ONE UPPER ESTIMATE
ON THE NUMBER OF LIMIT CYCLES
OF EVEN DEGREE LIÉNARD EQUATIONS
IN THE FOCUS CASE

GRISHA KOLUTSKY

Abstract. We give an explicit upper bound for a number of limit cycles of the Liénard equation \( \dot{x} = y - F(x), \dot{y} = -x \) of even degree in the case its unique singular point \((0, 0)\) is a focus.

M. Caubergh and F. Dumortier get an explicit linear upper estimate for the number of large amplitude limit cycles of such equations \([CD]\). We estimate the number of mid amplitude limit cycles of Liénard equations using the Growth-and-Zeros theorem proved by Ilyashenko and Yakovenko \([IYa]\).

Our estimate depends on four parameters: \(n, C, a_1, R\). Let \(F(x) = x^n + \sum_{i=1}^{n-1} a_i x^i\), where \(n = 2l\) is the even degree of the monic polynomial \(F\) without a constant term, \(\forall i |a_i| < C\), so \(C\) is the size of a compact subset in the space of parameters, \(R\) is the size of the neighborhood of the origin, such that there are no bigger than \(l\) limit cycles located outside of this neighborhood, \(|a_1|\) stands the distance from the equation linearization to the center case in the space of parameters and \(2 - |a_1|\) stands the distance from the equation linearization to the node case in the space of parameters.

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1. Hilbert-Smale problem

In 1977 A. Lins Neto, W. de Melo and C. C. Pugh [LMP] examined small perturbations of a linear center for a special class of polynomial vector fields on the plane. This class is called Liénard equations:

\[
\begin{align*}
\dot{x} &= y - F(x), \\
\dot{y} &= -x,
\end{align*}
\]

where \( F \) is a polynomial of odd degree. Actually, Liénard in 1928 introduced it for a modeling of the non-linear damping in electric circuits [L]. It was a generalization of the famous Van der Pol equation [V].

Authors of [LMP] proved the finiteness of limit cycles for a Liénard equation of odd degree \( n \). Let us remind that the Finiteness problem (also known as the "Dulac problem") was solved in full generality only in 1991 by Ilyashenko [I1] and in 1992 by Écalle [E] independently.

Also A. Lins Neto, W. de Melo and C. C. Pugh [LMP] conjectured that the number of limit cycles of (1) is not bigger than \( \frac{n-1}{2} \).

In 1998 S. Smale [S] suggested to consider a restriction of the second part of the Hilbert’s 16th problem to Liénard equations of odd degree. He conjectured that there exists an integer \( n \) and real \( C \) such that the number of limit cycles of (1) is not bigger than \( Cn^q \).

In 1999 Yu. Ilyashenko and A. Panov [IP] got an explicit upper bound for the number of limit cycles of Liénard equations through the (odd) degree of the monic polynomial \( F \) and magnitudes of its coefficients. Their result reclined on the theorem of Ilyashenko and Yakovenko that binds the number of zeros and the growth of a holomorphic function [IYa].

In 2007 F. Dumortier, D. Panazzolo and R. Roussarie [DPR] constructed a counterexample to the conjecture of A. Lins Neto, W. de Melo and C. C. Pugh. Namely, they presented an example of a Liénard equation of odd degree \( n \) with at least \( \frac{n+1}{2} \) limit cycles.

In 2008 Yu. Ilyashenko [I3] suggested to prove a result analogous to the one of Ilyashenko and Panov for Liénard equations of even degree.

In 2008 M. Caubergh and F. Dumortier in [CD] proved the following theorem for Liénard equations of even degree.

**Theorem 1.** Let \( K \) be a compact set of polynomials of degree exactly \( n = 2l \), then there exists \( R > 0 \) such that any system having an expression (1) with \( F \in K \) has at most \( l \) limit cycles having an intersection with \( \mathbb{R}^2 \setminus B_R \).

Here and bellow \( B_R \) denotes the ball around the origin with the radius \( R \).

2. Notations and the Ilyashenko strategy

From now on we will consider a system (1), where \( F \) is a monic polynomial of even degree \( n = 2l \) without a constant term.
**Remark 1.** The assumption $F(0) = 0$ does not reduce the generality; it may be fulfilled by a shift $y \mapsto y + a$. The assumption that $F$ is monic may be fulfilled by rescaling in $x$, $y$ and reversing the time if necessary.

Let $v$ be an analytic vector field in the real plane, that may be extended to $\mathbb{C}^2$. For any set $D$ in a metric space denote by $U_\varepsilon(D)$ the $\varepsilon$-neighborhood of $D$. The metrics in $\mathbb{C}$ and $\mathbb{C}^2$ are given by:

\[
\rho(z, w) = |z - w|, \quad z, w \in \mathbb{C};
\]

\[
\rho(z, w) = \max(|z_1 - w_1|, |z_2 - w_2|), \quad z, w \in \mathbb{C}^2.
\]

Denote by $|D|$ the length of the segment $D$. For any larger segment $D' \supset D$, let $\rho(D, \partial D')$ be the Hausdorff distance between $D$ and $\partial D'$.

We want to apply the next theorem proved by Ilyashenko and Panov [IP]. In fact, it is the easy corollary from the Growth-and-Zeros theorem for holomorphic functions proved by Ilyashenko and Yakovenko [IYa].

Consider the system

\[
\dot{x} = v(x), \quad x \in \mathbb{R}^2.
\]  

**Theorem 2.** Let $\Gamma$ be a cross-section of the vector field $v$, $D \subset \Gamma$ a segment. Let $P$ be the Poincaré map of (2) defined on $D$, and $D \subset D' = P(D)$. Suppose that $P$ may be analytically extended to $U = U_\varepsilon(D) \subset \mathbb{C}$, $\varepsilon < 1$, and $P(U) \subset U'(D') \subset \mathbb{C}$. Then the number $\#LC(D)$ of limit cycles that cross $D$ admits an upper estimate:

\[
\#LC(D) \leq e^{2|D|\varepsilon^{-1}} \log \frac{|D'| + 2}{\rho(D, \partial D')}. \tag{3}
\]

The same is true for $P$ replaced by $P^{-1}$.

Actually, the Ilyashenko strategy is the application of the previous theorem. It requires a purely qualitative investigation of a vector field, i.e. a construction of such $D$ for every nest of limit cycles. This strategy was applied before in papers [I2] and [IP].

We take $K$ from the Theorem 1 to be the space of monic polynomials of degree exactly $n$ with coefficients, which absolute values are bounded by some positive constant $C \geq 4$, i.e.

\[
F(x) = x^n + \sum_{i=1}^{n-1} a_i x^i, \quad \forall i : |a_i| < C.
\]

If $0 < |a_1| < 2$ then the unique singular point $(0, 0)$ of the system (1) is a focus. In our work we will consider only this case.

### 3. Bendixson trap from within

In this Section we construct an interval $D$, which lies inside $B_R$ and intersects transversally all limit cycles in $B_R$. Also we find an upper estimate for the Bernstein index, $b = \log \frac{|D'| + 2}{\rho(D, \partial D')}$. To do that we need
to estimate $\rho(D, \partial D')$ from below, where $D' = P(D) \subset D$ and $P$ is the Poincaré map defined on $D$ (see the Figure 1).

![Figure 1](image_url)

**Figure 1.** The inverse Poincaré map of the Liénard equation (1) inside the ball $B_R$.

Let $\varphi, r$ be polar coordinates on $\mathbb{R}^2$, $\dot{\varphi}, \dot{r}$ be derivatives with respect to (1).

First of all we need to determine the size of the domain, there the Poincaré map is defined.

**Lemma 1.** Put $\sigma = \frac{|a_1|}{8C} e^{\frac{-2\pi}{|a_1|}}$. In the focus case ($0 < |a_1| < 2$) the Poincaré map for the system (1) is well defined in $B_\sigma$.

**Proof.** Let us calculate $\dot{r}$.

$$\dot{r} = \frac{\dot{x}y + y\dot{x}}{r} = \frac{r \cos \varphi (r \sin \varphi - F(r \cos \varphi)) - r^2 \sin \varphi \cos \varphi}{r} =$$

$$= - \cos \varphi F(r \cos \varphi) = - r \cos^2 \varphi \sum_{i=1}^{n} a_i (r \cos \varphi)^{i-1} =$$

$$= - r \cos^2 \varphi (a_1 + O(r, \varphi)), \quad (4)$$

where $O(r, \varphi) = \sum_{i=2}^{n} a_i (r \cos \varphi)^{i-1}$.

Let us calculate $\dot{\varphi}$.

$$\dot{\varphi} = \frac{\dot{x}y - y\dot{x}}{r^2} = \frac{-r^2 \cos \varphi - r^2 \sin^2 \varphi + r \sin \varphi F(r \cos \varphi)}{r^2} =$$

$$= -1 + \frac{\sin \varphi F(r \cos \varphi)}{r} = -1 + 2\varphi \left( \frac{a_1}{2} + \frac{O(r, \varphi)}{2} \right). \quad (5)$$
The absolute value of the function \( O(r, \varphi) \) admits the following upper estimate in \( B_{\frac{1}{2}} \):

\[
|O(r, \varphi)| \leq \sum_{i=2}^{n} Cr^{i-1} = Cr \left( \frac{1 - r^{n-1}}{1 - r} \right) < \frac{Cr}{1 - r} \leq 2Cr. \quad (6)
\]

Therefore, in \( B_{2 - \frac{|a_1|}{4C}} \subset B_{\frac{2}{3}} \): \( \dot{\varphi} \leq \frac{|a_1| - 2}{4} \). Indeed,

\[
\dot{\varphi} \leq -1 + |\sin 2\varphi| \left( \frac{|a_1| + |O(r, \varphi)|}{2} \right) \leq -1 + \frac{|a_1|}{2} + Cr \leq \frac{|a_1| - 2}{2} + C \frac{2 - |a_1|}{4C} = \frac{|a_1| - 2}{4}.
\]

Also, in \( B_{2 - \frac{|a_1|}{4C}} \): \( |\dot{r}| \leq 2r \). Indeed,

\[
|\dot{r}| \leq r \cos^2 \varphi \left( |a_1| + |O(r, \varphi)| \right) \leq (|a_1| + 2Cr) r \leq \left( |a_1| + 2C \frac{2 - |a_1|}{4C} \right) r \leq \frac{2 + |a_1|}{2} r \leq 2r.
\]

Hence, any trajectory starting from any point from \( B_{\sigma} \) rotates around the origin on the angle not less than \( 2\pi \) before leaving \( B_{2 - \frac{|a_1|}{4C}} \).

Indeed, \( \dot{\varphi} \leq \frac{|a_1| - 2}{4} \) implies that during the time, \( \Delta t = \frac{2\pi}{\frac{4}{2} - |a_1|} \) the variation of the angle, \( \Delta \varphi \geq \frac{2\pi}{2} \) and the variation of the radius, \( \Delta r \leq e^{\frac{2\pi}{2} - |a_1|} \) during the same time \( \Delta t \), because \( |\dot{r}| \leq 2r \).

Finally, \( \sigma e^{\frac{2\pi}{2} - |a_1|} = \frac{|a_1| (2 - |a_1|)}{8C} \leq \frac{2 - |a_1|}{4C} \).

Let us denote by \( Y \) the maximal \( y \)-coordinate of the point of intersection between the most external limit cycle which lies inside \( B_R \) (if it exists, of course) and \( y \)-axis.

**Lemma 2.** If \( a_1 \) is negative, then \( \dot{r} > 0 \) in \( B_{\sigma} \). Let \( D = [\sigma, Y] \subset 0y \). Then \( d = \rho(D, \partial D') \geq \frac{2|a_1|}{\sigma} \).

**Proof.** If \( r < \sigma \), then \( r < \frac{1}{2} \) and by (6): \(|O(r, \varphi)| \leq 2Cr < \frac{|a_1|}{2} \).

Therefore by (I),

\[
\dot{r} > r \cos^2 \varphi \left( -a_1 - \frac{|a_1|}{2} \right) = -\frac{a_1}{2} \cos^2 \varphi > 0.
\]

This proves the first part of the Lemma.

Consider the orbit \( \gamma \) of the system (I) that passes through the point \((0, \sigma)\). Then the Hausdorff distance, \( d \) can be estimated as follows:

\[
d \geq \left| \int_{0}^{2\pi} \dot{r}(\gamma) \, d\varphi \right| > \int_{0}^{2\pi} -\frac{a_1}{2} \sigma \cos^2 \varphi \, d\varphi = \frac{\pi |a_1|}{2} \sigma.
\]

This inequality completes the proof of the Lemma. \( \square \)

**Remark 2.** For positive \( a_1 \) we can get the same results just by reversing of the time.
Now we can estimate $b$ from above:

$$b \leq \log \frac{R + 2}{d} \leq \log \frac{2(R + 2)}{\pi |a_1| \sigma} < \frac{R + 2}{|a_1| \sigma}.$$  \hfill (7)

4. Complex domain of the inverse Poincaré map

The Theorem 2 uses the width $\varepsilon$ of the complex domain $U^\varepsilon(D)$ to which the (inverse) Poincaré map may be extended. We will apply the following theorem to estimate this $\varepsilon$ from below.

**Theorem 3.** Let $P : D \to D'$ be the Poincaré map of (2). For any $x \in D$ denote by $\varphi_{x,P(x)}$ the arc of the phase curve of (2) starting at $x$ and ending at $P(x)$.

Let

$$\Omega(D) = \bigcup_{x \in D} \varphi_{x,P(x)},$$

and

$$1 \leq \mu = \max_{U^2(0)} |v|, \quad L = 2\mu.$$ \hfill (8)

Let $t(x)$ be the time length of the arc $\varphi_{x,P(x)}$, and

$$T_{\text{max}} = \max_{x \in D} t(x), \quad T = T_{\text{max}} + 1.$$

Let

$$\delta \leq e^{-LT}, \quad \lambda = \sqrt{\delta}, \quad \varepsilon = \delta^2.$$ \hfill (9)

Suppose that $(z_1, z_2)$ are coordinates in $\mathbb{C}^2$, $\mathbb{C}\Gamma = \{ z_1 = 0 \}$, $v = (v_1, v_2)$.

Let $\Pi_\delta = U^\delta(0) \times U^\lambda(D') \subset \mathbb{C}^2$. Suppose that

$$\left| \frac{v_2}{v_1} \right| \leq \mu \quad \text{in} \quad \Pi_\delta.$$ \hfill (10)

Then the Poincaré map $P : D \to D'$ of (2) may be analytically extended to $U^\varepsilon(D) \subset \mathbb{C}\Gamma$, and $P(U^\varepsilon(D)) \subset U^1(D)$.

The same is true for $P$ replaced by $P^{-1}$. In this case $P^{-1}(D) = D'$, $\Omega(D) = \bigcup_{x \in D'} \varphi_{x,P(x)}$.

For the proof see [IP]. \hfill $\square$

Bellow we will produce some preliminary calculations, which would allow us to apply the Theorem 3 later.

**Definition 1.** A $C$-monic polynomial is a real polynomial in one variable with the highest coefficient one and other coefficients no greater than $C$ in absolute value, with zero constant term.
Proposition 1 (Properties of \( C \)-monic polynomials). Let \( F \) be a \( C \)-monic polynomial of degree \( n, C \geq 2 \). Then
\[
\max_{x \in [0,X]} |F(x)| \leq 2X^n \quad \text{for } X \geq C + 1, \tag{11}
\]
\[
\max_{x \in [0,X]} |F'(x)| \leq CnX^{n-1} \quad \text{for } X \geq 1, \tag{12}
\]
\[
|F(z)| \leq 2C|z| \quad \text{for } z \in \mathbb{C}, |z| \leq \frac{1}{2}. \tag{13}
\]
For the proof see [IP]. \( \Box \)

Lemma 3. Let \( v \) be the vector field given by the system (1). Then \( \mu \) and \( L \) from the Theorem 3 admits the following estimates:
\[
\mu \leq 3(R + 2)^n \quad L \leq 6(R + 2)^n. \tag{14}
\]
Proof. By definition, \( U^2(\Omega) \subset B_{R+2} \). So
\[
|v| \leq |\dot{x}| + |\dot{y}| \leq |x| + |y| + |F(x)| \leq 2(R + 2) + 2(R + 2)^n,
\]
where the last inequality provided by (11). Hence,
\[
\mu \leq 2(R + 2 + (R + 2)^n) \leq 3(R + 2)^n, \quad L = 2\mu \leq 6(R + 2)^n,
\]
that proves the Lemma. \( \Box \)

Let \( G = B_R \setminus B_\sigma \). Then \( \Omega = \bigcup_{x \in D} \varphi_{x,P(x)} \subset G \).

Lemma 4. Let \( \gamma_y \) be the arc \( \varphi_{y,P^{-1}(y)} \) of the phase curve of (1), where \( y \in D \). Then \( t(y) \), the time length of \( \gamma_y \), admits an estimate
\[
T_{\text{max}} = \max_{y \in D} t(y) \leq \frac{25C^2n^2R^n}{\sigma}. \tag{15}
\]
Proof. The arcs \( \gamma_y, y \in D \) belongs to \( G \). We will split \( G \) into two domains: \( |\dot{x}| \leq \alpha \) and \( |\dot{x}| > \alpha \) for \( \alpha \) small to be chosen later. The second domain contains two parts of \( \gamma_y \): one with \( \dot{x} < -\alpha \), the other with \( \dot{x} > \alpha \). The time length of any of them is no greater than \( \frac{2R}{\alpha} \).

In the next Proposition we will choose \( \alpha \) so small that the curvilinear strip
\[
S_\alpha = \{(x,y) \in G : |y - F(x)| \leq \alpha\}
\]
is crossed by the orbits of (1) in the time no greater than 1.

Proposition 2. Let
\[
\omega = \frac{\sigma}{3C}, \quad \alpha = \frac{\omega}{2Cn^2R^{n-1}} = \frac{\sigma}{6C^2n^2R^{n-1}}. \tag{16}
\]
Then the time length of any arc of the orbit of (1) located in \( S_\alpha \) is no greater than 1.
**Proof.** By the symmetry arguments it is sufficient to prove that in \( S^+_\alpha = S_\alpha \cap \{ x > 0 \} \):

\[
\frac{d}{dt}(y - F(x)) \leq -2\alpha.
\]

Let us first prove that in \( S_\alpha \) we have: \( |x| > \omega \). Namely, let \( |x| \leq \omega \), \( |y - F(x)| \leq \alpha \). Then \((x, y) \in D_\sigma\). Indeed,

\[
|x| + |y| \leq \omega + \alpha + \max_{[0, \omega]} |F(x)|.
\]

By (16), \( \alpha < \omega < \frac{1}{2} \). By (13), \( |F(x)| \leq 2C\omega \). Hence, for \( x \in [0, \omega] \),

\[
|x| + |y| \leq (2C + 2)\omega < 3\sigma.
\]

By (12), \( |F'(x)| \leq Cn^2R^{n-1} \) in \( G \). Therefore, for \( x \) such that \((x, y) \in S^+_\alpha \) we have: \( x > \omega \), and

\[
\frac{d}{dt}(y - F(x)) = -x - F'(x)(y - F(x)) \leq -\omega + \alpha Cn^2R^{n-1} < -2\alpha,
\]

because \( \alpha = \frac{\omega}{2Cn^2R^{n-1}} < \frac{\omega}{2Cn^2R^{n-1} + 2} \). \hfill \square

Let us finish the proof of the Lemma 4.

The arc \( \gamma_y \) spends in \( S_\alpha \) no longer time than 2 (two crossings, each one no longer in time than 1, by the previous Proposition); in \( G \setminus S_\alpha \) no longer time than \( \frac{4R}{\sigma} \) (two crossings, one to the left, another to the right with \( |\dot{x}| \geq \alpha \)). Hence,

\[
T_{\max} \leq 2 + \frac{4R \cdot 6Cn^2R^{n-1}}{\sigma} < \frac{25Cn^2R^n}{\sigma}.
\]

This calculation completes the proof of the Lemma 4. \hfill \square

**Remark 3.** The same inequality holds for \( T_{\max} \) replaced by \( T_{\max} + 1 \).

Let us check the last assumption of the Theorem 3.

**Lemma 5.** Take

\[
\varepsilon = \exp \left( -\frac{300C^2nR^2(R + 2)^2}{\sigma} \right), \quad \delta = \sqrt{\varepsilon}, \quad \lambda = \sqrt{\delta}.
\]

Let, as in the Theorem 3, \( \Pi_\delta = U^\delta(0) \times U^\lambda(D') \subset \mathbb{C}^2 \). Then in \( \Pi_\delta \):

\[
\frac{|v_2|}{v_1} < \mu.
\]

**Proof.** By (13) and by definition of \( \Pi_\delta \),

\[
|v_1(z)| \geq ||z_1| - F(z)| \geq (\sigma - \lambda) - 2C\delta > \delta,
\]

where the last inequality is trivial. On the other hand, \( v_2 = -x \). In \( \Pi_\delta \), \( |v_2| \leq \delta \). Hence, \( \frac{|v_2|}{v_1} < 1 < \mu. \) \hfill \square
Lemma 6. The inverse Poincaré map of the Liénard equation (1) may be extended to the domain $U^\varepsilon(D) \subset \mathbb{C}$, where $\varepsilon = e^{-\frac{300C^2n^2R^n(R+2)^n}{\sigma}}$. Moreover, $P^{-1}(U^\varepsilon(D)) \subset U^1(D')$.

Proof. This Lemma following from the Theorem 3. Lemmas 3 and 5 verifies assumptions 8 and 10 respectively. We only should check the assumption 9. By the Remark 3, $T < \frac{25C^2n^2R}{\sigma}$. Hence,

$$\delta = \sqrt{\varepsilon} = \exp\left(-\frac{150C^2n^2R(R+2)^n}{\sigma}\right) = \exp\left(-6(R+2)^n\frac{25C^2n^2R}{\sigma}\right) \leq e^{-LT},$$

that proves the Lemma. \(\square\)

5. Final estimate

Theorem 4. The number $L(n, C, a_1, R)$ of limit cycles of (1) in the case when $n$ is even, $C \geq 4$ and $0 < |a_1| < 2$, admits the following upper bound:

$$L(n, C, a_1, R) < \exp\left(\exp\left(\frac{38400C^4n^2R^{n+1}(R+2)^{n+1}}{|a_1|^3(2 - |a_1|)^2} \frac{16\pi}{2 - |a_1|}\right)\right).$$

Proof. Now we can apply the Theorem 2. By definition, $|D|$ and $|D'|$ are less than $R$. The Lemma 6 provides us with the lower bound on $\varepsilon$. So estimates (3) and (7) imply:

$$L(n, C, a_1, R) < \exp\left(2R \exp\left(\frac{300C^2n^2R^n(R+2)^n}{\sigma}\right) \frac{R+2}{|a_1|\sigma} \right) < \exp\left(2R(R+2) \frac{600C^2n^2R^{n+1}(R+2)^{n+1}}{|a_1|\sigma^2}\right) = \exp\left(\exp\left(\frac{38400C^4n^2R^{n+1}(R+2)^{n+1}}{|a_1|^3(2 - |a_1|)^2} \frac{16\pi}{2 - |a_1|}\right)\right).$$

This calculation completes the proof of the Theorem. \(\square\)

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Grisha Kolutsky

Department of the Theory of Dynamical Systems
Faculty of Mechanics and Mathematics
Lomonosov Moscow State University
MSU, GSP, Glavnoe Zdanie, Leninckie Gory
119899 Moscow, Russia
e-mail: kolutsky AT mccme DOT ru