QUANTIZED NILRADICALS OF PARABOLIC SUBALGEBRAS
OF $\mathfrak{sl}(n)$ AND ALGEBRAS OF COINVARIANTS

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Abstract. Let $P_J$ be the standard parabolic subgroup of $SL_n$ obtained by deleting a subset $J$ of negative simple roots, and let $P_J = L_J U_J$ be the standard Levi decomposition. Following work of the first author, we study the quantum analogue $\theta : O_q(P_J) \to O_q(L_J) \otimes O_q(P_J)$ of an induced coaction and the corresponding subalgebra $O_q(P_J)^{co\theta} \subseteq O_q(P_J)$ of coinvariants. It was shown that the smash product algebra $O_q(L_J) \# O_q(P_J)^{co\theta}$ is isomorphic to $O_q(P_J)$. In view of this, $O_q(P_J)^{co\theta}$ – while it is not a Hopf algebra – can be viewed as a quantum analogue of the coordinate ring $O(U_J)$.

In this paper we prove that when $q \in \mathbb{K}$ is nonzero and not a root of unity, $O_q(P_J)^{co\theta}$ is isomorphic to a quantum Schubert cell algebra $U_q[w]$ associated to a parabolic element $w$ in the Weyl group of $\mathfrak{sl}(n)$. An explicit presentation in terms of generators and relations is found for these quantum Schubert cells.

1. Introduction and overview of results in the paper

Let $SL_n$ be the complex algebraic group of $n \times n$ matrices having determinant equal to one, and let $P_J$ be the standard parabolic subgroup of block upper triangular matrices of $SL_n$ obtained by deleting a subset $J$ of negative simple roots of $SL_n$. The group $P_J$ admits a Levi decomposition $P_J = L_J U_J$, where $L_J$ is the standard Levi factor of block diagonal matrices in $P_J$, and $U_J$ is the unipotent subgroup of matrices in $P_J$ having identity matrices along the block diagonal. Multiplication $L_J \times P_J \to P_J$ induces a coaction, $O(P_J) \to O(L_J) \otimes O(P_J)$, where $O(L_J)$ and $O(P_J)$ are the coordinate rings of $L_J$ and $P_J$ respectively.

With the classical case in mind, we turn our attention to the corresponding quantized coordinated rings, $O_q(L_J)$ and $O_q(P_J)$. Here and below, the base field for all algebras is an arbitrary field $\mathbb{K}$ that contains a nonzero element $q \in \mathbb{K}$ that is not a root of unity. Define

$$\hat{q} := q - q^{-1}.$$ 

Following [7,8], we focus on the quantum analogue of the coaction above,

$$\theta : O_q(P_J) \to O_q(L_J) \otimes O_q(P_J).$$

An element $x \in O_q(P_J)$ is a (left) coinvariant if $\theta(x) = 1 \otimes x$. It was shown in [8] Theorems 3.46 and 3.49] that the subalgebra of coinvariants $O_q(P_J)^{co\theta} \subseteq O_q(P_J)$ has a presentation as an iterated Ore extension $\mathbb{K}[t_{11}; t_{22}; \cdots; t_{MM}; \delta_{11}]$, where $M = \dim(U_J)$ and is, in fact, a Cauchon-Goodearl-Letzter extension. It was also shown that the smash product algebra $O_q(L_J) \# O_q(P_J)^{co\theta}$ is isomorphic as a $\mathbb{K}$-algebra to $O_q(P_J)$ [8] Theorem 3.19. In view of this, $O_q(P_J)^{co\theta}$ – while it is not a Hopf algebra – can be viewed as a quantized version of the coordinate ring.
\(\mathcal{O}(U_J)\). In fact, if \(q\) is put equal to 1 in the defining relations of \(\mathcal{O}_q(P_J)^{\mathrm{co}^{\theta}}\), we recover the defining relations of \(\mathcal{O}(U_J)\).

The generators of \(\mathcal{O}_q(P_J)^{\mathrm{co}^{\theta}}\) can be indexed by elements of the set

\[
\Phi_J := \{(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, n\} \mid \exists k \in J \text{ such that } i \leq k < j\}.
\]

We will denote the generators of \(\mathcal{O}_q(P_J)^{\mathrm{co}^{\theta}}\) by \(u_{ij}\), \(((i, j) \in \Phi_J\). Each \(u_{ij}\) is a certain ratio of quantum minors in the quantized coordinate ring \(\mathcal{O}_q(P_J)\) (see 3.3 in Section 3.2). Viewing \(u_{ij}\) as occupying the \((i, j)\)-position in an \(n \times n\) array, we observe that the full set of generators of \(\mathcal{O}_q(P_J)^{\mathrm{co}^{\theta}}\) forms a block upper triangular shape that depends on \(J\). We define the function \(r : \{1, \ldots, n\} \to J \cup \{0\}\),

\[
r(m) := \max\{k \in J \cup \{0\} \mid k < m\}.
\]

This function records which block a generator \(u_{ij}\) occupies. For instance, \(u_{ij}\) and \(u_{im}\) belong to the same block if and only if \(r(i) = r(\ell)\) and \(r(j) = r(m)\).

The specific form for the commutation relation between a pair of generators, say \(u_{ij}\) and \(u_{im}\), depends on the relative ordering on \(i, j, \ell, m, r(j), r(\ell),\) and \(r(m)\), and in some cases on the relative ordering of \(w_{j}^{-1}(i), w_{j}^{-1}(\ell), w_{j}^{0}(i),\) and \(w_{j}^{0}(\ell)\), where we have tacitly identified the Weyl group of \(\mathfrak{sl}(n)\) with the symmetric group on \(\{1, \ldots, n\}\).

**Theorem 1.1.** [8 Theorems 3.35 and 3.50]

The algebra \(\mathcal{O}_q(P_J)^{\mathrm{co}^{\theta}}\) is generated by \(u_{ij}\) (for \((i, j) \in \Phi_J\) and has the following defining relations:

\[
(1.1) \quad \begin{cases} 
qu_{\ell m}u_{ij} & (\ell = i < j < m) \\
u_{\ell m}u_{ij} & (j = m \text{ and } w_{j}^{-1}(\ell) < w_{j}^{-1}(i)) \end{cases}
\]

\[
u_{ij}u_{\ell m} = \begin{cases} 
u_{ij}u_{\ell m} + \hat{q}u_{\ell j}u_{im} & (r(i) < \ell < i < j < m) \\
q^{-1}u_{\ell m}u_{ij} - \hat{q}u_{(im), \ell} & (i < j = \ell < m) \end{cases}
\]

where \(u_{(im), \ell} := (-q)^{r(\ell)-w_{\ell}^{0}(\ell)}u_{im} + \sum_{r(\ell) < k < w_{\ell}^{0}(\ell)}(-q)^{r(k)-w_{\ell}^{0}(k)}w_{\ell}^{0}(k)mu_{i}, w_{\ell}^{0}(k)\).
We prove that \( \mathcal{O}_q(P_{J})^{\omega_{\emptyset}} \) is isomorphic to a quantized nilradical of a parabolic subalgebra of \( \mathfrak{sl}(n) \) (Theorem 1.3). In proving this result we first construct presentations for the quantized nilradicals. Then we compare these presentations with the presentations of \( \mathcal{O}_q(P_{J})^{\omega_{\emptyset}} \).

Quantized nilradicals belong to a larger family of algebras called quantum Schubert cell algebras, which were introduced by De Concini, Kac, and Procesi [3] and Lusztig [13]. Quantum Schubert cells play important roles in ring theory [16,19], crystal/canonical basis theory [11,14], and cluster algebras [5,8]. For a complex semisimple Lie algebra \( \mathfrak{g} \) with a root system \( \Delta = \Delta_{-} \sqcup \Delta_{+} \), triangular decomposition \( \mathfrak{g} = \mathfrak{n}_{-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+} \), and \( w \) an element in the Weyl group \( W_{\mathfrak{g}} \), the corresponding quantum Schubert cell algebras \( \mathcal{U}_q^{w} \) are quantizations of the universal enveloping algebra \( \mathcal{U}(\mathfrak{n}_{\pm} \cap \text{ad}(w)(\mathfrak{n}_{\mp})) \). The standard presentation of \( \mathcal{U}_q^{w} \) typically involves a generating set of variables \( \{ X_{\beta} \} \) indexed by roots \( \beta \) in \( \Delta_w := \Delta_{+} \cap w.(\Delta_{\pm}) \). With respect to a convex order on the roots in \( \Delta_w \), ordered monomials form a basis of \( \mathcal{U}_q^{w} \).

The quantum Schubert cell algebras of interest in this paper are those of the form \( \mathcal{U}_q^{w} \), where \( w \) is a parabolic element in the Weyl group and \( \mathfrak{g} = \mathfrak{sl}(n) \). Algebras of this type are quantizations of nilradicals \( \mathfrak{n}_{J} \) of parabolic subalgebras \( \mathfrak{p}_{J} \subseteq \mathfrak{sl}(n) \). We will refer to these particular quantum Schubert cell algebras as quantized nilradicals and denote them by \( \mathcal{U}_q^{(n,J)} \).

In Theorem 2.4 we give a presentation of the quantized nilradical \( \mathcal{U}_q^{(n,J)} \). In the extremal case, when \( J \) is the empty set, we have \( \mathcal{U}_q^{(n,\emptyset)} \cong \mathbb{K} \). At the other extreme, \( \mathcal{U}_q^{(n_{\{1,...,n-1\}},}) \cong \mathcal{U}_q^{(n_{+})} \). When \( J \) is a singleton, say \( J = \{ p \} \), we have an isomorphism \( \mathcal{U}_q^{(n_{\{p\}})} \cong \mathcal{O}_q(M_{p,n-p}) \), where \( \mathcal{O}_q(M_{p,n-p}) \) is the algebra of quantum \( p \times (n-p) \) matrices. In any case, the roots in \( \Delta_{w,J} \), as well as the generators of the algebra \( \mathcal{U}_q^{(n,J)} \), can be indexed by \( \Phi_J \). We will denote the generators of \( \mathcal{U}_q^{(n,J)} \) by

\[
X_{ij}, (i, j) \in \Phi_J.
\]

In Section 4 we prove the following theorem.

**Theorem 1.2.** The quantized nilradical \( \mathcal{U}_q^{(n,J)} \) is generated by the root vectors \( X_{ij} \) \( (i, j) \in \Phi_J \) and has the following defining relations:

\[
X_{ij} X_{km} = \begin{cases} qX_{km}X_{ij} & (\ell < i \text{ and } j = m) \\
X_{km}X_{ij} & (\ell < i < w^{-1}_j(m)) \\
X_{km}X_{ij} + \tilde{q}X_{\ell j}X_{im} & (\ell < i \leq r(m) < m) \text{ or } (\ell < m < r(j) < j) \\
q^{-1}X_{km}X_{ij} + X_{(j),m} & (\ell = m = i < j) \end{cases}
\]

where \( X_{(j),m} := (-q)^{m-r(m)-1}X_{\ell j} + \tilde{q}\sum_{r(m) < k < m} (-q)^{m-k-1}X_{kj}X_{\ell k} \).

For a subset \( J \subseteq \{ 1, \ldots, n-1 \} \), define

\[
\tilde{J} := \{ n-j \mid j \in J \} \subseteq \{ 1, \ldots, n-1 \}.
\]
There is a one-to-one correspondence between the generators of $\mathcal{U}_q(n_j)$ and $\mathcal{U}_q(n_j)$ given by “reflecting about the anti-diagonal”: $X_{ij} \leftrightarrow X_{w_0(j),w_0(i)}$. Using the defining relations of $\mathcal{U}_q(n_j)$, we can easily verify that there are algebra isomorphisms

$$\mathcal{U}_q(n_j) \cong \mathcal{U}_q(n_j)^{\text{op}}, \quad \mathcal{U}_q(n_j) \cong \mathcal{U}_{q^{-1}}(n_j)^{\text{op}},$$

given by $X_{ij} \mapsto X_{w_j(j),w_j(i)}$ and $X_{ij} \mapsto -qX_{ij}$, respectively, for all $(i,j) \in \Phi_J$.

Since we have presentations for $\mathcal{U}_q(n_j)$ and $\mathcal{O}_q(P_J)^{\text{co } \theta}$ (Theorems 1.1 and 1.2), it is routine to verify that there is an algebra isomorphism

$$\mathcal{U}_q(n_j) \cong \mathcal{O}_{q^{-1}}(P_J)^{\text{co } \theta}$$

defined by

$$X_{ij} \mapsto \frac{(-1)^{w_0(j) - w_0(i)} \bar{q}}{\bar{q}} u_{w_0(j),w_0(i)}$$

for all $(i,j) \in \Phi_J$. Composing the isomorphisms of 1.3 with the isomorphism 1.4 above gives us the following theorem.

**Theorem 1.3.** There is an algebra isomorphism

$$\Psi : \mathcal{U}_q(n_j) \to \mathcal{O}_q(P_J)^{\text{co } \theta}$$

given by $X_{ij} \mapsto \frac{q(-1)^{i+j}}{q} u_{w_0(j),w_0(i)}$ for all $(i,j) \in \Phi_J$.

In the extremal case, when $J = \{1, \ldots, n-1\}$, this isomorphism appears in [7, Theorem 17].

2. Quantized nilradicals of parabolic subalgebras of $\mathfrak{sl}(n)$

2.1. The quantum enveloping algebra $\mathcal{U}_q(g)$. Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank $\ell$, and let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a base of simple roots with respect to a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. We will denote the root lattice by $\mathcal{Q} = \mathbb{Z} \Pi$. Here and below, for $p \in \mathbb{N}$, we put $[p] := \{1, \ldots, p\}$. We will denote the Chevalley generators of the quantum universal enveloping algebra $\mathcal{U}_q(g)$ by

$$E_i, F_i, (i \in [\ell]), \quad \text{and } K_\mu (\mu \in \mathcal{Q}).$$

The algebra $\mathcal{U}_q(g)$ has a triangular decomposition

$$\mathcal{U}_q(g) \cong \mathcal{U}_q^+(g) \otimes \mathcal{U}_q^0(g) \otimes \mathcal{U}_q^-(g),$$

where $\mathcal{U}_q^+(g)$ is the subalgebra generated by the $E_i$'s, $\mathcal{U}_q^-(g)$ is the subalgebra generated by the $F_i$'s, and $\mathcal{U}_q^0(g)$ is the subalgebra generated by the $K_\mu$'s (see e.g. [2],[9],[12]).

While quantum enveloping algebras can be associated to semisimple Lie algebras, or more generally Kac-Moody Lie algebras, we focus on the case when $\mathfrak{g}$ is the special linear Lie algebra $\mathfrak{sl}(n)$. We use the standard realization of $\mathfrak{sl}(n)$ as the Lie algebra of traceless $n \times n$ matrices. Let $\mathfrak{h}$ be the Cartan subalgebra of diagonal matrices in $\mathfrak{sl}(n)$. The simple roots are $\alpha_i = e_i - e_{i+1}$ ($i \in [n-1]$), where $e_\ell$ ($\ell \in [n]$) is the linear functional on $\mathfrak{h}$ that returns the $\ell$-th entry along the diagonal. Let $\langle \cdot, \cdot \rangle$ be the symmetric bilinear form on $\mathfrak{h}^*$ defined by the rule $\langle e_i, e_j \rangle = \delta_{ij}$ for all $i, j \in [n]$. The quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(n))$ is the associative $\mathbb{K}$-algebra generated by $F_i, E_i, K_\mu$ ($i \in [n-1], \mu \in \mathcal{Q}$), and has the defining relations

\begin{align*}
(2.1) \quad K_0 &= 1, \quad K_\mu K_\nu = K_{\nu + \mu}, \\
(2.2) \quad K_\mu E_i &= q^{\langle \mu, \alpha_i \rangle} E_i K_\mu, \quad K_\mu F_i = q^{-\langle \mu, \alpha_i \rangle} F_i K_\mu.
\end{align*}
If \( \forall \) \( w \) \in \Pi \) and \( \alpha \in \Pi \), \( w(\alpha) \in \Pi \), then \( T_w(E_\alpha) = E_{w(\alpha)} \).

When \( g \) is the Lie algebra \( sl(n) \) and \( T_i = T_{i,1} \) \((i \in [n-1])\), Lusztig’s symmetries can be succinctly written as

\[
T_i(K_\mu) = K_{s_i(\mu)},
\]

\[
T_i(E_j) = \begin{cases}
E_j, & (|i - j| > 1), \\
E_i E_j - q^{-1} E_j E_i, & (|i - j| = 1), \\
-F_i K_{\alpha_i}, & (|i - j| = 0),
\end{cases}
\]

\[
T_i(F_j) = \begin{cases}
F_j, & (|i - j| > 1), \\
-q(F_i F_j - q^{-1} F_j F_i), & (|i - j| = 1), \\
-K_{\alpha_i}^{-1} E_i, & (|i - j| = 0).
\end{cases}
\]

for all \( i, j \in [n-1] \) and \( \mu \in Q \).

2.3. Quantum Schubert cells. Quantum Schubert cell algebras were introduced in [3] and [13]. They are a family of subalgebras of \( U_q(g) \) indexed by elements of the Weyl group of \( g \). To construct a quantum Schubert cell, first fix \( w \in W_\theta \) and \( \alpha \in \Pi \). Next define the positive roots

\[
\beta_1 = \alpha_{i_1}, \beta_2 = s_1 \alpha_{i_2}, \ldots, \beta_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}
\]

and the positive root vectors

\[
X_\beta_1 = E_{i_1}, X_\beta_2 = T_{s_{i_1}} E_{i_2}, \ldots, X_\beta_k = T_{s_{i_1}} \cdots T_{s_{i_{k-1}}} E_{i_k}.
\]

There is an analogous construction of negative root vectors \( X_{-\beta_1}, \ldots, X_{-\beta_k} \) by replacing the \( E_i \)’s with \( F_i \)’s in the above construction. Following [3] and [13] Section 40.2, the quantum Schubert cell algebra \( U_q^+\{w\} \) is defined to be the subalgebra of \( U_q(g) \) generated by the root vectors \( X_{\pm \beta_1}, \ldots, X_{\pm \beta_k} \). De Concini, Kac, and Procesi [3 Proposition 2.2] and Lusztig [13] proved that \( U_q^\pm\{w\} \) does not depend
on the reduced expression for $w$. It was conjectured by Berenstein and Greenstein in [11] Conjecture 5.3 that $\mathcal{U}_q^\pm[w]$ could equivalently be defined as

$$\mathcal{U}_q^\pm[w] = \mathcal{U}_q^\pm(\mathfrak{g}) \cap T_w(\mathcal{U}_q^\pm(\mathfrak{g}))$$

for any symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. They proved their conjecture in the case when $\mathfrak{g}$ is of finite type. The conjecture was later proven independently by Kimura [11] Theorem 1.1 (1) and Tanisaki [18] Proposition 2.10.

The algebra $\mathcal{U}_q^\pm[w]$ has a PBW-type basis of ordered monomials

$$(2.12) \quad X_{\pm \beta_1}^{n_1} \cdots X_{\pm \beta_t}^{n_t}, \quad n_1, \ldots, n_t \in \mathbb{Z}_{\geq 0}.$$  

The Levendorskii-Soibelmann Straightening Rule gives commutation relations in $\mathcal{U}_q^\pm[w].$

**Theorem 2.2.** [15, Prop. 5.5.2]  
For $i < j$,

$$(2.13) \quad X_{\beta_i} X_{\beta_j} = q^{(\beta_i, \beta_j)} X_{\beta_j} X_{\beta_i} + \sum_{n_{i+1}, \ldots, n_{j-1} \geq 0} z_{ij}(n_{i+1}, \ldots, n_{j-1}) X_{\beta_{i+1}}^{n_{i+1}} \cdots X_{\beta_{j-1}}^{n_{j-1}},$$

where $z_{ij}(n_{i+1}, \ldots, n_{j-1}) \in \mathbb{K}$, and $z_{ij}(n_{i+1}, \ldots, n_{j-1}) = 0$ whenever $\sum_{i<k<j} n_k \beta_k \neq \beta_i + \beta_j$.

An analogous straightening rule applies to $\mathcal{U}_q^-[w]$. The straightening law in conjunction with the PBW basis result (2.12) can be used to give finite presentations of quantum Schubert cell algebras.

2.4. **The quantum Schubert cell algebras $\mathcal{U}_q(n_J)$.** To construct the quantum Schubert cell algebras of interest in the remainder of this paper, we turn our attention to parabolic elements $w_J$ in the Weyl group $W$ of $\mathfrak{sl}(n)$. The algebras $\mathcal{U}_q^+[w_J]$ are quantizations of the nilradicals $n_J$ of standard parabolic subalgebras $\mathfrak{p}_J$ of $\mathfrak{sl}(n)$. For this reason we denote them instead by $\mathcal{U}_q(n_J)$. First let

$$J = \{i_1 < i_2 \cdots < i_t\} \subseteq [n - 1],$$

and let $W^J \subseteq W$ be the subgroup of $W$ generated by the simple reflections $\{s_i | i \notin J\}$. Let $w_0$ and $w_0'$ denote the longest elements in $W$ and $W^J$ respectively. Define

$$w_J := w_0' w_0 \in W.$$  

We can write $w_J = S_t S_{t-1} \cdots S_1$, where

$$(2.14) \quad S_k := (s_i \cdots s_{n-1})(s_{i_k-1} \cdots s_{n-2}) \cdots (s_{i_k-1}+1 \cdots s_{n-1-(i_k-1)}) \in W$$

for all $k \in [t]$ (where, by convention, $i_0 = 0$ and $i_{t+1} = n$). The expression written in (2.14) is a reduced expression, and one can obtain a reduced expression for $w_J$ by concatenating these reduced expressions for $S_t$, $S_{t-1}$, \ldots, $S_1$. We tacitly use this particular reduced expression for $w_J$ in our construction of $\mathcal{U}_q(n_J)$. The set of positive Lusztig roots is

$$\Delta_{w_J} = \{e_i - e_j \in Q \mid \exists k \in J \text{ such that } i \leq k < j\}.$$  

We find it convenient to define the set of tuples

$$\Phi_J := \{(i, j) \in [n] \times [n] \mid e_i - e_j \in \Delta_{w_J}\}.$$  

For brevity we let $X_{ij}$ ($(i, j) \in \Phi_J$) denote the positive Lusztig root vector of degree $e_i - e_j$. 


Example 2.3. Let \( n = 7 \) and suppose \( J = \{2, 5, 6\} \). Here,

\[
w_J = s_6(s_5s_6)(s_4s_5)(s_3s_4)(s_2s_3s_4s_5s_6)(s_1s_2s_3s_4s_5) \in W
\]
is a reduced expression. The positive Lusztig root vectors in the corresponding quantized nilradical \( \mathcal{U}_q(\mathfrak{n}_J) \) are \( X_{67}, X_{57}, X_{56}, X_{47}, X_{46}, X_{37}, X_{36}, X_{27}, X_{26}, X_{23}, X_{24}, X_{25}, X_{17}, X_{16}, X_{13}, X_{14}, X_{15} \).

If the root vector \( X_{ij} \) is viewed as occupying the \((i, j)\)-entry in an \( n \times n \) array, the entire set \( \{ X_{ij} \} \) of root vectors of \( \mathcal{U}_q(\mathfrak{n}_J) \) fills all entries in a block upper triangular shape that depends on \( J \). More precisely, the set of positive Lusztig roots \( \Delta_{w_J} \) can be characterized as the smallest set satisfying the conditions (1) \( e_j - e_{j+1} \in \Delta_{w_J} \) if and only if \( j \in J \), (2) if \( e_i - e_j \in \Delta_{w_J} \) with \( i > 1 \), then \( e_{i-1} - e_j \in \Delta_{w_J} \), and (3) if \( e_i - e_j \in \Delta_{w_J} \) with \( j < n \), then \( e_i - e_{j+1} \in \Delta_{w_J} \).

The Levendorskii-Solbelmam straightening rule together with the PBW basis result \( 2.12 \) implies that a finite presentation of a quantum Schubert cell algebra can be obtained from the commutation relations among the pairs of root vectors. In order to describe the defining relations of \( \mathcal{U}_q(\mathfrak{n}_J) \), we first define the function \( r : [n] \to J \cup \{0\} \) as

\[
r(m) := \max\{ k \in J \cup \{0\} \mid k < m \}.
\]

The specific form for the commutation relation between a pair of root vectors, say \( X_{ij} \) and \( X_{\ell m} \), depends on the relative ordering on \( i, j, \ell, m, r(j), \) and \( r(m) \), and in some cases the relative ordering on \( w_J^{-1}(j), w_J^{-1}(m), w_0^d(j), \) and \( w_0^d(m) \) plays a role, where we have identified the Weyl group of \( \mathfrak{sl}(n) \) with the symmetric group on \([n]\). Under this identification the simple reflection \( s_i \) corresponds to the transposition \((i, i+1)\), and \( w_J \) corresponds to the unique permutation satisfying the condition that for all \( 1 \leq i < j \leq n \), \( w_J(i) < w_J(j) \) if and only if \( r(i) = r(j) \).

**Theorem 2.4.** The quantized nilradical \( \mathcal{U}_q(\mathfrak{n}_J) \) is generated by the root vectors \( X_{ij} \) \((i, j) \in \Phi_J\) and has the following defining relations:

\[
X_{ij}X_{\ell m} = \begin{cases} 
qX_{\ell m}X_{ij} & (\ell < i \text{ and } j = m) \\
& \text{or } (\ell = i \text{ and } w_J^{-1}(j) < w_J^{-1}(m)) \\
X_{\ell m}X_{ij} & (\ell < i < w_0^d(j) < w_0^d(m)) \\
& \text{or } (\ell < m < i < j) \\
X_{\ell m}X_{ij} + qX_{\ell m}X_{ij} & (\ell < i < r(m) < j < m) \\
& \text{or } (\ell < i < r(m) < m \leq r(j) < j) \\
q^{-1}X_{\ell m}X_{ij} + X_{(\ell j), m} & (\ell < m = i < j)
\end{cases}
\]

where \( X_{(\ell j), m} := (-q)^{m-r(m)-1}X_{\ell j} + q^{r(m)-k}X_{k j}X_{\ell k} \).

**Proof.** See Section 4. □
3. The Algebra $O_q(P_J)^{co\, \theta}$ of Coinvariants

3.1. Crossed product algebras, $H$-cleft extensions, and coinvariants. Let $H$ be a bialgebra and let $A$ be a left $H$-comodule algebra with coaction $\xi : A \to H \otimes A$. An element $a \in A$ is a left coinvariant if $\xi(a) = 1 \otimes a$. The set $A^{co\, \xi}$ of left coinvariants is, in fact, a subalgebra of $A$. The $H$-extension $A^{co\, \xi} \subseteq A$ is called $H$-cleft if there exists a convolution invertible morphism of left $H$-comodules $\gamma : H \to A$ (with convolution inverse $\overline{\gamma}$) such that $\gamma(1) = 1$. In this setting, the vector space $H \otimes A^{co\, \xi}$ can be equipped with an associative multiplication giving it the structure of a left crossed product algebra $H \#^A A^{co\, \xi}$ \cite{11}. The multiplication in $H \#^A A^{co\, \xi}$ is constructed by using the cleavage map $\gamma$ to first define a right $H$-action on $A^{co\, \xi}$ given by

$$a \triangleright h := \sum_{(h)} \overline{\gamma}(h_1)a(\gamma(h_2))$$

and a linear map $\sigma : H \otimes H \to A^{co\, \xi}$ defined as

$$\sigma(h, h') = \sum_{(h), (h')} \overline{\gamma}(h_1')\gamma(h_2)\gamma(h_2'),$$

for all $a, a' \in A^{co\, \xi}$, $h, h' \in H$. The multiplication in $H \#^A A^{co\, \xi}$ is defined by

$$(h \otimes a)(h' \otimes a') = \sum_{(h), (h')} h_1h_1' \otimes \sigma(h_2, h_2')(a \triangleright h_3')a',$$

for all $a, a' \in A^{co\, \xi}$, $h, h' \in H$. In the case when $\gamma : H \to A$ is an algebra morphism, the multiplication in $H \#^A A^{co\, \xi}$ simplifies to

$$(h \otimes a)(h' \otimes a') = \sum_{(h)} hh_1' \otimes (a \triangleright h_2')a'$$

for all $h, h' \in H$ and $a, a' \in A^{co\, \theta}$, which is precisely the multiplication in the left smash product algebra, commonly denoted $H \# A^{co\, \xi}$. By a result of \cite{4}, shown in \cite{17} Proposition 7.2.3, there is an algebra isomorphism

$$\Psi : H \#^A A^{co\, \xi} \xrightarrow{\cong} A$$

given by $h \otimes a \mapsto \gamma(h)a$ for all $h \in H$ and $a \in A^{co\, \xi}$.

3.2. The algebra $O_q(P_J)^{co\, \theta}$. Recall we fix a subset

$$J \subseteq [n - 1].$$

Let $P_J$ be the standard parabolic subgroup of $SL_n$ obtained by deleting the negative simple roots $-\alpha_i$ for $i \in J$. Thus, $P_J$ is a group of block upper triangular matrices,

$$P_J = \{(a_{ij}) \in SL_n \mid a_{ij} = 0 \text{ if } (j, i) \in \Phi_J\}.$$

We will denote the Levi decomposition of $P_J$ by

$$P_J = L_J U_J,$$

where $L_J$ is the standard Levi factor of $P_J$ consisting of matrices with block entries off the main diagonal equal to 0, and $U_J$ is the unipotent subgroup of matrices in $P_J$ having block diagonal entries equal to identity matrices. Observe that matrix multiplication

$$L_J \times P_J \to P_J,$$

induces a coaction $O(P_J) \to O(L_J) \otimes O(P_J)$ among coordinate rings. Following \cite{18}, we turn our attention to the quantum analogue of this coaction

$$\theta : O_q(P_J) \to O_q(L_J) \otimes O_q(P_J),$$

where $O_q(L_j)$ and $O_q(P_j)$ are the quantized coordinate rings of $L_j$ and $P_j$ respectively. We recall $O_q(L_j)$ and $O_q(P_j)$ are obtained from $O_q(SL_n)$ by quotienting by certain two-sided ideals. First, the quantized coordinate ring $O_q(SL_n)$ is the $\mathbb{K}$-algebra generated by $x_{ij}$ $(i,j \in [n])$ and has defining relations

$$x_{ij}x_{\ell m} = \begin{cases} qx_{\ell m}x_{ij}, & (i < \ell \text{ and } j = m) \text{ or } (i = \ell \text{ and } j < m), \\ x_{\ell m}x_{ij}, & (i < \ell \text{ and } m < j), \\ x_{\ell m}x_{ij} + \tilde{q}x_{ij}x_{\ell m}, & (i < \ell \text{ and } j < m), \end{cases}$$

$$x_{ij}x_{\ell m}$$

inequality with cleavage map $\gamma$. In [8, Theorem 3.19] it was shown that

$$\text{det}_q := \sum_{\sigma \in \text{Sym}(n)} (-q)^{\ell(\sigma)}x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} = 1,$$

where $\ell(\sigma)$ is the number of inversions in $\sigma$ (see e.g. [12]). Furthermore $O_q(SL_n)$ is a Hopf algebra [12, Section 9.2.3, Proposition 10] with comultiplication $\Delta$, counit $\epsilon$, and antipode $S$; given by

$$\Delta(x_{ij}) := \sum_k x_{ik} \otimes x_{kj},$$

$$\epsilon(x_{ij}) := \delta_{ij},$$

$$S(x_{ij}) := (-q)^{i-j}[1, \ldots, \widehat{j}, \ldots, n][1, \ldots, \widehat{i}, \ldots, n],$$

where, for a pair of subsets $A = \{a_1 < \cdots < a_m\}$ and $B = \{b_1 < \cdots < b_n\}$ of $[n]$ of the same cardinality, the quantum minor $[A|B]$ with row set $A$ and column set $B$ is defined as

$$[A|B] := \sum_{\sigma \in \text{Sym}(m)} (-q)^{\ell(\sigma)}x_{a_1,\sigma(1)} \cdots x_{a_m,\sigma(m)} \in O_q(SL_n).$$

The quantized coordinate rings $O_q(P_j)$ and $O_q(L_j)$ are obtained from $O_q(SL_n)$ by quotienting by the two-sided ideals generated by $\{x_{ij} \mid (j,i) \in \Phi_j\}$ and $\{x_{ij} \mid (i,j) \in \Phi_j \text{ or } (j,i) \in \Phi_j\}$ respectively,

$$O_q(P_j) := O_q(SL_n)/\langle x_{ij} \mid (j,i) \in \Phi_j \rangle,$$

$$O_q(L_j) := O_q(SL_n)/\langle x_{ij} \mid (i,j) \in \Phi_j \text{ or } (j,i) \in \Phi_j \rangle.$$

With a slight abuse of notation, we use the symbol $x_{ij}$ to refer to the coset in $O_q(P_j)$ containing $x_{ij}$, whereas we will adopt the symbol $y_{ij}$ to refer to the coset in $O_q(L_j)$ containing $x_{ij}$. Hence,

$$\mathcal{G}_j := \{y_{ij} \mid (i,j) \in [n] \times [n], (i,j) \notin \Phi_j, (j,i) \notin \Phi_j\}$$

is a set of generators for $O_q(L_j)$. The quantized coordinate ring $O_q(L_j)$ inherits a Hopf algebra structure from $O_q(SL_n)$. We will denote the comultiplication, counit, and antipode of $O_q(L_j)$ by

$$\Delta_L, \epsilon_L, SL.$$

In [8, Theorem 3.19] it was shown that $O_q(P_j)^{co\theta} \subseteq O_q(P_j)$ is a left $O_q(L_j)$-cleft extension with cleavage map

$$\gamma : O_q(L_j) \rightarrow O_q(P_j)$$

being the algebra homomorphism given by $\gamma(y_{ij}) = x_{ij}$ for all generators $y_{ij} \in \mathcal{G}_j$. The convolution inverse of $\gamma$ is $\gamma = \gamma \circ SL$. Since $\gamma$ is an algebra homomorphism, the crossed product algebra $O_q(L_j)^{\#}\gamma \circ O_q(P_j)^{co\theta}$ is, in fact, the smash product algebra $O_q(L_j)\#O_q(P_j)^{co\theta}$ with multiplication given in [3,2]. By the above discussion, we have the following isomorphism.
Theorem 3.1. [8 Theorem 3.19]
There is an algebra isomorphism
\[ O_q(L_J) \# O_q(P_J)^{co\theta} \cong O_q(P_J) \]
given by \( y_{ij} \otimes u \mapsto x_{ij}u \), for all generators \( y_{ij} \in G_J \) and \( u \in O_q(P_J)^{co\theta} \).

To give a presentation of \( O_q(P_J)^{co\theta} \) in terms of generators and relations, first define the set
\[ C_i := \{ k \in [n] \mid r(k) = r(i) \}. \]
for \( i \in [n] \). It follows from the quantum determinant relation [3.3] in \( O_q(SL_n) \) that the quantum minor \( [C_i | C_j] \) is invertible in \( O_q(P_J) \) for all \( i \in [n] \). For each \((i, j) \in \Phi_J \), the ratio of quantum minors
\[ u_{ij} := [C_i | C_j]^{-1}[C_i | C_i \backslash \{i\} \cup \{j\}] \in O_q(P_J) \]
is a left coinvariant [8 Section 3.3]. These particular elements generate \( O_q(P_J)^{co\theta} \).

Theorem 3.2. [8 Theorems 3.35 and 3.50]
The algebra \( O_q(P_J)^{co\theta} \) is generated by \( u_{ij} \) (for \((i, j) \in \Phi_J \)) and has the following defining relations:
\[ u_{ij}u_{\ell m} - qu_{\ell m}u_{ij} = \begin{cases} 
q_{u_{\ell m}u_{ij}} & (\ell = i < j < m) \\
 & \text{or} \ (j = m \text{ and } w_{ij}^1(\ell) < w_{ij}^1(i)) \\
u_{\ell m}u_{ij} & \ (w_{ij}^1(\ell) < w_{ij}^1(i) < j < m) \\
 & \text{or} \ (i < j < \ell < m) \\
u_{\ell m}u_{ij} + \hat{q}u_{ij}u_{im} & \ (r(i) < \ell < i < j < m) \\
 & \text{or} \ (i \leq r(\ell) < \ell < j < m) \\
q^{-1}u_{\ell m}u_{ij} - \hat{q}u_{(im),\ell} & (i < j = \ell < m) \\
\end{cases} \]
where \( u_{(im),\ell} := (-q^{-r(\ell)})w_{ij}(\ell)u_{im} + \sum_{r(\ell) < k < w_{ij}(\ell)}(-q)^{r(\ell) - w_{ij}(k)}u_{\ell m}(k,m)u_{i,k}(k) \).

It was also shown in [8 Theorem 3.46] that \( O_q(P_J)^{co\theta} \) is an iterated skew polynomial ring
\[ \mathbb{K}[t_1; t_2; t_3; \cdots; t_M; \tau_M, \delta_M], \]
where \( M = \dim(U_J) \) and \( t_k \) (\( k \in [M] \)) is the \( k \)-th element in the sequence \( \{u_{ij}\} \) of generators ordered via the rule \( u_{ij} < u_{\ell m} \) if and only if either (1) \( w_{ij}^1(i) < w_{ij}^1(\ell) \), or (2) \( i = \ell \) and \( j < m \).

4. Proof of Theorem 2.4 The defining relations of \( \mathcal{U}_q(n_J) \)

In proving Theorem 2.4 which gives the defining relations in \( \mathcal{U}_q(n_J) \), we find it convenient to first introduce the \( q^{-1} \)-commutator; for \( x, y \in \mathcal{U}_q(\mathfrak{sl}(n)) \), define
\[ [x, y] := xy - q^{-1}yx \in \mathcal{U}_q(\mathfrak{sl}(n)). \]
Observe that for all \( x, y, z \in \mathcal{U}_q(\mathfrak{sl}(n)) \) such that \( xz = zx \), we have an associativity property,
\[ [[x, y], z] = [x, [y, z]]. \]
For $a_1, \ldots, a_m \in [n-1]$, we use the abbreviations
\[
E_{a_1, \ldots, a_m} = \left[ \left[ \cdots \left[ E_{a_1}, E_{a_2}, \cdots \right], E_{a_3} \right], \cdots \right], E_{a_m} \in \mathcal{U}_q(\mathfrak{sl}(n)),
\]
\[
\text{T}_{a_1, \ldots, a_m} = T_{a_1} \circ \cdots \circ T_{a_m} \in \text{Aut}(\mathcal{U}_q(\mathfrak{sl}(n))).
\]

The $q$-Serre relation \ref{2.6} implies that if a pair of consecutive indices, say $a_k$ and $a_{k+1}$, differ by more than 1, then those indices in the nested $q^{-1}$-commutator $E_{a_1, \ldots, a_m}$ can be interchanged,
\[
E_{a_1, \ldots, a_m} = E_{a_1, \ldots, a_{k-1}, a_{k+1}, a_k, a_{k+2}, \ldots, a_m}.
\]

The following lemma tells us how certain nested $q^{-1}$-commutators behave under Lusztig’s symmetries.

**Lemma 4.1.**

*For all $1 \leq k < \ell < n$,*

1. $T_k(E_{k+1, k+2, \ldots, \ell}) = E_{k, k+1, \ldots, \ell}$,
2. $T_k(E_{k, k+1, \ldots, \ell}) = E_{k, k+1, \ldots, \ell-1}$,
3. $T_k(E_{\ell-1, \ell-2, \ldots, \ell}) = E_{\ell, \ell-1, \ldots, k}$,
4. $T_k(E_{\ell, \ell-1, \ldots, k}) = E_{\ell, \ell-1, \ldots, k+1}$.

*For all $k, \ell, m \in [n-1]$ such that $k \leq m$ and $\ell \notin \{k-1, k, m, m+1\},$

5. $T_k(E_{k, k+1, \ldots, m}) = E_{k, k+1, \ldots, m}$,
6. $T_k(E_{m, m-1, \ldots, k}) = E_{m, m-1, \ldots, k}$.

**Proof.** Parts \ref{1} and \ref{3} follow from the definition of the braid group action \ref{2.8} together with the fact that the braid group acts via algebra automorphisms.

To prove part \ref{2} we first consider the case when $\ell = k + 1$. In this setting, the desired result follows directly from Proposition \ref{2.1} $T_\ell(E_{\ell-1, \ell}) = T_\ell T_{\ell-1}(E_\ell) = E_{\ell-1}$. However, if $\ell > k + 1$, we can use \ref{1} to write $E_{k, k+1, \ldots, \ell} = [E_{k, k+1, \ldots, \ell-2}, E_{\ell-1, \ell}]$. From \ref{2.8} $E_p \ (p = k, k+1, \ldots, \ell-2)$ is fixed by $T_\ell$. Thus
\[
T_\ell(E_{k, k+1, \ldots, \ell}) = T_\ell([E_{k, k+1, \ldots, \ell-2}, E_{\ell-1, \ell}])
= [T_\ell(E_{k, k+1, \ldots, \ell-2}), T_\ell(E_{\ell-1, \ell})]
= [E_{k, k+1, \ldots, \ell-2}, E_{\ell-1}]
= E_{k, k+1, \ldots, \ell-1}.
\]

Part \ref{4} can be proved in a manner similar to part \ref{2}.

For part \ref{5} consider first the case when $\ell < k-1$ or $\ell > m+1$. In this setting, the result follows directly from \ref{2.8}. Next, suppose $k < \ell < m$ and $\ell + 1 < m$. By \ref{1}
\[
E_{m, m-1, \ldots, k} = \left[ \left[ \cdots \left[ E_{m, m-1, \ldots, \ell+2}, E_{\ell+1, \ell}, E_{\ell-1}, \cdots \right], E_k \right].
\]
Thus,
\[
T_\ell(E_{m, m-1, \ldots, k}) = \left[ \left[ \cdots \left[ T_\ell(E_{m, m-1, \ldots, \ell+2}), T_\ell(E_{\ell+1, \ell}), T_\ell(E_{\ell-1}), \cdots \right], T_\ell(E_k) \right].
\]

By \ref{2.8} $E_{\ell-2}, E_{\ell-3}, \ldots, E_k$, and $E_{m, m-1, \ldots, \ell+2}$ are fixed by $T_\ell$ and $T_\ell(E_{\ell-1}) = E_{\ell, \ell-1}$, whereas by Proposition \ref{2.1} $T_\ell([E_{\ell+1, \ell}, E_\ell]) = T_{\ell+1}(E_\ell) = E_{\ell+1}$. Hence, we obtain
\[
T_\ell(E_{m, m-1, \ldots, \ell}) = \left[ \left[ \cdots \left[ E_{m, m-1, \ldots, \ell+1}, E_{\ell, \ell-1}, \cdots \right], E_k \right.
= E_{m, m-1, \ldots, k}.
\]
Now suppose $k < \ell < m$ and $m = \ell + 1$. In this case we have
\[
T_\ell(E_{m, m-1, \ldots, k}) = T_\ell([\cdots [E_{\ell+1, \ell}, E_{\ell-1}, \cdots], E_k])
= [\cdots [E_{\ell+1, \ell}, E_{\ell-1}, \cdots], E_k]
Part 2 can be proved in a manner similar to part 0. □

The following lemma gives some commutation relations among certain nested $q^{-1}$-commutators.

**Lemma 4.2.** For all $1 \leq k \leq \ell < m < n$,

1. $E_k E_{k+1,\ldots,\ell} E_{k+1,\ldots,m} = q E_{k+1,\ldots,m} E_{k+1,\ldots,\ell}$.

2. $E_{m,m-1,\ldots,k} E_{m,m-1,\ldots,k} = q E_{m,m-1,\ldots,k} E_{m,m-1,\ldots,\ell}$.

3. $E_{k,k+1,\ldots,\ell} E_{\ell+1,\ldots,m} E_{k+1,\ldots,\ell} = E_{\ell+1,\ldots,m} E_{k+1,\ldots,\ell}$.

4. $E_{p,p-1,\ldots,k} E_{m,m-1,\ldots,\ell} E_{p,p-1,\ldots,k} = E_{m,m-1,\ldots,\ell}$.

5. $E_{k,k+1,\ldots,\ell} E_{m,m-1,\ldots,\ell} E_{k+1,\ldots,\ell} = E_{m,m-1,\ldots,\ell} E_{k+1,\ldots,\ell}$.

6. $E_{p,p-1,\ldots,k} E_{\ell+1,\ldots,m} E_{p,p-1,\ldots,k} = E_{\ell+1,\ldots,m} E_{p,p-1,\ldots,k}$.

7. $E_k E_{k,k+1,\ldots,\ell} E_{k+1,\ldots,k+1} E_k = q E_{k+1,\ldots,k} E_{k,k+1,\ldots,\ell}$.

8. $E_{m+1, k, k-1, k} = q E_{m+1, k, k-1, k}$.

**Proof.** To prove part 1 we first define $\varphi := T_{k,k+1,\ldots,\ell}^{-1}$. From part 1 of Lemma 4.1, $\varphi(E_{k,k+1,\ldots,\ell}) = E_\ell$ and $\varphi(E_{k,k+1,\ldots,m}) = E_{m+1,\ldots,m+1}$. Observe that the q-Serre relation 2.4 is equivalent to $E_r E_s = q E_{r-s} E_r$ whenever $|r-s| = 1$. Thus $E_r E_{r+1} = q E_{r+1} E_r$. Furthermore, by 2.6 $E_r$ commutes with $E_{r+2}, E_{r+3}, \ldots, E_m$. Therefore $E_r E_{r+1,\ldots,m} = q E_{r+1,\ldots,m} E_r$. Since $\varphi$ is an automorphism of $U_q(\mathfrak{sl}(n))$, $E_{m+1, k, k-1, k} = q E_{m+1, k, k-1, k}$.

For part 2 let $\varphi := T_{m,m-1,\ldots,\ell}^{-1}$. From part 3 of Lemma 4.1, $\varphi(E_{m,m-1,\ldots,\ell}) = E_\ell$ and $\varphi(E_{m,m-1,\ldots,k}) = E_{m+1,\ldots,k}$. The q-Serre relations 2.4 and 2.6 imply $E_r E_{r+1,\ldots,k} = q E_{r+1,\ldots,k} E_r$. Thus, $0 = \varphi^{-1}(E_r E_{r+1,\ldots,k} - q E_{r+1,\ldots,k} E_r) = E_{m+1,\ldots,k} - q E_{m+1,\ldots,k} E_{m+1,\ldots,k}$. Observe that Proposition 4.1 implies $\theta(E_{m,m+1}) = E_{m+1,\ldots,k}$. Since $\varphi$, $\psi$, and $\theta$ are automorphisms of $U_q(\mathfrak{sl}(n))$, the composition $\theta \circ \psi \circ \varphi$ is also an automorphism of $U_q(\mathfrak{sl}(n))$. Since $E_{m+1}$ and $E_{m-1}$ commute and are the images of $E_\ell E_{\ell+1,\ldots,m}$ and $E_{k,k+1,\ldots,p}$ respectively under $\theta \circ \psi \circ \varphi$, this implies that $E_\ell E_{\ell+1,\ldots,m}$ and $E_{k,k+1,\ldots,p}$ also commute. Parts 4, 5, and 6 can be proved similarly.

To prove part 7 we first use the q-Serre relation 2.4 which is equivalent to $E_k E_{k,k-1} = q E_{k,k-1} E_k$ to get

$$E_k E_{k,k-1} E_{k+1} = q E_{k,k-1} E_k E_{k+1} - q^{-1} E_k E_{k+1} E_k.

Next we make the substitution $E_k E_{k+1} = E_{k+1} + q^{-1} E_k E_{k+1} E_k$ to obtain

$$E_k E_{k,k-1} E_{k+1} = q E_{k,k-1} E_{k+1} - q^{-1} E_k E_{k+1} E_k + E_k E_{k+1} E_k E_{k-1} E_k E_k.$$ (4.3)
Since $E_{k,k-1} = T_k(E_{k-1})$, $E_{k,k+1} = T_k(E_{k+1})$ and $E_{k-1}$ and $E_{k+1}$ commute, this implies that $E_{k,k-1}$ and $E_{k,k+1}$ commute also. Hence, the first two terms in (4.3) to obtain

\[ E_k E_{k,k-1} E_{k,k+1} = qE_k E_{k,k-1} E_{k,k+1} + E_{k,k+1} E_k E_{k,k-1} E_k - q^{-1}E_{k,k+1} E_k E_{k,k-1} E_k. \]

We again use $E_k E_{k,k-1} = qE_{k,k-1} E_k$ to obtain

\[ E_k E_{k,k-1} - E_{k,k+1} E_k = qE_{k,k-1} E_k + E_{k,k+1} E_k. \]

To prove part S, we first use the $q$-Serre relation to obtain

\[ E_k E_{k,k-1} = qE_{k,k-1} E_k + E_{k,k} E_k. \]

Next we substitute $E_k E_{k+1}$ with $-qE_{k+1} E_k + q E_{k+1} E_k$ to obtain

\[ E_k E_{k+1} E_k = -qE_{k+1} E_k E_{k,k} + q E_{k+1} E_k E_k + E_{k+1} E_{k,k} E_k. \]

Observe that $E_{k,k}$ and $E_{k+1,k}$ commute. Hence the first two terms in (4.4) above involving the products $E_{k,k} E_{k+1,k}$ and $E_{k+1,k} E_{k,k}$ can be combined to get

\[ E_k E_{k+1} E_k = -qE_{k+1} E_k E_{k,k} - q^{-1}E_{k,k+1} E_k E_{k,k} + E_{k+1} E_{k,k} E_k, \]

which is equivalent to the identity as written in part S.

\[ \square \]

Recall the definition of the function $r : [n] \to J \cup \{0\}$,

\[ r(m) := \max\{k \in J \cup \{0\} \mid k < m\}. \]

The following lemma tells us how each root vector in $\mathcal{U}_q(n_J)$ can be written as a nested $q^{-1}$-commutator.

**Lemma 4.3.** If $(i,j) \in \Phi_J$, then

\[ X_{ij} = E_r(j,r(j)+1,i) \cdot r(j)+2,j+1,\ldots,j-1. \]

**Proof.** Let $J = \{i_1 < \cdots < i_t\} \subseteq [n-1]$. For $k \in [t]$, let $w_k \in W$ be the initial segment of $w_J$ defined as $w_k := S_{i_1} S_{i_2} \cdots S_{i_{k+1}} \in W$ (recall the definition of $S_k$ in [2.14]). Observe that $w_k(p) = p$ whenever $p \leq i_k$, whereas $w_k(p) = w_J(p - i_k)$ if $p > i_k$.

For $p \leq n$, define $M(p) := \min\{r \in J \cup \{n\} \mid p \leq r\}$. Assume $(i,j) \in \Phi_J$. We use the abbreviation $\ell(i,j) = i + w_J^{-1}(j) - 1$. The restrictions imposed on $i$ and $j$ imply that $\ell(i,j) < n$. Define the Weyl group elements $v_{ij} := s_{i_1} S_{i_2} \cdots s_{i_{j+1}} \in W$ and $u_i := (s_{M(j)} \cdots s_{i_{j+1}})(s_{M(j)+1} \cdots s_{n-2})(s_{i+1} \cdots s_{n-M(i)+j+1}) \in W$. Let $N(p) := 1 + \#\{r \in J \mid r < p\}$ and let $W_{ij} \subseteq W$ be the initial segment of $w_J$ defined as $W_{ij} := w_N(i) u_i v_{ij}$. Since $W_{ij} s_{i_{j+1}}$ is also an initial segment of $w_J$, it follows that $W_{ij}(\alpha_{i_{j+1}})$ is a positive Lusztig root. In fact, we have

\[ W_{ij}(\alpha_{i_{j+1}}) = w_N(i) u_i (e_i - e_{i_{j+1}}) = w_N(i) (e_i - e_{M(i)+w_J^{-1}(j)}) = e_i - e_{j}. \]

Therefore, the simple root $\alpha_i$ is a Lusztig root because $w_{k,k+1} E_{\ell(i,k)} = e_{ik} - e_{ik+1} = \alpha_i$. Proposition 2.1 implies that $T_{W_{ik,i_k}}(E_{\ell(i,k+1)}) = E_{ik}$. Thus, $X_{ik,i_k} = E_{ik}$. More generally, $X_{ij} = T_{W_{ij}}(E_{\ell(i,j)})$. 

\[ \square \]
We assume now that $i > 1$ and $(i, j) \in \Phi_J$. Define the Weyl group elements
\[ y_{ij} := (s_{t(i,j)} \cdots s_{t-M(i,i)+1})v_{1,j} \]  
We have $W_{i-1,j} = W_{ij}y_{ij}$. Thus, $X_{i-1,j} = T_{w_{N(i)}} T_{v_{ij}} T_{y_{ij}} (E_{t(i,j)-1})$. Since
\[ T_{y_{ij}} (E_{t(i,j)-1}) = T_{v_{i,j}} T_{i,j} (E_{t(i,j)-1}) = T_{v_{i,j}} (\{ E_{t(i,j)}, E_{t(i,j)-1} \}) \]
and the braid group generators act via algebra automorphisms, we have $X_{i-1,j} = T_{w_{N(i)}} T_{v_{ij}} T_{y_{ij}} (\{ T_{v_{i,j}} (E_{t(i,j)}), T_{v_{i,j}} (E_{t(i,j)-1}) \})$. However, since $v_{i,j}^{-1} (\alpha_{t(i,j)}) = \alpha_i \alpha_{t(i,j)}$ and $v_{i,j} v_{i-1,j} (\alpha_i \alpha_{t(i,j)}^{-1}) = \alpha_{i-1}$, Proposition 2.1 implies
\[ X_{i-1,j} = T_{w_{N(i)}} T_{v_{ij}} (\{ T_{v_{i,j}} (E_{t(i,j)}), E_{t(i,j)-1} \}) \]
Finally, since $w_{N(i)} u_{i} (\alpha_{i-1}) = \alpha_{i-1}$, we have $X_{i-1,j} = [X_{ij}, E_{t(i,j)-1}]$. Hence, for all $r \in J$ and $1 \leq i \leq r$, we iteratively get $X_{i,r+1} = E_{r,r-1,\ldots,i}$.  
Next suppose $i$ and $j$ are a pair of integers in $\{ n-1 \}$ such that $j \notin J$ and $(i, j) \in \Phi_J$. Hence, $w_j^{-1}(j+1) = w_j^{-1}(j) + 1$, $\ell(i, j+1) = \ell(i, j) + 1$, and $W_{ij} = W_{ij} + W_{ij}$. Hence,
\[ X_{i,j+1} = T_{w_{ij}} (E_{t(i,j)+1}) = T_{w_{ij}} T_{s_{t(i,j)}} (E_{t(i,j)+1}) = T_{w_{ij}} (\{ E_{t(i,j)}, E_{t(i,j)+1} \}). \]
However, since
\[ W_{ij} (\alpha_{t(i,j)+1}) = w_{N(i)} u_{i} v_{ij} (\alpha_{t(i,j)+1}) = w_{N(i)} u_{i} (\alpha_{t(i,j)+1}) = w_{N(i)} (\alpha_{M(i)} + w_j^{-1}(j)) = \alpha_j, \]
It follows from Proposition 2.1 that $T_{w_{ij}} (E_{t(i,j)+1}) = E_j$. Thus,
\[ X_{i,j+1} = T_{w_{ij}} (\{ E_{t(i,j)}, E_{t(i,j)+1} \}) = [T_{w_{ij}} (E_{t(i,j)}), T_{w_{ij}} (E_{t(i,j)+1})] = [X_{ij}, E_j]. \]
If $r \in J$ and $r+1, \ldots, r+s \notin J$, we iteratively obtain
\[ X_{i,r+s+1} = [[[\cdots [X_{i,r+1}, E_{r+1}], E_{r+2}], \cdots], E_{r+s}] = [[[\cdots [E_{r-r-1,\ldots,i}, E_{r+1}], E_{r+2}], \cdots], E_{r+s}] = E_{r-r-1,\ldots,i,r+1, r+2, \ldots, r+s}. \]

The next lemma tells us how Lusztig’s symmetries act on the root vectors of $\mathcal{U}_q (\mathfrak{n}_J)$.

**Lemma 4.4.** Suppose $(i, j) \in \Phi_J$.

1. If $j > r(j) + 1$, then $T_{j-1} (X_{ij}) = X_{i,j-1}$.
2. If $i < r(j)$, then $T_{i} (X_{ij}) = X_{i+1,j}$.
3. If $k \in [n-1]$ and $k \notin \{ i-1, i, r(j), j-1, j \}$, $T_{k} (X_{ij}) = X_{ij}$.

**Proof.** For short, let $r = r(j)$. To prove part 1 we first consider the case when $j > r + 2$. In this setting we can use the associativity property 3.1 to write $X_{ij} = [E_{r-r-1,\ldots,i,r+1, r+2, \ldots, j-3}, E_{j-2,j-1}]$. From 2.8 $E_{r-r-1,\ldots,i,r+1, r+2, \ldots, j-3}$ is fixed by $T_{j-1}$, and by Proposition 2.1 $T_{j-1} (E_{j-2,j-1}) = T_{j-1,j-2} (E_{j-1}) = E_{j-2}$. Hence $T_{j-1} (X_{ij}) = [E_{r-r-1,\ldots,i,r+1, r+2, \ldots, j-3}, E_{j-2}] = X_{i,j-1}$. On the other hand, if $j = r + 2$, we can use 3.2 to rewrite $X_{ij}$ as $X_{ij} = E_{r-r-1,\ldots,i,r+1} = E_{r-r+1, r-1, \ldots, i}$. By Proposition 2.1 $T_{j-1} (E_{r+1}) = E_{r}$, whereas by part 6 of Lemma 4.1 we have $T_{j-1} (E_{r-2,\ldots,i}) = E_{r-1, r-2, \ldots, i}$. Therefore $T_{j-1} (X_{ij}) = E_{r-r-1, \ldots,i} = X_{ij}$. 

1. If $j > r(j) + 1$, then $T_{j-1} (X_{ij}) = X_{i,j-1}$.
2. If $i < r(j)$, then $T_{i} (X_{ij}) = X_{i+1,j}$.
3. If $k \in [n-1]$ and $k \notin \{ i-1, i, r(j), j-1, j \}$, $T_{k} (X_{ij}) = X_{ij}$.
In proving part 2 we first suppose $i + 1 < r$. We can use the associativity property 4.1 to write $X_{ij} = \lbrack \cdots | E_{r, \ldots, r+2}, E_{i+1,i}, E_{r+2}, \cdots, E_{j-1} \rbrack$. By 2.8 $T_i(E_k) = E_k$ ($k = r + 1, r + 2, \ldots, j - 1$), by Proposition 2.1 $T_i(E_{i+1,i}) = E_{i+1}$, and by part 6 of Lemma 4.1 $T_i(E_{r,\ldots,r+2}) = E_{r,\ldots,i+2}$. Hence $T_i(X_{ij}) = [\cdots | E_{r,\ldots,r+1}, E_{r+1}, \cdots, E_{j-1} | X_{i+1,j} = X_{i+1,j}$. On the other hand, if $i + 1 = r$ then $X_{ij}$ can be written as $X_{ij} = E_{i+1,i,r+1,r+2,\ldots,j-1}$. In this case the result follows, again, by using $E_k$ ($k = r + 1, r + 2, \ldots, j - 1$) is fixed by $T_i$ and $T_i(E_{i+1,i}) = E_{i+1}$.

For part 3 we consider first the case when $k < i - 1$ or $k > j$. Here, the result follows directly from the definition of the Lusztig symmetries 2.8. Now suppose $i < k < r(j)$. By part 6 of Lemma 4.1 $E_{r(j),r(j)-1,\ldots,i}$ is fixed by the automorphism $T_k$. Furthermore, by 2.8 $E_{r(j)+1}, E_{r(j)+2}, \ldots, E_{j-1}$ are also fixed by $T_k$. Thus $X_{ij}$, which can be written as $\lbrack \cdots | E_{r(j),r(j)-1,\ldots,i}, E_{r(j)+1}, \cdots, E_{j-1} \rbrack$, is also fixed by $T_k$. Finally suppose $r(j) < k < j$. We can use 1.2 to write $X_{ij}$ as $[\cdots | E_{r(j),r(j)+1,\ldots,j-1}, E_{r(j)-1}, E_{r(j)-2}, \cdots, E_i \rbrack$. From part 5 of Lemma 4.1 $E_{r(j),r(j)+1,\ldots,j-1}$ is fixed by the automorphism $T_k$, while by 2.8 $E_{r(j)-1}, E_{r(j)-2}, \ldots, E_i$ are fixed by $T_k$. Hence, $X_{ij}$ is also fixed by $T_k$.

$$\square$$

The following theorem is the main result of this section. It gives the defining relations in $\mathcal{U}_q(\mathfrak{n}_J)$.

**Theorem 4.5.** The quantized nilradical $\mathcal{U}_q(\mathfrak{n}_J)$ is generated by the Lusztig root vectors $X_{ij}$ ($(i,j) \in \Phi_J$) and has the following defining relations:

\begin{equation}
X_{ij}X_{\ell m} = \begin{cases}
qX_{\ell m}X_{ij} & (\ell < i \text{ and } j = m) \\
& \text{or } (\ell = i \text{ and } w_{j}^{-1}(j) < w_{j}^{-1}(m)) \\
X_{\ell m}X_{ij} & (\ell < i < w_{j}^{-1}(j) < w_{j}^{-1}(m)) \\
& \text{or } (\ell < m < i < j) \\
& \text{or } (\ell \leq r(m) < i < m) \\
X_{\ell m}X_{ij} + \hat{q}X_{ij}X_{\ell m} & (\ell < i \leq r(m) < j < m) \\
& \text{or } (\ell < i \leq r(m) < m \leq r(j) < j) \\
q^{-1}X_{\ell m}X_{ij} + X_{(ij),m} & (\ell < m = i < j)
\end{cases}
\end{equation}

where $X_{(ij),m} := (-q)^{m-r(m)-1}X_{ij} + \hat{q}\sum_{r(m) < k < m}(-q)^{m-k-1}X_{kj}X_{ik}$.

**Proof.** Suppose first $\ell < i$ and $j = m$. Let $r = r(j)$ and let $\varphi = T_{r+1,\ldots,j-2,j-1}$. By part 1 of Lemma 4.3 $\varphi(X_{ij}) = X_{i,r+1}$ and $\varphi(X_{\ell m}) = X_{\ell,r+1}$, and from Lemma 4.3 $X_{i,r+1} = E_{r,r-1,\ldots,i}$ and $X_{\ell,r+1} = E_{r,r-1,\ldots,\ell}$. By Lemma 4.2 part 2 $\varphi(X_{ij})\varphi(X_{\ell m}) = q\varphi(X_{\ell m})\varphi(X_{ij})$. Since $\varphi$ is an automorphism of $\mathcal{U}_q(\mathfrak{sl}(n))$, $X_{ij}X_{\ell m} = qX_{\ell m}X_{ij}$.

Now suppose $\ell = i$ and $w_{j}^{-1}(j) < w_{j}^{-1}(m)$. If $r(j) = r(m) = r$, then $j < m$. In this case, let $\varphi = T_{r-1,\ldots,r-2,\ldots}$. By part 2 of Lemma 4.3 $\varphi(X_{ij}) = X_{ij}$ and $\varphi(X_{\ell m}) = X_{\ell m}$, and from Lemma 4.3 $X_{ij} = E_{r,r+1,\ldots,j-1}$ and $X_{\ell m} = E_{r,r+1,\ldots,m-1}$. Part 1 of Lemma 4.2 implies $\varphi(X_{ij})\varphi(X_{\ell m}) = q\varphi(X_{\ell m})\varphi(X_{ij})$. Therefore $X_{ij}X_{\ell m} = qX_{\ell m}X_{ij}$. On the other hand, if $r(j) \neq r(m)$, then $r(m)$ <
\( m \leq r(j) < j \). Let

\[
\psi_1 := T_{r(m)+1,r(m)+2,...,m-1} \circ T_{r(j)+1,r(j)+2,...,j-1} \circ T_{r(m)-1,...,i+1,i}.
\]

From Lemma 4.4, \( \psi_1(X_{ij}) = X_{r(m),r(j)+1} \) and \( \psi_1(X_{\ell m}) = X_{r(m),r(m)+1} \). By Lemma 4.3, \( X_{r(m),r(j)+1} = E_{r(j),r(j)-1,...,r(m)} \) and \( X_{r(m),r(m)+1} = E_{r(m)} \). It follows from the \( q \)-commutativity relation (part 2 of Lemma 4.2) that \( X_{r(m),r(j)+1}X_{r(m),r(m)+1} = qX_{r(m),r(m)+1}X_{r(m),r(j)+1} \). Hence, \( X_{ij}X_{\ell m} = qX_{\ell m}X_{ij} \).

Now suppose \( \ell < i < w_0^j(j) < w_0^j(m) \). If \( r(j) = r(m) = r \), then we must have \( \ell < i \leq r < m < j \). Let

\[
\psi_2 := T_{r,-2,-3,...,i-1} \circ T_{r+1,r+2,...,m-1} \circ T_{r-1,r-2,...,i} \circ T_{r+1,m+2,...,j-1} \circ T_{i-2,i-3,...,\ell}.
\]

By Lemmas 4.3 and 4.4,

\[
\psi_2(X_{ij}) = X_{r,m-1} = E_{r,r+1,...,m},
\]
\[
\psi_2(X_{\ell m}) = X_{r-1,r+1} = E_{r,r-1}.
\]

Hence, from part 1 of Lemma 4.1, \( (T^{-1}_r \circ \psi_2)(X_{ij}) = E_{r+1,r+2,...,m} \). Furthermore, \( (T^{-1}_r \circ \psi_2)(X_{\ell m}) = E_{r-1} \). However, since \( E_{r+1,r+2,...,m} \) commutes with \( E_{r-1} \), it follows that \( X_{ij} \) and \( X_{\ell m} \) must also commute. On the other hand, if \( r(j) \neq r(m) \), then we must have \( \ell < i \leq r(j) < j \leq r(m) < m \). Let

\[
\psi_3 := T_{r(j)-1,r(j)-2,...,i} \circ T_{r(j)+1,r(j)+2,...,j-1} \circ T_{i-2,i-3,...,\ell} \circ T_{r(m)+1,r(m)+2,...,m-1}.
\]

By Lemmas 4.3 and 4.4,

\[
\psi_3(X_{ij}) = X_{r(j),r(j)+1} = E_{r(j)},
\]
\[
\psi_3(X_{\ell m}) = X_{r-1,r+1} = E_{r,r-1}.
\]

By part 2 of Lemma 4.2, \( E_{r(j)} \) and \( E_{r(r),r(m)-1,...,i-1} \) commute also. Hence, \( X_{ij} \) and \( X_{\ell m} \) commute also.

Next suppose \( \ell < m < i < j \). Since \( X_{ij} = E_{r(j),r(j)-1,...,i,r(j)+1,r(j)+2,...,j-1} \) and \( X_{\ell m} = E_{r(m),r(m)-1,...,\ell,r(m)+1,r(m)+2,...,m-1} \) (Lemma 4.3) and each of \( E_i, \ldots, E_{j-1} \) commutes with each of \( E_{i}, \ldots, E_{m-1} \) (2.6), it follows that \( X_{ij} \) commutes with \( X_{\ell m} \).

Next suppose \( \ell \leq r(m) < i < m < j \). Therefore, \( \ell \leq r(i) = r(m) < i \leq m \leq r(j) < j \). Let

\[
\psi_4 := T_{r(j)-1,r(j)-2,...,i} \circ T_{r(m)+1,r(m)+2,...,m-1} \circ T_{m-2,m-3,...,i} \circ T_{r(m)-1,r(m)-2,...,\ell} \circ T_{r(j)+1,r(j)+2,...,j-1}.
\]

From Lemmas 4.3 and 4.4,

\[
\psi_4(X_{ij}) = X_{r(j),r(j)+1} = E_{r(j)},
\]
\[
\psi_4(X_{\ell m}) = X_{r(m),r(m)+1} = E_{r(m)}.
\]

Since \( r(j) - r(m) > 1 \), the \( q \)-Serre relation 2.6 implies that \( E_{r(j)} \) and \( E_{r(m)} \) commute. Hence \( X_{ij} \) and \( X_{\ell m} \) commute also.

Next suppose \( \ell < i \leq r(m) < j < m \). Thus \( r(j) = r(m) = r \). Let

\[
\psi_5 = T_{r+2,r+3,...,j} \circ T_{r-2,r-3,...,i-1} \circ T_{r+1,r+2,...,j-1} \circ T_{r-1,r-2,...,i} \circ T_{j+1,j+2,...,m-1} \circ T_{i-2,i-3,...,\ell}.
\]

By Lemmas 4.3 and 4.4,

\[
\psi_5(X_{ij}) = X_{r,r+1} = E_r,
\]
\[
\psi_5(X_{\ell m}) = X_{r-1,r+2} = E_{r,r-1,r+1},
\]
\[
\psi_5(X_{\ell m}) = X_{r-1,r+1} = E_{r,r-1},
\]
\[
\psi_5(X_{\ell m}) = X_{r,r+2} = E_{r,r+1}.
\]
By Lemmas 4.3 and 4.4, \(\psi_5(X_{ij}) = \psi_5(X_{\ell m})\psi_5(X_{ij}) + \hat{\psi}_6\psi_5(X_{ij}) \psi_5(X_{im})\).

Since \(\psi_5\) is an automorphism of \(\mathcal{U}_q(\mathfrak{sl}(n))\), \(X_{ij} X_{\ell m} = X_{\ell m} X_{ij} + \hat{\psi}_6 X_{ij} X_{im}\).

Next suppose \(\ell < i \leq r(m) < m \leq r(j) < j\). Let \(\psi_6 = T_{r(m)-2,r(m)-3,...,i-1} \circ T_{r(m)-1,r(m)-2,...,i} \circ T_{r(m)+1,r(m)+2,...,m-1} \circ T_{r(j)+1,r(j)+2,...,j-1} \circ T_{i-2,i-3,...,\ell} \).

By Lemmas 4.3 and 4.4,
\[
\begin{align*}
\psi_6(X_{ij}) &= X_{r(m),r(j)+1} = E_{r(j),r(j)-1,...,r(m)}, \\
\psi_6(X_{\ell m}) &= X_{r(m)-1,r(m)+1} = E_{r(m),r(m)-1}, \\
\psi_6(X_{ij}) &= X_{r(m)-1,r(j)+1} = E_{r(j),r(j)-1,...,r(m)-1}, \\
\psi_6(X_{im}) &= X_{r(m),r(m)+1} = E_{r(m)}. 
\end{align*}
\]

Next let \(\xi = T_{r(j)-1}^{-1} T_{r(m)-1}(E_{r(m)+1,r(m)-1}) = E_{r(m)+1,r(m)-1}\). Since \(T_{r(m)-1}, \xi, \text{ and } \psi_6\) are automorphisms of \(\mathcal{U}_q(\mathfrak{sl}(n))\), the composition \(\theta := T_{r(m)-1} \circ \xi \circ \psi_6\) is also an automorphism of \(\mathcal{U}_q(\mathfrak{sl}(n))\). It follows from part 8 of Lemma 4.2 that \(X_{ij} X_{\ell m} = X_{\ell m} X_{ij} + \hat{\psi}_6 X_{ij} X_{im}\).

Finally suppose \(\ell < m = i < j\). Let \(\psi_7 = T_{r(j)+1,r(j)+2,...,j-1} \circ T_{r(m)-1,r(m)-2,...,\ell} \).

By Lemmas 4.3 and 4.4,
\[
\begin{align*}
\psi_7(X_{ij}) &= X_{r,j,r(j)+1} = E_{r(j),r(j)-1,...,r(j)}, \\
\psi_7(X_{\ell m}) &= X_{r,m} = E_{r(m),r(m)+1,...,m-1}, \\
\psi_7(X_{ij}) &= X_{r(j),r(j)+1} = E_{r(j),r(j)-1,...,r(m)}. 
\end{align*}
\]

We proceed now by induction on \(m - r(m)\). First suppose \(m - r(m) = 1\). Hence \(\psi_7(X_{im}) = E_{r(m)}\) and we have
\[
\begin{align*}
\psi_7(X_{ij} X_{\ell m} - q^{-1} X_{\ell m} X_{ij}) &= E_{r(j),r(j)-1,...,r(m)} - q^{-1} E_{r(m)} E_{r(j),r(j)-1,...,i} \\
&= E_{r(j),r(j)-1,...,i-1} \\
&= X_{i-1,r(j)+1} \\
&= \psi_7(X_{ij}). 
\end{align*}
\]

Next, suppose \(m - r(m) > 1\). From Lemmas 4.3 and 4.4,
\[
\psi_7(X_{i-1,j}) = X_{i-1,r(j)+1} = E_{r(j),r(j)-1,...,i-1}
\]
Therefore,

\[ \psi_7(X_{\ell,m-1}) = X_{r(m),m-1} = E_{r(m),r(m)+1,...,m-2}. \]

Therefore,

\[ \psi_7([X_{ij}, X_{km}]) = T_{r(m),r(m)+1,...,m-2}([X_{ij},r(j)+1,E_{m-1}]) \]

\[ = T_{r(m),r(m)+1,...,m-2}(E_{r(j),r(j)-1,...,m-1}) \]

\[ = T_{r(j),r(j)-1,...,m} \circ T_{r(m),r(m)+1,...,m-2}(E_{m-1}) \]

\[ = T_{r(j),r(j)-1,...,m}(E_{r(m),r(m)+1,...,m-1}) \]

\[ = [E_{r(m),r(m)+1,...,m-2}, E_{r(j),r(j)-1,...,m-1}] \]

\[ = [X_{r(m),m-1}, X_{m-1,r(j)+1}] \]

\[ = \psi_7([X_{\ell,m-1}, X_{i-1,j}]) \]

\[ = \psi_7(X_{\ell,m-1}X_{i-1,j} - q^{-1}X_{i-1,j}X_{\ell,m-1}). \]

By the inductive hypothesis, we can replace \( X_{\ell,m-1}X_{i-1,j} \) in the last line above with \( qX_{i-1,j}X_{\ell,m-1} + (-q)^{m-r(m)-1}X_{ij} - q\tilde{q}\sum_{r(m)<k<m}(-q)^{m-k-2}X_{kj}X_{ij} \). After making this substitution, the desired result follows.

Since all of the relations given in the statement of this theorem hold in the algebra \( U_q(n_j) \), these relations are defining relations of \( U_q(n_j) \) by the PBW basis theorem \([2,12]\).

\[ \square \]

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