Semi-classical scattering in two dimensions

Sadhan K. Adhikari\textsuperscript{1,\dagger} and Mahir S. Hussein\textsuperscript{2,1} \footnote{Martin Gutzwiller Fellow, 2007/2008}
\textsuperscript{1}Instituto de Física Teórica, UNESP – São Paulo State University, 01.405-900, SP, Brazil
\textsuperscript{2}Max-Planck-Institut für Physik komplexer Systeme, D-01187 Dresden, Germany
\textsuperscript{3}Instituto de Física, Universidade de São Paulo, C.P. 66318, 05315-970 São Paulo, SP, Brazil

The semi-classical limit of quantum-mechanical scattering in two dimensions (2D) is developed. We derive the Wentzel-Kramers-Brillouin and Eikonal results for 2D scattering. No backward or forward glory scattering is present in 2D. Other phenomena, such as rainbow or orbiting do show up.

I. INTRODUCTION

The quantum mechanical scattering problem in two space dimensions (2D) has been addressed Refs. \textsuperscript{1,2,3}. The inherent circular symmetry in 2D renders the free partial-wave radial function to be the ordinary Bessel (regular at the origin) and Neumann (irregular) functions, in contradistinction to the three-dimensional (3D) case where these functions become the spherical counterpart (spherical Bessel and Neumann functions). 2D scattering has application in surface physics, besides being an interesting pedagogical topic at the graduate level.

The semi-classical (SC) limit of the 3D scattering addressed in Refs. \textsuperscript{1,2} is an important subject as it shows the connection between quantum and classical mechanics in a particularly transparent form. It has also important application in atomic, molecular and nuclear scattering where one deals with the optics of matter waves. Such sums are converted into integrals and efficiently evaluated in SC scattering. Further, the use of semi-classical ideas is a common practice in the description of wave optics phenomena such as the meteorological rainbow and glory \textsuperscript{4}. The concepts employed in the wave optics field found a very natural adaptation in SC 3D scattering, where one deals with the optics of matter waves.

It would be only natural to extend the SC limit to 2D scattering, which should find interesting application in surface physics where the results of 2D quantum scattering are often applied. Here we develop the SC version of 2D scattering using the deflection function \textsuperscript{5,7} approach. The SC limit covers a situation where the potential varies very little over a distance of the order of de Broglie wave length. The deflection function \textsuperscript{1,2} is a central potential (depending only on the magnitude of \(r\)). Here we develop the SC version of 2D scattering using the deflection function \textsuperscript{5,7} approach. We present results for 2D scattering in the WKB and Eikonal approximations \textsuperscript{7} as well as a comprehensive discussion of rainbow and glory scattering and orbiting. From the asymptotic form of 2D SC wave function we extract the phase shift and construct the Eikonal result for scattering (cross section) using this phase shift. The same Eikonal result can also be obtained in 2D from an evaluation of the quantum Born approximation to scattering amplitude using the SC WKB wave function, as has been done in 3D \textsuperscript{7}. We leave this as an exercise for the reader, which can be performed in a straightforward fashion following the 3D analysis of Ref. \textsuperscript{2}.

In Sec. \textsuperscript{1} we give a brief account of 2D quantum scattering following Ref. \textsuperscript{2}. In Sec. \textsuperscript{3} a self-contained discussion of SC 2D scattering is presented. Specifically, we present results for WKB wave function and phase shift, Eikonal phase shift and scattering amplitude, SC cross section, and rainbow cross section. In Sec. \textsuperscript{4} a summary of our findings is given. In Appendix A we present a study of Rainbow scattering cross section at different angles where we also present a project for the reader. In Appendix B we have left an exercise.

II. QUANTUM SCATTERING IN 2D

The time-independent Schrödinger equation for reduced mass \(\mu\), potential \(V(r)\), energy \(E\), describing scattering is

\[ -\frac{\hbar^2}{2\mu}\nabla^2\psi(r) + V(r) = E\psi(r) \quad \text{(1)} \]

In polar coordinates [\(r \equiv (r, \theta)\)], Eq. (1) becomes \textsuperscript{1,2}

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \left[ k^2 - U(r) \right] \psi = 0, \quad \text{(2)} \]

where \(k^2 = 2\mu E/\hbar^2\) and \(U(r) = 2\mu V(r)/\hbar^2\). Here \(U(r)\) is a central potential (depending only on the magnitude of \(r\)). Equation (2) can be solved using the separation of variables method \textsuperscript{1,2}, namely, one writes \(\psi(r) = \sum_m R_m(r)T_m(\theta)\) with \(d^2T_m(\theta)/d\theta^2 = -m^2T_m(\theta)\), so that \(T_m(\theta) = \pi^{\frac{1}{2}} \cos(m\theta)\), thus yielding

\[ \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial R_m(k, r)}{\partial r} + \left[ k^2 - U(r) - \frac{m^2}{r^2} \right] R_m(k, r) = 0. \quad \text{(3)} \]
Introducing \( \hat{R}_m \equiv R_m e^{i\frac{\pi}{4}} \), Eq. (3) can be rewritten as
\[
\frac{d^2 \hat{R}_m(k, r)}{dr^2} + F(r)\hat{R}_m(k, r) = 0, \quad (4)
\]
\[
F(r) \equiv \left[ k^2 - U(r) - \left( \frac{m^2 - \frac{1}{4}}{r^2} \right) \right]. \quad (5)
\]

The function \( \hat{R}_m(t, r) \) should be a regular at \( r = 0 \) with asymptotic form similar to that of the free solution (no potential). Setting \( U(r) \) equal to zero in Eq. (3), one gets the ordinary Bessel equation, whose solutions are the regular Bessel \( [J_m(kr)] \) and irregular Neumann \( [N_m(kr)] \) functions, with asymptotic forms for \( kr \gg 1 \) [2]:
\[
J_m(kr) \to \left( \frac{2}{\pi kr} \right)^{\frac{1}{4}} \cos \left[ kr - \frac{m\pi}{2} - \frac{\pi}{4} \right], \quad (6)
\]
\[
N_m(kr) \to \left( \frac{2}{\pi kr} \right)^{\frac{1}{4}} \sin \left[ kr - \frac{m\pi}{2} - \frac{\pi}{4} \right]. \quad (7)
\]

The asymptotic form of radial function \( \hat{R}_m(kr) \), being a linear combination of \( J_m(kr) \) and \( N_m(kr) \), is taken as [2]
\[
\hat{R}_m(k, r) \to A_m(k) e^{i\delta_m(k)} e^{-bk \cos \left( \frac{\pi kr}{2} - \frac{\pi}{4} + \delta_m(k) \right)}, \quad (8)
\]
where \( \delta_m(k) \) is the scattering phase shift and the constant \( A_m \) is given by \( A_m = 2i^m \epsilon_m (2\pi)^{-\frac{1}{4}} e^{i\delta_m(k)} \), with \( \epsilon_m = 2 \) for \( m \neq 0 \) and \( \epsilon_0 = 1 \).

The scattering amplitude is written in terms of \( \delta_m(k) \) as [2]
\[
f(k, \theta) = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \cos(m\theta) \epsilon_m e^{i\delta_m(k)} \sin \delta_m(k). \quad (9)
\]

The differential scattering cross section is \( d\sigma_{2D}/d\theta = \vert f(k, \theta) \vert^2/k \) and the total cross section is \( \sigma_{2D}(k^2) = \int_0^{2\pi} (d\sigma_{2D}/d\theta) d\theta = 4 \sum_{m=0}^{\infty} \epsilon_m^2 \sin^2 \delta_m(k) \). In 2D the scattering amplitude has the dimension of \( L^{1/2} \) (L in 3D) and cross section has the dimension of \( L \) (\( L^2 \) in 3D).

**III. SEMI-CLASSICAL SCATTERING IN 2D**

When the local de Broglie wave length is short compared to the distance over which the potential changes appreciably, concepts of classical scattering can be used to describe quantum scattering. The problem is formulated in terms of classical quantities, such as the impact parameter, \( b \), and the deflection function, \( \Theta(k^2, b) \) [3, 8]: the validity criterion is \( \delta_{\text{opt}}/\Theta(k^2, b) \ll 1 \) for all \( b \) and \( E \). Here \( \delta_{\text{opt}} \) is the optimal angular dispersion of the scattering particle around a classical path. This corresponds to the high-energy region where many angular momenta contribute and angular momentum is often treated as a continuous variable and the quantum-mechanical partial-wave sum is replaced by an integral.

The useful functional relation of the deflection function \( \Theta(k^2, b) \), obtained from the Hamilton-Jacobi equation of motion, in both 2D and 3D is given by [6]:
\[
\Theta(k^2, b) = \pi - 2bk \int_{r_0}^{\infty} dr r^{-2} F^{-\frac{1}{4}}(r), \quad (10)
\]
where \( r_0 \) is the classical turning point: \( F(r_0) = 0 \).

At this stage we recall that Eqs. (4) and (5) can also be solved by the SC WKB approximation [2] with the ansatz \( \hat{R} = (-F)^{-1/4} \exp \left[ \int_{r_0}^r dr' (-F)^{-1/2}(r') \right] \) near \( r = 0 \). As we cross the turning point and reach the asymptotic region we take \( (-F)^{-1/2} \to i F^{1/2} \), \(( -F)^{-1/4} \to F^{-1/4} e^{-i\pi/4} \).

In this fashion the wave-function is real in both regions. Then for \( r > r_0 \) [2]
\[
\hat{R} = F^{-1/4} \cos \left( \int_{r_0}^r dr' F^{1/2} - \frac{\pi}{4} \right), \quad (11)
\]
\[
\approx \int_{r_0}^{\infty} dr (F^{1/2} - k) - kr_0 - \frac{\pi}{4} \quad (12)
\]

From Eqs. (5) and (12), the WKB phase shift becomes
\[
\delta_{2D}^{\text{WKB}}(k, m) = m\pi/2 + \int_{r_0}^{\infty} dr (F^{1/2} - k) - kr_0 \quad (13)
\]

Differentiating Eq. (10) with respect to \( m \equiv kb \) we get
\[
\frac{d\delta_{2D}^{\text{WKB}}(k, m)}{dm} = \frac{\pi}{2} - m \int_{r_0}^{\infty} dr r^{-2} F^{-1/2}. \quad (14)
\]

Comparing with Eq. (11) we establish
\[
\Theta = 2 \frac{d\delta_{2D}^{\text{WKB}}(k, m)}{dm}. \quad (15)
\]

At very high energies many angular momentum contribute and the so-called Eikonal approximation can be derived by replacing the \( m \)-sum in Eq. (13) by an integration with the identification \( m = kb \) [3]:
\[
f_{\text{Eik}}(k, b) = \frac{-ik}{(2\pi)^{1/2}} \int_0^{\infty} db \cos(kb) \left[ e^{2i\delta_{2D}^{\text{Eik}}} - 1 \right]. \quad (16)
\]

The Eikonal phase shift \( \delta_{2D}^{\text{Eik}} \) is now obtained from Eq. (13) by expanding \( F \) in a power series in \( U(r)/k^2 \), setting \( r_0 = b \), \( m = kb \) and \( m^2 - 1/4 \approx k^2 b^2 \), and keeping the lowest order term: [3]
\[
\delta_{2D}^{\text{Eik}} = -\frac{1}{2k} \int_b^{\infty} \frac{rdr U(r)}{(r^2 - b^2)^{1/2}}. \quad (17)
\]

Incidentally, in 3D the Eikonal approximation is similar to Eqs. (16) and (17) except for the replacement \( db \to (2\pi)^{1/2} b db \), \( \cos(kb) \to J_0(kb) \) [3].

We now recast the scattering amplitude into a sum of integrals [3] using the Poisson sum formula [4]:
\[
\sum_{\nu=-\infty}^{\infty} F_{\nu} = \sum_{\kappa=-\infty}^{\infty} e^{-i\pi \kappa} \int_{-\infty}^{\infty} dy F(y) e^{2i\pi \kappa y}. \quad (18)
\]
Formula (18) can be applied to scattering amplitude (9) once \( \cos(m\theta) \) is expressed in terms of exponentials:

\[
f(k, \theta) = f^{(+)}(k, \theta) + f^{(-)}(k, \theta) \tag{19}
\]

where the amplitudes \( f^{(+)}(k, \theta) \) and \( f^{(-)}(k, \theta) \) correspond to the \( \exp(im\theta) \) and \( \exp(-im\theta) \) branches of the cosine function, respectively. Then using Eq. (18)

\[
f^{(\pm)}(k, \theta) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\kappa} \times \int_{0}^{\infty} \exp \left[i\left(2\delta(k, m) + 2\pi\kappa m \pm m\theta\right)\right] dm, \tag{20}
\]

where now the discrete sum over angular momentum is replaced by an integration and the SC phase-shift \( \delta(k, m) \) is essentially the WKB phase shift. (We drop the label WKB in the following.)

The integrals in Eq. (20) can be evaluated in the SC regime of stationary phase approximation (SPA) when the phase shift, being an action divided over \( \hbar \), is very large. The formula of the SPA is based on evaluating integrals of the type \( \int dm \exp[i\zeta(m)] \), where the phase \( \zeta(m) \equiv [2\delta(k, m) + 2\pi\kappa m \pm m\theta] \) is real on the line of integration and analytic in some region surrounding it. If \( \zeta(m) \) varies rapidly with \( m \), the integral will be vanishingly small, except in cases where \( \zeta \) has minima on the line of integration. One thus expands \( \zeta(m) \) around the extremum and keeps terms up to second order in \( m \). For one extremum point, \( m_s \)

\[
\zeta(m) = \zeta(m_s) + \zeta''(m_s)(m - m_s)^2/2, \tag{21}
\]

where prime denotes derivative. The integral can now be performed to yield

\[
I = e^{i\zeta(m_s)} \int_{-\infty}^{\infty} \exp \left[i\zeta''(m_s)(m - m_s)^2/2\right] dm = [2\pi i/\zeta''(m_s)]^{1/2} e^{i\zeta(m_s)}, \tag{22}
\]

where the limits of integration have been extended to \( \pm \infty \) since the contribution of the far away points are strongly suppressed by oscillation. The above result can be generalized for several extrema, \( m_s^{(j)} \):

\[
I = \sum_{j} \left[2\pi i/\zeta''(m_s^{(j)})\right]^{1/2} e^{i\zeta(m_s^{(j)})} \tag{23}
\]

The power of the SPA is that it allows replacing an infinite sum, such as the \( m \)-sum in \( f \) by a sum of only few contributions arising from the stationary points. The Poisson sum can now be performed, by seeking the stationary points of the total phase \( \zeta(m) \) via \( d\zeta(m)/dm = 0 \), viz.

\[
\pm \theta = -2 \frac{\delta(k, m)}{dm} - 2\kappa \pi. \tag{24}
\]

In Eq. (24) \( \kappa \) represents the number of times the particle circles around the scatter for an attractive interaction.

This phenomenon is known as orbiting. Taking \( \kappa = 0 \) in Eq. (24) and comparing with Eq. (15) we find that the scattering angle \( \theta \) is related to the deflection function \( \Theta : \Theta = \mp \theta \) [5], where the minus sign refer to \( f^{(+)} \) and the plus sign to \( f^{(-)} \). If \( \Theta \) is positive \( f^{(-)} \) dominates and \( \text{vice versa} \). The case \( \Theta = 0 \) corresponds to a situation where \( f^{(+)} \) and \( f^{(-)} \) are comparable in size and the cross section should show a Fraunhofer-type interference. From Eqs. (20), (22), and (24) we get

\[
f^{(\pm)}(k, \theta) = \sum_{m_s^{(\pm)}} e^{i(m_s^{(\pm)} \pm 2\delta/\pi + \pi/4)} \left[ \frac{d\Theta}{dm} \right]^{-1/2}, \tag{25}
\]

where the \( m \)-derivative is evaluated at the stationary points \( m_s^{(\pm)} \). The differential cross section now is

\[
\frac{d\sigma_{2D}}{d\theta} = \frac{1}{k} \left| f^{(+)} + f^{(-)} \right|^2 \tag{26}
\]

\[
= \frac{1}{k} \sum_{m_s^{(\pm)}} \left[ \frac{d\sigma_{2D}^{\text{Classical}}}{d\theta} \right] m_s^{(\pm)} + I^{(\pm)}(k^2, \theta), \tag{27}
\]

where \( (d\sigma_{2D}^{\text{Classical}}/d\theta)_{m_s^{(\pm)}} \) is the classical differential cross section arising from stationary point \( m_s^{(\pm)} \)

\[
\left[ \frac{d\sigma_{2D}^{\text{Classical}}}{d\theta} \right]_{m_s^{(\pm)}} = \frac{1}{k} \left| \frac{d\Theta}{dm} \right|^{-1} \tag{28}
\]

and the interference terms \( I^{(\pm)}(k^2, \theta) \) are associated with the two branches of the amplitudes.

If only one stationary point contributes to, say, \( f^{(-)} \), then the SC differential scattering cross section is just the classical one (no interference term). This is similar to the 3D Coulomb scattering [5] where the exact quantum mechanical cross section is identical to the classical one, as the deflection function \( \Theta(m_s) \) is a monotonic function of the classical impact parameter, identified as \( b = km_s \). In the 2D case, this identification is just the same and one finally obtains for the 2D classical differential cross section

\[
\frac{d\sigma_{2D}^{\text{Classical}}}{d\theta} = \frac{d\Theta}{db} \bigg|_{\Theta = 0} \tag{29}
\]

\[
\frac{d\sigma_{3D}^{\text{Classical}}}{d\Omega} = \frac{b}{\sin\theta} \frac{d\Theta}{db} \bigg|_{\Theta = 0} \tag{30}
\]

If the differential cross sections are written as differentials with respect to \( \theta \), \( (d\sigma_{2D}/d\theta, d\sigma_{3D}/d\theta) \) we have from Eq. (24).

\[
\frac{d\sigma_{3D}^{\text{Classical}}}{d\theta} = 2\pi b \left| \frac{db}{d\theta} \right| = \frac{d(\pi b^2)}{d\theta} \tag{31}
\]

to be compared to the 2D cross section (29), namely

\[
\frac{d\sigma_{2D}^{\text{Classical}}}{d\theta} = \left| \frac{db}{d\theta} \right| \tag{32}
\]
where in Eqs. (31) and (32) we have set $|\Theta| = |\theta|$ as the stationary point condition requires. Correspondingly, the total cross sections integrated over all angles are $\sigma_{D}^{3D} = \pi b_{m}^{2}$ and $\sigma_{D}^{2D} = 2b_{m}^{2}$, where the integration is limited to a maximum $|b| = b_{m}$.

From Eq. (29) we see that when the deflection function is stationary with respect to variations of impact parameter $b$, e.g. $d\Theta/db = 0$, the differential cross section is infinite. Moreover, Eq. (18) shows that $\Theta(b)$ will not be a monotonic function of $b$ and the differential cross section may exhibit interference like in a rainbow [3]. Here angular deflection, as measured by the deflection function, can be positive or negative. The experimental scattering angle $\theta$ is limited by $0 < \theta < \pi$. Hence the values of impact parameter $b$, for which $\Theta > 0$, contributes more to $f^{(-)}$ and negligibly to $f^{(+)}$ and vice versa. If $d\Theta/db = 0$ for positive $\Theta$, $f^{(-)}$ will show rainbow characteristics and vice versa. If deflection function $\Theta(m)$, with $m = bk$, is expanded around the rainbow value $m_r$ of angular momentum $m_r \equiv kb_r$ at which $\Theta'(m_r) = 0$,

$$\Theta(m) = \Theta(m_r) + \Theta''(m_r)(m - m_r)^2/2 + \ldots,$$

then the phase $\zeta$ of Eq. (20) becomes $\zeta_{\pm} = 2b(m_r) + \Theta(m_r)(m - m_r) + \Theta''(m_r)(m - m_r)^2/6 + 2\pi km \pm \pi \delta$, [compare with Eq. (21)]. (For repulsive potential there is no orbiting and $\kappa$ should be set equal to zero. However, the results for amplitude and cross section below do not depend on $\kappa$.) Then the resulting integral in Eq. (20) has the same form as in Eq. (22) and can be expressed in terms of the Airy function $\text{Ai}(x) = (1/2\pi) \int_{-\infty}^{\infty} \exp[i(tx + t^3/3)]dt$, leading to

$$|f(k, \theta)| = \frac{\sqrt{2\pi}}{|\Theta''(m_r)/2|^{1/3}} \text{Ai} \left[ \frac{\theta - \theta_r}{|\Theta''(m_r)/2|^{1/3}} \right]$$

and the cross section is

$$\frac{d\sigma_{D}^{2D}}{d\theta} = \frac{2^{5/3}k^{1/3}}{(|d^2\Theta/db^2|_{b_r})^{1/3}} \text{Ai}^2 \left[ \frac{2^{1/3}k^{2/3}(|\theta - \theta_r|)}{|d^2\Theta/db^2|_{b_r}} \right]$$

where $\theta_r \equiv \Theta(m_r)$ is the rainbow angle. The Airy function being an oscillating function the rainbow-like interference pattern in the differential cross section is confirmed. For $\theta < \theta_r$, the expression (33) oscillates and the particle is scattered into the illuminated region. For $\theta > \theta_r$, the Airy function dies out rapidly, as the particle scatters into the dark side of the rainbow. This feature is of a purely quantum nature related to the phenomenon of tunneling; classically, there is no scattering for $\theta > \theta_r$, and thus the name "dark". Quantum mechanics supplies some "illumination" in this classically forbidden region.

Besides the rainbow, one has the effect of glory scattering in 3D, both near $\theta = 0$ (forward glory) and near $\theta = \pi$ (backward glory). For example, when back scattering is possible for impact parameter other than zero ($b \neq 0$), one has $\theta = \pi$ and cross section (30) can be infinite due to the vanishing of $\sin \theta$. It occurs whenever $\Theta$ of Eq. (10) goes smoothly through 0 or through an integral multiple of $\pi$ [5]. In 2D cross section (29) the $\sin \theta$ term is absent and one cannot have glory scattering. However, one can obtain oscillating cross section near $\theta = 0$ or $\pi$ owing to near equality of $f^{(+)}$ and $f^{(-)}$ [described by Bessel function(s)].

The phenomenon of orbiting will take place in both 2D and 3D when the effective potential $U(r) + b^2k^2/r^3$ has a maximum for a certain $b_0$ and the energy of the particle coincides with this maximum $\mathcal{E}$. One has to verify whether $d(U + b^2k^2/r^3)/dr = 0$ for a certain $b_0$ and $r_0$ and check if $k^2 = U(r_0) + k^2b_0^2/r_0^2$. From expression (10) for $\Theta$, one can show that in the case of orbiting [3]

$$\Theta(k^2, b) = \text{const.} + c\ln[(b-b_0)/b_0], \quad b > b_0 \quad (36)$$

$$\Theta(k^2, b) = \text{const.} + 2c\ln[(b_0-b)/b_0], \quad b < b_0. \quad (37)$$

As $b \to b_0$, $\Theta(k^2, b)$ can be infinitely large and for attractive interaction the particle can go around the center many times. Then the differential cross section made from many components – one from each orbit – could be greatly different from the classical results (29) and (30) and will show interference effect [3].

The interference terms of the SC 2D cross section (27) are absent in cross section (29) for a single stationary point $n_s$. The case of two stationary points in Eq. (27) leads to an interference term and deserves special attention. We assume that deflection function $\Theta(k^2, b)$ is a double-valued function. For a given $\theta$, there are two impact parameters, $b_1, b_2$ ($n_1 = kb_1$ and $n_2 = kb_2$), that contribute to Eq. (29). If there are two contributions to the term, say, $f^{(-)}$ in Eq. (25), then $d\Theta/d\theta$ will be positive at one of these points, and negative at the other. This results in an extra phase of $\pi/2$ in the contribution to the amplitude in Eq. (25) coming from the latter stationary point. Then from Eq. (25) the scattering cross section can be written as

$$\frac{d\sigma_{SC}^{2D}}{d\theta} = \frac{db_1}{d\theta} \left| + \frac{db_2}{d\theta} \right| + \frac{2}{|d\theta/d\theta|} \left[ \sin[k(b_1 - b_2)] \theta + 2\{\delta(b_1) - \delta(b_2)\} \right]. \quad (38)$$

Equation (38) clearly shows the wave nature of the scattering process. The interference term involving the sine term is purely quantum mechanical and would be absent in classical scattering. The classical scattering cross section is an incoherent sum of contributions. The interference pattern has an angular period, $P_0$ (angular distance between two adjacent maxima in the scattering cross section), which can be read off from the above, as $P_0 = 2\pi/(k(b_1 - b_2))$. When $b_1$ approaches $b_2$, the analysis leading to Eq. (38) breaks down and one has to resort to a more elaborate treatment, called the Uniform Approximation [10], which provides a generalization of Eqs. (31) and (33) involving the Airy function and its derivative.
IV. SUMMARY

We have presented a comprehensive discussion of semiclassical (SC) scattering theory in two dimensions. We have presented and discussed classical and SC cross section in 2D, WKB and Eikonal approximations. Although, rainbow scattering and orbiting take place in 2D, there is no forward and backward glory scattering.

Acknowledgments

The work was partially supported by CNPq and FAPESP of Brasil.

APPENDIX

A. An Application of Rainbow Scattering

The semiclassical cross section \( \sigma_{\text{CS}} \) exhibits the essential quantum effects of tunneling and interference. To see this more clearly we use the asymptotic form of the Airy function \( \text{Ai}(x) \) as \( x \to \pm \infty \):

\[
\text{Ai}(x) = \exp\left(-\frac{2x^{3/2}}{3}\right) \frac{1}{\sqrt{2\sqrt{\pi}x^{1/4}}} , \quad x > 0, \tag{39}
\]

\[
\text{Ai}(x) = \frac{\sin[2(-x)^{3/2}/3 + \pi/4]}{\sqrt{\pi(-x)^{1/4}}} , \quad x < 0. \tag{40}
\]

Using the asymptotic form \( \text{Ai}(x) \), we find that the cross section \( \sigma_{\text{CS}} \) exhibits a decay in angle that goes like

\[
\frac{d\sigma_{\text{CS}}}{d\theta} = \exp\left[ -\frac{4\sqrt{2}}{3} \frac{\left(\theta - \theta_{\text{r}}\right)^{3/2}}{\left(\frac{d^2\Theta}{d\theta^2}\right)_{\theta_{\text{r}}}} \right] \frac{2\sqrt{2}}{3} \frac{\sin^2}{\left(\frac{d^2\Theta}{d\theta^2}\right)_{\theta_{\text{r}}}} , \quad \theta > \theta_{\text{r}}. \tag{41}
\]

in the classically forbidden angle region, \( \theta > \theta_{\text{r}} \), which is reminiscent of tunneling effect [classically, there is zero scattering in this dark side of the rainbow, see Eq. \( \text{(29)} \)].

In the classically allowed, illuminated, region, \( \theta < \theta_{\text{r}} \), the cross section exhibits interference arising from the contributions of the two stationary phase points. Using the asymptotic form \( \text{Ai}(x) \) in Eq. \( \text{(35)} \), the oscillation in cross section is given by

\[
\frac{d\sigma_{\text{CS}}}{d\theta} = \exp\left[ -\frac{4\sqrt{2}}{3} \frac{\left(\theta - \theta_{\text{r}}\right)^{3/2}}{\left(\frac{d^2\Theta}{d\theta^2}\right)_{\theta_{\text{r}}}} \right] \frac{2\sqrt{2}}{3} \frac{\sin^2}{\left(\frac{d^2\Theta}{d\theta^2}\right)_{\theta_{\text{r}}}} , \quad \theta < \theta_{\text{r}}. \tag{42}
\]

One then finds the period of these oscillations as follows. Call the argument of the \( \sin^2 \) function above, \( g(\theta) = \frac{2\sqrt{2}}{3} \frac{\left(\theta - \theta_{\text{r}}\right)^{3/2}}{\left(\frac{d^2\Theta}{d\theta^2}\right)_{\theta_{\text{r}}}} \). Suppose at \( \theta = \theta_1 < \theta_{\text{r}} \), we have a maximum in cross section implying \( \sin^2[g(\theta_1)] = 1 \), and \( g(\theta_1) = \pi/2 \). At a slightly smaller angle, \( \theta = \theta_2 < \theta_1 \), one has another maximum implying \( \sin^2[g(\theta_2)] = 1 \) corresponding to \( g(\theta_2) = \pi/2 + \pi \). To obtain an estimate of the local period of oscillation \( P_{\text{theta}} \) of cross section \( \sigma_{\text{CS}} \), defined by \( P_{\text{theta}} \equiv \theta_1 - \theta_2 \), we expand \( g(\theta) \) near \( \theta = \theta_1 \) as \( g(\theta) = g(\theta_1) + (\theta - \theta_1)g'(\theta_1) \) (recall, \( \Theta(m) = \theta \), and \( \theta_{\text{r}} > \theta \)). Consequently, the period \( P_{\text{theta}} \) at \( \theta = \theta_1 \) is \( P_{\text{theta}} = -\pi/g'(\theta_1) \). Hence the generic local and angle-dependent period for any \( \theta < \theta_{\text{r}} \) is

\[
P_{\text{theta}} = -\frac{\pi}{\frac{d\Theta}{d\theta} |_{\theta_{\text{r}}}} = \frac{\pi}{k\sqrt{2 \left( \frac{\left(\theta - \theta_{\text{r}}\right)^{3/2}}{\left(\frac{d^2\Theta}{d\theta^2}\right)_{\theta_{\text{r}}}} \right)^2}}. \tag{43}
\]

Thus the smaller the angle \( \theta \) compared to the rainbow angle \( \theta_{\text{r}} \), the smaller will be the period.

The angle \( \theta = \theta_1 \) is called an Airy maximum; in the meteorological rainbow, it is the last maximum in the Airy function, just below \( \theta = \theta_{\text{r}} \), and corresponds to the primary bow, whereas the other maxima are the supernumeraries. The second maximum at an angle slightly smaller than that of the primary bow is called the secondary bow. One usually sees both of these bows in the clear sky after rain, the secondary bow being quite dim with the order of colors inverted compared to the more familiar primary bow.

The period formula \( \text{(43)} \) is an approximate one valid under condition \( \text{(33)} \) used in deriving the rainbow scattering \( \sigma_{\text{CS}} \). Keeping only two terms in Eq. \( \text{(35)} \), one obtains the solution for two turning points \( m_1 \) and \( m_2 \) as \( m_1(2) = m_r \pm \sqrt{2\sqrt{\pi}r_{\text{r}}/\sqrt{|\Theta''(m_r)|^{1/2}} \left(\Theta(m) = \theta \), and \( \theta_{\text{r}} > \theta \). Thus the period of the \( \text{Airy}^2 \) function in Eq. \( \text{(35)} \) is \( P_{\text{theta}} = 2\pi/|m_1 - m_2| \), which is the same as that derived in Eq. \( \text{(18)} \) (note that \( m = kb \)).

The correct period should be obtained by staying at angles much smaller than the rainbow angle [maximum of \( \Theta(m) \)] and using the semiclassical approximation employed in obtaining Eq. \( \text{(25)} \) and the quadratic approximation of the phase \( \zeta \), given by Eq. \( \text{(21)} \). [In the semiclassical approximation leading to Eq. \( \text{(25)} \), Eqs. \( \text{(22)} \) and \( \text{(23)} \) diverge at the rainbow caustic. The amplitude \( \text{(43)} \) is free from that divergence but the period of oscillation \( \text{(18)} \) obtained from this amplitude is not quite right.] The uniform approximation of Ref. \( \text{(10)} \) combines the good features of both approaches and produces the correct period of oscillation. However, the algebra is quite involved and without proof we quote the result for the angle-independent period so obtained: \( P_{\text{theta}} = 2\pi/|m_1 - m_2| \), where \( m_1 \) and \( m_2 \) are the \( m \) values at two stationary points. The period \( \text{(43)} \) can be shown to be equivalent to the result so obtained after using the relations \( m_1 = kb_1 \) and \( m_2 = kb_2 \). (We leave this as a project for an advanced reader.)

B. An Exercise for Reader

We consider an interaction of the form

\[
V(r) = \frac{A}{r}, \quad r > R_c, \tag{44}
\]

\[
V(r) = \frac{A}{2R_c} \left[ 3 - \left( \frac{r}{R_c} \right)^2 \right], \quad r < R_c. \tag{45}
\]
where $A$ and $R_c$ are constants. The above potential should not be confused with the Coulomb interaction between charged particles, which in 2D behaves as $\ln(r)$. The deflection function for the above interaction shows a rainbow maximum and exhibits the phenomenon of orbiting. The condition for rainbow is that the energy $E > \frac{3A}{2R_c}$. The calculation of the deflection function is presented in \ref{8}. We leave the calculation of 2D SC scattering with this potential as an exercise.

[1] I. R. Lapidus, “Quantum-mechanical scattering in 2 dimensions”, Am. J. Phys. 50, 45-47 (1982); “Scattering by two-dimensional circular barrier, hard circle, and delta-function ring potentials”, Am. J. Phys. 54, 459-461 (1986); M. J. Moritz and H. Friedrich, “Scattering by a Coulomb field in two dimensions”, Am. J. Phys. 66, 274-274 (1998).

[2] S. K. Adhikari, “Quantum scattering in 2 dimensions”, Am. J. Phys. 54, 362-367 (1986).

[3] G. Barton, “Rutherford scattering in 2 dimensions”, Am. J. Phys. 51, 420-422 (1983); P. A. Maurone and T. K. Lim, “More on two-dimensional scattering”, Am. J. Phys. 51, 856-857 (1983).

[4] M. J. Moritz, “Semiclassical methods in the long wave limit”, Am. J. Phys. 70, 663-663 (2002); D. Singleton, “Electromagnetic angular momentum and quantum mechanics”, Am. J. Phys. 66, 697-701 (1998); B. R. Holstein, Semiclassical treatment of above-barrier scattering”, Am. J. Phys. 52, 321-325 (1984).

[5] R. G. Newton, Scattering Theory of Waves and Particles, 2nd Edition, (Springer-Verlag, New York, 1982).

[6] R. L. Lee and A. B. Fraser, The Rainbow Bridge: Rainbows in Art, Myth and Science, (The Pennsylvania State University Press, University Park, 2001).

[7] J. J. Sakurai, Modern Quantum Mechanics, (Addison-Wesley, Reading, 1994), pp 392-394.

[8] M. S. Hussein, Y. T. Chen and F. I. A. Almeida, “The validity of the classical description of nuclear scattering”, Am. J. Phys. 52, 650-653 (1984).

[9] P. M. Morse and H. Feshbach, Methods of Theoretical Physics, (McGraw-Hill, New York, 1953).

[10] M. V. Berry, “Uniform approximation for potential scattering involving a rainbow”, Proc. Phys. Soc. London, 89, 479-490 (1966).

[11] M. Abromowitz and I. Stegun, Handbook of Mathematical Functions, (Dover, New York, 1970).