The rise of quantum information theory has led to an increased interest in simple representations of general mixed states of systems with finite dimensional Hilbert spaces [1] (see [2] for a large list of literature). Particularly well studied is the case of a single qubit, a two-level system, formally equivalent to a spin-1/2. The density matrix can then be expressed in a basis formed of Pauli matrices and the identity matrix, leading to a parametrization in terms of a vector in $\mathbb{R}^3$. Pure states correspond to points on the unit sphere, the so-called Bloch sphere, and mixed states fill the inside of the sphere, the “Bloch ball”. The simplicity of this representation allows one to visualize the action and geometry of all possible single qubit quantum channels [1]. Given the importance of this representation, there have been numerous attempts to generalize it to higher dimensional systems. Different matrix bases can be used to expand density matrices of d-dimensional quantum systems, so-called qudits, each yielding a different generalization of the Bloch vector. Examples are the $su(N)$-algebra generators [3,4], the polarization operator basis [5-7] or also the Weyl operator basis [8]. Mosseri et al. noted a connection to Hopf fibrations of higher dimensional spheres and used this insight for a parametrization of two-qubit states based on quaternions [9] that was later extended to three-qubit states using octonions [10]. Dietz introduced a parametrization for a subclass of m-qubit states that can be expressed in terms of a Clifford algebra [11]. Goyal et al. analyzed qutrit (three-level) states in a basis of Gell-Mann matrices [2].

In the present Letter we propose a new parametrization of a spin-j density matrix $\rho$ based on a set of $4^j$ covariant matrices introduced by Weinberg in 1964 in the context of relativistic spin theory [12]. The goal of this Letter is to show that these covariant matrices, constructed from products of components of $L = (J_1, J_2, J_3)$ with $J_a$ (1 ≤ $a$ ≤ 3), the usual (2 + 1)-dimensional representations of angular momentum operators, can serve to construct a generalization to arbitrary $j$ of the well-known Bloch sphere representation for spin-1/2 states. The remarkable features of this representation are especially reflected in the simple coordinates of coherent states, transformation under SU(2) operations, and the representation of reduced density matrices.

Defining the 4-vector $q = (q_0, q_1, q_2, q_3) = (q_0, \mathbf{q})$, Weinberg’s covariant matrices $S_{\mu_1\mu_2...\mu_2j}$, with $0 \leq \mu_i \leq 3$, can be obtained by expanding the square of the $(2j + 1)$-dimensional matrix corresponding to the $(j,0)$ representation of a Lorentz boost in direction $\mathbf{q}$, which can be put in the form [12]

$$\Pi^{(j)}(q) = (q_0^2 - |\mathbf{q}|^2)\lambda^{-2\eta_\eta} - \mathbf{q} \cdot \mathbf{J}$$

(1)

with $\eta_\eta = \text{arctanh}(-|\mathbf{q}|/q_0)$ and $\lambda = |\mathbf{q}|/q_0$. Matrices $S_{\mu_1\mu_2...\mu_2j}$ are defined in [12] by identifying the coefficients of the multivariate polynomial with variables $q_0, q_1, q_2, q_3$ in (1) with those of the polynomial

$$\Pi^{(j)}(q) = (-1)^{2j}q_{\mu_1}q_{\mu_2}...q_{\mu_2j}S_{\mu_1\mu_2...\mu_2j}$$

(2)

(we use Einstein summation convention for repeated indices). An explicit expression for $\Pi^{(j)}(q)$ is given in [12] as

$$\Pi^{(j)}(q) = (q_0^2 - |\mathbf{q}|^2)\lambda^j + \sum_{k=1}^{j} \frac{(q_0^2 - |\mathbf{q}|^2)^{j-k}}{(2k)!} (2\mathbf{q} \cdot \mathbf{J}) \left( \prod_{r=1}^{k-1} [(2\mathbf{q} \cdot \mathbf{J})^2 - (2r\mathbf{q})^2] \right) (2\mathbf{q} \cdot \mathbf{J} + 2kq_0)$$

(3)
for integer $j$, and
\[
\Pi^{(j)}(q) = (q_0^2 - q^2)^j(-q_0 - 2q \cdot J) - \sum_{k=1}^{j-1/2} \frac{(q_0^2 - q^2)^{j-1/2-k}}{(2k + 1)!} \left( \prod_{r=1}^{k} (2q \cdot J)^2 - ((2r - 1)q)^2 \right) (2q \cdot J + (2k+1)q_0)
\]

for half-integer $j$. The identity matrix is implicit in front of constant terms. For instance, identifying the coefficient of $q_0^2$ in these expressions we get that $S_{00\ldots0}$ is the $(2j + 1)$-dimensional identity matrix, $1_{2j+1}$.

The matrices $S_{\mu_1\mu_2\ldots\mu_{2j}}$ are Hermitian matrices, invariant under permutation of indices, and they obey the following linear relation:
\[
g_{\mu_1\mu_2}S_{\mu_1\mu_2\ldots\mu_{2j}} = 0,
\]
where $g \equiv \text{diag}(-, +, +, +)$.

Let us briefly consider the simplest examples. From (2), the explicit expression of $\Pi^{(j)}(q)$ for spin-1/2 reads
\[
\Pi^{(1/2)}(q) = -q_0 - 2q \cdot J
\]
where $J_\alpha$ are spin-1/2 representations of the angular momentum operators. Identifying with (2) directly gives $S_0 = \sigma_0$ and $S_2 = 2J_\alpha = \sigma_0$ where $\sigma_0$ is the $2 \times 2$ identity matrix and $\sigma_\alpha$ are the usual Pauli matrices. The usual Bloch sphere representation for an arbitrary spin-1/2 density matrix $\rho = \frac{1}{2} \sigma_0 + \frac{1}{2} \mathbf{x} \cdot \sigma$ can then be expressed in terms of the $S_{\mu_1}$ ($0 \leq \mu_1 \leq 3$) as
\[
\rho = \frac{1}{2} x_{\mu_1} S_{\mu_1}
\]
with the Bloch vector $\mathbf{x} = \text{tr}(\rho \sigma)$ and $x_0 = 1$.

For $j = 1$, the equality between expressions (2) and (3) for $\Pi^{(1/2)}(q)$ reads
\[
(q_0^2 - q^2) + 2q \cdot J (q \cdot J + q_0) = q_{\mu_1} q_{\mu_2} S_{\mu_1\mu_2}.
\]
Identifying coefficients of this quadratic form yields $S_0 = J_0$, $S_2 = 2J_\alpha = J_0$ and $S_1 = J_0 J_\alpha + J_\alpha J_0 - \delta_{\alpha\beta} J_0$ with $J_0$ the 3 x 3 identity matrix. Again, the set of $S_{\mu_1\mu_2}$ matrices can serve to express any spin-1 density matrix $\rho$ as
\[
\rho = \frac{1}{4} x_{\mu_1\mu_2} S_{\mu_1\mu_2}
\]
with coordinates
\[
x_{\mu_1\mu_2} = \text{tr}(\rho S_{\mu_1\mu_2}).
\]

Expressions (3)–(4) can be used to generalize this expansion to arbitrary $j$, as we will show in Theorem 2. The main property of the covariant matrices is given by Theorem 1 below. We first give a useful lemma.

**Lemma 1.** Let $|\alpha\rangle$ be a spin-$j$ coherent state, defined for $a = e^{-i\varphi} \cot(\theta/2)$ with $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ by
\[
|\alpha\rangle = \sum_{m = -j}^{j} \sqrt{\binom{2j}{j + m}} \left[ \sin \frac{\theta}{2} \right]^{j-m} \left[ \cos \frac{\theta}{2} e^{-i\varphi} \right]^{j+m} |j, m\rangle
\]
in the standard angular momentum basis $\{|j, m\rangle : -j \leq m \leq j\}$, and let $n = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. Then
\[
\langle \alpha| \Pi^{(j)}(q) |\alpha\rangle = (-1)^{2j} (q_0 + q \cdot n)^{2j}.
\]

The proof of this lemma is based on the SU(2) disentangling theorem and can be found in the Supplemental Material. One of its consequences is that, by identifying coefficients of the polynomial in $q_0$ in (12), we get
\[
\langle \alpha| S_{\mu_1 \mu_2 \ldots \mu_{2j}} |\alpha\rangle = n_{\mu_1} n_{\mu_2} \ldots n_{\mu_{2j}},
\]
with $n_0 = 1$.

Any spin-$j$ state can be seen as the state of a system of $N = 2j$ qubits that is symmetric under permutation of the qubits. The Hilbert space $\mathcal{H} \equiv \mathbb{C}^{2^N}$ of an $N$-qubit system has dimension $2^N$ but its symmetric subspace $\mathcal{H}_S$ has only dimension $N + 1 = 2j + 1$. It is spanned by the symmetric Dicke states
\[
|D_N^{(k)}\rangle = \sum_{\pi} \prod_{n = 0}^{\pi} |0, \ldots, 1, \ldots, 1\rangle, \quad k = 0, \ldots, N,
\]
where the sum runs over all permutations of the string with $N - k$ zeros and $k$ ones, and $N$ is the normalization constant. The Dicke state $|D_N^{(k)}\rangle$ corresponds to $|j, m\rangle$ with $j = N/2$ and $m = k - N/2$.

Let $B(\mathcal{H})$ be the Hilbert space of bounded linear operators acting on $\mathcal{H}$. An operator basis for $B(\mathcal{H})$ is given by the set of the $4^N$ generalized Pauli matrices defined as the $N$-fold tensor products of the $2 \times 2$ matrices $\sigma_0, \sigma_1, \sigma_2, \sigma_3$, [13],
\[
\sigma_{\mu_1 \mu_2 \ldots \mu_N} = \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \ldots \otimes \sigma_{\mu_N}.
\]

These Hermitian operators verify the relations $\text{tr}(\sigma_{\mu_1 \mu_2 \ldots \mu_N} \sigma_{\nu_1 \nu_2 \ldots \nu_N}) = 2^N \delta_{\mu_1 \nu_1} \delta_{\mu_2 \nu_2} \ldots \delta_{\mu_N \nu_N}$ and thus form an orthogonal basis with respect to the Hilbert-Schmidt inner product. Any $N$-qubit state $\rho$ can be expanded in this basis as
\[
\rho = \frac{1}{2^N} x_{\mu_1 \mu_2 \ldots \mu_N} \sigma_{\mu_1 \mu_2 \ldots \mu_N},
\]
where $x_{\mu_1 \mu_2 \ldots \mu_N}$ are real coefficients given by
\[
x_{\mu_1 \mu_2 \ldots \mu_N} = \text{tr}(\rho \sigma_{\mu_1 \mu_2 \ldots \mu_N}).
\]

We can now prove the following theorem:
Theorem 1. The Weinberg covariant matrices defined in Eq. (3) are given by the projection of tensor products of Pauli matrices into the subspace $H_S$ of states that are invariant under permutation of particles. Namely, denoting by $P_S = \sum_{k=0}^{N} |D_N^{(k)}\rangle \langle D_N^{(k)}|$ the projector onto $H_S$, the $S_{\mu_1\mu_2...\mu_N}$ matrix corresponds to the $(N+1)$-dimensional block spanned by the $|D_N^{(k)}\rangle$ of the matrix $P_S \sigma_{\mu_1\mu_2...\mu_N} P_S^\dagger$, i.e., in terms of matrix elements

$$\langle D_N^{(k)}|S_{\mu_1\mu_2...\mu_N}|D_N^{(\ell)}\rangle = \langle D_N^{(k)}|\sigma_{\mu_1\mu_2...\mu_N}|D_N^{(\ell)}\rangle,$$

with $0 \leq k, \ell \leq N$.

Proof. Let $\tilde{S}_{\mu_1\mu_2...\mu_N} = PS \sigma_{\mu_1\mu_2...\mu_N} P_S^\dagger$. Any spin-$j$ coherent state $|\alpha\rangle$ defined by Eq. (11) can also be written as the tensor product of identical spin-$1/2$ coherent states. As a symmetric state, $|\alpha\rangle$ is invariant under $P_S$, i.e., $|\alpha\rangle = P_S|\alpha\rangle$, so that $\langle \alpha|\tilde{S}_{\mu_1\mu_2...\mu_N}|\alpha\rangle = \langle \alpha|\sigma_{\mu_1\mu_2...\mu_N}|\alpha\rangle = n_{\mu_1}n_{\mu_2}...n_{\mu_N}$. Using Eq. (14), we thus have

$$\langle \alpha|\tilde{S}_{\mu_1\mu_2...\mu_N}|\alpha\rangle = \langle \alpha|S_{\mu_1\mu_2...\mu_N}|\alpha\rangle$$

(19)

for all $\alpha$, i.e., the Husimi functions of the two operators are identical. Therefore $S_{\mu_1\mu_2...\mu_N}$ and $\tilde{S}_{\mu_1\mu_2...\mu_N}$ coincide in $H_S$.

In other words, instead of obtaining the Weinberg matrices from the expansion of the rather complicated expressions (3–4), we can construct them simply by projecting the corresponding tensor-product of Pauli operators into the symmetric subspace. In order to fully exploit the consequences of this fact, we need some basic notions of frame theory (14).

A family of vectors $|\phi_i\rangle$, $i \in \{1,\ldots,M\}$, is called a frame for a Hilbert space $\mathcal{H}$ with bounds $A, B \in ]0, \infty[$, if

$$A\||\psi\||^2 \leq \sum_{i=1}^{M} \langle \psi|\phi_i\rangle^2 \leq B\||\psi\||^2, \quad \forall \ |\psi\rangle \in \mathcal{H}.$$  

(20)

If $A = B$, then the frame is called an $A$-tight frame.

Orthonormal bases are a special case of $A$-tight frames. In particular, the generalized Pauli matrices (14) form – up to normalizing – an orthonormal basis of $\mathcal{B}(\mathcal{H})$, and are in fact an $A$-tight frame with $A = 2^N$. According to proposition 22 in (14), a frame of $\mathcal{H}$ with bounds $A, B$ that is orthogonally projected to a subspace $PH$ is a frame of $PH$ with the same bounds $A, B$. Therefore we have as a corollary of Theorem 1 that the set of covariant matrices $S_{\mu_1\mu_2...\mu_N}$ forms a $2^N$-tight frame.

Tight frames are in a sense a generalization of orthonormal bases, as they allow an expansion over the elements of the frames with the same formulae as for an orthonormal basis, i.e., for all $|\psi\rangle \in \mathcal{H}$, we have $|\tilde{\psi}\rangle = A^{-1}\sum_{i=1}^{M} \langle \psi|\phi_i\rangle |\phi_i\rangle$ (proposition 20 in (14)). This immediately entails the following result, which provides a generalization of the Bloch sphere representation for spin-$1/2$, Eq. (17), to any spin:

Theorem 2. For general spin-$j$, the $4^N$ Hermitian matrices $S_{\mu_1\mu_2...\mu_N}$ (with $N \equiv 2j$) provide a $2^N$-tight frame over which $\rho$ can be expanded, that is, any state can be expressed as

$$\rho = \frac{1}{2^N} x_{\mu_1\mu_2...\mu_N} S_{\mu_1\mu_2...\mu_N},$$

(21)

with coefficients

$$x_{\mu_1\mu_2...\mu_N} = tr(\rho S_{\mu_1\mu_2...\mu_N})$$

(22)

real and invariant under permutation of the indices.

Since $S_{00...0}$ is the identity matrix, the condition $tr\rho = 1$ for density matrices is equivalent to $x_{00...0} = 1$. The tight frame property allows one to write the Hilbert-Schmidt scalar product of any two Hermitian operators $\rho$ and $\rho'$ with coordinates $x_{\mu_1\mu_2...\mu_N}$ and $x'_{\mu_1\mu_2...\mu_N}$ as the scalar product of coordinates, more precisely

$$tr(\rho\rho') = \frac{1}{2^N} x_{\mu_1\mu_2...\mu_N} x'_{\mu_1\mu_2...\mu_N}.$$  

(23)

The condition $tr\rho^2 \leq 1$ that every state must satisfy translates into $\sum_{\mu_1\ldots}\sum_{\mu_N} x_{\mu_1\mu_2...\mu_N}^2 \leq 2^N$. Note that from Eq. (22) and the definition of $S_{\mu_1\mu_2...\mu_N}$, the coordinates $x_{\mu_1\mu_2...\mu_N}$ appear as the coefficients of $(-1)^N(\Pi^{(j)}(q))$, which is a multivariate polynomial in variables $q_0, q_1, q_2, q_3$.

$$(-1)^N(\Pi^{(j)}(q)) = x_{\mu_1\mu_2...\mu_N} q_{\mu_1} \cdots q_{\mu_N}.$$  

(24)

Due to the over-completeness of the $S_{\mu_1\mu_2...\mu_N}$ the coordinates $x_{\mu_1\mu_2...\mu_N}$ in (21) are so far not unique. However, for a given spin-$j$ density matrix $\rho$, (22) is the unique choice of coordinates $x_{\mu_1\mu_2...\mu_N}$ such that these coordinates are real numbers, invariant under permutation of the indices, and verifying the condition $g_{\mu_1\mu_2} x_{\mu_1\mu_2...\mu_N} = 0$ (see Proposition 1 in the Supplemental Material).

The generalized Bloch representation (21) shares with the Bloch representation of a spin-$1/2$ several crucial properties. First of all, using Eq. (13) and Eq. (22), we see that coordinates of a coherent state are simply given by the product of components of the 4-vector $n = (1, n)$, namely $x_{\mu_1\mu_2...\mu_N} = n_{\mu_1}n_{\mu_2}...n_{\mu_N}$. This generalizes the fact that the Bloch vector representing a spin-1/2 state points in the direction given by the angles defining the coherent state. Secondly, under any $SU(2)$ transformation, the Bloch vector of a spin-1/2 simply rotates, i.e., transforms according to $x_\theta \rightarrow R_{\theta} x_\theta$, where $R$ is a rotation matrix. Similarly, for higher spins the tensor of coordinates of an arbitrary state transforms according to $x_{\mu_1...\mu_N} \rightarrow R_{\mu_1\nu_1} \cdots R_{\mu_N\nu_N}(x_{\nu_1}...x_{\nu_N})$, with $R_{\mu\nu}$ the 3 x 3 rotation matrix and $R_{\mu_1\nu_1}$ the 2 x 3 rotation matrix with $R_{\mu_1\nu_1} = R_{\nu_1\mu_1} = \delta_{\mu_1\nu_1}$. This is a consequence of a more general covariance property of the basis matrices $S_{\mu_1\mu_2...\mu_N}$. Indeed, they were constructed in such a way that for any element $\Lambda$ of the Lorentz group, with $D^{(j)}[\Lambda]$ the $(2j+1)$-dimensional matrix associated with $\Lambda$ in the $(j, 0)$ representation,

$$D^{(j)}[\Lambda] S_{\mu_1\mu_2...\mu_N} D^{(j)}[\Lambda]^\dagger \Lambda_{\mu_1}^\nu_1 \cdots \Lambda_{\mu_N}^\nu_N S_{\nu_1\nu_2...\nu_N}$$

(25)
in the covariant-contravariant notation of Eq. \text{[22]} this property translates to coordinates $x_{\mu_1\mu_2...\mu_N}$. For rotations $R_{\mu\nu}$, the distinction between upper and lower indices becomes irrelevant.

In addition to the shared advantages of a Bloch vector, our generalized Bloch sphere representation \text{[21]} enjoys additional convenient properties relevant for multi-qubit systems. For instance, coordinates of the spin-$k$ reduced density matrix obtained by tracing the spin-$j$ matrix over $j-k$ spins are simply given by

$$x_{\mu_1...\mu_2k} = x_{\mu_1...\mu_20...0}$$

(see Proposition 3 in the Supplemental Material). Note that in \text{[15]} a similar property was observed for the expansion of $\rho$ over generalized Pauli matrices, and a formal Lorentz invariance of that expansion was used very recently to generalize monogamy relations of entanglement \text{[16]}.

We now consider a few examples of states and give their coordinates in our representation. The maximally mixed state $\rho_0 = \frac{1}{N+1} I_{2j+1}$ has coordinates $x_{\mu_1\mu_2...\mu_j}$ given by

$$x_{\mu_1...\mu_2j} q_{\mu_1} \cdots q_{\mu_2j} = \sum_{k=0}^{j} \frac{(2k)!}{2k!} \frac{2^{j-k}}{N} (q_{\mu_1}^2)^k$$

(27)

(see Proposition 2 in the Supplemental Material). Another example is given by the Schrödinger cat states $|\psi_{cat}^{(j)}\rangle = (|j,j\rangle + |j,-j\rangle)/\sqrt{2}$. By linearity of the expansion \text{[21]} and of the trace, they have coordinates

$$x_{\mu_1...\mu_N}^{\text{cat}} = \frac{1}{2} \left(\prod_{i=1}^{N} n_{\mu_i} \frac{1}{2} \right) \left(\prod_{i=1}^{N} n_{\mu_i} \frac{1}{2} \right) + \text{Re} \left[ \prod_{i=1}^{N} n_{\mu_i} \frac{1}{2} \right]$$

(28)

where $n(\frac{1}{2}, -\frac{1}{2}) = (1, 0, 0, \pm 1)$ are the coordinates of the coherent states $|\frac{1}{2}, \pm 1\rangle$ and $|\frac{1}{2}, \pm \frac{1}{2}\rangle$ and $n(-\frac{1}{2}, \frac{1}{2}) = (0, 1, -1, 0)$ are the coordinates of the non-Hermitian operator $\left(\frac{1}{2}, -\frac{1}{2}\right) \left(\frac{1}{2}, -\frac{1}{2}\right)$.

While the complete characterization of the set of coordinates for which $\rho$ is positive is difficult in any representation \text{[20], \text{[8]}} our representation \text{[21]} allows one to solve this problem explicitly for $j = 1$. The set of all spin-$1$ states is characterized by $8$ real parameters. The transformation of tensor $x_{\mu_\nu}$ by rotation matrices under SU(2) operations allows one to diagonalize the $3 \times 3$ block $x_{ab}$ ($1 \leq a, b \leq 3$), and Eq. \text{[15]} imposes $\sum_{i=1}^{3} \mu_i = 1$ for the eigenvalues $\mu_i$, leaving five real parameters $\mu_1, \mu_2, \mathbf{x} \equiv (x_{01}, x_{02}, x_{03})$. In this case, $\mathbf{x}$ coincides with $\mathbf{u}$ in the representation found in \text{[17]}. We therefore immediately obtain that up to two special cases of measure zero the set of all spin-$1$ states can be represented as a two-parameter family of ellipsoids in the space of vectors $\mathbf{x}$ (Eq. (21) in \text{[17]} with $\mathbf{u} = \mathbf{x}$ and $w_{ab} = x_{ab}$), thus providing a simple geometrical picture of all spin-$1$ states.

As a direct application of our formalism, we give a simple necessary and sufficient criterion for anticoherence of spin states. Spin states are said to be anticoherent to order $t$ if $\langle \mathbf{n} \cdot \mathbf{J} \rangle^K$ is independent of the unit vector $\mathbf{n}$ for any $k$ with $0 \leq k \leq t$. Various characterizations have been given \text{[19]}. Very recently the case of pure but not necessarily symmetric states was considered in \text{[15, \text{[20]}}. The definition of matrices $S_{\mu_1\mu_2...\mu_N}$ via \text{[15, \text{[20]}} as a function of $\mathbf{J}$ makes them most convenient for the characterisation of anticoherent states. One can show the following result:

**Theorem 3.** A spin-$j$ state $\rho$, pure or mixed, is anticoherent to order $t$ if and only if its spin-$(t/2)$ reduced density matrix is the maximally mixed state $\rho_0 = \frac{1}{N+1} I_{2j+1}$.

The proof (see Supplemental Material for more detail) relies on the calculation of $\langle \mathbf{I}^{(j)}(q) \rangle$ for an anticoherent state, using the expansion \text{[23]–[24]} and identifying terms up to order $t$ with the expansion \text{[27]} of the maximally mixed state. For instance, spin-$j$ anticoherent states to order $1$ are characterized by $S_{\mu_000...0} = \delta_{\mu0}$ while anticoherent states to order $2$ are characterized by $S_{\mu_000...0} = \text{diag}(1, 1/3, 1/3, 1/3)$. From the characterization of anticoherence given by Theorem 3 one can easily obtain another characterization based on coefficients of the multipolar expansion of the density matrix. For a spin-$j$ density operator $\rho$, the expansion reads

$$\rho = \sum_{k=0}^{j} \sum_{q=-k}^{k} \rho_{kq} T_{kq}^{(j)}$$

(29)

with $\rho_{kq} = \text{tr}(T_{kq}^{(j)} \rho_{\text{cat}}) \delta_{q0}$, where $T_{kq}^{(j)}$ are the irreducible tensor operators \text{[3]}

$$T_{kq}^{(j)} = \frac{\sqrt{2k+1}}{2j+1} \sum_{m,m'=-j}^{j} C_{jm,kq}^{jm'} |j, m\rangle \langle j, m'|$$

(30)

and $C_{jm,kq}^{jm'}$ are Clebsch-Gordan coefficients. The following corollary of Theorem 3 can now be stated (see Supplemental Material for a proof).

**Corollary 1.** A spin-$j$ state $\rho$ is anticoherent to order $t$ if and only if $\rho_{kq} = 0$, $\forall k \leq t$, $\forall q : -k \leq q \leq k$.

Such anticoherent states have been studied in the context of quantum polarization of light (see e.g. \text{[21, \text{[22]}} as well as in the search for maximally entangled symmetric states \text{[20] where the current characterizations were obtained up to second order.

In summary, we have introduced a tensorial representation of spin states that leads to a natural generalization of the Bloch sphere representation to arbitrary spin $j$, based on Weinberg’s covariant matrices \text{[12, \text{[22]}}. We have found a new way of representing these matrices as projections of elements of the Pauli group into the symmetric subspace of $2j$ spins-1/2, proving that they form a tight frame. Our representation shares beautiful and essential properties with the one for spin-1/2, and provides additional insight for larger spins that we have used for a novel characterization of anti-coherent spin states.
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Supplementary Information

In this supplemental material, we provide the proofs of the propositions, lemmas and theorems 1 and 3 stated in the main text. We recall that covariant matrices $S_{\mu_1 \mu_2 \ldots \mu_N}$ are obtained by identifying coefficients of

$$\Pi^{(j)}(q) = (-1)^{2j} q_{\mu_1} q_{\mu_2} \cdots q_{\mu_{2j}},$$

with the expansion of the polynomials (multivariate in the $q_i$)

$$\Pi^{(j)}(q) = (q_0^2 - q^2)^j + \sum_{k=1}^j \frac{(q_0^2 - q^2)^{j-k}}{(2k)!} (2q \cdot J) \left( \prod_{r=1}^{k-1} [(2q \cdot J)^2 - (2rq^2)] \right) [2q \cdot J + 2kq_0]$$

for integer $j$, and

$$\Pi^{(j)}(q) = (q_0^2 - q^2)^{j-1/2}(q_0 - 2q \cdot J) - \sum_{k=1}^{j-1/2} \frac{(q_0^2 - q^2)^{j-1/2-k}}{(2k+1)!} \left( \prod_{r=1}^k [(2q \cdot J)^2 - (2r q^2)] \right) (2q \cdot J + (2k+1)q_0)$$

for half-integer $j$, with $q = (q_0, q_1, q_2, q_3) \equiv (q_0, q)$. We denote $N = 2j$. The operator $\Pi^{(j)}(q)$ is proportional to the square of the Hermitian operator associated to a Lorentz boost in direction $q$ for a particle of mass $m$,

$$\Pi^{(j)}(q) = m^N \exp(-2\eta_q \cdot q),$$

where $\eta_q = |q/|q|$, and $\eta_q$ and $m$ are defined by

$$q_0 = -m \cosh \eta_q$$

$$|q| = m \sinh \eta_q.$$  

We recall that the $S_{\mu_1 \mu_2 \ldots \mu_N}$ are linked by a linear relation, given by

$$g_{\mu_1 \mu_2} S_{\mu_1 \mu_2 \ldots \mu_N} = 0,$$

where $g \equiv \text{diag}(-, +, +, +)$. Theorem 2 which was proved in the paper states that for general spin–$j$, the Hermitian matrices $S_{\mu_1 \mu_2 \ldots \mu_N}$ provide an overcomplete basis over which $\rho$ can be expanded, that is, any state can be expressed as

$$\rho = \frac{1}{2^N} x_{\mu_1 \mu_2 \ldots \mu_N} S_{\mu_1 \mu_2 \ldots \mu_N},$$

with real coefficients given by

$$x_{\mu_1 \mu_2 \ldots \mu_N} = \text{tr}(\rho S_{\mu_1 \mu_2 \ldots \mu_N}).$$

We now turn to the proofs.

Lemma 1. Let $|\alpha\rangle$ be a spin–$j$ coherent state, defined for $\alpha = e^{-i\varphi} \cot(\theta/2)$ with $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ by

$$|\alpha\rangle = \sum_{m=-j}^j \sqrt{\binom{2j}{j+m}} \left[ \sin^2 \frac{\theta}{2} \right]^{j-m} \left[ \cos^2 \frac{\theta}{2} e^{-i\varphi} \right]^{j+m} |j, m\rangle,$$

$$= \frac{1}{(1 + |\alpha|^2)^j} e^{\alpha J_+} |j, -j\rangle,$$

in the standard angular momentum basis $\{|j, m\rangle : -j \leq m \leq j\}$, and let $n = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. Then

$$\langle \alpha | \Pi^{(j)}(q) | \alpha \rangle = (-1)^N (q_0 + q \cdot n)^N.$$
Applying (21) to the coherent state $|\alpha\rangle$, we have

$$\exp(a_+ J_+ + b J_z + a_- J_-) = e^{b J_z} e^{a_- J_-} e^{a_+ J_+}$$

with constants given by

$$b_\pm = 2a_\pm \frac{\sinh(\delta/2)}{\delta \cosh(\delta/2) + b \sinh(\delta/2)}$$
$$b_z = 2 \ln \left( \frac{\delta \cosh(\delta/2) + b \sinh(\delta/2)}{\delta} \right),$$

where $\delta = \sqrt{b^2 + 4a_+ a_-}$. From its definition, $\Pi^{(j)}(q)$ can be written as in (13) with $a_\pm = -\eta_q(q_x + i q_y)$ and $b = -2\eta_q q_z$. Thus, using (14) and (15) we get

$$\langle \alpha | \Pi^{(j)}(q) | \alpha \rangle = m^N \langle \psi | e^{b_\pm J_z} | \psi \rangle$$

with

$$|\psi\rangle = \frac{1}{(1 + |\alpha|^2)^2} e^{(\alpha + b_\pm) J_z} |j, -j\rangle.$$

If we parametrize $q = (\sin \gamma \cos \phi, \sin \gamma \sin \phi, \cos \gamma)$, we get from (14)–(15) that the constants are given by $\delta = 2\eta_q$ and

$$b_\pm = -\frac{\sin \gamma e^{\mp i \phi}}{\coth \eta_q - \cos \gamma},$$
$$b_z = 2 \ln (\cosh \eta_q - \sinh \eta_q \cos \gamma).$$

We see that, up to normalization, $|\psi\rangle$ is of the form (11). Namely, we have

$$|\psi\rangle = \left( \frac{1 + |\alpha + b_\pm|^2}{1 + |\alpha|^2} \right)^j |\alpha + b_\pm\rangle.$$

From Eq. (10) we get for a coherent state $|\alpha\rangle$ and a complex number $t$ the identity

$$\langle \alpha | e^{2t J_z} | \alpha \rangle = (\cosh t + \sinh t \cos \theta)^N.$$

Applying (21) to the coherent state $|\alpha + b_\pm\rangle$ gives

$$\langle \alpha | \Pi^{(j)}(q) | \alpha \rangle = m^N \left( \frac{1 + |\alpha + b_\pm|^2}{1 + |\alpha|^2} \right)^N \left( \cosh \frac{b_z}{2} + \frac{|\alpha + b_\pm|^2 - 1}{|\alpha + b_\pm|^2 + 1} \sinh \frac{b_z}{2} \right)^N.$$

On the right-hand side of (12) we have the $N$th power of

$$g_0 - q \cdot \alpha = m (\cosh \eta_q - \sinh \eta_q (\sin \theta \sin \gamma \cos(\varphi - \phi) + \cos \gamma \cos \theta)).$$

Using the explicit expressions of $\alpha$, $b_\pm$ and $b_z$ in Eq. (22), Eq. (12) is now equivalent to a trigonometric identity that is easily verified.

**Proposition 1.** For a given density matrix $\rho$, Eq. (3) is the unique choice of coordinates which are real numbers symmetric under permutation of the indices and verifying

$$g_{\mu_1 \mu_2} x_{\mu_1 \mu_2 \ldots \mu_N} = 0$$

**Proof.** The fact that Eq. (24) is verified follows immediately from Eqs. (7) and (9), and by linearity of the trace. It remains to show the uniqueness of this choice. Matrices $S_{\mu_1 \mu_2 \ldots \mu_N}$ are not linearly independent since they are invariant under permutation of indices and related through Eq. (7). The number of distinct sets $(\mu_1, \ldots, \mu_N)$ up to permutation of indices is $N+3\choose 3$, from which one has to subtract the $N+1\choose 3$ relations (7). That leaves

$$N+3 - N+1 = (N+1)^2$$

(25)
independent basis matrices, which coincides with the number of independent parameters in a \((N + 1) \times (N + 1)\) Hermitian matrix (here we disregard the fact that \(\text{tr} \rho = 1\), which would just correspond to imposing \(x_{00\ldots0} = 1\)). This means that there cannot be any other relation between the \(S_{\mu_1\mu_2\ldots\mu_N}\) than the permutation-symmetry relations and the relations (7) (otherwise there would not be enough parameters to describe all matrices \(\rho\)). In particular, if we fix \(\mu_1, \ldots, \mu_N\), the vector space \(V_{\mu_1\ldots\mu_N}\) generated by matrices \(S_{\nu_1\nu_2\ldots\nu_N}, 0 \leq \nu \leq 3\), is of dimension \(\leq 4\), and from Eq. (7) we get that \(V_{\mu_1\ldots\mu_N}\) is of dimension 3. So the only additional relation one can impose between the \(x_{\mu_1\mu_2\ldots\mu_N}\) is between the \(x_{\nu
u\nu\ldots\mu_N}\). Choosing Eq. (21) thus defines coordinates of a density matrix in a unique way.

**Proposition 2.** Coordinates of the maximally mixed state \(\rho_0 = \frac{1}{N+1} \mathbb{1}_{N+1}\) (with \(\mathbb{1}_{N+1}\) the \((N + 1)\)-dimensional identity matrix) are given by

\[
x_{\mu_1\mu_2\ldots\mu_N} q_{\mu_1} \ldots q_{\mu_N} = \sum_{k=0}^{j} \frac{N_k}{2k + 1} q_0^{2(j-k)} |q|^2k.
\]

**Proof.** We use the expansion of the identity in terms of coherent states,

\[
\rho_0 = \frac{1}{4\pi} \int d\alpha |\alpha\rangle \langle \alpha|.
\]  

According to the main text, the coordinates \(x_{\mu_1\mu_2\ldots\mu_N}\) are the coefficients of the multivariate polynomial \((-1)^N \langle \Pi^{(j)}(q) \rangle\). Using (27) we get

\[
\text{tr}(\rho_0 \Pi^{(j)}(q)) = \frac{1}{4\pi} \int d\alpha \langle \alpha | \Pi^{(j)}(q) | \alpha \rangle = (-1)^N \frac{1}{4\pi} \int d\alpha (q_0 + q \cdot n)^N,
\]

where the last equality comes from Lemma [1]. Since the integral runs over the whole sphere, one can take \(q = (0, 0, |q|)\) and rewrite the integral as

\[
\frac{(-1)^N}{2} \int_0^\pi d\theta (q_0 + |q| \cos \theta)^N \sin \theta
= (-1)^N \sum_{k=0}^{N} \frac{N_k}{2k + 1} q_0^{N-k} |q|^k \frac{1 + (-1)^k}{2k + 2}.
\]

Using Eq. [1], we get the result.

**Proposition 3.** Coordinates of the spin–\(j\) reduced density matrix obtained by tracing the spin–\(j\) matrix over \(j - k\) spins are given by

\[
x_{\mu_1\mu_2\ldots\mu_{2k}} = x_{\mu_1\mu_20\ldots0}
\]

**Proof.** Let \(\rho\) be a density matrix. Expanding \(\rho\) over coherent states as

\[
\rho = \int d\alpha P(\alpha) |\alpha\rangle \langle \alpha|,
\]

we get

\[
\text{tr}(\rho \Pi^{(j)}(q)) = \int d\alpha P(\alpha) \langle \alpha | \Pi^{(j)}(q) | \alpha \rangle.
\]

Using Lemma [1] and the expansion [11], Eq. [32] gives

\[
q_{\mu_1} \ldots q_{\mu_2} \text{tr}(\rho S_{\mu_1\mu_2\ldots\mu_{2j}}) = \int d\alpha P(\alpha) (q_0 + q \cdot n)^{2j},
\]

where \(n\) is the unit vector associated with \(|\alpha\rangle\). In the expansion [31], coherent states \(|\alpha\rangle\) are a 2\(j\)–fold tensor product of spin–1/2 coherent states \(|\alpha^{(1/2)}\rangle\). The trace of \(|\alpha\rangle \langle \alpha|\) over \(j - k\) spins is the 2\(k\)–fold tensor product of the projector on \(|\alpha^{(1/2)}\rangle\). From Eqs. [33] and [9], we get

\[
x_{\mu_1\mu_2\ldots\mu_{2k}} q_{\mu_1} \ldots q_{\mu_{2j}} = \int d\alpha P(\alpha) (q_0 + q \cdot n)^{2j}.
\]
and the coordinates \(x_{\mu_1 \mu_2 \ldots \mu_{2k}}\) of the spin–\(k\) reduced density matrix are thus given by

\[
x_{\mu_1 \mu_2 \ldots \mu_{2k}} q_{\mu_1} \cdots q_{\mu_{2k}} = \int d\alpha P(\alpha)(q_0 + q \cdot n)^{2k}.
\]

The \(x_{\mu_1 \mu_2 \ldots \mu_{2k}}\) defined by (35) can then be directly read off Eq. (34), which yields the result.

**Theorem 3.** A spin–\(j\) density matrix is anticoherent to order \(t\) if and only if its spin–(\(t/2\)) reduced density matrix is the maximally mixed state \(\rho_0 = \frac{1}{N+1}1_{t+1}\).

**Proof.** Let us first consider the case \(t = 2j\). The matrix \(\rho_0\) is entirely characterized by the quantities \(\langle \beta | \rho_0 | \beta \rangle\), where \(|\beta\rangle\) runs over coherent states. If one expands \(\rho_0\) as in (31), then

\[
\langle \beta | \rho_0 | \beta \rangle = \frac{1}{N+1} = \int d\alpha P(\alpha)(1 + n' \cdot n)^N,
\]

where \(n'\) denotes the unit vector corresponding to \(|\beta\rangle\). The maximally mixed state is thus characterized by the fact that for any \(n'\) its \(P\)-function verifies the right-hand equality in Eq. (36). Any state such that the right-hand side of (36) is independent of \(n'\) is thus proportional to \(\rho_0\).

Anticoherence to order \(2j\) means that \(\langle (n \cdot J)^k \rangle\) is independent of \(n\) for \(k\) up to \(2j\). The operator \(n \cdot J\) has eigenvalues \(-j, -j+1, \ldots, j\), and since its characteristic polynomial is also a minimal polynomial, one has

\[
\prod_{m=-j}^{j} (n \cdot J - m) = 0,
\]

which, by the way, is the reason why (41) can be expanded into a finite sum (2). From Eq. (37) this means that \(\langle (n \cdot J)^k \rangle\) is in fact independent of \(n\) for any \(k\), which in turn implies that \(\langle \Pi^{(j)}(q) \rangle\) is independent of \(q\). If \(P\) is the \(P\)-function associated with \(\rho\) as in (31), Eqs. (32)–(33) imply that

\[
\int d\alpha P(\alpha)(q_0 + q \cdot n)^N
\]

is independent of \(q\). In particular, for \(q_0 = 1\) and \(q = n'\) one recovers the condition (36) which characterizes \(\rho_0\) (up to a multiplicative constant, which is then fixed by the normalization condition \(tr\rho_0 = 1\)). Thus anticoherence to order \(2j\) implies that \(\rho = \rho_0\). The converse is true: since from Proposition 2 the coordinates of \(\rho_0\) do not depend on \(q\), Eq. (23) of the paper,

\[
(-1)^N \langle \Pi^{(j)}(q) \rangle = x_{\mu_1 \mu_2 \ldots \mu_N} q_{\mu_1} \cdots q_{\mu_N},
\]

implies that \(\langle \Pi^{(j)}(q) \rangle\) does not depend on \(q\), and thus the coefficients of its series expansion in powers of \(q_0\), obtained from (41), do not depend on \(q\) either.

Let us now consider a state \(\rho\) anticoherent to order \(t \leq 2j\), and let \(x_{\mu_1 \ldots \mu_N}\) be its coordinates. From (9) and proposition 3 the spin–(\(t/2\)) reduced density matrix of \(\rho\) has coordinates given by \(x_{\mu_1 \ldots \mu_0 \ldots 0}\). Following Proposition 2 we thus want to show that a state is anticoherent to order \(t\) if and only if its coordinates are such that

\[
x_{\mu_1 \mu_2 \ldots \mu_N} q_{\mu_1} \cdots q_{\mu_N} \equiv t/2 \prod_{k=0}^{t/2} \frac{t/2 + 1}{2k + 1} q_0^{t-2k} |q|^{2k}.
\]

or, equivalently,

\[
x_{\mu_1 \mu_2 \ldots \mu_N} q_{\mu_1} \cdots q_{\mu_N} \equiv t/2 \prod_{k=0}^{t/2} \frac{t/2 + 1}{2k + 1} q_0^{t-2k} |q|^{2k}.
\]

From the form (2), expanding the powers of \(q_0^2 - |q|^2\), one has, for integer \(j\),

\[
\langle \Pi^{(j)}(q) \rangle = \sum_{s=0}^{j} \binom{j}{s} q_0^{2(j-s)} |q|^{2s} + \sum_{k=1}^{j-k} \sum_{i=0}^{j-k} \binom{j-k}{i} q_0^{2(j-k-i)} (2k)! (q_0^2 - |q|^2)^i \langle 2q \cdot J \rangle \left( \prod_{r=1}^{k} [2q \cdot J - (2r |q|^2)] \right) (2q \cdot J + 2k q_0).
\]

(42)
Following Eq. (39), the \( x_{\mu_1\mu_2\ldots\mu_t0\ldots0} \) are obtained by considering the terms containing a factor \( q_0^k \) with \( k \geq N-t \) in \((-1)^N\langle j \rangle(q)\). The term in \( q_0^{2(j-s)} \) in (42) reads

\[
q_0^{2(j-s)} \left\{ \sum_{s=0}^{j} \frac{(j-k)}{(s-k)!(2k)!} \left( \langle 2q \cdot J \rangle \left( \prod_{r=1}^{k-1} [(2q \cdot J)^2 - (2r q)^2] \right) \right) \right\}
\]

and the term in \( q_0^{2(j-s)+1} \) in (42) reads

\[
q_0^{2(j-s)+1} \left\{ \sum_{s=0}^{j} \frac{(j-k)}{(s-k)!(2k-1)!} \left( \langle 2q \cdot J \rangle \left( \prod_{r=1}^{k-1} [(2q \cdot J)^2 - (2r q)^2] \right) \right) \right\}
\]

The largest power of \( q \cdot J \) in (43) corresponds to \( k = s \) and \( r = k-1 \), which gives a power \( (q \cdot J)^{2s} \). From the definition of an anticoherent state of order \( t \), we have that for \( 0 \leq s \leq t/2 \), all powers of \( q \cdot J \) appearing in (43) are such that their average does not depend on \( \hat{q} \). A similar reasoning holds for Eq. (44). One can thus rewrite (43) and (44) respectively as

\[
q_{0}^{N-2s}|q|^{2s} \sum_{k=0}^{s} \frac{(j-k)}{(s-k)!} \left( \prod_{r=0}^{k-1} [(2q \cdot J)^2 - (2r q)^2] \right)
\]

and

\[
q_{0}^{N-2s+1}|q|^{2s+1} \sum_{k=0}^{s+1} \frac{(j-k)}{(s-k)!(2k-1)!} \left( \langle 2q \cdot J \rangle \prod_{r=1}^{k-1} [(2q \cdot J)^2 - (2r q)^2] \right),
\]

where any \( \langle (q \cdot J)^k \rangle \) can be replaced by \( \langle (J_z)^k \rangle \).

When summing over \( \mu_1,\ldots,\mu_t \) in the left-hand side of (41), the coefficient of a given \( q_0^{N-t} \) is \( x_{\mu_1\mu_2\ldots\mu_t0\ldots0} \) multiplied by the number of permutations of \( \mu_1,\mu_2,\ldots,\mu_t \), while the coefficient of \( q_0^{N-t} \) in (41), or in (43)–(44), is obtained from (1) and (2) as \( x_{\mu_1\mu_2\ldots\mu_t0\ldots0} \) multiplied by the number of permutations of \( \mu_1,\mu_2,\ldots,\mu_t,0,\ldots,0 \). Let us group terms according to the number \( p_\nu \) of indices \( \mu_1,\mu_2,\ldots,\mu_t \) equal to \( \nu \). In order to identify terms containing a power \( p_\nu^{N-t} \) on both sides of Eq. (41) we must consider terms such that \( k = (t-p_0)/2 \) when \( t-p_0 \) is even (the contribution is 0 for odd \( t-p_0 \)). Similarly we must take terms such that \( s = (t-p_0)/2 \) in (43) when \( t-p_0 \) is even, and terms such that \( s = (t-p_0+1)/2 \) in (44) when \( t-p_0 \) is odd. In order to show equality between the \( x_{\mu_1\mu_2\ldots\mu_t0\ldots0} \) defined from (41) and those defined from (43) for \( t-p_0 \) even, one must show that the coefficient of \( q_0^{N-2s} \) in (43) with \( s = (t-p_0)/2 \) is equal to the coefficient of \( q_0^{N-2k} \) with \( k = (t-p_0)/2 \), multiplied by

\[
\#\text{perm.} \{\mu_1\mu_2\ldots\mu_p0\ldots0\} / \#\text{perm.} \{\mu_1\mu_2\ldots\mu_t\} = \frac{\left( \begin{array}{c} p_0+N-t-p_1-p_2-p_3 \\end{array} \right)}{(p_0+p_1+p_2+p_3)} = \frac{N!p_0!}{(p_0+N-t)!} = \frac{(N)}{(2k)} = \frac{(N)}{(2k)}.
\]

where \( \left( \begin{array}{c} n \end{array} \right) \) stands for the multinomial coefficient \( \frac{n!}{m_1!m_2!\ldots m_n!} \). From the right-hand side of (41), the coefficient of \( q_0^{N-2k} \) multiplied by the combinatorial factor (47) is readily seen to be

\[
\frac{(N)}{(2k)} = \frac{(N)}{(2k)}
\]

for \( (t-p_0)/2 \) integer. The equality between (47) and the coefficient obtained from (41) has to be shown for all \( s \) from 0 to \( t/2 \). But note that \( J_z \) has been eliminated both from (43) and (44), so that for fixed \( s \) and \( k \) these equations are the same as the ones between coefficients of a spin–\( j \) state anticoherent to order \( 2j \) (the only difference being that in the latter case Eq. (45) would hold for all \( s \) up to \( s = j \)). Since the case \( t = 2j \) has already been proved, the result ensues. The same argument applies to the case \( t-p_0 \) odd. Finally, the result for half-integer \( j \) can be derived along the same line of reasoning.

**Corollary 1.** A spin–\( j \) state \( \rho \) is anticoherent to order \( t \) if and only if \( \rho_{kq} = 0 \), \( \forall k \leq t, \forall q : -k \leq q \leq k \).

**Proof.** Let \( \rho \) be a spin–\( j \) density matrix with multipolar expansion

\[
\rho = \sum_{k=0}^{2j} \sum_{q=-k}^{k} \rho_{kq} \mathcal{T}_{kq}^{(j)}
\]
with $\rho_{kq} = \text{tr}(\rho T_{kq}^{(j)})$, where $T_{kq}^{(j)}$ are the irreducible tensor operators

$$T_{kq}^{(j)} = \sqrt{\frac{2k+1}{2j+1}} \sum_{m,m'=\pm j} C^{jm'}_{jm,kq} |j,m'\rangle \langle j,m|,$$

and $C^{jm'}_{jm,kq}$ are Clebsch-Gordan coefficients. For an anticoherent state to order $t$, $\rho_{kq} = 0 \ \forall \ k \leq t$ follows from the fact that coefficients $\rho_{kq}$ are proportional to coefficients $R_{kq}$ of the expansion of $\langle \alpha | \rho | \alpha \rangle$ over spherical harmonics (see e.g. [2]). From Eq. (25) of the paper,

$$\text{tr}(\rho \rho') = \frac{1}{2N} x_{\mu_1 \mu_2 \ldots \mu_N} x'_{\mu_1 \mu_2 \ldots \mu_N},$$

one gets

$$\langle \alpha | \rho | \alpha \rangle = \sum_{k,q} R_{kq} Y_{kq}(\theta, \phi) = \frac{1}{2N} x_{\mu_1 \mu_2 \ldots \mu_N} n_{\mu_1} n_{\mu_2} \ldots n_{\mu_N}$$

with $n = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. If a state $\rho$ is such that its spin–(t/2) reduced density matrix is the maximally mixed state then from [3] its coefficients $x_{\mu_1 \mu_2 \ldots \mu_0 \ldots 0}$ are given by [20], and all terms $x_{\mu_1 \mu_2 \ldots \mu_0 \ldots 0} n_{\mu_1} \ldots n_{\mu_t}$ on the right-hand side of (52) can be resummed to a constant independent of $\theta$ and $\phi$ given by

$$x_{\mu_1 \mu_2 \ldots \mu_0} n_{\mu_1} \ldots n_{\mu_t} = \sum_{k=0}^{t/2} \frac{(\frac{t}{2})}{2k+1} n_{0}^{t-2k} |n|^{2k} = \frac{2^t}{t+1}. \quad (53)$$

What remains in the sum (52) are the terms $x_{\mu_1 \mu_2 \ldots \mu_N} n_{\mu_1} n_{\mu_2} \ldots n_{\mu_N}$ with at least $N - t$ non-vanishing indices $\mu_i$, yielding trigonometric polynomials in $\theta$ and $\phi$ of order at least $t + 1$ and thus all coefficients $R_{k\ell}$ with $k \leq t$ (apart from $R_{00}$) vanish, and so do coefficients $\rho_{kq}$.

As an illustration, let us consider specific examples.

**Spin–1 anticoherent states to order 1.** The most general form of spin–1 states that are anticoherent to order 1 is obtained by setting $x_{00} = 1$, $x_{01} = x_{02} = x_{03} = 0$. The condition $x_{00} = 1$ imposes unit trace of $\rho$. The remaining conditions $x_{01} = x_{02} = x_{03} = 0$ imply, according to Eqs. (26) and (3), that the spin–1/2 reduced density matrix is maximally mixed. This yields a density matrix in the $|j,m\rangle$ basis of the form

$$\rho = \left( \begin{array}{ccc} \frac{1}{2} + a & \beta & \gamma \\ \beta^* & -2a & -\beta \\ \gamma^* & -\beta^* & \frac{1}{2} + a \end{array} \right) \quad (54)$$

with non-zero coordinates (up to permutations)

$$\begin{align*}
x_{00} &= 1 \\
x_{11} &= 2[\text{Re}(\gamma) - a], \quad x_{12} = 2\text{Im}(\gamma), \quad x_{13} = -2\sqrt{2} \text{Re}(\beta) \\
x_{22} &= -2[\text{Re}(\gamma) + a], \quad x_{23} = 2\sqrt{2} \text{Im}(\beta), \quad x_{33} = 4a + 1
\end{align*} \quad (55)$$

where $\beta, \gamma \in \mathbb{C}$ and $a \in \mathbb{R}$. Positivity of $\rho$ is however not yet guaranteed and imposes additional constraints on the values of $\beta, \gamma$ and $a$. If we ask that all principal minors of $\rho$ are nonnegative, which translates into the conditions

$$a(1 + 2a) \leq -|\beta|^2$$

$$2a \left( |\gamma|^2 - |\beta|^2 - \frac{1}{4} - (1 + a) \right) \geq |\beta|^2 + 2\text{Re}[\beta^2] \quad (56)$$

then the matrix $\rho$ is positive semi-definite and represents a possible spin–1 state anticoherent to order 1. Conversely, every spin–1 state anticoherent to order 1 has the form (54) with $\beta, \gamma \in \mathbb{C}$ and $a \in \mathbb{R}$ verifying conditions (56). An expression for spin–3/2 anticoherent state to order 2 has been given in [3].
Anticoherent states to order 2. From Theorem 3 and Eqs. (9) and (26), spin-\( j \) state anticoherent to order 2 is characterized by the fact that the matrix \( A_{\mu\nu} \equiv \langle S_{\mu\nu-00...0} \rangle \) is given by

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1}{3}
\end{pmatrix}.
\] (57)

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