Instanton representation of Plebanski gravity: XX.
Gravitational coherent states

Eyo Eyo Ita III
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Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road
Cambridge CB3 0WA, United Kingdom
eei20@cam.ac.uk

Abstract

In this paper we show that the instanton representation of Plebanski gravity exhibits a Hilbert space of harmonic oscillator-like coherent states. We put in place the formalism and carry out the construction of the states, and we elucidate on their physical interpretation. Additionally, we provide an invertible map between the Ashtekar variables and this Hilbert space of states, via the instanton representation.
1 Introduction

There are presently at least two unresolved issues in the full theory of general relativity. One issue regards the projection from the full phase space $\Omega$ to the physical degrees of freedom defined by the constraint surface. Secondly, the quantization of the theory in congruity with this projection to the present author’s knowledge remains to be consistently implemented. We have approached the first issue using the instanton representation of Plebanski gravity by projection from the unconstrained to the kinematic phase space, defined as the reduced phase space for gravity under gauge transformations and diffeomorphisms, and implementation of the Hamiltonian dynamics thereon. In this paper we will approach the second issue for certain sectors of GR from the standpoint of the algebra of observables. We will show that the representation of this algebra exhibits a natural coherent state structure for gravity.

In the instanton representation the basic momentum space variables are the densitized eigenvalues of the antiself-dual Weyl curvature (CDJ matrix), which constitute the physical degrees of freedom. The Hilbert space which we refer to is defined on the kinematic phase space, where these degrees of freedom are explicit. It is on this space where the quantization procedure and the implementation of the reality conditions have been defined. One feature of the Hilbert space as constructed in [1] is that the states resemble an infinite-dimensional analogue of harmonic oscillator-like coherent states, which are applicable for vanishing cosmological constant $\Lambda$. In [2] we have generalized the construction to include nonvanishing $\Lambda$, which entails the use of holomorphic hypergeometric functions. In the present paper we will carry out the construction of the states within the context of the coherent state formalism.

The organization of this paper follows a bottom-up rather than the conventional top-down approach, as we will first establish a system of coherent states and then provide a map from this system to the Ashtekar variables, via the instanton representation. The organization of this paper is as follows. In section 2 we provide a brief review of the oscillator formalism and coherent states, building on the relevant concepts from [4] and [5]. The purpose of this section is to put in place the formalism, and to introduce the constituents of some of the operators which which have direct analogues for gravity. Sections 3, 4 and 5 carry out the transformation from the coherent state basis and operators into the holomorphic Schrödinger representation, from which we derive the Ashtekar variables via the instanton represen-

\footnote{In [3] we have treated the implementation of reality conditions at the kinematic level both for $\Lambda = 0$ and for $\Lambda \neq 0$, including via adjointness relations on the Hilbert space.}
tation. We have also outlined the solution to the Hamiltonian constraint in the holomorphic Schrödinger representation in terms of hypergeometric functions. The association of the gravitational Hilbert space with oscillator coherent states uniquely picks out the Barmann representation and the accompanying adjointness relations. In section 6 we provide a brief physical interpretation of the states and what features of spacetime they describe. In section 7 we outline the construction of the hypergeometric solutions to the Hamiltonian constraint using a Lippman–Schwinger type expansion with respect to the coherent state basis. In this section we formalize the link from the coherent states to the gravitational degrees of freedom using the holomorphic Schrödinger representation. Section 8 is a brief discussion regarding the interpretation of uncertainties in terms of the Ashtekar variables.

2 Quantum harmonic oscillator formalism

We will first start with a simple system, where all of the the steps of the algebraic extension to Dirac’s quantization procedure, outlined in [6] can be carried out to completion. Our system consists of three simple harmonic oscillators with annihilation operators $a_1$, $a_2$ and $a_3$. From $a_f$ construct the following set $S$, given by

$$S = \{ a_1, a_2, a_3, a_1^*, a_2^*, a_3^*, 1 \}. \quad (1)$$

It is clear from (1) that $S$ is closed under complex conjugation. Additionally, $S$ is closed under the Poisson bracket since as one can easily verify from the harmonic oscillator algebra,

$$\{ a_f, a_g^* \} = \delta_{fg}; \quad \{ a_f, a_g \} = \{ a_f^*, a_g^* \} = \{ a_f, 1 \} = \{ a_f^*, 1 \} = 0. \quad (2)$$

From (2) the objects $a_f$ and $a_f^*$ may be regarded as the fundamental dynamical variables of a phase space $\Omega_{\text{Kin}}$. Define $F$ as the set of all suitably regular functions on $\Omega_{\text{Kin}}$ which can be obtained as a sum of products of elements $F^{(i)} \in S$. Some examples of elements of $F$ are given by

$$Q = a_3 a_3 + \frac{2}{3} (a_1 + a_2) a_3 + \frac{1}{3} a_1 a_2;$$

$$O = a_3 (a_3 + a_1) (a_3 + a_2); \quad \tau = a_3 + \frac{1}{3} (a_1 + a_2). \quad (3)$$

These particular functions will for gravity take on the interpretation as $SO(3,C)$ invariants which appear in the Hamiltonian constraint.
Next, we will associate with each element $F^{(i)}$ in $S$ an abstract operator $\hat{F}^{(i)}$, and construct the free algebra $A$ generated by these elementary quantum operators. This amounts to the promotion of (1) to

$$A = \{ \hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_3^\dagger, \hat{1} \},$$

whence the Poisson brackets (2) become promoted to commutators

$$[\hat{a}_f, \hat{a}_g^\dagger] = \delta_{fg}; \quad [\hat{a}_f, \hat{a}_g] = [\hat{a}_f^\dagger, \hat{a}_g^\dagger] = [\hat{a}_f, \hat{1}] = [\hat{a}_f^\dagger, \hat{1}] = 0. \quad (5)$$

Note that (5) can also be derived by application of an involution operation to (1). Additionally, the promotion $S \rightarrow A$ extends to the set $\{ F \}$, hence (3) become promoted to

$$\hat{Q} = \hat{a}_3\hat{a}_3 + \frac{2}{3}(\hat{a}_1 + \hat{a}_2)\hat{a}_3 + \frac{1}{3}\hat{a}_1\hat{a}_2;$$
$$\hat{O} = \hat{a}_3(\hat{a}_3 + \hat{a}_1)(\hat{a}_3 + \hat{a}_2); \quad \hat{\tau} = \hat{a}_3 + \frac{1}{3}(\hat{a}_1 + \hat{a}_2) \quad (6)$$

with adjoints

$$\hat{Q}^\dagger = \hat{a}_3^\dagger\hat{a}_3^\dagger + \frac{2}{3}(\hat{a}_1^\dagger + \hat{a}_2^\dagger)\hat{a}_3^\dagger + \frac{1}{3}\hat{a}_1^\dagger\hat{a}_2^\dagger;$$
$$\hat{O}^\dagger = \hat{a}_3^\dagger(\hat{a}_3^\dagger + \hat{a}_1^\dagger)(\hat{a}_3^\dagger + \hat{a}_2^\dagger); \quad \hat{\tau}^\dagger = \hat{a}_3^\dagger + \frac{1}{3}(\hat{a}_1^\dagger + \hat{a}_2^\dagger) \quad (7)$$

under the involution operation. As an aside, these operators satisfy the algebra

$$[\hat{a}_3, \hat{O}] = 3\hat{Q}; \quad [\hat{a}_3, \hat{Q}] = 2\hat{\tau}; \quad [\hat{a}_3, \hat{\tau}] = 1;$$
$$[\hat{O}, \hat{Q}] = [\hat{Q}, \hat{\tau}] = [\hat{\tau}, \hat{O}] = 0. \quad (8)$$

We will now construct a linear representation of the abstract algebra $A$ given by (4). Along with the algebra (5) comes a unique normalized ground state $|0, 0, 0\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle$ with $\langle 0, 0, 0 | 0, 0, 0 \rangle = 1$, such that

$$\hat{a}_f |0, 0, 0\rangle = \langle 0, 0, 0 | \hat{a}_f^\dagger = 0, \quad (9)$$

where the creation operator in (9) acts to the left on the bra state. Also, we have that

$$\hat{a}_1^\dagger |0, 0, 0\rangle = |1, 0, 0\rangle; \quad \hat{a}_2^\dagger |0, 0, 0\rangle = |0, 1, 0\rangle; \quad \hat{a}_3^\dagger |0, 0, 0\rangle = |0, 0, 1\rangle, \quad (10)$$
such that for an arbitrary state $|p,q,s\rangle$ with $p \geq 0$, $q \geq 0$ and $r \geq 0$,

$$C_{l,m,n}'(\hat{a}_1)^l(\hat{a}_2)^m(\hat{a}_3)^n(\hat{a}_1^\dagger)^{l'}(\hat{a}_2^\dagger)^{m'}(\hat{a}_3^\dagger)^{n'}|p,q,s\rangle$$

$$= |p-l+l',q-m+m',s-n+n'\rangle$$

(11)

where we have defined

$$C_{l,m,n}' = \sqrt{\frac{(l')!(m')!(n')!}{(l)!(m)!(n)!}}.$$  

(12)

These states form a mode number basis satisfying orthogonality relations

$$\langle l,m,n|p,q,s \rangle = \delta_{lp}\delta_{mq}\delta_{ns},$$

(13)

and $\hat{a}_f$ and $\hat{a}_f^\dagger$ are adjoints with respect to the inner product (13).

### 2.1 Coherent states

In this paper we will rather be utilizing a basis of coherent states, applying the formalism of [4] to our model for gravity. One way to define coherent states is states which are eigenstates of the annihilation operators $a_1$, $a_2$ and $a_3$ where

$$\hat{a}_1|\alpha,\beta,\lambda\rangle = \alpha|\alpha,\beta,\lambda\rangle; \quad \hat{a}_2|\alpha,\beta,\lambda\rangle = \beta|\alpha,\beta,\lambda\rangle; \quad \hat{a}_3|\alpha,\beta,\lambda\rangle = \lambda|\alpha,\beta,\lambda\rangle.$$

(14)

We will single out $a_3$ as special from $a_1$ and $a_2$, since the operators of interest (for example (7)) will be invariant under interchange of $a_1$ and $a_2$ but not with respect to $a_3$. Hence in terms of the individual states in the direct product, we have the definitions

$$|\alpha\rangle = e^{-|\alpha|^2} e^{\alpha a_1^\dagger}|0\rangle; \quad |\beta\rangle = e^{-|\beta|^2} e^{\beta a_2^\dagger}|0\rangle; \quad |\lambda\rangle = e^{\lambda a_1^\dagger}|0\rangle.$$  

(15)

where $\alpha$, $\beta$ and $\lambda$ are dimensionless quantities. It will be convenient to define the coherent states, in the sense of Perelemov [5], as the states obtained by application of a displacement operator to the vacuum state $|0,0,0\rangle$, or any appropriate fiducial state. This is given by

3We require for all states that $|p,r,s\rangle = 0$ for any of $p$, $r$, $s$ less than zero.
\[ |\alpha, \beta, \lambda \rangle = D(\alpha, \beta, \lambda) |0, 0, 0 \rangle, \]  
\hspace{1cm} (16)

where we have defined the displacement operator

\[ D(\alpha, \beta, \lambda) = e^{\alpha a_1^\dagger - \alpha^* a_1} e^{\beta a_2^\dagger - \beta^* a_2} e^{\lambda a_3^\dagger}. \]  
\hspace{1cm} (17)

The coherent states are obtained by displacing the vacuum state into \( C_3 \), a 3-dimensional complex space representing three copies of the complex plane. \( C_3 \) at the present level plays the role of the coset space for the group manifold of three copies of the complexified Heisenberg algebra \((H_4)^3\). Hence (17) is a typical representative in this coset space and there is a one-to-one correspondence between states \( |\alpha, \beta, \lambda \rangle \) and points in \( C_3 \).

There exists a natural flat metric on \( C_3 \) which can be used to define the distance between two states labelled by \( z \) and \( z' \), given by

\[ d(z, z') = \frac{1}{2} \left[ |\alpha - \alpha'|^2 + |\beta - \beta'|^2 + |\lambda - \lambda'|^2 \right]. \]  
\hspace{1cm} (18)

This metric induces the following overlap between coherent states

\[ |\langle z | z' \rangle|^2 = e^{-d(z, z')}. \]  
\hspace{1cm} (19)

However, for the purpose of the gravitational coherent states we will eliminate the last term of (18), since we will impose a constraint which reduces \( \lambda \rightarrow \lambda_{\alpha, \beta} \) to a function of just \( \alpha \) and \( \beta \). Hence \((\alpha, \beta) \in C_2 \) will coordinatize the physical degrees of freedom which makes normalization in \( \lambda \) redundant.\footnote{We have taken for granted the quotienting of the Heisenberg group \( H_4 \otimes H_4 \otimes H_4 \) by the identity group element 1 and the mode number operator \( \hat{N}_f = \hat{a}_f^\dagger \hat{a}_f \) in our notation, which yields the coset space \((H_4)^3/(U(1)) \).}

Therefore the overlap between two states for our purposes will involve only \( \alpha \) and \( \beta \), given by

\[ |\langle \alpha, \beta | \alpha', \beta' \rangle|^2 = e^{-|\alpha - \alpha'|^2} e^{-|\beta - \beta'|^2}, \]  
\hspace{1cm} (20)

and we will from now on omit \( \lambda \) from the labels in the anticipation of implementing the aforementioned constraint.

We will be using the following resolution of the identity for the states

\[ \int \frac{d^2 \alpha d^2 \beta}{\pi^2} |\alpha, \beta \rangle \langle \alpha, \beta | = I. \]  
\hspace{1cm} (21)
Note, since the states are labelled by continuous indices in a Hilbert space that has a countable basis, they are overcomplete. Note that any arbitrary state \( |\psi\rangle \) can be expanded in terms of these coherent states [4]

\[
|\psi\rangle = \int f(\alpha, \beta) (\alpha^*, \beta^*) e^{-|\alpha|^2/2} e^{-|\beta|^2/2} d^2\alpha d^2\beta / \pi^2,
\]

(22)

where the analytical function \( f(\alpha^*, \beta^*) \), the coherent state representation of \( |\psi\rangle \) is given by

\[
f(\alpha^*, \beta^*) = \langle \alpha, \beta | \psi \rangle e^{|\alpha|^2/2} e^{|\beta|^2/2} = \sum_{m,n} c_{m,n} (\alpha^*)^m (\beta^*)^n (m!)^{1/2} (n!)^{1/2}
\]

(23)

with \( c_{m,n} \) the mode basis expansion coefficients in the expansion

\[
|\psi\rangle = \sum_{m,n} c_{m,n} |m, n\rangle = \sum_{m,n} c_{m,n} (a_1^\dagger)^m (a_2^\dagger)^n |0, 0\rangle.
\]

(24)

2.2 Action of the constituent operators

We will now put in place the constituents of the operator which will we will use to impose constraints on our system to reduce the coset state manifold from \( |\alpha, \beta, \lambda\rangle \sim C_3 \) to \( |\alpha, \beta\rangle \sim C_2 \). The operators \( \hat{O}, \hat{Q} \) and \( \hat{\tau} \) from (7) have the following action on the coherent states

\[
\hat{Q}|\alpha, \beta, \lambda\rangle = (\lambda + \gamma^-)(\lambda + \gamma^+) |\alpha, \beta, \lambda\rangle,
\]

(25)

where we have defined

\[
\gamma^\pm = \frac{1}{3} (\alpha + \beta \pm \sqrt{\alpha^2 - \alpha\beta + \beta^2}) \equiv \lambda_{\alpha, \beta}
\]

(26)

as the roots of \( Q \), seen as a polynomial in \( a_3 \). Also we have the following actions

\[
\hat{O}|\alpha, \beta, \lambda\rangle = \lambda(\lambda + \alpha)(\lambda + \beta) |\alpha, \beta, \lambda\rangle;
\]

\[
\hat{\tau}|\alpha, \beta, \lambda\rangle = (\alpha + \beta + \frac{1}{3}\lambda) |\alpha, \beta, \lambda\rangle.
\]

(27)

From these operators construct the following Hamiltonian constraint operators for our theory, given by
\[ \hat{H}_1 = \hat{Q} + l\hat{O}e^{-\hat{a}_3^1}; \quad \hat{H}_2 = \hat{O} + r\hat{Q}e^{\hat{a}_3^1}; \quad l = \frac{1}{r} \]  

(28)

where \( r \neq 0 \) is a numerical constant. The aim of this paper will be to construct states annihilated by \( \hat{H}_1 \) and \( \hat{H}_2 \) using the coherent state basis. Part of this process will utilize the coherent states annihilated by \( \hat{Q} \) and \( \hat{O} \).

These are

\[ |\alpha, \beta, \lambda\rangle \in \text{Ker}\{\hat{Q}\} \]  

(29)

with \( \lambda = \lambda_{\alpha, \beta} \) given by (26), and

\[ |\alpha, \beta, 0\rangle, |\alpha, \beta, -\alpha\rangle, |\alpha, \beta, -\beta\rangle \in \text{Ker}\{\hat{O}\}. \]  

(30)

For those states annihilated by \( \hat{H}_1 \) and \( \hat{H}_2 \) it will be convenient to define the following states \( |\chi\rangle_{\alpha, \beta} \) by

\[ |\chi\rangle_{\alpha, \beta} \equiv |\chi\rangle \otimes |\alpha\rangle \otimes |\beta\rangle. \]  

(31)

We will replace the action of \( \hat{a}_1 \) and \( \hat{a}_2 \) on (31) by their eigenvalues, and leave the operator \( \hat{a}_3 \) in its present form since we have singled out \( \hat{a}_3 \) as special. Then the following relations ensue

\[ \hat{H}_1 |\chi\rangle_{\alpha, \beta} = \left( (\hat{a}_3 + \gamma^-)(\hat{a}_3 + \gamma^+) + l\hat{a}_3(\hat{a}_3 + \alpha)(\hat{a}_3 + \beta)e^{-\hat{a}_3^1} \right) |\chi\rangle_{\alpha, \beta} \]  

(32)

and

\[ \hat{H}_2 |\chi\rangle_{\alpha, \beta} = \left( \hat{a}_3(\hat{a}_3 + \alpha)(\hat{a}_3 + \beta) + r(\hat{a}_3 + \gamma^-)(\hat{a}_3 + \gamma^+)e^{\hat{a}_3^1} \right) |\chi\rangle_{\alpha, \beta}. \]  

(33)

Having defined the operators and algebra of our system, we will next associate the system to gravity. First let us associate to each point \( x \) in 3-space \( \Sigma \) a harmonic oscillator of the type (1), as in

\[ S = \left\{ a_1(x), a_2(x), a_3(x), a_1^*(x), a_2^*(x), a_3^*(x), 1 \right\}. \]  

(34)

Then all of the aforementioned formalism can be repeated for each \( x \in \Sigma \). If 3-space were continuous, then we would have an infinite number of representations of the oscillator algebra, one representation per point. But let us start with the assumption that space is discrete, and then we can always attempt to take the continuum limit of the resulting theory.
3 Holomorphic Schrödinger representation

Perform a 3+1 decomposition of 4-dimensional spacetime \( M = \Sigma \times R \) where \( \Sigma \) is a 3-dimensional spatial manifold, and define by \( \Delta_N(\Sigma) \) a discretization of \( \Sigma \) into a lattice of spacing \( \epsilon = \frac{l^3}{N} \), where \( l \) is the characteristic length scale of \( \Sigma \). For each \( x \in \Delta_N(\Sigma_t) \) on the final spatial hypersurface labelled by \( t \) define quantities \((X,Y,T)\), which are elements of the space of holomorphic functions, by

\[
(X(x,t), Y(x,t), T(x,t)) \in \Gamma_{Kin}
\]

where \( \Gamma_{Kin} \) is defined as the kinematic configuration space at point \( x \) on the hypersurface \( \Sigma_t \). Also define \( \forall x \in \Delta_N(\Sigma) \) a two dimensional complex space coordinatized by \((\tilde{\alpha}_x, \tilde{\beta}_x) \in \mathbb{C}^2 \) and associate with each \( C_2(x) \) a state

\[
\chi(T_x)\left| \tilde{\alpha}_x, \tilde{\beta}_x \right>.
\]

Hence we assume that the antiderivative in the exponential of (36) exists. The following mass dimensions are defined for the various quantities of interest

\[
[X] = [Y] = [T] = 0; \quad [\nu] = -3; \quad [\tilde{\alpha}] = [\tilde{\beta}] = 1
\]

Let us form the continuum limit of the part of the state dependent on \((X,Y)\) by the direct product

\[
\psi_{\alpha,\beta}[X,Y] = \langle X,Y \left| \tilde{\alpha}, \tilde{\beta} \right> = \lim_{\epsilon \to 0} \prod_x \langle X_x(t), Y_x(t) \left| \tilde{\alpha}_x, \tilde{\beta}_x \right)
\]

\[
= N(\tilde{\alpha}, \tilde{\beta}) e^{(\hbar G)^{-1}(\tilde{\alpha} \cdot X + \tilde{\beta} \cdot Y)}.
\]

In this limit we have \( \Delta_N(\Sigma) \to \Delta_\infty(\Sigma) \), and the dot product signifies a Riemannian integral over 3-space, as in

\[
U \cdot V = \int_\Sigma d^3x U(x)V(x) \forall U, V \in C^0(\Sigma).
\]

The quantity \( N(\tilde{\alpha}, \tilde{\beta}) \) in (38) is a normalization factor given by

\[
N(\tilde{\alpha}, \tilde{\beta}) = e^{-\nu(\hbar G)^{-1}(\tilde{\alpha}^* \cdot \tilde{\alpha} + \tilde{\beta}^* \cdot \tilde{\beta})}.
\]
Note that the states \( \psi_{\alpha,\beta} \in L_2(\Gamma_{\text{Kin}}, D\mu) \) are square-integrable with respect to the measure

\[
D\mu = \prod_x D(X,Y)_x e^{-\nu^{-1}(X\cdot X + Y\cdot Y)},
\]

(41)

where \( D(X,Y)_x = \delta X \delta Y \delta \delta Y \). The overlap between two states in the measure (41) is given by

\[
|\langle \tilde{\alpha}, \tilde{\beta} | \tilde{\alpha}', \tilde{\beta}' \rangle|^2 = \exp \left[ -\nu(hG)^{-2} \int_{\Sigma} d^3x \left( |\tilde{\alpha}(x) - \tilde{\alpha}'(x)|^2 + |\tilde{\beta}(x) - \tilde{\beta}'(x)|^2 \right) \right],
\]

(42)

which is inversely proportional to the Euclidean distance between the state labels in the two dimensional complex manifold \( C_2 \). Let us first consider a special case where \( \lambda(T) \) is independent of \( T \), given by

\[
\tilde{\lambda}(T) = \tilde{\lambda}_{\alpha,\beta}^{\pm} = -\frac{1}{3} \left( \tilde{\alpha} + \tilde{\beta} \pm \sqrt{\tilde{\alpha}^2 - \tilde{\alpha}\tilde{\beta} + \tilde{\beta}^2} \right).
\]

(43)

In this case (36) yields \( \chi(T) = e^{(hG)^{-1}\tilde{\lambda}_{\alpha,\beta}^{\pm} T} \) which produces a state

\[
\psi^0_{\alpha,\beta}[X,Y,T] = e^{(hG)^{-1}(\alpha X + \beta Y + \lambda_{\alpha,\beta}^{\pm} T)}.
\]

(44)

Define dynamical momentum space variables \( \Pi(x,t), \Pi_1(x,t) \) and \( \Pi_2(x,t) \) on the kinematic momentum space \( P_{\text{Kin}} \), which upon quantization become promoted to operators satisfying equal-time commutation relations

\[
[\hat{T}(x,t), \hat{\Pi}(y,t)] = [\hat{X}(x,t), \hat{\Pi}_1(y,t)] = [\hat{Y}(x,t), \hat{\Pi}_2(y,t)] = (hG)\delta^{(3)}(x,y).
\]

(45)

Also define the following function on the kinematic momentum space \( P_{\text{Kin}} \), given by

\[
Q = \Pi^2 + \frac{2}{3}(\Pi_1 + \Pi_2)\Pi + \frac{1}{3}\Pi_1\Pi_2.
\]

(46)

Equation (46) can be written in the equivalent form by dividing it by \( \Pi(\Pi + \Pi_1)(\Pi + \Pi_2) \neq 0 \), which yields

\[
\frac{1}{\Pi} + \frac{1}{\Pi + \Pi_1} + \frac{1}{\Pi + \Pi_2} = 0.
\]

(47)

Note that \( |\tilde{\alpha}, \tilde{\beta} \rangle \) are eigenstates of \( \hat{\Pi}_1 \) and \( \hat{\Pi}_2 \), given in the functional Schrödinger representation by
\[ \hat{\Pi}(x)\psi = (hG) \frac{\delta}{\delta T(x)} \psi; \]

\[ \hat{\Pi}_1(x)|\tilde{\alpha}\rangle \rightarrow (hG) \frac{\delta}{\delta X(x)} e^{(hG)^{-1}\tilde{\alpha}_X} \rightarrow \tilde{\alpha}(x)|\tilde{\alpha}\rangle; \]

\[ \hat{\Pi}_2(x)|\tilde{\beta}\rangle \rightarrow (hG) \frac{\delta}{\delta Y(x)} e^{(hG)^{-1}\tilde{\beta}_Y} \rightarrow \tilde{\beta}(x)|\tilde{\beta}\rangle. \]  

(48)

Also note that \( \psi^0_{\alpha,\beta} \in \text{Ker}\{\hat{Q}\} \), which can also be written as

\[ \left( (hG)^2 \frac{\delta^2}{\delta T(x)\delta T(x)} + \frac{2}{3}(\alpha + \beta)(hG) \frac{\delta}{\delta T(x)} + \frac{1}{3}\alpha\beta \right) \psi^0_{\alpha,\beta} = 0, \]  

(49)

where we have replaced the actions of \( \Pi_1 \) and \( \Pi_2 \) by their eigenvalues on the state. We have left the action of \( \Pi \) as a functional derivative, because we have singled \( T(x) \) as a time variable on \( \Gamma_{\text{Kin}} \) and we will be interested in the evolution of the state with respect to \( T \). Equations (48) are the continuum limit of the following discretized versions for \( x \in \Delta_N(\Sigma) \)

\[ \hat{\Pi}_x\psi = (hG)^{-1} \frac{\partial}{\partial T_x} \psi; \]

\[ (\hat{\Pi}_1)_x|\tilde{\alpha}\rangle \rightarrow (hG)^{-1} \frac{\partial}{\partial X_x} e^{(hG)^{-1}\tilde{\alpha}_X} \rightarrow \tilde{\alpha}_x|\tilde{\alpha}\rangle; \]

\[ (\hat{\Pi}_2)_x|\tilde{\beta}\rangle \rightarrow (hG)^{-1} \frac{\partial}{\partial Y_x} e^{(hG)^{-1}\tilde{\beta}_Y} \rightarrow \tilde{\beta}_x|\tilde{\beta}\rangle, \]  

(50)

whence the integration has been restricted to a single cell of volume \( \nu \) containing the point \( x \). The effect of the the factor \( \nu^{-1} \) in the partial derivative is the analogue of a delta function in the functional derivative of the continuum limit. Similarly, the discretized version of (49) is given by

\[ \left( (hG)^{-1} \frac{\partial^2}{\partial T_x^2} + \frac{2}{3}(\tilde{\alpha}_x + \tilde{\beta}_x)(hG^{-1}) \frac{\partial}{\partial T_x} + \frac{1}{3}\tilde{\alpha}_x\tilde{\beta}_x \right) \psi^0_{\alpha,\beta} = 0. \]  

(51)

We will now make an association from the holomorphic states \( \psi^0_{\alpha,\beta} \) constructed in this section to gravity in two stages. First we will show how the Hilbert space follows from the kinematic level of the instanton representation of Plebanski gravity. Secondly, we will provide an embedding map from the kinematic phase space to the unconstrained phase space which we will in turn map into the Ashtekar variables.
4 Transformation into the instanton representation action of Plebanski gravity

We will now construct the simplest action which upon quantization yields the commutation relations (45) and the constraint (47). This is given by

\[ I_{\text{Kin}} = \frac{i}{G} \int dt \int_{\Sigma} d^3x \left( \Pi \dot{T} + \Pi_1 \dot{X} + \Pi_2 \dot{Y} 
- iNK \sqrt{\Pi (\Pi + \Pi_1)(\Pi + \Pi_2)} \left( \frac{1}{\Pi} + \frac{1}{\Pi + \Pi_1} + \frac{1}{\Pi + \Pi_2} \right) \right), \tag{52} \]

where \( K = K(X, Y, T) \neq 0 \) is some function of the kinematic configuration space variables \( X, Y, T \in \Gamma_{\text{Kin}} \), which will be chosen appropriately. Note that (52) implies the symplectic two form

\[ \omega_{\text{Kin}} = \frac{i}{G} \int d^3x \left( \delta \Pi \wedge \delta T + \delta \Pi_1 \wedge \delta X + \delta \Pi_2 \wedge \delta Y \right) \]
\[ = \frac{i}{G} \left( \int_{\Sigma} d^4x (\Pi \delta T + \Pi_1 \delta X + \Pi_2 \delta Y) \right) \equiv \delta \theta_{\text{Kin}}, \tag{53} \]

where \( \theta_{\text{Kin}} \) is the canonical one form on the kinematic phase space \( \Omega_{\text{Kin}} \). We will now perform a change of variables. Define a mass scale \( a_0 = \text{const.} \) and define new momentum space variables \( (\lambda_1, \lambda_2, \lambda_3) \) such that

\[ \Pi_1 = a_0^3 e^T (\lambda_1 - \lambda_3); \quad \Pi_2 = a_0^3 e^T (\lambda_2 - \lambda_3); \quad \Pi = a_0^3 e^T \lambda_3, \tag{54} \]

and define new configuration space variables \( (a_1, a_2, a_3) \) such that\(^6\)

\[ a_1 = a_0 e^X; \quad a_2 = a_0 e^Y; \quad a_1 a_2 a_3 = a_0^3 e^T. \tag{55} \]

The ranges of the coordinates are \( -\infty < \vert X \vert, \vert Y \vert, \vert T \vert < \infty \) where

\[ |a| = \sqrt{(\text{Re}\{a\})^2 + (\text{Im}\{a\})^2}, \tag{56} \]

which corresponds to \( 0 < |a_f| < \infty \). Under the transformation (54) and (55), then the action (52) is given by

\[ I_{\text{Kin}} = \frac{i}{G} \int dt \int_{\Sigma} d^4x \left( \lambda_1 a_2 a_3 \dot{a}_1 + \lambda_2 a_3 a_1 \dot{a}_2 + \lambda_3 a_1 a_2 \dot{a}_3 
- iNK \sqrt{\lambda_1 \lambda_2 \lambda_3} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \right), \tag{57} \]

\(^6\)Note that \( a_1, a_2 \) and \( a_3 \) are not to be confused with the harmonic oscillator annihilation operators of the previous sections.
where now $K = K(a_1, a_2, a_3)$, which will be chosen appropriately. We will now adopt the following convention for indices, where symbols from the beginning of the Latin alphabet $a, b, c \ldots$ signify internal indices and symbols from the middle $i, j, k, \ldots$ signify spatial indices in $\Sigma$. We will associate the internal indices with $SO(3, C)$, the special complex orthogonal group in three dimensions. Let us now make the following identifications

$$\alpha^a_i = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}; \quad \beta^i_a = e^{ijk} \partial_j \alpha^a_k + \frac{1}{2} e^{ijk} f^{abc} \alpha^b_j \alpha^c_k,$$

where $\beta^i_a$ will play the role of a magnetic field for $\alpha^a_i$, seen as a nonabelian gauge field. Note for the diagonal $\alpha^a_i = \delta^a_i a$ that there are no spatial gradients in the canonical one form $\theta_{Kin}$. Let us define a new variable $\Psi_{ae}$, given by

$$\Psi_{ae} = (e^{\bar{\theta}T})_{af} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} (e^{-\bar{\theta}T})_{ge} + \epsilon_{aed} \psi^d,$$

where $\bar{\theta} = (\theta^1, \theta^2, \theta^3) \in C_3$ are a triplet of complex angles and $\psi^d$ is a $SO(3, C)$-valued 3-vector. Note for $\psi^d = 0$ that $\Psi_{ae}$ is symmetric in $a, e$, since it takes on the interpretation of an $SO(3, C)$ transformation of the diagonal matrix of eigenvalues. Define the following quantities

$$b^i_a = (e^{\bar{\theta}T})_{ae} \beta^i_e; \quad a^a_i = (e^{\bar{\theta}T})_{ae} \alpha^e_i + \frac{1}{2} e^{abc} (e^{\bar{\theta}T})_{bf} \partial_i (e^{\bar{\theta}T})_{cf}. \quad (58)$$

Note that $b^i_a = b^i_a(\bar{a}, \bar{\theta})$ is the result of rotating the internal index of $\beta^i_a$, which corresponds a $SO(3, C)$ transformation. It then follows that $a^a_i = a^a_i(\bar{a}, \bar{\theta})$, which now has six degrees of freedom, is the corresponding gauge transformed version of $\bar{a} = (a_1, a_2, a_3)$ which has just three degrees of freedom. The transformation (58) induces an embedding $\Omega_{Kin} \to \Omega_{diff}$, where $\Omega_{diff}$ is defined as a diffeomorphism invariant phase space with action

$$I_{diff} = \frac{i}{G} \int dt \int d^3x \left( \Psi_{(ae)} b^i_a \dot{a}^a_i - iN(\det b)^{1/2} \sqrt{\det \Psi \text{tr} \Psi^{-1}} \right) \Big|_{\text{Sym}(\Psi)}. \quad (59)$$

By the notation $\text{Sym}(\Psi)$ is meant that $\Psi_{ae} = \Psi_{ea}$ is symmetric. We can remove this restriction by allowing $\Psi$ to have an antisymmetric part while imposing the constraint that this antisymmetric part vanishes. We can also

---

\[7\]This is because, due to the antisymmetry of $\epsilon^{ijk}$ and the symmetry of a diagonal connection $\delta^a_i a_i$, that the spatial gradient terms drop out. Since the spatial gradients are still nonzero, we are dealing with the full theory and not minisuperspace. There are three degrees of freedom per point in the diagonal connection.
constrain the $SO(3, C)$ frame by imposing a constraint on $\tilde{\theta}$. In conjunction with the aforementioned constraints and the constraint on the eigenvalues $\lambda_f$ we will impose the following constraints on the unreduced phase space $\Omega_{\text{Inst}}$, given by

$$H = (\det b)^{1/2} \sqrt{\det \Psi} \text{tr} \Psi^{-1} = 0;$$
$$H_i = \epsilon_{ijk} b^j_e b^k_e \Psi_{ae} = 0;$$
$$G_a = b^e_e \partial_i \Psi_{ae} + (f_{abf} \delta_{ge} + f_{ebg} \delta_{af}) \Psi_{fg} = b^e_e D_i \Psi_{ae} = 0. \tag{60}$$

The constraints (60) can be obtained by the variation of Lagrange multipliers $(a_0, N, N^i)$ in the following action

$$I_{\text{Inst}} = \frac{i}{G} \int dt \int d^3 x \left( \Psi_{ae} b^j_e \delta a^a_i - a_0^a G_a - N^i H_i - iN H \right). \tag{61}$$

Note that there is no configuration space variable canonically conjugate to $\Psi_{ae}$, since the canonical one form $\theta = \int_{\Sigma} d^3 x \Psi_{ae} b^e_e \delta a^a_i$ is in general not exact.

The momentum space $\Psi_{ae}$ of (61) has nine degrees of freedom per point, but the connection $a^a_i$ has only six. We may lift this restriction, in conjunction with lifting the restriction to symmetric $\Psi_{ae}$, and make the identification $a^a_i \rightarrow A^a_i$ and $b^a_e \rightarrow B^a_e[A]$ where now $A^a_i$ and therefore $B^a_e$ now have nine degrees of freedom per point. We can then write the extended action as

$$I_{\text{Inst}} = \frac{i}{G} \int dt \int d^3 x \left( \Psi_{ae} B^j_e \delta A^a_i - A_0^a G_a - N^i H_i - iN H \right)$$
$$- \epsilon_{ijk} N^i B^j_e b^k_e \Psi_{ae} - iN (\det B)^{1/2} \sqrt{\det \Psi} \text{tr} \Psi^{-1} \right), \tag{62}$$

combined with a prescription for obtaining the diffeomorphism invariant phase space $\Omega_{\text{diff}}$. This prescription is to set to zero all components of $A^a_i$ not obtainable from a diagonal connection $\delta^a_i a_a$ by $SO(3, C)$ gauge transformation, in conjunction with setting $\Psi[ae] = 0$, when implementing the diffeomorphism constraint $H_i = 0$. Note, in direct analogy to (61), that the canonical one form $\theta_{\text{Inst}} = \int_{\Sigma} d^3 x \Psi_{ae} B^j_e \delta A^a_i$ is also not exact. The phase space variables satisfy

$$\left[ A^a_i(x, t), \Psi_{bf}(y, t) \right] = G \delta^a_b (B^{-1})_i^f \delta^{(3)}(x, y), \tag{63}$$

which are not canonical commutation relations owing to the field dependence on the right hand side. Note, however, that on the kinematic phase space $\Omega_{\text{Kin}}$ in (53) $\theta_{\text{Kin}}$ is exact and implies canonical commutation relations (45).
5 Transformation into the Ashtekar variables

We have performed an embedding map from the kinematic phase space $\Omega_{Kin}$, which has an exact symplectic two form $\omega_{Kin}$, to an unreduced phase space $\Omega_{Inst}$, whose symplectic two form $\omega_{Inst}$ is in general not exact. But we would like a theory which on its full unconstrained phase space admits an exact canonical one form, and we would like this theory to admit a well-defined sequence of transformations to $\Omega_{Kin}$ and its resulting Hilbert space. To deal with this let us make the change of variables

$$\Psi^{-1}_{ae} = B^i_e (\tilde{\sigma}^{-1})^a_i \bigg\rvert_{\det\tilde{\sigma} \neq 0},$$

(64)

which holds for nondegenerate variables. Substitution of (64) into (62) and defining $N = N(\det\tilde{\sigma})^{-1/2}$ yields an action

$$I_{Inst} \to \frac{i}{G} \int dt \int d^3x \left( \bar{\sigma}^i_a \dot{A}^a_i - A^a_0 D_i \bar{\sigma}^i_a - \epsilon_{ijk} N^i \bar{\sigma}^j_a B^k_a - \frac{i}{2} N \epsilon_{ijk} \epsilon^{abc} \bar{\sigma}^i_a \bar{\sigma}^j_b \bar{\sigma}^k_c \right),$$

(65)

with phase space variables $(\bar{\sigma}^i_a, A^a_i)$ which upon quantization would satisfy the canonical commutation relations

$$[A^a_i(x,t), \bar{\sigma}^j_b(y,t)] = G \delta^a_b \delta^j_i \delta^{(3)}(x,y),$$

(66)

Note that (64) is a noncanonical transformation from $\Omega_{Inst}$ into $\Omega_{Ash}$, the phase space of the Ashtekar variables, where $A^a_i$ is the self-dual Ashtekar connection. Indeed, (65) is the action for general relativity in the Ashtekar variables for vanishing cosmological constant (See e.g. [7], [8] and [9]). The symplectic two form corresponding to (65) is given by

$$\omega_{Ash} = \frac{i}{G} \int d^3x \delta \bar{\sigma}^i_a \wedge \delta A^a_i = \frac{i}{G} \delta \left( \int d^3x \delta \bar{\sigma}^i_a \delta A^a_i \right) = \delta \theta_{Ash},$$

(67)

which is the exact functional variation of the canonical one form $\theta_{Ash}$.

Let us now generalize to the case of a nonvanishing cosmological constant $\Lambda$. The only change to the action (65) occurs in the Hamiltonian constraint, which is now given by

$$H = \epsilon_{ijk} \epsilon^{abc} \bar{\sigma}^i_a \bar{\sigma}^j_b \bar{\sigma}^k_c + \frac{\Lambda}{3} \epsilon_{ijk} \epsilon^{abc} \bar{\sigma}^i_a \bar{\sigma}^j_b \bar{\sigma}^k_c.$$

(68)

Performing all of the previous steps from (35) to (61) in reverse to accomplish the projection $\Omega_{Ash} \to \Omega_{Inst} \to \Omega_{diff} \to \Omega_{Kin}$ from the full unconstrained
Ashtekar variables to the quantizable kinematic phase space of the instanton representation, we find that (47) for \( \Lambda \neq 0 \) is given by

\[
\frac{\Lambda}{a_0} + \left( \frac{1}{\Pi + \Pi_1} + \frac{1}{\Pi + \Pi_2} \right) e^T = 0 \quad \forall \ x \in \Delta_N(\Sigma). \tag{69}
\]

The effect of the cosmological constant is to bring a mass scale \( \sqrt{\Lambda} \) into the theory. Equation (69) can be written in polynomial form as

\[
\left( \frac{\Lambda}{3a_0^3} \right) \Pi(\Pi + \Pi_1)(\Pi + \Pi_2) + \left( \Pi^2 + \frac{2}{3}(\Pi_1 + \Pi_2)\Pi + \frac{1}{3}\Pi_1\Pi_2 \right) e^T = 0, \tag{70}
\]

obtained by multiplication by \( \Pi(\Pi + \Pi_1)(\Pi + \Pi_2) \neq 0 \). Upon quantization of (70) we have the following functional differential equation

\[
\hat{H}\psi = \left[ (hG)^3 \left( \frac{\Lambda}{3a_0^3} \right) \frac{\delta}{\delta T} \left( \frac{\delta}{\delta X} + \frac{\delta}{\delta Y} \right) \left( \frac{\delta}{\delta T} + \frac{\delta}{\delta T} \right) \right] \psi^\Lambda \alpha,\beta[T] = 0 \quad \forall x \in \Sigma. \tag{71}
\]

where \( \psi^\Lambda \alpha,\beta = \mid \tilde{\alpha}, \tilde{\beta} \rangle \otimes \chi(T) \). We can replace the action of the functional derivatives with respect to \( X \) and \( Y \) on the state with their eigenvalues \( \tilde{\alpha} \) and \( \tilde{\beta} \), yielding

\[
\hat{H}\psi = \left[ \left( \frac{\Lambda}{3a_0^3} \right)(hG) \frac{\delta}{\delta T} \left( hG \frac{\delta}{\delta T} + \tilde{\alpha} \right) \left( hG \frac{\delta}{\delta T} + \tilde{\beta} \right) \right] \psi^\Lambda \alpha,\beta[T] = 0. \tag{72}
\]

Whereas in the \( \Lambda = 0 \) case there was not a problem, one can see that for \( \Lambda \neq 0 \) one must deal with the multiple functional derivatives acting at the same point, which can now act on the factor of \( e^T \).

\[\text{At this point we will perform a discretization } \Delta_N(\Sigma) \text{ of 3-space } \Sigma. \text{ Then the functional derivatives turn into partial derivatives at a particular point, which are finite. In this process we must append the inverse volume of a cell in order to preserve the mass dimensions as in } \delta/\delta T(x) \rightarrow \nu^{-1}\partial/\partial T_x, \text{ and the Hamiltonian constraint reduces to the following equation}
\]

\[
\left[ \mu \frac{\partial}{\partial T} \left( \mu \frac{\partial}{\partial T} + \tilde{\alpha} \right) \left( \mu \frac{\partial}{\partial T} + \tilde{\beta} \right) + \left( \frac{3a_0^3}{\Lambda} \right) \left( \mu \frac{\partial}{\partial T} + \tilde{\lambda}_{\alpha,\beta} \right) \left( \mu \frac{\partial}{\partial T} + \tilde{\lambda}_{\alpha,\beta}^T \right) e^T \right] \chi(T) = 0. \tag{73}
\]

Note that this is not an issue for the \((X,Y)\) dependence, since the action on the state is finite without regularization, which as well highlights the reason why \( T \) is special.
where the following quantities are defined

\[ \mu = \frac{\hbar G}{\nu}; \quad z \equiv 3 \left( \frac{a_0^3}{\mu \Lambda} \right) e^T \]  

(74)

with mass dimensions \([\mu] = 1\) and \([z] = 0\). Additionally we will define the following dimensionless state labels from (15)

\[ \alpha = \frac{\bar{\alpha}}{\mu}; \quad \beta = \frac{\bar{\beta}}{\mu}; \quad \lambda_{\alpha,\beta} = \frac{\bar{\lambda}_{\alpha,\beta}}{\mu}, \]  

(75)

so that \([\alpha] = [\beta] = [\lambda_{\alpha,\beta}] = 0\). Dividing (73) by \(\mu^3\) and eliminating \(T\) in favor of \(z\), we obtain upon commuting the factor of \(z\) to the left the following differential equation

\[
\left[ z \frac{d}{dz} \left( z \frac{d}{dz} + \alpha \right) \left( z \frac{d}{dz} + \beta \right) + z \left( z \frac{d}{dz} + \lambda^-_{\alpha,\beta} + 1 \right) \left( z \frac{d}{dz} + \lambda^+_{\alpha,\beta} + 1 \right) \right] \chi(z) = 0. \tag{76}
\]

Equation (76) is a hypergeometric differential equation with solution

\[ \chi(z) = 2F_2(\bar{\lambda}^-_{\alpha,\beta} + 1, \bar{\lambda}^+_{\alpha,\beta} + 1; \alpha + 1, \beta + 1; z). \tag{77} \]

The state is then given by the direct product of these functions over a given discretization

\[ \Psi_{\alpha,\beta} = \prod_x \chi(T_x) |\alpha_x, \beta_x\rangle. \tag{78} \]

For \(\Lambda \neq 0\) there is a three to one correspondence between states and point in \(C_2\), whereas for \(\Lambda = 0\) there is a two to one correspondence.\(^9\) Later in this paper we will make the direct association from \(\alpha\) and \(\beta\) as defined in (75) to the labels of the harmonic oscillator coherent states derived in section 2.

\(^9\)It is shown in [2] that for \(\Lambda = 0\) the continuum limit in \(\Delta_\infty(\Sigma)\) exists as part of the same Hilbert space as each discretization \(\Delta_N(\Sigma)\), but for \(\Lambda = 0\) the Kodama state \(\psi_{Kod}^\prime\) is the only state with this property. In the latter case the discretized Hilbert space converges to elements \(\Psi \not\in Ker\{H\}\) in the continuum limit, which requires the inclusion of these elements \(\Psi\) to complete the Hilbert space.
6 Physical interpretation

We shall now elucidate upon the relation of the physical Hilbert space $H_{\text{phys}}$ to general relativity. Perform the following decomposition of $\Psi^{-1}_{ae}$

$$\Psi^{-1}_{ae} = -\frac{\Lambda}{3} \delta_{ae} + \psi_{ae},$$  \hfill (79)

where $\psi_{ae}$ is symmetric and traceless. In the language of $SL(2,\mathbb{C})^{\text{10}}$ Weyl, shorthand for the self-dual part of the Weyl curvature tensor, can be written in unprimed $SL(2,\mathbb{C})$ indices as

$$\psi_{ABCD} = \eta^a_{AB} \eta^e_{CD} \psi_{ae},$$  \hfill (80)

which is totally symmetric in uppercase indices. We have $A = 0, 1$ and $a = 1, 2, 3$, where $\eta^a_{AB}$ is an isomorphism from $SL(2,\mathbb{C})$ unprimed index pairs $AB = (00, 01, 11)$ to single $SO(3,\mathbb{C})$ indices $a = (1, 2, 3)$.

The eigenvalues of $\psi_{ae}$ encode the algebraic classification of spacetime [11], which are independent of coordinates and of tetrad frames [10]. These properties play a role in the determination of the principal null directions and the radiation properties of spacetime [12],[13]. These properties can be computed from the characteristic equation for $\psi_{ae}$ and the invariants $(I, J)$, given by

$$I = \psi_{ABCD} \psi^{ABCD}; \quad J = \psi_{ABCD} \psi_{EF} \psi^{EF}.$$  \hfill (81)

To make the link from these properties of spacetime to the degrees of freedom that have been quantized, equation (79) can be inverted. Since $\psi_{ae} = \psi_{ae}(I, J)$ encodes the classification of the spacetime, it follows that $\Psi_{ae} = \Psi_{ae}(I, J)$ also encodes this classification.

In the intrinsic frame $SO(3,\mathbb{C})$ frame, defined as the frame in which $\Psi_{ae}$ is diagonalized, the eigenvalues are given in terms of the state labels by

$$\tilde{\Psi}_{ae} = \Psi_{ae} \alpha^3e_T = \begin{pmatrix} \alpha + \lambda & 0 & 0 \\ 0 & \beta + \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$  

The states then imply the following classification scheme

$$\begin{align*}
\alpha = \beta = 0 : & \quad \text{Petrov Type O (Kodama state $\psi_{Kod}$)}; \\
\alpha = \beta \neq 0 : & \quad \text{Petrov Type D (Algebraically special)}; \\
\alpha \neq \beta \neq 0 : & \quad \text{Petrov Type I (Algebraically general)}.
\end{align*}$$  \hfill (82)

$^{10}$SL(2,\mathbb{C}) is the covering group for SO(3,\mathbb{C}).
To obtain a physical interpretation into the meaning of the densitized eigenvalues, let us examine them in the original variables

$$\lambda = \frac{\tilde{\lambda}}{\mu} = \left( \frac{\lambda \nu}{\hbar G} \right) = \left( \frac{a_0^3 \nu}{\hbar G} \right) \lambda_3 e^T. \quad (83)$$

The state labels depend on the mass space $a_0$ for the connection as well as the volume scale $\nu$ of the elementary cells of the discretization. Since these have so far remained unspecified, let us fix them by making the choice $a_0^3 \nu = 1$, which sets the mass scale $a_0$ to the inverse length scale $\nu^{1/3}$ of $\Sigma$. Then we have $\tilde{\lambda}_{\alpha,\beta} = (hG)^{-1} \lambda_3 e^T$, or that the state labels occur in multiples of the (undensitized) eigenvalues of the CDJ matrix $\Psi_{ae}$. Since $\Psi_{ae}^{-1}$ is the antiself-dual part of the Weyl curvature tensor, then it has the same dimensions as curvature which are inverse length squared. In our case the length scale referred to is the Planck length $l_P$. Hence $\tilde{\lambda}_{\alpha,\beta}$ can be seen as of the same order of magnitude of variations of the metric on the scale of the Planck length $l_P$. With this choice of $a_0$ the Hamiltonian constraint takes on the form

$$H = \nu \Lambda e^{-T} + \frac{1}{\Pi} + \frac{1}{\Pi + \Pi_1} + \frac{1}{\Pi + \Pi_2} = 0, \quad (84)$$

which as we have shown yields a solution for the states in terms of hypergeometric functions. In the undensitized variables this is given by

$$H = \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0, \quad (85)$$

which is transparent to the parameters introduced as a result of the quantization process.

7 Recapitulation: Lippman–Schwinger expansion on coherent state basis

Let us now expand upon the manifestation of the hypergeometric solutions to the Hamiltonian constraint in terms of the coherent state formalism of section 2, continuing from (32) and (33). The solution to the Hamiltonian constraint consists of states in the kernel of $\hat{H}_1$ and $\hat{H}_2$. We will build these states by expansion about $\text{Ker}\{\hat{Q}\}$ and $\text{Ker}\{\hat{O}\}$. For the first case we have

$$\hat{H}_1 |\psi\rangle = (\hat{Q} + l\hat{O}e^{-a_1}) |\psi\rangle = 0. \quad (86)$$
Now act on both sides of (86) with $\hat{Q}^{-1}$, yielding

$$(1 + l\hat{Q}^{-1}\hat{O}e^{-a_3^\dagger})|\psi\rangle = |\alpha, \beta, \lambda_{\alpha,\beta}\rangle.$$ \hspace{1cm} (87)

where $|\alpha, \beta, \lambda_{\alpha,\beta}\rangle \in Ker\{\hat{Q}\}$. Acting on (87) with the inverse of the operator in brackets, we have

$$|\psi\rangle_1 = (1 + l\hat{Q}^{-1}\hat{O}e^{-a_3^\dagger})^{-1}|\alpha, \beta, \lambda\rangle = \sum_{n=0}^{\infty} (-l)^n (\hat{Q}^{-1}\hat{O}e^{-a_3^\dagger})^n |\alpha, \beta, \lambda_{\alpha,\beta}\rangle.$$ \hspace{1cm} (88)

Likewise, for $|\psi\rangle_2 \in Ker\{\hat{H}_2\}$ we have

$$|\psi\rangle_2 = (1 + r\hat{H}^{-1}\hat{O}e^{a_3^\dagger})^{-1}|\alpha, \beta, \lambda\rangle = \sum_{n=0}^{\infty} (-r)^n (\hat{H}^{-1}\hat{O}e^{a_3^\dagger})^n |\alpha, \beta, \lambda\rangle.$$ \hspace{1cm} (89)

In (88) and (89), the states are eigenstates of all operators except for the action due to $a_3$, which is given by

$$e^{a_3^\dagger} |\lambda\rangle = |\lambda + 1\rangle; \quad e^{-a_3^\dagger} |\lambda\rangle = |\lambda - 1\rangle.$$ \hspace{1cm} (90)

This induces a raising and lowering action with respect to the $\lambda$ dependence of the state. Using the representation theory of the harmonic oscillator thus described, (88) can be written as

$$|\psi\rangle_1 = \sum_{n=0}^{\infty} (-l)^n \left(\frac{(\alpha + 1)n(\beta + 1)n(\lambda_{\alpha,\beta} + 1)n}{(\gamma^- + 1)n(\gamma^- + 1)n}\right) |\alpha, \beta, \lambda_{\alpha,\beta} - n\mu\rangle.$$ \hspace{1cm} (91)

Equation (91) is an infinite series with a zero radius of convergence unless we require the series to terminate at finite order. This leads to the restrictions $\alpha = n$, $\beta = n$ or $\lambda_{\alpha,\beta} = n$ for some $n$, which produces an infinite tower of states labelled by $\alpha$ and $n$, as shown in [2]. For the other states we have that

$$|\psi\rangle_2 = \sum_{n=0}^{\infty} (-r)^n \left(\frac{(\gamma^- + 1)n(\gamma^- + 1)n}{(\alpha + 1)n(\beta + 1)n(\lambda + 1)n}\right) |\alpha, \beta, \lambda + n\mu\rangle,$$ \hspace{1cm} (92)

which is convergent.
7.1 Association to quantum gravity

We will now provide the link from the coherent state formalism to the gravity, which follows from the holomorphic Schrödinger representation. Note that we have constructed states in the kernel of the Hamiltonian constraints, we will now transform the constraints and the corresponding states into the Schrödinger representation. First make the following associations

\[
\hat{a}_1 \equiv \frac{\delta}{\delta X}; \quad \hat{a}_2 \equiv \frac{\delta}{\delta Y}; \quad \hat{a}_3 \equiv \frac{\delta}{\delta T},
\]

(93)

where \(X, Y\) and \(T\) are holomorphic variables. Hence any arbitrary function \(f = f(X,Y,Z)\) is a holomorphic function. Note that the adjoints of (93) have a representation

\[
a_1^\dagger \equiv X; \quad a_2^\dagger \equiv y; \quad a_3^\dagger \equiv T,
\]

(94)

which fixes the measure for normalization essentially as (41). The harmonic oscillator coherent states then have a representation

\[
\psi(X,Y,T) = \langle \alpha, \beta, \lambda | X,Y,Z \rangle = e^{\alpha X + \beta Y + \lambda T},
\]

(95)

which are normalizable with respect to the Gaussian measure.

Making the identifications (93) and (94) in \(\hat{H}_1\) and \(\hat{H}_2\), we obtain the following Hamiltonian constraints in the new representation

\[
\hat{H}_1 = \frac{\delta^2}{\delta T^2} + \frac{2}{3} \left( \frac{\delta}{\delta X} + \frac{\delta}{\delta Y} \right) \frac{\delta}{\delta T} + \frac{1}{3} \frac{\delta^2}{\delta X \delta Y} + l \left( \frac{\delta}{\delta T} + \frac{\delta}{\delta X} \right) \left( \frac{\delta}{\delta T} + \frac{\delta}{\delta Y} \right) e^{-T},
\]

(96)

and

\[
\hat{H}_2 = \frac{\delta}{\delta T} \left( \frac{\delta}{\delta T} + \frac{\delta}{\delta X} \right) \left( \frac{\delta}{\delta T} + \frac{\delta}{\delta Y} \right) + r \left( \frac{\delta^2}{\delta T^2} + \frac{2}{3} \left( \frac{\delta}{\delta X} + \frac{\delta}{\delta Y} \right) \frac{\delta}{\delta T} + \frac{1}{3} \frac{\delta^2}{\delta X \delta Y} \right) e^{T}.
\]

(97)

The reason that \(a_3\) is special in relation to \(a_1\) and \(a_2\) is that the Hamiltonian constraint operators contain \(e^{a_3^\dagger}\), whose action causes a shift in \(\lambda\) by discrete steps. However, since there is no occurrence of \(a_1^\dagger\) or of \(a_2^\dagger\), then the state labels \((\alpha, \beta)\) remain intact under the Hamiltonian action.\(^{11}\) Therefore we may replace the action of \(\hat{a}_1\) and \(\hat{a}_2\) on the coherent states with their eigenvalues \(\alpha\) and \(\beta\), and focus on the dynamics of \(T\).

We will use the following notation for the states

\(^{11}\)They are in this sense time-independent, if one adopts the physical interpretation of \(a_3^\dagger \sim T\) as a time variable.
\[ \psi_{\alpha,\beta}^j[T] \equiv |\alpha, \beta\rangle \otimes \chi(T). \]  

(98)

The label \( j \) will be used to denote multiple states for the same \( \alpha, \beta \).

\[ \text{Note in (32) and (33) that the state } |\alpha \rangle \otimes |\beta \rangle \text{ can be omitted, leaving the following differential equation for } |\chi \rangle. \]

\[ \hat{H}_1 \chi_1 = \left[ \left( \frac{\delta}{\delta T} + \gamma^- \right) \left( \frac{\delta}{\delta T} + \gamma^+ \right) + l \left( \frac{\delta}{\delta T} + \alpha \right) \left( \frac{\delta}{\delta T} + \beta \right) e^{-T} \right] \chi[T] = 0 \]  

(99)

and

\[ \hat{H}_2 \chi_2 = \left[ \frac{\delta}{\delta T} \left( \frac{\delta}{\delta T} + \alpha \right) \left( \frac{\delta}{\delta T} + \beta \right) + r \left( \frac{\delta}{\delta T} + \gamma^- \right) \left( \frac{\delta}{\delta T} + \gamma^+ \right) e^T \right] \chi[T] = 0 \]  

(100)

Equations (99) and (100) are hypergeometric differential equations, with solution

\[ \chi_1 = _3F_2(\alpha - 1, \beta - 1, \lambda - 1; \gamma^- + 1, \gamma^+ + 1; (-le^{-T})); \]

\[ \chi_2 = _2F_2(\gamma^- + 1, \gamma^+ + 1; \alpha, \beta, \lambda; (-re^T)). \]

(101)

Let us now make the following identification

\[ r = \frac{3a^3 \nu}{hGA}, \]

(102)

where \( \Lambda \) is the cosmological constant, and \( \nu \) and \( a_0 \) are numerical constants of mass dimensions \([\nu] = -3\) and \([a_0] = 1\). Then for \( \alpha = \beta = 0 \) \( \chi_2 \) yields

\[ \psi_{0,0}^j = \exp\left[-3(hGA)^{-1}nu_0^3e^T\right] \equiv \psi_{Kod,}. \]

(103)

If we make the identifications

\[ X \equiv \ln\left(\frac{A_1}{a_0}\right); \quad Y \equiv \ln\left(\frac{A_2}{a_0}\right); \quad T \equiv \ln\left(\frac{A_1^2A_2^2A_3^3}{a_0^3}\right), \]

(104)

then one realizes that (103) is the Chern–Simons functional of a diagonal connection, and is nothing more than the Kodama state. The general state is given by

\footnote{For vanishing cosmological constant \( \Lambda = 0 \) \( j \) will take on the values 1 and 2, and for \( \Lambda \neq 0 \) it will have three possible values.}
\[ \psi^j_{\alpha,\beta} = e^{\alpha X + \beta Y} \mathcal{F}_2(\alpha - 1, \beta - 1, \lambda - 1; \gamma^+ + 1, \gamma^+ + 1; (-le^{-T})) ; \]
\[ \psi^j_{\alpha,\beta} = e^{\alpha X + \beta Y} \mathcal{F}_2(\gamma^+ + 1, \gamma^+ + 1; \alpha, \beta, \lambda; (-re^T)) , \] (105)

which are labelled by two arbitrary parameters. If we repeat the same construction at each point in 3-space \( \Sigma \) as in [2], then we obtain the functionals

\[ \Psi^j_{\alpha,\beta} = \prod_x e^{\alpha X + \beta Y} \mathcal{F}_2(\alpha - 1, \beta - 1, \lambda - 1; \gamma^+ + 1, \gamma^+ + 1; (-le^{-T(x)})) ; \]
\[ \Psi^j_{\alpha,\beta} = \prod_x e^{\alpha X + \beta Y} \mathcal{F}_2(\gamma^+ + 1, \gamma^+ + 1; \alpha, \beta, \lambda; (-re^T(x))) \] (106)

The wavefunctionals (106) correspond to the quantization of the algebraic classification of spacetime as encoded in the \( W_{cyl} \), the self-dual part of the Weyl curvature.\(^{13}\)

8 Discussion: Minimum uncertainties

Given that harmonic oscillator coherent states minimize the product of the uncertainties \( \Delta p \Delta q = \frac{\hbar}{2} \), an interesting question is what the corresponding properties represent in the case of gravity. We will limit our analysis to the diagonal sector of the Ashtekar variables. Recall that the self-dual Ashtekar connection is given by \( A_i^\alpha = \Gamma_i^\alpha - iK_i^\alpha \), where \( \Gamma_i^\alpha \) and \( K_i^\alpha \) are respectively the triad-compatible spin connection and the extrinsic curvature of a spatial slice \( \Sigma \). The diagonal components of the connection can be written in the form

\[ A_f^\alpha = a_0(\gamma_f - ik_f) \]

where \( \gamma_f \) and \( k_f \) are dimensionless. This can also be written in polar form

\[ A_f^\alpha = a_0 \sqrt{k_f^2 + \gamma_f^2} \exp\left[-i\tan^{-1}\left(\frac{k_f}{\gamma_f}\right)\right] \] (107)

for \( f = 1, 2, 3 \). The configuration space variables which have been quantized can then be written in terms of their real and imaginary parts as

\[ X = \ln \sqrt{k_1^2 + \gamma_1^2} - i\tan^{-1}\left(\frac{k_1}{\gamma_1}\right) ; \quad Y = \ln \sqrt{k_2^2 + \gamma_2^2} - i\tan^{-1}\left(\frac{k_2}{\gamma_2}\right) . \] (108)

\(^{13}\)The physical interpretation of the state labels and the canonical structure is treated in detail in [14].
The relation between the holomorphic Schrödinger and harmonic oscillator formalisms implies the relations (94). In terms of the real position-momentum space variables we have

\[ q_f = \frac{1}{2}(a_f + a_f^\dagger); \quad p_f = \frac{1}{2i}(a_f - a_f^\dagger), \quad f = 1, 2. \]  

(109)

Using (108) and (94), we have

\[ q_f = \ln \sqrt{k_f^2 + \gamma_f^2}; \quad p_f = \tan^{-1}\left(\frac{k_f}{\gamma_f}\right). \]  

(110)

The conclusion is that the coherent states of the instanton representation minimize the product of the uncertainties of the magnitude and the phase of the diagonal components of the Ashtekar connection.

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