Large mixing angles for neutrinos from infrared fixed points

To cite this article: J. Alberto Casas et al JHEP09(2003)048

View the article online for updates and enhancements.

Related content
- Hierarchical neutrino mass matrices, CP violation and leptogenesis
  Liliana Velasco-Sevilla
- Radiative corrections to neutrino mass matrix in the Standard Model and beyond
  Michele Frigerio and Alexei Yu. Smirnov
- Probing the seesaw mechanism with neutrino data and leptogenesis
  Evgeny Kh. Akhmedov, Michele Frigerio and Alexei Yu. Smirnov

Recent citations
- Precise predictions for Dirac neutrino mixing
  Gauhar Abbas et al
- Pieces of the flavour puzzle
  Ferruccio Feruglio
- High scale mixing unification for Dirac neutrinos
  Gauhar Abbas et al
Large mixing angles for neutrinos from infrared fixed points

J. Alberto Casas,\textsuperscript{a,b} José Ramón Espinosa\textsuperscript{b,c} and Ignacio Navarro\textsuperscript{d}

\textsuperscript{a}IEM (CSIC), Serrano 123, 28006 Madrid, Spain
\textsuperscript{b}IFT C-XVI, UAM., Cantoblanco, 28049 Madrid, Spain
\textsuperscript{c}IMAFF (CSIC), Serrano 113 bis, 28006 Madrid, Spain
\textsuperscript{d}IPPP, University of Durham, DH1 3LE, Durham, UK

E-mail: alberto@makoki.iem.csic.es, espinosa@makoki.iem.csic.es, ignacio.navarro@durham.ac.uk

Abstract: Radiative amplification of neutrino mixing angles may explain the large values required by solar and atmospheric neutrino oscillations. Implementation of such mechanism in the Standard Model and many of its extensions (including the Minimal Supersymmetric Standard Model) to amplify the solar angle, the atmospheric or both requires (at least two) quasi-degenerate neutrino masses, but is not always possible. When it is, it involves a fine-tuning between initial conditions and radiative corrections. In supersymmetric models with neutrino masses generated through the Kähler potential, neutrino mixing angles can easily be driven to large values at low energy as they approach infrared pseudo-fixed points at large mixing (in stark contrast with conventional scenarios, that have infrared pseudo-fixed points at zero mixing). In addition, quasi-degeneracy of neutrino masses is not always required.

Keywords: Renormalization Group, Supersymmetric Effective Theories, Beyond Standard Model, Neutrino Physics
1. Introduction

The experimental study of flavour non-conservation in diverse types of neutrino fluxes (solar, atmospheric and “man-made”) has produced in recent years considerable evidence in favour of oscillations among massive neutrinos [1]. Theoretically, the most economic scenario to accommodate the data (or at least the more firmly established data, therefore leaving aside the LSND anomaly [2]) assumes that the left-handed neutrinos of the Standard Model acquire Majorana masses through a dimension-5 operator [3], which is the low-energy trace of lepton-number violating physics at much higher energy scales (the simplest example being the see-saw [4]).
Neutrino masses are then described by a $3 \times 3$ mass matrix $M_\nu$ that is diagonalized by the PMNS \cite{1} unitary matrix $V$:

$$V^T M_\nu V = \text{diag}(m_1, m_2, m_3). \quad (1.1)$$

The masses $m_i$ are real (not necessarily positive) numbers. Following a standard convention we denote by $m_3$ the most split eigenvalue and choose $|m_1| \leq |m_2|$. For later use we define the quantities

$$\Delta m^2_{ij} \equiv m_i^2 - m_j^2, \quad \Delta_i \equiv \frac{m_i - m_j}{m_i + m_j}, \quad \nabla_{ij} \equiv \frac{m_i + m_j}{m_i - m_j}. \quad (1.2)$$

The latter plays an important rôle in the RG evolution of $V$. For simplicity we set CP-violating phases to zero throughout the paper, so $V$ can be parametrized by three successive rotations as

$$V = R_{31}(\theta_1) R_{31}(\theta_2) R_{12}(\theta_3) = \begin{pmatrix} c_2 c_3 & c_2 s_3 & s_2 \\ -c_1 s_3 - s_1 s_2 c_3 & c_1 c_3 - s_1 s_2 s_3 & s_1 c_2 \\ s_1 s_3 - c_1 s_2 c_3 & -s_1 c_3 - c_1 s_2 s_3 & c_1 c_2 \end{pmatrix}, \quad (1.3)$$

where $s_i = \sin \theta_i$, $c_i = \cos \theta_i$.

The experimental information on the neutrino sector is the following. For the CHOOZ angle: $\sin^2 \theta_2 < 0.052$; for the atmospheric neutrino parameters: $1.5 \times 10^{-3} < \Delta m^2_{\text{atm}}/\text{eV}^2 < 3.9 \times 10^{-3}$ and $0.45 < \tan^2 \theta_1 < 2.3$; and for the solar ones (with the MSW mechanism \cite{5} at work): $5.4 \times 10^{-5} < \Delta m^2_{\text{sol}}/\text{eV}^2 < 10^{-4}$ or $1.4 \times 10^{-4} < \Delta m^2_{\text{sol}}/\text{eV}^2 < 1.9 \times 10^{-4}$ and $0.29 < \tan^2 \theta_3 < 0.82$. These (3$\sigma$ CL) ranges arise from the global statistical analysis \cite{6} of many experimental data coming from neutrino fluxes of accelerator (K2K \cite{7}), reactor (CHOOZ, KAMLAND,…. \cite{8} \cite{9}), atmospheric (SK, MACRO, SOUDAN-2 \cite{10} \cite{11}) and solar (Kamiokande, SK, SNO,…. \cite{12} \cite{13} \cite{14} \cite{15}) origin. The smallness of $\theta_2$ and the hierarchy of mass splittings implies that the oscillations of atmospheric and solar neutrinos are dominantly two-flavour oscillations, described by a single mixing angle and mass splitting: $\theta_{\text{atm}} \equiv \theta_1$, $\Delta m^2_{\text{atm}} \equiv \Delta m^2_{31} \sim \Delta m^2_{32}$ and $\theta_{\text{sol}} \equiv \theta_3$, $\Delta m^2_{\text{sol}} \equiv \Delta m^2_{21}$.

Concerning the overall scale of neutrino masses, the non-observation of neutrinoless double $\beta$-decay requires the $ee$ element of $M_\nu$ to satisfy \cite{20}

$$M_{ee} \equiv |m_1 c_2 c_3|^2 + m_2 c_2 s_3^2 + m_3 s_2^2 | \lesssim 0.27 \text{ eV}. \quad (1.4)$$

In addition, Tritium $\beta$-decay experiments \cite{21}, set the bound $m_1 < 2.2 \text{ eV}$ for any mass eigenstate with a significant $\nu_e$ component. Finally, astrophysical observations of great cosmological importance, like those of 2dFGRS \cite{22} and especially WMAP \cite{23} set the limit $\sum_i |m_i| < 0.69 \text{ eV}$. This still allows three possibilities for the neutrino spectrum: hierarchical ($m_1^2 < m_2^2 < m_3^2$), inverted-hierarchical ($m_3^2 \approx m_2^2 \gg m_1^2$) and quasi-degenerate ($m_1^2 \approx m_2^2 \approx m_3^2$).

The nearly bi-maximal structure of the neutrino mixing matrix, $V$, is very different from that of the quark sector, where all the mixings are small. An attractive possibility to explain this is that some neutrino mixings are radiatively enhanced, i.e. are initially small and get large in the Renormalization Group (RG) running from high to low energy.
(RG effects on neutrino parameters have been discussed in [24, 22]). This amplification effect has been considered at large in the literature [25, 30, 51, 62]. Although it has been shown that the infrared pseudo-fixed points (IRFP) for $V$ are not compatible with the bi-maximal structure in the Standard Model (SM) and the Minimal Supersymmetric Standard Model (MSSM) [30, 55], radiative amplification has been nevertheless proposed as an attractive explanation of the observed large mixing angles [57, 58, 59, 62].

In this paper we carefully examine this mechanism for radiative amplification of mixing angles, paying particular attention to 1) a complete treatment of all neutrino parameters (to ensure that not only mixing angles but also mass splittings agree with experiment at low energy) and 2) the fine-tuning price of amplification. We perform this analysis in conventional scenarios, like the SM or the MSSM and confront them with unconventional supersymmetric scenarios, proposed recently, in which neutrino masses originate in the Kähler potential [46].

The sources of neutrino masses in both types of scenarios and their renormalization group equations (RGEs) are reviewed in section 2, which also includes a generic discussion of the presence of infrared pseudo-fixed points in the running of the mixing angles. Section 3 is devoted to the radiative amplification of mixing angles in the conventional scenarios (SM and MSSM): we start with an illustrative toy model of only two flavours and later we apply the mechanism first to the amplification of the solar angle, then to the atmospheric angle and finally to the simultaneous amplification of both. Section 4 deals with the amplification of the mixing angles in the unconventional supersymmetric model which looks quite promising due to its peculiar RG features. We collect some conclusions in section 5. Appendix A contains quite generic renormalization group equations for neutrino masses and mixing angles, while appendix B presents renormalization group equations for generic non-renormalizable operators in the Kähler potential (like the ones responsible for neutrino masses in the unconventional scenario discussed in this paper).

2. Sources of neutrino masses and RGEs

2.1 Conventional SM and MSSM

In the SM the lowest order operator that generates Majorana neutrino masses is

$$\delta \mathcal{L} = - \frac{1}{4M} \lambda_{\alpha\beta} (H \cdot L_\alpha)(H \cdot L_\beta) + \text{h.c.},$$

(2.1)

where $H$ is the SM Higgs doublet, $L_\alpha$ is the lepton doublet of the $\alpha^{th}$ family, $\lambda_{\alpha\beta}$ is a (symmetric) matrix in flavor space and $M$ is the scale of the new physics that violates lepton number, $L$. After electroweak symmetry breaking, the neutrino mass matrix is $M_\nu = \lambda v^2/(4M)$, where $v = 246$ GeV (with this definition $\lambda$ and $M_\nu$ obey the same RGE).

This scheme can be easily made supersymmetric. The standard SUSY framework has an operator

$$\delta \mathcal{W} = - \frac{1}{4M} \lambda_{\alpha\beta} (H_2 \cdot L_\alpha)(H_2 \cdot L_\beta),$$

(2.2)

We restrict our analysis to the simplest low-energy effective models for neutrino masses, with no other assumptions on the physics at high-energy. We therefore do not discuss RG effects in see-saw scenarios, which have been considered previously, e.g. in [25, 30, 57].
in the superpotential $W$, giving $\mathcal{M}_\nu = \lambda \langle H^0_2 \rangle^2 / (2M) = \lambda v^2 \sin^2 \beta / (4M)$ (with $\tan \beta = \langle H^0_1 \rangle / \langle H^0_2 \rangle$). Both in the SM and the MSSM the energy-scale evolution of $\mathcal{M}_\nu$ is governed by a RGE \cite{24,27} of the form ($t = \log Q$):

$$\frac{d\mathcal{M}_\nu}{dt} = -(u_M\mathcal{M}_\nu + c_M P_E\mathcal{M}_\nu + c_M \mathcal{M}_\nu P^T_E),$$  \hspace{1cm} (2.3)

where $P_E \equiv Y_\nu Y_\nu^\dagger / (16\pi^2)$ with $Y_\nu$ the matrix of leptonic Yukawa couplings. The model-dependent quantities $u_M$ and $c_M$ are given in appendix A. Notice that the non-renormalizable operator of eq. (2.1) (eq. (2.2) for the SUSY case) is the only L-violating operator in the effective theory, thus its presence in the right-hand side of eq. (2.3). The term $u_M\mathcal{M}_\nu$ gives a family-universal scaling of $\mathcal{M}_\nu$ which does not affect its texture, while the interesting non family-universal corrections, that can affect the neutrino mixing angles, appear through the matrix $P_E$.

A very important difference between the SM and the MSSM is the value of the squared tau-Yukawa coupling in $P_E$. One has:

$$\gamma^2 = \begin{cases} \frac{2m^2_\tau}{v^2}, & \text{(SM)} \\ \frac{2m^2_\tau}{v^2 \cos^2 \beta} = \frac{2m^2_\tau}{v^2} (1 + \tan^2 \beta), & \text{(MSSM)} \end{cases}$$  \hspace{1cm} (2.4)

Therefore, RG effects can be much larger in the MSSM for sizeable $\tan \beta$.

### 2.2 Neutrino masses from the Kähler potential

Operators that violate L-number in the Kähler potential, $K$, offer an alternative supersymmetric source of neutrino masses \cite{46}. The lowest-dimensional (non-renormalizable) operators of this kind (that respect $R$-parity) are

$$\delta K = \frac{1}{2M^2} \kappa_{\alpha \beta} (L_\alpha \cdot H_2)(L_\beta \cdot \overline{H}_1) + \frac{1}{4M^2} \kappa'_{\alpha \beta} (L_\alpha \cdot \overline{H}_1)(L_\beta \cdot \overline{H}_1) + \text{h.c.},$$  \hspace{1cm} (2.5)

where $\kappa$, $\kappa'$ are dimensionless matrices in flavour space and $\overline{H}_1 = -i\sigma_2 H^*_1$. While $\kappa'$ is a symmetric matrix, $\kappa$ may contain a symmetric and an antisymmetric part: $\kappa \equiv \kappa_S + \kappa_A$. These operators give a neutrino mass matrix \cite{46}

$$\mathcal{M}_\nu = \frac{\mu v^2}{M^2} \left[ \kappa_S \sin^2 \beta + \kappa' \sin \beta \cos \beta \right],$$  \hspace{1cm} (2.6)

where $\mu$ is the SUSY Higgs mass in the superpotential, $W \supset \mu H_1 \cdot H_2$. If $W$ also contains the conventional operator (2.2), the contribution (2.6) is negligible in comparison (by a factor $\mu / M \ll 1$). As shown in \cite{46} there are symmetries that can forbid the operator (2.2) and leave (2.6) as the only source of neutrino masses. This is our assumption for this scenario. Moreover, as we discuss below, interesting new effects appear through the matrix $\kappa$.

Therefore we focus on it as the main source of neutrino masses and set $\kappa' = 0$. This can also be the result of some symmetry \cite{46} or be a good approximation if the mass that
suppresses the \( \kappa' \) operator is much larger than that for \( \kappa \). Another possibility is that \( \tan \beta \) is large (as is common in this context), in which case the contribution of \( \kappa' \) to neutrino masses is suppressed by \( \sim 1/\tan \beta \). Finally, note that, due to the extra suppression factor \( \mu/M \), in this scenario \( M \) is much smaller than in the conventional case.

Appendix 3 presents the RGEs for some non-renormalizable couplings in the Kähler potential, of which \( \kappa \) and \( \kappa' \) are particular examples. The matrix \( \kappa' \) obeys a RGE of the form (2.3) and therefore behaves like the conventional case, while the RGE for \( \kappa \) has a remarkable structure \[ \frac{dk}{dt} = uk + P_{E \kappa} - \kappa P^T_E + 2 \left( P_{E \kappa} - \kappa^T P^T_E \right), \] (2.7)

where \( 16 \pi^2 u = \text{Tr}(3Y^T U_1 + 3Y^T D + Y^T E) - 3g_2^2 - g_1^2 \). Here \( Y_{U(D)} \) is the matrix of up (down) quark Yukawa couplings while \( g_2 \) and \( g_1 \) are the SU(2)\(_L\) and U(1)\(_Y\) gauge couplings, respectively.

Besides the usual universal piece, \( uk \), there are two different terms that can change the texture of \( \kappa \) and are therefore the most interesting. The first, \( P_{E \kappa} - \kappa P^T_E \), decomposes in a symmetric and an antisymmetric part. In that order:

\[ P_{E \kappa} - \kappa P^T_E = (P_{E \kappa A} - \kappa A P^T_E) + (P_{E \kappa S} - \kappa S P^T_E). \] (2.8)

The second texture-changing term, \( 2(P_{E \kappa} - \kappa P^T_E) \), is antisymmetric and, therefore, contributes only to the RG evolution of \( \kappa_A \), the antisymmetric part of \( \kappa \).

The diagrammatic origin of these contributions is explained with the help of figure 1. Diagram (a) is a tree-level supergraph for the coupling \( \kappa_{\alpha \beta} \). The order of subindices is important: \( L_\alpha \) is SU(2)-contracted with \( H_2 \); \( L_\beta \) with \( \overline{H}_1 \). This is depicted in figure 2 by the two “branches” of the vertex, with arrows indicating the order in the SU(2) product. We do not show the one-loop supergraphs that contribute to the universal renormalization of \( \kappa_{\alpha \beta} \) but focus on those that can change its texture. Diagrams (b) and (c) renormalize \( \kappa_{\alpha \beta} \) through the anomalous dimensions of the leptonic legs, \( L_\alpha \), \( L_\beta \). These kinds of diagrams are proportional to \( P_{E \kappa} \) and \( \kappa P^T_E \), as indicated, and are also present when neutrino mass operators arise from the superpotential. They contribute a \( P_{E \kappa} + \kappa P^T_E \) piece to the renormalization of \( \kappa \). Diagrams (d) and (e) are non zero only because \( \kappa \) involves chiral and anti-chiral fields. Similar vertex corrections are absent for the neutrino mass operator in \( W \), which involves only chiral fields, and is protected by SUSY non-renormalization theorems. Diagram (d) gives a contribution similar to that coming from diagram (e) but twice as large and with opposite sign. The net effect is to change \( P_{E \kappa} + \kappa P^T_E \) (from (b)+(c)) into \( P_{E \kappa} - \kappa P^T_E \). This is the origin of the first term in the RGE (2.7). Finally, diagram (e) gives only a correction to the operator \( L_\alpha \cdot L_\beta (\overline{H}_1 \cdot H_2) \), which is the antisymmetric part of \( \kappa_{\alpha \beta} \) by virtue of the identity \( r_{\alpha \beta} (L_\alpha \cdot L_\beta) (\overline{H}_1 \cdot H_2) = (r_{\beta \alpha} - r_{\alpha \beta}) (L_\alpha \cdot H_2) (L_\beta \cdot \overline{H}_1) \) and this is responsible for the last term \( 2(P_{E \kappa} - \kappa P^T_E) \) in (2.7).

In order to show more clearly the structure of the RGE for \( \kappa \), eq. (2.7), it is convenient to split it in two: one for the symmetric part, \( \kappa_S \), that is directly responsible for neutrino masses, and another for the antisymmetric part, \( \kappa_A \), that does not contribute to neutrino
Figure 1: The coupling $\kappa_{\alpha\beta}$ and one-loop texture-changing radiative corrections to it. The lines labelled $E$ represent right-handed (charged) lepton superfields.

masses. One gets

$$\frac{d\kappa_S}{dt} = u\kappa_S + P_E\kappa_A - \kappa_A P_E^T, \quad (2.9)$$

$$\frac{d\kappa_A}{dt} = u\kappa_A + P_E\kappa_S - \kappa_S P_E^T + 2(P_E\kappa - \kappa^T P_E^T). \quad (2.10)$$

As explained in [46], the RGE for $\kappa_S$ has the remarkable property of being dependent of $\kappa_S$ itself only through the universal piece. We have shown in more detail here how this arises from a cancellation involving corrections that are only present in supersymmetry for couplings in the Kähler potential. Non-supersymmetric two-Higgs-doublet models also have vertex corrections, but there is no such cancellation there. Some interesting implications that follow from the RGEs (2.9), (2.10) were presented in [46].

2.3 Infrared pseudo-fixed points (IRFP) for mixing angles

Equations (2.3) and (2.9) detail how $M_\nu$ receives a non-universal RG perturbation which is in general modest ($P_E$ is dominated by $y_{\nu_1}^2$, which is very small, unless $\tan\beta \gtrsim 50$). However,

\footnote{For instance, if initially $\kappa_S = 0$ the whole neutrino mass matrix is generated as a radiative effect through (2.3). Such matrix has precisely the texture of the Zee model [63] (actually, this possibility can be understood as the supersymmetrization of the Zee model).}
when $\mathcal{M}_\nu$ has (quasi-)degenerate eigenvalues ($m_i \simeq m_j$), even small perturbations can cause large effects in the eigenvectors (i.e. in the form of $V$). This can be easily understood: for exact degeneracy there is an ambiguity in the choice of the associated eigenvectors, and thus in the definition of $V$. When the perturbation due to RG running is added, the degeneracy is lifted and a particular form of $V$ is singled out \cite{55, 30}. If the initial degeneracy is not exact, the change of $V$ will be large or not depending on the size of the perturbation ($\delta_{\text{RG}} \Delta m_{ij}^2$) compared with the initial mass splitting at the scale $M$, $\Delta m_{ij}^2(M)$.

When the RG effect dominates, $V$ evolves quickly from its initial value $V(0)$ to a stable form (an infrared pseudo-fixed point, IRFP) $V(0) R_{ij}$ where $R_{ij}$ is a rotation in the plane of the two quasi-degenerate states $i, j$ such that the perturbation of $\mathcal{M}_\nu$ is diagonal in the rotated basis (as is familiar from degenerate perturbation theory). Scenarios in which this takes place are attractive for two main reasons: 1) some particular mixing angle will be selected as a result of $V$ approaching its IRFP form and 2) the $i$-$j$ mass-splitting will be essentially determined at low energy by RG effects.

When $\delta_{\text{RG}} \Delta m_{ij}^2 \sim \Delta m_{ij}^2(M)$, RG effects can produce substantial changes in $V$ without getting too close to the IRFP. This scenario is of interest because, as we show in section 3, the IRFP form of $V$ in the SM and the MSSM is not in agreement with experimental observations, while intermediate forms of $V$ can be.

3. Amplification of mixing angles: SM and MSSM

As explained above, when two neutrino masses are quasi-degenerate (and with the same sign) radiative corrections can have a large effect on neutrino mixing angles. This offers an interesting opportunity for generating large mixing angles at low energy as an effect of RG evolution, starting with a mixing angle that might be small. Here we explain why the implementation of this idea in the SM or the MSSM is not appealing. In order to show this we will make much use of the RGEs for physical parameters derived in refs. \cite{55, 30} and collected in appendix A for convenience. The present discussion intends to expose the involved fine-tuning as clearly and completely as possible. Some results have been obtained before (see \cite{30, 56}), concerning the two-flavour approximation and the atmospheric angle. The other cases, i.e. the fine-tuning required in amplifying the solar angle or in the simultaneous amplification of solar and atmospheric angles were not addressed previously.

Radiative corrections to $V$ are very small unless $|\epsilon_\tau \nabla_{ij} \log(M/M_Z)| \gtrsim 1$ for some $i, j$ (see eqs. (A.3), (A.13); here $\epsilon_\tau \sim y^2/(16\pi^2)$ and $\nabla_{ij}$ was defined in the Introduction) which generically requires mass degeneracy, both in absolute value and sign (i.e. $m_i \simeq m_j \Rightarrow |\nabla_{ij}| \gg 1$), except for the SUSY case with very large $\tan \beta$, and thus large $\epsilon_\tau$.\footnote{The possibility of big changes in the mixing angles due to RG evolution with degenerate masses was first pointed out in a $2 \times 2$ context in \cite{22}.} In general, if $m_i \simeq m_j$, so that $|\nabla_{ij}|$ dominates the RGEs of $V_{\tau i}, V_{\tau j}$, these quantities change

\footnote{If $|\epsilon_\tau \log(M/M_Z)| \sim \mathcal{O}(1)$ the validity of the one-loop approximation is in doubt and the analysis of RG evolution should be done by numerical integration of the RGEs in order to capture the leading-log effects at all loops.}
appreciably, but the following quantities will be approximately constant

\[
\begin{array}{l}
\Delta_{ij}V_{\tau_i}V_{\tau_j}, \\
V_{\alpha l}, (l \neq i, j)
\end{array}
\] \approx \text{RG-invariant},
\]

where \(\Delta_{ij} = 1/\nabla_{ij}\). The IRFP form for \(V\) can be deduced from eq. (A.3) and corresponds to \(T_{ij} = 0\), which for sizeable \(r_{ij}\) implies \(V_{\tau_i} = 0\) or \(V_{\tau_j} = 0\) (depending on the sign of \(\nabla_{ij}\)). Such \(V\) can not give the observed angles (it could if the SAMS solution were still alive).

Hence, the IRFP form for \(V\) should not be reached. Still, one may hope that the RG effects could amplify the atmospheric and/or the solar angles without reaching the IRFP form of \(V\). Such possibility would be acceptable only if 1) all mixing angles and mass splittings (which are also affected by the running) agree with experiment and 2) if this can be achieved with no fine-tuning of the initial conditions.

We explore in turn the possibility of RG amplification of the mixing angle in a two-flavour case and then for the solar or/and atmospheric angles.

### 3.1 The two-flavour approximation

There are several instances (see below) in which the evolution of a particular mixing angle is well approximated by a two-flavour model. This simple setting is very useful to understand the main features of the RG evolution of mixings and mass splittings, and thus the form of the infrared fixed points and the potential fine-tuning problems associated with mixing amplification. A discussion in this context can also be found in [30, 56]. Here we summarize the results, giving some intuitive and graphical understanding, and developing handful formulae that will be used in the later subsections 3.2 and 3.4, and in the comparison with the Kähler scenario in subsection 1.1.

In a two-flavour context we have flavour eigenstates, \((\nu_\alpha, \nu_\beta)\), a mixing angle, \(\theta\), (with \(V_{\alpha 2} = \sin \theta\) and \(\nu_\alpha = \nu_1\) for \(\theta = 0\)) and mass eigenstates (eigenvalues), \(\nu_i(m_i), i = 1, 2\). In a basis where the matrix of leptonic Yukawa couplings is diagonal, the RGE for the mixing angle (from eq. (A.3)) takes the form

\[
\frac{d\theta}{dt} = -\frac{1}{2} \epsilon_{\alpha\beta} \nabla_{21} \sin 2\theta,
\]

with

\[
\epsilon_{\alpha\beta} \equiv c_M \frac{y_\alpha^2 - y_\beta^2}{16\pi^2},
\]

where \(c_M\) is a model-dependent constant. As previously discussed, for \(|\nabla_{21}| \gg 1\) (i.e. for quasi-degenerate neutrinos), \(\theta\) can change appreciably. In such case, it will be driven towards the infrared (pseudo)-fixed point \(\theta^*\) determined by the condition \(d\theta/dt = 0\), which corresponds to \(\theta^* = 0, \pi/2\), that is, towards zero mixing, \(\sin 2\theta^* = 0\). The degree of approximation to this fixed point depends on the length of the running interval, \((\log(M/M_Z))\), on the values of the Yukawa couplings, and especially on \(\nabla_{21}\). On the other hand the relative splitting, \(\Delta_{21}\) satisfies the RGE (from eq. (A.11))

\[
\frac{d\Delta_{21}}{dt} = \frac{4m_1m_2}{(m_1 + m_2)^2} \epsilon_{\alpha\beta} \cos 2\theta \approx \epsilon_{\alpha\beta} \cos 2\theta,
\]
where, for the last approximation, we have assumed quasi-degenerate neutrinos, which is the case of interest. As a consequence, note that

$$\Delta_{21} \sin 2\theta \simeq \text{RG - invariant}. \quad (3.5)$$

There are two qualitatively different possibilities for the running of $\theta$ depending on the sign of $d\theta/dt$ at $M$ (see figure 2, where the fixed points for $\theta$ are indicated by dotted lines): if $\theta$ decreases with decreasing scale ($d\theta/dt > 0$ at $M$) and $\theta^{(0)} \equiv \theta(M)$ is small, the RG evolution drives $\theta$ to zero in the infrared, making it even smaller: the mixing never gets amplified. On the opposite case, if $\theta$ increases with decreasing scale ($d\theta/dt < 0$ at $M$), $\theta$ is driven towards $\theta^* = \pi/2$, and it may happen that the running stops (at $M_Z$) near $\theta \sim \pi/4$ so that large mixing is obtained.\footnote{For the solar angle, one should have $\tan^2 \theta_3(M_Z) < 1$ (with eigenvalues labelled such that $|m_1| < |m_2|$ holds at low energy), as needed for the MSW solution.}

In this second case the RG-evolution is illustrated by figure 2. The upper plot shows the running of $m_1/(m_1 + m_2)$ and $m_2/(m_1 + m_2)$ with the scale (this choice removes the universal part of the running and focuses on the interesting relative mass splitting) while the lower plot shows the running of $\sin^2 2\theta$. Notice that the evolution of the splitting is quite smooth (first decreasing and then increasing), while the change of $\theta$ is only important around the scale of maximal mixing ($\theta \sim \pi/4$) which corresponds to the scale of minimal splitting. A simple analytical understanding of this behaviour is possible in the case of interest, with quasi-degenerate masses. In that case the RGEs for $\theta$ and $\Delta_{21}$, eqs. (3.2), (3.4), can be integrated exactly (assuming also that the running of $y^2_{\tau}$ is neglected) to get

$$\sin^2 2\theta(Q) = \left[ \frac{\Delta_{21}^{(0)} \sin 2\theta^{(0)}}{\Delta_{21}^{(0)} \cos 2\theta^{(0)} - \epsilon_{\alpha\beta} \log \frac{M}{Q}} \right]^2 + \left[ \frac{\Delta_{21}^{(0)} \sin 2\theta^{(0)}}{\Delta_{21}^{(0)} \cos 2\theta^{(0)} - \epsilon_{\alpha\beta} \log \frac{M}{Q}} \right]^2; \quad (3.6)$$

$$\Delta_{21}^2(Q) = \left[ \frac{\Delta_{21}^{(0)} \cos 2\theta^{(0)} - \epsilon_{\alpha\beta} \log \frac{M}{Q}}{\Delta_{21}^{(0)} \cos 2\theta^{(0)} - \epsilon_{\alpha\beta} \log \frac{M}{Q}} \right]^2 + \left[ \frac{\Delta_{21}^{(0)} \sin 2\theta^{(0)}}{\Delta_{21}^{(0)} \cos 2\theta^{(0)} - \epsilon_{\alpha\beta} \log \frac{M}{Q}} \right]^2. \quad (3.7)$$

From these solutions we can immediately obtain the scale $Q_{\text{max}}$ at which maximal mixing occurs:

$$\log \frac{M}{Q_{\text{max}}} = \frac{\Delta_{21}^{(0)} \cos 2\theta^{(0)}}{\epsilon_{\alpha\beta}}; \quad (3.8)$$

the half-width, $\omega$, of the 'resonance' (defined at $\sin^2 2\theta = 1/2$)

$$\omega = \frac{\Delta_{21}^{(0)} \sin 2\theta^{(0)}}{\epsilon_{\alpha\beta}}; \quad (3.9)$$
Figure 3: Running of $m_1/(m_1 + m_2)$ and $m_2/(m_1 + m_2)$ (upper plot) and $\sin^2 2\theta$ (lower plot) from $M$ down to $M_Z$ in a two-flavour model width quasi-degenerate masses.

and the minimal splitting:

$$\Delta_{21,\text{min}} \equiv \Delta_{21}(Q_{\text{max}}) = \Delta_{21}^{(0)} \sin 2\theta^{(0)}. \quad (3.10)$$

These results make clear that amplification requires a fine-tuning of the initial conditions. Suppose one desires that the initially small value of the mixing, $\sin 2\theta^{(0)}$, gets amplified by a factor $F \gg 1$ at low energy due to the running. From (3.5), this requires the initial relative splitting, $\Delta_{21}^{(0)}$ to be fine-tuned to the RG shift, $\delta_{RG}\Delta_{21}$, as

$$\Delta_{21}^{(0)} = F\Delta_{21}(M_Z) = F[\Delta_{21}^{(0)} + \delta_{RG}\Delta_{21}]$$
$$\Rightarrow \delta_{RG}\Delta_{21} = -\left(1 - \frac{1}{F}\right)\Delta_{21}^{(0)}, \quad (3.11)$$

where$^6$

$$\delta_{RG}\Delta_{21} \simeq \epsilon_{\alpha\beta} \cos 2\theta \log \frac{M}{M_Z}. \quad (3.12)$$

$^6$This is a one-loop leading-log approximation valid when $\sin^2 2\theta^{(0)} \ll 1$. A more precise result is given by the exact expression in eq. (3.7), which includes all leading-log corrections.
Hence, eq. (3.11), which makes quantitative the arguments of the last paragraph of section 2.3, exposes a fine-tuning of one part in $F$ between two completely unrelated quantities. There is no (known) reason why these two quantities should be even of a similar order of magnitude, which stresses the artificiality of such coincidence.

Alternatively, this fine-tuning can be seen in the expressions for the scale $Q_{\text{max}}$ and the half-width $\omega$ (eqs. (3.8), (3.9)). The initial splitting, $\Delta_{21}^{(0)}$, and the strength of the radiative effect, $\epsilon_{\alpha\beta}$, have to be right to get $Q_{\text{max}}$ near $M_Z$: If $\Delta_{21}^{(0)}$ is small and/or $\epsilon_{\alpha\beta}$ is large, the angle goes through maximal mixing too quickly; if $\Delta_{21}^{(0)}$ is large and/or $\epsilon_{\alpha\beta}$ is small, the angle never grows appreciably. How delicate the balance must be is measured by the half-width $\omega$, or better its ratio to the running interval,

$$\frac{\omega}{\log(M/Q_{\text{max}})} = \tan 2\theta^{(0)}, \quad (3.13)$$

which is of order $1/F$ in agreement with the previous estimate of the fine-tuning. Finally, notice from (3.11) that the RG-shift must satisfy $\delta_{\text{RG}}\Delta_{21} = (1 - F)\Delta_{21}^{\text{exp}}$, which may be impossible or unnatural to arrange, as we show in some examples below.

### 3.2 Solar angle

To amplify only the solar angle, $\theta_{\text{sol}} \equiv \theta_3$, the RGE of $V$ must be dominated by $\nabla_{V_{21}}$ (see (A.19)), which requires a quasi-degenerate ($m_1 \simeq m_2 \simeq |m_3|$) or inversely-hierarchical ($m_1 \simeq m_2 \gg |m_3|$) spectrum. Then the RG-corrected $V$ is $V^{(0)}R_{12}(\phi)$ with $\phi$ evolving towards an IRFP such that $V_{\tau_1} = 0$ or $V_{\tau_2} = 0$, while $V_{a3} = (s_2, s_1c_2, c_1c_2)^T$ is almost unaffected by the running. This means, in particular, that $\theta_1$ and $\theta_2$ have to be determined by the physics at $M$, so as to have $V_{a3} \simeq (0, 1/\sqrt{2}, 1/\sqrt{2})^T$ as an initial condition. An attractive feature of this scenario is that the running would not upset such initial values as only $\theta_3$ is affected. This is most clearly seen by realizing that $V \to V^{(0)}R_{12}(\phi)$ amounts simply to $\theta_3 \to \theta_3^{(0)} + \phi$ (see (1.3)).

It is a good approximation to treat the running of $\theta_3$ in a two-flavour context (see eqs. (3.2) and (A.19)) with $(\nu_\alpha, \nu_\beta) = (\nu_e, \nu_\mu \cos \theta_1 - \nu_\tau \sin \theta_1)$, and $(m_i, m_j) = (m_1, m_2)$. Therefore we can apply the results obtained in the previous subsection (in particular eqs. (3.2), (3.4)) with $y_\alpha^2 \simeq 0$, $y_\beta^2 \simeq y_\tau^2 \simeq y_\tau^2/2$ to conclude that the amplification of $\sin 2\theta_3$ by a factor $F \gg 1$ requires a fine-tuning of one part in $F$ between two completely unrelated quantities: the relative mass splitting at the $M$ scale, $\Delta^{(0)}_{21}$, and the splitting generated radiatively, $\delta_{\text{RG}}\Delta_{21}$.

Moreover, from eqs. (3.11), (3.12), $\delta_{\text{RG}}\Delta_{21} \simeq (\epsilon_2/2) \log(M/M_Z)$ has to match $(1 - F)\Delta_{21}^{\text{exp}}$. In the SM $\delta_{\text{RG}}\Delta_{21}$ is quite small, which means that $F$ must be close to one: $F - 1 \simeq 10^{-3} \log(M/M_Z)(7 \times 10^{-5}\text{eV}^2/\Delta m_{\text{sol}}^2)/(0.2\text{eV})^2$. Consequently, it is not possible to amplify the solar angle in the SM, even with fine-tunings. Larger values of $\delta_{\text{RG}}\Delta_{21}$ can be achieved in the MSSM for large $\tan \beta$: amplification of $\sin 2\theta_3$ by a factor $F$ requires

$$\tan \beta \simeq 37 \sqrt{\frac{(F - 1)}{\log(M/M_Z)}} \left(\frac{\Delta m_{\text{sol}}^2}{7 \times 10^{-5}\text{eV}^2}\right)^{1/2} \left(\frac{0.2\text{eV}}{m}\right), \quad (3.14)$$

where $\log(M/M_Z) \sim 30$ is a typical value.
3.3 Atmospheric angle

This case was critically examined already in [30, 55]. We summarize here the main arguments and results and complete the analysis. From eqs. (A.17)–(A.19), the amplification of $\theta_{\text{atm}} \equiv \theta_1$ requires that $\nabla_{31}$ or $\nabla_{32}$ dominate the RGEs, and therefore $m_1 \simeq -m_2 \simeq \pm m_3$ is necessary.\(^7\)

Suppose that $\nabla_{31}$ is dominant (for $\nabla_{32}$ the argument is similar). The RG-corrected mixing matrix takes the form $V_\mu = V^{(0)} R_{31}(\phi)$, with $\phi$ evolving towards an IRFP such that $V_{r1} = 0$ or $V_{r3} = 0$ while $V_{r2}$ remains almost constant.\(^8\) Therefore, to agree with experiment we must assume as an initial condition $V_{e3} \neq 0$ or $V_{\mu 3} \neq 0$ while $V_{\tau 3}$ remains almost constant.

In figure 4 the green path is traversed as $\phi$ is varied, with IRFPs marked by solid red dots ($I_1$, $I_2$). By assumption we start the running at some point in this path near $\sin^2 2\theta_1 = 0$, i.e. near the intersections with the $e\tau$-side ($\theta_1 = 0$) or the $e\mu$-side ($\theta_1 = \pi/2$). Maximal $\theta_1$ mixing corresponds to points equidistant from these two sides. The goal would be to stop the running near the point $B$ marked by the open blue dot, which has maximal $\theta_1$ mixing and zero $\theta_2$. From the location of the IRFP points, this amplification can only work if we start near $\theta_1 = \pi/2$ (starting near $\theta_1 = 0$ we can never reach our goal) and have $|V_{e3}|^2 < 1/3$ (so as to take the right path). Schematically, this right path from one IRFP to the other going through the point of bi-maximal mixing is

\[
V_{I_1} \equiv \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{2\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix} \quad \rightarrow \quad V_B \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \rightarrow \quad V_{I_2} \equiv \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}.
\]

\(^7\)Another possibility, with $m_1 \simeq m_2 \simeq m_3$, is discussed in section 5.4.

\(^8\)The two-flavour approximation is not possible here: it requires $s_2 \simeq s_3 \simeq 0$, at odds with experiment.
Actually, it can be checked from eq. (A.17) that, since \( v \) can only work if \( \delta^2 \) the reasons explained in subsection 3.3. In consequence the possibility under consideration of the running, otherwise \( v \) and along most of the running.

Then \( r \) and

\[
\frac{\Delta m^2_{\text{atm}}}{m^2} = \left[ \frac{\Delta m^2_{\text{atm}}}{m^2} \right]^{(0)} + 4\epsilon_r \langle V^2_{r_3} - V^2_{r_1} \rangle \log \frac{M}{M_Z},
\]

(3.15)

\[
\frac{\Delta m^2_{\text{sol}}}{m^2} = \left[ \frac{\Delta m^2_{\text{sol}}}{m^2} \right]^{(0)} + 4\epsilon_r \langle V^2_{r_2} - V^2_{r_1} \rangle \log \frac{M}{M_Z},
\]

(3.16)

where \( \langle \cdot \rangle \) denote averages over the interval of running and \( m \) is the overall neutrino mass.

We see that both radiative corrections are of similar magnitude because, for rapidly changing \( V_{r_1} \) and \( V_{r_3} \), the averages of matrix elements in (3.15), (3.16) cannot be suppressed and are therefore of \( \mathcal{O}(1) \). On the other hand, \( \delta_{\text{RG}}(\Delta m^2_{31}/m^2) \) (last term in eq. (3.13)) must be \( (F - 1)\Delta m^2_{\text{atm}}/m^2 \), where \( F \) is defined as \( V_{r_1} V_{r_3} = F V_{r_1}^{(0)} V_{r_3}^{(0)} \) (see eq. (3.1)). For a sizeable amplification, \( (F - 1) \gtrsim 1 \), so unless there is an extremely accurate and artificial cancellation in (3.16), one naturally expects \( \Delta m^2_{\text{sol}} \sim \Delta m^2_{\text{atm}} \), which is not acceptable. In fact, there are further problems: in the SM \( \delta_{\text{RG}}(\Delta m^2_{31}/m^2) \) is not large enough to match \( (F - 1)\Delta m^2_{\text{atm}}/m^2 \). In the MSSM this is possible, but it requires, besides a certain tuning, a very large \( \tan \beta \) (\( \gtrsim 100 \) for \( |m| < 0.3 \text{eV} \)).

### 3.4 Solar and atmospheric angles simultaneously

There is still a possibility for mixing amplification not discussed in the previous subsections (3.3) and (3.4), namely when \( m_1 \approx m_2 \approx m_3 \), both in absolute value and sign.\(^9\) Then \( \nabla_{21} \gg \nabla_{31} \gg \nabla_{32} \gg 1 \) (in absolute value). Notice from eq. (A.17)–(A.19) that if \( \epsilon_r \nabla_{31} \log(M/M_Z) \sim \mathcal{O}(1) \), so that the atmospheric angle can run appreciably, then it is mandatory that \( V_{r_1} V_{r_2} \simeq 0 \); otherwise the running of \( V \) will be strongly dominated by the term proportional to \( T_{21} = \epsilon_r V_{r_1} V_{r_2} \nabla_{21} \), and thus rapidly driven to an IRFP (a phenomenological disaster). To avoid this, the condition \( V_{r_1} V_{r_2} \simeq 0 \) must be fulfilled initially and along most of the running.\(^10\) In addition one has to demand \( V_{r_2}^2 - V_{r_1}^2 \simeq 0 \) along most of the running, otherwise \( \Delta m^2_{21} \) gets radiative corrections of a size similar to \( \Delta m^2_{\text{atm}} \) for the reasons explained in subsection 3.3. In consequence the possibility under consideration can only work if \( V_{r_2} \simeq V_{r_1} \simeq 0 \Rightarrow \sin \theta_1 \simeq \sin \theta_2 \simeq 0 \) along most of the running.

The previous conclusion implies that \( \theta_1 \) must be radiatively amplified at low energy. Actually, it can be checked from eq. (A.17) that, since \( \nabla_{31} \approx \nabla_{32} \), the running of \( \theta_1 \) is well

---

\(^9\)This possibility has been recently used in [2], where the implicit fine-tuning we are about to show was not addressed.

\(^10\)This condition implies that one starts near one of the IRFPs. To avoid falling towards it, the signs of the splittings must be such that \( V \) is eventually driven towards a different IRFP, crossing in its way regions of parameter space with sizeable \( V_{r_1} \) and \( V_{r_2} \). (This is similar to the situation discussed after eq. (3.3).)
approximated by a two-flavour equation (see subsection 3.1)
\[
\frac{d\theta_1}{dt} \simeq \frac{1}{2} \epsilon_r \nabla_{31} \sin 2\theta_1 .
\] (3.17)

Hence, the results of subsection 3.1 apply here and we conclude that the amplification of \(\theta_1\) requires 1) a very large \(\tan \beta \simeq 100\) for \(|m| < 0.3\,\text{eV}\) to get a large enough \(\delta_{RG}\Delta_{31}\) and 2) a fine-tuning between the initial splitting \(\Delta_{31}^{(0)}\) and the radiative correction \(\delta_{RG}\Delta_{31}\):
\[
\delta_{RG}\Delta_{31} = -\left(1 - \frac{1}{F}\right)\Delta_{31}^{(0)} .
\] (3.18)

This tuning can also be seen by equations similar to (3.8) and (3.9) which in this particular case read
\[
\log \frac{M}{M_Z} \simeq \frac{\Delta_{31}^{(0)}}{\epsilon_r} \cos 2\theta_1^{(0)}, \quad \omega_1 \simeq \frac{\Delta_{31}^{(0)}}{\epsilon_r} \sin 2\theta_1^{(0)} .
\] (3.19)

Beside this, another problem affects the running of \(\theta_3\). The RGE of \(\theta_3\) is given by (see eq. (A.19))
\[
\frac{d\theta_3}{dt} \simeq \frac{1}{2} (\epsilon_r \sin^2 \theta_1) \nabla_{21} \sin 2\theta_3 ,
\] (3.20)

which is very similar to a two-flavour RGE except for the extra factor \(\sin^2 \theta_1\). Clearly, \(\sin 2\theta_3\) must be initially small. Otherwise, since \(|\nabla_{21}| \gg |\nabla_{31}|\), for moderate values of \(\sin^2 \theta_1, \theta_3\) runs much more quickly than \(\theta_1\) and therefore it is driven to the IRFP (with small \(\sin 2\theta_3\)) before \(\theta_1\) gets properly amplified. Consequently, \(\theta_3\) needs radiative amplification and this requires its own fine-tuning:
\[
\log \frac{M}{Q_{3,\text{max}}} \simeq \frac{\Delta_{31}^{(0)}}{\epsilon_r \sin^2 \theta_1^{(0)}} \cos 2\theta_3^{(0)}, \quad \omega_3 \simeq \frac{\Delta_{21}^{(0)}}{\epsilon_r \sin^2 \theta_1^{(0)}(Q_{3,\text{max}})} \sin 2\theta_3^{(0)} .
\] (3.21)

Notice that for estimating the position of the peak in the running of \(\sin^2 2\theta_3\) we have simply used the initial value of \(\sin^2 \theta_1\) as if it would not run, while for estimating the half-width, \(\omega_3\), a better choice is to use \(\sin^2 \theta_1\) at the peak \(Q_{3,\text{max}}\), which is assumed to be around \(M_Z\). This means in particular that \(\omega_3\) is not enlarged significantly by the initial smallness of \(\sin^2 \theta_1\) and, being controlled by \(\Delta_{21}^{(0)}\), it is in general even smaller than \(\omega_1\). This behaviour is shown in figure 5 where the solid lines give \(\sin^2 2\theta_1\) and the dashed ones \(\sin^2 2\theta_3\). The three different pairs of curves correspond to different initial conditions for \(\sin^2 2\theta_1\), with the \(\Delta_{31}^{(0)}\) mass splitting chosen so as to get maximal atmospheric mixing at \(M_Z\). As expected from eq. (3.21), when \(\sin^2 2\theta_1^{(0)}\) decreases, the narrow peak in \(\sin^2 2\theta_3\) moves rapidly to lower scales, making clear the need for an extra fine-tuning to ensure a large solar mixing angle at \(M_Z\).

4. Amplification of mixing angles: Kähler masses

Let us consider now the possibility of radiative amplification of mixing angles in the scenarios described in section 2.2, i.e. when neutrino masses in a supersymmetric model originate from non-renormalizable operators in the Kähler potential [46]. More precisely, we consider
only the operator $\kappa (L \cdot H_2)(L \cdot \overline{H}_1)$ as discussed in section 2.2. Then the RG-evolution of mixing angles is described by an equation of the usual form:

$$\frac{dV}{dt} = VT, \quad (4.1)$$

with the matrix $T$ given by (see appendix A):

$$T_{ij} = \frac{-1}{16\pi^2} (y_a^2 - y_b^2) \frac{V_{ai} V_{\beta j}}{m_i - m_j} m^A_{\alpha \beta}, \quad (4.2)$$

for $i \neq j$ and $T_{ii} = 0$. Here $m^A_{\alpha \beta} \equiv (M^A_\nu)_{\alpha \beta}$, where the matrix $M^A_\nu$ is related to the Kähler matrix $\kappa_A$ by $M^A_\nu \equiv \mu \kappa_A v^2 / M^2$. The neutrino mass eigenvalues run according to (no sum in $i$)

$$\frac{dm_i}{dt} = u m_i + \frac{y_a^2}{8\pi^2} V_{ai} m^A_{\alpha \beta} V_{\beta i}. \quad (4.3)$$

The generic condition required to have a significant change in the mixing angles is to have some sizeable $T_{ij}$, i.e.

$$\frac{y_a^2}{16\pi^2} \frac{m^A_{\alpha \beta}}{m_i - m_j} \log \frac{M}{M_Z} \lesssim \mathcal{O}(1). \quad (4.4)$$

This is consistent with the general arguments of subsection 2.3: notice from (4.3) that the relative splitting, $\Delta_{ij}$, typically gets a correction

$$\delta_{RG} \Delta_{ij} \sim \mathcal{O} \left( \frac{y_a^2}{16\pi^2} \frac{m^A_{\alpha \beta}}{m_i + m_j} \log \frac{M}{M_Z} \right). \quad (4.5)$$

Therefore, important effects in the mixing angles, eq. (4.4), occur when the RG-induced (relative) splittings are comparable (or larger) than the initial splitting ($\delta_{RG} \Delta_{ij} \simeq \Delta_{ij}$).

Comparing eqs. (4.2), (4.4) with the conventional $T_{ij}$, eq. (A.15), we see that it is possible to have now large effects even for neutrino spectra without quasi-degenerate masses,
provided the magnitude of the entries of $M_A$ is larger than the mass differences $m_i - m_j$. This implies that, in the new scenario, amplification of mixing angles is a more general phenomenon, which can occur also for spectra that cannot accommodate amplification in conventional models, e.g. normal hierarchy or inverted hierarchy with $m_1 \simeq -m_2$.

In parallel with the discussion of the conventional case (section 3) we consider in turn the amplification for a two-flavour case, for the solar angle and for the atmospheric angle.

4.1 Two-flavour scenario

As for the conventional case, the two-flavour model is very useful to understand in a simple setting the main features of the RG evolution of mixings and mass splittings, and the form of the infrared fixed points. In this new scenario the evolution of the mixing angle in a two flavour case does not follow an RGE of the form (3.2) but rather the following (no sum in $\alpha, \beta$):

$$\frac{d\theta}{dt} = -\epsilon_{\alpha\beta} \frac{m^A_{\alpha\beta}}{m_1 - m_2} \cos 2\theta,$$

where again $\epsilon_{\alpha\beta} \equiv (y^2_\alpha - y^2_\beta)/(16\pi^2)$ with $\alpha$ and $\beta$ the two flavours and $V_{\alpha2} \equiv \sin \theta$. Comparing eq. (4.6) to eq. (3.2) (i.e. the one for the conventional SM and MSSM cases) we note, beside numerical factors, two main differences: the quantity $\nabla_{12} = (m_1 + m_2)/(m_1 - m_2)$ is replaced by the ratio $m^A_{\alpha\beta}/(m_1 - m_2)$, and $\sin 2\theta$ by $\cos 2\theta$. The first difference implies that important changes on $\theta$ are achieved for $|m^A_{\alpha\beta}| \gg |m_1 - m_2|$, which does not require quasi-degeneracy of the eigenvalues. The second one implies that the running will drive $\cos 2\theta$ (rather than $\sin 2\theta$) to zero, i.e. IRFPs for $\theta$ are now at maximal mixing, $\theta^* = \pm \pi/4$ (dashed lines in figure 3). Therefore, $\theta$ will be amplified towards maximal

![Figure 6: Running of $\sin^2 2\theta$ from $M$ down to $M_Z$ in conventional scenarios (dashed lines) as compared with the Kähler case (solid lines). In each case, different lines correspond to different mass splittings at $M$.](image-url)
mixing in the infrared\footnote{Again, for the solar angle, one should have $\tan^2 \theta_s(M_Z) < 1$ (with eigenvalues labelled such that $|m_1| < |m_2|$ holds at low energy), as needed for the MSW solution. [3].} irrespective of the sign of $d\theta/dt$ at $M$ and with no fine-tuning required on $m_1 - m_2$.

For comparison with the conventional case, we have plotted in figure 6 the new evolution of $\sin^2 2\theta(Q)$ in solid lines, for different initial mass splittings; the dashed lines correspond to the running in the conventional scenario (with the same initial conditions). Clearly, in order to generate large neutrino mixing angles through RG evolution, scenarios that follow eq. (4.6) are more natural than those that follow eq. (3.2). Let us mention that, since $M$ is smaller now, the interval in energy available to amplify $\mu$ is also smaller.

Regarding the mass splittings, the quantity $(m_1 - m_2)/m_{\alpha\beta}^A$, which is now the relevant one, satisfies the RGE (from eqs. (2.9) and (2.10))

\[
\frac{d}{dt} \left( \frac{m_1 - m_2}{m_{\alpha\beta}^A} \right) = -2\epsilon_{\alpha\beta} \sin 2\theta \left( 1 - \frac{3}{4} \left( \frac{m_1 - m_2}{m_{\alpha\beta}^A} \right)^2 \right) - \frac{1}{8\pi^2} (y_\alpha^2 + y_\beta^2) m_1 - m_2 \frac{m_1 - m_2}{m_{\alpha\beta}^A}
\]

\[
\simeq -2\epsilon_{\alpha\beta} \sin 2\theta ,
\]

(4.7)

where, for the last approximation, we have assumed $|m_{\alpha\beta}^A| \gg |m_1 - m_2|$, which is the case of interest. As a consequence, note that

\[
\frac{m_1 - m_2}{m_{\alpha\beta}^A} \cos 2\theta \simeq \text{RG - invariant} ,
\]

(4.8)

in contrast with eq. (4.4).

\subsection*{4.2 Solar angle}

In the new scenario many of the requirements for amplification of the solar angle are the same as those in conventional cases: the RGE of $V$ should be dominated by $T_{21}$; $V_{\alpha3}$ does not run appreciably and should be fixed as an initial condition to be $\simeq (0, 1/\sqrt{2}, 1/\sqrt{2})^T$ while $\theta_3$ is the only angle that changes significantly. The difference with respect to the standard case is that $\theta_3$ is now driven towards a different IRFP, determined by the condition $T_{21} \to 0$. In terms of $\theta_3$ and neglecting all leptonic Yukawa couplings other than $y_\tau$ this IRFP condition reads

\[
\tan 2\theta_3 \to \frac{2 \sin \theta_2 \cos 2\theta_1 m_{\mu\tau}^A + 2 \sin \theta_1 \cos \theta_2 m_{e\tau}^A}{\sin 2\theta_1 (1 + \sin^2 \theta_2) m_{\mu\tau}^A - \cos \theta_1 \sin 2\theta_2 m_{e\tau}^A} .
\]

(4.9)

As this equation clearly shows, the IR behaviour of the running $\theta_3$ does not correspond in general to that expected in a two flavour case (discussed in the previous subsection) but is richer. Several cases of interest are the following:

- If we make the approximation $\sin \theta_2 \simeq 0$, $\theta_3$ evolves towards the IRFP $\tan 2\theta_3 \simeq m_{e\tau}^A/|\cos \theta_1 m_{\mu\tau}^A|$ and, for $m_{e\tau}^A/m_{\mu\tau}^A \sim O(1)$ one can easily get $\theta_3^* \in [0, \pi/2]$ inside the experimental range.
If, in the previous case with \( \sin \theta_2 \simeq 0 \) one has \( m_{\nu r}^A \gg m_{\mu r}^A \) (the precise condition is \( s_{2\theta_2} \ll m_{\nu r}^A / m_{\mu r}^A \ll 1 \)) the IRFP for \( \theta_3 \) is at maximal mixing. This case is indeed well described by a two-flavour approximation and can be acceptable if the running interval is not too long so that the low-energy value of \( \theta_3 \) is not too close to the IRFP.

If \( m_{\mu r}^A \simeq 0 \) (or, more precisely, \( m_{\mu r}^A \ll s_{2\theta_2} m_{\mu r}^A \)), the IRFP is simply \( \tan 2\mu_3 \approx -\tan \mu_1 / \sin \theta_2 \). That is, this scenario predicts a correlation between the neutrino mixing angles such that, given the experimental interval for \( \mu_3 \), the angle \( \mu_2 \) is predicted to be in the range \( 0.02 \lesssim \sin^2 \theta_2 \lesssim 0.5 \) (for \( \tan \theta_1 = 1 \)). In other words, for \( \theta_3 \) in the upper region of its experimentally allowed range, \( \theta_2 \) lies not far below the CHOOZ bound.

To end the discussion of the solar case one should also check that the requirement of phenomenological low-energy mass splittings is not in conflict with the requirements just described needed to amplify the solar angle. The solar mass splitting, in one-loop leading-log approximation,\(^{12} \) is now given by

\[
\Delta m_{\text{sol}}^2 \simeq \Delta m_{31}^2(M) \left[ 1 - 2u \log \frac{M}{M_Z} \right] + \frac{y_e^2}{4\pi^2} \left[(V_{e2}V_{\tau2}m_2 - V_{e1}V_{\tau1}m_1)m_{\nu r}^A + (V_{\mu 2}V_{\tau2}m_2 - V_{\mu 1}V_{\tau1}m_1)m_{\mu r}^A \right] \log \frac{M}{M_Z}. \tag{4.10}
\]

If the running is long enough to approach the IRFP, in which case the radiative correction of the splitting overcomes the initial value at the \( M \)-scale (see the discussion around eq. (4.4)), the above result gets simplified to

\[
\Delta m_{\text{sol}}^2 \simeq (m_1 + m_2)(M_Z) \frac{y_e^2}{8\pi^2} \left[(V_{e2}V_{\tau2} - V_{e1}V_{\tau1})m_{\nu r}^A + (V_{\mu 2}V_{\tau2} - V_{\mu 1}V_{\tau1})m_{\mu r}^A \right] \log \frac{M}{M_Z}. \tag{4.11}
\]

The most important aspect of this formula is that, in stark contrast to the standard cases of the SM or the MSSM, the radiative correction to the mass splitting is controlled by the elements of the matrix \( M_{\nu r}^A \), which do not contribute at tree level to neutrino masses. It is therefore quite easy to arrange the magnitude of the mass splitting at will, and this without disturbing the IRFP, which does not depend on the overall magnitude of \( M_{\nu r}^A \) but only on the ratio of its elements (see eq. (4.9)). Note also that the global scale of \( M_{\nu r}^A \), which is in principle an unknown in this kind of scenarios, is thus fixed to get the correct mass splitting. In conventional scenarios the latter is adjusted by tuning the initial mass splitting at high energy, as discussed in the previous section.

### 4.3 Atmospheric angle

As in the conventional cases, amplification of the atmospheric angle requires that \( T_{31} \) or \( T_{32} \) dominate the RGEs of the neutrino mixing angles (we will not discuss the case \( T_{31} \simeq T_{32} \) which in this scenario seems an unnatural coincidence in view of eq. (4.2)). Both cases are

\(^{12}\)Two-loop leading-log corrections \( \sim [y_e^2 m_{\nu r}^A \log(M/M_Z)/(16\pi^2)]^2 \) produce a non-vanishing splitting at low energy even if initially \( m_i = 0 \). Such corrections can be easily obtained from eq. (4.3).
in fact very similar, so we consider in some detail only the case of $T_{31}$-dominance and later explain what changes would apply to the other case. Incidentally, notice from (4.2) that a natural way to get $T_{31} \gg T_{21}$ is $|m_2 - m_1| \gg |m_3 - m_1|$, implying $m_1 \approx -m_2 \approx m_3$, as for the conventional case.

If $T_{31}$ dominates the evolution of the angles, the column $V_{\alpha 2}$ does not change much and should be chosen to agree with experimental data. This is imposed by hand as an initial condition, to be explained by physics at higher energy scales. By unitarity, this requirement amounts to only two conditions on the mixing angles. The IRFP is determined as usual by the condition $T_{31} \neq 0$, which leads to the third condition on the mixing angles:

$$ (c_1 c_3 c_2 \theta_2 + s_1 s_2 s_3) m^A_{e \tau} \simeq c_2 (s_3 c_2 \theta_1 + s_2 s_2 \theta_1) m^A_{\mu \tau}. \quad (4.12) $$

This IRFP condition amounts to one prediction for the angles in these scenarios.

One interesting possibility for the IRFP is the following. If $m^A_{e \tau} \ll m^A_{\mu \tau}$, the IRFP (4.12) reads $\tan 2 \theta_1 = -s_3/(s_2 c_3)$. This correlation between angles relates the smallness of the CHOOZ angle to the maximality of the atmospheric one and can be taken as an interesting prediction of this particular scenario.

The case of $T_{32}$-dominance is obtained from the previous one by the replacement $c_3 \to s_3$ and $s_3 \to -c_3$ (which interchanges the first and second columns of $V$). The IRFP condition is then

$$ (c_1 s_3 c_2 \theta_2 + s_1 s_2 s_3) m^A_{e \tau} \simeq c_2 (-c_3 c_2 \theta_1 + s_2 s_2 \theta_1) m^A_{\mu \tau}, \quad (4.13) $$

and all implications that follow are quite similar to the ones discussed above.

As for the solar case, provided the running reaches the IRFP, the low-energy mass splitting $\Delta m^2_{13}$ is of purely radiative origin and given by an expression similar to (4.11). Again the global magnitude of $M^A_{\nu}$ can be chosen to reproduce the atmospheric splitting, without modifying the previous discussion of IRFPs for the atmospheric angle, which are just controlled by the ratios of $M^A_{\nu}$ entries (see eqs. (4.12), (4.13)). Furthermore, in contrast with the conventional cases discussed in section 3.3, this is achieved without upsetting the solar mass splitting. To see this notice that $m_1 \approx -m_2 \approx \pm m_3$ implies that the radiative correction to the solar splitting, still given by eq. (11), is now of the order $\delta_{RG} \Delta m^2_{sol} \sim \Delta m^2_{atm} \Delta m^2_{sol}/m^2$.

5. Conclusions

The nearly bi-maximal structure of the neutrino mixing matrix, $V_{PMNS}$, is very different from that of $V_{CKM}$ in the quark sector, where all the mixings are small. An attractive possibility to understand this is that some neutrino mixings are radiatively enhanced, i.e. are initially small and get large in the RG running from high to low energy.

In this paper we have carefully examined this issue in two different contexts: conventional scenarios, in particular the SM and the MSSM, and supersymmetric scenarios in which neutrino masses originate in the Kähler potential. The RGEs are quite different in each case, and so are the results and conclusions. Our analysis is complete in the sense
that we have taken into account all neutrino parameters, to ensure that not only mixing angles but also mass splittings agree with experiment at low energy. Moreover we have investigated which scenarios require a fine-tuning in order to achieve the amplification, and quantified it. In order to explain in an intuitive way the main issues involved in the running (appearance of infrared fixed points, interplay between the running of the angles and that of the mass splittings, origin of the fine-tunings, etc.) we have first studied the two-flavour scenario, where all these features show up in a transparent way. Then, we have explored the physical cases exhaustively. Our main results and conclusions are the following.

- In the SM it is not possible to amplify either the solar or the atmospheric mixings, even with fine-tunings. Simply, the radiative effects that modify the mixings (which are proportional to the tau Yukawa coupling squared, $y^2_\tau$) are not large enough to do the job for the currently preferred range of masses ($m \leq 0.3$ eV). For the same reason, the mass splittings cannot have a radiative origin.

- For the MSSM the amplification is possible but only when (at least two) neutrinos are quasi-degenerate (in absolute value and sign), and always involves a fine-tuning between the initial mass splitting (solar or atmospheric) and its radiative correction: two physically unrelated quantities that are required to be close to each other. The magnitude of this fine-tuning is essentially the amplification factor achieved. Moreover, a precise value of $\tan \beta$ (very high for the atmospheric case) is required. The amplification of the atmospheric angle requires an additional and even more important fine-tuning, since the solar splitting gets a radiative correction of “atmospheric” size which should be compensated by an ad-hoc initial condition. Finally, simultaneous amplification of solar and atmospheric angles is possible but it is extremely fine-tuned.

All these problems come from the fact that in the SM and the MSSM the mixing matrix (when there is some initial quasi-degeneracy) approaches an infrared pseudo-fixed-point (IRFP) which implies a physically unacceptable mixing (solar or atmospheric). Therefore, parameters should be delicately chosen for the running to stop before reaching the IRFP.

- Things are much better for the scenario of neutrino masses arising from the Kähler potential. First of all, the infrared fixed points correspond here to maximal or quasi-maximal mixings, so there is no need of fine-tuning in order to amplify angles. This can work for both the solar and the atmospheric angles.

Moreover, the amplification mechanism can work even if the mass eigenvalues are not quasi-degenerate. The reason is that the strength of the RG effect is proportional to $m^A/(m_i - m_j)$, where $m^A$ is the scale of a coupling that arises from the Kähler potential and $m_i$ are the mass eigenvalues (in conventional cases the strength of the effect is proportional to $(m_i + m_j)/(m_i - m_j)$). So it is enough to have $m^A \gg m_i - m_j$ to get important radiative effects, thus reaching the IRFP. On the other hand, the presence of $m^A$ introduces an additional uncertainty, which is however removed taking
into account that here the low-energy splitting, $\Delta m^2_{ij}$, is essentially a pure radiative effect, whose magnitude can be adjusted with the value of $m^4$ without modifying the form of the IRFP.

We find very encouraging that the scenario of neutrino masses from the Kähler potential, which is attractive for different reasons (e.g. it implies that the scale of lepton number violation is much closer to the electroweak scale than in conventional scenarios) has this remarkable and nice behaviour regarding radiative corrections.

To conclude, two more comments are in order. First, radiative effects play a relevant role in neutrino physics that often cannot be ignored. e.g. in view of the scarce success of radiative amplification in the MSSM, one might think that radiative effects are not relevant for model building in that framework. However, especially for scenarios involving some quasi-degeneracy, radiative effects can have the (negative) effect of destabilizing the high-energy pattern of mixing angles and mass splittings. The formulae presented in this paper are useful to analyze these effects. Second, the radiative corrections studied in this paper are model-independent since they concern the running from the $M$-scale (the scale where the new physics that violates L appears) down to the electroweak scale and this running is determined by the effective theory valid in that range (the SM or the MSSM with a non-renormalizable operator responsible for neutrino masses). Besides these corrections there are others arising from physics beyond $M$,\(^\text{13}\) but they are much more model-dependent. Their role is to set the initial conditions at $M$ for the model-independent radiative effects analyzed here.

A. RGEs for neutrino physical parameters

The energy-scale evolution of a $3 \times 3$ neutrino (Majorana) mass matrix $\mathcal{M}_\nu$ is generically described by a RGE of the form $(t = \log Q)$:

$$
\frac{d\mathcal{M}_\nu}{dt} = -\left( u \mathcal{M}_\nu + P \mathcal{M}_\nu + \mathcal{M}_\nu P^T \right),
$$

(A.1)

In (A.1), $u$ is a number, so $u \mathcal{M}_\nu$ gives a family-universal scaling of $\mathcal{M}_\nu$ which does not affect its texture, while $P$ is a matrix in flavour space thus producing a non family-universal correction (the most interesting effect).

As explained in \([55, \text{30}]\) one can extract from (A.1) the RGEs for the physical neutrino parameters: the mass eigenvalues, the mixing angles and the CP phases.\(^\text{14}\) In this paper we focus for simplicity on real cases, with no phases. (General formulas for the case with all phases can be found in \([\text{30}]\).) Using the parametrization and conventions

\(^{13}\)E.g. threshold corrections at $M$ or corrections from the running between a fundamental scale, say $M_P$ or $M_{\text{GUT}}$, and $M$. In the see-saw model this corresponds to the running of the neutrino Yukawa couplings, $Y_\nu$, and the right-handed neutrino masses between $M_P$ and $M$. These effects can be quite important (for the see-saw they depends on the magnitude of $Y_\nu$ and have been analyzed elsewhere \([8, \text{49}]\).

\(^{14}\)The RGEs for the CKM matrix and quark masses were derived along similar lines in \([\text{83}]\).
of the Introduction, we get the following RGEs for the mass eigenvalues and the PMNS matrix

\[
\frac{dm_i}{dt} = -um_i - 2m_i\hat{P}_{ii}, \quad (A.2)
\]

\[
\frac{dV}{dt} = V T, \quad (A.3)
\]

We have defined

\[
\hat{P} = \frac{1}{2} V^T (P + P^T)V, \quad (A.4)
\]

while \(T\) is a \(3 \times 3\) matrix (anti-hermitian, so that the unitarity of \(V\) is preserved by the RG running) with

\[
T_{ii} = i\hat{Q}_{ii}, \quad T_{ij} = \nabla_{ij}\hat{P}_{ij} + i\hat{Q}_{ij}, \quad i \neq j, \quad (A.5)
\]

where

\[
\hat{Q} = i\frac{1}{2} V^T (P - P^T)V, \quad (A.6)
\]

and

\[
\nabla_{ij} = \frac{m_i + m_j}{m_i - m_j}. \quad (A.7)
\]

Note that the RGE for \(V\) does not depend on the universal factor \(u\), as expected.

From eqs. (A.3)-(A.7), we can derive the general RGEs for the mixing angles:

\[
\frac{d\theta_1}{dt} = \frac{1}{c_2} (s_3 T_{31} - c_3 T_{32}), \quad (A.8)
\]

\[
\frac{d\theta_2}{dt} = - (c_3 T_{31} + s_3 T_{32}), \quad (A.9)
\]

\[
\frac{d\theta_3}{dt} = -\frac{1}{c_2} T_{21} + \frac{s_2}{c_2} (s_3 T_{31} - c_3 T_{32}). \quad (A.10)
\]

In the next subsections we particularize the generic formulas above to scenarios of interest: first to the Standard Model and the MSSM and then to models with more sources of lepton number violation, among them supersymmetric scenarios with neutrino masses that are generated from the Kähler potential.

A.1 Conventional SM and MSSM

In the SM or the MSSM the RGE for the neutrino mass matrix (2.3) is of the form (A.1). The evolution of neutrino masses is then given by (no sum in \(i\))

\[
\frac{dm_i}{dt} = -u_M m_i - 2m_i c_M \hat{P}_{Eii}, \quad (A.11)
\]

where \(\hat{P}_E = V^T P_E V\) and \(P_E \equiv Y_E Y_E^d/(16\pi^2)\) with \(Y_E\) the matrix of leptonic Yukawa couplings, which can be well approximated by \(Y_E \simeq \text{diag}(0, 0, y_f)\) so that \(16\pi^2 \hat{P}_{Eij} \simeq y_f^2 V_{\tau i} V_{\tau j}\). The model-dependent quantities \(u_M\) and \(c_M\) are as follows [24]-[27]. In the SM

\[
u_M \simeq \frac{1}{16\pi^2} [3g_2^2 - 2\lambda - 6h_\tau^2], \quad (A.12)
\]
where $g_2$, $\lambda$, $h_t$ are the SU(2) gauge coupling, the quartic Higgs coupling and the top-Yukawa coupling (leptonic Yukawa couplings can be safely neglected here), while in the MSSM

$$u_M = \frac{1}{16\pi^2} \left[ \frac{3}{4} g_1^2 + 6 g_2^2 - 6 \frac{h_t^2}{\sin^2 \beta} \right], \tag{A.13}$$

where $g_1$ is the U(1) gauge coupling and $\tan \beta$ is the ratio of the vevs of the two supersymmetric Higgses. Finally,

$$c_M = \frac{3}{2} \quad \text{(SM)}, \quad c_M = -1 \quad \text{(MSSM)}. \tag{A.14}$$

The RGE for the mixing matrix is of the form (A.3) with

$$T_{ii} = 0, \quad T_{ij} = c_M \frac{\sin \bar{\theta}}{\sin^2 \beta} P_{Eij}, \quad i \neq j, \tag{A.15}$$

with

$$\epsilon_r \equiv c_M \frac{g_2^2}{16\pi^2}, \tag{A.16}$$

where electron and muon Yukawa couplings have been neglected. The running of the mixing angles, eqs. (A.8)–(A.9), is now given by ($s_i \equiv \sin \theta_i$, etc)

$$\frac{d\theta_1}{dt} = -\epsilon_1 c_1 (-s_3 V_{\tau 1} \bar{\nabla}_{31} + c_3 V_{\tau 2} \bar{\nabla}_{32}), \tag{A.17}$$

$$\frac{d\theta_2}{dt} = -\epsilon_2 c_1 c_2 (c_3 V_{\tau 1} \bar{\nabla}_{31} + s_3 V_{\tau 2} \bar{\nabla}_{32}), \tag{A.18}$$

$$\frac{d\theta_3}{dt} = -\epsilon_r (c_1 s_2 s_3 V_{\tau 1} \bar{\nabla}_{31} - c_1 s_2 c_3 V_{\tau 2} \bar{\nabla}_{32} + V_{\tau 1} V_{\tau 2} \bar{\nabla}_{21}). \tag{A.19}$$

A.2 More general models

Consider now a $3 \times 3$ neutrino mass matrix $\mathcal{M}_\nu$ that evolves with energy following a RGE of the form

$$\frac{d\mathcal{M}_\nu}{dt} = -(u_M \mathcal{M}_\nu + c_M P_E \mathcal{M}_\nu + c_M \mathcal{M}_\nu P_E^T) + P_E \mathcal{M}_\nu A - \mathcal{M}_\nu A^T P_E^T, \tag{A.20}$$

where, besides the usual terms there are new contributions coming from $M^A_\nu$, a $3 \times 3$ antisymmetric matrix that arises from lepton-violating physics but is not directly related to neutrino masses. Particular examples of RGEs of this form have been considered in the literature \[40, 46, 61\].

The explicit RGEs for neutrino mass eigenvalues, $m_i$, and mixing matrix, $V_{\alpha i}$, presented above for generic models with no $M^A_\nu$-terms (eqs. (A.2) and (A.3)) can be immediately extended also to eq. (A.20) simply making the substitution

$$P \rightarrow c_M P_E - P_E M^A_\nu M^{-1}_\nu, \tag{A.21}$$

which transforms (A.1) into (A.20). More explicitly, we obtain (no sum in $i$)

$$\frac{dm_i}{dt} = -u_M m_i - \frac{g_2^2}{8\pi^2} \left[ c_M m_i V_{\alpha i}^2 - V_{\alpha i} M^A_\nu V_{\beta i} \right], \tag{A.22}$$
and the usual
\[ \frac{dV}{dt} = VT, \] (A.23)
with \( T \) given by
\[ T_{ii} = 0, \]
\[ 16\pi^2 T_{ij} = c_M y_i^2 V_{\alpha i} V_{\alpha j} \nabla_{ij} - (y_\alpha^2 - y_\beta^2) \frac{V_{\alpha i} V_{\beta j}}{m_i - m_j} m_{\alpha \beta}^A. \] (A.24)

The explicit RGEs for the mixing angles, eqs. (A.8)–(A.10), are also valid in this case upon substitution of \( T_{ij} \) as above.

If we particularize these results to a case with just 2 flavors \((\alpha, \beta)\), with mass eigenvalues \(m_{1,2}\) and a single mixing angle \(\theta\) \((V_{\alpha 2} \equiv \sin \theta)\) we obtain the RGE (no sum in \(\alpha, \beta\))
\[ 16\pi^2 \frac{d\theta}{dt} = \left( \frac{y_\alpha^2 - y_\beta^2}{m_1 - m_2} \right) \left( \frac{c_M}{2} (m_1 + m_2) \sin 2\theta - m_{\alpha \beta}^A \cos 2\theta \right). \] (A.25)

From this equation we deduce the following. When \(M_\nu^A\) is absent, there is an infrared pseudo-fixed point at \(\theta^* = 0, \pi/2\) in the evolution of \(\theta\). This is the case in most scenarios usually considered, for example the SM and the MSSM (with neutrino masses obtained from the superpotential). On the other extreme case, with \(c_M=0\) and \(M_\nu^A\) present, the pseudo-fixed point is at maximal mixing, \(\theta^* = \pi/4\). The only instance that we know of such case is the MSSM with neutrino masses obtained through the Kähler potential [46] that we have considered in this paper. In cases with both \(c_M\) and \(M_\nu^A\) non-zero, the pseudo-fixed point is at
\[ \tan 2\theta^* = \frac{2 c_M m_{\alpha \beta}^A}{m_1 + m_2}. \] (A.26)

This latter case can be realized for instance in non-supersymmetric two Higgs doublet models and has been studied previously in ref. [41]. Note however that the analysis of fixed points in that reference is different from ours: in [41] it is assumed that neutrino masses also reach their fixed points, which is not usually the case in most examples of phenomenological interest. On the other hand, mixing angles quickly evolve to the fixed point if there is quasi-degeneracy of neutrino masses (with same sign masses, see e.g. the extended discussion in [30]).

\section{B. RGEs for couplings in the Kähler potential}

For a tree-level Kähler potential
\[ K = \sum_a |\phi_a|^2 + \frac{1}{2M} \left[ \kappa_{ab} \phi_a \phi_b \phi_c + h.c. \right] + \frac{1}{4M^2} \kappa_{cdef} \phi_a \phi_b \phi_c \phi_d + \frac{1}{6M^2} \left[ \kappa_{abcd} \phi_a \phi_b \phi_c \phi_d + h.c. \right], \] (B.1)
(with \(\phi^a \equiv \phi_a^*\)) and a superpotential that includes the Yukawa couplings
\[ W = \cdots + \frac{1}{3!} Y^{abc} \phi_a \phi_b \phi_c, \] (B.2)
one gets, from the condition $dK/dt = 0$ and using the one-loop corrected expression for non-renormalizable Kähler potentials computed in ref. [10], the RGEs

$$16 \pi^2 \frac{d\kappa^{ab}_{d}}{dt} = \left[ \frac{1}{2} \kappa^{ab}_{x} Y^{bmn} Y_{xmn} + 2 \kappa^{ab}_{y} Y^{mbn} Y_{xmc} + (b \leftrightarrow a) \right] + \frac{1}{2} \kappa^{ab}_{x} Y^{zmn} Y_{cmn} - 2g_{A}^{2} \kappa^{ab}_{c} \left[ C_{2}^{A}(a) + C_{2}^{A}(b) - C_{2}^{A}(c) \right],$$

$$16 \pi^2 \frac{d\kappa^{abc}_{d}}{dt} = \left[ \frac{1}{2} \kappa^{abc}_{d} Y^{cmm} Y_{xmn} + 2 \kappa^{abc}_{z} Y^{yzm} Y_{xmd} - \frac{4}{3} \kappa^{abc}_{y} Y^{yxx} Y_{nmn} - \frac{4}{3} \kappa^{abc}_{y} Y^{yxx} Y_{nmn} + (bca) + (cab) \right] + \frac{1}{2} \kappa^{abc}_{x} Y^{xmn} Y_{dnn} - 2g_{A}^{2} \kappa^{abc}_{c} \left[ C_{2}^{A}(a) + C_{2}^{A}(b) + C_{2}^{A}(c) - C_{2}^{A}(d) \right],$$

$$16 \pi^2 \frac{d\kappa^{abc}_{d}}{dt} = \frac{1}{2} \left[ \kappa^{abc}_{d} Y^{bmn} Y_{xmn} + (a \leftrightarrow b) \right] + \frac{1}{2} \left[ \kappa^{abc}_{d} Y^{zmn} Y_{cmn} + (c \leftrightarrow d) \right] + 2 \left[ \left[ \kappa^{abc}_{d} Y^{bmn} Y_{xmn} - \kappa^{abc}_{d} Y^{bmn} Y_{xmn} - \kappa^{abc}_{d} Y^{bmn} Y_{xmn} - \kappa^{abc}_{d} Y^{bmn} Y_{xmn} - \kappa^{abc}_{d} Y^{bmn} Y_{xmn} + (a \leftrightarrow b) \right] + \frac{1}{2} \left[ \left[ \kappa^{abc}_{d} Y^{abc} Y_{xmn} + \kappa^{abc}_{d} Y_{xmn} + \kappa^{abc}_{d} Y_{xmn} + \kappa^{abc}_{d} Y_{xmn} + \kappa^{abc}_{d} Y_{xmn} + (c \leftrightarrow d) \right] + 4g_{A}^{2} \left[ \kappa^{abc}_{d} (T_{Aa})^{x}_{c}(T_{Aa})^{y}_{d} + \kappa^{abc}_{d} (T_{Aa})^{a}_{x}(T_{Aa})^{b}_{y} \right],

where $C_{2}^{A}(a)$ is the quadratic Casimir invariant of the chiral field $\phi_{a}$ for the gauge group labelled by $A$ (with gauge coupling $g_{A}$), and the $T_{Aa}$'s are the group generators (with $A$ labelling the group and $a$ the generator). The diagrammatic techniques of [10] were useful to compute the non-abelian gauge contributions to these equations. If the superpotential contains mass terms $\mu^{ab}_{a} \phi_{a} \phi_{b}$ with $\mu^{ab} \sim O(1)$, the previous RGEs would receive additional contributions (e.g., terms like $\kappa^{abc}_{x} \kappa^{abc}_{y} \mu_{y} Y^{zyb}$ in the RGE of $\kappa^{abc}_{x}$) proportional to $\mu/M$. We neglect them on the assumption that $\mu/M \ll 1$.

References

[1] B. Pontecorvo, Neutrino experiments and the question of leptonic-charge conservation, Sov. Phys. JETP 26 (1968) 984 [Zh. Eksp. Teor. Fiz. 53 (1967) 1713]; Z. Maki, M. Nakagawa and S. Sakata, Remarks on the unified model of elementary particles, Prog. Theor. Phys. 28 (1962) 870.

[2] LSND collaboration, C. Athanassopoulos et al., Evidence for $\bar{\nu}_{e} \rightarrow \bar{\nu}_{e}$ oscillation from the LSND experiment at the Los Alamos meson physics facility, Phys. Rev. Lett. 77 (1996) 3082 [nucl-ex/9605003]; Evidence for $\nu_{\mu} \rightarrow \nu_{e}$ oscillations from pion decay in flight neutrinos, Phys. Rev. D 58 (1998) 2499 [nucl-ex/9709006]; Evidence for $\nu_{\mu} \rightarrow \nu_{e}$ neutrino oscillations from LSND, Phys. Rev. Lett. 81 (1998) 1774 [nucl-ex/9709006].

[3] S. Weinberg, Baryon and lepton nonconserving processes, Phys. Rev. Lett. 43 (1979) 1566.

[4] T. Yanagida, in proceedings of Workshop on unified theories and baryon number in the universe, Tsukuba, Japan 1979 O. Sawada and A. Sugamoto eds., KEK Report No. 79-18, Tsukuba; S.L. Glashow, in proceedings of Quarks and leptons, Cargèse 1979 M. Lévy et al. eds., Plenum, New York;
M. Gell-Mann, P. Ramond and R. Slansky, in proceedings of *Supergravity Stony Brook* workshop, New York, 1979, P. Van Nieuwenhuizen and D. Freedman eds., North-Holland, Amsterdam;

R.N. Mohapatra and G. Senjanovic, *Neutrino mass and spontaneous parity nonconservation*, *Phys. Rev. Lett.* **44** (1980) 912.

[5] L. Wolfenstein, *Neutrino oscillations in matter*, *Phys. Rev. D* **17** (1978) 2369; S.P. Mikheev and A.Y. Smirnov, *Resonance enhancement of oscillations in matter and solar neutrino spectroscopy*, *Sov. J. Nucl. Phys.* **42** (1985) 913; *Yad. Fiz.* **42** (1985) 144; S.P. Mikheev and A.Y. Smirnov, *Resonant amplification of neutrino oscillations in matter and solar neutrino spectroscopy*, *Nuovo Cim. C* **9** (1986) 17.

[6] M.C. González-García and C. Peña-Garay, *Three-neutrino mixing after the first results from K2K and KamLAND*, [hep-ph/0306001](http://arxiv.org/abs/hep-ph/0306001).

[7] K2K collaboration, M.H. Ahn et al., *Indications of neutrino oscillation in a 250 km long-baseline experiment*, *Phys. Rev. Lett.* **90** (2003) 041801 [hep-ex/0212007](http://arxiv.org/abs/hep-ex/0212007).

[8] CHOOZ collaboration, M. Apollonio et al., *Limits on neutrino oscillations from the CHOOZ experiment*, *Phys. Lett. B* **466** (1999) 413 [hep-ex/9907037](http://arxiv.org/abs/hep-ex/9907037).

[9] KamLAND collaboration, K. Eguchi et al., *First results from KamLAND: evidence for reactor anti-neutrino disappearance*, *Phys. Rev. Lett.* **90** (2003) 021802 [hep-ex/0212021](http://arxiv.org/abs/hep-ex/0212021).

[10] Super-Kamiokande collaboration, Y. Fukuda et al., *Measurement of a small atmospheric $\nu_e/\nu_x$ ratio*, *Phys. Lett. B* **433** (1998) 9 [hep-ex/9803006](http://arxiv.org/abs/hep-ex/9803006); *Study of the atmospheric neutrino flux in the multi-GeV energy range*, *Phys. Lett. B* **436** (1998) 33 [hep-ex/9805006](http://arxiv.org/abs/hep-ex/9805006); *Neutrino-induced upward stopping muons in super-Kamiokande*, *Phys. Lett. B* **467** (1999) 183 [hep-ex/9908045](http://arxiv.org/abs/hep-ex/9908045); *Measurement of the flux and zenith-angle distribution of upward through-going muons by super-Kamiokande*, *Phys. Rev. Lett.* **82** (1999) 2644 [hep-ex/9812014](http://arxiv.org/abs/hep-ex/9812014).

[11] MACRO collaboration, M. Ambrosio et al., *Matter effects in upward-going muons and sterile neutrino oscillations*, *Phys. Lett. B* **517** (2001) 59 [hep-ex/0106049](http://arxiv.org/abs/hep-ex/0106049); M. Goodman, *Other atmospheric neutrino experiments*, in XXth international conference on neutrino physics and astrophysics, Munich May 2002. [http://neutrino2002.ph.tum.de](http://neutrino2002.ph.tum.de) [hep-ex/0210055](http://arxiv.org/abs/hep-ex/0210055).

[12] SOUDAN-2 collaboration, D.A. Petryt, *Latest results on atmospheric neutrinos from SOUDAN 2*, *Nucl. Phys. 110 (Proc. Suppl.)* (2002) 349.

[13] Kamiokande collaboration, Y. Fukuda et al., *Solar neutrino data covering solar cycle 22*, *Phys. Rev. Lett.* **77** (1996) 1683.

[14] Super-Kamiokande collaboration, S. Fukuda et al., *Determination of solar neutrino oscillation parameters using 1496 days of super-Kamiokande-I data*, *Phys. Lett. B* **539** (2002) 179 [hep-ex/0205079](http://arxiv.org/abs/hep-ex/0205079).

[15] SNO collaboration, Q.R. Ahmad et al., *Measurement of the charged current interactions produced by $^8$B solar neutrinos at the Sudbury Neutrino Observatory*, *Phys. Rev. Lett.* **87** (2001) 071303 [nucl-ex/0106013](http://arxiv.org/abs/nucl-ex/0106013); *Direct evidence for neutrino flavor transformation from neutral-current interactions in the Sudbury Neutrino Observatory*, *Phys. Rev. Lett.* **89** (2002) 011301 [nucl-ex/0204008](http://arxiv.org/abs/nucl-ex/0204008).

[16] B.T. Cleveland et al., *Measurement of the solar electron neutrino flux with the homestake chlorine detector*, *Astrophys. J.* **496** (1998) 505.
[17] SAGE collaboration, J.N. Abdurashitov et al., Measurement of the solar neutrino capture rate by the russian-american Gallium solar neutrino experiment during one half of the 22-year cycle of solar activity, Sov. Phys. JETP 95 (2002) 181  Zh. Eksp. Teor. Fiz. 122 (2002) 211 astro-ph/0204243.

[18] GALLEX collaboration, W. Hampel et al., Gallex solar neutrino observations: results for Gallex IV, Phys. Lett. B 447 (1999) 12.

[19] GNO collaboration, M. Altmann et al., GNO solar neutrino observations: results for GNO I, Phys. Lett. B 490 (2000) 16 hep-ex/0006034; T. Kirsten, Progress in GNO, talk at the XXth international conference on neutrino physics and astrophysics (NU2002), Munich, May 25–30, 2002.

[20] H.V. Klapdor-Kleingrothaus et al., Latest results from the Heidelberg-moscow double-beta-decay experiment, Eur. Phys. J. A12 (2001) 147 hep-ph/0103062.

[21] J. Bonn et al., Limits on neutrino masses from tritium beta decay, Nucl. Phys. 110 (Proc. Suppl.) (2002) 395.

[22] O. Elgaroy et al., A new limit on the total neutrino mass from the 2df galaxy redshift survey, Phys. Rev. Lett. 89 (2002) 061301 astro-ph/0204152.

[23] D.N. Spergel et al., First year wilkinson microwave anisotropy probe (WMAP) observations: determination of cosmological parameters, Astrophys. J. Suppl. 148 (2003) 175 astro-ph/0302209.

[24] P.H. Chankowski and Z. Pluciennik, Renormalization group equations for seesaw neutrino masses, Phys. Lett. B 316 (1993) 312 hep-ph/9306333.

[25] K.S. Babu, C.N. Leung and J. Pantaleone, Renormalization of the neutrino mass operator, Phys. Lett. B 319 (1993) 191 hep-ph/9309223.

[26] S. Antusch, M. Drees, J. Kersten, M. Lindner and M. Ratz, Neutrino mass operator renormalization revisited, Phys. Lett. B 519 (2001) 235 hep-ph/0108008; S. Antusch, M. Drees, J. Kersten, M. Lindner and M. Ratz, Neutrino mass operator renormalization in two Higgs doublet models and the MSSM, Phys. Lett. B 525 (2002) 130 hep-ph/0110366.

[27] P.H. Chankowski and P. Wasowicz, Low energy threshold corrections to neutrino masses and mixing angles, Eur. Phys. J. C 23 (2002) 240 hep-ph/0110237.

[28] J.A. Casas, J.R. Espinosa, A. Ibarra and I. Navarro, Naturalness of nearly degenerate neutrinos, Nucl. Phys. B 556 (1999) 3 hep-ph/9904395; J.A. Casas, J.R. Espinosa, A. Ibarra and I. Navarro, Nearly degenerate neutrinos, supersymmetry and radiative corrections, Nucl. Phys. B 569 (2000) 82 hep-ph/9905381.

[29] J.A. Casas, J.R. Espinosa, A. Ibarra and I. Navarro, Theoretical constraints on the vacuum oscillation solution to the solar neutrino problem, J. High Energy Phys. 09 (1999) 015 hep-ph/9906281; R. Barbieri, G.G. Ross and A. Strumia, Vacuum oscillations of quasi degenerate solar neutrinos, J. High Energy Phys. 10 (1999) 020 hep-ph/9906470.

[30] J.A. Casas, J.R. Espinosa, A. Ibarra and I. Navarro, General RG equations for physical neutrino parameters and their phenomenological implications, Nucl. Phys. B 573 (2000) 652 hep-ph/9910420.
[31] A. Ibarra and I. Navarro, Impact of radiative corrections on sterile neutrino scenarios, J. High Energy Phys. 02 (2000) 031 [hep-ph/9912282].

[32] N. Haba, N. Okamura and M. Sugiura, The renormalization group analysis of the large lepton flavor mixing and the neutrino mass, Prog. Theor. Phys. 103 (2000) 367 [hep-ph/9810471]; N. Haba, Y. Matsui, N. Okamura and M. Sugiura, Energy-scale dependence of the lepton-flavor-mixing matrix, Eur. Phys. J. C 10 (1999) 67 [hep-ph/9904292]; The effect of Majorana phase in degenerate neutrinos, Prog. Theor. Phys. 103 (2000) 145 [hep-ph/9908429]; N. Haba and N. Okamura, Stability of the lepton-flavor mixing matrix against quantum corrections, Eur. Phys. J. C 14 (2000) 347 [hep-ph/9906481]; N. Haba, Y. Matsui and N. Okamura, Analytic solutions to the RG equations of the neutrino physical parameters, Prog. Theor. Phys. 103 (2000) 807 [hep-ph/9911481]; N. Haba, Y. Matsui, N. Okamura and T. Suzuki, Are lepton flavor mixings in the democratic mass matrix stable against quantum corrections?, Phys. Lett. B 489 (2000) 184 [hep-ph/0005064]; N. Haba, Y. Matsui and N. Okamura, The effects of majorana phases in three-generation neutrinos, Eur. Phys. J. C 17 (2000) 513 [hep-ph/0005075].

[33] E. Ma, Stability of neutrino mass degeneracy, J. Phys. G 25 (1999) L97 [hep-ph/9907400]; R. Adhikari, E. Ma and G. Rajasekaran, Three inequivalent mass-degenerate majorana neutrinos and a model of their splitting for neutrino oscillations, Phys. Lett. B 486 (2000) 134 [hep-ph/0004197].

[34] N.N. Singh, Effects of the scale-dependent vacuum expectation values in the renormalization group analysis of neutrino masses, Eur. Phys. J. C 19 (2001) 137 [hep-ph/0009211].

[35] T.K. Kuo, S.-H. Chiu and G.-H. Wu, Neutrino mixing in the seesaw model, Eur. Phys. J. C 21 (2001) 281 [hep-ph/0011053].

[36] P.H. Chankowski, A. Ioannisian, S. Pokorski and J.W.F. Valle, Neutrino unification, Phys. Rev. Lett. 86 (2001) 3488 [hep-ph/0011155].

[37] E.J. Chun, Lepton flavor violation and radiative neutrino masses, Phys. Lett. B 505 (2001) 153 [hep-ph/0101173].

[38] M.-C. Chen and K.T. Mahanthappa, Implications of the renormalization group equations in three neutrino models with two-fold degeneracy, Int. J. Mod. Phys. A 16 (2001) 3923 [hep-ph/0102215].

[39] R. Gonzalez Felipe and F.R. Joaquim, Is right-handed neutrino degeneracy compatible with the solar and atmospheric neutrino data?, J. High Energy Phys. 09 (2001) 015 [hep-ph/0106228].

[40] J.F. Oliver and A. Santamaria, Neutrino masses from operator mixing, Phys. Rev. D 65 (2002) 033003 [hep-ph/0108022].

[41] S. Lavignac, I. Masina and C.A. Savoy, Large solar angle and seesaw mechanism: a bottom-up perspective, Nucl. Phys. B 633 (2002) 139 [hep-ph/0202086].

[42] G. Bhattacharyya, A. Raychaudhuri and A. Sil, Can radiative magnification of mixing angles occur for two zero neutrino mass matrix textures?, Phys. Rev. D 67 (2003) 073004 [hep-ph/0211074].

[43] M.K. Parida, C.R. Das and G. Rajasekaran, Radiative stability of neutrino-mass textures, hep-ph/0203097.
[44] G. Dutta, Can radiative correction cause large neutrino mixing?, [hep-ph/0202097]. Stable bimaximal neutrino mixing pattern, [hep-ph/0203222].

[45] S.-H. Chang and T.K. Kuo, Renormalization invariants of the neutrino mass matrix, [Phys. Rev. D 66 (2002) 111302] [hep-ph/0205147].

[46] J.A. Casas, J.R. Espinosa and I. Navarro, New supersymmetric source of neutrino masses and mixings, [Phys. Rev. Lett. 89 (2002) 161801] [hep-ph/0206276].

[47] A.S. Joshipura, S.D. Rindani and N.N. Singh, Predictive framework with a pair of degenerate neutrinos at a high scale, [Nucl. Phys. B 660 (2003) 362] [hep-ph/0211378]; Phenomenology of pseudo Dirac neutrinos, [Phys. Lett. B 494 (2000) 114] [hep-ph/0007334]; Radiatively generated $\nu_e$ oscillations: general analysis, textures and models, [Phys. Rev. D 67 (2003) 073009] [hep-ph/0211404].

[48] M. Frigerio and A.Y. Smirnov, Radiative corrections to neutrino mass matrix in the Standard Model and beyond, [J. High Energy Phys. 02 (2003) 004] [hep-ph/0212263].

[49] S. Antusch, J. Kersten, M. Lindner and M. Ratz, Neutrino mass matrix running for non-degenerate see-saw scales, [Phys. Lett. B 538 (2002) 87] [hep-ph/0203233]; The lma solution from bimaximal lepton mixing at the GUT scale by renormalization group running, [Phys. Lett. B 544 (2002) 1] [hep-ph/0206078].

[50] S. Antusch, J. Kersten, M. Lindner and M. Ratz, Running neutrino masses, mixings and CP phases: analytical results and phenomenological consequences, [hep-ph/0305273].

[51] M. Tanimoto, Renormalization effect on large neutrino flavor mixing in the minimal supersymmetric Standard Model, [Phys. Lett. B 360 (1995) 41] [hep-ph/9508247].

[52] J.R. Ellis and S. Lola, Can neutrinos be degenerate in mass?, [Phys. Lett. B 458 (1999) 310] [hep-ph/9904279].

[53] M. Carena, J.R. Ellis, S. Lola and C.E.M. Wagner, Neutrino masses, mixing angles and the unification of couplings in the MSSM, [Eur. Phys. J. C 12 (2000) 507] [hep-ph/9906362].

[54] J.R. Ellis, G.K. Leontaris, S. Lola and D.V. Nanopoulos, Neutrino textures in the light of super-Kamiokande data and a realistic string model, [Eur. Phys. J. C 9 (1999) 389] [hep-ph/9808251].

[55] P.H. Chankowski, W. Krolikowski and S. Pokorski, Fixed points in the evolution of neutrino mixings, [Phys. Lett. B 473 (2000) 109] [hep-ph/9910231].

[56] P.H. Chankowski and S. Pokorski, Quantum corrections to neutrino masses and mixing angles, [Int. J. Mod. Phys. A 17 (2002) 573] [hep-ph/0110243].

[57] K.R.S. Balaji, A.S. Dighe, R.N. Mohapatra and M.K. Parida, Generation of large flavor mixing from radiative corrections, [Phys. Rev. Lett. 84 (2000) 5033] [hep-ph/0001310]; Radiative magnification of neutrino mixings and a natural explanation of the neutrino anomalies, [Phys. Lett. B 481 (2000) 33] [hep-ph/0002177].

[58] T. Miura, E. Takasugi and M. Yoshimura, Quantum effects for the neutrino mixing matrix in the democratic-type model, [Prog. Theor. Phys. 104 (2000) 1173] [hep-ph/0007066].

[59] K.R.S. Balaji, R.N. Mohapatra, M.K. Parida and E.A. Paschos, Large neutrino mixing from renormalization group evolution, [Phys. Rev. D 63 (2001) 113002] [hep-ph/0011263].
[60] T.K. Kuo, J. Pantaleone and G.-H. Wu, *Renormalization of the neutrino mass matrix*, Phys. Lett. B 518 (2001) 101 [hep-ph/0104131].

[61] J. Pantaleone, T.K. Kuo and G.-H. Wu, *Maximal neutrino mixing from an attractive infrared fixed point*, Phys. Lett. B 520 (2001) 279 [hep-ph/0108137].

[62] R.N. Mohapatra, M.K. Parida and G. Rajasekaran, *High scale mixing unification and large neutrino mixing angles*, hep-ph/0301234.

[63] A. Zee, *A theory of lepton number violation, neutrino Majorana mass and oscillation*, Phys. Lett. B 93 (1980) 389, erratum ibid. B 95 (1980) 461;

P.H. Frampton and S.L. Glashow, *Can the zee ansatz for neutrino masses be correct?*, Phys. Lett. B 461 (1999) 93 [hep-ph/9906373].

[64] G.L. Fogli, E. Lisi and G. Scioscia, *Accelerator and reactor neutrino oscillation experiments in a simple three generation framework*, Phys. Rev. D 52 (1995) 5334 [hep-ph/9506350];

*Matter-enhanced three-flavor oscillations and the solar neutrino problem*, Phys. Rev. D 54 (1996) 2043 [hep-ph/9605273].

[65] K.S. Babu, *Renormalization group analysis of the Kobayashi-Maskawa matrix*, Z. Physik C 35 (1987) 69.

[66] A. Brignole, *One-loop Kähler potential in non-renormalizable theories*, Nucl. Phys. B 579 (2000) 101 [hep-th/0001121].

[67] P. Cvitanovic, *Group theory for Feynman diagrams in nonabelian gauge theories: exceptional groups*, Phys. Rev. D 14 (1976) 1536.