Hybrid Decoding of Finite Geometry LDPC Codes

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Abstract

For finite geometry low-density parity-check codes, heavy row and column weights in their parity check matrix make the decoding with even Min-Sum (MS) variants computationally expensive. To alleviate it, we present a class of hybrid schemes by concatenating a parallel bit flipping (BF) variant with an Min-Sum (MS) variant. In most SNR region of interest, without compromising performance or convergence rate, simulation results show that the proposed hybrid schemes can save substantial computational complexity with respect to MS variant decoding alone. Specifically, the BF variant, with much less computational complexity, bears most decoding load before resorting to MS variant. Computational and hardware complexity is also elaborated to justify the feasibility of the hybrid schemes.
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I. INTRODUCTION

Low-density parity-check (LDPC) codes, given a sufficiently long block length, can approach Shannon limit with belief propagation (BP) decoding [1][2]. Hence, it remains a research focus among others in the coding field. Lately, a class of finite geometry (FG) LDPC codes have attracted great interest, by virtue of the fact that they are encodable in linear time with feedback shift registers [3][4]. However, compared to other classical LDPC codes, it require much more complexity to decode with standard BP algorithms for FG-LDPC codes, due to heavy row and column weights in their parity check matrix.

There exist many low complexity decoding schemes applicable for FG-LDPC codes. The hard decodings [5][6] have the least complexity but suffer severe performance loss. To alleviate the degradation, at the cost of moderate complexity, a class of bit flipping (BF) variants improve performance after taking into account the soft information of received sequences. In [7], a BF function was devised wherein both the most and the least reliable bits involved in one check sum are considered. Further improvement was reported [8] by weighting each term in the BF function. A bootstrapping step [9][10] was proposed to update those unreliable bits prior to calculating their BF function values. Based on [3], the methods presented in [11][12] achieved better performance, as a result of considering the impact of its received soft information on the BF function value of a specific bit. However, one common drawback of above BF variants is that only one bit is flipped per iteration, which is adverse to fast convergence requirement. To lower the decoding latency caused by such serial BF strategy, [13], [14] and [15] presented three decoding methods in the form of multi-bit flipping per iteration. In [13], when the flipping signal counter for each bit has reached a predesigned threshold, the pertinent bits flip immediately; in [14], the number of bits chosen to be flipped approximates the quotient of the number of unsatisfied check sums and the column weight of parity check matrix. In [15], it was suggested to flip those bits with positive flipping function values per iteration. Further decoding gain is obtained by adding into these multi-bit flipping algorithms a delay-handling procedure [16], which delays flipping those
bits whose soft information presents higher magnitude among others. With respect to the serial flipping, these parallel or multi-bit flipping methods show a significant convergence advantage at no cost of performance loss.

On the other hand, substantial complexity is saved by estimating complex $tanh$ function in standard BP with simple $min$ function, which leads to Min-Sum (MS) or BP-based algorithm [17][18]. Then MS variants such as normalized Min-Sum (NMS) and offset Min-Sum (OMS) [19] proves effective to fill most performance gap between MS and standard BP, at the cost of minor complexity increase.

Despite this, the heavy row and column weights of FG-LDPC codes may annoy the MS variants from perspective of complexity; while the BF variants present much less complexity but suffering some performance loss. To expect good performance and low complexity simultaneously, one natural way is to concatenate some component decoders to fulfill one decoding. This strategy was attempted in [15], wherein standard BP is called only when a multi-bit BF scheme failed. However, due to the serious performance mismatch between standard BP and the multi-bit BF scheme proposed in [15], such concatenation results in frequent invocations of standard BP in most of waterfall SNR region, which subsequently weakens the efforts of reducing complexity. Different from it, a gear shift decoding was presented in [20], it selects appropriate decoder among available ones at each iteration, according to the optimal trellis route obtained after extrinsic information transfer (EXIT) chart analysis. Theoretically, the gear shift decoding reaches the targets of reducing decoding latency while keeping performance. But several obstacles hinder its application for finite FG-LDPC codes. For one thing, the delicate optimal decoding route derived from EXIT chart analysis may deviate seriously from the real situation, since EXIT chart analysis is accurate largely for codes of large girth but FG-LDPC codes are known for the existence of abundant short loops. For another, the EXIT chart of BF variants remains unknown, but excluding such a class of decoding schemes may lead to an absence of a competitive decoder option for gear shift decoding.

In the paper, we adopt a similar framework to that of [15] where two two component decoders form an hybrid scheme. The former component decoder may be substituted with a newly proposed BF variant; the latter is an MS variant instead of standard BP, considering near BP performance is achieved with such an MS variant. At modest and high SNR regions, both decoding performance of the latter decoder and low computational complexity close to the former
decoder are achieved, which are verified via simulations and complexity analysis.

The remainder of the paper is organized as follows. Section II discusses the motivation of designing such a class of hybrid schemes. Section III describes its implementation using BF and MS variants. Simulation results, convergence rate and complexity analysis are presented in Section IV. Finally Section V concludes the work.

II. MOTIVATION OF HYBRID DECODING

With the goals of high performance and low complexity, a satisfying concatenation of two component decoders meets four conditions. First, the two decoders present distinct characteristics. Specifically, the former requires much less complexity than the latter. Secondly, the performance gap between them is within some limit to ensure performance match. In other words, while no gap wipes off the need of employing hybrid schemes, excessively large gap, manifested by no well overlapped waterfall regions for both decoders, results in frequent invocations of the second component decoder. Thirdly, in order not to worsen the whole decoding latency, it is beneficial that the convergence rates of two decoders are comparable by and large. Lastly, the hardware complexity of both decoders is shared to the greatest extent to lower implementation cost.

In [15], a multi-bit flipping scheme and standard BP are jointed to serve the purpose of decoding. However, the multi-bit flipping suffers serious performance loss when compared to standard BP. Such a concatenation violates the mentioned condition two and is less meaningful, since standard BP still takes a substantial load in most SNR region. Compared to the serial ones, the multi-bit BF variants requires much less decoding iterations [16]. According to condition three, multi-bit BF variant is thus preferable over serial one when selecting the first component decoder. Moreover, the multi-bit BF variant with the least complexity and the closest performance to its successor has the highest priority. On the other hand, for FG-LDPC codes, MS variants with proper correcting factors, present almost the same performance as standard BP, thus good candidates of the second component decoder.

III. IMPLEMENTATION OF HYBRID DECODERS

Assume a binary \((N, K)\) LDPC code with block length \(N\) and dimension \(K\). Its parity check matrix is of the form \(H_{M \times N}\), where \(M\) is the number of check sums. For high rate FG-LDPC codes, the relation \(M = N\) indicates there exist many redundant check sums in \(H\). The BPSK
modulation maps a codeword \( c = [c_1, c_2, \ldots, c_N] \) to a symbol sequence \( x = [x_1, x_2, \ldots, x_N] \) with \( x_i = 1 - 2c_i \), where \( i = 1, 2, \ldots, N \). After the symbols are transmitted through an additive white Gaussian noise (AWGN) memoryless channel, we obtain at the receiver a corrupted sequence \( y = [y_1, y_2, \ldots, y_N] \), where \( y_i = x_i + z_i \), \( z_i \) is an independent Gaussian random variable with zero mean and variance \( \sigma^2 \).

For convenience, the vectors below are treated as column or row vectors depending on the context. To differentiate each BF variant, the initials of the first two authors’ surname hyphened by the letters "WBF" make up a unique name, unless stated otherwise.

### A. BF variants

In LP-WBF [7], the BF function of variable node \( i \) at the \( l \)-th iteration is defined as

\[
 f_i^{(l)} = \sum_{k \in \mathcal{M}(i)} f_{i,k}^{(l)}, \ i \in [1, N]
\]  

\[
f_{i,k}^{(l)} = \begin{cases} 
|y_i| - \frac{1}{2}(\min_{j \in \mathcal{N}(k)} |y_j|) & \text{if } s_k^{(l)} = 0, \\
|y_i| - \frac{1}{2}(\min_{j \in \mathcal{N}(k)} |y_j|) - \max_{j \in \mathcal{N}(k)} |y_j| & \text{if } s_k^{(l)} = 1.
\end{cases}
\]  

where \( \mathcal{M}(i) \) denotes the neighboring check nodes of variable node \( i \), \( \mathcal{N}(k) \) is the neighboring variable nodes of check node \( k \), \( s_k^{(l)} \) is the \( k \)-th component of syndrome \( s \) at the \( l \)-th iteration.

With the intuition that the more reliable bits involved in a check sum, the more reliable the check will be, SZ-WBF [8] defines a BF function by weighting each term of the summation (1). That is,

\[
 f_i^{(l)} = \sum_{k \in \mathcal{M}(i)} w_{i,k} f_{i,k}^{(l)}, \ i \in [1, N]
\]  

\[
w_{i,k} = \max(0, \alpha_1 - \|\{j \mid |y_j| \leq \beta_1, j \in \mathcal{N}(k)\backslash i\}\|)
\]  

where \( \mathcal{N}(k)\backslash i \) denotes the neighboring variable nodes of check node \( k \) except variable node \( i \), \( \alpha_1 \) is an integer constant, \( \beta_1 \) is a real constant, \( \| \cdot \| \) is to obtain the set cardinality.

For serial BF variants such as SZ-WBF, only one bit with the smallest \( f_i^{(l)} \) is flipped at the \( l \)-th iteration. Hence, the maximum number of iterations \( I_m \) needs to be predesigned high enough to allow decoding convergence.
Due to a positive correlation between the number of erroneous bits and that of unsatisfied check sums, NT-WBF [14] suggests flipping $\lambda^{(l)}$ bits of the smallest $f_i^{(l)}$ defined by (1) at the $l$-th iteration,

$$\lambda^{(l)} = \left\lfloor \frac{w_h(s^{(l)})}{d_v} \right\rfloor$$

where $w_h(\cdot)$ denotes the calculation of Hamming weight, $d_v$ is the column weight of matrix $H$, $\lfloor x \rfloor$ is the integral part of $x$.

At each iteration, LZ-WBF [15] flips all the bits with flipping function values greater than zero, among which, the flipping function is defined as [11]

$$f_i^{(l)} = \sum_{k \in M(i)} (2s_k^{(l)} - 1)(\min_{j \in N(k)} |y_j|) - \beta_2 |y_i|, \ i \in [1, N] \quad (5)$$

where $\beta_2$ is a real weighting factor.

WZ-WBF [13] uses the same BF function as in [12], namely,

$$f_i^{(l)} = \sum_{k \in M(i)} (2s_k^{(l)} - 1)(\min_{j \in N(k) \setminus i} |y_j|) - \beta_3 |y_i|, \ i \in [1, N] \quad (6)$$

where $\beta_3$ is a real weighting factor. Then at each iteration, for each unsatisfied check sum, a flipping signal is assigned to some involved bit. And only those bits are flipped which have accumulated flipping signals more than a threshold.

To prevent some reliable bits from flipping hastily, improved parallel weighted bit flipping (IPWBF) [16] added a delay-handling step into the steps of WZ-WBF.

Compared with IPWBF, the proposed LF-WBF varies by utilizing the BF function of SZ-WBF, while keeping other steps largely unchanged. To be self-contained, LF-WBF is described as follows:

1) Preprocess: Assume a threshold $T$ be the value of the $\lfloor \beta_4 N \rfloor$-th smallest element among array $|y_i|, i \in [1, N]$, where $\beta_4$ is a real constant within $[0, 1]$, then those bits with $|y_i|$ greater than $T$ are marked reliable, otherwise unreliable.

2) Initialize: $l \leftarrow 0$; calculate initial values of $f_i^{(0)}, i \in [1, N]$ according to (2), (3). For the bits $\in \{i||y_i| > T, i \in [1, N]\}$, the delay-handling counters $a_i \leftarrow 0$; note hard-decision of $y$ as $\hat{c}^{(0)}$.

3) Syndrome and reset: Calculate $s^{(l)} = H\hat{c}^{(l)}$. If $s^{(l)} = 0$, stop to return $\hat{c}^{(l)}$ as the decoding result. If not, $b_i \leftarrow 0, i \in [1, N], b_i$ is a flipping counter which sums the flipping signals for bit $i$. 
4) Collect flipping signals: Update \( f_i^{(l)} \), \( i \in [1, N] \) based on (2), (3). For each \( k \in \{ k | s_k^{(l)} \neq 0, k \in [1, M] \} \), identify the index \( i^* = \arg \min_{i \in \mathcal{N}(k)} f_i^{(l)} \), then \( b_{i^*} \leftarrow b_{i^*} + 1 \), that is, a flipping signal is collected for bit \( i^* \).

5) Decide flipping bits: It is divided into two substeps.
   a) For the bits \( \{ i | b_i \geq \alpha_2, i \in [1, N] \} \), \( \alpha_2 \), as a positive integer, represents the flipping threshold, flip them if only the resulting syndrome \( s^{(l+1)} = 0 \). Otherwise turn to the next substep.
   b) Delay-handling: For the unreliable bits \( \{ i | b_i \geq \alpha_2, |y_i| \leq T, i \in [1, N] \} \), put them in a to-be-flipped list; for the reliable bits \( \{ i | b_i \geq \alpha_2, |y_i| > T, i \in [1, N] \} \), update by \( a_i \leftarrow a_i + 1 \). Subsequently, put the bits \( \{ i | a_i \geq \alpha_3, i \in [1, N] \} \) in the to-be-flipped list, where \( \alpha_3 \) is a small positive integer defining a delay-handling threshold. Obviously, it is meaningful only for \( \alpha_3 \geq 2 \). Relax \( \alpha_3 \leftarrow \alpha_3 - 1 \) if only the to-be-flipped list is empty, then flip the bits \( \{ i | b_i = \alpha_3, i \in [1, N] \} \). Declare failure if no bit is qualified yet.

Since delay-handling step may potentially increase the average number of decoding iterations, substep one reduces its impact effectively.

6) Flip and reset: Flip these bits in the to-be-flipped list. Reset all the bits \( \{ i | a_i \geq \alpha_3, i \in [1, N] \} \) by \( a_i \leftarrow 0 \). Noticeably, before the next resetting occurs, the duration of \( a_i \) may last several iterations while that of \( b_i \) is always one iteration.

7) \( l \leftarrow l + 1 \). If \( l < I_m \), goto step 3 to continue one more iteration; otherwise, declare failure.

B. MS variants

At the check nodes end, compared with standard BP implemented in Log-likelihood ratio (LLR) domain, NMS and OMS [19], approximate (7) with (8) and (9), respectively, thus saving most complexity,

\[
L_{j,i}^{(l)} = 2 \tanh^{-1} \left( \prod_{k \in \mathcal{N}(j) \backslash i} \tanh \left( \frac{Z_{j,k}^{(l-1)}}{2} \right) \right)
\]

\[
L_{j,i}^{(l)} = \frac{1}{\beta_5} \prod_{k \in \mathcal{N}(j) \backslash i} \sgn(Z_{j,k}^{(l-1)}) \cdot \min_{k \in \mathcal{N}(j) \backslash i} |Z_{j,k}^{(l-1)}|
\]

\[
L_{j,i}^{(l)} = \prod_{k \in \mathcal{N}(j) \backslash i} \sgn(Z_{j,k}^{(l-1)}) \cdot \max \left( \min_{k \in \mathcal{N}(j) \backslash i} |Z_{j,k}^{(l-1)}| - \beta_6, 0 \right)
\]
where \( I_{j,i}^{(l)} \) denotes the message sent from check node \( j \) to variable node \( i \) at the \( l \)-th iteration; \( Z_{j,k}^{(l-1)} \) denotes the message sent from variable node \( k \) to check node \( j \) at the \((l - 1)\)-th iteration; \( \beta_5 \) or \( \beta_6 \), being a real constant, functions as a scaling or offset factor, respectively.

To further reduce complexity, at the variable node end, the calculating of (10) is approximated with (11) in the normalized APP-based (NAB) algorithm [19].

\[
Z_{j,i}^{(l)} = F_i + \sum_{k \in \mathcal{M}(i) \setminus j} I_{k,i}^{(l)}
\]

(10)

\[
Z_{j,i}^{(l)} = F_i + \sum_{k \in \mathcal{M}(i)} I_{k,i}^{(l)}, \quad \forall j \in \mathcal{M}(i)
\]

(11)

Where \( F_i \) is the initial LLR of bit \( i \). For the difference-set cyclic (DSC) codes, it was reported NAB yields almost as good performance as NMS [19]. As shown in the simulation later, similar observation also holds for FG-LDPC codes.

C. Block graph of a hybrid decoding scheme

![Block graph of hybrid decoding scheme](image)

Fig. 1 Block graph of hybrid decoding scheme

There are many BF variants and MS variants, thus the combinations of BF variant plus MS variant is abundant. For instance of 'LF-WBF+NMS', as shown in Fig. 1, two component decoders are independent comparatively. The latter takes over decoding so long as the former failed.

D. Optimize parameters by differential evolution

It is hard to optimize the group of parameters involved in LF-WBF theoretically. Hence, differential evolution (DE), known as a heuristic search method, is exploited to approximate the optimality. Similar to the genetic algorithm, DE is a simple and reliable optimization tool
[21]. In DE, via various operations including mutation, combination and selection, a population of solution vectors are updated generation by generation, with those new vectors with small objective values survived, until the population converges to the global optimum.

To aid LF-WBF to optimize its parameter vector \((\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_4)\), the objective function of DE is designated to find the minimum bit error rate (BER) given a block of received sequences. In order to save computation, each parameter of LF-WBF is roughly assigned an evaluation interval beforehand. For instance, \(\alpha_1, \alpha_2\) are integers in \([1, \frac{d_v}{2}]\), \(\alpha_3\) is a small positive in \([1, 4]\), \(\beta_1, \beta_4\) are real numbers in \([0, 1]\).

For \((273, 191)\) and \((1023, 781)\) FG-LDPC codes [3], DE results are given in Table-I with varied channel variance \(\sigma^2\).

### IV. Simulation Results and Discussion

#### A. Parameters selection

It is verified that decoding performance of LF-WBF is largely insensitive to the minor change of its parameters, thus in all SNR region, we assume the settings as shown on the first row of Table-II after referring to Table-I. The additional advantage of such simplification is that the overall hybrid decoding requires no more a priori information about the channel, namely, holding as well the property of uniformly most powerful (UMP)[18] for MS variants.

For LZ-WBF and WZ-WBF, the data presented in Table-II come from the existing literature, as mentioned in the last column of Table-II.

After applying DE for MS variants, we select the settings as the last three rows of Table-II for NAB, NMS and OMS. Noticeably, the optimization results of the scaling factor for NAB and NMS are different.

Table-I: Parameters optimization of LF-WBF for \((273,191)\) (left) and \((1023,781)\) (right) FG-LDPC codes using differential evolution

| \(\sigma\) | \(\alpha_1\) | \(\alpha_2\) | \(\alpha_3\) | \(\beta_1\) | \(\beta_4\) |
|----|-----|-----|-----|-----|-----|
| 0.58 | 10   | 4   | 4   | 0.31 | 0.064 |
| 0.575 | 9    | 4   | 3   | 0.57 | 0.11 |
| 0.57 | 6    | 4   | 4   | 0.50 | 0.054 |
| 0.565 | 5    | 3   | 4   | 0.47 | 0.07 |

| \(\sigma\) | \(\alpha_1\) | \(\alpha_2\) | \(\alpha_3\) | \(\beta_1\) | \(\beta_4\) |
|----|-----|-----|-----|-----|-----|
| 0.565 | 5    | 9   | 2   | 0.38 | 0.036 |
| 0.56 | 12   | 8   | 3   | 0.51 | 0.071 |
| 0.555 | 10   | 8   | 3   | 0.41 | 0.075 |
| 0.55 | 6    | 6   | 3   | 0.32 | 0.025 |
Table-II : Parameters settings of various decoding schemes for (273,191) and (1023,781) FG-LDPC codes

| Scheme | Parameter(s)=those for (273,191); those for (1023,781) | Source |
|--------|------------------------------------------------------|--------|
| LF-WBF | \((\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2) = (6, 4, 2, 0.45, 0.07); (8, 7, 2, 0.4, 0.04)\) | DE     |
| LZ-WBF | \(\beta_2=1.5; 2.1\) | [15]   |
| WZ-WBF | \((\alpha_2, \beta_3) = (4, 1.3); (10, 1.8)\) | [22]   |
| SZ-WBF | \((\alpha_1, \beta_1)=N/A; (9,0.5)\) | [8]    |
| NAB    | \(\beta_5=5.7; 7.1\) | DE     |
| NMS    | \(\beta_5=2.9; 3.7\) | DE     |
| OMS    | \(\beta_5=0.22; 0.20\) | DE     |

B. Decoding performance

The frame error rate (FER) curves of some BF variants and MS variants are plotted in Fig. 2 for (273,191) code.

Fig. 2 FER curves for (273,191) FG-LDPC code under various BF or MS variants

In the legend, the number in the brackets stands for the maximum number of iteration \(I_m\). It is found that BF variants are in general inferior to MS variants from perspective of performance. Specifically, at the point FER=10\(^{-3}\), LF-WBF leads WZ-WBF, NT-WBF and LZ-WBF about 0.25, 0.58 and 0.6 dB, respectively. But it lags behind NAB, OMS and NMS about 0.2, 0.26,
0.32 dB, respectively. Further comparison between LF-WBF and IPWBF [16] shows that they present the similar decoding performance, thus exchangeable each other. Therefore, LF-WBF owns the most matched SNR region as that of MS variants among the available BF variants. Considering LDPC codes commonly have large enough minimum distance, the cases seldom occur where BF variant results in an undetectable error but MS variant decodes correctly. On the other hand, there exist a few cases where BF variant works but MS variant fails. Thus in the form of a BF variant plus an MS variant, the hybrid decoding will keep at least the same performance as the MS variant alone. However, for each combination, the matching degree between two component decoders impacts heavily the overall decoding complexity.

For (1023,781) code, the FER curves are plotted in Fig. 3. It is observed that when the block length increases from 273 to 1023, the curves relation within BF variants and MS variants still holds, except that the closeness among these curves slightly shifts. For instance, at the point FER=10^{-3}, there exists about 0.3 dB between LF-WBF and NMS, while LF-WBF exceeds LZ-WBF more than 0.3 dB. Also included are the curves of two serial approaches: LP-WBF and SZ-WBF. The performance of LP-WBF and SZ-WBF with $I_m=200$ approximates NT-WBF and LF-WBF with $I_m=20$, respectively. Meanwhile, the full loop detection [8], which proves
effective in avoiding decoding trappings for serial BF variants, is utilized for both LP-WBF and SZ-WBF.

C. Convergence rate

Since some applications require $I_m$ to be small, it is thus meaningful to investigate the convergence rate of various decoding schemes. At a typical point SNR=3.42 dB (or $\sigma=0.57$) of (273,191) code, Table-III gives performance comparison among each schemes under varied $I_m$.

It is seen that although $I_m = 3$ is too rigorous for all decoding schemes, each BF variant reaches its individual decoding capability at the specified point within $I_m = 20$. That is, more iterations after the 20-th iteration achieves no further decoding improvement; while MS variants require $I_m$ to be at least 50 to fully decode the received sequences. Also included in Table-III is the data of BP. Interestingly, at $I_m = 3$, BP yields the best decoding performance among others. But its convergence rate is not satisfying. It is shown that $I_m = 50$ is not even sufficient for BP decoding, because the performance improves from FER=1.6e-3 to 7.2e-4 when $I_m$ increases to 200. For this reason, given a small $I_m = 20$, LF-WBF even excels BP a little, as shown in Table-III. Further simulation shows that LF-WBF with $I_m = 20$ performs better than BP in the region where SNR is greater than 3.42 dB. Another noticeable point shown in Table-III is that the performance of BP is generally inferior to MS variants, despite its high complexity. Therefore, BP is less attractive to be selected as the second component decoder of a hybrid scheme. Taking into account the fact that serial BF variants require much more $I_m$ than above multi-bit BF

Table-III : Decoding performance of various schemes under varied $I_m$ for (273,191) FG-LDPC code at SNR=3.42 dB

| Scheme | $I_m = 3$ | $I_m = 10$ | $I_m = 20$ | $I_m = 50$ | $I_m = 200$ |
|--------|-----------|------------|-----------|-----------|------------|
| LZ-WBF | 9.2e-2    | 3.6e-2     | 3.6e-2    | 3.6e-2    | 3.6e-2     |
| NT-WBF | 5.9e-1    | 4.4e-2     | 3.8e-2    | 3.8e-2    | 3.8e-2     |
| WZ-WBF | 2.5e-2    | 9.6e-3     | 9.8e-3    | 9.8e-3    | 9.8e-3     |
| LF-WBF | 4.5e-2    | 4.2e-3     | 2.8e-3    | 2.4e-3    | 2.3e-3     |
| NAB    | 1.1e-1    | 2.1e-3     | 7.6e-4    | 4.4e-4    | 4.4e-4     |
| OMS    | 1.6e-2    | 9.6e-4     | 6.6e-4    | 5.0e-4    | 4.6e-4     |
| NMS    | 1.1e-2    | 5.2e-4     | 3.8e-4    | 3.6e-4    | 3.4e-4     |
| BP     | 9.4e-3    | 3.9e-3     | 2.9e-3    | 1.6e-3    | 7.2e-4     |
variants [14][16], and LF-WBF performs the best among existing multi-bit BF variants, LF-WBF plus some MS variant intuitively presents a competitive form of hybrid decoding scheme. The similar points are supported as well after generalized to other longer FG-LDPC codes.

Let \( A_{ni} \) denote average number of iterations for each decoding scheme, as seen in Fig. 4 for (1023,781) code, \( A_{ni} \) of NT-WBF sticks out prominently while that of LZ-WBF varies slowly with \( E_b/N_0 \), both are due to the algorithms themselves. In most SNR region of interest, all BF variants except NT-WBF present comparable \( A_{ni} \) as MS variants, thus meeting well the condition three discussed in Section II.

![Fig. 4 Average number of iterations \( A_{ni} \) of various decodings schemes for (1023,781) FG-LDPC code](image)

\[ 2.8 \quad 2.9 \quad 3 \quad 3.1 \quad 3.2 \quad 3.3 \quad 3.4 \quad 3.5 \]
\[ 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \quad 18 \quad 20 \]
\[ E_b/N_0 \text{ (dB)} \]

\( \text{LZ-WBF (20)} \)
\( \text{NT-WBF (20)} \)
\( \text{WZ-WBF (20)} \)
\( \text{LF-WBF (20)} \)
\( \text{NAB (20)} \)
\( \text{OMS (20)} \)
\( \text{NMS (20)} \)

**D. Computational complexity analysis**

Practically, any BF variant followed by an MS variant will yield the same decoding performance as the latter alone. For instance, LF-WBF plus MS variant performs almost equally as LZ-WBF plus MS variant, regardless of the fact that LF-WBF is far superior to LZ-WBF. However, computational complexity differs enormously with respect to each hybrid scheme. Generally, it is hard to accurately describe the required complexity for each decoding scheme, so data obtained in the simulations is presented to support our viewpoints if necessary.
Let $d_v, d_c$ individually denote the column and row weights of parity check matrix $H$, then for each BF variant, its complexity roughly consists of three parts: preprocessing, updating BF function and selecting flipping bits. To the best of our knowledge, the complexity of preprocessing and initialization is largely omitted in existing literature. However, the following analysis and simulation will show it contributes substantially to the complexity at very high SNR region. Ignoring simple binary operations and a small number of real multiplications involved sometimes, it suffice to address the dominant real additions for each BF variant, assuming one real comparison is treated as one real addition.

At the stage of preprocessing, for LP-WBF and NT-WBF, about $N(2d_c - 3)$ comparisons are needed in computing $\min$ and $\max$ terms of (2). Similarly, for LZ-WBF and WZ-WBF, about $N(d_c - 1)$ comparisons is required individually in computing the $\min$ term of (5) and (6). Besides that for LP-WBF, both SZ-WBF and LF-WBF require extra $N$ comparisons to obtain $w_{i,k}$ term of (4). With respect to SZ-WBF, LF-WBF requires about $N \log_2 \lfloor \beta_4N \rfloor + N$ more comparisons to determine the bit with the $\lfloor \beta_4N \rfloor$-th smallest magnitude and to mark the delay-flipping bits.

Prior to updating the BF function of each bit, it is initialized with $d_v - 1$ additions for each BF variant. For multi-bit BF variants, there are two ways of updating the BF function of pertinent bits since the second iteration. One is to invoke $d_v d_c$ additions per flipped bit to update its BF function; another is to update the BF function of each bit after comparing its column of $H$ with the syndromes before and after the latest iteration. The latter is more economical, considering two flipping bits in the same check sum result in two extra additions for the former, but avoidable for the latter. For serial BF variants, totally $d_v d_c$ terms are used to update the BF function of those affected bits per iteration.

To decide which bits to flip, each BF variant has its own approach. For LP-WBF and SZ-WBF, it just requires $N - 1$ comparisons to find the bit with the smallest BF function value; for WZ-WBF and LF-WBF, $d_c - 1$ comparisons are required per unsatisfied check to collect flipping signals for each bit; for NT-WBF, its complexity at this stage is equal to selecting the smallest say 5 elements in an unordered array. Noticeably, no computation is required for LZ-WBF, since it simply flips those bits with positive BF function values.

To sum up, Table-IV gives the complexity composition for each BF variant. In the table, $A_{nb}$ denotes the average number of selected bits per iteration for NT-WBF, $A_{nc}$ is the average number of updated BF function terms per bit per iteration, $A_{ns}$ is the average number of unsatisfied checks
Table-IV: Approximated real additions per sequence for various decoding schemes of FG-LDPC codes

| Schemes | Preprocess | Update BF function (include initialization) | Select bit(s) to flip |
|---------|------------|---------------------------------------------|----------------------|
| LZ-WBF  | $N(d_v - 1)$ | $N(d_v - 1) + (A_{ni} - 1)NA_{nc}$ | 0                   |
| NT-WBF  | $N(2d_c - 3)$ | $N(d_v - 1) + (A_{ni} - 1)NA_{nc}$ | $A_{ni}N \log_2 A_{nb}$ |
| WZ-WBF  | $N(d_v - 1)$ | $N(d_v - 1) + (A_{ni} - 1)NA_{nc}$ | $A_{ni}A_{na}(d_c - 1)$ |
| LF-WBF  | $N(2d_c - 1 + \log_2 \lfloor \beta_1 N \rfloor)$ | $N(d_v - 1) + (A_{ni} - 1)NA_{nc}$ | $A_{ni}A_{ns}(d_c - 1)$ |
| SZ-WBF  | $N(2d_c - 2)$ | $N(d_v - 1) + (A_{ni} - 1)d_v d_c$ | $A_{ni}(N - 1)$ |
| LP-WBF  | $N(2d_c - 3)$ | $N(d_v - 1) + (A_{ni} - 1)d_v d_c$ | $A_{ni}(N - 1)$ |
| NAB     | $A_{ns}(2Nd_v + M(\lceil \log_2 d_c \rceil) - 2))$ | $A_{ns}(N(4d_v - 3) + M(\lceil \log_2 d_c \rceil) - 2))$ |
| OMS, NMS| $A_{ns}(4d_v - 3) + M(\lceil \log_2 d_c \rceil) - 2))$ |

per iteration. Also included are the complexity expressions of NAB, OMS and NMS as reported in [7], wherein $\lceil \cdot \rceil$ is the ceiling function.

For (1023,781) code, $N = 1023$, $d_v = d_c = 32$ [3]. Assume $I_m = 20$ for multi-bit BF variants, $I_m = 200$ for MS variants and serial BF variants to ensure full decoding convergence, at a typical point of SNR=3.28 dB (or $\sigma = 0.555$), Table-V presents the figures observed in simulation, among which the last column is the number of real additions according to the expressions of Table-IV. Noticeably, the last two rows of Table-V gives complexity of two instances of hybrid decoding schemes as well.

After studying Table-V, we find that the class of BF variants demonstrates a substantial

Table-V: Complexity comparison per sequence for various decoding schemes of (1023,781) FG-LDPC code at SNR=3.28 dB

| Scheme   | $A_{ni}$ | $A_{ns}$ | $A_{nb}$ | $A_{nc}$ | number of real additions(e+5) |
|----------|----------|----------|----------|----------|------------------------------|
| LZ-WBF   | 4.70     | N/A      | N/A      | 8.11     | 0.94                         |
| NT-WBF   | 9.61     | N/A      | 9.73     | 7.72     | 1.94                         |
| WZ-WBF   | 4.48     | 348.01   | N/A      | 10.41    | 1.49                         |
| LF-WBF   | 4.74     | 373.63   | N/A      | 10.10    | 1.95                         |
| SZ-WBF   | 49.98    |          |          |          | 1.95                         |
| LP-WBF   | 68.66    |          |          |          | 2.34                         |
| NAB      | 5.53     |          |          |          | 3.79                         |
| OMS      | 4.47     |          |          |          | 5.78                         |
| NMS      | 3.77     |          |          |          | 4.93                         |
| LZ-WBF+NMS | (Data for LZ-WBF) + (NMS with $A_{ni}=1.88$) |          |          |          | 3.40                         |
| LF-WBF+NMS | (Data for LF-WBF) + (NMS with $A_{ni}=0.88$) |          |          |          | 3.10                         |
advantage over MS variants in terms of complexity. Among the BF variants, LZ-WBF presents
the least complexity due to its fast convergence, low-complexity preprocessing and no complexity
demand of selecting bits; despite its simplicity, at low and modest SNR regions, the combination
of LZ-WBF and NMS requires more complexity than that of LF-WBF and NMS, as a result
that the former combination demands one more iteration of NMS on average, as shown in
Table-V. Under the condition of offering equivalent performance, the last three rows of Table-V
illustrates that both hybrid decoding schemes can save much complexity, with respect to its
second component decoder alone.

To better illustrate complexity comparison in the whole SNR region, Fig. 5 present complex-
ity ratio curves for (273,191) and (1023,781) codes. Assuming the complexity of NMS is a
benchmark, then complexity ratio is defined as the ratio of the complexity of a specified hybrid
scheme and that of NMS. For NMS, since another $A_{ni}Nd_v$ divisions is actually required [7], we
roughly treat as total complexity the sum of this expression and the related formula in Table-IV.
At very low SNR region, any of the hybrid schemes shows no much advantage, due to the fact

![Complexity ratio curves for (273,191) and (1023,781) codes.](image-url)
that most decodings are up to NMS. However, with increased SNR, both hybrid schemes yield more and more complexity reduction, resulting from more involvements of LZ-WBF or LF-WBF in decoding. For short (273,191) code, the combination of ‘LZ-WBF+NMS’ exceeds that of ‘LF-WBF+NMS’ at the point SNR=3.05 dB. While the occurrence extends to SNR=3.45 dB for (1023,781) code. Hence, it suggests that the intersection of these two schemes will move to a higher SNR with longer block length.

Let $C_{NMS}, C_{LZN}, C_{LFN}$ denote the complexity of above three decoding schemes. For sufficiently long FG-LDPC codes, to seek the asymptotic performance ratios in very high SNR region, the following approximations are derived based on Table-IV,

\[
\frac{C_{LZN}}{C_{NMS}} = \frac{d_v + d_c - 2 + (A_{ni} - 1)A_{nc}}{A_{ni}(5d_v + \log_2 d_c - 5)} \approx \frac{2}{3A_{ni}},
\]

\[
\frac{C_{LFN}}{C_{NMS}} = \frac{2d_v + d_c - 2 + \log_2 \lceil \beta N \rceil + (A_{ni} - 1)A_{nc} + A_{ns}(d_c - 1)A_{ns}/N}{A_{ni}(5d_v + \log_2 d_c - 5)} \approx \frac{9 + A_{ns}}{15A_{ni}}.
\]

wherein the following simulation results are exploited: $d_v = d_c$, both are large numbers compared with other terms; $A_{ni}$ of various schemes ranges in [1, 2] and tends to be near each other; $A_{nc}$ of LZ-WBF or LF-WBF is small compared to $d_v$; $A_{ns}/N$ is about one third. Similar approach can be used to derive complexity ratios of other hybrid combinations.

### E. Hardware complexity

Seemingly, the proposed hybrid schemes add much more hardware complexity with respect to its second component decoder alone. However, most hardware complexity can be shared instead between two component decoders. For instance of ‘LF-WBF+NMS’, assuming NMS hardware is available, then $\min, \max$ operations at the preprocessing phase of LF-WBF, and collecting flipping signals at the selecting flipping bits phase of LF-WBF, can be accomplished via the check node logics of NMS, while the initialization step via the bit node logics of NMS. Thus compared with NMS, ‘LF-WBF+NMS’ only includes a few more integer counters and interconnection logics. Therefore, the extra hardware complexity of hybrid decoding schemes is largely ignorable.

### V. Conclusions

For finite FG-LDPC codes, the concatenation of BF variant and MS variant proves its effectiveness in decoding at a wide range of SNR region, by means of achieving performance of the MS
variant with substantial reduced computational complexity. While LZ-WBF plus MS variant has its advantage at high SNR region of interest; the proposed LF-WBF plus MS variant demonstrates better complexity saving at the rest of SNR region, due to the well overlapped waterfall regions between two component decoders. Evidently, if we can gear among these hybrid schemes based on varied SNRs, the decoding will be more powerful and robust.

For BP decoding, it is known that flooding schedule is not optimal. Sharon et al. [23][24][25] proved that serial message passing schedule, implemented by fully utilizing available updated messages, can halve the average number of iterations of flooding schedule without performance penalty. But it risks resulting in higher decoding latency. Contrary to it, our hybrid scheme yields a good tradeoff among performance, complexity and latency.

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