Multiple Killing horizons: the initial value formulation for $\Lambda$-vacuum

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Abstract

In Mars et al (2018 Class. Quantum Grav. 35 155015) we have introduced the notion of ‘multiple Killing horizon’ and analyzed some of its general properties. Multiple Killing horizons are Killing horizons for two or more linearly independent Killing vectors simultaneously. In this paper we focus on the vacuum case, possibly with cosmological constant, and study the emergence of multiple Killing horizons in terms of characteristic initial value problems for two transversally intersecting null hypersurfaces. As a relevant outcome, a more general definition of near horizon geometry is put forward. This new definition avoids the use of Gaussian null coordinates associated to the corresponding degenerate Killing vector and thereby allows for inclusion of its fixed points.

Keywords: Killing horizon, characteristic initial value problem, near horizon geometry, vacuum field equations, bifurcate horizon

(Some figures may appear in colour only in the online journal)

1. Introduction

In [18] we have introduced multiple Killing horizons (MKHs): these are null hypersurfaces (or portions thereof) which are simultaneously the Killing horizons of two or more independent Killing vectors. The order of a MKH is the number of linearly independent such...
Killing vectors. In [18] we focused on basic concepts and properties of MKHs. In particular we derived an equation which we called master equation, and which is satisfied by the proportionality function between two Killing vectors on their common horizon. We showed that there are two distinct types of MKHs, fully degenerate or not, the former having all surface gravities vanishing.

This paper is devoted to an analysis of the occurrence of MKHs if field equations are imposed. Specifically, we shall impose the Einstein’s vacuum field equations, possibly with a cosmological constant ≤ of any sign. As Killing horizons are null hypersurfaces, Λ-vacuum spacetimes with MKHs can be generated in terms of a characteristic initial value problem. While the analysis in [18] led, in form of the master equation, to necessary conditions for the existence of MKHs, this approach permits the derivation of necessary and sufficient conditions on the characteristic data to generate a vacuum spacetime with a MKH.

There is a strong relationship between MKHs and near-horizon geometries, as discussed in [19]. To put things in perspective, in [14] the relation between near-horizon geometries and stationary black-hole holographs has been studied. In this context the master equation is established as a necessary and sufficient condition on the bifurcation surface of a bifurcate horizon to be a non-degenerate MKH. Our purpose is to study the characteristic initial-value problem systematically and to analyze in detail the emergence of Killing vectors and MKHs, including fully degenerate ones, as well as their order.

The paper is organized as follows: in section 2 we recall the most important definitions and results from [18, 19]. In section 3 we study how Λ-vacuum spacetimes with MKHs arise via a characteristic initial value problem. More specifically, in section 3.1 we recall the characteristic initial value problem for two transversally intersecting null hypersurfaces à la Rendall [23]. In section 3.2 we recall the Killing initial data (KID) equations for this type of initial value problem [6]. The KID equations are analyzed in section 3.3 for characteristic data which generate a vacuum spacetime which admits a bifurcate Killing horizon.

This provides the basis to analyze in section 4 the emergence of bifurcate horizons which contain a (necessarily non-fully degenerate) MKH. It turns out that the master equation considered on the bifurcation surface provides a necessary and sufficient criterion in this setting. The remaining of the section contains several particular situations of relevance where non-degenerate MKHs arise. In particular, in section 4.1 we focus on the case where the non-degenerate MKH is at least of order 3. It turns out that in that case the spacetime needs to have a couple of additional Killing vectors. In section 4.2 we analyze the emergence of MKH in the case where the torsion one-form vanishes on the bifurcation surface (see section 3 for definitions). In particular we end up with data for Minkowski and (Anti-)de Sitter spacetime, respectively, if, in addition, the bifurcation surface is maximally symmetric.

We will show in section 4.4 that, in 3 + 1-dimensions, vacuum spacetimes with a bifurcate horizon which admits a MKH of order 3 do not exist, while examples where it is of order 4 are provided by maximally symmetric spacetimes. For Λ ≠ 0 the (Anti-) Nariai spacetime provides an example where the bifurcate horizon is a MKH of order 2. In section 4.3 we construct, in arbitrary dimensions, data which generate a Λ = 0-vacuum spacetime with a bifurcate horizon which contains a MKH of order 2.

Section 5 is devoted to the study of near-horizon geometries. In [19] we have shown that the near-horizon geometry of a MKH is unique: it does not depend on the Killing vector used to compute it. Here we provide a simpler proof of this result in the Λ-vacuum case, and discuss how the near-horizon limit is performed from the viewpoint of a characteristic initial-value problem. This allows for an improved definition of near-horizon geometries which can deal with fixed points of the associated Killing vector.
Finally, in section 6 we construct a family of characteristic initial data which generate vacuum spacetimes with fully degenerate horizons. It turns out that an analysis of the KID equations is more involved in this setting. In section 6 we will focus on the results while we have shifted most of the calculations to an appendix.

1.1. Notation

\((M, g)\) denotes a connected, oriented and time-oriented \((n + 1)\)-dimensional Lorentzian manifold with metric \(g\) of signature \((-\ldots, +,\ldots, +)\). \(\overline{A}\) is the topological closure of a set \(A\). For any vector (field) \(v\) in \(TM\), \(v\) denotes the metrically related one-form. Both index-free and index notation are used. Lowercase Greek letters \(\alpha, \beta, \ldots\) run from 1 to \(n + 1\), small Latin indices \(i, j, \ldots\) take values from 2 to \(n + 1\), \(a \in \{1, 2\}\), and capital Latin indices \(A, B, \ldots\) are running from 3 to \(n + 1\), unless otherwise stated.

2. Multiple Killing horizons

The purpose of this section is to recall the most relevant definitions and results for spacetimes which admit MKHs. For more details we refer the reader to [18].

Definition 1. A smooth null hypersurface \(H_\xi\) embedded in a spacetime \((M, g)\) is a Killing horizon of a Killing vector \(\xi\) of \((M, g)\) if and only if \(\xi\) is null on \(H_\xi\), nowhere zero on \(H_\xi\) and tangent to \(H_\xi\). We require that the interior of the closure of \(H_\xi\) is a smooth connected hypersurface.

Let us consider the case when the set of fixed points \(S\) of a Killing vector \(\xi\) on \((M, g)\) forms a connected and spacelike co-dimension two submanifold. Then, the null geodesics orthogonal to \(S\) generate two transversal null hypersurfaces \(H_1^+\) and \(H_2^+\) (as well as its past portions \(H_1^-\) and \(H_2^-\)) are connected Killing horizons. Notice that \(H_1^+ \cup H_1^- \subset H_1\) and \(H_2^+ \cup H_2^- \subset H_2\) are also Killing horizons according to our definition. That leads to the following definition:

Definition 2. Let \(\xi\) be a Killing vector on \((M, g)\) which has a connected and spacelike co-dimension two submanifold \(S\) of fixed points. Then, the set of points along all null geodesics orthogonal to \(S\) forms a bifurcate Killing horizon with respect to \(\xi\). It is given by the union of the sets \(H_1^+ \cup H_2^+ \cup H_1^- \cup H_2^- \cup S = H_1 \cup H_2\) defined above.

Recall that the surface gravity of a Killing horizon \(H_\xi\) is the function \(\kappa_\xi\) defined by

\[ \nabla_\xi\xi|_{H_\xi} = \kappa_\xi\xi. \]

Definition 3 ([18]). A null hypersurface \(\mathcal{H}\) embedded in a spacetime \((M, g)\) is a multiple Killing horizon (MKH) if \((M, g)\) admits Killing horizons \(H_\xi, i \in \{1, \ldots, m\}\) with \(m \geq 2\), associated to linearly independent Killing vectors \(\xi_i\) satisfying

\[ \overline{H} = \overline{H}_{\xi_1} = \cdots = \overline{H}_{\xi_m}. \]

We assume throughout that \(\overline{H}\) admits a global cross section. Obviously, all the results of local nature that we derive remain valid also when this global assumption is dropped.

Theorem 1 ([18]). All surface gravities of a multiple Killing horizon \(\mathcal{H}\) are necessarily constant on the entire \(\mathcal{H}\).

Let \(\mathcal{H}\) be a MKH and define \(\mathcal{A}_\mathcal{H}\) as the union of the trivial Killing vector and the collection of Killing vectors \(\xi\) which admit a Killing horizon \(H_\xi\) satisfying \(\overline{H} = \overline{H}_\xi\).
Theorem 2 ([18]). Let $\mathcal{H}$ be a MKH in an $(n + 1)$-dimensional spacetime $(\mathcal{M}, g)$ of dimension at least two. Then:

(i) $\mathcal{A}_{\mathcal{H}}$ is a Lie sub-algebra of the Lie algebra of all Killing vectors of $(\mathcal{M}, g)$. Its dimension $m \geq 2$ defines the order of the MKH.

(ii) $\mathcal{A}_{\mathcal{H}}$ contains an Abelian sub-algebra $\mathcal{A}^{\text{deg}}_{\mathcal{H}}$ of dimension at least $m - 1$ whose elements have vanishing surface gravities. If $\mathcal{A}^{\text{deg}}_{\mathcal{H}}$ has dimension $m - 1$, any element $\xi \in \mathcal{A}_{\mathcal{H}} \setminus \mathcal{A}^{\text{deg}}_{\mathcal{H}}$ has $\kappa_\xi \neq 0$ and satisfies $[\xi, \eta] = -\kappa_\xi \eta, \forall \eta \in \mathcal{A}^{\text{deg}}_{\mathcal{H}}$.

(iii) The maximum possible dimension of $\mathcal{A}^{\text{deg}}_{\mathcal{H}}$ is $n = \dim(\mathcal{M}) - 1$.

Definition 4 ([18]). A MKH $\mathcal{H}$ is said to be fully degenerate if $\mathcal{A}_{\mathcal{H}} = \mathcal{A}^{\text{deg}}_{\mathcal{H}}$, otherwise it is called non-fully degenerate, or in short non-degenerate.

By theorem 2 the maximum possible order of a MKH $\mathcal{H}$ is

(i) $m = n$ for fully degenerate MKHs,

(ii) $m = n + 1$ for non-degenerate MKHs.

Lemma 1 ([18]). Let $\mathcal{H}$ be a MKH, and let further $\eta \in \mathcal{A}^{\text{deg}}_{\mathcal{H}}$ and $\xi \in \mathcal{A}_{\mathcal{H}}$, so that $\kappa_\eta = 0$ and $\kappa_\xi$ may be zero or not. Then on $\mathcal{H}_\xi$ there exist smooth functions $\tau, f_\eta : \mathcal{H}_\xi \mapsto \mathbb{R}$ with $\xi(\tau) = 1$ and $\xi(f_\eta) = 0$ such that the following relation holds

$$\eta|_{\mathcal{H}_\xi} = f_\eta e^{-\kappa_\xi \tau} \xi.$$ \hfill (1)

The level sets of the function $\tau$ define a foliation $\{S_\tau\}$ of $\mathcal{H}_\xi$ by spacelike co-dimension 2 surfaces. Of course, there is a freedom, namely to apply shifts $\tau \mapsto \tau + \tau_0$ with $\xi(\tau_0) = 0$, which induces a change as $f_\eta \mapsto f_\eta e^{\kappa_\xi \tau_0}$. This freedom amounts to a change of the chosen foliation. However, for a Killing horizon its inherited first fundamental form $\gamma$ satisfies $L_\xi \gamma = 0$, and thus all possible spacelike cross sections are isometric to each other with positive-definite metric $\gamma_{AB}$ (we keep the same name for simplicity).

Pick up any particular cross section $S_0 \subset \mathcal{H}_\xi$, not necessarily belonging to the chosen foliation $\{S_\tau\}$. Let $D_\xi$ be the canonical covariant derivative on $(S_0, \gamma)$ and $\{e_A\}$ a basis of vector fields on $S_0$. Due to the fact that Killing horizons are totally geodesic null hypersurfaces, the following relation holds

$$D_\xi e_B - \nabla_{\xi} e_B = -K_{AB} \xi$$ \hfill (2)

which defines the unique non-vanishing second fundamental form $K_{AB}$ of $(S_0, \gamma)$. Then, the torsion one-form $s$ on $(S_0, \gamma)$ relative to the chosen Killing $\xi$ is defined in the given basis by

$$s_A^\xi = -e_A^B \nabla_\rho s^\rho_\. \hfill (3)$$

It is clear from (1) that the function $f : S_0 \mapsto \mathbb{R}$ defined by

$$f := f_\eta e^{-\kappa_\xi \tau}|_{S_0}$$

provides the proportionality between $\eta$ and $\xi$ on $S_0$: $\eta \equiv f \xi$. Then we have (we correct an unfortunate typo in [18, expression (60)])

Proposition 1 ([18]). Let $\mathcal{H}$ be a MKH and $\mathcal{H}_\xi, \mathcal{H}_\eta$ be Killing horizons satisfying $\mathcal{H}_\xi = \mathcal{H}_\eta = \mathcal{H}$. Then the following master equation holds on any spacelike cut $S_0$ of $\mathcal{H}_\xi$. 

Class. Quantum Grav. 37 (2020) 025010

M Mars et al
\[ D_A D_B f - 2s(\ell D_B f) + \left( \frac{1}{2} R_{AB} - \frac{1}{2} \gamma R_{AB} + s_A s_B - D(\ell D_B f) \right) f = 0 \]  

(4)

where \( R_{AB} := R_{\mu\nu}|_{S_0} e^A e^B \) is the pull-back of the Ricci tensor to \( S_0 \) and \( \gamma R_{AB} \) is the Ricci tensor of the cut \((S_0, \gamma)\) itself.

Remark 1. The \( \gamma \)-trace of (4) yields

\[ -\Delta f + 2s_A D^A f - \left( \frac{1}{2} \gamma^{AB} R_{AB} - \frac{1}{2} \gamma R + |s|^2 - D^A s_A \right) f = 0. \]

As \( \gamma^{AB} R_{AB} = R + 2R_{\mu\nu} k^\mu \ell^\nu = 2G_{\mu\nu} k^\mu \ell^\nu \), where \( k \) and \( \ell \) are future-directed null fields with \( g(k, \ell) = -1 \), the operator acting on \( f \) can be identified with the MOTS-stability operator [1]. Thus \( f \) is an eigenfunction of the MOTS-stability operator with eigenvalue \( \lambda = 0 \). If this is the eigenvalue with the smallest real part, the cut of the MKH is marginally stable and not strictly stable. Even more, this might indicate that there is a relation between the full master equation and stability of the vanishing of the null second fundamental form, which is to be analyzed elsewhere.

The linear homogeneous PDE (4) is of second order and written in normal form, i.e. with all second derivatives expressed in terms of lower order terms. Its integrability conditions were briefly considered in [18] and will be further analyzed in a forthcoming paper. It follows immediately from its structure that one can freely prescribe at most \( n - 1 \) independent solutions of (4) so that, in particular, adding \( \xi \) to the set of Killing generators arising from (4) does not increase the dimension. This can be checked explicitly because, from the Gauss equation for the cut \((S_0, \gamma)\) and general identities for Killing horizons, one can show [18] that (4) is fully equivalent to

\[ D_A D_B f - s_A D_B f - s_B D_A f + \kappa \xi K_{AB} = 0. \]  

(5)

Thus, if \( \mathcal{H} \) is fully degenerate, then in particular \( \kappa \xi = 0 \) and \( f = \text{const.} \) is a solution of the master equation (5) leading to the original Killing \( \xi \). Hence, there are at most \( n \) independent Killings in \( \mathcal{A}_H \) in this degenerate case, as it should.

3. Construction of spacetimes with MKHs via characteristic Cauchy problems

Let us now study the construction of \( \Lambda \)-vacuum spacetimes with MKHs in terms of characteristic initial value problems. For this we first recall some useful results. The null second fundamental form of a null hypersurface \( N \), relative to a field \( K \) of null tangents to \( N \), is defined as

\[ N_{\mathcal{K}}(X, Y) := g(X, \nabla_Y K), \quad \text{where} \quad X, Y \in T^N. \]

Note that this second fundamental form is different to the one introduced before which refers to the transversal direction (thus \( N_{\mathcal{K}} \) should not be confused with \( K \)).

\( N_{\mathcal{K}} \) is a symmetric tensor field on \( N \) and shares the same degeneracy as the first fundamental form \( \gamma \) of \( N \): \( \gamma(K, \cdot) = N_{\mathcal{K}}(K, \cdot) = 0 \). The trace-free part of \( N_{\mathcal{K}} \) gives the shear \( N_\tau := (N_{\mathcal{K}})^{\mu}_{\ell\ell} \) of \( N \) relative to \( K \), and its trace \( N_0 := \text{tr}(N_{\mathcal{K}}) \) is called expansion.

\[ \text{Remark 1.} \quad \text{The} \quad \gamma \text{-trace of (4) yields} \]

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The spacelike cross sections of \( N \) will usually be referred to as cuts in this paper. Given any cut \( S \subset N \), we consider its torsion one-form \( s \) relative to a null normal frame \( \{ k, \ell \} \): let \( k \) and \( \ell \) be two null normals to the codimension-2 spacelike cut \( S \) normalized by \( g(k, \ell) = -1 \). Then the torsion one form \( s \) of \( S \) with respect to \( \{ k, \ell \} \) is given by the formula
\[
 s(Z) := g(\nabla_Z k, \ell), \quad \text{where} \quad Z \in TS. \tag{6}
\]
For later use we recall that under a boost of the null basis \( \{ k', \ell' = \beta k, \beta^{-1} \ell \} \), \( \beta \) a smooth function on \( S \), the torsion one form transforms as
\[
 s' = s + \beta^{-1}d\beta. \tag{7}
\]

### 3.1. Characteristic initial value problem

We recall a fundamental result by Rendall [23], presented here in a slightly more geometrical version, along the lines of the discussion in section 4 in [5]. See Figure 1 for a schematic representation of the existence theorem.

**Theorem 3 ([23]).** Let \( M_1, M_2 \) be two smooth \( n \)-dimensional manifolds intersecting on a smooth \((n - 1)\)-dimensional manifold \( S \equiv M_1 \cap M_2 \). Let \( g^0 \) be a degenerate semi-positive symmetric tensor on \( M_1 \cup M_2 \) with one-dimensional radical and continuous and non-degenerate at \( S \). (Equivalently, with signature \((0, +, \ldots, +)\).) Let also \( \varsigma \) be a smooth one-form on \( S \), and \( \nu > 0, \omega > 0 \), \( \omega_0 \) and \( \omega_2 \) be given smooth functions on \( S \). Then, there exists an \((n + 1)\)-dimensional Lorentzian manifold with boundary \((U, g)\) whose smooth metric \( g \) is a solution of the \( \Lambda \)-vacuum Einstein field equations
\[
 R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0
\]
and a unique nowhere vanishing function \( \Omega \) on \( \partial U \), such that

- the boundary \( \partial U \) is formed by two intersecting null hypersurfaces \( N_1 \) and \( N_2 \) to the future of their intersection manifold \( S \subset U \) which are isometric to some open neighborhood of \((M_1, \Omega^0 g^0)\) and \((M_2, \Omega^2 g^0)\) around \( S \), respectively. We have kept the notation \( S \) for the intersection of \( N_1 \) and \( N_2 \) in \( U \).
- there exist representatives \( K \) and \( L \) of the null generators of \( N_1 \) and \( N_2 \), respectively, with \( g(K, L)|_S = \nu \)
- \( \varsigma \) is the torsion one-form relative to \( k = K/\sqrt{\nu} \) and \( \ell = -L/\sqrt{\nu} \) on \( S \)
- On \( S \), one has \( \Omega|_S = \omega, K(\Omega)|_S = \omega_0 \) and \( L(\Omega)|_S = \omega_2 \).

Moreover, any two such Lorentzian manifolds are isometric on some neighbourhood of \( S \).

**Remark 2.** In fact, the solution can be extended so that its boundary includes the whole initial hypersurfaces provided the Raychaudhuri equation for the initial-data expansions admits a global solution [3, 16].

We introduce (see [23]) an adapted null coordinate system \( (x^\alpha) = (u, v, x^A) \), \( A = 3, \ldots, n + 1 \), where \( N_1 = \{ v = 0 \} \) and \( N_2 = \{ u = 0 \} \). In particular, \( S = \{ u = 0, v = 0 \} \). Moreover, we assume that \( u \) (resp. \( v \)) parameterizes the null geodesic generators of \( N_1 \) (resp. \( N_2 \)), while the \( x^A \) are local coordinates on the \( \{ v = 0, u = \text{const.} \} \)- and \( \{ u = 0, v = \text{const.} \} \)-level sets, that is, the respective adapted cuts.

The induced metric on each cut \((1)S_u := \{ u = \text{const} \} \subset N_1 \) and \((2)S_v := \{ v = \text{const} \} \subset N_2 \) is denoted by \( \bar{g} \) and similarly the associated covariant derivative, connection coefficients,
Ricci tensor etc carry a hat. The induced Riemannian metric on the intersection surface $S$ will still be denoted by $\gamma$ and, correspondingly, connection coefficients, Ricci tensor etc will be decorated with $\gamma$. Its covariant derivative will be denoted by $D$ as before.

In adapted null coordinates we take $K := \partial_u$ on $N_1$ and $L := \partial_v$ on $N_2$. Then the non-vanishing components of the respective null second fundamental forms are given by

\begin{align}
(1) K_{AB} &= \frac{1}{2} \partial_u g_{AB}, \\
(2) K_{AB} &= \frac{1}{2} \partial_v g_{AB}.
\end{align}

Note that in these adapted coordinates, the metric component $g_{uv}$ on $S$ reads $g_{uv}|_S = \nu$ and the torsion one-form $\varsigma$ relative to $k$ and $\ell$ of theorem 3 is [5].

$$\varsigma_A = \frac{1}{2} (\Gamma^v_{vA} - \Gamma^u_{uA}). \quad (8)$$

**Remark 3.** For our purposes it is more convenient to prescribe the shears $^{(a)} \pi_{AB}$ of $N_a$, $a = 1, 2$, together with $\gamma_{AB} := g_{AB}|_S$ and the expansions of $N_1$ and $N_2$ at $S$, $^{(a)} \theta|_S$, rather than the family $g_{AB}^a$ together with $\Omega$, $\partial_u \Omega$ and $\partial_v \Omega$ at $S$.

Note for this that the shear is determined from $g_{AB}^a$ by a first-order ODE. The remaining freedom to prescribe $g_{AB}^a|_S$ together with that to choose $\Omega|_S$ can be combined into the prescription of $\gamma_{AB}$, i.e. the induced Riemannian metric on $S$. The freedom to prescribe $\partial_u \Omega$ and $\partial_v \Omega$ at $S$ can be identified with that to choose the expansions $^{(a)} \theta$ of $N_a$ at $S$.

Some remarks concerning the gauge freedom are in order. Usually when solving the evolution problem one imposes a (generalized) wave-map gauge condition [4, 8]. In this paper, though, it is irrelevant how the coordinates are extended off the initial hypersurface. We will nevertheless exploit some of the gauge freedom which arises from the freedom to prescribe the adapted null coordinates on the initial hypersurface: there remains the gauge freedom to reparametrize the null geodesic generators of $N_1$ and $N_2$, which can be used to prescribe the connection coefficients $\Gamma^u_{uv}|_{N_1}$ and $\Gamma^v_{uv}|_{N_2}$ [4]. They will vanish if and only if $u$ and $v$ are affine parameters on $N_1$ and $N_2$, respectively. The remaining reparametrization freedom can be used to prescribe $g_{uv}|_S > 0$ and add gradients to the torsion one-form $\varsigma$ [6].

The $\Lambda$-vacuum equations imply transport equations of certain fields along the null geodesic generators of $N_a$. Here we provide the relevant equations which will be used later on. It is convenient to set

\begin{figure}
\centering
\includegraphics{Characteristic_initial_value_problem.png}
\caption{Characteristic initial value problem.}
\end{figure}
\[ (1) \Xi_{AB} \overset{N_1}{=} 2\Gamma_{AB}^u + (1) \mathcal{K}_{AB} g_{uu}, \quad (2) \Xi_{AB} \overset{N_1}{=} 2\Gamma_{AB}^u + (2) \mathcal{K}_{AB} g_{uv}. \]

In geometrical terms, \( \Xi_{AB} ^{(1)} \) is the null second fundamental form of the sections \{ \( u = \text{const} \) \} along the null normal \( \mathbf{K} \) satisfying \( \langle \mathbf{K}, \mathbf{K} \rangle = -1 \), and similarly for \( \Xi_{AB} ^{(2)} \).

The following equations hold, which we present for reasons of definiteness on \( N_1 \), see [4, 5, 23],

\[
(\partial_u + \frac{(1)\theta}{n-1} - \Gamma_{uu} ^{(1)} \theta) + [(1)\pi]_1^2 N_1 = 0, \tag{9}
\]

\[
(\partial_u + (1)\theta) \Gamma_{uu} ^{(n-2)} n - (1)\theta - (1)\partial_A \Gamma_{uu} ^{(2)} - \hat{\nabla}_B ^{(1)} \pi_A B N_1 = 0, \tag{10}
\]

\[
(\partial_u + (1)\theta + \Gamma_{uu} ^{(2)} \text{tr}(\Xi) - 2g^{AB} (\hat{\nabla}_A + \Gamma_{uu} ^{(2)} g_{uu} + \hat{\mathbf{R}}) A N_1 = 0. \tag{11}
\]

As mentioned above, \( \hat{\nabla} \) and \( \hat{\mathbf{R}} \) refer to the Levi-Civita covariant derivative and Ricci scalar of the Riemannian family \( g = g_{AB} d^a d^b |_{N_1 \ll N_2} \), and \( [(1)\pi]_1^2 := (1)\pi_A B (1)\pi_B A \). Moreover, from \( (\mathcal{R}_{AB})_{\mu \nu} = 0 \) we find

\[
(\partial_u + \frac{n-5}{2(n-1)} (1)\theta + \Gamma_{uu} ^{(1)} \Xi_{AB}) \mu \nu - 2(1)\pi_A C (1)\Xi_{BC} \mu \nu - \frac{1}{2} \text{tr}(\Xi) (1)\pi_{AB} - 2(\hat{\nabla}_A \Gamma_{uu} ^{(2)}) \mu \nu - 2(\Gamma_{uu} ^{(2)} \Gamma_{uu} ^{(2)}) \mu \nu + (\hat{\mathbf{R}}_{AB}) \mu \nu \overset{N_1}{=} 0. \tag{12}
\]

The initial data for (9) are part of the free data. In contrast, the initial data for (10) are determined by the torsion one-form and \( g_{uv} | S \), while those for (11) and (12) are determined by expansion and shear at \( S \), respectively, of the other null hypersurface,

\[
(1) \Xi_{AB} \overset{S}{=} - \frac{2}{\nu} (2) \mathcal{K}_{AB}, \quad (2) \Xi_{AB} \overset{S}{=} - \frac{2}{\nu} (1) \mathcal{K}_{AB}, \tag{13}
\]

which follows immediately from the fact that, on \( S \), \( K = -\frac{1}{2} L \) and \( L = -\frac{1}{2} K \).

We also need the following fact concerning torsion one-forms on the cuts \( (1) s_A \) and \( (2) s_A \). Denoting by \( (1) s_A \) (resp. \( (2) s_A \)) the torsion one-form with respect to \{ \( K, K \) \} (resp. \{ \( L, L \) \}) one easily checks that

\[
(1) s_A \overset{(1)\mathbf{S}_A}{=} -\Gamma_{uu} ^{WA}, \quad (2) s_A \overset{(2)\mathbf{S}_A}{=} -\Gamma_{uv} ^{WA}. \tag{14}
\]

In particular, this means that these Christoffel symbols are in fact tensorial on each cut so \( \hat{\nabla} \)-covariant derivatives (as in e.g. (11) or (12)) make sense. The following identities on \( S \) follow directly from the definitions and from (8), respectively

\[
-\frac{1}{\nu} \partial_{\nu} s_A \overset{s_A}{=} (1) s_A + (2) s_A, \quad s_A \overset{s_A}{=} \frac{1}{2} (1) s_A - (2) s_A. \tag{15}
\]

### 3.2. KID equations

We are interested in generating \( \Lambda \)-vacuum spacetimes which admit Killing vectors \( \zeta \) in terms of an initial value problem. The spacetime Killing equation \( \mathcal{L} \zeta \mathcal{G} = 0 \) is then replaced by the Killing initial data (KID) equation. If and only if it admits \( m \) independent solutions, the emerging vacuum spacetime admits \( m \) independent Killing vectors. For a characteristic initial value problem they have been derived in [6]:

\[
\text{...}
\]
Theorem 4 ([6]). Consider two smooth null hypersurfaces \( N_1, N_2 \), in an \((n + 1)\)-dimensional manifold, \( n \geq 3 \), with transverse intersection along a smooth \((n - 1)\)-dimensional submanifold \( S \) in adapted null coordinates. Let \( \zeta \) be a continuous vector field defined on \( N_1 \cup N_2 \) such that \( \zeta|_{N_1} \) and \( \zeta|_{N_2} \) are smooth. Then \( \zeta \) extends to a smooth vector field on \( D^+ (N_1 \cup N_2) \) satisfying the Killing equation and coinciding with \( \zeta \) on \( N_1 \cup N_2 \) if and only if the KID equations hold by which we mean that on \( N_1 \) (note that the equations on \( N_1 \) and \( N_2 \) do not involve transverse derivatives of \( \zeta \))

\[
\nabla_u \zeta|_{N_1} = 0, \\
\nabla_{(\zeta \lambda)}|_{N_1} = 0, \\
(\nabla_{(\zeta \lambda)} u)|_{N_1} = 0, \\
\nabla_u \nabla_{\zeta \lambda} u|_{N_1} = R\lambda \mu u\zeta \mu,
\]

with identical corresponding conditions on \( N_2 \), while on \( S \) one further needs to assume

\[
\nabla_{(\zeta v)}|_{S} = 0, \\
g^{AB} \nabla_A \zeta_B|_{S} = 0, \\
\partial_b (g^{AB} \nabla_A \zeta_B)|_{S} = 0, \\
\partial_c (g^{AB} \nabla_A \zeta_B)|_{S} = 0, \\
\nabla_A \nabla_{[\zeta \lambda]}|_{S} = R\mu u\.\zeta \mu.
\]

3.3. Bifurcate Killing horizon

We consider a vacuum spacetime \((M, g)\) in \( n + 1 \) dimensions, \( n \geq 3 \), possibly with cosmological constant \( \Lambda \), which admits a Killing vector \( \xi \) which generates a bifurcate Killing horizon \( \mathcal{H} = \mathcal{H}_1^+ \cup \mathcal{H}_2^+ \cup \mathcal{H}_1^- \cup \mathcal{H}_2^- \cup S \) according to definition 2 and the foregoing comment. The associated surface gravity is constant, \( \kappa_\xi = \text{const.} \neq 0 \) on \( \mathcal{H} \). In fact, it is shown in [22] that, adding a certain technical condition, any spacetime with a non-degenerate Killing horizon with constant surface gravity can be extended to a spacetime where this Killing horizon forms a portion of a bifurcate Killing horizon.

For now, let us restrict attention to those parts of \( \mathcal{H} \) which lie in the causal future of \( S \), that is to say, \( \mathcal{H}_1^+ \) and \( \mathcal{H}_2^+ \). Then the spacetime \( D^+ (\mathcal{H}_1^+ \cup \mathcal{H}_2^+) \) can be generated in terms of a characteristic initial value problem for appropriate data specified on \( \mathcal{H}_1^+ \cup \mathcal{H}_2^+ \cup S \).

Theorem 5 ([6, 9, 10]).

(i) \( D^+ (\mathcal{H}_1^+ \cup \mathcal{H}_2^+) \) admits a Killing vector field \( \xi \) for which both, \( \mathcal{H}_1^+ \) and \( \mathcal{H}_2^+ \), are Killing horizons if and only if the null second fundamental forms \((\xi)K\) vanish on \( \mathcal{H}_1^+ \) and \( \mathcal{H}_2^+ \).

In that case \( \xi \) is unique (up to constant rescaling) and the associated surface gravity is constant and satisfies

\[
\kappa_\xi (\mathcal{H}_1^+) = -\kappa_\xi (\mathcal{H}_2^+) = \partial_{[\zeta \lambda]}|_{S} \neq 0.
\]
i.e. the bifurcate horizon in non-degenerate. Moreover, $\xi \neq 0$.

(ii) Assume that such a $\xi$ exists, then $D^\omega (\mathcal{H}^+ \cup \mathcal{H}^+_2)$ admits a second independent Killing vector $\zeta$, tangent to $S$, if and only if the bifurcation surface $S$ with the Riemannian metric $\gamma \equiv g_{AB} \, dx^A \otimes dx^B|_S$ admits a non-trivial (i.e. not the zero vector) Killing vector $\zeta = \partial_A$ such that the $\zeta$-Lie-derivative of the torsion one-form of $S$ is exact. The Killing vector $\zeta$ is then uniquely determined (up to an additive $c\xi$, $c \in \mathbb{R}$) by the condition $\zeta|_S = \zeta$, and is tangent to $\mathcal{H}^+ \cup \mathcal{H}^+_2$.

It follows immediately that there cannot exist a second independent Killing vector $\zeta$ with Killing horizons $\mathcal{H}^+_1$ and $\mathcal{H}^+_2$ as this would imply that $\zeta := \zeta|_S = 0$, which contradicts the fact that $\zeta$ has to be non-trivial (item (ii) of the theorem). However, $\zeta$ may have either $\mathcal{H}^+_1$ or $\mathcal{H}^+_2$ as Killing horizon. So let us analyze under which conditions one of the horizons, $\mathcal{H}^+_1$ say, is simultaneously the Killing horizon of a second Killing vector field, which we denote by $\eta$. Proportionality of $\eta$ and $\xi$ on $\mathcal{H}^+_1$ is equivalent, in adapted null coordinates, to

$$\eta^A|_{\mathcal{H}^+_1} = 0, \quad \eta^A|_{\mathcal{H}^+_1} \neq 0, \quad \eta^A|_{\mathcal{H}^+_1} = 0. \quad (25)$$

At $\mathcal{H}^+_1$ the metric in adapted null coordinates is of the form

$$ds^2 \equiv g^{\mathcal{H}^+} \, dv \left( g_{uv} \, dv + 2 g_{uv} \, du + 2 g_{uv} \, dx^A \right) + g_{AB} \, dx^A \, dx^B. \quad (26)$$

so an equivalent form of (25) is

$$\eta^u|_{\mathcal{H}^+_1} \neq 0, \quad \eta^u|_{\mathcal{H}^+_1} = 0, \quad \eta^u|_{\mathcal{H}^+_1} = 0. \quad (27)$$

We shall analyze to what extent these additional conditions are compatible with the KID equations. For this we shall first rewrite and simplify the system of KID equations in the special case of a bifurcate Killing horizon.

The null hypersurfaces $N_1$ and $N_2$ are required to form a bifurcate Killing horizon, i.e. the null second fundamental forms need to vanish (see theorem 5). With regard to the characteristic initial value problem this is achieved by the initial data

$$(a) \pi_{AB} = 0, \quad (a) \theta \neq 0, \quad a = 1, 2. \quad (28)$$

In this setup we denote $N_1$ and $N_2$ by $\mathcal{H}^+_1$ and $\mathcal{H}^+_2$. The Raychaudhuri equation then implies $\pi_{AB} = 0$. Equation (10) together with $(a) K_{AB} = 0$ and (14) becomes

$$\partial^a (1) \Gamma^A_{\alpha A} + \partial_A \Gamma^a_{\mu A} = 0. \quad (29)$$

From this equation it follows that the Riemann tensor components $R_{\alpha \beta \gamma \delta}^{\mu}$ vanish. Indeed, let $e_a := \partial_a \xi$ so that $[K, e_A] = 0$. The vanishing of $(a) K_{AB}$ and the definition of the torsion 1-form gives (see (3))

$$\nabla_{e_A} K \equiv (1) \xi_A K. \quad (30)$$

Inserting in the Ricci identity and using $\nabla_K K \equiv \Gamma^A_{\mu A} K$ yields, on $\mathcal{H}^+_1$,

$$R_{\alpha \beta \gamma \delta}^{\mu} = R_{\alpha \beta \gamma}^{\lambda} e_{\delta}^{\lambda} K^{\beta} \delta = (\nabla_K \nabla_{e_A} - \nabla_{e_A} \nabla_K - \nabla_{[K, e_A]}) K^{\mu} = -\left( K^{(1)}_{\lambda A} + \partial_A \Gamma^a_{\mu A} \right) K^{\mu} = 0. \quad (31)$$

This means in particular that the right hand side of (19) is $R_{\nu \mu \alpha}^{\mu} \xi^{\alpha} = R_{\nu \mu}^{\mu} \xi^{\nu} = g_{\mu \nu}^{-1} R_{\nu \mu}^{\mu} \xi^{\nu}$, the second equality following from

$$g^{\nu \mu} \equiv g_{\alpha \beta} \partial^{\alpha} \partial^{\beta}.$$
which is found by inverting the metric (26). Using the formulas in [4, appendix A] one checks that the KID equations (16)–(19) reduce to the following system on $H^+_1$ (and correspondingly on $H^+_2$):

\[
(\partial_u - \Gamma^u_{au})\zeta^a \overset{S}{=} 0, \tag{32}
\]

\[
\partial_u \zeta_A + \left(\nabla_A - 2\Gamma^a_{au}\right)\zeta_a \overset{H^+_1}{=} 0, \tag{33}
\]

\[
\left(\nabla_A (\zeta_B)\right)_a - \left(\Gamma^a_{AB}\right)\zeta_{ab} \overset{H^+_1}{=} 0, \tag{34}
\]

\[
\partial_u \left(\partial_u + \Gamma^u_{au}\right)\zeta^a - \frac{1}{2} \zeta_a \left(\partial_a + \Gamma^a_{au}\right)\zeta_a \overset{S}{=} 0, \tag{35}
\]

We next rewrite the KID equation on $S$. In particular we need to evaluate the curvature components in (24):

\[
R_{muAB} \zeta^u \overset{S}{=} R_{muAB} \zeta^v + R_{muAB} \zeta^B, \quad R_{muAB} \zeta^B = R_{muAB} \zeta^B,
\]

the first two terms being zero because of (31) and the corresponding identity on $H^+_2$. To evaluate the remaining term we first note that, on $H^+_1$,

\[
\langle K, \nabla_{e_A} \nabla_{e_B} K \rangle = \nabla_{e_A} \langle K, \nabla_{e_B} K \rangle - \langle \nabla_{e_A} K, \nabla_{e_B} K \rangle = \nabla_{e_A} (1) s_B - (1) s_A , \tag{36}
\]

where we used (30) and the corresponding $\nabla_{e_A} K = (1) s_A K$. From $\partial_u \overset{S}{=} -\nu K$ and $[e_A, e_B] = 0$ we have

\[
R_{muAB} \overset{S}{=} -\nu \langle K, \nabla_{e_A} \nabla_{e_B} K - \nabla_{e_A} \nabla_{e_B} K \rangle = \nu \left(\partial_a (1) s_A - (1) s_B \right) = \nu \left(\partial_u s_A - \partial_u s_B \right), \tag{37}
\]

where in the last equality we used that boosting a null basis changes the torsion one-form with an exact one-form (see (7)).

Set $\zeta := \zeta^A \partial_A S$. On $S$ we then have (see the corresponding computation in [6] but be aware that the Killing vector was assumed there to be tangential to $S$ whence some additional terms appear here)

\[
(\partial_u + \Gamma^u_{au} + \Gamma^v_{au}) \zeta^a + (\partial_u + \Gamma^v_{au} + \Gamma^w_{au}) \zeta^v + \nu^{-1} \zeta^w \overset{S}{=} 0, \tag{38}
\]

\[
D^A \zeta_A \overset{S}{=} 0, \tag{39}
\]

\[
D^A \partial_u \zeta_A - \left(\gamma^{AB} \Gamma^a_{AB} + D^A \Gamma^a_{ab} - \frac{\gamma R}{2} + \Lambda \right) \zeta_a \overset{S}{=} 0, \tag{40}
\]

\[
D^A \partial_u \zeta_A - \left(\gamma^{AB} \Gamma^a_{AB} + D^A \Gamma^a_{ab} - \frac{\gamma R}{2} + \Lambda \right) \zeta_v \overset{S}{=} 0, \tag{41}
\]

where indices on $D$ have been raised with $\gamma_{AB}$, and where (11) and (36) have been used. For convenience let us impose the following gauge conditions: require $u$ and $v$ to be affine parameters along the null geodesic generators of $H^+_1$ and $H^+_2$, respectively. Then (see [4])
This leaves the freedom to do affine transformations of these coordinates. With the gauge choice \( S = \{ u = 0, v = 0 \} \) there remains the freedom to do rescalings of the form \( u \mapsto f^{(i)}(x^4)u \) and \( v \mapsto f^{(v)}(x^4) \) which can be partially fixed [6] to achieve that
\[
\nu \equiv 1.
\]
The remaining coordinate freedom is \( \{ \tilde{u} = \phi^{-1}(x^4)u \} \) and \( \{ \tilde{v} = \phi(x^4)v \} \). Its effect on \( \zeta \) is
\[
\tilde{\zeta} = \zeta + \phi^{-1}d\phi,
\]
because of (7) together with
\[
\partial_{\tilde{u}} \equiv \phi \partial_u, \quad \partial_{\tilde{v}} \equiv \phi^{-1} \partial_v.
\]
Thus, all the remaining gauge freedom is to add gradients to \( \zeta \).

**Remark 4.** In fact, the non-gauge part of the torsion one-form \( \zeta \) together with the Riemannian metric \( \gamma \) on \( S \) provide the only ‘physical’, non-gauge data for a characteristic initial value problem for the vacuum equations on a bifurcate Killing horizon.

We want to compute the \( u \)-derivative of (34). With (12), (32) and (33) we obtain on \( \mathcal{H}_+^1 \)
\[
0 \overset{H_+^1}{=} \left( \nabla(A \partial_u \zeta_B) \right)_a - \partial_u (\Gamma_{aB}^u) u \zeta_a - (\Gamma_{AB}^u) u \partial_u \zeta_a
\]
\[
\overset{H_+^1}{=} - \left( \nabla(A \nabla_B \zeta_a) \right)_a + \left( \nabla(A \Gamma_{B}^u) u \zeta_a + 2(\Gamma_{AB}^u \nabla_B \zeta_a) u \right) - (\Gamma_{AB}^u) u \zeta_a + \frac{1}{2} (\hat{R}_{AB}) u \zeta_a.
\]
Observe that \( \hat{g}_{AB}, \Gamma_{aB}^u|_{\mathcal{H}_+^1} \) and \( \zeta_a|_{\mathcal{H}_+^1} \) are \( u \)-independent (see (10) and (32)), so this equation is independent of \( u \) and therefore only needs to be satisfied at \( S \) (similarly on \( \mathcal{H}_+^1 \)). In fact, this is a crucial property which allows to reduce the KID equations to a system of equations on \( S \) in the special case where \( S \) is a bifurcation surface (see the different setting in section 6).

Using (33), (34) and (14) with the fact that in the present gauge \( (1)_{x^4} s_A = - (5)_{x^4} s_A = \zeta_A \) on \( S \) (see (15)), one checks that the system
\[
D(A_s B) \overset{S}{=} 0,
\]
\[
\left( \Delta_{\gamma} - 2s_A D^4 - D^4 s_A + |s|^2 - \frac{7}{2} \frac{R}{2} + \Lambda \right) \zeta_a \overset{S}{=} 0,
\]
\[
\left( \Delta_{\gamma} + 2s_A D^4 + D^4 s_A + |s|^2 - \frac{7}{2} \frac{R}{2} + \Lambda \right) \zeta_a \overset{S}{=} 0,
\]
\[
\left( D_A D_B - 2s_A (D_B) - D_{(A s B)} + s_{A s B} - \frac{1}{2} \hat{R}_{AB} \right) \zeta_a \overset{S}{=} 0,
\]
\[
\left( D_A D_B + 2s_A (D_B) + D_{(A s B)} + s_{A s B} + \frac{1}{2} \hat{R}_{AB} \right) \zeta_a \overset{S}{=} 0,
\]
\[
\partial_a \zeta^a + \partial_{\nu} \zeta^\nu + \Gamma_{aB}^\nu \zeta^\nu + \Gamma_{aB}^\nu \zeta^\nu \overset{S}{=} 0,
\]
\[
D_A \left( \partial_{\nu} \zeta^\nu - \partial_{\nu} \zeta^u - \Gamma_{aB}^\nu \zeta^\nu + \Gamma_{aB}^\nu \zeta^u \right) + 2 \zeta_{sA} \overset{S}{=} 0.
\]
is equivalent to (37)–(41) plus the restriction of (34) to $S$ as well as its $u$- and $v$- derivatives. Supposing that this system admits a solution, the restriction of the Killing vector field $\zeta$ of the emerging vacuum spacetime to $H^+_1$ is computed by solving the ODEs (32), (33) and (35), and correspondingly on $H^+_2$. The initial data of the transport equations along the horizons is determined from the solution of the system on $S$ as follows: assume a solution of the system (46)–(49) has been selected. This fixes completely the initial data for (32) and (33) and the corresponding equations of $H^+_2$. Now, (50) can be read as an algebraic equation for $\zeta_u$ and $\zeta_v$ such that $\mathcal{L}_\zeta S$ is exact, and in that case it yields an equation algebraically solvable for $\partial_u \zeta^u - \partial_v \zeta^v|_S$ (equivalently $\partial_u \zeta^u|_S$) in terms of an arbitrary additive constant. The additive constant corresponds to the addition to $\zeta$ on $D^+ (H^+_1 \cup H^+_2)$ of a constant times $\xi$ (the Killing vector with respect to which $H^+_1 \cup H^+_2$ is a bifurcate horizon). Indeed, for any constant $\alpha$ the vector field along $H^+_1 \cup H^+_2$ given by

$$\zeta_u^H \equiv \zeta^H, \quad \zeta^w \equiv \alpha u, \quad \zeta_v^H \equiv \zeta^H, \quad \zeta^w \equiv -\alpha v,$$

solves the full set of KID equations. Denoting by $\xi$ the Killing vector it defines on $D^+ (H^+_1 \cup H^+_2)$, it is obvious that $\xi$ has $H^+_1 \setminus S$ and $H^+_2 \setminus S$ as Killing horizons and that it vanishes on $S$. It is also immediate from item (i) in theorem (5) that the surface gravities are

$$\kappa_\xi (H^+_1) = -\kappa_\xi (H^+_2) = \alpha.$$

Combining further (46)–(49), respectively, into a single equation we have established the following theorem:

**Theorem 6.** The data $(\gamma, \varsigma)$ on $S$, supplemented by vanishing null second fundamental forms on $H^+_1$ and $H^+_2$, generate an $(n+1)$-dimensional $\Lambda$-vacuum spacetime with bifurcation surface $S$ with $k+1$ Killing vectors if and only if the following set of KID equations admits $k$ independent solutions $(\zeta_u, \zeta^w, \zeta_v)_S$:

$$\zeta \equiv \zeta^w \partial_A|_S$$

such that $\mathcal{L}_\zeta S$ is exact,

$$\left( D_A D_B - 2 \varsigma(A D_B) - D_A \varsigma(B) + \varsigma_A \varsigma_B - \frac{1}{2} \gamma_{AB} R_{AB} + \frac{\Lambda}{n-1} \gamma_{AB} \right) \zeta^w |_S = 0,$$  

$$\left( D_A D_B + 2 \varsigma(A D_B) + D_A \varsigma(B) + \varsigma_A \varsigma_B - \frac{1}{2} \gamma_{AB} R_{AB} + \frac{\Lambda}{n-1} \gamma_{AB} \right) \zeta_v |_S = 0.$$  

**Remark 5.**

(i) The KID equations (53)–(54) are identical in form to the master equation (4) in $\Lambda$-vacuum. It should be emphasized however, that the two equations are not the same because the master equation holds for sections of the Killing horizon and $S$ is not part of the horizons. In fact, the torsion one-form $\omega$ in (4) is defined with respect to $\xi$, so it makes no sense on $S$, where $\xi$ vanishes. The underlying reason why the two equations are the same will be discussed in the next subsection.

(ii) As already mentioned, the trivial solution $\zeta|_S = 0$ of this system, supplemented by $\partial_u \zeta^u|_S = \text{const.} \neq 0$ (see (51)) corresponds to data for the Killing vector $\xi$ which generates the bifurcate horizon (a vanishing surface gravity would produce the trivial vector field).

(iii) For the emerging spacetime to be maximally symmetric, (52)–(54) need to admit $\frac{1}{2} (n+1)(n+2) - 1$ independent solutions. It is shown in [18] that each of (53)–(54)
admits at most $n$ solutions, so a necessary condition is that the bifurcation surface $(S, \gamma)$ admits $\frac{1}{2} n(n - 1)$ Killing vectors, i.e. is maximally symmetric.

3.3.1. Comparison with the master equation. There are two immediate differences between the initial data equation (53) and the master equation (4). First, the torsion-one forms are a priori different, and, second, the former equation involves one of the components of the Killing vector in adapted null coordinates and the latter involves the proportionality function between two Killings. Nevertheless, the two equations can be related to each other.

We discuss first the issue of the different torsion one-forms. In the gauge where $\nu = 1$ we have shown before that

$$\xi H_{\xi}^s = \kappa_{\xi} u \partial_u$$

were $\kappa_{\xi}$ is the surface gravity of the bifurcate Killing horizon with respect to $\xi$. On each cross section $(1)S$ the proportionality between $\xi$ and $K$ is constant, and by (7) we have $s = (1)s$. Moreover, $(1)s$ is independent of $u$ by (29) and since $(1)s = \zeta$ on $S$ we conclude that actually $s = \zeta$. Had we chosen another cross section of $H_{1}$, this is defined implicitly by a positive graph function $\psi : S \rightarrow \mathbb{R}$ as $S_{\psi} := \{u = \psi\}$. Defining $\tau : H_{1} \rightarrow \mathbb{R}$ as

$$\tau = \frac{1}{\kappa_{\xi}} \ln u$$

we have $\zeta(\tau) = 1$. Applying [17, lemma 3] the torsion one-form of $S_{\psi}$ is

$$s[\psi] = s + \psi^{-1} d\psi$$

and the freedom in changing the section corresponds to the freedom of adding differentials to $\zeta$. Indeed, the coordinate transformation $\{\tilde{u} = \psi^{-1} u, \tilde{v} = \psi v\}$ preserves $\nu = 1$, transforms the torsion $\zeta$ as in (44) and makes $S_{\psi} = \{\tilde{u} = 1\}$, so that now we have a constant $\tilde{u}$ section and by the previous argument $s[\psi] = \zeta$ must hold.

Let us compute how the KID equations on $S$ behave under the transformation (44). Clearly, $L_{\zeta}\zeta$ is exact if and only if $L_{\zeta}\zeta$ is exact, i.e. (52) remains invariant as is should be. One further finds that $\zeta_{\nu}$ is a solution of (53) if and only if $\psi\zeta_{\nu}$ is a solution of (53) with $\zeta$ replaced by $\zeta$. In particular, whenever (53) admits a solution with no zeros we can find a gauge where this solution is given by $\zeta_{\nu} = 1$, and in this gauge the following relation holds

$$D_{A(S\gamma)} - s A S\gamma + \frac{1}{2} \gamma R_{AB} - \frac{A}{n - 1} \gamma_{AB} = 0. \quad (55)$$

If $\zeta_{\nu}$ has zeros, the Killing vector it generates will vanish at these points, which therefore do not belong to its Killing horizon. Restricting $\zeta_{\nu}$ to the domain where it does not have zeros, the above rescaling can be done. In the gauge (55) the KID equation (53) becomes

$$\left(D_{A(D\gamma)} - 2 s A D\gamma\right) \zeta_{\nu} \overset{5}{=} 0. \quad (56)$$

Again we find a deep connection with the master equation [18] as given in (4) because the null second fundamental form vanishes on a bifurcation surface. We emphasize once more that, despite their identical form, the equations are intrinsically different as they involve different objects and are constructed on different types of surfaces.

We now turn into the issue that the master equation in [18] involves the proportionality factor $f$ between $\eta$ and $\xi$ while equation (53) is for a component of $\eta$ at the bifurcation surface. We use the proportionality (1) in lemma 1 together with $\tau = \kappa_{\xi}^{-1} \ln u$ to conclude that
\[ \eta H^+ = u^{-1} f_\eta \xi = \kappa f_\eta \partial_u \]  

(57)

and recall that \( f_\eta \) is constant along \( \xi \), hence independent of \( u \). Expression (57) extends to the bifurcation surface and we get \( \eta_0 = \eta^u = \kappa f_\eta \) on \( S \). On the other hand, evaluating on the cross section \( \{ u = u_0 \} \) with \( u_0 \) a positive constant, the proportionality function \( f \) between \( \eta \) and \( \xi \) is \( f = f_\eta u_0^{-1} = \eta_0 |s| \kappa^{-1} u_0^{-1} \). The master equation (4) is linear so the multiplicative constant \( \kappa^{-1} u_0^{-1} \) can be dropped. This, together with the equality (55) explains the full coincidence in form of the KID equation (53) and the master equation (4), while at the same times makes it clear the intrinsic difference between the two (observe that \( f = f_\eta u_0^{-1} \) makes no sense at \( u_0 = 0 \)).

4. Non-degenerate MKHs

Let us return to our original question. For \( \mathcal{H}_1^+ \) to be a multiple Killing horizon there needs to exist at least one Killing vector \( \eta \) which satisfies (27) in order to be tangential and null on \( \mathcal{H}_1^+ \) and independent of \( \xi \). It follows straightforwardly from (32)–(33) that this will be the case if and only if

\[ \eta_\xi |S \neq 0, \quad \eta_\eta |S = 0, \quad \eta_\lambda |S = 0. \]  

(58)

In that case the KID equations (52) and (54) become trivial. Whenever (53) admits a non-trivial solution it can be extended to a Killing vector field \( \eta \) for which \( \mathcal{H}_1^+ \) (but not \( \mathcal{H}_2^+ \)) is a Killing horizon and, solving the corresponding problem into the past, also \( \mathcal{H}_1^- \).

As consequence of theorem 6 we find that the master equation is in this setting not only necessary but also sufficient for the Killing horizon to be multiple.

**Corollary 1.** Let \((M, g)\) be an \((n + 1)\)-dimensional \( \Lambda \)-vacuum spacetime, \( n \geq 3 \), which admits a Killing vector \( \xi \) which generates a bifurcate Killing horizon \( \mathcal{H} = \mathcal{H}_1^+ \cup \mathcal{H}_2^+ \cup \mathcal{H}_1^- \cup \mathcal{H}_2^- \cup S \) (which is then non-degenerate). Let \((\gamma, \varsigma)\) be the free data on \( S \). Then \( \mathcal{H}_1^\pm \) is a MKH of order \( m \) if and only if the following equation admits \( m \) independent non-identically zero solutions \( f \) on \( S \),

\[ \left( D_\lambda D_\beta - 2 s_\lambda (D_\beta) - D_\lambda (s_\beta) + s_\lambda s_\beta - \frac{1}{2} \gamma R_{\lambda \beta} + \frac{\Lambda}{n - 1} \gamma_{\lambda \beta} \right) f = 0. \]  

(59)

Correspondingly, \( \mathcal{H}_2^\pm \) is a multiple Killing horizon of order \( m \) if and only if the following equation admits \( m \) independent non-trivial solutions \( f \) on \( S \),

\[ \left( D_\lambda D_\beta + 2 s_\lambda (D_\beta) + D_\lambda (s_\beta) + s_\lambda s_\beta - \frac{1}{2} \gamma R_{\lambda \beta} + \frac{\Lambda}{n - 1} \gamma_{\lambda \beta} \right) f = 0. \]  

(60)

**Remark 6.** As described in section 3.3.1, whenever (59) (similarly for (60)) admits a solution with no zeros, one can globally introduce a gauge where (55) holds. This equation appears in the context of near-horizon geometries and is sometimes called NHG equation. Actually, (59) (and (60)) supplemented with the condition that \( f \) has no zeroes has been established in [14] as a necessary and sufficient condition on the data on a bifurcate horizon to generate a near-horizon geometry (which contains a non-degenerate MKH). The solvability of this equation is analyzed in \( 3 + 1 \) dimensions if the bifurcation surface \( S \) is topologically a 2-sphere in [2, 13, 14], and in particular in [7] it is shown that in \( 3 + 1 \)-dimensions, and if the bifurcation...
surface $S$ is compact with positive genus, the only solutions $(\gamma, \varsigma)$ to (55) satisfy $\gamma R = 2\Lambda$ and $\varsigma = 0$. Note, though, that in our setting even for a compact $S$, if the solution to (59) has zeros the gauge where (55) holds can in general be realized only on non-compact subsets of $S$. We devote section 5 to discuss in depth the connection of the KID equations with near horizon geometries and, in particular, put forward a generalized definition of near-horizon geometries admitting zeros of the degenerate Killing vector.

The restriction of the emerging Killing vector to the bifurcate horizon is computed by integrating the ODEs (32), (33) and (35) on $H^+_1$, and the corresponding ones on $H^+_2$. If $\eta$ has $H^+_1$ as Killing horizon we have

$$\eta_u H^+_1 = 0, \quad \eta_H H^+_1 = 0,$$

$$\eta^u H^+_1 = f + \kappa_\eta u, \quad \kappa_\eta = \text{const.}$$

$$\eta_v H^+_2 = f,$$

$$\eta_H H^+_2 = -v(D_A - 2\varsigma_A)f.$$  

The ODE for $\eta^u|_{H^+_2}$ (equivalently $\eta_u|_{H^+_2}$) yields a more complicated solution, so let us just mention here that the initial data are given by $\eta^u S = 0$ and $\partial v \eta^u S = -\kappa_\eta - f \partial u \eta v (50)$.  

As the notation already suggests, the constant $\kappa_\eta$ can be identified with the surface gravity of $H^+_1$ associated to $\eta$. Adding to the candidate field $\eta$ an appropriate multiple of $\xi$, one may always assume $\kappa_\eta = 0$, i.e. that the associated Killing horizon is degenerate. In this sense a solution to (59) uniquely defines a Killing vector field $\eta$ for which $H^+_1$ is a degenerate horizon.

We also observe that

$$[\xi, \eta]^u \triangleq -\eta^u \partial \xi^u \delta^\mu u \triangleq -\kappa_\xi f \delta^\mu u \neq 0,$$

i.e. apart from multiples of $\xi$ no Killing vector which has $H_1$ or $H_2$ as Killing horizon can commute with $\xi$. This is in accordance with theorem 2.

As a straightforward consequence of theorem 6 and corollary 1 we have:

**Corollary 2.** Let $(M, g)$ be an $(n+1)$-dimensional $\Lambda$-vacuum spacetime, $n \geq 3$, with at least $\frac{1}{2}n(n-1) + 2$ Killing vectors. Then any bifurcate Killing horizon contains a non-degenerate MKH.

**Proof.** The KID equation (52) admits at most $\frac{1}{2}n(n-1)$ independent solutions so (53)–(54) must admit at least one non-trivial solution. $\square$

Another useful consequence of the existence result is the following lemma, to be used later.

**Lemma 2.** Let $(S, \gamma)$ be an $(n-1)$-dimensional Riemannian manifold ($n \geq 3$) endowed with a one-form $\varsigma$. Let $f : S \rightarrow \mathbb{R}$ be a non-identically zero solution of (59) or (60). Then $f$ is non-zero on a dense subset of $S$.

**Proof.** We consider only (59), the other case is similar. Let $(M, g)$ be a $\Lambda$-vacuum spacetime with bifurcate Killing horizon generated by the data $(S, \gamma, \varsigma)$ and $\eta$ the Killing vector
on $D^+(\mathcal{H}_1^+ \cup \mathcal{H}_2^+)$ satisfying, in corresponding null coordinates, (61)--(65) with $\kappa_\eta = 0$. Assume that the set $\{ p \in S; f(p) = 0 \}$ has a non-empty interior $S^{(0)}$. Then by theorem 5, $\eta$ is identically zero on some neighbourhood of $S$, which can only happen for a Killing if it is zero everywhere and hence $f$ was in fact identically zero on $S$.

4.1. Non-degenerate MKHs of order $m \geq 3$

Let us assume that a $\Lambda$-vacuum spacetime admits a bifurcate Killing horizon such that $\mathcal{K}_1^+$ is a multiple Killing horizon of order $m \geq 3$ and let $\mathfrak{q}_0$ be its torsion one-form. Assume that the bifurcation surface is connected. Then (59) (with $\varsigma$ replaced by $\mathfrak{q}_0$) admits at least two independent non-trivial solutions. Select one such solution $f^{(1)}$ and let $\tilde{S}$ be the set where $f^{(1)}$ is non-zero. By lemma 2, $\tilde{S}$ is dense on $S$. On this set we can realize a gauge where $f^{(1)} = 1$. Throughout this section we denote by $\varsigma$ the corresponding torsion-one-form on $\tilde{S}$ where (55) holds.

The second solution will be denoted by $f$. It cannot be constant on open sets (otherwise it is proportional to $f^{(1)}$ on this open set, hence everywhere by lemma 2 and we would contradict the linear independence) and in our current gauge solves the master equation (56),

$$
\left( D_A D_B - 2\varsigma_A D_B \right) f = 0.
$$

As a solution is uniquely determined by the values $f$ and $df$ at one point, and all constant functions solve (66), we have

$$
N := \gamma^D(df, df) > 0
$$

everywhere on $\tilde{S}$. Equation (66) has the following first integrability condition (see [18, equation (61)])

$$
-\gamma R_{ABCD} D^Df + 2D_C D_{[A} \varsigma_{B]} - 2\varsigma_C \varsigma_{[A} D_{B]} f - 2D_{[A} D_{B]} \varsigma_{C] = 0.
$$

We contract with $D_C f$ and $\gamma^{BC}$, respectively,

$$
2ND_{[A} \varsigma_{B]} + 2\varsigma^C D_C D_{[A} \varsigma_{B]} - D_{[A} D^C f D_{B]} \varsigma + D_{[A} D^C f D_{B]} \varsigma = 0.
$$

We insert the second equation into the first one,

$$
ND_{[A} \varsigma_{B]} - 2D_{[A} D^C f D_{B]} \varsigma + 2D_{[A} \gamma R_{C]D_{B]} = 0,
$$

and apply $D^B f$,

$$
ND^B f D_{[A} \varsigma_{B]} = 0 \implies D^B f D_{[A} \varsigma_{B]} = 0.
$$

From this we deduce via (71) and (67) that the torsion one-form $\varsigma$ must be closed on $\tilde{S}$. Thus, the same is true for the original torsion $\mathfrak{q}_0$ (as they differ by an exact form). Unlike $\varsigma$, $\mathfrak{q}_0$ is smooth on the whole of $\tilde{S}$, so it must be closed everywhere:
Lemma 3. Consider an \((n + 1)\)-dimensional \(\Lambda\)-vacuum spacetime which admits a bifurcate Killing horizon such that \(\mathcal{H}_1\) is a multiple Killing horizon of order \(m \geq 3\). Then its torsion 1-form is closed.

Remark 7. In the setup of this lemma (so that in particular \(\varsigma\) is closed), consider the domain \(\tilde{S}\) and define \(h: \tilde{S} \to \mathbb{R}\) by

\[
h := 2|\varsigma|^2 - D_A \varsigma^A + \frac{2}{n - 1} \Lambda = \frac{1}{2} \left( \gamma R - \frac{2(n - 3)}{n - 1} \Lambda \right) + |\varsigma|^2.
\]

Applying \(D^A\) to (55) and using the contracted Bianchi identity for \(\gamma\), it follows that (compare [12] where such an equation appears in the context of vacuum near-horizon geometries)

\[
D_A h - 2h_A = 0
\]

which obviously implies

\[
D_A h D^Af = 2h_A D^Af.
\]

Inserting (72) and (55) into (70) one obtains

\[
h D_a f = D_a (\varsigma^b D_b f) - 2\varsigma_A B A^f.
\]

Equations (66), (74) and (75) arise naturally also in a different but related context, and several consequences of them have been obtained in [19]. From those results one can extract the following lemma.

Lemma 4. Let \((\tilde{S}, \gamma)\) be an \((n - 1)\)-dimensional Riemannian manifold \((n \geq 3)\) with a one-form \(\varsigma\) satisfying (55). Let \(p + 1\) (necessarily \(\leq n\)) be the dimension of the space of solutions of (66). If \(p \geq 1\), then

(i) \(\varsigma\) is exact,

(ii) locally, \((\tilde{S}, \gamma)\) is a warped product \(\tilde{S} = V \times \Sigma, \gamma = \bar{\gamma} + \Omega \bar{\gamma}, \Omega: V \to \mathbb{R}\), where \((\Sigma, \bar{\gamma})\) is a \(p\)-dimensional maximally symmetric Riemannian manifold.

We emphasize that, in the context of multiple Killing horizons of order \(m \geq 3\) where this lemma automatically applies, the globally defined torsion one-form \(\varsigma_0\) is in general not exact.

Theorem 7. Let \((M, g)\) be an \((n + 1)\)-dimensional \(\Lambda\)-vacuum spacetime which admits a bifurcate Killing horizon with one of its Killing horizons being a non-degenerate MKH of order \(m \geq 3\). Then the spacetime admits (locally) at least \(\frac{1}{2} m(m + 1)\) Killing vectors.

Proof. Without restriction assume that \(\mathcal{H}_1^+\) is non-degenerate MKH of order \(m \geq 3\). We consider the initial data on the bifurcation surface which generate this spacetime. Moreover, we restrict attention to the subset \(\tilde{S} \subseteq S\) where the gauge (55) can be realized. In particular (66), (74) and (75) hold and the space of solutions of (66) is \(m - 1\). By item (ii) of lemma 4, \((\tilde{S}, \gamma|\tilde{S})\) admits at least \(\frac{1}{2} (m - 2)(m - 1)\) (local) Killing vectors. Moreover, since \(\varsigma\) is exact, equation (54) restricted to \(\tilde{S}\) admits the same number of independent solutions as (66). Applying theorem 6, the data on \(\tilde{S}\), and therefore on \(S\), generate a spacetime with

\[
\frac{1}{2} (m - 2)(m - 1) + 2(m - 1) + 1 = \frac{1}{2} m(m + 1)
\]

locally defined Killing vectors, as claimed. \(\square\)
4.2. Vanishing torsion one-form

A particular case of relevance for characteristic initial data corresponding to a bifurcate horizon is the case when the torsion one-form \( \varsigma \) is exact and \((S, \gamma)\) is Einstein, i.e.

\[
\gamma R_{AB} = \frac{\gamma R}{n-1} \gamma_{AB},
\]

(76)

Note that for \( n \geq 4 \) the second Bianchi identity then implies \( \gamma R = \text{const.} \), while for \( n = 2, 3 \) \( (76) \) imposes no restriction whatsoever on \( \gamma \). The case \( n = 2 \) is special since all KID equations become ODEs, so we assume \( n \geq 3 \). We also assume that \( S \) is connected.

We want to analyze the space of solutions of the KID equations for such data. When \( \varsigma \) is exact, there is a global gauge where \( \varsigma = 0 \), which we assume from now on. Note that when \( \varsigma \) is merely closed, this gauge can also be imposed locally, so all results below of a local nature also hold in this case.

The KID equation to be solved is (see (59) and (60))

\[
\left( D_A D_B - \frac{\gamma R}{2(n-1)} \gamma_{AB} + \frac{\Lambda}{n-1} \gamma_{AB} \right) f = 0.
\]

(77)

We split (77) into trace and trace-free part,

\[
(D_A D_B f)_{tr} = 0,
\]

(78)

\[
\left( \Delta_{\gamma} - \frac{1}{2} \gamma R + \Lambda \right) f = 0.
\]

(79)

We determine the divergence of the first equation,

\[
D_A \Delta_{\gamma} f + \frac{1}{n-2} \gamma R D_A f = 0.
\]

(80)

Inserting (79) we obtain

\[
\left( \frac{n}{n-2} \gamma R - 2 \Lambda \right) D_A f + f D_A \gamma R = 0.
\]

(81)

When the first parenthesis is not identically zero, which we write as

\[
\gamma R \neq \frac{2(n-2)}{n} \Lambda,
\]

(82)

the equation can be integrated, the general solution being

\[
f = C \left| \frac{n}{n-2} \gamma R - 2 \Lambda \right|^{\frac{n}{n-2}}, \quad C = \text{const.}
\]

(83)

We insert it into (77),

\[
\frac{2(n-1)}{n-2} D_A \gamma R D_B \gamma R - \left( \frac{n}{n-2} \gamma R - 2 \Lambda \right) D_A D_B \gamma R
\]

\[
- \frac{1}{n-1} \left( \frac{\gamma R}{2} - \Lambda \right) \left( \frac{n}{n-2} \gamma R - 2 \Lambda \right)^2 \gamma_{AB} = 0.
\]

(84)

Summarizing, the following result has been proved.

**Lemma 5.** Consider data \((S, \gamma, \varsigma)\) generating a bifurcate Killing horizon. Assume that \( S \) is connected of dimension at least two, \( \varsigma \) is exact and (76) and (82) hold. Then a MKH which
belongs to the bifurcate Killing horizon can be at most of order 2. It is of order 2 if and only if (84) holds and then the solution of the KID equation in the gauge $\zeta = 0$ is (83).

Remark 8. If $n \geq 4$ equation (84) implies by (76) and (82) that there will be an MKH of order 2 if and only if $\gamma R = 2\Lambda$. In that case $f$ is constant on $S$.

Let us now restrict the data further by imposing $\gamma R = \text{const.}$ This holds in particular if $(S, \gamma)$ is maximally symmetric which we recall to be equivalent to

$$\gamma R_{ABCD} = 2 \frac{\gamma R}{(n-1)(n-2)} \gamma_{[A}[\epsilon^{CD} B].$$

(85)

with $\gamma R = \text{const.}$ We compute the divergence of (80),

$$\Delta_\gamma \left( \Delta_\gamma + \frac{\gamma R}{n-2} \right) f = 0,$$

(86)

and plug in (79)

$$\left( \gamma R - 2\Lambda \right) \left( \frac{n}{n-2} \gamma R - 2\Lambda \right) f = 0,$$

(87)

which requires

$$\gamma R = 2\Lambda \quad \text{or} \quad \gamma R = \frac{2(n-2)}{n} \Lambda.$$

(88)

Remark 9. If $n \geq 4$ data of the form (76) always have constant scalar curvature so the same conclusion can be drawn: a necessary condition for the existence of a multiple Killing horizon is (88), and $\gamma R = 2\Lambda$ is, by remark 8, also sufficient.

4.2.1. Case $\gamma R = 2\Lambda$. Let us consider the case where $\gamma R = 2\Lambda$ first. It is different from the second case only for $\Lambda \neq 0$, which we assume here. Then lemma 5 applies and $f = \text{const.} \neq 0$ is the only candidate solution, and, indeed, solves (77). It then follows from theorem 6 that the number of Killings of the vacuum solution generated by the data is $3 + k$, where $k$ is the dimension of the Killing algebra of $(S, \gamma)$. Thus,

Lemma 6. With the same assumptions as in lemma 5, impose further $\gamma R = 2\Lambda \neq 0$. Let $k \geq 0$ be the number of independent Killing vectors of $(S, \gamma)$. Then the vacuum spacetime generated by this data admits $k + 3$ Killing vectors and both horizons $H^+_a$, $a = 1, 2$, which belong to the bifurcate Killing horizon are multiple Killing horizons of order 2.

It follows from the comment just after theorem 5 that the second Killing vector with $H^+_1$ as Killing horizon must be different from the second Killing vector with Killing horizon $H^+_2$. Lemma 6 confirms the existence (for $\Lambda \neq 0$) of non-maximally symmetric spacetimes with non-degenerate multiple Killing horizons, as was already shown in 4 dimensions in [18, section 4.2].

Since maximally symmetric spacetimes of dimension $n - 1$ admit $\frac{1}{2} n(n - 1)$ independent Killing we also have from lemma 6

Corollary 3. Let $(S, \gamma)$ be maximally symmetric, connected and of dimension at least two and assume $\gamma R = 2\Lambda \neq 0$. Then the spacetime generated by the data $(S, \gamma, \zeta = d\psi)$ has $\frac{1}{2} n(n - 1) + 3$ Killing vectors and both horizons $H^+_a$, $a = 1, 2$ are multiple Killing horizons of order 2.
Remark 10. The bifurcation surface of the (Anti-)Nariai spacetime [25] is, depending on the sign of $\Lambda$, a round sphere or hyperbolic space with Ricci scalar $\gamma R = 2\Lambda$, and has vanishing torsion, i.e. by uniqueness of solution to the characteristic Cauchy problem it must be the solution predicted by the corollary.

For $\Lambda > 0$ there is another way of seeing this: one may start with a round sphere as bifurcation surface. Then the emerging spacetime will be spherically symmetric and by Birkhoff’s theorem (see e.g. [24]) the emerging spacetime must locally be isometric to the (Anti-)Nariai solution (Schwarzschild-de Sitter can be excluded since it has only $\frac{1}{2}n(n-1) + 1$ Killing vectors).

4.2.2. Case $\gamma R = \frac{2(n-2)}{n} \Lambda$. Let us devote attention now to the second case where

$$\gamma R = \frac{2(n-2)}{n} \Lambda.$$  \hfill (89)

Maximally symmetric vacuum spacetimes have vanishing Weyl tensor, so the Gauss identity on $S$ gives $0 = C_{ABCD} = \gamma R_{ABCD} - \frac{4}{n(n-1)} \Lambda \gamma_A \gamma_C \gamma_D \gamma_B$. whence (89) is necessary for the spacetime to be maximally symmetric.

Now (77) becomes

$$(D_A D_B f)_t = 0,$$ \hfill (90)

$$(\Delta_\gamma + \frac{2}{n} \Lambda) f = 0.$$ \hfill (91)

Let us assume that $(S, \gamma)$ is maximally symmetric and, in addition, simply connected, connected and complete. Then, depending on the sign of the cosmological constant, the bifurcation surface can be identified with Euclidean space, hyperbolic space or a round sphere. In fact, if we require (90)–(91) to admit a solution, the simply connectedness-requirement is not needed if the cosmological constant is positive [20, 27].

Vanishing cosmological constant. If $\Lambda = 0$ then $(S, \gamma)$ needs to be the Euclidean space and in that case $n$ independent solutions are given, in standard coordinates $(x^A)$, by $f = 1$ and $f = x^A$, $A = 3, \ldots, n+1$.

Positive cosmological constant. If $\Lambda > 0$ let us rescale the metric such that $\Lambda = n(n-1)/2$. Then $\gamma R = (n-1)(n-2)$, while $(S, \gamma)$ is the standard $(n-1)$-sphere. In that case $n$ independent solutions to (91) are given by the $\ell = 1$-spherical harmonics $Y_{\ell}$, $\Delta_{\gamma} Y_{\ell} = -\ell(\ell + n-2) Y_{\ell}$. Since the gradient of each function with $\Delta_{\gamma} Y_{\ell} = -(n-1) Y_{\ell}$ is a conformal Killing one-form [20], all $Y_{\ell}$’s satisfy the full system (90)–(91).

Negative cosmological constant. If $\Lambda < 0$ we rescale the metric such that $\Lambda = -n(n-1)/2$. Then $\gamma R = -(n-1)(n-2)$, while $(S, \gamma)$ is the standard hyperbolic space. By analytic continuation of the spherical harmonics of degree $\ell$ on an $n-1$-dimensional standard sphere to the $n-1$-dimensional standard hyperbolic space, both regarded as subsets of the complex unit sphere in $\mathbb{C}^n$, one obtains spherical harmonics of degree $\ell$ on the hyperbolic space [26]. These spherical harmonics are eigenfunctions of $\Delta_{\gamma}$ with eigenvalue $\ell(\ell + n-2)$. In particular the $\ell = 1$-spherical harmonics provide $n$ independent solutions which solve (91).
When the conditions on \((S, \gamma)\) being connected, simply connected or complete are dropped, the KID equations \((90) - (91)\) still admit \(n\) linearly independent local solutions. Thus, the following result holds (note that since the statement is local we may assume that \(\varsigma\) is merely closed).

**Lemma 7.** Let \((S, \gamma)\) be maximally symmetric and satisfying \((89)\) and \(\varsigma\) be closed. Assume \(n \geq 3\) and let \((M, g)\) be a spacetime generated by the characteristic initial value problem corresponding to a bifurcate Killing horizon with data \((S, \gamma, \varsigma)\). Then any point \(q \in S \subset M\) admits a spacetime neighbourhood \(U_q \subset M\) with \(\frac{1}{2}(n+1)(n+2)\) Killing vectors. In particular \((M, g)\) is maximally symmetric in some neighbourhood of \(S\).

One may also consider whether the converse is true, i.e. given a spacetime with a bifurcate Killing horizon with bifurcation surface \(S\) and maximally symmetric near \(S\), whether the data \((S, \gamma, \varsigma)\) must be as in lemma 7. According to remark 5 (iii) the bifurcation surface \((S, \gamma)\) needs to be maximally symmetric, i.e. admit \(\frac{1}{2}n(n-1)\) independent (local) Killing vectors \(\zeta^{(b)}, b \in \{1, \ldots, n(n-1)/2\}\). Each of these Killing vectors need to extend (also locally) to a spacetime symmetry which requires, by theorem 5, that
\[
\mathcal{L}_{\zeta^{(b)}} D_{[A} [\varsigma_{B]} = 0 \quad \forall b \quad \implies D_{[A} [\varsigma_{B]} = 0.
\]
i.e. \(\varsigma\) needs to be closed. Finally, as already mentioned the Gauss identity on \(S\) forces \((89)\). Thus, indeed the converse of lemma 7 holds true.

Combining this converse with the uniqueness of solutions of the characteristic initial value problem near the initial hypersurface we conclude

**Lemma 8.** Let \((S, \gamma)\) be maximally symmetric, complete, connected and simply connected and \(\varsigma\) exact. Let \((M, g)\) be a spacetime generated by the characteristic initial value problem corresponding to a bifurcate Killing horizon with data \((S, \gamma, \varsigma)\). Then, \((M, g)\) is isometric to a portion of de Sitter \((\Lambda > 0)\), Minkowski \((\Lambda = 0)\) or anti-de Sitter \((\Lambda < 0)\).

### 4.3. Non-degenerate MKHs of order \(m = 2\) for \(\Lambda = 0\)

The (Anti-)Nariai solution provides an example of a \(\Lambda \leq 0\)-vacuum spacetime which admits a non-degenerate MKH of order 2. We therefore aim to construct characteristic initial data which generate \(\Lambda = 0\)-vacuum spacetimes with non-degenerate MKH of order 2. For this we consider, in \(n + 1\) dimensions, initial data with vanishing torsion 1-form. Then \((59)\) becomes
\[
\left(D_A D_B - \frac{1}{2} \gamma R_{AB}\right) f = 0. \tag{92}
\]
We want to find a metric \(\gamma\) for which these equations admit precisely one non-trivial solution. For this we assume \(n \geq 3\) (if \(n = 2\), then \((92)\) is a second order linear ODE which always has two independent solutions) and that \((S, \gamma)\) is a warped product metric with locally flat \(n - 2\) dimensional fibers, so that the metric can be written in the local form
\[
\gamma = dx^2 + \Omega^2(x) \delta, \quad \delta = \sum_{P=1}^{n-2} (dy^P)^2, \quad n \geq 3. \tag{93}
\]
Then
\[
\gamma R_{st} = -(n - 2) \frac{\partial^2 \Omega}{\Omega}, \quad \gamma R_{PQ} = - \left((n - 3)(\partial_s \Omega)^2 + \Omega \partial_s^2 \Omega\right) \delta_{PQ}. \tag{94}
\]
\[ \gamma R_{xp} = 0, \quad \gamma R = -(n - 2) \left( (n - 3) \frac{(\partial_x \Omega)^2}{\Omega^2} - 2 \partial_x^2 \Omega \right), \quad (95) \]

and (92) becomes

\[ \partial_x^2 f + \frac{(n - 2)}{2} \frac{\partial_x^2 \Omega}{\Omega} f = 0, \quad (96) \]

\[ \partial_x \partial_y f + \Omega \partial_x \Omega \partial_P \partial_x f + \frac{1}{2} \left( (n - 3)(\partial_x \Omega)^2 + \Omega \partial_x^2 \Omega \right) \partial_P \partial_x f = 0, \quad (97) \]

\[ \left( \partial_x - \frac{\partial_x \Omega}{\Omega} \right) \partial_x f = 0. \quad (98) \]

We assume that the metric \( \gamma \) is not flat, as otherwise (92) clearly admits \( n \) (hence more than one) linearly independent solutions. Thus, \( \partial_x \Omega \) is not identically zero and when \( n = 3 \) also \( \partial_x^2 \Omega \) is not identically zero. When \( n > 3 \) it is easy to show that if \( \Omega = \Omega_0 x + \Omega_1 \) with \( \Omega_0 \) a non-zero constant, the system (96)–(98) admits no non-trivial solutions. So, we may assume that neither \( \partial_x \Omega \) nor \( \partial_x^2 \Omega \) vanish identically.

Equation (98) integrates to \( f = a(y) \Omega + b(x) \), which inserted into (96) yields

\[ \frac{n \partial_x^2 \Omega}{2} a(y) + \partial_x^2 b + \frac{n - 2}{2} \partial_x^2 \Omega \Omega b = 0. \]

Thus, \( a(y) \) is necessarily constant and \( f \) only depends on \( x \). Thus the system to solve is

\[ \partial_x^2 f + \frac{(n - 2)}{2} \frac{\partial_x^2 \Omega}{\Omega} f = 0, \quad (99) \]

\[ \Omega \partial_x \Omega \partial_x f + \frac{1}{2} \left( (n - 3)(\partial_x \Omega)^2 + \Omega \partial_x^2 \Omega \right) f = 0. \quad (100) \]

This is a system of two ODEs for two functions. We aim at finding its general solution. Replacing \( x \to -x \) if necessary we may assume that \( \partial_x \Omega > 0 \). Equation (100) can be integrated to

\[ f = \frac{f_0 \Omega^{\frac{n-1}{4}}}{\Omega^{\frac{n-1}{4}}} \]

where \( f_0 \) is a constant. Inserting into the other equation gives a rather involved third order ODE for \( \Omega(x) \). It is more convenient to perform the change of variable

\[ dx = \frac{dz}{w(z)} \quad (101) \]

and use the freedom introduced by \( w(z) \) to impose the equation

\[ \frac{d\Omega}{dz} = \frac{\Omega}{zw}. \quad (102) \]

Note that in terms of this variable, \( f = f_0 \Omega^{-\frac{n-1}{4}} \). A straightforward computation shows that equation (99) becomes

\[ zw \frac{dw}{dz} = w^2 + w(n - 2) - \frac{n(n - 2)}{4} := P^{(n)}(w). \quad (103) \]
This is a separable equation which can be solved explicitly. Note that the second order polynomial $P(n)(w)$ vanishes at two real and non-zero values

$$w_{\pm} := \frac{-(n - 2) \pm \sqrt{2(n - 1)(n - 2)}}{2},$$

so that $w(z) := w_\pm$ solve the ODE. When $w(z)$ is not constant, $P(n)(w)$ is not identically zero and we get

$$z = \exp \left( \int \frac{dz}{z} \right) = \exp \left( \int \frac{w \, dw}{P(n)(w)} \right) = \frac{z_0 |w - w_+|^{\frac{p_+}{n}}}{|w - w_-|^{\frac{p_-}{n}}},$$

where $z_0$ is a constant. We elaborate further each case. When $w := w_\pm$, equation (101) integrates to

$$z^2 = 2w_\pm(x - x_0).$$

The constant $x_0$ may be set to zero by shifting $x$. Then (102) gives

$$\Omega(z) = c_0 |z|^{\frac{p_+}{n}} := c_0 |z|^{p_\pm} = \tilde{c}_0 |x|^{\frac{p_+}{n}},$$

where $c_0, \tilde{c}_0$ are non-zero constants and

$$p_\pm := \frac{2}{n} \left( 1 \pm \sqrt{\frac{2(n - 1)}{n - 2}} \right).$$

The constant $\tilde{c}_0$ can be absorbed into the coordinates \{y_P\} and we can also assume $x > 0$ without loss of generality. So, the metric and the function $f$ takes the form (redefining $f_0$ conveniently)

$$\gamma = dx^2 + x^{p_\pm} \delta, \quad f = f_0 x^{\frac{2 - (n - 2)p_\pm}{n}}.$$ Observe that $p_+ = 2$ only for the plus sign and $n = 3$ and this case has to be excluded as it corresponds to a flat metric. Thus, for $n = 3$ only $p_- = -\frac{2}{3}$ survives.

When $w(z)$ is not constant, we may use $w$ as coordinate. The equation for $\Omega(w)$ is, from (102) and (103)

$$\frac{d\Omega}{dw} = \frac{\Omega}{(w - w_+)(w - w_-)} \quad \implies \quad \Omega(w) = \Omega_0 \frac{w - w_+}{|w - w_-|^{\frac{1 + p_+}{n}}}, \quad \Omega_0 \in \mathbb{R}^+$$

so that the metric is, after absorbing a multiplicative constant in \{y_P\} and redefining a suitable constant $C > 0$,

$$\gamma = C \left| \frac{w - w_+}{w - w_-} \right|^{\frac{2(n - 2)p_+}{n(n - 2)}} \, dw^2 + \left| \frac{w - w_+}{w - w_-} \right|^{\frac{2(n - 2)p_-}{n(n - 2)}} \, \delta.$$ (104)

In this case the solution of (92) is

$$f := f_0 \frac{|w - w_+|^{\frac{2(n - 2)p_+}{n(n - 2)}}}{|w - w_-|^{\frac{2(n - 2)p_-}{n(n - 2)}}}.$$ Summarizing:
Lemma 9. Consider data \((S, \gamma, \zeta = 0)\), with \((S, \gamma)\) a warped product with locally flat \((n - 2)\)-dimensional fibers for the characteristic initial value problem for the \((\Lambda = 0)\)-vacuum equations on a bifurcate horizon in \((n + 1)\)-dimensions, \(n \geq 3\). Then the bifurcate horizon is a non-degenerate MKH of order 2 if and only if either (i) or (ii) hold:

(i) The metric takes the local form \(\gamma = dx^2 + x^p \delta\) where \(\delta\) is the flat \(n - 2\) dimensional metric, and
   
   (a) \(p = -\frac{2}{n}\) for \(n = 3\),
   
   (b) \(p = \frac{2}{n} \left(1 \pm \sqrt{\frac{2(n-1)}{n-2}}\right)\) for \(n \geq 4\).

(ii) The metric takes the local form \((104)\).

Remark 11. As any line element \(\gamma = F(x)dx^2 + H(x)\delta\) has at least \(\frac{1}{2}(n-2)(n-1)\) (local) Killing vectors, the emerging \((\Lambda = 0)\)-vacuum spacetimes in this lemma admit, by theorem 6, at least \(\frac{1}{2}(n-2)(n-1) + 3\) (local) Killing vectors.

Remark 12. The spacetime in case (i) can be fully determined. Recall the Kasner vacuum solution \([11]\) which can be written in the form

\[
g = -x^{2p_0}(dt)^2 + (dx)^2 + \sum_{A=2}^{n} x^{2p_A}(dy^A)^2, \quad p_0 + \sum_{A=2}^{n} p_A = 1, \quad (p_0)^2 + \sum_{A=2}^{n} (p_A)^2 = 1. \quad (105)
\]

Now choose \(p_A := p/2\), \(A = 3, \ldots, n\), and \(p_2 = p_0 = q/2\), and set \(u := (t - y^2)/\sqrt{2}\), \(v := (t + y^2)/\sqrt{2}\). Then

\[
g = 2^{1-\frac{2q}{n}} x^q du dv + (dx)^2 + x^{p_2} \delta \quad \text{with} \quad p_\pm = \frac{2}{n} \left(1 \pm \sqrt{\frac{n-1}{n-2}}\right). \quad (106)
\]

We conclude that the data constructed in lemma 9 (i) generate a subfamily of the Kasner metrics with non-degenerate MKH of order 2 (for \(n = 3\) we have \(p_+ = 2\) and this corresponds to the Minkowski metric which has a MKH of maximal order).

4.4. A no-go result

In section 4.1 we have shown that whenever a \(\Lambda\)-vacuum spacetime admits a bifurcate Killing horizon with a MKH of order \(m \geq 3\), the torsion one-form needs to be closed. We work on the dense domain \(\tilde{S}\) where the gauge condition \((55)\) can be imposed. The integrability condition \((68)\) becomes

\[
\gamma R_{ABCD}D^P f - D_A (\gamma R_{BC|D} - \frac{2}{n-1} \Lambda \gamma_{BC|D}) = 0. \quad (107)
\]

Let us assume now that \(m \geq n\), i.e. that the bifurcate horizon is a MKH of order \(n\) or \(n + 1\). Then \((66)\) admits at least \(n - 2\) independent non-constant solutions. The set of points \(q \in \tilde{S}\) where all the gradients of these solutions are non-zero and linearly independent is also a dense set. Locally near such points we can construct an orthonormal frame \((e_\tilde{A}) = (e_\hat{\gamma}, e_\hat{\emptyset})\), \(\hat{a} = 4, \ldots, n + 1\), such the \(e_\hat{\emptyset}\)'s are in the tangent plane generated by those gradients. Then \((107)\) can be written in these frame components as

\[
\gamma R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = \frac{1}{2} \delta_{\hat{\alpha}\hat{\beta}} \left(\gamma R_{\hat{\gamma}\hat{\delta}} - \frac{2}{n-1} \Lambda \delta_{\hat{\gamma}\hat{\delta}}\right) - \frac{1}{2} \delta_{\hat{\beta}\hat{\delta}} \left(\gamma R_{\hat{\gamma}\hat{\beta}} - \frac{2}{n-1} \Lambda \delta_{\hat{\gamma}\hat{\beta}}\right). \quad (108)
\]
\[
\gamma R_{\Lambda \Xi} = -\delta_{\Lambda \Xi} \left( \frac{\gamma R - 2(n-2)}{n-1} \Lambda \right), \quad \gamma R_{\Xi\Xi} = \frac{2(n-2)}{n(n-1)} \Lambda,
\]

whence
\[
\gamma R = \frac{2(n-2)}{n} \Lambda \implies \gamma R_{\Lambda \Xi} = \frac{2(n-2)}{n(n-1)} \Lambda \gamma_{\Lambda \Xi} \implies \gamma R_{A B C D} = \frac{4\Lambda}{n(n-1)} \gamma_{\Lambda (c^\gamma d)\beta},
\]
i.e. the bifurcation surface \((S, \gamma)\) needs to be maximally symmetric. Now we change the gauge. Instead of (55) we transform to a gauge where the torsion one-form vanishes, which can be done at least locally. Then we are in the setting of section 4.2, the emerging \(\Lambda\)-vacuum spacetime will have, at least locally, the maximum number of Killing vectors.

**Proposition 2.** Let \((M, g)\) be an \((n+1)\)-dimensional \(\Lambda\)-vacuum spacetime which admits a bifurcate Killing horizon such that one of its horizons is a MKH of order \(m \geq n\). Then \((M, g)\) is locally isometric to either Minkowski or (Anti-)de Sitter, depending on the sign of the cosmological constant. In particular a MKH of order \(m = n\) which belongs to a bifurcate horizon does not exist.

Summarizing this section, in \(3+1\) dimensions we have seen that non-degenerate MKHs of order 3 do not exist, at least not as part of a bifurcate Killing horizon, while the maximal order 4 is obtained by maximally symmetric spacetimes. The (Anti-)Nariai solution provides an example of a \(\Lambda \leq 0\)-vacuum spacetime which admits a non-degenerate MKH of order 2. An example of a \(\Lambda = 0\)-vacuum spacetime which admits a non-degenerate MKH of order 2 are certain Kasner metrics, see section 4.3.

5. Near-horizon geometries

Let us consider a \(\Lambda\)-vacuum spacetime \((M, g)\) which admits a Killing vector field \(\eta\) which has a degenerate Killing horizon \(H_1\). In a neighborhood of \(H_1\) and in Gaussian null coordinates with \(H_1 = \{v = 0\}\) and \(\eta = \partial_v\) the metric takes the form
\[
g = 2du dv + 4\mathcal{E}_A(v, x^B) dudA^A + v^2 h(v, x^A) dv^2 + g_{AB}(v, x^C) dx^A dx^B. \tag{109}
\]
The associated near-horizon geometry is
\[
g_{\text{NH}} = 2du dv + 4\mathcal{E}_A(0, x^B) dudA^A + v^2 h(0, x^A) dv^2 + g_{AB}(0, x^C) dx^A dx^B. \tag{110}
\]
Usually it is obtained as a certain limit of the original spacetime (109), see [12]. For our purposes, it is more enlightening how it can be obtained from a characteristic initial value problem. For this we consider the cut \(S = \{u = 0, v = 0\}\) of the horizon \(H_1\), and we keep the same notation for the corresponding objects in the metric (110). The null second fundamental forms \((K)\) induced by the line elements (109) or (110) on \(H_1^+ := H_1 \cap \{u > 0\}\) vanish. The data induced on \(S\) relevant for the characteristic initial-value problems are the \((n-1)\)-dimensional Riemannian manifolds \((S, \gamma_{AB} = g_{AB}|_S)\) and the torsion one-forms \(\zeta_A = \mathcal{E}_A|_S\). Hence, both metrics (109) and (110) induce the same data on \(H_1^+\) and \(S\). To generate either (109) or (110) these data need to be supplemented with the null second fundamental forms induced by either (109) or (110) on the null hypersurfaces \(H_2^+ := \{u = 0\} \cap \{v > 0\}\) (again we keep the same notation for both metrics) generated by the null geodesics intersecting \(H_1\) transversally on \(S\). These null second fundamental forms are in both cases proportional to \(\partial_v g_{AB}\). Thus, such a second fundamental form is in general different from zero for the case of the metric (109) but,
in contrast, it vanishes for the case of the metric (110). In other words, in terms of a characteristic initial-value problem, taking the near-horizon limit corresponds to replacing the induced null second fundamental form on the initial hypersurface surface \( N_2^+ \) by trivial data. It is worth stating this as a proposition.

**Proposition 3.** Let \((M, g)\) be a \(\Lambda\)-vacuum spacetime which contains a degenerate Killing horizon \( H_1 \), generated by a Killing vector \( \eta \). Then take any cut \( S \) of \( H_1 \), and let \( N_2 \) denote the null hypersurface which is generated by the null geodesics through \( S \) which are transversal to \( H_1 \). Denote by \((1) K_{AB} \cong 0, (2) K_{AB}, g_{AB}|_S, \varsigma\) the induced characteristic initial data. Then the associated NHG is the \(\Lambda\)-vacuum spacetime obtained as solution of the characteristic initial value problem for these data with \((2) K_{AB}\) replaced by trivial data.

It is clear from the construction (and theorem 5) that the intersection surface \( S \) becomes the bifurcation surface of a bifurcate Killing horizon. Thus the construction automatically implies that the near horizon spacetime admits an additional Killing vector field \( \xi \). In a sense, the present geometric reinterpretation of the near horizon limit explains why near horizon geometries always admit an additional Killing vector \( \xi = u\partial_u - v\partial_v \), which vanishes at a bifurcation surface and generates a bifurcation horizon an observation made in [12] (see also [15, 18, 19, 21]).

In view of the KID equations for the bifurcate Killing horizon data, the original Killing vector \( \eta \) corresponds to a solution of the KID equations (59) (or (60)) which vanishes nowhere. Thus, we can identify which characteristic data on a bifurcate horizon yield near horizon geometries.

**Corollary 4.** Characteristic initial data on a bifurcate horizon \((H_1 \cup H_2 \cup S)\) consist of a Riemannian metric \(\gamma\) and the torsion 1-form \(\varsigma\) on \( S \). The emerging \(\Lambda\)-vacuum spacetime is a NHG if and only if (59) or (60) admits a nowhere zero solution.

**Remark 13.** In the case \(\Lambda = 0\), this result was proved in [14] by explicit construction of the spacetime. Indeed, given such data on \( S \), there exists a global gauge where the torsion one-form satisfies (55). Then, one can define \( h \) as in (72), write down the metric (110) and check that it solves the \(\Lambda\)-vacuum field equations. This is a very direct way of establishing existence, but it is heavily limited by the condition that the solution of (59) is nowhere zero. Our existence results do not rely on this, so it seems reasonable to allow for fixed points of \( \eta \). In fact, one of the main interests of viewing the near horizon limit as a characteristic initial value problem is that it does not depend on the fact that the Killing vector \( \eta \) is non-zero on \( H_1 \), something which is crucial in the coordinate limit definition, since it relies on Gaussian null coordinates adapted to \( \eta \). This suggests the existence of a natural generalization of the concept of NHG, which we put forward in definition 5 below.

**Remark 14.** Another interesting characterization of vacuum near horizon geometries alternative to corollary 4 is given in [15], where it is shown that a vacuum spacetimes is a near horizon geometry if and only if admits a foliation of non-expanding horizons (supplemented by a transverse one).

It follows from the KID equations and the fact that the master equation (4) must hold on \( S \) for each Killing vector field \( \eta^{(k)} \) of the original spacetime for which \( H_1 \) is a degenerate horizon that the associated solutions \( f^{(k)} \) of (59) (which need to be independent) generate, by corollary 1, independent Killing vectors of the near-horizon geometry too, for which \( H_1^+ \) is also a degenerate horizon. We recover [18, theorem 4.1] in the \(\Lambda\)-vacuum case:
Corollary 5. Consider a \(\Lambda\)-vacuum spacetime which admits a fully degenerate (multiple) Killing horizon of order \(m \geq 1\). Then there exists an associated \(\Lambda\)-vacuum spacetime, the near-horizon limit, which has a non-degenerate MKH of order \(m + 1\).

5.1. Near-horizon limit if \(\text{dim}(\mathcal{A}_{\text{deg}}) \geq 2\)

Let us now assume that for the metric (109) \(\mathcal{H}_1 = \{v = 0\}\) is a MKH and that there exists another Killing vector \(\tilde{\eta}\) for which \(\mathcal{H}_{\tilde{\eta}}\) is a degenerate Killing horizon satisfying \(\mathcal{H}_{\tilde{\eta}} = \mathcal{H}_1\), i.e. we assume \(\text{dim}(\mathcal{A}_{\text{deg}}) \geq 2\) (recall its definition in theorem 2). Then the near-horizon limit can be taken with respect to both \(\eta\) or \(\tilde{\eta}\). In the latter case one would first need to transform to Gaussian null coordinates \((\tilde{u}, \tilde{v}, \tilde{x}^A)\). However, here we only need to observe the following:

with \(\tilde{\eta} = f\eta\), we find that, at points where \(f\) is not zero,

\[
\tilde{g}_{AB} = g_{AB}, \quad \tilde{s} = s - d\log |f|, \tag{111}
\]

and this implies that the characteristic Cauchy data for the near-horizon geometries associated to \(\eta\) and \(\tilde{\eta}\) coincide up to gauge transformations (see section 3.3). This way we recover [19, theorem 7] in the \(\Lambda\)-vacuum case:

Corollary 6. Let \((M, g)\) be a \(\Lambda\)-vacuum spacetime containing a MKH \(\mathcal{H}\) and let \(\eta, \tilde{\eta} \in \mathcal{A}_{\text{deg}}^{\mathcal{H}}\). Then the near-horizon geometries w.r.t. \(\eta\) and \(\tilde{\eta}\) are locally isometric.

5.2. Generalized near horizon geometry

Consider a spacetime admitting a null hypersurface \(\mathcal{N}\) and a (non-trivial) Killing vector \(\eta\) which, at \(\mathcal{N}\), is tangent and null. Since \(\eta\) vanishes at most on codimension-two submanifolds, there is a Killing horizon \(H_1\) of \(\eta\) satisfying \(\mathcal{H}_1 = \mathcal{N}\). Assume further that this horizon is degenerate. To determine the NHG in the usual way we need to restrict to (connected components of) \(H_1\) since fixed points of \(\eta\) have to be removed in order to construct Gaussian null coordinates. However, our construction in terms of characteristic initial data provides an immediate alternative definition of NHG for \(\Lambda\)-vacuum, which does not care about fixed points of \(\eta\):

Definition 5. Let \((M, g)\) be \(\Lambda\)-vacuum and admit a connected null hypersurface \(\mathcal{N}\) and a Killing vector \(\eta\) as described above. Consider any cut \(S\) of \(\mathcal{N}\) and let \(\gamma\) be the induced metric of \(S\) and \(\zeta\) the torsion one-form with respect to any choice of null frame. A generalized near horizon limit spacetime of \((M, g)\) is a spacetime obtained by solving the characteristic initial value corresponding to a bifurcate Killing horizon and data \((S, \gamma, \zeta)\).

Remark 15. It may appear from the wording of the definition that there are many possible generalized near horizon limits, but this is merely a consequence of the fact that, to the best of our knowledge, there is no existence result of a unique maximal Cauchy development in the characteristic case. All spacetimes generated by the data are isometric in some neighbourhood of \(S\).

Remark 16. When \(\eta\) has no fixed points, the generalized near horizon geometry is the same as the standard near horizon limit.
Remark 17. By corollary 1, the null hypersurface $N$ of the generalized near horizon limit is a non-fully degenerate multiple Killing horizon of order at least two.

6. Fully degenerate MKHs

So far we have studied the emergence of $\Lambda$-vacuum spacetimes with MKHs in terms of a characteristic initial value problem with data given on a bifurcate Killing horizon. However, since the bifurcate horizon is necessarily non-degenerate with respect to the bifurcate Killing vector this approach permits merely the construction of non-degenerate MKHs. In this section we consider the case of fully degenerate MKHs, and obtain partial results.

In order to construct spacetimes with fully degenerate MKHs we need to modify the initial data in such a way that a bifurcate Killing vector does not arise. For that purpose, one of the initial null hypersurfaces, $H^+_1$, say, needs to have a non-vanishing null second fundamental form. Since this null hypersurface will therefore not be a Killing horizon of the emerging spacetime anymore, we will denote it by $N^+_2$. It turns out, though, that an analysis of the KID equations becomes rather intricate for such a general class of data: while the KID equations (16), (17) and (19) still provide ODEs for the Killing candidate on $H^+_1$ and $N^+_2$, the main problem is to arrange the data in such a way that (18) holds. In the case of a bifurcate horizon we have seen, see the paragraph after remark 4, that its second-order $u$- (respectively, $v$-) derivative vanishes on $H^+_1$ (resp. $H^+_2$) whence (18) becomes a constraint which only provides restrictions on $S$. This is no longer true for fully degenerate MKHs.

Fortunately, an analysis of the equations shows that their behavior is similar if additional conditions are imposed on the initial data. Hence, the strategy we are going to apply is as follows: first, in section 6.1 we basically restrict attention to data where the second fundamental form is independent of the $x^A$-coordinates (in an appropriate gauge), i.e. it is constant along appropriately selected cuts. We will further impose that these cuts are intrinsically flat and set $\Lambda = 0$ and a vanishing torsion one-form. In this way, a class of data generating fully degenerate MKHs is identified, and these data include solutions corresponding to gravitational plane waves. In section 6.2 we will restore a general $\Lambda$ and explore the Ansatz where the shear tensor vanishes on $N^+_2$, with no conclusive results concerning fully degenerate MKHs.

6.1. KID equations for non-vanishing null second fundamental form

The following result is proved in appendix A.1:

**Proposition 4.** Consider two smooth hypersurfaces, $H^+_1$ and $N^+_2$ in an $(n + 1)$-dimensional manifold with transverse intersection along a smooth $(n - 1)$-dimensional submanifold $S$ in adapted null coordinates and where the gauge conditions (42)–(43) are fulfilled. Consider characteristic initial data which satisfy

\[ \begin{align*}
(1) \pi_{AB} H^+_1 &= 0, \\
(1) \theta &= 0, \\
(2) \partial_C (\pi_{AB}) N^+_2 &= 0, \\
(2) \theta &= S, \quad \theta_{S} = \text{const.},
\end{align*} \]

as well as

\[ \Lambda = 0, \quad (S, \gamma) \cong \text{Euclidean space}, \quad \varsigma = 0. \]

Then the emerging vacuum spacetime admits $k$ Killing vectors if and only if the following set of KID equations,
admits \( k \) independent solutions, determined by data \( \zeta_0 | S \) and an additive integration constant which arises from the \( \partial_\nu \zeta_0 | S \)-equation. The data on \( \mathcal{H}_1^+ \) are then uniquely determined by (A.8)–(A.10) in appendix A.1 below.

Now, we use the previous proposition to try and derive initial data producing MKHs (not necessarily degenerate). To that end, we add initial data for \( \zeta_0 \); recalling (25) and (27) we keep \( \zeta_0 \) as the only non-zero component on \( S \) (as this is necessary for the Killing vector field to have \( \mathcal{H}_1 \) as a Killing horizon) so that as a corollary of proposition 4 we obtain:

**Corollary 7.** Under the same hypotheses as in proposition 4, the emerging vacuum spacetime has \( \mathcal{H}_1^+ \) as a MKH of order \( m \) if and only if the following set of equations admits \( m \) independent solutions, parameterized by \( (\eta_\nu | S = \eta_\nu | S = 0, \eta_\nu | S, \partial_\nu \eta_0 | S = \kappa_\eta = \text{const.}) \)

\[
\partial_\nu \eta_0 N^\nu_+ \equiv 0, \tag{112}
\]

\[
\left( \partial_\nu - \frac{2}{n-1} (^{(2)} \bar{g}) \right) \eta_0 N^\nu_+ \equiv 2 (^{(2)} \pi_\alpha^\beta \eta_0) - \partial_\nu \eta_0, \quad \eta_0 \equiv 0, \tag{113}
\]

\[
\partial_\nu \partial_\nu (\eta^\nu - \frac{1}{2} g^{\nu\rho} \eta_0) N^\nu_+ \equiv 0, \quad (\eta^\nu - \frac{1}{2} g^{\nu\rho} \eta_0) \equiv 0, \quad \partial_\nu (\eta^\nu - \frac{1}{2} g^{\nu\rho} \eta_0) \equiv -\kappa_\eta, \tag{114}
\]

\[
(\tilde{\nabla}_{(A)} \eta_B)_0 N^+_+ \equiv - (^{(2)} \pi_{AB}(\eta^\nu - \frac{1}{2} g^{\nu\rho} \eta_0)), \tag{115}
\]

\[
\Delta_\gamma \eta_0 \equiv -\theta_3 \kappa_\eta. \tag{116}
\]

**Remark 18.** The condition on \( \partial_\nu \eta_0 | S \) in proposition 4 is relevant to determine the initial data on \( S \) for the second-order ODEs for \( \eta^\nu | N^+_\nu \) and \( \eta^\nu | N^+_2 \) and is thus contained in the initial conditions in (114).

As in previous cases, \( \kappa_\eta \) represents the surface gravity relative to \( \eta \) of the emerging horizon. Hence, in order to get a fully degenerate MKH we need to make sure that every solution
of this system satisfies $\kappa_\eta = 0$. To that end, it is convenient to assume data which satisfy

$$(2)\pi_{AB}|_S = 0.$$  

Then, we determine the first- and second-order $\varphi$-derivative of (115) which yields with (113)--(114) and evaluated on $S$

$$\left.\frac{1}{n-1}\delta_S(\partial_A\partial_B\eta_\nu)\nu S\right|_S = -\kappa_\eta \partial_\nu (2)\pi_{AB},$$

which implies

$$\kappa_\eta \partial_\nu (2)\pi_{AB}|_S = 0.$$  

Clearly, $\kappa_\eta = 0$ must hold whenever $\partial_\nu (2)\pi_{AB}|_S \neq 0$.

To enforce $\kappa_\eta = 0$ let us therefore restrict attention to initial data of the form

$$(2)\pi_{AB}|_S = 0, \quad \partial_\nu (2)\pi_{AB}|_S \neq 0.$$  

(117)

By the latter condition we mean that $\partial_\nu (2)\pi_{AB}|_S$ is not identically zero, i.e. that at least one component (which is, by assumption, constant on $S$) is non-zero.

We conclude from corollary 7 that the emerging vacuum spacetime has $H^+$ as a fully degenerate Killing horizon of order $\Gamma$ if the following set of equations admits $\Gamma$ independent solutions $\eta_\nu = (0)^N (\chi^A)$ (with $\eta_\nu|_S = 0$)

$$\left(\partial_\nu - \frac{2}{n-1} \Omega^{(2)} \right)\eta_\nu - 2(2)\pi^{AB}_A \eta_B \leq -\partial_\nu (0)^N,$$  

(118)

$$\delta_{(\nu)} (0)^N \gamma \equiv 0,$$  

(119)

$$\Delta (0)^N \gamma \equiv 0.$$  

(120)

Using (119), the anti-symmetrized derivative of (118) reads

$$\left(\partial_\nu - \frac{2}{n-1} \Omega^{(2)} \right)\partial_{(\nu)} \eta_\nu + (2)\pi^{ABC} \partial_{(\nu)} \eta_\nu - (2)\pi^{ABC} \partial_{(\nu)} \eta_\nu \leq 0.$$  

As the initial data on $S$ for this equation vanish (see (113)) we deduce

$$\partial_{(\nu)} \eta_\nu \leq 0.$$  

Using (119) we also compute the symmetrized trace-free part and the divergence of (118),

$$\hat{\nabla}_{(\nu)} \eta_\nu \equiv 0, \quad \left(\partial_\nu - \frac{2}{n-1} \Omega^{(2)} \right)\hat{\nabla}_{(\nu)} \eta_\nu \leq -\hat{\Delta} (0)^N \gamma.$$  

(121)

The first relation implies (note that the Christoffel symbols of $g_{AB}|_{N^+_2}$ vanish),

$$\partial_\nu \hat{\Delta} (0)^N \gamma \equiv \partial_\nu g^{AB} \partial_{(\nu)} \partial_{(\nu)} \eta_\nu \equiv -\frac{2}{n-1} \Omega^{(2)} \hat{\Delta} (0)^N \gamma$$  

(120)

$$\hat{\Delta} (0)^N \gamma \equiv 0 \implies \hat{\nabla}_{(\nu)} \eta_\nu \equiv 0.$$  

Any solution of (118)--(120) is therefore necessarily of the form

$$\partial_\nu \eta_\nu \equiv 0, \quad \partial_{(\nu)} \eta_\nu \equiv 0.$$  

(121)
The ODE (118) can be integrated for data $\eta_A^S = 0$. As the source and the coefficients do not depend on the $x^A$'s the same will be true for $\eta_A|_{N_2^+}$, so that there are no further obstructions by (119). We deduce that the above system has $m = n$ independent solutions which are given by

$$\begin{align*}
(0) N^+_1 \equiv 1 \quad \text{and} \quad (0) N^+_2 \equiv x^A, \ A = 3, \ldots, n + 1,
\end{align*}$$

together with the corresponding solution $\eta_A$ of (118) with $\eta_A|_{S} = 0$.

We thus have:

**Proposition 5.** Consider two smooth hypersurfaces $H^+_1 = \{v = 0\}$ and $N^+_2 = \{u = 0\}$ in an $(n + 1)$-dimensional manifold with transverse intersection along a smooth $(n - 1)$-dimensional submanifold $S$ in adapted null coordinates and where the gauge conditions (42)–(43) are fulfilled. Consider characteristic initial data which satisfy

$$\begin{align*}
(1^{(1)}) \pi_{AB}^H H^+_1 = 0, \quad (1^{(2)}) \pi_{AB}^H = 0, \quad \partial_u \pi_{AB}^H = 0, \quad \pi_{AB}^S = 0, \quad \partial_u \pi_{AB}|_{S} \neq 0, \quad (2^{(2)}) \pi_{AB}^S = \text{const.}
\end{align*}$$

and where $\Lambda = 0$, $(S, \gamma) \cong \text{Euclidean space}$, $\zeta = 0$.

Then the emerging vacuum spacetime has $H^+_1$ as a fully degenerate multiple Killing horizon of maximal order $n$.

In appendix A.2 we prove the following proposition which addresses the issue how many independent Killing vectors the spacetimes generated in proposition 5 have.

**Proposition 6.** Under the same hypotheses as in proposition 5 we have:

(i) The emerging vacuum spacetime has at least $2n - 1$ Killing vectors.

(ii) For appropriate data there may be a maximum of $\frac{1}{2}(n - 1)(n - 2)$ + 1 additional Killing vectors, depending on whether the candidate fields $\zeta_{AB}^H(v)$ determined by (A.48) and (A.51) fulfill (A.52), see appendix A.2.

Given the properties of gravitational plane waves concerning MKHs ([18, subsection 4.4]) we deduce that such wave solutions are generated by initial data of the type found in proposition 5.

6.2. Vanishing shear

Again we aim at the construction of MKHs where the initial data do not lead to a bifurcate horizon. Before doing that, though, let us simplify the KID equations (16)–(24) for a Killing vector for which $H^+_1$ is a Killing horizon in a more general setting. The Killing vector needs to satisfy $\zeta_u|_{S} = \zeta_A|_{S} = 0$ and one checks that the system (16)–(24) becomes (we assume that the gauge conditions (42)–(43) are fulfilled),

$$\begin{align*}
(122) \quad \zeta_u H^+_1 = 0, \quad \zeta_A H^+_1 = 0,
\end{align*}$$

2nd-order ODE for $\zeta^u|_{H^+_1}, \zeta^e|_{S} = \zeta^e(x^A) \neq 0, \quad \partial_u \zeta^u \equiv \partial_u[\zeta^e]$, (123)

$$\begin{align*}
(124) \quad \zeta^e H^+_1 \equiv \zeta^e,
\end{align*}$$
\[
\left( \partial_a - \frac{2}{n-1}(^{(2)}\theta) \right) \zeta_A ^{N^+_2} = 2 \left( ^{(2)}\pi_A ^B \zeta_B - \left( \partial_A - 2^{(0)}\Gamma^0_{aA} \right) \zeta \right), \tag{125}
\]
\[
\left( \nabla (A \zeta_B) \right)_A^{N^+_2} = \frac{1}{2} \left( \left( ^{(2)}\kappa_{AB} \right)_A^{(0)} + g^{(2)}_{vA} \right) \zeta - \left( ^{(2)}\pi_{AB} \zeta \right)^{v}, \tag{126}
\]

2nd-order ODE for \( \zeta^v|_{N^+_2} \), \( \zeta^v|_{S} = 0 \), \( \partial (u \zeta_B) = -\partial (u \zeta_B) - \partial_A g_{aw} \zeta \), \( u = (^{(0)}\pi_{AB} \zeta)^{v} \), \( D_A \partial (u \zeta_B) = (^{(0)}\pi_{AB} \zeta)^{v} \), \( D_A \partial (u \zeta_B) = 0 \).

Using (11) we rewrite the last equations as
\[
\Delta \gamma^v + \kappa \theta_S = 2 \zeta_A \left( D_A \zeta + \left( D_A - \gamma^v \right) \zeta_A + \frac{1}{2} \left( \gamma R - 2 \Lambda \right) \right)^{v}, \tag{129}
\]
\[
\partial (u \zeta_B) = \kappa = \text{const}. \tag{130}
\]

Here we choose to analyze the situation where \( N^+_2 \) has vanishing shear while its expansion \( \theta_S \) on the intersection manifold \( S \) as well as \( (S, \gamma) \) itself can be arbitrary, i.e. we consider data of the form
\[
(^{(1)}\pi_{AB} \zeta)^{H^+_1} = 0, \quad ^{(1)}\theta \leq 0, \quad ^{(2)}\pi_{AB} \zeta^{N^+_2} = 0, \quad ^{(2)}\theta \leq \theta_S \neq 0. \tag{131}
\]
Observe that the 2nd-order ODEs (123) and (127) admit unique solutions for \( \zeta^v|_{H^+_1} \) and \( \zeta^v|_{N^+_2} \) whose precise form is irrelevant whenever the shear tensors of both null hypersurfaces vanish, as they only arise in the other equations accompanied by a factor of \( ^{(0)}\pi_{AB} \). Thus, we do not need to consider the 2nd-order ODEs (123) and (127) any further. Let us further assume that the torsion one form is given by
\[
\zeta = d \log \theta_S. \tag{132}
\]
This choice makes the solution of (10) as given in appendix A.3 as simple as possible. In that appendix A.3 we prove:

**Proposition 7.** Consider two smooth hypersurfaces \( H^+_1 = \{ v = 0 \} \) and \( N^+_2 = \{ u = 0 \} \) in an \( (n+1) \)-dimensional manifold with transverse intersection along a smooth \( (n-1) \)-dimensional submanifold \( S \) in adapted null coordinates and where the gauge conditions (42)–(43) are fulfilled. Consider characteristic initial data which satisfy
\[
(^{(1)}\pi_{AB} \zeta)^{H^+_1} = 0, \quad ^{(1)}\theta \leq 0, \quad ^{(2)}\pi_{AB} \zeta^{N^+_2} = 0, \quad ^{(2)}\theta \leq \theta_S \neq 0, \quad \zeta = d \log \theta_S. \tag{133}
\]
Then \( H^+_1 = \{ v = 0 \} \) is a multiple Killing horizon of order \( m \) if and only if
\[
(^0 R_{AB})_A^{\prime} = 0, \tag{134}
\]
and the following system admits \( m \) independent solutions \((f, \kappa)\) on \( S \) with \( \kappa = \text{const.} \).
\[(D_A D_R f)_\alpha = 0,\]  
\[(D_A \theta_S D_R f)_\alpha = 0\]  
\[\Delta_s f + \kappa = \frac{1}{2} (\nabla R - 2\Lambda) f.\]

The multiple Killing horizon is fully degenerate if and only if all solutions satisfy \(\kappa = 0\).

Remark 19. At this stage, it is not clear to us whether or not there exist data for which proposition 7 produces fully degenerate MKHs.

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Appendix. Some calculations relevant for section 6

A.1. Proof of proposition 4

In order to establish proposition 4 let us start with data of the form

\((1)\pi_{AB} H^+ = 0, \quad (1)\theta^ S = 0, \quad \partial_A |^{(2)}\pi^ A \equiv 0, \quad \nabla_B |^{(2)}\pi_A \ N^+ = 0, \quad (2)\theta^ S = const.,\)  
\[(A.1)\]

and let us further assume that

\[
\hat{R}_{AB} = \frac{2}{n-1} \Lambda g_{AB}, \quad \varsigma = 0.
\]  
\[(A.2)\]

As the data will be more restricted later on, we do not care so much how data which fulfill (A.1) are constructed from a family of Riemannian metrics as in theorem 3.

We analyze the KID equations (16)–(24) for this class of data, where we again assume a gauge where (42)–(43) holds,

\[
\Gamma^ u_{uu} H^+ = 0, \quad \Gamma^ v_{vv} N^+ = 0, \quad g_{12} = 1.
\]  
\[(A.3)\]

First of all we observe that the constraint equations (9)–(12) yield,

\[
(1)\theta^ H^+ = 0, \quad \Gamma^ u_{uu} H^+ = 0, \quad \Gamma^ v_{vv} N^+ = 0, \quad \theta^ S = 0.
\]  
\[(A.4)\]

\[
\text{tr}(1)\Xi H^+ = -2\theta^ S, \quad (1)\Xi_{AB} H^+ = -2(2)\pi_{AB}|^ S.
\]  
\[(A.5)\]

\[
\left(\partial_S + \frac{(2)\theta}{n-1}\right)(2)\theta^ N^+ = -\ |^{(2)}\pi|^ 2, \quad (2)\theta^ S = \theta^ S.
\]  
\[(A.6)\]
\[ \Gamma^\nu_{\nu A} N^+_A = 0, \quad (2) \Xi_{AB} N^+_A = 0. \]  

Moreover, a computation reveals that
\[ R^a_{\mu A} \frac{\mathcal{H}^+_a}{\mu} \equiv 0, \]
\[ R^a_{\mu A} \frac{\mathcal{H}^+_a}{\mu} \equiv R^a_{\mu A} - R^a_{\mu A} A \frac{\mathcal{H}^+_a}{\mu} \equiv -\frac{1}{2} g_{\mu v} \left( \partial_{\mu A} (1) \Xi + (1) K^{AB} (1) \Xi_{AB} - \frac{4}{n - 1} A \right) \frac{\mathcal{H}^+_a}{\mu} \equiv \frac{2A}{n - 1} g_{\mu v}, \]
\[ R^A_{\mu v} N^+_A \equiv 0, \]
\[ R^A_{\mu v} N^+_A \equiv R^A_{\mu v} - R^A_{\mu v} A N^+_A \equiv \frac{2}{n - 1} \Lambda g_{\mu v}. \]

The KID equations (16)–(19) on \( \mathcal{H}^+_1 \) become
\[ \partial_u \zeta_u \frac{\mathcal{H}^+_1}{u} \equiv 0, \]  
\[ \partial_u \zeta_u \frac{\mathcal{H}^+_1}{u} \equiv -\partial_A \zeta_u, \]  
\[ \partial_u \partial_v (\zeta^u - \frac{1}{2} g^{uv} \zeta_u) \frac{\mathcal{H}^+_1}{u} \equiv \frac{2}{n - 1} \Lambda \zeta_u. \]

On \( N^+_2 \) they read
\[ \partial_v \zeta_v \frac{\mathcal{H}^+_2}{v} \equiv 0, \]  
\[ \left( \partial_v - \frac{2}{n - 1} (2) \theta \right) \zeta_A \equiv -2(2) \pi_A \frac{\mathcal{H}^+_2}{v} \equiv -\partial_A \zeta_v, \]  
\[ \partial_u \partial_v (\zeta^v - \frac{1}{2} g^{uv} \zeta_u) \frac{\mathcal{H}^+_2}{v} \equiv \frac{2}{n - 1} \Lambda \zeta_v, \]  
\[ (\hat{\nabla}_{(A} \zeta_{B)})_{\Delta A} \frac{\mathcal{H}^+_2}{v} \equiv -2 \pi_{AB} (\zeta^v - \frac{1}{2} g^{vv} \zeta_v). \]

For the KID equations (20)–(24) on \( s \) a computation shows (note that \( R^a_{\mu A} S \equiv 0 \))
\[ \partial_{(u} \zeta_{v)} \frac{S}{u} \equiv \Gamma^u_{uv} \zeta_u + \Gamma^v_{uv} \zeta_v, \]  
\[ D^A \zeta_A \equiv -\theta s \zeta_u, \]  
\[ \Delta_{\gamma} \zeta_u \equiv 0, \]  
\[ \Delta_{\gamma} \zeta_v \equiv -\theta s \partial_{[u} \zeta_{]v} - \frac{\theta^2}{n - 1} \zeta_u - (2) \pi [2] \zeta_v, \]  
\[ \partial_{(u} \partial_{[u} \zeta_{]v)} \frac{S}{u} \equiv -\theta s \partial_{(u} \zeta_{]v} - (2) \pi_A \partial_B \zeta_u. \]
Given data $\zeta_S, \partial(u\zeta)\mid_S$ and $\partial(u\zeta)\mid_S$ the equations (A.8)–(A.10) and (A.12)–(A.14) can be integrated. The data $\partial(u\zeta)\mid_S$ and $\partial(u\zeta)\mid_S$ are determined by (A.16) and (A.20), the latter one up to an additive constant. We further observe that the second-order $u$-derivative of (A.11) vanishes whence it simplifies to
\[
(D_A D_B \zeta)\mid_S = -\frac{2}{\pi} \zeta u,
\]
and the first equation is contained in (A.15).

As the main problematic equation remains (A.15). We do not want to impose a condition on $(2)\pi_{AB}$ which restricts its $v$-dependence. We rather want to make sure that $\zeta u\mid_S ^N = 0$ is a solution of (A.14) to get rid of $(2)\pi_{AB}$ in (A.15). Since we want $\zeta v\mid_S \neq 0$ in order to get a non-trivial Killing vector we are led to the condition $\Lambda = 0$. By (A.2) this requires $R_{AB}\mid_N = 0$, and the latter one holds if $\partial C (2)\pi_{AB} = 0$ and $\partial C g_{AB}\mid_S = 0$, as then $g_{AB}\mid_N$ is independent of $x^C$. This suggests to assume that $\zeta v\mid_S$ is the Euclidean space. Proposition 4 now follows immediately (note that the conditions on the initial data there imply (A.1)–(A.2)).

### A.2. Proof of proposition 6

Let us determine how many independent Killing vectors the spacetimes generated in proposition 4 have. Set $f := -\left(\zeta v - \frac{1}{2} g vv \zeta v\right)$ and note that $\zeta u\mid_S = -f$. For data as considered in proposition 5 the KID equations read for data $(\zeta\mid_S, \alpha\zeta = \text{const.})$ for the Killing vector,
\[
\partial_b \zeta^N = 0,
\]
\[
\left(\partial_v - \frac{2}{n-1} (2)\theta\right)\zeta_A - 2(2)\pi^A_{B} \zeta^N_{B} = -\partial_A \zeta_v,
\]
\[
\partial_v \partial_v f = 0,
\]
\[
(D_A (\zeta_B)\mid_S) = (2)\pi_{AB} f,
\]
\[
\partial_a \partial_v f = 0,
\]
\[
D^A \zeta_A = \theta_g f,
\]
\[
\Delta_v \zeta^S = -\theta_g \tilde{\kappa}_v.
\]
\[
\partial_v f = \frac{\theta_g}{n-1} f + \alpha_v.
\]
These equations are supplemented by an algebraic equation for $\partial_u \zeta\mid_S$ and the equations (A.8)–(A.10) for the evolution on $H^+_1\mid$, which, given the data on $S$, always admits unique solutions. Equivalently,
\[
\zeta_v \mid_S = \zeta v(x^A),
\]
\[
\left(\partial_v - \frac{2}{n-1} (2)\theta\right)\zeta_A - 2(2)\pi^A_{B} \zeta^N_{B} = -\partial_A \zeta ^v,
\]
\[ \mathfrak{f}^{N^+} = c + c_A x^A + \left( \frac{\theta_S}{n-1} (c + c_A x^A) + \alpha \zeta \right) v, \]  
(A.32)

\[(\partial_{(A}) \zeta_{B)})_{\mathfrak{f}} \overset{N^+}{=} (2) \pi_{AB} \mathfrak{f}, \]  
(A.33)

\[ D^A \zeta_A \overset{S}{=} \theta_S (c + c_A x^A), \]  
(A.34)

\[ \Delta^{(0)} \zeta_v \overset{S}{=} -\theta_S \alpha \zeta \]  
(A.35)

where \( c \) and the \( c_A \)'s are constants. The \( v \)-derivative of (A.33) yields with (A.31) and (A.35)

\[ \partial_A \partial_B \zeta_{v} \overset{S}{=} \theta_S n^{-1} \alpha \zeta \gamma_{AB} - (c + c_C x^C) \partial_{v}^{(2)} \pi_{AB}. \]  
(A.36)

Taking the \( x^C \)-derivative yields with (A.35)

\[ c^B \partial_v^{(2)} \pi_{AB} \overset{S}{=} 0, \quad c_C \partial_v^{(2)} \pi_{A[B} C] \overset{S}{=} 0, \]  
(A.37)

i.e.

\[ 0 \overset{S}{=} 2 \partial_v^{(2)} \pi_{AB} \partial_{v}^{(2)} \pi_{A[B} C] \overset{S}{=} |\partial_v^{(2)} \pi|^2 c_C - \partial_v^{(2)} \pi_{AC} \partial_v^{(2)} \pi_{AB} c_B. \]  
(A.38)

As, by assumption, \(|\partial_v^{(2)} \pi|^2 \}|S \neq 0 \) we necessarily have

\[ c_A = 0. \]  
(A.39)

By (A.33) that yields

\[ \partial_C (\partial_{(A} \zeta_{B)})_{\mathfrak{f}} \overset{N^+}{=} 0. \]  
(A.40)

Next, we evaluate the \( x^B \)-derivative of (A.31) on \( S \) and take its symmetric trace-free part. Using (A.33), we obtain

\[ \partial_v (\partial_{(A} \zeta_{B)})_{\mathfrak{f}} \overset{S}{=} -(\partial_{v} \zeta_{(A})_{\mathfrak{f}} - \partial_{v} \zeta_{B)}_{\mathfrak{f}}. \]  
(A.41)

Combining both equations and using (A.35) we obtain

\[ \zeta_v \overset{N^+}{=} d + d_A x^A + d_{AB} x^B, \quad \gamma^{AB} d_{AB} = -\theta_S \alpha \zeta, \quad d_{[AB]} = 0, \]  
(A.42)

in particular,

\[ \tilde{\Delta} \zeta_v \overset{N^+}{=} \gamma^{AB} d_{AB} \overset{N^+}{=} -\theta_S \alpha \zeta e^{-\frac{\pi}{\theta} f^{(1)} d \nu}. \]  
(A.43)

The divergence of (A.31) then gives

\[ \partial_v (\gamma^{AB} \partial_{v} \zeta_{B}) \overset{N^+}{=} -\tilde{\Delta} \zeta_v \overset{N^+}{=} \theta_S \alpha \zeta e^{-\frac{\pi}{\theta} f^{(1)} d \nu}, \]  
(A.44)

which we integrate with initial data given by (A.34),

\[ g^{AB} \partial_{v} \zeta_{B} \overset{N^+}{=} \theta_S \left( c + \alpha \zeta \int_{0}^{\nu} e^{-\frac{\pi}{\theta} f^{(1)} d \nu} \right). \]  
(A.45)
As the right-hand side does not depend on the $x^A$-coordinates it follows with (A.40) that
\[
\partial_c \partial(A\zeta^C) \mathrel{\overset{\text{N}}{\cong}} 0.
\] (A.46)
By taking the anti-symmetric part w.r.t. (AC) we find that also the $x^C$-derivatives of $\partial(A\zeta^B)$ need to vanish whence
\[
\partial_A \partial_B \zeta^C \mathrel{\overset{\text{N}}{\cong}} 0 \quad \implies \quad \zeta_A \mathrel{\overset{\text{N}}{\cong}} (0) + \zeta_{AB} x^B, \tag{A.47}
\]
where, by (A.33) and (A.45),
\[
(1) \quad \zeta_{(AB)}(v) \mathrel{\overset{\text{N}}{\cong}} \frac{\theta_s}{n-1} \left( c + \alpha_c \int_0^\theta e^{-\int (1)_{AB} dv} R_{AB} + \left[ c + \left( \frac{\theta_s}{n-1} c + \alpha_c \right) v \right] \right)^2 \pi_{AB}. \tag{A.48}
\]
We need to make sure that (A.31) is fulfilled. If we plug in the corresponding expressions we have found for $\zeta_A$ and $\zeta_C$ we find that (A.31) is equivalent to the following ODE system,
\[
\left( \partial_v - \frac{2}{n-1} (2) \theta \right) (0) \zeta_A(v) - 2 (2) \pi_A^B(0) \zeta_B(v) \mathrel{\overset{\text{N}}{\cong}} -d_A, \tag{A.49}
\]
\[
\left( \partial_v - \frac{2}{n-1} (2) \theta \right) (1) \zeta_{AB}(v) - 2 (2) \pi_A^C \zeta_B^C(v) \mathrel{\overset{\text{N}}{\cong}} -d_{AB}. \tag{A.50}
\]
Because of (A.48) it is useful to split the latter equation into symmetric and anti-symmetric part,
\[
\left( \partial_v - \frac{2}{n-1} (2) \theta \right) (1) \zeta_{AB}(v) - (2) \pi_A^C \zeta_{[CB]}(v) + (2) \pi_B^C \zeta_{[AC]}(v) - (2) \pi_A^C \zeta_{BC}(v) + (2) \pi_B^C \zeta_{AC}(v) \mathrel{\overset{\text{N}}{\cong}} 0, \tag{A.51}
\]
\[
\left( \partial_v - \frac{2}{n-1} (2) \theta \right) (1) \zeta_{(AB)}(v) - (2) \pi_A^C \zeta_{(CB)}(v) - (2) \pi_B^C \zeta_{(CA)}(v) \mathrel{\overset{\text{N}}{\cong}} -d_{AB}. \tag{A.52}
\]
We observe that the last equation requires
\[
d_{AB} = -\partial_v (1) \zeta_{(AB)}(0) + \frac{2}{n-1} \theta_S (1) \zeta_{(AB)}(0), \tag{A.53}
\]
\[
0 = \partial_c \partial_v (1) \zeta_{(AB)}(0) + \frac{2}{(n-1)^2} (2) \theta_S (1) \zeta_{(AB)}(0) - \frac{2}{n-1} \theta_S (1) \zeta_{(AB)}(0)
- \partial_v (2) \pi_A^C \zeta_{(CB)}(0) - \partial_v (2) \pi_B^C \zeta_{(CA)}(0)
- \partial_v (2) \pi_A^C \zeta_{(CB)}(0) - \partial_v (2) \pi_B^C \zeta_{(CA)}(0). \tag{A.54}
\]
As we have
\[
(1) \quad \zeta_{(AB)}(0) = \frac{\theta_S}{n-1} \gamma_{AB}. \tag{A.55}
\]
\[ \partial_{\gamma} \zeta_{(AB)}(0) = \alpha_{\gamma} \frac{\theta_S}{n-1} \gamma_{AB} + 2c \left( \frac{\theta_S}{n-1} \right)^2 \gamma_{AB} + c \partial_{\nu} \pi_{AB}, \] (A.56)

\[ \partial_{\nu} \partial_{\gamma} \zeta_{(AB)}(0) = 2 \left( \frac{\theta_S}{n-1} \right)^2 \alpha_{\gamma} \gamma_{AB} + 2c \left( \frac{\theta_S}{n-1} \right)^3 \gamma_{AB} + 2 \left( 2c \frac{\theta_S}{n-1} + \alpha_{\gamma} \right) \partial_{\nu} \pi_{AB} + c \partial_{\nu} \partial_{\nu} \pi_{AB}, \] (A.57)

that yields

\[ d_{AB} \equiv -\alpha_{\gamma} \frac{\theta_S}{n-1} \gamma_{AB} - c \partial_{\nu} \pi_{AB}, \] (A.58)

\[ 2\alpha_{\gamma} \partial_{\nu} \pi_{AB} \equiv \partial_{\nu} \pi_{A}^B \zeta_{[AB]}(0) + \partial_{\nu} \pi_{B}^A \zeta_{[AB]}(0) - c \partial_{\nu} \partial_{\nu} \pi_{AB}. \] (A.59)

To conclude, the data

\[ \zeta^\nu \equiv d + d_{\lambda} x^\lambda - \left( \alpha_{\gamma} \frac{\theta_S}{n-1} \gamma_{AB} + c \partial_{\nu} \pi_{AB} \right) x^A x^B, \] (A.60)

\[ \zeta_A \equiv \zeta_{A}^{(0)} + \left( \zeta_{[AB]} + c \frac{\theta_S}{n-1} \gamma_{AB} \right) x^B, \] (A.61)

\[ \Gamma^N_A \equiv c + \left( \frac{\theta_S}{n-1} + \alpha_{\gamma} \right) v, \] (A.62)

where \( d, d_{\lambda}, \zeta_{A}^{(0)}, \zeta_{[AB]}^{(1)} \) and \( c \) are constant on \( S \), and where \( \alpha_{\gamma} \) is determined by (A.59), exhaust all candidates to generate a unique solution to (A.22)–(A.29). They do whenever (A.52) holds with \( \zeta_{[AB]}^{(1)}(v) \) determined by (A.51) and \( \zeta_{(AB)}^{(1)}(v) \) as given by (A.48). In that case \( \zeta_A^{(0)}(v) \) is determined by (A.49).

This will certainly be the case for \( c = \zeta_{[AB]}^{(1)}(0) = 0 \) (which implies \( \alpha_{\gamma} = 0 \)), whence we have at least \( 2n - 1 \) independent solutions which are parameterized by \( d, d_{\lambda} \) and \( \zeta_{A}^{(0)}(0) \), while for specific choices of the data \( \pi_{AB} \) and \( \theta_S \) there might be \( \frac{1}{2} (n-1)(n-2) + 1 \) additional Killing vectors. This completes the proof of proposition 6.

A.3. Proof of proposition 7

First of all we observe that the constraint equations (9)–(12) yield (recall our assumptions (131)–(132) on the initial data)

\[ N^A_B \equiv 0, \]

\[ \Gamma^\nu_{\lambda A} N^1_{\lambda} \equiv -\zeta_A(x^B), \]

\[ N^1_{\lambda} \equiv \left( \frac{\nu}{n-1} + \frac{1}{\theta_S} \right)^{-1}, \]

\[ g_{AB} \equiv \left( 1 + \frac{\theta_S}{n-1} \nu \right)^2 \gamma_{AB}, \]

\[ \Gamma^\nu_{\lambda A} N^A_{\nu} \equiv \frac{n-1}{\theta_S(n-1 + \theta_S \nu)} \partial_A \theta_S. \]
Taking the behavior of the Ricci tensor under conformal transformations into account that yields
\[
(R_{ab})^\nu = n^\nu (\gamma R_{ab})^\nu - (n - 3)^2 (\gamma R_{ab})^\nu (D_a D_b \theta_5) u + 2(n - 3) \left(\frac{v}{n - 1 + \theta_5^\nu} \right)^2 (D_a \theta_5 D_b \theta_5) u.
\]
\[
\tilde{R} \equiv \left(\frac{n - 1}{n - 1 + \theta_5^\nu} \right)^2 (\gamma R - \frac{2(n - 2)^2}{n - 1 + \theta_5^\nu} \Delta_\theta \theta_5 - (n - 2)(n - 5) \left(\frac{v}{n - 1 + \theta_5^\nu} \right)^2 D_\theta \theta_5 D^\nu \theta_5).
\]

We then deduce that \((^{(2)}\Xi)_{ab}\) is determined by the following equations, with trivial initial data on \(S\),
\[
\left(\partial_\nu + \left(\frac{v}{n - 1} + \frac{1}{\theta_5^\nu} \right)^{-1} \right) \text{tr}(^{(2)}\Xi) - 2(n - 1)^2 n - 1 + (n - 2)^2 v \theta_5 \Delta_\theta \theta_5 \theta_5 \left(\frac{n - 1 + \theta_5^\nu}{(n - 1 + \theta_5^\nu)^2} \right) + (n - 1)^2 \left(\frac{v}{n - 1 + \theta_5^\nu} \right)^2 \gamma R \left(\frac{n - 1 + \theta_5^\nu}{(n - 1 + \theta_5^\nu)^2} \right)^2 \tilde{R} \equiv 2 \Lambda,
\]
\[
\left(\partial_\nu + \frac{n - 5}{2(n - 1)} \left(\frac{v}{n - 1} + \frac{1}{\theta_5^\nu} \right)^{-1} \right) \left(\frac{2(n - 1) + (n - 3)^2 \theta_5}{n - 1 + \theta_5^\nu} \right) \Delta_\theta \theta_5 \theta_5 \left(\frac{n - 1 + \theta_5^\nu}{(n - 1 + \theta_5^\nu)^2} \right) (D_\theta \Delta_\theta \theta_5) u + (^{(2)}\Xi)_{ab} \left(\frac{n - 1 + \theta_5^\nu}{(n - 1 + \theta_5^\nu)^2} \right) = 0.
\]

Next, we evaluate the KID equations \((122)-(130)\) in this setting, omitting the—for our purposes irrelevant—2nd-order ODEs for \(\zeta_\nu^{(1)}\) and \(\zeta_\nu^{(2)}\).

\[
\zeta_\nu^{(1)} \equiv 0, \quad \zeta_\nu^{(2)} \equiv 0, \quad (A.63)
\]

\[
\zeta_\nu \equiv \zeta (\nu^4), \quad (A.64)
\]

\[
\zeta_\nu \equiv \nu D_\nu \zeta = \theta_5^\nu \frac{2(n - 1) + \theta_5^\nu}{n - 1} D_\nu (\theta_5^{-1} \zeta), \quad (A.65)
\]

\[
0 \equiv \left(\frac{1}{2} \Xi_{ab} \right) u \equiv \frac{2v}{n - 1} \left(1 + \frac{\theta_5}{n - 1 - v} \right) \left(\frac{\theta_5}{n - 1 + \theta_5^\nu} \right)^{-1} \zeta (D_\nu \theta_5) u, \quad (A.66)
\]

\[
\partial_\nu \zeta_\nu = 0, \quad (A.67)
\]

\[
\Delta_\gamma (\theta_5^{-1} \zeta) + \kappa \equiv \frac{1}{2} \left(\gamma R - 2 \Lambda \left(\theta_5^{-1} \zeta \right), \quad (A.68)
\]

\[
\partial_\nu \kappa = \kappa, \quad \kappa = \text{const.} \quad (A.69)
\]

In fact, we observe that the above system admits a solution if and only if there exists a function \(\zeta\) on \(S\) and a constant \(\kappa\) such that
\[
0 \equiv \left(\frac{1}{2} \Xi_{ab} \right) u \equiv \frac{2v}{n - 1} \left(1 + \frac{\theta_5}{n - 1 - v} \right) \left(\frac{\theta_5}{n - 1 + \theta_5^\nu} \right)^{-1} \zeta (D_\nu \theta_5) u, \quad (A.70)
\]
\[ \Delta \gamma (\theta_S^{-1})^0_0(\zeta) + \kappa \leq \frac{1}{2} \left( \gamma R - 2 \Lambda \right)(\theta_S^{-1})^0_0(\zeta), \quad (A.71) \]

where
\[ \zeta_A^N = vD_A^0 \zeta - \frac{2(n-1) + \theta_S^2}{n-1} D_A(\theta_S^{-1})^0_0(\zeta). \quad (A.72) \]

The main obstruction comes from (A.70). Note that it is automatically satisfied on \( S \). We want to compute its \( v \)-derivative on \( S \). For this note that it follows from the constraint equations and (A.72) that
\[ \partial_v \zeta_A^N = -D_A \zeta + \frac{2}{\theta_S} \zeta D_A \theta_S, \]
\[ \partial_v \xi_{(2)}^{(0)} \Xi_{AB}^N = \frac{2}{\theta_S} (D_A D_B \theta_S)_{\alpha} - (\gamma R_{AB})_{\alpha}. \]
whence we find for the \( v \)-derivative of (A.70) on \( S \),
\[ \left( D_A D_B (\theta_S^{-1})^0_0(\zeta) \right)_{\alpha} = \frac{1}{2} \left( (\theta_S^{-1})^0_0(\zeta) \right)^\alpha R_{AB} \quad (A.73) \]

Similarly, from
\[ \partial_v^2 \zeta_A^N = -\frac{2}{n-1} \theta_S^2 D_A(\theta_S^{-1})^0_0(\zeta) \]
\[ \partial_v^2 \xi_{(2)}^{(0)} \Xi_{AB}^N = -\frac{8}{\theta_S} (D_A D_B \theta_S)_{\alpha} + \frac{n-5}{2(n-1)} D_A(\gamma R_{AB})_{\alpha}. \]
we find with (A.73) the equation \( (\gamma R_{AB})_{\alpha} = 0 \), for the 2nd-order \( v \)-derivative of (A.70). As we are interested in MKHs we require the existence of a solution where \( \zeta \neq 0 \) whence this condition becomes
\[ (\gamma R_{AB})_{\alpha} = 0. \quad (A.74) \]

Using this, we find from
\[ \partial_v^3 \zeta_A^N = 0, \]
\[ \partial_v^3 \xi_{(2)}^{(0)} \Xi_{AB}^N = \frac{24}{(n-1)^2} (D_A D_B \theta_S)_{\alpha}. \]
that the 3rd-order \( v \)-derivative of (A.70) yields the following condition
\[ \left( D_A D_B (\theta_S^{-1})^0_0(\zeta) \right)_{\alpha} = 0. \quad (A.75) \]

Let us analyze to what extent the conditions obtained so far are already sufficient, i.e. we consider again (A.70). We observe that with (A.74) the solution to the \( (\xi_{(2)}^{(0)} \Xi_{AB})_{\alpha} \)-constraint reads
\[ (\xi_{(2)}^{(0)} \Xi_{AB})_{\alpha} = \frac{2}{\theta_S} D_A D_B \theta_S - \frac{4\theta_S^2}{2\theta_S(n-1) + \theta_S^2} D_A D_B \theta_S \quad (A.76) \]
Inserting this and (A.72) into (A.70) and using (A.73)–(A.75) we find that (A.70) is fulfilled on \( N_2^+ \). This accomplishes the proof of proposition 7 (where we have set \( f := \theta_S^{-1} \zeta \)).
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References

[1] Andersson L, Mars M and Simon W 2005 Local existence of dynamical and trapping horizons Phys. Rev. Lett. 95 111102
[2] Ashtekar A, Beetle C and Lewandowski J 2002 Geometry of generic isolated horizons Class. Quantum Grav. 19 1195–225
[3] Cabet A, Christel P T and Tagne Wafo R 2016 On the characteristic initial value problem for nonlinear symmetric hyperbolic systems, including Einstein equations Dissertationes Math. 515 1–72
[4] Choquet-Bruhat Y, Christel P T and Martin-Garcia J M 2011 The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions Ann. Henri Poincaré 12 419–82
[5] Christel P T and Paetz T-T 2012 The many ways of the characteristic Cauchy problem Class. Quantum Grav. 29 145006
[6] Christel P T and Paetz T-T 2013 KIDs like cones Class. Quantum Grav. 30 235036
[7] Dobkowski-Rylko D, Kaminski W, Lewandowski J and Szereszewski A 2018 The near horizon geometry equation on compact 2-manifolds including the general solution for g > 0 Phys. Lett. B 785 381–5
[8] Friedrich H 1985 On the hyperbolicity of Einstein’s and other gauge field equations Commun. Math. Phys. 100 525–43
[9] Friedrich H, Racz I and Wald R M 1999 On the rigidity theorem for spacetimes with a stationary event horizon or a compact cauchy horizon Commun. Math. Phys. 204 691–707
[10] Hollands S, Ishibashi A and Wald R M 2007 A higher dimensional stationary rotating black hole must be axisymmetric Commun. Math. Phys. 271 699–722
[11] Kasner E 1921 Geometrical theorems on Einstein’s cosmological equations Am. J. Math. 43 217–21
[12] Kunduri H K and Lucietti J 2013 Classification of near-horizon geometries of extremal black holes Living Rev. Relativ. 16 8
[13] Lewandowski J and Pawłowski T 2003 Extremal isolated horizons: a local uniqueness theorem Class. Quantum Grav. 20 587–606
[14] Lewandowski J, Racz I and Szereszewski A 2017 Near horizon geometries and black hole holograph Phys. Rev. D 96 044001
[15] Lewandowski J, Szereszewski A and Waluk P 2016 Spacetimes foliated by non-expanding and Killing horizons: higher dimension Phys. Rev. D 94 064018
[16] Luk J 2012 On the local existence for the characteristic initial value problem in general relativity Int. Math. Res. Not. 2012 4625–78
[17] Mars M 2012 Stability of MOTS in totally geodesic null horizons Class. Quantum Grav. 29 145019
[18] Mars M, Paetz T-T and Senovilla J M M 2018 Multiple Killing horizons Class. Quantum Grav. 35 155015
[19] Mars M, Paetz T-T and Senovilla J M M 2018 Multiple Killing horizons and near horizon geometries Class. Quantum Grav. 35 245007
[20] Obata M 1962 Certain conditions for a Riemannian manifold to be isometric with a sphere J. Math. Soc. Japan 14 333–40
[21] Pawłowski T, Lewandowski J and Jezierski J 2004 Spacetimes foliated by Killing horizons Class. Quantum Grav. 21 1237–51
[22] Racz I and Wald R M 1992 Extensions of spacetimes with Killing horizons Class. Quantum Grav. 9 2643–56
[23] Rendall A D 1990 Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations Proc. R. Soc. A 427 221–39
[24] Schmidt H-J 2013 The tetralogy of Birkhoff theorems Gen. Relativ. Gravit. 45 395–410
[25] Stephani H, Kramer D, MacCallum M A H, Hoenselaers C and Herlt E 2003 Exact Solutions of Einstein’s Field Equations (Cambridge Monographs on Mathematical Physics) 2nd edn (Cambridge: Cambridge University Press)

[26] Strichartz R S 1989 Harmonic Analysis as spectral theory of laplacians J. Funct. Anal. 87 51–148

[27] Tashiro Y 1965 Complete Riemannian manifolds and some vector fields Trans. Am. Math. Soc. 117 251–75