Central extensions of groups of sections

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December 17, 2009

Abstract

If $K$ is a Lie group and $q: P \to M$ is a principal $K$-bundle over the compact manifold $M$, then any invariant symmetric $V$-valued bi-linear form on the Lie algebra $\mathfrak{k}$ of $K$ defines a Lie algebra extension of the gauge algebra by a space of bundle-valued 1-forms modulo exact 1-forms. In the present paper we analyze the integrability of this extension to a Lie group extension for non-connected, possibly infinite-dimensional Lie groups $K$. If $K$ has finitely many connected components, we give a complete characterization of the integrable extensions. Our results on gauge groups are obtained by specialization of more general results on extensions of Lie groups of smooth sections of Lie group bundles. In this more general context we provide sufficient conditions for integrability in terms of data related only to the group $K$.

Keywords: gauge group; gauge algebra; central extension; Lie group extension; integrable Lie algebra; Lie group bundle; Lie algebra bundle

Introduction

Affine Kac–Moody groups and their Lie algebras play an interesting role in various fields of mathematics and mathematical physics, such as string theory and conformal field theory (cf. [PS86] for the analytic theory of loop groups, [Ka90] for the algebraic theory of Kac–Moody Lie algebras and [Sch97] for connections to conformal field theory). For further connections to mathematical physics we refer to the monograph [Mi89] which discusses various occurrences of Lie algebras of smooth maps in physical theories (see also [Mu88], [DDS95]).
From a geometric perspective, **affine Kac–Moody Lie groups** can be obtained from gauge groups \( \text{Gau}(P) \) of principal bundles \( P \) over the circle \( \mathbb{S}^1 \) whose fiber group is a simple compact Lie group \( K \) by constructing a central extension and forming a semidirect product with a circle group corresponding to rigid rotations of the circle. Here the untwisted case corresponds to trivial bundles, where \( \text{Gau}(P) \cong C^\infty(\mathbb{S}^1, K) \) is a loop group, and the twisted case corresponds to bundles which can be trivialized by a 2- or 3-fold covering of \( \mathbb{S}^1 \).

In the present paper we address central extensions of gauge groups \( \text{Gau}(P) \) of more general bundles over a compact smooth manifold \( M \), where the structure group \( K \) may be an infinite-dimensional locally exponential Lie group. In particular, Banach–Lie groups and groups of smooth maps on compact manifolds are permitted. Since the gauge group \( \text{Gau}(P) \) is isomorphic to the group of smooth sections of the associated group bundle, defined by the conjugation action of \( K \) on itself, it is natural to address central extensions of gauge groups and their Lie algebras in the more general context of groups of sections of bundles of Lie groups, resp., Lie algebras.

In the following, \( K \) always denotes a locally trivial Lie group bundle whose typical fiber \( K \) is a locally exponential Lie group with Lie algebra \( \mathfrak{k} = L(K) \). Since we work with infinite-dimensional Lie algebras, we have to face the difficulty that, in general, the group \( \text{Aut}(\mathfrak{k}) \) does not carry a natural Lie group structure\(^1\). Therefore it is natural to consider only Lie algebra bundles which are associated to some principal \( H \)-bundle \( P \) with respect to a smooth action \( \rho_\mathfrak{k} : H \to \text{Aut}(\mathfrak{k}) \) of a Lie group \( H \) on \( \mathfrak{k} \), i.e., for which the map \( (h, x) \mapsto \rho_\mathfrak{k}(h)(x) \) is smooth.

Let \( \mathfrak{K} \) be such a Lie algebra bundle. Then the smooth compact open topology turns the space \( \Gamma\mathfrak{K} \) of its smooth sections into a locally convex topological Lie algebra. To construct 2-cocycles on this algebra, we start with a continuous invariant symmetric bilinear map

\[ \kappa : \mathfrak{k} \times \mathfrak{k} \to V \]

with values in a locally convex \( H \)-module \( V \) on which the identity component \( H_0 \) acts trivially. The corresponding vector bundle \( \nabla \) associated to \( P \) is flat, so that we have a natural exterior derivative \( \partial_\mathfrak{V} \) on \( \nabla \)-valued differential forms. If \( V \) is finite-dimensional or \( H \) acts on \( V \) as a finite group, then

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\(^1\)For Banach–Lie algebras, the group \( \text{Aut}(\mathfrak{k}) \) carries a natural Banach–Lie group structure with Lie algebra \( \text{der}(\mathfrak{k}) \), but if \( \mathfrak{k} \) is not Banach, this need not be the case (cf. [Mai39]).
\(d(\Gamma V)\) is a closed subspace of \(\Omega^1(M, V)\), so that the quotient \(\Omega^1(M, V) := \Omega^1(M, V)/d(\Gamma V)\) inherits a natural Hausdorff topology (see the introduction to Section 1).

We are interested in the cocycles on the Lie algebra \(\Gamma k\) with values in the space \(\Omega^1(M, V)\), given by
\[
\omega_\kappa^\nabla(f, g) := [\kappa(f, d\nabla g)].
\]

Here \(d\nabla\) is the covariant exterior differential on \(\Omega^\bullet(M, \mathfrak{k})\) induced by a principal connection \(\nabla\) on \(P\). For the special case of gauge algebras of principal bundles with connected compact structure group \(K\), cocycles of this form have also been discussed briefly in \([LMNS98]\). Clearly, (1) generalizes the well-known cocycles for Lie algebras of smooth maps, obtained from invariant bilinear forms and leading to universal central extensions of \(\Gamma k\) if \(\mathfrak{k}\) is trivial and \(\mathfrak{g}\) is semisimple (cf. \([NW08\text{a}], [KL82]\)). Since we are presently far from a complete understanding of the variety of all central extensions of \(\Gamma k\) or corresponding groups, it seems natural to study this class of cocycles first. For other classes of cocycles, which are easier to handle, and their integrability we refer to \([Ne09, \text{Sect. 4}]\) and \([Vi08]\).

It seems quite likely that if \(\mathfrak{g}\) is finite-dimensional semisimple, \(\mathfrak{k} = \text{Ad}(P)\) is the gauge bundle of a principal \(K\)-bundle and \(\kappa\) is universal, then the central extension of \(\text{gau}(P) \cong \Gamma k\) by \(\Omega^1(M, V)\) defined by \(\omega_\kappa^\nabla\) is universal. The analogous result for multiloop algebras has recently been obtained by E. Neher (\([Neh07, \text{Thm. 2.13}]\), cf. also \([PPS07, \text{p.147}]\)), so that one may be optimistic, at least if \(M\) is a torus.

Actually it is this class of examples that motivates the more complicated setting, where the group \(H\) acts non-trivially on \(V\). Already for twisted loop groups of real simple Lie algebras \(\mathfrak{g}\), one is lead to non-connected structure groups and the universal target space \(V(\mathfrak{g})\) is a non-trivial module for \(\text{Aut}(\mathfrak{g})\), on which \(\text{Aut}(\mathfrak{g})_0\) acts trivially.

The main goal of the present paper is to understand the integrability of the Lie algebra extension \(\hat{\Gamma k}\) of \(\Gamma k\) defined by the cocycle \(\omega := \omega_\kappa := \omega_\kappa^\nabla\) to a Lie group extension of the identity component of the Lie group \(\Gamma K\) (cf. Appendix A for the Lie group structure on this group). According to the general machinery for integrating central Lie algebra extensions described in \([Ne02a, \text{Thm. 7.9}]\), \(\hat{\Gamma k}\) integrates to a Lie group extension of the identity component \((\Gamma K)_0\) if and only if the image \(\Pi_\omega\) of the period homomorphism
\[
\text{per}_\omega : \pi_2(\Gamma K) \to \Omega^1(M, V)
\]
obtained by integration of the left invariant 2-form on \( \Gamma K \) defined by \( \omega \) is discrete and the adjoint action of \( \Gamma K \) on the central extension \( \Gamma \hat{K} \) integrates to an action of the corresponding connected Lie group \( (\Gamma K)_0 \) (cf. Appendix C for more details on these two conditions). Therefore our main task consists in verifying these two conditions, resp., in finding verifiable necessary and sufficient conditions for these conditions to be satisfied.

To obtain information on the period group \( \Pi_\omega \), it is natural to compose the cocycle with pullback maps \( \gamma^* : \Omega^1(M, V) \rightarrow \Omega^1(S^1, \gamma^* V) \), defined by smooth loops \( \gamma : S^1 \rightarrow M \). To make this strategy work, we need quite detailed information on the special case \( M = S^1 \), for which \( \Gamma K \) is a twisted loop group defined by some automorphism \( \varphi \in \text{Aut}(K) \):

\[
\Gamma K \cong C^\infty(\mathbb{R}, K)_{\varphi} := \{ f \in C^\infty(\mathbb{R}, K) : (\forall t \in \mathbb{R}) f(t + 1) = \varphi^{-1}(f(t)) \}.
\]

The structure of the paper is as follows. In Section 1 we introduce the cocycles \( \omega, \nabla \) and discuss the dependence of their cohomology class on the connection \( \nabla \). In particular, we show that they lift to \( \Omega^1(M, V) \)-valued cocycles if \( \kappa \) is exact in the sense that the 3-cocycle \( C(\kappa)(x, y, z) := \kappa([x, y], z) \) on \( \mathfrak{k} \) is a coboundary (cf. [Ne09], [NW08a]). In the end of this section we introduce an interesting class of bundles which are quite different from adjoint bundles and illustrate many of the phenomena and difficulties we encounter in this paper.

In Section 2 we analyze the situation for the special case \( M = S^1 \), where \( \Gamma K \) is a twisted loop group. It is a key observation that in this case the period map \( \text{per}_\omega \) is closely related to the period map of the closed biinvariant 3-form on \( K \), determined by the Lie algebra 3-cocycle \( C(\kappa) \). To establish this relation, we need the connecting maps in the long exact homotopy sequence of the fibration defined by the evaluation map \( \text{ev}_K^1 : C^\infty(\mathbb{R}, K)_{\varphi} \rightarrow K, f \mapsto f(0) \). Luckily, these connecting maps are given explicitly in terms of the twist \( \varphi \) and thus can be determined in concrete examples. Having established the relation between \( \text{per}_\omega \) and \( \text{per}_{C(\kappa)} \), we use detailed knowledge on \( \text{per}_{C(\kappa)} \) to derive conditions for the discreteness of the image \( \Pi_\omega \) of \( \text{per}_\omega \). In particular, we describe examples in which \( \Pi_\omega \) is not discrete. For the case where \( K \) is finite-dimensional, our results provide complete information, based on a detailed analysis of \( \text{per}_{C(\kappa_\text{ua})} \) for the universal invariant form \( \kappa_\text{ua} \) in Appendix B.

In Section 3 we turn to the integrability problem for a general compact manifold \( M \). Our strategy is to compose with pullback homomorphisms \( \gamma^* : \Gamma K \rightarrow \Gamma(\gamma^* K) \), where \( \gamma : S^1 \rightarrow M \) is a smooth loop, and to determine under which conditions the period homomorphism of the corresponding twisted
loop group $\Gamma(\gamma^* \mathcal{K})$ only depends on the homotopy class of $\gamma$. If this condition is not satisfied, then our examples show that the period groups cannot be controlled in a reasonable way. Fortunately, the latter condition is equivalent to the following requirement on the curvature $R(\theta)$ of the principal connection 1-form $\theta$ corresponding to $\nabla$ and the action $\mathcal{L}(\rho_k)$: For each derivation $D \in \text{im}(\mathcal{L}(\rho_k) \circ R(\theta))$, the periods of the 2-cocycle $\eta_D(x, y) := \kappa(x, Dy)$ have to vanish. This condition is formulated completely in terms of $K$ and it is always satisfied if $K$ is finite-dimensional because $\pi_2(K)$ vanishes in this case. If the curvature requirement is fulfilled, then $\Pi_\omega$ is contained in $H^1_{dR}(M, \mathcal{V})$, so that we can use integration maps $H^1_{dR}(M, \mathcal{V}) \to V$ to reduce the discreteness problem for $\Pi_\omega$ to bundles over $S^1$.

The second part of Section 3 treats the lifting problem for the important special case of gauge bundles $\mathfrak{r} = \text{Ad}(P)$ and $\Gamma \mathcal{K} = \text{Gau}(P)$. In this case we even show that the action of the full automorphism group $\text{Aut}(P)$ on $\Gamma \mathfrak{r} = \text{gau}(P)$ lifts to an action on the central extension $\hat{\text{gau}}(P)$, defined by $\omega$. We also give an integrability criterion for this action to central extensions of the identity component $\text{Gau}(P)_0$. Summarizing, we obtain for gauge bundles the following theorem:

**Theorem 0.1** If $\pi_0(K)$ is finite and $\text{dim } K < \infty$, then the following are equivalent:

1. $\omega_\kappa$ integrates for each principal $K$-bundle $P$ over a compact manifold $M$ to a Lie group extension of $\text{Gau}(P)_0$.

2. $\omega_\kappa$ integrates for the trivial $K$-bundle $P = S^1 \times K$ over $M = S^1$ to a Lie group extension of $C^\infty(S^1, K)_0$.

3. The image of $\text{per}_{\omega_\kappa}: \pi_3(K) \to V$ is discrete.

These conditions are satisfied if $\kappa$ is the universal invariant symmetric bilinear form with values in $V(\mathfrak{k})$.

In order to increase the readability of the paper, we present some background material in appendices. This comprises the Lie group structure on groups of sections of Lie group bundles, a discussion of the universal invariant form for finite-dimensional Lie algebras, the main results on integrating Lie algebra extensions to Lie group extensions and some curvature issues for principal bundles, needed in Section 3.
Notation and basic concepts

A Lie group \( G \) is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps (cf. \[Mil84\], \[Ne06\] and \[GN09\]). We write \( 1 \in G \) for the identity element and \( \lambda_g(x) = gx \), resp., \( \rho_g(x) = xg \) for the left, resp., right multiplication on \( G \). Then each \( x \in T_1(G) \) corresponds to a unique left invariant vector field \( x^l \) with \( x^l(g) := T_1(\lambda_g)x, g \in G \). The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on \( T_1(G) \) a continuous Lie bracket which is uniquely determined by \([x,y]^l = [x^l,y^l] \) for \( x,y \in T_1(G) \).

We write \( L(G) = g \) for the so obtained locally convex Lie algebra and note that for morphisms \( \varphi: G \to H \) of Lie groups we obtain with \( L(\varphi) := T_1(\varphi) \) a functor from the category of Lie groups to the category of locally convex Lie algebras. We write \( q_G: \tilde{G}_0 \to G_0 \) for the universal covering map of the identity component \( G_0 \) of \( G \) and identify the discrete central subgroup \( \ker q_G \) of \( \tilde{G}_0 \) with \( \pi_1(G) \cong \pi_1(G_0) \).

For a smooth map \( f: M \to G \) we define the (left) logarithmic derivative in \( \Omega^1(M, g) \) by \( \delta(f)v_m := f(m)^{-1} \cdot T_m(f)v_m \), where \( \cdot \) refers to the two-sided action of \( G \) on its tangent bundle \( TG \).

In the following, we always write \( I = [0,1] \) for the unit interval in \( \mathbb{R} \). A Lie group \( G \) is called regular if for each \( \xi \in C^\infty(I, g) \), the initial value problem

\[
\gamma(0) = 1, \quad \gamma'(t) = \gamma(t) \cdot \xi(t) = T_1(\lambda_{\gamma(t)})\xi(t)
\]

has a solution \( \gamma_{\xi} \in C^\infty(I, G) \), and the evolution map

\[
evol_G: C^\infty(I, g) \to G, \quad \xi \mapsto \gamma_{\xi}(1)
\]

is smooth (cf. \[Mil84\]). For a locally convex space \( E \), the regularity of the Lie group \( (E, +) \) is equivalent to the Mackey completeness of \( E \), i.e., to the existence of integrals of smooth curves \( \gamma: I \to E \). We also recall that for each regular Lie group \( G \), its Lie algebra \( g \) is Mackey complete and that all Banach–Lie groups are regular \((GN09)\).

A smooth map \( \exp_G: g \to G \) is said to be an exponential function if for each \( x \in g \), the curve \( \gamma_x(t) := \exp_G(tx) \) is a homomorphism \( \mathbb{R} \to G \) with \( \gamma_x'(0) = x \). Presently, all known Lie groups modelled on complete locally convex spaces possess an exponential function. For Banach–Lie groups, its existence follows from the theory of ordinary differential equations in Banach
spaces. A Lie group $G$ is called *locally exponential*, if it has an exponential function mapping an open 0-neighborhood in $\mathfrak{g}$ diffeomorphically onto an open neighborhood of 1 in $G$. For more details, we refer to Milnor’s lecture notes [Mil84], the survey [Ne06], and the forthcoming monograph [GN09].

If $q: E \to B$ is a smooth fiber bundle, then we write $\Gamma E$ for its space of smooth sections.

If $\mathfrak{g}$ is a topological Lie algebra and $V$ a topological $\mathfrak{g}$-module, we write $(C^\bullet(\mathfrak{g},V),d_\mathfrak{g})$ for the corresponding Lie algebra complex of continuous $V$-valued cochains ([ChE48]).

1 Central extensions of section algebras of Lie algebra bundles

We now turn to the details and introduce our notation. We write $P(M,H,q_P)$ for an principal $H$-bundle over the smooth manifold $M$ with structure group $H$ and bundle projection $q_P: P \to M$. To any such bundle $P$ and to any smooth action $\rho_\xi : H \to \text{Aut}(\mathfrak{k})$, we associate the Lie algebra bundle $\mathfrak{k}$, which is the set $(P \times \mathfrak{k})/H$ of $H$-orbits in $P \times \mathfrak{k}$ for the action $h.(p,x) = (p,h^{-1}\rho_\xi(h)x)$. We write $[(p,x)] := H.(p,x)$ for the elements of $\mathfrak{k}$ and $q_{\mathfrak{k}} : \mathfrak{k} \to M, [(p,x)] \mapsto q_P(p)$ for the bundle projection.

It is no loss of generality to assume that the bundle $P$ is connected. Indeed, if $P_1 \subseteq P$ is a connected component, then $q(P_1) = M$ and $H_1 := \{h \in H : P_1.h = P_1\}$ is an open subgroup, so that $P_1$ is a principal $H_1$-bundle over $M$. Further, the canonical map $P_1 \times \mathfrak{k} \to \mathfrak{k}, (p,x) \mapsto [(p,x)]$ is surjective and induces a diffeomorphism $(P_1 \times \mathfrak{k})/H_1 \cong \mathfrak{k}$. In the following we shall always assume that $P$ is connected. This implies that the connecting map $\delta_1 : \pi_1(M) \to \pi_0(H)$ of the long exact homotopy sequence of $P$ is surjective.

Further, let $V$ be a Fréchet $H$-module on which the identity component $H_0$ acts trivially and $\rho_V : H \to \text{GL}(V)$ be the corresponding representation, so that $H_0 \subseteq \ker \rho_V$ and $\rho_V$ factors through a representation $\overline{\rho}_V : \pi_0(H) \to \text{GL}(V)$ of the discrete group $\pi_0(H)$. Accordingly, the associated vector bundle $\mathbb{V} := (P \times V)/H$ is flat. It is also associated via $\overline{\rho}_V$ to the squeezed bundle $P_0 := P/H_0$, which is a principal $\pi_0(H)$-bundle over $M$. Due to the flatness of $\mathbb{V}$, we have a natural exterior derivative $d$ on the space $\Omega^\bullet(\mathbb{V}) \cong \Omega^\bullet(P_0,V)^{\pi_0(H)}$ of $\mathbb{V}$-valued differential forms and we define $\overline{\Omega}^\bullet(M,\mathbb{V}) := \Omega^1(M,\mathbb{V})/d(\Gamma \mathbb{V})$ and write its elements as $[\alpha]$,
\( \alpha \in \Omega^1(M, V) \). If \( V \) is finite-dimensional, then \( d \Gamma V \) is a closed subspace of the Fréchet space \( \Omega^1(M, V) \), so that the quotient inherits a natural Hausdorff locally convex topology. In fact, in Lemma 3.8 below we construct a continuous map \( \Omega^1(M, V) \to Z^1(\pi_1(M), V) \) (group cocycles with respect to the representation \( \rho_M := \rho_V \circ \delta_1 \)) and show that \( d \Gamma V \) is the inverse image of the space \( B^1(\pi_1(M), V) \) of coboundaries which is finite-dimensional if \( V \) is so, hence closed in the Fréchet space \( \Omega^1(M, V) \). Therefore \( d \Gamma V \) is closed.

If \( \rho_V(H) = \rho_M(\pi_1(M)) \) is finite, then \( \hat{M} := M/\ker \rho_M \) is a finite covering manifold of \( M \) and \( \mathbb{D} := \pi_1(M)/\ker \rho_M \) acts on \( \hat{M} \) by deck transformations. We then have \( \Omega^*(M, V) \cong \Omega^*(\hat{M}, V) \mathbb{D}^\mathbb{P} \) and the finiteness of \( \mathbb{D} \) implies that \( \mathbb{D} \Gamma V \cong B^1_{\text{dR}}(\hat{M}, V) \mathbb{D} \), so that \( \mathbb{D} \Gamma V \) is a closed subspace. We therefore assume in the following that either \( \rho_V(H) \) is finite or that \( V \) is finite-dimensional to ensure that \( \Omega^1(M, V) \) carries a natural Fréchet space structure (cf. Remark 3.9).

Now let \( \kappa : \mathfrak{k} \times \mathfrak{k} \to V \) be an \( H \)-invariant continuous symmetric bilinear map which is also \( \mathfrak{k} \)-invariant in the sense that

\[
\kappa([x, y], z) = \kappa(x, [y, z]) \quad \text{for} \quad x, y, z \in \mathfrak{k}.
\]

The \( H \)-invariance of \( \kappa \) implies that it defines a \( C^\infty(M, \mathbb{R}) \)-bilinear map

\[
\Gamma \mathfrak{h} \times \Gamma \mathfrak{h} \to \Gamma V, \quad (f, g) \mapsto \kappa(f, g), \quad \kappa(f, g)(p) := \kappa(f(p), g(p)).
\]

which defines a \( \Gamma V \)-valued invariant symmetric bilinear form on the Lie algebra \( \Gamma \mathfrak{h} \). To associate a Lie algebra 2-cocycle to this data, we choose a principal connection \( \nabla \) on the principal bundle \( P \) and also write \( \nabla \) for the associated connections on the vector bundles \( \mathfrak{h} \) and \( V \). Since \( H \) acts by automorphisms on \( \mathfrak{k} \), its Lie algebra \( \mathfrak{h} \) acts by derivations, which implies that the connection \( \nabla \) on \( \mathfrak{h} \) is a Lie connection, i.e.,

\[
\nabla_X[f, g] = \left[ \nabla_X f, g \right] + \left[ f, \nabla_X g \right] \quad \text{for} \quad X \in \mathcal{V}(M), f, g \in \Gamma \mathfrak{h}
\]

(cf. [Ma05] for more details on Lie connections on Lie algebra bundles). The \( H \)-invariance of \( \kappa \) and the fact that its Lie algebra \( \mathfrak{h} \) acts trivially on \( V \) imply that

\[
d(\kappa(f, g)(X)) = \kappa(\nabla_X f, g) + \kappa(f, \nabla_X g) \quad \text{for} \quad X \in \mathcal{V}(M), f, g \in \Gamma \mathfrak{h}.
\]

In the following we write \( d \nabla f \) for the \( \mathfrak{k} \)-valued 1-form defined for \( f \in \Gamma \mathfrak{k} \) by \( (d \nabla f)(X) := \nabla_X f \) for \( X \in \mathcal{V}(M) \). In the realization of \( \Gamma \mathfrak{k} \) as \( C^\infty(P, \mathfrak{k})^H \), we
have for $X \in \mathcal{V}(P)$:

$$(d^V f)(X) = df(X) + \theta(X)f,$$

where $\theta \in \Omega^1(P, \mathfrak{h})$ is the principal connection 1-form corresponding to the connection $\nabla$.

**Proposition 1.1** The prescription

$$\omega(f, g) := \omega^\nabla(f, g) := [\kappa(f, d^\nabla g)]$$

defines a Lie algebra cocycle on $\Gamma \mathfrak{K}$ with values in the trivial $\Gamma \mathfrak{K}$-module $\overline{\Omega}^1(M, \mathbb{V})$. If $\nabla' = \nabla + \beta$, $\beta \in \Omega^1(M, \text{Ad}(P))$, is another principal connection for which there exists some $\gamma \in \Omega^1(M, \mathfrak{K})$ with

$$L(\rho_k) \circ \beta(X) = \text{ad}(\gamma(X)) \in \text{End}(\Gamma \mathfrak{K}) \quad \text{for} \quad X \in \mathcal{V}(M),$$

then the corresponding cocycle $\omega'$ differs from $\omega$ by a coboundary.

**Proof.** From (3) we get $d(\kappa(f, g)) = \kappa(d^\nabla f, g) + \kappa(f, d^\nabla g)$, so that $\omega$ is alternating. In view of (2) and (3), we further have

$$d(\kappa([f, g], h)) = \kappa(d^\nabla [f, g], h) + \kappa([f, g], d^\nabla h) = \kappa([d^\nabla f, g], h) + \kappa([f, d^\nabla g], h) + \kappa([f, g], d^\nabla h) = \kappa([g, h], d^\nabla f) + \kappa([h, f], d^\nabla g) + \kappa([f, g], d^\nabla h),$$

showing that $\omega$ is a 2-cocycle.

If $\nabla$ is replaced by $\nabla' = \nabla + \beta$ and (4) is satisfied, then

$$\kappa(f, d^\nabla' g) = \kappa(f, d^\nabla g) + \kappa(f, [\gamma, g]) = \kappa(f, d^\nabla g) + \kappa(\gamma, [g, f])$$

implies that $\omega' - \omega = d_{\Gamma \mathfrak{K}}([\kappa(\gamma, \cdot)])$, where $\kappa(\gamma, \cdot)$ is an $\Omega^1(M, \mathbb{V})$-valued linear map on $\Gamma \mathfrak{K}$. 

**Remark 1.2** Since the space $\overline{\Omega}^1(M, \mathbb{V})$ is a quotient of the space $\Omega^1(M, \mathbb{V})$ of $\mathbb{V}$-valued 1-forms, it is natural to ask for the existence of $\Omega^1(M, \mathbb{V})$-valued cocycles on $\Gamma \mathfrak{K}$ lifting $\omega^\nabla$. To see when such cocycles exist, we consider the continuous bilinear map

$$\tilde{\omega}(f, g) := \kappa(f, d^\nabla g) - \kappa(g, d^\nabla f),$$
which is an alternating lift of \(2\omega_\kappa^\nabla\). Its Lie algebra differential is

\[
(d_{\Gamma_K}\omega)(f, g, h) = -\sum_{\text{cyc.}} \kappa([f, g], d\nabla h) - \kappa(h, d\nabla [f, g])) = \sum_{\text{cyc.}} \kappa([f, g], d\nabla h)
\]

\[= d(\kappa([f, g], h)),
\]

as we see with similar calculations as in the proof Proposition 1.1.

For the trivial \(\mathfrak{t}\)-module \(V\), we write \(\text{Sym}^2(\mathfrak{t}, V)^k\) for the space of \(V\)-valued symmetric invariant bilinear forms, and recall the Cartan map

\[
C: \text{Sym}^2(\mathfrak{t}, V)^k \to Z^3(\mathfrak{t}, V), \quad C(\kappa)(x, y, z) := \kappa([x, y], z).
\]

We say that \(\kappa\) is exact if \(C(\kappa)\) is a coboundary. If \(C(\kappa) = d_\mathfrak{t}\eta\) for some \(\eta \in C^2(\mathfrak{t}, V)\), then

\[
d(\kappa([f, g], h)) = d((d_\mathfrak{t}\eta)(f, g, h)) = -\sum_{\text{cyc.}} d(\eta([f, g], h)),
\]

so that

\[
\omega_{\kappa, \eta}(f, g) := \kappa(f, d\nabla g) - \kappa(g, d\nabla f) - d(\eta(f, g))
\]

is an \(\Omega^1(M, V)\)-valued 2-cocycle on \(\Gamma\mathfrak{r}\) lifting \(2\omega_\kappa^\nabla\) (cf. [Ne09, Sect. 2]).

Remark 1.3 If \(\beta \in \Omega^1(M, \text{Ad}(P))\) is a bundle-valued 1-form, then we obtain for each \(X \in \mathcal{V}(M)\) a derivation \(\beta_\kappa : \mathcal{V}(M) \to \mathcal{V}(M)\) of \(\Gamma\mathfrak{r}\) and this derivation preserves the symmetric bilinear \(\Gamma\mathcal{V}\)-valued map \((f, g) \mapsto \kappa(f, g)\), so that

\[
\eta_\beta(f, g) := \kappa(f, \beta g)
\]

defines an \(\Omega^1(M, \mathcal{V})\)-valued 2-cocycle on \(\Gamma\mathfrak{r}\). For \(\nabla' = \nabla + \beta\), we now have

\[
\omega' - \omega = q_\Omega \circ \eta_\beta,
\]

where \(q_\Omega : \Omega^1(M, \mathcal{V}) \to \Omega^1(M, \mathcal{V})\) denotes the quotient map. This argument shows that the dependence of the cohomology class \([\omega_\kappa^\nabla]\) on \(\nabla\) is described by elements of \(H^2(\Gamma\mathfrak{r}, \Omega^1(M, \mathcal{V}))\).

We may also consider \(\eta_\beta\) as a bundle map \(\mathfrak{r} \times \mathfrak{r} \to \text{Hom}(TM, \mathcal{V})\), which implies that \(\eta_\beta\) can also be used to define a central extension of Lie algebroids (cf. [Ma05]).
Example 1.4 Of particular importance is the special case where $K := H$ is a Lie group with Lie algebra $\mathfrak{k}$ and $\rho_K: K \to \text{Aut}(\mathfrak{k})$ is the adjoint action of $K$. Then $\mathfrak{g} = \text{Ad}(P)$ is the adjoint bundle of the principal $K$-bundle $P$ over $M$, $\Gamma\mathfrak{g} \cong \text{gau}(P)$, and we have the Lie algebra extension

$$0 \to \text{gau}(P) \cong \Gamma\mathfrak{g} \hookrightarrow \text{aut}(P) = \mathcal{V}(P)^K \to \mathcal{V}(M) \to 0.$$ 

Furthermore, the space $\Gamma\mathcal{V}$ of smooth sections of the flat vector bundle $\mathcal{V}$ carries a natural $\mathcal{V}(M)$-module structure which we may pull back to an $\text{aut}(P)$-module structure for which the ideal $\text{gau}(P)$ acts trivially.

Since $L(\rho_k) = \text{ad}$, Proposition 1.1 implies that in this situation the cohomology class

$$[\omega] = [\omega^\nabla] \in H^2(\text{gau}(P), \Omega^1(M, \mathcal{V}))$$

does not depend on the choice of the principal connection in $P$.

Remark 1.5 If $\kappa_u: \mathfrak{t} \times \mathfrak{t} \to V(\mathfrak{k})$ is the universal continuous invariant symmetric bilinear form on $\mathfrak{k}$ (cf. [MN03] and Appendix B below) and $K$ is a Lie group with Lie algebra $\mathfrak{k}$, then the universality of $\kappa_u$ implies that $K$ acts naturally on $V(\mathfrak{k})$, and since $\mathfrak{k}$ acts trivially on $V(\mathfrak{k})$, the identity component $K_0$ acts trivially ([GN09]; [Ne06, Rem. II.3.7]). This implies that the universal form $\kappa_u$ satisfies all assumptions required for our construction. For a detailed analysis of $\kappa_u$ and the period map of the corresponding closed 3-form on $K$, we refer to Appendix E.

The aim of this paper is to determine under which circumstances the Lie algebra extension defined by the cocycle $\omega$ from Proposition 1.1 integrates to an extension of Lie groups. The natural setting for this question is the case, where the action $\rho_k$ is induced by a smooth action $\rho_K: H \to \text{Aut}(K)$, i.e., $K$ is a Lie group with Lie algebra $\mathfrak{k}$ and we have $\rho_K = \text{L}(\rho_K)$. If $K$ is locally exponential, then the group of sections $\Gamma K$ of the adjoint Lie group bundle $\mathcal{K} := (P \times K)/H$ has a natural Lie group structure with $\text{L}(\Gamma K) \cong \Gamma\mathfrak{g}$ (cf. Appendix A). We therefore want to integrate our Lie algebra extension to the identity component $(\Gamma K)_0$ of this group.

From [Ne02a] (cf. Appendix C) we know that the Lie algebra cocycle $\omega$ defines a period map

$$\text{per}_\omega: \pi_2(\Gamma K) \to \overline{\Omega}^1(M, \mathcal{V}),$$

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and a necessary condition for the existence of a Lie group extension integrating \( \omega \) is that the image \( \Pi_\omega \) of the period map, the period group, is discrete ([Ne02a], Theorem VII.9). To obtain information on this period group, our strategy is first to take a closer look at the case \( M = S^1 \) and then to use this case to treat more general situations. The much simpler case of trivial bundles has been treated in a similar fashion in [MN03].

A class of examples

**Example 1.6** Let \( \pi : Q \to M \) be a compact locally trivial smooth bundle with (compact) fiber \( N \). Then \( Q \) is associated to the principal \( H := \text{Diff}(N) \)-bundle \( P \) with fiber \( P_m := \text{Diff}(N,Q_m) \) with the canonical \( H \) action by composition. For any locally convex Lie group \( G \), we have a canonical \( H \)-action on \( K := C^\infty(N,G) \) by \((\varphi, \gamma) \mapsto \gamma \circ \varphi^{-1}\) whose smoothness follows from the smoothness of the action of \( \text{Diff}(N) \) on \( N \) and the smoothness of the evaluation map of \( K \) (cf. [NW08], Lemma A.2) and we thus obtain an associated Lie group bundle \( K := P \times_H C^\infty(N,G) \). The sections of the Lie group bundle \( \Gamma K \) may be identified with the set \( C^\infty(P,K)^H \) of \( H \)-equivariant smooth functions \( P \to K \).

**Proposition 1.7** If \( G \) is locally exponential, then the map

\[
\begin{align*}
s : C^\infty(Q,G) &\to \Gamma K, \\
 s_f(p) &= f \circ p, \quad p \in P_m = \text{Diff}(N,P_m)
\end{align*}
\]

is an isomorphism of Lie groups.

**Proof.** In local coordinates one easily checks that \( s \) actually is an isomorphism of abstract groups compatible with the smooth compact open topology, so that it actually is an isomorphism of topological groups.

If, in addition, \( G \) is assumed to be locally exponential, then \( \Gamma K \) and \( C^\infty(Q,G) \) inherit this property (Theorem A.1), and now the general theory of locally exponential Lie groups ([Ne06, Thm. IV.1.18], [GN09]) implies that the topological isomorphism between these groups actually is a diffeomorphism, hence an isomorphism of Lie groups.

**Remark 1.8** If the bundle \( \pi : Q \to M \) in Example 1.6 is an principal \( H \)-bundle for some compact group \( H \), then the structure group can be reduced from the infinite-dimensional Lie group \( \text{Diff}(H) \) to the compact subgroup \( H \),
because the transition functions of the bundle charts have values in the group of left multiplications of \( H \). We then obtain an isomorphism of Lie groups

\[
s: C^\infty(P, G) \to \Gamma K \cong C^\infty(P, K)^H, \quad s_f(p)(h) := f(p.h).
\]

We thus associate to each principal \( H \)-bundle \( Q \) a Lie group bundle \( K \) with fiber \( K = C^\infty(H, G) \). This construction is particularly interesting for \( H = \mathbb{T} \). Then \( P \) is a circle bundle and \( K = C^\infty(\mathbb{T}, G) \) is the loop group of \( G \).

## 2 Lie group bundles over the circle

Throughout this section we consider the special case \( M = \mathbb{S}^1 \) and assume that the Lie group \( K \) is regular. Then every \( K \)-Lie group bundle over \( \mathbb{S}^1 \) is flat, hence determined by its holonomy \( \varphi \in \text{Aut}(K) \). Conversely, every automorphism \( \varphi \in \text{Aut}(K) \) leads to a Lie group bundle \( K_\varphi = \mathbb{R} \times_{\varphi} K \) over \( \mathbb{S}^1 \) with holonomy \( \varphi \). Indeed, \( K_\varphi \) is the Lie group bundle associated to the universal covering \( q_\mathbb{S}^1 : \mathbb{R} \to \mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z} \) by the action of \( H := \mathbb{Z} \cong \pi_1(\mathbb{S}^1) \) on \( K \) defined by \( \varphi \). The smooth sections of \( K_\varphi \) correspond to twisted loops:

\[
\Gamma K_\varphi \cong C^\infty(\mathbb{R}, K)_{\varphi} := \{ f \in C^\infty(\mathbb{R}, K) : (\forall t \in \mathbb{R}) \ f(t + 1) = \varphi^{-1}(f(t)) \}.
\]

From now on we identify \( \Gamma K_\varphi \) with \( C^\infty(\mathbb{R}, K)_{\varphi} \) and write

\[
\text{ev}^K_0 : C^\infty(\mathbb{R}, K)_{\varphi} \to K, \quad f \mapsto f(0),
\]

for the evaluation homomorphism in 0. On the Lie algebra level, we similarly get with \( \varphi_t = L(\varphi) \) a Lie algebra bundle \( \mathfrak{k}_{\varphi_t} \) with

\[
\Gamma \mathfrak{k}_{\varphi_t} \cong C^\infty(\mathbb{R}, \mathfrak{k})_{\varphi_t} := \{ f \in C^\infty(\mathbb{R}, \mathfrak{k}) : (\forall t \in \mathbb{R}) \ f(t + 1) = \varphi^{-1}_t(f(t)) \}
\]

and \( L(\text{ev}^K_0) = \text{ev}^\mathfrak{k}_0 \).

### 2.1 On the topology of twisted loop groups

In Section 3 we shall reduce the calculation of the period groups for \( \omega^\infty_\kappa \) essentially to the case \( M = \mathbb{S}^1 \), so that we need detailed information on the second homotopy group of twisted loop groups. A central tool is a simple description of the connecting maps in the long exact homotopy sequence defined by the evaluation map \( \text{ev}^K_0 \) for a twisted loop group \( C^\infty(\mathbb{R}, K)_{\varphi} \), which is based on the fact that the passage from smooth to continuous twisted loops is a weak homotopy equivalence.
Lemma 2.1 The image of the evaluation homomorphism \( \text{ev}_0^K \) is the open subgroup
\[
K^{[\varphi]} := \{ k \in K : \varphi(k) k^{-1} \in K_0 \} = \{ k \in K : k K_0 \in \pi_0(K) \varphi \}.
\]

Proof. For \( f \in C^\infty(\mathbb{R}, K)_\varphi \), we have \( f(1) = \varphi^{-1}(f(0)) \in f(0) K_0 \), so that the image of \( \text{ev}_0^K \) is contained in \( K^{[\varphi]} \). If, conversely, \( k \in K^{[\varphi]} \), then there exists a smooth curve \( \alpha : [0, 1] \to K \) with \( \alpha(0) = k \), \( \alpha(1) = \varphi^{-1}(k) \) such that \( \alpha \) is locally constant near 0 and 1. Then \( f(n + t) := \varphi^{-n}(\alpha(t)) \) for \( t \in [0, 1], n \in \mathbb{Z} \), defines a section of \( K_\varphi \) with \( f(0) = k \).

Lemma 2.2 The Lie group homomorphism \( \text{ev}_0^K \) has smooth local sections, hence defines a Lie group extension of \( K^{[\varphi]} \) by \( C^\infty(\mathbb{R}, K)_\varphi := \ker \text{ev}_0^K \).

Proof. That \( \text{ev}_0^K \) has smooth local sections can be seen as follows. Let \( (\psi, U) \) be a chart of \( K \), centered in \( 1 \) for which \( \psi(U) \) is convex. Let further \( h : [0, 1] \to \mathbb{R} \) be a smooth function with \( h(0) = 0 \) and \( h(1) = 1 \) which is constant in \( [0, \varepsilon] \) and \( [1 - \varepsilon, 1] \). Let \( W \subseteq U \cap \varphi(U) \) be a \( 1 \)-neighborhood in \( K \). For \( k \in W \) we then consider the smooth curve
\[
\gamma_k : [0, 1] \to K, \quad \gamma_k(t) := \psi^{-1}\left( (1 - h(t))\psi(k) + h(t)\psi(\varphi^{-1}(k)) \right).
\]
Then \( \gamma_k \) is constant near 0 and 1, \( \gamma_k(0) = k \), and \( \gamma_k(1) = \varphi^{-1}(k) \). We extend \( \gamma_k \) smoothly to \( \mathbb{R} \) in such a way that it defines an element on \( C^\infty(\mathbb{R}, K)_\varphi \). Then the smoothness of the map \( W \to C^\infty(\mathbb{R}, K)_\varphi, k \mapsto \gamma_k \) follows from the smoothness of the corresponding map \( W \times \mathbb{R} \to K, (k, t) \mapsto \gamma_k(t) \), which in turn follows from its smoothness on each subset \( W \times \mathbb{R}^\varepsilon, n + 1 + \varepsilon \) [cf. \cite{GN09}].

Proposition 2.3 The inclusion \( C^\infty(\mathbb{R}, K)_\varphi \hookrightarrow C(\mathbb{R}, K)_\varphi \) of the smooth twisted loop group into the continuous twisted loop group is a weak homotopy equivalence.

Proof. Let \( H := K \ltimes_{\varphi} \mathbb{Z} \), where the action of \( \mathbb{Z} \) on \( K \) is defined by \( n.k := \varphi^n(k) \) for \( n \in \mathbb{Z} \) and \( k \in K \). We write \( P_\varphi \) for the principal \( H \)-bundle over \( S^1 \) with holonomy \( (1, 1) \in H \). Then
\[
\text{Gau}(P_\varphi) \cong \{ f \in C^\infty(\mathbb{R}, H) : (\forall t \in \mathbb{R}) \ f(t + 1) = (1, 1)f(t)(1, -1) \},
\]
and this group contains the twisted loop group \( C^\infty(\mathbb{R}, K)_\varphi \) as an open subgroup. According to Prop. 1.20 in \cite{W07a}, the inclusion of the smooth gauge group \( \text{Gau}(P_\varphi) \) into the group \( \text{Gau}(P_\varphi)_c \) of continuous gauge transformations is a weak homotopy equivalence, and this property is inherited by the open subgroups of \( K \)-valued twisted loops.
Corollary 2.4 The inclusion \( C^\infty(\mathbb{R}, K)_\varphi \hookrightarrow C_*(\mathbb{R}, K)_\varphi \) of the smooth based twisted loop group into the continuous based twisted loop group is a weak homotopy equivalence.

Proof. In view of Lemma 2.2, the evaluation \( ev^K_0 \) defines a smoothly locally trivial fiber bundle \( C^\infty(\mathbb{R}, K)_\varphi \to K^{[\varphi]} \), and a similar (even simpler) argument shows that the same holds for the continuous twisted loop group. Since \( id_{K^{[\varphi]}} \) and the inclusion \( C^\infty(\mathbb{R}, K)_\varphi \hookrightarrow C(\mathbb{R}, K)_\varphi \) are weak homotopy equivalences, the 5-Lemma implies that the same holds for the inclusion \( C^\infty(\mathbb{R}, K)_\varphi \hookrightarrow C_*(\mathbb{R}, K)_\varphi \) of the fibers (cf. Prop. A.8 in [Ne02c]). ■

We have already determined the image of \( ev^K_0 \), showing that the long exact homotopy sequence ends with

\[
\ldots \to \pi_1(K) \xrightarrow{\delta_1} \pi_0(C^\infty(\mathbb{R}, K)_\varphi) \to \pi_0(C^\infty(\mathbb{R}, K)_\varphi) \to \pi_0(K)^\varphi \to 1.
\]

Now we turn to the connecting maps. For that we note that for continuous sections, the map

\[
\Phi : \Omega K := C_*(S^1, K) = C_*(\mathbb{R}/\mathbb{Z}, K) \to C_*(\mathbb{R}, K)_\varphi,
\]

\[
\Phi(f)(t) := \varphi^{-n}(f([t])) \quad \text{for} \quad t \in [n, n+1], n \in \mathbb{Z}
\]

defines an isomorphism of Lie groups.

Proposition 2.5 For \( j \geq 1 \), the connecting maps

\[
\delta_j : \pi_j(K) \to \pi_{j-1}(C_*(\mathbb{R}, K)_\varphi) \cong \pi_{j-1}(\Omega K) \cong \pi_j(K),
\]

are group homomorphisms given by \( \delta_j([f]) = [f] - [\varphi^{-1} \circ f] \).

Proof. For the adjoint action of \( K \) on itself, this formula is the Samelson product with \([k] \in \pi_0(K) \cong \text{Bun}(S^1, K)\), and the proof in [Wo07b, Thm. 2.4] implies the present assertion when applied to \( K \rtimes_\varphi \mathbb{Z} \) instead of \( K \). ■

Remark 2.6 (a) We have a short exact sequence

\[
1 \to \pi_1(K)_\varphi := \pi_1(K)/\text{im}(\pi_1(\varphi) - \text{id}) \hookrightarrow \pi_0(C^\infty(\mathbb{R}, K)_\varphi) \to \pi_0(K)^\varphi \to 1.
\]

If \( K \) is connected, we obtain in particular \( \pi_0(C^\infty(\mathbb{R}, K)_\varphi) \cong \pi_1(K)_\varphi \).
(b) For the evaluation of period maps, important information is contained in the short exact sequence

$$1 \rightarrow \pi_3(K) \varphi \hookrightarrow \pi_2(C^\infty(\mathbb{R}, K), \varphi) \rightarrow \pi_2(K) \varphi \rightarrow 1.$$ 

If \( \pi_2(K) \) vanishes, it follows that the corresponding map

$$\pi_3(K) \cong \pi_2(\Omega K) \rightarrow \pi_2(C^\infty(\mathbb{R}, K), \varphi)$$

is surjective.

**Example 2.7** We discuss some examples where \( \varphi \) acts non-trivially on \( \pi_2(K) \).

(a) A typical example of a Lie group \( K \) for which \( \pi_2(K) \) is non-trivial is the projective unitary group \( \text{PU}(\mathcal{H}) \) of an infinite-dimensional complex Hilbert space \( \mathcal{H} \) ([Ku65]). Each automorphism of this simply connected group either is induced by a unitary or an anti-unitary map. In fact, the simple connectedness of \( \text{PU}(\mathcal{H}) \) implies \( \text{Aut}(\text{PU}(\mathcal{H})) \cong \text{Aut}(\text{pu}(\mathcal{H})) \), and since the Lie algebra \( \text{u}(\mathcal{H}) \) is the universal central extension of \( \text{pu}(\mathcal{H}) \) ([Ne02b, Example III.6]), each automorphism of \( \text{pu}(\mathcal{H}) \) lifts to a unique automorphism of \( \text{u}(\mathcal{H}) \), so that

\[
\text{Aut}(\text{pu}(\mathcal{H})) \cong \text{Aut}(\text{u}(\mathcal{H})) \cong \text{Aut}_*(\text{gl}(\mathcal{H})) \cong \text{PU}(\mathcal{H}) \rtimes \mathbb{Z}/2,
\]

where the latter isomorphism follows from Prop. 3 in Section II.13 of [dH72] and \( \text{Aut}_* \) denote the group of all automorphism \( \varphi \) with \( \varphi(x^*) = \varphi(x)^* \). We conclude that

\[
\pi_0(\text{Aut}(\text{PU}(\mathcal{H}))) \cong \mathbb{Z}/2.
\]

Conjugation with an anti-unitary map induces the inversion on the center \( \text{Id}_\mathcal{H} \) of \( \text{U}(\mathcal{H}) \), and this implies that the action of \( \pi_0(\text{Aut}(\text{PU}(\mathcal{H}))) \) induces the inversion on \( \pi_2(\text{PU}(\mathcal{H})) \cong \pi_1(\text{Z}(\text{U}(\mathcal{H}))) \cong \mathbb{Z} \).

(b) Another example is the smooth loop group \( C^\infty(S^1, C) \) of a compact simple simply connected Lie group \( C \) which satisfies

\[
\pi_2(C^\infty(S^1, C)) \cong \pi_3(C) \cong \mathbb{Z}.
\]

Its automorphism group is

\[
\text{Aut}(C^\infty(S^1, C)) \cong C^\infty(S^1, \text{Aut}(C)) \rtimes \text{Diff}(S^1)
\]

(cf. [PS86] Prop. 3.4.2)) whose group of connected components is

\[
(\pi_1(\text{Aut}(C)) \rtimes \pi_0(\text{Aut}(C))) \rtimes \mathbb{Z}/2 \cong (\text{Z}(C) \rtimes \pi_0(\text{Aut}(C))) \rtimes \mathbb{Z}/2,
\]

hence finite. In particular, any orientation reversing diffeomorphism acts on \( \pi_2(C^\infty(S^1, C)) \cong \mathbb{Z} \) by inversion.
2.2 Period maps for twisted loop groups

Let \( \kappa : \mathfrak{k} \times \mathfrak{k} \to V \) be a \( \mathfrak{k} \)-invariant symmetric bilinear form and \( \varphi \in \text{GL}(V) \) (defining a \( \mathbb{Z} \)-module structure on \( V \)) with

\[
\varphi_V(\kappa(x, y)) = \kappa(\varphi x, \varphi y) \quad \text{for} \quad x, y \in \mathfrak{k}.
\]

We write \( V := V_{\varphi_V} \) for the vector bundle over \( S^1 \) with fiber \( V \) and holonomy \( \varphi_V \). Then the cocycle corresponding to the canonical connection \( \nabla \) on \( K_{\varphi} \) defined by \( d\nabla f = f'(t)dt \) is given by

\[
\omega_{\varphi}(f, g) := [\kappa(f, d\nabla g)] = \left[ \int_0^1 \kappa(f, g') \, dt \right] \in V_{\varphi_V} \cong \Omega^1(S^1, V),
\]

where the last isomorphism comes from the following lemma:

**Lemma 2.8** Let \( V \) be a vector bundle over \( S^1 \) with fiber \( V \) and holonomy \( \varphi_V \in \text{GL}(V) \). Identifying \( S^1 \) with \( \mathbb{R}/\mathbb{Z} \), the map

\[
\Omega^1(S^1, V) \cong H^1_{dR}(S^1, V) \to V_{\varphi_V} = \text{coker}(\varphi_V - \text{id}_V), \quad [f \cdot dt] \mapsto \left[ \int_0^1 f(t) \, dt \right]
\]

for \( f \in \Gamma V \cong C^\infty(\mathbb{R}, V)_{\varphi_V} \) is a linear isomorphism.

**Proof.** Write \( \frac{d}{dt} \) for the vector field generating the rigid rotations of \( S^1 \). Then \( \mathcal{V}(S^1) = C^\infty(S^1, \mathbb{R}) \frac{d}{dt} \) implies that evaluation in \( \frac{d}{dt} \) leads to an isomorphism

\[
\Omega^1(S^1, V) = \text{Hom}_{C^\infty(S^1, \mathbb{R})}(\mathcal{V}(S^1), \Gamma V) \cong \Gamma V \cong C^\infty(\mathbb{R}, V)_{\varphi_V},
\]

and under this identification, the canonical covariant derivative is given by

\[
d\nabla : \Gamma V \to \Omega^1(S^1, V), \quad d\nabla f = f'.
\]

We first observe that if \( f = g' \) for some \( g \in \Gamma V \), then

\[
\int_0^1 f(t) \, dt = g(1) - g(0) = \varphi_V^{-1}(g(0)) - g(0) \in \text{im}(\varphi_V^{-1} - \text{id}_V) = \text{im}(\varphi_V - \text{id}_V),
\]

and the map is well-defined. If, conversely, \( f : \mathbb{R} \to V \) is a smooth function, representing an element of \( \Omega^1(S^1, V) \), for which \( \int_0^1 f(t) \, dt = \varphi_V^{-1}(v) - v \) for some \( v \in V \), then

\[
g(t) := v + \int_0^t f(\tau) \, d\tau
\]
satisfies \( g' = f \) and

\[
g(t + 1) = v + \int_0^1 f(\tau) \, d\tau + \int_1^{t+1} f(\tau) \, d\tau = \varphi_V^{-1}(v) + \varphi_V^{-1} \left( \int_0^t f(\tau) \, d\tau \right) = \varphi_V^{-1}(g(t)).
\]

This proves injectivity. To obtain surjectivity, choose some \( \gamma : [0, 1] \to [0, 1] \) which is smooth, constant on a neighborhood of the boundary and satisfies \( \gamma(0) = 0, \gamma(1) = 1 \) and \( \int_0^1 \gamma(t) \, dt = \frac{1}{2} \). Then, for each \( v \in V \), the mapping \( t \mapsto (1 - \gamma(t)) \cdot v + \gamma(t) \varphi_V^{-1}(v) \) can be extended to an element \( f_v \) of \( \Gamma V \) with \( \left[ \int_0^1 f_v(t) \, dt \right] = [v] \).

**Remark 2.9** If \( V \) is infinite-dimensional, we further assume that the image of the operator \( \varphi_V - \text{id}_V \) is closed, so that \( \Omega^1(S^1, V_{\varphi_V}) \cong V_{\varphi_V} \) is Hausdorff.

To study the period map (cf. Appendix C) \(^2\)

\[ \text{per}_{\omega_{\varphi}} : \pi_2(C^\infty(\mathbb{R}, K)_{\varphi}) \to V_{\varphi_V}, \]

we first consider the subgroup \( C^\infty_*(\mathbb{R}, K)_{\varphi} \). Since \( C^\infty(\mathbb{R}, K)_{\varphi} \) is locally exponential (Appendix A), this is a Lie subgroup with Lie algebra \( C^\infty_*(\mathbb{R}, k)_{\varphi} := \ker \text{ev}^*_{\varphi} \). To evaluate the period map on \( \pi_2(C^\infty_*(\mathbb{R}, K)_{\varphi}) \), we note that on \( C^\infty_*(\mathbb{R}, k)_{\varphi} \),

\[
\tilde{\omega}_{\varphi}(f, g) := \int_0^1 \kappa(f, g')(t) \, dt \in V
\]

defines a Lie algebra cocycle. In fact, integration by parts shows that it is alternating, and with Remark 1.2 we obtain for \( f, g, h \in C^\infty_*(\mathbb{R}, k)_{\varphi} \)

\[
-(d\tilde{\omega}_{\varphi})(f, g, h) = \int_0^1 \kappa([f, g], h)' = \kappa([f, g], h)(1) - \kappa([f, g], h)(0) = 0.
\]

The following lemma reduces the period map of \( \tilde{\omega}_{\varphi} \) (cf. Appendix C) to the more accessible period map \( \text{per}_{C(\kappa)^l} : \pi_3(K) \to V \) of the closed 3-form \( C(\kappa)^l \) on \( K \) which is studied in detail in Appendix B for \( \dim K < \infty \).

\(^2\)Note that we do not have to impose any completeness condition on the quotient space \( V_{\varphi_V} \) to make sense of the period integrals because they can be calculated as \( V \)-valued integrals.
Lemma 2.10 Identifying $\pi_3(K)$ in the canonical way with the group $\pi_2(\Omega K) \cong \pi_2(C^\infty(\mathbb{R}, K)_\varphi)$, we have

$$\text{per}_{\omega_\varphi} = \frac{1}{2} \text{per}_{C(\kappa)}: \pi_3(K) \to V.$$ 

More generally, if $\sigma: S^2 \to C^\infty(\mathbb{R}, K)_\varphi$ is a smooth map and $\tilde{\sigma}: \mathbb{R} \times S^2 \to K$ defined by $\tilde{\sigma}(t, m) := \sigma(m)(t)$, then

$$\text{per}_{\omega_\varphi}(\sigma) = \frac{1}{2} \left[ \int_{[0,1] \times S^2} \tilde{\sigma}^*C(\kappa)^l \right].$$  \hspace{1cm} (5)

Proof. It suffices to verify that the $V$-valued Lie algebra 2-cochain $\tilde{\omega}_\varphi(f, g) := \frac{1}{2} \int_0^1 \kappa(f(g'), g') - \kappa(g, f') \, dt$ satisfies

$$\int_\sigma \tilde{\omega}_\varphi = \frac{1}{2} \int_{[0,1] \times S^2} \tilde{\sigma}^*C(\kappa)^l.$$ 

Since homotopy classes may be represented by smooth maps \cite{Ne02a, Sect. A.3}, both assertions follow from that.

First we note that $\sigma$ also defines a smooth curve in $\hat{\sigma} \in C^\infty(\mathbb{R}, C^\infty(S^2, K))$ by $\hat{\sigma}(t)(m) := \sigma(m)(t)$. We then identify its logarithmic derivative $\delta(\hat{\sigma})$ with a smooth curve with values in $L(C^\infty(S^2, K)) = C^\infty(S^2, \mathfrak{k})$, so that $d\delta(\hat{\sigma})$ is a smooth curve with values in $\Omega^1(S^2, \mathfrak{k})$.

We consider $\delta\sigma \in \Omega^1(S^2, C^\infty(\mathbb{R}, \mathfrak{k})) \cong C^\infty(\mathbb{R}, \Omega^1(S^2, \mathfrak{k}))$ as a smooth curve with values in $\Omega^1(S^2, \mathfrak{k})$ in the obvious fashion. Using the fact that $\delta\tilde{\sigma} \in \Omega^1(\mathbb{R} \times S^2, \mathfrak{k})$ satisfies the Maurer–Cartan equation

$$d\delta\tilde{\sigma} + \frac{1}{2} [\delta\tilde{\sigma}, \delta\tilde{\sigma}] = 0,$$

the derivative of this curve can be calculated by evaluating it on some smooth vector field $X \in \mathcal{V}(S^2)$:

$$(\delta\sigma)'(X) = L_{\partial t}(\delta\tilde{\sigma}(X)) = (L_{\partial t} \delta\tilde{\sigma})(X) = \left( i_{\partial t} d\delta\tilde{\sigma} + d i_{\partial t} \delta\tilde{\sigma} \right)(X)$$

$$= \left( -\frac{1}{2} i_{\partial t} [\delta\tilde{\sigma}, \delta\tilde{\sigma}] + d\delta(\tilde{\sigma}) \right)(X) = -[\delta\tilde{\sigma}(\partial_t), \delta\tilde{\sigma}(X)] + d\delta(\tilde{\sigma})(X)$$

$$= -[\delta\tilde{\sigma}, \delta\sigma(X)] + d\delta(\tilde{\sigma})(X) = (d\delta(\tilde{\sigma}) + [\delta\sigma, \delta\tilde{\sigma}]) (X).$$

This proves that

$$(\delta\sigma)' = d\delta(\tilde{\sigma}) + [\delta\sigma, \delta\tilde{\sigma}] \in C^\infty(\mathbb{R}, \Omega^1(S^2, \mathfrak{k})), \hspace{1cm} (6)$$
Using the Maurer–Cartan equation for $\delta \sigma$, we further get in $C^\infty(\mathbb{R}, \Omega^2(S^1, \frak{f}))$:

\[
\delta \sigma \land \kappa (\delta \sigma)' = \delta \sigma \land \kappa d\delta \tilde{\sigma} + \delta \sigma \land \kappa [\delta \sigma, \delta \tilde{\sigma}] = \delta \sigma \land \kappa d\delta \tilde{\sigma} + [\delta \sigma, \delta \sigma] \land \kappa \delta \tilde{\sigma} \\
= -d(\delta \sigma \land \kappa \delta \tilde{\sigma}) + d(\delta \sigma) \land \kappa \delta \tilde{\sigma} + [\delta \sigma, \delta \sigma] \land \kappa \delta \tilde{\sigma} \\
= -d(\delta \sigma \land \kappa \delta \tilde{\sigma}) + \frac{1}{2}[\delta \sigma, \delta \sigma] \land \kappa \delta \tilde{\sigma} \\
= -d(\delta \sigma \land \kappa \delta \tilde{\sigma}) + [\delta \sigma, \delta \sigma] \land \kappa \delta \tilde{\sigma} \\
= -d(\delta \sigma \land \kappa \delta \tilde{\sigma}) + C(\kappa)(\delta \sigma, \delta \sigma, \delta \tilde{\sigma})
\]

and hence

\[
\int \tilde{\sigma}^t = \int \int_{S^2} \tilde{\omega}_\varphi(\delta \sigma, \delta \sigma) = \frac{1}{2} \int_0^1 \int_{S^2} \delta \sigma \land \kappa (\delta \sigma)' dt \\
= \frac{1}{2} \int_0^1 \int_{S^2} C(\kappa)(\delta \sigma, \delta \sigma, \delta \tilde{\sigma}) dt = \frac{1}{2} \int_{[0,1] \times S^2} \tilde{\sigma}^* C(\kappa)^t.
\]

For the last equality we have used that $\tilde{\sigma}^* C(\kappa)^t = C(\kappa)(\delta \sigma, \delta \sigma, \delta \tilde{\sigma}) dt$, which is most easily verified by applying both sides to triples of tangent vectors of the form $(\frac{\partial}{\partial \tau}, v, w)$ for $v, w \in T_m(S^2)$.  

The preceding lemma shows in particular that the period homomorphism $\text{per}_{\tilde{\omega}_\varphi}$ does not depend on the pair $(\varphi, \varphi_V)$.

**Example 2.11** It is instructive to take a closer look at the example $K = \text{SU}_2(\mathbb{C})$. We realize $\text{SU}_2(\mathbb{C}) \cong S^3$ as the group of unit quaternions in $\mathbb{H}$ and write $\kappa(x, y) = -\frac{1}{4} \text{tr}(\text{ad} x \text{ad} y)$ for the normalized invariant symmetric bilinear form, satisfying $\kappa(x, x) = 2\|x\|^2$ for each $x \in \text{su}_2(\mathbb{C}) = \text{span}_\mathbb{R}\{I, J, K\}$. For the basis elements $I, J, K$, we then have $\kappa([I, J], K) = 2\kappa(K, K) = 4$, so that the left invariant 3-form defined by $C(\kappa)(x, y, z) := \kappa([x, y], z)$ on $\text{SU}_2(\mathbb{C}) \cong S^3$ is $4\mu_{S^3}$, where $\mu_{S^3}$ is the volume form of $S^3$. It follows in particular, that

\[
\text{per}_{C(\kappa)^t}([\text{id}_K]) = \int_{\text{SU}_2(\mathbb{C})} C(\kappa)^t = 4\text{vol}(S^3) = 8\pi^2.
\]
On the other hand, it has been shown in [PS86] (see also the calculations in Appendix IIa to Section II in [Ne01]) that 
\[ \frac{1}{2\pi} \Pi_\omega = \Pi_\frac{1}{2\pi} \omega = 2\pi \mathbb{Z} \]
for \( \omega := \tilde{\omega}_{id} \), so that 
\[ \Pi_\omega = 4\pi^2 \mathbb{Z} . \]

In view of the preceding lemma, this is a direct consequence of (7).

Let \( q_V : V \to V_{\varphi V} \) denote the projection map. Then the cocycle \( \tilde{\omega}_\varphi \) satisfies 
\[ q_V \circ \tilde{\omega}_\varphi = L(\iota)^* \omega_\varphi , \]
where \( \iota : C^\infty(\mathbb{R}, K)_{\varphi} \to C^\infty(\mathbb{R}, K)_{\varphi} \) denotes the inclusion map. Therefore Remark C.2 yields 
\[ \text{per}_{\omega_\varphi} \circ \pi_2(\iota) = \text{per}_{L(\iota)^* \omega_\varphi} = q_V \circ \text{per}_{\tilde{\omega}_\varphi} = q_V \circ \text{per}_{C(\kappa)}. \] (8)

If \( \pi_2(K) \) vanishes, then \( \pi_2(\iota) \) is surjective (Remark 2.6) and we thus obtain:

**Theorem 2.12** If \( \pi_2(K) \) vanishes, then \( \Pi_{\omega_\varphi} \subseteq q_V(\text{im}(\text{per}_{C(\kappa)})) \).

As a consequence, we obtain for finite dimensional groups with Theorem B.11 and Cartan’s Theorem that \( \pi_2(K) \) vanishes in this case (Remark B.8).

**Corollary 2.13** If \( K \) is finite-dimensional, \( V = V(\mathfrak{t}) \) and \( \kappa = \kappa_u \) is universal, then the period group \( \Pi_{\omega_\varphi} \) is discrete.

We now present an example where the period group depends significantly on the connection \( \nabla \).

**Example 2.14** We consider the special case of Remark 1.8 where 
\[ \pi : Q = \mathbb{T}^2 \to M = \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad \pi(t, s) = s, \]
\( H = \mathbb{T} \) and \( K = C^\infty(\mathbb{T}, G) \) for a compact simple Lie group \( G \). Then \( \Gamma K \cong C^\infty(\mathbb{T}^2, G) \) and \( K \cong \mathbb{T} \times K \) is a trivial bundle.

To a positive definite invariant symmetric bilinear form \( \kappa_{\mathfrak{g}} \) on \( \mathfrak{g} \), we associate the invariant bilinear form 
\[ \kappa(f, g) := \int_0^1 \kappa_{\mathfrak{g}}(f(t), g(t)) \, dt \]
on \( \mathfrak{k} \). The group \( H = \mathbb{T} \) acts on \( K \), resp., \( \mathfrak{k} \), by composition, and the action of the Lie algebra \( \mathfrak{h} \cong \mathbb{R} \) is given by \( Df = f' \), which leaves \( \kappa \) invariant. The Lie algebra cocycle

\[
\eta_D(f,g) := \kappa(f,Dg) = \int_0^1 \kappa_g(f(t),g'(t)) \, dt,
\]
on \( \mathfrak{k} \) is universal (cf. [PS86]). In particular, the period homomorphism

\[
\text{per}_{\eta_D} = \frac{1}{2} \text{per}_{C(\kappa_g)} : \pi_2(K) \cong \pi_3(G) \cong \mathbb{Z} \to \mathbb{R}
\]
is non-trivial.

The covariant exterior derivatives on the trivial bundle \( P = S^1 \times H \) take on \( \Gamma K \cong C^\infty(T, \mathfrak{k}) \) the form

\[
d\nabla f = f' + h \cdot Df = \frac{\partial f}{\partial s} + h(s) \cdot \frac{\partial f}{\partial t},
\]
for some \( h \in C^\infty(T, \mathbb{R}) \), determined by \( \nabla \). Accordingly, the cocycle \( \omega_\nabla^\kappa \) decomposes as

\[
\omega_\nabla^\kappa(f,g) = \int_0^1 \kappa(f, \frac{\partial g}{\partial s}) \, ds + \int_0^1 h(s) \kappa(f, \frac{\partial g}{\partial t}) \, ds.
\]

To calculate the period maps for the cocycles

\[
\omega_0(f,g) = \int_0^1 \int_0^1 \kappa_g(f, \frac{\partial g}{\partial s}) \, ds \, dt, \quad \eta(f,g) = \int_0^1 \int_0^1 h(s) \kappa_g(f, \frac{\partial g}{\partial t}) \, ds \, dt,
\]
we express them in terms of the universal cocycle

\[
\omega_u(f,g) = [\kappa_g(f, \partial g)] = [\kappa_g(f, \frac{\partial g}{\partial t}) \, dt + \kappa_g(f, \frac{\partial g}{\partial s}) \, ds]
\]
of \( \Gamma \mathfrak{k} = C^\infty(T^2, \mathfrak{g}) \) with values in \( \Omega^1(T^2, \mathbb{R}) \). If \( \kappa_g \) is suitably normalized, the period group of this cocycle is \( \Pi_{\omega_u} = \mathbb{Z}[dt] + \mathbb{Z}[ds] \) ([MN03]). We further find with Remark [B.8]

\[
\pi_2(C^\infty(T^2, G)) \cong \pi_2(G) \oplus \pi_3(G)^2 \oplus \pi_4(G) \cong \mathbb{Z}^2 \oplus \pi_4(G)
\]
(cf. [MN03, Rem. I.11]), and in these terms
\[ \text{per}_{\omega_u}(m, n, u) = m[dt] + n[ds]. \]
If \( \gamma_t(s) = \gamma^s(t) = (t, s) \) describes the vertical and horizontal circles in \( \mathbb{T}^2 \),
then
\[ \omega_0(f, g) = \int_0^1 \mathcal{I}_{\gamma_t} \circ \omega_u(f, g) \, dt, \]
where \( \mathcal{I}_{\gamma_t}[\alpha] := \int_{\mathbb{S}^1} \gamma_t^* \alpha \), so that the period map is given by
\[ \text{per}_{\omega_0}(m, n, u) = \int_0^1 \mathcal{I}_{\gamma_t} \circ \text{per}_{\omega_u}(m, n, u) \, dt = \int_0^1 n \, dt = n. \]
Similarly,
\[ \eta(f, g) = \int_0^1 h(s) \mathcal{I}_{\gamma_s} \circ \omega_u(f, g) \, ds, \]
and its period map is
\[ \text{per}_{\eta}(m, n, u) = \int_0^1 h(s) \mathcal{I}_{\gamma_s} \text{per}_{\omega_u}(m, n, u) \, ds = \int_0^1 h(s) m \, ds = m \int_0^1 h(s) \, ds. \]
This implies that the period group
\[ \Pi_\omega = \mathbb{Z} + \mathbb{Z} \cdot \int_0^1 h(s) \, ds \]
is discrete if and only if the integral \( \int_0^1 h(s) \, ds \) is rational.

**Remark 2.15** We have seen in Remark 2.6 that for a twisted loop group \( L_\varphi K := C^\infty(\mathbb{R}, K)_\varphi, \varphi \in \text{Aut}(K) \), the group \( \pi_2(L_\varphi K) \) is determined by a short exact sequence
\[ 1 \to \pi_3(K)_\varphi \to \pi_2(L_\varphi K) \to \pi_2(K) \varphi \to 1. \]
Accordingly, the period group \( \Pi_{\omega_\varphi} \) can be determined in a two-step process. The restriction to \( \pi_3(K)_\varphi \) is, up to the factor \( \frac{1}{2} \), the period map
\[ \text{per}_{C(\kappa), \varphi} : \pi_3(K)_\varphi \to V_{\varphi_V}, \quad [\sigma] \mapsto [\text{per}_{C(\kappa)}(\sigma)] \]
obtained by factorization of \( \text{per}_{C(\kappa)} \). If \( \Pi_{C(\kappa), \varphi} \subseteq V_{\varphi_V} \) denotes the image of this homomorphism, then \( \text{per}_{\omega} \) factors through a homomorphism
\[ \text{per}_{\omega} : \pi_2(K)_\varphi \to V_{\varphi_V}/\Pi_{C(\kappa), \varphi} \]
whose image determines the period group \( \Pi_\omega \) as its inverse image in \( V_{\varphi_V} \).
The following example shows that both parts \( \pi_3(K_\varphi) \) and \( \pi_2(K_\varphi) \) may contribute non-trivially to \( \Pi_\omega \) and that the period group depends seriously on \( \varphi \).

**Example 2.16** (a) Let \( G \) be a simply connected simple compact Lie group and \( K := C^\infty(S^1, G) \) be its loop group.

Let 

\[
\varphi_K = (h, \psi) \in \text{Aut}(K) \cong C^\infty(S^1, \text{Aut}(G)) \rtimes \text{Diff}(S^1)
\]

(cf. [PSS6], Prop. 3.4.2). Here \( \text{Aut}(K) \) actually carries a natural Lie group structure and the automorphism \( \varphi_V \) of \( V = \mathbb{R} \) induced by \( \varphi_K \) for which the form \( \kappa(f, g) = \int_{S^1} \kappa_g(f(t), g(t)) \, dt \) is invariant is \( \pm \text{id}_V \), depending on whether \( \psi \) preserves the orientation of \( S^1 \) or not. We also note that 

\[
\pi_0(C^\infty(S^1, \text{Aut}(G))) \cong \pi_1(\text{Aut}(G)) \rtimes \pi_0(\text{Aut}(G)) \cong Z(G) \rtimes \pi_0(\text{Aut}(G))
\]

is a finite group and that the subgroup 

\[
\pi_0(C^\infty(S^1, \text{Aut}(G))) \cong \pi_1(\text{Aut}(G))
\]

acts trivially on all higher homotopy groups of \( G \)\(^3\) hence in particular on \( \pi_3(G) \cong \pi_2(K) \). Moreover, \( \text{Aut}(G) \) preserves the Cartan–Killing form \( \kappa_g \) of \( g \), hence fixes the associated closed invariant 3-form, so that de Rham’s Theorem implies that it also acts trivially on \( \pi_3(G) \).

For \( \varphi_V = -\text{id}_V \) we obtain in particular \( V_{\varphi_V} = \{0\} \), so that all periods vanish. In the latter case, the natural identification of \( \pi_2(K) \) with \( \pi_3(G) \) shows that \( \varphi_K \) acts as \(-\text{id}\) on \( \pi_2(K) \cong \pi_3(G) \cong \mathbb{Z} \), so that \( \pi_2(K)_\varphi = \{0\} \).

If \( \varphi_V = \text{id}_V \), then \( V_{\varphi_V} = V = \mathbb{R} \) and \( \psi \) is orientation preserving. Then the action of \( \varphi_K \) on \( \pi_2(K) \cong \pi_3(G) \cong \mathbb{Z} \) is trivial. The action of \( \varphi_K \) on \( \pi_3(K) \cong \pi_3(G) \oplus \pi_4(G) \) is trivial on the first factor (coming from constant functions) and \( \pi_4(G) \) is finite, so that \( \pi_3(K) \varphi \) is of rank 1. Now the same arguments as in Example 2.14 show that \( \pi_2(L_\varphi K) \) is of rank 2 and both summands contribute to \( \Pi_\omega \).

\(^3\)Here we use that if a topological group acts on a space \( M \), then the corresponding action of \( \pi_1(G) \) on \( \pi_k(M, x_0) \), \( k \geq 1 \), is always trivial. One finds the special cases where \( G \) acts on itself by the multiplication map in [Hu59], Prop. 16.10. The general case is proved similarly.
3 Corresponding Lie group extensions

We now determine in which cases the central extension $\hat{\Gamma} K$ defined by the cocycle $\omega^\nabla \kappa$ integrates to a Lie group extension. To this end we analyze its period group $\Pi_\omega := \text{im} (\text{per}_{\omega^\nabla \kappa})$ and determine whether the adjoint action of $\Gamma K$ lifts to an action on $\hat{\Gamma} K$ (cf. Appendix C and [Ne02a]). Throughout, $M$ denotes a compact connected manifold.

3.1 On the image of the period map

Throughout this section, we fix a base point $p_0 \in P$ and put $m_0 := q_P(p_0)$. We also assume that the Lie group $H$ is regular.

It is convenient to consider an intermediate situation given by a covering manifold $\hat{q}_M : \hat{M} \to M$, defined as follows. Let $\delta_1 : \pi_1(M) \to \pi_0(H)$ denote the connecting map from the long exact homotopy sequence of the principal $H$-bundle $P$ that we used to define $K$. We write $\tilde{\rho}_V := \rho_V \circ \delta_1 : \pi_1(M) \to GL(V)$ for the corresponding pullback representation of $\pi_1(M)$ on $V$ and put $\hat{M} := \tilde{M} / \ker \tilde{\rho}_V$. Then $\hat{M}$ is a covering of $M$ with $\pi_1(\hat{M}) \cong \ker \tilde{\rho}_V$, and its group of deck transformations is $D := \pi_1(M)/\pi_1(\hat{M}) \cong \tilde{\rho}_V(\pi_1(M))$.

Since $P$ is connected, the connecting homomorphism $\delta_1$ is surjective and the squeezed bundle $P/H_0$ is a covering of $M$ associated to $\delta_1$, hence equivalent to $\hat{M} \times_{\delta_1} \pi_0(H) \cong \tilde{M} / \ker \delta_1$. This implies that

$$P / \ker \rho_V \cong (P/H_0) / \ker \tilde{\rho}_V \cong \hat{M} / \ker \tilde{\rho}_V \cong \tilde{M}.$$ 

**Remark 3.1** For the open subgroup $H_V := \ker \rho_V$ of $H$, we may also consider $P$ as an principal $H_V$-bundle $\hat{q}_V : P \to \hat{M}$. If $\hat{V} := \hat{q}_M^* V$ denotes the pullback of $V$ to $\hat{M}$, it follows that $\hat{V} \cong P \times_{\rho_V|H_V} V \cong \hat{M} \times V$ is a trivial vector bundle, which leads to a natural map

$$\Omega^1(M, V) \to \Omega^1(\hat{M}, V).$$

In the following our first step to the understanding of the period group of $\omega := \omega^\nabla \kappa$ is to investigate when it is contained in the subspace $H^1_{\text{dR}}(M, V)$ of $\Omega^1(M, V)$. If this condition is not satisfied, then one may not expect any simple criteria for discreteness, as the Examples 3.20 and 2.14 show.
Definition 3.2  (a) Fix a connection $\nabla$ on the principal $H$-bundle $P$. For any smooth loop $\alpha : [0, 1] \to M$, based in $m_0 \in M$, we define its holonomy $H_{p_0}(\alpha) \in H$ as follows. Since $H$ is assumed to be regular, the curve $\alpha$ has a unique smooth horizontal lift $\hat{\alpha} : [0, 1] \to P$ starting in $p_0$ (cf. [KM97]), and since $\hat{\alpha}(1)$ and $\hat{\alpha}(0)$ are both mapped to $\alpha(0) = \alpha(1) = m_0$, there exists a unique element $H_{p_0}(\alpha) \in H$ with

$$\hat{\alpha}(1) = \hat{\alpha}(0) . H_{p_0}(\alpha).$$

Changing the base point leads to the relation

$$H_{p_0 . h}(\alpha) = h^{-1} H_{p_0}(\alpha) h,$$

so that the holonomy depends on $p_0$. Since we keep the base point $p_0$ fixed, we may also write $H(\alpha) := H_{p_0}(\alpha)$. If $\alpha : \mathbb{R} \to M$ is a 1-periodic map with $\alpha(0) = m_0$, representing a smooth loop $\mathbb{R}/\mathbb{Z} \to M$, then we put $H(\alpha) := H(\alpha|_{[0, 1]}).

(b) We identify the group $\Gamma K$ and the Lie algebra $\Gamma \mathfrak{k}$ with the corresponding spaces of $H$-equivariant maps $C^\infty(P, K)^H$, resp., $C^\infty(P, \mathfrak{k})^H$.

Any smooth 1-periodic map $\alpha : \mathbb{R} \to M$ lifts to a unique smooth horizontal curve $\hat{\alpha} : \mathbb{R} \to P$ satisfying

$$\hat{\alpha}(t + 1) = \hat{\alpha}(t) . H(\alpha)$$

for each $t \in \mathbb{R}$ (both sides define horizontal curves and coincide in $t = 0$). We put $\varphi^\alpha_K := \rho_K(H(\alpha))$ and $\varphi^\alpha_t := L(\varphi^\alpha_K)$. Then we obtain a homomorphism of Lie groups

$$\hat{\alpha}_K^* : \Gamma K \to C^\infty(\mathbb{R}, K)_{\varphi^\alpha_K}, \quad f \mapsto f \circ \hat{\alpha},$$

and of Lie algebras

$$\hat{\alpha}_t^* = L(\hat{\alpha}_K^* : \Gamma \mathfrak{k} \to C^\infty(\mathbb{R}, \mathfrak{k})_{\varphi^\alpha_t}, \quad f \mapsto f \circ \hat{\alpha}.$$
Therefore we have a well-defined integration map

\[ I_\alpha : \Omega^1(M, V) \to V_{\psi^\alpha} = \text{coker}(\varphi^\alpha - \text{id}_V), \quad [\theta] \mapsto \left[ \int_0^1 \alpha^* \theta \right]. \]

(d) Let \( \tilde{q} : P \to \tilde{M} \) be the bundle projection and \( \tilde{\alpha}_M := \tilde{q} \circ \tilde{\alpha} \). This is a piecewise smooth (continuous) lift of \( \alpha \) to the covering space \( \tilde{M} \), starting in the base point \( \tilde{m}_0 := \tilde{q}(p_0) \). Since the fibers of \( \tilde{q} \) are the orbits of \( \ker \rho_V \), the condition \( \varphi^\alpha = \text{id}_V \) is equivalent to the path \( \tilde{\alpha}_M \) being closed. If this is the case, then \( I_\alpha \) has values in \( V_{\psi^\alpha} = V \).

**Remark 3.3** Let \( \omega_{\psi^\alpha} \) denote the canonical \( V_{\psi^\alpha} \)-valued cocycle on \( C^\infty(\mathbb{R}, k) \) (cf. Subsection 2.2). For the horizontal lift \( \tilde{\alpha} : \mathbb{R} \to P \) and \( f \in \Gamma_k \cong C^\infty(P, k)^H \), we have

\[
(d^\nabla f)(\tilde{\alpha}'(t)) = df(\tilde{\alpha}'(t)) = (f \circ \tilde{\alpha})'(t).
\]

Therefore

\[
(L(\tilde{\alpha}_K)^* \omega_{\psi^\alpha})(f, g) = \left[ \int_0^1 \kappa(f \circ \tilde{\alpha}, (g \circ \tilde{\alpha})') \, dt \right] = \left[ \int_0^1 \tilde{\alpha}^* \kappa(f, \nabla g) \right] = I_\alpha([\kappa(f, \nabla g)]).
\]

We thus obtain the important relation

\[
L(\tilde{\alpha}_K)^* \omega_{\psi^\alpha} = I_\alpha \circ \omega, \quad (9)
\]

and with Remark [C.2] this leads to

\[
I_\alpha \circ \text{per}_\omega = \text{per}_{I_\alpha \circ \omega} = \text{per}_{\omega_{\psi^\alpha}} \circ \pi_2(\tilde{\alpha}_K). \quad (10)
\]

**Remark 3.4** Let \( F : [0, 1] \times S^1 \to M \) be a smooth map which is a homotopy of loops based in \( m_0 \). Then

\[ \beta : [0, 1] \to H, \quad \beta(t) := \mathcal{H}(F_0)^{-1}\mathcal{H}(F_t) \]

is a smooth curve starting in \( 1 \). If \( F \) is chosen to be independent of the first variable on a neighborhood of \( \{0, 1\} \times S^1 \), then \( \beta \) is constant on a neighborhood of \( \{0, 1\} \), so that it can be extended to a smooth map

\[ \beta : \mathbb{R} \to H \quad \text{with} \quad \beta(t + 1) = \mathcal{H}(F_0)^{-1}\beta(t)\mathcal{H}(F_1) \quad \text{for} \quad t \in \mathbb{R}. \]
For $i = 0, 1$, put $\varphi_i := \rho_K(\mathcal{H}(F_i))$. Then
\[ \Phi_\beta : C^\infty(\mathbb{R}, K)_{\varphi_1} \to C^\infty(\mathbb{R}, K)_{\varphi_0}, \quad \Phi_\beta(f)(t) = (\beta.f)(t) := \rho_K(\beta(t))(f(t)) \]
is an isomorphism of Lie groups. The corresponding isomorphism on the Lie algebra level is similarly given by $\left( L(\Phi_\beta)\xi \right)(t) = \rho_t(\beta(t)).\xi(t)$. The curve $\varphi_{V,t} := \rho_V(\mathcal{H}(F_t))$ in $\text{GL}(V)$ is constant because $H_0$ acts trivially. We may thus put $\varphi_V := \varphi_{V,t}$, and the target spaces of the cocycles
\[ \omega_{\varphi_i}(f, g) = \left[ \int_0^1 \kappa(f, g') \, dt \right] \in V_{\varphi_V} \]
on coincide. Unfortunately, $\omega_{\varphi_1}$ does not coincide with $L(\Phi_\beta)^* \omega_{\varphi_0}$. Instead, the product rule $(\beta g)' = \beta.(\delta(\beta).g) + \beta g'$, and the $H_0$-invariance of $\kappa$ show that
\[ L(\Phi_\beta)^* \omega_{\varphi_0} - \omega_{\varphi_1} = \left[ \int_0^1 \kappa(f, \delta(\beta).g) \, dt \right]. \]
Here we identify $\Omega^1(\mathbb{R}, \mathfrak{h})$ with $C^\infty(\mathbb{R}, \mathfrak{h})$, so that $\delta(\beta)$ is interpreted as a smooth $\mathfrak{h}$-valued curve.

(b) If $q_V : V \to V_{\varphi_V}$ denotes the projection map, this means that
\[ L(\Phi_\beta)^* \omega_{\varphi_0} - \omega_{\varphi_1} = q_V \circ \eta_{\delta(\beta)}, \quad (11) \]
where we put
\[ \eta_\gamma(f, g) := \int_0^1 \kappa(f, \gamma.g) \, dt \quad \text{for} \quad \gamma \in C^\infty([0, 1], \mathfrak{h}). \]
Since $\mathfrak{h}$ preserves $\kappa$, each $\eta_\gamma$ is a $V$-valued 2-cocycle; actually
\[ \eta_\gamma = \int_0^1 \eta_\gamma(t) \, dt \quad \text{for} \quad \eta_\gamma(t)(x, y) := \kappa(x, \gamma(t).y), \quad \eta_\gamma(t) \in Z^2(\mathfrak{g}, V). \]

(c) Let $0 < \varepsilon < \frac{1}{2}$. In $C^\infty_{\ast}(\mathbb{R}, K)_{\varphi_1}$ we write $C^\infty_{\ast}(\mathbb{R}, K)_{\varphi_1}$ for the subgroup of those maps vanishing on the interval $[-\varepsilon, \varepsilon]$. From Corollary 2.31 it easily follows that the inclusion of $C^\infty_{\ast}(\mathbb{R}, K)_{\varphi_1}$ into $C^\infty_{\ast}(\mathbb{R}, K)_{\varphi_1}$ is a weak homotopy equivalence. On the other hand, restriction to $[0, 1]$ and periodic extension yields an isomorphism of Lie groups
\[ C^\infty_{\ast}(\mathbb{R}, K)_{\varphi_1} \to C^\infty_{\ast}(\mathbb{R}, K)_{\text{id}}. \]
Further, the isomorphism \( \Phi (\beta) \) induces an automorphism of \( C_c^\infty (\mathbb{R}, K)_{\text{id}} \). With \( \beta_I (s) := \beta (st) \), we even obtain by \( \Phi (\beta) \) a smooth family of automorphisms of \( C_c^\infty (\mathbb{R}, K)_{\text{id}} \) connecting \( \Phi (\beta) \) to the identity. Therefore \( \Phi (\beta) \) induces the identity on \( \tau_2 (C_c^\infty (\mathbb{R}, K)_{\text{id}}) \), and Lemma 2.10 implies that the period maps of \( \tilde{\omega}_{\varphi_1} \) and \( L (\Phi (\beta)) \tilde{\omega}_{\varphi_0} \) coincide. We conclude that the period map of the cocycle \( \eta_{\delta (\beta)} \) vanishes on the image of \( \tau_2 (C_c^\infty (\mathbb{R}, K)_{\varphi_1}) \). If, in addition, \( \tau_2 (K) \) vanishes, then the long exact homotopy sequence of the map \( \text{ev}_0^K \) shows that the period map of \( q^\nu \circ \eta_{\delta (\beta)} \) vanishes.

**Lemma 3.5** Let \( \alpha_i : S^1 \to M, i = 0, 1 \), be two smooth homotopic loops in \( m_0 \in M \) and \( \beta : [0, 1] \to H \) a smooth curve in \( H \) obtained from a smooth homotopy of \( \alpha_0 \) and \( \alpha_1 \) as in the preceding remark. Then the two morphisms of Lie groups

\[
\tilde{\alpha}_0, \quad \Phi \circ \tilde{\alpha}_0^* : \Gamma K \to C_c^\infty (\mathbb{R}, K)_{\varphi_0}
\]

are homotopic.

**Proof.** Let \( F : [0, 1] \times S^1 \to M \) be a smooth homotopy with \( F_i = \alpha_i \) for \( i = 0, 1 \), and assume w.l.o.g. that \( F \) is constant in a neighborhood of \( \{0, 1\} \times S^1 \). Define \( \beta \) as above. For \( f \in \Gamma K \cong C^\infty (P, K)^H \), we have

\[
(\Phi \circ \tilde{\alpha}_0^*)(f)(t) = \beta(t).f(\tilde{\alpha}_0(t)) = f(\tilde{\alpha}_0(t), \beta(t)^{-1}),
\]

so that it suffices to see that the curves \( \tilde{\alpha}_0, \tilde{\alpha}_1, \beta^{-1} : [0, 1] \to P \) with the same endpoints are homotopic with fixed endpoints, which is equivalent to the existence of a homotopy between \( \tilde{\alpha}_0, \beta \) and \( \tilde{\alpha}_1 \).

The homotopy \( F \) can be lifted to a smooth map \( \hat{F} : [0, 1] \times \mathbb{R} \to P \) such that the curves \( \hat{F}_i \) are horizontal lifts starting in the base point \( p_0 \). Then \( \hat{F}_i = \hat{\alpha}_i \) for \( i = 0, 1 \). If \( \circ \) denotes composition of paths and \( \sim \) the homotopy relation, then the restriction of \( \hat{F} \) to the boundary of \( [0, 1] \) shows that \( \tilde{\alpha}_1 \sim \hat{\alpha}_0 \circ (\hat{\alpha}_0 (1), \beta) \sim \hat{\alpha}_0, \beta \).

Putting all this information together, we now see with (\( \mathbb{I} \)), (\( \mathbb{II} \)) and (\( \mathbb{III} \)) how \( \mathcal{I}_{\alpha_1} \circ \text{per}_\omega \) and \( \mathcal{I}_{\alpha_0} \circ \text{per}_\omega \) differ:

\[
\mathcal{I}_{\alpha_1} \circ \text{per}_\omega = \text{per}_{\omega_{\varphi_1}} \circ \pi_2 (\hat{\alpha}_1^*) = \text{per}_{L (\Phi (\beta)) \omega_{\varphi_0}} \circ \pi_2 (\hat{\alpha}_1^*) - \text{per}_{q^\nu \circ \eta_{\delta (\beta)}} \circ \pi_2 (\hat{\alpha}_1^*)
\]

\[= \text{per}_{\omega_{\varphi_0}} \circ \pi_2 (\Phi (\beta) \circ \hat{\alpha}_1^*) - q^\nu \circ \text{per}_{\eta_{\delta (\beta)}} \circ \pi_2 (\hat{\alpha}_1^*)
\]

\[= \text{per}_{\omega_{\varphi_0}} \circ \pi_2 (\hat{\alpha}_0^*) - q^\nu \circ \text{per}_{\eta_{\delta (\beta)}} \circ \pi_2 (\hat{\alpha}_1^*)
\]

\[= \mathcal{I}_{\alpha_0} \circ \text{per}_\omega - q^\nu \circ \text{per}_{\eta_{\delta (\beta)}} \circ \pi_2 (\hat{\alpha}_1^*).
\]
Proposition 3.6 Let $\theta \in \Omega^1(P, \mathfrak{h})$ be a principal connection form corresponding to the connection $\nabla$ and $R(\theta) = d\theta + \frac{1}{2}[\theta, \theta] \in \Omega^2(P, \mathfrak{h})$ be its curvature. Then the homomorphisms

$$I_\alpha \circ \text{per}_{\omega_\alpha} : \pi_2(\Gamma K) \to V_{\varphi_V}$$

depend for each smooth loop $\alpha$ in $m_0$ only on the homotopy class $[\alpha] \in \pi_1(M, m_0)$ if and only if for each derivation $D \in \text{im}(L(\rho_t) \circ R(\theta))$, the periods of the cocycle $\eta_D(x, y) := \kappa(x, Dy)$ on $\mathfrak{h}$ are trivial.

Proof. Suppose first that for each derivation $D \in \text{im}(L(\rho_t) \circ R(\theta))$, the periods of $\eta_D$ vanish. Let $F : [0, 1] \times S^1 \to M$ be a smooth homotopy of the loops $F_0$ and $F_1$ based in $m_0$. We lift $F$ to a smooth map $\hat{F} : [0, 1]^2 \to P$ such that the curves $\hat{F}_t = \hat{F}(t, \cdot)$ are horizontal, start in the base point $p_0$ and define the smooth curve $\beta : [0, 1] \to H$ by $\hat{F}_t(1) = \hat{F}_0(1), \beta(t)$. This implies that

$$\delta(\beta)_t = \theta\left(\frac{d}{dt} \hat{F}_t(1)\right) = (\hat{F}^*\theta)(\frac{\partial}{\partial t})(t, 1).$$

Since the curves $\hat{F}_t$ are horizontal, $\hat{F}^*\theta(\frac{\partial}{\partial s}) = 0$, which leads to

$$\left(\hat{F}^* R(\theta)\right)\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = d(\hat{F}^*\theta)\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = \frac{\partial}{\partial s}(\hat{F}^*\theta)(\frac{\partial}{\partial t}),$$

so that we arrive with $\left(\hat{F}^*\theta\right)(\frac{\partial}{\partial s})(t, 0) = 0$ at

$$\delta(\beta)_t = \left(\hat{F}^*\theta\right)(\frac{\partial}{\partial s})(t, 1) = \int_0^1 \left(\hat{F}^* R(\theta)\right)\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)(t, s) \, ds. \tag{12}$$

We have to show that the periods of $\eta_{\delta(\beta)}$ vanish. As we have just seen, $\delta(\beta)_t = \int_0^1 D(s, t) \, ds$ holds for a smooth family $D(s, t)$ of derivations of $\mathfrak{h}$ for which the periods of $\eta_{D(t, s)}$ are trivial by assumption. Now the assertion follows from

$$\eta_{\delta(\beta)} = \int_0^1 \eta_{\delta(\beta)}_t \, dt = \int_0^1 \int_0^1 \eta_{D(s, t)} \, ds \, dt,$$

so that

$$\text{per}_{\eta_{\delta(\beta)}}(\sigma) = \int_0^1 \int_0^1 \text{per}_{\eta_{D(s, t)}}[\text{ev}_t^K \circ \sigma] \, ds \, dt = 0.$$
Now we prove the converse. To this end, we consider a family \((\alpha_t)_{0 \leq t \leq T}\) of smooth loops in \(m_0\) with \(\alpha_0 = m_0\) (the constant loop) for which the holonomy \(\beta(t) = \mathcal{H}(\alpha_t)\) satisfies
\[
\beta'(0) = 0 \quad \text{and} \quad \beta''(0) = D := 2R(\theta)_{p_0}(\bar{w}, \bar{v})
\]
for two horizontal vectors \(\bar{v}, \bar{w} \in T_{p_0}(P)\). Since each loop \(\alpha_t\) is contractible, the corresponding operator \(\varphi_V\) on \(V\) is \(\text{id}_V\) (cf. Appendix D). From the assumption and Remark 3.4 we now obtain that the cocycle
\[
\eta(T) := \eta_{\delta(\beta)} = \int_0^T \eta_{\delta(\beta)} t \, dt
\]
has trivial periods for each \(T > 0\). As a function of \(T\), we have \(\eta'(t) = \eta_{\delta(\beta)} t\), \(\eta'(0) = 0\) and \(\eta''(0) = \eta_{\beta''(0)} = \eta_D\), so that all periods of \(\eta_D\) must be trivial. 

**Corollary 3.7** The homomorphism \(\mathcal{I}_\alpha \circ \text{per}_\omega\) depends for each smooth loop \(\alpha\) only on the homotopy class \([\alpha] \in \pi_1(M, m_0)\) if one of the following conditions is satisfied:

(a) \(\pi_2(K)\) is a torsion group.

(b) \(\mathfrak{k} = \mathfrak{h}\) and \(\rho_{\mathfrak{k}} = \text{ad}\).

(c) The connection \(\nabla\) on \(P\) is flat.

**Proof.** (a) follows immediately from Proposition 3.6 because \(\text{tor} \pi_2(K)\) lies in the kernel of each period homomorphism.

(b) For each inner derivation \(D\) of \(\mathfrak{k}\), the cocycle \(\eta_D\) is a coboundary, so that its period map vanishes ([Ne02a, Rem. 5.9]). Therefore Proposition 3.6 applies.

(c) Proposition 3.6 applies because \(R(\theta) = 0\). 

**Lemma 3.8** Let \(\tilde{m}_0 \in \tilde{M}\) be a base point and realize \(\Omega^1(M, V)\) as the space \(\Omega^1(\tilde{M}, V)^{\pi_1(M)}\) and identify \(\pi_1(M)\) with the group of deck transformations of \(\tilde{M}\), acting from the right. Then the map
\[
\Psi: H^1_{\text{dR}}(M, V) \to H^1(\pi_1(M), V), \quad \Psi([\theta]) = [\psi_0], \quad \psi_0(\gamma) := \int_{\tilde{m}_0}^{\tilde{m}_0 \cdot \gamma^{-1}} \theta
\]
is a linear isomorphism.
Proof. The existence of the isomorphism \( \Psi \) follows from [CE56, p. 356]. Here we briefly argue that it is given as above. Let \( f_\theta \in C^\infty(\tilde{M}, V) \) be the unique function with \( df_\theta = \theta \) vanishing in \( \tilde{m}_0 \). From the observation that \( \psi_\theta(\gamma) = \gamma \cdot f_\theta - f_\theta \) is a constant function, it follows that \( \psi_\theta \) is a 1-cocycle and that \([\psi_\theta]\) only depends on the cohomology class \([\theta]\).

If \( \Psi([\theta]) = [\psi_\theta] \) vanishes, then there exists some \( v \in V \) with

\[
\psi_\theta(\gamma) = \gamma \cdot f_\theta - f_\theta = \gamma \cdot v - v.
\]

Then \( f_\theta - v \) is \( \pi_1(M) \)-invariant, so that \([\theta] = [d(f_\theta - v)] = 0\). Therefore \( \Psi \) is injective.

To see that \( \Psi \) is also surjective, let \( \chi \in Z^1(\pi_1(M), V) \). We consider the corresponding affine action of \( \pi_1(M) \) on \( V \), defined by \( \gamma \ast v := \tilde{\rho}_V(\gamma) \cdot v - \chi(\gamma) \). Then the associated bundle \( B := (\tilde{M} \times V)/\pi_1(M) \) is an affine bundle over \( M \). Using smooth partitions of unity, we see that this bundle has a smooth section \( s : M \to B \). We write \( s(q_M(m)) = [(m, f(m))] \) for some smooth function \( f : \tilde{M} \to V \). Then \([(m, \gamma, f(m, \gamma))] = [(m, \gamma \ast f(m, \gamma))] \) implies

\[
f(m) = \gamma \ast f(m, \gamma) = \tilde{\rho}_V(\gamma) \cdot f(m, \gamma) - \chi(\gamma) = (\gamma \cdot f)(m) - \chi(\gamma),
\]

so that \( \chi(\gamma) = \gamma \cdot f - f \) and thus \( \chi = \psi_d f \).

Remark 3.9 If \( B^1(\pi_1(M), V) \) is a closed subspace of \( Z^1(\pi_1(M), V) \) with respect to the topology of pointwise convergence, then it follows from the proof of the Lemma 3.8 that \( d(C^\infty(\tilde{M}, V)) \) is a closed subspace of \( \Omega^1(\tilde{M}, V) \), but in general this is not clear.

If \( \dim V < \infty \), then \( B^1(\pi_1(M), V) \) is finite-dimensional, hence closed, so that the quotient space \( H^1(\pi_1(M), V) \cong \Omega^1_{\text{dR}}(M, V) \) is Hausdorff.

Remark 3.10 There are compact 3-manifolds \( M \) whose fundamental group contains normal subgroups which are not finitely generated. In fact, by van Kampen’s Theorem, the fundamental group of the connected sum \( M \) of four copies of \( S^1 \times S^2 \) is the free group on 4 generators. In VA.23(iv) of [dH00] one finds an example of a group with four generators which is not finitely presented. This implies that the normal subgroup \( R \) of \( \pi_1(M) \), the free group on four generators, generated by these relations is not finitely generated. Therefore \( M/R \) is a 3-manifold whose fundamental group \( R \) is not finitely generated, although \( \tilde{M}/R \) covers the compact manifold \( M \).
Proposition 3.11 For each \( \alpha \in C^\infty(S^1, M) \), the homomorphism \( I_\alpha \circ \per_\omega \) depends only on the homotopy class \([\alpha] \in \pi_1(M, m_0)\) if and only if \( \Pi_\omega \subseteq H^1_{dR}(M, V) \).

Proof. We realize \( \Omega^1(M, V) \) as the space \( \Omega^1(\hat{M}, V)_{D} \) of \( D \)-invariant \( V \)-valued 1-forms on \( \hat{M} \). Let \( \hat{\alpha}_M \) denote the image of the horizontally lifted curve \( \hat{\alpha} : [0, 1] \rightarrow \hat{P} \) in \( \hat{M} = P/\ker \rho_V \) and observe that for two homotopic loops \( \alpha_0 \) and \( \alpha_1 \) in \( m_0 \), the curves \( \hat{\alpha}_{1, M} \) and \( \hat{\alpha}_{2, M} \) are homotopic with fixed endpoints. If, conversely, \( \beta_0, \beta_1 : [0, 1] \rightarrow \hat{M} \) are two smooth curves starting in \( \hat{m}_0 \) which are homotopic with fixed endpoints, such that the curves \( \alpha_i := \hat{q}_M \circ \beta_i \) are closed, then \( \beta_i = \hat{\alpha}_{i, M} \) for \( i = 1, 2 \).

Since \( \hat{M} \cong P/\ker \rho_V \) is connected, the homomorphisms \( I_\alpha \circ \per_\omega \) depend only on the homotopy class of \( \alpha \) if and only if each element \( [\theta] \in \Pi_\omega \subseteq \Omega^1(M, V) \subseteq \Omega^1(\hat{M}, V)_{D} \) has the property that the integral \( I_{\alpha}([\theta]) = \int_{\hat{\alpha}_M} \theta \) only depends on the homotopy class of \( \hat{\alpha}_M \) with fixed endpoints. This is equivalent to the 1-form \( \theta \) being closed, i.e., \( [\theta] \in H^1_{dR}(M, V) \).

Remark 3.12 If the group \( D \) is finite, then the fixed point functor \( H^0(D, \cdot) \) is exact on rational \( D \)-modules, so that

\[
H^1_{dR}(M, V) = Z^1_{dR}(\hat{M}, V)^D / d(C^\infty(\hat{M}, V)^D) = Z^1_{dR}(\hat{M}, V)^D / B^1_{dR}(\hat{M}, V)^D = H^1_{dR}(\hat{M}, V)^D.
\]

Since \( D \cong \pi_1(M)/\pi_1(\hat{M}) \) is finite and \( V \) is divisible, the surjective map

\[
H^1_{dR}(M, V) \cong \text{Hom}(\pi_1(M), V) \rightarrow H^1_{dR}(\hat{M}, V) \cong \text{Hom}(\pi_1(\hat{M}), V)
\]

is a linear isomorphism, and we thus obtain

\[
H^1_{dR}(M, V) \cong H^1_{dR}(\hat{M}, V)^D \cong H^1_{dR}(M, V)^D \cong H^1_{dR}(M, V^D).
\]  \hspace{1cm} (13)

Remark 3.13 For \( M = S^1 \cong \mathbb{R}/\mathbb{Z} \) and the \( m \)-fold covering \( \hat{M} := \mathbb{R}/m\mathbb{Z} \cong S^1 \), we have \( D \cong \mathbb{Z}/m \) and \( H^1_{dR}(M, V) \cong H^1_{dR}(\hat{M}, V)^D \cong V^D \).

Theorem 3.14 (Reduction Theorem) Assume that \( D \) is finite and that \( \Pi_\omega \subseteq H^1_{dR}(M, V) \). Then \( \Pi_\omega \) is discrete if this is the case for each \( \Pi_{\omega_\alpha} \), with \( \omega_\alpha := \omega_{\varphi_K} \) for \( \alpha \in C^\infty(S^1, \hat{M}) \).
Proof. Since $\mathbb{D} \cong \pi_1(M)/\pi_1(\hat{M})$ is finite, there exists a number $N \in \mathbb{N}$ such that the image of the homomorphism $H_1(\hat{M}) \to H_1(M)$ contains $N \cdot H_1(M)$. Suppose that $H_1(M)$ is finitely generated of rank $r$. Then the Universal Coefficient Theorem, combined with de Rham’s Theorem, yields

$$H^1_{dR}(M,V) \cong \text{Hom}(H_1(M),V) \cong V^r.$$ 

As we have seen above, there exist smooth loops $\bar{\alpha}_i \in C^\infty(S^1,\hat{M}), i = 1,\ldots,r$, whose images in $H_1(M)$ form a $\mathbb{Q}$-basis of $H_1(M) \otimes \mathbb{Q}$. We then obtain the concrete linear isomorphism

$$\Phi = (\mathcal{I}_{\alpha_i})_{i=1}^r : H^1_{dR}(M,V) \to V^r, \quad [\omega] \mapsto \left( \int_{\alpha_i} \omega \right)_{i=1}^r.$$ 

By (10), $\Phi(\Pi_\omega) \subseteq \prod_{i=1}^r \Pi_{\omega,\alpha_i}$, where the right hand side is a discrete subgroup of $V^r$. Therefore $\Pi_\omega$ is discrete. 

Remark 3.15 The assumption on $\mathbb{D}$ to be finite in the previous theorem was needed to ensure that the map

$$\Phi : H^1_{dR}(M,V) \to H^1_{dR}(\hat{M},V)^{\mathbb{D}}$$

is an isomorphism. The argument also works if $\Phi$ is injective and $H_1(\hat{M})$ is finitely generated. The kernel of $\Phi : H^1(\pi_1(M),V) \to H_1(\pi_1(\hat{M}),V)$ is the image of the natural map $H^1(\mathbb{D},V) \to H^1(\pi_1(M),V)$, hence vanishes whenever $H^1(\mathbb{D},V) = \{0\}$.

For a finite-dimensional orthogonal representation of $\mathbb{D}$ on $V$, this is the case if $\mathbb{D}$ has Kazhdan’s property (T) ([Pa07, Prop. 7 and Prop. 31]).

Combining the Reduction Theorem with Corollary 3.7, leads to:

Corollary 3.16 If $\mathbb{D}$ is finite, $K = H$ and $\mathfrak{R} = \text{Ad}(P)$, then $\Pi_\omega$ is discrete if this is the case for each $\Pi_{\omega,\alpha}, \alpha \in C^\infty(S^1,\hat{M})$.

Theorem 3.17 If $\pi_2(K)$ vanishes, then the following are equivalent:

1. $\Pi_\omega$ is discrete for each compact manifold $M$ and each connection $\nabla$, provided the group $\mathbb{D} \cong \rho_V(H)$ is finite.

2. $\Pi_\omega$ is discrete for the trivial bundle over $S^1$ and the canonical connection.
(3) The period group $\Pi_{C(\kappa)}$ of the 3-cocycle $C(\kappa)$ of $\mathfrak{k}$ is discrete.

**Proof.** The equivalence of (2) and (3) follows from Lemma 2.10, so that it remains to derive (1) from (2). If $\pi_2(K)$ vanishes, then Lemma 2.10 further implies that the period group of any cocycle $\omega_\alpha$, $\alpha \in C^\infty(S^1, \hat{M})$, is discrete if and only if this is the case for $\Pi_{C(\kappa)}$. Now the Reduction Theorem 3.14 applies.

With Corollary 2.13 we also get:

**Corollary 3.18** If $\mathfrak{k}$ is finite-dimensional, $V = V(\mathfrak{k})$, $\kappa = \kappa_u$ is universal, and $\mathbb{D}$ is finite, then the period group of the cocycle $\omega_\kappa$ is discrete for any connection $\nabla$.

**Remark 3.19** If $K$ is finite-dimensional and 1-connected, then $H := \text{Aut}(\mathfrak{k}) \cong \text{Aut}(K)$ has finitely many connected components because $\text{Aut}(\mathfrak{k})$ is a real algebraic group (OV90).

If $V^0(\mathfrak{k}) := V(\mathfrak{k}) / \text{der}(\mathfrak{k}).V(\mathfrak{k}) = V(\mathfrak{k}) / \text{der}(\mathfrak{k})$ denotes the quotient space, then the corresponding form $\kappa^0_u : \mathfrak{k} \times \mathfrak{k} \to V^0(\mathfrak{k})$ is the universal $\text{der}(\mathfrak{k})$-invariant symmetric bilinear form. Then $\kappa^0_u$ is invariant under $\text{Aut}(\mathfrak{k})_0$ and $\pi_0(\text{Aut}(\mathfrak{k}))$ is finite. Since the period groups of $C(\kappa_u)$ and $C(\kappa^0_u)$ coincide (Theorem 3.17), Theorem 3.17 implies that the period group $\Pi_{\omega_{\kappa^0_u}}$ is discrete.

**Example 3.20** Now we show that $\Pi_\omega$ is not always contained in $H^1_{\text{dR}}(M, V)$.

We consider a trivial bundle $K = M \times K$ and $H = \mathbb{R}$, so that $\mathfrak{h} = \mathbb{R}$ acts on $\mathfrak{k}$ by a derivation $D \in (\text{der}(\mathfrak{k})_\kappa$ and the bundle $\mathbb{V}$ is trivial. We then write any covariant exterior derivative as

$$d^\nabla f = df + \beta \cdot Df,$$

for some $\beta \in \Omega^1(M, \mathbb{R})$, and, accordingly, $\omega = \omega_0 + \eta_\beta = \omega_0 + \beta \cdot \eta_D$. Since all periods of $\omega_0$ are contained in $H^1_{\text{dR}}(M, V)$ (Corollary 3.7(d)), $\Pi_\omega$ is contained in $H^1_{\text{dR}}(M, V)$ if and only if this holds for the period group of $\eta_\beta$.

On $S^1$, each 1-form is closed, so that we consider $M = \mathbb{T}^2$. Then the range of

$$\text{per}_{\eta_\beta} = \beta \cdot \text{per}_{\eta_D} : \pi_2(K) \to \Omega^1(M, V)$$

does not lie in the space of closed forms if $\beta$ is not closed and $\text{per}_{\eta_D}$ is non-trivial, which is the case for $K = C^\infty(S^1, \mathfrak{g})$, $\mathfrak{g}$ simple compact and $Df = f'$ (cf. Example 2.14).
3.2 Integrating actions

In this section we show that for any principal \( K \)-bundle \( P \) \((K\text{ locally exponential})\), the action of the Lie group \( \text{Aut}(P) \) of bundle automorphism on the spaces \( \Omega^1(M, V) \) and the Lie algebra \( \text{gau}(P) \) combines to a smooth automorphic action on the central Lie algebra extension \( \hat{\text{gau}}(P) \), defined by the cocycle \( \omega = \omega_\nabla^\kappa \). Moreover, we show under which conditions this construction carries over to arbitrary Lie group bundles which are not necessarily gauge bundles.

Let \( \theta \in \Omega^1(P, \mathfrak{k}) \) be a principal connection 1-form corresponding to \( \nabla \). Realizing \( \text{gau}(P) \) in \( C^\infty(P, \mathfrak{k}) \), we have \( \nabla f = df + [\theta, f] \), so that
\[
\omega(f_1, f_2) = [\kappa(f_1, \nabla f_2)] = [\kappa(f_1, df_2) + \kappa(\theta, [f_2, f_1])].
\]
The Lie group \( \text{Aut}(P) \) acts smoothly on the affine space \( A(P) \subseteq \Omega^1(M, \text{Ad}(P)) \) of principal connection 1-forms by \( \varphi.\theta = (\varphi^{-1})^*\theta \) and on \( \text{gau}(P) \) by \( \varphi.f = f \circ \varphi^{-1} \) (cf. [Gl06, Prop. 6.4]). We then have
\[
\varphi.\nabla f = \varphi.(df + [\theta, f]) = d\varphi.f + [\varphi.\theta, \varphi.f] = d\nabla(\varphi.f) + [\varphi.\theta - \theta, \varphi.f].
\]
This leads to
\[
(\varphi.\omega)(f_1, f_2) = \varphi.\omega(\varphi^{-1}.f_1, \varphi^{-1}.f_2) = \omega(f_1, f_2) + [\kappa(\varphi.\theta - \theta, [f_2, f_1])].
\]
Note that
\[
\zeta: \text{Aut}(P) \rightarrow \Omega^1(M, \text{Ad}(P)), \quad \varphi \mapsto \varphi.\theta - \theta
\]
is a smooth 1-cocycle, so that
\[
\Psi: \text{Aut}(P) \rightarrow \text{Hom}(\text{gau}(P), \overline{\Omega}^1(M, V)), \quad \Psi(\varphi)(f) := [\kappa(\varphi.\theta - \theta, f)]
\]
is a 1-cocycle with \( d_{\text{gau}(P)}(\Psi(\varphi)) = \varphi.\omega - \omega \), defining a smooth map
\[
\text{Aut}(P) \times \text{gau}(P) \rightarrow \overline{\Omega}^1(M, V).
\]

**Theorem 3.21** The group \( \text{Aut}(P) \) acts smoothly by automorphisms on the centrally extended Lie algebra \( \hat{\text{gau}}(P) \) by
\[
\varphi.([\alpha], f) := ([\varphi.\alpha] + [\kappa(\varphi.\theta - \theta, \varphi.f)], \varphi.f). \quad (14)
\]

If, in addition, the period group \( \Pi_\omega \) is discrete and \( Z \hookrightarrow \hat{G} \rightarrow G \) is a central extension with 1-connected \( \hat{G} \), \( G = \text{Gau}(P)_0 \) and Lie algebra \( \hat{g} \), then this action integrates to a smooth action of \( \text{Aut}(P) \) on \( \hat{G} \).
Proof. First, [MN03, Lemma V.1] implies that we obtain automorphisms of \( \hat{\text{gau}}(P) \), and the smoothness of the action follows from the smoothness of \( \zeta \) and the smoothness of the actions of \( \text{Aut}(P) \) on \( \text{gau}(P) \) and \( \Omega^1(M, V) \).

Assume that \( \Pi_\omega \) is discrete. Since the action of \( \text{Aut}(P) \) on \( g := \text{gau}(P) \) and \( z := \Omega^1(M, V) \) preserves the cohomology class of \( \omega \) (cf. Example [1.4]), the period homomorphism \( \text{per}_\omega : \pi_2(G) \to z \) is \( \text{Aut}(P) \)-equivariant, which implies in particular that its image in \( z \) is invariant under the action of \( \text{Aut}(P) \). We therefore obtain a smooth action of \( \text{Aut}(P) \) on \( Z_0 := z / \Pi_\omega \). Now the group \( \hat{G} \) is a central extension of the universal covering group \( \tilde{G} \) of \( G \) by \( Z_0 \), and \( \pi_0(Z) \cong \pi_1(G) \) (cf. [Ne02a, Rem. 7.14]). Finally, we lift the \( \text{Aut}(P) \) action on \( G \) to a smooth action on \( \tilde{G} \) and apply the Lifting Theorem [C.3].

If \( \varphi_f \in \text{Gau}(P) \) is a gauge transformation corresponding to the smooth function \( f : P \to K \), then \( \varphi_f^* \theta = \delta(f) + \text{Ad}(f)^{-1} \theta \) implies
\[
\varphi_f \cdot \theta = \delta(f^{-1}) + \text{Ad}(f) \theta \quad \text{and} \quad \zeta(\varphi_f) = \delta(f^{-1}) + \text{Ad}(f) \theta - \theta.
\]

Corollary 3.22 The adjoint action of \( \text{gau}(P) \) on \( \hat{\text{gau}}(P) \) integrates to a smooth action of \( \text{Gau}(P) \) on \( \hat{\text{gau}}(P) \).

Theorem 3.23 If \( \pi_0(K) \) is finite and \( \pi_2(K) \) vanishes, then the following are equivalent:

1. \( \omega^\nabla \) integrates for each principal \( K \)-bundle \( P \) over a compact manifold \( M \) to a Lie group extension of \( \text{Gau}(P)_0 \).

2. \( \omega_\kappa \) integrates for the trivial \( K \)-bundle \( P = S^1 \times K \) over \( M = S^1 \) to a Lie group extension of \( \text{C}^\infty(S^1, K)_0 \).

3. The image of \( \text{per}_\kappa : \pi_3(K) \to V \) is discrete.

Proof. Since the existence of a Lie group extension of \( G := \text{Gau}(P)_0 \) integrating \( \omega_\kappa^\nabla \) is equivalent to the discreteness of \( \Pi_{\omega_\kappa^\nabla} \) and the integrability of the adjoint action to an action on \( \hat{\text{gau}}(P) \), this follows from Corollary 3.22 and Theorem 3.17.

Theorem 3.24 Let \( P \) be a finite-dimensional connected principal bundle with structure group \( K \) over the compact manifold \( M \). If \( V = V(\mathfrak{t}) \), \( \kappa = \kappa_u \) is universal and \( D = \mathfrak{p}_V(\pi_0(K)) \subseteq \text{GL}(V(\mathfrak{t})) \) is finite, then the central extension \( \hat{\text{gau}}(P) \) of \( \text{gau}(P) \) defined by \( \omega_\kappa^\nabla \) integrates for any connection \( \nabla \) to a central extension of the identity component \( \text{Gau}(P)_0 \) of the gauge group.
Proof. With Corollary 3.18 this follows as in the proof of Theorem 3.23.

For general Lie algebra bundles $\mathfrak{K}$, the action of $\Gamma \mathfrak{K}$ on $\hat{\Gamma} \mathfrak{K}$ is given by

$$h.(\alpha, f) = (\omega(h, f), [h, f]) = ([\kappa(-d^\nabla h, f), [h, f])$$

The fact that $\nabla$ is a Lie connection means that $d\nabla : \Gamma \mathfrak{K} \to \Omega^1(M, \mathfrak{K})$ is a 1-cocycle for the action of the Lie algebra $\Gamma \mathfrak{K}$ on $\Omega^1(M, \mathfrak{K})$ by $f.\alpha := [f, \alpha]$ (pointwise bracket). To integrate the action of $\Gamma \mathfrak{K}$ on $\hat{\Gamma} \mathfrak{K}$ to a group action, we therefore have to integrate $d\nabla$ to a Lie group cocycle $(\Gamma \mathfrak{K})_0 \to \Omega^1(M, \mathfrak{K})$.

This can be achieved as follows. We assume that $K$ is 1-connected. First we observe that $\operatorname{der}(\mathfrak{k}) = Z^1(\mathfrak{k}, \mathfrak{k})$, where $\mathfrak{k}$ acts on itself by the adjoint action. In this sense, each derivation $D \in \operatorname{der}(\mathfrak{k})$ is a 1-cocycle, hence defines an equivariant closed 1-form $D^{eq} \in \Omega^1(K, \mathfrak{k})$ which is exact since $K$ is 1-connected. Let $\chi^D : K \to \mathfrak{k}$ be the unique smooth function with $d\chi^D = D^{eq}$ and $\chi^D(1) = 0$. Then $\chi^D$ is a smooth 1-cocycle (cf. [GN09]), and the smoothness of the action $\rho_k$ of $\mathfrak{k}$ on $\mathfrak{k}$ implies that the function

$$\chi : \mathfrak{h} \times K \to \mathfrak{k}, \quad (x, k) \mapsto \chi^\rho_x(k)$$

is smooth.

If $\theta \in \Omega^1(P, \mathfrak{h})$ is a principal connection 1-form, we now define for $f \in \Gamma \mathcal{K}$ a 1-form $\chi^\theta(f)$ in $\Omega^1(P, \mathfrak{t})^H \cong \Omega^1(M, \mathfrak{K})$ by

$$\chi^\theta(f)v := \chi^\theta(v)(f(p)) \quad \text{for} \quad v \in T_p(P).$$

Then

$$\delta^\nabla(f) := \delta(f) + \chi^\theta(f^{-1})$$

is a covariant left logarithmic derivative on $\Gamma \mathcal{K}$ and

$$\Gamma \mathcal{K} \to \Omega^1(M, \mathfrak{K}), \quad f \mapsto \delta^\nabla(f^{-1})$$

is a 1-cocycle integrating $-d^\nabla$. We now calculate for $\varphi \in \Gamma \mathcal{K}$:

$$\varphi.d^\nabla f = \varphi.(df + \theta.f) = d(\varphi.f) + (\varphi.\theta).(\varphi.f)$$

$$= d^\nabla(\varphi.f) + (\varphi.\theta - \theta).(\varphi.f) = d^\nabla(\varphi.f) + \chi^\theta(\varphi).(\varphi.f).$$
This easily leads to
\[(\varphi, \omega)(f_1, f_2) = \varphi, \omega(\varphi^{-1}.f_1, \varphi^{-1}.f_2) = \omega(f_1, f_2) + [\kappa(\chi^\theta(\varphi), [f_2, f_1])],\]
and \(\chi^\theta: \Gamma K \to \Omega^1(M, \mathfrak{K})\) is a smooth 1-cocycle, so that
\[
\Psi: \Gamma K \to \text{Hom}(\Gamma \mathfrak{K}, \overline{\Omega}^1(M, \mathcal{V})), \quad \Psi(\varphi)(f) := [\kappa(\chi^\theta(\varphi), f)]
\]
is a 1-cocycle with \(d_{\Gamma \mathfrak{K}}(\Psi(\varphi)) = \varphi, \omega - \omega\), defining a smooth map
\[
\Gamma K \times \Gamma \mathfrak{K} \to \overline{\Omega}^1(M, \mathcal{V}).
\]

**Theorem 3.25** If \(K\) is 1-connected, then the group \(\Gamma K\) acts smoothly by automorphisms on the centrally extended Lie algebra \(\widehat{\Gamma \mathfrak{K}}\) by
\[
\varphi.(\alpha, f) := ([\varphi.\alpha] + [\kappa(\chi^\theta(\varphi), \varphi.f)], \varphi.f).
\]

**Theorem 3.26** If \(K\) is 2-connected, then the following are equivalent:

1. If \(\rho_V(H)\) is finite, then the Lie algebra defined by \(\omega_\nabla^\kappa\) integrates for each \(K\)-bundle \(K\) over a compact manifold \(M\) and each Lie connection \(\nabla\) on \(\mathfrak{K}\) to a Lie group extension of \((\Gamma K)_0\).

2. The extension defined by \(\omega_\kappa\) on \(C^\infty(S^1, \mathfrak{k})\) integrates to a Lie group extension of \(C^\infty(S^1, K)_0\).

3. The image of \(\text{per}_\kappa: \pi_3(K) \to V\) is discrete.

**Proof.** Since the existence of a Lie group extension of \((\Gamma K)_0\) integrating \(\omega_\nabla^\kappa\) is equivalent to the discreteness of \(\Pi_{\omega_\nabla^\kappa}\) and the integrability of the adjoint action of \(\Gamma \mathfrak{K}\) on \(\widehat{\Gamma \mathfrak{K}}\) to an action of \((\Gamma K)_0\), this follows from Theorem 3.25 and Theorem 3.17.

**A Appendix: The Lie group structure on \(\Gamma K\)**

In this section we explain how to obtain a locally exponential Lie group structure on the group \(\Gamma K\) of smooth sections of the (locally trivial) Lie group bundle \(K\) over the compact manifold \(M\) whose fiber is a locally exponential Lie group \(K\) with Lie algebra \(\mathfrak{k}\).
We further assume that the Lie group bundle $\mathcal{K}$ is associated to a principal $H$-bundle $P$ via a smooth action defined by $\rho_K : H \to \text{Aut}(K)$. We write $\rho_t(h) := L(\rho_K(h))$ for the corresponding smooth action of $H$ on $\mathfrak{k}$ and $\mathfrak{K} := L(K)$ for its Lie algebra bundle with fiber $\mathfrak{k} := L(K)$. We endow the space $\Gamma_{\mathcal{K}}$ of smooth sections of $\mathcal{K}$ with the smooth compact open topology. This turns $\Gamma_{\mathcal{K}}$ into a locally convex Lie algebra because over each open subset $U \subseteq M$ for which $\mathcal{K}_U$ is trivial, we have $\Gamma(\mathcal{K}_U) \cong C^\infty(U, \mathfrak{k})$, and the Lie bracket on the locally convex space $C^\infty(U, \mathfrak{k})$ is continuous since $U$ is finite-dimensional. Likewise, the smooth compact open topology turns the group $\Gamma_{\mathcal{K}}$ of smooth section of $\mathcal{K}$ into a topological group. Indeed, restriction defines a group homomorphism $\Gamma_{\mathcal{K}} \to \Gamma_{\mathcal{K}_U} \cong C^\infty(U, K)$, and the topology on $\Gamma_{\mathcal{K}}$ is defined by the embedding $\Gamma_{\mathcal{K}} \hookrightarrow \prod_U C^\infty(U, K)$, where $U$ runs through an open cover of $M$ consisting of trivializing open subsets (cf. [Ne06, Def. II.2.7]).

Since the exponential function $\exp_K : \mathfrak{k} \to K$ is natural, we have

$$\exp_K \circ L(\varphi) = \varphi \circ \exp_K$$

for every automorphism $\varphi \in \text{Aut}(K)$, and we obtain a fiberwise defined exponential map

$$\exp_{\mathcal{K}} : \mathfrak{K} \to \mathcal{K}.$$ 

Composing smooth sections with this exponential map, we obtain a map

$$\exp_{\Gamma_{\mathcal{K}}} : \Gamma_{\mathfrak{K}} \to \Gamma_{\mathcal{K}}.$$ 

**Theorem A.1** The topological group $\Gamma_{\mathcal{K}}$ carries a locally exponential Lie group structure with $L(\Gamma_{\mathcal{K}}) \cong \Gamma_{\mathfrak{K}}$. Moreover, this topology coincides with the smooth compact-open topology.

**Proof.** The proof of [Wo07a, Thm. 1.11] carries over from the case of the conjugation of $K$ on itself to an arbitrary action of some group $H$ on $K$. □

**B Appendix: The universal invariant bilinear form in finite dimensions**

Throughout this section, $K$ denotes a finite-dimensional Lie group and $\mathfrak{k} = L(K)$ its Lie algebra. We further choose a Levi decomposition $\mathfrak{k} = r \rtimes s$ and write $s = s_1^{m_1} \oplus \ldots \oplus s_r^{m_r}$, for the decomposition of $s$ into simple ideals $s_i$, where $s_i$ is supposed to be non-isomorphic to $s_j$ for $j \neq i$. 

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The action of \( Aut(\mathfrak{k}) \) on \( V(\mathfrak{k}) \)

**Definition B.1** We put \( V(\mathfrak{k}) := S^2(\mathfrak{k})/\mathfrak{k}S^2(\mathfrak{k}) \), where the action of \( \mathfrak{k} \) on \( S^2(\mathfrak{k}) \) is the natural action inherited by the one on the tensor product \( \mathfrak{k} \otimes \mathfrak{k} \) by \( x.(y \otimes z) = [x, y] \otimes z + y \otimes [x, z] \). There exists a natural invariant symmetric bilinear form

\[ \kappa_u : \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k}), \quad (x, y) \mapsto [x \vee y] \]

such that for each invariant symmetric bilinear form \( \beta : \mathfrak{k} \times \mathfrak{k} \to W \) there exists a unique linear map \( \varphi : V(\mathfrak{k}) \to W \) with \( \varphi \circ \kappa_u = \beta \). We call \( \kappa_u \) the universal invariant symmetric bilinear form on \( \mathfrak{k} \).

**Remark B.2** (cf. [MN03])

(a) The assignment \( \mathfrak{g} \to V(\mathfrak{g}) \) is a covariant functor from finite-dimensional Lie algebras to vector spaces.

(b) If \( \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \) and \( \mathfrak{a} \) is perfect, then \( V(\mathfrak{g}) \cong V(\mathfrak{a}) \oplus V(\mathfrak{b}) \) because for every symmetric invariant bilinear map \( \kappa : \mathfrak{g} \times \mathfrak{g} \to V \), we have for \( x, y \in \mathfrak{a}, z \in \mathfrak{b} \) the relation \( \kappa([x, y], z) = \kappa(x, [y, z]) = \kappa(x, 0) = 0 \).

(c) If \( \mathfrak{h} \leq \mathfrak{g} \) is an ideal and the quotient morphism \( q : \mathfrak{g} \to \mathfrak{q} := \mathfrak{g}/\mathfrak{h} \) splits, then \( \mathfrak{g} \cong \mathfrak{h} \times \mathfrak{q} \), and the natural map \( V(\mathfrak{q}) \to V(\mathfrak{g}) \) is an embedding. In fact, let \( \eta : \mathfrak{q} \to \mathfrak{g} \) be the inclusion map. Then \( q \circ \eta = \text{id}_{\mathfrak{q}} \) and this leads to \( V(q) \circ V(\eta) = \text{id}_{V(\mathfrak{q})} \), showing that \( V(\eta) \) is injective.

(d) If \( \mathfrak{s} \) is reductive with the simple ideals \( \mathfrak{s}_1, \ldots, \mathfrak{s}_n \), then (b) implies that

\[ V(\mathfrak{s}) \cong V(3(\mathfrak{s})) \oplus \bigoplus_{j=1}^n V(\mathfrak{s}_j). \]

(e) If \( \mathfrak{k} = \mathfrak{r} \times \mathfrak{s} \) is a Levi decomposition, then (c) shows that the natural map \( V(\mathfrak{s}) \to V(\mathfrak{k}) \) is an embedding.

**Remark B.3** As a consequence of our construction, the group \( Aut(\mathfrak{k}) \) and its Lie algebra \( \text{der}(\mathfrak{k}) \) act naturally on \( V(\mathfrak{k}) \). The Lie algebra \( \mathfrak{k} \) itself, resp., the subalgebra \( \text{ad} \mathfrak{k} \) of inner derivations acts trivially.

If all derivations are inner, as is the case if \( \mathfrak{k} \) is semisimple, it follows that the identity component \( Aut(\mathfrak{k})_0 \) acts trivially on \( V(\mathfrak{k}) \).

**Remark B.4** For a simple finite-dimensional real Lie algebra \( \mathfrak{s} \), its centroid

\[ \text{Cent}(\mathfrak{s}) := \{ \varphi \in \text{End}(\mathfrak{s}) : (\forall x \in \mathfrak{s}) [\varphi, \text{ad} x] = 0 \} \]

is a field, hence isomorphic to \( \mathbb{R} \) or \( \mathbb{C} \) ([Ja79 Theorem X.1]). If \( \text{Cent}(\mathfrak{s}) \cong \mathbb{C} \), then \( \mathfrak{s} \) actually carries the structure of a complex simple Lie algebra and if
Cent($\mathfrak{s}$) $\cong \mathbb{R}$, then its complexification $\mathfrak{s}_C$ is simple. In the latter case we call $\mathfrak{s}$ \textit{central simple}.

If $\beta(x, y) := \text{tr}(\text{ad} x \, \text{ad} y)$ is the Cartan–Killing form of $\mathfrak{s}$, then the map

$$\eta: \text{Cent}(\mathfrak{s}) \to \text{Sym}^2(\mathfrak{s}, \mathbb{R})^\mathfrak{s} \cong V(\mathfrak{s})^*$, \quad \eta(\varphi)(x, y) := \beta(\varphi(x), y)$$

is easily seen to be a linear isomorphism. This implies that $V(\mathfrak{s}) \cong \mathbb{C}$ if $\mathfrak{s}$ is complex and $V(\mathfrak{s}) \cong \mathbb{R}$ otherwise. In the latter case the Cartan–Killing form $\beta$ is already universal, and in the former case, we have the additional form $\beta'(x, y) = \beta(ix, y)$. Hence the Cartan–Killing form

$$\beta_C: \mathfrak{s} \times \mathfrak{s} \to \mathbb{C}, \quad \beta_C(x, y) = \frac{1}{2}(\beta(x, y) - i\beta(ix, y)) = \frac{1}{2}(\beta(x, y) - i\beta(x, iy))$$

of the complex simple Lie algebra $\mathfrak{s}$ is the universal invariant symmetric bilinear form for the real simple Lie algebra $\mathfrak{s}$.

\textbf{Example B.5} If $\mathfrak{t} = \mathfrak{gl}_n(\mathbb{R})$, then Remark B.2(d) implies that

$$V(\mathfrak{gl}_n(\mathbb{R})) \cong V(\mathfrak{sl}_n(\mathbb{R})) \oplus V(\mathbb{R}) \cong \mathbb{R}^2$$

because $\mathfrak{sl}_n(\mathbb{R})$ is central simple.

\textbf{Theorem B.6} Let $\mathfrak{t}$ be a finite-dimensional real Lie algebra with Levi decomposition $\mathfrak{t} = \mathfrak{r} \times \mathfrak{s}$ and $\mathfrak{s} = \bigoplus_{i=1}^r \mathfrak{s}_i^{m_i}$ the decomposition into simple ideals. With $V_0 := \kappa_u(\mathfrak{r}, \mathfrak{t})$ and $V_i := \kappa_u(\mathfrak{s}_i^{m_i}, \mathfrak{s}_i^{m_i}) \cong V(\mathfrak{s}_i)^{m_i}$, we obtain a direct sum decomposition

$$V(\mathfrak{t}) = V_0 \oplus V_1 \oplus \ldots \oplus V_r \cong V_0 \oplus V(\mathfrak{s}_1)^{m_1} \oplus \ldots \oplus V(\mathfrak{s}_r)^{m_r} \quad (16)$$

which is invariant under the group $\text{Aut}(\mathfrak{t})$.

\footnote{If $C$ is a complex linear endomorphism of a complex vector space, then the traces of $C$ with respect to $\mathbb{R}$ and $\mathbb{C}$ are related by $\text{tr}_C C = \frac{1}{2}(\text{tr}_\mathbb{R} C - i \text{tr}_\mathbb{R}(iC))$.}

\footnote{[MN03], Remark II.2(4) uses the invalid assumption that $V(\mathfrak{s})$ is one-dimensional for any real simple Lie algebra $\mathfrak{s}$. This has no serious consequence for the validity of the main results in that paper. The corresponding gap in the proof of Theorem II.9 loc. cit. is fixed by Proposition B.10 and Theorem B.11 below. Moreover, the assertion of Lemma II.11 loc. cit. should read $V(\mathfrak{t} \otimes A) \cong V(\mathfrak{t}) \otimes A$ for $\mathfrak{t}$ simple finite-dimensional.}

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Proof. Let \( q: \mathfrak{k} \to \mathfrak{k}/\mathfrak{r} \cong \mathfrak{s} \) denote the quotient map. Then Remark \[B.2\](c) implies that \( V(\mathfrak{s}) \) can be identified with a complement of the kernel of \( V(q) \). Clearly, \( \ker V(q) \supseteq \kappa_u(\mathfrak{r}, \mathfrak{k}) \), and since

\[
V(\mathfrak{k}) = \kappa_u(\mathfrak{r}, \mathfrak{k}) + \kappa_u(\mathfrak{s}, \mathfrak{s}) = V_0 + V(\mathfrak{s}),
\]

we see that \( \ker V(q) = V_0 \) and that the sum of \( V(\mathfrak{s}) \) and \( V_0 \) is direct. The decomposition of \( V(\mathfrak{s}) \) follows from Remark \[B.2\](d).

Now we show that the decomposition (16) is invariant under \( \text{Aut}(\mathfrak{k}) \). Let \( \text{Inn}(\mathfrak{k}) \subseteq \text{Aut}(\mathfrak{k})_0 \) denote the normal subgroup of inner automorphisms of \( \mathfrak{k} \). This subgroup acts trivially on \( V(\mathfrak{k}) \) because \( \mathfrak{k} \) acts trivially. Since all Levi complements are conjugate under the group \( \text{Inn}(\mathfrak{k}) \) of inner automorphisms (cf. [Bou89, Ch. I]), we obtain with

\[
\text{Aut}(\mathfrak{k}, \mathfrak{s}) := \{ \varphi \in \text{Aut}(\mathfrak{k}) : \varphi(\mathfrak{s}) = \mathfrak{s} \}
\]

that

\[
\text{Aut}(\mathfrak{k}) = \text{Aut}(\mathfrak{k}, \mathfrak{s}) \cdot \text{Inn}(\mathfrak{k}).
\]

Since \( \text{Inn}(\mathfrak{k}) \) acts trivially on \( V(\mathfrak{k}) \), it remains to see that the decomposition (16) is invariant under \( \text{Aut}(\mathfrak{k}, \mathfrak{s}) \). Clearly, this group preserves the Levi decomposition of \( \mathfrak{k} \), hence the subspaces \( V(\mathfrak{s}) \) and \( V_0 \) of \( V(\mathfrak{k}) \). Moreover, \( \text{Aut}(\mathfrak{s}) \) permutes the simple ideals of \( \mathfrak{s} \), hence preserves the isotypic ideals \( \mathfrak{s}_i^{m_i} \) for each \( i \). This completes the proof.

Remark B.7 (The action of \( \pi_0(\text{Aut}\mathfrak{s}) \) on \( V(\mathfrak{s}) \)) The group \( \text{Aut}(\mathfrak{k}) \) acts on the subspace \( V(\mathfrak{s}) \) on \( V(\mathfrak{k}) \) through the natural homomorphism

\[
\text{Aut}(\mathfrak{k}) \to \text{Aut}(\mathfrak{s}),
\]

obtained from \( \mathfrak{s} \cong \mathfrak{g}/\mathfrak{r} \) and the group \( \text{Aut}(\mathfrak{s})_0 \) act trivially on \( V(\mathfrak{s}) \) (Remark \[B.3\]). The product \( \mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2} \times \ldots \times \mathfrak{S}_{m_r} \) of symmetric groups acts naturally on \( \mathfrak{s} \cong \mathfrak{s}_1^{m_1} \oplus \ldots \oplus \mathfrak{s}_r^{m_r} \) by automorphisms, permuting the simple ideals of \( \mathfrak{s} \), and since each automorphism of \( \mathfrak{s} \) permutes the set of simple ideals, we obtain a semidirect decomposition

\[
\text{Aut}(\mathfrak{s}) \cong \left( \prod_{i=1}^r \text{Aut}(\mathfrak{s}_i)^{m_i} \right) \rtimes \prod_{i=1}^r \mathfrak{S}_{m_i}.
\]
This in turn leads to
\[
\pi_0(\text{Aut}(s)) \cong \left( \prod_{i=1}^{r} \pi_0(\text{Aut}(s_i))^{m_i} \right) \rtimes \prod_{i=1}^{r} \mathcal{G}_{m_i}.
\]

For \(V(s_i) \cong \mathbb{R}\), the invariance of the Cartan–Killing form under all automorphisms of \(s_i\) implies that \(\text{Aut}(s_i)\) acts trivially on \(V(s_i)\). For \(V(s_i) \cong \mathbb{C}\), the same argument implies that the index 2-subgroup of all complex linear isomorphisms acts trivially on \(V(s_i)\), and each antilinear isomorphism \(\varphi \in \text{Aut}(s_i)\) acts on \(V(s_i) \cong \mathbb{C}\) by complex conjugation.

If \(V(s_i)\) is one-dimensional, \(\mathcal{G}_{m_i}\) acts by permutations on \(V(s_i) \cong \mathbb{R}_{m_i}\), and if \(V(s_i) \cong \mathbb{C}\), then \((\mathbb{Z}/2)^{m_i} \rtimes \mathcal{G}_{m_i}\) acts by permutations on \(V(s_i) \cong \mathbb{C}_{m_i}\), combined with complex conjugation in the factors.

### B.2 The universal period map

Let \(\kappa_u : \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})\) be the universal invariant symmetric bilinear form. Then \(C(\kappa)(x, y, z) := \kappa([x, y], z)\) is a \(V(\mathfrak{k})\)-valued 3-cocycle, and the left invariant closed \(V(\mathfrak{k})\)-valued 3-form \(C(\kappa)^l\) on \(K\) specified by \(C(\kappa)^l = C(\kappa)\) defines a period homomorphism

\[
\text{per}_K : \pi_3(K) \to V(\mathfrak{k}), \quad [\sigma] \mapsto \int_{\sigma} C(\kappa)^l = \int_{S^3} \sigma^* C(\kappa)^l
\]

([Ne02a, Lem. 5.7 and Rem. 5.9]). We write \(\Pi_K := \text{im}(\text{per}_K)\) for its image.

To see that this subgroup is fixed by \(\text{Aut}(\mathfrak{k})_0 \cong \text{Aut}(K)_0\), we note that for each \(\varphi \in \text{Aut}(K)\), the relation

\[
\text{per}_K \circ \pi_3(\varphi) = V(L(\varphi)) \circ \text{per}_K
\]

implies that \(V(L(\varphi)) \circ \text{per}_K\) only depends on the class \([\varphi] \in \pi_0(\text{Aut}(K))\). Hence the image of \(\text{per}_K\) is fixed pointwise by \(\text{Aut}(K)_0\).

**Remark B.8** We recall some results on the homotopy groups of finite-dimensional Lie groups \(K\).

(a) If \(q : \hat{K} \to K\) is a covering of Lie groups, then for each \(j > 1\), the induced homomorphism \(\pi_j(q) : \pi_j(\hat{K}) \to \pi_j(K)\) is an isomorphism. This is an easy consequence of the long exact homotopy sequence of the principal \(ker q\)-bundle \(\hat{K}\) over \(K\).
(b) By E. Cartan’s Theorem, \( \pi_2(K) = 1 \) ([Mim95, Th. 3.7]).

(c) Bott’s Theorem asserts that for a compact connected simple Lie group \( K \) we have \( \pi_3(K) \cong \mathbb{Z} \) ([Mim95, Th. 3.9]). A generator of \( \pi_3(K) \) can be obtained from a suitable homomorphism \( \eta: SU_2(\mathbb{C}) \cong S^3 \to K \). More precisely, let \( \alpha \) be a long root in the root system \( \Delta_\mathfrak{k} \) of \( \mathfrak{k} \) and \( \mathfrak{k}(\alpha) \subseteq \mathfrak{k} \) be the corresponding \( \mathfrak{su}_2(\mathbb{C}) \)-subalgebra. Then the corresponding homomorphism \( SU_2(\mathbb{C}) \to K \) represents a generator of \( \pi_3(K) \) ([Bo58]).

**Remark B.9** Let \( K \) be a connected finite-dimensional Lie group, \( C \subseteq K \) a maximal compact subgroup, \( C_0 \) the identity component of the center of \( C \) and \( C_1, \ldots, C_m \) be the connected simple normal subgroups of \( C \). Every compact group is in particular reductive, so that the multiplication map

\[
C_0 \times C_1 \times \ldots \times C_m \to C
\]

has finite kernel, hence is a covering map. As \( C_0 \) is a torus, its universal covering group is a vector space, and therefore \( \pi_3(C_0) \cong \pi_3(\tilde{C}_0) \) is trivial. Since \( K \) is homotopy equivalent to \( C \), this leads with Remark [B.8] to

\[
\pi_3(K) \cong \pi_3(C) \cong \prod_{j=1}^m \pi_3(C_j) \cong \mathbb{Z}^m.
\]

**Proposition B.10** Let \( S \) be a simple connected Lie group with Lie algebra \( \mathfrak{s} \). Then

\[
\Pi_S \cong \begin{cases} 
\mathbb{Z} & \text{for } \mathfrak{s} \not\cong \mathfrak{sl}_2(\mathbb{R}) \\
0 & \text{for } \mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R}),
\end{cases}
\]

and this group is fixed pointwise by the action of \( \text{Aut}(\mathfrak{s}) \) on \( V(\mathfrak{s}) \).

**Proof.** Since \( \pi_3(S) \cong \pi_3(\tilde{S}) \) for the universal covering Lie group \( \tilde{S} \), we may w.l.o.g. assume that \( S \) is 1-connected.

If \( \mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R}) \), then \( S \) is diffeomorphic to \( \mathbb{R}^3 \), so that \( \pi_3(S) \) is trivial and therefore \( \Pi_S \) is trivial. If \( \mathfrak{s} \not\cong \mathfrak{sl}_2(\mathbb{R}) \), then the maximal compact subalgebra \( \mathfrak{c}_s \) is not abelian (cf. [Hel78, Prop. VIII.6.2]), so that the maximal compact subgroup \( C \) of \( S \) is non-abelian, hence contains non-trivial simple factors \( C_1, \ldots, C_m \). In view of Remark [B.9] \( \pi_3(S) \cong \mathbb{Z}^m \) is a non-trivial free group.

For \( K := SU_2(\mathbb{C}) \), pick \( x \in \mathfrak{k} \) with \( \text{Spec}(\text{ad}\, x) = \{ 0, \pm 2i \} \), where we view \( \text{ad}\, x \) as an endomorphism of the complexification \( \mathfrak{k}_C \cong \mathfrak{sl}_2(\mathbb{C}) \). The set of all such elements is a euclidean 2-sphere in the 3-dimensional Lie algebra \( \mathfrak{su}_2(\mathbb{C}) \).
which is an orbit of the adjoint action. Therefore \( v_t := 4\pi^2 \kappa_u (x, x) \in V(\mathfrak{t}) \cong \mathbb{R} \) is well-defined and with Example 2.11 we derive that \( \Pi_K = \mathbb{Z} v_t \).

Since \( \pi_3(S) \) is generated by the homotopy classes of the homomorphisms \( \eta_j : \text{SU}_2(\mathbb{C}) \twoheadrightarrow C_j \) specified in Remark B.8(c), we conclude that \( \Pi_S \subset V(\mathfrak{s}) \) is the subgroup generated by the corresponding elements \( v_1, \ldots, v_m \), coming from the basis elements \( v_j = 4\pi^2 \kappa_u (x_j, x_j) \in V(\mathfrak{c}_j) \), where \( x_j \) denotes an element in a suitable \( \mathfrak{su}_2 \)-subalgebra of the simple ideal \( \mathfrak{c}_j \) of the maximal compact subalgebra \( \mathfrak{c} \) of \( \mathfrak{s} \), which is normalized in such a way that \( \text{Spec}(\text{ad } x_j) = \{ \pm 2i, 0 \} \) holds on the \( \mathfrak{su}_2(\mathbb{C}) \)-subalgebra. The choice of the elements \( x_j \in \mathfrak{c}_j \) and the representation theory of \( \mathfrak{sl}_2(\mathbb{C}) \cong (\mathfrak{su}_2(\mathbb{C}))_{\mathbb{C}} \) imply that all eigenvalues of \( \text{ad } x_j \) on \( \mathfrak{t}_\mathbb{C} \) are contained in \( i\mathbb{Z} \), so that \( \text{tr}((\text{ad } x_j)^2) \in -\mathbb{N}_0 \). Therefore the values of the Cartan–Killing form of \( \mathfrak{s} \) on the \( x_j \) are integral.

If \( \dim V(\mathfrak{s}) = 1 \), then the Cartan–Killing form is universal (Remark B.4), and this already implies that the elements \( v_j \) generate a discrete non-trivial subgroup of \( V(\mathfrak{s}) \). If \( \dim V(\mathfrak{s}) = 2 \), then \( \mathfrak{s} \) is complex and \( \mathfrak{c} \) is a compact real form of \( \mathfrak{s} \), hence in particular simple. Therefore \( \pi_3(S) \cong \pi_3(C) \cong \mathbb{Z} \) (Remark B.8) implies that \( \Pi_S \cong \mathbb{Z} \).

To see that \( \text{Aut}(\mathfrak{s}) \) fixes \( \Pi_S \) pointwise, we observe that if \( \dim V(\mathfrak{s}) = 1 \), then the invariance of the Cartan–Killing form under all automorphisms of \( \mathfrak{s} \) implies that \( \text{Aut}(\mathfrak{s}) \) acts trivially on \( V(\mathfrak{s}) \). If \( \dim V(\mathfrak{s}) = 2 \), then the subgroup \( \text{Aut}_c(\mathfrak{s}) \) of all complex linear automorphisms of \( \mathfrak{s} \) acts trivially on \( V(\mathfrak{s}) \). Let \( \mathfrak{c} \subset \mathfrak{s} \) be a compact real form and \( \tau \in \text{Aut}(\mathfrak{s}) \) be the corresponding antilinear involution. Then \( \tau \) fixes \( \mathfrak{c} \) pointwise, so that the corresponding group automorphism fixes \( C \subset S \) pointwise, hence also the canonical image \( V(\mathfrak{c}) \subset V(\mathfrak{s}) \), generated by \( \Pi_S \cong \Pi_C \). Since \( \text{Aut}(\mathfrak{s}) \cong \text{Aut}_c(\mathfrak{s}) \rtimes \{ \text{id}, \tau \} \), the whole group \( \text{Aut}(\mathfrak{s}) \) fixes \( \Pi_S \) pointwise. □

**Theorem B.11** Let \( S_i \) be a connected Lie group with Lie algebra \( \mathfrak{s}_i \). Then \( \Pi_K \cong \prod_{i=1}^r \Pi_{S_i}^{m_i} \) is a discrete subgroup of \( V(\mathfrak{s}) \subset V(\mathfrak{t}) \), and if \( \varphi_V \in \text{GL}(V(\mathfrak{t})) \) is induced by an automorphism \( \varphi_t \in \text{Aut}(\mathfrak{t}) \), then the image of \( \Pi_K \) in \( V(\mathfrak{t})_{\varphi_V} \) is also discrete.

**Proof.** We may w.l.o.g. assume that \( K \) is 1-connected (Remark B.8). Then we have a Levi decomposition \( K \cong R \times S \), and \( S \cong S_1^{m_1} \times \ldots \times S_r^{m_r} \). The functoriality of the period group and the assignment \( \mathfrak{g} \mapsto V(\mathfrak{g}) \) now implies that \( \Pi_K \cong \Pi_S \cong \Pi_{S_1}^{m_1} \times \ldots \times \Pi_{S_r}^{m_r} \), where \( \Pi_{S_i} \subset V(\mathfrak{s}_i) \) is a cyclic subgroup, hence discrete (Proposition B.10).
From Theorem B.6 we recall the Aut($\mathfrak{t}$)-invariant decomposition $V(\mathfrak{t}) = V_0 \oplus \bigoplus_{i=1}^r V_i$ with $V_i \cong V(s_i)^{m_i}$. We have just seen that the period group is adapted to this decomposition with $\Pi_{S_i}^{m_i} \subseteq V_i$. For $i > 0$, $\varphi_i := \varphi_V|_{V_i}$ acts on $V_i$ as an element of Aut($s_i^{m_i}$), and since Aut($s_i^{m_i}$) acts trivially (Remark B.3), $\text{ord}(\varphi_i) < \infty$ (Remark B.7), so that $(V_i)_{\varphi_i} \cong V_i^{\varphi_i}$, and the projection to the cokernel corresponds to the projection to the subspace of fixed vectors of $\varphi_i$. Since $\varphi_i$ preserves $\Pi_{S_i}^{m_i}$ (Proposition B.10), the image of this group under the projection onto the $\varphi_i$-fixed space is contained in $\frac{1}{\text{ord}(\varphi_i)} \cdot \Pi_{S_i}^{m_i}$, hence discrete.

**Remark B.12** For each $i$, Proposition B.10 implies that the subgroup Aut($s_i^{m_i}$) of Aut($s_i^{m_i}$) fixing all simple ideals acts trivially on $\Pi_{S_i}^{m_i}$. Therefore only the permutation group $S_m$ acts on this discrete subgroup. Thus any automorphism of $\mathfrak{t}$ acts on $\Pi_K \cong \prod_{i=1}^r \Pi_{S_i}^{m_i}$ as an element of the product $S_{m_1} \times \ldots \times S_{m_r}$.

## C Appendix: Central extensions of Lie groups

In this appendix we recall some facts on the integration of Lie algebra 2-cocycles from [Ne02a]. They provide a general set of tools to integrate central extensions of Lie algebras to extensions of connected Lie groups.

Let $G$ be a connected Lie group and $V$ be a Mackey complete space. Further, let $\omega \in Z^2(\mathfrak{g}, V)$ be a $k$-cocycle and $\omega^l \in \Omega^k(G, V)$ be the corresponding left equivariant $V$-valued $k$-form with $\omega^l_1 = \omega$. Then each continuous map $S^k \to G$ is homotopic to a smooth map (cf. [Wo09] or [Ne02a]), and

$$\text{per}_\omega: \pi_k(G) \to V, \quad [\sigma] \mapsto \int_\sigma \omega^l = \int_{S^k} \sigma^* \omega^l = \int_{S^k} \omega(\delta \sigma, \ldots, \delta \sigma),$$

for $\sigma \in C^\infty(S^k, G)$, defines the *period homomorphism* whose values lie in the $G$-fixed part of $V$ ([Ne02a Lem. 5.7 and Rem. 5.9]).

**Theorem C.1** ([Ne02a Prop. 7.6 and Thm. 7.9]) Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. A central Lie algebra extension $\hat{\mathfrak{g}} = V \oplus_\omega \mathfrak{g}$ defined by $\omega \in Z^2(\mathfrak{g}, V)$ integrates to a Lie group extension of some covering group of $G$ if and only if the period group $\Pi_\omega := \text{per}_\omega(\pi_2(G)) \subseteq V$ is discrete. It integrates to an extension of $G$ if and only if the adjoint action of $G$ on $\mathfrak{g}$ lifts to an action of $G$ on $V \oplus_\omega \mathfrak{g}$.
Remark C.2 (a) To calculate period homomorphisms, it is often convenient to use related cocycles on different groups. So, let us consider a morphism \( \varphi: G_1 \to G_2 \) of Lie groups and \( \omega_2 \in Z^2(\mathfrak{g}_2, V) \), \( V \) a trivial \( G_1 \)-module. Then a straightforward argument shows that

\[
\text{per}_{\omega_2} \circ \pi_2(\varphi) = \text{per}_{L(\varphi)^* \omega_2} : \pi_2(G_1) \to V.
\]

(17)

(b) From (a) we obtain in particular for \( \omega \in Z^2(\mathfrak{g}, V) \) and \( \varphi \in \text{Aut}(G) \) the relation

\[
\text{per}_{\omega} \circ \pi_2(\varphi) = \text{per}_{L(\varphi)^* \omega} : \pi_2(G) \to V.
\]

(18)

If, in addition, \( \varphi \) is homotopic to the identity in the sense that \( \varphi = \gamma(1) \) for a curve \( \gamma: [0,1] \to G \) with \( \gamma(0) = \text{id}_G \) for which the map

\[
\tilde{\gamma}: [0,1] \times G \to G, \quad (t,g) \mapsto \gamma(t)(g)
\]

is smooth, then \( \pi_2(\varphi) = \text{id} \) implies that the periods of the 2-cocycle

\[
L(\varphi)^* \omega - \omega
\]

are trivial. In this case we further have a derivation \( D = \varphi'(0) \in \text{der}(\mathfrak{g}) \), and by applying the preceding relation to all automorphisms and taking derivatives in 1, it follows that the periods of the cocycle

\[
\omega_D(x,y) = \omega(Dx,y) + \omega(x, Dy) = \frac{d}{dt} \bigg|_{t=0} L(\varphi_t)^* \omega
\]

vanish.

The following theorem can be found in [MN03, Thm. V.9]:

Theorem C.3 (Lifting Theorem) Let \( q: \hat{G} \to G \) be a central Lie group extension of the 1-connected Lie group \( G \) by the Lie group \( Z \cong \mathfrak{z}/\Gamma_Z \). Let \( \sigma_G: H \times G \to G \), resp., \( \sigma_Z: H \times Z \to Z \) be smooth automorphic actions of the Lie group \( H \) on \( G \), resp., \( Z \) and \( \sigma_{\mathfrak{g}} \) be a smooth action of \( H \) on \( \mathfrak{g} \) compatible with the actions on \( \mathfrak{z} \) and \( \mathfrak{g} \). Then there is a unique smooth action \( \sigma_{\hat{G}}: H \times \hat{G} \to \hat{G} \) by automorphisms compatible with the actions on \( Z \) and \( G \), for which the corresponding action on the Lie algebra \( \mathfrak{g} \) is \( \sigma_{\mathfrak{g}} \).
D Appendix: Some facts on curvature and parallel transport

Let $M$ be a finite-dimensional manifold, $H$ a regular Lie group, $P(M, H, q)$ a principal $H$-bundle over $M$ and $\theta \in \Omega^1(P, \mathfrak{h})$ a principal connection 1-form.

For each piecewise smooth curve $\alpha: [a, b] \to M$, we then have an $H$-equivariant parallel transport map $\text{Pt}(\alpha): P_{\alpha(a)} \to P_{\alpha(b)}$ defined by $\text{Pt}(\alpha). p := \hat{\alpha}(1)$, where $\hat{\alpha}: [a, b] \to P$ is the horizontal lift of $\alpha$ starting in $p$. For a closed curve parallel transport and holonomy are connected by

$$\text{Pt}(\alpha). p_0 = p_0. \mathcal{H}(\alpha).$$

Let $v, w \in T_{m_0}(M)$ and consider an open connected neighborhood $U$ of $m_0$ in $M$ such that $P|_U$ is trivial and there exist smooth vector fields $X, Y \in \mathfrak{V}(U)$ with $X(m_0) = v$ and $Y(m_0) = w$ and $T > 0$ such that for $0 \leq t_i \leq T$ the points

$$\text{Fl}^Y_{-t_4} \circ \text{Fl}^X_{-t_3} \circ \text{Fl}^Y_{t_2} \circ \text{Fl}^X_{t_1}(m_0)$$

are defined and contained in $U$.

For $0 \leq t \leq T$,

$$\gamma(t) := \text{Fl}^Y_{-t} \circ \text{Fl}^X_{-t} \circ \text{Fl}^Y_{t} \circ \text{Fl}^X_{t}(m_0).$$

defines a smooth curve in $M$ with $\gamma(0) = m_0$, $\gamma'(0) = 0$, and

$$\gamma''(0) = 2[X, Y](m_0).$$

([BC64, Thm. 1.4.4]).

We write $\alpha_t: [0, 5t] \to M$ for the curve obtained by concatenating integral curves of $X, Y, -X$ and $-Y$ defined on $[0, t]$ with the reversed curve $\gamma$, so that we obtain a loop in $m_0$ which is piecewise smooth. Note that any piecewise smooth loop can be reparametrized as a smooth loop, so that $\beta(t) := \mathcal{H}(\alpha_t)$ also is the holonomy of a smooth loop, and it is clear that it is a smooth curve in $H$. We claim that

$$\beta'(0) = 0 \quad \text{and} \quad \beta''(0) = 2R(\theta)_{p_0}(\tilde{v}, \tilde{w}),$$

where $\tilde{v} \in T_{p_0}(P)$ denotes the unique horizontal lift of $v \in T_{m_0}(M)$ (cf. [BC64, Thm. 6.1.3]).
Since the bundle $P_U$ is trivial, we may w.l.o.g. assume that $P_U = U \times H$. Then the connection 1-form $\theta$ has the form

$$\theta = p^*_H \kappa_H + \text{Ad}(p_H)^{-1}(p^*_U A),$$

where $A \in \Omega^1(U, \mathfrak{h})$, and $p_H : U \times H \to H$ and $p_U : U \times H \to U$ are the projection maps. After adjusting the trivialization if necessary, we may w.l.o.g. assume that $A(m_0) = 0$, i.e., the subspace $T_{m_0}(M)$ is horizontal in $T_{(m_0,1)}(P)$.

Let $\tilde{X}, \tilde{Y} \in \mathcal{V}(P)^H$ denote the unique horizontal lifts of the vector fields $X, Y$ and $\hat{\gamma}(t) := \text{Fl}^{-1}_t \circ \text{Fl}^{-1}_{-t} \circ \text{Fl}_t \circ \text{Fl}_{-t} (m_0)$, which coincides with $\tilde{\alpha}(4t)$ for the horizontal lift $\tilde{\alpha}$ of $\alpha$, starting in $p_0$. In the product coordinates of $P_U = U \times H$, we now find with $p_0 = (m_0, 1)$:

$$\tilde{\gamma}(t) = (\gamma(t), \zeta(t)),$$

where $\zeta(0) = \zeta'(0) = 0$ and

$$\zeta''(0) = 2\theta([\tilde{X}, \tilde{Y}]) (p_0) = 2d\theta(\tilde{Y}, \tilde{X})(p_0) = 2R(\theta)(\tilde{w}, \tilde{v}).$$

Let $(\gamma(s), \rho(s)), 0 \leq s \leq t$, denote the horizontal lift of the curve $\gamma$ starting in $p_0$. Then $(\gamma, \rho, \beta(t) = (\gamma, \rho \cdot \beta(t))$ is the horizontal lift starting in $p_0, \beta(t)$, and we thus obtain

$$\rho(t) \cdot \beta(t) = \zeta(t).$$

Since $(\gamma, \rho)$ is horizontal, we have $\delta(\rho)_t = -A_{\gamma(t)}(\gamma'(t))$, which leads to $\rho(0) = 1, \rho'(0) = 0$ and further to

$$\rho''(0) = -A_{m_0}(\gamma''(t)) = 0.$$

Hence

$$\beta''(0) = (\rho \cdot \beta)''(0) = \zeta''(0) = 2R(\theta)(\tilde{w}, \tilde{v}).$$

We have thus constructed a family $(\alpha_t)_{0 \leq t \leq T}$ of (piecewise) smooth loops in $m_0$ for which the holonomy defines a smooth curve $\beta(t) = H(\alpha_t)$ in $H$ with $\beta'(0) = 0$ and $\beta''(0) = 2R(\theta)(\tilde{w}, \tilde{v})$. 

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