1 Introduction

In a letter dated September 4, 1751, Leonhard Euler proposed the following problem to his friend, Christian Goldbach [4, Appendix B, p. 178]: In how many ways, $T_n$, can a convex polygon of $n$ sides be partitioned into triangles by diagonals which do not intersect within the polygon? The statement of the problem is quite easy to understand, yet its general solution leads to extraordinary difficulties. Euler worked out the first few cases $T_3 = 1$, $T_4 = 2$, $T_5 = 5$ by actually drawing the triangulations and computed (without diagrams) $T_6 = 14$, $T_7 = 42$, $T_8 = 132$, $T_9 = 429$, $T_{10} = 1430$ and then conjectured the formula

$$T_n = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n - 10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (n-1)}.$$ (1)

At the end of his letter Euler guessed the generating function

$$T(x) := 1 + 2x + 5x^2 + 14x^3 + 42x^4 + 132x^5 + \cdots = \frac{1 - 2x - \sqrt{1 - 4x}}{2x^2},$$

and added:

"However, the induction that I employed was pretty tedious, and I do not doubt that this result can be reached much more easily."

Goldbach answered a month later, observing that the generating function satisfies the quadratic equation

$$1 + xT = T^{1/2},$$ (2)

which is equivalent to infinitely many equations in the coefficients and suggested that they may lead to a direct proof of Euler’s formula (1).

It will be convenient to change the notation. We let

$$C_{n-2} := T_n.$$
Thus, $C_n$ is the number of triangulations of a convex $(n+2)$-gon, and since a little algebra shows that

$$T_n = \frac{1}{n-1} \binom{2n-4}{n-2},$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ is the binomial coefficient, we obtain the famous formula:

$$C_n = \frac{1}{n+1} \binom{2n}{n} \quad (3)$$

for what today is called the $n$th Catalan number. We note that $C_0 = 1$. The generating function is now

$$C(x) := C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n + \cdots \quad (4)$$

and is given by the formula

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Some time later Euler suggested the problem to Johann Andreas von Segner who, in 1758, published the recursion formula [3]:

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + \cdots + C_{n-1} C_1 + C_n C_0, \quad (5)$$

together with a combinatorial proof. However, he apparently was unaware of Euler’s explicit product formula [1] since he never mentions it. Instead he uses [5] to directly compute $C_n$ for $n = 1, 2, \ldots, 18$.

Euler’s letter is the first known publication of the Catalan numbers, and with Segner’s recursion formula all the tools were available for the modern development.

Today, of course, the standard treatment of $C_n$ is to use Segner’s recursion formula [5] to deduce Goldbach’s quadratic equation [2] which leads to the generating function [4] and finally to the explicit formula [3], but eighty years would pass before Binet would publish such a proof [1].

The Catalan numbers continue to fascinate mathematicians, and Richard Stanley recently published a book devoted exclusively to them [4] in which he presents 214 (!) combinatorial interpretations of $C_n$.

In 1967, Marshall Hall published a text on combinatorics [2] and on page 28 we find the following comment (the notation has been slightly altered):

“We observe that an attempt to prove the convergence of [4] on the basis of [5] alone is exceedingly difficult.”

Hall offers no suggestions towards such a proof. Moreover, a search of the voluminous literature on Catalan numbers has failed to find such a proof. Therefore we offer a proof in this paper.
2 Heuristics

In order to prove the convergence of (4) we have to show that the power series \( C(x) \) has a positive radius of convergence, \( R \). By the Cauchy–Hadamard theorem from elementary analysis we have to show that

\[
\frac{1}{R} = \limsup_{n \to \infty} |C_n|^{1/n} > 0. \tag{6}
\]

Now, if we could show that there exists a positive constant \( M \) such that the inequality

\[
C_n \leq M^n \tag{7}
\]

holds for all \( n \geq 0 \), then we could conclude that \( R \geq \frac{1}{M} > 0 \). But Segner’s recursion formula (5) would give us

\[
C_{n+1} \leq M^0 \cdot M^n + M^1 \cdot M^{n-1} + M^2 \cdot M^{n-2} + \cdots + M^{n-1} \cdot M + M^n \cdot M^0
\]

which is a second upper bound. If we could show that it is smaller than \( M^{n+1} \) then we could conclude by induction that (7) holds for all \( n \). Unfortunately,

\[
M^n(n + 1) \leq M^{n+1} \implies n + 1 \leq M
\]

which is plainly false for all sufficiently large \( n \). So, the second upper bound is, in fact, larger than the first one and this shows us that the induction step does not work for a bound of the form (7). Thus, one has to alter (7) so as to somehow “cancel” the factor \( n + 1 \) which multiplies \( M^n \). This suggests that we should try an inequality of the form

\[
C_n \leq n^{-r} M^n
\]

for some positive integer \( r \), yet to be determined, with \( n \geq 1 \) and \( C_0 = 1 \), where the factor \( n^{-r} \) effects the cancellation.

If we try \( r = 1 \), i.e., if we assume that the inequality

\[
C_n \leq n^{-1} M^n \tag{8}
\]

holds for all \( n \geq 1 \), the recursion formula (5) gives us

\[
C_{n+1} \leq 1 \cdot \frac{M^n}{n} + M^1 \cdot \frac{M^{n-1}}{1} + \frac{M^2}{2} \cdot \frac{M^{n-2}}{(n-1)} + \cdots + \frac{M^{n-1}}{(n-1)} \cdot M + \frac{M^n}{n} \cdot M
\]

which is

\[
C_{n+1} \leq M^n \left( \frac{2}{n} + \sum_{k=1}^{n-1} \frac{1}{k(n-k)} \right)
\]

and by (8), we want

\[
M^n \left( \frac{2}{n} + \sum_{k=1}^{n-1} \frac{1}{k(n-k)} \right) \leq \frac{M^{n+1}}{(n+1)}
\]
But, if \( H_k := 1 + \frac{1}{2} + \cdots + \frac{1}{k} \) denotes the \( k \)-th harmonic number, then the identity

\[
\frac{1}{k(n-k)} \equiv \frac{1}{n} \left( \frac{1}{k} + \frac{1}{n-k} \right)
\]

shows us that

\[
\frac{2}{n} + \sum_{k=1}^{n-1} \frac{1}{k(n-k)} = \frac{1}{n}(2 + 2H_{n-1}) < \frac{M}{n+1},
\]

or \( H_{n-1} < \frac{n}{n+1} - 2 \), which is false for all \( n \).

However, if we try \( r = 2 \), i.e., if we assume that the inequality

\[
C_n \leq n^{-2} M^n
\]

holds for all \( n \geq 1 \), the recursion formula (5) gives us

\[
C_{n+1} \leq 1 \cdot \frac{M^n}{n^2} + \frac{M^1}{1^2} \cdot \frac{M^{n-1}}{(n-1)^2} + \frac{M^2}{2^2} \cdot \frac{M^{n-2}}{(n-2)^2} + \cdots + \frac{M^{n-1}}{(n-1)^2} \cdot \frac{M}{1^2} + \frac{M^n}{n^2} \cdot 1
\]

and by (9), we want

\[
M^n \left( \frac{2}{n^2} + \sum_{k=1}^{n-1} \frac{1}{\{k(n-k)\}^2} \right) \leq \frac{M^{n+1}}{(n+1)^2},
\]

or

\[
\frac{2}{n^2} + \sum_{k=1}^{n-1} \frac{1}{\{k(n-k)\}^2} \leq \frac{M}{(n+1)^2}.
\]

We will show that if \( M = 6 \) then the inequality (10) holds for all \( n \geq 37 \). By direct numerical computation it can be verified that (8) holds for \( n = 1, \ldots, 36 \) and that therefore the inequality (8) holds for all \( n \geq 1 \). Therefore \( R \geq \frac{1}{6} > 0 \).

We add that STIRLING’S formula and (4) show that

\[
C_n \sim \frac{4^n}{\sqrt{\pi} n^{3/2}},
\]

so that the true value of \( R \) is \( R = \frac{1}{4} \). However, this presupposes the knowledge of the explicit formula for \( C_n \), whereas our analysis makes no such assumption.

3 The Proof

Theorem 1. The following limit relation is valid:

\[
\lim_{n \to \infty} \left\{ \left( \frac{2}{n^2} + \sum_{k=1}^{n-1} \frac{1}{\{k(n-k)\}^2} \right) / \frac{1}{(n+1)^2} \right\} = 2 + \frac{\pi^2}{3}
\]
and, moreover, if the integer \( n \geq 4 \) then the quotient on the left decreases monotonically to its limit. This may also be written as the following asymptotic equality:

\[
\frac{2}{n^2} + \sum_{k=1}^{n-1} \frac{1}{k(n-k)}^2 \sim \frac{2 + \frac{n^2}{3}}{(n+1)^2}.
\]

Moreover, if \( n \geq 37 \), then

\[
\frac{2 + \frac{n^2}{3}}{(n+1)^2} < \frac{2}{n^2} + \sum_{k=1}^{n-1} \frac{1}{k(n-k)}^2 < \frac{6}{(n+1)^2}. \tag{12}
\]

**Proof.** We suppose that the integer \( n \) satisfies \( n \geq 4 \).

In what follows, we use the well-known elementary inequality

\[
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.
\]

We begin with the left-hand side of (11):

\[
\frac{2}{n^2} + \sum_{k=1}^{n-1} \frac{1}{k(n-k)}^2 = \frac{2}{n^2} + \sum_{k=1}^{n-1} \left\{ \frac{1}{n} \left( \frac{1}{k} + \frac{1}{n-k} \right) \right\}^2
\]

\[
= \frac{2}{n^2} + \frac{1}{n^2} \sum_{k=1}^{n-1} \left\{ \frac{1}{k^2} + \frac{2}{k(n-k)} + \frac{1}{(n-k)^2} \right\}
\]

\[
= \frac{2}{n^2} + \frac{1}{n^2} \sum_{k=1}^{n-1} \left\{ \frac{2}{k^2} + \frac{2}{n} \left( \frac{1}{k} + \frac{1}{n-k} \right) \right\}
\]

\[
= \frac{2}{n^2} + \frac{2}{n^2} \sum_{k=1}^{n-1} \frac{1}{k^2} + \frac{4}{n^3} H_{n-1}
\]

\[
= \frac{2}{n^2} \left\{ 1 + \frac{\pi^2}{6} - \sum_{k=n}^{\infty} \frac{1}{k^2} + \frac{2}{n} H_{n-1} \right\}. \tag{13}
\]

Let

\[
g(n) := \left( \frac{2}{n^2} + \sum_{k=1}^{n-1} \frac{1}{k(n-k)}^2 \right) / \frac{1}{(n+1)^2}.
\]

The computation (13) shows that

\[
g(n) = 2 \left( 1 + \frac{1}{n} \right)^2 \left\{ 1 + \frac{\pi^2}{6} - \sum_{k=n}^{\infty} \frac{1}{k^2} + \frac{2}{n} H_{n-1} \right\}.
\]

Therefore, \( \lim_{n \to \infty} g(n) = 2 + \pi^2/3 \), establishing (11).
To prove monotonicity, we have to show that
\[ g(n) > g(n + 1). \] (14)
But, since clearly
\[ 2 \left( 1 + \frac{1}{n} \right)^2 > 2 \left( 1 + \frac{1}{n+1} \right)^2, \]
it suffices to prove that
\[ 1 + \frac{\pi^2}{6} - \sum_{k=n}^{\infty} \frac{1}{k^2} + \frac{2}{n} H_{n-1} > 1 + \frac{\pi^2}{6} - \sum_{k=n+1}^{\infty} \frac{1}{k^2} + \frac{2}{n+1} H_n; \]
which, after a little algebra, reduces to proving
\[ H_{n-1} - \frac{n}{n+1} H_n > \frac{1}{2n}, \]
or
\[ \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} > \frac{1}{2} + \frac{1}{2n}, \]
which holds if and only if \( n \geq 4 \). This completes the proof of monotonicity.

To prove (12), we compute directly
\[ g(36) = 6.0150\ldots \quad \text{and} \quad g(37) = 5.9979\ldots \]
and by the monotonicity of \( g(n) \) we conclude that
\[ 2 + \frac{\pi^2}{3} < g(n) < 6 \quad \text{for} \quad n \geq 37. \]

\section*{Theorem 2.} The Catalan number generating function
\[ C(x) := C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n + \cdots \]
has a positive radius of convergence at least equal to \( \frac{1}{6} \).

\textit{Proof.} The previous theorem shows that (8) holds for \( n \geq 37 \), and by our earlier remarks, for \( n \geq 1 \). Therefore by the Cauchy–Hadamard theorem (6), we conclude that \( R \geq \frac{1}{6} > 0 \). \( \square \)

\section{Remarks}
Our method of proof is applicable to any integer \( r = 2, 3, \ldots \). That is, for any integer \( r \geq 2 \) there is a constant \( M_r \) such that the inequality
\[ C_n \leq \frac{M_n}{n^r} \] (15)
holds for all sufficiently large \( n \). Moreover, the validity of (15) for \( r = 3 \) shows that the inequality (15) holds for any \( r \) such that \( 2 \leq r \leq 3 \). Therefore, the method is applicable to any real number \( r \) which satisfies \( r \geq 2 \).

On the other hand, although the inequality (15) holds for every integer \( r \geq 2 \), our proof shows that it does not hold for \( r = 1 \). So, this suggests that there is a first value \( r = r_0 \) between 1 and 2 for which the inequality (15) is valid. The methodology of the proof and Stirling’s formula,

\[ C_n \sim \frac{4^n}{\sqrt{\pi n^{3/2}}} \]

show that, in fact,

\[ r_0 = \frac{3}{2}, \]

i.e., if \( r \geq \frac{3}{2} \) then there exists a constant \( M_r > 0 \) such that the inequality (15) holds for sufficiently large \( n \) and that therefore the radius of convergence is positive. On the other hand, if \( r < \frac{3}{2} \), the inequality (15) does not hold.

It would be interesting to have a direct proof of this property of \( r = r_0 \) without appealing to the explicit formula for the Catalan numbers.

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References

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