Proof of the Thin Sandwich Conjecture

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We prove that the Thin Sandwich Conjecture in general relativity is valid, provided that the data $(g_{ab}, \dot{g}_{ab})$ satisfy certain geometric conditions. These conditions define an open set in the class of possible data, but are not generically satisfied. The implications for the “superspace” picture of the Einstein evolution equations are discussed.

The “superspace” picture of the Einstein equations espoused by J. A. Wheeler and others [1,2], envisages a spacetime as a curve in superspace $S$, the space of Riemannian metrics on a fixed 3-dimensional manifold $M$ [3]. It was hoped that a useful analogy might be drawn with the classical dynamics of a point particle and its quantum mechanical generalisation using the Feynman path integral method, and with this objective Wheeler advanced the (Thick) Sandwich and Thin Sandwich conjectures. The full Sandwich Conjecture proposes that, given two 3-metrics $g_1, g_2 \in S$, there exists a unique spacetime $V$ in which $g_1, g_2$ arise as the induced metrics on (disjoint) Cauchy hypersurfaces in $V$ (“$V$ is a spacetime connecting $g_1, g_2$”). Foliating $V$ between these Cauchy surfaces determines a curve in $S$ connecting $g_1, g_2$, but perturbing the foliation changes the curve, so the local uniqueness of the classical connecting path that holds in particle dynamics is not valid here. The uniqueness assertion of the Sandwich Conjecture implies that this diffeomorphism degeneracy is the only cause of non-uniqueness of the Einstein flow in $S$.

However, the electromagnetic analogue of the Sandwich Conjecture has an ultraviolet degeneracy, with non-trivial non-uniqueness arising from high-frequency components — there are (spatially periodic) non-trivial solutions of Maxwell’s equations having vector potential $A_\mu$ vanishing at two times $t = 0, 1$ [4]. This implies that uniqueness must fail for solutions of the Sandwich Conjecture for Maxwell’s equations, so it is possible that the full conjecture for the Einstein evolution may also be false. Such problems do not arise with the Thin Sandwich Conjecture, which is analogous to the initial value problem in particle dynamics and asserts that given a point
\[ g \in S \] and a tangent vector \( \dot{g} \in T_gS \), there is a unique spacetime \( V \) realising the initial condition \((g, \dot{g})\). This means that \( V \) has a time function \( t \) and a time flow vector \( T \) such that the metric in adapted coordinates \((t, x)\) has the form

\[ ds^2 = -N^2 dt^2 + g_{ij}(dx^i + X^i dt)(dx^j + X^j dt), \]  

(1)

where \( g_{ij}(x, t) \) is the induced metric on the level sets of \( t \), \( N \) is the lapse and \( X^i \) is the shift, \( T = \partial/\partial t = \partial_t \), and

\[ g_{ij} = g_{ij}(0), \quad \dot{g}_{ij} = \frac{\partial}{\partial t} g_{ij}(0). \]  

(2)

The local existence and uniqueness theorems for the Einstein equations \([5,6]\) reduce this problem to one of constructing geometric initial data \((g_{ij}, K_{ij})\) satisfying the constraint equations

\[ 16\pi T_{00} \equiv 2\epsilon = R_g - K_{ij}K^{ij} + (K_k^k)^2, \]  

(3)

\[ 8\pi T_{0i} \equiv S^i = \nabla_j K_{ij} - \nabla_i K_j. \]  

(4)

and such that the extrinsic curvature \( K_{ij} \) is compatible with \((\dot{g}_{ij}, \dot{g}_{ij})\) through the metric \(1\), ie. there is \((N, X^i)\) such that

\[ K_{ij} = N^{-1} \left( \frac{1}{2} \dot{g}_{ij} - X_{(ij)} \right). \]  

(5)

Substituting \(3\) into \(4\) gives the Thin Sandwich Equations (TSE), a system of differential equations for \((N, X^i)\), where \((\dot{g}_{ij}, \dot{g}_{ij})\) and the (normalised) local energy and momentum densities \((\epsilon, S^i)\) are given fields. Formally solving \(3\) for \(N\)

\[ N = \left[ \left( (\gamma^k_k)^2 - \gamma_{ij}\gamma^{ij} \right) / (2\epsilon - R_g) \right]^{1/2}, \]  

(6)

where we use the notational shorthand \(\gamma_{ij} = \frac{1}{2} \dot{g}_{ij} - X_{(ij)}\), and then substituting into \(3\) gives the Reduced Thin Sandwich Equations (RTSE), for the shift vector \(X^i\) alone. Explicitly,

\[ \sqrt{2\epsilon - R_g \left( (\gamma^m_m)^2 - \gamma_{mn}\gamma^{mn} \right) \left( \gamma^{ij} - g^{ij}\gamma_k^k \right)}_{ij} = S^i. \]  

(7)

There are two basic conditions we must assume before this equation can make sense. The first is an a priori restriction on the solution \(X^i\) and hence on the data \((\dot{g}_{ij}, \dot{g}_{ij}, \epsilon, S^i)\), since we regard \(3\) as the definition of \(N\) in terms of the data and \(X^i\): we must assume

\[ N > 0 \quad \text{everywhere in } M. \]  

(8)

Physically, this is the superspace requirement that the spacetime slicing “advance in time”. Although it is elementary to construct examples \((\dot{g}_{ij}, \dot{g}_{ij}, N, X^i, \epsilon, S^i)\) satisfying \(3\), \(4\), \(5\) for which \(N\) changes sign (by choosing a 1-parameter family of intersecting Cauchy surfaces in \(V\)), such examples will not satisfy \(3\), \(4\) in the strict sense since the denominator in \(3\) will vanish at some
points; furthermore they cannot satisfy the conditions needed for the general results we describe below, and they do not satisfy the superspace “direction of time” requirement. The second condition does not have a clear-cut physical interpretation, but does have a geometric formulation solely in terms of the given data:

\[ 2\epsilon - R_g > 0 \quad \text{everywhere in } \mathcal{M}. \]  

(9)

In order to minimise technical complications, we shall also assume throughout that \( \mathcal{M} \) is a compact manifold without boundary. It is quite possible that a more detailed analysis could extend our results, for example, to the case of asymptotically hyperboloidal data.

The need for the apriori bound (8) means that our main result is perturbative:

**Theorem 1** If the TSE with data \((g_{ij}, \dot{g}_{ij}, \epsilon, S^i)\) admits a solution \((N, X^i)\) such that the conditions (8) and (9) are satisfied, and if furthermore, the equation

\[ M_{(ij)} = \mu K_{ij} \]  

(10)

has only the trivial solution \( \mu = 0, M_i = 0 \), then the TSE with data \((g_{ij} + \delta g_{ij}, \dot{g}_{ij} + \delta \dot{g}_{ij}, \epsilon + \delta \epsilon, S^i + \delta S^i)\) sufficiently close to \((g_{ij}, \dot{g}_{ij}, \epsilon, S^i)\), has a unique solution \((N + \delta N, X^i + \delta X^i)\) close to \((N, X^i)\).

In other words, there is an open neighbourhood of the data \((g_{ij}, \dot{g}_{ij}, \epsilon, S^i)\) in which the TSE is solvable. Our method of proof is to show that the linearisation of (7) is elliptic under suitable conditions, and then apply the Implicit Function theorem — it is here that the technical condition (10) arises. Note that the projections of a spacetime Killing vector satisfy (10).

The reduced equations (7) can be written formally as

\[ F^i(\Psi, X^j) = 0, \]  

(11)

where \( \Psi \) represents the prescribed fields, \( \Psi = (g_{ij}, \dot{g}_{ij}, \epsilon, S^i) \), and \( X^j \) is the shift vector. The linearised thin sandwich operator \( L = \delta F/\delta X \) is given explicitly by

\[ L^i(M^j) = \left[ \frac{1}{N} \left( g^{ij} M^k_{[j} - M^{(ij)} - \frac{\pi_{ij} \pi_{kl} M^{[kl]}}{2\epsilon - R_g} \right) \right]_{ij}, \]

where \( M^j = \delta X^j \) and \( \pi^{ij} \) is the (non-density) conjugate momentum,

\[ \pi_{ij} = K_{ij} - g_{ij} K^k = N^{-1} \left( \gamma_{ij} - g_{ij} \gamma^k \right). \]  

(12)

We now consider properties of the linearised operator \( L \); a simple computation first shows that \( L \) is self adjoint. Remarkably, \( L \) is elliptic if the conjugate momentum is either positive definite or negative definite: to see this we calculate the leading order symbol of \( L \),

\[ \sigma(L)^i_j(\xi) = \frac{1}{N} \left( \frac{1}{2} \xi^k \xi^j \delta^i_k - \frac{1}{2} \xi^i \xi^j + \frac{\pi^i_k \pi^j_l \xi^k \xi^l}{2\epsilon - R_g} \right). \]  

(13)
An operator $L$ is elliptic when the symbol $\sigma(L)$ is invertible for all $(\xi^i) \neq 0$. In an orthonormal frame in which $(\xi^i) = (\xi, 0, 0)$ we have
\[
\det (\sigma(L)(\xi)) = \frac{\xi^6}{4N^3(2\epsilon - R_g)}(\pi_{11})^2,
\]
and hence in general coordinates,
\[
\det (\sigma(L)(\xi)) = \frac{1}{4N^3(2\epsilon - R_g)}\xi^k \xi^l (\pi_{ij} \xi^i \xi^j)^2, \quad (14)
\]
which will be nonzero for all non-zero $\xi^i$ if and only if $\pi_{ij}$ is a definite matrix.

If condition (1) and the energy constraint (3) hold, then
\[
\frac{1}{2} (\pi_{ii})^2 - \pi_{ij} \pi_{ij} > 0.
\]
Introducing an orthonormal frame in which $\pi_{ij}$ is diagonal, $\pi_{ij} = \text{diag}(\alpha, \beta, \gamma)$ say, this gives
\[
\frac{1}{2} (\alpha + \beta + \gamma)^2 - (\alpha^2 + \beta^2 + \gamma^2) > 0,
\]
which may be rearranged as
\[
2\alpha\beta - \frac{1}{2}(\alpha + \beta - \gamma)^2 > 0.
\]
Clearly this requires that $\alpha\beta > 0$ and similarly $\beta\gamma > 0$, $\alpha\gamma > 0$, which implies that $\alpha$, $\beta$, $\gamma$ all have the same sign; in other words, $\pi_{ij}$ is definite. We have shown

**Proposition 1** If conditions (1), (3) hold, then the linearised RTS operator $L$ is self-adjoint and elliptic.

We note that a similar analysis can be made of the more complicated reparameterisations of the constraint equations proposed by Komar and Bergmann [7], see [4].

To apply the implicit function theorem, we need to show that $L$ is surjective, or equivalently, that the null space of $L$ is trivial.

**Proposition 2** Suppose $X^i$ is a solution of (3) with data $\Psi$ such that conditions (1), (3) are satisfied. Then $M^i$ is in the null space of $L$ iff there is a function $\mu$ such that $(M^i, \mu)$ satisfy (17).

**Proof:** Integrating $M^iL_i(M^j)$ and dropping divergence terms gives
\[
\int_\mathcal{M} \frac{1}{N} \left( M_{(ij)}M^{ij} - (M^i)_{(i)}^2 + \frac{(\pi^{ij}M_{ij})^2}{2\epsilon - R_g} \right) dv = 0.
\]
Using the scalar constraint (3), we can rewrite this integral in the homogeneous form,
\[
0 = \int_\mathcal{M} dv \frac{1}{N(2\epsilon - R_g)} \left\{ (\pi^{ij}M_{ij})^2 + \left( \frac{1}{2}(\pi^k)^2 - \pi^{kl}\pi_{kl} \right) (M_{(ij)}M^{ij} - (M^i)_{(i)}^2) \right\}.
\]
Choosing an orthonormal frame in which the extrinsic curvature is diagonal, $K_{ij} = \text{diag}(\alpha, \beta, \gamma)$, we have $0 < 2\epsilon - R_g = 2(\alpha\beta + \alpha\gamma + \beta\gamma)$. A simple calculation shows that the expression in braces can be written...
Proposition 3

They showed a global form of the previous result [9]:

It is easily verified that $M$ function two solutions of the RTSE (7), then there is a positive and $\tilde{\alpha}$ where $\tilde{\alpha}$ satisfies 

$\varepsilon_{\alpha M}$ and satisfies $\gamma_{M}$ negative and zero iff $\gamma_{M}$ $L$ satisfy (10), then it is easily verified that $X_i$ also satisfies (10) and conversely, if $X_i$ satisfies (10) and (15) then $L(M') = 0.$

The equation (10) also arose in the work of Belasco and Ohanian [8], who considered the RTSE as the Euler-Lagrange equations of the action functional

$$I = -2 \int_{\mathcal{M}} \left( \sqrt{\left( \gamma_k^2 - \gamma^{kl}\gamma_{kl} \right)} (2\varepsilon - R_g) + S_i X^i \right) dv.$$ 

They showed a global form of the previous result [8]:

**Proposition 3** Suppose (15) is satisfied. If $X^i$, $\tilde{X}^i$ are two solutions of the RTSE (7), then there is a positive function $\alpha$ such that

$$\tilde{g}_{ij} = \alpha g_{ij},$$

where $\tilde{g}_{ij} = \frac{1}{2} g_{ij} - \tilde{X}_{(ij)}$. Conversely, if $X^i$ is a solution, and $\tilde{X}^i$ satisfies (15), then $\tilde{X}^i$ also satisfies (7).

It is easily verified that $M^i = \tilde{X}^i - X^i$ and $\mu = N(1 - \alpha)$ satisfy (16) and conversely, if $X^i$ solves (7) and $(\mu', \mu)$ solves (11) (and $\mu > N$) then $\tilde{X}^i = X^i + M^i$ also solves (7).

Since the spacetime vector $(\mu, -M^i)$ satisfies the spatial part of the Killing equation if (11) holds, by choosing the lapse/shift $(\mu, -M^i)$ we see that the underlying data $(g_{ij}, K_{ij})$ admits a representation with $g_{ij} = 0$ (but this choice of lapse will not satisfy (7) in general).

To formally state our main result, let $H^n(T^i_j)$ denote the set of tensors of valence $(i, j)$ belonging to the Sobolev space $H^n = W^{n,2}$. The operator $F(\Psi, X^i)$ defines a smooth mapping

$$F : \mathcal{H}_n \times H^{n+2}(T^1_0) \to H^n(T^1_0).$$

(16)

$$\mathcal{H}_n = \bigtimes_{n \geq 2} \left( H^n(T^1_0) \times H^{n+1}(T^1_0) \times H^{n+1}(T^1_0) \times H^{n+1}(T^1_0) \right),$$

for any $n \geq 2$, assuming that $(\Psi, X^i)$ satisfies (7), (8), and if furthermore $H^n(T^1_0)$ has only the zero solution, then there is an open neighbourhood of $\Psi \in \mathcal{H}_n$ such that the RTSE (7) is uniquely solvable for any choice of data $\Psi$ in this neighbourhood.
There is a natural way of generating \((\Psi, N, X^i)\) satisfying the Thin Sandwich Equations. Any choice of spacelike hypersurface in an spacetime satisfying the Einstein equations induces a solution \((g_{ij}, K_{ij}, \epsilon, S^i)\) of the constraint equations. Then from
\[
\dot{g}_{ij} = 2NK_{ij} + 2X_{(i|j)},
\]
we see that an arbitrary choice of shift vector \(X^i\) and lapse function \(N\) fixes the time derivative of the spacelike metric \(\dot{g}_{ij}\) along the time flow vector field given by this shift and lapse. If \(N\) is chosen positive and if \(2\epsilon - R_g > 0\), then \(X^i\) is a solution of (7). In view of this we can rewrite our main theorem in a more natural form.

**Theorem 3** Suppose \(g_{ij} \in H^{n+2}(T_0^0), K_{ij} \in H^{n+1}(T_0^0),\) \(\epsilon \in H^{n+1}\) and \(S^i \in H^n(T_1^0)\) satisfy the constraint equations (3) and (4), and \(2\epsilon - R_g > 0\), where \(R_g\) is the scalar curvature of \(g_{ij}\). For any choice of shift vector \(X^i \in H^{n+2}(T_1^0)\) and positive lapse function \(N \in H^{n+1}\) we define \(\dot{g}_{ij}\) by (17). If the equation \(M_{(ij)} = \mu K_{ij}\) has only the trivial solution, then there exists a unique continuous map on an open neighbourhood of \(\Psi = (g_{kl}, \dot{g}_{kl}, \epsilon, S^k)\) in \(\mathcal{H}_n\), which assigns \(\bar{X}^k \in H^{n+2}(T_0^0)\) to \(\bar{\Psi}\) such that \(\bar{X}^k\) is a solution of the reduced thin-sandwich equations with data \(\Psi\).

Finally we show that there exist reference solutions \((\Psi, N, X^i)\) satisfying the conditions of these theorems. If \((\mu, M^i)\) satisfies (14) then
\[
M_{ij}^j + R_{ij}M^j - 2(\mu K_{ij})^j + (\mu K^j)_{ij} = 0,
\]
where \(R_{ij}\) is the Ricci tensor of \(g_{ij}\). Multiplying by \(M^i\), integrating over \(\mathcal{M}\), and substituting (14) and the energy constraint (3) gives
\[
\int_M \left\{-M^{i[j}M_{[i|j]} + M^iM^jR_{ij} - \mu^2(2\epsilon - R_g)\right\} dv = 0.
\]
If \(R_{ij}\) is negative definite, this implies \((\mu, M^i) \equiv (0, 0)\) and thus (14) has only the trivial solution. Since the \(k = -1\) Robertson-Walker spacetimes have spatial slices with \(R_{ij} = -2g_{ij}\), we conclude

**Corollary 1** The Thin Sandwich Equations are uniquely solvable for data \(\Psi\) in an \(\mathcal{H}_n\)-neighbourhood of the spatially compactified \(k = -1\) vacuum Robertson-Walker data \(\Psi_0\), for which \((M, \dot{g}_{ij})\) is a compact hyperbolic 3-manifold, \(\dot{g}_{ij} = 2g_{ij}, \epsilon = 0, S^i = 0,\) and we use the reference solution \(\bar{X}^i = 0, N = 1\).

A similar conclusion obtains for any initial data set \((g_{ij}, K_{ij})\) satisfying \(R_{ij} < 0\) and \(\epsilon \geq 0\), provided that the lapse \(N\) is chosen strictly positive.

**DISCUSSION**

We have shown that the Thin Sandwich Equations are solvable provided certain geometric conditions (3),
and a technical condition, related to the absence of Killing vectors, are satisfied. However, since the linearised equations are not elliptic if $\pi_{ij}$ is indefinite, we do not expect that the TSE will be well-posed in general, even if $N > 0$. For this reason, the underlying “super-space” metaphor must be considered as mathematically inadequate for the description of the general Einstein evolution. However, we note that on the basis of the results presented here, it is plausible that a suitably restricted form of the of the full (thick) sandwich conjecture may be valid.

It is useful to relate our thin sandwich results to the rather satisfactory ADM formulation of the Einstein equations. The constraint set $\mathcal{C} = \mathcal{C}(\epsilon, S^i)$, consisting of data $(g_{ij}, K_{ij})$ satisfying $\mathbf{5}$, $\mathbf{6}$, forms a natural (and general) phase space, but unlike $TS$, $\mathcal{C}$ is not easily parameterised. We can consider the solution of the TSE as defining a projection from $(g, \dot{g})$ to $(g, K) \in \mathcal{C}$, with the fibres of the projection consisting of classes of data $(g, \dot{g})$ which are equivalent under change of lapse and shift. This projection is well-defined on a region in $TS$ described by the two conditions $\mathbf{8}$, which is an implicit condition on $(g, \dot{g})$, and $\mathbf{9}$, which defines the image region in $\mathcal{C}$ of the projection, provided also that the “Killing” condition $\mathbf{10}$ holds. Thus, solving the TSE provides a parameterisation of that subset of the full constraint manifold $\mathcal{C}$ which is determined by the above restrictions.

The analysis by Arms, Fischer, Marsden and Moncrief $\mathbf{10}$ shows that the vacuum constraint manifold $\mathcal{C}$ (on a compact manifold $\mathcal{M}$), has a conical singularity when the resulting (vacuum) spacetime admits a Killing vector. As might have been expected, in this case $\mathbf{5}$ is not solvable in general, since $\mathbf{10}$ admits nontrivial solutions. However the converse is false, since the space of solutions of $\mathbf{5}$ is in general larger than the space of (projected) Killing vectors — indeed, if $K_{ij} \equiv 0$ then $\mu$ in $\mathbf{10}$ can be chosen arbitrarily, hence there is an infinite-dimensional space of solutions of $\mathbf{10}$. The correspondence between spacetime Killing vectors and singularities of the constraint manifold arises from the elegant relationship between the constraint equations and the ADM Hamiltonian formulation of the Einstein equations. The fact that solvability of the TSE requires the more restrictive condition that $\mathbf{10}$ have no non-trivial solutions, indicates that the thin sandwich equations (and, arguably, the underlying superspace approach) are not as well-adapted to the Einstein evolution.

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