MONIC MODULES AND SEMI-GORENSTEIN-PROJECTIVE MODULES

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Abstract. The category $\text{gp}(\Lambda)$ of Gorenstein-projective modules over tensor algebra $\Lambda = A \otimes_k B$ can be described as the monomorphism category $\text{mon}(B, \text{gp}(A))$ of $B$ over $\text{gp}(A)$. In particular, Gorenstein-projective $\Lambda$-modules are monic. In this paper, we find the similar relation between semi-Gorenstein-projective $\Lambda$-modules and $A$-modules, via monic modules, namely, $\text{mon}(B, {}^+A) = \text{mon}(B, A) \cap {}^+A$. Using this, it is proved that if $A$ is weakly Gorenstein, then $\Lambda$ is weakly Gorenstein if and only each semi-Gorenstein-projective $A$-modules are monic; and that if $B = kQ$ with $Q$ a finite acyclic quiver, then $\Lambda$ is weakly Gorenstein if and only if $A$ is weakly Gorenstein. However, this relation itself does not answer the question whether there exist double semi-Gorenstein-projective $\Lambda$-modules which are not monic. Using the recent discovered examples of double semi-Gorenstein-projective $A$-modules which are not torsionless, we positively answer this question, by explicitly constructing a class of double semi-Gorenstein-projective $T_2(A)$-modules with one parameter such that they are not monic, and hence not torsionless. The corresponding results are obtained also for the monic modules and semi-Gorenstein-projective modules over the triangular matrix algebras given by bimodules.

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1. Introduction

Monic modules, defined on tensor products $\Lambda = A \otimes_k B$, or on matrix algebras $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ of bimodule modules $M$, built a bridge between Gorenstein-projective $\Lambda$-modules and Gorenstein-projective $A$-modules (Theorems 2.6 and 2.13). In particular, Gorenstein-projective $\Lambda$-modules are monic, in the both cases. This paper is to show that they also play an important role in the study of semi-Gorenstein-projective modules. In the both cases, we will give sufficient and necessary conditions such that $\Lambda$ is weakly Gorenstein, and positively answer the question whether there exist double semi-Gorenstein-projective $\Lambda$-modules which are not monic, and hence not torsionless, and not Gorenstein-projective.

1.1. Let $A$ be an Artin algebra. All the modules in this paper are finitely generated, and we start from left modules. Let $A$-mod be the category of left $A$-modules. For $M \in A$-mod, denote by $\text{add}(M)$ the full subcategory of $A$-mod of direct summands of a direct sum of copies of $M$; by $^1M$ the full subcategory of $A$-mod of modules $X$ with $\text{Ext}^i_A(X, M) = 0$ for $i \geq 1$; and by $M^2$ the full subcategory of $A$-mod of modules $X$ with $\text{Ext}^i_A(M, X) = 0$ for $i \geq 1$.
Let $M^*$ denote the $A$-dual $\text{Hom}_A(M, A)$ of $M$. Denote by $\phi_M : M \rightarrow M^{**}$ the canonical $A$-map, defined by $\phi_M(m)(f) = f(m)$ for $m \in M$ and $f \in M^*$. A module $M$ is torsionless if it is a submodule of a projective module, or, equivalently, $\phi_M$ is a monomorphism; and $M$ is reflexive if $\phi_M$ is an isomorphism.

A module $M$ is semi-Gorenstein-projective if $M \in \mathcal{A}$; and $M$ will be called double semi-Gorenstein-projective, if both $M$ and $M^*$ are semi-Gorenstein-projective. By definition, a Gorenstein-projective module is double semi-Gorenstein-projective and reflexive. This is introduced by Auslander and Bridger [AB], under the name of modules of G-dimension zero, and it is equivalent to the definition in terms of complete projective resolution given by Enochs and Jenda ([EJ1], [EJ2]). For the equivalence we refer to [AM, p.398] (where it is called a total reflexive module) and [Chr, Theorem 4.2.6]. Denote by $\text{gp}(A)$ the full subcategory of $A$-mod of Gorenstein-projective modules. Thus $\text{add}(A) \subseteq \text{gp}(A) \subseteq \mathcal{A}$.

1.2. Avramov and Martsinkovsky [AM, p.398] has proposed the independence problem of the total reflexivity. In fact, the known examples of semi-Gorenstein-projective modules which are not Gorenstein-projective are few and complicated. The first examples of reflexive semi-Gorenstein-projective modules which are not Gorenstein-projective, and the first examples of reflexive modules $M$ with $M^*$ semi-Gorenstein-projective such that $M$ are not semi-Gorenstein-projective, are discovered by Jørgensen and Šega [JS]; and the first examples of double semi-Gorenstein-projective modules which are not torsionless, are recently found in [RZ2, RZ3]. Putting together, this solves the independence problem of the total reflexivity. Note that the first examples of semi-Gorenstein-projective modules which are not Gorenstein-projective over noncommutative algebras, are presented by Marczinzik [M2].

1.3. Let $A$ and $B$ be finite-dimensional algebras over field $k$. Since Cartan-Eilenberg [CE], modules over tensor algebra $\Lambda = A \otimes_k B$ have got interest. They are complicated in the sense that $\Lambda$-modules can not be controlled by $U \otimes_k V$ with $U \in A$-mod and $V \in B$-mod. However, if $B$ is given by a bound quiver $(Q, I)$, one can study $\Lambda$-modules by taking the advantage of the representations of quivers over algebra $A$ ([RS1-RS3], [S2-S5], [KLM1, KLM2], [LZ1, LZ2], [RZ1], [ZX]), i.e., any $\Lambda$-module can be identified with a representation $(X_i, \alpha, i \in Q_0, \alpha \in Q_1)$ of $(Q, I)$ over $A$, where each $X_i \in A$-mod, and each $X_\alpha : X_{s(\alpha)} \rightarrow X_{e(\alpha)}$ is an $A$-map, such that $X_\alpha$’s satisfy all the relations which generate $I$.

When $Q$ is finite acyclic and $I$ is generated by monomial relations, this identification permits us to define monic $\Lambda$-modules and monomorphism category $\text{mon}(B, \mathcal{C}) = \text{mon}(Q, I, \mathcal{C})$ ([LZ1, LZ2], [ZX]), for any additive full subcategory $\mathcal{C}$ of $A$-mod. This definition is combinatorial and constructive, and it admits a homological interpretation. In general, there is no longer the corresponding combinatorial definition of a monic module, but this homological interpretation still makes sense, and it is taken as the definition of the monomorphism category $\text{mon}(B, \mathcal{C})$ by Hu, Luo, Xiong and Zhou in [HLXZ]. See Subsection 2.1.

The study of the monomorphism categories can be traced to G. Birkhoff [Bir]. When $B$ is the path algebra of quiver $A_n$ with linear orientation, i.e., $B = T_n(k) = \begin{pmatrix} k & k & \cdots & k \\ 0 & k & \cdots & k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{pmatrix}$, then $\Lambda = A \otimes_k B = T_n(A)$ and the monomorphism category $\text{mon}(B, A) = \text{mon}(Q, A$-mod) is exactly the submodule category $\mathcal{S}_n(A)$ ([RS1-RS3]), or called the filtered chain category ([S1-S5]). They have Auslander-Reiten sequences ([RS2]) and the RSS equivalence ([ZX]). Simson ([S2]-[S5]) has studied their representation type. By Kussin, Lenzing and Meltzer [KLM1, KLM2] and Chen [Chen1], they are related to the singularity theory.
1.4. An important application of monomorphism categories is that they can describe Gorenstein-projective modules as $\text{gp}(\Lambda) = \text{mon}(B, \text{gp}(A))$ (cf. Theorem 2.6 below). Thus, Gorenstein-projective $\Lambda$-modules are monic $\Lambda$-modules over Gorenstein-projective $A$-modules. If $B$ is given by a finite acyclic quiver and monomial relations, by the combinatorial definition of monic modules, this gives in practice a reductive construction of Gorenstein-projective $\Lambda$-modules.

**Question 1.** Is there the similar relation between semi-Gorenstein-projective $\Lambda$-modules and $A$-modules?

**Theorem 1.1.** Let $A$ and $B$ be finite-dimensional $k$-algebras with $\text{gl.dim} B < \infty$, and $\Lambda = A \otimes_k B$. Then $$\text{mon}(B, \perp A) = \text{mon}(B, A) \cap \perp \Lambda.$$ Moreover, if $\text{inj.dim } A < \infty$, then $\perp \Lambda = \text{mon}(B, \perp A)$.

Theorem 1.1 will be proved in Section 3, as a special case of Theorem 3.1.

1.5. An Artin algebra $A$ is **Gorenstein**, if $\text{inj.dim } A < \infty$ and $\text{inj.dim } A < \infty$. An Artin algebra $A$ is left **weakly Gorenstein** ([M1], [RZ2]), if any left semi-Gorenstein-projective $A$-module is Gorenstein-projective, i.e., $\perp A = \text{gp}(A)$. It is open whether a left weakly Gorenstein algebra is right weakly Gorenstein ([M1, §5], [RZ2, 9.3]). However, if no confusions caused, we will omit the word “left”.

By Enochs and Jenda [EJ2, 11.5.3], Gorenstein algebras are weakly Gorenstein. By Yoshino [Y1, Theorem 5.5] and Beligiananis [Bel2, Corollary 5.11], if $\perp A$ is of finite type, then $A$ is weakly Gorenstein. By Marczinzik [M1, Theorem 3.5(3)], torsinless finite algebras are weakly Gorenstein. For more information on weakly Gorenstein algebras we refer to [Bel1, Bel2], [M1], and [RZ2, 1.2 - 1.4, 3.6].

**Question 2.** (i) Let $A$ and $B$ be Artin algebras, $M$ an $A$-$B$-bimodule such that $\Lambda = (\begin{array}{c|c} A & M \\ \hline 0 & B \end{array})$ is an Artin algebra. When $\Lambda$ is weakly Gorenstein?

(ii) Let $A$ and $B$ be finite-dimensional $k$-algebras with $\text{gl.dim } B < \infty$. When the tensor product $A \otimes_k B$ is weakly Gorenstein?

It turns out that, in the both cases, monic modules will play a crucial role.

**Theorem 1.2.** Let $A$ and $B$ be Artin algebras, $M$ an $A$-$B$-bimodule with $\text{proj.dim } A M < \infty$, and $\Lambda = (\begin{array}{c|c} A & M \\ \hline 0 & B \end{array})$.

1. If $\text{proj.dim } M_B < \infty$ and $D(M_B) \in (\perp B)^\perp$, then $\Lambda$ is weakly Gorenstein if and only if each semi-Gorenstein-projective $\Lambda$-module is monic respect to bimodule $M$, and $A$ and $B$ are weakly Gorenstein.

2. If $\text{proj.dim } M_B < \infty$ and $B$ is a Gorenstein algebra, then $\Lambda$ is weakly Gorenstein if and only if each semi-Gorenstein-projective $\Lambda$-module is monic respect to bimodule $M$ and $A$ is weakly Gorenstein.

3. If $A M$ is torsionless and $M_B$ is projective, then $\Lambda$ is weakly Gorenstein if and only if $A$ and $B$ are weakly Gorenstein. In particular, if $A M$ and $M_B$ are projective, then $\Lambda$ is weakly Gorenstein if and only if $A$ and $B$ are weakly Gorenstein.

Theorem 1.2 is the combination of Propositions 4.1, 4.2 and 4.4.

**Theorem 1.3.** Let $A$ and $B$ be finite-dimensional $k$-algebras.
Assume that $\text{gl.dim} B < \infty$ and $\Lambda = A \otimes_k B$. If $\Lambda$ is weakly Gorenstein, then so is $A$. Conversely, if $A$ is weakly Gorenstein, then a semi-Gorenstein-projective $\Lambda$-module $M$ is Gorenstein-projective if and only if $M$ is monic.

Thus, if $A$ is weakly Gorenstein, then $\Lambda$ is weakly Gorenstein if and only if each semi-Gorenstein-projective $\Lambda$-module is monic, or equivalently, $\Lambda = \text{mon}(B, \Lambda)$.

(2) Let $Q$ be a finite acyclic quiver. Then $A \otimes_k kQ$ is weakly Gorenstein if and only if $A$ is weakly Gorenstein.

The assumption $\text{gl.dim} B < \infty$ holds automatically if $B$ is given by a bound acyclic quiver. Theorem 1.3 is the combination of Propositions 4.7 and 4.8.

1.6 An Artin algebra $A$ will be called left semi-Gorenstein-projective-free, or in short, lsgp-free, provided that each left semi-Gorenstein-projective $A$-module is a projective module, i.e., $\Lambda = \text{add}(A)$. We do not know whether a lsgp-free algebra is right semi-Gorenstein-projective-free.

Recall that $A$ is left CM-free ([Chen 2]) if $\text{gp}(A) = \text{add}(A)$. Thus, $A$ is lsgp-free if and only if $A$ is left CM-free and weakly Gorenstein. It is open whether a left CM-free algebra is left weakly Gorenstein (or equivalently, lsgp-free). See [RZ2, 9.2]. Many algebras are lsgp-free. For example, this is the case if $\text{gl.dim} A < \infty$. There are also non Gorenstein algebras $A$ (thus, $\text{gl.dim} A = \infty$) which are lsgp-free.

**Theorem 1.4.** (1) Assume that $\Lambda = \text{add}(B)$ with proj.dim$M_B < \infty$ and that $A_M$ is torsionless with proj.dim$A_M < \infty$. Then $\Lambda = (A_M^0)_{0\in M}$ is weakly Gorenstein if and only if $A$ is weakly Gorenstein.

Moreover, if in addition $A_M$ is projective, then

$$\Lambda = \text{gp}(\Lambda) = \left\{ \left( \begin{array}{c} M \otimes_B P \\ P \end{array} \right) \mid P \in \text{add}(B), \ G \in \Lambda = \text{gp}(A) \right\}$$

and $\Lambda = \text{add}(\Lambda)$ if and only if $\Lambda = \text{add}(A)$.

(2) Let $I$ be an admissible ideal of $kQ$, and $\Lambda = A \otimes_k kQ/I$. Then $\Lambda = \text{add}(\Lambda)$ if and only if $\Lambda = \text{add}(A)$.

Theorems 1.4 is the combination of Propositions 4.9 and 4.11.

1.7 If $\Lambda = A \otimes_b B$ with gl.dim$B < \infty$, or if $\Lambda = (A_M^0)_{0\in M}$, then Gorenstein-projective $\Lambda$-modules are always monic (cf. Theorems 2.0 and 1.1). Is this true for semi-Gorenstein-projective $\Lambda$-modules? One may ask a stronger question:

**Question 3.** In the both cases, whether there exist double semi-Gorenstein-projective $\Lambda$-modules which are not monic?

The positive answer will in particular gives double semi-Gorenstein-projective modules which are not Gorenstein-projective. As mentioned in Subsection 1.2, this is highly nontrivial.

To answer Question 3, we consider $\Lambda = A \otimes_k k(\phi \rightarrow \phi) = (A_M^0) = T_2(A)$. Any left $\Lambda$-module $M$ can be identified with a triple $(\begin{array}{c} Y \\ X \end{array})_{\phi}$, where $\phi : Y \rightarrow X$ is a left $A$-map. Thus, one has the exact sequence of left $A$-modules $Y \rightarrow X \rightarrow \text{Coker}\phi \rightarrow 0$, and there is a unique $A$-map $\beta : \text{Coker}\pi^* \rightarrow Y^*$, such
that the diagram with exact rows

\[
\begin{array}{cccccccc}
0 & \longrightarrow & (\text{Coker}\varphi)^* & \overset{\pi^*}{\longrightarrow} & X^* & \overset{p}{\longrightarrow} & \text{Coker}\pi^* & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\text{Coker}\varphi)^* & \overset{\pi^*}{\longrightarrow} & X^* & \overset{\varphi^*}{\longrightarrow} & Y^* & \longrightarrow & 0 \\
\end{array}
\]

commutes, where \( p \) is the canonical \( A \)-epimorphism. So one has the left \( A \)-map \( \beta^* : Y^{**} \longrightarrow (\text{Coker}\pi^*)^* \), and the composition \( \beta^* \phi_Y : Y \longrightarrow (\text{Coker}\pi^*)^* \), where \( \phi_Y : Y \longrightarrow Y^{**} \) is the canonical map.

**Theorem 1.5.** Let \( A \) be an Artin algebra, \( \Lambda = T_2(A) = (A A_0 A) \), and \( (\overset{X}{Y})_\varphi \) a left \( \Lambda \)-module. Then

1. There is a left \( \Lambda \)-module isomorphism \( (\overset{X}{Y})_\varphi^{**} \cong \left( \overset{X^{**}}{(\text{Coker}\pi^*)^*} \right)_{p^*} \), where \( p^* : (\text{Coker}\pi^*)^* \longrightarrow X^{**} \) is the \( A \)-monomorphism induced by \( p : X^* \longrightarrow (\text{Coker}\varphi)^* \).

2. Taking this isomorphism as identity, then the canonical \( \Lambda \)-map \( \phi((\overset{X}{Y})_\varphi) \) is given by \( \phi((\overset{X}{Y})_\varphi) = (\overset{\phi_X}{\beta^* \circ \phi_Y}) \).

3. \( (\overset{X}{Y})_\varphi \) is torsionless and double semi-Gorenstein-projective if and only if \( (\overset{X}{Y})_\varphi \) is monic, \( X, Y, \) and \( \text{Coker}\varphi \) are double semi-Gorenstein-projective, and \( X \) and \( Y \) are torsionless.

4. \( (\overset{X}{Y})_\varphi \) is double semi-Gorenstein-projective with epimorphism \( \phi((\overset{X}{Y})_\varphi) \) if and only if \( \varphi^* : X^* \longrightarrow Y^* \) is an epimorphism, \( X \) and \( Y \) are double semi-Gorenstein-projective, \( (\text{Coker}\varphi)^* \) is semi-Gorenstein-projective, and \( \phi_X \) and \( \phi_Y \) are epimorphisms.

Theorem 1.5(1) is a summary of Lemma 5.3 and Proposition 5.4 and Theorem 1.5(2) and (3) will be clear after Proposition 5.7.

As remarked in [RZ4, 3.1], up to now, all the known examples have the following property:

Double semi-Gorenstein-projective modules \( M \) such that \( \phi_M \) is a monomorphism (an epimorphism, respectively) are Gorenstein-projective.

The following result shows that this property is preserved under the \( T_2 \)-extensions.

**Theorem 1.6.** Let \( A \) be an Artin algebra and \( \Lambda = T_2(A) = (A A_0 A) \). Then

1. Any torsionless and double semi-Gorenstein-projective \( A \)-module is Gorenstein-projective if and only if any torsionless and double semi-Gorenstein-projective \( \Lambda \)-module is Gorenstein-projective.

2. Any double semi-Gorenstein-projective \( A \)-module \( L \) with \( \phi_L \) an epimorphism is Gorenstein-projective if and only if any double semi-Gorenstein-projective \( \Lambda \)-module \( M \) with \( \phi_M \) an epimorphism is Gorenstein-projective.

Theorem 1.6 will be proved in Subsection 5.9.

1.8. The following result positively answers **Question 3**, and gives a construction of double semi-Gorenstein-projective \( T_2(A) \)-modules which are not monic.
Theorem 1.7. Suppose that \( Y \) is a double semi-Gorenstein-projective \( A \)-module which is not torsionless. Let \( \varphi : Y \to P \) be a left add(\( A \))-approximation of \( Y \). Then \( (\varphi, Y) \) is a double semi-Gorenstein-projective \( T_2(A) \)-module which is not monic. In particular, \( (\varphi, Y) \) is not torsionless.

Theorem 1.7 will be proved in Subsection 6.1. Using the algebra \( A \) in [RZ2] and the \( A \)-modules \( M(1, -q, c) \) in [RZ3], by Theorem 1.7 we obtain a class of double semi-Gorenstein-projective \( T_2(A) \)-modules with parameter \( c \) as

\[
X(c) := \left( M_{(1, -q, c)}^{A} \right)_{f_1}
\]

such that \( X(c) \) is not monic, and hence not torsionless; moreover, all the canonical maps \( \phi_{X(c)} : X(c) \to X(c)** \) are neither monomorphisms nor epimorphisms, and \( X(c)** \) are not semi-Gorenstein-projective. See Proposition 6.2.

2. Preliminaries: Monic modules with relations to Gorenstein-projective modules

2.1. Monic modules over tensor algebras.

Definition 2.1. ([HLXZ, 3.1]) Let \( A \) and \( B \) be finite-dimensional \( k \)-algebras, and \( \Lambda = A \otimes_k B \).

(1) A left \( \Lambda \)-module \( X \) is monic, if \( \text{Tor}_i^A(A \otimes V, X) = 0 \) for all \( i \geq 1 \) and for all right \( B \)-modules \( V \).

Denote by \( \text{mon}(B, A) \) the full subcategory of \( \Lambda \)-mod consisting of monic modules, which is called the monomorphism category of \( B \) over \( A \).

(2) Let \( \mathcal{C} \) be an additive full subcategory of \( A \)-mod. An object \( X \in \text{mon}(B, A) \) is a monic module over \( \mathcal{C} \), if \( (A \otimes_k V) \otimes_{\Lambda} X \in \mathcal{C} \) for all right \( B \)-module \( V \).

Denote by \( \text{mon}(B, \mathcal{C}) \) the full subcategory of \( \text{mon}(B, A) \) of monic modules over \( \mathcal{C} \), which is called the monomorphism category of \( B \) over \( \mathcal{C} \).

Lemma 2.2. ([HLXZ, Lemma 3.2(7)]; [ZX, Theorem 2.6(1)]) One has

\[
\text{mon}(B, A) = \{ X \in \Lambda \text{-mod} \mid \text{Tor}_i^A(A \otimes_k D(S), X) = 0, \forall i \geq 1, \forall \text{ simple left } B \text{-module } S \}
= \frac{1}{(D(A_A) \otimes_k B)}.
\]

Example 2.3. (1) If \( B \) is the path algebra of the quiver \( A_n \) (\( n \geq 2 \)) with linear orientation, then \( B = T_n(k) = \begin{pmatrix} k & \cdots & k \ 0 & \cdots & k \end{pmatrix} \), \( \Lambda = A \otimes_k B = \begin{pmatrix} A & \cdots & A \ 0 & \cdots & 0 \end{pmatrix} = T_n(A) \), and \( \text{mon}(B, A) \) turns out to be

\[
\mathcal{S}_n(A) = \{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \}_{(\varphi_i)} \in T_n(A)\text{-mod} \mid \varphi_i : X_{i+1} \to X_i \text{ is a monomorphism, } \forall 1 \leq i \leq n-1 \}.
\]

This submodular category has been studied in [A], [S1-S5], [RS1 - RS3], [Z1].

(2) If \( B \) is the path algebra \( kQ \), where \( Q = (Q_0, Q_1, s, e) \) is a finite acyclic quiver, then a monical \( \Lambda \)-module has been defined in [LZ1] as a representation \( (X_i, X_{\alpha}, i \in Q_0, \alpha \in Q_1) \) of \( Q \) over \( A \), such that for each \( i \in Q_0 \) the \( A \)-map

\[
(X_\alpha)_{\alpha \in Q_1, e(\alpha) = i} : \bigoplus_{\alpha \in Q_1, e(\alpha) = i} X_{s(\alpha)} \to X_i
\]
is a monomorphism.

For any additive full subcategory $\mathcal{C}$ of $A$-mod, a monic $\Lambda$-module over $\mathcal{C}$ has been defined in [ZX, 2.1], as a monic $\Lambda$-module $(X_i, \ X_\alpha, \ i \in Q_0, \ \alpha \in Q_1)$ satisfying

$$X_i/\text{Im}(X_\alpha)_{\alpha \in Q_1, e(\alpha) = i} \in \mathcal{C}, \ \forall \ i \in Q_0.$$ 

(3) If $B = kQ/I$ with $I$ generated by monomial relations, then $\text{mon}(B, \mathcal{C})$ has also been defined combinatorially. For details see [LZ2] and [ZX].

In all these monomorphism categories defined via quivers, “monomorphisms” are visible, and they also admit the homological description in Definition 2.6 ([Z1, Theorem 3.1], [LZ2, 2.1], [ZX, Theorem 2.6]).

Lemma 2.4. Let $\Lambda = A \otimes_k kQ$, where $Q$ is a finite acyclic quiver. Then torsionless $\Lambda$-modules are monic.

Proof. Let $X = (X_i, \ X_\alpha)$ be a torsionless $\Lambda$-module. Then $X$ is a submodule of a projective $\Lambda$-module, which is of the form $P \otimes_k L$, where $P$ is a projective left $A$-module, and $L = (L_i, \ L_\alpha)$ is a projective left $kQ$-module. Thus there is a monomorphism $\left((f_i)_{i \in Q_0} : (X_i, X_\alpha) \rightarrow (P \otimes_k L(i), \ Id_P \otimes_k L_\alpha)\right)$ of $\Lambda$-modules. Hence, for each $i \in Q_0$, the diagram of $A$-maps

$$\begin{array}{ccc}
X_{s(\alpha)} & \xrightarrow{f_{s(\alpha)}} & X_i \\
\bigoplus_{\alpha \in Q_1, e(\alpha) = i} X_{s(\alpha)} & \xrightarrow{(X_{s(\alpha)})_{\alpha \in Q_1, e(\alpha) = i}} & \bigoplus_{\alpha \in Q_1, e(\alpha) = i} P \otimes_k L_{s(\alpha)} \\
\bigoplus_{\alpha \in Q_1, e(\alpha) = i} P \otimes_k L_{s(\alpha)} & \xrightarrow{(Id_P \otimes_k L_\alpha)_{\alpha \in Q_1, e(\alpha) = i}} & P \otimes_k L_i
\end{array}$$

commutes. Since both $\bigoplus_{\alpha \in Q_1, e(\alpha) = i} f_{s(\alpha)}$ and $(Id_P \otimes_k L_\alpha)_{\alpha \in Q_1, e(\alpha) = i}$ are monomorphisms, it follows that $(X_{s(\alpha)})_{\alpha \in Q_1, e(\alpha) = i}$ is a monomorphism, i.e., $X$ is a monic $\Lambda$-module. $
$
Remark 2.5. Lemma 2.4 is not true for $\Lambda = A \otimes_k (kQ/I)$, even if $I$ is generated by monomial relations. For example, if $A = k$, $Q = 3 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 1$, and $I = \langle \alpha \rangle$, then the simple module $S(2) = \text{rad}P(3)$ is a torsionless $(kQ/I)$-module, but it is not monic.

2.2. Gorenstein-projective modules over tensor algebras. The relationship between Gorenstein-projective modules over $\Lambda = A \otimes_k B$ and monomorphism categories of $B$ over $A$ is

Theorem 2.6. ([HLXZ, Theorem 4.5]) Let $A$ and $B$ be finite-dimensional $k$-algebras with $\text{gl.dim}B < \infty$, and $\Lambda = A \otimes_k B$. Then $\text{gp}(\Lambda) = \text{mon}(B, \ \text{gp}(A))$. In particular, a Gorenstein-projective $\Lambda$-module is monic.

Theorem 2.6 is proved for $\Lambda = T_n(A) = A \otimes_k T_n(k)$ with $A$ Gorenstein in [Z1, Corollary 4.1(ii)]; it is proved for $B = kQ$ in [LZ1, Theorem 5.1], and for $B = kQ/I$ in [LZ2, Theorem 4.1], where $Q$ is any finite acyclic quiver, and $I$ is generated by monomial relations. In all these cases, since $\text{mon}(B, \ \text{gp}(A))$ are defined via the combinatorics of quivers, Theorem 2.6 provides in practice an inductive construction of Gorenstein-projective modules.
2.3. Monic modules respect to bimodules. Let $A$ and $B$ be Artin algebras, and $M$ an $A$-$B$-bimodule such that $\Lambda = (A \overset{M}{\underset{B}{\Lambda}})$ is an Artin algebra. This is equivalent to say that $A$ and $B$ are Artin $R$-algebra, and $M$ is finitely generated over $R$ which acts centrally on $M$, where $R$ is a commutative Artin ring ([ARS, Proposition 2.1, p.72]). Any left $\Lambda$-module is identified with a triple $(X, Y)\varphi$, where $X$ is a left $A$-module, $Y$ is a left $B$-module, and $\varphi : M \otimes_B Y \longrightarrow X$ is a left $A$-map.

**Definition 2.7.** ([XZZ, 2.1]) Let $\Lambda = (A \overset{M}{\underset{B}{\Lambda}})$ be an Artin algebra. A $\Lambda$-module $(X, Y)\varphi$ is monic respect to bimodule $\mathcal{A}M_B$, provided that $\varphi : M \otimes_B Y \longrightarrow X$ is a monomorphism.

Denote by $\mathcal{M}(A, M, B)$ the full subcategory of $\Lambda$-mod of monic $\Lambda$-modules respect to bimodule $\mathcal{A}M_B$, which is called the monomorphism category respect to bimodule $\mathcal{A}M_B$.

**Example 2.8.** The monomorphism category $\text{mon}(B, A)$ and the monomorphism category $\mathcal{M}(A, M, B)$ are in different setting. Even if $\Lambda = A \otimes_k B = (A \overset{M}{\underset{B}{\Lambda}})$, $\text{mon}(B, A) \neq \mathcal{M}(A', M, B')$ in general.

For example, consider $T_n(A) = A \otimes_k T_n(k)$. A $T_n(A)$-module $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ is a monic $T_n(A)$-module if and only if $\varphi_i : X_{i+1} \longrightarrow X_i$ is a monomorphism for all $1 \leq i \leq n-1$. Thus $\text{mon}(T_n(k), A) = S_n(A)$.

On the other hand, $T_n(A) = \left( T_{n-1}(A) \overset{M_{n-1}}{\underset{A}{\Lambda}} \right)$ for $n \geq 2$, where $M_{n-1} = \begin{pmatrix} A \\ \vdots \\ A \end{pmatrix}$ (n − 1 rows) is a $T_{n-1}(A)$-$A$-bimodule, and $X$ is a monic $T_n(A)$-module respect to bimodule $M_{n-1}$ if and only if

$$\varphi_i \cdots \varphi_{n-1} : X_n \longrightarrow X_i$$

is a monomorphism for all $1 \leq i \leq n-1$.

Thus, a $T_n(A)$-module $X$ is a monic $T_n(A)$-module if and only if

$$\begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}$$

is a monic $T_m(A)$-module respect to bimodule $M_{m-1}$ for all $2 \leq m \leq n$, where $T_m(A) = \left( T_{m-1}(A) \overset{M_{m-1}}{\underset{A}{\Lambda}} \right)$, and $M_{m-1} = \begin{pmatrix} A \\ \vdots \\ A \end{pmatrix}$ (m − 1 rows); and a monic $T_n(A)$-module $X$ respect to $M_{n-1}$ is not necessarily a monic $T_n(A)$-module. In some sense, $\mathcal{M}(T_{n-1}(A), M_{n-1}, A)$ can be seen as the local version of $\text{mon}(T_n(k), A)$.

For example, let $n \geq 3$. Consider $T_n(A)$-module $X = \begin{pmatrix} A \\ \vdots \\ A \oplus A \\ X_n \end{pmatrix}$, where

$$\varphi_{n-1} = \begin{pmatrix} \text{Id}_A \\ 0 \end{pmatrix} : A \longrightarrow A \oplus A, \quad \varphi_{n-2} = (\text{Id}_A, 0) : A \oplus A \longrightarrow A$$

and $\varphi_i = \text{Id}_A : X_{i+1} = A \longrightarrow A = X_i$ for all $1 \leq i \leq n-3$. Then $X \notin \text{mon}(T_n(k), A)$, but $X \in \mathcal{M}(T_{n-1}(A), M_{n-1}, A)$.

**Lemma 2.9.** Let $\Lambda = (A \overset{M}{\underset{B}{\Lambda}})$ be an Artin algebra, where $M_B$ is projective. Then torsionless $\Lambda$-modules are monic respect to bimodule $\mathcal{A}M_B$.

**Proof.** Let $L = \begin{pmatrix} X \\ Y \end{pmatrix}^\varphi$ be a torsionless $\Lambda$-module. Then $L$ is a submodule of a projective $\Lambda$-module, which is of the form $\begin{pmatrix} P \oplus (M \otimes_B Q) \end{pmatrix}_\text{Id}_{M \otimes_B Q}$, where $P$ is a projective left $A$-module, and $Q$ is a projective
left $B$-module. Thus, there is a monomorphism $\left( \begin{array}{c} 1 \\ y \end{array} \right) : \left( \begin{array}{c} X \\ Y \end{array} \right) \varphi \rightarrow \left( \begin{array}{c} P \oplus M \otimes_B Q \end{array} \right)_{(i)}$. Since $M_B$ is projective, $\text{Id}_M \otimes_B g$ is a monomorphism. By the commutative diagram

\begin{equation}
\begin{array}{c}
M \otimes_B Y \\
\downarrow \text{Id}_M \otimes_B g \\
M \otimes_B Q \\
\end{array}
\xrightarrow{\varphi}
\begin{array}{c}
X \\
\downarrow \\
P \oplus M \otimes_B Q \\
\end{array}
\end{equation}

$\varphi$ is a monomorphism, i.e., $L$ is a monic $\Lambda$-module.

**Remark 2.10.** Lemma 2.9 is not true if $M_B$ is not projective. For example, let $\Lambda = \left( \begin{array}{c} k \\ M \end{array} \right)$, where $A$ is the path algebra $k(2 \rightarrow 1)$, $kM_A = D(Ae_1) = \text{Hom}_k(Ae_1, k)$ is a $k$-$A$-bimodule. Since $M_{e_2} = 0$, $M \otimes_A Ae_2 = M e_2 \otimes_A e_2 = 0$. Thus $\left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ is a left projective $\Lambda$-module. Let $\sigma : Ae_1 \rightarrow Ae_2$ be the embedding. Then $\left( \begin{array}{c} 0 \\ 0 \end{array} \right) : \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \rightarrow \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ is a $\Lambda$-monomorphism, and hence $\left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ is not monic respect to bimodule $kM_A$.

2.4. (Semi-)Gorenstein-projective modules over triangular matrix algebras of bimodules.

For an Artin algebra $B$, let $D$ denote the duality of $B$ ([ARS, p.37]).

Let $A$ and $B$ be Artin algebras, $M$ an $A$-$B$-bimodule such that $\Lambda = \left( \begin{array}{c} A \\ B \end{array} \right)$ is an Artin algebra. Under suitable conditions, semi-Gorenstein-projective $\Lambda$-modules can be described as follows.

**Theorem 2.11.** ([Z2, Theorem 1.1]) Assume that $\text{proj.dim}_A M < \infty$ and $D(M_B) \in \left( \begin{array}{c} + \end{array} \right) \left( \begin{array}{c} + \end{array} \right)$. Then a $\Lambda$-module $\left( \begin{array}{c} X \\ Y \end{array} \right) \varphi \in \mathcal{L}(\Lambda)$ if and only if $Y \in \mathcal{L}(\Lambda)$, the left $A$-map $\varphi : M \otimes_B Y \rightarrow X$ induces isomorphisms $\text{Ext}^i_A(X, A) \cong \text{Ext}^i_A(M \otimes_B Y, A)$ for all $i \geq 1$, and $\varphi^* : X^* \rightarrow (M \otimes_B Y)^*$ is a right $\Lambda$-epimorphism.

An $A$-$B$-bimodule $M$ is compatible ([Z2, Definition 1.1]), if the following two conditions hold:

- If $Q^*$ is an exact sequence of projective $B$-modules, then $M \otimes_B Q^*$ is exact; and
- If $P^*$ is a complete $A$-projective resolution, then $\text{Hom}_A(P^*, M)$ is exact.

**Lemma 2.12.** ([Z2, Proposition 1.3(1)]) Let $M$ be an $A$-$B$-bimodule. If $\text{proj.dim}_A M < \infty$ and $\text{proj.dim}_B M < \infty$, then $M$ is compatible.

Under the condition of compatible bimodule, Gorenstein-projective $\Lambda$-modules can be described as follows. In particular, again, Gorenstein-projective $\Lambda$-modules are monic, but in the sense of respect to bimodule $A_M B$ (compare Theorem 2.6).

**Theorem 2.13.** ([Z2, Theorem 1.4]) Assume that $M$ is a compatible $A$-$B$-bimodule. Then $\left( \begin{array}{c} X \\ Y \end{array} \right) \varphi \in \text{gp}(\Lambda)$ if and only if $\varphi : M \otimes_B Y \rightarrow X$ is a monomorphism, $\text{Coker} \varphi \in \text{gp}(A)$, and $Y \in \text{gp}(B)$.

If this is the case, $X \in \text{gp}(A)$ if and only if $M \otimes_B Y \in \text{gp}(A)$.

**Corollary 2.14.** Let $A$ be an Artin algebra, and $\Lambda = T_2(A) = \left( \begin{array}{c} A \\ B \end{array} \right)$. Then

1. $\mathcal{L}(\Lambda) = \{(X, Y) \in \Lambda \text{-mod} \mid X \in \mathcal{L}(\Lambda), \ Y \in \mathcal{L}(\Lambda), \ \varphi^* : X^* \rightarrow Y^* \text{ is epic}\}$.

2. $\text{gp}(\Lambda) = \{(X, Y) \in \Lambda \text{-mod} \mid \varphi : Y \rightarrow X \text{ is monic}, \ \text{Coker} \varphi \in \text{gp}(A), \ Y \in \text{gp}(A)\}$

   $= \{(X, Y) \in \Lambda \text{-mod} \mid \varphi : Y \rightarrow X \text{ is monic}, \ \text{Coker} \varphi \in \text{gp}(A), \ Y \in \text{gp}(A), \ X \in \text{gp}(A)\}$. 

3. Monomorphism categories over perpendicular categories

Let $A$ and $B$ be finite-dimensional $k$-algebras, and $\Lambda = A \otimes_k B$. A relation between semi-Gorenstein-projective $A$-modules and semi-Gorenstein-projective $A$-modules is contained in the following general result.

**Theorem 3.1.** Let $A$ and $B$ be finite-dimensional $k$-algebra with $\text{gl.dim}B < \infty$, $T$ an $A$-module, and $\Lambda = A \otimes_k B$. Then

$$\text{mon}(B, \perp T) = \text{mon}(B, A) \cap \perp (T \otimes_k B).$$

Moreover, if there is an exact sequence of left $A$-modules

$$0 \longrightarrow T_m \longrightarrow \cdots \longrightarrow T_0 \longrightarrow D(A) \longrightarrow 0$$

with each $T_j \in \text{add}(T)$, then $\text{mon}(B, \perp T) = \perp (T \otimes_k B)$.

In particular, there holds $\text{mon}(B, \perp A) = \text{mon}(B, A) \cap \perp \Lambda$; and if $\text{inj.dim} A < \infty$, then $\perp \Lambda = \text{mon}(B, \perp A)$.

**Proof.** Let $X \in \text{mon}(B, A)$. Since by definition $\text{mon}(B, \perp T) \subseteq \text{mon}(B, A)$, it follows that, in order to prove $\text{mon}(B, \perp T) = \text{mon}(B, A) \cap \perp (T \otimes_k B)$, it suffices to prove that $X \in \perp (T \otimes_k B)$ if and only if $X \in \text{mon}(B, \perp T)$, i.e., $(A \otimes_k V) \otimes_\Lambda X \in \perp T$ for all right $B$-modules $V$.

Take a $\Lambda$-projective resolution

$$P_\bullet : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

**Claim 1:** $X \in \perp (T \otimes_k B)$ if and only if the complex

$$\text{Hom}_\Lambda(P_\bullet, \text{Hom}_k(S', T))$$

is exact, for each right simple $B$-module $S'$.

Since $\text{gl.dim}B < \infty$, it is clear that $\perp (T \otimes_k B) = \bigcap_S \perp (T \otimes_k S)$, where $S$ ranges over all the left simple $B$-modules. To use the Tensor-Hom adjoint pair later, we write a left simple $B$-module $S$ as $D(S')$, where $S'$ is a right simple $B$-module. Thus, $\perp (T \otimes_k B) = \bigcap_{S'} \perp (T \otimes_k D(S'))$, where $S'$ ranges over all the right simple $B$-modules. Therefore, $X \in \perp (T \otimes_k B)$ if and only if

$$\text{Hom}_\Lambda(P_\bullet, T \otimes_k D(S'))$$

is exact, for each right simple $B$-module $S'$. Note that the canonical $k$-linear isomorphism

$$T \otimes_k D(S') \longrightarrow \text{Hom}_k(S', T), \ t \otimes f \mapsto "s' \mapsto f(s')t", \ \forall t \in T, \ f \in D(S'), \ s' \in S'$$

is a left $\Lambda$-isomorphism. Thus, $X \in \perp (T \otimes_k B)$ if and only if $\text{Hom}_\Lambda(P_\bullet, \text{Hom}_k(S', T))$ is exact, for each right simple $B$-module $S'$.

**Claim 2:** $(A \otimes_k V) \otimes_\Lambda X \in \perp T$ for all right $B$-modules $V$ if and only if the complex

$$\text{Hom}_k((A \otimes_k S') \otimes_\Lambda P_\bullet, T)$$

is exact for each right simple $B$-module $S'$.
By assumption $X \in \text{mon}(B, A)$, i.e., $\text{Tor}_i^\Lambda(A \otimes_k V, X) = 0$ for all $i \geq 1$ and for all right $B$-modules $V$.

It follows that the functor

$$(A \otimes_k -) \otimes_A X : \text{mod}B \to \text{mod}A$$

is an exact functor. As a consequence, $(A \otimes_k V) \otimes_A X \in \perp T$ for all right $B$-modules $V$ if and only if $(A \otimes_k S') \otimes_A X \in \perp T$ for each right simple $B$-module $S'$, since $\perp T$ is extension closed. Since $\text{Tor}_i^\Lambda(A \otimes_k S', X) = 0$ for all $i \geq 1$, it follows that

$$(A \otimes_k S') \otimes_A P_\bullet : \cdots \to (A \otimes_k S') \otimes_A P_1 \to \cdots \to (A \otimes_k S') \otimes_A P_0 \to (A \otimes_k S') \otimes_A X \to 0$$

is an exact sequence of left $A$-modules. Since each $P_i$ is a projective left $A$-module, each $(A \otimes_k S') \otimes_A P_i \in \text{add}(A \otimes_k S')$. Thus each $(A \otimes_k S') \otimes_A P_i$ is projective as a left $A$-module, and hence $(A \otimes_k S') \otimes_A P_\bullet$ is an $A$-projective resolution of left $A$-module $(A \otimes_k S') \otimes_A X$, for each right simple $B$-module $S'$. Therefore, $(A \otimes_k S') \otimes_A X \in \perp T$ for each right simple $B$-module $S'$ if and only if $\text{Hom}_A((A \otimes_k S') \otimes_A P_\bullet, T)$ is exact for each right simple $B$-module $S'$.

**Claim 3:** There is an isomorphism of complexes

$$\text{Hom}_A((A \otimes_k S') \otimes_A P_\bullet, T) \cong \text{Hom}_A(P_\bullet, \text{Hom}_k(S', T))$$

for each right simple $B$-module $S'$.

Applying the Tensor-Hom adjoint pair $((A \otimes_k S') \otimes_A - , \text{Hom}_A(A \otimes_k S', -))$ between $\Lambda$-mod and $A$-mod, one has the following isomorphism of complexes of $k$-spaces

$$\text{Hom}_A((A \otimes_k S') \otimes_A P_\bullet, T) \cong \text{Hom}_A(P_\bullet, \text{Hom}_A(A \otimes_k S', T)).$$

Applying the adjoint pair $(A \otimes_k - , \text{Hom}_A(-, -))$ between $k$-mod and $A$-mod, one has the isomorphisms of $k$-spaces

$$\text{Hom}_A(A \otimes_k S', T) \cong \text{Hom}_k(S', \text{Hom}_A(A, T)) \cong \text{Hom}_k(S', T),$$

which is clearly also an isomorphism of left $A$-modules. All together we get an isomorphism of complexes

$$\text{Hom}_A((A \otimes_k S') \otimes_A P_\bullet, T) \cong \text{Hom}_A(P_\bullet, \text{Hom}_k(S', T))$$

for each right simple $B$-module $S'$.

It follows from **Claim 1**, **Claim 2** and **Claim 3** that $X \in \perp (T \otimes_k B)$ if and only if $(A \otimes_k V) \otimes_A X \in \perp T$ for all right $B$-module $V$. This proves $\text{mon}(B, \perp T) = \text{mon}(B, A) \cap \perp (T \otimes_k B)$.

Finally, assume that there is an exact sequence $0 \to T_m \to \cdots \to T_0 \to D(A_A) \to 0$ with each $T_j \in \text{add}(T)$. To show $\text{mon}(B, \perp T) = \perp (T \otimes_k B)$, it suffices to show $\perp (T \otimes_k B) \subseteq \text{mon}(B, A)$. By Lemma 2.2, $\text{mon}(B, A) = \perp (D(A_A) \otimes_k B)$. Thus, it suffices to show $\perp (T \otimes_k B) \subseteq \perp (D(A_A) \otimes_k B)$. This follows from the exact sequence $0 \to T_m \otimes_k B \to \cdots \to T_0 \otimes_k B \to D(A_A) \otimes_k B \to 0$ with each $T_j \otimes_k B \in \text{add}(T \otimes_k B)$. This completes the proof. 

4. **Weakly Gorenstein algebras: Proof of Theorems 1.2, 1.3, and 1.4**

4.1. **When triangular matrix algebras of bimodules are weakly Gorenstein?** Let $A$ and $B$ be Artin algebras, $M$ an $A$-$B$-bimodule such that $\Lambda = \left( \begin{array}{c|c} A & B \\ \hline 0 & M \end{array} \right)$ is an Artin algebra. We will give various conditions for $\Lambda$ being a left weakly Gorenstein algebra, i.e., $\perp \Lambda = \text{gp}(\Lambda)$. 

Proposition 4.1. Assume that \( \text{proj.dim}_A M < \infty \), \( \text{proj.dim}_B M < \infty \), and \( D(M_B) \in (\perp_{(B)} B)\perp \). Then \( \Lambda = (A \overset{M}{\underset{B}{\wedge}}) \) is weakly Gorenstein if and only if each semi-Gorenstein-projective \( \Lambda \)-module is monic respect to bimodule \( A M_B \), and \( A \) and \( B \) are weakly Gorenstein.

Proof. Since \( \text{proj.dim}_A M < \infty \) and \( \text{proj.dim}_B M < \infty \), the \( A \)-\( B \)-bimodule \( M \) is compatible (cf. Lemma 2.12). Thus, under the assumptions, one can apply Theorems 2.11 and 2.13.

Assume that each semi-Gorenstein-projective \( \Lambda \)-module is monic respect to bimodule \( A M_B \), and \( A \) and \( B \) are weakly Gorenstein. Let \( (\overline{X}_Y) \) \( \in \perp \Lambda \). We need to prove \( (\overline{X}_Y) \in \text{gp}(A) \). By the assumption, \( \varphi : M \otimes_B Y \rightarrow X \) is a monomorphism; thus \( \text{Coker} \varphi \rightarrow 0 \), since \( \text{projdim} \Lambda \). Applying Theorem 2.11 one gets the conclusions:

- \( \varphi : M \otimes_B Y \rightarrow X \) is a monomorphism;
- \( Y \in \perp B \), and hence \( Y \in \text{gp}(B) \) (since by assumption \( B \) is weakly Gorenstein);
- \( \varphi \) induces isomorphisms \( \text{Ext}^i_A(X, A) \cong \text{Ext}^i_A(M \otimes_B Y, A) \) for all \( i \geq 1 \);
- \( \varphi^* : X^* \rightarrow (M \otimes_B Y)^* \) is a right \( A \)-epimorphism.

Applying \( \text{Hom}_A(-, A) \) to the exact sequence \( 0 \rightarrow M \otimes_B Y \xrightarrow{\varphi} X \rightarrow \text{Coker} \varphi \rightarrow 0 \), since \( \varphi^* : X^* \rightarrow (M \otimes_B Y)^* \) is an epimorphism and \( \varphi \) induces isomorphisms \( \text{Ext}^1_A(X, A) \cong \text{Ext}^1_A(M \otimes_B Y, A) \) for all \( i \geq 1 \), it follows that \( \text{Coker} \varphi \in \perp A \). Hence \( \text{Coker} \varphi \in \text{gp}(A), \) since by assumption \( A \) is weakly Gorenstein. Thus, we get the following:

- \( \varphi : M \otimes_B Y \rightarrow X \) is a monomorphism;
- \( \text{Coker} \varphi \in \text{gp}(A); \) and
- \( Y \in \text{gp}(B). \)

Applying Theorem 2.13 one gets \( (\overline{X}_Y) \in \text{gp}(A) \). This proves the “if” part.

Conversely, assume that \( \Lambda \) is weakly Gorenstein. Thus, any semi-Gorenstein-projective \( \Lambda \)-module is Gorenstein-projective, and hence it is monic respect to bimodule \( A M_B \), by Theorem 2.13. It remains to prove that \( A \) and \( B \) are weakly Gorenstein. Let \( X \in \perp A \). Applying Theorem 2.11 one knows \( (\overline{X}_0) \in \perp \Lambda \), thus \( (\overline{X}_0) \in \text{gp}(A) \) by the assumption, and then by Theorem 2.13 one has \( X \in \text{gp}(A) \). This proves that \( A \) is weakly Gorenstein.

Similarly, let \( Y \in \perp B \). By Theorem 2.11 one knows \( (\overline{M \otimes_B Y})_{\text{Id}_{M \otimes_B Y}} \in \perp \Lambda \), and hence \( (\overline{M \otimes_B Y})_{\text{Id}_{M \otimes_B Y}} \in \text{gp}(A) \). Then by Theorem 2.13 \( Y \in \text{gp}(B) \). This proves that \( B \) is weakly Gorenstein.

Taking \( B \) to be a Gorenstein algebra in Proposition 4.1, we get

Proposition 4.2. Assume that \( \text{proj.dim}_A M < \infty \), \( \text{proj.dim}_B M < \infty \), and that \( B \) is a Gorenstein algebra. Then \( \Lambda = (A \overset{M}{\underset{B}{\wedge}}) \) is weakly Gorenstein if and only if each semi-Gorenstein-projective \( \Lambda \)-module is monic respect to bimodule \( A M_B \) and \( A \) is weakly Gorenstein.

Proof. Since by assumption \( B \) is a Gorenstein algebra, it follows that \( \perp (B) = \text{gp}(B) \). Recall that for a Gorenstein algebra \( B \), \( \text{gp}(B), p(B)^{< \infty} \) is a cotorsion pair (see e.g., [H], [EJ2], [BR]), where \( p(B)^{< \infty} \) is the full subcategory of \( B \)-mod consisting of modules of finite projective dimension. Thus

\[ \perp (B) = \text{gp}(B) = p(B)^{< \infty}. \]

Since by assumption \( \text{proj.dim}_B M < \infty \), it follows that \( \text{inj.dim} D(M_B) < \infty \). Since \( B \) is Gorenstein, it follows that \( \text{proj.dim} D(M_B) < \infty \), i.e., \( D(M_B) \in p(B)^{< \infty} = (\perp (B))^{\perp} \). Thus, the assertion follows from Proposition 4.1. \( \blacksquare \)
Taking $B$ to be a field $k$ in Proposition 4.12 we get

**Corollary 4.3.** Let $A$ be a finite-dimensional $k$-algebra.

1. Let $M$ be a finite-dimensional $A$-module. Assume that $\text{proj.dim}_A M < \infty$. Then $\Lambda = \begin{pmatrix} A & M \\ 0 & k \end{pmatrix}$ is weakly Gorenstein if and only if each semi-Gorenstein-projective $\Lambda$-module is monic respect to bimodule $AM_k$ and $A$ is weakly Gorenstein.

2. Let $P$ be a finite-dimensional projective left $A$-module, and $\Lambda = \begin{pmatrix} A & P \\ 0 & k \end{pmatrix}$. Then

$$\begin{aligned}
\perp \Lambda &= \left\{ \begin{pmatrix} G \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P \otimes_k V \\ V \end{pmatrix} \right\}_{\text{id}_{P \otimes_k V}} \mid G \in \perp A, V \in k\text{-mod} \\
\text{gp}(A) &= \left\{ \begin{pmatrix} G \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P \otimes_k V \\ V \end{pmatrix} \right\}_{\text{id}_{P \otimes_k V}} \mid G \in \text{gp}(A), V \in k\text{-mod}
\end{aligned}$$

and $\Lambda$ is weakly Gorenstein if and only if $A$ is weakly Gorenstein.

**Proof.** (2) Let $\begin{pmatrix} X \\ V \end{pmatrix}_ϕ \in \perp \Lambda$. By Theorem 2.11 one has

- $ϕ : P^{\oplus \dim V} \rightarrow X$ induces isomorphisms $\text{Ext}_A^i(X,A) \cong \text{Ext}_A^i(P^{\oplus \dim V},A) = 0$, $∀ i \geq 1$; and
- $ϕ^* : X^* \rightarrow (P^*)^{\oplus \dim V}$ is a right $A$-epimorphism.

Thus $X \in \perp A$, and $ϕ^*$ is a splitting epimorphism. Hence $ϕ^{**}$ is a splitting monomorphism. By the commutative diagram

$$\begin{array}{ccc}
P^{\oplus \dim V} & \xrightarrow{ϕ} & X \\
\cong & \downarrow{ϕ_X} & \downarrow{ϕ^*} \\
(P^{**})^{\oplus \dim V} & \xrightarrow{ϕ^{**}} & X^{**}
\end{array}$$

one sees that $ϕ$ is a splitting monomorphism. Thus $X = G \oplus P^{\oplus \dim V}$ where $G$ is semi-Gorenstein-projective, and hence $\begin{pmatrix} X \\ V \end{pmatrix}_ϕ \cong \begin{pmatrix} G \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P \otimes_k V \\ V \end{pmatrix}_{\text{id}_{P \otimes_k V}}$. This proves $\perp \Lambda = \left\{ \begin{pmatrix} G \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P \otimes_k V \\ V \end{pmatrix}_{\text{id}_{P \otimes_k V}} \mid G \in \perp A, V \in k\text{-mod} \right\}$. Since $\begin{pmatrix} P \otimes_k V \\ V \end{pmatrix}_{\text{id}_{P \otimes_k V}}$ is a projective $\Lambda$-module and $\begin{pmatrix} G \\ 0 \end{pmatrix}$ is a Gorenstein-projective $\Lambda$-module if and only if $G$ a Gorenstein-projective $A$-module, it follows that $\text{gp}(A) = \left\{ \begin{pmatrix} G \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P \otimes_k V \\ V \end{pmatrix}_{\text{id}_{P \otimes_k V}} \mid G \in \text{gp}(A), V \in k\text{-mod} \right\}$. Therefore $\Lambda$ is weakly Gorenstein if and only if $A$ is weakly Gorenstein.

**Proposition 4.4.** Assume that $AM$ is torsionless with $\text{proj.dim}_A M < \infty$ and $M B$ is projective. Then $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is weakly Gorenstein if and only if $A$ and $B$ are weakly Gorenstein.

In particular, if $AM$ and $MB$ are projective, then $\Lambda$ is weakly Gorenstein if and only if $A$ and $B$ are weakly Gorenstein.

**Proof.** Since $M_B$ is projective, $D(M_B)$ is an injective $B$-module, and hence $D(M_B) \in (B^+)$. Thus, the assumption that $\text{proj.dim}_A M < \infty$ and $M_B$ is projective guarantee that the conditions of Proposition 4.11 are satisfied. By Proposition 4.11 it suffices to prove that if $A$ and $B$ are weakly Gorenstein, then any semi-Gorenstein-projective $\Lambda$-module $\begin{pmatrix} X \\ V \end{pmatrix}_ϕ$ is monic respect to bimodule $AM_B$.

Applying Theorem 2.11 to $\begin{pmatrix} X \\ V \end{pmatrix}_ϕ \in \perp \Lambda$, one gets the following conclusions:

- $Y \in \perp B$, and hence $Y \in \text{gp}(B)$ (since by assumption $B$ is weakly Gorenstein);
- $ϕ : M \otimes_B Y \rightarrow X$ induces isomorphisms $\text{Ext}_A^i(X,A) \cong \text{Ext}_A^i(M \otimes_B Y, A)$, $∀ i \geq 1$; and
- $ϕ^* : X^* \rightarrow (M \otimes_B Y)^*$ is a right $A$-epimorphism.
Since $Y \in \text{gp}(B)$, $BY$ is a submodule of some projective $B$-module $_BP$. Since $M_B$ is projective, it follows that $A(M \otimes_B Y)$ is a submodule of $A(M \otimes_B P)$. Since $B_P$ is projective, $A(M \otimes_B P) \in \text{add}(AM)$. Since by assumption $AM$ is torsionless, it follows that $M \otimes_B P$ is a torsionless left $A$-module, and hence $M \otimes_B Y$ is a torsionless left $A$-module. Thus, the canonical map $\phi_{M \otimes_BY} : M \otimes_B Y \to (M \otimes_B Y)^{**}$ is a monomorphism.

Since $\varphi^* : X^* \to (M \otimes_B Y)^*$ is an epimorphism, it follows that $\varphi^{**} : (M \otimes_B Y)^{**} \to X^{**}$ is a monomorphism. From the commutative diagram with monomorphism $\phi_{M \otimes_BY}$

$$
\begin{array}{ccc}
M \otimes_B Y & \xrightarrow{\varphi} & X \\
\downarrow \phi_{M \otimes_B Y} & & \downarrow \phi_X \\
(M \otimes_B Y)^{**} & \xrightarrow{\varphi^{**}} & X^{**}
\end{array}
$$

one sees that $\varphi : M \otimes_B Y \to X$ is a monomorphism, i.e., $(X^Y)_\varphi$ is monic respect to bimodule $AM_B$. This completes the proof. 

\begin{remark}
The “only if” part in Proposition 4.4 does not need the condition that $AM$ is torsionless.
\end{remark}

4.2. When tensor algebras are weakly Gorenstein? Let $A$ and $B$ be finite-dimensional $k$-algebras with $\text{gl.dim}B < \infty$, and $\Lambda = A \otimes_k B$. We first look at some properties of a map $\text{mon}(B, -)$.

\begin{lemma}
Let $A$ and $B$ be finite-dimensional $k$-algebra, and $\Lambda = A \otimes_k B$.

(i) Let $\mathscr{C}$ be an additive full subcategory of $A$-mod closed under direct summands, and $M \in A$-mod. Then $M \otimes_k B \in \text{mon}(B, \mathscr{C})$ if and only if $M \in \mathscr{C}$.

(ii) If $M$ is a semi-Gorenstein-projective $A$-module which is not Gorenstein-projective, then $M \otimes_k B$ is a semi-Gorenstein-projective $\Lambda$-module which is not Gorenstein-projective.

(iii) Let $\Omega$ (respectively, $\Gamma$) be the set of additive full subcategories of $A$-mod (respectively, $\Lambda$-mod) closed under direct summands. Then the map

$$
\text{mon}(B, -) : \Omega \to \Gamma, \quad \mathscr{C} \mapsto \text{mon}(B, \mathscr{C})
$$

is an injective map.

\begin{proof}
(i) Assume that $M \in \mathscr{C}$. For any right $B$-module $V$, taking a $B$-projective resolution

$$
P_* : \cdots \to P_1 \to P_0 \to V \to 0
$$

of $V$, one has a projective resolution $A \otimes_k P_*$ of right $A$-module $A \otimes_k V$. By the isomorphisms

$$(A \otimes_k V) \otimes \Lambda (M \otimes_k B) \cong (A \otimes \Lambda M) \otimes_k (V \otimes_B B) \cong M \otimes_k V$$

one sees that there is an isomorphism of complexes

$$(A \otimes_k P_*) \otimes \Lambda (M \otimes_k B) \cong M \otimes_k P_*$$

and hence

$$\text{Tor}^\Lambda_i(A \otimes_k V, M \otimes_k B) \cong \text{Tor}_i^\Lambda(M, V) = 0.$$ 

This shows $M \otimes_k B \in \text{mon}(B, A)$. Further, by $(A \otimes_k V) \otimes \Lambda (M \otimes_k B) \cong M \otimes_k V \in \mathscr{C}$, one gets $M \otimes_k B \in \text{mon}(B, \mathscr{C})$.

\end{proof}
Conversely, if $M \otimes_k B \in \text{mon}(B, C)$, then by definition $(A \otimes_k B) \otimes_A (M \otimes_k B) \cong M \otimes_k B \in C$. Since $C$ is closed under direct summands, it follows that $M \in C$.

(ii) Assume that $M \in \perp A$ and $M \notin \text{gp}(A)$. By (i), $M \otimes_k B \in \text{mon}(B, \perp A) \subseteq \perp \Lambda$, where the inclusion follows from Theorem 3.1. Again by (i), $M \otimes_k B \notin \text{mon}(B, \text{gp}(A)) = \text{gp}(\Lambda)$, where the equality follows from Theorem 2.6.

(iii) Assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are additive full subcategories of $A\text{-mod}$ closed under direct summands, such that $\text{mon}(B, \mathcal{C}_1) = \text{mon}(B, \mathcal{C}_2)$. We need to prove $\mathcal{C}_1 = \mathcal{C}_2$. Let $M \in \mathcal{C}_1$. By (i), $M \otimes_k B \in \mathcal{C}_1$. Thus $M \otimes_k B \in \mathcal{C}_2$. Again by (i), $M \in \mathcal{C}_2$. This completes the proof.

**Proposition 4.7.** If $\Lambda$ is weakly Gorenstein, then so is $A$. Conversely, if $A$ is weakly Gorenstein, then a semi-Gorenstein-projective $\Lambda$-module is Gorenstein-projective if and only if it is monic.

Thus, if $A$ is weakly Gorenstein, then $\Lambda$ is weakly Gorenstein if and only if each semi-Gorenstein-projective $\Lambda$-module is monic, or equivalently, $\perp \Lambda = \text{mon}(B, \perp A)$.

**Proof.** If $\Lambda$ is weakly Gorenstein, then $A$ is weakly Gorenstein, by Lemma 4.6(ii).

Assume that $A$ is weakly Gorenstein and $M$ is a semi-Gorenstein-projective $\Lambda$-module. If $M$ is Gorenstein-projective, then $M$ is monic, by Theorem 2.6. If $M$ is monic, then by Theorem 3.1 and Theorem 2.6 one has $M \in \text{mon}(B, A) \cap \perp \Lambda = \text{mon}(B, \perp A) = \text{mon}(B, \text{gp}(A)) = \text{gp}(\Lambda)$.

**Proposition 4.8.** Let $Q$ be a finite acyclic quiver. Then $A \otimes_k kQ$ is weakly Gorenstein if and only if $A$ is weakly Gorenstein.

In particular, $T_n(A) = \left( \begin{array}{ccc} A & \cdots & A \\ 0 & \cdots & A \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{array} \right)$ is weakly Gorenstein if and only if $A$ is weakly Gorenstein.

**Proof.** By Proposition 4.7, it remains to prove the “if” part. Assume that $A$ is weakly Gorenstein. We will prove that $\Lambda = A \otimes_k kQ$ is weakly Gorenstein, by using induction on $|Q_0|$. If $|Q_0| = 1$, then $\Lambda = A$ is weakly Gorenstein, by the assumption.

Assume that $|Q_0| \geq 2$. We write the conjunction of paths of $Q$ from left to right. Since $Q$ is an acyclic quiver, $Q$ has a source vertex, say, $n$, and then

$$kQ = \left( \begin{array}{c} kQ' \text{rad}P(n) \\ 0 \end{array} \right)$$

where $Q'$ is the subquiver of $Q$ by deleting the source vertex $n$, and $P(n) = kQe_n$. Then $\text{rad}P(n)$ is a $kQ'$-$k$-bimodule. Thus

$$\Lambda = A \otimes_k kQ = \left( \begin{array}{c} A \otimes_k kQ' \text{M} \\ 0 \end{array} \right)$$

where $M = A \otimes_k \text{rad}P(n)$ is an $(A \otimes_k kQ')$-$A$-bimodule. Since $Q$ is acyclic, so is $Q'$. Hence $\text{rad}P(n)$ is a projective left $kQ'$-module. Thus $M = A \otimes_k \text{rad}P(n)$ is a projective right $(A \otimes_k kQ')$-module, and also $M = A \otimes_k \text{rad}P(n)$ is a projective right $A$-module.

Since $|Q'_0| = |Q_0| - 1$, by induction $A \otimes kQ'$ is weakly Gorenstein. Applying Proposition 1.3 to

$$\left( \begin{array}{c} A \otimes_{kQ'} \text{M} \\ 0 \end{array} \right) = \Lambda,$$ one sees that $\Lambda$ is weakly Gorenstein.

**4.3 Semi-Gorenstein-projective-free algebras.** Replacing the condition that $M_B$ is projective in Proposition 1.3 by $\perp B = \text{add}(B)$ and $\text{proj.dim}M_B < \infty$, we then get the following result on lsgp-free algebras.
Proposition 4.9. Assume that \( B = \text{add}(B) \) with \( \text{proj.dim} M_B < \infty \), and that \( A \) is torsionless with \( \text{proj.dim} A M < \infty \). Then \( \Lambda = (A M_B) / B \) is left weakly Gorenstein if and only if \( A \) is left weakly Gorenstein.

Moreover, if in addition \( A M \) is projective, then

\[
\bot \Lambda = \text{gp}(\Lambda) = \left\{ \left( \frac{M \otimes_B P}{P} \right)_{\text{id}_{M \otimes_B P}} \oplus \left( \begin{array}{c} G \\ 0 \end{array} \right) \mid P \in \text{add}(B), \ G \in \bot A = \text{gp}(A) \right\}
\]

and \( \bot A = \text{add}(A) \) if and only if \( \bot A = \text{add}(A) \).

Proof. Since \( \bot B = \text{add}(B) \), it follows that \( (\bot B) \bot = B \)-mod, and hence \( D(M_B) \in (\bot B) \bot \). So, the conditions of Proposition 4.11 are satisfied. To prove the first assertion, by Proposition 4.11 it suffices to prove that if \( A \) is left weakly Gorenstein, then any semi-Gorenstein-projective \( A \)-module \( \left( \frac{X}{Y} \right)_\varphi \) is monic respect to bimodule \( A M_B \). In fact, applying Theorem 2.13 to \( \left( \frac{X}{Y} \right)_\varphi \) one gets that \( Y \in \bot B \), that \( \varphi : M \otimes_B Y \rightarrow X \) induces isomorphisms \( \text{Ext}_A^1(X, A) \cong \text{Ext}_A^1(M \otimes_B Y, A) \) for all \( i \geq 1 \), and that \( \varphi^* : X^* \rightarrow (M \otimes_B Y)^* \) is a right \( A \)-epimorphism. Since \( Y \in \bot B = \text{add}(B) \), \( B Y \) is projective. Thus \( A(M \otimes_B Y) \in \text{add}(A M) \). Since by assumption \( A M \) is torsionless, it follows that \( A(M \otimes_B Y) \) is torsionless. Thus, the canonical map \( \phi_{M \otimes_B Y} : M \otimes_B Y \rightarrow (M \otimes_B Y)^* \) is a monomorphism. By the same argument as in the proof of Proposition 4.11 one concludes that \( \left( \frac{X}{Y} \right)_\varphi \) is monic respect to bimodule \( A M_B \).

Now, assume in addition that \( A M \) is projective. Continuing the argument above, one knows that \( A(M \otimes_B Y) \) is projective, thus, \( \phi_{M \otimes_B Y} : M \otimes_B Y \rightarrow (M \otimes_B Y)^* \) is an isomorphism. By \( \text{Ext}_A^1(X, A) \cong \text{Ext}_A^1(M \otimes_B Y, A) = 0 \) for all \( i \geq 1 \), one has \( X \in \bot A \). Since by assumption \( A \) is weakly Gorenstein, \( X \in \text{gp}(A) \), and hence \( \phi_X : X \rightarrow X^* \) is an isomorphism. Since \( \varphi^* : X^* \rightarrow (M \otimes_B Y)^* \) is an epimorphism and \( (M \otimes_B Y)^* \) is a right projective \( A \)-module, it follows that \( \varphi^* \) is a splitting epimorphism, and hence \( \varphi^* : X^* \rightarrow (M \otimes_B Y)^* \) is a splitting monomorphism. From the commutative diagram

\[
\begin{array}{ccc}
M \otimes_B Y & \xrightarrow{\varphi} & X \\
\phi_{M \otimes_B Y} \downarrow & \cong & \downarrow \phi_X \\
(M \otimes_B Y)^* & \xrightarrow{\varphi^*} & X^*
\end{array}
\]

one sees that \( \varphi : M \otimes_B Y \rightarrow X \) is also a splitting monomorphism. Thus \( X \cong (M \otimes_B Y) \oplus X' \) for some \( X' \in \text{gp}(A) \) and \( \left( \frac{X}{Y} \right)_\varphi = \left( \frac{M \otimes_B Y}{X'} \right)_{\text{id}_{M \otimes_B Y}} \oplus \left( \begin{array}{c} X' \\ 0 \end{array} \right) \).

Since \( A M \) is projective and \( \text{proj.dim} M_B < \infty \), the \( A-B \)-bimodule \( M \) is compatible (cf. Lemma 2.12). By Theorem 2.13 \( \left( \frac{X}{Y} \right)_\varphi \in \text{gp}(A) \), and hence \( \left( \frac{X}{Y} \right)_\varphi = \left( \frac{M \otimes_B Y}{X'} \right)_{\text{id}_{M \otimes_B Y}} \oplus \left( \begin{array}{c} X' \\ 0 \end{array} \right) \in \text{gp}(A) \). This proves

\[
\bot \Lambda = \text{gp}(\Lambda) = \left\{ \left( \frac{M \otimes_B P}{P} \right)_{\text{id}_{M \otimes_B P}} \oplus \left( \begin{array}{c} G \\ 0 \end{array} \right) \mid P \in \text{add}(B), \ G \in \bot A = \text{gp}(A) \right\}
\]

and from which one sees that \( \bot \Lambda = \text{add}(\Lambda) \) if and only if \( \bot A = \text{add}(A) \).

Remark 4.10. The “only if” part in Proposition 4.9 does not need the conditions that \( \bot B = \text{add}(B) \) and \( A M \) is torsionless.

Proposition 4.11. Let \( Q \) be a finite acyclic quiver, \( I \) an admissible ideal of \( kQ \), and \( \Lambda = A \otimes kQ/I \). Then \( \bot \Lambda = \text{add}(\Lambda) \) if and only if \( \bot A = \text{add}(A) \).
Proof. Assume that $\perp A = \text{add}(A)$. We will prove $\perp A = \text{add}(A)$, again by using induction on $|Q_0|$. If $|Q_0| = 1$, then $\Lambda = A$, thus the assertion holds, by the assumption $\perp A = \text{add}(A)$.

Assume that $|Q_0| \geq 2$. Similar as in the proof of Proposition 4.8 we write $\Lambda$ as a triangular matrix algebra. However, in order to apply Proposition 4.9 this time we need to use the subquiver $Q'$ of $Q$ by deleting a sink vertex, say, 1, and the corresponding algebra $kQ'/I'$. Then $kQ/I = \begin{pmatrix} k \text{rad}(e_1 kQ/I) & kQ'/I' \\ 0 & kQ'/I' \end{pmatrix}$

where $\text{rad}(e_1 kQ/I)$ is a $k$-$(kQ'/I')$-bimodule. Thus $\Lambda = \begin{pmatrix} A & M \\ 0 & \Lambda' \end{pmatrix}$

where $\Lambda' = A \otimes_k (kQ'/I')$, $M = A \otimes_k \text{rad}(e_1 kQ/I)$ is a $A$-$\Lambda'$-bimodule. Since $|Q_0| = |Q_0| - 1$, by induction one gets $\perp \Lambda' = \text{add}(\Lambda')$.

Since $\text{proj.dim.} \text{rad}(e_1 kQ/I)_{kQ'/I'} < \infty$, it follows that $\text{proj.dim.} \text{rad}(e_1 kQ/I)_{kQ'/I'} < \infty$.

Also, $A M$ is projective. Since we already known $\perp \Lambda' = \text{add}(\Lambda')$ by induction, thus we can apply Proposition 4.9 to $\Lambda = A \otimes_k (kQ/I) = \begin{pmatrix} A & M \\ 0 & \Lambda' \end{pmatrix}$ to get $\perp \Lambda = \text{gp}(\Lambda) = \{ \begin{pmatrix} M \otimes_{\Lambda'} P \\ P \end{pmatrix} | P \in \text{add}(\Lambda'), G \in \perp A \}$.

Since $G \in \perp A = \text{add}(A)$, it follows that $(0_0) \in \text{add}(\Lambda)$, and hence $\perp \Lambda = \text{add}(A)$.

Conversely, assume that $\perp A = \text{add}(A)$. Let $X$ be an indecomposable $A$-module with $X \in \perp A$. For any indecomposable projective $(kQ/I)$-module $P$, by the Cartan-Eilenberg isomorphism ([CE, Thm. 3.1, p.209, p.205]) one has $\text{Ext}^i_A(X \otimes_k P, A \otimes_k kQ/I) = \bigoplus_{p+q=i} (\text{Ext}^p_A(X, A) \otimes_k \text{Ext}^q_{kQ/I}(P, kQ/I)) = 0, \forall i \geq 1$.

So $X \otimes_k P \in \perp A = \text{add}(A)$, and hence $X \in \text{add}(A)$.

4.4. Example and Problem. Let $A$ be the algebra given by quiver $\begin{array}{cccc} \beta & \alpha \\ 2 & 1 \end{array}$ and relations $\beta^2$, $\alpha \beta$. The Auslander-Reiten quiver of $A$ is

\[
\begin{array}{cccc}
2 & 2 & 1 & 2 \\
2 & 1 & 2 & 2 \\
1 & 2 & 2 & 2 \\
\end{array}
\]

with indecomposable projective modules $P(1) = 1$ and $P(2) = 2^2 1$, and indecomposable injective modules $I(1) = \frac{1}{2}$ and $I(2) = \frac{3}{2}$. Since $\text{Ext}^1_A(2, 2^2 1) \neq 0$, $\text{Ext}^1_A(2, 1) \neq 0$, $\text{Ext}^2_A(1, 1) = \text{Ext}^2_A(1, 2) \neq 0$ one sees that $A$ is lsgp-free, i.e., $\perp A = \text{add}(A)$. Note that $A$ is not Gorenstein. By Proposition 4.11 $\perp (A \otimes_k kQ/I) = \text{gp}(A \otimes_k kQ/I) = \text{add}(A \otimes_k kQ/I)$, for any finite acyclic quiver $Q$ and any admissible ideal $I$. 
5.1. \textbf{Problem 1.} Are there a left weakly Gorenstein algebra $A$, a finite acyclic quiver $Q$, and an admissible ideal $I$ of $kQ$, such that $A \otimes_k kQ/I$ is not left weakly Gorenstein, or equivalently, such that there is a semi-Gorenstein-projective $(A \otimes_k kQ/I)$-module which is not monic?

This problem is different from \textbf{Question 4.} Such an algebra $A$ (if there exists) is not Gorenstein (otherwise, $A \otimes_k kQ/I$ is Gorenstein, by [AR, Proposition 2.2]); such an $I \neq 0$, by Proposition 4.8 and also $\perp A \neq \text{add}(A)$, by Proposition 4.11.

5. \textit{Canonical maps of modules over $T_2(A)$}

Let $A$ be an Artin algebra, $\Lambda = T_2(A) = (A, A)\otimes A$, and $M$ a $\Lambda$-module. We will give a sufficient and necessary condition, such that the canonical $\Lambda$-map $\phi_M : M \rightarrow M^{**}$ is a monomorphism (an epimorphism, and reflexive, respectively); and we will give a sufficient and necessary condition such that $M$ is double semi-Gorenstein-projective with $\phi_M$ a monomorphism (an epimorphism, respectively).

Recall that a left $\Lambda$-module $M$ is identified with the triple $(X, \phi, Y)$, where $\phi : Y \rightarrow X$ is a left $\Lambda$-map; and a right $\Lambda$-module is identified with a triple $(U, V, \psi)$, where $\psi : U \rightarrow V$ is a right $\Lambda$-map. Using the identifications, we will determine the right $\Lambda$-module $M^* = \text{Hom}_\Lambda(M, \Lambda)$, the left $\Lambda$-module $M^{**} = \text{Hom}_\Lambda(M^*, \Lambda)$, and $\phi_M : M \rightarrow M^{**}$.

5.1. \textbf{The $\Lambda$-dual of a left $\Lambda$-module.} For a left $\Lambda$-module $(X, \phi, Y)$, we will determine the right $\Lambda$-module $(X^*, \phi^*, Y^*) = \text{Hom}_\Lambda((X, \phi, Y), \Lambda)$. As a left $\Lambda$-module, $\Lambda = (\Lambda, \Lambda, \Lambda, \Lambda, \Lambda, \Lambda) = (A, A)\otimes A$. Thus, any $\Lambda$-map

$$f \in (X^*, \phi^*, Y^*) = \text{Hom}_\Lambda((X, \phi, Y), \Lambda)$$

is of the form $\left( \begin{array}{c} (\alpha_1) \\ \beta \end{array} \right)$, where $\alpha_1, \alpha_2 \in X^* = \text{Hom}_\Lambda(X, A)$, $\beta \in Y^*$, such that the square

$$\begin{array}{ccc}
Y & \xrightarrow{\phi} & X \\
\beta \downarrow & & \downarrow (\alpha_1) \\
A & \xrightarrow{(g, \alpha_2)} & A \oplus A
\end{array}$$

commutes. So $\alpha_1 \phi = 0$, $\beta = \alpha_2 \phi$. Thus, there is a unique $g \in (\text{Coker}\phi)^* = \text{Hom}_\Lambda(\text{Coker}\phi, A)$ such that $\alpha_1 = g \pi$, where $\pi : X \rightarrow \text{Coker}\phi$ is the canonical $A$-epimorphism.

\textbf{Lemma 5.1.} Let $(X, \phi, Y)$ be a left $\Lambda$-module with $\phi : Y \rightarrow X$ a left $\Lambda$-map. Then

(i) Any $f \in (X^*, \phi^*, Y^*)$ is of the form $\left( \begin{array}{c} (g \pi) \\ \alpha_2 \end{array} \right)$, where $g \in (\text{Coker}\phi)^*$, $\pi : X \rightarrow \text{Coker}\phi$ is the canonical $A$-epimorphism, and $\alpha_2 \in X^*$.

(ii) There is a unique right $\Lambda$-module isomorphism $h : (X^*, \phi^*, Y^*) \cong ((\text{Coker}\phi)^*, X^*)_{\pi^*}$, given by

$$f = \left( \begin{array}{c} (g \pi) \\ \alpha_2 \end{array} \right) \mapsto (g, \alpha_2)$$

where $\pi^* : (\text{Coker}\phi)^* \rightarrow X^*$ is the right $\Lambda$-monomorphism induced by $\pi$.

\textbf{Proof.} (ii) We claim that $h$ is a right $\Lambda$-map, i.e.,

$$h(f(a_1 a_2)) = h(f)(a_1 a_2), \quad \forall \ f = \left( \begin{array}{c} (g \pi) \\ \alpha_2 \end{array} \right) \in (X^*, \phi^*, Y^*), \quad \forall \ (a_1 a_2) \in \Lambda.$$
In fact, for any \( \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right) \), since \( \pi \varphi = 0 \), one has

\[
(f \left( \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right) \right)) \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} (g_1 \varphi)(x) \\ (g_2 \varphi)(y) \end{array} \right) \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right)
\]

\[
= \left( \begin{array}{c} (g_1 \varphi)(x) \\ (g_2 \varphi)(y) \end{array} \right) \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right)
\]

Thus \( f \left( \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right) \right) = \left( \begin{array}{c} (g_1 \pi)(x) \\ (g_2 \pi)(y) \end{array} \right) \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right) \), and hence

\[
h \circ f = (g_1 \pi)(x) + (g_2 \pi)(y) = (g_1, g_2) \pi + \alpha a_3.
\]

One the other hand, by the right \( \Lambda \)-module structure of \((\operatorname{Coker} \varphi)^*, X^* \), one has

\[
h(f \left( \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right) \right)) = (g, \alpha) \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right) = (g_1, \pi \varphi + \alpha a_3)
\]

This proves the claim.

Since the map

\[
((\operatorname{Coker} \varphi)^*, X^*) \rightarrow \left( \begin{array}{c} X^* \\ \varphi \end{array} \right), \quad (g, \alpha) \mapsto f = \left( \begin{array}{c} (g_1 \pi + \alpha a_3) \\ (g_2 \pi + \alpha a_3) \end{array} \right)
\]

is the inverse of \( h \), \( h \) is a right \( \Lambda \)-isomorphism. \( \square \)

### 5.2. The \( \Lambda \)-dual of a right \( \Lambda \)-module.

Similarly, one can determine the \( \Lambda \)-dual of a right \( \Lambda \)-module \((U, V)\), where \( \psi : U \rightarrow V \) is a right \( \Lambda \)-map. As a right \( \Lambda \)-module, \( \Lambda(A, A) \) is of the form \((\cdot) \rightarrow (\cdot)\), where \( \alpha \in U^* = \operatorname{Hom}_A(U, A) \), \( \beta_1, \beta_2 \in V^* \), such that

\[
\begin{array}{ccc}
U & \xrightarrow{\psi} & V \\
\alpha \downarrow & & \downarrow \beta_1 \\
A & \xrightarrow{\psi} & A \oplus A \\
\end{array}
\]

commutes. Thus \( \alpha = \beta_2 \psi, \beta_2 \psi = 0 \). Hence, there is a unique \( g \in (\operatorname{Coker} \psi)^* = \operatorname{Hom}_A(\operatorname{Coker} \psi, A) \) such that \( \beta_2 = g \pi \), where \( \pi : V \rightarrow \operatorname{Coker} \psi \) is the canonical \( \Lambda \)-map. By the similar argument one has

**Lemma 5.2.** Let \((U, V)\) be a right \( \Lambda \)-module with \( \psi : U \rightarrow V \) a right \( \Lambda \)-map. Then

(i) Any \( f \in (U, V)^* \) is of the form \( (\beta_1, \left( \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right)) \), where \( \beta_1 \in V^*, \beta_2 \in (\psi)^*, \) and \( \pi \in \psi \rightarrow \operatorname{Coker} \psi \) is the canonical \( \Lambda \)-map.

(ii) There is a unique left \( \Lambda \)-module isomorphism \( h' : (U, V)^* \cong \left( \begin{array}{c} V^* \\ \operatorname{Coker} \psi^* \end{array} \right) \), given by

\[
f = (\beta_1 \psi, \left( \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right)) \mapsto \left( \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right),
\]

where \( \pi^* : (\operatorname{Coker} \psi)^* \rightarrow V^* \) is the left \( \Lambda \)-monomorphism induced by \( \pi \).
5.3. The left $\Lambda$-module $(\overset{\sim}{X})^*_\varphi$. For any left $\Lambda$-module $(\overset{\sim}{X})_\varphi$ with left $\Lambda$-map $\varphi : Y \rightarrow X$, by Lemma 5.3, one has the right module isomorphism

$$h : (\overset{\sim}{X})_\varphi^* \cong ((\text{Coker}\varphi)^*, X^*)_{\pi^*}, \quad f = \left(\begin{pmatrix} g \\ \alpha \end{pmatrix}\right) \mapsto (g, \alpha)_2$$

where $\pi^* : (\text{Coker}\varphi)^* \rightarrow X^*$ is the right $\Lambda$-monomorphism induced by $\pi : X \rightarrow \text{Coker}\varphi$. Applying Lemma 5.2 to $(\text{Coker}\varphi)^*, X^*)_{\pi^*}$, we then get

**Lemma 5.3.** (i) Any $f \in ((\text{Coker}\varphi)^*, X^*)_{\pi^*}$ is of the form $(\beta_1 \pi^*, (\beta_2 g))$, where $\beta_1 \in X^{**}$, $g \in (\text{Coker}\pi^*)^*$, and $p : X^* \rightarrow \text{Coker}\pi^*$ is the canonical $\Lambda$-epimorphism.

(ii) There is a unique left $\Lambda$-module isomorphism $\tilde{h} : \left((\overset{\sim}{X})^{**}_{\varphi}\right)_p \cong (\overset{\sim}{X})^{**}_{\varphi}$, given by

$$\left(\begin{pmatrix} \beta_1 \\ g \end{pmatrix}\right) \mapsto h^*((\beta_1 \pi^*, (\beta_2 g)))$$

where $p^* : (\text{Coker}\pi^*)^* \rightarrow X^{**}$ is the $\Lambda$-monomorphism induced by $p$, $h : (\overset{\sim}{X})_\varphi^* \cong ((\text{Coker}\varphi)^*, X^*)_{\pi^*}$ is given in (5.1), and $h^* : ((\text{Coker}\varphi)^*, X^*)_{\pi^*} \rightarrow (\overset{\sim}{X})^{**}_{\varphi}$ is induced by $h$.

5.4. The canonical $\Lambda$-map $\phi(\overset{\sim}{X})_\varphi : (\overset{\sim}{X})_\varphi \rightarrow (\overset{\sim}{X})^{**}_{\varphi}$. For a left $\Lambda$-module $(\overset{\sim}{X})_\varphi$, one has an exact sequence $0 \rightarrow Y \xrightarrow{\varphi} X \xrightarrow{\varphi} \text{Coker}\varphi \rightarrow 0$ of left $\Lambda$-modules. Applying $\text{Hom}_\Lambda(-, A\Lambda)$, one gets an exact sequence of right $\Lambda$-modules

$$0 \rightarrow (\text{Coker}\varphi)^* \xrightarrow{\pi^*} X^* \xrightarrow{\varphi^*} Y^*$$

and the exact sequence

$$0 \rightarrow (\text{Coker}\varphi)^* \xrightarrow{\pi^*} X^* \xrightarrow{p} \text{Coker}\pi^* \rightarrow 0.$$

Thus, there is a unique $\Lambda$-map $\beta : \text{Coker}\pi^* \rightarrow Y^*$ such that the diagram

$$\begin{array}{ccc}
(\text{Coker}\varphi)^* & \xrightarrow{\pi^*} & X^* \\
\downarrow & & \downarrow \beta \\
0 & \xrightarrow{p} & \text{Coker}\pi^* \\
\downarrow & & \downarrow \beta \\
0 & \xrightarrow{\pi^*} & X^* \\
& & \varphi^* \\
& & Y^*
\end{array}$$

commutes, i.e., $\varphi^* = \beta p$. Thus, $\varphi^*$ is an epimorphism if and only if so is $\beta$, and if and only if $\beta$ is an isomorphism. So one has the $\Lambda$-map $\beta^* : Y^{**} \rightarrow ((\text{Coker}\pi^*)^*)^*$. Consider the composition

$$\beta^* \phi_Y : Y \rightarrow ((\text{Coker}\pi^*)^*)^*$$

where $\phi_Y : Y \rightarrow Y^{**}$ is the canonical map. By the definition of $\phi_Y$ and $\beta^*$, one knows that $\beta^* \phi_Y : Y \rightarrow ((\text{Coker}\pi^*)^*)^*$ is given by

$$y \mapsto "g \mapsto (\beta(g))(y)"., \quad \forall \ g \in \text{Coker}\pi^*$$

i.e., $((\beta^* \phi_Y)(y))(g) = (\beta(g))(y), \ \forall \ y \in Y$.

**Proposition 5.4.** For any left $\Lambda$-module $(\overset{\sim}{X})_\varphi$ with left $\Lambda$-map $\varphi : Y \rightarrow X$, with the notations above one has

(i) $(\overset{\sim}{X})_\varphi^* : (\overset{\sim}{X})_\varphi \rightarrow ((\overset{\sim}{X})^{**}_{\varphi})_p$, is left $\Lambda$-map, where $\phi_X : X \rightarrow X^{**}$ and $\phi_Y : Y \rightarrow Y^{**}$ are the canonical $\Lambda$-maps, $\beta : \text{Coker}\pi^* \rightarrow Y^*$ is the canonical $\Lambda$-map such that $\varphi^* = \beta p$, and $\beta^* : Y^{**} \rightarrow ((\text{Coker}\pi^*)^*)^*$ is induced by $\beta$. 
(ii) The canonical $\Lambda$-map $\phi_{(X)} : (X)_\varphi \rightarrow (X)_\varphi^{**}$ is given by
\[
\phi_{(X)} = \tilde{h} \circ (\phi_X)_{\beta^* \phi_Y}.
\]
where $\tilde{h} : (Coker\pi^*)^* \rightarrow (X)_\varphi^{**}$ is the isomorphism given in Lemma 5.3.

Proof. (i) One needs to prove the diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\varphi} & X \\
\downarrow{\beta^* \phi_Y} & & \downarrow{\phi_X} \\
(Coker\pi^*)^* & \xrightarrow{\pi^*} & X^{**}
\end{array}
\]
commutes, i.e., $p^* \beta^* \phi_Y = \varphi^{**} \phi_Y = \varphi X \varphi$. In fact, since $\varphi^* = \beta p$, one has $\varphi^{**} = p^* \beta^*$. By the functorial property of the canonical map $\phi_X : X \rightarrow X^{**}$ one has the commutative diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\varphi} & X \\
\downarrow{\phi_Y} & & \downarrow{\phi_X} \\
Y^{**} & \xrightarrow{\varphi^{**}} & X^{**}
\end{array}
\]
It follows that $p^* \beta^* \phi_Y = \varphi^{**} \phi_Y = \varphi X \varphi$.

(ii) We need to prove $\phi_{(X)}(X)_{\varphi}(x) = \tilde{h}(\phi_{X})_{\varphi}(x)$, $\forall (x) \in (X)_\varphi$. For this, let $f \in (X)_\varphi^*$. By Lemma 5.1(i), $f = \left(\frac{g}{\alpha_2}\right)_{\varphi}$, where $g \in (Coker\varphi)^*$, $\pi : X \rightarrow Coker\varphi$ is the canonical $\Lambda$-epimorphism, and $\alpha_2 \in X^*$.

By the definition of $\phi_{(X)}$, one has
\[
\phi_{(X)}(X)_{\varphi}(f) = f \left(\frac{x}{y}\right) = \left(\frac{g(x)}{\alpha_2(y)}\right) = \left(\frac{\alpha_2(x)}{\alpha_2(y)}\right) \in \Lambda.
\]

On the other hand, by the definitions of $\beta^* \phi_Y$ and $\tilde{h}$ one has
\[
\tilde{h}(\phi_X(\beta^* \phi_Y)(x, y))(f) = \tilde{h}(\phi_X(x))(f) = \tilde{h}(\phi_X(\beta^* \phi_Y)(y))(f)
\]
\[
= \tilde{h}((\phi_X(x)\pi^*, \left(\frac{\phi_X(x)}{(\beta^* \phi_Y)(y)}\right)))(f) = \tilde{h}(\phi_X(x)\pi^*, \left(\frac{\phi_X(x)}{(\beta^* \phi_Y)(y)}\right))(f)
\]
\[
= \pi^*(g(x), \alpha_2) = \left(\begin{array}{c}
\alpha_2(x) \\
\beta^* \phi_Y(\alpha_2(y))
\end{array}\right) = \left(\begin{array}{c}
\alpha_2(x) \\
\beta^* \phi_Y(\alpha_2(y))
\end{array}\right).
\]
This completes the proof.

5.5. Torsionless $\Lambda$-modules and reflexive $\Lambda$-modules.

**Corollary 5.5.** Let $(X)_\varphi$ be a left $\Lambda$-module, where $\varphi : Y \rightarrow X$ is a left $\Lambda$-map. Then

(i) $(X)_\varphi$ is a torsionless $\Lambda$-module if and only if it is monic (i.e., $\varphi$ is a monomorphism), $X$ and $Y$ are torsionless $\Lambda$-modules.

(ii) $\phi_{(X)}$ is a $\Lambda$-epimorphism if and only if $\phi_X$ and $\beta^* \phi_Y : Y \rightarrow (Coker\pi^*)^*$ are $\Lambda$-epimorphisms.
Lemma 5.6. Let \( \Lambda \)-module \( X \) be double semi-Gorenstein-projective if and only if the conditions (1), (2), (3) hold:

1. \( X \in \perp A \);
2. \( Y \in \perp A \);
3. \( \varphi^* : X^* \to Y^* \) is an epimorphism.

By Lemma 5.1, \( (Y)^* \varphi \cong (\text{Coker} \varphi)^* \) as right \( \Lambda \)-modules, where \( \pi^* : (\text{Coker} \varphi)^* \to X^* \) is the right \( \Lambda \)-monomorphism induced by \( \pi : X \to \text{Coker} \varphi \). Thus, by the right module version of Corollary 2.14(i), \( (Y)^* \varphi \in \perp A \) if and only if the following conditions (4)-(6) hold:

4. \( (\text{Coker} \varphi)^* \in \perp A \);
5. \( X^* \in \perp A \);
6. \( \pi^{**} : X^{**} \to (\text{Coker} \varphi)^{**} \) is an epimorphism.

Lemma 5.6. Let \( (Y)^\varphi \) be a left \( \Lambda \)-module, where \( \varphi : Y \to X \) is a left \( \Lambda \)-map. Then \( (Y)^\varphi \) is double semi-Gorenstein-projective if and only if the conditions (1) - (6) above hold, and if and only if the conditions (1) - (8) hold, where

7. \( Y^* \in \perp A \);
8. The canonical \( \Lambda \)-map \( \beta : \text{Coker} \pi^* \to Y^* \) is an isomorphism.
Proof. It remains to show that the conditions (1) - (6) imply the conditions (7) and (8).

Assume that the conditions (1) - (6) hold. Since \( \varphi^* : X^* \rightarrow Y^* \) is an epimorphism, \( \beta : \text{Coker}\pi^* \rightarrow Y^* \) is an isomorphism (cf. the diagram (5.2)). Applying \( \text{Hom}_A(-, A\Lambda) \) to the exact sequence \( 0 \rightarrow (\text{Coker}\varphi^*) \overset{\pi^*}{\rightarrow} X^* \overset{\varphi^*}{\rightarrow} Y^* \rightarrow 0 \), by the assumption that \( \pi^{**} : X^{**} \rightarrow (\text{Coker}\varphi)^* \) is an epimorphism and by the assumptions \( X^* \in \overset{\perp}{} A \) and \( (\text{Coker}\varphi)^* \in \overset{\perp}{} A \), one sees that \( Y^* \in \overset{\perp}{} A \). □

5.7. A double semi-Gorenstein-projective \( \Lambda \)-module \( M \) with \( \phi_M \) monomorphism or epimorphism.

Proposition 5.7. Let \( (\overset{\perp}{}Y, \overset{\perp}{}Y) \varphi \) be a left \( \Lambda \)-module with left \( \Lambda \)-map \( \varphi : Y \rightarrow X \). Then

(i) \( (\overset{\perp}{}Y, \overset{\perp}{}Y) \varphi \) is torsionless and double semi-Gorenstein-projective if and only if \( (\overset{\perp}{}Y, \overset{\perp}{}Y) \varphi \) is monic (i.e. \( \varphi \) is a monomorphism), \( X, Y, \) and \( \text{Coker}\varphi \) are double semi-Gorenstein-projective, and \( X \) and \( Y \) are torsionless.

(ii) \( (\overset{\perp}{}Y, \overset{\perp}{}Y) \varphi \) is double semi-Gorenstein-projective with epimorphism \( \phi(\overset{\perp}{}Y) \varphi \) if and only if the following conditions are satisfied:

- \( \varphi^* : X^* \rightarrow Y^* \) is an epimorphism;
- All the five modules \( X, Y, X^*, Y^* \), \( (\text{Coker}\varphi)^* \) are semi-Gorenstein-projective;
- \( \phi_X \) and \( \phi_Y \) are epimorphisms.

(iii) \( \text{(Corollary 5.14)} \) \( (\overset{\perp}{}Y, \overset{\perp}{}Y) \varphi \) is Gorenstein-projective if and only if \( \varphi \) is a monomorphism, \( Y \) and \( \text{Coker}\varphi \) are Gorenstein-projective. If this is the case, then \( X \) is Gorenstein-projective.

Proof. (i) Assume that \( (\overset{\perp}{}Y, \overset{\perp}{}Y) \varphi \) is torsionless and double semi-Gorenstein-projective. By Corollary 5.6(i), \( \varphi \) is a monomorphism, and \( X \) and \( Y \) are torsionless. By Lemma 5.6 all the conditions (1)-(8) hold. Applying \( \text{Hom}_A(-, A\Lambda) \) to the exact sequence \( 0 \rightarrow Y \overset{\varphi}{\rightarrow} X \overset{\pi}{\rightarrow} \text{Coker}\varphi \rightarrow 0 \), since \( \varphi^* : X^* \rightarrow Y^* \) is an epimorphism, and since \( X \) and \( Y \) are semi-Gorenstein-projective, it follows that \( \text{Coker}\varphi \) is semi-Gorenstein-projective.

Conversely, assume that \( \varphi \) is a monomorphism, \( X, Y \) and \( \text{Coker}\varphi \) are double semi-Gorenstein-projective, and that \( X \) and \( Y \) are torsionless. By Corollary 5.6(i), \( (\overset{\perp}{}Y, \overset{\perp}{}Y) \varphi \) is torsionless. Again applying \( \text{Hom}_A(-, A\Lambda) \) to the exact sequence \( 0 \rightarrow Y \overset{\varphi}{\rightarrow} X \overset{\pi}{\rightarrow} \text{Coker}\varphi \rightarrow 0 \), since \( \text{Coker}\varphi \) is semi-Gorenstein-projective, \( \varphi^* : X^* \rightarrow Y^* \) is an epimorphism and \( 0 \rightarrow (\text{Coker}\varphi)^* \overset{\pi^*}{\rightarrow} X^* \overset{\varphi^*}{\rightarrow} Y^* \rightarrow 0 \) is an exact sequence. Since \( Y^* \) is semi-Gorenstein-projective, \( \pi^{**} : X^{**} \rightarrow (\text{Coker}\varphi)^* \) is an epimorphism. Thus, all the conditions (1)-(6) hold. By Lemma 5.6 \( (\overset{\perp}{}Y, \overset{\perp}{}Y) \varphi \) is double semi-Gorenstein-projective.

(ii) Assume that \( (\overset{\perp}{}Y, \overset{\perp}{}Y) \varphi \) is double semi-Gorenstein-projective and \( \phi(\overset{\perp}{}Y) \varphi \) is an epimorphism. By Lemma 5.6 all the conditions (1)-(8) are satisfied. By Corollary 5.6(ii), \( \phi_X \) and \( \beta^*\phi_Y \) are epimorphisms, where \( \beta : \text{Coker}\pi^* \rightarrow Y^* \) is the canonical \( \Lambda \)-map such that \( \varphi^* = \beta p \), \( p : X^* \rightarrow \text{Coker}\pi^* \) is the canonical \( \Lambda \)-epimorphism, and \( \beta^* : Y^{**} \rightarrow (\text{Coker}\pi^*)^* \) is induced by \( \beta \). It remains to show that \( \phi_Y \) is an epimorphism. In fact, by Condition (8), \( \beta^* \) is an isomorphism, hence \( \phi_Y \) is an epimorphism.

Conversely, assume that \( \varphi^* : X^* \rightarrow Y^* \) is an epimorphism, all the five modules \( X, Y, X^*, Y^* \), \( (\text{Coker}\varphi)^* \) are semi-Gorenstein-projective, and that \( \phi_X \) and \( \phi_Y \) are epimorphisms. Since \( \varphi^* \) is an epimorphism, \( \beta : \text{Coker}\pi^* \rightarrow Y^* \) is an isomorphism (cf. Subsection 5.4), and hence \( \beta^* \) is an isomorphism. Thus \( \beta^*\phi_Y \) is an epimorphism. By Corollary 5.6(ii), \( \phi(\overset{\perp}{}Y) \varphi \) is an epimorphism.
Applying $\text{Hom}_A(-, A\Lambda)$ to the exact sequence $Y \xrightarrow{\varphi} X \xrightarrow{\pi} \text{Coker}\varphi \to 0$, since $\varphi^* : X^* \to Y^*$ is an epimorphism, it follows that $0 \to (\text{Coker}\varphi)^* \xrightarrow{\pi^*} X^* \xrightarrow{\varphi^*} Y^* \to 0$ is an exact sequence. Since $Y^*$ is semi-Gorenstein-projective, $\pi^{**} : X^{**} \to (\text{Coker}\varphi)^{**}$ is an epimorphism. Thus, all the conditions (1)-(6) hold. By Lemma 5.6, $(\overline{Y}_\varphi)$ is double semi-Gorenstein-projective.

(iii) This is just Corollary 2.14. We rewrite here, because in the setting of (i) and (ii), it admits a simple proof. The “if” part follows from (i) and (ii) and the fact that Gorenstein-projective modules are closed under extensions.

Assume that $(\overline{Y}_\varphi)$ is Gorenstein-projective. Then by (i) and (ii), $\varphi$ is a monomorphism, $X$ and $Y$ are Gorenstein-projective, and $\text{Coker}\varphi$ is double semi-Gorenstein-projective. Moreover, the diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{\varphi} & Y \\
& \searrow^{\phi_Y} & \downarrow \cong \\
& & X \\
& \searrow^{\phi_X} & \downarrow \cong \\
& & \text{Coker}\varphi \\
0 & \xrightarrow{\varphi^{**}} & Y^{**} \\
& \searrow^{\phi_{\text{Coker}\varphi}} & \downarrow \cong \\
& & X^{**} \\
& \searrow^{\pi^{**}} & \downarrow \cong \\
& & (\text{Coker}\varphi)^{**} \\
& \searrow & \downarrow \\
& & 0
\end{array}
$$

(5.3)

commutes with exact rows. So $\phi_{\text{Coker}\varphi}$ is an isomorphism, and thus $\text{Coker}\varphi$ is Gorenstein-projective. ■

5.8. Problems. As remarked in [RZ4, 3.1], all known examples of double semi-Gorenstein-projective modules $M$ such that $\phi_M$ is a monomorphism (an epimorphism, respectively) are Gorenstein-projective.

**Problem 2.** Is there a torsionless and double semi-Gorenstein-projective module $M$ such that $M$ is not Gorenstein-projective?

**Problem 3.** Is there a double semi-Gorenstein-projective module $M$ with $\phi_M$ an epimorphism such that $M$ is not semi-Gorenstein-projective?

Theorem 1.6 is a result in this direction.

5.9. Proof of Theorem 1.6. (1) Assume that any torsionless and double semi-Gorenstein-projective $A$-module is Gorenstein-projective. Let $M = (\overline{Y}_\varphi)$ be a torsionless and double semi-Gorenstein-projective $A$-module. We need to show that $M$ is Gorenstein-projective.

By Proposition 5.7(i), $\varphi$ is a monomorphism, $X$, $Y$, and $\text{Coker}\varphi$ are double semi-Gorenstein-projective, and $X$ and $Y$ are torsionless. By the assumption, $X$ and $Y$ are Gorenstein-projective.

By Lemma 5.6, $\varphi^*$ and $\pi^{**}$ are epimorphisms. Thus, one again has the commutative diagram (5.3) with exact rows, from which one knows that $\phi_{\text{Coker}\varphi}$ is also an isomorphism, and hence Coker$\varphi$ is Gorenstein-projective. Thus, $M$ is Gorenstein-projective, by Proposition 5.7(iii).

Conversely, assume that any torsionless and double semi-Gorenstein-projective $A$-module is Gorenstein-projective. Let $L$ be a torsionless and double semi-Gorenstein-projective $A$-module. We need to prove that $L$ is Gorenstein-projective.

Since $L$ is torsionless, a left add($A$)-approximation $\varphi : L \to P$ of $L$ is a monomorphism, where $P$ is a projective $A$-module. Since both $P$ and $L$ are semi-Gorenstein-projective and $\varphi$ is a left add($A$)-approximation, it follows that $\text{Coker}\varphi$ is also semi-Gorenstein-projective. Consider the $A$-module $(\overline{L}_\varphi)$. Since $P^*$ and $L^*$ are semi-Gorenstein-projective and

$$0 \to (\text{Coker}\varphi)^* \to P^* \xrightarrow{\varphi^*} L^* \to 0$$

is an exact sequence, it follows that $L^*$ is semi-Gorenstein-projective. Therefore, $\overline{L}_\varphi$ is double semi-Gorenstein-projective and, by the commutative diagram (5.3), $\phi_{\text{Coker}\varphi}$ is an isomorphism, and hence $\text{Coker}\varphi$ is Gorenstein-projective. Thus, $L$ is Gorenstein-projective, by Proposition 5.7(iii).
is an exact sequence, it follows that \((\text{Coker}\varphi)^*\) is also semi-Gorenstein-projective. Thus, by Proposition 5.7(i), \((\varphi_L)_\varphi^*\) is a torsionless and double semi-Gorenstein-projective \(A\)-module. By the assumption, \((\varphi_L)_\varphi\) is Gorenstein-projective. Hence \(L\) is Gorenstein-projective, by Proposition 5.7(ii).

(2) Assume that any double semi-Gorenstein-projective \(A\)-module \(L\) with \(\phi_L\) an epimorphism is Gorenstein-projective. Let \(M = (\bar{X})_\varphi\) be a double semi-Gorenstein-projective \(A\)-module such that \(\phi_M\) is an epimorphism. We need to prove that \(M\) is Gorenstein-projective.

By Proposition 5.7(ii), \(\varphi^* : X^* \to Y^*\) is an epimorphism, all the five modules \(X, Y, X^*, Y^*, (\text{Coker}\varphi)^*\) are semi-Gorenstein-projective, and \(\phi_X\) and \(\phi_Y\) are epimorphisms. By the assumption, \(X\) and \(Y\) are Gorenstein-projective, in particular, \(\phi_X\) and \(\phi_Y\) are isomorphisms.

We claim that \(\varphi : Y \to X\) is a monomorphism and \(\text{Coker}\varphi\) is reflexive. In fact, applying \(\text{Hom}_A(-, A)\) to \(Y \xrightarrow{\varphi} X \xrightarrow{\pi} \text{Coker}\varphi \to 0\), since \(\varphi^* : X^* \to Y^*\) is an epimorphism, it follows that

\[0 \to (\text{Coker}\varphi)^* \xrightarrow{\varphi^*} X^* \xrightarrow{\pi^*} Y^* \to 0\]

is an exact sequence. Since \(Y^*\) is semi-Gorenstein-projective,

\[0 \to Y^{**} \xrightarrow{\varphi^{**}} X^{**} \xrightarrow{\pi^{**}} (\text{Coker}\varphi)^{**} \to 0\]

is an exact sequence. Thus, by the functorial property of \(\varphi\) one has the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\varphi} & X \\
\Downarrow{\cong} & & \Downarrow{\cong} \\
0 & \xrightarrow{\pi} & \text{Coker}\varphi \\
\end{array}
\]

with exact rows. Since both \(\phi_Y\) and \(\varphi^{**}\) are monomorphisms, \(\varphi\) is a monomorphism. Also, this commutative diagram shows that \(\phi_{\text{Coker}\varphi}\) is an isomorphism, i.e., \(\text{Coker}\varphi\) is reflexive. This proves the claim.

Applying \(\text{Hom}_A(-, A)\) to the exact sequence \(0 \to Y \xrightarrow{\varphi} X \xrightarrow{\pi} \text{Coker}\varphi \to 0\), since \(\varphi^* : X^* \to Y^*\) is an epimorphism and \(X\) and \(Y\) are semi-Gorenstein-projective, it follows that \(\text{Coker}\varphi\) is also semi-Gorenstein-projective. So, \(\text{Coker}\varphi\) is double semi-Gorenstein-projective and reflexive, i.e., \(\text{Coker}\varphi\) is Gorenstein-projective. By Proposition 5.7(iii), \(M = (\bar{X})_\varphi\) is Gorenstein-projective.

Conversely, assume that any double semi-Gorenstein-projective \(A\)-module \(M\) with \(\phi_M\) an epimorphism is Gorenstein-projective. Let \(L\) be a double semi-Gorenstein-projective \(A\)-module such that \(\phi_L\) is an epimorphism. We need to show that \(L\) is Gorenstein-projective.

Take a left \(\text{add}(A)\)-approximation \(\varphi : L \to P\) of \(L\). Applying \(\text{Hom}_A(-, A)\) to the exact sequence \(L \xrightarrow{\varphi} P \xrightarrow{\pi} \text{Coker}\varphi \to 0\), since \(\varphi\) is left \(\text{add}(A)\)-approximation, \(\varphi^* : P^* \to L^*\) is an epimorphism and

\[0 \to (\text{Coker}\varphi)^* \xrightarrow{\varphi^*} P^* \xrightarrow{\pi^*} L^* \to 0\]

is an exact sequence. Since \(L^*\) and \(P^*\) are semi-Gorenstein-projective, so is \((\text{Coker}\varphi)^*\). Thus, by Proposition 5.7(ii), \((\varphi_L)_\varphi^*\) is a double semi-Gorenstein-projective \(A\)-module such that \(\phi_L\) is an epimorphism. By the assumption, \((\varphi_L)_\varphi\) is Gorenstein-projective. Hence by Proposition 5.7(iii), \(L\) is Gorenstein-projective. ■
6. Double semi-Gorenstein-projective modules which are not monic

6.1. Proof of Theorem 1.7 Since by assumption \( Y \) is not torsionless and \( \varphi : Y \to P \) is a left \( \text{add}(A) \)-approximation of \( Y \), it follows that \( \varphi \) is not a monomorphism. Thus \( (\ell^\varphi)_1 \) is not a monic \( T_2(A) \)-module. By Corollary 5.6(i), \( (\ell^\varphi)_1 \) is not torsionless.

Apply \( \text{Hom}_A(-, \text{add}(A)) \) to the exact sequence \( Y \xrightarrow{\varphi} P \xrightarrow{\pi} \text{Coker}\varphi \to 0 \). Since \( \varphi \) is a left \( \text{add}(A) \)-approximation of \( Y \), \( \varphi^* : P^* \to Y^* \) is an epimorphism and

\[
0 \to (\text{Coker}\varphi)^* \xrightarrow{\pi^*} P^* \xrightarrow{\varphi^*} Y^* \to 0
\]

is an exact sequence. Since \( Y^* \) and \( P^* \) are semi-Gorenstein-projective, it follows that \((\text{Coker}\varphi)^*\) is semi-Gorenstein-projective and \( \pi^{**} : P^{**} \to (\text{Coker}\varphi)^{**} \) is an epimorphism. Thus, all the conditions (1) - (6) in Subsection 5.6 are satisfied. By Lemma 5.6 (\( \ell^\varphi_1 \)) is a double semi-Gorenstein-projective \( T_2(A) \)-module.

6.2. A class of double semi-Gorenstein-projective modules which are not torsionless. From now on, \( A \) is the algebra \( \Lambda(q) \), which has been studied in [RZ2, RZ3], i.e.,

\[
A = k(x, y, z)/(x^2, y^2, z^2, yz, xy + qyx, xz - zy, zy - zx)
\]

where \( q \) is a non-zero element in field \( k \), and \( q \) is of multiplicative order \( \infty \). Then \( A \) is a semi-local algebra of wild representation type, with a basis \( 1, x, y, z, xy, xz \), and with Hilbert type \( (|J/J^2|, |J^2|) = (3, 2) \), where \( J \) is the Jacobson radical of \( A \). For the studies on short local algebras, we refer to e.g. [L], [Y2], [CV], [AIS], [RZ5], [RZ6].

When \( (a, b, c) \) ranges over \( k^3 \setminus \{0\} \), left \( A \)-modules

\[
M(a, b, c) := \text{add}(A)/[A(ax + by + cz) + \text{soc}A] = \overline{A}
\]

give all the 3-dimensional local \( A \)-modules. They are \( (A/J)^2 \)-modules. Since \( A/J^2 \) is commutative, \( \text{D}(M(a, b, c)) = \text{Hom}_k(M(a, b, c), k) \) are also left \( (A/J^2) \)-modules, and hence left \( A \)-modules. [RZ3, Proposition A.1] asserts that \( M(a, b, c) \) and \( D(M(a, b, c)) \) give all the indecomposable left \( A \)-modules of dimension 3.

For \( (a, b, c) \in k^3 \setminus \{0\} \), we also consider right \( A \)-modules

\[
M'(a, b, c) := A/[(ax + by + cz)A + \text{soc}A] = \overline{1}A.
\]

Lemma 6.1. ([RZ3, 1.7]) An indecomposable \( A \)-module \( M \) of dimension at most 3 is double semi-Gorenstein-projective which are not torsionless if and only if \( M \cong M(1, -q, c) \) for some \( c \in k \). Moreover,

\[
M(1, -q, c)^* \cong (x - y)A \cong A/(x - q^{-1}y)A = M'(1, -q^{-1}, 0)
\]

where the first isomorphism is given by \( f \mapsto f(\overline{1}) \), and the second isomorphism is given by \( x - y \mapsto \overline{1} \).

6.3. A class of double semi-Gorenstein-projective \( T_2(A) \)-modules which are not monic. In order to apply Theorem 1.7 to get a family of double semi-Gorenstein-projective \( T_2(A) \)-modules which are not monic, we look for a left \( \text{add}(A) \)-approximation of \( M(1, -q, c) = A/(x - qy + cz) = \overline{A} \). Any \( f \in M(1, -q, c)^* = \text{Hom}_A(A\overline{1}, A) \) is the right multiplication by \( f(\overline{1}) \). Since \( \text{Im}f \in J \), \( f(\overline{1}) = c_1x + c_2y + c_3z + c_4yx + c_5zx \) with \( c_i \in k \), such that \( (x - qy + cz)f(\overline{1}) = 0 \). Thus \( c_1 + c_2 = 0, c_3 = 0 \).
Hence \( f(\overline{1}) \in (x - y)A \) and \( M(1, -q, c)^* \) has a \( k \)-basis \( f_1, f_2, f_3 \), where \( f_i : M(1, -q, c) \to A \) is the left \( A \)-map given by

\[
\begin{align*}
f_1(\overline{1}) &= x - y, & f_2(\overline{1}) &= yx, & f_3(\overline{1}) &= zx.
\end{align*}
\]

Therefore \( M(1, -q, c)^* = f_1 A \) and

\[
f_1 : M(1, -q, c) \to A
\]

is a left \( \text{add}(A) \)-approximation of \( M(1, -q, c) \). Applying Theorem \ref{maintheorem}, one gets

**Proposition 6.2.** For all \( c \in k \), the \( T_2(A) \)-modules

\[
X(c) := \left( \frac{A}{M(1, -q, c)} \right)^A
\]

where \( f_1 : M(1, -q, c) \to A \) is the left \( A \)-map given by \( f_1(\overline{1}) = x - y \), are double semi-Gorenstein-projective, but not monic, and hence not torsionless.

Moreover, one has

\begin{itemize}
\item[(i)] \( X(c)^* \cong ((x - q^{-1}y)A, A)_\sigma \), where \( \sigma : (x - q^{-1}y)A \to A \) is the embedding; \( X(c)^{**} \cong \left( \frac{A}{A(x-y)A} \right)_\ell \) is not semi-Gorenstein-projective, where \( \ell : A(x-y) \to A \) is the embedding. In particular, \( X(c)^* \) is not Gorenstein-projective.
\item[(ii)] If one identifies \( X(c)^{**} \) with \( \left( \frac{A}{A(x-y)A} \right)_\ell \), then the canonical \( \Lambda \)-map \( \phi_{X(c)} : X(c) \to X(c)^{**} \) reads as

\[
\left( \frac{\text{Id}_A}{r_{x-y}} \right) : \left( \frac{A}{M(1, -q, c)} \right)^A \to \left( \frac{A}{A(x-y)A} \right)_\ell
\]

where \( r_{x-y} \) is the right multiplication by \( x - y \). Thus, \( \phi_{X(c)} \) is neither a monomorphism nor an epimorphism.
\end{itemize}

**Proof.** It remains to prove (i) and (ii).

(i) Note that there are isomorphisms

\[
\text{Coker} f_1) = (A/A(x-y))^* \cong (x - q^{-1}y)A \cong A/(x - q^{-2}y)A
\]

as right \( A \)-modules, where the first isomorphism is given by \( g \mapsto g(\overline{1}) \), and the second isomorphism is given by \( x - q^{-1}y \mapsto 1 \). We stress that

\[
A/A(x-y) \ncong M(1,-1,0) = A/[A(x-y) + \text{soc}A]
\]

and that \( (A/A(x-y))^* \cong (x - q^{-1}y)A \cong M(1,-1,0)^* \). By Lemma \ref{lemma1}ii), there is a right \( \Lambda \)-module isomorphism

\[
h : X(c)^* \cong ((x - q^{-1}y)A, A)_\sigma
\]

where \( \sigma : (x - q^{-1}y)A \to A \) is the embedding.

Note that \( (\text{Coker} \sigma)^* = (A/(x - q^{-1}y)A)^* \cong A(x-y)A \) as left \( A \)-modules, with the isomorphism given by \( g \mapsto g(\overline{1}) \). By Lemma \ref{lemma2}ii), there is a left \( \Lambda \)-module isomorphism

\[
\left( \frac{A}{A(x-y)A} \right)_\ell \cong X(c)^{**}
\]

where \( \ell : A(x-y)A \to A \) is the embedding.

Note that \( A(x-y)A = A(x-y) \oplus kzx \) is decomposable left \( A \)-module of dimension 3. By \cite[Theorem 1.5]{RZ3}, \( A(x-y)A \) not semi-Gorenstein-projective. It follows from Corollary \ref{corollary1} that \( \left( \frac{A}{A(x-y)A} \right)_\ell \) is not....
semi-Gorenstein-projective, and hence \( X(c)^{**} \) is not semi-Gorenstein-projective. In particular, \( X(c)^* \) is not Gorenstein-projective.

(ii) To get \( \phi_{X(c)} \), we apply Proposition 5.4 to \( X(c) = \left( \frac{A^A}{M(1,-q,c)} \right)_{f_1} \). It is clear that \( \phi_A = \text{Id}_A \).

Since \( X(c) \) is double semi-Gorenstein-projective, it follows from Lemma 5.6 that the canonical \( A \)-map \( \beta : \text{Coker} \sigma \rightarrow M(1,-q,c)^* \) appeared in Proposition 5.4 is an isomorphism. Without loss of generality, one may regard \( \beta \) as the identity. Note that

\[
M(1,-q,c)^{**} \cong M'(1,-q^{-1},0)^* \cong A(x-y)A
\]

where the first isomorphism is given in Lemma 6.1 and the second isomorphism is given by \( g \mapsto g(\bar{1}) \).

By Proposition 5.4(ii), if we identify \( X(c)^{**} \) with \( \left( \frac{A^A}{A(x-y)A} \right)_1 \), then \( \phi_{X(c)} \) reads as

\[
\left( \frac{\text{Id}_A}{\phi_{M(1,-q,c)}} \right) : \left( \frac{A^A}{M(1,-q,c)} \right)_{f_1} \rightarrow \left( \frac{A^A}{A(x-y)A} \right)_1.
\]

Since the diagram

\[
\begin{array}{ccc}
M(1,-q,c) & \xrightarrow{f_1} & A \\
\phi_{M(1,-q,c)} & \downarrow & \\
A(x-y)A & \xrightarrow{\iota} & A
\end{array}
\]

commutes, it follows that \( \phi_{M(1,-q,c)} : M(1,-q,c) \rightarrow A(x-y)A \) is just the right multiplication \( r_{x-y} \).

Thus, \( \phi_{M(1,-q,c)} \) is neither a monomorphism nor an epimorphism, and hence \( \phi_{X(c)} \) is neither a monomorphism nor an epimorphism.

\[\blacksquare\]

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