A Novel Application of Boolean Functions with High Algebraic Immunity in Minimal Codes

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Abstract

Boolean functions with high algebraic immunity are important cryptographic primitives in some stream ciphers. In this paper, two methodologies for constructing binary minimal codes from sets, Boolean functions and vectorial Boolean functions with high algebraic immunity are proposed. More precisely, a general construction of new minimal codes using minimal codes contained in Reed-Muller codes and sets without nonzero low degree annihilators is presented. The other construction allows us to yield minimal codes from certain subcodes of Reed-Muller codes and vectorial Boolean functions with high algebraic immunity. Via these general constructions, infinite families of minimal binary linear codes of dimension $m$ and length less than or equal to $m(m+1)/2$ are obtained. In addition, a lower bound on the minimum distance of the proposed minimal linear codes is established. Conjectures and open problems are also presented. The results of this paper show that Boolean functions with high algebraic immunity have nice applications in several fields such as symmetric cryptography, coding theory and secret sharing schemes.

Keywords: Boolean function, vectorial Boolean function, Reed-Muller code, secret sharing, minimal code.

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1. Introduction

Secret sharing, independently introduced in 1979 by Shamir \cite{Shamir1979} and Blakley \cite{Blakley1979}, is one of the most widely studied topics in cryptography. Relations between linear codes and secret sharing schemes were first investigated by McEliece and Sarwate in \cite{McEliece1980}. In theory every linear code can be employed to construct secret sharing schemes. Unfortunately, it is extremely hard to determine the access structures of secret sharing schemes based on general linear codes. However, the access structures of secret sharing schemes based on minimal linear codes are known and interesting \cite{Wang2017, Yang2018, Kreher2019}.

Minimal codes have already received a lot of attention. It was pointed out in \cite{Chen2019} and \cite{Mesnager2020} that minimal codes are close to blocking sets in finite geometry. Many minimal linear codes were

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obtained from codes with few weights [13, 14, 26, 27, 28, 32]. Recently, Ding, Heng and Zhou [15] constructed three infinite families of minimal binary linear codes using certain Boolean functions. They also constructed an infinite family of minimal ternary linear codes from ternary functions in [17]. Bartoli and Bonini [7] generalized the construction of minimal linear codes in [17] from the ternary case to the odd $p$ characteristic case via $p$-ary functions. Li and Yue [19] obtained some minimal binary linear codes with nonlinear Boolean functions. Xu and Qu [34] constructed minimal $q$-ary linear codes from some special functions. In the recent paper [29], the authors considered minimal codes from the supports of $p$-ary functions. Lu, Wu and Cao [20] obtained minimal codes with special subsets of vector spaces over finite fields. Bonini and Borello [8] presented a family of minimal codes arising from some blocking sets.

The main objective of this paper is to find connections among special sets, Boolean functions with high algebraic immunity and binary minimal codes. Two general constructions of minimal binary codes with minimal codes contained in the Reed-Muller codes, subsets of finite fields and vectorial Boolean functions with high algebraic immunity are proposed. Two families of minimal codes contained in the second-order Reed-Muller code with large dimension are presented. Sets and vectorial Boolean functions with high algebraic immunity are also demonstrated. By plugging these subcodes, special sets and vectorial Boolean functions into our general construction, some infinite classes of minimal binary linear codes of dimension $m$ and length less than or equal to $m(m+1)/2$ are produced. Finally, a lower bound on the minimum distance of the proposed minimal codes is derived. Conjectures and open problems are also presented.

The rest of this paper is organized as follows. In Section 2, we recall some standard facts about cyclic codes, Reed-Muller codes and vectorial Boolean functions. In Section 3, we establish some relations between binary minimal codes and subsets of finite fields without nonzero low degree annihilators. It enables us to yield minimal codes via certain subcodes of Reed-Muller codes and sets with high algebraic immunity. In Section 4, we present a general construction of minimal codes from subcodes of Reed-Muller codes and vectorial Boolean functions having high algebraic immunity. In Section 5, we conclude this paper.

2. Background

2.1. Boolean functions and vectorial Boolean functions

A Boolean function $f$ on $\text{GF}(2^m)$ is a $\text{GF}(2)$-valued function on the Galois field $\text{GF}(2^m)$ of order $2^m$. The set of all Boolean functions over $\text{GF}(2^m)$ forms a ring and is denoted by $\mathbb{B}_m$. The support of $f$, denoted by $\text{Supp}(f)$, is the set of elements of $\text{GF}(2^m)$ whose image under $f$ is 1, that is, $\text{Supp}(f) = \{ x \in \text{GF}(2^m) : f(x) = 1 \}$. The Hamming weight $\text{wt}(f)$ of a Boolean function is the size of its support $\text{Supp}(f)$. The characteristic function $f_D$ of a subset $D$ of $\text{GF}(2^m)$ is the Boolean function such that $f_D(x) = 1$ for all $x \in D$ and $f_D(x) = 0$ for all $x \in \text{GF}(2^m) \setminus D$. Thus $\text{Supp}(f_D) = D$. Every nonzero Boolean function $f$ on $\text{GF}(2^m)$ has a unique univariate polynomial expansion of the form

$$f(x) = \sum_{j=0}^{2^m-1} a_j x^j,$$

where $a_j \in \text{GF}(2^m)$. The algebraic degree $\text{deg}(f)$ of $f$ is then equal to the maximum 2-weight (or Hamming weight) of an exponent $j$ for which $a_j \neq 0$, with the usual convention that the degree of the zero function is the negative infinity.
For a nonempty proper subset \( D \) of \( \mathbb{GF}(2^m) \), a function \( g \in \mathbb{B}_m \) is called an annihilator of \( D \) if \( gf_D = 0 \). All annihilators of \( D \) form an ideal of \( \mathbb{B}_m \), denoted by \( \text{Ann}(D) \). The algebraic immunity of \( D \) is defined as

\[
\text{AI}(D) = \min \{ \deg(g) : g \in \text{Ann}(D) \setminus \{0\} \}.
\]

For convenience, we define \( \text{AI}(\emptyset) = -\infty \) and \( \text{AI}(\mathbb{GF}(2^m)) = +\infty \). It is easy to see that \( \text{AI}() \) is monotone, which means that \( \text{AI}(D_1) \leq \text{AI}(D_2) \) for any subsets \( D_1 \subseteq D_2 \) of \( \mathbb{GF}(2^m) \).

A vectorial Boolean \((m,r)\)-function \( F = (f_1, \ldots, f_r) \) is a function from \( \mathbb{GF}(2^m) \) to \( \mathbb{GF}(2)^r \). For any vector \( v = (v_1, \ldots, v_r) \in \mathbb{GF}(2)^r \), the component function \( v \cdot F \) is the Boolean function given by \( v_1 f_1 + \cdots + v_r f_r \). The algebraic immunity of \( F \) is defined as

\[
\text{AI}(F) = \min \{ \text{AI}(F^{-1}(y)) : y \in \mathbb{GF}(2)^r \},
\]

where \( F^{-1}(y) \) is the preimage of \( y \) under \( F \). It was shown in [12] that the Hamming weight \( \text{wt}(f) \) of a Boolean function \( f \) with prescribed algebraic immunity satisfies:

\[
\begin{align*}
\sum_{i=0}^{\text{AI}(f)-1} \binom{m}{i} &\leq \text{wt}(f) \leq \sum_{i=0}^{m-\text{AI}(f)} \binom{m}{i}.
\end{align*}
\]

It follows that \( \text{AI}(f) \leq \lceil \frac{m}{2} \rceil \). Thus, Boolean functions attaining this upper bound are often said to have the optimal algebraic immunity. For more information on vectorial Boolean functions, the reader is referred to [11].

The \( \tau \)-th order nonlinearity \( \text{NL}_\tau(f) \) of a Boolean function \( f \in \mathbb{B}_m \) is the minimum Hamming distance \( \text{dist}(f,g) = |\{x \in \mathbb{GF}(2^m) : f(x) \neq g(x)\}| \) between \( f \) and all functions \( g \) of algebraic degree at most \( \tau \). The \( \tau \)-th order nonlinearity \( \text{NL}_\tau(F) \) of a vectorial function \( F \) is the minimum \( \tau \)-th order nonlinearity of its component functions. It was shown in [9] that the \( \tau \)-th order nonlinearity of a vectorial \((m,r)\) function \( F \) with given algebraic immunity \( \text{AI}(F) = t \) satisfies

\[
\text{NL}_\tau(F) \geq \Upsilon_{m,r,t,\tau},
\]

where \( \Upsilon_{m,r,t,\tau} = 2^{1-r} \sum_{i=0}^{1-r} \binom{m}{i} + 2^{1-r} \sum_{i=2\tau}^{1-\tau} \binom{m-\tau}{i} \). In the particular case that \( r = \tau = 1 \), (1) says that

\[
\text{NL}_1(f) \geq 2^{\text{AI}(f)-2} \sum_{i=0}^{\text{AI}(f)-2} \binom{m-1}{i},
\]

where \( f \in \mathbb{B}_m \).

2.2. Minimal codes and cyclic codes

We assume that the reader is familiar with the basics of linear codes (see for instance [24] for detail). A linear code of length \( n \) and dimension \( k \) will be referred to as an \([n,k]\) code. Further, if the code has minimum distance \( d \), it will be referred to as an \([n,k,d]\) code.

The Hamming weight (for short, weight) of a vector \( v \) is the number of its nonzero entries and is denoted \( \text{wt}(v) \). The minimum (respectively, maximum) weight of the code \( C \) is the minimum (respectively, maximum) nonzero weight of all codewords of \( C \), \( w_{\min} = \min(\text{wt}(c)) \) (respectively, \( w_{\max} = \max(\text{wt}(c)) \)).
Let $c = (c_0, \cdots, c_{n-1})$ be a codeword in $C$. The support $\text{Supp}(c)$ of the codeword $c$ is the set of indices of its nonzero coordinates:

$$\text{Supp}(c) = \{i : c_i \neq 0\}.$$ 

A codeword $c$ of the linear code $C$ is called minimal if its support does not contain the support of any other linearly independent codeword. $C$ is called a minimal linear code if all codewords of $C$ are minimal. Minimal codes are a special class of linear codes. A sufficient condition for a linear code to be minimal is given in the following lemma [3].

**Lemma 1** (Ashikhmin-Barg). A linear code $C$ over $\text{GF}(q)$ is minimal if $\frac{w_{\text{min}}}{w_{\text{max}}} > \frac{q-1}{q}$.

Let $C$ be an $[n, k, d]$ linear code over $\text{GF}(q)$ and $T$ a set of $t$ coordinate locations of $C$. Then the code $C^T$ obtained from $C$ by puncturing at the locations in $T$ is the code of length $n-t$ consisting of codewords of $C$ which have their coordinate at the location $P$ deleted if $P \in T$ and left alone if $P \not\in T$, which is called the punctured code of $C$ on $T$. The shortened code $C_T$ is the set of codewords from $C$ that are zero at locations in $T$, with coordinates in $T$ deleted.

An $[n, k]$ linear code $C$ over $\text{GF}(q)$ is called cyclic if $(c_0, c_1, \cdots, c_{n-1}) \in C$ implies that the circular shift $(c_{n-1}, c_0, \cdots, c_{n-2}) \in C$. Clearly the vector space $\text{GF}(q)^n$ is isomorphic to the residue class ring $\text{GF}(q)[X]/(X^n - 1)$ (considered as an additive group). An isomorphism is given by

$$(c_0, c_1, \cdots, c_{n-1}) \leftrightarrow c_0 + c_1X + \cdots + c_{n-1}X^{n-1}.$$ 

From now on we do not distinguish between codewords of $C$ and polynomials of degree less than $n$ over $\text{GF}(q)$. Note that the multiplication by $X$ in $\text{GF}(q)[X]/(X^n - 1)$ amounts to the circular right shift $(c_0, c_1, \cdots, c_{n-1}) \rightarrow (c_{n-1}, c_0, \cdots, c_{n-2})$. From this it follows that a cyclic code $C$ corresponds to an ideal in $\text{GF}(q)[X]/(X^n - 1)$, which we also denote by $C$. Every $[n, k]$ cyclic code $C$ over $\text{GF}(q)$ is a principal ideal generated by some polynomial $g(X)$ of degree $n-k$ that divides $X^n - 1$. We shall call $g(X)$ and $h(X) = (X^n - 1)/g(X)$ the generator polynomial and the check polynomial of $C$, respectively. Note that the codewords $g(X), Xg(X), \cdots, X^{k-1}g(X)$ form a basis of $C$.

Let us recall the **BCH bound** on the minimum distance of cyclic codes [16].

**Theorem 2.** Let $h$ be an integer and $\delta$ be a positive integer with $1 \leq \delta < n$. Let $\alpha$ be a primitive $n$-th root of unity in the algebraic closure of $\text{GF}(q)$. Let $C$ be a cyclic code of length $n$ over $\text{GF}(q)$ with generator polynomial $g(X)$. If $g(X)$ has $\delta$ consecutive zeros $\alpha^h, \cdots, \alpha^{h+\delta-1}$, then the minimum distance of $C$ is greater than $\delta$.

### 2.3. Reed-Muller codes

Reed-Muller (RM) codes are classical codes that have enjoyed unabated interest since their introduction in 1954 due to their simple recursive structure.

Let $\alpha$ be a primitive element of $\text{GF}(2^m)$. Let $P_0 = 0$ and $P_j = \alpha^{j-1}$, where $1 \leq j \leq 2^m - 1$. Then $P_0, \ldots, P_{2^m-1}$ is an enumeration of the points of the vector space $\text{GF}(2^m)$. Under this enumeration, the Reed-Muller code $\text{RM}(\ell, m)$ of order $\ell$ in $m$ variables is defined as

$$\text{RM}(\ell, m) = \{(f(P_0), \cdots, f(P_{2^m-1})) : f \in \mathbb{B}_m, \text{deg}(f) \leq \ell\}.$$ 


In this paper, we index the coordinates of the code $RM(\ell, m)$ with the sequence $(P_0, P_1, \ldots, P_{2^m - 1})$. The general affine group over $GF(2^m)$, denoted by $GA(1, 2^m)$, is defined by

$$GA(1, 2^m) = \{ \pi_{a,b} : a \in GF(2^m)^*, b \in GF(2^m) \},$$

where $\pi_{a,b}$ is the permutation on $GF(2^m)$ defined by $x \mapsto ax + b$. Since $\deg(f(x)) = \deg(f(ax + b))$ for any $(a, b) \in GF(2^m)^* \times GF(2^m)$, the Reed-Muller code $RM(\ell, m)$ is invariant under the action by $GA(1, 2^m)$. We denote the codes obtained after the puncturing and shortening operation on $RM(\ell, m)$ at the coordinate location $P_0$ as $PRM(\ell, m)$ and $SRM(\ell, m)$, respectively. It is easy to see that the punctured code $PRM(\ell, m)$ and the shortened code $SRM(\ell, m)$ of the Reed-Muller code $RM(\ell, m)$ are cyclic codes of length $2^m - 1$. Let $g_{\ell, \alpha}(X)$ and $g^*_{\ell, \alpha}(X)$ denote the generator polynomials of the cyclic codes $PRM(\ell, m)$ and $SRM(\ell, m)$, respectively.

The following proposition describes the generator polynomials of the punctured Reed-Muller codes, which are not hard to prove \cite{2}.

**Proposition 3.** The punctured Reed-Muller code $PRM(\ell, m)$ is a cyclic code of dimension $\sum_{j=0}^{\ell} \binom{m}{j}$ with generator polynomial

$$g_{\ell, \alpha}(X) = \prod_{0 < i_{m-1} + \cdots + i_0 \leq m - 1 - \ell, \atop i_{m-1}, \cdots, i_0 \in \{0, 1\}} (X - \alpha^{i_{m-1}2^{m-1} + \cdots + i_02^0}).$$

The following proposition is taken from Corollary 4 of \cite{1}.

**Proposition 4.** The minimum weight of the Reed-Muller code $RM(\ell, m)$ is $2^{m-\ell}$ and the minimum-weight codewords are the incidence vectors of the $(m - \ell)$-flats of the affine space $AG(m, 2)$ of dimension $m$ over $GF(2)$. The minimum weight of the punctured code $PRM(\ell, m)$ is $2^{m-\ell} - 1$ and the minimum-weight codewords are the incidence vectors of the $(m - \ell - 1)$-dimensional subspaces of the projective space $PG(m - 1, 2)$ of dimension $m - 1$ over $GF(2)$.

**Lemma 5.** The minimum weight of the shortened Reed-Muller code $SRM(\ell, m)$ is $2^{m-\ell}$ and the minimum-weight codewords are the incidence vectors of the $(m - \ell)$-flats not passing through the origin in $AG(m, 2)$.

**Proof.** Note that the shortened code $SRM(\ell, m)$ consists of codewords of $RM(\ell, m)$ that are zero at the origin of $AG(m, 2)$. The desired result then follows from Proposition 4. \qed

### 3. Minimal codes from sets without nonzero low-degree annihilators

In this section we present a general construction of minimal codes using subcodes of Reed-Muller codes and subsets of finite fields without nonzero low-degree annihilators.

Here and hereafter, for any subset $D$ of $GF(2^m)$, let $D^c$ denote the set $D \setminus \{0\}$ and $\overline{D}$ stand for the complement of $D$ in $GF(2^m)$. In particular, if $D \subseteq GF(2^m)^*$, then $\overline{D}^c$ is the complement of $D$ in $GF(2^m)^*$. Let $\mathbb{B}_m^0$ denote the set $\{f \in \mathbb{B}_m : f(0) = 0\}$.

The following theorem presents a general approach to constructing binary minimal codes, and produces many classes of binary minimal codes by selecting some subcodes of Reed-Muller codes and subsets of $GF(2^m)$ with special annihilators.
Theorem 6. Let $C$ be a $k$-dimensional subcode of the Reed-Muller code $\text{RM}(\ell,m)$. Let $D$ be a subset of $\text{GF}(2^m)$. Then $C^D$ is a minimal code of dimension $k$ if and only if the following two conditions hold:

1. the code $C$ is minimal, and
2. for any two nonzero codewords $(f_1(P_0), \ldots, f_1(P_{2^m-1}))$ and $(f_2(P_0), \ldots, f_2(P_{2^m-1}))$ of $C$ (including the case $f_1 = f_2$), where $f_1, f_2 \in \mathbb{B}_m$, the product $f_1 f_2$ of $f_1$ and $f_2$ is not an annihilator of $D$.

Proof. Let $C$ be a linear code satisfying Conditions (1) and (2). Let $(f(P_0), \ldots, f(P_{2^m-1}))$ be any nonzero codeword of $C$, where $f \in \mathbb{B}_m$. Let $f_1 = f_2 = f$. Then $f_1 f_2 = f^2 = f$, and by Condition (2), $f$ is not an annihilator of $D$, i.e., $(f(P))_{P \in D} \neq 0$. Consequently the punctured code $C^D$ has the same dimension as the original code $C$. Suppose that $C^D$ is not minimal. Then there exist two distinct nonzero codewords $(f_1(P))_{P \in D}, (f_2(P))_{P \in D} \in C^D$, where $f_1, f_2 \in \mathbb{B}_m$, such that $\text{Supp} ((f_1(P))_{P \in D}) \not\subseteq \text{Supp} ((f_2(P))_{P \in D})$. This clearly forces

$$\text{Supp} ((f_2(P))_{P \in D}) = \text{Supp} (((f_1 + f_2)(P))_{P \in D}) \cup \text{Supp} ((f_1(P))_{P \in D}),$$

It follows that $f_1(f_1 + f_2) \in \text{Ann}(D)$, which is contrary to Condition (2). Therefore $C^D$ is minimal.

Conversely, assume $C^D$ is a minimal code with dimension $k$. It is clear that $C$ is minimal. It remains to show that Condition (2) holds. On the contrary, suppose that there exist two nonzero codewords $(f_1(P_0), \ldots, f_1(P_{2^m-1}))$ and $(f_2(P_0), \ldots, f_2(P_{2^m-1}))$ of $C$, where $f_1, f_2 \in \mathbb{B}_m$, such that $f_1 f_2 \in \text{Ann}(D)$. Then $\text{Supp} ((f_1(P))_{P \in D})$ and $\text{Supp} ((f_2(P))_{P \in D})$ are disjoint. This yields

$$\text{Supp} ((f_1(P))_{P \in D}) \not\subseteq \text{Supp} (((f_1 + f_2)(P))_{P \in D}),$$

which contradicts the minimality of nonzero codewords of $C^D$. This completes the proof.

Corollary 7. Let $C$ be a minimal code contained in the Reed-Muller code $\text{RM}(\ell,m)$. Let $D$ be a subset of $\text{GF}(2^m)^*$ with $\text{AI}(D) \geq 2\ell + 1$. Then $C^D$ is a minimal code of dimension $k$, where $k$ equals the dimension of $C$.

Proof. Let $(f_1(P_0), \ldots, f_1(P_{2^m-1}))$ and $(f_2(P_0), \ldots, f_2(P_{2^m-1}))$ be any two nonzero codewords of $C$, where $f_1, f_2 \in \mathbb{B}_m$. Since $C$ is minimal, $f_1 f_2$ cannot be the zero function. From $\deg(f_1 f_2) \leq 2\ell$ and $\text{AI}(D) \geq 2\ell + 1$, we conclude that $f_1 f_2 \not\in \text{Ann}(D)$. The desired result then follows from Theorem 6.

Corollary 8. Let $C$ be a minimal code contained in the shortened Reed-Muller code $\text{SRM}(\ell,m)$. Let $D$ be a subset of $\text{GF}(2^m)^*$ with $\text{AI}(D \cup \{0\}) \geq 2\ell + 1$. Then $C^D$ is a minimal code of dimension $k$, where $k$ equals the dimension of $C$.

Proof. Denote by $C'$ the codes $\{(0,c) : c \in C\}$. Since $\text{AI}(D \cup \{0\}) \geq 2\ell + 1$, it follows from Corollary 7 that $C^D_{\{0\}}$ is a minimal code of dimension $k$. The desired conclusion then follows from the definitions of $C'$ and shortened codes.

To deduce a lower bound on the minimum distance of the codes from sets with high algebraic immunity, we need some additional lemmas. Denote by $\text{Ann}_t(g)$ the vector space of those annihilators of degrees at most $t - 1$ of $\text{Supp}(g)$.
Lemma 9. Let $D \subseteq \GF(2^m)$ with $\AI(D) = t$ and $g \in \B_m$. Then
\[ \wt(gf_D) \geq \dim(\Ann_{t-1}(1+g)). \]

Proof. Let $w = \wt(gf_D)$ and let $Q$ be the set consisting of $w$ distinct points $Q_1, \ldots, Q_w$ in $D$ satisfying $g(Q_i) = 1$. Consider the evaluation map $\Ev_Q$
\[ \Ann_{t-1}(1+g) \rightarrow \GF(2)^w, \]
defined by $\Ev_Q(h) = (h(Q_1), \ldots, h(Q_w))$. Then $\Ev_Q$ is a linear transformation. Suppose the assertion of the lemma is false. Then $\Ev_Q$ is not injective. Thus there exists a nonzero function $h$ in $\Ann_{t-1}(1+g)$ such that $hf_Q = 0$. It follows easily that $h \in \Ann(D)$, which contradicts the condition that $\AI(D) = t$. This completes the proof.

Little is known about the behavior of the annihilators of a polynomial of a given degree. Mesnager [25] proved the following lower bound on the dimension of $\Ann_{t-1}(g)$
\[ \dim(\Ann_{t-1}(g)) \geq t - (m - \tau - 1) \sum_{i=0}^t m - \tau \]
where $g \in \B_m$ with $\deg(g) = \tau$. Lemma 9 indicates that the following lemma holds.

Lemma 10. Let $D \subseteq \GF(2^m)$ with $\AI(D) = t$ and $g \in \B_m \setminus \{0\}$ with $\deg(g) = \tau$. Then
\[ \wt(gf_D) \geq t - (m - \tau - 1) \sum_{i=0}^t m - \tau \]

A method of explicitly constructing minimal codes by puncturing the Simplex codes is given in the following theorem.

Theorem 11. Let $D$ be a subset of $\GF(2^m)^*$ with $\AI(D \cup \{0\}) = t \geq 3$. Let $C(D)$ be the linear code given by
\[ C(D) = \{ (\Tr^a(D))_{x \in D} : a \in \GF(2^m) \}. \]

Then $C(D)$ is a minimal code with parameters $[|D|, m, \geq \sum_{i=0}^t m - i]$. 

Proof. Let $C \subseteq \SRM(1,m)$ be the Simplex code defined by
\[ \{ (\Tr^a(D_1), \ldots, \Tr^a(D_{2^m-1})) : a \in \GF(2^m) \}. \]
It is well-known that $C$ is a minimal code of dimension $m$. The desired conclusions are immediate from Corollary 8 and Lemma 10.

Corollary 12. Let $m \geq 5$ be an integer and let $f \in \B_m$ be a Boolean function with $\AI(f) = t \geq 3$ and $\wt(f) \geq 2^{m-1}$. Let $D = \Supp(f) \setminus \{0\}$. Then the code $C(D)$ defined by (3) is an $m$-dimensional minimal code with minimum distance $d$ satisfying
\[ d \geq \sum_{i=0}^{t-2} \left( m - 1 \right)^i + \frac{1}{2} (\wt(f) - 2^{m-1}). \]
Proof. It follows from Theorem 11 that \( \mathcal{C}(D) \) is an \( m \)-dimensional minimal code. It remains to prove the lower bound on the minimum distance of \( \mathcal{C}(D) \). Denote by \( g \) the Boolean function \( \text{Tr}_1^n(ax) \), where \( a \in \text{GF}(2^m)^* \). Let \( w = \text{wt}\left((\text{Tr}_1^n(ax))_{x \in D}\right) \). Thus

\[
\begin{align*}
w &= |\{x \in \text{GF}(2^m) : f(x) = 1, g(x) = 1\}|.
\end{align*}
\]

By the definition of the Hamming distance between \( f \) and \( g \), we have

\[
\begin{align*}
\text{dist}(f, g) &= |\{x \in \text{GF}(2^m) : f(x) = 0, g(x) = 1\}| \\
&\quad + |\{x \in \text{GF}(2^m) : f(x) = 1, g(x) = 0\}| \\
&= \text{wt}(g) - w + \text{wt}(f) - w \\
&= \text{wt}(g) + \text{wt}(f) - 2w.
\end{align*}
\]

This yields

\[
\begin{align*}
w &= \left(\text{wt}(g) + \text{wt}(f) - \text{dist}(f, g)\right)/2 \\
&= \left(\text{dist}(f, 1+g) + \text{wt}(g) + \text{wt}(f) - 2^m\right)/2. \\
&= \left(\text{dist}(f, 1+g) + \text{wt}(g) + \text{wt}(f) - 2^m\right)/2.
\end{align*}
\]

The desired conclusion then follows from 2, 4 and the fact that \( \text{wt}(g) = 2^{m-1} \). \( \square \)

Now, consider balanced Boolean functions in Corollary 12. Then, we obtain the following result.

**Corollary 13.** Let \( m \geq 5 \) be an integer. Let \( f \in \mathbb{B}_m \) be a balanced Boolean function with \( \text{AI}(f) = t \geq 3 \). Let \( D = \text{Supp}(f) \setminus \{0\} \). Then the code \( \mathcal{C}(D) \) defined by (3) is a \( 2^{m-1}, m, \geq \sum_{i=0}^{t-2} \binom{m-1}{i} \) minimal code. Moreover, if \( f \) has optimum algebraic immunity, then \( \mathcal{C}(D) \) is a minimal code with parameters \( 2^{m-1}, m, \geq \sum_{i=0}^{\left\lceil \frac{m-4}{t} \right\rceil} \binom{m-1}{i} \).

In order to apply Theorem 6 to construct minimal codes, finding sets without low-degree nonzero annihilators is very important. Let \( \alpha \) be a primitive element of \( \text{GF}(2^m)^* \), \( h \) and \( \delta \) be two integers with \( \delta > 0 \). Denote then \([h; \delta]_\alpha = \{\alpha^h, \alpha^{h+1}, \ldots, \alpha^{h+\delta-1}\}\).

**Lemma 14.** Let \( \delta \) be an integer with \( \sum_{i=0}^{t} \binom{m}{i} \leq \delta < \sum_{i=0}^{t+1} \binom{m}{i} \). Then \( \text{AI}([h; \delta]_\alpha) = t+1 \).

**Proof.** Let \( f \) be a function of degree at most \( t \) in \( \text{Ann}([h; \delta]_\alpha) \) and \( \tilde{f}(X) \in \text{PRM}(t, m) \) be the codeword associated with \( f \). By assumption, we see that

\[
\tilde{f}(X) = X^h \sum_{i=\delta}^{2^m-2} c_i X^i,
\]

where \( c_i \in \text{GF}(2) \). By the definition of the generator polynomial \( g_{t,m}(X) \) of \( \text{PRM}(t, m) \), the codeword \( X^{-h}\tilde{f}(X) \) can also be written as

\[
X^{-h}\tilde{f}(X) = \left(a_0 + a_1 X + \cdots + a_{\delta_t-1} X^{\delta_t-1}\right) g_{t,m}(X),
\]

where \( \delta_t = \sum_{i=0}^{t} \binom{m}{i} \) and \( a_i \in \text{GF}(2) \). Combining (5) with (6) yields \( a_0 = \cdots = a_{\delta_t-1} = 0 \). We thus get \( f = 0 \). Hence \( \text{AI}([h; \delta]_\alpha) \geq t+1 \).
Let \( f \) be the function corresponding to the codeword \( \tilde{f}(X) \) of \( \text{PRM}(t+1,m) \), where \( \tilde{f}(X) \) is given by

\[
\tilde{f}(X) = X^{h} \left( X^{\delta} + X^{\delta+1} + \cdots + X^{\delta+t-1} \right) g_{t+1,m}(X),
\]

where \( \delta_{t+1} = \sum_{i=0}^{t-1} \binom{m}{i} \). It is easy to check that \( f \in \text{Ann}([h;\delta]_{\alpha}) \setminus \{0\} \). It follows that \( \text{AI}(\lfloor h;\delta \rfloor_{\alpha}) \leq t+1 \).

Summarising the discussions above yields \( \text{AI}(\lfloor h;\delta \rfloor_{\alpha}) = t+1 \), which is the desired conclusion.

The proof of the following lemma is similar to the proof of Lemma \([\text{4}]\) with punctured Reed-Muller codes replaced by shortened Reed-Muller codes, and therefore is omitted.

**Lemma 15.** Let \( \delta \) be an integer with \( \sum_{i=1}^{t} \binom{m}{i} \leq \delta < \sum_{i=1}^{t+1} \binom{m}{i} \). Then \( \text{AI}(\lfloor h;\delta \rfloor_{\alpha} \cup \{0\}) = t+1 \).

Now we recall some facts on Gauss sums which will be needed to derive an improved lower bound on the minimum distance of the codes from the sets \( \lfloor h;\delta \rfloor_{\alpha} \). Let \( \xi_{q-1} \) denote the complex primitive \((q-1)\)th root of unity \( e^{2\pi\sqrt{-1}/(q-1)} \). Let \( \alpha \) be a primitive element of \( \text{GF}(q) \), and let \( \chi \) be the character of \( \text{GF}(q)^* \) given by

\[
\chi(\alpha^j) = \xi_{q-1}^j,
\]

where \( 0 \leq j \leq q-2 \). The Gauss sum associated to \( \chi^j \) over \( \text{GF}(q) \) with \( q = 2^m \) is defined by

\[
G(\chi^j) = \sum_{i=0}^{q-2} (-1)^{\text{Tr}^m_i(\alpha^j)} \chi(\alpha^i) \text{, for } j = 0, \cdots, q-2.
\]

Then \( G(\chi^0) = -1 \) and \( G(\chi^j) \) \( 1 \leq j \leq q-2 \) satisfies the fundamental property \([\text{18, p. 132}]\)

\[
G(\chi^j)\overline{G(\chi^j)} = q, \tag{7}
\]

where the bar denotes complex conjugate. It is sometimes convenient to view the Gauss sum \( G(\chi^j) \) as a function of \( \chi^j \). This amounts to viewing \( \chi^j \mapsto G(\chi^j) \) as the multiplicative Fourier transformation of the function \( (-1)^{\text{Tr}^m_i(x)} \) on \( \text{GF}(q)^* \). The following Fourier inversion formula allows us to recover \( (-1)^{\text{Tr}^m_i(x)} \) from \( G(\chi^j) \) by

\[
(-1)^{\text{Tr}^m_i(\alpha^j)} = \frac{1}{q-1} \sum_{j=0}^{q-2} \overline{\chi}^j(\alpha^j) G(\chi^j). \tag{8}
\]

We will need the following lemma, whose proof can be found in \([\text{10}]\).

**Lemma 16.** Let \( q = 2^m \). It holds

\[
\sum_{j=1}^{2^{m-1}-1} \frac{1}{\sin(\pi j/(q-1))} \leq \frac{q-1}{2\pi} \ln \left( \frac{4(q-1)}{\pi} \right).
\]

Combining Simplex codes with the sets \( \lfloor h;\delta \rfloor_{\alpha} \), an infinite class of binary minimal codes is given in the following theorem.
Theorem 17. Let \( q = 2^m \). Let \( \delta \) be an integer with \( \sum_{i=1}^{m-1} \binom{m}{i} \leq \delta < \sum_{i=1}^{m} \binom{m}{i} \) and \( 3 \leq t \leq m \). Then the code \( C([h; \delta]_\alpha) \) defined by (3) is a minimal code with parameters \([\delta,m,d] \), where

\[
d \geq \max \left\{ \sum_{i=0}^{\delta-2} \binom{m-1}{i}, \frac{\delta - 1}{2} - \frac{\sqrt{q}}{2\pi} \ln \left( \frac{4(q - 1)}{\pi} \right) \right\}.
\]

Proof. Let \( c = (\text{Tr}_1^m(\lambda \alpha^h), \cdots, \text{Tr}_1^m(\lambda \alpha^{h+\delta-1})) \) be a nonzero codeword of \( C([h; \delta]_\alpha). \) Then

\[
\text{wt}(c) = \alpha^{\sum_{i=0}^{\delta-2} \binom{m}{i} \left( 1 - (-1)^{\text{Tr}_i^m(\lambda \alpha^h)} \right)}
\]

where \( F = \sum_{i=0}^{\delta-1} (-1)^{\text{Tr}_i^m(\lambda \alpha^h)} \). Set \( \lambda' = \lambda \alpha^h \). Substituting (8) into (9) yields

\[
|F| = \frac{1}{q-1} \sum_{i=0}^{\delta-2} \sum_{j=0}^{q-2} \chi(\lambda'^j G(\chi^j)) \leq \frac{1}{q-1} \sum_{i=0}^{\delta-1} \chi(G(\chi^i)) \sum_{j=0}^{q-2} \chi(\alpha^{ij}) \leq \frac{1}{q-1} \alpha j + \frac{\sqrt{q}}{q-1} \sum_{j=1}^{q-2} \left| \sum_{i=0}^{\delta-1} \chi(\alpha^{ij}) \right|,
\]

where the last inequality follows from (7). A simple calculation yields

\[
|F| \leq \frac{1}{q-1} \alpha j + \frac{\sqrt{q}}{q-1} \sum_{j=1}^{q-2} \left| \sum_{i=0}^{\delta-1} \frac{\sin(\pi j \delta/(q-1))}{\sin(\pi j/(q-1))} \right| \leq \frac{1}{q-1} \alpha j + \frac{\sqrt{q}}{q-1} \sum_{j=1}^{q-2} \left| \sum_{i=0}^{\delta-1} \frac{1}{\sin(\pi j/(q-1))} \right| \leq \frac{1}{q-1} \alpha j + \frac{\sqrt{q}}{\pi} \ln \left( \frac{4(q - 1)}{\pi} \right),
\]

where the last inequality follows from Lemma 16. Combining (9) with (10), we deduce that

\[
\text{wt}(c) \geq \frac{\delta - 1}{2} - \frac{\sqrt{q}}{2\pi} \ln \left( \frac{4(q - 1)}{\pi} \right).
\]

The desired conclusion then follows from Theorem 11 and Lemma 15.

\[\square\]

Corollary 18. Let \( m \geq 5 \) be an integer and \( \alpha \) a primitive element of \( \text{GF}(2^m) \). Let \( C \) be the set given by

\[C = \left\{ \left( \text{Tr}_1^m(a \alpha^0), \cdots, \text{Tr}_1^m(a \alpha^{m(m+1)/2-1}) \right) : a \in \text{GF}(2^m) \right\}.
\]

Then \( C \) is a binary minimal code of dimension \( m \) and length \( m(m+1)/2 \), which was introduced in [35]. These minimal codes can be described as follows:

\[
\left\{ \left( \text{Tr}_1^m(a \alpha_1), \cdots, \text{Tr}_1^m(a \alpha_m), \right. \right. \left. \text{Tr}_1^m(\alpha(\alpha_1 + \alpha_2)), \cdots, \text{Tr}_1^m(\alpha(\alpha_{m-1} + \alpha_m)) \right) : a \in \text{GF}(2^m) \right\}.
\]
where \( \alpha_1, \cdots, \alpha_m \) form a basis of \( \text{GF}(2^m) \) over \( \text{GF}(2) \). Obviously, the codes obtained in \([11]\) are unique, up to equivalence. Their minimum distance \( d(m) \) is equal to \( m \). Many infinite families of minimal codes of dimension \( m \) and length \( m(m+1)/2 \) can be produced from Corollary \([18]\).

Denote by \( d_{\text{max}}(m) \) and \( d_{\text{min}}(m) \) the largest and smallest values of the minimum distances of the codes of Corollary \([18]\) respectively. Figure 2 shows that the minimum distance of the minimal code from Corollary \([18]\) would be better than that of the code given in \([11]\), and suggests the following conjecture.

**Conjecture 19.** Let \( m \geq 7 \) be an integer. Then the minimal code in Corollary \([18]\) has minimum distance greater than \( m \). Moreover, if two primitive elements \( \alpha \) and \( \alpha' \) of \( \text{GF}(2^m) \) satisfy \( \alpha' \neq \alpha^{2^i} \) for any \( 0 \leq i \leq m-1 \), then the two codes corresponding to \( \alpha \) and \( \alpha' \) are inequivalent.

By the definition of shortened codes, Proposition \([8]\) shows that the generator polynomial of the shortened second-order Reed-Muller code \( \text{SRM}(2, m) \) can be expressed as

\[
g_{2, \alpha}^*(X) = \prod_{\sum_i i \alpha^i \leq m-3} (X - \alpha^{i_0 2^{i_0 + \cdots + i_{m-1}}}).
\]

For a positive integer \( \varepsilon \), let \( \mathcal{C}_\varepsilon \) be the code contained in the shortened second-order Reed-Muller code \( \text{SRM}(2, m) \) given by

\[
\mathcal{C}_\varepsilon = \left\{ \sum_{i=0}^{\varepsilon-1} c_i X^i g_{2, \alpha}^*(X) : c_i \in \text{GF}(2) \right\}.
\] (12)

The following theorem provides a way of constructing linear codes of dimension \( m \) and length less than \( m(m+1)/2 \) via sets of algebraic immunity 2.
Theorem 20. Let \( \varepsilon \) be an integer with \( 1 \leq \varepsilon < m(m-1)/2 \). Let \( \mathcal{C} \) be the set given by

\[
\mathcal{C} = \left\{ \left( \operatorname{Tr}_1^m(a\alpha^0), \ldots, \operatorname{Tr}_1^m(a\alpha^{m(m+1)/2-\varepsilon-1}) \right) : a \in \operatorname{GF}(2^m) \right\}.
\]

Then \( \mathcal{C} \) is a minimal code if and only if the minimum distance of the code \( \mathfrak{C}_\varepsilon \) in (12) is greater than \( 2^{m-2} \).

Proof. Since \( 1 \leq \varepsilon < m(m-1)/2 \), it follows immediately that \( \dim(\mathcal{C}) = m \).

Let us first prove the sufficient condition for \( \mathcal{C} \) to be a minimal code. Suppose the assertion of the theorem is false. There would be a codeword \( \left( \sum_{i=0}^{\varepsilon-1} c_i x_i^i \right) g_{2,\alpha}^*(X) \) of weight \( 2^{m-2} \) of \( \mathfrak{C}_\varepsilon \), where \( c_i \in \operatorname{GF}(2) \). Set \( g(X) = X^{m(m+1)/2-\varepsilon} \left( \sum_{i=0}^{\varepsilon-1} c_i x_i^i \right) g_{2,\alpha}^*(X) \) and let \( f \in \mathbb{B}_m^0 \) be the Boolean function corresponding to \( g(X) \). Then \( g(X) \) is a codeword of weight \( 2^{m-2} \) of \( \text{SRM}(2, m) \) and \( f \) satisfies

\[
f(\alpha^0) = 0,
\]

where \( 0 \leq i \leq m(m+1)/2 - \varepsilon - 1 \). Lemma 5 now leads to \( f(x) = \operatorname{Tr}_1^m(a_1 x) \operatorname{Tr}_1^m(a_2 x) \), where \( a_1 \neq a_2 \in \operatorname{GF}(2^m)^* \). Let \( c_i \) be the nonzero codeword of \( \mathcal{C} \) given by \( \left( \operatorname{Tr}_1^m(a_i \alpha^i) \right)_{i=0}^{m(m+1)/2-\varepsilon-1} \), where \( i = 1 \) or \( 2 \). From (13), it is a simple matter to check that \( \text{Supp}(c_1 + c_2) = \text{Supp}(c_1) \cup \text{Supp}(c_2) \), which contradicts the assumption that \( \mathcal{C} \) is a minimal code. Consequently, the minimum distance of \( \mathfrak{C}_\varepsilon \) is greater than \( 2^{m-2} \).

Conversely, let \( \mathfrak{C}_\varepsilon \) be a linear code of minimum distance greater than \( 2^{m-2} \). Suppose that \( \mathcal{C} \) is not a minimal code. Then we could find two distinct nonzero codewords \( c_0 \) and \( c_1 \) of \( \mathcal{C} \) such that \( \text{Supp}(c_1) \nsubseteq \text{Supp}(c_0) \), where \( c_i = \left( \operatorname{Tr}_1^m(a_i \alpha^i) \right)_{i=0}^{m(m+1)/2-\varepsilon-1} \). Let \( f \) be the quadratic Boolean function \( \operatorname{Tr}_1^m(a_1 x) \operatorname{Tr}_1^m(a_2 x) \), where \( a_2 = a_0 + a_1 \). A trivial verification shows that

\[
f(\alpha^0) = \cdots = f(\alpha^{m(m+1)/2-\varepsilon-1}) = 0,
\]
Table 1: Value Distribution of $\varepsilon_m(\alpha)$ ($5 \leq m \leq 9$)

| $m$ | $\varepsilon_m(\alpha)$ | Freq. | $m$ | $\varepsilon_m(\alpha)$ | Freq. |
|-----|--------------------------|-------|-----|--------------------------|-------|
| 5   | 0                        | 10    | 9   | 1                        | 36    |
| 5   | 1                        | 20    | 9   | 5                        | 18    |
| 6   | 0                        | 12    | 9   | 6                        | 18    |
| 6   | 1                        | 12    | 9   | 7                        | 18    |
| 6   | 3                        | 12    | 9   | 8                        | 36    |
| 7   | 0                        | 42    | 9   | 9                        | 36    |
| 7   | 3                        | 28    | 9   | 10                       | 72    |
| 7   | 4                        | 42    | 9   | 11                       | 36    |
| 7   | 5                        | 14    | 9   | 12                       | 54    |
| 8   | 6                        | 48    | 9   | 13                       | 18    |
| 8   | 7                        | 16    | 9   | 14                       | 54    |
| 8   | 10                       | 64    | 9   | 15                       | 18    |
| 9   | 0                        | 18    |     |                          |       |

and

$$\text{wt}(f) = 2^{m-2}. \quad (15)$$

Since $g_{2,\alpha}^*(X)$ is the generator polynomial of SRM$(2, m)$, the codeword $(f(P_i))_{i=1}^{2^m-1}$ can be uniquely expressed as $\left(\sum_{i=0}^{m(m+1)/2-1} c_i X^i\right) g_{2,\alpha}^*(X)$, where $c_i \in \text{GF}(2)$. From (14) it may be concluded that $c_i = 0$ for any $0 \leq i \leq m(m+1)/2 - \varepsilon - 1$. Combining this with (15) we deduce that

$$\text{wt}\left(\left(\sum_{i=0}^{\varepsilon-1} c_{i+m(m+1)/2-\varepsilon} X^i\right) g_{2,\alpha}^*(X)\right) = 2^{m-2}.$$ 

This contradicts our assumption about the minimum distance of $C_\varepsilon$. It completes the proof.

Let $\alpha$ be any primitive element of $\text{GF}(2^m)$. Let us denote by $\varepsilon_m(\alpha)$ the maximum $\varepsilon$ such that the code $C$ in Theorem 20 is an $m$-dimensional minimal code. The values of $\varepsilon_m(\alpha)$ and their corresponding frequencies are listed in Table 1 for $5 \leq m \leq 9$. It shows that a large number of minimal codes with dimension $m$ and length less than $m(m+1)/2$ can be produced from Theorem 20.

As a corollary of Theorem 20 we have the following.

**Corollary 21.** Let $C$ be the set given by

$$C = \left\{ \left(\text{Tr}_1^m (a\alpha^0), \cdots, \text{Tr}_1^m (a\alpha^{m(m+1)/2-2})\right) : a \in \text{GF}(2^m) \right\}.$$ 

Then $C$ is minimal if and only if the Hamming weight of the generator polynomial $g_{2,\alpha}^*(X)$ of the shortened second-order Reed-Muller code SRM$(2, m)$ is not equal to $2^{m-2}$. 

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Corollary 22. Let $C$ be the set given by

$$C = \left\{ \left( \text{Tr}_1^m (a\alpha^0), \ldots, \text{Tr}_1^m \left( a\alpha^{m(m+1)/2-3} \right) \right) : a \in \text{GF}(2^m) \right\}.$$ 

Then $C$ is a minimal code if and only if both $\text{wt} \left( g_{2,\alpha}^*(X) \right)$ and $\text{wt} \left( (1+X)g_{2,\alpha}^*(X) \right)$ are greater than $2^{m-2}$.

Example 23. Let $q = 2^5$ and $\alpha$ be a primitive element with minimal polynomial $\alpha^5 + \alpha^3 + 1 = 0$. Then $g_{2,\alpha}^*(X) = X^{16} + X^{12} + X^{11} + X^{10} + X^9 + X^4 + X + 1$. Clearly, $\text{wt} \left( g_{2,\alpha}^*(X) \right) = 8$. The binary linear code $C$ in Corollary 22 is a minimal code, but that in Corollary 27 is not.

Example 24. Let $q = 2^6$ and $\alpha$ be a primitive element with minimal polynomial $\alpha^6 + \alpha^5 + \alpha^3 + \alpha^2 + 1 = 0$. Then $g_{2,\alpha}^*(X) = X^{42} + X^{41} + X^{39} + X^{38} + X^{37} + X^{32} + X^{31} + X^{30} + X^{29} + X^{24} + X^{19} + X^{17} + X^{16} + X^{13} + X^{12} + X^{11} + X^{10} + X^9 + X^8 + X^7 + X^3 + X^2 + X + 1$ and $(1+X)g_{2,\alpha}^*(X) = X^{43} + X^{41} + X^{40} + X^{37} + X^{33} + X^{29} + X^{25} + X^{24} + X^{20} + X^{19} + X^{18} + X^{16} + X^{14} + X^7 + X^4 + 1$. Clearly, $\text{wt} \left( g_{2,\alpha}^*(X) \right) = 24$ and $\text{wt} \left( (1+X)g_{2,\alpha}^*(X) \right) = 16$. The binary linear code $C$ in Corollary 27 is a minimal code, but that in Corollary 22 is not.

Example 25. Let $q = 2^6$ and $\alpha$ be a primitive element with minimal polynomial $\alpha^6 + \alpha^5 + \alpha^4 + \alpha + 1 = 0$. Then $g_{2,\alpha}^*(X) = X^{42} + X^{41} + X^{39} + X^{37} + X^{36} + X^{35} + X^{33} + X^{30} + X^{27} + X^{26} + X^{24} + X^{21} + X^{19} + X^{17} + X^{16} + X^{15} + X^{13} + X^9 + X^8 + X^7 + X^5 + X^4 + X^3 + 1$ and $(1+X)g_{2,\alpha}^*(X) = X^{43} + X^{41} + X^{40} + X^{39} + X^{38} + X^{35} + X^{34} + X^{33} + X^{31} + X^{30} + X^{28} + X^{26} + X^{25} + X^{24} + X^{22} + X^{21} + X^{20} + X^{19} + X^{18} + X^{15} + X^{14} + X^{13} + X^{10} + X^7 + X^6 + X^3 + X + 1$. Clearly, $\text{wt} \left( g_{2,\alpha}^*(X) \right) = 24$ and $\text{wt} \left( (1+X)g_{2,\alpha}^*(X) \right) = 28$. Both the binary linear codes in Corollary 27 and 22 are minimal codes.

It would be interesting to know how the Hamming weight of the polynomial $g_{2,\alpha}^*(X)$ would be affected by selecting $\alpha$. Based on our numerical experiments, we pose the following conjecture and open problem.

Conjecture 26. For any integer $m \geq 5$, there exists a primitive element $\alpha$ of GF(2$^m$) such that the Hamming weight of the generator polynomial $g_{2,\alpha}^*(X)$ of the shortened second-order Reed-Muller SRM(2, m) is greater than $2^{m-2}$.

Open Problem 27. Are there infinitely many positive integers $m$ such that $\text{wt} \left( g_{2,\alpha}^*(X) \right) > 2^{m-2}$ for any primitive element $\alpha$ of GF(2$^m$)?

Let $\left( f(P_i) \right)_{i=0}^{q-1}$ be any codeword of the second-order Reed-Muller code RM(2, m). Then the corresponding Boolean function $f$ can be uniquely expressed as

$$f(x) = \begin{cases} \text{Tr}_1^{m/2} \left( a_m / 2X^{2m/2+1} \right) + \\ \sum_{i=1}^{m-2} \text{Tr}_1^m \left( a_i x^{2^i+1} \right) + \text{Tr}_1^m (a_0 x) + c, & \text{if } m \text{ is even,} \\ \\ \sum_{i=1}^{m-1} \text{Tr}_1^m \left( a_i x^{2^i+1} \right) + \text{Tr}_1^m (a_0 x) + c, & \text{if } m \text{ is odd,} \end{cases}$$

(16)
where $a_{m/2} \in \text{GF}(2^{m/2})$, $c \in \text{GF}(2)$ and $a_i \in \text{GF}(2^m)$ for $0 \leq i \leq \lfloor (m - 1)/2 \rfloor$. Berlekamp and Sloane [5] have shown that all possible weights of codewords of $\text{RM}(2, m)$ are of the forms $2^{m-1}$ and $2^{m-1} \pm 2^{m-1-j}$, where $0 \leq j \leq \lfloor m/2 \rfloor$. A compact formula of the weight distribution of $\text{RM}(2, m)$ can be found in [30].

Next we shall present several classes of minimal codes contained in the shortened second-order Reed-Muller code $\text{SRM}(2, m)$.

**Theorem 28.** Let $m \geq 3$ be an odd integer. Let $C$ be the cyclic subcode of the shortened second order Reed-Muller code $\text{SRM}(2, m)$ given by

$$\left\{ \left( \sum_{i=1}^{(m-1)/2} \text{Tr}^m_i \left(a_i \alpha^{(2^i+1)j}\right) \right)_{j=0}^{2^m-2} : a_i \in \text{GF}(2^m) \right\}.$$ 

Then $C$ is a minimal code with parameters $[2^m - 1, m(m-1)/2, \geq 3 \cdot 2^{m-3}]$.

**Proof.** Let us first prove that there is no codeword of weight $2^{m-2}$ or $3 \cdot 2^{m-2}$. If there existed a codeword $c \in C$ such that $\text{wt}(c) = 2^{m-2}$, by Lemma [5] there would be two distinct elements $a, b \in \text{GF}(2^m)^*$ such that $c = \left(\text{Tr}^m(a\alpha^j)\text{Tr}^m(b\alpha^j)\right)_{j=0}^{2^m-2}$. There is no loss of generality in assuming $a = 1$ and $b \in \text{GF}(2^m) \setminus \text{GF}(2)$ as $C$ is a cyclic code. A direct calculation shows

$$\text{Tr}^m_1(a\alpha^j)\text{Tr}^m_1(b\alpha^j)$$

$$= \text{Tr}^m_1(b\alpha^j\text{Tr}^m_1(a\alpha^j))$$

$$= \sum_{i=0}^{m-1} \text{Tr}^m_1(b\alpha^{2^i+1}j)$$

$$= \sum_{i=1}^{m-1} \text{Tr}^m_1(b\alpha^{2^i+1}j) + \text{Tr}^m_1(\sqrt{b}\alpha^j),$$

which is impossible. Thus there is no codeword of weight $2^{m-2}$. Suppose there was a codeword $c \in C$ of weight $3 \cdot 2^{m-2}$, then the codeword $1 + c$ of $\text{RM}(2, m)$ has weight $2^{m-2}$. Proposition [4] now implies $1 + c = \left((1 + \text{Tr}^m_1(a\alpha^j))(1 + \text{Tr}^m_1(b\alpha^j))\right)_{j=0}^{2^m-2}$, where $a \neq b \in \text{GF}(2^m)^*$. We can assume that $a = 1$ and $b \not\in \text{GF}(2)$ as in the previous discussion. Consequently, $c$ is just the codeword given by the Boolean function $\text{Tr}^m_1(x)\text{Tr}^m_1(bx) + \text{Tr}^m_1((b+1)x)$, which can be rewritten as:

$$\sum_{i=1}^{m-1} \text{Tr}^m_1(bx^{2^i+1}) + \text{Tr}^m_1((b+\sqrt{b}+1)x).$$

Since $m$ is an odd integer, we have $b + \sqrt{b} + 1 \neq 0$. This clearly forces $c \not\in C$, a contradiction. Hence the weight $\text{wt}(c)$ of any codeword $c$ of $C$ is not equal to $3 \cdot 2^{m-2}$. Consequently, we conclude that for any nonzero codeword $c$ of $C$ its weight satisfies the following

$$3 \cdot 2^{m-3} \leq \text{wt}(c) \leq 5 \cdot 2^{m-3}.$$ 

Therefore $C$ is a minimal code by Lemma [1] and has minimum distance at least $3 \cdot 2^{m-3}$. It is obvious that $\dim(C) = m(m-1)/2$ from the definition of $C$. This completes the proof. $\Box$
The proof above gives more, namely the code $C$ in Theorem 28 is not a minimal code if $m \geq 4$ is an even integer.

**Theorem 29.** Let $m \geq 3$ be an integer. Let $C$ be the cyclic subcode of the shortened second-order Reed-Muller code $\text{SRM}(2, m)$ given by

$$\left\{ \left( \sum_{i=2}^{\lfloor m/2 \rfloor} \text{Tr}_1^m \left( a_i \alpha^{(2^i+1)j} \right) + \text{Tr}_1^m(b \alpha^j) \right)_{j=0}^{2^m-2} : b, a_i \in \text{GF}(2^m) \right\}.$$ 

Then $C$ is a minimal code with parameters $[2^m - 1, m(m-1)/2, \geq 3 \cdot 2^{m-3}]$.

**Proof.** The statements will be proved once we prove that there are no codewords of weight $2^{m-2}$ or $3 \cdot 2^{m-2}$. Suppose, contrary to our claim, that there exists a codeword with weight $2^{m-2}$ or $3 \cdot 2^{m-2}$. By a similar argument in the proof of Theorem 28, we could find $b \in \text{GF}(2^m) \setminus \text{GF}(2)$ and $c \in \text{GF}(2)$ such that the codeword of $\text{SRM}(2, m)$ corresponding to the Boolean function $f(x) = \text{Tr}_1^m(x) \text{Tr}_1^m(bx) + c \text{Tr}_1^m((b+1)x)$ lies in $C$. A simple calculation yields

$$\text{Tr}_1^m(x) \text{Tr}_1^m(bx) + c \text{Tr}_1^m((b+1)x)
= \sum_{i=0}^{m-1} \text{Tr}_1^m(bx^{2^i+1}) + c \text{Tr}_1^m((b+1)x)
= \sum_{i=2}^{m-2} \text{Tr}_1^m(bx^{2^i+1}) + \text{Tr}_1^m((\sqrt{b} + bc + c)x)
+ \text{Tr}_1^m((b + b^2)x^{2^i+1}),$$

which contradicts the definition of $C$. This completes the proof. \hfill \Box

In the spirit of Theorems 28 and 29 we pose the following open problem.

**Open Problem 30.** Does there exist a minimal code $C$ contained in the second-order Reed-Muller code $\text{RM}(2, m)$ such that its dimension $\dim(C)$ is greater than $m(m-1)/2$?

The following two theorems describe two infinite classes of minimal codes obtained by puncturing of the minimal codes in Theorems 28 and 29.

**Theorem 31.** Let $m \geq 5$ be an odd integer. Let $\delta$ be an integer with $\sum_{i=1}^{\delta-1} \binom{m}{i} \leq \delta < \sum_{i=1}^{\delta} \binom{m}{i}$ and $5 \leq \delta < m$. Let $C$ be the binary code given by

$$\left\{ \left( \sum_{i=1}^{(m-1)/2} \text{Tr}_1^m \left( a_i \alpha^{(2^i+1)j} \right) \right)_{j=0}^{\delta-1} : a_i \in \text{GF}(2^m) \right\}.$$ 

Then $C$ is a minimal code with parameters $[\delta, m(m-1)/2, \geq \sum_{i=0}^{\delta-3} \binom{m-2}{i}]$.

**Proof.** Combining Theorem 28 and Lemma 15 with Corollary 7 proves the desired conclusion. \hfill \Box
Theorem 32. Let \( m \geq 5 \) be an integer. Let \( \delta \) be an integer with \( \sum_{i=1}^{t-1} \binom{m}{i} \leq \delta < \sum_{i=1}^{t} \binom{m}{i} \) and \( 5 \leq t \leq m \). Let \( C \) be the binary code given by

\[
\left\{ \left( \sum_{i=2}^{\lfloor m/2 \rfloor} \text{Tr}_1(a_i \alpha^{2^i+1}j) + \text{Tr}_1(a_0 \alpha^j) \right)_{j=0}^{\delta-1} : a_i \in \text{GF}(2^m) \right\}.
\]

Then \( C \) is a minimal code with parameters \( [\delta, m(m-1)/2, \geq \sum_{i=0}^{t-3} \binom{m-2}{i}] \).

Proof. Combining Theorem 29 and Lemma 15 with Corollary 7 yields the desired conclusion. \( \square \)

4. Minimal codes from vectorial Boolean functions with high algebraic immunity

In this section, we shall demonstrate that binary minimal codes can be obtained from the vector subspace spanned by certain subcodes of Reed-Muller and the component functions of vectorial Boolean functions with high algebraic immunity.

For a vectorial Boolean \((m, r)\)-function \( F = (f_1, \cdots, f_r) \) with \( \text{AI}(F) = t \geq 1 \), let \( \text{Span}(F) \) be the linear code defined by

\[
\text{Span}(F) = \left\{ \left( \sum_{j=1}^{r} a_j f_j(P_i) \right)_{i=0}^{2^m-1} : a_j \in \text{GF}(2) \right\}.
\] (17)

Let \( C \) be a linear code. The sum of two linear subcodes \( C_1 \) and \( C_2 \) of \( C \) is the set, denoted \( C_1 + C_2 \), consisting of all the elements \( c_1 + c_2 \), where \( c_1 \in C_1 \) and \( c_2 \in C_2 \). If \( C_1 \cap C_2 = \{0\} \), then the sum is also called the direct sum of \( C_1 \) and \( C_2 \), and is written by \( C_1 \oplus C_2 \). Note that the direct sum of linear subcodes of a linear code is not the same thing as the direct sum of some linear codes.

Lemma 33. Let \( r \geq 2 \) and \( F \) be a vectorial Boolean \((m, r)\)-function with \( \text{AI}(F) = t \geq 1 \). Let \( D \) be the subset of \( \text{GF}(2^m) \) given by

\[
D = \{ x \in \text{GF}(2^m) : v_1 \cdot F(x) = \epsilon_1, v_2 \cdot F(x) = \epsilon_2 \},
\]

where \( v_1 \) and \( v_2 \) are two distinct nonzero elements in \( \text{GF}(2)^r \) and \( \epsilon_1, \epsilon_2 \in \text{GF}(2) \). Then \( \text{AI}(D) \geq t \).

Proof. By assumption, \( v_1 \) and \( v_2 \) are linearly independent over \( \text{GF}(2) \). It follows that there exists \( y \in \text{GF}(2)^r \) such that \( v_1 \cdot y = \epsilon_1 \) and \( v_2 \cdot y = \epsilon_2 \). We thus get \( F^{-1}(y) \subseteq D \). By the definition of algebraic immunity, \( \text{AI}(D) \geq t \). This completes the proof. \( \square \)

Let \( F \) be a vectorial Boolean \((m, r)\)-function with \( \text{AI}(F) \geq t \). Then \( \text{Span}(F) \cap \text{RM}(t-1, m) = \{0\} \) from Lemma 33. The following theorem presents a new method to construct minimal codes from some subcodes of Reed-Muller codes and vectorial Boolean functions with high algebraic immunity.

Theorem 34. Let \( C \) be a \( k \)-dimensional subcode of the Reed-Muller code \( \text{RM}(\ell, m) \) such that \( k > 1 \). Let \( F \) be a vectorial Boolean \((m, r)\)-function with algebraic immunity \( \text{AI}(F) \geq 2\ell + 1 \). Then \( C \oplus \text{Span}(F) \) is a minimal code of dimension \( k + r \).
Proof. It is clear that \( \dim(\mathcal{C} \oplus \text{Span}(F)) = k + r \). It remains to prove that \( \mathcal{C} \oplus \text{Span}(F) \) is a minimal code.

Let \((v_1 \cdot F(P_i) + f_1(P_i))_{i=0}^{2m-1}\) and \((v_2 \cdot F(P_i) + f_2(P_i))_{i=0}^{2m-1}\) be any two nonzero codewords of \( \mathcal{C} \oplus \text{Span}(F) \), where \( f_1, f_2 \in \mathbb{B}_m \) are Boolean functions of algebraic degree at most \( \ell \), and \( v_1, v_2 \in \text{GF}(2)^r \). We will complete the proof of the theorem if we prove the following:

\[
(v_1 \cdot F(x) + f_1(x)) \cdot (v_2 \cdot F(x) + f_2(x)) \neq 0. \tag{18}
\]

To this end, consider the following four cases.

Case 1: \( (v_1 \cdot F + f_1) = (v_2 \cdot F + f_2) \). If \( f_1 \equiv 0 \), then \( v_1 \neq 0 \). Applying Lemma 33 we see that \( \text{Supp}(v_1 \cdot F) \) is not the empty set, which gives (18). Let \( f_1 \) be a nonzero function. By the assumption of the theorem, \( f_1 \) does not vanish on \( \{x \in \text{GF}(2^m) : v_1 \cdot F(x) = 0\} \), and (18) is proved.

Case 2: \( f_1 = f_2 \equiv 0 \). In this case, none of \( v_1 \) and \( v_2 \) is the zero vector. Lemma 33 now leads to \( \text{Supp}(v_1 \cdot F) \cap \text{Supp}(v_2 \cdot F) \neq \emptyset \). Then it follows that the product of \( v_1 \cdot F(x) \) and \( v_2 \cdot F(x) \) is not the zero function, which establishes (18).

Case 3: \( f_1 \equiv 0 \) and \( f_2 \neq 0 \), or \( f_1 \neq 0 \) and \( f_2 \equiv 0 \). By symmetry, we can assume \( f_1 \equiv 0 \) and \( f_2 \neq 0 \). Let \( D \) be the subset of \( \text{GF}(2^m) \) given by

\[
D = \begin{cases} 
\{x \in \text{GF}(2^m) : v_1 \cdot F = 1, v_2 \cdot F = 1\}, & \text{if } v_1 = v_2, \\
\{x \in \text{GF}(2^m) : v_1 \cdot F = 1, v_2 \cdot F = 0\}, & \text{if } v_1 \neq v_2.
\end{cases}
\]

Note that \( f_1 \neq 1 \) since \( \dim(\mathcal{C}) > 1 \). Therefore none of \( f_1 \) and \( (1 + f_1) \) vanishes on \( D \) from Lemma 33. Thus (18) holds.

Case 4: \( f_1 \neq 0 \) and \( f_2 \neq 0 \). Denote \( D = \{x \in \text{GF}(2^m) : v_1 \cdot F = v_2 \cdot F = 0\} \). It is obvious that \( f_1 f_2 \neq 0 \) because \( \mathcal{C} \) is a minimal code. Combining Lemma 33 with \( \text{deg}(f_1 f_2) \leq 2 \ell \) yields \( f_1 f_2 \notin \text{Ann}(D) \), which gives (18).

Summarising the discussions in the four cases completes the proof of (18) and the theorem. \( \square \)

In order to apply Theorem 34 to obtain minimal codes, we need to construct vectorial Boolean functions with high algebraic immunity. The following result is in this direction.

**Theorem 35.** Let \( n_0, n_1, \ldots, n_{2r} \) be integers satisfying \( 0 = n_0 < n_1 < \cdots < n_{2r} = 2^m - 1 \). Let \( y_0, y_1, \ldots, y_{2r-1} \) be an enumeration of the points of \( \text{GF}(2)^r \). Let \( F \) be the function defined by

\[
F(x) = \begin{cases} 
y_i, & \text{if } x \in [n_i, n_{i+1} - 1], \\
y_0, & \text{if } x = 0.
\end{cases}
\]

Then \( F \) is a vectorial Boolean \((m, r)\)-function with algebraic immunity \( t \), where \( t \) is the biggest integer \( t \) such that \( \sum_{j=1}^{2r-1} \binom{m}{j} \leq n_1 - n_0 \) and \( \sum_{j=0}^{t-1} \binom{m}{j} \leq n_{i+1} - n_i \) for \( 1 \leq i \leq 2r - 1 \).

*Proof.* The desired conclusion follows directly from Lemmas 14 and 15. \( \square \)

Note that the theorem is still true if the vector space \( \text{GF}(2)^r \) is replaced by the finite field \( \text{GF}(2^r) \). Using Theorem 35 we obtain the following explicit construction of vectorial functions with high algebraic immunity.
Corollary 36. Let \(n_0, n_1, \ldots, n_{2^r}\) be integers satisfying \(0 = n_0 < n_1 < \cdots < n_{2^r} = 2^m - 1\). Let \(F\) be the vectorial Boolean function defined by \(F(0) = 0\) and \(F(\alpha^j) = (y_0, \ldots, y_{r-1})\), where

\[
n^r j 
\]

Then \(F\) is a vectorial Boolean \((m, r)\)-function with algebraic immunity \(t\), where \(t\) is the biggest integer \(t\) such that \(\sum_{j=0}^{t-1} \binom{m}{j} \leq n_1 - n_0\) and \(\sum_{j=0}^{t-1} \binom{m}{j} \leq n_{i+1} - n_i\) for \(1 \leq i \leq 2^r - 1\).

Corollary 37. Let \(F\) be the vectorial Boolean \((m, r)\)-function of Theorem 35 with \(m^2 + m + 2 \leq 2^{m-r+1}\) and \(n_1 - n_0 + 1 = n_2 - n_1 = \cdots = n_{2^r} - n_{2^r-1} = 2^{m-r}\). Let \(C(F)\) be the binary code given by

\[
C(F) = \left\{(v \cdot F(P_t) + \text{Tr}_1(bP_t))_{t=1}^{2^m-1} : v \in \text{GF}(2^r), b \in \text{GF}(2^m)\right\}.
\]

Then \(C(F)\) is a minimal code of dimension \(m + r\).

Proof. Combining Theorem 35 with Theorem 34 proves the desired conclusion.

Corollary 38. Let \(\delta\) be an integer with \(\sum_{i=0}^{t-1} \binom{m}{i} \leq \delta < \sum_{i=0}^{t} \binom{m}{i}\) and \(3 \leq t \leq m - 3\). Let \(f\) be the Boolean function with \(\text{Supp}(f) = [0; \delta]\). Let \(C(f)\) be the binary code defined by

\[
C(f) = \left\{(f(P_t) + \text{Tr}_1(bP_t))_{t=1}^{2^m-1} : b \in \text{GF}(2^m)\right\}.
\]

Then \(C(f)\) is a minimal code of parameters \([2^m-1, m+1, \geq d]\) with \(d\) being the smaller of \(\delta\) and

\[
\max \left\{ \sum_{i=0}^{t-2} \binom{m-1}{i} + \sum_{i=0}^{t'-2} \binom{m-1}{i} , 2^{m-1} - 1 - \frac{\ln 2}{\pi} (m+1)\sqrt{2^m} \right\},
\]

where \(t' = m - t\) when \(\delta \neq \sum_{i=0}^{t-1} \binom{m}{i}\) and \(t' = m - t + 1\) when \(\delta = \sum_{i=0}^{t-1} \binom{m}{i}\).

Proof. An easy computation shows that

\[
\sum_{i=0}^{m-t-1} \binom{m}{i} < 2^m - \delta \leq \sum_{i=0}^{m-t} \binom{m}{i}.
\]

Note that \(\text{Supp}(f + 1) = 2^m - \delta\). From Lemmas 14 and 15 we conclude that \(\text{AI}(\text{Supp}(f)) = t\) and \(\text{AI}(\text{Supp}(f)) = t'\). Theorem 34 now implies that \(C(f)\) is a minimal code of dimension \(m + 1\). We are left with the task of determining the lower bound of \(d\). It is easy to see that for any \(b \in \text{GF}(2^m)\) the Hamming distance \(\text{dist}(f, \text{Tr}_1(b))\) of \(f\) and \(\text{Tr}_1(b)\) can be written as

\[
\text{dist}(f, \text{Tr}_1(b)) = \text{wt}(f(1 + \text{Tr}_1(bx))) + \text{wt}((1 + f)\text{Tr}_1(bx)).
\]

By Lemma 10 we deduce that

\[
\text{dist}(f, \text{Tr}_1(b)) \geq \sum_{i=0}^{t-2} \binom{m-1}{i} + \sum_{i=0}^{t'-2} \binom{m-1}{i}. \tag{19}
\]
An easy computation yields that
\[
\text{dist}(f, \text{Tr}_1^m(bx)) = 2^{m-1} + \sum_{x \in \text{Supp}(f)} (-1)^{\text{Tr}_1^m(bx)} \\
= 2^{m-1} + \sum_{i=0}^{\delta-1} (-1)^{\text{Tr}_1^m(bx)} \\
\geq 2^{m-1} - 1 - \frac{2m}{\pi} \ln \left( \frac{4(2^m-1)}{\pi} \right) \\
\geq 2^{m-1} - 1 - \frac{\ln 2}{\pi} (m+1) \sqrt{2^m},
\]
where the first inequality follows from (10). Combining (19) and (20) yields the desired conclusion.

\[\square\]

**Corollary 39.** Let \( m \) be an odd integer. Let \( F \) be the vectorial Boolean \((m, r)\)-function of Theorem 35 with \( \sum_{i=0}^{4} \binom{m}{i} \leq 2^{m-r} \) and \( n_1 - n_0 + 1 = n_2 - n_1 = \cdots = n_{2^r} - n_{2^{r-1}} = 2^{m-r} \). Let \( C(F) \) be the binary code given by

\[
C(F) = \left\{ \left( v \cdot F(P_i) + \sum_{j=1}^{(m-1)/2} \text{Tr}_1^m(b_j P_i^{2^j+1}) \right)_{i=1}^{2^m-1} : v \in \text{GF}(2)^r, b_j \in \text{GF}(2^m) \right\}.
\]

Then \( C(F) \) is a minimal code of dimension \( \frac{m(m+1)}{2} + r \).

**Proof.** Combining Theorems 28 and 35 with Theorem 34 proves the desired conclusion.

\[\square\]

**Corollary 40.** Let \( m \) be a positive integer. Let \( F \) be the vectorial Boolean \((m, r)\)-function of Theorem 35 with \( \sum_{i=0}^{4} \binom{m}{i} \leq 2^{m-r} \) and \( n_1 - n_0 + 1 = n_2 - n_1 = \cdots = n_{2^r} - n_{2^{r-1}} = 2^{m-r} \). Let \( C(F) \) be the binary code given by

\[
C(F) = \left\{ \left( v \cdot F(P_i) + \sum_{j=2}^{m/2} \text{Tr}_1^m(b_j P_i^{2^j+1}) + \text{Tr}_1^m(c P_i) \right)_{i=1}^{2^m-1} : v \in \text{GF}(2)^r, b_j, c \in \text{GF}(2^m) \right\}.
\]

Then \( C(F) \) is a minimal code of dimension \( \frac{m(m+1)}{2} + r \).

**Proof.** Combining Theorems 29 and 35 with Theorem 34 yields the desired conclusion.

\[\square\]

**Example 41.** Let \( q = 2^7 \) and \( \alpha \) be a primitive element with minimal polynomial \( \alpha^7 + \alpha + 1 = 0 \). Let \( f_1 \) be the Boolean function with \( \text{Supp}(f_1) = [63; 64]_{[\alpha]} \) and let \( f_2 \) be the function with \( \text{Supp}(f_2) = [31; 32]_{[\alpha]} \cup [95; 32]_{[\alpha]} \). Let \( F \) denote the vectorial Boolean function \((f_1, f_2)\). Then the algebraic immunity of \( F \) equals 3 and the binary linear code \( C(F) \) in Corollary 37 is a minimal code with parameters \([127, 9, 52]\).

5. **Summary and concluding remarks**

In this paper, a link between minimal linear codes and subsets of finite fields without nonzero low degree annihilators was established. This link allowed us to construct binary minimal codes with special sets, Boolean functions or vectorial Boolean with high algebraic immunity. A general construction of minimal binary linear codes from sets without nonzero low degree annihilators was
proposed. Employing this general construction, minimal codes of dimension $m$ and length less than or equal to $m(m-1)/2$ were obtained, and a lower bounder on the minimum distance of the proposed minimal codes was established. A explicit construction of minimal codes using certain subcodes of Reed-Muller codes and vectorial Boolean functions with algebraic immunity was also developed. These results show that there are natural connections among binary minimal codes, sets without nonzero low degree annihilators and Boolean functions with high algebraic immunity.

The results of this paper were presented in terms of univariate representations of functions and codes. The corresponding multivariate analogies can be easily worked out. It would be interesting to generalize the results of this paper to the nonbinary cases. It would be good if the open problems and conjectures proposed in this paper could be settled. The reader is cordially invited to join this adventure.

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