COMPRESSING THIN SPHERES IN THE COMPLEMENT OF A LINK

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Abstract. Let $L$ be a link in $S^3$ that is in thin position but not in bridge position and let $P$ be a thin level sphere. We generalize a result of Wu by giving a bound on the number of disjoint irreducible compressing disks $P$ can have, including identifying thin spheres with unique compressing disks. We also give conditions under which $P$ must be incompressible on a particular side or be weakly incompressible. If $P$ is strongly compressible we describe how a pair of compressing disks must lie relative to the link.

1. Introduction

For $L$ a link in $S^3$, the natural height function $p : (S^3 - \text{poles}) \to \mathbb{R}$ gives rise to level spheres which correspond to meridional planar surfaces in the complement of $L$. The link is said to be in thin position if the number of intersection points between the level spheres and the link has been minimized; the precise definition is given later. A level sphere is called thin if the highest critical point for $L$ below it is a maximum and the lowest critical point above it is a minimum. The link is said to be in bridge position if it does not have any thin spheres. In [4], Thompson has shown that if a knot is in thin, but not in bridge position, maximally compressing a thin level sphere results in a non-trivial incompressible meridional planar surface. Wu [5] has shown that a thin sphere of minimum width is itself incompressible in the link complement. A natural question to ask is what can be said about the compressing disks for other thin spheres in $S^3-L$ given information about the spheres’ widths. In this paper we give some sufficient conditions for a thin sphere to be incompressible on a particular side, to be weakly incompressible, or to have a unique compressing disk. If $P$ is strongly compressible we describe how a pair of compressing disks must be positioned relative to the link. We also prove a bound on the number of disjoint “simple” compressing disks a thin sphere can have based on its width, thus generalizing Wu’s result. In the last section we give an additional restriction on the compressibility of a thin sphere in the case where the link is prime or is a knot.

2. Thin position

We start with a quick review of thin position, a concept originally due to Gabai [1]. A detailed overview can be found in [2]. Let $p : S^3 \to \mathbb{R}$ be the standard height function and let $L$ be a link in $S^3$ such that $p$ restricts to a Morse function on $L$. If $t$ is a regular value of $p|L$, $p^{-1}(t)$ is called a level sphere with width $w(p^{-1}(t)) = |L \cap p^{-1}(t)|$. If $c_0 < c_1 < ... < c_n$ are all the critical values of $p|L$, 

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choose regular values $r_1, r_2, \ldots, r_n$ such that $c_{i-1} < r_i < c_i$. Then the width of $L$ with respect to $p$ is defined by $w(L, p) = \sum w(p^{-1}(r_i))$. The width of $L$, $w(L)$ is the minimum of $w(L', p)$ over all $L'$ isotopic to $L$. We say that $L$ is in thin position if $w(L, p) = w(L)$.

**Remark 2.1.** Isotoping $L$ so as to slide a minimum above a maximum decreases the width by 4. Sliding a minimum below a maximum increases the width by 4, while sliding a minimum past a minimum or a maximum past a maximum has no effect on the width.

If $\gamma \subseteq L$ is a 1-manifold (not necessarily connected) and $P$ is a level sphere, then we call $P$ a thin (resp. thick) level sphere for $\gamma$ if the lowest critical value of $p|\gamma$ above $P$ is a minimum (resp. maximum) and the highest critical value below $P$ is a maximum (resp. minimum). To avoid having to consider special cases later, we will define $P$ to be thin for $\gamma$ also when $\gamma \cap P = \emptyset$. If $\gamma = L$ we call $P$ thin or thick without specifying the 1-manifold. We say that $L$ is in bridge position if there are no thin spheres.

Since $L$ is in general position with respect to $p$ it is disjoint from both the minimum (south pole) and maximum (north pole) of $p$ on $S^3$. The complement of these poles in $S^3$ is of course diffeomorphic to $S^2 \times \mathbb{R}$ and, using $(-1, 1) \cong \mathbb{R}$, we can choose the diffeomorphism to preserve the foliation of $S^3 - \text{poles}$ by level spheres given by $p$. The width of $L$ could just as easily be computed via its diffeomorphic image in $S^2 \times \mathbb{R}$. So, with little risk of substantive confusion, we will often regard $L$ as contained in $S^2 \times \mathbb{R}$ and continue to use $p$ to denote the projection on the second factor $p : S^2 \times \mathbb{R} \to \mathbb{R}$.

### 3. Moving $L$ Around in 3-space

It will be useful to move parts of $L \subset S^2 \times \mathbb{R}$ vertically, that is without changing the projection of $L$ to $S^2$, but altering only the height function $p$ on those parts. Suppose, for example, $a < b$ are regular values for $p|L$. Take $\epsilon > 0$ so small that there are no critical values of $p|L$ in either of the intervals $[a, a + \epsilon]$ or $[b, b + \epsilon]$. Let $h : [a, b + \epsilon] \to [a, b + \epsilon]$ be the automorphism that consists of the union of the linear homeomorphisms $[a, a + \epsilon] \to [a, b]$ and $[a + \epsilon, b + \epsilon] \to [b, b + \epsilon]$.

**Definition 3.1.** Let $\beta$ be a collection of components of $L \cap (S^2 \times [a, b + \epsilon])$. The push-up of $\beta$ past $S^2 \times \{b\}$ is the image of $\beta$ under the homeomorphism $H : S^2 \times [a, b + \epsilon] \to S^2 \times [a, b + \epsilon] : (x, t) \mapsto (x, h(t))$. (Figure 1)
Notice that all critical points of $H(\beta)$ lie in $S^2 \times [b, b+\epsilon]$. Since there is a linear isotopy from $h$ to the identity, $\beta$ is properly isotopic to $H(\beta)$ in $S^2 \times [a, b+\epsilon]$. This isotopy from $\beta$ to $H(\beta)$ is called pushing the critical points of $\beta$ above the sphere $p^{-1}(b)$. There is an obvious symmetric isotopy that pushes the critical points of $\beta$ below the sphere $p^{-1}(a)$. These isotopies of $\beta$ only make sense as isotopies of $L$ if they do not move $\beta$ across any other part of $L$. To that end, we now enhance somewhat a fundamental construction due to Wu [5].

Lemma 3.2. Let $P$ be a level sphere for $L \subset S^3$ and let $B$ be a closed component of $S^3 - P$. Suppose $D$ is a compressing disk for $P - L$ in $B - L$. Then there is an isotopy of $\partial D \times [0, \infty) \subset S^2 \times [0, \infty) \subset S^3$.

Remark 3.3. An isotopy that, in the end, leaves $p|L$ unchanged is called an $h$-isotopy in [5]. Adding the adjacent pole to $B \cap (\partial D \times [0, \infty))$ creates a vertical disk in $B$ that is a compressing disk for $P - L$ in $B - L$ and that has the same boundary as $D$.

Proof. The proof extends the argument of Lemma 2 of [5]. With no loss of generality we may assume that $P = p^{-1}(0)$, $B$ and $D$ lie above $P$, and $\partial D$ is the equator of $P$, dividing $P$ into two disks, $D_w$ and $D_e$. Let $w$ and $e$ (the west and east poles) denote the centers of $D_w$ and $D_e$ respectively. Let $\gamma_w$ and $\gamma_e$ denote the unions of the north pole with respectively the vertical arcs $\{w\} \times [0, \infty)$ and $\{e\} \times [0, \infty)$ which are contained in $S^2 \times [0, \infty) \subset B$. (Figure 2 represents the situation one dimension lower.) By general position, we may assume that $\gamma_w$ and $\gamma_e$ are disjoint from $L$. By possibly piping points of $D \cap \gamma_e$ over the pole we may assume, without moving $L$, that $D$ is disjoint from $\gamma_e$. In particular, both $D$ and $L \cap B$ lie inside $S^2 \setminus \{e\} \times [0, \infty) \cong \mathbb{R}^2 \times [0, \infty)$. Choose the parameterization $(S^2 \setminus \{e\}) \cong \mathbb{R}^2$ so that $\partial D$ is the unit circle and $\{w\} = 0 \in \mathbb{R}^2$. (Then $\gamma_w$ corresponds to $\{0\} \times [0, \infty)$.)

We are now in a position to apply Wu’s argument almost verbatim, but adapted to our parameterization: Let $B_w$ and $B_e$ be the balls in $B$ bounded by $D \cup D_w$ and $D \cup D_e$ respectively. Let $\alpha = L \cap B_w$ and $\beta = L \cap B_e$. Perform a level-preserving isotopy (fixed near $L \cap P \cong L \cap (\mathbb{R}^2 \times \{0\})$) that shrinks at every level the radial distance from 0 in $\mathbb{R}^2$ so that, after the isotopy, $B_w \subset D_w \times [0, \infty)$. Next, without moving $\beta$, shrink the ball $B_e$ so that it is very close to $\mathbb{R}^2 \times \{0\}$. (This isotopy is not level-preserving on $\alpha$.)

Then perform a level-preserving isotopy on $\beta$, increasing at every level its radial distance from 0 in $\mathbb{R}^2$, until $\beta$ lies entirely outside of $D_w \times [0, \infty)$. Finally, undo the isotopy that shrunk $B_w$, restoring $\alpha$ to its original height, but leaving it entirely inside $D_w \times \mathbb{R}$.

It is straightforward to see that if a split link is placed in thin position then there is a splitting sphere that is also a level sphere. Thus little is lost in assuming henceforth that no link is split. For the remainder of this paper we will assume that $L$ is an unsplit link in $S^3$ in thin position and $P$ is a thin sphere compressible in $S^3 - L$ with compressing disk $D$. Without loss of generality we assume $P = p^{-1}(0)$ and $D \subset p^{-1}(0, \infty)$. Then $D$ bounds two balls, $B_e$ and $B_w$ in $p^{-1}(0, \infty)$. Let $\alpha = B_e \cap L$ and $\beta = B_w \cap L$. We will be examining the heights of the critical points for $\alpha$ and $\beta$ and studying how frequently $\alpha$ and $\beta$ intersect level spheres. Both are unaffected by $h$-isotopies of $L$, so, following Lemma 3.2, we may also assume that $\alpha$
and $\beta$ are separated by $\partial D \times \mathbb{R}$ and thus can be isotoped vertically independently of one another.

**Lemma 3.4.** Let $T = p^{-1}(t)$ be a thick sphere for $\alpha$ and let $S_+ = p^{-1}(s_+)$ and $S_- = p^{-1}(s_-)$ be the first thin spheres for $\alpha$ above and below $T$ respectively. Then, $T$ is a thin sphere for $\beta$ and $\beta$ has no critical points in $p^{-1}([s_-, s_+])$.

**Proof.** We first show that $\beta$ does not have a minimum in the region $p^{-1}(t, s_+)$. Isotope a small collar $\beta \cap p^{-1}(t, t + \epsilon)$ upwards, pushing all critical points of $\beta \cap p^{-1}(t, s_+)$ above $S^+$. Because $\alpha \cap p^{-1}(t, s_+)$ only has maxima, this isotopy slides all critical points of $\beta \cap p^{-1}(t, s_+)$ only above maxima of $\alpha$. If any critical point of $\beta$ is a minimum, this would decrease the width of $L$ by Remark 2.1 contradicting $L$ being in thin position. Similarly, $\beta \cap p^{-1}(s_-, t)$ cannot have any maxima.

It follows that $\beta \cap p^{-1}(s_-, s_+)$ has no critical points at all. For suppose it has a minimum, necessarily below $T$. Push $\beta \cap p^{-1}(s_-, s_- + \epsilon)$ upwards, above $S_+$. Since $\beta$ has no maximum below $T$ and $\alpha$ has no minimum above $T$ in $p^{-1}(s_-, s_+)$, no maximum is pushed up past a minimum and the minimum of $\beta \cap p^{-1}(s_-, t)$ is moved past the maxima in $\alpha \cap p^{-1}(t, s_+)$ thinning $L$, a contradiction. A similar argument shows that $\beta \cap p^{-1}(s_-, s_+)$ does not have any maxima and therefore $\beta \cap p^{-1}(s_-, s_+)$ is a product.

Finally, we show $T$ is a thin sphere for $\beta$. Suppose, to the contrary, that the highest critical point of $\beta$ below $S_-$ is a minimum. Isotope $\beta$ so the minimum moves above $S_+$ making use of the product structure of $\beta$ between $S_-$ and $S_+$. That would result in sliding a minimum up past the maxima for $\alpha$, thus thinning $L$. Therefore the highest critical point of $\beta$ below $T$ must be a maximum. Similarly, the lowest critical point of $\beta$ above $T$, if it exists, must be a minimum proving the lemma.

\[\square\]

4. **Alternating Levels**

Recall our assumption that $L$ is in thin position, $P = p^{-1}(0)$ is a compressible punctured sphere with compressing disk $D$ lying above $P$, and $\alpha$ and $\beta$ are the
stra"nds of $L$ on each side of $D$. In the following choose the labels $\alpha$ and $\beta$ so that the maximum height of $\beta$ is greater than the maximum height of $\alpha$. Let $A = p^{-1}(a)$ be the first thin sphere above $\alpha$. If $C = p^{-1}(c)$ is a thin sphere with $0 < c \leq a$ we call $C$ an alternating sphere for $D$ and $c$ an alternating level for $D$ if the minimum just above it and the maximum just below it are on different sides of the disk $D$.

Remark 4.1. It is always the case that $A$ is an alternating sphere for $D$.

Remark 4.2. Suppose $c' < c$ are two adjacent alternating levels. Then one of $\alpha \cap p^{-1}[c', c]$ or $\beta \cap p^{-1}[c', c]$ is a product. If $c$ is the lowest alternating level above $P$, then $\alpha \cap p^{-1}[0, c]$ or $\beta \cap p^{-1}[0, c]$ is a product.

Lemma 4.3. For any level sphere $S = p^{-1}(s)$ with $s \in [0, \alpha)$, $|A \cap \beta| \leq |S \cap \beta|$ with equality if and only if $\beta \cap p^{-1}(s, a)$ is a product. It follows that $|A \cap L| < |S \cap L|$.

Proof. If $S$ is a level sphere providing a counterexample to the claim, then at least one of the two adjacent thin spheres is also a counterexample, so without loss of generality we may assume that $S$ is thin. Take $S = p^{-1}(s)$ to be the highest thin sphere between $P$ and $A$ such that $|A \cap \beta| \geq |S \cap \beta|$ and $\beta \cap p^{-1}(s, a)$ is not a product. Let $Q_1, \ldots, Q_k$ be the level spheres coming from critical values for $\alpha$ above $S$ and let $R_1, \ldots, R_k$ be the level spheres coming from critical values for $\beta$ between $S$ and $A$. (See Figure 3). As $\beta \cap p^{-1}(s, a)$ was assumed not to be a product, $h \geq 1$. Push down the critical points of $\alpha \cap p^{-1}(s, a)$ past $S$. We will denote by $L'$, $\alpha'$, and $Q_i'$ the images of $L$, $\alpha$, and $Q_i$ after the isotopy. We will show that the width of $L'$ is lower than the width of $L$, leading to a contradiction. The widths of all level spheres other than $R_i$ and $Q_i$ remain unchanged by the isotopy. Notice that $|R_i \cap \alpha'| = 0$ as all $R_i$ are above $S$. So we have

$$|R_i \cap L| = |R_i \cap \alpha| + |R_i \cap \beta| > |R_i \cap \beta| = |R_i \cap \alpha'| + |R_i \cap \beta| = |R_i \cap L'|.$$  

The inequality here is strict as $R_i \cap \alpha \neq \emptyset$. To show $L'$ is thinner than $L$, we just need to show the intersection with the $Q_i$ has not increased. For each $Q_i$, 

$$|Q_i \cap L| = |Q_i \cap \alpha| + |Q_i \cap \beta| \geq |Q_i \cap \alpha| + |S \cap \beta| = |Q_i' \cap \alpha'| + |Q_i' \cap \beta| = |Q_i' \cap L'|.$$  

This inequality uses the fact that $S$ provides the highest counterexample. As $L$ was assumed to be in thin position, we have a contradiction. □

The following is a direct consequence of Lemma 4.3.

Corollary 4.4. Suppose $P$ is a thin sphere such that for all level spheres $S$ on one side of $P$ we have that $|P \cap L| \leq |S \cap L|$. Then $P$ is not compressible on that side.
Wu’s result that the level sphere of minimum width is incompressible \([5]\) is a special case of Corollary 4.3.

By extending the proof of Lemma 4.3 we can demonstrate that a statement similar to this lemma holds for all alternating thin level spheres.

**Theorem 4.5.** Suppose \(C = p^{-1}(c)\) is any alternating thin sphere for \(D\) and \(S = p^{-1}(s)\) is any level sphere with \(s \in [0, c)\). Then \(|C \cap \alpha| < |S \cap \alpha|\) unless \(\alpha \cap p^{-1}[s, c]\) is a product, and similarly for \(\beta\).

**Proof.** We know that the theorem holds for the highest alternating sphere \(A\). This follows for \(\beta\) from Lemma 4.3 and for \(\alpha\) from the observation that \(A \cap \alpha = \emptyset\).

Suppose \(C = p^{-1}(c)\) is the highest alternating sphere for which the statement is not true: i.e. there exists \(S = p^{-1}(s)\) with \(s \in [0, c)\) a level sphere such that either \(|C \cap \alpha| \geq |S \cap \alpha|\) and \(\alpha \cap p^{-1}[s, c]\) is not a product or \(|C \cap \beta| \geq |S \cap \beta|\) and \(\beta \cap p^{-1}[s, c]\) is not a product, say the former. Take \(S\) to be the highest such sphere.

As in Lemma 4.3 we may assume that \(S\) is thin by passing to an adjacent thin sphere. By hypothesis, if \(B = p^{-1}(b)\) is the first alternating thin sphere above \(C\) then \(|B \cap \beta| < |S \cap \beta|\). \(B\) and \(C\) are adjacent alternating spheres, so by Remark 4.2 one of \(\alpha \cap p^{-1}(c, b)\) and \(\beta \cap p^{-1}(c, b)\) has to be a product. If \(\alpha \cap p^{-1}(c, b)\) were a product, then \(|\alpha \cap B| = |\alpha \cap C| \geq |\alpha \cap S|\), contradicting the choice of \(C\) and \(S\). So \(\beta \cap p^{-1}(c, b)\) is a product. Let \(Q_1, \ldots, Q_k\) be the level spheres coming from critical values for \(\alpha\) between \(S\) and \(B\), and let \(R_1, \ldots, R_h\) be the level spheres coming from critical values for \(\beta\) in the same region. As \(C\) is alternating, \(\beta\) has a critical point directly below \(C\). Thus for all \(Q_i\) the region of \(\beta\) between \(B\) and \(Q_i\) is not a product (Figure 4), and so by the hypothesis \(|B \cap \beta| < |Q_i \cap \beta|\). Push down the critical points of \(\beta \cap p^{-1}[s, b]\) past \(S\). Let \(\beta', R_i'\) and \(L'\) be the images of \(\beta, R_i\) and \(L\) after the isotopy. The only level spheres whose widths are affected by the isotopy are the \(R_i\) and \(Q_i\).

Computing the width of \(L'\) we obtain, for each \(Q_i\) and \(R_i\),

\[
|Q_i \cap L| = |Q_i \cap \alpha| + |Q_i \cap \beta| > |Q_i \cap \alpha| + |B \cap \beta| = |Q_i \cap \alpha| + |Q_i \cap \beta'| = |Q_i \cap L'|
\]

and

\[
|R_i \cap L| = |R_i \cap \alpha| + |R_i \cap \beta| \geq |S \cap \alpha| + |R_i' \cap \beta'| = |R_i' \cap L'|.
\]

To see the second inequality, note that if for some \(R_i\) we have \(|R_i \cap \alpha| < |S \cap \alpha| \leq |C \cap \alpha|\) then a thin sphere \(S'\) adjacent to \(R_i\) should have been substituted for \(S\) which was assumed to provide the highest counterexample. Thus the isotopy thins \(L\) which was assumed to be in thin position, a contradiction.
Corollary 4.6. If $C = p^{-1}(c)$ is an alternating thin level sphere for $D$, then its width is lower than the widths of all level spheres lying in $p^{-1}(0, c)$.

Corollary 4.7. The widths of the alternating thin spheres above $P$ are monotone decreasing.

Corollaries 4.6 and 4.7 provide restrictions on which thin spheres can be alternating. As we know that the strands of the link on one side of a compressing disk must be a product between adjacent alternating thin spheres, we can now give a description of how pairs of compressing disks on opposite sides of $P$ must lie with respect to the link.

5. Thin level spheres compressible on both sides

The compressing disk $D$ for $P$ separates $p^{-1}[0, \infty)$ into two balls. Define the interior ball of $D$, denoted $B^{\text{int}}(D)$, to be the closed ball that contains the critical point of $L$ closest to $P$. The exterior ball of $D$ will be denoted $B^{\text{ext}}(D)$ and is also closed. Notice that $B^{\text{int}}(D)$ and $B^{\text{ext}}(D)$ intersect $P$ in disks with boundary $\partial D$ which we label $D^{\text{int}}$ and $D^{\text{ext}}$ respectively.

Theorem 5.1. If $P$ is a thin sphere compressible above and below with compressing disks $D_u$ and $D_l$ respectively, then $D_u^{\text{int}} \cap D_l^{\text{int}} \neq \emptyset$.

Proof. Suppose we have compressing disks $D_u$ above $P$ and $D_l$ below $P$ with $D_u^{\text{int}} \cap D_l^{\text{int}} = \emptyset$. By Remark 4.4, there is at least one alternating sphere on each side of $P$. Let $A_u = p^{-1}(a_u)$ be the lowest alternating thin sphere for $D_u$ and $A_l = p^{-1}(a_l)$ be the highest alternating thin sphere for $D_l$ (Figure 5). By Corollary 4.6,

$$w(P) > w(A_u).$$

By Remark 4.4 and after perhaps an $h$-isotopy described in Lemma 3.2, $B^{\text{ext}}(D_u) \cap p^{-1}[0, a_u]$ is a product, so we can extend $D_l$ up to $A_u$ using this product structure. Notice that $P$ is now an alternating thin sphere for the extended $D_l$ and therefore

$$w(P) < w(A_u).$$

Thus we have a contradiction. □
Definition 5.2. A level sphere is strongly compressible in the link complement if there exist compressing disks $D_u$ and $D_l$ above and below $P$ respectively, such that $\partial D_u \cap \partial D_l = \emptyset$. If a level sphere is not strongly compressible, then it is weakly incompressible.

Corollary 5.3. If a level sphere $P$ is strongly compressible with compressing disks $D_u$ and $D_l$, then the disks $D_u^{\text{int}}$ and $D_l^{\text{int}}$ are nested.

In fact, using the notation in the proof of Theorem 5.1, the relative widths of $A_u$ and $A_l$ determine the direction of the nesting as the next theorem shows.

Theorem 5.4. $D_u^{\text{int}} \subset D_l^{\text{int}}$ if and only if $w(A_l) < w(A_u)$.

Proof. If $D_u^{\text{int}} \subset D_l^{\text{int}}$ then $\partial D_l$ lies in $D_u^{\text{ext}}$, so just as in the proof of Theorem 5.1, we can extend $D_l$ to $A_u$. $A_l$ is still an alternating thin sphere for the extended disk which is now a compressing disk for $A_u$. Therefore $w(A_l) < w(A_u)$. The other direction is obtained by switching the labels $u$ and $l$. \hfill \Box

Corollary 5.5. If $P$ is a thin sphere with second lowest width among all thin spheres, then $P$ is weakly incompressible.

Proof. Suppose $P$ is strongly compressible and let $A_u$ and $A_l$ be as in the theorem. From Theorem 5.4 we can deduce that it is not possible to have $w(A_u) = w(A_l)$. If $P$ is a sphere of second lowest width then $A_u$ and $A_l$ will both be of minimal width and thus we would have $w(A_u) = w(A_l)$, a contradiction. \hfill \Box

The next theorem provides another sufficient condition for a thin sphere to be weakly incompressible. First notice the following:

Remark 5.6. Suppose $P$ is a thin sphere compressible above with compressing disk $D$ and let $Q$ be the first thin sphere above $P$ satisfying $w(Q) < w(P)$. Then there are no alternating thin spheres for $D$ below $Q$ and thus $\alpha$ or $\beta$ must be a product between $P$ and $Q$. The analogous statement holds for $P$ compressible below.

Theorem 5.7. Let $P_i$ and $P_k$ be two thin spheres with $P_i$ lying below $P_k$. Suppose $P_i$ is not compressible above and $P_k$ is not compressible below. Let $P_j$ be the thin sphere of minimal width among all spheres strictly between $P_i$ and $P_k$. Then $P_j$ is weakly incompressible.

Notice that in particular the hypothesis holds if $P_i$ and $P_k$ are both of minimal width.

Proof. Assume to the contrary that $P_j$ is strongly compressible with compressing disks $D_u$ and $D_l$ above and below $P_j$ respectively. By Corollary 5.3 the disks $D_u^{\text{int}}$ and $D_l^{\text{int}}$ are nested. We will assume $D_u^{\text{int}} \subset D_l^{\text{int}}$, the other case is similar. By Remark 5.6 we know that $L \cap B^{ext}(D_u)$ must have a product structure between $P_j$ and $P_k$. As $\partial D_l \subset D_u^{ext}$ we can use the product structure to extend $D_l$ up to $P_k$. But $P_k$ was assumed to be incompressible from below, so we reach a contradiction. \hfill \Box

Definition 5.8. The region between two level spheres is turbulent if every strand in the region has at least one critical point.

Corollary 5.9. If the region between two level spheres is turbulent then a thin sphere of minimal width between them is weakly incompressible.

In other words, a thinnest sphere between two other level spheres, if it is strongly compressible, “calms” the region between them.
6. Bound on the number of disjoint irreducible compressing disks

Once we have placed the link in thin position, the width function restricted to
the thin levels gives a sequence of even integers. Consider these integers as a set
discarding repeating values, arrange them in (strictly) increasing order and label
them \( w_0, w_1, \ldots, w_k \). Thus the lowest width thin sphere has \( w_0 \) intersection points
with the link, the second lowest width thin sphere has \( w_1 \) intersection points, etc.
Of course there may be several thin spheres all with width \( w_i \) and we have no
control in what order the \( w_i \) appear when we look at the natural ordering of the
thin spheres given by the height function on \( S^3 \).

Using this language Wu’s result can be restated as:

**Theorem (Wu, [5]).** If \( w(P) = w_0 \) then \( P \) has no compressing disks.

We can also restate Corollary 5.5:

**Corollary 5.5’.** If \( w(P) = w_1 \) then \( P \) is weakly incompressible.

For the remainder of the paper we will assume that \( D \) is a compressing disk for
\( P \) lying above, \( \alpha \) and \( \beta \) are the strands of a (unsplit) link \( L \) contained in each of
the two balls cobounded by \( D \) and \( P \) where \( \alpha \) is shorter than \( \beta \). Let the *short ball
for \( D \), \( B^{sh(D)} \), be the closed ball bounded by \( D \cup P \) containing \( \alpha \), and let \( B^{\ell(D)} \) be
the closed ball bounded by \( D \cup P \) containing \( \beta \). We will denote the disk \( B^{sh(D)} \cap P \)
by \( D_{sh} \) (See Figure 6).

**Definition 6.1.** A reducing disk for \( D, E \), is an embedded disk contained in \( B^{sh(D)} \)
such that \( \partial E = \tau \cup \omega \), where \( \tau \subset D \) and \( \omega \subset D_{sh} \) is essential in \( D_{sh} - L \). \( D \) is
reducible if such a disk exists and irreducible otherwise. (Figure 6)

We first prove two straightforward lemmas describing how two compressing disks
can lie relative to each other.

**Lemma 6.2.** Suppose \( D \) and \( D’ \) are two disjoint, nonparallel, compressing disks
for \( P \) such that \( D \subset B^{sh(D')} \). Then \( D' \) is reducible.

**Proof.** Consider an embedded arc \( \nu \) spanning the annulus \( D'_{sh} - D_{sh} \) and missing
the link. A regular neighborhood of \( B^{sh(D)} \cup \nu \) has boundary a 2-sphere which
intersects \( B^{sh(D') \cap P} \) in a reducing disk for \( D' \). \( \square \)
Lemma 6.3. Suppose $D_1$ and $D_2$ are two compressing disks for $P$. Then there is an isotopy of $D_2$ in the link complement, fixing $\partial D_2$, at the end of which the two disks do not have circles of intersection.

Proof. Suppose $c \in D_1 \cap D_2$ is a circle of intersection innermost on $D_1$. The disk on $D_1$ bounded by $c$, $F_1$, has an interior disjoint from $D_2$. There is also a disk $F_2$ in $D_2$ with boundary $c$. Because the link is not split the embedded sphere $F_1 \cup F_2$ bounds a ball on one side. Therefore $F_2$ can be isotoped across the ball to remove $c$ from the intersection.

If $P = p^{-1}(0)$ is a thin sphere we call a thin sphere $A = p^{-1}(a)$ above it potentially alternating for $P$ if it satisfies the condition for the level sphere $C$ in Corollary 4.6, i.e. its width is strictly lower than the width of any thin sphere in the region $p^{-1}(0, a)$. Notice that the definition of an alternating sphere depends on the disk under consideration while the definition of potentially alternating depends only on the initial sphere $P$. However, as the term suggests, a sphere can be alternating for some compressing disk for $P$ only if it is potentially alternating. We will denote the potentially alternating thin spheres for $P$ by $A_i$, $i \geq 1$, in order of ascending heights (and descending widths). Of course if $w(P) = w_k$ then $P$ can have at most $k$ potentially alternating thin spheres lying above it.

Remark 6.4. If $D$ is a compressing disk for $P$ and $\alpha$ intersects two adjacent potentially alternating thin spheres $A_{j-1}$ and $A_j$, then one of $\alpha$ or $\beta$ is a product in the region between them.

Definition 6.5. Suppose $D$ is a compressing disk for $P$ so that $\alpha \cap A_{i-1} \neq \emptyset$ but $\alpha \cap A_i = \emptyset$ then we say that $D$ has height $i$.

Now we can prove a theorem that has many corollaries including the main result of this section - a bound on the number of disjoint irreducible compressing disks a thin level sphere can have.

Theorem 6.6. Suppose $D$ and $D'$ are two irreducible compressing disks for $P$, and $\alpha$, $\alpha'$ are the strands of $L$ lying in the corresponding short balls. Then $\text{height}(D) = \text{height}(D')$ implies $\alpha = \alpha'$. Otherwise $\alpha \cap \alpha' = \emptyset$.

Proof. First we will show that either $\alpha = \alpha'$ or $\alpha \cap \alpha' = \emptyset$. If $D$ and $D'$ are disjoint the result follows from Lemma 6.2.

Let $\Lambda = D \cap D'$ and assume the disks have been isotoped in the link complement to minimize $|\Lambda|$; in particular, by Lemma 6.3 $\Lambda$ contains only arcs. By Theorem 6.2 we can assume $D'$ is vertical. Let $\{R_i\}$ be the set of all components of $D - \Lambda$ that lie inside $B^\text{sh}(D')$. Suppose $R_1$ is bounded by curves $\lambda_1, \ldots, \lambda_n \in \Lambda$ and $\omega_1, \ldots, \omega_n \subset \partial D \cap D'_{sh}$. We will assume $\omega_1$ is outermost of the $\omega_i$ in $D'_{sh}$ and thus cobounds a bigon with $\partial D'$. (See Figure 7h.) $R_1$ separates $B^\text{sh}(D')$ into closed balls which we will call $F_1$ and $E_1$ with the labels chosen so that the bigon cobounded by $\omega_1$ and $\partial D'$ lies in $F_1$. This bigon must intersect the link, otherwise we could decrease $|\Lambda|$ via an isotopy of $D'$ pushing $\omega_1$ and any other arcs of $\partial D \cap D'_{sh}$ contained in the bigon across it and thus eliminating at least two points of $\partial D' \cap \partial D'$. We can thus conclude that $\alpha' \cap F_1 \neq \emptyset$. We will show that in fact $\alpha \subset F_1$.

If we place the point at infinity in $D'_{sh} \cap E_1$ we obtain Figure 7h, depicting $\partial B^\text{sh}(D') - \infty$. The disk $F_1 \cap \partial B^\text{sh}(D')$ has been shaded.
Pick $\lambda_1$ to be outermost of the $\lambda_i$ in $D'$ so it cobounds a bigon with $\partial D'$. Isotope $R_1$ in the link complement pushing $\lambda_1$ across the bigon into $D_{sh}'$. No restriction is placed on the shading of the bigon. This process decreases $|\partial R_1 \cap \partial D'|$ by two, so by a series of such moves we can isotope $R_1$ so that $\partial R_1$ and $\partial D'$ are disjoint. Notice that $\omega_1$ was never isotoped so it must still lie in $D_{sh}'$ and thus all of $\partial R_1$ lies there. Now $R_1$ is a disk with $\partial R_1 \subset D_{sh}'$ and by Lemma 6.2 it is either not a compressing disk for $P$ or it is parallel to $D'$. In other words $\alpha'$ is contained entirely on one side of $R_1$, and as we noticed earlier that $\alpha' \cap F_1 \neq \emptyset$, we conclude that $\alpha' \subset F_1$.

Although we do not need this for the rest of the proof, it is interesting to observe that as $L \cap D_{sh}' \cap E_1 = \emptyset$, all outermost arcs $\omega_i$ must have bounded shaded bigons of $D_{sh}'$ or we could have decreased $|D \cap D'|$.

Now consider a second region $R_2$ separating $B^{sh}(D')$ into balls $F_2$ and $E_2$ with the labels chosen analogously to the labels for $R_1$ and thus $E_2 \cap L = \emptyset$. If $R_2$ is contained in $F_1$ we will say that $R_2$ is inside $R_1$. Observe that if $R_2$ is inside $R_1$, $F_2$ must be contained in $F_1$: an outermost arc of $\partial R_2 \cap D_{sh}'$ bounds a bigon of $D_{sh}'$ contained in $F_2$ (by definition of $F_2$) and also contained in $F_1$, which is on the side of $\partial R_2$ that does not contain $\partial R_1$. (See Figure 7.) Thus we can find $R_n$, an innermost region with $\alpha' \subset F_n$ and thus entirely on one side of $D$. We conclude that $\alpha' \subset \alpha$ or $\alpha' \subset \beta$. Switching the labels of $D$ and $D'$ we also have that $\alpha \subset \alpha'$ or $\alpha \subset \beta'$. As $\alpha \cap \beta = \alpha' \cap \beta' = \emptyset$ we must either have $\alpha = \alpha'$ or $\alpha \cap \alpha' = \emptyset$.

The height of a disk is determined by the strands of $L$ inside its short ball, therefore if $D$ and $D'$ have different heights we must have the case that $\alpha \cap \alpha' = \emptyset$. If $\text{height}(D) = \text{height}(D') = i$, as $D'$ was chosen to be vertical, $D' \cap A_{i-1}$ is a single circle bounding a subdisk $\hat{D}$ of $D'$. Let $\hat{\alpha} = L \cap B^{sh}(\hat{D})$ and $\hat{\beta} = L \cap B^{h}(\hat{D})$. As $B^{sh}(\hat{D}) \subset B^{sh}(D')$ we must have that $\hat{\alpha} \subset \alpha'$. $\hat{D}$ is a compressing disk for $A_{i-1}$ of height 1 so $\hat{\alpha}$ must have some critical points. Remark 6.4 implies that $\hat{\beta}$ cannot contain any critical points between $A_{i-1}$ and $A_i$ so all strands of $\hat{\beta}$ intersect $A_i$. By the definition of height we must have that $\alpha \cap A_i = \emptyset$. We can conclude that
\[ \alpha \cap \beta = \emptyset \] so \( \tilde{\alpha} \subset \alpha \) and therefore \( \alpha' \cap \alpha \supset \tilde{\alpha} \). Thus we must have the case that \( \alpha = \alpha' \).

\[ \square \]

**Corollary 6.7.** Any two distinct irreducible compressing disks for \( P \) of the same height must intersect.

**Proof.** Suppose \( D \) and \( D' \) are two disks of the same height. By Theorem 6.6 we know that \( \alpha = \alpha' \). So if \( D \cap D' = \emptyset \), possibly after switching labels, \( D \subset B^{sh}(D') \). Then by Lemma 6.2 we have that \( D \) is parallel to \( D' \).

\[ \square \]

**Corollary 6.8.** Suppose \( P \) is a thin sphere and \( w(P) = w_n \). Then \( P \) has at most \( n \) disjoint nonparallel irreducible compressing disks.

**Proof.** Let \( \Delta \) be a maximal collection of disjoint nonparallel irreducible compressing disks for \( P \). Corollary 6.7 implies that \( \Delta \) contains at most one disk of each of the \( n \) possible heights corresponding to the \( n \) potentially alternating thin spheres for \( P \). Therefore \( \Delta \) contains at most \( n \) disks.

\[ \square \]

Based on Theorem 6.6 if there is any compressing disk of height \( i \) we can, without ambiguity, introduce the notation \( \alpha_i \) and \( \beta_i \) for the strands of \( L \) inside and outside the short ball of the disk. By the same theorem we also know that \( \alpha_i \subset \beta_j \) for every \( i \neq j \).

**Corollary 6.9.** If there is some \( \alpha_k \) containing critical points of \( L \) between \( A_{j-1} \) and \( A_j \) with \( j < k \) then \( P \) does not have a compressing disk of height \( j \).

**Proof.** Suppose a disk of height \( j \) exists. Then \( \alpha_j \cap A_{j-1} \neq \emptyset \) but \( \alpha_j \cap A_j = \emptyset \), so \( \alpha_j \) must have critical points between \( A_{j-1} \) and \( A_j \). By hypothesis \( \alpha_k \) has critical points in this region so by Remark 6.4 we know that \( \beta_k \) must be a product there. But Theorem 6.6 gives \( \alpha_j \subset \beta_k \), leading to a contradiction.

\[ \square \]

Even though the height of the disk completely determines which strands of \( L \) its short ball will contain, it is not true that two irreducible disks can necessarily be isotoped to be disjoint in the link complement. Figure 8 shows an example in which two irreducible disks, \( D \) and \( D' \), both of which separate \( \alpha \) from \( \beta \), cannot be isotoped to be disjoint. However it is possible to pick a family of disjoint irreducible disks representing all different heights as the next theorem shows.
Theorem 6.10. There exists a collection of disjoint irreducible compressing disks for $P$ that contains one representative from each possible height.

Proof. Let $\Delta = \bigcup D_i$ be a maximal in size collection of distinct disjoint irreducible compressing disks for $P$. By Corollary 5.7 we know $\Delta$ can contain at most one compressing disk of each height. Suppose there is some height $k$ such that $P$ has a compressing disk $D$ of height $k$ but $\Delta$ does not contain any such disk. Isotope $D$ so that $|D \cap \Delta|$ is minimal. As before we will denote by $\alpha$ and $\alpha_i$ the intersections of $L$ with the short balls of $D$ and $D_i$ respectively. As $D \notin \Delta$ there exists a curve $\lambda \in D \cap \Delta$ outermost in $D$ and thus bounding a subdisk $E$ disjoint from all $D_i$. Let $j$ be such that $\lambda \in D \cap D_j$ and let $E_j$ be one of the components of $D_j - \lambda$. Consider the disk $\tilde{D} = E \cup E_j$ and let $B$ be the ball cobounded by $\tilde{D}$ and $P$ containing $\alpha$. Note that $\tilde{D} \cap \Delta = \emptyset$ so any disk in $\Delta$ is contained either entirely outside or entirely inside $B$. As each $D_i$ is irreducible, $\alpha \cap B = \emptyset$ implies $\alpha_i \subset B$. After renumbering we may assume $\alpha_{j_1}, \ldots, \alpha_{j_r}$ are contained inside $B$ (this collection may be empty). Reduce $\tilde{D}$ along a maximal collection of disjoint reducing disks contained in $B$ and let $\Delta'$ be the resulting collection of disjoint irreducible disks. Note that by Theorem 6.10 each $\alpha_{j_i}$, $1 \leq i \leq r$, is contained in the short ball of a unique disk in $\Delta'$. Let $D'_{j_i}$ be the disk in $\Delta'$ with short ball containing $\alpha_{j_i}$. Because $B$ was chosen so that $\alpha \subset B$ there must be an irreducible disk $D'_{j_0}$ inside $B$ with a short ball containing $\alpha$. Then the set $\{D'_{j_0}, \ldots, D'_{j_r}\} \cup (\Delta - \{D_1, \ldots, D_r\})$ contains $|\Delta| + 1$ disjoint irreducible compressing disks, thus contradicting the maximality of $\Delta$.

\[ \square \]

7. Thin spheres with unique compressing disks

In the previous section we restricted our attention to irreducible disks. However when discussing disks of height 1 we can drop this restriction.

Lemma 7.1. Any compressing disk of height 1 is irreducible.

Proof. Suppose now $P$ has a compressing disk $D$ of height 1 that is reducible. If $\bigcup_{i=1}^n E_i$ is a maximal collection of disjoint reducing disks for $D$, then the boundary of a regular neighborhood of $D \cup (\bigcup_{i=1}^n E_i)$ contains a collection of $n + 1$ compressing disks of height 1 with disjoint short balls. This contradicts Theorem 6.6. \[ \square \]

Theorem 7.2. If $P$ has only one potentially alternating thin sphere on some side, then $P$ has at most one compressing disk on that side.

Proof. If there are counterexamples to this theorem, choose $D_1$ and $D_2$ to be the pair which, among all counterexamples, can be isotoped to have the fewest intersection arcs. By the hypothesis any compressing disk for $P$ lying above $P$ must necessarily have height 1 so height$(D_1) = height(D_2) = 1$. By Lemma 6.1 $D_1$ and $D_2$ must be irreducible and by Corollary 6.7 they intersect. Isotope $D_1$ and $D_2$ so as to minimize their intersection, in particular all circles of intersection are removed by Lemma 6.3. Let $\lambda$ be an arc of intersection outermost on $D_1$. Then $\lambda$ bounds a disk $E_1$ in $D_1$ whose interior is disjoint from $D_2$. Let $E_2$ be one of the components of $D_2 - \lambda$. By minimality of $|D_1 \cap D_2|$ the disk $E = E_1 \cup E_2$ must be a compressing disk for $P$, necessarily of height 1. $E$ is disjoint from $D_2$ and has at least one fewer arc of intersection with $D_1$ than $D_2$ did. Therefore neither of the pairs $E, D_1$ nor $E, D_2$ are counterexamples to the theorem at hand, and so $E$ is
parallel to both $D_1$ and $D_2$. Thus $D_1$ and $D_2$ are parallel to each other and were also not a counterexample to the theorem. □

**Corollary 7.3.** If $w(P) = w_1$ then $P$ has at most one compressing disk on each side.

*Proof.* As $w(P) = w_1$, $P$ can have at most one potentially alternating thin sphere on each side. The result then follows from Theorem 7.2. □

It is straightforward to observe the following sufficient condition for $P$ to not have a compressing disk of height 1.

**Observation 7.4.** If there is a strand of $L$ that has a critical point between $P$ and $A_1$ and intersects $A_1$, then $P$ does not have a compressing disk of height 1 above.

8. **Application to Knots**

**Theorem 8.1.** Suppose $K$ is a knot or a prime link in thin position and suppose $w_1 = w_0 + 2$. If $w(P) = w_1$, then $P$ is incompressible.

*Proof.* Suppose there is a compressing disk $D$ for $P$ lying above it. $P$ has at most one potentially alternating thin sphere, $A = A_1$, with $w(A) = w_0$ thus $D$ must have height 1. For a disk of height 1 we always have that $B^{th(D)} = B^{int(D)}$ because $L$ can only have critical points between $P$ and $A_1$ on one side of the disk and that is necessarily the short ball. As before let $\alpha$ denote the strands of $K$ contained in $B^{th(D)} = B^{int(D)}$ and let $\beta$ denote those contained in $B^{ext(D)}$. Also let $D^{ext}$ and $D^{int}$ be the two disks that $\partial D$ bounds on $P$. (See Figure 9) By the product structure of $\beta$ between $P$ and $A$ we have that $|\beta \cap A| = |\beta \cap P|$. As $|\alpha \cap A| = 0$ we must have that $|\alpha \cap P| = 2$, and thus the sphere $D \cup D^{int}$ gives a decomposition of $K$ as a direct sum $K_1 \# K_2$ so $K$ cannot be a prime link. Suppose then that $K$ is a knot and let $K_1$ be the summand contained in $B^{int(D)}$, let $T$ be the level sphere directly below the highest maximum for $\alpha$ and let $S$ be the level sphere directly below the highest maximum for $\beta$. Imagine cutting the region of $\alpha$ between $P$ and $T$ and inserting it at $S$. As $K$ is a direct sum the same effect can be achieved via an isotopy making $K_1$ small and sliding it along $K_2$ to the desired position. No new critical points have been introduced and the only level spheres affected are
those between $P$ and $T$ which we denote by $Q_i$ before the isotopy and $Q'_i$ after the isotopy. The knot after the isotopy will be denoted by $K'$.

For all $i$ we have that

$$|Q_i \cap K| = |Q_i \cap \alpha| + |Q_i \cap \beta| > |Q_i \cap \alpha| = |Q'_i \cap K'|$$

so the width of $K$ has been decreased thus contradicting the assumption. □

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