C*-ALGEBRAS ASSOCIATED WITH THE $ax + b$-SEMIGROUP OVER $\mathbb{N}$

JOACHIM CUNTZ

Abstract. We present a C*-algebra which is naturally associated to the $ax + b$-semigroup over $\mathbb{N}$. It is simple and purely infinite and can be obtained from the algebra considered by Bost and Connes by adding one unitary generator which corresponds to addition. Its stabilization can be described as a crossed product of the algebra of continuous functions, vanishing at infinity, on the space of finite adeles for $\mathbb{Q}$ by the natural action of the $ax + b$-group over $\mathbb{Q}$.

Contents

1. Introduction 2
2. A canonical representation of the $ax + b$-semigroup over $\mathbb{N}$ 2
3. A purely infinite simple C*-algebra associated with the $ax + b$-semigroup 3
4. The canonical action of $\mathbb{R}$ on $\mathbb{Q}_N$ 6
5. The $K$-groups of $\mathbb{Q}_N$ 7
6. Representations as crossed products 8
7. The case of the multiplicative semigroup $\mathbb{Z}^\times$ 11
References 13

2000 Mathematics Subject Classification. Primary: 58B34, 46L05.
Key words and phrases. purely infinite, Bost-Connes algebra.
Research supported by the Deutsche Forschungsgemeinschaft.
1. Introduction

In this note we present a $C^*$-algebra (denoted by $Q_N$) which is associated to the $ax+b$-semigroup over $\mathbb{N}$. It is in fact a natural quotient of the $C^*$-algebra associated with this semigroup by some additional relations (which make it simple and purely infinite). These relations are satisfied in representations related to number theory. The study of this algebra is motivated by the construction of Bost-Connes in [1]. Our $C^*$-algebra contains the algebra considered by Bost-Connes, but in addition a generator corresponding to translation by the additive group $\mathbb{Z}$.

As a $C^*$-algebra, $Q_N$ has an interesting structure. It is a crossed product of the Bunce-Deddens algebra associated to $Q$ by the action of the multiplicative semigroup $\mathbb{N}^\times$. It has a unique canonical KMS-state. We also determine its $K$-theory, whose generators turn out to be determined by prime numbers.

On the other hand, $Q_N$ can also be obtained as a crossed product of the commutative algebra of continuous functions on the completion $\mathbb{Z}$ by the natural action of the $ax+b$-semigroup over $\mathbb{N}$. More interestingly, its stabilization is isomorphic to the crossed product of the algebra $C_0(A_f)$ of continuous functions on the space of finite adeles by the natural action of the $ax+b$-group $P^Q_+$ over $\mathbb{Q}$. Somewhat surprisingly, one obtains exactly the same $C^*$-algebra $Q_N$ (up to stabilization) working with the completion $\mathbb{R}$ of $\mathbb{Q}$ at the infinite place and taking the crossed product by the natural action of the $ax+b$-group $P^Q_+$ on $C_0(\mathbb{R})$.

In the last section we consider the analogous construction of a $C^*$-algebra replacing the multiplicative semigroup $\mathbb{N}^\times$ by $\mathbb{Z}^\times$, i.e. omitting the condition of positivity on the multiplicative part of the $ax+b$-(semi)group. We obtain a purely infinite $C^*$-algebra $Q_Z$ which can be written as a crossed product of $Q_N$ by $\mathbb{Z}/2$. The fixed point algebra for its canonical one-parameter group $(\lambda_t)$ is a dihedral group analogue of the Bunce-Deddens algebra. Its $K$-theory involves a shift of parity from $K_0$ to $K_1$ and vice versa. Its stabilization is isomorphic to the natural crossed product $C_0(A_f) \rtimes P_Q$.

2. A canonical representation of the $ax+b$-semigroup over $\mathbb{N}$

We denote by $\mathbb{N}$ the set of natural numbers including 0. $\mathbb{N}$ will normally be regarded as a semigroup with addition.

We denote by $\mathbb{N}^\times$ the set of natural numbers excluding 0. $\mathbb{N}^\times$ will normally be regarded as a semigroup with multiplication.

The natural analogue, for a semigroup $S$, of an unitary representation of a group is a representation of $S$ by isometries, i.e. by operators $s_g, g \in S$ on a Hilbert space that satisfy $s_g^*s_g = 1$.

On the Hilbert space $\ell^2(\mathbb{N})$ consider the isometries $s_n, n \in \mathbb{N}^\times$ and $v^k, k \in \mathbb{N}$ defined by

\[ v^k(\xi_m) = \xi_{m+k} \quad s_n(\xi_m) = \xi_{mn} \]

where $\xi_m, m \in \mathbb{N}$ denotes the standard orthonormal basis.
We have \( s_n s_m = s_{nm} \) and \( v^n v^m = v^{n+m} \), i.e. \( s \) and \( v \) define representations of \( \mathbb{N}^\times \) and \( \mathbb{N} \) respectively, by isometries. Moreover we have the following relation
\[
s_n v^k = v^{nk} s_n
\]
which expresses the compatibility between multiplication and addition.

In other words the \( s_n \) and \( v^k \) define a representation of the \( ax + b \)-semigroup
\[
P_\mathbb{N} = \left\{ \begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} \mid n \in \mathbb{N}^\times, k \in \mathbb{N} \right\}
\]
over \( \mathbb{N} \), where the matrix \( \begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} \) is represented by \( v^k s_n \).

Note that, for each \( n \), the operators \( s_n, vs_n, \ldots, v^{n-1} s_n \) generate a \( C^* \)-algebra isomorphic to \( \mathcal{O}_n \).

3. A purely infinite simple \( C^* \)-algebra associated with the
\( ax + b \)-semigroup

The \( C^* \)-algebra \( A \) generated by the elements \( s_n \) and \( v \) considered in section 2 contains the algebra \( \mathcal{K} \) of compact operators on \( l^2 \mathbb{N} \). Denote by \( u \), resp. \( v^k \) the image of \( v \), resp. \( v^k \) in the quotient \( A / \mathcal{K} \). We also still denote by \( s_n \) the image of \( s_n \) in the quotient. Then the \( u^k \) are unitary and are defined also for \( k \in \mathbb{Z} \) and they furthermore satisfy the characteristic relation \( \sum_{k=0}^{n-1} u^k e_n u^{-k} = 1 \) where \( e_n = s_n s_n^* \) denotes the range projection of \( s_n \). This relation expresses the fact that \( \mathbb{N} \) is the union of the sets of numbers which are congruent to \( k \mod n \) for \( k = 0, \ldots, n-1 \).

We now consider the universal \( C^* \)-algebra generated by elements satisfying these relations.

**Definition 3.1.** We define the \( C^* \)-algebra \( \mathcal{Q}_\mathbb{N} \) as the universal \( C^* \)-algebra generated by isometries \( s_n, n \in \mathbb{N}^\times \) with range projections \( e_n = s_n s_n^* \), and by a unitary \( u \) satisfying the relations
\[
s_n s_m = s_{nm}, \quad s_n u = u^n s_n, \quad \sum_{k=0}^{n-1} u^k e_n u^{-k} = 1
\]
for \( n, m \in \mathbb{N}^\times \).

**Lemma 3.2.** In \( \mathcal{Q}_\mathbb{N} \) we have
\[
\begin{align*}
\text{(a)} & \quad e_n = \sum_{i=0}^{m-1} u^{in} e_{nm} u^{-in} \text{ for all } n, m \in \mathbb{N}^\times. \\
\text{(b)} & \quad e_p s_q = e_{pq} s_p = s_q e_p \text{ and } e_p e_q = e_{pq} = e_q e_p \text{ when } p \text{ and } q \text{ are relatively prime.} \\
\text{(c)} & \quad s_n^* s_m = s_m s_n^* \text{ for all } n, m.
\end{align*}
\]

**Proof.** (a) This follows by conjugating the identity \( 1 = \sum u^i e_m u^{-i} \) by \( s_n \) and using the fact that \( s_n e_m s_n^* = e_{nm} \).

(b) Since the \( u^i e_{pq} u^{-i} \) are pairwise orthogonal for \( 0 \leq i < pq \), we see that \( u^p e_{pq} u^{-i} \perp u^{kq} e_{pq} u^{-kq} \) if \( 0 < l < q, 0 < k < p \). Thus, using (a),
\[
e_p s_q = \sum_{l=0}^{q-1} u^p e_{pq} u^{-lp} \sum_{k=0}^{p-1} u^{kq} e_{pq} u^{-kq} s_q = e_{pq} s_q
\]
This obviously implies $e_p e_q = e_{pq}$ and, by symmetry $e_q e_p = e_{pq}$. In particular, $e_p$ and $e_q$ commute.

(c) Using (b) we get

$$s_p (s_p s_q) s_q^* = e_p e_q = e_{pq} = s_p q s_{pq} = s_p (s_q s_p) s_q^*$$

Since $s_p, s_q$ are isometries this implies that the expressions in parentheses on the left and right hand side are equal. Here we have, in a first step, assumed that $p, q$ are prime. However any $s_n$ is a product of $s_p$’s with $p$ prime. \hfill \Box

From Lemma 3.2 (a) it follows that any two of the projections $u^i e_n u^{-i}$ commute. We denote by $D$ the commutative subalgebra of $Q_N$ generated by all these projections.

We also denote by $F$ the subalgebra of $Q_N$ generated by $u$ and the projections $e_n$, $n \in \mathbb{N}^\times$.

To analyze the structure of $Q_N$ further we write it as an inductive limit of the subalgebras $B_n$ generated by $s_{p_1}, s_{p_2}, \ldots, s_{p_n}$ and $u$, where $p_1, p_2, \ldots, p_n$ denote the $n$ first prime numbers. Each $B_n$ contains a natural (maximal) commutative subalgebra $D_n$ generated by all projections of the form $u^i e_{p_k} u^{-i}$ where $m$ is a product of powers of the $p_1, \ldots, p_n$ (i.e. a natural number that contains only the $p_i$ as prime factors).

Lemma 3.3. The spectrum $\text{Spec} D_n$ of $D_n$ can be identified canonically with the compact space

$$\{0, \ldots, p_1 - 1\}^N \times \ldots \times \{0, \ldots, p_n - 1\}^N \cong \hat{\mathbb{Z}}_{p_1} \times \ldots \times \hat{\mathbb{Z}}_{p_n}$$

Proof. $D_n$ is the inductive limit of the subalgebras $D_n^{(k)} \cong \mathbb{C}^{l_k}$ with $l_k = p_1^k p_2^k \ldots p_n^k$. The algebra $D_n^{(k)}$ is generated by the pairwise orthogonal projections $u^i e_{p_k} u^{-i}, 0 \leq i < l_k$ and in fact, by Lemma 3.2 (a), $D_n^{(k)}$ is the $k$-fold tensor product of $D_n^{(1)}$ by itself. \hfill \Box

Consider the action of $\mathbb{T}^n$ on $B_n$ given by

$$\alpha_{(t_1, \ldots, t_n)}(s_{p_i}) = t_i s_{p_i}$$

and denote by $F_n$ the fixed-point algebra for $\alpha$ (i.e. $F_n = F \cap B_n$). There is a natural faithful conditional expectation $E : B_n \to F_n$ defined by $E(x) = \int_{\mathbb{T}^n} \alpha_t(x) dt$.

Now $D_n$ is the fixed point algebra, in $F_n$, for the action $\beta$ of $\mathbb{T}$ on $F_n$ given by

$$\beta_t(e_k) = e_k \quad \beta_t(u) = e^{it} u$$

and there is an associated expectation $F : F_n \to D_n$ defined by $F(x) = \int_{\mathbb{T}} \beta_t(x) dt$.

The composition $G = F \circ E$ gives a faithful conditional expectation $A_n \to D_n$. These conditional expectations extend to the inductive limit and thus give conditional expectations $E : Q_N \to F$, $F : F \to D$ and $G : Q_N \to D$.

Theorem 3.4. The $C^*$-algebra $Q_N$ is simple and purely infinite.

Proof. Since inductive limits of purely infinite simple $C^*$-algebras are purely infinite simple [5], 4.1.8 (ii), it suffices to show that each $B_n$ is purely infinite simple.

For each $N$, denote, as above, by $D_n^{(N)}$ the subalgebra of $D_n$ generated by $\{u^k e_{l} u^{-k}, k \in \mathbb{Z}\}$ where $l = p_1^N p_2^N \ldots p_n^N$. The natural map $\text{Spec} D_n \to \text{Spec} D_n^{(N)}$ is surjective and, by the proof of Lemma 3.3, $D_n^{(N)} \cong \mathbb{C}^l$. 

Choose \(\xi_1, \xi_2, \ldots, \xi_l\) in \(\text{Spec } D_n\) such that \(\{\xi_1, \xi_2, \ldots, \xi_l\} \to \text{Spec } D_n^{(N)}\) is bijective and such that
\[\xi_i(s_k \uplus s_k^*) \neq \xi_i(\square)\]
for all \(k\) of the form \(k = p_1^{m_1} \cdots p_n^{m_n}\) with \(m_i \leq N\). We can choose pairwise orthogonal projections \(f_1, f_2, \ldots, f_l\) in \(D_n\) with sufficiently small support around the \(\xi_i\) such that \(f_i s_k f_i = 0\) for all \(1 \leq i \leq l\) and \(k\) as above and such that \(f_i x f_i = \xi_i(x) f_i\) for all \(x \in D_n^N\). Then the map \(\varphi: D_n^{(N)} \to C^*(f_1, \ldots, f_l) \cong \mathbb{C}^l\) defined by \(x \mapsto \sum f_i x f_i\) is an isomorphism.

Denote by \(P^{(N)}\) the set of linear combinations of all products of the form \(u^k s_{p_1}^{m_1} \cdots s_{p_n}^{m_n}\) with \(m_i \leq N\). For \(x \in P^{(N)}\) we have that
\[\varphi(G(x)) = \sum f_i x f_i\]
Let now \(0 \leq x \in B_n\) be different from 0. Since \(G\) is faithful, \(G(x) \neq 0\) and we normalize \(x\) such that \(\|G(x)\| = 1\). Let \(y \in P^{(N)}\), for sufficiently large \(N\), be such that \(\|x - y\| < \varepsilon \leq 1\). We may also assume that \(\|G(y)\| = 1\). Then there exists \(i_0, 1 \leq i_0 \leq l\) such that \(f_{i_0} y f_{i_0} = f_{i_0}\). Moreover, there exists an isometry \(s \in B_n\) (of the form \(s = u^k s_m\)) such that \(s^* f_{i_0} s = 1\).

In conclusion, we have \(s^* y s = 1\) and
\[\|s^* x s - 1\| = \|s^* x s - s^* y s\| = \|s^* (x - y) s\| < \varepsilon\]
This shows that \(s^* x s\) is invertible. \(\square\)

As a consequence of the simplicity of \(Q_n\) we see that the canonical representation on \(l^2(\mathbb{Z})\) by
\[s_n(\xi_k) = \xi_{nk}\quad u^n(\xi_k) = \xi_{k+n}\]
is faithful. Similarly, if we divide the \(C^*\)-algebra generated by the analogous isometries on \(l^2(\mathbb{N})\), discussed in section 2, by the canonical ideal \(K\), we get an algebra isomorphic to \(Q_n\).

The subalgebra \(F\) is generated by \(u\) and the projections \(e_n\), thus by a weighted shift. Thus we recognize \(F\) as the Bunce-Deddens algebra of the type where every prime appears with infinite multiplicity. This well known algebra has been introduced in [3] in exactly that form. It is simple and has a unique tracial state. It can also be represented as inductive limit of the inductive system \((M_n \mathcal{C}(S^1))\) with maps
\[M_n(\mathcal{C}(S^1)) \to M_{nk}(\mathcal{C}(S^1))\]
mapping the unitary \(u\) generating \(\mathcal{C}(S^1)\) to the \(k \times k\)-matrix
\[
\begin{pmatrix}
0 & 0 & \ldots & u \\
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
& & \ddots & \ddots \\
0 & \ldots & 1 & 0
\end{pmatrix}
\]
From this description it immediately follows that \(K_0(F) = \mathbb{Q}\) and \(K_1(F) = \mathbb{Z}\).
Remark 3.5. Just as $\mathcal{O}_n$ is a crossed product of a $UHF$-algebra by $\mathbb{N}$, we see that $\mathcal{Q}_\mathbb{N}$ is a crossed product $\mathcal{F} \rtimes \mathbb{N}^\times$ by the multiplicative semigroup $\mathbb{N}^\times$. The algebra $\mathcal{Q}_\mathbb{N}$ also contains the commutative subalgebra $D$. Since $D$ is the inductive limit of the $D_n$, we see from [3,3] that $\text{Spec } D = \hat{\mathbb{Z}} := \prod_p \hat{\mathbb{Z}}_p$. The Bost-Connes algebra $C_{\mathbb{Q}}$ [1] can be described as a crossed product $C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times$ and is a natural subalgebra of $\mathcal{Q}_\mathbb{N}$ (using the natural inclusion $C(\hat{\mathbb{Z}}) \hookrightarrow \mathcal{F}$).

We can also obtain $\mathcal{Q}_\mathbb{N}$ by adding to the generators $\mu_n$, $n \in \mathbb{N}$ (our $s_n$) and $e_\gamma$, $\gamma \in \mathbb{Q}/\mathbb{Z}$ for $C_{\mathbb{Q}}$, described in [1], Proposition 18, one additional unitary generator $u$ satisfying

\[ u e_\gamma = e_\gamma u \quad u^n \mu_n = \mu_n u \]

(here we identify an element $\gamma$ of $\mathbb{Q}/\mathbb{Z}$ with the corresponding complex number of modulus 1).

4. The canonical action of $\mathbb{R}$ on $\mathcal{Q}_\mathbb{N}$

So far, in our discussion, the isometries associated with each prime number appear as generators on the same footing and it is a priori not clear how to determine, for two prime numbers $p$ and $q$, the size of $p$ and $q$ (or even the bigger number among $p$ and $q$) from the corresponding generators $s_p$ and $s_q$. In fact, the $C^*$-algebra generated by the $s_n$, $n \in \mathbb{N}$ is the infinite tensor product of one Toeplitz algebra for each prime number and in this $C^*$-algebra there is no way to distinguish the $s_p$ for different $p$.

However, the fact that we have added $u$ to the generators allows to retrieve the $n$ from $s_n$ using the KMS-condition.

Definition 4.1. Let $(A, \lambda_t)$ be a C*-algebra equipped with a one-parameter automorphism group $(\lambda_t)$, $\tau$ a state on $A$ and $\beta \in (0, \infty]$. We say that $\tau$ satisfies the $\beta$-KMS-condition with respect to $(\lambda_t)$, if for each pair $x, y$ of elements in $A$, there is a holomorphic function $F_{x,y}$, continuous on the boundary, on the strip $\{z \in \mathbb{C} \mid \text{Im } z \in [0, \beta]\}$ such that

\[ F_{x,y}(t) = \tau(x \lambda_t(y)) \quad F_{x,y}(t + i\beta) = \tau(\lambda_t(y)x) \]

for $t \in \mathbb{R}$.

Proposition 4.2. Let $\tau_0$ be the unique tracial state on $\mathcal{F}$ and define $\tau$ on $\mathcal{Q}_\mathbb{N}$ by $\tau = \tau_0 \circ E$. Let $(\lambda_t)$ denote the one-parameter automorphism group on $\mathcal{Q}_\mathbb{N}$ defined by $\lambda_t(s_n) = n^t s_n$ and $\lambda_t(u) = u$. Then $\tau$ is a $1$-KMS-state for $(\lambda_t)$.

Proof. According to [2] 5.3.1, it suffices to check that

\[ \tau(x \lambda_t(y)) = \tau(yx) \]

for a dense *-subalgebra of analytic vectors for $(\lambda_t)$. Here $(\lambda_z)$ denotes the extension to complex variables $z \in \mathbb{C}$ of $(\lambda_t)$ on the set of analytic vectors.

However it is immediately clear that this identity holds for $x, y$ linear combinations of elements of the form $as_n$ or $s_n^* b$, $a, b \in \mathcal{F}$. Such linear combinations are analytic and dense. \qed
Theorem 4.3. There is a unique state \( \tau \) on \( Q_\mathbb{N} \) with the following property: there exists a one-parameter automorphism group \((\lambda_t)_{t \in \mathbb{R}}\) for which \( \tau \) is a 1-KMS-state and such that \( \lambda_t(u) = u, \lambda_t(e_n) = e_n \) for all \( n \) and \( t \). Moreover we have

- \( \tau \) is given by \( \tau = \tau_0 \circ E \) where \( \tau_0 \) is the canonical trace on \( F \).
- the one-parameter group for which \( \tau \) is 1-KMS, is unique and is the standard automorphism group considered above, determined by
  \[
  \lambda_t(s_n) = n^t s_n \quad \lambda_t(u) = u
  \]

Proof. Since \( \lambda_t \) acts as the identity on \( e_n \) and on \( u \), it is the identity on \( F = C^*(u, \{e_n\}) \). If \( \tau \) is a KMS-state for \((\lambda_t)\), it therefore has to be a trace on \( F \). It is well-known (and clear) that there is a unique trace \( \tau_0 \) on \( F \). For instance, the relation
  \[
  \sum_{0 \leq k < n} u^k e_n u^{-k} = 1
  \]
shows that \( \tau_0(e_n) = 1/n \) and \( \tau_0(u) = 0 \).

The relation
  \[
  \tau(s_n^* s_n) = n \tau(s_n s_n^*)
  \]
and the 1-KMS-condition show that \( \lambda_t(s_n) = n^t s_n \).

\[\square\]

5. The \( K \)-groups of \( Q_\mathbb{N} \)

The \( K \)-groups of \( Q_\mathbb{N} \) can be computed using the fact that \( Q_\mathbb{N} \) is a crossed product of the Bunce-Deddens algebra \( F \) by the semigroup \( \mathbb{N}^\times \). The \( K \)-groups of a Bunce-Deddens algebra are well known and easy to determine using its representation as an inductive limit of algebras of type \( M_n(C(S^1)) \). Specifically, for \( F \), we have

\[ K_0(F) = \mathbb{Q} \quad K_1(F) = \mathbb{Z} \]

We can consider \( Q_\mathbb{N} \) as an inductive limit of the subalgebras \( B_n = C^*(F, s_{p_1}, \ldots, s_{p_n}) \) where \( 2 = p_1 < p_2 < \ldots \) is the sequence of prime numbers in natural order. Now \( B_{n+1} \) can be considered as a crossed product of \( B_n \) by the action of the semigroup \( \mathbb{N} \) given by conjugation by \( s_{p_{n+1}} \) (in fact \( B_{n+1} \) is Morita equivalent to a crossed product by \( \mathbb{Z} \) of an algebra Morita equivalent to \( B_n \)), just as in [4]. Thus the \( K \)-groups of \( B_n \) can be determined inductively using the Pimsner-Voiculescu sequence.

Theorem 5.1. The \( K \)-groups of \( B_n \) are given by

\[
K_0(B_n) \cong \mathbb{Z}^{2^{n-1}} \quad K_1(B_n) \cong \mathbb{Z}^{2^{n-1}}
\]

Proof. The first application of the Pimsner-Voiculescu sequence gives an exact sequence

\[
\cdots \rightarrow K_1(B_1) \rightarrow \mathbb{Q} \xrightarrow{id-\alpha_{s}} \mathbb{Q} \rightarrow K_0(B_1) \rightarrow \mathbb{Z} \xrightarrow{id-\alpha_{s}} \mathbb{Z} \rightarrow K_1(B_1) \rightarrow \cdots
\]

where \( \alpha_{s} \) is the map induced by \( \text{Ad} s_1 \) on \( K_0, K_1 \).

Since \( \alpha_{s} = 2 \) on \( K_0(F) \) and \( \alpha_{s} = 1 \) on \( K_1(F) \) this gives \( K_0(B_1) = \mathbb{Z}, K_1(B_1) = \mathbb{Z} \).

In the following steps \( \alpha_{s} = \text{Ad} s_{p_i} \) induces 1 on \( K_0 \) and on \( K_1 \). Thus each consecutive prime \( p_i \) doubles the number of generators of \( K_0 \) and \( K_1 \). \[\square\]
6. Representations as crossed products

For each $n$, define the endomorphism $\varphi_n$ of $\mathcal{Q}_N$ by $\varphi_n(x) = s_n x s_n^*$. Since $\varphi_n \varphi_m = \varphi_{nm}$ this defines an inductive system.

Definition 6.1. We define $\overline{\mathcal{Q}}_N$ as the inductive limit of the inductive system $(\mathcal{Q}_N, \varphi_n)$.

By construction we have a family $\iota_n$ of natural inclusions of $\mathcal{Q}_N$ into the inductive limit $\overline{\mathcal{Q}}_N$ satisfying the relations $\iota_{nm} \varphi_n = \iota_m$. We denote by $1_k$ the element $\iota_k(1)$ of $\overline{\mathcal{Q}}_N$. We have that $1_k = \iota_{kl}(e_l) \leq e_{kl}$. The union of the subalgebras $1_k \mathcal{Q}_N 1_k$ is dense in $\overline{\mathcal{Q}}_N$ and $1_k \leq 1_{kl}$ for all $k, l$. In order to define a multiplier $a$ of $\overline{\mathcal{Q}}_N$ it therefore suffices to define $a1_k$ and $1_k a$ for all $k$.

We can extend the isometries $s_n$ naturally to unitaries $\bar{s}_n$ in the multiplier algebra $\mathcal{M}(\overline{\mathcal{Q}}_N)$ of $\overline{\mathcal{Q}}_N$ by requiring

$$\bar{s}_n 1_k = \iota_k(s_n) \quad 1_k \bar{s}_n = \iota_{kn}(e_k s_n)$$

Note that this is well defined because, using [3,2] we have

$$(1_k \bar{s}_n)1_l = \iota_{kl}(e_k s_n) \iota_{kl}(e_l) = \iota_{kl}(s_n e_k e_l) = \iota_{kl}(e_l s_k s_n s_k^*) = \iota_k(1) \iota_l(s_n) = 1_k(\bar{s}_n 1_l)$$

Proposition 6.2. The elements $\bar{s}_n$, $n \in \mathbb{N}$, define unitaries in $\mathcal{M}(\overline{\mathcal{Q}}_N)$ such that $\bar{s}_n \bar{s}_m = \bar{s}_{nm}$ and such that $\bar{s}_n \bar{s}_m^* = \bar{s}_m \bar{s}_n$.

Defining $\bar{s}_a = \bar{s}_n \bar{s}_m^*$ for $a = n/m \in \mathcal{Q}_+^x$ we define unitaries in $\mathcal{M}(\overline{\mathcal{Q}}_N)$ such that $\bar{s}_a \bar{s}_b = \bar{s}_a b$ for $a, b \in \mathcal{Q}_+^x$.

Proof. In order to show that $\bar{s}_n$ is unitary it suffices to show that $1_k \bar{s}_n \bar{s}_n^* = 1_k$ and $1_k \bar{s}_n^* \bar{s}_n = 1_k$ for all $k, n$. \hfill $\square$

We can also extend the generating unitary $u$ in $\mathcal{Q}_N$ to a unitary in the multiplier algebra $\mathcal{M}(\overline{\mathcal{Q}}_N)$. We define the unitary $\bar{u}$ in $\mathcal{M}(\overline{\mathcal{Q}}_N)$ by the identity

$$\bar{u} \ 1_k = \iota_k(u^k)$$

We can also define fractional powers of $\bar{u}$ by setting

$$\bar{u}^{1/n} \ 1_k = \iota_{kn}(u^k)$$

Proposition 6.3. For all $a \in \mathcal{Q}_+^x$ and $b \in \mathbb{Q}$ we have the identity

$$\bar{s}_a \bar{u}^b = \bar{u}^b \bar{s}_a$$

Proof. Check that $\bar{s}_a \bar{u}^b = \bar{u}^{1/n} \bar{s}_1$.

Following Bost-Connes we denote by $P_Q^+$ the $ax + b$-group

$$P_Q^+ = \{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a \in \mathcal{Q}_+^x, b \in \mathbb{Q} \}$$

It follows from the previous proposition that we have a representation of $P_Q^+$ in the unitary group of $\mathcal{M}(\overline{\mathcal{Q}}_N)$. Denote by $A_f$ the locally compact space of finite adeles over $\mathbb{Q}$ i.e.

$$A_f = \{ (x_p)_{p \in \mathcal{P}} \mid x_p \in \hat{\mathcal{Q}}_p \text{ and } x_p \in \hat{\mathbb{Z}}_p \text{ for almost all } p \}$$

where $\mathcal{P}$ is the set of primes in $\mathbb{N}$. \hfill $\square$
The canonical commutative subalgebra $D$ of $\mathcal{Q}_N$ is by the Gelfand transform isomorphic to $\mathcal{C}(X)$ where $X$ is the compact space $\hat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \hat{\mathbb{Z}}_p$. It is invariant under the endomorphisms $\varphi_n$ and we obtain an inductive system of commutative algebras $(D, \varphi_n)$. The inductive limit $D$ of this system is a canonical commutative subalgebra of $\mathcal{Q}_N$. It is isomorphic to $\mathcal{C}_0(\mathbb{A}_f)$. In fact the spectrum of $D$ is the projective limit of the system $(\text{Spec} D, \varphi_n)$ and $\varphi_n$ corresponds to multiplication by $n$ on $\hat{\mathbb{Z}}$.

**Theorem 6.4.** The algebra $\mathcal{Q}_N$ is isomorphic to the crossed product of $\mathcal{C}_0(\mathbb{A}_f)$ by the natural action of the $ax+b$-group $P^+_Q$.

**Proof.** Denote by $B$ the crossed product and consider the projection $e \in \mathcal{C}_0(\mathbb{A}_f) \subset B$ defined by the characteristic function of the maximal compact subgroup $\prod_{p \in \mathbb{P}} \hat{\mathbb{Z}}_p \subset \mathbb{A}_f$. Consider also the multipliers $s_n$ and $u$ of $B$ defined as the images of the elements \[
\begin{pmatrix}
1 & 0 \\
0 & n
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]
of $P^+_Q$, respectively. Then the elements $s_n = s_n e$ and $u = u e$ satisfy the relations defining $\mathcal{Q}_N$. Moreover, the $C^*$-algebra generated by the $s_n$ and $u$ contains the subalgebra $e\mathcal{C}_0(\mathbb{A}_f)$ of $B$. This shows that this subalgebra equals $e\mathcal{B}e$ and, by simplicity of $\mathcal{Q}_N$, it follows that $e\mathcal{B}e \cong \mathcal{Q}_N$. \hfill $\square$

It is a somewhat surprising fact that we get exactly the same $C^*$-algebra, even together with its canonical action of $\mathbb{R}$ if we replace the completion $\mathbb{A}_f$ of $\mathbb{Q}$ at the finite places by the completion $\mathbb{R}$ at the infinite place.

**Theorem 6.5.** The algebra $\mathcal{Q}_N$ is isomorphic to the crossed product of $\mathcal{C}_0(\mathbb{R})$ by the natural action of the $ax+b$-group $P^+_Q$ which carries the natural one parameter group $\lambda_t$ to a one parameter group $\lambda'_t$ such that $\lambda'_t(fu_g) = a^{it}fu_g$ where $u_g$ denotes the unitary multiplier of $\mathcal{C}_0(\mathbb{R}) \times P^+_Q$ associated with an element $g \in P^+_Q$ of the form

\[
\begin{pmatrix}
1 & b \\
0 & a
\end{pmatrix}
\]

In order to prove this we need a little bit of preparation concerning the representation of the Bunce-Deddens algebra $\mathcal{F}$ as an inductive limit. As noted above it is an inductive limit of the inductive system $(M_n(\mathcal{C}(S^1)))$ with maps sending the unitary generator $z$ of $\mathcal{C}(S^1)$ to a unitary $v$ in $M_k(\mathbb{C}) \otimes \mathcal{C}(S^1)$ satisfying $v^k = 1 \otimes z$. We observe now that such a $v$ is unique up to unitary equivalence.

**Lemma 6.6.** Let $v_1, v_2$ be two unitaries in $M_k(\mathbb{C}) \otimes \mathcal{C}(S^1)$ such that $v_1^k = v_2^k = 1 \otimes z$. Then there is a unitary $w$ in $M_k(\mathbb{C}) \otimes \mathcal{C}(S^1)$ such that $v_2 = vw_1w^*$.\hfill $\square$

**Proof.** The spectral projections $p_i(t)$ for the different $k-th$ roots of $e^{2\pi it}1$, given by $v_1(t)$ and $v_2(t)$, have to be continuous functions of $t$. This implies that all the $p_i(t)$ are one-dimensional and that, after possibly relabelling, we must have the situation where

\[
p_i(t + 1) = p_{i+1}(t)
\]

This means that the $p_i$ combine to define a line bundle on the $k$-fold covering of $S^1$ by $S^1$. However any two such line bundles are unitarily equivalent. \hfill $\square$
We now determine the crossed product $C_0(\mathbb{R}) \rtimes \mathbb{Q}$ where $\mathbb{Q}$ acts by translation.

**Lemma 6.7.** The crossed product $C_0(\mathbb{R}) \rtimes \mathbb{Q}$ is isomorphic to the stabilized Bunce-Deddens algebra $K \otimes F$.

**Proof.** The algebra $C_0(\mathbb{R}) \rtimes \mathbb{Q}$ is an inductive limit of algebras of the form $C_0(\mathbb{R}) \rtimes \mathbb{Z}$ with respect to the maps

$$
\beta_k : C_0(\mathbb{R}) \rtimes \mathbb{Z} \to C_0(\mathbb{R}) \rtimes \mathbb{Z}
$$

obtained from the embeddings $\mathbb{Z} \cong \mathbb{Z}_1 \to \mathbb{Q}$. It is well known that $C_0(\mathbb{R}) \rtimes \mathbb{Z}$ is isomorphic to $K \otimes C(S^1)$. An explicit isomorphism is obtained from the map

$$
\sum_{n \in \mathbb{Z}} f_n u^n \mapsto \sum_{n \in \mathbb{Z}, k \in \mathbb{Z}} \tau_k(f_n) e_{k,k+n}
$$

where $\tau_k$ denotes translation by $k$, $u^k$ denotes the unitary in the crossed product implementing this automorphism and $e_{ij}$ denote the matrix units in $K \cong K(\ell^2 \mathbb{Z})$. This map sends $C_0(\mathbb{R}) \rtimes \mathbb{Z}$ to the mapping torus algebra

$$
\{ f \in C(\mathbb{R}, K) \mid f(t+1) = U f(t) U^* \} \cong K \otimes C(S^1)
$$

where $U$ is the multiplier of $K = K(\ell^2 \mathbb{Z})$ given by the bilateral shift on $\ell^2(\mathbb{Z})$.

A projection $p$ corresponding, under this isomorphism, to $e \otimes 1$ in $K \otimes C(S^1)$ with $e$ a projection of rank 1 can be represented in the form $p = ug + f + gu^*$ with appropriate positive functions $f$ and $g$ with compact support on $\mathbb{R}$. Under the map $C_0(\mathbb{R}) \rtimes \mathbb{Z} \cong C_0(\mathbb{R}) \rtimes k\mathbb{Z} \to C_0(\mathbb{R}) \rtimes \mathbb{Z}$, the projection $p$ is mapped to $p' = u^k g_k + f_k + g_k u^k$ where $g_k(t) := g(t/k)$, $f_k(t) := f(t/k)$. Now, $p'$ corresponds to a projection of rank $k$ in $K \otimes C(S^1)$.

Let $z$ be the unitary generator of $C(S^1)$. Then the element $e \otimes z$ corresponds the function $e^{2\pi i t}(ug + f + gu^*)$ which is mapped to $v = e^{2\pi i t/k}(u^k g_k + f_k + g_k u^k)$. Thus $v^k = p'$ and $p'$ corresponds to a projection of rank $k$ in $K \otimes C(S^1)$.

On the other hand $F = \lim A_n$ where $A_n = M_n(C(S^1))$ and the inductive limit is taken relative to the maps $\alpha_k : M_n(C(S^1)) \to M_{kn}(C(S^1))$ which map the unitary generator $z$ of $C(S^1)$ to an element $v$ such that $v^k = 1 \otimes z$. Compare this now to the inductive system $A'_n = C_0(\mathbb{R}) \rtimes \mathbb{Z}_n$ with respect to the maps $\beta_k$ considered above. From our analysis of $\beta_k$ and from Lemma 6.6 we conclude that there are unitaries $W_k$ in $M(A'_{kn})$ such that the following diagram commutes

$$
\begin{array}{ccc}
A_n & \xrightarrow{\alpha_k} & A_{kn} \\
\downarrow & & \downarrow \\
A'_n & \xrightarrow{\text{Ad } W_k \beta_k} & A'_{kn}
\end{array}
$$

where the vertical arrows denote the natural inclusions $M_n(C(S^1)) \to K \otimes C(S^1)$.

We conclude that these natural inclusions induce an isomorphism from $K \otimes F = \lim K \otimes A_n$ to $C_0(\mathbb{R}) \rtimes \mathbb{Q} = \lim A'_n$. \qed
Proof. of Theorem 6.5. Consider the commutative diagram and the injection $F \hookrightarrow C_0(\mathbb{R}) \times Q$ constructed at the end of the proof of Lemma 6.7. Note also that the $\beta_k$ are $\sigma$-unital and therefore extend to the multiplier algebra. This shows that the injection transforms $\alpha_p$ into $\beta_p$ times an approximately inner automorphism, for each prime $p$. Therefore the injection transforms $\lambda_t$, which is the dual action on the crossed product for the character $n \mapsto n^{it}$ of $\mathbb{N}^\times$, to the restriction of the corresponding dual action on the crossed product $(C_0(\mathbb{R}) \times Q) \rtimes \mathbb{N}^\times$.

Finally, note that the endomorphisms $\beta_n$ of $C_0(\mathbb{R}) \times Q$ are in fact automorphisms and that therefore $\beta$ extends from a semigroup action to an action of the group $Q^\times$. This shows that $(C_0(\mathbb{R}) \times Q) \rtimes \mathbb{N}^\times = (C_0(\mathbb{R}) \times Q) \rtimes Q^\times = C_0(\mathbb{R}) \rtimes P^+$.

Remark 6.8. If we consider the full space of adeles $Q_A = A_f \times \mathbb{R}$ rather than the finite adeles $A_f$ or the completion at the infinite place $\mathbb{R}$, we obtain the following situation. By [6], IV, §2, Lemma 2, $Q$ is discrete in $Q_A$ and the quotient is the compact space ($\hat{Z} \times \mathbb{R})/\mathbb{Z}$. Thus, the crossed product $C_0(Q_A) \rtimes Q$ which would be analogous to the Bunce-Deddens algebra is Morita equivalent to $C((\hat{Z} \times \mathbb{R})/\mathbb{Z})$. Now, ($\hat{Z} \times \mathbb{R})/\mathbb{Z}$ has a measure which is invariant under the action of the multiplicative semigroup $\mathbb{N}^\times$. Therefore the crossed product $C_0(Q_A) \rtimes P^+_Q$ has a trace and is not isomorphic to $Q_N$.

7. The case of the multiplicative semigroup $\mathbb{Z}^\times$

We consider now the analogous $C^*$-algebras where we replace the multiplicative semigroup $\mathbb{N}^\times$ by the semigroup $\mathbb{Z}^\times$. This case is also important in view of possible generalizations from $Q$ to more general number fields. Thus, on $\ell^2(\mathbb{Z})$ we consider the isometries $s_n, n \in \mathbb{Z}^\times$ and the unitaries $u^m, m \in \mathbb{Z}$ defined by $s_n(\xi_k) = \xi_{nk}$ and $u^m(\xi_k) = \xi_{k+m}$. As above, these operators satisfy the relations

\begin{equation}
(1) \quad s_n s_m = s_{nm}, \quad s_n u^m = u^{nm} s_n, \quad \sum_{i=0}^{n-1} u^i e_n u^{-i} = 1
\end{equation}

The operator $s_{-1}$ plays a somewhat special role and we therefore denote it by $f$. Then the $s_n, n \in \mathbb{Z}$ and $u$ generate the same $C^*$-algebra as the $s_n, n \in \mathbb{N}$ together with $u$ and $f$. The element $f$ is a selfadjoint unitary so that $f^2 = 1$ and we have the relations

\begin{equation}
fs_n = s_n f, \quad fu = u^{-1}
\end{equation}

We consider now again the universal $C^*$-algebra generated by isometries $s_n, n \in \mathbb{Z}$ and a unitary $u$ subject to the relations (1). We denote this $C^*$-algebra by $Q_Z$. We see from the discussion above that we get a crossed product $Q_Z \cong Q_N \rtimes \mathbb{Z}/2$ where $\mathbb{Z}/2$ acts by the automorphism $\alpha$ of $Q_N$ that fixes the $s_n, n \in \mathbb{N}$ and $\alpha(u) = u^{-1}$.

Theorem 7.1. The algebra $Q_Z$ is simple and purely infinite.

Proof. Composing the conditional expectation $G : Q_N \to \mathcal{D}$ used in the proof of Theorem 3.4 with the natural expectation $Q_Z = Q_N \rtimes \mathbb{Z}/2 \to Q_N$ we obtain again a faithful expectation $G' : Q_Z \to \mathcal{D}$. The rest of the proof follows exactly the proof of 3.4, using in addition the fact that the $f_i$ in that proof can be chosen such that $f_i f_j f_i = 0$. 

\[ \square \]
Denote by \( P_Q \) the full \( ax + b \)-group over \( \mathbb{Q} \), i.e.

\[
P_Q = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Q}^\times, b \in \mathbb{Q} \right\}
\]

**Theorem 7.2.** \( \mathbb{Q}_z \) is isomorphic to the crossed product of \( C_0(\mathbb{A}_f) \) or of \( C_0(\mathbb{R}) \) by the natural action of \( P_Q \).

**Proof.** This follows from Theorems 6.4 and 6.5 since \( P_Q = P_Q^+ \times \mathbb{Z}/2 \). \( \square \)

On \( \mathbb{Q}_z \) we can define the one-parameter group \( (\lambda_t) \) by \( \lambda_t(s_n) = n^t s_n, n \in \mathbb{N}^\times \). The fixed point algebra is the crossed product \( \mathcal{F} \rtimes \mathbb{Z}/2 \) of the Bunce-Deddens algebra by \( \mathbb{Z}/2 \).

In order to compute the \( K \)-groups of \( \mathbb{Q}_z \) we first determine the \( K \)-theory for \( \mathcal{F}' = \mathcal{F} \rtimes \mathbb{Z}/2 \). This algebra is the inductive limit of the subalgebras \( A_n' = C^*(u, f, e_n) \).

**Lemma 7.3.**

(a) The \( C^* \)-algebra \( C^*(u, f) \) is isomorphic to \( C^*(D) \), where \( D \) is the dihedral group \( D = \mathbb{Z} \rtimes \mathbb{Z}/2 \) (\( \mathbb{Z}/2 \) acts on \( \mathbb{Z} \) by \( a \mapsto -a \)). For each \( n = 1, 2, \ldots \), the algebra \( A_n' \) is isomorphic to \( M_n(C^*(D)) \).

(b) The generators of \( K_0(C^*(D)) \) are given by the classes of the spectral projections \( f^+ \) and \( (uf)^+ \) of \( f \) and \( uf \), for the eigenvalue 1, and by 1. We have \( K_0(A_n') = \mathbb{Z}^3 \) and \( K_1(A_n') = 0 \) for all \( n \).

(c) Let \( p \) be prime. If \( p = 2 \), then the map \( K_0(A_n') \to K_0(A_{pn}') \) is described by the matrix

\[
\begin{pmatrix}
2 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

If \( p \) is odd, then the map \( K_0(A_n') \to K_0(A_{pn}') \) is described by the matrix

\[
\begin{pmatrix}
p & \frac{p-1}{2} & \frac{p-1}{2} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

**Proof.** (a) It is clear that the universal algebra generated by two unitaries \( u, f \) satisfying \( f^2 = 1 \) and \(ifu = u^* \) is isomorphic to \( C^*(D) \).

In the decomposition of \( C^*(u, f, e_2) \) with respect to the orthogonal projections \( e_2 \) and \( ue_2u^{-1} \), the elements \( u \) and \( f \) correspond to matrices

\[
\begin{pmatrix}
0 & w \\
1 & 0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
f_0 & 0 \\
0 & f_1
\end{pmatrix}
\]

where \( w \) is unitary and \( f_0, f_1 \) are symmetries (selfadjoint unitaries).

The relations between \( u \) and \( f \) imply that \( w f_1 = f_0 \) and that \( w f_1 \) is a selfadjoint unitary, whence \( f_1 w f_1 = w^* \). Thus \( A_2' \) is isomorphic to \( M_2(C^*(w, f_1)) \) and \( w, f_1 \) satisfy the same relations as \( u, f \).

If \( p \) is an odd prime, then in the decomposition of \( C^*(u, f, e_p) \) with respect to the pairwise orthogonal projections \( e_p, ue_pu^{-1}, \ldots, u^{p-1}e_pu^{-(p-1)} \), \( u \) and \( f \) correspond to
the following $p \times p$-matrices

\[
\begin{pmatrix}
0 & 0 & \ldots & w \\
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
0 & \ldots & \ldots & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
f_0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & f_1 \\
0 & 0 & \ldots & f_2 \\
f_1 & \ldots & 0 & 0
\end{pmatrix}
\]

where $w$ and the $f_1, \ldots, f_{\frac{p-1}{2}}$ are unitary and $f_0$ is a symmetry.

The relation $fuf = u^*$ implies that $f_0 = w f_1$, $f_1 = f_2 = \ldots = f_{p-1}$ and $f_i^2 = 1$ for all $i$. From $(wf_1)^2 = 1$ we derive that $f_1wf_1 = w^*$. Thus $A'_n \cong M_p(C^*(w, f_1))$ and $w, f_1$ satisfy the same relations as $u, f$.

The case of general $n$ is obtained by iteration using the fact that $C^*(u, f, e_{nm}) \cong M_n(C^*(u, f, e_{n}))$.

(b) This follows for instance from the well known fact that $C^*(D) \cong (\mathbb{C} \times \mathbb{C})^\infty$.

(c) Let $p = 2$. Using the description of $K_0(C^*(u, f))$ under (b) and the description of the map $C^*(u, f) \to C^*(u, fe_2)$ from the proof of (b), we see that the generators $[1]$, $[(uf)^+]$ and $[f^+]$ are respectively mapped to $[1]$, $[1]$ and $[f^+] + [(uf)^+]$.

Let $p$ be an odd prime. The matrix corresponding to $f$ in the proof of (b) is conjugate to the matrix where all the $f_i$, $i = 1, \ldots, (p-1)/2$ are replaced by 1. Thus the class of $f^+$ is mapped to $[(wf_1)^+] + [1]$.

The matrix corresponding to $uf$ is conjugate to a second diagonal matrix with all entries 1, except for the middle entry which is $f_1$. Thus the class of $(uf)^+$ is mapped to $[f_1^+] + \frac{p-1}{2}[1]$. □

**Proposition 7.4.** The $K$-groups of the algebra $\mathcal{F}' = \mathcal{F} \rtimes \mathbb{Z}/2$ are given by $K_0(\mathcal{F}') = \mathbb{Q} \oplus \mathbb{Z}$ and $K_1(\mathcal{F}') = 0$.

**Proof.** This follows immediately from the fact that $\mathcal{F}'$ is the inductive limit of the algebras $A'_n$ and the description of the maps $K_0(A'_n) \to K_0(A'_{pn})$ given in Lemma 7.3 (c). □

In analogy to the case over $\mathbb{N}$ we denote by $B'_n$ the $C^*$-subalgebra of $Q_{\mathbb{Z}}$ generated by $u, f, s_{p_1}, \ldots, s_{p_n}$. We now immediately deduce

**Theorem 7.5.** We have

\[
K_0(B'_n) = \mathbb{Z}^{2^{n-1}} \quad K_1(B'_n) = \mathbb{Z}^{2^{n-1}}
\]

and

\[
K_0(Q_{\mathbb{Z}}) = \mathbb{Z}^{\infty} \quad K_1(Q_{\mathbb{Z}}) = \mathbb{Z}^{\infty}
\]

**References**

[1] J.-B. Bost and A. Connes. Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory. *Selecta Math. (N.S.)*, 1(3):411–457, 1995.

[2] Ola Bratteli and Derek W. Robinson. *Operator algebras and quantum statistical mechanics. 2.* Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1997. Equilibrium states. Models in quantum statistical mechanics.
[3] John W. Bunce and James A. Deddens. A family of simple $C^*$-algebras related to weighted shift operators. *J. Functional Analysis*, 19:13–24, 1975.

[4] Joachim Cuntz. Simple $C^*$-algebras generated by isometries. *Comm. Math. Phys.*, 57(2):173–185, 1977.

[5] M. Rørdam. Classification of nuclear, simple $C^*$-algebras. In *Classification of nuclear $C^*$-algebras. Entropy in operator algebras*, volume 126 of *Encyclopaedia Math. Sci.*, pages 1–145. Springer, Berlin, 2002.

[6] André Weil. *Basic number theory*. Die Grundlehren der mathematischen Wissenschaften, Band 144. Springer-Verlag New York, Inc., New York, 1967.

**JOACHIM CUNTZ, MATHEMATISCHES INSTITUT, EINSTEINSTR.62, 48149 MÜNSTER, GERMANY**

*E-mail address: cuntz@math.uni-muenster.de*

*URL: http://www.math.uni-muenster.de/u/cuntz/cuntz*