UNIFIED FIELD EQUATIONS COUPLING FOUR FORCES AND PRINCIPLE OF INTERACTION DYNAMICS

TIAN MA AND SHOUHONG WANG

Abstract. The main objective of this article is to postulate a principle of interaction dynamics (PID) and to derive unified field equations coupling the four fundamental interactions based on first principles. PID is a least action principle subject to div A-free constraints for the variational element with A being gauge potentials. The Lagrangian action is uniquely determined by 1) the principle of general relativity, 2) the \( U(1) \), \( SU(2) \) and \( SU(3) \) gauge invariances, 3) the Lorentz invariance, and 4) principle of principle of representation invariance (PRI), introduced in [11]. The unified field equations are then derived using PID. The unified field model spontaneously breaks the gauge symmetries, and gives rise to a new mechanism for energy and mass generation. The unified field model introduces a natural duality between the mediators and their dual mediators, and can be easily decoupled to study each individual interaction when other interactions are negligible. The unified field model, together with PRI and PID applied to individual interactions, provides clear explanations and solutions to a number of outstanding challenges in physics and cosmology, including e.g. the dark energy and dark matter phenomena, the quark confinement, asymptotic freedom, short-range nature of both strong and weak interactions, decay mechanism of sub-atomic particles, baryon asymmetry, and the solar neutrino problem.

Contents

1. Introduction 2
2. Principle of Interaction Dynamics (PID) 8
  2.1. PID 8
  2.2. Principle of representation invariance (PRI) 10
3. Essentials of the Unified Field Theory Based on PID and PRI 11
  3.1. Symmetries 11
  3.2. Mechanism of fundamental interactions 12
  3.3. Geometry of unified fields 14
  3.4. Gauge symmetry breaking 17
4. Unified Field Equations Based on PID and PRI 18
  4.1. Unified field equations based on PID 18
  4.2. Coupling parameters and physical dimensions 19
  4.3. Standard form of unified field equations 21

Key words and phrases. principle of interaction dynamics (PID), principle of representation invariance, fundamental interactions, dark energy, dark matter, unified field equations, gauge symmetry breaking, Higgs mechanism, dual particle fields, quark confinement, asymptotic freedom, short-range nature of strong and weak interactions, solar neutrino problem, baryon asymmetry.

The authors are grateful for the referee’s insightful comments and suggestions. The work was supported in part by the Office of Naval Research, by the US National Science Foundation, and by the Chinese National Science Foundation.
1. Introduction

The four fundamental forces/interactions of Nature are the electromagnetic interaction, the strong interaction, the weak interaction and the gravity. Current successful theories describing these interactions include the Einstein general theory of relativity for gravitation, and the Standard Model, built upon an $SU(3) \otimes SU(2) \otimes U(1)$ gauge theory for electromagnetic, the weak and the strong interactions; see among many others [6, 17, 2, 7, 4]. Apparently a unified field theory will be built upon the success of gauge theory and the Einstein general theory of relativity. There are, however, a number of unexplained mysteries of Nature, including for example the dark energy and dark matter phenomena, quark confinement, asymptotic freedom, short-range nature of both strong and weak interactions, decay mechanism of sub-atomic particles, and the solar neutrino problem.

The main objectives of this article are 1) to postulate a new fundamental principle, which we call the principle of interaction dynamics (PID), and 2) to derive a unified field theory coupling all four interactions based on the following fundamental first principles:

- the principle of general relativity,
- the principle of gauge invariance,
- the principle of Lorentz invariance,
- the principle of interaction dynamics (PID), and
- the principle of representation invariance (PRI).

PID is introduced in this article, and PRI is postulated in [11]. Intuitively, PID takes the variation of the action functional under energy-momentum conservation constraint. PRI requires that physical laws be independent of representations of the gauge groups.

Hereafter we describe the main ideas and ingredients of this article.

Dark matter and dark energy as evidence for PID

There are strong physical evidences for the validity of PID. The first is the discovery of dark matter and dark energy. As the law of gravity, the Einstein gravitational field equations are inevitably needed to be modified to account for dark energy and dark matter. Over the years, there are numerous attempts, which can be classified into two groups: 1) $f(R)$ theories, and 2) scalar field theories. Also, the chief current attempts for dark matter are both direct and indirect searches for dark matter particles such as the weakly interactive massive particles (WIMPs).
Unfortunately, both dark energy and dark matter are still two greatest mysteries in modern physics.

In [12], we attack this problem in the fundamental level based on first principles. Since Albert Einstein discovered the general theory of relativity in 1915, his two fundamental first principles, the principle of equivalence (PE) and the principle of general relativity (PGR) have gained strong and decisive observational supports. These two principles amount to saying that the space-time manifold is a 4-dimensional (4D) Riemannian manifold $M$ with the Riemannian metric being regarded as the gravitational potentials.

The law of gravity is then represented by the gravitational field equations solving for the gravitational potentials $\{g_{\mu\nu}\}$, the Riemannina metric of the space-time manifold $M$. The principle of general relativity is a fundamental symmetry of Nature. This symmetry principle, together with the simplicity of laws of Nature, uniquely determines the Lagrangian action for gravity as the Einstein-Hilbert functional:

$$L_{EH}(\{g_{\mu\nu}\}) = \int_M \left( R + \frac{8\pi G}{c^4} S \right) \sqrt{-g} dx.$$  

Here $R$ stands for the scalar curvature of $M$, and $S$ is the energy-momentum density of matter field in the universe. In fact, this very symmetry is the main reason why the current $f(R)$ and scalar field theories can only yield certain approximations of the law of gravity.

We observe in [12] that due to the presence of dark energy and dark matter, the energy-momentum tensor $T_{\mu\nu}$ of normal matter is in general no longer conserved:

$$\nabla_\mu (T_{\mu\nu}) \neq 0.$$  

By the Orthogonal Decomposition Theorem [6.1] the Euler-Lagrange variation of the Einstein-Hilbert functional $L_{EH}$ is uniquely balanced by the co-variant gradient of a vector field $\Phi_\mu$:

$$\nabla_\mu \Phi_\nu,$$

leading to the following new gravitational field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} + \nabla_\mu \Phi_\nu,$$

which give rise to a unified theory for dark matter and dark energy [12].

Equivalently, as we have shown in [12] that the Euler-Lagranian variation of the Einstein-Hilbert functional $L_{EH}$ must be taken under energy-momentum conservation constraints; see [12] for details:

$$\frac{d}{d\lambda} \bigg|_{\lambda=0} L_{EH}(g_{\mu\nu} + \lambda X_{\mu\nu}) = (\delta L_{EH}(g_{\mu\nu}) , X) = 0 \quad \forall X = \{X_{\mu\nu}\} \text{ with } \nabla_\mu X_{\mu\nu} = 0.$$  

The term $\nabla_\mu \Phi_\nu$ does not correspond to any Lagrangian action density, and is the direct consequence of energy-momentum conservation constraint of the variation element $X$ in [1.3]. For the case given here, the vector field $\Phi_\nu$ is in fact the gradient of a scalar field $\varphi$: $\Phi_\nu = -\nabla_\nu \varphi$. However, if we take the cosmic microwave background radiation into consideration, the field equations are in a more general
form with the vector field $\Phi_\nu$; see \cite{5.5}:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu} + \left(\nabla_\mu + \frac{e}{\hbar c}A_\mu\right)\Phi_\nu,$$

where the term $\frac{e}{\hbar c}A_\mu\Phi_\nu$ represents the coupling between the gravitation and the microwave background radiation.

We have shown that it is the duality between the attracting gravitational field $\{g_{\mu\nu}\}$ and the repulsive dual field $\{\Psi_\mu\}$, and their nonlinear interaction that give rise to gravity, and in particular the gravitational effect of dark energy and dark matter; see \cite{12} and Section 5.2.

In a nutshell, the gravitational field equations (1.2) or (1.4) are derived based on the principle of equivalence and the principle of general relativity, which uniquely dictate the specific form of the Einstein-Hilbert action. The energy-momentum conservation constraint variation (1.3) is simply the direct and unique consequence of the presence of dark energy and dark matter. Hence it is natural for us to postulate PID for all four fundamental interactions, which amounts to variation of the Lagrangian action under the div$A$-free constraints, where $A$ represent the gauge potentials.

**Symmetries of fundamental interactions**

Fundamental laws of Nature are universal, and their validity is independent of the space-time location and directions of experiments and observations. The universality of laws of Nature implies that the Lagrange actions are invariant and the differential equations are covariant under certain symmetry.

As discussed early, the law of gravity is determined by the principle of general relativity. The electromagnetic, the weak and the strong interactions are dictated, respectively, by the $U(1)$, the $SU(2)$ and $SU(3)$ gauge invariances. In other words, following the simplicity principle of laws of Nature, the three basic symmetries—the Einstein general relativity, the Lorentz invariance and the gauge invariance—uniquely determine the interaction fields and their Lagrangian actions for the four interactions. For example, the $SU(2)$ gauge invariance for the weak interaction uniquely determines the Lagrangian density:

$$L_W = -\frac{1}{4}G_{ab}W^a_{\mu\nu}W^{\mu\nu b},$$

which represents the scalar curvature of the complex vector bundle $M \otimes_p (\mathbb{C}^4)^2$, of Dirac spinor fields. Here for $a = 1, 2, 3$,

$$W^a_\mu = (W^a_0, W^a_1, W^a_2, W^a_3), \quad W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g_w\lambda^a_\mu W^b_\nu W^c_\mu,$$

are weak gauge potentials and the corresponding curvature associated with the connection on $M \otimes_p (\mathbb{C}^4)^2$:

$$D_\mu = \nabla_\mu + ig_wW^a_\mu \sigma_a,$$

Throughout this article, we use the notation $\otimes_p$ to denote "gluing a vector space to each point of a manifold" to form a vector bundle. For example,

$$M \otimes_p \mathbb{C}^n = \cup_{p \in M}\{p\} \times \mathbb{C}^n$$

is a vector bundle with base manifold $M$ and fiber vector space $\mathbb{C}^n$. 


where $U = e^{i\theta(x)\sigma_a} \in SU(2)$, $\{\sigma_a \mid a = 1, 2, 3\}$ is a set of generators for $SU(2)$, $g_w$ is the gauge coupling constant, $\lambda^c_{bc}$ are the structure constants of $\{\sigma_a\}$, and

$$G^w_{ab} = \frac{1}{8} \lambda^c_{ad} \lambda^d_{cb} = \frac{1}{2} \text{Tr}(\tau_a \tau_b^\dagger).$$

The Lagrangian density (1.5) obeys also PRI, which was discovered and postulated by the authors in [11]. In other words, the physical quantities $W^a_{\mu}$, $W^a_{\mu\nu}$, $\lambda^c_{bc}$ and $G^w_{ab}$ are $SU(2)$-tensors under the following transformation of representation generators:

$$\tilde{\sigma}_a = x^b_a \sigma_b,$$

where $X = (x^b_a)$ is a nondegenerate complex matrix.

One profound consequence of PRI is that any linear combination of gauge potentials from two different gauge groups are prohibited by PRI. For example, the term $\alpha A_{\mu} + \beta W^3_{\mu}$ in the electroweak theory violates PRI, as this term does not represent a gauge potential for any gauge group. In fact, the term combines one component of a tensor with another component of an entirely different tensor with respect to the transformations of representation generators as given by (1.6).

**PID as a mechanism for gauge symmetry breaking and mass generation**

The principle of general relativity, the Lorentz invariance and PRI stand for the universality of physical laws, i.e., the validity of laws of Nature is independent of the coordinate systems expressing them. Consequently, these symmetries hold true for both the Lagrangian actions and their variational equations.

The physical implication of the gauge symmetry is different. Namely, the gauge symmetry holds true only for the Lagrangian actions for the electromagnetic, weak and strong interactions, and it will be broken in the field equations of these interactions. This is a general principle, which we call the principle of gauge symmetry breaking.

The principle of gauge symmetry breaking can be regarded as part of the spontaneous symmetry breaking, which is a phenomenon appearing in various physical fields. Although the phenomenon was discovered in superconductivity by Ginzburg-Landau in 1951, the mechanism of spontaneous symmetry breaking in particle physics was first proposed by Y. Nambu in 1960; see [14, 15, 16]. This mechanism amounts to saying that a physical system, whose underlying laws are invariant under a symmetry transformation, may spontaneously break the symmetry, if this system possesses some states that don’t satisfy this symmetry.

The Higgs mechanism is a special case of the Nambo-Jona-Lasinio spontaneous symmetry breaking, a gauge symmetry breaking mechanism, leading to the mass generation of sub-atomic particles. This mechanism was discovered at almost the same time in 1964 by three groups of six physicists [5, 1, 3].

**PID discovered in this article provides a new principle to achieve gauge symmetry breaking and mass generation. The difference between both PID and Higgs mechanisms is that the first one is a natural sequence from the first principle, and the second one is to add artificially a Higgs field in the Lagrangian action. In addition, the PID obeys PRI, and the Higgs violates PRI.**

**Interaction mechanism**
One of greatest revolutions in sciences is Albert Einstein’s vision on gravity: the gravitational force is caused by the space-time curvature. Yukawa’s viewpoint, entirely different from Einstein’s, is that the other three fundamental forces take place through exchanging intermediate bosons such as photons for the electromagnetic interaction, $W^\pm$ and $Z$ intermediate vector bosons for the weak interaction, and gluons for the strong interaction.

Based on the unified field theory presented in this article and in [11], in the same spirit as the Einstein’s principle of equivalence of gravitational force, it is natural for us to postulate an alternate mechanism for all four interactions: The gravitational force is the curved effect of the time-space, and the electromagnetic, weak, strong interactions are the twisted effects of the underlying complex vector bundles $M \otimes_\mu \mathbb{C}^n$.

**Unified field theory**

We have demonstrated that the Lagrangian actions for fundamental interactions are uniquely determined by

- the principle of general relativity,
- the principle of gauge invariance,
- the principle of representation invariance (PRI), and
- principle of Lorentz invariance.

Based on PRI, the coupled Lagrangian action for the four fundamental interactions is naturally given by

\[
L = \int_M \left[ \mathcal{L}_{EH} + \mathcal{L}_{EM} + \mathcal{L}_W + \mathcal{L}_S + \mathcal{L}_D + \mathcal{L}_{KG} \right] \sqrt{-g} dx
\]

where $\mathcal{L}_{EH}$, $\mathcal{L}_{EM}$, $\mathcal{L}_W$ and $\mathcal{L}_S$ are the Lagrangian densities for the gravity, the electromagnetism, the weak and the strong interactions, and $\mathcal{L}_D$ and $\mathcal{L}_{KG}$ are the actions for both Dirac spinor fields and Klein-Gordon fields.

This action is invariant under the following symmetries: general relativity, the Lorentz invariance, PRI, and the gauge invariance as given in Section 3.3.

With the Lagrangian action $L$ at our disposal, the unified field equations are then derived using PID; see equations (4.4)-(4.10). These equations are naturally covariant under the symmetries: general relativity, the Lorentz invariance and PRI. As indicated before, the unified field equations spontaneously break the $U(1)$, $SU(2)$ and $SU(3)$ gauge symmetries, giving rise to a new mass generation mechanism, entirely different from the Higgs mechanism.

**Duality of fundamental interactions**

In the unified field equations (4.15)-(4.18), there exists a natural duality between the interaction fields $(g_{\mu\nu}, A_\mu, W^a_\mu, S^k_\mu)$ and their corresponding dual fields $(\phi^G_\mu, \phi^E_\mu, \phi^w_a, \phi^s_k)$:

\[
\begin{align*}
g_{\mu\nu} &\leftrightarrow \phi^G_\mu, \\
A_\mu &\leftrightarrow \phi^E_\mu, \\
W^a_\mu &\leftrightarrow \phi^w_a \quad \text{for } 1 \leq a \leq 3, \\
S^k_\mu &\leftrightarrow \phi^s_k \quad \text{for } 1 \leq k \leq 8.
\end{align*}
\]

The duality relation (1.8) can be regarded as a duality between field particles for each interaction. It is clear that each interaction mediator possesses a dual field
particle, called the dual mediator, and if the mediator has spin-$k$, then its dual mediator has spin-($k-1$). Hence the dual field particles consist of spin-1 dual graviton, spin-0 dual photon, spin-0 charged Higgs and neutral Higgs fields, and spin-0 dual gluons. The neutral Higgs $H^0$ (the dual particle of $Z$) had been discovered experimentally.

In the weakton model [13], we realize that these dual particles possess the same weakton constituents, but different spins, as the mediators.

Thanks to the PRI symmetry, the $SU(2)$ gauge fields $W^a_\mu$ ($1 \leq a \leq 3$) and the $SU(3)$ gauge fields $S^k_\mu$ ($1 \leq k \leq 8$) are symmetric in their indices $a = 1,2,3$ and $k = 1,\ldots,8$ respectively. Therefore, the corresponding relation (1.8) can be considered as a duality of interacting forces: Each interaction generates both attracting and repelling forces. Moreover, for the corresponding pair of dual fields, the even-spin field generates an attracting force, and the odd-spin field generates a repelling force.

For the first time, we discovered such attracting and repelling property of each interaction derived from the field model. Such property plays crucial role in the stability of matter in the Universe. For example, repulsive behavior of gravity on a very large scale we discovered in [12] explains dark energy phenomena. Also, the strong interaction potentials in (1.9) below demonstrate, for example, that as the distance between two quarks increases, the strong force is repelling, diminishes (asymptotic freedom region), and then becomes attracting (confinement).

Decoupling

The unified field model can be easily decoupled to study each individual interaction when other interactions are negligible. In other words, PID is certainly applicable to each individual interaction. For gravity, for example, PID offers a new gravitational field model, leading to a unified model for dark energy and dark matter [12].

Interaction potentials and force formulas

With PRI and the duality for both strong and weak interactions, we are able to derive the long overdue strong and weak potentials and force formulas.

In fact, we have derived in [9] the layered formulas of strong interaction potentials for various level particles. In particular, the $w^*$-weakton potential $\Phi_0$, the quark potential $\Phi_q$, the nucleon/hadron potential $\Phi_n$ and the atom/molecule potential $\Phi_a$ are given as follows [9]:

$$\Phi_0 = g_s \left[ \frac{1}{r} - \frac{A_0}{\rho_w} (1 + k_0 r) e^{-k_0 r} \right], \quad \frac{1}{k_0} = 10^{-18} \text{ cm},$$

$$\Phi_q = \left( \frac{\rho_w}{\rho_q} \right)^3 g_s \left[ \frac{1}{r} - \frac{A_q}{\rho_q} (1 + k_1 r) e^{-k_1 r} \right], \quad \frac{1}{k_1} = 10^{-16} \text{ cm},$$

$$\Phi_n = 3 \left( \frac{\rho_w}{\rho_n} \right)^3 g_s \left[ \frac{1}{r} - \frac{A_n}{\rho_n} (1 + k_n r) e^{-k_n r} \right], \quad \frac{1}{k_n} = 10^{-13} \text{ cm},$$

$$\Phi_a = N \left( \frac{\rho_w}{\rho_a} \right)^3 g_s \left[ \frac{1}{r} - \frac{A_a}{\rho_a} (1 + k_a r) e^{-k_a r} \right], \quad \frac{1}{k_a} = 10^{-10} \sim 10^{-7} \text{ cm}.$$
the dual fields. This clearly demonstrates the need of the dual fields in the strong interaction as required by the strong force confinement property.

Also with PRI and the duality of weak interaction, we have derived in [10] the following layered weak interaction potential formulas:

$$\Phi_w = g_w(\rho)e^{-kr}\left[\frac{1}{r} - \frac{B}{\rho}(1 + 2kr)e^{-kr}\right], \quad \frac{1}{k} = 10^{-16}\text{ cm},$$

$$g_w(\rho) = N\left(\frac{\rho_w}{\rho}\right)^3 g_w,$$

where $\Phi_w$ is the weak force potential of a particle with radius $\rho$ and $N$ weak charges $g_w$ of each weakton [13], $\rho_w$ is the weakton radius, and $B$ is a parameter depending on the particles.

Quark confinement, asymptotic freedom, and short-range nature of weak and strong interactions

The above weak and strong potentials offer a clear mechanisms for quark confinement, for asymptotic freedom, and for short-range nature of both weak and strong interactions; see e.g. [9, 11] for details.

Weakton model, baryon asymmetry, and solar neutrino problem

Thanks also to these potentials, we are able to derive in [13] a weakton model of elementary particles, leading to an explanation of all known sub-atomic decays and the creation/annihilation of matter/antimatter particles, as well as the baryon asymmetry problem.

Remarkably, in the weakton model, both the spin-1 mediators (the photon, the W and Z vector bosons, and the gluons) and the spin-0 dual mediators introduced in the unified field model in this article have the same weakton constituents, differing only by their spin arrangements. The spin arrangements clearly demonstrate that there must be dual mediators with spin-0. This observation clearly provides another strong evidence of PID and the unified field model introduced in this article.

Also, the weakton model offer an alternate explanation to the longstanding solar neutrino problem. When the solar electron neutrinos collide with anti-electron neutrinos, which are abundant due to the $\beta$-decay of neutrons, they can form $\nu$ mediators and flying away, causing the loss of electron neutrinos. This is consistent with the experimental results on the agreement between the speed of light and the speed of neutrinos.

2. Principle of Interaction Dynamics (PID)

2.1. PID. The main objective in this section is to propose a fundamental principle of physics, which we call the principle of interaction dynamics (PID). Intuitively, PID takes the variation of the action functional under energy-momentum conservation constraint.

There are strong physical evidence and motivations for the validity of PID, including

1. the discovery of dark matter and dark energy,
2. the non-existence of solutions for the classical Einstein gravitational field equations in general cases,
3. the principle of spontaneous gauge-symmetry breaking, and
the theory of Ginzburg-Landau superconductivity.

It is remarkable that the term $\nabla \mu \Phi \nu$ in (1.2) plays a similar role as the Higgs field in the standard model in particle physics, and the constraint Lagrangian dynamics gives rise to a new first principle, which we call the principle of interaction dynamics (PID). This new first principle provides an entirely different approach to introduce the Higgs field, and leads to a new mass generation mechanism.

Let $(M, g_{\mu\nu})$ be the 4-dimensional space-time Riemannian manifold with $\{g_{\mu\nu}\}$ the Minkowski type Riemannian metric. For an $(r, s)$-tensor $u$ we define the $A$-gradient and $A$-divergence operators $\nabla_A$ and $\text{div}_A$ as

$$\nabla_A u = \nabla u + u \otimes A,$$

$$\text{div}_A u = \text{div} u - A \cdot u,$$

where $A$ is a vector field and here stands for a gauge field, $\nabla$ and div are the usual gradient and divergent covariant differential operators. Let $F = F(u)$ be a functional of a tensor field $u$. A tensor $u_0$ is called an extremum point of $F$ with the $\text{div}_A$-free constraint, if $u_0$ satisfies the equation

$$\frac{d}{d\lambda} \bigg|_{\lambda=0} F(u_0 + \lambda X) = \int_M \delta F(u_0) \cdot X \sqrt{-g} dx = 0 \quad \forall X \text{ with } \text{div}_A X = 0. \tag{2.1}$$

**Principle 2.1 (Principle of Interaction Dynamics).**

1. For all physical interactions there are Lagrangian actions

$$L(g, A, \psi) = \int_M L(g_{\mu\nu}, A, \psi) \sqrt{-g} dx, \tag{2.2}$$

where $g = \{g_{\mu\nu}\}$ is the Riemannian metric representing the gravitational potential, $A$ is a set of vector fields representing the gauge potentials, and $\psi$ are the wave functions of particles.

2. The action (2.2) satisfy the invariance of general relativity, Lorentz invariance, gauge invariance and the gauge representation invariance.

3. The states $(g, A, \psi)$ are the extremum points of (2.2) with the $\text{div}_A$-free constraint (2.1).

Based on PID and Theorems 7.2 and 7.3 the field equations with respect to the action (2.2) are given in the form

$$\frac{\delta}{\delta g_{\mu\nu}} L(g, A, \psi) = (\nabla_\mu + \alpha_b A^b_\mu) \Phi^\nu, \tag{2.3}$$

$$\frac{\delta}{\delta A^a_\mu} L(g, A, \psi) = (\nabla_\mu + \beta^a_b A^b_\mu) \varphi^a, \tag{2.4}$$

$$\frac{\delta}{\delta \psi} L(g, A, \psi) = 0 \tag{2.5}$$

where $A^a_\mu = (A^0_\mu, A^1_\mu, A^2_\mu, A^3_\mu)$ are the gauge vector fields for the electromagnetic, the weak and strong interactions, $\Phi_\nu = (\Phi_0, \Phi_1, \Phi_2, \Phi_3)$ in (2.3) is a vector field induced by gravitational interaction, $\varphi^a$ is the scalar fields generated from the gauge field $A^a_\mu$, and $\alpha_b, \beta^a_b$ are coupling parameters.

PID is based on variations with $\text{div}_A$-free constraint defined by (2.1). Physically, the conditions

$$\text{div}_A X = 0 \quad \text{in } (2.1)$$

stand for the energy-momentum conservation constraints.
2.2. Principle of representation invariance (PRI). We end this section by recalling the principle of representation invariance (PRI) first postulated in [11]. We proceed with the $SU(N)$ representation. In a neighborhood $U \subset SU(N)$ of the unit matrix, a matrix $\Omega \in U$ can be written as

$$\Omega = e^{i \theta^a \tau_a},$$

where

$$\tau_a = \{\tau_1, \cdots, \tau_K\} \subset T_c SU(N), \quad K = N^2 - 1,$$

is a basis of generators of the tangent space $T_c SU(N)$. An $SU(N)$ representation transformation is a linear transformation of the basis in (2.6) as

$$\tilde{\tau}_a = x_b^a \tau_b,$$

where $X = (x_b^a)$ is a nondegenerate complex matrix.

Mathematical logic dictates that a physically sound gauge theory should be invariant under the $SU(N)$ representation transformation (2.7). Consequently, the following principle of representation invariance (PRI) must be universally valid and was first postulated in [11].

**Principle 2.2 (Principle of Representation Invariance).** All $SU(N)$ gauge theories are invariant under the transformation (2.7). Namely, the actions of the gauge fields are invariant and the corresponding gauge field equations as given by (2.3)-(2.5) are covariant under the transformation (2.7).

Direct consequences of PRI include the following; see also [11] for details:

- The physical quantities such as $\theta^a, A^a_\mu$, and $\lambda^c_{ab}$ are $SU(N)$-tensors under the generator transformation (2.7).
- The tensor

$$G_{ab} = \frac{1}{4N} \lambda^c_{ad} \lambda^d_{cb} = \frac{1}{2} \text{Tr}(\tau_a \tau_b^\dagger)$$

is a symmetric positive definite 2nd-order covariant $SU(N)$-tensor, which can be regarded as a Riemannian metric on $SU(N)$.
- The representation invariant action is

$$L = \int_M - \frac{1}{4} G_{ab} g^{\mu \nu} g^{\alpha \beta} F^a_{\mu \nu} F^b_{\alpha \beta} + \bar{\Psi} \left[ i \gamma^\mu (\partial_\mu + i g A^a_\mu \tau_a) - m \right] \Psi,$$

and the representation invariant gauge field equations are

$$G_{ab} \left[ g^{\nu \rho} F^b_{\nu \rho} - g^{\lambda \rho} \delta_{cd} g^{\alpha \beta} F^c_{\alpha \beta} A^{d \rho}_a \right] - g \bar{\Psi} \gamma_\mu \tau_a \Psi = (\partial_\mu + \alpha_b A^b_\mu) \phi_a,$$

$$(i \gamma^\mu D_\mu - m) \Psi = 0.$$

As we indicated in [11], the field models based on PID appear to be the only model which obeys PRI. In particular, both the standard model and the electroweak theory violate PRI, and consequently they are approximate models of the fundamental interactions of Nature.
3. Essentials of the Unified Field Theory Based on PID and PRI

3.1. Symmetries. Symmetry plays a crucial role in physics. In fact, symmetry dictates and determines 1) the explicit form of differential equations governing the underlying physical system, 2) the space-time structure of the Universe, and the mechanism of fundamental interactions of Nature, and 3) conservation laws of the underlying physical system.

Each symmetry is characterized by three main ingredients: space, transformations, and tensors. For gravity, for example, the principle of general relativity consists of the space-time Riemannian manifold \( M \), the general coordinate transformation and the associated tensors. The following are two basic implications of a symmetry:

a) Fundamental laws of Nature are universal, and their validity is independent of the space-time location and directions of experiments and observations;

b) The universality of laws of Nature implies that the differential equations representing them are covariant. Equivalently the Lagrange actions are invariant under the corresponding coordinate transformations.

Laws of the fundamental interactions are dictated by the following symmetries:

\[
\begin{align*}
\text{gravity:} & \quad \text{general relativity,} \\
\text{electromagnetism:} & \quad U(1) \text{ gauge invariance,} \\
\text{weak interaction:} & \quad SU(2) \text{ gauge invariance,} \\
\text{strong interaction:} & \quad SU(3) \text{ gauge invariance,}
\end{align*}
\]

Also, the last three interactions in (3.1) obey the Lorentz invariance and PRI. As a natural outcome, the three charges \( e, g_w, g_s \) are the coupling constants of \( U(1), SU(2), SU(3) \) gauge fields.

Following the simplicity principle of laws of Nature, the three basic symmetries—the Einstein general relativity, the Lorentz invariance and the gauge invariance—uniquely determine the interaction fields and their Lagrangian actions for the four interactions, which we describe as follows.

**Gravity**

The gravitational fields are the Riemannian metric defined on the space-time manifold \( M \):

\[
(3.2) \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu,
\]

and \( g_{\mu\nu} \) stand for the gravitational potential. The Lagrange action for the metric (3.2) is the Einstein-Hilbert functional

\[
(3.3) \quad \mathcal{L}_{EH} = R + \frac{8\pi G}{c^4} S,
\]

where \( R \) stands for the scalar curvature of the tangent bundle \( TM \) of \( M \).

**Electromagnetism**

The field describing electromagnetic interaction is the \( U(1) \) gauge field

\[
A_\mu = (A_0, A_1, A_2, A_3),
\]

representing the electromagnetic potential, and the Lagrangian action is

\[
(3.4) \quad \mathcal{L}_{EM} = -\frac{1}{4} A_\mu A^{\mu\nu},
\]
which stands for the scalar curvature of the vector bundle $M \otimes_p \mathbb{C}$. Here

$$A_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  

**Weak interaction**

The weak fields are the $SU(2)$ gauge fields

$$W^a_\mu = (W^a_0, W^a_1, W^a_2, W^a_3) \quad \text{for } 1 \leq a \leq 3,$$

and their action is

$$(3.5) \quad \mathcal{L}_W = -\frac{1}{4} G_{ab}^w W^a_{\mu \nu} W^{\mu \nu b},$$

which also stands for the scalar curvature of spinor bundle: $M \otimes_p (\mathbb{C}^4)^2$. Here

$$W^a_{\mu \nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g^w_{\lambda bc} W^b_\mu W^c_\nu \quad \text{for } 1 \leq a \leq 3.$$  

**Strong interaction**

The strong fields are the $SU(3)$ gauge fields

$$S^k_\mu = (S^k_0, S^k_1, S^k_2, S^k_3) \quad \text{for } 1 \leq k \leq 8,$$

and the action is

$$(3.6) \quad \mathcal{L}_S = -\frac{1}{4} G_{kl}^s S^k_{\mu \nu} S^{\mu \nu l},$$

which corresponds to the scalar curvature of $M \otimes_p (\mathbb{C}^4)^3$. Here

$$S^k_{\mu \nu} = \partial_\mu S^k_\nu - \partial_\nu S^k_\mu + g^s_{\lambda rt} S^r_\mu S^t_\nu \quad \text{for } 1 \leq k \leq 8.$$  

3.2. **Mechanism of fundamental interactions.** Albert Einstein was the first physicist who postulated that the gravitational force is caused by the space-time curvature. However, Yukawa’s viewpoint, entirely different from Einstein’s, is that the other three fundamental forces take place through exchanging intermediate bosons such as photons for the electromagnetic interaction, $W^\pm$ and $Z$ intermediate vector bosons for the weak interaction, and gluons for the strong interaction.

Based on the unified field theory presented in this article and in [11], in the same spirit as the Einstein’s principle of equivalence of gravitational force, it is natural for us to postulate an alternate mechanism for all four interactions.

One crucial component of this viewpoint is to regard the coupling constants for gauge theories for the electromagnetic, the weak and the strong interactions. In fact, each interaction possesses a charge as follows:

- The gravitational force is due to the mass charge $m$, which is responsible for all macroscopic motions;
- The electromagnetic force is due to the electric charge $e$, and holds the atoms and molecules together.
- The strong force is due to the strong charge $g_s$, and mainly acts on three levels: quarks and gluons, hadrons, and nucleons.
- The weak force is due to the weak charge $g_w$, and provides the binding energy to hold the mediators, the leptons and quarks together.
Another crucial ingredient for each interaction is the corresponding interaction potential $\Phi$. The relation between each force $F$, its associated potential $\Phi$ and the corresponding charge is given by

$$(3.7) \quad F = -g \nabla \Phi,$$

where $\nabla$ is the gradient operator in the spatial directions, and $g$ is the interaction charge.

Regarding to the laws of Nature, physical states are described by functions $u = (u_1, \cdots, u_n)$ defined on the space-time $M$:

$$(3.8) \quad u : M \to M \otimes_p \mathbb{R}^n \quad \text{for non-quantum system},$$

$$(3.9) \quad u : M \to M \otimes_p \mathbb{C}^n \quad \text{for quantum system},$$

which are solutions of differential equations associated with the laws of the underlying physical system:

$$(3.10) \quad \delta L(Du) = 0,$$

where $D$ is a derivative operator, and $L$ is the Lagrange action. Consider two transformations for the two physical systems (3.8) and (3.9):

$$(3.11) \quad \bar{x} = Lx \quad \text{for (3.8)},$$

$$(3.12) \quad \bar{u} = e^{i\theta \tau} u \quad \text{for (3.9)},$$

where $x$ is a coordinate system in $M$, $e^{i\theta \tau} : \mathbb{C}^n \to \mathbb{C}^n$ is an $SU(n)$ transformation, and $\theta$ is a function of $x$, and $\tau$ is a Hermitian matrix.

One important consequence of the invariance of (3.10) under the transformations (3.11) and (3.12) is that the derivatives $D$ in (3.10) must take the following form:

$$(3.13) \quad D = \nabla + \Gamma \quad \text{for (3.8)},$$

$$(3.14) \quad D = \nabla + igA \quad \text{for (3.9)},$$

where $\Gamma$ depends on the metrics $g_{ij}$, $A$ is a gauge field, representing the interaction potential, and $g$ is the coupling constant, representing the interaction charge.

The derivatives defined in (3.13) and (3.14) are called connections respectively on $M$ and on the complex vector bundle $M \otimes_p \mathbb{C}^n$.

**Theorem 3.1.**

1. The space $M$ is curved if and only if $\Gamma \neq 0$ in all coordinates, or equivalently $g_{ij} \neq \delta_{ij}$ under all coordinate systems.
2. The complex bundle $M \otimes_p \mathbb{C}^n$ is geometrically nontrivial or twisted if and only if $A \neq 0$.

Consequently, by Principle of General Relativity, the presence of the gravitational field implies that the space-time manifold is curved, and, by Principle of Gauge Invariance, the presence of the electromagnetic, the weak and strong interactions indicates that the complex vector bundle $M \otimes_p \mathbb{C}^n$ is twisted.

This analogy, together with Einstein’s vision on gravity as the curved effect of space-time manifold, it is natural for us to postulate the following mechanism for all four interactions.

**Geometric Interaction Mechanism 3.2.** The gravitational force is the curved effect of the time-space, and the electromagnetic, weak, strong interactions are the twisted effects of the underlying complex vector bundles $M \otimes_p \mathbb{C}^n$. 
As mentioned earlier, traditionally one adopts Yukawa’s viewpoint that forces of the interactions of Nature are caused by exchanging the field mediators.

**Yukawa Interaction Mechanism 3.3.** The four fundamental interactions of Nature are mediated by exchanging interaction field particles, called the mediators. The gravitational force is mediated by the graviton, the electromagnetic force is mediated by the photon, the strong interaction is mediated by the gluons, and the weak interaction is mediated by the intermediate vector bosons $W^\pm$ and $Z$.

It is the Yukawa mechanism that leads to the $SU(2)$ and $SU(3)$ gauge theories for the weak and the strong interactions. In fact, the three mediators $W^\pm$ and $Z$ for the weak interaction are regarded as the $SU(2)$ gauge fields $W_a^\mu$ ($1 \leq a \leq 3$), and the eight gluons for the strong interaction are considered as the $SU(3)$ gauge fields $S_k^\mu$ ($1 \leq k \leq 8$). Of course, the three color quantum numbers for the quarks are an important fact to choose $SU(3)$ gauge theory to describe the strong interaction.

The two interaction mechanisms lead to two entirely different directions to develop the unified field theory. The need for quantization for all current theories for the four interactions are based on the Yukawa Interaction Mechanism. The new unified field theory in this article is based on the Geometric Mechanism, which focus directly on the four interaction forces as in (3.7), and does not involve a quantization process.

A radical difference for the two direction mechanisms is that the Yukawa Mechanism is oriented toward computing the transition probability for the particle decays and scatterings, and the Geometric Interaction Mechanism is oriented toward fundamental laws, such as interaction potentials, of the four interactions.

**3.3. Geometry of unified fields.** Hereafter we always assume that the manifold $M$ is the 4-dimensional space-time manifold of our Universe. We adopt the view that symmetry principles determine the geometric structure of $M$, and the geometries of $M$ associated with the fundamental interactions of Nature dictate all motion laws defined on $M$. The process that symmetries determine the geometries of $M$ is achieved in the following three steps:

1. The symmetric principles, such as the Einstein general relativity, the Lorentz invariance, and the gauge invariance, determine that the fields reflecting geometries of $M$ are the Riemannian metric $\{g_{\mu\nu}\}$ and the gauge fields $\{G^a_\mu\}$. In addition, the symmetric principles also determine the Lagrangian actions of $g_{\mu\nu}$ and $G^a_\mu$.
2. PID determines the field equations governing $g_{\mu\nu}$ and $G^a_\mu$.
3. The solutions $g_{\mu\nu}$ and $G^a_\mu$ of the field equations determine the geometries of $M$.

The geometry of unified field refers to the geometries of $M$, determined by the following known physical symmetry principles:

- principle of general relativity,
- principle of Lorentz invariance,
- $U(1) \times SU(2) \times SU(3)$ gauge invariance,
- principle of representation invariance (PRI).

The fields determined by the symmetries in (3.15) are given by

- general relativity: $g_{\mu\nu} : M \to T^0_0 M$, the Riemannian metric,
UNIFIED FIELD EQUATIONS COUPLING FOUR FORCES

- Lorentz invariance: \((\psi, \Phi) : M \to M \otimes_p [(\mathbb{C}^4)^N \times \mathbb{C}^N]\), the Dirac and Klein-Golden fields,
- \(U(1)\) gauge invariance: \(A_\mu : M \to T^*M\), the \(U(1)\) gauge field,
- \(SU(2)\) gauge invariance: \(W^a_\mu : M \to (T^*M)^3\), the \(SU(2)\) gauge fields,
- \(SU(3)\) gauge invariance: \(S^k_\mu : M \to (T^*M)^8\), the \(SU(3)\) gauge fields.

The Lagrange action for the geometry of the unified fields is given by

\[
L = \int_M [L_{EH} + L_{EM} + L_W + L_S + L_D + L_{KG}] \sqrt{-g} dx
\]

where \(L_{EH}, L_{EM}, L_W\) and \(L_S\) are the Lagrangian actions for the four interactions defined by (3.3)–(3.6), and the actions for both Dirac spinor fields and Klein-Gordon fields \(L_D\) and \(L_{KG}\) are given by

\[
L_D = \bar{\Psi}(i\gamma_\mu D_\mu - m)\Psi,
\]

\[
L_{KG} = \frac{1}{2} (D^\mu \Phi)^\dagger (D_\mu \Phi) + \frac{1}{2} m^2 \Phi^\dagger \Phi.
\]

Here

\[
A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]

\[
W^a_\mu = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g_w \lambda^a \rho W^b W^c \quad \text{for } 1 \leq a \leq 3,
\]

\[
S^k_\mu = \partial_\mu S^k_\nu - \partial_\nu S^k_\mu + g_s \lambda^k \rho S^\mu S^\nu \quad \text{for } 1 \leq k \leq 8,
\]

\[
\Psi = (\psi^E, \psi^w, \psi^s),
\]

\[
\Phi = (\phi^E, \phi^w, \phi^s),
\]

\[
m = (m_e, m_w, m_s),
\]

and

\[
\psi^E : M \to M \otimes_p \mathbb{C}^4 \quad \text{1-component Dirac spinor},
\]

\[
\psi^w : M \to M \otimes_p (\mathbb{C}^4)^2 \quad \text{2-component Dirac spinors},
\]

\[
\psi^s : M \to M \otimes_p (\mathbb{C}^4)^3 \quad \text{3-component Dirac spinors},
\]

\[
\phi^E : M \to M \otimes_p \mathbb{C} \quad \text{1-component Klein-Gordon field},
\]

\[
\phi^w : M \to M \otimes_p \mathbb{C}^2 \quad \text{2-component Klein-Gordon fields},
\]

\[
\phi^s : M \to M \otimes_p \mathbb{C}^3 \quad \text{3-component Klein-Gordon fields}.
\]

The derivative operators \(D_\mu\) are given by

\[
D_\mu(\psi^E, \phi^E) = (\partial_\mu + ieA_\mu)(\psi^E, \phi^E),
\]

\[
D_\mu(\psi^w, \phi^w) = (\partial_\mu + ig_w W^a_\mu \sigma_a)(\psi^w, \phi^w),
\]

\[
D_\mu(\psi^s, \phi^s) = (\partial_\mu + ig_s S^k_\mu \tau_k)(\psi^s, \phi^s).
\]

The geometry of unified fields consists of 1) the field functions and 2) the Lagrangian action (3.16), which is invariant under the following seven transformations:
(1) the general linear transformation on \( T_p M \):

\[
\mathcal{Q}_p = (a^p_\mu^\nu) : T_p M \to T_p M, \quad \mathcal{Q}_p^{-1} = (b^p_\mu^\nu)^T \quad \text{for } p \in M, \\
(\bar{g}_{\mu\nu}) = \mathcal{Q}_p(g_{\mu\nu})\mathcal{Q}_p^T, \quad \bar{A}_\mu = a^p_\mu A_\mu,
\]

(3.20)

\[
\bar{W}_\mu^a = a^p_\mu W_\mu^a \quad \text{for } 1 \leq a \leq 3, \\
\bar{S}^k = a^p_\mu S_\mu^k \quad \text{for } 1 \leq k \leq 8, \\
\bar{\gamma}^\mu = b^p_\mu \gamma^\mu, \quad \bar{\partial}_\mu = a^p_\mu \partial_\mu,
\]

and other fields do not change under this transformation.

(2) Lorentz transformation on \( T_p M \):

\[
L = (l^\nu_\mu) : T_p M \to T_p M, \quad L \text{ is independent of } p \in M, \\
(\bar{g}_{\mu\nu}) = L(g_{\mu\nu})L^T, \quad \bar{A}_\mu = l^\nu_\mu A_\nu,
\]

(3.21)

\[
\bar{W}_\mu^a = l^\nu_\mu W_\nu^a \quad \text{for } 1 \leq a \leq 3, \\
\bar{S}^k = l^\nu_\mu S_\nu^k \quad \text{for } 1 \leq k \leq 8, \\
\Psi = R_L \Psi, \quad R_L \text{ is the spinor transformation matrix}, \\
\bar{\partial}_\mu = l^\nu_\mu \partial_\nu,
\]

and other fields do not change under this transformation.

(3) \( U(1) \) gauge transformation on \( M \otimes_p \mathbb{C}_p^4 \) and \( M \otimes_p \mathbb{C}_p^1 \):

\[
\Omega : \mathbb{C}_p^4 \to \mathbb{C}_p^4 \text{ or } \mathbb{C}_p^1 \to \mathbb{C}_p^1 \quad \text{for } p \in M, \quad \Omega = e^{i \theta} \in U(1), \\
\bar{\psi}^E = e^{i \theta} \psi^E, \quad \bar{\phi}^E = e^{i \theta} \phi^E, \\
\bar{A}_\mu = A_\mu - \frac{1}{e} \partial_\mu \theta,
\]

(3.22)

(4) \( SU(2) \) gauge transformation:

\[
\Omega : (\mathbb{C}_p^1)^2 \to (\mathbb{C}_p^1)^2 \text{ or } \mathbb{C}_p^2 \to \mathbb{C}_p^2, \quad p \in M, \quad \Omega = e^{i \theta^a \sigma^a} \in SU(2), \\
\bar{\psi}^w = \Omega \psi^w, \quad \bar{\phi}^w = \Omega \phi^w, \\
\bar{W}\mu^a\sigma_a = W\mu^a\Omega\sigma_a\Omega^{-1} + \frac{i}{g_w} \partial_\mu \Omega^{-1}, \quad \bar{m}_w = \Omega m_w \Omega^{-1}.
\]

(3.23)

(5) \( SU(3) \) gauge transformation:

\[
\Omega : (\mathbb{C}_p^1)^3 \to (\mathbb{C}_p^1)^3 \text{ or } \mathbb{C}_p^3 \to \mathbb{C}_p^3, \quad p \in M, \quad \Omega = e^{i \theta^k \tau_k} \in SU(3), \\
\bar{\psi}^s = \Omega \psi^s, \quad \bar{\phi}^s = \Omega \phi^s, \\
\bar{S}^k_\mu \tau_k = S^k_\mu \Omega \tau_k \Omega^{-1} + \frac{i}{g_s} \partial_\mu \Omega^{-1}, \quad \bar{m}_s = \Omega m_s \Omega^{-1}.
\]

(3.24)

(6) \( SU(2) \) representation transformation on \( T_e SU(2) \):

\[
X = (x^b_\mu) : T_e SU(2) \to T_e SU(2), \quad (y^b_\mu)^T = X^{-1}, \\
\bar{\sigma}_s = x^b_\mu \sigma_b, \quad \bar{G}_w = X G_w X^T,
\]

(3.25)
(7) $SU(3)$ representation transformation on $T_eSU(3)$:

$$X = (x_k^l) : \ T_eSU(3) \rightarrow T_eSU(3), \ (y^k_l)^T = X^{-1},$$

(3.26) \hspace{1cm} \bar{\tau}_k = x_k^l \tau_l,

$$\bar{G}_{kl}^s = X(G_{kl})X^T,$$

$$\bar{S}_{k\mu} = y_k^l S_{l\mu}.$$ 

**Remark 3.1.** Here we adopt the linear transformations of the bundle spaces instead of the coordinate transformations in the base manifold $M$. In this case, the two transformations (3.20) and (3.21) are compatible. Otherwise, we have to introduce the Veibein tensors to overcome the incompatibility between the Lorentz transformation and the general coordinate transformation.

### 3.4. Gauge symmetry breaking.

In physics, symmetries are displayed in two levels in the laws of Nature:

(3.27) the invariance of Lagrangian actions $L$, 

(3.28) the covariance of variation equations of $L$.

The implication of the following three symmetries:

Einstein General Relativity,

Lorentz Invariance,

Gauge Representation Invariance,

stands for the universality of physical laws, i.e. the validity of laws of Nature is independent of the coordinate systems expressing them. Consequently, the symmetries in (3.29) cannot be broken at both levels of (3.27) and (3.28).

However, the physical implication of the gauge symmetry is different at the two levels (3.27) and (3.28):

1. The gauge invariance of the Lagrangian action, (3.27), amounts to saying that the energy contributions of particles in a physical system are indistinguishable.

2. The gauge invariance of the variation equations, (3.28), means that the particles involved in the interaction are indistinguishable.

It is clear that the first aspect (1) above is universally true, while the second aspect (2) is not universally true. In other words, the Lagrange actions obey the gauge invariance, but the corresponding variation equations break the gauge symmetry. This suggests us to postulate the following principle of gauge symmetry breaking for interactions described by a gauge theory.

**Principle 3.4** (Gauge Symmetry Breaking). *The gauge symmetry holds true only for the Lagrangian actions for the electromagnetic, weak and strong interactions, and it will be violated in the field equations of these interactions.*

The principle of gauge symmetry breaking can be regarded as part of the spontaneous symmetry breaking, which is a phenomenon appearing in various physical fields. In 2008, the Nobel Prize in Physics was awarded to Y. Nambu for the discovery of the mechanism of spontaneous symmetry breaking in subatomic physics. In 2013, F. Englert and P. Higgs were awarded the Nobel Prize for the theoretical discovery of a mechanism that contributes to our understanding of the origin of mass of subatomic particles.
Although the phenomenon was discovered in superconductivity by Ginzburg-Landau in 1951, the mechanism of spontaneous symmetry breaking in particle physics was first proposed by Y. Nambu in 1960; see [14, 15, 16]. The Higgs mechanism, discovered in [5, 1, 3], is a special case of the Nambu-Jona-Lasinio spontaneous symmetry breaking, leading to the mass generation of sub-atomic particles.

PID discovered in this article provides a new mechanism for gauge symmetry breaking and mass generation. The difference between both the PID and the Higgs mechanisms is that the first one is a natural sequence of the first principle, and the second is to add artificially a Higgs field in the Lagrangian action. Also, the PID mechanism obeys PRI, and the Higgs mechanism violates PRI.

4. Unified Field Equations Based on PID and PRI

4.1. Unified field equations based on PID. The abstract unified field equations (2.3)-(2.4) are derived based on PID. We now present the detailed form of this model, ensuring that these field equations satisfy both the principle of gauge-symmetry breaking and PRI.

By PID, the unified field model (2.3)-(2.4) are derived as the variation of the action (3.16) under the \( \text{div}_A \)-constraint

\[
\langle \delta L, X \rangle = 0 \quad \text{for any} \quad X \quad \text{with} \quad \text{div}_A X = 0.
\]

Here it is required that the gradient operator \( \nabla_A \) corresponding to \( \text{div}_A \) are PRI covariant. The gradient operators in different sectors are given as follows:

\[
\begin{align*}
D^G_\mu &= \nabla_\mu + \alpha^0 A_\mu + \alpha^1_b W^b_\mu + \alpha^2_k S^k_\mu, \\
D^E_\mu &= \nabla_\mu + \beta^0 A_\mu + \beta^1_b W^b_\mu + \beta^2_k S^k_\mu, \\
D^w_\mu &= \nabla_\mu + \gamma^0 A_\mu + \Gamma^1_b W^b_\mu + \gamma^2_k S^k_\mu - \frac{1}{4} m_w^2 \phi^w_\mu, \\
D^s_\mu &= \nabla_\mu + \delta^0 A_\mu + \delta^1_b W^b_\mu + \delta^2_k S^k_\mu - \frac{1}{4} m_s^2 \phi^s_\mu,
\end{align*}
\]

where

\[
\begin{align*}
m_w, m_s, \alpha^0, \beta^0, \gamma^0, \delta^0 & \quad \text{are scalar parameters,} \\
\alpha^1_a, \beta^1_a, \gamma^1_a, \delta^1_a & \quad \text{are first-order SU(2) tensors,} \\
\alpha^2_k, \beta^2_k, \gamma^2_k, \delta^2_k & \quad \text{are first-order SU(3) tensors.}
\end{align*}
\]

Thus, the PID equations (2.3)-(2.4) can be expressed as

\[
\begin{align*}
\frac{\delta L}{\delta g_{\mu\nu}} &= D^G_{\mu} \phi^G_\nu, \\
\frac{\delta L}{\delta A_\mu} &= D^E_{\mu} \phi^E, \\
\frac{\delta L}{\delta W^a_\mu} &= D^w_{\mu} \phi^w_a, \\
\frac{\delta L}{\delta S^k_\mu} &= D^s_{\mu} \phi^s_k,
\end{align*}
\]

where \( \phi^G_\mu \) is a vector field, and \( \phi^E, \phi^w, \phi^s \) are scalar fields.
With the PID equations (4.3), the PRI covariant unified field equations are then given as follows:

\[
\begin{align*}
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= -\frac{8\pi G}{c^4} T_{\mu\nu} + D^G_{\mu} \phi^G, \\
\partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) - e J_\nu &= D^E_\nu \phi^E, \\
G^w_{ab} \left[ \partial^\mu W^b_{\mu\nu} - g_w \lambda_{cd} g^{\alpha\beta} W^c_{\alpha\nu} W^d_{\beta\beta} \right] - g_w J_{\nu a} &= D^w_{\nu} \phi^w, \\
G^s_{kj} \left[ \partial^\mu S^j_{\mu\nu} - g_s \Lambda^j_{cd} g_{\alpha\beta} S^c_{\alpha\nu} S^d_{\beta\beta} \right] - g_s Q_{\nu k} &= D^s_{\nu} \phi^s, \\
(i\gamma^\mu D_{\mu} - m) \psi^E &= 0, \\
(i\gamma^\mu D_{\mu} - m_l) \psi^w &= 0, \\
(i\gamma^\mu D_{\mu} - m_g) \psi^s &= 0,
\end{align*}
\]

where \(D^G_{\mu}, D^E_\nu, D^w_{\nu}, D^s_{\nu}\) are given by (4.1), and

\[
\begin{align*}
J_\nu &= \bar{\psi}^E \gamma^\nu \psi^E, \\
J_{\nu a} &= \bar{\psi}^w \gamma^\nu \sigma_a \psi^w, \\
Q_{\nu k} &= \bar{\psi}^s \gamma^\nu \tau_k \psi^s, \\
T_{\mu\nu} &= \frac{\delta S}{\delta g_{\mu\nu}} + \frac{c^4}{16\pi G} g^{\alpha\beta} \left( G^w_{ab} W^a_{\alpha\mu} W^b_{\beta\nu} + G^s_{kj} S^j_{\alpha\mu} S^k_{\beta\nu} + A_{\alpha\mu} A_{\beta\nu} \right) \\
&\quad - \frac{c^4}{16\pi G} g_{\mu\nu} (L_{EM} + L_W + L_S).
\end{align*}
\]

As mentioned in Section 3, the action (3.16) for the unified field model is invariant under all seven transformations given in Section 3.3, including in particular the \(U(1)\), the \(SU(2)\), and the \(SU(3)\) gauge transformations. However, the equations (4.4)-(4.7) are not invariant under the gauge transformations, and spontaneous gauge symmetry breaking is caused by the presence of the terms \(D^G_{\mu} \phi^G, D^E_\nu \phi^E, D^w_{\nu} \phi^w, D^s_{\nu} \phi^s\) in the right-hand sides of (4.4)-(4.7) involving the gauge fields \(A_\mu, W^a_{\mu}, S^k_{\mu}\).

In other words, the unified field model based on PID and PRI obey

- principle of general relativity,
- the principle of Lorentz invariance,
- principle of representation invariance (PRI),
- the principle of spontaneous gauge-symmetry breaking.

### 4.2. Coupling parameters and physical dimensions.

There are a number of to-be-determined coupling parameters in the general form of the unified field equations (4.4)-(4.10), and the \(SU(2)\) and \(SU(3)\) generators \(\sigma_a\) and \(\tau_k\) are taken arbitrarily. With PRI we are able to substantially reduce the number of these to-be-determined parameters in the unified model to two \(SU(2)\) and \(SU(3)\) tensors

\[
\{\alpha^w_a\} = (\alpha^w_1, \alpha^w_2, \alpha^w_3), \quad \{\alpha^s_k\} = (\alpha^s_1, \cdots, \alpha^s_8),
\]

containing 11 parameters, representing the portions distributed to the gauge potentials by the weak and strong charges.

---

2We ignore the Klein-Gordon fields.
Also, if we take \( \sigma_a \) \((1 \leq a \leq 3)\) as the Pauli matrices and \( \tau_k = \lambda_k \) \((1 \leq k \leq 8)\) as the Gell-Mann matrices, then the two metrics \( G_{ab}^w \) and \( G_{kl}^s \) are Euclidian:

\[
G_{ab}^w = \delta_{ab}, \quad G_{kl}^s = \delta_{kl}.
\]

Hence, in general we usually take the Pauli matrices \( \sigma_a \) and the Gell-Mann matrices \( \lambda_k \) as the \( SU(2) \) and \( SU(3) \) generators.

For convenience, we first introduce dimensions of related physical quantities. Let \( E \) represent energy, \( L \) be the length and \( t \) be the time. Then we have

\[
(A_\mu, W^a_\mu, S^k_\mu) : \sqrt{E/L}, \quad (c, g_w, g_s) : \sqrt{EL},
\]

\[
(\tau^E, \phi^w_a, \phi^s_k) : \frac{\sqrt{E}}{\sqrt{LL}},
\]

\[
(J_\mu, J^a_\mu, Q^k_\mu) : \frac{1}{L^3}, \quad (\phi^E, \phi^w_a, \phi^s_k) : \frac{\sqrt{E}}{\sqrt{LL}},
\]

\[
(h, c) : \frac{E}{t}, \quad mc/\hbar : 1/L.
\]

In addition, for gravitational fields we have

\[
g_{\mu\nu} : \text{dimensionless}, \quad R : \frac{1}{L^2},
\]

\[
T_{\mu\nu} : \frac{E}{L^3}, \quad \phi^G_\mu : \frac{1}{L},
\]

gravitational constant \( G : \frac{L^5}{Et^4} \).

According to the dimensions above, we deduce the dimensions of the parameters in \((4.4)-(4.10)\) are as follows

\[
(m_w, m_s) : 1/L, \quad (\alpha^0, \beta^0, \gamma^0, \delta^0) : \frac{1}{\sqrt{EL}},
\]

\[
(\alpha^1_a, \beta^1_a, \gamma^1_a, \delta^1_a) : \frac{1}{\sqrt{EL}}, \quad (\alpha^2_k, \beta^2_k, \gamma^2_k, \delta^2_k) : \frac{1}{\sqrt{EL}}.
\]

Thus the parameters in \((4.2)\) can be rewritten as

\[
(m_w, m_s) = \left( \frac{m_H c}{\hbar}, \frac{m_\pi c}{\hbar} \right),
\]

\[
(\alpha^0, \beta^0, \gamma^0, \delta^0) = \frac{e}{\hbar c} (\alpha^E, \beta^E, \gamma^E, \delta^E),
\]

\[
(\alpha^1_a, \beta^1_a, \gamma^1_a, \delta^1_a) = \frac{g_w}{\hbar c} (\alpha^w_a, \beta^w_a, \gamma^w_a, \delta^w_a),
\]

\[
(\alpha^2_k, \beta^2_k, \gamma^2_k, \delta^2_k) = \frac{g_s}{\hbar c} (\alpha^s_k, \beta^s_k, \gamma^s_k, \delta^s_k),
\]

where \( m_H \) and \( m_\pi \) represent the masses of \( \phi^w \) and \( \phi^s \), and all the parameters \((\alpha, \beta, \gamma, \delta)\) on the right hand side of \((4.14)\) with different super and sub indices are dimensionless constants.
4.3. **Standard form of unified field equations.** Due to [4.14], the unified field equations (4.4)-(4.10) can be simplified in the form

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = -\frac{8\pi G}{c^4} T_{\mu \nu} + \left[ \nabla_\mu + e \frac{\alpha}{\hbar c} A_\mu + \frac{g_w}{\hbar c} \sigma_\mu W^a_\mu + \frac{g_s}{\hbar c} \sigma_\mu S_\mu \right] \phi^G, \]

(4.15) \[
\partial^\nu A_\mu - e J_\nu = \left[ \partial_\mu + \frac{e}{\hbar c} \gamma^E A_\mu + \frac{g_w}{\hbar c} \beta^a W^a_\mu + \frac{g_s}{\hbar c} \beta_\mu S_\mu \right] \phi^E, \]

(4.16) \[
\partial^\nu W^a_\mu - \frac{g_w}{\hbar c} \epsilon^{a \beta \lambda} W^\beta_\alpha W^\lambda_\mu - \frac{g_s}{\hbar c} J^a_\mu \]

\[ = \left[ \partial_\mu + \frac{e}{\hbar c} \gamma^E A_\mu + \frac{g_w}{\hbar c} \epsilon^{a \beta \lambda} W^\beta_\alpha W^\lambda_\mu + \frac{g_s}{\hbar c} \epsilon^{a \beta \lambda} S_\mu - \frac{1}{4} \left( \frac{m_H c^2}{\hbar} \right)^2 \right] \phi^a_\mu, \]

(4.17) \[
\partial^\nu S^k_\nu - \frac{g_s}{\hbar c} f^{k \beta \gamma} S^\beta_\alpha S^\gamma_\beta - g_s Q^k \]

\[ = \left[ \partial_\mu + \frac{e}{\hbar c} \gamma^E A_\mu + \frac{g_w}{\hbar c} \epsilon^{a \beta \lambda} W^\beta_\alpha W^\lambda_\mu + \frac{g_s}{\hbar c} \epsilon^{a \beta \lambda} S_\mu - \frac{1}{4} \left( \frac{m_H c^2}{\hbar} \right)^2 \right] \phi^k_\mu, \]

(4.18) \[
(i\gamma^\mu D_\mu - m)\Psi = 0, \]

(4.19) where \( \Psi = (\psi^E, \psi^s, \psi^a) \), and

\[
A_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]

\[
W^a_{\mu \nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + \frac{g_w}{\hbar c} \epsilon^{a \beta \lambda} W^\beta_\alpha W^\lambda_\nu, \]

\[
S^k_{\mu \nu} = \partial_\mu S^k_\nu - \partial_\nu S^k_\mu + \frac{g_s}{\hbar c} f^{k \beta \gamma} S^\beta_\alpha S^\gamma_\nu. \]

Equations (4.15)-(4.19) need to be supplemented with coupled gauge equations to compensate the new dual fields \( (\phi^E, \phi^a_\mu, \phi^k_\mu) \). In different physical situations, the coupled gauge equations may be different.

From the field theoretical point of view (i.e. not the field particle point of view), the coefficients in (4.15)-(4.19) should be

\[
(\alpha^w_1, \alpha^w_2, \alpha^w_3) = \alpha^w(\omega_1, \omega_2, \omega_3), \]

(4.21)

\[
(\beta^w_1, \beta^w_2, \beta^w_3) = \beta^w(\omega_1, \omega_2, \omega_3), \]

\[
(\gamma^w_1, \gamma^w_2, \gamma^w_3) = \gamma^w(\omega_1, \omega_2, \omega_3), \]

\[
(\delta^w_1, \delta^w_2, \delta^w_3) = \delta^w(\omega_1, \omega_2, \omega_3), \]

and

\[
(\alpha^s_1, \cdots, \alpha^s_8) = \alpha^s(\rho_1, \cdots, \rho_8), \]

(4.22)

\[
(\beta^s_1, \cdots, \beta^s_8) = \beta^s(\rho_1, \cdots, \rho_8), \]

\[
(\gamma^s_1, \cdots, \gamma^s_8) = \gamma^s(\rho_1, \cdots, \rho_8), \]

\[
(\delta^s_1, \cdots, \delta^s_8) = \delta^s(\rho_1, \cdots, \rho_8), \]

with the unit modules:

\[
|\omega| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = 1, \]

\[
|\rho| = \sqrt{\rho_1^2 + \cdots + \rho_8^2} = 1, \]

using the Pauli matrices \( \sigma_\alpha \) and the Gell-Mann matrices \( \lambda_k \) as the generators for \( SU(2) \) and \( SU(3) \) respectively.

The two \( SU(2) \) and \( SU(3) \) tensors in (4.21) and (4.22),

\[
\omega_a = (\omega_1, \omega_2, \omega_3), \quad \rho_k = (\rho_1, \cdots, \rho_8), \]

(4.23)
are very important, by which we can obtain $SU(2)$ and $SU(3)$ representation invariant gauge fields:

$$W_\mu = \omega_a W_\mu^a, \quad S_\mu = \rho_k S^k_\mu,$$

which represent respectively the weak and the strong interaction potentials.

In view of (4.21)-(4.24), the unified field equations for the four fundamental forces are written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{8\pi G}{c^4} T_{\mu\nu} = \left[ \nabla_\mu + \frac{e\alpha^E}{\hbar c} A_\mu + \frac{g_w\alpha^w}{\hbar c} W_\mu + \frac{g_s\alpha^s}{\hbar c} S_\mu \right] \phi^G_\mu,$$

$$\partial^\nu A_\nu - eJ_\mu = \left[ \partial_\mu + \frac{e\beta^E}{\hbar c} A_\mu + \frac{g_w\beta^w}{\hbar c} W_\mu + \frac{g_s\beta^s}{\hbar c} S_\mu \right] \phi^E,$$

$$\partial^\nu W_\alpha^a - \frac{g_w}{\hbar c} \epsilon^{abc}_w g^\alpha\beta W_\alpha^b W^\beta - g_w J_\mu^a = \left[ \partial_\mu - \frac{1}{4} k^2 x_\mu + \frac{e\gamma^E}{\hbar c} A_\mu + \frac{g_w\gamma^w}{\hbar c} W_\mu + \frac{g_s\gamma^s}{\hbar c} S_\mu \right] \phi^a_w,$$

$$\partial^\nu S_k^a - \frac{g_s}{\hbar c} \epsilon^{ijk}_s g^a\beta S_i^\alpha S_j^\beta - g_s Q_k^a = \left[ \partial_\mu - \frac{1}{4} k^2 x_\mu + \frac{e\delta^E}{\hbar c} A_\mu + \frac{g_w\delta^w}{\hbar c} W_\mu + \frac{g_s\delta^s}{\hbar c} S_\mu \right] \phi^a_s,$$

$$(i\gamma^\mu D_\mu - m)\Psi = 0.$$

5. Duality and Decoupling of Interaction Fields

The natural duality of four fundamental interactions to be addressed in this section is a direct consequence of PID. It is with this duality, together with the PRI invariant potentials $S_\mu$ and $W_\mu$ given by (5.21) and (5.33), that we establish a clear explanation for many longstanding challenging problems in physics, including for example the dark matter and dark energy phenomena, the formulas of the weak and strong forces, the quark confinement, the asymptotic freedom, and the strong potentials of nucleons. Also, this duality lay a solid foundation for the weakton model of elementary particles and the energy level theory of subatomic particles, and give rise to a new mechanism for sub-atomic decay and scattering.

The unified field model can be easily decoupled to study each individual interaction when other interactions are negligible. In other words, PID is certainly applicable to each individual interaction. For gravity, for example, PID offers to a new gravitational field model, leading to a unified model for dark energy and dark matter [12].

5.1. Duality. In the unified field equations (4.15)-(4.18), there exists a natural duality between the interaction fields $(g_{\mu\nu}, A_\mu, W^a_\mu, S^k_\mu)$ and their corresponding dual fields $(\phi^G_\mu, \phi^E, \phi^a_w, \phi^k_s)$:

$$g_{\mu\nu} \leftrightarrow \phi^G_\mu,$$

$$A_\mu \leftrightarrow \phi^E,$$

$$W^a_\mu \leftrightarrow \phi^a_w \quad \text{for } 1 \leq a \leq 3,$$

$$S^k_\mu \leftrightarrow \phi^k_s \quad \text{for } 1 \leq k \leq 8.$$
Thanks to PRI, the $SU(2)$ gauge fields $W^a_\mu$ ($1 \leq a \leq 3$) and the $SU(3)$ gauge fields $S^k_\mu$ ($1 \leq k \leq 8$) are symmetric in their indices $a = 1, 2, 3$ and $k = 1, \ldots, 8$ respectively. Therefore, the corresponding relation (5.1) can be also considered as the following dual relation

\begin{align*}
g_{\mu\nu} & \leftrightarrow \phi^{G}_\mu, \\
A_\mu & \leftrightarrow \phi^{E}_\mu, \\
\{W^a_\mu\} & \leftrightarrow \{\phi^a_w\}, \\
\{S^k_\mu\} & \leftrightarrow \{\phi^k_s\}. \\
\end{align*}

The duality relation (5.1) can be regarded as the correspondence between field particles for each interaction, and the relation (5.2) is the duality of interacting forces. We now address these two different dualities.

**Duality of field particles**

In the duality relation (5.1), if the tensor fields on the left-hand side are of $k$-th order, then their dual tensor fields on the right-hand side are of $(k - 1)$-th order. Physically, this amounts to saying that if a mediator for an interaction has spin $-k$, then the dual mediator for the dual field has spin $-(k - 1)$. Hence, (5.1) leads to the following important physical conclusion:

**Duality of Interaction Mediators 5.1.** Each interaction mediator possesses a dual field particle, called the dual mediator, and if the mediator has spin-$k$, then its dual mediator has spin-$-(k - 1)$.

The duality between interaction mediators is a direct consequence of PID used for deriving the unified field equations. Based on this duality, if there exist a graviton with spin $J = 2$, then there must exist a dual graviton with spin $J = 1$. In fact, for all interaction mediators, we have the following duality correspondence:

\begin{align*}
graviton (J = 2) & \leftrightarrow \text{dual graviton (} J = 1), \\
\text{photon (} J = 1) & \leftrightarrow \text{dual photon (} J = 0), \\
W^{\pm} \text{ bosons (} J = 1) & \leftrightarrow \text{charged Higgs } H^{\pm} \text{ (} J = 0), \\
Z \text{ boson (} J = 1) & \leftrightarrow \text{neutral Higgs } H^{0} \text{ (} J = 0), \\
gluons g^k (J = 1) & \leftrightarrow \text{dual gluons } \phi^k_g \text{ (} J = 0). \\
\end{align*}

The neutral Higgs $H^0$ (the adjoint particle of $Z$) had been discovered experimentally. We remark that the duality (5.3) can also be derived using the weakton model [13].

**Duality of interaction forces**

The correspondence (5.2) provides a dual relation between the attracting and repelling forces. In fact, from the interaction potentials we find that the even-spin fields yield attracting forces, and the odd-spin fields yield repelling forces.

**Duality of Interaction Forces 5.2.** Each interaction generates both attracting and repelling forces. Moreover, for each pair of dual fields, the even-spin field generates an attracting force, and the odd-spin field generates a repelling force.
This duality of interaction forces is illustrated as follows:

\begin{align}
\text{Gravitation force} &= \text{attraction due to } g_{\mu\nu} + \text{repelling due to } \phi^G_{\mu}, \\
\text{Electromagnetism} &= \text{attraction due to } \phi^E + \text{repelling due to } A_{\mu}, \\
\text{Weak force} &= \text{attraction due to } \phi^w + \text{repelling due to } W_{\mu}, \\
\text{Strong force} &= \text{attraction due to } \phi_s + \text{repelling due to } S_{\mu}.
\end{align}

\text{(5.4)}

5.2. \textbf{Gravitational field equations based on PID.} As we only consider the gravitational interaction, then the gravitational field equations can be decoupled from the unified field model \text{(4.15)}-\text{(4.19)}, and are given by

\begin{equation}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu} + (\nabla \mu + \frac{e}{\hbar c}A_{\mu})\Phi_{\nu},
\end{equation}

\text{(5.5)}

where the term \(\frac{e}{\hbar c}A_{\mu}\Phi_{\nu}\) represents the coupling between the gravitation and the cosmic microwave background (CMB) radiation.

Taking divergence on both sides of \text{(5.5)} yields

\begin{equation}
\nabla^\mu \nabla_\mu \Phi_{\nu} + \frac{e}{\hbar c} \nabla^\mu (A_{\mu}\Phi_{\nu}) = \frac{8\pi G}{c^4} \nabla^\mu T_{\mu\nu}.
\end{equation}

\text{(5.6)}

The duality of gravitation is based on the field equations \text{(5.5)} and \text{(5.6)}.

\textbf{Gravitons and dual gravitons}

It is known that as the equations describing field particles, \text{(5.5)} characterize the graviton as a massless, neutral bosonic particle with spin \(J = 2\), and \text{(5.6)} indicate that the dual graviton is a massless, neutral bosonic particle with \(J = 1\). Hence, the gravitational field equations induced by PID and PRI provide a pair of field particles:

\begin{align}
\text{graviton:} & \quad J = 2, \ m = 0, \ Q_e = 0, \\
\text{dual graviton:} & \quad J = 1, \ m = 0, \ Q_e = 0,
\end{align}

\text{(5.7)}

where \(Q_e\) is the electric charge.

It is the nonlinear interaction of these two field particles in \text{(5.7)} that lead to the dark matter and dark energy phenomena.

\textbf{Gravitational force}

We know that from the Schwarzschild solution of the classical Einstein field equations gives rise to the classical Newton’s gravitational force formula:

\begin{equation}
F = -\frac{mMG}{r^2},
\end{equation}

\text{(5.8)}

which is an attracting force generated by \(g_{\mu\nu}\).

However, the gravitational force by the field equations \text{(5.5)}, then we can deduce a revised formulas to \text{(5.8)}. Actually, as ignoring the microwave background radiation, the equations \text{(5.5)} become \text{[12]}:

\begin{equation}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu} - \nabla_\mu \nabla_\nu \Phi,
\end{equation}

\text{(5.9)}

where \(\Phi_{\nu} = -\nabla_\nu \varphi\), and \(\varphi\) is a scalar field. In \text{[12]}, we are able to derive from \text{(5.9)} that the gravitational force should be in the form

\begin{equation}
F = mMG \left[ -\frac{1}{r^2} + \frac{c^2}{2MG} \Phi r - \left( \frac{c^2}{MG} + \frac{1}{r} \right) \frac{d\varphi}{dr} \right],
\end{equation}

\text{(5.10)}
where \( \varphi \) is the dual field, representing the scalar potential, and
\[
\Phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi.
\]
The first term in the right-hand side of (5.10) is the Newton’s gravitational force, and the second term (5.11) represents the repelling force generated by the dual field \( \varphi \), and the third term
\[
- \left( \frac{c^2}{MG} + \frac{1}{r} \right) \frac{d\varphi}{dr}
\]
represents the force due to the nonlinear coupling of \( g_{\mu\nu} \) and its dual \( \varphi \). Formula (5.10) can be approximatively written as
\[
F = mMG \left( - \frac{1}{r^2} - \frac{k_0}{r} + k_1 r \right),
\]
\( k_0 = 4 \times 10^{-18} \text{ km}^{-1}, \quad k_1 = 10^{-57} \text{ km}^{-3} \).
The formulas (5.12) shows that a central gravitational field with mass \( M \) has an attracting force \( -k_0/r \) in addition to the normal gravitational force, explaining the dark matter, and has a repelling force \( k_1 r \), explaining the dark energy; see [12] for details.

5.3. Field equations for strong interactions. The decoupled field model from (4.15)-(4.19) for strong interactions describing field particles is given by
\[
\partial^\nu S_{k\mu}^{\nu} = \frac{g_s}{\hbar c} J_{ij} g^{\alpha\beta} S_{\alpha\mu}^{i} S_{\beta}^{j} - g_s Q_{\mu}^{k},
\]
\[
= \left[ \partial_\mu + \frac{g_s}{\hbar c} S_{\mu}^{i} \right] \phi_{s}^{k} \quad \text{for } 1 \leq k \leq 8,
\]
\[
i\gamma^\mu \left[ \partial_\mu + \frac{g_s}{\hbar c} S_{\mu}^{k} \tau_k \right] \psi - \frac{mc}{\hbar} \psi = 0,
\]
where \( \tau_k = \tau^k \) are the Gell-Mann matrices, and
\[
S_{\mu\nu}^{k} = \partial_\mu S_{\nu}^{k} - \partial_\nu S_{\mu}^{k} + \frac{g_s}{\hbar c} S_{\mu}^{i} S_{\nu}^{j},
\]
\[
Q_{\mu}^{k} = \bar{\psi} \gamma^\mu \tau_k \psi.
\]
Taking divergence on both sides of (5.13) and by
\[
\partial^\mu \partial^\nu S_{\mu\nu}^{k} = 0 \quad \text{for } 1 \leq k \leq 8,
\]
we deduce the following dual field equations for the strong interaction:
\[
\partial^\mu \partial_\mu \phi_{s}^{k} + \partial^\mu \left[ \left( \frac{g_s}{\hbar c} S_{\mu}^{i} \right) \phi_{s}^{k} - \frac{m^2 c^2}{R^2} x_\mu \right] \phi_{s}^{k} = g_s \partial^\mu Q_{\mu}^{k} - \frac{g_s}{\hbar c} J_{ij} g^{\alpha\beta} \partial^\mu (S_{\alpha\mu}^{i} S_{\beta}^{j}).
\]
The equations (5.13)-(5.14) also need 8 additional gauge equations to compensate the induced dual fields \( \phi_{s}^{k} \):
\[
P_{s}^{k} (S_{\mu}, \phi_{s}, \psi) = 0, \quad 1 \leq k \leq 8.
\]
We have the following duality for the strong interaction.

Gluons and scalar dual gluons

Based on quantum chromodynamics (QCD), the field particles for the strong interaction are eight massless gluons with spin \( J = 1 \), which are described by the
SU(3) gauge fields $S^k_\mu$ ($1 \leq k \leq 8$). By the duality (5.1), for the strong interactions we have the field particle correspondence

$$ S^k_\mu \leftrightarrow \phi^k_s \quad \text{for } 1 \leq k \leq 8. $$

It implies that corresponding to the 8 gluons $S^k_\mu$ ($1 \leq k \leq 8$) there should be 8 dual gluons represented by $\phi^k_s$, called the scalar gluons due to $\phi^k_s$ being scalar fields. Namely we have the following gluon correspondence

$$ \text{gluons } g^k \leftrightarrow \text{scalar gluons } \tilde{g}^k \quad (1 \leq k \leq 8). $$

\begin{align*}
\text{Strong force}
\end{align*}

The strong interaction forces are governed by the field equations (4.25)-(4.29). The decoupled field equations are given by

\begin{align*}
\partial^\mu S^k_{\nu \rho} - \frac{g_s}{\hbar c} f^k_{ij} g^{\alpha \beta} S^i_{\alpha \mu} S^j_{\beta \nu} - g_s Q^k_{\mu} &= \left[ \partial_\mu - \frac{1}{4} k_s^2 x_\mu + \frac{g_s \delta}{\hbar c} S^k_\mu \right] \phi^k_s, \\
\partial^\mu \partial_\mu \phi^k_s - k^2 \phi^k_s + \frac{1}{4} k_s^2 x_\mu \partial^\mu \phi^k_s + \frac{g_s \delta}{\hbar c} \partial^\mu (S^i_\mu S^j(\phi^k_s)) &= -g_s \partial^\mu Q^k_{\mu} - \frac{g_s}{\hbar c} f^k_{ij} g^{\alpha \beta} \partial^\mu (S^i_{\alpha \mu} S^j_{\beta \nu}), \\
i\gamma^\mu \left[ \partial_\mu + i \frac{g_s}{\hbar c} S^k_\mu \gamma^i \right] \psi - \frac{mc}{\hbar} \psi &= 0,
\end{align*}

for $1 \leq k \leq 8$, where $\delta$ is a parameter.

Remark 5.1. Usually, $k_s$ and $\delta$ are regarded as masses of the field particles. However, since (5.18)-(5.20) are the field equations for the interaction forces, the parameters $k_s$ and $\delta$ are no longer viewed masses. In fact, $k_s^{-1}$ represents the range of attracting force for the strong interaction, and $\left( \frac{2g_s \phi^0_s}{\hbar c \delta} \right)^{-1}$ is the range of the repelling force, where $\phi^0_s$ is a ground state of $\phi_s$.

Thanks to PRI, the strong interaction potential takes the following linear combination of the eight $SU(3)$ gauge fields:

\begin{align*}
(5.21) \quad S_\mu &= \rho_k S^k_\mu,
\end{align*}

where $\rho_k = (\rho_1, \cdots, \rho_8)$ is the $SU(3)$ tensor as given in (4.23).

Let $g_s$ be the strong charge of an elementary particle, equivalent to the strong charge of $w^*$ weakton as introduced in [13], and let

$$ \Phi_0 = S_0 \quad \text{the temporal-component of (5.21)} $$

be the strong charge potential of this particle. Then the strong force between two elementary particles with strong charges is

$$ F = -g_s \nabla \Phi_0. $$

However, the strong interactions are layered, i.e. the strong forces act only on particles at the same level, such as quarks and quarks, hadrons and hadrons, etc. Hence, the strong interaction potentials are also layered. In fact, we have derived
in [9] the layered formulas of strong interaction potentials. In particular, the weak toton potential $\Phi_0$, the quark potential $\Phi_q$, the nucleon/hadron potential $\Phi_n$ and the atom/molecule potential $\Phi_a$ are given as follows [9]:

$$
\Phi_0 = g_s \left[ \frac{1}{r} - \frac{A_0}{\rho_w} (1 + k_0 r) e^{-k_0 r} \right],
$$

$$
\Phi_q = \left( \frac{\rho_w}{\rho_q} \right)^3 g_s \left[ \frac{1}{r} - \frac{A_q}{\rho_q} (1 + k_1 r) e^{-k_1 r} \right],
$$

$$
\Phi_n = 3 \left( \frac{\rho_w}{\rho_n} \right)^3 g_s \left[ \frac{1}{r} - \frac{A_n}{\rho_n} (1 + k_n r) e^{-k_n r} \right],
$$

$$
\Phi_a = N \left( \frac{\rho_w}{\rho_a} \right)^3 g_s \left[ \frac{1}{r} - \frac{A_a}{\rho_a} (1 + k_a r) e^{-k_a r} \right].
$$

Here, $k_0, k_1, k_n, k_a$ are given by

$$
\frac{1}{k_0} = 10^{-18} \text{ cm}, \quad \frac{1}{k_1} = 10^{-16} \text{ cm}, \quad \frac{1}{k_n} = 10^{-13} \text{ cm}, \quad \frac{1}{k_a} = 10^{-10} \sim 10^{-7} \text{ cm}.
$$

5.4. **Weak interaction field equations.** Unified field model can be decoupled to study the weak interaction only, leading to the following weak interaction field equations:

$$
\partial'^{\nu} W^{a}_{\nu} - \frac{g_w}{\hbar c} \varepsilon_{bc} g^{\alpha\beta} W^{b}_{\alpha\mu} W^{c}_{\beta\mu} - g_w J^{a}_{\mu}
$$

$$
= \left[ \partial_{\mu} - \frac{1}{4} \left( \frac{m_H}{\hbar c} \right)^2 x_{\mu} + \frac{g_w}{\hbar c} \gamma^{\nu} W^{a}_{\nu} \right] \phi^{a}_{w},
$$

$$
i\gamma^{\mu} \left[ \partial_{\mu} + \frac{g_w}{\hbar c} W^{a}_{\mu} \sigma_{a} \right] \psi - \frac{m_c}{\hbar} \psi = 0,
$$

where $m_H$ represents the mass of the Higgs particle, $\sigma_{a} = \sigma^{a} (1 \leq a \leq 3)$ are the Pauli matrices and

$$
W^{a}_{\mu} = \partial^{a}_{\mu} W^{a} - \partial^{\nu} W^{a}_{\nu} + \frac{g_w}{\hbar c} \varepsilon_{bc} W^{b}_{\mu} W^{c}_{\nu},
$$

$$
J^{a}_{\mu} = \psi \gamma^{\nu} \sigma^{a} \psi.
$$

Taking divergence on both sides of (5.24) we get

$$
\partial^{\mu} \partial_{\mu} \phi^{a}_{w} - \left( \frac{m_H}{\hbar c} \right)^2 \phi^{a}_{w} + \frac{g_w}{\hbar c} \gamma^{\nu} \partial^{\mu} (W^{b}_{\mu} \phi^{a}_{w}) - \frac{1}{4} \left( \frac{m_H}{\hbar c} \right)^2 x_{\mu} \partial_{\mu} \phi^{a}_{w}
$$

$$
= - \frac{g_w}{\hbar c} \varepsilon_{bc} \gamma^{\nu} \partial^{\mu} (W^{b}_{\mu} \phi^{a}_{w}) - g_w \partial_{\mu} J^{a}_{\mu}.
$$

Also, we need to supplement (5.24)-(5.25) with three additional 3 gauge equations:

$$
F^{a}_{w} (W^{a}_{\mu}, \phi^{a}_{w}, \psi) = 0 \quad \text{for } 1 \leq a \leq 3.
$$

**Duality between $W^{\pm}, Z$ Bosons and Higgs Bosons $H^{\pm}, H^{0}$**

The three massive vector bosons, denoted by $W^{\pm}, Z^{0}$, has been discovered experimentally. The field equations (5.24) give rise to a natural duality:

$$
Z^{0} \leftrightarrow H^{0}, \quad W^{\pm} \leftrightarrow H^{\pm},
$$
where $H^0, H^\pm$ are three dual scalar bosons, called the Higgs particles. The neutral Higgs $H^0$ has been discovered by LHC in 2012, and the charged Higgs $H^\pm$ have yet to found experimentally.

**Weak force**

If consider the weak interaction force, we have to use the equations decoupled from (4.25)-(4.29):

\[
\partial^\nu W^a_{\nu \mu} - \frac{g_w}{\hbar c} \varepsilon_{abc} g^{\alpha \beta} W^b_{\alpha \mu} W^c_{\beta \mu} - g_w j^a_{\mu} = \left[ \partial_\mu - \frac{1}{4} k^2 x_\mu + \frac{g_w}{\hbar c} \gamma W_\mu \right] \phi^a_w, \tag{5.30}
\]

\[
\partial^\mu \partial_\mu \phi^a_w - k^2 \phi^2_w + \frac{g_w}{\hbar c} \gamma \partial^\mu (W_\mu \phi^a_w) - \frac{1}{4} k^2 x_\mu \partial^\mu \phi^a_w = -g_w \partial^\mu j^a_\mu - \frac{g_w}{\hbar c} \varepsilon_{abc} g^{\alpha \beta} \partial^\mu (W^b_{\alpha \mu} W^c_{\beta}), \tag{5.31}
\]

\[
i \gamma^\mu (\partial_\mu + i \frac{g_w}{\hbar c} W^a_\mu \sigma_a) \psi - \frac{m_c}{\hbar} \psi = 0, \tag{5.32}
\]

where $\gamma, k_w$ are constants.

As in the case for the strong interaction, the weak interaction potential is given by the following PRI invariant

\[
W^a_\mu = \omega^a W^a_\mu = (W_0, W_1, W_2, W_3), \tag{5.33}
\]

where $\omega^a (1 \leq a \leq 3)$ is the $SU(2)$ tensor as in (4.23). The weak charge potential and weak force are as

\[
\Phi_w = W_0 \quad \text{the time component of } W^a_\mu, \tag{5.34}
\]

\[
F^a_w = -g_w(\rho) \nabla \Phi_w, \tag{5.35}
\]

where $g_w(\rho)$ is the weak charge of a particle with radius $\rho$.

We have derived [10] from (5.30)-(5.32) the following layered weak interaction potential formulas:

\[
\Phi_w = g_w(\rho) e^{-kr} \left[ \frac{1}{r} - \frac{B}{\rho} (1 + 2kr) e^{-kr} \right], \tag{5.36}
\]

\[
g_w(\rho) = N \left( \frac{\rho_w}{\rho} \right)^3 g_w, \tag{5.36}
\]

where $\Phi_w$ is the weak force potential of a particle with radius $\rho$ and $N$ weak charges $g_w, g_w$ is the unit weak charge of weak charge $g_w$ for each weakton [13], $\rho_w$ is the weakton radius, $B$ is a parameter depending on the particles, and

\[
\frac{1}{k} = 10^{-16} \text{ cm}, \tag{5.36}
\]

represents the force-range of the weak interaction.

6. Orthogonal Decomposition for Tensor Fields

6.1. Orthogonal decomposition theorems. The aim of this section is to derive an orthogonal decomposition for $(k, r)$-tensor fields, with $k + r \geq 1$, into divergence-free and gradient parts. This decomposition plays a crucial role for the unified field theory introduced in this paper.
Remark 6.1. The above orthogonal decomposition theorem implies that
\begin{equation}
(6.6)
\end{equation}
Here \( \tilde{M} \) is a closed Riemannian manifold, and \( G = (g_{ij}) \) is the Riemannian metric of \( \tilde{M} \).

Let \( A \) be a vector field or a covector field, and \( u \in L^2(T^*_rM) \). We define the operators \( D_A \) and \( \text{div}_A \) by
\begin{equation}
(6.2)
\end{equation}
A tensor field \( v \in L^2(T^*_rM) \) \((k + r \geq 1)\) is \( \text{div}_A \)-free, denoted by \( \text{div}_A v = 0 \), if
\begin{equation}
(6.3)
\end{equation}
Here \( \psi \in H^1(T^*_rM) \) or \( H^1(T^{k-1}_rM) \), \( \nabla_A \) and \( \text{div}_A \) are as in (6.2).

We remark that if \( v \in H^1(T^*_rM) \) satisfies (6.3), then \( v \) is weakly differentiable, and \( \text{div} v = 0 \) in \( L^2 \)-sense. If \( v \in L^2(T^*_rM) \) is not differentiable, then (6.3) means that \( v \) is \( \text{div}_A \)-free in the distribution sense.

**Theorem 6.1 (Orthogonal Decomposition Theorem).** Let \( A \) be a given vector field or covector field, and \( u \in L^2(T^*_rM) \). Then the following assertions hold true:

1. The tensor field \( u \) can be orthogonally decomposed into
\begin{equation}
(6.4)
\end{equation}
where \( \varphi \in H^1(T^{k-1}_rM) \) or \( \varphi \in H^1(T^{k-1}_rM) \).

2. If \( M \) is a compact Riemannian manifold, then \( u \) can be orthogonally decomposed into
\begin{equation}
(6.5)
\end{equation}
where \( \varphi \) and \( v \) are as in (6.4), and \( h \) is a harmonic field, i.e.
\[ \text{div}_A h = 0, \quad \nabla_A h = 0. \]

In particular, the subspace of all harmonic tensor fields in \( L^2(T^*_rM) \) is of finite dimensional:
\begin{equation}
(6.6)
\end{equation}
\[ \text{div} H(T^*_rM) < \infty. \]

**Remark 6.1.** The above orthogonal decomposition theorem implies that \( L^2(T^*_rM) \) can be decomposed into
\begin{equation}
(6.7)
\end{equation}
\[ L^2(T^*_rM) = G(T^*_rM) \oplus L^2_D(T^*_rM) \quad \text{for general case}, \]
\[ L^2(T^*_rM) = G(T^*_rM) \oplus H(T^*_rM) \oplus L^2_N(T^*_rM) \quad \text{for } M \text{ compact Riemannian}. \]

Here \( H \) is as in (6.4), and
\[ G(T^*_rM) = \{ v \in L^2(T^*_rM) \mid v = \nabla_A \varphi, \varphi \in H^1(T^{k-1}_rM) \}, \]
\[ L^2_D(T^*_rM) = \{ v \in L^2(T^*_rM) \mid \text{div}_A v = 0 \}, \]
\[ L^2_N(T^*_rM) = \{ v \in L^2_D(T^*_rM) \mid \nabla_A v \neq 0 \}. \]
They are orthogonal to each other:
\[ L^2_D(T^k r M) \perp G(T^k r M), \quad L^2_N(T^k r M) \perp H(T^k r M), \quad G(T^k r M) \perp H(T^k r M). \]

**Remark 6.2.** The orthogonal decomposition (6.7) of \( L^2(T^k r M) \) implies that if a tensor field \( u \in L^2(T^k r M) \) satisfies that
\[ \langle u, v \rangle_{L^2} = \int_M (u, v) \sqrt{g} dx = 0 \quad \forall \ \text{div}_A v = 0, \]
then \( u \) must be a gradient field, i.e.
\[ u = \nabla_A \varphi \quad \text{for some } \varphi \in H^1(T^k_{r-1} M). \]

Likewise, if \( u \in L^2(T^k r M) \) satisfies that
\[ \langle u, v \rangle_{L^2} = 0 \quad \forall v \in G(T^k r M), \]
then \( u \in L^2_D(T^k r M) \). It is the reason why we define a \( \text{div}_A \)-free field by (6.3).

**Proof of Theorem 6.1.** We proceed in several steps as follows.

**Step 1. Proof of Assertion (1).** Let \( u \in L^2(E), E = T^k r M \ (k + r \geq 1) \).
Consider the equation
\[ \Delta \varphi = \text{div}_A u \quad \text{in } M, \]
where \( \Delta \) is the Laplace operator defined by
\[ \Delta = \text{div}_A \cdot \nabla_A. \]

Without loss of generality, we only consider the case where \( \text{div}_A u \in \tilde{E} = T^k_{r-1} M \). It is clear that if (6.8) has a solution \( \varphi \in H^1(\tilde{E}) \), then by (6.9), the following vector field must be \( \text{div}_A \)-free
\[ v = u - \nabla_A \varphi \in L^2(E). \]

Moreover, by (6.3), we have
\[ \langle v, \nabla_A \psi \rangle_{L^2} = \int_M (v, \nabla_A \psi) \sqrt{g} dx = 0. \]
Namely \( v \) and \( \nabla_A \varphi \) are orthogonal. Therefore, the orthogonal decomposition \( u = v + \nabla_A \varphi \) follows from (6.10) and (6.11).

It suffices then to prove that (6.8) has a weak solution \( \varphi \in H^1(\tilde{E}) \):
\[ \langle \nabla_A \varphi - u, \nabla_A \psi \rangle_{L^2} = 0 \quad \forall \psi \in H^1(\tilde{E}). \]

Obviously, if \( \phi \) satisfies
\[ \Delta \phi = 0, \]
where \( \Delta \) is as in (6.9), then, by integration by parts,
\[ \int_M (\Delta \phi, \phi) \sqrt{g} dx = - \int_M (\nabla_A \phi, \nabla_A \phi) \sqrt{g} dx = 0. \]
Hence (6.13) is equivalent to
\[ \nabla_A \phi = 0. \]

Therefore, for all \( \phi \) satisfying (6.13) we have
\[ \int_M (u, \nabla_A \phi) \sqrt{g} dx = 0. \]
By the Fredholm alternative theorem, we derive that the equation (6.8) has a unique weak solution $\phi \in H^1(\tilde{E})$.

For Minkowski manifolds, the existence of solutions for (6.8) is classical. Assertion (1) is proved.

**Step 2. Proof of Assertion (2).** Based on Assertion (1), we have

$$H^k(E) = H^k_D \oplus G^k, \quad L^2(E) = L^2_D \oplus G,$$

where

$$H^k_D = \{ u \in H^k(E) | \text{div}_A u = 0 \},$$

$$G^k = \{ u \in H^k(E) | u = \nabla_A \psi \}.$$

Define an operator $\tilde{\Delta} : H^2_D(\tilde{E}) \to L^2_D(\tilde{E})$ by

$$\tilde{\Delta} u = P\Delta u,$$

where $P : L^2(\tilde{E}) \to L^2_D(\tilde{E})$ is the canonical orthogonal projection.

We known that the Laplace operator $\Delta$ can be expressed as

$$\Delta = \text{div}_A \cdot \nabla_A = g^{kl} \frac{\partial^2}{\partial x^k \partial x^l} + B,$$

where $B$ is a lower-order differential operator. Since $M$ is compact, the Sobolev embeddings

$$H^2(E) \hookrightarrow H^1(E) \hookrightarrow L^2(E)$$

are compact. Hence the lower-order differential operator

$$B : H^2(M, \mathbb{R}^N) \to L^2(M, \mathbb{R}^N)$$

is a linear compact operator. Therefore the operator in (6.16) is a linear completely continuous field

$$\Delta : H^2(E) \to L^2(E),$$

which implies that the operator of (6.15) is also a linear completely continuous field

$$\tilde{\Delta} = P\Delta : H^2_D(\tilde{E}) \to L^2_D(\tilde{E}).$$

By the spectrum theorem of completely continuous fields [8], the space

$$\tilde{H} = \{ u \in H^2_D(\tilde{E}) | \tilde{\Delta} u = 0 \}$$

is finite dimensional, and is the eigenspace of the eigenvalue $\lambda = 0$. By integration by parts, for $u \in \tilde{H}$ we have

$$\int_M (\tilde{\Delta} u, u) \sqrt{g} dx = \int_M (\Delta u, u) \sqrt{g} dx \quad (\text{by div}_A u = 0)$$

$$= -\int_M (\nabla_A u, \nabla_A u) \sqrt{g} dx$$

$$= 0 \quad (\text{by } \tilde{\Delta} u = 0).$$

It follows that

$$u \in \tilde{H} \iff \nabla_A u = 0.$$
which implies that $\tilde{H}$ is the same as the harmonic space $H$ of (6.6), i.e. $\tilde{H} = H$.

Thus we have

$$
L^2_{\tilde{H}}(E) = H \oplus L^2_N(E),
$$

$$
L^2_N(E) = \{ u \in L^2_{\tilde{H}}(E) | \nabla Au \neq 0 \}.
$$

The proof of Theorem 6.1 is complete.

6.2. Orthogonal decomposition on manifolds with boundary. In the above subsections, we mainly consider the orthogonal decomposition of tensor fields on the closed Riemannian and Minkowski manifolds. In this subsection we discuss the problem on manifolds with boundary.

Orthogonal decomposition on Riemannian manifolds with boundaries

**Theorem 6.2.** Let $M$ be a Riemannian manifold with boundary $\partial M \neq \emptyset$, and

(6.17) \quad u : M \to T^k_M

be a $(k,r)$-tensor field. Then we have the following orthogonal decomposition:

(6.18) \quad u = \nabla_A \phi + v,

$$
\text{div}_A v = 0, \quad v \cdot n|_{\partial M} = 0, \quad \int_M (\nabla_A \phi, v) \sqrt{g} dx = 0,
$$

where $\partial v/\partial n = \nabla_A v \cdot n$ is the derivative of $v$ in the direction of outward normal vector $n$ on $\partial \Omega$.

**Proof.** For the tensor field $u$ in (6.17), consider

(6.19) \quad \text{div}_A \cdot \nabla_A \phi = \text{div}_A u \quad \forall x \in M,

$$
\frac{\partial \phi}{\partial n} = u \cdot n \quad \forall x \in \partial M.
$$

This Neumann boundary problem possesses a solution provided the following condition holds true:

(6.20) \quad \int_{\partial M} \frac{\partial \phi}{\partial n} ds = \int_{\partial M} u \cdot n ds,

which is ensured by the boundary condition in (6.19). Hence by (6.19) the field

(6.21) \quad v = u - \nabla_A \phi

is $\text{div}_A$-free, and satisfies the boundary condition

(6.22) \quad v \cdot n|_{\partial M} = 0.

Then it follows from (6.21) and (6.22) that the tensor field $u$ in (6.17) can be orthogonally decomposed into the form of (6.18). The proof is complete.

Orthogonal decomposition on Minkowski manifolds

Let $M$ be a Minkowski manifold in the form

(6.23) \quad M = \widetilde{M} \times (0,T),

with the metric (6.1).
In view of the Minkowski metric (6.1), we see that the operator \( \text{div}_A \cdot \nabla_A \) is a hyperbolic differential operator expressed as
\[
(6.24) \quad \text{div}_A \cdot \nabla_A = -\left( \frac{\partial}{\partial t} + A_0 \right)^2 + g^{ij} D_{Ai} D_{Aj}.
\]

Now a tensor field \( u \in L^2(T^k_r M) \) has an orthogonal composition if the following hyperbolic equation
\[
(6.25) \quad \text{div}_A \cdot \nabla_A \phi = \text{div}_A u \quad \text{in} \ M
\]
has a weak solution \( \phi \in H^1(T^{k-1} M) \) in the following sense:
\[
(6.26) \quad \int_M (D_A \phi, D_A \psi) \sqrt{-g} dx = \int_M (u, D_A \psi) \sqrt{-g} dx \quad \forall \psi \in H^1(T^{k-1} M).
\]

Theorem 6.3. Let \( M \) be a Minkowskian manifold as defined by (6.23)-(6.1), and \( u \in L^2(T^l_r M) \) \((k + r \geq 1)\) be an \((k,r)\)-tensor field. Then \( u \) can be orthogonally decomposed into the following form
\[
(6.27) \quad u = \nabla_A \phi + v, \quad \text{div}_A v = 0,
\]
\[
(6.28) \quad \int_M (\nabla_A \phi, v) \sqrt{-g} dx = 0
\]
if and only if equation (6.25) has a weak solution \( \phi \in H^1(T^{k-1} M) \) in the sense of (6.26).

7. Variations with \( \text{div}_A\)-Free Constraints

Let \( M \) be a closed manifold. A Riemannian metric \( G \) on \( M \) is a mapping
\[
G : M \to T^0_2 M = T^* M \otimes T^* M,
\]
which is symmetric and nondegenerate. Namely, in a local coordinate system, \( G \) can be expressed as
\[
(7.1) \quad G = \{g_{ij}\} \quad \text{with} \quad g_{ij} = g_{ji},
\]
and the matrix \( (g_{ij}) \) is invertible on \( M \):
\[
G^{-1} = (g^{ij}) = (g_{ij})^{-1} : M \to T^2_0 M = TM \otimes TM.
\]

If we regard a Riemannian metric \( G = \{g_{ij}\} \) as a tensor field on the manifold \( M \), then the set of all metrics \( G = \{g_{ij}\} \) on \( M \) constitute a topological space, called the space of Riemannian metrics on \( M \). The space of Riemannian metrics on \( M \) is defined by
\[
W^{m,2}(M, g) \equiv \{G \mid G \in W^{m,2}(T^0_2 M), \quad G^{-1} \in W^{m,2}(T^2_0 M), \quad G \quad \text{is the Riemannian metric on} \ M \quad \text{as in} \ (7.1)\}.
\]
The space \( W^{m,2}(M, g) \) is a metric space, but not a Banach space. However, it is a subspace of the direct sum of two Sobolev spaces \( W^{m,2}(T^0_2 M) \) and \( W^{m,2}(T^2_0 M) \):
\[
W^{m,2}(M, g) \subset W^{m,2}(T^0_2 M) \oplus W^{m,2}(T^2_0 M).
\]

A functional defined on \( W^{m,2}(M, g) \):
\[
(7.2) \quad F : W^{m,2}(M, g) \to \mathbb{R}
\]
is called the functional of Riemannian metric. In general, the functional (7.2) can be expressed as
\[ F(g_{ij}) = \int_M f(g_{ij}, \ldots, \partial^m g_{ij}) \sqrt{g} dx. \]

Since \((g^{ij})\) is the inverse of \((g_{ij})\), we have
\[ g_{ij} = \frac{1}{g} \times \text{the cofactor of } g^{ij}. \]

Therefore, \(F(g_{ij})\) in (7.4) also depends on \(g^{ij}\), i.e. putting (7.4) in (7.3) we get
\[ F(g^{ij}) = \int_M \tilde{f}(g^{ij}, \ldots, \partial^m g^{ij}) \sqrt{\tilde{g}} dx. \]

We note that although \(W^{m,2}(M, g)\) is not a linear space, for a given element \(g_{ij} \in W^{m,2}(M, g)\) and any symmetric tensor fields \(X_{ij}, X^{ij}\), there is a number \(\lambda_0 > 0\) such that
\[ g_{ij} + \lambda X_{ij} \in W^{m,2}(M, g) \quad \forall 0 \leq |\lambda| < \lambda_0, \]
\[ g^{ij} + \lambda X^{ij} \in W^{m,2}(M, g) \quad \forall 0 \leq |\lambda| < \lambda_0. \]

With (7.6), we can define the following derivative operators of the functional \(F\):
\[ \delta_s F : W^{m,2}(M, g) \rightarrow W^{-m,2}(T^2_0 M), \]
\[ \delta^* F : W^{m,2}(M, g) \rightarrow W^{-m,2}(T^2_0 M), \]
where \(W^{-m,2}(E)\) is the dual space of \(W^{m,2}(E)\), and \(\delta_s F, \delta^* F\) are defined by
\[ \langle \delta_s F(g_{ij}), X \rangle = \left. \frac{d}{d\lambda} \right|_{\lambda=0} F(g_{ij} + \lambda X_{ij}), \]
\[ \langle \delta^* F(g^{ij}), X \rangle = \left. \frac{d}{d\lambda} \right|_{\lambda=0} F(g^{ij} + \lambda X^{ij}). \]

For any given metric \(g_{ij} \in W^{m,2}(M, g)\), the value of \(\delta_s F\) and \(\delta^* F\) at \(g_{ij}\) are second-order contra-variant and covariant tensor fields:
\[ \delta_s F(g_{ij}) : M \rightarrow TM \times TM, \]
\[ \delta^* F(g^{ij}) : M \rightarrow T^* M \times T^* M. \]

**Theorem 7.1.** Let \(F\) be the functionals defined by (7.3) and (7.5). Then the following assertions hold true:

1. For any \(g_{ij} \in W^{m,2}(M, g)\), \(\delta_s F(g_{ij})\) and \(\delta^* F(g_{ij})\) are symmetric tensor fields.
2. If \(\{g_{ij}\} \in W^{m,2}(M, g)\) is an extremum point of \(F\), i.e. \(\delta F(g_{ij}) = 0\), then \(\{g^{ij}\}\) is also an extremum point of \(F\).
3. \(\delta_s F\) and \(\delta^* F\) have the following relation
   \[ (\delta^* F(g_{ij}))^{kl} = -g^{kr} g^{ls} (\delta_s F(g_{ij}))_{rs}, \]
   where \((\delta^* F)^{kl}\) and \((\delta_s F)_{kl}\) are the components of \(\delta^* F\) and \(\delta_s F\).

**Proof.** We only need to verify Assertion (3). In view of \(g_{ik} g^{kj} = \delta^i_j\), we have the variational relation
\[ \delta(g_{ik} g^{kj}) = g_{ik} \delta g^{kj} + g^{kj} \delta g_{ik} = 0. \]

It implies that
\[ \delta g^{kl} = -g^{ki} g^{lj} \delta g_{ij}. \]
In addition, in (7.7) and (7.8),
\[ \lambda X_{ij} = \delta g_{ij}, \quad \lambda X^{ij} = \delta g^{ij}, \quad \lambda \neq 0 \quad \text{small}. \]
Therefore, by (7.10) we get
\[ \langle (\delta \ast F)_{kl}, \delta g_{kl} \rangle = - \langle (\delta \ast F)_{kl}, g_{ki} g_{lj} \delta g_{ij} \rangle = - \langle g_{ki} g_{lj} (\delta \ast F)_{kl}, \delta g_{ij} \rangle = \langle (\delta \ast F)^{ij}, \delta g_{ij} \rangle. \]
Hence we have
\[ (\delta \ast F)^{ij} = - g^{ki} g^{lj} (\delta \ast F)_{kl}. \]
Thus Assertion (3) follows and the proof is complete. \( \square \)

We are now in position to consider the variation with div\(A\)-free constraints. We know that an extremum point \(g_{ij}\) of a metric functional is a solution of the equation (7.11)
\[ \delta F(g_{ij}) = 0, \]
in the sense that
\[ \langle \delta F(g_{ij}), X \rangle = \frac{d}{d\lambda} \bigg|_{\lambda=0} F(g_{ij} + \lambda X_{ij}) \bigg|_{\lambda=0} = \int_M (\delta \ast F)(g_{ij}) X_{kl} \sqrt{g} dx \]
(7.12)
\[ = 0 \quad \forall X_{kl} = X_{lk} \in L^2(T^0_2 M). \]
Note that the solution \(g_{ij}\) of (7.11) in the usual sense should satisfy
\[ \langle \delta F(g_{ij}), X \rangle = 0 \quad \forall X \in L^2(T^0_2 M). \]
Notice that (7.12) has a symmetric constraint on the variational elements \(X_{ij}: X_{ij} = X_{ji}\). Therefore, comparing (7.12) with (7.13), we may wonder if a solution \(g_{ij}\) satisfying (7.12) is also a solution of (7.12). Fortunately, note that \(L^2(T^0_2 M)\) can be decomposed into a direct sum of symmetric and anti-symmetric spaces as follows
\[ L^2(T^0_2 M) = L^2_s(T^0_2 M) \oplus L^2_c(T^0_2 M), \]
\[ L^2_s(T^0_2 M) = \{ u \in L^2(T^0_2 M) \mid u_{ij} = u_{ji} \}, \]
\[ L^2_c(T^0_2 M) = \{ u \in L^2(T^0_2 M) \mid u_{ij} = -u_{ji} \}, \]
and \(L^2_s(T^0_2 M)\) and \(L^2_c(T^0_2 M)\) are orthogonal:
\[ \int_M g^{ik} g^{jl} u_{ij} v_{kl} \sqrt{g} dx = - \int_M g^{ik} g^{jl} u_{ij} v_{kl} \sqrt{g} dx = 0 \quad \forall u \in L^2_s(T^0_2 M), \quad v \in L^2_c(T^0_2 M). \]
Thus, due to the symmetry of \(\delta F(g_{ij})\), the solution \(g_{ij}\) of (7.11) satisfying (7.12) must also satisfy (7.13). Hence the solutions of (7.11) in the sense of (7.12) are the solutions in the usual sense.

However, if we consider the variations of \(F\) under the div\(A\)-free constraint, then the extremum points of \(F\) are not solutions of (7.11) in the usual sense. Motivated by physical considerations, we now introduce variations with div\(A\)-free constraints.
Definition 7.1. Let $F(u)$ be a functional of tensor fields $u$. We say that $u_0$ is an extremum point of $F(u)$ under the $\text{div}_A$-free constraint, if

$$
\langle \delta F(u_0), X \rangle = \left. \frac{d}{d\lambda} \right|_{\lambda=0} F(u_0 + \lambda X) = 0 \quad \forall \ \text{div}X = 0,
$$

where $\text{div}_A$ is as defined in (6.2).

In particular, if $F$ is a functional of Riemannian metrics, and the solution $u_0 = g_{ij}$ is a Riemannian metric, then the differential operator $D_A$ in $\text{div}_A X$ in (7.14) is given by

$$
D_A = D + A, \quad D = \partial + \Gamma,
$$

and the connection $\Gamma$ is taken at the extremum point $u_0 = g_{ij}$.

We have the following theorems for $\text{div}_A$-free constraint variations.

Theorem 7.2. Let $F : W^{m,2}(M, g) \to \mathbb{R}$ be a functional of Riemannian metrics. Then there is a vector field $\Phi \in H^1(TM)$ such that the extremum points $\{g_{ij}\}$ of $F$ with the $\text{div}_A$-free constraint satisfy the equation

$$
\delta F(g_{ij}) = D\Phi + A \otimes \Phi,
$$

where $D$ is the covariant derivative operator as in (7.15).

Theorem 7.3. Let $F : H^m(TM) \to \mathbb{R}$ be a functional of vector fields. Then there is a scalar function $\varphi \in H^1(M)$ such that for a given vector field $A$, the extremum points $u$ of $F$ with the $\text{div}_A$-free constraint satisfy the equation

$$
\delta F(u) = (\partial + A)\varphi.
$$

Proof of Theorems 7.2 and 7.3. First we prove Theorem 7.2. By (7.14), the extremum points $\{g_{ij}\}$ of $F$ with the $\text{div}_A$-free constraint satisfy

$$
\int_M \delta F(g_{ij}) \cdot X \sqrt{g} dx = 0 \quad \forall X \in L^2(T^2_0 M) \text{ with } \text{div}_A X = 0.
$$

It implies that

$$
\delta F(g_{ij}) \perp L^2_D(T^2_0 M) = \{ v \in L^2(T^2_0 M) | \ \text{div}_A v = 0 \}.
$$

By Theorem 6.1 $L^2(T^2_0 M)$ can be orthogonally decomposed into

$$
L^2(T^2_0 M) = L^2_D(T^2_0 M) \oplus G^2(T^2_0 M), \quad G^2(T^2_0 M) = \{ D_A \varphi | \ \varphi \in H^1(T^1_0 M) \}.
$$

Hence it follows from (7.18) that

$$
\delta F(g_{ij}) \in G^2(T^2_0 M),
$$

which means that the equality (7.16) holds true.

To prove Theorem 7.3, for an extremum vector field $u$ of $F$ with the $\text{div}_A$-free constraint, we derive in the same fashion that $u$ satisfies the following equation

$$
\int_M \delta F(u) \cdot X \sqrt{g} dx = 0 \quad \forall X \in L^2(TM) \text{ with } \text{div}_A X = 0.
$$

In addition, Theorem 6.1 means that

$$
L^2(TM) = L^2_D(TM) \oplus G^2(TM), \quad L^2_D(TM) = \{ v \in L^2(TM) | \ \text{div}_A v = 0 \}, \quad G^2(TM) = \{ D_A \varphi | \ \varphi \in H^1(M) \}.
Then we infer from (7.19) that
\[ \delta F(u) \in G^2(TM). \]
Thus we deduce the equality (7.17).
The proofs of Theorems 7.2 and 7.3 are complete. □

References
[1] F. Englert and R. Brout, Broken symmetry and the mass of gauge vector mesons, Physical Review Letters, 13 (9) (1964), p. 32123.
[2] D. Griffiths, Introduction to elementary particles, Wiley-Vch, 2008.
[3] G. Guralnik, C. R. Hagen, and T. W. B. Kibble, Global conservation laws and massless particles, Physical Review Letters, 13 (20) (1964), p. 585587.
[4] F. Halzen and A. D. Martin, Quarks and leptons: an introductory course in modern particle physics, John Wiley and Sons, New York, NY, 1984.
[5] P. W. Higgs, Broken symmetries and the masses of gauge bosons, Physical Review Letters, 13 (1964), p. 508509.
[6] M. Kaku, Quantum Field Theory, A Modern Introduction, Oxford University Press, 1993.
[7] G. Kane, Modern elementary particle physics, vol. 2, Addison-Wesley Reading, 1987.
[8] T. Ma and S. Wang, Bifurcation theory and applications, vol. 53 of World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.
[9] Unified field theory and principle of representation invariance, arXiv:1212.4893; version 1 appeared in Applied Mathematics and Optimization, DOI: 10.1007/s00245-013-9226-0, 33pp., (2012).
[10] Gravitational field equations and theory of dark matter and dark energy, Discrete and Continuous Dynamical Systems, Ser. A, 34:2 (2014), pp. 335–366; see also arXiv:1206.5078v2.
[11] Weakton model of elementary particles and decay mechanisms, Indiana University Institute for Scientific Computing and Applied Mathematics Preprint Series, #1304: http://www.indiana.edu/~iscam/preprint/1304.pdf (May 30, 2013).
[12] Gauge theories of the strong, weak, and electromagnetic interactions, 2nd edition, Princeton University Press, 2013.

(TM) Department of Mathematics, Sichuan University, Chengdu, P. R. China

(SW) Department of Mathematics, Indiana University, Bloomington, IN 47405
E-mail address: showang@indiana.edu, http://www.indiana.edu/ fluid