LACUNARY POLYNOMIALS IN $L^1$: GEOMETRY OF THE UNIT SPHERE

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Abstract. Let $\Lambda$ be a finite set of nonnegative integers, and let $\mathcal{P}(\Lambda)$ be the linear hull of the monomials $z^k$ with $k \in \Lambda$, viewed as a subspace of $L^1$ on the unit circle. We characterize the extreme and exposed points of the unit ball in $\mathcal{P}(\Lambda)$.

1. Introduction

Let $\mathcal{P}_N$ stand for the set of polynomials (in one complex variable) of degree at most $N$. By a lacunary polynomial, or fewnomial, in $\mathcal{P}_N$ one loosely means a polynomial therein that has “gaps,” in the sense that it is spanned by some selected monomials from the family \( \{z^k : k = 0, \ldots, N\} \) rather than by all of them. Our plan is to look at the space of fewnomials generated by an arbitrary collection \( \{z^k : k \in \Lambda\} \) with $\Lambda \subset \{0, \ldots, N\}$, endow it with a suitable norm, and study the geometry of its unit sphere.

To be precise, suppose that $N$ and $k_1, \ldots, k_M$ are positive integers satisfying
\[
k_1 < k_2 < \cdots < k_M < N.
\]
We then consider the set
\[
\Lambda := \{0, \ldots, N\} \setminus \{k_1, \ldots, k_M\}
\]
and define $\mathcal{P}(\Lambda)$ as the space of polynomials of the form $\sum_{k \in \Lambda} c_k z^k$, with complex coefficients $c_k$. In the special case where $M = 0$, the “forbidden set” $\{k_1, \ldots, k_M\}$ is empty and $\mathcal{P}(\Lambda)$ reduces to $\mathcal{P}_N$.

We shall restrict our polynomials to the circle
\[
T := \{\zeta \in \mathbb{C} : |\zeta| = 1\}
\]
(without forgetting that they actually live on $\mathbb{C}$) and embed $\mathcal{P}(\Lambda)$ in $L^1 = L^1(T)$, the space of Lebesgue integrable complex-valued functions on $T$, with norm
\[
\|f\|_1 := \frac{1}{2\pi} \int_T |f(\zeta)| \, |d\zeta|.
\]
With a function $f \in L^1$ we associate the sequence of its Fourier coefficients
\[
\hat{f}(k) := \frac{1}{2\pi} \int_T \zeta^k f(\zeta) \, |d\zeta|, \quad k \in \mathbb{Z},
\]
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and the set
\[ \text{spec } f := \{ k \in \mathbb{Z} : \hat{f}(k) \neq 0 \}, \]
called the spectrum of \( f \). Thus,
\[ \mathcal{P}(\Lambda) = \{ f \in L^1 : \text{spec } f \subset \Lambda \}. \]

Next we recall, in the framework of a general (complex) Banach space \( X = (X, \| \cdot \|) \), the geometric concepts we shall be concerned with. We write
\[ \text{ball}(X) := \{ x \in X : \| x \| \leq 1 \} \]
for the closed unit ball of \( X \). As usual, a point in \( \text{ball}(X) \) is said to be extreme for the ball if it is not an interior point of any line segment contained in \( \text{ball}(X) \). Also, an element \( \xi \) of \( \text{ball}(X) \) is called an exposed point thereof if there exists a functional \( \phi \in X^* \) of norm 1 for which
\[ \{ x \in \text{ball}(X) : \phi(x) = 1 \} = \{ \xi \} \]
(so that \( \xi \) is the unique point of contact between the ball and the appropriate hyperplane). Clearly, every exposed point is extreme, and every extreme point lies on the sphere \( \{ x \in X : \| x \| = 1 \} \).

Our goal here is to determine the two types of points in the unit sphere of \( \mathcal{P}(\Lambda) = (\mathcal{P}(\Lambda), \| \cdot \|_1) \) for an arbitrary set \( \Lambda \) as above. This will be accomplished in subsequent sections.

Meanwhile, we proceed with a brief overview of what happens in other important—and better studied—subspaces of \( L^1 \). First of all, \( \text{ball}(L^1) \) has no extreme points and hence no exposed points. Next, we take a look at the Hardy space
\[ H^1 := \{ f \in L^1 : \text{spec } f \subset [0, \infty) \}. \]
Equivalently, \( H^1 \) is formed by the \( L^1 \) functions whose Poisson integral (i.e., harmonic extension) is holomorphic on the disk
\[ \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}. \]
Recall that an \( H^1 \) function is said to be inner if it has modulus 1 a.e. on \( \mathbb{T} \), while the non-null functions \( f \in H^1 \) satisfying
\[ \log |\hat{f}(0)| = \frac{1}{2\pi} \int_{\mathbb{T}} \log |f(\zeta)| \, |d\zeta| \]
are termed outer. We refer to [12, Chapter II] for these concepts and the theory around them, including the inner-outer factorization, etc.

Now, it was proved by de Leeuw and Rudin in [3] (see also [12, Chapter IV]) that the extreme points of \( \text{ball}(H^1) \) are precisely the outer functions \( f \in H^1 \) with \( \| f \|_1 = 1 \). By contrast, it is far from clear which unit-norm (and outer) functions in \( H^1 \) arise as exposed points therein. Such functions are also known as rigid; they turn up in various connections (e.g., in relation to Toeplitz operators, see [18, 20]) and have attracted quite a bit of attention. A number of their properties are established in [17, 18, 19] and [7, Section 3].
We further discuss an extension of the de Leeuw–Rudin result, which is relevant to our topic. Let \( \varphi \) be an essentially bounded function on \( T \), and put
\[
K_1(\varphi) := \{ f \in H^1 : z\varphi f \in H^1 \}.
\]
Assume, in addition, that \( K_1(\varphi) \neq \{0\} \) (We note that \( K_1(\varphi) \) is actually the kernel in \( H^1 \) of the Toeplitz operator with symbol \( \varphi \), a map we do not formally define here. One condition that ensures \( K_1(\varphi) \neq \{0\} \) is that \( \varphi \) be a nonconstant inner function; also, if \( \varphi \equiv 0 \) then \( K_1(\varphi) \) becomes the whole of \( H^1 \).) Now, by [5, Theorem 6], a function \( f \in K_1(\varphi) \) with \( \|f\|_1 = 1 \) is an extreme point of ball(\( K_1(\varphi) \)) if and only if
\[
\text{the inner factors of } f \text{ and } z\varphi f \text{ are relatively prime}
\]
(meaning that there is no nonconstant inner function \( J \) such that \( f/J \in H^1 \) and \( z\varphi f/J \in H^1 \)). Yet another de Leeuw–Rudin type theorem related to Toeplitz kernels can be found in [10]. As far as exposed points of ball(\( K_1(\varphi) \)) are concerned, no nice description is available; see, however, [4, Section 2] for some partial results in the case where \( \varphi \) is inner.

Finally, we turn to \( P_N \), the space of polynomials \( p \) with \( \deg p \leq N \), viewed again as a subspace of \( L^1 \). One easily checks that \( P_N \) is a special case of \( K_1(\varphi) \), as defined by (1.3), with \( \varphi(z) = z^{N+1} \). Therefore, to decide whether a unit-norm polynomial \( p \in P_N \) is an extreme point of ball(\( P_N \)), we only have to rewrite condition (1.4) with this specific \( \varphi \) plugged in, and with \( p \) in place of \( f \). The role of the function \( z\varphi f \) goes then to the polynomial \( p^* \in P_N \) defined (on \( T \)) by the formula
\[
p^*(z) := z^N p(1/z), \quad z \in \mathbb{C} \setminus \{0\}.
\]
Equivalently, if
\[
p(z) = \sum_{k=0}^N c_k z^k,
\]
then \( p^*(z) = \sum_{k=0}^N \overline{c_k} z^{N-k} \). The next fact now comes out readily.

**Theorem A.** Suppose that \( p \in P_N \) and \( \|p\|_1 = 1 \). The following conditions are equivalent.

(i.A) \( p \) is an extreme point of ball(\( P_N \)).

(ii.A) The polynomials \( p \) and \( p^* \) have no common zeros in \( \mathbb{D} \).

It should be noted that whenever \( p(a) = 0 \) for a point \( a \in \mathbb{C} \setminus \{0\} \), it follows that \( p^*(1/\overline{a}) = 0 \), the multiplicities of these respective zeros being the same. Also, we have \( p(0) = 0 \) if and only if \( \deg p^* < N \); moreover, the multiplicity of zero at 0 will be at least \( \ell \) if and only if \( \deg p^* \leq N - \ell \). The roles of \( p \) and \( p^* \) are of course interchangeable in these statements, since \( (p^*)^* = p \).

This said, we can rephrase condition (ii.A) in terms of \( p \) alone. Namely, a polynomial \( (1.6) \) satisfies (ii.A) if and only if it has the properties that
\[
|c_0| + |c_N| \neq 0
\]
and

\[(1.7) \quad p \text{ has no pair of symmetric zeros with respect to } T\]

(i.e., there is no point \(a \in \mathbb{D} \setminus \{0\} \) for which \(p(a) = p(1/\overline{a}) = 0\)). It was in this form that Theorem A appeared in [8, Section 3], where the next result was also established.

**Theorem B.** Suppose that \(p \in \mathcal{P}_N \) and \(\|p\|_1 = 1\). The following are equivalent.

(i.B) \(p\) is an exposed point of ball(\(\mathcal{P}_N\)).

(ii.B) Condition (ii.A) is fulfilled, and the zeros of \(p\) lying on \(T\) (if any) are all simple.

Thus, in particular, (ii.B) includes a stronger version of property (1.7): this time, even “degenerate pairs” of symmetric zeros (i.e., multiple zeros on \(T\)) are forbidden for \(p\).

We mention in passing that Theorem A has a counterpart in the case where \(\mathcal{P}_N\) is endowed with the supremum norm \(\|\cdot\|_\infty\). Indeed, the extreme points of the unit ball in \(\langle \mathcal{P}_N, \|\cdot\|_\infty \rangle\) were characterized by the author in [9]. No such counterpart of Theorem B seems to be currently available, except for a partial result (see [16, Theorem 1.2]) that settles the case of \(\langle \mathcal{P}(\Lambda), \|\cdot\|_\infty \rangle\) for a three-point set \(\Lambda\).

Going back to the \(L^1\) setting, we now seek to extend Theorems A and B to the “lacunary” situation, where \(\mathcal{P}_N\) gets replaced by \(\mathcal{P}(\Lambda)\). When \(\Lambda\) actually has gaps, so that the set \(\{k_1, \ldots, k_M\}\) in (1.1) is nonempty, the space \(\mathcal{P}(\Lambda)\) is no longer writable as \(K_1(\varphi)\) and a new method is needed. Anyhow, one quick observation is that whenever a polynomial \(p \in \mathcal{P}(\Lambda)\) is an extreme (resp., exposed) point of ball(\(\mathcal{P}_N\)), it will also extreme (resp., exposed) for ball(\(\mathcal{P}(\Lambda)\)). Thus, the interesting case is that where condition (ii.A) (resp., (ii.B)) is violated.

There are several types of difficulties we have to face when moving from \(\mathcal{P}_N\) to \(\mathcal{P}(\Lambda)\). Most notably, dividing a polynomial from \(\mathcal{P}(\Lambda)\) by one of its elementary factors, we cannot expect the quotient to be in \(\mathcal{P}(\Lambda)\). For instance, if \(\Lambda_N := \{0, N\}\) with an integer \(N \geq 2\), then the function \(z \mapsto 1 - z^N\) is in \(\mathcal{P}(\Lambda_N)\), but the ratio

\[
\frac{1 - z^N}{1 - z} = 1 + z + \cdots + z^{N-1}
\]

is not. For similar reasons, a polynomial from \(\mathcal{P}(\Lambda)\) that vanishes at a point \(a \in \mathbb{D}\) need not be divisible by the elementary Blaschke factor \((z - a)/(1 - \overline{a}z)\); again, the quotient may well be in \(\mathcal{P}_N \setminus \mathcal{P}(\Lambda)\). Nor is \(\mathcal{P}(\Lambda)\) preserved by the map \(p \mapsto p^*\), except when \(\Lambda\) has the appropriate symmetry property.

Besides, the \(L^1\) norms of lacunary polynomials tend to be rather unhandy. To illustrate this by a historical example, we mention the notoriously wicked conjecture of Littlewood concerning the magnitude of \(\|\sum_{k \in \Lambda} z^k\|_{L^1}\), with \(\Lambda\) as above. Specifically, it was conjectured—and eventually proved, after a few decades—that this quantity is bounded below by an absolute constant times \(\log(#\Lambda)\). Two different proofs were given, one in [14] and the other in [15]. (The latter actually deals with generic lacunary polynomials; see also [6] for related results.)

The plan for the rest of the paper is as follows. In Sections 2 and 3, we state our main theorems that characterize the extreme and exposed points, respectively,
among the unit-norm polynomials in $\mathcal{P}(\Lambda)$. In both cases, the statements are followed by a brief discussion, and some examples are provided. In Section 4, we collect a few auxiliary lemmas to lean upon later. Finally, we prove our results in Sections 5 and 6.

We conclude this introduction with a couple of open questions that puzzle us. First, what happens to our results in higher dimensions (i.e., with $\mathbb{T}^d$ in place of $\mathbb{T}$, the role of $\Lambda$ being played by a finite set of multi-indices in $\mathbb{Z}^d$)? Second, what about Paley–Wiener type analogues of $\mathcal{P}(\Lambda)$ on the real line? Here, the idea would be to replace $\Lambda$ by a compact set $K \subset \mathbb{R}$ and consider the space of all entire functions $f$ with $\int_{\mathbb{R}} |f(x)| \, dx < \infty$ whose Fourier transform is supported on $K$. So far, only the classical—i.e., nonlacunary—case (where $K$ is an interval) has been studied in this connection; see [8, Section 5].

2. Extreme points: criterion and examples

Throughout, $\Lambda$ will be a fixed set of the form (1.1) and $\mathcal{P}(\Lambda)$ will be viewed as a subspace of $L^1$, normed by (1.2). To determine whether a unit-norm polynomial $p$ from $\mathcal{P}(\Lambda)$ is an extreme point of the unit ball, we first cook up a certain matrix $M = M(p)$ from it; the construction will be described in a moment. This done, the answer will be stated in terms of the rank of $M$.

Given a polynomial $p \in \mathcal{P}(\Lambda)$ with $\|p\|_1 = 1$, consider also the associated polynomial $p^*(\in \mathcal{P}_N)$ defined by (1.3). Now let $a_1, \ldots, a_n$ be an enumeration of the common zeros of $p$ and $p^*$ lying in $\mathbb{D}$. It is understood that the $a_j$’s are pairwise distinct, and we attach to them the positive integers

$$m_j := \min \{ \text{mult}(a_j, p), \text{mult}(a_j, p^*) \}, \quad j = 1, \ldots, n,$$

where $\text{mult}(a_j, \cdot)$ is the multiplicity of zero at $a_j$ for the polynomial in question. Finally, we put

$$m := \sum_{j=1}^{n} m_j \quad \text{and} \quad s := N - 2m.$$

We then introduce the polynomial

$$G(z) := \prod_{j=1}^{n} (z - a_j)^{m_j} (1 - \overline{a_j} z)^{m_j} \quad \text{(which clearly divides both } p \text{ and } p^*)$$

along with the ratio

$$R(z) := \frac{p(z)}{G(z)},$$

which is also a polynomial. The factorization $p = GR$ that arises will occasionally be referred to as canonical. Furthermore, we observe that

$$\deg R \leq s.$$

Indeed, if none of the $a_j$’s is zero, then (2.3) follows from the facts that $\deg p \leq N$ and $\deg G = 2m$. Otherwise, we may assume that $a_1 = 0$, in which case $\deg p \leq N - m_1$ (because $p^*$ is divisible by $z^{m_1}$) and $\deg G = 2m - m_1$, so (2.3) is again valid.
Letting
\begin{equation}
C_k := \hat{R}(k), \quad k \in \mathbb{Z},
\end{equation}
we therefore have
\begin{equation}
R(z) = \sum_{k=0}^{s} C_k z^k,
\end{equation}
while \( C_k = 0 \) for all \( k \in \mathbb{Z} \setminus [0, s] \). We now define
\begin{equation}
A(k) := \text{Re} C_k, \quad B(k) := \text{Im} C_k \quad (k \in \mathbb{Z})
\end{equation}
and consider, for \( j = 1, \ldots, M \) and \( l = 0, \ldots, m \), the numbers
\begin{equation}
A_{j,l}^+ := A(k_j + l - m) + A(k_j - l - m), \quad B_{j,l}^+ := B(k_j + l - m) + B(k_j - l - m)
\end{equation}
and
\begin{equation}
A_{j,l}^- := A(k_j + l - m) - A(k_j - l - m), \quad B_{j,l}^- := B(k_j + l - m) - B(k_j - l - m),
\end{equation}
the integers \( k_j \) being the same as in (1.1). From these, we build the \( M \times (m + 1) \) matrices
\begin{equation}
\mathcal{A}^+ := \{ A_{j,l}^+ \}, \quad \mathcal{B}^+ := \{ B_{j,l}^+ \}
\end{equation}
and the \( M \times m \) matrices
\begin{equation}
\mathcal{A}^- := \{ A_{j,l}^- \}, \quad \mathcal{B}^- := \{ B_{j,l}^- \}.
\end{equation}
Here, the row index \( j \) always runs from 1 to \( M \), while the column index \( l \) runs from 0 to \( m \) for each of the “plus-matrices” (2.9), and from 1 to \( m \) for each of the “minus-matrices” (2.10).

Finally, we need the block matrix
\begin{equation}
\mathfrak{M} := \begin{pmatrix} \mathcal{A}^+ & \mathcal{B}^- \\ \mathcal{B}^+ & -\mathcal{A}^- \end{pmatrix},
\end{equation}
which has \( 2M \) rows and \( 2m + 1 \) columns.

**Theorem 2.1.** Suppose that \( p \in \mathcal{P}(\Lambda) \) and \( \|p\|_1 = 1 \). Then \( p \) is an extreme point of \( \text{ball}(\mathcal{P}(\Lambda)) \) if and only if \( \text{rank} \mathfrak{M} = 2m \).

Now, since \( \text{rank} \mathfrak{M} \leq \min(2M, 2m + 1) \), the following consequence is immediate.

**Corollary 2.2.** If \( p \) is a unit-norm polynomial in \( \mathcal{P}(\Lambda) \) satisfying \( M < m \), then \( p \) is a non-extreme point of \( \text{ball}(\mathcal{P}(\Lambda)) \).

Roughly speaking, this means that if \( p \) is not too lacunary (in the sense that \( M \), the number of forbidden frequencies in (1.1), is not too large), then the situation is similar to that in the nonlacunary case, as described by Theorem A.

On the other hand, whenever a unit-norm polynomial \( p \in \mathcal{P}(\Lambda) \) happens to be an extreme point of \( \text{ball}(\mathcal{P}_N) \), it will be extreme for \( \text{ball}(\mathcal{P}(\Lambda)) \) as well. Of course, this fact follows at once from the inclusion \( \mathcal{P}(\Lambda) \subset \mathcal{P}_N \), but we can also verify it by comparing the characterizations from Theorem A and Theorem 2.1. Indeed, in this case \( p \) and \( p^* \) have no common zeros in \( \mathbb{D} \), whence \( m = 0 \). Accordingly, the polynomials (2.1) and (2.2) in the canonical factorization take the form \( G = 1 \) and
$R = p$. In particular, the coefficients $C_{kj} = A(k_j) + iB(k_j)$ in (2.5) are then null for $j = 1, \ldots, M$. This means that the blocks $A^+$ and $B^+$ in (2.11) reduce to zero columns, whereas the other two blocks are absent, so we have $\text{rank} \ M = 0 (= 2m)$.

More interesting examples are to be found among those polynomials which are non-extreme points of ball($\mathcal{P}_N$) and satisfy $M \geq m$. Two such instances will now be considered.

**Example 2.1.** Let

$$p(z) = \gamma \left(z - \frac{1}{2}\right) (2 - z) \left(1 + z^4\right),$$

where $\gamma > 0$ is the number that ensures $\|p\|_1 = 1$. Clearly, $p \in \mathcal{P}_6$ and $\hat{p}(3) = 0$, so that $p \in \mathcal{P}(\Lambda)$ with $\Lambda = \{0, 1, 2, 4, 5, 6\}$. This last set can obviously be written as (1.1), where $N = 6$, $M = 1$ and $k_1 = 3$. Also, since $p \left(\frac{1}{2}\right) = p(2) = 0$ (both zeros being simple), the polynomials $p$ and $p^\ast$ have a common zero (of multiplicity 1) at the point $a_1 = \frac{1}{2}$; moreover, they have no other common zeros in $\mathbb{D}$. The corresponding parameters are, therefore, $n = m_1 = m = 1$. The canonical factorization $p = GR$, as determined by (2.1) and (2.2), is in this case obtained by taking

$$(2.12) \quad G(z) = \left(z - \frac{1}{2}\right) \left(1 - \frac{1}{2}z\right)$$

and

$$R(z) = 2\gamma \left(1 + z^4\right).$$

Recalling the notations (2.4) and (2.6), we now have $C_k = 0$ for all $k \in \mathbb{Z} \setminus \{0, 4\}$, while $C_0 = C_4 = 2\gamma$. Equivalently, $A(k) = 0$ for all $k \in \mathbb{Z} \setminus \{0, 4\}$ and

$$A(0) = A(4) = 2\gamma,$$

whereas $B(k) = 0$ for all $k \in \mathbb{Z}$. It follows immediately that the matrix elements (2.7) with $(j, l) \in \{(1, 0), (1, 1)\}$, as well as (2.8) with $(j, l) = (1, 1)$, are all null. Consequently, $M$ is the zero matrix (of size $2 \times 3$) and its rank is 0. This number being different from $2m (= 2)$, we see from Theorem 2.1 that $p$ is a non-extreme point of ball($\mathcal{P}(\Lambda)$).

**Example 2.2.** Now let

$$p(z) = \delta \left(z - \frac{1}{2}\right) \left(8 - z^3\right),$$

where $\delta > 0$ is the normalizing constant that ensures $\|p\|_1 = 1$. This time, we have $p \in \mathcal{P}_4$ and $\hat{p}(2) = 0$, so that $p \in \mathcal{P}(\Lambda)$ with $\Lambda = \{0, 1, 3, 4\}$. We also write this set $\Lambda$ in the form (1.1), putting $N = 4$, $M = 1$ and $k_1 = 2$. As in the previous example, the polynomials $p$ and $p^\ast$ have a common zero (of multiplicity 1) at the point $a_1 = \frac{1}{2}$ and no other common zeros in $\mathbb{D}$. Thus, $n = m_1 = m = 1$. As regards the canonical factorization $p = GR$, one factor is again given by (2.12), while the other is

$$R(z) = 2\delta \left(4 + 2z + z^2\right).$$
The coefficients (2.4) are thereby known, as are the numbers (2.6), and we use these to compute the matrix entries (2.7) and (2.8). Eventually, we find that
\[ M := 2 \delta \begin{pmatrix} 4 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \]
and so rank \( M = 2(= 2m) \). Consequently, Theorem 2.1 tells us that \( p \) is an extreme point of ball(\( P(\Lambda) \)).

3. Exposed points: criterion and examples

Now we move on to describing the exposed points of ball(\( P(\Lambda) \)). The criterion, to be stated in terms of the appropriate matrix \( \tilde{M} \) built from the polynomial in question, will be close in spirit to Theorem 2.1. However, there are some adjustments to be made and some complications to be dealt with.

Once again, we fix a unit-norm element \( p \) of \( P(\Lambda) \) and we recall the canonical factorization \( p = GR \) from the preceding section, the two factors being given by (2.1) and (2.2). Furthermore, let \( \zeta_1, \ldots, \zeta_\nu \) be the distinct zeros of \( p \) lying on \( T \), and let \( \lambda_1, \ldots, \lambda_\nu \) be their respective multiplicities. We also need the (nonnegative) integers
\[ \mu_j := \lfloor \lambda_j/2 \rfloor, \quad j = 1, \ldots, \nu, \]
where \( \lfloor \cdot \rfloor \) denotes integral part, as well as the numbers
\[ \mu := \sum_{j=1}^{\nu} \mu_j, \quad \tilde{m} := m + \mu, \quad \tilde{s} := N - 2\tilde{m}. \]
We then define the polynomials \( G_0 \) and \( \tilde{G} \) by putting
\[ G_0(z) := \prod_{j=1}^{\nu} (z - \zeta_j)^{\mu_j} (1 - \overline{\zeta}_j z)^{\mu_j} \]
and
\[ \tilde{G} := GG_0. \]
Rewriting (3.3) in the form
\[ G_0(z) = \prod_{j=1}^{\nu} (-\overline{\zeta}_j)^{\mu_j} (z - \zeta_j)^{2\mu_j}, \]
and noting that \( 0 \leq 2\mu_j \leq \lambda_j \), we see that \( p \) is divisible by \( G_0 \) and hence also by \( \tilde{G} \) (because \( G \) and \( G_0 \) are relatively prime). The function
\[ \tilde{R} := p/\tilde{G} \]
is therefore a polynomial; moreover, since \( \tilde{R} = R/G_0 \) and \( \deg G_0 = 2\mu \), we deduce from (2.3) that
\[ \deg \tilde{R} \leq s - 2\mu = N - 2m - 2\mu = \tilde{s}. \]
Our further steps towards constructing the matrix $\tilde{M}$ are quite similar to what we did previously to arrive at (2.11). Namely, we write $\tilde{C}_k$ (with $k \in \mathbb{Z}$) for the $k$th Fourier coefficient of $\tilde{R}$, so that

$$\tilde{R}(z) = \sum_{k=0}^{\tilde{m}} \tilde{C}_k z^k,$$

and put

$$\tilde{A}(k) := \text{Re} \tilde{C}_k, \quad \tilde{B}(k) := \text{Im} \tilde{C}_k \quad (k \in \mathbb{Z}).$$

Next we define, for $j = 1, \ldots, M$ and $l = 0, \ldots, \tilde{m}$, the numbers

$$\tilde{A}_{j,l}^+ := \tilde{A}(k_j + l - \tilde{m}) + \tilde{A}(k_j - l - \tilde{m}), \quad \tilde{B}_{j,l}^+ := \tilde{B}(k_j + l - \tilde{m}) + \tilde{B}(k_j - l - \tilde{m})$$

and

$$\tilde{A}_{j,l}^- := \tilde{A}(k_j + l - \tilde{m}) - \tilde{A}(k_j - l - \tilde{m}), \quad \tilde{B}_{j,l}^- := \tilde{B}(k_j + l - \tilde{m}) - \tilde{B}(k_j - l - \tilde{m}).$$

This done, we build the $M \times (\tilde{m} + 1)$ matrices

$$\tilde{A}^+ := \{\tilde{A}_{j,l}^+\}, \quad \tilde{B}^+ := \{\tilde{B}_{j,l}^+\}$$

and the $M \times \tilde{m}$ matrices

$$\tilde{A}^- := \{\tilde{A}_{j,l}^-\}, \quad \tilde{B}^- := \{\tilde{B}_{j,l}^-\}.$$

Here, the row index $j$ always runs from 1 to $M$, while the column index $l$ runs from 0 to $\tilde{m}$ for each of the two matrices in (3.12), and from 1 to $\tilde{m}$ for each of those in (3.13). Finally, the block matrix $\tilde{N}$, with $2M$ rows and $2\tilde{m} + 1$ columns, is defined by

$$\tilde{N} := \begin{pmatrix} \tilde{A}^+ & \tilde{B}^- \\ \tilde{B}^+ & -\tilde{A}^- \end{pmatrix}.$$
Theorem 3.1. Suppose that \( p \in \mathcal{P}(\Lambda) \) and \( \|p\|_1 = 1 \). Then \( p \) is an exposed point of ball(\( \mathcal{P}(\Lambda) \)) if and only if \( \dim_+ N = 1 \).

To make the analogy between this result and Theorem 2.1 more transparent, we may invoke the rank-nullity identity (see, e.g., [2, p. 63]) to rewrite the condition

\[
\text{rank } M = 2m
\]

from that theorem as \( \dim N = 1 \). Here, \( N \) is the kernel of the linear map \( \mathfrak{M} : \mathbb{R}^{2m+1} \to \mathbb{R}^{2M} \) with matrix (2.11), and \( \dim N \) is the (usual) dimension of \( N \). In the current setting, a suitably adjusted version of (3.18) provides a sufficient condition for \( p \) to be an exposed point.

Corollary 3.2. Suppose that \( p \in \mathcal{P}(\Lambda) \) and \( \|p\|_1 = 1 \). If \( \text{rank } \tilde{\mathfrak{M}} = 2\tilde{m} \), then \( p \) is an exposed point of ball(\( \mathcal{P}(\Lambda) \)).

A few remarks concerning the concepts of plus-vector and plus-dimension are in order. Once again, let \( d \) be a nonnegative integer. Along with a given vector (3.15) we consider the complex numbers

\[
\gamma_0 = 2\alpha_0, \quad \gamma_k = \alpha_k + i\beta_k \quad (k = 1, \ldots, d)
\]

and then extend this collection to a two-sided sequence \( \{\gamma_k\}_{k \in \mathbb{Z}} \) by setting \( \gamma_{-k} = \gamma_k \) for \( 1 \leq k \leq d \) and \( \gamma_k = 0 \) for \( |k| > d \). In this notation, (3.18) amounts to saying that the (real) trigonometric polynomial

\[
\tau(z) := \sum_{k=-d}^{d} \gamma_k z^k \quad (z \in \mathbb{T})
\]

satisfies \( \tau(z) \geq 0 \) everywhere on the circle; indeed, the left-hand side of (3.16) equals \( \frac{1}{2}\tau(e^{it}) \). Thus, plus-vectors are essentially the coefficient vectors of nonnegative trigonometric polynomials. In terms of the associated sequence \( \{\gamma_k\} \), they are characterized (see [1]) by the familiar condition that \( \{\gamma_k\} \) is positive definite, meaning that \( \sum_{j,k \geq 0} \gamma_{j-k}\xi_j\bar{\xi}_k \geq 0 \) for any finite sequence \( \{\xi_k\} \) of complex numbers.

It is easy to find a subspace \( V \) of \( \mathbb{R}^{2d+1} \) (for some, or any, \( d \)) with the property that \( \dim_+ V \neq \dim V \). To give a trivial example with \( d = 1 \), let \( V_1 \) be the one-dimensional subspace in \( \mathbb{R}^3 \) spanned by the vector \( (0, 1, 0) \). Then \( V_1 \) does not contain any nonzero plus-vector, so that \( \dim_+ V_1 = 0 \), while \( \dim V_1 = 1 \).

A more interesting example (with \( d = 2 \), which is also more relevant to our topic, can be produced as follows. Let \( V_2 \) be the two-dimensional subspace in \( \mathbb{R}^5 \) spanned by the vectors

\[
v_1 := (1, 0, -1, 0, 0) \quad \text{and} \quad v_2 := (0, 1, 0, 0, 0).
\]

The trigonometric polynomials representing \( v_1 \) and \( v_2 \) (in the sense of the above procedure, which leads from (3.15) to (3.19)) are

\[
\tau_1(z) = -z^{-2} + 2 - z^2 \quad \text{and} \quad \tau_2(z) = z^{-1} + z,
\]

respectively. Since \( \tau_1(z) = |z^2 - 1|^2 \geq 0 \) on \( \mathbb{T} \), whereas \( \tau_2(z) = 2\text{Re } z \) changes sign on \( \mathbb{T} \), we see that \( v_1 \) is a plus-vector and \( v_2 \) is not. Moreover, given real numbers \( c_1 \) and \( c_2 \), the linear combination \( c_1 v_1 + c_2 v_2 \) will be a plus-vector if and only if \( c_1 \geq 0 \).
and \(c_2 = 0\). Indeed, whenever \(c_2 \neq 0\), the corresponding trigonometric polynomial \(c_1 \tau_1 + c_2 \tau_2\) takes values of opposite signs (namely, \(2c_2\) and \(-2c_2\)) at the points 1 and \(-1\). This means that the only plus-vectors in \(V_2\) are scalar multiples of \(v_1\), and so \(\dim_+ V_2 = 1\).

There is one special case where Theorem 3.1 reduces to a simpler criterion (namely, to Theorem 2.1 from the preceding section) that does not involve the concept of plus-dimension.

**Proposition 3.3.** Let \(p \in \mathcal{P}(\Lambda)\) and \(\|p\|_1 = 1\). Assume, in addition, that \(p\) has no multiple zeros on \(\mathbb{T}\). Then \(p\) is an exposed point of \(\text{ball}(\mathcal{P}(\Lambda))\) if and only if it is an extreme point thereof.

**Proof.** We only have to verify the “if” part. Because the polynomial’s zeros \(\zeta_j\) lying on \(\mathbb{T}\) (if any) are all simple, their multiplicities \(\lambda_j\) do not exceed 1, and the \(\mu_j\)’s in (3.1) are all null. Hence \(\mu = 0\) and \(\tilde{m} = m\). The polynomial \(G_0\), as defined by (3.3), reduces then to the constant function 1; it follows that \(\tilde{G} = G, \tilde{R} = R\) and eventually \(\tilde{M} = M\). Now, assuming that \(p\) is an extreme point of ball(\(\mathcal{P}(\Lambda)\)), we use Theorem 2.1 to arrive at (3.18). Finally, we rewrite this last condition as \(\text{rank} \tilde{M} = 2\tilde{m}\) and invoke Corollary 3.2 to infer that \(p\) is exposed. \(\square\)

In subtler cases, however, the full strength of Theorem 3.1 may be needed. We conclude this section by looking at a couple of examples to that effect.

**Example 3.1.** Let

\[
p(z) = \frac{1}{2} \left(1 - z^2\right)^2.
\]

It is easy to check that \(\|p\|_1 = 1\) and \(p \in \mathcal{P}(\Lambda)\) with \(\Lambda = \{0, 2, 4\}\). When written in the form (1.1), this set \(\Lambda\) is determined by taking \(N = 4, M = 2, k_1 = 1\) and \(k_2 = 3\). The only zeros of \(p\) are \(\zeta_1 = 1\) and \(\zeta_2 = -1\), their multiplicities being \(\lambda_1 = \lambda_2 = 2\); the corresponding parameters in (3.1) are then \(\mu_1 = \mu_2 = 1\), while those in (3.2) are \(\mu = \tilde{m} = 2\) and \(\tilde{s} = 0\). The polynomials (3.3), (3.4) and (3.6) now take the form

\[
\tilde{G}(z) = G_0(z) = -(1 - z^2)^2
\]

and

\[
\tilde{R}(z) = -\frac{1}{2}.
\]

It follows that \(\tilde{A}(0) = -\frac{1}{2}\), whereas the numbers \(\tilde{A}(k)\) (resp., \(\tilde{B}(k)\)) are null for all \(k \in \mathbb{Z} \setminus \{0\}\) (resp., for all \(k \in \mathbb{Z}\)). Using this to compute the matrix entries (3.10) and (3.11) with the appropriate values of \(j\) and \(l\), we eventually find that

\[
\tilde{M} := -\frac{1}{2}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]

The rank of this matrix is 2, so its kernel \(\tilde{N}\) has dimension 3. In fact, we also have \(\dim_+ \tilde{N} = 3\) because the vectors

\((1, 0, 0, 0, 0), (1, 0, 1, 0, 0)\) and \((1, 0, 0, 0, 1)\)
are linearly independent plus-vectors from $\tilde{N}$. An application of Theorem 3.1 now shows that $p$ is not an exposed point of $\text{ball}(P(\Lambda))$. At the same time, from Theorem A we see that $p$ is an extreme point of $\text{ball}(P_4)$ and hence of $\text{ball}(P(\Lambda))$.

**Example 3.2.** Let 

$$p(z) = c(1 - z)^2(2 + z),$$

where the number $c > 0$ is chosen so as to make $\|p\|_1 = 1$. Equivalently, we have 

$$p(z) = c(2 - 3z + z^2),$$

so that $p \in P(\Lambda)$ with $\Lambda = \{0, 1, 3\}$. To write this set $\Lambda$ in the form (1.1), we take $N = 3$, $M = 1$ and $k_1 = 2$. The other relevant parameters are $\tilde{m} = \mu = 1$ and $\tilde{s} = 1$. Furthermore, the polynomials (3.3), (3.4) and (3.6) take the form

$$\tilde{G}(z) = G_0(z) = -(1 - z)^2$$

and

$$\tilde{R}(z) = -c(2 + z).$$

Using the coefficients of the latter polynomial to compute the entries (3.10) and (3.11), we find that

$$\tilde{M} := -2c \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

The kernel $\tilde{N}$ of this matrix is one-dimensional; moreover, it is spanned by the vector $(1, -1, 0)$ which is a plus-vector. Consequently, we have $\dim_{+}\tilde{N} = 1$, and an application of Theorem 3.1 reveals that $p$ is an exposed point of $\text{ball}(P(\Lambda))$. (Alternatively, since $\text{rank}\tilde{M} = 2$, we may arrive at the same conclusion via Corollary 3.2.) One final observation is that $p$ fails to be exposed in $\text{ball}(P_3)$, as Theorem B shows.

4. **Preliminaries**

This section contains three lemmas (at least two of them known) that will be employed later on. Below, we write $L^\infty_\mathbb{R}$ for the set of real-valued functions in $L^\infty = L^\infty(\mathbb{T})$, the space of essentially bounded functions on $\mathbb{T}$.

**Lemma 4.1.** Let $X$ be a subspace of $L^1$. Suppose also that $f \in X$ is a function with $\|f\|_1 = 1$ that does not vanish a.e. on $\mathbb{T}$. The following conditions are equivalent.

(i.1) $f$ is an extreme point of $\text{ball}(X)$.

(ii.1) Whenever $h \in L^\infty_\mathbb{R}$ and $fh \in X$, we have $h = \text{const } a.e. \text{ on } \mathbb{T}$.

A proof can be found in [11, Chapter V, Section 9], where the case $X = H^1$ was treated (but without using any specific properties of $H^1$). The same argument works for an arbitrary subspace $X \subset L^1$ as well.

**Lemma 4.2.** Under the assumptions of the preceding lemma, the following statements are equivalent.

(i.2) $f$ is an exposed point of $\text{ball}(X)$.

(ii.2) Whenever $h$ is a nonnegative measurable function on $\mathbb{T}$ for which $fh \in X$, we have $h = \text{const } a.e.$
This result appears—in a slightly more general form—as Lemma 1(B) in [5]. In fact, for $X = H^1$, the equivalence between (i.2) and (ii.2) is also implied by de Leeuw and Rudin’s work in [3, Subsection 4.2], even though their wording and notation may differ somewhat from ours. Moreover, their reasoning carries over to a generic subspace $X$ of $L^1$.

Our last lemma generalizes the classical fact (see [12, p. 92]) that any nonnegative function in the Hardy class $H^{1/2}$ is constant. Here, $H^{1/2}$ can be defined as the closure of $H^1$ in $L^{1/2} = L^{1/2}(\mathbb{T})$.

Lemma 4.3. Given an integer $k \geq 0$, suppose that $f \in H^{1/2}$ and $fz^{-k} \geq 0$ a.e. on $\mathbb{T}$. Then $f \in \mathcal{P}_{2k}$ (i.e., $f$ is a polynomial of degree at most $2k$).

Proof. We may assume that $f \not\equiv 0$. A standard factorization theorem for Hardy spaces (see [12, Chapter II]) tells us that $f$ has the form $Bg^2$, where $B$ is a Blaschke product and $g \in H^1$. In particular, since $|B| = 1$, we have

$$|f| = |g|^2 = \bar{g}g$$

(the identities involved are always assumed to hold a.e. on $\mathbb{T}$). On the other hand, the hypothesis that $fz^{-k} \geq 0$ yields

$$|f| = fz^{-k} = Bg^2z^{-k}.$$  

Comparing (4.1) and (4.2), we see that $\bar{g}g = Bg^2z^{-k}$, or equivalently,

$$\bar{g} = Bgz^{-k}.$$  

Now, the functions $g$ and $Bg$ are both in $H^1$, so their spectra are contained in $[0, \infty)$. It follows that

$$\text{spec } g \subset (-\infty, 0] \quad \text{and} \quad \text{spec } (Bgz^{-k}) \subset [-k, \infty).$$

At the same time, the two spectra in (4.4) are equal by virtue of (4.3), so they are actually contained in $[-k, 0]$. This in turn implies that

$$\text{spec } g \subset [0, k] \quad \text{and} \quad \text{spec}(Bg) \subset [0, k].$$

In other words, $g$ and $Bg$ are both in $\mathcal{P}_k$. Consequently, their product (which is $f$) lies in $\mathcal{P}_{2k}$, as required. \qed

5. Proof of Theorem 2.1

Let $p \in \mathcal{P}(\Lambda)$ and $\|p\|_1 = 1$. In view of Lemma 4.1, the issue boils down to deciding whether $p$ can be multiplied by a nonconstant function $h \in L^\infty_{\mathbb{R}}$ to produce another polynomial in $\mathcal{P}(\Lambda)$.

Our method consists essentially in parametrizing the class of eligible functions $h$. So let us assume that $h \in L^\infty_{\mathbb{R}}$ and the product $ph =: q$ is in $\mathcal{P}(\Lambda)$. We have then

$$h = \frac{q}{p} = \frac{\bar{q}}{\bar{p}} = \frac{z^N \bar{q}}{z^N \bar{p}} = \frac{z^N}{z^N p} = \frac{q^*}{p^*}$$

on $\mathbb{T}$, whence in particular

$$pq^* = p^*q.$$
This last identity must actually hold everywhere in $\mathbb{C}$, since both sides are polynomials.

Next, we recall the factorization $p = GR$, where the two factors are defined by (2.1) and (2.2), and we go on to claim that the ratio

$$(5.3) \quad Q := \frac{q}{R}$$

is a polynomial. To see why, let $\zeta_1, \ldots, \zeta_\nu$ be the distinct zeros of $p$ lying on $\mathbb{T}$, and let $\lambda_1, \ldots, \lambda_\nu$ be their respective multiplicities (we stick to the notation of Section 3). The $\zeta_j$'s are then also zeros for $p^*$, with the same multiplicities. Consequently, the polynomial

$$(5.4) \quad \mathcal{T}(z) := \prod_{j=1}^\nu (z - \zeta_j)^{\lambda_j}$$

divides both $p$ and $p^*$, while the product

$$(5.5) \quad G\mathcal{T} =: \Phi$$

is the greatest common divisor (GCD) of $p$ and $p^*$. For future reference, we write down the latter fact as

$$(5.6) \quad \Phi = \text{GCD}(p, p^*).$$

We further remark that $\mathcal{T}$ divides $R$ (because $R = p/G$ and $G$ has no zeros on $\mathbb{T}$), so that

$$(5.7) \quad R/\mathcal{T} =: \Psi$$

is a polynomial and

$$p = G\mathcal{T} \cdot \frac{R}{\mathcal{T}} = \Phi \Psi.$$

Plugging this into (5.2) yields

$$(5.8) \quad \Psi q^* = \frac{p^*}{\Phi} t,$$

and since the polynomials $p/\Phi(= \Psi)$ and $p^*/\Phi$ are relatively prime by virtue of (5.6), it follows from (5.8) that $\Psi$ divides $q$. At the same time, $q$ is divisible by $\mathcal{T}$, since otherwise the ratio $q/p(= h)$ would not be essentially bounded on $\mathbb{T}$. Finally, because the polynomials $\Psi$ and $\mathcal{T}$ are relatively prime (indeed, they have disjoint zero sets) and each of them divides $q$, we conclude that $q$ is divisible by their product, which is $R$. This proves our claim that the function $Q$ in (5.3) is a polynomial.

Going back to the identity $h = q/p$, we now combine it with the equalities $p = GR$ and $q = QR$ to find that

$$(5.9) \quad h = Q/G.$$

Furthermore, we have the elementary formula

$$(5.10) \quad G(z) = z^n \prod_{j=1}^n |z - a_j|^{2m_j}, \quad z \in \mathbb{T}.$$
which holds because \(1 - \bar{\alpha}_j z = z(\bar{\alpha} - \alpha_j)\) for all \(j\) and all \(z \in \mathbb{T}\), and together with (5.9) this shows that
\[
(5.11) \quad z^{-m}Q(z) = h(z) \prod_{j=1}^{n} |z - a_j|^{2m}, \quad z \in \mathbb{T}.
\]
Consequently, the function \(z \mapsto z^{-m}Q(z)\) is real-valued on \(\mathbb{T}\), so its spectrum, \(\text{spec}(z^{-m}Q)\), is symmetric with respect to the origin. This function is also a trigonometric polynomial (because \(Q\) is an analytic polynomial, as explained above); and since \(\text{spec} Q \subset [0, \infty)\), it follows easily that
\[
\text{spec}(z^{-m}Q) \subset [-m, m].
\]
The coefficients
\[
(5.12) \quad d_l := \hat{(z^{-m}Q)}(l) = \hat{Q}(l + m), \quad l \in \mathbb{Z},
\]
are therefore null for \(|l| > m\) and satisfy the relations
\[
(5.13) \quad d_{l-m} = \overline{d_l} \quad \text{for} \quad l = 0, \ldots, m
\]
(whence, in particular, \(d_0 \in \mathbb{R}\)). We have then
\[
(5.14) \quad Q(z) = \sum_{l=0}^{2m} d_{l-m}z^l,
\]
and there are further restrictions on the \(d_l\)'s coming from the fact that the polynomial \(q = QR\) is in \(\mathcal{P}(\Lambda)\).
To make these explicit, let us compute the Fourier coefficients \(\hat{q}(k)\) in terms of the numbers (2.4) and (5.12). For a fixed \(k \in \mathbb{Z}\), we get
\[
(5.15) \quad \hat{q}(k) = \sum_{l=0}^{2m} \hat{R}(k-l)\hat{Q}(l) = \sum_{l=0}^{2m} C_{k-l} d_{l-m} = \sum_{l=-m}^{m} C_{k-l-m} d_l
\]
where the last step relies on (5.13). Now, since \(q \in \mathcal{P}(\Lambda)\), we have the conditions
\[
(5.16) \quad \hat{q}(k_j) = 0 \quad \text{for} \quad j = 1, \ldots, M,
\]
and (5.15) allows us to rewrite them as
\[
(5.17) \quad \sum_{l=0}^{m} C_{k_j-l-m} d_l + \sum_{l=1}^{m} C_{k_j+l-m} \overline{d_l} = 0.
\]
We now introduce the real parameters \(\alpha_0, \alpha_1, \ldots, \alpha_m\) and \(\beta_1, \ldots, \beta_m\) setting
\[
(5.18) \quad d_0 = 2\alpha_0, \quad d_l = \alpha_l + i\beta_l \quad \text{for} \quad l = 1, \ldots, m.
\]
(This done, (5.14) takes the form
\[
(5.19) \quad Q(z) = 2z^m \left\{ \alpha_0 + \text{Re} \sum_{l=1}^{m} (\alpha_l + i\beta_l)z^l \right\}, \quad z \in \mathbb{T},
\]
a formula to be noted for later reference.) Also, in accordance with (2.6) we write
\[ C_r = A(r) + i B(r), \quad r \in \mathbb{Z}, \]
with \( A(r) \) and \( B(r) \) real. Plugging (5.18) and (5.20) into (5.17), we then split each of the resulting equations into a real and imaginary part to obtain
\[ \sum_{l=0}^{m} A_{j,l}^+ \alpha_l + \sum_{l=1}^{m} B_{j,l}^- \beta_l = 0 \quad (j = 1, \ldots, M) \]
(5.21) and
\[ \sum_{l=0}^{m} B_{j,l}^+ \alpha_l - \sum_{l=1}^{m} A_{j,l}^- \beta_l = 0 \quad (j = 1, \ldots, M), \]
(5.22)
where the notations (2.7) and (2.8) have been used.

The system of \( 2M \) real equations (5.21) \& (5.22), which has thus emerged, means that the vector
\[ (\alpha, \beta) := (\alpha_0, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m) \]
(5.23)
(or rather the column vector \((\alpha, \beta)^T\), where the superscript \( T \) denotes transposition) belongs to the subspace
\[ \mathcal{N} := \ker \mathfrak{M}, \]
(5.24)
the kernel of the linear map \( \mathfrak{M} : \mathbb{R}^{2m+1} \to \mathbb{R}^{2M} \) defined by (2.11).

To summarize, every function \( h \in L^\infty_\mathbb{R} \) for which \( ph \in \mathcal{P}(\Lambda) \) is given by (5.9), where \( Q \) is a polynomial of the form (5.19) whose coefficient vector (5.23) satisfies \((\alpha, \beta) \in \mathcal{N}\). Conversely, for every vector (5.23) from \( \mathcal{N} \), we may consider the associated polynomial (5.19) and use it to define the function \( h \) by (5.9). This \( h \) will be in \( L^\infty_\mathbb{R} \), thanks to (5.11), and the polynomial \( q = QR(= ph) \) will be in \( \mathcal{P}(\Lambda) \). Indeed, from (2.3) and (5.14) it follows that \( q \in \mathcal{P}_N \), while the conditions (5.16) are ensured by (5.21) and (5.22).

The constant function \( h = 1 \) corresponds to the choice \( Q = G \); the associated coefficient vector, say \((\alpha_G, \beta_G)\), is then a nonzero element of \( \mathcal{N} \), so we always have \( \dim \mathcal{N} \geq 1 \). Moreover, it is now clear that a nonconstant function \( h \in L^\infty_\mathbb{R} \) with \( ph \in \mathcal{P}(\Lambda) \) can be found if and only if \( \dim \mathcal{N} > 1 \). (In fact, for such an \( h \) to exist, there should be a vector \((\alpha, \beta) \in \mathcal{N} \) other than a scalar multiple of \((\alpha_G, \beta_G)\).) Consequently, in view of Lemma 4.1 this last condition characterizes the non-extreme points \( p \). In other words, the extreme points of \( \text{ball}(\mathcal{P}(\Lambda)) \) are precisely those unit-norm polynomials \( p \in \mathcal{P}(\Lambda) \) for which \( \dim \mathcal{N} = 1 \).

Finally, because the rank-nullity theorem tells us that
\[ \text{rank} \mathfrak{M} + \dim \mathcal{N} = 2m + 1, \]
the criterion just found can also be stated in the form \( \text{rank} \mathfrak{M} = 2m \). The proof is complete.
Proof of Theorem 3.1. Let \( p \in \mathcal{P}(\Lambda) \) and \( \|p\|_1 = 1 \). This time, in view of Lemma 4.2, we need to determine whether \( p \) can be multiplied by a nonconstant function \( h \geq 0 \) to produce another polynomial in \( \mathcal{P}(\Lambda) \). So let us assume that \( h \) is a nonnegative function on \( T \) and the product \( ph = q \) is in \( \mathcal{P}(\Lambda) \). This brings us to identities (5.1) exactly as before (indeed, to verify them, one only uses the fact that \( h \) is real-valued). As a consequence, (5.2) holds true on \( T \) and hence everywhere in \( \mathbb{C} \).

In what follows, we retain the notation established in Sections 2 and 3 above. In addition to the various polynomials (such as \( G, R, G_0, \tilde{G}, \tilde{R} \)) that were introduced there, we shall make use of the polynomials \( \mathcal{T}, \Phi \) and \( \Psi \) coming from the preceding proof; see formulas (5.4), (5.5) and (5.7) for definitions. In particular, the identities (6.1)

\[
p = \Phi \Psi = \tilde{G} \tilde{R}
\]

should be kept in mind.

We now recall the formula (3.5), along with the inequalities

\[
0 \leq 2\mu_j \leq \lambda_j, \quad j = 1, \ldots, \nu,
\]

to see that \( G_0 \) divides \( \mathcal{T} \). We then look at the polynomial

\[
S := \mathcal{T}/G_0
\]

and note that its zeros (if any) are all simple and contained among the \( \zeta_j \)'s; in fact, the zeros of \( S \) are precisely those \( \zeta_j \)'s for which \( \lambda_j \) is odd. The identity

(6.2)

\[
\mathcal{T} = G_0 S
\]

thus provides a (standard) representation of \( \mathcal{T} \) as the product of a square and a square-free factor. Furthermore, multiplying both sides of (6.2) by \( G \) yields

\[
\Phi = G \mathcal{T} = GG_0 S = \tilde{G} S,
\]

and we may combine the resulting equality \( \Phi = \tilde{G} S \) with (6.1) to infer that

(6.3)

\[
\tilde{R} = S \Psi.
\]

On the other hand, going back to (5.2) and rewriting it as (5.8), we couple this with (6.5) to deduce (just as we did in the preceding section) that \( \Psi \) divides \( q \). The ratio \( q/\Psi =: Q_0 \) is therefore a polynomial. Yet another function we need is

(6.4)

\[
\tilde{Q} := \frac{Q_0}{S} \left( = \frac{q}{S \Psi} \right).
\]

Using (6.1), (6.3) and (6.4), we now obtain

\[
h = \frac{q}{p} = \frac{q}{GR} = \frac{q}{G S \Psi} = \frac{\tilde{Q}}{G},
\]

so that

(6.5)

\[
h = \frac{\tilde{Q}}{G}.
\]

Also, from (6.3) and (6.4) we see that

(6.6)

\[
q = \tilde{Q} \tilde{R}.
\]
This said, let us pause to take a closer look at \( \tilde{Q} \). Because \( Q_0 \) and \( S \) are both polynomials, whereas the zeros of \( S \) are all simple and contained in \( \mathbb{T} \), it follows that \( \tilde{Q}(=Q_0/S) \) belongs to every Hardy space \( H^{1-\varepsilon} \) with \( 0 < \varepsilon < 1 \). In particular,

(6.7) \[
\tilde{Q} \in H^{1/2}.
\]

Besides, we claim that

(6.8) \[
 z^{-\tilde{m}} \tilde{Q}(z) \geq 0, \quad z \in \mathbb{T},
\]
a fact to be verified in a moment. Indeed, we know from (5.10) that \( z^{-m}G(z) \geq 0 \) for all \( z \in \mathbb{T} \); similarly, the identity

\[
G_0(z) = z^\mu \prod_{j=1}^{\nu} |z - \zeta_j|^{2\mu_j}, \quad z \in \mathbb{T},
\]
implies that \( z^{-\mu}G_0(z) \geq 0 \) on \( \mathbb{T} \). The two inequalities together yield

(6.9) \[
 z^{-\tilde{m}} \tilde{G}(z) = z^{-m}G(z) \cdot z^{-\mu}G_0(z) \geq 0
\]
(recall that \( \tilde{m} = m + \mu \) and \( \tilde{G} = G G_0 \)). At the same time, (6.5) tells us that \( \tilde{Q} = \tilde{G} \tilde{h} \) and so

(6.10) \[
 z^{-\tilde{m}} \tilde{Q}(z) = z^{-\tilde{m}} \tilde{G}(z) \tilde{h}(z).
\]

Finally, our claim (6.8) follows from (6.10), thanks to (6.9) and the hypothesis that \( \tilde{h} \geq 0 \).

Now that we have (6.7) and (6.8) at our disposal, an application of Lemma 4.3 shows that \( \tilde{Q} \in \mathcal{P}_{2\tilde{m}} \). Consequently, the function \( z \mapsto z^{-\tilde{m}} \tilde{Q}(z) \) is a nonnegative trigonometric polynomial with spectrum in \( [-\tilde{m}, \tilde{m}] \). Denoting its \( l \)th Fourier coefficient by \( \tilde{d}_l \), we have

(6.11) \[
\tilde{Q}(z) = \sum_{l=0}^{2\tilde{m}} \tilde{d}_{l-\tilde{m}} z^l.
\]

We also know that \( \tilde{d}_{-l} = \overline{\tilde{d}_l} \) for all \( l \in \mathbb{Z} \). Setting

(6.12) \[
\tilde{d}_0 = 2\tilde{\alpha}_0 \quad \text{and} \quad \tilde{d}_l = \tilde{\alpha}_l + i\tilde{\beta}_l \quad (l = 1, \ldots, \tilde{m}),
\]
with \( \tilde{\alpha}_l \) and \( \tilde{\beta}_l \) real, we now rewrite (6.11) in terms of these real parameters to get

(6.13) \[
\tilde{Q}(z) = 2z^{\tilde{m}} \left\{ \tilde{\alpha}_0 + \text{Re} \sum_{l=1}^{\tilde{m}} (\tilde{\alpha}_l + i\tilde{\beta}_l) z^l \right\}, \quad z \in \mathbb{T}.
\]

We may therefore rephrase condition (6.8) by saying that the vector

(6.14) \[
(\tilde{\alpha}, \tilde{\beta}) := (\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{\tilde{m}}, \tilde{\beta}_1, \ldots, \tilde{\beta}_{\tilde{m}})
\]
is a plus-vector in \( \mathbb{R}^{2\tilde{m}+1} \).
To gain full information about the $\tilde{\alpha}_l$'s and $\tilde{\beta}_l$'s, we should also recall the relation (6.6) and use the fact that $q \in \mathcal{P}(\Lambda)$. This amounts to recasting conditions (5.16) in terms of the coefficients of $\tilde{Q}$ and $\tilde{R}$. Eventually, these conditions take the form

$$\sum_{l=0}^{\tilde{m}} \tilde{A}_{j,l}^+ \tilde{\alpha}_l + \sum_{l=1}^{\tilde{m}} \tilde{B}_{j,l}^- \tilde{\beta}_l = 0 \quad (j = 1, \ldots, M)$$

and

$$\sum_{l=0}^{\tilde{m}} \tilde{B}_{j,l}^+ \tilde{\alpha}_l - \sum_{l=1}^{\tilde{m}} \tilde{A}_{j,l}^- \tilde{\beta}_l = 0 \quad (j = 1, \ldots, M),$$

where the notations (3.10) and (3.11) are being used. (The calculations leading to (6.15) and (6.16) are almost identical to those made in the preceding section, when moving from (5.16) to (5.21) and (5.22). We only have to replace the factorization $q = QR$, used at that stage, by $q = \tilde{Q}\tilde{R}$ and proceed accordingly with the coefficients of the polynomials involved. Formally, we just switch to the tilde notation wherever applicable.)

Now, equations (6.15) and (6.16) tell us that the vector (6.14), or rather the column vector $(\tilde{\alpha}, \tilde{\beta})^T$, belongs to the subspace $\tilde{N}$, defined (in accordance with (3.17)) as the kernel of the linear map $\tilde{\mathfrak{M}} : \mathbb{R}^{2\tilde{m}+1} \to \mathbb{R}^M$ with matrix (3.14).

In summary, every nonnegative function $h$ with $ph \in \mathcal{P}(\Lambda)$ is given by (6.5), where $\tilde{Q}$ is a polynomial of the form (6.13) such that (6.14) is a plus-vector from $\tilde{N}$. Conversely, for every plus-vector (6.14) from $\tilde{N}$, we may consider the associated polynomial (6.13) and then define the function $h$ by (6.5). This $h$ will be nonnegative, thanks to (6.8) and (6.9), and the polynomial $q = \tilde{Q}\tilde{R}(= ph)$ will be in $\mathcal{P}(\Lambda)$. Indeed, from (3.7) and (6.11) it follows that $q \in \mathcal{P}_N$, while the conditions (5.16) are ensured by (6.15) and (6.16).

The constant function $h = 1$ corresponds to the choice $\tilde{Q} = \tilde{G}$. Letting $(\tilde{\alpha}_*, \tilde{\beta}_*)$ stand for the coefficient vector (6.14) that arises in this special case, we see that $(\tilde{\alpha}_*, \tilde{\beta}_*)$ is a nonzero plus-vector in $\tilde{N}$. Thus, we always have

$$\dim_+ \tilde{N} \geq 1.$$  

Moreover, the above discussion reveals that a nonconstant function $h \geq 0$ with $ph \in \mathcal{P}(\Lambda)$ can be found if and only if $\tilde{N}$ contains plus-vectors that are not scalar multiples of $(\tilde{\alpha}_*, \tilde{\beta}_*)$. This last condition, meaning that $\dim_+ \tilde{N} > 1$, is therefore equivalent to saying that $p$ is a non-exposed point of ball($\mathcal{P}(\Lambda)$). In other words, the exposed points are characterized by the property that $\dim_+ \tilde{N} = 1$. The proof is complete.

**Proof of Corollary 7.2.** We know from the rank-nullity theorem that

$$\text{rank } \tilde{\mathfrak{M}} + \dim \tilde{N} = 2\tilde{m} + 1.$$
Consequently, the hypothesis $\text{rank} \tilde{M} = 2 \tilde{m}$ implies that $\dim \tilde{N} = 1$ and hence $\dim_+ \tilde{N} \leq 1$. In conjunction with (6.17), this means that $\dim_+ \tilde{N}$ is actually equal to 1, and the result follows by Theorem 3.1. □

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