The $m^4 \ln m$ Contribution to the Nucleon Mass in CHPT

A. Kallen

Institut für Theoretische Physik, Universität Bern
Sidlerstrasse 5, CH-3012 Bern, Switzerland

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Abstract

In CHPT the hadron masses obtain loop corrections from the pseudoscalar mesons which are identified with the Goldstone bosons of broken chiral symmetry. An expansion of the baryon mass in the quark masses therefore includes non-analytic terms. We calculate the nucleon mass in the one-loop approximation to order $m^4 \sim m_q^2$. We compare the result in the relativistic and in the heavy mass formulation of the theory and derive matching relations between corresponding low-energy constants. We calculate the pion-nucleon loop in old-fashioned perturbation theory and find a contribution to the $m^4 \ln m$ term from an intermediate state which does not occur in the heavy baryon theory.
1 Introduction

The QCD Lagrangian for \( n \) flavours of massless quarks is invariant under chiral rotations of the fields. The spontaneous breaking of the \( SU(n)_R \times SU(n)_L \) chiral symmetry to \( SU(n)_V \) gives rise to \( n^2 - 1 \) Goldstone particles which lead to infrared divergences. In the presence of small quark masses the Goldstone bosons also become massive and are identified with the light pseudoscalar mesons. The square of the meson mass \( m^2 \) is proportional to the quark mass \( m_q \). The mass of a hadronic bound state is determined by the QCD scale and by the quark masses. Due to the Goldstone nature of the mesons the expansion of the hadron mass in the quark masses is not a simple Taylor series but contains non-analytic terms proportional to \( m_q^{3/2} \) and \( m_q^2 \ln m_q \).

Hadronic low-energy processes cannot be computed directly from the QCD Lagrangian but they can be described by an effective Lagrangian which reproduces the correct symmetry properties. Chiral Perturbation Theory (CHPT) is based on a simultaneous expansion in powers of derivatives and of quark masses \([1]\). In a relativistic formulation of the effective theory the loop expansion coincides with the expansion in small momenta and masses. In the mesonic sector higher dimension operators are typically suppressed by factors of \( m^2/\Lambda^2 \) where \( \Lambda = 4\pi F_\pi \) is the QCD scale of order 1 GeV.

In the baryonic sector the chiral expansion is complicated by the additional scale introduced by the mass \( M \) in the baryon propagator which is of the same order as \( \Lambda \) itself. The loop expansion no longer coincides with the small momentum expansion. A consistent power counting is only possible if the baryon kinematics is treated in a non-relativistic framework \([2]\).

In this article we consider the dependence of the nucleon mass on the masses of the two lightest quark flavours, the \( u \) and \( d \) quark, keeping \( m_s \) fixed. In the two-flavour sector the low-energy particle spectrum consists of nucleons and pions. We work in the isospin limit, setting \( m_u = m_d \). Then the proton and neutron form a mass degenerate isospin doublet while the pions belong to a triplet. The expansion of the pion mass squared starts with \( m^2_\pi = (m_u + m_d)B + O(m_q^2 \ln m_q) \) where \( B \) is related to the quark condensate. It is convenient to use the abbreviation \( m^2 \equiv (m_u + m_d)B \). The
expansion of the nucleon mass is given by

\[ M_{\text{phys.}} = a_0 + a_1 m^2 - \frac{3g_A^2}{32\pi F^2} m^3 + a_2 m^4 \ln \frac{m^2}{\mu^2} + a_3 m^4 + O(m^5). \]  

(1)

\( a_0 \) is the value of the nucleon mass in the chiral limit where \( m = 0 \). The coefficient \( a_2 \) receives contributions from pion-nucleon loops and pion tadpoles. In this article we compare these contributions in CHPT and in the heavy baryon expansion. This comparison leads to matching conditions for corresponding low-energy constants. With the help of non-relativistic old-fashioned perturbation theory we can then show that the pion-nucleon loop contribution to \( a_2 \) in CHPT is due to two different intermediate states. One of them contains an anti-nucleon and has no corresponding counterpart in the heavy baryon theory.

This article is organized in the following way. In section 2 we list those terms in the pion-nucleon Lagrangian which are relevant for the calculation of the nucleon mass. We give them in the relativistic theory and in the heavy baryon approximation. Section 3 contains the results for the expansion coefficients \( a_i \) in both approaches and matching conditions for the corresponding low-energy constants. In section 4 we calculate the pion-nucleon loop contribution in old-fashioned perturbation theory and discuss the origin of the non-analytic terms, in particular of \( m^4 \ln m \), in the different approaches. Section 5 contains a short summary of the results.

2 The Effective Lagrangian

The chiral effective Lagrangian for baryons has been given by Gasser et al. [3]. It is given by a string of interaction Lagrangians of increasing chiral power,

\[ \mathcal{L}_{\pi N} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \cdots. \]

The superscript \((i)\) in \( \mathcal{L}^{(i)} \) gives the number of derivatives and/or powers of quark mass. \( \mathcal{L}^{(1)} \) contains the free nucleon Lagrangian. The nucleon mass to order \( m^4 \) in the one-loop approximation is determined by tree graphs and one-loop graphs with vertices from \( \mathcal{L}^{(1)} \) and \( \mathcal{L}^{(2)} \). From the higher order Lagrangians \( \mathcal{L}^{(3)} \) and \( \mathcal{L}^{(4)} \) we only have to consider tree diagrams. In the calculation of the self-energy there arises a term of dimension zero which
requires the introduction of a counterterm Lagrangian $\mathcal{L}^{(0)}$. Explicitly we have

\[
\begin{align*}
\mathcal{L}^{(0)} &= \Delta M \bar{\psi}\psi, \\
\mathcal{L}^{(1)} &= \frac{g_A}{2} \bar{\psi} \gamma_5 \psi + \cdots, \\
\mathcal{L}^{(2)} &= \frac{M}{F^2} \bar{\psi} \left[ c_1 m^2 <U^\dagger + U> - \frac{c_3}{4} <u \cdot u> \right] \psi + \cdots, \\
\mathcal{L}^{(3)} &= d m^4 \bar{\psi}\psi + \cdots.
\end{align*}
\]

Brackets denote the trace in isospin space, $M$ is the bare nucleon mass, $F \approx 93$ MeV the pion decay constant and $g_A \approx 1.25$ is the axial-vector coupling constant. The three pions are parametrized with the help of the Pauli matrices $\tau$,

\[
U = u^2 = \exp i \frac{\vec{\pi} \cdot \vec{\tau}}{F},
\]

\[
u_\mu = i u^\dagger \partial_\mu U u^\dagger.
\]

There is no term in $\mathcal{L}^{(3)}$ which can give a contribution to the nucleon mass up to order $m^4$. The term proportional to $c_2$ in $\mathcal{L}^{(2)}$ does not contribute to the nucleon mass. Thus we have omitted it, together with other interaction terms ($+ \cdots$) which are irrelevant for the determination of the coefficients $a_2$ and $a_3$ of equation (1). The constants $c_1$, $c_3$ and $d$ are divergent and have to be renormalized.

\[
c_i = c_i^R(\mu) + \gamma_i L \quad (i = 1, 3) \quad \gamma_1 = -\frac{3g_A^2}{4}, \quad \gamma_3 = 4g_A^2 \]

\[
d = d^R(\mu) + \delta L \quad \delta = 4c_1^R(\mu) + c_3^R(\mu)
\]

\[
L = \frac{\mu^{d-4}}{16\pi^2} \left[ \frac{1}{d-4} - \frac{1}{2} (\ln 4\pi + \Gamma'(1) + 1) \right]
\]

$\mu$ is the renormalization scale introduced by dimensional regularization. The finite low-energy constants $c_i^R(\mu)$ are a priori undetermined but can in principle be fixed from phenomenology. Their corresponding counterparts $c_i'$ in the heavy baryon approximation have been calculated (see equation (2) below).

In section 3 we derive matching conditions between the low-energy constants of both theories at the renormalization scale $\mu = M$ and give numerical values for $c_1^R(M)$ and $c_3^R(M)$ (equation (4) in section 3).
For an introduction to heavy baryon chiral perturbation theory, see e.g. refs. \cite{2, 4}. The nucleon momentum is written as \( p^\mu = M v^\mu + k^\mu \) with \( v^2 = 1 \) and \( v \cdot k \ll M \). After decomposing the wave-function \( \psi \) into two components \( B \) and \( b \) and integrating out the heavy field \( b \) we are left with the velocity-dependent baryon field

\[
B(v, x) = \exp \left( i M v \cdot x \right) \frac{1 + v}{2} \psi(x) .
\]

This redefinition transforms the free nucleon Lagrangian for the massive field \( \psi \) into a free Lagrangian for \( B \) plus a string of terms which are suppressed by factors of \( 1/M \). The Dirac equation for the massless field \( B \) to leading order is given by

\[
i v \cdot \partial B = 0 .
\]

Explicitly the relevant interaction terms for the calculation of the nucleon mass are

\[
L^{(1)}_{hb} = g_A \bar{B} S \cdot u B ,
\]

\[
L^{(2)}_{hb} = \frac{1}{2M} \bar{B} \left[ (v \cdot \partial)^2 - \partial^2 - ig_A \{ S \cdot \partial, v \cdot u \} + c'_1 m^2 < U^\dagger + U > + \left( c'_2 - \frac{g_A^2}{2} \right) < (v \cdot u)^2 > + \left( c'_3 + \frac{g_A^2}{4} \right) < u \cdot u > \right] B ,
\]

\[
L^{(4)}_{hb} = \frac{d'}{M} m^4 \bar{B} B .
\]

Note that the Dirac matrices \( \gamma^\mu \) have been expressed in terms of \( v^\mu \) and the velocity-dependent spin operator \( S^\mu = \frac{i}{2} \gamma_5 \sigma^{\mu\nu} v_\nu \) which obeys \( S \cdot v = 0 \). In the heavy baryon approximation the nucleon mass vanishes in the chiral limit, therefore there is no \( L^{(0)}_{hb} \). The terms in \( L^{(2)}_{hb} \) with have fixed coefficients are due to the \( 1/M \) expansion of the Lagrangian \( L^{(1)} \). The first two contributions are corrections to the kinetic energy. The term proportional to \( (v \cdot \partial)^2 \) does not contribute to matrix elements at order \( 1/M \) because of the leading order Dirac equation for \( B \). The counterterm \( d' \) has to be renormalized.

\[
d' = d'^R(\mu) + \delta' L \quad \delta' = \frac{1}{2F^2} \left( -c'_1 + \frac{1}{4} c'_2 + c'_3 - \frac{1}{16} g_A^2 \right)
\]

The low-energy coefficients \( c'_i \) are finite. They are determined from the pion-nucleon \( \sigma \)-term and from \( \pi N \) scattering lengths. We take the numerical
values given by Meißner in ref. [6] where one can find a critical discussion of these numbers and their errors.

\[
\begin{align*}
c'_1 &= -1.63 \pm 0.21 \\
c'_2 &= 6.20 \pm 0.38 \\
c'_3 &= -9.86 \pm 0.41
\end{align*}
\]

(2)

3 The Nucleon Mass

The relativistic calculation yields the following results for the coefficients \(a_i\) of equation (1)

\[
\begin{align*}
a_0 &= M, \\
a_1 &= \frac{M}{F^2} \left[ \frac{3 g^2_A}{32 \pi^2} \left( 1 - \ln \frac{M^2}{\mu^2} \right) - 4 c_1^R(\mu) \right], \\
a_2 &= \frac{1}{32 \pi^2 F^2} \left[ \frac{M}{F^2} \left( 4 c_1^R(\mu) + c_3^R(\mu) \right) - \frac{3 g^2_A}{2M} \right], \\
a_3 &= \frac{3 g^2_A}{32 \pi^2 F^2 M} \left( 1 + \frac{1}{2} \ln \frac{M^2}{\mu^2} \right) - d^R(\mu).
\end{align*}
\]

We have chosen the value of \(\Delta M\) in such a way that \(M_{\text{phys.}} = M = a_0\) in the chiral limit. The result in the heavy baryon limit is

\[
\begin{align*}
a_0 &= 0, \\
a_1 &= -\frac{2}{M} c'_1, \\
a_2 &= \frac{1}{32 \pi^2 F^2} \frac{1}{2M} \left( -c'_1 + \frac{1}{4} c'_2 + c'_3 - \frac{1}{16} g^2_A \right), \\
a_3 &= -\frac{1}{M} d^R(\mu).
\end{align*}
\]

Comparing the results for the \(a_i\) of the two theories leads to the following matching conditions at the renormalization scale \(\mu = M\).
\[ c_1^R(M) = \frac{F^2}{2M^2} c'_1 + \frac{3g_A^2}{128\pi^2}, \]
\[ c_3^R(M) = \frac{F^2}{2M^2} \left(-5c'_1 + \frac{1}{4}c'_2 + c'_3\right) + g_A \left(\frac{F^2}{M^2} \frac{47}{32} - \frac{3}{32\pi^2}\right), \tag{3} \]
\[ d^R(M) = \frac{1}{M} \left( d'^R(M) + \frac{3g_A^2}{32\pi^2 F^2} \right). \]

Note that the condition for \( c_1^R(M) \) is the one given by Bernard et al. in ref. \[4\]. With the values of equation (2) for the \( c'_i \) and taking \( F = 93 \text{ MeV}, \ M = 940 \text{ MeV} \) and \( g_A = 1.25 \) we obtain the numerical results
\[ c_1^R(M) = (-4.27 \pm 1.03) \cdot 10^{-3}, \]
\[ c_3^R(M) = (6.84 \pm 7.61) \cdot 10^{-3}. \tag{4} \]

The huge error of \( c_3^R(M) \) is due to the fact that the central values almost cancel while the errors sum up: \((-5c'_1 + \frac{1}{4}c'_2 + c'_3) = (-0.16 \pm 1.56).\)

4 The Origin of the \( m^4 \ln m \) term

The coefficient \( a_2 \) of the non-analytic term proportional to \( m^4 \ln m \) receives contributions both from the pion-nucleon loop and from pion tadpoles. However, depending on which theory one uses to calculate the nucleon mass, the loop and the counterterms play a different role. The coefficient of the leading non-analytic term proportional to \( m^3 \) on the other hand is entirely due to the pion-nucleon loop in both models. In the relativistic theory we have
\[ a_{2,\text{loop}} = \frac{3g_A^2}{32\pi^2 F^2 M} \left(-\frac{1}{2}\right), \]
\[ a_{2,\text{tadpole}} = \frac{1}{32\pi^2 F^2} \frac{M}{F^2} M \left(4c_1^R(\mu) + c_3^R(\mu)\right). \]

We can calculate \( a_{2,\text{loop}} \) from the pion-nucleon interaction in \( \mathcal{L}^{(1)} \) by using old-fashioned perturbation theory. There one looks for a solution of the eigenvalue problem
\[ (H_0 + g_A H_1)|\phi\rangle = \left(\sum_n g_A^n E_n\right)|\phi\rangle. \]
To second order \((n = 2)\) the solution is given by the standard formula

\[
E = E_0 + g_A^2 \sum_{\text{int.}} \frac{\langle \text{int.} | H_1 | \phi_0 \rangle^2}{E_0 - E(\text{int.})}.
\]

where \(H_0, E_0\) and \(\phi_0\) are the unperturbed quantities and \(E(\text{int.})\) is the energy of the intermediate state \(|\text{int.}\rangle\). The energy-shift directly translates into the mass-shift and we can therefore investigate the contributions of the possible intermediate states to the coefficients \(a_i\). \(H_1\) is proportional to \(L^{(1)}\) and we find two intermediate states which give a non-vanishing matrix element. They are depicted in figure 1. State (a) contains one pion and one nucleon, state (b) contains one pion, two nucleons and one anti-nucleon. Their matrix elements are determined by three-dimensional integrals,

\[
\frac{|\langle(a) | H_1 | \phi_0 \rangle|^2}{E_0 - E(a)} \leftrightarrow \int \frac{d^3 l}{(2\pi)^3} \frac{1}{e(m) e(M)} \frac{1}{M - e(m) - e(M)},
\]

\[
\frac{|\langle(b) | H_1 | \phi_0 \rangle|^2}{E_0 - E(b)} \leftrightarrow \int \frac{d^3 l}{(2\pi)^3} \frac{1}{e(m) e(M)} \frac{(-1)}{M + e(m) + e(M)},
\]

\[
e(x) = \sqrt{x^2 + \vec{l}^2}.
\]

The sum of the two contributions is just the relativistic loop integral in the rest frame of the nucleon, \(p = (M, \vec{0})\), after integration over the time component \(l^0\).

\[
\int \frac{d^4 l}{(2\pi)^4} \frac{1}{m^2 - l^2} \frac{1}{M^2 - (p - l)^2} \sim \int \frac{d^3 l}{(2\pi)^3} \frac{1}{e(m) e(M)} \left[ \frac{1}{M - e(m) - e(M)} - \frac{1}{M + e(m) + e(M)} \right]
\]

We expand the integrands in terms of \(m\) and use dimensional regularization to evaluate the integrals. The leading non-analytic term proportional to \(m^3\) in equation (1) is entirely due to the intermediate state (a) as we would expect. The contributions to the coefficient \(a_{2,\text{loop}} = a_{2}^{(a)} + a_{2}^{(b)}\) from the two states are given by

\[
a_{2}^{(a)} = \frac{3g_A^2}{32\pi^2 F^2 M} \left( -\frac{3}{4} \right),
\]

\[
a_{2}^{(b)} = \frac{3g_A^2}{32\pi^2 F^2 M} \left( \frac{1}{4} \right).
\]
The reason that state (a) contributes to the logarithmic divergence is clearly due to the singular behaviour of the denominator in the integral: 
\[ M - e(m) - e(M) \] is infrared divergent for vanishing \( m^2 \) (chiral limit). Still it does not give the full contribution to \( a_2 \).

The surprising result is, that there is also a non-zero \( a_2^{(b)} \). The integral which gives the contribution from state (b) has a denominator proportional to \( 2M \) in the chiral limit, it is finite. Obviously, its derivative with respect to \( m^2 \) is not and the singularity comes from the expansion of the integrand in terms of \( m \). The intermediate state (b) contains an anti-particle, a configuration which does not occur in the framework of the heavy baryon expansion. Here the loop contribution from \( L^{(1)}_{hb} \times L^{(1)}_{hb} \) is independent of \( M \). It gives just the non-analytic term proportional to \( m^3 \) and no contribution to \( a_2 \) or \( a_3 \). The non-zero contribution to \( a_2 \) comes from the \( L^{(1)}_{hb} \times L^{(2)}_{hb} \) loop. We have

\[
a_{2,\text{loop}} = \frac{3g_A^2}{32\pi^2 F^2 M} \left( -\frac{1}{32} \right),
\]

\[
a_{2,\text{tadpole}} = \frac{1}{32\pi^2 F^2} \frac{1}{2M} \left( -c_1' + \frac{1}{4}c_2' + c_3' + \frac{1}{8}g_A^2 \right).
\]

For completeness we also give the results for the coefficient \( a_3 \). In the relativistic calculation we have \( a_{3,\text{loop}} = a_3^{(a)} + a_3^{(b)} \) with

\[
a_3^{(a)} = \frac{3g_A^2}{32\pi^2 F^2 M} \left( -\frac{1}{4} + 2\ln 2 + \frac{3}{4} \ln \frac{M^2}{\mu^2} \right),
\]

\[
a_3^{(b)} = \frac{3g_A^2}{32\pi^2 F^2 M} \left( \frac{5}{4} - 2\ln 2 - \frac{1}{4} \ln \frac{M^2}{\mu^2} \right),
\]

while the heavy baryon calculation gives \( a_{3,\text{loop}} = 0 \).

We have the following results. The pion-nucleon loop is always responsible for the leading non-analytic term proportional to \( m^3 \). In the relativistic theory, the loop of \( L^{(1)} \times L^{(1)} \) contributes to the coefficients \( a_2 \) and \( a_3 \). In old-fashioned perturbation theory it is the intermediate state (a) with one meson and one baryon which gives the leading singularity. The perturbative treatment shows that state (b), which contains an anti-baryon, gives also a contribution to \( a_2 \). We would not have expected this, since the integral expression which describes its contribution to the nucleon mass looks perfectly innocuous. In the heavy mass theory the loop of \( L^{(1)}_{hb} \times L^{(1)}_{hb} \) can only contribute to order \( m^3 \). It gives no contribution to \( a_2 \) or \( a_3 \). The \( m^4 \ln m \) terms
are due to pion tadpole contributions of $\mathcal{L}_{hb}^{(2)}$ and to the loop of $\mathcal{L}_{hb}^{(1)} \times \mathcal{L}_{hb}^{(2)}$, which are both suppressed by a factor of $1/M$.

5 Summary

We have calculated the nucleon mass in the one-loop approximation to order $m^4 \sim m_q^2$ in CHPT and in the heavy baryon approximation. We reproduce the matching condition for the low-energy constant $c_R^1(M)$ of Bernard et al. given in [4] and give similar conditions for $c_R^3(M)$ and the counterterm $d$. We have used old-fashioned perturbation theory to decompose the contribution of the relativistic pion-nucleon loop into the sum of terms corresponding to two different intermediate loop states. The leading singularity proportional to $m^3$ is always due to the pion-nucleon loop. The coefficient $a_2$ of the $m^4 \ln m$ term in equation (1) can have contributions from the pion-nucleon loop and from pion tadpole diagrams. These contributions are model dependent.

- In the relativistic theory there is a contribution to the logarithmic divergence from the loop of $\mathcal{L}_{(1)}^{(1)} \times \mathcal{L}_{(1)}^{(1)}$. Other contributions come from the tadpole diagrams of $\mathcal{L}_{(2)}^{(2)}$, they are proportional to the counterterms $c_1$ and $c_3$.

- In the heavy baryon approximation the loop of $\mathcal{L}_{hb}^{(1)} \times \mathcal{L}_{hb}^{(1)}$ gives no contribution to the logarithmic term. To leading order in $1/M$ there is just the leading singularity proportional to $m^3$ [4]. Consequently, all contributions to $a_2$ are suppressed by a factor of $1/M$.

- In old-fashioned non-relativistic perturbation theory there are two intermediate states (see figure [4]). State (a) contains one pion and one nucleon. The relevant integral gives an $m^4 \ln m$ contribution due to the vanishing of the denominator in the chiral limit. State (b) contains one pion, two nucleons and one anti-nucleon and also contributes to the $m^4 \ln m$ term although the denominator in the corresponding integral is finite. In this case the logarithmic singularity arises from the expansion of the integrand in $m$.

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Figure 1: The two intermediate states which can contribute to the nucleon mass to order $g_A^2$ in the perturbative approach. In (a) we have one pion (dashed line) and one nucleon (solid line) while in (b) we have three intermediate nucleon fields. Situation (b) is absent in the heavy baryon approximation since anti-particles do not occur in that framework.