On the sharpness of the zero-entropy-density conjecture

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The zero-entropy-density conjecture states that the entropy density defined as \( s := \lim_{N \to \infty} S_N / N \) vanishes for all translation-invariant pure states on the spin chain. Or equivalently, \( S_N \), the von Neumann entropy of such a state restricted to \( N \) consecutive spins, is sublinear. In this paper it is proved that this conjecture cannot be sharpened, i.e., translation-invariant states give rise to arbitrary fast sublinear entropy growth. The proof is constructive, and is based on a class of states derived from quasifree states on a CAR algebra. The question whether the entropy growth of pure quasifree states can be arbitrary fast sublinear was first raised by Fannes \textit{et al.} \cite{J. Math Phys. 44, 6005 (2003)}. In addition to the main theorem it is also shown that the entropy asymptotics of all pure shift-invariant nontrivial quasifree states is at least logarithmic.

I. INTRODUCTION

Quantum spin chains belong to the most studied models of quantum statistical mechanics.\textsuperscript{1} Still, only for a few types of models have the thermal and ground state structures been determined. This is mainly the consequence of the complicated correlations that can appear in quantum states. These strong correlations can even be present in pure states, while classical pure states can only have a trivial product state structure. Unlike the classical case, the restrictions of pure states on the quantum spin chain to local subsystem are typically mixed states. This type of correlation between subsystems is commonly referred to as entanglement. The von Neumann entropy, defined as \( S := -\text{Tr} \rho \log \rho \), is a natural measure of the nonpurity of the restricted density matrix \( \rho \), thus it is a very useful quantity in the description of entanglement.

The entropy of a restricted density matrix is also a basic measure when mixed states are treated, however, in this case it cannot be interpreted as a measure of

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entanglement. Let $S_N$ denote the von Neumann entropy of a translation-invariant state restricted to $N$ consecutive spins. The entropy density $s := \lim_{N \to \infty} S_N / N$ is considered to quantify the "strong nonpurity" of the entire mixed state, and it plays a central role in the characterization of Gibbs states.\(^1\) A natural and long-standing conjecture is that the entropy density vanishes for all translation-invariant pure states on a quantum spin chain, i.e., for such states, $S_N$ is sublinear. In the present paper we will prove that if this zero-entropy-density conjecture is true, then it is sharp in the sense that for any sublinear function $f_N$ ($\lim_{N \to \infty} f_N / N = 0$), there exists a translation-invariant state so that $S_N \geq f_N$, for every sufficiently large $N$. This has already been conjectured by Fannes, Haegeman, and Mosonyi in Ref. 2. Moreover, they proved that any sublinear power function can be exceeded by the entropy growth of an appropriate pure translation-invariant state.

It also should be mentioned that there is a revived interest in studying entropy asymptotics for two other reasons. First, $S_N$ seems to be a good indicator of quantum criticality. Several ground states of Hamiltonians with local interactions were studied, and in these models, $S_N$ was found to be bounded for noncritical systems, while for critical systems, it turned out to diverge logarithmically.\(^3\)–\(^6\) The prefactor of this logarithmic growth was argued to be one-third of the central charge.\(^7\) Also higher dimensional lattice models have been investigated in this respect.\(^8\)–\(^11\)

Second, entropy is supposed to play an important role in the quantification of the "essential subspace" of a restricted density matrix. The possibility of compressing the restricted density matrix from its full dimension to a much smaller subspace without loss of much information is the starting point of the DMRG calculations.\(^12\) For ergodic translation-invariant states, with non-vanishing entropy density $s$, the density matrix pertaining to $N$ consecutive spins, $\rho_N$, is essentially concentrated on a subspace of dimension proportional to $\exp(N s)$.\(^13\) In more general situations, numerical calculations suggest that the dimension of the "essential subspace" of $\rho_N$ is proportional to $\exp(S_N)$.\(^14\) This could lead to a very efficient compression of states with bounded or slowly diverging entropy asymptotics.

In the present paper we give a constructive proof of the sharpness of the zero-entropy-density conjecture. The states that are studied are translation-invariant pure states on the spin chain derived from quasifree states on the CAR algebra. In Sec. II we recapitulate the construction of such states in order to be self-contained. In Sec. III we prove our main theorem. The argument is based on the approach to quasifree states developed in Ref. 2. Finally, we include a proof of the statement that the entropy growth for all nontrivial gauge-invariant quasifree states are bounded from below by a logarithmic growth.
II. QUASIFREE STATES ON THE SPIN CHAIN

A. The Araki-Jordan-Wigner construction

The algebra of observables of a quantum spin chain is the UHF algebra

\[ \mathcal{M} := \bigotimes_{k=-\infty}^{+\infty} M_2, \]

where \( M_2 \) denotes the algebra of \( 2 \times 2 \) matrices. Let \( \sigma^k_a \) \((a = 1, 2, 3; k \in \mathbb{Z})\) denote the Pauli matrices embedded into the \( k \)th factor of \( \mathcal{M} \). They satisfy the well-known relations:

\[ \sigma^k_a \sigma^l_b = \sigma^l_b \sigma^k_a, \quad \text{when} \quad k \neq l, \]
\[ \sigma^l_b \sigma^k_a = i \epsilon_{abc} \sigma^l_c + \delta_{ab} 1. \]

The Pauli matrices and \( 1 \) generate \( \mathcal{M} \). The translation automorphism \( \tau \) on \( \mathcal{M} \) is defined by \( \tau(\sigma^k_a) = \sigma^{k+1}_a \).

The states we are investigating in this paper are translation-invariant pure states on \( \mathcal{M} \) derived from quasifree states of a fermion chain. The \( C^* \) algebra describing a fermion chain is the CAR algebra corresponding to the one-particle Hilbert space \( \ell^2(\mathbb{Z}) \), i.e. it is the \( C^* \) algebra generated by \( \{c_k \mid k \in \mathbb{Z}\} \) satisfying the canonical anticommutation relations:

\[ c_k c_l^* + c_l^* c_k = \delta_{k,l} 1, \quad c_k c_l + c_l c_k = 0. \]

Denote this \( C^* \) algebra by \( \mathcal{A} \). The translation automorphism \( \gamma \) is defined by \( \gamma(c_k) = c_{k+1} \).

The \( C^* \) algebras \( \mathcal{M} \) and \( \mathcal{A} \) are isomorphic. However, there exists no isomorphism \( \iota: \mathcal{M} \to \mathcal{A} \) that satisfies the property \( \iota \circ \tau = \gamma \circ \iota \). This intertwining property is needed to derive the translation invariance of a state \( \omega \circ \iota \) on \( \mathcal{M} \) from that of \( \omega \) on \( \mathcal{A} \). This problem can be circumvented by Araki’s construction.15 In this section we will present a modified but equivalent formulation of this construction.

First, let us introduce the parity automorphism \( \pi \) on \( \mathcal{A} \). It is defined by \( \pi(c_k) = -c_k \). The elements of \( \mathcal{A}_+ := \{a \in \mathcal{A} \mid \pi(a) = a\} \) are called even, while those of \( \mathcal{A}_- := \{a \in \mathcal{A} \mid \pi(a) = -a\} \) are called odd. Any element \( a \in \mathcal{A} \) can uniquely be written in the form \( a = a_+ + a_- \), where \( a_+ \in \mathcal{A}_+ \), and \( a_- \in \mathcal{A}_- \). Thus, \( \mathcal{A} = \mathcal{A}_+ + \mathcal{A}_- \).

Second, let \( \mathcal{M}_+ \) be the \( C^* \) subalgebra of \( \mathcal{M} \) generated by \( 1, \sigma^k_3, \) and \( \sigma^k_1 \sigma^l_1 \) \((k, l \in \mathbb{Z})\). \( \mathcal{M}_+ \) is isomorphic to \( \mathcal{A}_+ \), an explicit isomorphism \( \alpha \) is given by the

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a) More precisely, spin \( \frac{1}{2} \) chain.

b) This is clear if we note that \( (\mathcal{M}, \tau) \) is asymptotically Abelian, while \( (\mathcal{A}, \gamma) \) is not.
restricted Jordan-Wigner transformation:
\[ \alpha(\sigma_3^k) := 2c_k^* c_k - 1, \]
\[ \alpha(\sigma_i^k \sigma_j^l) := -\prod_{m=k}^{l-1} (2c_m^* c_m - 1)(c_k^* + c_k)(c_j^* + c_j) \quad \text{when} \quad k < l. \]

Since \( M_+ \) and \( A_+ \) are invariant under the translations, \( \tau \) and \( \gamma \) can be restricted to \( M_+ \) and \( A_+ \), respectively. Let us denote these restrictions by \( \tau_+ \) and \( \gamma_+ \). Although there is no isomorphism that intertwines the translations on \( M \) and \( A \), \( \alpha \) is an isomorphism that intertwines the translations on the subalgebras \( M_+ \) and \( A_+ : \alpha \circ \tau_+ = \gamma_+ \circ \alpha. \)

Now, let \( \omega \) be the restriction of a state \( \omega \) on \( A \) to \( A_+ \). If \( \omega \) is a translation-invariant state, i.e., \( \omega \circ \gamma = \omega \), then \( \omega \circ \alpha \), which is a state on \( M_+ \), is invariant under \( \tau_+ \). The state \( \omega_+ \circ \alpha \) can be extended to a state \( \tilde{\omega} \) on \( M \) by \( \tilde{\omega}(a) = \tilde{\omega}(a_+ + a_-) := \omega_+(\alpha(a_+)) \), where \( a_+ \in A_+ \), and \( a_- \in A_- \). This way a translation-invariant state \( \tilde{\omega} \) on \( M \) is obtained. Moreover, if \( \omega \) is an even state, i.e., \( \omega \circ \pi = \omega \), then \( \tilde{\omega} \) is pure if and only if \( \omega \) is pure.\(^2\)

To summarize, a translation intertwining automorphism \( \alpha \) has been given not between the algebras \( M \) and \( A \) but between their appropriate subalgebras \( M_+ \) and \( A_+ \). Any translation-invariant state on \( M_+ \) can be straightforwardly extended to a translation-invariant state on \( M \). Thus the isomorphism \( \alpha \) makes it possible to transport the translation-invariant states from \( A \) to \( M \).

B. Quasifree states on CAR algebras

Following Araki’s construction presented in the previous section, a class of states will be derived from quasifree states on the CAR algebra \( A \). In this section we will shortly recapitulate the most important definitions and facts concerning these states, more details and the proofs of the statements can be found in Ref. 1 and Ref. 16.

Let \( Q \) be an operator on the Hilbert space \( \ell^2(\mathbb{Z}), 0 \leq Q \leq 1 \). Let \( Q_{ij} := \langle \delta_i, Q \delta_j \rangle \) be the matrix elements of \( Q \) in the standard basis \( \{ \delta_k \mid k \in \mathbb{Z} \} \) of \( \ell^2(\mathbb{Z}) \), where \( \delta_k \) is the characteristic function of the number \( k \). The (gauge-invariant) quasifree state \( \omega_Q \) on \( A \) is defined through the following formula:

\[ \omega_Q(c_{i_1}^* \cdots c_{i_m}^* c_{j_1} \cdots c_{j_l}) = \delta_{m,n} \det \left( [Q_{0,j_{l+1}}]_{k,l=1}^n \right), \]

The operator \( Q \) is called the symbol of the state. Quasifree states are by definition even states.

A quasifree state \( \omega_Q \) is translation-invariant if and only if its symbol \( Q \) is a Toeplitz operator in the basis \( \delta_k \), i.e., there exists a sequence \( (q_k)_{k \in \mathbb{Z}} \) such that \( Q_{kl} = \langle \delta_k, Q \delta_l \rangle = q_{k-l} \). Let us introduce the Fourier transform:

\[ \tilde{q}(\theta) := \sum_{k \in \mathbb{Z}} q_k e^{i2\pi k\theta}, \quad \text{where} \quad \theta \in [0, 1). \]
The function $\tilde{q}$ satisfies $0 \leqslant \tilde{q}(\theta) \leqslant 1$ almost everywhere. A translation-invariant quasifree state $\omega_Q$ is pure if and only if the Fourier transform $\tilde{q}$ is a characteristic function, i.e., there exists a measurable set $K \subset [0, 1)$ such that $\tilde{q}(\theta) = \chi_K(\theta)$.

Now, applying the Araki-Jordan-Wigner construction to a translation-invariant quasifree state $\omega_Q$, one obtains a translation-invariant state $\tilde{\omega}_Q$ on the spin chain algebra $\mathcal{M}$. Since quasifree states are even, the state $\tilde{\omega}_Q$ is pure if and only if the corresponding quasifree state $\omega_Q$ is also pure.

Let $\rho_N$ denote the reduced density matrix obtained by restricting the state $\tilde{\omega}_Q$ to an interval of $N$ spins. The von Neumann entropy of the restricted state is defined as

$$S_N = -\text{Tr} \left( Q_N \log Q_N + (1 - Q_N) \log(1 - Q_N) \right),$$

(1)

where $Q_N$ is the restriction of $Q$ to the $N$-dimensional space spanned by the set $\{\delta_k \mid 0 \leqslant k \leqslant N - 1\}$.

On the basis of the Szegő theorem one can prove that the entropy density $s := \lim_{N \to \infty} S_N/N$ of pure translation-invariant quasifree states vanishes. In the next section we will prove that this statement is sharp in the sense that for any $f_N$ sublinear function there is a quasifree state for which $S_N \geqslant f_N$ for sufficiently large $N$.

III. QUASIFREE STATES GIVE RISE TO ARBITRARY FAST SUBLIN-EAR ENTROPY GROWTH

An explicit formula of the entropy function $S_N$ for quasifree states was given in the previous section by equation (1). In order to simplify further computations, we work with a quadratic lower bound of $S_N$ introduced in Ref. 2:

$$q_N := \text{Tr} \, Q_N \, (1 - Q_N)$$

That $q_N$ is a lower bound of $S_N = -\text{Tr}(Q_N \log Q_N + (1 - Q_N) \log(1 - Q_N))$ can be proved by the aid of the inequality $x(1 - x) \leqslant -x \ln x - (1 - x) \ln(1 - x)$, which holds for $0 \leqslant x \leqslant 1$.

As derived in Ref. 2, $q_N$ can be rewritten in the form:

$$q_N = \int_0^1 d\phi \, \frac{\sin^2 N\pi\phi}{\sin^2 N\pi} \Lambda_K(\phi),$$

(2)

where $K$ denotes the measurable set $K$ that characterizes the symbol $Q$, and $\Lambda_K$ is the function:

$$\Lambda_K(\phi) = |(K + \phi) \setminus K|.$$
|·| denotes the Lebesgue measure. By reducing the region of the integration in (2) to [0, 1/(2N)], and substituting the trigonometric factor with its lower bound on this restricted region, we obtain a lower estimate for $q_N$:

$$q_N \geq \frac{4N^2}{\pi^2} \int_0^{\pi} \Lambda_K(\phi) d\phi.$$  (3)

This is the starting point in the proof of the following proposition.

**Theorem.** For any sublinear function $f : \mathbb{N} \to \mathbb{R}^+$, there exists a pure translation-invariant quasifree state for which $S_N$ is bounded from below by $f_N$, that is $S_N \geq f_N$ for every sufficiently large $N$.

**Proof.** By (3), the problem has been reduced to showing the existence of a set $K \subset [0, 1)$ for which the right hand side of (3) grows not slower than the given $f_N$ as $N$ goes to infinity.

The construction of $K$ is based on two non-negative sequences: a sequence of integers $(n_i)_{i \in \mathbb{N}}$ and another one of real numbers $(\ell_i)_{i \in \mathbb{N}}$, where $\ell_i \geq 2\ell_{i+1}$. Let $K$ be the union of infinitely many disjoint intervals, the end points of which are determined by these two sequences as follows:

$$K = \bigcup_{i \in \mathbb{N}} \bigcup_{k=1}^{n_i} I^k_i, \quad I^k_i = [a^k_i, b^k_i], \quad b^k_i - a^k_i = \ell_i;$$

$$a^1_0 = 0; \quad a^1_i = b^{n_{i-1}}_{i-1} + \ell_{i-1}, \quad \text{if } i > 0;$$

$$a^k_i = b^{n_{i-1}}_{i-1} + \ell_{i}, \quad \text{if } k > 1.$$  (4)

The $(\ell_i)_{i \in \mathbb{N}}$ and $(n_i)_{i \in \mathbb{N}}$ are chosen so that the set $K$ above-constructed is bounded, and for convenience, we suppose additionally that $n_i\ell_i$ is monotonically decreasing, and:

$$\sum_{i=0}^{\infty} n_i\ell_i < \frac{1}{4}.$$  

Thus $K \subset [0, 1/2]$. With construction (4), $\Lambda_K$ takes the form

$$\Lambda_K(\phi) = \sum_{i=0}^{\infty} \sum_{k=1}^{n_i} |(I^k_i + \phi) \setminus K| \geq \sum_{i=0}^{\infty} \sum_{k=1}^{n_i} |(I^k_i + \phi) \setminus K|,$$

where $i_\phi$ is the smallest index for which $2n_i\ell_i < \phi$ for all $i \geq i_\phi$. Each translated interval $(I^k_i + \phi)$ with $i \geq i_\phi$ is situated in a region where the original intervals in the construction of $K$ and the gaps between them are not longer than $\ell_i/2$ (or where $K$ has no point at all). For this reason $|(I^k_i + \phi) \setminus K| \geq \ell_i/3$ for every term in the last summation. Therefore we obtain

$$\Lambda_K(\phi) \geq \frac{1}{3} \sum_{i=0}^{\infty} n_i\ell_i.$$  

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Now, let \( f_N \) be an arbitrary sublinear function, i.e., \( \lim_{N \to \infty} f_N/N = 0 \). Obviously, there exists a monotonically increasing continuously differentiable function \( g: [0, 1/2] \to \mathbb{R}^+ \) with the properties:

\[
g(0) = 0, \quad \frac{\pi^2}{2} g \left( \frac{1}{2N} \right) > f_N/N.
\]

Let us define function \( h \) as \( h(x) = \frac{d}{dx}(xg(x)) \). \( h \) is continuous, and \( h(0) = 0 \). We suppose that \( h \) is strictly monotonically increasing in the neighborhood of zero. If not, we choose a continuous, strictly monotonically increasing \( \hat{h} \) such that \( \hat{h} \geq h \), and \( \hat{h}(0) = 0 \).

This \( \hat{h} \) can be derived from a \( \hat{g} \) for which \( \hat{g} \geq g \), and then the argument can be continued with \( \hat{h} \) instead of \( h \).

The next step is to specify \((n_i)_{i \in \mathbb{N}}\) and \((\ell_i)_{i \in \mathbb{N}}\) so that

\[
\Lambda_K(\phi) \geq \frac{1}{3} \sum_{i=1}^{\infty} n_i \ell_i \geq h(\phi) \tag{5}
\]

should hold for sufficiently small \( \phi \).

Let \( s_i \) be the solution of the following recursive equation, starting from a given \( s_0 \) (\( 0 < s_0 < 1/2 \)):

\[
h(6(s_i - s_{i+1})) = s_{i+1}. \tag{6}
\]

It is clear from the required properties of \( h \) that there is a solution that satisfies the equalities \( 0 \leq s_{i+1} \leq s_i \) for every \( i \). Since \((s_i)_{i \in \mathbb{N}}\) is bounded from below and monotonically decreasing, it has a limit at infinity. Suppose that this limit differs from zero, say it is \( s_\infty > 0 \). Taking an arbitrary small \( \epsilon > 0 \), there is an \( i \) for which \( \epsilon > 6(s_i - s_{i+1}) \), and we find that \( h(\epsilon) \geq h(6(s_i - s_{i+1})) = s_{i+1} \geq s_\infty \) for any \( \epsilon \), so \( h(0) \geq s_\infty \) in contradiction with \( h(0) = 0 \). Thus \( \lim_{i \to \infty} s_i = 0 \).

Now we are ready to specify the values of \( \ell_i \) and \( n_i \) by

\[
s_i = \frac{1}{3} \sum_{j=1}^{\infty} n_j \ell_j \tag{7}
\]

Considering that \((s_i)_{i \in \mathbb{N}}\) is a monotonically decreasing sequence tending to zero, these equalities can be satisfied by some series \((n_i)_{i \in \mathbb{N}}\) and \((\ell_i)_{i \in \mathbb{N}}\). Starting with a particular \( \ell_i \), we can always determine the next term by choosing some \( \ell_{i+1} \leq \ell_i/2 \). The only restriction on the choice of \( \ell_i \) is that \( s_i - s_{i+1} \) should be an integral multiple of \( \ell_i \). This requirement can undoubtedly be met, and then \( s_i - s_{i+1} = \frac{1}{3} n_i \ell_i \) yields the value of \( n_i \). The inclusion \( K \subset [0, 1/2] \) can be assured by choosing sufficiently small \( s_0 \).

Recall that \((n_i \ell_i)_{i \in \mathbb{N}}\) has been required to be monotonic. We can easily convince ourselves that \((n_i \ell_i)_{i \in \mathbb{N}}\) constructed from \((s_i)_{i \in \mathbb{N}}\) has this property. Indeed, it follows

\[c\]A possible choice is \( \hat{h}(x) := \max(h(y) \mid y \in [0, x]) + x \).
immediately from the strict monotonicity of $h$: $h(2n_i\ell_i) = h(6(s_i - s_{i+1})) = s_{i+1} \leq s_i = h(6(s_{i-1} - s_i)) = h(2n_{i-1}\ell_{i-1})$.

Monotonicity of $(s_i)_{i \in \mathbb{N}}$ and its behavior at infinity entail that for any $\phi$ below a certain bound, there is an index $i$ for which $6(s_i - s_{i+1}) \leq \phi \leq 6(s_{i-1} - s_i)$. Notice that this index is nothing but $i_\phi$. Thus putting together (6), and (7), we arrive at the desired estimate (5). Consequently, for sufficiently large $N$, in the region of the integration in (3), $\Lambda_K(\phi) \geq h(\phi)$ holds. Performing the integration in (3) completes the proof:

$$S_N \geq q_N \geq 4N^2 \int_0^{\frac{1}{\pi}} \Lambda_K(\phi)d\phi \geq 4N^2 \int_0^{\frac{1}{\pi}} h(\phi)d\phi =$$

$$\frac{4N^2}{\pi^2} \int_0^{\frac{1}{\pi}} \frac{d}{d\phi}(\phi g(\phi))d\phi = \frac{2N}{\pi^2} g\left(\frac{1}{2N}\right) \geq f_N.$$

□

We have just shown that pure translation-invariant quasifree states give rise to arbitrary fast sublinear entropy growth. In the trivial cases, $|K| = 0$ or $|K| = 1$, the entropy is identically zero. The question naturally arises whether it is possible to achieve arbitrary slow nonbounded entropy growth by such states.

**Proposition.** Apart from the trivial cases, pure (gauge- and) translation-invariant quasifree states give at least logarithmic entropy growth.

**Proof.** It has been shown in Ref. 2 that if

$$\Lambda_K(\phi) \geq c\phi, \quad \text{for some } c > 0 \quad (8)$$

in the vicinity of zero, then $S_N$ is bounded from below by a logarithmic growth. We will prove that (8) holds for any measurable set $K \subset [0, 1)$ (apart from the trivial cases, where $\Lambda_K(\phi) = 0$).

It is known from Lebesgue density theorem that for any measurable set $K$, $|K| = |K^d|$ holds, where $K^d$ denotes the set of the density points of $K$:

$$K^d = \left\{ x \in K \left| \lim_{\delta \to 0} \frac{1}{2\delta} \frac{(x-\delta, x+\delta) \cap K}{(x-\delta, x+\delta)} = 1 \right. \right\}.$$

It can be inferred from this theorem that for any $K$ of positive measure, there is such a point $x \in K$ that

$$\forall \epsilon > 0 : \exists \delta > 0 \text{ so that for every interval } I \text{ that satisfies } x \in I, \text{ and } |I| < \delta,$$

$$|K \cap I| > (1 - \epsilon)|I|. \quad (9)$$

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Disregarding the trivial cases, the measure of $K^c$ (the complement of $K$), is also positive: $|K^c| > 0$. It means that $K^c$ also has a point that satisfies (9). We denote this point by $y$. For a given $\epsilon$, we can chose a common $\delta$ to $x$ and $y$. Let $I$ be an interval shorter than this $\delta$: $|l| < \delta$, and $x \in I$. There is an integer $n$ such that $y \in (I + n|l|)$. The set $(I + n|l|)$ can be assured to be disjoint from $I$ by choosing a sufficiently small $\delta$. The following inequalities hold for $I$:

$$|K \cap I| > (1 - \epsilon)|l|, \quad |K^c \cap (I + n|l|)| > (1 - \epsilon)|l|, \quad (10)$$

The following estimate, though seemingly weak, is the core of the proof:

$$\Lambda_K(|l|) \geq \left| \left( \bigcup_{k=0}^{n-1} (I + k|l|) \cap (K + |l|) \right) \setminus K \right|$$

$$= \sum_{k=0}^{n-1} \left| (I + k|l|) \cap (K + |l|) \right| \setminus (I + (k + 1)|l|) \cap K$$

$$\geq \sum_{k=0}^{n-1} \left| (I + k|l|) \cap (K + |l|) \right| - |(I + (k + 1)|l|) \cap K|$$

$$= |I \cap K| - |(I + n|l|) \cap K|$$

Having a look at (10), we obtain that for arbitrary positive $\epsilon$,

$$\Lambda_K(|l|) \geq (1 - 2\epsilon)|l|,$$

if $|l|$ is sufficiently small. This inequality entails (8). \qed

**Conclusion and Outlook**

We have shown that for any sublinear growth $f_N$, there exist shift-invariant pure states that have faster entropy growths than $f_N$. However, the question if the entropy asymptotics of any translation-invariant pure state is sublinear, that is whether they have a vanishing entropy density, is still unsolved. It is difficult to address this problem generally. One can instead take into consideration only special classes of translation-invariant states. For instance, finitely correlated states turn out to lead to bounded entropy growth.\(^{17}\) A further step could be to explore the entropy growths and the entropy densities of pure algebraic states, which are the generalizations of the finitely correlated states.

Another question that can be raised is whether there exists for each sublinear growth $f_N$ a state with local von Neumann entropies $S_N$ such that $\lim_{N\to\infty} S_N / f_N = c$, where $c > 0$. As we can learn from the last proposition in Sec. III, in the case of pure translation-invariant quasifree state the answer is negative.

In the case of local Hamiltonians only ground states with bounded or logarithmic entropy growth have been found. From a mathematical point of view, our
construction is not sophisticated. Any given sublinear asymptotics is exceedable by the entropy growth of a state characterized by a set of rather simple structure: a set built from countably many intervals. Nevertheless, it is still an open question whether these asymptotics can be physically realized, or entropy asymptotics stronger than logarithmic (or some other sublinear) function can never occur for ground states in the presence of only local interactions.

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