THE QUATERNIONIC KP HIERARCHY AND CONFORMALLY IMMERSED 2-TORI IN THE 4-SPHERE.

IAN MCINTOSH

Abstract. The quaternionic KP hierarchy is the integrable hierarchy of p.d.e obtained by replacing \( \mathbb{C} \) with \( \mathbb{H} \), \textit{mutatis mutandis}, in the standard construction of the KP hierarchy equations and solutions; it is equivalent to what is often called the Davey-Stewartson II hierarchy. This article studies its relationship with the theory of quaternionic holomorphic 2-tori in \( \mathbb{H}P^1 \) (which are equivalent to conformally immersed 2-tori in \( S^4 \)). After describing how the Sato-Segal-Wilson construction of KP solutions (particularly solutions of finite type) carries over to this quaternionic setting, we compare three different notions of “spectral curve”: the QKP spectral curve, which arises from an algebra of commuting differential operators; the (unnormaised) Floquet multiplier spectral curve for the related Dirac operator; and the curve parameterising Darboux transforms of a conformal 2-torus in \( S^4 \) (in the sense of Bohle, Leschke, Pedit and Pinkall \[3\]). The latter two are shown to be images of the QKP spectral curve, which need not be smooth. Moreover, it is a singularisation of this QKP spectral curve, rather than the normalised Floquet multiplier curve, which determines the classification of conformally immersed 2-tori of finite spectral genus.

1. Introduction.

The purpose of this article is to better understand the connection between the Davey-Stewartson hierarchy, an integrable system of non-linear p.d.e, and conformal immersions of tori in \( S^4 \). The connection began with observations about tori in \( \mathbb{R}^3 \) and the modified Novikov-Veselov equations \[16, 13, 28, 29, 10, 25\] and extended to \( \mathbb{R}^4 \) and the Davey-Stewartson equations later \[14, 15, 31, 33\] (see also the survey article \[34\]). One compelling reason for the interest shown in this correspondence is that the simplest “first integral” of these hierarchies represents the Willmore functional, and there is some hope that a strategy based upon integrable systems will succeed in settling the Willmore conjecture (see \[25\] and \[9\] for two different perspectives on this).

In the early papers the hierarchy itself was viewed as a system of p.d.e describing deformations of a Dirac operator. This is linked to the conformal immersion

\[ \text{Date: February 20, 2009.} \]
\[ 1991 \text{ Mathematics Subject Classification.} \ 14H70, 35Q53, 53A30, 53C42. \]
\[ \text{Key words and phrases.} \ \text{Integrable systems, conformally immersed tori, quaternionic holomorphic curves, spectral curves.} \]
via the so-called generalised Weierstrass representation of surfaces in \( \mathbb{R}^3 \) and \( \mathbb{R}^4 \). However, this point of view makes it difficult to see the conformal invariance and presents other problems in \( \mathbb{R}^4 \) (see [33]). These difficulties are avoided when we adopt a different approach, using the theory of quaternionic holomorphic curves developed by the Berlin school [22] [5] [9] [1] [3] [2]. The conformal 4-sphere is thought of as \( \mathbb{H}P^1 \); a conformal immersion of a surface \( M \) becomes a quaternionic line bundle over \( M \) whose dual bundle possesses a canonical quaternionic holomorphic structure \( D \), the quaternionic analogue of a \( \bar{\partial} \)-operator.

This operator \( D \) is essentially the Dirac operator which plays the central role in the alternative interpretation for the Davey-Stewartson hierarchy described here. This hierarchy is exactly what is obtained by replacing the field \( \mathbb{C} \) with the division algebra \( \mathbb{H} \) in the standard constructions of the KP hierarchy and its solutions (particularly, following [35] [36] [26]). This point of view is sufficiently rewarding that I feel the new name “quaternionic KP” (QKP) hierarchy is justified.

The two features of integrable p.d.e which make them so special are: (i) the presence of infinitely many, independent, commuting flows, often called the “higher flows”; (ii) the existence of solutions possessing a “spectral curve” of finite genus, so that those solutions (called “finite type/gap/zone”) are part of a finite dimensional completely integrable system whose Liouville tori are something like the Jacobi varieties of the spectral curves. These features are well understood for the KP hierarchy and the relevant properties for QKP are described in this article (although we have no interest in Hamiltonian structures and say nothing about them). As a general procedure this is “well-known” for multi-component KP hierarchies, although the detail requires some care.

The challenge is to understand how these features translate into the geometry of the immersed torus. There is at present no direct definition of the flows as deformations of a torus (although, see [6] for an attempt at this); there is only an indirect construction for tori of finite type via the linear motion on the spectral curve (and here it matters very much which definition of the spectral curve is chosen). For the spectral curve, Taimanov initially proposed, in [29], that this should be the normalisation of the analytic curve of Floquet multipliers of the Dirac operator: generically this has “infinite genus”. Recently this was examined from the quaternionic perspective [3] [2] [4], which provides an intrinsically conformally invariant construction of Taimanov’s spectral curve, and also gives a geometric interpretation to the points on this curve, as parameterising Darboux transformations of the immersed torus (where the notion of a Darboux transformation has been defined to give the flexibility necessary).

The main contribution of this article lies in clarifying relationship between different notions of “spectral curve” which arise in the case of finite genus: from QKP (following KP), from the multiplier spectral curve, and from the curve of

\[1\] The idea that there should be a “quaternionic KP” hierarchy is due to Ulrich Pinkall.
Darboux transformations. To achieve this, we study first the QKP hierarchy itself.

Properly thought of, the QKP hierarchy is a collection of Lax equations \( L_t = [P, L] \) for a pseudo-differential operator with quaternionic coefficients

\[
L = i\partial_y + U_0 + U_1\partial_y^{-1} + \ldots,
\]

and a differential operator \( P \). Because the construction of these Lax equations is fundamentally algebraic we follow Wilson \[35, 36\] and take the point of view that it is done formally in a real differential algebra, so that \( \partial_y \) is a real derivation and the components of the coefficients \( U_0, U_1, \ldots \), together with their formal \( y \)-derivatives, generate this algebra. The Lax equations are obtained by the familiar “dressing” method. In particular, one of these provides the real derivation \( \partial_x \): the operator

\[
D = \frac{1}{2}(\partial_x + i\partial_y + U_0),
\]

will play the role of the Dirac operator. The QKP equations which involve the derivation \( \partial_x \) imply (but are not equivalent to) equations of the form

\[
D_t = [D, P].
\]

To reconnect with analysis, we realise the QKP equations as flows on a manifold. Here we follow the approach taken by Segal & Wilson \[26\] with the KP hierarchy: the manifold is a quotient of an infinite dimensional Grassmannian and contains all the finite type solutions of the QKP hierarchy. This Grassmannian, \( Gr(\mathbb{H}) \), possesses pretty much all the properties that hold for KP. In particular:

(a) the QKP flows correspond to the action of an infinite dimensional abelian Lie group \( \Gamma_+ \) on \( Gr(\mathbb{H}) \) and the QKP solutions are parameterized by a quotient manifold \( \mathcal{M} \). This manifold is the disjoint union of two manifolds, \( \mathcal{M}_{KP} \), which is a copy of the ordinary KP phase space and parameterises QKP solutions for which the Dirac potential \( U_0 \) is zero, and \( \mathcal{M}_{QKP} \), which parameterizes solutions with non-zero Dirac potential.

(b) the points of \( \mathcal{M} \) which have finite dimensional \( \Gamma_+ \)-orbits correspond precisely to the solutions which can be constructed from spectral data (i.e., a complete algebraic curve and other algebraic data on it), and the orbits themselves are open subvarieties of (generalised) Jacobi varieties. We call these “of finite type”.

The object which mediates between Lax equations and points of \( Gr(\mathbb{H}) \) is the (quaternionic) Baker function. This is an \( \mathbb{H} \)-valued solution to the spectral problem \( L\psi = -\psi\zeta \) which admits a left \( \mathbb{H} \) Fourier series expansion of the form

\[
\psi(x, y, \zeta) = (1 + a_1(x, y)\zeta^{-1} + \ldots)\exp[(x + iy)\zeta], \quad \zeta \in S^1.
\]

This Baker function converges for \( |\zeta| > 1 \) for all QKP solutions in \( \mathcal{M} \). Points of the Grassmannian \( Gr(\mathbb{H}) \) parameterise Baker functions. For the QKP solutions of finite type there is a representative Baker function which extends to a globally
meromorphic function on the spectral curve. We call this the global Baker function: it plays an important role in translating between the different notions of spectral curve.

The QKP spectral curve $X$, and the other data it carries, is the object which we spend the most time studying. It is not required to be smooth, but it must possess a real involution $\rho$ which has no fixed points. Part of the data on it is a marked point $P$ (and therefore its partner $\rho P$). This makes it the natural quaternionic analogue of the pointed curve which appears in KP theory. Indeed, although this idea is not pursued here except incidentally, it seems certain that the QKP Grassmannian plays the same role for the moduli space of such curves that the KP Grassmannian plays for the moduli space of all pointed complete irreducible algebraic curves.

The Baker function satisfies $D\psi = 0$ for all $\zeta$ (i.e., it provides a complex analytic family of quaternionic holomorphic functions) and as a consequence it can be used to construct quaternionic holomorphic curves in $\mathbb{H}P^1$ (or, more generally, $\mathbb{H}P^n$). When $\psi$ is the global Baker function evaluation of it at points of $X$ provides the homogeneous quaternionic coordinates for these curves. We describe the conditions under which these quaternionic holomorphic curves are doubly periodic and are therefore immersed tori: these conditions involve the Jacobi variety of a singularisation $X_\mathfrak{q}$ of the QKP spectral curve $X$, obtained by collapsing into one singular point the divisor $\mathfrak{q}$ of points at which the global Baker function is evaluated to obtain the quaternionic holomorphic curve.

In this passage between QKP and quaternionic holomorphic curves two problems remain unresolved.

(a) The periodicity conditions on the QKP spectral data imply that the whole QKP operator $L$ is “periodic” (the precise condition is that it have unimodular monodromy over the torus, so it acts on sections of a flat, but not necessarily trivial, bundle). By contrast, if one begins with a conformal torus then at first sight one only obtains “periodicity” of the Dirac operator $D$. There is a Baker-type function in the kernel of the Dirac operator when its multiplier spectral curve has finite type (i.e., the normalised curve has finite genus). The problem, which I was unable to solve, is to show that the two Baker functions coincide without knowing a priori that the QKP operator $L$ is “periodic”.

(b) When the multiplier spectral curve is not of finite type it seems there can be no convergent multiplier Baker function, which raises the problem of how to make the passage between QKP and conformal tori at all. In particular, it is not clear how to describe integrable deformations in this case. This may be possible using the theory of infinite genus Riemann surfaces [8, 24], but that would take us out of the Grassmannian class of solutions discussed here.
In the case where the two Baker functions can be shown to be the same we construct both the Darboux spectral curve and the multiplier spectral curve from the QKP spectral, as images which are not necessarily biholomorphic. The Darboux spectral curve has a particularly neat characterization via an Abel image of a punctured version of the QKP spectral curve $X$: when $X$ is smooth the punctured curve is just $X \setminus q$, whose Abel image lies in $\text{Jac}(X_q)$. The intention here is to show that it is the singular curve $X_q$ which carries the spectral data required to understand quaternionic holomorphic curves.

The article concludes with a short discussion of the topics which I believe deserve further attention: a geometric understanding of integrable deformations; tori which are not of finite type; non-isospectral Darboux transformations. It seems to me that the proper arena for discussing all of these topics is not the theory of conformally immersed tori in $S^4$, but rather the more general study of quaternionic holomorphic curves in $\mathbb{HP}^n$.

Acknowledgments. I would like to thank Christoph Bohle, Katrin Leschke, Franz Pedit and Ulrich Pinkall for letting me see preliminary versions of \cite{3, 4} and for explaining their work on quaternionic holomorphic curves to me over a number of years. A preliminary, and far less comprehensive, version of this article (containing a few misunderstandings) appeared in the conference proceedings \cite{20}.

2. Quaternionic holomorphic curves.

We begin by summarising the correspondence between conformally immersed surfaces in $S^4 \simeq \mathbb{HP}^1$ (as the space of left quaternionic lines in $\mathbb{H}^2$) and quaternionic holomorphic curves (based on \cite{3, 9}). Each immersed surface $f : M \to S^4$ can be equated with a smooth (quaternionic) line subbundle $L$ of the trivial bundle $\mathbb{H}^2 = M \times \mathbb{H}^2$ via $f(p) = L_p$.

The reader is warned at this point that we do not adopt the convention of \cite{9} that all quaternionic bundles are right $\mathbb{H}$ bundles, since this does not fit well with our construction of the QKP hierarchy using differential operators, with coefficients over $\mathbb{H}$, acting on the left on $\mathbb{H}$-valued functions. Therefore $L$ will be a left $\mathbb{H}$-bundle its quaternionic dual $L^* = \text{Hom}_{\mathbb{H}}(L, \mathbb{H})$ a right $\mathbb{H}$-bundle.

The theory of quaternionic holomorphic curves encodes the conformality of the map $f$ as follows. The differential $df$ of any immersed surface $f : M \to \mathbb{HP}^1$ can be represented, after the standard manner for maps into projective spaces, by $\delta f \in \Omega^1_M(\text{Hom}(L, \mathbb{H}^2/L))$ where

$$\delta f : \psi \mapsto d\psi \mod L, \quad \psi \in \Gamma(L).$$

Here $\Gamma(L)$ denotes the space of smooth sections of $L$. We also adopt the convention\footnote{Note that in this convention $*dz = idz$ for the standard complex structure on the complex plane, contrary to the usual convention for the Hodge $*$-operator.} that, for any vector bundle valued 1-form $\omega$, $*\omega = \omega \circ J_M$ where $J_M$ is the complex structure on $M$. Now $f$ is a conformal immersion if and only if there...
exists $J \in \text{End}_\mathbb{H}(L)$ with $J^2 = -I$ for which $*\delta f = \delta f \circ J$. The pair $(L, J)$ is called a quaternionic holomorphic line bundle.

Further, $L^*$ inherits a complex structure, which we will also call $J$, and a canonical quaternionic holomorphic structure

$$D : \Gamma(L^*) \to \Omega^1(L^*).$$

This is an $\mathbb{H}$-linear operator satisfying, for any $\Psi \in \Gamma(L^*)$ and $\mu : M \to \mathbb{H}$:

$$*D\Psi = -JD\Psi,$$

$$D(\Psi \mu) = (D\Psi)\mu + \frac{1}{2}(\Psi d\mu + J\Psi * d\mu).$$

The canonical structure is characterised by the property that the natural inclusion $(\mathbb{H}^2)^* \subset \Gamma(L^*)$ maps into the kernel of $D$, i.e., into the space $H^0_D(L^*)$ of quaternionic holomorphic sections (see [5, §4.3]).

The definitions above apply equally well to line subbundles of $\mathbb{H}^{n+1}$ and provide the notion of a quaternionic holomorphic curve in $\mathbb{H}P^n$. Conversely, given a complex quaternionic line bundle $(E, J)$ equipped with a quaternionic holomorphic structure $D$, define $H^0_D(E)$ to be the kernel of $D$. When $M$ is compact this space is finite dimensional (indeed, a Riemann-Roch formula can be derived to calculate this dimension: see [9]). The “Kodaira construction” follows through to give a quaternionic holomorphic curve

$$f : M \to \mathbb{P}H^0_D(E)^*; \quad p \mapsto E_p^*.$$

A two dimensional $\mathbb{H}$-subspace $H \subset H^0_D(E)$ can be thought of as a linear system, in the sense of algebraic curve theory. Provided $H$ is well-positioned (has no base points) the dual projection $H^0_D(E)^* \subset H^*$ gives us a conformal immersion $f : M \to \mathbb{P}H^* \simeq \mathbb{H}P^1$.

2.1. The Dirac operator for a conformal torus in $S^4$ with flat normal bundle. For any complex quaternionic line bundle $(L, J)$ we define $\hat{L}$ to be the complex line bundle whose fibres are

$$\hat{L}_P = \{ \sigma \in L_P | J\sigma = i\sigma \}.$$

By definition, $\deg(L)$ is the degree of the complex line bundle $\hat{L}$. Bohle [11 p19] showed that when $M$ is a compact Riemann surface the degree of the normal bundle of $f$ is $2\deg(\hat{L}) - \deg(K_M)$, where $K_M$ is the canonical bundle.

Suppose we have a complex (right) quaternionic line bundle $(E, J)$ over a torus $M$, with quaternionic holomorphic structure $D$. Let us assume this has degree zero, i.e., $\deg(\hat{E}) = 0$. The complex bundle $\hat{E}$ inherits the complex holomorphic structure

$$\bar{\partial}_J = \frac{1}{2}(D - JJD).$$

The moduli space of degree zero holomorphic line bundles is isomorphic to the moduli space $H^1(M, S^1)$ of Hermitian line bundles possessing a flat Hermitian
connexion, so $\partial_J$ can be extended in a unique way to a flat connexion $\nabla$ which is also Hermitian with respect to some Hermitian inner product on $\hat{E}$. Now, if we represent $M$ as $\mathbb{C}/\Lambda$ for some lattice $\Lambda$ and let $\pi: \mathbb{C} \to M$ denote the universal cover then we can trivialise $\pi^*\hat{E}$ with a smooth $\nabla$-parallel section $\Phi$ which is therefore $\partial_J$-holomorphic. This section is unique up to right multiplication by a complex constant and has unimodular monodromy $\mu \in \text{Hom}(\Lambda, S^1)$, i.e., for each $\lambda \in \Lambda$ and all $z \in \mathbb{C}$

$$\Phi(z + \lambda) = \Phi(z)\mu(\lambda).$$

For any $\Psi \in \Gamma(E)$ we therefore have some $\mathbb{H}$-valued function $\psi$ on $\mathbb{C}$ for which $\Psi = \Phi\psi$ and we observe that $\psi$ has monodromy $\mu^{-1}$ along the lattice $\Lambda$. It follows that

$$D\Psi = (D\Phi)\psi + \Phi \frac{1}{2}(d\psi + i* d\psi).$$

Now recall the decomposition

$$D = \bar{\partial}_J + Q,$$

where $Q = (D + JDJ)/2$ is called the “Hopf field”, and write $Q\Phi = \Phi dzU$ for some $U: \mathbb{C} \to \mathbb{H}$. Since $Q$ anti-commutes with $J$ it follows that $U$ anti-commutes with $i$. Then

$$D(\Phi\psi) = (\Phi dz)D\psi, \quad D = \partial/\partial \bar{z} + U.$$

We call $D$ the Dirac operator. For simplicity let us define

$$\ker(D) = \{ \psi \in C^\infty(\mathbb{C}, \mathbb{H}) | D\psi = 0 \}.$$

Then we have an identification

$$H^0_D(E) \simeq \{ \psi \in \ker(D) : \psi(z + \lambda) = \mu^{-1}(\lambda)\psi(z) \ \forall \lambda \in \Lambda \}$$

Notice that $D$ does not in general preserve the space of functions on the torus $M$, since for any $\lambda \in \Lambda$ and all $z \in \mathbb{C}$

$$U(z + \lambda) = \mu(\lambda)^{-1}U(z)\mu(\lambda) = U(z)\mu(\lambda)^2.$$

Therefore $D$ is doubly periodic if and only if $\hat{E}$ is a spin bundle. We know from [22] that this is the case which corresponds to immersions into $\mathbb{R}^3$ (see also [1]). By (4) $|U|$ is always a function on $M$ itself and the $L^2$-norm of $U$ over $M$ is called the Willmore energy of $D$: it is essentially the Willmore functional for the corresponding conformally immersed torus.

### 2.2. The multiplier spectrum and the Baker function.

For any degree zero quaternionic holomorphic (right $\mathbb{H}$) line bundle $(E, D)$ over a torus $\mathbb{C}/\Lambda$ let $(\pi^*E, D)$ denote its pull-back to the universal cover $\pi: \mathbb{C} \to \mathbb{C}/\Lambda$. Using this we construct the multiplier spectrum of $(E, D)$:

$$\text{Sp}(E, D) = \{ \chi \in \text{Hom}(\Lambda, \mathbb{C}^\times) : \exists \Psi \in H^0_D(\pi^*E), \ \Psi \neq 0, \ \lambda^*\Psi = \Psi\chi(\lambda) \ \forall \lambda \in \Lambda \}.$$  

By fixing two generators for $\Lambda$ we can identify $\text{Sp}(E, D)$ with a subset of $(\mathbb{C}^\times)^2$. It has been shown (see [34] for a survey of results) that $\text{Sp}(E, D)$ is a complex

---

3In the spinor representation $\mathbb{H} = \mathbb{C} + j\mathbb{C}$ the operator $D$ has the form of a Dirac operator.
analytic curve. Right multiplication on \( H^0_D(\pi^*E) \) by \( j \) induces on \( \text{Sp}(E, D) \) a real involution \( \chi \mapsto \bar{\chi} \). Using the isomorphism \( (4) \) we can write

\[
\text{Sp}(E, D) = \{ \chi \in \text{Hom}(\Lambda, \mathbb{C}^\times) : \exists \psi \in \ker(D), \psi \neq 0, \lambda^* \psi = \mu(\lambda)^{-1} \psi(\lambda) \forall \lambda \in \Lambda \}. 
\]

Here \( \mu \) is the monodromy for the flat bundle \( L^* \).

The asymptotic structure of this spectrum is quite well understood (see [3, 34]) and is described by comparison with the spectrum of the “vacuum” operator \( D_0 = \bar{\partial}_J \), with vacuum Dirac operator \( D_0 = \partial/\partial \bar{z} \). Let us assume for the moment that \( E \) is trivial (i.e., \( \mu = 1 \)). Taking \( \psi(z, \zeta) = e^{z\zeta} \), for \( \zeta \in \mathbb{C} \), we observe that \( D_0 \psi = 0 \) for all \( \zeta \) and

\[
\psi(z + \lambda, \zeta) = \psi(z, \zeta)e^{\lambda\zeta}, \quad \forall z \in \mathbb{C}, \lambda \in \Lambda.
\]

Hence \( \text{Sp}(E, D_0) \) contains an analytic curve of monodromies

\[
C = \{ (e^{\lambda_1\zeta}, e^{\lambda_2\zeta}) \in (\mathbb{C}^\times)^2 : \zeta \in \mathbb{C} \}, \quad \Lambda = \mathbb{Z}(\lambda_1, \lambda_2).
\]

The full multiplier spectrum of \( D_0 \) is \( C \cup \bar{C} \). These two branches of \( \text{Sp}(E, D_0) \) intersect infinitely often in double points. When the monodromy \( \mu \) is nontrivial the structure is the same but with the branches \( C \) and \( \bar{C} \) shifted by appropriate factors.

The spectrum \( \text{Sp}(E, D) \) is asymptotic to \( \text{Sp}(E, D_0) \) in the sense that outside a compact subset of \( (\mathbb{C}^\times)^2 \) the former is contained in an arbitrarily small tube around the latter. Away from the double points of \( \text{Sp}(E, D_0) \) the curve \( \text{Sp}(E, D) \) is a graph over \( \text{Sp}(E, D_0) \), while near each double point \( \text{Sp}(E, D) \) either has a double point itself or is annular: in the latter case \( \text{Sp}(E, D) \) resolves the double point into a handle. Now let us consider what happens to \( \text{Sp}(E, D) \) as it follows the curve \( C \) along \( |\zeta| \to \infty \). Either for every \( R > 0 \) the domain \( |\zeta| > R \) contains at least one handle, or there is some \( R \) for which \( \text{Sp}(E, D) \) only contains double points. In the latter case \( \text{Sp}(E, D) \) must have two intersecting branches, one of which can be parameterised by \( \Delta = \{ \zeta : |\zeta| > R \} \), thought of as a punctured parameter disc about the point at infinity of \( C \). Thus we have a map

\[
\chi : \Lambda \times \Delta \to \mathbb{C}^\times, \quad \chi(\cdot, \zeta) \in \text{Hom}(\Lambda, \mathbb{C}^\times),
\]

which is holomorphic in \( \Delta \) for each \( \lambda \). The following result is a direct consequence of Theorem 4.1 and Lemma 5.2 in [34] (cf. [34, §4]).

**Theorem 2.1.** When \( \text{Sp}(E, D) \) has only double points along \( \Delta \) there exists a function

\[
\psi : \mathbb{C} \times \Delta \to \mathbb{H}
\]

satisfying:

(a) \( D \psi = 0 \) for all \( \zeta \in \Delta \),

(b) \( \psi \) is holomorphic in \( \zeta \) and \( \lim_{\zeta \to \infty} \psi(z, \zeta)e^{-z\zeta} = 1 \),

(c) \( \psi(z + \lambda, \zeta) = \mu(\lambda)^{-1} \psi(z, \zeta) \chi(\lambda, \zeta) \) for all \( z \in \mathbb{C}, \lambda \in \Lambda \).
Further, $\psi$ is uniquely determined by $\psi(0, \zeta)$.

We will call this function $\psi(z, \zeta)$ the multiplier Baker function for $\text{Sp}(E, D)$. By the properties above it has the expansion (in left Fourier series)

$$
\psi(z, \zeta) = (1 + \sum_{j=1}^{\infty} a_j(z)\zeta^{-j}) \exp(z\zeta), \quad a_j : \mathbb{C} \to \mathbb{H}, \ |\zeta| > R.
$$

By rescaling $\zeta$ we may as well assume $\Delta$ is the punctured disc $|\zeta| > 1$. In the next section we will introduce the QKP Baker function, and later on we will consider under what conditions we can show that the two agree.

We finish this section with an example which will help us understand, later, in section 6, the difference between the multiplier spectrum and the QKP spectral curve. The example comes from the study of Hamiltonian stationary Lagrangian (HSL) tori in $\mathbb{R}^4$.

**Example 2.2.** Fix a torus $\mathbb{C}/\Lambda$ and equip $\mathbb{C}$ with its standard metric $\langle z, w \rangle = \frac{1}{2}(z\bar{w} + \bar{z}w)$. With this we think of the dual lattice $\Lambda^*$ as

$$
\Lambda^* = \{ \alpha \in \mathbb{C} : \langle \alpha, \lambda \rangle \in \mathbb{Z} \ \forall \lambda \in \Lambda \}.
$$

Fix some non-zero $\beta_0 \in \Lambda^*$ and define $\beta(z) = 2\pi \langle \beta_0, z \rangle$. Now consider the complex quaternionic line bundle $(E, J)$ where $\pi : E \to \mathbb{C}/\Lambda$ is the trivial right $\mathbb{H}$-bundle and $J\sigma = N\sigma$ for $N = e^{j\beta i}$. This has quaternionic holomorphic structure

$$
D\sigma = \frac{1}{2}(d\sigma + N \ast d\sigma).
$$

From [17] we know that $f : \mathbb{C}/\Lambda \to \mathbb{H}$ is HSL with Maslov class $\beta_0$ whenever $Df = 0$. It is easy to check that $\Phi = e^{j\beta/2i}$ is a parallel section of $\pi^*\hat{E}$ for the unique flat Hermitian connexion $\hat{\nabla}$ on $\hat{E}$ for which $\hat{\nabla}'' = \partial J$, and the Dirac operator given by (2) is

$$
(6) \quad D = \frac{\partial}{\partial \bar{z}} + \frac{\pi}{2} \beta_0 j.
$$

Notice that $(\hat{E}, J)$ is trivial if $\beta_0/2 \in \Lambda^*$ but otherwise a spin bundle, since the monodromy of $\Phi$ is given by $\mu(\lambda) = e^{j\beta(\lambda)/2} = \pm 1$ for $\lambda \in \Lambda$.

We can explicitly calculate $\text{Sp}(E, D)$ for this example using (5). It is easiest to do this by writing any solution of $D\psi = 0$ in the form

$$
\psi(z) = \mu^{-1}(z)\varphi_1(z)\chi(z) + j\mu^{-1}(z)\varphi_2(z)\chi(z),
$$

where $\mu(z) = e^{j\beta(z)/2}$, $\varphi_m$ are both $\Lambda$-periodic complex valued functions and

$$
\chi(z) = \exp[\pi i(\xi z + \eta \bar{z})],
$$
for complex parameters $\xi, \eta$. These will parameterise the logarithmic spectrum. Now $D\psi = 0$ is equivalent to the equations

\begin{align*}
\frac{1}{\pi i} \frac{\partial \varphi_1}{\partial \bar{z}} + (\eta - \beta_0/2) \varphi_1 + (i\beta_0/2) \varphi_2 &= 0 \\
\frac{1}{\pi i} \frac{\partial \varphi_2}{\partial z} + (\xi - \bar{\beta}_0/2) \varphi_2 - (i\bar{\beta}_0/2) \varphi_1 &= 0.
\end{align*}

Since the complex valued functions $\varphi_m$ are both $\Lambda$-periodic they can be represented by Fourier series

$$
\varphi_m = \sum_{\alpha \in \Lambda^*} \varphi_{m\alpha} e^{2\pi i \langle \alpha, z \rangle}.
$$

There is a non-trivial solution to (7) if and only if there exists $\alpha \in \Lambda^*$ for which the linear system

\begin{equation}
\begin{pmatrix}
\alpha + \eta - \beta_0/2 & i\beta_0/2 \\
-i\bar{\beta}_0/2 & \bar{\alpha} + \xi - \bar{\beta}_0/2
\end{pmatrix}
\begin{pmatrix}
\varphi_{1\alpha} \\
\varphi_{2\alpha}
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{equation}

has a non-trivial solution. Thus if we set

$$
F_\alpha(\eta, \xi) = (\alpha + \eta - \beta_0/2)(\bar{\alpha} + \xi - \bar{\beta}_0/2) - |\beta_0|^2/4
$$

we can describe the logarithmic spectrum $\tilde{\Sigma}$ of $(E, D)$ as the union of infinitely many irreducible rational curves:

$$
\tilde{\Sigma} = \bigcup_{\alpha \in \Lambda^*} C_\alpha, \quad C_\alpha = \{ (\eta, \xi) \in \mathbb{C}^2 : F_\alpha(\eta, \xi) = 0 \}.
$$

Even though each component $C_\alpha$ is smooth the curve $\tilde{\Sigma}$ possesses infinitely many singularities caused by the intersections of components. The dual lattice $\Lambda^*$ acts on $\tilde{\Sigma}$ by $(\eta, \xi) \mapsto (\eta + \alpha, \xi + \bar{\alpha})$ and it is easy to see that this identifies all components $C_\alpha$ with one, say $C_0$. Thus

$$
\text{Sp}(E, D) \simeq \tilde{\Sigma}/\Lambda^* \simeq C_0/\sim
$$

where $\sim$ is the equivalence relation on $C_0$ defined by

$$(\eta, \xi) \sim (\eta', \xi') \iff (\eta', \xi') = (\eta + \alpha, \xi + \bar{\alpha}) \exists \alpha \in \Lambda^*.$$

This identification creates the singularities of $\text{Sp}(E, D)$, which are of two types.

(a) For each non-zero $\alpha \in \Lambda^*$ the points $(\eta, \xi)$ and $(\eta + \alpha, \xi + \bar{\alpha})$ are identified whenever

$$
\frac{\alpha}{\bar{\alpha}} Z^2 + \alpha Z + \frac{|\beta_0|^2}{4} = 0, \quad Z = \xi - \frac{\bar{\beta}_0}{2}.
$$

This gives a double point when the discriminant (which is proportional to $|\alpha|^2 - |\beta_0|^2$) is non-zero but a cusp when it vanishes (the two components $C_0$ and $C_\alpha$ of $\tilde{\Sigma}$ meet tangentially when the discriminant vanishes).

(b) The point $(0, 0)$ is identified with every point in the set

$$
S = \{ (\alpha, \bar{\alpha}) : |\alpha - \frac{\beta_0}{2}| = |\frac{\beta_0}{2}, \alpha \in \Lambda^* \}. 
$$
This includes \((β_0, \bar{β}_0)\) and therefore one of the above cusps is folded into this singularity.

The simplest Baker function is obtained by taking \(\varphi_{iα} = 0\) for \(α \neq 0\). Then we have a non-trivial solution to (3) given by

\[
(\varphi_{10}, \varphi_{20}) = (1, -\frac{π \bar{β}_0}{2} \zeta^{-1}), \quad \zeta = π i (\bar{α} - \bar{β}_0/2).
\]

This leads to the multiplier Baker function

\[
ψ(z, ζ) = (1 - j \frac{π |β_0|^2 z}{2} \zeta^{-1}) \exp(-\frac{π^2 |β_0|^2 z}{4} ζ^{-1}) e^{ζ},
\]

which is uniquely determined by its initial value \(ψ(0, ζ) = 1 - j \frac{π β_0}{2} \zeta^{-1}\). It follows that the space \(H^0(E)\) of global quaternionic holomorphic sections is spanned by the sections \(Φψ\) obtained by evaluating \(ψ\) at each \(ζ = π i (\bar{α} - \bar{β}/2)\) for \((α, \bar{α}) \in S\) (cf. [17, Thm 2.9]).

### 3. The QKP Hierarchy.

#### 3.1. Formal construction of the QKP hierarchy.

The QKP hierarchy is a real form of the two component KP hierarchy (and is referred to elsewhere, for example [29, 31], as the Davey-Stewartson II hierarchy). In this subsection we will see how this hierarchy is constructed by the formal dressing method, working entirely within a quaternionic framework, so that the comparison between KP and QKP is quite literally the replacement of \(C\) by \(H\). I will follow the purely algebraic approach given in two papers by George Wilson [35, 36].

We begin by fixing a real differential algebra \(B\), with derivation \(\partial_y\), of the form

\[
B = \mathbb{R}[u^{(k)}_{αβ}], \quad \text{for} \quad α, k = 0, 1, 2, \ldots, \beta = 1, 2, 3, 4.
\]

These generators are algebraically free but related under the derivation by

\[
\partial_y u^{(k)}_{αβ} = u^{(k+1)}_{αβ}.
\]

With this we construct a formal pseudo-differential operator with coefficients in \(B \otimes H\)

\[
L = i \partial_y + U_0 + U_1 \partial_y^{-1} + \ldots
\]

where

\[
U_0 = j(u_{03} + i u_{04})
\]

\[
U_α = u_{α1} + i u_{α2} + j(u_{α3} + i u_{α4})
\]

The reader should note that when working with the algebra of such operators we use the identities

\[
\partial_y a = a \partial_y + a', \quad \partial_y^{-1} a = \sum_{m=0}^{∞} (-1)^m a^{(m)} \partial_y^{-m-1},
\]
where \( a^{(m)} \) denotes the \( m \)-th derivative: the second identity follows from the first. In this way the inverse of \( \partial_y \) preserves the algebra of pseudo-differential operators over \( \mathcal{B} \).

It is important for this construction that the leading coefficient \( i \) of \( L \) is regular for the Lie algebra structure on \( \mathbb{H} \) (i.e. the commutator of \( i \) has minimal dimension 2). As a consequence, we ignore the component of \( U_0 \) which commutes with \( i \).

The aim is to construct an infinite family of independent derivations on \( \mathcal{B} \) each of which commutes with \( \partial_y \). One approach is to use the formal dressing construction to produce a subalgebra of the commutative algebra \( Z(L) \) of all pseudo-differential operators (over \( \mathcal{B} \otimes \mathbb{H} \)) which commute with \( L \). The basis of this method lies in the following theorem.

**Theorem 3.1** ([35]). There exists a formal operator of the form

\[
K = 1 + \sum_{k \geq 1} a_k (i\partial_y^{-1})^k
\]

such that \( K^{-1}LK = i\partial_y \). Moreover, \( K \) is unique up to right multiplication by operators of the form \( 1 + \sum_{k \geq 1} c_k \partial_y^{-k} \) where each \( c_k \) is a complex constant.

The components of the coefficients \( a_k \) do not belong to \( \mathcal{B} \) but generate an extension algebra \( \hat{\mathcal{B}} \).

By “dressing” we mean the homomorphism of \( \mathbb{R} \)-algebras

\[
\mathbb{C}[\partial_y] \to Z(L); \ P_0 \mapsto P = KP_0K^{-1}.
\]

A crucial point here, and one of the main results in [35], is that the coefficients of \( P \) all lie in \( \mathcal{B} \otimes \mathbb{H} \). We will denote the image of this homomorphism by \( Z_0(L) \).

By splitting \( P \) into \( P_+ + P_- \), where \( P_+ \) is the differential operator part of \( P \), we obtain a derivation \( \partial_P \) on \( \mathcal{B} \), characterised by

\[
\partial_P L = [L, P_+].
\]

This is extended to all of \( \mathcal{B} \) by insisting that it commute with \( \partial_y \). Right now we single out one of these for special attention: the derivation \( \partial_L \) will be renamed \( \partial_x \). These derivations also extend to \( \hat{\mathcal{B}} \) via

\[
\partial_P K = P_- K.
\]

The second principal result of [35] is that all these derivations commute. Further, we deduce from [35, p137] the following equations relating the derivations.

**Theorem 3.2.** For every \( P, Q \in Z_0(L) \) we have the equation

\[
\partial_P Q_+ = \partial_Q P_+ + [Q_+, P_+].
\]

In particular, this includes a family of p.d.e of the form

\[
\partial_P L_+ = \partial_x P_+ + [L_+, P_+].
\]

Historically these are the equations referred to as the hierarchy, since these are the equations which occur in practical applications to physics and fluid dynamics.
(for example, the $t_2$ flow yields the Davey-Stewartson II equations). But this is misleading: the coefficients of $P_\pm$ cannot in general be expressed as differential polynomials in the coefficients of $L_\pm$ and therefore the equations (13) alone do not carry the information contained in the definition (11). Therefore we follow Sato’s nomenclature, also used in [26], and call the system of equations (11) the QKP hierarchy.

Dressing permits us to introduce the formal Baker function. In the current algebraic setting it is a formal power series

$$\psi = (1 + a_1 \zeta^{-1} + \ldots) \exp(x \zeta + iy \zeta) = K \exp(x \zeta + iy \zeta)$$

where $\zeta$ is a formal parameter (to make sense of this we can take an appropriate extension of the differential algebra $\hat{B}$). From the definition of $L$ it follows that $L \psi = -\psi$. Therefore

$$\partial_x \psi + L_+ \psi = (\partial_x K e^{(x+iy)\zeta} + Ke^{(x+iy)\zeta} \zeta + (L - L_-) \psi = L_- \psi + \psi \zeta - \psi \zeta - L_\psi = 0$$

Now we can extend the definition of $\psi$ to

$$\psi = K \exp\left[ \sum_{k \in \mathbb{N}} (s_k + it_k) \zeta^k \right]$$

for real variables $s_k, t_k$. It is not hard to see that for every $P_0 \in \mathbb{C}[\partial_y]$ we can find a sequence $t = (x, y, s_2, t_2, \ldots)$ for which, with the relabelling of $\partial_P$ as $\partial_t$, we have

$$\partial_t \psi + P_+ \psi = 0$$

3.2. The Grassmannian class of solutions. Following Segal & Wilson [26] we can construct an infinite dimensional Grassmannian $Gr(\mathbb{H})$ whose points essentially parameterise the set of all formal Baker functions which actually converge for $|\zeta| = 1$. It is well-known that for the two component KP hierarchy this class of solutions can be studied using a Grassmannian $Gr(\mathbb{C}^2)$ of subspaces of the Hilbert space $H = L^2(S^1, \mathbb{C}^2)$. Indeed, the equations “linearise” on this space: they correspond to the flows of an abelian group action. Since the QKP hierarchy is a real form of two component KP, we should obtain it by imposing a reality condition: this can be achieved by fixing a left $\mathbb{H}$ action on $H$ for which $j$ acts conjugately linearly.

The quickest way to do this is to realise $H$ as $L^2(S^1, \mathbb{H})$. We view $\mathbb{H}$ as having two complex structures: the first arises from left multiplication by $i \in \mathbb{H}$, the second comes from right multiplication by $i$. These structures are inherited by any space of $\mathbb{H}$-valued functions. We view $H$ as a complex vector space with respect to the first complex structure. Then it has the Hermitian inner product

$$<f, g> = \int_{S^1} (f \bar{g}) \zeta$$
where \( \bar{g} \) is the quaternionic conjugate, and if \( q = a + bj \in \mathbb{H} \) for \( a, b \in \mathbb{C} \) then \( q_{\mathbb{C}} = a \). The integral is normalised in the usual way so that \( <1,1> = 1 \). To make calculations for differential operators acting on the left, we represent each element of \( H \) in its left Fourier series

\[
f(\zeta) = \sum_{m \in \mathbb{Z}} f_m \zeta^m, \quad |\zeta| = 1, \quad f_m \in \mathbb{H}.
\]

The identification with \( L^2(S^1, \mathbb{C}^2) \) uses the complete complex orthonormal system for \( L^2(S^1, \mathbb{H}) \) given by \( \{ \zeta^m, j\zeta^m | m \in \mathbb{Z} \} \), so that

\[
(u, v) \mapsto (u + j\bar{v}),
\]

where \( \bar{v}(\zeta) = v(\bar{\zeta}) \). Both left \( \mathbb{H} \) multiplication and right complex multiplication preserve the polarization of \( H \) into orthogonal subspaces \( H^+ \) and \( H^- \) which consist of functions whose left Fourier series have, respectively, only non-negative and only negative powers of \( \zeta \).

Now define \( G(\mathbb{C}^2) \) to be the space of all complex subspaces \( W \subset H \) (i.e., \( iW = W \)) for which the projections \( pr_+ : W \to H^+ \) and \( pr_- : W \to H^- \) are respectively Fredholm (of index zero) and Hilbert-Schmidt. Then \( G(\mathbb{C}^2) \) is a complex Hilbert manifold modelled on the space of Hilbert-Schmidt operators from \( H^+ \) to \( H^- \) with the Hilbert-Schmidt inner product. Inside this we have the real submanifold

\[
G(\mathbb{H}) = \{ W \in G(\mathbb{C}^2) | jW = W \},
\]

the fixed point subspace of the real involution \( W \mapsto jW \). One knows (from [23] or [26]) that \( G(\mathbb{C}^2) \) is a homogeneous space for an infinite dimensional Lie group \( GL_{res} \) of certain complex endomorphisms of \( H \), so we can view \( G(\mathbb{H}) \) as a homogeneous space for the real subgroup \( GL_{res}(\mathbb{H}) \) of left \( \mathbb{H} \)-linear endomorphisms. As with the KP equations, we can produce an abelian subgroup whose flows correspond to the QKP equations.

Define \( \Gamma \subset C^\omega(S^1, \mathbb{C}^* \) to be the abelian subgroup of all non-vanishing analytic functions with winding number zero, and let \( \mathcal{G} \) denote its Lie algebra \( C^\omega(S^1, \mathbb{C}) \). \( \Gamma \) acts \( \mathbb{H} \)-linearly on \( H \) by

\[
\gamma : H \to H; \quad f \mapsto f\gamma,
\]

Thus we have a representation \( \Gamma \subset GL_{res}(\mathbb{H}) \), hence \( \Gamma \) acts on \( G(\mathbb{H}) \) as a real group: to emphasize the definition we will write this action as \( \gamma \circ W = W\gamma \). The group \( \Gamma \) factorises into the product \( \Gamma_- \Gamma_+ \) of two subgroups: \( \Gamma_+ \), whose elements extend holomorphically into the disc \( |\zeta| < 1 \) and are unimodular at \( \zeta = 0 \), and \( \Gamma_- \), whose elements extend holomorphically into the disc \( |\zeta| > 1 \) and take a real
positive value at $\zeta = \infty$. We write $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ for the corresponding Lie algebra splitting. We will parameterise elements of $\Gamma_+$ by writing each in the form

$$\gamma(t) = \exp[it_0 + \sum_{k \geq 1} (s_k + it_k)\zeta^k], \quad t = (s_1, t_1, \ldots) \in \mathcal{G}_+.$$  

We will tend to use $x, y$ instead of $s_1, t_1$ below, and write $z = x + iy$.

In the same manner as [26], we wish to assign to $W \in Gr(\mathbb{H})$ a convergent Baker function and thereby obtain solutions to the QKP equations. Such an assignment only works when the $\Gamma_+$-orbit of $W$ meets the big cell: this is the open dense subset of $Gr(\mathbb{C}^2)$ consisting of all $W \in Gr(\mathbb{C}^2)$ for which $pr_+: W \to H_+$ is invertible. The following result shows that, like the KP case, this condition is always satisfied.

**Theorem 3.3.** Let $W \in Gr(\mathbb{H})$ and define

$$\Gamma_1 = \{\exp((x + iy)\zeta) : x, y \in \mathbb{R}\} \simeq \mathbb{R}^2.$$  

Then the $\Gamma_1$-orbit of $W$ meets the big cell off a real analytic (proper) subvariety of $\mathbb{R}^2$. Consequently the $\Gamma_+$-orbit of $W$ meets the big cell on the complement of a real analytic (proper) subvariety of $\Gamma_+$.

The proof is in appendix [A]. As a consequence, to every $W \in Gr(\mathbb{H})$ we can assign a convergent Baker function $\psi_W$ as follows. Since $\Gamma_+$-orbits meet the big cell on open sets there is a function, defined for almost all $t \in \mathcal{G}_+$, taking values in $W$: it is characterised by the property that

$$pr_+(\psi_W(t)\gamma(t)^{-1}) = 1.$$  

Moreover, $\psi_W$ uniquely determines $W$, since the set $\psi_W(0), \psi_W'(0), \psi_W''(0), \ldots$ of all $y$ derivatives spans an $\mathbb{H}$-subspace of $H$ whose closure is $W$.

From now on we will set $z = x + iy$ and, for notational convenience, we will use, for example, $\gamma(t_0)$ to denote setting every parameter except $t_0$ equal to zero.

**Theorem 3.4.** To every $W \in Gr(\mathbb{H})$ we can assign a formal pseudo-differential operator $L_W$ of the form (10) satisfying $L_W\psi_W = -\psi_W\zeta$. Consequently, for $t = s_k$ or $t = t_k$, $k \in \mathbb{N}$ there exists a differential operator $P_+$ for which

$$\partial\psi_W/\partial t + P_+\psi_W = 0,$$  

hence $L_t = [P_+, L]$. In particular, for each $W$ we obtain a Dirac operator $D = \partial/\partial \bar{z} + U_W$ for which $U_W = -(a_1 + ia_1i)/2$, $D\psi_W = 0$ and the equations for $\partial U_W/\partial t$ are (13).

---

4This slightly unusual normalisation for $\Gamma_-$ and $\Gamma_+$ makes our discussion of the $\Gamma$-orbits easier later on, since the real scaling action is trivial but the action of the unimodular scaling plays an important role.
The proof is identical to that for the KP hierarchy given in [26]. In particular, from $\psi_W$ we extract

$$\tilde{\psi}_W(t) = 1 + \sum_{k>0} a_k(t)\zeta^{-k},$$

from which we obtain $L_W = K_W i\partial_y K_W^{-1}$ using

$$K_W = 1 + \sum_{k>0} a_k(t)(i\partial_y)^k.$$

**Remark 3.5.** Let us use $Gr(C)$ to denote the Segal-Wilson Grassmannian for $L^2(S^1, \mathbb{C})$. We can embed this in $Gr(H)$ using $V \mapsto V \oplus \bar{V}$, where the latter is the space \(\{(u, \bar{v}) : u, v \in V\} \subset Gr(C^2)\). Points of $Gr(H)$ of this type yield solutions to the (complex scalar) KP hierarchy of equations, since it is clear that the complex Baker function $\psi_V$ which Segal & Wilson assign to $V \in Gr(C)$ is also our quaternionic Baker function $\psi_W$, for $W = V \oplus \bar{V}$. Therefore all calculations reduce to those of [26]. We will use $Gr_{KP}$ to denote the image of $Gr(C)$ in $Gr(H)$ and denote its complement by $Gr_{QKP}$. We can characterise the points of $Gr_{KP}$ as follows.

**Lemma 3.6.** $W \in Gr_{KP}$ if and only if $Wi = W$, equally, if and only if the Dirac potential $U_W$ is trivial.

**Proof.** If $W \in Gr_{KP}$ then clearly we have both $Wi = W$ and $U_W = 0$. Now if $Wi = W$ then $W = V + jV$ where

$$V = \left\{ \frac{1}{2}(f - ifi) : f \in W \right\}.$$

In particular, $\frac{1}{2}(\psi_W - i\psi_W i)$ belongs to $V$ and by uniqueness of the Fourier expansion for Baker functions must equal $\psi_W$, hence $\psi_W$ takes values in $V$. This means $V \in Gr(C)$, since derivatives of $\psi_W$ generate $V$ over $\mathbb{C}$. Thus $W \in Gr_{KP}$.

Now suppose $U_W = 0$ and set $\partial = \partial/\partial z$. Then $\partial\psi_W = 0$ and so if we write $\psi_W = \psi_1 + j\psi_2$, where both $\psi_1, \psi_2$ commute with $i$, we have $\partial\psi_1 = 0$ and $\partial\psi_2 = 0$. We notice that, restricting $\psi_W$ to $x, y$,

$$\psi_1 = (1 + p_1\zeta^{-1} + \ldots) \exp(z\zeta), \quad \psi_2 = (q_1\zeta^{-1} + \ldots) \exp(z\zeta);$$

for some complex valued functions $p_1, q_1, \ldots$. Now $\partial\psi_2 = 0$ means

$$(q_1 + (\partial q_1 + q_2)\zeta^{-1} + \ldots + (\partial q_k + q_{k+1})\zeta^{-k} + \ldots)e^{z\zeta} = 0.$$

So $\partial\psi_2 = 0$ if and only if $q_k = 0$ for all $k$, i.e., $\psi_2 = 0$. Therefore $\psi_W = \psi_1$ and since $\partial\psi_1 = 0$ the closure $V$ of the complex subspace of $W$ generated by $\psi_W(0), \partial\psi_W(0), \ldots$ belongs to $Gr(C)$. But $V$ generates $W$ over $\mathbb{H}$ and therefore $W = V + jV$, which is $V \oplus \bar{V}$ in our notation. □
3.3. QKP flows. An important consequence of the construction is that the action of $\Gamma_+$ generates the QKP flows, in the following very precise sense. Let $t, t' \in \mathcal{G}_+$. Then, by comparing Fourier series, we deduce that

$$
\psi_{W\gamma(t')}(t) = \psi_W(t - t')\gamma(t'),
$$

and therefore, treating $t$ as variable and $t'$ as a constant,

$$
\tilde{\psi}_{W\gamma(t')}(t) = \tilde{\psi}_W(t - t'),
$$

whence

$$
L_{W\gamma(t')}(t) = L_W(t - t').
$$

Now, if our interest is purely in the QKP operator $L_W$ then, as with the KP hierarchy \cite{26}, we observe that for any $\gamma \in \Gamma_-$ we have $\psi_{W\gamma} = \psi_W\gamma$, hence $L_{W\gamma} = L_W$. In fact we have the following result.

**Theorem 3.7.** The quotient space $\mathcal{M} = \text{Gr}(\mathbb{H})/\Gamma_-$ is a manifold, and the map

$$
\mathcal{M} \rightarrow \{L_W : W \in \text{Gr}(\mathbb{H})\}; \quad \Gamma_- \circ W \mapsto L_W
$$

is bijective.

**Proof.** It is easy to extrapolate from the proof of the analogous result in Segal & Wilson \cite[2.4]{26} that the subgroup of elements of $\Gamma_-$ of the form $1 + O(\zeta^{-1})$ acts freely on $\text{Gr}(\mathbb{H})$, while the constant scalars act trivially, hence $\mathcal{M}$ is a manifold. By theorem 3.1 $L_W$ uniquely determines $\psi_W$ up to right multiplication by an element of $\Gamma_-$, and $\psi_W$ determines $W$, so the map is bijective. $\square$

Since $\Gamma$ is abelian the group $\Gamma_+$ acts on $\mathcal{M}$ and it is clear from (17) that the orbits correspond to the flows of the QKP hierarchy. One particular subgroup of $\Gamma_+$ acts in a significant way. Let $\Gamma_0 = \{\exp(it_0) : t_0 \in \mathbb{R}\} \subset \Gamma_+$ be the subgroup of unimodular constants, so $\Gamma_0 \simeq S^1$. In the next result we will see that the orbits of $\Gamma_0$ are either points or circles depending upon whether or not $W \in \text{Gr}_{\text{KP}}$. Since the disjoint submanifolds $\text{Gr}_{\text{KP}}$ and $\text{Gr}_{\text{QKP}}$ are both $\Gamma$ invariant the phase space $\mathcal{M}$ can be written as a disjoint union $\mathcal{M}_{\text{KP}} \cup \mathcal{M}_{\text{QKP}}$ of quotient manifolds under the action of $\Gamma_-$. 

For ease of notation, for any $W \in \text{Gr}(\mathbb{H})$, set $W(t_0, z) = We^{it_0+z\zeta}$, with the usual convention that the absence of either variable denotes that it is zero.

**Theorem 3.8.**

(a) For any $W \in \text{Gr}(\mathbb{H})$

$$
L_{W(t_0)} = e^{-it_0}L_W e^{it_0}.
$$

(b) The group $\Gamma_0$ acts trivially on $\mathcal{M}_{\text{KP}}$, while the space $\mathcal{M}_{\text{QKP}}/\Gamma_0$ of $\Gamma_0$-orbits in $\mathcal{M}_{\text{QKP}}$ is the quotient of $\mathcal{M}_{\text{QKP}}$ by a free action of $S^1$. 

(c) Let $\Lambda \subset \mathbb{C}$ be a lattice, then $L_W(z + \lambda) = \mu(\lambda)^{-1} L_W(z) \mu(\lambda)$ for some $\mu \in \text{Hom}(\Lambda, S^1)$ if and only if the map

$$\mathbb{C} \to \mathcal{M}/\Gamma_0; \ z \mapsto [L_W(z)]$$

is $\Lambda$-periodic, where $[L_W]$ denotes the $\Gamma_0$-orbit of $L_W \in \mathcal{M}$.

Proof. (a) First we observe that

$$e^{-it_0} \psi_W(z) e^{it_0} = (1 + e^{-it_0} a_1(z) e^{it_0} \zeta^{-1} + \ldots) \exp(z \zeta),$$

and that this takes values in $W(t_0)$. By the uniqueness of the Baker function, we deduce that $\psi_W(t_0) = e^{-it_0} \psi_W(z) e^{it_0}$. The formula (18) follows at once.

(b) The definition of the $\Gamma_+$ action, combined with (18), yields

$$e^{it_0} \circ L_W = L_W(t_0) = e^{-it_0} L_W e^{it_0}.$$ 

This is trivial if $L_W$ is purely complex, which by lemma 3.6 is the case precisely when $W \in \text{Gr}_{\text{KP}}$, so $\Gamma_0$ acts trivially on $\mathcal{M}_{\text{KP}}$. Now observe that for any $q \in \mathbb{H}$, $e^{-ir} \psi_W(z) e^{ir} = q$. Therefore if we define an $S^1$-action on $\mathcal{M}_{\text{KP}}$ by $e^{it_0} L_W = L_W(t_0/2)$ this action is free, since the points in the subspace $\mathcal{M}_{\text{KP}}$ are those for which $\psi_W$ is not purely complex. Clearly the quotient of $\mathcal{M}_{\text{KP}}$ by this action equals $\mathcal{M}_{\text{KP}}/\Gamma_0$.

(c) Suppose $L_W(z + \lambda) = \mu(\lambda)^{-1} L_W(z) \mu(\lambda)$ for some $\mu \in \text{Hom}(\Lambda, S^1)$ and all $z \in \mathbb{C}$. By (17) and (18) this means

$$L_W(z + \lambda)(z') = L_W(z)(z' - \lambda) = L_W(z) \mu(-\lambda)(z')$$

thinking of $\mu(\lambda) \in \Gamma_0$. But the last has the same $\Gamma_0$-orbit as $L_W(z)$, so $[L_W(z + \lambda)] = [L_W(z)]$.

Conversely, suppose that (19) is $\Lambda$-periodic. Let

$$G = \{\exp(it_0 + z \zeta) \in \Gamma\} = \Gamma_0 \Gamma_1,$$ 

and let $S \subset \mathcal{M}$ be the $G$-orbit of $L_W$. The projection $\mathcal{M} \to \mathcal{M}/\Gamma_0$ makes $S$ a bundle over the $\Gamma_1$-orbit $M \simeq \mathbb{C}/\Lambda$ of $[L_W]$. When $L_W \in \mathcal{M}_{\text{KP}}$ this projection is a bijection, so that $L_W(z)$ is $\Lambda$-periodic and $\mu$ is trivial. Otherwise $S \to M$ is an $S^1$-bundle, by (b), with a natural flat connexion for which the action of $\Gamma_1 \subset G$ is horizontal. This makes $z \mapsto L_W(z)$ a flat section over the universal cover $\mathbb{C}$ of $M$ and hence it has a monodromy $\mu' \in \text{Hom}(\Lambda, S^1)$. The action of $S^1$ is via $\Gamma_0$ and therefore

$$L_W(z + \lambda) = \mu'(\lambda)^{-1} L_W(z) \mu'(\lambda),$$

which implies

$$L_W(z' - \lambda) = \mu'(\lambda)^{-1} L_W(z') \mu'(\lambda).$$

Taking $\mu = (\mu')^{-1}$ gives the required result. \qed
3.4. **Solutions of finite type.** By adapting the construction in \[26\] we can construct many points \( W \in Gr(H) \) corresponding to spectral data and thereby construct Baker functions (using, for example, Riemann \( \theta \)-functions). Our spectral data will be a collection \((X, \rho, P, \zeta, L, \varphi)\) of the following.

(a) \( X \) is a complete, reduced, algebraic curve of arithmetic genus \( g \), with fixed-point free anti-holomorphic involution \( \rho \). \( X \) need not be irreducible, but if it is reducible it must have no more than two irreducible components and these must be swapped by \( \rho \).

(b) \( P \in X \) is a smooth point and \( \zeta^{-1} \) is a local parameter about \( P \).

(c) \( L \) is a holomorphic line bundle of degree \( g + 1 \) for which \( \rho^* L \cong L \) and \( L(-P - \rho P) \) is non-special. This induces a unique, up to sign, conjugate linear isomorphism \( \tilde{\rho}^* : L \to L \) covering \( \rho \) and satisfying \( (\tilde{\rho}^*)^2 = -1 \) (so \( \tilde{\rho}^* \) is a quaternionic involution).

(d) \( \varphi \) is a holomorphic trivialising section of \( L \) over the disc \( \Delta_P = \{ Q : |\zeta(Q)^{-1}| < 1 \} \) and its boundary circle \( C_P \), both of which we assume contain no singular points of \( X \).

Note very well that we do not insist that \( X \) be smooth (a great many interesting solutions would be ignored if we did so). In general we let \( X^{sm} \) denote the open variety of smooth points on \( X \).

**Remark 3.9.** It is possible for \( X \) to be disconnected but we will see below (in example 3.13) that this will only lead to points in \( Gr_{KP} \). When \( X \) is singular the condition (c) is more strict than necessary. As is the case in \[26\] p38, we can allow \( L \) to be a maximal torsion free coherent sheaf of rank 1 with \( \chi(L) = 2 \), where maximal means that \( L \) is not the direct image of a sheaf over some less singular curve, and therefore the \( \text{Jac}(X) \)-orbit of \( L \) in the Mumford compactification of the Picard group \( \text{Pic}(X) \) is isomorphic to \( \text{Jac}(X) \), i.e., \( \text{Jac}(X) \) acts freely on this orbit.

From this data we construct a point \( W \in Gr(H) \): the points constructed this way will be called of finite type (this terminology is justified by theorem 3.15 below). To do this, first identify \( S^1 \) with the circle \( C_P \) (and also with \( \rho C_P \) by the map \( \zeta \mapsto \rho^* \zeta \)). Let \( X_0 \) denote the closed non-compact surface \( X \setminus (\Delta_P \cup \rho(\Delta_P)) \). We define

\[
(20)\quad w : H^0(X_0, L) \to H; \quad \sigma \mapsto (\sigma - j\tilde{\rho}^* \sigma)/\varphi.
\]

This is clearly \( \mathbb{C} \)-linear. Now define \( W \) to be the closure of the image of \( w \). A simple generalisation of the Mayer-Vietoris argument in \[26\] \S 6 shows that \( W \in Gr(\mathbb{C}^2) \). Observe that since \( \tilde{\rho}^* \) is quaternionic on \( H^0(X_0, L) \) we have

\[
(jw(\sigma) = w(\tilde{\rho}^* \sigma)
\]

\( ^5 \)Since it has degree \( g - 1 \) this is equivalent to saying it possesses no non-trivial globally holomorphic sections.
and therefore $W \in Gr(\mathbb{H})$. The non-speciality condition on $L(-P - \rho P)$ ensures that $W$ lies in the big cell. This is not essential, as remarked above, but we may as well assume this, since under the QKP flow we can always satisfy this condition.

From the discussion above, we are most interested in the $\Gamma$-orbits in $Gr(\mathbb{H})$. The effect of this group action on the spectral data is easily calculated. Clearly
\[ w(\sigma)\gamma^{-1} = (\sigma - j\rho^*\sigma)/(\varphi\gamma). \]
This twists the trivialisation $\varphi$ into $\varphi\gamma$, which we interpret as trivialisation for the line bundle $L \otimes \ell(\gamma)$. Here $L(\gamma)$ is a degree zero line bundle obtained by gluing the trivial bundles over $X_0$ and $clos(\Delta P)$ together using $\gamma$ as a transition function, and gluuing that to the trivial bundle over $clos(\rho(\Delta P))$ using $\gamma$ (after identifying $\zeta$ with $\rho^*\zeta$). Since we will need to work with these trivialisations later on we will be more precise about this by introducing better notation (and then suppress this in most of what follows until we require it).

The construction of $\ell(\gamma)$ equips it with trivialisations $\tau_P(\gamma)$ and $\tau_0(\gamma)$, over $\Delta_P \cup \rho\Delta_P$ and $X \setminus \{P, \rho P\}$ respectively, satisfying the transition relation
\[ \tau_P(\gamma) = \gamma\tau_0(\gamma) \text{ over } \Delta_P \setminus \{P\}, \]
and the $\rho^*$-conjugate of this about $\rho P$. So by "$\varphi\gamma$" we really mean the trivialisation $\varphi\tau_P(\gamma)$. Using this we obtain
\[ H^0(X_0, L \otimes \ell(\gamma)) \to C^\omega(S^1, \mathbb{C}); \quad \sigma \mapsto \sigma/\varphi\tau_P(\gamma). \]
Now we observe that
\[ \frac{\sigma}{\varphi\tau_P(\gamma)} = \frac{\sigma\tau_0(\gamma)^{-1}}{\varphi}\gamma^{-1}, \]
and $\sigma\tau_0(\gamma)^{-1} \in H^0(X_0, L)$, hence $W\gamma^{-1}$ corresponds to replacing $L$ by $L \otimes \ell(\gamma)$ and $\varphi$ by $\varphi\tau_P(\gamma)$. We have therefore proved the following lemma.

**Lemma 3.10.** If $W$ corresponds to the spectral data $(X, \rho, P, \zeta, L, \varphi)$ and $\gamma \in \Gamma$ then $W\gamma^{-1}$ corresponds to the same data with $L$ and $\varphi$ replaced by $L \otimes \ell(\gamma)$ and $\varphi\gamma$.

Notice that when $\gamma \in \Gamma_-$ the two line bundles are isomorphic and only the trivialisation changes.

A consequence of this construction is the epimorphism of real groups $\ell : \Gamma \to J_{\mathbb{R}}(X)$, where the target here is the connected component of the identity of the real subgroup
\[ \{L \in \text{Jac}(X) : \rho^*L \simeq L\} \]
of the Jacobi variety of $X$. The restriction of $\ell$ to $\Gamma_+$ is still onto.

We can also characterise the Baker function in terms of the spectral data. Since $L(-P - \rho P) \otimes \ell(\gamma)$ is almost always non-special (by theorem 3.3) for almost all $\gamma \in \Gamma$ there is a unique $\sigma_\gamma \in H^0(X, L(-\rho P) \otimes \ell(\gamma))$ for which $\sigma_\gamma/(\varphi\gamma)$ has value
1 at $P$: then $\sigma/(\bar{\rho}^* \varphi \bar{\gamma})$ is locally holomorphic about $\rho P$ with a simple zero there. It follows from these properties that $\psi_W(\gamma) = w(\sigma_\gamma)$.

**Remark 3.11.** This construction is at its most concrete in the case where the trivialising section $\varphi$ extends to a globally holomorphic section on the whole of $X$. Such a choice is always possible for our spectral data (for example, using a non-zero section vanishing at $\rho P$) and we can choose $\Delta_P$ so that it contains no zeroes of $\varphi$. Then $\psi_1 = \sigma_\gamma/\varphi$ and $\psi_2 = \bar{\rho}^* \sigma_\gamma/\varphi$ are both meromorphic functions on $X \setminus \{P, \rho P\}$. It follows that $\psi_W$ extends to $X \setminus \{P, \rho P\}$ and we have

$$\psi_W : \Gamma_+ \times X \setminus \{P, \rho P\} \mapsto \mathbb{H}; \, \psi_W = \psi_1 - j \psi_2.$$  

(23)

Notice that these conditions on $\varphi$ do not uniquely specify it. For if $D$ denotes its divisor of zeroes we are free to multiply $\varphi$ by a rational function on $X$ which does not vanish at $P$ and whose divisor of poles lies in the linear system $|D|$. Thus we cannot talk of a unique global Baker function, but any two differ by such a rational function. Nevertheless, in what follows we will use the phrase “the global Baker function” to refer to any one of these.

**Remark 3.12.** Every other trivialisation $\varphi'$ of $L$ over $\Delta_P$ is of the form $s\varphi$, where $s$ is a non-vanishing holomorphic function on $\Delta_P$. This can be factorised into $s = \gamma' e^{it_0}$ where $\gamma' \in \Gamma_-$. By previous observations we conclude that this change of trivialisation has the effect

$$\psi_W \mapsto e^{-it_0} \psi_W e^{it_0} \gamma'.$$

It follows that the quotient map $Gr(\mathbb{H}) \to \mathcal{M}$ amounts to discarding from the spectral data almost all the information given by the choice of trivialisation: the remainder being the identification it gives between fibres of $L$ over $P$ and $\rho P$. The further quotient $\mathcal{M} \to \mathcal{M}/\Gamma_0$ discards even this information. In particular, when dealing with QKP solutions we may as well work with global Baker functions.

**Example 3.13.** Consider taking $X = Y \cup \bar{Y}$, a disjoint union of two copies of the same irreducible complete algebraic curve, with $\bar{Y}$ equipped with the opposite complex structure to $Y$. Then $\rho : (P_1, P_2) \to (P_2, P_1)$ is a fixed point free anti-holomorphic involution. It is not hard to see that if we equip $Y$ with the spectral data described in [26], i.e., fix $P \in Y$ with local parameter $\zeta^{-1}$, a line bundle $\mathcal{L}_Y$ over $Y$ of degree equal to the genus of $Y$ and a local trivialisation $\varphi$ about $P$, then we automatically obtain our quaternionic spectral data, with $\mathcal{L}$ given by $\mathcal{L}|Y = \mathcal{L}_Y$ and $\mathcal{L}|\bar{Y} = \rho^* \mathcal{L}_Y$. It follows that the point $W \in Gr(\mathbb{H})$ we obtain has the form $W = V \oplus \bar{V}$, where $V$ is the point in the Grassmannian $Gr(\mathbb{C})$ built from the data on $Y$. Therefore, by remark [33] spectral curves of this type only result in solutions to the KP hierarchy.

Now let us consider the $\Gamma_+$-orbits in $\mathcal{M}_{QKP}$ and $\mathcal{M}_{QKP}/\Gamma_0$. Let $X_p$ be the singular curve obtained by identifying the two points $P$ and $\rho(P)$ together to
form an ordinary double point and define $J_{R}(X_{p})$ to be the connected component of the identity of the real group
\[ \{ L' \in \text{Jac}(X_{p}) | \bar{\rho}^* L' \simeq L' \}. \]

**Lemma 3.14.** $J_{R}(X_{p})$ is an $S^1$-bundle over $J_{R}(X)$.

**Proof.** Think of $L'$ as $L \in \text{Jac}(X)$ equipped with an isomorphism between the fibres $L_{\rho}$ and $L_{\rho^{P}}$. An isomorphism between $\bar{\rho}^* L'$ and $L'$ is a conjugate linear isomorphism of $\rho^* L$ with $L$ which identifies the fibre isomorphisms. Therefore, given a fixed isomorphism $\bar{\rho}^* L \simeq L$ we can only vary the fibre identification by a unimodular scaling, hence the fibres of $J_{R}(X_{p})$ are free $S^1$-orbits. \qed

As a corollary to remark 3.12 and the previous lemma we see how the real groups $J_{R}(X_{p})$ and $J_{R}(X)$ sit geometrically with respect to the QKP phase space $\mathcal{M}_{\text{QKP}}$. The equivalent theorem for the KP hierarchy is well-known and can be deduced from [20].

**Theorem 3.15.** Suppose $W \in Gr_{\text{QKP}}$ is of finite type, then the $\Gamma_+$-orbit of $L_{W} \in \mathcal{M}_{\text{QKP}}$ can be identified with the real group $J_{R}(X_{p})$, while the $\Gamma_+$-orbit of $[L_{W}]$ in $\mathcal{M}_{\text{QKP}}/\Gamma_0$ can be identified with $J_{R}(X)$. Moreover, $L_{W}$ is of finite type if and only if it admits only finitely many independent non-stationary QKP flows.

The last claim in this theorem is explained in [6.3] where we briefly discuss the reconstruction of the spectral curve from the QKP operator $L_{W}$ via its ring of commuting differential operators.

**Remark 3.16.** It is a natural question to ask whether there are any $W \in Gr_{\text{QKP}}$ whose $\Gamma_+$-orbit lies entirely in the big cell. We can deduce the existence of such points by the chain of reasoning which connects certain QKP solutions with harmonic tori in $S^3$ and CMC tori in $R^3$, but there is a more easily accessible collection of examples available. Namely, I claim that when the spectral curve is smooth and has genus $g \leq 2$ we can obtain QKP solutions which are globally defined (for $g = 1$ one must avoid certain choices of $\mathcal{L}$). This is quite easy to see.

We wish is to find $\mathcal{L}$ of degree $g + 1$ for which $\mathcal{L}(-P - \rho P) \otimes L$ is non-special for all $L \in J_{R}(X)$. This is trivial for $g = 0$. More generally, this $J_{R}(X)$-orbit lies in the real slice Pic$_{g-1}(X)^{\rho}$ of Pic$_{g-1}(X)$. For $g = 1$ the orbit must avoid the unique special line bundle, namely the trivial bundle. Since Pic$_{g-1}(X)^{\rho}$ has two connected components for $g = 1$ we can take any $\mathcal{L}(-P - \rho P)$ to be any point lying on the component not containing the trivial bundle. For $g = 2$ the special line bundles of degree $g - 1$ lie in the image of the map $X \rightarrow$ Pic$_{g-1}(X)$ which sends $Q$ to $O_{X}(Q)$. This image cannot intersect the real slice Pic$_{g-1}(X)^{\rho}$, for if $O_{X}(Q) \simeq O_{X}(\rho Q)$ then $Q = \rho Q$, but $\rho$ has no fixed points. Hence for $g = 2$ any choice of real line bundle $\mathcal{L}$ of degree 3 provides global solutions to the QKP hierarchy.

Let us finish this section by discussing the $\Gamma_1$ orbits. Let us identify $\Gamma_1$ with $C$ by $e^{i \xi} \mapsto z$. We wish to characterise the homomorphism $\ell : C \rightarrow J_{R}(X)$. This
is straightforward: \( \ell(z) \) is obtained from the trivial bundle by twisting it by \( e^{z \zeta} \) about \( P \) and \( \rho^* e^{z \zeta} \) about \( \rho(P) \). It follows (cf., for example, [18]) that \( \ell \) is uniquely determined by the property that

\[
(24) \quad \frac{\partial \ell}{\partial z} |_{z=0} = \frac{\partial A_P}{\partial \zeta^{-1}} |_{\zeta^{-1}=0}
\]

where \( A_P : X^\text{sm} \to \text{Jac}(X) \) is the Abel map with base point \( P \) and we interpret this equation by identifying \( T^1,0 \text{Jac}(X) \) with the tangent space \( T^0 \text{Jac}(X) \) of real tangent vectors. As a corollary to this and theorem 3.8 we deduce the following.

**Theorem 3.17.** Suppose \( W \in \text{Gr}_{\text{QKP}} \) arises from spectral data and its \( \Gamma_1 \)-orbit lies entirely in the big cell. Then there is a lattice \( \Lambda \subset \mathbb{C} \) for which \( L^\mu_W(z+\lambda) = \mu(\lambda)^{-1} L^\mu_W(z) \mu(\lambda) \) for some \( \mu \in \text{Hom}(\Lambda, S^1) \) if and only if \( \ell(z) \) is \( \Lambda \)-periodic.

**Remark 3.18.** This monodromy \( \mu \in \text{Hom}(\Lambda, S^1) \) corresponds to a flat \( S^1 \)-bundle \( S \) over \( \mathbb{C}/\Lambda \), the principal \( S^1 \)-bundle for the dual \( L^* \) to our quaternionic holomorphic curve \( L \). But we can also interpret this as follows. Using \( \ell \) we can pull back the \( S^1 \)-bundle \( J^R(X_p) \) to an \( S^1 \)-bundle over \( \mathbb{C}/\Lambda \), which is a Lie group and isomorphic to \( S \). This comes equipped with a flat connexion as follows. By the construction above we have a natural homomorphism of real groups

\[
\Gamma_+ \to J^R(X_p) \to J^R(X).
\]

When this is restricted to \( \Gamma_1 \) it gives \( \ell \) and a lift \( \ell^p : \mathbb{C} \to S \) on its universal cover. This determines a flat connexion on \( S \) with monodromy which is, by theorem 3.8, \( \mu \).

### 4. Construction of quaternionic holomorphic curves.

4.1. **Construction via the Baker function.** In this section we will suppose that \( W \in \text{Gr}(\mathbb{H}) \) has been chosen so that the \( \Gamma_1 \)-orbit of \( [L_W] \) is a torus \( \mathbb{C}/\Lambda \): let \( \mu \in \text{Hom}(\Lambda, S^1) \) be the corresponding monodromy of \( L_W \). In that case there is a quaternionic holomorphic line bundle \((E, J, D)\) over \( \mathbb{C}/\Lambda \) whose Dirac operator \( D \) has Dirac potential \( U_W \).

For any left \( \mathbb{H} \)-linear form \( \alpha \in W^* \) the function \( \psi : \mathbb{C} \to \mathbb{H} ; \psi = \alpha(\psi_W) \) lies in the kernel of \( D \). It corresponds to a section in \( H^0_D(E) \) whenever \( \psi(z+\lambda) = \mu(\lambda)^{-1} \psi(z) \) for all \( z \in \mathbb{C}, \lambda \in \Lambda \). The dimension of \( H^0_D(E) \) will be bounded below by the number of independent \( \alpha \in W^* \) possessing the same monodromy. In particular, with sufficiently many of these we obtain quaternionic holomorphic immersions of \( \mathbb{C}/\Lambda \) in \( \mathbb{H}P^n \).

Equivalently, to obtain a coordinate free perspective, let \( V \subset W \) be an \( \mathbb{H} \)-subspace of \( \mathbb{H} \)-codimension \( n + 1 \), and suppose it has the two properties:

(a) \( \psi_W(z) \) does not belong to \( V \) for any \( z \) (we will say \( V \) is well-positioned in \( W \)),

...
(b) for every $\lambda \in \Lambda$ we have
\[ \psi_W(z + \lambda) - \mu(\lambda)^{-1} \psi_W(z) \in V, \forall z \in \mathbb{C}. \]

Then we may define a map
\begin{equation}
(25) \quad f : \mathbb{C}/\Lambda \to \mathbb{P}(W/V) \simeq \mathbb{HP}^n; \quad z + \Lambda \mapsto [\psi_W(z) + V],
\end{equation}
where the square brackets denote the corresponding left $\mathbb{H}$-line in $W/V$.

**Theorem 4.1.** The map $f : \mathbb{C}/\Lambda \to \mathbb{HP}^n$ given by (25) is a quaternionic holomorphic curve.

The proof is just an instance of the general principle linking quaternionic holomorphic curves to quaternionic holomorphic sections of the dual bundle, but we will give it here since it makes explicit the complex structure $J$ on $L$.

**Proof.** Over $\mathbb{C}$ the map $f$ has a global lift $\tilde{f} = \psi_W + V$: we can think of this as a section of the pullback to $\mathbb{C}$ of the line bundle $L$ corresponding to $f$. This has complex structure $J$ for which $J\tilde{f} = i\tilde{f}$. Now $\star \delta f = \delta f \circ J : \tilde{f} \mapsto -2id\bar{z} \frac{\partial \tilde{f}}{\partial \bar{z}} \mod L$.

But since $\partial\psi_W/\partial\bar{z} = -U_W\psi_W$ this is identically zero, whence $f$ is a quaternionic holomorphic curve. $\square$

**Remark 4.2.** For every $\gamma \in \Gamma_-$ we know that $\psi_{W\gamma} = \psi_W\gamma$. Therefore the quaternionic holomorphic curve determined by $(W, V)$ is congruent (i.e., equivalent up to the right action of $\mathbb{P}GL(n+1, \mathbb{H})$ on $\mathbb{HP}^n$, created by the choice of isomorphism $W/V \simeq \mathbb{H}^{n+1}$) to the one determined by $(W\gamma, V\gamma)$ when $\gamma \in \Gamma_-$. Similarly, since $\psi_{We^{it_0}} = e^{-it_0}\psi_{We^{it_0}}$ the curve (25) is unchanged, up to congruence of $\mathbb{HP}^n$, by the action of $\Gamma_0$.

To study the periodicity conditions let us first consider the more general map
\begin{equation}
(26) \quad f : \Gamma \to \mathbb{P}(W/V); \quad \gamma \mapsto [\psi_W(\gamma) + V].
\end{equation}
Here we have extended $\psi_W$ to a function on $\Gamma$ by identifying $\Gamma_+ \cong \Gamma_0\Gamma_P V$ cosets. Now let
\[ \Gamma_V = \{ \gamma \in \Gamma : W\gamma = W, V\gamma = V \}. \]

Then $\Gamma_V$ acts on $W/V$ and we define $\Gamma_P V \subset \Gamma_V$ to be the subgroup of those elements which fix $\mathbb{P}(W/V)$ pointwise. For example, when $W = H_+$ and $V = H_+\zeta^2$ we have
\[ \Gamma_V = \{ a_0 + a_1\zeta + \ldots \in \Gamma : a_0 \in \mathbb{C}^\times \}, \quad \Gamma_P V = \{ a_0 + a_2\zeta^2 + \ldots \in \Gamma : a_0 \in \mathbb{R}^\times \}. \]

**Lemma 4.3.** The map $f : \Gamma \to \mathbb{P}(W/V)$ in (26) is constant on $\Gamma_-\Gamma_0\Gamma_P V$ cosets, and therefore it descends to $J_V = \Gamma/(\Gamma_-\Gamma_0\Gamma_P V)$. 
Proof. It suffices to check the invariance of \( f \) along \( \Gamma_0 \Gamma_\gamma^0 \) cosets. Let \( \gamma' \in \Gamma \) and \( \gamma \in \Gamma_\gamma^0 \), then by (16) we have
\[
\psi_W(\gamma'\gamma^{-1}) = \psi_W(\gamma')\gamma^{-1}.
\]
Therefore, modulo \( V \) this is independent of \( \gamma \). Similarly, by (16) and the proof of theorem 3.8(a),
\[
(27) \quad \psi_W(\gamma'e^{-i\theta_0}) = e^{-i\theta_0}\psi_W(\gamma')
\]
and therefore the projective map \( f \) is constant along \( \Gamma_0 \)-orbits. \( \square \)

**Corollary 4.4.** Let \( \ell_V : \mathbb{C} \to J_V \) be the homomorphism (of real groups) obtained by the composition \( \mathbb{C} \simeq \Gamma_1 \to \Gamma \to J_V \). Then \( f : \mathbb{C} \to \mathbb{P}(W/V) \) factors through this. Hence a sufficient condition for (25) to be \( \Lambda \)-periodic is that \( \ell_V \) be \( \Lambda \)-periodic.

### 4.2. A construction for solutions of finite type.

Let us now consider a more concrete version of the above construction which applies to solutions of finite type. The spectral data \( (X, \rho, P, \zeta, \mathcal{L}, \phi) \) provides us with \( W \); we can choose a \( \mathbb{H} \)-codimension \( n + 1 \) \( \mathbb{H} \)-subspace \( W_q \) of \( W \) as follows. Fix a \( \rho \)-invariant divisor \( q \) consisting of \( 2n + 2 \) distinct smooth points in \( X \setminus \{ P, \rho P \} \). We will write
\[
q = Q_0 + \ldots + Q_n + \rho(Q_0) + \ldots + \rho(Q_n).
\]
We may choose the local parameter disc \( \Delta_P \) so that \( q \subset X_0 \). Let \( W_q \) correspond to the subspace \( H^0(X_0, \mathcal{L}(-q)) \) of holomorphic sections of \( \mathcal{L} \) which vanish on \( q \): it is a left \( \mathbb{H} \)-subspace since \( q \) is \( \rho \)-invariant. Since \( X_0 \) is a Stein manifold it follows (by calculating the cohomology of the sheaf exact sequence \( 0 \to \mathcal{L}(-q) \to \mathcal{L} \to \mathcal{L}_q \to 0 \)) that \( \dim_{\mathbb{C}}(W/W_q) = \deg(q) = 2n + 2 \).

**Remark 4.5.** I see no reason why singular points of \( X \) should be excluded from the divisor \( q \), even in the case where \( \mathcal{L} \) is not locally free, but since this adds complications to the subsequent discussion we will ignore this possibility in this article.

Now consider the dependence of \( f \) on the choice of trivialisation \( \phi \). By remark 3.12 any change of trivialisation has the form \( \phi \mapsto \varphi e^{a_0 + i\theta_0} \gamma' \) where \( \gamma' \in \Gamma_\gamma \). By remark 4.2 this only alters \( f \) by congruence. Further, a change of local parameter \( \zeta \) amounts to rescaling the parameter \( z \) in the Dirac operator. Therefore it is enough to be given \( (X, \rho, P, \mathcal{L}, q) \) to have \( f \) well-defined up to congruence in \( \mathbb{H}\text{P}^n \): we may as well take \( \psi_W \) to be the global Baker function. The condition above that \( f \) be well-defined is that \( \psi_W \) does not vanish on \( q \) for any \( z \in \mathbb{C} \). We can think of \( W_q \) as the kernel of \( n + 1 \) left \( \mathbb{H} \)-linear forms \( \text{ev}_{Q_0}, \ldots, \text{ev}_{Q_n} \) on \( W \), where \( \text{ev}_Q \) is obtained by extending the map “evaluation at \( Q \)” to all of \( W \) (it is straightforward to check that \( \text{ev}_{\rho Q} = \text{ev}_Q u \) for some \( u \in \mathbb{H}^\times \)). In this case, in homogeneous coordinates, and up to congruence in \( \mathbb{H}\text{P}^n \),
\[
(28) \quad f(z) = [\psi_W(z, Q_0), \ldots, \psi_W(z, Q_n)].
\]
Further, for any $\mathbb{H}$-codimension two subspace $V$ satisfying $W_q \subset V \subset W$ we obtain a projection of $f$ onto $\mathbb{P}(W/V) \simeq \mathbb{H}\mathbb{P}^1$, which will be a conformal immersion provided $V$ is well-positioned in $W$. Equally, this arises from a well-positioned two $\mathbb{H}$-dimensional linear system in the dual space $(W/W_q)^*$. 

To study the periodicity conditions for $f$ when it arises from this construction we begin by showing that when $V = W_q$ the group $J_{W_q}$ appearing in lemma 4.4 is isomorphic to a real subgroup of a generalised Jacobi variety. Let $X_q$ be the singularization of $X$ obtained by identifying the points of $q$ together simply. The real involution $\rho$ descends to $X_q$, so its generalised Jacobian $\text{Jac}(X_q)$ has a real subgroup $J_{\mathbb{R}}(X_q)$ which is the connected component of the identity of the real subgroup of $\tilde{\rho}^*$-fixed line bundles over $X_q$. Let $X_{q}^{\text{sm}}$ denote the curve of smooth points on $X_q$ (this is just $X_{q}^{\text{sm}} \setminus q$) and let $\mathcal{A}^q_p : X_{q}^{\text{sm}} \rightarrow \text{Jac}(X_q)$ denote the Abel map for $X_q$, with base point $P$. Let $\ell^q : \mathbb{C} \rightarrow J_{\mathbb{R}}(X_q)$ be the unique homomorphism of real groups determined by the property

\[ \frac{\partial \ell^q}{\partial z}|_{z=0} = \frac{\mathcal{A}^q_p}{\partial \zeta^{-1}}|_{\zeta^{-1}=0}. \]

As earlier, we use the natural isomorphism $T_e J_{\mathbb{R}}(X_q) \simeq T_{1,0} \text{Jac}(X_q)$ to equate the left hand side of this equation with the right.

**Lemma 4.6.** For a pair $(W, W_q)$ given by spectral data $(X, \rho, P, \mathcal{L}, q)$ in the manner above, we have $J_{W_q} \simeq J_{\mathbb{R}}(X_q)$ and $\ell_{W_q} = \ell^q$.

**Proof.** First we note that an equivalent way of describing $\ell^q$ above is that the line bundle $\ell^q(z)$ is obtained using transition functions $e^{z\zeta}$ and $e^{z\rho^* \zeta}$ to glue together trivial bundles as in (21), but where $X \setminus \{P, \rho P\}$ is replaced by its singularisation $X_q \setminus \{P, \rho P\}$. This extends naturally to a real homomorphism $\ell^q : \Gamma \rightarrow J_{\mathbb{R}}(X_q)$ by using $\gamma$ and $\tilde{\rho}^* \gamma$ as transition functions.

Now $\gamma \in \ker(\ell^q)$ precisely when $\gamma$ factorises into a product $\gamma = \alpha \beta$ where $\alpha$ extends holomorphically to $\Delta_p$ and $\beta$ extends holomorphically to $(X_q)_0$, i.e., $\beta$ represents the boundary of a holomorphic function on $X_0$ which takes the same value at each point of $q$. Clearly $\Gamma \setminus \Gamma_0$ equals the group of all boundaries of the type $\alpha$, while the boundaries of the type $\beta$ are exactly those which, by multiplication, fix

\[ H^0(X_0, \mathcal{L})/H^0(X_0, \mathcal{L}(\mathcal{q})) \simeq W/W_q \]

projectively, since they act by scaling. Hence $\ker(\ell^q) = \Gamma_0 \Gamma_{W_q}$, whence the result. \hfill $\square$

**Remark 4.7.** One knows (from e.g., [27]) that $\text{Jac}(X_q)$ is a group extension of $\text{Jac}(X)$:

\[ 1 \rightarrow K \rightarrow \text{Jac}(X_q) \xrightarrow{\pi_q} \text{Jac}(X) \rightarrow 1, \]
where $K$ is a linear algebraic group. Specifically, if $O^\times_q$ denotes the group of germs of non-zero holomorphic functions about $q \subset X$ then

$$K \simeq O^\times_q / C^\times \simeq (C^\times)^{2n+2} / C^\times,$$

where $C^\times$ here represents the subgroup of non-zero constants in $O^\times_q$. The real automorphism $\bar{\rho}^*$ acts on $\text{Jac}(X_q)$ and preserves $K$. Define $K_R = K \cap J_R(X_q)$, then

$$K_R \simeq (C^\times)^{n+1} / R^\times.$$ 

This is the kernel of the restriction of $\pi_q$ to $J_R(X_q)$.

We can now achieve a more precise understanding of how the map $f$ factors through $\text{Jac}(X_q)$ by considering the natural twistor lift which every quaternionic holomorphic curve $f : M \to \mathbb{H}P^n$ possesses. This is a map $\hat{f} : M \to \mathbb{CP}^{2n+1}$ for which $T \circ \hat{f} = f$, where $T : \mathbb{CP}^{2n+1} \to \mathbb{H}P^n$ is the twistor projection assigning to every left $\mathbb{C}$-line in $\mathbb{H}^{n+1}$ its left $\mathbb{H}$-line. The lift arises as follows: the corresponding line bundle $L \subset \mathbb{H}^{n+1}$ possesses a unique complex structure $J$ for which $\delta f = \delta fJ$. The twistor lift $\hat{f}$ is given by the complex line subbundle $\hat{L} \subset L$ of $i$-eigenspaces of $J$. Now let $\tilde{f} : C \to \mathbb{H}^{n+1}$ represent $\psi_W + W_q$ in some choice of coordinates on $W/W_q$ and write $f = \mathbb{H}\tilde{f}$. It follows from the proof of theorem 4.1 that $\hat{f} = \mathbb{C}\tilde{f}$.

**Theorem 4.8.** Given spectral data as above, the natural twistor lift of $f : C \to \mathbb{H}P^n$ is a composite of the form

$$\hat{f} : C \xrightarrow{\hat{f}} \text{Jac}(X_q) \xrightarrow{\theta} \mathbb{CP}^{2n+1},$$

where $\theta$ is a rational map.

**Proof.** We will show that $\hat{f}$ comes from a construction similar to the one given in [19] for harmonic tori. To this end, let $E \to \text{Jac}(X)$ denote the complex rank $2n + 2$ vector bundle with fibres $E_L = H^0(X, (L \otimes L)_q)$. It was shown in [19] that we can embed $\text{Jac}(X_q)$ into the bundle of (complex) projective frames of $E$ and therefore the tautological section of $\pi_q^* \text{Jac}(X_q)$ globally (and algebraically) trivialises bundle $\mathbb{PE}'$ of complex projective space of $E' = \pi_q^* E$, by canonically identifying each fibre of $\mathbb{PE}'$ with $\mathbb{P}H^0(X, L_q) \simeq \mathbb{CP}^{2n+1}$. This works as follows: each point of $\text{Jac}(X_q)$ should be thought of as a line bundle $L \in \text{Jac}(X)$ equipped with a trivialising section of $L_q$ determined up to scaling. This fixes a projective identification of $(L \otimes L)_q$ with $L_q^2$ by tensor product. We will denote this canonical trivialisation by

$$\tau : \mathbb{PE}' \to \text{Jac}(X_q) \times \mathbb{P}H^0(X, L_q).$$

Now define

$$\mathcal{U} = \{ L' \in \text{Jac}(X_q) : \mathcal{L}(-P - \rho P) \otimes \pi_q(L') \text{ is non-special} \}.$$
This is an affine open subvariety of $\text{Jac}(X_q)$ and it is a consequence of theorem 3.3 that $J_R(X_q) \cap U$ is the complement of a real analytic subvariety of $J_R(X_q)$. For each $L \in \pi_q(U)$ the vector space $H^0(X, \mathcal{L}(-\rho P) \otimes L)$ is one-dimensional and therefore the bundle $\mathbb{P}\mathcal{E}'$ possesses an algebraic section $s : U \to \mathbb{P}\mathcal{E}'$ corresponding to the injection

$$H^0(X, \mathcal{L}(-\rho P) \otimes L) \to H^0(X, \mathcal{L} \otimes L) \to H^0(X, (\mathcal{L} \otimes L)_q).$$

Now we define $\theta : \text{Jac}(X_q) \to \mathbb{CP}^{2n+1}$ by $\theta = \tau \circ s$, having fixed some projective linear identification of $\mathbb{P}H^0(X, \mathcal{L}_q)$ with $\mathbb{CP}^{2n+1}$. This is algebraic on the affine open subvariety $U$ and therefore rational. The theorem is proved once we have shown that, via the identification

$$H^0(X, \mathcal{L}_q) = H^0(X_0, \mathcal{L})/H^0(X_0, \mathcal{L}(-q)) \xrightarrow{w} W/W_q,$$

derived from (20), the map $\hat{f}$ corresponds to $\theta \circ \ell^q$. To see this we note that $\ell^q(z)$ is defined using the transition relation (21) to glue together the trivial bundles over $X \setminus X_0$ and $(X_q)_0$, and therefore can be thought of as $\ell(z)$ equipped with the trivialisation $\tau_0(z)$ restricted to $q$. But now we recall that the global Baker function in (23) is the result of applying (20) to the gl obal section $\sigma_0 \in H^0(X, (\mathcal{L}(-\rho P) \otimes \ell(\gamma))$ pulled back to a section of $\mathcal{L}$ over $X_0$ by $\sigma_z \to \sigma_z\tau_0(z)^{-1}$. When we restrict this to $\mathcal{L}_q$ we see that this generates the line $\tau \circ s \circ \ell^q(z)$ in $H^0(X, \mathcal{L}_q)$. Hence $\hat{f} = \theta \circ \ell^q$.}

**Remark 4.9.** From the previous proof we see that if $f = T \circ \hat{f}$ is directly obtained from the $\mathbb{H}$-line subbundle of $\mathcal{E}'$ whose fibres are the left $\mathbb{H}$-lines $H^0(X, \mathcal{L} \otimes L) \subset H^0(X, (\mathcal{L} \otimes L)_q)$. Notice that if $W_q \subset V \subset W$ for a well-positioned $\mathbb{H}$-codimension two subspace $V$ and $f_V : C \to \mathbb{H}P^1$ is the corresponding projection of $f$ then the twistor lift of $f_V$ is the projection of $\hat{f}$ onto $\mathbb{CP}^3$, i.e., the map

$$\hat{f}_V : C \to \mathbb{P}_{\mathbb{C}}(W/V); \quad z \mapsto \mathbb{C}.(\hat{f}(z) + V).$$

Now we return to the question of when the map $f$ is doubly periodic.

**Theorem 4.10.** Let $f : C \to \mathbb{H}P^n$ correspond to the spectral data above and suppose that it is linearly full (i.e., its image does not lie in some $\mathbb{H}P^{n-1}$). Then $f$ and $\ell : C \to \text{Jac}(X)$ are simultaneously $\Lambda$-periodic if and only if $\ell^q$ is $\Lambda$-periodic.

This implies that the monodromy of the corresponding Dirac potential ($\mathbb{H}$) is that of the flat $S^1$-bundle $S$ over $\mathbb{C}/\Lambda$ described in remark 3.18.

**Proof.** That the $\Lambda$-periodicity of $\ell^q$ is sufficient follows at once from corollary 4.3. Now suppose both $f$ and $\ell$ are $\Lambda$-periodic. Then $\hat{f}$ is $\Lambda$-periodic, and $\ell^q$ determines a homomorphism

$$h : \Lambda \to K_R \subset \ker(\pi_q); \quad h(\lambda) = \ell^q(\lambda).$$

In this situation we can use [19 Lemma 1] to deduce that there is an injective homomorphism of $K_R$ into $\mathbb{P}GL(n + 1, \mathbb{H})$ which allows us to write $f(z + \lambda) = \ldots$
$f(z)h(\lambda)$ for all $\lambda \in \Lambda$. Now since $f$ is linearly full we can, after possibly a congruence, find homogeneous coordinates for $f(z)$ so that these are all non-zero at some $z$, hence $h$ must be the identity, whence $\ell^4$ is $\Lambda$-periodic. \hfill \Box

Remark 4.11. I was unable to find a way to remove the assumption that $\ell$ be $\Lambda$-periodic: it amounts to the difference between knowing that the Dirac potential has $\Lambda$-monodromy \cite{4} (which follows from the periodicity of $f$) and knowing that the full QKP operator $L_W$ has $\Lambda$-monodromy (i.e., $[L_W]$ is $\Lambda$-periodic). The former implies the latter if one knows a priori that the QKP Baker function has property (c) of theorem (2.1). We will see later in the article that under the assumption that $\ell$ is $\Lambda$-periodic we do get agreement of the two Baker functions.

Remark 4.12. In the case where $X$ has genus 0 or 1 the map $\ell : \mathbb{C} \to \text{Jac}(X)$ is not injective, so the periodicity condition on $\ell$ does not uniquely determine a lattice $\Lambda \subset \mathbb{C}$. But the arithmetic genus of $X_q$ is at least $2n + 1 \geq 3$ hence $\ell_q$ is always injective, therefore it uniquely determines the lattice $\Lambda$ when such a lattice exists.

Finally, let us note a condition under which the conformal torus in $\mathbb{HP}^1$ can be immersed in $\mathbb{R}^3$, at least in the case where $X$ is a smooth curve. Recall from the discussion at the end of §2.1 that $f : M \to S^4$ lies in some $\mathbb{R}^3$ if and only if $\hat{E}$ is a spin bundle, hence if and only if $S$ is a spin bundle.

Proposition 4.13. When $X$ is a smooth curve the bundle $S$ above is a spin bundle if the divisor $P - \rho P$ has order two (in which case $X$ is hyperelliptic).

Proof. One knows (e.g., from \cite{27}) that pulling back line bundles along the Abel map gives an isomorphism between $\text{Jac}(X)$ and its dual $\text{Jac}(X)^*$ (the moduli space of flat line bundles on $\text{Jac}(X)$). One also knows from \cite{27, VII} that the flat line bundle over $\text{Jac}(X)$ with principal bundle $\text{Jac}(X_p)$ is pulled back to $O_X(P - \rho P)$. Hence $S$ is a spin bundle whenever $O_X(2P - 2\rho P)$ is trivial, i.e., when $P - \rho P$ is a divisor of order two. \hfill \Box

5. Darboux transformations.

We will follow the notion of Darboux transformations introduced by Bohle et al. in \cite{3}. It generalises the classical notion of a Darboux transform between isothermic surfaces in $S^3$, in which both surfaces envelope the same sphere congruence.

Let us recall first, from \cite{5}, that we can identify the set of all oriented round 2-spheres in $S^4$ with the set

$$Z = \{ S \in \text{End}_H(\mathbb{H}^2) : S^2 = -I \}.$$ 

This identification gives to each $S \in Z$ the 2-sphere $\{ L \in \mathbb{HP}^1 : SL = L \}$, which we will also denote by $S$. The orientation is given by the complex structure each 2-sphere inherits from $S$. Given a Riemann surface, $M$, a sphere congruence is a
map $S : M \to \mathcal{Z}$. Now let $f : M \to S^4$ be a conformal map. We say $f$ envelopes a sphere congruence $S : M \to \mathcal{Z}$ if $f(p) \in S(p)$ and the oriented tangent plane of $f(M)$ at $f(p)$ agrees with that of $S(p)$. In terms of the line subbundle $L \in \mathbb{H}^2$ corresponding to $f$, these two conditions can be expressed as

$$SL = L, \quad \text{and} \quad \ast \delta f = S\delta f = \delta f S.$$ 

It is a classical result that two conformal maps which envelope the same sphere congruence must both be isothermic (see [1] Cor. 67, p78 for a modern proof), therefore Bohle et al. [3] (see also [1]) relax the condition slightly to achieve a broader class of transformations. We will say a conformal map $f : M \to S^4$ left-envelopes a sphere congruence $S : M \to \mathcal{Z}$ if $f(p) \in S(p)$ and their oriented tangent planes agree up to action of $SU_2$ on $T_{f(p)}S^4$ as the left component in the epimorphism $SU_2 \times SU_2 \to SO_4$ (the notion of a right-envelope is defined similarly). In terms of the line bundle $L$ the property of being a left-envelope can be expressed as

$$SL = L, \quad \text{and} \quad \ast \delta f = S\delta f.$$ 

**Definition 5.1** ([3]). Let $f : M \to S^4$ be a conformal map of a Riemann surface $M$. Another conformal map $f^\sharp : M \to S^4$ is a Darboux transform of $f$ if $f(p) \neq f^\sharp(p)$ for all $p \in M$ and there exists a sphere congruence $S : M \to \mathcal{Z}$ which is enveloped by $f$ and left-enveloped by $f^\sharp$. In terms of the line bundles $L, L^\sharp$ this means

$$\mathbb{H}^2 = L \oplus L^\sharp, \quad SL = L, \quad SL^\sharp = L^\sharp, \quad \ast \delta f = S\delta f = \delta f S, \quad \text{and} \quad \ast \delta f^\sharp = S\delta f^\sharp.$$ 

Strictly speaking, we want to allow singular Darboux transformations of a torus: those for which $L \cap L^\sharp$ is trivial except at finitely many points (see [3]). We want to understand what Darboux transformations look like for maps arising from a pair $(W, V)$ of the type above. First we invoke a simple result from [3] which gives a neat characterization for Darboux transforms.

**Lemma 5.2** ([3]). Let $f, f^\sharp : M \to \mathbb{HP}^1$, with corresponding line bundles $L, L^\sharp$, be conformal immersions so that $\mathbb{H}^2 = L \oplus L^\sharp$. Then $f^\sharp$ is a Darboux transform for $f$ if and only if $\ast \delta f^\sharp = J\delta f^\sharp$ where $\ast \delta f = \delta f J$ and we identify $\mathbb{H}^2/L^\sharp$ with $L$ using projection along the splitting.

It is easy to check that the sphere congruence which is enveloped by $f$ and left-enveloped by $f^\sharp$ is given by $S|_L = J$ and $S|_{L^\sharp} = J$ where $\ast \delta f = J\delta f$, again using the splitting to identify $L^\sharp$ with $\mathbb{H}^2/L$.

**Lemma 5.3.** Let $f : M \to \mathbb{P}(W/V) \simeq \mathbb{HP}^1$ be the conformal map defined by equation (25). Given $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$, define $W_\lambda = W(1-\lambda \zeta)$, $V_\lambda = V(1-\lambda \zeta)$ and let $f^\lambda : M' \to \mathbb{HP}^1$ be the map corresponding to the pair $(W_\lambda, V_\lambda)$. Then $f^\lambda$ is a Darboux transform of $f$ over their common domain $M \cap M'$. 

Proof. We assume, without loss of generality, that $M = M'$ is an open domain in $\mathbb{C}$. Let $V = W_\lambda/V_\lambda$ and let $f, f^\lambda : M \to \mathbb{P}V$ denote the conformal maps with lifts

$$\tilde{f} = \psi_W(1 - \lambda \zeta) + V_\lambda, \quad \tilde{f}^\lambda = \psi_{W_\lambda} + V_{\lambda},$$

and line subbundles $L, L^\lambda \subset M \times V$. From the proof of theorem 4.1 we know that $*\delta f = \delta f J$ for $J \tilde{f} = i \tilde{f}$. I claim that there is a function $b : M \to \mathbb{H}$ for which

$$(35) \quad \tilde{f}_z^\lambda = b \tilde{f}^\lambda - \frac{1}{\lambda} \tilde{f}.$$ 

Therefore

$$\delta f^\lambda(\tilde{f}^\lambda) = -dz \frac{1}{\lambda} \tilde{f}.$$ 

It follows that, since $\lambda$ is complex, $*\delta f^\lambda = J\delta f^\lambda$. In light of the previous lemma, this proves the theorem.

It remains to verify equation (35). This is a direct result of an identity for Baker functions. Set

$$\psi = \psi_W(1 - \lambda \zeta), \quad \psi^\lambda = \psi_{W_\lambda}.$$ 

These have respective Fourier series expansions

$$\psi = (-\lambda \zeta + (1 - a\lambda) + \ldots) e^{z\zeta},$$

$$\psi^\lambda = (1 + a^\lambda \zeta^{-1} + \ldots) e^{z\zeta}.$$ 

Therefore there is an $\mathbb{H}$-valued function $b$ for which

$$(\psi^\lambda + \frac{1}{\lambda} \psi - b\psi^\lambda)e^{-z\zeta} \equiv 0 \mod O(\zeta^{-1}).$$ 

Since $W_\lambda e^{-z\zeta}$ is transverse to $H_-$ almost everywhere (theorem 3.3), the right hand side must be identically zero. Equation (35) follows.

Notice that $(1 - \lambda \zeta) \in \Gamma_+$, since its only zero is at $\lambda^{-1}$. It is clear that the proof above still works if we replace $(1 - \lambda \zeta)$ by any representative in the coset $(1 - \lambda \zeta)\Gamma_- \in \Gamma/\Gamma_-$. It follows that in the case where $W_q \subset V \subset W$ for some $\rho$-invariant divisor $q \subset X \setminus \{P, \rho P\}$ this Darboux transform acts on the spectral data by fixing everything except the pair $\mathcal{L}, \varphi$, which it transforms by

$$(36) \quad (\mathcal{L}, \varphi) \mapsto (\mathcal{L}^Q, \varphi^Q), \quad \mathcal{L}^Q = \mathcal{L}(Q + \rho Q - P - \rho P),$$

where $\zeta(Q) = \lambda^{-1}$ and $\varphi^Q$ is a local trivialising section of $\mathcal{L}^Q$ over $\Delta_P$. In fact, provided $Q$ is not in the support of $q$, the transform (36) always yields a Darboux transform.

Theorem 5.4. Let $f : \mathbb{C}/\Lambda \to \mathbb{H}P^1$ be a quaternionic holomorphic torus corresponding to the spectral data $(X, \rho, P, \mathcal{L}, q)$ and a choice of well-positioned two dimensional linear system $H \subset (W/W_q)^*$. Assume that $\ell : \mathbb{C} \to \text{Jac}(X)$ is $\Lambda$-periodic. Let $X^\text{sm}_q$ be the variety of smooth points in $X_q$. Then for every $Q \in X^\text{sm}_q \setminus \{P, \rho P\}$ there is a Darboux transform $f^Q$ of $f$ arising from the spectral data obtained from the transformation (36).
The linear system $H$ is spanned by two forms each of which is a $\mathbb{H}$-linear combination of evaluation maps $ev_{Q_0}, \ldots, ev_{Q_n}$. Therefore $f^Q$ corresponds to the linear system $H^Q \subset (W/W_q)^*$ determined by the same combination of evaluations. Since both $\ell$ and $f$ are assumed $\Lambda$-periodic theorem 4.10 ensures that $f^q_\ell$ is $\Lambda$-periodic, and therefore the $\Gamma_1$-orbit of $L^Q$ in $Pic(X_q)$ is isomorphic to $\mathbb{C}/\Lambda$. The proof now follows from theorem 5.3 by choosing a coordinate disc on $X$ centred at $P$ and containing $Q$ but not any points in $q$. The singularities of such a Darboux transform correspond to points where the $\Gamma_1$-orbit of $W^Q$, the point in $Gr_{\text{QKP}}$ corresponding to the transformed spectral data, leaves the big cell.

6. Spectral curves.

One of the main observations made in [3] is that the family of all Darboux transforms of a quaternionic holomorphic torus is essentially parameterised by the normalisation of the multiplier spectral curve and gives an immersion of it in $\mathbb{CP}^3$. In this section I will explain the relationship between the three types of spectral curve: the QKP spectral curve $X$ which we have been working with, the curve of Darboux transforms and the multiplier spectrum.

6.1. The QKP spectral curve. In the construction of $W \in Gr_{\text{QKP}}$ from spectral data we use the map $w$ in [20]. When this is restricted to the algebraic sections of $L$ over $X_0$ (i.e., those which have only poles at $P, \rho P$) we obtain an open dense subspace $W^{\text{alg}} \subset W$. The elements of $W^{\text{alg}}$ are algebraic in the sense that their projections onto $H_+$ are polynomial in $\zeta$. Now if $A$ denotes the coordinate ring of $X \setminus \{P, \rho P\}$ its real subalgebra $A^\rho = \{h \in A : \overline{\rho^*h} = h\}$, acts on $W^{\text{alg}}$ by right multiplication, since $w(h\sigma) = w(\sigma)h$ (where on the right hand side we restrict $h$ to the circle $C_P$). Therefore

$$A^\rho \hookrightarrow A_W = \{h \in C^\omega(S^1, \mathbb{C}) : W^{\text{alg}} h \subset W^{\text{alg}}\}.$$ 

In fact it is easily shown that this inclusion is onto, so that $A^\rho \simeq A_W$. Thus we recover $X \setminus \{P, \rho P\}$ as $\text{Spec}(A^\rho_W)$, where $A^\rho_W$ is the complex subalgebra of $C^\omega(S^1, \mathbb{C})$ generated by $A_W$. Now $(W^{\text{alg}})^C$, the complexification $W^{\text{alg}} + W^{\text{alg}}i \subset H$, is a torsion free $A^\rho_W$-module and this recovers the rank one torsion free coherent sheaf $L$ equipped with the trivialisation implicit in the inclusion $(W^{\text{alg}})^C \subset H^{\text{alg}}_+ \oplus H_-$. Now, in analogy with the KP case, it is clear that we can assign a commutative algebra $A_W$ to any $W \in Gr_{\text{QKP}}$, but in general this will not be very useful: typically $A_W = \mathbb{R}$, and even if it is not so trivial we only obtain spectral data of the type we desire when $(W^{\text{alg}})^C$ is locally rank one. Nevertheless, we can always obtain $A^\rho$ as a commutative algebra of differential operators over $\mathbb{H}$, and this connects us to the stationary QKP flows. The proof of the following lemma is obtained mutatis mutandis from [20, Remark 6.4].
Lemma 6.1. Given $W \in \mathcal{G}_{\text{QKP}}$ to each $h \in \mathcal{A}_W$ there is a unique pseudo-differential operator $P(h) \in Z_0(L_W)$ for which the differential operator part $P(h)_+$ satisfies $P(h)_+ \psi_W = \psi_Wh$. The $\mathbb{R}$-algebra $\{P(h)_+: h \in \mathcal{A}_W\}$ is isomorphic to $\mathcal{A}_W$. It follows that $[P(h)_+, L_W] = 0$ and therefore the QKP solution corresponding to $W$ is stationary for every flow $\partial_{P(h)}$.

One consequence of this lemma is that $L_W$ admits only finitely many independent non-stationary QKP flows precisely when the algebra $\mathcal{A}_W$ possesses an element of every sufficiently high order, and therefore $(W_{\text{alg}})^\mathbb{C}$ is locally rank one. Hence $L_W$ is a solution of finite type.

Example 6.2. Consider, in light of proposition 4.13, the case when $W$ comes from a smooth $X$ for which the divisor $P - \rho P$ has order two. Then there exists a rational function $\lambda$ of degree two on $X$ with divisor $2P - 2\rho P$ and we may normalise this so that $\rho^*\lambda = \pm \lambda^{-1}$. Let us examine the case $\rho^*\lambda = \lambda^{-1}$ (the other sign choice leads to similar conclusions). This means both $\lambda + \lambda^{-1}$ and $i(\lambda - \lambda^{-1})$ belong to $\mathcal{A}_W$. Now we may choose $\zeta = \sqrt{\lambda}$ about $P$ and therefore

$$W\zeta^2(1 + \zeta^{-4}) \subset W, \quad W i\zeta^2(1 - \zeta^{-4}) \subset W.$$ 

From the previous lemma we deduce that every “even” flow in the QKP hierarchy $s_2, t_2, s_4, t_4, \ldots$ leaves $L_W$ stationary. This corresponds to a subhierarchy of the QKP hierarchy which I believe is called the modified Novikov-Veselov hierarchy: it sits in relationship to the QKP hierarchy just as the KdV hierarchy sits in relationship to the KP hierarchy.

6.2. The curve of Darboux transforms. Assume we have a quaternionic holomorphic torus $f : \mathbb{C}/\Lambda \to \mathbb{HP}^n$ obtained from spectral data $(X, \rho, P, L, q)$ as described above. According to Bohle et al. [3] there will be a holomorphic curve in $\mathbb{CP}^3$ given by the Darboux transforms of any torus in $\mathbb{HP}^1$ obtained from $f$ by projection. We are going to use the interpretation of the Darboux transform given in (36) to work directly with $f$ and view the curve of Darboux transforms as an algebraic curve in $\mathbb{CP}^{2n+1}$. For this purpose, let $f^Q$ be the quaternionic holomorphic torus obtained from the transformation (36). Any projection of $f$ onto $\mathbb{HP}^1$ then has a Darboux transform by applying the same projection to $f^Q$ (see remark 4.9).

Following [3], since $f^Q = f^\rho Q$, a direct consequence of theorem 5.4 is a geometric realisation of the Klein surface $X_q^\text{sm}/\rho$ (recall that $X_q^\text{sm} \subset X \setminus q$ is the subvariety of smooth points), since we obtain from it a map

$$F : (\mathbb{C}/\Lambda) \times (X_q^\text{sm}/\rho) \to \mathbb{HP}^n; \quad (p, Q + \rho Q) \mapsto f^Q(p).$$

Here we define $f^P = f$. An immediate consequence of theorems 4.8 and 5.4 is that this factors through the generalised Jacobian $\text{Jac}(X_q)$ via

$$(\mathbb{C}/\Lambda) \times (X_q^\text{sm}/\rho) \to \text{Jac}(X_q); \quad (z, Q + \rho Q) \mapsto \ell^q(z) \otimes \mathcal{O}_{X_q}(Q + \rho Q - P - \rho P).$$
Post-composition of this with $T \circ \theta : \text{Jac}(X_q) \to \mathbb{HP}^n$ (cf. remark 4.9) gives us $F$. For each $p \in \mathbb{C}/\Lambda$ let us define

$$\xi_p : X_q^{\text{sm}}/\rho \to \mathbb{HP}^n; \quad \xi_p(Q) = F(p, Q) = f^Q(p).$$

Generalising the discussion in [3], since each $f^Q$ is equivalent to a complex quaternionic line subbundle $L^Q \subset \mathbb{HP}^{n+1}$ the complex lines $\hat{L}^Q_p$ (which we recall are the $i$-eigenspaces of the complex structure on $L^Q$) provide a twistor lift

$$\hat{\xi}_p : X_q^{\text{sm}} \to \mathbb{CP}^{2n+1}, \quad Q \mapsto \hat{L}^Q_p \subset \mathbb{HP}^{n+1}. \tag{38}$$

The image of $\hat{\xi}_p$ will be called the Darboux spectral curve. This twistor lift has an elegant description, in terms of the generalised Jacobian construction of $\xi_p$, which follows from theorem 4.8.

Let $S^2 X_q^{\text{sm}}$ denote the symmetric product of $X_q^{\text{sm}}$ with itself; equally, think of it as the set of all divisors of degree two supported on $X_q^{\text{sm}}$. Because $X_q^{\text{sm}}$ excludes $q$ and all singularities of $X$, this symmetric product admits an Abel map

$$A_{P+\rho P} : S^2 X_q^{\text{sm}} \to \text{Jac}(X_q); \quad A + B \mapsto O_{X_q}(A + B - P - \rho P). \tag{39}$$

We can embed $X_q^{\text{sm}}$ algebraically in $S^2 X_q^{\text{sm}}$ via $Q \mapsto Q + \rho P$; we can also embed $X_q^{\text{sm}}/\rho$ real algebraically by thinking of it as the curve of pairs $Q + \rho Q$ for $Q \in X_q^{\text{sm}}$. By post-composing each of these with the Abel map (39) we obtain

$$\alpha : X_q^{\text{sm}} \to \text{Jac}(X_q); \quad Q \mapsto O_{X_q}(Q - P),$$
$$\beta : X_q^{\text{sm}}/\rho \to \text{Jac}(X_q); \quad Q + \rho Q \mapsto O_{X_q}(Q + \rho Q - P - \rho P).$$

From the discussion above we see that, if $\mathcal{L}$ corresponds to the base point $p \in \mathbb{C}/\Lambda$,

$$\xi_p = T \circ \theta \circ \beta.$$

**Theorem 6.3.** The twistor lift $\hat{\xi}_p$ of $\xi_p$ defined in (38) equals $\theta \circ \alpha$ and is therefore complex algebraic.

This can be summarised by the following commutative diagram, in which the top line is the algebraic map $\hat{\xi}_p$ and the bottom line is $\xi_p$.

$$\begin{array}{ccc}
X_q^{\text{sm}} & \xrightarrow{\alpha} & \text{Jac}(X_q) \xrightarrow{\theta} \mathbb{CP}^{2n+1} \\
\downarrow & & \downarrow T \\
X_q^{\text{sm}}/\rho & \xrightarrow{\beta} & J_{\mathbb{R}}(X_q) \xrightarrow{T \circ \theta} \mathbb{HP}^n \end{array} \tag{40}$$

**Proof.** First we note that $O_{X_q}(A + B - P - \rho P)$ can be thought of as the bundle $O_X(A + B - P - \rho P)$ together with the fibre identification over $q$ uniquely determined by the rational section with divisor $A + B - P - \rho P$ (there is only one of these, up to scaling). Consequently the canonical trivialisation of the complex projective bundle $\mathbb{PE}^\ast$ over $\text{Jac}(X_q)$ (described in the proof of theorem 4.8) works as follows over $\alpha(X_q^{\text{sm}})$. Let $\sigma_Q$ be a non-zero rational section of $O_X(Q - P)$
with divisor $Q - P$. Then the canonical trivialisation of $\mathbb{P}\mathcal{E}'$ identifies fibres over $\alpha(X_q^*)$ via

$$H^0(X, \mathcal{L}(Q - P)_q) \rightarrow H^0(X, \mathcal{L}_q); \quad s \mapsto s/\sigma_Q,$$

thinking of $\sigma_Q$ restricted to $q$. Similarly, over $\beta(X_q^*)$ we have

$$H^0(X, \mathcal{L}^Q_q) \rightarrow H^0(X, \mathcal{L}_q); \quad s \mapsto (s/\sigma_Q\bar{\rho}^*\sigma_Q).$$

Now recall from (33) that $\theta$ is the result of applying this trivialisation to the line subbundle of $\mathcal{E}'$ which picks out the complex line

$$(41) \quad H^0(X, \mathcal{L}(-\rho P \otimes L) \subset H^0(X, (\mathcal{L} \otimes L)_q) = \mathcal{E}'_q, \quad L = \pi_q(L'),$$

while its twistor projection $T \circ \theta$ corresponds to the quaternionic line

$$H^0(X, \mathcal{L} \otimes L) \subset H^0(X, (\mathcal{L} \otimes L)_q).$$

We claim that

$$(42) \quad \theta \circ \alpha(Q) = \theta \circ \beta(Q + \rho Q),$$

and this proves the theorem since the proof of theorem 4.8 shows us that $\hat{\xi}_p(Q) = \theta \circ \beta(Q + \rho Q)$. To prove the claim, follow the composition

$$H^0(X, \mathcal{L}(Q - P)_q) \rightarrow H^0(X, \mathcal{L}_q) \rightarrow H^0(X, \mathcal{L}^Q_q); \quad s \mapsto s/\sigma_Q \mapsto (s/\sigma_Q)\sigma_Q\bar{\rho}^*\sigma_Q = s\bar{\rho}^*\sigma_Q.$$

When $\theta$ is restricted to $\alpha(X_q^*)$, i.e., when $L = \mathcal{O}_X(Q - P)$ in (41), the result of this composition is (the restriction to $q$ of) a globally holomorphic section of $\mathcal{L}^Q(-\rho P)$, since the simple pole of $\bar{\rho}^*\sigma_Q$ lies at $\rho P$ where $s$ has a zero. This proves (42).

6.3. The multiplier spectrum. Here we will examine how the multiplier spectrum $\text{Sp}(L^*, D)$ is related to the spectral curve $X$ in the case of a quaternionic holomorphic curve $L$ arising from spectral data in the manner of section 4.2.

Suppose that we have a conformally immersed torus $f : \mathbb{C}/\Lambda \rightarrow \mathbb{H}\mathbb{P}^1$ of finite type, with spectral data $(X, \rho, P, \mathcal{L}, q)$, i.e., we suppose that the map $\ell^q : \mathbb{C} \rightarrow J_{\mathbb{R}}(X)$ is $\Lambda$-periodic. We may as well assume $q$ is the largest $\rho$-invariant divisor of distinct smooth points which we can choose with this property. We will show how the multiplier spectrum $\text{Sp}(L^*, D)$ arises from the sections of $\pi^*L^*$ obtained by evaluating the global Baker function at different points of $X \setminus \{P, \rho P\}$.

To $X$ we can assign the subgroup

$$\Gamma_X = \{ \gamma \in \Gamma : \gamma \text{ extends hol. to } h : X \setminus \{P, \rho P\} \rightarrow \mathbb{C}^*, \quad \overline{\rho^*h} = h \};$$

Notice that $\Gamma_- \cap \Gamma_X = \mathbb{R}^+$ while $\Gamma_0 \cap \Gamma_X = \{1\}$. Clearly we have an exact sequence

$$1 \rightarrow \Gamma_- \Gamma_0 \Gamma_X \rightarrow \Gamma \rightarrow J_{\mathbb{R}}(X) \rightarrow 1.$$

In particular this gives us homomorphisms

$$\mu : \ker(\ell) \rightarrow \Gamma_0; \quad \chi : \ker(\ell) \rightarrow \Gamma_X.$$
defined by the unique factorisation

\[ \gamma = \gamma_\mu(\gamma)^{-1}\chi(\gamma), \quad \gamma \in \ker(\ell), \]

where \( \gamma_\mu \) is normalised by \( \gamma_\mu = 1 + O(\zeta^{-1}) \).

**Lemma 6.4.** The global Baker function \( \psi_W \) for this spectral data satisfies

\[
(43) \quad \psi_W(t' + t) = \mu(\gamma)^{-1}\psi_W(t')\chi(\gamma).
\]

whenever \( \gamma = \gamma(t) \in \Gamma_+ \cap \ker(\ell) \).

**Proof.** The transformation due to \( \Gamma_0 \) comes directly from equation (27). So let us assume \( \gamma \) has trivial \( \Gamma_0 \) factor and that \( \ell(\gamma) \simeq O_X \). Since we are dealing with a global Baker function there is a trivialisation \( \varphi \) over \( P \) which extends holomorphically to \( X \). Let \( \varphi(\gamma) \) be the trivialisation for \( L \otimes \ell(\gamma) \) obtained by twisting the 1-cocycle for \( (L, \varphi) \) by \( \gamma \). Then

\[
\psi_W(\gamma) = \left( \frac{\sigma}{\varphi(\gamma)} - j\rho^*\sigma \right)\gamma = \psi_W(1)\chi(\gamma).
\]

Equation (43) follows easily from this by replacing \( L \) with \( L \otimes \ell(\gamma') \). \( \square \)

Now let us restrict \( \ell \) to \( \Gamma_1 \simeq \mathbb{C} \). Here it has kernel \( \Lambda \) and we obtain homomorphisms

\[
\mu : \Lambda \to S^1; \quad \chi : \Lambda \to \Gamma_X.
\]

We may think of \( \chi \) as a function on \( \Lambda \times X \setminus \{P, \rho P\} \). Thus to each point \( Q \in X \setminus \{P, \rho P\} \) we get a function \( \psi_W(z, Q) : \mathbb{C} \to \mathbb{H} \) with the properties

\[
D\psi_W(z, Q) = 0, \quad \psi_W(z + \lambda, Q) = \mu(\lambda)^{-1}\psi_W(z, Q)\chi(\lambda, Q).
\]

Hence \( \chi(\lambda, Q) \in \text{Sp}(L^*, D) \).

**Theorem 6.5.** Let \( f : \mathbb{C}/\Lambda \to \mathbb{H}^1 \) be a non-constant conformal immersion of finite type from \( \text{Gr}_{QKP} \), with global Baker function \( \psi(z, Q) \) on \( \mathbb{C} \times X \setminus \{P, \rho P\} \), and for which the flat bundle \( L^* \) has monodromy \( \mu \in \text{Hom}(\Lambda, S^1) \). Define

\[
\chi : \Lambda \times X \setminus \{P, \rho P\}; \quad \chi(\lambda, Q) = \psi(0, Q)^{-1}\mu(\lambda)\psi(\lambda, Q).
\]

For any pair of generators \( \lambda_1, \lambda_2 \) of \( \Lambda \) the holomorphic map

\[
(44) \quad X \setminus \{P, \rho P\} \to \text{Sp}(L^*, D); \quad Q \mapsto (\chi(\lambda_1, Q), \chi(\lambda_2, Q))
\]

is surjective onto \( \text{Sp}(L^*, D) \). Moreover, this map factors through the covering map \( X \setminus \{P, \rho P\} \to X_q \setminus \{P, \rho P\} \).
Proof. This map is clearly holomorphic and non-constant when $f$ is non-constant, and therefore the image is an analytic subvariety of $\text{Sp}(L^*, D)$. The image also contains annuli about each of $P, \rho P$, by the symmetry of the real involution. Since $\text{Sp}(L^*, D)$ has at most two irreducible components, one about each of $P, \rho P$, the image must agree with $\text{Sp}(L^*, D)$. At each point of $Q \in q$ we know from theorem 4.10 that $\psi(z, Q)$ corresponds to a quaternionic holomorphic section of $L^*$, i.e., it has trivial multiplier. Hence the map (44) factors through $X_q$. \hfill \square

6.4. The spectral curve of a conformally immersed torus in $\mathbb{HP}^1$ with flat normal bundle. The multiplier spectrum $\text{Sp}(L^*, D)$ of a conformally immersed torus is a non-compact analytic curve. The immersion is said to be of finite type when the normalisation of this curve can be smoothly compactified into a Riemann surface $\Sigma$ with finite genus. Taimanov [31] initially proposed that $\Sigma$ should be the spectral curve, and this is the definition used in [3, 4]. But $\Sigma$ lacks the subtlety necessary to be useful, because it throws away crucial information. It does this at two levels: (i) it throws away the information contained in the divisor $q$, (ii) unless $X$ is smooth, which it need not be, it throws away the information of singularities in $X$. Examples of the latter case have been discussed by Taimanov himself, in the context of the generalised Weierstrass representation, in [30, 32, 34].

A potent illustration of this lies in the case where $\Sigma$ is the Riemann sphere $\mathbb{C}_\infty$. Here pretty much all useful information is lost in the normalisation. For example, as we saw in example 2.22 every Hamiltonian stationary Lagrangian torus has $\Sigma \simeq \mathbb{C}_\infty$. In this case we also have $X \simeq \mathbb{C}_\infty$ (see example 6.6 below) so all the relevant information about the map is contained in the divisor $q$. There are also conformally immersed tori for which $X$ is rational but not smooth. For example, there are rational curves with only double points for which $\ell^q : \mathbb{C} \to \text{Jac}(X_q)$ is $\Lambda$-periodic for some lattice $\Lambda$, and it seems the arithmetic genus of these curves can be arbitrarily large. I expect these will not be the only examples, so that $\Sigma \simeq \mathbb{C}_\infty$ hides a great deal of inequivalent information.

The virtue of the unnormalised multiplier spectral curve $\text{Sp}(L^*, D)$ is that it is directly constructed from the Dirac operator, equally, the quaternionic holomorphic structure $(L^*, D)$. This means that it is more properly an invariant of the quaternionic holomorphic curve $f : \mathbb{C}/\Lambda \to \mathbb{HP}^n$ given by (1) (where $n + 1 = \dim H^0_B(E)$). Provided we take $q$ to be the maximal divisor on which the multiplier of the Baker function is trivial, $X_q$ is likewise an invariant of this quaternionic holomorphic curve. Congruence of $f$ in $\mathbb{HP}^n$, which is the natural equivalence relation on such curves, gives a broader equivalence than congruence of any of the projections of $f$ into $S^4$, but it is clear from all the discussions above that this is the correct notion of equivalence from the point of view of spectral data. The virtue of $X_q$ over $\text{Sp}(L^*, D)$ is that it is algebraic and has no spurious singularities. The next example illustrates the importance of the divisor $q$ to the spectral data: we see that when $X = \mathbb{C}_\infty$ the choice of $q$ governs everything.
Example 6.6. Set $X = \mathbb{C}_\infty$, which comes equipped with a rational parameter $\eta$. Define the real involution $\rho$ by $\rho^* \eta = -\eta^{-1}$ and set $\eta(P) = \infty$, so $\eta(\rho P) = 0$. All holomorphic line bundles of degree $g + 1 = 1$ on $X$ are isomorphic, so we may as well choose $\mathcal{L} = \mathcal{O}(\rho P)$. Fix $\varepsilon \in \mathbb{C}$ to have $\varepsilon < 1$ and now define $\zeta = \varepsilon \eta$, so that $|\zeta| = 1$ on the circle $C_P = \{ \eta : |\eta| = |\varepsilon|^{-1} \}$. It follows that $C_P \cap \rho(C_P) = \emptyset$, so that we have an admissible collection of spectral data $(X, \rho, P, \zeta)$. We choose $\varphi$ to be the unique global section of $\mathcal{L}$ vanishing at $\rho P$. With this data we obtain the global QKP Baker function

$$
\psi(z, \zeta) = (1 - j\varepsilon \zeta^{-1}) \exp(z\zeta - \bar{z}|\varepsilon|^2 \zeta^{-1}) = (1 - j\eta^{-1}) \exp(z\varepsilon \eta - \bar{z}\varepsilon \eta^{-1}).
$$

Consequently the Dirac potential is $U = j\varepsilon$. As we noted in remark 4.12 there is no lattice $\Lambda$ determined by this data; it appears when we choose points on $X$ at which to evaluate $\psi$.

For $Q_0 \in X \setminus \{ P, \rho P \}$ with $\eta(Q_0) = q_0$ the function $\psi(z, Q_0)$ has monodromy $\mu \in \text{Hom}(\Lambda, S^1)$ for some lattice $\Lambda$ if and only if

$$
\mu(\lambda) = \exp(\lambda \varepsilon q_0 - \bar{\lambda} \varepsilon q_0^{-1}) = \pm 1, \quad \lambda \in \Lambda.
$$

In that case

$$(45) \quad z\varepsilon q_0 - \bar{z}\varepsilon q_0^{-1} = \pi i \langle \beta_0, z \rangle, \quad \exists \beta_0 \in \Lambda^*.
$$

As in example 2.2 the inner product on the right is the standard one on $\mathbb{C}$. This implies two conditions, $\varepsilon q_0 = \pi i \beta_0 / 2$ and $|\varepsilon| = |\beta_0| / 2$, which together require $|q_0| = 1$. Given this, the choice of $Q_0$ fixes one generator for the dual lattice $\Lambda^*$. Now choose another point $Q_1$, with $\eta(Q_1) = q_1$ satisfying $|q_1| = 1$. We obtain a dual lattice $\Lambda^*$ with generators $\beta_0, \beta_1$ (where $\beta_1$ is obtained from $q_1$ via (45) mutatis mutandis) for which $\psi(z, Q_0)$, $\psi(z, Q_1)$ have the same monodromy provided $(\beta_1 - \beta_0) / 2 \in \Lambda$. This lattice is completely determined by $Q_0, Q_1$ provided $Q_1 \neq \rho Q_0$. Without loss of generality we may assume $q_0 = i$, and therefore the Dirac potential is $U = \pi i / 2 \beta_0 j$. Now let $q$ be the divisor of all points in the set

$$
S = \{ Q_m \in X : \eta(Q_m) = i\alpha / \beta_0, \quad \exists \alpha \in \Lambda^*, \quad |\alpha| = |\beta_0|, \quad (\alpha - \beta_0) / 2 \in \Lambda^* \}.\n$$

This is the largest set of points on which all functions $\psi_m(z) = \psi(z, Q_m)$ have monodromy $\mu$ and from it we obtain a quaternionic holomorphic curve

$$
f : \mathbb{C}/\Lambda \to \mathbb{H} \mathbb{P}^n; \quad f = [\psi_0, \ldots, \psi_n], \quad \psi_m(z) = \psi(z, Q_m),
$$

where $\deg(q) = 2n + 2$.

Remark 6.7. It has already been observed (in [1, 17]) that $\Sigma \simeq \mathbb{C}_\infty$ for every Hamiltonian stationary Lagrangian torus in $\mathbb{R}^4$, whereas the approach in [21], working through Riemannian geometry, provides a spectral curve for a HSL torus which is a much richer object, subtle enough to provide a complete classification of HSL tori in $\mathbb{R}^4$ as Riemannian submanifolds.
A comparison with example 2.2 earlier shows that $X_q$ is obtained from $\text{Sp}(L^*, D)$ by a partial resolution of singularities: it keeps the essential information about where the multipliers are trivial but discards the singularities caused by coincidental agreement of the multiplier functions $\chi(\lambda_1, Q)$ and $\chi(\lambda_2, Q)$.

This prompts the question of how to describe $X_q$ as a cover of $\text{Sp}(L^*, D)$ without passing through the QKP construction. I suspect the answer is something like the following: the kernel of the Dirac operator should determine over $\text{Sp}(L^*, D)$ a coherent analytic sheaf $\mathcal{E}$ whose sections represent functions $\psi(z, \zeta)$ which satisfy $D\psi = 0$ for each $\zeta$, are holomorphic in $\zeta$, and have the appropriate multiplier at $\zeta$. The sheaf of algebras $\text{Hom}(\mathcal{E}, \mathcal{E})$ would be the model for the structure sheaf of $X \setminus \{P, \rho P\}$. Notice that this is consistent with the construction of $X$ as the compactification of $\text{Spec}(A^{\mathbb{C}}W)$ since $A_{W}$ is isomorphic to an algebra of operators preserving the kernel of $D$. This instantly makes $\mathcal{E}$ a maximal sheaf over $X \setminus \{P, \rho P\}$ (see remark 3.9) which should be both rank one and torsion free.

7. Final comments and questions.

Integrable deformations. In the theory above, and in the literature in general, there no satisfactory description of how the QKP flows provide deformations of conformal tori. The principal problem is the geometric meaning, if any, of the Lax equations $D_t = [D, P]$. There are several aspects to this:

(a) these equations at best define a deformation for $\ker(D) \subset \Gamma(E)$ and therefore one expects the proper place for understanding these equations lies in the theory of quaternionic holomorphic curves in $\mathbb{HP}^n$ rather than conformal tori in $S^4$,

(b) each differential operator $P$ needs to have the same monodromy as $D$ in order for the equations $\partial_t \psi + P \psi = 0$ to induce equations on $\Gamma(E)$. At the bottom of this is the question of whether or not $D$ determines a unique QKP operator $L$ with the same monodromy as $D$ (and for which $[D, L] = 0$). Calculations suggest this is true, but I have not seen a proof of this (even in the special case of the Dirac operator for conformal tori in $\mathbb{R}^3$) and I was unable to construct a proof myself.

Tori which are not of finite type. The convergent Baker function requires an open parameter disc about $P \in X$ on which to exist. When the conformal torus is not of finite type it seems there are two possibilities: (a) that the branch points of the multiplier spectral curve converge fast enough “at infinity” for a theory of infinite genus spectral curves (e.g. [8, 24]) to apply, (b) that the branch points cannot be controlled near infinity sufficiently well to allow such a theory. Unless one can prove that case (a) holds for all conformal tori, one has to question whether there is any application of integrable systems techniques to tori for which case (b) holds. A related question, raised by Taimanov [34], is whether tori of finite type are dense in the space of all tori (in some suitable topology).
Non-isospectral Darboux transformations. Theorem 5.4 is reminiscent of the theory of Darboux-Bäcklund transforms for finite type solutions of the KdV hierarchy [7] (this is no accident: the two are related via the theory of periodic discrete conformal maps [12]). In [7] Darboux-Bäcklund transforms could also be obtained by choosing data at singular points of the spectral curve, but such transforms are not isospectral. Discussions with K Leschke reveal that there is a similar phenomenon for the Darboux transformations of conformal tori: when a point on the multiplier spectral curve corresponds to \( \text{dim}( \ker(D)) > 1 \) one can find non-isospectral Darboux transforms. Such points correspond to singularities on \( X \). An important property of the non-isospectral transformations for KdV solutions is that singularities can be both added and removed: it would be very interesting to see if a similar situation applies to the Darboux transforms of conformal tori.

Appendix A. \( \tau \)-functions.

Our aim is to prove theorem 3.3. To do this we draw heavily on similar results proven in [26], but rather than use the theory of \( \tau \)-functions for \( \text{Gr}(\mathbb{C}^2) \) we will reduce everything to expressions using the \( \tau \)-functions for \( \text{Gr}(\mathbb{C}) \) since this gets us more quickly the result we desire. I will assume some familiarity with [26] and where possible use identical notation.

Let us quickly recall what we need. Let \( \mathcal{H} \) denote \( L^2(S^1, \mathbb{C}) \). To each \( S \subset \mathbb{Z} \) for which both \( S \setminus \mathbb{N} \) and \( \mathbb{N} \setminus S \) have the same finite cardinality (so \( S \) has zero virtual cardinal, in the terminology of [26]) we assign the closed subspace \( \mathcal{H}_S \subset \mathcal{H} \) spanned by \( \{ \zeta^s_j : s_j \in S \} \). A closed subspace \( V \subset \mathcal{H} \) belongs to the Segal-Wilson Grassmannian \( \text{Gr}(\mathbb{C}) \) precisely when it is the graph of a Hilbert-Schmidt operator \( \mathcal{H}_S \rightarrow \mathcal{H}_S^\perp \) for some \( S \). To any admissible basis of \( V \) (which we think of as a linear map \( v : \mathcal{H}_+ \rightarrow V \)) we can assign Plücker coordinates

\[
v^S = \det(v_{ij})_{i,j \in \mathbb{N}}, \quad \text{for } v_{ij} = \langle v(\zeta^j), \zeta^s_i \rangle.
\]

We recall that \( v^S \) vanishes if and only if the orthogonal projection \( V \rightarrow \mathcal{H}_S \) fails to be invertible. Our main tool will be the the \( \tau \)-function:

\[
(46) \quad \tau_V : \Gamma_+ \rightarrow \mathbb{C}; \quad \tau_V(\gamma) = \det(v_+ + a^{-1}bv_-).
\]

where \( \gamma = \exp(z_1\zeta + z_2\zeta^2 + \ldots) \), the action (by multiplication) of \( \gamma \) on \( \mathcal{H}_+ \oplus \mathcal{H}_- \) is represented by the block matrix

\[
\begin{pmatrix}
a & b \\
0 & d
\end{pmatrix}.
\]

This function vanishes precisely when \( \gamma \circ V \) leaves the big cell. A crucial result for our purposes is [26, Prop 8.3], which states that

\[
\tau_V(\gamma) = \sum_S v^S \tau_S(\gamma).
\]
Segal & Wilson showed that \( \tau_S(\gamma) \) is a polynomial in \( z_1, z_2, \ldots \) related to the Schur polynomials and consequently proved (in [26, Prop 8.6]) that for any \( V \in Gr(\mathbb{C}) \)

\[
\tau_V(z_1, 0, \ldots) = cz_1^m + \text{higher terms},
\]

where \( c \neq 0, m \in \mathbb{N} \).

**Proof of theorem**. We will think of \( H \) as \( L^2(S^1, \mathbb{C}^2) \). First recall the isometric isomorphism \( \iota : H \to H, (u(\zeta), v(\zeta)) \mapsto u(\zeta^2) + \zeta v(\zeta^2) \). This identifies \( Gr(\mathbb{C}^2) \) with \( Gr(\mathbb{C}) \) and we deduce that: (a) each \( W \) also has Plücker coordinates \( w^S \) by this identification, (b) \( W \in Gr(\mathbb{C}^2) \) if and only if it is the graph of a Hilbert-Schmidt operator \( H_S \to H^S_\delta \) where \( \iota(H_S) = H_S \). We can write

\[
H_S = H_{S_0} \oplus H_{S_1} \subset H \oplus H = H,
\]

where \( S_0 = \{ j : 2j \in S \} \) and \( S_1 = \{ j : 2j + 1 \in S \} \). We cannot claim that \( S_0, S_1 \) have equal virtual cardinal (although when they do this must clearly be zero). However, when \( W \in Gr(\mathbb{H}) \) it follows from \( jW = W \) that \( w^S \neq 0 \) precisely when \( S_0 = S_1 \). Now let us consider the action of \( \Gamma_+ \times \Gamma_+ \) on \( H \):

\[
(\gamma_0, \gamma_1) \circ (u, v) \mapsto (\gamma_0 u, \gamma_1 v).
\]

We define a \( \tau \)-function for this group by setting

\[
\hat{\tau}_W : \Gamma_+ \times \Gamma_+ \to \mathbb{C}; \quad \hat{\tau}_W(\gamma_0, \gamma_1) = \det(w_+ + a^{-1}b w_-)
\]

where \( w : H_+ \to W \) is some choice of admissible basis and \( a, b \) are the appropriate entries in the matrix representation for the action of \( (\gamma_0, \gamma_1) \). This function vanishes precisely when the \( \Gamma_+ \times \Gamma_+ \) orbit of \( W \) leaves the big cell. Further, when \( W = H_S \) the determinant reduces to one of a finite rank matrix. By analogy with the results from [26] it is easy to see that we have

\[
\hat{\tau}_W(\gamma_0, \gamma_1) = \sum_S w^S \hat{\tau}_S(\gamma_0, \gamma_1).
\]

Now, the action of \( (\gamma_0, \gamma_1) \) is diagonal for \( H = H \oplus H \) and so, writing \( w = w^0 \oplus w^1 \) (and so forth) to represent the summands in this splitting, we obtain

\[
\hat{\tau}_S(\gamma_0, \gamma_1) = \det(w^0_+ + a^{-1}_0 b_0 w^0_-) \det(w^1_+ + a^{-1}_1 b_1 w^1_-).
\]

This is achieved by making a finite number of row and column changes to bring into block diagonal form the finite rank determinant which computes \( \hat{\tau}_S \). The subgroup along which \( \gamma_1 = \gamma_0 \) preserves \( Gr(\mathbb{H}) \). Therefore, assuming \( W \in Gr(\mathbb{H}) \),

\[
\hat{\tau}_S(\gamma_0, \gamma_0) = \pm \tau_{S_0}(\gamma_0) \tau_{S_0}(\gamma_0).
\]

It follows that for some \( c \neq 0 \) and \( m \in \mathbb{N} \) we have

\[
\hat{\tau}_W(z, 0, \ldots) = c|z|^{2m} + \text{higher terms}.
\]

Thus \( \hat{\tau} \) cannot vanish identically along \( \Gamma_1 \)-orbits in \( Gr(\mathbb{H}) \).
[1] C Bohle, *Möbius invariant flows of tori in $S^4*, doctoral thesis, Technische Universität Berlin, 2003.
[2] C Bohle, *Constrained Willmore tori in the 4-sphere*, arXiv 0803.06331v1.
[3] C Bohle, K Leschke, F Pedit & U Pinkall, *Conformal maps from a 2-torus to the 4-sphere*, arXiv 0712.2311v1 (2007).
[4] C Bohle, F Pedit & U Pinkall, *Spectral curves of quaternionic holomorphic line bundles over 2-tori*, in preparation.
[5] F E Burstall, D Ferus, K Leschke, F Pedit & U Pinkall, *Conformal geometry of surfaces in $S^4$ and quaternions*, Lecture Notes in Mathematics 1772, Springer, 2002.
[6] F E Burstall, F Pedit & U Pinkall, *Schwarzian derivatives and flows of surfaces*, Differential geometry and integrable systems (Tokyo, 2000), 39–61, Contemp. Math., 308, Amer. Math. Soc., Providence, RI, 2002.
[7] F Ehlers & H Knörrer, *An algebro-geometric interpretation of the Bäcklund transformation for the Kortweg-de Vries equation*, Comment. Math. Helvetici 57 (1982), 1–10.
[8] J Feldman, H Knörrer & E Trubowitz, *Riemann surfaces of infinite genus*. CRM Monograph Series, 20. AMS, Providence, RI, 2003.
[9] D Ferus, K Leschke, F Pedit & U Pinkall, *Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic tori*, Invent. Math. 146 (2001), 507–593.
[10] P G Grinevich & M U Schmidt, *Conformal invariant functionals of immersions of tori in $\mathbb{R}^3**, J. Geometry Phys. 26 (1998) 51–78.
[11] F Hélein & P Romon, *Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2*$, Comm. Anal. Geom. 10 (2002), 79–126.
[12] U Hertrich-Jeromin, I McIntosh, P Norman & F Pedit, *Periodic discrete conformal maps*, J. reine angew. Math. 534, 129–153 (2001).
[13] B G Konopelchenko, *Induced surfaces and their integrable dynamics*, Stud. Appl. Math. 96 (1996), no. 1, 9–51.
[14] B G Konopelchenko, *Weierstrass representations for surfaces in 4D spaces and their integrable deformations via DS hierarchy*, Ann. Global Anal. Geom. 18 (2000), no. 1, 61–74.
[15] B G Konopelchenko & G Landolfi, *Induced surfaces and their integrable dynamics II. Generalized Weierstrass representations in 4D spaces and deformations via DS hierarchy*, Stud. Appl. Math. 104 (2000), no. 2, 129–169.
[16] B G Konopelchenko & I A Taimanov, *Generalized Weierstrass formulae, soliton equations and Willmore surfaces. I. Tori of revolution and the mKdV equation*, arXiv dg-ga/9511005 (1995).
[17] K Leschke & P Romon, *Darboux transforms and spectral curves of Hamiltonian Stationary Lagrangian tori*, arXiv 0806.1848v1 (2008).
[18] I McIntosh, *On the existence of superconformal 2-tori and doubly periodic affine Toda fields*, J. Geometry Phys. 24 (1998), 223–243.
[19] I McIntosh, *Harmonic tori and generalised Jacobi varieties*, Comm. Anal. Geom. 9 (2001), 423–449.
[20] I McIntosh, *Quaternionic holomorphic curves and the QKP hierarchy*, proceedings of the international workshop on Integrable Systems, Geometry and Visualization, ed. R Miyaoka (2004) Kyushu University, Fukuoka, Japan.
[21] I McIntosh & P Romon, *The spectral data for Hamiltonian stationary Lagrangian tori in $\mathbb{R}^4*$, arXiv
[22] F Pedit & U Pinkall, *Quaternionic analysis on Riemann surfaces and differential geometry*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 389–400 (electronic).

[23] A Pressley & G Segal, *Loop Groups*, Oxford Math. Monographs, O.U.P. (1986).

[24] M U Schmidt, *Integrable systems and Riemann surfaces of infinite genus*. Mem. Amer. Math. Soc. 122 (1996), no. 581.

[25] M U Schmidt, *A proof of the Willmore conjecture*, preprint arXiv, math.DG/0203224 (2002).

[26] G Segal & G Wilson, *Loop groups and equations of KdV type*, Publ. IHES 61 (1985) 5–65.

[27] J-P Serre, *Algebraic groups and class fields*, Graduate Texts in Math. 117, Springer, New York (1988).

[28] I A Taimanov, *Modified Novikov-Veselov equation and differential geometry of surfaces*, Solitons, geometry, and topology: on the crossroad, 133–151, Amer. Math. Soc. Transl. Ser. 2, 179, Amer. Math. Soc., Providence, RI, 1997.

[29] I A Taimanov, *The Weierstrass representation of closed surfaces in $\mathbb{R}^3$*, (Russian) Funktsional. Anal. i Prilozhen. 32 (1998), no. 4, 49–62, 96; translation in Funct. Anal. Appl. 32 (1998), no. 4, 258–267 (1999)

[30] I A Taimanov, *On two-dimensional finite gap potential Schrödinger and Dirac operators with singular spectral curves*. Siberian Math. J. 44 (2003), 686–694.

[31] I A Taimanov, *Dirac operators and conformal invariants of tori in 3-space. I*. (Russian) Tr. Mat. Inst. Steklova 244 (2004), Din. Sist. i Smezhnye Vopr. Geom., 249–280 (English) Proceedings of the Steklov Institute of Math. 244 (2004), 233–263.

[32] I A Taimanov, *Finite gap theory of the Clifford torus*, Int. Math. Res. Notices (2005), 103–120.

[33] I A Taimanov, *Surfaces in the four-space and the Davey-Stewartson equations*, J. Geometry Phys. 56 (2006), 1235–1256.

[34] I A Taimanov, *Two-dimensional Dirac operator and surface theory*. Russian Math. Surveys 61:1 (2006), 79–159.

[35] G Wilson, *Commuting flows and conservation laws for Lax equations*, Math. Proc. Camb. Phil. Soc. 86 (1979), 131–143.

[36] G Wilson, *On two constructions of conservation laws for Lax equations*. Quart. J. Math. Oxford 32 (1981), 491–512.

Department of Mathematics, University of York, Heslington, York YO10 5DD, UK

E-mail address: im7@york.ac.uk