Convergence of Dynamic Programming on the Semidefinite Cone

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Abstract—The goal of this paper is to investigate new and simple convergence analysis of dynamic programming for linear quadratic regulator problem of discrete-time linear time-invariant systems. In particular, bounds on errors are given in terms of both matrix inequalities and matrix norm. Under a mild assumption on the initial parameter, we prove that the Q-value iteration exponentially converges to the optimal solution. Moreover, a global asymptotic convergence is also presented. These results are then extended to the policy iteration. We prove that in contrast to the Q-value iteration, the policy iteration always converges exponentially fast. An example is given to illustrate the results.

Index Terms—Dynamic programming, optimal control, convergence, linear time-invariant system, reinforcement learning

I. INTRODUCTION

The optimal control theory for linear systems has a long tradition [1]–[4]. At the core of the optimal control problem is the dynamic programming: it offers a general and effective paradigm for finding optimal policies for the optimal control problem. Early works of dynamic programming [2], [3], [5] clarified many issues, such as asymptotic convergence and precise conditions to guarantee the convergence. More recently, progresses have been made for nonlinear systems [6], [7] and switched linear systems [8]–[10] to name just a few. A natural next step is to understand its non-asymptotic behavior: does the algorithm make a consistent and quantifiable progress toward the optimal solution? Although an exponential convergence has been established in [8] under a stronger assumption on the weighting matrix, the question still remains unsettled.

Motivated by the discussions, in this paper, we revisit the classical results on the convergence analysis of dynamic programming with different angles for discrete-time linear time-invariant systems. In particular, the classical analysis usually focuses on the value function iteration. On the other hand, in this paper, we pay more attentions to the Q-function iteration, which is relevantly more popular in the field of reinforcement learning [11]–[13], in particular, Q-learning [13], [14]. However, the proposed analysis can be directly applied to the value function-based dynamic programming as well. Most importantly, we study the convergence of Q-value iteration (Q-VI) and Q-function-based policy iteration (Q-PI) in terms of matrix inequality bounds and matrix norm. We prove that the error of Q-VI has an upper bound expressed in terms of matrix inequalities, while the lower bound is proven to converge asymptotically. The overall convergence rate is dominated by the asymptotic behavior of the lower bound. However, it turns out that when the initial parameter lies on a certain semidefinite cone, the error matrix is upper and lower bounded by matrices that exponentially converge to zero. Therefore, under this scenario, an exponential convergence of the error can be derived in terms of some matrix norm. The validity of the results is demonstrated through an example. As a next step, the results for Q-VI are extended to the analysis of Q-PI. In particular, similar analysis can be applied to Q-PI except for one aspect. In contrast to Q-VI, Q-PI always guarantees exponential convergence independently of the initial parameters. This improvement comes from the additional initial information of Q-PI: the stabilizing gain initially given to Q-PI. We expect that the present work sheds new light on more exact analysis of dynamic programming with different angles, which inherit the simplicity and elegance.

Notation: The adopted notation is as follows: \( \mathbb{R} \): set of real numbers; \( \mathbb{R}^n \): \( n \)-dimensional Euclidean space; \( \mathbb{R}^{m×n} \): set of all \( n \times m \) real matrices; \( A^T \): transpose of matrix \( A \); \( A^{-T} \): transpose of matrix \( A^{-1} \); \( A \succ 0 \) (\( A \prec 0 \), \( A \succeq 0 \), and \( A \preceq 0 \), respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix \( A \); \( I \): identity matrix with appropriate dimensions; \( \mathbb{S}^n \): symmetric \( n \times n \) matrices; \( \mathbb{S}_+^n := \{ P \in \mathbb{S}^n : P \succeq 0 \} \); \( \mathbb{S}_+^{n,+} := \{ P \in \mathbb{S}^n : P \succ 0 \} \); \( \rho(\cdot) \): spectral radius; \( \lambda_{\text{max}}(\cdot) \): maximum eigenvalue; \( \lambda_{\text{min}}(\cdot) \): minimum eigenvalue.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the discrete-time linear time-invariant (LTI) system

\[
x(k + 1) = Ax(k) + Bu(k), \quad x(0) = z \in \mathbb{R}^n,
\]

where the integer \( k \geq 0 \) is the time, \( x(k) \in \mathbb{R}^n \) is the state vector, \( u(k) \in \mathbb{R}^m \) is the input vector, and \( z \in \mathbb{R}^n \) is the initial state. Assuming the input, \( u(k) \), is given by a state-feedback control policy, \( u(k) = Fx(k) \), we denote by \( x(k; F, z) \) the solution of (1) starting from \( x(0) = z \). Under the state-feedback control policy, the cost function for the classical linear quadratic regulator (LQR) problem is denoted by

\[
J(F, z) := \sum_{k=0}^{\infty} \gamma^k \left[ \frac{x(k; F, z)}{F x(k; F, z)} \right]^T \Lambda \left[ \frac{x(k; F, z)}{F x(k; F, z)} \right],
\]

where \( \Lambda := \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \succeq 0 \) is the weight matrix and \( \gamma \in (0, 1) \) is called the discount factor. By introducing the augmented
state vector \( v(k) := \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \), we consider the augmented system throughout the paper

\[
v(k + 1) = A(F)v(k), \quad v(0) = v_0 \in \mathbb{R}^{n+m}, \tag{2}\]

where

\[
A(F) := \begin{bmatrix} A & B \\ FA & FB \end{bmatrix} \in \mathbb{R}^{(n+m) \times (n+m)}.
\]

If \( v_0 = \begin{bmatrix} z^T \\ z^T FT \end{bmatrix}^T \), then the state and input parts of \( v(k) \) are identical to \( x(k) \) and \( u(k) \) in (1). A useful property of \( A(F) \) is that its spectral radius, \( \rho(A(F)) \), is identical to that of \( A + BF \).

**Lemma 1** ([15]). \( \rho(A + BF) = \rho(A(F)) \) holds.

Define \( \mathcal{F}_\gamma \) as the set of all stabilizing state-feedback gains of system \( (\gamma^{1/2}A, \gamma^{1/2}B) \), i.e., \( \mathcal{F}_\gamma := \{ F \in \mathbb{R}^{m \times n} : \rho(\gamma^{1/2}A + \gamma^{1/2}BF) < 1 \} \). Then, \( \mathcal{F}_\gamma \) is an open set, not necessarily convex [16, Lemma 2]; however, finding a state feedback gain \( F \in \mathcal{F}_\gamma \) can be reduced to a simple convex problem. Notice that with \( \gamma = 1, \mathcal{F}_1 \) is the set of all stabilizing state-feedback gains of \((A, B)\). From the standard LQR theory, although \( J^*(F, z) \) has different values for different \( z \in \mathbb{R}^n \), the minimizer \( F^* = \arg\min_{F \in \mathbb{R}^{m \times n}} J(F, z) \) is not dependent on \( z \). Based on this notion, the infinite-horizon LQR problem is formalized below.

**Problem 1** (Infinite-horizon LQR problem). For any \( z \in \mathbb{R}^n \), solve \( F^* = \arg\min_{F \in \mathbb{R}^{m \times n}} J(F, z) \) if the optimal value of \( \inf_{F \in \mathbb{R}^{m \times n}} J(F, z) \) exists and is attained.

For a given \( z \in \mathbb{R}^n \), if the optimal value of \( \inf_{F \in \mathbb{R}^{m \times n}} J(F, z) \) exists and is attained, then the optimal cost is denoted by \( J^*(z) = J(F^*, z) \). Assumptions that will be used throughout the paper are summarized below.

**Assumption 1.** Throughout the paper, we assume that

- \( Q \geq 0, R \succ 0 \);
- \((A, B)\) is stabilizable and \( Q \) can be written as \( Q = C^T C \), where \((A, C)\) is detectable.

Under **Assumption 1**, the optimal value of \( \inf_{F \in \mathbb{R}^{m \times n}} J(F, z) \), exists, is attained, and \( J^*(z) \) is a quadratic function, i.e., \( J^*(z) = z^T X z^* \), where \( X^* \) is the unique solution of the algebraic Riccati equation (ARE) [3, Proposition 4.4.1] for \( X \):

\[
X = \gamma A^T X A - \gamma A^T X B (R + \gamma B^T X B)^{-1} \gamma B^T X A + Q,
\]

\( X \succeq 0 \).

In this case, \( J^*(z) \) as a function of \( z \in \mathbb{R}^n \) is called the optimal value function, which satisfies the Bellman equation

\[
J^*(z) = \min_{w \in \mathbb{R}^m} \left\{ z^T \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} 0 & \gamma J^*(A z + B w) \\ \gamma J^*(A z + B w) & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} + \gamma J^*(A z + B w) \right\}
\]

The reader can refer to [3] and [17] for more details of the classical LQR results. The corresponding optimal control policy is \( u^*(z) = F^* z \), where

\[
F^* := -(R + \gamma B^T X^* B)^{-1} \gamma B^T X^* A \in \mathcal{F}_\gamma \tag{3}
\]
is the unique optimal gain. Note that \( F^* \in \mathcal{F}_\gamma \), i.e., it stabilizes \((\gamma^{1/2}A, \gamma^{1/2}B)\). Alternatively, the optimal \( Q \)-function [3] is defined as

\[
Q^*(z, u) := z^T Q z + u^T R u + \gamma J^*(A z + B u) = z^T u^T P^* \begin{bmatrix} z \\ u \end{bmatrix},
\]

where

\[
P^* := \begin{bmatrix} Q + \gamma A^T X^* A & \gamma A^T X^* B \\ R + \gamma B^T X^* A & R + \gamma B^T X^* B \end{bmatrix}.
\]

Once the optimal \( Q \)-function is found, then the optimal policy can be expressed as

\[
u^*(z) = F^* z = \arg\min_{u \in \mathbb{R}^m} Q^*(z, u).
\]

The optimal \( Q \)-function is known to satisfy the \( Q \)-Bellman equation

\[
Q^*(z, u) = \begin{bmatrix} z \\ u \end{bmatrix}^T \Lambda \begin{bmatrix} z \\ u \end{bmatrix} + \gamma \min_{w \in \mathbb{R}^m} Q^*(A z + B u, w)
\]

and its parametric form is

\[
P^* = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}
\]

\[
+ \gamma \begin{bmatrix} A & B \end{bmatrix}^T (P_{11}^* - P_{12}^* (P_{12}^*)^{-1} (P_{12}^*)^T) \begin{bmatrix} A & B \end{bmatrix}
\]

or more compactly,

\[
P^* = \Lambda + \gamma A (F^*)^T P^* A (F^*).
\]

We close this section by introducing some additional definitions and lemma. Throughout the paper, we will use the partition \( P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \) for any matrix \( P \in \mathbb{S}^{n+m} \), where \( P_{11} \in \mathbb{S}^n, P_{12} \in \mathbb{R}^{n \times m}, P_{22} \in \mathbb{S}^m \). For convenience, we introduce the set

\[
\mathcal{P} := \left\{ \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} : P_{11} \in \mathbb{S}^n, P_{22} \in \mathbb{S}^m \right\}
\]

**Lemma 2** ([15]). For \( P \in \mathcal{P} \), it holds that

\[
A(F)^T P A(F) \preceq A(-P_{22}^{-1} P_{12}^T) P A(-P_{22}^{-1} P_{12}^T), \quad \forall F \in \mathbb{R}^{m \times n}
\]

**III. VALUE ITERATION**

In this section, we provide analysis of \( Q \)-value iteration (Q-VI). Define the Bellman operator

\[
\mathcal{T}(P) = \Lambda + \gamma A(-P_{22}^{-1} P_{12}^T) P A(-P_{22}^{-1} P_{12}^T)
\]

Note that the operator is well defined only for \( P \in \mathcal{P} \) because in this case, \( P_{22}^{-1} \) exists in the definition of \( \mathcal{T} \). In this paper, for \( P = 0 \), the operator is defined as \( \mathcal{T}(P) = \Lambda \). Then, \( \mathcal{T}(0) \in \mathcal{P} \) holds because \( \Lambda \in \mathcal{P} \). Moreover, \( P^* \in \mathcal{P} \) as well.

**Lemma 3** (Positiveness of \( \mathcal{T} \)). For any \( P \in \mathcal{P} \cup \{0\} \), \( \mathcal{T}(P) \in \mathcal{P} \) holds.

Using this matrix operator, Q-VI can be briefly summarized as in **Algorithm 1**.

As a first step toward our goal, an important property of \( \mathcal{T} \) is its monotonicity.
Lemma 2 is given in Appendix II. We are now for an induction.

Lemma 4 Algorithm 1 tells us that if the initial parameter satisfies \( P \), and the lower bound is applied for the second inequality. This only provides an upper bound on the error, \( \epsilon \).

Proof. Suppose \( \tilde{P} \leq P \). Then,

\[
\mathcal{T}(P) = \Lambda + \gamma A(-P_{22}^{-1} T P A(-P_{22}^{-1} T) \geq \Lambda + \gamma A(-P_{22}^{-1} P_{12}^T T P A(-P_{22}^{-1} P_{12}^T) \geq \Lambda + \gamma A(-P_{22}^{-1} P_{12}^T T P A(-P_{22}^{-1} P_{12}^T) = \mathcal{T}(P)
\]

where the first inequality is due to the hypothesis, \( \tilde{P} \leq P \), and Lemma 2 is applied for the second inequality. This completes the proof.

The analyze the convergence of Algorithm 1, we now focus on the error matrix

\[
\mathcal{T}^k(P) - \mathcal{T}^k(P^*) = \mathcal{T}^k(P) - P^*.
\]

In the following, an upper bound on the error is given in terms of the matrix bound on the semidefinite cone.

Theorem 1 (Upper bound). For any \( P \in \mathcal{P} \cup \{0\} \), we have

\[
\mathcal{T}^k(P) - P^* \leq \gamma^k(A(F^*)^T (P - P^*) A(F^*) \rightarrow \infty
\]

as \( k \rightarrow \infty \).

Proof. First of all, we can derive the following bounds

\[
\mathcal{T}(P) - \mathcal{T}(P^*) = \gamma A(-P_{22}^{-1} P_{12}^T T P A(-P_{22}^{-1} P_{12}^T) \leq \gamma A(F^*)^T T P A(F^*)
\]

where the inequality is due to Lemma 2. For an induction argument, suppose that

\[
\mathcal{T}^k(P) - \mathcal{T}^k(P^*) \leq \gamma^k(A(F^*)^T (P - P^*) A(F^*)
\]

holds. To proceed, let \( \mathcal{T}(P) = P_k \) and \( F_k = -P_{k,22}^{-1} P_{k,12}^T \). Then, using \( P^* = \mathcal{T}(P^*) \), we have

\[
\mathcal{T}^k(P) - \mathcal{T}^k(P^*) = \gamma A(F_k-1)^T T \mathcal{T}^k(P) A(F_k-1) - \gamma A(F^*)^T T \mathcal{T}^k(P^*) A(F^*)
\]

The desired conclusion is obtained by induction. Since \( \gamma^{1/2} A(F^*) \) is Schur, \( \gamma^{1/2} A(F^*) \rightarrow 0 \) as \( k \rightarrow \infty \) (see [18, Theorem 5.6.12, pp. 348]), and the proof is completed.

Theorem 1 only provides an upper bound on the error, \( \mathcal{T}^k(P) - P^* \). On the other hand, its lower bound cannot be established in this way. However, under a special condition on the initial point \( P \), a trivial lower bound can be found, and we can obtain an exponential convergence of the Q-value iteration to \( P^* \). To proceed further, define the positive semidefinite cone

\[
C(P) := \{ P' \in S^{n+m} : P' \succeq P \} \subseteq P,
\]

which will play an important role in this paper. We can prove that with \( P \in C(P^*) \), a complete exponential convergence can be obtained.

Theorem 2 (Local convergence on the semidefinite cone). Suppose \( P \in C(P^*) \), then

\[
0 \leq \mathcal{T}^k(P) - P^* \leq \gamma^k(A_{F^*})^k(P - P^*) A_{F^*}
\]

and \( \gamma^k(A_{F^*})^k(P - P^*) A_{F^*} \rightarrow 0 \) as \( k \rightarrow \infty \).

Proof. The upper bound comes from Theorem 1, and the lower bound is due to the monotonicity in Lemma 4. This completes the proof.

Theorem 2 tells us that if the initial parameter satisfies \( P \in C(P^*) \), then the value iteration error is over bounded by a matrix which vanishes as \( k \rightarrow \infty \), and under bounded by the zero matrix. A natural question arising here is if we can also derive the error bounds in terms of some matrix norm. To answer this question, some mathematical ingredients should be prepared. First of all, let us choose a proper matrix norm. For any \( P \succ 0 \), define the following norm:

\[
\|\cdot\|_P = \sqrt{\lambda_{\max}(\cdot^T T P(\cdot))}
\]

which is the induced matrix norm of the vector norm \( \|\cdot\|_P = \sqrt{\gamma^T T P(\gamma)} \). An important property of the norm is the property called the monotonicity.

Definition 1 ([19, pp. 57]). A matrix norm \( ||\cdot|| \) is monotone if for any \( A, B \in S^{n+m} \) such that \( A \succeq B \), \( ||A|| \leq ||B|| \) holds.

We can easily prove that \( ||\cdot||_P \) is monotone, presented in the following lemma. The proof is given in Appendix I.

Lemma 5. For any \( P \in S^{n+m}_+ \), the norm \( ||\cdot||_P \) is monotone.

For such a norm, \( ||\cdot||_P \), to meet our purpose, the matrix \( P \) needs to be properly chosen. One can conclude that the matrix can be chosen as a Lyapunov matrix. In the sequel, we establish some results related to the Lyapunov inequality.

Lemma 6. Suppose that \( F \in \mathcal{F}_1 \) so that \( A(F) \) is Schur or equivalently, \( \rho(A(F)) < 1 \). For any \( \varepsilon > 0 \), there exists \( P_\varepsilon \in S^{n+m}_+ \) such that the following Lyapunov inequality holds:

\[
A(F) T P_\varepsilon A(F) \preceq (\rho(A(F)) + \varepsilon)^2 P_\varepsilon
\]

The desired conclusion is obtained by induction. Since \( \gamma^{1/2} A(F^*) \) is Schur, \( \gamma^{1/2} A(F^*) \rightarrow 0 \) as \( k \rightarrow \infty \) (see [18, Theorem 5.6.12, pp. 348]), and the proof is completed.

Theorem 3. For any \( P \in C(P^*) \), we have

\[
\|\mathcal{T}(P) - P^*\|_P \leq (\rho(\gamma^{1/2} A(F^*)) + \varepsilon)^2 \| P - P^* \|_P^2
\]
Lemma 6.

Moreover, for all \( P \in \mathcal{C}(P^*) \) and \( k \geq 0 \), we have
\[
\|T^k(P) - P^*\|_{P^*_\varepsilon} \leq (\rho(\gamma^{1/2}A(F^*)) + \varepsilon)^2k\|P - P^*\|_{P^*_\varepsilon},
\]
for any \( \varepsilon > 0 \) such that \( \rho(\gamma^{1/2}A(F^*)) + \varepsilon < 1 \).

Proof. We first conclude that
\[
\|\gamma^{1/2}A(F^*)\|_{P^*_\varepsilon} = \sqrt{\lambda_{\max}(\gamma^{1/2}A(F^*)^TP^*_\varepsilon\gamma^{1/2}A(F^*))} 
\leq (\rho(\gamma^{1/2}A(F^*)) + \varepsilon)^2 \sqrt{\lambda_{\max}(P^*_\varepsilon)}
\]
where we used \( \gamma^{1/2}A(F^*)^TP^*_\varepsilon\gamma^{1/2}A(F^*) \preceq (\rho(\gamma^{1/2}A(F^*))) + \varepsilon^2 P^*_\varepsilon \) in the first inequality, and \( \lambda_{\max}(P^*_\varepsilon) \leq 1 \) in the second inequality. On the other hand, taking the norm, \( \|\cdot\|_{P^*_\varepsilon} \), to the inequality in Theorem 2, we have
\[
\|T(P) - T(P^*)\|_{P^*_\varepsilon} \leq \|\gamma^{1/2}A(F^*)^T(P - P^*)\gamma^{1/2}A(F^*)\|_{P^*_\varepsilon} 
\leq \|P - P^*\|_{P^*_\varepsilon}\|\gamma^{1/2}A(F^*)\|_{P^*_\varepsilon} 
\leq \|P - P^*\|_{P^*_\varepsilon}(\rho(\gamma^{1/2}A(F^*)) + \varepsilon)^2
\]
where the last inequality comes from (10), which is (8). Recursively combining (11) yields (9). This completes the proof.

The bound in Theorem 3 can be readily expressed in terms of the spectral norm, which is summarized below.

**Corollary 1.** For any \( P \in \mathcal{C}(P^*) \), we have
\[
\|T^k(P) - P^*\|_2 
\leq \lambda_{\max}(P^*_\varepsilon)\rho(\gamma^{1/2}A(F^*)) + \varepsilon)^2 k\|P - P^*\|_2,
\]
for any \( \varepsilon > 0 \) such that \( \rho(\gamma^{1/2}A(F^*)) + \varepsilon < 1 \), where \( P^*_\varepsilon \in \mathbb{S}^{n+m} \) is a matrix satisfying the conditions in Lemma 6.

The convergence of Theorem 3 requires that the initial parameter \( P \) is within \( \mathcal{C}(P^*) \) with an unknown \( P^* \). In the general case where \( P \) may not be in \( \mathcal{C}(P^*) \), such a lower bound is hard to be established. Instead, we can obtain a bound with asymptotic convergence in the following result.

**Theorem 4** (Global convergence). For any \( P \in \mathcal{P} \cup \{0\} \), we have
\[
T^k(0) - T^k(P^*) \preceq T^k(P) - T^k(P^*) 
\preceq \gamma^k(\rho(\gamma^{1/2}A(F^*)))^k(P - P^*)A(F^*)^k, \forall k \geq 0,
\]
where \( \gamma^k(\rho(\gamma^{1/2}A(F^*)))^k(P - P^*)A(F^*)^k \rightarrow 0 \) and \( T^k(0) - T^k(P^*) \rightarrow 0 \) as \( k \rightarrow \infty \).

Proof. The upper bound is due to Theorem 1. For the lower bound, note that from the monotonicity of \( T \) in Lemma 4, \( T(0) \preceq T(P) \), and hence, \( T^k(0) - T^k(P^*) \preceq T^k(P) - T^k(P^*) \). It remains to prove that \( \gamma^k(\rho(\gamma^{1/2}A(F^*)))^k(P - P^*)A(F^*)^k \rightarrow 0 \) and \( T^k(0) - T^k(P^*) \rightarrow 0 \) as \( k \rightarrow \infty \). The former is true because \( \gamma^{1/2}A(F^*) \) is Schur (see [18, Theorem 5.6.12, pp. 348]). To prove \( T^k(0) \rightarrow T^k(P^*) = P^* \), note that \( 0 \preceq T(0) \) from the monotonicity of \( T \), one concludes \( T^k(0) \preceq T^{k+1}(0) \), and hence, \( T^k(0) \) is monotonically non-decreasing. Moreover, from the upper bound, \( T^k(0) \preceq T^k(P) \preceq \gamma^k(\rho(\gamma^{1/2}A(F^*)))^k(P - P^*)A(F^*)^k + P^* \), we conclude that \( T^k(0) \rightarrow S \) as \( k \rightarrow \infty \) for some matrix \( S \in \mathbb{S}^{n+m} \) such that \( T(S) = S \), implying that \( S = P^* \). This completes the proof.

Theorem 4 offers a global convergence result of the error in terms of upper and lower bounds on the semidefinite cone. The upper bound is applied to the general case, and it can provide a finite-time analysis. On the other hand, the convergence of the lower bound is asymptotic. Overall convergence in this case is dominated by the asymptotic behavior of the lower bound. Besides, the upper bound can be further analyzed, and can be proven to converge exponentially after a certain number \( N \geq 0 \) of iterations.

**Corollary 2.** Consider any \( P \in \mathcal{P} \cup \{0\} \). Then, for any \( \varepsilon > 0 \), there exists an integer \( N \geq 0 \) such that
\[
\|T^k(P) - T^k(P^*)\|_2 \leq |\lambda_{\max}(P - P^*)||\gamma^{1/2}A(F^*) + \varepsilon|^k||2\|I
\]
for all \( k \geq N \), where \( N \) is such that
\[
\|\gamma^{1/2}A(F^*)^k\|_2 \leq \rho(\gamma^{1/2}A(F^*)) + \varepsilon, \forall k \geq N.
\]
Proof. From Theorem 4, one gets
\[
T^k(P) - T^k(P^*) \preceq \gamma^k(\rho(\gamma^{1/2}A(F^*)))^k(P - P^*)A(F^*)^k 
\preceq |\lambda_{\max}(P - P^*)||\gamma^{1/2}A(F^*)^k||2\|I = |\lambda_{\max}(P - P^*)||\gamma^{1/2}A(F^*)^k||2\|k^2 I.
\]
From the Gelfand’s formula [18, Corollary 5.6.14, pp. 349], there exists a finite \( N \geq 0 \) such that
\[
\|\gamma^{1/2}A(F^*)^k\|_2 \leq \rho(\gamma^{1/2}A(F^*)) + \varepsilon, \forall k \geq N.
\]
Plugging the bounds into the previous inequalities yields the desired conclusion.

Although Theorem 4 does not provide a finite-time lower bound, we can prove that after a sufficient number, \( N \), of iterations, the lower bound also converges exponentially fast. The result is presented in the sequel, and the proof is given in Appendix III.

**Proposition 1.** Consider any \( P \in \mathcal{P} \cup \{0\} \). Then, for any \( \varepsilon_1, \varepsilon_2 > 0 \), there exist integers \( N_1, N_2 \geq 0 \) and a constant, \( \eta > 0 \), such that
\[
T^{N_1+k}(P) - T^{N_1+k}(P^*) \geq -\eta((\rho(\gamma^{1/2}A(F^*)))^{2k} + \varepsilon_1^{k} + \varepsilon_1^{k})I
\]
for all \( k \geq N_2 \).

Proposition 1 suggests that although the lower bound on the error can progress with a sublinear speed, it could eventually converge with linear rates. This behavior will be demonstrated in the example section. Finally, in this section, an analysis of Q-VI has been established in terms of the matrix bounds and matrix norm bounds. Especially, under the condition, \( P \in \mathcal{C}(P^*) \), on the initial point, an exponential convergence has been derived. Before closing this section, we briefly discuss
the value function counterpart of Q-VI (Algorithm 1), which is the well-known Riccati recursion: for all \( k \in \{0, 1, \ldots \} \)
\[
X_{k+1} = \gamma A^T X_k A - \gamma A^T X_k B (R + \gamma B^T X_k B)^{-1} \gamma B^T X_k A + Q
\]
with any initial \( X_0 \in S_+^n \). For this case, noting that
\[
X_k - X^* = \begin{bmatrix} I \\ 0 \end{bmatrix}^T \{ T^k(P) - T^k(P^*) \} \begin{bmatrix} I \\ 0 \end{bmatrix}
\]
we can conclude that all the results corresponding to Q-VI can be directly obtained for the convergence of the VI as well.

IV. POLICY ITERATION

The policy iteration, summarized in Algorithm 2, is another class of the dynamic programming algorithm which iterates policies together with values. Especially, we will consider a policy iteration based on the Q-function, which will be called Q-PI throughout this paper. Note that in Q-PI, an additional information is required initially, namely, the initially stabilizing gain \( F_0 \in F_\gamma \) for \((\gamma^{1/2} A, \gamma^{1/2} B)\). At each iteration, one needs to solve the Bellman equation in (12), which is linear in \( P_k \), to evaluate the given gain \( F_k \), which can be achieved by using the following recursion
\[
P_{k+1} = \Lambda + \gamma A(F_k)^T P_k A(F_k)
\]
for \( i \in \{0, 1, \ldots \} \) with \( P_0 \in S_+^{n+m} \). The global exponential convergence of the iteration can be easily proved using the same lines as in the proof of Theorem 3, so omitted here. Therefore, we will focus on the outer iteration given in Algorithm 2. To analyze Q-PI, define the mapping \( \mathcal{H}(P) \) for any \( P \in \mathcal{P} \) such that
\[
\mathcal{H}(P) = S,
\]
where
\[
S = \Lambda + \gamma A(-P_{22}^{-1}P_{12}^T) A(-P_{22}^{-1}P_{12}^T)
\]
Then, Q-PI in Algorithm 2 can be viewed as the recursion \( \mathcal{H}^k(P) = P_k \). To proceed, for any given \( F \in F_\gamma \), define the mapping
\[
\mathcal{L}_F(\cdot) = \Lambda + \gamma A(F)^T (\cdot) A(F),
\]
which will play an important role in the analysis of Q-PI. Based on these definitions, some useful properties of \( \mathcal{H} \) are summarized below.

**Lemma 7.** Suppose that for any given \( F \in F_\gamma \), \( P \) satisfies \( P = \Lambda + \gamma A(F)^T P A(F) \). Then, \( P \geq \mathcal{H}(P) \) holds.
be obtained under the special initial point, \( P \in C(P^*) \) as in Q-VI case. However, in the policy iteration case, we can prove that the initial \( P \) always satisfies \( P \in C(P^*) \). The benefit comes from the additional knowledge on the initial gain \( F_0 \) which is stabilizing for \((\gamma^{1/2}A, \gamma^{1/2}B)\). Therefore, a global exponential convergence can be derived for Q-VI as follows.

**Theorem 6** (Global convergence). Suppose that for any given \( F \in \mathcal{F}_\gamma \), \( P \) satisfies \( P = \Lambda + \gamma A(F)^T P A(F) \). Then, \( P \in C(P^*) \), and we have

\[
0 \preceq H^k(P) - P^* \preceq \gamma^k(A(F^*)^T k(P - P^*) A(F^*),
\]

for all \( k \geq 0 \). Moreover, we have

\[
\|H^k(P) - P^*\|_{P^*} \preceq \left( \rho(\gamma^{1/2}A(F^*)) + \varepsilon \right) 2^k \|P - P^*\|_{P^*},
\]

for all \( k \geq 0 \) and for any \( \varepsilon > 0 \) such that \( \rho(\gamma^{1/2}A(F^*)) + \varepsilon < 1 \) and \( P^* \in \mathbb{S}^{n+m}_+ \) is a matrix satisfying the conditions in Lemma 6 with \( \gamma^{1/2}A(F^*) \).

Proof. The proof of the error bounds follows the same lines as in the proof of Theorem 2 and Theorem 3, so it is omitted in this paper. We only prove the fact that under the initialization scheme in Algorithm 2, \( P \in C(P^*) \) holds. The initial point, \( P \), of the policy iteration satisfies \( P = \Lambda + \gamma A(F)^T P A(F) \) for any given \( F \in \mathcal{F}_\gamma \). Define \( z_k(F, x, u) \in \mathbb{R}^{n+m} \) as the augmented state trajectory, 

\[
\begin{bmatrix}
  x(0) \\
  u(0)
\end{bmatrix} = \begin{bmatrix} x \\ u \end{bmatrix} \quad \text{and under the input } u(k) = Fx(k). \]

Then, from the definition of \( F^* \) and using the relation

\[
P = \sum_{k=0}^{\infty} \gamma^k(A(F^*)^T k A(F),
\]

we have

\[
J(x; F^*) = \min_{u \in \mathbb{R}^{m \times n}} \sum_{k=0}^{\infty} \gamma^k z(k; F^*, x, u)^T \Lambda z(k; F^*, x, u)
\]

\[
= \min_{u \in \mathbb{R}^{m \times n}} Q^*(x, u)
\]

\[
= \min_{u \in \mathbb{R}^{m \times n}} \begin{bmatrix} x \\ u \end{bmatrix}^T P^* \begin{bmatrix} x \\ u \end{bmatrix}
\]

\[
\preceq \sum_{k=0}^{\infty} \gamma^k z(k; F, x, u)^T \Lambda z(k; F, x, u)
\]

\[
= \begin{bmatrix} x \\ u \end{bmatrix}^T P \begin{bmatrix} x \\ u \end{bmatrix}
\]

for any \( u \in \mathbb{R}^m \). Therefore, it follows that

\[
\begin{bmatrix} x \\ u \end{bmatrix}^T P^* \begin{bmatrix} x \\ u \end{bmatrix} \preceq \begin{bmatrix} x \\ u \end{bmatrix}^T P \begin{bmatrix} x \\ u \end{bmatrix}
\]

for any \( u, \bar{u} \in \mathbb{R}^m \), or equivalently, \( P^* \preceq P \). This completes the proof.

Contrary to Q-VI, Q-PI always guarantees \( P \succeq P^* \), and hence, the exponential convergence in Theorem 6 always holds. Compared to Q-VI, this improvement comes from the additional information on the initially stabilizing gain \( F_0 \in \mathcal{F}_\gamma \). If \( F_0 \in \mathcal{F}_\gamma \) is initially given, then Q-VI can be improved as well. In particular, one can develop the following two-phase algorithm: 1) For initial \( F \in \mathcal{F}_\gamma \), find \( \tilde{P} \in \mathbb{S}^{n+m}_+ \) such that \( \tilde{P} = \Lambda + I + \gamma A(F)^T \tilde{P} A(F) =: \mathcal{D}(\tilde{P}) \). One can prove that \( P \succeq P^* + I \), and \( D^k(P) \) converges to \( \tilde{P} \) as \( k \to \infty \) exponentially fast. Therefore, there exists a finite \( k > 0 \) such that \( D^k(P) \in C(P^*) \). Since there exists an explicit gap between \( P \) and \( P^* \), we can find a lower bound on the number of iteration such that \( D^k(P) \in C(P^*) \). 2) Once \( P_k = D^k(P) \) enters \( C(P^*) \), then run Q-VI with \( P_k \) as an initial parameter. The two phases process guarantees the exponential convergence to \( P^* \).

Although the convergence is given in terms of the Q-function parameter \( P_k \), if it converges to \( P^* \), this implies that the corresponding gain \( F_{k+1} = -(P_{2k}^{-1}P_{2k})^T \) also converges to \( F^* \). Therefore, results in this section also establish the convergence of the gains. Finally, the standard value function-based policy iteration (PI)

**Algorithm 3** Value function based policy iteration (PI)

1: Initialize \( F_0 \in \mathcal{F}_\gamma \).
2: for \( k \in \{0, 1, \ldots\} \) do
3: Solve for \( X_k \) the linear equation

\[
X_k = \gamma A^T X_k A - F_k^T (R + \gamma B^T X_k B) F_k + Q
\]

4: Update \( F_{k+1} = -(R + \gamma B^T X_k B)^{-1} \gamma B^T X_k A \)
end for

In the sequel, an example is studied to demonstrate the validity of the analysis given throughout the paper.

V. EXAMPLE

Consider the randomly generated system \((A, B)\)

\[
A = \begin{bmatrix} 0.4527 & 0.9648 \\ 0.9521 & 0.6309 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2871 \\ 0.5994 \end{bmatrix}
\]

and

\[
Q = 0.1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad R = 100, \quad \gamma = 0.9.
\]

The optimal \( P^* \) is

\[
P^* = \begin{bmatrix} 2604.8 & 2877.2 & 1643.4 \\ 2877.2 & 3178.1 & 1815.3 \\ 1643.4 & 1815.3 & 2036.9 \end{bmatrix}
\]

and the corresponding optimal gain is

\[
F^* = \begin{bmatrix} -0.8068 & -0.8912 \end{bmatrix}
\]

The spectral radius of \( \gamma^{1/2}A(F^*) \) is \( \rho(\gamma^{1/2}A(F^*)) = 0.7006 < 1 \). We run Q-VI in Algorithm 1 with the two initial parameters \( P_0 = \lambda_{\min}(P^*) I \) so that \( P_0 \preceq P^* \), and \( P_0 = \lambda_{\max}(P^*) I \) so that \( P_0 \succeq P^* \), where \( \lambda_{\min}(P^*) = 0.0005 \) and \( \lambda_{\max}(P^*) = 6992.8 \). The evolution of the error, \( \|P_k - P^*\|_2 \), of Q-VI for different initial parameter \( P_0 = \lambda_{\min}(P^*) I \).
(blue line) and $P_0 = \lambda_{\max}(P^*)I$ (red line) are depicted in Figure 1(a), and its log-scale plot is given in Figure 1(b). The figures suggest that the evolution of $||P_k - P^*||_2$ with $P_0 \leq P^*$ has sublinear convergence, while it has linear (or exponential) convergence with $P_0 \geq P^*$, which match with the proposed analysis.

We also investigate the evolution of $[P_k - P^*]_+$, which denotes the projection of the error onto the positive semidefinite cone, and $[P_k - P^*]_-$, the projection of the error onto the negative semidefinite cone. If $P_0 \leq P^*$ initially, then $P_k = T^k(P_0) \leq P^* = T^k(P^*)$ for any $k \geq 0$, and hence, $||[P_k - P^*]_+||_2 = 0$ for all $k \geq 0$. Therefore, we consider an indefinite initial point by setting

$$P_0 = \frac{1}{2}(\lambda_{\min}(P^*) + \lambda_{\max}(P^*))\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

so that neither $P_0 \leq P^*$ nor $P_0 \geq P^*$. The evolutions of $||[P_k - P^*]_+||_2$ (red line) and $||[P_k - P^*]_-||_2$ (blue line) of Q-VI are shown in Figure 2, which suggests that the positive semidefinite part, $||[P_k - P^*]_+||_2$, converges faster with an exponential rate, while the negative semidefinite part, $||[P_k - P^*]_-||_2$, converges with a sublinear rate. These results empirically demonstrate the theoretical analysis in this paper.

\section*{Conclusion}

In this paper, we have studied the convergence of Q-VI and Q-PI for discrete-time LTI systems. Bounds on errors have been given in terms of both matrix inequalities and matrix norm. In particular, we have proved that Q-VI exponentially converges to the optimal solution if the initial parameter lies in a certain semidefinite cone. A simple analysis of convergence in general cases has also been presented. These results have been then extended to Q-PI. Finally, an example has been given to illustrate the validity of the proposed analysis. Potential future works include analysis for generalized dynamic programming with errors incurred in each update step, extensions to switching linear systems, and analysis for approximate dynamic programming and reinforcement learning algorithms.

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Lemma 6
We will use the following two lemmas:

Lemma 9 ([20, Chap. 2]). For any $A \in S^{n+m}_+$, $\lambda_{\text{max}}(A)$ can be characterized by the following optimization:

$$\lambda^* := \arg\inf_{\lambda \in \mathbb{R}} \lambda$$
subject to $A \preceq \lambda I$.

Lemma 10 ([18, Corollary 7.7.13]). Let $A, B \in S^{n+m}_+$. The following statements are equivalent:

1) $A \succeq B$
2) $\begin{bmatrix} A & B \\ B & A \end{bmatrix} \succeq 0$

Consider any $A, B \in S^{n+m}_+$ such that $A \succeq B$. Then, $\|A\|_F \leq \|B\|_F$ is equivalent to $\lambda_{\text{max}}(A^T PA) \leq \lambda_{\text{max}}(B^T PB)$. Using Lemma 9, the inequality can be cast as $\lambda_A^* \leq \lambda_B^*$, where

$$\lambda_A^* := \arg\inf_{\lambda \in \mathbb{R}} \lambda$$
subject to $A^T PA \prec \lambda I$ and

$$\lambda_B^* := \arg\inf_{\lambda \in \mathbb{R}} \lambda$$
subject to $B^T PB \prec \lambda I$.

Using the Schur complement [20, Chap. 2], the linear matrix inequality constraints can be equivalently written as

$$\begin{bmatrix} -\lambda I & 0 \\ 0 & -P^{-1} \end{bmatrix} + \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} \prec 0 \quad (14)$$
and

$$\begin{bmatrix} -\lambda I & 0 \\ 0 & -P^{-1} \end{bmatrix} + \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \prec 0 \quad (15)$$
On the other hand, by Lemma 10, $A \succeq B \iff A - B \preceq 0$ implies

$$\begin{bmatrix} 0 & B - A \\ B - A & 0 \end{bmatrix} \preceq 0 \iff \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \preceq \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$$

The above inequality implies that if (15) is satisfied, then so is (14). Therefore, it is easy to prove that $\lambda_A^* \leq \lambda_B^*$ holds. This completes the proof.

APPENDIX B
PROOF OF LEMMA 6
Define

$$P = \alpha \sum_{k=0}^{\infty} \left( \frac{1}{\rho(A(F)) + \varepsilon} \right)^{2k} (A(F)^T)^k A(F)^k$$
where $\alpha > 0$ is any number. We will prove that it is the Lyapunov matrix stated in Lemma 6. From the above definition, it is clear that $P$ satisfies

$$A(F)^T P A(F) + \alpha I = (\rho(A(F)) + \varepsilon)^2 P$$

It remains to prove the existence of such a matrix $P$, and the second statement. We have

$$\lambda_{\text{max}}(P) \leq \alpha \sum_{k=0}^{\infty} \left( \frac{1}{\rho(A(F)) + \varepsilon} \right)^{2k} \lambda_{\text{max}}((A(F)^T)^k A(F)^k)$$

$$= \alpha \sum_{k=0}^{\infty} \left( \frac{1}{\rho(A(F)) + \varepsilon} \right)^{2k} \| A(F)^k \|_2^2$$

$$\leq \alpha \sum_{k=0}^{\infty} \left( \frac{\|A(F)^k\|_2^{1/k}}{\rho(A(F)) + \varepsilon} \right)^{2k}$$

$$= \alpha \sum_{k=0}^{N-1} \left( \frac{\|A(F)^k\|_2^{1/k}}{\rho(A(F)) + \varepsilon} \right)^{2k}$$

$$+ \alpha \sum_{k=N}^{\infty} \left( \frac{\|A(F)^k\|_2^{1/k}}{\rho(A(F)) + \varepsilon} \right)^{2k}$$

From the Gelfand’s formula [18, Corollary 5.6.14, pp. 349], there exists a finite $N > 0$ such that

$$\| A(F)^k \|_2^{1/k} \leq \rho(A(F)) + 0.5\varepsilon, \quad \forall k \geq N$$

Combining the two inequalities leads to

$$\lambda_{\text{max}}(P) \leq \alpha \sum_{k=0}^{N-1} \left( \frac{\|A(F)^k\|_2^{1/k}}{\rho(A(F)) + \varepsilon} \right)^{2k}$$

$$+ \alpha \sum_{k=N}^{\infty} \left( \frac{\rho(A(F)) + 0.5\varepsilon}{\rho(A(F)) + \varepsilon} \right)^{2k}$$

$$= \alpha \sum_{k=0}^{N-1} \left( \frac{\|A(F)^k\|_2^{1/k}}{\rho(A(F)) + \varepsilon} \right)^{2k}$$

$$+ \alpha \left( \frac{\rho(A(F)) + 0.5\varepsilon}{\rho(A(F)) + \varepsilon} \right)^{2N} \sum_{k=0}^{\infty} \left( \frac{\rho(A(F)) + 0.5\varepsilon}{\rho(A(F)) + \varepsilon} \right)^{2k}$$

$$= \alpha \sum_{k=0}^{N-1} \left( \frac{\|A(F)^k\|_2^{1/k}}{\rho(A(F)) + \varepsilon} \right)^{2k}$$

$$+ \alpha \left( \frac{\rho(A(F)) + 0.5\varepsilon}{\rho(A(F)) + \varepsilon} \right)^{2N} \frac{1}{1 - \left( \frac{\rho(A(F)) + 0.5\varepsilon}{\rho(A(F)) + \varepsilon} \right)^2}$$

$$= \alpha \sum_{k=0}^{N-1} \left( \frac{\|A(F)^k\|_2^{1/k}}{\rho(A(F)) + \varepsilon} \right)^{2k}$$
Therefore, \( \lambda_{\text{max}}(P) \) is finite. By setting \( \alpha \) to be the inverse of the upper bound, we have the desired result.

**APPENDIX C**

**PROOF OF PROPOSITION 1**

Let \( P_k = \mathcal{T}^k(P) \) and \( F_k = -P_{k,22}^{-1} P_{k,12}^T \). First of all, we have

\[
T(P) - T(P^*) = \gamma A(-P_{22}^{-1}P_{12}^T)PA(-P_{22}^{-1}P_{12}) - \gamma A(F^*)^TP^*A(F^*) \\
\geq \gamma A(-P_{22}^{-1}P_{12}^T)PA(-P_{22}^{-1}P_{12}) - \gamma A(-P_{22}^{-1}P_{12}^T)P^*A(-P_{22}^{-1}P_{12}) \\
= \gamma A(-P_{22}^{-1}P_{12}^T)(P - P^*)A(-P_{22}^{-1}P_{12}),
\]

where the inequality is due to Lemma 2. In general, using \( T(P^*) = P^* \), it follows that

\[
\mathcal{T}^{k+1}(P) - \mathcal{T}^{k+1}(P^*) \\
= \gamma A(F_k)^T\mathcal{T}^k(P)A(F_k) - \gamma A(F^*)^TP^*A(F^*) \\
\geq \gamma A(F_k)^T\mathcal{T}^k(P)A(F_k) - \gamma A(F_k)^TP^*A(F_k) \\
= \gamma A(F_k)^T(\mathcal{T}^k(P) - \mathcal{T}^k(P^*))A(F_k)
\]

for all \( k \geq 0 \), where the inequality is due to Lemma 2. By recursively applying the last inequality, one gets

\[
\mathcal{T}^k(P) - \mathcal{T}^k(P^*) \geq \left( \prod_{i=0}^{k-1} \gamma^{1/2} A(F_i) \right)^T (P - P^*) \left( \prod_{i=0}^{k-1} \gamma^{1/2} A(F_i) \right)
\]

Since \( P_k \to P^* \) as \( k \to \infty \), \( F_k \to F^* \) as well. Therefore, for any \( \varepsilon_1 > 0 \), there exists a sufficiently large \( N_1 > 0 \) such that

\[
\left\| \gamma^{1/2} A(F_k) - \rho(\gamma^{1/2} A(F^*)) \right\|_2 \leq \varepsilon_1, \quad \forall k \geq N
\]

holds, which implies

\[
\left\| \gamma^{1/2} A(F_k) \right\|_2 \leq \left\| \rho(\gamma^{1/2} A(F^*)) \right\|_2 + \varepsilon_1, \quad \forall k \geq N
\]

using the inverse triangular inequality. Now, by letting

\[
M = \left( \prod_{i=0}^{N-1} \gamma^{1/2} A(F_i) \right)^T (P - P^*) \left( \prod_{i=0}^{N-1} \gamma^{1/2} A(F_i) \right)
\]

it follows that

\[
\mathcal{T}^{k+N_1}(P) - \mathcal{T}^{k+N_1}(P^*) \\
\geq \left( \prod_{i=1}^{k} \gamma^{1/2} A(F_i) \right)^T M \left( \prod_{i=1}^{k} \gamma^{1/2} A(F_i) \right) \\
\geq -I\lambda_{\text{max}}(M) \left\| \prod_{i=1}^{k} \gamma^{1/2} A(F_i) \right\|_2^2 \\
\geq -I\lambda_{\text{max}}(M) \left( \left\| \gamma^{1/2} A(F^*) \right\|_2 + \varepsilon_1 \right)^2 \\
= -I\lambda_{\text{max}}(M) \left( \left\| \gamma^{1/2} A(F^*) \right\|_2^{1/k} + \varepsilon_1 \right)^2
\]

for all \( k \geq 0 \). From the Gelfand’s formula [18, Corollary 5.6.14, pp. 349], there exists a finite \( N_2 \geq 0 \) such that

\[
\left\| \gamma^{k/2} A(F^*)^k \right\|_2 \leq \rho(\gamma^{1/2} A(F^*)) + \varepsilon, \quad \forall k \geq N_2,
\]

for any \( \varepsilon_2 > 0 \). Plugging this bound into the previous inequality leads to

\[
\mathcal{T}^{k+N_1}(P) - \mathcal{T}^{k+N_1}(P^*) \geq -I\lambda_{\text{max}}(M) \left( \left\| \gamma^{1/2} A(F^*) \right\|_2^{1/k} + \varepsilon_1 \right)^2
\]

for all \( k \geq N_2 \). Therefore, the desired result is obtained.