Constructive Liouville Conformal Field Theory

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Abstract

These lectures give an introduction to a probabilistic approach to Liouville Quantum Field Theory developed in a joint work with F. David, R. Rhodes and V. Vargas.

1 Probabilistic Liouville Theory

One of the simplest and at the same time most intriguing of the Conformal Field Theories (CFT) is the Liouville CFT (LCFT). It first appeared in Polyakov’s formulation of String Theory [14] and then in the work of Knizhnik, Polyakov and Zamolodchikov [8] on the relationship between CFT’s in fixed background metric and in a random metric (2d gravity) or in other terms on the relationship between statistical mechanics models on fixed lattices and on random lattices. Decisive progress in LCFT came in the 90’s as Dorn and Otto [5] and Zamolodchikov and Zamolodchikov [18] produced an explicit formula for Liouville three point functions, the celebrated DOZZ-formula. More recently, the Liouville three point functions were shown to have a deep relationship to four dimensional Yang Mills theories [1].

Unlike most CFT’s the LCFT has an explicit functional integral formulation. However the exact results for Liouville correlations are not derived from this functional integral but rather from general principles of CFT (BPZ equations, crossing symmetry) coupled to assumptions about the spectrum of LCFT [15]. A rigorous probabilistic formulation of the LCFT was given in [3]. In that work it was shown that the LCFT functional integral can be defined using the theory of Gaussian Multiplicative Chaos, a well studied chapter of probability theory. In this paper I will review in Section 1 this construction as well as the proof of local conformal invariance given in [9]. In Section 2 it is shown how the Quantum Field Theory structure and a representation of the Virasoro algebra can be derived from the probabilistic theory.

1.1 Scaling Limits

One of the motivations for the study of LCFT comes from the study of scaling limits of discrete models of random surfaces. This subsection gives a brief summary of the discrete objects that LCFT aims to describe.

Random triangulations

By a triangulation of the unit sphere we mean a finite connected graph $T$ s.t. there is an embedding of $T$ to the two dimensional sphere $S^2$ s.t. each connected component of $S^2 \setminus T$ (a face) has a boundary consisting of 3 edges (we denote the embedding of $T$ by $T$ again). A marked triangulation is a triangulation $T$ together with a choice of three vertices $v_1, v_2, v_3$ of $T$. We denote by $T$ the set of marked triangulations, by $\mathcal{V}(T)$ the set of vertices of $T$ and by $|T|$ the number of faces in $T$.
Next we want to consider probability measures on $\mathcal{T}$. The simplest example is the case of "pure gravity". We define the probability
\[
P_{\mu_0,\sqrt{3}}(T) = \frac{1}{Z_{\mu_0,\sqrt{3}}} e^{-\mu_0|T|}
\]
(for the index $\sqrt{3}$, see below) where $Z_{\mu_0,\sqrt{3}} = \sum_{T \in \mathcal{T}} e^{-\mu_0|T|}$. For other examples we add "matter" to the gravity model. Given a triangulation $T$ one may consider statistical mechanics models on it. For example, for the Ising model one defines "spin" variables $\sigma_v \in \{1, -1\}$ indexed by the vertices $v \in \mathcal{V}(T)$ of $T$ and considers the joint probability distribution on triangulations $T$ and spin configurations $\sigma = \{\sigma_v\}_{v \in \mathcal{V}(T)}$ defined by
\[
P_{\mu_0,\sqrt{3}}(T, \sigma) = \frac{1}{Z_{\mu_0,\sqrt{3}}} e^{-\mu_0|T|} e^{\beta_0 \sum_{\sigma = \{\sigma_v\}_{v \in \mathcal{V}(T)}}}
\]
where $v \sim v'$ means $v, v'$ share an edge. The parameter $\beta_0$ is the critical value for the inverse temperature of the Ising model, known to exist in this setup. More generally there are many other critical statistical mechanical models one can define on $T$. Their marginal distributions on $T$ are all of the form
\[
P_{\mu_0,\gamma}(T) = \frac{1}{Z_{\mu_0,\gamma}} e^{-\mu_0|T|} Z_\gamma(T)
\]
(1.1)
Here $Z_\gamma(T)$ is the partition function of the statistical model on the graph $T$ and $\gamma$ is a parameter depending on that model. It is related to its central charge $c$ by $c = 25 - 6Q^2$, $Q = \frac{\gamma}{2} + \frac{3}{2}$. In particular $\gamma \in [\sqrt{2}, 2]$ corresponding to $c \in [-2, 1]$. Examples of models covering this whole range are given by the $O(N)$ loop models. The $\gamma = 2$ case is discrete Gaussian Free Field where $Z_\gamma(T) = \det(-\Delta_T)^{-\frac{1}{2}}$ with $\Delta_T$ the Laplacean on the graph $T$ and the $\gamma = \sqrt{2}$ is the uniform spanning tree with $Z_\gamma(T) = \det(-\Delta_T)$.

It is known that
\[
Z_N := \sum_{T \in \mathcal{T}; |T| = N} Z_\gamma(T) = N^{1 - \frac{\gamma}{12}} e^{\bar{\mu} N} (1 + o(1))
\]
(1.2)
where $\bar{\mu}$ depends on the model. In particular this implies that $P_{\mu_0,\gamma}$ is defined for $\mu_0 > \bar{\mu}$ and $\lim_{\mu_0 \downarrow \bar{\mu}} Z_{\mu_0,\gamma} = \infty$ if $\gamma > \sqrt{2}$. Hence as $\mu_0 \to \bar{\mu}$ the measure samples large triangulations.

Conformal structure

For each $T$ we may associate a conformal structure on $S^2$ as follows. Assign to each face $f$ a copy $\Delta_f$ of an equilateral triangle $\Delta$ of unit area and let $M_T = \sqcup \Delta_f / \sim$ where in the disjoint union of the $\Delta_f$ we identify the common edges. $M_T$ is a topological manifold homeomorphic to $S^2$.

We can make $M_T$ a complex manifold by the following atlas. It consists of the following coordinate patches. First, interiors of $\Delta_f$ are mapped by identity to $\Delta$. Second, for each pair of faces $f$ and $f'$ that share an edge we map the interiors of $\Delta_f \cup \Delta_{f'}$ by identity to two copies of the standard triangle $\Delta$ sitting next to each other in $\mathbb{C}$. Finally for each vertex $v \in M$ we map its neighbourhood to $\mathbb{C}$ as follows. First, list the faces sharing $v$ in consecutive order: $f_0, \ldots, f_{n-1}$. Then parametrize the set $\Delta_{f_j} \cap U$ by $z_j = r e^{2\pi i \theta_j}$ with $\theta_j \in [6j/n, 6(j+1)/n]$. Then $z \to z^{n/6}$ provides a complex coordinate for a neighborhood of $v$. This atlas makes $M_T$ a complex manifold homeomorphic to $S^2$. Picking three points $z_1, z_2, z_3 \in \mathbb{C}$ there is a unique conformal map $\psi_T: M_T \to \hat{\mathbb{C}}$ s.t. $\psi(v_j) = z_j$ where $\hat{\mathbb{C}}$ is the Riemann sphere, see Section 1.2.

Let $\lambda_T$ be the area measure on $M_T$ i.e. $\lambda_T$ is the Lebesgue measure in the local coordinate on $\Delta_f$. Let $\gamma_T$ be the Riemannian metric on $M_T$ which in the local coordinate on $\Delta_f$ is given by $dx \otimes dx + dy \otimes dy$. We may transport these objects to $S^2$ by the conformal map $\psi_T$. If we now sample $T$ from $P_{\mu_0,\gamma}$, these become a random measure $\nu_{\mu_0,\gamma}$ and a random Riemannian metric $G_{\mu_0,\gamma}$ on $\hat{\mathbb{C}}$. In the standard coordinate of $\hat{\mathbb{C}}$ they are given by $\nu_{\mu_0,\gamma} = \gamma_{\mu_0,\gamma}(z) dz^1$ and $G_{\mu_0,\gamma} = \gamma_{\mu_0,\gamma}(z)(dx \otimes dx + dy \otimes dy)$ where the density $\gamma_{\mu_0,\gamma}$ is singular at the images of the vertices with $n > 6$.

\footnote{We denote the Lebesgue measure on $\mathbb{C}$ by $dz$}
Scaling limit

Consider now a scaling limit as follows. Recalling that as $\mu_0 \downarrow \bar{\mu}$ typical size of triangulation diverges we define for $\mu > 0$

$$\rho_{\mu,\gamma}^{(e)} := e^{\nu_\mu + \epsilon_\mu,\gamma} \quad g_{\mu,\gamma}^{(e)} := \epsilon^{\alpha, \gamma} \Gamma_{\mu,\gamma}$$

Then it is conjectured that $\rho_{\mu,\gamma}^{(e)}$ converges as $\epsilon \to 0$ to a random measure $\rho_{\mu,\gamma}$ on $\hat{C}$ and the metric space defined by $g_{\mu,\gamma}^{(e)}$ converges to a random metric space. In the case $\gamma = \sqrt{8/3} a_\gamma = \hat{z}$ and this was proven by Gall [10] and Miermont [11] and the random metric was constructed directly in the continuum by Miller and Sheffield [12]. Since $\nu_\mu(S^2) = \nu N$ the asymptotics (1.2) implies that the law of $\rho_{\mu,\gamma}(S^2)$ is given by

$$E[F(\rho_{\mu,\gamma}(S^2))] = \lim_{\epsilon \to 0} \frac{1}{Z_{\epsilon}} \sum_N e^{-\mu N} N^{1-\frac{\hat{z}}{4}} F(\epsilon N)$$

i.e. the law is $\Gamma(2 - \frac{\hat{z}}{4}, \mu)$. In what follows we will construct a random measure that has this law for its total mass and is a candidate for the scaling limit.

As another example of a limiting object consider the case of Ising model ($\gamma = \sqrt{3}$). We can transport the Ising spins $\sigma_v = \pm 1$ sitting at vertices $v$ of $T$ to $\hat{C}$. Define the distribution

$$\Phi_T^{(e)}(z) = \epsilon^\frac{\hat{z}}{2} \sum_{v \in \mathcal{V}(T)} \sigma_v \delta(z - \psi_T(v)).$$

Then under $\mathbb{P}_{\mu_0 + \epsilon_\mu,\gamma}$ this becomes a random field on $\hat{C}$ and we will get a conjecture for its distribution as $\epsilon \to 0$ in terms of the correlation functions of the Liouville QFT, see Section 1.7.

1.2 KPZ Conjecture

Locality and coordinate invariance are the basic principles of relativistic physics. Locality means that the basic objects are fields that are functions on the space-time manifold $M$ (string theory is an exception to this) and their dynamics is determined by an action functional that is local in the fields and their derivatives, e.g. the free scalar field has

$$S(\phi) = \int_M (\nabla \phi)^2 dx.$$ 

(General) relativity enters also through a local field, (pseudo) Riemannian metric $g(x) = \sum g_{\alpha\beta}(x) dx^\alpha \otimes dx^\beta$. In (Euclidean) quantum gravity one looks for a probability law in the space of fields. Coordinate invariance means that this law should be invariant under coordinate transformations i.e. under the action of the group of diffeomorphisms $Diff(M)$. Hence in particular this law lives on the space of metrics modulo diffeomorphisms $Met(M)/Diff(M)$. In two dimensions this space is particularly simple. In particular on the sphere $S^2$ any two smooth metrics $g, g'$ are, modulo a diffeomorphism, conformally equivalent, i.e. $f^* g' = e^{2 \varphi} g$ for $\varphi : M \to \mathbb{R}$.

Conformal metrics

Recall that the Riemann sphere $\hat{C} = \mathbb{C} \cup \{\infty\}$ can be covered by two coordinate patches $\hat{C} \setminus \{\infty\}$ and $\hat{C} \setminus \{0\}$ with the coordinates $z$ and $z^{-1}$. A conformal metric on $C$ is given by $g(z)(dz \otimes dz + d\bar{z} \otimes d\bar{z})$ which becomes $g(1/z)|z\bar{z}|^{-2}$ on the other patch. Hence if $g$ is continuous on $\hat{C}$ this means $g(z) = \mathcal{O}(|z\bar{z}|^{-2})$ at infinity. The round metric is given by

$$\tilde{g}(z) = 4(1 + |z\bar{z}|)^{-2}$$

(1.4)
and it has the area is \( \int_C g = 4\pi \) and the scalar curvature \( R_g := -4g^{-1}\partial z\partial\bar{z}\ln g \) is constant: \( R_g = 2 \). For all smooth conformal metrics one computes \( \int gR_g = \frac{1}{2} \int \partial\bar{\partial}\ln g = 8\pi \), an instance of the Gauss-Bonnet theorem. Given a conformal metric on \( \hat{C} \) we can define the Sobolev space \( H^1(\hat{C}, g) \) with the norm 

\[
\|f\|_2^2 := \int (|\partial_z f|^2 + g(z)|f|^2)dz.
\]

These norms are equivalent for all continuous conformal metrics. Finally we define \( H^{-1}(\hat{C}, g) \) as the dual space and denote the dual pairing by \( \langle X, f \rangle \). Formally \( \langle X, f \rangle = \int X(z) f(z) g(z) dz \).

### Liouville QFT

For (Euclidean) quantum gravity on \( \hat{C} \) one is thus looking for the probability law of the conformal metric 

\[
\frac{1}{2} e^{\phi} dz \otimes d\bar{z} + d\bar{z} \otimes dz
\]

i.e. for a law for a random real valued field \( \phi \). To state the KPZ conjecture for this law we fix a conformal metric \( g(z) \) ("background metric") on \( \hat{C} \) and then the KPZ conjecture \([8, ?, ?]\) states that the random measure \( \rho_{\mu, \gamma} \) is given by

\[
\rho_{\mu, \gamma}(dz) = e^{\gamma\phi_g(z)} dz (1.5)
\]

where \( \phi_g \) is the Liouville field

\[
\phi_g := X + \frac{Q}{2} \ln g
\]

and \( X \) is a random field whose law is formally given by

\[
E_{\gamma, \mu} f(X) = Z^{-1} \int_{\text{Map}(\mathbb{C} \to \mathbb{R})} f(X) e^{-S_L(X,g)} DX. (1.7)
\]

Here \( S_L \) is action functional of the Liouville model:

\[
S_L(X,g) := \frac{1}{\pi} \int_C (\partial_z X \partial\bar{z} X + \frac{Q}{4} gR_g X + \pi \mu e^{\gamma\phi}) dz. (1.8)
\]

Here \( Q \) is related to \( \gamma \) by

\[
Q = 2/\gamma + \gamma/2.
\]

Furthermore the heuristic integration over \( X \) in (1.7) is supposed to include "gauge fixing" due to the marked points \( z_1, z_2, z_3 \). Our aim is to give precise meaning to the law (1.7) and study its properties that include conformal invariance.

**Remark 1.1.** Note that for a conformally equivalent metric \( g' = e^\varphi g \)

\[
g' R_{g'} = g R_g - 4\partial_z \partial_{\bar{z}} \varphi
\]

so that

\[
S_L(X, e^\varphi g) = S_L(X + \frac{Q}{2} \varphi, g) - \frac{Q^2}{4\pi} \int_C (\partial_z \varphi \partial_{\bar{z}} \varphi + \frac{1}{2} gR_g \varphi) dz (1.9)
\]

Thus, modulo an additive constant, the Weyl transformation \( g \to e^\varphi g \) is a shift in \( X \).

**Remark 1.2.** \( \hat{C} \) has a nontrivial automorphism group \( SL(2, \mathbb{C}) \) which acts as Möbius transformations \( \psi(z) = \frac{az+b}{cz+d} \). By change of variables one can compute

\[
S_L(X \circ \psi^{-1}, g) = S_L(X + \frac{Q}{2} \varphi, g)
\]

where \( e^\varphi = |\psi'|^2 g \circ \psi \).
1.3 Massless Free Field

Let us first keep only the quadratic term in the action functional (1.8) and try to define the linear functional

\[ \langle F \rangle = \int_{\text{Map}(\mathbb{C} \rightarrow \mathbb{R})} F(X) e^{-\frac{1}{2} \int_{\mathbb{C}} |\partial_z X|^2 dz} dX \]  

(1.10)

We may define this in terms of the Gaussian Free Field (GFF).

GFF

In general the GFF is a Gaussian random field whose covariance is the Green function of the Laplacean. In our setup the Laplace operator is given by \( \Delta_g = 4g(z)^{-1} \partial_z \partial_{\bar{z}} \). Some care is needed here since \( \Delta_g \) is not invertible. Indeed, \(-\Delta_g\) is a non-negative self-adjoint operator on \( L^2(\mathbb{C}, g) \) (whose inner product we denote by \( (f, h)_g = \int \bar{f}h g dz \)). It has a point spectrum consisting of eigenvalues \( \lambda_n \) and orthonormal eigenvectors \( e_n \) which we take so that \( \lambda_0 > 0 \) except for \( \lambda_0 = 0 \) with \( e_0 = 1/\|1\|_g \). We define the GFF \( X_g \) as the random distribution

\[ X_g(z) = \sqrt{2\pi} \sum_{n > 0} \frac{x_n}{\sqrt{\lambda_n}} e_n(z) \]  

(1.11)

where \( x_n \) are i.i.d. \( N(0, 1) \). The covariance \( G_g(z, z') := \mathbb{E} X_g(z) X_g(z') \) is easily computed: we have (for real \( f, h \))

\[ \frac{1}{2\pi} \mathbb{E} (X_g, (-\Delta_g)f)(X_g, h)_g = (f, h)_g - (e_0, f)_g (e_0, h)_g \]

which implies that

\[ -\Delta_g G_g(z, z') = 2\pi (g(z)^{-1} \delta(z - z') - (\int g(w) dw)^{-1}) \]  

(1.12)

Since \( (e_0, X_g)_g = 0 \) we have \( \int G_g(z, z') g(z) dz = 0 = \int G_g(z, z') g(z') dz' \) and we end up with

\[ G_g(z, z') = \mathbb{E} X_g(z) X_g(z') = \ln |z - z'|^{-1} - c_g(z) - c_g(z') + C_g \]  

(1.13)

where

\[ c_g(z) = m_g(\ln |z - \cdot|^2) = \ln |z| + O(1) = \frac{1}{4} \ln g(z) + O(1) \]  

(1.14)

and \( C_g = (1, g)^{-2} \int \ln |u - v|^{-1} g(u) g(v) du dv \). We used the notation for the average in \( g \)

\[ m_g(f) := \int_{\mathbb{C}} f(z) g(z) dz / \int_{\mathbb{C}} g(z) dz \]

For the round metric we have

\[ c_g = \frac{1}{4} \ln \hat{g} - \frac{1}{2} \ln 2, \quad C_g = -\frac{1}{2}. \]  

(1.15)

One should think about the \( X_g \) as we vary \( g \) as obtained from the same field \( X \) by \( X_g = X - m_g(X) \). Although there is no such \( X \) this makes the following fact evident. If \( g' \) is another conformal metric then

\[ X_{g'} \overset{\text{law}}{=} X_g - m_g(X_g). \]  

(1.16)

Moreover the GFF \( X_g \) transforms simply under Möbius trasformation \( \psi \) of \( \hat{\mathbb{C}} \):

\[ \mathbb{E} X_g(\psi(x)) X_g(\psi(y)) = \mathbb{E} X_{g\circ}(x) X_{g\circ}(y) \]
where the transformed metric is
\[ g_\psi := |\psi'|^2 g \circ \psi. \] (1.17)
Indeed (1.13) may be written as
\[ G_g(z, z') = \left( \int_C g(v)dv \right)^{-2} \int_{C^2} \frac{|z - u||z' - v|}{|z - z'||u - v|} g(u)g(v)dudv. \] (1.18)
A change of variables \( u = \psi(u'), v = \psi(v') \) and invariance of cross ratios under Möbius maps give the claim. We may state this as
\[ X_g \circ \psi \overset{law}{=} X_{g_\psi}. \] (1.19)
The random field \( X_g \) determines probability measure \( P_g \) on \( H^{-1}(\hat{C}, g) \) through its generating function
\[ E e^{i(X_g + f)g} = \int e^{i(X + f)g} P_g(dX). \]
We define the Massless Free Field as the Borel measure \( \nu_{MFF} \) on \( H^{-1}(\hat{C}, g) \) as the push-forward of the measure \( P_g \times dc \) on \( H^{-1}(\hat{C}, g) \times \mathbb{R} \) to \( H^{-1}(\hat{C}, g) \) under the map \( (X, c) \to X + c \). Concretely
\[ \int F(X)\nu_{MFF}(dX) = \int (E F(X_g + c)) dc \]
Note that \( \nu_{MFF} \) is not a probability measure: \( \int \nu_{MFF}(dX) = \infty \). Using (1.16) we see that this measure is independent of the chosen metric in the conformal class of \( \hat{g} \) since the random constant \( m_g(X_g) \) can be absorbed to a shift in \( c \).

We can now give a tentative definition of the measure in (1.7) by defining
\[ \nu_g(dX) = e^{-\frac{1}{2} \int \left( \int_{\hat{C}} R_q X + \pi \mu e^{\gamma \phi} \right) dz} \nu_{MFF}(dX). \] (1.20)
However, now we encounter the problem of renormalization as \( e^{\gamma X_g} \) is not defined since \( X_g \) is not defined pointwise.

### 1.4 Multiplicative Chaos
To define \( e^{\gamma X_g} \) we proceed by taking a mollified version of GFF
\[ X_{g,\epsilon} := \rho_\epsilon \ast X_g \] (1.21)
where \( \rho_\epsilon(z) = \epsilon^{-2} \rho(z/\epsilon) \) and \( \rho \) is a smooth rotation invariant mollifier. We have from (1.13)
\[ EX_{g,\epsilon}(z)^2 = \ln \epsilon^{-1} + a(\rho) - 2c_g(z) + C_g + o(1) \] (1.22)
uniformly on \( \mathbb{C} \) where the constant \( a(\rho) = \int \rho(z)\rho'(z) \ln |z - z'|^{-1}dzdz' \) depends on the regularization function \( \rho \). Hence
\[ e^{-\frac{1}{2} \int \left( \int_{\hat{C}} R_q X + \pi \mu e^{\gamma \phi} \right) dz} \nu_{MFF}(dX) \]
for a constant \( A \). Hence it is natural to renormalize by defining the random measure on \( \mathbb{C} \)
\[ M_g,\epsilon(dz) := e^{-\frac{1}{2} \int \left( \int_{\hat{C}} R_q X + \pi \mu e^{\gamma \phi} \right) dz} \nu_{MFF}(dX) \] (1.24)
In particular for the round metric we get
\[ M_{\hat{g},\epsilon}(dz) = (a + o(1))e^{\gamma X_{\hat{g},\epsilon} - \frac{1}{2} \int \left( \int_{\hat{C}} R_q X + \pi \mu e^{\gamma \phi} \right) dz} \hat{g}(z)dz \] (1.25)
with \( a = e^{\frac{1}{2} \gamma (2 - \frac{1}{2})} \).
Proposition 1.3. \[ M_{g,\gamma,\epsilon} \rightarrow M_{g,\gamma} \] weakly in probability as \( \epsilon \rightarrow 0 \). The limit is independent of the mollifier and nonzero if and only if \( \gamma < 2 \). It satisfies

(a) Let \( B_r \) a ball of radius \( r \) and \( p > 0 \). Then \( E M_{g,\gamma}(B_r)^p < \infty \) if and only if \( p < 4/\gamma^2 \) and then
\[
EM_{g,\gamma}(B_r)^p \leq C r^{\xi(p)}
\]
where \( \xi(p) = \gamma Qp - \frac{4}{\gamma^2}p^2 \).

(b) \( EM_{g,\gamma}(B_r)^p < \infty \) for all \( p > 0 \).

The limit is an example of Gaussian multiplicative chaos (see [7] for a review), a random multifractal measure on \( \mathbb{C} \). We will use the notation \( M_{g,\gamma}(f) = \int f(z) M_{g,\gamma}(dz) \). From (1.24) we have \( EM_{g,\gamma}(1) = \int_B g(z)dz \) where \( B \) is bounded. Hence \( M_{g,\gamma}(\mathbb{C}) < \infty \) a.s. We may now define (1.27) as
\[
\nu_g = e^{-\frac{1}{2} \int \frac{Q}{2} R_g dz - \mu e^{\gamma c} M_{g,\gamma}(1)} \nu_{MFF}.
\]
We will often use the notations
\[
e^{\gamma(X_g + \frac{Q}{2\ln g})}dz := M_{g,\gamma}(dz) \quad e^{\gamma \phi_g}dz := e^{\gamma c} M_{g,\gamma}(dz)
\]
but the reader should be aware that \( M_{g,\gamma} \) is not absolutely continuous w.r.t. the Lebesque measure.

The chaos measure has a nice transformation law under conformal maps:

Proposition 1.4. Let \( \psi \) be a Möbius map of \( \hat{\mathbb{C}} \). Then
\[
\int f e^{\gamma X_g} dz = \int f \circ \psi e^{\gamma X_g \circ \psi} |\psi'|^2 + \frac{\phi^2}{\pi} dz
\]
\[
\text{Proof. Making a change of variables we get}
\[
\int f e^{\gamma X_g} dz = \lim_{\epsilon \to 0} \int f e^{\frac{Q}{2\ln g}} e^{\gamma X_{g,\epsilon}} dz = \lim_{\epsilon \to 0} \int f \circ \psi e^{\frac{Q}{2\ln \psi'(g)}} e^{\gamma X_{g,\epsilon} \circ \psi} |\psi'|^2 dz.
\]
Suppose first \( \psi \) is the scaling \( \psi(z) = \lambda z \). Then
\[
X_{g,\epsilon}(\psi(z)) = \epsilon^2 \int \rho(|\lambda z - u|/\epsilon) X_g(u)du = (X_g \circ \psi)_{\epsilon/|\lambda|}(z)
\]
and the claim follows by setting \( \epsilon' = \epsilon/|\lambda| \). For the general case one notes that
\[
\lim_{\epsilon \to 0} \left( E X_{g,\epsilon}(\psi(z))X_{g,\epsilon}(\psi(u)) - E(X_g \circ \psi)_{\epsilon/|\psi'(z)|}(z)(X_g \circ \psi)_{\epsilon/|\psi'(u)|}(u) \right) = 0
\]
uniformly on compacts in \( \mathbb{C} \setminus \{ \psi^{-1}(\infty) \} \) and invokes uniqueness of the chaos measure under such condition.

Note that by (1.19) we get in particular
\[
\int_{\mathbb{C}} e^{\gamma \phi_g} dz \xrightarrow{\text{law}} \int_{\mathbb{C}} e^{\gamma \phi_\psi} dz.
\]
1.5 Weyl and Möbius invariance

We saw that $X$ is metric independent under $\nu_{MFF}$. For the Liouville field we have (compare with (1.9))

**Proposition 1.5.** Let $F \in L^1(\nu_g)$ and $g' = e^{c}g$. Then

$$\int F(\phi_{g'})d\nu_{g'} = e^{\frac{c_L}{4\pi}}\int F(\phi_g)d\nu_g$$

where $c_L = 1 + 6Q^2$.

**Proof.** By metric independence of $X$ we replace $c + X_{g'}$ by $c + X_g$ so that

$$\int F(\phi_{g'})d\nu_{g'} = \int F(\phi_g + \frac{Q}{4\pi} \varphi)e^{-\frac{1}{8\pi} \int (\Delta g' + \pi g e^{-(\varphi + \frac{Q}{2\pi} \varphi)})dz}d\nu_{MFF}.$$

Use $R_{g'}g' = -\Delta \ln g' = R_gg - \Delta \varphi$ and Gauss-Bonnet theorem $\int R_{g'}g' = 8\pi = \int R_gg$ to get

$$\int R_{g'}g' (c + X_g)dz = \int R_gg (c + X_g)dz - \int \Delta \varphi X_g dz.$$

Hence

$$\int F(\phi_{g'})d\nu_{g'} = \int F(\phi_g + \frac{Q}{4\pi} \varphi)e^{-\frac{1}{8\pi} \int (\Delta g + \pi g e^{-(\varphi + \frac{Q}{2\pi} \varphi)})dz}e^{\frac{Q}{8\pi} \int (X_g, \Delta_g \varphi) g \nu_{MFF}}.$$

The result then follows by a shift in the Gaussian integral $X_g \to X_g - \frac{Q}{4\pi} (\varphi - m_g \varphi)$ and $c \to c - m_g \varphi$. Indeed, by Girsanov theorem the law of $X_g - \frac{Q}{4\pi} G \Delta \varphi$ under the measure

$$e^{\frac{Q}{8\pi} (X_g, \Delta_g \varphi) g - \frac{Q^2}{32\pi^2} \mathbb{E}((X_g, \Delta_g \varphi)_g)^2}) \nu_g$$

equals the law of $X_g$ under $\nu_g$. We use here the notation $(G_g f)(z) := \int G_g(z, w) f(w) g(w) dw$. Since from (1.12) $\Delta_g G_g(x, y) = -2\pi \delta(x - y) + 2\pi g(x)$ we have $G_g \Delta_g \varphi = -2\pi (\varphi - m_g \varphi)$ so that

$$X_g - \frac{Q}{4\pi} G \Delta_g \varphi = X_g + \frac{Q}{2\pi} (\varphi - m_g \varphi)$$

and we end up with

$$\int F(\phi_{g'})d\nu_{g'} = e^{A(\varphi, g)} \int F(\phi_g)d\nu_g$$

where

$$A(\varphi, g) = \frac{Q^2}{32\pi^2} \mathbb{E}(X_g, \Delta_g \varphi)^g + \frac{Q^2}{8\pi} \int R_g g \varphi$$

The claim follows since $\mathbb{E}(X_g, \Delta_g \varphi)^g = (\varphi, \Delta_g G_g \Delta_g \varphi) g = -2\pi (\varphi, \Delta_g \varphi)_g = 8\pi \int |\partial_x \varphi|^2 dz$. \hfill \qed

**Remark 1.6.** The multiplicative factor is called the Weyl anomaly in physics literature and $c_L$ is the central charge of Liouville theory. Usually the Weyl transformation law in CFT has the factor $c_L$ and not $c_L - 1$. The reason for this discrepancy is that we have used the normalized law $\nu_g$ for the GFF instead of the unnormalized one as in (1.10). To get the unnormalized law one needs to multiply by the partition function $Z_g$ of the GFF $X_g$ which formally is given by

$$\det(-\Delta_g^{\frac{1}{2}})^{\frac{1}{2}} = \prod_{n > 0} \lambda_n^{\frac{1}{2}}$$

While this is not defined its variation under Weyl transformation can be defined and the upshot is that $c_L - 1$ gets replaced by $c_L$. 

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As a consequence of the Proposition we get Möbius transformation rule:

**Corollary 1.7.** Let \( \psi \) be a Möbius map of \( \hat{\mathbb{C}} \). Then

\[
\int F(\phi_g \circ \psi) d\nu_g = \int F(\phi_g - Q \ln |\psi'|) d\nu_g
\]

**Proof.** From (1.28) we get

\[
\int e^{\gamma \phi_g} dz = \int e^{\gamma(c + X_g \circ \psi + \hat{\Phi} \ln g \circ \psi)} dz
\]

For the curvature term we get

\[
\int g_R_g X_g = - \int \Delta \ln g X_g = - \int \Delta(\ln g \circ \psi) X_g \circ \psi = - \int \Delta(\ln g \circ \psi) X_g \circ \psi = \int g_R_g X_g \circ \psi
\]

where in the third step we used that \( \ln |\psi'| \) is harmonic. Recalling that \( X_g \circ \psi \stackrel{law}{=} X_{g\circ \psi} \) and combining with Proposition 1.5 we obtain

\[
\int F(\phi_g \circ \psi) d\nu_g = \int F(\phi_g - Q \ln |\psi'|) d\nu_g = e^{A(\varphi,g)} \int F(\phi_g - Q \ln |\psi'|) d\nu_g
\]

where \( \varphi = \ln |\psi|^2 + \ln g \circ \psi - \ln g \). Using the fact that \( \ln |\psi|^2 \) is harmonic we get \( A(\varphi,g) = A(\ln g \circ \psi - \ln g,g) \) and some algebra shows this vanishes.

### 1.6 Vertex operators

Since the Möbius group is non-compact the Corollary makes one suspect that the measure \( \nu_g \) does not have a finite mass. Let us consider its Laplace transform. By Proposition 1.5 we may work with the round metric \( \hat{g} \) where \( R_{\hat{g}} = 2 \). Then \[ \frac{1}{2\pi} \int R_g (c + X_g)^2 dz = 2c \] since \( \int X_g dz = 0 \). We get

\[
\int e^{(X,f)s} d\nu_{\hat{g}} = \int e^{((1,f)\hat{s} - 2Q)c} E_{\hat{g}}(X_{\hat{f}} s) e^{-\mu e^{c} \int C M_{\hat{g},\gamma}(dz) dc}
\]

Since \( \int M_{\hat{g},\gamma}(dz) < \infty \) a.s. the integral converges if and only if \( (1,f)_{\hat{g}} > 2Q \). In particular taking \( f = 0 \) we see that the total mass of \( \nu_{\hat{g}} \) is infinite. We can do the \( c \)-integral to get

\[
\int e^{(X,f)s} \nu_{\hat{g}}(dX) = \gamma^{-1} \mu^{-s_f} \Gamma(s_f) E_{\hat{g}}(X_{\hat{f}} s) \int C M_{\hat{g},\gamma}(dz)^{-s_f})
\]

where we denoted \( s_f = \gamma^{-1}((1,f)_{\hat{g}} - 2Q) \). We may further simplify this my a shift in the gaussian integral i.e. by a use of the Girsanov theorem:

\[
\int e^{(X,f)s} \nu_{\hat{g}}(dX) = \gamma^{-1} \mu^{-s_f} \Gamma(s_f) e^{\frac{1}{2}(f,G_{\hat{f}})s} E_{\hat{g}}( \int e^{\gamma(G_{\hat{f}})(z) M_{\hat{g},\gamma}(dz))^{-s_f}
\]

(1.30)

Note how the Laplace transform of the Gaussian measure is modified in the Liouville theory.

We define (regularized) vertex operators

\[
V_{\alpha,\epsilon}(z) := e^{\frac{\alpha^2}{2} e^{\alpha \Phi \circ \epsilon(z)}
\]

and consider their correlation function

\[
\langle \prod_{i=1}^{n} V_{\alpha,\epsilon}(z_i) \rangle_{g} := \lim_{\epsilon \to 0} \int \prod_{i=1}^{n} V_{\alpha,\epsilon}(z_i) d\nu_g.
\]

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Plugging in (1.30) \( f = \hat{g}^{-1} \sum_{i} \alpha_i p_i \star \delta_{z_i} \) and recalling (1.25) we get

\[
\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \rangle_{\hat{g}} = C(\alpha) \gamma^{-1} \mu^{-s} \Gamma(s)e^{\sum_{i<j} G_{\beta}(z_i,z_j) \alpha_i \alpha_j} e^{\sum_{i} \alpha_i \hat{\mu} \log \hat{g}(z_i) E_{g} \left( \int e^{\gamma \sum_{i} G_{\beta}(z_i,z_j) M_{\beta,\gamma}(dz)} \right)^{-s}}
\]

with \( s = \gamma^{-1}(\sum_{i} \alpha_i - 2Q) \) and we need the condition

\[
\sum_{i} \alpha_i > 2Q \tag{1.31}
\]

for convergence of the \( c \)-integral. Using the expression for the Green function (1.14), (1.18) we arrive at

\[
\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \rangle_{\hat{g}} = C'(\alpha) \prod_{j<k} \frac{1}{|z_j - z_k|^{\gamma_{\alpha_j \alpha_k}}} \mu^{-s} \gamma^{-1} \Gamma(s)E \left( \int F(z) M_{\hat{g},\gamma}(dz) \right)^{-s}
\]

where

\[
F(z) = \prod_{i} \frac{1}{|z - z_i|^{\gamma_{\alpha_i}}} \hat{g}(z)^{-\frac{1}{\gamma} \alpha_i}.
\]

Note that the expectation is finite due to Proposition 1.3 since

\[
M_{\hat{g},\gamma}(F)^{-s} \leq M_{\hat{g},\gamma}(F1_{B_r})^{-s} \leq (\inf_{z \in B_r} F(z))^{-s} M_{\hat{g},\gamma}(B_r)^{-s}.
\]

However, the expectation may be zero due to blowup of the integral. This is the content of

**Proposition 1.8.** \( 0 < \langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \rangle_{\hat{g}} \leq \infty \) if and only if \( \sum_{i} \alpha_i > 2Q \) and \( \alpha_i < Q \).

These bounds for \( \alpha_i \) are called Seiberg bounds. Note that they imply that we need at least three vertex operators to have a finite correlation function.

**Sketch of proof.** Let \( Z = M_{\hat{g},\gamma}(F) \) and write \( Z = \sum_{i=0}^{n} Z_i \) where in \( Z_i \) the integration is over a small ball around \( z_i \) if \( i > 0 \) and in the complement of all the balls if \( i = 0 \). Then, for \( 0 < p < 1 \)

\[
EZ^{-s} \geq r^{-s} P(Z < r) = r^{-s}(1 - P(Z > r)) \geq r^{-s}(1 - r^{-p} EZ^p) \geq r^{-s}(1 - r^{-p} \sum_{i} E Z_i^p)
\]

where in the last step we used subadditivity. By Proposition 1.3 \( EZ_i^p < \infty \) for \( p < 4/\gamma^2 \) since \( F < C \) on the support. Thus \( EZ^{-s} > 0 \) follows from \( EZ_i^p < \infty \) for some \( 0 < p < 1 \) and all \( i \). On the other hand,

\[
EZ^{-s} \leq EZ_i^{-s}
\]

so \( EZ^{-s} = 0 \) follows from \( EZ_i^{-s} = 0 \) for some \( i \).

For the first case, we use Kahane convexity (see [16]). In \( B_i \) we can bound

\[
\ln |z - u|_{+}^{-1} \leq G_{\beta}(z, u) + A
\]

so that comparing chaos with field \( X_+ \) to \( X_{\beta} + n \) where \( n \) is normal with variance \( A \) Kahane convexity gives us for \( 0 < p < 1 \)

\[
EZ_i^p \leq CE(\int_{\beta} |z|^{-\gamma_{\alpha_1} dM_+})^p
\]

Write \( \mathcal{D} = \bigcup_{n=0}^{\infty} A_n \) where \( A_n \) is annulus with radi \( 2^n \) and \( 2^{n+1} \). The field \( X_+ \) satisfies

\[
X_+(2^{-n} \cdot) \xrightarrow{law} X_+(\cdot) + x_n
\]

where the summands are independent and \( x_n \) is normal with variance \( \ln 2^n \). Hence

\[
I_n := \int_{A_n} |z|^{-\gamma_{\alpha_1} dM_+} \xrightarrow{law} e^{\gamma x_n - \frac{1}{2} \gamma^2 \ln 2^n 2^{(\gamma_{\alpha_1} - 2)n} I_0 = e^{\gamma x_n 2^{\gamma_{\alpha_1} - (Q)n} I_0}
\]
and
\[ EZ_i^p \leq C \sum_n E_{R_n} \leq C \sum_n 2^{(\gamma \alpha_i - \gamma Q)p} E_{I_0}^{\gamma p x_n} \leq C \sum_n 2^{(\gamma \alpha_i - \gamma Q)p + \frac{1}{2} \gamma^2 p^2 n} \]
which converges if \( p < \frac{2(Q - \alpha)}{\gamma} \).

For the second claim by Möbius invariance it suffices to suppose \( z_i = 0 \) and \( B_i = \mathbb{D} \). We use the following "radial" decomposition of the GFF (see [6, 4]). Let
\[ X_{\tilde{g}, r} := \frac{1}{2\pi} \int X_{\tilde{g}}(re^{i\theta}) d\theta. \]
Then
\[ X_{\tilde{g}}(z) = X_{\tilde{g}, |z|} + Y(z) \]
where the fields on the RHS are independent and \( Y \) has the covariance
\[ \mathbb{E}Y(re^{i\theta})Y(r'e^{i\theta'}) = \ln \frac{r \lor r'}{|re^{i\theta} - r'e^{i\theta'}|} \]
and the process \( B_t := X_{\tilde{g}, e^{-t}(0)} \)
is a Brownian motion starting at \( B_0 = X_{\tilde{g}, 1}(0) \), an independent gaussian variable of variance \( O(1) \). This leads to the following expression for the chaos
\[ \int_{\mathbb{D}} \frac{1}{|x|^{\gamma \alpha}} M_x(dx) = \int_0^\infty \int_0^{2\pi} e^{\gamma B_t - (Q - \alpha) t} \mu_Y(dt, d\theta). \]
where
\[ \mu_Y(dt, d\theta) = \text{const} \cdot e^{\gamma Y(e^{-r+i\theta}) - 2 \mathbb{E}Y(e^{-r+i\theta})^2} g(e^{-r}) d\theta dr \]
is a chaos measure independent of the process \( B_t \). Note that the drift term vanishes as \( \alpha \to Q \). Let
\[ Z_t := \int_0^t \int_0^{2\pi} e^{\gamma B_s} \mu_Y(ds, d\theta). \]
Recall that \( \mathbb{P}(\sup_{s \leq t} B_s < k) \leq kt^{-\frac{1}{4}} \). This leads us to expect that \( \mathbb{E}Z_t^{-\frac{s}{4}} = O(t^{-\frac{1}{2}}) \). Indeed, this is true:
\[ \lim_{t \to \infty} t^{\frac{s}{2}} \mathbb{E}Z_t^{-s} = 0 \]
extits and is nonzero. Therefore the correct normalization of the vertex operators for \( \alpha = Q \) is
\[ V_{Q, \epsilon} = (\ln \epsilon^{-1})^{\frac{s}{8}} \epsilon^{\frac{Q^2}{8}} e^{Q \phi_y}. \]
Then Proposition 1.8 holds also for \( \alpha_i \leq Q \).

1.7 KPZ Conjecture for Measure and Correlations

Let us now return the scaling limit of random triangulations. Define the probability measure
\[ d\mathbb{P}_{\mu, \gamma} := \prod_{i=1}^3 \frac{1}{V_\gamma(z_i)} \prod_{i=1}^3 V_\gamma(z_i) d\nu \]
We may then state the KPZ conjecture for the random measure \( \rho_{\mu, \gamma} \) obtained from scaling limit of triangulations. We conjecture that the law of \( \rho_{\mu, \gamma} \) equals the law of the measure \( e^{\gamma \phi_y} dz := e^{\gamma} M_{\mu, \gamma}(dz) \) under
\( P_{\mathbf{z},\gamma} \). Let us check that the law for the total volume \( A = \int_{C} e^{\gamma \phi_{g}} dz \) matches. By a simple change of variables in the \( c \)-integration \( e^{\gamma c} M_{\gamma, \gamma}(C) = A \) we obtain

\[
E F(A) = \frac{\mu^{s}}{\Gamma(s)} \int_{0}^{\infty} F(y) y^{s-1} e^{-\mu y} dy.
\]

where \( s = (3\gamma - 2Q)/\gamma = 2 - 4/\gamma^{2} \) i.e. under \( P_{\mathbf{z},\gamma} \) the law of \( A \) is \( \Gamma(2 - 4/\gamma^{2}, \mu) \). This agrees with the result in random surfaces. Note that the conformal weight \( \Delta_{\gamma} = 1 \) (see next Section) so the vertex operator \( e^{\gamma \phi_{g}} \) transforms under conformal maps as a density.

For the Ising model random field (1.3) the KPZ conjecture says that its correlation functions converge

\[
\lim_{\epsilon \to 0} E \Phi_{\epsilon}(u_{1}) \ldots \Phi_{\epsilon}(u_{n}) = E \sigma(u_{1}) \ldots \sigma(u_{n}) E_{\mathbf{z},\gamma} V_{\alpha}(u_{1}) \ldots V_{\alpha}(u_{n})
\]

where \( E \sigma(u_{1}) \ldots \sigma(u_{n}) \) are the correlation functions of the Ising model in the scaling limit on \( \mathcal{C} \) and \( \alpha \) is determined from the requirement \( \frac{1}{16} + \Delta_{\alpha} = 1 \) which means that \( \sigma(z) e^{\alpha \phi_{g}(z)} \) transforms under conformal maps as a density.

### 1.8 Conformal Ward Identities

So far we have motivated the Liouville model through its conjectural relationship to scaling limits of random triangulations. However, the Liouville model is also an interesting Conformal Field Theory by itself. This way of looking we view the vertex operators as (Euclidean) quantum fields.

First, using the M"{o}bius invariance (Corollary 1.7) of \( \nu_{g} \) and taking care with the transformation of the \( \epsilon \) in the vertex operator one gets

\[
\langle \prod_{i=1}^{n} V_{\alpha_{i}}(\psi(z_{i})) \rangle_{g} = \lim_{\epsilon \to 0} \langle \prod_{i=1}^{n} \epsilon^{\frac{\gamma}{2}} e^{\frac{\epsilon}{2} \phi_{g}} \psi(\psi(z_{i})) \rangle_{g} = \lim_{\epsilon \to 0} \langle \prod_{i=1}^{n} (\epsilon \psi'(z_{i}))^{\frac{\gamma}{2}} e^{\frac{\epsilon}{2} \phi_{g}} \psi(\psi(z_{i})) \rangle_{g} = \prod_{i} |\psi'(z_{i})|^{-2\Delta_{\alpha_{i}}} \prod_{i=1}^{n} V_{\alpha_{i}}(z_{i})
\]

where \( \Delta_{\alpha} = \frac{1}{2}(Q - \frac{1}{2}) \). In CFT parlance, \( V_{\alpha} \) is a primary field with conformal weight \( \Delta_{\alpha} \).

Second, the Liouville model has also local conformal symmetry. In CFT this derives from the energy-momentum tensor which encodes the variations of the theory with respect to the background metric. In classical field theory this is defined as follows. Let \( S(g, X) \) be an action functional where \( g = g_{\alpha \beta} dx^{\alpha} \otimes dx^{\beta} \) (we use summation convention of repeated indices) is a smooth Riemannian metric. In Liouville case

\[
S(g, X) = \int (g^{\alpha \beta} \partial_{\alpha} X \partial_{\beta} X + Q R_{g} X + \mu e^{\gamma X}) \sqrt{\det g} dx
\]

where \( g^{\alpha \beta} \) is the inverse matrix \( g^{\alpha \beta} g_{\beta \gamma} = \delta_{\alpha}^{\gamma} \). Then the EM tensor \( T_{\alpha \beta}(x) \) is defined by

\[
\partial_{\epsilon} |_{\epsilon=0} S(g_{\epsilon}, X) = \int T_{\alpha \beta}(x) f^{\alpha \beta}(x) \sqrt{\det g} dx
\]

where \( g^{\alpha \beta}_{\epsilon} = g^{\alpha \beta} + \epsilon f^{\alpha \beta} \). For Liouville model one finds that the only interesting component of \( T \) in complex coordinates is \( T_{zz} := T(z) \) which is classically analytic \( \partial_{z} T = 0 \) if \( X \) satisfies the Euler-Lagrange equations. In quantum theory one defines in the same way

\[
\frac{d}{d\epsilon} |_{\epsilon=0} \prod_{l} V_{\alpha_{l}}(z_{l}) g_{\epsilon} := \frac{1}{4\pi} \int f^{\alpha \beta}(z) \langle T_{\alpha \beta}(z) \prod_{l} V_{\alpha_{l}}(z_{l}) \rangle_{g} dz.
\]

for \( f \) a smooth function with support in \( \mathbb{C} \setminus \cup_{l} z_{l} \).
A simple formal computation then yields the following heuristic formula

\[ T(z) = Q\partial^2_\gamma \phi(z) - ((\partial_\gamma \phi(z))^2 - \mathbb{E}(\partial_\gamma X_\gamma(z))^2) \]  

(1.33)

where \( \phi \) is the Liouville field.

\( T(z) \) encodes local conformal symmetries through the Conformal Ward Identities. The first Ward identity says the correlation function is meromorphic in the argument of \( T(z) \) with prescribed singularities:

\[ \langle T(z) \prod_l V_{\alpha_l}(z_l) \rangle = \frac{\Delta_{\alpha_k}}{(z-z_k)^2} \langle \prod_l V_{\alpha_l}(z_l) \rangle - \sum_k \frac{1}{z-z_k} \partial_{z_k} \langle \prod_l V_{\alpha_l}(z_l) \rangle \]  

(1.34)

and the second identity controls the singularity when two \( T \)-insertions come close

\[ \langle T(z)T(z') \prod_l V_{\alpha_l}(z_l) \rangle = \frac{\Phi_\gamma}{(z-z')^2} \langle T(z) \prod_l V_{\alpha_l}(z_l) \rangle + \frac{2}{(z-z')^2} \langle T(z') \prod_l V_{\alpha_l}(z_l) \rangle + \frac{1}{z-z'} \partial_{z'} \langle T(z') \prod_l V_{\alpha_l}(z_l) \rangle + \ldots \]  

(1.35)

where the dots refer to terms that are bounded as \( z \to z' \). To prove these identities we need to define what we mean by the LHS. Let \( \phi_\epsilon \) be a regularization of the Liouville field. Set

\[ T_\epsilon(z) = Q\partial^2_\gamma \phi_\epsilon(z) - ((\partial_\gamma \phi_\epsilon(z))^2 - \mathbb{E}(\partial_\gamma X_\gamma(z))^2) \]  

(1.36)

and define \( \langle T(z) \prod_l V_{\alpha_l}(z_l) \rangle \) as the limit of \( \langle T_\epsilon(z) \prod_l V_{\alpha_l}(z_l) \rangle \) and similarly for the two \( T \) insertions. Let us see how the first Ward identity follows by formal calculation before commenting on the mathematical problems in actually making it rigorous.

The basic formula is the following identity:

\[ \langle \partial_\gamma \phi(z) \prod_k V_{\alpha_k}(z_k) \rangle = -\frac{1}{2} \sum_i \alpha_i \frac{1}{z-z_i} \langle \prod_k V_{\alpha_k}(z_k) \rangle + \frac{\mu \gamma}{2} \int \frac{1}{z-y} \langle \gamma \partial_\gamma V_\gamma(y) \prod_k V_{\alpha_k}(z_k) \rangle dy \]  

(1.37)

To prove this first note that by integration by parts in the Gaussian measure:

\[ \langle X_\gamma(z) \prod_k V_{\alpha_k}(z_k) \rangle = \sum_i \alpha_i G_\gamma(z, z_i) \langle \prod_k V_{\alpha_k}(z_k) \rangle - \mu \gamma \int G_\gamma(z, y) \langle \gamma \partial_\gamma V_\gamma(y) \prod_k V_{\alpha_k}(z_k) \rangle dy. \]

Recalling the definition of the Liouville field (1.6) we then get

\[ \langle \partial_\gamma \phi(z) \prod_k V_{\alpha_k}(z_k) \rangle = -\frac{1}{2} \sum_i \alpha_i \frac{1}{z-z_i} \langle \prod_k V_{\alpha_k}(z_k) \rangle + \frac{\mu \gamma}{2} \int \frac{1}{z-y} \langle \gamma \partial_\gamma V_\gamma(y) \prod_k V_{\alpha_k}(z_k) \rangle dy \]

\[ -\frac{1}{2} \partial_\gamma \ln \hat{g}(z) (2Q - \sum_i \alpha_i \prod_k V_{\alpha_k}(z_k)) + \mu \gamma \int \langle \gamma \partial_\gamma V_\gamma(y) \prod_k V_{\alpha_k}(z_k) \rangle dy \]

The metric dependent term actually vanishes due to the following identity

**Lemma 1.9.**  (KPZ-identity)

\[ \mu \gamma \int \langle \gamma \partial_\gamma V_\gamma(y) \prod_k V_{\alpha_k}(z_k) \rangle = (\sum_i \alpha_i - 2Q) \langle \prod_k V_{\alpha_k}(z_k) \rangle. \]

(1.38)

**Proof.** By a simple change of variables \( \gamma^{-1} \ln \mu + c = c' \), we get

\[ \langle \prod_k V_{\alpha_k}(z_k) \rangle = \int_{\mathbb{R}} e^{-2Qc'} \mathbb{E}[\prod_k V_{\alpha_k}(z_k) e^{-\mu \int_c c' \gamma \partial_\gamma V(y) dy}] dc' \]

\[ = \mu^{-\frac{\sum_i \alpha_i - 2Q}{2}} \int_{\mathbb{R}} e^{-2Qc'} \mathbb{E}[\prod_k V_{\alpha_k}(z_k) e^{-\int_c c' \gamma \partial_\gamma V(y) dy}] dc'. \]
The identity follows by differentiating in $\mu$.

Using the integration by parts formula (1.37) we get

$$
\langle \partial^2_x \phi(z) \rangle \prod_l V_{\alpha_l}(z_l) = \frac{1}{2} \sum_i \alpha_i \frac{1}{(z-z_i)^2} \langle \prod_l V_{\alpha_k}(z_k) \rangle - \frac{1}{2} \mu \gamma \int \frac{1}{(z-y)^2} \langle V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k) \rangle dy
$$

and

$$
((\partial_z \phi(z))^2 - E(\partial_z X(z))^2) \prod_l V_{\alpha_l}(z_l) = \frac{1}{4} \sum_{j,k} \frac{\alpha_j \alpha_k}{(z-z_k)(z-z_j)} \langle \prod_l V_{\alpha_k}(z_k) \rangle
$$

$$
- \frac{1}{4} \mu \gamma \sum_k \alpha_k \frac{1}{z-z_k} \int \frac{1}{z-y} \langle V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k) \rangle dy - \frac{1}{4} \mu \gamma^2 \int \frac{1}{(z-y)^2} \langle V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k) \rangle dy
$$

$$
+ \frac{1}{4} \mu^2 \gamma^2 \int \frac{1}{z-y} \frac{1}{z-x} \langle V_{\gamma}(y) V_{\gamma}(x) \prod_l V_{\alpha_k}(z_k) \rangle dy dx
$$

Combining we get

$$
\langle T(z) \prod l V_{\alpha_l}(z_l) \rangle = \left( \frac{Q}{4} \sum_i \alpha_i \frac{1}{(z-z_i)^2} - \frac{1}{4} \sum_{j,k} \frac{\alpha_j \alpha_k}{(z-z_k)(z-z_j)} \right) \langle \prod_l V_{\alpha_k}(z_k) \rangle
$$

$$
+ \frac{1}{2} \mu \gamma \sum_k \alpha_k \frac{1}{z-z_k} \int \frac{1}{z-y} \langle V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k) \rangle dy
$$

$$
- \mu \int \frac{1}{(z-y)^2} \langle V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k) \rangle dy - \frac{1}{4} \mu^2 \gamma^2 \int \frac{1}{z-y} \frac{1}{z-x} \langle V_{\gamma}(y) V_{\gamma}(x) \prod_l V_{\alpha_k}(z_k) \rangle dy dx
$$

Integrating by parts we have

$$
- \mu \int \frac{1}{(z-y)^2} \langle V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k) \rangle dy = - \mu \int \partial_y \frac{1}{z-y} \langle V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k) \rangle dy = \mu \int \frac{1}{z-y} \partial_y (V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k)) dy
$$

$$
= - \frac{1}{2} \gamma \mu \sum_i \alpha_i \int \frac{1}{z-y} \frac{1}{z-z_i} \langle V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k) \rangle dy + \frac{1}{2} \mu \gamma^2 \int \frac{1}{z-y} \frac{1}{z-x} \langle V_{\gamma}(y) V_{\gamma}(x) \prod_l V_{\alpha_k}(z_k) \rangle dy dx
$$

$$
= - \frac{1}{2} \gamma \mu \sum_j \alpha_j \frac{1}{z-z_j} \int \frac{1}{z-y} \langle V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k) \rangle dy + \mu \int \frac{1}{z-y} \frac{1}{z-x} \langle V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k) \rangle dy
$$

$$
+ \frac{1}{4} \mu^2 \gamma^2 \int \frac{1}{z-y} \frac{1}{z-x} \langle V_{\gamma}(y) V_{\gamma}(x) \prod_l V_{\alpha_k}(z_k) \rangle dy dx
$$

so that

$$
\langle T(z) \prod l V_{\alpha_l}(z_l) \rangle = \left( \frac{Q}{4} \sum_i \alpha_i \frac{1}{(z-z_i)^2} - \frac{1}{4} \sum_{j,k} \frac{\alpha_j \alpha_k}{(z-z_k)(z-z_j)} \right) \langle \prod_l V_{\alpha_k}(z_k) \rangle
$$

$$
- \frac{1}{2} \mu \gamma \sum_k \alpha_k \frac{1}{z-z_k} \int \frac{1}{y-z_k} \langle V_{\gamma}(y) \prod_l V_{\alpha_k}(z_k) \rangle dy
$$

On the other hand using (1.37)

$$
\partial_z \langle \prod_k V_{\alpha_k}(z_k) \rangle = \alpha_i \langle \partial_z \phi(z_i) \prod_k V_{\alpha_k}(z_k) \rangle
$$

$$
= - \frac{1}{2} \sum_{j \neq i} \frac{\alpha_i \alpha_j}{z_i-z_j} \langle \prod_k V_{\alpha_k}(z_k) \rangle + \frac{1}{2} \alpha_i \mu \gamma \int \frac{1}{z_i-y} \langle V_{\gamma}(y) \prod_k V_{\alpha_k}(z_k) \rangle dy
$$

(1.40)
so that the 1st Ward identity follows. Let us make some remarks regarding this calculation.

First, for the proof one needs to work with regularized correlations. Then some of the identities used in this calculation are not exact. Worse, some of the resulting integrals are only conditionally convergent. Using multiplicative chaos techniques one can study the divergence of the vertex operator correlations as two or more points come together. For instance for two points one gets

\[ \langle V_{\gamma}(y) \prod_k V_{\alpha_k}(z_k) \rangle \leq C |y - z_i|^{-2+\delta} \]

with \( \delta > 0 \). Hence this singularity is integrable (as is also evident from Lemma 1.38). However above we need to control \( (y - z_i)^{-1} \) times this and the result is not absolutely integrable. The clue what to do is in equation (1.39). The LHS is the Beltrami transform of the correlator computed at \( z \neq z_i \). This is pointwise defined provided the correlator is Hölder continuous which can be shown by multiplicative chaos techniques. The first and third terms are absolutely convergent. As a result this identity relates the potentially divergent integral to finite ones. For the proof one needs to work with a regularized version of the identity. As an upshot one obtains using (1.41) that the correlation functions are \( C^1 \). For the second Ward identity one needs to control singular integrals such as

\[ \int \frac{1}{(z - y)^3} \langle V_{\gamma}(y) \prod_k V_{\alpha_k}(z_k) \rangle dy \]

which are related by identities to less singular expressions. Upshot is that the correlations are \( C^2 \).

2 Quantum Liouville Theory

The Liouville model gives rise to Quantum Field Theory. This means in particular that there is a canonical construction of a Hilbert space \( \mathcal{H} \) and a representation of the symmetries of the theory as operators acting on \( \mathcal{H} \). This reconstruction of quantum fields is very general and is based on a peculiar positivity property of the random field, the reflection positivity (or Osterwalder-Schrader positivity, [13]).

2.1 Liouville functional

Recall that in the round metric \( \hat{g} \) the Liouville field is given as

\[ \phi = c + X_{\hat{g}} + \frac{Q}{2} \log \hat{g}. \]

and the Liouville "expectation" is given by

\[ \int F(\phi) d\nu_{\hat{g}} = \int dc \ e^{-2Qc} E e^{-\mu \int c e^{\gamma \phi} dz} F(\phi). \] (2.1)

It will be convenient to make a change of variables from the zero average field \( X_{\hat{g}} \) to one that has zero average on \( \partial \mathbb{D} \). Let

\[ m_{\partial \mathbb{D}}(X_{\hat{g}}) := \frac{1}{2\pi} \int X_{\hat{g}}(e^{i\theta}) d\theta. \]

Making a shift in the \( c \)-integral we get

\[ \int F(\phi) d\nu_{\hat{g}} = \int dc e^{-2Qc} E e^{2Qm_{\partial \mathbb{D}}(X_{\hat{g}})} F(c + X_{\hat{g}} - m_{\partial \mathbb{D}}(X_{\hat{g}}) + Q/2 \ln \hat{g}) e^{-\mu \int c e^{\gamma(c + X_{\hat{g}} - m_{\partial \mathbb{D}}(X_{\hat{g}}) + Q/2 \ln \hat{g})} dz} \]

By the Girsanov theorem \( X_{\hat{g}} \) under \( e^{2Qm_{\partial \mathbb{D}}(X_{\hat{g}})} e^{Q^2 E m_{\partial \mathbb{D}}(X_{\hat{g}})} P_{\hat{g}} \) equals in law \( X_{\hat{g}} + 2Q \frac{1}{2\pi} \int G_{\hat{g}}(z, e^{i\theta}) d\theta \) under \( P_{\hat{g}} \). We have

\[ \frac{1}{2\pi} \int G_{\hat{g}}(z, e^{i\theta}) d\theta = k(z) - \frac{1}{4} \ln \hat{g}(z) + C_{\hat{g}}. \]
where we have set
\[ k(x) = \ln \frac{1}{|x|^{1\{|x| \geq 1\}}} \]
and \( E[m_{\partial \Omega}(X_{\tilde{g}})^2] = C_{\tilde{g}} \). Hence we get the following expression after a further shift of \( c \) by \(-2QC_{\tilde{g}}\)
\[
\int F(\phi) d\nu_{\phi} = e^{6C_{\tilde{g}}Q^2} \int dc e^{-2Qc^{2}} F(c + \Phi + 2Qk) e^{-\mu \int e^{\gamma \phi(z)} dz} \]
(2.2)
We have defined the field
\[ \Phi = X_{\tilde{g}} - m_{\partial \Omega}(X_{\tilde{g}}). \]
We have arrived to a new representation of the Liouville field as
\[ \phi = c + \Phi + 2Qk \]
(2.3)
where \( \Phi \) is a gaussian field with covariance
\[
G(z, z') := E[\Phi(x)\Phi(y)] = \ln \frac{1}{|z - z'|} - k(z) - k(z').
\]
We will construct the quantum theory starting with the linear functional
\[
\langle F \rangle = \int dc e^{-2Qc^{2}} E[F(\phi)] e^{-\mu \int e^{\gamma \phi} dz}
\]
(2.4)
with \( \phi \) given by (2.3).

### 2.2 Osterwalder-Schrader positivity

For \( A \subset \mathbb{C} \) let \( \mathcal{F}_A \) be the \( \sigma \)-algebra generated by \( \int_{\mathbb{C}} \phi f \) with \( \text{supp} f \subset A \). The Hilbert space is constructed out of \( \mathcal{F}_\Omega \). Let \( \theta : \mathbb{C} \to \mathbb{C} \) be the reflection from the unit circle \( \theta(z) = 1/\bar{z} \). Define \( \Theta : \mathcal{F}_\Omega \to \mathcal{F}_{\Omega^c} \) by
\[
\Theta F(\phi) := \overline{F(\theta \phi - 2Q \ln |z|)}
\]
(2.5)
where \((\theta \phi)(z) := \phi(\theta z) = \phi(1/\bar{z})\). Consider now the following sesquilinear form
\[
(F, G) := \langle \Theta FG \rangle.
\]
(2.6)
for \( F, G \in \mathcal{F}_\Omega \). OS-positivity is the following statement:

**Proposition 2.1.** The form (2.15) is positive semidefinite:
\[
\langle \Theta FF \rangle \geq 0.
\]
(2.7)

The main ingredient in the proof is the corresponding statement for MFF i.e. \( \mu = 0 \) case. Let
\[
\langle F \rangle_0 := \int e^{-2Qc^{2}} E[F(\phi)] dc = \int e^{-2Qc^{2}} E[F(c + \Phi + 2Qk)] dc.
\]
We will decompose \( \Phi \) to independent fields on \( \Omega \), \( \Omega^c \) and \( S^1 = \partial \Omega \). For this let \( \Phi_{\Omega}(z) \) be the Dirichlet GFF on \( \Omega \), i.e.
\[
G_{\Omega}(z, z') := E[\Phi_{\Omega}(z) \Phi_{\Omega}(z')] = \ln \frac{|1 - z\bar{z}'|}{|z - z'|}.
\]
(2.8)
and \( \Phi_{\Omega^c}(z) \) the Dirichlet GFF on \( \Omega^c \) i.e.
\[
\Phi_{\Omega^c} \overset{\text{law}}{=} \theta \Phi_{\Omega}.
\]
Next note that $\varphi := \Phi |_{\partial \mathbb{D}}$ is the GFF on circle (with zero average) i.e. concretely

$$\varphi \overset{\text{law}}{=} \sum_{n \neq 0} \varphi_n e^{in\theta}$$

where

$$\varphi_n = \frac{1}{2\sqrt{n}}(\alpha_n + i\beta_n) \quad n > 0, \quad \varphi_{-n} = \overline{\varphi_n}$$

with $\alpha_n, \beta_n$ i.i.d. $N(0, 1)$.

Let $P\varphi$ be the Harmonic extension of $\varphi$ defined on $\mathbb{D}$ by

$$P\varphi(z) = \sum_{n>0} (\varphi_n \overline{z^n} + \varphi_{-n} z^n) \quad (2.10)$$

On $\mathbb{D}^c$, $P\varphi$ is given by

$$P\varphi(z) = (\theta P\varphi)(z), \quad z \in \mathbb{D}^c. \quad (2.11)$$

The we have

**Proposition 2.2.** We may decompose as sum of independent fields:

$$\Phi \overset{\text{law}}{=} \Phi_{\mathbb{D}} + P\varphi + \Phi_{\mathbb{D}^c}.$$ 

*Proof. We have*

$$E\varphi_n \varphi_m = \frac{1}{2|n|} \delta_{n,-m}$$

so that for $z, u \in \mathbb{D}$

$$E P\varphi(z) P\varphi(u) = \frac{1}{2} \sum_{n>0} ((z\overline{u})^n + (\overline{z}u)^n) = -\ln |1 - z\overline{u}|$$

and then for $z \in \mathbb{D}$, $u \in \mathbb{D}^c$

$$E P\varphi(z) P\varphi(u) = -\ln |1 - z/u| = \ln |z - u|^{-1} - \ln |u|^{-1}$$

It is then straightforward to check the equality of covariances. \qed

Using this decomposition we then get for $F, G \in \mathcal{F}_D$:

$$\langle \Theta F \rangle_0 = \int e^{-2Qc} E(\Theta F)(c + \Phi_{\mathbb{D}} + P\varphi + 2Qk)G(c + \Phi_{\mathbb{D}} + P\varphi)dc$$

$$= \int e^{-2Qc} E_F(c + \Phi_{\mathbb{D}} + P\varphi)^2 E_D G(c + \Phi_{\mathbb{D}} + P\varphi)dc. \quad (2.12)$$

Hence

$$\langle \Theta F \rangle_0 \geq 0. \quad (2.13)$$

The Proposition 2.1 follows then from
Lemma 2.3. Let $I(X) := \int_D e^{\gamma \phi(z)} dz$. Then

$$\int_C e^{\gamma \phi} dz = I(\phi) + (\Theta I)(\phi)$$  \hspace{1cm} (2.14)

Proof. In the same way as Proposition 1.4 one proves the change of variables formula

$$(e^{\gamma \theta \phi})(z) = |z|^{\gamma^2} (e^{\gamma \phi})(1/\bar{z})$$

and thus

$$(\Theta I)(\phi) = \int_D |z|^{\gamma^2} (e^{\gamma \phi}(1/\bar{z})) e^{-2\gamma Q \ln |z|} dz = \int_D |z|^{-4} (e^{\gamma \phi}(1/\bar{z})) dz = \int_D e^{\gamma \phi(z)} dz$$

2.3 Hilbert space

The Liouville Hilbert space $\mathcal{H}$ is defined as the completion of $\mathcal{F}_D/\mathcal{N}$ where

$$\mathcal{N} = \{F \in \mathcal{F}_D | (F, F) = 0\}.$$

Then

$$(F, G) = (UF, UG)_{L^2(d\mathcal{P}(\varphi)dc)}$$  \hspace{1cm} (2.15)

where

$$UF := e^{-QcE_D} e^{-\mu \int_\phi e^{\gamma X} dz} F$$  \hspace{1cm} (2.16)

Thus

$$U : \mathcal{H} \rightarrow L^2(D \mathcal{P}(\varphi)dc)$$

is an isometry and we may identify $\mathcal{H}$ with a subspace of $L^2(d\mathcal{P}(\varphi)dc)$.  

2.4 Q-Free Field

Let us consider the $\mu = 0$ case in more detail. This is the Free field with "background charge" $iQ$. We may realize $L^2(d\mathcal{P}(\varphi)dc)$ as

$$d\mathcal{P}(\varphi) = \prod_{n>0} \frac{1}{2\pi} e^{-\frac{1}{2}(\alpha_n^2 + \beta_n^2)} d\alpha_n d\beta_n.$$

We have then

Proposition 2.4. The map

$$(UF)(c, \varphi) = e^{-QcE_D} F(c + \Phi_D + P_{\varphi}).$$

extends to a unitary map $U : \mathcal{H} \rightarrow L^2(d\mathcal{P}(\varphi)dc)$.

Proof. By (2.12) $U$ is an isometry from $\mathcal{F}_D$ to a subspace of $L^2(d\mathcal{P}(\varphi)dc)$. To show $U$ is onto note that $c = \frac{1}{2\pi} \int \phi(e^{i\theta}) d\theta$ and consider $F$ of the form

$$F(\phi) = \psi(\int \phi(e^{i\theta}) \frac{d\theta}{2\pi}) e^{(\phi, f)} = \psi(c) e^{(1, f)} e^{(\Phi, f)}$$  \hspace{1cm} (2.17)

where $f \in C_0^\infty(\mathbb{C})$ and we use in this Chapter the notation $(\phi, f) := \int_D \phi f$. Then

$$(UF)(c, \varphi) = \psi(c) e^{-Qc} e^{(1, f)} e^{(\Phi, f)} + \frac{1}{2} (f, G_{\varphi} f) e^{(P_{\varphi}, f)}$$  \hspace{1cm} (2.18)
From (2.10) we get

\[(P \phi, f) = \sum_{n>0} (\phi_n \int_D \bar{z}^n f + \phi_{-n} \int_D z^n f).\]

so that by (2.2)

\[(P \phi, f) = \sum_{n \neq 0} \phi_n \pi(f)_{-n} := (\phi; \pi(f)) \quad (2.19)\]

where we defined for \(n > 0:\)

\[\pi(f)_n = \int_D \bar{z}^n f, \quad \pi(f)_{-n} = \int_D z^n f \quad (2.20)\]

and we denote the scalar product in \(L^2(\partial D)\) also by \((\cdot, \cdot)\). Thus

\[UF = \psi(c)e^{((1, f) - Q)c}e^{\frac{1}{2}(f, Gz)f}e^{(\phi, \pi(f))}.\]

The linear span of such functions is dense in \(L^2(d\mu_\phi)\).

In particular, for the vertex operator

\[V_\alpha(z) = e^{\alpha \phi(z) - \frac{1}{2} \alpha^2 E\Phi(z)^2}\]

we get

\[UV_\alpha(z) = e^{(\alpha - Q)c} : e^{\alpha(P\phi)(z)}, \quad (2.21)\]

where

\[: e^{\alpha(P\phi)(z)} : = e^{\alpha(P\phi)(z) - \frac{1}{2} \alpha^2 E(P\phi)(z)^2} = (1 - |z|^2)^\frac{1}{2} e^{\alpha(P\phi)(z)}\]

### 2.5 Hamiltonian of Q-free field

From now on we use for \(\phi\) the representation

\[\phi = c + \Phi + 2Qk.\]

For \(q \in \mathbb{C}\) define dilation

\[(s_q f)(z) = f(qz)\]

and

\[S_q F(\phi) = F(s_q \phi + Q \log |q|).\]

Hence \(S_q : \mathcal{F}_D \to \mathcal{F}_D\) if \(|q| \leq 1\). We have

**Proposition 2.5.** The adjoint of \(S_q\) is \(S^*_q = S^*_q\) i.e.

\[(F, S_q G) = (S_q F, G).\]

**Proof.** By Möbius invariance of the Liouville expectation, Corollary 1.7, we get

\[(F, S_q G) = \langle (\Theta F)(S_q G)(\phi) \rangle = \langle (\Theta F)(\phi)G(s_q \phi + Q \log |q|) \rangle = \langle (\Theta F)(s_q^{-1} \phi - Q \log |q|)G(\phi) \rangle\]

By the definition of \(\Theta\), (2.5) we may write this as

\[(F, S_q G) = \langle \bar{G}(\theta(s_q^{-1} \phi - 2Qk) - Q \log |q|)G(\phi) \rangle \]

\[= \langle \bar{G}(s_q \theta(\phi - 2Qs_q k) - Q \log |q|)G(\phi) \rangle \]

\[= \langle (\Theta S_q F)(\phi - 2Qs_q k + 2Qk - 2Q \log |q|)G(\phi) \rangle \]

\[= (S_q F, G) \quad (2.22)\]

since \(\Theta S_q F\) is supported in \(q^{-1} \mathbb{D}^c\) and \(1_{q^{-1} \mathbb{D}^c}(k - s_q k) = \log |q|\). \qed
$S_q$ gives rise two semi groups. Taking $q = e^{-t}$ we define

$$US_{e^{-t}}F = e^{-tH}UF.$$  

This is a contraction semigroup with generator $H \geq 0$, the Hamiltonian of the GFF. Taking $q = e^{i\alpha}$ we define

$$US_{e^{i\alpha}}F = e^{i\alpha PUF},$$

where $P$ is the momentum operator of the GFF. It is a generator of an unitary group. To compute them explicitly we use the complex coordinates $\{\varphi_n\}_{n \in \mathbb{Z}}$ in $L^2(\mathbb{P}(d\varphi))$ given in the representation (2.2) and define for $n > 0$:

$$a_n = \frac{1}{2} \frac{\partial}{\partial \varphi_{-n}}, \quad a_n = n\varphi_{-n} - \frac{1}{2} \frac{\partial}{\partial \varphi_n},$$

and

$$\tilde{a}_n = \frac{1}{2} \frac{\partial}{\partial \bar{\varphi}_n}, \quad \tilde{a}_n = n\varphi_n - \frac{1}{2} \frac{\partial}{\partial \bar{\varphi}_{-n}}.$$ 

$a_n$ and $\tilde{a}_n$ are $(n > 0)$ called the annihilation operators (for analytic and anti analytic modes) and $a_{-n}$ and $\tilde{a}_{-n}$ the creation operators. They are densely defined closable operators in $L^2(\mathbb{P}(d\varphi))$ and their closures satisfy $a_n^* = a_{-n}$, $\tilde{a}_n^* = \tilde{a}_{-n}$. Furthermore we have $a_n1 = 0$ and $\tilde{a}_n1 = 0$ for $n > 0$ and we have the commutation relations

$$[a_n, a_m] = n \frac{1}{2} \delta_{n,-m} = [\tilde{a}_n, \tilde{a}_m], \quad [a_n, \tilde{a}_m] = 0.$$ 

**Proposition 2.6.** We have

$$H = \frac{1}{2} \left( -\frac{d^2}{dc^2} + Q^2 \right) + 2 \sum_{n>0} (a_{-n}a_n + \tilde{a}_{-n}\tilde{a}_n)$$

$$P = 2 \sum_{n>0} (a_{-n}a_n - \tilde{a}_{-n}\tilde{a}_n)$$

**Proof.** It suffices to compute

$$US_qF = e^{-Qe(c+Q\log |q|)(1,f)E_0 e^{i\Phi_D + s_qP\varphi + Q\log |q|})}$$

for

$$F = e^{(\phi,f)}.$$ 

We get

$$US_qF = e^{-Qe(c+Q\log |q|)(1,f)E_0 e^{i\Phi_D + P\varphi, f_1}} = e^{-Qe(c+Q\log |q|)(1,f) + \frac{1}{2} (f_0, G_0 f_0) e^{(\phi, \pi(f_1))}},$$

with

$$f_q(z) = |q|^{-2} f(z/q).$$

Now from (2.8)

$$(f_q, G_0 f_q) = (f, G_0 f) - \ln |q|(1, f)^2 + \int \log \frac{1 - \bar{q}z\bar{z}'}{|1 - z\bar{z}'|} f(z)f(z')dzdz'.$$

We compute next $\partial_q US_qF$ and $\partial_q US_qF$ at $q = 1$. First

$$\partial_q|q=1 (f_q, G_0 f_q) = -\frac{1}{2} (1, f)^2 - \frac{1}{2} \int \left( \frac{zz'}{1-\bar{z}z'} + \frac{z'z}{1-\bar{z}'z} \right) f(z)f(z')dzdz'.$$

$$= -\frac{1}{2} (1, f)^2 - \sum_{n>0} \pi (f)n \pi (f)_{-n}.$$
Next,
\[ \pi(f_q)_n = \int_D z^n f_q = q^n \pi(f)_n, \quad n > 0 \]
and
\[ \pi(f_q)_n = \int_D \bar{z}^{-n} f_q = \bar{q}^{-n} \pi(f)_n, \quad n < 0 \]
so that
\[ \partial_q |_{q=1} \pi(f_q)_n = n \pi(f)_n 1_{n>0}. \]
Altogether we get
\[ \partial_q |_{q=1} U \mathcal{S}_q F = \left( \frac{1}{2} Q(1, f) - \frac{1}{4} (1, f)^2 + \sum_{n>0} ((n \pi(f)_n \varphi - \frac{1}{2} \frac{\partial}{\partial \varphi} n \pi(f)_n) \right) UF \]
Using \((\partial_c + Q)UF = (1, f)UF\) and \(\frac{\partial}{\partial \varphi_n} UF = \pi(f)_{-n} UF\) this becomes
\[ \partial_q |_{q=1} UF = \frac{1}{4} (-\partial_c^2 + Q^2) + \sum_{n>0} (n \varphi_{-n} - \frac{1}{2} \frac{\partial}{\partial \varphi} n \varphi_{-n}) \frac{\partial}{\partial \varphi_{-n}} UF \]
\[ = \frac{1}{4} (-\partial_c^2 + Q^2) + 2 \sum_{n>0} a_{-n} a_n \]
In the same way, using
\[ \partial_q |_{q=1} \pi(f_q)_n = -n \pi(f)_n 1_{n<0}. \]
we get
\[ \partial_q |_{q=1} UF = \frac{1}{4} (-\partial_c^2 + Q^2) + \sum_{n>0} (n \varphi_{-n} - \frac{1}{2} \frac{\partial}{\partial \varphi} n \varphi_{-n}) \frac{\partial}{\partial \varphi_{-n}} UF \]
\[ = \frac{1}{4} (-\partial_c^2 + Q^2) + 2 \sum_{n>0} \bar{a}_{-n} \bar{a}_n \]

### 2.6 Virasoro algebra

The Energy-Momentum tensor field is given by
\[ T(z) = Q \partial_c^2 \phi - ((\partial_c \phi)^2 - E(\partial_c \phi)^2). \]
Let the support of \(F\) be in the ball \(B_r\) of radius \(r < 1\) centred at origin. We define for \(n \in \mathbb{Z}\)
\[ L_n UF : = \frac{1}{2\pi i} U \oint z^{n+1} T(z) F \]
\[ \bar{L}_n UF : = \frac{1}{2\pi i} U \oint \bar{z}^{n+1} T(\bar{z}) F \]
where the contour is in \(B_r^c\). Let us compute these operators explicitly on a dense domain. Again, it suffices to take \(F = e^{(\phi,f)}\) with \(f\) supported in \(B_r\). We compute first \(U \partial_c \phi(z) F\). First note that
\[ U((\phi,g) F) = \partial_{\lambda(0)} U e^{(\phi,f+\lambda g)} = (\partial (1, g) + (P \varphi, g) + (g, G_B f)) UF. \]
Now take \(g = -\partial \delta_z\) so that
\[ U \partial \phi(z) F = (\partial P \varphi(z) + \partial (G_B f)(z)) UF. \]
From (2.8) we get
\[ \partial_z G_D(z,u) = -\frac{1}{2} \left( \frac{1}{z-u} - \frac{1}{\bar{u}} \right) = -\frac{1}{2} \sum_{n=0}^{\infty} (u^n z^{-n-1} + \bar{u}^{n+1} z^n) \]
which converges since \(|u| < |z| < 1\). Hence we get
\[ \partial(G_D f)(z) = -\frac{1}{2} \sum_{n=0}^{\infty} (z^{-n-1} \pi(f)_n + z^n \pi(f)_{-n-1}) = -\frac{1}{2} \sum_{n \in \mathbb{Z}} \pi(f)_{-n-1} z^n \]
where we defined \( \pi(f)_0 = (1,f) \). Since
\[ \partial_P \varphi(z) = \sum_{n=1}^{\infty} n z^{n-1} \varphi_{-n} \]
we end up with
\[ U \partial \phi(z) F = \sum_{n \neq 0} z^{-n-1} (n \varphi_{-n} 1_{n>0} - \frac{1}{2} \partial \varphi_n) UF - \frac{1}{2} (\partial_c + Q) UF = \sum_{n \in \mathbb{Z}} z^{-n-1} a_n UF \]
provided we define
\[ a_0 = -\frac{1}{2} (\partial_c + Q). \]
Next consider \( U : (\partial \phi(z))^2 : F \) where \( : (\partial \phi(z))^2 := (\partial \phi(z))^2 - E_D(\partial \Phi(z))^2 - E(\partial P \varphi(z))^2 \). Noting that \( E(\partial P \varphi(z))^2 = 0 \) we then compute
\[ U : (\partial \phi(z))^2 : F = \sum z^{-n-1} z^{-m-1} (a_n a_m + \frac{1}{2} m \delta_{n,-m}) UF = \sum z^{-n-1} z^{-m-1} : a_n a_m : UF \]
where \( : a_n a_m : = a_n a_m \) if \( m > 0 \) and \( a_m a_n \) if \( n > 0 \) (i.e. annihilation operators are on the left). Combining we then get
\[ L_n = -(n+1) Q a_n - \sum_{m \in \mathbb{Z}} : a_{n-m} a_m :. \]
In particular we get
\[ L_0 = -Q a_0 - a_0^2 - 2 \sum_{n>0} a_{-n} a_n = \frac{1}{4} (-\partial_c^2 + Q^2) - 2 \sum_{n>0} a_{-n} a_n. \]
Comparing with Proposition 2.6 we get
\[ H = L_0 + \bar{L}_0 \]
\[ P = L_0 - \bar{L}_0 \]
\( L_n \) satisfy the commutation relations of the \textit{Virasoro Algebra}:
\[ [L_n, L_m] = (n-m)L_{n+m} + \frac{c_L}{12} (n^3 - n) \delta_{n,-m} \]
where the Central charge is
\[ c_L = 1 + 6Q^2. \]
The operators \( L_n \) are densely defined in \( \mathcal{H} \) and closable. They satisfy \( L_{-n} = L_n^* \) i.e. we have a unitary representation \( \mathcal{R} \) of the Virasoro Algebra on \( \mathcal{H} \).
2.7 Spectrum

By Fourier transform in the $c$ variable we represent $\Psi \in \mathcal{H}$ as $\{\hat{\Psi}(p)\}_{p \in \mathbb{R}}$ with $\hat{\Psi}(p) \in L^2(dP)$:

$$\langle \Psi, \Psi \rangle = \int_{\mathbb{R}} dp |\hat{\Psi}(p)|^2$$

We have

$$\hat{(a_0 \Psi)}(p) = -\frac{1}{2}(Q + ip)\hat{\Psi}(p)$$

Hence our representation is reducible

$$\mathcal{R} = \int_{\mathbb{R}}^{\oplus} \mathcal{R}_p.$$

We can formally write this as

$$\Psi = \int_{\mathbb{R}} dp \hat{\Psi}(p)|p\rangle$$

with

$$\langle p, p' \rangle = \delta(p - p')$$

and $|p\rangle$ is a highest weight state

$$L_0|p\rangle = \Delta_p|p\rangle, \quad L_n|p\rangle = 0, n > 0$$

where $\Delta_p = \frac{1}{4}(Q^2 + p^2)$. From (2.21) we get

$$|p\rangle = UV_{Q+ip}(0).$$

2.8 Liouville Hamiltonian

Recall that for $\mu > 0$

$$UF := e^{-Qc}E_0e^{-\mu \int_0^t e^{\gamma \theta}dz}F$$

provides an isometry

$$U_L : \mathcal{H} \to L^2(d\mathcal{P}(\varphi)dc).$$

which allows us to identify $\mathcal{H}$ with a subspace of $L^2(d\mathcal{P}(\varphi)dc)$. The Liouville semigroup is defined by

$$e^{-tH_L}U = US_{-t}.$$

Since the Liouville functional also satisfies (1.7) the calculation (2.22) may be repeated to conclude

$$(S_q F, G) = (F, S_q G)$$

(2.24)

Proposition 2.7. $e^{-tH_L}$ is a contraction semigroup on $\mathcal{H}$ with a positive generator $H_L$.

Proof. (sketch). Let $F \in \mathcal{F}_0$ satisfy $\langle F^2 \rangle < \infty$. Such $F$ form a dense set in $\mathcal{H}$. Let $\psi = U_L F$. We have

$$\|S_{-t}F\| = \langle S_{-t}F, S_{-t}F \rangle^{\frac{1}{2}} = \langle F, S_{-2t}F \rangle^{\frac{1}{2}} \leq \|F\|^{1-2^{-k}} \|S_{-2^k}F\|^{2^{-k}} \leq \cdots \leq \|F\|^{1-2^{-k}} \|S_{-2^k}F\|^{2^{-k}}$$

Now

$$\|S_{-2^k}F\|^2 = \langle S_{-2^k}F \Theta S_{-2^k}F \rangle \leq \langle (S_{-2^k}F)^2 \rangle = \langle F^2 \rangle$$

by conformal invariance of $\langle \cdot \rangle$. Taking $k \to \infty$ we conclude $\|S_{-t}F\| \leq \|F\|$.

\hfill $\Box$
2.9 Feynman-Kac formula

In this section we proceed heuristically. Let

\[ \phi_t(\theta) = \phi(e^{-\tau+i\theta}) - Q\tau \]

and consider observable

\[ F_\tau(\phi) = f(\phi_\tau). \]

Then

\[ S_{e^{-\tau}} F_\tau = F_{\tau+t}. \]

Taking \( \tau = 0 \) we get

\[ UF_t = e^{-tH_L} f. \]

Recall that

\[ e^{\gamma \phi} = \lim_{\epsilon \to 0} \epsilon^{\gamma^2} e^{\gamma \phi}. \]

We get, taking into account scaling of \( \epsilon \)

\[ \mu \int_D e^{\gamma \phi} dz = \mu \int_0^\infty dt \int_0^{2\pi} d\theta e^{-(2+\gamma^2)\tau} e^{\gamma \phi(e^{-t+i\theta})} = \mu \int_0^\infty dt \int_0^{2\pi} d\theta e^{\gamma \phi}(\theta) := \int_0^\infty v(\phi_t) dt. \]

Now we get by Trotter product formula, denoting by \( U_0 \) the \( \mu = 0 \) map in Proposition 2.4

\[ U_0 e^{-\int_0^T v(\phi_s) ds} S_{e^{-t}} = \lim_{n \to \infty} U_0 e^{-\frac{T}{n} \sum_t \frac{\dot{v}(\phi_t)}{2} S_{e^{-t}} = \lim_{n \to \infty} U_0 (e^{-\frac{T}{n} v(\phi_0) S_{e^{-t/n}}})^n = \lim_{n \to \infty} (e^{-\frac{T}{n} v(\phi_0) e^{-t/n H}} U_0 = e^{-t(H+v)} U_0. \]

Hence in particular

\[ U_0 e^{-\int_0^T v(\phi_s) ds} S_{e^{-t}} F = U_0 e^{-\int_0^T v(\phi_s) ds} S_{e^{-t}} F e^{-\int_0^T v(\phi_s) ds} = e^{-t(H+v)} (U_0 e^{-\int_0^T v(\phi_s) ds} F). \]

Taking \( T \to \infty \) in (2.25) we then get

\[ US_{e^{-t}} = e^{-t(H+v)} U \quad \text{(2.26)} \]

i.e. the Liouville Hamiltonian is given by

\[ H_L = H + v \quad \text{(2.27)} \]

This is only formal. What is the precise definition of \( v \)? Formally it is just

\[ v = \int : e^{\gamma \varphi(\theta)} : d\theta. \]

Note that

\[ E : e^{\gamma \varphi(\theta)} : e^{\gamma \varphi(\theta')} := |e^{i\theta} - e^{i\theta'}|^{-\gamma^2}. \]

Hence \( v \) is not defined in the Fock space for \( \gamma \geq 1 \) since \( \|v1\| = \infty. \)
2.10 Representation Theory

Let $\mathcal{V}$ be the linear span of the vectors $U(\prod_{i=1}^n V_{\alpha_i}(z_i))$ with $|z_i| < 1$. Then

$$L_n = \oint_{|z|=r} z^{n+1} T(z).$$

acts on $\mathcal{V}$ by taking $1 - r$ small enough. The Ward identities imply the Virasoro algebra commutation rules on $\mathcal{V}$:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} \delta_{m,-n}.$$

The operators satisfy $L_0^+ = L_{-n}$ on $\mathcal{V}$. A major challenge is to find a common dense domain in $\mathcal{H}$ for these operators so that the representation is unitary and then study its reduction to irreducibles. It is conjectured [7] that

$$\mathcal{H} = \int_{\mathbb{R}^+} \mathcal{H}_P dP$$

where $\mathcal{H}_P$ is a highest weight module $M_P = \text{span}\{L_n\psi_P, n \leq 0\}$, $L_0\psi_P = \Delta_Q + i\psi_P, L_n\psi_P = 0, n > 0$.

As in the $\mu = 0$ case $\psi_P$ would be a generalized eigenfunction for $L_0$ i.e. not a vector in $\mathcal{H}$. It formally corresponds to the vertex operator $V_{Q+i\beta}$ which saturates the Seiberg bound. In [4] these were constructed for $P = 0$. It would be nice to understand the complex case.

Note that the spectrum for Liouville is $\mathbb{R}^+$ and not $\mathbb{R}$ as for the MFF. This is due to the potential barrier at positive $c$. Consider a toy model where we keep only the $c$ degree of freedom i.e. the operator

$$H = \frac{1}{2}(-\frac{d^2}{dx^2} + Q^2) + \mu e^{\gamma c}$$

on $L^2(\mathbb{R})$. The spectrum of $H$ is determined by the $c \to \infty$ asymptotics of $H$ i.e. it is $[\frac{1}{4}Q^2, \infty)$ like in the $\mu = 0$ case. In the $\mu = 0$ case the spectrum has degeneracy two: the (generalized) eigenfunctions are $e^{\pm ipc}$. In the $\mu > 0$ case as $c \to \infty$ the eigenfunctions are linear combinations of these plane waves but as $c \to \infty$ they have to vanish. This means the degeneracy is one and asymptotics at $c \to \infty$ is $\psi_p(c) = e^{ipc} + R(p)e^{-ipc}$ and $p \in \mathbb{R}^+$. It is a challenge to extend this analysis to the full Liouville Hamiltonian.

2.11 DOZZ-conjecture

In conformal field theory it is believed [2] that all correlation functions are determined by the knowledge of primary fields (i.e. spectrum of representations) and their three point functions. For the latter there is a remarkable conjecture due to Dorn, Otto, Zamolodchikov and Zamolodchikov [5, 18] in Liouville theory. By Möbius invariance

$$(V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)) = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{31}} C_\gamma(\alpha_1, \alpha_2, \alpha_3)$$

where $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$ etc. The three point structure constants and be obtained from

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) = \lim_{z_3 \to \infty} |z_3|^{4\Delta_3} \langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(z_3) \rangle$$

(2.28)

which leads to the following expression in terms of Multiplicative chaos

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) = \text{const.} \mu^{-s} \Gamma(s) E Z^{-s}$$

with

$$Z = \int |z|^{-\alpha_1\gamma} |z - 1|^{-\alpha_2\gamma} \hat{g}(z)^{-\hat{\beta}} \sum_{i=1}^n \alpha_i M_{\alpha_i\gamma}(dz).$$

The DOZZ Conjecture gives an explicit formula for $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$. It is based on analyticity and symmetry assumptions that lack proofs. One of the ingredients in its derivation was recently proved in [9] namely the
so-called BPZ equations \([2]\) for the vertex operators \(V_{-\frac{1}{2}}\) and \(V_{\frac{1}{2}}\) (in the language of CFT, \(V_{-\frac{1}{2}}\) and \(V_{\frac{1}{2}}\) are called a degenerate fields). More precisely, setting \(F(z, z) := (V_{-\frac{1}{2}}(z) \prod_i V_{\alpha_i}(z_i))\), we prove

\[
\left(\frac{4}{z}\partial^2 + \sum_k \left(\frac{\Delta_{\alpha_k}}{(z-z_k)^2} + \frac{1}{z-z_k}\partial_{z_k}\right)\right)F = 0.
\]

and the same equation for \(\langle V_{-\frac{1}{2}}(z) \prod_i V_{\alpha_i}(z_i)\rangle\) with \(\frac{1}{z}\) replaced by \(\frac{1}{z^2}\).

The proof of these equations proceeds as the proof of the Ward identities: differentiation of the correlation function brings down a \(\partial_z \phi(z)\) which then can be integrated by parts in the Gaussian measure. Note that we are not postulating that e.g. \(V_{-\frac{1}{2}}(z)\) is a degenerate field but rather proving it in the sense that it satisfies the expected equation.

Using the BPZ equation, we recover an explicit formula found earlier in the physics literature for the 4 point correlation functions \(\langle V_{-\frac{1}{2}}(z) \prod_{i=1}^3 V_{\alpha_i}(z_i)\rangle\) and \(\langle V_{-\frac{1}{2}}(z) \prod_{i=1}^2 V_{\alpha_i}(z_i)\rangle\) in terms of the 3-point structure constants. Any four point function is fixed by Möbius invariance up to a single function depending on the cross ratio of the points. Specializing to the case we are interested in, we get

\[
\langle V_{-\frac{1}{2}}(z) \prod_{i=1}^3 V_{\alpha_i}(z_i)\rangle = |z_3 - z|^{-4\Delta_{-\frac{1}{2}}} |z_2 - z_1|^{2(\Delta_{1} - \Delta_{-\frac{1}{2}})} |z_3 - z_1|^{2(\Delta_{2} + \Delta_{-\frac{1}{2}} - \Delta_{3} - \Delta_{1})} \times |z_3 - z_2|^{2(\Delta_{1} + \Delta_{-\frac{1}{2}} - \Delta_{3} - \Delta_{2})} G\left(\frac{(z - z_1)(z_2 - z_3)}{(z_3 - z_2)(z - z_1)}\right)
\]

where \(\Delta_{-\frac{1}{2}} = -\frac{\gamma}{2}(Q + \frac{1}{Q})\) and \(\Delta_i = \frac{\alpha_{\mu}}{2}(Q - \frac{\mu}{Q})\). We can recover \(G(z)\) as the following limit

\[
G(z) = \lim_{z_3 \to \infty} |z_3|^{4\Delta_3} \langle V_{-\frac{1}{2}}(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(z_3)\rangle
\]

and this becomes in terms of multiplicative chaos

\[
G(z) = |z|^{-\frac{\gamma}{2}} |z - 1|^{-\frac{\gamma}{2}} T(z)
\]

where \(T(z)\) is given by

\[
T(z) = \text{const.} \mu^{-s + \frac{1}{\gamma}} \gamma^{-1} \Gamma(s - \frac{1}{\gamma}) E[R(z)^{\frac{1}{\gamma} - s}]
\]

and

\[
R(z) = \int e^{\gamma x(z)} - \frac{\gamma}{2} E[x(z)^2] \frac{|x - z|^{\frac{2}{\gamma}}}{|x|^{\gamma \alpha_1} |x - 1|^{\gamma \alpha_2} g(x)^{1 + \frac{\gamma}{2} - \frac{1}{2} \sum_i \alpha_i}} dx.
\]

In particular one has

\[
T(0) = C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3).
\]

A little algebra then shows that the BPZ equation becomes the hypergeometric equation for \(T\)

\[
(1 - z)\partial^2_{zz} T(z) + (a_1 + a_2 z) \partial_z T(z) + a_3 T(z) = 0
\]

where \(a_i\) are are explicit constants depending on \(\alpha_i\) and \(\gamma\). By analysing positive solutions to the hypergeometric equation one finds that

\[
T(z) = \lambda_1 |F_{-}(z)|^2 + \lambda_2 \text{Re}(F_{-}(z)F_{+}(z)) + \lambda_3 \text{Im}(F_{-}(z)F_{+}(z)) + \lambda_4 |F_{+}(z)|^2
\]

where \(\lambda_i \in \mathbb{R}\) and \(F_{\pm}\) are definite hypergeometric functions. It remains to determine the coefficients \(\lambda_i\). One has from the asymptotics of \(F_{\pm}\) as \(z \to 0\)

\[
T(z) = \lambda_1 + \lambda_2 \text{Re}(z^\alpha) + \lambda_3 \text{Im}(z^\alpha) + \lambda_4 |z|^{2\alpha} + o(|z|^{2\alpha})
\]
where \( a = \frac{1}{4}\gamma(Q - \alpha_1) \). On the other hand we may study the explicit expression (2.32) as \( z \to 0 \) by multiplicative chaos techniques. This gives

\[
T(z) = C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) - \mu \frac{\pi}{l(-\frac{a}{\gamma})l(\frac{a\alpha_1}{\gamma})l(\frac{a\alpha_2}{\gamma})l(\frac{a\alpha_3}{\gamma})} C(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3) |z|^{2a} + o(|z|^{2a}). \quad (2.35)
\]

where \( l(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \). We conclude that \( \lambda_1 = C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) \), \( \lambda_2 = \lambda_3 = 0 \) and

\[
\lambda_4 = -\mu \frac{\pi}{l(-\frac{a}{\gamma})l(\frac{a\alpha_1}{\gamma})l(\frac{a\alpha_2}{\gamma})l(\frac{a\alpha_3}{\gamma})} C(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3).
\]

From this we can deduce the following corollary on the 3 point structure constants (this argument is called Teschner’s trick [17] in the literature):

\[
\frac{C(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)}{C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)} = -\frac{1}{\pi \mu} \frac{l(-\frac{a}{\gamma})l(\frac{a\alpha_1}{\gamma})l(\frac{a\alpha_2}{\gamma})l(\frac{a\alpha_3}{\gamma})}{l(\frac{a\alpha_1}{\gamma})(\frac{a\alpha_2}{\gamma})(\frac{a\alpha_3}{\gamma})} \frac{l(-\alpha_1 + \alpha_2 + \alpha_3 - \frac{\gamma}{2})}{l(\frac{1}{4}(\alpha_1 + \alpha_2 + \alpha_3 - \frac{\gamma}{2}))}.
\]

Similarly the other BPZ equation gives rise to a formula for \( \frac{C(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)}{C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)} \). Our derivation of these identities holds only if the Seiberg bounds are satisfied i.e. \( \alpha_1 < Q - \frac{\gamma}{2} \) and \( \sum \alpha_i > 2Q + \frac{\gamma}{2} \) and similarly for the other identity. If one makes the assumption that \( C(\alpha_1, \alpha_2, \alpha_3) \) is an analytic function in the \( \alpha_i \) and these identities hold then one may derive an explicit formula for 3 point structure constants, the DOZZ formula.

The DOZZ conjecture and the representation content of Liouville theory are signals of its integrability. They remain the main challenge for the probabilistic approach to the Liouville model.

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