On Sectional Curvature Operator
of 3-dimensional Locally Homogeneous
Lorentzian Manifolds*

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Abstract
The main purpose of this paper is to determine the admissible forms of the sectional curvature operator on a three-dimensional locally homogeneous Lorentzian manifolds.

Keywords: locally homogeneous Lorentzian manifold, sectional curvature operator, Segre type.

1 Introduction
The problem of the restoring of (pseudo)Riemannian manifold on prescribed curvature operator is the actual direction in the research of curvature operators. The Riemannian locally homogeneous spaces with the prescribed values of the spectrum of Ricci operator have been identified by O. Kowalski and S. Nikcevic in [1]. The problem of the existence of locally homogeneous Lorentzian manifold and prescribed Ricci operator was investigated by G. Calvaruso and O. Kowalski in [2]. There are also some papers about this problem in nonhomogeneous case (see [3, 4]).

Similar results were obtained by D.N. Oskorbin, E.D. Rodionov, O.P. Khromova for the one-dimensional curvature operator and the sectional curvature operator in the case of three-dimensional Lie groups with left-invariant Riemannian metrics [5, 6].

The main purpose of this paper is to consider the problem of the prescribed sectional curvature operator $\mathcal{K}$ on the three-dimensional Lorentzian locally homogeneous manifolds.

Unlike the case of the Riemannian metric, there always exist an orthonormal basis, in which the matrix of the curvature operator is diagonal, in the case of Lorentzian metric different cases can occur known as Segre types (see. [7]). Namely, the following cases can occur:

1) Segre type $\{111\}$: the operator $\mathcal{K}$ has three real eigenvalues (possibly coincident), each associated to a one-dimensional eigenspace.

2) Segre type $\{1\bar{z}z\}$: the operator $\mathcal{K}$ has one real and two complex the conjugate eigenvalues.

3) Segre type $\{21\}$: the operator $\mathcal{K}$ has two real eigenvalues (possibly coincident), the first of which has algebraic multiplicity 2, each associated to a one-dimensional eigenspace.

4) Segre type $\{3\}$: the operator $\mathcal{K}$ has one real eigenvalue of algebraic multiplicity 3, associated one-dimensional eigenspace.

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2 Three-dimensional homogeneous Loretzian manifolds

Let \((M, g)\) be three-dimensional homogeneous manifold, with the Lorenzian metric \(g\) of signature \((+, +, -)\). We denote by \(\nabla\) its Levi-Civita connection and by \(R\) its curvature tensor, which defined by following

\[
R(X, Y) Z = [\nabla_Y, \nabla_X] Z + \nabla_{[X,Y]}Z.
\]

The Lorenzian metric \(g\) induces a scalar product \(\langle \cdot, \cdot \rangle\) in the bundle \(\Lambda^2 M\) by the rule

\[
\langle X_1 \wedge X_2, Y_1 \wedge Y_2 \rangle = \det (\langle X_i, Y_i \rangle).
\]

The curvature tensor \(R\) at any point can be considered as an operator \(K: \Lambda^2 M \rightarrow \Lambda^2 M\), called the sectional curvature operator and defined by the equation

\[
\langle X \wedge Y, K(Z \wedge T) \rangle = R(X, Y, Z, T).
\]

The studying of curvature operators on three-dimensional locally homogeneous Lorentzian spaces is based on the following fact, which was proved by G. Calvaruso in [8].

**Theorem 2.1.** Let \((M, g)\) be a three-dimensional connected, simply connected, complete locally homogeneous Lorentzian manifold. Then, either \((M, g)\) is locally symmetric, or it is locally isometric to a three-dimensional Lie group equipped with a left-invariant Lorentzian metric.

Further classification results for three-dimensional Lorentzian Lie groups was obtained in [9].

**Theorem 2.2.** Let \(G\) be a three-dimensional Lie group with left-invariant Lorentzian metric. Then

- If \(G\) is unimodular, then there exists a pseudo-orthonormal frame field \(\{e_1, e_2, e_3\}\), such that the metric Lie algebra of \(G\) is one of the following:

  1) \(\mathcal{A}_1:\)

     \[
     [e_1, e_2] = \lambda_3 e_3, \\
     [e_1, e_3] = -\lambda_2 e_2, \\
     [e_2, e_3] = \lambda_1 e_1,
     \]

     with \(e_1\) timelike;

  2) \(\mathcal{A}_2:\)

     \[
     [e_1, e_2] = (1 - \lambda_2) e_3 - e_2, \\
     [e_1, e_3] = e_3 - (1 + \lambda_2) e_2, \\
     [e_2, e_3] = \lambda_1 e_1,
     \]

     with \(e_3\) timelike;

  3) \(\mathcal{A}_3:\)

     \[
     [e_1, e_2] = e_1 - \lambda e_3, \\
     [e_1, e_3] = -\lambda e_2 - e_1, \\
     [e_2, e_3] = \lambda_1 e_1 + e_2 + e_3,
     \]

     with \(e_3\) timelike;
4)\[ e_1, e_2 = \lambda_3 e_2, \]
\[ e_1, e_3 = -\beta e_1 - \alpha e_2, \]
\[ e_2, e_3 = -\alpha e_1 + \beta e_2, \]

with \( e_1 \) timelike and \( \beta \neq 0 \).

- If \( G \) is non-unimodular, then there exists a pseudo-orthonormal frame field \( \{ e_1, e_2, e_3 \} \), such that the metric Lie algebra of \( G \) is one of the following:

1) \[ e_1, e_2 = 0, \]
\[ e_1, e_3 = \lambda \sin \varphi e_1 - \mu \cos \varphi e_2, \]
\[ e_2, e_3 = \lambda \cos \varphi e_1 + \mu \sin \varphi e_2, \]

with \( e_3 \) timelike and \( \sin \varphi \neq 0 \), \( \lambda + \mu \neq 0 \), \( \lambda \geq 0 \), \( \mu \geq 0 \);

2) \[ e_1, e_2 = 0, \]
\[ e_1, e_3 = te_1 - se_2, \]
\[ e_2, e_3 = pe_1 + qe_2, \]

with \( \langle e_2, e_2 \rangle = -\langle e_1, e_3 \rangle = 1 \) and otherwise zero, and \( q \neq t \);

3) \[ e_1, e_2 = 0, \]
\[ e_1, e_3 = se_1 + pe_2, \]
\[ e_2, e_3 = pe_1 + qe_2, \]

with \( e_2 \) timelike and \( q \neq s \);

4) \[ e_1, e_2 = 0, \]
\[ e_1, e_3 = qe_1 - re_2, \]
\[ e_2, e_3 = pe_1 + qe_2, \]

with \( e_2 \) timelike and \( q \neq 0 \), \( p + r \neq 0 \).

Remark 2.3. There are exactly six nonisomorphic three-dimensional unimodular Lie algebras and the corresponding types of three-dimensional unimodular Lie groups (see [10]). All of them are listed in the Table 1 together with conditions on structure constants for which the Lie algebra has this type. If there is a “−” in the Table 1 at the intersection of the row, corresponding to the Lie algebra, and the column, corresponding to the type, then it means that this type of the basis is impossible for given Lie algebra. For the case of Lie algebra \( A_1 \) we give only the signs of the triple \((\lambda_1, \lambda_2, \lambda_3)\) up to reorder and sign change.

Remark 2.4. We note that similar bases was also constructed by G. Calvaruso, L.A. Cordero and P.E. Parker in [8, 11].
Table 1: Three-dimensional unimodular Lie algebras

| Lie algebra | Restrictions on the structure constants |
|-------------|----------------------------------------|
| su(2)       | (+, +, +)                              |
| sl(2, ℝ)    | (+, +, -)                               |
| e(2)        | (+, +, 0)                               |
| e(1, 1)     | (+, - , 0)                              |
| h           | (+, 0, 0)                               |
| ℝ³          | (0, 0, 0)                               |

The following classification result for the case of three-dimensional Lorentzian locally symmetric space was proved in [8].

**Theorem 2.5.** A connected, simply connected three-dimensional Lorentzian locally symmetric space \((M, g)\) is locally isometric to

1) a Lorentzian space form \(ℝ^3_1, S^3_1\) or \(H^3_1\) (with zero, positive and negative sectional curvature respectively), or

2) a direct product \(ℝ × S^3_1, ℝ × H^3_1, S^2 × ℝ_1\) or \(H^2 × ℝ_1\), or

3) a space with a Lorentzian metric \(g\), which admitted a local coordinate system \((u_1, u_2, u_3)\) such, that the metric tensor has the following form

\[
g = \begin{pmatrix}
0 & 0 & 1 \\
0 & \varepsilon & 0 \\
1 & 0 & f(u_2, u_3)
\end{pmatrix},
\]

where \(\varepsilon = \pm 1\), \(f(u_2, u_3) = u_2^2 \alpha + u_2 \beta(u_3) + \xi(u_3), \alpha \in ℝ, \beta, \xi\) are arbitrary smooth functions.

### 3 Three-dimensional Lorentzian Lie groups

Further, by “a three-dimensional Lorenzian Lie group \((G, g)\)” we shall mean a three-dimensional Lie group \(G\), which equipped with a left-invariant Lorentzian metric \(g\) and having metric Lie algebra \(g\). Now we can prove the following

**Theorem 3.1.** A three-dimensional unimodular Lorentzian Lie group \((G, A_2)\) with the sectional curvature operator \(K\) exist if and only if

1) \(K\) has the Segre type \(\{111\}\) with the eigenvalues \(k_1 = -k_2 = -k_3 \geq 0\) up to renumeration, or

2) \(K\) has the Segre type \(\{12\}\) with the eigenvalues

(a) \(k_1 = k_2 = 0\), or

(b) \(k_2 < 0\).
Proof. In this case the matrix of the sectional curvature operator $\mathcal{K}$ has the following form

$$\mathcal{K} = \begin{pmatrix} \frac{3}{4}\lambda_1^2 - \lambda_1 \lambda_2 & 0 & 0 \\ 0 & 2\lambda_2 - \lambda_1 - \frac{1}{4}\lambda_1^2 & 2\lambda_2 - \lambda_1 \\ 0 & \frac{1}{4}\lambda_1^2 & \frac{1}{4}\lambda_1^2 \end{pmatrix}.$$ 

If $\lambda_1 = 2\lambda_2$, then the matrix of the sectional curvature operator has the diagonal form with the eigenvalues $k_1 = -k_2 = -k_3 \geq 0$. Else, the matrix of the sectional curvature operator $\mathcal{K}$ has the following Jordan form:

$$\mathcal{K} = \begin{pmatrix} -\lambda_1 \lambda_2 + \frac{3}{4}\lambda_1^2 & 0 & 0 \\ 0 & -\frac{1}{4}\lambda_1^2 & 1 \\ 0 & 0 & -\frac{1}{4}\lambda_1^2 \end{pmatrix},$$

and the eigenvalues are equal to

$$k_1 = -\lambda_1 \lambda_2 + \frac{3}{4}\lambda_1^2, \quad k_2 = -\frac{1}{4}\lambda_1^2 \leq 0.$$ 

If $k_2 = 0$, then $\lambda_1 = 0$ and all of the eigenvalues are equal to zero. Suppose that $k_2 < 0$. Then, this follows that $\lambda_1 = \pm 2\sqrt{-k_2}$. Expressing $\lambda_2$, we find

$$\lambda_2 = \frac{k_1 + 3k_2}{2\sqrt{-k_2}}.$$

The remaining cases are concerned in a similar way.

4 Three-dimensional Lorentzian locally symmetric spaces

The Theorem 2.5 allows us to divide the problem of studying the curvature operators on three-dimensional locally symmetric Lorentzian manifolds by three subtasks. At the same time, it is obvious that the sectional curvature operator $\mathcal{K}$ is diagonalizable for Lorentzian manifolds of constant sectional curvature $\mathbb{R}^3_1$, $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$ (i.e. $\mathcal{K}$ has the Segre type \{111\}) and $\mathcal{K}$ has three equal eigenvalues (zero, positive or negative respectively).

In the case of direct products (case 2 of the Theorem 2.5) the sectional curvature operator $\mathcal{K}$ has the Segre type \{111\} with two zero and third non-zero eigenvalues.

Therefore, only the case 3 of Theorem 2.5 is of interest, in which the metric tensor has the following form in local coordinate system

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f(u_2, u_3) \end{pmatrix},$$

where $\varepsilon = \pm 1$, $f(u_2, u_3) = u_2^2 \alpha + u_2 \beta(u_3) + \xi(u_3)$, $\alpha \in \mathbb{R}$, $\beta, \xi$ are arbitrary smooth functions.

Calculating the matrix of the sectional curvature operator $\mathcal{K}$, we have

$$\mathcal{K} = \begin{pmatrix} 0 & 0 & \frac{2}{\lambda} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$

5
Therefore, either the sectional curvature operator $K$ has the Segre type $\{111\}$ with the eigenvalues $k_1 = k_2 = k_3 = 0$ for $\alpha = 0$, or $K$ has the Segre type $\{12\}$ with the eigenvalues $k_1 = k_2 = 0$ for $\alpha \neq 0$. Hence, the following theorem holds.

**Theorem 4.1.** A connected, simply connected three-dimensional Lorentzian locally symmetric space with the sectional curvature operator $K$ exist if and only if

1) $K$ has the Segre type $\{111\}$ with the equal eigenvalues, or

2) $K$ has the Segre type $\{111\}$ with the two zero eigenvalues and one nonzero, or

3) $K$ has the Segre type $\{12\}$ with the zero eigenvalues.

## 5 Sectional curvature operator of locally homogeneous Lorentzian 3-manifolds

In this section, using the results of the previous sections, we determine under which conditions the different Segre types occur for the sectional curvature operator of a three-dimensional locally homogeneous Lorentzian manifold. Next theorems follows from the results on the cases of metric Lie groups and of the locally symmetric spaces.

**Theorem 5.1.** A connected, simply connected three-dimensional Lorentzian locally homogeneous manifold $(M, g)$ exist if and only if $K$ satisfies one of the following conditions:

1) all eigenvalues are equal to each other;

2) two eigenvalues are equal to zero and third is nonzero;

3) exactly two of $k_1 + k_2, k_1 + k_3$ and $k_2 + k_3$ are equal to zero;

4) $(k_1 + k_2)(k_1 + k_3)(k_2 + k_3) < 0$;

5) up to renumeration

$$k_2k_3 \leq k_1^2 < \left(\frac{k_2 + k_3}{2}\right)^2 \quad \text{and} \quad k_1 < \frac{k_2 + k_3}{2};$$

**Theorem 5.2.** A connected, simply connected three-dimensional Lorentzian locally homogeneous manifold $(M, g)$ with sectional curvature operator $K$ with Segre type $\{111\}$ exist if and only if eigenvalues $k_1, k_2, k_3$ satisfy one (or more than one) of the following conditions:

1) all eigenvalues are equal to each other;

2) two eigenvalues are equal to zero and third is nonzero;

3) exactly two of $k_1 + k_2, k_1 + k_3$ and $k_2 + k_3$ are equal to zero;

4) $(k_1 + k_2)(k_1 + k_3)(k_2 + k_3) < 0$;

5) up to renumeration

$$k_2k_3 \leq k_1^2 < \left(\frac{k_2 + k_3}{2}\right)^2 \quad \text{and} \quad k_1 < \frac{k_2 + k_3}{2};$$
6) up to renumeration

\[ k_2 < 0, \, k_3 < 0, \, |k_1| \leq \sqrt{k_2 k_3}; \]

7) up to renumeration

\[ k_1 < -\left| \frac{k_2 + k_3}{2} \right|. \]

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