Double Coverage with Machine-Learned Advice

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Abstract

We study the fundamental online $k$-server problem in a learning-augmented setting. While in the traditional online model, an algorithm has no information about the request sequence, we assume that there is given some advice (e.g. machine-learned predictions) on an algorithm’s decision. There is, however, no guarantee on the quality of the prediction and it might be far from being correct.

Our main result is a learning-augmented variation of the well-known Double Coverage algorithm for $k$-server on the line (Chrobak et al., SODA 1991) in which we integrate predictions as well as our trust into their quality. We give an error-dependent competitive ratio, which is a function of a user-defined confidence parameter, and which interpolates smoothly between an optimal consistency, the performance in case that all predictions are correct, and the best-possible robustness regardless of the prediction quality. When given good predictions, we improve upon known lower bounds for online algorithms without advice. We further show that our algorithm achieves for any $k$ an almost optimal consistency-robustness tradeoff, within a class of deterministic algorithms respecting local and memoryless properties.

Our algorithm outperforms a previously proposed (more general) learning-augmented algorithm. It is remarkable that the previous algorithm crucially exploits memory, whereas our algorithm is memoryless. Finally, we demonstrate in experiments the practicability and the superior performance of our algorithm on real-world data.

1 Introduction

The $k$-server problem is one of the most fundamental online optimization problems. Manasse et al. [39, 40] introduced it in 1988 as a generalization of other online problems, such as the prominent paging problem, and since then, it has been a corner stone for developing new models and techniques. We follow this line and investigate the $k$-server problem in the recently evolving framework of learning-augmented online computation.

We consider the $k$-server problem on the line, in which there are given $k$ distinct servers $s_1, \ldots, s_k$ located at initial positions on the real line. A sequence of requests $r_1, \ldots, r_n \in \mathbb{R}$ is revealed online one-by-one, that is, an algorithm only knows the current (unserved) request, serves it and only then sees the next request; it has no knowledge about future requests. To serve a request, (at least) one of the servers has to be moved to the requested point. The cost of serving a request is defined as the distance traveled by the server(s). The task is to give an online strategy of minimum total cost for serving a request sequence.

In standard competitive analysis, an online algorithm $\mathcal{A}$ is called $\mu$-competitive if for every instance $I$, there is some constant $c$ depending only on the initial configuration such that $\mathcal{A}(I) \leq \mu \cdot \text{Opt}(I) + c$.

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where $\mathcal{A}(I)$ denotes the cost of $\mathcal{A}$ on $I$ whereas $\text{Opt}(I)$ is the cost of an optimal solution that can be obtained when having full information about $I$ in advance.

Manasse et al. [40] gave a strong lower bound which rules out any deterministic online algorithm with a competitive ratio better than $k$. They also stated the famous $k$-server conjecture in which they conjecture that there is a $k$-competitive online algorithm for the $k$-server problem in any metric space and for any $k$. The conjecture has been proven to be true for special metric spaces such as the line [17], considered in this paper, the uniform metric space (paging problem) [47] and tree metrics [18]. For the $k$-server problem on the line, Chrobak et al. [17] devised the DoubleCoverage algorithm and proved a best possible competitive ratio $k$. For a given request, DoubleCoverage moves the (at most) two adjacent servers towards the requested point until the first of them reaches that point.

The past decades have witnessed a rapid advancement of machine learning (ML) methods, which nowadays can be expected to predict often—but not always—uncertain data with good accuracy. The lack of guarantees on the predictions and the need for trustable performance guarantees lead to the area of learning-augmented online algorithms. This recently emerging research area investigates online algorithms that have access to predictions, e.g., on parts of the instance or the algorithm’s execution, while not making any assumption on the quality of the predictions. Formally, we assume that a prediction has a certain quality $\eta \geq 0$. In the context of learning theory one may think of the loss of a prediction with respect to the ground truth. Accordingly, $\eta = 0$ refers loosely speaking to the case where the prediction was correct. In the field of learning-augmented algorithm this quantity is called prediction error. An algorithm does not know what quality a prediction has, but we can use it in the analysis to measure an algorithm’s performance depending on $\eta$. If a learning-augmented algorithm is $\mu(\eta)$-competitive for some function $\mu$, we say that the algorithm is $\alpha$-consistent if $\alpha = \mu(0)$ and $\beta$-robust if $\mu(\eta) \leq \beta$ for any prediction with prediction error $\eta$ [44].

Very recently, Antoniadis et al. [3] proposed learning-augmented online algorithms for general metrical task systems, a generalization of our problem. Their algorithm relies on simulating several online algorithms in parallel and keeping track of their solutions and cost. This technique crucially employs additional memory which can be a serious drawback in practice when decisions must be made without access to the history.

In this work, we introduce memory-constrained learning-augmented algorithms for the $k$-server problem on the line. An algorithm $\mathcal{A}$ is intuitively memory-constrained, if the decision for the next move of $\mathcal{A}$ only depends on the current situation (server positions, request and prediction). It is especially independent of previous requests. However, as the algorithm is allowed to move a server to any point of the real line, it could use its position to encode any information at a negligible cost. This issue is often addressed by forbidding algorithms to move several servers per request (hence, restricting to so-called lazy algorithms) which leads to the classical memoryless property, although variations of this definition exist [28]. A downside of this restriction is that deterministic memoryless algorithms cannot be competitive, and there is no distinction between the type of information gathered by DoubleCoverage and unconstrained information encoding. This difference has been nevertheless acknowledged by informally considering DoubleCoverage as memoryless [26], although noting immediately that such a definition for a non-lazy algorithm is cumbersome. In order to allow the behavior of DoubleCoverage, we formally define memory-constrained algorithms as algorithms allowed to move several servers, making decisions independently of previous requests, but with an erasable memory: for any set of $k$ distinct points and any starting configuration, there exists a finite sequence of requests among these $k$ points after which each point contains exactly one server. We will refer to such a sequence as a force to these $k$ points. This definition is quite general as it allows to pre-move some servers as DoubleCoverage does, and even allows information encoding, but provides a possibility to erase any information gathered. The algorithms we design will not abuse information encoding, but our lower bounds will hold in this context.
Further related work The past few years have exhibited several demonstrations of the power of learning-augmented algorithms improving on traditional online algorithms. Studied online problems include caching [3, 37, 45, 51], paging [24], ski rental [9, 20, 44, 50, 52], TCP acknowledgement [8, 9], bin packing [2], scheduling [6, 7, 22, 31, 43, 44, 52], secretary problems [5, 19], linear search [1], matching [30, 32], sorting [36], online covering problems [8], and possibly more by now. Learning-augmented algorithms have proven to be successful also in other areas, e.g., to speed up search queries [29], in revenue optimization [41], to compute low rank approximations [23], frequency estimation [21] and bloom filters [42].

More than a decade ago, Mahdian et al. [38] demonstrated performance improvements for online allocation algorithms when there is access to an accurate solution estimation. They further bounded the case where the estimation is inaccurate. While these bounds essentially correspond to consistency and robustness, they did not precisely measure the prediction quality. Yet they introduced a parameter to express the tradeoff between both bounds. In the recent field of learning-augmented algorithms, Kumar et al. [44] initiated the use of a similar parameter $\lambda \in [0, 1]$. It can be interpreted as an algorithm’s indicator of trust in the given predictions: smaller $\lambda$ indicates stronger trust and gives a higher priority to a better consistency at the cost of a worse robustness, and vice versa. Such parameterized consistency-robustness tradeoff has become standard for expressing the performance of learning-augmented algorithms when aiming for constant factors [2, 5, 8, 9, 22, 44, 50, 52].

As mentioned, Antoniadis et al. [3] provide a general learning-augmented framework for any metrical task systems which includes the $k$-server problem. Applied to the line metric, they devise a learning-augmented algorithm that crucially requires memory and obtains a 9-consistent and 9k-robust algorithm.

The $k$-server problem has been studied also in the context of reinforcement learning (RL), originating at [25] and including hierarchical RL learning [33] as well as deep RL learning [35].

The classical online $k$-server problem without access to predictions has been studied extensively, also in general metric spaces. The best known deterministic algorithm is the WORKFUNCTION algorithm [27] with a competitive ratio of $2k - 1$. For several special metric spaces there are even tighter bounds known for this algorithm [11, 53]. When allowing randomization, a $\Omega(\log k / \log \log k)$ lower bound holds [10] and a $(\log k)^{O(1)}$-competitive randomized algorithm is conjectured [26]. Restricting further to memoryless randomized algorithms increases the lower bound on the competitive ratio exponentially to $k$ [26] and some recent efforts focus on a more general variant in this setting [15].

The power of DOUBLECOVERAGE goes beyond its optimality for the $k$-server problem in tree metrics [18]. Recently, Buchbinder et al. [14] showed that it is a best possible deterministic algorithm for the more general $k$-taxi problem, even in general metric spaces using an embedding into hierarchically separated trees.

Our contribution We design learning-augmented memory-constrained online algorithms for the $k$-server problem on the line. Firstly, we define some more notation and the precise prediction model. We denote a server’s name as well as its position on the line by $s_i$, for $i \in \{1, 2, \ldots, k\}$. A configuration $C_t = (s_1, \ldots, s_k) \in \mathbb{R}^k$ is a snapshot of the server positions at a certain point in time. For a given instance, a $k$-server algorithm outputs a sequence of configurations $C_1, \ldots, C_n$ (also called schedule) such that for every $t = 1, \ldots, n$, we have $r_t \in C_t$. We denote the initial configuration by $C_0$. The objective function can be expressed as $\sum_{t=1}^n d(C_{t-1}, C_t)$, where $d(C_{t-1}, C_t)$ denotes the cost for moving the servers from $C_{t-1}$ to $C_t$. We assume w.l.o.g. $s_1 \leq \ldots \leq s_k$, as server overtakings can be uncrossed without increasing the total cost.

We employ a prediction model that predicts algorithmic choices of an optimal algorithm, that is predicting which server should serve a certain request. Given an instance $I$ composed of the request sequence $r_1, \ldots, r_n$, we define a prediction for $I$ as a sequence of indices $p_1, \ldots, p_n$ from the set $\{1, \ldots, k\}$. If $s_1, \ldots, s_k$ are the servers of some learning-augmented algorithm, we call $s_{p_t}$ the predicted server for the
Let \( t \)-th request. We call the algorithm that simply follows the predictions FrP, that is, it serves each request by the predicted server (to simplify computations, we still remove overtakings as mentioned above, which is equivalent to relabel servers by their position order). We denote its cost by FrP(\( I \)). We define the prediction error \( \eta = \text{FrP}(I) - \text{Opt}(I) \) as quality measure for our predictions. Note that this error definition is independent of our algorithm.

Our main result is a parameterized algorithm for the \( k \)-server problem on the line with an error-dependent performance guarantee that—when having access to good-quality predictions—beats the known lower bound for deterministic online algorithms.

**Theorem 1.** Let \( \lambda \in [0, 1] \). We define \( \beta(k) = \sum_{i=0}^{k-1} \lambda^{-i} \), for \( \lambda > 0 \), and \( \beta(k) = \infty \), for \( \lambda = 0 \). Further, let

\[
\alpha(k) = \begin{cases} 
1 + 2\lambda + 2\lambda^2 + \ldots + 2\lambda^{(k-1)/2} & \text{if } k \text{ is odd} \\
1 + 2\lambda + 2\lambda^2 + \ldots + 2\lambda^{k/2-1} + \lambda^{k/2} & \text{if } k \text{ is even.} 
\end{cases}
\]

Let \( \eta \) denote the total prediction error and \( \text{Opt} \) the cost of an optimal solution. Then, there exists a learning-augmented memory-constrained online algorithm for the \( k \)-server problem on the line with a competitive ratio of at most

\[
\min \left\{ \alpha(k) \left( 1 + \frac{\eta}{\text{Opt}} \right), \beta(k) \right\}.
\]

In particular, the algorithm is \( \alpha(k) \)-consistent and \( \beta(k) \)-robust, for \( \lambda > 0 \).

Interpreting both bounds as functions of \( \lambda \in [0, 1] \) illustrates that \( \alpha(k) \) interpolates monotonously between 1 and \( k \) while \( \beta(k) \) grows from \( k \) as \( \lambda \) decreases. This matches our expectation on a learning-augmented online algorithm, as it improves in consistency but loses in robustness compared to the best possible online algorithm. From another perspective, for a fixed value of \( \lambda \), \( \alpha(k) \) is bounded by a constant (equal to \( 1 + \frac{1}{\sqrt{\lambda^{-1}}} \)) which highlights the algorithm consistency but this comes at the price of an exponential dependency on \( k \) for \( \beta(k) \).

To show this result, we design an algorithm that carefully balances between (i) the wish to simply follow the predictions (FrP) which is obviously optimal if the predictions are correct, i.e. is \( 1 \)-consistent, and (ii) the best possible online algorithm when not having access to (good) predictions DOUBLECOVERAGE [17], which is \( k \)-robust. An additional challenge is to preserve the memory-constrained property. We achieve this, by generalizing the classical DOUBLECOVERAGE [17] in an intuitive way. Essentially, our algorithm LAMBDA\( DC \) includes the information about predicted servers and our trust into them by varying server speeds.

The analysis of our algorithm is tight. On the technical side, our analysis builds on the powerful potential function method, as does the analysis of the classical DOUBLECOVERAGE [17]. While LAMBDA\( DC \) is quite simple (a precise definition follows), the analysis is much more intricate and requires a careful re-design for the learning-augmented setting. Our main technical contribution is the definition and analysis of different parameterized potential functions for proving robustness and consistency, that capture the different speeds for moving servers and the accordingly more difficult tracing of the server moves.

We remark that our performance bound also holds (with an additional factor of 2 on the error) using the error measure of Antoniadis et al. [3] for our problem [34]. Their error definition sums up the distances between the configurations of \( \text{Opt} \) and FrP after every request, thus, it may seem more intuitive as server positions are compared instead of solution costs. However, our error definition allows to establish learnability results and also simplifies some analyses.

While our result is tailored to the \( k \)-server problem, the framework by Antoniadis et al. [3] is designed for more general metrical task systems. Interestingly, one of their methods is a deterministic combination of DOUBLECOVERAGE and FrP, we refer to it as FrP\&DC. It is shown that FrP\&DC is \( 9 \)-consistent and \( 9k \)-robust.
Our methods differ substantially. While FrP&DC carefully tracks states and costs of the simulated individual algorithms, LAMBDADC is a simple algorithm that only requires knowledge of the current configuration. Further, LAMBDADC has a better performance for \( k < 20 \) and an appropriate parameter \( \lambda \) (e.g., \( k = 19 \) and \( \lambda = 0.83 \)), but does not offer such a good tradeoff for larger \( k \). Actually, this is unavoidable for a certain class of memory-constrained algorithms, that includes LAMBDADC.

Indeed, we complement our main result with an almost matching lower bound on the consistency-robustness tradeoff. We construct a non-trivial bound for the class of memory-constrained algorithms that satisfy an additional locality property; its precise definition is formulated in Section 5. Intuitively, the locality property enforces an algorithm to achieve a better competitive ratio for a subinstance served by fewer servers. Other locality restrictions have been required before to establish lower bounds, e.g., for matching on the line, see [4].

**Theorem 2.** Let \( \lambda \in (0, 1) \), \( \rho(k) = \sum_{i=0}^{k-1} \lambda^i \) and \( \beta(k) = \sum_{i=0}^{k-1} \lambda^{-i} \). Let \( \mathcal{A} \) be a learning-augmented locally-consistent and memory-constrained deterministic online algorithm for the \( k \)-server problem on the line. Then, if \( \mathcal{A} \) is \( \rho(k) \)-consistent, it is at least \( \beta(k) \)-robust.

Algebraic transformations (see Lemma 20) show that \( a(k) < 2\rho(k) \), which implies that LAMBDADC achieves a tradeoff within a factor of at most 2 of the optimal consistency-robustness tradeoff (among locally-consistent and memory-constrained algorithms). For \( k = 2 \), LAMBDADC achieves the optimal tradeoff (among memory-constrained algorithms).

We demonstrate the power of our approach in empirical experiments on real-world data. We show that for a reasonable choice of \( \lambda \) our method outperforms the classical online algorithm DOUBLEOVERAGE as well as the algorithm in [3] for nearly all prediction errors.

Finally, we address the learnability of our predictions, even though this is not the focus of our work. We show that a static prediction sequence is PAC-learnable [48, 49]. We show a bound on the sample complexity that is polynomial in the number of requests, \( n \), and the number of servers, \( k \), and we give a learning algorithm with a polynomial running time in \( n, k \) and the number of samples.

## 2 Algorithm and Roadmap for the Analysis

### The Algorithm LAMBDADC

We generalize the classical DOUBLEOVERAGE [17] by including the information about predicted servers as well as our trust into this advice, in an intuitive way. If a request \( r_t \) appears between two servers, the one closer to the predicted server \( p_t \) moves by a greater distance towards the request—as if it traveled at a higher speed.

Formally, we define LAMBDADC for a given \( \lambda \in [0, 1] \) as follows. If \( r_t < s_i \) or \( r_t > s_k \), then LAMBDADC only moves the closest server. Otherwise, we have \( s_i < r_t < s_{i+1} \). If \( p_t \leq i \), then LAMBDADC moves \( s_i \) with speed 1 and \( s_{i+1} \) with speed \( \lambda \) towards \( r_t \) until one server reaches the request. If \( p_t \geq i + 1 \), the speeds of \( s_i \) and \( s_{i+1} \) are swapped. Hence, LAMBDADC equals FrP (with shortcuts) for \( \lambda = 0 \), and DOUBLEOVERAGE for \( \lambda = 1 \). Using nonintegral values for \( \lambda \) gives an algorithm that interpolates between both.

### Potential Function Analysis

The analysis of our algorithm builds on the powerful potential function method, as does the analysis of the classical DOUBLEOVERAGE [17].

Our potential analysis follows the well-known interleaving moves technique [12]. To compare two algorithms \( \mathcal{A} \) and \( \mathcal{B} \) in terms of competitiveness, we simulate both in parallel on some instance \( I \). Then, we employ a potential function \( \Phi \) which maps at every time \( t \) the state of both algorithms (i.e. the algorithms current configurations) to a value \( \Phi_t \geq 0 \), the potential at time \( t \). We define \( \Delta \Phi_t = \Phi_t - \Phi_{t-1} \). Let \( \Delta \mathcal{B}_t(I) \)
resp. \( \Delta A_t(I) \) denote the cost \( A \) resp. \( B \) charges for serving the request at time \( t \) and let \( \mu > 0 \). For every request \( r_t \), we assume that first \( B \) serves the request, and second \( A \). If

(i) the move of \( B \) increases \( \Phi \) by at most \( \mu \cdot \Delta B_t(I) \), whereas

(ii) the move of \( A \) decreases \( \Phi \) by at least \( \Delta A_t(I) \),

we can use a telescoping sum argument to conclude \( \mathcal{A}(I) \leq \mu \cdot \mathcal{B}(I) + \Phi_0 \). Note that if \( B \) is the optimal algorithm, \( \mu \) is equal to the competitive ratio of \( A \) since \( \Phi_0 \) only depends on \( C_0 \).

To show an error-dependent competitive ratio in the learning-augmented setting, we follow three steps. We show first that the cost of LAMBDADC is close to the cost of FrP, that is \( \text{ALG}(I) \leq \alpha(k) \cdot \text{FrP}(I) + c \) for some \( c > 0 \) and for every instance \( I \). Note that this corresponds to the consistency case as FrP is the optimal algorithm if \( \eta = 0 \). Second we plug in the definition of our prediction error \( \eta \) to bound the cost of FrP by the cost of the fixed optimal solution (fixed with respect to the definition of \( \eta \)) and \( \eta \). Combining both results yields the first part of the competitive ratio of Theorem 1. Lastly we prove a robustness bound, i.e. a general bound independent of the prediction, on the cost of LAMBDADC with respect to \( \text{OPT} \). All additive constants in the competitive ratios only depend on the initial configuration of the servers, being zero if all servers start at the same position.

The potential functions we use to analyze LAMBDADC are inspired by the potential function in the classical analysis of DoubleCoverage [17]. It is composed of a matching part \( \Psi \), summing the distances between the server positions of an algorithm and the reference algorithm (\( \text{OPT} \), FrP) and a spreadness part \( \Theta \), summing the distances between an algorithms server positions. To incorporate the more sophisticated server moves at different speeds, we introduce multiplicative coefficients to both parts. The main technical contribution lies in identifying the proper weights and performing the much more involved analysis.

**Lower Bounds for LAMBDADC** In Appendix A we show that our analysis is tight.

**Lemma 3.** LAMBDADC is at least \( \alpha(k) \)-consistent and \( \beta(k) \)-robust.

**Organization of the paper** For ease of exposition, we first consider the setting of 2 servers in Section 3. Then, we extend the techniques to the general setting in Sections 4 and 5 while maintaining the same structure as for \( k = 2 \). We illustrate and discuss the results of computational experiments in Section 6, and, finally, talk about PAC learnability of our predictions in Section 7.

### 3 Full Analysis for Two Servers

#### 3.1 Error-dependent Competitive Ratio of LAMBDADC

We show the theoretical guarantees of LAMBDADC claimed in Theorem 1 restricted to two servers. We denote the cost of LAMBDADC for some instance \( I \) by \( \text{ALG}(I) \), and the cost for serving a request \( r_t \) by \( \Delta \text{ALG}_t(I) \). If \( t \) is clear from the context then we omit the index.

**Theorem 4.** For any parameter \( \lambda \in [0, 1] \), LAMBDADC has a competitive ratio of at most

\[
\min \left\{ (1 + \lambda) \left( 1 + \frac{\eta}{\text{OPT}} \right), 1 + \frac{1}{\lambda} \right\}.
\]

Thus, it is \((1 + \lambda)\)-consistent and \((1 + 1/\lambda)\)-robust.
For any instance $I$ and $\lambda \in [0, 1]$, there is some $c \geq 0$ that only depends on the initial configuration such that $\text{ALG}(I) \leq (1 + \lambda) \cdot \text{FtP}(I) + c$.

Proof. Let $I$ be an arbitrary instance and let servers start at positions $s_1^0$ and $s_2^0$. If $\lambda = 0$, LAMBADC only skips the request, hence $\text{ALG}(I) \leq \text{FtP}(I)$. Now assume that $\lambda > 0$. Let $s_1, s_2$ be LAMBADC’s servers and $x'_1, x'_2$ be FtP’s servers. We simulate $I$ in parallel for both algorithms. At every time $t$, we map the configurations of both algorithms to a non-negative value using the potential function

$$\Phi = \frac{1 + \frac{\lambda}{\lambda}}{\lambda} \left( |s_1 - x'_1| + |s_2 - x'_2| \right) + \Theta(s_{\text{spread}}).$$

Suppose that a new request arrives. First, FtP serves the request. Assume that $x'_1$ moves and costs $\Delta\text{FtP}$. Since LAMBADC remains in its previous configuration, $|x'_1 - s_1|$ increases by at most $\Delta\text{FtP}$, and $\Phi$ increases by at most $(1 + \lambda)/\lambda \cdot \Delta\text{FtP}$. Second, LAMBADC moves. Assume by scaling the instance that the algorithm serves the request after exactly one time unit, i.e., the fast server moves distance 1 and the slow server distance $\lambda$. We distinguish whether the request is between the algorithm’s servers or not, and prove in each case that $\Phi$ decreases by at least $1/\lambda \cdot \Delta\text{ALG}$.

(a) Suppose the request is not between the servers $s_1$ and $s_2$; say, it is left of $s_1$. Then LAMBADC moves only $s_1$ and $\Delta\text{ALG} = 1$. Either $x'_1$ or $x'_2$ covers the request, hence moving $s_1$ decreases $\Psi$ by $(1 + \lambda)/\lambda$ while it increases $\Theta$ by 1. Thus,

$$\Delta\Phi \leq -\frac{1 + \lambda}{\lambda} + 1 = -\frac{1}{\lambda} = -\frac{1}{\lambda} \cdot \Delta\text{ALG}.$$

(b) Suppose the request is between $s_1$ and $s_2$, and suppose that $s_1$ is predicted. LAMBADC moves both servers and $\Delta\text{ALG} = 1 + \lambda$. This means that $x'_1$ already covers the request. Thus, moving $s_1$ towards the request decreases $\Psi$ by $(1 + \lambda)/\lambda$, while $s_2$ increases $\Psi$ by at most $(1 + \lambda)/\lambda \cdot \lambda$. Also, $\Theta$ decreases by $1 + \lambda$. We can conclude that

$$\Delta\Phi \leq \frac{1 + \lambda}{\lambda} (-1 + \lambda) - (1 + \lambda) = -\frac{1}{\lambda} (1 + \lambda) = -\frac{1}{\lambda} \cdot \Delta\text{ALG}.$$

Summing over all rounds, we obtain $\text{ALG}(I) \leq (1 + \lambda)\text{FtP}(I) + \lambda|s_1^0 - s_2^0|$.

Finally, we give a robustness guarantee for LAMBADC’s performance independently of the prediction quality.

Lemma 6. For any instance $I$ and $\lambda \in (0, 1]$, there is some $c \geq 0$ that only depends on the initial configuration such that $\text{ALG}(I) \leq (1 + 1/\lambda) \cdot \text{Opt}(I) + c$.

The proof of this claim is similar to the proof of Lemma 5 with the crucial difference that the reference algorithm is unknown. Hence, the multiplicative factor is larger but relative to the optimal solution and, thus, independent of the prediction error.
Proof. Let \( I \) be an arbitrary instance and let \( \lambda \in (0, 1] \). Let \( s_1, s_2 \) be LAMBDADC’s servers and \( x_1, x_2 \) the servers of an optimal algorithm. We define
\[
\Phi = (1 + \lambda) \left( |s_1 - x_1| + |s_2 - x_2| + |s_1 - s_2| \right).
\]
Upon arrival of a request, first the optimal algorithm moves and \( \Phi \) increases by at most \((1 + \lambda) \cdot \Delta \text{Opt}\). Second LAMBDADC moves and, by scaling the instance, we assume that the request is served after exactly one time unit. We distinguish whether the request is between the algorithm’s servers or not, and show that in each case \( \Phi \) decreases by at least \( \lambda \cdot \Delta \text{Alg} \).

(a) Let the request be not between the servers, say on the left of \( s_1 \). Either \( x_1 \) or \( x_2 \) covers the request, hence moving \( s_1 \) decreases \( \Psi \) by \( 1 + \lambda \) while it increases \( \Theta \) by 1. Thus,
\[
\Delta \Phi \leq -(1 + \lambda) + 1 = -\lambda = -\lambda \cdot \Delta \text{Alg}.
\]

(b) Let the request be between \( s_1 \) and \( s_2 \), and suppose that \( s_1 \) is predicted. The request is covered by \( x_1 \) or \( x_2 \). In the worst case (\( x_2 \) covers the request), moving \( s_1 \) towards the request increases \( \Psi \) by at most \( 1 + \lambda \), while \( s_2 \) decreases \( \Psi \) only by \((1 + \lambda)\lambda \). Also, \( \Theta \) decreases by \( 1 + \lambda \). Put together,
\[
\Delta \Phi \leq (1 + \lambda)(1 - \lambda) - (1 + \lambda) = -\lambda(1 + \lambda) = -\lambda \cdot \Delta \text{Alg}. \quad \Box
\]

3.2 Optimality of LAMBDADC: the Consistency-Robustness Tradeoff

We now show that LAMBDADC is optimal for two servers, in the sense that no memory-constrained algorithm can achieve a better robustness-consistency tradeoff. As we target memory-constrained algorithms, at any time, we can use force requests, cf., Section 1, to enforce the algorithm to place its servers at prescribed locations.

Theorem 7. Let \( \mathcal{A} \) be a learning-augmented memory-constrained algorithm for the 2-server problem on the line and let \( \lambda \in (0, 1] \). If \( \mathcal{A} \) is \((1 + \lambda)\)-consistent, it is at least \((1 + 1/\lambda)\)-robust.

Proof. Let \( \lambda \in (0, 1] \) and \( \mathcal{A} \) be a \((1 + \lambda)\)-consistent, memory-constrained algorithm for the 2-server problem on the line. This means for every instance \( I \), \( \mathcal{A}(I) \leq (1 + \lambda) \cdot \text{Opt}(I) + \nu \) if \( \eta = 0 \), where \( \nu \) depends on the initial configuration. Let \( a, b \) and \( c \) be consecutive points on the line at position \(-1, 0\) and \( L \geq 1 + 1/\lambda \), and \((a, b)\) the algorithm’s initial configuration.

Consider the instance \( I^\infty \) which is composed of a force to \((a, c)\), followed by arbitrarily many alternating requests at \( b \) and \( a \). Clearly, an optimal solution for instance \( I^\infty \) is to move the right server to \( c \) and then immediately back to \( b \) with a total cost of \( 2L \).

Assume that \( \mathcal{A} \) gets this optimal solution as prediction. \( \mathcal{A} \) moves one server to \( c \) for the first request. Since the consistency implies that \( \mathcal{A}(I^\infty) \leq (1 + \lambda) \cdot \text{Opt} \), at some point in time \( \mathcal{A} \) has to move the right server to \( b \). Denote the instance which ends at this point in time by \( I \). Note that \( \mathcal{A}(I) \geq \mathcal{A}(I) \). Let \( n_L \) denote the number of times in instance \( I \) where the left server moves from \( a \) to \( b \) and back to \( a \) (cost of 2). Since the right server pays at least \( L \) for moving from \( c \) to \( b \) and back to \( a \) (cost of 2), we conclude \( \mathcal{A}(I) \geq 2n_L + 2L \). The consistency of \( \mathcal{A} \) leads to \( 2n_L + 2L \leq (1 + \lambda)2L + \nu \), which means \( n_L \leq \lambda L + \nu /2 \).

We now construct another instance \( I^\omega \) by concatenating \( \omega \) copies of instance \( I \), each starting by the force to \((a, c)\). We call such a copy an iteration, and in each iteration we use the same predictions as in instance \( I \). \( \mathcal{A} \) has to pay at least \( L \) for the force, as the right server was previously on \( b \), and then \( \mathcal{A} \) follows
Figure 1: Visualization of all incident \( \delta_{ij} \)-weights of the servers \( s_1 \) and \( s_2 \). The thickness (resp. color) of an arc indicates the influence of the corresponding distance in \( \Phi \).

the same behavior as in \( I \) in each iteration. So \( \mathcal{A}(I^\omega) \geq \omega \cdot (2n_L + 2L) \). Another solution for instance \( I^\omega \) is to move the right server to \( c \) in the beginning with cost \( L \) and leave it there, while the left server alternates between \( a \) and \( b \). Hence, \( \text{Opt}(I^\omega) \leq L + \omega \cdot 2(n_L + 1) \). Indeed, \( b \) is requested \( n_L + 1 \) times per iteration: \( n_L \) where \( \mathcal{A} \) uses the left server and one where it uses the right server. The ratio is then

\[
\frac{\mathcal{A}(I^\omega)}{\text{Opt}(I^\omega)} \geq \frac{\omega \cdot (2n_L + 2L)}{L + \omega \cdot 2(n_L + 1)} \xrightarrow{\omega \to \infty} \frac{2n_L + 2L}{2(n_L + 1)} = 1 + \frac{L - 1}{n_L + 1} \geq 1 + \frac{L - 1}{\lambda L + \frac{k}{2} + 1} \xrightarrow{L \to \infty} 1 + \frac{1}{\lambda},
\]

which implies that \( \mathcal{A} \) is at least \((1 + 1/\lambda)\)-robust. \( \square \)

4 The General Case with \( k \) Servers: Upper Bound

We present two lemmas which imply Theorem 1. The novelty lies in designing appropriate potential functions that capture the server movements at different speeds. This takes substantially more technical care than in the 2-server case but builds on the same ideas.

In the first step of the analysis, we compare the performance of LAMBdacDC and FtP.

**Lemma 8.** For every instance \( I \) and \( \lambda \in [0, 1] \), there is some \( c > 0 \) that only depends on the initial configuration such that \( \text{ALG}(I) \leq \alpha(k) \cdot \text{FtP}(I) + c \).

Let \( I \) be an arbitrary instance. Note that \( \lambda = 0 \) implies \( \text{ALG}(I) \leq \text{FtP}(I) \) as LAMBdacDC can only shortcut FtP’s moves. So, we now assume that \( \lambda \in (0, 1] \). We define a new potential function \( \Phi \) as follows. Let \( s_1, \ldots, s_k \) be the servers of LAMBdacDC and let \( x'_1, \ldots, x'_k \) be the servers of FtP. For \( 1 \leq i < j \leq k \) and \( \ell = \min\{j - i, k - (j - i)\} - 1 \) we define \( \delta_{ij} = \lambda^\ell \), see Figure 1. Then,

\[
\Phi = \frac{\alpha(k)}{\lambda} \cdot \sum_{i=1}^{k} |s_i - x'_i| + \sum_{i<j} \delta_{ij} |s_i - s_j|.
\]

Intuitively, the leading coefficient of \( \Psi \) comes from the targeted competitive ratio. Then, in \( \Theta \), the coefficient in front of each term depends on the number of interleaving servers. Following the idea of Lemma 3, when LAMBdacDC moves a server by a distance of 1 as in \( \text{Opt} \), its neighbor moves by a distance of \( \lambda \). Hence,
We demonstrate in the appendix that these values correspond to the change of this neighbor moves by a distance $\lambda^2$. Therefore, this geometric decrease in the consequences of a movement also appears in the expression of $\Theta$. The symmetric increase when $j - i$ grows is more difficult to explain intuitively, but is required to compensate the modifications of $\Psi$. The coefficients of $\Theta$ are illustrated in Figure 1.

We carefully analyze in Appendix B how the potential changes when FrP and LAMBDA DC move servers. Further, we give a robustness guarantee for LAMBDA DC for any error.

**Lemma 9.** For any instance $I$ and $\lambda \in (0, 1]$, there is some $c \geq 0$ that only depends on the initial configuration such that $\text{ALG}(I) \leq \beta(k) \cdot \text{OPT}(I) + c$.

Proving the general upper bound on the competitive ratio, independent of the prediction error, is much more intricate than in the two-server case and than the consistency proof. Again, our key ingredient is a carefully chosen potential function $\Phi$. We generalize the function used for the consistency bound even further by refining the weights, in particular, adding server-dependent weights to the term $\Psi$ measuring the distance between the positions of the algorithm’s servers and the optimal servers.

Let $\lambda \in (0, 1]$. Fix $k$, let $\beta = \beta(k) = \sum_{i=0}^{k-1} \lambda^{-i}$, and let $s_1, \ldots, s_k$ be the servers of LAMBDA DC and let $x_1, \ldots, x_k$ be the servers of an optimal solution. The potential function is

$$\Phi = \beta \left( \sum_{i=1}^{k} \omega_i |s_i - x_i| \right) + \sum_{i < j} \delta_{ij} |s_i - s_j| .$$

We specify the weights in this function as follows. For a pair of servers $s_i, s_j$ with $1 \leq i < j \leq k$, let $\ell = \min\{j - i, k - (j - i)\} - 1$ and $\delta_{ij} = (\lambda^\ell + \lambda^{k-2-\ell})/(1 + \lambda^{k-2})$.

The intuition of the weights in the spreadness part $\Theta$ is the same as in the consistency potential function above. However, the new weights $\omega_i$ in the matching part $\Psi$ (defined below) require the more complex weights $\delta_{ij}$ compared to the simpler $\lambda^\ell$ weights.

Further, we define $d_{\lfloor k/2 \rfloor} = 0$ if $k$ is odd and for all $1 \leq i \leq \lfloor k/2 \rfloor$ let

$$d_i = d_{k+1-i} = \frac{2}{1 + \lambda^{k-2}} \sum_{\ell=1}^{k-1-i} \lambda^\ell.$$

We demonstrate in the appendix that these values correspond to the change of $\Theta$ when a server of LAMBDA DC moves. Let $\gamma = d_1/(\beta - 1)$, $\omega_1 = \omega_k = 1$ and for $2 \leq i \leq \lfloor k/2 \rfloor$ we define the server-individual weights

$$\omega_i = \omega_{k+1-i} = \begin{cases} 
2\lambda \sum_{j=1}^{i/2-1} d_{2j} - 2 \sum_{j=1}^{i/2-1} d_{2j+1} + \lambda d_i + (2 + \lambda)\gamma & \text{if } i \text{ is even,} \\
2\lambda \sum_{j=1}^{(i-1)/2} d_{2j} - 2 \sum_{j=1}^{(i-1)/2} d_{2j+1} - d_i + \gamma & \text{if } i \text{ is odd.}
\end{cases}$$

We finally prove Lemma 9 in Appendix B by exhaustively reviewing all possible moves and bounding the corresponding change of $\Phi$. Establishing a constant upper bound of the $\omega$-weights yields a general upper bound on the increase of $\Phi$ independently of the choice of the optimal solution’s server. We further choose the scaling parameter $\gamma$ such that the decrease of $\Phi$ exactly matches the required lower bound for the case where the request is outside of the convex hull of LAMBDA DC’s servers. The remaining cases are split among the possible locations where a request can appear between two servers of LAMBDA DC, and
we show in each case that $\Phi$ decreases enough. Intuitively, the $\omega$ values are defined such that a wrong prediction gives a tight bound on the decrease of $\Phi$ for LAMBDA DC’s move, while a correct prediction still guarantees a loose bound.

5 The Consistency-Robustness Tradeoff

In this section we give a bound on the consistency-robustness tradeoff, as stated in Theorem 2. Our bound holds for memory-constrained algorithms that satisfy a certain locality property, which includes LAMBDA DC. Informally, we require that a $k$-server algorithm with a certain consistency $\mu(k)$ shall have a consistency $\mu(k')$ on a sub-instance that it serves with $k' < k$ servers. The rationale is to prevent the mere presence of additional unused workers to allow the algorithm to perform poorly on a sub-instance served by few servers, as $\mu(k') < \mu(k)$. Hence, such algorithms are expected to present a better performance on a modified instance where some extreme servers are removed and side-effects due to their presence are simulated. In the following, we make this intuition precise and sketch our worst-case construction.

Given an algorithm $\mathcal{A}$ which is $\mu(k)$-consistent for the $k$-server problem, we define the notion of locally-consistent. Given an instance of the $k$-server problem served by algorithm $\mathcal{A}$, consider any subset $S'$ of $k'$ consecutive servers. We construct an instance $I'$ of the $k'$-server problem based on $I$ and $S'$: If a request of $I$ is predicted to be served by a server in $S'$ then this request is replicated in $I'$. Otherwise, $I'$ requests the position of the closest server among $S'$ after $\mathcal{A}$ served this request in $I$ (in order to take into account side-effects due to additional servers in the original instance). Let $FP(I')$ be the cost of solving $I'$ following the original predictions of $I'$ (using the closest server among $S'$ if a server outside of $S'$ was initially predicted). An algorithm is locally-consistent if its total cost on $I$ restricted to the servers in $S'$ is at most $\mu(k') \cdot FP(I') + c$, where $c$ can be upper bounded based only on the initial configuration. We further require that if the initial and final configurations differ by a total distance of $\varepsilon$, then $c = O(k'\varepsilon)$. Note that LAMBDA DC is locally-consistent as its behavior in $I$ restricted to the servers in $S'$ is equal to its behavior in $I'$ with $k'$ servers.

The proof of Theorem 2 generalizes ideas from the 2-server case (Section 3.2) in a highly non-trivial way. We only sketch the main idea and refer to Appendix C for details. Let $\mathcal{A}$ be a memory-constrained and locally-consistent deterministic algorithm. We construct an instance that starts with $k$ equidistant servers. First, a point far on the right is requested. Then the initial server locations are requested following specific rules until the rightmost server comes back. Predictions correspond to the server initially at the point requested. The consistency of $\mathcal{A}$ limits the possible cost paid before the rightmost server comes back. The locally-consistent definition allows, with technical care, to link the distance traveled by two neighboring servers: the left one travels a total distance at most $\lambda$ times the right one (plus negligible terms). An offline solution can afford to initially shift all servers to the right, and then move only the leftmost server, which $\mathcal{A}$ could not move much. We then repeat this instance, and use the memory-constrained and deterministic characteristics of $\mathcal{A}$ to eliminate constant costs and show the desired robustness lower bound, again with technical care.

6 Experiments

We supplement our theoretical results by empirically comparing our learning-augmented algorithm LAMBDA DC with the classical online algorithm ignoring predictions DOUBLECOVERAGE [17] and the previously proposed prediction-based algorithm FrP&DC [3] on real world data. We generate instances with 1000 requests based on the BrightKite-Dataset [16], which is composed of sequences of coordinates of app
check-ins. This dataset was used previously to evaluate and compare learning-augmented algorithms for caching problems [3, 37]. We further generate predictions in a semi-random fashion aiming for large and evenly distributed prediction errors. All algorithms are implemented in lazy and non-lazy variants.

The results for non-lazy implementations are displayed in Figure 2. They show well that, for a reasonable choice of \( \lambda (0.1 \leq \lambda \leq 0.5) \), LAMBDA DC outperforms both DOUBLE COVERAGE and FtP&DC for almost all generated relative prediction errors. This is true even if laziness is allowed as we show in Appendix E. We give also more details on the generation of instances and predictions, as well as an overview over all results.

7 PAC Learnability of Predictions

While our results show the applicability of untrusted predictions, it is a natural question whether such predictions are actually learnable.

In Appendix D, we show that for our model a static prediction sequence is PAC learnable in an agnostic sense using empirical risk minimization. That is, given an unknown distribution over request sequences which we can sample, we can find a prediction that is close to the best possible prediction for this distribution in terms of prediction error using a bounded number of samples.

**Theorem 10.** For any \( \epsilon, \delta \in (0, 1) \), a known initial configuration \( C_0 \) and any distribution \( D \) over the sequences of \( n \) requests of known extent, there exists an algorithm which, given an i.i.d. sample of \( D \) of size \( m \in O \left( \frac{1}{\epsilon^2} \cdot (n \log k - \log \delta) \eta_{\text{max}}^2 \right) \), returns a prediction \( \tau_p \in \mathcal{H} \) in polynomial time depending on \( k, n \) and \( m \), such that with probability of at least \( (1 - \delta) \) it holds \( \mathbb{E}_{\sigma \sim \mathcal{D}} \left[ \eta_\sigma(\tau_p) \right] \leq \mathbb{E}_{\sigma \sim \mathcal{D}} \left[ \eta_\sigma(\tau^*) \right] + \epsilon \), where \( \tau^* = \arg \min_{\tau \in \mathcal{H}} \mathbb{E}_{\sigma \sim \mathcal{D}} \left[ \eta_\sigma(\tau) \right] \).

We remark that a pre-computed static prediction does not include information about the partially revealed input. Thus, this is a rather weak prediction and may not help LAMBDA DC much. The existence of an adaptive prediction policy which can be efficiently learned remains an open question. Such a policy would provide much more valuable information to our learning-augmented online algorithm.
8 Conclusion

We show the power of (untrusted) predictions in designing online algorithms for the $k$-server problem on the line. Our algorithm generalizes the classical DOUBLECOVERAGE algorithm [17] in an intuitive way and admits a (nearly) tight error-dependent competitive analysis, based on new potential functions, and outperforms other methods from the literature. While we can show PAC learnability for static predictions, we leave open whether possibly more powerful adaptive prediction models are learnable.

Clearly, it would be interesting to see whether our results generalize to more general metric spaces than the line. In fact, in a related version we show that our upper bounds for the 2-server problem can be extended to tree metrics [34] and we expect that an extension to $k$ servers is possible. However, for more general metrics our current approach seems not to generalize well. Further, we focused on memory-constrained algorithms, leaving open a more precise quantification of the power of memory. Finally, the recent success on randomized k-server algorithms [13] raises the question whether and how randomized algorithms can benefit from (ML) predictions.
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A Proofs for Section 2

Lemma 3. \textsc{Lambdadc} is at least $\alpha(k)$-consistent and $\beta(k)$-robust.

Proof. We give two separate instances for consistency and robustness.

(i) Consider $k$ servers initially at positions 0, 1, $-1$, 2, $-2$, \ldots and the request sequence of length $k + 1$ at positions 0.5, 0, 1, $-1$, 2, $-2$, \ldots. There is a solution of cost 1 that only moves the server that is initially at 0.

\textsc{Lambdadc} serves the first request by moving the optimal server from 0 to 0.5 and additionally the one from 1 to $1 - \lambda/2$. With the second request, the first server is moved back to 0, having moved a total distance of 1, and the server from $-1$ moves to $-1 + \lambda/2$. For the third request, the server from original position 1 returns to this position, etc. Each server moves back to its initial position $i$ after moving a total distance of $\lambda^{|i|}$. Repeating this example gives the lower bound on the consistency.

(ii) Consider $k$ servers initially at positions $\beta(1)$, $\beta(2)$, \ldots, $\beta(k)$, and the request sequence of length $k + 1$ at positions $0$, $\beta(1)$, $\beta(2)$, $\beta(3)$, \ldots. There is a solution of cost 2 that only moves the server that is initially at 1. Consider predictions corresponding always to the rightmost server at the highest position.

\textsc{Lambdadc} serves the second request by moving both servers from 0 and $\beta(2)$ to $\beta(1)$ as the closest server moves by a distance of 1 and the furthest server, which is predicted, moves by a distance of $1/\lambda$. Similarly, for each request except the last one, both servers neighboring the request end up serving the request simultaneously. So the $i$-th server moves by a total distance of $2/\lambda^{i-1}$. Repeating this example gives the lower bound on the robustness. \hfill \square

B Proofs for Section 4

The Consistency Bound

Lemma 8. For every instance $I$ and $\lambda \in [0, 1]$, there is some $c > 0$ that only depends on the initial configuration such that $\text{Alg}(I) \leq \alpha(k) \cdot \text{FrP}(I) + c$. 

Before proving Lemma 8, we need a few preliminary results. Let $I$ be an arbitrary instance. Note that $\lambda = 0$ implies $\text{Alg}(I) \leq \text{FrP}(I)$ as \textsc{Lambdadc} can only shortcut \textsc{FrP}'s moves, so we now assume that $\lambda \in (0, 1]$.

Observation 11. For every $k > 3$, we have $\frac{\alpha(k)}{\lambda} = \alpha(k - 2) + \frac{1}{\lambda} + 1$.

Proof. We prove this statement depending on the parity of $k$. If $k$ is odd, $k - 2$ is also odd. By definition of $\alpha(k)$,

$$1 + \frac{1}{\lambda} + \alpha(k - 2) = \frac{1}{\lambda} + 2 + 2 \sum_{i=1}^{(k-3)/2} \lambda^i = \frac{1}{\lambda} + 2 \sum_{i=1}^{(k-1)/2} \lambda^{i-1} = \frac{\alpha(k)}{\lambda}.$$
Figure 3: Visualization of all incident $\delta_{ij}$-weights of the servers $s_1$ and $s_2$. The thickness (resp. color) of an arc indicates the influence of the corresponding distance in $\Phi$.

If $k$ is even, $k-2$ is also even, and we conclude

\[
1 + \frac{1}{\lambda} + \alpha(k - 2) = 1 + \frac{1}{\lambda} + 2 + 2 \sum_{i=1}^{(k-2)/2-1} \lambda^i + \lambda^{(k-2)/2} \\
= 1 + 2 \sum_{i=1}^{k/2-1} \lambda^{i-1} + \lambda^{(k-2)/2} \\
= \frac{\alpha(k)}{\lambda}.
\]

We defined our potential function $\Phi$ as follows. Let $s_1, \ldots, s_k$ be the servers of LAMDADC and let $x'_1, \ldots, x'_k$ be the servers of FrP. For $1 \leq i < j \leq k$ and $\ell = \min\{j - i, k - (j - i)\} - 1$ we define $\delta_{ij} = \lambda^\ell$. Then,

\[
\Phi = \frac{\alpha(k)}{\lambda} \cdot \sum_{i=1}^{k} |s_i - x'_i| + \sum_{i < j} \delta_{ij} |s_i - s_j|.
\]

The analysis of LAMDADC requires evaluating the evolution of $\Phi$ after each request. The following lemma characterizes how a move of LAMDADC influences $\Theta$.

**Lemma 12.** Let $i \leq \lfloor k/2 \rfloor$. If $s_i$ moves from $p$ to $p + x$, $\Theta$ changes by

\[
(-x) \cdot \left(1 + \alpha(k - 2) - 2 \sum_{j=0}^{i-2} \lambda^j\right).
\]

**Proof.** Assume w.l.o.g. that $x = -1$, that is, server $s_i$ moves one unit to the left. Consider servers $s_j, s'_j$ such that $j' + \ell = i = j - \ell$ for some $1 \leq \ell \leq i - 1$. Since $\delta_{j'} = \delta_{ij}$ we observe that the changes to the terms $\delta_{ij} |s_i - s_j|$ and $\delta_{j'} |s_i - s'_j|$ of $\Theta$ cancel out. Hence, as $i \leq \lfloor k/2 \rfloor$, the change of $\Theta$ due to the move of $s_i$ is equal to $\sum_{j=2i}^k \delta_{ij}$. We now prove the statement depending on the parity of $k$.  

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[19]
(i) If $k$ is odd, $k - 2i + 1$ is even. By definition,

$$
\sum_{j=2i}^{k} \delta_{ij} = \sum_{j=2i}^{k} \lambda^{\min\{j-i,k-(j-i)\}-1} = 2 \sum_{j=2i}^{(k-1)/2-i} \lambda^{j-i-1} = 2 \sum_{j=i}^{(k-1)/2-1} \lambda^j
$$

$$
= 2 + 2 \sum_{j=1}^{(k-3)/2} \lambda^j - 2 \sum_{j=0}^{i-2} \lambda^j = 1 + \alpha(k - 2) - 2 \sum_{j=0}^{i-2} \lambda^j.
$$

(ii) If $k$ is even, $k - 2i + 1$ is odd, and there is a single term where the minimum in the definition of $\delta_{ij}$ in achieved by both conditions. Hence,

$$
\sum_{j=2i}^{k} \lambda^{\min\{j-i,k-(j-i)\}-1} = 2 \sum_{j=2i}^{k/2-1+i} \lambda^{j-i-1} + \lambda^{k/2-1} = 2 \sum_{j=i}^{k/2-2} \lambda^j + \lambda^{k/2-1}
$$

$$
= 2 + 2 \sum_{j=1}^{k/2-2} \lambda^j + \lambda^{k/2-1} - 2 \sum_{j=0}^{i-2} \lambda^j
$$

$$
= 1 + \alpha(k - 2) - 2 \sum_{j=0}^{i-2} \lambda^j.
$$

**Proof of Lemma 8.** Suppose that the next request appears. First FrP moves some server $x'_i$ towards the request, and the distance to $s_i$ increases by at most $\Delta_{FrP}$. Since this move only affects $\Psi, \Delta \Phi \leq \alpha(k)/\lambda \cdot \Delta_{FrP}$. Second LAMBDA DC moves. We distinguish whether the request is between two servers or not, and assert for both cases $\Delta \Phi \leq -1/\lambda \cdot \Delta_{ALG}$.

(a) Let the request be located w.l.o.g on the left of $s_1$. Thus, $s_1$ moves towards it and charges cost $\Delta_{ALG}$. The fact that some server $x'_j$ must already be on $r_t$ implies with Lemma 12 that

$$
\Delta \Phi \leq -\frac{\alpha(k)}{\lambda} \Delta_{ALG} + \left(1 + \alpha(k - 2) - 2 \sum_{j=0}^{i-2} \lambda^j\right) \Delta_{ALG}.
$$

Rearranging and using Observation 11 gives the claimed bound, that is

$$
\left(-\alpha(k - 2) - \frac{1}{\lambda} - 1 + \alpha(k - 2)\right) \Delta_{ALG} = -\frac{1}{\lambda} \Delta_{ALG}.
$$

(b) Let the request be between $s_i$ and $s_{i+1}$. Assume w.l.o.g. that FrP serves it with $x'_i$ and $j \leq i$. For ease of exposition, we assume that $s_i$ travels distance 1 and $s_{i+1}$ distance $\lambda$. Hence, $\Delta_{ALG} = 1 + \lambda$. Since $j \leq i$, we know that $x'_j$ must be located on the right of $x'_i$. Hence, the distance between $s_i$ and $x'_i$ decreases by 1, but the distance between $s_{i+1}$ and $x'_{i+1}$ increases by at most $\lambda$. Thus, $\Delta \Psi \leq \alpha(k)/\lambda \cdot (\lambda - 1)$. The change of $\Theta$ is clearly bounded from above by the case where $s_i$ moves distance $\lambda$ and $s_{i+1}$ moves distance 1 for $i + 1 \leq \lfloor k/2 \rfloor$. Combining Lemma 12 for both servers gives

$$
\Delta \Theta = 1 + \alpha(k - 2) - 2 \sum_{j=0}^{i-1} \lambda^j - \lambda \left(1 + \alpha(k - 2) - 2 \sum_{j=0}^{i-2} \lambda^j\right)
$$

$$
= -1 - \lambda + \alpha(k - 2) - \lambda \alpha(k - 2).
$$

20
Using this and Observation 11, we can bound the increase of the potential by
\[ \Delta \Phi \leq \frac{\alpha(k)}{\lambda} (\lambda - 1) - 1 - \lambda + \alpha(k - 2) - \lambda \alpha(k - 2) \]
\[ = -\frac{\alpha(k)}{\lambda} + \alpha(k - 2) = -1 - \frac{1}{\lambda} \]
\[ = -\frac{1}{\lambda} \Delta \text{ALG}. \]

The Robustness Bound

**Lemma 9.** For any instance \( I \) and \( \lambda \in (0, 1] \), there is some \( c \geq 0 \) that only depends on the initial configuration such that \( \text{ALG}(I) \leq \beta(k) \cdot \text{OPT}(I) + c \).

We start by defining a potential function \( \Phi \). Let \( \lambda \in (0, 1] \). Fix \( k \), let \( \beta = \beta(k) = \sum_{i=0}^{k-1} \lambda^{-i} \), and let \( s_1, \ldots, s_k \) be the servers of LAMBDA DC and let \( x_1, \ldots, x_k \) be the servers of an optimal solution. The potential function is
\[ \Phi = \beta \gamma \left( \sum_{i=1}^{k} \omega_i |s_i - x_i| \right) + \sum_{i<j} \delta_{ij} |s_i - s_j| \]

We specify the weights in this function as follows. For a pair of servers \( s_i, s_j \) with \( 1 \leq i < j \leq k \), let \( \ell = \min\{j - i, k - (j - i)\} - 1 \) and
\[ \delta_{ij} = \frac{\lambda^\ell + \lambda^{k-2-\ell}}{1 + \lambda^{k-2}}. \]

The intuition of the weights in the spreadness part \( \Theta \) is the same as in the consistency potential function. However, the new weights \( \omega_i \) in the matching part \( \Psi \) (defined below) require the more complex weights \( \delta_{ij} \) compared to the simpler \( \lambda^\ell \) weights.

Further, we define \( d_{[k/2]} = 0 \) if \( k \) is odd and for all \( 1 \leq i \leq [k/2] \) let
\[ d_i = d_{k+1-i} = \frac{2}{1 + \lambda^{k-2}} \sum_{\ell=1}^{k-1-i} \lambda^\ell. \]

Let \( \gamma = d_1/(\beta - 1) \), \( \omega_1 = \omega_k = 1 \) and for \( 2 \leq i \leq [k/2] \) we define the server-individual weights
\[ \omega_i = \omega_{k+1-i} = \begin{cases} \frac{2\lambda \sum_{j=1}^{i-1/2} d_{2j} - 2 \sum_{j=1}^{i-1/2} d_{2j+1} + \lambda d_i + (2 + \lambda)\gamma}{\beta \gamma \lambda} & \text{if } i \text{ is even}, \\
\frac{2\lambda \sum_{j=1}^{(i-1)/2} d_{2j} - 2 \sum_{j=1}^{(i-3)/2} d_{2j+1} - d_i + \gamma}{\beta \gamma} & \text{if } i \text{ is odd}. \end{cases} \]

This finishes the definition of the potential function \( \Phi \). To prove a robustness guarantee for LAMBDADC, we show bounds on the change of \( \Phi \) when the algorithms (LAMBDADC and OPT) move their servers. To that end, several preliminary results will become handy. We first observe that the values \( d_1, \ldots, d_k \) correlate with the change of \( \Theta \) when LAMBDADC moves a server.
Observation 13. Let $i \leq \lfloor k/2 \rfloor$. If server $s_i$ moves from $p$ to $p + x$, $\Theta$ changes by $(-x) \cdot d_i$.

Proof. Assume w.l.o.g. that $x = -1$, that is, the server $s_i$ moves one unit to the left. Consider servers $s_j, s_j'$ such that $j' + \ell = i = j - \ell$ for some $1 \leq \ell \leq i - 1$. Since $\delta_{ij} = \delta_{ij'}$ we observe that the changes to the terms $\delta_{ij}|s_i - s_j|$ and $\delta_{ij'}|s_i - s_j'|$ of $\Theta$ cancel out. Hence, as $i \leq \lfloor k/2 \rfloor$, it suffices to consider the distances of $s_i$ to servers $s_j$ with $j \geq 2i$. Therefore,

$$\Delta \Theta = \sum_{j=2i}^{k} \delta_{ij} = \begin{cases} \sum_{t=1}^{(k-3)/2} 2\zeta_t & \text{if } k \text{ is even, and} \\ \sum_{t=1}^{(k-3)/2} k/2 \zeta_t & \text{if } k \text{ is odd.} \end{cases}$$

The definition of $\zeta_t$ implies that this is indeed equal to $d_i$. \hfill \Box

Next, we give several algebraic transformations of $\gamma$.

Lemma 14. The following statements are true:

(i) $\gamma = 2\lambda^{k-1}/(1 + \lambda^{k-2})$.

(ii) For all $1 \leq i \leq \lfloor k/2 \rfloor$, it holds $(1 + \lambda)\gamma = \lambda^{i+1}d_i - \lambda^{i}d_{i+1}$.

(iii) If $k$ is even, it holds $\gamma = \lambda d_1 + (1 + \lambda) \sum_{j=2}^{k/2} (-1)^{j-1} d_j$.

Proof. (i) Since

$$\lambda^{k-1}(\beta - 1) = \lambda^{k-1} \sum_{t=1}^{k-1} \lambda^{t-i} = \sum_{t=0}^{k-2} \lambda^t,$$

we conclude by the definition of $\gamma$ and $d_i$ that

$$\gamma = \frac{d_1}{\beta - 1} = \frac{2}{(1 + \lambda^{k-2})(\beta - 1)} \sum_{t=0}^{k-2} \lambda^t = \frac{2}{1 + \lambda^{k-2}} \lambda^{k-1}.$$

(ii) Simplifying the right-hand side gives

$$\lambda^{i+1}d_i - \lambda^{i}d_{i+1} = \frac{2\lambda^i}{1 + \lambda^{k-2}} \left( \lambda^{k-i} \sum_{t=1}^{i} \lambda^t - \sum_{t=i}^{k-2} \lambda^t \right) = \frac{2\lambda^i}{1 + \lambda^{k-2}} \left( \sum_{t=1}^{i} \lambda^t - \sum_{t=i}^{k-2} \lambda^t \right)$$

$$= \frac{2\lambda^i}{1 + \lambda^{k-2}} \left( \lambda^{k-i} + \lambda^{k-i-1} \right) = 2 \left( \frac{\lambda^k + \lambda^{k-1}}{1 + \lambda^{k-2}} \right)$$

Then, Lemma 14(i) concludes the proof.

(iii) Assume that $k$ is even. The right-hand side is equal to

$$(-1)^{k/2-1} \lambda d_{k/2} + \sum_{j=1}^{k/2-1} (-1)^{j-1} (\lambda d_j - d_{j+1}).$$

By Lemma 14(ii),

$$(-1)^{k/2-1} \lambda d_{k/2} + (1 + \lambda) \sum_{j=1}^{k/2-1} (-1)^{j-1} \frac{Y}{\lambda^j},$$

22
which is equal to
\[
\frac{1}{\lambda^{k/2-1}} \left( (-1)^{k/2-1} \lambda^{k/2} d_{k/2} + \sum_{j=0}^{k/2-2} (-1)^{k/2-j}(\lambda^j + \lambda^{j+1})y \right).
\]

We proceed by applying a telescoping sum argument. Since \(k/2 - (k/2 - 2) = 2\), the last term of the sum \(\lambda^{k/2-1}y\) is positive. Similarly, the first term \(\lambda^0y\) has the same sign as \((-1)^{k/2-0} = -(\lambda^{k/2-1})\).

The remaining terms of the sum cancel out. Thus, it remains
\[
\frac{1}{\lambda^{k/2-1}} \left( \lambda^{k/2-1}y + (-1)^{k/2-1}(\lambda^{k/2}d_{k/2} - y) \right).
\]

By definition, \(d_{k/2} = 2\lambda^{k/2-1}/(1 + \lambda^{-2})\). Hence, \(\lambda^{k/2}d_{k/2}\) is equal to \(y\) by Lemma 14(i). We conclude that the expression is indeed equal to \(y\).

These preliminary results enable us to prove two more involved observations about the weights chosen for our potential function. The proofs are deferred to Appendix B. The first observation is important for all cases where a request appears between two servers. Recall the definition of \(\Phi\). If \(s_i\) moves with speed \(\lambda\) and \(s_{i+1}\) with speed 1, the changes to \(\Phi\) (increase or decrease) are scaled by \(\beta \gamma \lambda \omega_i\) regarding \(s_i\) and \(\beta \gamma \omega_{i+1}\) regarding \(s_{i+1}\). If \(i\) is even, we can easily use the definition of \(\omega\), since the denominators cancel. However, if \(i\) is odd, we use the following alternative representation of the \(\omega\)-weights.

**Observation 15.** For \(2 \leq i \leq \lfloor k/2 \rfloor\), \(\omega_i\) is equal to

\[
\begin{cases}
2\lambda \sum_{j=1}^{i/2} d_{2j-1} - 2\sum_{j=1}^{i/2-1} d_{2j} - d_i - y & \text{if } i \text{ is even, and} \\
2\lambda \sum_{j=1}^{(i-1)/2} d_{2j-1} - 2\sum_{j=1}^{(i-1)/2} d_{2j} + \lambda d_i + \lambda y & \text{if } i \text{ is odd.}
\end{cases}
\]

**Proof.** We first note that for every \(1 \leq j \leq \lfloor k/2 \rfloor - 1\), applying Lemma 14(ii) with \(j\) and \(j+1\) yields

\[
\lambda d_j - d_{j+1} = \frac{1 + \lambda}{\lambda^j} \gamma = \lambda^2 d_{j+1} - \lambda d_{j+2}.
\]

We now prove the statement separately for all even and all odd values of \(2 \leq i \leq \lfloor k/2 \rfloor\) by induction.

As induction base for the even case, we first prove the claim for \(i = 2\). Indeed,

\[
\omega_2 = \frac{\lambda d_2 + (2 + \lambda) \gamma}{\beta \gamma \lambda} = \frac{d_2 + (2 + \lambda) \gamma / \lambda}{\beta \gamma} = \frac{2\lambda d_1 - d_2 - y}{\beta \gamma}.
\]

Note that the last equality derives from Lemma 14(ii). Now assume that \(i > 2\) is even. The induction hypothesis for \(i - 2\) yields in this case

\[
\beta \gamma \lambda \cdot \omega_{i-2} = 2\lambda^2 \sum_{j=1}^{i/2-1} d_{2j-1} - 2\lambda \sum_{j=1}^{i/2-2} d_{2j} - \lambda d_{i-2} - \lambda y.
\]

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We want to prove that $\beta \gamma \lambda \cdot \omega_i$ is equal to

$$2\lambda^2 \sum_{j=1}^{i/2} d_{2j-1} - 2\lambda \sum_{j=1}^{i/2-1} d_{2j} - \lambda d_i - \lambda \gamma,$$

which can be rearranged to

$$2\lambda^2 \sum_{j=1}^{i/2-1} d_{2j-1} - 2\lambda \sum_{j=1}^{i/2-2} d_{2j} - \lambda d_{i-2} - \lambda \gamma - \lambda d_i - \lambda d_{i-2} + 2\lambda^2 d_{i-1}.$$

Replacing the right side of (2) in the above expression yields

$$\beta \gamma \lambda \cdot \omega_{i-2} - \lambda d_i - \lambda d_{i-2} + 2\lambda^2 d_{i-1}.$$  

Since (1) gives $2(1 + \lambda^2) d_{i-1} = 2\lambda(d_{i-2} + d_i)$, and by the definition of $\omega_{i-2}$, this can be rewritten to

$$2\lambda \sum_{j=1}^{i/2-2} d_{2j} - 2\sum_{j=1}^{i/2-2} d_{2j+1} + \lambda d_{i-2} + (2 + \lambda) \gamma - 2d_{i-1} + \lambda d_{i-2} + \lambda d_i = 2\lambda \sum_{j=1}^{i/2-1} d_{2j} - 2\sum_{j=1}^{i/2-1} d_{2j+1} + (2 + \lambda) \gamma + \lambda d_i,$$

which is indeed equal to $\beta \gamma \lambda \cdot \omega_i$ by definition.

As induction base for the odd case, we start by proving the claim for $i = 3$, that is

$$\omega_3 = \frac{2\lambda d_2 - d_3 + \gamma}{\beta \gamma} = \frac{2\lambda^2 d_2 - \lambda d_3 + \lambda \gamma}{\beta \gamma \lambda} = \frac{2\lambda d_1 - 2d_2 + \lambda d_3 + \lambda \gamma}{\beta \gamma \lambda}.$$  

In the last equality we used that $2(1 + \lambda^2) d_2 = 2\lambda(d_1 + d_3)$ by (1). Now assume that $i > 3$ is odd. By induction hypothesis for $i - 2$,

$$\beta \gamma \lambda \cdot \omega_{i-2} = 2\lambda \sum_{j=1}^{(i-1)/2-1} d_{2j-1} - 2\sum_{j=1}^{(i-1)/2-1} d_{2j} + \lambda d_{i-2} + \lambda \gamma.$$  

(3)

Consider the claimed expression for $\beta \gamma \lambda \cdot \omega_i$, that is

$$2\lambda \sum_{j=1}^{(i-1)/2} d_{2j-1} - 2\sum_{j=1}^{(i-1)/2} d_{2j} + \lambda d_i + \lambda \gamma,$$

which we can rearrange to

$$2\lambda \sum_{j=1}^{(i-1)/2-1} d_{2j-1} - 2\sum_{j=1}^{(i-1)/2-1} d_{2j} + \lambda d_{i-2} + \lambda \gamma + \lambda d_{i-2} + \lambda d_i - 2d_{i-1}.$$  

Replacing the right side of (3) in the above expression gives

$$\beta \gamma \lambda \cdot \omega_{i-2} + \lambda d_{i-2} + \lambda d_i - 2d_{i-1}.$$  

24
Noting that (1) gives $2(1 + \lambda^2)d_{i-1} = 2\lambda(d_{i-2} + d_i)$ yields together with the definition of $\omega_{i-2}$ the equivalent expression

$$2\lambda^2 \sum_{j=1}^{(i-1)/2-1} d_{2j} - 2\lambda \sum_{j=1}^{(i-3)/2-1} d_{2j+1} - \lambda d_{i-2} + \lambda y + 2\lambda^2 d_{i-1} - \lambda d_{i-2} - \lambda d_i$$

$$= 2\lambda^2 \sum_{j=1}^{(i-1)/2} d_{2j} - 2\lambda \sum_{j=1}^{(i-3)/2} d_{2j+1} - \lambda d_i + \lambda y.$$

Since this is by definition equal to $\beta y \lambda \cdot \omega_i$, we can also conclude this case. 

The second observation is an upper and a lower bound of the $\omega$-weights regardless of the corresponding server. The lower bound is necessary to show that $\Phi \geq 0$, while we use the upper bound to give an easy upper bound on the increase of the potential when the optimal solution moves, independently of its chosen server.

**Observation 16.** The values $\omega_1, \ldots, \omega_k$ are at least 0 and at most 1.

**Proof.** By definition, $\omega_1 = \omega_k = 1$. We now show this property for $\omega_i$ depending on whether $2 \leq i \leq k - 1$ is even or odd.

Assume that $i$ is even. By definition, the numerator of $\omega_i$ is equal to

$$2\lambda \sum_{j=1}^{i/2-1} d_{2j} - 2\lambda \sum_{j=1}^{i/2-1} d_{2j+1} + \lambda d_i + (2 + \lambda)y.$$

Using the definition of $d_i$ and Lemma 14(i) gives

$$\frac{2}{1 + \lambda^{k-2}} \left( \lambda \sum_{j=1}^{i/2-1} \sum_{t=2j+1}^{k-1} 2\lambda^t - \sum_{j=1}^{i/2-1} \sum_{t=2j}^{k-1} 2\lambda^t + \lambda \sum_{t=1}^{k-1-i} \lambda^t \right) + (2 + \lambda)y$$

$$= \frac{2}{1 + \lambda^{k-2}} \left( \sum_{j=1}^{i/2-1} 2(\lambda^{k-2j} + \lambda^{k-2j-2}) + \sum_{t=1}^{k-1-i} \lambda^t \right) + (2 + \lambda)y$$

$$= \frac{2}{1 + \lambda^{k-2}} \left( \sum_{t=k-i}^{k-2} 2\lambda^t + \sum_{t=1}^{k-1-i} \lambda^t \right) + (2 + \lambda)y$$

$$= \frac{2}{1 + \lambda^{k-2}} \left( \sum_{t=k-i}^{k-1} 2\lambda^t + \sum_{t=1}^{k-1-i} \lambda^t \right) + \lambda y$$

Since $\beta y \lambda = \lambda y + \lambda d_1 \geq 0$, we conclude that $\omega_i \geq 0$. Further, using the fact that $\sum_{t=k-i+1}^{k-1} \lambda^t \leq \sum_{t=1}^{k-1} \lambda^t$ yields

$$\frac{2}{1 + \lambda^{k-2}} \left( \sum_{t=k-i+1}^{k-1} 2\lambda^t + \sum_{t=1}^{k-1-i} \lambda^t \right) + \lambda y \leq \frac{2}{1 + \lambda^{k-2}} \sum_{t=1}^{k-1} \lambda^t + \lambda y = \lambda d_1 + \lambda y,$$

and we conclude that $\omega_i \leq 1$. 

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Assume that $i$ is odd. By definition, the numerator of $\omega_i$ is equal to
\[
2\lambda \sum_{j=1}^{(i-1)/2} d_{2j} - 2 \sum_{j=1}^{(i-3)/2} d_{2j+1} - d_i + \gamma.
\]

Using definitions gives
\[
\frac{2}{1 + \lambda^{k-2}} \left( \lambda \sum_{j=1}^{(i-1)/2} \sum_{t=2j}^{k-1} 2\lambda^t - \sum_{j=1}^{(i-3)/2} \sum_{t=2j}^{k-2} 2\lambda^t - \sum_{t=1}^{k-1-i} \lambda^t \right) + \gamma
\]
\[
= \frac{2}{1 + \lambda^{k-2}} \left( \sum_{j=1}^{(i-1)/2} \sum_{t=2j}^{k-2} 2\lambda^t - \sum_{j=1}^{(i-3)/2} \sum_{t=2j}^{k-2} 2\lambda^t - \sum_{t=1}^{k-1-i} \lambda^t \right) + \gamma
\]
\[
= \frac{2}{1 + \lambda^{k-2}} \left( \sum_{t=k-i}^{k-2} 2\lambda^t + \sum_{t=1}^{k-1-i} \lambda^t \right) + \gamma.
\]

Since $\beta \gamma = \gamma + d_i \geq 0$, we conclude that $\omega_i \geq 0$. Further, using the fact that $\sum_{t=k-i}^{k-2} \lambda^t \leq \sum_{t=0}^{i-2} \lambda^t$ yields
\[
\frac{2}{1 + \lambda^{k-2}} \left( \sum_{t=k-i}^{k-2} 2\lambda^t + \sum_{t=1}^{k-1-i} \lambda^t \right) + \gamma \leq \frac{2}{1 + \lambda^{k-2}} \sum_{t=0}^{k-2} \lambda^t + \gamma = d_i + \gamma,
\]
and we conclude that $\omega_i \leq 1$. \qed

Before proving formally our robustness bound by exhaustively reviewing all possible moves and bounding the corresponding changes of $\Phi$, we give some intuition.

We choose the scaling parameter $\gamma$ such that the decrease of $\Phi$ exactly matches the required lower bound for the case where the request is outside the convex hull of LAMBDA’s servers. The remaining cases are split among the possible locations where a request can appear between two servers of LAMBDA, and we show in each case that $\Phi$ decreases enough. The definition of the $\omega$ values ensures that a wrong prediction gives a tight bound on the decrease of $\Phi$ for LAMBDA’s move, while a correct prediction still guarantees a loose bound.

Proof of Lemma 9. Note that Observation 16 implies $\Phi \geq 0$. Suppose that the next request arrives. First the optimal solution increases due to Observation 16 the potential by at most $\beta \gamma \Delta \text{OPT}$ while LAMBDA remains in its previous configuration. Second LAMBDA moves. In the remaining proof we demonstrate that the potential decreases by at least $\gamma \Delta \text{ALG}$, which proves the Lemma. We look at the following set of exhaustive cases that occur when LAMBDA makes its move. Assume by scaling that in each case the fast server moves distance 1.

(a) Let the request w.l.o.g. be on the left of $s_1$. Hence, $\Delta \text{ALG} = 1$, and $\Theta$ increases by $d_i$ due to Observation 13. Since $x_1$ cannot be on the right side of the request, the potential changes by
\[
\Delta \Phi = -\beta \gamma + d_i = -(d_i - \gamma) + d_i = -\gamma \Delta \text{ALG}.
\]
The remaining cases tackle the situations where the request is located between the two servers $s_i$ and $s_{i+1}$. Without loss of generality we only look at those cases where $i \leq \lfloor k/2 \rfloor$, since the others hold by the symmetry of the line and by the symmetry of $\Phi$.

(b) Let $1 \leq i \leq \lfloor k/2 \rfloor - 1$ and suppose that $s_i$ is predicted while the optimal solution serves the request with $x_j$ for some $j > i$. Note that $\Delta \text{ALG} = 1 + \lambda$. The change of $\Phi$ is at most

$$\Delta \Phi \leq \beta \gamma (\omega_i - \lambda \omega_{i+1}) - d_i + \lambda d_{i+1}.$$  

By using the definition of $\omega_i$ if $i$ is odd and Observation 15 if $i$ is even, this is equal to

$$d_i - \lambda d_{i+1} - (1 + \lambda) \gamma - d_i + \lambda d_{i+1} = -\gamma (1 + \lambda) = -\gamma \Delta \text{ALG}.$$  

(c) Let $1 \leq i \leq \lfloor k/2 \rfloor - 1$ and suppose that $s_{i+1}$ is predicted while the optimal solution serves the request with $x_j$ for some $j \leq i$. The change of $\Phi$ is at most

$$\Delta \Phi \leq \beta \gamma (\omega_{i+1} - \lambda \omega_i) - \lambda d_i + d_{i+1}.$$  

By using the definition of $\omega_i$ if $i$ is even and Observation 15 if $i$ is odd, this is equal to

$$\lambda d_i - d_{i+1} - (1 + \lambda) \gamma - \lambda d_i + d_{i+1} = -\gamma (1 + \lambda) = -\gamma \Delta \text{ALG}.$$  

(d) Let $1 \leq i \leq \lfloor k/2 \rfloor - 1$ and suppose that $s_i$ is predicted while the optimal solution serves the request with $x_j$ for some $j \leq i$. The change of $\Phi$ is at most

$$\Delta \Phi \leq \beta \gamma (\lambda \omega_{i+1} - \omega_i) - d_i + \lambda d_{i+1}.$$  

By using the definition of $\omega_i$ if $i$ is odd and Observation 15 if $i$ is even, this is equal to

$$-d_i + \lambda d_{i+1} + (1 + \lambda) \gamma - d_i + \lambda d_{i+1}$$

$$= -\gamma (1 + \lambda) + 2(\lambda d_{i+1} - d_i + (1 + \lambda) \gamma)$$

$$= -\gamma (1 + \lambda) + 2(\lambda d_{i+1} + \lambda^2 d_i - \lambda^2 d_i - d_i + (1 + \lambda) \gamma).$$

Applying Lemma 14(ii) yields

$$-\gamma (1 + \lambda) + 2 \left( (\lambda^2 - 1) d_i - (1 + \lambda) \frac{\gamma}{\lambda^2 - 1} + (1 + \lambda) \gamma \right)$$

$$\leq -\gamma (1 + \lambda) = -\gamma \Delta \text{ALG}.$$  

(e) Let $1 \leq i \leq \lfloor k/2 \rfloor - 1$ and suppose that $s_{i+1}$ is predicted while the optimal solution serves the request with $x_j$ for some $j > i$. The change of $\Phi$ is at most

$$\Delta \Phi \leq \beta \gamma (\lambda \omega_i - \omega_{i+1}) - \lambda d_i + d_{i+1}.$$  

By using the definition of $\omega_i$ if $i$ is even and Observation 15 if $i$ is odd, we can conclude

$$-\lambda d_i + d_{i+1} + (1 + \lambda) \gamma - \lambda d_i + d_{i+1}$$

$$= -(1 + \lambda) \gamma + 2((1 + \lambda) \gamma - \lambda d_i + d_{i+1}).$$

Using Lemma 14(ii) gives

$$- (1 + \lambda) \gamma + 2 \left( (1 + \lambda) \gamma - (1 + \lambda) \frac{\gamma}{\lambda^2} \right)$$

$$\leq -\gamma (1 + \lambda) = -\gamma \Delta \text{ALG}.$$  

27
If \( k \) is even, there are two additional cases which occur when the request is located between the two middle servers \( s_{k/2} \) and \( s_{k/2+1} \). Note that these cases cannot be covered by the previous ones, since the \( \omega \)-weights of the servers on both sides of the request are equal.

(f) Let the request be between \( s_{k/2} \) and \( s_{k/2+1} \), and suppose that \( s_{k/2} \) is predicted while the optimal solution serves \( r \) with \( x_j \) for some \( j > k/2 \). The change of \( \Phi \) is at most

\[
\Delta \Phi \leq \beta \gamma (\omega_{k/2} - \lambda \omega_{k/2}) - \lambda d_{k/2} - d_{k/2}.
\]  

(4)

For the rest of this case, we distinguish two cases according to the parity of \( k/2 \), and show that \( \Delta \Phi \leq -\gamma \Delta \text{ALG} \).

If \( k/2 \) is even, (4) is by Observation 15 and the definition of \( \omega_{k/2} \) equal to

\[
2\lambda \sum_{j=1}^{k/4} d_{2j-1} - 2 \sum_{j=1}^{k/4-1} d_{2j} - d_{k/2} - \gamma
\]

\[
- \left( 2\lambda \sum_{j=1}^{k/4-1} d_{2j} - 2 \sum_{j=1}^{k/4-1} d_{2j+1} + \lambda d_{k/2} + (2 + \lambda) \gamma \right) - \lambda d_{k/2} - d_{k/2}.
\]

Noting that \( 2\lambda \sum_{j=1}^{k/4} d_{2j-1} = 2\lambda d_1 + 2\lambda \sum_{j=1}^{k/4-1} d_{2j} \) gives

\[
-(3 + \lambda) \gamma + 2 \left( \lambda d_1 + (1 + \lambda) \sum_{j=2}^{k/2} (-1)^{j-1} d_j \right).
\]

We can conclude that this is equal to \(-\gamma(1 + \lambda)\) by Lemma 14(iii).

Similarly, if \( k/2 \) is odd, (4) is by Observation 15 and the definition of \( \omega_{k/2} \) equal to

\[
2\lambda \sum_{j=1}^{(k/2-1)/2} d_{2j-1} - 2 \sum_{j=1}^{(k/2-3)/2} d_{2j} - d_{k/2} + \gamma
\]

\[
- \left( 2\lambda \sum_{j=1}^{(k/2-1)/2} d_{2j-1} - 2 \sum_{j=1}^{(k/2-1)/2} d_{2j} + \lambda d_{k/2} + \lambda \gamma \right) - \lambda d_{k/2} - d_{k/2}.
\]

Noting that \( 2\lambda \sum_{j=1}^{(k/2-1)/2} d_{2j-1} = 2\lambda d_1 + 2\lambda \sum_{j=1}^{(k/2-3)/2} d_{2j+1} \) yields

\[
\gamma - \lambda d_1 + (1 + \lambda) \sum_{j=2}^{k/2} (-1)^{j-1} d_j = \gamma - \lambda \gamma - 2 \left( \lambda d_1 + (1 + \lambda) \sum_{j=2}^{k/2} (-1)^{j-1} d_j \right).
\]

This is equal to \(-\gamma(1 + \lambda)\) by Lemma 14(iii).

(g) Let the request be between \( s_{k/2} \) and \( s_{k/2+1} \), and suppose that \( s_{k/2} \) is predicted while the optimal solution serves \( r \) with \( x_j \) for some \( j \leq k/2 \). The change of \( \Phi \) is at most

\[
\Delta \Phi \leq \beta \gamma (\omega_{k/2} - \omega_{k/2}) - \lambda d_{k/2} - d_{k/2},
\]

which is bounded from above by the previous case. Hence, \( \Delta \Phi \leq -\gamma \Delta \text{ALG} \).
Theorem 2. Let $A$ (see the memory-constrained definition). The instance terminates when $A$.

This section is dedicated to the proof of Theorem 2, which we restate below. The proof is a generalization of the one proposed in Section 3.2 for two servers. However, for proving the general case we need a more sophisticated construction rule and a more involved argumentation.

Theorem 2. Let $\lambda \in (0, 1]$, $\rho(k) = \sum_{i=0}^{k-1} \lambda^i$ and $\beta(k) = \sum_{i=0}^{k-1} \lambda^{-i}$. Let $A$ be a learning-augmented locally-consistent and memory-constrained deterministic online algorithm for the $k$-server problem on the line. Then, if $A$ is $\rho(k)$-consistent, it is at least $\beta(k)$-robust.

Let $\lambda \in (0, 1]$. Recall that $\rho(k) = \sum_{i=0}^{k-1} \lambda^i$. Let $A$ be a $\rho(i)$-consistent locally-consistent and memory-constrained deterministic online algorithm for the $i$-server problem on the line, for all $i \leq k$. The objective is to show that $A$ is then at least $\beta(k)$-robust, with $\beta(k) = \sum_{i=0}^{k-1} \lambda^{-i}$.

Let $p_1 \leq \ldots \leq p_{k+1}$ be points on the line with inter-distances $d_1, \ldots, d_k$, where for $1 \leq i \leq k - 1$, $d_i = 1$, and $d_k > 1$ is arbitrarily large. See Figure 4 for an illustration. We also define an arbitrarily small constant $\varepsilon > 0$ and say that a server covers a point $p_i$ if it is at most a distance $\varepsilon$ away from it. We refer to smaller positions on the line as left. Let $P := \{p_1, \ldots, p_k\}$. In the following we inductively construct an instance. In their initial configuration, i.e. at time $t = 0$, the $k$ servers, $s_1, \ldots, s_k$, are located at $p_1, \ldots, p_k$. We assume that servers never overpass each other to simplify the notations. Then, we force the servers to $p_1, \ldots, p_{k-1}, p_{k+1}$ (see the memory-constrained definition). The instance terminates when $A$ places $s_k$ to cover $p_k$. At any time $t > 0$, the next requested point $r_t$ is the leftmost point (i.e. the point with the smallest index) which is not covered by any server of $A$. If $p_1$ is not covered and $s_1$ is on the left of $p_2 - \varepsilon$ then $r_t$ is the second leftmost uncovered point. If $p_1$ is not covered and $s_1$ covers $p_2$, but did not serve it since leaving $p_1$, then $r_t$ is $p_2$ and $r_{t+1}$ (next in time) is $p_1$. At any time $t > 0$, we denote the instance composed of $r_1, \ldots, r_t$ by $I_t$.

At every point in time, we give $A$ the prediction that suggests serving a request at some point $p_t$ with the server $s_t$. An exception is the first request, where $s_k$ is predicted (note that the first request is always located at $p_{k+1}$). We now show that this construction rule is well-defined.

Lemma 17. The construction ends after a finite number of requests.

Proof. For the sake of contradiction, assume that the construction does not end after a finite number of requests. Hence, every request $r$ except the first one must be in the set $P$, and by construction, no server
covered \( r \) in the previous configuration. Thus, the server that serves \( r \) must have been moved with some cost at least \( \epsilon \), which implies that \( \mathcal{A} \) has unbounded cost.

Now consider any infinite instance \( I^p \) which starts with a request at \( p_{k+1} \) followed by requests contained in \( P \). An optimal solution for \( I^p \) is to serve the first request with \( s_k \) and then to move it immediately back to the set \( P \), such that every point in \( P \) contains a server. Hence, the total cost of an optimal solution is constant. Therefore the consistency of \( \mathcal{A} \) would be infinite, as the prediction given to \( \mathcal{A} \) corresponds to the optimal solution, which is a contradiction. \( \square \)

Due to Lemma 17, we assume for the rest of this section that the construction ends after \( n \) steps, and we define \( I = I_n \), see Figure 4.

We first focus on the cost that \( \mathcal{A} \) charges for \( I \). Let \( D_i \) be the distance traveled by the server \( s_i \) in \( \mathcal{A} \). Using the locally-consistent definition, we show the following relation between \( D_i \)'s:

**Lemma 18.** For all \( i \leq k \), for \( \epsilon \) small enough, we have \( D_1 \leq \lambda^{i-1}D_i + O_k(\epsilon D_i + d_i) \), where the notation \( O_k(\cdot) \) treats \( k \) as a constant.

**Proof.** Let \( i \in \{2, 3, \ldots, k\} \) and assume by induction that the relation is true for all \( j < i \). Note that it is trivial for \( i = 1 \).

We denote by \( \mathcal{A}_1(I) \) the cost of \( \mathcal{A} \) restricted to the \( i \) leftmost servers. Consider the \( i \) leftmost servers and we apply the locally-consistent property of \( \mathcal{A} \) on these servers. Let \( I' \) be the corresponding instance on \( i \) servers, where requests not served by \( \{s_1, \ldots, s_i\} \) are replaced by requests to the new position of \( s_i \).

Consider the algorithm \( \text{FrP} \) serving \( I' \) following the initial predictions as in the locally-consistent definition. There are two types of requests: a point \( p_\ell \) for \( \ell < i \) is served at no cost by \( s_\ell \), and any other request is served by \( s_i \). The objective is to upper bound \( \text{FrP}(I') \) by \( D_i \) plus negligible terms. Consider all requests different from \( p_i \) served by \( s_i \) in \( \text{FrP} \), and let \( r_1 \) and \( r_2 \) be two consecutive requests in this set (there can be other requests not belonging to this set between \( r_1 \) and \( r_2 \)). These requests are based on requests of \( I \) outside of \( \{p_1 \ldots p_i\} \), which means that each of these points (except \( p_i \)) is covered by a server of \( \mathcal{A} \) before the request, and that \( s_i \) also went to \( r_1 \) and \( r_2 \) in \( \mathcal{A} \), at the time at which they are requested in \( I' \). A technical difficulty here is that \( s_i \) does not need to be exactly at \( p_i \) before these requests: it can be within a distance of \( \epsilon \). There are several cases to analyze.

- If \( p_i \) is not requested between \( r_1 \) and \( r_2 \), then \( \text{FrP} \) pays the shortest path between \( r_1 \) and \( r_2 \), so at most how much \( s_i \) travels in \( \mathcal{A} \).
- If \( s_i \) goes on \( p_i \) between \( r_1 \) and \( r_2 \) in \( \mathcal{A} \), then \( \text{FrP} \) also pays at most how much \( s_i \) travels in \( \mathcal{A} \).
- If \( p_i \) is requested between \( r_1 \) and \( r_2 \) and \( s_i \) does not go on \( p_i \) in \( \mathcal{A} \), we focus on the subinstance \( I^* \) starting from the request \( r_1 \) and ending just before \( r_2 \) is requested. Let \( \text{FrP}(I^*) \), \( \mathcal{A}_i(I^*) \) and \( D_i^* \) be the restrictions of \( \text{FrP}(I) \), \( \mathcal{A}_i(I) \) and \( D_i \) to \( I^* \). Note that \( \text{FrP}(I^*) \leq D_i^* + \epsilon \) as \( \text{FrP} \) moves \( s_i \) to \( r_1 \) then back to \( p_i \) whereas \( \mathcal{A} \) needs only to move \( s_i \) to \( r_1 \) and then near \( p_i \). The objective is now to show that this additive term \( \epsilon \) is negligible compared to \( \mathcal{A}_i(I^*) \), for which we need a further case distinction.
  - If \( r_1 \) is at least a distance \( \sqrt{\epsilon} \) away from \( p_i \), then \( \text{FrP}(I^*) \) moves \( s_i \) by a distance which is close to \( D_i^* \). Specifically, we have \( D_i^* \geq 2\sqrt{\epsilon} - 2\epsilon \geq \sqrt{\epsilon} \) for \( \epsilon \) small enough, and the relation \( \text{FrP}(I^*) \leq D_i^* + \epsilon \) implies \( \text{FrP}(I^*) \leq (1 + \sqrt{\epsilon})D_i^* \).
  - If \( r_1 \) is at most a distance \( \sqrt{\epsilon} \) away from \( p_i \), we get \( \text{FrP}(I^*) \leq 2\sqrt{\epsilon} + \epsilon \) and we distinguish two cases which are slightly different if \( i = 2 \) or \( i > 2 \).
• If \(i > 2\) then the cost of \(\mathcal{A}_I\) on \(I^*\) is at least \(A_I(I^*) \geq D_{i-1}' > d_{i-1} - \epsilon = 1 - \epsilon\) as \(p_i\) must have been served by \(s_{i-1}\) (previously located near \(p_{i-1}\)) if it was not served by \(s_i\). We therefore obtain \(\text{FrP}(I^*) \leq 3 \sqrt{\epsilon} \cdot D_{i-1}'\).

• If \(i = 2\), the difference is that \(s_1\) may be initially located anywhere between \(p_1\) and \(p_2\). \(s_1\) serves \(p_i = p_2\) when it is requested (as this case assumes \(s_2\) does not serve \(p_2\) in \(I^*\)), and then must serve \(p_1\) by the definition of the instance \(I\). Therefore, the cost of \(\mathcal{A}_I\) on \(I^*\) is at least \(A_I(I^*) \geq D_1' \geq d_1 = 1\). We thus obtain \(\text{FrP}(I^*) \leq 3 \sqrt{\epsilon} \cdot D_1'\).

Summing over all subinstances, we obtain the following inequality:

\[
\text{FrP}(I^*) \leq (1 + \sqrt{\epsilon})D_1 + 3 \sqrt{\epsilon} \cdot \sum_{i=1}^{i-1} D_i \leq D_1 + 3 \sqrt{\epsilon} \cdot \mathcal{A}_I(I).
\]

As the initial and final configurations are identical up to a distance of \(d_1\) for \(s_1\) and \(\epsilon\) for other servers, the locally-consistent property for \(I^*\) yields

\[
\mathcal{A}_i(I) \leq \rho(i)D_i + 3 \sqrt{\epsilon} \rho(i) \cdot \mathcal{A}_i(I) + O(\epsilon k^2 + d_1 k).
\]

For \(\epsilon\) small enough, we have \(3 \sqrt{\epsilon} \rho(i) < 1/2\), which implies that \(\mathcal{A}_i(I) \leq 2\rho(i)D_i + O(\epsilon k^2 + d_1 k)\). Using this new bound on \(\mathcal{A}_i(I)\) on the right-hand side of the above inequality leads to the following:

\[
\sum_{i=1}^{i-1} D_i \leq (\rho(i) - 1)D_i + O(\epsilon k^2 + d_1 k + \sqrt{\epsilon} \rho(i)^2 D_i)
\]

We now use the induction hypothesis to lower bound \(D_i\) by \(\lambda^{1-\ell}D_1 + O_k(\epsilon D_1 + d_1)\) and replace \(\rho(i)\) by its expression, before dividing all sides by \(\lambda^{1-\ell} \lambda^{-\ell}\). We use the notation \(O_k(\cdot)\) to avoid detailing the irrelevant dependencies on \(k\), note that \(\rho(i)\) depends only on \(\lambda\) and \(k\) so does not appear inside the notation \(O_k(\cdot)\).

\[
\sum_{i=1}^{i-1} \frac{1}{\lambda^{i-1}}D_i \leq \sum_{i=1}^{i-1} \lambda^\ell D_i + O_k(\epsilon D_1 + d_1)
\]

\[
D_1 \leq \lambda^{i-1}D_i + O_k(\epsilon D_1 + d_1)
\]

We build the instance \(I^\omega\) repeating the instance \(I\) \(\omega\) times, starting directly by the force to \(p_1, \ldots, p_{k-1}, p_{k+1}\), see Figure 5. The predictions for each iteration correspond to the predictions defined in instance \(I\). We now bound the optimal cost for this instance.

**Lemma 19.** \(\text{OPT}(I^\omega) \leq 2d_k + \omega \cdot (D_1 + 2 \sum_{i=2}^{k-1} d_i)\).
Proof. Consider the following schedule for $I^\omega$: at each iteration, move $k-1$ servers to $p_2, \ldots, p_{k+1}$ and alternate between $p_1$ and $p_2$ with $s_1$. We now analyze how many alternations we need to do in each iteration. By definition of the instance, $p_1$ is only requested if $s_1$ has served $p_2$ since it last left $p_1$. Therefore, the distance traveled by $s_1$ equals $D_1$. At the end of the iteration, we move back the $k-2$ middle servers, giving the target cost.

Proof of Theorem 2. As $A$ is memory-constrained, its behavior on each iteration of $I$ is identical, $s_k$ is at $p_k$ initially, then the $k$ servers are forced to the points $p_1, \ldots, p_k$ before continuing the requests. Therefore $A$ must pay at least $d_k$ to serve the first force operation, and then must make the same decisions in all iterations.

Using Lemma 19, the competitive ratio of $A$ for instance $I^\omega$ is therefore at least

$$\frac{A(I^\omega)}{\text{OPT}(I^\omega)} \geq \frac{\omega \cdot \sum_{i=1}^{k} D_i}{2d_k + \omega \cdot (D_1 + 2 \sum_{i=2}^{k-1} d_i)} \xrightarrow{\omega \to \infty} \frac{\sum_{i=1}^{k} D_i}{D_1 + 2 \sum_{i=2}^{k-1} d_i}.$$

Consider $d_k$ arbitrarily large (but still small compared to $\omega$). If $D_1$ is bounded by a constant, then the competitive ratio is unbounded, so $A$ is not robust. Otherwise, the terms $d_i$ become negligible compared to $D_1$, and we show that the limit of the competitive ratio is lower bounded by the desired robustness expression, using Lemma 18 (which implies that $d_1$ is also negligible compared to any $D_i$):

$$\frac{A(I^\omega)}{\text{OPT}(I^\omega)} \xrightarrow{d_k \to \infty} \frac{\sum_{i=1}^{k} D_i}{D_1} \geq \sum_{i=0}^{k-1} \frac{1}{\lambda^i} + O_k(\epsilon + \frac{d_1}{D_1\epsilon}) \xrightarrow{\epsilon \to 0} \sum_{i=0}^{k-1} \lambda^{-i}. \qed$$

In the following we show that the consistency of LAMBDA0 is best possible up to a factor of 2.

Lemma 20. For every $\lambda \in [0, 1]$, $\alpha(k) < 2\rho(k)$. 

32
We show that our predictions are PAC learnable in an agnostic sense with a sample complexity polynomial

\[ \rho(k) = \frac{1 - \lambda^k}{1 - \lambda}. \]

We now prove the result based on the parity of \( k \). Assume that \( k \) is odd. Recall that

\[ \alpha(k) = 1 + 2 \sum_{i=1}^{k/2-1} \lambda^i + \lambda^{k/2} = 1 + 2 \frac{\lambda - \lambda^{k/2}}{1 - \lambda} + \lambda^{k/2} \]

and, thus,

\[ \frac{\alpha(k)}{\rho(k)} = \frac{(1 + \lambda^{k/2})(1 - \lambda) + 2(\lambda - \lambda^{k/2})}{1 - \lambda^k} = \frac{1 + \lambda - \lambda^{k/2} - \lambda^{k/2+1}}{1 - \lambda^k} < 2. \]

Assume that \( k \) is odd, then

\[ \alpha(k) = 1 + 2 \sum_{i=1}^{(k-1)/2} \lambda^i = 1 + 2 \frac{\lambda - \lambda^{(k+1)/2}}{1 - \lambda}, \]

and we conclude that

\[ \frac{\alpha(k)}{\rho(k)} = \frac{1 - \lambda + 2(\lambda - \lambda^{(k+1)/2})}{1 - \lambda^k} = \frac{1 + \lambda - 2\lambda^{(k+1)/2}}{1 - \lambda^k} < 2. \]

\[ \square \]

### D PAC Learnability of Predictions

We show that our predictions are PAC learnable in an agnostic sense with a sample complexity polynomial in the number of requests and we give an efficient learning algorithm. Let \( D \) be an unknown distribution of sequences of \( n \) requests represented by points in the interval [0, 1]. Here we assume a bounded line as a metric (scaled to [0, 1]), which is a restriction but natural in most applications. Further, we assume that we can sample i.i.d. sequences from \( D \).

Let \( H = \{1, \ldots, k\}^n \) denote a hypothesis class containing all possible static predictions, i.e., the set of all \( k \)-server solutions for request sequences of length \( n \). Let \( C_0 \) be a known initial configuration. The prediction error for a prediction \( \tau \in H \) on a request sequence \( \sigma \) is defined as \( \eta_\sigma(\tau) = \text{FtP}(\sigma, \tau) - \text{OPT}(\sigma) \), where \( \text{FtP}(\sigma, \tau) \) is the total cost of following the prediction \( \tau \) on the sequence \( \sigma \) starting in \( C_0 \), and \( \text{OPT}(\sigma) \) is the cost of an optimal solution on \( \sigma \) starting in \( C_0 \). Then, \( \eta_\sigma(\tau) \leq \eta_{\max} \leq n \) for all possible sequences \( \sigma \) and for all \( \tau \in H \).

We argue that we can use a classical empirical risk minimization (ERM) learning method, see, e.g., [46]. The ERM method uses a training set \( S = \{\sigma_1, \ldots, \sigma_m\} \) of i.i.d. samples from \( D \). Then, it determines a prediction \( \tau_p \in H \) that minimizes the empirical error \( \eta_S(\tau) = \frac{1}{m} \sum_{j=1}^{m} \eta_{\sigma_j}(\tau) \). Since our hypothesis class is finite and the error function bounded, classical results imply that our predictions are PAC learnable in an agnostic sense with a polynomial sample complexity. Further, we show that the problem of finding the prediction minimizing the empirical error within the training set can be reduced to an offline \( k \)-server problem on a modified request sequence \( \tilde{\sigma} \) of length \( n \), where the distance between the \( \ell \)th and \( i \)th request in \( \tilde{\sigma} \) is given by \( \frac{1}{m} \sum_{j=1}^{m} d(\sigma_j(\ell), \sigma_j(i)) \). This problem can be solved efficiently [17].
Theorem 10. For any $\epsilon, \delta \in (0, 1)$, a known initial configuration $C_0$ and any distribution $\mathcal{D}$ over the sequences of $n$ requests of known extent, there exists an algorithm which, given an i.i.d. sample of $\mathcal{D}$ of size $m \in O \left( \frac{1}{\epsilon^2} \cdot (n \log k - \log \delta)\eta_{\max}^2 \right)$, returns a prediction $\tau_p \in \mathcal{H}$ in polynomial time depending on $k$, $n$, and $m$, such that with probability of at least $(1 - \delta)$ it holds $\mathbb{E}_{\sigma \sim \mathcal{D}}[\eta_\sigma(\tau_p)] \leq \mathbb{E}_{\sigma \sim \mathcal{D}}[\eta_\sigma(\tau^*)] + \epsilon$, where $\tau^* = \arg\min_{\tau \in \mathcal{H}} \mathbb{E}_{\sigma \sim \mathcal{D}}[\eta_\sigma(\tau)]$.

Proof. Since the hypothesis class $\mathcal{H}$ is finite with $|\mathcal{H}| = k^n$, and our non-negative error function is bounded by $\eta_{\max}$, classical results, see e.g. [46], imply that $\mathcal{H}$ is agnostically PAC-learnable using the ERM algorithm with a sample complexity of

$$m \leq \left[ \frac{2 \log(2|\mathcal{H}|/\delta)}{\epsilon^2} \right] \leq O \left( \frac{(n \log k - \log \delta)\eta_{\max}^2}{\epsilon^2} \right).$$

That is, given a sample of size at least $m$, the ERM algorithm outputs with a probability of at least $(1 - \delta)$ a prediction $\tau_p$ such that $\mathbb{E}_{\sigma \sim \mathcal{D}}[\eta_\sigma(\tau_p)] \leq \mathbb{E}_{\sigma \sim \mathcal{D}}[\eta_\sigma(\tau^*)] + \epsilon$, holds, where $\tau^* = \arg\min_{\tau \in \mathcal{H}} \mathbb{E}_{\sigma \sim \mathcal{D}}[\eta_\sigma(\tau)]$.

It remains to describe an efficient implementation of the ERM algorithm for our setting. Let $S = \{\sigma_1, \ldots, \sigma_m\}$ be a sample drawn i.i.d. from $\mathcal{D}$. We assume that this can be done in polynomial time in $m$. For a sequence $\sigma_j \in S$ let $\sigma(j(i))$ be the position of the $i$th request in $\sigma_j$. We further define for $1 \leq \ell \leq i \leq n$ and $1 \leq k' \leq k$ the distance functions $\delta_j(\ell, i) = |\sigma_j(\ell) - \sigma_j(i)|$ and $y_j(k', i) = |C_0(k') - \sigma_j(i)|$. The empirical error of a prediction $\tau$ is in our setting defined as

$$\eta_S(\tau) = \frac{1}{m} \sum_{j=1}^m \eta_{\sigma_j}(\tau) = \frac{1}{m} \sum_{j=1}^m \text{FrP}(\sigma_j, \tau) - \text{OPT}(\sigma_j).$$

The ERM algorithm outputs the prediction $\tau_p \in \mathcal{H}$ that minimizes $\eta_S(\tau)$ as a function over $\mathcal{H}$. Since iterating over all predictions in $\mathcal{H}$ takes exponential time, we compute $\tau_p$ differently. To do so, we first observe that $\frac{1}{m} \sum_{j=1}^m \text{OPT}(\sigma_j)$ is independent of $\tau$, thus minimizing $\eta_S(\tau)$ can be reduced to minimizing

$$\frac{1}{m} \sum_{j=1}^m \text{FrP}(\sigma_j, \tau) = \frac{1}{m} \sum_{j=1}^m \sum_{k'=1}^k \sum_{i=1}^n \xi^\tau_{k', i} \cdot y_j(k', i) + \frac{1}{m} \sum_{j=1}^m \chi^\tau_{k', \ell, i} \cdot \delta_j(\ell, i) = \sum_{k'=1}^k \sum_{i=1}^n \xi^\tau_{k', i} \cdot \left( \frac{1}{m} \sum_{j=1}^m y_j(k', i) \right) + \sum_{\ell=1}^i \sum_{j=1}^m \chi^\tau_{k', \ell, i} \cdot \frac{1}{m} \sum_{j=1}^m \delta_j(\ell, i),$$

where $\chi^\tau_{k', \ell, i} \in \{0, 1\}$ indicates (i.e. is equal to 1) that server $k'$ serves the $i$th request of $\sigma_j$ directly after the $\ell$th request of $\sigma_j$ in $\tau$ and $\xi^\tau_{k', i} \in \{0, 1\}$ indicates that the $i$th request of $\sigma_j$ is the first one that server $k'$ serves in $\tau$.

We now demonstrate that we can efficiently compute a prediction $\tau \in \mathcal{H}$ that minimizes (5). Indeed, observe that (5) is equal to the total cost of the solution $\tau$ for the $k$-server instance that starts in $C_0$ and serves a sequence $\tilde{\sigma}$ of length $n$, where the distance between the $\ell$th and $i$th request in $\tilde{\sigma}$ is given by $\delta(\ell, i) = \frac{1}{m} \sum_{j=1}^m \delta_j(\ell, i)$ and the distance between the $i$th request in $\tilde{\sigma}$ and the initial position of server $k'$ is given by $y'(k', i) = \frac{1}{m} \sum_{j=1}^m y_j(k', i)$. But this means that any optimal solution $\overline{\tau}$ for this instance also minimizes (5). Clearly, $\overline{\tau} \in \mathcal{H}$, and an optimal solution for a $k$-server instance with known distance functions can be computed in $O(kn^2)$ time using a min-cost flow algorithm [17].

\[ \Box \]
Table 1: Relative improvements of the largest mean empirical competitive ratio for any bin due to lazy implementations.

|      | DOUBLECOVERAGE | LAMBDA DC (0.1) | LAMBDA DC (0.5) | FrP&DC |
|------|----------------|----------------|----------------|--------|
|      | non-lazy | lazy | non-lazy | lazy | non-lazy | lazy | non-lazy | lazy |
| $k = 2$ | Improvement | 35% | 16% | 22% | 27% |
| $k = 10$ | Improvement | 21% | 12% | 19% | 24% |
| $k = 50$ | Improvement | 21% | 12% | 19% | 28% |

E Experiments

This section gives a detailed overview over the empirical experiments. The simulation software is written in Rust (version 1.51.0, 2018 edition). We executed all experiments in Ubuntu 18.04.5 on a machine with two AMD EPYC ROME 7542 CPUs (64 cores in total) and 1.96 TB RAM.

We implemented FrP&DC [3] with the hyperparameter $\gamma$ equal to 1. The instances are based on the BrightKite-Dataset [16]. We extract sequences with a length of 1000 checkins, normalize the scaling of latitudes to the interval $[0, 4000]$, and use these values as the positions of the requests on the line. All servers start at the same initial random position.

We generate predictions in a semi-random fashion. Fix two parameters $p$, the number of bins, and $b$, the bin size, and an instance. Our goal is to generate evenly distributed predictions, i.e., in each bin $i \in \{1, \ldots, p\}$ there are at least five predictions with relative error between $(i-1)b$ and $ib$. Additionally, we use an optimal solution of the instance as the perfect prediction.

Given those parameters and an instance, we iteratively sample many predictions with an increasing number of wrong choices with respect to the optimal solution. While this procedure does not find all predictions, especially these with the largest relative error, it gives a good tradeoff between running time and range of prediction error. We set $p = 10$ and $b$ as high as we find for at least 40 instances these evenly distributed predictions. Other instances are discarded.

The results for $k = 2$, $k = 10$ and $k = 50$ are displayed in Figures 6 to 8. We first observe that for a reasonable choice of $\lambda$ ($0.1 \leq \lambda \leq 0.5$) LAMBDA DC outperforms FrP&DC throughout almost all generated relative prediction errors in both lazy and non-lazy settings. This is also the case compared to DOUBLECOVERAGE with the exception of its strong performance for $k = 2$ in the lazy implementation. Further, all algorithms except LAMBDA DC for $\lambda = 0.0$ improve by a lazy implementation. This is no surprise, as this is the only algorithm that only moves a single server in the non-lazy setting, so there are no postponed moves that can possibly be improved by a lazy implementation. The actual improvements of the largest mean empirical ratio for any bin of all algorithms which we discovered in our experiments are given in Table 1. Observe that LAMBDA DC benefits more from the lazy implementation when $\lambda$ gets closer to 1, whereas the improvements for FrP&DC are between 24% and 28%. We suspect that FrP&DC only makes few expensive resets in our instances, while LAMBDA DC benefits from many cheap improvements.
Figure 6: Results for $k = 2$ and $b = 1$.

Figure 7: Results for $k = 10$ and $b = 2$.

Figure 8: Results for $k = 50$ and $b = 3$. 