Quantum violation of variants of LGIs upto algebraic maximum for qubit system

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In 1985, Leggett and Garg formulated a class of inequalities for testing the compatibility between macrorealism and quantum mechanics. In this paper, we point out that based on the same assumptions of macrorealism that are used in the derivation of Leggett-Garg inequalities (LGIs), there is a scope of formulating another class of inequalities different from standard LGIs. By considering the three-time measurement scenario in a dichotomic system, we first propose an interesting variant of standard LGIs and show that its quantum violation is larger than the standard LGI. By extending this formulation to n-time measurement scenario, we found that the quantum violations of variants of LGIs for a qubit system increase with n, and for a sufficiently large n algebraic maximum can be reached. Further, we compare the quantum violations of our formulated LGIs with the standard LGIs and no-signaling in time formulation of macrorealism.

I. INTRODUCTION

Since the inception of quantum mechanics (QM), it remains a debatable question how our everyday world view of macrorealism can be reconciled with the quantum formalism. Historically, this question was first pointed out by Schrödinger [1] through his famous cat experiment. Since then, quite a number of attempts have been made to pose the appropriate questions relevant to this issue and to answer that questions. One effective approach to encounter this issue is to experimentally realize the quantum coherence of Schrödinger cat-like states of large objects [2]. Another approach within the formalism of QM is the decoherence program [3]. It explains how interaction between quantum systems and environment leads to classical behavior, but does not by itself provide the desired ‘cut’ (à la Heisenberg [8]). It is also argued that even if the decoherence effect is made negligible, the quantum behavior can be disappeared by the effect of coarse-graining of measurements [4]. Proposal has also been put forwarded [5] to modify the dynamics of standard formalism of QM allowing an unified description of microscopic and macroscopic systems.

However, the above mentioned attempts do not exactly address the fundamental question whether macrorealism is, in principle, compatible with the formalism of QM. Macrorealism is a classical world view that asserts that the properties of macro-objects exist independently and irrespective of ones observation. Motivated by the Bell’s theorem [9], in 1985, Leggett and Garg [9] formulated a class of inequalities based on the notions of macrorealism, which provides an elegant scheme for experimentally testing the compatibility between the macrorealism and QM.

To be more specific, the notion of macrorealism consists of two main assumptions [9,10] are the following:

- **Macrorealism per se (MRps):** If a macroscopic system has two or more macroscopically distinguishable ontic states available to it, then the system remains in one of those states at all instant of time.

- **Non-invasive measurement (NIM):** The definite ontic state of the macroscopic system is determined without affecting the state itself or its possible subsequent dynamics.

It is reasonable to assume that the systems in our everyday world, in principle, obeys the aforementioned assumptions of a macrorealistic theory. Based on these assumptions, the standard Leggett-Garg inequalities (LGIs) are derived. Such inequalities can be shown to be violated in certain circumstances, thereby implying that either or both the assumptions of MRps and NIM is not compatible with all the quantum statistics. In recent times, a flurry of theoretical studies on macrorealism and LGIs have been reported [12–23] and a number of experiments have been performed by using various systems [24–29].

Let us encapsulate the simplest LG scenario. Consider that the measurement of a dichotomic observable $M$ is performed at three different times $t_1$, $t_2$ and $t_3$ ($t_3 \geq t_2 \geq t_1$). In Heisenberg picture, this in turn implies the sequential measurement of the observables $\hat{M}_1$, $\hat{M}_2$ and $\hat{M}_3$ corresponding to $t_1$, $t_2$ and $t_3$ respectively. From the assumption of MRps and NIM, one can derive the standard LGI is given by

$$K_3 = \langle \hat{M}_1 \hat{M}_2 \rangle + \langle \hat{M}_2 \hat{M}_3 \rangle - \langle \hat{M}_1 \hat{M}_3 \rangle \leq 1 \tag{1}$$

Here $\langle M_1 M_2 \rangle = \sum_{m_1,m_2=\pm 1} m_1 m_2 P(M_1^m_1, M_2^m_2)$ and similarly for other temporal correlation terms. By relabeling the measurement outcomes of each $M_i$ as $M_i = -M_i$ with $i = 1, 2$, and 3, three more standard LGIs can be obtained.

Instead of three times, if the measurement of $M$ is performed $n$ times, then the standard LGI for the $n$-measurement LG strings can be written as

$$K_n = \langle \hat{M}_1 \hat{M}_2 \rangle + \ldots + \langle \hat{M}_{n-1} \hat{M}_n \rangle - \langle \hat{M}_1 \hat{M}_n \rangle \tag{2}$$

The inequality (2) is bounded as [23] follows. If $n$ is odd, $-n \leq K_n \leq n - 2$ for $n \geq 3$ and if $n$ is even, $-(n-2) \leq K_n \leq n - 2$ for $n \geq 4$. For $n = 3$, one simply recovers inequality (1).

For a two-level system, the maximum quantum value of $K_n$ is $(K_n)^{\max}_Q = n \cos \frac{\pi}{n}$. For $n = 3$, $(K_3)^{\max}_Q = 3/2$. Thus for a three-time standard LG scenario involving a dichotomic observable, the temporal Tsirelson bound of...
$K_3$ is $3/2$. It is proved \[12\] that this bound is irrespective of the system size.

Within the standard framework of QM, the maximum violation of CHSH inequality \[6\] is restricted by the Tsirelson bound\[7\], which is significantly less than the algebraic maximum of the inequality. The algebraic maximum may be achieved in post-quantum theory but not in QM. LGIs are often considered to be the temporal analog of Bell’s inequality. However, it has been shown \[19\] that for a degenerate dichotomic observables in a qutrit system, the quantum value of $K_3$ goes up to 2.21 and can even reach to algebraic maximum 3 in the asymptotic limit of the system size. Such amount of violation is achieved by invoking a degeneracy breaking projective measurement which they termed as von Neumann rule. Recently, two of us have argued \[31\] that such a violation of temporal Tsirelson bound has no relevance to the usual violation of LGIs.

The purpose of the present paper is to provide improved quantum violation of macrorealism for qubit system. We argue that by keeping the assumptions of macrorealism intact, there is scope for formulating inequalities different from the standard LGIs. We note here an important observation that due to the sequential nature of the measurement, the LG scenario is flexible than CHSH one. Such flexibility allows us to formulate new variants of standard LGIs. For the simplest case of three-time measurement scenario, we first formulate an interesting variant of LGI and show that our proposed inequality provides considerably larger quantum violation compared to the standard LGIs. We then formulate more variants of standard LGIs by increasing number of measurements $n$ and show that the quantum violation increases with $n$. For sufficiently large $n$, the quantum values of variants of LGIs reach its algebraic maximum, even for qubit system. Such variants of LGIs thus provide improved test of macrorealism than standard LGIs. Further, in terms of no-disturbance (coined as no-signaling in time in LG scenario), we discuss how the variants of LGIs are conceptually elegant and can be considered better candidates for experimentally testing the macrorealism compared to standard LGIs.

This paper is organized as follows. In Sec.II, we propose variant of LGI for three-time measurement scenario and demonstrate that it provide larger quantum violation compared to standard LGI. By increasing the number of measurements ($n$), in Sec.III, we formulate two more variants of LGIs. We show that for a qubit system, the quantum violation of our variants of LGIs increase with $n$ and can even reach algebraic maximum for large $n$ limit. In Sec.IV, we compare variant of LGIs with standard LGIs and no-signaling in time conditions. We summarize our results in Sec.V.

II. VARIANTS OF LGIS IN THREE-TIME MEASUREMENT SCENARIO

We start by noting that the standard LGIs is a particular class of inequalities but is not unique one. The flexibility of LG scenario allows us to formulate variants of LGIs different from the standard LGI given by Eq. (1). We ensure that the assumptions of MRps and NIM used in the derivation of standard LGI remains the same.

Let us again consider the three-time LG scenario involving measurement of dichotomic observables $M_1, M_2$ and $M_3$ in sequence. Now, instead of three two-time correlation functions used in Eq. (1), we consider a three-time correlation function $\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle$, a two-time function $\langle \hat{M}_1 \hat{M}_2 \rangle$ and finally $\langle \hat{M}_3 \rangle$. Using them, we propose an inequality is given by

$$K_3^3 = \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle + \langle \hat{M}_i \hat{M}_j \rangle - \langle \hat{M}_k \rangle \leq 1 \quad (3)$$

where $i, j = 1, 2, 3$ with $j > i$. We call those inequalities as variant of LGIs. It is crucial to note again that, the assumptions of MRps and NIM remain same as in the derivation of standard LGIs.

The inequalities (3) are violated by QM. In order to showing this, we take one inequality by choosing $i, j$ and $k$ are $1, 2$ and $3$ respectively, and consider the qubit state is given by

$$|\psi(t_1)\rangle = \cos \theta |0\rangle + \exp(-i\phi) \sin \theta |1\rangle \quad (4)$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The measurement observable at initial time $t_1$ is taken to be Pauli observable $\sigma_x$. The unitary evolution is given by $U_{ij} = \exp^{-i\omega(t_j-t_i)\sigma_x}$ and $\omega$ is coupling constant. For simplicity, we consider $\tau = |t_{i+1} - t_i|$ and $g = \omega \tau$.

The quantum mechanical expression of $K_3^3$ is given by

$$(K_3^3)_Q = \cos 2g(4 \cos^2 g \cos 2\theta) + \sin 4g \sin 2\theta \sin \phi - 2 \cos^2 g \cos 2\theta \quad (5)$$

which is state-dependent in contrast to the quantum value of standard LGI.

To compare with the standard LGIs, let us write the quantum expression of $K_3$ is given by

$$(K_3)_Q = 2 \cos 2g - \cos 4g \quad (6)$$

which is independent of the state.

If the values of the relevant parameters are taken as $g = 1.72$, $\theta = 2.04$ and $\phi = \pi/2$, the quantum value of $K_3^3$ is 1.93, thereby violating the inequality (3). The maximum quantum value $(K_3^3)_{max}$ can be shown to be 2 for different coupling constants in between the evolutions. For simplicity, here we take same coupling constant $g$. The quantum value of $K_3^3$ is then larger than $(K_3)_Q^{max} = 3/2$. The expressions $(K_3)_Q$ and $(K_3^3)_Q$ are plotted in Fig. (1).

Thus, if the larger violation of an inequality is considered to be an indicator of more non-classicality, then the variant of LGI captures the notion of macrorealism better than the standard LGIs.
But, both the $n$ belongs to the other class of inequalities given by (8).

If number of measurements is further increased, one finds more variants of LGIs. Similar to the earlier case three more inequalities can be obtained. If number of measurements is further increased, one finds more variants of LGIs.

Interestingly, for $n = 4$, there can be another variant of LGI can be proposed as

$$\hat{L}_4^3 = \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \hat{M}_4 \rangle + \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle - \langle \hat{M}_4 \rangle \leq 1 \quad (8)$$

Similar to the earlier case three more inequalities can be obtained. If number of measurements is further increased, one finds more variants of LGIs.

Now, by generalizing the above formulation for $n$-time measurement scenario we propose the following two inequalities

$$K_n^3 = \langle \hat{M}_1 \hat{M}_2 \cdots \hat{M}_n \rangle + \langle \hat{M}_1 \hat{M}_2 \cdots \hat{M}_{n-1} \rangle - \langle \hat{M}_n \rangle \leq 1 \quad (9)$$

and

$$\hat{L}_n^3 = \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \hat{M}_4 \hat{M}_5 \hat{M}_6 \cdots \hat{M}_{n-1} \rangle + \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \hat{M}_4 \hat{M}_5 \hat{M}_6 \rangle - \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \hat{M}_4 \hat{M}_5 \hat{M}_6 \hat{M}_7 \rangle \leq 1 \quad (10)$$

where $\langle \hat{M}_1 \cdots \hat{M}_n \rangle = \sum_{m_1, \ldots, m_n} P(M_1^{m_1}, \ldots, M_n^{m_n})$ and similarly for other correlation. While inequality (9) belongs to the class of [3] and [4], the inequality (10) belongs to the other class of inequalities given by [7]. But, both the $n$-time inequalities are derived from the same assumptions of macrorealism.

Next, we examine the quantum violation of inequalities [7] and [8] for the state given by Eq. (4). The quantum mechanical expressions of $K_n^3$ and $L_n^3$ are respectively given by

$$K_n^3_Q = \frac{1}{2} \left( 1 + \cos 4g + 8 \cos 2g \sin^2 2g \cos 2\theta - 2 \sin 6g \sin \theta \sin \phi \right) \quad (11)$$

$$L_n^3_Q = 2 \cos^2 g \cos 2g \cos 2\theta - \cos 6g \sin 4g \sin 2\theta \sin \phi \quad (12)$$

The value of $(K_n^3)_Q$ is 2.12 at $g = 1.24$, $\theta = 1.90$ and $\phi = \pi/2$ and of $(L_n^3)_Q$ is 2.03 at $g = 0.42$, $\theta = 0.21$, and $\phi = \pi/2$. However, the above values of $(K_n^3)_Q$ and $(L_n^3)_Q$ are not temporal Tsirelson bound of [7] and [8], which is not very important to our present purpose.

Now, if a qubit system, the maximum quantum value of standard four-time LGI is $2\sqrt{2}$ and its macrorealist bound is 2. Then, in four-time measurement scenario, the difference between quantum and macrorealist values is 0.82. But, in the case of our variant of LGIs, we have $(K_n^3)_Q - K_n^3 = 1.12$ and $(L_n^3)_Q - L_n^3 = 1.03$. It can also be seen that $(K_n^3)_Q > (K_3^3)_Q > (K_4^3)_Q > (K_5^3)_Q$. Thus, by increasing the number of measurements the quantum violation of the variants of LGIs can be improved compared to the quantum violation of standard three or four-time LGIs.

Further, we demonstrate that when $n$ is sufficiently large, the quantum values of $(K_n^3)_Q$ and $(L_n^3)_Q$ reach algebraic maximum of $K_n^3$ and $L_n^3$ respectively. For $n$-time sequential measurement, the calculation of correlation function in QM seems difficult task. In order to tackle this problem, we derive a compact formula for $n$-time sequential correlation given in Eq. (A7) of Appendix A.

For the qubit state given by (4), the quantum expressions of $K_n^3$ for even $n$ is given by

$$(K_{n\text{even}}^3)_Q = (\cos 2g)^{\frac{n-1}{2}} + (\cos 2g)^{\frac{n-1}{2}} - (\cos 2(n-1)g \cos 2\theta + \sin 2(n-1)g \sin 2\theta \sin \phi) \quad (13)$$

and for odd $n$

$$(K_{n\text{odd}}^3)_Q = (\cos 2g)^{\frac{n-1}{2}} \cos 2\theta + (\cos 2g)^{\frac{n-1}{2}} - (\cos 2(n-1)g \cos 2\theta + \sin 2(n-1)g \sin 2\theta \sin \phi) \quad (14)$$

By considering $g = \pi/n$, Eqs. (13) and (14) take the form

$$(K_{n\text{even}}^3)_Q = \left( \cos \frac{\pi}{n} \right)^{\frac{n-1}{2}} + \left( \cos \frac{\pi}{n} \right)^{\frac{n-1}{2}} \cos 2\theta + \cos \frac{\pi}{n} \cos 2\theta - \sin \frac{\pi}{n} \sin 2\theta \sin \phi \quad (15)$$

and

$$(K_{n\text{odd}}^3)_Q = \left( \cos \frac{\pi}{n} \right)^{\frac{n-1}{2}} \cos 2\theta + \left( \cos \frac{\pi}{n} \right)^{\frac{n-1}{2}} + \cos \frac{\pi}{n} \cos 2\theta - \sin \frac{\pi}{n} \sin 2\theta \sin \phi \quad (16)$$
respectively. In the large \( n \) limit, both of them reduces to
\[
(K_{n_{even}}^3)Q = (K_{n_{odd}}^3)Q \approx 1 + 2 \cos 2\theta \tag{17}
\]
Thus, when \( \theta \approx 0 \), the quantities \((K_{n_{even}}^3)Q = (K_{n_{odd}}^3)Q \approx 3\), i.e., the algebraic maximum of the inequalities \([9,10]\).

Next, we calculate the quantum violation of the other variant of LGI given by \([10]\) for the state in Eq. (4). The quantum expression of \( L_n^3 \) for even \( n \) is given by
\[
(L_{n_{even}}^3)Q = (\cos 2g)^{n-1} \cos 2\theta + (\cos 2g)^{n-1} (\cos 2g \cos 2\theta + \sin 2g) - \cos 2(n-1)g \tag{18}
\]
If \( n \) is odd, we have
\[
(L_{n_{odd}}^3)Q = (\cos 2g)^{\frac{n-1}{2}} + (\cos 2g)^{\frac{n-3}{2}} - \cos 2(n-1)g \tag{19}
\]
which is independent of the state.

Similar to the earlier case, again by taking \( g = \frac{\pi}{2n} \), from Eqs. (18) and (19), we have
\[
(L_{n_{even}}^3)Q = (\cos \frac{\pi}{n})^{\frac{n-1}{2}} \cos 2\theta + \cos \frac{\pi}{n} (\frac{n-1}{2}) (\cos 2\theta + \sin 2\theta \sin \phi) \tag{20}
\]
and
\[
(L_{n_{odd}}^3)Q = 2 (\cos \frac{\pi}{n})^{\frac{n+1}{2}} + \cos \frac{\pi}{n} \tag{21}
\]
For large \( n \), the quantum value of \((L_{n_{odd}}^3)Q\) is 3 which is independent of the state and the quantity \((L_{n_{even}}^3)Q\) approaches the algebraic maximum 3 when \( \theta \approx 0 \). The Eqs. (20) and (21) are plotted in Figure 2 to demonstrate how the quantum values of \((L_{n_{even}}^3)Q\) and \((L_{n_{odd}}^3)Q\) approach algebraic maximum with increasing the number of measurements \( n \).

IV. COMPARING VARIANTS OF LGIS WITH OTHER FORMULATIONS OF MACROREALISM

Fine \([33]\) theorem states that the CHSH inequalities are necessary and sufficient condition for local realism. Since standard LGIs are often considered to be the temporal analogue of CHSH inequalities one may expect that they also provide the necessary and sufficient condition for macrorealism. In recent works, Clemente and Kofler \([30]\) showed that no set of standard LGIs can provide the necessary and sufficient condition for macrorealism. However, a suitable conjunction of no-signaling in time (NSIT) conditions provides the same. In this connection, two of us \([20]\) have shown that the Wigner formulation of LGIs are stronger than standard LGIs but they also do not provide necessary and sufficient condition for macrorealism. Against this backdrop, in this section, we shall analyze the state of our variant of LGIs for three-time measurement scenario. For this, let us first find the connection between standard LGIs, NSIT condition and macrorealism.

NSIT condition is the statistical version of NIM condition. It is analogous to the no-signaling condition in Bell’s theorem, however violation of NSIT condition does not provide any inconsistency with physical theories. It simply assumes that the probability of a outcome of measurement remains unaffected due to prior measurement. Clearly, the satisfaction of all NSIT conditions in any operational theory ensures the existence of global joint probability condition \( P(M_1^{m_1}, M_2^{m_2}, M_3^{m_3}) \) where \( m_1, m_2, m_3 = \pm 1 \) and in such a case no violation of any LGI can occur.

A two-time NSIT condition can be written as
\[
NSIT_{(1)2} : P(M_2^{m_2}) = \sum_{m_1} P_{12}(M_1^{m_1}, M_2^{m_2}) \tag{22}
\]
which means that the probability \( P(M_2^{m_2}) \) is unaffected by the prior measurement of \( M_1 \). Similarly, a three-time NSIT condition is given by
\[
NSIT_{(1)23} : P(M_2^{m_2}, M_3^{m_3}) = \sum_{m_1} P_{123}(M_1^{m_1}, M_2^{m_2}, M_3^{m_3}) \tag{23}
\]
Here \( P_{123}(M_1^{m_1}, M_2^{m_2}, M_3^{m_3}) \) denotes the joint probabilities when all the three measurements are performed.

Clemente and Kofler \([30]\) have shown that a suitable conjunction of two-time and three-time NSIT conditions provides the necessary and sufficient condition for macrorealism, i.e.,
\[
NSIT_{(2)3} \land NSIT_{(1)23} \land NSIT_{(2)3} \Leftrightarrow MR \tag{24}
\]
where MR denotes macrorealism. We first show how standard LGIs do not provide necessary and sufficient
condition for macrorealism. Such an argument was first initiated in [14] and discussed in detail in [20]. But for making the present work self-contained we encapsulate the essence of the argument.

Let us consider the pairwise marginal statistics of the experimental arrangement when all three measurements ($M_1$, $M_2$ and $M_3$) are performed and introduce the following quantity

$$D_1(M_{m2}^1, M_{m3}^3) = P(M_{m2}^1, M_{m3}^3) - \sum_{m_1} P_{123}(M_{m1}^1, M_{m2}^m, M_{m3}^3)$$

which quantifies the amount of disturbance created (in other words, degree of violation of NSIT condition) by the measurement $M_1$ at $t_1$ to the measurements of $M_2$ and $M_3$ at $t_2$ and $t_3$ respectively. Similarly,

$$D_2(M_{m1}^m, M_{m2}^{m2}) = P(M_{m1}^m, M_{m2}^{m2}) - \sum_{m_3} P_{123}(M_{m1}^m, M_{m2}^{m2}, M_{m3}^3)$$

$$D_3(M_{m1}^m, M_{m2}^{m2}) = P(M_{m1}^m, M_{m2}^{m2}) - \sum_{m_3} P_{123}(M_{m1}^m, M_{m2}^{m2}, M_{m3}^3)$$

Note that, since no information can travel backward in time, $D_3(M_{m1}^m, M_{m2}^{m2}) = 0$ in any physical theory. For two-time measurements, we can define similar quantity, for example, $D_1(M_{m2}^m)$.

Standard LGIs are derived by assuming the satisfaction of all NSIT conditions. But, in QM, the NSIT conditions are, in general, not satisfied. This, in fact, is the reason of the violation of LGIs in QM. It is then straightforward to understand that the difference between $K_3$ and $(K_3)_{123}$ plays an important role for the violation of LGI. Clearly, if $K_3 = (K_3)_{123}$ is satisfied, the LGI will not be violated. When all the three measurements are performed for measuring each correlation, the expression of $K_3$ in inequality [1] can be written

$$(K_3)_{123} = \langle M_1 M_2 \rangle_{123} + \langle M_2 M_3 \rangle_{123} - \langle M_1 M_3 \rangle_{123} = 1 - 4\alpha$$

Using Eqs. (25) and (26) we can write

$$K_3 - (K_3)_{123} = \sum_{m_2=m_3} D_1(M_{m2}^m, M_{m3}^m) - \sum_{m_1=m_3} D_2(M_{m1}^m, M_{m3}^3)$$

$$- \sum_{m_2 \neq m_3} D_1(M_{m2}^m, M_{m3}^m) + \sum_{m_1 \neq m_3} D_2(M_{m1}^m, M_{m3}^3)$$

Since $\sum_{m_2=m_3} D_1(M_{m2}^m, M_{m3}^m) = 0$, $\sum_{m_1=m_3} D_2(M_{m1}^m, M_{m3}^3) = 0$ and $K_3 \leq 1$, from Eq. (29) we obtain

$$2 \sum_{m_2=m_3} D_1(M_{m2}^m, M_{m3}^m) - 2 \sum_{m_2=m_3} D_2(M_{m1}^m, M_{m3}^3) + (K_3)_{123} \leq 1$$

By putting the value of $(K_3)_{123}$ from Eq. (28) we have

$$\sum_{m_2=m_3} D_1(M_{m2}^m, M_{m3}^m) - \sum_{m_1=m_3} D_2(M_{m1}^m, M_{m3}^3) \leq 2\alpha$$

We have thus written down the standard LGIs in terms of NSIT conditions. For the violation of standard LGI in [1] the relation

$$\sum_{m_2=m_3} D_1(M_{m2}^m, M_{m3}^m) - \sum_{m_1=m_3} D_2(M_{m1}^m, M_{m3}^3) > 2\alpha$$

needs to be satisfied in QM. This implies that for violation of standard LGI at least one of the two three-time NSIT conditions ($NSIT_{(1)23}$ and $NSIT_{(2)3}$) required to be violated. However, mere violations of NSIT conditions do not guarantee the violation of LGIs which depends on the interplay between the violations of two NSIT conditions and on a threshold value $2\alpha$. Thus, NSIT conditions are necessary for LGI but not sufficient [13] [20].

Next, we compare our variant of LGIs with standard LGIs and NIST conditions. We found that violation of variant of LGIs can be shown to be larger than the standard LGIs ($(K_3^3)_{123}$). Before writing variant of LGI in terms of NIST conditions, we note the following interesting points. Let us write one of the variant of LGIs for three-time measurement scenario is given by

$$K_3^3 = \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle + \langle \hat{M}_1 \hat{M}_2 \rangle - \langle \hat{M}_3 \rangle \leq 1$$

Since $\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle = (\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle)_{123}$ and $\langle \hat{M}_1 \hat{M}_2 \rangle = (\langle \hat{M}_1 \hat{M}_2 \rangle)_{123}$, then disturbance can only come due to the term $\langle \hat{M}_3 \rangle$. Intuitively one may then expect that whenever the quantity $D_{12}(M_{m3}^3)$ is defined as

$$D_{12}(M_{m3}^3) = P(M_{m3}^m) - \sum_{m_1=m_2} P_{123}(M_{m1}^m, M_{m2}^{m2}, M_{m3}^3)$$

is positive, one may expect the violation of the variant of LGI given by Eq. (33). Thus, one may infer that the NSIT condition $NSIT_{(12)3}$ provides the necessary and sufficient condition for variant of LGI. But, we shall shortly see that similar to the case of standard LGI, $NSIT_{(12)3}$ provides the necessary but not the sufficient condition.

Using similar approach adopted for standard LGIs, we express the variant of LGI given by (33) in terms of NSIT condition. Then, the expression of $K_3^3$ in inequality (33) can be written when all three measurements are performed is given by

$$(K_3^3)_{123} = \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle + \langle \hat{M}_1 \hat{M}_2 \rangle_{123} - \langle \hat{M}_3 \rangle_{123} = 1 - 4\beta$$

where $\beta = P(M_{\hat{1}}, M_{\hat{2}}, M_{\hat{3}}) + P(M_{\hat{1}}, M_{\hat{2}}, M_{\hat{3}})$. Using Eq. (34), we can write

$$K_3^3 - (K_3^3)_{123} = \sum_{m_3} D_{12}(M_{m3}^m)$$

(36)
Since $K_3^2 \leq 1$, using Eq. (35) we obtain

$$\sum_{m_3} D_{12}(M_{3}^{m_3}) \leq 4\beta$$  \hspace{1cm} (37)$$

For the violation of variants in inequality (3) the following relation needs to be satisfied in QM is given by

$$\sum_{m_3} D_{12}(M_{3}^{m_3}) > 4\beta$$  \hspace{1cm} (38)$$

which is the variant of LGI written in terms of the NSIT(123) condition. It can be seen from (38) that mere violation of NSIT(123) do not provide the violation of inequality (33), the value of $\sum_{m_3} D_{12}(M_{3}^{m_3})$ needs to greater than a non-zero threshold value $4\beta$. Thus, NSIT condition is necessary for the violation of variant of LGI but not sufficient.

Using the similar argument we can derive the condition of violation of inequality (9) for $n$-number of measurements in terms of NSIT condition as

$$\sum_{m_n} D_{1,2,..(n-1)}(M_{n}^{m_n}) > 4\gamma$$  \hspace{1cm} (39)$$

where $\gamma = P(M_{1}^{m_1}, M_{2}^{m_2},..,M_{n}^{m_n}) + ...2^{n-2}$ terms. The quantity $D_{1,2,..(n-1)}(M_{n}^{m_n})$ denotes the amount of disturbance caused by $n-1$ number of prior measurements. Intuitively, it increases with the number of measurements and becomes maximum when quantum value of the inequality (30) reaches its algebraic maximum.

V. SUMMARY AND DISCUSSION

The quantum violation of standard LGIs for a dichotomic system is restricted by temporal Tsirelson bound which is significantly lower than the algebraic maximum. In this paper, we note an important observation that the standard LGIs are a class of inequalities but not the unique one. There is a scope of formulating new variant of inequalities based on the assumptions of MRps and NIM. For the simplest case of three-time measurement scenario, we first proposed new variants of LGIs which are different from the standard LGIs. For a qubit system, we demonstrated that such macrorealist inequalities provide larger quantum violation than standard LGIs. By increasing the number of measurements $n$, we proposed more variants of LGIs. We found that the quantum violation of variants of LGIs increase with the increment of $n$. Interestingly, for a sufficiently large value of $n$, the quantum violation of variant of LGBs reach their algebraic maximum. Thus, we obtained the quantum violation of LGIs up to its algebraic maximum, even for a state in qubit system. Further, we have compared the variants of LGIs proposed in our paper with the standard LGIs and NSIT condition.

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Appendix A: General formula for calculating n-time sequential measurement

Here we provide a general formula for calculating sequential correlation of n-time measurements of a dichotomic observable. In LG scenario, the measurement of a dichotomic observable $M$ having outcomes $±1$ is performed at time $t_1, t_2,..., t_n$ ($t_1 < t_2 < ... < t_n$), which, in turn, can be considered as the sequential measurement of the observables $\hat{M}_1, \hat{M}_2,..., \hat{M}_n$ respectively.

Given a density matrix $\rho$, the correlation function for the sequential measurement of two observables $\hat{M}_1$ and $\hat{M}_2$ can be calculated by using the formula [19]

$$\langle \hat{M}_1 \hat{M}_2 \rangle_{seq} = \frac{1}{2} \text{Tr} \left[ \rho \left\{ \hat{M}_2, \hat{M}_3 \right\} \right]$$  \hspace{1cm} (A1)

where $\{\}$ denotes anti-commutation.

Here we generalize the above formula for n-time measurement scenario. For this, let us first consider the three-measurement scenario. The correlation function for three-time measurement can be written as,

$$\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle_{seq} = \sum_{m_1, m_2, m_3 = \pm 1} m_1 m_2 m_3 P(M_{m_1} M_{m_2} M_{m_3})$$  \hspace{1cm} (A2)

Let $\Pi_{M_1}^{m_1}$, $\Pi_{M_2}^{m_2}$ and $\Pi_{M_3}^{m_3}$ are projectors of observables $\hat{M}_1$, $\hat{M}_2$ and $\hat{M}_3$ corresponding to the to eigenvalues $m_1, m_2$ and $m_3$ respectively. In QM, Eq. (A2) can then be written as,

$$\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle_{seq} = \sum_{m_1, m_2, m_3 = \pm 1} m_1 m_2 m_3 \text{Tr}[\Pi_{M_2}^{m_2} \Pi_{M_1}^{m_1} \rho \Pi_{M_1}^{m_1} \Pi_{M_2}^{m_2} \Pi_{M_3}^{m_3}]$$  

$$= \sum_{m_1, m_2 = \pm 1} m_1 m_2 \text{Tr}[\Pi_{M_2}^{m_2} \Pi_{M_1}^{m_1} \rho \Pi_{M_1}^{m_1} \Pi_{M_2}^{m_2} \Pi_{M_3}^{m_3} + \Pi_{M_2}^{m_2} \Pi_{M_1}^{m_1} \rho \Pi_{M_1}^{m_1} \Pi_{M_2}^{m_2} \Pi_{M_3}^{m_3}]$$  \hspace{1cm} (A3)

Using $\hat{M}_3 = \Pi_{M_3}^{m_3} - \Pi_{\bar{M}_3}^{m_3}$ and putting the value of $m_2 = \pm 1$, we have

$$\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle_{seq} = \sum_{m_1 = \pm 1} m_1 \text{Tr}[(\Pi_{M_2}^{m_2} \Pi_{M_1}^{m_1} \rho \Pi_{M_1}^{m_1} \Pi_{M_2}^{m_2}) \hat{M}_3] - \sum_{m_1 = \pm 1} m_1 \text{Tr}[(\Pi_{M_2}^{m_2} \Pi_{M_1}^{m_1} \rho \Pi_{M_1}^{m_1} \Pi_{M_2}^{m_2}) \hat{M}_3]$$  \hspace{1cm} (A4)

Since $\Pi_{M_2}^{m_1} = (\mathbb{1} \pm \hat{M}_2)/2$, Eq. (A4) can be simplified as

$$\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle_{seq} = \frac{1}{2} \sum_{m_1 = \pm 1} m_1 \text{Tr} \left[ (\Pi_{M_1}^{m_1} \rho \Pi_{M_1}^{m_1}) \left\{ \hat{M}_2, \hat{M}_3 \right\} \right]$$  \hspace{1cm} (A5)

Adopting the similar to the procedures adopted above, further simplification provides

$$\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle_{seq} = \frac{1}{4} \text{Tr} \left[ \rho \left\{ \hat{M}_1, \left\{ \hat{M}_2, \hat{M}_3 \right\} \right\} \right]$$  \hspace{1cm} (A6)

For the case of n-time measurements, we derive

$$\langle \hat{M}_1 \hat{M}_2 \ldots \hat{M}_{n-1} \hat{M}_n \rangle_{seq} = \frac{1}{2^{n-1}} \text{Tr} \left[ \rho \left\{ \hat{M}_1, \left\{ \hat{M}_2, \ldots, \left\{ \hat{M}_{n-2}, \left\{ \hat{M}_{n-1}, \hat{M}_n \right\} \right\} \right\} \right\} \right]$$  \hspace{1cm} (A7)