TOWARD QUANTITATIVE MODEL FOR SIMULATION AND FORECAST OF SOLAR ENERGETIC PARTICLES PRODUCTION DURING GRADUAL EVENTS - II: KINETIC DESCRIPTION OF SEP

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ABSTRACT

Solar Energetic Particles (SEPs) possess a high destructive potential as they pose multiple radiation hazards on Earth and onboard spacecrafts. The present work continues a series started with the paper by Borovikov et al. (2018) describing a computational tool to simulate and, potentially, predict the SEP threat based on the observations of the Sun. Here we present the kinetic model coupled with the global MHD model for the Solar Corona (SC) and Inner Heliosphere (IH), which was described in the first paper in the series. At the heart of the coupled model is a self-consistent treatment of the Alfvén wave turbulence. The turbulence not only heats corona, powers and accelerates the solar wind, but also serves as the main agent to
scatter the SEPs and thus controls their acceleration and transport. The universal character of the turbulence in the coupled model provides a realistic description of the SEP transport by using the level of turbulence as validated with the solar wind and coronal plasma observations. At the same time, the SEP observations at 1 AU can be used to validate the model for turbulence in the IH, since the observed SEPs have witnessed this turbulence on their way through the IH.

Keywords: shock waves—acceleration of particles—Sun: magnetic fields—Sun: coronal mass ejections (CMEs)

1. INTRODUCTION

The kinetic transport of energetic particle population through the inter-planetary space is an important problem in space science. It was studied since the discovery of Galactic Cosmic Rays (GCR), energetic particles originating from beyond the Solar system. A comprehensive summary of the problem can be found in the review by Parker (1965). Although results in the said review are obtained in a different context, some can be readily applied for SEP transport.

In the present paper we discuss the numerical methods and tools to solve the realistic kinetic equations in application to the solar energetic particle acceleration and transport.

The present paper is structured as follows. In Section 2 we state the basic concepts that both theoretical and numerical aspects of the kinetic description of SEP rely upon.

2. BASIC CONCEPTS

2.1. SEP distribution function

As SEP population forms a suprathermal tail of particle distribution in the solar wind, we need to start developing the kinetic treatment from a distribution function of a general form. We char-
acterize SEPs by a (canonical) distribution function $F(R, p, t)$ of coordinates, $R$, and momentum, $p$, as well as time, $t$, such that the number of particles, $dN$, within the elementary volume, $d^3R$, is given by the following normalization integral:

$$dN = d^3R \int d^3p F(R, p, t).$$

In a magnetized moving plasma, it is convenient to consider the distribution function at any given point, $R$, in the co-moving frame of reference, which moves with the local plasma velocity, $u(R, t)$. Also, we introduce spherical coordinates, $(p = |p|, \mu = b \cdot p/p, \varphi)$, in the momentum space with its polar axis aligned with the direction, $b = B/B$, of the magnetic field, $B(R, t)$. Herewith, $\mu$ is the cosine of pitch-angle. The normalization integral in these new variables becomes:

$$dN = d^3R \int_0^\infty p^2 dp \int_{-1}^1 d\mu \int_0^{2\pi} d\varphi F(R, p, \mu, \varphi, t).$$

Using the canonical distribution function, one can also define a gyrotropic distribution function, $f(R, p, \mu, t) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi F(R, p, \mu, \varphi, t)$. This function is designed to describe the particle motion averaged over the phase of gyration about the magnetic field. The isotropic (omnidirectional) distribution function, $f_0(R, p, t) = \frac{1}{2} \int_{-1}^1 d\mu f(R, p, \mu, t)$ is additionally averaged over pitch angle. The normalization integrals are:

$$dN = 2\pi d^3R \int_0^\infty p^2 dp \int_{-1}^1 d\mu f(R, p, \mu, t) = 4\pi d^3R \int_0^\infty p^2 dp f_0(R, p, t)$$

The kinetic equation for the isotropic part of the distribution function, $f_0(R, p, t)$, was introduced in Parker (1965):

$$\frac{\partial f_0}{\partial t} + (u \cdot \nabla) f_0 - \frac{1}{3} (\nabla \cdot u) \frac{\partial f_0}{\partial \ln p} = \nabla \cdot (\kappa \cdot \nabla f_0) + S. \quad (2.1)$$

where $\kappa = D_{xx} b b$ is the tensor of parallel (spatial) diffusion along the magnetic field, $S$ is the source term. In this approximation, the cross-field diffusion of particles is neglected. The Parker Eq. 2.1 captures effects that Interplanetary Magnetic Field (IMF) and other background parameters of the solar wind on the SEP transport and acceleration. The term proportional to the divergence of $u$ is the adiabatic cooling, for $(\nabla \cdot u) > 0$, or (the first order Fermi) acceleration in compression...
or shock waves. In the companion paper Borovikov et al. (2018) we provided preliminary results for the SEP acceleration and transport obtained by solving Eq. 2.1 numerically.

2.2. Flux/Lagrangian Coordinates

Our model of SEP transport and acceleration is based on the assumption that particles don’t decouple from their field lines. In other words, we assume that particle motion in physical space consists of: (a) displacement of particle’s guiding center along some IMF line; and (b) joint advection of both the guiding center and the IMF line together with plasma into which the field is frozen. Mathematically, the method employs Lagrangian coordinates, $R_L$, which stay with advecting fluid elements rather than with fixed positions in space. As each fluid element moves, its Lagrangian coordinates, $R_L$, remain unchanged, while its spatial location, $R (R_L, t)$, changes in time in accordance with the local velocity of plasma, $u(R, t)$:

$$\frac{D R(R_L, t)}{Dt} = u(R, t)$$  \hfill (2.2)

Herewith, the partial time derivative at constant Lagrangian coordinates (also referred to as substantial derivative), $R_L$, is denoted as $\frac{\partial}{\partial t}$, while the notation $\frac{\partial}{\partial t}$ is used to denote the partial time derivative at constant Eulerian coordinates, $R$. The two are related as $\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla$.

Certain terms in Parker Eq. 2.1 as well as other equations considered in this paper may be expressed in terms of the Lagrangian derivatives and spatial derivative along lines ($\partial/\partial s = b \cdot \nabla$) using equations of the plasma motion. Particularly, the continuity equation for the plasma density, $\rho(R, t)$, can be represented as follows:

$$\nabla \cdot u = - \frac{D \ln \rho}{Dt}$$  \hfill (2.3)
Expressing the induction equation using the substantial derivative, $\frac{D}{Dt}$, we obtain

$$\mathbb{I} - \mathbb{b} : \nabla \mathbf{u} = -\frac{D \ln B}{Dt}$$  \hspace{1cm} (2.4)

where $\rho$ is the plasma density, $\mathbb{I}$ is the identity matrix. The time-dependent changes in the distance between two neighboring Lagrangian meshes, $\delta s$, is described by the following evolutionary equation (e.g. Landau and Lifshitz 1959):

$$\frac{D \ln \delta s}{Dt} = \mathbb{b} : \nabla \mathbf{u}$$  \hspace{1cm} (2.5)

Using Eqs. 2.3, 2.4 the latter can be also written as:

$$\frac{D \ln \delta s}{Dt} = \frac{D \ln (B/\rho)}{Dt}$$  \hspace{1cm} (2.6)

Eq. 2.6 may be applied to derive relation between Lagrangian and Eulerian distances, $s_L$ and $s$. With the initial condition $\frac{\partial s}{\partial s_L} = 1$ at $t = 0$ we have:

$$\frac{\partial s(s_L, t)}{\partial s_L} = \frac{B(s_L, t)\rho(s_L, 0)}{B(s_L, 0)\rho(s_L, t)}$$  \hspace{1cm} (2.7)

From the solenoidal constraint, $\nabla \cdot \mathbf{B} = 0$, one can also find that:

$$\nabla \cdot \mathbf{b} = -\frac{\ln B}{\partial s}$$  \hspace{1cm} (2.8)

We apply the formalism presented above whenever possible. For example, Parker equation may be rewritten as follows:

$$\frac{D f_0}{Dt} + \frac{1}{3} \frac{D \ln \rho}{Dt} \frac{\partial f_0}{\partial \ln p} = \nabla \cdot (\mathbf{z} \cdot \nabla f_0) + S$$  \hspace{1cm} (2.9)

Such formulation of mathematical problem is particularly convenient for translating it into a numerical model and has been used in development of M-FLAMPA (see Section 7.1).

3. FOCUSED TRANSPORT EQUATION AND DIFFUSIVE LIMIT
As mentioned in Section 2.1 the Parker equation captures major effects that the background has on SEP population. The Parker equation was used to develop a Diffuse Shock Acceleration (DSA) theory (Axford et al. 1977; Krymskii 1977; Bell 1978a,b; Blandford and Ostriker 1978; Axford 1981), which predicts the power-law spectrum of galactic cosmic rays, close to the observed one. However, due to pitch-angle dependent part of the distribution function not being featured in the Parker equation explicitly, its importance to formulation of DSA may be lost in the context.

In this section we consider a distribution function \( f(R, p, \mu, t) \) and demonstrate how diffusive behavior derives from its pitch-angle-dependant part.

When pitch angles of particles are taken into account, one needs to consider the appropriate scattering in the momentum space. The equation for a non-relativistic gyrotropic distribution function \( f(R, p, \mu, t) \) can be found in, for example, Skilling (1971):

\[
\frac{\partial f}{\partial t} + (u + \mu v b) \cdot \nabla f + \\
\left[ \frac{1-3\mu^2}{2} (bb : \nabla u) - \frac{1-\mu^2}{2} (\nabla \cdot u) - \frac{\mu}{v} \left( b \cdot \frac{Du}{Dt} \right) \right] \frac{\partial f}{\partial \ln p} + \\
\frac{1-\mu^2}{2} \left[ v (\nabla \cdot b) - 3\mu (bb : \nabla u) + \mu (\nabla \cdot u) - \frac{2}{v} \left( b \cdot \frac{Du}{Dt} \right) \right] \frac{\partial f}{\partial \mu} = \frac{\delta f}{\delta t} + S, \tag{3.1}
\]

The particle scattering rate, \( \frac{\delta f}{\delta t} \), in this model is due to the particle interaction with the Alfvén wave turbulence. An important physical effect related to particles’ pitch angle distribution is the focusing effect (Earl 1976, and references therein), also referred to as focused transport. The effect takes places under conditions that constrain pitch-angle scattering across \( \mu=0 \). In the extreme case, when particles can’t change the direction of their propagation along their field lines, the whole population is effectively split into two independent hemispheric subpopulations, one of particles propagating inward, the other of particles propagating outward. The implications of such splitting have been explored in Isenberg (1997). Effects of interaction of particles with solar wind
plasma and IMF such as adiabatic cooling/heating on the focused transport have been studied in, for example, Ruffolo (1995). A detailed view on different aspects of evolution of distribution of particles propagating along magnetic-field lines, i.e. convection, cooling/heating, and focusing, can be found in Kóta and Jokipii (1997).

Relations in Eqs. 2.3, 2.4 and 2.8 allow a new treatment of Eq. 3.1 for acceleration and field-aligned transport of SEPs (Kóta and Jokipii 2004; Kóta et al. 2005):

\[
\frac{Df}{Dt} + v_B \frac{\partial f}{\partial s} + \left[ \frac{1}{3} \frac{D \ln \rho}{Dt} + \frac{1 - 3 \mu^2}{6} \frac{D \ln (B^3/\rho^2)}{Dt} - \frac{\mu}{v} b \cdot \frac{Du}{Dt} \right] \frac{\partial f}{\partial p} + \frac{1 - \mu^2}{2} \left[ -v \frac{\partial \ln B}{\partial s} + \mu \frac{D \ln (\rho^2/B^3)}{Dt} - \frac{2}{v} b \cdot \frac{Du}{Dt} \right] \frac{\partial f}{\partial \mu} = \left( \frac{\delta f}{\delta t} \right)_{\text{scat}} \tag{3.2}
\]

Or, in the conservative form, which is convenient both for solving the equation numerically using the conservative scheme and for analytical derivations:

\[
\frac{Df}{Dt} + v_B \frac{\partial f}{\partial s} \left\{ \frac{1 - \mu^2}{2B} \frac{\partial f}{\partial \mu} + \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 \left( \frac{1 - 3 \mu^2}{6} \frac{D \ln (B^3/\rho^2)}{Dt} - \frac{\mu}{v} b \cdot \frac{Du}{Dt} \right) f - \frac{2}{v} b \cdot \frac{Du}{Dt} \right] \right\} + \frac{1}{3} \frac{D \ln \rho}{Dt} \frac{\partial f}{\partial \mu} = \left( \frac{\delta f}{\delta t} \right)_{\text{scat}} \tag{3.3}
\]

The diffusive limit of Eq. 3.3 is less accurate but widely used. In general case, scattering integral may be applied (see, e.g. Sokolov et al. 2006) in the Fokker-Planck form (note that $D_{\mu \mu} = D_{p \mu}$):

\[
\left( \frac{\delta f}{\delta t} \right)_{\text{scat}} = \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 \left( D_{pp} \frac{\partial f}{\partial p} + D_{p \mu} \frac{\partial f}{\partial \mu} \right) \right] + \frac{\partial}{\partial \mu} \left[ D_{\mu p} \frac{\partial f}{\partial p} + D_{\mu \mu} \frac{\partial f}{\partial \mu} \right] \tag{3.4}
\]

One can assume the particle speed to be large compared to the Alfvén speed, as well as the plasma speed $u \ll v$, and suppose that $D_{\mu \mu}^{-1}$ is small compared to any hydrodynamic time. Under these assumption one can treat the pitch-angle dependant part of the distribution function as a small correction, $f_1$, to its isotropic part, $f_0$. In other words, $f(R, p, \mu, t) = f_0(R, p, t) + f_1(R, p, \mu, t)$, where $f_1 \ll f_0$. To obtain the evolutionary equation for $f_0$ (i.e. the Parker equation, Eq. 2.1) let us
average Eq. 3.3 with respect to the particle pitch angle:

\[
\frac{Df_0}{Dt} + B \frac{\partial}{\partial s} \left[\frac{v}{B} \left\langle \frac{(1 - \mu^2)}{2} \frac{\partial f_1}{\partial \mu} \right\rangle_{\mu} \right] + \frac{1}{3} \frac{D \ln \rho}{Dt} p \frac{\partial f_0}{\partial p} = \\
= \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 \left( D_{pp} \frac{\partial f_0}{\partial p} + \left\langle D_{p\mu} \frac{\partial f}{\partial \mu} \right\rangle_{\mu} \right) \right],
\]

(3.5)

where \( \langle ... \rangle_{\mu} = \frac{1}{2} \int d\mu (...). \) The perturbation of the distribution function, \( \partial f_1 / \partial \mu, \) may be found by claiming that the flux along \( \mu \) coordinate in Eq. 3.3 vanishes. Keeping in the expression for this flux only large terms, proportional to the scattering frequency or particle speed, we find:

\[
\frac{\partial f_1}{\partial \mu} = -v \frac{1 - \mu^2}{2D_{\mu\mu}} \frac{\partial f_0}{\partial s} - \frac{D_{p\mu} \partial f_0}{D_{\mu\mu} \partial p}.
\]

(3.6)

The particle flux, \( J, \) may be found by averaging the parallel velocity,

\[
J = \langle \mu v f_1 \rangle_{\mu} = v \left\langle \frac{(1 - \mu^2)}{2} \frac{\partial f_1}{\partial \mu} \right\rangle_{\mu} = -D_{zz} \frac{\partial f_0}{\partial s} - \frac{1}{3} \tilde{V} p \frac{\partial f_0}{\partial p},
\]

with the following expressions for a spatial diffusion coefficient, \( D_{zz}, \) and average ion speed, \( \tilde{V}: \)

\[
D_{zz} = v^2 \left\langle \frac{(1 - \mu^2)^2}{4D_{\mu\mu}} \right\rangle_{\mu}, \quad \tilde{V} = \frac{3v}{p} \left\langle \frac{(1 - \mu^2)}{2D_{\mu\mu}} \right\rangle_{\mu}.
\]

(3.7)

From here, we obtain the equation of the diffuse approximation,

\[
\frac{Df_0}{Dt} + B \frac{\partial}{\partial s} \left( \frac{J}{B} \right) + \frac{p D \ln \rho \partial f_0}{3 Dt \partial p} = \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 \left( D_{F2} \frac{\partial f_0}{\partial p} - \frac{p \tilde{V}}{3} \frac{\partial f}{\partial s} \right) \right],
\]

(3.8)

where the second order Fermi acceleration coefficient is defined as \( D_{F2} = \left\langle D_{pp} - \frac{D_{\mu\mu}^2}{D_{\mu\mu}} \right\rangle_{\mu}. \) Eq. 3.8 reduces to the Parker equation 2.1 in Lagrangian coordinates, if \( D_{p\mu} = D_{pp} \equiv 0. \)

4. WAVE-PARTICLE INTERACTION

The kinetic equation of SEP propagation and acceleration includes pitch-angle scattering, which plays a crucial role. Specifically, according to DSA, during gradual SEP events particle acceleration occurs near the Sun at the CME-driven shock waves. Fast DSA requires that particles experience frequent scattering back and forth across the shock-wave front. This scattering may be caused
by the turbulence preexisting in the solar wind or, unless the shock wave is entirely perpendicular, it may be enhanced by the Alfvén waves that are generated by the accelerated particles streaming from the shock (e.g. Bell 1978a,b; Lee 1983). Therefore, a complete SEP model needs to be coupled with a realistic model of Alfvén turbulence, including the self-excited one, as well as a model of particle transport in realistic turbulent IMF.

Within the quasi-linear (QL) approach, the turbulence is thought of as an ensemble of linear circularly polarized Alfvén waves with a harmonic electric field:

$$\delta \mathbf{E} = (E_x, E_y) = \delta E \left( \cos(kz - \omega t), \pm \sin(kz - \omega t) \right) = \delta E (1, \pm i) \exp(-i\omega t + i k z), \quad (4.1)$$

where $\delta E$ is the field’s amplitude. Hereafter, only the real part is implied in complex expressions for real physical quantities. $z$-axis of the Cartesian coordinate frame, $(x, y, z)$, is aligned with the magnetic field direction, $b$. Herewith, while the frequency, $\omega$, is always positive, the wave number, $k$, is positive or negative for the wave modes propagating parallel or anti-parallel to the magnetic field respectively. A choice of sign of $\delta E_y$ accounts for the two types of circular polarization. Thus, there are 4 distinct wave modes. We denote quantities for measured for each mode with index $\sigma = 1, 4$. For example, the phase speed is $V_\sigma = \omega / k$. Note that $V_\sigma$ has the same sign as $k$.

The oscillating magnetic field, $\delta \mathbf{B} = (b \times \delta \mathbf{E}) / V_\sigma$, may be found from the induction equation, which gives us an expression for the magnetic field amplitude, $\delta B$, in terms of that for the electric field:

$$\left( \delta B \right)^2 = \frac{(\delta E)^2}{V_\sigma^2}, \quad (4.2)$$

as well as an expression for the Lorentz force:

$$\mathbf{F}_L^{(\omega)} = qZ_i (\delta \mathbf{E} + \mathbf{v} \times \delta \mathbf{B}) = qZ_i \left[ \left( 1 - \frac{v_\parallel}{V_\sigma} \right) \delta \mathbf{E} + \frac{(v_\perp \cdot \delta \mathbf{E})}{V_\sigma} \mathbf{b} \right], \quad (4.3)$$

$qZ_i$ being the ion charge and $q = |q|$ being the elementary charge. The effect of this force on a
distribution function $F(R, p_\perp, p_\parallel, t)$ is described by the term $F_L \cdot \frac{\partial F}{\partial p}$ in the Boltzmann equation,

$$F_L^{(w)} \cdot \frac{\partial F}{\partial p} = qZ_i \delta E_L \cdot G\{F\}, \quad G\{f\} = \left(1 - \frac{v_\parallel}{V_\sigma}\right) \frac{\partial F}{\partial p_\perp} + \frac{v_\perp}{V_\sigma} \frac{\partial F}{\partial p_\parallel}, \quad (4.4)$$

where the differential operator, $G\{F\}$, is only by a numerical factor different from that used in Ng et al. (2003). The perpendicular components may be expressed terms of the polar angle, $\varphi$:

$$p_\perp = p_\perp (\cos \varphi, \sin \varphi), \quad \frac{\partial f}{\partial p_\perp} = \frac{\partial f}{\partial p_\perp} (\cos \varphi, \sin \varphi) + \frac{1}{p_\perp} \frac{\partial f}{\partial \varphi} (-\sin \varphi, \cos \varphi). \quad (4.5)$$

Combining Eqs. 4.1, 4.4 and B3 we obtain:

$$F_L^{(w)} \cdot \frac{\partial F}{\partial p} = qZ_i \delta E \exp(-i\omega t + ikz \pm i\varphi) \left[\left(1 - \frac{v_\parallel}{V_\sigma}\right) \left(\frac{\partial F}{\partial p_\perp} \pm \frac{i}{p_\perp} \frac{\partial F}{\partial \varphi}\right) + \frac{v_\perp}{V_\sigma} \frac{\partial F}{\partial p_\parallel}\right]. \quad (4.6)$$

4.1. Kinetic Response Function

The perturbed ion distribution function satisfies the Boltzmann equation:

$$\frac{\partial}{\partial t} (f + \delta f) + (v \cdot \nabla) (f + \delta f) - \omega_{ci} \frac{\partial}{\partial \varphi} (f + \delta f) + F_L^{(w)} \cdot \frac{\partial F}{\partial p} (f + \delta f) = 0, \quad (4.7)$$

where $\omega_{ci} = \frac{qZ_i B}{m_i}$ is ion-cyclotron frequency, $\delta f$ is the perturbation of the distribution function due to the turbulence. Naturally, if we keep only the term that are of zeroth order in $\delta E$, Eq. 4.7 yields the equation for the unperturbed distribution function:

$$\frac{\partial f}{\partial t} + (v \cdot \nabla)f - \omega_{ci} \frac{\partial f}{\partial \varphi} = 0, \quad (4.8)$$

By definition from Section 2.1, $\frac{\partial f}{\partial \varphi} = 0$. In other words, the solution of Eq. 4.8 is a gyrotropic function, $f(p_\perp, p_\parallel)$. To evaluate both ion scattering by Alfvén waves and wave excitation one needs to find the perturbation, $\delta f$, of the distribution function. In the first order approximation, $\delta f$ obeys the following equation:

$$\frac{\partial \delta f}{\partial t} + (v \cdot \nabla) \delta f - \omega_{ci} \frac{\partial \delta f}{\partial \varphi} = -qZ_i \delta E_L \exp(-i\omega t + ikz \pm i\varphi)G\{f\}, \quad (4.9)$$
where:

\[
G\{\psi\} = \left(1 - \frac{v_y}{V_\sigma}\right) \frac{\partial \psi}{\partial p_\perp} + \frac{v_y}{V_\sigma} \frac{\partial \psi}{\partial p_\parallel} = \sqrt{1 - \mu^2} \left[ \frac{\partial \psi}{\partial p} + \left(\frac{1}{m_i V_\sigma} - \frac{\mu}{p}\right) \frac{\partial \psi}{\partial \mu} \right],
\] (4.10)

is a linear differential operator acting on a gyrotrropic function \(\psi(p_\perp, p_\parallel)\). Eq. 4.9 can be solved:

\[
\delta f = \frac{q Z_i \delta E}{i (\omega - k v_\parallel \pm \omega_{ci})} \exp(-i\omega t + ik z \pm i \varphi) G\{f\}.
\] (4.11)

### 4.2. Excitation of Turbulence

The self-generated Alfvén waves produced in the vicinity of a shock-wave front have been demonstrated to have important consequences for SEP elemental abundance variations (Ng et al. 1999; Tylka et al. 1999) and the evolution of SEP anisotropies (Reames et al. 2001).

The dispersion relation, i.e. relation between wave frequency, \(\omega\), and wave number, \(k\), is:

\[
\left(\frac{ck}{\omega}\right)^2 = \varepsilon_{r(l)}(k, \omega),
\] (4.12)

where \(r\) and \(l\) denote right and left polarizations. Here, \(c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\) is the speed of light, \(\varepsilon_0, \mu_0\) are the vacuum dielectric and magnetic permeabilities and \(\varepsilon_{r(l)}(k, \omega)\) is a dielectric response function, which is complex in general. Eq. 4.12 determines the wave phase speed, \(V_\sigma = \omega/k = \pm c/\sqrt{\varepsilon_{r(l)}}\).

For low frequency Alfvén waves, \(\omega \ll \omega_{ci}\), the dispersion relation gives: \(|V_\sigma| = V_A = \sqrt{B^2/\mu_0 \rho}\).

However, for higher frequency \(V_\sigma\) depends on both \(|k|\) and polarization. A small wave growth rate due to interaction with energetic ions can be expressed in terms of the small imaginary part of the dielectric response function via Eq. 4.12 (see Ichimaru 1973): \(-2\Im(\omega)c^2/(\omega V_\sigma^2) = \Im(\varepsilon_{r(l)})\). For a harmonic time-dependence for the wave amplitude \(\propto \exp(-i\omega t) = \exp[-i\Re(\omega)t + \Im(\omega)t]\), the wave intensity (\(\propto\) square of wave amplitude) grows/decays as \(\propto \exp[\gamma_{r(l)}(k)t]\), where:

\[
\gamma_{r(l)}(k) = 2\Im(\omega) \approx -\Im(\varepsilon_{r(l)})\omega V_\sigma^2/c^2.
\]

The imaginary part of the dielectric response function can be conveniently expressed in terms of
the conductivity, \( \epsilon_r(l) = 1 + \frac{\Sigma_{\nu}(l)}{\omega_{ci}} \), so that \( \gamma_r(l) = -\Re(\Sigma_{\nu}(l)) \mu_0 V^2 \). The contribution from ions to the conductivity can be found from Eq. 4.11 by calculating the current density, \( j_\perp = qZ_i \int d^3p v_\perp \delta f \).

Averaging \( \delta f \) over \( \varphi \) using the easy-to-derive formula, \( \frac{1}{2\pi} \int_0^{2\pi} d\varphi v_\perp e^{\pm i\varphi} = \frac{v_\perp}{2} (1, \pm i) \), shows that the current is parallel to \( \delta E_\perp \) (see Eq 4.1):

\[
\Sigma_\perp = \int d^3p \frac{q^2Z_i^2}{2i} \frac{v_\perp G\{f\}}{\omega - kv_\parallel \pm i\omega_{ci}}.
\] (4.13)

In agreement with Eq. 7.59 in Ichimaru (1973), a formula for the growth rate is as follows:

\[
\gamma_r(l) = \int d^3p \frac{q^2Z_i^2}{2} \frac{v_\perp G\{f\}}{\omega - kv_\parallel \pm i\omega_{ci}} \left( 1 - \frac{1}{k v_\parallel - \omega \mp i\omega_{ci} - i\omega} \right) v_\perp G\{f\}.
\] (4.14)

The pole in Eq. 4.14 should be bypassed using the Landau rule (e.g. Ginzburg and Rukhadze 1975), so that Eq. 4.14 can be re-written in terms of the Dirac \( \delta \)-function as follows:

\[
\gamma_\sigma = \int d^3p K_\sigma(|k|, p) v_\perp G\{f\},
\] (4.15)

\[
K_\sigma(|k|, p) = \frac{\pi}{2} q^2Z_i^2 \mu_0 V_\perp^2 \delta (|k| (\mu v - V_\sigma) - g_\sigma \omega_{ci}).
\] (4.16)

Here, \( g_\sigma = \pm 1 \), where index \( \sigma = 1, 2, 3, 4 \) enumerates all combinations of signs of \( V_\sigma \) and \( g_\sigma \).

Note, Eq. 4.16 is only valid in the QL approximation, whereas Eq. 4.15 is general and holds true in the non-linear theory (see Ng et al. 2003). An explicit expression in spherical coordinates reads:

\[
\gamma_\sigma = 2\pi \int p^2 dp d\mu K_\sigma(|k|, p) v (1 - \mu^2) \left[ \frac{\partial f}{\partial p} + \left( \frac{1}{m_\perp V_\sigma} - \frac{\mu}{p} \right) \frac{\partial f}{\partial \mu} \right]
\] (4.17)

4.3. Particle Scattering

The influence of the Alfvén turbulence on the supra-thermal particles can be described via the collision integral, in the second order approximation of Eq. 4.7 for the gyration-averaged distribution function, \( f \):

\[
\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f - \omega_{ci} \frac{\partial f}{\partial \varphi} = \left( \frac{\delta f}{\delta t} \right)_{\text{scat}} = -F_L^{(w)} \cdot \frac{\partial \delta f}{\partial \mathbf{p}},
\] (4.18)
where the bi-linear product of rapidly oscillating multipliers should be time-averaged in a usual manner as: \( \left( \frac{\delta f}{\delta t} \right)_{\text{scat}} = -\frac{qZi}{2}\Re (\delta E_\perp \cdot G\{\delta f\}) \), the superscript asterisk denoting the complex conjugation. On substituting \( \delta f \) from Eq. 4.11 and using 4.6 the scattering integral becomes:

\[
\left( \frac{\delta f}{\delta t} \right)_{\text{scat}} = G^T \left\{ \Re \left( \frac{q^2Z_i^2(\delta E)^2}{2t(kv_\parallel - \omega \mp \omega_{ci} - i0)} \right) G\{f\} \right\},
\]

where

\[
G^T\{\psi\} = G\{\psi\} \pm \left( \frac{ie^{\pm i\varphi}}{p_\perp} \right) \left[ 1 - \frac{v_\parallel}{V_\sigma} \right] \frac{\partial (e^{\pm i\varphi} \psi)}{\partial \varphi} = \left( 1 - \frac{v_\parallel}{V_\sigma} \right) \frac{1}{p_\perp} \frac{\partial (p_\perp \psi)}{\partial p_\perp} + \frac{v_\perp}{V_\sigma} \frac{\partial \psi}{\partial p_\parallel} =
\]

\[
= \frac{1}{p^2} \frac{\partial (p^2 \sqrt{1 - \mu^2} \psi)}{\partial p} + \frac{\partial}{\partial \mu} \left[ \left( \frac{\sqrt{1 - \mu^2}}{m_iV_\sigma} - \frac{\mu \sqrt{1 - \mu^2}}{p} \right) \psi \right]
\]

(4.20)
is another operator acting on a gyrotropic function, \( \psi (p_\perp, p_\parallel) \). This operator differs from \( G\{\psi\} \), because \( \delta f \) in contrast with \( f \) includes \( \varphi \)-dependent multiplier, \( e^{\pm i\varphi} \). The electric field amplitude may be expressed in terms of that of magnetic field using 4.2:

\[
(\delta E)^2 = V_\sigma^2(\delta B)^2.
\]

Since the energy density in the Alfvén wave is twice that of the magnetic field, \((\delta B)^2/2\mu_0\), the quantity \((\delta B)^2/\mu_0\) is the energy density in the Alfvén wave, which for a turbulent spectrum of harmonics can be represented as the integral over spectrum and sum over four wave branches: \((\delta B)^2/\mu_0 = \sum_\sigma \int_0^\infty d|k| I_\sigma(|k|) \). We arrive at the particle scattering operator in the QL approximation:

\[
\left( \frac{\delta f}{\delta t} \right)_{\text{scat}} = \sum_\sigma G^T \left\{ \int_0^\infty d|k| I_\sigma(|k|) K_\sigma'(|k|, p) G\{f\} \right\},
\]

(4.21)

where the contribution from the pole can be again expressed in terms of the Dirac function:

\[
K_\sigma'(|k|, p) = \frac{\pi}{2} q^2Z_i^2\mu_0V_\sigma^2 \delta (|k| (\mu v - V_\sigma) - g_\sigma \omega_{ci}).
\]

(4.22)
The kernel \( K_\sigma' \) in Eq. 4.21 appears to be identical to \( K_\sigma \) in Eq. 4.16. Such relation is to be expected since the kernels must be coupled due to energy conservation in the system comprising all ions and
all waves (see Ng et al. 2003). The latter can be verified using a remarkable conjugation property:

$$\int d^3 p \psi_1 G\{\psi_2\} = - \int d^3 p \psi_2 G^T\{\psi_1\},$$

(4.23)

which is true for any pair of gyrotropic functions $\psi_1$ and $\psi_2$. Using 4.21,B10 and an easy-to-check identity, $G\{\mathcal{E}(p)\} = v_\perp$, one can express the total ion energy growth rate due to scattering:

$$\int d^3 p \mathcal{E}(p) \left( \frac{\delta f}{\delta t} \right)_{\text{scat}} = - \sum_\sigma \int d^3 p v_\perp \int_0^\infty d|k| I_\sigma(|k|) K'_\sigma(|k|, p) G\{f\},$$

(4.24)

$\mathcal{E}(p)$ being the ion energy, relativistic in general case. In turn, the total wave energy growth rate may be formulated in terms of $\gamma_\sigma$ from Eq. 4.15:

$$\sum_\sigma \int d|k| I_\sigma(|k|) \gamma_\sigma(|k|) = \sum_\sigma \int d|k| I_\sigma(|k|) \int d^3 p v_\perp K_\sigma(|k|, p) G\{f\}$$

The total energy conservation is controled by the following equation:

$$\sum_\sigma \int d|k| I_\sigma(|k|) \gamma_\sigma(|k|) + \int d^3 p \mathcal{E}(p) \left( \frac{\delta f}{\delta t} \right)_{\text{scat}} = 0,$$

(4.25)

which holds as long as the following relation between the kernels of the integrals is fulfilled:

$$K_\sigma(|k|, p) = K'_\sigma(|k|, p).$$

Note, Eq. 4.25 is much more general than any existing model for wave generation and/or particle scattering. For comparison, a similar formula was obtained by Ng et al. (2003) using the non-linear growth rate for Alfvén turbulence. As long as Eq. 4.25 is true, the momentum conservation can be also proven based upon another easy-to-check identity, $G\{p_\parallel\} = \frac{v_\parallel}{V_\sigma}$:

$$\sum_\sigma \int d|k| \frac{I_\sigma(|k|)}{V_\sigma} \gamma_\sigma(|k|) + \int d^3 p p_\parallel \left( \frac{\delta f}{\delta t} \right)_{\text{scat}} = 0,$$

(4.26)

Using the conjugation property Eq. B10 one can also prove that the isotropic part of the distribution function, $f_0(\mathcal{E})$ always contributes to dissipation rather than excitation for all Alfvén wave branches as long as $df_0/d\mathcal{E} < 0$. 
The expression for the collision integral for waves of a given branch becomes very simple and easy to compute in a special frame of reference moving with the phase speed of the wave. A transformed particle velocity in this frame of reference is \( v_\sigma = v - V_\sigma b \), so that the distribution function, \( f(\mathbf{R}, p_\perp, p_\parallel, t) \), transforms as follows: 
\[
\begin{align*}
  f(\mathbf{R}, p_\perp, p_\parallel + m_\sigma V_\sigma, t) & \quad, (4.27)
\end{align*}
\]
being taken at constant \( p_\sigma \). Eq. 4.15 in this frame of reference reads:
\[
\begin{align*}
  \gamma_\sigma = 2\pi \int p_\sigma^2 dp_\sigma d\mu_\sigma \frac{\pi \omega_{ci}^2}{2 (B^2 / \mu_0)} \delta (|k| \mu_\sigma v_\sigma - g_\sigma \omega_{ci}) \left[ (1 - \mu_\sigma^2) m_\sigma V_\sigma \frac{\partial f}{\partial \mu_\sigma} \right], \quad (4.28)
\end{align*}
\]
If each term in the collision integral is calculated in frame of reference moving with the Alfvén corresponding Alfvén wave, the use of Eq. 4.27 gives:
\[
\begin{align*}
  \left( \frac{\delta f}{\delta t} \right)_{scat} = \sum_\sigma \frac{\partial}{\partial \mu_\sigma} \left( D_\sigma^{\mu\mu} \frac{\partial f}{\partial \mu_\sigma} \right), \quad (4.29)
\end{align*}
\]
where
\[
\begin{align*}
  D_\sigma^{\mu\mu} = \frac{\pi \omega_{ci}^2}{2 (B^2 / \mu_0)} (1 - \mu_\sigma^2) \int_0^\infty dk I_\sigma(k) \delta (|k| \mu_\sigma v_\sigma - g_\sigma \omega_{ci}) \left( \frac{\omega_{ci}}{v_\sigma |\mu_\sigma|} \right), \quad (4.30)
\end{align*}
\]
In the QL approximation, the integral by \( k \) can be taken using the presence of \( \delta \)-function in \( K_\sigma \) (see Eq. 4.16). Thus, for given \( \sigma \) and \( \mu_\sigma \), Eq. 4.30 becomes (Jokipii 1966; Lee 1982, 1983):
\[
\begin{align*}
  D_\sigma^{\mu\mu} = \frac{\pi \omega_{ci}^2}{2 (B^2 / \mu_0)} (1 - \mu_\sigma^2) \left( \frac{\omega_{ci}}{v_\sigma |\mu_\sigma|} \right) I_\sigma \left( \frac{\omega_{ci}}{v_\sigma |\mu_\sigma|} \right), \quad (4.31)
\end{align*}
\]
for two wave branches with \( g_\sigma = \text{sign}(\mu_\sigma) \), while for the two other branches \( D_\sigma^{\mu\mu} \) vanishes.

5. APPLICATION TO THE DIFFUSIVE LIMIT

Since both types of diffusion, spatial and pitch-angle, are different representations of the same physical process, scattering on the magnetic field irregularities, the spatial diffusion coefficient
along the magnetic field, $D_{xx}$, is expressed in terms of $D_{\mu\mu}$ (Jokipii 1966; Earl 1974):

$$D_{xx} = \frac{v^2}{8} \int_{-1}^{1} \frac{(1 - \mu^2)^2}{D_{\mu\mu}} d\mu$$  \hspace{1cm} (5.1)

In turn, the pitch-angle diffusion coefficient, $D_{\mu\mu}$, may be expressed in terms of the Alfvén wave turbulence spectrum, as discussed in Section 4.3. In the QL approximation, using Eqs. 4.29 and 4.31 one can obtain a closed form of Eq. 3.7 for the spatial diffusion coefficient in the diffusive approximation:

$$D_{xx} = \frac{v^3 B^2}{2\pi \mu_0 \omega_{ci}^2} \int_{-1}^{1} \frac{(1 - \mu^2) |\mu|}{I_- \left(\frac{\omega_{ci}}{v|\mu|}\right) + I_+ \left(\frac{\omega_{ci}}{v|\mu|}\right)} d\mu. \hspace{1cm} (5.2)$$

For the two wave branches contributing to Eq. 4.31, the propagation directions are opposite for each $\mu$. We can assume that in the Alfvén wave turbulence the left and right polarized waves are balanced and that their total wave energy for a given $k$ equals $I_+(k)$ for waves propagating along the field direction ($V_\sigma = +V_A$) and $I_- (|k|)$ for the oppositely propagating waves ($V_\sigma = -V_A$). One can notice that under this assumption for any positive $\mu$, hence, for a given $g_\sigma = 1$, the contribution to Eq. 4.31 is proportional $\frac{1}{2} \left[I_- \left(\frac{\omega_{ci}}{v|\mu|}\right) + I_+ \left(\frac{\omega_{ci}}{v|\mu|}\right)\right]$, i.e. the half of the total wave spectral energy, while the other half would contribute to scattering the particles with the negative $\mu$. Herewith, we consider only the high-energy particles with $v_\sigma \gg V_A$ and thus neglect the difference between $v_\sigma$ and $v$, which allows us to write the total scattering rate as follows:

$$D_{\mu\mu} = \frac{\pi \omega_{ci}}{4 (B^2 / \mu_0)} \left(1 - \mu^2\right) \left[\frac{\omega_{ci}}{v|\mu|}\right] \left[I_- \left(\frac{\omega_{ci}}{v|\mu|}\right) + I_+ \left(\frac{\omega_{ci}}{v|\mu|}\right)\right], \hspace{1cm} (5.3)$$

In terms of an integral over the turbulence spectrum, the spatial diffusion coefficient can be written as:

$$D_{xx} = \frac{v B^2}{\pi \mu_0} \int_{k_r}^{\infty} \frac{dk \left(k^2 - k_r^2\right)}{k^5 \left[I_- (k) + I_+ (k)\right]}, \hspace{1cm} (5.4)$$

where the resonant wave number, $k_r$, is the inverse of the Larmor radius, i.e., $k_r = eZ_i B/p$, and
One can also use Eq. 3.6 to evaluate the pitch-angle dependence of the distribution function in the expression for the wave growth rate (see Eq. 4.28). Again, by neglecting the difference between \( \mu \) and \( \mu_\sigma \), one obtains:

\[
\gamma_{\sigma} = 2\pi \int dp d\mu \mu^2 (1 - \mu^2) \left(-kv^2 \frac{1 - \mu^2}{2D_{\mu\mu}} \frac{\partial f_0}{\partial s}\right) K_\sigma(k; \omega, \mathbf{p}).
\] (5.5)

In the QL limit, using Eq. 4.31 this becomes:

\[
\gamma_{\sigma} = -\frac{\pi V_{\sigma}}{|k| (I_+(k) + I_-(k))} \int_{p_{\text{res}}}^{\infty} dp pp^3 \frac{p_{\text{res}}(k)}{m_i} \left(1 - \frac{p_{\text{res}}^2(k)}{p^2}\right) \frac{\partial f_0}{\partial s},
\] (5.6)

where the resonant value of momentum, \( p_{\text{res}} \), for a given \( k \), is defined as \( p_{\text{res}}(k) = m_i \omega_{ci}/k \).

### 6. KOLMOGOROV’S SPECTRUM OF TURBULENCE

Some further evaluations can be performed, if one assumes the Kolmogorov’s spectrum for turbulence: \( I_-(k) \propto k^{-5/3} \), \( I_+(k) \propto k^{-5/3} \), at \( k > k_0 \). We take the total spectrum to be

\[
I_-(k) + I_+(k) = \frac{I_C}{k^{5/3}},
\] (6.1)

the parameter \( I_C \) characterizes the turbulence level and is specified below. In this section, we calculate both the scattering rate, \( D_{\mu\mu} \), and the spatial diffusion coefficient, \( D_{xx} \), for this kind of turbulence spectrum. Eq. 5.3 yields the following scattering rate:

\[
D_{\mu\mu} = \frac{v}{\lambda_{\mu\mu}} (1 - \mu^2) |\mu|^{2/3}, \quad \lambda_{\mu\mu} = \frac{4B^2/\mu_0}{I_C} r_{L}^{1/3},
\] (6.2)

\( r_L = v/\omega_{ci} \) being the Larmor radius and \( \lambda_{\mu\mu} \) being the characteristic value of the mean free path with respect to pitch-angle scattering.

An alternative and more consistent way to parameterize the turbulence level is to take into account an energy integral. By assuming, as stated above, a negligible level of turbulence below some minimum wave number, i.e. at \( k \leq k_0 \), which correspond to large spatial scales, we floor an
integration span by condition, \( k \geq k_0 \):

\[
    w_- + w_+ = \frac{(\delta B)^2}{\mu_0} = \int_{k_0}^{\infty} dk \left[ I_-(k) + I_+(k) \right] = \frac{3}{2} I_C k_0^{-2/3}. \tag{6.3}
\]

In this way, the mean free path can be expressed in terms of the turbulent energy density and \( k_0 \):

\[
    \lambda_{\mu\mu} = \frac{6}{\pi} \frac{B^2}{(\delta B)^2} \frac{r_L^{1/3}}{k_0^{2/3}}, \quad (\delta B)^2 = \mu_0 (w_- + w_+). \tag{6.4}
\]

In agreement with the Bohm-like estimate, \( \lambda \sim \frac{B^2}{(\delta B)^2} r_L \), i.e. mean free path is proportional to a (large) factor, \( \frac{B^2}{(\delta B)^2} \). However, the distinction is in a different dependence on the particle momentum (via the Larmor radius): \( \propto p^{1/3} \) with the derived formula versus \( \propto p \) in the Bohm-like estimate.

The spatial diffusion coefficient, \( D_{xx} \), in terms of the energy spectrum of turbulence is given by Eq. 5.4. It may be also expressed in terms of the mean free path, \( \lambda_{xx} \):

\[
    D_{xx} = \frac{1}{3} \lambda_{xx} v, \quad \lambda_{xx} = \frac{3B^2}{\pi \mu_0} \int_{k_r}^{\infty} dk \frac{(k^2 - k_r^2)}{k^5 \left[ I_-(k) + I_+(k) \right]}, \tag{6.5}
\]

where \( k_r(p) = \frac{eZB}{p} \) is the inverse of the Larmor radius. With ansatz (6.1) this mean free path is only by a numerical factor different from above introduced \( \lambda_{\mu\mu} \) and equals:

\[
    \lambda_{xx} = \frac{54}{7\pi} \frac{B^2/\mu_0}{I_C} r_L^{1/3}. \tag{6.6}
\]

Particularly, one can choose \( I_C \) in such way, that the mean free path estimate Eq. 6.6 would agree with that provided by Li et al. (2003), which had been also used by Sokolov et al. (2004):

\[
    \lambda_{xx} = \lambda_0 \frac{R}{1AU} \left( \frac{pc}{1GeV} \right)^{1/3}, \tag{6.7}
\]

where \( \lambda_0 \sim 0.1 \div 0.4 \) AU is a free parameter. The same dependence on the particle momentum, but a different dependence, \( \lambda_{xx} \propto (R/1AU)^{2/3} \), on the heliocentric distance was assumed by Zank et al. (2007). The mean free path in Eq. 6.7 corresponds to the choice of \( I_C \) as follows:

\[
    I_C = \frac{54B^2}{7\pi \mu_0 \lambda_0 R/1AU} r_L^{1/3}. \tag{6.8}
\]
\( r_{L0} = \frac{1 \text{ GeV}}{\gamma c B} \) being the Larmor radius for the particle momentum \( 1 \text{ GeV}/c \). Sokolov et al. (2009) employed the Kolmogorov spectrum with \( I_C \) from Eq. 6.8 to provide a seeding level of the Alfvén wave turbulence upstream the shock wave, which is strongly enhanced by the SEPs accelerated by the DSA mechanism and up-streaming the shock. Far upstream, the turbulence is not affected by the SEP of low intensity, so that the mean free path as in Eq. 6.5 with \( I_C \) from Eq. 6.8 correspond to the estimate in Eq. 6.7. We see that with the use of the Kolmogorov’s spectrum of turbulence, the dependence of mean free path on the particle momentum \( \lambda \propto p^{1/3} \) is achieved which can be found in literature and the spatial modulation of the turbulence spectrum may be applied to achieve a desired spatial modulation of the mean free path.

On the other hand, by expressing the mean free path in terms of \( k_0 \) and \((\delta B)^2\)

\[
\lambda_{xx} = \frac{81}{4\pi} \frac{B^2}{(\delta B)^2} \frac{r_{L0}^{1/3}}{k_0^{2/3}} \equiv \frac{81}{4\pi} \frac{B^2}{(\delta B)^2} \frac{r_{L0}^{1/3}}{k_0^{2/3}} \frac{p c}{1 \text{ GeV}}^{1/3},
\]

in the last identity we separated the momentum-dependent factor, same as in (Li et al. 2003; Sokolov et al. 2004; Zank et al. 2007; Sokolov et al. 2009). At the same time we keep the dependence on large Bohm-like factor \( \frac{B^2}{(\delta B)^2} = \frac{B^2}{\mu_0 (w_- + w_+)}, \) which can be consistently obtained from the turbulence-driven model for IH and SC.

To close the model, we need the estimate for \( k_0 \). As the first trial of the model in Borovikov et al. (2018), we performed simulations with

\[
k_0 = \text{const} \sim 0.1/R_S.
\]

However, more realistic seems to be an observation-based constraint for the maximum spatial scale in the turbulence, \( L_{\text{max}} \), which relates to the minimum wave vector and scales about linearly with the heliocentric distance:

\[
k_0^{-1} = \frac{L_{\text{max}}(R)}{2\pi}, \quad L_{\text{max}}(R) \sim 0.03 R,
\]
so that, on evaluating the numerical factor \( \frac{81}{7\pi(2e)^{2/3}} \approx 0.92 \) we arrive at the following formulae:

\[
\lambda_{xx} \approx 0.9 \frac{B^2/\mu_0}{w_- + w_+} \left( \frac{L^2 \rho}{L_{\text{max}} r_L} \right)^{1/3} \left( \frac{pc}{1\text{GeV}} \right)^{1/3}, \quad D_{xx} = \frac{1}{3} \lambda_{xx} v
\]

and, with another factor of \( 14/27 \),

\[
\lambda_{\mu\mu} \approx 0.5 \frac{B^2/\mu_0}{w_- + w_+} \left( \frac{L^2 \rho}{L_{\text{max}} r_L} \right)^{1/3} \left( \frac{pc}{1\text{GeV}} \right)^{1/3}, \quad D_{\mu\mu} = \frac{v}{\lambda_{\mu\mu}} (1 - \mu^2) |\mu|^{2/3}
\]

7. NUMERICAL IMPLEMENTATION

7.1. M-FLAMPA

To solve the Parker Eq. 2.1, Borovikov et al. (2018) developed the Multiple Field Line Advection Model for Particle Acceleration (M-FLAMPA). M-FLAMPA is based on the method first proposed by Sokolov et al. (2004) and reduces a 3-D problem of particle propagation in the IMF to a multitude of much simpler 1-D problems of the particle transport along a single line of the Interplanetary Magnetic Field (IMF).

We choose the Lagrangian coordinates for a given fluid element equal to the Eulerian coordinates of this element at the initial time instant, \( R_L = R |_{t=0} \). For numerical simulations, the initial grid is chosen as follows. Let the points, \( (R^{l\lambda}_l) |_{t=0} \), form a grid on a segment of a spherical heliocentric surface of the radius of \( R = 2.5 R_{\odot} \). The indices \( l, \lambda \) enumerate both this spherical grid’s points and the magnetic field lines passing through these points. For each \( l, \lambda \) one can solve numerically for the field line passing through the point \( (R^{l\lambda}_l) |_{t=0} \) by solving the following ordinary differential equation with the boundary condition:

\[
\frac{dR^{l\lambda}(s)}{ds} = (b \left( R^{l\lambda}(s), t \right)) |_{t=0}, \quad R^{l\lambda}(0) = (R^{l\lambda}_l) |_{t=0},
\]

where \( s \) is the curve length along the magnetic field line. When all the lines are constructed, one can introduce a grid \( s_i \) along each line. Now, the choice of the grid in Lagrangian coordinates,
(R^i_{1\lambda})_L = R^{1\lambda}(s_i), ensures that for fixed l, \lambda all points with different i initially belong to the magnetic field line. Then, one can numerically solve the multitude of ordinary differential equations, Eq. 2.2, to trace the spatial location for all Lagrangian grid points in the evolving fluid velocity field, u(R, t), as long as the latter is known. Since the magnetic field lines are frozen into a moving plasma, still all the grid points with fixed l, \lambda belong to the same magnetic field line and the kinetic equation for these points is independent and effectively one-dimensional in space. In this way, the three-dimensional kinetic equation for waves reduces to a two-dimensional multitude of one-dimensional equations.

**APPENDIX**

**A. FOCUSED TRANSPORT EQUATION AND SINGLE PARTICLE DYNAMICS**

The kinetic treatment of SEP was used throughout the present paper. In other words, the particle population and its properties were encapsulated in the distribution function. However, there exists another approach, which suggests solving equations of single particle dynamics directly for a relatively small yet representative set of particles. In this appendix we want to emphasize a deep connection between these very different approaches by revealing how equations of single particle dynamics are actually a part of the kinetic equation governing the particle population.

We reproduce the focused transport equation below (see Section 3):

\[
\begin{align*}
\frac{Df}{Dt} + v\mu \frac{\partial f}{\partial s} + \left[ \frac{1}{3} \frac{D\ln \rho}{Dt} + \frac{1}{6} \frac{D\ln (B^3/\rho^2)}{Dt} - \frac{\mu}{v} \mathbf{b} \cdot \frac{Du}{Dt} \right] \frac{\partial f}{\partial p} + \\
+ \frac{1 - \mu^2}{2} \left[ -v \frac{\partial \ln B}{\partial s} + \mu \frac{D\ln (\rho^2/B^3)}{Dt} - \frac{2}{v} \mathbf{b} \cdot \frac{Du}{Dt} \right] \frac{\partial f}{\partial \mu} = \left( \frac{\delta f}{\delta t} \right)_{\text{scat}} \tag{A1}
\end{align*}
\]

We introduce the parallel, \(p_{||} = \mu p\), and perpendicular, \(p_{\perp} = (1 - \mu^2)^{1/2} p\), components of the momentum instead of pitch-angle, \(\mu\). On substituting \(\frac{\partial f}{\partial \ln p} = p_{\perp} \frac{\partial f}{\partial p_{\perp}} + p_{||} \frac{\partial f}{\partial p_{||}}\) and \(\frac{\partial f}{\partial \mu} =
\[-\frac{\mu}{(1-\mu^2)^{1/2}} \frac{\partial f}{\partial p_{\perp}} + p \frac{\partial f}{\partial p_{||}},\]

one can rewrite Eq. A1 in the form:

\[
\frac{D f}{Dt} + \left( \frac{ds}{dt} \right)_{p} \frac{\partial f}{\partial s} + \left( \frac{dp_{\perp}}{dt} \right)_{p} \frac{\partial f}{\partial p_{\perp}} + \left( \frac{dp_{||}}{dt} \right)_{p} \frac{\partial f}{\partial p_{||}} = \left( \frac{\delta f}{\delta t} \right)_{\text{scat}}.
\]

Here, the coefficients \( \left( \frac{d}{dt} \right)_{p} \) in the kinetic equation are time-derivatives of canonical variables of a particle along its trajectory and are given by the following equations (cf. Northrop 1963):

\[
\left( \frac{ds}{dt} \right)_{p} = v_{||} \tag{A3}
\]

\[
\left( \frac{dp_{\perp}}{dt} \right)_{p} = \frac{1}{2} \left( \frac{D \ln B}{Dt} + \frac{\partial \ln B}{\partial s} v_{||} \right) p_{\perp} = \frac{1}{2} \left( \frac{d \ln B}{dt} \right)_{p} p_{\perp} \tag{A4}
\]

\[
\left( \frac{dp_{||}}{dt} \right)_{p} = - \frac{p_{\perp}^2}{2m_{i}} \frac{\partial \ln B}{\partial s} + \frac{D \ln (\rho/B)}{Dt} p_{||} - m_{i} \mathbf{b} \cdot \frac{Du}{Dt} \tag{A5}
\]

Here, \( m_{i} = \frac{p}{v} \) is the ion mass, which in application to relativistic particles should be substituted with the relativistic mass \( \sqrt{m_{i}^2 + p^2/c^2} = m_{i} \sqrt{1-v^2/c^2} \). The terms in Eqs. A3-A5 have simple and straightforward physical meaning. Thus, we see that evolution of the distribution function in Eq. A2 is governed by: (1) particle’s guiding center displacement along the field line, see Eq. A3; (2) conservation of the magnetic moment, \( p_{\perp}^2/(2m_{i}B) \), see Eq. A4; (3) magnetic mirror force, see first term on RHS of Eq. A5; (4) first-order Fermi acceleration, with the conservation of another adiabatic invariant, \( p_{||} \delta s \) (clear if rewritten as \( -\frac{D \ln \delta s}{Dt} p_{||} \)), see the second term on RHS of Eq. A5; (5) action of a non-inertial force \( \propto -D u /Dt \), see Eq. A5; and (6) particle scattering and sources (RHS of Eq. A2). Regarding process (3), the term in Eq. A5 is the force repelling the particle from a magnetic mirror. For a time-independent magnetic field (i.e., \( DB/Dt = 0 \)), the action of this force balances the energy change due to the perpendicular momentum increase (adiabatic focusing), thus ensuring the energy conservation.

The above Eqs. A3-A5 are convenient for computations using particle methods, especially within the Monte-Carlo approach. Similarly, the \( \mu \)-dependent form of Eq. A1 can be also be
solved in this way by integrating the equations for \((\frac{d\ln p}{dt})_p\) and \((\frac{d\mu}{dt})_p\) with the RHSs being the factors by \(\frac{\partial f}{\partial \ln p}\) and \(\frac{\partial f}{\partial \mu}\) terms in Eq. A1.

B. QUASI-LINEAR PERTURBATION OF DISTRIBUTION FUNCTION

Electro-magnetic fields of Alfvén waves exert Lorentz force on ions in solar wind. The effect on the distribution function is described by the term in the Boltzmann equation:

\[
F_L^{(w)} \cdot \frac{\partial F}{\partial p} = qZ\delta E_\perp \cdot G\{F\}
\]  

(B1)

with the differential operator

\[
G\{F\} = \left(1 - \frac{v_\parallel}{V_\sigma}\right) \frac{\partial F}{\partial p_\perp} + \frac{v_\perp}{V_\sigma} \frac{\partial F}{\partial p_\parallel}
\]

(B2)

A number of relations involving \(G\) are used in derivations in Section 4.

Operator \(G\) can be expressed in several ways. The perpendicular components may be expressed terms of the polar angle, \(\varphi\):

\[
p_\perp = p_\perp (\cos \varphi, \sin \varphi), \quad \frac{\partial f}{\partial p_\perp} = \frac{\partial f}{\partial p_\perp} (\cos \varphi, \sin \varphi) + \frac{1}{p_\perp} \frac{\partial f}{\partial \varphi} (-\sin \varphi, \cos \varphi).
\]

(B3)

Then

\[
G\{F\} = \left(1 - \frac{v_\parallel}{V_\sigma}\right) \left(\frac{\partial F}{\partial p_\perp} \mathbf{e}_\perp + \frac{1}{p_\perp} \frac{\partial F}{\partial \varphi} \mathbf{e}_\varphi\right) + \frac{v_\perp}{V_\sigma} \frac{\partial F}{\partial p_\parallel} \mathbf{e}_\perp
\]

\[
= G\{F\} \mathbf{e}_\perp + \left(1 - \frac{v_\parallel}{V_\sigma}\right) \frac{1}{p_\perp} \frac{\partial F}{\partial \varphi} \mathbf{e}_\varphi
\]

(B4)

Here we introduced a new scalar differential operator

\[
G\{\psi\} = \left(1 - \frac{v_\parallel}{V_\sigma}\right) \frac{\partial \psi}{\partial p_\perp} + \frac{v_\perp}{V_\sigma} \frac{\partial \psi}{\partial p_\parallel}
\]

(B5)

If we express \(p_\perp\) and \(p_\parallel\) in terms of magnitude of momentum, \(p\), and cosine of pitch-angle, \(\mu\), we obtain the following expression:

\[
G\{\psi\} = \sqrt{1 - \mu^2} \left[ \frac{\partial \psi}{\partial p} + \left(\frac{1}{m_i V_\sigma} - \frac{\mu}{p}\right) \frac{\partial \psi}{\partial \mu}\right]
\]

(B6)
In the course of determining the effect of waves on particle scattering, the following bilinear form appears:

\[ F_L^{(w)} \cdot \frac{\partial \delta f}{\partial p} = q Z_i \Re \left[ \delta E_\perp \right] \cdot G\{ \delta f \} \]

\[ = (q Z_i)^2 \Re \left[ \delta E_\perp \right] \cdot G \left\{ \frac{G\{ f \}}{i \left( \omega - k v_\parallel \pm \omega_\text{ci} \right)} \right\} \cdot \Re \left[ \delta E_\perp \right] \]

\[ = \Re \left[ \delta E_\perp \right] \cdot \mathbf{A} \cdot \Re \left[ \delta E_\perp \right] \quad \text{(B7)} \]

Note that here we explicitly stated that only the real part of the electric field is used in the expression. The equation that contains this bilinear form is averaged over period of rapid oscillations of the vector \( \delta E_\perp \). The following holds:

\[ \langle \delta E_\perp^* \cdot \mathbf{A} \cdot \delta E_\perp \rangle = \langle \Re \left[ E_\perp \right] \cdot \mathbf{A} \cdot \Re \left[ \delta E_\perp \right] \rangle + \langle \Im \left[ E_\perp \right] \cdot \Im \left[ \delta E_\perp \right] \rangle + \\
\quad i \left( \langle \Re \left[ E_\perp \right] \cdot \mathbf{A} \cdot \Im \left[ \delta E_\perp \right] \rangle - \langle \Im \left[ E_\perp \right] \cdot \mathbf{A} \cdot \Re \left[ \delta E_\perp \right] \rangle \right) \]

\[ = 2 \langle \Re \left[ E_\perp \right] \cdot \mathbf{A} \cdot \Re \left[ \delta E_\perp \right] \rangle, \quad \text{(B8)} \]

where the last transition is possible due to \( \delta E_\perp \) being a simple harmonic field. Thus, we have:

\[ F_L^{(w)} \cdot \frac{\partial \delta f}{\partial p} = \frac{q Z_i}{2} \langle \delta E_\perp^* \cdot G\{ \delta f \} \rangle \]

\[ = \frac{q Z_i}{2} \langle \delta E_\perp^* \cdot \left( G\{ \delta f \} e_\perp + \left( 1 - \frac{v_\parallel}{V_\sigma} \right) \frac{1}{p_\perp} \frac{\partial \delta f}{\partial \phi} e_\phi \right) \rangle \]

\[ = \frac{(q Z_i \delta E)^2}{2} \ldots \quad \text{(B9)} \]

A remarkable conjugation property:

\[ \int d^3 p \psi_1 G\{ \psi_2 \} = - \int d^3 p \psi_2 G^T \{ \psi_1 \}, \quad \text{(B10)} \]

which is true for any pair of gyrotrropic functions \( \psi_1 \) and \( \psi_2 \).

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