like theories, could imply a large hierarchy, (11.23). Such a possibility appears to be worth further studies, both from theoretical and phenomenological points of view.

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A  a theorem and the ACS criterion

For free theory of bosons and fermions, the $a$ and $c$ coefficients are given by

$$a = \frac{1}{360}(N_S + \frac{11}{2}N_f + 62N_V), \quad c = \frac{1}{120}(N_S + 6N_f + 12N_V), \quad (A.1)$$

where $N_S$ is the number of scalar particles, $N_f$ is the number of Weyl fermions, and $N_V$ is the number of vector bosons. The $a$-theorem tells

$$a_{IR} \leq a_{UV}. \quad (A.2)$$

On the other hand, the free-energy is

$$f = N_B + \frac{7}{4}N_f, \quad (A.3)$$

where $N_f$ is the number of the Weyl fermions and $N_B$ is the number of bosons. The ACS criterion is that \[17, 8\]

$$f_{IR} \leq f_{UV}. \quad (A.4)$$

For simplicity we shall use $\tilde{a} = 360a$. For $(N_\psi, N_\chi)$ model,

$$\tilde{a}_{UV} = 62(N^2 - 1) + \frac{33}{4}N_\psi N(3 + N) - \frac{11}{4}N_\chi N(N - 7), \quad (A.5)$$

$$\tilde{a}_{IR} = N_S + \frac{11}{2}N_f + 62N_V, \quad (A.6)$$

where $N_V, N_S, N_f$ are the number of vector bosons, scalars, and Weyl fermions in the infrared. For the ACS free energy,

$$f_{UV} = 2(N^2 - 1) + \frac{7N_\psi}{8}(N^2 + 3N + 8) + \frac{7N_\chi}{8}(N^2 - 3N + 8), \quad (A.7)$$

$$f_{IR} = N_B + \frac{7}{4}N_f. \quad (A.8)$$

We put those two criteria to the test in Tables [17] and [18] for the theories and their possible IR phases discussed in the paper. In all cases the $a$ theorem is satisfied, the ACS criterion fails only for the $(3, 0)$ and $(0, 3)$ models.
| Model             | $\tilde{a}_{UV}$       | $\tilde{a}_{IR}$       | Status |
|-------------------|------------------------|------------------------|--------|
| (1, 1) CFL ($N \geq 8$) | $\frac{135N^2}{2} + 44N - 62$ | $106N - 538$            | ✓      |
| (1, 1) CFL ($N \leq 8$) | $\frac{135N^2}{2} + 44N - 62$ | $-2N^2 + \frac{109N}{2} + 2$ | ✓      |
| (1, 1) Abelianiz. | $\frac{135N^2}{2} + 44N - 62$ | $\frac{223N}{2} - 61$          | ✓      |
| (1, 0) No XSB    | $\frac{281N^2+99N-248}{4}$ | $\frac{11N^2+77N+132}{4}$    | ✓      |
| (1, 0)           | $\frac{281N^2+99N-248}{4}$ | $\frac{11N^2+109N+4}{4}$     | ✓      |
| (2, 0) (symm)    | $\frac{1}{4}(157N^2 + 99N - 124)$ | $\frac{1}{2}(17N^2 + 141N + 2)$ | ✓      |
| (2, 0)           | $\frac{1}{4}(157N^2 + 99N - 124)$ | $\frac{1}{2}(17N^2 + 141N + 8)$ | ✓      |
| (3, 0)           | $\frac{347N^2}{8} + 297N - 62$ | $\frac{65N}{4} + \frac{19N}{4} + 1$ | ✓      |
| (0, 1)           | $\frac{281N^2 - 99}{4} - 62$ | $\frac{11}{2}(N - 3)(N - 4)$ | ✓      |
| (0, 2)           | $\frac{1}{4}(157N^2 - 297N - 248)$ | $\frac{1}{2}(17N^2 - 125N + 228)$ | ✓      |
| (0, 3)           | $\frac{1}{4}(347N^2 - 297N - 248)$ | $\frac{1}{2}(65N^2 - 487N + 876)$ | ✓      |
| (2, 1)           | $\frac{1}{4}(303N^2 + 275N - 248)$ | $\frac{1}{2}(33N^2 + 297N + 16)$ | ✓      |
| (1, -1)          | $\frac{157N^2}{8} - 62$ | $\frac{15N^2}{4} + 2$    | ✓      |

Table 17: The $a$ theorem.

| Model             | $f_{UV}$       | $f_{IR}$       | Status |
|-------------------|----------------|----------------|--------|
| (1, 1) CFL ($N \geq 8$) | $\frac{15N^2}{8} + 14N - 2$ | $4(4N - 7)$            | ✓      |
| (1, 1) CFL ($N \leq 8$) | $\frac{15N^2}{8} + 14N - 2$ | $2 + \frac{113N}{4} - 2N^2$ | ✓      |
| (1, 1) Abelianiz. | $\frac{15N^2}{8} + 14N - 2$ | $\frac{21N}{4} - 1$ | ✓      |
| (1, 0) No XSB    | $\frac{1}{8}(37N^2 + 63N - 16)$ | $\frac{7}{8}(N^2 + 7N + 12)$ | ✓      |
| (1, 0)           | $\frac{1}{8}(37N^2 + 63N - 16)$ | $\frac{1}{4}(7N^2 + 113N + 8)$ | ✓      |
| (2, 0) symm      | $\frac{1}{4}(29N^2 + 63N - 8)$ | $\frac{1}{4}(19N^2 + 177N + 4)$ | ✓      |
| (2, 0)           | $\frac{1}{4}(29N^2 + 63N - 8)$ | $\frac{1}{4}(19N^2 + 177N + 16)$ | ✓      |
| (3, 0)           | $\frac{79N^2}{8} - \frac{189N}{8} - 2$ | $\frac{85N^2}{8} + \frac{723N}{8} + 8$ | ✓      |
| (0, 1)           | $\frac{37N^2}{8} - \frac{63N}{8} - 2$ | $\frac{7}{8}(N - 3)(N - 4)$ | ✓      |
| (0, 2)           | $\frac{1}{8}(29N^2 - 63N - 8)$ | $\frac{1}{8}(19N^2 - 145N - 276)$ | ✓      |
| (0, 3)           | $\frac{1}{8}(79N^2 - 189N - 16)$ | $\frac{1}{8}(85N^2 - 659N + 1212)$ | ✓      |
| (2, 1)           | $\frac{1}{8}(51N^2 + 175N - 16)$ | $\frac{1}{8}(21N^2 + 189N + 32)$ | ✓      |
| (1, -1)          | $\frac{29N^2}{4} - 2$ | $\frac{15N^2}{4} + 2$ | ✓      |

Table 18: The ACS Criterion.