Transition to Shocks in TASEP and Decoupling of Last Passage Times

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Abstract

We consider the totally asymmetric simple exclusion process in a critical scaling parametrized by $a \geq 0$, which creates a shock in the particle density of order $at^{-1/3}$, $t$ the observation time. When starting from step initial data, we provide bounds on the limiting law which in particular imply that in the double limit $\lim_{a \to \infty} \lim_{t \to \infty}$ one recovers the product limit law and the degeneration of the correlation length observed at shocks of order 1 in [13]. This result is shown to apply to a general last-passage percolation model. We also obtain bounds on the two-point functions of several Airy processes.

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1 Introduction

We consider the totally asymmetric simple exclusion process (TASEP). In this model, particles move on $\mathbb{Z}$ and jump one step to the right with rate 1, subject to the exclusion constraint that there is at most one particle on each site, see [20], [21] for details of the construction and properties of this process. In the past two decades, TASEP has become the poster child of an exactly solvable model belonging to the KPZ universality class of growth models (see [9] for a review of this class of models). With $(x,t) \rightarrow h(x,t)$ (where $x \in \mathbb{R}$ is space and $t \geq 0$ time) the height function associated to TASEP, one has a determinsitic limit shape $h_{ma}$

$$\lim_{t \to \infty} \frac{h(\xi t, t)}{t} = h_{ma}(\xi). \quad (1.1)$$

When $h_{ma}$ is differentiable, the KPZ correlation exponent is $2/3$ and the fluctuation exponent is $1/3$, i.e. the rescaled height function

$$h_{resc}(u,t) = \frac{h(\xi t + ut^{2/3}, t) - th_{ma}(\xi + ut^{-1/3})}{t^{1/3}} \quad (1.2)$$

converges to some non-trivial limit process as $t \to \infty$. Depending on the initial data, one obtains different processes, and recently, for arbitrary initial data, explicit formulas for the law of $h_{resc}(u,t)$ have been found in [22].

However, depending on the initial data (or by altering the jump rate of some particles), TASEP can develop shocks, which, macroscopically, are discontinuities in the particle density, where $h_{ma}$ is no longer differentiable. When the shock is created by random Bernoulli initial data, the fluctuations in the initial data supersede those of TASEP itself and one observes gaussian fluctuations under diffusive scaling (see [21], Chapter 3, and the references therein). When the initial data is determinstic, it was shown in [13], that at the shock, the correlation exponent degenerates to $1/3$ (the fluctuation exponent remains $1/3$) and that the limit law of $(1.2)$ is given by a product, see Theorem 2.1 in [13] and its applications.

In [13], the result was obtained by working in the last passage percolation (LPP) picture (see Section 2 for the definition of LPP). In LPP, studying the fluctuations at the shock corresponds to study the maximum of two last passage times $L_{\mathcal{E}^+ \rightarrow (\eta t, t)}, L_{\mathcal{E}^- \rightarrow (\eta t, t)}$ for certain $\mathcal{E}^+, \mathcal{E}^- \subset \mathbb{Z}^2$ and $\eta > 0$ is chosen to be at the shock. Now in [13], the maximizing paths from $\mathcal{E}^+$ to $(\eta t, t)$ and from $\mathcal{E}^-$ to $(\eta t, t)$ tend to start in points at distance $O(t)$. The maximizing paths have transversal fluctuations of order $t^{2/3}$ (17) around their characteristic lines, hence the two paths will only meet at distance $t^\nu, \nu \in (2/3, 1)$ from $(\eta t, t)$. By the slow decorrelation phenomenon (12),
the fluctuations built up along the characteristics between points with distance $t', \nu < 1$, vanish under the $t^{1/3}$ scaling. This leads to the asymptotic independence of $L_{\mathcal{L} \rightarrow (p,t)}$, $L_{\mathcal{L} \rightarrow (p',t)}$ as $t \to \infty$ and hence to the product limit law. Later, this procedure was also applied to study the competition interface and multipoint distributions at shocks, see [14].

The main topic of this paper is the transition of fluctuations when $h_{\text{max}}$ is smooth to the fluctuations when there is a shock. In terms of LPP, this means to study the maximum of two last passage times which remain correlated for all $t > 0$, but which, as we show, decouple in a double limit $\lim_{a \to \infty} \lim_{t \to \infty}$, where $a$ is an extra parameter in the TASEP/LPP model. As a concrete example, in Theorem 2.3 we consider for $a \geq 0$ $\mathcal{L}^+ = (-[at^{2/3}], 0)$, $\mathcal{L}^- = (0, -[at^{2/3}])$ and an end point $E = ([t + \frac{a}{n}t^{2/3}, [t])$. In terms of TASEP, for $a = 0$ one has the step initial data, by taking $a = \mathcal{O}(t^{1/3})$, one gets the macroscopic shock between two regions of decreasing density studied in Corollary 2.7 of [13]. For $a > 0$, we are in a critical scaling, which, for flat TASEP, had previously been studied in [11] (without showing the transition of fluctuations, and we shortly discuss this at the end of Section 5). A lower bound for $\mathbb{P}(\max\{L_{\mathcal{L} \rightarrow E_1}, L_{\mathcal{L} \rightarrow E_2}\} \leq s)$ is provided by the FKG inequality, so the main work is to find a suitable upper bound, which we do in Theorems 2.1 and 2.3. This in particular implies that one recovers, in the double limit $\lim_{a \to \infty} \lim_{t \to \infty}$, the product structure of [13], and shows that the correlation length degenerates in this double limit. In the concrete case at hand, this leads to a transition from $F_{\text{GUE}}$ to a product of two $F_{\text{GUE}}$ distributions.

One can translate Theorem 2.3 in a statement about the decoupling of the two-point function of the Airy$_2$ process, see Corollary 2.4. While more precise statements than ours are available (see [25], [26], [1]), our proof is new, and probabilistic in that we make use of the convergence in LPP; which gives some intuition as to why the decoupling happens.

Theorem 2.3 can be seen as an instance of a general Theorem about the decoupling of last passage times under some assumptions, see Theorem 2.6. Theorem 2.6 could also be used to improve the results of [13], in that it provides some finite time estimates, rather than merely showing the convergence to a product, as was done in [13]. Also, Theorem 2.6 is much simpler than Theorem 2.1 of [13]. Furthermore, Theorem 2.6 gives the framework to show the decoupling of the Airy$_1$, Airy$_2$ processes (see [6], [5] for definitions), which has not been done before, see Theorem 2.5. The decoupling of these processes corresponds to the decoupling of last passage times $L_{\mathcal{L} \rightarrow E_1}, L_{\mathcal{L} \rightarrow E_2}$ where $\mathcal{L}$ is now a (half-) line and the points $E_1, E_2$ have distance $at^{2/3}$ from each other.

One feature of our approach to obtain bounds on the limiting objects (in our case, the Airy processes) is that it makes no use of their explicit form.
Therefore it can also be used in cases where no explicit formula is available, which we illustrate in Theorem 5.2 where we provide bounds for LPP times along the time-like direction. These in particular imply a decoupling of two Airy\(_2\) processes. In the case of brownian directed percolation, [19] had provided an explicit formula for the one-point two-time distribution and expected such a decoupling to occur, see [19], Remark 2.3.

As a first main ingredient to prove Theorem 2.3, and unlike in [13], one needs a control of transversal fluctuations of size \(kt^{2/3}\). To our knowledge, the first result in this direction was Theorem 2.5 in [2], which bounds polynomially in \(k\) the probability of having \(kt^{2/3}\) transversal fluctuations on a single horizontal line. Theorem 2.5 of [2] applies, with \(\eta_0 > 0\), to maximizing paths from \((0,0)\) to \((\lfloor \eta_0 t \rfloor, \lfloor t \rfloor)\) in LPP models with boundary weights bounded by the stationary LPP, and i.i.d. \(\exp(1)\) weights on \(\mathbb{Z}_{\geq 1}^2\). Later, in Theorem 11.1 of [4], for Poisson LPP (corresponding to i.i.d. \(\exp(1)\) weights on \(\mathbb{Z}_{\geq 0}^2\)) an exponential bound for \(kt^{2/3}\) transversal fluctuations of the entire path was obtained. Here we show that the result from [2] can be combined with the ideas from [4] to control the transversal fluctuations of the entire path. As a result, we can bound in Theorem 3.1 the transversal fluctuations of maximizing paths from \((0,0)\) to \((\lfloor \eta_0 + ct^{2/3} \rfloor, \lfloor t \rfloor)\). Theorem 2.5 in [2] a priori only provides an upper bound dependent on \(c\) in this case, in the Appendix A we remove this problem in the case of i.i.d. weights, which is the case we need for our main purposes, see Theorem 3.5.

The second main ingredient to Theorem 2.3 is an extended slow decorrelation result. Namely, since we have two maximizers which start in points with distance \(O(t^{2/3})\) and go to \(E\), the maximizers will come together already at distance \(O(t)\) from \(E\). Consequently, we wish to replace \(L_{E^- \to E^+}\) by \(L_{E^- \to E^+}\), with \(E^+\) on the straight (characteristic) line from \(L^+\) to \(E\) and at distance \(ct\) from \(E\). If \(\varepsilon\) is not too small, the probability that the maximizers of \(L_{E^+ \to E^-}\) and \(L_{E^- \to E^+}\) cross will vanish for \(a, t\) large. As mentioned, in the usual slow decorrelation, \(E_{+}\) is at distance \(t^\nu, \nu < 1\) from \(E\) such that the fluctuations from \(E^+\) to \(E\) vanish under the \(t^{1/3}\) scaling. In our situation, however, they do not vanish as \(t \to \infty\), but are only of order \(\varepsilon^{1/3}t^{1/3}\); in particular, they vanish in the double limit \(\lim_{t \to 0} \lim_{t \to \infty}\). This can be phrased in terms of a general statement about slow decorrelation on the \(O(t)\) scale, see Theorem 2.7. We show that it is possible to choose \(\varepsilon = \varepsilon(a)\) in such a way that \(\varepsilon(a)\) goes to zero with \(a\), but is large enough so that the maximizers stay in disjoint sets with high probability (see Section 4), leading to Theorem 2.3.
To show the decoupling of the Airy\(_1\), Airy\(_2\rightarrow 1\) processes, no slow decorrelation result is needed, but, next to controlling transversal fluctuations, the extra ingredient required here is the control over the (random) starting point of the maximizing path, which was obtained recently in [24], Lemma 1.1, Lemma 1.2 and also in (4.18) in [15].

Outline. In Section 2 we define our models and state our main results. In Section 3 we prove Theorem 3.5 and 3.1 which bound the probability of having large transversal fluctuations. In Section 4 we prove our main result about the transition to shock fluctuations, Theorem 2.1. Finally, in Section 5 we prove our bounds on the two point functions of several Airy processes as well as of the two time distribution in exponential LPP. We also briefly discuss the transition to shock fluctuations for flat TASEP. Finally, in Appendix A we prove a lemma needed for the transversal fluctuations.

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Notation We denote for \(x \in \mathbb{R}\) by \(\lfloor x \rfloor\) the largest \(z \in \mathbb{Z}\) with \(z \leq x\). By \(C(p_1,\ldots,p_k)\) we denote a constant which depends on \(p_1,\ldots,p_k \in \mathbb{R}\).

2 Model and Results

We consider TASEP with particles labelled from right to left, i.e., when \(x_n(t)\) denotes the position of particle number \(n \in \mathbb{Z}\) at time \(t\) we have

\[
\cdots < x_2(0) < x_1(0) < x_0(0) < x_{-1}(0) < x_{-2}(0) \cdots ,
\]

note this order is preserved in time. TASEP is in one-to-one correspondence with last passage percolation, which we define next. Fix \((m,n) \in \mathbb{Z}^2\) (the end point) and \(\mathcal{L} \subseteq \mathbb{Z}^2\) (the starting set). Let \(\{\omega(i,j)\}_{(i,j) \in \mathbb{Z}^2}\) be nonnegative random variables, seen as weights at the point \((i,j)\)\(^1\). An up-right path \(\pi = (\pi(0),\ldots,\pi(k))\) from \(\mathcal{L}\) to \((m,n)\) is a sequence of points with \(\pi(0) \in \mathcal{L}, \pi(k) = (m,n), \pi(i) - \pi(i-1) \in \{(0,1),(1,0)\}\). Then the LPP time from \(\mathcal{L}\) to \((m,n)\) is defined as

\[
L_{\mathcal{L}\to(m,n)} = \max_{\pi: \mathcal{L}\to(m,n)} \sum_{(i,j) \in \pi} \omega_{i,j}
\]

where the maximum in (2.2) is taken over all up-right paths from \(\mathcal{L}\) to \((m,n)\). When there are no or infinitely many such paths, we set, say, \(L_{\mathcal{L}\to(m,n)} = \infty\),

\(^1\) With the exception of Theorem 3.1 and weights which may be zero, we only consider nonnegative, independent, continuous weights in this paper.
also \([22]\) straightforwardly generalizes to several end points. Given an initial data \(\{x_n(0)\}_{n \in I}, I \subset \mathbb{Z}\) of TASEP we set \(\mathcal{L} = \{(x_n(0) + n, n), n \in I\}\). Assuming all particles have an exponential clock with parameter 1 we take \(\{\omega_{i,j}\}_{(i,j) \in \mathbb{Z}^2}\) independent, and \(\omega_{i,j} \sim \exp(1)\) if \((i, j) \notin \mathcal{L}\) and \(\omega_{i,j} = 0\) for \((i, j) \in \mathcal{L}\). With this choice, the link between TASEP and LPP is given by
\[
P(x_n(t) \geq m - n) = P(I_{\mathcal{L}(m, n)} \leq t).
\] (2.3)

In the following, we will only consider LPP times and particle positions, not the height function mentioned in the introduction. The following Theorem provides the transition from the fluctuations of TASEP with step initial data to the fluctuations at the GUE – GUE shock of \([13]\).

**Theorem 2.1.** Let \(x_n(0) = -n\) for \(-\lfloor aT^{2/3}\rfloor \leq n \leq 0\) and \(x_n(0) = -n - \lfloor aT^{2/3}\rfloor\) for \(n \geq 1\). Then, for any \(0 < k < a, \delta > 0\) and \(\frac{b}{a} < \varepsilon(a) < 1\) we may bound
\[
F_{\text{GUE}}(s)F_{\text{GUE}}(s - u2^{4/3})
\]
\[
\leq \lim_{T \to \infty} P(x_{\lfloor T - T^{2/3}k \rfloor}(T) \geq \frac{u}{a}T^{2/3} + T^{1/3}\frac{(\frac{u}{a} + a)^2}{2} - T^{1/3}\frac{1}{2^{1/3}s})
\]
\[
\leq F_{\text{GUE}}\left(\frac{s + \delta}{1 - \varepsilon(a)^{1/3}}\right)F_{\text{GUE}}(s - u2^{4/3}) + F_{\text{GUE}}(\delta\varepsilon(a)^{-1/3}) + \psi(k),
\] where, for any \(\alpha \in (0, 1)\) there is a constant \(C(\alpha)\) such that \(\psi(k) \leq C(\alpha)k^{-3\alpha}\).

In particular, Theorem \([21]\) implies
\[
\lim_{a \to \infty} \lim_{T \to \infty} P(x_{\lfloor T - T^{2/3}k \rfloor}(T) \geq \frac{u}{a}T^{2/3} + T^{1/3}\frac{(\frac{u}{a} + a)^2}{2} - T^{1/3}\frac{1}{2^{1/3}s}) = F_{\text{GUE}}(s)F_{\text{GUE}}(s - u2^{4/3}).
\] (2.4)

By taking \(u = a\tilde{u}\) in \([21]\) such that \(u/a = \tilde{u}\) and then setting \(a = 0\), one has the usual step initial data and the \(T \to \infty\) limit in \([21]\) gives the Airy_2 process \(\mathcal{A}_2(\tilde{u})\). To recover the shock situation, one should transfer the \(T^{1/3}\frac{(\frac{u}{a} + a)^2}{2}\) term in the particle number, i.e. consider
\[
P\left(x_{\lfloor T - T^{2/3}\frac{a\tilde{u}}{a} + T^{1/3}\frac{(\frac{u}{a} + a)^2}{2}}(T) \geq \frac{u}{a}T^{2/3} - T^{1/3}\frac{1}{2^{1/3}s}\right).
\] (2.5)

To create a macroscopic shock, set, for \(\beta \in (0, 1)\), \(a = \beta T^{1/3}, u = \xi, \tilde{U} = \frac{\xi}{2}\beta^{-1}\), so that \([2.5]\) becomes
\[
P\left(x_{T^{1/3}(\xi + \tilde{U})}(T) \geq T^{1/3}\xi/\beta - T^{1/3}\frac{1}{2^{1/3}s}\right).
\] (2.6)
Now Corollary 2.7 of [13] implies
\[
\lim_{\beta \to 0} \lim_{T \to \infty} \mathbb{P}(x_{(\alpha - \beta)^{2} + \xi T^{1/3}}(T) \geq T^{1/3}\xi/\beta - \frac{T^{1/3}}{2^{1/3}}s) = F_{\text{GUE}}(s)F_{\text{GUE}}(s - \xi 2^{1/3}),
\]
which coincides with the double limit of (2.4). In this sense we thus have a transition to shock fluctuations as \(a \to \infty\), as well as a degeneration of the correlation length.

**Remark 2.2.** The bound \(\psi(k) \leq C(\alpha)k^{-3\alpha}\), which also appears in Theorems 2.3 and Corollary 2.4, could be improved to \(\psi(k) \leq Ce^{-ck}, C,c > 0\) by adapting the proof of Theorem 11.1 of [4], see the beginning of Section 3 for a discussion.

The LPP counterpart of Theorem 2.1 is as follows.

**Theorem 2.3.** Set \(L^{+} = (-\lceil at^{2/3}\rceil, 0), L^{-} = (0, -\lceil at^{2/3}\rceil), L = L^{+} \cup L^{-}\) and define
\[
\mu^{a}t = 4t + 2t^{2/3}(a + u/a) - \left(a + \frac{u}{a}\right)^{2} t^{1/3}/4.
\]
Let \(a > k > 0\). Then for any \(\delta > 0\) and \(k/a < \varepsilon(a) < 1\) we may bound
\[
F_{\text{GUE}}(s)F_{\text{GUE}}\left(s - \frac{u}{2^{1/3}}\right) \leq \lim_{t \to \infty} \mathbb{P}\left(\frac{L_{\xi \to t^{1/3} + \frac{u}{2^{1/3}}}}{2^{1/3}t^{1/3}} \leq s\right)
\]
\[
\leq F_{\text{GUE}}\left(s + \frac{\delta}{(1 - \varepsilon(a))^{1/3}}\right) F_{\text{GUE}}\left(s - \frac{u}{2^{1/3}}\right)
\]
\[
+ F_{\text{GUE}}(-\delta \varepsilon(a)^{-1/3}) + \psi(k),
\]
where, for any \(\alpha \in (0, 1)\) there is a constant \(C(\alpha)\) such that \(\psi(k) \leq C(\alpha)k^{-3\alpha}\).

Theorem 2.3 gives some estimates on the decay of the two point function of the Airy_2 process \(A_2\). The two point function \(\mathbb{P}(A_2(0) \leq s_1, A_2(a) \leq s_2)\) has already been studied in detail (see in particular (7) in [25], and also the previous works [26, 1]). In particular, it is known that \(\mathbb{P}(A_2(0) \leq s_1, A_2(a) \leq s_2) = F_{\text{GUE}}(s_1)F_{\text{GUE}}(s_2) + \mathcal{O}(a^{-2})\) as \(a \to \infty\). However, the works [25, 26, 1] all are based on Fredholm determinant (in [26, 25]) or PDE expression (in [1]) for the two point function, whereas we use that the Airy_2 process arises as limit in LPP.
Corollary 2.4. Let $a > k > 0$. Then for any $\delta > 0$ and $1 > \varepsilon(a) > k/a$ we may bound
\[
F_{\text{GUE}}(s)F_{\text{GUE}}(s - 4u) \leq \mathbb{P}\left(A_2\left(-a - \frac{u}{a}\right) \leq s, A_2\left(a - \frac{u}{a}\right) \leq s - 4u\right)
\leq F_{\text{GUE}}\left(s + \delta \left(\frac{1}{1 - \varepsilon(a)}\right)^{1/3}\right) F_{\text{GUE}}(s - 4u)
+ F_{\text{GUE}}(-\delta \varepsilon(a)^{-1/3} + \psi(k),
\]
where, for any $\alpha \in (0, 1)$ there is a constant $C(\alpha)$ such that $\psi(k) \leq C(\alpha)k^{-3\alpha}$.

In Section 5, we also study the decay of the joint distribution of the Airy$_1$, Airy$_{2\to 1}$ processes, which we denote by $A_1$, $A_{2\to 1}$. This decoupling does not correspond to a transition to shock fluctuations, rather one has two maximizers which start and end in points with distance $at^{2/3}$. The starting point is random, and controlling it is an extra ingredient required here, which was obtained in recently in [15, 24].

Theorem 2.5. There are $C, c, a_0 > 0$ such that for $a > a_0, b \in \mathbb{R}$ we have
\[
F_{\text{GOE}}(2s_1)F_{\text{GOE}}(2s_2) \leq \mathbb{P}(A_1(0) \leq s_1, A_1(a) \leq s_2)
\leq F_{\text{GOE}}(2s_1)F_{\text{GOE}}(2s_2) + Ce^{-ca^2} + \psi(a)
\tag{2.9}
\]
and that for the Airy$_{2\to 1}$ process we may bound
\[
\mathbb{P}(A_{2\to 1}(b) \leq s_1)\mathbb{P}(A_{2\to 1}(|b| + a) \leq s_2)
\leq \mathbb{P}(A_{2\to 1}(b) \leq s_1, A_{2\to 1}(|b| + a) \leq s_2)
\leq \mathbb{P}(A_{2\to 1}(b) \leq s_1)\mathbb{P}(A_{2\to 1}(|b| + a) \leq s_2) + Ce^{-ca^2} + \psi(a),
\tag{2.12}
\]
where, for any $\alpha \in (0, 1)$ there is a constant $C(\alpha)$ such that $\psi(a) \leq C(\alpha)a^{-3\alpha}$.

The previous results concern, in terms of LPP, last passage times along so-called space like paths (see e.g. [5]), i.e. LPP times $L_{\mathcal{C} \to (m_1,n_1)}, L_{\mathcal{C} \to (m_2,n_2)}$ with $m_2 \geq m_1, n_2 \leq n_1$. In this case, the convergence to the limiting objects (the Airy processes) is widely established. Along the so-called time-like direction however, i.e. for $m_1 > m_2, n_1 > n_2$, much less is known, an explicit formula for the limiting object was obtained by Johansson [19] in the case of the brownian directed percolation. Since we do not rely on exact formulas, we can in Theorem 2.2 show the decoupling along the time-like direction in exponential LPP.

Finally, the preceding results can all be phrased in a simple Theorem about a general LPP model, which improves the general framework given in Theorem 2.1 of [13]. Let $\mathcal{L}^+, \mathcal{L}^- \subset \mathbb{Z}^2$ and let $\{\omega_{i,j}, i, j \in \mathbb{Z}\}$ be independent exponentially distributed weights. We make three assumptions on our model.
Assumption 1. Let \( t, a > 0 \) and assume there are \( E_1 = E_1(t, a), E_2 = E_2(t, a) \in \mathbb{Z}^2 \) and \( \mu_1^a, \mu_2^a > 0 \) such that

\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{L_{\mathcal{L}+ \to E_1} - \mu_1^a t}{t^{1/3}} \leq s \right) = G_1^a(s) \tag{2.13}
\]
\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{L_{\mathcal{L}+ \to E_2} - \mu_2^a t}{t^{1/3}} \leq s \right) = G_2^a(s), \tag{2.14}
\]

where \( G_1^a(s), G_2^a(s) \) are some distribution functions.

In Theorem 2.3, \( G_1^a, G_2^a \) will be (shifted) \( F_{\text{GUE}} \) distributions, in Theorem 2.5, they will be \( F_{\text{GOE}} \) distributions.

Assumption 2. Assume there is a point \( E^+ = E_1 - (\kappa \varepsilon(a) t + dl^{2/3}, \varepsilon(a) t) \) with \( \kappa, \varepsilon(a) \geq 0, d \in \mathbb{R} \) such that for a \( \mu_\varepsilon(a) \geq 0 \) we have

\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{L_{\mathcal{L}+ \to E_1} - \mu_\varepsilon(a) t}{t^{1/3}} \leq s \right) = G_0^a(s) \tag{2.15}
\]
\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{L_{\mathcal{L}+ \to E_2} + \mu_\varepsilon(a) t - \mu_1^a t}{t^{1/3}} \leq s \right) = G_1^a(c_\varepsilon(a) s) \tag{2.16}
\]

where \( G_0^a \) is a distribution function, \( c_\varepsilon(a) \) is a constant and \( G_1^a \) is from Assumption 1.

In the context of Theorem 2.3, we will take \( \varepsilon(a) > 0, \lim_{a \to \infty} \varepsilon(a) = 0 \).

Then, with \( E \) as in Theorem 2.3, \( \frac{L_{\mathcal{L}+ \to E_1} - \mu_\varepsilon(a) t}{t^{1/3}} \) will vanish in the double limit

\[
\lim_{a \to \infty} \lim_{t \to \infty} \mathbb{P} \left( \frac{L_{\mathcal{L}+ \to E_1} - \mu_\varepsilon(a) t}{t^{1/3}} \leq s \right) \leq \tilde{s}.
\]

Assumption 3. Assume there are independent random variables \( \tilde{L}_{\mathcal{L}+ \to E_1}, \tilde{L}_{\mathcal{L}+ \to E_2} \) such that for a \( \tilde{\psi} \geq 0 \)

\[
\lim_{t \to \infty} \sup \mathbb{P} \left( \{ \tilde{L}_{\mathcal{L}+ \to E_1} \neq \tilde{L}_{\mathcal{L}+ \to E_2} \} \cup \{ \tilde{L}_{\mathcal{L}+ \to E_2} \neq \tilde{L}_{\mathcal{L}+ \to E_2} \} \right) \leq \tilde{\psi}. \tag{2.17}
\]

In Theorem 2.3, \( \tilde{L}_{\mathcal{L}+ \to E_1}, \tilde{L}_{\mathcal{L}+ \to E_2} \) will be last passage times with restricted transversal fluctuations, in Theorem 2.5, they will additionally have restricted starting points.

We denote by

\[
L_{\mathcal{L}+ \to E_1}^\text{rec} = \frac{L_{\mathcal{L}+ \to E_1} - \mu_1^a t}{t^{1/3}} \tag{2.18}
\]

and similarly denote by \( L_{\mathcal{L}+ \to E_1}^\text{rec}, L_{\mathcal{L}+ \to E_1}^\text{rec} \) the LPP times rescaled as in Assumptions 1, 2.
Theorem 2.6. Under Assumptions \[\textup{[1],[2],[3]}\] we have for any \(\delta \geq 0\)
\[
G_1^a(s_1)G_2^a(s_2) \leq \lim_{t_k \to \infty} \mathbb{P}(L_{L^+ \to E_1} \leq s_1, L_{L^- \to E_2} \leq s_2)
\]
\[
\leq G_1^a((s_1 + \delta)c_{\varepsilon(a)}G_2^a(s_2) + G_0^a(-\delta) + 3\tilde{\psi},
\]
where \(\lim_{t_k \to \infty}\) is any subsequential limit.

Proof. The lower bound follows from the fact that \(\{L_{L^+ \to E_1} \leq s_1\}, \{L_{L^- \to E_2} \leq s_2\}\) are decreasing events, the FKG inequality and Assumption \[\textup{[1]}\]. Denote \(A^\delta = \{L_{E^+ \to E_1} \leq -\delta\}\). Noting that \(L_{E^+ \to E_1} \geq L_{L^+ \to E_1} + L_{L^- \to E^+}\) we get from Assumptions \[\textup{[2],[3]}\]
\[
\lim_{t_k \to \infty} \mathbb{P}(L_{L^+ \to E_1} \leq s_1, L_{L^- \to E_2} \leq s_2)
\]
\[
\leq \lim_{t_k \to \infty} \mathbb{P}(\{L_{E^+ \to E_1} + L_{L^+ \to E^+} \leq s_1\} \cap \{L_{L^- \to E_2} \leq s_2\} \cap (A^\delta \cup (A^\delta)^c))
\]
\[
\leq G_0^a(-\delta) + \lim_{t_k \to \infty} \mathbb{P}(\{-\delta + L_{L^- \to E^+} \leq s_1\} \cap \{L_{L^- \to E_2} \leq s_2\} \cap (A^\delta)^c)
\]
\[
\leq G_0^a(-\delta) + \lim_{t_k \to \infty} \mathbb{P}(\{L_{L^+ \to E^-} \leq s_1 + \delta\} \cap \{L_{L^- \to E_2} \leq s_2\})
\]
\[
\leq G_0^a(-\delta) + 3\tilde{\psi} + \lim_{t_k \to \infty} \mathbb{P}(\{L_{E^+ \to E_1} \leq s_1 + \delta\})\mathbb{P}(\{L_{L^- \to E_2} \leq s_2\})
\]
\[
= G_0^a(-\delta) + 3\tilde{\psi} + G_1^a((s_1 + \delta)c_{\varepsilon(a)}G_2^a(s_2)
\]
\[
(2.19)
\]

Clearly, a version of Theorem 2.6 without taking the \(t_k \to \infty\) limit also holds. This could be used to refine the results of \[\textup{[13]}\] by obtaining upper and lower bounds for finite \(t\) in Theorem \[\textup{2.1}\] in \[\textup{[13]}\] and its applications, instead of showing only the convergence to a product as \(t \to \infty\).

Under some extra Assumptions, one obtains a general statement about slow decorrelation on the \(O(t)\) scale. This is what we had proved and used in the first version of this paper, \[\textup{[23]}\]. By using the FKG inequality and subadditivity in Theorem 2.6 the following Theorem is no longer directly used, but we believe it might be instructive to the reader, and refer to \[\textup{[23]}\] for a proof.

Theorem 2.7 (Slow Decorrelation on the \(O(t)\) scale). Assume \(\textup{[2],[13]}\) holds with \(\lim_{a \to \infty} G_1^a = G_1\), \(G_1\) some distribution function. Let Assumption 2 hold with \(\lim_{a \to \infty} \limsup_{t \to \infty} \mathbb{P}(\{L_{E^+ \to E_1} \geq m\} = 0\) for all \(m > 0\) and \(\lim_{a \to \infty} c_{\varepsilon(a)} = 1\). Then, for any \(\delta > 0\) we have
\[
\lim_{a \to \infty} \limsup_{t \to \infty} \mathbb{P}(\{L_{E^+ \to E_1} - L_{E^{-} \to E^+} - \mu_{\varepsilon(a)}t \geq \delta t^{1/3}\} = 0.
\]
\[
(2.20)
\]
3 Transversal Fluctuations

In this section, we provide bounds on the probability of having \(kt^{2/3}, k > 0\) transversal fluctuations in LPP, see Theorems 3.1 and 3.5. We take Theorem 2.5 of [2] (cited here as Theorem 3.2) as key ingredient and bootstrap the polynomial bound it provides to the entire path. For this bootstrapping we adapt the strategy and (mostly) the notation of [4], Lemmas 11.4 - 11.6 where, for Poisson LPP, an exponential bound is bootstrapped to the entire path. In the case of i.i.d. weights, one could adapt the entire proof of Theorem 11.1 of [4] to obtain an upper bound \(Ce^{-ck}\), see also Section 13 of [4]. We have a few reasons why we did not do this. First, we thus obtain, with a short proof, Theorem 3.1, which applies to LPP models with more general weights on the coordinate axes, which include the stationary LPP not considered in [4]. The transversal fluctuations of the stationary LPP are genuinely different from in the i.i.d. case: As was shown recently in [3], Theorem 2, in the case of i.i.d. weights, on a horizontal line \((\cdot, \ell)\), the transversal fluctuations are at most \(O(\ell^{2/3})\), whereas in the stationary case, the maximizer has \(O(t^{2/3})\) fluctuations right away, see Theorem 2.2 b) of [2]. Second, by using [2], the proof of Theorem 3.1 is entirely probabilistic and makes no use of exact (determinantal) formulas and moderate deviation bounds, in contrast to [4].

We start with the relevant definitions. Let \((m, n) \in \mathbb{Z}_2\). Denote for \(l \leq n\)

\[
Z_l^0(m, n) = Z_l^0 = \max\{i: (i, l) \in \pi_{0\to(i(m,n))}^{\max,0}\}
\]

(3.1)

where, with \(A, B \in \mathbb{Z}_{\geq 0}^2\), \(\pi_{A\to B}^{\max,0}\) is the maximizing path from \(A\) to \(B\) in the LPP model with independent weights given \(\omega_{i,j} = 0\) if \(i = 0\) or \(j = 0\), \(\omega_{i,j} \sim \exp(1)\) else. Similarly, define \(Y_{r,TOP,0}(m, n)\) to be the top-most point of \(\pi_{0\to(i(m,n))}^{\max,0}\) on the vertical line \(j = r\). We denote by \(Z_l(m, n), Y_{r,TOP}(m, n)\) the analogous objects for the LPP model with all \(\omega_{i,j} \sim \exp(1), i, j \geq 0\) and independent. For an \(\eta_0 > 0\) we write \(g(\eta_0) = \varrho = \frac{1}{1 + \sqrt{\eta_0}}\) such that

\[
\frac{(1 - \varrho)^2}{\varrho^2} = \eta_0.
\]

(3.2)

For this \(\varrho\) we define independent weights \(\omega_{i,j}, i, j \geq 0\) with distribution

\[
\omega_{i,j} \sim \left\{
\begin{array}{ll}
0 & \text{if } i = j = 0, \\
\exp(1 - \varrho) & \text{if } i \geq 1, j = 0, \\
\exp(\varrho) & \text{if } j \geq 1, i = 0, \\
\exp(1) & \text{if } i, j \geq 1.
\end{array}
\right.
\]

(3.3)

Let now \(\hat{\omega}_{i,j}, i, j \geq 0\) be random variables with \(\hat{\omega}_{i,j} = \omega_{i,j}\) if \(i, j \geq 1\) and

\[
\hat{\omega}_{0,0} = 0, \quad \hat{\omega}_{i,0} \leq \omega_{i,0} \text{ for } i \geq 1, \quad \hat{\omega}_{0,j} \leq \omega_{0,j} \text{ for } j \geq 1.
\]

(3.4)
The weights $\hat{\omega}_{i,j}, i,j \geq 0$, are the only ones in this paper which are not assumed to be independent and continuous (with the exception of weights which may be zero) and where in consequence, one need not have a unique maximizing path. Let $(m, n) \in \mathbb{Z}^2_{\geq 0}$ and $l \leq n$. Denote by $\hat{Z}_i(m,n)$ the right-most point on the horizontal line $j = l$ of the right most maximizing path from $(0,0)$ to $(m,n)$ and denote for $r \leq m$ by $\hat{Y}^{\text{TOP}}_r(m,n)$ the top-most point on the vertical line $i = r$ of the top-most maximizing path from $(0,0)$ to $(m,n)$. The result we get is as follows.

**Theorem 3.1.** Let $m = \lfloor \eta_0 t \rfloor$, $n = \lfloor t \rfloor$. Fix $\alpha \in (2/3,1)$. There is a constant $C(\eta_0, \alpha)$ such that for $k > 0$

$$\limsup_{t \to \infty} \mathbb{P}(\max_{\kappa \in [0,1]} \{\hat{Z}_{[\kappa t]}(m, n) - \kappa \eta_0 t\} \geq kt^{2/3}) \leq \frac{C(\eta_0, \alpha)}{k^{3\alpha}} \quad (3.5)$$

$$\limsup_{t \to \infty} \mathbb{P}(\max_{\kappa \in [0,1]} \{\hat{Y}^{\text{TOP}}_{[\kappa \eta_0 t]}(m, n) - \kappa t\} \geq kt^{2/3}) \leq \frac{C(\eta_0, \alpha)}{k^{3\alpha}}. \quad (3.6)$$

Theorem 3.1 takes the following result from [2] as key ingredient.

**Theorem 3.2 (Part of Theorem 2.5 from [2]).** Let $t \geq 1, \eta_0 > 0$ and $\varphi(\eta_0) \in (0,1)$ be as in (3.2). Set $\tilde{t} = \frac{t}{\varphi(\eta_0)^2}$ and

$$(m, n) = ([\eta_0 t], [t]) = ([1 - \varphi(\eta_0)^2 \tilde{t}], [\varphi(\eta_0)^2 \tilde{t}]).$$

For all $\alpha \in (0,1)$ there is a constant $C(\varphi(\eta_0), \alpha)$ such that for all $k > 0, s \leq \tilde{t}$ and $(r,l) = ([1 - \varphi(\eta_0)^2 s], [\varphi(\eta_0)^2 s])$ we have

$$\mathbb{P}(\hat{Z}_i(m, n) \geq r + kt^{2/3}) \leq \frac{C(\varphi(\eta_0), \alpha)}{k^{3\alpha}}. \quad (3.7)$$

Theorem 3.2 has the following corollary.

**Corollary 3.3 (Corollary of Theorem 3.2).** Let $t > 1, m = \lfloor \eta_0 t \rfloor, n = \lfloor t \rfloor$. Fix $\kappa \in [0,1], \alpha \in (0,1)$. There is a constant $C(\eta_0, \alpha)$ such that for $k > 0$

$$\mathbb{P}(\hat{Z}_{[\kappa t]}(m, n) \geq \kappa \eta_0 t + kt^{2/3}) \leq \frac{C(\eta_0, \alpha)}{k^{3\alpha}} \quad (3.8)$$

$$\mathbb{P}(\hat{Y}^{\text{TOP}}_{[\kappa \eta_0 t]}(m, n) \geq \kappa t + kt^{2/3}) \leq \frac{C(\eta_0, \alpha)}{k^{3\alpha}}. \quad (3.9)$$

**Proof.** Only (3.9) is not obvious, but follows by noting that transposing $(i,j) \to (j,i)$ $\hat{Y}^{\text{TOP}}_{[\kappa \eta_0 t]}(m, n)$ becomes $\hat{Z}_{[\kappa \eta_0 t]}(n, m)$ and that the transposed weights satisfy the assumptions (3.4) with $\tilde{\varphi} = 1 - \varphi(\eta_0)$. \qed
In Theorem 3.5 below, we bound the transversal fluctuations of maximizing paths from \((0, 0)\) to \([\eta_0 t + ct^{2/3}, [t]]\) for some \(c \neq 0\). This requires to show that the constant \(C(\cdot, \alpha)\) in Corollary 3.3 can be taken uniformly bounded in a neigboorhood of \(\eta_0\). We provide a proof of this (for zero weights on the boundary) in the Appendix A, which then shows that \(C(\cdot, \alpha)\) can actually be taken uniformly for all \(g(\eta_0) \in [b_1, b_2]\), for \(0 < b_1 < b_2 < 1\). This results in the following Lemma, which essentially is Lemma 7.4 of [2] with a uniform constant.

**Lemma 3.4.** Let \(\eta_0 > 0\) and \(0 < b_1 < g(\eta_0) < b_2 < 1\). For \(t > 0\) let \(c = c(t) = c_0 + O(t^{-1/3}), c_0 \in \mathbb{R}\). Set \(m = \lfloor \eta_0 t + ct^{2/3} \rfloor, n = \lfloor t \rfloor\). Fix \(\kappa \in [0, 1], \alpha \in (0, 1)\). There are constants \(C = C(b_1, b_2, \alpha), t_0 = t_0(b_1, b_2, c_0, \alpha)\) such that for \(k > 0, t > t_0\)

\[
\mathbb{P}(Z^0_{\lfloor \kappa t \rfloor}(m, n) \geq \kappa(\eta_0 t + ct^{2/3}) + kt^{2/3}) \leq \frac{C}{k^{3\alpha}} \tag{3.10}
\]

\[
\mathbb{P}(Y^{\text{TOP}, 0}_{\lfloor \kappa(\eta_0 t + ct^{2/3}) \rfloor}(m, n) \geq \kappa t + kt^{2/3}) \leq \frac{C}{k^{3\alpha}}. \tag{3.11}
\]

Furthermore, the same statements hold for \(Z_{\lfloor \kappa t \rfloor}, Y^{\text{TOP}}_{\lfloor \kappa(\eta_0 t + ct^{2/3}) \rfloor}\).

Given Corollary 3.3 and Lemma 3.4, we can now prove the following Theorem, as well as Theorem 3.1.

**Theorem 3.5.** Let \(0 < b_1 < g(\eta_0) < b_2 < 1, c = c(t) = c_0 + O(t^{-1/3}), c_0 \in \mathbb{R}\). Set \(m = \lfloor \eta_0 t + ct^{2/3} \rfloor, n = \lfloor t \rfloor\). Fix \(\alpha \in (2/3, 1)\). There is a constant \(C(b_1, b_2, \alpha)\) such that for \(k > 0\)

\[
\limsup_{t \to \infty} \mathbb{P}(\max_{\kappa \in [0, 1]} \{Z^0_{\lfloor \kappa t \rfloor}(m, n) - \kappa(\eta_0 t + ct^{2/3}) \} \geq kt^{2/3}) \leq \frac{C(b_1, b_2, \alpha)}{k^{3\alpha}}. \tag{3.12}
\]

\[
\limsup_{t \to \infty} \mathbb{P}(\max_{\kappa \in [0, 1]} \{Y^{\text{TOP}, 0}_{\lfloor \kappa(\eta_0 t + ct^{2/3}) \rfloor}(m, n) - \kappa t \} \geq kt^{2/3}) \leq \frac{C(b_1, b_2, \alpha)}{k^{3\alpha}}. \tag{3.13}
\]

Clearly, the same statements hold for \(Z_{\lfloor \kappa t \rfloor}(m, n), Y^{\text{TOP}}_{\lfloor \kappa(\eta_0 t + ct^{2/3}) \rfloor}(m, n)\).

**Proof of Theorem 3.5.** We take \(k \geq 1\), for \(k < 1\) we can simply take \(C(b_1, b_2, \alpha) > 1\) to obtain the result. Choose \(j_0 > 0\) such that \(2^{-j_0} \eta_0 t = kt^{2/3}/20\), implying \(2^{-\lfloor j_0 \rfloor} \eta_0 t \in [kt^{2/3}/20, kt^{2/3}/10]\). We define \((j \geq 1)\)

\[
k_j = \frac{k}{10^{10}} \prod_{i=0}^{j-1} (1 + 2^{-i/10}), k_0 = \frac{k}{10^{10}}. \tag{3.14}
\]

Define for \(0 \leq j \leq \lfloor j_0 \rfloor\)

\[
A_j = \left\{ \max_{l=0,\ldots,2} \{Z^0_{\lfloor l(2^{-j} t) \rfloor}(m, n) - l2^{-j}(\eta_0 t + ct^{2/3}) \} \leq k_j t^{2/3} \right\}, \tag{3.15}
\]
note $A_0$ is the full set (for $t$ sufficiently large, such that $k_0t^{2/3} \geq 1$). We note that

$$
\bigcap_{j=0}^{[j_0]} A_j \subseteq \left\{ \max_{\kappa \in [0,1]} Z_{[\kappa t]}^0 - \kappa(\eta_0 t + ct^{2/3}) \leq kt^{2/3} \right\}. \quad (3.16)
$$

Indeed, if $A_{[j_0]}$ holds, then for $\kappa \in [l2^{-[j_0]}, (l+1)2^{-[j_0]}], l \in \{0, \ldots, 2^{[j_0]} - 1\}$ we have for $t$ large enough

$$
Z_{[\kappa t]}^0 - \kappa(\eta_0 t + ct^{2/3}) \leq Z_{[(l+1)2^{-[j_0]}]}^0 - (l+1)2^{-[j_0]}(\eta_0 t + ct^{2/3})
+ (l+1)2^{-[j_0]}(\eta_0 t + ct^{2/3}) - \kappa(\eta_0 t + ct^{2/3})
\leq k_{[j_0]}t^{2/3} + kt^{2/3}/9
\leq kt^{2/3}.
$$

Let $j < [j_0]$ and $0 \leq h \leq 2^j - 1$ and set

$$
u_h = (\lfloor \eta_0 t + ct^{2/3} \rfloor, \lfloor h2^{-j}t \rfloor).
$$

Denote now

$$
Z^{h,j} = \max\{i : (i, \lfloor (h+1/2)2^{-j}t \rfloor) \in \pi_{\nu_h \rightarrow \nu_{h+1}}^{\max}\}
$$

and define

$$
A^{h,j} = \left\{ Z^{h,j} - (h+1/2)2^{-j}(\eta_0 t + ct^{2/3}) \leq k_{j+1}t^{2/3} \right\}. \quad (3.19)
$$

By translation invariance, with $T = 2^{-j}t, E_T = ([\eta_0 T + c2^{-j/3}T^{2/3}], \{T\})$

$$
Z^{h,j} = \max\{i : (i, \lfloor T/2 \rfloor) \in \pi_{[0,T]}^{\max} + \lfloor h2^{-j}(\eta_0 t + ct^{2/3}) + k_jt^{2/3} \rfloor\}
$$

(3.20)

(where $=^d$ denotes equality in distribution). In $[3.20]$, $\pi_{[0,T]}^{\max}$ is the maximizing path from $(0,0)$ to $E_T$ in an LPP model with, for $h \geq 1$, all weights i.i.d. $\exp(1)$ and, for $h = 0$, with weights 0 on the $x$–axis, i.i.d. $\exp(1)$ else. Noting that extra weights on the vertical axis cannot increase the maximal fluctuations to the right, we obtain

$$
P((A^{0,j})^c) \leq P(Z_{[T/2]}^0 (E_T) - \frac{1}{2}(\eta_0 T + c2^{-j/3}T^{2/3}) \geq (k_{j+1} - k_j)^2/3 - 1)
\leq P(Z_{[T/2]}^0 (E_T) - \frac{1}{2}(\eta_0 T + c2^{-j/3}T^{2/3}) \geq k_j2^{j/2}T^{2/3})
$$

(3.21)

For $h \geq 1$ we obtain an upper bound with $Z^0$ replaced by $Z$ in $[3.21]$. So by the definition of $k_j$ and Lemma 3.4 we thus get that there is a $t_0$ such that for all $t > t_0$

$$
P((A^{h,j})^c) \leq \frac{C(b_1, b_2, \alpha)}{2^{5\alpha j/2}k^{3\alpha}}
$$

(3.22)
and consequently, taking \( 3\alpha/2 - 1 =: f(\alpha) > 0 \)
\[
\sum_{h=0}^{2^i-1} \mathbb{P}((A^{h,j})^c) \leq \frac{C(b_1, b_2, \alpha)}{2^{if(\alpha)}k^{3\alpha}}. \tag{3.23}
\]
Note now that for \( 0 \leq j \leq \lfloor j_0 \rfloor - 1 \) we have \( A_{j+1}^c \cap A_j \subseteq \bigcup_{h=0}^{2^j-1} (A^{h,j})^c \).
Consequently by (3.23),
\[
\sum_{j=0}^{\lfloor j_0 \rfloor - 1} \mathbb{P}(A_{j+1}^c \cap A_j) \leq \frac{C_1(b_1, b_2, \alpha)}{k^{3\alpha}}. \tag{3.24}
\]
We have \( \bigcup_{j=1}^{\lfloor j_0 \rfloor} A_j^c \subseteq \bigcup_{j=1}^{\lfloor j_0 \rfloor} A_j \cap A_{j-1} \), thus (3.12) follows from (3.10).

The result for \( Y_{\text{TOP}}^{\lfloor \kappa t \rfloor}(m,n) \) follows by the transposition \( \omega_{i,j} \rightarrow \omega_{j,i} \) under which \( Y_{\text{TOP}}^{\lfloor \kappa t + ct^2/3 \rfloor}(m,n) \) becomes \( Z_{\lfloor \kappa t + ct^2/3 \rfloor}(n,m) \). We can now proceed as in the proof of (3.11), by maximizing over \( \kappa \) in (A.22), to reach the result.

Finally, the result for \( Z_{\lfloor \kappa t \rfloor}(m,n) \) follows by noting \( Z_{\lfloor \kappa t \rfloor}(m,n) = d Z_{\lfloor \kappa t \rfloor+1}(m+1,n+1) \) and then proceeding as in the proof of (3.10). The result for \( Y_{\text{TOP}}^{\lfloor \kappa t \rfloor}(m,n) \) follows again by transposition and the result for \( Z_{\lfloor \kappa t \rfloor}(m,n) \).

Proof of Theorem 3.1. The proof of Theorem 3.1 is similar to the proof of Theorem 3.5 but a little simpler. We use Corollary 3.3 instead of Lemma 3.4 and set \( c = 0 \) in the proof of Theorem 3.5. We define \( \hat{A}_j \) as \( A_j \) in (3.15) but with \( \hat{Z} \) instead of \( Z^0 \). Defining \( \hat{A}^{h,j} \) as \( A^{h,j} \) in (3.19) (but with the boundary weights of Theorem 3.1), we can bound \( \mathbb{P}((\hat{A}^{h,j})^c) \) for all \( h \) by Corollary 3.3 since, now the boundary weights for \( \pi_{0 \rightarrow ET}^{\max} \) of (3.20) can via coupling be bounded by the \( \omega_{i,0}, \omega_{0,j}, i, j \geq 1 \), from (3.3) (for \( h \geq 1 \) the LPP model for \( \pi_{0 \rightarrow ET}^{\max} \) has a non-zero weight at the origin, but this does not affect the transversal fluctuations). We obtain a bound on \( \mathbb{P}(\bigcup_{j=1}^{\lfloor j_0 \rfloor} \hat{A}_j^c) \) as in the proof of Theorem 3.5 but here we have to use Corollary 3.3 to also bound \( \mathbb{P}((\hat{A}_0)^c) \), and thus obtain
\[
\limsup_{t \to \infty} \mathbb{P}(\bigcup_{j=0}^{\lfloor j_0 \rfloor} \hat{A}_j^c) \leq \frac{C(\eta_0, \alpha)}{k^{3\alpha}}, \tag{3.25}
\]
proving (3.5). The result for \( Y_{\text{TOP}}(m,n) \) is easily obtained, since after transposition, the transposed weights satisfy (3.4) for \( \tilde{\varphi} = 1 - \varphi \), and \( Y_{\text{TOP}}(m,n) \) becomes \( \hat{Z}(n,m) \), giving the result.
4 Proof of Theorems 2.1 and 2.3

We start by recalling the following result for point-to-point LPP.

Proposition 4.1 (Theorem 1.6 of [16], Theorem 2 of [7]). Let $0 < \eta < \infty$, $\eta = \eta_0 + c\ell^{-1/3}$. Then,
\[
\lim_{t \to \infty} \mathbb{P} \left( L_{0 \to ([\eta], [\ell])] \leq \mu_{pp} t + \sigma_\eta \ell^{1/3} \right) = \mathbb{F}_{\text{GUE}}(s) \tag{4.1}
\]
where $\mu_{pp} = (1 + \sqrt{\eta})^2$, and $\sigma_\eta = \eta^{-1/6}(1 + \sqrt{\eta})^{4/3}$. In particular, with $\mathcal{L}^+, \mathcal{L}^-, \mu^a t$ as in Theorem 2.3, we have
\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{L_{\mathcal{L}^+(t + ut^{2/3}/a, t)} - \mu^a t}{2^{4/3} t^{1/3}} \leq s \right) = \mathbb{F}_{\text{GUE}}(s) \tag{4.2}
\]
Next we choose the point $E^+$ from Assumption 2. From Proposition 4.1 one can easily compute that $E^+$ should lie on the line segment from $\mathcal{L}^+$ to $E = ([t + ut^{2/3}, [t])$, so it remains to choose $\varepsilon(a)$. To motivate this choice, note that by Theorem 3.5 we can control the probability that $\pi_{\mathcal{L}^+ \to E}^{\max}, \pi_{\mathcal{L}^- \to E}^{\max}$ have transversal fluctuations of order $kt^{2/3}$. In particular, we have a good upper bound for the probability that $\pi_{\mathcal{L}^+ \to E}^{\max}$ contains no point of the straight line $\mathcal{R}^+$ which joins (in $\mathbb{Z}^2$) the points $([a t^{2/3} + kt^{2/3}, [t])$ and $E + ([kt^{2/3}, [0])$ and for the probability that $\pi_{\mathcal{L}^- \to E}^{\max}$ contains no point of the straight line $\mathcal{R}^-$ joining $(0, [a t^{2/3} + kt^{2/3}])$ and $E + ([0, [kt^{2/3}])$. Now an elementary calculation reveals that $\mathcal{R}^-$ and $\mathcal{R}^+$ cross in a point
\[
\left( \left[t \left(1 - \frac{k}{a}\right) + O(t^{2/3})\right], \left[t \left(1 - \frac{k}{a}\right) + O(t^{2/3})\right] \right), \quad (4.3)
\]
see Figure 1. In view of Assumption 3 we thus should choose $\varepsilon(a) > \frac{k}{a}$, though to satisfy Assumption 2 this is not necessary, as the following result shows.

Proposition 4.2. Let $1 > \varepsilon(a) > 0$. Then Assumption 2 holds with
\[
E^+ = ([t(1 - \varepsilon(a)) + t^{2/3}(u/a - \varepsilon(a)(u/a + a)), [t(1 - \varepsilon(a))]) \tag{4.4}
\]
\[
\mu^{\varepsilon(a)} t = 4\varepsilon(a)t + 2\varepsilon(a)(u/a + a)t^{2/3} - \frac{\varepsilon(a)(a + u/a)^2}{4} t^{1/3} \tag{4.5}
\]
\[
c^{\varepsilon(a)} = (1 - \varepsilon(a))^{-1/3}. \tag{4.6}
\]
\[
G^\varepsilon_a(s) = F_{\text{GUE}}(s\varepsilon(a)^{-1/3}) \tag{4.7}
\]
Figure 1: The maximizing path (blue) from $\mathcal{L}^- = (0, -\lfloor at^{2/3} \rfloor)$ to $E = ([t + \frac{u}{a}t^{2/3}], [t])$ crosses the line segment $R_-(k)$ (dotted) with vanishing probability as $\lim_{k \to \infty} \lim_{t \to \infty}$. The point $E^+$ is at distance $\frac{u}{a}t$ from $E$ on the line connecting $E$ with $\mathcal{L}^+ = (-\lfloor at^{2/3} \rfloor, 0)$ (see (4.4)). The maximizer from $\mathcal{L}^+ = \mathcal{L}^+ + (-\lfloor at^{2/3} \rfloor, 0)$ crosses $R_+(k)$ (dashed) with vanishing probability. So the two maximizers do not cross asymptotically, leading to the decoupling.

Proof. We have $L_{\mathcal{L}^+ \to E^+} = d L_{0 \to ([t(1-\varepsilon(a)) + \lfloor t(1-\varepsilon(a)) \rfloor, [t])}$ for $r_1 = (u/a + a)(1 - \varepsilon(a))$. The $\mu_{pp}$ of Proposition 4.1 for $L_{\mathcal{L}^+ \to E^+}$ is given by $\mu_{pp} = 4t(1 - \varepsilon(a)) + 2t^{2/3}r_1 - \frac{r_1}{4(1 - \varepsilon(a))}t^{1/3}$ and the one, with $E = ([t + \frac{u}{a}t^{2/3}], [t])$, of $L_{E^+ \to E}$ equals $\mu_{pp} = 4t\varepsilon(a) + 2t^{2/3}r_2 - \frac{r_2}{4t\varepsilon(a)}t^{1/3}$, with $r_2 = -r_1 + u/a + a$ and since the two terms need to sum up to $\mu^a$ from Theorem 2.3 we obtain the condition

$$\frac{r_1}{4t} + \frac{r_2}{4t} = \frac{(u/a + a)^2}{2},$$

which is precisely solved by our $r_1$. Finally, $c_{\varepsilon(a)}$ and $G^a_0$ are immediately obtained from Proposition 4.1.

Let now $E^+$ be as in (4.14) and denote $E^{+, k} = E^+ + (kt^{2/3}, 0)$. Define $\mathfrak{R}_+(k) = (\lfloor at^{2/3} + kt^{2/3} \rfloor, 0)E^{+, k}$ as the line segment (in $\mathbb{R}^2$ ) from $(-at^{2/3} + kt^{2/3}, 0)$ to $E^{+, k}$, and denote

$$R_+(k) = \{x \in \mathbb{Z}^2 : |x - y| \leq 2 \text{ for a } y \in \mathfrak{R}_+(k)\}$$

(4.9)

a discrete approximation. See Figure III. Denote by $\Pi^{+, k}$ the set of up-right paths from $\mathcal{L}^+$ to $E^+$ which do not contain any point of $R_+(k)$. Set

$$\bar{L}_{\mathcal{L}^+ \to E^+} = \bar{L}_{\mathcal{L}^+ \to E^+}(k) = \max_{\pi \in \Pi^{+, k}} \sum_{(i,j) \in \pi} \omega_{i,j}.$$  

(4.10)
Let now $E = ([t + \frac{u}{a}t^{2/3}, [t])$ and $E^k = E + (0, kt^{2/3})$. Write $\mathcal{R}_-(k) = (0, [-at^{2/3} + kt^{2/3}]) \mathbb{E}^k$ for the line segment in $\mathbb{R}^2$ joining $(0, [-at^{2/3} + kt^{2/3}])$ and $E^k$ and set

$$R_-(k) = \{x \in \mathbb{Z}^2 : |x - y| \leq 2 \text{ for a } y \in \mathcal{R}_-(k)\}. \quad (4.11)$$

Define $\Pi^{-k}$ to be the set of up-right paths from $\mathcal{L}^-$ to $E$ which do not contain any point of $R_-(k)$. We define

$$\tilde{L}_{\mathcal{L}^- \to E} = \tilde{L}_{\mathcal{L}^- \to E}(k) = \max_{\pi \in \Pi^{-k}} \sum_{i,j} \omega_{i,j}. \quad (4.12)$$

**Proposition 4.3.** Let $k > 0$ and let $E^+$ be given by (4.4) with $\varepsilon(a) > \frac{\varepsilon}{a}$ and let $\tilde{L}_{\mathcal{L}^- \to E}(k), \tilde{L}_{\mathcal{L}^+ \to E^+}(k)$ be given by (4.12), (4.10). Then for any $\alpha \in (0, 1)$ there is a constant $C(\alpha) > 0$ such that Assumption [3] holds with $\psi = C(\alpha)k^{-3\alpha}$.

**Proof.** Note that we have $L_{\mathcal{L}^+ \to E^+} = dL_{0 \to ([t(1 - \varepsilon(a)) + t^{2/3}r_1], [t(1 - \varepsilon(a))])}$ for $r_1 = (u/a + a)(1 - \varepsilon(a))$. Write $([t(1 - \varepsilon(a)) + t^{2/3}r_1], [t(1 - \varepsilon(a))] = (m_+, n_+)$. Thus by translation invariance and Theorem 3.5 (fix some $b_1 < 1/2 < b_2$)

$$\limsup_{t \to \infty} \mathbb{P}(L_{\mathcal{L}^+ \to E^+} \neq \tilde{L}_{\mathcal{L}^+ \to E^+}(k)) \leq \limsup_{t \to \infty} \mathbb{P}(\max_{\kappa \in [0, 1]} Z_{\kappa m_+}(m_+, n_+) - \kappa m_+ \geq kt^{2/3}) \leq C(\alpha)k^{-3\alpha}. \quad (4.14)$$

Furthermore,

$$L_{\mathcal{L}^- \to E} = dL_{0 \to ([t + \frac{u}{a}t^{2/3}, [t]) + (aT^{2/3})} \quad (4.15)$$

Setting $T = [t] + [at^{2/3}]$ we have $t + \frac{u}{a}t^{2/3} = T + (\frac{u}{a} - a)T^{2/3} + O(T^{1/3}) = T + c_-(T)T^{2/3}$ for a $c_-(T) = \frac{u}{a} - a + O(T^{-1/3})$ such that

$$L_{\mathcal{L}^- \to E} = dL_{0 \to ([T + c_-(T)T^{2/3}], [T])}. \quad (4.16)$$

We thus get

$$\limsup_{t \to \infty} \mathbb{P}(L_{\mathcal{L}^- \to E} \neq \tilde{L}_{\mathcal{L}^- \to E}) \leq \limsup_{t \to \infty} \mathbb{P}\left(\max_{\kappa \in [0, 1]} \{Y_{\kappa T}^{\text{TOP}} - \kappa T\} \geq kT^{2/3}/2\right) \leq C(\alpha)k^{-3\alpha}. \quad (4.13)$$

Finally, the independence of $\tilde{L}_{\mathcal{L}^- \to E}(k), \tilde{L}_{\mathcal{L}^+ \to E^+}(k)$ follows from choosing $\varepsilon(a) > k/a$ and (4.3): The admissible paths for $\tilde{L}_{\mathcal{L}^+ \to E^+}(k)$ do not cross
$R_+(k)$ from (1.9), and the admissible paths for $\hat{L}_{\mathcal{L}\rightarrow E}(k)$ do not cross $R_-(k)$ from (1.11), and since $\varepsilon(a) > k/a$, we have by (4.3) that $R_+(k), R_-(k)$ do not cross each other, see also Figure 1. So $\hat{L}_{\mathcal{L}\rightarrow E}(k), \hat{L}_{\mathcal{L}\rightarrow E}(k)$ may only use points from disjoint, (deterministic) subsets of $\mathbb{Z}^2$, leading to the independence.

\[\square\]

**Proof of Theorem 2.6.** Assumptions 1, 2, 3 of Theorem 2.6 have been verified in Propositions 4.1, 4.2, 4.3, such that the result follows.

Next we proof Theorem 2.1.

**Proof of Theorem 2.1.** Define $c_1 = -\frac{\xi_1 a}{2}, c_2 = \frac{\xi_2 a}{2}$ and $\xi_2 = (\frac{a}{a + a})^2 - 2^{1/3}s$.

Note that (see e.g. Theorem 5 in [8]) for $K \in \mathbb{N}, v \in \mathbb{R}, \gamma \in [0, 1/3]$

\[
\lim_{K \rightarrow \infty} \frac{L_0 - (K + [K + v], K)}{1/3} - L_0 - (K, K) - 2^{1/3} = 0
\]  

(4.17)

In particular, since we are only interested in asymptotic results, any shift of order 1 of the end/starter point for a point-to-point LPP time will be asymptotically irrelevant.

We set

\[
t = \left\lfloor \frac{T}{4} + c_1 T^{2/3} \right\rfloor.
\]

\[
M = t + [c_2 T^{2/3} + \xi_2 T^{1/3}]
\]

Then

\[
T = 4t - c_1 t^{2/3} 4^{5/3} + c_1 2 t^{1/3} 4^{7/3} + O(1)
\]

\[
T^{2/3} = (4t)^{2/3} - c_1 2 t^{1/3} 4^{4/3} + O(1)
\]

(4.18)

\[
T^{1/3} = (4t)^{1/3} + O(1)
\]

We define furthermore

\[
\hat{L}^+ = (\lfloor -a((4t)^{2/3} - c_1 2 t^{1/3} 4^{4/3}) \rfloor, 0) \quad \hat{L}^- = (0, \lfloor -a((4t)^{2/3} - c_1 2 t^{1/3} 4^{4/3}) \rfloor)
\]  

(4.19)
and \( \hat{L} = \hat{L}^+ \cup \hat{L}^- \). Then by the link (2.3)
\[
\lim_{T \to \infty} \mathbb{P}\left( x_{\left[ T^{2/3}s^{2/3} + \frac{x}{2} \right]}(T) \geq \frac{u}{a}T^{2/3} + T^{1/3}\left( \frac{u}{a} + a \right)^2 - \frac{T^{1/3}}{21/3} \right)
= \lim_{T \to \infty} \mathbb{P}\left( L_{\left\{ t \in [T^{2/3}aT^{1/3} + c_2T^{2/3} + \xi_2T^{1/3}], \left[ T^{2/3}aT^{1/3} + c_2 \right] \right\}}(t) \leq T \right)
= \lim_{t \to \infty} \mathbb{P}\left( \hat{L}_{\to(M,t)} \leq 4t - c_1t^{2/3}4^{5/3} + c_3^2t^{1/3}4^{7/3} \right)
\] (4.20)

We now check the Assumptions 1, 2, 3 for the LPP times \( L_{\hat{L}^+ \to(M,t)} \), \( L_{\hat{L}^- \to(M,t)} \). By Proposition 4.1 and (4.17), we have with \( \mu_{\hat{L}^+ \to(M,t)} \), \( \mu_{\hat{L}^- \to(M,t)} \) defined by
\[
\mu_{\hat{L}^+ \to(M,t)} = 4t + 2(c_2 + a)(4t)^{2/3} - 41/3(c_2 + a)^2t^{1/3} + 2(4t)^{1/3}(\xi_2 - 8c_1(c_2 + a)/3)
\]
\[
\mu_{\hat{L}^- \to(M,t)} = \mu_{\hat{L}^+ \to(M,t)} + u4^{4/3}t^{1/3}
\] (4.21)

the convergence
\[
\lim_{t \to \infty} \mathbb{P}\left( L_{\hat{L}^+ \to(M,t)} \leq \mu_{\hat{L}^+ \to(M,t)} + s2^{4/3}t^{1/3} \right) = F_{\text{GUE}}(s)
\]
\[
\lim_{t \to \infty} \mathbb{P}\left( L_{\hat{L}^- \to(M,t)} \leq \mu_{\hat{L}^- \to(M,t)} + s2^{4/3}t^{1/3} \right) = F_{\text{GUE}}(s).
\] (4.22)

The choice of \( c_1, c_2, \xi_2 \) is precisely such that
\[
\mu_{\hat{L}^+ \to(M,t)} + s2^{4/3}t^{1/3} = 4t - c_1t^{2/3}4^{5/3} + c_3^2t^{1/3}4^{7/3}.
\] (4.23)

So (4.22) verifies Assumption 1. Next we choose the point \( \hat{E}^+ \) of Assumption 2. Note that with \( \tilde{a} = a4^{2/3}, \tilde{u} = 4^{4/3}u \) we have
\[
L_{\hat{L}^+ \to(M,t)} = L_{\left\{ t \in [-\tilde{a}t^{2/3} + O(t^{1/3})], 0 \right\} \to \left( \left[ t + \tilde{a}t^{2/3} + O(t^{1/3}) \right], t \right)}.
\] (4.24)

This is, with \( \tilde{a}, \tilde{u} \) instead of \( a, u \) and up to an \( O(t^{1/3}) \) horizontal shift in the starting and end point, the same LPP time for which we chose \( E^+ \) in (4.4). By (4.17), any shift of order \( t^{1/3} \) in \( E^+ \) just leads, in the \( t \to \infty \) limit, to deterministic shifts in \( L_{\hat{L}^+ \to E^+}, L_{\hat{E}^+ \to(M,t)} \), which cancel each other out. Hence we can take \( \hat{E}^+ = E^+ \) in (4.4), only with \( \tilde{a}, \tilde{u} \) instead of \( a, u \). Finally, Assumption 3 can be verified as in the proof of Theorem 2.3 the \( O(t^{1/3}) \) shifts in (4.24) (and in \( L_{\hat{L}^- \to(M,t)} \) ) not affecting the argument.

5 Decoupling of Airy processes and Two-Time Distribution

In this Section we prove Corollary 2.1 and Theorem 2.3 as well as Theorem 5.2 about the decoupling along time-like directions. We also shortly discuss
Lemma 5.1. Let $\mathcal{L}$ be as in Theorem 2.3. We have

$$
\lim_{t \to \infty} \mathbb{P}\left( \frac{L_{(t, t + \frac{u}{2^{5/3}}/a^2)} - \mu^a t}{2^{4/3} a^{1/3}} \leq s \right) = \mathbb{P}\left( A_2\left( \frac{-a - u}{2^{5/3}} \right) \leq s \right), \quad (5.1)
$$

$$
A_2\left( \frac{a - u}{2^{5/3}} \right) \leq s - \frac{u}{2^{5/3}} \right). \quad (5.2)
$$

Proof (Outline). By exchanging the end point and $\mathcal{L}$ we see that

$$
\lim_{t \to \infty} \mathbb{P}(L_{(t, t + \frac{u}{2^{5/3}}/a^2)} = \mathbb{P}(L_{(0, 0)} \rightarrow \{(t + u/a + at^2/3, t), (t + at^2/3/a, t + at^2/3)\}) \quad (5.3)
$$

where $=^d$ denote equality in distribution. One can now translate (5.3) back to TASEP with step initial data, and use (2.23) of [5], which treats a generalization of TASEP (though some details of the asymptotics leading to (2.23) in [5] were not carried out); for geometric LPP (of which the exponential LPP is a limit), the convergence to the Airy$_2$ process was shown in Theorem 1.1 of [18].

Proof of Corollary 2.4. It is an immediate Corollary of Theorem 2.3 and Lemma 5.1 by a simple change of variable.

Proof of Theorem 2.5. We start by proving (2.9). We apply Theorem 2.6 with $\mathcal{L} = \mathcal{L}^+ = \mathcal{L}^- = \{(-k, k) : k \in \mathbb{Z}\}, E_+ = E_1 = ([t], [t]), E_2 = ([t] - |at^{2/3}|, [t] + |at^{2/3}|)$. One obtains from Theorem 2.2 of [5] and the link between TASEP and LPP (all weights i.i.d., $\omega_{i,j} \sim \exp(1)$)

$$
\lim_{t \to \infty} \mathbb{P}(\cap_{i=1}^2 \{L_{\mathcal{L} \rightarrow E_i} \leq 4t + st^{1/3}\}) = \mathbb{P}(A_1(0) \leq 2^{5/3}s_1, A_1(at^{-2/3}) \leq 2^{-5/3}s_2). \quad (5.4)
$$

Also for $i = 1, 2$ we have

$$
\lim_{t \to \infty} \mathbb{P}(L_{\mathcal{L} \rightarrow E_i} \leq 4t + st^{1/3}) = F_{\text{GOE}}(2^{-2/3}s). \quad (5.5)
$$

Now (5.5) shows Assumption 1 and Assumption 2 is trivially fulfilled with $c_{(a)} = 1, G_0 = 1_{[0, \infty)}$. Set now for $k \in \mathbb{Z}$

$$
\mathcal{F}^1(k) = \{(-i, i), i = -\lfloor kt^{2/3}\rfloor, \ldots, \lfloor kt^{2/3}\rfloor\},
$$

$$
\mathcal{F}^2(k) = \{(-|at^{2/3}| - i, |at^{2/3}| + i), i = -\lfloor kt^{2/3}\rfloor, \ldots, \lfloor kt^{2/3}\rfloor\}.
$$

Denote by $\pi_{\mathcal{L} \rightarrow E_i}^\text{max}$ the maximizing path from $\mathcal{L}$ to $E_i$ and by $\pi_{\mathcal{L} \rightarrow E_i}(0)$ the point of $\pi_{\mathcal{L} \rightarrow E_i}^\text{max}$ which belongs to $\mathcal{L}$. By a simple shift one sees $\mathbb{P}(\pi_{\mathcal{L} \rightarrow E_i}^\text{max}(0) \in \mathcal{L} \rightarrow E_i) \leq 4t + st^{1/3}) = F_{\text{GOE}}(2^{-2/3}s). \quad (5.5)$
\(F^i(k)\) is the same for \(i = 1, 2\). Consequently, by (4.18) of \([15]\) one gets that for \(k\) sufficiently large and some constants \(C, c\)

\[
P(\bigcup_{i=1}^{2} \{ \pi_{\mathcal{L} \rightarrow E_i}(0) \notin F^i(k) \}) \leq Ce^{-ck^2}. \tag{5.6}
\]

Let \(E_3 = (\lfloor \frac{3\alpha t^{2/3}}{4}, \lfloor \frac{3\alpha t^{2/3}}{4} \rfloor \rfloor, \lfloor \frac{3\alpha t^{2/3}}{4}, \lfloor \frac{3\alpha t^{2/3}}{4} \rfloor \rfloor)\) and \(E_4 = E_3 + E_1\). Define also \(E_5 = (\lfloor \frac{3\alpha t^{2/3}}{4}, \lfloor \frac{3\alpha t^{2/3}}{4} \rfloor \rfloor, \lfloor \frac{3\alpha t^{2/3}}{4}, \lfloor \frac{3\alpha t^{2/3}}{4} \rfloor \rfloor)\), \(E_6 = E_5 + E_1\). Denote by \(R_1(k)\) the straight line (in \(\mathbb{Z}^2\)) which goes through \(E_3 + (\lfloor kt^{2/3} \rfloor, \lfloor kt^{2/3} \rfloor)\) and \(E_4 + (\lfloor kt^{2/3} \rfloor, \lfloor kt^{2/3} \rfloor)\) and by \(R_2(k)\) the straight line going through \(E_5 + (\lfloor kt^{2/3} \rfloor, -\lfloor kt^{2/3} \rfloor)\) and \(E_6 + (\lfloor kt^{2/3} \rfloor, -\lfloor kt^{2/3} \rfloor)\). Denote by \(U_1\) the event \(\{ \pi_{\mathcal{L} \rightarrow E_i}(a/10) = \emptyset \}\) and by \(U_2\) the event \(\{ \pi_{\mathcal{L} \rightarrow E_i} \cap R_1(a/10) = \emptyset \}\).

It follows from Theorem 3.1 that for any \(\alpha \in (0, 1)\) there is a \(C(\alpha) > 0\) such that

\[
\limsup_{t \to \infty} P(\bigcup_{i=1}^{2} U_i^c) \leq C(\alpha)a^{-3\alpha}. \tag{5.7}
\]

Now the event

\[
\bigcap_{i=1}^{2} \{ \pi_{\mathcal{L} \rightarrow E_i}(0) \in F^i(a/10) \} \cap \bigcap_{i=1}^{2} U_i \tag{5.8}
\]

is a subset of

\[
\bigcap_{i=1}^{2} \{ \pi_{\mathcal{L} \rightarrow E_i} \cap R_1(a/10) = \emptyset \} \tag{5.9}
\]

(taking \(a, t\) sufficiently large). Denote now for \(i = 1, 2\) by \(\Pi^i\) the set of up-right paths from \(\mathcal{L}\) to \(E_i\) which contain no point of \(R_i(a/10)\). Set

\[
\tilde{L}_{\mathcal{L} \rightarrow E_i} = \max_{\pi \in \Pi^i} \sum_{(i,j) \in \pi} \omega_{i,j}. \tag{5.10}
\]

Note that \(\tilde{L}_{\mathcal{L} \rightarrow E_i}, i = 1, 2\) are independent, and by (5.6), (5.7) Assumption 3 is fulfilled with

\[
\tilde{\psi} = \tilde{\psi}(a) = Ce^{-c\alpha^2} + C(\alpha)a^{-3\alpha} \tag{5.11}
\]

(as mentioned earlier, the \(C(\alpha)a^{-3\alpha}\) improves to \(Ce^{-c\alpha}\) by adapting Theorem 11.1 of [2] to the exponential case). This finishes the proof.

Next we come to the proof of (2.12). Set \(\mathcal{L}^{\text{half}} = \{(-k, k) : k \geq 0\}\). It follows from Theorem 2 of [2] that with \(E(k) = ([t - kt^{2/3}], [t + kt^{2/3}])\) and \(b_1, b_2 \in \mathbb{R}\)

\[
\lim_{t \to \infty} P(\bigcap_{i=1}^{2} \{ L_{\mathcal{L}^{\text{half}} \rightarrow E(b_i)} \leq 4t + (s_i - 2^{4/3} \min(0, b_i)^{2} t^{1/3}) \})
\]

\[
= P(\mathcal{A}_{2 \to 1}(b_1 2^{-2/3}) \leq 2^{-4/3}s_1, \mathcal{A}_{2 \to 1}(b_2 2^{-2/3}) \leq 2^{-4/3}s_2) \tag{5.12}
\]

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To localize \( \pi_{L^{\text{half}} \to E(b+a)}^{\text{max}}(0) \) note that by a simple coupling, with \( E_7 = (-\lfloor (|b| + a)t^{2/3} \rfloor, \lfloor (|b| + a)t^{2/3} \rfloor) \) and

\[
\mathcal{F}_3^3(k) = \{ E_7 + (-i, i) : i = -\lfloor kt^{2/3} \rfloor, \lfloor kt^{2/3} \rfloor \}
\]

we have \( \{ \pi_{L^{\text{half}} \to E(b+a)}^{\text{max}}(0) \in \mathcal{F}_3^3(a/10) \} \subseteq \{ \pi_{L^{\text{half}} \to E(b+a)}^{\text{max}}(0) \in \mathcal{F}_3^3(a/10) \} \) such that \( \mathbb{P}(\{ \pi_{L^{\text{half}} \to E(b+a)}^{\text{max}}(0) \notin \mathcal{F}_3^3(a/10) \}) \leq Ce^{-ca^2} \) by (4.18) in [15]. Similarly, one can control \( \mathbb{P}(\pi_{L^{\text{half}} \to E(b+a/5)}^{\text{max}}(0) \notin \mathcal{F}_4(a/20)) \leq Ce^{-ca^2} \), where \( \mathcal{F}_4(k) = \{ E_8 + (-i, i) : i = -\lfloor kt^{2/3} \rfloor, \lfloor kt^{2/3} \rfloor \} \) with \( E_8 = (-\lfloor (|b| + a/5)t^{2/3} \rfloor, \lfloor (|b| + a/5)t^{2/3} \rfloor) \). Let \( R_3(k) \) be the line which connects \( E_8 + (- \lfloor kt^{2/3} \rfloor, \lfloor kt^{2/3} \rfloor) \) with \( E_7 + (\lfloor kt^{2/3} \rfloor, -\lfloor kt^{2/3} \rfloor) \) and \( R_4(k) \) the line which connects \( E_7 + (\lfloor kt^{2/3} \rfloor, -\lfloor kt^{2/3} \rfloor) \) with \( E(|b| + a) + (\lfloor kt^{2/3} \rfloor, -\lfloor kt^{2/3} \rfloor) \). As was done above, we can bound

\[
\begin{align*}
\mathbb{P}(\pi_{L^{\text{half}} \to E(b+a/5)}^{\text{max}}(0) \cap R_3(a/10) \neq \emptyset) & \leq Ce^{-ca^2} + C(\alpha)a^{-3\alpha} \\
\mathbb{P}(\pi_{L^{\text{half}} \to E(b+a)}^{\text{max}}(0) \cap R_4(a/10) \neq \emptyset) & \leq Ce^{-ca^2} + C(\alpha)a^{-3\alpha}.
\end{align*}
\]

(5.13)

Note now that if \( \pi_{L^{\text{half}} \to E(b+a/5)}^{\text{max}}(0) \) contains no point of \( R_3(k) \) then this is also true for \( \pi_{L^{\text{half}} \to E(b)}^{\text{max}}(0) \). Let \( \Pi_3 \) be the up-right paths from \( L^{\text{half}} \) to \( E(b) \) which contain no point of \( R_3(a/10) \), and \( \Pi_4 \) be the up-right paths from \( L^{\text{half}} \) to \( E(|b| + a) \) which contain no point of \( R_4(a/10) \). We define the independent random variables

\[
\tilde{L}_{L^{\text{half}} \to E(b)} = \max_{\pi \in \Pi_3} \sum_{(i,j) \in \pi} \omega_{i,j} \quad \tilde{L}_{L^{\text{half}} \to E(b+a)} = \max_{\pi \in \Pi_4} \sum_{(i,j) \in \pi} \omega_{i,j}.
\]

(5.14)

Now we take \( L^{\text{half}} = L^+ = L^- \), \( E_1 = E_+ = E(b) \), \( E_2 = E(|b| + a) \). Then Assumption [1] holds by (5.12), and Assumption [2] holds trivially with \( c_{(\phi)} = 1 \). Finally, by (5.13), Assumption [3] holds with \( \tilde{\psi} \) as in (5.11). \( \square \)

Finally, we show the decoupling of last passage percolation times along the time-like direction. Denote for \( x, y \in \mathbb{R} \) \( P(x, y) = (\lfloor -y(xt)^{2/3} \rfloor, 0) \) and \( \mu(x, y) = 4xt - 2y(xt)^{2/3} + \frac{x^2}{3}(xt)^{1/3} \) and denote

\[
L_{P(x, y) \to ([xt], [xt])}^{\text{resc}} = \frac{L_{P(x, y) \to ([xt], [xt])} - \mu(x, y)t}{2^{4/3}(xt)^{1/3}}
\]

(5.15)

For e.g. points lying on a line with slope 1, the decoupling we consider corresponds to look for \( c < a \) at

\[
\lim_{t \to \infty} \mathbb{P}(\{ L_{0 \to ([at], [at])}^{\text{resc}} \leq s \} \cap \{ L_{0 \to ([at], [at])}^{\text{resc}} \leq \zeta \})
\]

(5.16)
and then let $a$ go to infinity. It is a priori not clear if (5.2) exists, hence we work with an arbitrary subsequential limit in (5.18). For the case of brownian directed percolation, Johansson proved in [19] an explicit formula for (5.16). We expect (see Remark 2.3 in [19]) that (the analogue of) (5.16) converges to $F_{\text{GUE}}(s)F_{\text{GUE}}(\zeta)$ as $a \to \infty$ and notes that this can be checked heuristically but that it appears rather subtle. Here we show that this decoupling occurs for the multipoint two time distribution in exponential LPP (where no explicit formulas are available), by a soft probabilistic argument.

**Theorem 5.2.** Let $a > c > 0$, and let the $\{\omega_{i,j}, i, j \in \mathbb{Z}\}$ be i.i.d. $\exp(1)$ distributed. Let $r_1 < \cdots < r_l$ and $u_1 < \cdots < u_k$. Denote by $\lim_{t \to \infty}$ any subsequential limit. Then for any $\delta > 0$

\[
\begin{align*}
\mathbb{P} \left( \bigcap_{j=1}^l \mathcal{A}_2(r_j) \leq s_i \right) \mathbb{P} \left( \bigcap_{i=1}^k \mathcal{A}_2(u_i) \leq \zeta_i \right) & \leq \lim_{t \to \infty} \mathbb{P} \left( \bigcap_{j=1}^l \{L_{P(c,r_j)}^{\text{resc}}([ct_j],[ct_j]) \leq s_i \} \cap \bigcap_{i=1}^k \{L_{P(a,u_i)}^{\text{resc}}([at_j],[at_j]) \leq \zeta_i \} \right) \\
& \leq \mathbb{P} \left( \bigcap_{j=1}^l \mathcal{A}_2(r_j) \leq s_i \right) \mathbb{P} \left( \bigcap_{i=1}^k \mathcal{A}_2(u_i(1 - c/a)^{1/3}) \leq (\zeta_i + \delta) \frac{a^{1/3}}{(a-c)^{1/3}} \right) \\
& + kF_{\text{GUE}}(-\delta a^{1/3}c^{-1/3}).
\end{align*}
\]

Note that the preceeding Theorem implies in particular that

\[
\lim_{a \to \infty} \lim_{t \to \infty} \mathbb{P} \left( \bigcap_{j=1}^l \{L_{P(c,r_j)}^{\text{resc}}([ct_j],[ct_j]) \leq s_i \} \cap \bigcap_{i=1}^k \{L_{P(a,u_i)}^{\text{resc}}([at_j],[at_j]) \leq \zeta_i \} \right) = \mathbb{P} \left( \bigcap_{j=1}^l \mathcal{A}_2(r_j) \leq s_i \right) \mathbb{P} \left( \bigcap_{i=1}^k \mathcal{A}_2(u_i) \leq \zeta_i \right).
\]

**Proof.** The lower bound in (5.17) follows from the FKG inequality and the known convergence to the Airy$_2$ process, see Theorem 2 in [7]. For the upper bound, define the points

\[
P_2(u) = ([ct + ut^{2/3}(ca^{-1/3} - u^{2/3})], [ct + 1]).
\]

and set $\mu_u t = 4ct + 2ct^{2/3}au^{-1/3} - u^2 \frac{at^{1/3}}{4a^{1/3}}$. Then for any $\delta > 0$

\[
\lim_{t \to \infty} \mathbb{P} \left( \bigcup_{i=1}^k \frac{L_{P(a,u_i)} - \mu_u t}{2^{4/3}(at)^{1/3}} \leq -\delta \right) \leq kF_{\text{GUE}}(-\delta a^{1/3}c^{-1/3}).
\]

(5.22)
Denote for brevity $F = \bigcap_{i=1}^{l} \{ L_{P(c,r)}^{resc \to ([ct], [ct])} \leq s_i \}$. Then, using subadditivity and (5.22), we get

$$\lim_{t_j \to \infty} \mathbb{P} \left( F \cap \bigcap_{i=1}^{k} \{ L_{P(a,u)}^{resc \to ([at], [at])} \leq \zeta_i \} \right)$$

$$\leq \lim_{t_j \to \infty} \mathbb{P} \left( F \cap \bigcap_{i=1}^{k} \left\{ \frac{L_{P(a,u)}^{r,s} - \mu_{ui}t_j}{2^{1/3}(at_j)^{1/3}} + \frac{L_{P(c,u)}^{r,s} - \mu(a,u)_{t_j} + \mu_{ui}t_j}{2^{1/3}(at_j)^{1/3}} \leq \zeta_i \right\} \right)$$

$$\leq \lim_{t_j \to \infty} \mathbb{P} \left( F \cap \bigcap_{i=1}^{k} \left\{ \frac{L_{P(c,u)}^{r,s} - \mu(a,u)_{t_j} + \mu_{ui}t_j}{2^{1/3}(at_j)^{1/3}} \leq \zeta_i + \delta \right\} \right)$$

$$+ kF_{\text{GUE}}(-\delta a^{1/3}c^{-1/3})$$

Note now that $L_{P(c,r)}^{r,s}$ and $L_{P(c,u)}^{r,s}$ are independent for all $r, u \in \mathbb{R}$. Hence we get that

$$\lim_{t \to \infty} \mathbb{P} \left( F \cap \bigcap_{i=1}^{k} \left\{ \frac{L_{P(c,u)}^{r,s} - \mu(a,u)_{t_j} + \mu_{ui}t_j}{2^{1/3}(at_j)^{1/3}} \leq \zeta_i + \delta \right\} \right)$$

$$= \mathbb{P} \left( \bigcap_{i=1}^{l} A_{2}(r_i) \leq s_i \right) \mathbb{P} \left( \bigcap_{i=1}^{k} A_{2}(u_i(1-c/a)^{1/3}) \leq (\zeta_i + \delta) \frac{a^{1/3}}{(a-c)^{1/3}} \right),$$

finishing the proof.

Finally, we believe that our Theorem 2.6, together with a control over maximizing paths as in the proof of Theorem 2.5, could be used to study the transition to shock fluctuations when starting at $a = 0$ from flat initial data. Since the needed ideas and arguments - transversal fluctuations, extended slow decorrelation and localization of the starting point - all have already appeared, we decided not to carry out the details, but just to give the following outline. Consider $q_1, q_2 \in (0, 1)$ $x_n(0) = -\lfloor n/q_1 \rfloor, n \geq 0$ and $x_n(0) = -\lfloor n/q_2 \rfloor, n < 0$. This means we have TASEP with so-called Riemann initial data and densities $q_1, q_2$. If $q_2 > q_1$, at time $t$ we have a macroscopic shock at $t(1 - q_1 - q_2)$ and particle $x_{[q_2 t]}(t)$ is located at it (in a law of large number sense). Choosing $q_2 = q_1 + at^{1/3}, a > 0$, we are in a critical scaling. To observe the transition of fluctuations, we thus look at
$P(x_{[b_1 \leq t]}(t) > t(1 - q_1 - q_2) - st^{1/3})$. Translating this into LPP, this corresponds to have $\mathcal{L}^+ = \{(n + x_n(0), n) : n \geq 0\}$, $\mathcal{L}^- = \{(n + x_n(0), n) : n < 0\}$ and $E = \{(t(1 + q_1 q_2 - q_1 - q_2) - st^{1/3}), [q_1 q_2 t]\}$. Now one should choose $E^+$ to lie on the characteristic line from $\mathcal{L}^+ \to E$. Controlling the location of $\pi_{\mathcal{L}^+ \to E^+}, \pi_{\mathcal{L}^+ \to E}$ similarly as in the proof of Theorem 2.3 ((4.18) of [15] localizes the starting point for lines with arbitrary slope), it should be possible to check, with some asymptotic analysis, Assumptions 1, 2, 3. This would then lead to an analogue of Theorem 2.1 where one obtains in the double limit a product of two $F_{\text{GOE}}$ distributions.

A Proof of Lemma 3.4

In this appendix, we follow closely [2] to obtain a uniform constant. Consider the weights (3.3). Following [2], we denote

$$L_{0 \to ((1-\rho)^2t], [\rho^2t)} = G^\rho((1-\rho)^2t], [\rho^2t)) = G^\rho(t). \quad (A.1)$$

Denote by $Z^\rho(t)$ the signed exit point from the axes of the maximizing path $\pi_{0 \to ((1-\rho)^2t], [\rho^2t)}$ in this model, such that if $Z^\rho > 0$, $\max \{i : (i, 0) \in \pi_{0 \to ((1-\rho)^2t], [\rho^2t)}\} = Z^\rho$, and if $Z^\rho < 0$, $\max \{j : (0, j) \in \pi_{0 \to ((1-\rho)^2t], [\rho^2t)}\} = -Z^\rho$. Furthermore we define for $x \geq 0$, $U^\rho_x = \sum_{i=0}^x \omega_{i,0}$ and for $x < 0$, $U^\rho_x = \sum_{j=1}^{-x} \omega_{0,j}$. Let $0 < b_1 < b_2 < 1$ be fixed. Finally, we define $T(b_1, b_2) = \max_{0 \leq t \leq b_1, b_2} \frac{\alpha \rho}{2} (1 - \rho)^2$.

**Lemma A.1** (Lemma 5.5 of [2] with uniform constant). There exists a constant $C_1(b_1, b_2)$ such that for any $u \geq T(b_1, b_2)$, $t > 0, \rho \in [b_1, b_2]$,

$$P(Z^\rho(t) > u) \leq C_1(b_1, b_2) \left( \frac{t^2}{u^4} E(U^\rho_{Z^\rho(t)}) + \frac{t^2}{u^3} \right) \quad (A.2)$$

**Proof.** This is an imminent corollary of the explicit upper bound for $P(Z^\rho(t) > u)$ provided in the proof of Lemma 5.5 of [2].

**Lemma A.2** (Lemma 5.7 of [2] with uniform constant). For $\alpha \in (0, 1)$ there exist constants $C_2(b_1, b_2, \alpha)$ and $C_3(b_1, b_2, \alpha)$ such that, for all $\rho \in [b_1, b_2]$ and

$$r \geq \frac{8(1 - \rho)}{\alpha \rho^2 E(U^\rho_{Z^\rho(t)})} \quad (A.3)$$

we have the bound

$$P(U^\rho_{Z^\rho(t)} > r E(U^\rho_{Z^\rho(t)})) \leq C_2(b_1, b_2, \alpha) \frac{t^2}{E(U^\rho_{Z^\rho(t)})} \left( \frac{1}{r^3} + \frac{1}{r^4} \right) + e^{-C_3(b_1, b_2, \alpha) r E(U^\rho_{Z^\rho(t)})} \quad (A.4)$$
Theorem A.3. For all \( G \) and define, with this replacement, \( u \geq \) (with \( \eta \)). Following the computation on the bottom of [2], p. 1128, we see that, setting \( \rho \) which, by taking \( R \) such that for all \( R > 0 \) and \( t_0 > 0 \) there is a \( t > t_0 \) and a \( \varrho \in [b_1, b_2] \) such that \( \mathbb{E}(U^2_{t_0}(t)) \geq R t^{2/3} \). Then there are sequences \( t_N \to \infty \) and \( g_N \in [b_1, b_2] \) with \( \mathbb{E}(U^2_{t_N}(t_N)) \geq R t_N^{2/3} \). Thus by Lemma A.2 and dominated convergence

\[
\limsup_{N \to \infty} \int_0^\infty dt \mathbb{P}(U^2_{t_N}(t)) > R \mathbb{E}(U^2_{t_N}(t_N)) \leq \frac{\tilde{C}_4(b_1, b_2, \alpha)}{R^3},
\]

which, by taking \( R^3 \) \( \tilde{C}_4(b_1, b_2, \alpha) \) leads to the contradiction

\[
1 = \limsup_{N \to \infty} \mathbb{E} \left( \frac{U^2_{t_N}(t_N)}{\mathbb{E}(U^2_{t_N}(t_N))} \right) < 1.
\]

Proof of Lemma 7.4. We follow closely the proof of Lemma 7.4 in [2]. Define \( \varrho = \varrho(t, c) \) through \((1 - \varrho(t, c))^2 t = m\), and set \( \tilde{t} = t/\varrho^2, \tilde{s} = \kappa \tilde{t}, \tilde{k} = (1 - \varrho)^2 \tilde{s} = \kappa(\varrho^2 t + c t^{2/3}), l = \varrho^2 \tilde{s} \). We go through the proof of Lemma 7.4 in [2] and replace the \( k, l, m, n, s, t \) in [2] by our \( \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n}, \tilde{s}, \tilde{t} \). In particular, we set for \( u > 0 \)

\[
\frac{(1 - \lambda)^2}{\lambda^2} = \frac{\tilde{k} + u + 1}{l} \quad \frac{(1 - \lambda)^2}{\lambda^2} = \frac{m - \tilde{k} - u}{n - l}
\]

and define, with this replacement, \( \tilde{G}, \tilde{G} \) as in the proof of Lemma 7.4. Following the computation on the bottom of [2], p. 1128, we see that, setting (with \( u > 4 \)) \( Q_1 = \tilde{G} \tilde{k} + u + 1, l), Q_2 = \tilde{G} (m - \tilde{k} - u, n - l), Q = Q_1 + Q_2 \) we can bound

\[
\mathbb{E}(Q) \leq \tilde{t} - C_1(\varrho) u^2 / \tilde{t} + C_2(\varrho)
\]
Furthermore, \( C_2(\theta) = \frac{2\theta}{1 - \theta} + \frac{1 - \theta}{\theta} + 2 \) and \( C_1(\theta) = \frac{\theta}{8(1 - \theta)^3} \). Note now, for \( t > t_0(c_0) \), 
\[ C_5(b_1, b_2)t^{1/3} \leq \tilde{t}^{1/3} \leq C_5(b_1, b_2)t^{1/3} \]
for some \( C_5(b_1, b_2), C_6(b_1, b_2) > 0 \). Thus by taking \( t > t_0(c_0) \), \( u = kt^{2/3} \) we obtain for a \( C_4(b_1, b_2) > 0 \) and, say, \( k > 1 \), 
\[ \mathbb{E}(Q) \leq \tilde{t} - C_4(b_1, b_2)k^2t^{1/3}. \]  
(A.10)

One obtains as in Lemma 7.4 of \cite{2}
\[ P(Z^0_{[\kappa t]} \geq \kappa(\eta t + ct^{2/3}) + kt^{2/3}) \leq P(Q - \tilde{t} \geq -C_4(b_1, b_2)k^2t^{1/3}/2) \]  
(A.11)
\[ + P(A_0(\tilde{t}) - \tilde{t} \leq -C_4(b_1, b_2)k^2t^{1/3}/2), \]  
(A.12)
where \( A_0(\tilde{t}) \) is the LPP time from \((0, 0)\) to \((m, n)\) with \( \omega_{i,j} = 0 \) if \( i = 0 \) or \( j = 0 \), \( \omega_{i,j} \sim \exp(1) \) else.

We now need to bound the variance of \( Q_1, Q_2 \). While these are stationary LPP times to points on their respective characteristics, Theorem \[ A.3 \] does not apply here since e.g. \( \lambda \) need not be bounded away from 0 (note that \( \tilde{k}, l \) can be of order 1 (i.e. independent of \( t \)) so if one takes \( u = kt^{2/3} \) in this case then \( \lambda \) will converge to 0 as \( t \) goes to infinity.) Instead, one notes first that \( 1 - \lambda > 1 - \rho \). Then, we may apply Lemma 4.7 of \cite{2} to bound
\[ \text{Var}(Q_1) = \text{Var}(G^{1-\lambda}(l, \tilde{k} + u + 1)) \]  
(A.13)
\[ \leq \frac{(1 - \rho)^2}{(1 - \lambda)^2} \text{Var}(G^{1-\rho}(l, \tilde{k} + u + 1)) + l \left( \frac{1}{\lambda^2} - \frac{(1 - \rho)^2}{(1 - \lambda)^2 \rho^2} \right). \]  
(A.14)

Next, note that \( G^{1-\rho}(l, \tilde{k} + u + 1) - G^{1-\rho}(l, \tilde{k}) \) is a sum of \( u + 1 \) i.i.d. random variables which are \( \exp(1 - \rho) \) distributed. Hence by Theorem \[ A.3 \] we may bound
\[ \text{Var}(G^{1-\rho}(l, \tilde{k} + u + 1)) \leq C(b_1, b_2, \alpha)t^{2/3} + \frac{u + 1}{(1 - \rho)^2} + 2\sqrt{(u + 1)C(b_1, b_2, \alpha)}t^{1/3}. \]  
(A.15)

Furthermore,
\[ l \left( \frac{1}{\lambda^2} - \frac{(1 - \rho)^2}{(1 - \lambda)^2 \rho^2} \right) = \frac{u + 1}{(1 - \lambda)^2}. \]  
(A.16)

In total this yields for \( u = kt^{2/3} \) and some \( C_0(b_1, b_2, \alpha) > 0 \)
\[ \text{Var}(Q_1) \leq C_0(b_1, b_2, \alpha)t^{2/3}k \]  
(A.17)
Thus we have, with a

\[ \text{Var}(Q_2) \leq C_6^0(b_1, b_2, \alpha) t^{2/3} k \]

(A.18)

Consequently, by (A.10) and Chebychev’s inequality, for \( t \) large enough

\[ \mathbb{P}(\hat{Q} - \hat{t} \geq -C_4(b_1, b_2) k^2 t^{1/3}/2) \leq C_7(b_1, b_2, \alpha)/k^3. \]  

(A.19)

Finally, we could bound \( \mathbb{P}(A_0(\hat{t}) - \hat{t} \leq -C_4(b_1, b_2) k^2 t^{1/3}) \) by adapting Theorem 2.4 of [2]. An alternate way is to use Theorem 13.2 of [4] (see also Proposition 4.3 in [13]) to show that for \( t \) large enough, \( k > k_0(b_1, b_2) > 1 \)

\[ \mathbb{P}(A_0(\hat{t}) - \hat{t} \leq -C_4(b_1, b_2) k^2 t^{1/3}/2) \leq C_{10}(b_1, b_2)/k^{3\alpha}. \]  

(A.20)

Thus (A.19), (A.20) prove (3.10) for \( k > k_0(b_1, b_2) \), and smaller \( k \) can be included by enlarging the constant.

To prove (3.11), use the transposition \((i, j) \to (j, i)\) which shows that \( Y^{\text{TOP,0}}_{[\kappa(\eta_0 t + ct^{2/3})]}(m, n) \) equals in distribution \( Z^0_{[\kappa\eta_0 t + ct^{2/3}]}(n, m) \). With \( T = \eta_0 t + c t^{2/3} \) we see that (recall \( c = c_0 + \mathcal{O}(t^{-1/3}) \)) \( t = T/\eta_0 - c_0 T^{2/3}/\eta_0 + \mathcal{O}(T^{1/3}) \) thus we can write \( Z^0_{[\kappa\eta_0 t + ct^{2/3}]}(n, m) \) as

\[ Z^0_{[\kappa T]}([T/\eta_0 - c_0 T^{2/3}/\eta_0 + \mathcal{O}(T^{1/3})], [T]). \]  

(A.21)

Thus we have, with a \( \hat{c} = -c_0/\eta_0^{5/3} + \mathcal{O}(T^{-1/3}) \)

\[ \mathbb{P}(Y^{\text{TOP,0}}_{[\kappa(\eta_0 t + ct^{2/3})]}(m, n) \geq \kappa t + k t^{2/3}) \leq \mathbb{P}(Z^0_{[\kappa T]}(n, m) \geq \kappa(T/\eta_0 + \hat{c} T^{2/3}) + k C_{11}(b_1, b_2) T^{2/3}) \leq \frac{C_{12}(b_1, b_2)}{k^{3\alpha}} \]

(A.22)

In the last step we bounded \( k t^{2/3} \geq C_{11}(b_1, b_2) k T^{2/3} \) (possible for \( t \) large enough). Finally, to prove the statement for the LPP model with all weights i.i.d. exp(1)–distributed, simply note that \( Z_{[\kappa t]}(m, n) \overset{d}{=} Z^0_{[\kappa t]}(m + 1, n + 1) \). Writing \( \tilde{T} = [t] + 1 \) and \( [\kappa t] + 1 = [\kappa' \tilde{T}] \), we get, with \( \eta_0 t + c t^{2/3} = \eta_0 \tilde{T} + \hat{c} T^{2/3} \)

\[ \mathbb{P}(Z_{[\kappa t]} \geq \kappa(\eta_0 t + c t^{2/3}) + k t^{2/3}) \leq \mathbb{P}(Z^0_{[\kappa T]} \geq \kappa'(\eta_0 \tilde{T} + \hat{c} T^{2/3}) + k \tilde{T}^{2/3}/2). \]

The result follows from (3.10), the proof for \( Y^{\text{TOP}} \) can now be given as the one for \( Y^{\text{TOP,0}} \).
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