CONSTRUCTION OF UNIQUE MILD SOLUTION AND CONTINUITY OF SOLUTION FOR THE SMALL INITIAL DATA TO 1-D KELLER-SEGEL SYSTEM

YUMI YAHAGI
Department of Mathematical Informatics
Tokyo University of Information Sciences
4-1 Onaridai, Wakaba-ku, Chiba, 265-8501 Japan

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ABSTRACT. In this paper, a one-dimensional Keller-Segel system of parabolic-parabolic type which is defined on the bounded interval with the Dirichlet boundary condition is considered. Under the assumption that initial data is sufficiently small, a unique mild solution to the system is constructed and the continuity of solution for the initial data is shown, by using an argument of successive approximations.

1. Introduction. Keller-Segel system was firstly introduced by Keller and Segel [12] in 1970 as a mathematical model that describes the biological phenomenon of the cellular slime molds. In Hillen and Painter [9], there is a detailed exploration of variations of the system. Here, we consider the following one-dimensional Keller-Segel system (KS) which is the classical model dealt in [9] with the Dirichlet boundary condition:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - \chi \nabla(u \nabla v), & (x, t) &\in I \times (0, T), \\
\frac{\partial v}{\partial t} &= \Delta v - \gamma v + \alpha u, & (x, t) &\in I \times (0, T), \\
u(x, t) &= v(x, t) = 0, & (x, t) &\in \partial I \times (0, T), \\
u(x, 0) &= u_0(x), v(x, 0) &= v_0(x), & x &\in I,
\end{align*}
\]

where \(I \equiv (-\frac{\pi}{2}, \frac{\pi}{2})\) is a bounded open interval, \((u_0, v_0)\) is a given nonnegative initial data, and \(\alpha > 0, \gamma > 0, \chi > 0\) are given constants, and \(T (0 < T < 1)\) is also given a constant.

The solution \(u\) of (KS) denotes the density of the cell population and \(v\) resp., the concentration of chemical substance which is called “cyclic AMP”. We remark that the first equation in (KS) denotes that the cellular slime molds move at the speed \(\chi \nabla v\).

In this paper, let us assume that \(u_0\) and \(\nabla v_0\) are sufficiently small. Then, we can construct a unique mild solution \((u, v)\) of (KS), and the continuity of the solution for...
the initial data \((u_0, v_0)\) is shown, by using an argument of successive approximations, which are our main theorems.

In recent years, Keller-Segel systems have been studied by many researchers. The systems are defined on the various domains, as whole space \(\mathbb{R}^n\), (cf. eg. Corrias, Escobedo and Matos [5], Kozono, Sugiyama and Wachi [13], Sugiyama and Yahagi [18]), half space \(\mathbb{R}^n_+\) (cf. eg. Carrillo, Li and Wang [4], Hou and Wang [10]), and bounded domain (cf. eg. Albeverio, Yahagi and Yoshida [1], Cao [3], Hou, Liu, Wang and Wang [11], Osaki and Yagi [16], Winkler [19], Yahagi [22]). As in the above references, the systems defined on the bounded domain are often investigated under the Neumann boundary condition (in [4], [11] and [22], numerical simulations are also given). There are some researchs on the systems with mixed boundary conditions. Miura [15] deals with the system on the no-flux and Neumann boundary conditions. In Lee, Wang and Yan [14], the system with the velocity field of a fluid flow governed by the incompressible Navier-Stokes equation is considered with zero-flux/Dirichlet/no-slip boundary conditions, and Winkler [20] is discussed about the small-mass solutions with no-flux/no-flux/Dirichlet boundary conditions in the system coupled to the Navier-Stokes equations. On the other hand, there is a little research (although, cf. Aruchamy and Tyagi [2]) on the systems with the Dirichlet boundary condition. Under the Neumann boundary condition, (KS) has the mass conservation law, which tells us that the amounts of the cellular slime molds do not change in time. Although under the Dirichlet boundary condition, (KS) does not have the mass conservation law, in our case, the condition means that the cellular slime molds die at the boundary, which seems be interesting. Furthermore, by introducing the Brownian motion process, we can extend the Keller-Segel system to the Stochastic Keller-Segel system (SKS), and we are interested in the study on (SKS) with the Dirichlet boundary condition.

This paper is organized as follows: In Section 2, we introduce some properties which will play an important role in this paper. Section 3 is the main section, in which we will give our main theorems and their proofs. Section 4 is devoted as an appendix.

2. Preliminaries. In this section, we introduce two basic properties which are used in the proofs of our main theorems: Firstly, Proposition 2.1 below, is the standard application of the Beta function.

**Proposition 2.1.** Let \(p, q\) be positive constants. Then for any \(t > 0\), it holds that
\[
\int_0^t (t - \tau)^{p-1} \tau^{q-1} \, d\tau = t^{p+q-1} B(p, q),
\]
where \(B(p, q)\) denotes the Beta function.

Secondly, let us denote \(L^r(I)\) as the Lebesgue space on \(I\) with norm \(\|u\|_{L^r(I)} \equiv (\int_I |u(x)|^r \, dx)^{1/r}\) for \(1 \leq r < \infty\), and \(\|u\|_{L^\infty(I)} = \text{ess sup}_{x \in I} |u(x)|\). The Sobolev space \(W^{m,r}(I)\), \(m = 1, 2, \ldots, 1 < r < \infty\) is the space of all functions \(u\) on \(I\) such that \(\|u\|_{W^{m,r}(I)} \equiv \sum_{i=0}^m \|D^i u\|_{L^r(I)} < \infty\), with the derivative \(D\) with respect to the variable \(x\). \(W^{1,2}_0(I)\) is a subspace of \(W^{1,2}(I)\), composed with the functions \(u\) satisfying \(\text{supp}[u] \subset I\).

In the following Lemma 2.1, we give some inequalities for the Dirichlet heat semigroup \(e^{t\Delta}\) (cf. Lemma 1.3 in [19], Lemma 2.1 in [3], Proposition 1 in [1]):
Lemma 2.1. Suppose $(e^{t\Delta})_{t>0}$ is the Dirichlet heat semigroup in $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. Then, there exist positive constants $k_1$ and $k_2$ such that the following estimates hold:

**For $1 \leq p \leq q \leq \infty$,**

\[
\|e^{t\Delta} w\|_{L^p(I)} \leq k_1 t^{\ell(\frac{1}{2} - \frac{1}{q})}\|w\|_{L^p(I)} \quad \text{for all } t > 0,
\]

holds for all $w \in L^p(I)$.

**For $1 \leq p \leq q \leq \infty$,**

\[
\|\nabla e^{t\Delta} w\|_{L^q(I)} \leq k_2 t^{\ell(\frac{1}{2} - \frac{1}{q})}\|w\|_{L^p(I)} \quad \text{for all } t > 0,
\]

is true for each $w \in L^p(I)$.

\[
\|\nabla e^{t\Delta} w\|_{L^q(I)} \leq \|\nabla w\|_{L^q(I)} \quad \text{for all } t > 0,
\]

is valid for any $w \in W^{1,2}(I)$.

\[
\|\Delta e^{t\Delta} w\|_{L^2(I)} \leq t^{-\frac{\ell}{2}}\|\nabla w\|_{L^2(I)} \quad \text{for all } t > 0,
\]

holds for any $w \in W^{1,2}_0(I)$.

**Proof of Lemma 2.1.** (2) is the well-known $L^p - L^q$ estimate (cf., eg., Chapter 3 in Davies [7]), and the proof of (5) is given in Appendix. Then we just give here the proofs of (3) and (4).

For each $w \in L^p(I)$ ($1 \leq p \leq \infty$), it holds that (cf. eg. Georgiev and Taniguchi [8])

\[
\|\nabla e^{t\Delta} w\|_{L^p(I)} \leq kt^{-\frac{\ell}{2}}\|w\|_{L^p(I)} \quad \text{for all } t > 0,
\]

where $k > 0$ is a constant. By (2) we obtain

\[
\|\nabla e^{t\Delta} w\|_{L^q(I)} = \|\nabla e^{\frac{t}{2}\Delta} (e^{\frac{t}{2}\Delta} w)\|_{L^q(I)} \leq k\left(\frac{t}{2}\right)^{\ell} \|e^{\frac{t}{2}\Delta} w\|_{L^q(I)} \leq k_2 t^{\ell(\frac{1}{2} - \frac{1}{q})} \|w\|_{L^p(I)},
\]

where $k_2 = kk_1$, which proves (3).

We set $z \equiv e^{t\Delta} w$ ($w \in W^{1,2}(I)$). Multiplying $z_t = \Delta z$ by $-\Delta z$ and integrating shows that

\[
\frac{d}{dt}\|\nabla z(t)\|_{L^2(I)}^2 = -\|\Delta z(t)\|_{L^2(I)}^2 \leq 0,
\]

which implies (4) (cf. Proposition A.2 in Da Prato and Zabczyk [6]).

$\square$

3. **Main results.** For given $T$ ($0 < T < 1$), we firstly present Banach spaces $X_T$ and $Y_T$ which are used to state our main results.

\[\begin{align*}
X_T &\equiv \{ u \in L^\infty(0, T; L^2(I)); \ t^{\ell} (\nabla u) \in L^\infty(0, T; L^2(I)), u = 0 \text{ on } \partial I \}, \\
Y_T &\equiv \{ v \in L^\infty(0, T; W^{1,2}_0(I)); \ t^{\ell} (\Delta v) \in L^\infty(0, T; L^2(I)) \},
\end{align*}\]

with the norms

\[
\|u\|_{X_T} \equiv \sup_{0 < t < T} \|u(t)\|_{L^2(I)} + \sup_{0 < t < T} t^{\ell}\|\nabla u(t)\|_{L^2(I)},
\]

\[
\|v\|_{Y_T} \equiv \sup_{0 < t < T} \|v(t)\|_{L^2(I)} + \sup_{0 < t < T} \|\nabla v(t)\|_{L^2(I)} + \sup_{0 < t < T} t^{\ell}\|\Delta v(t)\|_{L^2(I)}.
\]

Then we give a definition of the mild solution of (KS) on $I \times (0, T)$.
Definition 3.1. Let $X_T$, $Y_T$ be the Banach spaces defined by (7) and (8) respectively, and $u_0$, $v_0$ be the nonnegative functions. A pair $(u,v) \in X_T \times Y_T$ is a mild solution of (KS) on $I \times (0,T)$ if the identities
\[
u(t) = e^{-t(-\Delta + \gamma)} v_0 + \alpha \int_0^t e^{-(t-\tau)(-\Delta + \gamma)} u(\tau) \, d\tau,
\]
hold for $0 < t < T$, where $\chi, \alpha, \gamma$ are the constants appearing in (KS).

Our first result is on existence and uniqueness of mild solution to (KS), which is so curious as shown in [5], [13].

Theorem 3.2. Let $0 < T < 1$. We assume that nonnegative initial data $(u_0, v_0)$ satisfies $u_0 \in L^2(I)$ and $v_0 \in W^{1,2}_0(I)$. Furthermore we assume that $\|u_0\|_{L^2(I)}$ and $\|\nabla v_0\|_{L^2(I)}$ are sufficiently small. Then, there exists a unique mild solution $(u_*, v_*)$ of (KS) on $I \times (0,T)$.

The second result is on the continuity of solution for the initial data, which is also often investigated (cf. [18]).

Theorem 3.3. Let $0 < T < 1$. For every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if nonnegative functions $(u_0, v_0), (\tilde{u}_0, \tilde{v}_0) \in L^2(I) \times W^{1,2}_0(I)$ satisfy
\[
u_0 - \tilde{u}_0\|_{L^2(I)} + \|v_0 - \tilde{v}_0\|_{W^{1,2}_0(I)} < \delta,
\]
and if $(u, v)$ and $(\tilde{u}, \tilde{v})$ are the mild solutions of (KS) on $I \times (0,T)$ with $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, $\tilde{u}(x,0) = \tilde{u}_0(x)$, $\tilde{v}(x,0) = \tilde{v}_0(x)$ respectively, then it holds that
\[
u - \tilde{u}\|_{Y_T} + \|v - \tilde{v}\|_{Y_T} < \epsilon.
\]

Remark 1. In the proofs of the above theorems, an argument of successive approximations plays an important role. Indeed, as we show below, we introduce the sequences of numbers \{m\}, \{b_n\}, \{c_m\} which are bounded above and non-decreasing, and the assumption $0 < T < 1$ is necessary to show that (see e.g. inequality (18) and the above it). It would be possible to extend $0 < T < \infty$, which will be discussed in forthcoming papers.

Proof of Theorem 3.2. Before giving a proof, we define the sequences \{m\}, \{b_m\} as follows:

For $t \ (0 \leq t \leq T),$
\[
u_1(t) = e^{t\Delta} u_0, \quad \nu_2(t) = e^{-(t-\Delta + \gamma)} v_0,
\]
\[
u_{m+1}(t) = e^{t\Delta} u_0 - \chi \int_0^t e^{-(t-\tau)\Delta \nabla (u_m \nabla v_m)(\tau)} \, d\tau,
\]
\[
u_{m+1}(t) = e^{-(t-\Delta + \gamma)} v_0 + \alpha \int_0^t e^{-(t-\tau)(-\Delta + \gamma)} u_m(\tau) \, d\tau,
\]
where $\alpha, \gamma, \chi > 0$ are the constants appearing in (KS).

I. Existence part of the proof

To prove the existence part of Theorem 3.2, we go through three steps (cf. [1], Yahagi [21]).
Step 1. By mathematical induction, we shall show that there exist three sequences of numbers \( \{a_m\}, \{b_m\} \) and \( \{c_m\} \) satisfying
\[
\sup_{0 < t < T} \|u_m(t)\|_{L^2(I)} \leq a_m, \tag{11}
\]
\[
\sup_{0 < t < T} t^{\frac{1}{2}} \|\nabla u_m(t)\|_{L^2(I)} \leq a_m, \tag{12}
\]
\[
\sup_{0 < t < T} \|v_m(t)\|_{L^2(I)} \leq c_m, \tag{13}
\]
\[
\sup_{0 < t < T} \|\nabla v_m(t)\|_{L^2(I)} \leq b_m, \tag{14}
\]
\[
\sup_{0 < t < T} t^{\frac{1}{2}} \|\Delta v_m(t)\|_{L^2(I)} \leq b_m. \tag{15}
\]
Indeed, by using (2), (6), since it holds that
\[
\|u_1(t)\|_{L^2(I)} \leq k_1\|u_0\|_{L^2(I)}, \quad t^{\frac{1}{2}}\|\nabla u_1(t)\|_{L^2(I)} \leq k_2\|u_0\|_{L^2(I)},
\]
we can define that
\[
a_1 \equiv \max\{k_1, k_2\}\|u_0\|_{L^2(I)}. \tag{16}
\]
As well as the above, we can define that
\[
c_1 \equiv k_1\|v_0\|_{L^2(I)}, \quad b_1 \equiv \|\nabla v_0\|_{L^2(I)}, \tag{17}
\]
where \( k_1 \) and \( k_2 \) are the constants appearing in (2) and (3), respectively. Thus, (11) - (15) hold for \( m = 1 \).

Next, we assume that (11) - (15) are true for some \( m \), and we will show that they are also true for \( m + 1 \). In fact, by (1) and (2) we have
\[
\|u_{m+1}(t)\|_{L^2(I)} \leq \|u_1(t)\|_{L^2(I)} + \chi \int_0^t \left| e^{(t-\tau)}\Delta \nabla (u_m \nabla v_m) (\tau) \right|_{L^2(I)} \, d\tau
\]
\[
\leq \|u_1(t)\|_{L^2(I)} + \chi \int_0^t k_1 (t-\tau)^{-\frac{1}{2}} \|\nabla u_m \nabla v_m (\tau) \|_{L^2(I)} \, d\tau
\]
\[
+ \left( u_m \Delta v_m (\tau) \right)_{L^2(I)} \, d\tau
\]
\[
\leq a_1 + \chi \int_0^t k_1 (t-\tau)^{-\frac{1}{2}} \|\nabla u_m (\tau)\|_{L^2(I)} \|\nabla v_m (\tau)\|_{L^2(I)} \, d\tau
\]
\[
+ \|u_m (\tau)\|_{L^2(I)} \|\Delta v_m (\tau)\|_{L^2(I)} \, d\tau
\]
\[
\leq a_1 + 2\chi k_1 \int_0^t (t-\tau)^{-\frac{1}{2}} a_m \tau^{-\frac{1}{2}} b_m \, d\tau
\]
\[
\leq a_1 + 2\chi k_1 B\left(\frac{3}{4}, \frac{1}{2}\right) a_m b_m t^{\frac{1}{2}},
\]
where \( k_1 \) is the constant in (2) and \( B\left(\frac{3}{4}, \frac{1}{2}\right) \) is also the constant in (1). With the assumption \( 0 < T < 1 \), we have
\[
\sup_{0 < t < T} \|u_{m+1}(t)\|_{L^2(I)} \leq a_1 + 2\chi k_1 B\left(\frac{3}{4}, \frac{1}{2}\right) a_m b_m. \tag{18}
\]
As well as the above, we obtain the followings:
\[
\sup_{0 < t < T} t^{\frac{1}{2}} \|\nabla u_{m+1}(t)\|_{L^2(I)} \leq a_1 + 2\chi k_2 B\left(\frac{1}{4}, \frac{1}{2}\right) a_m b_m, \tag{19}
\]
\[
\sup_{0 < t < T} \|v_{m+1}(t)\|_{L^2(I)} \leq c_1 + \alpha k_1 a_m, \tag{20}
\]
\[
\sup_{0 < t < T} \|\nabla v_{m+1}(t)\|_{L^2(I)} \leq b_1 + 2\alpha k_2 a_m, \tag{21}
\]
\[
\sup_{0 < t < T} t^{\frac{1}{2}} \|\Delta v_{m+1}(t)\|_{L^2(I)} \leq b_1 + \alpha \pi a_m, \tag{22}
\]
where \( k_1 \) and \( k_2 \) are the constant in (2), (3), respectively, and \( B\left(\frac{1}{4},\frac{1}{2}\right) \) is also the constant in (1). By (18) – (22), we set
\[
\rho \equiv 2 \max \left\{ k_1 B\left(\frac{3}{4},\frac{1}{2}\right), k_2 B\left(\frac{1}{4},\frac{1}{2}\right) \right\}, \quad \eta \equiv \max\{2k_2, \pi\},
\]
and we define
\[
a_{m+1} \equiv a_1 + \chi \rho a_m b_m, \quad c_{m+1} \equiv c_1 + \alpha k_1 a_m, \quad b_{m+1} \equiv b_1 + \alpha \eta a_m.
\]
Thus, we have proved that (11) – (15) are true for \( m + 1 \).

Step 2. In the following, by the inductive method, we shall prove that the sequences of numbers \( \{a_m\}, \{b_m\}, \{c_m\} \) defined by (16), (17) and (24), (25), (26) are bounded above and non-decreasing.

Indeed, we consider the following equations:
\[
\begin{aligned}
x &= a_1 + \chi \rho x, \\
z &= c_1 + \alpha k_1 x, \\
y &= b_1 + \alpha \eta x.
\end{aligned}
\]

From the above first and third equations, we have the following quadratic equation for \( x \):
\[
\chi \alpha \rho \eta x^2 + (\chi \rho b_1 - 1)x + a_1 = 0.
\]

Here, we can choose the initial data \( u_0 \) and \( \nabla v_0 \) sufficiently small such that
\[
b_1 < \frac{1}{\chi \rho}, \quad a_1 < \frac{(\chi \rho b_1 - 1)^2}{4\chi \alpha \rho \eta},
\]
where \( a_1 \equiv \max\{k_1, k_2\}||u_0||_{L^2(I)}, \quad b_1 \equiv ||\nabla v_0||_{L^2(I)} \) (cf. (16), (17)). Then, it holds that
\[
\chi \rho b_1 - 1 < 0, \quad (\chi \rho b_1 - 1)^2 - 4\chi \alpha \rho \eta a_1 > 0.
\]
(29) means that (27) has two positive roots for \( x \). Let \( x_* \) be the smallest root of (27) under the condition (28), that is,
\[
x_* \equiv -\frac{(\chi \rho b_1 - 1) - \sqrt{(\chi \rho b_1 - 1)^2 - 4\chi \alpha \rho \eta a_1}}{2\chi \alpha \rho \eta}.
\]

Further we define
\[
\begin{aligned}
z_* &\equiv c_1 + \alpha k_1 x_*, \\
y_* &\equiv b_1 + \alpha \eta x_*.
\end{aligned}
\]

By induction, it is easily proved that
\[
a_m \leq x_*, \quad b_m \leq y_*, \quad c_m \leq z_*.
\]

Step 3. We shall show that there exist \( (u_*, v_*) \in X_T \times Y_T \) such that
\[
\begin{aligned}
&\sup_{0 < t < T} \|u_m(t) - u_*(t)\|_{L^2(I)} \to 0 \quad (m \to \infty), \\
&\sup_{0 < t < T} \int_0^t \|\nabla u_m(t) - \nabla u_*(t)\|_{L^2(I)} \to 0 \quad (m \to \infty), \\
&\sup_{0 < t < T} \|v_m(t) - v_*(t)\|_{L^2(I)} \to 0 \quad (m \to \infty), \\
&\sup_{0 < t < T} \|\nabla v_m(t) - \nabla v_*(t)\|_{L^2(I)} \to 0 \quad (m \to \infty),
\end{aligned}
\]
\[
\sup_{0 < t < T} t^\frac{1}{2} \| \Delta v_m(t) - \Delta v_*(t) \|_{L^2(I)} \to 0 \quad (m \to \infty).
\]  

(37)

The proofs of (33) - (37) can be carried out as follows: Firstly, we define the sequences \( \{U_m\}, \{V_m\} \) by

\[
\begin{align*}
U_{m+1}(t) &\equiv u_{m+1}(t) - u_m(t) \quad (m \geq 1), \quad U_1(t) \equiv u_1(t), \\
V_{m+1}(t) &\equiv v_{m+1}(t) - v_m(t) \quad (m \geq 1), \quad V_1(t) \equiv v_1(t).
\end{align*}
\]

Then, we shall show that there exist sequences of numbers \( \{A_m\}, \{B_m\}, \{C_m\} \) such that

\[
\begin{align*}
\sup_{0 < t < T} ||U_m(t)||_{L^2(I)} &\leq A_m, \quad (38) \\
\sup_{0 < t < T} t^\frac{1}{2} ||\nabla U_m(t)||_{L^2(I)} &\leq A_m, \quad (39) \\
\sup_{0 < t < T} ||V_m(t)||_{L^2(I)} &\leq C_m, \quad (40) \\
\sup_{0 < t < T} ||\nabla V_m(t)||_{L^2(I)} &\leq B_m, \quad (41) \\
\sup_{0 < t < T} t^\frac{1}{2} ||\Delta V_m(t)||_{L^2(I)} &\leq B_m. \quad (42)
\end{align*}
\]

Indeed, by setting

\[
A_1 \equiv a_1, \quad B_1 \equiv b_1, C_1 \equiv c_1,
\]

(38) - (42) are true for \( m = 1 \). Then, we assume that (38) - (42) are true for some \( m \). we will show that they are also true for \( m+1 \). In fact,

\[
\begin{align*}
||U_{m+1}(t)||_{L^2(I)} &\leq \chi \int_0^t \| e^{(t-\tau)\Delta} \left( \nabla U_m \nabla v_m + \nabla u_m - \nabla \Delta v_m + U_m \Delta v_m + u_m - v_m \right)(\tau) \|_{L^2(I)} \, d\tau \\
&\leq \chi k_1 \int_0^t (t-\tau)^{-\frac{1}{2}} \left( \| \nabla U_m \nabla v_m + \nabla u_m - \nabla \Delta v_m + U_m \Delta v_m + u_m - v_m \|_{L^2(I)} \right) \| \nabla \Delta v_*(\tau) \|_{L^2(I)} \| \nabla \Delta v_m(\tau) \|_{L^2(I)} \| \Delta v_m(\tau) \|_{L^2(I)} \| \nabla v_*(\tau) \|_{L^2(I)} \| \Delta v_*(\tau) \|_{L^2(I)} \right) \, d\tau \\
&\leq 2k_1 \int_0^t (t-\tau)^{-\frac{1}{2}} \left\{ A_m y_s + x_s B_m \right\} \tau^{-\frac{1}{2}} \, d\tau \\
&\leq 2k_1 B \left( 4 - \frac{1}{3} \right) \left\{ A_m y_s + x_s B_m \right\} t^\frac{1}{2},
\end{align*}
\]

where \( k_1 \) is the constant in (2) and \( B \left( \frac{3}{4}, \frac{1}{2} \right) \) is also the constant in (1). With the assumption \( 0 < T < 1 \), we have

\[
\sup_{0 < t < T} ||U_{m+1}(t)||_{L^2(I)} \leq 2\chi k_1 B \left( \frac{3}{4}, \frac{1}{2} \right) (A_m y_s + x_s B_m). \quad (43)
\]

As well as the above, we have the followings:

\[
\begin{align*}
\sup_{0 < t < T} t^\frac{1}{2} ||\nabla U_{m+1}(t)||_{L^2(I)} &\leq 2\chi k_3 B \left( \frac{1}{4}, \frac{1}{2} \right) (A_m y_s + x_s B_m). \quad (44) \\
\sup_{0 < t < T} ||V_{m+1}(t)||_{L^2(I)} &\leq \alpha k_1 A_m. \quad (45) \\
\sup_{0 < t < T} ||\nabla V_{m+1}(t)||_{L^2(I)} &\leq 2\alpha A_m. \quad (46) \\
\sup_{0 < t < T} t^\frac{1}{2} ||\Delta V_{m+1}(t)||_{L^2(I)} &\leq \pi \alpha A_m. \quad (47)
\end{align*}
\]
By (43) – (47), we set
\[ A_{m+1} = \rho(A_m y_s + x_s B_m), \quad (48) \]
\[ C_{m+1} = \alpha k_1 A_m, \quad (49) \]
\[ B_{m+1} = \alpha \eta A_m, \]
where \( \rho \) and \( \eta \) are given in (23). Then, we have proved that the estimates (38) – (42) hold for \( m + 1 \).

By (48) and (49), we have
\[ A_{m+1} = \rho(y_s + x_s \alpha \eta) A_m. \quad (50) \]
Here, we set
\[ r \equiv \rho(y_s + x_s \alpha \eta)(> 0). \]
By substituting (30) and (31) into (50), we obtain
\[ r = 1 - \frac{\sqrt{(\chi \rho b_1 - 1)^2 - 4 \chi \rho \eta a_1}}{\chi}, \quad (51) \]
In the case of \( \chi > 1 \), (51) implies that \( r < 1 \) holds. On the other hand, in the case of \( 0 < \chi \leq 1 \), we can choose \( u_0 \) such that
\[ a_1 < \frac{\chi \rho^2 b_1^2 - 2 \rho b_1 + 2 - \chi}{4 \alpha \rho \eta}, \quad (52) \]
where \( a_1 \equiv \max\{k_1, k_2\} \|u_0\|_{L^2(I)} \), \( b_1 \equiv \|\nabla v_0\|_{L^2(I)} \) (cf. (16), (17)), and (52) means that \( r < 1 \).

Under the condition \( 0 < r < 1 \), it holds that \( A_m \) converges to 0 as \( m \to \infty \), and also \( B_m, C_m \).

Hence, we have the followings:
\[ \sup_{0 < t < T} \|u_{m+1}(t) - u_m(t)\|_{L^2(I)} = \sup_{0 < t < T} \|U_{m+1}(t)\|_{L^2(I)} \leq A_{m+1} \to 0, \quad (53) \]
as \( m \to \infty \).
\[ \sup_{0 < t < T} t^{-\frac{1}{2}} \|\nabla u_{m+1}(t) - \nabla u_m(t)\|_{L^2(I)} = \sup_{0 < t < T} t^{-\frac{1}{2}} \|\nabla U_{m+1}(t)\|_{L^2(I)} \leq A_{m+1} \to 0, \quad (54) \]
as \( m \to \infty \).
\[ \sup_{0 < t < T} \|v_{m+1}(t) - v_m(t)\|_{L^2(I)} = \sup_{0 < t < T} \|V_{m+1}(t)\|_{L^2(I)} \leq C_{m+1} \to 0, \quad (55) \]
as \( m \to \infty \).
\[ \sup_{0 < t < T} \|\nabla v_{m+1}(t) - \nabla v_m(t)\|_{L^2(I)} = \sup_{0 < t < T} \|\nabla V_{m+1}(t)\|_{L^2(I)} \leq B_{m+1} \to 0, \quad (56) \]
as \( m \to \infty \).
\[ \sup_{0 < t < T} t^{-\frac{1}{2}} \|\Delta v_{m+1}(t) - \Delta v_m(t)\|_{L^2(I)} = \sup_{0 < t < T} t^{-\frac{1}{2}} \|\Delta V_{m+1}(t)\|_{L^2(I)} \leq B_{m+1} \to 0, \quad (57) \]
as \( m \to \infty \).

The above results (53) – (57) mean that \( \{u_m\} \) and \( \{v_m\} \) are the Cauchy sequences in the Banach spaces \( X_T \) and \( Y_T \) respectively. Indeed, we set
\[ S_n = \sum_{k=1}^{n} A_k = \frac{A_1(1 - r^n)}{1 - r}. \]
Then, since the number sequence \( \{S_n\} \) is convergent, it follows that \( \{S_n\} \) is also a Cauchy sequence. Thus we have
\[ \sup_{0 < t < T} \|u_{m+n}(t) - u_m(t)\|_{L^2(I)} \leq S_{m+n} - S_m \to 0 \quad (n, m \to \infty). \]
In the same way, it holds that
\[
\sup_{0 < t < T} t^\frac{1}{2} \|\nabla u_{m+n}(t) - \nabla u_n(t)\|_{L^2(I)} \to 0 \quad (m, n \to \infty),
\]
\[
\sup_{0 < t < T} \|v_{m+n}(t) - v_n(t)\|_{L^2(I)} \to 0 \quad (m, n \to \infty),
\]
\[
\sup_{0 < t < T} \|\nabla v_{m+n}(t) - \nabla v_n(t)\|_{L^2(I)} \to 0 \quad (m, n \to \infty),
\]
\[
\sup_{0 < t < T} t^\frac{1}{2} \|\Delta v_{m+n}(t) - \Delta v_n(t)\|_{L^2(I)} \to 0 \quad (m, n \to \infty).
\]

Since \(X_T\) and \(Y_T\) are the Banach spaces, there exist \((u_*, v_*)\) \(\in X_T \times Y_T\) such that
\[
\sup_{0 < t < T} \|u_m(t) - u_*(t)\|_{L^2(I)} \to 0 \quad (m \to \infty),
\]
\[
\sup_{0 < t < T} t^\frac{1}{2} \|\nabla u_m(t) - \nabla u_*(t)\|_{L^2(I)} \to 0 \quad (m \to \infty),
\]
\[
\sup_{0 < t < T} \|v_m(t) - v_*(t)\|_{L^2(I)} \to 0 \quad (m \to \infty),
\]
\[
\sup_{0 < t < T} \|\nabla v_m(t) - \nabla v_*(t)\|_{L^2(I)} \to 0 \quad (m \to \infty),
\]
\[
\sup_{0 < t < T} t^\frac{1}{2} \|\Delta v_m(t) - \Delta v_*(t)\|_{L^2(I)} \to 0 \quad (m \to \infty).
\]

Finally, in the proof of existence part, we can certify that \((u_*, v_*)\) is a mild solution of (KS) on \(I \times (0, T)\). For \(u_*,\) it holds that
\[
\|u_*(t) - (e^{t\Delta}u_0 - \chi \int_0^t e^{(t-\tau)\Delta} \nabla (u_* \nabla v_*) (\tau) \, d\tau)\|_{L^2(I)}
\leq \|u_*(t) - u_{m+1}(t)\|_{L^2(I)} + \chi \int_0^t \|e^{(t-\tau)\Delta} \nabla (u_* \nabla v_*) (\tau) - \nabla (u_m \nabla v_m) (\tau)\|_{L^2(I)} \, d\tau
\leq \|u_*(t) - u_{m+1}(t)\|_{L^2(I)} + \chi \int_0^t (t - \tau)^{-\frac{1}{2}} \|\nabla (u_* \nabla v_*) (\tau) - \nabla (u_m \nabla v_m) (\tau)\|_{L^2(I)} \, d\tau
\leq \|u_*(t) - u_{m+1}(t)\|_{L^2(I)}
+ \chi \int_0^t (t - \tau)^{-\frac{1}{2}} \|\nabla (u_* \nabla v_*) (\tau) - \nabla (u_m \nabla v_m) (\tau)\|_{L^2(I)} \, d\tau
+ \|u_m(\tau)\|_{L^2(I)} \|\nabla (\Delta u_\tau - \Delta v_\tau)\|_{L^2(I)} \, d\tau
\leq \sup_{0 < t < T} \|u_*(t) - u_{m+1}(t)\|_{L^2(I)}
+ \chi \int_0^t \|\nabla (u_* - u_m)(\tau)\|_{L^2(I)} \, d\tau
+ \|u_\tau(\tau)\|_{L^2(I)} \|\nabla (\Delta u_\tau - \Delta v_\tau)\|_{L^2(I)} \, d\tau
\rightarrow 0 \quad (m \to \infty).
\]

This means that
\[
\sup_{0 < t < T} \|u_*(t) - (e^{t\Delta}u_0 - \chi \int_0^t e^{(t-\tau)\Delta} \nabla (u_* \nabla v_*) (\tau) \, d\tau)\|_{L^2(I)} = 0. \quad (58)
\]
As well as the above, we can easily see that
\[
\sup_{0 < t < T} t^\frac{1}{2} \| \nabla u_*(t) - \nabla \left( e^{t\Delta} u_0 - \chi \int_0^t e^{(t-\tau)\Delta} \nabla (u_* \nabla u_*) (\tau) \, d\tau \right) \|_{L^2(I)} = 0. \tag{59}
\]

From (58) and (59), we obtain
\[
u_*(t) = e^{t\Delta} u_0 - \chi \int_0^t e^{(t-\tau)\Delta} \nabla (u_* \nabla u_*) (\tau) \, d\tau \quad \text{in} \quad X_T. \tag{60}
\]

For \( v_* \), it holds that
\[
\| v_*(t) - \left( e^{-t(-\Delta+\gamma)} v_0 + \alpha \int_0^t e^{-(t-\tau)(-\Delta+\gamma)} u_m (\tau) \, d\tau \right) \|_{L^2(I)} \\
\leq \| v_*(t) - v_{m+1} (t) \|_{L^2(I)} + \alpha \int_0^t \| e^{-(t-\tau)(-\Delta+\gamma)} (u_*(t) - u_m (\tau)) \|_{L^2(I)} \, d\tau, \\
\leq \sup_{0 < t < T} \| v_*(t) - v_{m+1} (t) \|_{L^2(I)} + \alpha k_1 \sup_{0 < t < T} \| u_*(t) - u_m (\tau) \|_{L^2(I)} \\
\rightarrow 0 \quad (m \to \infty).
\]

Also, it holds that
\[
\sup_{0 < t < T} \| v_*(t) - \left( e^{-t(-\Delta+\gamma)} v_0 + \alpha \int_0^t e^{-(t-\tau)(-\Delta+\gamma)} u_m (\tau) \, d\tau \right) \|_{L^2(I)} = 0, \tag{61}
\]
\[
\sup_{0 < t < T} \| \nabla v_*(t) - \nabla \left( e^{-t(-\Delta+\gamma)} v_0 + \alpha \int_0^t e^{-(t-\tau)(-\Delta+\gamma)} u_m (\tau) \, d\tau \right) \|_{L^2(I)} = 0, \tag{62}
\]
\[
\sup_{0 < t < T} t^\frac{1}{2} \| \nabla v_*(t) - \Delta \left( e^{-t(-\Delta+\gamma)} v_0 + \alpha \int_0^t e^{-(t-\tau)(-\Delta+\gamma)} u_m (\tau) \, d\tau \right) \|_{L^2(I)} = 0. \tag{63}
\]

(61), (62) and (63) imply that
\[
u_*(t) = e^{-t(-\Delta+\gamma)} v_0 + \alpha \int_0^t e^{-(t-\tau)(-\Delta+\gamma)} u_m (\tau) \, d\tau \quad \text{in} \quad Y_T. \tag{64}
\]

(60) and (64) show that \((u_*, v_*)\) is a mild solution of (KS) on \( I \times (0, T) \).

II. Uniqueness part of the proof

We shall prove that \((u_*, v_*)\) is the unique mild solution of (KS) on \( I \times (0, T) \) with the same initial data \((u_0, v_0)\). Let \((u, v)\) and \((u_*, v_*)\) be the mild solutions of (KS) on \( (0, T) \) with the initial data \((u_0, v_0)\), where \( u_* = \lim_{m \to \infty} u_m, \quad v_* = \lim_{m \to \infty} v_m \). Then by using Proposition 2.1, Lemma 2.1, it holds that
\[
\| (u - u_*) (t) \|_{L^2(I)} \\
\leq \chi \int_0^t \| e^{(t-\tau)\Delta} \left( \nabla (u_0 v) - \nabla (u_* v) \right) (\tau) \|_{L^2(I)} \, d\tau \\
\leq \chi k_1 \int_0^t (t-\tau)^{-\frac{1}{2}} \left| \left( \nabla (u_0 v) + \nabla u_* (v - v_*) + (u - u_*) \Delta v + u_* (v - v_*) \right) (\tau) \right|_{L^1(I)} \, d\tau \\
\leq \chi k_1 \int_0^t (t-\tau)^{-\frac{1}{2}} \left\{ \| \nabla (u_0 v) (\tau) \|_{L^2(I)} \| \nabla v (\tau) \|_{L^2(I)} + \| \nabla u_* (\tau) \|_{L^2(I)} \| \nabla (v - v_*) (\tau) \|_{L^2(I)} + \| u_0 (\tau) \|_{L^2(I)} \| \Delta (v - v_*) (\tau) \|_{L^2(I)} \right\} \, d\tau \\
\leq \chi k_1 \int_0^t (t-\tau)^{-\frac{1}{2}} r^{-\frac{1}{2}} \left\{ \sup_{0 < r < T} \tau^\frac{1}{2} \| \nabla (u - u_*) (\tau) \|_{L^2(I)} \sup_{0 < r < T} \| \nabla v (\tau) \|_{L^2(I)} \\
+ x_* \sup_{0 < r < T} \| \nabla (v - v_*) (\tau) \|_{L^2(I)} + \sup_{0 < r < T} \| (u - u_*) (\tau) \|_{L^2(I)} \sup_{0 < r < T} \| \nabla v (\tau) \|_{L^2(I)} \right\} \right\} \\
\leq \chi k_1 B \left( \frac{3}{2} \frac{1}{4} \frac{1}{2} \right) \times \left\{ \sup_{0 < r < T} \tau^\frac{1}{2} \| \nabla (u - u_*) (\tau) \|_{L^2(I)} \sup_{0 < r < T} \| \nabla v (\tau) \|_{L^2(I)} \right\} \]
\[ + x_\ast \sup_{0 < \tau < T} \| \nabla (v - v_\ast) (\tau) \|_{L^2(I)} + \sup_{0 < \tau < T} \| (u - u_\ast) (\tau) \|_{L^2(I)} \sup_{0 < \tau < T} \tau^{\frac{1}{2}} \| \Delta v (\tau) \|_{L^2(I)} \\
\leq \chi k_1 B \left( \frac{4}{4}, \frac{1}{2} \right) A(v) T^{\frac{1}{2}} \left( \| u - u_\ast \|_{X_T} + \| v - v_\ast \|_{Y_T} \right). \]

Then for any \( t \) \((0 < t < T)\), it holds that

\[ \sup_{0 < t < T} \| (u - u_\ast) (t) \|_{L^2(I)} \leq \chi k_1 B \left( \frac{4}{4}, \frac{1}{2} \right) A(v) T^{\frac{1}{2}} \left( \| u - u_\ast \|_{X_T} + \| v - v_\ast \|_{Y_T} \right). \] (65)

As well as the above estimates, for any \( t \) \((0 < t < T)\), we obtain the followings:

\[ \sup_{0 < t < T} t^{\frac{1}{2}} \| \nabla (v - v_\ast) (t) \|_{L^2(I)} \leq \chi k_2 B \left( \frac{4}{4}, \frac{1}{2} \right) A(v) T^{\frac{1}{2}} \left( \| u - u_\ast \|_{X_T} + \| v - v_\ast \|_{Y_T} \right), \] (66)

\[ \sup_{0 < t < T} \| (v - v_\ast) (t) \|_{L^2(I)} \leq \alpha k_1 A(v) T^{\frac{1}{2}} \left( \| u - u_\ast \|_{X_T} + \| v - v_\ast \|_{Y_T} \right), \] (67)

\[ \sup_{0 < t < T} \| \nabla (v - v_\ast) (t) \|_{L^2(I)} \leq \alpha k_2 A(v) T^{\frac{1}{2}} \left( \| u - u_\ast \|_{X_T} + \| v - v_\ast \|_{Y_T} \right), \] (68)

\[ \sup_{0 < t < T} t^{\frac{1}{2}} \| \Delta (v - v_\ast) (t) \|_{L^2(I)} \leq \alpha \pi A(v) T^{\frac{1}{2}} \left( \| u - u_\ast \|_{X_T} + \| v - v_\ast \|_{Y_T} \right). \] (69)

As a result, by setting

\[ \mu \equiv \max \{ \chi k_1 B \left( \frac{4}{4}, \frac{1}{2} \right), \chi k_2 B \left( \frac{4}{4}, \frac{1}{2} \right), \alpha k_1, 2 \alpha k_2, \alpha \pi \}, \]

we have

\[ \| u - u_\ast \|_{X_T} + \| v - v_\ast \|_{Y_T} \leq \mu A(v) T^{\frac{1}{2}} \left( \| u - u_\ast \|_{X_T} + \| v - v_\ast \|_{Y_T} \right). \] (70)

Since (70) holds for any \( T \) \((0 < T < 1)\), we can choose \( T_1 \) such that

\[ 1 - \mu A(v) T_1^{\frac{1}{2}} > 0, \]

Then it follows that

\[ u = u_\ast \text{ in } X_{T_1}, \ v = v_\ast \text{ in } Y_{T_1}. \]

Finally, starting from the first step \( u \equiv u_\ast \) on \([0, T_1]\), and then repeating after \([T_1, 2T_1]\), within \( N \) steps such that \( NT_1 > T \), we conclude that \( u \equiv u_\ast \) on \([0, T]\). This proves the uniqueness part of Theorem 3.2. \( \square \)

**Proof of Theorem 3.2.** Let \((u, v)\), and \((\tilde{u}, \tilde{v})\) be the mild solutions of (KS) with the initial data \((u_0, v_0)\), and \((\tilde{u}_0, \tilde{v}_0)\) respectively. From the proof of Theorem 3.2 (cf. (18) - (22), and (32)), there exist \( x_\ast, \tilde{y}_\ast > 0 \) such that

\[ \sup_{0 < t < T} \| u(t) \|_{L^2(I)} \leq x_\ast, \]

\[ \sup_{0 < t < T} t^{\frac{1}{2}} \| \nabla u(t) \|_{L^2(I)} \leq x_\ast, \]

\[ \sup_{0 < t < T} \| \nabla \tilde{v}(t) \|_{L^2(I)} \leq \tilde{y}_\ast, \]

\[ \sup_{0 < t < T} t^{\frac{1}{2}} \| \Delta \tilde{v}(t) \|_{L^2(I)} \leq \tilde{y}_\ast. \]
By similar estimates to the proof of Theorem 3.2 (cf. (65) – (69)), we obtain
\[
\sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2(I)} \leq k_1 \| u_0 - \tilde{u}_0 \|_{L^2(I)} + 2\chi k_1 B \left( \frac{3}{4}, \frac{1}{2} \right) (x_* + \tilde{y}_*) T^{\frac{3}{2}} \left( \| u - \tilde{u} \|_{X_T} + \| v - \tilde{v} \|_{Y_T} \right),
\]
(71)
\[
\sup_{0 < t < T} \tau^2 \| \nabla (u - \tilde{u})(t) \|_{L^2(I)} \leq k_2 \| u_0 - \tilde{u}_0 \|_{L^2(I)}
\]
\[
+ 2\chi k_2 B \left( \frac{1}{4}, \frac{1}{2} \right) (x_* + \tilde{y}_*) T^{\frac{1}{2}} \left( \| u - \tilde{u} \|_{X_T} + \| v - \tilde{v} \|_{Y_T} \right),
\]
(72)
\[
\sup_{0 < t < T} \| (v - \tilde{v})(t) \|_{L^2(I)} \leq k_1 \| v_0 - \tilde{v}_0 \|_{L^2(I)} + \alpha k_1 T^{\frac{1}{2}} \left( \| u - \tilde{u} \|_{X_T} + \| v - \tilde{v} \|_{Y_T} \right),
\]
(73)
\[
\sup_{0 < t < T} \| \nabla (v - \tilde{v})(t) \|_{L^2(I)} \leq k_1 \| v_0 - \tilde{v}_0 \|_{L^2(I)} + 2\alpha k_2 T^{\frac{1}{2}} \left( \| u - \tilde{u} \|_{X_T} + \| v - \tilde{v} \|_{Y_T} \right),
\]
(74)
\[
\sup_{0 < t < T} \tau^2 \| \Delta (v - \tilde{v})(t) \|_{L^2(I)} \leq \alpha \pi T^{\frac{1}{2}} \left( \| u - \tilde{u} \|_{X_T} + \| v - \tilde{v} \|_{Y_T} \right). \tag{75}
\]
From (71) – (75), we obtain (cf. (9), (10))
\[
\| u - \tilde{u} \|_{X_T} + \| v - \tilde{v} \|_{Y_T} \leq \alpha (\| u_0 - \tilde{u}_0 \|_{L^2(I)} + \| v_0 - \tilde{v}_0 \|_{W^{1,2}(I)}) + b \tau \left( \| u - \tilde{u} \|_{X_T} + \| v - \tilde{v} \|_{Y_T} \right), \tag{76}
\]
where
\[ a = k_1 + k_2 + 2, \quad b = 2\chi \left( k_1 B \left( \frac{3}{4}, \frac{1}{2} \right) + k_2 B \left( \frac{1}{4}, \frac{1}{2} \right) \right) (x_* + \tilde{y}_*) + \alpha (k_1 + 2k_2 + \pi). \]
Here, we choose \( \tilde{T} \) such that \( b \tilde{T}^{\frac{1}{2}} < \frac{1}{2} \). Then (76) derives
\[
\| u - \tilde{u} \|_{X_T} + \| v - \tilde{v} \|_{Y_T} \leq 2a (\| u_0 - \tilde{u}_0 \|_{L^2(I)} + \| v_0 - \tilde{v}_0 \|_{W^{1,2}(I)}).
\]
We set \( N \) that satisfies \( N \tilde{T} > T \), and we can take \( \delta = \delta(\epsilon) \) such that
\[ \delta(\epsilon) \equiv \frac{\epsilon}{(2a)^N}. \]
Then it holds that
\[ \| u - \tilde{u} \|_{X_T} + \| v - \tilde{v} \|_{Y_T} \leq \frac{\epsilon}{(2a)^{N-1}}. \]
Finally, starting from the first step on \([0, \tilde{T}]\), and then repeating after \([\tilde{T}, 2\tilde{T}]\), within \( N \) steps such that \( N \tilde{T} > T \), we conclude that \( \| u - \tilde{u} \|_{X_T} + \| v - \tilde{v} \|_{Y_T} \leq \epsilon \). This proves Theorem 3.3. \( \square \)

4. Appendix. This section is devoted to the proof of (5) (cf. Appendix in Albeverio, Yahagi, Yoshida [1]).
\[
\phi_0(x) = \frac{1}{\sqrt{\pi}}, \quad \phi_n(x) = \frac{2}{\sqrt{\pi}} \cos n(x + \frac{\pi}{2}) \quad \text{for } n = 1, 2, 3, \ldots,
\]
\[
\varphi_n(x) = \frac{2}{\sqrt{\pi}} \sin n(x + \frac{\pi}{2}) \quad \text{for } n = 1, 2, 3, \ldots,
\]
then \( \phi_n, n = 0, 1, \ldots \) and \( \varphi_n, n = 1, 2, \ldots \) are eigenfunctions of the Neumann Laplacian and the Dirichlet Laplacian respectively, and both \( \{ \phi_n \}_{n=0,1,2,\ldots} \) and \( \{ \varphi_n \}_{n=1,2,\ldots,} \) form orthonormal bases of \( L^2([-\frac{\pi}{2}, \frac{\pi}{2}]) \rightarrow \mathbb{R} \):
\[
\frac{d^2}{dx^2} \phi_n = n^2 \phi_n, \quad \| \phi_n \|_{L^2(I)} = 1, \quad n = 0, 1, 2, \ldots,
\]
\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi_n(x)\phi_m(x)\,dx = 0 \text{ for } n \neq m,
\]
\[
\frac{d^2}{dx^2}\varphi_n = n^2\varphi_n, \quad \text{with } \|\varphi_n\|_{L^2(I)} = 1, \quad n = 1, 2, \ldots,
\]
\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi_n(x)\varphi_m(x)\,dx = 0 \text{ for } n \neq m.
\]

For the Dirichlet Laplacian \(\Delta\), the corresponding semigroup \(e^{t\Delta} : L^2(I) \ni f \mapsto e^{t\Delta}f \in W^{1,2}_0(I) (t > 0)\), is explicitly given by (cf. the argument of spectral projection, cf. eg. chapter VIII of Reed, Simon [17])

\[
(e^{t\Delta}f)(x) = \int_I \left( \sum_{n=0}^{\infty} e^{-n^2t}\varphi_n(x)\varphi_n(\xi) \right) f(\xi) \, d\xi,
\]
and

\[
G_t(x, \xi) = \sum_{n=0}^{\infty} e^{-n^2t}\varphi_n(x)\varphi_n(\xi), \quad t > 0,
\]
is the fundamental solution by which

\[
\partial_t (G_t f) = \Delta(G_t f), \quad t > 0,
\]
holds for any \(f \in L^2(I)\).

For \(f \in W^{1,2}_0(I)\), by the integration by parts formula, we see that

\[
\int_I nf(\xi)\varphi_n(\xi) \, d\xi = \int_I f(\xi)\{-\varphi_n'(\xi)\} \, d\xi = \int_I f(\xi)\varphi_n'(\xi) \, d\xi.
\]

Hence, by the Schwarz’s inequality, we see that

\[
\|\Delta e^{t\Delta}f\|_{L^2(I)} = \| \sum_{n=1}^{\infty} -ne^{-n^2t}\varphi_n(x) \int_I nf(\xi)\varphi_n(\xi) \, d\xi \|_{L^2(I)}
\]
\[
= \left( \int_{n=1}^{\infty} n^2e^{-2n^2t}\varphi_n^2(x) \sum_{n=1}^{\infty} b_n^2(f') \, dx \right)^{\frac{1}{2}}
\]
\[
= \left( \sum_{n=1}^{\infty} n^2e^{-2n^2t}b_n^2(f') \int_I \varphi_n^2(x) \, dx \right)^{\frac{1}{2}}
\]
\[
= \left( \sum_{n=1}^{\infty} n^2e^{-2n^2t}b_n^2(f') \right)^{\frac{1}{2}}
\]
\[
\leq \sqrt{\frac{1}{\sqrt{\eta}}} \|f'\|_{L^2(I)} < t^{-\frac{1}{2}} \|f'\|_{L^2(I)},
\]

where \(b_n(f') \equiv \int_I f'(\xi)\varphi_n(\xi) \, d\xi\), which proves (5).

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E-mail address: yumi.yahagi.yoshida@gmail.com