Local entanglement and quantum phase transition in spin models

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\textit{New Journal of Physics} 8 (2006) 61
Received 21 November 2005
Published 27 April 2006
Online at http://www.njp.org/
doi:10.1088/1367-2630/8/4/061

\textbf{Abstract.} In this paper, we study quantum phase transitions in both the one- and two-dimensional XXZ models with either spin $S = 1/2$ or $S = 1$ by a local entanglement $E_v$. We show that the behaviour of $E_v$ is dictated by the low-lying spin excitation spectra of these systems. Therefore, the anomalies of $E_v$ imply their critical points. This recalls the well-known fact in optics: the three-dimensional image of one subject can be recovered from a small piece of holograph, which records the interference pattern of the reflected light beams from it. Similarly, we find that the local entanglement, which is rooted in the quantum superposition principle, provides us with a deep insight into the long-range spin correlations in these quantum spin systems.

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1. Introduction

In recent years, the phenomenon of quantum entanglement in the strongly correlated electron models has been intensively studied by many physicists [1]–[28]. The main research interest is focused on its strong influence on quantum phase transition in these systems. In particular, as a measure of entanglement between two qubits, concurrence [29] was widely used to investigate several typical localized spin models. At the corresponding quantum critical points [30], this quantity shows either singular or extreme behaviours. For example, Osterloh et al [2] calculated concurrence between two spins located on a pair of nearest-neighbour sites for the transverse-field Ising model [1]. They found that, as a function of the coupling constants, the quantity is singular at the transition point of the model and obeys scaling law in its vicinity. On the other hand, for the antiferromagnetic XXZ model, the same measurement of entanglement is a continuous function of the anisotropic parameter and reaches its maximum at the critical point [4, 5]. However, in either case, concurrence is very short-ranged. It vanishes very quickly as the distance between two spins increases. As an alternative, Verstraete et al [17] proposed the concept of localizable entanglement, which is defined as the maximum entanglement that can be localized for two qubits by doing local measurement on all other qubits. Unlike concurrence, this quantity has been shown to be long-ranged for the above-mentioned quantum spin models [17]–[19].

On the other hand, investigation on the block–block entanglement between two parts of the system near the critical points of these models establishes a close connection between conformal field theory and the quantum phase transition phenomena in condensed matter physics [20]–[22]. Furthermore, some authors generalized this concept to itinerant fermion systems [25]–[27]. For example, for the extended Hubbard model, we showed that its global phase diagram can be identified by the on-site local entanglement [25]. To understand why such a localized measurement can be used to uncover some global properties, such as the quantum phase transition in a many-body system, let us recall a well-known example. In classical optics, by recording interference pattern of two laser beams reflected from one object, the whole image of it can be recovered from a small piece of holograph, although the resolution may be reduced. In other words, due to interference of electromagnetic waves, the main global information of the object can be reserved in a localized spatial area. Similarly, for a quantum many-body system, the reduced density matrix, which is constructed by keeping some local variables intact and integrating out all the rest degrees of freedom, contains information not only on the subset itself, but also on the correlation between this part and the rest of the system. This observation is dictated by the superposition principle of quantum mechanics. Therefore, one is allowed to investigate some global properties of the system by a local measurement, such as the block–block entanglement [20, 21] for the spin models and the on-site local entanglement for the extended Hubbard model [25]. In other words, it is now possible for us to locate the critical points of a quantum many-body system with data obtained by numerical calculations on small size samples. Therefore, these local measurements are very useful in practice.

Conceptually, the block–block, the on-site and the two-site local entanglement, which we shall introduce in the following, are different representations of the same measurement. However, for numerical calculations, the latter are more convenient and economic to use. For the block–block entanglement with each one containing dozens of spins [20, 21], the size of the reduced density matrix, which is required for calculation, grows exponentially as diameter of the block increases [20, 21]. On the other hand, from the physical point of view, it is obviously unnecessary
to consider a block whose size is larger than the characterizing length of the system. In other words, one expects that entanglement between a block, which contains one or two spins, and the rest of the system should be sufficient to reveal the most important information on the system. In the present paper, we shall show that, indeed, a two-site local entanglement can be effectively used for this purpose.

To define the two-site local entanglement, we trace out all the spin degrees of freedom of the system except two, which occupy a pair of nearest-neighbour sites of the lattice. This process yields a reduced two-spin density matrix. Then, we use the von Neumann entropy of its eigenvalues to determine the corresponding entanglement, as shown explicitly in the next section. Obviously, this quantity measures the entanglement between these two spins and the rest of the system. We shall show that, although it is the simplest form of the block–block entanglement with one block having only two sites, the two-site local entanglement can provide us with sufficient information on the quantum phase transitions in both the one- and two-dimensional XXZ models with spin \( S = 1/2 \), which have been previously studied in terms of concurrence \([4, 5]\). Then, we apply it to explore quantum phase transition in a generalized one-dimensional XXZ model with \( S = 1 \), for which concurrence is not well defined. We show that, even in this case, the two-site local entanglement is still applicable. In fact, our results are not only consistent with the previous ones derived by different approaches, but also give us some new information. For instance, based on analysis of the two-site local entanglement behaviour, we propose the possible existence of an additional critical point, which might have been overlooked in the previous investigations.

This paper is organized as follows. In section 2, we introduce first the definition of the two-site local entanglement. Then, for the one-dimensional XXZ model with \( S = 1/2 \), we show that the extreme and singular points of the quantity are the critical ones of the system. Therefore, like concurrence, the local entanglement can be also used to describe properly quantum phase transition in this model; in section 3, we calculate the local entanglement of the same model in two dimensions. We show that existence of the antiferromagnetic long-range order in the system makes its behaviour at the transition point more singular; in section 4, we apply the local entanglement to study quantum phase transition in some spin models with \( S = 1 \). We show that, although concurrence is not well defined in this case, the local entanglement still provides us with useful information on quantum phase transition in the systems; finally, in section 5, we make some remarks and summarize the content of the paper.

2. One-dimensional XXZ model with \( S = 1/2 \)

First, let us consider the one-dimensional XXZ model with \( S = 1/2 \). Its Hamiltonian is of the following form

\[
\hat{H} = \sum_{i=1}^{N} (\hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y + \Delta \hat{S}_i^z \hat{S}_{i+1}^z ),
\]

(1)

where \( \hat{S}_i^x, \hat{S}_i^y \) and \( \hat{S}_i^z \) are \( S = 1/2 \) operators at site \( i \), \( N \) is the total number of sites, and \( \Delta = J_z / J_x \) (\( J_x = J_y \)) is a dimensionless parameter characterizing the anisotropy of the model. The summation is over all the lattice sites. Furthermore, we impose the periodic boundary condition on the system. Consequently, the system is translational invariant. Therefore, one can choose any pair of nearest-neighbouring spins as the two-site block.
It is easy to check that Hamiltonian (1) commutes with $\hat{S}_i^z = \sum_j \hat{S}_j^z$, the z-component of the total spin operator. Therefore, $S_z$ is a conserved quantity. As a result, in terms of the standard basis vectors $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, the reduced two-spin density matrix, which is constructed by tracing out the rest spin degrees of freedom, can be written as

$$
\hat{\rho}_{i,i+1} = \begin{pmatrix}
u^* & 0 & 0 & 0 \\
0 & u_1 & w & 0 \\
0 & z^* & w_2 & 0 \\
0 & 0 & 0 & u^{-1}
\end{pmatrix}.
$$

(2)

In terms of the spin correlation functions, the elements of $\hat{\rho}_{i,i+1}$ in equation (2) are given by [4]

$$
u^\pm = \frac{1}{2} \pm (\hat{S}_i^x \hat{S}_{i+1}^x), \quad z = (\hat{S}_i^x \hat{S}_{i+1}^z) + (\hat{S}_i^y \hat{S}_{i+1}^y), \quad w_1 = w_2 = \frac{1}{2} - (\hat{S}_i^z \hat{S}_{i+1}^z).
$$

(3)

Obviously, by its definition, $\hat{\rho}_{i,i+1}$ is a semi-positive definite matrix. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$ be its eigenvalues. We define the two-site local entanglement $E_v$ of the system to be the von Neumann entropy of $\hat{\rho}_{i,i+1}$, i.e., we have

$$
E_v = -\lambda_1 \ln \lambda_1 - \lambda_2 \ln \lambda_2 - \lambda_3 \ln \lambda_3 - \lambda_4 \ln \lambda_4.
$$

(4)

As is well known, on a $d$-dimensional simple cubic lattice, the phase diagram of the antiferromagnetic XXZ model is divided into three parts by two phase transition points $\Delta_1 = -1$ and $\Delta_2 = 1$. For $\Delta < -1$, the system is ferromagnetic. In other words, its ground state is fully spin polarized and has degeneracy $D = 2S + 1 \approx \frac{1}{4} N_A$, where $N_A$ is the number of lattice sites. In this case, we find that the local entanglement $E_v = 0$. Namely, the two localized spins are not entangled with the rest spins in the system at all. Therefore, the spin correlation in the ferromagnetic phase of this model can be thought of as classical.

On the other hand, for $\Delta > -1$, it can be shown that the global ground state of the XXZ model is non-degenerate and has spin $S = 0$ [31, 32]. Consequently, $\langle \hat{S}_i^z \rangle = 0$ holds true at any lattice site $i$ under the periodic boundary condition. Therefore, the local entanglement $E_v$ defined in equation (4) can be now reduced to

$$
E_v = -2 u \log_2 u - \lambda^+ \log_2 \lambda^+ - \lambda^- \log_2 \lambda^-,
$$

(5)

where $u = \frac{1}{4} + (\hat{S}_i^z \hat{S}_{i+1}^z)$ and $\lambda^\pm = w_1 \pm z$. Obviously, $E_v$ depends on all the three spin correlation functions $\langle \hat{S}_i^x \hat{S}_{i+1}^x \rangle$, $\langle \hat{S}_i^y \hat{S}_{i+1}^y \rangle$ and $\langle \hat{S}_i^z \hat{S}_{i+1}^z \rangle$. Their behaviours are dictated by the symmetries of the system in parameter regions $\Delta < 1$ and $\Delta > 1$. Our task is to find out how the local entanglement is affected by change of these symmetries.

To obtain the spin correlation functions of the one-dimensional XXZ model, we can apply the Feynman–Hellman theorem. In fact, the ground-state energy density of the model has been already calculated with the quantum inverse scattering method [33]–[35]. The corresponding Bethe–ansatz equations read [34]

$$
\left( \frac{\sinh \gamma (\lambda_j + i)}{\sinh \gamma (\lambda_j - i)} \right)^N = \prod_{l \neq j}^{M} \frac{\sinh \gamma (\lambda_j - \lambda_l + 2i)}{\sinh \gamma (\lambda_j - \lambda_l - 2i)}.
$$

(6)

where $\lambda_j$, $j = 1, \ldots, M$ are spin rapidities describing the kinetics of a state with $M$ down spins. The parameter $\gamma$ is related to $\Delta$ by $\Delta = \cos 2\gamma$. For $-1 < \Delta < 1$, it can be chosen real and
Figure 1. The local entanglement (left) of one-dimensional XXZ model and its first derivative (right) as a function of $\Delta$. The results are obtained by solving Bethe–ansatz equations of $N = 1280$ sites system numerical, and the integral equation for infinite length system. (We obtained the same results.)

positive, while, for $1 < \Delta$, $\gamma$ is an imaginary number. Solving equation (6) yields the ground-state energy density $e(\Delta)$ as a function of the anisotropic parameter $\Delta$. Therefore, by applying the Feynman–Hellman theorem, we are able to write the spin correlation functions as

$$\langle \hat{S}_i^z \hat{S}_{i+1}^z \rangle = \frac{\partial e(\Delta)}{\partial \Delta}, \quad \langle \hat{S}_i^x \hat{S}_{i+1}^x \rangle = \langle \hat{S}_i^y \hat{S}_{i+1}^y \rangle = \frac{1}{2}(e(\Delta) - \Delta \langle \hat{S}_i^x \hat{S}_{i+1}^x \rangle). \quad (7)$$

Then we calculate the local entanglement $E_v$ by substituting these correlation functions into equation (5). Our results are shown in figure 1.

First, we notice that, as $\Delta$ tends to infinity, $E_v \to 1$. In this limit, the ground state of the system becomes doubly degenerate and can be approximately expressed as $\frac{1}{\sqrt{2}}(\Psi_1 \pm \Psi_2)$, where $\Psi_1$ and $\Psi_2$ represent the two degenerate Néel states. Consequently, the transverse spin correlation functions $\langle \hat{S}_i^x \hat{S}_{i+1}^x \rangle$ and $\langle \hat{S}_i^y \hat{S}_{i+1}^y \rangle$ vanish but the longitudinal spin correlation function $\langle \hat{S}_i^z \hat{S}_{i+1}^z \rangle$ takes on its minimal value $-\frac{1}{4}$. That makes $E_v = 1$.

On the other hand, as $\Delta$ decreases, spin flipping interactions become more and more important. As a result, the contribution to $E_v$ from the transverse spin correlation functions $\langle \hat{S}_i^x \hat{S}_{i+1}^x \rangle$ and $\langle \hat{S}_i^y \hat{S}_{i+1}^y \rangle$ grows. It leads to enhancement of the quantum correlation between the two localized spins and the rest part of the system. Therefore, although the contribution to $E_v$ from the longitudinal spin correlation function $\langle \hat{S}_i^z \hat{S}_{i+1}^z \rangle$ is now suppressed, we still see an increment in $E_v$, as shown in figure 1. Then, $E_v$ reaches its maximum at $\Delta = 1$, where the ground-state energy is $1/4 - \ln 2$ [34] in the thermodynamic limit, then $\langle \hat{S}_i^x \hat{S}_{i+1}^x \rangle = \langle \hat{S}_i^y \hat{S}_{i+1}^y \rangle = \langle \hat{S}_i^z \hat{S}_{i+1}^z \rangle = (1/4 - \ln 2)/3$ and hence, $E_v = 1.3759$. It is due to the fact that, on the left-hand side of $\Delta = 1$, the contribution by the longitudinal spin correlation function $\langle \hat{S}_i^z \hat{S}_{i+1}^z \rangle$ is reduced very quickly as $\Delta$ decreases further to zero. It causes $E_v$ itself also to decrease. Therefore, one of the critical points of the model, $\Delta = 1$ is recognized as the maximal point of the local entanglement. Moreover, we observe that $E_v(\Delta)$ is continuous at this point.
For a double check, we calculate also these spin correlation functions at $\Delta = 0$. In this case, the XXZ model can be transformed into a free spinless fermion model through the Jordan–Wigner transformation. A little algebra gives us
\[
\langle \hat{S}_i^x \hat{S}_{i+1}^x \rangle = \langle \hat{S}_i^y \hat{S}_{i+1}^y \rangle = -1/2, \quad \text{and} \quad \langle \hat{S}_i^z \hat{S}_{i+1}^z \rangle = -1/\pi^2
\]
for an infinite system. Obviously, the spin-flipping interaction is now dominant. We have $E_v(\Delta = 0) = 1.3675$.

When $\Delta$ becomes negative, the longitudinal antiferromagnetic correlation is further suppressed. However, interestingly, it results also in a larger value of $E_v$. In particular, as $\Delta \to -1^+$, $E_v$ becomes singular, as shown in figure 1. We interpret it due to the infinite ferromagnetic degeneracy of the ground state at $\Delta = -1$. Then if the symmetry of the ground state is broken, we can have $\langle \hat{S}_i^z \hat{S}_{i+1}^z \rangle = 1/4$ and $E_v = 0$. In other words, unlike the point $\Delta = 1$, where the ground state of the system is non-degenerate, the transition point between the ferromagnetic and antiferromagnetic phase of the model is characterized by singularity of the local entanglement.

Here, we would like to emphasize that the ground-state energy, which is usually used in investigating quantum phase transition, is not singular at all at the transition point $\Delta = 1$ for the one-dimensional XXZ model. In fact, it is continuous at the critical point, as shown in [36]. Therefore, in this case, the ground-state energy is not an effective means to locate the critical point of the system.

For comparison, we compute also concurrence $C$ and entanglement of formation $E_f$ of the XXZ chain as a function of parameter $\Delta$. According to [29], the concurrence of two spins is defined by
\[
C = \max(0, \sqrt{\mu_1} - \sqrt{\mu_2} - \sqrt{\mu_3} - \sqrt{\mu_4}),
\]
where $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 \geq 0$ are the eigenvalues of the semi-positive definite matrix
\[
\hat{\rho}_{i,i+1} \hat{\rho}_{i,i+1}^\dagger \equiv \hat{\rho}_{i,i+1}(\sigma_i^y \otimes \sigma_{i+1}^y \hat{\rho}_{i,i+1}^\dagger \sigma_i^y \otimes \sigma_{i+1}^y).
\]
There exists a monotonous relation between the concurrence and the entanglement of formation $E_f = -x \log_2 x - (1 - x) \log_2 (1 - x)$ with $x = 1/2 + \sqrt{1 - C^2}/2$, and the entanglement of formation share same unit with the local entanglement. The results are shown in figure 2. We find that its behaviour is quite different from $E_v$ as $\Delta \to -1^+$. In fact, while the local entanglement increases dramatically, concurrence drops quickly to zero in this limit. This difference can be explained on the basis of the monogamy property of the quantum entanglement [37]. It tells us that the maximal entanglement between two parties will restrict their entanglement with a third party, and vice versa. On the other hand, both $E_v$ and $E_f$ have their maximum at $\Delta = 1$. As explained above, it is a result of competition between the transverse and longitudinal magnetic correlations in the parameter region around the point.

In summary, our results show clearly that, like concurrence, the two-site local entanglement is a suitable measurement to determine the quantum phase points of the one-dimensional XXZ model with $S = 1/2$.

3. Two-dimensional XXZ model with $S = 1/2$

Next, we turn to the two-dimensional antiferromagnetic XXZ model with $S = 1/2$. Unlike its one-dimensional counterpart, this model cannot be exactly solved. Therefore, one has to use some approximate techniques, such as the spin-wave theory [38, 39] or numerical calculations on a
The concurrence (—) and the entanglement of formation (–––) between two nearest-neighbouring spins in one-dimensional XXZ model as a function of $\Delta$ in thermodynamic limit.

The local entanglement of two-dimensional XXZ model as a function of $\Delta$ on $4 \times 4$ (circle line) and $6 \times 6$ (square line) square lattices. Two insets are for a simple comparison between two systems with different size.

finite lattice [40]. In general, for the localized spin models, these methods provide very reliable information on their properties. For instance, with proper scaling analysis, the results derived by exact diagonalization, the quantum Monte Carlo method [41] and other analytical techniques [38, 39] are fully consistent with each other. In the following, we apply exact diagonalization technique to calculate the local entanglement of two-dimensional XXZ model on both $4 \times 4$ and $6 \times 6$ square lattices under the periodic boundary condition. Our results are shown in figures 3.
Quantitatively, the behaviour of $E_\tau$ for both one- and two-dimensional $XXZ$ models is quite similar. For example, in the region of $\Delta > 1$, they are decreasing functions of $\Delta$. As explained above, this is caused by suppression of the transverse spin correlation as $\Delta$ increases. Therefore, the entanglement between the localized spin pair and the rest part of the system is weakened. On the other hand, for $0 < \Delta < 1$, $E_\tau$ is increasing as $\Delta$ grows. Consequently, the local entanglement has a maximum at the critical point $\Delta = 1$.

However, we would like to emphasize that, while the local entanglement is continuous at the critical point in one dimension, it has a cusp-like behaviour around the point in the two-dimensional case. The same phenomenon was also observed in our previous work [5], in which we used concurrence as measurement of entanglement between two spins. Considering the fact that the antiferromagnetic order exists in the two-dimensional Heisenberg model [42], but is absent in one dimension, we attribute this difference to the long-range correlation effect in the system, as explained in the following.

It is well known that, in one dimension, the spin correlation functions of the $XXZ$ chain decay by power-law when $\Delta > -1$. Therefore, influence of a remote spin on the local spin pair decreases rather rapidly as the distance between them tends to infinity. Consequently, all the local quantities, such as the energy density and the nearest-neighbouring spin correlation functions are not affected very much by growth of the system size. More precisely, even in the thermodynamic limit, these quantities will change smoothly as parameter $\Delta$ varies, as long as the ground state is non-degenerate. On the other hand, the existence of the antiferromagnetic long-range correlation in the two-dimensional model leads to a strong dependence of these local quantities on the system size. In particular, it causes their non-analytical behaviours at the quantum phase transition points in the thermodynamic limit. In other words, it is the strong dependence of the local entanglement on infinitely large degrees of spin freedom makes it singular at $\Delta = 1$ of two-dimensional $XXZ$ model, as mentioned above. In fact, the similar critical behaviours were also found in concurrence of the two-dimensional $XXZ$ model [5, 38, 41].

4. One-dimensional spin models with $S = 1$

In the above sections, we have shown that the two-site local entanglement can be successfully used to characterize the quantum critical points for both the one- and two-dimensional $XXZ$ models with $S = 1/2$. In this section, we shall go further and study spin models with $S = 1$ on a chain. For these models, concurrence is not well defined [43, 44]. However, as we show in the following, the two-site local entanglement can be still applied and produces meaningful results.

Let us consider the following Hamiltonian

$$\hat{H} = \sum_i [\hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y + \Delta \hat{S}_i^z \hat{S}_{i+1}^z - \beta (\hat{S}_i \cdot \hat{S}_{i+1})^2],$$

(10)

where $\beta$ is a real parameter and $\hat{S}_i^x$, $\hat{S}_i^y$, and $\hat{S}_i^z$ are the spin operators of $S = 1$. Unlike the $XXZ$ model with $S = 1/2$, the spin excitation spectrum of this system is gapped at the isotropic point. This fact was originally realized by Haldane [45] for the isotropic antiferromagnetic Heisenberg chain with integer spin. Then, it was confirmed by quantum Monte Carlo simulation [46] and exact diagonalization numerical calculation [47]. Moreover, when $\beta = 0$, this gap vanishes around $\Delta \simeq 1.18$ as $\Delta$ increases.
By exact diagonalization calculation, we compute the two-site entanglement in the ground state of Hamiltonian (10). First, we set $\beta = 0$, and show the results in figure 4. We find that $E_v$ behaves in a quite similar way to that of the one-dimensional XXZ model of $S = 1/2$. For instance, it takes on its maximum at the isotropic point. But, we would like to emphasize that, in the current case, $\Delta = 1$ is not a quantum phase transition point of the system at all. It is due to the fact that the spin excitation spectrum of the model is now gapped on both sides of the point. A more careful analysis reveals that the extreme behaviour of local entanglement is actually caused by the first-excited state level-crossing, which happens also at $\Delta = 1$ for both the one- and two-dimensional XXZ models with $S = 1/2$.

Instead, for the one-dimensional XXZ model with $S = 1$, a Kosterlitz–Thouless type of phase transition occurs at $\Delta_0 \approx 1.18$, where the Haldane gap is closed. More precisely, on the left-hand side of $\Delta_0$, the spin excitation spectrum is gapped. However, on the right-hand side of $\Delta_0$, it is gapless. A similar spectrum structure is also observed around the critical point of the transverse field Ising model with $S = 1/2$ [30]. Since the first derivative of the concurrence of this model is singular at the transition point and obeys a scaling law [2], we speculate that the same behaviour should be also seen in the local entanglement around the transition point of the XXZ model with $S = 1$. Indeed, by taking the first derivative of $E_v$ with respect to $\Delta$, we find a minimal point at $\Delta \approx 1.3$ in figure 5. Moreover, the position of this extreme point varies as the size of the system increases. Obviously, it is caused by the finite-size effect. By fitting data with respect to the sample size, we find that it tends to $\Delta_0 \approx 1.18$ in the thermodynamic limit.

Finally, we explore the effect of parameter $\beta$ on quantum phase transition in the one-dimensional model. For that purpose, we set $\Delta = 1$ and rewrite equation (10) as

$$\hat{H} = \sum_i \left[ \cos \theta (\hat{S}_i \cdot \hat{S}_{i+1}) + \sin \theta (\hat{S}_i \cdot \hat{S}_{i+1})^2 \right].$$

(11)

Here, by introducing trigonometric functions, we can also take the effect of sign of the coupling constants into our consideration. The phase diagram of Hamiltonian (11) consists of the Haldane phase trimerized phase, and dimerized phase at zero temperature [48]. The two-site local entanglement as a function of $\theta$ is shown in figure 6.
First, we observe that $E_v$ has a local minimum at $\theta = \pi/4$ though it is not obvious from the figure. This minimum point separates the Haldane phase and the trimerized phase. On the other hand, for $\pi/2 < \theta < 5\pi/4$, the ground state is ferromagnetic and degenerate, then the local entanglement cannot be well defined from the entropy of a thermal ground state which comprises all states of lowest energy with equal weight. Indeed, a fully polarized state is separable, then if the ground state is symmetry broken, $E_v = 0$. Therefore, as we explained in section 2, the spin correlation in the ground state of the system is now purely classical. As a result, the local entanglement is singular at both $\theta = \pi/2$ and $\theta = 5\pi/4$. It is caused by the ground-state level-crossing. However, we do not find any obvious anomalous structure of $E_v$ around $\theta = 7\pi/4$. 

\begin{figure}[ht]
\centering
\includegraphics[width=0.8\textwidth]{figure5.png}
\caption{Left: the first derivative of the local entanglement of one-dimensional XXZ model as a function of $\Delta$ for various system size $L = 8, 10, 12, 14$ and 16. Right: scaling analysis of the minimal point of $E'_v(\Delta)$.}
\end{figure}

\begin{figure}[ht]
\centering
\includegraphics[width=0.8\textwidth]{figure6.png}
\caption{The local entanglement of one-dimensional bilinear–biquadratic model as a function of $\theta$ with $L = 6$.}
\end{figure}
which is known to be a critical point, separating the dimerized phase and the Haldane phase of the system. This problem may be caused by our calculation scheme. In our calculation, we only considered the samples with $L = 6n$ ($n = 1, 2, 3, \ldots$) sites such that, the ground state of the system has both the trimerized and dimerized orders. Therefore, the rapid increment of sample size makes scaling analysis more difficult. But we are sure that the derivatives of the local entanglement will show extremum around $\theta = 7\pi/4$ when the system size becomes larger and larger. At another controversial critical point related to nematic phase [49]–[51], the properties of the local entanglement still needs further investigation.

Interestingly, we also find a local minimum in the figure of $E_v$ at $\theta = 3\pi/2$, where $\cos \theta = 0$ and the Hamiltonian is reduced to $\hat{H} = -\sum_i (\hat{S}_i \cdot \hat{S}_{i+1})^2$. It can be shown that the first excited state of Hamiltonian (10) is highly degenerate around $\theta = 3\pi/2$. For instance, the degeneracy is three-fold on one side of the point and is five-fold on the other side. At $\theta = 5\pi/2$, the first excited state is exactly eight-fold degenerate. Moreover, around $3\pi/2$, the system is dimerized [48]. Consequently, a level-crossing between the lowest excited states must occur. Moreover, unlike the case of $\Delta = 1$ in $XXZ$ model with $S = 1$ where the system is gapped from a non-degenerate ground state, the ground state of pure biquadratic model is degenerate [52]. Therefore, we propose that $\theta = 3\pi/2$ is also a quantum phase transition point, which separates two ordered phases of the system.

5. Summary

In the present paper, we study the global phase diagram of quantum spin models with either spin $S = 1/2$ or $S = 1$ by the two-site local entanglement $E_v$. We show that, indeed, important information on the quantum phase transition in these systems can be uncovered by such a local measurement. For instance, we find that the local entanglement is singular at one critical point $\Delta = -1$ and takes on its maximum at another critical point $\Delta = 1$ of the one-dimensional $XXZ$ model with $S = 1/2$. The same behaviour is also observed for the two-dimensional $XXZ$ model with $S = 1/2$, although the slope of $E_v$ becomes sharper at $\Delta = 1$ due to existence of the antiferromagnetic long-range order. These results are consistent with the conclusions derived previously by using concurrence as a measurement of entanglement [4, 5].

Moreover, the local entanglement can be also applied to study quantum phase transition in spin models with integer spin. For a concrete example, we consider a generalized $XXZ$ model with spin $S = 1$. To these models, the quantity of concurrence is not well defined. However, the local entanglement is still applicable. We find that, around the Kosterlitz–Thouless type of phase transition point $\Delta_0 \approx 1.18$ of this model, the first derivative of $E_v$ obeys a scaling law. Furthermore, it reaches also its minimum at the critical point. It characterizes a transition from a gapped phase to a gapless one. We further explore the rich phase diagram of the same model with an extra bilinear–biquadratic interaction, whose Hamiltonian is given by equation (11). We observe that the local entanglement is a minimum at $\theta = 3\pi/2$. Therefore, we propose that this point is also a transition point, which might have been overlooked in the previous studies. Definitely, this issue deserves further investigation.

At the same time, we also find some other extremum points in the figure, such as a minimum between 0 and $\pi/4$. However, no obvious level reconstructions are found at these points, then they are not considered as critical points.

In summary, by studying these examples, we conclude that the phenomena of quantum phase transitions, which was previously described by the long-range spin correlations of these systems
in the thermodynamic limit, can be also successfully studied by using a local entanglement measurement. Consequently, one can study the transition points of the systems from data obtained by numerical calculations on small size samples. This recalls the well-known fact in optics: the whole three-dimensional image of one subject can be recovered from a small piece of holograph, which records interference pattern of the reflected light beams from it, although the image resolution may be reduced. Similarly, we see that the local entanglement, which is rooted in the quantum superposition principle, can provide us with information on long-range correlations in a quantum many-body system.

However, we also acknowledge some limitations of the local entanglement. For example, some extremum points are not related to the known critical behaviour, such as $\Delta = 1$ in the XXZ model with $S = 1$ and a minimum point between $0$ and $\pi/4$ for the bilinear–biquadratic model. Therefore, it is still an interesting question whether we can witness new critical points simply from the behaviour of the entanglement, including another type of entanglement measure, such as concurrence, or how the entanglement can help us to explore new critical behaviour. Obviously, the answer needs further investigation, and the future is exciting and worthy of expectation.

**Acknowledgments**

This work was supported by a grant from the Research Grants Council of the HKSAR, China (project no 401703) and the Chinese National Science Foundation under grant nos 10329403 and 90403003.

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