SolutionsoftheGinzburg–Landauerquationsconcentratingoncodimension-2
minimalsubmanifolds

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Abstract
WeconsiderthemagneticGinzburg–Landauequationsonaclosedmanifold\(N\)
\[
\begin{cases}
-\varepsilon^2 \Delta A u = \frac{1}{2} (1-|u|^2)u, \\
\varepsilon^2 d^* dA = \langle \nabla^A u, iu \rangle,
\end{cases}
\]
formallycorrespondingtotheEuler–Lagrangeequationsfortheenergyfunctional
\[
E(u, A) = \frac{1}{2} \int_N |\nabla^A u|^2 + \varepsilon^2 |dA|^2 + \frac{1}{4\varepsilon^2} (1-|u|^2)^2,
\]
where\(u: N \to \mathbb{C}\)and\(A\)isa1-formon\(N\).Giventacodimension-2minimalsubmanifold\(M \subset N\),whichis
alsoorientedandnon-degenerate,weconstructsolutions\((u_\varepsilon, A_\varepsilon)\)suchthat\(u_\varepsilon\)hasazerosetconsistingofasmooth
surfacecloseto\(M\).Awayfrom\(M\),wehave
\[
u_1(y) + z_2 v_2(y))
\]
as\(\varepsilon \to 0\),forallsufficientlysmall\(z \neq 0\)andy\(\in M\).Here,
\({v_1, v_2}\)isnormalframeto\(M\)in\(N\).Thisimprovesa
recent result by De Philippis and Pigati (2022), who built a solution for which the concentration phenomenon holds in an energy, measure-theoretical sense.

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1 INTRODUCTION

Let $n \geq 3$ and consider, for $\varepsilon > 0$, the magnetic Ginzburg–Landau energy on a smooth, closed (compact and without boundary) $n$-dimensional Riemannian manifold $(N, g)$,

$$E(u, A) = \frac{1}{2} \int_N |\nabla^A u|^2 + \varepsilon^2 |dA|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2,$$

where $u : N \to \mathbb{C}$ is a complex valued function and $A \in \Omega^1(N)$ is the magnetic potential. Here $d$ denotes the exterior differential and $\nabla^A := d - iA$ is the covariant derivative. Explicitly, if \{x^1, ..., x^n\} are local coordinates of some open set on $N$, then we write $A = A_j dx^j$ and

$$|\nabla^A u|^2 = \sum_{j,k=1}^n g^{jk}(\partial_j u - iA_j u)(\partial_k \bar{u} + iA_k \bar{u}),$$

$$|dA|^2 = \frac{1}{2} \sum_{j,k,s,t=1}^n g^{ks} g^{jl} (\partial_k A_j - \partial_j A_k)(\partial_s A_t - \partial_t A_s).$$

The corresponding Euler–Lagrange system is given by

$$\begin{cases}
-\varepsilon^2 \Delta^A u = \frac{1}{2} (1 - |u|^2) u, \\
\varepsilon^2 d^* dA = \langle \nabla^A u, iu \rangle
\end{cases} \quad \text{on } N,
$$

which can be also written as $S(u, A) = 0$, where $S$ is the Fréchet derivative of the energy. In (1.2), $\langle \cdot, \cdot \rangle$ is the usual inner product between complex numbers, so that $\langle \nabla^A u, iu \rangle$ is the 1-form $\langle \partial_j^A u, iu \rangle dx^j$. We also denoted with $-\Delta^A = (\nabla^A)^* \nabla^A$ the connection Laplacian, being $(\nabla^A)^* = d^* + iA$ the adjoint operator to $\nabla^A$. The operators in (1.2) explicitly read

$$-\Delta^A u = -\frac{1}{\sqrt{\det g}} (\partial_j - iA_j) \left[ \sqrt{\det g} g^{jk}(\partial_k - iA_k) u \right],$$

$$d^* dA = -\frac{1}{\sqrt{\det g}} g_{jk} \partial_i \left( \sqrt{\det g} g^{li} g^{jk}(\partial_l A_i - \partial_i A_l) \right) dx^j.$$ 

Energy (1.1) appears in the classical theory of superconductivity by Ginzburg and Landau. In [30], Taubes found solutions of (1.2), with $\varepsilon = 1$, in the planar case $\mathbb{R}^2$ with isolated zeros (vortices).
of \( u \), and in particular in [4] one of these solution is proven to have a degree 1 radial symmetry, namely \( U_0 = (u_0, A_0) \) where

\[
    u_0(\xi) = f(r)e^{i\theta}, \quad A_0(\xi) = a(r)d\theta, \quad \xi = re^{i\theta}.
\]  

The functions \( f(r) \) and \( a(r) \) are positive solutions to the system of ordinary differential equations

\[
    \begin{cases}
        -f'' - \frac{f'}{r} + \frac{(1-a)^2 f}{r^2} - \frac{1}{2}f(1-f^2) = 0, \\
        -a'' + \frac{a'}{r} - f^2(1-a) = 0
    \end{cases}
\]

in \((0, +\infty)\) (1.4)

with \( f(0) = a(0) = 0 \). This is the unique solution of (1.2) with \( \varepsilon = 1 \) and exactly one zero with degree 1 at the origin. Also, \( U_0 \) is linearly stable as established in [19, 29]. In addition,

\[
    f(r) - 1 = O(e^{-r}), \quad a(r) - 1 = O(e^{-r}), \quad \text{as } r \to \infty,
\]

see [4, 27]. Remarkably, (1.4) is equivalent to a system of first order differential equations

\[
    \begin{cases}
        f' = \frac{(1-a)f}{r} \\
        a' = \frac{r}{2}(1-f^2)
    \end{cases}
\]

This is a consequence of a phenomenon called self-duality [6, 31], which holds for energy (1.1) but in general it fails to hold for other choices of double-well type potentials, see [26, Remark 1.2].

In this paper, we look for solutions \( U_\varepsilon = (u_\varepsilon, A_\varepsilon) \) of (1.2) concentrating in the limit \( \varepsilon \to 0 \) around a given codimension-2 minimal submanifold \( M \subset N \), in the form of \( \varepsilon \)-scalings of (1.3). More precisely, suppose the existence of a global frame \( \{\nu_1, \nu_2\} \) on \( M \) and normal to \( M \), namely an orthonormal basis for \( T_{\perp}M \). We describe a tubular neighbourhood of \( M \) in \( N \) by local Fermi coordinates

\[
    x = X(y, z) = \exp_y(z_1\nu_1(y) + z_2\nu_2(y)), \quad y \in M, \quad \|z\| < \tau
\]

for some \( \tau > 0 \) sufficiently small. Then, the sought solution \( U_\varepsilon(x) = (u_\varepsilon(x), A_\varepsilon(x)) \) behaves as

\[
    u_\varepsilon(x) \approx f\left(\frac{z}{\varepsilon}\right) \frac{z}{|z|}, \quad A_\varepsilon(x) \approx a\left(\frac{z}{\varepsilon}\right) \frac{1}{|z|^2}(-z_2dz_1 + z_1dz_2)
\]

and in particular, it satisfies

\[
    u_\varepsilon(x) \to \frac{z}{|z|}, \quad A_\varepsilon(x) \to \frac{1}{|z|^2}(-z_2dz_1 + z_1dz_2), \quad \varepsilon \to 0.
\]

We consider a closed, \( n \)-dimensional manifold \( N^n \) and a closed \((n-2)\)-dimensional minimal submanifold \( M^{n-2} \subset N^n \). We require a topological condition on \( M \), adapted from [8]: we say that \( M \subset N \) is admissible if

(H) \( M \) is the boundary of a \((n-1)\)-dimensional, oriented, embedded submanifold \( B^{n-1} \subset N^n \).
Admissibility guarantees the existence of a normal frame \( \{ \nu_1, \nu_2 \} \) in a way that will be made precise below. Finally, we require that \( M \) is non-degenerate, in the sense that the Jacobi operator has trivial bounded kernel, namely,

\[
h \in L^\infty(M), \quad J[h] = 0 \Rightarrow h = 0,
\]

where \( J \) is the second variation of the area functional around \( M \), whose components are given below in (2.8). Now we state the main result.

**Theorem 1.** Let \((N, g)\) be a closed, \( n \)-dimensional Riemannian manifold and let \( M \subset N \) be a complete, embedded, admissible and non-degenerate codimension-2 minimal submanifold of \( N \). Then, for every \( \sigma \in (0, 1) \) there exist \( \varepsilon_0, \delta > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) there exists a solution \((u_\varepsilon, A_\varepsilon)\) to (1.2), which as \( \varepsilon \to 0 \) satisfies

\[
\begin{align*}
u_\varepsilon(x) &= u_0 \left( \frac{z - \varepsilon^2 h_0(y)}{\varepsilon} \right) + O\left( \varepsilon^2 e^{-\frac{\|z\|}{\varepsilon}} \right), \\
A_\varepsilon(x) &= A_0 \left( \frac{z - \varepsilon^2 h_0(y)}{\varepsilon} \right) + O\left( \varepsilon^2 e^{-\frac{\|z\|}{\varepsilon}} \right),
\end{align*}
\]

for all points \( x = X(y, z) \) of the form (1.6) and where \( h_0 \) is a smooth function on \( M \), explicitly characterised in (2.21). Moreover, \( |u_\varepsilon| \to 1 \) uniformly on compact subsets of \( N \setminus M \) and the zero set of \( u_\varepsilon \) consists of a smooth codimension-2 manifold, \( \varepsilon^2 \)-close to \( M \).

In the interesting paper [11], De Philippis and Pigati consider a setting similar to that of Theorem 1 and construct a solution exhibiting energy concentration near \( M \). Their approach, variational in nature, does not yield the refined local description (1.8) nor the manifold character of the zero set, but the same technique successfully resolves the challenging case of Ginzburg–Landau equations without magnetic field, to which our techniques do not extend directly.

Our strategy relies on the construction of an accurate approximation to a solution, the use of linearisation and the so-called inner–outer gluing method to set up a suitable fixed-point formulation. We have recently used this approach in a similar context in \( \mathbb{R}^4 \) obtaining concentration for a special class of non-compact minimal surfaces embedded in \( \mathbb{R}^3 \) that includes a catenoid and the Costa–Hoffman–Meeks minimal surfaces [2]. Liu, Ma, Wei and Wu [22] have obtained the existence of a solution in \( \mathbb{R}^4 \) with precise asymptotics and concentration on a special non-compact codimension-2 symmetric minimal manifolds discovered by Arezzo and Pacard [1]. See also [10] for a recent construction in \( \mathbb{R}^3 \) of interacting helicoidal vortex filaments in Ginzburg–Landau without magnetic field.

In the related Allen–Cahn equation, analogues of these constructions via gluing methods appear in [12–14, 25]. In the context of the Ginzburg–Landau equation, with or without magnetic fields related variational constructions have been performed in [5, 8, 9, 20, 21, 24, 28].

Theorem 1 relates to a program of construction of codimension-2 minimal submanifolds as limits of critical points of scaling energies Ginzburg-Landau type, see Pigati–Stern [26]. This program has already been successful in the codimension-1 case, see [3, 7, 16, 18] where the Allen–Cahn energy is used as an alternative to the Almgren–Pitts min–max theory to construct minimal surfaces.
We now recall some facts that will be relevant in the rest of the paper. As already mentioned, we work with pairs $W = (u, A)$ given by a complex-valued function $u$ and a 1-form $A$ defined on $N$. We define an $\varepsilon$-dependent $L^2$-inner product on these pairs by setting

$$
\langle W_1, W_2 \rangle = \int_N \left( \frac{u_1}{A_1} \right) \cdot \left( \frac{u_2}{A_2} \right)
$$

$$
:= \int_N \langle u_1, u_2 \rangle + \varepsilon^2 A_1 \cdot A_2,
$$

where $\langle u_1, u_2 \rangle := \text{Re}(u_1 \bar{u}_2)$ and $A_1 \cdot A_2 := g^{ij}(A_1)_i(A_2)_j$, being $g$ the metric on $N$. Energy (1.1) is invariant under $U(1)$-gauge transformations $G_\gamma$, given by

$$
G_\gamma \left( \frac{u}{A} \right) := \left( \frac{ue^{i\gamma}}{A + d\gamma} \right),
$$

for any smooth, real-valued function $\gamma$ on $N$. We remark that by gauge-transforming the trivial global minimiser of the energy (1,0) we obtain a family of solutions, called pure gauges, of the form $(e^{i\gamma}, d\gamma)$. Later, the following formulation will be useful: by defining $\psi := e^{i\gamma}$, we can rewrite the pure gauge as

$$
\left( \frac{\psi}{d\gamma} \right).
$$

Indeed, it holds that $(i\psi)^{-1}d\psi = -ie^{-i\gamma}ie^{i\gamma}d\gamma = d\gamma$. So, formally, (1.9) is a pure gauge, provided that $\psi$ is a smooth $S^1$-valued map.

Gauge invariance creates an infinite dimensional part of the kernel of the linearised operator $S'(W)$ around any solution $W$ to $S(W) = 0$, generated by

$$
\Theta_W[\gamma] = \left( \frac{iu\gamma}{d\gamma} \right)
$$

and where again $\gamma$ varies among all smooth functions defined on $N$, see [2]. By computing the formal adjoint of $\Theta_W$ we can characterise orthogonality to all elements in the form (1.10): if $\Phi = (\phi, \omega)$

$$
\int \Phi \cdot \Theta_W[\gamma] = \int \left( \frac{\phi}{\omega} \right) \cdot \left( \frac{iu\gamma}{d\gamma} \right)
$$

$$
= \int \langle \phi, iu\gamma \rangle + \varepsilon^2 \omega \cdot d\gamma
$$

$$
= \int \gamma [\varepsilon^2 d^* \omega + \langle \phi, iu \rangle],
$$

so that

$$
\Phi \perp \Theta_W[\gamma], \forall \gamma \iff \Theta_W[\Phi] := \varepsilon^2 d^* \omega + \langle \phi, iu \rangle = 0.
$$
With this notation, we introduce the gauge-corrected linearised

\[ L_W := S'(W) + \Theta_W \Theta_W^* \]  

(1.11)
given by

\[ L_W [\Phi] = \begin{pmatrix} -\varepsilon^2 \Delta^A \phi - \frac{1}{2} (1 - 3|u|^2) \phi + 2i\varepsilon^2 V^A u \cdot \omega \\ -\varepsilon^2 \Delta \omega + |u|^2 \omega - 2 \langle V^A u, i\phi \rangle \end{pmatrix}, \]  

(1.12)

where \(-\Delta \omega = (d^* d + dd^*) \omega\) the Hodge Laplacian on 1-forms. Observe that, in contrast to \(S'(W)\), \(L_W\) is an elliptic operator. The correction in (1.11) is reminiscent of the DeTurck trick [15] used in the prescribed Ricci curvature problem.

The natural space in which \(L_W\) is defined and continuous is the space \(H^1(W)(N)\) of pairs \(\Phi = (\phi, \omega)\) such that

\[ \|\Phi\|_{H^1(W)(N)} := \|\nabla^A \phi\|_{L^2(N)} + \|\phi\|_{L^2(N)} + \|\nabla \omega\|_{L^2(N)} + \|\omega\|_{L^2(N)} < \infty, \]  

(1.13)

where \(\nabla \omega\) is the Levi–Civita connection applied to the 1-form \(\omega\).

Finally, we introduce some notation that will allow us to express the operator \(L_W\) in compact form. Define, for \(W = (u, A)\), the gradient-like operator

\[ \nabla_W \begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} \nabla^A \phi \\ d\omega + d^* \omega \end{pmatrix}, \]  

(1.14)

and the corresponding Laplacian-like operator \(-\Delta_W = \nabla^*_W \nabla_W\), which explicitly reads

\[ -\Delta_W \begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} -\Delta^A \phi \\ -\Delta \omega \end{pmatrix}. \]

Using this notation, we can rewrite (1.12) as

\[ L_W [\Phi] = -\varepsilon^2 \Delta_W \Phi + \Phi + T_W \Phi, \]  

(1.15)

where

\[ T_W \begin{pmatrix} \phi \\ \omega \end{pmatrix} := \begin{pmatrix} -\frac{3}{2} (1 - |u|^2) \phi + 2i\varepsilon^2 V^A u \cdot \omega \\ -(1 - |u|^2) \omega - 2 \langle V^A u, i\phi \rangle \end{pmatrix}. \]

The rationale behind this expression is that it highlights the behaviour of \(L_W\) in regions where \(W\) is a low energy configuration: if \(|u| \approx 1\) and \(V^A u \approx 0\), then \(T_W \approx 0\) and \(L_W \approx -\varepsilon^2 \Delta + \text{Id}\).

Of fundamental importance for us is the case in which the space is the plane \(\mathbb{R}^2\). There is a rich literature devoted to this problem, see, for instance, [4, 19, 27, 29–31]. We will denote with \(L\) the gauge-corrected linearised around degree +1 solution \(U_0\) defined in (1.3),

\[ L := L_{U_0} = S'(U_0) + \Theta_{U_0} \Theta_{U_0}^*, \]  

(1.16)
where \( \varepsilon = 1 \). The solution \( U_0 \) is non-degenerate \([19]\), in the sense that the kernel of the (gauge-corrected) linearised around it is only given by the elements corresponding to natural invariances of the plane, namely the translations. More precisely,

\[
Z_{U_0} = \ker L = \text{span}\{V_1, V_2\}
\]

being \( V_\alpha := (\nabla U_0 U_0)(e_\alpha) \), explicitly given by

\[
V_1 = \begin{pmatrix} f' \\ \frac{a'}{r} dt^2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} i f' \\ -\frac{a'}{r} dt^1 \end{pmatrix},
\]

(1.17)

where \( f \) and \( a \) are the solutions to (1.4). As shown by Stuart \([29]\), the following coercivity estimate holds

\[
\langle L[\Phi], \Phi \rangle_{L^2} \geq c \|\Phi\|_{H^1_{U_0}}^2, \quad \forall \Phi \perp V_\alpha, \alpha = 1, 2
\]

(1.18)

for some \( c > 0 \), where \( \| \cdot \|_{H^1_{U_0}} \) is as in (1.13). This coercivity estimate along with Riez’s theorem implies the validity of the following existence result.

**Lemma 1.1.** There is a \( C > 0 \) such that for every \( \Psi \in L^2(\mathbb{R}^2) \) satisfying

\[
\int_{\mathbb{R}^2} \Psi(t) \cdot V_\alpha(t) dt = 0, \quad \alpha = 1, 2,
\]

(1.19)

there exists a unique solution \( \Phi \in H^1_{U_0}(\mathbb{R}^2) \) to

\[
L[\Phi] = \Psi
\]

satisfying

\[
\int_{\mathbb{R}^2} \Phi(t) \cdot V_\alpha(t) dt = 0, \quad \alpha = 1, 2,
\]

and such that

\[
\|\Phi\|_{H^1_{U_0}(\mathbb{R}^2)} \leq C \|\Psi\|_{L^2(\mathbb{R}^2)}.
\]

2 | THE FIRST APPROXIMATION

The setting of Theorem 1 allows us to choose in a canonical way a basis for the normal bundle \( T^\perp M \) for the immersion \( M \subset N \). Recall that, by the admissibility hypothesis (H), \( M = \partial B \) where \( B \) is oriented. Let \( \nu_2 \) be the normal field of the immersion \( B \subset N \) (which exists because of the orientability assumption) and let \( \nu_1 \) the vector field of \( TB \) restricted to \( M \) that is normal to \( TM \). In this way, \( \{\nu_1, \nu_2\} \) is a basis of \( T^\perp M \) (see Figure 1). In particular, given a basis \( \{e_1, ..., e_{n-2}\} \) for \( TM \) then \( \{e_1, ..., e_{n-2}, \nu_1, \nu_2\} \) is a basis of \( TN \) defined on \( M \).
In what follows, we use Einstein’s summation convention on repeated indices; letters $i, j, k, \ldots$ are used for tangential coordinates to the manifold $M$, while $\alpha, \beta, \gamma, \ldots$ denote coordinates in the normal directions. We use $a, b, c \ldots$ to indicate all coordinates at once, so in particular,

$$1 \leq i, j, k, \ldots \leq n - 2, \quad 1 \leq \alpha, \beta, \gamma, \ldots \leq 2, \quad 1 \leq a, b, c \ldots \leq n.$$ 

We describe a tubular neighbourhood $\mathcal{T}$ of $M$ in $N$ with coordinates defined by the exponential map,

$$x = X(y, z) := \exp_{y}(z^\beta \nu_{\beta}(y)), \quad (y, z) \in M \times B(0, \tau),$$

(2.1)

where $B(0, \tau) \subset \mathbb{R}^2$ and $\tau$ is sufficiently small (see Figure 2). We consider the change of coordinates

$$z = \varepsilon(t + h(y))$$

(2.2)

for some pair $h = (h^1, h^2)$ of functions defined on $M$. This produces new coordinates

$$x = X_h(y, t) := \exp_{y} \left( \varepsilon(t^\beta + h^\beta(y))\nu_{\beta}(y) \right)$$

(2.3)
defined on
\[ \mathcal{O}_h = \{(y, t) \mid y \in M, \|t + h(y)\| < \tau / \varepsilon\} \]
and we set \( \mathcal{T} := X_h(\mathcal{O}_h) \). With this choice of coordinates, we define the first local approximation \( W_0 \) to a solution of Problem (1.2) as
\[ W_0(x) = U_0(t), \quad x = X_h(y, t) \tag{2.4} \]
with \( U_0(t) = (u_0, A_0) \) defined in (1.3). The pair \( W_0 \) will describe the main order of the concentration profile around \( M \). The parameter \( h = h(\varepsilon) \) identifies the nodal set of the solution \( u_\varepsilon \) to be constructed (as a normal graph over \( M \)) and serves as an extra degree of freedom that will be fundamental in the analysis.

Below, we shall estimate the error of approximation \( S(W_0) \) in \( \mathcal{T} \), where
\[ S(u, A) = \left( \begin{array}{c} -\varepsilon^2 \Delta A u - \frac{1}{2}(1 - |u|^2)u \\ \varepsilon^2 d^* dA - \langle \nabla A u, iu \rangle \end{array} \right). \tag{2.5} \]

To do so, we make explicit our assumption on the size of \( h \). First, we set in a standard way, for \( 0 < \gamma < 1 \),
\[ \|\varphi\|_{C^0, \gamma(\Omega)} := \|\varphi\|_{L^\infty(\Omega)} + \sup_{p \in \Omega} [\varphi]_{\gamma, B(p, 1)}, \]
where \( \varphi \) is a function defined on a manifold \( \Omega \), \( B(p, 1) \) is the geodesic ball of radius 1 around \( p \in \Omega \), and
\[ [\varphi]_{\gamma, X} := \sup_{x, y \in X, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{\gamma}}. \]

We also set, for \( k \geq 1 \),
\[ \|\varphi\|_{C^k, \gamma(\Omega)} := \|\varphi\|_{C^0, \gamma(\Omega)} + \|D^k \varphi\|_{C^0, \gamma(\Omega)}. \]

With this notation, we state our hypothesis on \( h \): we assume that for some fixed \( K > 0 \)
\[ \|h\|_{C^{2, \gamma}(M)} \leq K\varepsilon. \tag{2.6} \]

**2.1 The operators in coordinates**

It is convenient to express the operators \( -\Delta_N^A \) and \( d^* d \) in coordinates \((y, t)\), and to do so we first express them in coordinates \((y, z)\) in (2.1). Recall the expression for the connection Laplacian relative to the 1-form \( \omega \in \Omega^1(N) \)
\[ -\Delta_N^\omega = \frac{1}{\sqrt{|g|}} \partial_a^\omega \left( \sqrt{|g|} g^{ab} \partial_b^\omega \right), \]
where $\bar{\omega}_a = \delta_a - i \omega_a$. The volume form can be expanded in Fermi coordinates (see [17])

$$
\sqrt{|g|} = 1 - \langle H, \nu_\beta \rangle z^\beta - \frac{1}{2} \Omega_{\beta \gamma} z^\beta z^\gamma - \frac{1}{6} \Omega_{\alpha \beta \gamma} z^\alpha z^\beta z^\gamma + \mathcal{O}(|z|^4)
$$

$$
= 1 - \frac{1}{2} \Omega_{\beta \gamma} z^\beta z^\gamma - \frac{1}{6} \Omega_{\alpha \beta \gamma} z^\alpha z^\beta z^\gamma + \mathcal{O}(|z|^4),
$$

where $H$ is the mean curvature vector of $M$, which vanishes by assumption. The functions $\Omega_{\beta \gamma}$ and $\Omega_{\alpha \beta \gamma}$ are defined on $M$ and are given by

$$
\Omega_{\beta \gamma} = R_{i \beta i \gamma} + \sqrt{\beta_{ij}} \sqrt{\gamma_{ij}} + \frac{1}{3} R_{\alpha \gamma \beta \delta} \delta(0) z^\gamma z^\delta + \mathcal{O}(|z|^4),
$$

and

$$
\Omega_{\alpha \beta \gamma} = \frac{2}{3} \nabla_\alpha R_{\beta \gamma} - \frac{2}{3} \nabla_\beta R_{\alpha \gamma} + 2 R_{\alpha \beta \gamma} j_{ij} A_{ijkl} + 2 A_{ij} A_{jkl} A_{i kl},
$$

where $R_{abcd}$ and $R_{ab}$ are the coordinates of the Riemann and Ricci curvature tensors, respectively, and $A$ is the second fundamental form, whose coefficients are given by

$$
A_{ij} := -\langle \delta_i \nu_\gamma, e_j \rangle.
$$

We remark that in the case $n = 3$, in which $M$ is a geodesic, the second fundamental form $A$ is equal to its trace $H$, thus $A \equiv 0$. Recall the convention we set on indices, so for instance the expression $R_{i \beta \gamma}$ denotes the trace of the Riemann tensor on the tangent bundle to $M$. Observe also that in $\Omega_{\beta \gamma}$ and $\Omega_{\alpha \beta \gamma}$ all instances of the Riemann or Ricci tensors’ coefficients are evaluated in $(y, 0)$, namely on the manifold $M$. Recall also the formula for the Jacobi operator acting on a normal vector field $h = h^\beta \nu_\beta$

$$
J^\nu[h] = \Delta_M h^\nu + R_{i \beta \gamma} h^\beta + A_{ij} A_{ijkl} h^\beta.
$$

The coefficients of the metric tensor can be expanded

$$
g_{ij}(y, z) = (g_z)_{ij}(y),
$$

$$
g_{i\alpha}(y, z) = g_{j\alpha}(y, z) = \mathcal{O}(|z|^2),
$$

$$
g_{\alpha \beta}(y, z) = \delta_{\alpha \beta} + \frac{1}{3} R_{\alpha \gamma \beta \delta}(y, 0) z^\gamma z^\delta + \mathcal{O}(|z|^3),
$$

where $g_z$ is a family of metrics on $M$ smoothly depending on $z = (z^1, z^2) \in B(0, \tau)$, see [23]. The inverse metric has the expansion

$$
g^{ij}(y, z) = g_z^{ij}(y) + \mathcal{O}(|z|^4),
$$

$$
g^{i\alpha}(y, z) = g_z^{i\alpha}(y, z) = \mathcal{O}(|z|^2),
$$

$$
g^{\alpha \beta}(y, z) = \delta_{\alpha \beta} - \frac{1}{3} R_{\alpha \gamma \beta \delta}(y, 0) z^\gamma z^\delta + \mathcal{O}(|z|^3).$$

(2.9)
Using (2.7) and (2.9), we can compute an expansion in $z$ for the connection Laplacian. For instance,

\[ |g|^{-1/2} \partial_i \left( |g|^{1/2} g^{ij} \partial_j \right) = \Delta_M + \mathcal{O}(|z|) \partial_{ij} + \mathcal{O}(|z|) \partial_j, \]

where we left implicit a sum over $i, j$ in the right-hand side. Similarly, we find

\[ |g|^{-1/2} \partial_i \left( |g|^{1/2} g^{i\beta} \partial_\beta \right) = \mathcal{O}(|z|) \partial_\beta + \mathcal{O}(|z|^2) \partial_\beta, \]

\[ |g|^{-1/2} \partial_\alpha \left( |g|^{1/2} g^{\alpha j} \partial_j \right) = \mathcal{O}(|z|) \partial_{\alpha j} + \mathcal{O}(|z|) \partial_j. \]

Finally,

\[ |g|^{-1/2} \partial_\alpha \left( |g|^{1/2} g^{\alpha \beta} \partial_\beta \right) = g^{\alpha \beta} \partial_{\alpha \beta} + \partial_\alpha g^{\alpha \beta} \partial_\beta + g^{\alpha \beta} \partial_\alpha \frac{\sqrt{|g|}}{|g|} \partial_\beta. \]

Using the symmetries of the curvature tensor, we expand

\[ \partial_\alpha g^{\alpha \beta} = -\frac{1}{3} R_{\alpha \alpha \beta \delta}(y, 0) z^\delta - \frac{1}{3} R_{\alpha \gamma \beta \alpha}(y, 0) z^\gamma + \mathcal{O}(|z|^2) \]

\[ = \frac{1}{3} R_{\alpha \beta \alpha \gamma}(y, 0) z^\gamma + \mathcal{O}(|z|^2) \]

and by (2.7)

\[ g^{\alpha \beta} \partial_\alpha \frac{\sqrt{|g|}}{|g|} = -\Omega_{\beta \gamma} z^\gamma - A^\delta_{ij} A^\gamma_{jk} A^\gamma_{ki} z^\gamma + \mathcal{O}(|z|^2). \]

Putting all pieces together, we find

\[ \Delta_N^{\omega} = \Delta_M^{\omega} + \mathcal{O}(|z|) \partial_{ij}^{\omega} + \mathcal{O}(|z|) \partial_j^{\omega} + \mathcal{O}(|z|^2) \partial_\beta^{\omega} + \partial_\beta^{\omega} \]

\[ + \frac{1}{3} R_{\alpha \gamma \beta \delta}(y, 0) z^\delta \partial_\omega_{\alpha \beta} - \left( \Omega_{\beta \gamma} z^\gamma - \frac{1}{3} R_{\alpha \beta \alpha \gamma}(y, 0) z^\gamma \right) \partial_\beta^{\omega} \]

\[ + \left( A^\delta_{ij} A^\gamma_{jk} A^\gamma_{ki} z^\gamma + \mathcal{O}(|z|^2) \right) \partial_\beta^{\omega}, \]

(2.10)

where we denoted $\partial_\alpha^{\omega}_{ab} := \partial_\alpha^{\omega}_a \partial_\alpha^{\omega}_b$. Observe that

\[ \Omega_{\beta \gamma} - \frac{1}{3} R_{\alpha \beta \alpha \gamma} = R_{\beta i\gamma} + A^\beta_{ij} A^\gamma_{ij} \quad \text{on } M. \]

(2.11)

Next, we express the operator $\varepsilon^2 \Delta_N$ in the variables $(y, t)$ given by (2.2), expanding in powers of $\varepsilon$ up to the third order, recalling also that $h$ and its derivatives are $\mathcal{O}(\varepsilon)$. For instance, it is direct to see that if $v(y, t) = u(y, z/\varepsilon - h(y))$

\[ \partial_{ij} v = \partial_{ij} u - \partial_i h^\beta \partial_j u - \partial_j h^\beta \partial_i u + \partial_i h^\gamma \partial_j h^\beta \partial_\gamma u - \partial_j h^\beta \partial_i u. \]
With similar calculations for the other derivatives, we obtain

\[
\varepsilon^2 \Delta_N = \varepsilon^2 \Delta_M - \varepsilon^2 \Delta_M h^\beta \partial_{\partial \beta} \varepsilon + O(\varepsilon^3) \partial_{ij} + O(\varepsilon^3) \partial_i + O(\varepsilon^3) \partial_i \\
+ \partial_{\partial \beta} + \frac{1}{3} \varepsilon^2 R_{\gamma \rho \delta}(y, 0)(t^\gamma + h^\gamma)(t^\rho + h^\rho) \partial_{\partial \alpha} + \varepsilon^2 (R_{i \beta \gamma} + A_{ij}^\beta A_{ij}^\gamma)(t^\gamma + h^\gamma) \partial_{\partial \beta} \\
+ \varepsilon^3 A_{ij}^\beta A_{jk}^\gamma A_{k\ell}^\delta (t^\gamma + h^\gamma)(t^\rho + h^\rho) + O(\varepsilon^3) \partial_{\partial \beta}.
\]

Thus, using Equation (1.2), we find

\[
\varepsilon^2 \Delta_N u - \frac{1}{2} (1 - |u|^2) u = \varepsilon^2 \left( \frac{1}{3} R_{\gamma \rho \delta}(y, 0) t^\gamma t^\rho A_{\partial \beta \alpha} dt^\sigma + (R_{i \beta \gamma} + A_{ij}^\beta A_{ij}^\gamma) t^\gamma A_{\partial \beta \alpha} dt^\alpha \right) \\
+ \varepsilon^2 J^{\beta} [h] A_{\partial \beta \alpha} dt^\alpha + \varepsilon^3 A_{ij}^\beta A_{jk}^\gamma A_{k\ell}^\delta t^\gamma A_{\partial \beta \alpha} dt^\alpha + O(\varepsilon^3).
\] (2.12)

Similar calculations can be carried out for the operator \(d^*d\), which acts on \(\omega = \omega d x^a \in \Omega^1(N)\) as

\[
d^* d\omega = - \frac{1}{\sqrt{|g|}} g_{ab} \partial_c \left( \sqrt{|g|} g^{de} g^{eb} \omega_{de} \right) dx^a,
\]

where \(\omega_{de} := \partial_d \omega_e - \partial_e \omega_d\). After expanding, using the change of variable (2.2) and (1.2), we find

\[
\varepsilon^2 d^* dA - \langle \nabla^A u, i u \rangle = \varepsilon^2 \left( \frac{1}{3} R_{\gamma \rho \delta}(y, 0) t^\gamma t^\rho \partial_{\partial \alpha} A_{\partial \beta \alpha} dt^\sigma + (R_{i \beta \gamma} + A_{ij}^\beta A_{ij}^\gamma) t^\gamma A_{\partial \beta \alpha} dt^\alpha \right) \\
+ \varepsilon^2 J^{\beta} [h] A_{\partial \beta \alpha} dt^\alpha + \varepsilon^3 A_{ij}^\beta A_{jk}^\gamma A_{k\ell}^\delta t^\gamma A_{\partial \beta \alpha} dt^\alpha + O(\varepsilon^3).
\] (2.13)

Putting together (2.12) and (2.13), we get

\[
S(W_0) = \varepsilon^2 \left( \frac{1}{3} R_{\gamma \rho \delta}(y, 0) t^\gamma t^\rho \nabla_{\partial \alpha \beta} U_0(t) + (R_{i \beta \gamma} + A_{ij}^\beta A_{ij}^\gamma) t^\gamma V_{\partial \beta}(t) \right) \\
+ \varepsilon^2 J^{\beta} [h] V_{\partial \beta}(t) + \varepsilon^3 A_{ij}^\beta A_{jk}^\gamma A_{k\ell}^\delta t^\gamma V_{\partial \beta}(t) + O(\varepsilon^3).
\] (2.14)

At this point, we can try to improve the approximation by setting \(W_1 = W_0 + \Lambda_1\) and observing that

\[
S(W_1) = S(W_0) + L[\Lambda_1] + (S'(W_0) - L)[\Lambda_1] + \mathcal{N}_0(\Lambda_1),
\]

where

\[
\mathcal{N}_0(\Lambda_1) = S(W_0 + \Lambda_1) - S(W_0) - S'(W_0)[\Lambda_1],
\]

where we recall that the 2-dimensional linearised operator \(L\), given by (1.16), is an operator in the \(t\)-variable only. The largest term of \(S(W_0)\), namely that of order \(\varepsilon^2\), is given by

\[
Q_2(\xi, t) = \varepsilon^2 \left( \frac{1}{3} R_{\gamma \rho \delta}(y, 0) t^\gamma t^\rho \nabla_{\partial \alpha \beta} U_0(t) + (R_{i \beta \gamma} + A_{ij}^\beta A_{ij}^\gamma) t^\gamma V_{\partial \beta}(t) \right).
\] (2.15)
Therefore, if we solve

\[ L[\Lambda_1] = -Q_2(y, t), \]

the biggest part of the error in terms of \( \varepsilon \) is cancelled. Such \( \Lambda_1 \) exists thanks to Lemma 1.1, using that

\[ \int_{\mathbb{R}^2} Q_2(y, t) \cdot \nabla \alpha(t) \, dt = 0, \quad \forall y \in M, \quad (2.16) \]

which follows from a direct calculation. Also, as the right-hand side (2.15) is \( \mathcal{O}(e^{-|t|}) \) for \( |t| \) large a standard barrier argument along with the fact that \( L \sim -\Delta + \text{Id} \) at infinity ensures that

\[ \sup_{t \in \mathbb{R}^2} e^{\sigma |t|} |Q_2(y, t)| < \infty \quad \forall y \in M \]

for any \( \sigma < 1 \). Finally, we observe that the error created

\[ \mathcal{E}(y, t) = (S'(W_0) - L)[\Lambda_1] + \mathcal{N}_0(\Lambda_1) \]

satisfies

\[ |\mathcal{E}(y, t)| \leq C\varepsilon^4 e^{-\sigma |t|}. \quad (2.17) \]

Indeed, it holds an equation of the form

\[ (S'(W_0) - L)[\Lambda_1] + \mathcal{N}_0(\Lambda_1) = \alpha^{ij} D_{ij} \Lambda_1 + \beta^j D_j \Lambda_1 + O(\varepsilon^4) \]

with \( \alpha^{ij}, \beta^j = O(\varepsilon^2) \) and hence, using the exponential decay in \( |t| \) and the fact that \( \Lambda_1 = O(\varepsilon^2) \), we check the validity of (2.17). Thus, with the above improvement for the first approximation the following estimate holds

\[ ||S(W_1)||_{C^{0, \gamma}} \leq C\varepsilon^3. \]

We can obtain a further improvement of the approximation by eliminating the \( \varepsilon^3 \)-terms in \( S(W_1) \). To do this, the non-degeneracy assumption (1.7) becomes crucial. Proceeding as before, we look for a bounded function \( \Lambda_2 \) such that

\[ L[\Lambda_2] = -Q_3(y, t), \quad (2.18) \]

where \( Q_3 \) are terms of size \( \varepsilon^3 \) in \( S(W_1) \), given by (recall assumption (2.6))

\[ Q_3(y, t) = \varepsilon^2 J_{[h]} [\nabla \rho(t)] + \varepsilon^3 A_{ij}^\delta A_{jk}^\gamma A_{kl}^\delta \gamma^\delta \nabla \rho(t) \]
and then set $W_2 = W_1 + \Lambda_2$. This can be done by virtue of Lemma 1.1 if the right-hand side satisfies the orthogonality condition (1.19), namely if the quantities

$$\int_{\mathbb{R}^2} Q_3(y, t) \cdot \nabla_\alpha(t) \, dt = c \varepsilon^2 J^\alpha(h)(y) + \varepsilon^3 q^\alpha(y) \quad (2.19)$$

vanish for $\alpha = 1, 2$, where $c = \int_{\mathbb{R}^2} |\nabla_\alpha(t)|^2 \, dt$ (independent of $\alpha = 1, 2$) and

$$q^\alpha(y) = A_{ij}^\beta A_{jk}^\gamma A_{kl}^\delta \int_{\mathbb{R}^2} t^\gamma t^\delta |\nabla_\alpha(t)|^2 \, dt. \quad (2.20)$$

Assumption (1.7) of non-degeneracy and Fredholm alternative for elliptic operators imply the following result.

**Lemma 2.1.** Let $f \in C^{0,\gamma}(M, \mathbb{R}^2)$. Then the system

$$J(h) = f \quad \text{on } M$$

admits a solution $h = H(f)$ satisfying

$$\|h\|_{C^{2,\gamma}(M)} \leq C \|f\|_{C^{0,\gamma}(M)}. \quad (2.21)$$

Lemma 2.1 guarantees the existence of a bounded $h_0(y)$ with

$$J(h_0) = -(q^1, q^2)^T, \quad \text{on } M. \quad (2.22)$$

Choosing $h = \varepsilon h_0$, the right-hand side of (2.19) vanishes for $\alpha = 1, 2$. Thus, (2.18) has a unique solution $\Lambda_2(y, t)$ with

$$\int_{\mathbb{R}^2} \Lambda_2(y, t) \cdot \nabla_\alpha(t) \, dt = 0.$$  

The role of the invertibility on the Jacobi operator $J$ is fairly transparent in the above argument, but as the operator $L$ does not include the differential operator in the $y$-variable, a linear theory for the full linearised problem needs to be made. This is the content of Proposition 3.1. The solvability conditions needed on the right-hand side only involve the $t$-variable and read very much as (1.19) or (2.16). The approximation $W_1(y, t)$ will actually be sufficient for our purposes. The function, $\Lambda_2(y, t)$ will be part of the expansion of the full solution.

The approximation obtained is only local and we need to extend it to the entire ambient manifold $N$, in a way that creates only very small errors terms. To do so, we use the following lemma, which holds thanks to the admissibility hypothesis (H) and will be proved in the Appendix.

**Lemma 2.2.** Let $M^{n-2}$ be an embedded, closed, oriented, smooth submanifold of a closed ambient manifold $N^n$ satisfying the admissibility hypothesis (H). Given Fermi coordinates $(y, t)$ defined on a sufficiently small tubular neighbourhood $T$ of $M$, let $\theta = \text{Arg}(t^1 + it^2)$ be the angle in the normal
frame. Then, the map $e^{i\theta}$ defined on $\mathcal{T}$ can be extended to the whole ambient space $N$ to a map $\psi : N \to \mathbb{S}^1$ that is smooth outside of $M$.

By such lemma, the map $e^{i\theta}$ defined in $\mathcal{T}$, where $\theta$ is the polar angle in coordinates $(t^1, t^2)$, can be extended to an $\mathbb{S}^1$-valued function $\psi$ defined on the whole ambient manifold $N$. Remark that we are applying Lemma 2.2 to $M_h := \{ \exp_y(h(\nu(y)) : y \in M \}$ rather then $M$. By (1.9), the pair

$$\Psi = \begin{pmatrix} \psi \\ \frac{d\psi}{i\psi} \end{pmatrix}$$

is a pure gauge, namely a solution to $S(\Psi) = 0$. The pure gauge $\Psi$ will be our ‘outer extension’. Let $\delta > 0$ and consider for every positive integer $m$ the cut-off functions defined by

$$\zeta_m(x) = \begin{cases} \zeta\left(\frac{\zeta}{\delta} |t + h(y)| - m \right) & \text{if } x = X_h(y, t) \in \mathcal{T} \\ 0 & \text{otherwise,} \end{cases} \quad (2.22)$$

where $\zeta$ is a smooth cut-off function such that $\zeta(s) = 1$ if $s < 1$ and $\zeta(s) = 0$ if $s > 2$. Define the global approximation $W$ to a solution of (1.2) as

$$W = \zeta_3 W_1 + (1 - \zeta_3)\Psi.$$

A lengthy but straightforward calculation similar to that of [2, section 5.5] shows that

$$S(W) = \zeta_3 S(W_1) + (1 - \zeta_3)S(\Psi) + E,$$

where

$$|E| \leq C \chi_{\{0 < \zeta_3 < 1\}}(1 + |W_1|^2)(|W_1 - \Psi| + |D(W_1 - \Psi)| + |\nabla^{A_1} u_1|).$$

Now, by construction of $\Psi$ and the properties of the function $f$ and $a$ introduced in (1.3), we have, for $\sigma \in (0, 1)$

$$|W_1 - \Psi| + |D(W_1 - \Psi)| + |\nabla^{A_1} u_1| \leq Ce^{-\sigma |t|} \quad \text{on } \{0 < \zeta_3 < 1\}.$$

On such set we have

$$e^{-\sigma |t|} \leq e^{-\frac{4\sigma \delta}{\epsilon}}$$

we obtain that $|E|$ is exponentially small in $\epsilon$. 
3 | PROOF OF MAIN RESULT

Starting from the global approximation just constructed we look for an actual solution as a small perturbation of \( W \), namely we want to solve

\[
S(W + \Phi) = 0
\]  

for some \( \Phi \) that is small in a suitable sense. Let us now define

\[
\mathcal{N}(\Phi) := S(W + \Phi) - S(W) - S'(W)[\Phi],
\]

where \( S' \) is the Fréchet derivative of \( S \). Equation (3.1) can be rewritten as

\[
0 = S'(W)[\Phi] + S(W) + \mathcal{N}(\Phi),
\]

which is in turn equivalent, using (1.11), to

\[
0 = L_W[\Phi] + S(W) + \mathcal{N}(\Phi) - \Theta_W \Theta^*_W[\Phi].
\]  

(3.2)

We claim that, if we find a solution \( \Phi \) to

\[
L_W[\Phi] + S(W) + \mathcal{N}(\Phi) = 0,
\]  

(3.3)

which is small with respect to \( W \), then also \( \Theta_W \Theta^*_W[\Phi] = 0 \) automatically and (3.2) holds. Indeed, let \( \gamma := -\Theta^*_W[\Phi] \); if \( \Phi \) solves (3.3) then

\[
S(W + \Phi) = \Theta_W[\gamma].
\]

Now consider a parameter \( s > 0 \) and the gauge transformation \( G_s \gamma \). By gauge invariance and after an integration by parts, we find

\[
0 = \frac{d}{ds} \bigg|_{s=0} E(G_s \gamma(W + \Phi))
= \int_N S(W + \Phi) \cdot \Theta_{W+\Phi}[\gamma]
= \int_N \Theta_W[\gamma] \cdot \Theta_{W+\Phi}[\gamma]
= \int_N \gamma \cdot [(-\varepsilon^2 \Delta + |u|^2 + \langle u, \varphi \rangle)\gamma],
\]

where \( u \) and \( \varphi \) are the first components of \( W \) and \( \Phi \), respectively. As \( \Phi \) is very small with respect to \( W \) (we will see that \( \Phi = \mathcal{O}(\varepsilon^3) \)), we find that \(-\varepsilon^2 \Delta + |u|^2 + \langle u, \varphi \rangle\) is a positive operator, thus \( \gamma = 0 \) and the claim is proved. Hence, the problem is reduced to finding a solution to (3.3). Rather than that, we initially solve a corrected version, given by

\[
L_W[\Phi] + S(W) + \mathcal{N}(\Phi) = \xi^2 b^2(\gamma)\mathcal{V}_\lambda(t).
\]  

(3.4)
The adjustment on the right-hand side provides unique solvability with estimates for the corresponding linear problem, with estimates, in terms of $\Phi$ for a precise choice of $b = (b_1, b_2)$, in the sense of the following result.

**Proposition 3.1.** Let $0 < \gamma < 1$ and let $\Lambda \in C^{0,\gamma}(N)$. Then there exists $b \in C^{0,\gamma}(M)$ and a unique solution $\Phi = \mathcal{G}(\Lambda)$ to

$$L_W[\Phi] = \Lambda + \zeta_2 b^2(y)\mathcal{V}_\alpha(t)$$

satisfying

$$\|\Phi\|_{C^{2,\gamma}(N)} + \|b\|_{C^{0,\gamma}(M)} \leq C \|\Lambda\|_{C^{0,\gamma}(N)}$$

for some $C > 0$.

Proposition 3.1 allows us to write (3.4) as a fixed-point problem

$$\Phi = -\mathcal{G}(S(W) + \mathcal{N}(\Phi))$$  \hspace{1cm} (3.5)

on the space

$$X_{K_0} := \{ \Phi \in C^{2,\gamma}(N) \mid \|\Phi\|_{C^{2,\gamma}(N)} \leq K_0 \varepsilon^3 \},$$

where $K_0 > 0$ is sufficiently large. Using the Lipschitz character of $N$ on $X_{K_0}$ (see section 6.1), namely

$$\|N(\Phi_1) - N(\Phi_2)\|_{C^{2,\gamma}(N)} \leq C \varepsilon^3 \|\Phi_1 - \Phi_2\|_{C^{2,\gamma}(N)}, \quad \Phi_1, \Phi_2 \in X_{K_0},$$

we find a solution $\Phi \in X_{K_0}$ of (3.5). We also highlight the Lipschitz dependence of $\Phi$ on $h$, namely

$$\|\Phi(h_1) - \Phi(h_2)\|_{C^{2,\gamma}(N)} \leq C \varepsilon^2 \|h_1 - h_2\|_{C^{2,\gamma}(M)},$$

which will be proved in Subsection 6.1. The only step left is to adjust $h$ in a way that the correction $\zeta_2 b^2 \mathcal{V}_\alpha$ vanishes. To do so, we multiply (3.4) by $\zeta_4 \mathcal{V}_\gamma(t)$, $\gamma = 1, 2$, and integrate on $\mathbb{R}^2$ to find an expression for $b^\gamma$,

$$b^\gamma(y) = \frac{1}{\int_{\mathbb{R}^2} \zeta_2 |V_\gamma|^2} \int_{\mathbb{R}^2} [\zeta_4 S(W) + \zeta_4 N(\Phi) + \zeta_4 L_W[\Phi]] \cdot V_\gamma.$$

By expression (2.14), we find that, also thanks to assumption (2.6), the biggest term of the error $S(W)$ within the support of $\zeta_4$ is $\varepsilon^2 J^\gamma[h]V_\gamma$, where $J^\gamma$ are the components of the Jacobi operator given by (2.8). Let us set

$$q_m(y) = \int_{\mathbb{R}^2} \zeta_m(y,t)|V_\alpha(t)|^2, \quad m = 1, 2, ...$$
and observe that the above quantity does not depend on $\alpha = 1, 2$. Let

$$B[\Phi] : = (L_W - L_{U_0})[\Phi] \quad \text{in supp } \xi_4.$$ 

We define the non-linear operator $G$ in the following way: for $\alpha = 1, 2$, let

$$G^1_\alpha (h) : = q_4 J^\alpha [h] - \varepsilon^{-2} \int_{\mathbb{R}^2} \xi_4 S(W_1) \cdot V_\alpha,$$

$$G^2_\alpha (h) : = -\varepsilon^{-2} \int_{\mathbb{R}^2} \xi_4 [N(\Phi) + B[\Phi]] \cdot V_\alpha,$$

$$G^3_\alpha (h) : = -\varepsilon^{-2} \int_{\mathbb{R}^2} \xi_4 L_{U_0} \cdot V_\alpha,$$

and set

$$G(h) = \left( G^1_1(h), G^2_1(h), G^3_1(h), G^2_2(h), G^3_2(h) \right) = q_4^{-1} \sum_{k=1}^{3} \left( G^k_1(h), G^k_2(h) \right).$$

With this notation, the system $b^{\gamma} = 0$, $\gamma = 1, 2$, reads

$$J(h) = G(h). \quad (3.6)$$

To solve (3.6), we use Lemma 2.1 and restate it as a fixed-point problem

$$h = H(G(h)). \quad (3.7)$$

To conclude the proof, we use the following lemma, which will be proved in Subsection 6.1.

**Lemma 3.1.** The map $G$ satisfies

$$\|G(0)\|_{C^{0,\gamma}(M)} \leq C\varepsilon$$

and

$$\|G(h_1) - G(h_2)\|_{C^{0,\gamma}(M)} \leq C\varepsilon \|h_1 - h_2\|_{C^{2,\gamma}(M)},$$

for some $C > 0$.

Thanks to Lemma 3.1, we see that by contraction mapping principle Equation (3.7) admits a solution in the space

$$X_\Lambda = \left\{ h : \|h\|_{C^{2,\gamma}(M)} \leq \Lambda\varepsilon \right\},$$

for any $\Lambda$ sufficiently big, which concludes the proof. Let us observe that, looking at the expansion of the terms of $G(h)$ in (3.6), we have $h(y) = \varepsilon h_0(y) + O(\varepsilon^2)$ where $h_0(y)$ is the unique solution of Equations (2.20)–(2.21).
Finally, we consider the zero-set of \( u_\varepsilon(x) \). The function \( u_\varepsilon \) does not vanish away from \( M \) as it is approaching an \( S^1 \)-valued map. Using (1.8), the equation \( u_\varepsilon(x) = 0 \) becomes near \( M \)

\[
u_0(\varepsilon^{-1}(z - \varepsilon h(y))) + \varepsilon^2 \hat{\vartheta}(\varepsilon^{-1}z, y) = 0,
\]

where \( \hat{\vartheta} \) is smooth, with uniformly bounded derivatives. The fact that \( \partial_z u_0(0) \neq 0 \) and the implicit function theorem yield that the nodal set is a smooth manifold parametrised as

\[ z = \varepsilon h(y) + O(\varepsilon^3), \quad y \in M. \]

The proof is complete. \( \square \)

4 \quad PROOF OF PROPOSITION 3.1

To prove Proposition 3.1, we will use the fact that on a region close to \( M \) the linearised operator \( L_W \) can be approximated by \( L_{U_0} \), namely the (scaled) linearised operator on \( M \times \mathbb{R}^2 \) around the building block \( U_0(y, t) := U_0(t) \), which in the scaled coordinates \( (y, t) = (y, z/\varepsilon) \) is given by

\[
L_{U_0}[\Phi] = -\Delta_{t, U_0} \Phi - \varepsilon^2 \Delta_M \Phi + \Phi + T_{U_0}(t) \Phi,
\]

being

\[
T_{U_0} = \begin{pmatrix}
-\frac{3}{2}(1 - |u_0|^2)\phi + 2i \nabla A_0 u_0 \cdot \omega \\
-(1 - |u_0|^2)\omega - 2\langle \nabla A_0 u_0, i\phi \rangle
\end{pmatrix}.
\]

First, we consider the cut-off functions introduced in (2.22) and look for a solution to

\[
L_W[\Phi] = -\varepsilon^2 \Delta_W \Phi + \Phi + T_W \Phi = \Lambda + \zeta^2 b^\alpha \nabla \alpha \quad \text{on } N
\]

of the form

\[ \Phi(x) = \zeta^2(x)\Phi(y, t) + \Psi(x), \]

where \( \Phi \) is defined on \( M \times \mathbb{R}^2 \) and \( \Psi \) is defined on \( N \). In what follows, it will be useful the following definition: for a real valued function \( f \) we set

\[
R_W[f, \Phi] = -\Delta_W(f\Phi) + f\Delta_W \Phi.
\]

More precisely, if \( \Phi = (\phi, \omega) \) and \( W = (u, A) \), we have

\[
R_W[f, \Phi] = \begin{pmatrix}
-2 \nabla A \cdot df - \phi \Delta f \\
d^\ast(df \wedge \omega) + d^\ast \omega df - d(df \cdot \omega) - \omega \Delta f
\end{pmatrix}
\]

(4.2)
and it holds
\[ |R_W[f, \Phi]| \leq C(|df| + |D^2f|)(|\Phi| + |\nabla W \Phi|). \]

Equation (4.1) becomes
\[
L_W[\Phi] = \zeta_2 L_{U_0}[\Phi] + L_W[\Psi] + \zeta_2 (L_W - L_{U_0})[\Phi] + \varepsilon^2 R_W[\zeta_2, \Phi]
= \Lambda + \zeta_2 b^2 \nabla \alpha
\]
and such equation is solved if the pair \((\Phi, \Psi)\) solves the system
\[
L_{U_0}[\Phi] + (L_W - L_{U_0})[\Phi] + \varepsilon^2 T_W \Psi = \Lambda + b^2 \nabla \alpha \quad \text{on supp } \zeta_2,
\]
\[
-\varepsilon^2 \Delta_W \Psi + \Psi + (1 - \zeta_1) T_W \Psi + \varepsilon^2 R_W[\zeta_2, \Phi] = (1 - \zeta_2) \Lambda \quad \text{on } N,
\]
where we used that \(\zeta_2 \zeta_1 = \zeta_1\). The following lemma will allow us to reduce the above system to an equation depending only on \(\Phi\).

**Lemma 4.1.** For \(\gamma \in (0, 1)\) and every \(\varepsilon > 0\) and \(\Gamma \in C^{0,\gamma}(N)\) there exists only one \(\Psi\) satisfying
\[
-\varepsilon^2 \Delta_W \Psi + \Psi + (1 - \zeta_1) T_W \Psi = \Gamma \quad \text{on } N
\]
and such that
\[
\|\Psi\|_{C^{2,\gamma}(N)} \leq C \|\Gamma\|_{C^{0,\gamma}(N)}.
\]

Lemma 4.1 follows from the positivity of the operator on the left-hand side on the space \(H^1_W(N)\) defined by the norm in (1.13), the maximum principle and elliptic estimates. We define the weighted norm \(C^{0,\gamma}_\sigma\) on functions \(\psi(y, t)\) defined on \(M \times \mathbb{R}^2\) as
\[
\|\psi\|_{C^{0,\gamma}_\sigma(M \times \mathbb{R}^2)} = \|e^{-\sigma |t|} \psi\|_{C^{0,\gamma}(M \times \mathbb{R}^2)},
\]
where \(k \geq 0\) and \(\gamma, \sigma \in (0, 1)\). Using the fact that on the region \(\{\zeta_2 < 1\}\), given also hypothesis (2.6), it holds \(|t| > \delta' / \varepsilon\) we find that
\[
e^{-\sigma |t|} \leq e^{-\frac{\sigma \delta'}{\varepsilon}} \quad \text{on } \{0 < \zeta_2 < 1\}
\]
and hence
\[
\|R_W[\zeta_2, \Phi]\|_{C^{0,\gamma}(N)} \leq C e^{-\frac{\sigma \delta'}{\varepsilon}} \|\Phi\|_{C^{0,\gamma}_\sigma(M \times \mathbb{R}^2)}.
\]

Using Lemma 4.1, we find a solution \(\Psi = \Psi_1 + \Psi_2\) to (4.4), where \(\Psi_1, \Psi_2\) satisfy
\[
-\varepsilon^2 \Delta_W \Psi_1 + \Psi_1 + (1 - \zeta_1) T_W \Psi_1 = -R_W[\zeta_2, \Phi],
-\varepsilon^2 \Delta_W \Psi_2 + \Psi_2 + (1 - \zeta_1) T_W \Psi_2 = (1 - \zeta_2) \Lambda
\]
on $N$. Also, $\Psi$ satisfies

$$\|\Psi\|_{C^{2,\gamma}(N)} \leq C \left( \| (1 - \xi_2) A \|_{C^{0,\gamma}(N)} + e^{-\sigma \frac{\gamma}{\tau}} \| \Phi \|_{C^{0,\gamma}_\sigma(M \times \mathbb{R}^2)} \right).$$

At this point, we reduce the whole system (4.3)-(4.4) to a single equation on $M \times \mathbb{R}^2$. We define

$$\mathcal{B}[\Phi] := \zeta_4 \mathcal{B}[\Phi] = \zeta_4 (L_W - L_{U_0})[\Phi], \quad \mathcal{A} = \zeta_4 \Lambda, \quad (y, t) \in M \times \mathbb{R}^2.$$

We claim that $\mathcal{B}$ satisfies

$$\| \mathcal{B}[\Phi] \|_{C^{0,\gamma}(M \times \mathbb{R}^2)} \leq C \delta \| \Phi \|_{C^{2,\gamma}(M \times \mathbb{R}^2)}, \quad (4.5)$$

where $\delta$ is the one from the definition of $\zeta_4$ in (2.22). For instance, we readily find from (2.10) that

$$|\zeta_4 \left[ \varepsilon^2 \Delta_N \Phi(y, t) - \varepsilon^2 \Delta_M \Phi(y, t) + \delta_{\alpha\alpha} \Phi(y, t) \right]| \leq C \delta \| \Phi \|_{C^{2}(M \times \mathbb{R}^2)},$$

and similarly for the other terms. With this notation, the system is equivalent to

$$L_{U_0}[\Phi] + \mathcal{B}[\Phi] + \zeta_1 T_W \Psi = \mathcal{A} + b \psi_{\alpha} \quad \text{on } M \times \mathbb{R}^2. \quad (4.6)$$

In Section 5, we will prove the following result.

**Proposition 4.1.** Let $\gamma \in (0, 1)$ and $\sigma > 0$ sufficiently small. Then for every $\Lambda \in C^{0,\gamma}_\sigma(M \times \mathbb{R}^2)$ there exists $b \in C^{0,\gamma}(M)$ such that the problem

$$L_{U_0}[\Phi] = \Lambda + b \psi_{\alpha} \quad \text{on } M \times \mathbb{R}^2$$

admits a unique solution $\Phi = \mathcal{P}(\Lambda)$ satisfying

$$\| \Phi \|_{C^{2,\gamma}(M \times \mathbb{R}^2)} + \| b \|_{C^{0,\gamma}(M)} \leq C \| \Lambda \|_{C^{0,\gamma}_\sigma(M \times \mathbb{R}^2)}$$

for some $C > 0$.

Using Proposition 4.1, we can rephrase (4.6) as

$$\Phi + G[\Phi] = H, \quad (4.7)$$

where

$$G[\Phi] = \mathcal{P}(\mathcal{B}[\Phi] + \zeta_1 T_W \Psi_1[\Phi])$$

$$H = \mathcal{P}(\Lambda - \zeta_1 T_W \Psi_2[\Lambda]).$$

We observe that, thanks to (4.5)

$$\| \mathcal{B}[\Phi] \|_{C^{0,\gamma}_\sigma(M \times \mathbb{R}^2)} \leq C \delta \| \Phi \|_{C^{2,\gamma}(M \times \mathbb{R}^2)},$$

where we obtain the control on exponential decay from the fact that $\mathcal{B}[\Phi]$ is supported on $\text{supp} \, \zeta_4$. Also

$$\| \zeta_1 T_W \Psi_1[\Phi] \|_{C^{0,\gamma}_\sigma(M \times \mathbb{R}^2)} \leq C \| \Psi_1[\Phi] \|_{C^{0,\gamma}(N)} \leq C e^{-\frac{\gamma}{\tau}} \| \Phi \|_{C^{2,\gamma}(M \times \mathbb{R}^2)},$$

$$\| \zeta_1 T_W \Psi_2[\Lambda] \|_{C^{0,\gamma}_\sigma(M \times \mathbb{R}^2)} \leq C \| \Psi_2[\Lambda] \|_{C^{0,\gamma}(N)} \leq C e^{-\frac{\gamma}{\tau}} \| \Lambda \|_{C^{0,\gamma}_\sigma(M \times \mathbb{R}^2)}.$$
where we used the fact that $\zeta_1 T_w \sim e^{-|t|} \chi_{[\zeta_1 > 0]}(t)$ to control the exponential decay in the weighted norm. Thus, we have

$$
\|G[\Phi]\|_{C^2_\gamma (M \times \mathbb{R}^2)} \leq C \left( \|B[\Phi]\|_{C^{0,\gamma}_\sigma (M \times \mathbb{R}^2)} + \|\zeta_1 T_w \Psi_1[\Phi]\|_{C^{0,\gamma}_\sigma (M \times \mathbb{R}^2)} \right)
$$

$$
\leq C \left( \delta + e^{-\frac{\sigma}{\tau}} \|\Phi\|_{C^2_\gamma (M \times \mathbb{R}^2)} \right)
$$

and hence, choosing $\varepsilon$ and $\delta$ sufficiently small, we find a unique solution to (4.7), from which it follows the existence of a unique solution $(\Phi, \Psi)$ to system (4.3)–(4.4). In conclusion, $\Phi = \zeta_2 \Phi + \Psi$ solves (4.1) and it follows directly that

$$
\|\Phi\|_{C^2_\gamma (N)} \leq C \|\Lambda\|_{C^{0,\gamma}(N)},
$$

which concludes the proof of Proposition 3.1.

\[\Box\]

5 \hspace{1cm} INVERTIBILITY OF $L_{U_0}$

In this section, we prove a solvability theory for the equation

$$
L_{U_0} [\Phi] = -\Delta_t U_0 \Phi - \varepsilon^2 \Delta_M \Phi + \Phi + T_{U_0}(t)\Phi = \Psi \quad \text{on } M \times \mathbb{R}^2.
$$

(5.1)

In general, it is not possible to solve (5.1), with estimates, for every choice of right-hand side $\Psi$. We solve instead the projected problem

$$
\begin{align*}
L_{U_0} [\Phi] &= \Psi + b^\alpha(y)V_{\alpha}(t) \quad \text{on } M \times \mathbb{R}^2 \\
\int_{\mathbb{R}^2} \Phi(y,t) \cdot V_{\alpha}(t)dt &= 0 \quad \alpha = 1, 2,
\end{align*}
$$

(5.2)

where

$$
b^\alpha(y) = -\frac{1}{\int_{\mathbb{R}^2} |V_{\alpha}(t)|^2 dt} \int_{\mathbb{R}^2} \Psi(y,t) \cdot V_{\alpha}(t) dt, \quad y \in M, \alpha = 1, 2.
$$

(5.3)

This variation will provide unique solvability in the sense of Proposition 4.1. To prove this result we restrict to an open cover $\{U_k\}$ of $M$ and solve the problem locally on $U_k \times \mathbb{R}^2$ for every $k$, finding then a global solution by gluing of all the local solutions. First, we need an invertibility theory for the same operator on the flat space $\mathbb{R}^n$

$$
\begin{align*}
-\Delta_t U_0 \Phi - \varepsilon^2 \Delta_\gamma \Phi + \Phi + T_{U_0}(t)\Phi &= \Psi + b^\alpha(y)V_{\alpha}(t) \quad \text{on } \mathbb{R}^n \\
\int_{\mathbb{R}^2} \Phi(y,t) \cdot V_{\alpha}(t)dt &= 0 \quad \alpha = 1, 2,
\end{align*}
$$

(5.4)

where $b^\alpha$ is given by (5.3), $\alpha = 1, 2$ and $(y,t) \in \mathbb{R}^{n-2} \times \mathbb{R}^2$. It holds the following lemma, whose proof is postponed to Section 6.
Lemma 5.1. Let $\Psi \in L^2(\mathbb{R}^n) \cap C^{0,\gamma}(\mathbb{R}^n)$ and let $b^\alpha$ be given by (5.3) for $\alpha = 1, 2$. Then Problem (5.4) admits a unique solution $\Phi = Q(\Psi) \in H^1_{U_0}(\mathbb{R}^n)$ satisfying

$$\|\Phi\|_{C^2,\gamma(\mathbb{R}^n)} \leq C\|\Psi\|_{C^{0,\gamma}(\mathbb{R}^n)}$$

for some $C > 0$.

5.1 Proof of Proposition 4.1

Given any point $p \in M$ we can find a local parametrisation

$$Y_p : B(0, 1) \subset \mathbb{R}^{n-2} \to M, \quad \xi \mapsto Y_p(\xi)$$

such that

$$g_{ij}(\xi) = \langle \partial_i Y_p(\xi), \partial_j Y_p(\xi) \rangle = \delta_{ij} + \theta_p(\xi), \quad \xi \in B(0, 1),$$

where $\theta_p(0) = 0$ and

$$|D^2 \theta_p| \leq C \quad \text{in } B(0, 1).$$

Locally, the Laplace–Beltrami operator reads

$$\Delta_M = \frac{1}{\sqrt{\det g}} \partial_i \left( \sqrt{\det g} g^{ij} \partial_j \right),$$

where $B_p$ has the form

$$B_p = b^{ij}(\xi) \partial_{ij} + b^j(\xi) \partial_j, \quad |\xi| < 1,$$

and $b^{ij}, b^j$ and their derivatives are uniformly bounded. Let us fix $\delta > 0$ sufficiently small and choose a sequence of points $\{p_1, \ldots, p_\ell\} \subset M$ such that, if we define

$$V_j = Y_p(B(0, \delta/2)), \quad j = 1, \ldots, \ell'$$

then $\{V_j\}_{j=1}^{\ell'}$ covers $M$. For a cut-off $\eta$ such that $\eta(s) = 1$ if $s < 1$ and $\eta(s) = 0$ if $s > 2$ define the following set of cut-offs on $M$

$$\eta^k_m(y) = \eta \left( \frac{|\xi|}{m\delta} \right), \quad y \in Y_{p_k}(\xi),$$

which are supported in $U_{k,m} := Y_{p_k}(\{\xi \in \mathbb{R}^2 : |\xi| \leq 2m\delta\})$ and extended as 0 outside of $U_{k,m}$.

Observe that, given our choice of $\{V_j\}$, there exists a $C > 0$ for which

$$1 \leq \eta_1 := \sum_{k=1}^{\ell'} \eta^k_1 \leq C.$$
We look for a solution to (5.2) of the form
\[ \Phi = \Phi_0 + \eta^k \Phi_k, \quad b^\alpha = \eta^k_1 b^\alpha_k, \quad \alpha = 1, 2, \]
where we left implicit the sum over \( k \in 1, \ldots, \ell \) and the functions \( \Phi_k, b^\alpha_k \) are defined in \( \mathbb{R}^n \). With this ansatz, the equation can be written as
\[ \begin{align*}
-\Delta_{t,U_0} \Phi - \varepsilon^2 \Delta_M \Phi + \Phi + T_{U_0}(t)\Phi &= -\Delta_{t,U_0} \Phi_0 - \varepsilon^2 \Delta_M \Phi_0 + \Phi_0 + T_{U_0}(t)\Phi_0 \\
+ \eta^k_1 \left[ -\Delta_{t,U_0} \Phi_k - \varepsilon^2 \Delta_M \Phi_k + \Phi_k + T_{U_0}(t)\Phi_k \right] \\
+ \varepsilon^2 \mathcal{R}_{U_0}[\eta^k_1, \Phi_k] \\
= \Psi + \eta^k_1 b^\alpha V_\alpha,
\end{align*} \]
where \( \mathcal{R} \) is as in (4.2). The above equation is satisfied if we solve the system
\[ \begin{align*}
-\Delta_{t,U_0} \Phi_0 - \varepsilon^2 \Delta_M \Phi_0 + \Phi_0 & = \Psi, \\
-\Delta_{t,U_0} \Phi_0 - \varepsilon^2 \Delta_M \Phi_0 + \Phi_0 & = - \varepsilon^2 \mathcal{R}_{U_0}[\eta^k_1, \Phi_k] \quad \text{in } M \times \mathbb{R}^2.
\end{align*} \] (5.9)

We first solve (5.9) with the following lemma, which will be proved in Subsection 6.3.

**Lemma 5.2.** There exists \( C > 0 \) independent on \( \varepsilon \) such that for every \( H \in C^{0,\gamma}(M \times \mathbb{R}^2) \) there exists a solution \( \Phi = \Phi(H) \) to the equation
\[ -\Delta_{t,U_0} \Phi - \varepsilon^2 \Delta_M \Phi + \Phi = H \quad \text{in } M \times \mathbb{R}^2 \]
defining a linear operator in \( H \), such that
\[ \| \Phi \|_{C^{2,\gamma}(M \times \mathbb{R}^2)} \leq C \| H \|_{C^{0,\gamma}(M \times \mathbb{R}^2)}. \] (5.10)

We use Lemma 5.2 to solve (5.9). Precisely, consider the two separate problems
\[ \begin{align*}
-\Delta_{t,U_0} \Phi_0^1 - \varepsilon^2 \Delta_M \Phi_0^1 + \Phi_0^1 &= \Psi, \\
-\Delta_{t,U_0} \Phi_0^2 - \varepsilon^2 \Delta_M \Phi_0^2 + \Phi_0^2 &= - \varepsilon^2 \mathcal{R}_{U_0}[\eta^k_1, \Phi_k]
\end{align*} \]
and let \( \Phi = (\Phi_1, \ldots, \Phi_\ell) \in (C^{2,\gamma}(\mathbb{R}^n))^\ell \). Then, using Lemma 5.2 we find \( \Phi_0^1, \Phi_0^2 \) solving the above equations and hence \( \Phi_0 = \Phi_0^1 + \Phi_0^2 \) solving (5.9), satisfying
\[ \begin{align*}
\| \Phi_0 \|_{C^{2,\gamma}(M \times \mathbb{R}^2)} & \leq C(\| \Psi \|_{C^{0,\gamma}(M \times \mathbb{R}^2)} + \varepsilon^2 \| \mathcal{R}_{U_0}[\eta^k_1, \Phi_k] \|_{C^{0,\gamma}(M \times \mathbb{R}^2)}) \\
& \leq C(\| \Psi \|_{C^{0,\gamma}(M \times \mathbb{R}^2)} + \varepsilon^2 \| \Phi \|_{(C^{2,\gamma}(\mathbb{R}^n))^{\ell}})
\end{align*} \]
If we plug \( \Phi_0 \) in (5.8) we obtain, for \( k = 1, \ldots, \ell \),
\[ -\Delta_{t,U_0} \Phi_k - \varepsilon^2 \Delta_s \Phi_k - \varepsilon^2 B_{p_k} \Phi_k + \Phi_k + T_{U_0}(t)\Phi_k = -\eta^{-1}_1 \eta^k_1 T_{U_0}(t)\Phi_0 + b^\alpha V_\alpha \quad \text{in } B(0, 2\delta) \times \mathbb{R}^2, \]
which can be extended to the whole space by using $\eta_2^k$

$$-\Delta_{t,U_0} \Phi_k - \varepsilon^2 \Delta \Phi_k - \varepsilon^2 \eta_2^k B_{p_k} \Phi_k + \Phi_k + T_{U_0}(t) \Phi_k = -\eta_1^{-1} \eta_1^k T_{U_0}(t) \Phi_0 + b^a \nabla \alpha \quad \text{in } \mathbb{R}^n, \quad (5.11)$$

where we used that $\{\eta_2^k = 1\} \subset supp \eta_1^k$. By applying the inversion operator $Q$ of Lemma 5.1, the system (5.11) can be formulated as

$$\Phi + S(\Phi) = \mathcal{W}, \quad (5.12)$$

where

$$S(\Phi)_k = Q(\eta_1^{-1} \eta_1^k T_{U_0}(t) \Phi_0^2(\Phi) - \varepsilon^2 \eta_2^k B_{p_k} \Phi_k)$$

and

$$\mathcal{W}_k = -Q(\eta_1^{-1} \eta_1^k T_{U_0}(t) \Phi_0^1(\Psi)).$$

We claim that $\|S\| < 1$, so that we can find a solution to (5.12). This concludes the existence part of Proposition 4.1 and the sought solution $\Phi = \Phi_0 + \eta_1^k \Phi_k$ satisfies

$$\|\Phi\|_{C^2(\mathcal{M} \times \mathbb{R}^2)} \leq C \|\Psi\|_{C^0(\mathcal{M} \times \mathbb{R}^2)}.$$

To show the claim, we check that

$$\|S(\Phi)_k\|_{C^2(\mathbb{R}^n)} \leq C \|\eta_1^{-1} \eta_1^k T_{U_0}(t) \Phi_0^2(\Phi) - \varepsilon^2 \eta_2^k B_{p_k} \Phi_k\|_{C^0(\mathbb{R}^n)}$$

$$\leq C \varepsilon^2 \|\Phi\|_{C^\infty(\mathbb{R}^n)},$$

hence taking the supremum over $k$ and choosing $\varepsilon$ sufficiently small the claim holds. To show the validity of the weighted estimates let $\varphi(t) = e^{-\sigma |t|}$ and let $\Phi = \varphi^{-1} \Phi$, where $\Phi$ is the solution to (5.2) just found. In terms of $\Phi$, it holds

$$-\Delta_{t,U_0} \tilde{\Phi} - \varepsilon^2 \Delta_M \tilde{\Phi} + \varphi^{-1} R_{U_0}[\varphi, \tilde{\Phi}] + \tilde{\Phi} + T_{U_0} \tilde{\Phi} = \tilde{\Psi} + \tilde{b}^\alpha \nabla \alpha,$$

where $\tilde{b}^\alpha = \varphi^{-1} b^\alpha$, $\tilde{\Psi} = \varphi^{-1} \Psi$ and $R$ is as in (4.2). Observe that

$$\|\varphi^{-1} R_{U_0}[\varphi, \tilde{\Phi}]\|_{C^0(\mathcal{M} \times \mathbb{R}^2)} \leq C \sigma \|\tilde{\Phi}\|_{C^2(\mathcal{M} \times \mathbb{R}^2)}$$

and thus, up to reducing $\sigma$, the invertibility theory developed in Proposition 4.1 applies together with a fixed-point argument providing the corresponding estimates for $\tilde{\Phi}$, namely

$$\|\tilde{\Phi}\|_{C^2(\mathcal{M} \times \mathbb{R}^2)} \leq C \|\tilde{\Psi}\|_{C^0(\mathcal{M} \times \mathbb{R}^2)}.$$

The proof is concluded by observing that

$$\|\varphi^{-1} D^2 \Phi\|_{C^0(\mathcal{M} \times \mathbb{R}^2)} + \|\varphi^{-1} D \Phi\|_{C^0(\mathcal{M} \times \mathbb{R}^2)} + \|\varphi^{-1} \Phi\|_{C^0(\mathcal{M} \times \mathbb{R}^2)} \leq C \|\tilde{\Phi}\|_{C^2(\mathcal{M} \times \mathbb{R}^2)}.$$

□
6 | PROOFS OF LEMMAS 3.1, 5.1, AND 5.2

6.1 | Proof of Lemma 3.1

We will prove the statement for each term $G^k$, for $k = 1, 2, 3$. The claim for $G^1$ follows from a direct calculation using expression (2.14). Define

$$N(h) = -\varepsilon^{-2} [\xi_k N(\Phi(h)) + \bar{B}(h, \Phi(h))]$$

so that we can write

$$G_2^2(h) = \int_{\mathbb{R}^2} N(h) \cdot V_\alpha.$$ 

We claim that the following Lipschitz dependence hold

$$\|N(\Phi_1) - N(\Phi_2)\|_{C^0,\gamma(N)} \leq C \varepsilon^3 \|\Phi_1 - \Phi_2\|_{C^{2,\gamma(N)}}, \tag{6.1}$$

$$\|\bar{B}(h, \Phi_1) - \bar{B}(h, \Phi_2)\|_{C^0,\gamma(N)} \leq C \varepsilon^3 \|\Phi_1 - \Phi_2\|_{C^{2,\gamma(N)}}, \tag{6.2}$$

$$\|\bar{B}(h_1, \Phi) - \bar{B}(h_2, \Phi)\|_{C^0,\gamma(N)} \leq C \varepsilon^4 \|h_1 - h_2\|_{C^{2,\gamma(M)}} \|\Phi\|_{C^{2,\gamma(N)}}, \tag{6.3}$$

$$\|\Phi_1 - \Phi_2\|_{C^{2,\gamma(N)}} \leq C \varepsilon^2 \|h_1 - h_2\|_{C^{2,\gamma(M)}}, \tag{6.4}$$

Assuming the validity of (6.1)–(6.4) and denoting $\Phi_l = \Phi(h_l), \ l = 1, 2$, we check that

$$\left\|G_2^2(h_1) - G_2^2(h_2)\right\|_{C^0,\gamma(M)} = \left\|\int_{\mathbb{R}^2} [N(h_1) - N(h_2)] \cdot V_\alpha\right\|_{C^0,\gamma(M)}$$

$$\leq C \varepsilon^{-2} \|N(\Phi_1) - N(\Phi_2)\|_{C^0,\gamma(N)}$$

$$+ C \varepsilon^{-2} \|B(h_1, \Phi_1) - B(h_2, \Phi_2)\|_{C^0,\gamma(N)}$$

$$\leq C \varepsilon \|\Phi_1 - \Phi_2\|_{C^{2,\gamma(N)}} + C \varepsilon^{-2} \|B(h_1, \Phi_1) - B(h_1, \Phi_2)\|_{C^0,\gamma(N)}$$

$$+ C \varepsilon^{-2} \|B(h_1, \Phi_2) - B(h_2, \Phi_2)\|_{C^0,\gamma(N)}$$

$$\leq C \varepsilon \|\Phi_1 - \Phi_2\|_{C^{2,\gamma(N)}} + C \varepsilon^2 \|h_1 - h_2\|_{C^{2,\gamma(M)}} \|\Phi\|_{C^{2,\gamma(N)}}$$

$$\leq C (\varepsilon^3 + \varepsilon^5) \|h_1 - h_2\|_{C^{2,\gamma(M)}},$$

which proves the Lipschitz dependence of $G^2$. The validity of (6.1), follows directly from the definition of $N$ as a second order expansion of $S$ in the direction of $\Phi$. Precisely, it holds

$$\|N(\Phi_1) - N(\Phi_2)\|_{C^0,\gamma(N)} \leq C (\|\Phi_1\|_{\infty} + \|\Phi_2\|_{\infty}) \|\Phi_1 - \Phi_2\|_{C^{2,\gamma(N)}}$$

$$\leq C \varepsilon^3 \|\Phi_1 - \Phi_2\|_{C^{2,\gamma(N)}},$$

where we used the fact that $\|\Phi_\varepsilon\|_{C^{2,\gamma(N)}} = O(\varepsilon^3)$. Next we prove (6.4) and to do so we recall that $\Phi$ solves (3.4). The term $R(h) = -S(W(h))$ has a Lipschitz dependence

$$\|R(h_1) - R(h_2)\|_{C^0,\gamma(N)} \leq C \varepsilon^2 \|h_1 - h_2\|_{C^{2,\gamma(M)}}$$
as shown by calculations analogous to those for the Lipschitz behaviour of $G^1$. The estimates from Proposition 3.1 yield to

$$
\|\Phi_1 - \Phi_2\|_{C^{2,\gamma}(N)} \leq C\|R(h_1) - R(h_2)\|_{C^{0,\gamma}(N)} + \|\mathcal{N}(\Phi_1) - \mathcal{N}(\Phi_2)\|_{C^{0,\gamma}(N)}
$$

$$
\leq C\varepsilon^2 \|h_1 - h_2\|_{C^{2,\gamma}(M)} + C\varepsilon^3 \|\Phi_1 - \Phi_2\|_{C^{2,\gamma}(N)}
$$

and hence, up to choosing $\varepsilon$ sufficiently small, we have (6.4). Then, we observe that (6.2) is a direct consequence of the smallness in $\varepsilon$ of all the coefficients of the second order operator $B$, while (6.3) is a consequence of the mild Lipschitz dependence in $h$ of such coefficients. Finally, we prove that the term

$$
G_3^\alpha(h_1) = -\varepsilon^{-2} \int_{\mathbb{R}^2} \zeta_4 L_U^0 [\Phi] \cdot V_\alpha
$$

is Lipschitz with a constant that is exponentially small in $\varepsilon$. This is straightforward using the self-adjointness of $L_U^0$ and the fact that $V_\alpha \in \ker L_U^0$. Indeed, integrating by parts yields to

$$
\int_{\mathbb{R}^2} \zeta_4 L_U^0 [\Phi] \cdot V_\alpha dt = \int_{\mathbb{R}^2} \zeta_4 \left( -\Delta t, U_0 \Phi - \Delta_M \Phi + \Phi + T_U^0 \Phi \right) \cdot V_\alpha dt
$$

$$
= -\int_{\mathbb{R}^2} (\Delta t \zeta_4) V_\alpha \cdot \Phi dt - 2 \int_{\mathbb{R}^2} \left( \nabla t \zeta_4 \cdot \nabla t, U_0 V_\alpha \right) \Phi dt,
$$

where we also used the orthogonality between $\Phi$ and $V_\alpha$. Using the exponential decay of $V_\alpha$ and the usual argument with the support of $\zeta_4$ we obtain

$$
\|G_3^\alpha(h_1) - G_3^\alpha(h_2)\|_{C^{0,\gamma}(N)} \leq C\varepsilon^{-2} e^{-\delta / \varepsilon} \|\Phi_1 - \Phi_2\|_{\infty}
$$

$$
\leq C e^{-\delta / \varepsilon} \|h_1 - h_2\|_{C^{2,\gamma}(M)}.
$$

Finally, we can put all the estimates just found together to obtain that $G$ has an $O(\varepsilon)$-Lipschitz constant, concluding the proof.

\[\square\]

### 6.2 Proof of Lemma 5.1

We claim that for every $\Psi \in L^2(\mathbb{R}^n)$ we can find a solution to the system

$$
\begin{cases}
-\Delta_{t,U_0} \Phi - \varepsilon^2 \Delta_y \Phi + \Phi + T_U^0 \Phi = \Psi + b^\alpha(y)V_\alpha(t) & \text{on } \mathbb{R}^n \\
\int_{\mathbb{R}^2} \Phi(y,t) \cdot V_\alpha(t) dt = 0 & \alpha = 1, 2.
\end{cases}
$$

(6.5)

This result has already been established in [2, Lemma 7]. We sketch the proof for completeness. The differential equation in (6.5) reads explicitly, denoting $\Phi = (\phi, \omega_1 dt + \omega_2 dy)$ and
\[ \Psi = (\psi, \eta_t dt + \eta_y dy), \]

\[
\begin{aligned}
-\Delta_t^\vartheta \phi - \varepsilon^2 \Delta_y \phi - \frac{1}{2} (1 - 3 |u_0|^2) \phi + 2i \nabla A_0 u_0 \cdot \omega_t &= \psi + b^\alpha V_\alpha^1 \\
-\Delta_t \omega_t - \varepsilon^2 \Delta_y \omega_t + |u_0|^2 \omega_t + 2 \langle \nabla A_0 u_0, i \phi \rangle &= \eta_t + b^\alpha V_\alpha^2 \\
-\Delta_t \omega_y - \varepsilon^2 \Delta_y \omega_y + |u_0|^2 \omega_y &= \eta_y,
\end{aligned}
\]

(6.6)

where we have that the last equation is not coupled with the first two. Therefore, we need to solve two separate problems:

\[
\begin{aligned}
-\Delta_t^\vartheta \phi - \varepsilon^2 \Delta_y \phi - \frac{1}{2} (1 - 3 |u_0|^2) \phi + 2i \nabla A_0 u_0 \cdot \omega_t &= \psi + b^\alpha V_\alpha^1 \\
-\Delta_t \omega_t - \varepsilon^2 \Delta_y \omega_t + |u_0|^2 \omega_t + 2 \langle \nabla A_0 u_0, i \phi \rangle &= b^\alpha V_\alpha^2,
\end{aligned}
\]

(6.7)

where \( \omega = \omega(x, y)d x \), and the second one is

\[ -\Delta_t \omega_y - \varepsilon^2 \Delta_y \omega_y + f^2 \omega_y = \eta_y \quad \text{in} \quad \mathbb{R}^n. \]

(6.8)

On (6.7), we apply Fourier transform on the \( y \) variable; the transformed system reads

\[
\left( \mathcal{L} + \varepsilon^2 |\xi|^2 \right) \tilde{\Phi} = \tilde{\Psi} + b^\alpha V_\alpha \quad \text{in} \quad \mathbb{R}^n,
\]

(6.9)

where \( \mathcal{L} \) is the two-dimensional linearised around \( U_0(t) \) and \( \xi \) is the Fourier variable. Using the coercivity of \( \mathcal{L} \) given by (1.18) we solve system (6.9), for any \( \xi \in \mathbb{R}^{n-2} \) fixed, by Lax–Milgram theorem. Remark that this is the point in which the correction \( b^\alpha V_\alpha \) is needed because the coercivity is only true under the orthogonality condition with \( V_\beta, \beta = 1, 2 \). Applying the inverse Fourier transform we then obtain a solution to (6.7). Equation (6.8) is solved in a similar, easier manner. This procedure also yields to

\[
\| \Phi \|_{H^1_{U_0}(\mathbb{R}^n)} \leq C \| \Psi \|_{L^2(\mathbb{R}^n)},
\]

from which is simply proven, using elliptic interior regularity, the validity of the Hölder estimates

\[
\| \Phi \|_{C^{2,\gamma}(\mathbb{R}^n)} \leq C \| \Psi \|_{C^{0,\gamma}(\mathbb{R}^n)},
\]

(6.10)

which concludes the proof.

\[ \square \]

### 6.3 Proof of Lemma 5.2

For integers \( k = 1, 2, \ldots \) consider the problem

\[
\begin{aligned}
-\Delta_t U_0 \Phi - \varepsilon^2 \Delta_M \Phi + \Phi &= H \quad \text{in} \quad (M \times \mathbb{R}^2) \cap B_k \\
\Phi &= 0 \quad \text{on} \quad (M \times \mathbb{R}^2) \cap \partial B_k,
\end{aligned}
\]

(6.11)
which admits a solution in $H^{1}_{U_{0},0}(M \times \mathbb{R}^{2} \cap B_{k})$, defined as the closure on $C_{c}^{\infty}(M \times \mathbb{R}^{2} \cap B_{k})$ pairs under the $H^{1}_{L_{0}}$, by Riez’s theorem. By using $\|H\|_{L_{\infty}}$ as a barrier, we obtain the uniform bound

$$\|\Phi_{k}\|_{\infty} \leq \|H\|_{\infty} \quad \forall k \in \mathbb{N}.$$ 

Thus, using elliptic estimates we obtain the presence of a subsequence of $\Phi_{k}$ that converges uniformly over compact sets as $k \to \infty$ to a limit $\Phi$ that solves

$$-\Delta_{t,U_{0}}\Phi - \varepsilon^{2}\Delta_{M}\Phi + \Phi = H \quad \text{in} \ M \times \mathbb{R}^{2},$$

and the estimate

$$\|\Phi\|_{C^{2,\gamma}(M \times \mathbb{R}^{2})} \leq C\|H\|_{C^{0,\gamma}(M \times \mathbb{R}^{2})}$$

for some constant $C > 0$ independent on $\varepsilon$. □

**APPENDIX: CONSTRUCTION OF THE OUTER FUNCTION $\psi$**
Here we prove Lemma 2.2, namely that the map $\psi_{0} := e^{i\theta}$, where $\theta$ is the angle in the normal frame, can be extended beyond $\mathcal{T}$ to a map $\psi$ such that

$$\psi \in C^{\infty}(N \setminus M).$$

To do so, we use the admissibility hypothesis (H), namely the existence of an oriented manifold $B$ of dimension $n - 1$ such that $M = \partial B$. Up to composing with an homotopy, we can assume that

$$\psi_{0}^{-1}(\{1\}) = B \cap \mathcal{T}. \quad \text{(A.1)}$$

Consider, for some number $\eta < \delta/2$ an $\eta$-thickening of $B$, given by

$$B_{\eta} = \{\exp_{q}(t\nu_{B}(q)) : t \in (-\eta, \eta)\},$$

where $\nu_{B}$ is a normal field on $B$, which exists thanks to the orientability assumption. Observe that, according to this definition the set $A_{1} := B_{\eta} \cup \mathcal{T}$ is not smooth as the intersection of the two sets is transversal, but it is direct to modify slightly the definition of $B_{\eta}$ (for instance, by inserting an appropriate dependence $t = t(q)$) in order to avoid this issue. Thus, up to this modification, we will assume that $A_{1}$ is a smooth manifold of dimension $n$. It is then possible to choose $0 < \eta_{0} < \pi$ and a map $\psi_{1} : A_{1} \to S^{1}$ such that $\psi_{1}$ is a smooth extension of $\psi_{0}$ and

$$\psi_{1}(B_{\eta} \setminus \mathcal{T}) \subset \{e^{i\theta} : \theta \in (-\eta_{0}, \eta_{0})\}.$$

Now, define

$$A_{2} := \{x \in N : \text{dist}(x, B) < \eta/2\}$$

and observe that $\psi_{1}^{-1}(\{1\}) \subset A_{2}$. The map

$$\psi_{1}|_{A_{1} \setminus A_{2}} : A_{1} \setminus A_{2} \to S^{1} \setminus \{1\}$$
can be extended smoothly to a function $\psi_2 : N \setminus A_2 \to S^1 \setminus \{1\}$ by Tietze’s theorem, which applies by virtue of the fact that a punctured $S^1$ is diffeomorphic to $\mathbb{R}$. Finally, setting

$$
\psi = \begin{cases} 
\psi_1 & \text{in } A_2 \\
\psi_2 & \text{in } N \setminus A_2
\end{cases},
$$

we obtain the sought extension.

\[\Box\]

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