Field-induced superfluids and Bose liquids in Projected Entangled Pair States

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In two-dimensional incompressible quantum spin liquids, a large enough magnetic field generically induces “doping” of polarized S=1 triplons or S=1/2 spinons. We review a number of cases such as spin-3/2 AKLT or spin-1/2 Resonating Valence Bond (RVB) liquids where the Projected Entangled Pair States (PEPS) framework provides very simple and comprehensive pictures. On the bipartite honeycomb lattice, simple PEPS can describe Bose condensed triplons (AKLT) or spinons (RVB) superfluids with transverse staggered (Néel) magnetic order. On the Kagome lattice, doping the RVB state with deconfined spinons or triplons (i.e. spinon bound pairs) yields uncondensed Bose liquids preserving U(1) spin-rotation symmetry. We find that spinon (triplon) doping destroys (preserves) the topological $Z_2$ symmetry of the underlying RVB state. We also find that spinon doping induces longer range interactions in the entanglement Hamiltonian, suggesting the emergence of (additive) log-corrections to the entanglement entropy.

Magnetic frustration in quantum SU(2)-invariant spin systems of low dimensionality – typically in two dimensions (2D) - has the potential to stabilize spin liquids with no magnetic order and gapped magnetic (i.e. spin-1 “triplon”) excitations. Two distinct important classes of such states are Affleck-Kennedy-Lieb-Tasaki (AKLT) states and topological spin liquids. The AKLT ground state (GS) is simply constructed out of valence bonds (VB), is non-degenerate, and breaks no symmetries (in 2D). The spin-3/2 AKLT state on the honeycomb lattice has been proposed as a universal quantum computation resource. Nearest neighbor (NN) resonating valence bond (RVB) states – where neighboring spins 1/2 are paired up in resonating singlet dimers – offer simple ansätze of a new type of spin liquid, with $Z_2$ symmetry on the kagome lattice. A remarkable feature of gapped topological ($Z_2$) liquid is that triplons spontaneously fractionalize into deconfined spins 1/2 dubbed “spinons”. Other gapped (but spinless) topological excitations are “visons” vortexlike excitations which carry half a quantum of flux of the (underlying) $Z_2$ gauge field. An external magnetic field plays the role of a chemical potential both for the triplons or the spinons polarized along the field, and hence controls their densities. For liquids with fractional spinon excitations (which can only be created by pairs), this issue has been investigated using simplified “doped” quantum dimer models (QDM) representing a mixture of fluctuating dimers (mimicking singlet VB) and mobile vacancies (representing spinons). Despite their apparent simplicity, these models exhibit very rich phase diagrams with i) superfluid (or supersolid) phases – breaking spontaneously the $U(1)$ symmetry associated to the spin rotation around the magnetic field direction (equivalent to spinon number conservation in these models) – and ii) Bose or Fermi liquid phases where the $U(1)$ symmetry is preserved. They also provide a microscopic system where “statistical transmutation” is realized: vacancies can bind to visons and change their mutual statistics (from bosons to fermions or vice versa) as originally proposed by Kivelson. Such a scenario in a real quantum spin systems has not been observed so far.

In recently synthesized Bi$_3$Mn$_4$O$_{12}$ (NO$_3$) (named BiMnO), Heisenberg-like spin-3/2 moments on the Mn$^{4+}$ ions form a bilayer honeycomb lattice. Despite the bipartite structure and the large antiferromagnetic in-plane coupling, BiMnO behaves as a spin liquid down to very low temperatures. A transition towards a Néel state (antiferromagnetic order) is induced by a moderate magnetic field. Theoretically, it has been suggested that the S=3/2 honeycomb AKLT model might well describe the properties of this material in zero and finite magnetic field. However the behavior under magnetic field of the S=3/2 AKLT model on the hexagonal lattice has not been investigated theoretically so far.

In this work we investigate the behavior of 2D gapped spin liquids under an applied magnetic field. At small field before the spin gap vanishes, the system remains in the same singlet GS. Therefore the magnetization curve (i.e. magnetization versus field) of spin-gapped systems generically starts with a zero-magnetization “plateau” up to a lower critical field $h_{c,1}$ (which equals the spin gap in appropriate units) before the magnetization starts to raise continuously. In other words, the zero-magnetization (gapped) phase has zero magnetic susceptibility and can therefore be viewed as an “incompressible” liquid, in contrast to the compressible (i.e. gapless) finite magnetization phase occurring at fields between $h_{c,1}$ and the saturation field $h_{s,2}$ above which the spin system is fully polarized. To describe the gapless phase, we construct Projected Entangled Pair States (PEPS) carrying a finite magnetization and originating from a simple PEPS representation of the corresponding zero-field spin liquid. Note that for fields below $h_{c,1}$ the PEPS representation of the GS is not changing. The properties of these PEPS are investigated on infinitely long cylinders using standard methods. By construction, these gapless spin liquid ansätze preserve full space group symmetry but may or may not break the $U(1)$ spin-
rotation symmetry around the magnetic field axis leading to (spin) superfluids or Bose liquids, respectively. Note that the translation symmetry-breaking "crystals" leading to magnetization plateaux at special commensurate values\(^{18,19}\) of the magnetization are not addressed here.

**AKLT state:** we start with the S=3/2 AKLT Hamiltonian under an applied magnetic field:

\[
H = H_{\text{AKLT}} - hS^2 = \sum_{\langle ij \rangle} \mathcal{P}^{(S_T=3)}_{ij} - h \sum_{i=1}^{N} S_i^z \tag{1}
\]

where the sum is over all nearest neighbor (NN) bonds, \(\mathcal{P}^{(S_T=3)}_{ij}\) is the projector on total spin \(S_T = 3\) acting on the product Hilbert spaces of sites \(i\) and \(j\), and \(h = g\mu_B H\) is the reduced effective field. The AKLT ground state for \(h = 0\) can be understood by viewing each spin-3/2 as being composed of three "virtual" spin-1/2 moments which are symmetrised on-site, with each spin-1/2 moment forming a singlet with its neighbor; thus, it can be written exactly as a simple \(D = 2\) PEPS. After turning on a finite \(h\), the magnetization \(m = \langle \sum_i S_i^z \rangle\) starts to rise above the critical field \(h_{c,1}\) for which the Zeeman energy overcomes the spin energy gap. Intuitively, an increasing density \(x = m/m_{\text{sat}}\) of singlets is turned into polarized triplets until the saturation field \(h_{c,2}\) is reached, where all singlets are converted into triplets and \(m = m_{\text{sat}} = SN\), where \(N\) is the number of sites. In such a picture, the triplets "resonate" to gain energy and form a resonating triplet bond (RTB) AKLT state. One way (i) to construct a RTB AKLT state is to extend the \(D = 2\) AKLT PEPS to a \(D = 3\) PEPS with bonds \(|01\rangle - |01\rangle + |22\rangle\), where \(|1\rangle\) and \(|2\rangle\) are both assigned \(S_z^i = \frac{1}{2}\) (i.e., \(|22\rangle\) represents a triplet). The three spins are then first symmetrized as in the AKLT state, and subsequently projected onto a configuration with \(p\) \(|2\rangle\)’s with relative weight \(\alpha_p\), as indicated in Fig. 1(b). We expect the triplets to be weakly interacting, so that for simplicity, we choose the coefficients \(\alpha_p\) to be equal to their statistical probabilities \(\alpha_p^* = (1 - \lambda)^3\), \(\alpha_1^* = 3\lambda(1 - \lambda^2)\), \(\alpha_2^* = 3\lambda^2(1 - \lambda)\) and \(\alpha_3^* = \lambda^3\), depending on a single parameter \(\lambda \in [0, 1]\) playing the role of a fugacity for the triplets. Another (independent) way (ii) to introduce fluctuating \(S_z = +1\) triplets is to allow for an admixture of a triplet component on every bond, i.e., to replace the virtual \(|01\rangle - |10\rangle\) singlets by \(|01\rangle - |10\rangle + \beta|11\rangle\) states before the symmetrization, keeping the bond dimension \(D = 2\). Of course this admixture can be performed as well on the extended \(D = 3\) state above, resulting in a two-parameter family of PEPS. In fact, both of these constructions can be understood as special cases of a more general 9-parameter RTB AKLT construction, as explained in Appendix A. These states are generically not invariant under \(\exp(i a S_z)\) where \(S_z\) is the total spin, and thus can have a finite magnetization in the plane.

We have placed the square lattice of tensors on infinite cylinders with \(N_c\) unit cells in the periodic (vertical) direction as shown in Fig. 1(d) and use standard techniques (involving exact tensor contractions and iterations of the transfer operator) to compute relevant observables. We have investigated the variational energy \(E_{\text{AKLT}}(x) = \frac{1}{N_c} \langle H_{\text{AKLT}} \rangle\) of the RTB-AKLT PEPS for \(N_c = 6\). Choosing \(\beta = 0\) and varying \(\lambda\) provides (approximately) the best energy for \(x > 0.2\), while for \(x < 0.2\) the PEPS with \(\lambda = 0\) and \(\beta \neq 0\) has lower energy. The overall energy curve crudely obtained from these two separate PEPS is already quite accurate as shown in Fig. 2 when compared to Lanczos exact diagonalisations (ED)\(^{21}\). By optimizing w.r.t. \(\lambda\) and \(\beta\) simultaneously, one can lower the energy even further down, especially for \(x < 0.3\) (see Appendix B). The magnetization curve \(m(h)/m_{\text{sat}}\) can be obtained by minimizing \(E_{\text{AKLT}}(x) - (hS)x\) w.r.t. \(x\). The slopes \(\frac{1}{D} dE_{\text{AKLT}}/dx\) at \(x = 0\) and \(x = 1\) hence provide the lower and upper critical fields \(h_{c,1}\) and \(h_{c,2}\) as indicated on Fig. 2. Note that, in our units, \(h_{c,1}\) equals the zero-field spin gap \(\Delta_S\). The physics close to saturation \(m = m_{\text{sat}}\) is captured exactly by our PEPS (with \(\lambda \to 1\)). Also, our estimate \(h_{c,1} \simeq 0.113\) (for \(\lambda, \beta \to 0\)) is quite close to the extrapolated (zero-field) spin gap \(\Delta_S \simeq 0.10\)\(^{14}\).

We have found that, generically, the doped AKLT PEPS exhibit a transverse staggered magnetization in the
Squared tranverse magnetization ($<S_x>/S$)
i
try. This state is the "negative" of the familiar S=1/2 (spin

gapped) "Bose liquid" with restored U(1) symme-
ty is the SF order parameter between triplets. Interestingly, as seen in Fig. 3(b),
α̸= 0.2 providing the best ansatz). We have also constructed a RTB-AKLT state with
α̸= 0.113 (a) Magnon-doped S=3/2 AKLT $D = 2$ and $D = 3$ states and $D = 3$ spinon-doped S=1/2 NN RVB state. (b) Comparison between S=3/2 AKLT PEPS with doped softcore [as in (a)] and hardcore triplons. (c) Fugacity of the kagome RVB $D = 3$ PEPS vs $m/m_{sat}$.

FIG. 2. Energy per site (different symbols are used when either $\lambda$ or $\beta$ is fixed to 0) of the triplon-doped S=3/2 AKLT state versus reduced magnetization computed on an infinite cylinder with perimeter $N_c = 6$. The PEPS gives the exact asymptotic behavior close to saturation (i.e. slope $h_{c,2} = 3$ at $m = m_{sat}$) and the low-field slope ($h_{c,1} \approx 0.113$) is in good agreement with Ref. 14. Inset: comparison between the energies of AKLT PEPS with softcore and hardcore triplons. PEPS results are also compared with ED data\textsuperscript{14}.

plane perpendicular to the field, i.e. $\langle S_x^i \rangle = (-1)^i \times \text{cst}$
(where $(-1)^i = \pm 1$ depending on the sublattice) and 
$\langle S_y^i \rangle = 0$, hence breaking $U(1)$ symmetry. This property is generic for any choice of the coefficients $\alpha_p$ and $\beta$ (except, possibly, for isolated points). When $m \to 0$
($m \to m_{sat}$), the system can be understood in terms of a
low concentration $x (1-x)$ of interacting bosonic triplets
(singlets) undergoing a Bose condensation and forming a
correlated superfluid (SF). In the semi-classical approach applied to spins $1/2$ forming dimers\textsuperscript{20}, the quantity
$\langle (S_x^i)^2 \rangle$ is the condensate density of triplons. We believe it is also a good (but approximate) indicator of Bose condensation for $S > 1/2$, and plot it in Fig. 3(a)
versus the reduced magnetization for $\alpha_p = \alpha_1^x \lambda$, $\beta = 0$
and for $\lambda = 0$, $\beta \neq 0$. When $x \to 1$, approaching saturation,
$\langle (S_x^i)^2 \rangle \to 2(1-x)$ corresponding exactly to the
effective singlet density. In contrast, in the low magnetization limit $x \to 0$, $\langle (S_x^i)^2 \rangle \to 19x$ (for $\lambda = 0$ and $\beta \neq 0$
providing the best ansatz).

We have also constructed a RTB-AKLT state with
$\alpha_2 = \alpha_3 = 0$, enforcing by hand an infinite repulsion
between triplets. Interestingly, as seen in Fig. 3(b),
The SF order parameter $\langle S_x^i \rangle$ now vanishes at exactly
$m/m_{sat} = 1/3$ (again as a square root) giving rise to a
(spinn gapped) "Bose liquid" with restored U(1) symme-
try. This state is the "negative" of the familiar S=1/2 (al-
gebraic) RVB state: the singlet ($m = 0$) AKLT state can be viewed as the new quantum "vacuum" where polar-
ized hardcore triplets at 1/3-density resonate. We believe that such a topological state (despite its poor variational energy for the AKLT Hamiltonian – see Fig. 2) could still

be stabilized when the (effective) triplet repulsion is large
enough (although not infinite).

**RVB wave functions:** The second class of wave func-
tions we now investigate are NN S=1/2 RVB states, which we consider on honeycomb and kagome lattices as depicted in Fig. 4(a,b). Both states have short range spin-spin correlations\textsuperscript{4} but, on a bipartite lattice like the
honeycomb lattice, (singlet) dimer-dimer correlations are expected to be critical\textsuperscript{22,23}. On the kagome lattice, the RVB state has $\mathbb{Z}_2$ topological structure\textsuperscript{5,23}. Interestingly, strong numerical evidence has been provided that the ground state of the NN $S=1/2$ quantum Heisenberg model (QHM) is indeed a gapped $\mathbb{Z}_2$ spin liquid\textsuperscript{24–26}.

One of the most remarkable properties of such RVB states is that magnetic excitations are gapped deconfined $S=1/2$ spinons (marginally confined on bipartite lattices) instead of $S=1$ triplons. Turning on a magnetic field larger that the spin gap, $h > \Delta_S$, will therefore dope spinons into the system. Simple extension of the RVB $D = 3$ PEPS\textsuperscript{9} can be realized to include a finite density of spinons as shown in Fig. 4(c,d). Note that incompressible phases such as those discussed in the literature\textsuperscript{18,19} for special fractional values of the magnetization (so-called "magnetization plateaux") are not addressed here.

Computations are done on the same cylinder geometry (see Fig. 1(d)) as above. By increasing the fugacity $\lambda$ from 0 to 1, one can tune the magnetization between 0 and $m_{\text{sat}}$ as shown in Fig. 3(c). As for the AKLT state, a finite transverse staggered magnetization is found for the honeycomb RVB state as shown in Fig. 3(b). Approaching saturation $m \rightarrow m_{\text{sat}}$, the linear behaviors of the condensate density are identical, corresponding to the same condensation of singlets in a polarized ferromagnetic background. Above $h_{c1} = \Delta_S$, the condensate density grows also linearly but the slope is smaller for the RVB state.

A strikingly different behavior is found on the kagome lattice: no transverse order is observed in the RVB PEPS, i.e. $\langle S^z_x \rangle \neq \langle S^z_y \rangle = 0$ (up to small finite size effects) for all magnetizations. Therefore, the $U(1)$ symmetry (spin rotation around $z$-axis) is preserved so that this state can be viewed as a new type of gapless Bose liquid.

We have also computed the variational energy of the QHM $H_{\text{Heis}} = \sum_{ij} S_i \cdot S_j$. First, the energy $E_{\text{QHM}}(x)$ compares poorly to Lanczos ED\textsuperscript{21} and DMRG data\textsuperscript{24} at low field and, in addition, it has a slight negative curvature at $x \ll 1$ which signals an unphysical (small) jump in the magnetization curve (see Appendix B for details). Our simple PEPS wave function then might not describe very well the physics of the QHM under magnetic field because i) the RVB state is only a poor ansatz for the ground state of the QHM at zero field\textsuperscript{28}; ii) another source of difference might be that spinons could form bound NN pairs (triplons) in the QHM even though they are deconfined at long distance. We have tested this scenario by adding to the rank-3 $R$ tensors (see Fig. 4(c)) the new non-zero elements $R(1, 1, 2) = R(1, 2, 1) = R(2, 1, 1)$ which control the density of $S_z = 1$ triplons on NN sites. The triplon-doped RVB has indeed a lower energy than the spinon-doped RVB (see Appendix B) but the corresponding slope $\partial E/\partial x$ at $x = 0$ remains too large compared to DMRG or ED. Note that, when only triplons are doped, the $\mathbb{Z}_2$ topological sectors are preserved, since the doping keeps the $\mathbb{Z}_2$ symmetry of the tensors. iii) Thirdly, it is known from the studies of QDMs that spinons in $\mathbb{Z}_2$ spin liquids\textsuperscript{10,11} can bind a topological vortex (vison) changing their mutual statistics from fermions to bosons, or vice versa. Further studies with more elaborate PEPS would be needed to investigate these possibilities.

**Entanglement Hamiltonian:** Entanglement measures offer new tools for characterizing exotic states like topological liquids. If the (infinite) cylinder of Fig. 1(d) is partitioned in two $A$ and $B$ halves, the 2D reduced density matrix $\rho_A = \text{Tr}_B(\ket{\Psi_{\text{PEPS}}}\bra{\Psi_{\text{PEPS}}})$ of any $\ket{\Psi_{\text{PEPS}}}$ PEPS can be simply mapped, via a spectrum conserving isometry $U$, onto an operator $\sigma_\nu^n$ acting only on the $D^{\otimes N_b}$ edge (virtual) degrees of freedom\textsuperscript{17}, i.e. $\rho_A = U^n\sigma_\nu^n U$. Therefore, it is convenient to define an entanglement (or boundary) Hamiltonian $H_b$ as $\sigma_\nu^n = \exp(-H_b)$. As $\sigma_\nu^n$, $H_b$ is one-dimensional and its spectrum – the so-called entanglement spectrum (ES) – is the same as the one of $-\ln \rho_A$. In a $\mathbb{Z}_2$ topological liquid, the ES (and the associated $H_b$) depends on the choice of the boundary conditions $B_L (= B_R)$ – due to the existence of two "even" and "odd" disconnected topological sectors – and on the existence/absence of a $\mathbb{Z}_2$ flux through the cylinder\textsuperscript{23,29}. Adding any magnetization in the PEPS breaks topological order, since it break the gauge symmetry of the tensors which is responsible for that. Therefore, at any (arbitrary small) doping, the two topological sectors are mixed and $H_b$ become independent of $B_L (= B_R)$ provided $N_b \rightarrow \infty$.

In fact, the entanglement Hamiltonian of the $h = 0$ RVB PEPS belongs to the $1/2 \oplus 0$ representation of $SU(2)$ and its Hilbert space is the same as the one of a bosonic $t$–$J$ model\textsuperscript{23}. In the presence of a finite magnetization in the bulk, the $SU(2)$ symmetry is broken but $H_b$ keeps the unbroken $U(1)$ symmetry of the bulk Bose liquid. To have a better insight of the $U(1)$ entanglement Hamiltonian, we expand it in terms of a

![Weights](image-url)
where site indices have been added and we restrict to parts of different amplitudes, spins (hole hoppings. In this basis

where transverse antiferromagnetic order is in-

numerically when

though the long-range "tails" are more difficult to detect

Appendix C). For large enough magnetization, e.g. on

area law for the entanglement entropy can indeed occur

this work we became aware of a related work on AKLT Hamiltonians by Ar-

Note added: after completion of this work we became

1 Ian Affleck, Tom Kennedy, Elliott H. Lieb and Hal Tasaki, Phys. Rev. Lett. 59, 799 (1987).

2 X. G. Wen, Int. J. Mod. Phys. B 5, 1641 (1991).
Appendix A: PEPS for resonating triplon AKLTs

In this appendix, we discuss the general form of a resonating triplon doped AKLT, and explicitly explain how the constructions described in the main text fit into this picture. To this end, we start from virtual maximally entangled bonds of the form \(|01⟩−|10⟩+|22⟩\). The idea is that the \(\{|0⟩,|1⟩\}\) subspace holds the singlets, while the \(\{|2⟩\}\) holds the triplets. Thus, both \(|1⟩\) and \(|2⟩\) are understood to have \(S_z = \frac{1}{2}\), but they are distinguished by a “triplet-ness” quantum number \(t\): \(|1⟩ \equiv |S_z = \frac{1}{2}, t = 0⟩\), and \(|2⟩ \equiv |S_z = \frac{1}{2}, t = 1⟩\).

To obtain a triplon-doped AKLT from these bonds, we now need to do two things: (i) we need to symmetrize the virtual spins; (ii) we need to choose the relative probabilities for having a certain number \(p\) of triplets at each site and subsequently erase the “triplet-ness” quantum number, i.e., make \(|1⟩\) and \(|2⟩\) indistinguishable. These steps can be carried out in either ordering, which will generally give different outcomes (since converting \(2⇒1\) changes the norm of vectors). In the most general framework, this can be expressed by decomposing the total PEPS projector \(P\) as (i) a symmetrization map

\[
S = \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \sum_{p=0}^{\frac{1}{2}+m} w_{m,p} |S_z = m; t = p⟩\langle \sum_{i_1,j_2,i_3} |i_1,i_2,i_3⟩ (A1)
\]

where the r.h.s. sum symmetrizes over all \(|i_1,i_2,i_3⟩\) with \(\frac{1}{2} - m\) 0’s (i.e., \(S_z = m\)) and \(p\) 2’s; and (ii) a map projecting onto a given relative weight of different triplet numbers \(p\),

\[
T = \sum_{p=0}^{3} \gamma_p \sum_{m=p-\frac{1}{2}}^{\frac{1}{2}} |S_z = m⟩⟨S_z = m; t = p| (A2)
\]
such that $\mathcal{P} = T S$. Clearly, the $\gamma_p$ can be absorbed into the $w_{m,p}$, leaving us with a $(10-1) = 9$-parameter family of PEPS.

We will now show that the families of triplon-doped AKLT PEPS studied in the paper both fall into this family. The first one (with $\beta = 0$) is obtained by choosing the symmetrization map $S$ to be a projector, i.e., the $w_{m,p}$ are equal to the square root of the number of terms in the r.h.s. sum in (A1),

$$w_{3/2,0} = 3/2, 1, \quad w_{3/2,1} = w_{3/2,2} = 1/\sqrt{3},
$$

$$w_{1/2,0} = w_{1/2,2} = 1/\sqrt{3}, \quad w_{1/2,1} = 1/\sqrt{6},
$$

$$w_{-1/2,0} = w_{-1/2,1} = 1/\sqrt{3}, \quad w_{-3/2,0} = 1,$$

and by setting $\gamma_p = \alpha_p$ in (A2), leaving us with the four parameter family described in the main text.

The second variant described in the main text – using a bond $|01\rangle \pm |10\rangle + \beta |11\rangle$, while leaving the $\alpha_p = 0$ for $p > 0$ – can be understood as first using the $|2\rangle$ level to pick a triplet on each bond with weight $\sqrt{3}$ on each end, turning the $|2\rangle$ into a $|1\rangle$, and subsequently projecting on the symmetric subspace. Since turning the $|2\rangle$ into a $|1\rangle$ effectively changes the number of terms in the r.h.s. sum of (A1), this leads to different weights $w_{m,t}$, and one finds that this PEPS can be described by choosing $w_{3/2,0} = 1, w_{1/2,0} = 1/\sqrt{3}$ in (A1), and $\gamma_p = \beta p^2/2$ in (A2).

Clearly, due to linearity the ansatz where both the $\alpha_p$ and $\beta$ are non-zero can also be described using the full family defined by Eqs. (A1) and (A2).

### Appendix B: Spin gaps and magnetization jumps

The derivative $\partial E/\partial x|_{x=0}$ provides an estimate of $S \Delta_S$, the spin gap (times $\ln N$) of the incompressible phase at $x = 0$. We have plotted the quantity $(E(x) - E(0)/x)$ in Fig. 6 for doped $S = 3/2$ AKLT and RVB states which gives the above derivative when taking the limit $x \to 0$. The agreement with the gap estimate of the $S=3/2$ AKLT model is fairly good for the triplon-doped AKLT state, with a proper choice of the parameters. In contrast, the spinon-doped RVB state and, to a lesser extent, the triplon-doped RVB state overestimates the small gap of the kagome quantum antiferromagnet by a large amount.

A close look at the variational PEPS energies $E(x)$ reveals a small negative curvature when $x \to 0$, both for the $\beta = 0$ AKLT and the spinon-doped kagome RVB states. $\partial^2 E/\partial x^2$ is the inverse spin susceptibility $\chi^{-1}$ and $\chi^{-1} < 0$ signals a (weak) instability towards phase separation between a $x = 0$ phase and a phase with $x = x_{\text{min}}$ where $x_{\text{min}}$ is given by the minimum of $(E(x) - E(0)/x)$ (Maxwell construction). When increasing the field $h$, $x(h)$ then jumps from 0 to $x_{\text{min}}$ at $h = h_{c,1}$. As shown in Fig. 6, $x_{\text{min}} \approx 0.2$ ($x_{\text{min}} \approx 0.1$) for the AKLT state (RVB state) on the (infinite) $N_v = 6$ cylinder. However, comparison with Lanczos ED suggests that this is in fact a finite size effect of the variational ansatz on finite cylinder. More precisely, comparing $N_v = 4$ and $N_v = 6$ cylinders we get $x_{\text{min}} \sim 1/N_v$ suggesting that phase separation disappears when $N_v \to \infty$. Note that, in contrast, the $\beta = 0$ AKLT state and kagome triplon-doped RVB (with improved variational energies) do not show phase separation for finite $N_v$.

### Appendix C: Entanglement spectrum

By definition the ES is the spectrum of $-\ln \rho_A$. Since $\rho_A$ and $\sigma_Z^z = \exp ( -H_b )$ are related by an isometry, it is also the spectrum of the boundary Hamiltonian $H_b$. ES are shown in Fig. 7 for 3 values of the fugacity $\lambda$, as a function of the momentum along the cut. The $h = 0$ SU(2) spin multiplets are split by an arbitrary small spinon concentration as shown in Fig. 7(a). For increasing $\lambda$ (and magnetization), the splittings of the Kramers multiplets increase (see Fig. 7(b,c)) due to the relative increase of the amplitudes of an effective Zeeman term and of new SU(2)-symmetry breaking many-body terms in the boundary Hamiltonian (see main text).
FIG. 7. Entanglement spectrum of a bi-partitioned $N_v = 6$ kagome RVB cylinder as a function of the momentum along the cut, for different values of the spinon fugacity $\lambda$ corresponding to $m/m_{\text{sat}} \sim 10^{-3}, 0.007, 0.02$, respectively. Different symbols are used for different $S_z$ sectors of the edge. The same $S_z = 0, K = 0$ state is used as energy reference $\xi_0$ (bold + symbol).