Coherence for modalities

Kosta Došen *, Zoran Petrić
Mathematical Institute, SANU, Knez Mihailova 36, p.f. 367, 11001 Belgrade, Serbia

1. Introduction

A modality is a finite (possibly empty) sequence of the modal operators of necessity □ and possibility ◊. Negation is usually also allowed to occur in a modality, and the definition we just gave would cover only positive modalities, but in this paper we do not consider negation (for reasons mentioned towards the end of this introduction), and we take modality to be synonymous with positive modality. Our aim is to investigate modalities for logics in the vicinity of S₄ and S₅ in terms of categorial proof theory.

The modalities in S₄ and S₅ are pretty well known, and one could imagine that there is nothing new to say about this topic. This is indeed so if one wants to say just what modalities are equivalent, and which implies which (this structure, which reduces to just three nonequivalent positive modalities: □, the empty modality and ◊, is very simple for S₅). If, however, one approaches this topic from the point of view of general proof theory, or categorial proof theory, where one is interested in identity of deductions, there are quite interesting facts about the modalities of S₄, S₅ and logics in their vicinity, facts that are not very well known, or are not known at all.

We consider deductions involving only modalities, and define categories whose objects are these modalities, and whose arrows may be taken as these deductions. For the logic S₄, these happen to be freely generated categories that have the structure of a monad (or triple) or a comonad (for these notions, see [37], Section V.1.1, and Sections 3 and 8). The connection between S₄ and the notions of monad and comonad is known ([30], Section 1, should be the first reference for this connection, which was exploited in particular in papers dealing with categorial models of deductions in linear logic, starting with [42]), but here we present this matter in a new, gradual, detailed and systematic manner, concentrating on coherence results, some of which are presumably new. (We will, of course, give references concerning results for which we know that they have been previously established.)

Roughly speaking, a coherence result is a result that characterizes a category C, freely generated in a class of categories, in terms of a manageable category M. More precisely, in a coherence result one establishes that there is a faithful functor G from C to M. One may take that C is syntax and M a model. Coherence then amounts to proving a completeness theorem: the existence of the functor G is soundness, while its faithfulness is completeness proper. As happens often with completeness theorems, coherence results yield usually through the manageability of M an easy decision procedure for equality of arrows in C.
In this paper, and in general in our approach to coherence (see [13]), the syntactic categories \( C \) are indeed constructed out of syntactic material. They correspond to logical systems, but not to the usual systems of theorems; we have instead systems of equations between deductions. (The usual logical systems correspond here to the inductive definitions of terms that stand for deductions.)

The model category \( \mathcal{M} \) often has a geometrical inspiration, and its arrows can be drawn. In this paper, the arrows of \( \mathcal{M} \) will be relations of some kind, which can always be drawn. In the first part of the paper, for categories in Sections 2–5, these relations are either relations between finite ordinals or split equivalences between finite ordinals. A split equivalence is an equivalence relation on the union of two disjoint source and target sets (see [14], Section 2.3, [10] and [11]), which here we take to be finite ordinals. For the categories in Sections 6–8, our relations are always split equivalences between finite ordinals.

In contradistinction to coherence such as it is treated in [13], the relations of \( \mathcal{M} \) in this paper do not link occurrences of propositional letters, but occurrences of modal operators. This approach may suggest finer coherence results for predicate logic than those obtained in [16], where quantifiers were not linked, but only predicate letters.

In this paper we deal only with modalities, which is a preparatory process for a treatment of wider fragments of modal logic, involving other connectives. We believe that concerning this basic matter we have enough new material to present, especially in connection with \( S5 \), and that it is unwise to rush to wider fragments without having settled the fundamentals first. If in these wider fragments we link both occurrences of propositional letters and occurrences of modal operators, hoping for coherence, we enter into a largely unchartered territory. Let us only mention that in the presence of a lattice conjunction, which corresponds to binary product, or a lattice disjunction, which corresponds to binary coproduct, we should not expect straightforward coherence results if we have both kinds of link. Problems arise with the distribution of \( \Box \) over such a conjunction and the distribution of \( \Diamond \) over such a disjunction. An approach to \( S5 \) following the present paper would require that these distributions be isomorphisms, and this does not square with coherence (for much the same reasons that prevent a straightforward approach to coherence with the isomorphic distribution of product over coproduct, which one has in bicartesian closed categories; see [13], Sections 1.2 and 14.3).

The finite ordinals that are the objects of \( \mathcal{M} \) in this paper may be replaced by modalities so as to make \( C \) isomorphic to a subcategory of \( \mathcal{M} \) (see the parenthetical remark in the first paragraph of Section 3 of [15]), but when the objects of \( C \) are modalities built only of \( \Box \) or only of \( \Diamond \), we need no further adjustments of \( \mathcal{M} \) to have as a consequence of coherence that \( C \) is isomorphic to a subcategory of \( \mathcal{M} \), and this subcategory may happen to be an important and interesting concrete category. As an example of such an important concrete category, we find in this paper the simplicial category, whose arrows are the order-preserving functions between finite ordinals, which is isomorphic to the category \( C \) whose arrows may be taken as the deductions in the modal logic \( S4 \) involving the modalities built out only of \( \Diamond \). As another example, we have the skeleton of the category \( \text{Finset} \) of finite sets, whose arrows are all the functions between finite ordinals, which is isomorphic to the category \( \mathcal{C} \) whose arrows may be taken as the deductions involving the modalities built out only of \( \Box \) in an extension of \( S4 \). The isomorphisms with \( \mathcal{C} \) provide an axiomatic presentation in terms of generating arrows and equations between arrows of these important concrete categories.

Before we reach \( S4 \), we have in Section 2 a basic underlying category that we call \( S \). We prove for \( S \) a basic simple coherence result, which is an essential ingredient of the proofs of coherence in the subsequent two sections (Sections 3 and 4) dealing with categories related to \( S4 \), and in later sections. The arrows of the category \( S \) and closely related categories may be taken as the deductions involving the modalities in the modal logics \( T \) and \( K4 \). As a consequence of coherence for \( S \), we obtain the isomorphism of categories closely related to \( S \) with respect to the concrete categories whose arrows are respectively the order-preserving injections and the order-preserving surjections between finite ordinals. These isomorphisms yield axiomatic presentations in terms of generating arrows and equations of these concrete categories. They show also that the notions of injectivity and surjectivity are almost the same in this context.

After \( S4 \), we concentrate in Section 5 on modal logics with deductions permuting modalities. Some of these, which permute \( \Box \) with \( \Diamond \), or \( \Box \) with \( \Diamond \), would not be distinguished from \( S4 \) in ordinary modal logic, where we are interested only in theorems, and not in deductions. From our proof-theoretical point of view, we obtain, however, new logics, for whose categories of modalities we prove coherence results. As an interesting consequence of these results, one obtains through the isomorphism with the category \( S4_{\Box} \) of Section 5 an axiomatic presentation in terms of generating arrows and equations of the skeleton of \( \text{Finset} \) mentioned above. In this context we also have the modal logic \( S4.2 \) (new from anybody’s point of view), for which we also prove coherence. The related category, combining a monad and a comonad, was remarked independently in attempts to describe an algebra and a coalgebra with mixed distributive laws (see Section 5 for references).

The first part of the paper (up to Section 6) is to a great extent of an introductory character. It systematizes matters, many of which are already known, and lays the ground for our main results in the remainder of the paper.

In Sections 6 and 7, we consider categories that correspond to \( S5 \) and a dual system, usually not considered in modal logic, which we call \( S5 \). These categories are about combining a monad and a comonad structure as in situations where a functor has both a left and a right adjoint (for the notion of adjunction, see [37], Section IV.1, and the beginning of Section 10). These common adjoint situations do not seem to have a standard name. In Section 8, we give them the name trijunction, while the corresponding monad–comonad structures, exemplified in \( S5 \) and \( S5 \), will be called dyad and codyad.

The dyad and codyad structures are closely related to Frobenius algebras, a topic that has recently become rather prominent with the proof of the equivalence between the category of commutative Frobenius algebras and two-dimensional topological quantum field theories (see [24]). Our coherence results for the free dyad and codyad are related to these...
topological results. The difference is that with Frobenius monads, which correspond to Frobenius algebras, $\Box$ and $\Diamond$ are not distinguished any more, but the gist of the matter is in the results of this paper. It is an interesting connection between modal logic and topology, found on a different level from the well-known connection between $\mathbf{S}4$ operators and the topological interior and closure operators. Here the connection with topology arises for $\mathbf{S}5$, and its dual $\mathbf{S}5$. (The roots of topology and modal logic are intermingled: one of the earliest papers in modal logic – in some sense the first one from the modern point of view – is [26]; in that paper, Kuratowski actually introduced $\mathbf{S}4$, algebraically treated, for the first time, and investigated its modalities.)

For our coherence results concerning categories that correspond to $\mathbf{S}5$ and $\mathbf{S}5$, the model category $\mathcal{M}$ is a category whose arrows are split equivalences between finite ordinals. With arrows being relations between finite ordinals, we would obtain different categories that correspond to $\mathbf{S}5$ and $\mathbf{S}5$, with which we do not deal in this paper.

In the final sections of the paper (Sections 9–11), we deal with the property of maximality for our categories of modalities. This is a kind of syntactic completeness for the systems of equations of arrows that define these categories, a property analogous to the Post completeness (which should be called Bernays completeness; see [44]) of classical propositional logic. Maximality is important because it shows that not only our categories with relations, but any nontrivial category modelling our categories of modalities could serve as a faithful model.

Beyond our nontrivial categories, for which we have coherence and maximality, we find preorders, where all arrows with the same source and target, i.e. all deductions with the same premise and conclusion, are equal. These categories are trivial from the point of view of general proof theory, but it is not trivial to find systems of equations that guarantee that they are preorders, as we do in the sections on maximality at the end of the paper. These are also coherence results, in the sense of the earliest coherence result there is; namely, Mac Lane's coherence result for monoidal categories in [35].

Matters pertaining to coherence for modalities involving classical negation would not significantly change the picture we present, and this is why we concentrate on positive modalities only. In the presence of binary connectives, conjunction, disjunction, or implication, where we would not deal only with modalities any more, matters would, however, change considerably. The distributivity of the necessity operator $\Box$ over conjunction, or, dually, of the possibility operator $\Diamond$ over disjunction, which normal modal logics require, introduces particular problems for our model categories $\mathcal{M}$ with relations. We leave these problems for a separate treatment.

In this paper we do not deal with categories of modalities that correspond to Frobenius monads, where $\Box$ and $\Diamond$ are isomorphic (they actually coincide), and where these modal operators lose the meaning they have usually in modal logic. These categories are very interesting, in particular because of their relationship with topological quantum field theories mentioned above, but we prefer not to extend further a sufficiently long paper. We leave these matters, which are at the limits of logic in the strict sense, for [17].

For the proof of our coherence results we rely on normal forms. Although these normal forms are similar to those found in proof theory, they are not inspired by cut elimination in the style of Gentzen. Cut elimination, however, would work too, at least in some cases (see the comments in the next section). These normal forms may be easier to connect with natural deduction than with Gentzen's sequent systems. The possibility to obtain these normal forms is a proof-theoretical justification that our equations between deductions are well chosen. Our coherence and our maximality results provide other such justifications. (For an extended discussion of these matters see [13].)

We assume for this paper an acquaintance with only rather basic notions of category theory, which may all be found in [37]. Practically no knowledge of modal logic is assumed, except for the sake of motivation, which may be gathered from [23]. Some further references concerning category theory and modal logic will be given later in the paper.

2. The category $\mathbf{S}$

We define in this section a basic category called $\mathbf{S}$, and prove for it a basic simple coherence result, which will be an essential ingredient of the proofs of coherence in later sections. We introduce first some terminology and notation.

Every arrow term, i.e. term for an arrow in a category, has a type assigned to it; a type is a pair of objects $(A, B)$ where $A$ is the source and $B$ the target. We use $f$, $g$, $h$, . . . , sometimes with indices, as variables for arrow terms, and $f : A \rightarrow B$ indicates that the arrow term $f$ is of type $(A, B)$. (The turnstile $\vdash$ reminds us here that our arrows may be taken as deductions.)

The objects of the category $\mathbf{S}$ are the finite ordinals. The primitive arrow terms of $\mathbf{S}$ are

$$1_n : n \vdash n,$$
$$\xi_n : n + 1 \vdash n.$$ The arrow terms of $\mathbf{S}$ are closed under the following operations:

- if $f : n \vdash m$ and $g : m \vdash k$ are arrow terms, then so is $(g \circ f) : n \vdash k$;
- if $f : n \vdash m$ is an arrow term, then so is $Mf : n + 1 \vdash m + 1$.

We take for granted the outermost parentheses of arrow terms, and omit them. (Further omissions of parentheses will be permitted by the associativity of $\circ$, namely, (cat 2) below.)

The arrows of the category $\mathbf{S}$, and of analogous syntactic categories considered in this paper, will be made of this syntactic material in the manner described in detail in [13] (Chapter 2). The arrows of $\mathbf{S}$ are equivalence classes of arrow terms such
that the following equations (which always have arrow terms of the same type on the two sides of =) are satisfied for \( f : n \vdash m \).

**Categorial equations:**

\[
\begin{align*}
\text{(cat 1)} & \quad f \cdot 1_n = 1_m \cdot f = f, \\
\text{(cat 2)} & \quad h \cdot (g \cdot f) = (h \cdot g) \cdot f.
\end{align*}
\]

**Functorial equations:**

\[
\begin{align*}
\text{(M1)} & \quad M1_n = 1_{n+1}, \\
\text{(M2)} & \quad M(g \cdot f) = Mg \cdot Mf.
\end{align*}
\]

**Naturality equation:**

\[
\begin{align*}
(\xi \text{ nat}) & \quad \xi_m \cdot Mf = f \cdot \xi_n.
\end{align*}
\]

The functorial equations say that \( M \), where \( Mn = n+1 \), is an endofunctor of \( S \) (i.e. a functor from \( S \) to \( S \)). The naturality equation \((\xi \text{ nat})\) can be replaced for \( S \) by the two equations

\[
\begin{align*}
(\xi^{MM} \text{ nat}) & \quad \xi_{m+1} \cdot Mmf = Mf \cdot \xi_{n+1}, \\
(\xi M) & \quad \xi_n \cdot M\xi_m = \xi_n \cdot \xi_{n+1}.
\end{align*}
\]

(For other categories, to be considered in later sections, the last two equations will not necessarily yield \((\xi \text{ nat})\), because of the presence of arrows different from \(\xi\).)

The category \( S \) can be presented as a strict monoidal category (where associativity arrows are identity arrows), with tensor product given by addition of natural numbers. So presented, it would be a product category \((\text{PRO})\) without permutation in the sense of [36] (Chapter V); for a more recent reference see [27]. Many of the categories considered later in this paper have analogously the structure of a product category, or a product category with permutation, i.e. symmetry \((\text{PROP})\).

For \( k \geq 0 \), let \( M^k \) be the sequence of \( k \) occurrences of \( M \). Every arrow term of the form \( M^k \xi_n \) is called a \( \xi \)-term. For \( n \geq 1 \), an arrow term of the form \( f_0 \cdot \cdots \cdot f_i \), where \( f_i \) is \( 1_m \) for some \( m \) and for every \( i \in \{2, \ldots, n\} \) we have that \( f_i \) is a \( \xi \)-term, is called a developed arrow term.

It is easy to show by using categorial and functorial equations that the following lemma holds for \( S \), and, with an appropriate understanding of “developed arrow term”, for all the categories that will be considered in this paper.

**DEVELOPMENT LEMMA.** For every arrow term \( f \) there is a developed arrow term \( f' \) such that \( f = f' \).

Next we define inductively two functors, \( G^\xi \) and \( G^\nu \), from \( S \) to the category \( \text{Rel} \), whose objects are again the finite ordinals, and whose arrows are the relations between finite ordinals; composition in \( \text{Rel} \) is composition of relations, and the identity arrows are identity relations. For \( \alpha \in \{\xi, \nu\} \), let \( G^\alpha \) be the identity relation on \( n \), and let \( G^\xi \emptyset_1 \) be the empty relation between 1, which is equal to \( \{\emptyset\} \), and 0, which is equal to \( \emptyset \). For \( n \geq 1 \), we have clauses corresponding to the following pictures:

\[
\begin{align*}
& G^\xi \emptyset_n \\
& G^\xi \emptyset_n \\
& G^\xi \emptyset_n
\end{align*}
\]

We have \( G^\alpha(g \cdot f) = G^\alpha g \cdot G^\alpha f \), where \( \cdot \) on the right-hand side is composition of relations, and for every \( f : n \vdash m \) we have that the relation \( G^\alpha Mf \subseteq (n+1) \times (m+1) \) is obtained by adding the pair \((n, m)\) to the relation \( G^\alpha f \subseteq n \times m \).

We easily check by induction on the length of derivation that, if \( f = g \) in \( S \), then \( G^\xi f = G^\xi g \) in \( \text{Rel} \). Hence \( G^\xi \) so defined is indeed a functor. Our purpose is to show that the functors \( G^\alpha \) are faithful functors.

A developed arrow term of \( S \) is said to be in normal form when it has no subterm of the form \( M^k \xi_n \cdot M^{k+l} \xi_{n-l+1} \) for \( l \geq 1 \). That every arrow term of \( S \) is equal in \( S \) to an arrow term in normal form follows from the Development Lemma and from the following equations of \( S \) for \( l \geq 1 \), which for \( k = 0 \) and \( l = 1 \) give the equation \((\xi M)\), and which could replace \((\xi \text{ nat})\) in the axiomatization of the equations for \( S \):

\[
(\xi M^l) \quad M^k \xi_n \cdot M^{k+l} \xi_{n-l+1} = M^{k+l-1} \xi_{n-l+1} \cdot M^k \xi_{n+1}.
\]

Note that the sum of the superscripts of \( M \) on the right-hand side is strictly smaller than that sum on the left-hand side.

We can easily establish the following lemma.

**AUXILIARY LEMMA.** If \( f \) and \( g \) are in normal form and \( G^\xi f = G^\xi g \), then \( f \) and \( g \) are the same arrow term.

To prove this lemma, we proceed by induction on the number of \( \xi \)-terms in \( f \) and \( g \), which must be equal.
We infer immediately from the Auxiliary Lemma that the normal form of an arrow term is unique. This fact can, however, easily be established directly by confluence (i.e. the Church–Rosser property) of reductions that consist in passing from the left-hand side of \((\xi M')\) to the right-hand side. We can then easily infer the following result.

**S Coherence. The functors** \(G^\circ\) **and** \(G^\Delta\) **from** \(S\) **to Rel are faithful.**

This coherence result could alternatively be established by relying on a sequent presentation in the style of Gentzen (as in [20]) of the category \(S\). Instead of the primitive arrow terms \(\xi_n\) we would have the operation on arrow terms:

\[
\text{if } f : n \vdash m \text{ is an arrow term, then so is } M_n f : n + 1 \vdash m,
\]

which is easily defined in terms of \(\xi_n\), and vice versa, in the presence of \(\circ\) and \(1_n\); namely, we have \(M_n f =_{df} f \circ \xi_n\) and \(\xi_n =_{df} M_n 1_n\). The following equations,

\[
\begin{align*}
g \circ (\xi_n f) &= M_n (g \circ f), \\
M_n g \circ f &= M_n (g \circ f),
\end{align*}
\]

(together with (M2) and (cat 1), enable us to find for every arrow term \(f\) a composition-free arrow term \(f'\) such that \(f = f'\). The Auxiliary Lemma then holds if we replace “in normal form” by “composition free”, and this yields \(S\) Coherence.

So there are two ways to obtain a normal form. The first is to “draw compositions out”, as we did first, and as Mac Lane does in the Lemma of Section VII.5 of [37] (see the next section of this paper). The second way is to “push compositions inside”, until they disappear, as Gentzen would do. This is the gist of his cut-elimination method.

Let the category \(S_n\) be defined like \(S\), save that we have \(\xi_n\) only for \(n \geq 1\). It is easy to show \(S_n\) Coherence; namely, the assertion that the functors \(G^\circ\) and \(G^\Delta\) from \(S_n\) to \(Rel\) defined in the same way as before, are faithful.

Let \(S^{op}\) be the category opposite to \(S\), and let the functor \(G^\circ\) from \(S^{op}\) to \(Rel\) be defined by taking that \(G^\circ f^{op} = (G^\circ f)^{-1}\), where \(R^{-1}\) is the relation converse to \(R\); on objects, \(G^\circ\) is again identity. Then out of \(S\) Coherence and \(S_n\) Coherence we can infer \(S^{op}\) Coherence, which says that these new functors \(G^\circ\) are faithful, and \(S^{op}_+\) Coherence, which says that analogously defined functors from \(S^{op}\) to \(Rel\) are faithful.

The category \(S\) could be called \(T\), because its arrows may be taken as the deductions in the modal logic \(T\) (the normal modal logic with the axiom \(\Box p \rightarrow p\) or \(p \rightarrow \Diamond p\), which is characterized by reflexive frames; see [23], p. 42) involving the modalities built out only of \(\Box\), provided that \(M\) is replaced by \(\Box\). The category \(S_n\) could analogously be called \(K_4\), because its arrows may be taken as the deductions in the modal logic \(K_4\) (the normal modal logic with the axiom \(\Box p \rightarrow \Box \Box p\) or \(\Diamond \Box p \rightarrow \Box \Diamond p\), which is characterized by transitive frames; see [23], p. 64) involving the modalities built out only of \(\Diamond\), provided that \(M\) is replaced by \(\Diamond\). For analogous reasons, \(S^{op}\) could be called \(T^{op}\), and \(S^{op}_+\) could be called \(K_4^{op}\). The interesting coherence results here are then the \(G^\circ\) instances of \(S\) Coherence and \(S^{op}\) Coherence, and the \(G^\Delta\) instances of \(S_n\) Coherence and \(S^{op}_+\) Coherence, as will become clear in the next section.

By combining the assumptions for \(T\) and \(T_n\), for two distinct modal operators \(\Box\) and \(\Diamond\), we would obtain the category \(T_{\Box\Diamond}\), whose arrows may be taken as the deductions in the modal logic \(T\) involving all the positive modalities (see Section 4). We may combine the assumptions for \(K_4\) and \(K_4\), analogously, to obtain the category \(K_{4\Box\Diamond}\), whose arrows may be taken as the deductions in the modal logic \(K_4\) involving all the positive modalities. Since \(\Box\) and \(\Diamond\) do not “cooperate” in \(T_{\Box\Diamond}\) and \(K_{4\Box\Diamond}\), we can prove easily coherence for the first with respect to a \(G^\circ\) functor, and coherence for the second with respect to a \(G^\Delta\) functor, which are the interesting forms of coherence here (see Section 4). The arrows of the category defined like \(S\), save that we omit the arrows \(\xi_n\) and the equation (\(\xi n\at\) nat) may be taken as the deductions in the minimal normal modal logic \(K\) involving the modalities built out only of \(\Box\), or only of \(\Diamond\). This is, however, a discrete category; all its arrows are identity arrows, and coherence for it, which is very easy to establish, is a trivial result.

(The category whose arrows may be taken as the deductions in \(K\) involving all the positive modalities is also discrete; \(\Box\) and \(\Diamond\) do not cooperate in this category.) The \(G^\circ\) instance of \(S^{op}\) Coherence and the \(G^\Delta\) instance of \(S\) Coherence, together with easily established facts about the generation of order-preserving injections and surjections between finite ordinals, yield that \(S^{op}\) is isomorphic to the category whose arrows are the order-preserving injections between finite ordinals, and \(S\) is isomorphic to the category whose arrows are the order-preserving surjections between finite ordinals. All this shows that the notions of injectivity and surjectivity are up to duality almost the same.

### 3. The categories \(S_+\) and \(S_+\)

We now introduce the category \(S_+\), whose arrows may be taken as the deductions in the modal logic \(S_4\) involving the modalities built out only of \(\Box\). We identify these modalities with their lengths, and so we take as the objects of \(S_+\) not these modalities, but the natural numbers, i.e. finite ordinals. The category \(S_+\) is isomorphic to the category \(\Delta^{op}\) for \(\Delta\) being the simplicial category, i.e. the category whose arrows are the order-preserving functions between finite ordinals (see [37], Section VII.5, and the end of this section). The category \(S_+\) is the free comonad generated by a single object, and the opposite category \(S_+\) is isomorphic to \(\Delta\), which we will consider later in this section, is the free monad generated by a single object (see the beginning of Section 8).
The objects of \( S_{4□} \) are the finite ordinals. The primitive arrow terms of \( S_{4□} \) are \( \mathbf{1}_n : n \vdash n \) plus
\[
\begin{align*} 
\varepsilon_{n^\square} : n+1 & \vdash n, \\
\delta_{n^\square} : n+1 & \vdash n+2.
\end{align*}
\]
In the notation for comonads of [37] (Section VI.1), our \( \varepsilon^□ \) and \( \delta^□ \) correspond respectively to \( \varepsilon \) and \( \delta \). (We write the superscripts because we introduce in this paper a systematic notation for comonads, monads and their combinations; see \( \varepsilon^\circ \) and \( \delta^\circ \) towards the end of this section, and also the notation of Sections 6 and 7.) In [16], whose subject matter is related to the subject matter of the present paper, \( \varepsilon \) (derived from instantiation) corresponds to \( \varepsilon \) as it is used in this paper.

The operations on arrow terms are as for \( S \), with \( M \) replaced by \( □ \). The arrows of \( S_{4□} \) are obtained by assuming the following equations besides the categorical and functorial equations:
\[
\begin{align*} 
(\varepsilon^□ \text{ nat}) & \quad \varepsilon_{m^\square} \circ □f = f \circ \varepsilon_{n^\square}, \\
(\delta^□ \text{ nat}) & \quad □f \circ \delta_{n^\square} = \delta_{m^\square} \circ □f, \\
(\delta^□) & \quad \delta_{m^\square} \circ \delta_{n^\square} = \delta_{m^\square} \circ \delta_{n^\square}, \\
(□□β) & \quad \varepsilon_{n^\square+1} \circ \delta_{n^\square} = 1_{n+1}, \\
(□□η) & \quad □f_{n^\square} \circ \delta_{n^\square} = 1_{n+1}.
\end{align*}
\]

The naturality equation \( (\varepsilon^□ \text{ nat}) \) is the instance of \( (ξ \text{ nat}) \) for \( ξ \) being \( \varepsilon^□ \), while the naturality equation \( (δ^□ \text{ nat}) \) and the equation \( (δ^□) \) are obtained from the equations \( (ξ^{MM} \text{ nat}) \) and \( (ξ \text{ M}) \) adapted to \( \varepsilon_{n+1} : n+1 \vdash n+2 \), which has the type of \( δ_{n^\square} \). We may take that the equations for \( S_{4□} \), except the new equations \( (□□β) \) and \( (□□η) \), are obtained from those for \( S \) and \( S_{□}^\oplus \), provided that in the presentation of \( S_{□}^\oplus \) we have instead of \( (ξ^{\oplus} \text{ nat}) \) the two equations \( (ξ^{MM} \text{ nat}) \) and \( (ξ \text{ M}) \) adapted to \( ξ_{n+1} \), which we mentioned above. The equations for \( S_{4□} \) above correspond exactly to the equations for the category \( Δ^{\oplus} \) obtained from the equations \( (11), (12) \) and \( (13) \) for \( Δ \) in [37] (Section VII.5).

The functor \( G \) from \( S_{4□} \) to \( Rel \) is defined by the clauses
\[
\begin{align*} 
G\varepsilon_{n^\square} & = G\varepsilon_{n}, \\
G\delta_{n^\square} & = (Gδ_{n+1})^{-1};
\end{align*}
\]
otherwise, \( G \) is defined like \( G^\circ \) from the preceding section. These clauses correspond to the following pictures:
\[
\begin{align*} 
G\varepsilon_{n^\square} & \quad \begin{array}{c} n \circ \n \circ \cdots \circ 0 \circ 1 \circ 0 \circ 1 \circ 0 \circ \cdots \circ 0 \circ \cdots \circ 0 \\
\end{array} \\
G\delta_{n^\square} & \quad \begin{array}{c} n+1 \circ \n \circ \cdots \circ 0 \circ 1 \circ 0 \circ 1 \circ 0 \circ \cdots \circ 0 \circ \cdots \circ 0 \\
\end{array}
\end{align*}
\]
where the parts of the pictures involving 0, . . . , \( n \) do not exist if \( n = 0 \).

It is well known that \( G \) so defined is a faithful functor (see [21], Appendix, [31], pp. 148ff, [30], p. 95, [1], p. 10, [8], Section 5.9, and [29], Section 2.2; among these references, [30] and [8] rely on Gentzen’s cut-elimination method). We will, however, prove this again by relying on the coherence results of the preceding section. This proof is otherwise like Mac Lane’s proof of an analogous result in [37] (Section VII.5).

\( S_{4□} \) Coherence. The functor \( G \) from \( S_{4□} \) to \( Rel \) is faithful.

Proof. We say that an arrow term \( f_2 \circ f_1 \) of \( S_{4□} \) is in normal form when \( δ \) does not occur in \( f_1 \) and \( ε \) does not occur in \( f_2 \). By using the equations of \( S_{4□} \), it is easy to establish that every arrow term of \( S_{4□} \) is equal to an arrow term in normal form.

For \( f \) and \( g \) arrow terms of \( S_{4□} \) of the same type, let \( Gf = Gg \). Then \( f = f_2 \circ f_1 \) and \( g = g_2 \circ g_1 \) for \( f_2 \circ f_1 \) and \( g_2 \circ g_1 \), in normal form. So \( Gf_2 \circ Gf_1 = Gg_2 \circ Gg_1 \). It is easy to see that, for every arrow term \( f \) of \( S_{4□} \), the relation converse to \( Gf \) is an order-preserving function. Every order-preserving function \( h : m \to n \) is equal to the composition \( h_2 \circ h_1 : m \to n \) for a unique order-preserving surjection \( h_1 : m \to k \) and a unique order-preserving injection \( h_2 : k \to n \), where \( k \) is the cardinality of the image of \( h \) (see [38], Section IV.6, Propositions 1 and 2, for a more general categorial result, with the help of which this can be inferred). For future reference, we call this the surjection-injection decomposition of order-preserving functions between finite ordinals.

We use this surjection-injection decomposition to establish that \( Gf_1 = Gg_1 \) and \( Gf_2 = Gg_2 \). Then we use the \( G^\circ \) instance of \( S \) Coherence to establish that \( f_1 = g_1 \), and the \( C^\circ \) instance of \( S_{\oplus}^{\oplus} \) Coherence to establish that \( f_2 = g_2 \), from which it follows that \( f = g \) in \( S_{4□} \). ☐

The normal form introduced in this proof, which is suggested by the surjection-injection decomposition, could be replaced in our proof by a normal form suggested by another decomposition of order-preserving functions between finite ordinals, which should be called the injection-surjection decomposition. In this other decomposition we have that every order-preserving function \( h : m \to n \) is equal to \( h_2 \circ h_1 : m \to n \) for a unique order-preserving injection \( h_1 : m \to k \) and a
unique order-preserving surjection $h_2: k \to n$, where $k$ is $m+n$ minus the cardinality of the image of $h$. This new normal form is obtained from the previous one $f_3 \circ f_1$ by applying naturality equations until we obtain $f_1 \circ f_2$ such that $\varepsilon$ does not occur in $f_2$ and $\delta$ does not occur in $f_1$. The old normal form is thin: the cardinality of the interpolated $k$ is the least possible; the new normal form is thick: the cardinality of the interpolated $k$ can now be greater than in the thin normal form, and is in a certain sense maximal (see [8], Section 0.3.5).

Let the category $S_4$, be $S_4^\text{op}$, where $\square$ is written $\Diamond$, while $(\varepsilon_n^\text{op})^\eta: n \vdash n+1$ and $(\delta_n^\text{op})^\eta: n+2 \vdash n+1$ are written

\[
\varepsilon_n^\text{op}: n \vdash n+1, \\
\delta_n^\text{op}: n+2 \vdash n+1,
\]

respectively. (In the notation for monads of [37], Section VI.1, our $\varepsilon^\text{op}$ and $\delta^\text{op}$ correspond respectively to $\eta$ and $\mu$.) The arrows of the category $S_4$, may be taken as the deductions in the modal logic $S_4$ involving the modalities built out only of $\Diamond$.

Let the functor $G$ from $S_4^\text{op}$ to Rel be defined by taking that $Gf^\text{op} = \varepsilon Gf^{-1}$, where on the right-hand side $G$ is the functor from $S_4$ to Rel; on objects, $G$ is identity. This means that we have clauses corresponding to the following pictures, obtained from the pictures given above for $G\varepsilon_n^\text{op}$ and $G\delta_n^\text{op}$ by putting them upside down (and taking for granted the line involving $n-1$ in the right picture):

\[
\begin{array}{c}
G\varepsilon_n^\text{op} \\
\text{n-1} \\
\text{n-1} \\
\text{n-1} \\
\text{n+1} \\
\text{n+1} \\
\text{n+1} \\
\text{n+1} \\
\end{array}
\begin{array}{c}
\text{0} \\
\text{0} \\
\text{0} \\
\text{0} \\
\text{0} \\
\end{array}
\]

Then out of $S_4$, Coherence we can infer $S_4$, Coherence, which says that this new functor $G$ is faithful.

This faithfulness result, together with the surjection-injection decomposition of order-preserving functions between finite ordinals, and the isomorphisms involving $S^\text{op}$ and $S_4$, mentioned at the end of the preceding section, yields that $S_4^\text{op}$ is isomorphic to the category whose arrows are the order-preserving functions between finite ordinals, i.e. the simplicial category $\Delta$.

4. The category $S_4^{\Delta}$

We now introduce the category $S_4^{\Delta}$, whose arrows may be taken as the deductions in the modal logic $S_4$ involving all the positive modalities; namely, all the modalities built out of both $\Box$ and $\Diamond$. The category $S_4^{\Delta}$ will have the structures of a comonad and a monad.

The objects of $S_4^{\Delta}$ are finite (possibly empty) sequences of $\Box$ and $\Diamond$, sequences that we call modalities, and denote by $A$, $B$, $C$, . . .. The primitive arrow terms of $S_4^{\Delta}$ are

\[
I_A: A \vdash A, \\
\varepsilon_A: \Box A \vdash A, \\
\delta_A: \Diamond A \vdash A, \\
\eta_A: \Box \Diamond A \vdash A.
\]

The operations on the arrow terms of $S_4^{\Delta}$ are defined like the operations on the arrow terms of the category $S$ in Section 2, save that $n$, $m$ and $k$ are replaced respectively by $A$, $B$ and $C$, while $n+1$ and $m+1$ are replaced respectively by $MA$ and $MB$, where $M$, as in the preceding section, stands either for $\Box$ or for $\Diamond$.

The arrows of $S_4^{\Delta}$, satisfy the categorial and functorial equations of Section 2, provided that we make the replacements just mentioned. We have, moreover, the equations taken over from $S_4^\text{op}$ and $S_4$, namely, the equations $(\varepsilon_n^\text{op} \Diamond)$, $(\delta_n^\text{op} \Diamond)$, $(\delta_n^\text{op} \varepsilon_n^\text{op})$, $(\Box \Diamond \beta)$ and $(\Box \Diamond \eta)$, and the equations for $S_4$, dual to these where $\Box$ is replaced by $\Diamond$. (Some of these equations of $S_4$ are mentioned in Section 6 when we give the equations for $S_5^{\Delta}$.) This concludes the definition of the equations for $S_4^{\Delta}$.

Note that in these equations $\Box$ and $\Diamond$ do not “cooperate”.

We define a functor $G$ from $S_4^{\Delta}$ to Rel by stipulating first that $GA$ is the length of the object $A$. For $\alpha \in I = \{\epsilon, \delta, \varepsilon_n^\text{op}, \delta_n^\text{op}\}$ and $\alpha A = n$, let $G\alpha A$ be defined like $G\alpha n$, where $G$ in $G\alpha n$ is either $G$ in $S_4^\text{op}$ to $\text{Rel}$ or $G$ from $S_4^\text{op}$ to $\text{Rel}$ (see the preceding section); otherwise, $G$ is defined like $G^\text{op}$ in Section 2. We are now going to prove the following.

$S_4^{\delta}$ Coherence. The functor $G$ from $S_4^{\Delta}$ to Rel is faithful.

**Proof.** We say that an arrow term $f$ of $S_4^{\Delta}$ is an $\alpha$ arrow term when no $\beta \in I$ \{-\alpha\} occurs in $f$. The equations of $S_4^{\Delta}$ enable us to find for every arrow term $f$ of $S_4^{\Delta}$, an arrow term equal to $f$ in the normal form $f_4 \circ f_3 \circ f_2 \circ f_1$, where $f_1$ is an $\varepsilon$ arrow term, $f_2$ is a $\delta^\text{op}$ arrow term, $f_3$ is a $\delta$ arrow term, and $f_4$ is an $\epsilon$ arrow term. Suppose now that for $f$ and $g$ arrow terms of $S_4^{\Delta}$ of the same type we have $Gf = Gg$. For $f_1 \circ f_2 \circ f_3 \circ f_4$ and $g_1 \circ g_2 \circ g_3 \circ g_4$ being respectively the normal forms of $f$ and $g$, it is easy to see that $f_i$ and $g_i$, for $1 \leq i \leq 4$, are of the same type, and that $Gf_i = Gg_i$.

Roughly speaking, $Gf_4$ and $Gg_4$ tell us which occurrences of $\Diamond$ in the target of $f_4$ and $g_4$ disappear in their sources, and since according to $Gf = Gg$ the same of these occurrences disappear, these sources must be the same, as well as $Gf_4$ and $Gg_4$. So the targets of $f_4$, and $g_4$ are the same. Since $G(f_4 \circ f_3 \circ f_2 \circ f_1) = G(g_4 \circ g_3 \circ g_2 \circ g_1)$ and $Gf_4 = Gg_4$, while $Gf_4$ and $Gg_4$ are one-one functions, and hence left cancellable (see [37], Section I.5), we conclude that $G(f_3 \circ f_2 \circ f_1) = G(g_3 \circ g_2 \circ g_1)$.
Roughly speaking, $G_f$ and $G_g$ tell us which occurrences of $\square$ in the target of $f_1$ and $g_3$ are amalgamated in their sources, and since according to $G(f_1 \circ f_2 \circ f_1) = G(f_3 \circ G_2 \circ G_1)$ the same of these occurrences are amalgamated, these sources must be the same, as well as $G_f$ and $G_g$. We reason analogously in the two remaining dual cases, where $i$ is 1 and 2, starting from the source of $f$ and $g$.

Then we can conclude out of $S$ Coherence and $S_+$ Coherence, and their $op$ variants, that $f_i = g_i$, from which it follows that $f = g$ in $S4_{\square \circ}$.

Note that the normal form in this proof is thin, in the sense that the target of $f_2$, which is also the source of $f_3$, is a minimal interpolant for decomposing $f$. Various other thicker normal forms, with interpolants being modalities of greater length, can be envisaged (among these there is a thickest one). A thicker normal form, for which we will find analogues later (see the normal forms for $S4_{\square \circ}$ and $S4.2.2_{\circ}$ in the next section), is $f_4 \circ f_2 \circ f_4 = f_4$ for $f_3$ being as above. Note that $\diamond$ does not occur in the superscripts of $f_4 \circ f_4$, which hence becomes an arrow term of $S4_{\circ}$, when $\diamond$ is replaced by $\square$, while $\diamond$ does not occur in the superscripts of $f_4 \circ f_2$, which hence becomes an arrow term of $S4_{\diamond}$, when $\square$ is replaced by $\diamond$.

In every situation where we have an endoadjunction, i.e., where we have two functors $F$ and $G$ from a category $\mathcal{A}$ to $\mathcal{A}$ such that $F$ is left adjoint to $G$ (for the notion of adjunction, see the beginning of Section 10), the composite functors $FG$ and $GF$, for $FG$ being $\square$ and $GF$ being $\diamond$, together with the associated natural transformations $\epsilon^F$ and $\delta^{MM}$, defined in terms of the canonical arrows of the adjunction (as in [37], Section VI.1), have the structure of $S4_{\square \circ}$.

5. Modalities and permutation

We will envisage in this section categories with arrows that permute modalities, whose image by the functor $G$ will correspond to the picture

$$\begin{array}{ccc}
M_1 & \rightarrow & M_2 \\
\downarrow & & \downarrow \\
M_2 & \leftrightarrow & M_1
\end{array}$$

Here, $M_1$ and $M_2$, which are either $\square$ or $\diamond$, may be either equal or not.

The category $S_\chi$ is defined like $S$ of Section 2, where $M$ is $\square$ and $\xi$ is $\epsilon^\square$, with the additional primitive arrow terms

$$\chi_{n+1} \mathbin{\square} \chi_n \vdash \chi_n \mathbin{\square} \chi_{n+1},$$

with $n \geq 0$, for which we assume the additional equations

$$
\begin{align*}
(\chi \mathbin{\square} nat) & \quad \square f \circ \chi_n = \chi_{n+1} \circ \square f, \\
(\chi \mathbin{\diamond} nat) & \quad \chi_n \circ \chi_n = 1_{n+2}, \\
(\chi \mathbin{\square} \chi \mathbin{\square}) & \quad \chi_{n+1} \circ \chi_n \circ \chi_{n+1} = \square \chi_n \circ \chi_{n+1} \circ \square \chi_n, \\
(\epsilon^\square \mathbin{\square} \chi^{\square \square}) & \quad \epsilon_{n+1} \circ \chi_n = \square \epsilon_n.
\end{align*}
$$

The first three of these four equations are analogous to the equations commonly used to present symmetric groups (see [7], Section 6.2).

We define the functor $G$ from $S_\chi$ to $Rel$ like $G^\square$ in Section 2 with an additional clause for $\chi^{\square \square}$ that corresponds to the following picture:

$$G\chi_n \begin{array}{cccccc}
n+1 & n & n-1 & \cdots & 0 \\
\downarrow & & & & \downarrow \\
n & & & & 0
\end{array}$$

where the part of the picture involving $0, \ldots, n-1$ does not exist if $n \neq 0$.

To show that this functor $G$ is faithful, i.e. to show $S_\chi$ Coherence, we establish first that every arrow term $f$ of $S_\chi$ is equal in $S_\chi$ to an arrow term in the normal form $f_2 \circ f_1$, where $\chi^{\square \square}$ does not occur in $f_1$ and $\epsilon^{\square}$ does not occur in $f_2$. Here $f_1$ is an arrow term of $S$ (with $M$ being $\square$ and $\xi$ being $\epsilon^\square$), while $f_2$ should be called a $\chi^{\square \square}$ arrow term. Note that $GF$ uniquely determines $Gf_1$ and $Gf_2$, as well as the target of $f_1$, which is also the source of $f_2$. To obtain $S_\chi$ Coherence we rely then on $S$ Coherence and on the standard presentation of symmetric groups mentioned above, which we call Symmetric Coherence.

The category $S_{\chi^{\square \square}}$ is defined like $S_{\chi^{\square}}$ of Section 2, where $M$ is $\square$ and $\xi^{\square}$ is $\delta^{\square \square}$, with $(\delta^{\square \square} nat)$ and $(\delta^{\square \square})$ assumed instead of $(\epsilon^{\square \square} nat)$; we have the additional primitive arrow terms $\chi^{\square \square}$, for which we assume the additional equations $(\chi^{\square \square} nat)$, $(\chi^{\square \square} \diamond \square)$ and

$$
\begin{align*}
(\delta^{\square \square} \mathbin{\square} \chi^{\square \square}) & \quad \delta_{n+1} \circ \chi_n = \square \chi_n \circ \chi_{n+1} \circ \square \delta_n, \\
(\chi^{\square \square} \mathbin{\diamond} \delta^{\square \square}) & \quad \chi_n \circ \delta_n = \delta_n.
\end{align*}
$$
corresponds to the picture given above for $G$.

Distributivelaws stem from $S$, involving the positivemovalities in the modal system for which we assume the additional equations obtained by extending what we have assumed for $G$, from $\delta^{\text{nat}}_n$, such that the number of inversions in the permutation $Gf_2$ is minimal. We can omit all the inversions involving $i$ and $j$ such that applying the function $(Gf_1)^{-1}$ to $i$ and $j$ gives the same value. By relying on that, together with the equation $(\chi^{\text{nat}}, \delta^{\text{nat}})$, we can finish our proof of $S^{\text{op}}_{\delta}$, Coherence by appealing to $S^{\text{op}}_{\delta}$ Coherence and Symmetric Coherence.

The category $S^{\text{op}}_{\delta}$ is isomorphic to the category whose arrows are arbitrary injections between finite ordinals, while $S_{\delta}$, which is $(S^{\text{op}}_{\delta})^{\text{op}}$, is isomorphic to the category whose arrows are arbitrary surjections between finite ordinals. For that we rely on the possibility to decompose every such injection into a bijection followed by an order-preserving injection, and the same when “injection” is replaced by “surjection”.

The category $S_{\delta}$ is obtained by combining what we have assumed for $S$ and $S^{\text{op}}_{\delta}$, as $S_{\delta}$ may be obtained from $S$ and $S^{\text{op}}_{\delta}$, and the functor $G$ from $S_{\delta}$ to $\text{Rel}$ is obtained by combining what we have assumed for the functors $G$ from $S$ and $S^{\text{op}}_{\delta}$. This category is interesting because, as we will see below, its opposite category is isomorphic to the category whose arrows are arbitrary functions between finite ordinals.

We can prove $S^{\text{op}}_{\delta}$ Coherence with respect to the functor $G$ we have just defined by relying for every arrow term $f$ of $S^{\text{op}}_{\delta}$ on its normal form $f_1 \circ f_2$, where $f_1$ is an arrow term of $S$ and $f_2$ is an arrow term of $S^{\text{op}}_{\delta}$. Then we apply $S$ Coherence and $S^{\text{op}}_{\delta}$ Coherence (for related proofs of $S^{\text{op}}_{\delta}$ Coherence see [4], Section 2.2, [28], [22], [29], Section 2.3, and [24], p. 195).

The category $S^{\text{op}}_{\delta}$ is $S^{\text{op}}_{\delta}$. We use for $S^{\text{op}}_{\delta}$ the same notation that we used for $S$, (see Section 3), and we write $\chi^{\text{nat}}_n$ for $(\chi^{\text{nat}}, \chi^{\text{nat}}_n)$. From $S^{\text{op}}_{\delta}$ Coherence we can infer, of course, $S^{\text{op}}_{\delta}$ Coherence, for an appropriately defined functor $G$ whose definition extends the definition of the functor $G$ from $S$ to $\text{Rel}$ (see Section 3) with a clause for $\chi^{\text{nat}}_n$ like the clause for $\chi^{\text{nat}}_n$.

As $S^{\text{op}}_{\delta}$ is isomorphic to the category whose arrows are the order-preserving functions between finite ordinals, so $S^{\text{op}}_{\delta}$ is isomorphic to the category whose arrows are arbitrary functions between finite ordinals, which is, up to isomorphism, the skeleton of the category $\text{Finset}$ of finite sets. This is shown by relying on the decomposition of every such function into a bijection followed by an order-preserving function.

We define the category $S^{\text{op}}_{\delta}$ like $S_{\delta}$ with the additional arrows $\chi^{\text{nat}}_n$ and $\chi^{\text{nat}}_n$, for which we have the equations we have assumed for $S_{\delta}$ and $S^{\text{op}}_{\delta}$ with $n$ and $m$ replaced respectively by $A$ and $B$, while $n+1$ and $n+2$ are replaced respectively by $\square A$ and $\square B$, or $\square A$ and $\square B$, as appropriate. The definition of the functor $G$ from $S^{\text{op}}_{\delta}$ to $\text{Rel}$ extends the definition of the functor $G$ from $S_{\delta}$ to $\text{Rel}$ (see the preceding section) with the clauses for $\chi^{\text{nat}}_n$ and $\chi^{\text{nat}}_n$. Since $\square A$ and $\square B$ do not “cooperate” in $S^{\text{op}}_{\delta}$ Coherence and $S_{\delta}$ Coherence to establish $S^{\text{op}}_{\delta}$ Coherence with respect to this functor $G$.

The normal form in which we rely here is analogous to the thicker normal form for $S^{\text{op}}_{\delta}$ mentioned in the penultimate paragraph of the preceding section.

In the category $S^{\text{op}}_{\delta}$, which we are now going to define, $\square A$ and $\square B$ will “cooperate” for the first time. This category is obtained by extending what we have assumed for $S^{\text{op}}_{\delta}$ with the additional primitive arrow terms

\[\chi^{\text{nat}}_n: \square A \vdash \square B,\]

for which we assume the additional equations

\[(\chi^{\text{nat}}_n \text{ nat})\]
\[\square f \circ \chi^{\text{nat}}_n = \chi^{\text{nat}}_n \circ \square f,\]

\[(\epsilon^{\square} \chi^{\text{nat}})\]
\[\epsilon^{\square} \circ \chi^{\text{nat}}_n = \square \epsilon^{\square},\]

\[(\epsilon^{\text{nat}} \chi^{\text{nat}})\]
\[\chi^{\text{nat}}_n \circ \epsilon^{\square} = \square \chi^{\text{nat}}_n,\]

\[(\delta^{\text{nat}} \chi^{\text{nat}})\]
\[\delta^{\text{nat}} \circ \chi^{\text{nat}}_n = \square \delta^{\text{nat}} \circ \chi^{\text{nat}}_n,\]

\[(\delta^{\text{nat}} \chi^{\text{nat}})\]
\[\chi^{\text{nat}}_n \circ \delta^{\text{nat}} = \square \delta^{\text{nat}} \circ \chi^{\text{nat}}_n,\]

analogous to the equations $(\chi^{\text{nat}}_n \text{ nat})$, $(\epsilon^{\square} \chi^{\text{nat}})$, and $(\delta^{\text{nat}} \chi^{\text{nat}})$ above. The arrows of $S^{\text{op}}_{\delta}$ may be taken as the deductions involving the positive modalities in the modal system $S$ which extends $S$ with a principle corresponding to the type of $\chi^{\text{nat}}_n$ (see [23], p. 134). All the equations assumed above for $S^{\text{op}}_{\delta}$ may be found in [43] (Section 5.3), in connection with mixed distributive or entwining natural transformations in structures that combine a comonad and a monad; these distributive laws stem from [3].

We define the functor $G$ from $S^{\text{op}}_{\delta}$ to $\text{Rel}$ as the functor $G$ from $S_{\delta}$ to $\text{Rel}$ with an additional clause for $\chi^{\text{nat}}_n$ that corresponds to the picture given above for $G^{\text{nat}}_n$. We can then show the following.

$S^{\text{op}}_{\delta}$ Coherence. The functor $G$ from $S^{\text{op}}_{\delta}$ to $\text{Rel}$ is faithful.
Proof. We establish first that every arrow term $f$ of $S_{2}$ is equal in $S_{2}$ to an arrow term in the normal form $f_{1} \circ f_{2} \circ f_{1}$, where $\delta$, $\delta^{\infty}$ and $\chi^{\infty}$ do not occur in $f_{1}$, while $\delta$, $\delta^{\infty}$ and $\chi^{\infty}$ do not occur in $f_{3}$, and $\delta$, $\delta^{\infty}$ and $\delta^{\infty}$ do not occur in $f_{2}$. If we replace $\diamond$ by $\square$, then $f_{1}$, in whose superscripts $\diamond$ does not occur, becomes an arrow term of $S_{5}$, and if we replace $\square$ by $\diamond$, then $f_{3}$, in whose superscripts $\square$ does not occur, becomes an arrow term of $S_{4}$. This normal form is analogous to the thicker normal form for $S_{4}$, mentioned in the penultimate paragraph of the preceding section. Note that $Gf$ uniquely determines $Gf_{1}$, $Gf_{2}$ and $Gf_{3}$, as well as the targets of $f_{1}$ and $f_{2}$. Then we can apply $S_{4}$, $S_{5}$, $\chi^{\infty}$, and appropriate equations analogous to those of $S_{4}$, which deliver coherence. The equations explicitly involving $\chi^{\infty}$ are obtained from the equations $(\chi^{\infty})_{nat}$, $(\varepsilon^{\infty})_{nat}$, $(\varepsilon^{\infty})_{nat}$, $(\delta^{\infty})_{nat}$ and $(\delta^{\infty})_{nat}$ by taking $\chi^{\infty}$ as the inverse of $\chi^{\infty}$. For example, from $(\delta^{\infty})_{nat}$ we obtain the equation

$$X_{\square}^{\infty} \circ \diamond X_{\square}^{\infty} \circ \delta^{\infty}_{\square} = \delta^{\infty}_{\square} \circ \chi^{\infty}.$$  

The arrows of this category may be taken as the deductions involving the positive modalities in McKinsey’s modal system $S_{4}$, also called $S_{4}M$ (for historical comments see [23], p. 143, note 7), whose theorems are not included in $S_{5}$. Coherence for this category is demonstrated quite analogously to what we had for $S_{4}$.

We can also envisage the category with both $\chi^{\infty}$ and $\chi^{\infty}$ arrows, which would be isomorphisms inverse to each other. Coherence for that category is again shown analogously. To this last category we can also add the arrows $\chi^{\infty}$ and $\chi^{\infty}$, and again easily obtain a coherence result.

6. The category $S_{5}$

We now introduce the category $S_{5}$, whose arrows may be taken as the deductions in the modal logic $S_{5}$ involving the positive modalities. As $S_{4}$, this category will have the structures of a comonad and a monad, which, however, will now “cooperate”.

We define the category $S_{5}$ like the category $S_{4}$ with the following additions. We have the additional primitive arrow terms

$$\delta^{\infty}_{\square} : \Diamond A \vdash \Diamond A, \quad \delta^{\infty}_{\square} : \Box A \vdash \Box A.$$  

We use $\delta^{\infty}_{\square}$ for either $\delta^{\infty}_{\square}$ or $\delta^{\infty}_{\square}$, and likewise $\delta^{\infty}_{\square}$ for either $\delta^{\infty}_{\square}$ or $\delta^{\infty}_{\square}$. The equations of $S_{5}$ are obtained by assuming those assumed for $S_{4}$ and the following additional equations:

$$(\delta^{\infty}_{\square} \text{ nat}) \quad \Diamond Mf \circ \delta^{\infty}_{\square} = \delta^{\infty}_{\square} \circ Mf, \quad (\delta^{\infty}_{\square} \text{ nat}) \quad \delta^{\infty}_{\square} \circ \Box Mf = Mf \circ \delta^{\infty}_{\square}.$$  

$$(\delta^{\infty}_{\square} \text{ nat}) \quad \Box M\beta \circ \delta^{\infty}_{\square} = 1_{MA}, \quad (\delta^{\infty}_{\square} \text{ nat}) \quad \delta^{\infty}_{\square} \circ \Box M\beta = 1_{MA}.$$  

$$(\delta^{\infty}_{\square} \text{ nat}) \quad \Box M\beta = 1_{MA}, \quad (\delta^{\infty}_{\square} \text{ nat}) \quad \delta^{\infty}_{\square} \circ \Box M\beta = 1_{MA}.$$  

The equations $(\delta^{\infty}_{\square} \text{ nat})$, $(\delta^{\infty}_{\square})$ and $(\Box M\beta)$ for $M$ being $\Box$ were already assumed for $S_{4}$, while the equations $(\delta^{\infty}_{\square} \text{ nat})$, $(\delta^{\infty}_{\square})$ and $(\Box M\beta)$ for $M$ being $\Diamond$ were already assumed for $S_{4}$ and $S_{4}$. There is no generalization with $M$ of the equation $(\Box M\eta)$ of Section 3, and of the dual equation for $\Box$. These equations are assumed for $S_{4}$ as they were assumed before for $S_{4}$.

The names of the equations $(\delta N)$ and $(\delta \eta)$ are derived from graphs related to their left-hand sides (as will be explained below). These equations are related to the Frobenius equations of Frobenius algebras (see [24]; for some history concerning the Frobenius equations, see [25], which traces the equations to [5], where they occur in a different context). The difference is that in the Frobenius equations $\Box$ and $\Diamond$ are not distinguished. The equations

$$\Box \delta^{\infty}_{\square} \circ \delta^{\infty}_{\square} = \delta^{\infty}_{\square} \circ \delta^{\infty}_{\square} = \delta^{\infty}_{\square} \circ \delta^{\infty}_{\square},$$  

or, alternatively, the dual equations

$$\Box \delta^{\infty}_{\square} \circ \delta^{\infty}_{\square} = \delta^{\infty}_{\square} \circ \delta^{\infty}_{\square} = \delta^{\infty}_{\square} \circ \delta^{\infty}_{\square},$$  

suggested by Lawvere (see [31], p. 152, where $\Box$ and $\Diamond$ are not distinguished), could replace the equations $(\delta N)$ and $(\delta \eta)$ in our axiomatization of the equations of $S_{5}$. 
The equations \((\delta^{\Delta M})\) and \((\delta^M)\) are redundant in this axiomatization. For \((\delta^{\Delta M})\), we have
\[
\square\delta^{\Delta M}_A \circ \delta^{\Delta M}_A = \square\delta^{\Delta M}_A \circ \delta^{\Delta M}_A \circ \delta^{\Delta M}_A \circ e^{M_A}, \quad \text{with } (\Delta M\beta),
\]
\[
= \square\delta^{\Delta M}_A \circ \square\delta^{\Delta M}_A \circ \delta^{\Delta M}_A \circ e^{M_A}, \quad \text{with } (\Delta N),
\]
\[
= \square\delta^{\Delta M}_A \circ \square\delta^{\Delta M}_A \circ \delta^{\Delta M}_A \circ e^{M_A}, \quad \text{with } (\Delta M),
\]
\[
= \square\delta^{\Delta M}_A \circ \square\delta^{\Delta M}_A \circ \delta^{\Delta M}_A \circ e^{M_A}, \quad \text{with } (\Delta N),
\]
\[
= \square\delta^{\Delta M}_A \circ \square\delta^{\Delta M}_A \circ \delta^{\Delta M}_A \circ e^{M_A}, \quad \text{with } (\Delta M),
\]
\[
= \square\delta^{\Delta M}_A \circ \square\delta^{\Delta M}_A \circ \delta^{\Delta M}_A \circ e^{M_A}, \quad \text{with } (\Delta N),
\]
and we proceed analogously for \((\delta^M)\) (for an analogous derivation see \[24\]), Proposition 2.3.24, which in Section 2.3.25 is credited to \[41\]). The equations \((\delta^{\Delta M})\) and \((\delta^M)\), however, do not seem to be redundant if we replace \((\Delta N)\) and \((\Delta M)\) by the equations suggested by Lawvere.

For \(S5_{\Delta 0}\), we derive from \((\Delta N)\) and \((\Delta M\beta)\) the equation
\[
\delta^{\Delta 0}_A = \square\delta^{\Delta 0}_A \circ \delta^{\Delta 0}_A \circ e^{M_A},
\]
and we derive analogously from \((\Delta M)\) and \((\Delta M\beta)\) the equation
\[
\delta^{\Delta 0}_A = \square\delta^{\Delta 0}_A \circ \delta^{\Delta 0}_A \circ e^{M_A},
\]
which means that the arrows \(\delta^{\Delta 0}_A\) and \(\delta^{\Delta 0}_A\) may be defined in terms of other arrows, and need not be taken as primitive.

We will now define a category called \(Gen\), which will replace \(Rel\) to denote a functor \(G\) from \(S5_{\Delta 0}\). The objects of \(Gen\) are again the finite ordinals. An arrow of \(Gen\) from \(n\) to \(m\) is an equivalence relation defined on the disjoint union of \(n\) and \(m\), which is called a split equivalence. The identity arrow from \(n\) to \(n\) is the split equivalence that corresponds to the following picture:

\[
\begin{array}{cccccccc}
\cdots & n & 0 & \cdots & n & 0 \\
\end{array}
\]

which is empty if \(n = 0\). We do not draw in such pictures the loops corresponding to the pairs \((x, x)\). Composition of arrows is defined, roughly speaking, as the transitive closure of the union of the two relations composed, where we omit the ordered pairs one of whose members is in the middle (see \[10\], Section 2, and \[11\], Section 2, for a detailed definition). For example, the split equivalences \(R_1\) and \(R_2\) corresponding to the following pictures

\[
R_1:
\begin{array}{c}
1 \quad 0 \\
2 \quad 1 \quad 0
\end{array}
\]

\[
R_2:
\begin{array}{c}
2 \quad 1 \quad 0 \\
1 \quad 0
\end{array}
\]

are composed as follows, so as to yield the split equivalence \(R_2 \circ R_1\) that corresponds to the picture on the right-hand side:

\[
R_2 \circ R_1:
\begin{array}{c}
2 \quad 1 \quad 0 \\
1 \quad 0
\end{array}
\]

We define the functor \(G\) from \(S5_{\Delta 0}\) to \(Gen\) by stipulating first that \(G\alpha\) is the length of the object \(A\). On arrows, we have first that \(G1\) is the identity arrow of \(Gen\) from \(G\alpha\) to \(G\alpha\). For \(GA = n\), let \(G\alpha^{\Delta M}_A\) and \(G\alpha^\Delta A\) be the split equivalences that correspond respectively to the pictures given for \(G\alpha^{\Delta M}_A\) and \(G\alpha^\Delta A\) in Section 3. We have next, for \(GA = n\), the clauses that correspond to the following pictures:

\[
\begin{array}{c}
n \quad \cdots \quad n \\
0 \quad \cdots \quad 0
\end{array}
\]

\[
\begin{array}{c}
n \quad \cdots \quad n \\
0 \quad \cdots \quad 0
\end{array}
\]

The semicircle joining \(n\) and \(n+1\) at the bottom (in the target) in the left picture is the cap \((n, n+1)\), and the semicircle joining \(n\) and \(n+1\) at the top (in the source) in the right picture is the cup \((n, n+1)\). These two pictures are like those we had in Section 3 for \(G\alpha^{\Delta M}_A\) and \(G\alpha^\Delta A\), but with the cap and the cup added.

As before, we have \(G(f \circ g) = Gg \circ Gf\), and for \(Gf: n \to m\) the partition induced by the split equivalence \(GMf\) is obtained from the partition induced by the split equivalence \(Gf\) by adding the equivalence class \([n, m]\), where \(n\) is in the source and \(m\) in the target. We easily check by induction on the length of derivation that, if \(f = g\) in \(S5_{\Delta 0}\), then \(Gf = Gg\) in \(Gen\); hence \(G\) so defined is indeed a functor.
The split equivalences $R_1$ and $R_2$ in the example above may be taken to be respectively $G\delta\beta^G_{\delta A}$ and $G\delta\beta^\delta A_{\delta}$ for $A$ being empty. Then $R_2 \cdot R_1$ is the $G$ image of an instance of the left-hand side of $(\delta \mathbb{N})$, and when in the left picture corresponding to $R_2 \cdot R_1$ we omit the cup $(0, 1)$ and the cap $(1, 2)$ in the middle, we obtain the form of $\mathbb{N}$. (This explains $\mathbb{N}$ in the name of $(\delta \mathbb{N})$; horizontally, we would obtain $\mathbb{Z}$, and in the comments in [25] this horizontal look at the matter is favoured. The word of the name of $(\delta \mathbb{M})$ arises analogously.)

Before proving that this functor is faithful, note that the coherence results established in the preceding text with respect to $\text{Rel}$ could be established with respect to $\text{Gen}$, by relying on functors $G$ obtained by appropriately restricting the functor $G$ from $S_5^{\text{Rel}}$ to $\text{Gen}$. For that we have to check first that these restricted functors are indeed functors, which is done by induction on the length of derivation (the essential ingredient in this induction is to go through the axiomatic equations). This is nearly all we have to check, because the faithfulness of these functors can next be established by proceeding as before, via the same normal forms. Roughly speaking, adding the cups and caps to the pictures we had before does not change matters. (For a more detailed treatment of the relationship between $\text{Rel}$ and $\text{Gen}$, see [18].)

Next, as an auxiliary result, we establish coherence with respect to $\text{Gen}$ for the category $S_5^{\text{Gen}}$, defined by omitting from the definition of $S_5^{\text{Rel}}$ the arrow terms $\epsilon_5^A$ and $\delta_5^A$, and all the equations involving them explicitly. This means that we have in $S_5^{\text{Gen}}$ only the primitive arrow terms $\mathbf{1}_A$, $\epsilon_5^A$ and $\delta_5^A$, for which we assume the categorial and functorial equations plus $(\epsilon^A \text{ nat})$, $(\delta^A \text{ nat})$ and $(\delta^A \text{ M})$. The functor $G$ from $S_5^{\text{Gen}}$ to $\text{Gen}$ is defined by restricting the definition of $G$ from $S_5^{\text{Rel}}$ to $\text{Gen}$. We have the following.

$S_5^{\text{Gen}}$ Coherence. The functor $G$ from $S_5^{\text{Gen}}$ to $\text{Gen}$ is faithful.

Proof. Suppose that for $f$ and $g$ arrow terms of $S_5^{\text{Gen}}$ of the same type we have $Gf = Gg$. We prove that $f = g$ in $S_5^{\text{Gen}}$ by induction on the number $n$ of occurrences of $\delta^A$ in $f$, $g$ must be equal to that number for $g$. If $n = 0$, then we rely on $\text{S}$ Coherence of Section 2. If $n > 0$, then we rely on $\mathbb{S}$ Coherence that says that, if in the picture corresponding to $Gf$ we have a cup ($i, i+1$) in the source, then $f$ is equal in $S_5^{\text{Gen}}$ to an arrow term of the form $f' \circ A\delta_B$ such that $GB = 1$. This lemma is sufficient because, if there are no cups ($i, i+1$) in the source, then $f$ and $g$ are equal respectively to $f' \circ h$ and $g' \circ h$ for $h$ without $\delta^A$, and a cup ($i, i+1$) in the source of $Gf'$, which is equal to $Gg'$.

Here is a sketch of the proof of this lemma. We first transform $f$ into the developed form $f_1 \circ \cdots \circ f_1$ (see Section 2), and then we find the $f_i$ "responsible" for the cup ($i, i+1$). We use then the equations of $S_5^{\text{Gen}}$, and we may rely in particular on $(\delta^A \text{ M})$, to permute this $f_i$ to the right, until a descendent of it becomes the rightmost factor. 

Let $S_5^{\text{Gen}}$ be the category isomorphic to $(S_5^{\text{Gen}})^{op}$, where $\epsilon^A$ and $\delta^A$ are replaced by $\epsilon^\circ A$ and $\delta^A \text{ M}$, respectively. The equations for $S_5^{\text{Gen}}$ are dual to those for $S_5^{\text{Gen}}$ (instead of $(\epsilon^A \text{ nat})$, $(\delta^A \text{ nat})$ and $(\delta^A \text{ M})$ we have $(\epsilon^\circ A \text{ nat})$, $(\delta^A \text{ M} \text{ nat})$ and $(\delta^A \text{ M})$). Coherence for $S_5^{\text{Gen}}$, which we have established above, delivers of course coherence for $S_5^{\text{Gen}}$. We can then establish the following.

$S_5^{\text{Gen}}$ Coherence. The functor $G$ from $S_5^{\text{Gen}}$ to $\text{Gen}$ is faithful.

Proof. We verify that every arrow term $f$ of $S_5^{\text{Gen}}$ is equal in $S_5^{\text{Gen}}$ to an arrow term in the normal form $f_2 \circ f_1$, where $f_1$ is an arrow term of $S_5^{\text{Gen}}$ and $f_2$ is an arrow term of $S_5^{\text{Gen}}$. It is easy to see that $Gf$ uniquely determines $Gf_1$ and $Gf_2$, as well as the target of $f_1$. To conclude the proof of $S_5^{\text{Gen}}$ Coherence, we rely then on coherence for $S_5^{\text{Gen}}$ and $S_5^{\text{Gen}}$. 

The normal form we have used in this proof is of the thin kind (see Section 4).

Suppose that in the definition of $G$ for $S_5^{\text{Gen}}$ we omit from the picture corresponding to the clause for $G\delta^A \text{ M}$ the cap $(n, n+1)$, and from the picture corresponding to the clause for $G\delta^A \text{ nat}$ the cup $(n, n+1)$. The target category for that $G$ would be $\text{Rel}$, but we could not show that this defines a functor from $S_5^{\text{Gen}}$, because of the equations $(\delta \mathbb{N})$ and $(\delta \mathbb{M})$. These equations require the caps and cups, and the split equivalences of $G\delta^A \text{ M}$.

We can prove coherence for $S_5^{\text{Gen}}$ with respect to a functor $G^d$ from $S_5^{\text{Gen}}$ to $\text{Gen}$ that is a kind of dual of the functor $G$ we had above. It interchanges the role of $\epsilon$ and $\delta$ in the following manner. On objects, $G^d A = GA + 1$. On arrows, we have

$G^d \epsilon^A = G\epsilon_A \circ \delta^A \beta$,  
$G^d \delta^A = G\delta_A \circ \beta \epsilon^A$,  
$G^d \epsilon^A = GM \epsilon_A \circ \delta^A \beta$,  
$G^d \delta^A = GM \delta_A \circ \beta \epsilon^A$,  
$(G^d g \cdot f) = G^d (g \cdot f)$

for $G^d Mf$, we have a clause exactly analogous to the clause for $GMf$ for $S_5^{\text{Gen}}$. Graphically, for the length of $A$ being $n$, we have the following:

$G^d \epsilon^A_A$ and analogously for $G^d \delta^A_A$. 

and analogously for $G^d \epsilon^A_A$ and $G^d \delta^A_A$. 

}$
That $C^G$ is indeed a functor is checked by induction on the length of derivation of the equations of $5S_{\Box_0}$. The only problematic case arises with the equations $(\varepsilon^M \text{ nat})$, where we rely on the fact that the pair $(n-1, m-1)$ belongs to $C^G_{n-1}$. That $C^G$ is a faithful functor can be shown either directly, as $5S_{\Box_0}$ Coherence above, via the same normal form, or, alternatively, we can rely on the Maximality of $5S_{\Box_0}$ of Section 11 (which presupposes $5S_{\Box_0}$ Coherence).

This duality between $\varepsilon$ and $\delta$, exhibited by $C^G$, was already present in the category $S_+$ of Section 2, whose arrows $\varepsilon_{n+1}$ could be interpreted either as $\varepsilon_{n+1}$ or as $\delta_n$ arrows. Functors dual to the functors $G$ from $S_4$, $S_5$, and $S_4$ to Rel, as $C^G$ is dual to $G$ from $S_{\Box_0}$ to Gen, can be defined analogously (just omit the cups and caps from the $G^G$ images). The faithfulness of these dual functors can be proved either directly, via normal forms used previously, or for $S_4$ and $S_5$, we could rely on their maximality (see Section 9). We could also rely on a result about the duality of the simplicial category, analogous to the duality between $G$ and $C^G$, which is explained in [17] (end of Section 6).

Note that in $5S_{\Box_0}$, we have the arrows
\[ \begin{align*}
\varepsilon^G_\Box \circ \varepsilon^G_{\Box, A} : & A \not\vdash \Box \Box A, \\
\varepsilon^G_{\Box, A} \circ \varepsilon^G_{\Box} : & \Box \Box A \not\vdash \Box A,
\end{align*} \]
which are of the same type as the arrows $\chi_{\Box, G}^G$ of $S_{4.2\Box}$ (see the preceding section), but the equation $(\varepsilon^G \chi_{\Box, G}^G)$ fails for the first arrow, and the equation $(\varepsilon^G \chi_{\Box}^G)$ fails for the second, as can be easily verified with the help of the functor $G$ form $S_{\Box_0}$ to Gen. That these arrows of $5S_{\Box_0}$ do not amount to $\chi_{\Box, G}^G$ is clear from their interpretation via $G$. So, although, as far as theorems and provable sequents are concerned, the modal logic $S_{4.2}$ is included in the modal logic $S_5$, from a proof-theoretical point of view we should not assume that $S_{4.2}$ is a subsystem of $S_5$. Our $5S_{\Box_0}$ does not cover $S_{4.2\Box}$. There are deductions in $S_{4.2}$ (i.e. arrows of $S_{4.2\Box}$) absent from $S_5$.

In $5S_{\Box_0}$, the endofunctor $\Box$ is left adjoint to the endofunctor $\Box$ (for the notion of adjunction, see the beginning of Section 10). The members of the unit and counit of this adjunction are respectively the arrows
\[ \begin{align*}
\delta^G_{\Box, A} : & A \not\vdash \Box, \\
\delta^G_{\Box} : & \Box A \not\vdash A,
\end{align*} \]
which correspond to modal laws found in the modal system $B$ (see [23], p. 62). We will treat of matters pertaining to this adjunction in Section 10.

7. The category $5S_{\Box_0}$

We consider now a category isomorphic to $S_{\Box_0}$, a kind of mirror image of it. We define this category like $S_{4\Box_0}$, save that instead of $\varepsilon$ we write $\sigma$, and we have the following additions. We have the additional primitive arrow terms
\[ \begin{align*}
\sigma^G_{\Box, A} : & \Box A \not\vdash \Box A, \\
\sigma^G_{\Box, A} : & \Box A \not\vdash \Box A.
\end{align*} \]
The modal laws corresponding to the types of these arrow terms were investigated in [34] (p. 67).

The equations of $S_{\Box_0}$ are obtained by assuming those assumed for $S_{\Box_0}$, with $\delta$ replaced by $\sigma$, and the following additional equations:
\[ \begin{align*}
(\sigma^{M_4} \text{ nat}) & M \Box f \sigma^{M_4} = \sigma^{M_4} \Box M f, \\
(\sigma^{M_0} \text{ nat}) & \sigma^{M_0} M \Box f = M f \sigma^{M_0}, \\
(\Box M) & M \sigma^G_{\Box, A} \sigma^{M_4} = 1_{A_M}, \\
(\Box M) & \sigma^{M_0} \sigma^G_{\Box, A} \sigma^{M_4} = 1_{A_M}, \\
(\Box N) & \sigma^G_{\Box, A} \sigma^{M_4} \sigma^{M_0} = 1_{A_M}, \\
(\Box N) & \sigma^{M_0} \sigma^G_{\Box, A} \sigma^{M_4} = 1_{A_M}.
\end{align*} \]
The following equations can be derived (see the derivation of $(\delta^{M_4})$ in the preceding section):
\[ \begin{align*}
(\sigma^{M_4}) & \sigma^{M_4} \sigma^{M_0} = \sigma^{M_4} \sigma^{M_0}, \\
(\sigma^{M_0}) & \sigma^{M_0} \sigma^{M_0} = \sigma^{M_0} \sigma^{M_0}.
\end{align*} \]

It is not difficult to show that the categories $5S_{\Box_0}$ and $5S_{\Box_0}$ are isomorphic. In this isomorphism, the object $A$ is mapped to $\Box A$ read from right to left. (This isomorphism does not preserve the functors $\Box$ and $\Box$.)

It follows that for $S_{\Box_0}$ we can establish coherence with respect to the functor $G$ from $S_{\Box_0}$ to Gen defined like $G$ from $S_{\Box_0}$ to Gen; namely, $G_{\Box, A}^{M_2} = G_{\Box, A}^{M_2}$. In $S_{\Box_0}$, the endofunctor $\Box$ is right adjoint to the endofunctor $\Box$, while in $S_{\Box_0}$ it was left adjoint, as we noted at the end of the preceding section.

Note that in $S_{\Box_0}$ we do not have an arrow of the type $\Box \not\vdash \Box \Box$ for $\Box$ being the empty sequence. Analogously, we do not have an arrow of the type $\Box \not\vdash \Box$. This is because, for every arrow $f$ of $S_{\Box_0}$, every occurrence of $\Box$ in the target of $f$ must be linked by $G^G$ to an occurrence of $\Box$ in the source of $f$, or an occurrence of $\Box$ in the target of $f$, and every occurrence of $\Box$ in the source of $f$ must be linked by $G^G$ to an occurrence of $\Box$ in the target of $f$ or an occurrence of $\Box$ in the source of $f$. Another way to conclude that arrows of the type $\Box \not\vdash \Box \Box$ for $\Box \not\vdash \Box$ do not exist in $S_{\Box_0}$ is to appeal to the isomorphism of $S_{\Box_0}$ with $S_{\Box_0}$, and the well-known fact that in the modal logic $S_5$ we do not have modal laws corresponding to these types. However, in the extension of the modal logic $T$ (namely, the normal modal logic with the axiom $\Box p \rightarrow p$ or $p \rightarrow \Box p$ with the axiom $\Box p \rightarrow \Box p$ or $\Box \Box p \rightarrow \Box p$, we can derive $p \rightarrow \Box p$ and $\Box p \rightarrow p$. We have
\[\square p \rightarrow \diamond \Box p, \quad \text{by } \alpha \rightarrow \diamond \alpha,\]
\[\diamond (p \rightarrow \Box p), \quad \text{by laws of normal modal logics},\]
\[\square (p \rightarrow \Box p), \quad \text{by necessitation},\]
\[\Box (p \rightarrow \Box p), \quad \text{by } \Box \diamond \alpha \rightarrow \Box \alpha,\]
\[p \rightarrow \Box p, \quad \text{by } \Box \alpha \rightarrow \alpha.\]

This may be the reason why the modalities of \(S5_{\square} \) are not usually considered, though the laws governing these modalities are as interesting as those of \(S5_{\square} \), whose faithful image they are.

8. Trijunctions, dyads and codyads

In this section, we show that the assumptions made for the category \(S5_{\square} \) can be justified by adjunctions underlying the comonad and monad structures of that category.

A comonad on a category \(\mathcal{C} \) is a structure \((\mathcal{C}, \Box, \varepsilon, \delta)\), where \(\Box\) is an endofunctor of \(\mathcal{C}\), while \(\varepsilon: \Box \rightarrow I_{\mathcal{C}}\) and \(\delta: \Box \rightarrow \Box\), for \(I_{\mathcal{C}}\) being the identity functor of \(\mathcal{C}\), are natural transformations that satisfy the equations of \(S4_{\Box}\), (provided that \(n\) and \(n + 1\) are replaced respectively by \(A\) and \(\square A\), for \(A\) an object of \(\mathcal{C}\)). The category \(S4_{\Box}\) is the free comonad generated by a single object (understood as an arrowless one-node graph, or the trivial one-object category; for details, see [8], Chapter 5, and [9], Section 4). A monad on \(\mathcal{C}\) is a structure \((\mathcal{C}, \diamond, \eta, \mu)\) defined analogously by reference to \(S4_{\diamond}\), which is the free monad generated by a single object.

We call a dyad on \(\mathcal{C}\) a structure that includes a comonad on \(\mathcal{C}\), a monad on \(\mathcal{C}\), and two additional natural transformations \(\varepsilon^{\diamond}: \Box \rightarrow \diamond \Box\) and \(\delta^{\diamond}: \diamond \Box \rightarrow \Box\) that satisfy the equations of \(S5_{\Box,\diamond}\). The category \(S5_{\Box,\diamond}\) is the free dyad generated by a single object.

We call a codyad on \(\mathcal{C}\) a structure that includes a comonad on \(\mathcal{C}\), a monad on \(\mathcal{C}\), and two additional natural transformations \(\varepsilon^{\Box}: \Box \rightarrow \Box \diamond\) and \(\delta^{\Box}: \Box \diamond \rightarrow \diamond\) that satisfy the equations of \(S5_{\Box,\Box}\). The category \(S5_{\Box,\Box}\) is the free codyad generated by a single object.

A trijunction is a structure made of the categories \(\mathcal{A}\) and \(\mathcal{B}\), the functor \(U\) from \(\mathcal{A}\) to \(\mathcal{B}\), and the functors \(L\) and \(R\) from \(\mathcal{B}\) to \(\mathcal{A}\), such that \(L\) is left adjoint to \(U\), with the counit \(\psi^L: UL \rightarrow IA\) and unit \(\gamma^L: IA \rightarrow UL\), and \(R\) is right adjoint to \(U\), with the counit \(\phi^R: UR \rightarrow IB\) and unit \(\gamma^R: IB \rightarrow RU\). The notion of a trijunction is very well known, but no special name seems to be commonly used for it. An important example of a trijunction is obtained when \(\mathcal{A}\) is a category with products and coproducts; then \(\mathcal{B}\) is the product category \(\mathcal{A} \times \mathcal{A}\), the functor \(U\) is the diagonal functor, and the functors \(L\) and \(R\) are respectively the coproduct and product bifunctors. Another example of a trijunction, interesting for logic, which involves the functor of substitution and the existential and universal quantifiers, may be found in Lawvere’s hyperdoctrines (see [32] and [33]). A trijunction involving the category of adjunctions, the category of monads (or comonads), and the Eilenberg–Moore and the Kleisli constructions is investigated in [40] (see also [8], Sections 5.2.3–4; see also [1]). Trijunctions, and in connection with them the adjunction from the end of Section 6, are mentioned in [2] (Section 10.4). Particular trijunctions are called quasi-Frobenius triples of functors in [6]. In [17], the trijunctions where the functors \(L\) and \(R\) are the same functor are called bijunctions, and trijunctions where \(U\), \(L\), and \(R\) are all the same endofunctor are self-adjunctions (examples of such structures may be found in [12]; see also [17]).

The relationship between the notions of trijunction, dyad and codyad is analogous to a certain extent to the relationship between the notions of adjunction, monad and comonad. Every trijunction gives rise to a dyad on \(\mathcal{B}\) with \(\Box\) being \(UR\) and \(\Box\) being \(UL\); for \(B\) an object of \(\mathcal{B}\), we have

\[\varepsilon^\Box_B = \psi^R_B, \quad \delta^\Box_B = \gamma^L_B, \quad \varepsilon^\Box_B = \psi^R_B, \quad \delta^\Box_B = \gamma^L_B,\]

Every trijunction analogously gives rise to a codyad on \(\mathcal{A}\) with \(\Box\) being \(UL\) and \(\Box\) being \(RU\). Conversely, every dyad or codyad gives rise to a trijunction by a construction analogous to the Eilenberg–Moore construction of an adjunction out of a monad or comonad (see [37], Sections VI.2, and [8], Section 5.1.7). We present this construction here.

For a dyad on \(\mathcal{C}\), let \(\mathcal{C}_{\Box}^\Box\) be the category whose objects are of the form \((A, d, g)\) for \(d: A \rightarrow \Box A\) and \(g: \Box A \rightarrow A\) arrows of \(\mathcal{C}\) that satisfies the conditions below. Strictly speaking, the mentioning of the object \(A\) is here superfluous, but it is kept to be in tune with common usage concerning the Eilenberg–Moore construction. The conditions for \(d\) and \(g\) are the following equations, analogous to the similarly named equations of \(S5_{\Box,\Box}\) in Section 6:

\[(\Box M \beta) d = \delta^\Box_A \circ d = 1_A, \quad (\Box M \beta) g = \varepsilon^\Box_A = 1_A,\]
\[(\Box N g) \circ \varepsilon^\Box_A = d \circ g = d \circ g, \quad (\Box N d) = \delta^\Box_A \circ d = d \circ g,\]

The equations

\[(\delta^\Box M) d = \delta^\Box_A \circ d, \quad (\delta^\Box M) g = \varepsilon^\Box_A = \delta^\Box_A\]
can be derived (see the derivation of \((\delta^\text{M}_\text{triv})\) in Section 6). An arrow of \(e^\ominus_{\text{triv}}\) from \((A_1, d_1, g_1)\) to \((A_2, d_2, g_2)\) is an arrow \(h:\ A_1 \rightarrow A_2\) of \(C\), indexed by \((A_1, d_1, g_1)\) and \((A_2, d_2, g_2)\), such that the following equations hold:
\[
(\delta^\text{M}_\text{triv} \circ h) \circ d_1 = d_2 \circ h, \quad (\delta^\text{M}_\text{triv} \circ h) \circ g_1 = h \circ g_2.
\]

We define two functors \(R\) and \(L\) from \(C\) to \(C^\ominus\) in the following manner. The object \(R\) is \((\square \! A, \delta^\ominus_A, \delta^\ominus_A^\ominus)\), while \(Rf = \square f\), appropriately indexed. Dually, \(L\) is \((\triangle A, \delta^\ominus_A, \delta^\ominus_A^\ominus)\), while \(Lf = \triangle f\), appropriately indexed. We define next a functor \(U\) from \(C^\ominus\) to \(C\) by stipulating that \(U(A, d, g) = A\) and \(Uh = h\). Then it can be shown that \(L\) is left adjoint to \(U\), while \(R\) is right adjoint to \(U\). We need the equation \((\delta Ng)\) to check that the counit of the adjunction involving \(L\) and \(U\) satisfies \((\delta^\text{M}_\text{triv} \circ h)\). Dually, we need the equation \((\star \! \omega d)\) to check that the unit of the adjunction involving \(U\) and \(R\) satisfies \((\delta^\text{M}_\text{triv} \circ h)\). The endofunctors \(UR\) and \(UL\) are equal respectively to \(\square\) and \(\triangle\).

We have a trijunction with the categories \(C^\ominus\) and \(C\) above, together with the functors \(L\), \(R\), \(U\) and \(U\) between them, and the dyad to which this trijunction gives rise is the dyad on \(C\). One can prove a theorem that says that this trijunction is terminal, in an appropriate sense, among all the trijunctions that give rise to the dyad on \(C\), which is analogous to a theorem about the adjunction involving the Eilenberg–Moore category (see [37], Section VI.3, and [8], Section 5.2.4).

Consider the full subcategory \((C^\ominus)_{\text{free}}\) of \(C^\ominus\) whose objects are of the form \((MA, \delta^\ominus_M, \delta^\ominus_M^\ominus)\). It is clear that there is a trijunction involving \((C^\ominus)_{\text{free}}\) and \(C\), but it is not immediately clear how to obtain from \((C^\ominus)_{\text{free}}\) an analogue of the Kleisli category, such that the trijunction involving it and \(C\) would be initial among all the trijunctions that give rise to the dyad on \(C\) (see [37], Section VI.5, and [8], Sections 5.1.6 and 5.2.4). We leave this matter for another occasion.

We can prove coherence for trijunctions with respect to a functor \(G\) into \(\text{Gen}\) such that the counits and units of the trijunction are mapped into the split equivalences corresponding to the following pictures:

\[
\begin{align*}
G\varphi^L_A & \quad \text{LUA} & \quad G\varphi^R_B & \quad \text{URB} \\
G\gamma^R_A & \quad \text{RUA} & \quad G\gamma^L_B & \quad \text{ULB}
\end{align*}
\]

and, for \(F\) being \(U\), \(L\) or \(R\), we have
\[
G(Ff) = F^\ominus
\]

(Related functors may be found in [8], Section 4.10, [9], Section 7, [12] and [17], Section 6; in contradistinction to what we have in [12] and [17], circles cannot arise with trijunctions, as they do not arise in [8] and [9]).

Our coherence results for \(S5_{\text{triv}}\) and \(S5_{\text{triv}}\), established in the preceding sections, are closely related to this coherence result for trijunctions. The connection of the functors \(G\) from \(S5_{\text{triv}}\) and \(S5_{\text{triv}}\) to \(\text{Gen}\) with the functor \(G\) for trijunctions is explained in [17] (end of Section 6). The trijunctional split equivalences are an isomorphic image of the split equivalences of \(S5_{\text{triv}}\) and \(S5_{\text{triv}}\). (In the terminology of [17], Sections 6–7, the split equivalences of \(S5_{\text{triv}}\) and \(S5_{\text{triv}}\) arise out of the even equivalence classes, i.e. the black regions, of trijunctional split equivalences.)

If we freely generate a trijunction with a single generating object of the category \(B\), then \(B\) is isomorphic to the free dyad generated by a single object, i.e. to the category \(S5_{\text{triv}}\). If we freely generate our trijunction with a single generating object of the category \(A\), then \(A\) is isomorphic to the free codyad generated by a single object, i.e. to the category \(S5_{\text{triv}}\). This is shown by relying on the coherence results for trijunctions, \(S5_{\text{triv}}\) and \(S5_{\text{triv}}\) mentioned in the preceding paragraph. Related matters are considered at the end of the paper in connection with the square of trijunctions (see Section 11).

9. Maximality in the context of \(S4\)

Let \(S4_{\text{triv}}\) be the category defined like \(S4_{\text{triv}}\), save that for every \(n\) we have the additional equation
\[
(\varepsilon \! n \circ \varepsilon \! n) = \varepsilon \! n + 1.
\]

It is shown in [8] (Section 5.8.2) that the category \(S4_{\text{triv}}\) is a preorder; namely, for every \(f\) and \(g\) of the same type we have \(f = g\). In \(S4_{\text{triv}}\), we have that \(\square\) is isomorphic to \(\square\).

To define \(S4_{\text{triv}}\), we could use instead of \((\varepsilon \! n \circ \varepsilon \! n)\) the equation
\[
(\delta \! n \circ \delta \! n) = \delta \! n + 1.
\]
which would make the assumption of the equation $(\delta □ □)$ superfluous. In fact, to define $S4_{\text{triv}}$, we could add to $S4_4$, instead of $(\epsilon □ □)$ any other equation between arrow terms of $S4_4$ that does not hold in $S4_{\text{triv}}$, provided that we assume this equation universally. This means that besides this equation we also assume all the equations obtained from it by increasing the subscripts of $1, \epsilon^3$ and $\delta □ □$ by a natural number $k$. For example, if we assume the following instance of $(\epsilon □ □)$:

$$\Box^k \epsilon^1_A = \epsilon^k_A,$$

we must also assume $\Box^k \epsilon^2_A = \epsilon^k_A$ for every $k \geq 0$. We do not assume thereby $\Box^0 \epsilon^2_A = \epsilon^1_A$, but it can be shown that this last instance of $(\epsilon □ □)$ is derivable from $\Box^1 \epsilon^2_A = \epsilon^2_A$, and so we obtain the whole of $(\epsilon □ □)$.

In defining the categories of this paper we always assume the axiomatic equations universally. So when for the extensions we assume new equations universally, we proceed as usual in our definitions.

The maximality of $S4_4$ is the result which says that any extension of the definition of that category with a new universally holding equation for the arrow terms of that category (new meaning that it does not hold in $S4_{\text{triv}}$) leads to collapse, i.e. to a category that is a preorder. (For a proof of this result, see [8], Section 5.10.) We will speak of maximality for other categories later on in the same sense. (The notion of maximality in [13], Section 9.3, is related, but stronger; it requires not only that the newly obtained category, like $S4_{\text{triv}}$, be a preorder, but also that any category in the class in which the newly obtained category is the freely generated one be also a preorder.)

The category $S4_{\text{triv}}$ is defined like $S4_k$ with the additional equation

$$(\epsilon \Box □)^{\text{triv}} \quad \Diamond \epsilon^k_A = \epsilon^k_A + 1.$$ 

We can say for $S4_{\text{triv}}$, mutatis mutandis, whatever we said for $S4_{\text{triv}}$. The category $S4_k$ is maximal in the same sense in which $S4_4$ is maximal.

When we consider extensions with new equations for categories like $S4_{\text{triv}}$, whose objects are not finite ordinals but modalities, assuming an equation universally means that besides this equation we also assume all the equations obtained from it by appending to the subscripts of the primitive arrow terms an arbitrary modality $A$ on the right-hand side. For example, the equation $(\epsilon □ □)$ now becomes the following equation:

$$(\epsilon \Box □)^{\text{triv}} \quad \Box \epsilon^k_A = \epsilon^k_A + 1.$$ 

If we assume the following instance of this equation:

$$\Box \epsilon_A = \epsilon_A,$$

we must also assume $\Box \epsilon_A = \epsilon_A$ for every modality $A$.

The category $S4_{\text{triv}}$ is not maximal in the sense in which $S4_k$ and $S4_A$ were maximal. We can add to $S4_{\text{triv}}$ one of the equations $(\epsilon □ □)$ or $(\epsilon □ □)$, where $n$ and $n + 1$ are replaced respectively by $A$ and $\Box A$, or $A$ and $\Box A$, without thereby obtaining the other. This is shown with the help of appropriate modifications of the functor $G$ from $S4_{\text{triv}}$ to $\text{Rel}$ (we may omit the pairs involving $\Box$ without omitting those involving $\Diamond$, and vice versa).

Let the category $S4_{\Box □}$ be defined like $S4_{\text{triv}}$, save that we have the additional equations $(\epsilon □ □)$ and $(\epsilon □ □)$, with the replacement mentioned in the preceding paragraph. This category is not a preorder because the equation

$$(\Box □)^{\text{triv}} \quad \Diamond \Box \Box \epsilon_A = \epsilon_A + 1 \Box □ \epsilon_A,$$

does not hold in it, as we are going to show now. Consider the pictures

$$\begin{array}{c}
G(\Box \Box \epsilon_A) \quad \Box □ \epsilon_A \\
\Box □ □ \epsilon_A \\
\Box □ □ □ \epsilon_A.
\end{array}$$

which are yielded by the functor $G$ from $S4_{\Box □}$ to $\text{Rel}$, but also by a modification $G'$ of that functor, which goes from $S4_{\Box □}$ to $\text{Rel}$, and takes into account that $MM$ is isomorphic to $M$, for $M$ being $\Box$ or $\Diamond$.

To define $G'$, we define first inductively a function $\Box$ on the objects of $S4_{\Box □}$, which are also objects of $S4_{\Box □}$, i.e. the modalities. For $M, M_1, M_2 \in \{\Box, \Diamond\}$, we have

$$M^\text{triv} = M,$$

$$(M_1M_2A)^{\text{triv}} = \begin{cases} (M_2A) & \text{if } M_1 = M_2, \\
M_1(M_2A) & \text{if } M_1 \neq M_2.\end{cases}$$

Next we define the arrow terms $j_A: A \prec A^\text{triv}$ and $j^A: A^\text{triv} \rightarrow A$ of $S4_{\Box □}$ inductively:

$$j_M = j_M^\text{triv} = 1_M,$$

$$j_\Box □ A = j_\Box □ A + j_\Box □ A,$$

$$j_\Box □ □ A = j_\Box □ □ A + j_\Box □ □ A,$$

for $M_1$ different from $M_2$,

$$j_{M_1M_2A} = j_{M_1M_2A}, \\
j_{M_1M_2A} = j_{M_1M_2A}.$$
It is easy to see that $j_A$ and $j^A$ are isomorphisms of $S_{4\triangledown\triangledown}$, inverse to each other. Then, for $G$ being the functor from $S_{4\triangledown\triangledown}$ to $\text{Rel}$, we have that $G^2A$ is $G\delta^2$, and for $f : A \vdash B$ an arrow term of $S_{4\triangledown\triangledown}$, i.e. of $S_{4\triangledown\triangledown}$, we have that $G^2f$ is $G(j_B \circ f \circ j^A)$. It is easy to verify that $G^2$ is indeed a functor, which is sufficient to show that the equation $(\square\triangledown\triangledown)$ does not hold in $S_{4\triangledown\triangledown}$.

Then we can infer from $S_{4\triangledown\triangledown}$ Coherence that $G^2$ is a faithful functor, i.e. $S_{4\triangledown\triangledown}$ Coherence. Suppose for $f, g : A \vdash B$ that $G^2f = G^2g$; by $S_{4\triangledown\triangledown}$ Coherence we have $j_B \circ f \circ j^A = j_B \circ g \circ j^A$ in $S_{4\triangledown\triangledown}$, and hence also in $S_{4\triangledown\triangledown}$. Since $j^A$ and $j_B$ are isomorphisms in $S_{4\triangledown\triangledown}$, it follows that $f = g$ in $S_{4\triangledown\triangledown}$.

Let $S_{4\triangledown\triangledown}$ be defined like $S_{4\triangledown\triangledown}$, save that we have the additional equation $(\square\triangledown\triangledown)$. In $S_{4\triangledown\triangledown}$, besides having that $MM$ is isomorphic to $M$, for $M$ being $\square\triangledown\triangledown$ or $\Box\triangleleft\triangledown\triangledown$, we also have this isomorphism for $M$ being $\Box\square\triangledown\triangledown$ or $\square\Box\triangledown\triangledown$. For $M$ being $\Box\square\triangleleft\triangledown\triangledown$, let

$i =_{df} \Box\square\Box\delta^\Box \ast \Box\Box\Box \varepsilon^\Box : MM \vdash M,$

$i^{-1} =_{df} \Box\Box\Box \delta^\Box \ast \Box\square\Box \varepsilon^\Box : M \vdash MM.$

To show that $i \circ i^{-1} = 1_{\Box\Box\Box\triangledown\triangledown}$, we may apply $S_{4\triangledown\triangledown}$ Coherence. To show that $i^{-1} \circ i = 1_{\Box\Box\Box\triangledown\triangledown}$, we have

$i^{-1} \circ i = \Box\Box\Box \delta^\Box \ast \Box\Box\Box \varepsilon^\Box \ast \Box\Box\Box \delta^\Box \ast \Box\Box\Box \varepsilon^\Box,$

by $S_{4\triangledown\triangledown}$ Coherence,

$= \Box\Box\Box \delta^\Box \ast \Box\Box\Box \varepsilon^\Box \ast \Box\Box\Box \delta^\Box \ast \Box\Box\Box \varepsilon^\Box,$

by $(\square\triangledown\triangledown)$,

$= 1_{\Box\Box\Box\triangledown\triangledown},$ by $S_{4\triangledown\triangledown}$ Coherence.

We proceed analogously for $M$ being $\Box\square\triangleleft\triangledown\triangledown$.

For the proposition below, we need the following diagram of arrows of $S_{4\triangledown\triangledown}$, which without the arrow terms may be found in [26], and is commonly used to classify the modalities of $S4$ (see [23], p. 56):

![Diagram of arrows](image_url)

**Preorder of $S_{4\triangledown\triangledown}$**. The category $S_{4\triangledown\triangledown}$ is a preorder, and its skeleton is given by the diagram above.

**Proof**. Note first that the isomorphisms of $S_{4\triangledown\triangledown}$ yield just the seven objects in the diagram above. Next, for $(M_1, M_2)$ being a pair of these seven modalities that is not $(\Box\square\Box, \square\Box\Box\Box)$, we may conclude from $S_{4\triangledown\triangledown}$ Coherence, and the properties of the functor $G$ from $S_{4\triangledown\triangledown}$ to $\text{Rel}$, that there is at most one arrow from $M_1$ to $M_2$ in $S_{4\triangledown\triangledown}$, and hence also in $S_{4\triangledown\triangledown}$, and $(\square\Box\Triangledown\Triangledown)$ does not hold in $S_{4\triangledown\triangledown}$.

We can then show the following.

**Maximality of $S_{4\triangledown\triangledown}$**. The category $S_{4\triangledown\triangledown}$ is maximal.

**Proof**. Suppose that we have the arrow terms $f, g : A \vdash B$ of $S_{4\triangledown\triangledown}$ such that $f = g$ does not hold in $S_{4\triangledown\triangledown}$. By $S_{4\triangledown\triangledown}$ Coherence, we have $G^2f \neq G^2g$, where $G^2$ is the functor from $S_{4\triangledown\triangledown}$ to $\text{Rel}$ defined above. Then it can be inferred that $G^2f$ corresponds to the picture with solid lines, while $G^2g$ corresponds to the picture with dotted lines

$A \quad \square\Box\Box\Box\triangledown\triangledown \quad \square\Box\Box\Box\triangledown\triangledown$

$B \quad \square\Box\Box\Box\triangledown\triangledown \quad \square\Box\Box\Box\triangledown\triangledown$

($G^2f$ and $G^2g$ can of course switch places). This is because in our pictures we cannot have crossings. Let $h_A : \Box\square\Box \vdash A$ and $h_B : B \vdash \Box\Box\Box\triangledown\triangledown$ be the arrows of $S_{4\triangledown\triangledown}$ such that $G^2h_A$ and $G^2h_B$ correspond respectively to the pictures.
In the left picture, any $\Box$ in $A$ to the left of the displayed $\Diamond$ is tied to the left $\square$ in $\Box\Box$, and analogously when “left” is replaced by “right”. We interpret the right picture analogously, replacing $\Box$ by $\Diamond$. Then, by $S4_{\Box\Box\Delta}$ Coherence, we can conclude that

$$h_B \circ f \circ h_A = \Box\Box \Diamond \circ \epsilon_{\Box\Box} \circ \epsilon_{\Box\Box},$$

$$h_B \circ g \circ h_A = \Box\Box \Box \circ \Box\Box \Diamond,$$

and this, together with appending modalities on the right-hand side in the subscripts of $f$ and $g$, yields the equation $(\Box\Box)$. So if we assume $f = g$ universally, we will also have $(\Box\Box)$, and hence we will be in $S4_{\Box\Box\Box\Delta}$, which is a preorder.

Let the category $S4_{\Box\Box\Box\Delta}$ be defined like $S4_{\Box\Box\Box}$, save that we have the additional equations $(\epsilon\Box\Box)$ and $(\epsilon\Box\Box\Box)$ we used to obtain $S4_{\Box\Box\Box}$ out of $S4_{\Box\Box}$. We define a functor $G'$ from $S4_{\Box\Box\Box\Delta}$ to $Rel$ with the help of the functor $G$ from $S4_{\Box\Box\Box}$ to $Rel$, as we did for $S4_{\Box\Box\Box}$, and by relying on $S4_{\Box\Box\Box}$ Coherence we establish that this new functor $G'$ is faithful, i.e. $S4_{\Box\Box\Box\Delta}$ Coherence (see above). Then we can show that $S4_{\Box\Box\Box\Delta}$ is not a preorder. Besides the equation $(\Box\Box)$, we do not have in $S4_{\Box\Box\Box\Delta}$ the equations

$$(\Box\Box\Diamond) \, \Box\Box \Diamond \circ \Box\Box \Diamond = \Box\Box \Diamond \circ \Box\Box \Diamond,$$  
$$(\Box\Box\Diamond) \, \Box \circ \Box \Diamond = \Box \circ \Box \Diamond.$$  

as is easily shown with the help of $G'$.

When we add $(\Box\Box)$ to $S4_{\Box\Box\Box}$, we can derive $(\Box\Box\Diamond)$ as follows:

$$\Box\Box \Diamond \circ \Box\Box \Diamond = \Box \circ \Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond,$$

by $(\Box\Box)$. Next, when we add $(\Box\Box\Diamond)$ to $S4_{\Box\Box\Box}$, we can derive $(\Box\Box\Diamond\Diamond)$ as follows:

$$\Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond = \Box \circ \Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond,$$

by $(\Box\Box\Diamond)$. We proceed analogously to derive $(\Box\Box\Diamond\Diamond)$ from $(\Box\Box\Diamond)$. When we add $(\Box\Box\Diamond\Diamond)$ and $(\Box\Box\Diamond\Diamond)$ to $S4_{\Box\Box\Box}$, we can derive $(\Box\Box\Diamond\Box)$ as follows:

$$\Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond \circ \Box\Box \Diamond,$$

by $(\Box\Box\Diamond\Diamond)$. With the help of modifications of $G'$ in which we omit all $\Box$-links, i.e. links involving $\Box$ (which are here links joining occurrences of $\Box$, without omitting $\Diamond$-links, i.e. links involving $\Diamond$ (which are here links joining occurrences of $\Diamond$), and vice versa, we can show that none of $(\Box\Box\Diamond)$ and $(\Box\Box\Diamond\Diamond)$ implies the other, and the same for $(\Box\Box\Diamond\Diamond)$ and $(\Box\Box\Diamond\Diamond)$. Let the category $S4_{\Box\Box\Box\Box\Box\Box\Box}$ be defined like $S4_{\Box\Box\Box\Box\Box}$, save that we have the additional equation $(\Box\Box)$. We can show that $S4_{\Box\Box\Box\Box\Box\Box\Box}$ is a preorder, and that its skeleton is given by the following diagram:
Note that in $\mathcal{S}4_{\mathrm{Conn}}$, the modalities $\Box\Diamond$ and $\Diamond\Box$ on the one hand, and $\Diamond\Diamond$ and $\Box\Box$ on the other hand, are isomorphic. Note also that the arrows $\chi_{\mathcal{A}\mathcal{B}}$ are isomorphisms in $\mathcal{S}4_{\mathrm{Conn}}$.

It can be shown in extending $\mathcal{S}4_{\mathrm{Conn}}$, that if we have universally assumed any new equation for arrow terms of $\mathcal{S}4_{\mathrm{Conn}}$, then we will obtain one of the equations \((\Box\Diamond\chi)\) and \((\Diamond\Box\chi)\), and hence also one of the equations \((\Box\Box\chi)\) and \((\Diamond\Diamond\chi)\). This is not maximality as we had for $\mathcal{S}4_{\mathrm{Conn}}$, but it is not very far from it. A more precise result, which yields this relative maximality, is stated as follows.

If the new equation $f = g$, which does not hold in $\mathcal{S}4_{\mathrm{Conn}}$, is such that $G^f$ differs from $G^g$ in the $M$-links, for $M$ being $\Box$ or $\Diamond$, then we can derive $(M\chi\Box)$ and $(M\chi\Diamond)$. If $M$ is $\Box$, then we proceed in a manner analogous to what we had in the proof of the Maximality of $\mathcal{S}4_{\mathrm{Conn}}$, with $h_A : \Box\Diamond\Box \rightarrow A$ as there and $h_B : B \rightarrow \Box\Diamond$ replaced by an arrow of the type $B \rightarrow \Diamond\Box$. If $M$ is $\Diamond$, then we proceed dually by replacing $h_A$.

If $M$ is here only $\Box$, then we cannot derive $(\Box\Diamond\chi)$ and $(\Diamond\Box\chi)$, and if it is only $\Diamond$, then we cannot derive $(\Box\Box\chi)$ and $(\Diamond\Diamond\chi)$. If $M$ stands here for both $\Box$ and $\Diamond$, i.e., $G^f$ differs from $G^g$ both in $\Box\Diamond$-links and $\Diamond\Box$-links, then we can derive $(\Box\Diamond)$.

(The point in the proof of the Maximality of $\mathcal{S}4_{\mathrm{Conn}}$ is that $G^f$ and $G^g$ cannot differ in $\Diamond\Box$-links without differing also in $\Box\Diamond$-links, and vice versa.)

10. The square of adjunctions

In this section, we consider some elementary facts concerning adjunctions, which we need for the exposition later on.

That a functor $f$ from $\mathcal{B}$ to $\mathcal{A}$ is left adjoint to a functor $g$ from $\mathcal{A}$ to $\mathcal{B}$ (alternatively, $g$ is right adjoint to $f$) means that, for $I_\mathcal{B}$ and $I_\mathcal{A}$ being respectively the identity functors of $\mathcal{B}$ and $\mathcal{A}$, we have a natural transformation $\gamma : I_\mathcal{B} \rightarrow GF$, the unit of the adjunction, and a natural transformation $\phi : FG \rightarrow I_\mathcal{A}$, the counit of the adjunction, which satisfy the following triangular equations for every object $B$ of $\mathcal{B}$ and every object $A$ of $\mathcal{A}$:

\[ \gamma_{I_{\mathcal{B}}} \circ f = I_{\mathcal{B}}, \quad G\phi_{A} \circ \gamma_{A} = I_{\mathcal{A}}. \]

An adjunction is a structure made of such functors $F$ and $G$, and such natural transformations $\gamma$ and $\phi$ (for more details, see [37], Chapter IV, and [8], Chapter 4).

Every adjunction generates four adjunctions involving functor categories, which we display in the following picture, where left adjoints have solid arrows, and right adjoints have dotted arrows:

\[
\begin{array}{cccc}
\mathcal{A}^C & \overset{G^C}{\longrightarrow} & \mathcal{B}^C \\
\mathcal{A} & \overset{G}{\longrightarrow} & \mathcal{B} \\
\mathcal{A} & \overset{F}{\longrightarrow} & \mathcal{B} \\
\mathcal{A}^C & \overset{G^C}{\longrightarrow} & \mathcal{B}^C \\
\end{array}
\]

For the functors $H$, $H_1$, and $H_2$ from $\mathcal{B}$ to $\mathcal{A}$, and for $\alpha$ a natural transformation from $H_1$ to $H_2$, we have

\[ C^B H = GH, \quad (G^B \alpha)_B = G\alpha_B, \]
\[ A^C H = HG, \quad (A^C \alpha)_A = \alpha_{CA}; \]

we define the other functors involved in the adjunctions above analogously.

In this square of adjunctions, the members of the units for the two horizontal adjunctions are the natural transformations $\gamma_H : H \rightarrow GH$, and the members of the counits are $\phi_H : FG \rightarrow H$. For the two vertical adjunctions, the members of the units are $\gamma_H : H \rightarrow HGF$, and the members of the counits are $\phi_H : HFG \rightarrow H$. In the horizontal adjunctions, the functors involving $F$ and $G$ behave like $F$ and $G$, while, in the vertical adjunctions, the functor involving $F$ becomes right adjoint, and that involving $G$ left adjoint. The horizontal adjunctions are images of the original adjunction by two covariant 2-endofunctors of the 2-category $\mathsf{Cat}$ of categories with functors and natural transformations, while the vertical adjunctions are such images by two contravariant 2-endofunctors (for the notions of 2-category and 2-functor, see [37], Sections XII.3–4).

For $C_1, C_2 \in \{\mathcal{A}, \mathcal{B}\}$, let a canonical functor from $C_1$ to $C_2$ be any functor from $C_1$ to $C_2$ defined in terms of the identity functors $I_\mathcal{A}$ and $I_\mathcal{B}$, the functors $F$ and $G$, and composition of functors. Let $C_2^{C_1}$ be the subcategory of the functor category $C_2^C$ whose objects are the canonical functors from $C_1$ to $C_2$, and whose arrows are the canonical natural transformations, defined in terms of the identity natural transformations, the unit $\gamma$ and counit $\phi$ of the adjunction, the functors $F$ and $G$, and composition.

So the objects of $C_2^{C_1}$ are $I_\mathcal{B}, GF, GFG, etc.,$ those of $C_2^{C_1}$ are $F$, $FG$, $GFG$, etc., those of $C_2^{C_1}$ are $I_\mathcal{A}, FC$, $FGF$, etc., and finally those of $C_2^{C_1}$ are $G$, $GFG$, $GFGF$, etc. Then from the square of adjunctions above we obtain an analogous square by replacing $C_2^{C_1}$ with $C_2^{C_1}$. Yet another analogous square of adjunctions is obtained when $C_2^{C_1}$ is understood as the full subcategory of $C_2^{C_1}$ whose objects are the canonical functors from $C_1$ to $C_2$. (The four preordering equations of [8], Section 4.6.2, are connected by the bijections between hom-sets of the horizontal and vertical adjunctions in the square of adjunctions.)

For every category $\mathcal{A}$ treated in this paper, whose objects are either finite ordinals or modalities, let a canonical functor from $\mathcal{A}$ to $\mathcal{A}$ be a functor definable in terms of the functors assumed for defining $\mathcal{A}$ and composition of functors. Then these canonical functors may be identified with the objects of $\mathcal{A}$, and, for $\mathcal{A}^{\mathcal{A}}$ being the full subcategory of $\mathcal{A}^{\mathcal{A}}$ whose objects are the canonical functors from $\mathcal{A}$ to $\mathcal{A}$, we have that $\mathcal{A}$ is isomorphic to $\mathcal{A}^{\mathcal{A}}$. 
If $A$ is $S5_{\square}$, then, as we have seen in Section 6, the endofunctor $\diamond$ is left adjoint to the endofunctor $\Box$. Since $C A^x$ is isomorphic to $A$, the $C C^2_2$ variant of the square of adjunctions reduces to

$$
\begin{array}{c}
S5_{\diamond} \\
\downarrow \theta \\
S5_{\square}
\end{array}
$$

with the two sides omitted being exact replicas of those drawn. In the horizontal adjunction here, $\Box f$ and $\diamond f$ are just $\diamond$ and $\Box$, respectively, and this adjunction is the original adjunction mentioned at the end of Section 6.

The functors involved in the vertical, contravariant, adjunction, for $M$ being $\square$ or $\diamond$, and $\alpha_A$ a primitive arrow term of $S5_{\Box}$, are defined by

$$
l^M A = AM, \quad l^M \alpha_A = \alpha AM, \quad l^M (g \circ f) = l^M g \circ l^M f.
$$

That these are indeed functors is guaranteed by the fact that the equations of $S5_{\Box}$ are assumed universally. These functors will hence also exist when we extend $S5_{\Box}$ with new equations, assumed universally. Note that they exist in the free dyad $S5_{\Box}$, but they need not exist in an arbitrary dyad. (Analogous functors exist in $S4_{\Box}$, $S4_{\Diamond}$, etc., but they need not exist in arbitrary comonads and monads.)

11. Maximality in the context of $S5$

Consider the following equations, which do not hold in $S5_{\Box}$:

$$
\begin{align*}
\epsilon_A^{\square} &= \epsilon_A^{\Box}, \\
\epsilon_A^{\diamond} &= \epsilon_A^{\Box}, \\
\epsilon_A^{\circ} &= \delta_A^{\Box} \circ \epsilon_A^{\Box}.
\end{align*}
$$

In the left upper corner and the right lower corner we have the equations $(\epsilon_{\square} \text{ triv})$ and $(\epsilon_{\diamond} \text{ triv})$. The left-hand sides of these six equations correspond to the six pictures on the left, while the right-hand sides correspond to the six pictures on the right:

$$
\begin{array}{cccc}
\square & \Box & \diamond & \circ \\
\Box & \square & \diamond & \circ \\
\diamond & \square & \Box & \not\circ \\
\circ & \square & \Box & \not\diamond \\
\end{array}
$$

The bijections between hom-sets of the horizontal adjunction of $S5_{\Box}$, mentioned at the end of the preceding section stand behind the horizontal connections in the six pictures on the left. The same holds when we replace “horizontal” by “vertical”, or “left” by “right”. From that we can conclude that any of the six equations above when added to $S5_{\Box}$ yields the five remaining ones. Anticipating matters, we call all of these equations a preordering equation of $S5_{\Box}$.

Let $S5_{\Box,\text{triv}}$ be the category defined like $S5_{\Box}$, save that we have as an additional equation one of the preordering equations of $S5_{\Box}$ (universally assumed). To show that $S5_{\Box,\text{triv}}$ is a preorder, we need to consider first some properties of the functor $G$ from $S5_{\Box}$ to $Gen$.

For every arrow $f$ of $S5_{\Box}$, the partition corresponding to the split equivalence $Gf$ induces a partition on the occurrences of $\Box$ and $\diamond$ in the source and target of $f$, and we call the members of the latter partition the equivalence classes of $f$. An element of an equivalence class of $f$ is either a source element or a target element, and also every such element is either a $\Box$ element or a $\diamond$ element.

From the normal form for the arrow terms of $S5_{\Box}$ in the proof of $S5_{\Box}$, coherence in Section 6, we can conclude that for every arrow $f$ of $S5_{\Box}$ the equivalence classes of $f$ are of one of the following two kinds:

- $\Box$ there is a $\Box$ element that is the rightmost source element in the class, and is called the head of the class; all the other source elements (if any) are $\diamond$ elements, and all the target elements (if any) are $\Box$ elements;
- $\diamond$ there is a $\diamond$ element that is the rightmost target element in the class, and is called the head of the class; all the other target elements (if any) are $\Box$ elements, and all the source elements (if any) are $\diamond$ elements.
Every source □ element and every target ◊ element is a head. Let an element of an equivalence class that is not its head be called subordinate. Every source ◊ element and every target □ element is subordinate. The number of equivalence classes of an arrow depends only on the type of that arrow.

Take an arrow \( f : A \rightarrow B \) of \( S_{5,\text{triv}} \), and consider an equivalence class \( E \) of \( f \). For an arbitrary subset \( E' \) of \( E \) that contains the head of \( E \), there is an arrow \( k_A : A' \rightarrow A \) built by using essentially \( e' \), and there is an arrow \( k_B : B \rightarrow B' \) built by using essentially \( e'' \), such that \( k_B \circ f \circ k_A \) has equivalence classes exactly like \( f \), save that \( E \) is replaced by \( E' \). As a limit case, we may take \( E' \) to be the singleton whose only element is the head of \( E \). We say that \( k_B \circ f \circ k_A \) is obtained by reducing \( E \) in \( f \) to \( E' \). Next we show the following.

**Preorder of \( S_{5,\text{triv}} \).** The category \( S_{5,\text{triv}} \) is a preorder, and its skeleton is given by the following diagram:

\[
\begin{array}{ccc}
\square & \xrightarrow{e'} & \square \\
\downarrow & & \downarrow e \\
\Diamond & & \Diamond \\
\end{array}
\]

**Proof.** Note first that, for \( M_1, M_2 \in \{\square, \Diamond\} \), we have in \( S_{5,\text{triv}} \) that \( M_1 M_2 \) is isomorphic to \( M_2 \). To prove these isomorphisms, besides equations we have encountered previously, we have

\[
\delta^{\text{com}} \circ e_M = e_M \circ \square^{\text{com}}, \text{ by } (e_M \text{ nat}),
\]

\[
= 1_{\square^{\text{com}}}, \text{ by } (e_M \text{ triv}) \text{ and } (\square^{\text{com}} \beta),
\]

and we derive \( e_M^{-1} \circ \delta^{\text{com}} = 1_{\square^{\text{com}}} \) analogously. Next, if \( M_1, M_2 \in \{\square, \emptyset, \Diamond\} \), then from \( S_{5,\emptyset} \) Coherence and the form of the equivalence classes of the arrows of \( S_{5,\emptyset} \), we may conclude that there is at most one arrow from \( M_1 \) to \( M_2 \) in \( S_{5,\emptyset} \), and hence also in \( S_{5,\text{triv}} \). □

We define \( S_{5,\emptyset} \), analogously, and prove in the same manner that it is a preorder, with an isomorphic skeleton.

The category \( S_{5,\emptyset} \), as well as \( S_{5,\text{triv}} \), is maximal in the sense in which \( S_{4,\text{triv}} \) was shown maximal in Section 9.

**Maximality of \( S_{5,\emptyset} \).** The category \( S_{5,\emptyset} \) is maximal.

**Proof.** Suppose that we have the arrow terms \( f_1, f_2 : A \rightarrow B \) of \( S_{5,\emptyset} \) such that \( f_1 = f_2 \) does not hold in \( S_{5,\emptyset} \). By \( S_{5,\emptyset} \) Coherence, we have \( Gf_1 \neq Gf_2 \). Then it can be inferred that there are three distinct occurrences \( x, y_1, y_2 \) of \( \square \) or \( \Diamond \) in \( A \) or \( B \) such that \( y_1 \) and \( y_2 \) are heads of equivalence classes both in \( f_1 \) and in \( f_2 \), and \( x \) is in the same class \( E_1 \) as \( y_1 \) in \( f_1 \), and in the same class \( E_2 \) as \( y_2 \) in \( f_2 \). So \( x \) is a subordinate element both in \( f_1 \) and \( f_2 \).

Let \( S_{5,\emptyset}^{\text{nat}} \) be obtained by extending \( S_{5,\emptyset} \) with \( f_1 = f_2 \), universally assumed. By reducing \( E_1 \) in \( f_1 \) to \( \{x, y_1\} \), and every other equivalence class of \( f_1 \) to a singleton, we obtain the arrow \( f_1' = k'(f_1 \ast k') \), which is equal in \( S_{5,\emptyset}^{\text{nat}} \) to \( f_2' = k''(f_2 \ast k') \). In \( f_2' \) the subordinate element \( x \) belongs to the equivalence class \( \{x, y_2\} \), while all the other equivalence classes of \( f_2' \) are singletons.

The number \( n \) of equivalence classes in \( f_1, f_2, f_1' \) and \( f_2' \) is the same, and we proceed by induction on \( n \) to show that we can derive one of the preordering equations in \( S_{5,\emptyset}^{\text{nat}} \), i.e. that \( S_{5,\emptyset} \) is \( S_{5,\text{triv}} \). The basis of this induction is when \( n = 2 \), and then we have cases that are covered by the six preordering equations of \( S_{5,\emptyset} \). If \( n \geq 4 \), then either \( f_1' \ast C\delta^{\text{com}} = f_2' \ast C\delta^{\text{com}} \), and we can apply the induction hypothesis, or \( C\delta^{\text{com}} \cdot f_1' = C\delta^{\text{com}} \cdot f_2' \), and we can apply the induction hypothesis (here \( C \) is a modality, possibly empty).

If \( n = 3 \), we proceed either as when \( n \geq 4 \), or we have an additional case in which we rely also on the vertical adjunction involving \( f' \) and \( f'' \), which obtains also in \( S_{5,\emptyset} \) (see the preceding section). For example, if we find ourselves in the situation that corresponds to the following pictures:

\[
\begin{array}{ccc}
\&\& \\
\downarrow & & \downarrow \\
\& & \\
\end{array}
\]

\[
\begin{array}{ccc}
\&\& \\
\downarrow & & \downarrow \\
\& & \\
\end{array}
\]

by the vertical adjunction, we pass first to

\[
\begin{array}{ccc}
\&\& \\
\downarrow & & \downarrow \\
\& & \\
\end{array}
\]

\[
\begin{array}{ccc}
\&\& \\
\downarrow & & \downarrow \\
\& & \\
\end{array}
\]

and then by precomposing with \( \Diamond \square \Diamond \) we obtain

\[
\begin{array}{ccc}
\&\& \\
\downarrow & & \downarrow \\
\& & \\
\end{array}
\]

\[
\begin{array}{ccc}
\&\& \\
\downarrow & & \downarrow \\
\& & \\
\end{array}
\]

i.e. the preordering equation in the right upper corner. This is enough to show that \( S_{5,\emptyset} \) is maximal. □

The category \( S_{5,\emptyset} \) is shown to be maximal in the same manner.

It is shown in [8] (Addenda and Corrigenda, Section 5.11) that the maximality of comonads, i.e. of \( S_{4,\text{triv}} \), entails an analogous maximality of adjunction. In the same way, the maximality of \( S_{5,\text{triv}} \) or \( S_{5,\emptyset} \) entails the maximality of trijunction, as we will show below. We cannot extend this notion with new equations in the canonical language of trijunctions, equations being assumed universally (see Section 9), without trivializing the notion: any equation in the canonical language will hold.
To infer the maximality of adjunction from the maximality of comonads, or the maximality of monads, we can proceed not as in the reference mentioned above, but by appealing to the square of adjunctions of the preceding section. The category $A^A$ corresponds to the comonad, and $B^B$ to the monad. Any arrow of the freely generated adjunction is in one of four disjoint categories, which correspond to the categories $C A^A$, $C B^B$, $C A^A$ and $C B^B$ (see the preceding section). By the horizontal and vertical adjunctions, any such equation can be reduced to a new equation of comonads or monads.

There is a square of trijunctions analogous to the square of adjunctions. Suppose we have a trijunction given by the categories $A$ and $B$, a functor $U$ from $A$ to $B$, and the functors $L$ and $R$ from $B$ to $A$, with $L$ being left adjoint and $R$ right adjoint to $U$. Then, with arrows of right adjoints being more finely dotted, we have

The category $B^B$ here corresponds to dyads, i.e., $S_5^{2,2}$, and $A^A$ to codyads, i.e., $S_5^{5,5}$. Any arrow of the freely generated trijunction is in one of four disjoint categories, which correspond to the four categories in the square of trijunctions above. For example, to $A^B$ there corresponds a category $C A^B$ whose objects are $L$, $R$, $UL$, $UR$, $ULUL$, etc., to $B^B$ there corresponds a category $C B^B$ whose objects are $U$, $R$, $ULU$, $LU$, etc., to $A^A$ there corresponds a category $C A^A$ whose objects are $\emptyset$, $U$, $ULU$, $LU$, etc., and, finally, to $B^A$ there corresponds a category $C B^A$ whose objects are $U$, $UL$, $UR$, $ULUL$, $UL$, etc. Here, $\emptyset$ corresponds to identity functors.

By horizontal and vertical adjunctions, any new equation of trijunctions can be reduced to a new equation of dyads or codyads. So the maximality of trijunction can be inferred from the maximality of $S_5^{2,2}$, or the maximality of $S_5^{5,5}$.

To make this inference, we could also proceed as in [8] (Addenda and Corrigenda, Section 5.11). The category $C A^B$ is isomorphic by the functor $U$ to a subcategory $B'$ of $C B^B$, and $C B^B$ and $B'$ together with the functors $LU$, $RU$ and the inclusion functor from $B'$ to $C B^B$ make a trijunction isomorphic to the original trijunction. The category $B'$ is isomorphic to the category $(C B^B)|_{U'}$ of Section 8. Any new equation for trijunctions corresponds by this isomorphism to a new equation of dyads.

We will not consider here the extension of $S_5^{2,2}$ with the arrows $\chi_A^{\square\circ}$ or $\chi_A^{\square\bullet}$ of Section 5. With $\chi_A^{\square\circ}$ we would obtain a $\square\circ$-structure that is both $S_5^{5,5}$ and $S_5^{2,2}$, at the same time. With this structure, we come close to the Frobenius monads of [31] (pp. 151–152); namely, dyads where $\square$ and $\circ$ coincide, and where $\delta^{\square\circ}$ and $\delta^{\square\bullet}$ coincide respectively with $\delta^{\square\square}$ and $\delta^{\square\bullet}$ (alternatively, these are codyads where $\square$ and $\circ$ coincide). We deal with them in [17].

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