EXTENDED FERMIONIC $p$-ADIC INTEGRALS ON $\mathbb{Z}_p$

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Abstract. In the paper, using the extended fermionic $p$-adic integral on $\mathbb{Z}_p$, the authors find some applications of the umbral calculus. From these applications, the authors derive some identities on the weighted Euler numbers and polynomials. In other words, the authors investigate systematically the class of Sheffer sequences in connection with the generating function of the weighted Euler polynomials.

1. Preliminaries

Let $\mathbb{C}$ denote the set of complex numbers, $\mathcal{F}$ the set of all formal power series in $t$ over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\},$$

$\mathcal{P} = \mathbb{C}[x]$, $\mathcal{P}^*$ the vector space of all linear functionals on $\mathcal{P}$, and $\langle L|p(x) \rangle$ the action of the linear functional $L$ on the polynomial $p(x)$.

It is well-known that the vector space operation on $\mathcal{P}^*$ is defined by

$$\langle L + M|p(x) \rangle = \langle L|p(x) \rangle + \langle M|p(x) \rangle \quad \text{and} \quad \langle cL|p(x) \rangle = c\langle L|p(x) \rangle,$$

where $c$ is a complex constant.

The formal power series is defined by

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F},$$

which describes a linear functional on $\mathcal{P}$ as $\langle f(t)|x^n \rangle = a_n$ for all $n \geq 0$. In particular,

$$\langle t^k|x^n \rangle = n! \delta_{n,k},$$

where $\delta_{n,k}$ stands for the Kronecker delta. If we take

$$f_L(t) = \sum_{k=0}^{\infty} \langle L|x^k \rangle \frac{t^k}{k!},$$

then

$$\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle.$$

Additionally, the map $L \to f_L(t)$ is a vector space isomorphism from $\mathcal{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of the formal power series in $t$ and the vector space of all linear functionals on $\mathcal{P}$. So an element $f(t)$ of $\mathcal{F}$ will be thought
of as both a formal power series and a linear functional and $F$ will be called an umbral algebra.

It is well-known that $\langle e^y | x^n \rangle = y^n$, which implies that

$$\langle e^y | p(x) \rangle = p(y).$$

We note that for all $f(t)$ in $F$

$$f(t) = \sum_{k=0}^{\infty} \langle f(t) | x^k \rangle \frac{t^k}{k!},$$

and for all polynomials $p(x)$

$$p(x) = \sum_{k=0}^{\infty} \langle t^k | p(x) \rangle \frac{x^k}{k!}.$$

The order $o(f(t))$ of the power series $f(t) \neq 0$ is the smallest integer $k$ for which $a_k$ does not vanish. We say that $o(f(t)) = \infty$ if $f(t) = 0$. It is clear that

$$o(f(t)g(t)) = o(f(t)) + o(g(t)) \quad \text{and} \quad o(f(t) + g(t)) \geq \min\{o(f(t)), o(g(t))\}.$$

A series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$, if $o(f(t)) = 0$. Such a series is called an invertible series. A series $f(t)$ satisfying $o(f(t)) = 1$ is called a delta series. For $f(t), g(t) \in F$, we have

$$\langle f(t)g(t) | p(x) \rangle = \langle f(t) | g(t)p(x) \rangle.$$

A delta series $f(t)$ has a compositional inverse $\tilde{f}(t)$ such that

$$f(\tilde{f}(t)) = \tilde{f}(f(t)) = t.$$

By (1.8), it follows that

$$p^{(k)}(x) = \frac{d^k p(x)}{dx^k} = \sum_{\ell=k}^{\infty} \langle t^\ell | p(x) \rangle \frac{k-1}{\ell!} \prod_{i=0}^{\ell-i-1}(\ell-i)x^{\ell-k},$$

and

$$p^{(k)}(0) = \langle t^k | p(x) \rangle = \langle 1 | p^{(k)}(x) \rangle.$$

The relation (1.10) implies that

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k},$$

and

$$e^{yt} p(x) = p(x + y).$$

Let $S_n(x)$ denote a polynomial with degree $n$. Let $f(t)$ be a delta series and $g(t)$ an invertible series. Then there exists a unique sequence $S_n(x)$ such that

$$\langle g(t)f^k(t) | S_n(x) \rangle = n! \delta_{n,k}$$

for all $n, k \geq 0$. Such a sequence $S_n(x)$ is called a Sheffer sequence for $(g(t), f(t))$ or say that $S_n(t)$ is Sheffer for $(g(t), f(t))$.

The Sheffer sequence for $(1, f(t))$ is called an associated sequence for $f(t)$ or say that $S_n(x)$ is associated to $f(t)$. The Sheffer sequence for $(g(t), t)$ is called an Appell sequence for $g(t)$ or say that $S_n(x)$ is Appell for $g(t)$.

Let $p(x) \in P$. Then

$$\langle f(t) | xp(x) \rangle = \langle \partial_t f(t) | p(x) \rangle = \langle f'(t) | p(x) \rangle$$
and 
\[ (e^{ty} + 1)p(x) = p(y) + p(0). \]

Let \( S_n(x) \) be Sheffer for \((g(t), f(t))\). Then

\[
\begin{align*}
\quad h(t) &= \sum_{k=0}^{\infty} \frac{\langle h(t)|S_k(x) \rangle}{k!} g(t) f^k(t), \quad h(t) \in \mathcal{F}, \\
p(x) &= \sum_{k=0}^{\infty} \frac{\langle g(t)f^k(t)|p(x) \rangle}{k!} S_k(x), \quad p(x) \in \mathcal{P}, \\
\quad e^{y\tilde{f}(t)} &= \sum_{k=0}^{\infty} S_k(y) \frac{t^k}{k!}, \quad y \in \mathbb{C}, \\
f(t)S_n(x) &= nS_{n-1}(x).
\end{align*}
\]

Moreover, we have

\[
\langle f_1(t)f_2(t) \cdots f_m(t)|x^n \rangle = \sum_{i_1, \ldots, i_m} \binom{n}{i_1, \ldots, i_m} \prod_{j=1}^{m} \langle f_j(t)|x^{i_j} \rangle,
\]

where \( f_1(t), f_2(t), \ldots, f_m(t) \in \mathcal{F} \) and the sum is taken over all nonnegative integers \( i_1, \ldots, i_m \) such that \( i_1 + \cdots + i_m = n \).

For details on the above knowledge, please refer to [8, 9, 22, 23, 24, 25, 26] and plenty of references therein.

Let \( p \) be a fixed odd prime number. In what follows, we use \( \mathbb{Z}_p \) to denote the ring of \( p \)-adic rational integers, \( \mathbb{Q} \) the field of rational numbers, \( \mathbb{Q}_p \) the field of \( p \)-adic rational numbers, and \( \mathbb{C}_p \) the completion of algebraic closure of \( \mathbb{Q}_p \). Let \( \mathbb{N} \) be the set of natural numbers and \( \mathbb{N}^* = \{0\} \cup \mathbb{N} \). The \( p \)-adic absolute value is defined by \( |p|_p = p^{-1} \). We also assume that \( |q-1|_p < 1 \) is an indeterminate. Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable functions on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) is defined by Kim (see [1, 2, 3, 4]) as

\[
I_{-1}(f) = \int_{\mathbb{Z}_p} f(a) \, d\mu_{-1}(a) = \lim_{n \to \infty} \sum_{a=0}^{p^n-1} f(a)(-1)^a.
\]

Hence, we have

\[
I_{-1}(f_1) + I_{-1}(f) = 2f(0),
\]

where \( f_1(a) = f(a+1) \). For detailed information on these notions, see [5, 7, 11, 12, 13, 14, 15, 16, 17, 18, 19].

Now let us consider Kim’s \( p \)-adic fermionic integral on \( \mathbb{Z}_p \). For \( |1-w|_p < 1 \),

\[
I_{-1}^w(f) = \int_{\mathbb{Z}_p} w^a f(a) \, d\mu_{-1}(a) = \lim_{n \to \infty} \sum_{a=0}^{p^n-1} w^a f(a)(-1)^a,
\]

where \( I_{-1}^w(f) \) is the extended fermionic \( p \)-adic integral on \( \mathbb{Z}_p \). Letting \( f(x) = e^{t(x+a)} \in UD(\mathbb{Z}_p) \) in this equation yields

\[
\int_{\mathbb{Z}_p} w^a e^{t(x+a)} \, d\mu_{-1}(a) = \frac{2}{we^{t}+1}e^{tx} = \sum_{n=0}^{\infty} E_{n,w}(x) \frac{t^n}{n!},
\]
where \( E_{n,w}(x) \) is the weighted Euler polynomials defined in [20]. Specially, the quantity \( E_{n,w}(0) = E_n,w \) is the weighted Euler numbers. The relation between weighted Euler numbers and weighted Euler polynomials is given by

\[
E_{n,w}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} x^\ell E_{n-\ell,w} = (x + E_w)^n,
\]

with the usual convention of replacing \((E_w)^n\) by \(E_n,w\). Combining this with (1.22) leads to

\[
E_{n,w} = \int_{\mathbb{Z}_p} w^a a^n d\mu_{-1}(a) \quad \text{and} \quad E_{n,w}(x) = \int_{\mathbb{Z}_p} w^a(x + a)^n d\mu_{-1}(a).
\]

In [9, 10], the authors studied applications of the umbral algebra to special functions. In [21], the author gave some new interesting links to works of many mathematicians in the analytic number theory and the modern classical umbral calculus. In [22, 23, 24], the authors established some properties of the umbral calculus for Frobenius-Euler polynomials, Euler polynomials, and other special functions. In [21], the authors investigated some new applications of the umbral calculus associated with \(p\)-adic invariants integral on \(\mathbb{Z}_p\).

In this paper, by the same motivation as in [21] and using the extended fermionic \(p\)-adic integral on \(\mathbb{Z}_p\), we will give some applications of the umbral calculus and, from these applications, derive some identities concerning weighted Euler numbers, weighted Euler polynomials, and weighted Euler polynomials of order \(k\).

2. On the Extended Fermionic \(p\)-Adic Integral on \(\mathbb{Z}_p\)

Now we start out to state and prove our main results.

**Theorem 2.1.** If \( n \geq 0 \), then \( E_{n,w}(x) \) is an Appell sequence for \( g(t) = \frac{we^t + 1}{2} \).

**Proof.** Suppose that \( S_n(x) \) is an Appell sequence for \( g(t) \). Then, by (1.16), we have

\[
\frac{1}{g(t)} x^n = S_n(x) \quad \text{if and only if} \quad x^n = g(t)S_n(x)
\]

for \( n \geq 0 \). Let

\[
g(t) = \frac{we^t + 1}{2} \in \mathcal{F}.
\]

It is clear that \( g(t) \) is an invertible series. By (2.1), we have

\[
\sum_{n=0}^{\infty} E_{n,w}(x) \frac{t^n}{n!} = \frac{1}{g(t)} e^{xt}.
\]

This means that

\[
\frac{1}{g(t)} x^n = E_{n,w}(x).
\]

Making use of (1.16) gives

\[
t E_{n,w}(x) = E'_{n,w}(x) = n E_{n-1,w}(x).
\]

Combining (2.3) and (2.4) results in Theorem 2.1. \( \square \)

**Theorem 2.2.** Let \( g(t) = \frac{we^t + 1}{2} \in \mathcal{F} \). Then for \( n \geq 0 \)

\[
E_{n+1,w}(x) = x - \frac{g'(t)}{g(t)} E_{n,w}(x).
\]
Proof. By (1.24), we derive that
\[ \sum_{n=1}^{\infty} E_{n,w}(x) \frac{t^n}{n!} = \frac{xg(t)e^{xt} - g'(t)e^{xt}}{g(t)^2} = \sum_{n=0}^{\infty} \left[ \frac{1}{g(t)} x^n - \frac{g'(t)}{g(t)} \frac{1}{g(t)} x^n \right] \frac{t^n}{n!}. \]
Considering (2.3) and the above equality, we discover
\[ E_{n+1,w}(x) = xE_{n,w}(x) - \frac{g'(t)}{g(t)} E_{n,w}(x). \]
Theorem 2.2 is thus proved. \(\square\)

Theorem 2.3. For \(n \geq 0\),
\[ (2.6) \quad E_{n+1,w}(x) = \left[ x - \frac{g'(t)}{g(t)} \right] E_{n,w}(x), \]
where \(g'(t) = \frac{d}{dt} g(t)\).
Proof. From (1.24), it is easy to see that
\[ \sum_{n=0}^{\infty} \left[ wE_{n,w}(x+1) + E_{n,w}(x) \right] \frac{t^n}{n!} = \sum_{n=0}^{\infty} (2x^n) \frac{t^n}{n!}. \]
Comparing the coefficients on the both sides, we find
\[ (2.7) \quad wE_{n,w}(x+1) + E_{n,w}(x) = 2x^n. \]
From Theorem 2.2, it follows that
\[ (we^t + 1)E_{n+1,w}(x) = (we^t + 1)xE_{n,w}(x) - we^t E_{n,w}(x). \]
Consequently, we have
\[ wE_{n+1,w}(x+1) + E_{n+1,w}(x) = w(x+1)E_{n,w}(x+1) + xE_{n,w}(x) - wE_{n,w}(x+1). \]
Combining this with (2.7) and (2.8), we acquire the required conclusions. \(\square\)

Corollary 2.3.1. For \(n \geq 0\), we have
\[ wE_{n+1}(x+1) + E_{n+1,w}(x) = 2x^{n+1}. \]

Theorem 2.4. For \(n \geq 0\), we have
\[ (2.9) \quad \langle f(t)|p(x) \rangle = \int_{\mathbb{Z}_p} w^a p(a) \, d\mu_{-1}(a), \]
\[ (2.10) \quad \left\langle \frac{2}{we^t + 1} | p(x) \right\rangle = \int_{\mathbb{Z}_p} w^a p(a) \, d\mu_{-1}(a), \]
\[ (2.11) \quad E_{n,w} = \left\langle \int_{\mathbb{Z}_p} w^a e^{at} \, d\mu_{-1}(a) \left| x^n \right\rangle. \]
Proof. Let us consider the linear functional $f(t)$ satisfying
\begin{equation}
\langle f(t) | p(x) \rangle = \int_{\mathbb{Z}} w^a p(a) \, d \mu_{-1}(a)
\end{equation}
for all polynomials $p(x)$. Then we readily see that
\begin{equation}
f(t) = \sum_{n=0}^{\infty} \frac{\langle f(t) | x^n \rangle}{n!} t^n = \sum_{n=1}^{\infty} \left[ \int_{\mathbb{Z}} w^a a^n \, d \mu_{-1}(a) \right] \frac{t^n}{n!} = \int_{\mathbb{Z}} w^a e^{at} \, d \mu_{-1}(a).
\end{equation}
Thus, we have
\begin{equation}
f(t) = \int_{\mathbb{Z}} w^a e^{at} \, d \mu_{-1}(a) = \frac{2}{we^t + 1}.
\end{equation}
Therefore, by (2.12) and (2.13), we arrive at the theorem. □

**Theorem 2.5.** For $p(x) \in \mathcal{P}$, we have
\begin{equation}
\int_{\mathbb{Z}} w^a p(x + a) \, d \mu_{-1}(a) = \int_{\mathbb{Z}} w^a e^{at} \, d \mu_{-1}(a)p(x) = \frac{2}{we^t + 1} p(x).
\end{equation}
Equivalently,
\begin{equation}
E_{n,w}(x) = \int_{\mathbb{Z}} w^a e^{at} \, d \mu_{-1}(a)x^n = \frac{2}{we^t + 1} x^n.
\end{equation}

Proof. From (1.24) and (2.11), we see that
\begin{equation}
\sum_{n=0}^{\infty} \left[ \int_{\mathbb{Z}} w^a (x + a)^n \, d \mu_{-1}(a) \right] \frac{t^n}{n!} = \int_{\mathbb{Z}} w^a e^{(x+a)t} \, d \mu_{-1}(a)
\end{equation}
\begin{equation}
= \sum_{n=0}^{\infty} \left[ \int_{\mathbb{Z}} w^a e^{at} \, d \mu_{-1}(a)x^n \right] \frac{t^n}{n!}.
\end{equation}
By this equality and (1.24), we see that for $n \in \mathbb{N}^*$
\begin{equation}
E_{n,w}(x) = \int_{\mathbb{Z}} (x + a)^n \, d \mu_{-1}(a) = \int_{\mathbb{Z}} w^a e^{at} \, d \mu_{-1}(a)x^n.
\end{equation}
As a result, we obtain the theorem. □

**Theorem 2.6.** For $p(x) \in \mathcal{P}$ and $k \in \mathbb{N}$, we have
\begin{equation}
\int_{\mathbb{Z}} \cdots \int_{\mathbb{Z}} w^{a_1 + \cdots + a_k} p(a_1 + \cdots + a_k + x) \prod_{j=1}^{k} d \mu_{-1}(a_j) = \left( \frac{2}{we^t + 1} \right)^k p(x).
\end{equation}
In particular,
\begin{equation}
E_{n,w}^{(k)}(x) = \left( \frac{2}{we^t + 1} \right)^k x^n
\end{equation}
\begin{equation}
= x^n \int_{\mathbb{Z}} \cdots \int_{\mathbb{Z}} w^{a_1 + \cdots + a_k} e^{(a_1 + \cdots + a_k)t} \prod_{j=1}^{k} d \mu_{-1}(a_j).
\end{equation}
Consequently, 
\[ E^{(k)}_{n,w}(x) \sim \left( \frac{we^t + 1}{2} \right)^k t. \]

**Proof.** For \(|1 - w|_p < 1\), we consider the weighted Euler polynomials of order \(k\).

\[ \int_{Z_p} \cdots \int_{Z_p} w^a_1 + \cdots + a_k e^{(a_1 + \cdots + a_k + x)t} \prod_{j=1}^k d \mu_{-1}(a_j) \]

\[ = \left( \frac{2}{we^t + 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} E^{(k)}_{n,w}(x) \frac{t^n}{n!}. \]

where \(E^{(k)}_{n,w}(0) = E^{(k)}_{n,w}\) are the weighted Euler numbers of order \(k\). Accordingly, 

\[ \int_{Z_p} \cdots \int_{Z_p} w^a_1 + \cdots + a_k (a_1 + \cdots + a_k)^n \prod_{j=1}^k d \mu_{-1}(a_j) \]

\[ = \sum_{i_1 + \cdots + i_k = n} \binom{n}{i_1, \ldots, i_m} \prod_{j=1}^k \int_{Z_p} w^{a_j} i_j^i d \mu_{-1}(a_j) \]

\[ = \sum_{i_1 + \cdots + i_k = n} \binom{n}{i_1, \ldots, i_m} \prod_{j=1}^k E_{i_j} = E^{(k)}_{n,w}. \]

Thanks to (2.19) and (2.20), we have 

\[ E^{(k)}_{n,w}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} x^{\ell} E^{(k)}_{n-\ell,w}. \]

From (2.20) and (2.21), we notice that \(E^{(k)}_{n,w}(x)\) is a monic polynomial of degree \(n\) with coefficients in \(\mathbb{Q}\). For \(k \in \mathbb{N}\), let us assume that

\[ g^{(k)}(t) = \left[ \int_{Z_p} \cdots \int_{Z_p} w^a_1 + \cdots + a_k e^{(a_1 + \cdots + a_k + x)t} \prod_{j=1}^k d \mu_{-1}(a_j) \right]^{-1} \]

\[ = \left( \frac{we^t + 1}{2} \right)^k. \]

From this, we see that \(g^{(k)}(t)\) is an invertible series. Due to (2.19) and (2.22), we readily derive that

\[ \frac{1}{g^{(k)}(t)} e^{xt} = \int_{Z_p} \cdots \int_{Z_p} w^a_1 + \cdots + a_k e^{(a_1 + \cdots + a_k + x)t} \prod_{j=1}^k d \mu_{-1}(a_j) \]

\[ = \sum_{n=0}^{\infty} E^{(k)}_{n,w}(x) \frac{t^n}{n!}. \]

Taking account of this and

\[ tE^{(k)}_{n,w}(x) = nE^{(k)}_{n-1,w}(x) \]
yields that $E^{(k)}_{n,w}(x)$ is an Appell sequence for $g^{(k)}(t)$. Theorem 2.6 is proved. \(\square\)

**Theorem 2.7.** For $p(x) \in \mathcal{P}$, we have

$$
(2.24) \quad \left\langle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1 + \cdots + a_k} e^{(a_1 + \cdots + a_k)t} \prod_{j=1}^{k} d\mu_{-1}(a_j) \bigg| p(x) \right\rangle = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1 + \cdots + a_k} p(a_1 + \cdots + a_k) \prod_{j=1}^{k} d\mu_{-1}(a_j).
$$

Furthermore,

$$
\left\langle \left( \frac{2}{we^t + 1} \right)^k \bigg| p(x) \right\rangle = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1 + \cdots + a_k} p(a_1 + \cdots + a_k) \prod_{j=1}^{k} d\mu_{-1}(a_j),
$$

equivalently,

$$
E^{(k)}_{n,w} = \left\langle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1 + \cdots + a_k} e^{(a_1 + \cdots + a_k)t} \prod_{j=1}^{k} d\mu_{-1}(a_j) \bigg| x^n \right\rangle.
$$

**Proof.** Let us take the linear functional $f^{(k)}(t)$ fulfilling

$$
(2.25) \quad \left\langle f^{(k)}(t) \big| p(x) \right\rangle = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1 + \cdots + a_k} p(a_1 + \cdots + a_k) \prod_{j=1}^{k} d\mu_{-1}(a_j)
$$

for all polynomials $p(x)$. Then

$$
f^{(k)}(t) = \sum_{n=0}^{\infty} \frac{\left\langle f^{(k)}(t) \bigg| x^n \right\rangle}{n!} t^n = \sum_{n=0}^{\infty} \left[ \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1 + \cdots + a_k} (a_1 + \cdots + a_k)^n \prod_{j=1}^{k} d\mu_{-1}(a_j) \right] \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1 + \cdots + a_k} e^{(a_1 + \cdots + a_k)t} \prod_{j=1}^{k} d\mu_{-1}(a_j) = \left( \frac{2}{we^t + 1} \right)^k.
$$

Therefore, we procure Theorem 2.7. \(\square\)

**Remark 2.1.** From (1.18), we notice that

$$
\left\langle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1 + \cdots + a_k} e^{(a_1 + \cdots + a_k)t} \prod_{j=1}^{k} d\mu_{-1}(a_j) \bigg| x^n \right\rangle = \sum_{i_1 + \cdots + i_k = n} \binom{n}{i_1, \ldots, i_k} \prod_{\ell=1}^{k} \left\langle \int_{\mathbb{Z}_p} w^{a_\ell} e^{a_\ell t} d\mu_{-1}(a_\ell) \bigg| x^{i_\ell} \right\rangle.
$$

Therefore, we have

$$
E^{(k)}_{n,w} = \sum_{i_1 + \cdots + i_k = n} \binom{n}{i_1, \ldots, i_k} E_{i_1,w} \cdots E_{i_k,w}.
$$
Remark 2.2. Our applications to the weighted Euler polynomials, the weighted Euler numbers, and the weighted Euler polynomials of order \( k \) seem to be interesting, because evaluating at \( w = 1 \) leads to Euler polynomials and Euler polynomials of order \( k \) defined respectively by

\[
\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{xt} \quad \text{and} \quad \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!} = \left( \frac{2}{e^t + 1} \right)^k e^{xt}.
\]

It is also well known that

\[
E_n(x) = \int_{\mathbb{Z}_p} (x + a)^n \ d\mu_{-1}(a)
\]

and

\[
E_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1 + \cdots + a_k + x)^n \prod_{j=1}^{k} \ d\mu_{-1}(a_j).
\]

See [5, 6, 11, 13, 16] and related references therein.

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