Compatibility condition for the Eulerian left Cauchy–Green deformation tensor field

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Abstract

An explicit compatibility condition formula is presented for the Eulerian left Cauchy–Green deformation tensor field. It is shown to be the appropriate finite-strain counterpart of Saint-Venant’s compatibility condition. The difference between the Eulerian problem and the Lagrangian one is highlighted.

1 Introduction

The left Cauchy–Green deformation tensor field of continuum mechanics is a quantity related to the motion of a body in a Euclidean space. The motion defines a so-called deformation gradient tensor field, and the left and right Cauchy–Green tensor fields are defined in terms of the deformation gradient. Therefore, given the motion, the Cauchy–Green tensors are determined (and are used, for example, as the variable of a constitutive relationship that tells the elastic stress tensor emerging in a given arrangement/configuration of the body).

The right Cauchy–Green tensor field is defined corresponding to the Lagrangian description of the continuum, i.e., when the position of a given material point at the chosen reference time is used as the space variable for the tensor field. In parallel, for the left Cauchy–Green tensor field, a natural setting is the Eulerian description, where, at a given time, the position of the material point at that instant is used as the space variable.

It is both a fundamental question (as emphasized, e.g., in [1]) and a practical one (related to applications) whether a given tensor field can be regarded as a left—or a right—Cauchy–Green tensor field corresponding to a motion. If the answer is yes then we call the given Cauchy–Green tensor field compatible (since it is compatible with a possible motion of the continuum). In the small-strain or infinitesimal-strain approximation, where the deformation gradient is not far from the unit tensor and the
corresponding Cauchy–Green tensors are also near the unit tensor (more closely, up to linear order, both differ from the unit tensor by the Cauchy strain tensor), the condition for compatibility reduces to Saint-Venant’s compatibility condition (see, e.g., [2], Sect. 4.6). This formula says that the Cauchy strain tensor field should have zero left+right curl in order to stem from a motion. This condition is sufficient as long as a simply connected and complete spatial domain is considered, which is going to be assumed throughout this paper. The fact that compatibility has already been discussed by Saint-Venant in itself indicates how elementary the need is for deciding whether a given tensor field can correspond to a continuum motion.

Without the small-strain or linearization approximation, the question is considerably more involved. The condition for the right Cauchy–Green deformation tensor field has long been well-studied—see the thorough investigation [3].

The first aim of this writing is to highlight that the case of the left Cauchy–Green tensor actually raises two questions: compatibility for the field in the Lagrangian description and compatibility in the Eulerian one. Regarding the former, results and discussions can be found in [4, 5, 1]. Here, the latter case is investigated.

Namely, as the second and main aim of this article, a compatibility formula is determined and presented for the Eulerian left Cauchy–Green tensor field. At this point, it is important to emphasize that the principles behind the formula are not novel at all: as pointed out in [3], the problem of the Eulerian left Cauchy–Green tensor can be treated analogously to the one of the Lagrangian right Cauchy–Green tensor. Some virtues of the form of the here-devised compatibility condition are that

- it is given explicitly in terms of the left Cauchy–Green tensor (not in terms of the Christoffel symbols as intermediate quantities), and
- it contains derivatives of the left Cauchy–Green tensor, not derivatives of its inverse like in the related Christoffel symbols.

Correspondingly, the formula is intended to be user-friendly for applications, for example, for implementing it in numerical simulations in programming languages that are fast but with the price of no symbolic manipulations. The formula can help in validating and controlling the numerical calculations: e.g., when the left Cauchy–Green tensor is determined from some evolution or other equation, numerical errors make compatibility gradually violated, and finding where violation is the most apparent can offer insight into which part of the numerical scheme (or other aspect) could be improved.

As the third aim, the formula can be exhibited as the explicit Eulerian finite-strain counterpart of Saint-Venant’s one, with which it agrees in the linear approximation, as is shown here (as an “obligatory” check).

2 Notations and conventions

Throughout the paper, customary notations are used; exceptions are where some essential aspect is to be emphasized—in those cases, the notation helps highlighting that aspect.
Since the essence of the problem, a differential geometric question, can also be formulated without explicitly using coordinate systems (parametrizations), a coordinate-free tensorial description (see, e.g., [3]) is possible. Formulae with indices below can also be read this way, as expressions in the abstract index notation introduced by Penrose. Naturally, all indices can also be read as classic coordinate indices.

For the problem at hand, distinction between vectors and covectors (elements of the dual vector space) plays an important role so upper and lower indices are distinguished; when indices are read as coordinate indices then this distinction is the customary one between contravariant and covariant components (see [7], for example).

Motion is considered in a three-dimensional Euclidean point space $E$, where Euclidean vectors serve as the tangent vectors at each point of $E$, in other words, all the tangent spaces are the same vector space $E$. Correspondingly, all cotangent spaces are also the same, namely, the dual space of Euclidean vectors, $E^*$. These enable a convenient, unindexed, tensorial notation (denoted in upright boldface as usual) for tensors of various kind: tensors, cotensors and mixed ones. This notation also expresses the coordinate free content. Formulae below are provided in both the tensorial and the indexed notation since both have advantages.

In order to highlight the differential geometric essence of the problem to discuss—and also for emphasizing the difference between vectors and covectors,—instead of the dot-product notation $v \cdot w$ for the scalar product of Euclidean vectors $v$ and $w$, $h_{KL} v^K w^L$ is written, i.e., the cotensor behind (denoted by $h$) is explicitly displayed. $h$ is a symmetric, nonsingular, and positive definite cotensor, in other words, it is a Riemann metric; $E$ as a manifold, equipped with $h$, forms a Riemannian manifold $(E, h)$. This specific $h$ is constant along $E$ (the notion of constantness is meaningful since all tangent spaces along $E$ are the same $E$). Correspondingly, the associated Riemann curvature tensor field $\text{Ric}(h)$ is zero [$\text{Ric}(h)_{KLMN} = 0$], in other words, this Riemannian manifold is flat. As a consequence, the Ricci tensor field $\text{Ric}(h)$ is also zero [$\text{Ric}(h)_{KL} = 0$]; the Riemannian manifold at hand is Ricci-flat as well.

It is $h$ [$h_{KL}$] and its inverse $h^{-1}$ [$h^{KL}$] via which the customary ‘index raising and lowering’ (in the coordinate-free abstract index language, the distinguished and natural identification between Euclidean covectors and vectors) is accomplished [$v_K = h_{KL} v^L$, $k^M = h^{MN} k_N$]. Here and hereafter, repeated indices involve summation (tensorial contraction in the abstract index language) [Einstein convention]. Note also that, in the indexed version, inverse is not indicated explicitly, according to the custom.

At a reference time $t_{\text{ref}}$, a material point of the body is at a point $X$ of $E$, and, along its motion, at time $t$ it is at a point $x$ in $E$. Having a space origin chosen in $E$, $X$ is described by position vector $X$ [in index notation: $X^K$] and $x$ by vector $x$ [$x^i$].

Material points starting from different positions $X$ arrive at different positions $x$, and the motion-characterizing map

$$\chi: X \mapsto x = \chi(X)$$

is assumed to be invertible and smooth (smooth enough for all subsequent considerations to hold).

The definition of the deformation gradient tensor field $F$ is

$$F^i_K = \partial_K \chi^i, \quad F = \chi \otimes \nabla,$$
where in the tensorial form \( \odot \) stands for tensorial/dyadic product, and \( \nabla_X \) differentiates in the variable \( X \) and acts to the left, to reflect the proper tensorial order.

The left Cauchy–Green tensor field is defined as
\[
B_{ij} = F^i_K h^{KL} F^j_L, \quad B = F h^{-1} F^T,
\]
with \(^T\) denoting the transpose. Note that, although customarily \( h^{-1} \) is omitted from the notation, i.e., the index-raising role of \( h^{-1} \) is usually not displayed explicitly, the distinction between superscripts and subscripts (vectors and covectors) requires showing all appearances of \( h^{-1} \) (and, analogously, of \( h \) in index-lowering situations).

Due to its definition (3), \( B \) is \( X \)-variabled, following from that \( F \) is \( X \)-variabled and \( h \) is constant. Via
\[
X = \chi^{-1}(x),
\]
quantities given in the \( X \)-variable based—so-called Lagrangian—description can be transformed to the \( x \)-variable based—so-called Eulerian—description. For example, rewriting (3) with
\[
f^i(x) = F^i \chi(x),
\]
one obtains
\[
B(x) = B\left(\chi^{-1}(x)\right), \quad B = fh^{-1}f^T, \quad b_{ij} = f^i_K h^{KL} f_L^j.
\]
It is this tensor field \( B \) on which the present paper focuses.

Before proceeding, a few comments becoming useful below are as follows. As a consequence of (4), the formula that is the counterpart to (2) is
\[
f^i_K = \partial_i (\chi^{-1})^K, \quad f^{-1}(x) = \chi^{-1}(x) \odot \nabla_X.
\]
Also, \( F \) and \( f^{-1} \) act as the Jacobi tensor (or matrix) of the variable transformations (1) and (4), respectively:
\[
\nabla_X = \nabla_x F, \quad \nabla_x = \nabla_X f^{-1}, \quad \partial_K = F^i_K \partial_i; \quad \partial_i = f^K_i \partial_K.
\]
At this point, it is worth noting that, since the same \( E \) is the tangent space at any point of \( E \), and the same \( E^* \) is the same cotangent space at any point, the \( \nabla \) notation (i.e., \( \nabla_X \) in the Lagrangian description and \( \nabla_x \) in the Eulerian one) has a tensorial, coordinate-free, meaning (a.k.a. the operator Grad, i.e., the Fréchet derivative). This derivative coincides with the differential geometric covariant derivative accompanied to the constant metric \( h \) of \( E \)—plain partial derivatives appear in the indexed notation [see (8)]. In contrast, the other Riemann metric \( g \) introduced below will be different—the related nontrivial Christoffel symbols will be shown in (14).
3 Explicit compatibility condition formula for the Eulerian left Cauchy–Green field

From (3)—considered in the variable \( x \)—and (5)–(6), we find

\[ b^{-1} = \left(f^T\right)^{-1} hf^{-1}, \quad b_{ij} = f^K h_{KL} f^L, \tag{9} \]

or, involving (7),

\[ b^{-1} = \left(\bar{\nabla}_x \otimes \bar{x}^{-1} \right) h \left(\bar{x}^{-1} \otimes \bar{\nabla}_x \right), \quad b_{ij} = \left(\partial_i (\bar{x}^{-1})^K \right) h_{KL} \left(\partial_j (\bar{x}^{-1})^L \right). \tag{10} \]

(This tensor has its own names, e.g., the Finger deformation tensor [8] and the left Piola deformation tensor [9], and it provides an alternative way to introduce \( b \) as the inverse of this tensor defined directly via (10)).

One consequence that can be read off from this is that, similarly to \( h \), \( b^{-1} \) is also a symmetric, nonsingular, and positive definite cotensor; in other words, it is a Riemann metric. Therefore, \( \mathcal{E} \) as a manifold, equipped with

\[ g = b^{-1}, \quad g_{ij} = b_{ij}, \tag{11} \]

forms a Riemannian manifold.

The other consequence is that, in differential geometric language, the smooth map \( \chi^{-1} \) gives rise to a pullback of \( h \) to \( g \) (see, e.g., [6]). Putting these two together, we find that \( \chi^{-1} \) is an isometry between the Riemannian manifold \( (\mathcal{E}, g) \) and the Riemannian manifold \( (\mathcal{E}, h) \).

Now, an isometry brings a zero Riemann tensor to a zero Riemann tensor. Accordingly, although \( g \) is not a constant in general, we know that it is flat:

\[ \text{Ric}(g) = 0, \quad \text{Ric}(g)_{ijkl} = 0. \tag{12} \]

Then it also follows that it is Ricci-flat as well:

\[ \text{Ric}(g) = 0, \quad \text{Ric}(g)_{ij} = 0. \tag{13} \]

To summarize, we have that, for any \( b \) that stems from some motion (i.e., for any \( b \) that is compatible with some motion), \( g = b^{-1} \) is a flat (hence, Ricci-flat) Riemannian metric. Note that Ricci flatness is a technically more advantageous property than flatness since it involves a second-order tensor rather than a fourth-order one.

Let us now consider the opposite direction. Given a symmetric, nonsingular, and positive definite \( b \), can it stem from a motion?

We have just seen that Ricci flatness of \( g = b^{-1} \) is a minimal necessary—and technically favourable—requirement. In fact, it turns out to be a sufficient condition as well. Namely, in two and three dimensions, Ricci flatness implies that the so-called sectional curvature is zero (as can be found, e.g., by analysing the topmost formula in page 88 of [6]). Zero sectional curvature leads to flatness (see, e.g., [6], 3.41 Proposition). Finally, the Killing–Hopf theorem (see, e.g., [6], 8.25 Corollary) ensures that any two (simply-connected, complete) flat Riemannian manifolds of the same dimension are isometric, which in the present context says that \( b \) can stem from a motion.
Therefore, if we express Ricci flatness of \( g = b^{-1} \) in terms of \( b \) then we have a necessary and sufficient (hence, compatibility) condition for \( b \).

In terms of the Christoffel symbols of the second kind—actually, a tensor in our case\[^1\]—

\[
\Gamma = \frac{1}{2} g^{-1} \left[ (g \circ \nabla_x)_{T_{2, 1}} + (g \circ \nabla_x) - (\nabla_x \circ g) \right], \quad \Gamma^{k}_{ij} = \frac{1}{2} g^{kl} \left( \partial_l g_{ij} + \partial_j g_{il} - \partial_i g_{lj} \right),
\]

where \( T_{2, 1} \) indicates transpose in the second and third indices (tensorial slots in the abstract-index language), the Ricci tensor reads

\[
\text{Ric}(g) = \nabla_x \cdot \Gamma - (\text{tr}_{1, 3} \Gamma) \circ \nabla_x + (\text{tr}_{1, 2} \Gamma) \Gamma - \text{tr}_{1, 3} (\Gamma \Gamma),
\]

\[
\text{Ric}(g)_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{km} \Gamma^m_{ij} - \Gamma^k_{im} \Gamma^m_{kj},
\]

with \( \text{tr}_{1, 3} \) denoting contraction in the first and third indices (tensorial slots). The compatibility condition for \( b \) is

\[
\text{Ric}(b^{-1}) = 0, \quad \text{Ric}_{(b^{-1})ij} = 0,
\]

which yields, when written explicitly in terms of \( b \),

\[
\frac{1}{4} \left\{ \text{tr}_{1, 2, 3, 5} \left[ b^{-1} (b \circ \nabla_x) (b \circ \nabla_x) \circ b^{-1} \right] - 2 b^{-1} \text{tr}_{2, 4} \left[ (b \circ \nabla_x) (b \circ \nabla_x) b^{-1} \right] \right. \\
+ 2 \text{tr}_{1, 3, 4} \left[ b^{-1} \circ b^{-1} (b \circ \nabla_x) (b \circ \nabla_x) \right] + 2 b^{-1} \text{tr}_{1, 2} \left[ (b \circ \nabla_x) (b \circ \nabla_x) b^{-1} \right] \right. \\
+ 2 \text{tr}_{2, 3} \left[ b^{-1} \circ (\nabla_x \cdot b) (b \circ \nabla_x) \right] + 2 b^{-1} \text{tr}_{2, 4} \left[ (b \circ \nabla_x) b^{-1} (b \circ \nabla_x) \right] \right. \\
+ 2 \text{tr}_{1, 2} \left[ b^{-1} (b \circ \nabla_x) \circ b^{-1} \right] - 2 b^{-1} \text{tr}_{2, 4} \left[ (b \circ \nabla_x) \circ b^{-1} \right] \right. \\
- 3 \text{tr}_{2, 3, 4} \left[ (\nabla_x \circ b) b^{-1} (b \circ \nabla_x) \circ b^{-1} \right] - 2 \left[ (b \circ \nabla_x) \circ b^{-1} \right] \right. \\
- 2 b^{-1} (b \cdot \nabla_x) \circ \nabla_x + 2 \text{tr}_{2, 3, 4, 5} \left[ b^{-1} \circ b^{-1} (b \circ \nabla_x) (b \circ \nabla_x) \right] \right. \\
- \text{tr}_{3, 4, 2, 5} \left[ b^{-1} \circ b^{-1} (b \circ \nabla_x) b (b \circ \nabla_x) \circ b^{-1} \right] = 0,
\]

\[
\frac{1}{4} \left\{ b_{kij} b_{mj} \partial_n b^{nk} \partial_i b^{mn} - 2 b_{ik} b_{lj} \partial_m b^{nl} \partial_n b^{mk} + b_{ik} b_{mn} \partial_j b^{nm} \partial_j b^{lk} \\
+ 2 b_{ik} b_{lj} b^{mn} \partial_n b^{kl} + 2 b_{ik} b_{lj} \partial_n b^{mn} \partial_n b^{kl} \right. \\
+ 2 b_{kl} b_{mj} \partial_n b^{lm} \partial_i b^{mk} + 2 b_{kl} \partial_l \partial_i b^{kl} \\
- 2 b_{ik} b_{lj} b^{pq} \partial_p b^{nl} \partial_q b^{km} - 3 b_{kl} b_{mn} \partial_i b^{lm} \partial_j b^{nk} - 2 b_{kj} \partial_i b^{kl} , \\
- 2 b_{ik} \partial_l \partial_j b^{kl} + 2 b_{ik} b_{lm} \partial_n b^{mk} \partial_j b^{nl} - b_{ik} b_{lj} b^{pq} \partial_p b^{nm} \partial_q b^{kl} \right\} = 0,
\]

where, for example, \( \text{tr}_{1, 5, 3, 4} \) indicates tensorial contraction in the first and fifth slots and another contraction in the third and fourth slots. \([^19\] has been calculated by hand, and

\[^1\]In general, the Christoffel symbols of the second kind corresponding to the metric of a Riemannian manifold do not form a tensor. It is only the present particular case, where the same manifold \( E \) is considered with two metric tensors \( b \) and \( b \), being a flat metric and making \( E \) an Euclidean affine space over \( b \), where the Christoffel symbols of the second kind of \( g \) form a tensor from the point of view of the Euclidean affine space \( (E, h) \).
verified by the xAct packages for Wolfram Mathematica.] Notably, (18)—or (19)—has been devised to be in a form that contains $b$, $b^{-1}$, and derivatives of $b$ but no derivatives of $b^{-1}$. Accordingly, this form is (relatively) friendly for applications.

As a remark, formula (18) can also be considered as a case study for how far one can get with the index-free tensorial notation.

### 4 The small-strain regime

Saint-Venant’s compatibility condition says that a symmetric cotensor field $\varepsilon$ can be expressed as the symmetric derivative of a covector field if and only if its left+right curl is zero,

$$\nabla_X \times \varepsilon(X) \times \nabla_X = 0, \quad \epsilon_{KMP} \epsilon_{LNQ} \partial_M \partial_N \epsilon_{PQ} = 0, \quad (20)$$

or, equivalently [10]—as is straightforward to check (e.g., taking a contraction of the left+right curl of either of (20) and (21) to derive the other)—, if and only if

$$\partial_M \partial_M \epsilon_{KL} - \partial_K \partial_M \epsilon_{ML} - \partial_M \partial_L \epsilon_{KM} + \partial_K \partial_L \epsilon_{MM} = 0, \quad (21)$$

where $\epsilon$ is the Levi-Civita permutation symbol (pseudotensor).

In continuum mechanical applications, one wishes to use this condition for a vector field, namely, the displacement vector field

$$u(X) = \chi(X) - X, \quad \dot{u}^K = \dot{\chi}^K - X^K. \quad (22)$$

The Euclidean structure $h$ provides an identification between vectors and covectors (index lowering) so $\varepsilon$ satisfying the compatibility condition can be written as the symmetric derivative of the covector field $u_K = h_{KL} u^L$, that is,

$$\epsilon_{KL} = \frac{1}{2} \left[ \partial_K \left( h_{LM} u^M \right) + \partial_L \left( h_{KM} u^M \right) \right], \quad \varepsilon = \frac{1}{2} \left[ \nabla_X \otimes (hu) + (hu) \otimes \nabla_X \right]. \quad (23)$$

When a version in terms of the Eulerian differentiation $\nabla_{\chi}$ is needed then vector–covector identification can be done via $g$:

$$\varepsilon_{ij} = \frac{1}{2} \left[ \partial_i \left( g_{jk} u^k \right) + \partial_j \left( g_{ik} u^k \right) \right], \quad \varepsilon = \frac{1}{2} \left[ \nabla_{\chi} \otimes (gu) + (gu) \otimes \nabla_{\chi} \right]. \quad (24)$$

and actually in a different way as well:

$$\varepsilon_{ij} = \frac{1}{2} \left[ \partial_i u^k g_{jk} + g_{ik} \left( \partial_j u^k \right) \right], \quad \varepsilon = \frac{1}{2} \left[ \nabla_{\chi} \otimes u + g(u) \otimes \nabla_{\chi} \right]. \quad (25)$$

The two versions (24) and (25) differ because $g \equiv g(\chi)$ is not constant. Fortunately, at a material point, each of the $\varepsilon$’s provided by (23), (24) and (25) agree in the linearized leading order of $F - I$, i.e., when we are in the small-strain regime

$$F \approx I, \quad \|F - I\| \ll 1, \quad \Rightarrow \quad g \approx h, \quad b \approx h^{-1}. \quad (26)$$
More closely, each of these three $\varepsilon$’s agree in the linearized leading order with the Almansi tensor $e = \frac{1}{2}(h - b^{-1})$, $\varepsilon_{ij} = \frac{1}{2}(h_{ij} - b_{ij})$.

Now, taking the linearized leading order of (19) yields

$$\frac{\partial_k}{\partial k} \varepsilon_{ij} - \frac{\partial_l}{\partial l} \varepsilon_{kj} - \frac{\partial_k}{\partial l} \varepsilon_{lk} + \frac{\partial_j}{\partial k} \varepsilon_{kk} + \text{(higher-order terms)} = 0. \quad (27)$$

This, in light of (21), tells that compatibility condition (19) can be considered as the finite-strain Eulerian counterpart of the classic small-strain compatibility condition.

## 5 Discussion

In the Eulerian description, the question of compatibility of the left Cauchy–Green deformation tensor field admits a differential geometric interpretation and, putting various differential geometric facts together, leads to the question of vanishing of a two-indexed cotensor field. This resulting condition turns out to be the Eulerian finite-strain generalization of Saint-Venant’s one applicable for small strain.

In the Lagrangian variable, the left Cauchy–Green deformation tensor field does not have such a differential geometric background. The difference is nicely depicted by (10): the two derivative-related tensorial slots/indices are the two outward (most leftward and most rightward) ones, enabling isometric mapping of this cotensor field from one Riemannian manifold to another one, via the derivative map. On the other side, the Lagrangian derivative slots/indices in (5) are inward—cf. (2). This “insulation” prevents analogous manipulations.

This observation adds to why the left Cauchy–Green deformation tensor field is typically found suitable for Eulerian problems while, for Lagrangian ones, the right Cauchy–Green deformation tensor field is preferred—and, hopefully, also to how the compatibility problem of the Lagrangian left Cauchy–Green deformation tensor field could be successfully treated.

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