A recursive coloring function without $\Delta^0_3$ solution for Hindman theorem

Liao Yuke

Abstract

We show that there exists a recursive coloring function $c$ such that any $\Delta^0_3$ set is not a witness of it for Hindman theorem.

1 Introduction

Hindman theorem is a Ramsey-type combinatorics theorem about finite sum.

Theorem 1 (Hindman, 1974 [2]) For any coloring function $c : \mathbb{N} \to r$, where $r \geq 2$ is a natural number, there exists an infinite set $H \subset \mathbb{N}$ such that $FS(H)$ is monochromatic. Here, $FS(H)$ is the set consisting of all finite sums of different numbers from $H$:

$$FS(H) = \{\sum_{x \in F} x : F \subset H, F \text{ is finite}\}.$$ 

For convenience, we call such a witness $H$ in the above theorem the solution of $c$ for Hindman theorem.

The reverse question of Hindman theorem starts from the paper of Blass, Hirst, and Simpson in 1987 [1]. They gave some basic results about the reverse question of Hindman theorem:

Theorem 2 (Blass, Hirst, Simpson 1987 [1])

1. Any coloring function $c : \mathbb{N} \to r$ has a solution for Hindman theorem recursive in $\omega^{\omega+1}$. This is an upper bound.
2. There exists a recursive coloring function $c$ such that any solution for $c$ can compute $0'$; this result can be relativized to any degree. This is a lower bound.

In terms of subsystems of Second Order Arithmetic, the strength of Hindman theorem is between ACA$_0$ and ACA$_0^\omega$. $\square$

Before their main results, they gave another theorem about lower bounds of the complexity of a solution for Hindman theorem as an example. That is:

Theorem 3 (Blass, Hirst, Simpson 1987 [1]) There exists a recursive coloring function $c$ in two colors without any $\Delta^0_3$ solutions for Hindman theorem. $\square$

In this paper, we will prove similar results for $\Delta^0_3$ sets:

Theorem 4 There exists a recursive coloring function $c$ in finitely many colors without any $\Delta^0_3$ (II$^0_3$) solutions for Hindman theorem.

2 Proof of $\Delta^0_3$ case

We list some notations. Some of them are from Blass, Hirst, Simpson’s paper, or elder papers.

Definition 1

1. [1] Define two functions $\mu(x), \lambda(x)$ on $\mathbb{Z}^+$ as follows:
   
   For $x = \sum_{i=0}^{k} 2^{m_i} \in \mathbb{Z}^+$ where $m_0 < \cdots < m_k$, define $\mu(x) = n_k, \lambda(x) = n_0$.
2. For two positive integers $x < y$, we write that $x \ll y$ or $y \gg x$ if $\mu(x) < \lambda(y)$.

One set $A$ has apartness (or is with apartness) if for any $x < y$ in $A$, $x \ll y$. 

3. For each natural number \( n \), define the set \( B^n = \{ x \in \mathbb{Z}^+ : \mu(x) = n \} \) and \( B^{\leq n} = \{ x \in \mathbb{Z}^+ : \mu(x) \leq n \} \).

For convenience, for a set \( A \), we use \( A(x) \) to denote its characteristic function. We say that a coloring \( c \) kills one set \( A \) if \( F S(A) \) is not monochromatic for \( c \). And in this paper, any quantifier without specified range is over the set of natural numbers \( \mathbb{N} \). For example, \( \forall x \phi \) means that for any \( x \in \mathbb{N} \), \( \phi \) holds. And \( n \in (a,b) \) means that the integer \( n \) is in the open interval between \( a \) and \( b \) and it still can only be an integer.

**General strategy**

A naive strategy is to construct a recursive set of pairs \( \{(x, y)\} \subseteq \mathbb{N}^2 \) and let \( x \) be the Red and \( y \) be Green for every pair in this set. But this strategy does not succeed. We will use a different strategy.

We will construct a recursive set of pairs \( R \subseteq \{(w, x) \in \mathbb{N}^2 : x << w\} \) which satisfies two conditions:

C1. There exists a recursive coloring \( c : \mathbb{N} \to \{0,1\} \) such that for any \( (w, x) \) in \( R \), \( c(w) \neq c(x+w) \).

C2. For every infinite \( \Delta^0_3 \) set \( A \) with apartness, there exists a pair \( (w, x) \) in \( R \) such that both \( w, x \) are in \( F S(A) \).

If such \( R \) exists, then by C1, we can get a coloring function \( c \) and by C2, this coloring function kills all infinite \( \Delta^0_3 \) sets with apartness.

When constructing \( R \), to satisfy the second condition C2, we hope to put more pairs \( (w, x) \) into \( R \) as many as possible, while to satisfy the first condition C1, we cannot put too many pairs into \( R \). For example, if \( (4, 2), (4, 3), (6, 1) \) are all in \( R \), then three numbers 4, 6, 7 need to have pairwise different colors, which contradicts the condition that \( c \) has only two colors. So we need to explore which kind of set \( R \) satisfies C1.

**Proposition 1** Let \( R \) be a recursive subset of \( \{(w, x) \in \mathbb{N}^2 : x << w\} \). If for any \( n \in \mathbb{N} \), \( B^n \cap \text{ran}(R) \) contains at most one element, then \( R \) satisfies C1: there exists a recursive coloring \( c : \mathbb{N} \to \{0,1\} \) such that for any \( (w, x) \) in \( R \), \( c(w) \neq c(x+w) \).

**Example 1** Consider \( R = \{(w, 2^n) : \lambda(w) > n\} \), then \( B^n \cap \text{ran}(R) = \{2^n\} \).

The condition C1 now becomes: there exists a coloring function \( c : \mathbb{N} \to \{0,1\} \) such that for any two numbers \( w, n \), if \( \lambda(w) > n \) (i.e., \( 2^{n+1} \mid w \)), then \( w \) and \( w + 2^n \) have different colors.

We can find a simple witness \( c \): let \( c(w) \) be the number of digits \( 1 \mod 2 \) in the binary representation of \( w \) : \( c(2) = c(10_2) = 1, c(3) = c(11_2) = 0, c(4) = c(100_2) = 1, c(5) = c(101_2) = 0 \) and so on.

We omit the proof of Proposition 1 because it is a special case of Lemma 1 later.

For the set \( R \) in Proposition 1, we can use a function to represent it: define a function \( R \) from \( \mathbb{N} \) to \( \mathbb{N} \) by letting \( R(n) \) be the unique element of \( B^n \cap \text{ran}(R) \) (and \( 2^n \) if \( B^n \cap \text{ran}(R) \) is empty).

However, the assumption of Proposition 1 is too strong. Such a set \( R \) does not contain enough elements to satisfy C2. We will use a lemma similar to Proposition 1 with a weaker assumption.

**Proposition 2** Let \( R \) be a recursive subset of \( \{(w, x) \in \mathbb{N}^2 : x << w\} \). If for any \( w \), any \( n < \lambda(w) \), and \( R_w = \{ x : (w, x) \in R \} \subseteq \text{ran}(R) \), \( B^n \cap R_w \) contains at most one element, then \( R \) satisfies C1.

Similar to Prop 1, we can use a function to represent it. Define \( R : \mathbb{N}^2 \to \mathbb{N} \) by letting \( R(n, w) \) be the unique element \( x \) in \( B^n \cap R_w \) (and \( 2^n \) if \( B^n \cap R_w \) is empty). In term of function \( R(n, w) \), C1 now becomes: there exists a coloring \( c \) such that for any \( n \) and any \( w \) that \( \lambda(w) > n \), \( c(R(n, w)) \neq c(w) \).

Now we show that any such function \( R(n, w) \) and hence the set \( R \) satisfy C1.

**Lemma 1** Let \( R : \mathbb{N}^2 \to \mathbb{N} \) be a recursive function such that \( R(n, w) \in B^n \). There exists a coloring function \( c : \mathbb{N} \to \{0,1\} \) uniformly recursive in \( R \) such that for any \( w \) and any \( n < \lambda(w) \),

\[
    c(w) \neq c(R(n, w) + w).
\]
Proof: We construct the coloring function \( c \) stage by stage. At stage \( s \), we color numbers in \( B^s \). For each \( s \), we will create a graph with vertex set \( B^s \) and edges depending on the function \( R \) such that this graph is a tree. So for any \( w_1, w_2 \in B^s \), there exists the unique path \( P \) from \( w_1 \) to \( w_2 \).

Fix a stage \( s \). For each \( w \in B^s \) and \( n < \lambda(w) \), we add one edge between \( w + R(n, w) \) and \( w \) and call it \( e_{w, w + R(n, w)} \). This graph does not contain any multiple edges or self-loops. We show that this graph is a tree by induction on some subgraphs. For every \( w \in B^s \) and \( n < \lambda(w) \), define the vertex set \( T_{w, n} = \{ w + x : 0 \leq x < 2^{n+1} \} \) and consider the corresponding full subgraph. By definition, \( T_{w_1, n_1} = T_{w_2, n_2} \) iff \((w_1, n_1) = (w_2, n_2)\).

If we write integers in binary representation, then the vertex set \( T_{w, n} \) looks like

\[
T_{w, n} = \{(w_s w_{s-1} \cdots w_n + 1 x_{n-1} \cdots x_0) : x_i \in \{0, 1\}\},
\]

where \( w_s w_{s-1} \cdots w_n + 1 \) is highest \( s - n \) many digits of \( w \) and it is also the common part of all numbers in the vertex set. All other digits of \( w \) are 0. And by definition, for \( n \geq 1 \),

\[
T_{w, n} = \{w + x : 0 \leq x < 2^{n+1}\} = \{w + x : 0 \leq x < 2^n\} \cup \{w + x : 2^n \leq x < 2^{n+1}\} = T_{w, n-1} \cup T_{w, 2^n, n-1}.
\]

The two sets \( T_{w, n-1} \) and \( T_{w+2^n, n-1} \) look like

\[
T_{w, n-1} = \{(w_s w_{s-1} \cdots w_n + 1 x_{n-1} \cdots x_0) : x_i \in \{0, 1\}\},
\]

\[
T_{w+2^n, n-1} = \{(w_s w_{s-1} \cdots w_n + 1 x_{n-1} \cdots x_0) : x_i \in \{0, 1\}\},
\]

Hence the full subgraph of \( T_{w, n} \) is the union of the full subgraphs of \( T_{w, n-1}, T_{w+2^n, n-1} \) and the edges between vertices in \( T_{w, n-1} \) and vertices in \( T_{w+2^n, n-1} \). We claim that there exists exactly one edge between two vertex sets \( T_{w, n-1}, T_{w+2^n, n-1} \).

An edge \( e' \) between \( T_{w, n-1}, T_{w+2^n, n-1} \) has two vertices \( w' \) and \( w' + R(n', w') \) for some \( w', n' \) that \( \lambda(w') > n' \). We show that \( w' = w, n' = n \). Because \( w' + R(n', w') > w' \) and any numbers in \( T_{w+2^n, n-1} \) is greater than all numbers in \( T_{w, n-1} \), for two vertices of the edge \( e' \), it must be that \( w' \in T_{w, n-1} \) and \( w' + R(n', w') \in T_{w+2^n, n-1} \). So \( w' = w + x \) where \( 0 \leq x < 2^n \).

If \( w' > w \), then \( x > 0 \) and \( \lambda(w') = \lambda(w + x) = \lambda(x) < n \). Because \( \mu(R(n', w')) < \lambda(w') = \lambda(x), x + R(n', w') < 2^n \) and so \( w' + R(n', w') = w + x + R(n', w') < w + 2^n \), contradicting our assumption that \( w' + R(n', w') \in T_{w+2^n, n-1} \). So \( x = 0 \) and \( w' = w \).

If \( n' < n \), then \( R(n', w') = n' < n \), \( R(n', w) < 2^n \), and \( w' + R(n', w) = w + R(n', w) < 2^n \), contradicting our assumption. If \( n' > n \), then \( R(n', w) \geq 2^n + 1 \) and \( w + R(n', w) \) is not in \( T_{w+2^n, n-1} \). So \( n' = n \).

So \( w' = w, n' = n \) and the two vertices are \( w \) and \( w + R(n, w) \). This means any edge \( e' \) between two vertex set \( T_{w, n-1}, T_{w+2^n, n-1} \) must have two vertices \( w \) and \( w + R(n, w) \). Hence such edge is unique and the claim is true.

We show that every such subgraph is a tree by induction on \( n \). For \( n = 0 \) and every \( w \) such that \( \lambda(w) > 0 \), \( T_{w, 0} \) has a single vertex. For \( n > 0 \), by induction hypothesis, for any \( w \) that \( \lambda(w) > n \), \( T_{w, n} \) is a tree because it is the union of two trees (full graphs of \( T_{w, n-1} \) and \( T_{w+2^n, n-1} \)) plus one edge between two trees. Finally, since \( B^s = T_{2^{s-1}, s-1} \), the graph of \( B^s \) is a tree.

Definition of the coloring: Let \( c(2^s) = 0 \). For another vertex \( w \), because the graph is a tree, find the unique path \( P \) from \( 2^s \) to \( w \) and let \( c(w) = \) the length of \( P \mod 2 \). This process is uniformly recursive in \( R \).

Verification: Let \( P_w \) be the path from \( 2^s \) to \( w \), and \( x = R(n, w) \). If \( w + x \) is a vertex in \( P_w \), then it must be the vertex in \( P_w \) exactly before \( w \); otherwise there will be two paths from \( w \) to \( w + x \). So \( c(w + x) = |P_w| - 1 \mod 2 \neq c(w) \). If \( w + x \) is not in the path, then the path connecting \( P_w \) and \( e_{w, w+x} \) is a path from \( 2^s \) to \( w + x \), and \( c(w + x) = |P_w| + 1 \mod 2 \neq c(w) \). In both cases, \( c(w) \neq c(w + x) \), and this coloring satisfy the requirement of the lemma.

Once we have Lemma 1, our task becomes to construct a function \( R \) satisfying the second condition C2. The construction will use approximations of \( \Delta_0^3 \) sets. Every \( \Delta_0^3 \) set \( A \) has a recursive approximation \( A(x, y, s) \) with two extra variables.
Proposition 3 There exists a recursive list of total recursive functions \( \{A_i(x, y, s)\}_{i \in \mathbb{N}} \) such that for any \( \Delta_0^3 \) set \( A \) and its characteristic function \( A(x) \), there exists a function \( A_i(x, y, s) \) in the list such that

\[
\lim_{y} \lim_{s} A_i(x, y, s) = A(x).
\]

Proof: For any \( \Delta_0^3 \) set \( A \), we have that \( A \leq_T \emptyset'' \) and there exists a function \( A'(x, y) \leq_T \emptyset' \) such that \( \lim_y A'(x, y) = A(x) \). Let \( A'(x, y) = \Phi_i^y(x, y) \), where \( \Phi_i \) is the \( i \) th recursive function. Consider \( \Phi_i^y[s](x, y)[s] \) and we will show that for every \( x \), \( \lim_y \lim_s \Phi_i^y[s](x, y)[s] \) exists and equals \( A(x) \).

There exists \( y_0 \) such that for any \( y > y_0 \), \( A'(x, y) \downarrow = A(x) \). Now, for those \( y \), we check if \( \lim_s \Phi_i^y[s](x, y)[s] = A'(x, y) \). Because \( A'(x, y) = \Phi_i^y(x, y) \) is defined, there exists \( s \) such that \( \Phi_i^y[s](x, y)[s] \downarrow = A'(x, y) \) and the usage of \( \emptyset'' \) in the computation will never be changed since stage \( s \). Now we could see for any \( s' > s \), the computation still holds and so \( \Phi_i^{y}[s'](x, y)[s'] = A'(x, y) \). So \( \lim_s \Phi_i^y[s](x, y)[s] = A'(x, y) \), and \( \lim_y \lim_s \Phi_i^y(x, y) = A(x) \).

So we define the list as follows. For each natural number \( j \), define \( A_j(x, y, s) = \Phi_j^y[s](x, y)[s] \) if it is 0 or 1, and \( A_j(x, y, s) = 0 \) otherwise. This is a total recursive function and this list is also recursive.

Strategy of killing one set

Assuming \( A \) is an infinite \( \Delta_0^3 \) set with apartness, we show how to kill \( A \) by constructing the function \( R \). In terms of \( R(n, w) \), C2 now becomes: there exist numbers \( w \in FS(A) \) and \( n < \lambda(w) \) such that \( R(n, w) \in A \cap B^n \). If this condition holds, then by lemma 1, we can get a coloring function \( c \) which kills set \( A \).

Our strategy is to guess which element is in \( A \cap B^n \) (by apartness, \( A \cap B^n \) contains at most one element) and let \( R(n, w) \) be this element. We can use its approximation \( A(x, k, s) \) as a guess function.

We use the guess function to construct the function \( R(n, w) \). For numbers \( n, w \), compute \( A(x, \lambda(w), \mu(w)) \) for all \( x \in B^n \), and check if there exists any \( x \) in \( B^n \) such that

\[
A(x, \lambda(w), \mu(w)) = 1,
\]

which suggests that \( x \in B^n \cap A \); if such \( x \) exists, then we guess \( B^n \cap A \neq \emptyset \) and let \( R(n, w) \) be the minimal such \( x \); if such \( x \) does not exist, let \( R(n, w) = 2^n+1-1 \), i.e.:

\[
R(n, w) = \min\{x \in B^n : A(x, \lambda(w), \mu(w)) = 1\} \cup \{2^n+1-1\}.
\]

We show this construction can kill set \( A \). Because \( A \) is infinite with apartness, there exists \( n \) such that \( B^n \cap A \) is nonempty. Let \( x \) be the unique element in \( B^n \cap A \). By Proposition 3, for any number \( y \) in \( B^n \), \( \lim_y \lim_s A(y, k, s) = A(y) \). Because \( B^n \) is finite, \( \lim_y \lim_s A(\cdot, k, s) \uparrow B^n = A(\cdot) \uparrow B^n \). This means that for sufficiently large \( k, s \), for any \( y \in B^n \setminus \{x\}, A(y, k, s) = 0 \) and \( A(x, k, s) = 1 \).

Now, look for a number \( K \) such that for any \( k > K \), \( \lim_y \lim_s A(\cdot, k, s) \uparrow B^n = A(\cdot) \uparrow B^n \) and choose a number \( w_1 \in A \) such that \( \lambda(w_1) = k > K, n \). Next, for the fixed \( k \), look for a bound \( s_k \) such that for any \( s > s_k \), \( A(\cdot, k, s) \uparrow B^n = A(\cdot) \uparrow B^n \) and choose another number \( w_2 >> w_1 \) in \( A \) such that \( \mu(w_2) > s_k \).

Let \( w = w_1 + w_2 \), then \( \lambda(w) = \lambda(w_1) = k > K \) and \( \mu(w) = \mu(w_2) > s_k \). Combining all discussions above, \( A(\cdot, \lambda(w), \mu(w)) \uparrow B^n = A(\cdot) \uparrow B^n \). This means \( A(x, \lambda(w), \mu(w)) = 1 \) while for any \( y \in B^n \setminus \{x\}, A(y, \lambda(w), \mu(w)) = 0 \). By the definition of function \( R(n, w) \), \( R(n, w_1 + w_2) = x \in A \). All three numbers \( x << w_1 << w_2 \) are in \( A \). By Lemma 1, there exists a recursive function \( c \) such that \( c(w_1 + w_2) \neq c(x + w_1 + w_2) \) where both \( w_1 + w_2 \) and \( x + w_1 + w_2 \) are in \( FS(A) \).

Kill infinitely many sets

The strategy of killing infinitely many sets is based on the strategy of killing one set. Now, condition C2 becomes: for any \( \Delta_0^3 \) infinite set \( A_i \) with apartness, there exist numbers \( w \in FS(A_i) \) and \( n < \lambda(w) \) such that \( R(n, w) \in A_i \). We cannot use the
construction of killing one set directly. Before using such construction, we need to choose a set \( A_i \) among all \( \Delta^0_3 \) sets, and then apply that construction.

We use some priority arguments to choose a set \( A_i \). It is to guess for which \( A_i, A_i \cap B^n \) is nonempty and then choose one such set \( A_i \). This argument needs the second guess function guessing that: for a number \( n \) and a set \( A_i \), whether \( A_i \cap B^n \) is empty or not. For a set \( A_i \), a number \( w \), and a number \( n < \lambda(w) \), we check if there exists \( x \in B^n \) such that \( A_i(x, \lambda(w), \mu(w)) = 1 \). We use a function \( a_i(n, k, s) \) to denote this check:

\[
a_i(n, k, s) = \max\{A(x, k, s) : x \in B^n\}.
\]

Because \( B^n = \{x : \mu(x) = n\} \) is finite, this function is also recursive and total. If \( a_i(n, \lambda(w), \mu(w)) = 1 \), which means such \( x \in B^n \) that \( A(x, \lambda(w), \mu(w)) = 1 \) exists, then we guess \( A_i \cap B^n \) is nonempty; otherwise, we guess \( A_i \cap B^n \) is empty.

We cannot recursively list all infinite \( \Delta^0_3 \) sets, but by Proposition 3, we have a recursive list of some total recursive functions \( A_i(x, k, s) \) such that every \( \Delta^0_3 \) set is represented in the list. This list contains approximations of all infinite \( \Delta^0_3 \) sets. Some functions in the list may not correspond to an infinite \( \Delta^0_3 \) set, a set with apartness or even not have limits.

So we need to use some restrictions similar to Blass, Hirst and Simpson’s paper. During the construction, for any function \( A_i(x, k, s) \) and any pair \( k, s \), we will choose at most \( 2^i \) many numbers \( n \in (i, s) \) and call them the candidates of \( A_i \) for \( k, s \). For any \( k, s \), any set \( A_i \) has at most \( 2^i \) many candidates. When defining \( R(n, w) \), only if \( n \) is a candidate of \( A_i \) for \( k = \lambda(w), s = \mu(w) \), \( A_i \) may be chosen to apply the construction of killing one set.

Construction:

We use the recursive list \( \{A_i(x, k, s)\}_{i \in \mathbb{N}} \) in Proposition 3. For each \( A_i(x, y, s) \) in the list, define the function \( a_i(n, k, s) = \max\{A(x, k, s) : x \in B^n\} \) and for each \( k, s \) define the candidate set \( C_i(k, s) \) to be the set of first \( 2^i \) numbers \( n \in (i, s) \) such that \( a_i(n, k, s) = 1 \). If there are not \( 2^i \) many such number \( n \), then simply let \( C_i(k, s) \) be all such \( n \).

For each \( n, w \), compute the set \( \{i < n : i \in C_i(\lambda(w), \mu(w))\} \), the index set of sets \( A_i \) that \( C_i(\lambda(w), \mu(w)) \) contains \( n \), which means that we guess \( A_i \cap B^n \neq \emptyset \).

Next, let \( j = \min\{i < n : i \in C_i(\lambda(w), \mu(w))\} \cup \{n\} \) and let

\[
R(n, w) = \min(\{x \in B^n : A_j(x, \lambda(w), \mu(w)) = 1\} \cup \{2^{n+1} - 1\})
\]
as killing one set in previous subsection. This is the construction of the function \( R(n, w) \). \( R(n, w) \) is a recursive function.

By Lemma 1, we can construct a coloring function \( c \) recursively in \( R(n, w) \) and that coloring function \( c \) satisfies the condition C1: for any \( w \) and \( n < \lambda(w) \), \( c(w) \neq c(w + R(n, w)) \). The only remaining task is verifying condition C2 for \( R(n, w) \) and \( c \).

Verification

In this subsection, we show that the above construction satisfies condition C2. Because by Proposition 3, every \( \Delta^0_3 \) set is contained in the list, we only need to consider sets in the list. We use \( A_i \) to denote the set of function \( A_i(x, k, s) \) if this set exists. First, we have the following lemma.

Lemma 2 For any \( \Delta^0_3 \) infinite set \( A_i \) with apartness, there exists a finite set \( C_i \) of size \( 2^i \) such that \( \lim_k \lim_s C_i(k, s) \downarrow C_i \).

Proof: Because \( A_i \) is infinite and has apartness, there are infinitely many \( n > i \) such that \( A_i \cap B^n \) is nonempty. Let \( N \) be the \( 2^i \) th number \( n \) among them.

By definition, for any \( x \) in \( B^{\leq N} \), \( \lim_k \lim_s A_i(x, k, s) = A_i(x) \). Because \( B^{\leq N} \) is finite, by properties of limits, we also have:

\[
\lim_k \lim_s (A_i(\cdot, k, s) \upharpoonright B^{\leq N}) = A_i(\cdot) \upharpoonright B^{\leq N}
\]

Hence for any sufficiently large \( k \), fix \( k \), and for this fixed \( k \), for any sufficiently large \( s \) and every \( n \leq N, a_i(n, k, s) = \max\{A(x, k, s) : x \in B^n\} = \max\{A(x) : x \in B^n\} \) is nonempty.

This means, \( a_i(n, k, s) = 1 \) if \( B^n \cap A_i \) is nonempty for \( n \leq N \). Because \( N \) is the \( 2^i \)th
number $n > i$ such that $B^n \cap A_i$ is nonempty, there exists $2^i$ numbers $n \leq N$ greater than $i$ such that $B^n \cap A_i$ is nonempty and hence $a_i(n, k, s) = 1$. By the definition of $C_i(k, s)$, the first $2^i$ elements $n \in (i, s)$ that $a_i(n, k, s) = 1$, for these sufficiently large $k, s$, $C_i(k, s)$ is equal to the set of first $2^i$ number $n$ that $B^n \cap A_i \neq \emptyset$.

So we can conclude that, for any sufficiently large $k$, there exists $s_k$ such that for any $s > s_k$, $C_i(k, s)$ exactly contains first $2^i$ number $n$ that $B^n \cap A_i \neq \emptyset$. This is just $\lim_k \lim_s C_i(k, s) \downarrow C_i$ for a fixed set $C_i$.

Next, we have the following lemma:

**Lemma 3** For any $\Delta^0_3$ infinite set $A_i$ with apartness, there exists a finite set $X_i \subset A_i$ such that, there exists a number $K$ such that for any $k > K$, there exists a number $s_k$ such that for any number $s > s_k$ and for any number $w$ that $\lambda(w) = k, \mu(w) = s$, there exists $x \in X_i$ such that $R(\mu(x), w) = x$.

Shortly and inaccurately, there exists $X_i \subset A_i$ such that $\lim_{\lambda(w)} \lim_{\mu(w)} R(n, w) \in X_i$.

**Proof:** Consider limits $\lim_{\lambda(w)} \lim_{\mu(w)} C_i(k, s) \downarrow C_i$ and $\lim_{\lambda(w)} \lim_{\mu(w)} A_i(\cdot, k, s) \uparrow B^{\leq N} = A_i(\cdot) \uparrow B^{\leq N}$ in the proof of lemma 2 and let $X_i = \cup_{n \in C_i} B^n \cap A_i$. By the two limits, there exists a number $s_k$ such that for any $k > K$, there exists a number $s_k$ such that for any $s > s_k$, $C_i(k, s) \downarrow C_i$ and $A_i(\cdot, k, s) \uparrow B^{\leq N} = A_i(\cdot) \uparrow B^{\leq N}$. We show that if these two equations hold for numbers $k, s$, then for any $w$ that $k = \lambda(w), s = \mu(w)$, there exists $x \in X_i$ such that $R(\mu(x), w) = x$ and it proves this lemma.

We claim that for these $k, s$, for at least one $n \in C_i(k, s) = C_i$,

\[ i = \min\{\{j < n : n \in C_j(k, s)\} \cup \{n\}\}. \]

By definition, for any $n < j$, $|C_j(k, s)| < 2^i$. So $|\cup_{j < i} C_j(k, s)| < 2^i$. If the claim is not true, then for every $n \in C_i(k, s)$, there exists $j < i$ that $n \in C_j(k, s)$. This means that $\cup_{j < i} C_j(k, s) \supseteq C_i(k, s)$, which is impossible because $|\cup_{j < i} C_j(k, s)| < 2^i = |C_i(k, s)|$. So the claim is true.

Let $n \in C_i(k, s) = C_i$ be a number that $i = \min\{\{j < n : n \in C_j(k, s)\} \cup \{n\}\}$. For any $w$ that $\lambda(w) = k \land \mu(w) = s$, by definition,

\[ R(n, w) = \min\{\{x \in B^n : A_i(x, \lambda(w), \mu(w)) = 1\} \cup \{2^{n+1} - 1\}\}. \]

Because for $k, s$, $A_i(x, k, s) = A_i(x)$ for all $x \in B^n$, we have that $\min\{x \in B^n : A_i(x, k, s) = 1\}$ is defined and is just the unique number in $B^n \cap A_i$. Hence $R(n, w)$ is the unique element in $B^n \cap A_i$ and is in $X_i$.

So for these $k, s$ that $C_i(k, s) \downarrow C_i$ and $A_i(\cdot, k, s) \uparrow B^{\leq N} = A_i(\cdot) \uparrow B^{\leq N}$, for any $w$ that $\lambda(w) = k, \mu(w) = s$, there exists $x \in X_i$ such that $R(\mu(x), w) = x$. This finishes the proof.

Now we can get the following result.

**Proposition 4** There exists a recursive coloring function $c$ such that for any infinite $\Delta^0_3$ set $A_i$ with apartness, there exist $x << w_1 << w_2$ in $A_i$ such that $c(w_1 + w_2) \neq c(x + w_1 + w_2)$. Hence $FS(A_i)$ is not monochromatic for $c$.

**Proof:** Choose the coloring function $c$ in our construction. For an infinite $\Delta^0_3$ set $A_i$ with apartness, we use set $X_i \subset A_i$, and numbers $K, s_k$ in lemma 3 for $A_i$.

Find $K$ as in the previous lemma, and choose a sufficiently large number $w_1 >> X_i$ in $A_i$ such that $\lambda(w_1) = k > K$. For the fixed $k$, find $s_k$ as in the previous lemma and choose a sufficiently large $w_2 >> w_1$ in $A_i$ such that $\mu(w_2) = s > s_k$. Because $w_2 >> w_1, \lambda(w_1 + w_2) = \lambda(w_1) = k, \mu(w_1 + w_2) = \mu(w_2) = s$. Hence by lemma 3, there exists a number $x \in X_i$ such that $R(\mu(x), w_1 + w_2) = x$. By the construction and lemma 1, $c(w_1 + w_2) \neq c(x + w_1 + w_2)$.

Proposition 4 shows that, there exists a recursive coloring function $c$ such that for any infinite $\Delta^0_3$ set $A$ with apartness, $FS(A)$ is not monochromatic. This proposition is close to our aim. To prove it is also true for sets without apartness, we use the following fact proved by Hindman:
Lemma 4 ([2]) For any infinite set $A$, there exists infinite $B \subseteq T A$ such that $B$ has apartness and $FS(B) \subseteq FS(A)$.

So for the same coloring function $c$, if it has any infinite $\Delta^0_3$ set solution then it must also have an infinite $\Delta^0_3$ set solution with apartness, but this is impossible by Proposition 4. Now we have proved the aim of this section.

Corollary 1 There exists a recursive coloring function $c$ without any $\Delta^0_3$ solution for Hindman theorem.

References

[1] Stephen G. Simpson Andreas R. Blass, Jeffry L. Hirst. Logical analysis of some theorems of combinatorics and topological dynamic. Contemporary Mathematics, 1987.

[2] Neil Hindman. Finite sums from sequences within cells of a partition of $n$. Journal of Combinatory Theory (A), 1974.