Asymptotically Universal Crossover in Perturbation Theory with a Field Cutoff

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Abstract. We discuss the crossover between the small and large field cutoff (denoted $x_{\text{max}}$) limits of the perturbative coefficients for a simple integral and the anharmonic oscillator. We show that in the limit where the order $k$ of the perturbative coefficient $a_k(x_{\text{max}})$ becomes large and for $x_{\text{max}}$ in the crossover region, $a_k(x_{\text{max}}) \propto \int_{-\infty}^{x_{\text{max}}} e^{-A(x-x_0(k))^2} \, dx$. The constant $A$ and the function $x_0(k)$ are determined empirically and compared with exact (for the integral) and approximate (for the anharmonic oscillator) calculations. We discuss how this approach could be relevant for the question of interpolation between renormalization group fixed points.

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1. Introduction

The existence of different behaviors at different scales is a central difficulty in quantum field theory. This is clearly the case for QCD where the short distance behavior can be described using perturbation theory, while confinement and other non-perturbative phenomena appear at large distance. A similar situation is encountered in scalar field theory and spin models where, for instance, descriptions of the renormalization group (RG) flows near the high-temperature fixed point in terms of perturbative expansions about the Gaussian fixed point are in general problematic.

In general, the fact that aspects of the RG flows cannot be described using perturbation theory is not surprising given that perturbative series usually have a zero radius of convergence. The convergence of perturbative series change drastically if the large field configurations are removed from the path integral. This can be implemented, for instance, by removing configurations such that the norm of local fields exceeds some cutoff value. The field cutoff modifies the original problem, however, it has been found that for nontrivial $\phi^4$ problems the modified series apparently converge toward values which are exponentially close to the exact ones. In addition, at fixed order, it is possible in simple examples to choose the field cutoff in order to optimize the accuracy.

The calculation of the perturbative series for the modified theories where a large field cutoff is introduced is non-trivial. In the case of the anharmonic oscillator, approximate analytical methods have been developed in the limits of large and small field cutoffs. Except at very low order, these methods are not accurate in the crossover region between these two regimes and it is imperative to develop new methods that are accurate in the crossover region. Fortunately, remarkable regularities were observed empirically: with appropriate rescalings and translations, the functions expressing the perturbative coefficients in terms of the field cutoff approximately collapse along a single curve. This suggests that despite the complexity of the perturbative coefficients, a simple description of the crossover seems possible.

In this article, we refine the description of the approximate collapse of the crossover functions discussed above. An important feature is that the collapse improves as the order increases and a universal (order independent) function is reached asymptotically. In order to model properly this approach, we start with a simpler example, namely the perturbative series associated with a one variable integral. This is done in Sec. where a very good agreement between an empirical polynomial parametrization and a saddle point approximation is found. Generalizations of these results are discussed in Sec. Our main focus in this article is to discuss the case of the anharmonic oscillator “with a field cutoff”. This model is introduced in Sec. and recent results regarding the behavior of the perturbative coefficients are briefly reviewed. Approximate Gaussian parametrizations of the crossover are introduced in Sec. Higher orders corrections are discussed in Sec. where a simple description of the crossover for large order emerges. The possible relevance of this approach for the description of RG flows is discussed in the conclusions.

2. The simple integral

In this section, we discuss the calculation of the simple integral

$$Z(\lambda) = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2 - \lambda x^4}.$$
In this example, introducing a field cutoff amounts to remove the integration tails where \(|x| > x_{\text{max}}\). Their contributions to the integral is less than \(\sqrt{2\pi e^{-\lambda x_{\text{max}}} x_{\text{max}}} \) and if \(x_{\text{max}}\) is large enough, we can approximate \(Z(\lambda)\) with

\[
Z(\lambda, x_{\text{max}}) \approx \int_{-x_{\text{max}}}^{x_{\text{max}}} dx e^{-\frac{1}{2} x^2 - \lambda x^4}.
\]  

If \(x_{\text{max}}\) is kept at a fixed finite value, the Taylor expansion of the exponential converges absolutely and uniformly over the domain of integration and it is legitimate to interchange the sum and the integral. The resulting series converges over the entire \(\lambda\) complex plane and reads:

\[
Z(\lambda, x_{\text{max}}) = \sum_{k=0}^{\infty} a_k(x_{\text{max}}) \lambda^k,
\]

with

\[
a_k(x_{\text{max}}) = \frac{(-1)^k}{k!} \int_{-x_{\text{max}}}^{x_{\text{max}}} dx e^{-\frac{1}{2} x^2} x^{4k}.
\]

On the other hand, in the limit where \(x_{\text{max}} \to \infty\), we have

\[
a_k(\infty) = \frac{(-1)^k}{k!} \Gamma(2k + \frac{1}{2})(2)^{2k+\frac{1}{2}},
\]

and the coefficients grow factorially with the order which implies that the series diverges for any non-zero value of \(\lambda\). In Fig. 1 we display the ratios

\[
R_k(x_{\text{max}}) = \frac{a_k(x_{\text{max}})}{a_k(\infty)}
\]

for \(k = 1\) to \(k = 10\). As we can see, the curves corresponding to the different orders have a similar shape and move to the right as the order increases. Each curve can be characterized by three regimes: the low field cutoff regime, where the coefficient becomes very small, the large field cutoff regime where the coefficient reaches its asymptotic value and the crossover region. We now discuss each case separately.

2.1. The \(x_{\text{max}} \to \infty\) limit

When \(x_{\text{max}} \to \infty\), we can use integration by part recursively for the missing tails of integration in Eq. 1:

\[
1 - R_k(x_{\text{max}}) \simeq e^{-\frac{1}{2} x_{\text{max}}^2} \left[ \frac{\left(\frac{1}{2} x_{\text{max}}^2\right)^{2k+\frac{1}{2}}}{\Gamma(2k + \frac{1}{2})} + \frac{\left(\frac{1}{2} x_{\text{max}}^2\right)^{2k-\frac{1}{2}}}{\Gamma(2k - \frac{1}{2})} + \cdots \right].
\]

At each integration by part, the argument of the \(\Gamma\) function at the denominator decreases by one. Since the arguments are half integers, the process does not terminate. Using \(1/\Gamma(x-1) = (x-1)/\Gamma(x)\), we end up with a factorial growth for the coefficients of the power series.

2.2. The \(x_{\text{max}} \to 0\) limit

In the opposite limit where \(x_{\text{max}} \to 0\), we expand the exponential in Eq. 4 and we obtain the converging series in \(x_{\text{max}}^2\):

\[
R_k(x_{\text{max}}) = \frac{2^{\frac{1}{2} x_{\text{max}}^2} (2k + \frac{1}{2})}{\Gamma(2k + \frac{1}{2})} \sum_{l=0}^{\infty} \frac{(-1)^l (2k+2l+1)!}{(4k+2l+1)!}.
\]
universal crossover in improved perturbation theory

Figure 1. The ratio $R_k(x_{\text{max}})$ as defined in Eq. (6). As $k$ goes from 1 to 10, the curves move to the right.

Figure 2. The ratio $R_k(x_{\text{max}})$ in the large $x_{\text{max}}$ and small $x_{\text{max}}$ approximations. The dot stands for the accurate numerical result. The dash lines correspond to the large $x_{\text{max}}$ approximation. The solid lines correspond to small $x_{\text{max}}$ approximation. The labels 1, 2, 3 correspond to the number of terms in Eq. (7). The six lines on the bottom left correspond to polynomial expansions when $x_{\text{max}} \to 0$ from Eq. (8) with truncations at order 0, 5, 10, 15, 20, 25.

It is clear that the factorial growth of the denominator implies that this series converges over the entire $x_{\text{max}}$ complex plane.

In Fig. 2 we show the curves corresponding to different orders in the two expansions discussed above, for $k = 5$. The small $x_{\text{max}}$ region needs to be resolved logarithmically. If we restrict the expansion in Eq. (8) to $l \leq L$, for some given $L$, we notice that the truncated series becomes a poor approximation when $x_{\text{max}}$ reaches some critical value (almost independent of the order $k$ in $\lambda$). This is illustrated in Fig. 3 for $L = 10$, where the critical value is near $x_{\text{max}} \approx 2.7$. One can roughly estimate
this critical value by requiring that the $L$-th term is the same order as the $L-1$-th term. This yield the critical value $x_{\text{max}} \simeq \sqrt{2L}$. A more accurate way to proceed is to determine the value of $x_{\text{max}}$ for which the $L$-th order approximation becomes poor. The critical values of $x_{\text{max}}$ can be fitted reasonably well with $0.63 + 0.64\sqrt{L}$. We can thus conclude that despite the fact that the series is convergent, many terms need to be calculated when the order becomes large. The number of terms that we need to calculate in order to get an accurate answer for a given $x_{\text{max}}$ grows like $x_{\text{max}}^2$.

2.3. Approximate universality

Fig. 2 and the above discussion show that it is not easy to use the two expansions in the crossover region, which corresponds to the region where the integrand of Eq. (4) peaks. Using a saddle point approximation about the maximum of the integrand $x = \sqrt{4k}$, we obtain

$$R_k(x_{\text{max}}) \simeq \int_0^{x_{\text{max}}} dx_0 e^{-2k+4k\ln(\sqrt{4k})} e^{-(x_0-\sqrt{4k})^2 + \cdots}.$$  \hspace{1cm} (9)

If we only retain the quadratic term in the above expansion of the phase, we obtain what we call hereafter the Gaussian approximation. This approximation fails to reproduce the small $x_{\text{max}}$ behavior of Eq. (8). It is clear that far from the peak of the integrand, the Gaussian approximation is not good but it is in regions where the original integrand is very small. One can simplify this expression by adding the left tails of integration. This makes sense for $k$ not too small, where it generates errors of the order $(4\sqrt{k})^{-1}e^{-4k}$. For sufficiently large $k$, the following approximation holds for $x_{\text{max}}$ not to far from $\sqrt{4k}$

$$R_k(x_{\text{max}}) \simeq \pi^{-1/2} \int_{-\infty}^{x_{\text{max}}} dx e^{-(x-\sqrt{4k})^2}. \hspace{1cm} (10)$$

![Figure 3. The numerical values (dots) of log$_{10}$ $R_k$ and the approximate values (lines) obtained with the small $x_{\text{max}}$ expansion Eq. (8) with 10 terms for $k = 1$ to $k = 10$.](image)
In Eq. (10), the only dependence on $k$ is in the argument of the exponential. Its net effect is a translation in $x_{\text{max}}$ and the shape of the function is universal ($k$-independent). In other words,

$$R_k(x_{\text{max}}) \approx U_{\text{int}}(x_{\text{max}} - \sqrt{4k})$$  \hspace{1cm} (11)

with

$$U_{\text{int}}(x) \equiv \int_{-\infty}^{x} dy e^{-y^2/2}. \hspace{1cm} (12)$$

If the assumption of universality expressed in Eq. (11) is approximately correct, the data should collapse into the universal curve when the argument of the $k$-th order curve is shifted by $\sqrt{4k}$. Fig. 4 confirms that this is approximately the case. A more detailed look at Fig. 4 indicates that as the order increases, the shifted curve gets closer and closer to $U_{\text{int}}$. This limit is studied in the next section.

2.4. Order-dependent corrections

The basic approximation used to obtain $U_{\text{int}}$ in the previous section is the saddle point approximation of the integral. We expanded the phase of the integrand

$$\Phi_k(x_{\text{max}}) \equiv \ln R_k'(x_{\text{max}})$$  \hspace{1cm} (13)

about its maximum up to second order. It is possible to improve this approximation by expanding to higher order

$$-\frac{1}{2}x^2 + 4k \ln x = 2k(2 \ln(2\sqrt{k}) - 1) - (x - \sqrt{4k})^2 - 4k \sum_{n=3}^{\infty} (1/n) \left( \frac{\sqrt{4k} - x}{\sqrt{4k}} \right)^n. \hspace{1cm} (14)$$

The series converges for $0 < x < 2\sqrt{k}$. The even terms are negative and if we truncate at even orders, the integral of the exponential of the truncated sum converges when
the bounds of integration are sent to infinity. As the order increases, the contribution of the two regions outside the radius of convergence are effectively cutoff, because the phase $\Phi_k$ becomes large and negative in these regions. For $x < 0$, this is desirable because there is no contribution from this region in the original integral. On the other hand, contributions from $x > 2\sqrt{4k}$ exist in the original integral. However, as we move from the maximum of the phase at $\sqrt{4k}$ to the edge of the radius of convergence at $2\sqrt{4k}$, the phase of the integrand drops by $(-6 + 4\ln 2)k \simeq -3.2k$ and the contribution of the right tail in the original integral is small, even for relatively small values of $k$.

In order to check the validity of this approach, we have calculated the phase $\Phi_k$ by using a discretized approximation of the derivative of $R_k$. We then fitted a set of points near the maximum with polynomials. The results were quite robust under changes in the number of points used, the degree of the polynomial and the spacing used to calculate the derivative. The second coefficient stabilized rapidly between -0.99 and -1 in agreement with Eq. (14). The coefficients of order 3 and 4 are shown in Fig. 5 where the good agreement with Eq. (14) is quite obvious.

The inclusion of the corrections of order 3 and 4 in the phase significantly reduces the small differences between $U_{int}$ and the data in Fig. 4 and, for instance for $k = 5$, it is impossible to distinguish between the corrected integral and the data with the naked eye.
It is now clear how and why the agreement between the data and the simple guess $U_{int}$ of Eq. (12) gets better as the order increases. The size of the crossover region is of order 1 and controlled by the quadratic term in the phase. In this region, for $|x - \sqrt{4k}| \sim 1$, $\Phi_k \simeq \text{const.}$ because corrections of order $n > 2$ in Eq. (14) are suppressed by a factor $k^{1-n/2}$. On the other hand the argument does not hold far away from the crossover, for $|x - \sqrt{4k}| \sim \sqrt{k}$, where a large number of terms is necessary to approximate logarithms in the phase that reproduce the correct power behavior of the integral.

3. Remarks about the saddle point approximation

In this section, we generalize some aspects of the saddle point approximation used in the previous section. One important feature of the expansion of the phase given in Eq. (14) is that the coefficient of the quadratic term is $k$-independent and twice the value of the coefficient of the quadratic term in the original integral. This has a very simple explanation. In general, $e^{-Ax^2}x^B$ is maximal at $x^* = \sqrt{B/(2A)}$ and we have the expansion

$$e^{-Ax^2}x^B \propto e^{-2A(x-x^*)^2+...}. \quad (15)$$

In doing this calculation, one realizes that in the second derivative of the log estimated at $x^*$, the factors $B$ cancel exactly. We want to emphasize that this “non-renormalization” is due to the fact that the exponential is multiplied by a single power. If we consider instead $e^{-Ax^2}(x^B + c x^B + \delta)$, then the exact cancellation does not hold. This can be checked at first order in $\epsilon$, where $x^*$ is changed according to

$$\sqrt{B/(2A)} \rightarrow \sqrt{B/(2A)} \left[1 + (\epsilon \delta/(2B))(B/(2A))^{\delta/2} \right], \quad (16)$$

and the coefficient of the quadratic term in the expansion of the phase is changed according to

$$-2A \rightarrow -2A \left[1 - (\epsilon \delta/(2B))(B/(2A))^{\delta/2} \right], \quad (17)$$

and becomes $B$-dependent (except if $\delta = 2$ or 0).

4. The anharmonic oscillator with a field cutoff

Our main objective in this article is to understand the crossover for the anharmonic oscillator. The model has been studied in detail in Ref. [7]. We introduce some notations and briefly review some basic results. For convenience, we use quantum mechanical notations instead of field theoretical ones. The “field” will be denoted $x$ instead of $\phi$ and the field cutoff will be denoted $x_{max}$. In quantum mechanics language, it means that the potential becomes infinite at $\pm x_{max}$. We use units such that $\hbar$, $\omega$, and the “mechanical mass” $m$ are 1. If the dependence on these 3 quantities were restored, dimensionless quantities would be expressed in terms of $\sqrt{\omega m/\hbar x_{max}}$ and $\hbar \lambda/m^2\omega^3$. The Hamiltonian reads

$$H = \frac{p^2}{2} + V(x) \quad (18)$$

with

$$V(x) = \begin{cases} \frac{1}{2}x^2 + \lambda x^4 & \text{if } |x| < x_{max} \\ \infty & \text{if } |x| \geq x_{max} \end{cases}, \quad (19)$$
We will consider the perturbative expansion of the ground state:

\[ E_0(x_{\text{max}}) = \sum_{k=0}^{\infty} E_0^{(k)}(x_{\text{max}}) \lambda^k . \]  

(20)

As in section 2, we use the notation \( R_k \) for the \( k \)-th coefficient in units of its usual value:

\[ R_k(x_{\text{max}}) \equiv E_0^{(k)}(x_{\text{max}})/E_0^{(k)}(\infty) . \]  

(21)

We have shown [7] that in the limit of small \( x_{\text{max}} \),

\[ R_k(x_{\text{max}}) \propto x_{\text{max}}^{6k-2} , \]  

(22)

while in the limit of large \( x_{\text{max}} \),

\[ 1 - R_k(x_{\text{max}}) \propto x_{\text{max}}^{4k+1} e^{-x_{\text{max}}^2} . \]  

(23)

5. Gaussian approximations of the crossover

In this section, we discuss Gaussian approximations of the crossover in a way similar to what was done in Sec. 2.3 for the integral. From Eq. (23), we expect that in the large \( x_{\text{max}} \) limit, we have at leading order

\[ R_k'(x_{\text{max}}) \propto x_{\text{max}}^{4k+2} e^{-x_{\text{max}}^2} . \]  

(24)

If we now extrapolate backward this behavior at intermediate values of \( x_{\text{max}} \) (this is of course an approximation), \( R_k'(x_{\text{max}}) \) has a maximum at

\[ x^*(k) \equiv \sqrt{2k+1} . \]  

(25)

Expanding the logarithm up to order 2 about this point, we obtain the Gaussian approximation:

\[ R_k'(x_{\text{max}}) \propto e^{-2(x_{\text{max}}-x^*(k))^2} . \]  

(26)

With this approximation, we have \( R_k'(x^*(k)) = 0 \). This approximate prediction for the value of \( x \) for which the second derivative vanishes can be compared with the accurate numerical value, denoted \( x_0(k) \), for which it actually vanishes. The numerical values can be found in Table 2 of Ref. [7] and are shown in Fig. 6 together with approximate analytical formulas. Typically, \( x_0(k) \approx x^*(k) \) with a 10 percent accuracy for \( k \leq 10 \). A more accurate formula can be found by fitting \( x_0(k) \) with a linear function. Using the data for \( k = 2 \) to 10, we obtain

\[ x_0(k) \approx \sqrt{1.61k + 3.48} . \]  

(27)

In Ref. [7], we found that if \( R_k(x_{\text{max}}) \) is translated by \( x_0(k) \), the data for \( k = 2, \ldots 10 \) approximately collapses. In other words,

\[ R_k(x + x_0(k)) \simeq U_{\text{anh}}(x) . \]  

(28)

An approximate form for \( U_{\text{anh}}(x_{\text{max}}) \) is suggested by Eq. (20). Integrating and normalizing by requiring \( \lim_{x \to +\infty} U_{\text{anh}}(x) = 1 \), we obtain

\[ U_{\text{anh},1}(x) \simeq \sqrt{2/\pi} \int_{-\infty}^{\infty} dy e^{-2y^2} . \]  

(29)
We have used the subscript $anh, 1$ to indicate that it is a first approximation and that a better form will be found later. This proposal is compared with the empirical data for $k = 7, \ldots, 10$ in Fig. 7. We only displayed a few values of $k$ so that it is possible to distinguish the various orders on the graph. One can see that the collapse of the various curves is reasonably good. The function $U_{anh, 1}$ fits the collapsed data quite well on the upper right, but appears to be slightly above the data on the lower left. However, as the order increases (in the Figure from 7 to 10), the difference diminishes.

The reason why $\sqrt{2k + 1}$ is not a very good approximation of $x_0(k)$ is that the large $x_{max}$ behavior given by Eq. (24) is not accurate for intermediate values of $x_{max}$. Studying numerically the behavior of $R_k(x_{max}) e^{x_{max}^2}$, near $x_0(k)$, ones obtain a very good linear behavior on a log-log plot. If Eq. (24) were correct, the slope would be $4k + 2$. However, collecting the slopes for various values of $k$, one concludes that $4k + 2$ should be replaced by $3.2k + 7.8$ in Eq. (24). This is consistent with the empirical form of $x_0(k)$ of Eq. (27) which is approximately the square root of one half of this fit (see Sec. 3 with $A=1$.) When studying the difference between Eq. (30) and its approximation by a single power, one notices small quadratic corrections. We will not report all the details of this investigation, but only mention that these quadratic corrections can be approximately removed by replacing Eq. (30) by $R'_k(x_{max}) e^{\beta_k x_{max}^2}$ and adjusting the value of $\beta_k$ to the value for which the quadratic corrections change sign.

We conclude that a more realistic version of Eq. (24) is

$$R'_k(x_{max}) \propto x_{max}^{3.2k + 7.8} e^{-\beta_k x_{max}^2},$$

The results of Sec. 3 suggest that $e^{-2y^2}$ should be replaced by $e^{-2\beta_k y^2}$ in Eq. (29). This brings an explicit dependence on $k$, and universality can only be reached asymptotically if $\beta_k$ tends to a limit when $k$ becomes large.
Empirically, the values of $\beta_k$ are slightly larger than 1 for $k < 10$ and slightly smaller than 1 for $k > 10$. This modification allows to reduce the small discrepancies of the Gaussian approximation used before. Note that in Fig. 7, the closeness of $k = 9$ and 10 to $U_{\text{anh},1}$ reflects the fact that $\beta_9$ and $\beta_{10}$ are very close to 1. $\beta_k$ is closely related to $A_k^{(2)}/2$, a quantity that will be studied in the next section. Anticipating the results presented there, we expect that as $k$ increases, $\beta_k$ stabilizes to a value close to 0.75.

The limitations of the Gaussian approximation, or of any other expression of a universal function $U$ as the integral of an even function, can be seen from the duality relation which exchanges the small and large field cutoff regions

$$U(-x) = 1 - U(x),$$

which cannot be exact since the approach of 0 when $x_{\text{max}} \to 0$ is power like while in the large $x_{\text{max}}$ limit, the approach of 1 is exponentially small. In at least one of these two limits, $U$ should fail to provide the correct behavior.

6. Numerical study of the higher order corrections

The complexity of the exact form of the terms of the perturbative series increases rapidly with the order (see for instance [8] for exact expression up to order 4). The number of graphs grows factorially [9]. It seems unlikely, that at this point, one could guess approximate equations such as Eq. (31), or an improved version of it, from analytical expressions at large order. For this reason, we will describe $\ln(R_k)$ near $x_0(k)$ using polynomial approximations. This method is numerically robust and has been tested successfully in the case of the integral where the expansion can be
calculated analytically. Our main working assumption is that in the vicinity of $x_0(k)$:

$$\ln(R'_k(x)) \simeq \sum_{q=0}^Q A_k^{(q)}(x - x_0(k))^q$$

(33)

Note that the presence of odd powers in the expansion breaks the duality expressed in Eq. (32). The fitted value for the largest order ($q = Q$) coefficient is typically not very stable and as we want to focus on the terms with $q = 2$, $3$ and $4$, we have used $Q = 5$ and $6$. We then checked that the terms $A_k^{(q)}$ for $q = 2$, $3$ and $4$ were quite stable under small changes as in Sec. 2. They are displayed in Fig. 8.

As the $A_k^{(q)}$ seem to reach asymptotic values, we have fitted the larger $k$ values with the parametric form

$$A_k^{(q)} = a^{(q)} + b^{(q)}k^{-c^{(q)}}$$

(34)

Other parametric forms such as an exponential decay have been tried too but have been shown to have a significantly larger chi-square. We first estimated $c^{(q)}$ by making a log-log plot of $|A_k^{(q+1)} - A_k^{(q)}|$ and then obtained $a^{(q)}$ and $b^{(q)}$ with standard linear fit methods. We used different sets of $A_k^{(q)}$ with $k$ running from $k_0$ to $20$. The results showed a regular dependence on $k_0$: as $k_0$ increased, the three fitted parameters showed small, monotonous changes and we then attempted an extrapolation (to large $k_0$) using the same method as above. This second extrapolation is of course much more delicate and used only to estimate the errors. The stability of the procedure was checked by using five different sets of $A_k^{(q)}$ obtained with slightly different procedures: changes in the range of $x_{max}$ used in the fits, changes in the $\Delta x_{max}$ used to calculate the derivative of $R_k$, changes in $Q = 5$ or $6$.

For $q = 2$, the fits at fixed $k_0$ were found to be quite consistent among the five sets and the second extrapolation was in some cases possible. Our final estimate is

$$A_k^{(2)} \simeq -1.5(1) + 2.0(1)k^{-0.6(1)}$$

(35)

with the parenthesis indicating the estimated error on the last written digit. For sufficiently large $k$, we found that $A_k^{(2)} \simeq -2\beta_k$ as defined in the previous section. It is clear form Fig. 8 that for $k \simeq 10$, $A_k^{(2)} \simeq -2$ in agreement with the observation $\beta_1 \simeq 1$. To give an idea about the accuracy of the agreement, for $k = 15$, $A_1^{(2)} \simeq -1.866$ while $\beta_{15} = 0.938 = 1.876/2$.

For $q = 3$, more variations among the five data sets were observed. A tentative value for $a^{(3)}$ is 0.02(1). However this is more than 4 time less than $A_2^{(3)} \simeq 0.089(1)$. More stable results with consistent extrapolations were obtained by setting $a^{(3)} = 0$. Our final estimate is

$$A_k^{(3)} \simeq 1.3(3)k^{-0.9(1)}$$

(36)

We repeat that it is difficult to rule out a small positive value asymptotically. The decay rate seems faster than in the single power case studied in Sec. 3 where $A_k^{(3)} \propto k^{-1/2}$. In our fits the average value of the power was -0.94 for $k_0 = 17$ and the extrapolated value the farther away from this average was -0.86.

For $q = 4$, stronger variations among the five sets were observed. In all cases $a^{(4)}$ was much smaller than $A_2^{(4)} \simeq -0.017(1)$ and with no consistent sign. More stable results with reasonably consistent extrapolations were obtained by setting $a^{(4)} = 0$. Our final estimate is

$$A_k^{(4)} \simeq -1.5(6)k^{-1.3(3)}$$

(37)
We have also considered the coefficients for larger values of $q$ and found that they also appear to reach zero asymptotically. Their values are in general quite small. For instance, $A_{20}^{(5)} \simeq 0.004$ and $A_{20}^{(6)} \simeq -0.0007$.

We conclude that for $k$ sufficiently large and $x_{\max} = x_0(k) + x$, with $x$ of order 1, we have in good approximation that

$$R_k(x_0(k) + x) \simeq U_{\text{anh}}(x) \simeq \sqrt{2A/\pi} \int_{-\infty}^{x} dy e^{-2A y^2},$$

(38)
with $A \simeq 0.75$. The words of caution regarding this “limit” are the same as in the case of the integral: if $|x|$ is too large, the non-Gaussian corrections need to be taken into account.

7. Conclusions

In summary, we have considered the simplest field theoretical models with a field cutoff, in zero and one dimension. We found in both cases, that the crossover behavior of the perturbative coefficient can, for orders large enough, be described in good approximation by rescaling and translating a universal (order-independent) function. Furthermore, this function can be expressed as a Gaussian integral. This statement is exact for the simple integral (in the sense of Eq. (14)). For the anharmonic oscillator, small non-Gaussian corrections persisting at large order cannot be ruled out.

We expect similar features in higher dimensional scalar field theory with a UV regulator. Generically, a modified perturbative calculation involves a few coefficients with values close to their usual ones, a few coefficients in the crossover region, and the rest of the coefficients taking values much smaller than in regular perturbation theory. These three regimes are reminiscent of the three regimes encountered when calculating renormalization group flows between two fixed points. One should notice the similarities between the graphs presented here and those of Refs. [10] [11] where this type of crossover behavior is studied. In the same way, the approximate duality of Eq. (32) reminds of the duality between fixed points of a simplified renormalization group equation [12] or the exchange between the perturbative and non-perturbative sectors of quasi-exactly solvable periodic potentials [13].

We plan to make these analogies more specific (or disprove these ideas), by making explicit calculations with Dyson’s hierarchical model. Namely, we plan to compare the relative weights of the three types of perturbative terms discussed above with the relative increases in the three regions of the renormalization group flows discussed above for the zero-momentum $n$-point functions. Our hope is that we will be able to trade the very complicated problem of the interpolation between RG fixed points to a more tractable one.

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