Temperature effect in the Casimir attraction of a thin metal film

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Abstract. The Casimir effect for conductors at arbitrary temperatures is theoretically studied. By using the analytical properties of the Green functions and applying the Abel-Plan formula to Lifshitz’s equation, the Casimir force is presented as sum of a temperature dependent and vacuum contributions of the fluctuating electromagnetic field. The general results are applied to the system consisting of a bulk conductor and a thin metal film. It is shown that a characteristic frequency of the thermal fluctuations in this system is proportional to the square root of a thickness of the metal film. For the case of the sufficiently high temperatures when the thermal fluctuations play the main role in the Casimir interaction, this leads to the growth of the effective dielectric permittivity of the film and to a disappearance of the dependence of Casimir’s force on the sample thickness.

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1. Introduction

The theoretical (see, for example, Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9]) and experimental [10, 11, 12, 13] study of the Casimir effect has more than fifty years of history. The kernel of this phenomenon consists in the fluctuation electromagnetic interaction of uncharged bodies. For metals with a high value of the conductivity $\sigma$, the Casimir interaction manifests itself as the attractive force $f_0$ which varies as the inverse fourth power of the distance $a$ between the plates [1],

$$f_0 = -\frac{\pi^2 \hbar c}{240 a^4} \quad (T = 0, \sigma \to \infty),$$

where $c$ is the speed of light and $T$ is the temperature.

With an increase of the temperature, but under the condition

$$\frac{kT}{\hbar} \ll \frac{c}{a},$$

an additional term proportional to the fourth power of the temperature appears in the Casimir force (see Ref. [14] and references therein),

$$\Delta f(T) = \frac{\pi^2 (kT)^4}{45 (\hbar c)^3}.$$

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This term was derived under an assumption that the thermal equilibrium between the
matter and the radiation takes place. The additional \( a \)-independent attractive force
between the metal plates arises as a result of the pressure of the thermal radiation
outside of the plates. Under the inverse inequality,

\[
\frac{kT}{\hbar} \gg \frac{c}{a},
\]  

(4)

the Casimir force is completely defined by the temperature and is described by the
formula

\[
f(T) = -1.2 \frac{kT}{8\pi a^3}
\]  

(5)

with the exponential accuracy. At \( T = 300 \) K, the parameter \( \hbar c/kT \) is about \( 30 \) \( \mu m \).

So, the Casimir force between the bulk metal plates displays a weak temperature
dependence (Eq. (3)) in the range of the realistic separations \( a \) about \( 0.1 \) – \( 1 \) \( \mu m \).

The temperature effects in the Casimir force could be in the forefront if the
interacting objects are thin metal films. Indeed, formula (1) is obtained under an
assumption that the thickness \( d \) of the plates is the greatest parameter with the
dimension of the length. As was shown in Refs. [15, 16], the asymptotical formula (1)
appears to be invalid for the thin metal plates if the inequality

\[
\omega_p \sqrt{\frac{d}{a}} = \omega_c \ll \omega_p, \frac{c}{a}
\]  

(6)

is fulfilled. Here \( \omega_p \) is the plasma frequency, \( \omega_c \) is the characteristic frequency of the
fluctuating electromagnetic field. In this case the collective properties of the electron
subsystem of the metal are important in forming the Casimir force. Specifically, the
evaluation

\[
f_0 \propto -\frac{\hbar \omega_c^2}{\nu + \omega_c} \frac{1}{a^3} \quad (T = 0),
\]  

(7)

is valid if the metal film of a thickness \( d \) with a weak reflecting power interacts with
the bulk metal (\( \nu \) is the frequency of the electron bulk collisions). The decrease of the
absolute value of the Casimir force \( f(T = 0) \) allows one to emphasize the temperature
dependence \( f(T) \) even in the range of the realistic separations \( a \sim 0.1 \) – \( 1 \) \( \mu m \). The
study of this dependence is a subject of the present paper.

2. Statement of the problem. The basic equations

The general formula for the force of Casimir interaction between dielectric slabs
with arbitrary dielectric constants \( \varepsilon \) was originally derived by E. Lifshitz [2] (see,
also, Refs. [3, 4, 7, 17, 18]). The Casimir force is presented in this formula as a
functional defined on the set of functions \( \varepsilon(\omega_n) \) of a discrete variable \( \omega_n = 2\pi nkT \n = 0, 1, 2, \ldots \). We use Lifshitz’s formula for the system comprising a bulk conductor
and a thin metal film of thickness \( d \) separated by a distance \( a \). The system of
coordinates is chosen so that the \( x \)-axis is perpendicular to the plane of interacting
plates. The conductivity of our system as the function of the coordinate \( x \) is

\[
\sigma(x) = \sigma[\theta(-x + a + d) - \theta(-x + a)] + \sigma_\infty \theta(x) =
\begin{cases} 
\sigma, & x \in S_{II}, \\
0, & x \in S_I, x \in S_{III}, \\
\sigma_\infty, & x \in S_{IV}, 
\end{cases}
\]  

(8)
where $\theta(x)$ is the Heaviside function; $S_I = (-\infty, -a - d)$ and $S_{II} = (-a - d, -a)$ are vacuum domains; $S_{II} = (-a - d, -a)$ and $S_{IV} = (0, \infty)$ are regions occupied by the metal film and the bulk conductor, respectively. We do not make specific supposition about the conductivity $\sigma_\infty$ of the bulk conductor because its value does not affect the final result. As for the conductivity of the metal film, we take it in the $\tau$-approximation with the following frequency dependence:

$$\sigma(\omega) = \frac{\omega^2}{\nu - i\omega}. \quad (9)$$

Lifshitz’s formula for the Casimir force can be expressed in terms of the conductivity (8) of our system,

$$F[\sigma] = -kT \sum_{n=0}^{\infty} \int d\vec{r} \frac{\delta \sigma^{(M)}(x|\omega_n)}{\delta a} \Gamma_{ii}(\vec{r}, \vec{r}|\omega_n), \quad (10)$$

where Matsubara’s conductivity $\sigma^{(M)}(\omega_n)$ is related to the frequency dispersion of the metal conductivity $\sigma(\omega)$ by the relation,

$$\sigma^{(M)}(\omega_n) = \sigma(i\omega_n); \quad (11)$$

$\Gamma^{(M)}$ is the temperature Green function of the electromagnetic field; the prime on the sum symbol indicates that the term with $n = 0$ is taken with half the weight. By using the analytical properties of the Green functions and the Abel-Plan formula for summing up the series, Eq. (10) can be rewritten in the integral form (see Appendix A),

$$F[\sigma] = -\frac{1}{2\pi} \int d\vec{r} \left\{ \int_0^{\infty} d\xi \frac{\delta \sigma(x|\xi)}{\delta a} \Gamma_{ii}(\vec{r}, \vec{r}|\xi) \right\} + 2 \int_0^{\infty} d\omega \Im \left\{ \frac{\delta \sigma(x|\omega)}{\delta a} \Gamma_{ii}(\vec{r}, \vec{r}|\omega) \right\} \left( e^{\hbar\omega/kT} - 1 \right)^{-1}. \quad (12)$$

The first term in Eq. (12) describes the Casimir force at zeroth temperature and is obtained from Eq. (10) through the simple change of the summation by integrating over the imaginary frequency. The second term provides the temperature-dependent contribution to the Casimir force which is suppressed by the small exponential factor $\exp(-\hbar\omega/kT)$ at $T \to 0$. Contrary to the low-temperature case, this term can be governing in the force (12) at sufficiently high temperatures.

In order to simplify the general formula (10), we introduce the transverse spatial Fourier transformation

$$\Gamma^{(M)}_{ik}(\vec{r}, \vec{r}'|\omega_n) = \int \frac{dq}{(2\pi)^2} \exp[iq(\vec{r} - \vec{r}')] \Gamma^{(M)}_{ik}(x, x'|q^2, \omega_n).$$

At infinitesimal displacement $\delta a$ of the bulk conductor, the conductivity $\sigma(x)$ changes by

$$\delta \sigma(x) = -\sigma_\infty \delta(x) \delta a.$$

Therefore, formula (10) for the Casimir force can be rewritten in the final form,

$$f = \frac{F}{A} = kT \int_0^{\infty} \frac{dq^2}{4\pi} \sum_{n=0}^{\infty} \sigma_\infty \Gamma^{(M)}_{ii}(x, x'|q^2, \omega_n) \left|_{x\to 0, x'\to 0} \right. \quad (13)$$
where \( A \) is the area of slabs. We interpret the limiting process in Eq. (13) as one in which \( x \) and \( x' \) tend to the interface from opposite sides, \( x \to -0 \) and \( x' \to +0 \). The formula (13) defines the force acting on the unit area of the bulk conductor from the metal film. The positive force corresponds to the repulsion of bodies and the negative one to the attraction.

In terms of "transverse electric" and "transverse magnetic" Green's functions \( g^e \) and \( g^m \), we have

\[
\lim_{x,x' \to 0} \Gamma_{ii}^{(M)}(x, x') = \lim_{x,x' \to 0} \left[ \omega_n g^e(x, x') - \omega_n^{-1} \left( \partial_x \frac{1}{\epsilon_\infty} \partial_{x'} + \frac{q^2}{\epsilon_\infty} \right) g^m(x, x') \right],
\]

where \( g^e \) and \( g^m \) are defined by

\[
[-\partial_x^2 + q^2 + \omega_n^2 \epsilon(x|\omega_n)]g^e(x, x') = \delta(x - x')
\]

and

\[
\left[ -\partial_x \frac{1}{\epsilon(x|\omega_n)} \partial_x + \frac{q^2}{\epsilon(x|\omega_n)} + \omega_n^2 \right] g^m(x, x') = \delta(x - x').
\]

Here

\[
\epsilon(x|\omega_n) = 1 + \frac{\sigma(x|\omega_n)}{\omega_n},
\]

is the effective dielectric permittivity of a metal taken at the imaginary frequency.

Thus, to analyze the temperature dependence of the Casimir force we should solve the set of Eqs. (14), (15) and substitute the obtained function \( \Gamma_{ii}^{(M)}(x = 0, x' = 0|q^2, \omega_n) \) into Eq. (13).

### 3. Temperature dependence of the Casimir force

While solving the set of Eqs. (14) and (15), we are interested in Green's functions with the argument \( x' \) within the region \( S_{IV} \) occupied by the bulk conductor. For \( x' \in S_{IV} \), the general solutions of Eqs. (14) and (15) have the following form:

\[
g^{(e,m)}_I = A e^{kx - \kappa x'}, \quad g^{(e,m)}_{II} = (B_1 e^{\kappa x} + B_2 e^{-\kappa x}) e^{-\kappa x'}; \\
g^{(e,m)}_{III} = (C_1 e^{kx} + C_2 e^{-kx}) e^{-\kappa x'}; \\
g^{IV} = (2\kappa)\epsilon^{-1} \left( e^{-\kappa |x-x'|} + r e^{-\kappa(x+x')} \right); \\
g^{IV}_m = \epsilon\kappa\epsilon^{-1} \left( e^{-\kappa |x-x'|} + r e^{-\kappa(x+x')} \right),
\]

where

\[
k = \sqrt{q^2 + \omega_n^2}, \quad \kappa = \sqrt{q^2 + \epsilon\omega_n^2}, \quad \kappa = \sqrt{q^2 + \epsilon\omega_n^2}.
\]

Determining constants \( A, B_{1,2}, C_{1,2} \), and \( r \) from the boundary conditions to Eqs. (14) and (15), we obtain for the difference

\[
\Gamma_{ii}(a) - \Gamma_{ii}(a \to \infty) \equiv \text{reg} \Gamma_{ii}(a) \quad (17)
\]

the expression

\[
\text{reg} \Gamma_{ii}(a) = -k\kappa^{-1} \frac{2\epsilon k d \exp(-2ka)}{2 + \epsilon k d[1 - \exp(-2ka)]} + \cdots. \quad (18)
\]
This formula is derived under inequalities (19) and
\[ d \ll \delta_0 = c/\omega_p. \]  
(Dots in Eq. (11) denote terms of higher order of the small parameter \( d/\delta_0 \). The procedure (17) of the regularization of the Green function allows us to avoid the "surface" divergence in the Casimir force. The divergent term does not depend on the separation \( a \) between the plates and represents an addition to the renormalized Casimir force. According to Eqs. (13) and (18), the expression for this force is
\[ f = -kT \int_0^\infty \frac{dq^2 2 \pi}{2} \sum_{n=0}^{\infty} \frac{ek^2 d \exp(-2ka)}{2 + ekd[1 - \exp(-2ka)]}. \]  
(20)

For Casimir’s interaction of sufficiently thin films, the characteristic frequencies of the fluctuating electromagnetic field turn out to be much less than the parameter \( c/a \),
\[ \omega_c \ll c/a. \]  
(21)

In this case, one can neglect the relativistic retarding effect and pass to the limit \( c \to \infty \). This allows us to assume \( k = q \) in Eq. (21) for the characteristic frequencies and to approximate the Casimir force as
\[ f = -\frac{B}{4\pi a^3} \int_0^\infty dx x^3 I(x) e^{-x}, \quad \beta = \frac{1}{kT}. \]  
(22)

where \( x = 2qa \) is the new variable of integration, symbol \( I(x) \) denotes the sum of the series,
\[ I(x) = \sum_{n=0}^{\infty} \frac{1}{n(n + C) + BF(x)}, \]  
with the parameters
\[ B = \frac{\omega_p^2 \beta^2 d}{(4\pi)^2 a}, \quad C = \frac{\beta \nu}{2\pi} \]
and the function \( F(x) = x(1 - e^{-x}) \).

In the case (21), the temperature-dependent part of the Casimir force can be calculated by means of Eq. (22) without using summation formula (12) for the Casimir force. However, the analysis of the spectral integrals in (12) is useful in order to define the characteristic frequencies giving the main contribution to the Casimir force \( f \). Let us recall that the contribution \( f_0 \) of the vacuum fluctuating electromagnetic field to the Casimir force is defined by the spectral density of energy taken at the imaginary frequency whereas the contribution \( \Delta f(T) \) of the thermal radiation of the system is related to the imaginary part of the spectral density at the real frequencies. The force \( f_0 \) connected to the vacuum fluctuations is evaluated by Eq. (5). It disappears with the decrease of the film thickness, \( d \to 0 \).

Now consider the temperature-dependent part of the Casimir force. According to Eqs. (12) and (22), we have
\[ \Delta f(T) = -\frac{B}{4\pi a^3} \int_0^\infty dx x^3 e^{-x} \int_0^\infty d\tau (e^{2\pi \tau} - 1)^{-1} \frac{\tau}{(\tau^2 - BF(x))^2 + C^2 \tau^2}. \]  
(24)
where \( \tau = 2\pi\beta\omega \). At \( \rightarrow 0 \), this expression can be approximated as

\[
\Delta f(T) = \frac{B}{4a^3} \int_0^\infty dxx^3e^{-x} \int_0^\infty d\tau(e^{2\pi\tau} - 1)^{-1}\delta[\tau^2 - BF(x)]
\]

\[
= -\frac{\omega_c\beta}{32\pi a^3} \int_0^\infty dxx^3e^{-x}F^{-\frac{1}{2}}(x) \left(e^{\frac{\beta\omega_c}{F(x)}} - 1\right)^{-1}.
\]

(25)

The corresponding characteristic frequency \( \omega_c = \omega_p \sqrt{d/a} \) of the Casimir interaction is less than the parameter \( kT/\hbar \) (\( \beta\omega_c \ll 1 \)) for sufficiently small thicknesses \( d \). In this case

\[
\Delta f(T) = -1.2 \frac{kT}{8\pi a^3} - f_0 + \cdots,
\]

(26)

where the symbol \( \cdots \) denote terms of higher order of the smallness. Thus,

\[
f(T) = f_0 + \Delta f(T) = -1.2 \frac{kT}{8\pi a^3} + \cdots.
\]

(27)

Asymptotics (27) can be shown to be valid at \( B, C \ll 1 \) as well as for \( B, 1 \ll C \). In terms of the characteristic frequencies these inequalities give

\[
\omega_c \ll \max\left(\frac{kT}{\hbar}, \sqrt{\nu \frac{kT}{\hbar}}\right).
\]

(28)

If the inverse inequality is fulfilled we get evaluation (7).

The surprising things are that formula (27) coincides with Eq. (5) for the Casimir force of the bulk metals and that \( f(T) \) does not disappear even at \( d \rightarrow 0 \). The difference between the cases of the bulk materials and the thin films consists only in the inequality defining the high-temperature regime. This is the specific feature of the Casimir attraction of the metals. Contrary to the dielectrics, a significant reduction of the characteristic frequencies of the thermal fluctuations in a metal film occurs if the film thickness decreases. In its turn, this leads to the growth of the effective dielectric permittivity (16) of the conductor and, as it follows from Eq. (18), to a disappearance of the dependence of temperature Green’s function \( \text{reg}\Gamma \) on the sample thickness. It is precisely this fact that results eventually in the unexpected insensitivity of the Casimir force to the thickness \( d \) in the high-temperature regime (28).

Using the asymptotics (7) and (27), we can obtain the following evaluating formula for the Casimir force:

\[
f \propto -\left(\frac{kT}{\hbar} + \frac{\hbar\omega_c^2}{\nu + \omega_c}\right) \frac{1}{a^3}.
\]

(29)

It is necessary to keep in mind that Eq. (29) is obtained under conditions (6) and (21) for the frequency \( \omega_c \) and for sufficiently thin films with \( d \) satisfying the inequality (19).

4. Discussion

The theoretical description of the temperature dependence of the Casimir force between a bulk conductor and a thin metal film is given in the present paper. In
the general case, the Casimir force can be presented as a sum (12) of temperature-dependent and a vacuum contributions of fluctuating electromagnetic field. We have obtained the surprising result for the situation of reasonably thin films (or reasonably high temperatures) Eq. (28). The Casimir force Eq. (27) has proved to be independent of the sample thickness in the main approximation. This fact is characteristic precisely for the metals because the Casimir force for dielectric films with a constant value of \( \epsilon \) vanishes at \( d \to 0 \).

Mathematically, the mentioned above \( d \)-independence of the Casimir attraction of the metal film in the regime (28) is connected to the proportionality of the characteristic frequency \( \omega_c \) of the thermal fluctuations to the thickness of the film and to the significant reduction of \( \omega_c \) with the decrease of \( d \). The appearance of the characteristic low-frequency regime in the Casimir attraction of the sufficiently thin metal films is physically caused by the strong classical long wavelength fluctuations of the conduction current and the plasmic shielding of the electromagnetic modes. These peculiarities add to the list of the characteristic features \([15, 16]\) of the Casimir effect for metals.

In conclusion, let us emphasize the entropy origin of the Casimir force (27). As it follows from (27), the free energy of the Casimir interaction has a form

\[
\mathcal{F} = -1.2 \frac{kT}{16\pi a^2} A. \tag{30}
\]

Hence, the entropy of interaction is

\[
S = 1.2 \frac{A}{16\pi a^2}. \tag{31}
\]

Contrary to the free energy and entropy, the energy \( E \) of interaction calculated by the formula,

\[
E = \frac{\partial}{\partial \beta}(\beta \mathcal{F}), \tag{32}
\]

vanishes at \( d \to 0 \).

Appendix A. Summation formula for the Casimir force

We transform the sum Eq. (10) over Matsubar’s frequencies \( \omega_n \) into the ”spectral” integrals using the Abel-Plan formula for summing up a series,

\[
\lim_{n \to \infty} \left\{ \sum_{k=1}^{n} F(k) - \int_{\theta}^{\theta + \infty} dx F(x) \right\} = \int_{\theta}^{\theta + i\infty} dz F(z) (e^{2i\pi z} - 1)^{-1} + \int_{\theta}^{\theta - i\infty} dz F(z) (e^{-2i\pi z} - 1)^{-1}, \tag{A1}
\]

where the function \( F(z) \) is regular in the half-plane \( \text{Re} z > 0 \) and satisfy the inequality

\[ |F(x + iy)| < f(x)e^{a|y|} \quad (a < 2\pi); \]

the function \( f(x) \) is bounded at \( x \to \infty; 0 < \theta < 1 \). The applicability of the Abel-Plan formula (A1) to the expression for Casimir’s force \([11]\) is ensured by the analytical
properties of the conductivity $\sigma^{(M)}$ and of the Green function $\Gamma^{(M)}$ in the upper half-plane of the complex frequency $\omega$. Then, we make the limiting transition $\theta \to 0^+$ in Eq. (A1) ($0^+$ is an infinitesimal positive parameter). The final transformations are carried out with regard to the formulae

$$\Gamma^{(M)}(\omega_n) = \Gamma(i|\omega_n|), \quad \Gamma(\omega) = \Gamma^{(M)} \left( \frac{\omega}{i} + 0^+ \right)$$

(A2)

where $\Gamma$ is the retarding Green function of the electromagnetic field, and the symmetry relations

$$\sigma^*(-\omega) = \sigma(\omega), \quad \Gamma^*(-\omega) = \Gamma(\omega).$$

(A3)

As a result, we get Eq. (12) for the Casimir force expressed in terms of the spectral integrals.

References

[1] Casimir H B G 1948 *Konink. Nederl. Acad. Wetens., Proc. Sec. Sci.* 51 793
[2] Lifshitz E M 1955 *Zh. Eksp. Teor. Fiz.* 29 94 (Engl. Transl. 1956 *Sov. Phys.–JETP* 2 73)
[3] Schwinger J, DeRaad L L and Milton K A 1978 *Ann. Phys., NY* 115 1
[4] Milton K A 1980 *Ann. Phys., NY* 127 49
[5] Elizalde E, Bordag M and Kirsten K 1998 *J. Phys. A: Math. Gen.* 31 1743
[6] Brevik I, Nesterenko V V and Pirozhenko I G 1998 *J. Phys. A: Math. Gen.* 31 8661
[7] Schwinger J 1992 *Proc. Natl. Acad. Sci. USA* 89 4091
[8] Milton K A and Ng Y J 1997 *Phys. Rev. E* 55 4209; 1998 *Phys. Rev. E* 57 5504
[9] Barton G 1999 *J. Phys. A: Math. Gen.* 32 525
[10] Sparnaay M J 1958 *Physica* 24 751
[11] Onofrio R 1995 *Phys. Lett. A* 198 367
[12] Lamoreaux S K 1997 *Phys. Rev. Lett.* 78 5
[13] Mohideen U and Roy A 1998 *Phys. Rev. Lett.* 81 4549
[14] Mostepanenko V M and Trunov N N 1997 *The Casimir Effect and its Applications* (London: Clarendon) p 231
[15] Dubrava V N and Yampol’skii V A 1999 *Fiz. Nizk. Temp.* 25 1304 (Engl. Transl. 1999 *Sov. J. Low. Temp. Phys.* 25 979)
[16] Dubrava V N, Yampol’skii V A and Lyubimov O I 1999 *Radiofizika i Electronika* 4 70 (Engl. Transl. *Telecommunications and Radio Engineering*, to be published)
[17] Lifshitz E M and Pitaevskii L P 1960 *Statistical Physics* Part 2 (Oxford: Pergamon) p 463
[18] Abrikosov A A, Gorkov L P and Dzyaloshinski I E 1975 *Methods of Quantum Field Theory in Statistical Physics* (NY: Dover Publ.) p 429
[19] Dubrava V N and Yampol’skii V A 2000 *Dop. Natz. Akad. Nauk Ukr.* in press
[20] Nesterenko V N, Yampol’skii V A and Lyubimov O I 1997 *Proceedings of 3-rd International conference on physical phenomena in solids* (Kharkov: Kharkov state university press) p 47