Approximation and bounds for the Wallis ratio

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October 15, 2018

Abstract

In this paper, we present an improved continued fraction approximation of the Wallis ratio. This approximation is fast in comparison with the recently discovered asymptotic series. We also establish the double-side inequality related to this approximation. Finally, some numerical computations are provided for demonstrating the superiority of our approximation.

1 Introduction

The Wallis ratio is defined as

\[ W(n) = \frac{(2n - 1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)}, \]

where \( \Gamma \) is the classical Euler gamma function which may be defined by

\[ \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \text{Re}(x) > 0. \]

The study and applications of \( W(n) \) have a long history, a large amount of literature, and a lot of new results. For detailed information, please refer these papers [1, 2, 3, 4] and references cited therein.

Chen and Qi [1] presented the following inequalities for the Wallis ratio for every natural number \( n \):

\[ \frac{1}{\sqrt{\pi(n + \frac{4}{\pi} - 1)}} \leq \frac{(2n - 1)!!}{(2n)!!} \leq \frac{1}{\sqrt{\pi(n + \frac{1}{2})}}, \]

where the constants \( \frac{4}{\pi} - 1 \) and \( \frac{1}{2} \) are the best possible.

\[ \begin{array}{l}
2010 \text{ Mathematics Subject Classification: 33B15, 26A48, 26D07} \\
\text{Key words and phrases: Wallis ratio, Gamma function, Inequalities, Multiple-correction method}
\end{array} \]
Guo, Xu and Qi proved in \[6\] that the double inequality
\[
\sqrt{e \pi \left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}} < W(n) \leq \frac{4}{3} \left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n},
\]
for \(n \geq 2\) is valid and sharp in the sense that the constants \(\sqrt{e \pi}\) and \(\frac{4}{3}\) are best possible. They also proposed the approximation formula
\[
W(n) \sim \chi(n) := \sqrt{e \pi \left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}}, n \to \infty.
\]

Recently, Qi and Mortici \[5\] improved the approximation formula (1.3) as following,
\[
W(n) \sim \sqrt{e \pi \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n + \frac{1}{3}} \frac{1}{\sqrt{n}} \exp \left(\frac{1}{n^2 n + b_1 + \frac{a_1}{n + b_2 + \frac{a_2}{n + b_3}}}ight)},
\]
where \(a_1 = \frac{1}{144}, b_1 = \frac{1}{60}; a_2 = \frac{781}{3600}, b_2 = -\frac{4309}{109340}; a_3 = \frac{51396085}{89664267}, b_3 = \frac{25682346121}{449571834712}\).

Motivated by these works, in this paper we will apply the multiple-correction method \[7, 8, 9\] to construct an improved continued fraction asymptotic expansion for the Wallis ratio as follows:

**Theorem 1.** For the Wallis ratio \(W(n) = \frac{(2n-1)!!}{(2n)!!}\), we have
\[
W(n) \sim \sqrt{e \pi \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n + \frac{1}{3}} \frac{1}{\sqrt{n}} \exp \left(\frac{1}{n^2 n + b_1 + \frac{a_1}{n + b_2 + \frac{a_2}{n + b_3}}}ight)},
\]
where \(a_1 = \frac{1}{144}, b_1 = \frac{1}{60}; a_2 = \frac{781}{3600}, b_2 = -\frac{4309}{109340}; a_3 = \frac{51396085}{89664267}, b_3 = \frac{25682346121}{449571834712}\).

Using Theorem 1, we provide some inequalities for the Wallis ratio.

**Theorem 2.** For every integer \(n > 1\), it holds:
\[
\sqrt{e \pi \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n + \frac{1}{3}} \frac{1}{\sqrt{n}} \exp \left(\frac{1}{n^2 n + b_1 + \frac{a_1}{n + b_2 + \frac{a_2}{n + b_3}}}ight)} > W(n) > \sqrt{e \pi \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n + \frac{1}{3}} \frac{1}{\sqrt{n}} \exp \left(\frac{1}{n^2 n + b_1 + \frac{a_1}{n + b_2 + \frac{a_2}{n + b_3}}}ight)},
\]
where \(a_1 = \frac{1}{144}, b_1 = \frac{1}{60}; a_2 = \frac{781}{3600}, b_2 = -\frac{4309}{109340}; a_3 = \frac{51396085}{89664267}, b_3 = \frac{25682346121}{449571834712}\).

To obtain Theorem 1, we need the following lemma which was used in \[10, 11, 12\] and is very useful for constructing asymptotic expansions.

**Lemma 1.** If the sequence \((x_n)_{n \in \mathbb{N}}\) is convergent to zero and there exists the limit
\[
\lim_{n \to +\infty} n^s (x_n - x_{n+1}) = l \in [-\infty, +\infty]
\]
with \(s > 1\), then
\[
\lim_{n \to +\infty} n^{s-1} x_n = \frac{l}{s-1}.
\]
Lemma 1 was proved by Mortici in [10]. From Lemma 1, we can see that the speed of convergence of the sequences \((x_n)_{n \in \mathbb{N}}\) increases together with the values \(s\) satisfying (1.8).

The rest of this paper is arranged as follows. In section 2, we will apply the multiple-correction method to construct a new asymptotic expansion for the Wallis ratio and prove Theorem 1 by the multiple-correction method. In section 3, we established the double-side inequality for the Wallis ratio. In section 4, we give some numerical computations which demonstrate the superiority of our new series over some formulas found recently.

2 Proof of Theorem 1

According to the argument of Theorem 5.1 in [5], we can introduce a sequence \((u_n)_{n \geq 1}\) by the relation

\[
W(n) = \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n + \frac{1}{3}} \frac{1}{\sqrt{n}} \exp u(n),
\]

and to say that an approximation \(W(n) \sim \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n + \frac{1}{3}} \frac{1}{\sqrt{n}}\) is better if the speed of convergence of \(u(n)\) is higher.

(Step 1) The initial-correction. When \(n \to \infty\), we define a sequence \((u_0(n))_{n \geq 1}\)

\[
W(n) = \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n + \frac{1}{3}} \frac{1}{\sqrt{n}} \exp u_0(n).
\]

From (2.2), we have

\[
u_0(n) = \ln W(n) - \ln \sqrt{\frac{e}{\pi}} - \left(n + \frac{1}{3}\right) \ln \left(1 - \frac{1}{2(n + \frac{1}{3})}\right) - \frac{1}{2} \ln \frac{1}{n}.
\]

Thus,

\[
u_0(n) - u_0(n + 1) = \ln \frac{2n + 2}{2n + 1} - \left(n + \frac{1}{3}\right) \ln \left(1 - \frac{1}{2(n + \frac{4}{3})}\right) - \frac{1}{2} \ln \frac{1}{n} + \left(n + \frac{4}{3}\right) \ln \left(1 - \frac{1}{2(n + \frac{4}{3})}\right) + \frac{1}{2} \ln \frac{1}{n + 1}.
\]

Developing (2.4) into power series expansion in \(1/n\), we have

\[
u_0(n) - u_0(n + 1) = \frac{1}{48 n^4} + O\left(\frac{1}{n^5}\right),
\]

By Lemma 1, we know that the rate of convergence of the sequence \((u_0(n))_{n \geq 1}\) is \(n^{-3}\).
(Step 2) The first-correction. We define the sequence \((u_1(n))_{n \geq 1}\) by the relation
\[
W(n) = \sqrt{\frac{c}{\pi}} \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n+\frac{1}{3}} \frac{1}{\sqrt{n}} \exp \left(\frac{1}{n^2} a_1 \frac{n^2 + b_1}{n + b_1}\right) \exp u_1(n).
\]
From (2.6), we have
\[
u_1(n) - u_1(n + 1) = \ln \frac{2n + 2}{2n + 1} - \left(n + \frac{1}{3}\right) \ln \left(1 - \frac{1}{2(n + \frac{1}{3})}\right) - \frac{1}{2} \ln \frac{n}{n + 1} + \frac{1}{2} \ln \frac{1}{n + 1} + \frac{n + 4}{3} \ln \left(1 - \frac{1}{2(n + \frac{1}{3})}\right) - \frac{1}{n^2} a_1 + \frac{1}{(n + 1)^2} n + 1 + b_1.
\]
Developing (2.7) into power series expansion in \(1/n\), we have
\[
u_1(n) - u_1(n + 1) = \left(\frac{1}{48} - 3a_1\right) \frac{1}{n^6} + \left(-\frac{91}{2160} + a_1(6 + 4b_1)\right) \frac{1}{n^9} + O\left(\frac{1}{n^{12}}\right).
\]
By Lemma 1, we know that the fastest possible sequence \((u_1(n))_{n \geq 1}\) is obtained as the first item on the right of (2.8) vanishes. So taking \(a_1 = \frac{1}{144}, b_1 = \frac{1}{60}\), we can get the rate of convergence of the sequence \((u_1(n))_{n \geq 1}\) is at least \(n^{-5}\).

(Step 3) The second-correction. We define the sequence \((u_2(n))_{n \geq 1}\) by the relation
\[
u_2(n) = \sqrt{\frac{c}{\pi}} \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n+\frac{1}{3}} \frac{1}{\sqrt{n}} \exp \left(\frac{1}{n^2} a_1 \frac{n^2 + b_1 + \frac{42}{n+b_2}}{n+b_3}\right) \exp u_2(n).
\]
Using the same method as above, we obtain that the sequence \((u_2(n))_{n \geq 1}\) converges fasted only if \(a_2 = \frac{781}{3600}, b_2 = -\frac{4399}{109540}\).

(Step 4) The third-correction. Similarly, define the sequence \((u_3(n))_{n \geq 1}\) by the relation
\[
u_3(n) = \sqrt{\frac{c}{\pi}} \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n+\frac{1}{3}} \frac{1}{\sqrt{n}} \exp \left(\frac{1}{n^2} a_1 \frac{n^2 + b_1 + \frac{a_2}{n+b_2} + \frac{a_4}{n+b_3}}{n+b_3}\right) \exp u_3(n)
\]
Using the same method as above, we obtain that the sequence \((u_3(n))_{n \geq 1}\) converges fasted only if \(a_3 = \frac{51396085}{35964256}, b_3 = \frac{2562346121}{149571834712}\).
The new asymptotic (1.5) is obtained.

3 Proof of Theorem 2

The double-side inequality (1.6) may be written as follows:
\[
f(n) = \ln W(n) - \frac{1}{2} \ln \frac{1}{2} - \left(n + \frac{1}{3}\right) \ln \left(1 - \frac{1}{2(n + \frac{1}{3})}\right) - \frac{1}{2} \ln \frac{1}{n^2} n + 1 + \frac{4309}{144} \frac{1}{n^9} + O\left(n^{-12}\right)
\]
and

\[ g(n) = \ln W(n) - \frac{1}{2} + \frac{1}{2} \ln \pi - \left( n + \frac{1}{3} \right) \ln \left( 1 - \frac{1}{2(n + \frac{1}{3})} \right) \]

\[ - \frac{1}{n} \ln \frac{1}{n} - \frac{1}{n^2} \ln \frac{1}{n} + \frac{1}{2}\frac{1}{n + \frac{1}{6}} + \frac{1}{n + \frac{1}{12}}. \]

Suppose \( F(n) = f(n+1) - f(n) \) and \( G(n) = g(n+1) - g(n) \). For every \( x > 1 \), we can get

\[ F''(x) = \frac{A(x-1)}{160x^4(1+x)^4(1+2x)^2(1+3x)(4+3x)(6x-1)^2(5+6x)^2\Psi_1^2(x;2)\Psi_2^2(x;2)} < 0 \]

and

\[ G''(x) = \frac{B(x)}{32x^4(1+x)^4(1+2x)^2(1+3x)(4+3x)(-1+6x)^2(5+6x)^2\Psi_1^2(x;3)\Psi_2^2(x;3)} > 0, \]

where

\[ \Psi_1(x;2) = 10642 - 1119x + 49203x^2, \]
\[ \Psi_2(x;2) = 58726 + 97287x + 49203x^2, \]
\[ \Psi_1(x;3) = 113505180 + 4083418255x + 178132605x^2 + 5180725368x^3, \]
\[ \Psi_2(x;3) = 9555781408 + 19981859659x + 15720308709x^2 + 5180725368x^3, \]

\[ A(x) = -461003, ..., 512n^{18} - ... \] is a polynomial of 18th degree with all negative coefficients and \( B(x) = 212876, ..., 696x^{22} + ... \) is a polynomial of 22nd degree with all positive coefficients.

It shows that \( F(x) \) is strictly concave and \( G(x) \) is strictly convex on \((0, \infty)\). According to Theorem 1, when \( n \rightarrow \infty \), it holds that \( \lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} g(n) = 0 \); thus \( \lim_{n \rightarrow \infty} F(n) = \lim_{n \rightarrow \infty} G(n) = 0 \). As a result, we can make sure that \( F(x) < 0 \) and \( G(x) > 0 \) on \((0, \infty)\). Consequently, the sequence \( f(n) \) is strictly increasing and \( g(n) \) is strictly decreasing while they both converge to 0. As a result, we conclude that \( f(n) > 0 \), and \( g(n) < 0 \) for every integer \( n > 1 \).

The proof of Theorem 2 is complete.
### Table 1: Simulations for $\alpha(n)$, $\beta(n)$, $\gamma(n)$ and $\sigma(n)$

| $n$ | $W(n) - \alpha(n)$ | $W(n) - \beta(n)$ | $W(n) - \gamma(n)$ | $W(n) - \sigma(n)$ |
|-----|--------------------|--------------------|--------------------|--------------------|
| 50  | $-6.1876 \times 10^{-6}$ | $7.3576 \times 10^{-14}$ | $5.5532 \times 10^{-8}$ | $-3.8082 \times 10^{-19}$ |
| 500 | $-6.2438 \times 10^{-8}$ | $7.1643 \times 10^{-20}$ | $5.5554 \times 10^{-11}$ | $-3.8138 \times 10^{-28}$ |
| 1000| $-1.5617 \times 10^{-8}$ | $1.1177 \times 10^{-21}$ | $6.9443 \times 10^{-12}$ | $-7.4489 \times 10^{-31}$ |
| 2000| $-3.9053 \times 10^{-9}$ | $1.7452 \times 10^{-23}$ | $8.6805 \times 10^{-13}$ | $-1.4549 \times 10^{-33}$ |

### 4 Numerical computations

In this section, we give Table 1 to demonstrate the superiority of our new series respectively. From what has been discussed above, we found out the new asymptotic function as follows:

$W(n) \sim \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n+\frac{1}{2}} \frac{1}{\sqrt{n}} \exp \left(\frac{1}{n^2} \frac{1}{69} + \frac{1}{n^{10}} \sum_{k=4}^{4n} k\right) = \sigma(n)$.  

Chen and Qi \[2\] gave:

$W(n) \sim \frac{1}{\sqrt{\pi (n + \frac{1}{4})}} = \alpha(n)$.  

Qi and Mortici \[5\] gave the improved formula:

$W(n) \sim \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \sqrt{n} \exp \left(\frac{1}{24n^2} + \frac{1}{48n^3} + \frac{1}{160n^4} + \frac{1}{960n^5}\right) = \beta(n)$

and

$W(n) \sim \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2(n + \frac{1}{3})}\right)^{n+\frac{1}{2}} \frac{1}{\sqrt{n}} = \gamma(n)$.  

We can easily observe that the new formula converges fastest of the other three formulas.

### 5 Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant No. 61403034), Beijing Municipal Commission of Education Science and Technology Program (KM201510017002).
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