Circular Coloring and Mycielski Construction

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Abstract

In this paper, we investigate circular chromatic number of Mycielski construction of graphs. It was shown in [20] that the Mycielskian of the Kneser graph KG(m, n) has the same circular chromatic number and chromatic number provided that m + t is an even integer. We prove that if m is large enough, then χ(Mt(KG(m, n))) = χc(Mt(KG(m, n))) where Mt is the tth Mycielskian. Also, we consider the generalized Kneser graph KG(m, n, s) and show that there exists a threshold m(n, s, t) such that χ(Mt(KG(m, n, s))) = χc(Mt(KG(m, n, s))) for m ≥ m(n, s, t).

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1 Introduction

In this section we elaborate on some basic definitions and facts that will be used throughout the paper. Throughout this paper we only consider finite graphs. A homomorphism f : G −→ H from a graph G to a graph H is a map f : V(G) −→ V(H) such that uv ∈ E(G) implies f(u)f(v) ∈ E(H). Also, the symbol Hom(G, H) is used to denote the set of all homomorphisms from G to H.

For a given graph G, the notation g(G) stands for the girth of graph G. We denote the neighborhood of a vertex v ∈ V(G) by N(v) and N[v] stands for the closed neighborhood of v, i.e., N[v] = N(v) ∪ {v}. We denote by [m] the set {1, 2, . . . , m}, and denote by (m \[=\] n) the collection of all n-subsets of [m]. The Kneser graph KG(m, n) is the graph with vertex set (m \[=\] n), in which A is connected to B if and only if A ∩ B = ø. It was conjectured by Kneser [14] in 1955, and proved by Lovász [16] in 1978, that χ(KG(m, n)) = m − 2n + 2. A subset S of [m] is called 2-stable if 2 ≤ |x − y| ≤ m − 2 for all distinct elements x and y of S. The Schrijver graph SG(m, n) is the subgraph of KG(m, n) induced by all 2-stable n-subsets of [m]. It was proved by Schrijver [19] that χ(SG(m, n)) = χ(KG(m, n)) and every proper subgraph of SG(m, n) has a chromatic number smaller than that of SG(m, n). The fractional chromatic number, χf(G), of a graph G is defined as

χf(G) def = inf{\frac{m}{n} | Hom(G, KG(m, n)) ≠ ø}.

The so-called generalized Kneser graphs are generalized from the Kneser graphs in a natural way. Let m ≥ 2n be positive integers. The generalized Kneser graphs

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\( KG(m, n, s) \) is the graph whose vertex-set is the set of \( n \)-subsets of \([m] = 1, 2, \ldots , m \) where two \( n \)-subsets \( A \) and \( B \) are joined by an edge if \( |A \cap B| \leq s \). Some properties of generalized Kneser graphs has been studied in several papers, see [3, 5, 8].

The Erdős-Ko-Rado theorem [6] states that if \( F \) is an independent set in the Kneser graph \( KG(m, n) \), then \( |F| \leq \binom{m-1}{n-1} \). If \( m > 2n \) and equality holds, then \( F \) is trivial, i.e., \( |\bigcap_{F \in F} F| = 1 \). Here is a generalization of the Erdős-Ko-Rado theorem.

**Theorem A.** [22] Let \( n > s \) be non-negative integers. If \( m \geq (s + 2)(n - s) \) and \( F \) is an independent set of the generalized Kneser graph \( KG(m, n, s) \), then \( |F| \leq \binom{m-s-1}{n-s-1} \).

Ahlswede and Khachatrian [1] determined the maximum size of independent sets for all \( m \), by proving their complete intersection theorem.

Let \( G \) be a graph. A \( k \)-coloring of \( G \) is the partition of the vertex set into independent sets \( V_1, V_2, \ldots, V_k \). Also, \( V_1, V_2, \ldots, V_k \) is termed a \( k \)-coloring of \( G \). If \( n \) and \( d \) are positive integers with \( n \geq 2d \), then the circular complete graph \( K_n^d \) is the graph with vertex set \( \{0, 1, \ldots, n-1\} \) in which \( i \) is connected to \( j \) if and only if \( d \leq |i - j| \leq n - d \). A graph \( G \) is said to be \((n, d)\)-colorable if \( G \) admits a homomorphism to \( K_n^d \). The circular chromatic number (also known as the star chromatic number [21]) \( \chi_c(G) \) of a graph \( G \) is the minimum of those ratios \( \frac{n}{d} \) for which \( \gcd(n, d) = 1 \) and such that \( G \) admits a homomorphism to \( K_n^d \). It can be shown that one may only consider onto-vertex homomorphisms [23]. It is known [21, 23] that for any graph \( G \), \( \chi(G) - 1 < \chi_c(G) \leq \chi(G) \), and hence \( \chi(G) = \lceil \chi_c(G) \rceil \).

The circular chromatic number of Kneser graphs has been studied by Johnson, Holroyd, and Stahl [13]. They proved that \( \chi_c(KG(m, n)) = \chi(KG(m, n)) \) if \( m \leq 2n + 2 \) or \( n = 2 \), and conjectured that equality holds for all Kneser graphs.

**Conjecture 1.** [13] For all \( m \geq 2n + 1 \), \( \chi_c(KG(m, n)) = \chi(KG(m, n)) \).

This conjecture has been studied in several papers [4, 10, 17, 20]. It was proved in [10], if \( m \) is sufficiently large, then the conjecture holds. Later, it was shown in [17, 20], if \( m \) is an even positive integer, then the conjecture is true.

**Theorem B.** [10] If \( m \geq 2n^2(n - 1) \), then \( \chi_c(KG(m, n)) = \chi(KG(m, n)) \).

The following notations will be needed throughout the paper. Let \( G \) be a graph with vertex set \( \{v_1, v_2, \ldots, v_n\} \). Recall that the Mycielskian \( M(G) \) of \( G \) is the graph defined on \( \{v_1, v_2, \ldots, v_n\} \cup \{v'_1, v'_2, \ldots, v'_n\} \cup \{z\} \) with edge set \( E(M(G)) = E(G) \cup \{v'_i v_j : v_i, v_j \in E(G)\} \cup \{zv'_i : i = 1, 2, \ldots, n\} \). The vertex \( v'_i \) is called the twin of the vertex \( v_i \) (\( v_i \) is also called the twin of the vertex \( v'_i \)); and the vertex \( z \) is called the root of \( M(G) \). If there is no ambiguity we shall always use \( z \) as the root of \( M(G) \). For \( t \geq 2 \), let \( M^t(G) \) be a graph whose vertex-set is the set of \( t \)-roots, the vertices of the twins, the twins of the twins, and etc., and the set of the roots of \( M^t(G) \) is equal to the vertex set of \( M^{t-1}(z) \) where \( z \) is the root of \( M(G) \). Mycielski [13] used this construction to increase the chromatic
number of a graph while keeping the clique number fixed: \( \chi(M(G)) = \chi(G) + 1 \) and \( \omega(M(G)) = \omega(G) \).

The problem of calculating the circular chromatic number of the Mycielskian of graphs has been investigated in the literature \([2, 7, 9, 12, 15, 20]\). It turns out that the circular chromatic number of \( M(G) \) is not determined by the circular chromatic number of \( G \) alone. Rather, it depends on the structure of \( G \). Even for graph \( G \) with very simple structure, it is still difficult to determine \( \chi_c(M(G)) \). The problem of determining the circular chromatic number of the iterated Mycielskian of complete graphs was discussed in \([2]\). It was conjectured in \([2]\) that if \( n \geq t + 2 \geq 3 \), then \( \chi_c(M^t(K_n)) = n + t \).

2 Free chromatic number

In this section, we introduce the concept of free coloring of graphs. Section 2 establishes the relationship between circular chromatic number and free chromatic number. Section 3 contains a generalization of the concept of free coloring. In section 4, we introduce some sufficient conditions for equality of circular chromatic number and chromatic number of graphs in terms of \((a, b)\)-free chromatic number. In the last section, we consider the generalized Kneser graph \( KG(m, n, s) \) and show that there exists a threshold \( m(n, s, t) \) such that \( \chi(M^t(KG(m, n, s))) = \chi_c(M^t(KG(m, n, s))) \) for \( m \geq m(n, s, t) \). Also, We show that if \( m \geq 2n^2(n - 1) + \min\{2^{t+1} - 2, 2^t + 3\}n - \min\{0, 2n - t - 3\} \), then \( \chi(M^t(KG(m, n))) = \chi_c(M^t(KG(m, n))) \) where \( M^t \) is \( t^{th} \) Mycielskian.
**Definition 3.** Free chromatic number of a graph $G$, denoted by $\phi(G)$, is the minimum positive integer $t$ such that there exists a partitioning of the vertices of $G$ such as $V(G) = V_1 \cup V_2 \cup \cdots \cup V_t$ where $V_i$ is a free independent set for any $1 \leq i \leq t$. If $G$ is not free, then we define the free chromatic number of $G$ to be $\infty$. ♠

The following simple lemma provides a necessary condition for the existence of graph homomorphism based on the free chromatic number of graphs.

**Lemma 1.** Let $G$ and $H$ be connected free graphs. If there exists an onto edge homomorphism from $G$ to $H$, then $\phi(G) \leq \phi(H)$.

An easy computation shows that if $(n, d) = 1$ and $d \geq 2$, then the free chromatic number of circular complete graph $K_\frac{n}{d}$ is less than twice of its chromatic number.

**Lemma 2.** Let $G$ be a graph and $\chi_c(G) = \frac{n}{d}$ such that $(n, d) = 1$. If $d \geq 2$ or equivalently $\chi_c(G) \neq \chi(G)$, then $\phi(G) \leq \lceil \chi_c(G)(1 + \frac{1}{d-1}) \rceil \leq 2\chi(G) - 1$.

**Proof.** Since the circular chromatic number of $G$ is $\frac{n}{d}$, Hence there exists a homomorphism from graph $G$ to the circular complete graph $K_{\frac{n}{d}}$. It was proved that for any $i$, the edge between the vertices $i$ and $i + d$ (in $K_{\frac{n}{d}}$) should be in the range of homomorphism. Consequently, the inverse image of any $d - 1$ consecutive vertices of $K_{\frac{n}{d}}$ is a free independent set of $G$. Hence, the free chromatic number of graph $G$ is less than or equal to $\lceil \frac{n}{d-1} \rceil = \lceil \frac{n}{d} \cdot \frac{1}{d-1} \rceil = \lceil \chi_c(G)(1 + \frac{1}{d-1}) \rceil$. Also, $\lceil \chi_c(G)(1 + \frac{1}{d-1}) \rceil \leq 2\chi(G) - 1$ which completes the proof. ■

The aforementioned lemma provides a sufficient condition for the equality of chromatic number and circular chromatic number of graphs. Hence, it can be of interest to have bounds for the free chromatic number of graphs.

Suppose $G$ is a free graph with $n$ vertices. Let $\bar{\alpha}(G)$ be the size of the largest free independent set in $G$. It is obviously that $\phi(G) \geq \frac{n}{\bar{\alpha}(G)}$. Also, for any edge $e = uv$ in $G$, let $d(e)$ be the number of vertices that lie in the neighborhood of $u$ or $v$. Define $d(G) = \min\{d(e) : e \in E(G)\}$. It is obviously that $\bar{\alpha} \leq n - d(G)$. So we have the next lemma.

**Lemma 3.** Let $G$ be a graph with $n$ vertices, then $\phi(G) \geq \frac{n}{\bar{\alpha}(G)} \geq \frac{n}{n - d(G)}$.

Next theorem gives an upper bound for the free chromatic number of a graph $G$ in terms of chromatic number and $d(G)$. Also, it is shown when the girth of a graph $G$ is greater than 4, then the difference between its free chromatic number and its chromatic number is at most 4.

**Theorem 1.** Let $G$ be a free graph. Then $\phi(G) \leq \chi(G) + d(G)$. Also, if $g(G) \geq 5$, then $\phi(G) \leq \chi(G) + 4$.

**Proof.** Assume that $e = uv$ is an edge in $G$ such that $d(e) = d(G)$. Color the vertices of the induced graph $G \setminus (N(u) \cup N(v))$ by using at most $\chi(G)$ colors. Assign to each vertex of $N(u) \cup N(v)$ a new color. One can easily check that this coloring is a free coloring for $G$. Consequently, $\phi(G) \leq \chi(G) + d(G)$. 

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Assume that the girth of $G$ is greater than 4. First, suppose $G$ is a tree. In view of the assumption, since $G$ is a free graph, the diameter of $G$ should be greater than 4. Let $u_1, u_2, \ldots, u_d$ be the longest path in $G$. Let $V_1, V_2$ be a 2-coloring for $G \setminus N[u_{d-1}]$. Set $V_3 \overset{\text{def}}{=} \{u_{d-1}\}$ and $V_4 \overset{\text{def}}{=} N(u_{d-1})$. One can check that $V_1, V_2, V_3, V_4$ is a free coloring for $G$. Now, suppose that $G$ is not a tree. Let $C$ be an induced cycle of $G$ such that it has $g \overset{\text{def}}{=} g(G)$ vertices. Assume that $V(C) = \{u_1, u_2, \ldots, u_g\}$ and $E(C) = \{u_iu_{i+1} | 1 \leq i \leq g \pmod{g}\}$. Set $V_1 \overset{\text{def}}{=} \{u_1\}$, $V_2 \overset{\text{def}}{=} \{u_2\}$, $V_3 \overset{\text{def}}{=} N(u_1)$, $V_4 \overset{\text{def}}{=} N(u_2)$. Also, suppose $V_5, V_6, \ldots, V_{\chi(G)+4}$ is a $\chi(G)$-coloring of $G \setminus (V_1 \cup V_2 \cup V_3 \cup V_4)$. It is easy seen that $V_1, V_2, \ldots, V_{\chi(G)+4}$ is a free coloring for $G$.

Similarly, if $G$ is a free graph and $g(G) \geq 7$, then set $V_1 \overset{\text{def}}{=} N(u_1)$ and $V_2 \overset{\text{def}}{=} N(u_2)$. Moreover, assume that $V_3, V_4, \ldots, V_{\chi(G)+2}$ is a $\chi(G)$-coloring of $G \setminus (V_1 \cup V_2)$. One can check that $V_1, V_2, \ldots, V_{\chi(G)+2}$ is a free coloring for $G$. Thus, $\phi(G) \leq \chi(G) + 2$.

Since $d(G) \leq \Delta + \delta$ we have the next corollary.

**Corollary 1.** Let $G$ be a free graph. Then $\phi(G) \leq \chi(G) + \Delta + \delta$.

This question naturally arises in mind that how much free chromatic number can be larger than chromatic number. Consider $K_m \times K_m$ which is the Categorical product of the complete graphs $K_m$ and $K_m$. It is easily seen that the chromatic number of this graph is $\min\{m, n\}$. On the other hand, every independent set of size at least 2 can be extended to a unique maximal independent set. Hence, the free chromatic number of this graph is $mn$. In special case, when $n = 2$ and $m \geq 2$, $\phi(K_2 \times K_m) = 2m = \chi(K_2 \times K_m) + d(K_2 \times K_m)$. Therefore, the bound in the previous corollary is sharp.

In view of proof of Theorem 1 if $G$ is a graph which contains two adjacent vertices $u$ and $v$ such that the induced subgraph on $N(u) \cup N(v)$ is a free graph with girth greater than 4, then one can similarly show that $\phi(G) \leq \chi(G) + 4$. Hence, if $G$ contains an induced sparse subgraph, then Lemma 2 is not fruitful. This leads us to generalize the definition of free coloring.

### 3 Generalization of Free Coloring

In this section, we generalize the concept of free coloring in order to show that the circular chromatic number of the $t^\text{th}$ Mycielskian of the generalized Kneser graph $KG(m, n, s)$ is equal to its chromatic number whenever $m$ is sufficiently large.

**Definition 4.** For a free independent set $F$, we say $F$ is supported by the edge $e = uv$ or $e$ supports $F$ if $F \cap (N(u) \cup N(v)) = \emptyset$. The set of all edges which support $F$ is noticed by $\text{supp}(F)$.

Here is a generalization of free chromatic number.

**Definition 5.** Let $a \geq 0$ and $b \geq 1$ be integers. The $(a, b)$-**free chromatic number** of a graph $G$, denoted by $\phi^a_b(G)$, is the minimum natural number $t$ (if there is not such $t$, we define $\phi^a_b(G) = \infty$) such that
1. There exists a partition of the vertices of $G$ into independent sets $V_1, V_2, \ldots, V_t$ where all but at most $a$ of $V_i$’s are free independent sets. For convince, let $V_i$ be a free independent sets for $i = 1, 2, \ldots t - a$.

2. There exist the edges $e_1, e_2, \ldots, e_{t-a}$ such that for any $1 \leq i \leq t - a$, $e_i \in supp(V_i)$ and also for every vertex $v \in V(G)$ the number of edges among $e_1, e_2, \ldots, e_{t-a}$ which are incident with $v$ is less than or equal to $b$. ♠

If the partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_t$ and edges $e_1, e_2, \ldots, e_{t-a}$ satisfy the conditions 1 and 2 in the previous definition where $e_i \in supp(V_i)$, then $V_1, V_2, \ldots, V_t$ together $e_1, e_2, \ldots, e_{t-a}$ be noticed as an $(a, b)$-free coloring for $G$. It should be noted there is no obligation that the edges $e_1, e_2, \ldots, e_{t-a}$ are distinct.

Here are some elementary properties of $(a, b)$-free chromatic number of graphs. The bellow lemma can be concluded immediately.

**Lemma 4.** Let $G$ be a graph and $\alpha(G)$ be the independence number of $G$.

a) For any integer $b \geq 1$, $\phi^a_b(G) \geq \phi(G)$. Moreover, $\lim_{b \to \infty} \phi^0_b(G) = \phi(G)$.

b) For any integers $a' \geq a \geq 0$ and $b' \geq b \geq 1$, $\phi^{a'}_{b'}(G) \geq \phi^a_b(G)$.

c) For any integers $a \geq 0$ and $b \geq 1$, $\phi^0_b(G) \geq \frac{|V(G)|-\alpha_0(G)}{a(G)}$.

One can deduce the following lemma whose proof is almost identical to that of Lemma 2 and the proof is omitted for the sake of brevity.

**Lemma 5.** If there are some integers $a \geq 0$ and $b \geq 2$ such that $\phi^a_b(G) \geq 2\chi(G)$, then $\chi_c(G) = \chi(G)$.

### 4 Mycielski construction and circular chromatic number

In this section we find some relationships between the circular chromatic number and $(a, b)$-free chromatic number of graphs. The following lemma was observed in [7].

**Lemma A.** [7] Let $M(G)$ be the Mycielski construction of $G$. Assume that $z$ is the root. Also, for any $v \in V(G)$, let $v'$ be the twin of $v$. Suppose $\chi_c(M(G)) = \frac{n}{d}$, $\gcd(n, d) = 1$, and $d \geq 2$. Then there is a homomorphism $c \in \text{Hom}(M(G), K_2)$ such that $c(z) = 0$, and $c(v) = c(v')$ if $c(v) \notin [n - d + 1, d - 1] \pmod n$.

**Lemma 6.** Let $G$ be a graph. Assume that $a \geq 0$ and $b \geq 1$ are integers. Then $\phi^a_b(M(G)) \geq \phi^{a+1}_{2b}(G)$. Moreover, for any positive integer $t$, $\phi^a_b(M^t(G)) \geq \phi^{2t+1}_{2t+1}(G)$.

**Proof.** Assume that $V(M(G)) = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n, z\}$ where $V(G) = \{v_1, v_2, \ldots, v_n\}$, $v'_i$ is the twin of $v_i$ and $z$ is the root of $M(G)$. If $\phi^a_b(M(G)) = \infty$, then there is nothing to prove. Suppose that $\phi^a_b(M(G)) = t$. Let $V_1, V_2, \ldots, V_t$
together }_{i=1}^{t-a} e_i \text{ be an } (a,b) \text{-free coloring for } M(G) \text{ where } e_i \in \text{supp}(V_i). \text{ Set } U_i \overset{\text{def}}{=} V_i \cap V(G).

Note that at most } b \text{ edges of } e_i \text{'s are incident with } z. \text{ Without loss of generality, assume that } e_i, \text{ for } i = 1, 2, \ldots, t-a-b, \text{ is not incident with } z. \text{ If there is an } 1 \leq i \leq t-a-b \text{ such that } e_i = v_sv_k', \text{ then we define } e_i' = v_kv_s', \text{ otherwise, set } e_i' = e_i. \text{ It is easy to see that } e_i' \text{ supports } U_i \text{ for any } 1 \leq i \leq t-a-b. \text{ Also, it is straightforward to check that for every vertex } v \in V(G) \text{ the number of edges among } e_1', e_2', \ldots, e_{t-a-b}' \text{ which are incident with } v \text{ is less than or equal to } 2b. \text{ Therefore, } U_1, U_2, \ldots, U_t \text{ together } e_1', e_2', \ldots, e_{t-a-b}' \text{ is an } (a+b, 2b) \text{-free coloring for } G. \text{ The other assertion follows by induction on } t. 

In view of Lemma 5 and the preceding lemma, the following corollary yields.

**Corollary 2.** Let } G \text{ be a graph. } \text{ If } \chi_c(M^t(G)) \neq \chi(M^t(G)), \text{ then } \phi_{2^{t+1} - 2}(G) \leq 2\chi(M^t(G)) - 1 = 2\chi(G) + 2t - 1.

The aforementioned corollary and the next Lemma can be considered as generalizations of Lemma 2.

**Lemma 7.** Let } G \text{ be a graph and } t \text{ be a positive integer. } \text{ If } \chi_c(M^t(G)) \neq \chi(M^t(G)), \text{ then } \phi_{2^{t+1}}(G) \leq 2\chi(M^t(G)) - 1.

**Proof.** Suppose } \chi_c(M^t(G)) = \frac{n}{d}, \gcd(n,d) = 1, \text{ and } d \geq 2. \text{ In view of Corollary 2 if } t = 1, \text{ the assertion holds. Hence, assume that } t \geq 2. \text{ Consider } M^t(G) \text{ as the Mycielskian of } H \overset{\text{def}}{=} M^{t-1}(G) \text{ and let } c \in \text{Hom}(M(H), K_\frac{n}{d}) \text{ which satisfies Lemma A. Assume that } n = k(d-1) + s \text{ where } 0 \leq s \leq d-2. \text{ Set } V_i = c^{-1}(\{(i-1)(d-1) + 1, i(d-1) + 2, \ldots, i(d-1)\}) \text{ for } i = 1, 2, \ldots, k \text{ and } V_{k+1} = c^{-1}(\{k(d-1) + 1, k(d-1)+2, \ldots, n\}). \text{ It is well-known that for any } i, \text{ the edge between vertices } i \text{ and } i+d \text{ (in } K_{\frac{n}{d}}) \text{ should be in the range of } c. \text{ Hence, for any } i \in \{1, 2, \ldots, k+1\}, \text{ the set } E_i = c^{-1}(\{(i-1)(d-1), i(d-1)+1\}) \text{ is not empty. For any } i \in \{1, 2, \ldots, k+1\}, \text{ choose an } e_i \in E_i. \text{ Note that for each } 1 \leq j \leq n, \text{ the number of edges among } e_1, e_2, \ldots, e_{k+1} \text{ which are incident with some vertices in } e_j^{-1}(j) \text{ is at most } 2 \text{ (for } d > 2 \text{ this number is at most } 1). \text{ One can check that } V_1, V_2, \ldots, V_{k+1} \text{ together } e_1, e_2, \ldots, e_{k+1} \text{ is a } (0,b) \text{-free coloring (} b \leq 2 \text{) for } M^t(G). \text{ Obviously, } 2\chi(M^t(G)) - 1 \geq k + 1 = \left\lceil \frac{n}{d} \right\rceil. \text{ We consider two cases:}

Case I) } d > 2.

In this case, } V_1, V_2, \ldots, V_{k+1} \text{ together } e_1, e_2, \ldots, e_{k+1} \text{ is a } (0,1) \text{-free coloring and by Lemma 6 we have }

\[ k + 1 \geq \phi_1^0(M^t(G)) \geq \phi_2^1(M^{t-1}(G)) \geq \cdots \geq \phi_{2^t-1}^t(G). \]

On the other hand, we know that \( \phi_2^{t-1}(G) \geq \phi_{2^{t+1}}^t(G). \)

Case II) } d = 2

Assume that } R = \{y_1, y_2, \ldots, y_{2^{t-1}-1}\} \cup \{y_1', y_2', \ldots, y_{2^{t-1}-1}'\} \cup \{z\} \text{ be the roots of } M^t(G) \text{ where } T = \{y_1, y_2, \ldots, y_{2^{t-1}-1}\} \text{ are the roots of } H = M^{t-1}(G) \text{ and } T' = \{y_1', y_2', \ldots, y_{2^{t-1}-1}'\} \text{ are the twins of the vertices of } T (y_i' \text{ is the twin of } y_i).
Set $U_i^{t-1} \overset{\text{def}}{=} V_i \cap V(M_i^{t-1}(G))$, for $1 \leq i \leq k+1$. For any vertex $v \in V(M_i^{t}(G))$, let $n(v)$ denote the number of edges among $e_1, e_2, \ldots, e_{k+1}$ which are incident with $v$. The number of edges among $e_1, e_2, \ldots, e_{k+1}$ which are incident with $z$ is $n(z)$.

Without loss of generality, assume that $e_i$, for $i = 1, 2, \ldots, k+1-n(z)$, is not incident with $z$. If there is an $1 \leq i \leq k + 1 - n(z)$ such that $e_i = v_s v_k'$ where $v_s$ and $v_k$ are the vertices of $H$ ($v_k'$ is the twin of $v_k$), then we define $e_i^{t-1} \overset{\text{def}}{=} v_s v_k$, otherwise, set $e_i^{t-1} \overset{\text{def}}{=} e_i$. It is readily seen that $e_i^{t-1}$ supports $U_i^{t-1}$ for any $1 \leq i \leq k + 1 - n(z)$.

Inductively, after iterating the aforementioned procedure $t - 1$ times more, we obtain $U_1^0, U_2^0, \ldots, U_{k+1}^0$ together $e_1^0, e_2^0, \ldots, e_{k+1}^0$ which is a $(p, 2^{t+1})$-free coloring for $G$.

It is a simple matter to check that $p$ is the number of edges among $e_1, e_2, \ldots, e_{k+1}$ which are incident with at least one vertex of $\mathcal{R}$.

Now we show that $p \leq 2t + 3$. To see this, let $S \subseteq T$ be the set of vertices whose color (in the coloring $c$) is not in $[n-1, 1]$ (mod $n$). Define $S' = \{ y_j' \mid y_j \in S \} \subseteq T'$. Note that $c$ satisfies Lemma A, hence, for any $y_j \in S$, $y_j$ and $y_j'$ have the same color. Therefore, for every vertex $y_j \in S$, $n(y_j) + n(y_j') \leq 2$. Also, it is easy to check that the number of edges among $e_1, e_2, \ldots, e_{k+1}$ which are incident with vertices in $(R \setminus S) \cup \{ z \}$ is at most $5$. Furthermore, for any vertex $v \in V(M_i^{t}(G))$, $n(v) \leq 2$; consequently, $p \leq \sum_{y \in T} n(y_i) + \sum_{y_j' \in T'} n(y_i') + n(z) \leq 2|S| + 2|T \setminus S| + 5 \leq 2t + 3$. \hfill \Box

### 4.1 Mycielski Construction of Generalized Kneser Graphs

Here we investigate the circular chromatic number of the Mycielski construction of generalized Kneser graphs. Although, the exact value of $\chi(KG(m, n, s))$ is unknown in general, we show that $\chi_c(M_i^{t}(KG(m, n, s))) = \chi(M_i^{t}(KG(m, n, s)))$ whenever $m$ is large enough.

**Theorem 2.** For any fixed integers $n > s \geq 0$ and $t \geq 0$, if $m$ is large enough, then $\chi_c(M_i^{t}(KG(m, n, s))) = \chi(M_i^{t}(KG(m, n, s)))$.

**Proof.** First, we show that $\chi(M_i^{t}(KG(m, n, s))) \leq \begin{pmatrix} m \\ s+1 \end{pmatrix} + t = O(m^{s+1})$. To see this, for every vertex $A \in V(KG(m, n, s))$, choose an arbitrary subset $B \subseteq A$ of size $s+1$ and define $c(A) \overset{\text{def}}{=} B$. Obviously, $c$ is a proper coloring for $KG(m, n, s)$. Therefore, $\chi(KG(m, n, s)) \leq \begin{pmatrix} m \\ s+1 \end{pmatrix}$ which implies that $\chi(M_i^{t}(KG(m, n, s))) \leq \begin{pmatrix} m \\ s+1 \end{pmatrix} + t$.

Now we show that $\alpha(KG(m, n, s)) \leq \begin{pmatrix} 2n \\ s+2 \end{pmatrix} \begin{pmatrix} m-s-2 \\ n-s-2 \end{pmatrix} = O(m^{n-s-2})$. Let $\mathcal{F}$ be a free independent set in $KG(m, n, s)$. Since $\mathcal{F}$ is a free independent set, there is an edge $AB \in E(KG(m, n, s))$ such that $\mathcal{F} \cup \{ A \}$ and $\mathcal{F} \cup \{ B \}$ are still independent. Thus, for any $F \in \mathcal{F}$, $|F \cap A| \geq s + 1$ and $|F \cap B| \geq s + 1$; consequently, $|F \cap (A \cup B)| \geq s + 2$. Now by counting, the desired inequality holds. If $\chi_c(M_i^{t}(KG(m, n, s))) \neq \chi(M_i^{t}(KG(m, n, s)))$, then by Corollary 2 we have

$$2(\chi(M_i^{t}(KG(m, n, s))) - 1 \geq \phi_2^{t+1-2}(KG(m, n, s)).$$

In view of Theorem A and Lemma 4(c), when $m \geq (s + 2)(n - s)$ we have

$$\phi_2^{t+1-2}(KG(m, n, s))) \geq \begin{pmatrix} m \\ n-s \end{pmatrix} - \begin{pmatrix} 2t+1-2 \\ n-s-1 \end{pmatrix} \begin{pmatrix} m-s-1 \\ n-s-1 \end{pmatrix} = O(m^{s+2}).$$

8
Since \( \chi(M^t(KG(m,n,s))) \leq \binom{m}{s+1} + t = O(m^{s+1}) \), there exists a threshold \( m(n,s,t) \) such that for \( m \geq m(n,s,t) \), (1) does not hold, as desired. ■

The chromatic number of the generalized Kneser graph \( KG(m,n,1) \) has been specified in [8]. The chromatic number of \( KG(m,n,s) \) remains open for \( s \geq 2 \). Motivated by the aforementioned theorem, we propose the following question.

**Question 1.** Let \( m, n, \) and \( s \) be non-negative integers where \( m > n > s \). Is it true that \( \chi_c(KG(m,n,s)) = \chi(KG(m,n,s)) \)?

It was proved by Hilton and Milner [11] that if \( X \) is an independent set of \( KG(m,n) \) of size

\[
\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 2,
\]

then

\[
\bigcap_{A \in X} A = \{i\},
\]

for some \( i \in [m] \). Therefore, any independent set of size greater than \( \binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 1 \) can be extended to a unique maximum independent set. This leads us to the following lemma.

**Lemma 8.** Let \( m > 2n \) be positive integers. The size of any free independent set in the Kneser graph \( KG(m,n) \) is less than or equal to \( \binom{m-1}{n-1} - \binom{m-n-1}{n-1} \). Also, for any integers \( a \geq 0 \) and \( b \geq 1 \), the \((a, b)\)-free chromatic number of the Kneser graph \( KG(m,n) \) is at least

\[
\phi^b_a(KG(m,n)) \geq \frac{\binom{m}{n} - a\binom{m-1}{n-1}}{\binom{m-1}{n-1} - \binom{m-n-1}{n-1}}.
\]

**Proof.** Let \( \mathcal{F} \) be a free independent set of \( KG(m,n) \). In view of the Hilton and Milner theorem, we should have \( |\mathcal{F}| \leq \binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 1 \). Since \( \mathcal{F} \) is a free independent set, it can be extended to two distinct maximal independent sets. If \( |\mathcal{F}| = \binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 1 \), by considering the Hilton and Milner theorem, there exist two positive integer \( i \) and \( j \) such that \( \bigcap_{A \in X} A = \{i, j\} \). Hence, \( |\mathcal{F}| \leq \binom{m-2}{n-2} \).

On the other hand, it is easy to check that \( \binom{m-2}{n-2} \leq \binom{m-1}{n-1} - \binom{m-n-1}{n-1} \) which is a contradiction. ■

The next theorem was shown in [10].

**Theorem C.** [10] For any positive integer \( n \), if \( m \) is sufficiently large, then we have \( \chi_c(SG(m,n)) = \chi(SG(m,n)) \).

Here is a generalization of the previous theorem.

**Theorem 3.** For any integers \( n \geq 1 \) and \( t \geq 0 \), there is a threshold \( m(n,t) \) such that

\[
\chi_c(M^t(SG(m,n))) = \chi(M^t(SG(m,n))) = m - 2n + 2 + t
\]

whenever \( m \geq m(n,t) \).
Proof. Let \( \chi_c(M^t(SG(m,n))) \neq m - 2n + 2 + t \). Set \( k \overset{\text{def}}{=} \min\{2^{t+1} - 2, 2^t + 3\} \). By Corollary 2 and Lemma 7 we have
\[
\phi_{2t+1}^k(SG(m,n)) \leq \phi_2^0(M^t(SG(m,n))) \leq 2(m - 2n + 2 + t) - 1. \tag{2}
\]
On the other hand, the vertex set of \( V(SG(m,n)) \) has cardinality \( \binom{m-n-1}{n-1} \frac{m}{n} = O(m^n) \) (each 2-stable \( n \)-subsets of \( \{m\} \) containing 1 corresponds to an integral solution of the equation \( x_1 + x_2 + \cdots + x_n = m \) with \( x_i \geq 2 \). So there are \( \binom{m-n-1}{n-1} \), 2-stable \( n \)-subsets of \( \{m\} \) containing 1). In view of Lemma 8 we have
\[
\phi_{2t+1}^k(SG(m,n)) \geq \frac{\binom{m-n-1}{n-1} \frac{m}{n} - k\binom{m-1}{n-1}}{(\frac{m}{n}) - \binom{m-n-1}{n-1}} \overset{\text{def}}{=} O(m^2).
\]
Therefore, there is a threshold \( m(n,t) \) such that if \( m \geq m(n,t) \), then
\[
\phi_2^0(M^t(SG(m,n))) \geq 2\chi(M^t(SG(m,n))) = 2(m - 2n + 2 + t).
\]
This contradicts (2). \( \blacksquare \)

Here is a generalization of Theorem 3.

Theorem 4. For any integers \( n \geq 1 \) and \( t \geq 0 \), if \( m \geq 2n^2(n-1) + \min\{2^{t+1} - 2, 2^t + 3\}n - \min\{0,2n-t-3\} \), then \( \chi_c(M^t(KG(m,n))) = \chi(M^t(KG(m,n))) \).

Proof. The case \( n = 1 \) was proved in [10], hence, assume that \( n \geq 2 \). For convince, set \( k \overset{\text{def}}{=} \min\{2^{t+1} - 2, 2^t + 2\} \). In view of the proof of Theorem 3 it is sufficient to show that the following inequality holds for \( m \geq 2n^2(n-1) + kn - \min\{0,2n-t-3\} \)
\[
\binom{m-1}{n-1} - \binom{m-n-1}{n-1} \leq \frac{\binom{m}{n} - k\binom{m-1}{n-1}}{2(m - 2n + 2 + t)}.
\]
By double counting we have
\[
\binom{m-1}{n-1} - \binom{m-n-1}{n-1} \leq n\binom{m-2}{n-2}.
\]
It is straightforward to check that
\[
n\binom{m-2}{n-2} \leq \frac{\binom{m}{n} - k\binom{m-1}{n-1}}{2(m - 2n + 2 + t)}
\]
for \( m \geq 2n^2(n-1) + kn - \min\{0,2n-t-3\} \). This completes the proof. \( \blacksquare \)

It was shown in [2], if \( G \) is a graph with chromatic number \( t \), then \( \chi_c(M^{t-1}(G)) \leq \chi(M^{t-1}(G)) - \frac{1}{2} \). Also, it was conjectured in [2] that if \( n \geq t + 2 \geq 3 \), then \( \chi_c(M^t(K_n)) = \chi(M^t(K_n)) = n + t \).

Question 2. Let \( m,n \), and \( t \) be non-negative integers where \( m > 2n \) and \( 0 \leq t \leq m - 2n \). Is it true that \( \chi_c(M^t(KG(m,n))) = \chi(M^t(KG(m,n))) \)?
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