On reconstruction of an unknown polygonal cavity in a linearized elasticity with one measurement

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Abstract. In this paper we consider a reconstruction problem of an unknown polygonal cavity in a linearized elastic body [16]. For this problem, an extraction formula of the convex hull of the unknown polygonal cavity is established by means of the enclosure method introduced by Ikehata [2, 4]. The advantages of our method are that it needs only a single set of boundary data and we do not require any a priori assumptions for the unknown polygonal cavity and any constraints on boundary data. The theoretical formula may have possibility of application in nondestructive evaluation.

1. Introduction
Inverse problems have been received a great deal of attention in various fields of science and engineering. Typical examples of inverse problems are to extract information about discontinuity embedded in a medium, such as cracks, cavities, obstacles and inclusions, from observed data. This kind of problems arises in geophysics, medical imaging such as Computed Tomography (CT), Electrical Impedance Tomography (EIT) and Magnetic Resonance Imaging (MRI), Nondestructive testing, etc.

In the mathematical model, the problems are often described as inverse boundary value problems of partial differential equations called the governing equations and the solution is to extract information about location, shape and size of unknown discontinuity from boundary data. However, inverse problems are typically nonlinear and ill posed which means the lack of at least one of three conditions; existence, uniqueness, stability of the solution or solutions. Therefore, one cannot expect to obtain the desired solution in general. Then it is quite important to impose reasonable a priori assumptions on the problems in a practical sense and to extract the useful information of unknown discontinuity from fewer boundary data. From their standpoints we consider the following problem [16].

Let $\Omega$ be a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary and represent a homogeneous isotropic linearized elastic plate in both states of plane stress and plane strain. Let $D$ denote cavities such that $\overline{\mathcal{D}} \subset \Omega$ and $\Omega \setminus \overline{\mathcal{D}}$ is connected. As a priori assumption for unknown $D$ we require that

(A1) $D = D_1 \cup D_2 \cup \cdots \cup D_m$, $\overline{D_j} \cap \overline{D_k} = \emptyset$ for $j \neq k$, where each $D_j$ is a simply connected open set and polygon.

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Inverse Problem  Reconstruct unknown $D$ satisfying (A1) from a single set of a surface force and the corresponding displacement field on the boundary of $\Omega$.

In order to solve the Problem we employ the enclosure method introduced by Ikehata [4]. By virtue of this method we show that one can reconstruct the convex hull of $D$ uniquely from a single set of boundary data, stated as Theorem in Section 3. As a remark, for the case that $D$ is a linear crack we have already considered the reconstruction problem in an isotropic and anisotropic linearized elasticity, respectively (e.g. [13, 14, 15]).

2. The Enclosure Method
The enclosure method is a methodology in inverse problems for partial differential equations. The method yields a partial information about the location of unknown discontinuity which appears as discontinuity of the coefficients of a partial differential equation or a part of the boundary of the common domain of definition of solutions of the equation. Although the original enclosure method makes use of infinitely many observation data, the single measurement version has been introduced in [2] and can be divided into the following three steps.

(i) Find a special solution of the formal adjoint of the governing equation for the background medium which is parameterized by a large parameter $\tau$ and divides the whole space into two parts: in one part the absolute value of the solution decays as $\tau \to \infty$; in another part the solution grows as $\tau \to \infty$.

(ii) Construct an indicator function of independent variable $\tau$ by multiplying the governing equation of the medium by the special solution constructed in (i), integrating over the domain of definition and extracting only the integral on the known boundary of the domain.

(iii) Study the asymptotic behaviour of the indicator function as $\tau \to \infty$.

There are several existing applications of the enclosure method to a wide variety of inverse problems. Some typical examples of them are listed below.

- Electrical Impedance Tomography ([2, 3, 4, 5, 9])
- Inverse obstacle scattering problems of acoustic wave ([6, 7, 11]).
- Reconstruction problems of obstacles in an electromagnetic body ([25]).
- Recently the enclosure method has found its applications in inverse problems for heat and wave equations. Therein the data is observed in a limited time. Some results are listed as follows.
– Inverse obstacle scattering problems with dynamical data over a finite time interval ([10]).
– Reconstruction problems of inclusions in a heat conductive body from dynamical data over a finite time interval ([8, 17, 18, 19, 12]).

3. Statement of the main result
Let \( n \) be the unit outward normal vector to \( \partial(\Omega \setminus \overline{D}) \). Let the displacement vector \( u = (u_1, u_2)^T \in \{H^1(\Omega \setminus \overline{D})\}^2 \) satisfy the Navier equation in the absence of any body forces

\[
\frac{\tilde{E}}{2(1 + \nu)} \Delta u + \frac{\tilde{E}}{2(1 - \nu)} \nabla(\nabla \cdot u) = 0 \quad \text{in} \quad \Omega \setminus \overline{D}
\]

(3.1)

and the traction free boundary condition

\[
Tu = 0 \quad \text{on} \quad \partial D
\]

(3.2)

where \( Tu \) is the stress vector expressed by

\[
Tu = \frac{\nu \tilde{E}}{1 - \nu^2} (\nabla \cdot u) n + \frac{\tilde{E}}{2(1 + \nu)} \{\nabla u + (\nabla u)^T\} n
\]

and

\[
\tilde{E} = \begin{cases} 
E & \text{(plane stress)}, \\
\frac{E}{1 - \nu^2} & \text{(plane strain)}
\end{cases}
\]

\[
\tilde{\nu} = \begin{cases} 
\nu & \text{(plane stress)}, \\
\frac{\nu}{1 - \nu} & \text{(plain strain)}
\end{cases}
\]

Here \( E \) and \( \nu \) is Young’s modulus and Poisson’s ratio of the elastic medium, respectively. Since both the shear modulus and the bulk modulus are required to be positive, we suppose \( E > 0 \) and \( -1 < \nu < \frac{1}{2} \).

We denote by \( h_D \) the support function of \( D \):

\[
h_D(\omega) = \sup_{x \in D} x \cdot \omega \quad \text{for} \quad \omega \in S^1.
\]

The values of \( h_D(\cdot) \) give the signed distances from the origin of coordinates to the support lines of \( D \).

To state the main result, we introduce the following assumptions for \( \omega \).

(W1) \( \omega \) satisfies that the intersection of the line \( x \cdot \omega = h_D(\omega) \) with \( \partial D \) consists of only one point,

(W2) \( \omega \) satisfies (W1) and that the interior angle bisector of \( D \) at the point \( \{x \in \mathbb{R}^2 \mid x \cdot \omega = h_D(\omega)\} \cap \partial D \) is not perpendicular to the line \( x \cdot \omega = h_D(\omega) \).

As shown in Figure 2 in Section 4, (W2) is equivalent to a condition \( p + q + \pi \neq 0 \). Then our main result is the following formula:

**THEOREM ([16])** Let \( u \) be not a rigid displacement. Under the assumptions of (A1) and (W2) the formula

\[
h_D(\omega) = \lim_{\tau \to -\infty} \frac{1}{\tau} \log \left| \int_{\partial \Omega} (Tu \cdot v - Tv \cdot u) \, dS \right|,
\]

is valid, where \( v(x) = (\omega + i\omega^\perp)e^{\tau x \cdot (\omega + i\omega^\perp)} \), \( \tau > 0 \); \( \omega^\perp \in S^1 \) is perpendicular to \( \omega \) and satisfies \( \det (\omega^\perp \omega) > 0 \).
Some remarks on Theorem are in order.

- For given $D$ satisfying (A1) the set of all $\omega$ that does not satisfy (W2) (or (W1)) is a finite set. Since the support function is continuous, it follows from Theorem that a single set of the data $u, Tu$ on $\partial \Omega$ uniquely determine $h_D(\omega)$ for all $\omega \in S^1$ and thus the convex hull of $D$ via the formula

$$\bigcap_{\omega \in S^1} \{ x \in \mathbb{R}^2 | x \cdot \omega < h_D(\omega) \}.$$  

This is the origin of the name “the enclosure method” and simultaneously the solution of Problem.

- In contrast to the related results, we do not require any other a priori assumptions for $D$ excepting (A1) (e.g. [2]) and any constraints on boundary data (e.g. [13, 14, 15]).

- If we assume (W1) instead of (W2), then the formula is valid by replacing $\lim_{\tau \to \infty}$ with $\limsup_{\tau \to \infty}$, for the detail see the end of Section 4.

- In two-dimensional case rigid displacements can be described in the form

$$F(x)k = (k_0 + k_2x_2, k_1 - k_2x_1)^T$$

with an arbitrary constant vector $k = (k_0, k_1, k_2)^T$.

- It follows from the compatibility condition on the data that

$$\int_{\partial \Omega} Tu \cdot F(x)k \, dS = 0$$

for arbitrary $k \in \mathbb{R}^3$.

### 4. Outline of the proof of Theorem

The proof of Theorem proceeds along the same lines as results [2, 13, 14, 15].

The first part of the enclosure method mentioned in Section 2 is to find the special solution for the Navier equation (3.1), which corresponds to $v(x)$ in Theorem.

The indicator function in the second part is just

$$\tau \mapsto \int_{\partial \Omega} (Tu \cdot v - Tv \cdot u) \, dS.$$  

The third part of the enclosure method is to study the asymptotic behaviour of the indicator function as $\tau \to \infty$. This part is the most important in this analysis, because if one can see that

$$I_\omega(\tau) \equiv e^{-\tau h_D(\omega)} \int_{\partial \Omega} (Tu \cdot v - Tv \cdot u) \, dS$$

is truly algebraic decaying as $\tau \to \infty$, then it is easy to see that

$$\lim_{\tau \to \infty} \frac{1}{\tau} \log |I_\omega(\tau)| = 0 \quad (4.1)$$

and then the formula in Theorem automatically follows.

The assumptions (A1) and (W1) yield that there exists a unique point $Q$ on $\partial D$ such that $Q \cdot \omega = h_D(\omega)$, that is, $Q$ has to be a vertex of a connected component of $D$. Hereafter we always assume that $\omega$ satisfies (W1). Using the special form of $v$ and integration by parts, we see that $I_\omega(\tau)$ coincides with the function

$$e^{-\tau h_D(\omega)} \int_{\partial D \cap B_R(Q)} Tu \cdot u \, dS, \quad B_R(Q) = \{ x \in \mathbb{R}^2 | |x - Q| < R \} \quad R > 0,$$

modulo exponentially decaying as $\tau \to \infty$. Thus it is quite important to know the behaviour of $u$ in a neighbourhood of $Q$.  

4
4.1. A convergent series expansion near a corner

In order to derive the convergent series expansion of $u$ around $Q$, first let us fix some notation.

For $R > 0$ we set $D_R = B_R(Q) \cap (\Omega \setminus \mathcal{D})$. We see that if $R < 1$ is sufficiently small, then $D_R$ becomes a sector with center angle $2\alpha > \pi$, see Figure 2. We now fix such $R$.

Next, we identify each point $x$ in $D_R$ with $X$ in a local coordinate system $X = (X_1, X_2)^T = (r \cos \theta, r \sin \theta)^T$ with respect to the origin $Q$ as follows, see Figure 2.

$$x = Q + r (\cos \theta \mathbf{a} + \sin \theta \mathbf{a}^\perp)$$

where $\frac{\pi}{2} < \alpha < \pi$,

$$\mathbf{a} = \sin (\alpha + p) \omega + \cos (\alpha + p) \omega^\perp = \sin (q - \alpha) \omega + \cos (q - \alpha) \omega^\perp,$$

$$\mathbf{a}^\perp = \cos (\alpha + p) \omega - \sin (\alpha + p) \omega^\perp = \cos (q - \alpha) \omega - \sin (q - \alpha) \omega^\perp,$$

$-\pi < q < p < 0$ and $p - q = 2\pi - 2\alpha$.

Figure 2. Polar coordinates

Then, we can derive the convergent series expansion of $u$ near $Q$ in $D_\rho$ for $0 < \rho < R$.

**PROPOSITION 1 ([16])** If $2\alpha \neq \tan 2\alpha$, then there exist complex numbers $A_{j,m}$, $B_{j,m}(s)$ for $j = 1, 2, m = 1, 2, 3, \ldots$ and a constant vector $\mathbf{k}$ such that

$$u(r, \theta) = \sum_{j,m} A_{j,m} r^{-s_{j,m}} \Phi_j(s_{j,m}, \theta) + \sum_{j,m} \frac{\partial}{\partial s} \left( r^{-s_{j,m}} B_{j,m}(s) \Phi_j(s, \theta) \right)_{s=s_{j,m}} + F(X) \mathbf{k}, \quad (4.2)$$

where $\Phi_j$ is the following vector field:

$$\Phi_j(s, \theta) = \begin{pmatrix} (1 + \tilde{v}) s \sin (s + 2) \alpha \sin (s + 1) \theta + (\tilde{v} - 3) \sin s \alpha \sin (s + 1) \theta \\ -(1 + \tilde{v}) s \alpha \sin s \alpha \sin (s + 1) \theta \\ -(1 + \tilde{v}) s \cos (s + 2) \alpha \cos (s + 1) \theta + (\tilde{v} - 3) \sin s \alpha \cos (s + 1) \theta \\ +(1 + \tilde{v}) s \alpha \cos s \alpha \cos (s + 1) \theta \end{pmatrix},$$
\[
\Phi_2(s, \theta) = \begin{pmatrix}
(1 + \tilde{\nu})s \cos((s + 2)\alpha) \cos((s + 1)\theta) + (\tilde{\nu} - 3) \cos \alpha \cos((s + 1)\theta) \\
- (1 + \tilde{\nu})(s + 1) \cos \sigma \cos((s + 3)\theta) \\
(1 + \tilde{\nu})s \cos((s + 2)\alpha) \sin((s + 1)\theta) - (\tilde{\nu} - 3) \cos \alpha \sin((s + 1)\theta) \\
- (1 + \tilde{\nu})(s + 1) \cos \sigma \sin((s + 3)\theta)
\end{pmatrix}
\]

and \(s_{j,m}, \tilde{s}_{j,m}\) which are not \(-2\), are the simple and double roots of \(h_j(\alpha, s) = 0\);

\[
\begin{align*}
 h_1(\alpha, s) &= (s + 1) \sin 2\alpha - \sin(2(s + 1)\alpha), \\
 h_2(\alpha, s) &= (s + 1) \sin 2\alpha + \sin(2(s + 1)\alpha),
\end{align*}
\]

respectively, following decreasing of \(\text{Res}\) order for \(m = 1, 2, 3, \cdots\) in \(\text{Res} < -1\). Moreover, the series (4.2) is convergent, absolutely in \(H^1(D_\rho)\) and uniformly in \(D_\rho\).

**REMARK 1** The coefficients \(A_{j,m}, B_{j,m}(s)\) in (4.2) have the following properties:

(i) If \(s_{j,m} \in \mathbb{R}\), then the corresponding \(A_{j,m} \in \mathbb{R}\).

(ii) If \(s_{j,m} \in \mathbb{C} \setminus \mathbb{R}\) such that \(s_{j,m+1} = \overline{s}_{j,m}\) which denotes the complex conjugate of \(s_{j,m}\), then it holds \(A_{j,m+1} = \overline{A}_{j,m}\).

(iii) \(B_{j,m}(\overline{s}_{j,m}) \in \mathbb{R}\) and \(\frac{\partial}{\partial s} B_{j,m}(\overline{s}_{j,m}) \in \mathbb{R}\).

**REMARK 2** If \(2\alpha = \tan 2\alpha\), then the following additional term with a real constant \(\lambda^*\) should be taken into account to the formula (4.2):

\[
 r\lambda^* \left( (1 + \tilde{\nu})(1 - \cos 2\alpha) \cos \theta + 2(1 - \tilde{\nu})(\cos 2\alpha) \cos \theta \cos \theta + 4 \log r \cos 2\alpha \sin \theta \right)
\]

**Sketch of the proof of Proposition 1**

Here we give the essence of the proof of Proposition 1, for the detail see [16]. First we construct Airy’s stress function \(U \in H^2(D_R)\) by using Poincaré’s lemma. Then, the problem (3.1)–(3.2) in \(D_R\) can be reduced to the Dirichlet boundary value problem for the biharmonic equation of \(U\). Next, applying the Mellin transform to the problem for \(U\), we have a boundary value problem for a fourth order ordinary differential equation in the transformed complex domain. For the problem we represent the solution by Green’s function (e.g. [24]) and investigate the property. Since according to [1] and [22] it is well known that \(U\) multiplying by a cut-off function belongs to a weighted Sobolev space, we can apply the inverse Mellin transform to the solution for \(r < \rho\). Furthermore, the residue theorem enable us to change the integration path. Here the residue of the integrand is calculated by the roots of the transcendental equations \(h_j(\alpha, s) = 0\), which are not explicitly solvable and are complex. This is completely different from that of the case that the governing equation is the Laplace equation [2] and the case that \(D\) is a linear crack [13]. In the matter of the transcendental equations the following results are known, e.g. [1], [16] and [23].

**LEMMA**

(i) When \(\frac{\alpha}{2} < \alpha < \alpha_0\), it holds \(-2 < \text{Res}_{2,1} < -\frac{3}{2}\).

When \(\alpha_0 < \alpha < \pi\), it holds \(-2 < \text{Res}_{1,1} < -\frac{\pi}{2\alpha_0} - 1 < \text{Res}_{2,1} < -\frac{3}{2}\).

(ii) In \(\text{Res} < -1\) the lines \(\text{Res} = \ell_{k+1} \equiv -k\frac{\pi}{2\alpha} - 1\) \((k = 1, 2, 3, \cdots)\) contain no roots of each \(h_j(\alpha, s) = 0\).

(iii) If \(k\) is an odd number, then the strip \(\ell_{k+2} < \text{Res} < \ell_{k+1}\) contains only two roots of \(h_1(\alpha, s) = 0\) (including complex conjugate and multiplicity) for \(\frac{\pi}{2} < \alpha < \pi\).
(iv) If $k$ is an even number, then the strip contains only two roots of $h_2(\alpha, s) = 0$ for $\frac{\pi}{2} < \alpha < \pi$.
(v) The multiplicity of the roots of each $h_j(\alpha, s) = 0$ is not greater than two. Moreover, the double roots are all real and at most two.
(vi) For $\frac{\pi}{2} < \alpha < \pi$ the roots of each $h_j(\alpha, s) = 0$ in $\text{Res} < -2$ are not integers.
(vii) For $\frac{\pi}{2} < \alpha < \pi$ the roots of each $h_j(\alpha, s) = 0$ in $s(s + 2) \neq 0$ satisfy $\sin \alpha \cos \alpha \neq 0$.

Thus we obtain an asymptotic expansion of $U$ with the remainder term expressed by the integral form.

Finally, by using the estimate of the Green's function we have the estimate of the remainder term and we can derive convergent series of $U$. Since $u$ is uniquely determined by $U$ up to any rigid displacements $F(X)k$, we obtain Proposition 1.

4.2. The asymptotic behaviour of $I_\omega(\tau)$ as $\tau \to \infty$

Next, to show (4.1) we consider the asymptotic behaviour of $I_\omega(\tau)$ by using Proposition 1. The arbitrary truncation of the expansion (4.2) together with a remainder estimate yields the complete asymptotic expansion of $I_\omega(\tau)$.

PROPOSITION 2 Assume $D$ contains no corner of angle satisfying $2\alpha = \tan 2\alpha$. For each $k \geq 1$, as $\tau \to \infty$,

$$I_\omega(\tau) = \sum_{\text{Res}_j, m > \ell_k, 1} C_j(s, m) A_{j, m} + \sum_{\text{Res}_j, m > \ell_k, 1} \frac{\partial}{\partial s} \left( \frac{C_j(s) B_{j, m}(s)}{\tau^{-s - \frac{1}{2}}} \right) \bigg|_{s = \delta_j, m} + O \left( \frac{1}{\tau^\frac{\pi}{2}} \right),$$

where

$$C_1(s) = 4 \frac{E}{1 + p} \sin \alpha e^{i\pi - i\frac{\pi}{2}} e^{-i\frac{\pi}{2} s} e^{-i(s + 2)(p + \alpha)} \Gamma(-s) \left\{ 1 - e^{i\pi \alpha \omega^2} \right\},$$
$$C_2(s) = 4 \frac{E}{1 + p} \cos \alpha e^{i\pi - i\frac{\pi}{2}} e^{-i\frac{\pi}{2} s} e^{-i(s + 2)(p + \alpha)} \Gamma(-s) \left\{ 1 - e^{i\pi \alpha \omega^2} \right\}.$$

REMARK 3 If $D$ contains some corners of angle satisfying $2\alpha = \tan 2\alpha$, then Proposition 2 should be modified by adding the following term to (4.3);

$$-\frac{\pi}{2} \left( \frac{E}{1 + p} \cos 2\alpha e^{i\pi - i\frac{\pi}{2}} \right) \frac{1}{\tau}.$$

However, since Proposition 2 only implies $I_\omega(\tau)$ decays at most algebraically as $\tau \to \infty$, this is not enough to prove (4.1) without any restrictions on boundary data in itself.

For this we need to show that there exist a positive constants $\tau_0$, $c$ independent of $\tau$ and a real number $\mu_0 < -1$ such that for all $\tau \geq \tau_0$

$$|I_\omega(\tau)| \geq c \tau^{\mu_0}.$$  

(4.5)

Now we set

$$P_1 = \{ s, m \in C \mid A_{j, m} \neq 0 \}, \quad P_2 = \{ \delta_j, m \in R \mid B_{j, m}(\delta_j, m) \neq 0 \text{ or } \frac{\partial}{\partial s} B_{j, m}(\delta_j, m) \neq 0 \}.$$

Since $u$ is not a rigid displacement from the hypothesis in Theorem , the unique continuation theorem ensures $P_1 \cup P_2 \neq \emptyset$. Note here that the convergence of (4.2) is indispensable in this part because $P_1 \cup P_2 = \emptyset$ follows $u(r, \theta) = F(X)k$ in $D_\rho$ directly.
Accordingly, we can choose \( \lambda \in P_1 \cup P_2 \) such that \( \Re \lambda \geq \Re s \) for any \( s \in P_1 \cup P_2 \). Then the following three cases are considered.

**Case 1.** \( \lambda = s_{j,m} \in \mathbb{R} \).

In this case, from \( \theta \) and \( \gamma \) in Lemma one sees that \( C_j(s_{j,m}) \) in Proposition 2 never vanishes. Namely, it holds that

\[
I_\omega(\tau) = C_j(s_{j,m})A_{j,m}\tau^{s_{j,m}+1} + o(\tau^{s_{j,m}+1}),
\]

which simultaneously means (4.5).

**Case 2.** \( \lambda = \tilde{s}_{j,m} \in \mathbb{R} \).

Similarly to *Case 1*, one sees \( C_j(\tilde{s}_{j,m}) \neq 0 \). From \( \beta \) in Remark 1, it holds that

\[
I_\omega(\tau) = \frac{\partial}{\partial s} \left( C_j(s)B_{j,m}(s)\tau^{s+1} \right) \bigg|_{s=\tilde{s}_{j,m}} + o(\tau^{s_{j,m}+1}),
\]

which leads to (4.5).

**Case 3.** \( \lambda = s_{j,m} \in \mathbb{C} \setminus \mathbb{R} \) such that \( s_{j,m+1} = \overline{s}_{j,m} \). Namely,

\[
I_\omega(\tau) = C_j(s_{j,m})A_{j,m}\tau^{s_{j,m}+1} + C_j(s_{j,m+1})A_{j,m+1}\tau^{s_{j,m+1}+1} + o(\tau^{s_{j,m}+1}). \tag{4.6}
\]

In this case, it follows from \( 6, 7 \) in Lemma and \( 2 \) in Remark 1 that \( C_j(s_{j,m})A_{j,m} \neq 0 \) and \( C_j(s_{j,m+1})A_{j,m+1} \neq 0 \). And one sees \( |C_j(s_{j,m})A_{j,m}| = |C_j(s_{j,m+1})A_{j,m+1}| \) if and only if \( p + q + \pi = 0 \). Hence, when \( \omega \) satisfies (W2) which means \( p + q + \pi \neq 0 \), from (4.6) we obtain (4.5).

Summing up the above, (4.1) is verified in the case that \( D \) contains no corner of angle satisfying \( \tan 2\alpha = \pm \alpha \). If \( \tan 2\alpha = 2\alpha \), then note that (4.4) in Remark 3 vanishes if and only if \( A^* = 0 \). In consequence, by the same argument as above, (4.1) is still valid even in the case that some angles of \( D \) satisfy \( \tan 2\alpha = 2\alpha \) and the proof of Theorem is completed.

Lastly, if we weaken the assumption (W2) to (W1), then the formula in Theorem does not always hold. Now let consider the case \( p + q + \pi = 0 \) in *Case 3*. Then, it can be easily seen that there exists a sequence \( \{\tau_N\}_{N=1}^\infty \) such that \( \tau_N \to \infty \) as \( N \to \infty \) and it holds

\[
\lim_{N \to \infty} \tau_N^{(\Re s_{j,m}+1)} \left| C_j(s_{j,m})A_{j,m}\tau_N^{s_{j,m}+1} + C_j(s_{j,m})\overline{A_{j,m}}\tau_N^{s_{j,m}+1} \right| > 0.
\]

Namely, it follows from (4.6) that

\[
\lim_{\tau \to \infty} \frac{1}{\tau_N} \log |I_\omega(\tau_N)| = 0.
\]

Hence, one concludes that the formula

\[
h_D(\omega) = \lim_{\tau \to \infty} \frac{1}{\tau} \log \left| \int_{\partial \Omega} (T\mathbf{u} \cdot \mathbf{v} - T\mathbf{v} \cdot \mathbf{u}) \, dS \right|
\]

is valid even when \( p + q + \pi = 0 \), that is, under the assumption (W1) not (W2).
5. Future works

The enclosure method still has many possibilities for the further applications to inverse problems other than the problems listed in section 2.

Some open problems are in order.

- In the present result consider an inclusion instead of cavity and establish an extraction formula of its support function.
- Consider the scattering problem of an elastic plane wave.
- Extend results in [2], [6] and [16] to the three dimensional case.
- Consider inverse scattering problems for elastic or electromagnetic waves with use of dynamical data.
- Construct and test numerical methods based on our theoretical formulae had established up to now. At present they have succeeded in only a few cases (cf. [20], [21]), therefore further development of studies in this sides is expected.

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