Higher dimensional Thompson groups have subgroups with infinitely many relative ends

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Abstract

The Thompson group $V$ is a subgroup of the homeomorphism group of the Cantor set $C$. Brin [3] defined higher dimensional Thompson groups $nV$ as generalizations of $V$. For each $n$, $nV$ is a subgroup of the homeomorphism group of $C^n$. We prove that the number of ends of $nV$ is equal to 1, and there is a subgroup of $nV$ such that the relative number of ends is $\infty$. This is a generalization of the corresponding result of Farley [8], who studied the Thompson group $V$. As a corollary, $nV$ has the Haagerup property and is not a Kähler group.

1 Introduction

Higher dimensional Thompson groups $nV$ were introduced by Brin in [3] as generalizations of the Thompson group $V$. The Thompson group $V$ is an infinite simple finitely presented group, which is described as a subgroup of the homeomorphism group of the Cantor set $C$. Basic facts about $V$ are found in a paper by Cannon, Floyd and Parry [6].

Brin first studied the case of $n = 2$ in detail, and showed that $V$ and $2V$ are not isomorphic ([3]), $2V$ is simple ([3]) and $2V$ is finitely presented ([3]). These properties also hold true for general $nV$. The simplicity of $nV$ was shown by Brin later in [5]. Bleak and Lanoue showed $n_1V$ and $n_2V$ are isomorphic if and only if $n_1 = n_2$ in [2]. Hennig and Matucci gave a finite presentation for each $nV$ ([10]).

In this paper we prove that for each $n$, the number of ends of $nV$ is equal to 1 and there is a subgroup of $nV$ such that the relative number of ends is $\infty$. We also show that $nV$ has the Haagerup property and does not have the structure of a Kähler group. These are properties related to the number of relative ends of the group. These theorems are the generalizations of the corresponding results of Farley [8], who studied $V$.

Section 2 contains a brief summary of the definition and properties of the higher dimensional Thompson group. We have compiled some well-known facts on the number of ends in Section 3.1. We will look more closely at the relative number of ends in Section 3.2. Sections 4 and 5 are devoted to the proof of main theorems.
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2 Higher dimensional Thompson groups $nV$

In this section, we give the definition of higher dimensional Thompson groups according to Brin’s paper [3]. The symbol $I$ denotes $[0,1)$ throughout this paper.

An $n$-dimensional rectangle is defined inductively as follows. First, $I^n$ is a rectangle.

If $R = [a_1, b_1] \times \cdots \times [a_i, b_i] \times \cdots \times [a_n, b_n]$ is a rectangle, then for all $i \in \{1, \ldots, n\}$, the “$i$-th left half” and “the $i$-th right half” defined by

$$R_{l,i} = [a_1, b_1] \times \cdots \times [a_i, (a_i + b_i)/2] \times \cdots \times [a_n, b_n] \quad (2.1)$$

$$R_{r,i} = [a_1, b_1] \times \cdots \times [(a_i + b_i)/2, b_i] \times \cdots \times [a_n, b_n] \quad (2.2)$$

are again rectangles.

Throughout this paper, $I_l$ denotes $[0, 1/2) \times I^{n-1}$. Similarly, $I_r$ denotes $[1/2, 1) \times I^{n-1}$.

Let $R = [a_1, b_1] \times \cdots \times [a_i, b_i] \times \cdots \times [a_n, b_n]$ be a rectangle. A corner of $R$ is a point in $\text{cl}(R)$, whose $i$-th coordinate is either $a_i$ or $b_i$. Here $\text{cl}(R)$ denotes the closure of $R$ in $\mathbb{R}^n$. An $n$-dimensional pattern is a finite set of $n$-dimensional rectangles, with pairwise disjoint, non-empty interiors and whose union is $I^n$. A numbered pattern is a pattern with a one-to-one correspondence to $\{0, 1, \ldots, r - 1\}$ where $r$ is the number of rectangles in the pattern.

From now on, we will identify $n$-dimensional rectangle with a subset of $C^n$ and use the common symbol. First we identify $I^n$ and $C^n$. $I$ denotes
both \([0, 1]\) and \(C\). Let \(R\) be a rectangle which is identified with a subset of \(C^n\),

\[
R' = C^n \cap [a'_1, b'_1] \times \cdots \times [a'_i, b'_i] \times \cdots \times [a'_n, b'_n].
\]  
(2.3)

Define rectangles \(R_{l,i}\) and \(R_{r,i}\) in the same way as we obtained (2.1) and (2.2). These rectangles are identified respectively with the “\(i\)-th left third” and the “\(i\)-th right third” of \(R'\), which is defined by

\[
C^n \cap [a'_1, b'_1] \times \cdots \times [a'_i, (2a' + b'_i)/3] \times \cdots \times [a'_n, b'_n],
\]  
(2.4)

\[
C^n \cap [a'_1, b'_1] \times \cdots \times [a'_i, (a' + 2b'_i)/3] \times \cdots \times [a'_n, b'_n].
\]  
(2.5)

We proceed by induction. In the same manner, every pattern describes a division of \(C^n\).

We will construct a self-homeomorphism of \(C^n\) from a pair of numbered patterns with the same number of rectangles. Let \(P = \{P_i\}_{0 \leq i \leq r-1}\) and \(Q = \{Q_i\}_{0 \leq i \leq r-1}\) be numbered patterns. We define \(g(P, Q) : I^n \to I^n\) which takes each \(P_i\) onto \(Q_i\) affinely so as to preserve the orientation. Namely, the restriction of \(g(P, Q)\) to each \(P_i\) has the form \((x_1, \ldots, x_n) \mapsto (a_1 + 3^{j_1}x_1, \ldots, a_n + 3^{j_n}x_n)\) for some integers \(j_1, \ldots, j_n\).

With the former identification of rectangles with subsets of \(C^n\), above construction defines a self-homeomorphism of \(C^n\). We again write \(g(P, Q)\) for this homeomorphism.

When \(n = 2\), we illustrate \(g(P, Q)\) as follows. First we draw \(P\) and \(Q\) as divisions of \(I^2\). Next we add an arrow from \(P\) to \(Q\), which indicates the domain and the range.
The $n$-dimensional Thompson group $nV$ is the set of self-homeomorphisms of $C^n$ of the form $g(P,Q)$. Every element of $nV$ is identified with a partially affine, partially orientation preserving bijection from $I^n$ to itself.

Next is an important property which will be used in later discussion.

Theorem 2.1 (Brin [5]). For all $n \in \mathbb{N}$, $nV$ is simple.

The following fact does not appear in the rest of this paper, but it is important to justify studying $nV$ for general $n$ aside from the well discussed case, $n = 1$.

Theorem 2.2 (Bleak and Lanoue [2]). $n_1V$ and $n_2V$ are isomorphic if and only if $n_1 = n_2$.

3 Ends of groups

Let $\Gamma$ be a path-connected locally finite CW complex. For a compact subset $K$, $\|\Gamma - K\|$ denotes the number of unbounded connected components of $\Gamma - K$. The number of ends of $\Gamma$, $e(\Gamma)$, is defined to be the supremum of $\|\Gamma - K\|$ taken over all the compact subsets.

When $\Gamma$ is a graph, we equip $\Gamma$ with graph metric. $B(m)$ denotes a ball of radius $m$ in $\Gamma$, based at some fixed vertex. For simplicity, we ignore the dependence of $B(m)$ on the base point in notation.

Throughout this section, $G$ denotes a finitely generated group and $S$ denotes a finite generating set of $G$. The Cayley graph $\Gamma_{G,S}$ is a graph whose vertex set is $G$, and there is an oriented edge from $g \in G$ to $h \in G$ if some $s \in S$ satisfies $g \cdot s = h$. $G$ acts freely on $\Gamma_{G,S}$ from the left.

3.1 The number of ends

The number of ends of $G$, $e(G)$, is the number of ends of $\Gamma_{G,S}$.

Theorem 3.1 (cf. Geoghegan [9, Corollary 13.5.12]). Let $\Gamma$ be a path-connected locally finite CW complex on which $G$ acts freely. Further suppose that the quotient space $\Gamma/G$ is a finite CW-complex. Then $e(\Gamma) = e(G)$.

Proposition 3.2. (1) $e(G)$ does not depend on the choice of $S$.

(2) (The Freudenthal-Hopf Theorem) $e(G)$ is 0, 1, 2 or $\infty$.

(3) $e(G) = 0$ if and only if $G$ is finite.

(4) $e(G) = 2$ if and only if $G$ has an infinite cyclic subgroup of finite index.

The following result, Stallings’ theorem, provides a group-theoretical characterization of the case where $e(G) \geq 2$. 

4
Theorem 3.3 (Stallings [15], Bergman [1]). \( e(G) \geq 2 \) if and only if \( G \) has a structure of an amalgamated product or an HNN-extension on some finite subgroup.

In the light of this theorem, we can characterize the case of \( e(G) = 1 \) in terms of group actions on trees. We say that \( G \) has property \( FA \) if every simplicial action of \( G \) on a simplicial tree without edge-inversions has a fixed point. Here, a fixed point means \( x \in T \) such that \( g(x) = x \) for every \( g \in G \).

Theorem 3.4 (Serre [14]). If \( e(G) \geq 2 \), \( G \) does not have property \( FA \).

3.2 The relative number of ends

Let \( H \) be a subgroup of \( G \). \( H \) acts freely on \( \Gamma_{G,S} \), since \( G \) does. The relative number of ends \( e(G, H) \) is the number of ends of a quotient graph \( H \backslash \Gamma_{G,S} \).

Let \( \Gamma \) be a path-connected CW complex. The symmetric difference of two sets \( A \) and \( B \), denoted by \( A \Delta B \), is defined to be \( (A \cup B) - (A \cap B) \).

Lemma 3.5 (cf. Meier [11, Lemmas 11.30, 11.31]). Suppose that \( G \) is infinite and acts freely on a path-connected locally finite graph \( \Gamma \) with 2 ends. Let \( m \) be a number such that \( \Gamma - B(m) \) has two unbounded connected components, \( \Gamma^+ \) and \( \Gamma^- \). Let \( X \) be the vertex set of \( \Gamma^+ \).

(1) For all \( g \in G \), either \( gX \Delta X \) or its complement is finite.

(2) \( H := \{ g \in G \mid |gX \Delta X| < \infty \} \) is a subgroup of \( G \) with index at most two.

Lemma 3.6 (cf. Geoghegan [9, Theorem 13.5.9, Exercise 7 on page 308]). Suppose that \( G \) is infinite and acts freely on a path-connected locally finite graph \( \Gamma \). If \( e(\Gamma) = 2 \) then \( e(G) = 2 \).

Proof. It is enough to show that \( \Gamma/G \) is a finite graph. The symbols \( m, \Gamma^+, \Gamma^- \) and \( X \) are the same as in Lemma 3.5.

We first show that there is an element of \( G \) with infinite order. By Lemma 3.5, there is an infinite subgroup \( G' \) whose element \( g' \) makes \( g'X \Delta X \) finite. \( G' \) also acts on \( \Gamma \) freely. Since \( \Gamma \) is locally finite and \( G' \) is infinite, there is some \( g \in G' \) such that \( g(B(m+1)) \cap B(m+1) = \emptyset \) and \( g(B(m+1)) \subset \Gamma^+ \).

We next claim that \( g(\text{cl}(\Gamma^+)) \) is contained in \( \Gamma^+ \). Suppose that this were false. We can take \( x \in \text{cl}(\Gamma^+) \) such that \( g(x) \in \Gamma^- \cup B(m) \). Since \( g(B(m+1)) \subset \Gamma^+ \), this \( x \) is in \( \Gamma^+ \). Let \( \omega : [0, \infty) \to \Gamma \) be an isometry which satisfies \( \omega(0) = x \) and \( \omega([0, \infty)) \subset \Gamma^+ \). Since \( \omega([0, \infty)) \) does not meet \( B(m) \), \( g \cdot \omega([0, \infty)) \) does not meet \( g(B(m)) \). Let \( \tau : [0, \infty) \to \Gamma \) be another isometry which satisfies \( \tau(0) = g(x) \), \( \tau([0, \infty)) \subset \Gamma^- \cup B(m) \), and \( \tau([t, \infty)) \subset \Gamma^- \) for some \( t > 0 \). By definition, \( \omega \) and \( \tau \) define the opposite ends, \( \Gamma^+ \) and \( \Gamma^- \) respectively. We have assumed that \( gX \Delta X \) is finite, which implies that \( g \)
fixes two ends. Therefore, \( g(\omega) \) and \( \tau \) define the opposite ends while both do not meet \( g(K) \). This proves that \( g(\omega) \cap \tau = \emptyset \), contradiction.

Hence \( g(\Gamma^+) \subseteq \Gamma^+ \), \( g^i(\Gamma^+) \neq \Gamma^+ \) for all \( i \neq 0 \). This shows that \( g \) is an element of infinite order.

Finally, we prove that there is a finite subgraph \( L \) of \( \Gamma \) such that \( \{g^j(L)\}_{j \in \mathbb{Z}} \) covers \( \Gamma \). Let \( L \) be the closure of \( \Gamma - (\Gamma^- \cup g(\Gamma^+)) \). Since the vertex set of \( L \) is included in \( g(X\Delta X) \), \( L \) is finite. For any \( y \in \Gamma \), there is the greatest integer \( l(y) \) such that \( y \in g^{l(y)}(B(m) \cup \Gamma^+) \). This \( y \) is included in \( g^{l(y)}(L) \). Therefore, \( \bigcup_{j \in \mathbb{Z}} g^j(L) = \Gamma \).

The quotient of \( \Gamma \) by \( G \) is included in \( L \), which is finite. Applying Theorem 3.1, we obtain \( e(G) = 2 \). \( \square \)

**Proposition 3.7** (cf. Geoghegan [9, Theorem 13.5.21, Proposition 10.1.12]). Let \( H \) be a subgroup of \( G \).

1. If \( H \) has infinite index in its normalizer \( N_G(H) = \{ g \in G \mid g^{-1}Hg = H \} \), then \( e(G, H) = 1, 2 \) or \( \infty \).

2. Further suppose that \( e(G, H) = 2 \). Then the quotient group \( N_G(H)/H \) is virtually infinite cyclic.

**Proof.** The group \( N_G(H)/H \) acts freely on the quotient graph \( H \setminus \Gamma_{G,S} \) from the right. By Theorem 3.1, \( e(G, H) = 0, 1, 2 \) or \( \infty \). Since \( [N_G(H) : H] \) is infinite, (1) is proved. We now suppose that \( e(G, H) = 2 \). By Lemma 3.6, we conclude that \( N_G(H)/H \) is finitely generated and has two ends. \( \square \)

The following result will be needed in the later section.

Suppose \( G \) acts on a set. A subset \( X \) is said to be almost invariant if the symmetric difference \( gX\Delta X \) is finite for any \( g \in G \).

**Theorem 3.8** (Sageev [13, Theorem 2.3]). Let \( H \) be a subgroup of \( G \). \( G \) acts on the set of left (or right) cosets from the the left (or right).

\( e(G, H) \geq 2 \) if and only if there is a subset \( X \subset G/H \) (or \( H \setminus G \)) such that both \( X \) and its complement \( X^c \) are infinite, and \( X \) is almost invariant.

**Proof.** We discuss the case where \( G \) acts on \( H \setminus G \) from the right. Let \( d \) and \( d_H \) be distances on \( \Gamma_{G,S} \) and \( H \setminus \Gamma_{G,S} \) respectively. First, suppose that \( e(G, H) \geq 2 \). Let \( m \) be a number such that \( \Gamma_{G,S} - B(m) \) has at least two unbounded components. Let \( X \) be the vertex set of one of such components. \( X \) and \( X^c \) are infinite. It remains to prove that \( X \) is almost invariant. Fix \( g \in G \) and consider \( X - Xg = \{ Hk \mid Hk \in X, Hkg^{-1} \notin X \} \). For all \( Hk \in X - Xg \), \( d_H(Hk, B(m)) \leq d_H(Hk, Hkg^{-1}) \leq d(g, \text{the base point}) \). By the local finiteness of \( H \setminus \Gamma_{G,S} \), \( X - Xg \) is finite. This observation shows that \( X \) is almost invariant.

We next suppose that there is an almost invariant subset \( X \) such that \( X \) and \( X^c \) are infinite. Let \( K \) be the set of edges of \( H \setminus \Gamma_{G,S} \) having exactly one
endpoint in $X$. $K$ is finite, because the endpoints of edges in $K$ are included in $sX\Delta X$. Since $X$ and $X^c$ are infinite, there are at least two unbounded components in $(H\setminus \Gamma_{G,S}) - K$. Therefore, $H\setminus \Gamma_{G,S}$ has at least two ends.

To treat the case of the left action, we may convert the definition of the Cayley graph to get another graph $\Gamma'$ on which $G$ acts from the right. The vertex set of $\Gamma'$ is again $G$, and the edges correspond to left multiplications of generators. Since $e(H\setminus \Gamma) = e(\Gamma'/H)$, similar arguments apply to this case.

\[\square\]

4 \hspace{1cm} nV has one end

In this section, we prove that $nV$ has one end, using a finite presentation of $nV$. Throughout this section, $T$ denotes a simplicial tree. For $x, y \in T$, we write $[x : y]$ for the geodesic joining $x$ to $y$. An action of a group on $T$ is assumed to be simplicial and to act without edge inversions.

Let $G$ be a group acting on $T$. Let $g \in G$. If $\text{Fix}(g)$ is non-empty, $g$ is said to be elliptic. Otherwise, we say that $g$ is hyperbolic.

The following proposition is a basic fact about group actions on trees.

**Proposition 4.1** (Serre \[14\]). Let $G$ be a group acting on $T$. Let $g \in G$.

1. $\text{Fix}(g) = \{x \in T \mid g(x) = x\}$ is either empty or a subtree of $T$.

2. If $g$ is hyperbolic, $g$ acts on a unique simplicial line in $T$ by translation.

   This line is called the axis of $g$.

3. (Serre’s lemma) Assume that $G$ is generated by a finite set of elements $\{s_j\}_{1 \leq i \leq m}$ such that every element and the multiplication of every two elements are elliptic. Then there is $x \in T$ which is fixed by every element of $G$.

**Lemma 4.2.** Let $G$ be a group acting on $T$. If $g$ and $h$ are elliptic and satisfy $gh = hg$, then $g$ and $h$ have a common fixed point.

**Proof.** Let $g$ and $h$ be elliptic elements which satisfy $gh = hg$. Assume to the contrary that $g$ and $h$ do not have a common fixed point. Fix $y \in \text{Fix}(h)$. Let $[y : x]$ be the shortest geodesic joining $y$ to $\text{Fix}(g)$. The composition of $g^{-1}([y : x])$ and $[y : x]$ is $[y : g^{-1}(y)]$. Now $g^{-1}(y) \in \text{Fix}(h)$, because $h^{-1}g^{-1}(y) = g^{-1}h^{-1}(y) = g^{-1}(y)$. By Lemma 4.1(1), $[y : g^{-1}(y)] \subset \text{Fix}(h)$. Therefore $x \in \text{Fix}(h)$. This contradicts our assumption. \[\square\]

We define $X_{1,0}, X_{d',0}, C_{d',0}, \pi_0, \overline{\pi}_0 \in nV$ ($2 \leq d' \leq n$) as shown in the following figure. For $i \geq 1$, $X_{d,i}$ ($1 \leq d \leq n$) is defined inductively. On $I_r$, $X_{d,i}$ restricts to the identity. For $x \in I_l$, we write $x = (x_1, x_2)$ where $x_1 \in [0,1/2)$ and $x_2 \in \mathbb{R}^{n-1}$. We define $\phi : I_l \rightarrow I^n$ by $\phi(x_1, x_2) = (2x_1, x_2)$. On $I_l$, $X_{d,i} = X_{d,i-1} \phi$. Similarly, $C_{d',i}, \pi_i$ and $\overline{\pi}_i$ restricts to the identity on $I_r$ and $C_{d',i-1} \phi$, $\pi_{i-1} \phi$ and $\overline{\pi}_{i-1} \phi$ respectively on $I_l$.
Theorem 4.3 (Hennig and Matucci [10, Theorem 23]). Let

\[ \Sigma = \{ X_{d,i}, C_{d',i}, \pi_i, \pi_j \} \text{ where } 1 \leq d \leq n, 2 \leq d' \leq n, i \geq 0. \]  

(1) \( \Sigma \) is a generating set of \( nV \).

(2) The elements of \( \Sigma \) satisfy the following relations.

\begin{align*}
X_{d'',j}X_{d,i} &= X_{d,i}X_{d'',j+1} & (i < j, 1 \leq d, d'' \leq n) \quad (4.2) \\
C_{d',j}X_{d,i} &= X_{d,i}C_{d',j+1} & (i < j, 1 \leq d \leq n, 2 \leq d' \leq n) \quad (4.3) \\
Y_jX_{d,i} &= X_{d,i}Y_{j+1} & (i < j, Y \in \{ \pi_i, \pi_j \}, 1 \leq d \leq n) \quad (4.4) \\
\pi_jX_{d,i} &= X_{d,i}\pi_j & (i > j + 1, 1 \leq d \leq n) \quad (4.5) \\
\pi_jC_{d',i} &= C_{d',i}\pi_j & (i > j + 1, 2 \leq d' \leq n) \quad (4.6) \\
\pi_j\pi_i &= \pi_i\pi_j & (|i - j| > 2) \quad (4.7) \\
\pi_j\pi_i &= \pi_i\pi_j & (j > i + 1) \quad (4.8) \\
\pi_iX_{1,i} &= \pi_i\pi_{i+1} & (i \geq 0) \quad (4.9) \\
C_{d',i}X_{d,i} &= X_{d,i}C_{d',i+2\pi_{i+1}} & (i \geq 0, 2 \leq d' \leq n) \quad (4.10) \\
\pi_iX_{d,i} &= X_{d,i+1}\pi_i\pi_{i+1} & (i \geq 0, 1 \leq d \leq n) \quad (4.11)
\end{align*}

Corollary 4.4. Let

\[ S = \{ X_{d,1}, X_{d,1}(X_{d,0})^{-1}, C_{d',2}, \pi_0, \pi_3, \pi_3 \} \text{ where } 1 \leq d \leq n, 2 \leq d' \leq n. \]  

This is a generating set of \( nV \).
Proof. $\langle S \rangle$ denotes a subgroup generated by $S$. $X_{d,0} \in \langle S \rangle$. For $i \geq 2$, the relation (4.2) shows that $X_{d,i} = (X_{d,0})^{-(i-1)}X_{d,1}(X_{d,0})^{i-1} \in \langle S \rangle$.

Similarly, the relation (4.3) shows that $Y_i = (X_{d,0})^{-(i-3)}Y_3(X_{d,0})^{i-3} \in \langle S \rangle$ for $i \geq 1$, where $Y$ is $\pi$ or $\overline{\pi}$. By the relation (4.2), $\pi_0 \in \langle S \rangle$.

The relation (4.4) shows that $C_{d,i} = (X_{d,0})^{-(i-2)}C_{d,2}(X_{d,0})^{i-2} \in \langle S \rangle$ for $i \geq 1$. By the relation (4.10), $C_{d,0} \in \langle S \rangle$.

The next lemma is a generalization of Lemma 4.2 in [8].

**Lemma 4.5.** Let $g \in nV$ which acts identically on some rectangle. If $nV$ acts on $T$, $g$ is elliptic.

**Proof.** Let $g \in nV$ be an element with a rectangle $R$ on which $g$ acts as the identity. Assume to the contrary that $g$ is hyperbolic. We write $l_g$ for the axis of $g$. Let

$$H_g = \{ h \in nV \mid \text{supp}(h) \subseteq R \} \cong nV. \quad (4.13)$$

For every $h \in H_g$, $hg = gh$ and $g$ acts on $h(l_g)$ as a translation. By the uniqueness of the axis, $h(l_g) = l_g$. Restricting the action of $h$ on $l_g$, we regard $h$ as an element of the infinite dihedral group $D_\infty$. In this way we obtain a homomorphism $\Phi : H_g \to D_\infty$. By the simplicity of $H_g$, $\ker \Phi$ is $H_g$ or the trivial subgroup. We claim that $\ker \Phi$ is not trivial. Indeed, $H_g$ has the subgroup which is isomorphic to the Thompson group $F$. $\ker \Phi$ contains the commutator subgroup of this subgroup, because every proper quotient of $F$ is abelian (8). Hence $\ker \Phi = H_g$.

There is $k \in nV$ such that $k \cdot \text{supp}(g) \subseteq R$. For this $k$, $kgk^{-1} \in H_g$. Therefore, $kgk^{-1}$ is elliptic, which contradicts our assumption that $g$ is hyperbolic.

The following theorem is the main result in this section.

**Theorem 4.6.** $nV$ has property FA. Especially, $e(nV) = 1$. 
Proof. Let $S$ be the generating set of $\{4.12\}$. By Serre’s lemma, it is enough to show that every element and the product of every two elements of $S$ are elliptic. By Lemma 4.5 every element of $S$ is elliptic.

$S = S_1 \cup S_2$, where

\[
S_1 = \{X_{d,1}, C_{d',2}, \pi_3, \pi_3\}_{1 \leq d \leq n, 2 \leq d' \leq n}, \tag{4.14}
\]

\[
S_2 = \{X_{d,1}(X_{d,0})^{-1}, \pi_0\}_{1 \leq d \leq n} \tag{4.15}
\]

Every element of $S_1$ acts as the identity on $I_r$. Every element of $S_2$ acts as the identity on the “left quarter” of the unit cube, $[0, 1/4) \times I^{n-1}$. Therefore, Lemma 4.5 shows that the product of every two elements in $S_1$ ($i = 1, 2$) is elliptic.

Next we consider $S_1' = \{C_{d',2}, \pi_3, \pi_3\}_{2 \leq d' \leq n} \subset S_1$. The relations $\{4.3\}$, $\{4.4\}$ and $\{4.5\}$ imply that $X_{d,1}(X_{d,0})^{-1}$ and $Z \in S_1'$ are commutative. In fact,

\[
X_{d,1}(X_{d,0})^{-1}Z(X_{d,1}(X_{d,0})^{-1})^{-1}Z^{-1} = (X_{d,1}(X_{d,0}^{-1}ZX_{d,0})X_{d,1}^{-1})Z^{-1} = 1.
\]

The relations $\{4.6\}$, $\{4.7\}$ and $\{4.8\}$ say that $\pi_0$ and the elements of $S_1$ are commutative. Thus Lemma 4.5 shows the products of $Z \in S_1'$ and $X_{d,1}(X_{d,0})^{-1}$ or $\pi_0$ are elliptic.

The rest of the proof is to show the following lemma. \hfill $\Box$

**Lemma 4.7.** For every $d, d'' \in \{1, \ldots, n\}$,

1. $X_{d,1}$ and $X_{d'',1}(X_{d'',0})^{-1}$ have a common fixed point.
2. $X_{d,1}$ and $\pi_0$ have a common fixed point.

**Proof.** (1) $X_{d'',1}$ and $X_{d,2}$ act identically on $I_r$. By Lemma 4.5 and Serre’s lemma, there exists $y \in T$ which is fixed by $X_{d'',1}$ and $X_{d,2}$.

To obtain a contradiction, suppose $X_{d,1}$ and $X_{d'',1}(X_{d'',0})^{-1}$ do not have a common fixed point. Take the shortest geodesic $[y : x]$ joining $y$ to $\text{Fix}(X_{d'',1}(X_{d'',0})^{-1})$. The composition of $(X_{d'',1}(X_{d'',0})^{-1})^{-1}[y : x]$ and $[y : x]$ is $[y : X_{d'',0}(y)]$. By the relation $\{4.2\}$,

\[
X_{d,1}X_{d'',0}(y) = X_{d'',0}(X_{d'',0})^{-1}X_{d,1}X_{d'',0}(y) = X_{d'',0}X_{d,2}(y) = X_{d'',0}(y).
\]

Therefore, $[y : X_{d'',0}(y)] \subset \text{Fix}(X_{d,1})$ and $x \in \text{Fix}(X_{d,1})$. This contradicts our assumption.

(2) We first show that $\pi_0$ and $X_{d,0}$ have a common fixed point. To obtain a contradiction, suppose $\pi_0$ and $X_{d,0}$ do not have a common fixed point. By (1), $\text{Fix}(X_{d,0})$ is not empty. We consider a new element $(X_{d,0})^{-1}X_{d,1}$, which acts as the identity on the left one-eighth of $I^n$. Since $\pi_0$ and $\pi_1$ also act as the identity on this rectangle, we can take $y$ as a common fixed point of $(X_{d,0})^{-1}X_{d,1}, \pi_0$ and $\pi_1$. There is the shortest geodesic $[y : x]$ joining $y$ to
Fix$(X_{d,0})$. The composition of $[y : x]$ and $X_{d,0}([x : y])$ is $[y : X_{d,0}(y)]$. By the relation \(4.11\),

\[\pi_0 X_{d,0}(y) = X_{d,1} \pi_0 \pi_1(y) = X_{d,1}(y) = X_{d,1}((X_{d,0})^{-1} X_{d,1})^{-1}(y) = X_{d,0}(y).\]

Therefore, $[y : X_{d,0}(y)] \subset \text{Fix}(\pi_0)$ and $x \in \text{Fix}(\pi_0)$. This contradicts our assumption.

We consider a subgroup generated by \(\{\pi_0, (X_{d,0})^{-1} X_{d,1}, X_{d,0}\}\). By Serre’s lemma, this subgroup has a fixed point. Therefore, \(\text{Fix}(\pi_0) \cap \text{Fix}(X_{d,1})\) is not empty. \(\square\)

5 \(nV\) has subgroups with infinitely many relative ends

The argument in this section follow Farley’s paper [8] closely.

**Lemma 5.1** (Farley [7, Lemma 2.2]). Let $P$ be a 1-dimensional pattern. $C_P = \{x_i\}_{0 \leq i \leq r}$ is the set of corners of rectangles in $P$. Let $R = (a, b)$ be a 1-dimensional rectangle.

If $R$ has nonempty intersection with some rectangle in $P$, then $a, b \in C_P$.

The next theorem is a generalization of Proposition 3.2 in [8].

**Theorem 5.2.** Let $H = \{h \in nV \mid h \text{ acts as the identity on } I_r\}$. $H$ is a subgroup of $nV$. Let $X = \{kH \in nV/H \mid k \in nV, k \text{ is affine on } I_r\}$.

1. $H$ has infinite index in its normalizer $N_G(H)$. The quotient group $N_G(H)/H$ is not virtually infinite cyclic.

2. Both $X$ and its complement are infinite, and $X$ is almost invariant under the left action of $nV$ on $nV/H$.

Therefore, we get $e(G, H) = \infty$.

**Proof.** (1) We first prove that $H$ has infinite index in $N_G(H)$. Let $\overline{H}$ denote the subgroup of $nV$ which acts as the identity on $I_r$. For every $\overline{h} \in \overline{H}$, $\overline{h}^{-1} \overline{H} \overline{h} = H$ and $\overline{H} \subset N_G(H)$. If $\overline{h}_1$ and $\overline{h}_2$ in $\overline{H}$ satisfy $\overline{h}_1 H = \overline{h}_2 H$, $\overline{h}_1 = \overline{h}_2$. Therefore, $\overline{H}$ can be regarded as a subgroup of $N_G(H)/H$. Since $\overline{H}$ is infinite, $H$ has infinite index in $N_G(H)$.

Suppose $N_G(H)$ has a finite index subgroup which is isomorphic to the infinite cyclic group. We write $\mathbb{Z}$ for this subgroup. According to a basic fact of the group theory, $[N_G(H)/H : \mathbb{Z}] \geq [\overline{H} : \overline{H} \cap \mathbb{Z}]$. Hence $\overline{H} \cap \mathbb{Z}$ is a finite index subgroup of $\overline{H}$. By definition, $\overline{H}$ is isomorphic to $nV$, and $\overline{H}$ is simple. Since $nV$ is an infinite simple group, it does not contain a proper finite index subgroup. Therefore, $\overline{H} \cap \mathbb{Z} = \overline{H}$. This is impossible, because $\overline{H}$ is not abelian.
(2) It is easily seen that $X$ and $X^c$ are infinite. In fact, an element $kH$ of $nV/H$ is characterized by $k(I_l)$, and there are infinitely many choices of a rectangle $k(I_l)$. This shows that $X$ is infinite. Similar considerations apply to $X^c$.

Fix $g \in nV$ arbitrarily. Let $\pi_g = \chi_{gX} - \chi_X$ be a function on $nV/H$, where $\chi$ denotes the characteristic function.

We will rewrite $\pi_g$, using $X_P$ and $X_Q$ defined below. Let $(P, Q)$ be the pair of patterns which represents $g$. As in Lemma 5.1 we write $CP$ (resp. $C_Q$) for the set of corners in $P$ (resp. $Q$). Let

$$X_P = \{ kH \in X \mid k(I_l) \cap CP = \emptyset \}, \quad (5.1)$$

$$X_Q = \{ kH \in X \mid k(I_l) \cap C_Q = \emptyset \}. \quad (5.2)$$

Since every element of $nV$ is a bijection, $h \in H$ maps the $I^n - I_l$ onto itself. Therefore, the image of $I_l$ by $k$ depends only on the coset $kH$. This shows that $X_P$ and $X_Q$ are well-defined.

For $kH \in X_P$, $g \cdot k(I_l)$ does not contain an element of $Q$ and $gkH \in X_Q$. Similarly, the multiplication of $g^{-1}$ from the left defines a map from $X_Q$ to $X_P$. It follows that $g(X_P) = X_Q$. Therefore,

$$\pi_g = (\chi_{g(X-X_P)} + \chi_{g(X_P)}) - (\chi_{(X-X_Q)} + \chi_X)$$

$$= \chi_{g(X-X_P)} - \chi_{(X-X_Q)}. \quad (5.3)$$

If $X - X_P$ and $X - X_Q$ are finite sets, then $X$ is almost invariant. To prove that $X - X_P$ is finite, it is enough to show that the following function $f_P$ is injective. Let $Pr_i(C_P) = \{ \text{the } i\text{-th coordinate of a corner of } R \mid R \in P \}$. Define a function $f_P : X - X_P \to (Pr_1(C_P) \times \cdots \times Pr_n(C_P))^n$ by

$$f_P(kH) = (kH(\alpha_1), \ldots, kH(\alpha_i), \ldots, kH(\alpha_n)). \quad (5.4)$$

Here, $\alpha_1 \in I^n$ is the element whose first coordinate is 1/4, and the other ones are all 0. For $i \geq 2$, $\alpha_i$ is the element whose $i$-th coordinate is 1/2, and the other ones are 0. By applying Lemma 5.1 repeatedly, $kH(\alpha_i) \in Pr_1(C_P) \times \cdots \times Pr_n(C_P)$ for all $i \geq 1$. This shows that $f_P$ is well-defined.

Since $k$ is affine on $I_l$, $k(I_l)$ is characterized by $f_P(kH)$. Therefore, if there exist $k_1H$ and $k_2H$ which satisfy $f_P(k_1H) = f_P(k_2H)$, then $k_1(I_l) = k_2(I_l)$, which implies $k_1H = k_2H$. This proves $f_P$ is injective and that $X - X_P$ is a finite set.

By the same discussion, $X - X_Q$ is also finite. Therefore, $|X \Delta gX| < \infty$ for all $g \in X$.

6 Relevant properties

A finitely generated group $G$ has property $(T)$ if every affine isometric action of $G$ on any Hilbert space has a fixed point.
$G$ has the *Haagerup property* if there exists a proper affine isometric action of $G$ on a Hilbert space.

Let $S$ be a countable set, equipped with the counting measure $\mu$. For $p > 0$, we define the norm space $l^p(S)$ to consist of all functions $f : S \to \mathbb{C}$ satisfying $\|f\|_p = \left( \int |f|^p d\mu \right)^{1/p} < \infty$. $l^2(S)$ is a Hilbert space. One can also define $l^\infty(S)$ as the set of essentially bounded $\mathbb{C}$-valued functions on $S$.

**Corollary 6.1.** For all $n \geq 1$,

1. $nV$ does not have the property $(T)$.
2. $nV$ has the Haagerup property.

**Proof.** (1) We use the same notations as in the proof of Theorem 5.2. The function $\pi : nV \to l^\infty(nV/H)$ is defined by $\pi(g) = \pi_g$. For each $g \in nV$, the support of $\pi(g)$ is finite. Since $\|\pi(g)\|_2 = |X_\Delta g X|^1/2$, $\pi(g) \in l^2(nV/H)$. Therefore, $\pi$ gives an affine isometric action of $nV$ on $l^2(nV/H)$ by $f \mapsto f + \pi(g)$. If $\pi$ has a global fixed point, $X = gX$ for all $g \in X$. Since $H \in X$, for all $g \in nV$, there exists $k_g H \in X$ which satisfies $g k_g H = H$. Therefore, $g^{-1} H \in X$ for all $g$ and $X = nV/H$; contradiction.

(2) We would confirm that the action on $l^2(nV/H)$ defined by $\pi$ is proper. It is enough to see that for all $m > 0$, there are only finitely many $g \in nV$ such that $\|\pi(g)\|_2 < m$.

Let $\alpha_i$ be the same elements as we used in (5.4). We consider a function $f_1 : X - X_P \to \text{Pr}_1(C_P) - \{0, 1, 1/2, 1/4\}$, which maps $kH$ to the first coordinate of $kH(\alpha_1)$. Similarly, for all $i \geq 2$ we consider the surjection $f_i : X - X_P \to \text{Pr}_i(C_P) - \{0, 1, 1/2\}$ which maps $kH$ to the $i$-th coordinate of $kH(\alpha_i)$.

For all $i \geq 1$, $f_i$ is a surjection. It follows that the number of rectangles in $P$ is less than $(m^2 + 4)^n$. When $m$ is fixed, there are finitely many $g = (P, Q)$ with such condition. Combined with (1), this shows that $\pi$ gives a proper affine isometric action of $nV$ on $l^2(nV/H)$. $$\square$$

A group $G$ is a *Kähler group* if there is a connected compact Kähler manifold $M$ whose fundamental group is $G$.

**Theorem 6.2.** $nV$ is not Kähler.

We recall the following result by Napier and Ramachandran.

**Theorem 6.3** (Napier and Ramachandran [12, Corollary 3.7(a)]). Suppose $M$ is a connected complete Kähler manifold which has bounded geometry. If $M$ has at least three ends, then there exists a proper holomorphic mapping of $M$ onto a hyperbolic surface.

In our context, this theorem can be used to derive the following.
Theorem 6.4 (Napier and Ramachandran \[12\]). If a Kähler group $G$ has a subgroup $H$ such that $e(G, H) \geq 3$, then $H$ must have a quotient that is isomorphic to a hyperbolic surface group.

Proof. Suppose that $G$ is the fundamental group of a connected compact Kähler manifold $M$, and that $G$ has a subgroup $H$ satisfying $e(G, H) \geq 3$. According to \[9\], page 303, $e(M(H)) = e(G, H)$, where $M(H)$ is the covering space of $M$ whose fundamental group is $H$. We may confirm that $M(H)$ is a connected complete Kähler manifold with bounded geometry. By Theorem 6.3, there is a holomorphic map from $M$ onto a hyperbolic surface $S$. The kernel of this map gives the normal subgroup $N$ of $\pi_1(M(H)) = H$ such that $H/N$ is isomorphic to a hyperbolic surface group $S$.

Proof of 6.2. Suppose that $nV$ is the fundamental group of a compact Kähler manifold $M$. We take $H$ which satisfies $e(nV, H) = \infty$, as in the proof of Theorem 6.2. By Theorem 6.4, there is a normal subgroup $N$ of $H$ such that $H/N$ is isomorphic to a hyperbolic surface group $S$. Since $H$ is simple, $H/N$ is $H$ and its commutator subgroup is trivial or $H$. In fact $[H, H] = H$, because $[nV, nV]$ contains non-identity elements, such as the elements of $[F, F]$ where $F$ is the Thompson group $F$. Therefore, the first homology group is trivial, $H_1(S) = \pi_1(S)/[\pi_1(S), \pi_1(S)] = 0$. This means the Euler number of $S$ is non-negative, which contradicts to the supposition that $S$ is a hyperbolic surface. This completes the proof.

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