Hybrid ideals in semigroups

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Abstract: The notions of hybrid subsemigroups and hybrid left (resp., right) ideals in semigroups are introduced, and several properties are investigated. Using these notions, characterizations of subsemigroups and left (resp., right) ideals are discussed. The concept of hybrid product is also introduced, and characterizations of hybrid subsemigroups and hybrid left (resp., right) ideals are considered by using the notion of hybrid product. Relations between hybrid intersection and hybrid product are displayed.

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1. Introduction

The study of the fuzzy algebraic structures has started with the introduction of the concepts of fuzzy (subgroups) subgroups and fuzzy (left, right) ideals in the pioneering paper of Rosenfeld (1971). Since then, several authors applied fuzzy set theory to semigroups, and Mursaleen, Srivastava, and Sunil (2016) studied certain new spaces of statistically convergent and strongly summable sequences of fuzzy numbers. As a parallel circuit of fuzzy sets and soft sets (or, hesitant fuzzy set), Jun, Song, and Muhiuddin (in press) introduced the notion of hybrid structure in a set of parameters over an initial universe set, and applied it to BCK / BCI-algebras and linear spaces.

As a new mathematical tool for dealing with uncertainties, Molodtsov (1999) introduced the soft set theory. Torra introduced the concept of a hesitant fuzzy set (Torra, 2010; Torra & Narukawa, 2011). Using the notions and results in this paper, we will study the hybrid structures in related algebraic structures and decision making problems etc.
which is a generalization of Zadeh’s fuzzy set (Zadeh, 1965). The hesitant fuzzy set is very useful to express peoples hesitancy in daily life, and it is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers.

In this paper, we apply the notion of hybrid structure to semigroups. We introduce the notions of hybrid subsemigroups and hybrid left (resp., right) ideals in semigroups, and investigate several properties. Using these notions, we consider characterizations of subsemigroups and left (resp., right) ideals. We also introduce the concept of hybrid product, and discuss characterizations of hybrid subsemigroups and hybrid left (resp., right) ideals by using the notion of hybrid product. We provide relations between hybrid intersection and hybrid product.

2. Preliminaries

2.1. Fundamentals on semigroups

Let $L$ be a semigroup. Let $A$ and $B$ be subsets of $L$. Then the multiplication of $A$ and $B$ is defined as follows:

$AB = \{ ab \in L \mid a \in A \text{ and } b \in B \}.$

A semigroup $L$ is said to be regular if for every $x \in L$ there exists $a \in L$ such that $xax = x$.

A nonempty subset $A$ of $L$ is called

- a subsemigroup of $L$ if $AA \subseteq A$, i.e. $ab \in A$ for all $a, b \in A$,
- a left (resp., right) ideal of $L$ if $LA \subseteq A$ (resp. $AL \subseteq A$), i.e. $xa \in A$ (resp. $ax \in A$) for all $x \in L$ and $a \in A$.
- a two-sided ideal of $L$ if it is both a left and a right ideal of $L$.

2.2. Fundamentals on hybrid structures

In what follows, let $I$ be the unit interval, $L$ a set of parameters and $\mathcal{P}(U)$ denote the power set of an initial universe set $U$.

**Definition 2.2** (Jun et al., in press) A hybrid structure in $L$ over $U$ is defined to be a mapping $\bar{f}_L : (\bar{f}, \lambda)_L : \mathcal{P}(U) \times I \rightarrow (\mathfrak{f}(x), i(x))$ where $\bar{f}_L : \mathcal{P}(U)$ and $\lambda : L \rightarrow I$ are mappings.

Let us denote by $H(L)$ the set of all hybrid structures in $L$ over $U$. We define an order $\ll$ in $H(L)$ as follows:

$$\left( \forall \bar{f}_L, \bar{g}_L \in H(L) \right) \left( \bar{f}_L \ll \bar{g}_L \iff \bar{f}_L \subseteq \bar{g}_L, \lambda \geq \gamma \right) \tag{2.1}$$

where $\bar{f}_L \subseteq \bar{g}_L$ means that $f(x) \subseteq g(x)$ and $\lambda \geq \gamma$ means that $\lambda(x) \geq \gamma(x)$ for all $x \in L$. Note that $(H(L), \ll)$ is a poset.

**Definition 2.2** (Jun et al., in press) Let $\bar{f}_L$ be a hybrid structure in $L$ over $U$. Then the sets

\begin{align*}
\bar{f}_L(a, t] &:= \{ x \in X \mid f(x) \geq a, i(x) \leq t \}, \\
\bar{f}_L(a, t) &:= \{ x \in X \mid f(x) \geq a, i(x) = t \}, \\
\bar{f}_L(a, t] &:= \{ x \in X \mid f(x) = a, i(x) \leq t \}, \\
\bar{f}_L(a, t) &:= \{ x \in X \mid f(x) = a, i(x) = t \}, \\
\bar{f}_L(a, t) &:= \{ x \in X \mid f(x) \geq a, i(x) < t \}, \\
\bar{f}_L(a, t) &:= \{ x \in X \mid f(x) = a, i(x) < t \}
\end{align*}
are called the \{a, t\}-hybrid cut, \{a, t\}-hybrid cut, and \{(a, t)\}-hybrid cut of \(\tilde{f}_\nu\), respectively, where \(a \in \mathcal{P}(U)\) and \(t \in I\). Obviously,
\[
\tilde{f}_\nu(a, t) \subseteq \tilde{f}_\nu(a) \subseteq \tilde{f}_\nu[a, t] \quad \text{and} \quad \tilde{f}_\nu(a, t) \subseteq \tilde{f}_\nu[a, t] \subseteq \tilde{f}_\nu(a, t).
\]

**Definition 2.3** (Jun et al., in press) Let \(\tilde{f}_\nu\) and \(\tilde{g}_\gamma\) be hybrid structures in \(L\) over \(U\). Then the hybrid intersection of \(\tilde{f}_\nu\) and \(\tilde{g}_\gamma\) is denoted by \(\tilde{f}_\nu \cap \tilde{g}_\gamma\) and is defined to be a hybrid structure
\[
\tilde{f}_\nu \cap \tilde{g}_\gamma : L \to \mathcal{P}(U) \times I, \quad x \mapsto (\tilde{f}_\nu \cap \tilde{g}_\gamma)(x),
\]
where
\[
\tilde{f}_\nu \cap \tilde{g}_\gamma : L \to \mathcal{P}(U), \quad x \mapsto \tilde{f}_\nu(x) \cap \tilde{g}_\gamma(x),
\]
\[
\lambda \lor \gamma : L \to I, \quad x \mapsto \bigvee \{\lambda(x), \gamma(x)\}.
\]

**3. Hybrid subsemigroups and ideals**

**Definition 3.1** Let \(L\) be a semigroup. A hybrid structure \(\tilde{f}_\nu\) in \(L\) over \(U\) is called a hybrid subsemigroup of \(L\) over \(U\) if the following assertions are valid:
\[
(\forall x, y \in X) \quad \left( \tilde{f}(xy) \supseteq \tilde{f}(x) \cap \tilde{f}(y), \quad \lambda(xy) \leq \bigvee \{\lambda(x), \lambda(y)\} \right).
\]

**Example 3.2** Let \(L = \{0, 1, 2, 3, 4, 5\}\) be a semigroup with the following Cayley table:

\[
\begin{array}{cccccc}
\cdot & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 \\
2 & 0 & 1 & 2 & 3 & 1 & 1 \\
3 & 0 & 1 & 1 & 1 & 2 & 3 \\
4 & 0 & 1 & 4 & 5 & 1 & 1 \\
5 & 0 & 1 & 1 & 1 & 4 & 5 \\
\end{array}
\]

Let \(\tilde{f}_\nu\) be a hybrid structure in \(L\) over \(U = \mathbb{Z}\) which is given by Table 1.

| \(L\) | \(\tilde{f}_\nu\) | \(\lambda\) |
|------|----------------|--------|
| 0    | \(\mathbb{Z}\) | 0.2    |
| 1    | \(2\mathbb{Z}\) | 0.5    |
| 2    | \(8\mathbb{N}\) | 0.9    |
| 3    | \(4\mathbb{N}\) | 0.7    |
| 4    | \(8\mathbb{N}\) | 0.9    |
| 5    | \(4\mathbb{Z}\) | 0.6    |
It is easy to verify that \( \tilde{f} \) is a hybrid subsemigroup of \( L \) over \( U = \mathbb{Z} \).

Also the hybrid structure \( \tilde{g} \) in \( L \) over \( U = \mathbb{Z} \) which is given by Table 2 is a hybrid subsemigroup of \( L \) over \( U = \mathbb{Z} \).

**Definition 3.3** Let \( L \) be a semigroup. A hybrid structure \( \tilde{f} \) in \( L \) over \( U \) is called a hybrid left (resp., right) ideal of \( L \) over \( U \) if the following assertions are valid:

\[
(\forall x, y \in X) \left( \tilde{f}(xy) \supseteq \tilde{f}(y) \ (\text{resp.,} \ \tilde{f}(xy) \supseteq \tilde{f}(x)) \right).
\]

If a hybrid structure \( \tilde{f} \) in \( L \) over \( U \) is both a hybrid left ideal and a hybrid right ideal of \( L \) over \( U \), we say that \( \tilde{f} \) is a hybrid two-sided ideal of \( L \) over \( U \).

**Example 3.4** Let \( L = \{a, b, c, d\} \) be a semigroup with the following Cayley table:

|       | a  | b  | c  | d  |
|-------|----|----|----|----|
| a     | a  | a  | a  | a  |
| b     | a  | a  | a  | a  |
| c     | a  | a  | b  | a  |
| d     | a  | a  | b  | b  |

Then the hybrid structure \( \tilde{f} \) in \( L \) over an initial universe set \( U = \{u_1, u_2, u_3, u_4, u_5\} \) which is given by Table 3 is a hybrid two-sided ideal of \( L \) over \( U \).

**Table 2. Tabular representation of the hybrid structure \( \tilde{g} \)***

| \( L \) | \( \tilde{g} \) | \( \gamma \) |
|--------|----------------|-------------|
| 0      | \( \mathbb{Z} \) | 0.2         |
| 1      | \( \mathbb{Z} \) | 0.2         |
| 2      | 4Z             | 0.6         |
| 3      | 2Z             | 0.4         |
| 4      | 4N             | 0.8         |
| 5      | 4N             | 0.8         |

**Table 3. Tabular representation of the hybrid structure \( \tilde{f} \)***

| \( L \) | \( \tilde{f} \) | \( \lambda \) |
|--------|----------------|-------------|
| a      | \{u_1, u_2, u_3, u_4\} | 0.2         |
| b      | \{u_2, u_3, u_4\}         | 0.5         |
| c      | \{u_1\}                    | 0.9         |
| d      | \{u_2, u_3\}               | 0.7         |
Obviously, every hybrid left (resp., right) ideal is a hybrid subsemigroup, but the converse is not true in general. In fact, the hybrid subsemigroup $\tilde{f}_1$ in Example 3.2 is not a hybrid left ideal of $L$ over $U = Z$ since $\tilde{f}(3 \cdot 5) = \tilde{f}(3) = 4N4Z = \tilde{f}(5)$ and/or $\lambda(3 \cdot 5) = \lambda(3) = 0.7 \neq 0.6 = \lambda(5).$ Also the hybrid subsemigroup $g_f$ in Example 3.2 is not a hybrid right ideal of $L$ over $U = Z$ since $\tilde{g}(3 \cdot 4) = \tilde{g}(2) = 4N2Z = \tilde{g}(3)$ and/or $\gamma(3 \cdot 4) = \gamma(2) = 0.6 \neq 0.4 = \gamma(3).$

For a nonempty subset $A$ of $L$ and $\varepsilon, \delta \in \mathcal{P}(U)$ with $\varepsilon \supseteq \delta,$ and $s, t \in [0, 1]$ with $t < s,$ consider a hybrid structure $X^{\varepsilon, \delta}_{A, t, s}(f) = \left( X^{\varepsilon, \delta}_{A, t}, X^{t, s}_{A, \delta} \right),$ where

$$X^{\varepsilon, \delta}_{A, t}: L \to \mathcal{P}(U), \ x \mapsto \begin{cases} \varepsilon & \text{if } x \in A, \\ \delta & \text{otherwise,} \end{cases}$$

and

$$X^{t, s}_{A, \delta}: L \to I, \ x \mapsto \begin{cases} t & \text{if } x \in A, \\ s & \text{otherwise,} \end{cases}$$

which is called the $(\varepsilon, \delta)$-characteristic hybrid structure in $L$ over $U.$ The hybrid structure $X^{\varepsilon, \delta}_{A, t, s}(f) = \left( X^{\varepsilon, \delta}_{A, t}, X^{t, s}_{A, \delta} \right)$ is called the $(\varepsilon, \delta)$-identity hybrid structure in $L$ over $U.$ The $(\varepsilon, \delta)$-characteristic (resp., identity) hybrid structure in $L$ over $U$ with $\varepsilon = U, \delta = \emptyset, t = 0$ and $s = 1$ is called the characteristic (resp., identity) hybrid structure in $L$ over $U,$ and is denoted by $X^{\varepsilon, \delta}_{A, t, s}(f) = \left( X^U_A, X^\emptyset_A \right)$ (resp., $X^{t, s}_{A, \delta}(f) = \left( X^0_A, X^1_A \right)$).

**Theorem 3.5** For any nonempty subset $A$ of a semigroup $L,$ the following are equivalent:

(i) $A$ is a left (resp., right) ideal of $L.$

(ii) The characteristic hybrid structure $X^{\varepsilon, \delta}_{A, t, s}(f)$ in $L$ over $U$ is a hybrid left (resp., right) ideal of $L$ over $U.$

**Proof** Assume that $A$ is a left ideal of $L.$ For any $x, y \in L,$ if $y \not\in A$ then $X^{\varepsilon, \delta}_{A, t}(xy) \supseteq \emptyset = X^{\varepsilon, \delta}_{A, t}(y)$ and $X^{\varepsilon, \delta}_{A, t}(xy) \subseteq 1 = X^{\varepsilon, \delta}_{A, t}(\lambda(y)).$ If $y \in A,$ then $x \in A$ and so $X^{\varepsilon, \delta}_{A, t}(xy) = U = X^{\varepsilon, \delta}_{A, t}(y)$ and $X^{\varepsilon, \delta}_{A, t}(xy) = 0 = X^{\varepsilon, \delta}_{A, t}(\lambda(y)).$ Therefore $X^{\varepsilon, \delta}_{A, t, s}(f)$ is a hybrid left ideal of $L$ over $U.$ Similarly, $X^{\varepsilon, \delta}_{A, t, s}(f)$ is a hybrid right ideal of $L$ over $U$ when $A$ is a right ideal of $L.$

Conversely, suppose that $X^{\varepsilon, \delta}_{A, t, s}(f)$ is a hybrid left ideal of $L$ over $U.$ Let $x \in L$ and $y \in A.$ Then $X^{\varepsilon, \delta}_{A, t, s}(f)(y) = U$ and $X^{\varepsilon, \delta}_{A, t, s}(f)(\lambda(y)) = 0,$ and so $X^{\varepsilon, \delta}_{A, t, s}(f)(xy) \supseteq X^{\varepsilon, \delta}_{A, t, s}(f)(y)$ and $X^{\varepsilon, \delta}_{A, t, s}(f)(xy) \subseteq 0 = X^{\varepsilon, \delta}_{A, t, s}(f)(\lambda(y)).$ Hence $xy \in A$ and therefore $A$ is a left ideal of $L.$ Similarly, we can show that if $X^{\varepsilon, \delta}_{A, t, s}(f)$ is a hybrid right ideal of $L$ over $U,$ then $A$ is a right ideal of $L.$ \hfill $\square$

**Corollary 3.6** For any nonempty subset $A$ of a semigroup $L,$ the following are equivalent:

(i) $A$ is a two-sided ideal of $L.$

(ii) The characteristic hybrid structure $X^{\varepsilon, \delta}_{A, t, s}(f)$ in $L$ over $U$ is a hybrid two-sided ideal of $L$ over $U.$

**Theorem 3.7** A hybrid structure $\tilde{f}_1$ in $L$ over $U$ is a hybrid subsemigroup of $L$ over $U$ if and only if the nonempty sets

$$L^1_e = \{ x \in L \mid \tilde{f}(x) \supseteq \varepsilon \} \quad \text{and} \quad L^1_{\lambda} = \{ x \in L \mid \lambda(x) \leq t \}$$

are subsemigroups of $L$ for all $(\varepsilon, t) \in \mathcal{P}(U) \times I.$

**Proof** Suppose that a hybrid structure $\tilde{f}_1$ in $L$ over $U$ is a hybrid subsemigroup of $L$ over $U.$ Assume that $L^1_e \neq \emptyset \neq L^1_{\lambda}$ for all $(\varepsilon, t) \in \mathcal{P}(U) \times I.$ Let $x, y \in L^1_e \cap L^1_{\lambda}.$ Then $\tilde{f}(x) \supseteq \varepsilon, \tilde{f}(y) \supseteq \varepsilon, \lambda(x) \leq t$ and $\lambda(y) \leq t.$ It follows from (3.1) that
\[ \hat{f}(xy) \supseteq \hat{f}(x) \cap \hat{f}(y) \supseteq \epsilon, \quad \lambda(xy) \leq \bigvee (\lambda(x), \lambda(y)) \leq t. \]  

(3.3)

Hence \( xy \in L'_1 \cap L'_2 \) and so \( L'_1 \) and \( L'_2 \) are subsemigroups of \( L \).

Conversely, assume that the nonempty sets \( L'_1 \) and \( L'_2 \) are subsemigroups of \( L \) for all \( (\epsilon, t) \in \mathcal{P}(U) \times I \)

For any \( x, y \in L \), let \( \hat{f}(x) = \epsilon_x \) and \( \hat{f}(y) = \epsilon_y \). If we put \( \epsilon := \epsilon_x \cap \epsilon_y \), then \( x, y \in L'_1 \) and so

\[ \hat{f}(xy) \supseteq \epsilon = \epsilon_x \cap \epsilon_y = \hat{f}(x) \cap \hat{f}(y). \]

Now, for any \( a, b \in L \), let \( \lambda(a) = t_a \) and \( \lambda(b) = t_b \). Taking \( t := \bigvee (t_a, t_b) \) implies that \( a, b \in L'_1 \). Thus \( ab \in L'_1 \), which implies that \( \lambda(ab) \leq t = \bigvee (t_a, t_b) = \bigvee (\lambda(a), \lambda(b)) \). Therefore \( \hat{f}_1 \) is a hybrid subsemigroup of \( L \) over \( U \).

\[ \square \]

Note that \( f_1[x, t] = L'_1 \cap L'_2 \) for all \( (\epsilon, t) \in \mathcal{P}(U) \times I \). Hence we have the following corollary.

**Corollary 3.8** If a hybrid structure \( \hat{f}_1 \) in \( L \) over \( U \) is a hybrid subsemigroup of \( L \) over \( U \), then the nonempty \( (\epsilon, t) \)-hybrid cut \( f_1[x, t] \) is a subsemigroup of \( L \) for all \( (\epsilon, t) \in \mathcal{P}(U) \times I \).

**Theorem 3.9** A hybrid structure \( \hat{f}_1 \) in \( L \) over \( U \) is a hybrid left (resp., right) ideal of \( L \) over \( U \) if and only if the nonempty sets

\[ L'_1 := \{ x \in L \mid \hat{f}(x) \supseteq \epsilon \} \quad \text{and} \quad L'_2 := \{ x \in L \mid \lambda(x) \leq t \} \]

are left (resp., right) ideals of \( L \) for all \( (\epsilon, t) \in \mathcal{P}(U) \times I \).

**Proof** It is the same as the proof of Theorem 3.7.

\[ \square \]

For any hybrid structures \( \hat{f}_1 \) and \( \hat{g}_1 \) in \( L \) over \( U \), the hybrid product of \( \hat{f}_1 \) and \( \hat{g}_1 \) is defined to be a hybrid structure \( \overline{\lambda \circ f} \) in \( L \) over \( U \) where

\[ (\overline{\lambda \circ f})(x) = \begin{cases} \bigcup_{x=yz} \{ f(y) \cap g(z) \} & \text{if } \exists y, z \in L \text{ such that } x = yz \\ \emptyset & \text{otherwise} \end{cases} \]

and

\[ (\overline{\lambda \circ g})(x) = \begin{cases} \bigwedge_{x=yz} \bigvee (\lambda(y), \lambda(z)) & \text{if } \exists y, z \in L \text{ such that } x = yz \\ 1 & \text{otherwise} \end{cases} \]

for all \( x \in L \).

**Proposition 3.10** Let \( \hat{f}_1 \) and \( \hat{g}_1 \) be hybrid structures in \( L \) over \( U \). If \( \hat{f}_1 \ll \hat{g}_1 \) and \( \hat{g}_1 \ll \hat{g}_2 \), then \( \hat{f}_1 \circ \hat{g}_1 \ll \hat{g}_1 \circ \hat{g}_2 \).

**Proof** Let \( x \in L \). If \( x \) is not expressed as \( x = yz \) for \( y, z \in L \), then clearly

\[ \hat{f}_1 \circ \hat{g}_1 \ll \hat{g}_1 \circ \hat{g}_2. \]

Assume that \( x = yz \) for some \( y, z \in L \). Then

\[ (\overline{\lambda \circ f})(x) = \bigcup_{x=yz} \{ f_1(y) \cap f_2(z) \} \]

\[ \subseteq \bigcup_{x=yz} \{ g_1(y) \cap g_2(z) \} = (\overline{\lambda \circ g})(x), \]

\[ (\overline{\lambda \circ g})(x) = \bigwedge_{x=yz} \bigvee (\overline{\lambda_1(y)}, \overline{\lambda_2(z)}) \]

\[ \geq \bigwedge_{x=yz} \bigvee (\overline{\lambda_1(y)}, \overline{\lambda_2(z)}) = (\overline{\lambda \circ g})(x). \]  

(3.4)
Therefore $\tilde{f}_1 \circ \tilde{f}_2 \prec \tilde{g}_1 \circ \tilde{g}_2$.

\[\Box\]

**Lemma 3.11** For subsets $A$ and $B$ of $L$, let $X_A(\tilde{f}_i)$ and $X_B(\tilde{f}_j)$ be characteristic hybrid structures in $L$ over $U$. Then

(i) $X_A(\tilde{f}_i) \cup X_B(\tilde{f}_j) = X_{A \cap B}(\tilde{f}_j)$.

(ii) $X_A(\tilde{f}_i) \cap X_B(\tilde{f}_j) = X_{A \cap B}(\tilde{f}_j)$.

**Proof**

(i) Let $x \in L$. If $x \in A \cap B$, then

$$
(X_A(\tilde{f}_i) \cap X_B(\tilde{f}_j))(x) = X_A(\tilde{f}_i)(x) \cap X_B(\tilde{f}_j)(x) = U = X_{A \cap B}(\tilde{f}_j)(x),
$$

$$
(X_A(\tilde{f}_i) \cup X_B(\tilde{f}_j))(x) = \bigvee \{ X_A(\tilde{f}_i)(x), X_B(\tilde{f}_j)(x) \} = 0 = X_{A \cap B}(\tilde{f}_j)(x).
$$

If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$. Hence

$$
(X_A(\tilde{f}_i) \cap X_B(\tilde{f}_j))(x) = X_A(\tilde{f}_i)(x) \cap X_B(\tilde{f}_j)(x) = \emptyset = X_{A \cap B}(\tilde{f}_j)(x),
$$

$$
(X_A(\tilde{f}_i) \cup X_B(\tilde{f}_j))(x) = \bigvee \{ X_A(\tilde{f}_i)(x), X_B(\tilde{f}_j)(x) \} = 1 = X_{A \cap B}(\tilde{f}_j)(x).
$$

It follows that

$$
(X_A(\tilde{f}_i) \cap X_B(\tilde{f}_j))(x) = (X_A(\tilde{f}_i) \cap X_B(\tilde{f}_j))(x) = (X_A(\tilde{f}_i) \cup X_B(\tilde{f}_j))(x)
$$

$$
= (X_{A \cap B}(\tilde{f}_j)(x), X_{A \cap B}(\tilde{f}_j)(x))
$$

$$
= X_{A \cap B}(\tilde{f}_j)(x).
$$

Hence

(i) is valid.

(ii) For any $x \in L$, if $x \in AB$ then $x = ab$ for some $a \in A$ and $b \in B$. Thus

$$
(X_A(\tilde{f}) \circ X_B(\tilde{f}))(x) = \bigcup_{x=yz} \{ X_A(\tilde{f})(y) \cap X_B(\tilde{f})(z) \}
$$

$$
\supseteq X_A(\tilde{f})(a) \cap X_B(\tilde{f})(b) = U,
$$

and

$$
(X_A(\tilde{f}) \circ X_B(\tilde{f}))(x) = \bigvee_{x=yz} \{ X_A(\tilde{f})(y), X_B(\tilde{f})(z) \}
$$

$$
\supseteq X_A(\tilde{f})(a) \cap X_B(\tilde{f})(b) = U,
$$

and so $(X_A(\tilde{f}) \circ X_B(\tilde{f}))(x) = U = X_{A \cap B}(\tilde{f})(x)$ and $(X_A(\tilde{f}) \circ X_B(\tilde{f}))(x) = 0 = X_{A \cap B}(\tilde{f})(x)$. Suppose $x \notin AB$. Then $x \neq ab$ for all $a \in A$ and $b \in B$. If $x = yz$ for some $y, z \in L$, then $y \not\in A$ or $z \not\in B$. Thus

$$
(X_A(\tilde{f}) \circ X_B(\tilde{f}))(x) = \bigcup_{x=yz} \{ X_A(\tilde{f})(y) \cap X_B(\tilde{f})(z) \} = \emptyset = X_{A \cap B}(\tilde{f})(x),
$$

$$
(X_A(\tilde{f}) \circ X_B(\tilde{f}))(x) = \bigvee_{x=yz} \{ X_A(\tilde{f})(y), X_B(\tilde{f})(z) \} = 0 = X_{A \cap B}(\tilde{f})(x).
$$

If $x \neq yz$ for all $y, z \in L$, then

$$
(X_A(\tilde{f}) \circ X_B(\tilde{f}))(x) = \emptyset = X_{A \cap B}(\tilde{f})(x),
$$

$$
(X_A(\tilde{f}) \circ X_B(\tilde{f}))(x) = 1 = X_{A \cap B}(\tilde{f})(x).
$$
In any case, we have \((X_A(f) \circ X_B(f))(x) = X_{AB}(f)(x)\) and \((X_A(\lambda) \circ X_B(\lambda))(x) = X_{AB}(\lambda)(x)\) for all \(x \in L\). Therefore

\[
X_A(f) \circ X_B(f) = (X_A(f) \circ X_B(f), X_A(\lambda) \circ X_B(\lambda)) = (X_{AB}(f), X_{AB}(\lambda)) = X_{AB}(f).
\]

This completes the proof. \(\square\)

**Theorem 3.12** A hybrid structure \(\tilde{f}_s\) in a semigroup \(L\) over \(U\) is a hybrid subsemigroup of \(L\) over \(U\) if and only if \(\tilde{f}_s \circ \tilde{f}_s \preccurlyeq \tilde{f}_s\).

**Proof** Assume that \(\tilde{f}_s\) is a hybrid subsemigroup of \(L\) over \(U\). Then

\[
\tilde{f}(x) \supseteq \tilde{f}(y) \cap \tilde{f}(z) \quad \text{and} \quad \lambda(x) \leq \bigvee \{\lambda(y), \lambda(z)\}
\]

for all \(x \in L\) with \(x = yz\). Hence

\[
\tilde{f}(x) \supseteq \bigcup_{x = yz} (\tilde{f}(y) \cap \tilde{f}(z)) = (\tilde{f} \circ \tilde{f})(x),
\]

\[
\lambda(x) \leq \bigwedge_{x = yz} \bigvee \{\lambda(y), \lambda(z)\} = (\lambda \circ \lambda)(x)
\]

for all \(x \in L\). Thus \(\tilde{f} \circ \tilde{f} \subseteq \tilde{f}\) and \(\lambda \circ \lambda \geq \lambda\), and so \(\tilde{f}_s \circ \tilde{f}_s \preccurlyeq \tilde{f}_s\).

Conversely suppose that \(\tilde{f}_s \circ \tilde{f}_s \preccurlyeq \tilde{f}_s\) and let \(x, y \in L\). Then

\[
\tilde{f}(xy) \supseteq (\tilde{f} \circ \tilde{f})(xy) \supseteq \tilde{f}(x) \cap \tilde{f}(y),
\]

\[
\lambda(xy) \leq (\lambda \circ \lambda)(xy) \leq \bigvee \{\lambda(x), \lambda(y)\},
\]

and so \(\tilde{f}_s\) is a hybrid subsemigroup of \(L\) over \(U\). \(\square\)

**Theorem 3.13** For a hybrid structure \(\tilde{g}_s\) and the identity hybrid structure \(X_f\) in a semigroup \(L\) over \(U\), the following assertions are equivalent.

(i) \(\tilde{g}_s\) is a hybrid left ideal of \(L\) over \(U\).

(ii) \(X_f \circ \tilde{g}_s \preccurlyeq \tilde{g}_s\).

**Proof** Assume that \(\tilde{g}_s\) is a hybrid left ideal of \(L\) over \(U\). Let \(x \in L\). If \(x = yz\) for some \(y, z \in L\), then

\[
(x \circ y)(x) = \bigcup_{x = yz} \{x(y) \cap y(z)\} \subseteq \bigcup_{x = yz} (U \cap yz) = \tilde{g}(x),
\]

\[
(x \circ y)(x) = \bigwedge_{x = yz} \bigvee \{x(y), y(z)\} \geq \bigwedge_{x = yz} \bigvee \{0, y(z)\} = \gamma(x).
\]

Otherwise, we have \(\left(x \circ y\right)(x) = \emptyset \subseteq \tilde{g}(x)\) and \(\left(x \circ y\right)(x) = \emptyset \geq \gamma(x)\). Hence \(x \circ y \preccurlyeq \tilde{g}\) and \(x \circ y \preccurlyeq \gamma\). Therefore \(X_f \circ \tilde{g}_s \preccurlyeq \tilde{g}_s\).

Conversely, suppose \(X_f \circ \tilde{g}_s \preccurlyeq \tilde{g}_s\). For any \(x, y \in L\), we have

\[
\tilde{g}(xy) \supseteq \left(x \circ y\right)(xy) \supseteq x_f(x) \cap \tilde{g}(y) = U \cap y = \tilde{g}(y),
\]

\[
\gamma(xy) \leq \left(x \circ y\right)(xy) \leq \bigvee \{x_f(x), \gamma(y)\} = \bigvee \{0, \gamma(y)\} = \gamma(y).
\]

Consequently, \(\tilde{g}_s\) is a hybrid left ideal of \(L\) over \(U\). \(\square\)

Similarly, we have the following theorem.
**Theorem 3.14**  For a hybrid structure $\hat{g}$, and the identity hybrid structure $\chi_L$ in a semigroup $L$ over $U$, the following assertions are equivalent.

(i) $\hat{g}$ is a hybrid right ideal of $L$ over $U$.

(ii) $\hat{g} \odot \chi_L \preceq \hat{g}$.

**Theorem 3.15**  If $\hat{f}$ and $\hat{g}$ are hybrid subsemigroups of a semigroup $L$ over $U$, then so is the hybrid intersection $\hat{f} \cap \hat{g}$.

**Proof**  For any $x, y \in L$, we have

\[(\hat{f} \cap \hat{g})(xy) = \hat{f}(xy) \cap \hat{g}(xy) \]

\[\supseteq (\hat{f}(x) \cap \hat{g}(y)) \cap (\hat{g}(x) \cap \hat{g}(y)) \]

\[= (\hat{f}(x) \cap \hat{g}(x)) \cap (\hat{f}(y) \cap \hat{g}(y)) \]

\[= (\hat{f} \cap \hat{g})(x) \cap (\hat{f} \cap \hat{g})(y)\]

and

\[(\lambda \lor \gamma)(xy) = \bigvee (\lambda(xy), \gamma(xy)) \]

\[\leq \bigvee \{ \bigvee (\lambda(x), \lambda(y)), \bigvee (\gamma(x), \gamma(y)) \} \]

\[= \bigvee \{ \bigvee (\lambda(x), \gamma(x)), \bigvee (\lambda(y), \gamma(y)) \} \]

\[= \bigvee (\lambda \lor \gamma)(x), (\lambda \lor \gamma)(y).\]

Hence $\hat{f} \cap \hat{g}$ is a hybrid subsemigroup of a semigroup $L$ over $U$.

By the similar way, we can prove the following theorem.

**Theorem 3.16**  If $\hat{f}$ and $\hat{g}$ are hybrid left (resp., right) ideals of a semigroup $L$ over $U$, then so is the hybrid intersection $\hat{f} \cap \hat{g}$.

**Theorem 3.17**  Let $\hat{f}$ and $\hat{g}$ be hybrid structures in a semigroup $L$ over $U$. If $\hat{f}$ is a hybrid left ideal of $L$ over $U$, then so is the hybrid product $\hat{f} \odot \hat{g}$.

**Proof**  Assume that $\hat{f}$ is a hybrid left ideal of $L$ over $U$ and let $x, y \in L$. If there exist $a, b \in L$ such that $y = ab$, then $xy = x(ab) = (xa)b$. Then

\[(\hat{f} \odot \hat{g})(y) = \bigcup_{y = ab} (\hat{f}(a) \cap \hat{g}(b)) \subseteq \bigcup_{xy = xab} (\hat{f}(a \odot b) \cap \hat{g}(b)) \]

\[\subseteq \bigcup_{xy = xab} (\hat{f}(x) \cap \hat{g}(b)) = (\hat{f} \odot \hat{g})(xy),\]

\[(\lambda \odot \gamma)(y) = \bigwedge_{y = ab} \bigvee (\lambda(a), \gamma(b)) \geq \bigwedge_{xy = xab} \bigvee (\lambda(a \odot b), \gamma(b)) \]

\[\geq \bigwedge_{xy = xab} \bigvee (\lambda(c), \gamma(b)) = (\lambda \odot \gamma)(xy).\]

If $y$ is not expressible as $y = ab$ for $a, b \in L$, then $(\hat{f} \odot \hat{g})(y) = \emptyset \subseteq (\hat{f} \odot \hat{g})(xy)$ and $(\lambda \odot \gamma)(y) = 1 \geq (\lambda \odot \gamma)(xy)$. Hence

\[(\hat{f} \odot \hat{g})(y) \subseteq (\hat{f} \odot \hat{g})(xy)\]

and

\[(\lambda \odot \gamma)(y) \geq (\lambda \odot \gamma)(xy)\]

for all $x, y \in L$. Therefore $\hat{f} \odot \hat{g}$ is a hybrid left ideal of $L$ over $U$.

Similarly, we have the following theorem.
THEOREM 3.18 Let $\hat{f}_s$ and $\hat{g}_r$ be hybrid structures in a semigroup $L$ over $U$. If $\hat{g}_r$ is a hybrid right ideal of $L$ over $U$, then so is the hybrid product $\hat{f}_s \circ \hat{g}_r$.

Given a hybrid structure $\hat{f}_s$ in $L$ over $U$, define a new hybrid structure

$$\hat{f}_s^* : = (\hat{f}^*, \lambda^*): L \rightarrow \mathcal{P}(U) \times I, \ x \mapsto (\hat{f}^*(x), \lambda^*(x))$$

in which $\hat{f}^*$ and $\lambda^*$ are given as follows:

$$\hat{f}^*: L \rightarrow \mathcal{P}(U), \ x \mapsto \begin{cases} \hat{f}(x) & \text{if } x \in L^1_s, \\ \delta & \text{otherwise}, \end{cases}$$

$$\lambda^*: L \rightarrow I, \ x \mapsto \begin{cases} \lambda(x) & \text{if } x \in L^1_s, \\ s & \text{otherwise} \end{cases}$$

where $(\epsilon, t), (\delta, s) \in \mathcal{P}(U) \times I$ with $L^1_s \neq \emptyset \neq L^1_t, \delta \subseteq \hat{f}(x)$ and $s \geq \lambda(x)$.

THEOREM 3.19 If $\hat{f}_s$ is a hybrid subsemigroup of a semigroup $L$ over $U$, the so is $\hat{f}_s^*$.

Proof Let $x, y \in L$. If $x, y \in L^1_s \cap L^1_t$, then $xy \in L^1_s \cap L^1_t$ since $L^1_s$ and $L^1_t$ are subsemigroups of $L$ by Theorem 3.7. Thus

$$\hat{f}(xy) = \hat{f}(xy) \supseteq \hat{f}(x) \cap \hat{f}(y) = \hat{f}^*(x) \cap \hat{f}^*(y),$$

$$\lambda(xy) = \lambda(xy) \leq \bigvee \{\lambda(x), \lambda(y)\} = \bigvee \{\lambda^*(x), \lambda^*(y)\}.$$  

If $x \notin L^1_s$ or $y \notin L^1_t$, then $\hat{f}(x) = \delta$ or $\hat{f}(y) = \delta$. Hence $\hat{f}(xy) \supseteq \delta = \hat{f}^*(x) \cap \hat{f}^*(y)$. If $x \notin L^1_s$ or $y \notin L^1_t$, then $\lambda(xy) = s$ or $\lambda(xy) = s$. Hence $\lambda^*(xy) \leq s = \bigvee \{\lambda^*(x), \lambda^*(y)\}$. Therefore $\hat{f}_s^*$ is a hybrid subsemigroup of $L$ over $U$.

In a similar way, we can prove the following theorem.

THEOREM 3.20 If $\hat{f}_s$ is a hybrid left (resp., right) ideal of a semigroup $L$ over $U$, the so is $\hat{f}_s^*$.

Proposition 3.21 Let $L$ be a semigroup. If $\hat{f}_s$ is a hybrid right ideal of $L$ over $U$ and $\hat{g}_r$ is a hybrid left ideal of $L$ over $U$, then $\hat{f}_s \circ \hat{g}_r \ll \hat{f}_s \circ \hat{g}_r$.

Proof Let $x \in L$. Assume that $x = ab$ for some $a, b \in L$. Then

$$(\hat{f} \circ \hat{g})(x) = \bigcup_{x=ab} \{\hat{f}(a) \cap \hat{g}(b)\} \subseteq \bigcup_{x=ab} \{\hat{f}(ab) \cap \hat{g}(ab)\}$$

$$= \hat{f}(x) \cap \hat{g}(x) = (\hat{f} \circ \hat{g})(x)$$

and

$$(\lambda \circ \gamma)(x) = \bigwedge_{x=ab} \{\lambda(a), \gamma(b)\} \supseteq \bigwedge_{x=ab} \{\lambda(ab), \gamma(ab)\}$$

$$= \bigvee \{\lambda(x), \gamma(x)\} = (\lambda \circ \gamma)(x).$$

If $x$ is not expressible as $x = ab$ for $a, b \in L$, then

$$(\hat{f} \circ \hat{g})(x) = \emptyset \subseteq (\hat{f} \circ \hat{g})(x) \text{ and } (\lambda \circ \gamma)(x) = 1 \geq (\lambda \circ \gamma)(x).$$

In any case, we have $\hat{f} \circ \hat{g} \ll \hat{f} \circ \hat{g}$ and $\lambda \circ \gamma \geq \lambda \lor \gamma$, i.e. $\hat{f}_s \circ \hat{g}_r \ll \hat{f}_s \circ \hat{g}_r$.

If we strengthen the condition of the semigroup $L$, then we can induce the reverse inclusion in Proposition 3.21 as follows.
Proposition 3.22  Let $L$ be a regular semigroup. If $\tilde{f}_i$ is a hybrid right ideal of $L$ over $U$, then $\tilde{f}_i \circ \tilde{g}_j \ll \tilde{f}_i \circ \tilde{g}_j$, for every hybrid structure $\tilde{g}_j$ in $L$ over $U$.

Proof  Let $x \in L$. Then there exists $a \in L$ such that $xax = x$ since $L$ is regular. Hence

$$\begin{align*}
(\tilde{f} \circ \tilde{g})(x) &= \bigcup_{x=yz} \{ \tilde{f}(y) \cap \tilde{g}(z) \}, \\
(\tilde{f} \circ \tilde{g})(x) &= \bigwedge_{x=yz} \{ \tilde{f}(y) \cap \tilde{g}(z) \}.
\end{align*}$$

On the other hand, we have

$$\begin{align*}
(\tilde{f} \circ \tilde{g})(x) &= \tilde{f}(x) \cap \tilde{g}(x) \subseteq \tilde{f}(xa) \cap \tilde{g}(x), \\
(\tilde{f} \circ \tilde{g})(x) &= \bigvee \{ \tilde{f}(y), \tilde{g}(z) \}. \quad (3.5)
\end{align*}$$

since $\tilde{f}_i$ is a hybrid right ideal of $L$ over $U$. Since $xax = x$, we get

$$\begin{align*}
\tilde{f}(xa) \cap \tilde{g}(x) \subseteq \bigcup_{x=yz} \{ \tilde{f}(y) \cap \tilde{g}(z) \} &= (\tilde{f} \circ \tilde{g})(x), \\
\bigvee \{ \tilde{f}(xa), \tilde{g}(x) \} &\geq \bigwedge_{x=yz} \{ \tilde{f}(y), \tilde{g}(z) \} = (\tilde{f} \circ \tilde{g})(x). \quad (3.6)
\end{align*}$$

It follows from (3.5) and (3.6) that $(\tilde{f} \circ \tilde{g})(x) \subseteq (\tilde{f} \circ \tilde{g})(x)$ and $(\tilde{f} \circ \tilde{g})(x) \geq (\tilde{f} \circ \tilde{g})(x)$ for all $x \in L$. Therefore $\tilde{f}_i \circ \tilde{g}_j \ll \tilde{f}_i \circ \tilde{g}_j$.

In a similar way, we obtain the following.

Proposition 3.23  Let $L$ be a regular semigroup. If $\tilde{g}_j$ is a hybrid left ideal of $L$ over $U$, then $\tilde{f}_i \circ \tilde{g}_j \ll \tilde{f}_i \circ \tilde{g}_j$, for every hybrid structure $\tilde{f}_i$ in $L$ over $U$.

Combining Propositions 3.22 and 3.23, we have the following theorem.

THEOREM 3.24  If a semigroup $L$ is regular, then $\tilde{f}_i \circ \tilde{g}_j = \tilde{f}_i \circ \tilde{g}_j$ for every hybrid right ideal $\tilde{f}_i$ and hybrid left ideal $\tilde{g}_j$ of $L$ over $U$.

4. Conclusion

We have introduced the notions of hybrid subsemigroups and hybrid left (resp., right) ideals in semigroups, and have investigated several properties. Using these notions, we have considered characterizations of subsemigroups and left (resp., right) ideals. We also have introduced the concept of hybrid product, and have discussed characterizations of hybrid subsemigroups and hybrid left (resp., right) ideals by using the notion of hybrid product. We have provided relations between hybrid intersection and hybrid product. Using the notions and results in this paper, we will study the hybrid structures in related algebraic structures and decision making problems etc.

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