TIME IN COSMOLOGY

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Abstract  The notion of time in cosmology is revealed through an examination of transition matrix elements of radiative processes occurring in the cosmos. To begin with, the very concept of time is delineated in classical physics in terms of correlations between the succession of configurations which describe a process and a standard trajectory called the clock. The total is an isolated system of fixed energy. This is relevant for cosmology in that the universe is an isolated system which we take to be homogeneous and isotropic. Furthermore, in virtue of the constraint which arises from reparametrization invariance of time, it has total energy zero. Therefore the momentum of the scale factor is determined from the energy of matter. In the quantum theory this is exploited through use of the WKB approximation for the wave function of the scale factor, justified for a large universe. The formalism then gives rise to matrix elements describing matter processes. These are shown to take on the form of usual time dependent quantum amplitudes wherein the temporal dependence is given by a background which is once more fixed by the total energy of matter.
Preface

This paper is conceived as a pedagogical review to the problem of time in quantum cosmology. However, rather than providing a complete survey of the topic, it aims to present in a concise manner the line of thought developed in refs. [8]-[11] and [19]-[23]. In particular, by applying the usual analysis of statistical fluctuations in large systems, it confirms that matter evolution is parametrized by a mean time which is determined by its mean energy. This is in contrast with the alternative approach [7, 14, 15] which is based on the fact that the Planck mass is much larger than any other mass.

The paper is organized as follows.

1. Introduction.
We present the problem of time in cosmology and time in general, and then motivate the procedure brought to bear.

2. Classical cosmology.
This section contains the Hamilton-Jacobi formulation of classical cosmology. In general relativity, the Hamilton-Jacobi equation is a constraint which specifies that the total energy in the universe must vanish. Matter is here represented schematically in the form of uniformly distributed “stars”.

3. Time.
The concept of time in classical physics is developed. The key point is the use of the conservation the total energy to correlate trajectories of subsystems. Application to cosmology is then presented in terms of a model containing heavy atoms and radiation.

4. Quantum cosmology.
The quantized version of the constraint equation (called the Wheeler-DeWitt equation) and its solutions as WKB wave functions for the gravitational part are displayed. The matter degrees of freedom are not yet coupled among themselves; so this section, like the previous one, is preparatory to deriving the usefulness of time, a parameter which describes the rates of processes.

5. Quantum transitions in cosmology.
The perturbation theoretic calculation of transition probabilities induced by coupling radiation to heavy atoms is carried out and the Golden Rule derived. It is most interesting here to understand the vital role of the gravitational wave functions in delivering the familiar time dependent phases. This Section is the heart of the paper in that the various threads of formal development are woven together to give rise to time as we commonly understand it, a temporal parameter describing evolution.

6. Concluding remarks.
This paper attacks the question of time from the point of view of parametrizing rates of matter processes whereas heretofore emphasis has been put on the wave function of the universe. We add a remark on this subject and in particular contrast the cosmological case with a seemingly similar situation, the Stern-Gerlach experiment.

Our approach suggests a way to look at Mach’s principle from a quantum point of view. We close the paper with a remark on this point.
1 Introduction

A fundamental problem in physics is posed by the concept of temporality in cosmology, both in verbal and technical terms. Concerning the former: what is the clock that times cosmological processes? First of all, what is a clock?

The physicist’s answer (and perhaps, in the last analysis, the only one) to this latter query is: a clock executes a standard classical trajectory and to define time is to establish a correspondence, or better a correlation, between any trajectory or any process—the totality of changes in a complicated system—and points on this standard. A preferred clock is one whose trajectory allows for the expression of the laws of physics in simple and intelligible terms. This is accomplished through the use of perfect periodic systems. These clocks play a privileged role because they execute a “wrapped-up” inertial trajectory—a piecewise recording of this latter. [One may think of a long ribbon that is wound up by attaching one of its ends to the needle of an ordinary clock. The other end is on an inertial trajectory, in the idealization of neglecting the spatial dimensions of the clock. In Section 3 more detail will be given on the nature of periodic systems which will lead to some qualifications concerning the precise definition of a preferred clock.] Because inertial trajectories are simply represented in an inertial system, such clocks are convenient. Indeed Newton’s law $F = m\alpha$ relates quantities, $F$ and $\alpha$, which express first order deviations from inertiality. Its simplicity and intelligibility are consequences of adopting what we can well call “inertial” time—since it issues, as we have said, from use of a standard trajectory which, at bottom, is inertial.

In this paper cosmology is reduced to its most essential expression, the study of processes in homogeneous and isotropic geometries. We develop the concept of cosmological time in that context since already at that level of simplification non-trivial conceptual questions arise. Generalization to situations wherein all gravitational degrees of freedom are taken into account is very difficult and has not yet arrived at the same degree of clarity. For homogeneous geometries, our neat convenient clocks are not available as timers. Indeed in the primitive universe, before the formation of structure, such clocks are nowhere present. Upon a little thought, one realizes that the only available clock is the scale factor $a$, or any dynamical function of $a$ which also shows semi-classical behavior. After all, cosmological data are generally delivered in terms of the red shift $z$ where $z + 1 = a_{\text{reception}}/a_{\text{emission}}$. Cosmology may be summed up as the succession of configurations of matter which evolve as $a$ does. In its simplest version this is encoded in the cosmological temperature, $T(a)$. The problem is thus: how does one first obtain the correlations to $a$ and then translate this correlation into the usual language of temporal evolution?

The technical aspect of the problem is the one that will deliver the answer. At first sight one is rather perplexed because in general relativity the problem of cosmology is reduced to the realization of the single mathematical constraint: the total energy, matter plus gravity, is strictly equal to zero. This proposition is a direct consequence of the fact that the temporal coordinate in general relativity is not prescribed, there being no preferred frame. (In technical jargon this is called reparametrization invariance.) Consequently, on the most fundamental of grounds, one loses the ordinary sense of
temporality. Translated into quantum mechanical terms, the universe is in an energy eigenstate (or perhaps a mixed state) of zero total energy\(^2\), literally a stationary state\(^3\). No time! The equation of constraint imposed on this state vector is the Wheeler-DeWitt equation (WDW).

But it is precisely for the same mathematical reason–vanishing total energy–that one can recover time since it is this constraint that brings about the correlations between \(a\) and the configurations of matter. In its quantum form, it is the wave function that encodes these correlations. So it is the job of the physicist to use the formal theory of constraints, in a perspicacious way, to convert it into the usual description of temporality. It is noteworthy that almost all the efforts on this problem have been carried out in the quantum framework. This for the good reason that, for the quantum physicist, it is the fact that the wave function is a stationary state which gives rise to the acuity of the problem in no uncertain terms. This is not to say that the problem cannot also be analyzed in purely classical terms such as in \(^4\) or as in Section 3 of the present paper\(^5\).

It is the quantum point of view which has been adopted in the seminal works of Lachitski and Rubakov\(^6\), of Banks\(^7\) and of Halliwell and Hawking\(^8\). To put into evidence time as a measure of matter correlations to \(a\), these authors postulate that the gravitational part of the wave function of the universe (that part that depends on \(a\) alone) is described by a WKB function. From the action which appears in its phase one makes contact with classical mechanics in its Hamilton-Jacobi guise. And this, in turn, gives rise to time (\(\equiv t(a)\)) as a definition. The usefulness of this definition is that, by trading correlation to \(a\) against correlation to \(t(a)\), one recovers the usual temporal description of the evolution of matter. Banks calls it a “maquillage”.

There has been considerable effort on the part of several people \(^9\) to implement the program of refs. \(^6\), \(^7\), \(^8\) so as to take into account the fact that gravity is driven by matter at the quantum level as well as to determine how matter evolution is modified by this coupling. The present paper is written in the same vein but whereas previous work has been concentrated on elucidating “the wave function of the universe” we here pursue, following \(^9\)–\(^15\), a more operational approach; to wit: time is a variable which is used to parametrize rates of material processes. In quantum mechanics, these are given by the “Golden Rule” or its non-perturbative generalization. We focus our effort on this aspect of the problem. How is it that the “maquillage” encoded in \(t(a)\) yields the conventional temporal description of material processes?

In answering this question, certain approximations must be made (in addition to the idealization of working with homogeneous isotropic universes, a problem which we do not touch upon, but which clearly requires further critical analysis\(^3\)\(^15\)\(^19\)). We show that all these approximations are justified when the universe contains a large number of particles \(N\). Indeed, by dropping correction terms of order \(N^{-1}\) in matrix elements describing matter processes, the metric which appears is driven by the mean matter energy. And this is precisely what is necessary to establish the concept of classical time in cosmology. In this approach the difficult points\(^12\)\(^17\)\(^20\)\(^23\) of interpreting the wave

\(^1\) We are grateful to Julian Barbour for pointing us his work\(^\ddagger\) wherein the point of view exposed is similar to that expressed in Section 3.
function of the universe are not encountered.

In summary, the substance of this paper is to show how the WDW equation encodes changes in matter configurations as the universe expands. To this end, we introduce interactions among matter degrees of freedom and then show, perturbatively, how to recover time dependent perturbation theory in terms of \( t(a) \), wherein this latter is determined from Einstein’s equations driven by the mean matter energy.

The main physical principle which we use in this analysis is adiabaticity. By this we mean in its extended sense, such as the theory of non-elastic electronic transitions induced by molecular motion or the quantum description of the Stern-Gerlach experiment, most elegantly presented by Gottfried[24]. In the present case, the validity of the adiabatic scheme requires that the universe must contain a large number of particles. Indeed, this is used in three different ways:

a. To justify the WKB approximation for the gravitational wave functions.

b. To legitimize the conventional expression for transition amplitudes. The key point here is that the phase differences which govern these amplitudes contain differences of macroscopic quantities (i.e. \( O(N) \)) which expanded in power of \( 1/N \) yield the usual time dependent phase evaluated in the mean background geometry.

c. In order for this mean to have meaning, it is necessary that the total matter energies \( E \) be dispersed such that \( \langle (\Delta E)^2 \rangle / \langle E \rangle^2 \ll 1 \), as is the case in usual statistical mechanics. In this case, the background geometry is indeed the solution of the semi-classical theory of gravity wherein gravity is driven by the mean energy \( \langle E \rangle \).

In this we recover, in a quantum setting, a deep appreciation and a precise statement of Mach’s principle. It is the use of the fact that the universe is macroscopic that allows for the reduction of the interacting gravity-matter system to a description of material events in a gravitational background, wherein history is given in terms of inertial time. Specifically, microscopic processes involving a few degrees of freedom are so described. Were the universe not large, this would not be possible. This is as Mach would have it. The background, our background, is determined by the myriad of fixed stars. And the laws of physics, expressed in their usual spatio-temporal form, follow accordingly.

2 Classical Cosmology

In this section classical cosmology is presented in a manner that prepares the way for its quantization. Our treatment is based on Hamilton-Jacobi theory at fixed energy.

What follows are the bare bones necessary to construct the simple cosmological models, Friedman type universes characterized by Robertson-Walker geometries, homogeneous and isotropic. These are conceived as an idealization introduced to describe the universe in the mean, i.e. spatial average.

General relativity is built upon the dynamics of the space-time metric[1]. The length element in 3+1 dimensions is
\[
\begin{align*}
ds^2 &= -g_{00}(\xi) d\xi^2 + 2g_{0i}(\xi) dx^i + g_{ij}(\xi) dx^i dx^j \\
\end{align*}
\]

\(\xi\) is the temporal coördinate, \(x^i\) is spatial with \(i = 1, 2, 3\).

The Einstein-Hilbert action, \(S_G = \int R \sqrt{|g|} d^3 x d\xi\) wherein \(R\) is the scalar curvature, governs the dynamics of the metric. One finds that \(g_{00}(\xi, x)\) and \(g_{0i}(\xi, x)\) appear without derivatives with respect to \(\xi\). This is analogous to electrodynamics wherein the time component of the vector potential \(A_0\) appears without temporal derivatives, essentially for reasons of gauge invariance. As a result, in the Hamiltonian formalism, Gauss’s law emerges as a condition of constraint. One should not then be surprised to find similar constraints in the present case. Indeed the momenta conjugate to \(g_{00}\) and \(g_{0i}\) are identically zero, thereby giving rise to 4 local constraints \(H_G(\xi, x) = 0, H^i_G(\xi, x) = 0\).

When matter is present one must add to \(S_G\) its action (≡ \(S_m\)). And, once again, the absence of temporal derivatives delivers the constraint conditions, now enlarged to

\[
\begin{align*}
H_T(\xi, x) &= H_G(\xi, x) + H_m(\xi, x) = 0 \\
H^i_T(\xi, x) &= H^i_G(\xi, x) + H^i_m(\xi, x) = 0
\end{align*}
\]

The quantities \(H_m(\xi, x)\) and \(H^i_m(\xi, x)\) are respectively the energy and the momentum densities of matter. Eq. (2) may thus be interpreted as the vanishing of the sum of the gravitational and matter energy-momentum densities. (For the geometric interpretation of these constraints the reader is referred to [1]). The consequence of this is that gravitational degrees of freedom are the components of the spatial metric \(g_{ij}(\xi, x)\), those which determine the 3-geometry at fixed \(\xi\). The space in which these degrees of freedom live is called superspace.

The dynamics of homogeneous and isotropic cosmological models is carried out in a subspace called mini-superspace. In this much smaller space, the only part of the \(g_{ij}(\xi, x)\) degrees of freedom which is considered is the scale factor, \(a(\xi)\), of cosmology which depends on \(\xi\) only. It is defined through the reduced form of the metric

\[
ds^2 = -N^2(\xi) d\xi^2 + a^2(\xi) d\Omega_3^2
\]

The quantity \(d\Omega_3^2\) is the square of the length element on the unit 3-dimensional hypersurface that is characterized by homogeneous isotropic geometry. There are only three such: flat, open, closed [1]. Their differences will play but a slight rôle in what follows. In eq. (3), the variable \(\xi\) is the temporal parameter which parametrizes the different scales realized in the cosmos. Very important is that \(\xi\) may be reparametrized at will, i.e. \(N(\xi)\) is arbitrary. This residual reparametrization invariance is seen by the explicit expression for the gravitational action when limited to mini superspace:

\[
S_G = \frac{1}{2G} \int d\xi N \left[ a \left( \frac{da}{d\xi} \right)^2 + Ka \right]
\]

where \(K = 0, \pm 1\) for the flat, closed and open cases. \(G\) is Newton’s gravitational constant. This identification comes about by adding to \(S_G\) the action of matter \(S_m\).
To display cosmological dynamics in more general terms, two routes can be chosen. The first is to fix \( N(\xi) \) [said to be gauge fixing] and then solve for \( a(\xi) \) in terms of that particular choice of temporal parameter. The second method is superior in that it is more physical since it does not resort to a particular choice of an arbitrary (hence unphysical) temporal parameter. We shall follow this second method since it will lead us directly to the concept of time presented in the Introduction. The correlations introduced in that discussion will be seen to come about from the constraint which arises precisely because \( N(\xi) \) is arbitrary.

The technique used is the Hamilton-Jacobi procedure. One first expresses the total action \( S_T = S_G + S_m \) in terms of a Hamiltonian. The momentum conjugate to \( a \) is \( p_a = -\dot{a}a/NG \) where \( \dot{a} = da/d\xi \). For simplicity we take matter to be described by a uniform distribution of \( n_{\text{star}} \) massive stars of mass \( M \). Its action is \( S_m = -\int d\xi NH_m = -\int d\xi Nn_{\text{star}}M \). Then the total action takes the form

\[
S_T = \int d\xi [p_a\dot{a} - N(H_G + H_m)]
\]

wherein the gravitational piece is obtained from eq. \( (4) \). In general there will appear as well, the momenta of matter degrees of freedom (in the form of \( p\dot{q} \)). They are absent in this simple model but the structure of eq. \( (\xi) \) wherein \( N \) is multiplied by the total Hamiltonian \( H_T = H_G + H_m \) will still hold.

Explicitly, one obtains

\[
H_G = -\frac{G^2p_a^2 - Ka^2}{2Ga}
\]

One sees that \( H_G \) no longer contains \( N(\xi) \) and the same is true for \( H_m \). This will be seen when we adopt a specific model later in this section. Notice the unusual negative sign of the kinetic energy of \( a \). This is characteristic of theories having reparametrization invariance. This negative sign guarantees that there exists always a solution for any positive matter energy.

Variation with respect to \( N(\xi) \) then yields the residual constraint that the total energy must vanish

\[
H_T = H_G + H_m = 0
\]

an equation which delivers \( p_a \) as a function of \( a \) and the matter energy. Because of the quadratic character of the kinetic energy of \( H_G \), \( p_a \) is determined up to a sign. Positive momenta correspond to contracting universes and negative ones to expanding universes, since \( p_a = -a\dot{a}/NG \).

The Hamiltonian-Jacobi formalism is obtained by working within the domain of possible classical solutions, i.e. which satisfy the constraint. In that case, it is seen from eq. \( (\xi) \) that the stationary action is \( S_G = \int a p(a)da' \) whence \( p(a) = \partial_a S_G \) and the constraint, eq. \( (\xi) \), is the Hamiltonian-Jacobi equation

\[
-G^2(\partial_a S_G)^2 - Ka^2 + 2GaH_m = 0
\]

This equation is valid for any matter system possessed of a Hamiltonian.

We momentarily postpone further explicitization of the cosmological model of \( H_m \) which we shall adopt to handle quantum transitions in order to first make contact with the epistemological discussion concerning time which was presented in the Introduction.
3 Time

Whereas, in the cosmological case, energy conservation is derived on the fundamental grounds of reparametrization invariance, we shall here work in a more general context wherein energy is conserved (at any value rather than zero as in gravity) due to the isolation of the dynamical system. Thus, time dependent forces are absent, so the hamiltonian has no explicit time dependence.

We wish to display how it comes about that in such systems one degree of freedom can serve as a clock, see \[25\]. It is not amiss here to think of our earth-sun system idealized to be decoupled from all else in the cosmos. The clock degree of freedom is the angle of rotation $\phi$ from which one extracts time, $t(\phi)$, in the conventional sense. This is then the temporal parameter that is used to describe the changing configurations in all the other degrees of freedom on earth, as has been done since time immemorial. Indeed before ”quartz” time or ”cesium” time it was the mean solar time that served as a standard.

We shall resort to Hamiltonian-Jacobi theory at fixed energy $E$ to display the conversion of correlation of positions into temporality when working without external time. Before this, we remind the reader the way the Hamiltonian-Jacobi function $S(x, E)$ is used to recover the Newtonian temporal description of a trajectory of a single particle of mass $m$ moving in a one-dimensional potential $V(x)$. It is through the identification of the temporal parameter defined by

$$
t(x, E) = \partial_E S(x, E)|_x = \partial_E \int^x p(x', E)dx' = \partial_E \int^x dx' \sqrt{2m(E - V(x'))} = \int^x dx' \frac{m}{p(x', E)} = \int^x dx' \frac{1}{v(x'; E)}
$$

(9)

We have introduced the velocity, $v(x, E)$, which is here defined by the inverse derivative of the $t(x, E)$ with respect to $x$. This is to impress upon the reader that this function which expresses temporality is a derivative concept for isolated systems. This is not how Newton conceived time which he stated was “absolute”, but whose absolute character was neither used nor exhibited. Nevertheless, in classical mechanics it is tacitly taken that $t(x, E)$ is a universal parameter, used to parametrize all trajectories. Thus all trajectories become correlated one to the other as time flows. Our aim is to investigate how these correlations come about using the fundamental principle of dynamics without introducing time. But first let us return briefly to the concepts at issue.

As such, Eq. (9) is simply the definition of some dynamical quantity. Its operational sense emerges only in the context of its measurement. And for this purpose, the trajectory in question has to be compared to some other, which, by convention may be chosen to be the standard, the clock. For example when I am in a moving train, in order to answer the question when am I at such and such location, I consult my watch. Otherwise the question is senseless. Any operational definition of time involves such a comparison. The fact that the two trajectories are correlated is a wonderful fact and one must ask why. The answer given below is that the total energy, (clock + system), of an isolated system is conserved. This makes very good sense. For example if my watch is subject to
uncontrollable forces from the outside (i.e. time dependent forces) its needle will move erratically as far as I am concerned and it will no longer be a faithful clock. Let us now formalize this idea.

Consider two degrees of freedom \( x_1 \) (system) and \( x_2 \) (clock) which comprise the total isolated system of total energy \( E \). Their action is

\[
S_{Total} = S_1(x_1, E_1) + S_2(x_2, E_2)
\]  (10)

with \( E_1 + E_2 = E \), in the limit where the coupling between them is negligible (i.e. infinitesimal). Nevertheless we want to find out why their trajectories are correlated, i.e. for given initial condition (which henceforth will not be explicitly written) why a specification of \( x_2 \) fixes \( x_1 \). From these actions one finds two à priori independent temporal functions

\[
t_1(x_1, E_1) = \partial E_1 S_1 \big|_{x_1}
\]

\[
t_2(x_2, E_2) = \partial E_2 S_2 \big|_{x_2}
\]  (11)

So the question we are asking is why \( t_1(x_1, E_1) = t_2(x_2, E_2) \) when \( E_1 + E_2 = E \) is kept fixed?

The answer lies in the principle of least action that is intrinsic to classical mechanics –how to get trajectories from the action. Indeed, in this case it gives

\[
\partial_\epsilon S_{Total} \big|_{x_1, x_2, E} = \partial_\epsilon S_1(x_1, \epsilon) + \partial_\epsilon S_2(x_2, E - \epsilon) = 0
\]  (12)

where \( E_1 = \epsilon, E_2 = E - \epsilon \). Eq. (12) is the answer to the question: for what values of the energy split, \( \epsilon \), do two dynamical systems of total energy \( E \) pass through the points \( x_1 \) and \( x_2 \) at fixed initial conditions?

The crucial concept that has been exploited is the notion of a single isolated system, albeit comprised of components parts. A priori, these parts could be completely independent mechanical systems, with no possibility of communications between them throughout their history. Were this so, the conditions we have set down would be operationally meaningless in that there would be no way of specifying initial conditions simultaneously. We have tacitly assumed this to be possible, either through contact or exchange of light rays or what not. Once this is admitted, then we are dealing with a single system, hence with one independent specification less than that which is required for two completely independent systems. In short, some interaction among the component parts is assumed, be it ever so small. Thus the partitioning of energy \( \epsilon \) introduced in eq. (12) is one too many to determine the action of the isolated system. Therefore, the least action principle requires that the total action be stationary with respect to its variation, i.e. eq. (12). This is the dynamical principle which is operationally meaningful, that one must call upon to replace the Newtonian sense of absolute time.

In this fashion one sees that the description of the trajectory of the system (1) is given by the function \( x_1(x_2) \). Moreover, it is delivered through eq. (12) in the form \( x_1(t_2(x_2)) \). The usefulness of \( t_2(x_2, E_2) \) rather that \( x_2 \) itself, is that it opens the way to express the dynamics of \( x_1 \) without appeal to the clock’s dynamics. For periodic systems with one
degree of freedom, this is accomplished by adopting, for this latter, action-angle variables
\((\equiv J_2, \theta_2)\) since \(\theta_2\) is directly proportional to \(\partial S_2/\partial E_2\) because the angular velocity
\((= (\partial E/\partial J)^{-1})\) is independent of \(\theta_2\). Hence the previously displayed correlation of \(x_1\) to \(x_2\)
delivers the trajectory directly in terms of \(\theta_2\) instead of some complicated function \(t_2(x_2)\).
Were there many clocks, each described by an angle \(\theta_i\), then the stationarity condition correlates them so that any angle, \(\theta_i\), can be used to parametrize the trajectory of \(x_1\).
Indeed this is a universal function given by \(x_1(\theta_i \partial E_i J_i)\). It is \((\theta_i \partial E_i J_i)\) that is called
time\(^2\). Its universal character is now demonstrated. For simplicity, the above has been
presented for the case where system (1) has one degree of freedom. Generalization to
systems possessing many degrees of freedom is straightforward.

We now apply this approach to matter evolution in a cosmological situation. For
purposes of describing typical quantum transitions in cosmology, we take matter to be
composed of conformal radiation and heavy particles. These are explained in turn.
Conformal radiation refers to massless fields. This is exemplified in nature by photons
and we shall take a scalar version of them. Conformality has as consequence that the
photon’s energy, as measured in “conformal” time \(\eta\), see below for its definition, is
proportional to its inverse wavelength
\[
\omega_k = \sqrt{\sum_i k_i^2}
\]  
(13)
For simplicity we have taken the case of flat 3-geometries so that the components \(k_i\) label
the non dimensional wave numbers. Thus \(\omega_k^2\) is the eigenvalue of the 3-laplacian, \(\triangle_3\),
non dimensionalized by the scale factor \(a\), corresponding to the metric of eq. (3). The
definition of conformal time is obtained from this equation with the choice \(N(\eta) = a(\eta)\)
so that the d’Alembertian operator is \(\partial^2_\eta - \triangle_3\). The solutions for the flat case are thus
\[exp\ i(\omega_k \eta - k \cdot r)\] with the dispersion relation eq. (13) where \(r = (r \cdot r)^{1/2}\) is the non
dimensional length obtained from \(d\Omega_3\) of eq. (3).

We now turn to the description of “heavy” particles. By heavy we mean that they
are sufficiently massive so as not to be produced by the cosmological expansion\(^2\). For
this to be true it is required that
\[
M \gg (\frac{da/dt}{a})
\]  
(14)
where \(h = c = 1\) and \(t\) is the proper time corresponding to the choice \(N = 1\) in eq. (3).
The energy relative to \(t\) of a heavy particle whose conserved wave number components
are \(p_i\) is given by
\[
\Omega(t) = \sqrt{M^2 + \sum_i p_i^2/a(t)^2}
\]  
(15)
\(^2\) For periodic systems (2) of more than one degree of freedom, one must replace the above consideration by more complicated expressions which take into account the shape of the trajectory, for example the shape of the earth’s trajectory around the sun. Indeed the angle traversed by the earth with respect to fixed stars does not define “time”. However \(n\) complete revolutions of the earth is a faithful measure of time.
where the dependence on $a$ is understood in the light of eq. (13) valid for massless particles when it is noticed that for that case, $\Omega = \omega/a$. It is the time dependence of $\Omega$ through $a(t)$ which generates pair production in that $M$ and $p/a$ "feel" the expansion differently. For simplicity in this and the next section we shall set $p = 0$.

In resumé the action of matter which is the subject of Sections 4 and 5 is

$$S_{\text{matter}} = S_M + S_\gamma = \frac{1}{2} \int d\xi N \left\{ \frac{\dot{\psi}^2}{N^2} - M^2 \psi^2 \right\} + a^{-1} \left[ \frac{a^2 \dot{\phi}^2}{N^2} - \omega_k^2 \phi^2 \right]$$

(Dots mean $d/d\xi$. As compared to conventional notation, we have simplified by rescaling the matter field $\psi$ by a factor of $a^{3/2}$ (hence $\psi$ has dimension $L^{1/2}$) and the radiation field $\phi$ by a factor of $a$ (hence $\phi$ is dimensionless). The scaling of $\phi$ results in the simple form, eq. (16), in consequence of conformal symmetry. Instead that of $\psi$ introduces terms in $\dot{a}$ which lead to pair production, see [26, 21]. In this paper, due to eq. (14), they have been neglected. For simplicity we work here with a single value of $\omega_k$ and introduce its continuum spectrum in subsequent sections to display transitions in usual terms.

With these simple choices, $\psi(\phi)$ are harmonic oscillators in $t(\eta)$ respectively. Denoting their constant amplitude by $A_M(A_\gamma)$, their energy (with respect to $t$) is

$$E_m(a) = M |A_M|^2 + \omega_k |A_\gamma|^2 / a = \epsilon_M + \epsilon_\gamma / a$$

Since both $\epsilon_M$ and $\epsilon_\gamma$ are conserved, i.e. independent of $a$, the total action is a sum of three actions

$$S_T(a, \psi, \phi) = S_G(a; \epsilon_M, \epsilon_\gamma) + S_M(\psi, \epsilon_M) + S_\gamma(\phi, \epsilon_\gamma)$$

wherein the gravitational part satisfies the Hamilton-Jacobi equation, eq. (8), with $H_m = E_m(a)$. The solution is

$$S_G(a; \epsilon_M, \epsilon_\gamma) = \int_a^a da' p(a'; \epsilon_M, \epsilon_\gamma)$$

where $p(a; \epsilon_M, \epsilon_\gamma)$ is the momentum of gravity driven by the matter energy specified by $\epsilon_M$ and $\epsilon_\gamma$:

$$p(a; \epsilon_M, \epsilon_\gamma) = -G^{-1} \left[ -Ka^2 + 2aG(\epsilon_M + \epsilon_\gamma / a) \right]^{1/2}$$

The sign is chosen to correspond to $\dot{a} > 0$, an expanding universe.

To obtain cosmological time we refer to eq. (12) but generalized to include matter in two forms, $\psi$ and $\phi$ coupled to gravity through their energy, eq. (17). Because the matter action is a sum of two terms it is now necessary to consider extremization with respect to the independent constants $\epsilon_M$ and $\epsilon_\gamma$. Variation with respect to the former correlates $\psi$ to $a$. As emphasized before, this correlation between dynamical variables is delivered in terms of the temporal parameter conjugate to $\epsilon_M$, hence given by

$$t(a; \epsilon_M, \epsilon_\gamma) = -\partial_{\epsilon_M} S_G(a; \epsilon_M, \epsilon_\gamma) = -\int_a^a da' \frac{a'}{Gp(a'; \epsilon_M, \epsilon_\gamma)}$$
Similarly variation with respect to $\epsilon_\gamma$ correlates $\phi$ to $a$ through $\eta(a; \epsilon_M, \epsilon_\gamma)$ given by

$$\eta(a; \epsilon_M, \epsilon_\gamma) = -\partial_\epsilon S_G(a; \epsilon_M, \epsilon_\gamma) = -\int^a da' \frac{1}{Gp(a'; \epsilon_M, \epsilon_\gamma)}$$  \hspace{1cm} (22)

To conclude, we have made explicit how time arises through correlation of the motion of different degrees of freedom within an isolated system. Among these there is one which can be used to define the time. This is called the clock. Periodic clocks are most useful, but they are not always available such as in cosmology where the only natural clock is $a$. The price to pay is that it entails a quadrature which encodes the unfolding of the dynamics up to the present. Indeed, this is one of the central points of debate in realistic cosmology. What is the function $p(a; \epsilon_M, \epsilon_\gamma)$ and hence what is the age of the universe?

In the next Sections we shall show how in quantum developments the classical times introduced in eqs. (21, 22) emerge from the formalism quite naturally once transitions are induced by interactions. 

### 4 Quantum Cosmology

The classical constraint, eq. (8), can be quantized in the usual way, up to an operator order ambiguity often encountered in passing from classical to quantum theory. This ambiguity occurs in $H_G$ and we shall adopt the simple rule of setting $p_a^2$ to $\partial^2_a$. There is some polemic in the literature on this point which is tied up with the interpretation of the “wave function of the universe” [12, 17, 20, 23]. This review is not concerned with this profound aspect of quantum cosmology, but rather with the behavior of matter in the universe [13, 14]. We postulate that the gravitational wave functions are WKB and justify this à posteriori by showing that corrections are negligible in a universe containing many particles. Since WKB wave functions are insensitive to the ordering problem we need no further into the question. We may remark that on this basis questions concerning interpretation and ordering seem more to do with the very beginning of the universe, rather than that which is covered by the part of cosmological history which is, at least partially, accessible to experimental and theoretical analysis. And of course one must bear in mind that in any quantum theory there are always the deeper epistemological problems which, at least at present, seem to have nothing to do with cosmology. These will not be touched upon in this either. As Gottfried says[24] we operate as “quantum mechanics”.

The quantum version of eq. (8), called the Wheeler-DeWitt (WDW) equation, is thus

$$[H_G + H_m] \Xi(a, \psi, \phi) = [G^2 \partial^2_a - Ka^2 + 2GaH_m] \Xi(a, \psi, \phi) = 0$$  \hspace{1cm} (23)

$\psi$ and $\phi$ are oscillator coördinates appearing in the matter hamiltonian $H_m$, see eqs. (16, 17). Matter is quantized in the usual way. The eigenstates of our simple model for $H_m$

---

3 This raises the question whether one can implement the same program purely classically. How does one find the history of the effects of interactions in classical terms? This has been carried out in [22].
are products of states of two harmonic oscillators. Thus, the eigenvalues are
\[
E(a; n_M, n_\gamma) = \langle n_M | \langle n_{\gamma} | H_m(a) | n_{\gamma} \rangle | n_M \rangle \\
= (n_M + 1/2)M + (n_\gamma + 1/2)(\omega_\gamma/a)
\] (24)

Since these eigenstates are stationary, i.e. \( \partial_a |n\rangle = 0 \), the general solution of eq. (23) can be written in the form
\[
\Xi(a, \psi, \phi) = \sum_{n_M, n_\gamma} c_{n_M, n_\gamma} \Psi(a, n_M, n_\gamma) \langle \psi | n_M \rangle \langle \phi | n_\gamma \rangle 
\] (25)

where the gravitational wave function \( \Psi(a, n_M, n_\gamma) \) corresponding to the matter state \(|n_\gamma\rangle |n_M\rangle\) obeys
\[
[G^2 \partial_a^2 - Ka^2 + 2GaE(a; n_M, n_\gamma)] \Psi(a, n_M, n_\gamma) = 0 
\] (26)

It is a second order equation of an operator having a continuous spectrum, so the amplitude of the solutions is determined by the Wronskian (W) (often called the current, here the expansion of the universe). The sign of W is chosen to correspond to an expanding universe and its absolute value fixed to be unity. At this stage, this is convention, however upon introducing interactions among the matter states \(|n_\gamma\rangle |n_M\rangle\), it will be seen that the Wronskians generated by \( \Psi(a, n_M, n_\gamma) \) determine \( a \)-dependent probability amplitudes which give rise to the usual statistical interpretation of quantum mechanics subject to the condition
\[
W = \Psi^* (a; n_M, n_\gamma) i \partial_a \Psi (a; n_M, n_\gamma) = 1 
\] (27)

The WKB solution of eq. (26) with unit Wronskian is
\[
\Psi(a; n_M, n_\gamma) = \frac{1}{\sqrt{2p(a; n_M, n_\gamma)}} \exp \left[ i \int_{a_0}^{a} p(a'; n_M, n_\gamma) da' \right] 
\] (28)

for all \( n_M, n_\gamma \).

The value \( a = a_0 \) is an arbitrary reference point. The value of the classical momentum \( p(a, n_M, n_\gamma) \) is read off from eq. (20) with \( E_m(a) = \epsilon_M + \epsilon_\gamma/a \) replaced its quantum expression given in eq. (24). We do not enter here into the question of how to handle closed universes near turning points \( (p(a; n_M, n_\gamma) = 0 \text{ when } K = 1) \). The material presented here is preparatory to the description of transitions in matter immersed in an expanding universe (hence the expanding phase of a closed universe). So what we have in hand is sufficient provided \( n_M \) and \( n_\gamma \) are large enough to keep the universe expanding at the value of \( a \) in question. We postpone the discussion of the legitimacy of the WKB approximation in eq. (28) to the end of next section.

Let us now briefly show how usual quantum mechanical ideas and formulae emerge in this scheme. Instead of using the configuration representatives \( \langle \psi | n_M \rangle \) and \( \langle \phi | n_\gamma \rangle \) as in eq. (23), it will be convenient to work with \( |\Xi\rangle \) as a state (a ket) in Fock space: \( |\Xi(a)\rangle = \sum_{n_M, n_\gamma} c_{n_M, n_\gamma} \Psi(a; n_M, n_\gamma) |n_M\rangle |n_\gamma\rangle \). From eq. (27), we find
\[
c_{n_M, n_\gamma} = \langle n_M | \langle n_\gamma | \Psi^* (a; n_M, n_\gamma) i \partial_a |\Xi(a)\rangle
\] (29)
and one checks $\partial_a c_{nM,n\gamma} = 0$. This of course is a consequence of our model with no matter interactions so that $nM, n\gamma$ are constants independent of $a$. Thus the normalization condition $\sum_{nM,n\gamma} |c_{nM,n\gamma}|^2 = 1$, being conserved, can and will be adopted. In order for matrix elements to take on the familiar form consistent with the statistical interpretation of quantum mechanics, we make use of eqs. (25) and (27). For example, the mean value of the energy is given by

$$\langle H_m(a) \rangle_\Xi = \langle \Xi(a) | \left[ -i \partial_a H_m(a) + i H_m(a) \partial_a \right] | \Xi(a) \rangle = \sum_{nM,n\gamma} |c_{nM,n\gamma}|^2 E(a; nM, n\gamma) \quad (30)$$

In this way, the $c_{nM,n\gamma}$ are the conventional probability amplitudes to find the matter in state $|nM\rangle |n\gamma\rangle$, in the universe. With our choice of Wronskians, gravity does not enter into these diagonal elements, save for the $a$ dependence of the eigenvalues $E(a; nM, n\gamma)$. This is in analogy to $e^{-iEt}$ dropping out in quantum mechanics. That we can construct this conventional result from the solutions of the WDW equation is to be expected. Indeed, when applying the same procedure in more conventional quantum settings such as atomic physics in a Stern Gerlach experiment[24], or the transitions of accelerated systems[27] giving rise to the Unruh effect, the same types of expressions follow.

5 Quantum Transitions in Cosmology.

Our aim is to apply the formalism of the previous Section to recover the time dependent perturbative theory of transition rates –the Golden Rule.

In quantum cosmology however, since there is no time, there is no Schrödinger equation which delivers the sense of evolution of amplitudes. This physical idea of propagation must somehow or other be encoded in the constraint in its quantum version, the WDW equation. In Section 4, we have seen that, in the absence of interaction, the formalism was set up to express the physical idea that, the $c_{nM,n\gamma}$, the probability amplitudes to find a certain number of matter quanta were fixed, i.e. independent of $a$. We now ask the following question. Suppose we know these amplitudes at a certain value of $a (= a_0)$. From this datum, how does the constraint equation deliver the values of the amplitudes at other values of $a$? If the hamiltonian interaction $H_{int}$ depends on $a$, the linear combination which represents the complete vector state $|\Xi(a_0)\rangle$ in Fock space of free matter states at $a_0$ will contain different amplitudes in this space at different values of $a$. We seek the law which governs the variation of these amplitudes, i.e. the description of evolution, a succession of events which are quantum transitions.

As shown in [19, 20], the manipulation of the WDW equation in the presence of $H_{int}$ leads to expressions which resemble conventional time dependent theory. Indeed, the momentum of gravity becomes operator valued in Fock space –in total analogy to the interaction representation of the evolution operator in usual time dependent perturbation theory. Then in the integrals over $a$ that come up (which are in fact the usual integrals over time necessary to pick up a resonance) in transition amplitudes, the difference of WKB phases of the gravitational wave functions associated with the initial and final matter states delivers what is usually called the virtual energy. Necessarily, since it is
the energy constraint that is sacrosanct in all of this. In this way, it will be seen that the use of a classical background in usual time dependent theory is replaced by the recoil effects of gravity necessitated by the energy constraint.

The model used in this section to describe matter is a 2-level atom of masses $m$ and $M$ where $(M - m)/M \ll 1$ and positive. The atom interacts with conformal radiation (scalar photons). Thus

$$\bar{H}_m = H_m + H_{int}$$

$$H_m = Md_M^dM + md_m^dm + \sum_k (\omega_k/a)d_k^d d_k$$

$$H_{int} = g \sum_k \psi_M \psi_m \phi_k$$

(31)

where the two atom states have been described by two massive fields and where $\phi_k$ is the $k$th component of the photon field, see eq. (14). The $d_i$ ($d_i^d$) are annihilation and creation operators of the three fields, c.f. eq. (24). The momentum of the atom is neglected, as is often done in atomic physics for heavy atoms wherein $\Delta m \ll M$.

We remind the reader how to recover the Golden Rule in lowest order perturbation theory. In a fixed cosmological background described by $a(t)$, the Schrödinger equation for the expansion coefficients, $c_n$, of the wave function in the basis of the eigenfunctions of $H_m$ is

$$i \frac{dc_n}{dt} = \sum_{n'} c_{n'} \langle n' | e^{i \int_{t_i}^t H_m(t') dt'} H_{int} e^{-i \int_{t_i}^t H_m(t') dt'} | n \rangle$$

(32)

where $n$ and $n'$ designate triplets of occupation numbers $n_m, n_M, n_k$ characterizing un-interacting matter states. Solving eq. (32) to lowest order in $g$ subject to $c_n(t_i) = \delta_{ni}$ gives the transition amplitude

$$A_{fi} = \int_{t_i}^{t_f} dt' \langle n_f | H_{int} | n_i \rangle e^{i \int_{t_i}^t [E_f(t') - E_i(t')]} dt'$$

(33)

For example, the amplitude of emission of photon $k$ is proportional to

$$ig \sqrt{n_k + 1} \int_{t_i}^{t_f} dt \ e^{i[\Delta m(t-t_i) - \omega_k(\eta(t)-\eta(t_i))]}$$

(34)

where the phase referring to the photon is $\int_{t_i}^{t_f} dt'/a(t') = \eta(t)-\eta(t_i)$. Conventional Golden Rule physics is recovered if $a(t)$ varies sufficiently slowly during the time necessary to pick up resonant conditions, i.e. $(\dot{a}/a) \ll \Delta m$. Then one can set $a = a(t_i)$ and the usual technique of extracting a rate applies. One squares and integrates over $k$ with the correct phase space factor. The resonant values of $\omega_k$ dominate $\omega_k/a(t_i) = \Delta m$ corresponding to energy conservation around $t_i$ and the Golden Rule rate formula ensues upon integrating over $k$.

Our aim now is to show how the formalism of Section 4 recovers eq. (32). The rest is conventional quantum mechanics as given in the above example. To this end one introduced the following trick. Rather than working with the expansion coefficients $c_n$
of eq. (24), which now will depend on \(a\) owing to \(H_{\text{int}} \neq 0\), we introduce new coefficients \(\tilde{c}_n\) defined by

\[
\tilde{c}_n(a) \equiv \langle n | \Psi^*(a, n) \rangle \langle \tilde{\Xi}(a) | \rangle
\]  

One verifies that \(\tilde{c}_n\) and \(c_n\) differ by a term in \(dc_n/da\) hence by terms in \(O(H_{\text{int}})\). We shall estimate this difference subsequently and show that it is negligible for a macroscopic universe. The advantage of working with \(\tilde{c}_n\) is the simplicity of its equation of motion which then leads directly to eq. (32). Indeed, one has

\[
-i\partial_a \tilde{c}_n = \langle n | \partial^2_a \Psi^*(a, n) - \Psi^*(a, n) \partial^2_a | \Xi(a) \rangle
\]  

We now substitute eq. (26) for \(\partial^2_a \Psi^*\) and eq. (23) for \(\partial^2_a \Xi(a)\) where \(H_m\) in that equation is here replaced by \(\tilde{H}_m\) of eq. (31). Whereas the former does not contain \(H_{\text{int}}\), the latter does, since the constraint equation (WDW) contains the full matter hamiltonian. Thus the momentum \(p_a\) carried by \(\Xi(a)\) is operator valued in Fock space. It contains the non diagonal matrix, \(H_{\text{int}}\). Gravity is thus participating in a highly non-trivial way in matter dynamics through the constraint condition. Substituting eq. (26) and eq. (23) into eq. (36), one notices that the term in \(-K a^2\) drops out. Similarly were a cosmological constant present it would cancel as well. The result is

\[
i\partial_a \tilde{c}_n = 2a \frac{\sum_{n'} c_{n'} \langle n'|H_{\text{int}}(t)|n \rangle \Psi^*(a, n) \Psi(a, n')}{G} 
\approx 2a \frac{\sum_{n'} \tilde{c}_{n'} \langle n'|H_{\text{int}}(t)|n \rangle \Psi^*(a, n) \Psi(a, n')}{G}
\]

In the second equality of eq. (37) we have written \(\tilde{c}_{n'}\) for \(c_{n'}\) in anticipation that the difference is small.

Eq. (37) is a nice equation in that \(\sum |\tilde{c}_n(a)|^2 = 1\) is conserved owing to the hermiticity of the matrix \(\langle n'|H_{\text{int}}(t)|n \rangle \Psi^*(a, n) \Psi(a, n')\). And since \(\tilde{c}_n(a)\) is to very good approximation equal to \(c_n(a)\) we have unitary evolution.

Adopting the WKB approximation for \(\Psi(a, n)\) we then have

\[
i\partial_a c_n = \sum_{n'} c_{n'} \langle n'|H_{\text{int}}(a)|n \rangle \frac{a}{G \sqrt{p(a; n)p(a, n')}} \exp [-i \int_{a_i}^{a} \left( p(a'; n') - p(a'; n) \right) da']
\]

In an objective sense, eq. (38) is the central result of this paper: It describes the unitary evolution of material processes as a function of \(a\). Its derivation turns around the Wheeler-DeWitt energy constraint. This is used in two different ways: 1) the appearance of \(H_{\text{int}}\) which has arisen from the constraint for \(|\Xi(a)\rangle\), 2) the function \(\Psi(a, n)\) which encodes unperturbed propagation. The oscillations in the integral on the phase of eq. (38) are those of the gravitational wave functions which occur during the time is takes to sort them out to get resonances \(O(1/\Delta m)\).

This derivation makes clear the essential physics of the nature of temporality in cosmology. The phase \([= S_G = \int p_a da]\) contains the macroscopic quantity \(p(a; n_M, n_m, n_r)\) of eq. (21) enlarged to include the contribution of the second massive field of mass \(m\).
For a flat universe it is proportional to $\sqrt{E_{\text{tot}}}$, $E_{\text{tot}}$ being the energy of all the matter in the universe. This sounds ridiculous, but the essential point is that it is the difference $\Delta S_G = \int \Delta p_a da$ associated with the initial and final matter states which figures in the description of the physical process and this latter is microscopic in character, albeit taking place in a macroscopic universe. Similarly $da$ is the differential of a quantity of cosmological proportions. But the particular function of $a$ which multiples $\Delta E$, is the microscopic quantity $\partial E P_a da = dt$, given by eq. (21), simply the differential of proper time, the temporal parameter which is intrinsic to the description of the rate process studied in the rest frame of our comoving heavy particles.

In a subjective historical sense, it nevertheless is important to make contact with conventional time dependent Schrödinger theory. Precisely, what are the conditions to realize the replacement of $a$ by $\bar{t}(a)$ where $\bar{t}(a)$ is the mean time, i.e. the solution of eq. (21) evaluated with the mean energy $E(\bar{n}, a)$. Unlike the previous considerations which are of purely dynamical character, these new conditions involve the specification of the initial data: the set of $c_n(a = a_0)$. Indeed, when the fluctuations in $n$ around the mean values $\bar{n}$ are small, one may expand both the prefactors and the phases of eq. (38) to first order in the changes $n - \bar{n}$ and $n' - \bar{n}$, or equivalently in $E(n, a) - E(\bar{n}, a)$. (For a discussion of higher order terms, see [19].) Upon the further use of eq. (21) to express $ada/Gp(a; \bar{n})$ as $d\bar{t}$, we obtain

$$i\left(\frac{Gp(a; \bar{n})}{a}\right)\partial_a c_n = i\partial_a c_n = \sum_{n'} c_{n'} \langle n'| H_{\text{int}} |n \rangle \ e^{-i\int_{t_0}^{t(\bar{a})}|E(a(t'), n') - E(a(t'), n)|dt'}$$

(39)

It is to be noted that the momentum prefactor ($= p(a; \bar{n})^{-1}$ to order $1/N$) is precisely what is necessary to convert $\partial_a$ to $\partial_\bar{t}$. Its origin stems from the second order character of the “free” WDW equation, eq. (21), and our choice of the Wronskians, eq. (27).

Therefore using the restriction on initial data that the fluctuations of $n_M, n_m$ and $n_s$ around their mean values are small, we can replace the $n$-dependent functions of eq. (38), by the conventional expressions governed by the mean time $\bar{t}(a)$. In this way we have recovered the Schrödinger equation, eq. (32), governed by a single time determined by the evolution of the mean gravitational background. In conclusion we emphasize that this result so familiar from fixed background physics was entirely encoded in the product of the two gravitational wave functions $\Psi^*(n_f, a)\Psi(n_i, a)$, see eq. (37). This is what delivers the space-time history of events.

It is appropriate here to clarify a confusing point that exists in the literature. This is concerned with the specification of the gravitational background that is used to extract time from the WKB wave functions. Based on Bank’s work[7], many authors make an expansion in the inverse square Planck mass. In the absence of a cosmological constant, the background so obtained is that of an empty universe, i.e. characterized by $n_M = n_s = 0$. But from eqs. (38, 39) one sees that this is incorrect: In the microscopic transition $n \rightarrow n'$, it is the background driven by the total matter energy $E(n, a)$ (or $E(\bar{n}, a)$), they differ by $O(\sqrt{N})$ out of $O(N)$ which determines the time dependent phases. The value of the Planck mass is completely irrelevant.

It remains to estimate the difference between $c_n$ and $\bar{c}_n$. From eq. (27) and eq. (35),
one has
\[ \left| \tilde{c}_n - c_n \right| = \left| (\partial_a \ln c_n) / p(a; n) \right| \cong (\partial_a \ln \tilde{c}_n) / p(a; n) \] (40)
where in the second equality we have replaced \( c_n \) by \( \tilde{c}_n \) in anticipation that the difference is small. From eq. (37), the order of magnitude of the r.h.s of eq. (40) is \( O(g M^1 N^1) \) where \( g \) is the coupling constant. Since our whole perturbative scheme depends on the smallness of \( g \), one sees that the correction is smaller than \( 1/N \).

It still remains to justify the use of the WKB approximation. As emphasized above, the physics of processes in the universe makes use of the functions \( \Psi(a, n) \), eq. (28), through eq. (37), i.e. in the form \( \Psi^*(a, n) \Psi(a, n') \) where \( n' - n \) is microscopic in character. Therefore it suffices to estimate the error induced by WKB in this quantity. After some algebra\(^4\), one finds that the local relative corrections scale like \( 1/N^4 \). Thus these corrections are also completely negligible in a macroscopic universe.

It is also worth mentioning the close analogy between the preceding development and the derivation of the canonical distribution from a microcanonical ensemble in statistical mechanics. In the latter one considers the distribution of \( \epsilon \), the energy of a small system immersed in a large one (the reservoir) of energy \( E_{\text{tot}} - \epsilon \). Its determination follows the same procedure as the determination of the lapse of time in a universe as it comes in the passage of eq. (38) to eq. (39), to wit: a first development in \( \Delta \epsilon \), the energy fluctuation of the small system. In both cases the big system is a reservoir and first order expansions in the small energy fluctuations yield the parameter that characterizes on one hand evolution, \( t = \partial_E \text{Action}_{\text{gravity}} \big|_{E=E_m} \), and on the other the distribution, \( \beta = \partial_E \text{Entropy}_{\text{reservoir}} \big|_{E_{\text{total}}-\langle \epsilon \rangle} \). More detail is given in Appendix B of [19].

6 Concluding Remarks

In this essay we have shown how the evolution of matter in the cosmos is parametrized by the cosmological time \( \bar{t}(a) \) defined by the expansion of the universe, see eq. (21). Events in the cosmos are so analyzed and indeed this is the only operational sense that one can ascribed to the word “time”.

The outcome of our analysis is the unitary evolution described by eq. (37) wherein the expanding universe is encoded in the WKB approximation to \( \Psi(a; n) \). The all important point is the dependence of \( \Psi(a; n) \) on the state of matter, the index \( n \), which is determined by the constraint eq. (26). For a macroscopic universe further reduction is possible since the microscopic transition described by \( \langle n' | H_{\text{int}} | n \rangle \) engenders but infinitesimal differences in the WKB phases displayed on the r.h.s. of eq. (38). To first order in these differences, one finds an equation which generalizes eq. (39) in that the mean time \( \bar{t}(a) \) is replaced by a “state dependent time” \( t(a, n) \) defined by eq. (21) with \( E_{\text{matter}} = E_n(a) \). Only for a macroscopic universe whose fluctuations in energy density

\[ \text{When abandoning the WKB approximation, there is also a “global” effect namely the production of back-scattered wave functions describing contracting universes, see [13]. In [21], it has been proven that eq. (38) is correct upon neglecting precisely the contribution of these back-scattered waves. The physical consequences of backscattering transitions will be presented in [23].} \]
are typically statistical (i.e. $O(N^{-1/2})$) can one replace $t(a,n)$ by $\bar{t}(a)$, for then $t(a,n)$, being a microscopic variable, also has relative fluctuation $O(N^{-1/2})$. On the contrary, when the energy density is widely distributed, i.e. far from equilibrium, such a reduction is not possible and one must fall back on the initial formulation parametrized by $a$.

It is appropriate to compare this operational approach with that of [11, 13] which focussed on the wave function $\Xi(a, \phi, \psi)$ rather than on processes encoded in it. In these works, interactions among matter degrees of freedom were not considered; hence these works are preparatory in the sense of Section 4. Thus, as in eq. (20), each matter sector, $n$, defines its corresponding gravitational wave function. When these sectors are crowded around energy eigenfunctions of nearly the same energy, one may expand the gravitational wave functions around the mean value, so as to express the dependence of $\Xi$ on $a$ in terms of the mean time $\bar{t}(a)$. However, in the absence of interactions, this procedure of making use of the quantum spread is physically empty since the sectors will never interfere.

We may note the difference between the above consideration and the analysis of the Stern Gerlach experiment [24]. In that case the center of mass motion is encoded in the construction of a wave packet of components with different asymptotic momenta. Hence, the total wave function is also a superposition of energy eigenstates. However, this wave function does not describe everything and in particular it has not been taken into account the wave function of the position measuring apparatus (that one can always introduce). It is the apparatus which is sensitive to interferences of the different components of the wave packet, thereby giving physical substance to the expansion coefficients, i.e. bringing about a localized description of the experiment, see [23] for more details. Clearly in cosmology such a procedure is inoperative since it cannot be carried out.

We wish to conclude this essay by suggesting a way to look at Mach’s principle from a quantum point of view. Mach’s principle is often stated “the distant stars determine our local inertial frame.” We should like to enlarge upon this thought so as to give it more dynamical content. Provided that first, a mean description in the sense of mini-superspace is meaningful and second, the universe is macroscopic, we have shown that one can describe cosmological events in terms of inertial concepts (i.e. the cosmological time $\bar{t}(a)$ is used in the same way as inertial time). This remarkable result, emerging from the Wheeler-DeWitt constraint, obtains because the phase differences of the WKB gravitational wave functions which occur in the description of transitions are small, i.e. energy differences induced by transitions are infinitesimal in the almost infinite sea of matter that fills the universe. It is, as it were, that gravity (described by WKB waves) possesses an enormous “inertiality” due to the enormity of the mass in the universe. Thus we are led to enlarge upon Mach’s principle in that use of the conventional notion of inertial time to describe cosmological events requires a macroscopic universe.

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