A stability result for purely radiative spacetimes

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Abstract

An existence and stability result for a class of purely radiative vacuum spacetimes arising from hyperboloidal data is given. This result generalises semiglobal existence results for Minkowski-like spacetimes to the case where the reference solution contains gravitational radiation. The analysis makes use of the extended conformal field equations and a gauge based on conformal geodesics so that the location and structure of the conformal boundary of the perturbed solutions is known a priori.

1 Introduction

In this article we analyse the stability of the hyperboloidal initial value problem for a class of solutions to the vacuum Einstein field equations —the so-called purely radiative vacuum spacetimes. Purely radiative vacuum spacetimes describe gravitational radiation which comes from infinity, interacts with itself in a non-linear way and then disperses to infinity. In order to encode that the spacetime is made only of incoming gravitational radiation, one assumes that it admits a Penrose conformal extension such that the null generators of future and past null infinity (I+ and I−) are complete and that future and past timelike infinity are represented in the conformal extension by two points i+ and i−. The points i± are required to be regular —that is, the conformally rescaled unphysical spacetime admits a smooth extension which is regular at i±. More precisely, given a vacuum spacetime (M, gμν) it will be assumed that there exists a conformally related spacetime (M, gμν) such that

\[ g_{\mu\nu} = \Theta^2 \tilde{g}_{\mu\nu}, \] (1)

where Θ is a suitable conformal factor. Null infinity, I±, is characterised as the set of points for which

\[ \Theta = 0, \quad d\Theta \neq 0. \] (2)

On the other hand, the points i± in a purely radiative spacetime are such that

\[ \Theta = 0, \quad d\Theta = 0, \quad \text{Hessian } \Theta \text{ is non-degenerate.} \] (3)

It is perhaps worth noting that the corresponding points i± in a spacetime with a black hole (say, the Schwarzschild spacetime) do not satisfy these conditions.

It has been known for a long time that the question of the existence of non-trivial examples of purely radiative solutions to the Einstein field equations hinges very delicately on the structure of the spacetimes in a neighbourhood of the so-called spatial infinity, i0 —see e.g. [10] [12] [14] [15]. A way of getting around the so-called i0-problem is to consider initial value problems posed on hyperboloidal hypersurfaces, that is, spacelike hypersurfaces which can be thought of as intersecting null infinity in a transversal manner. If the initial data prescribed on the hyperboloid

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is suitably close to a canonical hyperboloidal data implied by the Minkowski spacetime—that is, the data is a perturbation of Minkowski data—then it has been proved that the resulting development has a conformal structure similar to that of Minkowski spacetime [9]. More precisely, the future development of initial data admits a conformal compactification such that the locus of points, $\mathcal{I}^+$ satisfying the condition (4) has the topology $S^2 \times [0, 1)$, and that there is a single point, $i^+$, satisfying the conditions (5) where the null generators of null infinity converge. The original proof of this result was carried out using the so-called conformal Einstein field equations discussed in [5, 6, 7, 8]. These equations have the property of being regular even at the points where the conformal factor vanishes and are such that a solution to them implies a solution to the vacuum Einstein field equations. A property of the evolution systems implied by these conformal field equations is that it includes the conformal factor as one of the unknowns, and thus any discussion of the structure of the conformal boundary has to be performed a posteriori. The results of [9] have been generalised to the case of the Einstein field equations are coupled to Maxwell and Yang-Mills fields [11].

A more general system of conformal field equations has been introduced in [13]. This extended conformal Einstein field equations are expressed in terms of so-called Weyl connections—torsion free connections (not necessarily Levi-Civita) which preserve the conformal structure. Thus, the field equations contain more gauge freedom. In particular, using the extended conformal equations it is possible to introduce gauges—conformal Gaussian gauges—which are based on certain type of conformal invariants—the so-called conformal geodesics. The advantages of these conformal Gaussian systems are twofold: combined with the extended conformal field equations, they imply simpler systems of evolution equations; and, if working in vacuum, they provide an a priori canonical conformal factor so that the location and properties of the conformal boundary are known and can be adjusted according to need before the propagation equations are solved. In [27], the extended conformal Einstein field equations and conformal Gaussian gauge systems have been used to provide a new version of the proof of the stability results for Minkowski and de Sitter spacetime. In particular, the analysis of the properties of the conformal boundary of the spacetimes obtained is much simpler and transparent.

The proofs of the stability of perturbations of the Minkowski spacetime in [9] and in [27] make use of an adaptation of a very general result on the existence and stability of symmetric hyperbolic systems by Kato [24, 25, 26]. The possibility of using these theorems to directly prove the global existence of solutions to the Einstein field equations stems from the use of a conformally compactified picture where timelike infinity is at a finite position. As a result, Kato’s theorems allow to provide very compact global existence results which require almost no PDE (partial differential equations) analysis. The latter should be contrasted, for example, with the analysis of the non-linear stability of the Minkowski spacetime carried out by Christodoulou & Klainerman [3] where precise asymptotic estimates of the solution are obtained.

In the present article we generalise the methods and result of [27] to reference solutions which are purely radiative spacetimes in the sense discussed in the first paragraphs of this introduction. A procedure to construct an infinite number of these purely radiative spacetimes has been given in [10]. The possibility of generalising the existence and stability results of the Minkowski spacetime using this class of solutions has been suggested in [11]. This construction depends upon a remarkable connection between stationary solutions to the Einstein field equations and spacetimes with vanishing mass. The reference spacetimes thus obtained are only known in an abstract sense and without much information about their geometry at hand. It is remarkable that although one has very little explicit information about the reference solutions, it is still possible to use the methods of [27] to obtain a stability result. Our main result can be summarised in the following form.

**Theorem.** Given a sequence of multipole moments for a static solution to the Einstein vacuum equations subject to a suitable convergence condition, there exists a hyperboloidal initial data set whose future development is a purely radiative spacetime admitting a conformal compactification with a future null infinity satisfying conditions (4) and a point representing timelike infinity satisfying conditions (5). A small enough perturbation of the hyperboloidal initial data has a
future development whose conformal boundary has the same properties as that of the unperturbed solution.

In order to prove this theorem one starts by considering the asymptotic end of an initial data set for a vacuum static solution to the Einstein field equations. This asymptotic end can be compactified to a 3-dimensional manifold, \( \bar{S} \), such that the point at infinity corresponds to a certain point \( i \). The subsequent analysis will be restricted to a suitable neighbourhood of this point. The observations in [10] allow to construct an initial data set for the conformal Einstein field equations which is analytical in a neighbourhood of \( i \). This initial data has the peculiarity of having a vanishing ADM mass. As our analysis is local to \( i \) the later feature causes no problem. A local existence for this data follows directly. Our interest is then concentrated to the region in the development which is included in \( \Gamma^{-}(i) \) (the causal past of \( i \)). As a consequence of our conformal setting, this region implies the existence a portion of a purely radiative spacetime. The latter can be regarded as the future development of some data for a purely radiative spacetime prescribed on a fiduciary hyperboloid \( \hat{H} \). The point \( i \) in \( \bar{S} \) corresponds to the point \( i^{+} \) of the radiative spacetime. In order to prove the existence of the reference radiative spacetime one employs a conformal Gaussian system based on a congruence of conformal geodesics leaving the hypersurface \( \bar{S} \) orthogonally.

It should be noted that this congruence will, in general, intersect the fiduciary hyperboloid in a non-orthogonal manner. This feature hinders a direct application of Kato’s theorems in order to conclude the stability of the hyperboloidal data. The reason for this is that in order to make use of Kato’s theorems one has to make use of a Gaussian system in which the conformal geodesics leave the initial hyperboloid with the same angle and orientation as the congruence in the reference solution meets the hyperboloid \( \hat{H} \). It turns out that the data for such an evolution are not really intrinsic to the initial hyperboloid but contain information along the direction normal to the initial hypersurface. Thus, one needs to show, first, local existence for the perturbed hyperboloidal data —using, for example, a congruence of conformal geodesics which is orthogonal to the hyperboloid. Once this local existence has been established, one can construct initial data for the evolution along the tilted congruence —using a Lorentz transformation. From here Kato’s theorems and the requirement of the perturbed data being close enough to the reference solution yields the required stability result.

Outline of the article

We start by setting our conventions in section 2 and recalling the physical and the conformal vacuum constraints. As we are working on \( S^{3} \) and subsets thereof, we fix coordinates and vector fields on \( S^{3} \) and briefly review the norm and the extension operator for Sobolev spaces associated to \( S^{3} \). In section 3 we discuss the conformal tools, which we will be using later on. These include conformal geodesic, their associated conformal Gaussian gauge and the conformal Einstein field equations. We follow the standard approach by Friedrich and derive an evolution system form the field equation by using space spinors. Section 4 is concerned with the construction of a purely radiative reference spacetime from time symmetric static vacuum data with non-vanishing mass given on a Cauchy surface passing through the point \( i \) at spacelike infinity. In section 3.3.3 the resulting radiative spacetime is analysed from the viewpoint of hyperboloidal data on a surface not passing through \( i \). We note at this stage that the initial data on such a hypersurface is not proper hyperboloidal data due to the fact that the conformal geodesic congruence does not intersect the hyperboloid orthogonally. Thus we analyse the relationship between this so named tilted hyperboloidal data and the proper hyperboloidal data. Section 5 is concerned with the existence and stability result for radiative spacetimes. The structure of the conformal boundary is analysed prior to the evolution of the data. In section 5.3 we use a modified version of a theorem by T.Kato employed in [2] and [27] to establish the existence and stability for purely radiative spacetimes in theorems 3 and 1.
2 Basics and conventions

We shall consider spacetimes \((\mathcal{M}, \tilde{g}_{\mu \nu})\) satisfying the vacuum Einstein field equations

\[ \tilde{R}_{\mu \nu} = 0, \tag{4} \]

where \(\tilde{R}_{\mu \nu}\) denotes the Ricci tensor of a Lorentzian metric \(\tilde{g}_{\mu \nu}\), \((\mu, \nu = 0, 1, 2, 3)\) with signature \((+, -, -, -)\). The discussion of the solutions to equation (4) will be carried out in terms of a conformally rescaled metric \(g_{\mu \nu}\) related to \(\tilde{g}_{\mu \nu}\) according to equation (1). In what follows, let \(\nabla_{\mu}\) denote the Levi-Civita covariant derivative of \(g_{\mu \nu}\), while \(\tilde{R}^\mu_{\lambda \rho}, C_{\mu \lambda \rho}, \tilde{R}_{\mu \nu}, R\) denote the associated Riemann, Weyl and Ricci tensors and the Ricci scalar of \(g_{\mu \nu}\).

In the stability analysis we will compare a spacetime \((\mathcal{M}, g_{\mu \nu})\), evolved from some chosen initial data, with a given reference spacetime. Throughout the article, the latter will be denote by \((\mathcal{M}, \tilde{g}_{\mu \nu})\). We will also adapt this notation for all quantities related to the reference spacetime.

We will make use of two types of hyperboloidal data. \textit{Proper} hyperboloidal data is given with respect to a frame consisting of three tangent vectors to the hypersurface and its normal, while \textit{tilted} hyperboloidal data depends on a frame not intrinsic in the hypersurface. In our setup this frame will be oriented along the congruence. We will distinguish between the two frames and the corresponding frame components, by using an underscore for proper hyperboloidal data.

The conformal factor on the spacetime will be denoted by \(\Theta\), while \(\Omega\) is the conformal factor on an initial hypersurface.

2.1 The constraint equations

Let \(\tilde{S}\) denote a timelike hypersurface of the spacetime \((\mathcal{M}, \tilde{g}_{\mu \nu})\). Let \(\tilde{h}_{\alpha \beta}, \tilde{K}_{\alpha \beta}\) denote, respectively, the first and second fundamental forms induced by the metric \(\tilde{g}_{\mu \nu}\) on \(\tilde{S}\) \((\alpha, \beta = 1, 2, 3)\). The vacuum Einstein field equations (4) imply the following constraint equations on \(\tilde{S}\)

\[ \begin{align*}
\tilde{r} - \tilde{K}^2 + \tilde{K}^{\alpha \beta} \tilde{K}_{\alpha \beta} &= 0, \\
\tilde{D}^\alpha \tilde{K}_{\alpha \beta} - \tilde{D}_\beta \tilde{K} &= 0,
\end{align*} \tag{5a} \tag{5b} \]

where \(\tilde{D}\) denotes the Levi-Civita and \(\tilde{r}\) the Ricci scalar of the metric \(\tilde{h}_{\alpha \beta}\). The first and second fundamental forms determined by the metrics \(g_{\mu \nu}\) and \(\tilde{g}_{\mu \nu}\) on \(\tilde{S}\) are related by

\[ h_{\alpha \beta} = \Omega^2 \tilde{h}_{\alpha \beta}, \quad K_{\alpha \beta} = \Omega \left( \tilde{K}_{\alpha \beta} + \Sigma \tilde{h}_{\alpha \beta} \right), \]

where \(\Omega \equiv \Theta|_{\tilde{S}}\) and the function \(\Sigma\) on \(\tilde{S}\) denotes the derivative of \(\Theta\) in the direction of the future directed \(g\)-normal of \(\tilde{S}\). The traces \(K = h^{\alpha \beta} K_{\alpha \beta}\) and \(\tilde{K} = \tilde{h}^{\alpha \beta} \tilde{K}_{\alpha \beta}\) are related by

\[ \Omega K = \tilde{K} + 3\Sigma. \]

The Hamiltonian and momentum constraints for vacuum, equations (5a) and (5b), written in terms of the conformal fields \(\Omega, \Sigma, h_{\alpha \beta}\) and \(K_{\alpha \beta}\) read

\[ \begin{align*}
2\Omega D_\alpha D^\alpha \Omega - 3D_\alpha \Omega D^\alpha \Omega + \frac{1}{2} \Omega^2 r - 3\Sigma^2 - \frac{1}{2} \Omega^2 (K^2 - K_{\alpha \beta} K^{\alpha \beta}) + 2\Omega \Sigma K &= 0, \tag{6a} \\
\Omega^3 D^\alpha (\Omega^{-2} K_{\alpha \beta}) - \Omega (D_\beta K - 2\Omega^{-1} D_\beta \Sigma) &= 0, \tag{6b}
\end{align*} \]

where \(D\) denotes the Levi-Civita connection and \(r\) the Ricci scalar of the metric \(h_{\alpha \beta}\). In particular if \(\Omega = 0\) one has that

\[ \Sigma^2 + D_\alpha \Omega D^\alpha \Omega = 0. \tag{7} \]

In the sequel we will consider two different classes of solutions to equations (6a) and (6b). The first class will consist of initial data sets for the Einstein vacuum field equations which are \textit{asymptotically Euclidean}, and thus, the relevant initial hypersurfaces are Cauchy hypersurfaces of an asymptotically flat spacetime. Further details will be discussed in section 4.1. The second class
of solutions to be considered are hyperboloidal data sets. In this case the initial hypersurfaces are not Cauchy hypersurfaces of a globally hyperbolic spacetimes. Further details will be given in section 5.1. The character of the solutions to the conformal constraint equations (asymptotically Euclidean or hyperboloidal) is determined through the boundary conditions which supplement equations (6a) and (6b).

2.2 Coordinates on $S^3$ and submanifolds thereof

In the sequel we shall work with conformal spacetimes with time slices diffeomorphic to simply connected subsets of the 3-sphere, $S^3$. Therefore, it will be important to have a good coordinatisation of $S^3$ and to have a frame which is globally defined over this manifold. Following the work in [10, 27] we regard $S^3$ as a submanifold of $\mathbb{R}^4$:

$$S^3 = \left\{ x^A \in \mathbb{R}^4 \left| (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^4 = 1 \right. \right\}.$$ 

The restrictions of the functions $x^A$, $A = 1, 2, 3, 4$ on $\mathbb{R}^4$ to $S^3$ will again be denoted by $x^A$. If we view the point $N \equiv (0, 0, 0, 1)$ as the North pole and $D \equiv \{x^4 > 0\} \cap S^3$ as the open upper half sphere, then for any subset $U$ of $D$ with $N \in U$ the functions $(x^1, x^2, x^3)$ provide coordinates on $U$ centered at $N$. The vector fields

$$c_1 \equiv x^1 \partial_4 - x^4 \partial_1 + x^2 \partial_3 - x^3 \partial_2,$$

$$c_2 \equiv x^1 \partial_3 - x^3 \partial_1 + x^4 \partial_2 - x^2 \partial_4,$$

$$c_3 \equiv x^1 \partial_2 - x^2 \partial_1 + x^4 \partial_3 - x^3 \partial_4.$$ 

on $\mathbb{R}^4$ are tangent to $S^3$ and provide a globally defined frame field on $S^3$.

Given a simply connected 3-manifold $\mathcal{H}$ with boundary $\partial \mathcal{H}$, there exists a diffeomorphism $\phi : \mathcal{H} \to U \equiv \{x^4 \geq 1/2\} \subset D$, such that $\phi : \partial \mathcal{H} \to \{x^4 = 1/2\}$. Since $\phi$ is a diffeomorphism we can map any tensor from $\mathcal{H}$ to $U$ and vice versa. For convenience we denote tensors on $\mathcal{H}$ and $U$ by the same symbols and the underlying manifold is to be understood from the context. In particular we can pull-back the functions $x^A$ to provide coordinates on $\mathcal{H}$ and use the vectors $c_1$, $c_2$, $c_3$, as given by (8a)-(8c), as a globally defined frame field on $\mathcal{H}$.

2.3 Extending fields to $S^3$

In the sequel it will be necessary to extend certain (vector valued) functions defined on a subset $U \subset S^3$ to the whole of $S^3$. For this we follow the approach pursued in [9]. On the spaces $C^\infty(U, \mathbb{R}^N)$ and $C^\infty(S^3, \mathbb{R}^N)$ of smooth $\mathbb{R}^N$-valued functions on $U$ and $S^3$ define for $m \in \mathbb{N}$ the Sobolev-like norm

$$\| w \|_{m, Q} = \left( \sum_{k=0}^m \int_Q \left( \sum_{r_1, \ldots, r_k=1}^3 |D_{r_1} \cdots D_{r_k} w|^2 \right) d\mu \right)^{1/2}, \quad (9)$$

Figure 1: The mapping $\mathcal{H}$ onto $S^3$. 
with \( Q = U, S^3 \). Let \( H^m(U, \mathbb{R}^N) \) and \( H^m(S^3, \mathbb{R}^N) \) be the Hilbert spaces obtained, respectively, as the completion of the spaces \( C^\infty(U, \mathbb{R}^N) \) and \( C^\infty(S^3, \mathbb{R}^N) \) on the norms given by formula (9). Given \( w \in H^m(U, \mathbb{R}^N) \), there exists a linear extension operator \( E \) such that

\[
(Ew)(x) = w(x) \quad \text{almost everywhere in } U
\]

(10a)

\[
\|Ew\|_{m,S^3} \leq K \|w\|_{m,U},
\]

(10b)

with \( K \) a constant which is universal for fixed \( m \). It is important to notice that if, for example, \( w = (h_{ij}, \chi_{ij}, \Omega, \Sigma) \) is a solution to the conformal constraints, equations (6a) and (6b), on \( U \) then \( Ew \) will not satisfy the constraints in \( S^3 \setminus U \).

3 The conformal Einstein field equations and conformal Gaussian gauges

When discussing the existence of solutions to the Einstein field equations in terms of conformally rescaled spacetimes, it is convenient to introduce gauges based on conformally invariant structures. The particular conformal structures to be used in our analysis are conformal geodesics. These curves are autoparallel with respect to Weyl connections — that is, torsion-free connections which preserve the conformal structure, but are not necessarily the Levi-Civita connection of a metric. Conformal geodesics were first applied in the context of General Relativity in [20]. The idea of conformal Gaussian coordinates based on congruences of conformal geodesics has been introduced in [13, 14]. It has been shown in [17] that conformal Gaussian coordinates can be used to cover even the strong field regions of the Schwarzschild spacetime.

In this section we present a brief discussion of the properties of gauges based on conformal Gaussian coordinates, and of the associated propagation equations implied by the extended conformal Einstein field equations. For further details we refer the reader to [17, 27, 31] and to the references mentioned in the previous paragraph.

3.1 Conformal geodesics

Given a spacetime \( (\tilde{M}, \tilde{g}_{\mu\nu}) \), a conformal geodesic \( x^\mu(\tau) \) and its associated 1-form \( b_\mu(\tau) \) are solutions to the system of equations

\[
\tilde{x}^\nu \tilde{\nabla}_\nu \dot{x}^\mu + 2(b, \dot{x}) \dot{x}^\mu - \tilde{g}((\dot{x}, \dot{x}), b) = 0, \quad (11a)
\]

\[
\tilde{x}^\nu \tilde{\nabla}_\nu b_\mu - (b, \dot{x}) b_\mu + \frac{1}{2} \tilde{\tilde{g}}((b, b), \dot{x}) = \tilde{L}_{\lambda\mu} \dot{x}^\lambda, \quad (11b)
\]

where \( \tilde{L}_{\mu\nu} \) denotes the Schouten tensor of the Levi-Civita connection \( \tilde{\nabla} \), given by

\[
\tilde{L}_{\mu\nu} \equiv \frac{1}{2} \tilde{R}_{\mu\nu} - \frac{1}{12} \tilde{\tilde{R}} \delta_{\mu\nu}.
\]

The notation

\[
(b, \dot{x}) \equiv b_\mu \dot{x}^\mu, \quad \tilde{g}((b, b), b) \equiv \tilde{g}^{\mu\nu} b_\mu b_\nu
\]

has been used. Associated to the pair \( (x^\mu(\tau), b_\mu(\tau)) \), there is a preferred family of conformal factors, \( \Theta \), which are developed along the congruence using

\[
\dot{x}^\mu \tilde{\nabla}_\mu \Theta = \Theta(b, \dot{x}), \quad \Theta|_{\tilde{S}} = \Theta|_{\infty}. \quad (12)
\]

Consistent with equation (11), define \( g_{\mu\nu} \equiv \Theta^2 \tilde{g}_{\mu\nu} \), and denote the corresponding Levi-Civita connection by \( \nabla \). In this conformal gauge the 1-form along the conformal geodesics is given by

\[
f \equiv b - \Theta^{-1} d\Theta
\]

and satisfies \( \langle f, \dot{x} \rangle = 0 \). In the following, we will work with the unphysical metric \( g_{\mu\nu} \) and the 1-form

\[
d_\mu \equiv \Theta f_\mu + \nabla_\mu \Theta. \quad (13)
\]
The 1-forms $b_\mu$ and $f_\mu$ are used to define a Weyl connection, $\hat{\nabla}$, via

$$\hat{\Gamma}^\nu_{\mu \rho} = \Gamma^\nu_{\mu \rho} + \left( \delta^\nu_{\mu} b_\rho + \delta^\nu_{\rho} b_\mu - g_{\mu \rho} g^{\nu \lambda} b_\lambda \right).$$  (14)

In the gauge corresponding to the connection $\hat{\nabla}$ the equations (11a) and (11b) take the simpler form

$$\dot{x}^\nu \hat{\nabla}_\nu \dot{x}^\mu = 0, \quad \hat{L}_{\lambda \mu} \dot{x}^\lambda = 0,$$

where $\hat{L}_{\mu \nu}$ is the Schouten tensor of the Weyl connection $\hat{\nabla}$. It is given by

$$\hat{L}_{\mu \nu} = L_{\mu \nu} - \nabla_{\mu} f_{\nu} + f_{\mu} f_{\nu} - \frac{1}{2} g_{\mu \nu} f_{\lambda} f^{\lambda}. $$

Note that generally one will have that $\hat{L}_{[\mu \nu]} \neq 0$.

We transport an orthogonal frame $e_k$, $k = 0, 1, 2, 3$, such that $e_0 = \dot{x}$, along the congruence of conformal geodesics according to

$$\dot{x}^\mu \hat{\nabla}_\mu e_k = 0,$$  (15)

which implies that the frame is automatically $g$-orthonormal. If $\bar{g}$ is a vacuum metric, then the conformal factor, $\Theta$, can be written down explicitly by

$$\Theta = \Theta_* + \dot{\Theta}_*(\tau - \tau_*) + \frac{1}{2} \ddot{\Theta}_*(\tau - \tau_*)^2, $$  (16)

where $\Theta_*, \dot{\Theta}_*$ and $\ddot{\Theta}_*$ are functions which are constant along a given conformal geodesic. They are subject to the constraint

$$2 \ddot{\Theta}_* \Theta_* = g^a(d_a, d_a). $$  (17)

Furthermore, along each conformal geodesic

$$d_a = \langle d, e_a \rangle,$$

are constant for $a = 1, 2, 3$ — see e.g. [17].

Analogous to the case of metric geodesic congruences there exist deviation vector fields for congruences of conformal geodesics [17], referred to as conformal Jacobi fields and denote them by $\eta^a$. Let $\eta_k \equiv \eta_\mu e^{\mu}_k$. These fields satisfy

$$\hat{\nabla}_\dot{x} \eta_k = g(\hat{\nabla}_\eta \dot{x}, e_k). $$  (18)

We say the congruence has a conjugate point if $\eta_a$ ($a = 1, 2, 3$) vanish at that point.

### 3.2 Conformal Gaussian gauges and space spinors

The coordinates $x^A$ described in section 2.2 will be extended off a fiduciary hypersurface by dragging them along a congruence of conformal geodesics. Hence, setting $x^0 = \tau$ one obtains a coordinatisation of a slab of the fiduciary hypersurface. In these coordinates one has that

$$e_0 = \dot{x} = \partial_{\tau}. $$  (19)

Let $\hat{\Gamma}_{0 \rho}^j$ and $\hat{L}_{ij}$ denote, respectively, the components of $\hat{\nabla} \mu^\nu$ and $\hat{L}_{\mu \nu}$ with respect to the frame $e_k$ propagated according to the rule (15). One then has that

$$\hat{\Gamma}_{0 \rho}^j = 0, \quad \hat{L}_{0k} = 0, $$  (20)

— see e.g. [13, 14]. This choice of coordinates, frame field and conformal will be referred to as a conformal Gaussian gauge system based on the fiduciary hypersurface. The appearance of
conjugate points in the congruence of conformal geodesics implies a breakdown of the conformal Gaussian gauge.

Let \( \tau^\mu = \sqrt{2} \rho^\mu \), and let \( \tau^{\AA'} \) be its spinorial counterpart. In terms of spinors, the gauge conditions \([19, 20]\) can be written as —see \([18]\) for details of the definition of \( \Theta_{AA'BB'} \)—

\[
\tau^{AA'} \epsilon_{AA'} = \sqrt{2} \partial_\tau, \quad \tau^{AA'} \hat{\Gamma}_{AA'B}^C = 0, \quad \tau^{BB'} \Theta_{AA'BB'} = 0.
\]

The frame vectors \( e_{AA'} \) will be expressed with respect to the vectors \( \{e_0, e_1, e_2, e_3\} \) where \( e_0 \) is the normal to the surfaces of constant \( \tau \) and \( e_1, e_2 \) and \( e_3 \) are given by \([8a]-[8c]\). We will write

\[
e_{AA'} = e_{AA'}^s e_s, \quad s = 0, 1, 2, 3.
\]

In order to obtain a suitable set of evolution equations we consider a space-spinor formalism —see e.g. \([30, 18, 27]\) — as it allows us to obtain evolution equations involving only unprimed spinors. Introduce a spin dyad \( \{o, \iota\} \) such that

\[
\tau^{AA'} = o^A o^{A'} + \iota^A \iota^{A'}.
\]

We decompose the connection as follows

\[
\nabla_{AA'} = \frac{1}{2} \tau_{AA'} \nabla - \tau^C_A \nabla_{AC}, \quad (21a)
\]

\[
\nabla_{AB} = \tau_{(A}^{B'} \nabla_{B')}, \quad \nabla \equiv \tau^{CC'} \nabla_{CC'}, \quad (21b)
\]

Note that \( \nabla_{AB} \) is not the Levi-Civita connection of the surfaces of constant \( \tau \), but the so-called \( \hat{\Sen} \) connection. The frame fields \( e_{AA'}^s \) are similarly decomposed using

\[
e_{AA'}^s = \frac{1}{2} \epsilon_{CC'} e_{CC'} \tau_{AA'} - \tau^C_A e_{AC}^s, \quad (22a)
\]

\[
e_{AB}^s = \tau_{(A}^{B'} e_{B')}. \quad (22b)
\]

The fields \( e_{AB}^s \) are associated to spatial vectors, and hence, they satisfy the reality conditions

\[
e_{AB}^s = -\tau_{A'}^{A} \tau_{B'}^{B} \iota^{A'B'}. \quad (23)
\]

Contracting with \( \tau^{AA'} \) and symmetrising, one decomposes the remaining connection and curvature spinors as follows. The space spinor \( \Theta_{ABCD} = \Theta_{AB(CD)} \) is defined by

\[
\Theta_{ABCD} = \Theta_{AA'CC'} \tau^{A'B'CD} = \Theta_{(AB)(CD)} + \frac{1}{2} \epsilon_{AB} \Theta_{G^G}^{\, (CD)}.
\]

One observes that \( \hat{\Gamma}_{AA'BC} = \Gamma_{AA'BC} + \epsilon_{AB} f_{CA'} \) and \( \tau^{AA'} f_{AA'} = 0 \). Define \( \Gamma_{ABCD} \equiv \tau_{B'}^{B} \Gamma_{AB'CD} \), which in turn, will be decomposed as

\[
\Gamma_{ABCD} = \frac{1}{\sqrt{2}} \left( \xi_{ABCD} - \chi_{(AB)(CD)} \right) - \frac{1}{2} \epsilon_{AB} f_{CD}.
\]

The spinors in the latter equation possess the following symmetries

\[
\Gamma_{ABCD} = \Gamma_{AB(CD)}, \quad \chi_{ABCD} = \chi_{AB(CD)}, \quad \xi_{ABCD} = \xi_{(AB)(CD)}.
\]

### 3.3 The conformal Einstein evolution equations

In the sequel we will make use of propagation equations which are derived from the extended conformal field equations of \([13]\) —see also \([19, 18]\). These equations are a generalisation of the original conformal equations which allow the use of Weyl connections, as defined in \([14]\). Weyl connections make it possible to consider more general gauges in the derivation of systems of
propagation equations out of the conformal field equations. In particular, it makes it possible to make use of the conformal Gaussian gauge systems discussed in section 3.2.

Using the space spinor formalism and suitable contractions of the extended conformal field equations with $\tau^{BB'}$, one finds that the conformal Gaussian gauge system implies the following propagation equations for the unknowns $\xi_{AB}$, $\xi_{ABCD}$, $f_{AB}$, $\chi_{(AB)CD}$, $\Theta_{(AB)CD}$ and $\Theta_{G}^{G CD}$:

\begin{align*}
\partial_{\tau} e_{AB}^{0} &= -\chi^{(AB)}_{(AB)} e_{EF}^{0} - f_{AB}, & (24a) \\
\partial_{\tau} e_{AB}^{\xi} &= -\chi^{(AB)}_{(AB)} e_{EF}^{\xi}, & (24b) \\
\partial_{\tau} \xi_{ABCD} &= -\chi_{(AB)}^{EF} \xi_{EFCD} + \frac{1}{\sqrt{2}} (\epsilon_{AC} \chi_{BD}^{EF} + \epsilon_{BD} \chi_{AC}^{EF}) f_{EF}^C - \sqrt{2} \chi_{(AB)C}^{E} f_{D}^{EF} - \frac{1}{2} (\epsilon_{AC} \Theta_{F}^{BD} + \epsilon_{BD} \Theta_{F}^{AC}) - i \Theta \mu_{ABCD}, & (24c) \\
\partial_{\tau} f_{AB} &= -\chi_{(AB)}^{EF} f_{EF}^C + \frac{1}{\sqrt{2}} \Theta_{F}^{F AB}, & (24d) \\
\partial_{\tau} \chi_{(AB)CD} &= -\chi_{(AB)}^{EF} \xi_{EFCD} - \Theta_{(CD)AB} + \Theta_{ABCD}, & (24e) \\
\partial_{\tau} \Theta_{(AB)CD} &= -\chi_{(CD)}^{EF} \Theta_{(CD)AB} - \partial_{\tau} \Theta_{ABCD} + i \sqrt{2} d_{(AB)CDE}, & (24f) \\
\partial_{\tau} \Theta_{G}^{G AB} &= -\chi_{(AB)}^{EF} \Theta_{G}^{G EF} + \sqrt{2} d_{EF}^{AB} \eta_{ABEF}, & (24g)
\end{align*}

where $\eta_{ABCD}$ and $\mu_{ABCD}$ denote, respectively, the electric and magnetic parts of $\phi_{ABCD}$ and $d_{AB}$ is the spinor representation of $d_{A'}$ — for further details see e.g. [14, 18, 27]. Note that the above equations involve an unspecified conformal factor $\Theta$ and the space spinor representation of the 1-form $d_{\mu}$ ($d_{AB}$) introduced in [13]. For $\Theta$ we will use [12] and [16].

The evolution equations for the spinor $\phi_{ABCD}$ are derived from a space-spinor decomposition of the Bianchi identity

$$\nabla^{AA'} \phi_{ABCD} = 0. \quad (25)$$

The resulting propagation equations are not unique as one can always add to them a multiple of the constraint equations implied by (25), $\nabla^{AB} \phi_{ABCD} = 0$. In [13, 14] the following Bianchi propagations equations

$$\sqrt{2} \partial_{\tau} \phi_{ABCD} - 2 \nabla_{(D}^{F} \phi_{ABC)F} = 0. \quad (26)$$

are considered — to the so called standard system. Note that equation (26) is expressed with respect to the Levi-Civita connection $\nabla$ and not the Weyl connection $\tilde{\nabla}$.

In order to keep track of the possible appearance of conjugate points in the solutions to the propagation equations, we append to (24a)-(24g) and (26) an evolution equation for the Jacobi field. The conformal Jacobi field $\eta^{\mu}$ has a spinorial counterpart $\eta_{AA'}$ which can be split as

$$\eta_{AA'} = \frac{1}{2} \eta_{\tau AA'} - \tau_{A'}^{B'} \eta_{AB}, \quad \eta \equiv \eta_{AA'} \tau_{AA'}, \quad \eta_{AB} \equiv \tau_{(A}^{B'} \eta_{B')}. \quad (27a)$$

Conjugate points in the congruence of conformal geodesics arise if $\eta_{AB} = 0$. Equation (18) implies that the fields $\eta$, $\eta_{AB}$ satisfy the propagation equations

$$\sqrt{2} \partial_{\tau} \eta = f_{AB} \eta^{AB}, \quad (27a)$$

$$\sqrt{2} \partial_{\tau} \eta_{AB} = \chi_{CD(AB)} \eta^{CD}. \quad (27b)$$

Besides the above propagation equations (24a)-(24g), and (26) one also obtains a set of equations referred to as the extended conformal constraint equations involving the operators $e_{AB}$ and $\nabla_{AB}$ (respectively $\nabla_{AB}$). It should be noticed that despite their name, the constraint equations, are not intrinsic to the hypersurfaces of constant $\tau$ — except, possibly, for an initial fiducial slice — as generically, they contain $\partial_{\tau}$ derivatives. The crucial observation for our purposes concerns their propagation properties as given by the following lemma proved in [13]. Let $D^{+}(S)$ denote the future domain of dependence of $S$ — see e.g. [22, 32] for definitions.
Lemma 1. Assume that the constraint equations are satisfied on an initial hypersurface $S$. If the conformal propagation equations are satisfied on $D^+(S)$, then the conformal constraint equations are also satisfied on $D^+(S)$.

3.4 Structural properties of the evolution equations

We discuss now some general structural properties of the equations (24a)-(24g), (26), (27a)-(27b), which will be used systematically in the sequel. Let

$$\phi_i \equiv \phi_{(ABCD)}i,$$

where the subscript $(ABCD)$ indicates that after symmetrisation $i$ indices are set to 1. Introduce the notation

$$v \equiv (e^{s}_{AB}, \Gamma_{ABCD}, \Theta_{ABCD}, \eta_i, \eta_{iAB}) \quad \phi \equiv (\phi, \phi_1, \phi_2, \phi_3, \phi_4),$$

where it is understood that $v$ contains only the independent components of the respective spinor—which are obtained by writing linear combinations of irreducible spinors. In terms of $v$ and $\phi$, the propagation equations (24a)-(24g) and (26), (27a)-(27b) can be written as:

$$\partial_\tau v = K + Q(v, v) + L\phi,$$

where $K$ and $Q$ denote, respectively, a linear constant matrix-valued function and a bilinear vector-valued function both with constant entries and $L$ is a linear matrix-valued function with coefficients depending on the coordinates. Similarly equation (26) can be written in the form

$$\sqrt{2}E\partial_\tau \phi + A^{AB}e^s_{AB}\partial_\tau \phi = B(\Gamma_{ABCD})\phi,$$

where $E$ denotes the $5 \times 5$ identity matrix and $A^{AB}e^s_{AB}, s = 0, \ldots, 3$, are $5 \times 5$ matrices depending on the coordinates, while $B(\Gamma_{ABCD})$ denotes a constant matrix-valued linear function of the connection coefficients $\Gamma_{ABCD}$. For later reference it is noted that

$$\sqrt{2}E + A^{AB}e^0_{AB} = \begin{pmatrix}
\sqrt{2} - 2e^0_{11} & 2e^0_{00} & 0 & 0 & 0 \\
-e^0_{11} & \sqrt{2} & e^0_{02} & 0 & 0 \\
0 & -e^0_{11} & \sqrt{2} & e^0_{02} & 0 \\
0 & 0 & -e^0_{11} & \sqrt{2} & e^0_{02} \\
0 & 0 & 0 & -2e^0_{11} & \sqrt{2} + 2e^0_{01} \\
\end{pmatrix},$$

and that

$$A^{AB}e^\tilde{r}_{AB} = \begin{pmatrix}
-2e^\tilde{r}_{01} & 2e^\tilde{r}_{00} & 0 & 0 & 0 \\
-e^\tilde{r}_{11} & \sqrt{2} & e^\tilde{r}_{02} & 0 & 0 \\
0 & -e^\tilde{r}_{11} & \sqrt{2} & e^\tilde{r}_{02} & 0 \\
0 & 0 & -e^\tilde{r}_{11} & \sqrt{2} & e^\tilde{r}_{02} \\
0 & 0 & 0 & -2e^\tilde{r}_{11} & 2e^\tilde{r}_{01} \\
\end{pmatrix},$$

with $\tilde{r} = 1, 2, 3$. From the reality condition (23) one has that

$$e^0_{00} = -\overline{e^0_{11}}, \quad e^\tilde{r}_{00} = -\overline{e^\tilde{r}_{11}},$$

so that in particular $e^0_{01}$ and $e^\tilde{r}_{01}$ are real. The Hermitian matrices $\sqrt{2}E + A^{AB}e^0_{AB}, A^{AB}e^\tilde{r}_{AB}$ imply real symmetric matrices for the propagation equations obtained from splitting the system (28)-(29) into real and imaginary parts of the propagation equations. These matrices are explicitly given in (27).

Let $u \equiv (\text{Re}(v), \text{Im}(v), \text{Re}(\phi), \text{Im}(\phi))$. The unknown $u$ takes values in $\mathbb{R}^N$ for some $N \in \mathbb{N}$. In this article $u$ will be defined over $S^3$ or subsets thereof. From equations (28) and (29) it follows that $u$ satisfies a system of quasilinear partial differential equations for $u$ of the form

$$A^0(u) \cdot \partial_\tau u + \sum_{\tilde{r}=1}^3 A^\tilde{r}(u) \cdot e^\tilde{r}(u) + B(\tau, x^A, u) \cdot u = 0,$$
with $c_v(u)$ denoting the vector fields (8a)-(8c) acting on the unknown $u$. Given any $z \in \mathbb{R}^N$, the matrix valued functions $A^\xi(z)$, $\bar{s} = 0, 1, 2, 3$ have entries which are polynomial in $z$. These polynomials are at most of degree one and have constant coefficients. The matrices are symmetric $\bar{i}(A^\xi(z)) = A^\xi(z)$, $z \in \mathbb{R}^N$. The matrix valued function $B = B(\tau, x^A, z)$ with $(\tau, x^A, z) \in \mathbb{R} \times S^3 \times \mathbb{R}^N$ has entries which are polynomials in $z$ (of at most degree 1) with coefficients which are analytic functions on $\mathbb{R} \times S^3$.

4 Purely radiative reference spacetimes

In what follows we will construct purely radiative vacuum spacetimes using the ideas of [10]. These spacetimes are required to allow a conformal compactification such that the resulting conformal metric is smooth at null infinity including the point past or future timelike infinity. The purely radiative spacetimes in [10] have been obtained from of static, asymptotically flat solutions to the Einstein field equations. Before discussing the results in [10] which are crucial for our analysis, we introduce some definitions.

4.1 Asymptotically Euclidean hypersurfaces

Asymptotically flat spacetimes, $(\bar{\mathcal{M}}, \bar{g}_{\mu\nu})$, can be obtained as the development of asymptotically Euclidean initial data sets. The initial data $(\bar{\mathcal{S}}, \bar{h}_{\alpha\beta}, \bar{K}_{\alpha\beta})$ will be said to be asymptotically Euclidean if there is a compact subset of $\bar{\mathcal{S}}$ whose complement for some positive number $r_0$ is diffeomorphic to $\{y^a \in \mathbb{R}^3 \mid |y| > r_0\}$, where $|y|^2 \equiv (y^1)^2 + (y^2)^2 + (y^3)^2$. In terms of the coordinates $y^a$ introduced by this identification a standard asymptotic flatness requirement is

$$\bar{h}_{\alpha\beta} = - \left(1 + \frac{2m}{|y|}\right) \delta_{\alpha\beta} + \mathcal{O}\left(\frac{1}{|y|^2}\right), \quad \text{as } |y| \to \infty,$$

with $m$ a constant —the ADM mass of $\bar{\mathcal{S}}$. An initial data set $(\bar{\mathcal{S}}, \bar{h}_{\alpha\beta}, \bar{K}_{\alpha\beta})$ is said to be time symmetric if $\bar{K}_{\alpha\beta} = 0$. It follows from the Einstein field equations that its development will have a time reflection symmetry with respect to the surface $\bar{\mathcal{S}}$.

As we shall be working with conformally compactified spacetimes, it will convenient to consider a compactified version of the initial hypersurface $\bar{\mathcal{S}}$. To this end, it will be assumed that there is a 3-dimensional, orientable, smooth compact manifold $(\bar{\mathcal{S}}, h)$, a point $i \in \bar{\mathcal{S}}$, a diffeomorphism $\Phi : \bar{\mathcal{S}} \setminus \{i\} \to \bar{\mathcal{S}}$ and a function $\Omega \in C^2(\bar{\mathcal{S}}) \cap C^\infty(\bar{\mathcal{S}})$ with the properties

$$\Omega(i) = 0, \quad D_a \Omega(i) = 0, \quad D_a D_\beta \Omega(i) = -2h_{\alpha\beta}(i), \quad \Omega > 0 \text{ on } \bar{\mathcal{S}} \setminus \{i\},$$

$$h_{\alpha\beta} = \Omega^2 \Phi^\ast \bar{h}_{\alpha\beta}. \quad (32a)$$

The last condition shall be, sloppily, written as $h_{\alpha\beta} = \Omega^2 \bar{h}_{\alpha\beta}$ —that is, $\bar{\mathcal{S}} \setminus \{i\}$ will be identified with $\bar{\mathcal{S}}$. Under these assumptions $(\bar{\mathcal{S}}, \bar{h}_{\alpha\beta}, \bar{K}_{\alpha\beta})$ will be said to be asymptotically Euclidean and regular. Suitable, punctured neighbourhoods of the point $i$ will be mapped into the asymptotic end of $\bar{\mathcal{S}}$.

In the sequel, we will consider initial data sets for the Einstein field equations which satisfy the boundary conditions (32a)-(32c) but have vanishing mass ($m = 0$). It follows from the mass positivity theorem that if $m = 0$ then either $(\bar{\mathcal{S}}, \bar{h}_{\alpha\beta})$ is data for the Minkowski spacetime, or the initial data is singular somewhere in $\bar{\mathcal{S}}$ —see e.g. [28] [29]. Our discussion of initial data sets with vanishing mass will be concentrated on a small neighbourhood $\mathcal{B}_a(i)$ of the point $i$ where the relevant objects are suitably smooth.

4.2 Time symmetric initial data sets with vanishing mass

The construction of purely radiative spacetimes of [10] can be summarised as follows. Let $V \equiv (\xi^\mu \xi_\mu)^{1/2}$ be the norm of the timelike Killing vector, $\xi^\mu$, of some static, asymptotically flat,
vacuum spacetime. Assume that the spacetime has non-vanishing mass. In a static spacetime, there are coordinates \((t, x^α)\) for which the spacetime metric takes the form

\[
\tilde{g}^{\text{static}} = V^2 dt^2 + V^{-2} \tilde{\gamma}_{\alpha\beta} dx^\alpha dx^\beta,
\]

where \(\tilde{h}^{\text{static}} \equiv V^{-2} \tilde{\gamma}_{\alpha\beta}\) denotes the metric of the hypersurfaces of constant \(t\), whereas \(\tilde{\gamma}_{\alpha\beta}\) is the metric of the quotient manifold —in the case of static spacetimes the quotient manifold and any surface of constant \(t\) can be identified. Denote by \(\tilde{\mathcal{S}}\) one of these (time symmetric) hypersurfaces of constant \(t\). The metric \(\tilde{h}_{\alpha\beta}\) satisfies on \(\tilde{\mathcal{S}}\) the time symmetric Hamiltonian constraint \(r[\tilde{h}^{\text{static}}] = 0\), whereas \(\tilde{\gamma}_{\alpha\beta}\) and \(V\) satisfy the \textit{vacuum static Einstein field equations}. Static spacetimes are usually analysed in terms of the quotient metric \(\tilde{\gamma}_{\alpha\beta}\). The asymptotic behaviour of the quotient metric \(\tilde{\gamma}_{\alpha\beta}\) is best understood in terms the behaviour of a conformally rescaled quotient metric \(\gamma_{\alpha\beta} \equiv (\Omega^{\text{static}})^2 \tilde{\gamma}_{\alpha\beta}\) in a neighbourhood, \(\mathcal{B}_0(i) \subset \tilde{\mathcal{S}}\), of the point at infinity \(i\). As before we have \(\tilde{\mathcal{S}} = \tilde{\mathcal{S}} \cup \{i\}\). Using the so-called \textit{conformal static equations} \textit{Beig & Simon} [2] have shown that there is a choice of \(\Omega^{\text{static}}\) and coordinates for which \(\tilde{h}_{\alpha\beta}\) and \(\tilde{\gamma}_{\alpha\beta}\) are analytic in \(\mathcal{B}_0(i)\) and satisfy the boundary conditions (32a)-(32c). The crucial observation in [10] is that the conformally rescaled metric

\[
\tilde{h}_{\alpha\beta} = \frac{1}{2} (1 + V)^4 V^{-2} \tilde{\gamma}_{\alpha\beta},
\]

also satisfies the time symmetric Hamiltonian constraint. The metric \(\tilde{h}_{\alpha\beta}\) is asymptotically Euclidean with vanishing mass. Let

\[
\bar{\Omega} \equiv \frac{1}{\sqrt{2}} (1 + V)^2 V^{-1} \Omega^{\text{static}}, \quad \tilde{h}_{\alpha\beta} \equiv (\Omega^{\text{static}})^2 \tilde{h}_{\alpha\beta} = \bar{\Omega}^2 \tilde{\gamma}_{\alpha\beta}.
\]

The pair \((\tilde{h}_{\alpha\beta}, \bar{\Omega})\) satisfies the time symmetric conformal Hamiltonian constraint, equation (6a), with \(\Sigma = 0\). Both \(\bar{\Omega}\) and \(\tilde{h}_{\alpha\beta}\) are analytic in \(\mathcal{B}_0(i)\) and, furthermore, they satisfy the boundary conditions (32a)-(32c). The pair \((\tilde{h}_{\alpha\beta}, \bar{\Omega})\) can be used to construct initial data for the conformal Einstein field equations of section 3.3 —this data consists essentially of the value of the rescaled Weyl tensor and the Schouten tensor on the initial hypersurface. This initial data is obtained out of algebraic manipulations of the conformal constraint equations. Although \(\tilde{h}_{\alpha\beta}\) and \(\bar{\Omega}\) are given in a gauge which makes them analytic around \(i\), the data for the conformal field equations need not be. It is a non-trivial result that the data constructed out of this procedure is indeed analytic around \(i\). The time evolution of this data yields —using, say, the Cauchy-Kowalevska theorem—in a 4-dimensional neighbourhood of \(i\) an analytic spacetime metric \(g_{\mu\nu}\) and an analytic conformal factor \(\Theta\).

In the sequel this local existence problem will be solved by a different method. Under these circumstances, the physical metric, \(\tilde{g}_{\mu\nu} = \Theta^{-2} g_{\mu\nu}\), obtained on the timelike future \(J^+(i)\) of \(i\) in this evolution is a solution of the Einstein vacuum field equations with an analytic structure at past null infinity, for which the point \(i\) represents a regular past timelike infinity \(i^-\). These solutions of the Einstein field equations are \textit{radiative spacetimes} in the sense discussed in the introduction. Note that due to the time reflection symmetry, the same holds for the pair \((I^-(i), g_{\mu\nu})\) where \(i\) now represents the regular future timelike infinity \(i^+\). On the complement of \(J^+(i) \cup J^-(i)\) —these are the points in the development which are spacelike related to \(i\) —one also has a solution of Einstein’s field equations in a neighbourhood of \(i\), for which the point \(i\) now represents the point spatial infinity \(i^0\). Definitions of the causal sets \(J^\pm\) and \(I^\pm\) can be found in [22, 32].

The results of [10] relevant for our analysis are presented in the following theorem.

**Theorem 1.** Given an analytic solution to the conformal static equations on \(\mathcal{B}_0(i) \subset \tilde{\mathcal{S}}\), there exists a solution, \((\tilde{h}_{\alpha\beta}, \bar{\Omega})\), to the time symmetric conformal Hamiltonian constraint on \(B_0(i)\) satisfying

(i) it is asymptotically Euclidean and regular in a neighbourhood \(B_0(i)\),

(ii) it has vanishing ADM mass,
(iii) the spinorial fields $\phi_{ABCD}$ and $\Theta_{ABCD}$ constructed out of $(\bar{\eta}_{\alpha\beta}, \bar{\Omega})$ are analytic in $\mathcal{B}_a(i)$.

It was long conjectured that given a series of multipole moments satisfying an appropriate convergence condition, there exists a static solution of the Einstein equations with precisely those moments. This conjecture was proved in full generality in [19] —see also [1, 23] for partial results and an alternative approach. These results show that the metric, once written in a certain gauge, is uniquely determined by the multipole moments. Thus there is an infinite family of static metrics analytic in a neighbourhood of $i$. This family is parametrised by the multipole moments. As a consequence of Theorem 1 it follows that we have an infinite family radiative spacetimes constructed in the above mentioned way that will serve as reference spacetimes for our stability analysis.

4.3 Construction of radiative reference spacetimes using conformal Gaussian gauges

In this section we discuss how conformal Gaussian gauge systems can be used to construct radiative spacetimes out of the Cauchy initial data provided by Theorem 1. The original construction discussed in [11] makes use of a propagation system derived from the “original” conformal field equations of [5, 6, 8]. As mentioned before, due to the analyticity of the setting, existence of a unique analytic solution in a spacetime neighbourhood of the point $i$ follows from the Cauchy-Kowalewska theorem —see e.g. [4].

Here we follow a different approach. Our intention is to use the development of the data given by Theorem 1 as a reference solution from which a stability result will be derived. As seen in [27], the discussion of the properties of the conformal boundary of radiative spacetimes simplifies with the use of a conformal Gaussian system. Thus, it is natural to discuss the structure and existence of the reference solution using the same type of gauges. In our approach, existence of a solution of the propagation equations implied by the extended conformal field equations and the conformal Gaussian gauge follows from a variation of a theorem of Kato —see [9, 26, 27].

4.3.1 Setting up conformal Gaussian coordinates

Let $(\bar{\eta}_{\alpha\beta}, \bar{\Omega})$ on $\bar{\mathcal{S}}$ be one of the solutions to the time symmetric conformal Hamiltonian constraint, equation (6a), given by Theorem 1. It is assumed that $\bar{\Sigma} = 0$. In what follows all quantities derived from this initial set and from its time development will be distinguished with an overbar. Although it is customary to make use of a conformal factor $\bar{\Omega}$ which is non-negative, it will be convenient to consider a non-positive conformal factor on $\bar{\mathcal{S}}$ —the reason for this will be evident in section 4.3.3. Accordingly, if one takes the conformal factor of Theorem 1 with an extra factor of $-1$, so that it is negative at least in a neighbourhood away from $i$. One has that $\bar{\Omega}$ satisfies the appropriate modification of the boundary conditions (32a)-(32b). Namely,

\[ \bar{\Omega} < 0 \text{ on } \bar{\mathcal{S}} \setminus \{i\}, \quad \bar{\Omega}(i) = 0, \quad D_A\bar{\Omega}(i) = 0, \quad D_A D_B \bar{\Omega}(i) = 2h_{AB}(i). \]  

(33)

On a suitably small neighbourhood, $\mathcal{B}_a(i) \subset \bar{\mathcal{S}}$, we use the coordinates $x^A$, $A = 1, 2, 3$ centered at $i$ and consider the following initial data for a congruence of conformal geodesics:

\[ \bar{\tau}_* = 0, \quad \dot{x}^\mu = \bar{\eta}^\mu, \quad \bar{\Theta}_* = \bar{\Omega}, \quad \dot{\bar{\Theta}}_* = 0, \quad \bar{d}_* \equiv \bar{\Theta}_* \bar{b}_* = (d\bar{\Theta})_* \] 

(34)

The coordinates $x^A$ are extended off $\bar{\mathcal{S}}$ by dragging along the congruence of conformal geodesics to obtain conformal Gaussian coordinates as discussed in section 3.2. It follows that $\bar{d}_* = 0$ at $i$ and using $\bar{d}_a(\bar{\tau}) = \bar{d}_a(0)$, we get $\bar{d}_a = D_a(\bar{\Theta})$. The constraint (17) applied to our situation gives the initial data for $\bar{\Theta}$ on $\mathcal{B}_a \setminus \{i\}$, which must hence be positive there. The initial value at $i$ is obtained taking the corresponding limit. Making use of normal coordinates centred in $i$, one finds that since $\bar{\Omega} = O(|x|^2)$, we have $h^{AB}(\bar{\Theta}, \bar{D}_A\bar{\Theta}) = O(|x|^2)$ and thus $\bar{\Theta} = O(1)$. Hence $\bar{\Theta}_* > 0$ everywhere on $\bar{\mathcal{S}}$. It follows from equation (16) that along each conformal geodesic the conformal factor $\bar{\Theta}$ is given by

\[ \bar{\Theta}(\bar{\tau}) = \bar{\Omega} + \frac{1}{2} \bar{\Theta}_* \bar{\tau}^2 = \bar{\Omega} \left( 1 - \frac{\bar{\tau}^2}{\omega^2} \right) \] 

(35)
where

\[ \bar{\omega} \equiv \sqrt{\frac{2\Omega}{\Theta}} \quad \bar{\omega}(i) = 0. \]

Define now the conformal boundary, \( \mathcal{I} \), in a natural way as the locus of points in the development of the data on \( \bar{S} \) for which \( \bar{\Theta} = 0 \) and \( d\bar{\Theta} \neq 0 \). It is easy to see that a conformal geodesic with data given by (34) passes through \( \mathcal{I} \) whenever \( \bar{\tau} = \pm \bar{\omega} \). Note that every conformal geodesic, except for the one passing through \( i \), intersects \( \mathcal{I} \) twice, once to the past of \( \bar{S} \) and once to the future.

**Remark.** We shall always work on a neighbourhood \( \mathcal{B}_0(i) \subset \bar{S} \) which is assumed to be sufficiently small to ensure \( h^{\alpha\beta}D_{\alpha}\bar{\Omega}D_{\beta}\bar{\Omega} \neq 0 \) in a neighbourhood of \( i \).

### 4.3.2 Existence and conformal structure of the reference spacetime

We discuss now the existence of solutions to the propagation system (31) with data prescribed on \( \mathcal{B}_0(i) \subset \bar{S} \). The initial data for this initial value problem is obtained through the conformal constraint equations on \( \bar{S} \) \([8, 10]\) and satisfies on \( \mathcal{B}_0(i) \subset \bar{S} \)

\[
\begin{align*}
e^0_{AB} &= 0, \quad &\zeta_{ABCD} &= \zeta_{ABCD}[\bar{h}], \quad &\chi_{(AB)CD} &= 0, \quad &\Theta_{ABCD} &= -\frac{1}{\Omega} D_{(AB}D_{CD)}\bar{\Omega} + \frac{1}{12} h_{ABCD}, \quad &\phi_{ABCD} &= \frac{1}{\Omega^2} D_{(AB}D_{CD)}\bar{\Omega} + \frac{1}{\Omega} s_{ABCD}, \quad &\eta = 0, \quad \eta_{AB} = \sigma^e_{AB},
\end{align*}
\]

where \( h_{ABCD} \equiv -\epsilon_{ACDEB} \) and \( \sigma^e_{AB} \) are the spatial Infeld-van der Waerden symbols. The spinorial fields \( \zeta_{ABCD}[\bar{h}] \) and \( s_{ABCD} \) are the connection coefficients and the trace-free Ricci curvature of the 3-metric \( \bar{h}_{\alpha\beta} \).

Since \( \mathcal{B}_0(i) \) is assumed to be a simply connected neighbourhood of \( i \), we can find a diffeomorphism \( \phi_B \) onto \( \mathcal{U} \subset S^3 \). We extend the initial data to \( S^3 \) in the following way. The initial values for

\[
\begin{align*}
e^0_{AB}, \quad f_{AB}, \quad \chi_{(AB)CD}, \quad \eta, \quad \eta_{AB}
\end{align*}
\]

are constant and are extended with their respective values to all of \( S^3 \). For

\[
\Theta_s, \quad \bar{d}_s, \quad \zeta_{ABCD}, \quad \Theta_{ABCD}, \quad \phi_{ABCD}
\]

we use the linear extension operator \( E \) given in section 2.3 to obtain fields defined over \( S^3 \). The extended fields will be denoted on \( S^3 \) in the same way as in (37).

The vectorial unknown

\[
u(\tau, x) \equiv (\text{Re}(v)(\tau, x), \text{Im}(v)(\tau, x), \text{Re}(\phi)(\tau, x), \text{Im}(\phi)(\tau, x))
\]

is viewed as a function of \( \tau \) with values in the Sobolev space \( H^s(S^3, \mathbb{R}^N) \). Let \( u_0(x) = u(0, x) \) denote the extended initial data on \( S^3 \) as described in the previous paragraphs. For \( \delta \in \mathbb{R}, \ m \in \mathbb{N}, \ m \geq 2, \) define set

\[
D_\delta^m = \left\{ w \in H^m(S^3, \mathbb{R}^N) \mid (z, A^0(w)z) > \delta(z, z), \forall z \in \mathbb{R}^N \right\}, \quad (38)
\]
where $A^0(w)$ denotes the matrix valued function defined by the symmetric hyperbolic system \((31)\), and $(\cdot, \cdot)$ is the standard scalar product on $\mathbb{R}^N$. It can be verified that

$$A^0(u_0) = \text{diag}\left(1, \ldots, 1, \frac{1}{\sqrt{2}}, \sqrt{2}, \frac{1}{\sqrt{2}}, \sqrt{2}, \frac{1}{\sqrt{2}} \right).$$

and hence $u_0 \in D_\delta^m$ for $0 < \delta < 1/\sqrt{2}$.

One has the following local existence result.

**Theorem 2.** Suppose $m \geq 4$. There exists a $\bar{T}_0 > 0$, a subset $D_{\bar{T}_0} \subset D_\delta^m$ and a unique solution $u(\tau)$ of equation \((31)\) defined on $[-\bar{T}_0, \bar{T}_0]$ with initial data $u_0$ on $\bar{S} \simeq S^3$ and such that

$$u \in C(-\bar{T}_0, \bar{T}_0; D_{\bar{T}_0}) \cap C^1(0, T; H^{m-1}(S^3, \mathbb{R}^N)),$$

Furthermore, $u \in C^\infty([-\bar{T}_0, \bar{T}_0] \times S^3)$ and $\bar{T}_0$ can be chosen such that $u(\tau)$ has non-vanishing Jacobi fields $\eta_{AB}$, so that the solution is free of conjugate points in $[-\bar{T}_0, \bar{T}_0]$.

**Proof.** The first part of the theorem follows from the generalisation given in \cite{31} —see also \cite{27}— of Kato’s existence and stability result for quasilinear symmetric hyperbolic systems \cite{26}. For the Jacobi fields we observe that $\eta_{AB}$ satisfies an ordinary differential equation along the curves of the congruence. Hence there is a minimum interval for which the fields do not vanish. The lack of conjugate points then follows from our earlier discussion. \(\square\)

**Remark 1.** It follows that on $(-\bar{T}_0, \bar{T}_0)$ the solution, as given by the theorem, is of class

$$H^m((-\bar{T}_0, \bar{T}_0) \times S^3) \subset C^{m-2}((-\bar{T}_0, \bar{T}_0) \times S^3).$$

**Remark 2.** Let $\mathcal{M} \equiv [-\bar{T}_0, \bar{T}_0] \times S^3$ and denote the metric on $\mathcal{M}$ by $\bar{g}_{\mu\nu}$. The spacetime $(\mathcal{M}, \bar{g}_{\mu\nu})$ will be used as the reference spacetime for our stability analysis later on. Let $D(B_{a}(i))$ denote the domain of dependence of $B_{a}(i)$ —see e.g. \cite{22, 32} for definitions. The value of $\bar{T}_0$ can be chosen small enough so that

$$\mathcal{M} \equiv \mathcal{M} \cap J^-(i) \subset D(B_{a}(i)) \quad \text{for } a = 0, 1.$$ \hfill (39)

The finite reduction of $\bar{T}_0$ is not a problem in our scheme as any $\bar{T}_0 > 0$ corresponds to an infinite time interval in the physical description. As it will be seen in the sequel, it may be necessary to further reduce $\bar{T}_0$ by a finite (non-negative) amount.

The spacetime $(\mathcal{M} \setminus \mathcal{S}, \bar{g}_{\mu\nu})$ is conformally related to a vacuum spacetime, $(\mathcal{M} \setminus \mathcal{S}, \Theta^{-2}g_{\mu\nu})$, with vanishing cosmological constant. The spacetime $(\mathcal{M} \setminus \mathcal{S}, \Theta^{-2}g_{\mu\nu})$ is a radiative spacetime as the conformal metric $g_{\mu\nu}$ and associated fields extend smoothly through the conformal boundary $\mathcal{S}$ and furthermore, there exists a point $i^+ = (0, i) \in \{0\} \times \mathcal{S} \subset \mathcal{M}$ satisfying (by construction)
the conditions to be a future timelike infinity —cfr. the discussion in the introduction. As a result of Lemma [1] it follows that the conformal constraints are only propagated in $D(B_s(i))$. Consequently, the region $[-T_0, T_0] \times S^3 \setminus D(B_s(i))$ is of no physical interest as it does not describe a vacuum spacetime. It is nevertheless relevant to use the whole of $(\mathcal{M}, \check{g}_{\mu\nu})$ for the correct formulation of the stability part of our work. In the following, in a slight abuse of language, we will refer to $(\mathcal{M}, \check{g}_{\mu\nu})$ as the purely radiative reference solution.

### 4.3.3 Hyperboloidal data for the radiative reference spacetime

The conformal affine parameter $\tau$ defines, in a natural way, a foliation of the manifold $\mathcal{M}$. Let $S_\tau$ denote the surfaces of constant $\tau$. For fixed $\tau$ one has that $S_\tau$ is diffeomorphic to $S^3$. Let $\tau_0 \in (0, T_0)$ and define

$$S_0 \equiv \{-\tau_0\} \times S^3, \quad Z \equiv \{p \in S_0|\bar{\Theta} = 0\}.$$  

The set $S_0$ intersects null infinity in a hyperboloidal way. More precisely, $Z$ divides $S_0$ into two regions: one where $\bar{\Theta} < 0$ and one where $\bar{\Theta} > 0$. The latter contains the origin and will be denoted by $\hat{\mathcal{H}}$. Hence

$$\hat{\mathcal{H}} \equiv \{p \in S_0|\Theta > 0\}.$$  

The latter definition justifies the choice of a negative conformal factor $\hat{\Omega}$ made in the previous sections. We rewrite the conformal parameter $\tau$ and the conformal factor of the reference solution given by equation (35), so that their initial surface is $S_0$. Define

$$\tau \equiv \bar{\tau} + \tau_0, \quad \check{\Theta}(\tau) \equiv \bar{\Theta}(\tau - \tau_0)$$  

so that $\tau = 0$ on $S_0$ and

$$\check{\Theta}(\tau) = \hat{\Omega} \left(1 - \frac{\tau_0^2}{\bar{\omega}^2}\right) - \check{\omega}_s \tau_0 \tau + \frac{1}{2} \tau^2 \check{\omega}_s$$  

$$= \hat{\Omega} \left(1 - \frac{\tau_0^2}{\bar{\omega}^2}\right) + 2 \frac{\tau_0}{\bar{\omega}^2} \tau - \frac{1}{2} \tau^2 \check{\omega}_s.$$  

The initial value of $\check{\Theta}$ on $S_0$ will be denoted by $\hat{\Omega}$. Recalling that $\tau_\pm(x^4) = \pm \bar{\omega}(x^4)$ gives the location of null infinity, we can see that

$$\hat{\Omega} > 0 \iff |\bar{\omega}| > |\tau_0|,$$

$$\hat{\Omega} = 0 \iff |\bar{\omega}| = |\tau_0|,$$

$$\hat{\Omega} < 0 \iff |\bar{\omega}| < |\tau_0|,$$

and hence $\hat{\Omega}$ is the correct boundary defining function on $S_0$ with $Z = \{p \in S_0|\hat{\Omega} = 0\} = \partial \hat{\mathcal{H}}$. One can briefly verify that the new conformal factor $\check{\Theta}$ gives the correct location of null infinity. If $\Omega \neq 0$ we get that $\check{\Theta} = 0$ whenever

$$\tau = \frac{-2\tau_0}{\bar{\omega}^2} \pm \sqrt{\frac{4\tau_0^2}{\bar{\omega}^2} + \frac{4}{\bar{\omega}^2} \left(1 - \frac{\tau_0^2}{\bar{\omega}^2}\right)} = \tau_0 \pm \hat{\omega}.$$  

Along the unique conformal geodesic $\gamma$ passing through the point $N$ on $S^3$ at all times, we have $\hat{\Omega} = 0$ and

$$\check{\Theta} = \frac{1}{2} \check{\omega}_s (\tau - \tau_0)^2.$$  

Thus, along $\gamma$ one has that $\check{\Theta} = 0$ only when $\tau = \tau_0$. This corresponds to the point $i$ in the Cauchy hypersurface $\tilde{S}$.

In what follows, let

$$\tilde{u}(\tau, x) \equiv u(\tau - \tau_0, x).$$
For consistency, we shall denote entries of \( \ddot{u} \), as defined above, also with \( ^* \). The entries of \( \ddot{u} \) imply on \( S_0 \) data for the symmetric hyperbolic system \([31]\). We shall write \( \dot{u}_0(x) = \ddot{u}(0, x) \). The data \( \dot{u}_0 \) are not truly hyperboloidal data. The reason for this is the following. The frame vector \( \dot{e}_0 \) which on \( \mathcal{S} \) is normal to this surface, ceases to be normal to the surfaces \( S_{\tau} \), of constant \( \tau \) if \( \tau \neq 0 \). This is a feature that differentiates conformal geodesics from standard geodesics. This phenomenon manifests itself in the fact that the components \( e_{0}^{AB} \) of the space spinor decomposition of \( \dot{e}_0 \) according to formul\( e^{(22a)} \) and \( e^{(22b)} \), which are zero on \( \mathcal{S} \), become, in general non-zero off \( \mathcal{S} \).\(^1\) As a consequence, the spatial vectors of the frame \( \dot{e}_a \), \( a = 1, 2, 3 \) which span the tangent bundle to \( \mathcal{S} \), \( T\mathcal{S} \), in general, pick up components off the tangent bundle \( T\mathcal{S}_{\tau} \). Accordingly, the data \( \dot{u}_0 \), which is expressed in terms of the spatial frame \( \dot{e}_a \) \( (\dot{e}_{AB}) \) is not intrinsic to \( S_0 \). We shall call this type of data tilted hyperboloidal data.

In order to relate the tilted data \( \dot{u}_0 \) to proper hyperboloidal data one proceeds as follows. Denote the normal vector field to \( S_0 \) by \( \hat{n}^\mu \), \( \hat{h}_{\alpha\beta} \) and \( \hat{K}_{\alpha\beta} \) respectively. Note that \( K_{\alpha\beta} \) and \( \chi_{\alpha\beta} \) are two conceptually different quantities — the former is related to the hypersurface, the later to the conformal geodesic congruence. Choose a \( h \)-orthonormal frame \( \dot{\hat{e}}_i \), \( i = 1, 2, 3 \) and set \( \dot{\hat{e}}_0 = n \), so that \( \dot{\hat{e}}_k \), \( k = 0, 1, 2, 3 \), forms a \( g \)-orthonormal frame on \( S_0 \). The frames \( \{v, \dot{e}_a\} \) and \( \{n, \dot{\hat{e}}_a\} \) are related by a Lorentz transformation \( \Lambda^i_j = \Lambda^i_j(x) \), which will be regarded as a matrix valued function over \( \mathcal{S}^3 \), as follows

\[
\Lambda^i_j : \dot{e}_i \rightarrow \ddot{\hat{e}}_j, \quad \Lambda^i_j\Lambda^j_k = \delta^i_k, \quad (40a)
\]
\[
\dot{\hat{e}}_j = \Lambda^k_j \ddot{\hat{e}}_k, \quad (40b)
\]
\[
\ddot{\hat{e}}_k = \Lambda^j_k \dddot{\hat{e}}_j. \quad (40c)
\]

The fields \( \dot{f}_i, \dot{\Gamma}^i_{jk}, \dot{L}_{ij}, \dot{d}_{ijkl} \) are the frame components of spacetime quantities and the Lorentz transformation induces transformed components \( \ddot{f}_i, \ddot{\Gamma}^i_{jk}, \ddot{L}_{ij}, \ddot{d}_{ijkl} \) in the canonical way. For example

\[
\ddot{d}_{ijkl} = \Lambda^m_i \Lambda^n_j \Lambda^p_k \Lambda^q_l \ddot{d}_{mnqp}.
\]

The SL(2,C)-spinorial counterparts of the above tensors can be obtained by suitable contractions with the constant Infeld-van der Waerden symbols. These, in turn, can be space-spinor split with respect to \( \ddot{\hat{e}}^\mu \equiv \sqrt{2}\ddot{\hat{e}}_0 \) to obtain proper hyperboloidal data on \( \mathcal{H} \).

## 5 Radiative spacetimes from perturbed hyperboloidal data

Given one of the radiative spacetimes discussed in section 4, it is natural to ask whether it is possible to establish for these spacetimes an analogue of the semiglobal stability results for the Minkowski spacetime discussed in [10, 27]. In this section we establish such a result. This result generalises the aforementioned stability results to the case where the reference solutions are not flat.

### 5.1 On hyperboloidal data

In order to discuss appropriately the notion of closeness of two hyperboloidal initial data sets, we make use of the notion of \textit{limit of spacetimes} as discussed in e.g [21]. Assume that one has a smooth family of hyperboloidal initial data sets \( (\mathcal{H}^\varepsilon, h^\varepsilon_{\alpha\beta}, K^\varepsilon_{\alpha\beta}, \Omega^\varepsilon, \Sigma^\varepsilon) \), parametrised by \( \varepsilon \geq 0 \), such that

\[
(\mathcal{H}^0, h^0_{\alpha\beta}, K^0_{\alpha\beta}, \Omega^0, \Sigma^0) = (\mathcal{H}, h_{\alpha\beta}, K_{\alpha\beta}, \Omega, \Sigma),
\]

where \( \mathcal{H}^\varepsilon \) are simply connected 3-dimensional manifolds with boundary \( \partial \mathcal{H}^\varepsilon \). The fields \( (h^\varepsilon_{\alpha\beta}, K^\varepsilon_{\alpha\beta}, \Omega^\varepsilon, \Sigma^\varepsilon) \) satisfy the vacuum constraints, equations \([6a] \) and \([6b] \), with

\[
\Omega^\varepsilon = 0, \quad d\Omega^\varepsilon \neq 0 \text{ on } \partial \mathcal{H}^\varepsilon, \quad \Omega^\varepsilon > 0 \text{ on the interior of } \mathcal{H}^\varepsilon.
\]

\(^1\)Note, however, that in the case of the conformal Minkowski spacetime in the form discussed in [27], tangent vectors to the congruence of conformal geodesics are always normal to the surfaces of constant affine conformal parameter.
As the manifolds $\mathcal{H}^\varepsilon$ are taken to be simply connected, there exists a family of diffeomorphisms

$$\phi^\varepsilon : \mathcal{H}^\varepsilon \to \mathcal{U} \subset S^3, \quad \phi^\varepsilon(\partial \mathcal{H}^\varepsilon) = \partial \mathcal{U},$$

with $\mathcal{U}$ as in section 2.2. In particular,

$$\Phi^\varepsilon \equiv \phi^{-1} \circ \phi^\varepsilon : \mathcal{H}^\varepsilon \to \tilde{\mathcal{H}} \quad \text{with} \quad \phi \equiv \phi^0,$$

is a smooth family of diffeomorphisms between $\mathcal{H}^\varepsilon$ and $\tilde{\mathcal{H}}$, such that $\Phi^0$ is the identity map. Note that the locus of points for which $\Omega^\varepsilon = 0$ coincides as sets on $S^3$, by construction, for all $\varepsilon \geq 0$.

Let $\{\xi^a_\varepsilon\}, \ a = 1, 2, 3$ denote a smooth family of $h^\varepsilon$-orthonormal frames such that

$$\xi^a_\varepsilon = \tilde{\xi}^a, \ a = 1, 2, 3,$$

with $\tilde{\xi}^a$ as given in the end of section 4.3.3. The Levi-Civita connection $D^\varepsilon$ of the metric $h^\varepsilon_{\alpha\beta}$, together with the frame $\xi^a_\varepsilon$, imply the 3-dimensional Ricci tensor $\mathcal{R}[h^\varepsilon]_{\alpha\beta}$ and spin connection coefficients $\gamma^b_\alpha$, as well as their spinorial counterparts $\mathcal{R}^{ABC\varepsilon}_{\gamma} = \xi^a_\varepsilon K^{ABC\varepsilon}_{\gamma}$ and $\mathcal{R}^{A_1B_1C_1D_1}_{\xi} = \xi^a_\varepsilon \epsilon^{abc} \xi^{A_1B_1C_1D_1}_{\varepsilon}$, which are determined by the Schouten tensor and $\epsilon^0$-electric and $\epsilon_3$-magnetic parts of the rescaled Weyl tensor for the spacetime on $\mathcal{H}^\varepsilon$—see e.g. [8, 9]. Using the frame $\xi^a_\varepsilon$ and $\mathcal{R}^\varepsilon \equiv \sqrt{2}\epsilon^0_\varepsilon$, their spinorial counterparts can be readily calculated. This procedure produces proper hyperboloidal data—in the sense discussed in section 4.3.3. We note that so far the data is only given on $\mathcal{H}^\varepsilon$, respectively, $\phi^\varepsilon(\mathcal{H}^\varepsilon) = \mathcal{U} \subset S^3$. We use the extension operator $E$ given in section 2.3 to extend the proper hyperboloidal data to the whole of $S^3$, bearing in mind that outside $\mathcal{U}$ the data may violate the vacuum constraints (6a) and (6b).

In the following, the parameter $\varepsilon$ will be dropped from the expressions, in order to ease the presentation.

### 5.1.1 Local existence for hyperboloidal data

From studying the proper hyperboloidal data on the hypersurface $S_0$ in the reference solution one can see that this data does not include all the quantities that are needed for the tilted hyperboloidal data used for (31). We are missing the data for $\epsilon^0_{AB}, f_{AB}, \xi_{ABCD}, \chi_{ABCD}$.

In order to consider the construction of tilted hyperboloidal data for the conformal propagation equations we shall first develop a small part of the perturbed spacetime. For this we need a local existence result for the development of data constructed using an auxiliary congruence of conformal geodesics which departs the initial hyperboloid orthogonally. The information thus obtained complements the proper hyperboloidal data constructed above.

For the construction of the auxiliary congruence we use the following initial setup. Let $\dot{x} = n$, $\epsilon_k|S = \xi_k, \xi_{ABCD}|S = \xi_{ABCD} | h)$, $\chi_{(AB)CD}|S = K_{ABCD}$. As long as the data for $\mathcal{I}_{AB}$ is smooth and satisfies $(\mathcal{I}_{AB}, n) = 0$, i.e. $\mathcal{I}_{AB} = \mathcal{I}_{AB}$ it can be chosen arbitrarily, since we are only interested in local existence in some small neighbourhood of the initial surface. In particular, we can set $\mathcal{I}_{AB} = 0$ again.

We can now proceed to our local existence result for the unknown

$$\underline{u} = (\xi^0_{AB}, \xi_{AB}, f_{AB}, \xi_{ABCD}, \chi_{ABCD}, \chi_{ABCD}, \chi_{ABCD}, \chi_{ABCD}, \chi_{ABCD}).$$

The underline in the above spinors indicates that they are expressed with respect to the frame $\xi^a_\varepsilon$. The proposition below is essentially very similar to Theorem 2. This is not a stability result, but purely a local existence for the perturbed data. For this we define $D^m_\varepsilon$ analogously to (38).

**Proposition 1.** Suppose $m \geq 4$. There exists a $T_0 > 0$, a subset $D^m_\varepsilon \subset D^m_\varepsilon$ and a unique solution $\underline{u}(\tau)$ of equation (31) defined on $[-T_0, T_0]$ with initial data $\underline{u}_{0}\varepsilon$ on $S' \simeq S^3$ and such that

$$u \in C(-T_0, T_0 ; D^m_\varepsilon) \cap C^1 (0, T; H^{m-1}(\mathbb{R}^3, \mathbb{R}^N)).$$

Furthermore, $\underline{u} \in C^\infty([-T_0, T_0] \times S^3)$ and $\underline{u}(\tau)$ has non-vanishing Jacobi fields $\mathcal{J}_{AB}$, so that the solution is free of conjugate points in $[-T_0, T_0]$. 

Remark: The local existence theorem clearly applies to \((\mathcal{H}, \dot{h}_{\alpha\beta}, \dot{K}_{\alpha\beta}, \dot{\Omega}, \dot{\Sigma})\) and the resulting spacetime coincides locally with the reference spacetime \((\mathcal{M}, \dot{g}_{\mu\nu})\). One could then consider the derivation of a result similar to the stability of Minkowski in [27]. However it should be noted that there is no guarantee that this setup covers \(i^+\) or \(\mathscr{I}^+\) of \((\mathcal{M}, \dot{g}_{\mu\nu})\) and its perturbations. Thus a semiglobal existence and stability result can not be guaranteed.

5.1.2 Construction of tilted hyperboloidal data

As a result of Proposition 1 we have a spacetime slab \(N \equiv [-T_0, T_0] \times S^3\) with metric \(g_{\mu\nu}\) which describes a vacuum spacetime on \(\mathcal{M} \equiv N \cap D(\mathcal{H})\). For sufficiently small \(\varepsilon\), one can regard the 3-manifold \(S\) as a hypersurface of a spacetime \((N, g_{\mu\nu})\). On this hypersurface we now want to give tilted hyperboloidal data for (31). For this we will evolve a conformal geodesic congruence, that departs \(S\) in the same way as in the reference solution. Thus we can locally obtain all the components of the unknown \(u\) on \(S\) and use this as initial data \(u_0\) for (31).

We know fix the initial data for the frame and the congruence. As before, let \(n\) denote the unit normal to \(S\). Then \(\{e_i\} \equiv \{n, e_a\}\) is an orthonormal tetrad on \(S\). From \(e_j\) we construct a tilted orthonormal frame \(e_i\) on \(T N|_S\) via
\[
e_i = \Lambda^i_j e_j,
\]
where \(\Lambda^i_j\) is the same matrix valued function associated to the Lorentz transformation via \((40a)-(40c)\). It follows then that the timelike vector \(e_0\) has the same projections on the orthonormal frame \(e_j\) as \(\dot{e}_0\) has on \(\dot{e}_j\). That is \(g(e_0, \dot{e}_j) = \dot{g}(\dot{e}_0, \dot{e}_j)\). Note that \(\dot{e}_0\) is by construction tangent to a congruence of conformal geodesics. For discussions of the development of the perturbed initial data \((\mathcal{H}, \gamma_{\alpha\beta}, K_{\alpha\beta}, \Omega, \Sigma)\) which go beyond local existence, it will be important to consider a congruence of conformal geodesics that departs the hypersurface \(\mathcal{H}\) in the same way as the congruence of conformal geodesics in the reference spacetime arrives at \(\mathcal{H}\). For this reason we will use \(\dot{x} = e_0\).

As in the case of the reference spacetime \((\mathcal{M}, \dot{g}_{\mu\nu})\), we want to cover the development of the initial data set \((\mathcal{H}, h_{\alpha\beta}, K_{\alpha\beta}, \Omega, \Sigma)\) with a conformal geodesic coordinate system. The formulation of the evolution equations is dependent on a gauge for which \((f, \nu) = 0\). Taking this as well as \(d_{\mu} = \Theta f_{\mu} + \nabla_{\mu} \Theta\) into account, we have
\[
d_0 \equiv \langle d, e_0 \rangle = \Lambda^0_a \Sigma + \Lambda^a_0 \xi_a (\Omega), \tag{41}
\]
\[
d_a \equiv \langle d, e_a \rangle = \Omega f_a + \Lambda^0_a \Sigma + \Lambda^b_a \xi_b (\Omega). \tag{42}
\]
Assume that the perturbed hyperboloidal data \((\mathcal{H}, h_{\alpha\beta}, K_{\alpha\beta}, \Omega, \Sigma)\) satisfies the technical condition
\[
\dot{d}_a - \Lambda^b_a \Sigma - \Lambda^b_a \xi_b (\Omega) = O(\Omega), \tag{43}
\]
close to \(\partial \mathcal{H}\), where \(\dot{d}_a\) denote the value of the components of the 1-form \(d_{\mu}\) for the reference solution and \(O(\Omega)\) indicates that the quotient of the left hand side divided by \(\Omega\) goes to zero as
Ω goes to zero. Initial data for the congruence of conformal geodesics on which the conformal Gaussian system hinges will be prescribed such that:

\[ \dot{x} = e_0, \quad d_a = \hat{d}_a, \quad f_a = \Omega^{-1} \left( \hat{d}_a - \Lambda_a^0 \Sigma - \Lambda_a^b \mathcal{L}_b(\Omega) \right) \]

on \( \mathcal{H} \). As a result of the technical condition \((43)\) one has that that \( f_a \) has a finite limit at \( \partial \mathcal{H} \) —and is thus well defined there. This particular choice of initial data for the congruence of conformal geodesics provides an easy analysis of the structure of the conformal boundary of the spacetime —as will be seen in 5.2.

Initial values for \( \Theta_\star, \dot{\Theta}_\star, \ddot{\Theta}_\star \) which characterise the canonical conformal factor \( \Theta \) associated to the congruence of conformal geodesics defined by \((44a)-(44c)\) are given on \( \mathcal{H} \) by

\[ \Theta_\star = \Omega, \quad \dot{\Theta}_\star = \langle d, e_0 \rangle, \quad 2\Omega \ddot{\Theta}_\star = g^t(d, d)_\star, \]

with the corresponding limits for points on \( \partial \mathcal{H} \). As usual, the subscript \( \star \) indicates that the value of the functions are extended of the initial hyperboloid by requiring them to be constant along a given conformal geodesic with initial data given by \((44a)-(44c)\). It follows from the conformal Hamiltonian constraint, equation \((6a)\) that \( d \) is null on \( \partial \mathcal{H} \). Note that as a consequence of the conformal Hamiltonian constraint it follows that when \( \Omega = 0 \)

\[ \eta^{ij} d_i d_j = (\Lambda_0^0)^2 \Sigma^2 - \delta^{ab} \Lambda_a^c \mathcal{L}_c(\Omega) \Lambda_b^d \mathcal{L}_d(\Omega), \]

\[ = \Sigma^2 + D^a \Omega D_a \Omega = 0, \]

consistently with the choice of initial data for the conformal factor in \((45)\).

Initial data for the Jacobi field associated to the congruence of conformal geodesics arising from \((44a)-(44c)\) is set by

\[ \eta_k = \hat{\eta}_k = \langle \hat{\eta}, \hat{e}_k \rangle, \quad \text{on } \mathcal{H}. \]

Finally, one needs to construct tilted data for the rescaled Weyl \((d_{\mu\nu\lambda\rho})\) and Schouten \((L_{\mu\nu})\) tensors. For this, one starts with the standard hyperboloidal data for these tensors implied by the conformal constraint equations. As a consequence of the local existence result given by Proposition \( \text{[1]} \), one can regard \( \mathcal{H} \) rightfully as a hypersurface of a spacetime. Let \( \hat{d}_{ijkl} \) and \( \hat{L}_{ij} \) denote the components of the Weyl and Schouten tensors with respect to the frame \( \{ \mathcal{L}_k \} \). The tilted initial data for the Schouten and rescaled Weyl tensors on \( \mathcal{H} \) is given by

\[ L_{ij} = \Lambda_i^m \Lambda_j^n \hat{L}_{mn}, \quad d_{ijkl} \equiv \Lambda_i^m \Lambda_j^n \Lambda_k^p \Lambda_l^q \hat{d}_{mnpq}. \]

The required spinorial data is then obtained by means of a space spinor decomposition of the spinorial counterpart of the above tensor.

### 5.2 Structure of the conformal boundary

In this subsection we analyse the structure of the conformal boundary of the development of hyperboloidal data which is close to hyperboloidal data of the reference radiative spacetime. Attention is focused, in particular, on the location of timelike infinity \( i^+ \). This investigation is possible as in the vacuum case the conformal factor can be calculated \textit{a priori} from the initial data.

Along the congruence of conformal geodesics with initial data on \( \mathcal{H} \) given by \((44a)-(44c)\), the canonical conformal factor, \( \Theta \), takes the form

\[ \Theta(\tau) = \Omega + \dot{\Theta}_\star \tau + \frac{1}{2} \ddot{\Theta}_\star \tau^2, \]
with \( \dot{\Theta} \) and \( \ddot{\Theta} \) given by the conditions in \( (45) \). As it is customary, denote by \( \mathcal{I} \) the locus of points in the development of the data \( (h_{ij}, \chi_{ij}, \Theta, \Sigma) \) for which \( \Theta = 0 \). For \( q \in \mathcal{I} \) to qualify as a candidate to be the timelike infinity of the development it needs to satisfy

\[
\Theta(q) = 0, \quad d\Theta(q) = 0, \quad \text{Hess}\Theta(q) \text{ non-degenerate.} \tag{47}
\]

We recall that the 1-form \( d\mu = \theta f_\mu + \nabla_\mu \Theta \) and that \( 2\Theta \dot{\Theta} = g^i(d, d) \) hold. Thus, for smooth \( f_\mu \) and \( \Theta \) we observe that on \( \mathcal{I} \), \( d\mu \) is null and can only vanish there if \( d\Theta = 0 \). Therefore the only point on \( \mathcal{I} \) at which all \( d\mu = \langle d, e_a \rangle \) can vanish, is at a candidate for \( i^+ \).

Define the function

\[
\Delta \equiv g^i(d, d_a) - \dot{\Theta}^2 = -\delta^{ab}d_ad_b,
\]

and observe that \( \Delta = 0 \) if and only if \( d_a = 0 \). We recall here that \( d_a \), and hence \( \Delta \), are constant along conformal geodesics —see e.g. \([13, 17]\).

Now, by construction of the reference solution from the initial Cauchy data given on \( \mathcal{S} \) and our choice of \( \mathcal{B}_0(i) \), there is a unique point in \( \mathcal{B}_0(i) \subset \mathcal{S} \) such that \( \Delta = 0 \), namely \( i \) itself. By our choice of coordinates, \( i \) lies at the origin of our spatial coordinate chart and is identified with the north pole of \( \mathcal{S}^+ \). Thus by Theorem \([2]\) and remark 2 following it, the point \( (\bar{\tau}, x^A) = (\bar{\tau}_0, 0, 0, 0) \) is the unique point in \( D(\mathcal{B}_0(i)) \) satisfying the conditions for \( i^+ \). This corresponds to \( (\tau, x^A) = (\bar{\tau}_0, 0, 0, 0) \) in the \( \mathcal{S}_\nu \)-adapted coordinates. Furthermore the points \( (\tau, 0, 0, 0) \) are by construction the only points in \( D(\mathcal{B}_0(i)) \) where \( \Delta = 0 \) for the reference solution. Given our choice for the initial data of \( d\mu \) in \([44b]\) for the perturbed spacetime, it follows immediately that the only point at which \( \Delta \) vanishes on \( \mathcal{H} \) is the origin. Therefore the only candidate for \( i^+ \) is the point, assuming it exists, where the conformal geodesic \( \gamma(\tau) = (\tau, 0, 0, 0) \) intersects \( \mathcal{I} \). Note that for general initial data it is possible for the development to contain a singularity before reaching \( i^+ \) —as in the case of black hole spacetimes.

The first condition in \( (47) \) implies

\[
\tau = -\frac{\dot{\Theta}_* \pm \sqrt{-\Delta}}{\Theta_*}, \tag{48}
\]

while the second one can be rewritten as

\[
d\Omega + d\Theta_* \tau + \frac{1}{2}d\dot{\Theta}_* \tau^2 + (\dot{\Theta}_* + \ddot{\Theta}_*)d\tau = 0. \tag{49}
\]

Together with \( \Delta = 0 \), both conditions imply that

\[
\tau = \tau_+ = -\frac{\dot{\Theta}_*}{\Theta_*} = -\frac{\Omega}{\Theta_*}. \tag{49}
\]

For sufficiently small perturbations, \( \dot{\Theta}_* = (d, v) = \Lambda_0^0 \Sigma + \Lambda_0^a \Sigma_a (\Omega) \) does not vanish and \( \tau_+ \) is finite. In fact later on the initial data will be restricted such that \( \tau_+ \) lies in the interval \((0, 2T_0)\).

Note that for the reference solution we have \( \tau_+ = T_0 \) and \( q = i \in \mathcal{S} \). By construction we have that Hess\(\Theta(q)\) is non-degenerate, and hence the \( i^+ \)-candidate in the reference spacetimes can be regarded, rightfully, as future timelike infinity.

### 5.2.1 Behaviour of the Hessian of \( \Theta \)

The point \( q = (\tau_+, 0, 0, 0) \) satisfies the first two conditions of \( (47) \). We now verify that the conditions discussed in the previous paragraphs imply that the Hessian of \( \Theta \) is non-degenerate at \( q \). For this we make use of the following expression:

\[
\nabla_\mu \nabla_\nu \Theta = \nabla_\mu \nabla_\nu \Omega + \nabla_\mu \nabla_\nu \dot{\Theta}_* \tau + \frac{1}{2} \nabla_\mu \nabla_\nu \ddot{\Theta}_* \tau^2 + 2 \nabla_\mu (\dot{\Theta}_* \nabla_\nu) \tau + 2 \tau \nabla_\mu (\dot{\Theta}_* \nabla_\nu) \tau + \nabla_\mu \nabla_\nu \Theta + \dot{\Theta}_* \nabla_\mu \tau + \dot{\Theta}_* \nabla_\nu \tau + \ddot{\Theta}_* \nabla_\mu \nabla_\nu \tau.
\]
Using the conformal Gaussian coordinates \((\tau, x^A)\) for our calculations and \(\alpha = \langle b, v \rangle_* = \hat{\Theta}_* / \Omega\) as well as \([49]\) we find that

\[
\begin{align*}
\nabla_0 \nabla_0 \Theta |_{i^+} &= \frac{1}{2} \Omega \alpha^2, \\
\nabla_0 \nabla_A \Theta |_{i^+} &= -\Omega \nabla_A \alpha, \\
\nabla_A \nabla_B \Theta |_{i^+} &= \frac{2}{\Omega \alpha} \eta^{ab} \nabla_A d_a \nabla_B d_b + \frac{2 \Omega}{\alpha^2} \nabla_A \alpha \nabla_B \alpha,
\end{align*}
\]

(50a) (50b) (50c)

where the right hand side is given in terms of initial data and spatial derivatives. Observe that \(\nabla_A d_a\) are identical to the ones in the reference solution and thus the only change arises from the perturbation of \(\alpha\). For data \(\hat{\Theta}_*\), \(\Omega\) close enough to the reference solution, in the sense that at the spatial origin \(\alpha\) and \(\nabla_A \alpha\) are sufficiently close to their counterparts in the reference solution, the Hessian will not be degenerate and thus \(q\) can be identified as the timelike infinity \(i^+\) of the perturbed spacetime. Note that one may rescale the initial data such that \(\Sigma\) remains fixed, while \(\Omega = \hat{\Omega}\). This greatly simplifies the analysis of the effect of perturbations of \(\alpha\) on the Hessian.

### 5.3 The stability result for purely radiative spacetimes

We are now in the position of presenting our main result —a stability result for purely radiative spacetimes arising from hyperboloidal data. The proof of this result follows the ideas of \([10, 27]\).

Let \(\hat{\alpha}_0\) and \(u_0\) denote, respectively, the tilted initial data for the radiative reference spacetime and for a perturbation thereof, as discussed in the previous section. Using the diffeomorphism \(\phi\), we can think of \(\hat{\alpha}_0\) and \(u_0\) as a vector-valued function over \(\mathbb{S}^3\). On the other hand, using \(\phi\), \(u_0\) can only be regarded as a vector valued function over \(U \subset \mathbb{S}^3\). On \(U\), let

\[
w \equiv u_0 - \hat{u}_0.
\]

Using the linear extension operator given in section 2.3 with \(Ew\) satisfying \([10b]\), we set on \(\mathbb{S}^3\)

\[
\hat{u}_0 \equiv Ew, \quad u_0 = \hat{u}_0 + \bar{u}_0.
\]

Consistently with the previous discussion of the initial data, one writes

\[
u = \hat{u} + \bar{u},
\]

(51)

where we recall that both \(\hat{u}\) and \(\bar{u}\) are regarded as vector valued functions over \(\mathbb{R} \times \mathbb{S}^3\). In particular, the quantities in

\[
\hat{u} \equiv (\hat{e}^A_{AB}, \hat{f}_{AB}, \hat{\xi}_{ABCD}, \hat{\chi}_{(AB)CD}, \hat{\Theta}_{ABCD}, \hat{\phi}_{ABCD}, \hat{\eta}, \hat{\bar{\eta}}_{AB}),
\]

describe the (non-linear) perturbations of the reference radiative spacetime. Substituting the Ansatz \([51]\) into the evolution system \([51]\) one obtains a symmetric hyperbolic system for \(\bar{u}\) of the form

\[
A^0(\hat{u} + \bar{u}) \cdot \partial_r \bar{u} + \sum_{r=1}^3 A^r(\hat{u} + \bar{u}) \cdot c_r(\bar{u}) + \hat{B}(\tau, x^A, \hat{u}, \bar{u}) \cdot \bar{u} = 0,
\]

(52)

with \(\hat{B}(\tau, x^A, \hat{u}, \bar{u})\) a matrix valued function with entries which are polynomials in \(\bar{u}\) of at most degree one and coefficients which are smooth functions on \(\mathbb{R} \times \mathbb{S}^3\). As discussed earlier, the matrix valued functions \(A^r(z)\) are symmetric, have entries which are polynomial in \(z\) of at most degree one and have constant coefficients. In contrast to the situation discussed in \([27]\), the matrix \(A^0(\hat{u}_0)\) is not diagonal. Nevertheless, Theorem \([2]\) guarantees that there exists a neighbourhood \(D_{\hat{T}_0} \subset D_\hat{y}^0\) such that \(\hat{u}_0 \in D_{\hat{T}_0}\). Hence,

\[
(z, A^0(\hat{u}_0)z) > \delta(z, z), \quad \forall z \in \mathbb{R}^N.
\]

The previous discussion allows us to formulate an existence and stability result for the solutions to equation \([52]\).
Figure 4: The perturbed radiative spacetime and its conformal boundary. Note that this figure is not a conformal diagram.

**Theorem 3.** Suppose $m \geq 4$. Let $u_0 = ˚u_0 + ˘u_0$ be hyperboloidal initial data satisfying the technical condition (43). Given $T \in (\bar{\tau}_0, 2\bar{\tau}_0)$, there exists $\varepsilon > 0$ such that:

(i) For $||\bar{u}_0||_m < \varepsilon$ there exist a unique solution $u = \bar{u} + \bar{u}$ to the conformal propagation equations (24a)-(24g) and (26) with minimal existence interval $\tau \in [0, T]$ and $u \in C^{m-2}([0, T] \times S^3)$.

(ii) The associated congruence of conformal geodesics contains no conjugate points in $[0, T]$.

(iii) At the origin one has $\tau_+ = -\Omega/\dot{\Theta} \in [0, T]$.

(iv) The Hessian as given by (50a)-(50c) is nondegenerate at $(\tau_+, 0, 0, 0)$.

The solution $u = \bar{u} + \bar{u}$ on $D^+(\mathcal{S})$ implies a $C^{m-2}$ solution $(\mathcal{M}, \tilde{g})$ to the vacuum Einstein field equations with vanishing cosmological constant, where $\tilde{g}_{\mu\nu} = \Theta^{-2}g_{\mu\nu}$ with $\Theta$ given by (46).

**Proof.** By hypothesis, we know that $\bar{u}_0 \in D_T \subset D^T_\Theta$. We thus satisfy the conditions of the variation of Kato’s theorem given in [10, 27]. Parts (i) and (ii) follow directly from Kato’s theorem. Parts (iii) and (iv) follow from the discussion in section 5.2. Due to Lemma 1 on the propagation of the extended conformal constraints, one knows that the existence of a solution of the propagation system (24a)-(24g) and (26) implies a solution to the full extended conformal Einstein field equations which in turns implies a solution to the vacuum Einstein field equations with vanishing cosmological constant.

One has the following corollary.

**Corollary 1.** Suppose the condition of Theorem 3 are satisfied, then a solution $(\mathcal{M}, \tilde{g})$, as given above, has a conformal boundary given by the set $\Theta = 0$. The conformal boundary consists of the set $\mathcal{I}^+$, which represents future null infinity, and the point $i^+$ given by $(\tau_+, 0, 0, 0)$, which represents timelike infinity.

Hence the conformal boundary of radiative spacetimes is shown to be stable, subject to the conditions of Theorem 3.

**Remark.** Note that the evolution of $u_0$ implies a unique solution $u$ and a spacetime, which is locally isometric with $(\mathcal{N}, g_{\mu\nu})$ as obtained in Proposition 1.

### 6 Conclusions

The conformal Einstein field equations have been used to formulate an hyperboloidal initial value problem by means of which one can address the nonlinear stability of a class of purely radiative spacetimes. Our analysis is particularly concerned with the structure of null infinity and the location of future timelike infinity, $i^+$. The use of a conformal Gaussian gauge system coordinates allows to calculate the location of the conformal boundary directly from the initial data, as well as analyse its structure. The results are a generalisation of previous results on Minkowski-like
In particular the conditions on the location of $i^+$ are given in a more general form, while the choice of initial data for the perturbed conformal geodesic congruence facilitates the stability analysis of the conformal boundary in comparison to [27].

The reference purely radiative spacetimes are constructed from static initial data using procedure introduced by Friedrich [10]. Recent results on the necessary and sufficient convergence conditions required on a series of multipolar data, one can conclude that there is an infinite family of reference purely radiative spacetimes.

We observe that we were able to prove the stability of radiative spacetimes and study their conformal boundary without requiring any explicit knowledge of the reference spacetime. All our analysis has been carried out in the abstract only relying on general properties of these spacetimes. This raises the hope that similar results may be possible for other classes of spacetimes.

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