Evolutionary game on networks with high clustering coefficient

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Abstract

This study investigates the influence of lattice structure in evolutionary games. The snowdrift games is considered in networks with high clustering coefficients, that use four different strategy-updating. Analytical conjectures using pair approximation were compared with the numerical results. Results indicate that general statements asserting that the lattice structure enhances cooperation are misleading.
I. INTRODUCTION

Evolutionary games in complex networks have recently attracted attention in evolutionary biology, behavioral science and statistical physics [1–3]. One of the most important questions in these fields is how network structure affects the evolution of cooperative behavior [4–9]. Nowak and May noted that the lattice structure enhances cooperative behavior in the prisoner’s dilemma game [4]. Currently lattice structure is considered one of the mechanisms that support cooperation [1, 10]. However, Hauert and Doebeli found that lattice structure often inhibits cooperative behavior in the snowdrift game [11, 12]. Thus, it is not clear how lattice structure affects the evolution of cooperation in general situation. Lattice structure are characteristically predisposed to high clustering. The purpose of this study was to establish a theoretical formula that describes the influence of the clustering coefficient in evolutionary games. Moreover, the effects of the lattice structure have been clarified. The pair approximation technique was applied to obtain an analytical solution [13, 14].

The clustering coefficient is used to measure the tendency of nodes in a network to cluster together [15]. The clustering coefficient of a single node is defined as the probability that two randomly selected neighbors are connected to each other. The clustering coefficient $C$ of the entire network is determined by averaging the clustering coefficients of all nodes. For many social networks, such as file actor collaborations [15], telephone calls [16], e-mails [17], sexual relationships [18], and citation networks [19], the clustering coefficients are greater than those of randomly established networks. Although several studies have examined the effects of clustering on the organization of cooperation [8, 9], there is little agreement as to whether clustering promotes or inhibits the evolution of cooperation. This study considers models with four different strategy-updating rules and presents analytical predictions.

II. MODELS

Consider a static network with $n$ nodes. An individual occupies each node. Individuals play games with all neighbors and their reproduction depends on the average payoff of a sequence of games. The snowdrift game is considered as an example. An individual chooses one of the two strategies: cooperation (C) or defection (D). The payoff matrix is given by
where the positive parameters $b$ and $c$ represent the benefit and cost of cooperation, respectively. The cost-to-benefit ratio of mutual cooperation is defined by $r = c/(2b - c)$. When $r < 1$ (i.e., $b > c$), the snowdrift game has an inner Nash equilibrium, where the cooperator frequency is $1 - r$. In this case, the two strategies coexist in a well-mixed population. This type of game is also known as the hawk-dove or chicken game.

Next, the networks on which this evolutionary game is performed are defined. All notes were assumed to have same degree $z$ (the number of neighbors) to focus on the network cluster coefficient effects. We used a random regular graph with a high clustering coefficient. The edge exchange method [21] that selects two links randomly and repetitively was used to construct the graph. The links were rewired only when the new network configuration was connected and had a larger clustering coefficient. In addition, three types of regular lattices with periodic boundary conditions were used: square lattice with von Neumann neighborhood ($z = 4$), hexagonal lattice ($z = 6$), and square lattice with Moore neighborhood ($z = 8$). The clustering coefficient $C$ is calculated as zero for the von Neumann lattice, although it is highly clustered.

The strategy was assumed to be updated stochastically and asynchronously. These are natural assumptions because strategy selection is not deterministic and occurs simultaneously in the population. Four different strategy-updating rules were selected [7, 11].

1. Birth-death (BD). Choose an individual $i$ with probability proportional to its fitness $f_i$. Then, choose another individual $j$ among the neighbors of individual $i$. Individual $j$ adopts the strategy of individual $i$.

2. Death-birth (DB). Choose an individual $i$ at random. Then, choose another individual $j$ among the neighbors of individual $i$ with probability proportional to fitness $f_j$. Individual $i$ adopts the strategy of individual $j$.

3. Imitation (IM). Choose an individual $i$ at random. Then, choose another individual $j$ among individual $i$ and its neighbors with probability proportional to its fitness. Individual $i$ adopts the strategy of individual $j$. 
4. Local competition (LC). Choose an individual $i$ at random and then choose another individual $j$ among its neighbors randomly. Individual $j$ adopts the strategy of individual $i$ with probability $f_i/(f_i + f_j)$.

The fitness $f_i$ of individual $i$ is given by $1 - w + wP_i$, where $P_i$ is the average payoff of all its neighbors. The parameter $w$ is the intensity of selection. We assumed a weak selection with small $w$. This weak selection assumption allowed the pair approximation to be performed analytically within reason of what occurs in the biological world.

III. ANALYTICAL PREDICTIONS

A pair approximation was used to calculate the equilibrium state. Let $p_C$ and $p_D$ be the densities of cooperators (C) and defectors (D), respectively. The pair densities taken into consideration $p_{CC}$, $p_{CD}$ and $p_{DD}$ represented the frequency that two neighboring pairs were CC, CD or DD. Pairs CD and DC were not distinguished from each other. Thus, $p_{CC} + p_{DD} + p_{CD} = 1$. The conditional probabilities $p_{C|C}$ and $p_{D|D}$ is given by

$$p_{C|C} = p_{CC}/p_C, \quad p_{D|D} = p_{DD}/p_D.$$  

(2)

Considering $p_C = p_{CC} + p_{CD}/2$ and $p_D = p_{DD} + p_{CD}/2$, the densities $p_C$, $p_D$, $p_{CC}$, $p_{CD}$ and $p_{DD}$ are represented as functions of $p_{C|C}$ and $p_{D|D}$:

$$p_C = \frac{1 - p_{D|D}}{2 - p_{C|C} - p_{D|D}},$$

$$p_D = \frac{1 - p_{C|C}}{2 - p_{C|C} - p_{D|D}},$$

$$p_{CC} = \frac{p_{C|C}(1 - p_{D|D})}{2 - p_{C|C} - p_{D|D}},$$

(3)

$$p_{CD} = \frac{2(1 - p_{C|C})(1 - p_{D|D})}{2 - p_{C|C} - p_{D|D}},$$

$$p_{DD} = \frac{p_{D|D}(1 - p_{C|C})}{2 - p_{C|C} - p_{D|D}}.$$  

A triplet, which includes three nodes, can be one of two different configurations. The triad has a node connected with two other nodes that do not connect with each other. The triangle configuration is when all three nodes connected. The standard pair approximation
for a triad \[13\] leads to
\[
\begin{align*}
  p_{\angle CCD} & \approx \frac{P_{CC}P_{CD}}{2P_c} \\
  p_{\angle CDD} & \approx \frac{P_{DD}P_{CD}}{2P_D}.
\end{align*}
\]

However, the use of an extended pair approximation \[14\] for a triangle provides
\[
\begin{align*}
  p_{\triangle CCD} : p_{\triangle CDD} & \approx \frac{P_{CC}P_{CD}P_{CD}}{P_{CD}P_{CD}P_{CD}} = p_{C|C} : p_{D|D}.
\end{align*}
\]

The first approximate equality was calculated from the Kirkwood superposition approximation \[22, 23\]. Therefore, we obtained
\[
\begin{align*}
  p_{C|CD} & \approx (1 - C)p_{C|C} + C \frac{p_{C|C}}{p_{C|C} + p_{D|D}}, \\
  p_{D|DC} & \approx (1 - C)p_{D|D} + C \frac{p_{D|D}}{p_{C|C} + p_{D|D}},
\end{align*}
\]
where \(C\) represents the clustering coefficient, \(p_{C|CD}\) is the probability that a neighbor of the end cooperator of a CD pair is a cooperator, and \(p_{D|DC}\) is the probability that a neighbor of the end defector of a CD pair is a defector.

A cooperator can become a defector only when at least one defector exists in the neighborhood of the cooperator, and vice versa for all four strategy-updating rules. Thus, a strategy can be replaced only in CD pairs. In the strategy-updating cases of BD and LC, the probability to choose C among a CD pair is proportional to the average fitness
\[
1 - w + w[b - c + \frac{c}{2}(1 - 1/z)p_{C|CD}],
\]
while the probability to choose D is proportional to
\[
1 - w + w[b - b(1 - 1/z)p_{D|DC}].
\]

The necessary condition for equilibrium is that \(5\) equals to \(6\). This condition is simplified as
\[
\frac{c}{2} p_{C|CD} + b p_{D|DC} = \frac{c}{1 - 1/z}
\]
If eq. \(7\) is correct, the strategy changing rates coincide with each other:
\[
P_{C\rightarrow D} = P_{D\rightarrow C}.
\]

In addition, in the equilibrium state, the rate at which CD pairs become CC needs to equal the rate at which CC pairs become CD. The rate of change of the doublet density is
FIG. 1. The density $p_c$ of cooperators plotted as a function of the cost-to-benefit ratio $r = c/(2b-c)$ for four different updating rules. The clustering coefficient was set to $C = 0, 0.25, 0.5$ for fixed $z = 4$ and $w = 0.5$. The system size was 10,000. In all simulations, $p_c$ was obtained by averaging the last 10,000 time steps after the first 10,000 ones, and each point resulted from 10 different realizations. The lines represent the predictions \(12\) for BD and LC, \(13\) for DB, and \(14\) for IM.

Given by

$$P_{CD \to CC} = [1 + (z - 1)(1 - p_{D|DC})]P_{D \to C}$$  \(9\)

$$P_{CC \to CD} = (z - 1)p_{C|CD}P_{C \to D}.\tag{10}$$

Since $P_{CD\to CC} = P_{CC\to CD}$ and eq. \(8\), another condition is

$$p_{C|CD} + p_{D|DC} = \frac{z}{z - 1}.\tag{11}$$

Solving the system of eqs. \(7\) and \(11\), yields the equilibrium solutions of $p_{C|CD}$ and $p_{D|DC}$.

Using \(3\) and \(4\), the cooperator equilibrium density was calculated as

$$p_c = -\frac{z - C(z - 1)}{z - 2 - C(z - 1)} \left( r - \frac{1}{2} \right) + \frac{1}{2}.\tag{12}$$

The result \(12\) is valid for BD and LC.
FIG. 2. The density $p_c$ of cooperators plotted for four different updating rules. The degree was set to $z = 4$, 6 and 8 for fixed $C = 0$ and $w = 0.5$. Lines represent predictions (12) for BD and LC, (13) for DB, and (14) for IM. Other parameter values are the same as in Fig. 1.

It is more complicated to obtain a pair approximation that involves the effect of triangles in the case of DB and IM updating. The condition (11) is also valid in these cases. The following are conjecture equations

$$p_c = -\frac{[z - C(z - 1)](z - 1)}{[z - 2 - C(z - 1)](z + 1)} \left( r - \frac{1}{2} - \frac{1}{z - 1} \right) + \frac{1}{2}$$

(13)

for DB updating, and

$$p_c = -\frac{[z - C(z - 1)](z - 1)}{[z - 2 - C(z - 1)](z + 1)} \left[ r - \frac{1}{2} - \frac{z}{(z + 2)(z - 1)} \right] + \frac{1}{2}$$

(14)

for IM updating. Although eqs. (13) and (14) can be calculated by analogy with (12), proper deviation do not exist yet.

IV. NUMERICAL RESULTS

To confirm the predictions presented in the previous section, numerical results were performed for random networks with high clustering coefficients (Figs. 1 and 2). The predictions
FIG. 3. The density $p_c$ of cooperators plotted for four different updating rules. The simulations were performed for the von Neumann square lattice ($z = 4$), hexagonal lattice ($z = 6$) and Moore lattice ($z = 8$). Reference lines are superimposed for BD and LC, DB, and IM, where the clustering coefficient $C$ was set to $C = 0.5$ for the von Neumann square lattice, $C = 0.65$ for the hexagonal lattice, and $C = 0.7$ for the Moore lattice. Other parameter values are the same as in Fig. 1 agree well with the numerical results. Figure 1 shows that when the clustering coefficient increases, the frequency of the major strategy increases for all four updating rules. Predictions suggest the interval of the coexisting region is \[ \frac{[z - 2 - C(z - 1)]/[z - C(z - 1)]}{[z - C(z - 1)](z + 1)}/[[z - C(z - 1)](z - 1)] \] for DB and IM. Thus, if $C > (z - 2)/(z - 1)$ the only one strategy can survive for all four updating rules. Figure 2 shows that the frequency of the majority increases when the degree $z$ decreases for BD and LC updating rules. Cooperation is enhanced for small $z$ for DB and IM updating rules.

Figure 3 shows numerical results for three types of two-dimensional lattices. The predictions for BD and LC, DB, and IM, were superimposed for reference where the parameters were set as $z = 4$ and $C = 0.5$ for the von Neumann lattice, $z = 6$ and $C = 0.65$ for the hexagonal lattice, and $z = 8$ and $C = 0.7$ for the Moore lattice in Fig 3.
This result suggests that the ‘effective’ cluster coefficients are approximately 0.5, 0.65, and 0.7 rather than the “nominal” values 0, 0.4, and 0.43. The clustering coefficient measures the density of triangles in a network. This deviation is because of the effect of loops of length four and above. Cooperator density increases when the degree \( z \) decreases for DB and IM updating rules.

V. DISCUSSION AND CONCLUSION

In conclusion, the frequency of the majority increases with the clustering coefficient. In situations where cooperators and defectors coexist and cooperators are the majority, clustering enhances cooperative behavior. When cooperators are the minority where cooperators and defectors coexist, clustering inhibits cooperative behavior. These results are independent of the strategy-updating rule. We can explain this tendency intuitively by using the heterophilicity \[24\] as follows. From eqs. (3), (4) and (11), the heterophilicity \[24\] is calculated as

\[
H := \frac{p_{CD}}{2p_{CP}p_D} = 1 - \frac{1}{(z - 1)(1 - C)},
\]

for all four updating rules. It is obvious \( H < 1 \), meaning that C have more connections to D than expected randomly. When the clustering coefficient \( C \) increases, heterophilicity \( H \) decreases. In this case, the population is more exclusive, and it is more difficult for strategies to coexist. Consequently, the parameter region where two strategies can coexist becomes narrow. We performed numerical simulations for a small world network \[15\] and a scale-free network on geographical space \[25\] (not shown) to confirm the generality of this result, Essentially the same results were obtained. Unfortunately, a rigorous derivation of (13) and (14) is not provided and it remains an open problem.

Lastly, we considered the prisoner’s dilemma game, where the payoff matrix is \[10\]

\[
\begin{pmatrix}
C & D \\
C & b - c & -c \\
D & b & 0
\end{pmatrix}.
\]

In this case, mutual defection is the only strong Nash equilibrium, regardless of the values of the parameters \( b \) and \( c \). Thus, only defectors can survive in well-mixed population. In addition, cooperators cannot survive for BD and LC updating rules. The standard pair
approximation for the DB updating rule shows if $b/c > z$, only cooperators can survive; conversely, if $b/c < z$, only defectors can survive. The result is the same in the case of IM updating, except the threshold is $b/c = z + 2$. In any case, there is no parameter region where the two strategies can coexist. Thus, the clustering coefficient has no influence on the density of cooperators in the prisoner’s dilemma game. In conclusion, the assertion that lattice structure enhances cooperative behavior is misleading.

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