LIOUVILLE TYPE THEOREMS FOR THE STEADY AXIALLY SYMMETRIC NAVIER-STOKES AND MAGNETOHYDRODYNAMIC EQUATIONS

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Abstract. In this paper we study Liouville properties of smooth steady axially symmetric solutions of the Navier-Stokes equations. First, we provide another version of the Liouville theorem of [14] in the case of zero swirl, where we replaced the Dirichlet integrability condition by mild decay conditions. Then we prove some Liouville theorems under the assumption

\[ \| \frac{\partial u}{\partial r} \|_{L^{3/2}(\mathbb{R}^3)} < C_\sharp \]

where \( C_\sharp \) is a universal constant to be specified. In particular, if \( u_r(r, z) \geq -\frac{1}{r} \) for \( \forall (r, z) \in [0, \infty) \times \mathbb{R} \), then \( u \equiv 0 \). Liouville theorems also hold if \( \lim_{|x| \to \infty} \Gamma = 0 \) or \( \Gamma \in L^q(\mathbb{R}^3) \) for some \( q \in [2, \infty) \) where \( \Gamma = ru_\theta \). We also established some interesting inequalities for \( \Omega := \partial_r u_r - \partial_z u_z \), showing that \( \nabla \Omega \) can be bounded by \( \Omega \) itself. All these results are extended to the axially symmetric MHD and Hall-MHD equations with

\[ u = u_r(r, z)e_r + u_\theta(r, z)e_\theta + u_z(r, z)e_z, \quad h = h_\theta(r, z)e_\theta, \]

indicating that the swirl component of the magnetic field does not affect the triviality. Especially, we establish the maximum principle for the total head pressure \( \Phi = \frac{1}{2}(|u|^2 + |h|^2) + p \) for this special solution class.

1. Introduction. The Steady Navier-Stokes takes the following form.

\[
\begin{aligned}
(u \cdot \nabla)u + \nabla p &= \Delta u, \quad \forall x \in \mathbb{R}^3 \\
\text{div } u &= 0, \\
\lim_{|x| \to \infty} u(x) &= 0.
\end{aligned}
\]  

Consider a weak solution to (1.1) with finite Dirichlet integral

\[ \int_{\mathbb{R}^3} |\nabla u(x)|^2 dx < \infty. \]

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It is well-known that a weak solution to (1.1) belonging to \( \mathcal{W}^{1,2}_{\text{loc}}(\mathbb{R}^3) \) is indeed smooth. Here we include the result stated in Galdi’s book [9].

**Theorem 1.1.** (Theorem X.5.1 in [9]). Let \( \mathbf{u}(x) \) be a weak solution of (1.1) satisfying (1.3) and \( p(x) \) be the associated pressure, then there exists \( p_1 \in \mathbb{R} \) such that

\[
\lim_{|x| \to \infty} |\nabla^\alpha \mathbf{u}(x)| + \lim_{|x| \to \infty} |\nabla^\alpha (p(x) - p_1)| = 0
\]

uniformly for all multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in [\mathbb{N} \cup \{0\}]^3 \).

We remark here that from (1.3) and (1.4), one has \( \mathbf{u} \in L^6(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \). So \( \mathbf{u} \in L^p(\mathbb{R}^3) \) for any \( p \in [6, \infty) \).

One of outstanding open problems for steady Navier-Stokes equations is: Is a smooth solution to (1.1) satisfying (1.3) identically zero? For the two dimensional case, the uniqueness result had been proved in [10]. For the three dimensional case, Galdi [9] first established the triviality, under an additional integrability condition \( \mathbf{u} \in L^{6/2}(\mathbb{R}^3) \). Chae and Yoneda [7] also obtained a Liouville theorem if \( \mathbf{u} \in X \cap Y \), where \( X, Y \) are two function spaces and \( X \) controls the high oscillation in large part of \( \mathbf{u} \) and \( Y \) gives control on the decay rate of \( \mathbf{u} \) for sufficiently large \( \mathbf{x} \). In [2], Chae showed that the condition \( \Delta \mathbf{u} \in L^{6/3}(\mathbb{R}^3) \) is enough to guarantee the triviality. Note that this condition is stronger than the finite Dirichlet condition \( \nabla \mathbf{u} \in L^2(\mathbb{R}^3) \), but both of them have the same scaling. In [14], the authors proved that any axially symmetric smooth solution to (1.1) without swirl \( u_\theta \equiv 0 \), but with finite Dirichlet integral must be zero.

To simplify the problem, in this article, we will consider the solution \( \mathbf{u} \) with additional axially symmetric property. More precisely, we introduce the cylindrical coordinate

\[
r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3.
\]

We denote \( \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z \) the standard basis vectors in the cylindrical coordinate:

\[
\mathbf{e}_r = (\cos \theta, \sin \theta, 0), \quad \mathbf{e}_\theta = (-\sin \theta, \cos \theta, 0), \quad \mathbf{e}_z = (0, 0, 1).
\]

A function \( f \) is said to be axially symmetric if it does not depend on \( \theta \). A vector-valued function \( \mathbf{u} = (u_r, u_\theta, u_z) \) is called axially symmetric if \( u_r, u_\theta \) and \( u_z \) do not depend on \( \theta \). A vector-valued function \( \mathbf{u} = (u_r, u_\theta, u_z) \) is called axially symmetric with no swirl if \( u_\theta = 0 \) while \( u_r \) and \( u_z \) do not depend on \( \theta \). For more information about smooth axially symmetric vector fields, one may refer to [17].

Assume that \( \mathbf{u}(\mathbf{x}) = u_r(r,z)\mathbf{e}_r + u_\theta(r,z)\mathbf{e}_\theta + u_z(r,z)\mathbf{e}_z \) is a smooth solution to (1.1). The corresponding asymmetric steady Navier-Stokes equations read as follows.

\[
\begin{aligned}
(u_r\partial_r + u_z\partial_z)u_r - \frac{u_r^2}{r} + \partial_r p &= \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u_r, \\
(u_r\partial_r + u_z\partial_z)u_\theta + \frac{u_\theta u_r}{r} &= \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u_\theta, \\
(u_r\partial_r + u_z\partial_z)u_z + \partial_z p &= \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 \right) u_z, \\
\partial_r u_r + \frac{u_r}{r} + \partial_z u_z &= 0.
\end{aligned}
\]  

(1.5)

In this paper, we want to investigate that what kinds of decay and integrability conditions we should prescribe on \( u_r, u_\theta \) and \( u_z \) to guarantee the triviality of axially symmetric smooth solution to (1.1). Some of our conditions are weaker than the finite Dirichlet integral condition.
To our purpose, we also need the vorticity \( \omega(x) = \text{curl } u(x) = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z \), where
\[
\omega_r = -\partial_z u_\theta, \quad \omega_\theta = \partial_z u_r - \partial_r u_z, \quad \omega_z = \frac{1}{r} \partial_r (ru_\theta).
\]
The equations satisfied by \( \omega_r, \omega_\theta \) and \( \omega_z \) are listed as follows.
\[
\begin{align*}
(\omega_r \partial_r + u_r \partial_z) \omega_r - (\omega_r \partial_r + \omega_z \partial_z) u_r &= \left( \frac{\partial^2}{r^2} + \frac{1}{r} \partial_r + \frac{\partial^2_z}{r^2} - \frac{1}{r^2} \right) \omega_r, \\
(\omega_r \partial_r + u_\theta \partial_z) \omega_\theta - \frac{u_r u_\theta}{r^2} - \frac{1}{r^2} \partial_z (u_\theta^2) &= \left( \frac{\partial^2}{r^2} + \frac{1}{r} \partial_r + \frac{\partial^2_z}{r^2} - \frac{1}{r^2} \right) \omega_\theta, \\
(\omega_r \partial_r + u_z \partial_z) \omega_z - (\omega_r \partial_r + \omega_z \partial_z) u_z &= \left( \frac{\partial^2}{r^2} + \frac{1}{r} \partial_r + \frac{\partial^2_z}{r^2} \right) \omega_z.
\end{align*}
\]

This paper is structured as follows. In section 2, we first provide another version of the Liouville theorem of [14] for axially symmetric flows with no swirl, replacing the Dirichlet integral condition by mild decay condition at infinity. Then we establish some Liouville theorems under \( \| \frac{\omega_\theta}{r} \|_{L^{1/2}(\mathbb{R}^3)} \leq C_r \), where \( C_r \) is a universal constant to be specified later. In particular, if \( u_r(r, z) \geq -\frac{1}{r} \) for any \((r, z) \in [0, \infty) \times \mathbb{R}\), then \( u \equiv 0 \). The Liouville theorem also holds under some decay or \( L^q(\mathbb{R}^3) \ (q \in [2, \infty)) \) integrability conditions on \( \Gamma = ru_\theta \). We also establish some interesting inequalities for \( \Omega \), use the equation for \( \frac{u_r}{r} \) and the relation between \( \frac{\omega_\theta}{r} \) and \( \Omega \). These indicate that \( \nabla \Omega \) can be bounded by \( \Omega \) itself, which is not so clear at first sight due to the additional term \( -\frac{1}{r^2} \partial_z (u_\theta)^2 \). All these results will be extended to the axially symmetric MHD case in section 3, where we consider a special solution form \( u(x) = u_r(r, z) e_r + u_\theta(r, z) e_\theta + u_z(r, z) e_z \) and \( h(x) = h_\theta(r, z) e_\theta \). Our results show that the swirl component of the magnetic field does not affect the triviality. Especially, we establish a maximum principle for the total head pressure \( \Phi = \frac{1}{2} (|u|^2 + |h|^2) + p \). In section 4, we investigate the corresponding problem for steady resistive, viscous Hall-MHD equations and obtain similar results as the MHD case.

2. Liouville type theorems for steady Navier-Stokes equations.

2.1. The Liouville theorem for axially symmetric flows with no swirl. In [14], the authors have showed that the axially symmetric smooth solutions to (1.1) satisfying (1.3) must be zero in the absence of swirl. In the following, we provide another version of their Liouville theorem, replacing the Dirichlet integrability condition by mild decay conditions on \( u \).

**Theorem 2.1.** Let \( u(x) \) be an axially symmetric smooth solution to (1.1)-(1.2) with no swirl. Then \( u \equiv 0 \).

Indeed this follows from Theorem 5.2 in [13] immediately. In Theorem 5.2 in [13], they proved that for any bounded weak solution \( u \) of the unsteady Navier-Stokes equations in \( \mathbb{R}^3 \times (-\infty, 0) \), if \( u \) is axially symmetric with no swirl, then \( u = (0, 0, b_3(t)) \) for some bounded measurable function \( b_3: (-\infty, 0) \to \mathbb{R} \). By (1.2), \( u \equiv 0 \). Here we provide another proof of Theorem 2.1 by using the maximum principle of \( \Omega \).

**Proof of Theorem 2.1.** Since \( u \) is smooth, by (1.2), one can conclude that \( \nabla^\alpha u \in L^\infty(\mathbb{R}^3) \) and
\[
\lim_{|x| \to \infty} |\nabla^\alpha u(x)| + \lim_{|x| \to \infty} |\nabla^\alpha (\rho(x) - p_1)| = 0 \quad (2.1)
\]
uniformly for all multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in [\mathbb{N} \cup \{0\}]^3 \).
In the case \( u_\theta \equiv 0 \), the equation for \( \omega_\theta \) is reduced to
\[
(u_r \partial_r + u_z \partial_z) \omega_\theta - \frac{u_r \omega_\theta}{r} = \left( \frac{\partial_r^2}{r^2} + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) \omega_\theta. \tag{2.2}
\]

It is easy to derive the equation for \( \Omega \):
\[
(u_r \partial_r + u_z \partial_z) \Omega = \left( \frac{\partial_r^2}{r^2} + \frac{3}{r} \partial_r + \partial_z^2 \right) \Omega. \tag{2.3}
\]

As is well known, \((\partial_r^2 + \frac{2}{r} \partial_r + \partial_z^2) \Omega \) can be written as the Laplacian \( \Delta_5 \) in \( \mathbb{R}^5 \) (see [13]). Write \( r = \sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2} \) and \( y_5 = z \), then (2.3) becomes
\[
(u_r \partial_r + u_z \partial_z) \Omega = \Delta_5 \Omega.
\]

By (2.1), \( \lim_{|x| \to \infty} \Omega(x) = 0 \), one can employ the maximum principle arising from (2.3) to show that \( \Omega \equiv 0 \). Since \( \text{curl } u = \omega_\theta(r, z) e_\theta \), we have \( \text{curl } u \equiv 0 \). Since \( \text{div } u \equiv 0 \), we have \( u(x) = \nabla \phi(x) \) for a harmonic function \( \phi \). The decay condition (1.2) implies \( u \equiv 0 \) by the Liouville theorem for a harmonic function.

### 2.2. Liouville type theorems conditioned on \( u_r \) and \( u_\theta \)

We look at the equation \( u_\theta \) directly:
\[
(u_r \partial_r + u_z \partial_z) u_\theta + \frac{u_r}{r} u_\theta = \left( \frac{\partial_r^2}{r^2} + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u_\theta. \tag{2.4}
\]

**Theorem 2.2.** Let \( u(x) \) be an axially symmetric smooth solution to (1.1)-(1.2). Suppose there exists \( q \in [1, \infty) \) such that
\[
u_\theta \in L^q(\mathbb{R}^3), \tag{2.5}
\]
and
\[
\| \frac{u_r}{r} 1_{\{ u_r < -\frac{1}{r} \}} \|_{L^{3/2}(\mathbb{R}^3)} \leq \frac{4(q+1)}{(q+2)^2} C^*_\ast \tag{2.6}
\]
where \( C^*_\ast \) is the optimal constant in the Sobolev inequality \( \| f \|_{L^5(\mathbb{R}^3)} \leq C^*_\ast \| \nabla f \|_{L^1(\mathbb{R}^3)} \) for any \( f \in C^\infty_0(\mathbb{R}^3) \) and \( 1_E \) is the characteristic function for a set \( E \subset \mathbb{R}^3 \). Then, we have \( u \equiv 0 \). In particular, if
\[
u_r \geq -\frac{1}{r}, \text{ for all } (r, z) \in [0, \infty) \times \mathbb{R}, \tag{2.7}
\]
then (2.6) is automatically satisfied and \( u \equiv 0 \).

**Proof.** Since \( u \) is smooth, by (1.2) we have \( \| u \|_{L^\infty(\mathbb{R}^3)} < \infty \). Moreover, one can conclude that \( \nabla^\alpha u \in L^\infty(\mathbb{R}^3) \) and
\[
\lim_{|x| \to \infty} |\nabla^\alpha u(x)| + \lim_{|x| \to \infty} |\nabla^\alpha(p(x) - p_1)| = 0 \tag{2.8}
\]
uniformly for all multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in [\mathbb{N} \cup \{0\}]^3 \).

Introduce the radial cut-off function \( \sigma \in C^\infty_0(\mathbb{R}^3) \) such that
\[
\sigma(x) = \sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases}
\]
and \( 0 \leq \sigma(x) \leq 1 \) for \( 1 < |x| < 2 \). Without loss of generality, we may assume that \( \sigma(|x|) \) is monotonic decreasing in \([0, +\infty)\). Then, for each \( R > 0 \), we define \( \sigma_R(x) = \sigma(|x|/R) \), then the support of \( \nabla \sigma_R(x) \) is contained in \( Q_R := B_{2R}(0) \setminus B_R(0) \).
For $q \in [0, \infty)$, multiplying (2.4) by $\sigma_R|u_\theta|^q u_\theta$, and integrating over $\mathbb{R}^3$, then we obtain

$$
\int_{\mathbb{R}^3} \sigma_R|u_\theta|^q u_\theta (u_r \partial_r + u_z \partial_z + \frac{u_r}{r}) u_\theta \, dx
= \int_{\mathbb{R}^3} \sigma_R|u_\theta|^q u_\theta \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u_\theta \, dx. \tag{2.9}
$$

We estimate both sides as follows.

\[
\text{LHS} = \int_{\mathbb{R}^3} \sigma_R(x) (u_r \partial_r + u_z \partial_z)|u_\theta|^q + 2 \, dx + \int_{\mathbb{R}^3} \sigma_R(x) u_r |u_\theta|^q + 2 \, dx
\]

\[
= \left[ \frac{2\pi}{q+2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \sigma'(\sqrt{\frac{r^2 + z^2}{R}}) |u_\theta|^q + 2 \left( r u_r + z u_z \right) \frac{R}{r \sqrt{r^2 + z^2}} \, dr \, dz \right]
\]

\[
+ \int_{\mathbb{R}^3} \sigma_R(x) u_r |u_\theta|^q + 2 \, dx
:= A_1(R) + A_2(R).
\]

\[
\text{RHS} = \int_{\mathbb{R}^3} \sigma_R(x) |u_\theta|^q u_\theta \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u_\theta \, dx
\]

\[
= \left[ \frac{2\pi}{q+2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \sigma'(\sqrt{\frac{r^2 + z^2}{R}}) |u_\theta|^q + 2 \left( |\partial_r| u_\theta \frac{q+2}{2} |u_\theta|^{q+2} |u_\theta|^{q+2} \right) \, dr \, dz \right]
\]

\[
+ \int_{\mathbb{R}^3} \sigma_R(x) |u_\theta|^q + 2 \, dx
:= B_1(R) + B_2(R) + B_3(R).
\]

By the condition (1.2), we have (2.8) and

\[||u||_{L^\infty} + \frac{|u_\theta|}{r} ||u_\theta||_{L^\infty(\mathbb{R}^3)} < +\infty.\]

Since $\nabla |u_\theta|^{\frac{q+2}{2}} = 2 \frac{q+2}{2} |u_\theta|^{\frac{q+2}{2}-1} u_\theta \nabla u_\theta$ and $u_\theta \in L^\infty(\mathbb{R}^3)$, (2.5) implies $u_\theta \in L^{q+2}(\mathbb{R}^3)$ and $\nabla |u_\theta|^{\frac{q+2}{2}} \in L^{2}(\mathbb{R}^3)$. Now let $R$ tends to infinity, we obtain

\[|A_1(R)| \leq \frac{1}{q+2} \left( ||u_r||_{L^\infty} + ||u_z||_{L^\infty} \right) \frac{1}{R} \int_{Q_R} |u_\theta|^q + 2 \, dx \to 0, \quad \text{as } R \to \infty,\]

\[A_2(R) \to \int_{\mathbb{R}^3} \frac{u_r}{r} |u_\theta|^q + 2 \, dx, \quad \text{as } R \to \infty,\]

\[B_1(R) = -\frac{4(q+1)}{(q+2)^2} \int_{\mathbb{R}^3} |\nabla |u_\theta|^{\frac{q+2}{2}}|^2 \, dx, \quad \text{as } R \to \infty,\]

\[|B_2(R)| \leq \frac{C}{q+2} \left( ||u||_{L^\infty} + ||u||_{L^\infty} \right) \frac{1}{R^2} \int_{Q_R} |u_\theta|^q + 2 \, dx \to 0, \quad \text{as } R \to \infty,\]

\[B_3(R) = - \int_{\mathbb{R}^3} \sigma_R(x) \frac{u_\theta}{r} ||u_\theta||_{L^\infty} + 2 \, dx \to - \int_{\mathbb{R}^3} \frac{|u_\theta|^q}{r^2} \, dx, \quad \text{as } R \to \infty.\]

Combining all these calculations, we conclude that

\[\frac{4(q+1)}{(q+2)^2} \int_{\mathbb{R}^3} |\nabla |u_\theta|^{\frac{q+2}{2}}|^2 \, dx + \int_{\mathbb{R}^3} |u_\theta|^q + 2 \, dx + \int_{\mathbb{R}^3} \frac{u_r}{r} |u_\theta|^q + 2 \, dx = 0. \tag{2.10}\]
If \(\|u_{r}^{1/2}1_{(u_-, -\frac{1}{r})}\|_{L^{3/2}(\mathbb{R}^3)} < \frac{4(q+1)}{(q+2)^2} C^2\), then rewrite (2.10) as
\[
\frac{4(q + 1)}{(q + 2)^2} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \frac{|u_{\theta}|^{q+2}}{r^2} \, dx + \int_{\mathbb{R}^3} \frac{u_{r}}{r} 1_{(u_+ , -\frac{1}{r})} |u_{\theta}|^{q+2} \, dx = - \int_{\mathbb{R}^3} \frac{u_{r}}{r} |u_{\theta}|^{q+2} 1_{(u_-, -\frac{1}{r})} \, dx \leq \|u_{r}^{1/2}1_{(u_-, -\frac{1}{r})}\|_{L^{3/2}(\mathbb{R}^3)} \|u_{\theta}\|_{L^q(\mathbb{R}^3)}^2 + \frac{4(q + 1)}{(q + 2)^2} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 < 4(q + 1) \|\nabla u\|_{L^2(\mathbb{R}^3)},
\]
yielding \(\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 = 0\) and \(u_{\theta} \equiv 0\). Hence the problem reduces to the case of axially symmetric flows with no swirl, the theorem follows from Theorem 2.1.

In the case of smooth axially symmetric solutions to (1.1)-(1.2) with finite Dirichlet integral (1.3) one can have the optimal constant in the estimate (2.6).

**Theorem 2.3.** Let \(u(x)\) be an axially symmetric smooth solution to (1.1)-(1.2) with finite Dirichlet integral (1.3). Assume that
\[
\|u_{r}^{1/2}1_{(u_-, -\frac{1}{r})}\|_{L^{3/2}(\mathbb{R}^3)} < \frac{8}{9C^2}.
\]
Then \(u \equiv 0\). In particular, if
\[
u_{r} \geq -\frac{1}{r}, \quad \text{for all } (r, z) \in [0, \infty) \times \mathbb{R},
\]
then (2.11) is automatically satisfied and \(u \equiv 0\).

**Proof.** Note that \(f(q) := \frac{4(q+1)}{(q+2)^2}\) is monotonic decreasing for \(q \in [1, \infty)\), and sup \(\frac{4(q + 1)}{(q + 2)^2} = f(1) = \frac{8}{9}\). In the following, we will verify that (2.10) holds for \(q = 1\). Since \(\nabla u \in L^2(\mathbb{R}^3)\), then by Sobolev embedding theorem \(u \in L^6(\mathbb{R}^3)\) and since
\[
\nabla u = (e_r \partial_r + \frac{e_\theta}{r} \partial_\theta + e_z \partial_z)(u, e_r + u_{\theta} e_\theta + u_z e_z)
\]
\[
= \partial_r u, e_r \otimes e_r + \frac{u_{\theta}}{r} e_\theta \otimes e_\theta + \partial_z u, e_z \otimes e_r + \partial_r u_{\theta} e_r \otimes e_\theta
\]
\[
- \frac{u_{\theta}}{r} e_\theta \otimes e_r + \partial_z u_{\theta} e_z \otimes e_\theta + \partial_r u_z e_r \otimes e_\theta + \partial_r u_z e_z \otimes e_z + \partial_z u_z e_z \otimes e_z,
\]
we also have \(\|\nabla u\|_{L^2(\mathbb{R}^3)} + \|\frac{u_{\theta}}{r}\|_{L^2(\mathbb{R}^3)} + \|\frac{u_{\theta}}{r}\|_{L^2(\mathbb{R}^3)} \leq \|\nabla u\|_{L^2(\mathbb{R}^3)}.\) These implies that
\[
\left| \int_{\mathbb{R}^3} \frac{u_{r}}{r} |u_{\theta}|^2 \, dx \right| \leq \|\frac{u_{r}}{r}\|_{L^2(\mathbb{R}^3)} \|u_{\theta}\|_{L^6(\mathbb{R}^3)} < +\infty,
\]
\[
\|\nabla u\|_{L^2(\mathbb{R}^3)} \leq C \|u_{\theta}\|_{L^{\infty}(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)} < +\infty,
\]
\[
\left\| \frac{u_{\theta}}{r} \frac{1}{r} \right\|_{L^2(\mathbb{R}^3)} \leq \|\frac{u_{\theta}}{r}\|_{L^{\infty}(\mathbb{R}^3)} \left\| \frac{u_{\theta}}{r} \right\|_{L^2(\mathbb{R}^3)} < +\infty.
\]
Proof. Multiplying the first three equations in (1.5) by \( \frac{u_r}{r} |u_\theta|^3 \), and adding them together, and then integrating by parts, we obtain

\[
A_2 \to \int_{\mathbb{R}^3} \frac{u_r}{r} |u_\theta|^3 \, dx, \quad \text{as } R \to \infty,
\]
\[
B_1 \to -\frac{8}{9} \int_{\mathbb{R}^3} |\nabla |u_\theta|^3/2|^2 \, dx, \quad \text{as } R \to \infty,
\]
\[
B_3 \to -\int_{\mathbb{R}^3} |u_\theta|^3 \, dx, \quad \text{as } R \to \infty,
\]
\[
|A_1| \leq C(\sigma) \|u_\theta\|_{L^3(Q_R)}^2 \|u_r, u_z\|_{L^6(Q_R)}, \quad \text{as } R \to \infty,
\]
\[
|B_2| \leq C(\sigma) \|u_\theta\|_{L^6(Q_R)}^2 R^{-1/2} \to 0, \quad \text{as } R \to \infty.
\]

Finally, we obtain (2.10) for \( q = 1 \). \( \square \)

Remark 1. It is well-known that if \( u \) is a smooth solution to (1.1), then so is \( u^\lambda(x) = \lambda u(\lambda x) \), for any \( \lambda > 0 \). It should be emphasized here that the conditions (2.11), (2.12) and (2.6) are scaling invariant.

Finally we give a simple vanishing criteria based on the \( L^3(\mathbb{R}^3) \) integrability conditions for \( u_r \) and \( u_z \). Note that we do not need any additional conditions on \( u_\theta \).

Theorem 2.4. Let \( (u, p) \) be an axially symmetric smooth solution to (1.1)-(1.2) with finite integral (1.3). If \( (u_r, u_z) \in L^3(\mathbb{R}^3) \), then \( u \equiv 0 \).

Proof. Multiplying the first three equations in (1.5) by \( \sigma_R(x) u_r, \sigma_R(x) u_\theta, \sigma_R(x) u_z \) and adding them together, and then integrating by parts, we obtain

\[
\int_{\mathbb{R}^3} \sigma_R(x) \left( |\nabla u_r|^2 + |\nabla u_\theta|^2 + |\nabla u_z|^2 + \frac{u_r^2}{r^2} + \frac{u_\theta^2}{r^2} \right) \, dx \tag{2.13}
\]

\[
= \pi \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{1}{R} |u|^2 \left[ \partial_r \left( r \sigma' \left( \frac{\sqrt{r^2 + z^2}}{R} \right) \frac{r}{\sqrt{r^2 + z^2}} \right) + \partial_z \left( r \sigma' \left( \frac{\sqrt{r^2 + z^2}}{R} \right) \frac{z}{\sqrt{r^2 + z^2}} \right) \right] \, r \, dz \, dr
\]

\[
+ \int_{\mathbb{R}^3} \left( \frac{1}{2} |u|^2 + p \right) \frac{1}{R} \sigma' \left( \frac{\sqrt{r^2 + z^2}}{R} \right) \frac{r u_r + z u_z}{\sqrt{r^2 + z^2}} \, dr \, dz := J_1 + J_2.
\]

Since \( \nabla u \in L^2(\mathbb{R}^3) \), then \( \frac{1}{2} |u|^2 + p \in L^3(\mathbb{R}^3) \) and the left hand side of (2.13) tends to

\[
\int_{\mathbb{R}^3} \left( |\nabla u_r|^2 + |\nabla u_\theta|^2 + |\nabla u_z|^2 + \frac{u_r^2}{r^2} + \frac{u_\theta^2}{r^2} \right) \, dx, \quad \text{as } R \to \infty.
\]

If \( (u_r, u_z) \in L^3(\mathbb{R}^3) \), then

\[
|J_1| \leq \frac{C(\sigma)}{R} \|u\|_{L^3(Q_R)}^2 |Q_R|^{2/3} \to 0, \quad \text{as } R \to \infty,
\]
\[
|J_2| \leq \frac{C(\sigma)}{R} \left( \frac{1}{2} |u|^2 + p \right)_{L^3(Q_R)} \|u_r, u_z\|_{L^3(Q_R)} |Q_R|^{1/3} \to 0, \quad \text{as } R \to \infty.
\]

Hence letting \( R \to \infty \) in (2.13), we can conclude that \( u \equiv 0 \). \( \square \)
2.3. Liouville type theorems conditioned on $\Gamma = ru_\theta$. It follows from (1.5) that

$$\left(u_r \partial_r + u_z \partial_z\right) \Gamma = \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2\right) \Gamma - \frac{2}{r} \partial_r \Gamma. \tag{2.14}$$

As is well known, for any harmonic function $h$ in $\mathbb{R}^3$, if $\lim_{|x| \to \infty} h(x) = 0$ or $h \in L^q(\mathbb{R}^3)$ for $q \in [1, \infty)$, then $h \equiv 0$. Inspired by this, in the following we show that some decay or integrability conditions on $\Gamma$ are also enough to ensure that $\Gamma \equiv 0$.

**Theorem 2.5.** Let $u(x)$ be an axially symmetric smooth solution to (1.1)-(1.2). If one of the following conditions holds

(i) $\lim_{|x| \to \infty} \Gamma(x) = 0$,

(ii) $\Gamma \in L^q(\mathbb{R}^3)$ for some $q \in [2, \infty)$,

then $u \equiv 0$.

**Proof.** Assume (i), then the conclusion follows immediately from the maximum and minimum principle by $\Gamma$. Note that $\Gamma(0+, z) = 0$ for all $z \in \mathbb{R}$, so $r = 0$ does not cause any trouble. Now we assume (ii). Multiplying the equation (2.14) by $\sigma_r \Gamma$ and integrating over the whole space, we obtain that

$$\int_{\mathbb{R}^3} \sigma_R(x) |\Gamma|^{q-2} \Gamma \left(u_r \partial_r + u_z \partial_z\right) \Gamma dx \tag{2.15}$$

We estimate both sides as follows.

**LHS**

$$= \frac{2\pi}{q} \int_{-\infty}^{\infty} \int_{0}^{\infty} r \sigma \left(\frac{\sqrt{r^2 + z^2}}{R}\right) (u_r \partial_r + u_z \partial_z) |\Gamma|^q(r, z) dr dz$$

$$\tag{2.16}$$

$$= -\frac{2\pi}{q} \int_{-\infty}^{\infty} \int_{0}^{\infty} r |\Gamma|^q \sigma' \left(\frac{\sqrt{r^2 + z^2}}{R}\right) \frac{ru_r + zu_z}{R \sqrt{r^2 + z^2}} dr dz := -I(R),$$

**RHS**

$$= 2\pi \int_{-\infty}^{\infty} \int_{0}^{\infty} r \sigma \left(\frac{\sqrt{r^2 + z^2}}{R}\right) |\Gamma|^{q-2} \Gamma(r, z) \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{2}{r} \partial_r\right) \Gamma(r, z) dr dz \tag{2.17}$$

$$= -\frac{8\pi (q - 1)}{q^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} r \sigma \left(\frac{\sqrt{r^2 + z^2}}{R}\right) |\nabla |\Gamma|^{q/2}|^2 dr dz$$

$$+ \frac{4\pi}{q} \int_{-\infty}^{\infty} \int_{0}^{\infty} \sigma' \left(\frac{\sqrt{r^2 + z^2}}{R}\right) |\Gamma|^q \frac{r}{R \sqrt{r^2 + z^2}} dr dz$$

$$- \frac{4\pi}{q} \int_{-\infty}^{\infty} \int_{0}^{\infty} r \sigma' \left(\frac{\sqrt{r^2 + z^2}}{R}\right) \frac{ru_r + zu_z |\Gamma|^q}{R \sqrt{r^2 + z^2}} dr dz$$

$$= -\frac{8\pi (q - 1)}{q^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} r \sigma \left(\frac{\sqrt{r^2 + z^2}}{R}\right) |\nabla |\Gamma|^{q/2}|^2 dr dz \tag{2.18}$$
Combining these together, we see that for any $0 < R < R_1 < \infty$:

\[ J_1(R) = I(R) + J_2(R) + J_3(R) \]
\[ \leq J_1(R_1) = I(R_1) + J_2(R_1) + J_3(R_1) \]
\[ \leq |I(R_1)| + |J_2(R_1)| + |J_3(R_1)|, \tag{2.19} \]

where we have used $J_1(R)$ is positive and monotonic increasing with respect to $R$.

Assume that $\Gamma \in L^q(\mathbb{R}^3)$ for some $q \in [2, \infty)$, then by Hölder’s inequality,

\[ |I(R_1)| \leq \frac{C(\sigma)}{R_1^3} \|u_r, u_z\|_{L^\infty(\mathbb{R}^3)} \|\Gamma\|_{L^q(Q_{R_1})} \to 0, \quad \text{as} \quad R_1 \to \infty, \]
\[ |J_2(R_1)| \leq CR_1^{-2} \|\sigma'\|_{L^\infty} \int_{Q_{R_1}} |\Gamma|^q dx \]
\[ \leq C(\sigma)R_1^{-2} \|\Gamma\|_{L^q(Q_{R_1})} \to 0, \quad \text{as} \quad R_1 \to \infty, \]
\[ |J_3(R_1)| \leq CR_1^{-2} (\|\sigma''\|_{L^\infty} + \|\sigma''\|_{L^\infty}) \int_{Q_{R_1}} |\Gamma|^q dx \]
\[ \leq C(\sigma)R_1^{-2} \|\Gamma\|_{L^q(Q_{R_1})} \to 0, \quad \text{as} \quad R_1 \to \infty. \]

In (2.19), we first fix $R > 0$ and let $R_1$ tends to $\infty$, we arrive at $\int_{B_R} |\nabla|\Gamma|^\frac{q}{2}|^2 dx = 0$, which implies that $\nabla|\Gamma|^\frac{q}{2} \equiv 0$ in $B_R$. Since $R$ is arbitrary, we have $\nabla|\Gamma|^\frac{q}{2} \equiv 0$ in the whole space. Since $\Gamma \in L^q(\mathbb{R}^3)$ for some $q \in [2, \infty)$, then $\Gamma \equiv 0$, i.e. $u_\theta \equiv 0$.

The problem is reduced to the case of axially symmetric flows with no swirl, then the theorem follows from Theorem 2.1. The proof is completed. 

\[ \square \]

**Remark 2.** The condition in (i) is equivalent to $u_\theta(x) = o(\frac{1}{r})$ as $|x| \to \infty$. The bound $\frac{1}{r}$ is the optimal decay one can expect by the fundamental solution of the steady Stokes equations.

**Remark 3.** To understand the condition in (ii), we take $q = 2$, then (ii) becomes $\Gamma \in L^2(\mathbb{R}^3)$, i.e.

\[ \int_0^\infty r^3 \int_{-\infty}^\infty |u_\theta(r, z)|^2 dzdr < +\infty. \tag{2.20} \]

In [8], the authors have proved the following inequality by using $(u_\theta, \nabla u_\theta) \in L^2(\mathbb{R}^3)$:

\[ \int_{-\infty}^\infty |u_\theta(r, z)|^2 dz < +\infty, \quad \forall r > 0. \tag{2.21} \]

Comparing with (2.20) and (2.21), heuristically, one may guess that the integrability of $u_\theta$ in the $z$-direction is enough, the decay rate of $u_\theta$ in the radial direction maybe a key issue. Unfortunately, the decay rates obtained in [8] seemed not good enough, this issue will be further investigated in [18].
From Theorem 2.2, it is natural to conject that a condition like \[ \|u_\theta\|_X \leq c_3, \]
maybe enough to guarantee that \( u \equiv 0 \), where \( X \) is a function norm and \( c_3 > 0 \) is a constant depending only on the dimension \( n = 3 \). Right now, we can not prove such a conjecture (see Corollary 1). However, we have some interesting conclusions.

**Theorem 2.6.** Let \( u(x) \) be an axially symmetric smooth solution to (1.1)-(1.2) with finite Dirichlet integral (1.3). If

\[
\left\| \frac{u_\theta}{r} \right\|_{L^4(\mathbb{R}^3)}^2 < \| \nabla \Omega \|_{L^2(\mathbb{R}^3)},
\]

then \( u \equiv 0 \).

**Proof.** It follows from (1.6), then \( \Omega \) satisfies

\[
(u_r \partial_r + uz \partial_z)\Omega - \frac{1}{r^2} \partial_z (u_\theta^2) = \left( \frac{\partial_r^2}{r^2} + \frac{1}{r} \partial_r + \partial_z^2 + \frac{2}{r} \partial_r \right) \Omega. \tag{2.23}
\]

We first verify that \( \Omega \in L^2(\mathbb{R}^3) \) and \( \nabla \Omega \in L^2(\mathbb{R}^3) \). Note that for any axially symmetric vector field \( f = (f_r, f_\theta, f_z) \), we have

\[
|\nabla f|^2 = |\partial_r f_r|^2 + |\partial_z f_z|^2 + |\partial_r f_\theta|^2 + |\partial_z f_\theta|^2 + \frac{|f_r|^2 |f_\theta|^2}{r^2} + \frac{|f_r|^2 |f_z|^2}{r^2} + |\partial_r f_z|^2 + |\partial_z f_z|^2.
\]

Hence by the definition of \( \Omega \), it suffices to verify that \( \nabla^2 u \) and \( \nabla^3 u \) belong to \( L^2(\mathbb{R}^3) \). These indeed follow from a standard bootstrap argument by regarding \( (u \cdot \nabla) u \) as a forcing term and using \( L^p \) estimates for the Stokes system (see Theorem IV.2.1 in [9]). By using the cut-off function and integration by parts as in the proof of Theorem 2.5, we can conclude that

\[
\int_{\mathbb{R}^3} |\nabla \Omega|^2 dx + 2\pi \int_{-\infty}^{\infty} \Omega^2(0, z) dz = \int_{\mathbb{R}^3} \frac{1}{r^2} \partial_z (u_\theta^2) \Omega dx \tag{2.24}
\]

\[
= - \int_{\mathbb{R}^3} \frac{u_\theta^2}{r^2} \partial_z \Omega dx \leq \left\| \frac{u_\theta}{r} \right\|_{L^4(\mathbb{R}^3)}^2 \| \nabla \Omega \|_{L^2(\mathbb{R}^3)}
\]

\[
< \| \nabla \Omega \|_{L^2(\mathbb{R}^3)}^2, \quad \text{if (2.22) holds.}
\]

Then \( \Omega \equiv 0 \) if (2.22) holds. That is, \( \omega_\theta = \partial_z u_r - \partial_r u_z \equiv 0 \). Together with \( \partial_r u_r + \frac{\omega_\theta}{r} + \partial_z u_z = 0 \), we can conclude that \( u_r = u_z \equiv 0 \). The equation for \( u_\theta \) in (1.5) reduces to

\[
\left( \frac{\partial_r^2}{r^2} + \frac{1}{r} \partial_r + \frac{1}{r^2} \right) u_\theta = 0.
\]

Setting \( \Lambda = \frac{u_\theta}{r} \), then

\[
\left( \frac{\partial_r^2}{r^2} + \frac{1}{r} \partial_r + \frac{2}{r} \partial_r \right) \Lambda = 0.
\]

Same argument as in the proof of Theorem 2.2, we can show that \( \Lambda \equiv 0 \) and \( u_\theta \equiv 0 \).

**Remark 4.** This theorem shows that if the swirl component is smaller than the other two components in the sense of (2.22), then \( u \equiv 0 \).

In the following, we will use the equation for \( \Lambda := \frac{u_\theta}{r} \) and the relation between \( \frac{u_\theta}{r} \) and \( \Omega \), to derive some interesting inequalities for \( \Omega \), showing that \( \nabla \Omega \) can be bounded by \( \Omega \) itself in some senses. Although \( \Omega \) satisfies an elliptic equation (2.23),...
but due to the extra term \(-\frac{1}{r^2} \partial_z(u_\theta^2)\) in (2.23), this property is not so clear at first sight.

**Theorem 2.7.** Let \( u(x) \) be an axially symmetric smooth solution to (1.1)-(1.2) with finite Dirichlet integral (1.3). Then

\[
\|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2 + 2\pi \|\Omega(0, \cdot)\|_{L^2(\mathbb{R})}^2 \leq 2^{5/3} C_5 u^2 \| \frac{u_\theta}{r} \|_{L^\infty} \| \frac{u_\theta}{r} \|_{L^2(\mathbb{R}^3)} \| \Omega \|_{L^{5/3}(\mathbb{R}^3)}^{5/3}, \tag{2.25}
\]

Furthermore, if we assume \( \frac{u_\theta}{r} \in L^{3/2}(\mathbb{R}^3) \), then

\[
\|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2 + 2\pi \|\Omega(0, \cdot)\|_{L^2(\mathbb{R})}^2 \leq 2 C_6 \| \frac{u_\theta}{r} \|_{L^{3/2}(\mathbb{R}^3)}^2 \| \nabla \Omega\|_{L^2}^2 + 2 \| \frac{u_\theta}{r} \|_{L^2} \| \partial_z u_r \|_{L^3} + \| \partial_z u_z \|_{L^3} \| \Omega\|_{L^{5/2}(\mathbb{R}^3)}^{5/2} \| \nabla \Omega\|_{L^2}^2, \tag{2.26}
\]

**Proof.** It follows from (2.24) that

\[
\|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2 + 2\pi \|\Omega(0, \cdot)\|_{L^2(\mathbb{R})}^2 = -2 \int_{\mathbb{R}^3} \Lambda \partial_z \Lambda \cdot \Omega \, dx \leq 2 \| \Lambda \|_{L^\infty} \| \partial_z \Lambda \|_{L^2(\mathbb{R}^3)} \| \Omega\|_{L^2(\mathbb{R}^3)}. \tag{2.27}
\]

It follows from (1.5) that \( \Lambda \) satisfies the following equation

\[
(u_r, \partial_z u_z) + \frac{2 u_r}{r} \Lambda = \left( \partial_r^2 + \frac{3}{r^2} \partial_r + \frac{1}{r^2} \right) \Lambda. \tag{2.28}
\]

Since \( \Lambda \in L^2 \) and \( \nabla \Lambda \in L^2(\mathbb{R}^3) \), then same as before, we can conclude

\[
\|\nabla \Lambda\|_{L^2(\mathbb{R}^3)}^2 \leq 2 \pi \| \Lambda(0, \cdot)\|_{L^2(\mathbb{R})}^2 \leq 2 \| \frac{u_\theta}{r} \|_{L^2(\mathbb{R}^3)}^2 \| \nabla \Lambda\|_{L^2(\mathbb{R}^3)}^2 \leq 2 \| \frac{u_\theta}{r} \|_{L^2(\mathbb{R}^3)}^2 \| \nabla \Lambda\|_{L^2(\mathbb{R}^3)}^2, \tag{2.29}
\]

where we have used the classical Gagliardo-Nirenberg inequality on \( \mathbb{R}^n \):

\[
\| f \|_{L^p} \leq C \| \nabla f \|_{L^q} \| f \|_{L^r}^{1-s},
\]

with \( p, q, s \geq 1 \) and \( \alpha \in [0, 1] \) satisfy the identity

\[
\alpha = \left( \frac{1}{s} - \frac{1}{q} \right) \left( \frac{1}{n} - \frac{1}{p} + \frac{1}{s} \right)^{-1}.
\]

Then we obtain

\[
\| \nabla \Lambda\|_{L^2(\mathbb{R}^3)}^2 \leq 2^{2/3} C_5 \| \nabla \left( \frac{u_\theta}{r} \right) \|_{L^2(\mathbb{R}^3)}^{2/3} \| \Lambda \|_{L^2(\mathbb{R}^3)}. \tag{2.30}
\]

We still need to explore the relation between \( \frac{u_\theta}{r} \) and \( \Omega \) by introducing the stream function, this relation was already known in the unsteady case \([5, 16]\). By the divergence free condition, \( \partial_r (ru_r) + \partial_z (ru_z) = 0 \), one can introduce a stream function \( \psi_\theta \) such that

\[
u_r = -\partial_z \psi_\theta, \quad u_z = \frac{1}{r} \partial_r (r \psi_\theta).
\]

Since \( \omega_\theta = \partial_z u_r - \partial_r u_z \), we have

\[-\left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \right) \psi_\theta = \omega_\theta.\]
Setting $\varphi = \frac{w_0}{r}$, then it is easy to see that

$$-\left(\partial_r^2 + \frac{3}{r}\partial_r + \frac{2}{r^2}\right)\varphi = \Omega.$$ 

The second order operator $\left(\partial_r^2 + \frac{3}{r}\partial_r + \frac{2}{r^2}\right)$ can be interpreted as the Laplace operator in $\mathbb{R}^5$, see [12, 13]. Introduce

$$y = (y_1, y_2, y_3, y_4, y_5), \quad r = \sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2}, \quad \Delta_y = (\partial_r^2 + \frac{3}{r}\partial_r + \frac{2}{r^2}).$$

Hence we have $\varphi = (-\Delta_y)^{-1}\Omega$ and $\frac{\partial}{\partial r} \varphi = -\partial_r \varphi$. By simple calculations, one has

$$|\nabla_y^2 \varphi|^2 \simeq |\partial_y^2 \varphi|^2 + \frac{1}{r}\partial_r \varphi|^2 + |\partial_y^2 \varphi|^2 + |\partial_y \varphi|^2$$

and

$$\int_{\mathbb{R}^5} |\nabla_y^2 \varphi|^2 dx \leq C_3 \int_{-\infty}^{\infty} \int_{0}^{\infty} \left(|\partial_y^2 \varphi|^2 + \frac{1}{r}\partial_r \varphi|^2 + |\partial_y^2 \varphi|^2 + |\partial_y \varphi|^2\right) r dr dz$$

$$= C_3 \int_{-\infty}^{\infty} \int_{0}^{\infty} \left(|\partial_y^2 \varphi|^2 + \frac{1}{r}\partial_r \varphi|^2 + |\partial_y^2 \varphi|^2 + |\partial_y \varphi|^2\right) w(r) r^3 dr dz$$

$$\leq C_4 \int_{-\infty}^{\infty} \int_{0}^{\infty} |\nabla_y^2 \varphi|^2 w(r) r^3 dr dz$$

$$= C_4 \int_{-\infty}^{\infty} \int_{0}^{\infty} |\nabla_y^2 (-\Delta_y)^{-1}\Omega|^2 w(r) r^3 dr dz$$

$$= C_4 \int_{\mathbb{R}^5} |\nabla_y^2 (-\Delta_y)^{-1}\Omega|^2 w(r) dy$$

$$\leq C_5 \int_{\mathbb{R}^5} |\Omega|^2 w(r) dy = C_5 \int_{\mathbb{R}^5} |\Omega|^2 dx,$$  \hspace{1cm} (2.31)

where $w(r) = r^{-2}$ and in the last step we have used the boundedness of Riesz operators in weighted Sobolev spaces (Lemma 2 in [12]). See also Corollary 2 in [4] for a similar weighted estimate for a singular integral operator.

Similarly, we also have

$$\int_{\mathbb{R}^5} |\nabla_y^2 \partial_y \varphi|^2 dx \leq C_6 \int_{\mathbb{R}^5} |\partial_y \Omega|^2 dx.$$  \hspace{1cm} (2.32)

Then by (2.31), we have

$$\left\|\nabla \left(\frac{u_r}{r}\right)\right\|_{L^2(\mathbb{R}^5)} \leq C_5 \|\Omega\|_{L^2(\mathbb{R}^5)}, \quad \left\|\nabla \partial_y \left(\frac{u_r}{r}\right)\right\|_{L^2(\mathbb{R}^5)} \leq C_6 \|\partial_y \Omega\|_{L^2(\mathbb{R}^5)}.$$  \hspace{1cm} (2.33)

Then (2.25) follows from (2.27), (2.30) and (2.33).

To derive (2.26), we estimate (2.27) as follows.

$$\|\nabla \Omega\|^2_{L^2(\mathbb{R}^5)} + 2\pi \|\Omega(0, \cdot)\|^2_{L^2(\mathbb{R})}$$

$$\leq \|\Lambda\|_{L^2(\mathbb{R}^5)} \|\partial_y \Lambda\|_{L^6(\mathbb{R}^5)} \|\Omega\|_{L^6(\mathbb{R}^5)}$$

$$\leq C_2^2 \|\Lambda\|_{L^2(\mathbb{R}^5)} \|\nabla \partial_y \Lambda\|_{L^2(\mathbb{R}^5)} \|\nabla \Omega\|_{L^2(\mathbb{R}^5)}.$$  \hspace{1cm} (2.34)

To estimate $\nabla \partial_y \Lambda$, we first derive the equation for $\partial_y \Lambda$:

$$\left(\partial_r^2 + \frac{3}{r}\partial_r + \frac{2}{r^2}\right) \partial_y \Lambda = (u_r \partial_r + u_x \partial_x) \partial_y \Lambda + 2u_r \partial_x \partial_y \Lambda + \partial_x \left(\frac{2u_r}{r}\right) \Lambda$$

$$+ (\partial_x u_r \partial_r + \partial_x u_x \partial_x) \Lambda.$$  \hspace{1cm} (2.35)
Then integrating by parts, we have
\[
\|\nabla \partial_z A\|_{L^2}^2 + 2\pi \|\partial_z A(0, \cdot)\|_{L^2}^2
\]
\[
= \left| \int_{\mathbb{R}^3} \left[ 2\frac{u_r}{r} (\partial_z A)^2 + 2\partial_z \left( \frac{u_r}{r} \right) \partial_z \Lambda + \partial_z u_r \partial_z \Lambda + \partial_z u_\perp (\partial_z \Lambda)^2 \right] dx \right|
\]
\[
\leq 2\|\frac{u_r}{r}\|_{L^3} \|\partial_z A\|_{L^2} d_{\Lambda} \|\partial_z A\|_{L^6} + 2 \left\| \partial_z \left( \frac{u_r}{r} \right) \right\|_{L^6} d_{\Lambda} \|\partial_z A\|_{L^6}
\]
\[
+ \|\partial_z u_r\|_{L^3} \|\partial_z A\|_{L^2} + \|\partial_z u_\perp\|_{L^3} \|\partial_z A\|_{L^2} \|\partial_z A\|_{L^6}
\]
\[
\leq 2C_s \|\frac{u_r}{r}\|_{L^3} \|\partial_z A\|_{L^2} \|\nabla \partial_z A\|_{L^2} + 2C_s^2 \left\| \partial_z \left( \frac{u_r}{r} \right) \right\|_{L^2} \|\partial_z A\|_{L^2} \|\nabla \partial_z A\|_{L^2}
\]
\[
+ C_s \|\partial_z u_r\|_{L^3} \|\partial_z A\|_{L^2} \|\nabla \partial_z A\|_{L^2} + C_s \|\partial_z u_\perp\|_{L^3} \|\partial_z A\|_{L^2} \|\nabla \partial_z A\|_{L^2},
\]  \quad (2.36)

By (2.32), (2.30), (2.36) yields
\[
\|\nabla \partial_z A\|_{L^2}
\]
\[\leq 2C_s^2 C_6 \|\partial_z A\|_{L^2} \|\nabla \Omega\|_{L^2} + C_s \left( 2\|\frac{u_r}{r}\|_{L^3} \|\partial_z u_r\|_{L^3} + \|\partial_z u_\perp\|_{L^3} \right) \|\nabla \Lambda\|_{L^2}
\]
\[\leq 2C_s^2 C_6 \|\partial_z A\|_{L^2} \|\nabla \Omega\|_{L^2} + 2C_s^2 C_6 \|\nabla \Omega\|_{L^2} \left( 2\|\frac{u_r}{r}\|_{L^3} \|\partial_z u_r\|_{L^3} + \|\partial_z u_\perp\|_{L^3} \right) \|\Omega\|_{L^\frac{3}{2}}^\frac{3}{2}.
\]

Hence (2.26) follows from (2.34) and (2.37). \hfill \Box

**Corollary 1.** Let \( u(x) \) be an axially-symmetric smooth solution to (1.1)- (1.2) with finite Dirichlet integral (1.3). If
\[
\begin{align*}
\|\frac{u_r}{r}\|_{L^3}^2 &< \frac{1}{4C_s^2 C_6}, \\
2C_s^2 \|\frac{u_r}{r}\|_{L^2} \|\frac{u_r}{r}\|_{L^2} \left( 2\|\frac{u_r}{r}\|_{L^3} \|\partial_z u_r\|_{L^3} + \|\partial_z u_\perp\|_{L^3} \right) \|\Omega\|_{L^\frac{3}{2}}^\frac{3}{2} &< \frac{1}{2} \|\nabla \Omega\|_{L^2},
\end{align*}
\]
then \( \Omega \equiv 0 \), which implies that \( u \equiv 0 \).

**Remark 5.** We remark here all the quantities in (2.25) and (2.26) have same scaling properties.

3. **Liouville type theorem for steady MHD equations.** In this section, we will investigate the steady Magnetohydrodynamics equations (MHD). The steady MHD equations are listed as follows.
\[
\begin{align*}
(u \cdot \nabla)u + \nabla p &= (h \cdot \nabla)h + \Delta u, & \forall x \in \mathbb{R}^3, \\
\text{div } u &= 0, \\
(u \cdot \nabla)h - (h \cdot \nabla)u &= \Delta h, & \forall x \in \mathbb{R}^3, \\
\text{div } h &= 0, \\
\text{lim } \frac{u(x)}{|x|} &= \text{lim } \frac{h(x)}{|x|} = 0.
\end{align*}
\]  \quad (3.1)

We also consider the weak solution to (3.1) with finite Dirichlet integral:
\[
\int_{\mathbb{R}^3} |\nabla u(x)|^2 + |\nabla h(x)|^2 dx < \infty.
\]  \quad (3.2)

Then following the argument developed in [9], one can show that any weak solution to (3.1) satisfying (3.2) is smooth. A natural problem is whether this solution is zero or not. We first present the following simple Liouville theorem for MHD equations,
by prescribing the $L^3$ integrability only on $u$. Similar regularity criteria has been developed in [11] and [20] for unsteady MHD equations.

**Theorem 3.1.** Let $(u, h)$ be a smooth solution to (3.1) in $\mathbb{R}^3$ with finite Dirichlet integral. If $u \in L^3(\mathbb{R}^3)$, then $u = h \equiv 0$.

**Proof.** Multiplying the first and third equation in (3.1) by $\sigma_R u$ and $\sigma_R h$ respectively, and integrating by parts, we finally obtain

$$
\int_{\mathbb{R}^3} \sigma_R(x)|\nabla u|^2 + |\nabla h|^2 \, dx + \frac{1}{R} \int_{\mathbb{R}^3} (u_i \partial_j u_i + h_i \partial_j h_i) \partial_j \sigma(x/R) \, dx
- \int_{\mathbb{R}^3} \frac{1}{R} \left( \frac{1}{2}|u|^2 + |h|^2 \right) u \cdot \nabla \sigma(x/R) \, dx + \int_{\mathbb{R}^3} \frac{1}{R} (u \cdot h) (h \cdot \nabla) \sigma(x/R) \, dx
\leq H_1 + H_2 + H_3 + H_4 = 0.
$$

It is well-known that $u \in L^6(\mathbb{R}^3), h \in L^6(\mathbb{R}^3)$ and $p \in L^3(\mathbb{R}^3)$. Since we assume that $u \in L^q(\mathbb{R}^3)$ for $q \in [1, 3]$, then $u \in L^3(\mathbb{R}^3)$. It is easy to see that $H_1 \to \|\nabla u\|_{L^2}^2 + \|\nabla h\|_{L^2}^2$ as $R \to \infty$, and also

$$
|H_2| \leq \|u\|_{L^6(Q_R)} \|\nabla u\|_{L^2(Q_R)} \to 0, \quad \text{as } R \to \infty,
$$

$$
|H_3| \leq \left( \|u\|_{L^6(Q_R)}^2 + \|h\|_{L^6(Q_R)}^2 \right) \|u\|_{L^3(Q_R)} \to 0, \quad \text{as } R \to \infty,
$$

$$
|H_4| \leq \|u\|_{L^3(Q_R)} \|h\|_{L^6(Q_R)}^2 \to 0, \quad \text{as } R \to \infty.
$$

Hence letting $R \to \infty$ in (3.3), we obtain

$$
\|\nabla u\|_{L^2}^2 + \|\nabla h\|_{L^2}^2 = 0
$$

Hence $u = h \equiv 0$. \qed

In the following, we derive the equation for the total head pressure $\Phi := \frac{1}{2}(|u|^2 + |h|^2) + p$. Assume that $(u, h)$ is a smooth solution to steady MHD equations (3.1), then by simple calculations, we obtain

$$
\Delta p = -\text{div} [(u \cdot \nabla)u] + \text{div} [(h \cdot \nabla)h] = -\sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i + \sum_{i,j=1}^{3} \partial_i h_j \partial_j h_i.
$$

This yields

$$
(u \cdot \nabla)\Phi - (h \cdot \nabla)(u \cdot h)
= u \cdot \Delta u + h \cdot \Delta h = \Delta \left( \frac{1}{2}(|u|^2 + |h|^2) \right) - |\nabla u|^2 - |\nabla h|^2
$$

$$
= \Delta \Phi - \left( |\nabla u|^2 - \sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i \right) - \left( |\nabla h|^2 + \sum_{i,j=1}^{3} \partial_i h_j \partial_j h_i \right).
$$

Since

$$
|\nabla u|^2 - \sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i = |\text{curl } u|^2
$$

$$
|\nabla h|^2 + \sum_{i,j=1}^{3} \partial_i h_j \partial_j h_i = 2 \sum_{i=1}^{3} (\partial_i h_1)^2 + (\partial_2 h_3 + \partial_3 h_2)^2
$$

$$
+ (\partial_3 h_1 + \partial_1 h_3)^2 + (\partial_1 h_2 + \partial_2 h_1)^2,
$$

we have

$$
(u \cdot \nabla)\Phi - (h \cdot \nabla)(u \cdot h) - \Delta \Phi \leq 0.
$$

(3.6)
Consider the special case, where \((\mathbf{u}, h)\) are axi-symmetric, and of the following special form
\[
\begin{aligned}
\mathbf{u}(x) &= u_r(r, z)e_r + u_\theta(r, z)e_\theta + u_z(r, z)e_z, \\
h(x) &= h_\theta(r, z)e_\theta,
\end{aligned}
\tag{3.7}
\]
then \((h \cdot \nabla)(\mathbf{u} \cdot h) = \hat{h}_\theta \partial_\theta (u_\theta h_\theta) \equiv 0\). Then we obtain the following important maximum principle.

**Lemma 3.2.** Let \((\mathbf{u}, h)\) be a axially symmetric smooth solution to (3.1) with the form (3.7). Then we have the following important inequality
\[
(\mathbf{u} \cdot \nabla)\Phi - \Delta \Phi \leq 0.
\tag{3.8}
\]
Hence we have the following maximum principle for \(\Phi\) in any bounded domains \(\Omega\):
\[
\max_{x \in \Omega} \Phi(x) \leq \max_{x \in \partial \Omega} \Phi(x).
\tag{3.9}
\]
In our cases, as shown in [9], by adding a constant if necessary, one has \(\lim_{|x| \to \infty} p(x) = 0\), so \(\lim_{|x| \to \infty} \Phi(x) = 0\). Hence by maximum principle, we have
\[
\Phi(x) \leq 0, \quad \forall x \in \mathbb{R}^3.
\tag{3.10}
\]

If \((\mathbf{u}, h)\) are axi-symmetric with the form (3.7), then (3.1) can be rewritten as
\[
\begin{aligned}
(u_r \partial_r + u_z \partial_z)u_r - \frac{u_\theta^2}{r} + \partial_r p &= -\frac{h_\theta^2}{r} + \left(\partial_\theta^2 + \frac{1}{r} \partial_r + \partial_r^2 - \frac{1}{r^2}\right) u_r, \\
(u_r \partial_r + u_z \partial_z)u_\theta + \frac{u_r u_\theta}{r} &= \left(\partial_\theta^2 + \frac{1}{r} \partial_r + \partial_r^2 - \frac{1}{r^2}\right) u_\theta, \\
(u_r \partial_r + u_z \partial_z)u_z + \partial_z p &= \left(\partial_\theta^2 + \frac{1}{r} \partial_r + \partial_r^2\right) u_z, \\
\partial_r u_r + \frac{u_r h_\theta}{r} &= \left(\partial_\theta^2 + \frac{1}{r} \partial_r + \partial_r^2 - \frac{1}{r^2}\right) h_\theta,
\end{aligned}
\tag{3.11}
\]

**Theorem 3.3.** Let \((\mathbf{u}, h)\) be an axially symmetric smooth solution to (3.1) with the form (3.7). Then \(h \equiv 0\).

**Proof.** Define \(\Pi(r, z) = \frac{h_\theta(r, z)}{r}\), then it follows from (3.11) that
\[
(u_r \partial_r + u_z \partial_z)\Pi = \left(\partial_\theta^2 + \frac{1}{r} \partial_r + \partial_r^2\right) \Pi + \frac{2}{r} \partial_\theta \Pi.
\tag{3.12}
\]
Since \(\lim_{|x| \to \infty} \Pi(x) = 0\), by the equation (3.12), we have the maximum and minimum principle, which implies that \(\Pi(x) \equiv 0\), i.e. \(h \equiv 0\).

Now we consider the non-resistive, inviscid MHD equations:
\[
\begin{aligned}
(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= (h \cdot \nabla)h, \quad \forall x \in \mathbb{R}^3, \\
\text{div} \mathbf{u} &= 0, \\
(\mathbf{u} \cdot \nabla)h - (h \cdot \nabla)\mathbf{u} &= 0, \quad \forall x \in \mathbb{R}^3, \\
\text{div} \ h &= 0, \\
\lim_{|x| \to \infty} \mathbf{u}(x) &= \lim_{|x| \to \infty} h(x) = 0.
\end{aligned}
\tag{3.13}
\]
For smooth axially symmetric solution \((\mathbf{u}, h, p)\) to (3.13) with the form (3.7), we can derive the Bernoulli’s law for the total head pressure:
\[
(\mathbf{u} \cdot \nabla)\Phi = 0.
\tag{3.14}
\]
One can use this Bernoulli’s law to establish the existence of weak solutions to steady axially symmetric MHD equations with nonhomogeneous boundary conditions by following the approach developed in [15], for the details please refer to [19].

By slightly modifying the proof in [14], we can derive the following Liouville theorem for (3.13).

**Theorem 3.4.** Let \((u, h, p)\) be an axially symmetric solution to (3.13) with the form (3.7) and finite Dirichlet integral. If \(u\) has no swirl and the corresponding head pressure \(\Phi \in L^3(\mathbb{R}^3)\) satisfies the condition \(\Phi(x) \leq 0\) for any \(x \in \mathbb{R}^3\), then \(u = h \equiv 0\).

**Proof.** Suppose \(\nabla u, \nabla h \in L^2(\mathbb{R}^3)\). Then \((u, h) \in L^6(\mathbb{R}^3) \times L^6(\mathbb{R}^3)\). As argued in [14], we can also derive that

\[ p \in D^{2,1}(\mathbb{R}^3) \cap D^{1,3/2}(\mathbb{R}^3). \tag{3.15} \]

In particular, \(\partial_r p \in L^1(P_+)\) with \(P_+ := \{(r, 0, z) \in \mathbb{R}^3 : r > 0, z \in \mathbb{R}\}\). Moreover,

\[ p(r, \cdot) \in L^1(\mathbb{R}), \quad |u(r, \cdot)|^2 + |h(r, \cdot)|^2 \in L^1(\mathbb{R}), \quad \forall r > 0. \tag{3.16} \]

From the steady MHD system (3.1) it follows by direct calculation that for any smooth vector function \(g\) we have

\[ \text{div } [pg + (u \cdot g)u - (h \cdot g)h] = pd\text{iv } g + [(u \cdot \nabla)g] \cdot u - ((h \cdot \nabla g) \cdot h). \tag{3.17} \]

We choose \(g = g(r)e_r\), where \(e_r\) is the unit vector parallel to the \(r\)-axis. Then by simple calculations, we obtain

\[
\begin{align*}
\text{div } [pg + (u \cdot g)u - (h \cdot g)h] &= \text{div } [(p + u^2)g(r)e_r + g(r)u_ux_z e_z] \\
&= \partial_r \left( g(r)(p + u^2) \right) + \frac{g(r)}{r} \partial_r (p + u^2) + \partial_z (g(r)u_ux_z),
\end{align*}
\]

\[ pd\text{iv } g + [(u \cdot \nabla)g] \cdot u - ((h \cdot \nabla g) \cdot h) = \left( g'(r) + \frac{1}{r} g(r) \right) p + g'(r)u^2 + \frac{g(r)}{r} u^2 - \frac{1}{r} g(r) h_\theta^2. \]

(i) Let \(g(r) = r\). Then for axially symmetric \(u\) and \(p\) we get

\[ pd\text{iv } g + [(u \cdot \nabla)g] \cdot u - ((h \cdot \nabla g) \cdot h) = 2p + u_\theta^2 + u_r^2 - h_\theta^2. \]

Integrating this identity over the three dimensional infinite cylinder \(C_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : r = \sqrt{x_1^2 + x_2^2} < t, x_3 \in \mathbb{R}\}\), we obtain

\[ t^2 \int_\mathbb{R} [p(t, z) + u_\theta^2(t, z)] dz = \int_{P_t} r[2p + u_\theta^2 + u_r^2 - h_\theta^2] dr dz \tag{3.18} \]

\[ = \int_{P_t} r(2\Phi - u_\theta^2 - h_\theta^2) dr dz \leq 0 \]

where \(P_t = \{(r, z) \in P_+ : r < t\}\). Here we use \(\Phi(x) \leq 0\) for all \(x \in \mathbb{R}^3\).

(ii) Let \(g(r) = \frac{1}{r}\). Then

\[ pd\text{iv } g + [(u \cdot \nabla)g] \cdot u - ((h \cdot \nabla g) \cdot h) = \frac{1}{r^2}(u_\theta^2 - u_r^2 - h_\theta^2). \]

Since we have an essential singularity at \(r = 0\), we need to integrate this identity over the cylindrical annulus \(C_{t0} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : r = \sqrt{x_1^2 + x_2^2} \in (t_0, t), x_3 \in \mathbb{R}\}\) to obtain

\[
\int_\mathbb{R} [p(t, z) + u_\theta^2(t, z)] dz - \int_\mathbb{R} [p(t_0, z) + u_\theta^2(t_0, z)] dz = \int_{P_{t0}} \frac{1}{r} (u_\theta^2 - u_r^2 - h_\theta^2) dr dz.
\]
where \( P_{tot} = \{ (r, z) \in P_+ : r \in (t_0, t), z \in \mathbb{R} \} \).

Since \( \int_{\mathbb{R}} [p(t, z) + u^2_r(t, z)] dz \to 0 \) as \( t \to +\infty \) and
\[
\int \int_{P_+} \left| \frac{1}{r} (u^2_\theta - u^2_r - h^2_\theta) \right| dz dr < \infty,
\]
we obtain immediately from the above formulas that
\[
\int [p(t, z) + u^2_r(t, z)] dz = \int \int_{P_{t,\infty}} \left( \frac{1}{r} (h^2_\theta + u^2_r - u^2_\theta) \right) dz dr \geq 0, \quad \text{if } u_\theta(r, z) \equiv 0,
\]
where \( P_{t,\infty} = \{ (r, z) \in P_+ : r \in (t, +\infty), z \in \mathbb{R} \} \).

If \( u \) is axially symmetric with no swirl, then the formulas (3.18) and (3.19) imply \( u \equiv 0 \). Indeed, from the last two inequalities it follows that \( \int_{\mathbb{R}} [p(t, z) + u^2_r(t, z)] dz \equiv 0 \) for all \( t > 0 \), and thus, by virtue of (3.19) and \( u_\theta(x) = 0 \), we obtain \( u_r \equiv h_\theta \equiv 0 \). Therefore, by (3.13) we conclude that \( u_z \equiv 0 \).

4. Liouville theorem for steady viscous resistive Hall-MHD equations.

The steady Hall-MHD equations read as follows.
\[
\begin{align*}
(u \cdot \nabla)u + \nabla p &= (h \cdot \nabla)h + \Delta u, \quad \forall x \in \mathbb{R}^3, \\
\text{div } u &= 0, \\
(u \cdot \nabla)h - (h \cdot \nabla)u + \nabla \times ((\nabla \times h) \times h) &= \Delta h, \quad \forall x \in \mathbb{R}^3, \\
\text{div } h &= 0, \\
\lim_{|x| \to \infty} u(x) &= \lim_{|x| \to \infty} h(x) = 0.
\end{align*}
\]

In this section, we consider the smooth solution to (4.1) with finite Dirichlet integral (3.2). Comparing with the well-known MHD system, the Hall term \( \nabla \times ((\nabla \times h) \times h) \) is included due to the Ohm’s law, which is believed to be a key issue for understanding magnetic reconnection. Note that the Hall term is quadratic in the magnetic field and involves the second order derivatives. A derivation of Hall-MHD system from a two-fluids Euler-Maxwell system for electrons and ions was presented in [1], through a set of scaling limits. They also provided a kinetic formulation for the Hall-MHD, and proved the existence of global weak solutions for the incompressible viscous resistive Hall-MHD system. The authors in [5] have showed unsteady Hall-MHD system without resistivity may develop finite time singularity for a special class of axially symmetric datum. Different from the steady MHD case, a weak solution to (4.1) with finite Dirichlet integral may not be smooth. Chae and Wolf [6] have investigated the partial regularity of suitable weak solutions to steady Hall-MHD equations, showing the set of possible singularities has Hausdorff dimension at most one. The authors in [3] had derived a Liouville theorem for steady Hall-MHD under the conditions \( (u, h) \in L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \).

**Theorem 4.1.** Let \( (u, h) \) be a smooth solution to (4.1) in \( \mathbb{R}^3 \) with finite Dirichlet integral. If \( u \in L^3(\mathbb{R}^3) \), then \( u = h \equiv 0 \).

**Proof.** Similar to the proof in Theorem 3.1, we only need to check the effect of the Hall-term \( H = \int_{\mathbb{R}^3} \sigma_R(x) h \cdot (\nabla \times ((\nabla \times h) \times h)) dx \). Indeed,
\[
|H| = \left| - \int_{\mathbb{R}^3} \frac{1}{R} h \cdot (\nabla \sigma(x/R) \times ((\nabla \times h) \times h)) dx \right|
\leq \| \nabla h \|_{L^2(Q_R)} \| h \|_{H^1(Q_R)}^2 \to 0, \quad \text{as } R \to \infty.
\]
Then we finish the proof.

If we also consider the axially symmetric smooth solution \((u, h)\) to (4.1) with the form

\[
 u(x) = u_r(r, z)e_r + u_\theta(r, z)e_\theta + u_z(r, z)e_z, \quad h(x) = h_\theta(r, z)e_\theta,
\]

then (4.1) can be rewritten as

\[
 \begin{aligned}
 (u_r \partial_r + u_z \partial_z) u_r - \frac{u_r^2}{r} + \partial_r p &= -\frac{h_\theta^2}{r} + \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u_r, \\
 (u_r \partial_r + u_\theta \partial_\theta) u_\theta + \frac{u_r u_\theta}{r} &= \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u_\theta, \\
 (u_r \partial_r + u_z \partial_z) u_z + \partial_z p &= \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u_z, \\
 (u_r \partial_r + u_\theta \partial_\theta) h_\theta - \frac{u_r h_\theta}{r} - \frac{2h_\theta^2}{r} \partial_r h_\theta &= \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) h_\theta, \\
 \partial_r u_r + \frac{u_r}{r} + \partial_z u_z &= 0.
 \end{aligned}
\]

Due to Hall term, we cannot arrive at any maximum principle for the total head pressure. However, we still have the following Liouville theorem.

**Theorem 4.2.** Let \( u \) and \( h \) be an axially symmetric smooth solution to (4.1) with the form (4.2). Then \( h \equiv 0 \).

**Proof.** Setting \( \Pi := \frac{h_\theta}{r} \) as before, then \( \Pi \) satisfies the following equation

\[
 (u_r \partial_r + u_z \partial_z) \Pi - 2 \Pi \partial_z \Pi = \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 \right) \Pi + \frac{2}{r} \partial_r \Pi.
\]

Since \( \lim_{|x| \to \infty} h(x) = 0 \), then \( \lim_{|x| \to \infty} \Pi(x) = 0 \). Then by the maximum principle of \( \Pi \) from (4.4), we conclude that \( \Pi \equiv 0 \), i.e. \( h_\theta \equiv 0 \). \qed

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