STRICHARTZ TYPE ESTIMATES FOR FRACTIONAL HEAT EQUATIONS

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Abstract. We obtain Strichartz estimates for the fractional heat equations by using both the abstract Strichartz estimates of Keel-Tao and the Hardy-Littlewood-Sobolev inequality. We also prove an endpoint homogeneous Strichartz estimate via replacing $L^\infty_x(\mathbb{R}^n)$ by $\text{BMO}_x(\mathbb{R}^n)$ and a parabolic homogeneous Strichartz estimate. Meanwhile, we generalize the Strichartz estimates by replacing the Lebesgue spaces with either Besov spaces or Sobolev spaces. Moreover, we establish the Strichartz estimates for the fractional heat equations with a time dependent potential of an appropriate integrability. As an application, we prove the global existence and uniqueness of regular solutions in spatial variables for the generalized Navier-Stokes system with $L^r(\mathbb{R}^n)$ data.

1. Introduction

This paper studies Strichartz type estimates for the inhomogeneous initial problem associated with the fractional heat equations

\begin{equation}
\begin{aligned}
\partial_t v(t,x) + (-\Delta)^\alpha v(t,x) &= F(t,x), & (t,x) &\in \mathbb{R}^{1+n} = (0, \infty) \times \mathbb{R}^n, \\
v(0,x) &= f(x), & x &\in \mathbb{R}^n,
\end{aligned}
\end{equation}

where $\alpha \in (0, \infty)$ and $n \in \mathbb{N}$. The main goal is to determine pairs $(q,p)$ and $(q_1,p_1)$ ensuring

\begin{equation}
\|e^{-t(-\Delta)^\alpha}f\|_{L^q_t(I;L^p_x)} \lesssim \|f\|_{L^2_x},
\end{equation}

\begin{equation}
\left\| \int_0^t e^{-(t-s)(-\Delta)^\alpha}F(s,x)ds \right\|_{L^q_t(I;L^{p_1}_x)} \lesssim \|F\|_{L^{q'_1}_t(I;L^{p'_1}_x)},
\end{equation}

where $I$ is either $[0, \infty)$ or $[0,T]$ for some $0 < T < \infty$, and $p'_1 = \frac{p_1}{p_1 - 1}$ is the conjugate of a given number $p_1 \geq 1$. Here $\partial_t$ and $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ are the partial derivative with respect to $t$ and the Laplacian with respect to $x = (x_1, \cdots, x_n)$, respectively. Furthermore,

\[ (-\Delta)^\alpha v(t,x) = \mathcal{F}^{-1}(|\xi|^{2\alpha} \mathcal{F}(v(t,\xi)))(x), \]

where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ denotes its inverse. By the Fourier transform and Duhamel’s principle, the solution of (1.1) can be written as

\[ v(t,x) = e^{-t(-\Delta)^\alpha}f(x) + \int_0^t e^{-(t-s)(-\Delta)^\alpha}F(s,x)ds, \]
where
\[ e^{-t(-\triangle)^{\alpha}} f(x) = \mathcal{F}^{-1}(e^{-t|\xi|^{2\alpha}} \mathcal{F} f(\xi))(x) =: K_t^{\alpha}(x) * f(x) \]
and * stands for the convolution operating on the space variable.

The Strichartz type estimates for equation (1.1) have just been studied by few experts. Pierfelice [18] concerned such estimates for equation (1.1) with \( \alpha = 1 \) and small potentials of very low regularity. Miao, Yuan and Zhang [10] studied the non-endpoint case of (1.2) for equation (1.1).

For the Schrödinger and wave equations, the Strichartz estimates have been well studied in recent years, see, for example, Blair-Smith-Sogge [2], Burq-Gérard-Tzvetkov [3], Cazenave [4], Kapitanski [5], Kenig-Merle [12], Kenig-Ponce-Vega [13], D'Ancona-Pierfelice-Visciglia [6]. The Strichartz estimates for the Schrödinger and wave equations can be directly derived from the abstract Strichartz estimates of Keel-Tao [11] since the solution groups of these two equations act as unitary operators on \( L^2(\mathbb{R}^n) \) and such operators obey both the energy estimate and the untruncated decay estimate. While, since \( \{e^{-t(-\triangle)^{\alpha}}\}_{t \geq 0} \) is a semigroup and acts as a self-adjoint operator on \( L^2(\mathbb{R}^n) \)—see Lemma 2.1, we can only apply the abstract Strichartz estimates of Keel-Tao directly to obtain (1.2) if we have the energy estimate and untruncated decay estimate. But for (1.3), we can make use of the \( L^p \)– decay estimates and the Hardy-Littlewood-Sobolev inequality.

In this paper, we also establish an endpoint case of (1.2) by replacing \( L^\infty(\mathbb{R}^n) \) with the spaces of functions of bounded mean oscillation (\( BMO_x(\mathbb{R}^n) \)). Meanwhile, we obtain a parabolic homogeneous Strichartz estimate for equation (1.1), the two dimensional case of which is very useful for dealing with the global regularity of wave maps when combined with Lemma 2.1 for \( \alpha = 1 \) and the comparison principle for the heat equation, see Tao [24]. Moreover, we generalize (1.2) and (1.3) via replacing \( L^p(\mathbb{R}^n) \) with either Besov spaces or Sobolev spaces. These function spaces will be made precise later.

If equation (1.1) has a time dependent potential \( V(t, x) \), then it becomes
\[
\partial_t v(t, x) + (-\triangle)^{\alpha} v(t, x) + V(t, x) v(t, x) = F(t, x), \quad (t, x) \in \mathbb{R}_+^{1+n},
\]
\[
v(0, x) = f(x), \quad x \in \mathbb{R}^n.
\]
We can obtain the Strichartz estimates for equation (1.4) by using the Banach contraction mapping principle and assuming an appropriate integrability condition in space and time on \( V(t, x) \). A similar idea was used by D’Ancona-Pierfelice-Visciglia in [6] to get analogous estimates for the Schrödinger equations.

As an application, we establish the global existence and uniqueness of regular solutions in spatial variables for the generalized Navier-Stokes system on the half-space \( \mathbb{R}_+^{1+n}, n \geq 2 \):
\[
\left\{ \begin{array}{ll}
\partial_t v + (-\triangle)^{\alpha} v + (v \cdot \nabla)v - \nabla p = h, & \text{in } \mathbb{R}_+^{1+n}; \\
\nabla \cdot v = 0, & \text{in } \mathbb{R}_+^{1+n}; \\
v(0, x) = g(x), & \text{in } \mathbb{R}^n
\end{array} \right.
\]
with \( \alpha \in \left( \frac{1}{2}, \frac{1}{2} + \frac{n}{4} \right) \). For system (1.5), Lions [15] proved the global existence of the classical solutions when \( \alpha \geq \frac{5}{4} \) in dimensional 3. Similar result holds for general
We choose \( \phi \) where \( P \) are \((1.6)\). Moreover, \( S \) for a function space \( x \) with \( \delta \) being the Kronecker symbol and \( R_j = \partial_j (\Delta)^{-1/2} \) being the Riesz transform. When \( \alpha = 1 \), system \((1.5)\) becomes the classical Navier-Stokes system which is a celebrated nonlinear partial differential system.

In the above and below, \( U \leq V \) denotes \( U \leq CV \) for some positive constant \( C \) which is independent of the sets or functions under consideration in both \( U \) and \( V \); for a Banach space \( X \), \( L^p(X) \) (where \( p \in [1, \infty) \)) is used as the space of functions \( f : X \to \mathbb{R} \) with

\[
\|f\|_{L^p(X)} = \left( \int_X |f(x)|^p dx \right)^{1/p} < \infty;
\]

for a function space \( F(\mathbb{R}^n) \) on \( \mathbb{R}^n \), \( L^q(I; F(\mathbb{R}^n)) \) (where \( q \in [1, \infty) \)) represents the set of functions \( f : I \times \mathbb{R}^n \to \mathbb{R} \) for \( I \subseteq \mathbb{R} \) with

\[
\|f\|_{L^q(I; F(\mathbb{R}^n))} = \left( \int_I \|f(t, x)\|^q_{F(\mathbb{R}^n)} dt \right)^{1/q} < \infty.
\]

To state our main results, let us recall the definitions of some function spaces.

We use \( \mathcal{S}_0 \) to denote the following subset of the Schwartz class of rapidly decreasing functions \( \mathcal{S} \),

\[
\mathcal{S}_0 = \left\{ \phi \in \mathcal{S} : \int_{\mathbb{R}^n} \psi(x)x^\gamma dx, |\gamma| = 0, 1, 2, \cdots \right\},
\]

where \( x^\gamma = x_1^{\gamma_1}x_2^{\gamma_2} \cdots x_n^{\gamma_n}, |\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n \). Its dual \( \mathcal{S}'_0 = \mathcal{S}' / \mathcal{S}_0^\perp = \mathcal{S}' / \mathcal{P} \), where \( \mathcal{P} \) is the space of multinomials.

We introduce a dyadic partition of \( \mathbb{R}^n \). For each \( j \in \mathbb{Z} \), we let

\[
D_j = \{ \xi : 2^{j-1} < |\xi| \leq 2^{j+1} \}.
\]

We choose \( \phi_0 \in \mathcal{S}(\mathbb{R}^n) \) such that

\[
\text{supp}(\phi_0) = \{ \xi : 2^{-1} \leq |\xi| \leq 2 \} \text{ and } \phi_0 > 0 \text{ on } D_0.
\]

Let

\[
\phi_j(\xi) = \phi_0(2^{-j} \xi) \text{ and } \Psi_j(\xi) = \frac{\phi_j(\xi)}{\sum \phi_j(\xi)}.
\]

Then \( \Psi_j \in \mathcal{S} \) and

\[
\hat{\Psi}_j(\xi) = \hat{\Psi}_0(2^{-j} \xi), \text{ supp}(\hat{\Psi}_j) \subset D_j, \Psi_j(x) = 2^{jn} \Psi_0(2^j x).
\]

Moreover,

\[
(1.6) \quad \sum_{k=-\infty}^{\infty} \tilde{\Psi}_k(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}
\]

Let \( \Phi \in C_0^\infty(\mathbb{R}^n) \) be even and satisfy

\[
\hat{\Phi}(\xi) = 1 - \sum_{k=0}^{\infty} \tilde{\Psi}_k(\xi).
\]
Then, for any $\psi \in \mathcal{S}$,
\[ \Phi * \psi + \sum_{k=0}^{\infty} \Psi_k * \psi = \psi \]
and for any $f \in \mathcal{S}$,
\[ \Phi * f + \sum_{k=0}^{\infty} \Psi_k * f = f. \]

To define the homogeneous Besov spaces, we let
\[ \triangle_j f = \Psi_j * f, \quad j = 0, \pm 1, \pm 2, \ldots. \]
For $s \in \mathbb{R}^n$ and $1 \leq p, q \leq \infty$, we define the homogeneous Besov space $\dot{B}^s_{p,q}$ as the set of all $f \in \mathcal{S}'$ with
\[ \|f\|_{\dot{B}^s_{p,q}} = \left( \sum_{j=-\infty}^{\infty} (2^{js} \|\triangle_j f\|_{L^p})^q \right)^{1/q} < \infty, \text{ for } q < \infty, \]
\[ \|f\|_{\dot{B}^s_{p,\infty}} = \sup_{-\infty < j < \infty} 2^{js} \|\triangle_j f\|_{L^p} < \infty, \text{ for } q = \infty. \]

To define the inhomogeneous Besov spaces, we define
\begin{equation}
\triangle_j f = \begin{cases} 
0, & \text{if } j \leq -2, \\
\Phi * f, & \text{if } j = -1, \\
\Psi_j * f, & \text{if } j = 0, 1, 2, \ldots
\end{cases}
\end{equation}
For $s \in \mathbb{R}^n$ and $1 \leq p, q \leq \infty$, we define the inhomogeneous Besov space $B^s_{p,q}$ as the set of all $f \in \mathcal{S}'$ with
\[ \|f\|_{B^s_{p,q}} = \|\triangle_{-1} f\|_{L^p} + \left( \sum_{j=0}^{\infty} (2^{js} \|\triangle_j f\|_{L^p})^q \right)^{1/q} < \infty, \text{ for } q < \infty, \]
\[ \|f\|_{B^s_{p,\infty}} = \|\triangle_{-1} f\|_{L^p} + \sup_{0 \leq j < \infty} 2^{js} \|\triangle_j f\|_{L^p} < \infty, \text{ for } q = \infty. \]

On the other hand, Besov spaces can be defined by interpolation between the Lebesgue spaces and the Sobolev spaces of integer order (see Triebel [25]). Moreover, it follows from Bergh and Lofstrom [1] that for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$,
\[ B^s_{p,q}(\mathbb{R}^n) = [H^{s_1,p}, H^{s_2,p}]_{\theta,q}, \text{ and } \dot{B}^s_{p,q}(\mathbb{R}^n) = [\dot{H}^{s_1,p}, \dot{H}^{s_2,p}]_{\theta,q}, \]
where $s_1 \neq s_2$, $0 < \theta < 1$ and $s = (1-\theta)s_1 + \theta s_2$. Here $H^{s,p}(\mathbb{R}^n)$ and $\dot{H}^{s,p}(\mathbb{R}^n)$ are the inhomogeneous and homogeneous Sobolev spaces which are the completion of all infinitely differential functions $f$ with compact support in $\mathbb{R}^n$ with respect to the norms
\[ \|f\|_{H^{s,p}(\mathbb{R}^n)} = \|(I - \triangle)^{s/2} f\|_{L^p(\mathbb{R}^n)}, \text{ and } \|f\|_{\dot{H}^{s,p}(\mathbb{R}^n)} = \|(-\triangle)^{s/2} f\|_{L^p(\mathbb{R}^n)} \]
respectively, where $(I - \triangle)^{s/2} f = \mathcal{F}^{-1}((1 + |\xi|^2)^{s/2} \mathcal{F} f(\xi))$.
\[ BMO(\mathbb{R}^n) \text{ is the set of locally integrable functions } f \text{ with semi-norm } \]
\[ \|f\|_{BMO} = \left( \sup_Q \mathcal{L}(Q)^{-n} \int_Q |f(x) - f_Q|^2 dx \right)^{1/2} < \infty, \]
where $Q$ is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, $\mathcal{L}(Q)$ is the sidelength of $Q$ and $f_Q = \mathcal{L}(Q)^{-n} \int_Q f(x)\,dx$.

**Definition 1.1.** The triplet $(q, p, r)$ is called a $\sigma$–admissible triplet provided

$$\frac{1}{q} = \sigma \left( \frac{1}{r} - \frac{1}{p} \right),$$

where $1 < r \leq p \leq \infty$ and $\sigma > 0$.

**Proposition 1.2.** Let $(q, p, 2)$ be $\frac{n}{2\alpha}$–admissible. If $q \geq 2$ and $(q, p, \frac{n}{2\alpha})$ is not $(2, \infty, 1)$, then $(1.3)$ holds.

**Remark 1.3.** Proposition 1.2 extends Miao-Yuan-Zhang’s [16, Lemma 3.2] to the cases: $\begin{cases} (q, p, r) = (2, \frac{2n}{n+2\alpha}, 2) \text{ when } n > 2\alpha; \quad (q, p, r) = (\frac{2n}{n-2\alpha}, \infty, 2) \text{ when } n < 2\alpha. \end{cases}$

It is well known that for the Schrödinger equations, there are pairs $(q, p)$ and $(q_1, p_1)$ such that $(q, p, 2)$ and $(q_1, p_1, 2)$ are not $n/2$–admissible but the inhomogeneous Strichartz estimates hold (see Cazenave-Weissler [5], Kato [10] and Vilela [20]). Similarly, we will prove that $(1.3)$ holds for some pairs $(q, p)$ and $(q_1, p_1)$ satisfying the property

$$\frac{1}{q_1} - \frac{1}{q} + \frac{n}{2\alpha} \left( \frac{1}{p_1} - \frac{1}{p} \right) = 1. \tag{1.8}$$

This property is weaker than the $\frac{n}{2\alpha}$–admissibility of $(q, p, 2)$ and $(q_1, p_1, 2)$.

**Theorem 1.4.** Let $1 \leq p_1 < p \leq \infty$ and $1 < q_1 < q < \infty$. If $(q, p)$ and $(q_1, p_1)$ satisfy $(1.8)$, then $(1.3)$ holds.

**Remark 1.5.** Since $e^{-t(-\Delta)\alpha}$ commutes with $(-\Delta)^\beta$ and $(I-\Delta)^\beta$ for $\beta > 0$, if $(q, p)$ satisfies the assumption of Theorem 1.2 then $(1.2)$ holds with $\| \cdot \|_{L^p(\mathbb{R}^n)}$ replaced by either $\| \cdot \|_{L^p(\mathbb{R}^n)}$ or $\| \cdot \|_{H_0^s(\mathbb{R}^n)}$. Similarly, if $(q, p)$ and $(q_1, p_1)$ satisfy the assumption of Theorem 1.4 then $(1.3)$ holds with the same replacement.

**Theorem 1.6.** Let $n = 2\alpha$. Then

$$\| e^{-t(-\Delta)\alpha} f \|_{L^2_t((0,\infty); \text{BMO}_x(\mathbb{R}^n))} \lesssim \| f \|_{L^2}. \tag{1.9}$$

**Theorem 1.7.** (a) Let $1 \leq r \leq p \leq \infty$ and $0 < T < \infty$. If $n < 2\alpha$, then

$$\int_0^T s^{-\frac{n}{2\alpha}} \| e^{-s(-\Delta)\alpha} f \|_{L^2_r(\mathbb{R}^n)} \, ds \lesssim T^{-\frac{n}{2\alpha}} \| f \|_{L^r(\mathbb{R}^n)}. \tag{1.10}$$

(b) Let $2 \leq p \leq \infty$. If $n = 2\alpha$, then

$$\int_0^\infty s^{-2/p} \| e^{-s(-\Delta)\alpha} f \|_{L^2_r(\mathbb{R}^n)} \, ds \lesssim \| f \|_{L^2(\mathbb{R}^n)}. \tag{1.11}$$

**Remark 1.8.** We can refer to (1.11) as a parabolic homogeneous Strichartz estimate.

The special case $n = 2$ of (1.11) was proved by Tao in [24]. On the other hand, according to Miao-Yuan-Zhang’s [16, Proposition 2.1], (1.11) amounts to the fact that $L^2(\mathbb{R}^n)$ is embedded in the homogeneous Besov space

$$\dot{B}^s_{p,2}(\mathbb{R}^n), \quad s = \frac{(2-p)n}{2p}, \quad 2 < p \leq \infty.$$

Using the imbedding of $\dot{B}^{s,2}(\mathbb{R}^n)$ into $L^{\frac{2n}{n-2\alpha}}$ when $0 < 2\alpha < n$, we prove the following result.
Theorem 1.9. Let $n > 2\alpha > 0$, $p \in [1, 2)$, $q \in (1, 2)$. If \( \frac{1}{q} + \frac{n}{2\alpha} \left( \frac{1}{p} - \frac{1}{2} \right) = 0 \) then
\[
\left\| \int_0^t e^{-\frac{t-s}{\alpha} } F(s, x) ds \right\|_{L_t^2(I; L_x^{\infty, 2-\alpha})} \lesssim \| F \|_{L_t^r(I; Z)}
\]
holds with $Z = \dot{H}^{\alpha, p}_x$ or $H^{\alpha, p}_x$.

Using the Littlewood-Paley decomposition, we establish the following estimates in Besov spaces.

Corollary 1.10. (a) Let $(q, p, 2)$ be $\frac{2}{2\alpha}$-admissible. If $q \geq 2$ and $(q, p, \frac{2}{2\alpha})$ is not $(2, \infty, 1)$, then
\[
\| e^{-t(-\Delta)\alpha} f \|_{L_t^q(I; X_1)} \lesssim \| f \|_{X_2}
\]
holds with $(X_1, X_2) = (B^s_{p, 2}, B^s_{2, 2})$ or $(\dot{B}^s_{p, 2}, \dot{B}^s_{2, 2})$.

(b) Let $1 \leq p' \leq p \leq \infty$ and $1 < q'_1 < q < \infty$. If $(q, p)$ and $(q_1, p_1)$ satisfy (1.8) and $q_1 \geq 2$, then
\[
\left\| \int_0^t e^{-\frac{t-s}{\alpha} } F(s, x) ds \right\|_{L_t^q(I; Y_1)} \lesssim \| F \|_{L_t^{q'_1}(I; Y_2)}
\]
holds with $(Y_1, Y_2) = (B^{s}_{p', 2}, B^{s}_{2, 2})$ or $(\dot{B}^{s}_{p', 2}, \dot{B}^{s}_{2, 2})$.

Corollary 1.11. Let $n \geq 2\alpha$, $I = [0, T]$ or $[0, \infty)$. Suppose $V$ is a real potential and
\[
V \in L^r_t(I; L^2_x), \quad \frac{1}{r} + \frac{n}{2\alpha} s = 1,
\]
for some fixed $r \in (1, 2) \cup (2, \infty)$ and $s \in \left( \frac{2}{\alpha r}, \frac{n}{\alpha} \right) \cup \left( \frac{n}{\alpha}, \infty \right)$. Let $f \in L^2$ and $F \in L_t^q(I; L^p_x)$ for some $\frac{2}{2\alpha}$-admissible triplet $(q_1, p_1, 2)$ with $p'_1 \in [1, 2)$ and $q'_1 \in (1, 2)$. Then equation (1.4) has a unique solution $v(t, x)$ satisfying
\[
\| v \|_{L_t^q(I; L^p_x)} \lesssim \| f \|_{L^2} + \| F \|_{L_t^{q'_1}(I; L^p_x)},
\]
for all $\frac{2}{2\alpha}$-admissible triplets $(q, p, 2)$ with $2 \leq q < \infty$.

We can prove the following estimate by estimating $K^\alpha_2(x)$ in mixed norm spaces.

Theorem 1.12. Let $\alpha > 0$, $0 < T < \infty$, $1 \leq p'_1 < p \leq \infty$, $1 < q'_1 < q \leq \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{p'_1}$ and $\frac{1}{s} = \frac{1}{q} + \frac{1}{q'_1}$. If
\[
0 < \frac{ns}{2\alpha} \left( 1 - \frac{1}{r} \right) < 1,
\]
then
\[
\left\| \int_0^t e^{-\frac{(t-s)(-\Delta)\alpha}{\alpha} } F(s, x) ds \right\|_{L_t^q([0, T]; X)} \lesssim T^\frac{s}{1 - \frac{ns}{2\alpha} \left( 1 - \frac{1}{r} \right)} \| F \|_{L_t^{q'_1}([0, T]; Y)}
\]
holds with $(X, Y) = (L^r_x, L^{p'_1}_x)$, $(\dot{H}^{\beta, p}_x, \dot{H}^{\beta, p'_1}_x)$ or $(H^{\beta, p}_x, H^{\beta, p'_1}_x)$ for all $\beta > 0$.

In the rest of this paper, we use the notation $L^p$ indiscriminately for scalar and vector valued functions.
Proposition 1.13. Let $\alpha > 1/2$ and $T > 0$. Assume that $u, v \in L^q([0, T]; L^p)$ with $p, q$ satisfying
\[
\max \left\{ \frac{n}{2\alpha - 1}, 2 \right\} < p < \infty, \quad 2\alpha - 1 = \frac{2\alpha}{q} + \frac{n}{p}.
\]
Then the operator
\[
B(u, v) = \int_0^t e^{-(t-s)(-\Delta)^{\alpha}} P\nabla \cdot (u \otimes v) \, ds
\]
is bounded from $L^q([0, T]; L^p) \times L^q([0, T]; L^p)$ to $L^q([0, T]; L^p)$ with
\[
\|B(u, v)\|_{L^q([0, T]; L^p)} \lesssim \|u\|_{L^q([0, T]; L^p)} \|v\|_{L^q([0, T]; L^p)}.
\]
Applying Theorems 1.4 & 1.12, Proposition 1.13 and Lemma 2.4, we obtain the global existence and uniqueness of solutions for system (1.5).

Proposition 1.14. Let $\alpha \in \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2}^+\right)$, $0 < T < \infty$, $p > \frac{n}{2\alpha - 1}$ and $\frac{n}{p} + \frac{2\alpha}{q} = 2\alpha - 1$.

(a) Assume that $\frac{n}{2\alpha - 1} < r \leq p$, $1 \leq p_1' < p < \infty$, $1 \leq q_1' < q \leq \infty$,
\[0 < \frac{n}{2\alpha} \left( \frac{1}{q_1} + \frac{1}{q_1'} \right) \left( 1 - \frac{1}{p} - \frac{1}{p_1} \right) < 1,
\]
g \in L^r(\mathbb{R}^n) with $\nabla \cdot g = 0$ and $h \in L^{q_1'}([0, T]; L^{p_1'}(\mathbb{R}^n))$. If there exists a suitable constant $C > 0$ such that
\[
T^{1 - \frac{n}{2\alpha} \left( \frac{1}{q_1} + \frac{1}{q_1'} \right)} \|g\|_{L^r(\mathbb{R}^n)} + T^{\frac{1}{q_1} + \frac{n}{2\alpha} \left( \frac{1}{p_1} - \frac{1}{p} \right)} \|h\|_{L^{q_1'}([0, T]; L^{p_1'}(\mathbb{R}^n))} \leq C,
\]
then (1.3) has a unique strong solution $v \in L^q_t([0, T]; L^p_x(\mathbb{R}^n))$ in the sense of
\[
v = e^{-t(\Delta)^{\alpha}} g(x) + \int_0^t e^{-(t-s)(\Delta)^{\alpha}} P[h(s, x) - \nabla \cdot (v \otimes v)(s, x)] \, ds,
\]
(b) Assume that $g \in L^{\frac{n}{n-2\alpha}}(\mathbb{R}^n)$ with $\nabla \cdot g = 0$ and $h \in L^{q_1'}([0, \infty); L^{p_1'}(\mathbb{R}^n))$ with $q_1'$ and $p_1'$ satisfying $1 < q_1' < q < \infty$,
\[1 \leq p_1' < p < \begin{cases} \frac{n^2}{(n-2\alpha)(2\alpha-1)}, & 2\alpha < n, \\ \frac{n}{2\alpha} \leq n, & 2\alpha \geq n, \quad \text{and} \quad \frac{n}{p_1'} + \frac{2\alpha}{q_1'} = 4\alpha - 1. \end{cases}
\]
If $\|g\|_{L^{\frac{n}{n-2\alpha}}(\mathbb{R}^n)} + \|h\|_{L^{q_1'}([0, \infty); L^{p_1'}(\mathbb{R}^n))}$ is small enough, then (1.3) has a unique strong solution $v \in L^q_t([0, \infty); L^p_x(\mathbb{R}^n))$.

We show that the solution established in proposition 1.14 is smooth in spatial variables. For a non-negative multi-index $k = (k_1, \ldots, k_n)$ we define
\[
D^k = \left( \frac{\partial}{\partial x_1} \right)^{k_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{k_n}
\]
and $|k| = k_1 + \cdots + k_n$.

Corollary 1.15. Under the hypothesis of Corollary 1.14 we assume further that for a non-negative multi-index $k$
\[
D^k g \in L^r \quad \text{and} \quad D^k h \in L^{q_1'}([0, T]; L^{p_1'}).
\]
Then the solution $v$ established in Corollary 1.14 satisfies
\[
D^k v \in L^q([0, T]; L^p), \quad \text{for} \quad k \leq k_1 + \cdots + k_n.
\]
for any non-negative multi-index \( j \) with \( |j| \leq |k| \).

The rest of this paper is organized as follows. In the next section, we give some basic lemmas: Lemma 2.1 states that \( e^{-t(-\triangle)^{\alpha}} \) commutes with fractional derivatives and is self-adjoint as an operator on \( L^2(\mathbb{R}^n) \); Lemmas 2.2-2.3 provide us the \( L^p \)-decay estimates and non-endpoint strichartz estimates of the fractional heat equation established by Miao-Yuan-Zhang in [10]; Lemma 2.4 gives another mixed norm estimate of \( e^{-t(-\triangle)^{\alpha}} f \); Lemma 2.5 is the well known abstract Strichartz estimates of Keel-Tao [11]. In the third section, we prove the main results of this paper: Proposition 1.12 is derived from the abstract Strichartz estimates. Theorem 1.3 is proved by the Hardy-Littlewood-Sobolev inequality and Lemmas 2.1 & 2.2. Theorem 1.6 is verified by the Young inequality and estimating \( \frac{x}{|x|^2} \) and \( \frac{|x|^2}{x} \). Theorem 1.9 is demonstrated according to the imbedding of \( \tilde{H}^{\alpha,2}(\mathbb{R}^n) \) into \( L^{\frac{\alpha}{n-2\alpha}}(\mathbb{R}^n) \) when \( \alpha \in (0, \frac{4}{n}) \) and the Hardy-Littlewood-Sobolev inequality. Corollary 1.10 is showed by Proposition 1.12 and Theorem 1.3 via the definition of Besov spaces. Corollary 1.11 is proved form Proposition 1.12 Theorem 1.3 and the Banach contraction mapping principle. Theorem 1.13 is showed by using the Young inequality and estimating \( \tilde{K}_n(x) \) in mixed norm spaces. Proposition 1.14 is proved via Lemma 2.2 and the Hardy-Littlewood-Sobolev inequality. Proposition 1.15 is established by applying Proposition 1.13 Lemma 2.4 the Banach contraction mapping principle and our main results. Corollary 1.15 is verified by induction and the Banach contraction mapping principle.

2. Lemmas

This section contains five results needed for proving the main results of this paper. The first one states that \( e^{-t(-\triangle)^{\alpha}} \) commutes with \( (-\triangle)^{\beta} \) and \( (I - \triangle)^{\beta} \), and it is a self-adjoint bounded linear operator on \( L^2(\mathbb{R}^n) \).

**Lemma 2.1.** For all \( t > 0 \) and \( \beta, \alpha > 0 \), we have
(a) \( e^{-t(-\triangle)^{\alpha}} (-\triangle)^{\beta} = (-\triangle)^{\beta} e^{-t(-\triangle)^{\alpha}} \).
(b) \( e^{-t(-\triangle)^{\alpha}} (I - \triangle)^{\beta} = (I - \triangle)^{\beta} e^{-t(-\triangle)^{\alpha}} \).
(c) \( \langle e^{-t(-\triangle)^{\alpha}} f, g \rangle = \langle f, e^{-t(-\triangle)^{\alpha}} g \rangle \), \( \forall f, g \in L^2(\mathbb{R}^n) \).

**Proof.** The proofs of (a) and (b) will follow form the definition of \( e^{-t(-\triangle)^{\alpha}} \), \( (-\triangle)^{\beta} \) and \( (I - \triangle)^{\beta} \). For (b), let \( f, g \in L^2(\mathbb{R}^n) \). According to the Fourier transform and the Plancherel’s identity we have

\[
\langle e^{-t(-\triangle)^{\alpha}} f, g \rangle = \int (e^{-t(-\triangle)^{\alpha}} f) \overline{g(x)} \, dx
= \int \mathcal{F}^{-1} \left[ (e^{-t|\xi|^{2\alpha}} \mathcal{F}f(\xi)) \overline{g(x)} \right] \, dx
= \int e^{-t|\xi|^{2\alpha}} \mathcal{F}f(\xi) \overline{\mathcal{F}g(\xi)} \, d\xi
= \int \mathcal{F}f(\xi) e^{-t|\xi|^{2\alpha}} \mathcal{F}g(\xi) \, d\xi
= \int f(x) \mathcal{F}^{-1} \left( e^{-t|\xi|^{2\alpha}} \mathcal{F}g(\xi) \right)(x) \, dx
= \langle f, e^{-t(-\triangle)^{\alpha}} g \rangle.
\]
This finishes the proof of Lemma 2.1.

Miao-Yuan-Zhang in [16] established the forthcoming two lemmas.

Lemma 2.2. [16] Let \( 1 \leq r \leq p \leq \infty \) and \( f \in L^r(\mathbb{R}^n) \). Then
\[
\|e^{-t(-\triangle)^\alpha}f\|_{L^p_x} \lesssim t^{-\frac{n}{2\alpha}(\frac{1}{r} - \frac{1}{p})}\|f\|_{L^r_x},
\]
\[
\|\nabla e^{-t(-\triangle)^\alpha}f\|_{L^p_x} \lesssim t^{-\frac{n}{2\alpha} - \frac{\alpha}{2}(\frac{1}{r} - \frac{1}{p})}\|f\|_{L^r_x}.
\]

Lemma 2.3. [16] Let \((q,p,r)\) be any \(\frac{n}{2\alpha}-\)admissible triplet satisfying
\[
p < \left\{ \begin{array}{ll}
\frac{nr}{n-2\alpha}, & n > 2\alpha, \\
\frac{nr}{\infty}, & n \leq 2\alpha,
\end{array} \right.
\]
and let \(\varphi \in L^r(\mathbb{R}^n)\). Then \(e^{-t(-\triangle)^\alpha}\varphi \in L^q(I;L^p(\mathbb{R}^n))\) with the estimate
\[
\|e^{-t(-\triangle)^\alpha}\varphi\|_{L^q_t(I;L^p_x)} \lesssim \|\varphi\|_r,
\]
for \(I = [0,T), 0 < T \leq \infty\).

We can obtain the following estimate from Lemma 2.2

Lemma 2.4. Let \(1/2 < \alpha, T > 0\), and \(p,q\) satisfy
\[
p > \frac{n}{2\alpha - 1}, \quad 2\alpha - 1 = \frac{2\alpha}{q} + \frac{n}{p}.
\]
Assume that \(f \in L^r(\mathbb{R}^n)\) with \(\frac{n}{2\alpha - 1} < r \leq p\). Then we have
\[
\|e^{-t(-\triangle)^\alpha}f\|_{L^q_t([0,T];L^p_x)} \lesssim T^{1 - \frac{n}{2\alpha - 1}}(\frac{1}{r} - \frac{1}{p})\|f\|_{L^r_x}.
\]

Lemma 2.5. [11] Let \(H\) be a Hilbert space and \(X\) be a Banach space. Suppose that \(U(t) : H \rightarrow L^2(X)\) obeys the energy estimate:
\[
\|U(t)f\|_{L^2(X)} \lesssim \|f\|_H
\]
and the untruncated decay estimate, that is for some \(\sigma > 0\),
\[
\|U(t)(U(s))^*f\|_{L^\infty} \lesssim |t - s|^{-\sigma}\|f\|_{L^1}, \ \forall s \neq t.
\]

Then the estimates
\[
\|U(t)f\|_{L^q_tL^p_x} \lesssim \|f\|_H,
\]
\[
\left\|\int(U(s))^*F(s)ds\right\|_H \lesssim \|F\|_{L^q_tL^p_x'},
\]
\[
\left\|\int_{s<t}U(t)(U(s))^*F(s)ds\right\|_{L^q_tL^p_x} \lesssim \|F\|_{L^q_tL^p_x},
\]
hold for all \(\sigma\)-admissible triplets \((q,p,2)\) and \((q_1,p_1,2)\) with \(q,q_1 \geq 2\), \((q,p,\sigma)\) and \((q_1,p_1,\sigma)\) are not \((2,\infty,1)\).
3. Proofs of Main Results

3.1. Proof of Proposition 1.2 We only need to prove (1.2) for $I = [0, \infty)$ since the proofs for other cases are similar. Assume that $(q, p, 2)$ is a $\frac{1}{q_1}$-admissible triplet with $q \geq 2$ and $(q, p, \frac{1}{p'})$ is not $(2, \infty, 1)$. It follows from Lemma 2.2 that we have the energy estimate

$$
\| e^{-t(-\Delta)\alpha} f \|_{L^2} \lesssim \| f \|_{L^2}, \forall t > 0,
$$

and untruncated decay estimate

$$
\| e^{-(t+s)(-\Delta)\alpha} f \|_{L^\infty} \lesssim |t + s|^{-\frac{\alpha}{p}} \| f \|_{L^1} \lesssim |t - s|^{-\frac{\alpha}{p}} \| f \|_{L^1}, \forall s \neq t, s, t \in (0, \infty).
$$

By (3.1), (3.2) and Lemma 2.1 we can apply Lemma 2.5 with $U = \phi_t > 0$ (3.2) and untruncated decay estimate (1.3). This finishes the proof of (1.3).

3.2. Proof of Theorem 1.4 We only need to prove (1.3) for $I = [0, \infty)$, the proofs for other cases being similar. Assume that $(q, p, 2)$ and $(q_1, p_1, 2)$ satisfy $1 \leq p_1 < p \leq \infty$, $1 < q_1 < q < \infty$ and $\frac{1}{q_1} + \frac{1}{p_1} + \left(\frac{1}{p_1} - \frac{1}{p}\right) = 1 + \frac{1}{q}$. It follows from Lemma 2.2 that

$$
\| e^{-(t-s)(-\Delta)\alpha} F(s, x) \|_{L^2_x} \lesssim |t - s|^{-\frac{\alpha}{p_1}\left(\frac{1}{p_1} - \frac{1}{p}\right)} \| F(s, x) \|_{L^{p_1'}_x}, \forall s < t.
$$

Then the Hardy-Littlewood-Sobolev inequality implies that

$$
\left\| \int_0^t e^{-(t-s)(-\Delta)\alpha} F(s, x) ds \right\|_{L^2_x(I; L^2_x)} \lesssim \left\| \int_0^t \| e^{-(t-s)(-\Delta)\alpha} F(s, x) \|_{L^2_x} ds \right\|_{L^1_x(I)} \lesssim \left\| \int_0^t |t - s|^{-\frac{\alpha}{p_1}\left(\frac{1}{p_1} - \frac{1}{p}\right)} \| F(s, x) \|_{L^{p_1'}_x} ds \right\|_{L^1_x(I)} \lesssim \| F \|_{L^{p_1'}_x(I; L^2_x)}.
$$

This finishes the proof of (1.3).

3.3. Proof of Theorem 1.6 Let $n = 2\alpha$. Define $\varphi \in C^\infty(\mathbb{R})$ with $\text{supp}(\varphi) \subseteq (1/2, 2)$, $\varphi(x) = 1$ for $x \in (3/4, 9/8)$ and $\sum_{k \in \mathbb{Z}} \varphi(2^{-k}t) = 1$ for all $t > 0$. Let $\varphi_k(t) = \varphi(2^{-k}t)$. Define $P_k f = \mathcal{F}^{-1}(\mathcal{F} f \cdot \varphi_k(\cdot))$ be a Littlewood-Paley decomposition with respect to $\varphi_k$ (see [22]). Since $BMO = F^{q_1, 2}$ (see Frazier-Jawerth-Weiss [7]),

$$
\| g \|_{BMO} \approx \left\| \left( \sum_{k \in \mathbb{Z}} |P_k g|^2 \right)^{1/2} \right\|_{L^\infty}.
$$
Let $M_k = B(0, 2^{k+1}) \setminus B(0, 2^{k-1})$ and $\chi_{M_k}$ its characteristic function. Since $\varphi$ is supported in $(1/2, 2)$ and $n = 2\alpha$, we have

$$
\|e^{-t(-\Delta)^\alpha} P_k f\|_{L^2_x L^\infty_t}^2 \leq \int_0^\infty \sup_x \left| \int_{\mathbb{R}^n} e^{-t|\xi|^\alpha} e^{i(\xi, x)} f(\xi) \varphi(2^{-k} |\xi|) d\xi \right|^2 dt
$$

$$
\leq \int_0^\infty \int_{\mathbb{R}^n} \chi_{M_k}(\xi) d\xi \sup_x \int_{\mathbb{R}^n} \left| e^{-t|\xi|^\alpha} e^{i(\xi, x)} f(\xi) \varphi(2^{-k} |\xi|) \right|^2 d\xi dt
$$

$$
\lesssim (2^{k-1})^n (2^{2n} - 1) \int_0^\infty \int_{M_k} e^{-t2^{(k-1)n+1}} dt \|f\|_{L^2}^2
$$

$$
\lesssim (2^{2n-1} - 1/2) \|f\|_{L^2}^2
$$

$$
\lesssim \|f\|_{L^2}^2.
$$

Take $\psi \in C^\infty(\mathbb{R})$ with $\text{supp}(\psi) \subseteq (1/4, 4)$ and $\psi(x) \varphi(x) = \varphi(x)$. Define

$$
\tilde{P}_k f = \mathcal{F}^{-1}(\mathcal{F} f \psi_k).
$$

Then we have

$$
\|e^{-t(-\Delta)^\alpha} f\|_{L^2_x L^\infty_t}^2 \lesssim \int_0^\infty \sup_x \left( \sum_k \|e^{-t(-\Delta)^\alpha} P_k f\|_{L^2_x L^\infty_t}^2 \right) dt
$$

$$
\lesssim \sum_k \|\tilde{P}_k f\|_{L^2_x L^\infty_t}^2
$$

$$
\lesssim \|f\|_{L^2}^2.
$$

That is, (1.9) holds.

3.4. Proof of Theorem 1.7 (a). Let $1 \leq r \leq p \leq \infty$ and $n < 2\alpha$. It follows from Lemma 2.2 that

$$
s^{-\frac{2\alpha}{p}} \|e^{-s(-\Delta)^\alpha} f\|_{L^p(\mathbb{R}^n)} \lesssim s^{-\frac{2\alpha}{m}} \|f\|_{L^r(\mathbb{R}^n)}.
$$

On the other hand, $n < 2\alpha$ implies that

$$
\int_0^T s^{-\frac{2\alpha}{2\alpha - n}} ds = \frac{2\alpha}{2\alpha - n} T^{1 - \frac{2\alpha}{2\alpha - n}}.
$$

Thus (1.10) holds.

(b). The following proof is essentially the same as the proof Tao’s [24, Lemma 2.5]. For the sake of completeness, it is provided here. We use the $TT^*$ method. Thus, by duality and the self-adjointness of $e^{-t(-\Delta)^\alpha}$ it suffices to verify

$$
\left( \int_0^\infty s^{-1/p} e^{-s(-\Delta)^\alpha} F(s, x) ds \right)^2 \leq \int_0^\infty \|F(s, x)\|_{L^2_x L^\infty_t(\mathbb{R}^n)}^2 ds,
$$

for all test functions $F$. The left hand side of (3.3) can be written as

$$
\int_0^\infty \int_0^\infty s^{-1/p} s^{-1/p} \left( e^{-s(-\Delta)^\alpha} F(s, x), e^{-s(-\Delta)^\alpha} F(s_1, x) \right)_x dsds_1.
$$
Let $g(s) = \|F(s, x)\|_{L^p_w(\mathbb{R}^n)}$. According to Lemma 2.2 we have
\[
\left| \left< e^{-\frac{\sqrt{2m+1}}{\sqrt{2}}(-\triangle)^\alpha} F(s, x), e^{-\frac{\sqrt{2m+1}}{\sqrt{2}}(-\triangle)^\alpha} F(s_1, x) \right> \right| \lesssim (s+s_1)^{-2\left(\frac{1}{p'}-\frac{1}{2}\right)} g(s)g(s_1).
\]
Hence, it suffices to prove that
\[
\left(3.4\right) \quad \int_0^\infty \int_0^\infty \frac{g(s)g(s_1)}{(s+s_1)^{1-2/p}2^{m/p}s_1/p} dsds_1 \lesssim \int_0^\infty g(s)^2 ds.
\]
On the other hand, by symmetry we can only consider the region $s_1 \leq s$ which can be decomposed into the dyadic ranges $2^{-m}s \leq s_1 \leq 2^{-m+1}s$. Hence the left hand side of (3.4) can be bounded by
\[
\lesssim \sum_{m=1}^\infty 2^{m/p} \int_0^\infty \int_{2^{-m}s \leq s_1 \leq 2^{-m+1}s} \frac{g(s)g(s_1)}{s} dsds_1 ds
\]
\[
\lesssim \sum_{m=1}^\infty 2^m \left(\frac{1}{p}-\frac{1}{2}\right) \int_0^\infty g(s)ds
\]
\[
\lesssim \int_0^\infty g(s)ds
\]
with the second inequality using the Schur’s test of Tao [23].

3.5. Proof of Theorem 1.9 We only need to prove (1.12) for $Z = \dot{H}_x^{\alpha,p}$. Suppose $n > 2\alpha > 0$, $p \in [1, 2)$, $q \in (1, 2)$, $\frac{1}{q} + \frac{n}{2\alpha} \left(\frac{1}{p} - \frac{1}{2}\right) = \frac{3}{2}$.

Thus $\frac{1}{q} + \frac{n}{2\alpha} \left(\frac{1}{p} - \frac{1}{2}\right) \in (0, 1)$. According to the imbedding of $\dot{H}_x^{\alpha,2}$ into $L^{\frac{2\alpha}{n-2\alpha}}$, Lemmas 2.1 & 2.2 and the Hardy-Littlewood-Sobolev inequality, we obtain
\[
\left\| \int_0^t e^{-\left(t-s\right)\left(-\triangle\right)^\alpha} F(s, x) ds \right\|_{L^q_t(L^p_w)} \lesssim \left\| \int_0^t e^{-\left(t-s\right)\left(-\triangle\right)^\alpha} F(s, x) ds \right\|_{L^q_t(\dot{H}_x^{\alpha,2})}
\]
\[
\lesssim \left\| \left(-\triangle\right)^{\alpha/2} \int_0^t e^{-\left(t-s\right)\left(-\triangle\right)^\alpha} F(s, x) ds \right\|_{L^q_t(L^2)}
\]
\[
\lesssim \left\| \int_0^t e^{-\left(t-s\right)\left(-\triangle\right)^\alpha} \left(-\triangle\right)^{\alpha/2} F(s, x) ds \right\|_{L^q_t(L^2)}
\]
\[
\lesssim \left\| \int_0^t ||e^{-\left(t-s\right)\left(-\triangle\right)^\alpha}|| \left(-\triangle\right)^{\alpha/2} F(s, x) ||ds \right\|_{L^q_t(L^2)}
\]
\[
\lesssim \left\| \int_0^t \left|t-s\right|^{-\frac{n}{2\alpha}\left(\frac{1}{p}-\frac{1}{2}\right)} \left(-\triangle\right)^{\alpha/2} F(s, x) ds \right\|_{L^q_t(L^2)}
\]
\[
\lesssim \left\| \left(-\triangle\right)^{\alpha/2} F \right\|_{L^q_t(L^p_w)} \lesssim \left\| F \right\|_{L^q_t(\dot{H}_x^{\alpha,p})}.
\]

This finishes the proof of (1.12).

3.6. Proof of Corollary 1.10 We only check (1.13) with $(X_1, X_2) = (\dot{B}_p^{\alpha,2}, \dot{B}_p^{\alpha,2})$ and (1.14) with $(Y_1, Y_2) = (\dot{B}_p^{\alpha,2}, \dot{B}_p^{\alpha,2})$ because the proofs of other cases are similar. We assume that $p < \infty$ since the case $p = \infty$ is similar. We will use the following
equivalent norms in Besov spaces. Let \( \eta \) be an infinitely differential function with compact support in \( \mathbb{R}^n \) satisfying
\[
\eta(\xi) = \begin{cases} 
1, & |\xi| \leq 1, \\
0, & |\xi| \geq 2,
\end{cases}
\]
define the sequence \( \{\psi_j\}_{j \in \mathbb{Z}} \) in \( \mathcal{S}(\mathbb{R}^n) \) by
\[
\psi_j(\xi) = \eta\left(\frac{\xi}{2^j}\right) - \eta\left(\frac{\xi}{2^{j-1}}\right).
\]
Through this sequence, the norms in the inhomogeneous and homogeneous Besov spaces \( B^s_{p,q}(\mathbb{R}^n) \) and \( \dot{B}^s_{p,q}(\mathbb{R}^n) \) for \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R}^n \) are equivalent to
\[
\|f\|_{B^s_{p,q}} = \|F^{-1}(\eta F(f))\|_{L^p(\mathbb{R}^n)} + \left\{ \sum_{j=1}^{\infty} (2^{sj}\|F^{-1}(\psi_j F(f))\|_{L^p(\mathbb{R}^n)})^q \right\}^{1/q} \quad \text{if } q < \infty,
\]
and
\[
\|f\|_{\dot{B}^s_{p,q}} = \left\{ \sup_{j \in \mathbb{Z}} (2^{sj}\|F^{-1}(\psi_j F(f))\|_{L^p(\mathbb{R}^n)})^q \right\}^{1/q} \quad \text{if } q = \infty.
\]
\[\textbf{Part 1. Proof of (1.13)}. \] We assume that \( q < \infty \), note that the case \( q = \infty \) is obvious. Define \( u(t) = e^{-t(-\Delta)^{\alpha}} f \). Then
\[
F^{-1}(\psi_j F(u)) = F^{-1}(e^{-t|\xi|^{2\alpha}} \psi_j F(f)) = e^{-t(-\Delta)^{\alpha}} (F^{-1}(\psi_j F(f))).
\]
Hence
\[
\|u\|_{L^q(I; \dot{B}^s_{p,q})}^2 = \left( \int I \left( \sum_j 2^{sj}\|e^{-t(-\Delta)^{\alpha}} (F^{-1}(\psi_j F(f)))\|_{L^p}^2 \right)^{q/2} dt \right)^{2/q}.
\]
Letting \( A_j(t) = 2^{sj}\|e^{-t(-\Delta)^{\alpha}} (F^{-1}(\psi_j F(f)))\|_{L^p}^2 \) and \( k = q/2 \geq 1 \), we have
\[
\|u\|_{L^q(I; \dot{B}^s_{p,q})}^2 \leq \left( \int I \left( \sum_j A_j(t) \right)^k dt \right)^{1/k}
\leq \sum_j \|A_j(\cdot)\|_{L^k(I)}
\leq \sum_j \|A_j(\cdot)\|_{L^k(I)}
= \sum_j 2^{sj}\|e^{-t(-\Delta)^{\alpha}} (F^{-1}(\psi_j F(f)))\|_{L^q(I; L^p)}^2.
\]
Using Proposition \([12]\), we deduce
\[
\|u\|_{L^q(I; \dot{B}^s_{p,q})} \lesssim \left( \sum_j 2^{sj}\|F^{-1}(\psi_j F(f))\|_{L^2}^2 \right)^{1/2} \lesssim \|f\|_{B^s_{2,2}}.
\]
Thus (1.14) holds.

**Part 2. Proof of (1.14)** Let \( u(t) = \int_0^t e^{-(t-s)(-\Delta)\alpha} F(s, x) ds \). Then

\[
2^{s_j} \mathcal{F}^{-1}(\psi_j \mathcal{F}(u)) = 2^{s_j} \mathcal{F}^{-1} \left( \int_0^t \psi_j \mathcal{F}(e^{-(t-s)(-\Delta)\alpha} F(s, x)) ds \right)
\]

\[
= 2^{s_j} \mathcal{F}^{-1} \left( \int_0^t e^{-(t-s)|\xi|^{2\alpha}} \psi_j \mathcal{F}(F(s, \xi)) ds \right)
\]

\[
= 2^{s_j} \int_0^t \mathcal{F}^{-1} \left( e^{-(t-s)|\xi|^{2\alpha}} \psi_j \mathcal{F}(F(s, \xi)) \right) ds
\]

\[
= \int_0^t e^{-(t-s)(-\Delta)\alpha} \left( 2^{s_j} \mathcal{F}^{-1} \left( \psi_j \mathcal{F}(F(s, \xi)) \right) \right) ds
\]

\[
= \int_0^t e^{-(t-s)(-\Delta)\alpha} v_j(t) ds,
\]

where \( v_j(t) = 2^{s_j} \mathcal{F}^{-1} \left( \psi_j \mathcal{F}(F(s, \xi)) \right) \). Thus

\[
\|u\|_{L^2_q(T, \dot{B}^{\alpha}_{p,2})}^2 \lesssim \left( \int I \left( \sum_j \left\| \int_0^t e^{-(t-s)(-\Delta)\alpha} v_j(t) ds \right\|_{L_p}^{2/2} \right)^{2/q} dt \right)^{2/q}.
\]

In a similar manner to verify (1.13), we have

\[
\|u\|_{L^2_q(T, \dot{B}^{\alpha}_{p,2})}^2 \lesssim \sum_j \left\| \int_0^t e^{-(t-s)(-\Delta)\alpha} v_j(t) ds \right\|_{L^2_q(T, L^p)}^2.
\]

Applying Theorem 1.4, we get

\[
\|u\|_{L^2_q(T, \dot{B}^{\alpha}_{p,2})}^2 \lesssim \sum_j \|v_j\|_{L^2_q(T, L^p)}^2 \lesssim \sum_j \left( \int I \|R_j(t) dt \right)^k,
\]

where \( R_j(t) = \|v_j(t)\|_{L^p_{r_i}} \) and \( k = 2/q_1 \geq 1 \). An application of the Minkowski inequality yields

\[
\|u\|_{L^2_q(T, \dot{B}^{\alpha}_{p,2})}^{2/k} \lesssim \left\| \int I \|R_j(t) dt \right\|_{L^p_{1}(\mathbb{Z})}
\]

\[
\lesssim \int I \|R_j(t) dt \|_{L^p_{1}(\mathbb{Z})} dt
\]

\[
\lesssim \left( \int I \left( \sum_j \|v_j(t)\|_{L^p_{r_i}}^2 \right)^{q_i/2} dt \right)
\]

\[
\lesssim \|F\|_{L^q_{1}(T, \dot{B}^{\alpha}_{p,2})}.
\]

Thus (1.14) holds.

**3.7. Proof of Corollary 1.11** We shall prove this theorem for \( n > 2\alpha \). In the case \( n = 2\alpha \), we can replace in the sequel the space \( L^2_q(J; \dot{B}^{\alpha}_{p,2}) \) by any \( L^2_q(J; L^p_x) \) for \( 1 \)-admissible \((q, p, 2)\) with \( p \) arbitrarily large.

We consider the following two cases.
where \( \varepsilon > 0 \) will be determined later and \( (k, l, 2) \) be \( \frac{n}{2\alpha} \)-admissible with \( q \leq k < \infty \), and set
\[
X = L_t^k(J; L_x^2) \cap L_t^2(J; L_x^{2n/2\alpha}) \quad \text{with} \quad \|v\|_X := \max \left\{ \|v\|_{L_t^k(J; L_x^1)}, \|v\|_{L_t^2(J; L_x^{2n/2\alpha})} \right\}.
\]
By interpolation (see Triebel [25]), \( X \) can be embedded into \( L_t^{q_0}(J; L_x^{p_0}) \) for each \( \frac{n}{2\alpha} \)-admissible triplet \((q_0, p_0, 2)\) with \( 2 \leq q_0 \leq k \). Define \( T(v) \) on \( X \) by
\[
T(v) = e^{-t(-\triangle)^s} f + \int_0^t e^{-(t-s)(-\triangle)^s} (F(s, x) - V(s, x)v(s, x)) \, ds, \quad \forall v = v(t, x) \in X.
\]
Applying Proposition 1.2 and Theorem 1.4 we have
\[
\|T(v)\|_{L_t^{q_0}(J; L_x^{p_0})} \leq C\|f\|_2 + C\|F\|_{L_t^{q_1}(J; L_x^{p_1})} + C\|V\|_{L_t^{q_2}(J; L_x^{p_2})},
\]
for all \( \frac{n}{2\alpha} \)-admissible triplets \((q_0, p_0, 2), (q_1, p_1, 2), \) and \((q_2, p_2, 2)\) satisfying \( 2 \leq q_0 \leq k, \ q_1 \in (1, 2), \ q_2 \in (1, 2), \ 1 \leq p_1 < p_0 \leq \infty, \ 1 \leq p_2 < p_0 \leq \infty. \)
Here and later \( C > 0 \) is a constant. Clearly, Hölder’s inequality implies
\[
\|T(v)\|_{L_t^{q_0}(J; L_x^{p_0})} \leq C\|f\|_2 + C\|F\|_{L_t^{q_1}(J; L_x^{p_1})} + C\|V\|_{L_t^{q_2}(J; L_x^{p_2})}\|v\|_{L_t^1(J; L_x^{2n/2\alpha})},
\]
provided
\[
\frac{1}{q_0} = \frac{1}{2} - \frac{1}{r} - \frac{1}{p_2} = \frac{n + 2\alpha}{2n} - \frac{n}{s}.
\]
This and the assumption on \( r \) and \( s \) imply that \( q_0' \in (1, 2), \ p_2' \in [1, 2) \) and
\[
\frac{1}{q_0'} + \frac{1}{2\alpha} \frac{1}{p_2'} = \frac{1}{2} + \frac{n + 2\alpha}{2n} - \left( \frac{n}{2\alpha} \frac{1}{s} + 1 \right) = \frac{n}{4\alpha}.
\]
Taking \((q_0, p_0, 2)\) be \((k, l, 2)\) and \((2, \frac{n}{2\alpha}, 2)\), we get
\[
\|T(v)\|_X \leq C\|f\|_2 + C\|F\|_{L_t^{q_1}(J; L_x^{p_1})} + C\|V\|_{L_t^{q_2}(J; L_x^{p_2})}\|v\|_X.
\]
Hence \( T(v) \in X \) and \( T \) is a operator from \( X \) to \( X \). Since \( r < \infty \), we may choose such an \( \varepsilon > 0 \) that
\[
(3.9) \quad C\|V\|_{L_t^1(J; L_x^2)} \leq \frac{1}{2}
\]
This fact yields that
\[
\|T(v_1) - T(v_2)\|_X \leq \frac{1}{2}\|v_1 - v_2\|_X, \quad \forall v_1, v_2 \in X.
\]
Thus \( T \) is a contraction operator on \( X \), and \( T \) has a unique fixed point \( v(t, x) \) which is the unique solution of equation (1.4) and \( v \) satisfies
\[
\|v\|_X \lesssim \|f\|_2 + \|F\|_{L_t^{q_1}(J; L_x^{p_1})},
\]
Since \( X \) is embedded in \( L_t^k(J; L_x^2) \), one finds
\[
\|v\|_{L_t^k(J; L_x^2)} \lesssim \|f\|_2 + \|F\|_{L_t^{q_1}(J; L_x^{p_1})}.
\]
Now, we can apply the previous arguments to any subinterval \( J = [t_1, t_2] \) on which a condition like (3.9) holds and obtain
\[
(3.10) \quad \|v\|_{L_t^k(J; L_x^2)} \lesssim \|v(t_1)\|_2 + \|F\|_{L_t^{q_1}(J; L_x^{p_1})},
\]
Note that (3.10) implies

\[(3.11) \quad \|v(t_2)\|_{L^2} \lesssim \|v(t_1)\|_{L^2} + \|F\|_{L^q_t(L^{r'_1})}.
\]

If \(I = [0, T]\) for \(0 < T < \infty\), we can partition \(I\) into a finite many of subintervals on which the condition (3.9) holds. If \(I = [0, \infty)\), since \(V \in L^1_t(I; L^q(\mathbb{R}^n))\) we can find\(T_1 > 0\) such that \(C\|V\|_{L^1_t((T_1, \infty); L^q(\mathbb{R}^n))} < \frac{\tau}{\tau'}\) and partition \([0, T_1]\) similarly. Thus we can prove (1.15) by inductively applying (3.10) and (3.11).

**Case 2, \(r \in (1, 2)\).** Since \((r, \frac{2r}{s+2})\) is the dual of \((r', \frac{2s}{s+2})\), our assumption on \(r, s\) implies

\[
\frac{1}{r'} + \frac{n - 2}{2s} = \frac{n}{2\alpha s} + \frac{n - 2}{2s} = \frac{n}{4\alpha}.
\]

Thus \((r', \frac{2s}{s+2})\) is \(\frac{p}{\alpha}\)-admissible with \(r \in (1, 2)\). In a fashion analogous to handling Case 1, we use Theorems [1.2 & 1.4] to obtain

\[
\|T(v)\|_{L^p_t(L^p_{X,Y})} \leq C\|f\|_2 + C\|F\|_{L^q_t(L^{r'_1}} + C\|V\|_{L^r_t(L^{r'_1})}.
\]

Again, by Hölder’s inequality we have

\[
\|T(v)\|_{L^p_t(L^p_{X,Y})} \leq C\|f\|_2 + C\|F\|_{L^q_t(L^{r'_1})} + C\|V\|_{L^r_t(L^{r'_1})}\|v\|_{L^p_t(L^p_{X,Y})}.
\]

Similarly, taking \((q_0, p_0, 0)\) be \((k, l, 2)\) and \((2, \frac{2n}{n-2\alpha}, 2)\), we have

\[
\|T(v)\|_{X} \leq C\|f\|_2 + C\|F\|_{L^q_t(L^{r'_1})} + C\|V\|_{L^r_t(L^{r'_1})}\|v\|_{X}.
\]

The rest of the proof is similar to that of the first case.

### 3.8. Proof of Theorem [1.2 & 1.4]

We only prove the case \((X, Y) = (L^p_t, L^{p_0})\) since similar arguments apply to other cases. Assume that \(T \in (0, \infty), 1 \leq p_0 < p \leq \infty, 1 \leq q_0 < q \leq \infty, \frac{1}{p} = \frac{1}{p} + \frac{1}{p_0}, \frac{1}{q} = \frac{1}{q} + \frac{1}{q_0}\) and \(\frac{1}{q_0}(1 - \frac{1}{p}) \in (0, 1)\). Let \(I = [0, T]\).

According to the Young’s inequality and the definition of \(e^{-t(-\Delta)^\alpha}\), we have

\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^\alpha} F(s, x) ds \right\|_{L^p_t(L^{p_0})} \lesssim \left\| \int_0^t e^{-(t-s)(-\Delta)^\alpha} F(s, x) ds \right\|_{L^p_t(L^{p_0})} \lesssim \left\| \int_0^t K^{\alpha}_{2-s}(x) * F(s, x) ds \right\|_{L^p_t(L^{p_0})} \lesssim \left\| \int_0^t K^{\alpha}_{2-s}(x) ds \right\|_{L^p_t(L^{p_0})} \lesssim \left\| K^{\alpha}_{2-s}(x) \right\|_{L^p_t(L^{p_0})} \|F(s, x)\|_{L^p_t(L^{p_0})}.
\]

Thus it suffices to prove \(\|K^{\alpha}_{2-s}(x)\|_{L^p_t(L^{p_0})} \lesssim T^\frac{1}{2} \|x\|^{1 - \frac{1}{p} + \frac{1}{p_0}}\). In fact, it follows from Miao-Yuan-Zhang's [16] Lemma 2.1 that \(K^{\alpha}_{2-s}(x) \in L^k\) for all \(1 \leq k \leq \infty\). Since
\[ \frac{1}{r} = \frac{1}{p} + \frac{1}{p_1} \text{ and } p_1' < p \text{ imply that } r > 1, \ K_1^\alpha(x) \in L^r. \text{ Hence} \]

\[
\| K_1^\alpha(x) \|_{L^r(I; L^r)} = \left( \int_0^T \left( \int_{\mathbb{R}^n} e^{ix \cdot \xi - \frac{t}{t^2}} (\xi^{2\alpha})^{\frac{r}{p}} \, dx \right)^{\frac{1}{r}} \right)^{\frac{1}{h}}
\]

\[
= \left( \int_0^T t^{-\frac{\alpha}{2}} (1 - \frac{1}{h}) \, dt \right)^{\frac{1}{h}} \| K_1^\alpha \|_{L^r}
\]

\[
\leq T^{\frac{1}{h} -\frac{\alpha}{2}} (1 - \frac{1}{h}).
\]

This finishes the proof of Theorem 1.12.

3.9 Proof of Proposition 1.13: By Lemma 2.2 and \( L^p \)-boundness of Riesz transform, we have

\[
\| B(u, v) \|_{L^p} \lesssim \int_0^t \| \nabla e^{-t/(\xi^\beta)} P(u(s, \cdot) \otimes v(s, \cdot)) \|_{L^p} \, ds
\]

\[
\lesssim \int_0^t \frac{1}{|t - s|^{\frac{n}{2\alpha} + \frac{n}{2\alpha}}} \| (u(s, \cdot) \otimes v(s, \cdot)) \|_{L^{p'/2}} \, ds
\]

\[
\lesssim \int_0^t \frac{1}{|t - s|^{\frac{n}{2\alpha} + \frac{n}{2\alpha}}} \| u(s, \cdot) \|_{L^p} \| v(s, \cdot) \|_{L^p} \, ds.
\]

Since \( \alpha > \frac{1}{2} \) and \( p > \frac{n}{2\alpha - 1} \),

\[
0 < \frac{1}{2\alpha} + \frac{n}{2p\alpha} < 1.
\]

It follows from \( 2\alpha - 1 = \frac{2\alpha}{p} + \frac{n}{p} \) and the Hardy-Littlewood-Sobolev inequality that

\[
\| B(u, v) \|_{L^q[0, T]; L^p)} \lesssim \| (\| u(s, \cdot) \|_{L^p} \| v(s, \cdot) \|_{L^p}) \|_{L^{p'/2}(0, T)}
\]

\[
\lesssim \| u \|_{L^q(0, T); L^p)} \| v \|_{L^q(0, T); L^p)}.
\]

3.10 Proof of Proposition 1.14 (a) Under the assumption of (a), let \( X = L^q([0, T]; L^p(\mathbb{R}^n)) \). Define

\[
Tv = e^{-t/(\Delta^\alpha)} g + \int_0^t e^{-t/(\xi^\beta)} P(h - \nabla \cdot (v \otimes v))(s, x) \, ds.
\]

We will prove that if

\[
a := T^{1 - \frac{\alpha}{2}} (1 - \frac{1}{h}) \| g \|_{L^r(\mathbb{R}^n)} + T^{\frac{\alpha}{2} + \frac{n}{2\alpha}} (\frac{\alpha}{2} + \frac{n}{2\alpha}) \| h \|_{L^r(0, T); L^r(\mathbb{R}^n)}
\]

is bounded by an appropriate constant, then \( T \) is a contraction operator on the ball \( B_R \) in \( X \) with radius \( R = 2a \). For any \( v_1, v_2 \in B_R \), we have

\[
\| T(v_1) - T(v_2) \|_X = \left\| \int_0^t e^{-t/(\xi^\beta)} P \nabla \cdot (v_1 \otimes v_1) \, ds - \int_0^t e^{-t/(\xi^\beta)} P \nabla \cdot (v_2 \otimes v_2) \, ds \right\|_X
\]

\[
= \| B(v_1 - v_1, v_1) - B(v_2, v_1 - v_2) \|_X
\]

\[
\leq \| B(v_1 - v_2, v_1) \|_X + \| B(v_2, v_1 - v_2) \|_X,
\]

where \( B \) is a contraction operator on the ball \( B_R \) in \( X \).
where

\[ B(u, v) = \int_0^t (e^{-(t-s)(-\triangle)^{\alpha}}) P\nabla \cdot (u \otimes v)(s) \, ds. \]

It follows from Proposition 1.13 that \( B \) is bounded on \( X \). Thus

\[ \|T(v_1) - T(v_2)\|_X \leq C \|v_1 - v_2\|_X \|v_1\|_X + C \|v_2\|_X \|v_1 - v_2\|_X, \]

where \( C > 0 \) is only dependent on \( \alpha, p \) and \( q \). Thus

\[ \|T(v_1) - T(v_2)\|_X \leq C(\|v_1\|_X + \|v_2\|_X)\|v_1 - v_2\|_X \leq CR\|v_1 - v_2\|_X. \]

To estimate \( \|Tv\|_X \) for \( v \in B_R \), we use

\[ T(0) = e^{-t(-\triangle)^{\alpha}}g + \int_0^t e^{-(t-s)(-\triangle)^{\alpha}} Ph(s, x) \, ds \]

to obtain \( \|T(0)\|_X \leq Ca \) according to Theorem 1.12 and Lemma 2.3. Consequently,

\[ \|T(v)\|_X = \|T(v) - T(0) + T(0)\|_X \leq \|T(v - 0)\|_X + \|T(0)\|_X \leq CR\|v\|_X + Ca. \]

Since \( a \) is bounded by a suitable constant, then we have

\[ \|T(v^1) - T(v^2)\|_X \leq \frac{1}{2}\|v^1 - v^2\|_X \quad \text{and} \quad \|T(v)\|_X \leq R. \]

It follows from the Banach contraction mapping principle that there exists a unique \( v \in X = L^q_t([0, T]; L^p_s(\mathbb{R}^n)) \).

(b) Note that \( \frac{p}{p} + \frac{\alpha}{q} = 2\alpha - 1 \) implies that \( (q, p, \frac{\alpha}{q}) \) is \( \frac{\alpha}{q} \)-admissible. By Lemma 2.3, we get

\[ \|e^{-t(-\triangle)^{\alpha}}g\|_{L^q_t([0, \infty); L^p_s)} \lesssim \|g\|_{L^{\frac{\alpha}{q}}}. \]

On the other hand, Theorem 1.13 implies

\[ \left\| \int_0^t e^{-(t-s)(-\triangle)^{\alpha}} h(s, x) \, ds \right\|_{L^q_t([0, \infty); L^p_s)} \lesssim \|h\|_{L^{\frac{\alpha}{\alpha}}_t([0, \infty); L^{\alpha}_s)}. \]

Applying Proposition 1.13 for \( T = \infty \) and the Banach contraction mapping principle, we can prove (b) since \( \|g\|_{L^{\frac{\alpha}{q}}}, \|h\|_{L^{\frac{\alpha}{\alpha}}_t([0, \infty); L^{\alpha}_s)} \) is small enough.

### 3.11 Proof of Corollary 1.13

The proof is similar to that of Proposition 1.14. We only demonstrate the case \( |j| = 1 \), since similar arguments apply to the cases \( |j| = 2, 3, \ldots, |k| \). Define

\[ (3.13) \quad \overline{T}(Dv) = e^{-t(-\triangle)^{\alpha}}(Dg) + \int_0^t e^{-(t-s)(-\triangle)^{\alpha}} P(Dh) - B(Dv, v) - B(v, Dv). \]

Consider the integral equation \( Dv = \overline{T}(Dv) \). Then \( \overline{T} \) is a mapping of the space \( X \) of function \( v \) with

\[ v \in L^q([0, T]; L^p) \quad \text{and} \quad Dv \in L^q([0, T]; L^p). \]

The norm in \( X \) is defined by

\[ \|v\|_X = \|v\|_{L^q([0, T]; L^p)} + \|Dv\|_{L^q([0, T]; L^p)}. \]

The assumption on \( Dg \) and \( Dh \) implies that the first two terms in the right hand side of (3.13) are bounded in \( X \). The boundness of the other terms follows from Proposition 1.13. So, \( \overline{T} \) is a contraction mapping of \( X \) into itself and has a unique fixed point in \( X \). Therefore, the solution \( v \) established in Proposition 1.14 satisfies \( Dv \in L^q([0, T]; L^p) \).
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References

1. J. Bergh, J. Löfström, Interpolation Spaces: An Introduction. Springer, Heidelberg, 1976.
2. M.D. Blair, H.F. Smith, C.D. Sogge, On Strichartz estimates for Schrödinger operators in compact manifolds with boundary. Proc. Amer. Math. Soc. 136 (2008) 247-256.
3. N. Burq, P. Gérard, N. Tzvetkov, Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. Amer. J. Math. 126 (2004) 569-605.
4. T. Cazenave, Semilinear Schrödinger Equations. Courant Lecture Notes in Mathematics, Vol. 10. New York University Courant Institute of Mathematical Sciences, New York, 2003.
5. T. Cazenave, F.B. Weissler, Rapidly decaying solutions of nonlinear Schrödinger equation. Commun. Math. Phys. 147 (1992) 75-100.
6. P. D’Ancona, V. Pierfelice, N. Visciglia, Some remarks on the Schrödinger equation with a potential in $L^p L^q$. Math. Ann. 333 (2005) 271-290.
7. M. Frazier, B. Jawerth, G. Weiss, Littlewood-Paley Theory and The Study of Function Spaces. CBMS Regional Conference Series in Mathematics, Vol. 79. American Mathematical Society, Providence, R.I., 1991.
8. J. Ginibre, G. Velo, Generalized Strichartz inequalities for the wave equation. J. Funct. Anal. 133 (1995) 50-68.
9. L. Kapitanski, Some generalizations of the Strichartz-Brenner inequality. Leningrad Math. J. 1 (1990) 693-676.
10. T. Kato, An $L^{r,s}$—theory for nonlinear Schrödinger equations, spectral and scattering theory and applications. Adv. Stud. Pure Math., vol 23, Math. Soc. Japan, Tokyo, (1994) 223-238.
11. M. Keel, T. Tao, Endpoint Strichartz estimates. Amer. J. Math. 120 (1998) 955-980.
12. C.E. Kenig, R. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. Invent. Math. 166 (2006) 645-675.
13. C.E. Kenig, G. Ponce, L. Vega, Global well-posedness for semi-linear wave equations. Comm. Partial Differential Equations 25 (2000) 1741-1752.
14. H. Lindblad, C.D. Sogge, On the existence and scattering with minimal regularity for semilinear wave equations. J. Funct. Anal. 130 (1995) 357-426.
15. J. L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, (French) Paris: Dunod/Gauthier-Villars, 1969.
16. C. Miao, B. Yuan, B. Zhang, Well-posedness of the Cauchy problem for the fractional power dissipative equations. Nonlinear Anal. TMA 68 (2008) 461-484.
17. G. Mockenhaupt, A. Seeger, C.D. Sogge, Local smoothing of fourier integrals and Carleson-Sjölin estimates. J. Amer. Math. Soc. 6 (1993) 65-130.
18. V. Pierfelice, Strichartz estimates for the Schrödinger and heat equations perturbed with singular and time dependent potentials. Asymptot. Anal. 47 (2006) 1-18.
19. G. Staffilani, D. Tataru, Strichartz estimates for a Schrödinger operator with nonsmooth coefficients. Comm. Partial Differential Equations 27 (2002) 1337-1372.
20. A. Stefanov, Strichartz estimates for the Schrödinger equation with radial data. Proc. Amer. Math. Soc. 129 (2001) 1395-1401.
21. E.M. Stein, Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton, New Jersey, 1970.
22. E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton, New Jersey, 1993.
23. T. Tao, Multilinear weighted convolution of $L^2$-functions, and applications to nonlinear dispersive equations. Amer. J. Math. 123 (2001) 839-908.
24. T. Tao, Global regularity of wave maps IV. Absence of stationary or self-similar solutions in the energy class. http://lanl.arxiv.org/abs/0806.3592.
25. H. Triebel, Interpolation Theory, Function Spaces, Differential Operators. North-Holland, 1978.
26. M.C. Vilela, Inhomogeneous Strichartz estimates for the Schrödinger equations. Trans. Amer. Math. Soc. 359 (2007) 2123-2136.
27. J. Wu, Generalized MHD equations, J. Differ. Eq. 195 (2003) 284-312.
28. J. Wu, The generalized incompressible Navier-Stokes equations in Besov spaces, Dyn. Partial Differ. Eq. 1 (2004) 381-400.
29. K. Yajima, G. Zhang, Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity. J. Differential Equations 202 (2004) 81-110.

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