Inequalities of relative weighted metrics

Peter A. Hästö,
Department of Mathematics, University of Helsinki, P.O. Box 4, 00014, Helsinki, Finland,
E-mail: peter.hasto@helsinki.fi.
3rd of January, 2002. Bullv9.tex

Abstract

In this paper we present inequalities between two generalizations of the hyperbolic metric and the \( j_G \) metric. We also prove inequalities between generalized versions of the \( j_G \) metric and Seittenranta’s metric.

1. Introduction

This paper contains various inequalities between metrics defined in subdomains \( G \) of the Möbius space \( \mathbb{R}^n := \mathbb{R}^n \cup \{\infty\}, n \geq 2 \). In what follows all topological operations are with respect to \( \mathbb{R}^n \) (see Section 2, for further reference e.g. [5]). We will always denote by \( G \subset \mathbb{R}^n \) a domain (i.e. open and connected set) with at least two boundary points and by \( x \) and \( y \) points in \( G \) similarly for \( G' \), \( x' \) and \( y' \).

This section contains the definitions of the metrics studied as well as the statement of the main results. The main results are two blocks of inequalities, Theorems 1.3 and 1.5 which concern two different generalizations of the hyperbolic metric. Section 2 describes the notation used in this paper, which conforms to that used in [5]. The two main theorems are proved in Sections 3 and 4, respectively.

Our first result is a comparison between the generalized hyperbolic metric which was introduced in [3, (3.28)], and proven to be a metric in domains with at least two boundary points in [4] with the generalized hyperbolic metric introduced by Pasi Seittenranta in [4, Definition 1.1] and the well-known \( j_G \) metric defined for \( G \subset \mathbb{R}^n \) by

\[
j_G(x, y) := \log \left(1 + \frac{|x - y|}{\min\{d(x), d(y)\}}\right).
\]

For simplicity, the generalized hyperbolic metric from [3] will be called the generalized hyperbolic metric or the \( \rho_G \) metric, whereas that from [4] will be called Seittenranta’s metric or...
the \( \delta_G \) metric. For domains \( G \) with at least two boundary points, the generalized hyperbolic metric is defined by

\[
\rho_G(x, y) := \sup_{a, b \in \partial G} \operatorname{arch}\{1 + |a, x, b, y||a, y, b, x|/2\}
\]

and Seittenranta’s metric is defined by

\[
\delta_G(x, y) := \sup_{a, b \in \partial G} \log\{1 + |a, x, b, y|\},
\]

where \(|a, x, b, y|\) denotes the cross-ratio, see (2.2).

We cite some basic desirable properties of \( \rho_G \) from [5], as this may help motivate studying this metric. Note that \( \delta_G \) also has all of these properties except the third which is replaced by \( \delta_G(x, y) \geq \exp\{(q(\partial G)q(x, y))\} - 1 \). (Theorem 3.1 and Remark 3.2(2) of [4], [5, 8.38(3)])

1.2 Lemma. ([5, 3.25 & 3.26])

(i) \( \rho_G \) is Möbius invariant (see [1, p. 32]).

(ii) \( \rho_G \) is monotone in \( G \), that is, if \( G \subseteq G' \) then \( \rho_G(x, y) \leq \rho_G'(x, y) \) for all \( x, y \in G \).

(iii) \( \rho_G(x, y) \geq \cosh\{(q(\partial G)q(x, y))^2\} - 1 \).

(iv) For \( G = B^n \) and \( G = H^n \) (the upper half-space), \( \rho_G \) equals the hyperbolic metric.

In this paper we prove the following inequalities of \( \rho_G \):

1.3 Theorem. Let \( G \) be a domain with \( \operatorname{card} \partial G \geq 2 \). Then

(i) \( \delta_G \leq \rho_G \leq \frac{\operatorname{arch} 3}{\log 3} \delta_G \).

Assume additionally that \( G \subseteq \mathbb{R}^n \). Then

(ii) \( \bar{j}_G \leq \rho_G \leq \frac{\operatorname{arch} 3}{\log 2} \bar{j}_G \).

Both inequalities in (i) and the former inequality in (ii) are sharp.

1.4 Remark. Note that the term “sharp” when applied to an inequality means that the constant cannot be improved, i.e. there exists points \( x_i, y_i \in G, i = 1, 2, \ldots \), such that

\[
\lim_{i \to \infty} d_1(x_i, y_i)/d_2(x_i, y_i) = c,
\]

for the inequality \( d_1 \leq cd_2 \).
It was shown in [2, Corollary 6.1] that $\delta_G$ can be embedded in the following family of metrics ($0 < p < \infty$):

$$\delta^p_G(x, y) := \sup_{a,b \in \partial G} \log \left\{ 1 + \left( |x, a, y, b|^p + |x, b, y, a|^p \right)^{1/p} \right\}, \quad \delta^\infty_G(x, y) := \lim_{p \to \infty} \delta^p_G(x, y).$$

With this notation $\delta^\infty_G = \delta_G$, Seittenranta’s metric. It likewise follows directly from Remark 6.1 and Corollary 6.1 in [2] that for $G \subset \mathbb{R}^n$, $j_G$ can be embedded in the family

$$j^p_G(x, y) := \sup_{a \in \partial G} \log \left( 1 + \left( \frac{|x - y|^p}{|x - a|^p} + \frac{|x - y|^p}{|y - a|^p} \right)^{1/p} \right), \quad j^\infty_G(x, y) := \lim_{p \to \infty} j^p_G(x, y).$$

where $0 < p < \infty$. Here then $j^\infty_G = j_G$, the classical $j_G$ metric. We note that if we fix $b = \infty$ in the expression for $\delta^p_G$ then we get the expression for $j^p_G$.

In this paper we prove the following inequalities of the generalized $j_G$ and $\delta_G$ metrics.

1.5 Theorem. Let $G$ be a domain with at least two boundary points. If $0 < q \leq p \leq \infty$ then

(i) $\delta^p_G \leq \delta^q_G \leq 2^{1/q - 1/p} \delta^p_G$.

If additionally $G \subset \mathbb{R}^n$ then

(ii) $j^p_G \leq j^q_G \leq 2^{1/q - 1/p} j^p_G$.

If $p \in [1, \infty]$ and $G \subset \mathbb{R}^n$ then

(iii) $j^p_G \leq \delta^p_G \leq 2 j^p_G$.

All the inequalities are sharp.

Note that inequality (iii) of the previous theorem is a generalization of [4, Theorem 3.4].

2. Notation

The notation adopted here mostly corresponds to that of [4, Chapter 2], the same material is also presented in e.g. [1, Chapter 3]. We denote by $\{e_1, e_2, ..., e_n\}$ the standard basis of $\mathbb{R}^n$ and by $n$ the dimension of the Euclidean space under consideration and assume that $n \geq 2$. For $x \in \mathbb{R}^n$ we denote by $x_i$ the $i$th coordinate of $x$. We will identify $\mathbb{R}$ with the subspace $\mathbb{R}e_1$ of $\mathbb{R}^n$. Hence if $x$ is a real number then the expression “the point $x$” means the point $xe_1$ etc. We will use the notation $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$ and $\overline{\mathbb{R}}^n := \mathbb{R}^n \cup \{\infty\}$ for the two and one point compactifications of $\mathbb{R}$ and $\mathbb{R}^n$, respectively.

By $\partial G$ we will denote the boundary and by $G^c$ the complement of $G$ with respect to $\overline{\mathbb{R}}^n$. The following notation will be used for balls, spheres and the upper half-space:

$$B^n(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}, \quad S^{n-1}(x, r) := \partial B^n(x, r), \quad H^n := \{x \in \mathbb{R}^n : x_n > 0\}.$$
We define the spherical metric $q$ in $\mathbb{R}^n$ by means of the canonical projection onto the Riemann sphere, hence
\[
q(x, y) := \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad q(x, \infty) := \frac{1}{\sqrt{1 + |x|^2}}.
\]

We will consider $\mathbb{R}^n$ as the metric space $(\mathbb{R}^n, q)$, hence its balls are the (open) balls of $\mathbb{R}^n$ and complements of closed balls of $\mathbb{R}^n$ as well as half-spaces. The cross-ratio $[a, b, c, d]$ is defined by
\[
[a, b, c, d] := \frac{q(a, c)q(b, d)}{q(a, b)q(c, d)}
\]
for $a, b, c, d \in \mathbb{R}^n$, $a \neq b$ and $c \neq d$. If $a, b, c, d \in \mathbb{R}^n$ then the cross-ratio can be expressed in terms of Euclidean distances as
\[
[a, b, c, d] := \frac{|a - c||b - d|}{|a - b||c - d|}.
\]
A mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ is a Möbius mapping if
\[
|f(a), f(b), f(c), f(d)| = |a, b, c, d|
\]
for every quadruple $a, b, c, d \in \mathbb{R}^n$ with $a \neq b$ and $c \neq d$ ([1, p. 32]).

We denote the inverses of the hyperbolic sine and cosine by
\[
\text{arsh}(x) = \log(x + \sqrt{x^2 + 1}), \quad x \geq 0,
\]
and
\[
\text{arch}(x) = \log(x + \sqrt{x^2 - 1}), \quad x \geq 1,
\]
respectively. Note that $\sinh(\text{arch}(x)) = \sqrt{x^2 - 1}$.

3. The proof of Theorem 1.3

Proof of Theorem 1.3(i). We start by proving the first of the inequalities. Fix the points $x, y \in G$ and $a, b \in \partial G$ such that $\delta_G(x, y) = \log\{1 + |a, x, b, y|\}$. The points $a$ and $b$ can be chosen, since $\partial G$ is a compact set in $\overline{\mathbb{R}^n}$. Then it suffices to prove the first inequality in
\[
\log\{1 + |a, x, b, y|\} \leq \text{arch}\{1 + |a, x, b, y||a, y, b, x|/2\} \leq \rho_G(x, y),
\]
since the second follows directly from the definition of $\rho_G$. Moreover, since both $\delta_G$ and $\rho_G$ are Möbius invariant we may assume that $a = \infty$ and $b = 0$. Denote $s := |x - y|/\sqrt{|x||y|}$ and $k := \sqrt{|x||y|}$ and assume that $|x| \geq |y|$. Then (3.1) becomes
\[
\log\{1 + ks\} \leq \log\{1 + s^2/2 + \sqrt{s^4/4 + s^2}\}
\]
which reduces to $k - s/2 \leq \sqrt{s^2/4 + 1}$. Squaring this, we see that the inequality holds, since $s \geq k - 1/k$ by the definitions of $k$ and $s$ using the Euclidean triangle inequality. We see that there is equality for $G = \mathbb{R}^n \setminus \{0\}$, $x = e_1$ and $y = re_1$, $r \in \mathbb{R}$. 
In proving the second inequality it again suffices to assume $a = \infty$ and $b = 0$. Let $s$ and $k$ be as before and set $c := \text{arch}\{3\}/\log\{3\}$. The second inequality is equivalent to

\[(3.2) \quad c \log\{1 + ks\} - \log\{1 + s^2/2 + \sqrt{s^4/4 + s^2}\} \geq 0.\]

The derivative with respect to $s$ of the left hand side of the above inequality equals

\[
\frac{c}{s + 1/k} - \frac{1}{\sqrt{s^2/4 + 1}} = \frac{1}{s + 1/k} \left( c - \frac{s + 1/k}{\sqrt{s^2/4 + 1}} \right).
\]

Since the term in the parenthesis is decreasing in $s$, the derivative has at most one zero, which is a maximum. Therefore we need only check that (3.2) holds at the end-points, $s = 0$ and $s = k + 1/k$, which correspond to $|x - y| = 0$ and $|x - y| = |x| + |y|$, respectively. For $s = 0$ the inequality (3.2) obviously holds. In the case $s = k + 1/k$, since $k = s/2 + \sqrt{s^2 - 1}/2$, we need to show that

\[c \log\{1 + s^2/2 + \sqrt{s^4/4 - s^2}\} \geq \log\{1 + s^2/2 + \sqrt{s^4/4 + s^2}\}.
\]

Clearly equality holds for $s = 2$. The claim then follows when we show that the left hand side has greater derivative than the right hand side for $s \geq 2$. Let us change variable, $t = s^2$, and differentiate with respect to $t$:

\[
\frac{ct^2 + t\sqrt{t^2 - 4} - 1}{\sqrt{t^2 - 4(t^2 + 1)}} \geq \frac{1}{\sqrt{t^2 + 4}}.
\]

Since $c \geq 1$, we may drop it. Multiplying by $\sqrt{t^2 - 4(t^2 + 1)}\sqrt{t^2 + 4}$ and squaring gives, after rearranging and dividing by 2,

\[t(t^2 - 1)(t^2 + 4)\sqrt{t^2 - 4} \geq t^6 - 8t^4 + 4t^2 - 4.
\]

To see that this holds, observe the following chain of inequalities (note that $t \geq 4$):

\[t(t^2 - 1)(t^2 + 4)\sqrt{t^2 - 4} \geq t^5\sqrt{t^2 - 4} \geq t^6 - 7t^4 \geq t^6 - 8t^4 + 4t^2 - 4.
\]

For the sharpness of this inequality we choose $G = \mathbb{R}^n \setminus \{0\}$, $x = 1$ and $y = -1$. Then there is equality in the inequality, and hence the constant cannot be improved. □

**Proof of Theorem 1.3(ii).** The first inequality follows from Theorem 1.3(i) ($\delta_G \leq \rho_G$) and [1, Theorem 3.4] ($j_G \leq \delta_G$). Its sharpness follows by letting $G = H^n$, $x = se_n$ and $y = re_n$ (see [1, Remark 3.5]).

We turn to the second inequality. The metric $j_G$ as it is normally defined is not Möbius invariant, and indeed $\infty$ is a special point in the sense that it may not belong to the domain $G$ in which the metric is defined. We may, however, think of $j_G$ as the member $j_{G,\infty}$ of the following family:

\[(3.3) \quad j_{G,b}(x, y) := \sup_{a \in \partial G} \log\{1 + \max\{|x, a, y, b|, |x, b, y, a|\}\},\]

where $G \subset \mathbb{R}^n$ is a domain not containing $b$ with at least two boundary points. Since $j_{G,b}$ is defined in terms of cross ratios, it is clear that it is M"{o}bius invariant. Hence we may apply an auxiliary M"{o}bius transform to both sides of the inequality $\rho_G \leq j_{G,\infty}$, as long as we keep track of where $\infty$ is mapped and use the appropriate $j_{G,b}$.

As before we may then assume that the boundary points $a$ and $b$ occurring in the definition of $\rho_G$ equal 0 and $\infty$. We need to prove

$$
\text{arch} \left( 1 + \frac{|x' - y'|^2}{2|x'||y'|} \right) \leq \frac{\text{arc} \ 3}{\log 2} \sup_{a \in \partial G'} \log \left(1 + \max \{ \infty, a, y', b, \infty, b, y', a \} \right)
$$

$$
\leq \frac{\text{arc} \ 3}{\log 2} \sup_{a \in \partial G'} \log \left(1 + \frac{|x' - y'||b - a|}{\min \{|x' - b||y' - a|, |x' - a||y' - b|\}} \right),
$$

where the supremum is over the boundary point $a$ only; $b$ is some fixed point in the complement of $G$. If $b = 0$ or $b = \infty$, we may proceed exactly as in the proof of Theorem 1.3 (i) and arrive at the better constant $\text{arch}(3)/\log 3$. Assume then that $b \notin \{0, \infty\}$. We may then assume without loss of generality that $b = 1$ (recall that 1 denotes the point $e_1$) by scaling and rotating. Since both sides are M"{o}bius invariant, we may assume that $|x'||y'| \leq 1$ by performing an inversion in the unit sphere, since this leaves $b$ fixed. We then forget about the original $x$ and $y$ and denote $x'$ by $x$ and $y'$ by $y$, to simplify the notation.

We may restrict the supremum from $\partial G$ to $\{0, \infty\} \subseteq \partial G$ in the right hand side of (3.4), since this only makes the supremum smaller. Moreover, we can move the supremum to the denominator of the fraction inside the logarithm, changing it to minimum, since it is taken over finitely many terms. We then see that it suffices to prove that

$$
\sup_{a \in \partial G} \log \left(1 + \frac{|x - y||a - 1|}{\min \{|x - 1||y - a|, |x - a||y - 1|\}} \right) \geq \log \left(1 + \frac{|x - y|}{\min \{|x - 1||y|, |x||y - 1|, |y - 1|, |x - 1|\}} \right).
$$

Let us estimate $|y - 1| \leq |y| + 1$ and $|x - 1| \leq |x| + 1$. Then

$$
\min \{|x - 1||y|, |x||y - 1|, |y - 1|, |x - 1|\} \leq \min \{|1, |y|\}(1 + |x|), \min \{|1, |x|\}(1 + |y|) = \min \{|x|, |y|\} + \min \{1, |x||y|\}.
$$

Recall that we assumed that $|x||y| \leq 1$. By symmetry, we may assume that $|x| \leq |y|$. Then we need to prove

$$
\text{arch} \left( 1 + \frac{|x - y|^2}{2|x||y|} \right) \leq \frac{\text{arc} \ 3}{\log 2} \log \left(1 + \frac{|x - y|}{|x| + |x||y|} \right).
$$

Denote $S_c := \{z \in \mathbb{R}^n : |z - y| = c|z|\}$. For fixed $y$ and $c > 0$ consider how the inequality (3.5) varies as $x$ varies over $S_c$:

$$
\text{arch} \left( 1 + \frac{c|x - y|}{2|y|} \right) \leq \frac{\text{arc} \ 3}{\log 2} \log \left(1 + \frac{c}{1 + |y|} \right).
$$
We see that the right hand side does not depend on $|x - y|$, which means that it suffices to consider points $x \in S_c$ which maximize this distance, since this yields the “hardest” inequality.

Observe that for all $c > 0$ the sphere $S_c$ intersects the segment $[0, y]$ and $S_c$ encloses $y$ if and only if $c \in (0, 1)$ and $0$ if and only if $c > 1$. Note also that $S_c$ is a subset of $B^n(|y|)$ if and only if $c > 2$. Since we need only consider points $x$ that satisfy $|x| \leq |y|$, we see that for $c \in (0, 2]$, the distance $|x - y|$ is maximized by some $x$ satisfying $|x| = |y|$. If $c > 2$ $|x - y|$ is maximized by the choice $x = -y/(c - 1)$.

Let $\lambda := \sqrt{|x||y|} \leq 1$. If $\lambda < 1$ then we can consider the points $x' := x/\lambda$ and $y' := y/\lambda$. The left hand side of (3.3) is the same for the points $x$ and $y$ as for the points $x'$ and $y'$, however the right hand side is smaller for the latter points. Hence we see that it suffices to prove (3.3) for points $x$ and $y$ with $|x||y| = 1$.

Combining the conclusions of the previous two paragraphs, we see that if $c \leq 2$ we need to consider only the case $|x| = |y| = 1$, i.e.

$$\text{arch}\left\{1 + s^2/2\right\} \leq \frac{\text{arch} 3}{\log 2} \log\{1 + s/2\},$$

where we have denoted $s := |x - y|$. For $s = 0$ there is equality in the inequality, and since the left hand side has lesser derivative than the right hand side we are done with this case.

In the case $c < 2$ we need to consider points $x$ and $y$ with $|x||y| = 1$ such that $x$, $0$ and $y$ lie on some line in this order. Hence we need to show that

$$\text{arch}\left(1 + \frac{(|x| + |y|)^2}{2|x||y|}\right) \leq \frac{\text{arch} 3}{\log 2} \log\left(1 + \frac{|x| + |y|}{|x| + |x||y|}\right).$$

Let us write $t := |y| = 1/|x| \geq 1$. The previous inequality becomes

$$\text{arch}\{1 + (t + 1/t)^2/2\} \leq \frac{\text{arch} 3}{\log 2} \log\{1 + (t^2 + 1)/(t + 1)\}.$$

For $t = 1$ there is clearly equality in this inequality. We show that the right hand side has larger derivative than the left hand side for all $t > 1$, which is equivalent to

$$\frac{2}{t} \frac{t^2 - 1}{\sqrt{t^4 + 6t^2 + 1}} \leq \frac{\text{arch} 3}{\log 2} \frac{t^2 + 2t - 1}{(t + 1)(t^2 + t + 2)}.$$

We use the estimate $\text{arch} 3/(2 \log 2) \geq 5/4$ and multiply both sides by the denominators:

$$4(t^2 - 1)(t + 1)(t^2 + t + 2) \leq 5(t^2 + 2t - 1)t\sqrt{t^4 + 6t^2 + 1}.$$ 

We then use the estimates $t^2 + 2t - 1 \geq t(t + 1)$ and $\sqrt{t^4 + 6t^2 + 1} \geq t^2 + 1$ and cancel the term $t + 1$ from both sides:

$$4(t^4 + t^3 + t^2 - t - 2) \leq 5(t^4 + t^2).$$

With the substitution $u := t + 1$ this is equivalent to $u^4 - 5u^2 - 2u + 10 \geq 0$. Since $2u \leq u^2 + 1$ we have $u^4 - 5u^2 - 2u + 10 \geq u^4 - 6u^2 + 9 \geq (u^2 - 3)^2 \geq 0$. □
4. Proof of inequality 1.5

**Proof of Theorem 1.5 (i) and (ii).** It suffices to prove each of the claims for some fixed boundary point(s), since we may choose it (them) to correspond to the point(s) where the supremum is attained in the quantity whose upper bound we want to establish. Hence it suffices to prove the real-number inequality

\[
\log(1 + (x^p + y^p)^{1/p}) \leq \log(1 + (x^q + y^q)^{1/q}) \leq 2^{1/p-1/q} \log(1 + (x^p + y^p)^{1/p})
\]

in order to prove both of the claims. Since \((x^p + y^p)^{1/p} \leq (x^q + y^q)^{1/q}\) the first inequality is clear. Let us denote \(s := 2^{1/q-1/p} \geq 1\). Then \(\log(1 + xs) \leq s \log(1 + x)\) for \(x \geq 0\) by the Bernoulli inequality. Hence it suffices to prove the first inequality in

\[
\log(1 + (x^q + y^q)^{1/q}) \leq \log(1 + s(x^p + y^p)^{1/p}) \leq s \log(1 + (x^p + y^p)^{1/p}).
\]

However, this is immediately clear, since \((x^q + y^q)^{1/q} \leq s(x^p + y^p)^{1/p}\) by the power-mean inequality.

We still need to show that the inequalities are sharp: Let \(G := \mathbb{R}^n \setminus \{0\}\). Then

\[
\delta_G^p(x, y) = j_G^p(x, y) = \log \left(1 + \left(\frac{|x-y|}{|x|^p} + \frac{|x-y|^p}{|y|^p}\right)^{1/p}\right).
\]

Fix \(y\) and let \(x \to \infty\). Then

\[
\lim_{x \to \infty} \frac{j_G^p(x, y)}{\log |x|} \to \log 2
\]

irrespective of the value of \(p\), which shows that the first inequalities are sharp. If \(|x| = |y|\) then

\[
\delta_G^p(x, y) = \log(1 + 2^{1/p}|x-y|/|x|).
\]

As \(x \to y\) we see that the second inequalities are also sharp. □

**4.1 Remark.** Note that since we are using a point-wise estimate, we need not consider the cases \(p = \infty\) and \(q = \infty\) separately, since \(j_G^p\) and \(\delta_G^p\) are both extended to \(p = \infty\) continuously.

**Proof of Theorem 1.5 (iii).** The first inequality follows since \(b\) can equal \(\infty\) in the definition of \(\delta_G\), in which case \(\delta_G^p = j_G^p\). In the domain \(\mathbb{R}^n \setminus \{0\}\) we have \(\delta_G^p(x, y) = j_G^p(x, y)\) for every pair of points \(x, y \in G\), hence the inequality is sharp. It remains to consider the second inequality.

Fix \(x\) and \(y\) in \(G\) and the boundary points \(a\) and \(b\) for which the supremum is attained. We may assume without loss of generality that \(|x-y| = 1\). Then

\[
\delta_G^p(x, y) = \log\{1 + (|x, a, y, b|^p + |x, b, y, a|^p)^{1/p}\} \leq \log\{1 + ((s + t + st)^p + (u + v + uv)^p)^{1/p}\},
\]

where we have denoted

\[
s := \frac{1}{|x-a|}, \quad t := \frac{1}{|y-b|}, \quad u := \frac{1}{|x-b|}, \quad v := \frac{1}{|y-a|}.
\]
and used the estimates

\[ |a - b| \leq |a - x| + |x - y| + |y - b| \quad \text{and} \quad |a - b| \leq |a - y| + |y - x| + |x - b|. \]

in \(|x, a, y, b|\) and \(|x, b, y, a|\), respectively. Now

\[ j^p_G(x, y) \geq \sup_{w \in \{a, b\}} \log \left\{ 1 + (|x - w|^{-p} + |y - w|^{-p})^{1/p} \right\} = \log \left\{ 1 + \max \{s^p + v^p, t^p + u^p\}^{1/p} \right\}. \]

By symmetry we may assume that the \(t^p + u^p \leq s^p + v^p\). If we apply the exponential function to both sides of the inequality

\[
\delta^p_G(x, y) \leq \log \left\{ 1 + ((s + t + st)^p + (u + v + uv)^p)^{1/p} \right\} \leq 2 \log \left\{ 1 + \max \{s^p + v^p, t^p + u^p\}^{1/p} \right\} \leq 2j^p_G(x, y),
\]

we see that it suffices to show that

\[ 1 + ((s + t + st)^p + (u + v + uv)^p)^{1/p} \leq (1 + (s^p + v^p)^{1/p})^2. \]  

(4.2)

We see that the left hand side can be increased by increasing \(t\) while keeping the right hand side constant if \(t^p + u^p < s^p + v^p\). Hence we may assume that \(t^p + u^p = s^p + v^p =: \alpha^p\).

We will show that (4.2) holds for every quadruple \(s, t, u, v \in \mathbb{R}^+\) with \(t^p + u^p = s^p + v^p\) for \(p \geq 1\). For fixed \(s, t, u\) and \(v\) let us consider how the inequality varies under the transformation \(x \mapsto wx, y \mapsto wy, u \mapsto wu\) and \(v \mapsto uv\). Then the equation (4.2) becomes, after we divide it by the common factor \(w\),

\[ f(w) := 2\alpha + w\alpha^2 - ((s + t + stw)^p + (u + v + uvw)^p)^{1/p} \geq 0. \]

We will show that \(f\) increases in \(w\). The derivative \(f'(w)\) equals

\[
\alpha^2 - \{(s + t + stw)^p + (u + v + uvw)^p\}^{1/p-1}\{(s + t + stw)^{p-1}st + (u + v + uvw)^{p-1}uv\} - \alpha^2 - ((s + t + stw)^p + (u + v + uvw)^p)^{1/p-1}uv
\]

\[ = \alpha^2 - ((1 + \zeta^p)^{1/p-1}st + (1 + \zeta^{-p})^{1/p-1}uv\), \]

where \(\zeta := (u + v + uvw)/(s + t + stw)\). We will now consider how

\[ g(\zeta) = (1 + \zeta^p)^{1/p-1}st + (1 + \zeta^{-p})^{1/p-1}uv \]

varies with \(\zeta\). The derivative \(g'(\zeta)\) equals

\[ -(p - 1)((1 + \zeta^p)^{1/p-2}\zeta^{p-1}st - (1 + \zeta^{-p})^{1/p-2}\zeta^{-p-1}uv) = - (p - 1)(1 + \zeta^p)^{1/p-2}\zeta^{p-2}(st\zeta - uv). \]

We see that \(g\) has a maximum at \(\zeta = uv/(st)\) for \(p > 1\). Hence

\[
\frac{df}{dw} \geq \alpha^2 - g\left(\frac{uv}{st}\right) = \alpha^2 - \left(\left(\frac{(st^p + (uv)^p)}{(st)^p}\right)^{1/p-1}st + \left(\frac{(st^p + (uv)^p)}{(uv)^p}\right)^{1/p-1}uv\right)
\]

\[ = \alpha^2 - ((st^p + (uv)^p)^{1/p} = (t^p + u^p)^{1/p}(s^p + v^p)^{1/p} - ((st^p + (uv)^p)^{1/p} \geq 0. \]
Now since $f$ is increasing in $w$, it suffices to show that $f(0) \geq 0$ in order to obtain $f(w) \geq 0$, which is equivalent with (4.2). In other words we must show that $2(s^p + v^p)^{1/p} - ((s + t)^p + (u + v)^p)^{1/p} \geq 0$.

Recall that $t^p + u^p = s^p + v^p =: \alpha^{1/p}$ and denote additionally $\beta := s + t$. The previous inequality becomes

$$2\alpha - \{\beta^p + [(\alpha^p - s^p)^{1/p} + (\alpha^p - (\beta - s)^p)^{1/p}]^{1/p}\}.$$

For fixed $\alpha$ and $\beta$, $(\alpha^p - s^p)^{1/p} + (\alpha^p - (\beta - s)^p)^{1/p} \leq 2(\alpha^p - (\beta/2)^p)^{1/p}$ and so it suffices to show that $2\alpha - (\beta^p + 2^p(\alpha^p - (\beta/2)^p))^{1/p} \geq 0$, which is obvious.

We still have to show that the inequality is sharp. Consider then the domain $G = \mathbb{R}^n \setminus \{-e_1, e_1\}$ and the point $\varepsilon e_2$ and $-\varepsilon e_2$. We have

$$\delta^p_G(\varepsilon e_2, -\varepsilon e_2) = \log \left(1 + \frac{2^{1/p+2}\varepsilon}{\sqrt{1 + \varepsilon^2}}\right)$$

and

$$j^p_G(\varepsilon e_2, -\varepsilon e_2) = \log \left(1 + \frac{2^{1/p+1}\varepsilon}{\sqrt{1 + \varepsilon^2}}\right).$$

It is then clear that

$$\lim_{\varepsilon \to 0} \frac{\delta^p_G(\varepsilon e_2, -\varepsilon e_2)}{j^p_G(\varepsilon e_2, -\varepsilon e_2)} = 2.$$

4.3 Remark. It is not immediately clear whether the inequality from Theorem 1.3 (iii) holds for $0 < p < 1$ as well. It is clear that (4.2) does not hold in this case for arbitrary $s, t, u, v \in \mathbb{R}^+$, however, these variables are not really arbitrary but rather related by various triangle inequalities.

Acknowledgement. I would like to thank Matti Vuorinen for numerous comments and suggestions during the life-span of this manuscript as well as Pentti Järvi for his comments on a previous version of the manuscript.

References

[1] Alan F. Beardon, *The Geometry of Discrete Groups*, Graduate Texts in Math, Vol. 91, (Springer-Verlag, Berlin-Heidelberg-New York, 1982).

[2] Peter A. Hästö, 'A New Weighted Metric: the Relative Metric I', preprint, [http://www.arXiv.org/ math.MG/0108023](http://www.arXiv.org/ math.MG/0108023), to appear in *J. Math. Anal. Appl*.

[3] Peter A. Hästö, 'A New Weighted Metric: the Relative Metric II', preprint, [http://www.arXiv.org/math.MG/0108023](http://www.arXiv.org/math.MG/0108023).

[4] Pasil Seitteenraata, 'Möbius-invariant metrics', *Math. Proc. Cambridge Philos. Soc.*, 125 (1999) 511–533.

[5] Matti K. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Mathematics 1319, (Springer-Verlag, Berlin-Heidelberg-New York, 1988).