A TYPE II HALF LOGISTIC EXPONENTIATED-G FAMILY OF DISTRIBUTIONS WITH APPLICATIONS TO SURVIVAL ANALYSIS

Olalekan Akanji Bello, Sani Ibrahim Doguwa, Abukakar Yahaya and Haruna Mohammed Jibril

Department of Statistics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria.

Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria.

*Corresponding author; e-mail: olalekan4sure@gmail.com

ABSTRACT

Statisticians have created and proposed new families of distribution by extending or generalizing existing distributions. These families of distributions are made more flexible in fitting different types of data by adding one or more parameters to the baseline distributions. In this article, we present a new family of distributions called Type II half-logistic exponentiated-G family of distributions. We discuss some of the statistical properties of the proposed family such as explicit expressions for the quantile function, probability weighted moments, moments, generating function, survival and order statistics. The new family’s sub-models were discussed. We discuss the estimation of the model parameters by maximum likelihood. Two real data sets were employed to show the usefulness and flexibility of the new family.

Keywords: Hazard rate, Survival, Exponentiated–G, Type II Half Logistic-G, Maximum likelihood, Moments, Order Statistics.

INTRODUCTION

Real-life events are frequently described using statistical distributions. Because of its efficiency, the theory of statistical distributions is extensively studied, and new distributions are created. There is a strong desire among statisticians to create statistical distributions that are more flexible. There have been many different types of generalized distributions developed and applied to various phenomena. In many fields, such as economics, engineering, biological studies, and environmental sciences, several continuous univariate distributions have been extensively used for data modeling. However, applied areas such as finance, lifetime analysis and insurance clearly require extended forms of these distributions. So, various classes of distributions have been developed by extending common families of continuous distributions. A new improvement for creating and extending traditional distributions is the generated family of continuous distributions. The newly generated families have been extensively researched in a variety of fields, and they provide greater application flexibility. Gupta et al., (1998) pioneered the exponentiated-G (E-G) class, Eugene et al., (2002) pioneered beta-G, Marshall and Olkin (1997) pioneered Marshall-Olkin-G, Zografos and Balakrishnan (2009) pioneered gamma G distributions, Kumaraswamy Weibull G distribution by Cordeiro et al., (2010), Ristic and Balakrishnan (2011) an Alternative Gamma G distribution, Kumaraswamy-G family by Cordeiro and Castro(2011), Kummer beta generalized family by Cordeiro et al., (2012), a new methods for generating families of continuous distributions by Alzaatreh et al., (2013). Exponentiated T-X family by Alzaghah et al., (2013), Weibull-G family by Silva et al., (2014), Exponentiated half-logistic family by Cordeiro et al., (2014a), Lomax Generator by Cordeiro et al., (2014b), Logistic-generated distribution by Torabi and Montazeri (2014), beta Marshall-Olkin family of distributions by Alizadeh et al., (2015a), Kumaraswamy OddLog-Logistic-G by Alizadeh et al., (2015b), the logistic-X family by Tahir et al., (2016), the Topp-Leone family of distributions by Al-Shomrani et al., (2016), Exponentiated Marshall-Olkin family of distributions by Dias et al., (2016), Generalized Burr-G family of distributions by Nasir et al.,(2017), Odd Exponentiated Half-Logistic-G family by Afify et al., (2017), Type II Half-Logistic Family of Distributions by Hassan et al., (2017), Weibull-X family of distributions by Ahmad et al., (2018), Marshall-Olkin generalized-G (MOG-G) family of distribution by Yousouf et al., (2018), Exponentiated Kumaraswamy-G by Silva et al., (2019) More recently, Topp-Leone exponentiated-G(TLEX-G) family of distributions by Ibrahim et al., (2020a), Topp Leone Kumaraswamy-G family of distribution by Ibrahim et al., (2020b), Burr X Exponential-G Family by Sanusi et al., (2020a), Topp-Leone Exponential-G family of distributions by Sanusi et al., (2020b), Kumaraswamy-Odd Rayleigh-G family by Falgore and Doguwa (2020a), Inverse Lomax-G (IL-G) family of distributions by Falgore and Doguwa (2020b), Inverse Lomax Exponentiated G family of distributions by Falgore and Doguwa (2020c), Weibull-Odd Frechet family of distributions by Usman et al., (2020), Type II Exponentiated Half-logistic family of distributions by Al-Mofleh et al., (2020), Odd Weibull-Topp-Leone-G power series family of distributions by Broderick et al., (2021), Exponentiated Half Logistic Odd Lindley-G (EHLOG) distribution by Whatmore et al.,(2021). The ability to model both monotonically and non-monotonically increasing, decreasing and constant or more importantly with bathtub-shaped failure rates, even if the baseline failure rate is monotonic, is the motivation for extending distributions for modeling lifetime data. The following are the basic vindications for creating a new family of distributions in practice: to generate skewness for symmetrical models; to generate distributions with negatively skewed, positively skewed, and symmetric; to define special models with all types of hazard rate functions; to make the kurtosis more flexible than that of the baseline distribution, to
construct tail weight distributions for modeling various real data sets; to provide consistently better fits than other generated distributions with the same underlying model.

The aim of this paper is to develop and investigate the Type II Half-Logistic Exponentiated-G family of distributions (TIIHLEt-G).

MATERIALS AND METHOD
Type II Half-Logistic Exponentiated-G Family
The Exponentiated G (Et-G) family of distributions as defined in Ibrahim et.al., (2020a) has cumulative distribution function (cdf) given as

\[
F_{EG}(x; \alpha, \beta) = \left[ H(x; \beta) \right]^\alpha
\]

and its corresponding probability density function (pdf) is defined as

\[
f_{EG}(x; \alpha, \beta) = \alpha h(x; \beta) \left[ H(x; \beta) \right]^{\alpha - 1}, \quad x > 0
\]

where \( \alpha > 0 \) is the shape parameter and \( h(x; \beta) \) and \( H(x; \beta) \) are the pdf and cdf of the baseline distribution with parameter vector \( \beta \).

Hassan et.al., (2017) proposed the Type II half logistic family (TIIHL) of distributions with cdf defined as:

\[
F_{TIIHL}(x; \lambda, \xi) = \frac{2[1 - G(x; \xi)]^\lambda}{1 + [G(x; \xi)]^\lambda}
\]

and its corresponding pdf is

\[
f_{TIIHL}(x; \lambda, \xi) = \frac{2\lambda g(x; \xi)[G(x; \xi)]^{\lambda - 1}}{[1 + [G(x; \xi)]^\lambda]^2}
\]

where \( \lambda > 0 \), \( x > 0 \) and \( G(x; \xi) \) and \( g(x; \xi) \) are the cdf and pdf of the baseline distribution with parameter vector \( \xi \).

Proposition
The cdf of a new family of distribution that extends the TIIHL family called Type II Half-Logistic Exponentiated-G Family of distributions is given as

\[
F_{TIIHLEt-G}(x; \lambda, \alpha, \beta) = \frac{2H^{\alpha\lambda}(x; \beta)}{1 + H^{\alpha\lambda}(x; \beta)}
\]

and the pdf is derived as

\[
f_{TIIHLEt-G}(x; \lambda, \alpha, \beta) = \frac{\partial F_{TIIHLEt-G}(x; \lambda, \alpha, \beta)}{\partial x}
\]

\[
= \frac{2\lambda h(x; \beta)H^{\alpha\lambda - 1}(x; \beta)\left[ H^{\alpha(\lambda - 1)}(x; \beta) \right]}{\left[ 1 + H^{\alpha\lambda}(x; \beta) \right]^2}, \quad x > 0, \lambda, \alpha > 0 \text{ and } \beta \text{ is parameter vector}
\]

Proof:
Let the Exponentiated G family be the baseline family with cdf and pdf given in equations (1) and (2) respectively, then the proposed Type II Half Logistic Exponentiated-G family of probability distributions has cdf given as:

\[
F_{TIIHLEt-G}(x; \lambda, \alpha, \beta) \int_0^x \frac{2\lambda h(t; \beta)H^{\alpha\lambda - 1}(t; \beta)\left[ H^{\alpha}(t; \beta) \right]^{\lambda - 1}}{\left[ 1 + H^{\alpha}(t; \beta) \right]^2} \, dt
\]

Let \( y = \left[ H(t; \beta) \right]^\alpha \); when \( t = 0, y = 0 \); when \( t = x, y = \left[ H(x; \beta) \right]^\alpha \); \( \partial y = \alpha h(t; \beta)\left[ H(t; \beta) \right]^{\alpha - 1} \)

Then
A Type II Half Logistic…

\[ F_{THILEt-G}(x;\lambda,\alpha,\beta) = \int_0^{[H(x;\beta)]^{\gamma}} \frac{2\lambda y^{\gamma-1}}{[1 + y^{\gamma}]} \, dy \]  

(7)

Let

\[ M = \left[1 + y^{\gamma}\right]^2 \] when \( y = 0 \), \( m = 1 \); when \( H^{\alpha}(x;\beta), m = \left[1 + H^{\alpha}(x;\beta)\right]^2\); \( \partial m = 2\left[1 + y^{\gamma}\right]\lambda y^{\gamma-1} \partial y \)

\[ \therefore \partial y = \frac{2\lambda y^{\gamma-1}}{\left[1 + y^{\gamma}\right]} 

F_{THILEt-G}(x;\lambda,\alpha,\beta) = \int_0^{[1 + H^{\alpha}(x;\beta)]^2} \frac{3}{2} \partial m = -2m^{-\frac{1}{2}} \left[1 + H^{\alpha}(x;\beta)\right]^{-1} + 2

= 2 - \frac{2}{1 + H^{\alpha}(x;\beta)} = \frac{2 + 2H^{\alpha}(x;\beta) - 2}{1 + H^{\alpha}(x;\beta)}

Thus

\[ F_{THILEt-G}(x;\lambda,\alpha,\beta) = \frac{2H^{\alpha}(x;\beta)}{1 + H^{\alpha}(x;\beta)} \]  

(8)

Important Representation

In this section, we derived a useful representation for the TIHLEt−G pdf and cdf. Due to the fact that the generalized binomial series is

\[ (1 + Z)^{-\beta} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta + i - 1}{i} z^i, \]  

(9)

For \(|z| < 1\), and \( \beta \) is a positive real non integer. The density function of the TIHLEt-G family is then obtained by using the binomial theorem (9) to (6).

\[ f(x) = \sum_{i=0}^{\infty} 2\lambda ah(x)(-1)^i(i + 1)H_{(x)}^{\alpha(i-1)+1} \]  

(10)

Then, the pdf (10) can be written as:

\[ f(x) = \sum_{i=0}^{\infty} \partial_i h(x)H_{(x)}^{\alpha(i)+1} \]  

(11)

Where

\[ \partial_i = 2\lambda \alpha(-1)^i(i + 1) \]

In addition, an expansion for the \([F(x)]^h\) is produced, with \( h \) being an integer, and the binomial expansion is worked out once more.

\[ [F(x)]^h = \sum_{j=0}^{h} 2^j (-1)^j \binom{h + j - 1}{j} H_{(x)}^{\alpha(j+h)} \]  

(12)

Moreso, the binomial expansion is applied to \( H^{\alpha(j+h)}(x) \) by adding and subtracting 1, so \([F(x)]^h\) can now be expressed as follows

\[ [F(x)]^h = \sum_{j=0}^{h} \sum_{m=0}^{\infty} 2^j (-1)^j \binom{h + j - 1}{j} \binom{\alpha(j+h)}{m} \left[1 - H(x)\right]^m \]  

(13)

For \( m \) is a real, then \([F(x)]^h\) is as follows
A Type II Half Logistic…

\[
[F(x)]^h = \sum_{p=0}^{\infty} t_p H(x)^p
\]  
(14)

Where

\[
t_p = \sum_{j=0}^{h} \sum_{m=0}^{\infty} 2^h (-1)^{j+m+p} \binom{h+j-1}{j} \binom{\alpha \lambda (j+h)}{m} (p)
\]

Statistical Properties
In this section, we derived statistical properties of the new family of distribution.

Probability weighted moments

Greenwood et.al., (1979) introduced a class of moments known as probability weighted moments (PWMs). This class is used to derive inverse form estimators for the parameters and quantiles of a distribution. The PWMs, represented by \( \tau_{r,s} \), can be derived for a random variable \( X \) using the following relationship.

\[
\tau_{r,s} = E\left[ X^r F(X)^s \right] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s dx
\]  
(15)

The PWMs of TIIHLEt-G is derive by substituting (11) and (14) into (15), and replacing \( h \) with \( s \), as follows

\[
\tau_{r,s} = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} t_p \partial_x^r h(x)H^{p+\alpha \lambda (i+1)-1} dx
\]  
(16)

Now ,

\[
\tau_{r,s} = \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} t_p \partial_x^r \tau_{r,p+\alpha \lambda (i+1)-1}
\]  
(17)

Where,

\[
\tau_{r,p+\alpha \lambda (i+1)-1} = \int_{0}^{\infty} x^r h(x)H^{p+\alpha \lambda (i+1)-1} (x) dx
\]

Another formula will be obtained by applying the quantile function, as follows:

\[
\tau_{r,s} = \frac{1}{\partial u} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} t_p \partial_u (Q_H(u))^r u^{p+\alpha \lambda (i+1)-1} du
\]  
(18)

Moments
Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, we derive the \( r^{th} \) moment for the new family.

\[
\mu_r = E(x^r) = \int_{0}^{\infty} x^r f(x) dx
\]  
(19)

By using the important representation of the pdf in equation (11), we have

\[
\mu_r = \int_{0}^{\infty} \partial_x^r h(x) H^{\alpha \lambda (i+1)-1} (x) dx
\]  
(20)
Then, \( \mu_i = \sum_{r=0}^{\infty} \theta_r \tau_r,\alpha (i+1)-1 \)

Where \( \tau_r,\alpha (i+1)-1 \) is the PWMs.

Also, moment can be obtained, based on the parent quantile function, as follow;

\[
\mu_i \hat{=} \sum_{r=0}^{\infty} \theta_r \left( \int_0^1 (Q_r(u))^r u^{\alpha (i+1)-1} du \right)
\]  

(21)

Moment generating function (mgf)

The Moment Generating Function of \( x \) is given as:

\[
M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_i \hat{=} \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^r}{r!} \theta_r \tau_r,\alpha (i+1)-1
\]  

(22)

Another formula will be obtained by applying the quantile function, as follows:

\[
M_u(t) = \sum_{i=0}^{\infty} \theta_i \left[ \int_0^1 e^{(Q_u(u))^r} u^{\alpha (i+1)-1} du \right]
\]  

(24)

Mean deviation

The mean deviation about the mean \( \sigma_2 = E(|x - \mu|) \) and about the median \( \sigma_1 = E(|x - M|) \) of \( x \) given as

\[
\sigma_1 = 2 \mu_i F(\mu_i) - 2 J(\mu_i)
\]  

(25)

\[
\sigma_2 = \mu_i - 2 J(M)
\]  

(26)

Where \( \mu_i = E(x) \), is the mean, \( M = \text{Median}(x) = Q(0.5) \), is the median and \( J(c) = \int_0^c x f(x) dx \) which is the first incomplete moment.

Additional form is derived as follows, depending on the parent quantile function.

\[
j(c) = \sum_{i=0}^{\infty} \theta_i \left[ \int_0^1 Q_i(u) u^{\alpha (i+1)-1} du \right]
\]  

(27)

Reliability function

The reliability function which is also known as survivor function that gives the probability that a patient will survive longer than specified period of time. It is defined as

\[
R(x; \lambda, \alpha, \beta) = 1 - F(x; \lambda, \alpha, \beta)
\]  

(28)

\[
R(x; \lambda, \alpha, \beta) = 1 - \frac{H^{\alpha-1}(x; \lambda, \alpha, \beta)}{1 + H^{\alpha-1}(x; \lambda, \alpha, \beta)}
\]  

(29)

Obtaining survival probabilities for different values of time \( x \) provides crucial summary information from survival data.

Hazard function

The hazard function is the probability of an event of interest occurring within a relatively short time frame and is defined as:

\[
T(x; \lambda, \alpha, \beta) = \frac{2 \lambda \alpha \lambda h(x; \lambda, \alpha, \beta) H^{\alpha-1}(x; \lambda, \alpha, \beta) H^{\alpha (\lambda-1)}(x; \lambda, \alpha, \beta)}{1 - H^{2\alpha-1}(x; \lambda, \alpha, \beta)}
\]  

(30)

The hazard function also known as conditional failure rate, gives the instantaneous potential per unit time for the event of interest to occur, given that the individual has survived up to time \( x \).

Quantile function

The quantile function is vital tool to create random variables from any continuous probability distribution. As a result, it has a significant position in probability theory. For \( x \), the quantile function is \( F(x) = u \), where \( u \) is distributed as \( U(0,1) \). The TIHLEt-G family is easily simulated by inverting equation (5) which yields the Quantile function \( Q(u) \) defined as:
\[ Q(u) = H^{-1} \left( \frac{u}{2-u} \right)^{1/\alpha} \]  

(31)

Where \( H^{-1} \) is the quantile function of the baseline cdf \( H(x; \beta) \). The first quartile, the median and the third quartile are obtained by putting \( u = 0.25, 0.5 \) and 0.75, respectively in equation (4.6).

**Order Statistics**

Many areas of statistics including reliability and life testing, have made substantial use of order statistics. Let \( X_1, X_2, \ldots, X_n \) be independent and identically distributed random variables with their corresponding continuous distribution function \( F(x) \). Let \( X_1, X_2, \ldots, X_n \) be independently distributed and continuous random variables from the TIIHLEt-G family of distribution. Let \( F_{r,n}(x) \) and \( f_{r,n}(x) \), \( r = 1, 2, 3, \ldots, n \) denote the cdf and pdf of the \( r \)th order statistics \( X_{r:n} \) respectively. David (1970) gave the probability density function of \( X_{r:n} \) as:

\[ f_{r,n}(x) = \frac{1}{B(r, n-r+1)} F^{r-1}(x) \left[ 1 - F(x) \right]^{n-r} f(x) \]  

(32)

By substituting equation (5) and equation (6) into equation (32), we have,

\[ f_{r,n}(x; \lambda, \alpha, \beta) = \frac{1}{B(r, n-r+1)} \left[ \frac{2H^{\alpha\lambda}(x; \lambda, \alpha, \beta)}{1 + H^{\alpha\lambda}(x; \lambda, \alpha, \beta)} \right]^{r-1} \left[ \frac{1 - H^{\alpha\lambda}(x; \lambda, \alpha, \beta)}{1 + H^{\alpha\lambda}(x; \lambda, \alpha, \beta)} \right]^{n-r} \times \frac{2\lambda \alpha h(x) H^{\alpha-1}(x; \lambda, \alpha, \beta) H^{\alpha(\lambda-1)}(x; \lambda, \alpha, \beta)}{\left[ 1 + H^{\alpha\lambda}(x; \lambda, \alpha, \beta) \right]^2} \]  

(33)

The equation above is called the \( r \)th order statistics for the TIIHLEt-G family of distributions. Let \( r = n \), then the probability density function of the maximum order statistics is

\[ f_{n,n}(x; \lambda, \alpha, \beta) = \frac{2n\lambda \alpha h(x; \lambda, \alpha, \beta) H^{\alpha-1}(x; \lambda, \alpha, \beta) H^{\alpha(\lambda-1)}(x; \lambda, \alpha, \beta)}{\left[ 1 + H^{\alpha\lambda}(x; \lambda, \alpha, \beta) \right]^2} \left[ \frac{2H^{\alpha\lambda}(x; \lambda, \alpha, \beta)}{1 + H^{\alpha\lambda}(x; \lambda, \alpha, \beta)} \right]^{n-1} \]  

(34)

Also, let \( r = 1 \), then the probability density function of the minimum order statistics is

\[ f_{1,n}(x; \lambda, \alpha, \beta) = \frac{2n\lambda \alpha h(x; \lambda, \alpha, \beta) H^{\alpha-1}(x; \lambda, \alpha, \beta) H^{\alpha(\lambda-1)}(x; \lambda, \alpha, \beta)}{\left[ 1 + H^{\alpha\lambda}(x; \lambda, \alpha, \beta) \right]^2} \left[ \frac{1 - H^{\alpha\lambda}(x; \lambda, \alpha, \beta)}{1 + H^{\alpha\lambda}(x; \lambda, \alpha, \beta)} \right]^{n-1} \]  

(35)

**Sub-Models**

In this section, we describe two sub-models of the TIIHLEt-G family namely, TIIHLEt- Exponential and TIIHLEt-Logistic respectively.

**Type II Half-Logistic Exponentiated Exponential (TIIHLEtE) Distribution**

The cdf and pdf of Exponential distribution which is our baseline distribution with parameter \( \theta \) are:

\[ H(x; \theta) = 1 - e^{-\theta x} \]  

(36)

And

\[ h(x; \theta) = \theta e^{-\theta x}, \quad x > 0, \theta > 0 \]  

(37)

The TIIHLEtE has cdf and pdf as follows:

\[ F_{TIIHLEtE}(x; \lambda, \alpha, \theta) = \frac{2 \left[ 1 - e^{-\theta x} \right]^{\alpha \lambda}}{1 + \left[ 1 - e^{-\theta x} \right]^{\alpha \lambda}}, \quad x > 0, \lambda, \alpha, \theta > 0 \]  

(38)

And

\[ f(x; \lambda, \alpha, \theta) = \frac{2\lambda \alpha \theta e^{-\theta x} \left[ 1 - e^{-\theta x} \right]^{\alpha - 1} \left[ 1 - e^{-\theta x} \right]^{\alpha(\lambda - 1)}}{\left[ 1 + \left( 1 - e^{-\theta x} \right)^{\alpha \lambda} \right]^2} \]  

(39)

Furthermore, the following are the reliability function, hazard rate function and the quantile function respectively:
A Type II Half Logistic...

\[ R(x; \lambda, \alpha, \theta) = \frac{1 - \left[1 - e^{-\theta x}\right]^{\alpha \lambda}}{1 + \left[1 - e^{-\theta x}\right]^{\alpha \lambda}} \]  

(40)

\[ T(x; \lambda, \alpha, \theta) = \frac{2\lambda \alpha \theta e^{-\theta x}\left[1 - e^{-\theta x}\right]^{\alpha - 1}\left[1 - e^{-\theta x}\right]^\alpha (\lambda - 1)}{1 - \left[1 - e^{-\theta x}\right]^{2\alpha \lambda}} \]  

(41)

\[ x = -\frac{1}{\theta} \ln \left[1 - \left(\frac{u}{2 - u}\right)^{\frac{1}{\alpha \lambda}}\right] \]  

(42)

Figure 1: Plots of PDF of the TIIHLEtE distribution for different parameter values.

It can be seen from figure 1 that the TIIHLEtE distribution has a positively skewed shape.
Type II Half-Logistic Exponentiated Log-Logistic (TIIHLEtLL) Distribution

cdf and pdf of Log-Logistic distribution with parameter $\theta$ as our baseline distribution are:

$$H(x; \theta) = \frac{x^\theta}{1 + x^\theta}$$  \hspace{1cm} (43)

$$h(x; \theta) = \frac{\theta x^{\theta-1}}{(1 + x^\theta)^2}, \quad x > 0, \theta > 0$$  \hspace{1cm} (44)

Now

$$H^\alpha(x; \theta, \alpha) = \left[ \frac{x^\theta}{1 + x^\theta} \right]^{-\alpha \lambda} = \left[ x^{-\lambda} + 1 \right]^{-\alpha \lambda}$$
The TIIHLEtLL distribution has CDF and PDF given as:

\[
F_{\text{TIIHLEtLL}}(x; \lambda, \alpha, \theta) = \frac{2\left(x^{-\theta} + 1\right)^{-\alpha \lambda}}{1 + \left(x^{-\theta} + 1\right)^{-\alpha \lambda}}, \quad x > 0, \lambda, \alpha, \theta > 0
\]  

(45)

\[
f_{\text{TIIHLEtLL}}(x; \lambda, \alpha, \theta) = \frac{2\lambda \alpha \theta x^{\theta - 1}\left(1 + x^{\theta}\right)^{\frac{1}{2}} x^{-\theta - 1} x^{\theta} + 1}{1 + \left(x^{-\theta} + 1\right)^{-\alpha \lambda}^2}
\]  

(46)

Moreso, the following are the reliability function, hazard rate function and the quantile function respectively:

\[
R(x; \lambda, \alpha, \theta) = \frac{1 - \left(x^{-\theta} + 1\right)^{-\alpha \lambda}}{1 + \left(x^{-\theta} + 1\right)^{-\alpha \lambda}}
\]  

(47)

\[
T(x; \lambda, \alpha, \theta) = \frac{2\lambda \alpha \theta x^{\theta - 1}\left(1 + x^{\theta}\right)^{\frac{1}{2}} x^{-\theta - 1} x^{\theta} + 1}{1 - \left(x^{-\theta} + 1\right)^{-\alpha \lambda}^2}
\]  

(48)

\[
x = \left[\frac{u}{2 - u}\right]^{-\alpha \lambda - 1}
\]  

(49)

Figure 3: Plots of Pdf of the TIIHLEtLL distribution for different parameter values.

It can be seen from figure 3 that the TIIHLEtLL distribution has a positively skewed shape.
The TIHLEtLL distribution hazard plots in figure 4 illustrate that the hazard function has a bathtub-shaped failure rate.

Parameter Estimation
In this paper, we explore the maximum likelihood technique to estimate the unknown parameters of the TIHLEt-G family for complete data. Maximum likelihood estimates (MLEs) have appealing qualities that may be used to generate confidence ranges and provide simple approximations that function well in finite samples. In distribution theory, the resulting approximation for MLEs is easily handled, either analytically or numerically. Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample of size n from the TIHLEt-G family. Then, the likelihood function based on observed sample for the vector of parameter $(\lambda, \alpha, \beta)^T$ is given by

$$L(\phi) = n \log(2) + n \log(\lambda) + n \log(\alpha) + \sum_{i=0}^{n} \log h(x, \beta) + (\alpha - 1) \sum_{i=0}^{n} \log H(x, \beta)$$

$$+ \alpha(\lambda - 1) \sum_{i=0}^{n} \log H(x, \beta) - 2 \sum_{i=0}^{n} \log \left[ 1 + H^{\alpha \lambda}(x, \beta) \right]$$

(50)

The components of score vector $U = (U_\lambda, U_\alpha, U_\beta)^T$ are given as

$$U_\lambda = \frac{n}{\lambda} + \sum_{i=0}^{n} \log \left[ H(x, \beta) \right] - 2 \sum_{i=0}^{n} \frac{H^{\alpha \lambda}(x, \beta) \log \left[ H^\alpha(x, \beta) \right]}{1 + H^{\alpha \lambda}(x, \beta)}$$

(51)

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=0}^{n} \log H(x, \beta) + (\lambda - 1) \sum_{i=0}^{n} \log H(x, \beta) - 2 \lambda \sum_{i=0}^{n} \frac{H^{\alpha(\lambda - 1)}(x, \beta)H^\alpha(x, \beta) \log \left[ H(x, \beta) \right]}{1 + H^{\alpha \lambda}(x, \beta)}$$

(52)

$$U_\beta = \sum_{i=0}^{n} \frac{h(x, \beta)^\beta}{h(x, \beta)} + (\alpha - 1) \sum_{i=0}^{n} \frac{H(x, \beta)^\beta}{H(x, \beta)} + \alpha(\lambda - 1) \sum_{i=0}^{n} \frac{H(x, \beta)^\beta}{H(x, \beta)}$$

$$- 2\alpha \lambda \sum_{i=0}^{n} \frac{H^{\alpha(\lambda - 1)}(x, \beta)H^\alpha(x, \beta) \log \left[ H(x, \beta) \right]}{1 + H^{\alpha \lambda}(x, \beta)}$$

(53)

The MLEs are obtained by setting $U_\lambda, U_\alpha$ and $U_\beta$ to zero and solving these equations simultaneously. These Equations cannot be solved analytically, necessitating the use of analytical tools to solve them numerically.

Applications to Real Data
In this section, we fit the TIHLEtE distribution to two real data sets and give a comparative study with the fits of the Type II half logistic exponential (TIHLE) by Elgarhy et al. (2018), Topp-Leone exponential distribution (TLEx) by Al-Shomrani et al. (2016), Kumaraswamy exponential distribution (KE) by Adepoju and Chukwu (2015), Exponentiated exponential distribution (ExEx) by Gupta and Kundu (1999), and Logistic-X exponential distribution (LoEx) by Oguntunde et al. (2018), as comparator distributions for illustrative purposes.

The TIHLE distribution proposed by Elgarhy et al. (2018) has probability density function given as:

\[
 f(x; \lambda, \alpha) = \frac{2\lambda \theta e^{-\lambda x} \left[1 - e^{-\theta x}\right]^{\lambda - 1}}{\left[1 + \left[1 - e^{-\theta x}\right]^\alpha\right]^2} \tag{54}
\]

The TLEx distribution proposed by Al-Shomrani et al. (2016) has pdf defined as:

\[
 f(x; \alpha, \theta) = 2\alpha \theta e^{-2\alpha x} \left[1 - e^{-2\alpha x}\right]^\theta - 1 \tag{55}
\]

The KE distribution developed by Adepoju and Chukwu (2015) has pdf defined as:

\[
 f(x; \alpha, \lambda, \theta) = \alpha \lambda \theta e^{-\alpha x} \left(1 - e^{-\alpha x}\right) \left[1 - \left[1 - e^{-\alpha x}\right]^\lambda\right]^{\lambda - 1} \tag{56}
\]

The ExEx distribution pioneered Gupta and Kundu (1999) has pdf given as:

\[
 f(x; \alpha, \theta) = \alpha \theta e^{-\alpha x} \left[1 - e^{-\alpha x}\right]^\theta - 1 \tag{57}
\]

And the LoEx distribution developed by Oguntunde et al. (2018) has pdf given as:

\[
 f(x; \alpha, \lambda) = \frac{\lambda}{\alpha^2} x^{-\left(\lambda + 1\right)} \left[1 + \left(\alpha x\right)^{-\lambda}\right]^{-2} \tag{58}
\]

The two datasets that were used as examples in the application demonstrate the new family of distributions’ flexibility, applicability and ‘best fit’ compared to the above comparator distributions in modeling the data sets experimentally. The R programming language is used to carry out all of the computations.

**Data set 1**

The first data set as listed below represents the daily confirmed cases of COVID-19 positive cases record in Pakistan from March 24 to April 28, 2020, previously used by Al-Marzouki, et al., (2020):

108, 102, 133, 170, 121, 99, 236, 178, 250, 161, 258, 172, 407, 577, 210, 243, 281, 186, 254, 336, 342, 269, 520, 414, 463, 514, 427, 796, 555, 742, 642, 785, 783, 605, 751, 806.

![Figure 5: Fitted pdfs for the TIHLEtE, TIHLE, TLEx, KE, ExEx, and LoEx distributions to the data set 1](image-url)
Table 1: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 1

| Distributions | \( \alpha \) | \( \lambda \) | \( \theta \) | LL   | AIC  |
|---------------|-------------|-------------|-------------|------|------|
| TIIHLEtE      | 1.4789      | 2.2610      | 0.003       | -242.8838 | 491.7676 |
| TIIHLE        | 5.3268      | 0.0049      | -244.8945   | 493.7891 |
| ToLE          | 1.3304      | 0.0016      | -247.2426   | 498.4852 |
| KE            | 0.1284      | 0.0521      | 0.0431      | -254.0471 | 514.0942 |
| ExEx          | 0.0016      | 0.4800      | -262.3582   | 528.7164 |
| LoEx          | 0.0032      | 0.9270      | -263.1668   | 530.3336 |

Table 1 presents the results of the Maximum Likelihood Estimation of the parameters of the proposed distribution and the five comparator distributions. Based on the goodness of fit measure, the proposed distribution reported the minimum AIC value, though followed closely by the TIIHLE. The visual inspection of the fit presented in Figure 5, also confirms the superiority of the proposed distribution amongst its comparators. Thus the proposed distribution ‘best fit’ COVID 19 data set amongst the range of distributions considered.

Data set 2

The second data set shown below represents the mortality rate of the COVID-19 patients in Canada, previously used by Liu et al., (2021):

3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091, 3.2823, 3.8594, 4.0480, 4.1685, 3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592, 2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141, 1.9048.

Figure 6: Fitted pdfs for the TIIHLEtE, TIIHLE, TLEx, KE, ExEx, and LoEx distributions to the data set 2

Table 2: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 2

| Distributions | \( \alpha \) | \( \lambda \) | \( \theta \) | LL   | AIC  |
|---------------|-------------|-------------|-------------|------|------|
| TIIHLEtE      | 4.9661      | 4.9266      | 0.9995      | -48.8348 | 103.6695 |
| TIIHLE        | 6.5782      | 0.5936      | -54.5742    | 113.1484 |
| ToLE          | 2.9205      | 0.2653      | -62.4905    | 128.981 |

FUDMA Journal of Sciences (FJS) Vol. 5 No.3, September, 2021, pp 177 - 190
Table 2 shows the results of the Maximum Likelihood Estimation of the parameters of the TIIHLEtE distribution and the five comparator distributions. Based on the goodness of fit statistic AIC, the new distribution reported the minimum AIC value suggesting that the distribution is the ‘best fit’ to the mortality rate of the COVID-19 patients. The visual inspection of the fit presented in Figure 6, also reaffirms the superiority of the new distribution amongst its comparators.

**Conclusion**

A new family of continuous distributions called the Type II Half Logistic Exponentiated-G (TIIHLEt-E) family was proposed and studied. Some of the statistical properties of the proposed family, such as explicit quantile function expressions, probability weighted moments, moments, generating function, survivor functions, and order statistics were investigated. Some of the new family’s sub-models were discussed. The method of maximum likelihood was used to estimate the model parameters. In comparison to well-known models, two real data sets were used to highlight the importance and flexibility of one of the sub model. The findings reveal that the new model appears to be superior to the existing models considered and, therefore, provides new distribution to model data in many applications.

**REFERENCES**

Adepoju, K. A. and Chukwu, O. I. (2015). Maximum Likelihood Estimation of the Kumaraswamy Exponential Distribution with Applications, *Journal of Modern Applied Statistical Methods*, 14, 1, 208-214.

Afify, A. Z., Altun, E., Alizadeh, M., Ozel, G. and Hamedani, G.G.(2017). The Odd Exponentiated Half-Logistic-G Family: Properties, Characterizations and Applications. *Chilean Journal of Statistics*, 8(2), 65-91.

Ahmad, Z., Elgarhy, M., and Hamedani, G. G. (2018). A new Weibull-X family of distributions: properties, characterizations and applications. *Journal of Statistical Distributions and Applications*, 5:5.

Alizadeh, M., Cordeiro, G. M., de Brito, E. and Demtrio, C. G. B. (2015a). The beta Marshall-Olkin family of distributions. *Journal of Statistical Distributions and Applications*, 2(4), 1-18.

Alizadeh, M., Emadi, M., Doostparast, M., Cordeiro, G. M., Ortega, E. M. M. and Pescim, R. R. (2015b). A new family of distributions: The Kumaraswamy odd log-logistic properties and applications. *Hacettepe Journal of Mathematics and Statistics*, 44, 1491-1512.

Al-Marzouki, S., Jamal, F., Chesneau, C., and Elgarhy, M. (2020). Topp-Leone Odd Frechet Generated Family of Distributions with Applications to COVID-19 Data Sets, *Computer Modeling in Engineering and Sciences*, 125(1), 437-458.

Al-Mofleh Hazem, Elgarhy Mohamed, Afify Ahmed Z., and Zannon Mohammad.(2020). Type II Exponentiated Half Logistic generated family of distributions with applications. *Electronic Journal of Applied Statistical Analysis*, 13(2), 536-561 DOI: 10.1285/i20705948v13n2p536

Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S. and Shahbaz, M. Q. (2016). Topp-Leone family of distributions: Some properties and application. *Pak.j.stat.oper.res*, 12(3), 443-451.

Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71, 63-79.

Alzaghal, A., Lee, C. and Famoye, F. (2013). Exponentiated T-X family of distributions with some applications. *International Journal of Probability and Statistics*, 2, 3149.

Cordeiro, G. M., Ortega, E. M., and Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute*, 347(8), 1399-1429.

Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distribution. *Journal of Statistical Computations and Simulation*, 81, 883-898.

Cordeiro, G. M., Alizadeh, M. and Ortega, E. M. M. (2014a). The exponentiated half-logistic family of distributions: Properties and applications. *Journal of Probability and Statistics*, 1-21.

Cordeiro, G. M., Ortega, E. M., Popovic, B. V., and Pescim, R. R. (2014b). The Lomax generator of distributions: Properties, mini cation process and regression model. *Applied Mathematics and Computation*, 247, 465-486.

Cordeiro, G. M., Pescim, R. R., Demetrio, C. G. B., Ortega, E. M. M. and Nadarajah, S. (2012). The new class of Kummer beta generalized distributions. *Statistics and Operations Research Transactions*, 36, 153-180.

David, H. A. (1970). Order statistics, Second edition. Wiley, New York.

Dias, C. R. B., Cordeiro, G. M., Alizadeh, M., Marinho, P. R. D. and Colhó, H. F. C. (2016). Exponentiated Marshall-Olkin family of distributions. *Journal of Statistical Distributions and Applications*, 3:15

Elgarhy M., Muhammad Ahsan ul Haq and Ismat Perveen.(2018). Type II Half Logistic Exponential Distribution with Applications, *Annals of Data Science*.
Eugene, N., Lee, C. and Famoye, F. (2002). The beta-normal distribution and its applications. Communications in Statistics Theory and Methods, 31, 497-512.

Falgore, J.Y. and Doguwa, S.I. (2020a). Kumaraswamy-Old Rayleigh-G Family of Distributions with Applications. Open Journal of Statistics, 10, 719-734.

Falgore, J.Y. and Doguwa, S.I. (2020b). Inverse Lomax-Exponentiated G (IL-EG) Family of Distributions: Properties and Applications. Asian Journal of Probability and Statistics, 9(4), 48-64.

Falgore, J.Y. and Doguwa, S.I. (2020c). Inverse Lomax G Family of Distributions with applications to Breaking Strength Data. Asian Journal of Probability and Statistics 8 (2) 49-60.

Greenwood, J.A., Landwehr, J.M., and Matalas, N.C. (1979). Probability weighted moments: Definitions and relations of parameters of several distributions expressible in inverse form. Water Resources Research, 15, 1049-1054.

Gupta, R.C., Gupta, P.L. and Gupta, R.D. (1998). Modeling the survival times of the failure time data by Lehmann alternatives. Biometrika, 85, 161-168.

Hassan, A. S., Elgarhy, M. and Shakil, M. (2017). Type II Half Logistic Family of Distributions with Applications. Pakistan Journal of Statistics and Operation Research 13, 245-264.

Ibrahim, S., Doguwa, S.I., Audu, I. and Muhammad, J.H. (2020a). On the Topp Leone exponentiated-G Family of Distributions: Properties and Applications. Asian Journal of Probability and Statistics, 7(1), 1-15.

Ibrahim S, Doguwa S.I, Isah A and Haruna J.M.(2020b). The Topp Leone Kumaraswamy-G Family of Distributions with Applications to Cancer Disease. Journal of Biostatistics and Epidemiology, 6(1), 37-48.

Liu X., Ahmad Z., Gmeay A. M., Abdulrahman A. T., Hafez E. H., and Khalil N. (2021). Modeling the survival times of the COVID-19 patients with a new statistical model: A case study from China, PLoS ONE 16(7): e0254999. https://doi.org/10.1371/journal.pone.0254999

Marshall, A.W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the Exponential and Weibull families. Biometrika, 84, 641-652.

Nasir, M. A., Tahir, M. H., Jamal, F. and Ozel, G. (2017). New Generalized Burr Family of Distributions for the Lifetime Data, Journal of Statistics Applications and Probability, 6(2), 401-417.

Oguntunde, P. E., Khaleel, M. A., Adejumo, A. O., Okagbue, H. I., (2018). A study of an extension of the exponential distribution using logistic-x family of distributions, International Journal of Engineering and Technology, 7 (4), 5467-5471

Ristic, M. M. and Balakrishnan, N. (2011). The gamma-exponentiated exponential distribution. Journal of Statistical Computation and Simulation, 82, 1191-1206.

Sanusi, A., Doguwa, S., Audu, I. and Baraya, Y. (2020a). Burr X Exponential-G Family of Distributions: Properties and Application. Asian Journal of Probability and Statistics, 7, 58-75.

Sanusi, A., Doguwa, S., Audu, I. and Baraya, Y. (2020b). Topp Leone Exponential G Family of Distributions: Properties and Application. Journal of Science and Technology Research, 2(4), 83-96.

Silva, R., Silva, F.G., Ramos, M., Cordeiro, G., Marinho, P., De Andrade, A. N. T. (2019). The Exponentiated Kumaraswamy-G Class: General Properties and Application. Revista Colombiana de Estadistica 42(1), 1-33.

Silva, R.B., Bourguignon, M. and Cordeiro, G.M. (2014). The Weibull-G family of probability distributions. Journal of Data Science, 12, 53-68.

Tahiri , M.H., Zubair, M., Mansoor , M., Cordeiro , G. M., Alizadeh, M. and Hamedani, G. G. (2016). A New Weibull-G Family of Distributions. Hacettepe Journal of Mathematics and Statistics, 45(2), 629-647.

Torabi, H., Montazari, N.H. (2014). The logistic-uniform distribution and its application. Communications in Statistics Simulation and Computation, 43, 25512569.

Usman, A. , Doguwa, S. I. S., Alhaji, B. B. and Imam, A. T.(2020). A New Generalized Weibull-Odd Frechet Family of Distributions: Statistical Properties and Applications. Asian Journal of Probability and Statistics, 9(3), 25-43.

Whatmore Sengweni, Brodrick Olyude and Boikanyo Makutabe (2021). Journal of Nonlinear Sciences and Applications, 287309.

Yousof, H. M., Afy, A. Z., Nadarajah, S., Hamedani, G., and Aryal, G. R. (2018). The Marshall-Olkin generalized-G family of distributions with Applications. Statistica, 78(3), 273-295.

Zografos, K. and Balakrishnan, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference. Statistical Methodology, 6, 344-362.