Conformal Mapping, Power Corrections, and the QCD Bound State Spectrum

H. F. Jones\textsuperscript{a}	extsuperscript{*}, A. Ritz\textsuperscript{a}	extsuperscript{†}, and I.L. Solovtsov\textsuperscript{ab}	extsuperscript{‡}

\textsuperscript{a} Theoretical Physics Group, Blackett Laboratory, Imperial College, South. Kensington, London, SW7 2BZ, U.K.

\textsuperscript{b} Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow Region, 141980, Russia

(March 26, 2022)

Abstract

We analyze the heavy quark bound state spectrum using an order-dependent conformal mapping to re-sum the perturbative expansion for current correlators. The procedure consists of two main steps. Firstly, the Borel plane structure of the truncated perturbative expression is modified to ensure consistency with the operator product expansion. This is analogous to a resummation of infrared renormalon chains. Secondly, this perturbative expansion is conformally mapped to a new series with improved convergence properties. This approach may be shown to induce power corrections consistent with existing condensates, and the resulting expansion may be ordered in powers of an infrared-analytic effective coupling. The technique is then applied to $c\bar{c}$ and $b\bar{b}$ sum rules without any explicit introduction of vacuum condensate parameters. Ground state masses for the vector, axial–vector and $A'$ channels are well reproduced, while results for the scalar–pseudoscalar mass splitting are less impressive.

PACS Numbers : 11.55.Hx, 12.38.-t, 12.38.Lg, 11.10.Hi

*email: h.f.jones@ic.ac.uk
†email: a.ritz@ic.ac.uk
‡email: solovtso@thsun1.jinr.dubna.su
I. INTRODUCTION

The Wilson operator product expansion, within the framework of QCD sum rules [4–6] (see also [1] for a review), constitutes the standard continuum approach for the systematic treatment of nonperturbative corrections to perturbation theory in QCD, and study of the bound state resonances. Taking asymptotic freedom and perturbative QCD as a starting point, this formalism extends the validity of the operator product expansion (OPE) to the scale at which resonances appear in the spectrum as a manifestation of confinement. This generalised operator product expansion for the correlators of various electromagnetic currents includes, in addition to the perturbative contribution, non-perturbative power corrections associated with vacuum condensates of the fields.

For a given current $j_{\Gamma}$, we have the correlator

$$T_{\mu\nu\ldots\Pi}(Q^2) = i \int d^4xe^{iqx}\left\langle 0|T\{j_{\mu\ldots}(x)j_{\nu\ldots}(0)\}|0\right\rangle,$$

where $T_{\mu\nu\ldots}$ is a tensor which depends on the Lorentz structure of the particular current $j_{\Gamma}$. The validity of the OPE at the resonance scale requires the assumption that one may separate the short distance (perturbative) and long distance (condensate) contributions in a consistent manner. The expansion then has the form,

$$i \int d^4xe^{iqx}T(j_{\mu\ldots}^{\Gamma}(x)j_{\nu\ldots}^{\Gamma}(0)) = C_{\Gamma}^{\text{pert}}(Q^2)1 + \sum_{n \geq 2} C_{\Gamma}^{W}(Q^2)Q^{2n}O_{2n},$$

where $C_{\Gamma}^{\text{pert}}(Q^2)$ is the perturbative contribution, and $O_{2n}$ are $2n$-dimensional gauge invariant operators with $C_{\Gamma}^{W}$ the corresponding perturbatively calculable Wilson coefficients.

While the perturbative contribution is calculable in a consistent manner in the short distance regime, provided one imposes an infrared (IR) cutoff, the fact that the split is not necessarily unique or unambiguous is signalled by the perturbative large order divergences associated with infrared renormalons [6–9]. Indeed, it is well known that the Borel resummation ambiguity of the perturbative series for correlators associated with inclusive processes has precisely the form, if not the magnitude [11], of the power corrections determined by the non-perturbative condensate contribution to the OPE. The perturbative ambiguity may then be cancelled by a similar ambiguity in the condensates, rendering the combination well-defined. Consequently, estimating the numerical value of condensates requires detailed knowledge of the perturbative ambiguity itself, hence the recent interest in renormalon asymptotics (see e.g. [12–24] and references therein).

Our aim in this paper is to consider the calculation of the mass of heavy quark bound states within the sum rules formalism via an alternative approach which is to try to resum the asymptotic perturbative series into a convergent form. There are two clear problems that arise in this endeavour which need to be overcome: (1) The Borel non-summability of the perturbative series, due to the presence of IR renormalons, implies that such a resummation technique must be powerful enough to induce power corrections to counteract the

1We define $Q^2 = -q^2$, with $q$ the current momentum.
fundamental ambiguity in the asymptotic perturbative series; (2) The Borel resummation ambiguity, while having the correct functional form to hint at the structure of the nonperturbative corrections required, cannot itself provide the normalisation of these terms from within perturbation theory, as the condensates have a non-perturbative origin associated with confinement to which perturbation theory itself is ostensibly blind.

Nevertheless, with regard to point (1), order-dependent mapping techniques, based on the resummation of a perturbative series in the coupling \( g \), \( G(g) \), via a conformal mapping

\[
g \rightarrow \left( \frac{a}{C(1 - a)^\alpha} \right)
\]

(3)

to a new series in \( a \), \( G(g(a)) \), [25,20] are powerful enough to resum Borel non-summable series into convergent form. An example is the quantum mechanical double-well potential, and we illustrate the structure of the induced corrections in a \( \phi^4 \) model in Appendix A to indicate their connection with field theoretic power corrections.

The nonperturbative input stressed in point (2) can, however, only arise in such an approach in a summation to all orders, and thus cannot obviously be obtained with knowledge of only a finite truncation of the perturbative series, which is the practical reality in field theoretic situations. Furthermore, rigorous proofs of convergence require detailed knowledge of the complex-analytic structure of \( G(g) \), which is again generally lacking in 4D field theories. Thus, in this case one needs extra information to try and “optimise convergence” of the conformally mapped series. This is possible since in Eq. (3) there is a free positive parameter \( C \) (\( \alpha \) is usually fixed via constraints on the mapping) which may be fixed order by order to improve convergence. Generically, a necessary condition is that \( C \) must scale with the order of the expansion in a specific manner. In QCD, with our knowledge limited to only the lowest order terms, a natural way to achieve this is to use some infrared data related to confinement, as this non-perturbative input allows us to estimate the appropriate scaling of \( C \) for the resummed series. Note, however, that our implicit parametrisation of the nonperturbative corrections which may be relevant is directly related to the structure of the conformal mapping, and not just the constant \( C \) itself.

An order-dependent mapping approach of the kind discussed above has been developed by Solovtsov [27,28] (see also [21,31] for applications) which we briefly review in Section 2, indicating how, at any finite order, the conformally mapped series may be re-ordered into a form structurally equivalent to perturbation theory, but with an infrared analytic effective coupling which alters the IR behaviour of correlation functions.

For the application of this technique in situations where the numerical value of condensate related power corrections is significant, and only one- and two-loop perturbative coefficients are known, as in the sum-rules case, we also require a modification of the perturbative result to ensure the correct all-order resummation ambiguity, i.e. the correct position of the first IR renormalon pole. This then ensures consistency with the OPE prior to resummation in the sense that the perturbative ambiguity may be consistently cancelled by ambiguities in the OPE condensates. Or in the present context, that any induced power corrections compensating the perturbative ambiguity can be associated with OPE condensates. An important point is that this modification is only necessary to ensure the correct momentum dependence of the ambiguity, and thus a full resummation is not required. The normalisation is provided by the nonperturbative parameter \( C \), and not the renormalon residue.
We illustrate this idea in the massless case in Section 2, which was also discussed in [34], while in Section 3 we apply the technique in the determination of the masses of the $cc$ and $b\bar{b}$ bound state families. In this case we also require a resummation of threshold Coulomb singularities and the particular resummation procedure adopted here is discussed in Appendix B. We then show that one may closely approximate the experimental estimates in all channels other than the scalar and pseudoscalar. This suggests additional sources of power–behaved contributions in those channels to which this technique is insensitive. In Section 4, we discuss our results and consider the possible interpretations.

II. AN ORDER-DEPENDENT MAPPING IN QCD

In this section we briefly review how an order-dependent mapping of Green functions may be obtained via direct manipulation of the functional integral in QCD using the approach of Solovtsov [27,28] (see also [29–34]), and then proceed to discuss the developments which facilitate later application to heavy quark sum rules.

At a formal level, the mapping may be implemented along the ideas of Seznec and Zinn-Justin [25] via a direct resummation of a perturbative series $G(...)$ = $\sum_n c_n \lambda^n$ as an expansion in a new parameter $a$, $G = G(\lambda(a))$, related to $\lambda = \alpha_s / (4\pi)$, by the conformal mapping,

$$\lambda \equiv \frac{g^2}{(4\pi)^2} = \frac{a^2}{C (1-a)^3},$$

(4)

where $C$ is a positive constant. Note that this is particular case of the conformal mapping (3) with $\alpha = 3/2$, and one observes that $0 \leq a < 1$ for all values of the gauge coupling.

However, in the case of QCD, to ensure gauge invariance, and to allow implementation of the renormalization group, it is convenient to have an explicit realisation of this mapping. Such a realisation has been developed by Solovtsov [27,28], and becomes possible with the introduction of an auxiliary field $\chi^a_{\mu\nu}$ used to split up the quartic gauge field interaction term, which we denote $S_4(A)$.

$$\exp(i g^2 S_4(A)) = \int [d\chi] \exp \left[ \frac{i}{2} \int dx \chi^a_{\mu\nu} \left( \chi^{\mu\nu a} + i \frac{g}{\sqrt{2}} f^{abc} A_{\mu b} A_{\nu c} \right) \right].$$

(5)

Use of this auxiliary field reduces all the interaction terms in the QCD action to Yukawa form, and enables a new split between free and interaction parts, $S_\chi = \tilde{S}_0 + \tilde{S}_I$, parametrised by the variables $\xi$ and $\zeta$, where

$$\tilde{S}_0 = \zeta^{-1}[S(A, \chi) + S_2(\psi) + S_2(c)] + \xi^{-1} S_2(\chi),$$

(6)

and

\footnote{We note that the theory is still consistent off-shell. One may readily verify that if $\chi$ has a gauge transformation consistent with its on-shell constraint, $\chi \sim gfAA$, then the full effective action still satisfies the functional Slavnov-Taylor identities.}
\[
\tilde{S}_I = gS_3(A, \psi, c) - (\zeta^{-1} - 1)[S(A, \chi) + S_2(\psi) + S_2(c)] - (\xi^{-1} - 1)S_2(\chi). \tag{7}
\]

We use a compact notation for the standard kinetic and Yukawa interaction terms of the gauge \((A)\), quark \((\psi)\), and ghost \((c)\) fields. and there are also terms fixing the covariant \(\alpha_G\) gauge. \(S_2(\chi)\) is a mass term for the \(\chi\) field introduced in (3), while \(S(A, \chi)\) is the gauge propagator in the \(\chi\) background (see [28]). A Green function for an even number of fields then takes the form,

\[
G(\ldots) = \frac{1}{Z} \int [d\chi][d(A, \psi, c)] \sum_{n=0}^{\infty} \frac{1}{n!} (\ldots)(i\tilde{S}_I)^n e^{i\tilde{s}_0}, \tag{8}
\]

where \([d(A, \psi, c)]\) denotes the conventional functional integral over gauge, quark, and ghost fields, and \(Z\) is the partition function.

As discussed in [27,28], further expansion of \(\tilde{S}_I\), and a rescaling of the fields, allows \(\chi\) to integrated out, restoring the standard free action of QCD in the exponential and the correct relationship between the 3-point and 4-point gauge couplings, provided that \(\xi = \zeta^3\). However, on inspection of the subsequent series, one observes that it may be ordered in terms of a new parameter, \(a \equiv 1 - \zeta\), provided one identifies \(a\) with the gauge coupling precisely via the conformal mapping (4).

Expansion of \(G\) as a power series in \(a\), up to \(O(a^5)\), then results in the expression

\[
G^{(5)}(\ldots) = g_0(\ldots) + \frac{a^2}{C}g_2(\ldots) + \frac{a^3}{C^2}g_2(\ldots) + \frac{a^4}{C^2}[6Cg_2(\ldots) + g_4(\ldots)]
+ \frac{a^5}{C^2}[10Cg_2(\ldots) + 6g_4(\ldots)] + O(a^6), \tag{9}
\]

where \(g_{2n}(\ldots)\) are the corresponding terms in a standard perturbative series with coefficient \(\lambda^n\) [28]. Re-ordering this series in terms of the perturbative factors \(g_{2n}(\ldots)\) we find

\[
G^{(5)}(\ldots) = g_0(\ldots) + \frac{a^2}{C}(1 + 3a + 6a^2 + 10a^3 + \cdots)g_2(\ldots)
+ \frac{a^4}{C^2}(1 + 6a + \cdots)g_4(\ldots) + \cdots \tag{10}
\]

\[
= g_0(\ldots) + \lambda_{\text{eff}}[5g_2(\ldots) + (\lambda_{\text{eff}})^2]g_4(\ldots) + \cdots, \tag{11}
\]

where we have introduced the effective coupling, \(\lambda_{\text{eff}}|_n\), which is the expansion of (4) to \(O(a^n)\). Therefore, at a given finite order, the conformal mapping may be realised by a direct replacement of the coupling \(\lambda\) in the perturbative series with \(\lambda_{\text{eff}}\) defined to the appropriate order in \(a\). We shall henceforth work at \(O(a^3)\), and thus

\[
\lambda_{\text{eff}} = \frac{a^2}{C}(1 + 3a). \tag{12}
\]

While the equation (4) implies a formally nonlinear relationship between the power series in \(\lambda\) and \(a\), for the truncated series truly non-perturbative input arises via the ability to fix the variational parameter \(C\) at each order. One thus obtains a sequence of approximants for the quantity \(G\) under study, \(\{G_1(O(a), C_1), G_2(O(a^2), C_2), \ldots, G_N(O(a^N), C_N)\}\), and the choice of \(C\) is dictated by the aim of optimising the convergence of this sequence.
In the present case the normalisation of the result depends on the value of condensates related to confinement effects which are invisible at any finite order of perturbation theory. The structure of these corrections is implied by the large order ambiguity, but this does not assist in providing the correct normalisation. Therefore we fix $C$ at each order by fitting a particular low-energy quantity, which ensures the correct normalisation at that scale. The universality of this choice can then be checked by applying the same method to different processes, and the relative magnitude of successive terms in the series can still be monitored, even though a formal proof of convergence is lacking.

Given this philosophy, the approach which has been used (with previous success [29–34]) is to ensure that the running coupling has a singular infrared asymptotic behaviour consistent with the linear part of the static quark potential, $V_{\text{lin}}(r) = \sigma r$. As discussed in [28], this corresponds to the requirement that the $\beta$–function obey $\beta(\lambda) \rightarrow -\lambda$ for large $\lambda$. Enforcement of this constraint requires us to consider the renormalisation group evolution of the parameters. A convenient check on the gauge invariance of the expansion is provided by calculating the charge renormalisation constant with an arbitrary covariant gauge and ensuring that dependence on the gauge parameter drops out at each order in $a$. This is indeed the case [28], and to $O(a^3)$ the $\beta$–function is given by [28]

$$\beta(a) = -\frac{b_0}{C^2} \frac{a^4}{(2 + a)(1-a)^2} (2 + 9a),$$

(13)

where $b_0$ is the first coefficient of the perturbative $\beta$–function. The resulting RG equation has the form

$$f(a) = f(a_0) + \frac{2b_0}{C} \ln \frac{Q^2}{Q_0^2},$$

(14)

where at this order

$$f(a) = \frac{2}{a^2} - \frac{6}{a} - 48 \ln a - \frac{18}{11} \frac{1}{1-a} + \frac{624}{121} \ln (1-a) + \frac{5184}{121} \ln \frac{1 + 9}{2} \frac{Q^2}{Q_0^2}.$$

(15)

The parameter $a_0$ and the momentum $Q_0$ in Eq. (14) are defined at a normalisation point for the effective coupling constant [12]. Explicitly, using phenomenological data for the string tension $\sigma$, the above analysis of the $\beta$–function has been shown to lead to $C = 4.1$ [28] at this order. The running coupling is then given by (14), with the running parameter $a(Q^2)$ determined implicitly by (14). As noted earlier, $a(Q^2)$ is finite at all scales and thus $\lambda_{\text{eff}}$ does not exhibit a Landau pole [28].

Having fixed parameters in the infrared, the first consistency check is that we recover the perturbative expression for the running coupling at large energy scales. Indeed this is the case. If, using (12), (14), and (13), we consider the regime $Q^2 \gg Q_0^2 = \Lambda_{\text{QCD}}^2$ where $\lambda \sim a^2/C' \ll 1$, we recover the one–loop perturbative result, $\lambda(Q^2) \rightarrow 1/(b_0 \ln(Q^2/\Lambda_{\text{QCD}}^2))$.

---

3Note that such series have been proven to converge in simpler models [35–37] for which the corresponding perturbative series is at best asymptotic, and also as discussed in Appendix A in some situations where the perturbative series is non-Borel-summable [28].
The IR running of the coupling is, however, significantly modified \[28\]. The running quark mass \[27\], given by

\[
m(Q^2) = m_0 \left( \frac{\lambda(Q^2)}{\lambda(Q_0^2)} \right)^{\frac{\gamma_0}{\gamma_0}},
\]

where \(m_0 = m(Q_0^2)\) and \(\gamma_0 = 4\) is the first coefficient of the anomalous dimension, similarly coincides with perturbation theory at large scales but has a modified IR behaviour.

We now consider the application of this approach to the current correlators introduced in section 1. For illustration it is convenient to consider first massless correlators \[34\], and in particular the Adler \(D\)-function, \(D(Q^2) \equiv -Q^2 d\Pi(Q^2)/dQ^2\). These arguments will then be extended to the massive case in Section 3.

We begin with the standard spectral representation,

\[
D(Q^2, \lambda) = Q^2 \int_0^\infty ds \frac{1}{(s + Q^2)^2} R(s, \lambda),
\]

where \(R(s) \equiv Im\Pi(s + i\epsilon)/\pi\) is given in a convenient normalisation at \(O(\lambda)\) by

\[
R(s) = 1 + 4\lambda.
\]

Conventional RG improvement of the \(D\)-function would break its analytic properties by introducing the perturbative Landau pole at \(Q^2 = \Lambda_{QCD}^2\). However, this may be circumvented within the spectral representation using RG improvement of the integrand \[38\], with the knowledge that \(R(s)\) obeys the same homogeneous RG equation as \(D\). This corresponds to choosing a more general solution of the RG equation for \(D\) with the same UV asymptotics. However, the integral in \(L7\) is then undefined, as it now runs over the Landau pole. The structure of the virtuality distribution in \(L7\) implies that there is an infrared renormalon ambiguity of \(O(\Lambda_{QCD}^2/Q^2)\). However, there is no corresponding condensate operator in the OPE, and thus this first IR pole is expected to be absent, at least within the current perturbative analysis. An all-order resummation in the large \(b_0\) limit would indeed recover the expected branch structure, removing the first IR renormalon pole.

We shall make the assumption that the first IR pole should be absent from the point of view of perturbative asymptotics, to ensure consistency with the OPE\[7\]. However, as discussed in Section 1, a full resummation is unnecessary in the present context, as the normalisation is fixed elsewhere. Thus we can use a simple trick to remove the first IR pole and ensure that the large order asymptotics are consistent with the OPE. A convenient way to achieve this is to perform an integration by parts in \(L7\) \[34\], obtaining

\[
D(Q^2) = 2Q^2 \int \frac{sd\lambda}{(s + Q^2)^3} R(s) - Q^2 \frac{d}{dQ^2} D(Q^2).
\]

Note that this excludes the possibility that such a correction could arise as an exponentially suppressed correction to the coefficient of the identity operator in the OPE.
FIGURES

FIG. 1. The virtuality distribution functions $\tau \omega(\tau)$ taken from Ref. [23] (solid line) and the function (21) multiplied by a factor of $\tau$ (dashed line) versus $\ln \tau$.

At the order to which we are working the second term vanishes, and we find

$$D(t, \lambda) = 1 + 4 \int_0^{\infty} d\tau \omega(\tau) \lambda(\mu^2),$$  \hspace{1cm} (20)

where $t = Q^2/\mu^2$. The weight function, given by

$$\omega(\tau) = \frac{2\tau}{(1+\tau)^3},$$ \hspace{1cm} (21)

describes the distribution of virtuality which we shall comment on shortly. Performing RG improvement on the integrand, using for consistency the one–loop \(\beta\)–function $\beta(\lambda) = -b_0 \lambda^2$, we obtain

$$D(t, \lambda) = 1 + 4 \int_0^{\infty} d\tau \omega(\tau) \frac{\bar{\lambda}}{1 + \bar{\lambda} b_0 \ln \tau},$$ \hspace{1cm} (22)

where $\bar{\lambda} = \bar{\lambda}(t)$. The modified structure of the virtuality distribution, which at low scales now has a linear dependence on $\tau$, $\omega(\tau) \sim 2\tau + O(\tau^2)$, is now consistent with an ambiguity associated with the first IR renormalon pole of $O(\Lambda_{QCD}^4/Q^4)$ which can be consistently cancelled by an associated ambiguity in the gluon condensate in the OPE.

We observe that the virtuality distribution function (21) coincides with the function used in [19] and remarkably is numerically very close to that obtained in [23] (see Figure 1) which, in contrast to the present naive modification, corresponds to an all orders resummation of renormalon contributions in the large $b_0$ limit.

---

We denote perturbative running parameters in the $\overline{\text{MS}}$ scheme with a bar, as opposed to the unbarred running parameters in the variational series.
This connection with renormalons is illustrated more clearly by performing a formal Borel transformation on (22), whereby we obtain

\[ D(t, \lambda) = 1 + 4 \int_0^\infty db \exp \left( -\frac{b}{\lambda(t, \lambda)} \right) B(b), \tag{23} \]

with

\[ B(b) = \Gamma(1 + bb_0)\Gamma(2 - bb_0). \tag{24} \]

This Borel function exhibits the correct infrared and ultraviolet renormalon poles for the \( D \)-function, although not the full branch structure \[10\]. Nonetheless, as mentioned above, the crucial point here in using the partial integration in (21) is to obtain the correct positioning of the poles and thus an ambiguity consistent with the OPE.

Having massaged the perturbative series into a form whose large order divergence may be consistently cancelled by ambiguities in the OPE condensates we may now perform the conformal mapping, as discussed earlier, by replacing the perturbative running coupling with \( \lambda_{\text{eff}} \) (12). In the modified spectral representation, this leads to the replacement of (22) by

\[ D(t, \lambda) = 1 + 4 \int_0^\infty d\tau \omega(\tau) \lambda_{\text{eff}}(t\tau), \tag{25} \]

where, to \( O(a^3) \), \( \lambda_{\text{eff}} \) is given by (12), with the running expansion parameter \( a = a(Q^2) \) determined via (14).

Within a field theory such as QCD it is not possible to directly assess the convergence of this series, and therefore to determine whether indeed OPE type power corrections of the appropriate magnitude are induced when compared to the initial perturbative series\[6\]. Although analysis of a \( \phi^4 \) model \[26\] (see Appendix A) indicates that the mapping is able to resum the appropriate corrections, within QCD it is not always possible to assess even the magnitude of higher order terms. Therefore we propose instead to test the technique by analysing the massive case, and comparing the results directly with experiment.

### III. MOMENTS FOR HEAVY \( Q\bar{Q} \) BOUND STATES

We shall make use of the standard sum rules approach for the determination of the lowest mass resonances \[1\] (see also \[4\] for a review), wherein one assumes the validity of the narrow resonance approximation for the lowest mass contribution to the imaginary part of the two-point current correlator. The other side of the sum rule is conventionally determined directly from QCD up to parameters associated with vacuum condensate operators. However, in the present context, as elaborated earlier, there is no distinct split between perturbative and nonperturbative contributions. The non-perturbative parameter \( C \) does not have any direct connection with vacuum condensates.

\[6\] Note that a \( 1/Q^2 \) correction may also be induced in the running coupling by removal of the Landau pole \[18\]. However, this has a purely perturbative short distance origin. For recent work see \[32,40\].
We now repeat the analysis of the preceding section in the case of massive correlators. To \(O(\lambda)\) in perturbation theory we have

\[
\text{Im} \Pi^\Gamma(s) = \frac{1}{4\pi} \left[ \Pi_\Gamma^{(0)}(s) + 4\lambda \Pi_\Gamma^{(1)}(s) \right],
\]

(26)

where \(\Gamma\) denotes the current in question, and \(s = q^2\). The one- \((\Pi^{(0)})\) and two-loop \((\Pi^{(1)})\) components have been obtained previously and are enumerated in Appendix B to fix our notation. The first convergent moment is defined by

\[
M^{(\Gamma)}_{1+ N_\Gamma}(Q^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds}{(s + Q^2)^2} \text{Im} \Pi^\Gamma(s),
\]

(27)

which is also the Adler \(D\)-function up to a factor of \(Q^2\). Introducing \(\sigma = s - 4m^2\), which we associate with the virtuality, and the quantity \(u^2 = \sigma/(\sigma + 4m^2)\), we have

\[
M^{(\Gamma)}_{1+ N_\Gamma}(Q^2) = \frac{1}{4\pi^2} \int_{0}^{\infty} d\sigma \frac{(\sigma + 4m^2)^{N_\Gamma} \left( \Pi_\Gamma^{(0)}(u) + 4\lambda \Pi_\Gamma^{(1)}(u) \right)}{(Q^2 + \sigma + 4m^2)^{2+N_\Gamma}}. \tag{28}
\]

An additional problem encountered in studying moments for heavy \(q\bar{q}\) systems is the dominant Coulomb interaction, which leads to the perturbative series having an effective expansion parameter given by \(\lambda/u\). In the conventional approach, higher order moments are dominated by the nonrelativistic region of small \(u\) and thus the perturbative expansion breaks down due to the size of the expansion parameter \(\lambda/u\). Since our approach still makes use of these perturbative coefficients a resummation is required in order to consider the moments for large \(n\).

When, in analogy with the massless case, we perform an integration by parts to remove the first IR renormalon pole, a particular resummation which places higher order corrections inside the virtuality distribution becomes very natural. This resummation may be performed exactly, and in Appendix C we describe its implementation in a simplified model illustrating its effectiveness in resumming Coulomb singularities, and its connection with the familiar Sommerfeld-Sakharov factor \[12\].

Implementation of the resummation in the present case results in the following expression:

\[
M^{(\Gamma)}_{1+ N_\Gamma}(Q^2) = \frac{1}{4\pi^2} \int_{0}^{\infty} d\sigma \frac{(\sigma + 4m^2)^{N_\Gamma} \left( \Pi_\Gamma^{(0)}(u) + 4\lambda \Pi_\Gamma^{(1)}(u) \right)}{(Q^2 + W(\sigma))^{2+N_\Gamma}}, \tag{29}
\]

where to \(O(\lambda)\)

\[
W(\sigma) = (\sigma + 4m^2) \left( 1 - 4\lambda \frac{\psi_\Gamma(u)}{\Pi_\Gamma^{(0)}(u)} \right), \tag{30}
\]

and

\[
\psi_\Gamma(u) = (1 - u^2)^{1+N_\Gamma} \int_{0}^{u} du \frac{2u}{(1 - u^2)^{2+N_\Gamma}} \Pi_\Gamma^{(1)}(u). \tag{31}
\]
Carrying out renormalisation group improvement of the integrand, the solution of the RG equation corresponds to (29) with $\lambda$ replaced by $\lambda(\sigma)$ and $m$ replaced by $m(\sigma)$, where in the $\overline{MS}$ scheme the arguments of these running parameters are scaled by $k_\lambda = \exp(-5/3)$ [43].

The conformal mapping is then realised via the replacement of perturbative running parameters with the non-perturbative running parameters in (12) and (16), and in place of (30) we have

$$W(\sigma) = (\sigma + 4m^2(k\sigma)) \left( 1 - 4\lambda_{\text{eff}}(k\sigma) \frac{\psi_r(u)}{\Pi_r^{(0)}(u)} \right),$$

where now $u^2 = \sigma/(\sigma + 4m^2(ks))$.

Finally, we may obtain the power moments for arbitrary $n$, which are given by

$$M_n^{(\Gamma)}(Q^2) = \frac{1}{4\pi^2} \int_0^\infty ds \frac{(\sigma + 4m^2(k\sigma))^{N_r} \Pi_r^{(0)}(u)}{(Q^2 + W(\sigma))^{1+n}}.$$  (33)

As in the standard sum rules approach, the mass of the first resonance is then obtained by considering ratios of the moments, $R_n^{\Gamma} = M_{n-1}^{\Gamma}/M_n^{\Gamma}$, for large $n$.

Using the perturbative formulae for the imaginary parts of the two-point correlators listed in Appendix B the integral for the moments can be evaluated numerically for each current. There are only three parameters in this calculation: a reference value for the QCD coupling $\alpha_0$, taken at the $\tau$ mass scale; the quark mass $m_c$ (or $m_b$); and the variational parameter $C$, constrained to be near 4.1 via data from the quark-antiquark potential. We have previously presented an analysis of $\tau$–decay using this technique [34] which resulted in a particular normalisation of the coupling at the $\tau$ scale, $\alpha_0(M_\tau)$. For consistency we use the value extracted at $O(a^3)$, $\alpha_0 = 0.379$, which corresponds to the order of the perturbative expressions with which we are working.

An important aspect of the modification of the virtuality distribution (Fig. 1) invoked to ensure consistency with the OPE, is that the peak is shifted to higher scales. Consequently, implementation of the Coulomb resummation produces a saddle point for the moments which dominates the expression for large $n$ (see Appendix C). The existence of the saddle point is a result of the function $W(\sigma)$ attaining a minimum, at say $\sigma = \bar{\sigma}$. This scale $\bar{\sigma}$ then dominates the expression for the moment as $n$ becomes large. Analysis of the various channels indicates that this minimum occurs in the range $\bar{\sigma} \sim 4 - 6$ GeV$^2$ for $c\bar{c}$, and $\bar{\sigma} \sim 20 - 30$ GeV$^2$ for $b\bar{b}$. The fact that the saddle point occurs well above $\Lambda_{QCD}$ provides additional justification for the validity of the perturbative expressions with which we are working. It is then clear that for large $n$, the moment ratio tends asymptotically to its saddle point approximation, i.e.

$$R_n(Q^2) \overset{n \to \infty}{\sim} Q^2 + W(\bar{\sigma}).$$  (34)

7 Note that for currents with a non-zero anomalous dimension the modification takes the form of an overall factor in the integral which will not contribute to the moment ratios for large $n$, and thus not to the asymptotic estimates for the bound state masses. For this reason we may ignore any anomalous dimensions.
The corresponding ratio, $R_n^{\text{had}}(Q^2)$, which arises from assuming the validity of the narrow resonance hypothesis, has the form $Q^2 + M_R^2$ for large $n$, where $M_R$ is the mass of the first resonance in the relevant channel. Thus we obtain the prediction

$$M_R = \sqrt{W(\tilde{\sigma})},$$

and as a consequence there is no requirement to fix $Q^2$ explicitly, for example at $Q^2 = 0$, in order to determine the mass of the resonance.

Results of the numerical calculations for the full moment ratios, and the asymptotic estimates $\sqrt{W(\tilde{\sigma})}$, for all currents under consideration and for both $c\bar{c}$ and $b\bar{b}$ bound states are shown in Figures 2–5. The quark mass parameters giving the optimal fit for all channels in each family are $m_c = 1.51$ GeV, and $m_b = 4.72$ GeV, which agree well with experimental constraints [44]. As a consequence of the resummation, we observe that in all cases the moment ratios are stable as $n$ becomes large, in contrast to the conventional results obtained by explicitly adding power corrections to a truncated perturbative series for condensates up to dimension six or eight. For clarity a summary of the results presented in the figures is given in Tables 1 and 2, where the asymptotic estimates for the moment ratios are compared with the experimental results [44] where available. For the vector, axial–vector, and $A'$ channels the magnitudes of the bound state masses, and also the inter–channel splitting, are very well reproduced. For the scalar and pseudo–scalar channels the magnitudes are reasonably well approximated but the splitting between these states is not well described. Possible interpretations for the results obtained will be presented in the next section.
FIG. 2. On the left we plot ratios of the vector (solid curve), axial vector (dashed curve) and $A'$ (dot-dashed curve) moments for $c\bar{c}$ bound states, and on the right we plot $\sqrt{W(\sigma)}$ versus $\sigma$ for the same currents, the minimum being the asymptotic limit of the moment ratios for large $n$. For comparison, the straight lines are, in order of decreasing mass, the corresponding experimental $c\bar{c}$, $A'$, axial-vector, and vector bound state masses [44].

FIG. 3. On the left we plot ratios of the pseudo-scalar (solid curve), and scalar (dashed curve) moments for $c\bar{c}$ bound states, and on the right we plot $\sqrt{W(\sigma)}$ versus $\sigma$ for the same currents, the minimum being the asymptotic limit of the moment ratios for large $n$. For comparison, the straight lines are, in order of decreasing mass, the corresponding experimental $c\bar{c}$ scalar and pseudo-scalar bound state masses [44].
FIG. 4. On the left we plot ratios of the vector (solid curve), axial vector (dashed curve) and $A'$ (dot-dashed curve) moments for $b\bar{b}$ bound states, and on the right we plot $\sqrt{W(\sigma)}$ versus $\sigma$ for the same currents, the minimum being the asymptotic limit of the moment ratios for large $n$. For comparison, the straight lines are, in order of decreasing mass, the corresponding experimental $b\bar{b}$ bound state masses for the axial-vector and vector channels [44].

FIG. 5. On the left we plot ratios of the pseudo-scalar (solid curve), and scalar (dashed curve) moments for $b\bar{b}$ bound states, and on the right we plot $\sqrt{W(\sigma)}$ versus $\sigma$ for the same currents, the minimum being the asymptotic limit of the moment ratios for large $n$. For comparison, the straight line is the corresponding experimental scalar $b\bar{b}$ bound state mass [44].
TABLE I. Summary of the estimates obtained for the $c\bar{c}$ bound state masses, compared to the experimental values [44]. Note that $\tilde{\sigma}$ denotes the virtuality scale at which the asymptotic result was obtained.

| current | $M_{\text{expt}}$ (GeV) | $\sqrt{W(\tilde{\sigma})}$ (GeV) | $\tilde{\sigma}$ (GeV$^2$) |
|---------|----------------------|---------------------------------|------------------|
| $j_V$   | 3.10                 | 3.06                            | 4.6              |
| $j_A$   | 3.51                 | 3.52                            | 4.8              |
| $j_A'$  | 3.53                 | 3.56                            | 4.8              |
| $j_S$   | 3.42                 | 3.27                            | 6.0              |
| $j_P$   | 2.98                 | 3.19                            | 3.9              |

TABLE II. Summary of the estimates obtained for the $b\bar{b}$ bound state masses, compared to the experimental values [44], with $\tilde{\sigma}$ again denoting the virtuality scale at which the asymptotic result was obtained.

| current | $M_{\text{expt}}$ (GeV) | $\sqrt{W(\tilde{\sigma})}$ (GeV) | $\tilde{\sigma}$ (GeV$^2$) |
|---------|----------------------|---------------------------------|------------------|
| $j_V$   | 9.46                 | 9.43                            | 27               |
| $j_A$   | 9.89                 | 10.02                           | 26               |
| $j_A'$  | –                    | 10.07                           | 26               |
| $j_S$   | 9.86                 | 9.80                            | 30               |
| $j_P$   | –                    | 9.56                            | 24               |
IV. DISCUSSION

In this paper we have presented an explicit application of the order-dependent mapping approach to the study of the heavy quark QCD resonance spectrum. While it proved convenient to study and modify the perturbative asymptotics to ensure consistency with ambiguities in the OPE power corrections, it is clear that subsequent to the conformal mapping any obvious split between perturbative (short distance) and nonperturbative (long-distance) contributions is lost. Thus, other than by fitting data, it does not appear possible to extract values for the condensates themselves or even to determine the dimension of the dominant contribution, in contrast to related approaches such as that of Dokshitzer, Marchesini, and Webber [45]. This requires a clear and well-defined split between the perturbative and nonperturbative contributions, a distinction which does not emerge naturally within the framework of order-dependent mappings.

Nevertheless, at a purely calculational level, applying this formalism to the bound state spectrum, without the need for explicit introduction of condensate parameters, we found quite remarkable agreement with experimental results for the vector, axial-vector, and $A'$ channels in both $c\bar{c}$ and $b\bar{b}$ families. Indeed if we use a more accurate extraction of the coupling at the $\tau$ scale at $O(a^5)$ [34] the results are even more impressive for these channels, although for consistency one should then include the next order perturbative coefficients.

However, the results for the scalar and pseudo–scalar channels, while again having approximately correct magnitudes, were unsatisfactory in terms of the relative splitting between the states. This was not improved by a more accurate extraction of the coupling. It may be that higher order corrections within this approach could resolve this discrepancy. However, considering the success in the case of the vectorial channels, it appears likely that there are contributions to the scalar and pseudo–scalar channels which are missed by the variational approach, e.g. matrix elements not associated with vacuum condensates. Whether this is due to the particular averaging over non-perturbative contributions implicit here, or more closely related to use of the static quark potential model for normalisation, is currently under investigation.

V. ACKNOWLEDGEMENTS

The authors would like to express their gratitude to D. J. Broadhurst for helpful comments, and supplying a copy of Ref. [47] and to Yu. L. Dokshitzer and C. J. Maxwell for helpful discussions. The financial support of A.R. by the Commonwealth Scholarship Commission in the U.K. and the British Council, and of I.L.S. by the Royal Society and RFBR (grant 96-02-16126-a) is gratefully acknowledged. I.L.S. also thanks Prof. T.W.B. Kibble and the Theoretical Physics Group for their warm hospitality at Imperial College where most of this work was performed.
APPENDIX A:

Induced Power Corrections in a $\phi^4_{D=0}$ Model

In this appendix we use a simple zero dimensional model discussed by Guida, Konishi, and Suzuki [26] to illustrate concretely the power of order-dependent mappings in generating power corrections to a perturbative series, and consequently producing a convergent expansion in a situation where the corresponding perturbative series is non-Borel summable.

Consider the zero dimensional $\phi^4_{D=0}$ model with partition function

$$Z(g) = \int_{-\infty}^{\infty} d\phi e^{-\phi^2 - g\phi^4},$$

(A1)

and analytically continue the result for $g \in \mathbb{C}$. With the change of variables $z = \sqrt{2g} \phi^2$, this is represented in terms of a parabolic cylinder function $D_{\nu}(a)$ [46],

$$Z(g) = \left(\frac{\pi^2}{2g}\right)^{1/4} e^{\frac{1}{2g} D_{-1/2} \left(\frac{1}{\sqrt{2g}}\right)}.$$

(A2)

An $N^{th}$ order approximant to this partition function in the order-dependent mapping may be written in the form

$$Z_N(a) = (1 - a)^{1/2} \psi_N(a, C_N),$$

(A3)

where $\psi_N$ is the $N^{th}$-order truncation of the Taylor series for a function $\psi$. The order-dependent mapping between the coupling $g$ and the parameter $a$ is given by

$$g = \frac{a}{C(1 - a)^2},$$

(A4)

where this relation holds for complex $g$. Restricted to the positive real axis this $N^{th}$ order approximant may be written explicitly as

$$Z_N(a) = \sum_{k=0}^{N} \sum_{n=0}^{k} \frac{\omega_n C_n^N}{(1 - a)^{1/2} a^k} \frac{\Gamma(n + k + 1/2)}{\Gamma(2n + 1/2) \Gamma(k - n + 1)};$$

(A5)

where $\omega_n$ are the perturbative expansion coefficients.

Convergence of the sequence of approximants,

$$\{Z_N(a, C_N)\} \xrightarrow{N \to \infty} Z(g)$$

(A6)

has been proven in [26] for the analytically continued model, and the details will not be recalled here. The importance of this result in the present context for investigating induced power corrections follows from the structure of the asymptotic series for $Z(g)$ for $|g| \ll 1$. This expansion has the form

$$Z(g) \sim \sum_{n=0}^{\infty} \frac{\Gamma(2n + 1/2)}{n!}((-1)^n + m \sqrt{2} e^{1/(4g)}) g^n,$$

(A7)
with \( m = \pm 1 \) for \( \mp 5\pi/2 < \text{Arg}(g) < \mp \pi/2 \) and \( m = 0 \) otherwise. If we consider the structure of this expansion for \( g = e^{i\pi} g_R \) with \( g_R \) real and positive, then the first term is the standard perturbative series about \( g_R = 0 \), which at large orders has coefficients exhibiting the factorial growth \( (\sqrt{e/2\pi\Gamma(n)2^{2n}}) \) associated with a Borel non-summable series. The second term can be interpreted as a complex nonperturbative “power correction” accounting for the imaginary part introduced in resolving the perturbative ambiguity (c.f. [15]).

This may be seen more explicitly by performing a Borel transform on the perturbative series for \( Z(e^{i\pi} g_R) \). From

\[
Z^{\text{pert}}(e^{i\pi} g_R) = \sum_{n=0}^{\infty} \frac{g_R^n}{n!} \Gamma(2n + 1/2),
\]

(A8)

representing the \( \Gamma \)-function as an integral and transforming variables, we have formally

\[
Z^{\text{pert}}(e^{i\pi} g_R) = \sqrt{2} \int_0^{\Lambda} db e^{-b/g_R} B(b),
\]

(A9)

where

\[
B(b) = \frac{1}{\sqrt{1 - 4b} \sqrt{1 - \sqrt{1 - 4b}}},
\]

(A10)

exhibiting branch points at \( b = 0 \) and \( b = 1/4 \). Strictly, the transformation of variables we have performed is only valid for \( b \leq 1/4 \) and some regularization is required to integrate beyond \( \Lambda = 1/4 \). Nonetheless, the position of the branch point at \( 1/4 \) can also be inferred directly from the structure of the large order coefficients \( \sqrt{e/2\pi\Gamma(n)2^{2n}} \).

The branch point at \( b = 0 \) may be removed by renormalisation, while the ambiguity associated with regularising the integral at the branch point \( b = 1/4 \) is given by the residue as (c.f. the second term in (A7))

\[
\text{Res}(Z^{\text{pert}}_{b=1/4}) = \sqrt{2} \frac{1}{g_R} e^{-\frac{1}{4g_R}}.
\]

(A11)

For illustration, we may symbolically represent this ambiguity as it would appear in a field theory with

\[
g_R(Q^2) = \frac{1}{b_0 \ln(Q^2/\Lambda^2)},
\]

(A12)

where \( \beta(g_R) = -b_0 g_R^2 + \cdots \). The ambiguity then takes the form

\[
\exp \left( -\frac{1}{4g_R} \right) \rightarrow \left( \frac{\Lambda^2}{Q^2} \right)^{b_0/4},
\]

(A13)

which we may loosely interpret as an OPE type “power correction”.

Thus we can conclude that the weak coupling expansion is only consistent when one includes a power correction of this form to the perturbative series. More generally the regularisation of the Borel singularity will also result in an imaginary contribution (e.g. by
passing the contour above or below the branch point) counteracted by the imaginary part in (A7).

Therefore at least in this rather formal example, the proven convergence of the order dependent mapping ensures that it can generate a power correction of this form. Note that it is likely that one can perturb this result away from the branch cut by considering $Z(e^{i(\pi - \epsilon)} g_R)$. The imaginary part of the weak coupling expansion still exists for any $\epsilon < \pi/2$, so one might expect that a power correction is still required in this case.
APPENDIX B:

Perturbative Wilson Coefficients

For completeness, and also to fix our notation, in this appendix we list the perturbative Wilson coefficients for the relevant correlators. The currents considered are listed below in the format $j_{\Gamma} = \ldots (J^{PC})$:

- **scalar**: $j_S = \bar{\psi}_i \psi_j (0++)$
- **pseudo–scalar**: $j_P = i \bar{\psi}_i \gamma_5 \psi_j (0-)$
- **vector**: $j_V = \bar{\psi}_i \gamma_\mu \psi_j (1-)$
- **axial vector**: $j_A = (q_\mu q_\nu / q^2 - g_{\mu\nu}) \bar{\psi}_i \gamma_\mu \gamma_5 \psi_j (1++)$
- **$A'$**: $j_{A'} = \bar{\psi}_i \partial_\mu \gamma_5 \psi_j (1++)$

Using the notation of [4] we introduce the following generic components for the correlators,

\[ A(u) = (1 + u^2) \left[ \frac{\pi^2}{6} + \ln \frac{1+u}{1-u} \ln \frac{1+u}{2} + 2l \left( \frac{1-u}{1+u} \right) + 2l \left( \frac{1+u}{2} \right) 
- 2l \left( \frac{1-u}{2} \right) - 4l(u) + l(u^2) \right] + 3u \ln \frac{1-u^2}{4u} - u \ln u \] (B1)

\[ A'(u) = (1 + u^2) \left[ 2l \left( \frac{1-u}{1+u} \right)^2 - 2l \left( \frac{u-1}{u+1} \right) 
- 3 \ln \frac{1-u}{1+u} \ln \frac{1+u}{2} + 2 \ln \frac{1-u}{1+u} \ln u \right], \] (B2)

where

\[ l(x) = - \int_0^x dt \frac{1}{t} \ln(1-t), \] (B3)

is the Spence function. Then following the notation of Section 3, the relevant formulae for each current under consideration are given by:

**Vector Current** [4] ($\Gamma = \gamma_\mu$)

\[ N_V = 0 \] (B4)

\[ \Pi^0_V = \frac{1}{2} u(3 - u^2) \] (B5)

\[ \Pi^1_V = 2 \left[ \left( 1 - \frac{u^2}{3} \right) A(u) + P_V(u) \ln \frac{1+u}{1-u} + Q_V(u) \right] \] (B6)

\[ P_V(u) = \frac{1}{24}(33 + 22u^2 - 7u^4) \] (B7)

\[ Q_V(u) = \frac{1}{4}(5u - 3u^3). \] (B8)
Axial-vector Current \[\Gamma = \gamma_5 \gamma_\mu (q_\mu q_\nu / q^2 - g_{\mu \nu})\]

\[N_A = 1\] \hspace{1cm} (B9)
\[\Pi_A^0 = u^3\] \hspace{1cm} (B10)
\[\Pi_A^1 = \frac{4}{3} \left[ u^2 A(u) + P_A(u) \ln \frac{1+u}{1-u} + Q_A(u) \right]\] \hspace{1cm} (B11)
\[P_A(u) = \frac{1}{32} (21 + 59u^2 - 19u^4 - 3u^6)\] \hspace{1cm} (B12)
\[Q_A(u) = \frac{1}{16} (-21u + 30u^3 + 3u^5).\] \hspace{1cm} (B13)

\[A' \text{ Current}\] \[\Gamma = \partial_{\mu} \gamma_5\]

\[N_{A'} = 2\] \hspace{1cm} (B14)
\[\Pi_{A'}^0 = \frac{1}{2} u^3\] \hspace{1cm} (B15)
\[\Pi_{A'}^1 = \frac{2}{3} \left[ u^2 A'(u) + P_{A'}(u) \ln \frac{1+u}{1-u} + Q_{A'}(u) \right]\] \hspace{1cm} (B16)
\[P_{A'}(u) = \frac{1}{16} (13 + 28u^2 + 17u^4 - 2u^6)\] \hspace{1cm} (B17)
\[Q_{A'}(u) = \frac{1}{24} (-39u + 47u^3 + 6u^5).\] \hspace{1cm} (B18)

Scalar Current \[\Gamma = 1\]

\[N_S = 1\] \hspace{1cm} (B19)
\[\Pi_S^0 = \frac{3}{2} u^3\] \hspace{1cm} (B20)
\[\Pi_S^1 = 2 \left[ u^2 A'(u) + P_S(u) \ln \frac{1+u}{1-u} + Q_S(u) \right]\] \hspace{1cm} (B21)
\[P_S(u) = \frac{1}{16} (3 + 34u^2 - 13u^4)\] \hspace{1cm} (B22)
\[Q_S(u) = \frac{1}{8} (21u - 3u^3).\] \hspace{1cm} (B23)

Pseudo-scalar Current \[\Gamma = \gamma_5\]

\[N_P = 1\] \hspace{1cm} (B24)
\[\Pi_P^0 = \frac{3}{2} u\] \hspace{1cm} (B25)
\[\Pi_P^1 = 2 \left[ A'(u) + P_P(u) \ln \frac{1+u}{1-u} + Q_P(u) \right]\] \hspace{1cm} (B26)
\[P_P(u) = \frac{1}{16} (19 - 48u + 2u^2 + 3u^4)\] \hspace{1cm} (B27)
\[Q_P(u) = \frac{1}{8} (21u - 3u^3).\] \hspace{1cm} (B28)
APPENDIX C:

Resummation of Coulomb Singularities

In this appendix we justify the resummation procedure used in Eq. (29) to avoid the adverse effects of threshold Coulomb singularities in the moment ratios for large \( n \).

As is well known \cite{41} (see also \cite{48,49}), the perturbative series for \( \text{Im}\Pi(s) \) is generically an expansion in powers of \( \lambda/u \), where \( u \) is the quark velocity introduced in Section 3. The series suffers from threshold singularities for small \( u \), and this is accentuated in moment space, where the dominant contribution to the \( n^{th} \) moment comes from the region \( u \sim 1/\sqrt{n} \). In other words, for higher order moments the system appears more and more nonrelativistic, and the perturbative series is poorly behaved; the expansion parameter is now \( \sqrt{n}\lambda \).

This behaviour is usually accounted for by treating the nonrelativistic Coulomb system exactly, leading to a resummation, e.g. in the form of the Sommerfeld-Sakharov factor \cite{12},

\[
\text{Im}\Pi(X) \rightarrow \frac{X}{1 - \exp(-X)} \quad \text{with} \quad X = \frac{16\pi^2 \lambda}{3} \frac{1}{u}, \quad (C1)
\]

Since our conformal mapping procedure makes use of the same perturbative coefficients, a resummation of this form is still necessary. However, the modified virtuality distribution introduced in Section 2 actually allows an alternative resummation to that mentioned above.

In order to illustrate the procedure, we consider a simple model and study first the way in which the standard perturbative approach breaks down for large \( n \). Consider the \( n^{th} \) moment,

\[
M_n(Y) = \int_0^\infty d\sigma \frac{\rho(\sigma)}{(\sigma + Y)^{n+1}}, \quad (C2)
\]

where \( \sigma = s - 4m^2 \) is the virtuality, and \( Y = Q^2 + 4m^2 \). Now assume that the full spectral density \( \rho(\sigma) \) is known and has the form

\[
\rho(\sigma) = \left( \frac{\sqrt{\sigma}}{\lambda + \sqrt{\sigma}} \right)^p, \quad (C3)
\]

with \( p \) a positive integer, and normalized to \( \rho(\sigma, \lambda = 0) = 1 \). A perturbative expansion of \( \rho(\sigma) \) has the form

\[
\rho_{\text{pert}}(\sigma) \sim 1 - p \frac{\lambda}{\sqrt{\sigma}} + O \left( \frac{\lambda}{\sqrt{\sigma}} \right)^2 \sim 1 - p \frac{\lambda}{2m_u} + O \left( \frac{1}{u} \right)^2, \quad (C4)
\]

and thus exhibits Coulomb singularities for small \( u \).

For large \( n \) we may rewrite the moments in the form \cite{13}:

\[
M_n(Y) \sim Y^{-(n+1)} \int_0^\infty d\sigma \rho(\sigma) \exp \left( -n \frac{\sigma}{Y} \right) \left( 1 + O \left( \frac{1}{n} \right) \right), \quad (C5)
\]

\footnote{We assume that \( \rho(\sigma) \) is smooth \cite{13}.}
and observe that the dominant contribution to the integral corresponds to virtualities in the region $\sigma \sim Y/n$ (this is just the threshold behaviour $u \sim 1/\sqrt{n}$ noted earlier). Then, for the exact moments, $M_n^{ex}$, given by (C2) with the exact spectral density (C3), the dominant threshold behaviour at small $u$ (or large $n$) corresponds to

$$\rho_{ex} \rightarrow \left(1 + \sqrt{n} \frac{\lambda}{\sqrt{Y}}\right)^{-p},$$

which is positive and finite for all $n$.

If instead we consider the $O(\lambda)$ perturbative contribution only, we find

$$\rho_{pert} \rightarrow 1 - p\sqrt{n} \frac{\lambda}{\sqrt{Y}},$$

which implies that the approximation must break down, and hence the moments will become unreliable, for large $n$ when $\rho_{pert}$ passes through zero and becomes negative.

We can now introduce a resummation which restores the correct behaviour of the moments for large $n$. Performing the integration by parts which ensures consistency of the perturbative ambiguities with the OPE naturally suggests that we subsume higher order corrections to the spectral function into the denominator of the virtuality distribution. If we consider this as a formal transformation, we rewrite the moments in the form

$$M_n^{re}(Y) = \int_0^\infty d\sigma' \frac{1}{(f(\sigma') + Y)^{n+1}},$$

where the function $f(\sigma')$ obeys the implicit equation

$$\sigma' = \int_0^{f(\sigma')} d\sigma \rho(\sigma).$$

This representation follows from considering the change of variables $\sigma = f(\sigma')$ and assuming that $\rho(\sigma)$ satisfies certain conditions which ensure the limits $f(\infty) = \infty$ and $f(0) = 0$.

Although this transformation is exact, in this paper we work to $O(\lambda)$ and, solving (C9) to this order, we find

$$f(\sigma') = \sigma' + 2p\lambda\sqrt{\sigma'}.$$  

Note that at this order, the present resummation essentially coincides with the Sommerfeld-Sakharov factor (C1). The corrections are of $O(\lambda^2/u^2)$ and are thus neglected. Inserting the leading order result in (C8), and representing it in the standard form (C2) with a spectral density $\rho_{re}(\sigma')$, one finds that for small $u \sim 1/\sqrt{n}$,

$$\rho_{re} \rightarrow \left(1 + 2\sqrt{n} \frac{\lambda}{\sqrt{Y}}\right)^{-p}.$$

This clearly indicates an approximate recovery of the correct large $n$ dependence associated with $\rho_{ex}(\sigma)$, despite using only the $O(\lambda)$ terms in $\rho(\sigma)$.

To illustrate the success of this resummation we plot moment ratios $R_n = M_n/M_{n-1}$ using the exact ($R_n^{ex}$), perturbative ($R_n^{pert}$), and resummed ($R_n^{re}$) spectral functions, in Fig. 6.
FIG. 6. Moment ratios are plotted using the exact ($R_n^{\text{ex}}$), perturbative ($R_n^{\text{pert}}$), and resummed ($R_n^{\text{re}}$) spectral functions. We use the parameters $Y = 1$, $\lambda = 0.1$, and $p = 1$. Other choices lead to similar qualitative results.

One clearly observes the close agreement between $R_n^{\text{ex}}$ and $R_n^{\text{re}}$ as $n$ becomes large, while the perturbative approximation without any resummation breaks down for large $n$.

Furthermore, it is important to note that in the resummed case $u \sim 1/\sqrt{n}$ is not the dominant contribution to the integral. This representation allows a saddle point to appear for a finite value of $\sigma$ which is independent of $n$. This is associated with the shift in the peak of the virtuality distribution observed in Fig. 1 for the massless case. Therefore, in practice the dominant contribution to the moments does not move closer to the threshold region, and as $n$ becomes large the saddle point dominates the moments, leading to a finite limit. This behaviour is also observed in the QCD discussion presented in Section 3.
REFERENCES

[1] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B 147, 385 (1979).
[2] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B 147, 448 (1979).
[3] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B 147, 519 (1979).
[4] L. J. Reinders, H. R. Rubinstein, and S. Yazaki, Phys. Rep. 127, 1 (1985).
[5] G. 't Hooft, In “The Whys of Subnuclear Physics”, Proceedings of the 15th Int. School on Subnuclear Physics, Erice, Sicily (1977), ed. by A. Zichichi, 943 (1979).
[6] B. Lautrup, Phys. Lett. B 69, 109 (1980).
[7] G. Parisi, Phys. Lett. B 76, 65 (1978).
[8] G. Parisi, Nucl. Phys. B 150, 163 (1979).
[9] F. David, Nucl. Phys. B 234, 237 (1984).
[10] A. H. Mueller, Nucl. Phys. B 250, 327 (1985).
[11] Y. L. Dokshitser and N. G. Uraltsev, Phys. Lett. B 380, 141 (1996).
[12] V. I. Zakharov, Nucl. Phys. B 385, 452 (1992).
[13] M. Beneke and V. I. Zakharov, Phys. Rev. Lett. 69, 2472 (1992).
[14] M. Beneke, Nucl. Phys. B 405, 424 (1993).
[15] G. Grunberg, Phys. Lett. B 325, 441 (1994).
[16] M. Beneke and V. M. Braun, Nucl. Phys. B 426, 301 (1994).
[17] M. Beneke and V. M. Braun, Phys. Lett. B 348, 513 (1995).
[18] P. Ball, M. Beneke, and V. M. Braun, Nucl. Phys. B 452, 563 (1995).
[19] I. I. Bigi, M. A. Shifman, N. G. Uraltsev, and A. I. Vainshtein, Phys. Rev. D 50, 2234 (1994).
[20] C. N. Lovett-Turner and C. J. Maxwell, Nucl. Phys. B 432, 147 (1994).
[21] G. Altarelli, P. Nason, and G. Ridolfi, Z. Phys. C 68, 257 (1995).
[22] C. N. Lovett-Turner and C. J. Maxwell, Nucl. Phys. B 452, 188 (1995).
[23] M. Neubert, Phys. Rev. D 51, 5924 (1995).
[24] M. Neubert and C. T. Sachrajda, Nucl. Phys. B 438, 235 (1995).
[25] R. Seznec and J. Zinn-Justin, J. Math. Phys. 20, 1398 (1979).
[26] R. Guida, K. Konishi, and H. Suzuki, Ann. Phys. 249, 109 (1996).
[27] I. L. Solovtsov, Phys. Lett. B 327, 335 (1994).
[28] I. L. Solovtsov, Phys. Lett. B 340, 245 (1994).
[29] H. F. Jones and I. L. Solovtsov, Phys. Lett. B 349, 519 (1995).
[30] I. L. Solovtsov and O. P. Solovtsova, Phys. Lett. B 344, 377 (1995).
[31] H. F. Jones, I. L. Solovtsov, and O. P. Solovtsova, Phys. Lett. B 357, 441 (1995).
[32] H. F. Jones and I. L. Solovtsov, in HEP95, ed. J. Lemonne, C. Vander Velde and F. Verbeure, (World Scientific), p.242, (1995).
[33] H. F. Jones, A. N. Sissakian, and I. L. Solovtsov, in HEP96 vol. II, ed. Z. Ajduk and A.K. Wroblewski, (World Scientific), p.1650, (1996).
[34] H. F. Jones, A. Ritz, and I. L. Solovtsov, Mod. Phys. Lett. A 12, 1361 (1997).
[35] I. R. C. Buckley, A. Duncan, and H. F. Jones, Phys. Rev. D 47, 2554 (1993).
[36] R. Guida, K. Konishi, and H. Suzuki, Ann. Phys. 241, 152 (1995).
[37] C. Arvanitis, H. F. Jones, and C. S. Parker, Phys. Rev. D 52, 3704 (1995).
[38] I. F. Ginzburg and D. V. Shirkov, Sov. Phys. JETP 22, 234 (1966).
[39] G. Grunberg, Preprint: CPTH-S505-0597, hep-ph/9705290 (1997).
[40] D. V. Shirkov and I. L. Solovtsov, Phys. Rev. Lett. 79, 1209 (1997).
[41] V. A. Novikov et al., Phys. Rev. Lett. 38, 626 (1977); Phys. Rep. 41C, 1 (1978); M. B. Voloshin and Yu. M. Zaitsev, Sov. Phys. (Usp.) 30, 553 (1987).
[42] A. Sommerfeld, *Atombau und Spektrallinien* Vol.2, (Vieweg, Braunschweig, 1939); A. D. Sakharov, Sov. Phys. JETP 18, 631 (1948).
[43] M. Neubert, Nucl. Phys. B 463, 511 (1996).
[44] R. Barnett et al., Phys. Rev. D 54, 1 (1996).
[45] Y. L. Dokshitser, G. Marchesini, and B. R. Webber, Nucl. Phys. B 469, 93 (1996).
[46] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, London, 1980).
[47] D. J. Broadhurst and S. C. Generalis, Open University preprint: OUT-4102-8/R, (1982).
[48] M. B. Voloshin, Int. J. Mod. Phys. A10, 2865 (1995); M. Jamin and A. Pich, Nucl. Phys. B 507, 334 (1997).
[49] J. H. Kuhn, A. A. Penin, and A. A. Pivovarov, Preprint: hep-ph/9801356 (1998).