NEW INEQUALITIES OF OSTROWSKI TYPE FOR MAPPINGS 
WHOSE DERIVATIVES ARE $s$–CONVEX IN THE SECOND 
SENSE VIA FRACTIONAL INTEGRALS

ERHAN SET

Abstract. New identity similar to an identity of [13] for fractional integrals have been defined. Then making use of this identity, some new Ostrowski type inequalities for Riemann-Liouville fractional integral have been developed. Our results have some relationships with the results of Alomari et. al., proved in [13] [published in. Appl. Math. Lett. 23 (2010) 1071-1076] and the analysis used in the proofs is simple.

1. Introduction and Preliminary Results

In 1938, A.M. Ostrowski proved the following interesting and useful integral inequality ([30], see also [29, page 468]):

**Theorem 1.** Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior $I^0$ of $I$, and let $a,b \in I$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a,b]$, then the following inequality holds:

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t) dt| \leq M (b-a) \left[ \frac{1}{4} + \frac{(x-a+b)^2}{(b-a)^2} \right]
$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

This inequality gives an upper bound for the approximation of the integral average $\frac{1}{b-a} \int_a^b f(t) dt$ by the value $f(x)$ at point $x \in [a,b]$. In recent years, such inequalities were studied extensively by many researchers and numerous generalizations, extensions and variants of them appeared in a number of papers see ([1]-[13]).

In [15], the class of functions which are $s$–convex in the second sense has been introduced by Hudzik and Maligranda as the following:

**Definition 1.** A function $f : [0, \infty) \to \mathbb{R}$ is said to be $s$–convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x,y \in [0, \infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$. This class of $s$-convex functions is usually denoted by $K_s^2$. 

2000 Mathematics Subject Classification. 26A33, 26A51, 26D07, 26D10, 26D15.

Key words and phrases. Ostrowski type inequality, $s$–convex function, , Riemann-Liouville fractional integral.
It can be easily seen that for \( s = 1 \), s-convexity reduces to ordinary convexity of functions defined on \([0, \infty)\).

In [14], S.S. Dragomir and S. Fitzpatrick proved a variant of Hadamard’s inequality which holds for s-convex functions in the second sense:

**Theorem 2.** Suppose that \( f : [0, \infty) \to [0, \infty) \) is an s-convex function in the second sense, where \( s \in (0, 1) \), and let \( a, b \in [0, \infty) \), \( a < b \). If \( f' \in L^1([a, b]) \), then the following inequalities hold:

\[
2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1}.
\]

The constant \( k = \frac{s}{s+1} \) is the best possible in the second inequality in (1.2).

The following identity is proved by Alomari et.al. (see [13])

**Lemma 1.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^\circ \) where \( a, b \in I \) with \( a < b \). If \( f' \in L^1([a, b]) \), then we have the equality:

\[
f(x) - \frac{1}{b-a} \int_a^b f(t)dt = \frac{(x-a)^2}{b-a} \int_0^1 tf'(tx + (1-t)a)dt - \frac{b-x}{b-a} \int_0^1 tf'(tx + (1-t)b)dt
\]

for each \( x \in [a, b] \).

Using the Lemma 1 Alomari et al. in [14] established the following results which holds for s-convex functions in the second sense.

**Theorem 3.** Let \( f : I \subset [0, \infty) \to \mathbb{R} \) be a differentiable mapping on \( I^\circ \) such that \( f' \in L^1([a, b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'| \) is s-convex in the second sense on \([a, b]\) for some fixed \( s \in (0, 1] \) and \( |f'(x)| \leq M, x \in [a, b] \), then we have the inequality:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{M}{b-a} \left[ \frac{(x-a)^2 + (b-x)^2}{s+1} \right],
\]

for each \( x \in [a, b] \).

**Theorem 4.** Let \( f : I \subset [0, \infty) \to \mathbb{R} \) be a differentiable mapping on \( I^\circ \) such that \( f' \in L^1([a, b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is s-convex in the second sense on \([a, b]\) for some fixed \( s \in (0, 1], q > 1 \), \( p = \frac{2}{q} \) and \( |f'(x)| \leq M, x \in [a, b] \), then we have the inequality:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{M}{b-a} \left[ \frac{(x-a)^2 + (b-x)^2}{s+1} \right],
\]

for each \( x \in [a, b] \).

**Theorem 5.** Let \( f : I \subset [0, \infty) \to \mathbb{R} \) be a differentiable mapping on \( I^\circ \) such that \( f' \in L^1([a, b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is s-convex in the second sense on \([a, b]\) for some fixed \( s \in (0, 1], q \geq 1 \) and \( |f'(x)| \leq M, x \in [a, b] \), then we have the inequality:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq M \left( \frac{2}{s+1}\right)^{\frac{1}{q}} \left[ \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right],
\]
for each $x \in [a, b]$.

**Theorem 6.** Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on $I$ such that $f' \in L^1[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is $s$-concave in the second sense on $[a, b]$ for some fixed $s \in (0, 1)$, $q > 1$ and $p = \frac{q}{q+1}$, then we have the inequality:

$$
(1.6) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| 
\leq \frac{2^{(s-1)/q}}{(1 + p)^{1/p} (b-a)} \left[ (x-a)^2 \left| f' \left( \frac{x+a}{2} \right) \right| + (b-x)^2 \left| f' \left( \frac{b+x}{2} \right) \right| \right],
$$

for each $x \in [a, b]$.

For other recent results concerning $s$-convex functions see [13-17].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 2.** Let $f \in L^1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$
J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a
$$

and

$$
J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,
$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$. Here, $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Some recent results and properties concerning this operator can be found ([18]-[28]).

In spired and motivated by the recent results given in [13], [18]-[23], and [27], in the present note, we establish new Ostrowski type inequalities for $s$-convex functions in the second sense via Riemann-Liouville fractional integral. An interesting feature of our results is that they provide new estimates on these types of inequalities for fractional integrals.

2. **Ostrowski Type Inequalities via Fractional Integrals**

In order to prove our main results we need the following identity:

**Lemma 2.** Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ with $a < b$. If $f' \in L^1[a, b]$, then for all $x \in [a, b]$ and $\alpha > 0$ we have:

$$
(2.1) \quad \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_{a+}^\alpha f(a) + J_{x+}^\alpha f(b) \right]
= \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f' (tx + (1-t)a) \, dt - \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f' (tx + (1-t)b) \, dt
$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$. 


Proof. By integration by parts, we can state
\[
\int_0^1 t^\alpha f' (tx + (1 - t)a) \, dt
\]
\[
= t^\alpha f \left(\frac{tx + (1 - t)a}{x - a} \right)^1_0 - \int_0^1 t^\alpha \left(\frac{f (tx + (1 - t)a)}{x - a} \right) \, dt
\]
(2.2)
\[
= \frac{f (x)}{x - a} - \frac{\alpha}{(x - a)} \int_a^x \frac{(a - u)^{x-1} f (u)}{(a - x)^{x-1}} \, du
\]
\[
= \frac{f (x)}{x - a} - \frac{\alpha \Gamma (a)}{(x - a)^{x+1}} \Gamma (a) \int_a^x (u - a)^{x-1} f (u) \, du
\]
and
\[
\int_0^1 t^\alpha f' (tx + (1 - t)b) \, dt
\]
\[
= t^\alpha f \left(\frac{tx + (1 - t)b}{x - b} \right)^1_0 - \int_0^1 t^\alpha \left(\frac{f (tx + (1 - t)b)}{x - b} \right) \, dt
\]
(2.3)
\[
= \frac{f (x)}{x - b} - \frac{\alpha}{(x - b)} \int_b^x \frac{(b - u)^{x-1} f (u)}{(b - x)^{x-1}} \, du
\]
\[
= \frac{f (x)}{x - b} + \frac{\alpha \Gamma (a)}{(b - x)^{x+1}} \Gamma (a) \int_x^b (b - u)^{x-1} f (u) \, du.
\]
Multiplying the both sides of (2.2) and (2.3) by \(\frac{(x - a)^{x+1}}{b - a}\) and \(\frac{(b - x)^{x+1}}{b - a}\), respectively, we have
\[
\frac{(x - a)^{x+1}}{b - a} \int_0^1 t^\alpha f' (tx + (1 - t)a) \, dt = \frac{(x - a)^{x+1}}{b - a} f (x) - \frac{\Gamma (a + 1)}{b - a} J_x^\alpha f (a)
\]
and
\[
\frac{(b - x)^{x+1}}{b - a} \int_0^1 t^\alpha f' (tx + (1 - t)b) \, dt = -\frac{(b - x)^{x+1}}{b - a} f (x) + \frac{\Gamma (a + 1)}{b - a} J_x^\alpha f (b).
\]
From (2.4) and (2.5), it is obtained desired result. □

Using Lemma 2 we can obtain the following fractional integral inequalities:

Theorem 7. Let \(f : [a, b] \subset [0, \infty) \to \mathbb{R}\), be a differentiable mapping on \((a, b)\) with \(a < b\) such that \(f' \in L [a, b]\). If \(|f'|\) is \(s\)-convex in the second sense on \([a, b]\) for some fixed \(s \in (0, 1]\) and \(|f' (x)| \leq M, x \in [a, b]\), then the following inequality for fractional integrals with \(\alpha > 0\) holds:
\[
\left| \left(\frac{(x - a)^\alpha + (b - x)^\alpha}{b - a}\right) f (x) - \frac{\Gamma (a + 1)}{b - a} \left[J_x^\alpha f (a) + J_x^\alpha f (b)\right] \right|
\]
(2.6)
\[
\leq \frac{M}{b - a} \left(1 + \frac{\Gamma (a + 1) \Gamma (s + 1)}{\Gamma (a + s + 1)}\right) \left[\frac{(x - a)^{x+1} + (b - x)^{x+1}}{\alpha + s + 1}\right]
\]
where \( \Gamma \) is Euler Gamma function.

**Proof.** From (2.1) and since \( |f'| \) is a \( s \)-convex mapping in the second sense on \([a, b]\), we have

\[
\left| \left( \frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a} \right) f(x) - \frac{\Gamma(\alpha + 1)}{(b-a)} \left[ J_x^\alpha f(a) + J_x^\alpha f(b) \right] \right|
\leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha} |f'(tx + (1-t)a)| dt
\]

\[
\quad + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha} |f'(tx + (1-t)b)| dt
\]

\[
\leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha+s} |f'(x)| + t^{\alpha}(1-t)^s |f'(a)| dt
\]

\[
\quad + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha+s} |f'(x)| + t^{\alpha}(1-t)^s |f'(b)| dt
\]

\[
\leq \frac{M}{b-a} \left( \frac{1}{\alpha + s + 1} + \frac{\Gamma(\alpha + 1)\Gamma(s + 1)}{\Gamma(\alpha + s + 2)} \right) \left[ (x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right]
\]

where we have used the fact that

\[
\int_0^1 t^{\alpha+s} dt = \frac{1}{\alpha + s + 1}, \quad \text{and} \quad \int_0^1 t^{\alpha}(1-t)^s dt = \frac{\Gamma(\alpha + 1)\Gamma(s + 1)}{\Gamma(\alpha + s + 2)}.
\]

Hence, using the reduction formula \( \Gamma(n+1) = n\Gamma(n) \) for Euler Gamma function, the proof is complete. \( \square \)

**Remark 1.** In Theorem 7, if we choose \( \alpha = 1 \), then (2.6) reduces the inequality (1.3) of Theorem 3.

**Theorem 8.** Let \( f : [a, b] \subset [0, \infty) \to \mathbb{R} \), be a differentiable mapping on \((a, b)\) with \( a < b \) such that \( |f'|^q \) is \( s \)-convex in the second sense on \([a, b]\) for some fixed \( s \in (0, 1] \), \( p, q > 1 \) and \( |f'(x)| \leq M \), \( x \in [a, b] \), then the following inequality for fractional integrals holds:

\[
\left| \left( \frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a} \right) f(x) - \frac{\Gamma(\alpha + 1)}{(b-a)} \left[ J_x^\alpha f(a) + J_x^\alpha f(b) \right] \right|
\leq \frac{M}{(1 + pa)^{\frac{1}{q}}} \left( \frac{2}{s + 1} \right)^{\frac{1}{q}} \left[ \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \right],
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \), \( \alpha > 0 \) and \( \Gamma \) is Euler Gamma function.
Proof. From Lemma 2 and using the well known Hölder inequality, we have
\[
\left| \left( \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_x^\alpha f(a) + J_x^\alpha f(b) \right] \right|
\]
\[
\leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 t^\alpha |f'(tx + (1-t)a)| dt
\]
\[
+ \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^\alpha |f'(tx + (1-t)b)| dt
\]
\[
\leq \frac{(x-a)^{\alpha+1}}{b-a} \left( \int_0^1 t^{p\alpha} dt \right)^\frac{1}{\beta} \left( \int_0^1 |f'(tx + (1-t)a)|^q dt \right)^\frac{1}{q}
\]
\[
+ \frac{(b-x)^{\alpha+1}}{b-a} \left( \int_0^1 t^{p\alpha} dt \right)^\frac{1}{\beta} \left( \int_0^1 |f'(tx + (1-t)b)|^q dt \right)^\frac{1}{q}.
\]
Since \(|f'|^q\) is \(s\)-convex in the second sense on \([a, b]\) and \(|f'(x)| \leq M\), we get (see [13, p. 1073])
\[
\int_0^1 |f'(tx + (1-t)a)|^q dt \leq \frac{2M^q}{s+1} \quad \text{and} \quad \int_0^1 |f'(tx + (1-t)b)|^q dt \leq \frac{2M^q}{s+1}
\]
and by simple computation
\[
\int_0^1 t^{p\alpha} dt = \frac{1}{p\alpha + 1}.
\]
Hence, we have
\[
\left| \left( \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_x^\alpha f(a) + J_x^\alpha f(b) \right] \right|
\]
\[
\leq \frac{M}{(1+p\alpha)^\frac{1}{\beta}} \left( \frac{2}{s+1} \right)^\frac{1}{q} \left[ \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \right]
\]
which completes the proof. \(\square\)

Remark 2. In Theorem 8, if we choose \(\alpha = 1\), then (2.4) reduces the inequality (1.4) of Theorem 4.

Theorem 9. Let \(f : [a, b] \subset [0, \infty) \rightarrow \mathbb{R}\), be a differentiable mapping on \((a, b)\) with \(a < b\) such that \(f' \in L[a, b]\). If \(|f'|^q\) is \(s\)-convex in the second sense on \([a, b]\) for some fixed \(s \in (0, 1]\), \(q \geq 1\), and \(|f'(x)| \leq M\), \(x \in [a, b]\), then the following inequality for fractional integrals holds:
\[
\left| \left( \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_x^\alpha f(a) + J_x^\alpha f(b) \right] \right|
\]
\[
\leq \frac{M}{(1+p\alpha)^\frac{1}{\beta}} \left( \frac{2}{s+1} \right)^\frac{1}{q} \left[ \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \right],
\]
where \(\alpha > 0\) and \(\Gamma\) is Euler Gamma function.
Ostrowski type inequalities for s-convex functions via fractional integrals

Proof. From Lemma[2] and using the well known power mean inequality, we have

\[
\left| \left( \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_{x-a}^\alpha f(a) + J_{x+b}^\alpha f(b) \right] \right|
\]

\[
\leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 t^\alpha |f'(tx + (1-t)a)| dt
\]

\[
+ \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^\alpha |f'(tx + (1-t)b)| dt
\]

\[
\leq \frac{(x-a)^{\alpha+1}}{b-a} \left( \int_0^1 t^\alpha dt \right)^{1-\frac{\alpha}{\alpha+1}} \left( \int_0^1 t^\alpha |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}}
\]

\[
+ \frac{(b-x)^{\alpha+1}}{b-a} \left( \int_0^1 t^\alpha dt \right)^{1-\frac{\alpha}{\alpha+1}} \left( \int_0^1 t^\alpha |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}.
\]

Since \(|f'|^q\) is s-convex in the second sense on \([a, b]\) and \(|f'(x)| \leq M\), we get (see [13] p. 1073)

\[
\int_0^1 t^\alpha |f'(tx + (1-t)a)|^q dt \leq \int_0^1 \left[ t^{s+\alpha} |f'(x)|^q + t^\alpha (1-t)^s |f'(a)|^q \right] dt
\]

\[
= \frac{|f'(x)|^q}{\alpha + s + 1} + |f'(a)|^q \int_0^1 t^\alpha (1-t)^s dt
\]

\[
= \frac{|f'(x)|^q}{\alpha + s + 1} + |f'(a)|^q \beta(\alpha + 1, s + 1)
\]

\[
= \frac{|f'(x)|^q}{\alpha + s + 1} + |f'(a)|^q \frac{\Gamma(\alpha + 1)\Gamma(s + 1)}{(\alpha + s + 1)\Gamma(\alpha + s + 1)}
\]

\[
\leq M^q \frac{1}{\alpha + s + 1} \left( 1 + \frac{\Gamma(\alpha + 1)\Gamma(s + 1)}{\Gamma(\alpha + s + 1)} \right)
\]

and similarly

\[
\int_0^1 t^\alpha |f'(tx + (1-t)b)|^q dt \leq \int_0^1 \left[ t^{s+\alpha} |f'(x)|^q + t^\alpha (1-t)^s |f'(b)|^q \right] dt
\]

\[
= \frac{|f'(x)|^q}{\alpha + s + 1} + |f'(b)|^q \int_0^1 t^\alpha (1-t)^s dt
\]

\[
= \frac{|f'(x)|^q}{\alpha + s + 1} + |f'(b)|^q \beta(\alpha + 1, s + 1)
\]

\[
= \frac{|f'(x)|^q}{\alpha + s + 1} + |f'(b)|^q \frac{\Gamma(\alpha + 1)\Gamma(s + 1)}{(\alpha + s + 1)\Gamma(\alpha + s + 1)}
\]

\[
\leq M^q \frac{1}{\alpha + s + 1} \left( 1 + \frac{\Gamma(\alpha + 1)\Gamma(s + 1)}{\Gamma(\alpha + s + 1)} \right).
\]
where $\beta$ is Euler Beta function defined by

$$
\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (x, y > 0).
$$

We used the fact that

$$
\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{and} \quad \Gamma(n+1) = n\Gamma(n)(n > 0).
$$

Hence, we have

$$
\left| \left( \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_{x-}^\alpha f(a) + J_x^\alpha f(b) \right] \right|
$$

\leq M \left( \frac{1}{1+\alpha} \right)^{1-\frac{1}{q}} \left[ \frac{1}{\alpha + s + 1} \right]^{\frac{1}{s}} \times \left( 1 + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)} \right) \left[ \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \right]
$$

which completes the proof.

\[\square\]

**Remark 3.** In Theorem 2, if we choose $\alpha = 1$, then (2.8) reduces the inequality (1.5) of Theorem 5.

The following result holds for $s$-concavity:

**Theorem 10.** Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$, be a differentiable mapping on $(a, b)$ with $a < b$ such that $f' \in L[a, b]$. If $|f'|^q$ is $s$-concave in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$ and $p, q > 1$, then the following inequality for fractional integrals holds:

$$
\left| \left( \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_{x-}^\alpha f(a) + J_x^\alpha f(b) \right] \right|
$$

\leq \frac{2(s-1)/q}{(1 + pac)^{\frac{1}{p}} (b-a)} \left( x-a \right)^{\alpha+1} \left( \left| f' \left( \frac{x+a}{2} \right) \right| + (b-x)^{\alpha+1} \left| f' \left( \frac{b+x}{2} \right) \right| \right),
$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$ and $\Gamma$ is Euler Gamma function.
Using (2.11) and (2.12) in (2.10), we have

\[ \left| \left( \left( \frac{x-a}{a} \right)^{\alpha} + \frac{b-x}{b} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_{x}^{\alpha} f(a) + J_{x+b}^{\alpha} f(b) \right] \right| \]

\[ \leq \frac{(x-a)^{\alpha+1}}{b-a} \int_{0}^{1} t^{\alpha} \left| f'(tx+(1-t)a) \right| dt \]

(2.10)

\[ + \frac{(b-x)^{\alpha+1}}{b-a} \int_{0}^{1} t^{\alpha} \left| f'(tx+(1-t)b) \right| dt \]

Since \( |f'| \) is concave, using the inequality (1.2) we get (see [13, p. 1074])

\[ \int_{0}^{1} |f'(tx+(1-t)a)|^{q} dt \leq 2^{q-1} \left| f' \left( \frac{x+a}{2} \right) \right|^{q} \]

(2.11)

and

\[ \int_{0}^{1} |f'(tx+(1-t)b)|^{q} dt \leq 2^{q-1} \left| f' \left( \frac{b+x}{2} \right) \right|^{q} \]

(2.12)

Using (2.11) and (2.12) in (2.10), we have

\[ \left| \left( \left( \frac{x-a}{a} \right)^{\alpha} + \frac{b-x}{b} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_{x}^{\alpha} f(a) + J_{x+b}^{\alpha} f(b) \right] \right| \]

\[ \leq \frac{\alpha^{\alpha+1}}{(1+a)^{q}} \frac{\alpha^{\alpha+1}}{(b-a)} \left[ (x-a)^{\alpha+1} \left| f' \left( \frac{x+a}{2} \right) \right| + (b-x)^{\alpha+1} \left| f' \left( \frac{b+x}{2} \right) \right| \right] \]

which completes the proof.

Remark 4. In Theorem [10], if we choose \( a = 1 \), then (2.1) reduces the inequality (1.0) of Theorem [6].

References

[1] S.S. Dragomir, On the Ostrowski’s integral inequality for mappings with bounded variation and applications, Math. Ineq. & Appl., 1(2) (1998).
[2] S.S. Dragomir, The Ostrowski integral inequality for Lipschitzian mappings and applications, Comput. Math. Appl., 38 (1999), 33-37.
[3] S. S. Dragomir, S. Wang, A new inequality of Ostrowski’s type in L1-norm and applications to some special means and to some numerical quadrature rules, Tamkang J. of Math., 28 (1997), 239-244.
[4] Z. Liu, Some companions of an Ostrowski type inequality and application, J. Inequal. in Pure and Appl. Math, 10(2), 2009, Art. 52, 12 pp.
[5] M.E. Özdemir, H. Kavurmacı and E. Set, Ostrowski’s type inequalities for \( (\alpha,m) \)-convex functions, Kyungpook Math. J., 50 (2010), 371-378.
[6] B. G. Pachpatte, On an inequality of Ostrowski type in three independent variables, J. Math.Anal. Appl., 249(2000), 583-591.
[7] B. G. Pachpatte, On a new Ostrowski type inequality in two independent variables, Tamkang J. of Math., 32(1), (2001), 45-49.
[8] A. Rafiq, N.A. Mir and F. Ahmad, Weighted Čebyšev-Ostrowski type inequalities, Applied Math. Mechanics (English Edition), 2007, 28(7), 901-906.
[9] M. Z. Sarikaya, On the Ostrowski type integral inequality, Acta Math. Univ. Comenianae, Vol. LXXIX, 1(2010), pp. 129-134.
[10] N. Ujević, Sharp inequalities of Simpson type and Ostrowski type, Comput. Math. Appl. 48 (2004) 145-151.
[11] L. Zhongxue, On sharp inequalities of Simpson type and Ostrowski type in two independent variables, Comput. Math. Appl., 56 (2008) 2043-2047.
[12] M. Alomari and M. Darus, Some Ostrowski type inequalities for convex functions with applications, RGMA 13 (1) (2010) article No. 3, Preprint.
[13] M. Alomari, M. Darus, S.S. Dragomir, P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, Appl. Math. Lett. 23 (2010) 1071-1076.
[14] S. S. Dragomir and S. Fitzpatrick, The Hadamard’s inequality for s-convex functions in the second sense, Demonstratio Math. 32(4), (1999), 687-696.
[15] H. Hudzik and L. Maligranda, Some remarks on s−convex functions, Aequationes Math. 48 (1994), 100-111.
[16] U.S. Kirmaci, M.K. Bakula, M.E. Özdemir, J. Pečarić, Hadamard-type inequalities for s-convex functions, Appl. Math. and Comp., 193 (2007), 26-35.
[17] M.Z. Sarikaya, E. Set and M.E. Özdemir ”On new inequalities of Simpson’s type for s-convex functions”, Comput. Math. Appl., 60(8), (2010) 2191-2199.
[18] G. Anastassiou, M.R. Hooshmandasl, A. Ghasemi and F. Moftakharzadeh, Montgomery identities for fractional integrals and related fractional inequalities, J. Ineq. Pure and Appl. Math., 10(4) (2009), Art. 97.
[19] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, J. Ineq. Pure and Appl. Math., 10(3) (2009), Art. 86.
[20] Z. Dahmani, New inequalities in fractional integrals, International Journal of Nonlinear Science, 9(4) (2010), 493-497.
[21] Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, Ann. Funct. Anal. 1(1) (2010), 51-58.
[22] Z. Dahmani, L. Tabharit, S. Taf, Some fractional integral inequalities, Nonl. Sci. Lett. A, 1(2) (2010), 155-160.
[23] Z. Dahmani, L. Tabharit, S. Taf, New generalizations of Gruss inequality using Riemann-Liouville fractional integrals, Bull. Math. Anal. Appl., 2(3) (2010), 93-99.
[24] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order, Springer Verlag, Wien (1997), 223-276.
[25] S. Miller and B. Ross, An introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, USA, 1993, p.2.
[26] I. Podlubni, Fractional Differential Equations, Academic Press, San Diego, 1999.
[27] M.Z. Sarikaya, E. Set, H. Yaldız and N. Başak, Hermite-Hadamard’s inequalities for fractional integrals and related fractional inequalities, Submitted.
[28] M.Z. Sarikaya and H. Ogunmez, On new inequalities via Riemann-Liouville fractional integration, [arXiv:1005.1167v1, submitted].
[29] D.S. Mitrović, J.E. Pečarić and A.M. Fink, Inequalities Involving Functions and Their Integrals and Derivatives, Kluwer Academic Publishers, Dordrecht, 1991
[30] A.M. Ostrowski, Über die Absolutabweichung einer differentierbaren Funktion von ihren Integralmittelwert, Comment. Math. Helv., 10, 226-227, (1938).
[31] D.S. Mitrović, J.E. Pečarić and A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, 1993, p. 106.

Department of Mathematics, Faculty of Science and Arts, Düzce University, DÜZCE-TURKEY

E-mail address: erhanset@yahoo.com