The Sample Complexity of Meta Sparse Regression

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Abstract
This paper addresses the meta-learning problem in sparse linear regression with infinite tasks. We assume that the learner can access several similar tasks. The goal of the learner is to transfer knowledge from the prior tasks to a similar but novel task. For \( p \) parameters, size of the support set \( k \), and \( l \) samples per task, we show that \( T \in O((k \log p)/l) \) tasks are sufficient in order to recover the common support of all tasks. With the recovered support, we can greatly reduce the sample complexity for estimating the parameter of the novel task, i.e., \( l \in O(1) \) with respect to \( T \) and \( p \). We also prove that our rates are minimax optimal. A key difference between meta-learning and the classical multi-task learning, is that meta-learning focuses only on the recovery of the parameters of the novel task, while multi-task learning estimates the parameter of all tasks, which requires \( l \) to grow with \( T \). Instead, our efficient meta-learning estimator allows for \( l \) to be constant with respect to \( T \) (i.e., few-shot learning).

1 Introduction

Current machine learning algorithms have shown great flexibility and representational power. On the downside, in order to obtain good generalization, a large amount of data is required for training. Unfortunately, in some scenarios, the cost of data collection is high. Thus, an inevitable question is how to train a model in the presence of few training samples. This is also called \textbf{Few-Shot Learning} (Wang and Yao, 2019). Indeed, there might not be much information about an underlying task when only few examples are available. A way to tackle this difficulty is \textbf{Meta-Learning} (Vanschoren, 2018): we gather many similar tasks instead of several examples in one task, and use the data from different tasks to train a model that can generalize well in the similar tasks. This hopefully also guarantees a good performance of the model for a novel task, even when only few examples are available for the new task. In this sense, the model can rapidly adapt to the novel task with prior knowledge extracted from other similar tasks.

As a meta-learning example, for the particular model class of neural networks, researchers have developed algorithms such as Matching Networks (Vinyals et al., 2016), Prototypical Networks (Snell et al., 2017), long-short term memory-based meta-learning (Ravi and Larochelle, 2016), Model-Agnostic Meta-Learning (MAML) (Finn et al., 2017), among others. These algorithms are experimental works that have been proved to be successful in some cases. Unfortunately, there is a lack of theoretical understanding of the generalization of meta-learning, in general, for any model class. Some of the algorithms can perform very well in tasks related to some specific applications,
but it is still unclear why those methods can learn across different tasks with only few examples given for each task. For example, in few shot learning, the case of 5-way 1-shot classification requires the model to learn to classify images from 5 classes with only one example shown for each class. In this case, the model should be able to identify useful features (among a very large learned feature set) in the 5 examples instead of building the features from scratch.

There has been some effort on building the theoretical foundation of meta-learning. For MAML, Finn et al. (2019) showed that the regret bound in the online learning regime is $O(\log T)$, and Fallah et al. (2019) showed that MAML can converge and find an $\epsilon$-first order stationary point with $O(1/\epsilon^2)$ iterations. A natural question is how we can have a theoretical understanding of the meta-learning problem for any algorithm, i.e., the lower bound of the sample complexity of the problem.

The upper and lower bounds of sample complexity is commonly analyzed in simple but well-defined statistical learning problems. Since we are learning a novel task with few samples, meta-learning falls in the same regime than sparse regression with large number of covariates $p$ and a small sample size $l$, which is usually solved by $\ell_1$ regularized (sparse) linear regression such as LASSO, albeit for a single task. Even for a sample efficient method like LASSO, we still need the sample size $l$ to be of order $O(k \log(p))$ to achieve correct support recovery, where $k$ is the number of non-zero coefficients among the $p$ coefficients. The $l \in O(k \log p)$ rate has been proved to be optimal (Wainwright, 2009). If we consider meta-learning, we may be able to bring prior information from similar tasks to reduce the sample complexity of LASSO. In this respect, researchers have considered the multi-task problem, which assumes similarity among different tasks, e.g., tasks share a common support. Then, one learns for all tasks at once. While it seems that considering many similar tasks together can bring information to each single task, the noise or error is also introduced.

In the results from previous papers, e.g., (Yuan and Lin, 2006), (Negahban and Wainwright, 2008), (Lounici et al. 2009), (Jalali et al., 2010), in order to achieve good performance on all $T$ tasks, one needs the number of samples $l$ to scale with the number of tasks $T$. More specifically, one requires $l \in O(T)$ or $l \in O(\log T)$ for each task, which is not useful in the regime where $l \in O(1)$ with respect to $T$.

Our contribution in this paper is as follows. First, we proposed a meta-sparse regression problem and a corresponding generative model that are amenable to solid statistical analysis and also capture the essence of meta-learning. Second, we prove the upper and lower bounds of the sample complexity of this problem, and show that they match in the sense that $l \in O((k \log p)/T)$ and $l \in \Omega((k \log p)/T)$. Here $p$ is the number of coefficients in one task, $k$ is the number of non-zero coefficients among the $p$ coefficients, and $l$ is the sample size of each task. In short, we assume that we have access to possibly an infinite number of tasks from a distribution of tasks, and for each task we only have limited number of samples. Our goal is to first recover the common support of all tasks and then use it for learning a novel task. We prove that simply by merging all the data from different tasks and solving a $\ell_1$ regularized (sparse) regression problem (LASSO), we can achieve the best sample complexity rate for identifying the common support and learning the novel task. To the best of our knowledge, our results are the first to give upper and lower bounds of the sample complexity of meta-learning problems.

2 Method

In this section, we present the meta sparse regression problem as well as our $\ell_1$ regularized regression method.
2.1 Problem Setting

We consider the following meta sparse regression model. The dataset containing samples from multiple tasks \( \{(X_{i,j}, y_{t,j}, t_i) | i = 1, 2, \cdots, T, T + 1; j = 1, 2, \cdots, l\} \) is generated in the following way:

\[
y_{t,j} = X^T_{t,j}(w^* + \Delta^*_i) + \epsilon_{t,j},
\]

where, \( t_i \) indicates the \( i \)-th task, \( w^* \in \mathbb{R}^p \) is a constant across all tasks, and \( \Delta^*_i \) is the individual parameter for each task. Note that the tasks \( \{t_i | i = 1, 2, \cdots, T\} \) are the related tasks we collect for helping solve the novel task \( t_{T+1} \). Each task contains \( l \) training samples. The sample size of task \( t_{T+1} \) is denoted by \( l_{T+1} \), which is equal to \( l \) in the setting above, but generally it could also be larger than \( l \).

Tasks are independently drawn from one distribution, i.e., \( t_i \) i.i.d \( \sim F(t) \), or equivalently \( \Delta^*_i \) i.i.d \( \sim F^*(\Delta) \). We assume \( F^*(\Delta) \) is a sub-Gaussian distribution with mean 0 and variance proxy \( \sigma^2_\Delta \). The latter is a very mild assumption, since the class of sub-Gaussian random variables includes for instance Gaussian random variables, any bounded random variable (e.g. Bernoulli, multinomial, uniform), any random variable with strictly log-concave density, and any finite mixture of sub-Gaussian variables. We denote the support set of each task \( t_i(i = 1, 2, \cdots, T) \) as \( S_i = \text{Supp}(w^* + \Delta^*_i) \). For simplicity, here we consider the case that \( S_i = S = \text{Supp}(w^*) \) and \(|S| = k \ll p, k \leq Tl, S_{T+1} \subseteq S, |S_{T+1}| = k_{T+1} \leq k \).

We assume that \( \epsilon_{t,j} \) are i.i.d and follow a sub-Gaussian distribution with mean 0 and variance proxy \( \sigma^2_\epsilon \). Sample covariates \( X_{i,j} \in \mathbb{R}^p \) are independent and all entries in them are independent. These entries are i.i.d from a sub-Gaussian distribution with mean 0 and variance proxy no greater than \( \sigma^2_\epsilon \).

2.2 Our Method

In meta sparse regression, our goal is to use the prior \( T \) tasks and their corresponding data to recover the common support of all tasks. We then estimate the parameters for the novel task. For the setting we explained above, this is equivalent to recover \( (w^*, \Delta^*_{T+1}) \).

First, we determine the common support \( S \) over the prior tasks \( \{t_i | i = 1, 2, \cdots, T\} \) by the support of \( \hat{w} \) formally introduced below, i.e., \( \hat{S} = \text{Supp}(\hat{w}) \), where

\[
\ell(w) = \frac{1}{2Tl} \sum_{i=1}^{T} \sum_{j=1}^{l} \|y_{t,j} - X^T_{t,j}w\|_2^2
\]

\[
\hat{w} = \arg \min_w \{ \ell(w) + \lambda \|w\|_1 \}
\]

Note that we have \( T \) tasks in total, and \( l \) samples for each task.

Second, we use the support \( \hat{S} \) as a constraint for recovering the parameters of the novel task \( t_{T+1} \). That is

\[
\ell_{T+1}(w) = \frac{1}{2l} \sum_{j=1}^{l} \|y_{t_{T+1},j} - X^T_{t_{T+1},j}w\|_2^2
\]

\[
\hat{w}_{T+1} = \arg \min_{w, \text{Supp}(w) \subseteq \hat{S}} \{ \ell_{T+1}(w) + \lambda_{T+1} \|w\|_1 \}
\]

We point out that our method makes a proper application of \( \ell_1 \) regularized (sparse) regression, and in that sense is somewhat intuitive. In what follows, we show that this method correctly recovers
the common support and the parameter of the novel task. At the same time, our method is minimax optimal, i.e., it achieves the optimal sample complexity rate.

3 Main Results

First, we state our result for the recovery of the common support among the prior $T$ tasks.

**Theorem 3.1.** Let $\hat{w}$ be the solution of the optimization problem $\text{(5)}$. If $\lambda \in O\left(\sqrt{\frac{\log p}{Tl}}\right)$, $\epsilon \in (0, 1/2)$ and

$$T \in O\left(\max\left\{\frac{(\log p)^2}{k^2l^6}, \frac{k \log(p - k)}{le^2}\right\}\right),$$

with probability greater than $1 - c_1 \exp(-c_2 \min\{k, \log(p - k), (\frac{\log p}{k^2l^2}, \frac{\log k}{l^2e^2})\})$, we have that

1. the support of $\hat{w}$ is contained within $S$ (i.e., $S(\hat{w}) \subseteq S$);
2. $\|\hat{w} - w^\ast\|_\infty \leq \frac{c_3 \lambda}{\sigma_2^2\epsilon} \left(\sqrt{\frac{k}{Tl}} + 1\right) + c_4 \sqrt{\frac{\sigma_2^2 \log k}{\sigma_2^2 Tl}} := g(\lambda),$

where $c_1, c_2, c_3, c_4$ are constants.

Note that in Theorem 3.1, the term $k \log(p - k)$ in $T$ is typically encountered in the analysis of the single-task sparse regression or LASSO (Wainwright [2009]). The additional term $\frac{(\log p)^2}{l^2}$ in $T$ is due to the difference in the coefficients among tasks.

Next, we state our result for the recovery of the parameters of the novel task.

**Theorem 3.2.** Let $\hat{w}_{T+1}$ be the solution of the optimization problem $\text{(6)}$. With the support $\hat{S}$ recovered from Theorem 3.1, if $k' = k_{T+1}$, $\lambda' = \lambda_{T+1} \in O\left(\sqrt{\frac{\log k'}{Tl}}\right)$, $\epsilon \in (0, 1/2)$ and $l \in O\left(k' \log(k - k')/e^2\right)$, with probability greater than $1 - c'_1 \exp(-c'_2 \min\{k', \log(k - k')\})$, we have that

1. the support of $\hat{w}_{T+1}$ is contained within $S_{T+1}$ (i.e., $S(\hat{w}_{T+1}) \subseteq S_{T+1} \subseteq S$);
2. $\|\hat{w}_{T+1} - w^\ast\|_\infty \leq \frac{c'_3 \lambda'}{\sigma_2^2\epsilon} \left(\sqrt{\frac{k'}{Tl}} + 1\right) + c'_4 \sqrt{\frac{\sigma_2^2 \log k'}{\sigma_2^2 Tl}},$

where $c'_1, c'_2, c'_3, c'_4$ are constants.

The theorems above provide an upper bound of the sample complexity, which can be achieved by our method. The lower bound of the sample complexity is an information-theoretic result, and it relies on the construction of a restricted class of parameter vectors. We consider a special case of the setting we previously presented: all non-zero entries in $w^\ast$ are 1, and all non-zero entries in $w^\ast + \Delta_i$, are also 1. We use $\Theta$ to denote the set of all possible parameters $\theta = (w^\ast, \Delta^\ast_{i_{T+1}})$. Therefore the number of possible outcomes of the parameters $|\Theta| = \binom{p}{k} \binom{k}{k_{T+1}} \in O(p^k k^{k_{T+1}})$.

If the parameter $\theta^\ast$ is chosen uniformly at random from $\Theta$, for any algorithm estimating this parameter by $\hat{\theta}$, the answer is wrong (i.e., $\hat{\theta} \neq \theta^\ast$) with probability greater than $1/2$ if $O(Tl + l_{T+1}) \leq k \log p + k_{T+1} \log k$. Here we use $l_{T+1}$ to denote the sample size of task $t_{T+1}$. This fact is proved in the following theorem.
Theorem 3.3. Let $\Theta := \{\theta = (w, \Delta_{T+1})|w \in \{0,1\}^p, \|w\|_0 = k, \Delta_{t_i} \in \{0, -1\}^p \subseteq \text{Supp}(w), \|w + \Delta_{t_i}\|_0 = k_i\}$. Furthermore, assume that $\theta^* = (w^*, \Delta_{T+1}^*)$ is chosen uniformly at random from $\Theta$. We have:

$$\Pr[\hat{\theta} \neq \theta^*] \geq 1 - \frac{\log 2 + c'_1 \cdot Tl + c'_2 \cdot l_{T+1}}{\log |\Theta|}$$

where $c'_1, c'_2$ are constants.

In the following section, we prove that the mutual information $I(\theta^*, S)$ between the true parameter $\theta^*$ and the data $S$ is bounded by $c'_1 \cdot Tl + c'_2 \cdot l_{T+1}$. In order to prove Theorem 3.3, we use Fano's inequality and the construction of a restricted class of parameter vectors. The use of Fano's inequality and restricted ensembles is customary for information-theoretic lower bounds (Wang et al., 2010; Santhanam and Wainwright, 2012; Tandon et al., 2014).

Note that from Theorem 3.3, we know if $T \lesssim \frac{\log p}{l_{T+1}}$ and $l_{T+1} \lesssim k_{T+1} \log k$, then any algorithm will fail to recover the true parameter very likely. On the other hand, if we have $T \in O(\frac{\log p}{l_{T+1}})$ and $l_{T+1} \in O(k_{T+1} \log k)$, by Theorem 3.1 and 3.2 we can recover the support of $w^*$ and $\Delta_{T+1}^*$ (by $w_{T+1}^* - w^*$). Therefore we claim that our rates of sample complexity is minimax optimal.

4 Sketch of the proofs

In this section, we provide details about the proofs of our main results.

4.1 Proof of Theorem 3.1

We use the primal-dual witness framework (Wainwright, 2009) to prove our results. First we construct the primal-dual candidate; then we show that the construction succeeds with high probability. Here we outline the steps in the proof. (See the supplementary materials for detailed proofs.)

We first introduce some useful notations:

$X_{t_i} \in \mathbb{R}^{l \times p}$ is the matrix of collocated $X_{t_i,j}$ (covariates of all samples in the $i$-th task). Similarly, $y_{t_i} \in \mathbb{R}^l$ and $c_{t_i} \in \mathbb{R}^l$.

$X_{[T]} \in \mathbb{R}^{T \times p}$ is the matrix of collocated $X_{t_i}$ (covariates of all samples in all tasks). Similarly, $c_{[T]} \in \mathbb{R}^{T}$.

$X_{t_i,S} \in \mathbb{R}^{l \times k}$ is the sub-matrix of $X_{t_i}$ containing only the rows corresponding to the support of $w^*$, i.e., $S$ with $|S| = k$. Similarly, $X_{[T],S} \in \mathbb{R}^{T \times k}$, $\Delta_{t_i,S}^* \in \mathbb{R}^k$, and $w_S \in \mathbb{R}^k$.

$A_{S,S} \in \mathbb{R}^{k \times k}$ is the sub-matrix of $A \in \mathbb{R}^{p \times p}$ containing only the rows and columns corresponding to the support of $w^*$.

4.1.1 Primal-dual witness

Step 1: Prove that the objective function has positive definite Hessian when restricted to the support, i.e., $w_S = 0$.

$$(\forall w_S \in \mathbb{R}^{|S|}) \quad [\nabla^2 \ell((w_S, 0))]_{S,S} > 0$$  \hspace{1cm} (6)

We know that

$$[\nabla^2 \ell((w_S, 0))]_{S,S} > 0 \iff \frac{1}{Tl} [X_{[T],S}^T X_{[T]}]_{S,S} > 0$$  \hspace{1cm} (7)

We prove the above condition in the following lemma.
Lemma 4.1. For $k \leq n$, assume that each element in $X \in \mathbb{R}^{n \times k}$ is i.i.d. sub-Gaussian random variable with mean 0 and variance proxy $\sigma_x^2$. We have

$$\Pr \left[ \left\| \frac{1}{n}X^TX - \sigma_x^2I \right\|_2 \geq \sigma_x^2\delta(n,k,t) \right] \leq 2e^{-nC_1t^2} \quad (8)$$

where

$$\delta(n,k,t) := \left( C_2\sqrt{k/n + t} \right) + \left( C_2\sqrt{k/n + t} \right)^2$$

and $C_1, C_2$ are constants.

Using the lemma above, we show the minimum singular value of $\hat{\Sigma}_{S,S} = \frac{1}{Tl}[X_T^T X_T]_{S,S}$ is larger than 0 with high probability. We let $t = 1/4$, $n = Tl \geq 16C_2^2k$ to have $\delta(n,k,t) \leq 3/4$.

Thus, with probability $1 - 2e^{-nC_1/16} = 1$, we have

$$\lambda_{\min}(\hat{\Sigma}_{S,S}) = x_0^T \hat{\Sigma}_{S,S} x_0 = x_0^T (\hat{\Sigma}_{S,S} - \sigma_x^2 I)x_0 + \sigma_x^2 \geq \sigma_x^2 - \delta(n,k,t)\sigma_x^2 \geq \sigma_x^2/4 > 0$$

where we set $x_0 = \arg \min_{x \in S^{-1}} x^T \hat{\Sigma}_{S,S} x$

**Step 2:** Set up a restricted problem:

$$\hat{w}_S = \arg \min_{w_S \in \mathbb{R}^{[n]}} \ell((w_S, 0)) + \lambda \|w_S\|_1 \quad (9)$$

**Step 3:** Choose the corresponding dual variable $\hat{z}_S$ to fulfill the complementary slackness condition:

$\forall i \in S, \hat{z}_i = sign(\hat{w}_i)$ if $\hat{w}_i \neq 0$, otherwise $\hat{z}_i \in [-1, +1]$

**Step 4:** Solve $\hat{z}_{S^c}$ to let $(\hat{w}, \hat{z})$ fulfill the stationarity condition:

$$[\nabla \ell((\hat{w}_S, 0))]|_{S} + \lambda \hat{z} = 0 \quad (10)$$

$$[\nabla \ell((\hat{w}_S, 0))]|_{S^c} + \lambda \hat{z}_{S^c} = 0 \quad (11)$$

**Step 5:** Verify that the strict dual feasibility condition is fulfilled for $\hat{z}_{S^c}$:

$$\|z_{S^c}\|_\infty < 1$$

To prove support recovery, we only need to show that **step 5** holds. In the next subsection we indeed show that this holds with high probability.

4.1.2 Strict dual feasibility condition

We first rewrite (10) as follows:

$$\frac{1}{Tl} \sum_{i=1}^{T} X_{t_i,S}^T X_{t_i,S}(\hat{w}_S - w_S^*) = -\lambda \hat{z} + \frac{1}{Tl} \sum_{i=1}^{T} X_{t_i,S}^T \epsilon_{t_i} + \frac{1}{Tl} \sum_{i=1}^{T} X_{t_i,S}^T X_{t_i,S} \Delta_{t_i,S}$$

Then we solve for $(\hat{w}_S - w_S^*)$. That is

$$\hat{w}_S - w_S^* = \left( \frac{1}{Tl} \sum_{i=1}^{T} X_{t_i,S}^T X_{t_i,S} \right)^{-1} \left( \frac{1}{Tl} \sum_{i=1}^{T} X_{t_i,S}^T \epsilon_{t_i} - \lambda \hat{z} + \frac{1}{Tl} \sum_{i=1}^{T} X_{t_i,S}^T X_{t_i,S} \Delta_{t_i,S} \right)$$
and plug it in the equation below (by rewriting (11)).

\[ \tilde{z}_{S^c} = \frac{1}{\lambda T I} \sum_{i=1}^{T} (X^T_{t_i, S^c} \epsilon_{t_i} - X^T_{t_i, S^c} X_{t_i, S^c} (\tilde{w}_S - \bar{w}_S^* - \Delta_{t_i, S})) \]

We have

\[ \tilde{z}_{S^c} = X^T_{[T], S^c} \left\{ \frac{1}{T} X_{[T], S} (\hat{\Sigma}_{S, S})^{-1} \tilde{z}_S + \Pi_{X^T_{[T], S}} \left( \frac{\ell}{\lambda TI} \right) \right\} + \frac{1}{\lambda T I} \sum_{i=1}^{T} X^T_{t_i, S^c} X_{t_i, S^c} \Delta_{t_i, S}^* \]

\[ - \frac{1}{\lambda (TI)^2} X^T_{[T], S^c} X_{[T], S} (\hat{\Sigma}_{S, S})^{-1} \left( \sum_{i=1}^{T} X^T_{t_i, S} X_{t_i, S} \Delta_{t_i, S}^* \right) \]

where \( \Pi_{X^T_{[T], S}} := I_{n \times n} - X_{[T], S}(X^T_{[T], S} X_{[T], S})^{-1} X^T_{[T], S} \) is an orthogonal projection matrix, and \( \tilde{z}_S \) is the dual variable we choose at step 3.

One can bound the \( \ell_\infty \) norm of \( \tilde{z}_{S^c, 1} \) by the techniques used in \cite{Wainwright, 2009}. Specifically, if we set \( T \in O(k \log (p - k)/(\epsilon^2)) \), \( \epsilon \in (0, 1/2) \) and \( \lambda \in O(\sqrt{\frac{\log (p)}{T}}) \), with the mutual incoherence condition being satisfied (i.e., \( ||\Sigma_{S^c, S}(\hat{\Sigma}_{S, S})^{-1}||_{\infty} = 0 \leq 1 - \gamma, \gamma \in (0, 1) \)), we have

\[ \mathbb{P}[||\tilde{z}_{S^c, 1}||_{\infty} \geq \gamma] \leq c_1 \exp(-c_2 \min\{k, \log (p - k)\}) \]

Note that the remaining two terms \( \tilde{z}_{S^c, 2}, \tilde{z}_{S^c, 3} \) containing \( \Delta_{t_i} \) are new to the meta-learning problem and need to be handled with novel proof techniques.

We first rewrite \( \tilde{z}_{S^c, 2} \) with respect to each of its entries (denoted by \( \tilde{z}_{n, 2} \)) as follows: \( \forall n \in S^c \), we have

\[ \tilde{z}_{n, 2} = \frac{1}{\lambda T I} \sum_{i=1}^{T} \sum_{j=1}^{l} \sum_{m \in S} X_{t_i, j, n} X_{t_i, j, m} \Delta_{i, m}^* \]

(12)

We know that \( X_{t_i, j, n}, X_{t_i, j, m}, \Delta_{i, m}^* \) are sub-Gaussian random variables. It is well-known that the product of two sub-Gaussian is sub-exponential (whether they are independent or not). To characterize the product of three sub-Gaussians and the sum of the i.i.d. products, we need to use Orlicz norms and a corresponding concentration inequality.

### 4.1.3 Orlicz norm

Here we introduce the concept of exponential Orlicz norm. For any random variable \( X \) and \( \alpha > 0 \), we define the \( \psi_\alpha \) (quasi-) norm as

\[ ||X||_{\psi_\alpha} = \inf \left\{ t > 0 : \mathbb{E} \exp \left( \frac{|X|^\alpha}{t^\alpha} \right) \leq 2 \right\} \]

We define \( \inf \emptyset = \infty \). This concept is a generalization of sub-Gaussianity and sub-exponentiality since the random variable family with finite exponential Orlicz norm \( || \cdot ||_{\psi_\alpha} \) corresponds to the
α-sub-exponential tail decay family which is defined by

\[ \mathbb{P}(\lvert X \rvert \geq t) \leq c \exp\left(-\frac{t^\alpha}{C}\right) \quad \forall t \geq 0. \]

where \(c, C\) are constants. More specifically, if \(\|X\|_{\psi_\alpha} = k\), we set \(c = 2, C = k^\alpha\) so that \(X\) fulfills the α-sub-exponential tail decay property above. We have two special cases of Orlicz norms: \(\alpha = 2\) corresponds to the family of sub-Gaussian distributions and \(\alpha = 1\) corresponds to the family of sub-exponential distributions.

A good property of the Orlicz norm is that the product or the sum of many random variables with finite Orlicz norm has finite Orlicz norm as well (possibly with a different \(\alpha\)). We state this property in the two lemmas below.

**Lemma 4.2.** [Lemma A.1 in (Götz et al., 2019)] Let \(X_1, \cdots, X_k\) be random variables such that \(\|X_i\|_{\psi_{\alpha_i}} < \infty\) for some \(\alpha_i \in (0, 1]\) and let \(t = \frac{1}{\sum_{i=1}^k \alpha_i}\). Then \(\|\prod_{i=1}^k X_i\|_{\psi_t} < \infty\) and

\[ \mathbb{E}^2 \left( \prod_{i=1}^k X_i \right) \leq \prod_{i=1}^k \mathbb{E}^2 (X_i). \]

**Lemma 4.3.** [Lemma A.3 in (Götz et al., 2019)] For any \(0 < \alpha \leq 1\) and any random variables \(X_1, \cdots, X_l\), we have

\[ \| \sum_{i=1}^l X_i \|_{\psi_{\alpha}} \leq \left( \sum_{j=1}^l \mathbb{E}^2 (X_j) \right)^{1/\alpha}. \]

By the lemmas above, we know that the sum (with respect to \(j, m\)) of the products in (12) is a \(\frac{2}{3}\)-sub-exponential tail decay random variable. The details are shown in the next subsection. This result does not require any independence conditions, thus we will use this fact later for bounding \(\tilde{z}_{S^c,3}\).

### 4.1.4 \(\frac{2}{3}\)-sub-exponential tail decay random variable

For \(j \in S^c, q \in S\), we have

\[ \tilde{z}_{j,2} := \frac{1}{\lambda T l} \sum_{i=1}^T X_{t, i, j}^T X_{t, i, S} \Delta_{t, i, S}^* \]

\[ = \frac{1}{\lambda T l} \sum_{i=1}^T \sum_{m=1}^l X_{t, i, j, m} X_{t, i, S, m} \Delta_{t, i, S}^* \]

\[ = \frac{1}{\lambda T l} \sum_{i=1}^T \sum_{q \in S} \Delta_{t, i, q}^* \left( \sum_{m=1}^l X_{t, i, j, m} X_{t, i, q, m} \right) \]

From Lemma 4.2 we know

\[ \|X_{t, i, j, m} X_{t, i, q, m}\|_{\psi_1} \leq \|X_{t, i, j, m}\|_{\psi_2} \|X_{t, i, q, m}\|_{\psi_2} = M_X^2 \]

where \(M_X = c_3 \sigma_x\) and \(c_3\) is a constant.
From Lemma 4.3, we have
\[ \left\| \sum_{m=1}^{l} X_{t,j,m} X_{t,q,m} \right\|_{\psi_1} \leq l^2 M_X^2 \]

From Lemma 4.2 again, we know
\[ \left\| \Delta_{t,q}^* \left( \sum_{m=1}^{l} X_{t,j,m} X_{t,q,m} \right) \right\|_{\psi_2} \leq \left\| \Delta_{t,q}^* \psi_2 \left( \sum_{m=1}^{l} X_{t,j,m} X_{t,q,m} \right) \right\|_{\psi_1} \]
\[ \leq M \Delta^2 M_X^2 = M_S \]
where \( M_S = c_4 l^2 \).

4.1.5 Concentration inequality for \( \tilde{z}_{S^c,2} \)

Recall that
\[ \tilde{z}_{j,2} = \frac{1}{XT} \sum_{i=1}^{T} \sum_{q \in S} \Delta_{t,i,q}^* \left( \sum_{m=1}^{l} X_{t,i,m} X_{t,q,m} \right) \]
\[ : = \frac{1}{T} \sum_{i=1}^{T} \sum_{q \in S} \tilde{z}_{j,2,i,q} \]

We know that for different \( q \in S \) and \( i \), the random variables \( \tilde{z}_{j,2,i,q} \) are independent with \( \mathbb{E} \tilde{z}_{j,2,i,q} = 0 \) and \( \| \tilde{z}_{j,2,i,q} \psi_{2/3} \| \leq \frac{M_S}{M} = c_4 l / \lambda \). Now we use a concentration inequality to bound \( \tilde{z}_{j,2} \).

**Lemma 4.4** (Theorem 1.4 in (Götze et al., 2019)). Let \( X_1, \ldots, X_k \) be a set of independent random variables satisfying \( \| X_i \psi_{2/3} \| \leq M \) for some \( M > 0 \). There is a constant \( C_3 \) such that for any \( t > 0 \), we have
\[ \mathbb{P} \left( \left\| \frac{1}{n} \sum_{i=1}^{n} (X_i - \mathbb{E} X_i) \right\| \geq t \right) \leq 2 \exp \left( -\frac{f_3(M,t,n)}{C_3} \right) \]
where
\[ f_3(M,t,n) = \min \left( \frac{t^2 n}{M^2}, \frac{tn}{M}, \left( \frac{tn}{M} \right)^{\frac{2}{3}} \right) \]

We set \( \omega = \frac{\lambda}{k c_4} \). Then we have
\[ \mathbb{P} (|\tilde{z}_{j,2}| \geq \gamma) = \mathbb{P} \left( \left\| \frac{1}{Tk} \sum_{i=1}^{T} \sum_{q \in S} \tilde{z}_{j,2,i,q} \right\| \geq \frac{\gamma}{k} \right) \]
\[ \leq 2 \exp \left( -\frac{\min \left( Tk \omega^2, Tk \omega, (Tk \omega)^{\frac{2}{3}} \right)}{C_3} \right) \]
When $\lambda \in O(\sqrt{\frac{\log(p)}{T^2}}), T \in O((\log(p)^2)/(k^3l^6))$, we have

$$\omega \in O((\log(p))^{1/2}/(kT^{1/2}l^{3/2}))$$

$$Tk\omega^2 \in O((\log(p))/(k^3l^3))$$

$$Tk\omega \in O((\log(p))^{3/2}/(k^3l^3))$$

$$(Tk\omega)^3 \in O((\log(p))/(k^3l^3)).$$

Therefore, $\|\tilde{z}_{S',2}\|_\infty$ can be bounded by $\gamma$ with probability

$$\mathbb{P}[\|\tilde{z}_{S',2}\|_\infty \geq \gamma] \leq c_5\exp(-c_6 \log(p)/(k^3l^3))$$

### 4.1.6 Bound on $\|\tilde{z}_{S',3}\|_\infty$

By definition,

$$\tilde{z}_{S',3} = \frac{1}{\lambda(Tl)}X_{[T],s'}X_{[T],s}(\hat{\Sigma}_{S,S})^{-1}\left(\sum_{i=1}^{T} X_{t_i,s}^T X_{t_i,s} \Delta_{t_i,S}\right)$$

$$= \frac{1}{Tl}X_{[T],s'}X_{[T],s}(\hat{\Sigma}_{S,S})^{-1}\left(\sum_{i=1}^{T} X_{t_i,s}^T X_{t_i,s} \Delta_{t_i,S}\right)$$

$$:= \frac{1}{Tl}X_{[T],s'}X_{[T],s}(\hat{\Sigma}_{S,S})^{-1}\zeta_S$$

Here we define

$$\zeta_S := \frac{1}{\lambda(Tl)}\left(\sum_{i=1}^{T} X_{t_i,s}^T X_{t_i,s} \Delta_{t_i,S}\right)$$

Since the independence between random variables is not necessary in Lemma 4.2, we use the same technique for bounding $\tilde{z}_{S',2}$ to bound $\zeta_S$. More specifically, $\|\zeta_S\|_\infty$ can be bounded by $\gamma$ with probability

$$\mathbb{P}[\|\zeta_S\|_\infty \geq \gamma] \leq c_5\exp(-c_6 \log(p)/(k^3l^3))$$

We define event $T_2 = \{\|\zeta_S\|_\infty \geq \gamma\}$. Then we know

$$\mathbb{P}[\|\tilde{z}_{S',3}\|_\infty \geq \gamma'] \leq \mathbb{P}[\|\tilde{z}_{S',3}\|_\infty \geq \gamma' | T_2] + \mathbb{P}[T_2]$$

We bound $\|\tilde{z}_{S',3}\|_\infty$ by breaking it into two terms $A'_j$ and $B'_j$.

$$\tilde{z}_{S',3} = A'_j + B'_j$$

$$A'_j := E_j \frac{1}{Tl}X_{[T],s}(\hat{\Sigma}_{S,S})^{-1}\zeta_S$$

$$B'_j := \Sigma_{jS}(\Sigma SS)^{-1}\zeta_S$$

Here we know $B'_j = 0$.

$$\mathbb{P}[\max |A'_j| \geq \gamma'] \leq \mathbb{P}[\max |A'_j| \geq \gamma' | T_2] + \mathbb{P}[T_2]$$

$$\leq 2(p-k)\exp\left(-\frac{\gamma'^2}{2\rho_a(\Sigma_{S\cdot S})M_1'(\epsilon)}\right) + 4\exp(-c_1k) + c_5\exp(-c_6 \log(p)/(k^3l^3))$$
\[ M'_1(\epsilon) = \left( 1 + \frac{8}{\sigma^2} \sqrt{\frac{k}{T_l}} \right) \frac{k\gamma^2}{TI\sigma^2} \]

We set \( \lambda \in O\left( \frac{\log(p)}{T_l} \right) \), \( T \in O((\log(p)^2)/(k^3\gamma^6)) \).

Then \( \Pr[|A'_j| \geq \gamma'] \leq c_7\exp(-c_8 \min\{k, \log(p-k), \log(p)/(k\gamma^3)\}) \), where \( c_7, c_8 \) are constants.

4.1.7 Bound on \( \|\hat{z}_{S^c}\|_\infty \)

Since we have bounded each part of \( \hat{z}_{S^c} \), we now prove Theorem 3.1 by bounding \( \|\hat{z}_{S^c}\|_\infty \).

If \( \lambda \in O\left( \frac{\log(p)}{T_l} \right) \), \( T \in O((\log(p)^2)/(k^3\gamma^6)) \), we have

\[
\Pr[\|\hat{z}_{S^c}\|_\infty \geq 2\gamma + \gamma'] \leq \Pr[\|\hat{z}_{S^c,1}\|_\infty + \|\hat{z}_{S^c,2}\|_\infty + \|\hat{z}_{S^c,3}\|_\infty \geq 2\gamma + \gamma'] \\
\leq \Pr[\|\hat{z}_{S^c,1}\|_\infty \geq \gamma] + \Pr[\|\hat{z}_{S^c,2}\|_\infty \geq \gamma] + \Pr[\|\hat{z}_{S^c,3}\|_\infty \geq \gamma'] \\
\leq c_9\exp(-c_{10} \min\{k, \log(p-k), \log(p)/(k\gamma^3)\}) \\
:= P_0
\]

where \( c_9, c_{10} \) are constants that do not depend on \( T, p, k, l \).

We can set \( \gamma \) and \( \gamma' \) such that \( 2\gamma + \gamma' < 1 \). By primal-dual witness, we can guarantee that

\[ \Pr[S(\hat{w}) \subseteq S] \geq 1 - P_0 \]

4.1.8 Estimation error bound

The second part in Theorem 3.1 is about the estimation error. We first write the estimation error in the form below.

\[
\hat{w}_S - w^*_S = \hat{S}_{S,S}^{-1} \left( \frac{1}{T_l} \sum_{i=1}^T X^T_{t_i,S} \epsilon_{t_i} - \lambda \hat{z}_S + \frac{1}{T_l} \sum_{i=1}^T X^T_{t_i,S} X_{t_i,S} \Delta^*_{t_i,S} \right) \\
= \hat{S}_{S,S}^{-1} \frac{1}{T_l} \sum_{i=1}^T X^T_{t_i,S} \epsilon_{t_i} - \hat{S}_{S,S}^{-1} \lambda \hat{z}_S + \hat{S}_{S,S}^{-1} \lambda \frac{1}{T_l} \sum_{i=1}^T X^T_{t_i,S} X_{t_i,S} \Delta^*_{t_i,S}
\]

Similar to the proof of Theorem 2 in [Negahban and Wainwright, 2011], with probability greater than \( 1 - 2\exp(-c_{11}k) - 4\exp(-c_{12}T_l) \), we can bound the first two parts, which do not contain \( \Delta^*_{t_i,S} \):

\[
\|F_1 + F_2\|_\infty \leq \lambda \frac{\sigma^2}{\sigma^2} \left( \frac{c_{13}k}{\sqrt{T_l}} + 1 \right) + c'_{13} \sqrt{\frac{\sigma^2 \log k}{\sigma^2 T_l}}
\]

We also use the technique for bounding \( F_2 \) and \( \hat{z}_{S^c,2} \) to bound \( F_3 \).

**Lemma 4.5.** Assume we have \( T = O(k \log(p-k)/l) \). With probability greater than

\[ 1 - 2\exp(-c_{11}k) - c_8 \exp\left(-c_6 \log(p)/k\gamma^3\right), \]

we have

\[ \|F_3\|_\infty \leq \gamma \frac{\lambda}{\sigma^2} \left( \frac{c_{13}k}{\sqrt{T_l}} + 1 \right) \]
Therefore, we define the bound

\[ g(\lambda) := (1 + \gamma) \frac{\lambda}{\sigma^2} \left( \frac{c_{13} k \sqrt{p}}{\sqrt{Tl}} + 1 \right) + c'_{13} \sqrt{\frac{\sigma^2 \log k}{\sigma^2 Tl}} \]

Finally, we have

\[ \mathbb{P}[\|\hat{w}_S - w^*_S\|_\infty \geq g(\lambda)] \leq 1 - c_{14} \exp \left( - c_{15} \min \left\{ k, \frac{(\log(p))^2}{k^3 Tl}, \frac{\log(p)}{kl^3} \right\} \right) \]

### 4.2 Proof of Theorem 3.3

We first introduce Fano’s inequality (Fano, 1952; Yu, 1997) (the version below can also be found directly in Scarlett and Cevher, 2019).

**Lemma 4.6.** (Fano’s inequality) With input dataset \( S \), for any estimator \( \hat{\theta}(S) \) with \( k \) possible outcomes, i.e., \( \hat{\theta} \in \Theta, |\Theta| = k \), if \( S \) is generated from a model with true parameter \( \theta^* \) chosen uniformly at random from the same \( k \) possible outcomes \( \Theta \), we have:

\[ \mathbb{P}[\hat{\theta}(S) \neq \theta^*] \geq 1 - \frac{I(\theta^*, S) + \log 2}{\log k} \]

Now we show that \( I(\theta^*, S) \leq Tl \cdot c_1 + l_{T+1} \cdot c_2 \), where \( c_1, c_2 \) are constants, and \( \theta^* \) represents the parameter \((w^*, \Delta^*_t)\) we want to recover. Here \( S \) is all the data in the \( T+1 \) tasks, \( S_{[T]} \) is the data in the first \( T \) tasks, and \( S_t \) is the data of task \( t_i \). The mutual information is bounded by the following steps.

\[ I(\theta^*, S) = \frac{1}{k} \sum_{\theta^* \in \Theta} \int_S p_{S|\theta^*}(S) \log \left( \frac{p_{S|\theta^*}(S)}{p_{S}(S)} \right) dS \]

\[ = \frac{1}{k} \sum_{\theta^* \in \Theta} \int_S p_{S|\theta^*}(S) \log \left( \frac{p_{S|\theta^*}(S)}{\sum_{\theta' \in \Theta} p_{S|\theta'}(S)} \right) dS \]

\[ \leq \frac{1}{k^2} \sum_{\theta^* \in \Theta} \sum_{\theta' \in \Theta} \int_S p_{S|\theta^*}(S) \log \left( \frac{p_{S|\theta^*}(S)}{p_{S|\theta'}(S)} \right) dS \]

\[ = \frac{1}{k^2} \sum_{\theta^* \in \Theta} \sum_{\theta' \in \Theta} \text{KL}(P_{S|\theta^*} || P_{S|\theta'}) \]

Given the common coefficient \( w^* \), the data for each task is independent from each other. Therefore we have

\[ \text{KL}(P_{S|\theta^*} || P_{S|\theta'}) = \text{KL}(P_{S_{[T]}|\theta^*} || P_{S_{[T]}|\theta'}) + \text{KL}(P_{S_{T+1}|\theta^*} || P_{S_{T+1}|\theta'}) \]  \hspace{1cm} (13)

First, we consider the first part of equation (13). We use \( S' \) to denote \( S_{[T]} \). Let \( P_{S'} = P_{S_{[T]}|\theta^*}, P'_{S'} = P_{S_{[T]}|\theta^*} \). Note that

\[ \text{KL}(P_{S'} || P'_{S'}) = \int_{S'} P'_{S'} \log \left( \frac{P_{S'}(S')}{P'_{S'}(S')} \right) dS' \]
Furthermore

\[ P_{S'} = \int_{\Delta_{t_1}^*, \ldots, \Delta_{t_T}^*} P_{S'|w^*, \Delta_{t_1}^*, \ldots, \Delta_{t_T}^*} d\Delta_{t_1}^*, \ldots, d\Delta_{t_T}^* \]
\[ = \int_{\Delta_{t_1}^*} P_{S_1|w^*, \Delta_{t_1}^*} d\Delta_{t_1}^* \cdot \int_{\Delta_{t_T}^*} P_{S_T|w^*, \Delta_{t_T}^*} d\Delta_{t_T}^* \]

This is because conditioning on \( w^*, \Delta_{t_1}^*, \ldots, \Delta_{t_T}^* \), the data for each task is independent and therefore

\[ P_{S'|w^*, \Delta_{t_1}^*, \ldots, \Delta_{t_T}^*} = P_{S_1|w^*, \Delta_{t_1}^*} \cdots P_{S_T|w^*, \Delta_{t_T}^*} \]
\[ = P_{S_1|w^*, \Delta_{t_1}^*} \cdots P_{S_T|w^*, \Delta_{t_T}^*} \]

If we set \( a_i = P_{S_i|w^*, \Delta_{t_i}^*} \), \( a_i' = P_{S_i|w^*, \Delta_{t_i}^*} \), we have

\[ P_{S'} = a_1 a_2 \cdots a_T, \quad P_{S'}' = a_1' a_2' \cdots a_T'. \]

Therefore

\[ \mathbb{KL}(P_{S'}||P_{S'}) = \int_{S_T} a_1 \cdots a_T \left( \log \frac{a_1}{a_1'} + \cdots + \log \frac{a_T}{a_T'} \right) dS' \]

We know \( a_i \) is a function of \( S_j \) only when \( i = j \), and \( \int_{S_j} a_j dS_j = 1 \). Therefore, we have

\[ \int_{S_T} a_1 a_2 \cdots a_T \left( \log \frac{a_i}{a_i'} \right) dS' = \int_{S_T} a_i \log \frac{a_i}{a_i'} dS_i \]

Therefore

\[ \mathbb{KL}(P_{S'}'||P_{S'}) = \sum_{i=1}^{T} \int_{S_i} a_i \log \frac{a_i}{a_i'} dS_i \]
\[ \leq T \max_{i \in \{1,2,\ldots,T\}} \int_{S_i} a_i \log \frac{a_i}{a_i'} dS_i \]

For any task \( t_i \), conditioning on \( (w^*, \Delta_{t_i}^*) \), we know all samples in \( S_i \) are i.i.d. If we set \( S_{i,j} \) to be the \( j \)-th sample in the task \( t_i \), and \( a_{i,j} = P_{S_{i,j}|w^*, \Delta_{t_i}^*} \), we have

\[ \int_{S_i} a_i \log \frac{a_i}{a_i'} dS_i = \int_{S_{i,1}} a_{i,1} \log \frac{a_{i,1}}{a_{i,1}} dS_{i,1} \]

Therefore,

\[ \mathbb{KL}(P_{S'}||P_{S'}) \leq TL \max_{i} \int_{S_{i,1}} P_{S_{i,1}|w^*, \Delta_{t_i}^*} \frac{P_{S_{i,1}|w^*, \Delta_{t_i}^*}}{P_{S_{i,1}|w^*, \Delta_{t_i}^*}} dS_{i,1} \]
\[ = TL \cdot c_1 \]
Then we consider the second part of equation (13). For the task $t_{T+1}$, conditioning on $(w^*, \Delta^*_{t_{T+1}})$, since we know all samples in $S_{T+1}$ are i.i.d., we have

$$\text{KL}(P_{S_{T+1}|\theta^*} || P_{S_{T+1}|\theta'}) = l_{T+1} \int_{S_{T+1}} P_{S_{T+1}|w^*, \Delta^*_{t_{T+1}}} \frac{P_{S_{T+1}|w^*, \Delta^*_{t_{T+1}}}}{P_{S_{T+1}|w^*, \Delta^*_{t_{T+1}}}} dS_{T+1,1}$$

$$= l_{T+1} \cdot c_2$$

Combining the results above, we have

$$I(\theta^*, S) \leq Tl \cdot c_1 + l_{T+1} \cdot c_2.$$ 

Finally, from Fano’s inequality, we know

$$\mathbb{P}[\hat{\theta} \neq \theta^*] \geq 1 - \frac{\log 2 + Tl \cdot c_1 + l_{T+1} \cdot c_2}{\log |\Theta|}$$

5 DISCUSSIONS

Our problem setting and method are amenable to solid statistical analysis.

By focusing on sparse regression, our analysis shows clearly the difference between meta-learning and multi-task learning. In meta-learning, we only need to recover $w^*$ and $\Delta^*_{t_{T+1}}$, thus the number of samples needed for each task (including the novel task) is $l \in O((k \log p)/T + k_{T+1} \log k)$. When $T \to \infty$, meta-learning only needs $l \in O(k_{T+1} \log k)$. For multi-task learning, one needs to recover $(w^* + \Delta^*_{t_i})$ for all $t_i$, which requires the sample size $l$ of each task to be of order $O(\max(k \log(pT), kT \log(p)))$ (see, e.g., Theorem 2 in (Jalali et al., 2010)). When $T \to \infty$, the sample size of multi-task learning also goes to infinity.

While meta sparse regression might apparently look similar to the classical sparse random effect model (Bondell et al., 2010), a key difference is that in the random effect model, the experimenter is interested on the distribution of the estimator $w^*$ instead of support recovery. To the best of our knowledge, our results are the first to give upper and lower bounds of the sample complexity of meta-learning problems.

Although our paper shows that a proper application of $\ell_1$ regularized (sparse) regression achieves the minimax optimal rate, it is still unclear whether there is a method that can improve the constants in our results. To have further theoretical understanding of meta-learning, one could consider other algorithms, such as nonparametric regression or neural networks. We believe that our results are a solid starting point for the sound statistical analysis of meta-learning.

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A  Proof of Lemma 4.1  

Proof. According to equation (5.26) in [Vershynin, 2012], if we assume Σ^{1/2}A_i are isotropic sub-Gaussian random vectors, where A_i ∈ R^k, A = [A_1 A_2 \cdots A_n]^T ∈ R^{n×k}, with probability at least 1 − 2\exp(−C_1nt^2), we have

\[ \left\| \frac{1}{n} A^T A - \Sigma \right\|_2 \leq \max(\delta, \delta^2)\|\Sigma\|_2 \leq (\delta + \delta^2)\|\Sigma\|_2, \text{ where } \delta = C_2 \sqrt{\frac{k}{n} + t} \]

where C_1, C_2 only depend on Σ. □

B  Proof of Lemma 4.5

Proof. In Subsection 4.1.6 we have shown that

\[ \mathbb{P}[\|\zeta_S\|_\infty \geq \gamma] \leq c_5\exp(-c_6\log(p)/(kl^3)) \]

Since we have \( F_3 = \hat{\Sigma}_{S,S}^{-1}\lambda\zeta_S \), with probability at least \( 1 - 2\exp(-c_{11}k) - c_5\exp(-c_6\log(p)/(kl^3)) \), we have

\begin{align*}
\| F_3 \|_\infty & \leq \| (\hat{\Sigma}_{S,S}^{-1} - \sigma_x^{-2}I)\lambda\zeta_S \|_\infty + \| \sigma_x^{-2}I\lambda\zeta_S \|_\infty \\
& \leq \| (\hat{\Sigma}_{S,S}^{-1} - \sigma_x^{-2}I)\lambda\zeta_S \|_2 + \sigma_x^{-2}\lambda \gamma \\
& \leq \| (\hat{\Sigma}_{S,S}^{-1} - \sigma_x^{-2}I)\|_2 \| \lambda\zeta_S \|_2 + \sigma_x^{-2}\lambda \gamma \\
& \leq \| (\hat{\Sigma}_{S,S}^{-1} - \sigma_x^{-2}I)\|_2 \| \lambda \sqrt{k}\|\zeta_S \|_\infty + \sigma_x^{-2}\lambda \gamma \\
& \leq c_{13} \frac{\sqrt{k/Tl}}{\sigma_x} \lambda \sqrt{k}\gamma + \sigma_x^{-2}\lambda \gamma \\
& = \gamma \frac{\lambda}{\sigma_x^2} \left( c_{13} \frac{k}{Tl} + 1 \right) \\
\end{align*}

(14)

To have the inequality (14), we use Theorem 5.39 in [Vershynin, 2012]. Here we let \( N = Tl, n = k, t = \Omega(\sqrt{k}) \), \( A = X_{[Tl] \gamma} \). Furthermore, we let \( T = O(k \log(p - k)/l) \) such that \( C\sqrt{k/(Tl)} \in (0, 1/2) \) in order to bound the eigenvalues of \( \hat{\Sigma}_{S,S}^{-1} \): 

With probability \( 1 - 2\exp(-c_{11}k) \), we have

\[ s_{\max}(\hat{\Sigma}_{S,S}^{-1}) = \frac{\sigma_x^{-2}}{Tl} s_{\max}(A)^2 \leq \frac{\sigma_x^{-2}}{(1 - C\sqrt{Tl})} \leq \left( 1 + 6C\sqrt{\frac{k}{Tl}} \right) \sigma_x^{-2} \]

\[ s_{\min}(\hat{\Sigma}_{S,S}^{-1}) = \frac{\sigma_x^{-2}}{Tl} s_{\min}(A)^2 \geq \frac{\sigma_x^{-2}}{(1 + C\sqrt{Tl})} \geq \left( 1 - 2C\sqrt{\frac{k}{Tl}} \right) \sigma_x^{-2} \]

where \( c_{11}, C \) are constants.
Therefore
\[
\|\hat{\Sigma}_{S,S}^{-1} - \sigma_x^2 I\|_2 = \max \left( a_1^T (\hat{\Sigma}_{S,S}^{-1} - \sigma_x^2 I) a_1, a_2^T (\sigma_x^2 I - \hat{\Sigma}_{S,S}^{-1}) a_2 \right)
\leq \max \left( 1 + 6C \sqrt{\frac{k}{Tl}}, 1 - \left( 1 - 2C \sqrt{\frac{k}{Tl}} \right) \right) \sigma_x^{-2}
\leq \max \left( 6C \sqrt{\frac{k}{Tl}}, 2C \sqrt{\frac{k}{Tl}} \right) \sigma_x^{-2} = c_{13} \sqrt{\frac{k}{Tl}} \sigma_x^{-2}
\]

where we set \(a_1 = \arg \max_{a \in \mathbb{S}^{n-1}} a^T (\hat{\Sigma}_{S,S}^{-1} - \sigma_x^2 I) a\), \(a_2 = \arg \max_{a \in \mathbb{S}^{n-1}} a^T (\sigma_x^2 I - \hat{\Sigma}_{S,S}^{-1}) a\).