Short comment about the lattice gluon propagator at vanishing momentum

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Abstract

We argue that all evidences point towards a finite non-vanishing zero momentum renormalised lattice gluon propagator in the infinite volume limit. We argue that different simulations with different lattice setups end-up with fairly compatible results for the gluon propagator at zero momentum, with different positive slopes as a function of the inverse volume.

1 Introduction

The lattice gluon propagator at small or vanishing momentum in the Landau gauge has recently been frequently addressed as it is related to several studies in the small momentum regime using non-lattice methods. It is often advocated that the zero momentum gluon propagator should vanish, while we have \textsuperscript{11} shown a Slavnov-Taylor based argument in favor of a divergence when the momentum goes to zero. Notwithstanding these extraneous arguments we observe that the

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genuine lattice data point towards a finite non-vanishing gluon propagator at zero momentum in the infinite volume limit. Our second claim is that, once a well defined renormalisation procedure has been defined, the different available results are close enough, despite several systematic effects, to suggest an agreement.

Our aim in this note is simply to gather the arguments in favor of this claim, without discussing the relationship with any non-lattice claim. We will not present any new result but only quote published results and add a reanalysis of our old data. We concentrate on $SU(3)$ pure Yang-Mills theory in the Landau gauge.

There are two approaches to the problem.

• One is to simply compute the gluon propagator at zero momentum and perform a well defined renormalisation. It is well known that the result is a finite non-vanishing value. But it might happen that the vanishing only happens in the infinite volume limit. Therefore an extrapolation to infinite volume is needed.

• The second approach uses a set of small non-vanishing momenta and tries a fit of the propagator in terms of a power law $(p^2)^{\alpha G} - 1$, or equivalently of the dressing function in terms of $(p^2)^{\alpha G}$. The fit gives some range of value for $\alpha G$. The value $\alpha G = 1$ – which corresponds to a non-vanishing of the gluon propagator at vanishing momentum – has obviously zero measure and it is thus impossible to be assertive with this second method. It is nevertheless important to check that $\alpha G = 1$ is compatible with the result and to check that the gluon propagator at vanishing momentum is in continuity with the result at small non-vanishing momenta.

2 Definitions and notations

In Landau gauge the gluon propagator writes

$$G_{\mu \nu}(p) = (\delta_{\mu \nu} - \frac{p_{\mu} p_{\nu}}{p^2}) G^{(2)}(p^2),$$

which implies

$$G^{(2)}(p^2) = \frac{1}{3} \sum_{\mu} G_{\mu \mu}(p) \quad \text{for} \quad p \neq 0$$

$$G^{(2)}(0) = \frac{1}{4} \sum_{\mu} G_{\mu \mu}(0).$$

The factor 1/4 for zero momentum is due to an additional degree of freedom (clearly the orthogonality to the momentum in Landau gauge does not provide
any constraint for $p_\mu = 0$) related to the fact that the Landau gauge fixing algorithm keeps unconstrained the global gauge transformations $^3$.

In order to be able to compare the results from different gauge actions and different lattice spacings one needs to renormalise the gluon propagator. The standard method on the lattice is the Momentum substraction scheme (MOM) which amounts to define the renormalised propagator $G^{(2)}_R$ from the bare one $G^{(2)}$ according to

$$G^{(2)}_R(p, \mu) = Z_3(\mu, a) G^{(2)}_R(p, \mu)$$

where the renormalisation condition is

$$G^{(2)}_R(\mu, \mu) \equiv \frac{1}{\mu^2}, \quad \text{whence} \quad Z_3(\mu, a) = \mu^2 G^{(2)}(\mu, a)$$

where $a$ is the lattice spacing, i.e. the ultraviolet cut-off.

This renormalisation can thus be done non perturbatively from lattice data provided that $\mu$ is in the available range for the given lattice spacing $a$. If this is not the case it is necessary to match the $Z_3$'s with different lattice spacings. To illustrate this let us give an example: if we take $\mu = 4 \text{ GeV}$ it is not possible to compute directly $Z_3$ for the Wilson gauge action with $\beta = 6.0$ ($a^{-1} = 1.97$ GeV) $^4$. We will thus use the results at $\beta = 6.4$ ($a^{-1} = 3.58$ GeV). We then need to compute $Z_3(\mu, 6.0)/Z_3(\mu, 6.4)$. This ratio is independent of $\mu$ at leading order. It can thus be computed non perturbatively for momenta in which both lattice spacings provide data. An analytic approach is to rely on the one loop perturbative formula

$$\frac{Z_3(\mu, a')}{Z_3(\mu, a)} = \left(\frac{\beta(\mu)}{\beta(a)}\right)^{13/22},$$

which is valid for small enough lattice spacings (in the perturbative regime).

### 3 The gluon propagator at vanishing momentum

The Adelaide group has performed a systematic study $^2$ of the gluon propagator in the infinite volume limit. They use the mean-field (tadpole) improved version of the tree-level, $O(a^2)$ Symanzik improved gauge action. They choose a MOM renormalisation at $\mu = 4 \text{ GeV}$ i.e. $G^{(2)}_R(4 \text{ GeV}, \mu = 4 \text{ GeV}) = 1/(4 \text{ GeV})^2$. They fit the volume dependence of the zero momentum gluon propagator on several lattice spacings and lattice volumes up to a volume of 2000 fm$^4$, with always a

$^3$Notice that this theoretically justified 3/4 factor is numerically confirmed as it ensures the continuity of the gluon propagator at $p^2 \to 0$ which will be discussed later on.

$^4$It is advisable to keep $p < (\pi/2) a^{-1}$. 

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spatial cubic lattice and a length in the time direction twice the spatial length. Their fitting formula is

$$G_R^{(2)}(0, \mu = 4 \text{ GeV}) = G_{R\infty}^{(2)}(0, \mu = 4 \text{ GeV}) + \frac{c}{V}, \quad (6)$$

and gives

$$G_{R\infty}^{(2)}(0, \mu = 4 \text{ GeV}) = 7.95 \pm 0.13 \text{ GeV}^{-2}, \quad c = 245 \pm 22 \text{ fm}^4 \text{ GeV}^{-2}. \quad (7)$$

This result clearly indicates a finite non-vanishing $G_R^{(2)}(0, \mu)$. It is strange that nobody objects to this published result but that, nevertheless, one repeatedly reads that the zero momentum gluon propagator vanishes.

| $\beta$ | $V$ in units of $a$ | bare propagator $G^{(2)}(p, a)$ | $1/L$ in GeV |
|---------|-------------------|--------------------------|-------------|
| 5.7     | $16^4$            | 16.81 ± 0.13             | 0.0672      |
| 5.7     | $24^4$            | 15.06 ± 0.29             | 0.0448      |
| 5.8     | $16^4$            | 19.12 ± 0.16             | 0.0841      |
| 5.9     | $24^4$            | 18.12 ± 0.30             | 0.0685      |
| 6.0     | $32^4$            | 17.70 ± 0.59             | 0.0615      |
| 6.0     | $24^4$            | 19.67 ± 0.35             | 0.0821      |

Table 1: Physical lattice sizes and raw data for the gluon propagator at zero momentum $G^{(2)}(p, a)$ from our old data.

This is why, waiting for a systematic and extensive reanalysis, we have simply dug out our old results for the gluon propagator which have been obtained from simulations with the Wilson pure gauge action on hypercubic lattices [3, 4]. Table 1 lists the normalized raw data of the gluon propagator at zero momentum for our largest physical volumes (some of these data have never been published). No rescaling, perturbative (Eq. 5) or non-perturbative, has yet been applied to these data. Our volumes are not very large as this was not the aim of our simulations, and we do not claim our study to be an improvement over ref. [2] but simply an independent check. Using the same renormalisation as ref. [2] we find

$$G_{R\infty}^{(2)}(0, \mu = 4 \text{ GeV}) = 9.1 \pm 0.2 \pm 0.2 \text{ GeV}^{-2}, \quad c = 140 \pm 30 \pm 40 \text{ fm}^4 \text{ GeV}^{-2}. \quad (8)$$

where the first error is statistical and the second is a systematic one estimated from different choices of the fitting points.

More recently ref. [5] provides additional information on the same issue. In their table 2 the authors report fits of the zero momentum gluon propagator as a function of $1/V$, using only data at $\beta = 6.0$ with Wilson gauge action obtained on very anisotropic lattices [5]. The results are given in lattice units and concern bare propagators.

\footnote{The time length is typically 16 times the spatial one}
We shall assume, as has been done up to now, that the volume dependence is polynomial in $1/V$ for large volumes. The very asymmetric shape is meant to provide very low values of the momentum; it is interesting to check whether the zero momentum propagator depends on the geometry. We therefore convert the authors’ fit of the 0-momentum propagator to physical units and perform a MOM renormalisation at 4 GeV for which we use $a^{-1}(\beta = 6.0) = 1.97\,\text{GeV}$ and, from our non-perturbative fits:

$$Z_3(4\,\text{GeV}, \beta = 6.0)) = 1.648,$$

and we get

$$G^{(2)}_R(0, \mu = 4\,\text{GeV}) = 11.3\,\text{GeV}^{-2} \text{ and } 10.9\,\text{GeV}^{-2} ,$$

$$c = 47\,\text{fm}^4\,\text{GeV}^{-2} \text{ and } 65\,\text{fm}^4\,\text{GeV}^{-2}.$$

where the two results correspond to a linear/quadratic fit in $1/V$. We do not know the statistical errors.

Concerning $G^{(2)}_R(0, \mu = 4\,\text{GeV})$ the three results are in the same ballpark and it may be conjectured that the systematic errors are not all taken into account: $O(a)$ effects, effect of the shape, insufficiently large volumes (for the second and third lines), uncertainty in the estimate of the lattice spacing in physical units, etc. Altogether it seems that, not only there is a clear indication in favor of a finite non vanishing zero momentum gluon propagator, but that different simulations agree on the value. Of course a more extensive study is necessary.

Concerning the slope $c$ the numbers clearly differ, they only agree in order of magnitude and are all positive. We expect that the slope is much more sensitive to systematic effects such as the shape.

We turn now to the second approach, namely a fit of the $p^2$ dependence of the propagator at small momenta. We first claim that the gluon propagator is continuous and smooth at $p = 0$. This has been observed in several references (see for instance figure 17 in [2]). This can also be seen in figure 2 in [7].

Concerning $G^{(2)}_R(0, \mu = 4\,\text{GeV})$ also use non polynomial fits in $1/V$ which can lead to vanishing or infinite zero momentum propagators. But they have themselves noticed that this destroys the smoothness.

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| reference | $G^{(2)}_R(0, \mu = 4\,\text{GeV})$ in GeV$^{-2}$ | $c$ in GeV$^{-2}\,\text{fm}^4$ | max vol in fm$^4$ |
|-----------|-------------------------------------------------|-----------------|------------------|
| [2]       | 7.95 $\pm$ 0.13                                 | 245 $\pm$ 22    | 2000             |
| table [1] | 9.1 $\pm$ 0.3                                   | 140 $\pm$ 50    | 90               |
| [2]       | 10.9 - 11.3                                     | 47 - 65         | 110              |

Table 2: Summary of the infinite volume zero momentum propagator and its slope in terms of $1/V$ for three different simulations. The largest volume used in the fit is also indicated. The statistical error is not quoted in ref [5].
The latter paper also compares the gluon propagator with periodic or twisted boundary conditions and concludes that the twisted propagator is smaller than the periodic one but that the difference vanishes in the large volume limit. Let us now comment on the fit as a power law \((G^{(2)}(p) \propto (p^2)^{\alpha_G-1})\) which necessarily discards the zero-momentum. In section 3.1 of ref. [1] we have shown with similar fitting formulae that \(\alpha_G\) is compatible with 1 on the examples of \(SU(2)\) and \(SU(3)\). But we have experienced instabilities and we do not know of any convincing results obtained with this method [7]. This instability may be due to the fact that, if such a power law applies in the small momentum limit, it can only be isolated at very small momenta which have not yet been reached.

4 Conclusions

The renormalised gluon propagator at zero momentum converges, in the infinite volume limit, towards a non vanishing finite value [2] if one uses a volume dependence which is polynomial in \(1/V\) independently of the boundary conditions [7]. We have shown that different studies with different gauge actions, different parameters and different shapes of the volume agree rather well on the value of the renormalised gluon propagator while the slopes in \(1/V\) agree only in sign and order of magnitude.

No discontinuity of the gluon propagator is seen when the momentum goes to zero. It results that the infrared exponent \(\alpha_G\) (often named \(2\kappa\)) must be equal to 1. We have stressed the instability of the fits of the propagator without the point at \(p = 0\) assuming a power law dependence: different fitting functions, which are equivalent in the infrared limit, give incompatible results. If this is duly taken into account in the systematic errors the value \(\alpha_G = 1\) lies within the error bars in agreement with the claim about a non vanishing finite gluon propagator at zero momentum.

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7We do not understand the fits of [6] and we obviously disagree with their conclusion that the zero momentum propagator vanishes.
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