Stringy Dyonic Solutions and Clifford Structures

A. Belfakir\textsuperscript{1}, A. Belhaj\textsuperscript{1}, Y. El Maadi\textsuperscript{1}, S. E. Ennadifi\textsuperscript{2}, Y. Hassouni\textsuperscript{1}, A. Segui\textsuperscript{3}

\textsuperscript{1} Equipe des Sciences de la Matière et du Rayonnement(ESMAR), Département de Physique
Faculté des Sciences, Université Mohammed V, Rabat, Morocco

\textsuperscript{2} Département de Physique, Faculté des Sciences, Université Mohammed V, Rabat, Morocco

\textsuperscript{3} Departamento de Física Teórica, Universidad de Zaragoza, E-50 009-Zaragoza, Spain

Abstract

Using the toroidal compactification of string theory on $n$-dimensional tori, $\mathbb{T}^n$, we investigate dyonic objects in arbitrary dimensions. First, we present a class of dyonic black solutions formed by two different D-branes using a correspondence between toroidal cycles and objects possessing both magnetic and electric charges, belonging to $U(1)^{2n-1} \times U(1)^{2n-1}$ dyonic gauge symmetry. This symmetry could be associated with electrically charged magnetic monopole solutions in stringy model buildings of the standard model extensions. Then, we consider in some details such black hole classes obtained from even dimensional toroidal compactifications, and we find that they are linked to $\mathcal{Cl}(n)$ Clifford algebras using the vee product. It is believed that this analysis could be extended to dyonic objects which can be obtained from local Calabi-Yau manifold compactifications.

Keywords: Superstring theory; Toroidal compactification; Standard model; Clifford Algebras; Dyonic black solutions.
1 Introduction

Recently, black holes and their extensions have been extensively investigated in connection with higher dimensional supergravity models compactified on Calabi-Yau (CY) manifolds\[1, 2, 3, 4, 5\]. In certain compactifications, the scalar field contributions can be fixed in terms of the black brane potential by optimizing the stringy moduli parameterized by physical and geometrical deformations. In this way, the corresponding entropy functions have been computed using the duality symmetries which act on the invariant black object charges. Considering D-brane physics, several compactifications producing some black brane solutions embedded in type II superstrings have been studied \[6, 7, 8\]. It has been shown that these black objects can be linked to many subjects including quantum information theory by exploiting the qubit mathematical formalism \[9, 10, 11, 12, 13, 14, 15, 16\]. Concretely, a remarkable correspondence between qubit systems and black holes in superstring theory has been established. The relevant findings concern a nice link between the $N = 2$ STU black hole with eight charges and three-qubit system states \[13, 17, 18, 19\]. The analysis has been developed to describe structures going beyond such extremal black hole solutions in terms of higher dimensional qubit systems. Using string dualities between type IIA and heterotic superstrings, four-qubit systems in the context of type II superstring compactifications have been investigated \[20, 21\]. It has been revealed that such qubit systems are related to a stringy moduli space $\frac{SO(4,4)}{SO(4) \times SO(4)}$ considered as a reduction of the moduli space of six dimensional supergravity model obtained from the compactification of the heterotic superstring on $T^4$ \[22\]. Moreover, the string/string duality has been exploited to show that there exists an interplay between four-qubit states and a particular ordinary dyonic black hole in six dimensions having eight electric and eight magnetic charges.

More recently, electric and magnetic charges have been reconsidered to discuss a dark matter sector using the $SL(2, \mathbb{Z})$ duality \[23\]. It is recalled that such a sector involves a massive dark photon and dark magnetic monopoles investigated in terms of a kinetic mixing of the dark and usual photons. These activities could be related to black objects in the presence of Dark Energy being the hardest puzzles in modern and new physics. In this regard, any serious attempt corresponding to such an ambiguous dark sector is welcome.

The aim of this work is to contribute to this program by reconsidering the study of dyonic black objects in arbitrary dimensions obtained from the toroidal compactification of string theory on $T^n$. Precisely, we first present a class of dyonic black solutions, given by doublets, formed by two different D-branes using a correspondence between toroidal cycles and objects carrying electric and magnetic charges. These charges belong to $U(1)^{2n-1}_e \times U(1)^{2n-1}_m$ dyonic gauge symmetry. This symmetry could be associated with electrically charged magnetic monopole solutions in stringy model buildings of the standard model (SM) extensions. Then, we pay a particular attention to black object classes obtained from lower even dimensional toroidal
compactifications. Notably, we find that they are linked to $Cl(n)$ Clifford algebra structures using the vee product of real differential forms on $T^n$. It is suggested that this analysis could be extended to dyonic objects obtained from local Calabi-Yau manifold compactifications. The paper is organized as follows. After a study of stringy dyonic black objects in section 2, we build the associated toroidal compactifications in section 3. Then, we emphasize the situation for even dimensional compactifications. A link with dark sectors is provided. Section 4 concerns a relation with $Cl(n)$ Clifford algebras using the vee ($\lor$) product associated with real differential forms dual to cycles on which D-branes can be wrapped on to generate dyonic black objects. Concluding remarks and discussions are presented in section 5.

2 Stringy dyonic black objects in arbitrary dimensions

In this section, we reconsider the study of dyonic solutions in type II superstrings compactified on $n$-dimensional real manifolds $X^n$. It has been remarked that each compact manifold possesses some geometric information which can play a crucial role in the determination of the superstring theory spectrum in $10 - n$ dimensions [22]. In particular, they can be exploited to produce all physical data in lower dimensional superstring models. In connections with extremal black objects, it has been suggested that the black $p$-branes can be built using a system involving $(p + k)$-branes wrapping appropriate $k$-cycles of $X^n$. It has been remarked that the corresponding near horizon geometries are given by a product of AdS spaces and spheres $AdS_{n+2} \times S^{8-n-p}$ where $n$ and $p$ are integers constrained by $1 \leq n$ and $2 \leq 8 - n - p$. It is noted that these black objects are coupled to $(p + 1)$-gauge field $C_{p+1}$. To measure their electric charges, one should use the field strength $dC_{p+1}$. This field will play the same role of the field strength $F = dA$, where $A$ is the 1-form gauge potential coupled to the ordinary particle associated with the following action term

$$S \sim \int F \wedge \star F,$$

(2.1)

where $\star F$ is the dual of $F$. It is known that the electric charge is measured by the integration

$$Q_e = \int_{S^{10-n-2}} \star F,$$

(2.2)

where $S^{10-n-2}$ is the $(10 - n - 2)$-dimensional real sphere. Similarly, one can compute the magnetic charge using the relation

$$P_m = \int_{S^2} F.$$

(2.3)

These particle charge equations can be generalized to $p$-branes being $p$-dimensional objects moving in the string theory space-time. In this case, the electric charge is calculated as follows

$$Q_e = \int_{S^{10-n-p-2}} \star dC_{p+1}$$

(2.4)
where \( \ast dC_{p+1} \) is the dual of the \( dC_{p+1} \) gauge field. However, the magnetic charge can be given by the following integration

\[
P_m = \int_{\mathbb{S}^{p+2}} dC_{p+1},
\]

where \( \mathbb{S}^{p+2} \) is the \((p+2)\)-dimensional real sphere. In higher dimensional theories including string theory, one may classify the black brane solutions using the extended electric/magnetic duality connecting a \( p \)-dimensional electrical black brane to a \( q \)-dimensional magnetic one via the following relation

\[
p + q = 6 - n.
\]

This constraint can be solved in different manners using appropriate \((p, q)\) couple values including the ones representing dyonic solutions in arbitrary dimensions. However, a deeper inspection in string compactifications shows that dyonic solutions carrying electric and magnetic charges can be arranged as object like doublets

\[
\begin{pmatrix}
p \\
q
\end{pmatrix}.
\]

This doublet configuration is motivated and supported by the \( SL(2, \mathbb{Z}) \) electric-magnetic duality \cite{23}. It has been remarked that these dyonic solutions are associated with a gauge field symmetry involving two factors corresponding to electric and magnetic sectors

\[
G_{dyon} = G_e \times G_m.
\]

In the compactification on \( n \)-dimensional real manifolds, two different dyonic solutions could appear with electric and magnetic charges. They are classified in what follows.

### 2.1 Ordinary dyonic solution

This solution is described by the constraint

\[
p = q.
\]

It turns out that such a solution finds a place only in even dimensional string theory compactifications. A simple calculation shows that one has

\[
p = q = 3 - \frac{n}{2}.
\]

This generates dyons represented by the following doublets

\[
\begin{pmatrix}
p \\
q
\end{pmatrix} = \begin{pmatrix} 3 - \frac{n}{2} \\ 3 - \frac{n}{2} \end{pmatrix}
\]
consisting of the same object which appears in even dimensional superstring models. This solution can be described only by a single factor related to both electric and magnetic charges. In this way, the above dyonic symmetry can be reduced to

\[ G_{dyon} = G_e. \quad (2.12) \]

### 2.2 Non ordinary dyonic solution

The second solution concerns the case

\[ p \neq q \quad (2.13) \]

associated with two different D-branes. This involves dyonic objects consisting of an electrically charged black object and its magnetic dual one. In such a case, \( p \) and \( q \) are constrained by

\[ q = 6 - n - p, \quad p \neq 3 - \frac{n}{2}. \quad (2.14) \]

This solution generates doublets of the following form

\[
\left( \begin{array}{c}
p \\
q 
\end{array} \right) = \left( \begin{array}{c}
p \\
6 - n - p 
\end{array} \right)
\quad (2.15)
\]

carrying electric and magnetic charges associated with two gauge symmetry factors given in \((2.8)\). This new dyonic solution could be worked out to provide a possible support of monopoles in the context of string theory compactifications.

We wish to add some comments on these dyonic black solutions. The first comment concerns the fact that the non ordinary solution is considered as a single object sharing similar features of the ordinary one described by the same object, appearing in even dimensional string theory. The associated dyonic physics is invariant under the mapping

\[ p \rightarrow 6 - n - p \quad (2.16) \]

interpreted as an electric-magnetic symmetry in string theory compactifications. The second comment concerns the brane configurations of such dyonic black solutions. In string theory, they should be constructed from the following dyonic brane doublets living in ten dimensional space time

\[
\left( \begin{array}{c}
D0 \\
D6 
\end{array} \right), \quad \left( \begin{array}{c}
NS1 \\
NS5 
\end{array} \right), \quad \left( \begin{array}{c}
D1 \\
D5 
\end{array} \right), \quad \left( \begin{array}{c}
D2 \\
D4 
\end{array} \right), \quad \left( \begin{array}{c}
D3 \\
D3 
\end{array} \right).
\quad (2.17)
\]

Before examining a particular supergravity theory solution, we should recall that the black object charges depend on the choice of the internal space. It turns out that each compactification involves a Hodge diagram carrying not only geometric information but also physical data providing the above mentioned dyonic black solutions.
3 Dyonic black solutions from toroidal compactifications

For simplicity reasons, we consider the toroidal compactification having a nice real Hodge diagram playing a primordial role in the elaboration of dyonic stringy solutions in $10 - n$ dimensions. It is recalled that $\mathbb{T}^n$ is a flat compact space which can be constructed using different approaches. One of them is to exploit the trivial circle fibrations given by the identifications $x_i \equiv x_i + 1, \quad i = 1, \ldots, n$. It is remarked that the general real Hodge diagram of $\mathbb{T}^n$ can encode all possible non trivial cycles describing geometric data including the size and the shape parameters. Using a binary number notation, the number $h^{e_1 \ldots e_n}$ will be associated with a real differential form of degree $k$ ($k$-differential form) $\prod_{\ell=1}^n (\overline{e}_\ell + e_\ell dx_\ell)$, where $e_\ell$ takes either 0 or 1, and where $\overline{e}_\ell$ is its conjugate. In this way, $k$ equals to $\sum_{i=1}^n e_i$. To illustrate such real Hodge diagrams, one may consider lower dimensional cases corresponding to $n = 1$, $n = 2$ and $n = 3$. They are given by, respectively

| $n = 1$ | $h^1$ | $h^0$ | 1 |
|---|---|---|---|
| $n = 2$ | $h^{1,0}$ | $h^{0,1}$ | 1 1 1 |
| $n = 3$ | $h^{1,1,0}$ | $h^{1,0,1}$ | 1 1 1 |

3.1 Dyonic black objects on $S^1$

To establish a link with string theory, we consider some dyonic black objects obtained from the compactification on 1-dimensional circle $S^1$. Indeed, the nine dimensional dyonic black objects involve one electric charge $Q_0$ and one magnetic charge $P_0$ associated with the dyonic gauge symmetry

$$G_{dyon} = U(1)_e \times U(1)_m. \quad (3.1)$$

The corresponding brane representations can be obtained from the ten dimensional dyonic ones illustrated in the previous section. Using the Hodge diagram representation, the brane configurations are constrained by

$$p + q = 5. \quad (3.2)$$
In this case, the dyonic solutions can be classified as

\[
\begin{pmatrix}
D0 \\
D5
\end{pmatrix}, \quad 
\begin{pmatrix}
D1 \\
D4
\end{pmatrix}, \quad 
\begin{pmatrix}
D2 \\
D3
\end{pmatrix}. \tag{3.3}
\]

These objects can be built from the following wrapped D-branes, respectively

\[
\begin{align*}
D0 & \quad D1 & \quad D2 \\
D6/\Sigma^1 & \quad D5/\Sigma^1 & \quad D4/\Sigma^1.
\end{align*} \tag{3.4}
\]

The charges of these solutions are associated with the cycles dual to the following real forms on the circle

\[
1, \quad dx. \tag{3.5}
\]

### 3.2 Dyonic black objects on $T^2$

Eight dimensional dyonic black solutions can be obtained from the compactification on $T^2$ constrained by

\[
p + q = 4. \tag{3.6}
\]

In this case, the corresponding type II superstrings involve three dyonic black solutions given by

\[
\begin{align*}
\begin{pmatrix}
0 \\
4
\end{pmatrix}, \quad 
\begin{pmatrix}
1 \\
3
\end{pmatrix}, \quad 
\begin{pmatrix}
2 \\
2
\end{pmatrix}.
\end{align*} \tag{3.7}
\]

For instance, the \( \begin{pmatrix} 0 \\ 4 \end{pmatrix} \) dyonic solution can be represented by the following D-brane charge configurations

\[
\begin{align*}
D0 & \quad D1/\Sigma^1 & \quad D5/\Sigma^1 \\
D6/T^2
\end{align*} \tag{3.8}
\]

Exploiting string theory links including the S-duality, an equivalent D-brane configuration can be obtained by the following mapping

\[
\begin{align*}
D1 & \leftrightarrow \text{NS1} \\
D5 & \leftrightarrow \text{NS5}.
\end{align*} \tag{3.9}
\]

However, the remaining \( \begin{pmatrix} p \\ q \end{pmatrix} \) dyonic solutions such that \( p + q = 4 \) can be represented by the following D-brane charge configurations

\[
\begin{align*}
Dp & \quad Dp + 1/\Sigma^1 & \quad Dq + 1/\Sigma^2 \\
Dq + 2/T^2
\end{align*} \tag{3.10}
\]
having two electric charges and two magnetic charges. These charges associated with the cycles belonging to $T^2 = S^1_1 \times S^1_2$ dual to the following real differential forms on $T^2$ can be organized as follows

$$
\begin{align*}
1 & \quad dx_1 \\
& \quad dx_2 \\
& \quad dx_1dx_2.
\end{align*}
$$

In this representation, the forms $(1, dx_1)$ correspond to the $(D_p, D_p + 1/S^1_1)$ brane system involving two electric charges associated with $G_e = U(1)^2_e$. However, the dual forms $(dx_2, dx_1dx_2)$ are linked to the $(Dq + 1/S^1_2, Dq + 2/T^2)$ brane configuration having two magnetic charges corresponding to $G_m = U(1)^2_m$. For $p = q = 2$, it is worth noting that the dyonic gauge symmetry $G_{dyon} = U(1)^2_e \times U(1)^2_m$ reduces to $G_{dyon} = U(1)^2$ for both electric and magnetic charges.

### 3.3 Dyonic black objects on $T^3$

In this subsection, we consider three copies of $S^1$ corresponding to the compactification of superstring theory on $T^3$. The manifold is determined by three circles $S^1_1, S^1_2$ and $S^1_3$ coordinated by $x_1, x_2$ and $x_3$, respectively. This compactification produces seven dimensional dyonic black solutions having four electric charges and four magnetic ones, under $G_{dyon} = U^4(1)_e \times U^4(1)_m$ dyonic gauge symmetry. In this scenario, one has two fundamental solutions given by the following doublets

$$
\begin{pmatrix}
0 \\
3
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
2
\end{pmatrix}.
$$

Using the T-duality and the real Hodge diagram of $T^3$, the solution $\begin{pmatrix}
0 \\
3
\end{pmatrix}$ can be built using the following D-brane configuration

$$
\begin{align*}
& \text{D0} \\
& \text{D1}/S^1_1 \quad \text{D1}/S^1_2 \quad \text{D1}/S^1_3 \\
& \text{D5}/\ast S^1_1 \quad \text{D5}/\ast S^1_2 \quad \text{D5}/\ast S^1_3 \\
& \text{D6}/T^3
\end{align*}
$$
where $\star S_1^i$ are dual to $S_1^i$ in $T^3$. However, the solution \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) involves only one Hodge D-brane configuration given by

\[
\begin{align*}
D1 & \\
D2/S_1^1 & D2/S_2^1 & D2/S_3^1 \\
D4/\star S_1^1 & D4/\star S_2^1 & D4/ \star S_3^1 \\
D5/T^3.
\end{align*}
\]

The four states \( \{D1, D2/S_1^1, D2/S_2^1, D2/S_3^1\} \) represent seven dimensional dyonic solutions carrying electric charges. The remaining four states having magnetic charges are obtained using the Hodge duality which has been used to build the real Hodge diagram illustrated in section 2. An examination reveals that these states can be linked to the following real Hodge diagram of $T^3$

\[
\begin{array}{ccc}
\star dx_1 & \star dx_2 & \star dx_3 \\
\end{array}
\]

\[
dx_1 dx_2 dx_3
\]

where $\star dx_i$ are Hodge duals to $dx_i$ in $T^3$.

### 3.4 Dyonic black objects on $T^n$ and dark sectors

The compactification of superstring theory on $T^n$ generates dyonic black solutions in $10 - n$ dimensions. In this way, the corresponding real Hodge diagram can be built in terms of the forms

\[
1, \quad dx_1, \quad dx_2, \ldots, \quad dx_1 \ldots dx_n.
\]

Concretely, the dyonic black object states can be associated with such a set of differential forms dual to cycles in which D-branes can be wrapped on. The presence of dyonic black objects \( \begin{pmatrix} p \\ 6 - n - p \end{pmatrix} \), in such a superstring theory compactification, suggests that there are $2^{n-1}$ states associated with the electric charges. By using the Hodge mapping between forms

\[
k - \text{forms} \Leftrightarrow (n - k) - \text{forms},
\]

\[
(3.17)
\]
we can show that there are also $2^{n-1}$ states corresponding to the magnetic charges assured by the decomposition

$$2^n = 2^{n-1} + 2^{n-1}$$

(3.18)
supported by the dyonic symmetry

$$G_{dyon} = U(1)^{2_{n-1}}_e \times U(1)^{2_{n-1}}_m.$$  

(3.19)

This symmetry, associated with the electrically charged magnetic monopole solutions within the context of superstring theory, could be relevant in stringy model buildings of the standard model (SM) extensions [25, 26]. Precisely, interesting extensions of SM can be constructed from compactified superstring models where the geometry and the topology of the internal space give rise to extra abelian gauge symmetries $U_{X_i}(1)$’s corresponding to some conserved charges $X_i$ along with new scalar fields $S_i$. This can be usually achieved in the context of intersecting D-branes wrapping non trivial cycles embedded in compact manifolds [27, 28, 29]. Thus, it could be shown that there are many roads to handle the physics underlying such a dyonic symmetry $G_{dyon}$ in the stringy inspired extended SM’s. For the charge identification $X_i \equiv Q_e, P_m$, we can consider the assumed symmetry such as

$$U_{X_i}(1)^{2^n} \equiv U(1)^{2_{n-1}}_e \times U(1)^{2_{n-1}}_m,$$  

(3.20)

where the new scalar fields $S_i$ can play an important role in the underlying scale as well as the resulting mass spectrum. On the basis of such motivations, one can deal with the extended gauge symmetry

$$G_{SM+dyon} = SU_C(3) \times SU_L(2) \times G_{dyon}$$  

(3.21)

where the first piece $SU_C(3)$ refers to the strong interaction associated with the color charge $C$. The second one $SU_L(2)$ refers to the weak interaction corresponding to the left isospin charge $L$. However, the sector $SU_L(2) \times G_{Dyon}$ refers now to the extended electroweak dark sector. Roughly, this extended electroweak symmetry is expected to be broken by the vacuum expectation value (vev) of a new scalar $S$ down to the four dimensional SM one. This, in turn, is broken down to the electromagnetic symmetry $U_{Qem}(1)$ by the vev of the standard Higgs scalar field $H$ in the following scheme

$$SU(2)_L \times U(1)^{2_{n-1}}_e \times U(1)^{2_{n-1}}_m \overset{\langle S \rangle}{\rightarrow} SU(2)_L \times U(1)_Y \overset{\langle H \rangle}{\rightarrow} U_{Qem}(1),$$

(3.22)

where now the new complex scalar $S$, associated with the dyonic symmetry breaking, corresponds to a $2^{n-1}$-tuple with $2^n$ real components as $S = s_i^1 + is_i^2$, $i = 1, \ldots, 2^{n-1}$. At this point, seen that the dyonic symmetry breaking scale is expected to be above the SM electroweak scale $\langle S \rangle > \langle H \rangle \sim 10^2 GeV$, this would make such dyons heavy as being proportional somehow to the new scalar vev. Therefore, they are far to be observed at the current experiments unless
the corresponding proportionality is highly suppressing. In this case, the best way of probing the existence of such dyons remains their observation in cosmic rays as recently inspected by the actual telescopes [30, 31, 32, 33].

Having discussed how a possible link with extended SM is provided by using extra symmetries and scalar fields. It has been shown that these symmetries can be considered as structures of division algebraic ladder operators. In what follows, we elaborate a relation with Clifford algebraic structures reported in [34, 35].

4 Link with differential form structures

A close inspection shows that the dyonic black objects obtained from toroidal compactifications can be linked to non trivial structures corresponding to the real differential forms on $\mathbb{T}^n$.

To do so, let us first recall such a structure [34, 35, 36, 37, 38]. Let $V$ be a vector space over the field of real numbers. In this space, we can define a quadratic form using the map $Q : V \to R$, where $Q$ satisfies

$$Q(\alpha v) = \alpha^2 Q(v)$$

(4.1)

for all $\alpha$ in $R$ and $v$ in $V$. One can also define the Grassmann-Cartan exterior product $\wedge$ of differential forms as

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu, \quad \mu \neq \nu, \quad dx^\mu \wedge dx^\mu = 0.$$  

(4.2)

The Grassman product is an associative product $(dx^\mu \wedge dx^\nu) \wedge dx^\lambda = dx^\mu \wedge (dx^\nu \wedge dx^\lambda)$. However, the $\vee$ product of two differential forms is defined as

$$dx^\mu \vee dx^\nu = (dx^\mu, dx^\nu) + dx^\mu \wedge dx^\nu.$$  

(4.3)

Here, $(dx^\mu, dx^\nu)$ denotes the scalar product between $dx^\mu$ and $dx^\nu$ and $(dx^\mu, dx^\nu) = g^{\mu \nu}$ where $g^{\mu \nu}$ is the vector space metric. In what follows, we will take $g^{\mu \nu} = \delta^{\mu \nu}$ by considering only the euclidian metrics associated with the toroidal compactification using the vee product. The Clifford algebra $Cl(n)$ generated by the differential forms is defined as

$$dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu = 2\delta^{\mu \nu} Q(dx^\mu)$$  

(4.4)

where $Q(dx^\mu) = \{dx^\mu, dx^\mu\} = 1$, $\mu = 1, \ldots, \frac{n}{2}$ and $n$ is an even number. Using the string theory toroidal compactification, we will see that one can construct a set of operators $a_\ell$ satisfying

$$\{a_\ell, a_{\ell'}^\dagger\} = \{a_\ell^\dagger, a_{\ell'}\} = 0, \quad \{a_\ell, a_{\ell'}\} = \delta_{\ell \ell'}.$$  

(4.5)
4.1 \( Cl(n) \) and differential forms on \( \mathbb{T}^n \)

To make contact with the above structure, we will be interested in the complex geometry associated with \( n \) even. For \( n = 2 \), the operators \( a \) and \( a^\dagger \) are given in terms of one-forms on \( \mathbb{T}^2 \). Indeed, one has the following identifications

\[
a = \frac{1}{2} (dx^1 + idx^2), \quad a^\dagger = \frac{1}{2} (-dx^1 + idx^2). \quad (4.6)
\]

It is remarked that \( \dagger \) maps \( i \rightarrow -i \) and \( dx^j \rightarrow -dx^j \) for \( j = (1, 2) \). The \( Cl(4) \) Clifford algebra could be built using the real differential form set \( \{dx^1, dx^2, dx^3, dx^4\} \). From these forms, we construct a set of operators satisfying the above structure. In this case, we have 4 generators which are \( a_1, a_2 \) and their adjoint \( a_1^\dagger, a_2^\dagger \). Once again the operation \( \dagger \) maps \( i \rightarrow -i \) and \( dx^i \rightarrow -dx^i \). Using the compactification on \( \mathbb{T}^4 \), the generators can be constructed as follows

\[
a_1 = \frac{1}{2} (-dx^1 + idx^2), \quad a_2 = \frac{1}{2} (-dx^3 + idx^4) \quad (4.7)
\]

and

\[
a_1^\dagger = \frac{1}{2} (dx^1 + idx^2), \quad a_2^\dagger = \frac{1}{2} (dx^3 + idx^4). \quad (4.8)
\]

Similarly as \( Cl(2) \) and \( Cl(4) \), the set of ladder operators of \( Cl(6) \) is \( \{a_1, a_2, a_3, a_1^\dagger, a_2^\dagger, a_3^\dagger\} \) where

\[
a_1 = \frac{1}{2} (-dx^5 + idx^4), \quad a_2 = \frac{1}{2} (-dx^3 + idx^1) \quad a_3 = \frac{1}{2} (-dx^6 + idx^2) \quad (4.9)
\]

and

\[
a_1^\dagger = \frac{1}{2} (dx^5 + idx^4), \quad a_2^\dagger = \frac{1}{2} (dx^3 + idx^1) \quad a_3^\dagger = \frac{1}{2} (dx^6 + idx^2) \quad (4.10)
\]

satisfying the above Clifford structures.

4.2 \( Cl(n) \) and dyonic black objects

The above dyonic black object construction can be made using D-branes moving on non trivial cycles. In the associated compactification, one can distinguish two types of cycles namely electric and magnetic cycles noted by \( C^\alpha \) and \( D^\alpha \), respectively

\[
\{[C^\alpha], \alpha = 0, \ldots, 2^{n-1} - 1\}, \quad \{[D^\alpha], \alpha = 0, \ldots, 2^{n-1} - 1\}. \quad (4.11)
\]

These cycles are dual to \( \zeta \) and \( \eta \) forms on \( \mathbb{T}^n \) defined by

\[
\int_{C^\alpha} \zeta^\beta = \delta^\alpha_\beta, \quad \int_{D^\alpha} \eta^\beta = \delta^\alpha_\beta. \quad (4.12)
\]

such that

\[
\int_{\mathbb{T}^n} \zeta^\alpha \wedge \eta^\beta = \delta^\alpha_\beta. \quad (4.13)
\]

The electric vectors determine a basis for the \( 2^{n-1} \) abelian \( (p + 1) \)-gauge field \( C_{p+1} \) obtained by integrating the ten dimensional forms on \( C^\alpha \). Similarly, the magnetic vectors determine a
basis for the $2^{n-1}$ abelian vector fields $(q+1)$-gauge field $C_{q+1}$ obtained by integrating the ten dimensional forms on $D^\alpha$. Under these abelian gauge fields, associated with $G_{dyon} = U(1)^{2^{n-1}} \times U(1)$ dyonic gauge symmetry, the dyonic black objects in generic cohomology class $[X]$, on $\mathbb{T}^m$, are given by

$$[X] = \sum_{\alpha=0}^{2^{n-1}-1} P_\alpha [C^\alpha] + Q_\alpha [D^\alpha]$$

(4.14)

which carry $P_\alpha$ electric and $Q_\alpha$ magnetic charges. The dual of this cycle is a general element of $Cl(n)$ with real coefficient associated with physical charges. Motivated by ideals of $Cl(n)$ used in SM of particle physics [34, 35], we associate to each $[X]$ the following black object state

$$|\psi\rangle = Cl(n) |0\rangle$$

(4.15)

where $|0\rangle$ is considered here as a vacuum state. To be precise, the state $|\psi\rangle$ is mapped to the class $[X]$

$$[X] \leftrightarrow |\psi\rangle.$$

(4.16)

In terms of the ladder operators $a_\ell$, this state can take the following form

$$|\psi\rangle = \sum_{e_1,...,e_n=0,1} p_{e_1...e_n} \prod_{\ell=1}^{n} a_\ell^{e_\ell} \vee a_\ell^{\ell+1/2}|0\rangle,$$

(4.17)

where $p_{e_1...e_n}$ are real coefficients describing electric and magnetic charges of dyonic solutions. It should be noted that the ordering problem can be absorbed by negative charges carried by D-brane objects. Moreover, the electric/magnetic charge duality can be assured by

$$p_{e_1...e_n} \leftrightarrow p_{\overline{e_1...e_n}}$$

(4.18)

originated from the Hodge duality of $\mathbb{T}^m$. In this way, these charges can be nicely organized as follows

$$\left(\begin{array}{c} Q_\alpha \\ P_\alpha \end{array}\right) \equiv \left(\begin{array}{c} p_{e_1...e_n} \\ p_{\overline{e_1...e_n}} \end{array}\right), \quad \alpha = 0, \ldots, 2^{n-1} - 1.$$

(4.19)

In order to illustrate this procedure in detail we consider lower dimensional cases. We believe that the general case can be dealt with without ambiguities. For $n = 2$ associated with the $Cl(2)$ structure, the situation is simple. The charges of dyonic black objects can be organized as follows

$$p_{e_1e_2} \equiv (Q_\alpha, P_\alpha), \quad \alpha = 0, 1, \quad e_i = 0, 1.$$

(4.20)

A quick examination shows that one can use the following ordering index

$$Q_\alpha = p_{\alpha 0} \quad P_\alpha = p_{1\alpha}, \quad \alpha = 0, 1.$$

(4.21)
This involves two electric charges and two magnetic charges corresponding to the following graphic operator representation

\[
\begin{align*}
1 & \\
1 \vee 1 & \quad 1 \vee a^\dagger \\
a \vee a^\dagger
\end{align*}
\] (4.22)

In this graphic representation, the operators \((1, a \vee 1)\) correspond to the \((D^p, D^p + 1/S^1)\) brane system carrying electric charges, under the \(U(1)^2\) electric gauge symmetry. However, the dual operators \((1 \vee a^\dagger, a \vee a^\dagger)\) are associated with the \((D^q + 1/S^1_2, D^q + 2/T^2)\) brane configuration having two magnetic charges, under the \(U(1)^2\) magnetic gauge symmetry.

For \(n = 4\), the situation is quite different involving extra indices. It is not obvious to find an elegant index notation. However, we will consider a simple one inspired by the previous case. The charges are indexed as

\[
p_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4} \equiv (Q_\alpha, P_\alpha), \quad \alpha = 0, \ldots, 7.
\] (4.23)

In this way, they can be arranged as follows

\[
\begin{pmatrix}
p_{0000} & p_{1000} & p_{0100} & p_{0010} & p_{1100} & p_{1010} & p_{1001} \\
p_{1111} & p_{0111} & p_{1101} & p_{1110} & p_{0011} & p_{0111} & p_{0101}
\end{pmatrix}
\equiv
\begin{pmatrix}
Q_0 & Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 & Q_7 \\
P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7
\end{pmatrix}
\] (4.24)

generating 6-dimensional dyonic black objects obtained from the compactification of string theory on \(T^4\). They are charged under gauge fields associated with the \(G_{dyon} = U^8(1)_e \times U^8(1)_m\) dyonic gauge symmetry.

5 Conclusion and discussions

In this work, we have investigated dyonic black objects in arbitrary dimensions. Inspired by real Hodge diagrams of the toroidal compactifications on \(T^n\), we have proposed a new class of doublets of dyonic black solutions formed by two different D-branes given by

\[
\begin{pmatrix}
p \\
6 - n - p
\end{pmatrix}
\]

objects. In particular, we have pointed out a correspondence between toroidal cycles and the black solutions carrying electric and magnetic charges, under \(G_{dyon} = U^2(1)_e \times U^2(1)_m\) dyonic gauge symmetry. It has been suggested that this symmetry could be associated with electrically charged magnetic monopole solutions in stringy model buildings of the SM extensions. More precisely, we have considered in some details such dyonic black classes obtained from lower dimensional toroidal compactification, and we have found that they are linked to \(Cl(n)\) Clifford algebras using the vee product. It has been observed that the analysis presented here might be extended to a class of local Calabi-Yau manifolds. These manifolds have been
extensively studied in the geometric engineering method of quantum field theory, obtained from the compactification of higher dimensional models including string theory, M-theory, and F-theory [39, 40]. Some of them are known by canonical bundles on \( n \)-dimensional compact spaces given by trivial fibrations of \( n \) copies of the projective space \( \mathbb{CP}^1 \) namely

\[
\mathcal{M}^n = \bigtimes^n \mathbb{CP}^1.
\]

The compactification of string theory on such manifolds produces dyonic black objects in \( 8-2n \)-dimensional space-time. Like toroidal compactifications, we are thus expecting that a similar analysis can take place by replacing the circle \( S^1 \) by the one-dimensional complex space \( \mathbb{CP}^1 \)

\[
S^1 \rightarrow \mathbb{CP}^1.
\]

Concretely, one should also expect to be able to engineer dyonic black objects in terms of even dimensional D-branes producing models with less supersymmetric charges.

This work opens up for further discussions. We could cut some of them. First, it would be interesting to investigate geometries based on odd dimensional toroidal compactifications. Second, it should be nice to find a link with fermionic coherent state theory shearing certain similarities with the dyonic black objects dealt with in the present work. On the other hand, one should also try to establish a relation or a link between graph theory and such dyonic black objects via Clifford algebra structures.

Before closing this discussion, we would like to make a remark on monopoles. It is recalled that the search of such objects is the subject of many scientific efforts including the ATLAS-LHC experiment. In string theory compactification however, the monopole/dyon sectors cannot be separated and could be studied in realistic string-inspired SM extensions. In this framework, one may think about a possible doublet decomposition

\[
\begin{pmatrix}
p \\
6 - n - p
\end{pmatrix} = \begin{pmatrix}
p \\
p
\end{pmatrix} + \begin{pmatrix}
0 \\
6 - n - 2p
\end{pmatrix}.
\]

In this way, the last factor involves only magnetic charges which could be useful in such investigations from stringy monopole analysis using D-branes wrapping non trivial magnetic cycles. We leave these investigations for future works.

References

[1] S. Ferrara, R. Kallosh, A. Strominger, \( N = 2 \) Extremal Black Holes, Phys. Rev. D52 (1995) 5412, \texttt{hep-th/9508072}.

[2] S. Ferrara and R. Kallosh, Supersymmetry and Attractors, Phys. Rev. D54 (1996) 1514, \texttt{hep-th/9602136}. 

14
[3] L. Borsten, M. J. Duff, J. J. Fernandez-Melgarejo, A. Marrani, E. Torrente-Lujan, Black holes and general Freudenthal transformations, \texttt{arXiv:1905.00038}.

[4] P. Bueno, R. Davies, C. S. Shahbazi, Quantum black holes in Type-IIA String Theory, \texttt{arXiv:1210.2817}.

[5] H. Ooguri, A. Strominger, C. Vafa, Black Hole Attractors and the Topological String, Phys. Rev. D\textbf{70}(2004)106007, \texttt{arXiv:hep-th/0405146}.

[6] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, Mirror Fermat Calabi-Yau threefolds and Landau-Ginzburg Black Hole Attractors, Riv. Nuov o Cim. \textbf{029} (2006)1, \texttt{hep-th/0608091}.

[7] A. Belhaj, On Black Objects in Type IIA Superstring Theory on Calabi-Yau Manifolds, African Journal Of Math. Phys. Vol. \textbf{6} (2008)49, \texttt{arXiv:0809.1114}.

[8] M. J. Duff, S. Ferrara, A. Marrani, $D=3$ Unification of Curious Supergravities, JHEP \textbf{1701} (2017) 023, \texttt{arXiv:1610.08800} [hep-th].

[9] A. Belhaj, M. Bensed, Z. Benslimane, M. B. Sedra, A. Segui, Qubit and Fermionic Fock Spaces from Type II Superstring Black Hole, Int. J. Geom. Methods Mod. Phys. \textbf{14} (2017)1750087 \texttt{arXiv:1604.03998}.

[10] A. Belhaj, Z. Benslimane, M. B. Sedra, A. Segui, Qubits from Black Holes in M-theory on K3 Surface, Int. J. Geom. Methods Mod. Phys. \textbf{13} (2016)1650075 \texttt{arXiv:1601.07610}.

[11] M. J. Duff, S. Ferrara, Four curious supergravities, Phys. Rev. \textbf{D83} (2011)046007, \texttt{arXiv:1010.3173}.

[12] P. Levay, Qubits from extra dimensions, Phys. Rev. \textbf{D84} (2001)125020.

[13] M. J. Duff, String triality, black hole entropy and Cayley's hyperdeterminant, Phys. Rev. \textbf{D76} (2007) 025017, \texttt{hep-th/0601134}.

[14] P. Levay, Stringy Black Holes and the Geometry of Entanglement, Phys. Rev. \textbf{D74}, 024030 (2006), \texttt{arXiv:0603136}.

[15] P. Levay, F. Holweck, Embedding qubits into fermionic Fock space, peculiarities of the four-qubit case, (2015), \texttt{arXiv:1502.04537}.

[16] P. Levay, F. Holweck, M. Saniga, The magic three-qubit Veldkamp line: A finite geometric underpinning for form theories of gravity and black hole entropy, \texttt{arXiv:1704.01598}.
[17] L. Borsten, M. J. Duff, A. Marrani, W. Rubens, *On the Black-Hole/Qubit Correspondence*, Eur. Phys. J. Plus **126** (2011) 37, [arXiv:1101.3559 [hep-th]].

[18] Y. Aadel, A. Belhaj, M. Bensed, Z. Benslimane, M. B. Sedra, A. Segui, *Qubit Systems from Colored Toric Geometry and Hypercube Graph Theory*, Commun. Theor. Phys. **68** (2017) 285.

[19] A. Belhaj, M. B. Sedra, A. Segui, *Graph Theory and Qubit Information Systems of Extremal Black Branes*, J. Phys. **A48** (2015) 045401, [arXiv:1406.2578]

[20] P. Levay, *STU Black Holes as Four Qubit Systems*, Phys. Rev. D**82**(2010)026003, [arXiv:1004.3639]

[21] P. Levay, M. Planat, M. Saniga, *Grassmannian Connection Between Three- and Four-Qubit Observables, Mermin’s Contextuality and Black Holes*, JHEP **09** (2013) 037, [arXiv:1305.5689]

[22] A. Belhaj, M. Bensed, Z. Benslimane, M. B. Sedra, A. Segui, *Four-qubit Systems and Dyonic Black Hole-Black Branes in Superstring Theory*, International Journal of Geometric Methods in Modern Physics, **15** (2018) 1850065.

[23] J. Terning, C. B. Verhaaren, *Dark Monopoles and SL(2,Z) Duality*, JHEP **12** (2018) 123, [arXiv:1808.09459]

[24] C. Vafa, *Lectures on Strings and Dualities*, arXiv:hep-th/970220.

[25] G.’t Hooft, *Magnetic monopoles in unified gauge theories*, Nuc. Phys. B **79** (1974) 276.

[26] C. Montonen and D. Olive, *Magnetic monopoles in unified gauge theories*, Phys. Lett. B **72** (1977) 117.

[27] R. Blumenhagen, M. Cvetic, P. Langacker, G. Shiu, *Toward Realistic Intersecting D-Brane Models*, Annu. Rev. Nucl. Part. Sci. **55**, (2005) 71.

[28] F. Marchesano, Fortsch. Phys. **55** (2007) 491.

[29] S-E. Ennadifi, On the D-branes Standard-Like Models, APPB **48** (2017) 13.

[30] A. Achterberg, et al., (IceCube Collaboration), Astropart. Phys. **26** (2006) 155.

[31] M. Ageron et al., *ANTARES: the first undersea neutrino telescope*, Nucl. Instr. and Meth. A **656** (2011) 11.

[32] M. G. Aartsen et al., *Searches for Relativistic Magnetic Monopoles in IceCube*, EPJC **76** (2016) 133.
[33] S. Adrian-Martinez et al., *Search for Relativistic Magnetic Monopoles with the ANTARES Neutrino Telescope*, Astropart. Phys. 35 (2012) 634.

[34] C. Furey, Eur. Phys. J. C 78 (2018)375.

[35] C. Furey, *Standard model physics from an algebra*, arXiv:1611.09182.

[36] N. A. Salingaros and G. P. wene, Acta Applicandae Mathematicae 4 (1985) 1.

[37] N. Salingaros, M. Dresden, Advances in applied Mathematics 4 (1983)1.

[38] J. A. Emanuello, *Analysis of Functions of Split-Complex, Multicomplex, and SplitQuat ernionic Variables and Their Associated Conformal Geometries*. PhD thesis, The Florida State University, 2015.

[39] S. Katz, P. Mayr, C. Vafa, *Mirror symmetry and Exact Solution of 4D \( N=2 \) Gauge Theories I*, Adv. Theor. Math. Phys. 15(1998)53, arXiv:hep-th/9706110.

[40] A. Belhaj, *F-theory Duals of M-theory on G2 Manifolds from Mirror Symmetry*, J. Phys. A36 (2003) 4191, hep-th/0207208.