THE KONTSEVICH INTEGRAL AND ALGEBRAIC STRUCTURES ON THE SPACE OF DIAGRAMS

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ABSTRACT

This paper is part expository and part presentation of calculational results. The target space of the Kontsevich integral for knots is a space of diagrams; this space has various algebraic structures which are described here. These are utilized with Le’s theorem on the behaviour of the Kontsevich integral under cabling and with the Melvin-Morton Theorem, to obtain, in the Kontsevich integral for torus knots, both an explicit expression up to degree five and the general coefficients of the wheel diagrams.

Keywords: Vassiliev invariants, torus knots, Kontsevich integral.

INTRODUCTION

The Kontsevich integral, $Z$, is a map from the set of oriented knots to a space, $\mathcal{A}$, of “diagrams”. For example it will be seen in Section 8 that the value taken on a negative trefoil — the $(2, -3)$-torus knot — begins

$$Z \left( \begin{array}{c} \circ \end{array} \right) = \begin{array}{c} \circ \end{array} - \frac{1}{2} \begin{array}{c} \circ \end{array} - \frac{1}{2} \begin{array}{c} \circ \end{array} - \frac{31}{48} \begin{array}{c} \circ \end{array} + \frac{21}{24} \begin{array}{c} \circ \end{array} + \frac{1}{8} \begin{array}{c} \circ \end{array} + \ldots \right.$$

The Kontsevich integral was introduced in [2, 14]; it is universal for Vassiliev knot invariants in the sense that every finite-type invariant factors through the Kontsevich integral, and it is universal in a certain sense for knot invariants coming from quantum groups. Other approaches to defining this invariant have been introduced but none give in easily to attempts at actual calculation.

The space of diagrams has a rich algebraic structure, the following will be described in this paper: it has two products making it a graded Hopf algebra in two ways, it has a set of simultaneously diagonalizable “Adams operations” and the space of primitive elements in $\mathcal{A}$ has the structure of a module over Vogel’s algebra $\Lambda$. How all of these structures interact and how they relate to topological properties of knots is not yet properly understood. However, they can be utilized to describe the Kontsevich integral for torus knots and to identify the Alexander polynomial in terms of wheel diagrams. These last two things are taken in this paper as motivations for the algebraic structures described herein.

The paper is organized as follows. The first section contains a brief sketch of the original definition of the Kontsevich integral. The second section introduces the target space $\mathcal{A}$ and gives an explicit basis up to degree five — the specific choice
Figure 1. A Morse knot together with a set of pairings $P$ for $m = 3$, and the corresponding diagram $D_P$.

of basis is made to interact nicely with the algebraic structure described later. The Hopf algebra structure of $A$ coming from the connect sum product is presented in Section 3, together with some consequences of the Hopf algebra structure theorems, such as identifying $A$ as being freely generated by the space spanned by connected diagrams. Section 4 shows how the theory naturally extends to framed knots and how the target space is extended to $\mathcal{I}A$ by the addition of an extra generator to encode the framing. In Section 5 the Adams operations are introduced, as are symmetrized diagrams which are eigenvectors for these operations; also wheel diagrams are defined and used with the disjoint union product to give a description of the element $\Omega$. The module structure of the primitive elements of $A$ over Vogel’s algebra $\Lambda$ is the subject of the sixth section. Cabling knots and Le’s theorem on the behaviour of the Kontsevich integral are contained in the seventh section. Section 8 presents the result of calculating the Kontsevich integral of torus knots by using the machinery so far introduced, and also gives a criterion for Vassiliev invariants of torus knots. The final section shows how the Melvin-Morton Theorem allows the identification of the wheels coefficients from the Alexander polynomial and, in particular, does this explicitly for torus knots. Appendix A lists the change of basis data from round diagrams to symmetrized diagrams used in Section 8 and Appendix B compares the results here with those of Alvarez and Labastida.

This paper is an extended version of the talk that I gave at Knots in Hellas ’98, and this supersedes the earlier preprint [27].

1. The Kontsevich integral.

Here I will try to give a flavour of the original definition of the Kontsevich integral — it still remains somewhat mysterious — one of the goals of this paper is to show that it can be calculated for certain knots without performing any sorts of integral. The target space $A$ will not be defined until the next section, so here it should be thought of as “linear combinations of diagrams”.

Suppose that $K$ is a Morse knot, that is a knot embedded in $\mathbb{R}^3 = \mathbb{R}_t \times \mathbb{C}_z$ such that the projection to the $\mathbb{R}$ co-ordinate is a Morse function. Then define

$$\tau(K) := \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\text{pairings}} \sum_{P=\{(z_j, z'_j)\}} (-1)^{\#P}[D_P] \prod_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j}.$$
where a set of pairings \( P \) is a set of unordered pairs \( \{z_j, z'_j\} \) so that \( (t_j, z_j), (t_j, z'_j) \in \mathbb{R} \times \mathbb{C} \) are distinct points on the knot on the same level, where \( \# \downarrow P \) is the number of points in the pairing such that the orientation is pointing downwards and where \( D_P \) is the diagram obtained by drawing the preimage of the knot — a circle — and drawing chords connecting the points corresponding to the pairs. See Figure 1.

Considered as an element of \( \mathcal{A} \), this gives an invariant of Morse knots, but is not invariant if one introduces a “hump”, i.e. a minima-maxima pair with respect to the \( \mathbb{R} \)-axis, which increases the number of critical points. It is necessary to introduce the following correction term: define \( \Omega := \tau \left( \infty \infty \right)^{-1} \in \mathcal{A} \), where this is a Morse embedding of the unknot with four critical points. \( \Omega \) is an important element which will also be returned to later in the paper. Now the Kontsevich integral for a Morse knot \( K \) with \( c(K) \) critical points is defined to be

\[
Z(K) := \tau(K).\Omega^{c(K)/2-1} \in \mathcal{A}.
\]

For a pleasant survey going into some of the finer points of this definition, the reader is directed [8].

It is important to note here that there are two normalizations of the Kontsevich integral in common use: the other can be defined by \( Z_q(K) := \tau(K).\Omega^{c(K)/2} \). It is clear from this that the two normalizations differ by a factor of \( \Omega \) and that \( Z(\text{unknot}) = \bigcirc \), while \( Z_q(\text{unknot}) = \Omega \), and so \( \Omega \) is often referred to as “the Kontsevich integral of the unknot”. Both normalizations have their advantages: \( Z \) is multiplicative under connect sum, while \( Z_q \) is better behaved under cabling operations. In this work I will use exclusively the multiplicative normalization.

There are a couple of other ways of defining such a knot invariant (see [6]): one is via quasi-Hopf algebras and Drinfeld associators, this is essentially a discretized version of the above; the other way is the so-called Bott-Taubes approach using configuration space integrals, this is believed to give the same answer as the Kontsevich integral above.

2. The Space of Diagrams \( \mathcal{A} \).

The target space \( \mathcal{A} \) of the Kontsevich integral will be defined here, and a basis given up to degree five.

First it is necessary to define the notion of a diagram (which I will also call a round diagram) — this was called a Chinese character diagram in [2]. A diagram is a trivalent connected graph such that each vertex is cyclically oriented and such that there is a distinguished oriented cycle called the Wilson loop. A diagram is graded by half the number of vertices that it has. By convention a diagram is drawn such that the Wilson loop is an anticlockwise oriented circle, and such that each vertex is also oriented anticlockwise. See Figure 2. The complement of Wilson loop is termed the internal graph.

Define the rational vector space \( \mathcal{A}_n \) to be the vector space spanned by the diagrams of degree \( n \) and subject to the so-called STU and 1T relations:

\[
\mathcal{A}_n := \langle \text{diagrams of degree } n \rangle / \left\{ \begin{array}{c}
\begin{array}{c}
\includegraphics{STUT.png}
\end{array}
\end{array} \right\} \quad 1T
\]

\[
\begin{array}{c}
\includegraphics{STUT.png}
\end{array} = \begin{array}{c}
\includegraphics{STUT.png} - \includegraphics{STUT.png}
\end{array} : \quad \text{STU}.
\]
Figure 2. A degree three diagram. Note that the four-valent vertex is not a vertex of the diagram, but just a result of the planar drawing.

Note that the 1T relation means that a diagram vanishes if it has a chord which is not intersected by any other part of the internal graph. It is also worth noting that the antisymmetry (AS) relation is a direct consequence of the STU relation: antisymmetry says that reversing the orientation of one vertex in a diagram is the same as multiplying the diagram by minus one; pictorially,

\[
\begin{array}{c}
\text{AS.}
\end{array}
\]

Look now at the spaces of low order, and find explicit bases. Degree zero is very easy:

\[
\mathcal{A}_0 := \langle \begin{array}{c}
\end{array} \rangle_Q.
\]

Degree one is almost as easy:

\[
\mathcal{A}_1 := \langle \begin{array}{c}
\end{array} \rangle_Q / \left( \begin{array}{c}
\end{array} = 0 \right) \cong 0.
\]

It starts getting interesting at degree two:

\[
\mathcal{A}_2 := \langle \begin{array}{c}
\end{array} \rangle_Q / \left\{ \begin{array}{c}
\end{array} = 0,
\end{array} = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array},
\end{array} = 2 \begin{array}{c}
\end{array} \right\}.
\]

That has four generators and three relations, so pick the following generator:

\[
\mathcal{A}_2 \cong \langle \begin{array}{c}
\end{array} \rangle_Q.
\]

Degree three is more awkward:

\[
\mathcal{A}_3 := \langle \begin{array}{c}
\end{array} \rangle_Q / \text{relations}.
\]

It transpires that \( \mathcal{A}_3 \) is also one dimensional and the generator can be chosen to be as follows:

\[
\mathcal{A}_3 \cong \langle \begin{array}{c}
\end{array} \rangle_Q.
\]

For later use, we will just write down bases for \( \mathcal{A}_4 \) and \( \mathcal{A}_5 \):

\[
\mathcal{A}_4 \cong \langle \begin{array}{c}
\end{array} \rangle_Q ; \quad \mathcal{A}_5 \cong \langle \begin{array}{c}
\end{array} \rangle_Q.
\]


| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|
| dim $A_n$ | 1 | 0 | 1 | 1 | 3 | 4 | 9 | 14 | 27 | 44 | 80 | 132 | 232 |

Table 1. The dimensions of the spaces $A_n$ up to $n = 12$ as given in [2, 12].

A rationale for this choice of basis should become apparent during the course of the paper; it has primarily been chosen to simplify the calculation of the results in Appendix A. The dimension of $A_n$ is only known for $n \leq 12$ and these dimensions are tabulated in Table 1. See [10] for some results on asymptotic lower bounds on the dimensions of these spaces.

Define $A$ to be the projective limit of the direct sum $\bigoplus_{n=0}^{\infty} A_n$, so an element of $A$ is a possibly infinite linear combination of diagrams which is finite in each degree. This space has lots of algebraic structure which will be described in the sections below.

3. HOPF ALGEBRAIC STRUCTURE OF $A$.

In this section I will introduce a Hopf algebra structure for $A$ coming from the connect sum product, and show consequences of the Hopf algebra structure theorems, notably that the Kontsevich integral of a knot can be expressed as an exponential of a sum of connected diagrams.

3.1. Product and coproduct on $A$. Firstly, $A$ has a natural product $A \otimes A \to A$. This is defined on diagrams in terms of a connect sum operation.

E.g.

$$ \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array}. 
\end{array} $$

The resulting diagram is, of course, dependent on where the cuts are made in the two diagrams; however, the result is well-defined modulo the STU relation (see [2]). This makes $A$ into a graded, commutative algebra. The unit is the diagram which has no internal graph.

This product is related to the connect sum operation, $\#$, on knots via the Kontsevich integral, as the following theorem demonstrates.

**Theorem 1** (Kontsevich). For $K$ and $L$ knots: $Z(K\#L) = Z(K).Z(L)$. 

Secondly, $A$ has a coproduct $\Delta: A \to A \otimes A$. This is defined on a diagram by taking all of the ways that the internal graph can be decomposed into an ordered pair of complementary sets of components.

E.g.

$$ \Delta \left( \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array} \right) = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{connect_sum}
\end{array}
\end{array}. 
\end{array} $$

---

1This means commutative in the genuine sense and not the commutative-graded sense. Philosophically this is because $A$ all lives in homological degree zero. The structure theorems of [2] deal with commutative-graded algebras, but this is not a problem as, if necessary, the grading of $A$ can be doubled so that everything lives in even degree.
The counit is the function \( A \rightarrow \mathbb{Q} \) which maps the diagram with no internal graph to 1 and which vanishes on all other diagrams. This coproduct is graded and cocommutative, and is also compatible with the product in the sense that it is a map of algebras: \( \Delta(a \cdot b) = \Delta(a) \cdot \Delta(b) \). This makes \( A \) into a connected, graded Hopf algebra \( \text{la Milnor and Moore}^{2} \).

3.2. Consequences of Hopf algebra structure theorems. Now we can utilize the nice structure theorems of commutative, cocommutative Hopf algebras.

There are two special subsets of a Hopf algebra. The first is the vector space of primitive elements; this is defined to be the set of elements which satisfy \( \Delta(x) = x \otimes 1 + 1 \otimes x \). This space naturally forms a Lie algebra under the commutator bracket, but in the commutative case of \( A \) the bracket is trivial. The space of primitive elements of \( A \) certainly contains those diagrams whose internal graph is connected — the so-called connected diagrams.

Connected diagrams: \( \otimes \), \( \otimes \). Non-connected diagrams: \( \otimes \), \( \otimes \).

In fact by the following theorem it suffices to know these diagrams.

**Theorem 2 (Bar-Natan).** The vector space of primitive elements of \( A \) is (projectively) spanned by connected diagrams.

One of the key structure theorems of Milnor and Moore is that a commutative and cocommutative Hopf algebra is (naturally isomorphic to) the polynomial algebra over its primitive elements. Translating this into the current context we obtain:

**Theorem 3.** The algebra \( A \) is the completed polynomial algebra on the connected diagrams.

In particular this means that if one has a basis for the space of connected diagrams then this can be extended to a basis of \( A \) by taking the monomials in the basis of connected diagrams. Now look back to the bases given in Section 2 and see that this is precisely what I have done. The basis elements are all connected except one in degree four and one in degree five which are products of lower degree connected basis elements.

The second important subset of a Hopf algebra is the set of group-like elements; these are the elements which satisfy \( \Delta(g) = g \otimes g \). This set forms a group under the product; the inverse operation is just the antipode map. This set is important in the current context because of the following theorem.

**Theorem 4 (Kontsevich).** The Kontsevich integral maps knots to group-like elements.

A theorem of Quillen \( ^{2} \) asserts that, under reasonable hypotheses, the usual exponential map, \( \exp(x) := \sum_{i=0}^{\infty} x^i / i! \), defines a bijection of sets from the primitive elements to the group-like elements, with the inverse being given by logarithm. Combining with Theorem 4 this gives

**Theorem 5.** The Kontsevich integral of a knot is expressible as the exponential of a sum of connected diagrams.

\(^{2}\) This implies that it is uniquely equipped with an antipode, but this is not really of concern here.
Figure 3. This framed trefoil can be drawn in the blackboard framing by adding a kink.

For instance the example from the introduction can be rewritten as

\[ Z \left( \includegraphics[scale=0.5]{framed_trefoil} \right) = \exp \left( -\frac{1}{2} \left( -\frac{5}{31} - \frac{1}{2} \left( -\frac{41}{38} + \frac{5}{24} + \ldots \right) \right) \right). \]

Here \( \exp \) means that the connect sum product is used in the definition of the exponential map; this is to prevent confusion when the disjoint union product, \( \sqcup \), is introduced later.

The Baker-Campbell-Hausdorff formula describes how the product on the group-like elements is pulled back to the primitive elements; in the commutative case this reduces to just addition on the space of primitive elements. Together with Theorem 1, this means that not only is \( \ln \circ Z \) well defined on knots, but that it is additive under connect sum of knots: \( \ln \circ Z(K \# L) = \ln \circ Z(K) + \ln \circ Z(L) \).

4. Framed version of the theory.

The Kontsevich integral can be naturally defined for framed knots. The target space of this has one extra generator which lives in degree one — this keeps count of the framing.

4.1. Framed knots. A framed knot is a knot equipped with a homotopy class of sections of the normal bundle. This can be thought of as a parallel strand running very close to the knot. A knot diagram is said to be blackboard framed if the framing curve lies parallel to the knot in the plane of the paper. Any framed knot can be drawn in the blackboard framing, possibly by adding kinks, see Figure 3.

For a given knot, the framings are classified by the framing number (also called the writhe) which is the linking number of the knot and its framing curve: for a framed knot \( K \), this will be denoted by \( F(K) \).

4.2. The Hopf algebra \( \mathcal{L}A \). The Kontsevich integral for framed knots will be defined below to take values in a Hopf algebra of diagrams defined similarly to \( A \), but without the 1T relation, namely:

\[ \mathcal{L}A := \langle \text{diagrams} \rangle_{\mathbb{Q}} / \text{STU}. \]

Thus \( A \cong A/1T \). In fact, \( A \) can be identified as a sub-Hopf algebra of \( \mathcal{L}A \) by an inclusion \( \varphi : A \hookrightarrow \mathcal{L}A \). Furthermore \( \mathcal{L}A \) has just one extra primitive generator — this is the diagram in degree one: \( \includegraphics[scale=0.5]{writhe} \). So there is the natural isomorphism \( \mathcal{L}A \cong A \oplus \includegraphics[scale=0.5]{writhe} \mathcal{L}A \).
4.3. The framed Kontsevich integral. The Kontsevich integral for oriented framed knots, taking values in $\mathcal{A}$, can be defined by

$$\mathcal{A}(K) := Z(K) \cdot \exp\left(\frac{F(K)}{2}\right).$$

This looks a little *ad hoc*, but it is equivalent to other definitions, see [18]. Note that this extra generator of $\mathcal{A}$ encodes precisely the framing data and on the level of logarithms

$$\ln \circ \mathcal{A}(K) = \ln \circ Z(K) + F(K)\exp(1)/2.$$

5. Adams operations and symmetrized diagrams.

5.1. Adams operations. On the algebra $\mathcal{A}$ there is a set of operations, $\{\psi^m\}_{m \in \mathbb{Z}}$, called the Adams operations. For $m$ a positive integer, $\psi^m$ is defined on a diagram by lifting the vertices on the Wilson loop to the $p$th connected cover of the Wilson loop.

E.g.

$$\psi^3\left(\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}\right) = \left(\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}\right) = 33 \bigcirc + 48 \bigcirc.$$

For $m$ a negative integer, one takes $\psi^{-m}$ of the diagram with the orientation of the Wilson loop reversed — note it is unknown whether reversing the orientation of the Wilson loop induces the identity map on $\mathcal{A}$ or not.

The Adams operations satisfy the relation $\psi^m \circ \psi^p = \psi^{mp}$, but they are not algebra maps, so in general $\psi^m(a,b) \neq \psi^m(a)\psi^m(b)$. However the $\{\psi^m\}$ are co-algebra maps and in fact $\mathcal{A}$ can be equipped with a different multiplication for which the $\{\psi^m\}$ are Hopf algebra maps — see below.

5.2. Symmetrized diagrams and the disjoint union product. In certain situations, such as doing calculations involving the Adams operations or for expressing the element $\Omega$, it is convenient to use a basis of $\mathcal{A}$ which is an eigenbasis for the Adams operations; such a basis can be given in terms of symmetrized diagrams (in [2] they are called Chinese characters).

A symmetrized diagram is a graph with univalent vertices, and cyclicly ordered, trivalent vertices, such that each connected component of the graph has at least one univalent vertex. The univalent vertices are also called legs. The element of $\mathcal{A}$ that such a graph represents is obtained by averaging all possible ways of attaching the univalent vertices to a Wilson loop.

E.g.

$$\bigodot = \frac{1}{11} \left(8 \bigodot + 16 \bigodot\right).$$

It is not difficult to show that these are eigenvectors for the Adams operations: a symmetrized diagram with $u$ univalent vertices, is an eigenvector for the Adams operation $\psi^p$ with eigenvalue $p^u$. Also it is not difficult to show that symmetrized diagrams span $\mathcal{A}$: one can use the easy fact that a diagram can be written as

\footnote{Note that this will not quite give a universal Vassiliev invariant of framed knots, a term involving the parity of the framing is required — see [1].}
the sum of the symmetrized diagram obtained by removing the Wilson loop and symmetrized diagrams with fewer legs. If one defines the following IHX relation:

\[ \begin{array}{c}
\text{symmetrized diagram} \\
\end{array} - \begin{array}{c}
\text{symmetrized diagram}
\end{array} : \text{IHX}, \]

then one can in fact take the vector space generated by symmetrized diagrams modulo the IHX and AS relations, and the above map to \( \mathcal{A} \) induces an isomorphism. Note that the connected symmetrized diagrams span the primitive subspace of \( \mathcal{A} \).

The disjoint union of symmetrized diagrams, \( \bigcup \), gives \( \mathcal{A} \) a second product which is also compatible with the co-product; thus it gives \( \mathcal{A} \) a second Hopf algebra structure. With respect to this second structure the Adams operations are Hopf-algebra maps. There seems not to be any topological interpretation of this disjoint union product; however, it can be related to the connect sum product by the (complicated) wheeling map [5].

5.3. Wheels. Certain symmetrized diagrams — the wheels — play a particularly important rôle in the theory, both in the Alexander polynomial (see Section 9) and the explicit formula for the element \( \Omega \) described below.

If one considers the connected symmetrized diagrams of degree \( n \geq 2 \) (i.e. with a total of \( 2n \) vertices) then those with more than \( n \) univalent vertices are trivial in \( \mathcal{A} \). If \( n \) is odd then those with \( n \) univalent vertices also vanish; but if \( n \) is even then define the non-trivial diagram \( w_n \) to be the wheel with \( n \) legs. E.g.

\[ w_2 := \begin{array}{c}
\text{wheel with 2 legs}
\end{array} \quad w_4 := \begin{array}{c}
\text{wheel with 4 legs}
\end{array}. \]

Each \( w_n \) spans the “top” eigenspace of the degree \( n \) primitive diagrams for the Adams operations.

These wheels can be used together with the disjoint union product to give a closed formula for the element \( \Omega \in A \) defined in Section 1. Firstly define the modified Bernoulli numbers, \( \{ b_{2n} \}_{n=1}^{\infty} \) via

\[ \sum_{n=1}^{\infty} b_{2n} x^{2n} = \frac{1}{2} \ln \frac{\sinh(x/2)}{x/2}. \]

The following was conjectured in [4] and proved in [5]:

**Theorem 6** (Bar-Natan, Le, Thurston). \( \Omega = \exp_\Pi \left( \sum_{n=1}^{\infty} b_{2n} w_{2n} \right) \).

For instance, writing 1 for the empty symmetrized diagram of degree zero,

\[ \Omega = 1 + \frac{1}{48} \begin{array}{c}
\text{wheel with 2 legs}
\end{array} - \frac{1}{5760} \begin{array}{c}
\text{wheel with 4 legs}
\end{array} + \frac{1}{22680} \begin{array}{c}
\text{wheel with 6 legs}
\end{array} + \{ \text{terms of degree } \geq 6 \}. \]

This will be needed in the calculation in Section 8.

6. Primitive elements as a module over Vogel’s algebra \( \Lambda \).

The space of primitive elements of \( A \) admits a module structure, introduced in [19], which turns out to be very useful for performing low order calculations by hand; it is also very useful in the context of weight systems coming from Lie algebras, and it was used by Vogel to prove that not all weight systems come from simple Lie super-algebras — this will not be addressed further here, but the reader is directed to [18, 12, 13, 21].
It is important to note that I am working over the rationals, so I denote by $\Lambda$ what is denoted $\Lambda \otimes \mathbb{Q}$ in \cite{27}.

Define a Vogel diagram to be a graph with cyclicly oriented trivalent vertices and three univalent vertices labelled bijectively with the set \{1, 2, 3\}. A Vogel diagram has degree \((\# \text{trivalent vertices} - 1)/2\). Consider the space of linear combinations of Vogel diagrams modulo the IHX and AS relations, then $\Lambda$ is defined to be the subspace consisting of all elements $u$ such that if $\sigma$ is a permutation of the label set then $\sigma(u) = \text{signature}(\sigma)u$.

Given a Vogel diagram and a connected diagram in $\mathcal{A}$, one can form a new connected diagram by putting the Vogel diagram in the place of a trivalent vertex, taking note of orientations. For example

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {1};
\node (b) at (1,0) {2};
\node (c) at (2,0) {3};
\draw (a) edge[bend left=30] (b);
\end{tikzpicture}
\end{array}, \quad \begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {1};
\node (b) at (1,0) {2};
\node (c) at (2,0) {3};
\draw (a) edge[bend left=30] (b);
\end{tikzpicture}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {1};
\node (b) at (1,0) {2};
\node (c) at (2,0) {3};
\draw (a) edge[bend left=30] (b);
\end{tikzpicture}
\end{array}.
\]

The constraint on the elements of $\Lambda$ ensures precisely that this descends to a well-defined action of $\Lambda$ on the primitive elements of $\mathcal{A}$, for instance it does not depend on which trivalent vertex is replaced. Similarly $\Lambda$ acts on itself, giving it a graded algebra structure.

Up to degree six, the algebra $\Lambda$ is freely generated by the following elements:

\[
\frac{1}{2} x_1 := t := \begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {1};
\node (b) at (1,0) {2};
\node (c) at (2,0) {3};
\draw (a) edge[bend left=30] (b);
\end{tikzpicture}
\end{array}; \quad x_3 := \begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {1};
\node (b) at (1,0) {2};
\node (c) at (2,0) {3};
\draw (a) edge[bend left=30] (b);
\end{tikzpicture}
\end{array}; \quad x_5 := \begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {1};
\node (b) at (1,0) {2};
\node (c) at (2,0) {3};
\draw (a) edge[bend left=30] (b);
\end{tikzpicture}
\end{array}.
\]

It was conjectured by Vogel that $\Lambda$ is freely generated by $\{x_{2i-1}\}_{i=1}^{\infty}$ but Kneissler \cite{12} proved a relation between them in degree ten: it is unknown whether these elements generate $\Lambda$.

Kneissler also showed that in degrees less than six, the primitive space of $\mathcal{A}$ is freely generated over $\Lambda$ by the diagrams (\(\bigcirc\)) and (\(\otimes\)); although more than just the round wheel diagrams are needed in degree six. The reader can check that this is precisely the form that my basis of primitive elements takes.

I do not know how to sensibly extend this to a $\Lambda$ action on the whole of $\mathcal{A}$, nor how to interpret the action topologically.

7. Cabling knots and the theorem of Le

Cabling is a natural operation on framed knots, with respect to which the Kontsevich integral is particularly well behaved, as is demonstrated by the theorem of Le described below.

For $m$ and $p$ coprime integers and $K$ a framed knot, the $(m, p)$-cable, $K^{(m, p)}$, of $K$ can be defined by taking a blackboard framed knot diagram of $K$, drawing $m$ parallels of the knot in the plane of the paper and inserting $p$ copies of the tangle $T_m$ shown in Figure 4(i), to obtain a blackboard framed knot diagram representing $K^{(m, p)}$ — see Figure 4(ii).

The algebraic structure of $\mathcal{A}$ described in the previous sections, together with the element $\Omega$, permits the following description of how the framed Kontsevich integral behaves under cabling:

\footnote{If one is working over a ring in which 2 is not invertible then an extra condition is required on the elements of $\Lambda$, see \cite{27}.}
Theorem 7 (Le [17]). For $m$ and $p$ coprime integers, and $K$ a framed knot,

$$\int Z(K^{(m,p)}).\Omega = \psi^m (\int Z(K).\exp.(\frac{1}{m^2})).$$

Note that if any of the weight system coming from a semi-simple Lie algebras is applied to both sides, then the consequent equality follows from an analysis of the theorem in [21] which Morton attributes to Strickland [24] and Rosso and Jones [23], and it was because of this that I conjectured this theorem in an earlier version of this note before discovering that Le had already proved it.

This theorem will be used in the next section to calculate the Kontsevich integral up to degree five for torus knots.

8. The Kontsevich integral and Vassiliev invariants of torus knots.

If $m$ and $p$ are coprime integers then the $(m,p)$-torus knot, denoted $T^{(m,p)}$, is the knot type that has a representative which is embedded on the standard torus, wrapping $m$ times around longitudinally and $p$ times around meridinally. For instance, the positive trefoil is the $(2,3)$-trefoil. The $(m,p)$-torus knot can equivalently be described as the $(m,p)$-cable of the zero-framed unknot. This latter description allows the calculation of the Kontsevich integral of the $(m,p)$-torus knot via the previous theorem, as the framed Kontsevich integral of the zero-framed unknot is just the diagram in degree zero.

To calculate $Z(T^{(m,p)})$ proceed as follows: convert the expression for $\Omega$ given in Section 5 to round diagrams using the change of basis data in Appendix A; form the connect sum product with $\exp.(\frac{1}{m^2})$; convert back to symmetrized diagrams; apply $\psi^m$, recalling that symmetrized diagrams are eigenvectors; convert back to round diagrams and form the connect sum product with $\Omega^{-1}$; finally change to the zero framing — this will require multiplying by $\exp.(-mp/2)$.

This calculation is easily performed by hand up to degree four and the reader is strongly encouraged to do this. The following degree five result was obtained with a little bit of help from **maple**:

![Figure 4](image-url)
\[ Z(T(m,p)) = \exp \left( (m^2 - 1)(p^2 - 1) \left( -\frac{1}{11520} \frac{m p}{288} + \frac{9 m^2 p^2 - m^2 - p^2 - 1}{11520} \frac{3}{48} \right) \right. \]
\[ + \left. \frac{(m^2 + 1)(p^2 + 1)}{5760} \frac{m p(71m^2 p^2 - 19m^2 - 19p^2 - 9)}{345600} \right) \]
\[ + \frac{mp(m^2 + 1)(p^2 + 1)}{345600} \left( \frac{mp(2m^2 p^2 + m^2 + p^2)}{12280} \right) + \ldots \)

Note that the coefficients of the round wheels, \( \), and \( \), are respectively \(-\frac{(m^2 - 1)(p^2 - 1)}{48}\) and \(\frac{(m^4 - 1)(p^4 - 1)}{5760}\); this will be generalized in the next section. First however, the methodology of this section leads to the next theorem which I had not been able to prove using more naïve methods — for example the methods discussed in [28].

**Theorem 8.** If \( v \) is a rational, type \( n \), Vassiliev knot invariant then, as a function of \( m \) and \( p \), \( v(T(m,p)) \) is a polynomial, symmetric in \( m \) and \( p \) and of degree at most \( n \) in each of \( m \) and \( p \).

**Proof.** The symmetry in \( m \) and \( p \) follows from the fact that the \((m,p)\)-torus knot and the \((p,m)\)-torus knots are of the same knot-type (see [7]). Because the Kontsevich integral is universal for Vassiliev invariants it suffices to prove that in the Kontsevich integral the coefficient of a basis symmetrized diagram has the properties described in the theorem. Fix a basis of symmetrized diagrams of degree \( n \). Pick any element of this basis. By the linearity of change of basis from round diagrams to symmetrized diagrams, the coefficient of this element in \( \Omega. \exp \left( \frac{m}{p} \frac{1}{2} \frac{1}{2} \right) \) is a polynomial in \( \frac{m}{p} \) of degree at most \( n \). The action of \( \psi^m \) will just be to multiply by some power of \( m \). By the symmetry in \( m \) and \( p \) the result must be a symmetric polynomial of degree at most \( n \) in \( m \) and \( p \). \( \square \)

9. The **Alexander Polynomial, Wheel Coefficients and Torus Knots.**

From work on the Melvin-Morton Theorem it is known that the Alexander polynomial of a knot corresponds to the coefficients of the wheels in its Kontsevich integral. This is used to identify the coefficients of the wheels for the torus knots.

9.1. **Identifying the Alexander Polynomial in the Kontsevich Integral.**

For a knot \( K \), let \( A_K(t) \) be the Alexander polynomial normalized so that \( A_K(t) = A_K(-t) \) and \( A_K(1) = 1 \). Define the weight system \( W_{AC} : \mathcal{A} \to \mathbb{Q}[\llbracket h \rrbracket] \) by first defining it on connected symmetrized diagrams as

\[ W_{AC}(D) = \begin{cases} -2h^{2n} & \text{if } D = w_{2n}, \\ 0 & \text{if } D \text{ has less than } \deg D \text{ legs}, \end{cases} \]

then extend it linearly to the space of primitive elements and multiplicatively, with respect to the connect sum product, to the whole of \( \mathcal{A} \) (or equivalently \[15\], extend \( W_{AC} \) multiplicatively with respect to the \( \sqcup \) product).

The AC in \( W_{AC} \) stands for Alexander-Conway, the reason being given by the next theorem which follows from work on the Melvin-Morton Theorem.
Theorem 9 \((\text{[3, 16]}\)\). If \(K\) is a knot then, as formal power series in \(h\),
\[ A_K(e^h) = W_{AC} \circ Z(K). \]

This can be used to identify the coefficients of the wheels in the Kontsevich integral in the following manner. Take logarithms of both sides, as \(W_{AC}\) is multiplicative with respect to the disjoint union product, one finds
\[ \ln A_K(e^h) = W_{AC} \circ \ln \circ Z(K). \]

From the definition of \(W_{AC}\) and the fact that \(\ln \circ Z(K)\) can be written as the sum of connected symmetrized diagrams, one deduces:

Theorem 10. With respect to a polynomial basis of symmetrized diagrams, the coefficient of \(w_{2n}\) in \(Z(K)\) is equal to \(-\frac{1}{2}\) times the coefficient of \(h^{2n}\) in \(\ln A_K(e^h)\).

Note that if a basis of connected round diagrams is taken which in degree \(2n\) consists of the round wheel together with diagrams with fewer univalent vertices, then the coefficient of the round wheel with \(n\) legs is the same as the coefficient of \(w_{2n}\) in the above theorem. This can now be applied to the case of torus knots.

9.2. Wheels coefficients for torus knots. For the torus knot \(T(m,p)\) it is known that the Alexander polynomial is given by
\[ A_{T(m,p)}(t) = \frac{(t^{mp/2} - t^{-mp/2})(t^{1/2} - t^{-1/2})}{(t^{m/2} - t^{-m/2})(t^{p/2} - t^{-p/2})}. \]

Substituting \(t = e^h\), rewriting the bracketted terms as \(\sinh\)s, dividing the numerator and the denominator of the right hand side by \(h^{2mp}\), taking logarithms of both sides and recalling the definition of the modified Bernoulli numbers \(\{b_{2n}\}\) from Section 5, one obtains
\[ \ln \left(A_{T(m,p)}(e^h)\right) = \sum b_{2n}(m^{2n} - 1)(p^{2n} - 1)2h^{2n}. \]

It then follows from the theorem above that the coefficient of \(w_{2n}\), the wheel with \(2n\) legs, in \(Z(T(m,p))\) is precisely \(-(m^{2n} - 1)(p^{2n} - 1)b_{2n}\). This can be seen to be in accord with the calculation of Section 8.

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References

1. M. Alvarez and J. M. F. Labastida, Vassiliev invariants for torus knots, J. Knot Theory Ramifications 5 (1996), 779–803.
2. D. Bar-Natan, On the Vassiliev knot invariants, Topology 34 (1995), no. 2, 423–472.
3. D. Bar-Natan and S. Garoufalidis, On the Melvin-Morton-Rozansky conjecture, Invent. Math. 125 (1996), 103–133.
4. D. Bar-Natan, S. Garoufalidis, L. Rozansky, and D. P. Thurston, Wheels, wheeling, and the Kontsevich integral of the unknot, preprint, see q-alg/9703025, March 1997.
5. D. Bar-Natan, T. Q. T. Le, and D. Thurston, in preparation.
Appendix A. Change of basis data.

Here is the change of basis data, up to degree five, between round diagrams and symmetrized diagrams, used in the calculation of Section 8. It was worked out by hand by expanding each symmetrized diagram and then using the STU relations to rewrite in terms of the round diagram basis. The $\Lambda$-module structure certainly simplified the calculations. The calculations took more than a week and the degree six result seems to be beyond the limit of my stamina. I don’t know how much could be gained by automating the procedure, but I think there is still much structure to be exploited, for instance there is the following.
Note how sparse these matrices are. The presence of identity matrices along the diagonal is explained by the fact mentioned above that a round diagram can be written as the symmetrized diagram obtained by removing the Wilson loop plus diagrams with fewer legs. Le conjectured that the connect sum of $l$ round diagrams can be written as a linear combination of symmetrized diagrams with at least $l$ legs: this would explain some of the zeroes in the bottom left corners. The bottom row could be calculated if it was known, as was also suggested by Le, that $\exp(\Omega/2) = \Omega \Pi \exp(\Omega/2)$. Also some of the zeroes seem to be reflecting $\Lambda$-module structure, but I can not yet make that more precise.

Degree zero:

$$\begin{array}{c}
\mathcal{O} = 1.
\end{array}$$

Degree one:

$$\begin{array}{c}
\mathcal{O} = \bigg| \bigg|
\end{array}$$

Degree two:

$$\begin{array}{c}
\left( \begin{array}{c}
\mathcal{O}
\end{array} \right) = \left( \begin{array}{c}
1 \\
\frac{1}{2}
\end{array} \right) \left( \begin{array}{c}
\frac{1}{2}
\end{array} \right).
\end{array}$$

Degree three:

$$\begin{array}{c}
\left( \begin{array}{c}
\mathcal{O}
\end{array} \right) = \left( \begin{array}{c}
1 \\
\frac{1}{3}
\end{array} \right) \left( \begin{array}{c}
\frac{1}{3}
\end{array} \right).
\end{array}$$

Degree four:

$$\begin{array}{c}
\left( \begin{array}{c}
\mathcal{O}
\end{array} \right) = \left( \begin{array}{c c c c c c c c c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{2}{3} & 1 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{15} & \frac{1}{12} & 0 & 1 & 1 & 0 & 0 & 0 \\
\end{array} \right) \left( \begin{array}{c}
\frac{1}{2}
\end{array} \right).
\end{array}$$
Degree five:

\[
\begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
  2 & -\frac{1}{12} & \frac{3}{8} & 0 & \frac{1}{4} & 1 & 0 & 0 \\
  0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 1 & 0 \\
  0 & 0 & 0 & \frac{1}{6} & \frac{2}{3} & 0 & 0 & 1 \\
  0 & 0 & -\frac{2}{15} & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 1 \\
  0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{5}{12} & 0 \\
  0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{5}{12} & 1
\end{pmatrix}
\]

Appendix B. Comparison with Alvarez and Labastida's results

In [1], Alvarez and Labastida calculate Vassiliev invariants up to degree six for the torus knots by using the values on the torus knots of the HOMFLY, Kauffman and Akutsu-Wadati polynomials, together with a Chern-Simons approach to the universal Vassiliev invariant. Their method will not give all Vassiliev invariants of torus knots because the invariants coming from Lie algebras do not span the space of Vassiliev invariants [25] (although they do so up to at least degree twelve [12]). However, up to degree five, their results suggest that \(\ln Z(T(m,p))\) might be the following:

\[
(m^2 - 1)(p^2 - 1)\left(\frac{1}{6} \gamma + \frac{mp}{18} \gamma + \frac{9m^2p^2-m^2-p^2-1}{360} \gamma + \frac{(m^2+1)(p^2+1)}{360} + \frac{mp(69m^2p^2-21m^2-21p^2-11)}{5400} + \frac{mp(11m^2p^2+m^2+p^2-9)}{5400} + \frac{mp(m^2+1)(p^2+1)}{900}\right).
\]

Most of the basis element for \(\mathcal{A}\) which they use differ only by a scalar from the basis elements used in the calculations above; the only non-trivial change of basis relation is\footnote{5One of the signs given in this expression in [1] is incorrect.}

\[
\gamma = \frac{3}{7} \gamma - \frac{1}{6} \gamma - \frac{1}{12} \gamma.
\]
Rewriting, in the basis of Alvarez and Labastida, the expression derived in Section 4, the following expression for $\ln Z(T(m, p))$ up to degree five is obtained:

$$(m^2 - 1)(p^2 - 1)\left(-\frac{1}{\pi} + \frac{mp}{144} - \frac{9m^2p^2 - m^2 - p^2 - 1}{5760} + \frac{(m^2+1)(p^2+1)}{5760} + \frac{mp(69m^2p^2 - 21m^2 - 21p^2 - 11)}{172800} - \frac{mp(11m^2p^2 + m^2 + p^2 - 9)}{172800} - \frac{mp(m^2+1)(p^2+1)}{28800}\right).$$

Note that this expression differs from that of Alvarez and Labastida in the following two respects: for a diagram of degree $n$, there is a factor of $2^n$ difference — this is probably explained taking a different norm on the Lie algebras; and diagrams with an odd number of external vertices differ in sign — this is probably explained by an orientation convention.