On the Energy Issue for a Class of Modified Higher Order Gravity Black Hole Solutions

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January 19, 2013

Abstract

In the case of a large class of static, spherically symmetric black hole solutions in higher order modified gravity models, an expression for the associated energy is proposed and identified with a quantity proportional to the constant of integration, which appears in the explicit solution. The identification is achieved making use of derivation of the First Law of black hole thermodynamics from the equations of motion, evaluating independently the entropy via Wald method and the Hawking temperature via quantum mechanical methods in curved space-times. Several non trivial examples are discussed, including a new topological higher derivative black hole solution, and the proposal is shown to work in all examples considered.

PACS: 04.50.Kd; 04.70.Dy; 97.60Lf; 95.30.Sf

1 Introduction

Recent observational data imply an accelerating expansion of the visible universe, which gives rise to the so called Dark Energy issue.

There exist several descriptions of this acceleration. Among them, the simplest one consists in the introduction of a small positive cosmological constant in the framework of General Relativity (GR), the so called Λ-CDM model. A generalization of this simple modification of GR consists in considering modified gravitational theories, in which the action is described by a function $F(R)$ of the Ricci scalar $R$ (see for example [1, 2]). Typically these modified models admit the de Sitter space as a solution and the stability of this solution has been investigated in several places (see for example [3, 4, 5, 6]). Furthermore, viable $F(R)$ models, that is the ones which are able to pass the local gravitational GR tests, as well as to describe the inflation with dark energy in a unified way, have been recently discussed [7, 8, 9, 10]. Another very interesting class of modified gravitational models in which the square root of the quadratic Weyl scalar appears have been investigated in Ref. [11].

Static, spherically symmetric solutions have been investigated in several papers (the simplest one being the Schwarzschild-de Sitter solution), and they have been discussed for example in Refs. [12, 13, 14, 11]. Within this class of higher order gravitational models, the issue associated with the energy (mass) of black hole solutions is problematic, and several attempts in order to find a satisfactory answer to that problem have been proposed (see for example [15, 16, 17, 18] and references therein).

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To start with, let us remind the case of GR, in which several notions of quasi-local energies may be introduced. In particular we mention the so called Misner-Sharp mass, which has the important property to be defined for dynamical, spherically symmetric space-time \[19\], where the use of invariant quantities plays a crucial role \[20, 21\]. For the sake of completeness, we recall that in four dimensions, any spherically symmetric metric can locally be expressed in the form

\[
ds^2 = \gamma_{ij}(x^i) dx^i dx^j + R^2(x^i) d\Omega^2_2, \quad i, j \in \{0, 1\} ,
\]

(1)

where \(d\Omega^2_2\) here is the usual metric on the two sphere \(S^2\), but it could be be the metric of a generic two-dimensional maximally symmetric space. Of course, in such cases the black hole will have a different topology. The two-dimensional metric

\[
d\gamma^2 = \gamma_{ij}(x^i) dx^i dx^j
\]

(2)

is referred to as the normal one. The related coordinates are \(\{x^i\}\), while \(R(x^i)\) is the areal radius, considered as a scalar field in the two dimensional normal space. A relevant scalar quantity in the reduced normal space is

\[
\chi(x) = \gamma^{ij}(x) \partial_i R(x) \partial_j R(x),
\]

(3)

since the dynamical trapping horizon, if it exists, is located in correspondence of

\[
\chi(x)\big|_H = 0,
\]

(4)

provided that \(\partial_i \chi|_H \neq 0\). (We use the suffix \(|_H\) for all quantities evaluated on the horizon). The quasi-local Misner-Sharp gravitational energy is defined by

\[
E_{MS}(x) = \frac{1}{2} R(x) [1 - \chi(x)] .
\]

(5)

This is an invariant quantity on the normal space. Note also that, on the horizon, \(E_{MS}|_H = \frac{1}{2} R_H \equiv E\), \(E\) being the energy of black hole. Furthermore, one can introduce the Hayward surface gravity associated with this dynamical horizon, which is given by the normal-space scalar

\[
\kappa_H = \frac{1}{2} \Box_{\gamma} R|_H ,
\]

(6)

\(\Box_{\gamma}\) being the Laplacian corresponding to the \(\gamma\) metric. In the spherical symmetric, dynamical case, it is also possible to introduce the Kodama vector field \(K\). Given the metric \(1\) it is defined by

\[
K^i(x) = \frac{1}{\sqrt{-\gamma}} \varepsilon^{ij} \partial_j R, \quad K^\theta = 0 = K^\phi ,
\]

(7)

\(\varepsilon^{ij}\) being the completely antisymmetric Levi-Civita tensor on the normal space.

Assuming Einstein equations, in a generic four-dimensional spherically symmetric space-time, a geometric dynamical identity holds true in general. This can be derived as follows. Let us introduce the normal space invariant

\[
T^{(2)} = \gamma^{ij} T_{ij} ,
\]

(8)

which is the reduced trace of the stress energy tensor \(T_{\mu\nu}\). Then, making use of Einstein equations, it is possible to show that, on the dynamical horizon (see for example\[19\])

\[
\kappa_H = \frac{1}{2 R_H} + 2 \pi R_H T^{(2)}_H .
\]

(9)
Introducing the horizon area $A_H$ and the (formal) three-volume $V_H$ enclosed by the horizon, with their respective “thermodynamical” differentials $dA_H = 8\pi R_H dR_H$, and $dV_H = 4\pi R_H^2 dR_H$ (we are assuming a horizon with the topology of a sphere), we get

$$\frac{\kappa_H}{8\pi} dA_H = d \left( \frac{R_H}{2} \right) + \frac{T_H^{(2)}}{2} dV_H . \quad (10)$$

This equation can be recast in the form of a geometrical identity, once the Misner-Sharp energy at the horizon has been introduced. It reads

$$dE = \frac{\kappa_H}{2\pi} d \left( \frac{A_H}{4} \right) - \frac{T_H^{(2)}}{2} dV_H . \quad (11)$$

In the following, we shall restrict the discussion to the static case in the absence of matter. This means that we shall consider only vacuum static solutions. In such a case the metric in (1) can be written in the simpler form

$$ds^2 = -B(r)e^{2\alpha(r)} dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_2^2 , \quad (12)$$

where $\alpha(r)$ and $B(r)$ are functions of $r$. Of course the general formalism is also valid in the static case, and leads to the horizon condition

$$B(r_H) = 0 , \quad B'(r_H) \neq 0 , \quad e^{\alpha(r_H)} \neq 0 . \quad (13)$$

The Kodama vector reduces to

$$K^\mu = \left( e^{-\alpha(r)}, 0 \right) . \quad (14)$$

When $\alpha(r) = 0$, which corresponds to case of GR in vacuum, the static Kodama vector coincides with the usual Killing vector $(1, 0)$, and Hawking temperature of the related black hole reads

$$T_K = \frac{1}{4\pi} \left| \frac{dB(r_H)}{dr} \right|_{r=r_H} . \quad (15)$$

This is a well known result, and it can be justified in several ways, for example making use of standard derivations of Hawking radiation [22], or by eliminating the conical singularity in the corresponding Euclidean metric, or making use of the tunneling method, recently introduced in Refs. [23] [24] [25], and discussed in details in several papers.

However, as we shall see in explicit examples, within modified gravity it happens to deal with black hole solutions with $\alpha(r) \neq 0$. In this case, the Kodama vector does not coincide with the Killing vector. Then one may introduce two Hawking temperatures, the Killing temperature (see, Appendix I)

$$T_K = \frac{1}{4\pi} \left| \frac{dB(r_H)}{dr} \right|_{r=r_H} = \frac{1}{4\pi} e^{\alpha(r_H)} \left| \frac{dB(r)}{dr} \right|_{r=r_H} , \quad (16)$$

and, making use of (13) the Hayward temperature

$$T_H = \frac{\kappa_H}{2\pi} = \frac{1}{4\pi} \left| \frac{dB(r)}{dr} \right|_{r=r_H} , \quad (17)$$

which is trivially related to the previous one by $T_K = e^{\alpha(r_H)} T_H$. If $\alpha(r) = 0$ we recover Eq.(15), namely $T_K = T_H$. A detailed discussion about this issue can be found in Refs. [20] [21], in which also the dynamical case is discussed.

In the static case, all derivations of Hawking radiation (for example, the tunneling method in Appendix I) leads to a semi-classical expression for the black hole radiation rate

$$\Gamma \equiv e^{-\frac{\Delta E_{\text{H}}}{k}} , \quad (18)$$
in terms of the change $\Delta E_K$ of the Killing energy $E_K$ \cite{24}, but if one uses the Kodama energy $E_H$ for the emitted particle, one has
\[ \Gamma \equiv e^{-\Delta E_K/T_H}. \] (19)

From the Eqs. (18) and (19) one arrives at the identity
\[ \frac{\Delta E_H}{T_H} = \frac{\Delta E_K}{T_K}, \] (20)
which may interpreted as the First Law of black hole thermodynamics as soon as $\Gamma \equiv e^{-\Delta S}$, with $S$ the entropy of the black hole itself. As a result, in the static case the two temperatures $T_K$ and $T_H$ are equivalent.

With regard to entropy of the black hole, it is well known that in GR the so called Area Law is satisfied, and we have
\[ S_W = \frac{A_H}{4G}. \] (21)

In GR and in the static case, the First Law of black hole thermodynamics in vacuum reduces to
\[ dE = T_H dS_W, \] (22)
where $E$ is the Misner-Sharp energy evaluated on the horizon.

Now we come to the key point of our proposal. For a generic modified gravity theories, for example the $F(R)$ models, where $R$ is the Ricci curvature, it seems very difficult to define in a reasonable way the analogue of the local Misner-Sharp mass (see Ref. \cite{18}). As we will see, an exception is the higher-dimensional Lovelock gravity \cite{26}.

For this reason, in this paper, an attempt is made for obtaining an expression of energy associated with black holes solutions in higher order modified gravitational models. The proposal consists in the identification of the black hole energy with a quantity proportional to the constant of integration, which appears in the explicit solution. The identification is achieved making use of derivation of the First Law of black hole thermodynamics from the equations of motion, evaluating in an independent way the related black hole entropy via Wald method \cite{27} (see the Appendix II) and the Hawking temperature via the quantum mechanics in curved space-time, for example the tunneling method \cite{23} or other standard equivalent methods.

This approach is also supported by the results obtained in Refs. \cite{28, 29}, where, on quite general grounds, generalizing the Jacoboison results on GR (see the seminal paper \cite{30}), the equations of a modified gravitational theories are shown to be equivalent to the First Law of black hole thermodynamics. As it is well known, this issue may be of high relevance in substantiating the idea that gravitation might be a manifestation of thermodynamics of quantum vacuum \cite{31}.

The paper is organized as follows. In Section 2, the Lovelock gravity \cite{26} is revisited, and the approach here proposed is shown to work for such a case. In Section 3, the four dimensional modified gravity models of the $F(R)$ type are investigated, and the method proposed is applied to several cases in Sections 4 and 5. In Section 6 a new topological black hole solution is discussed and the method is shown to work, as in Section 7, where the conformal Weyl gravity black holes are considered. Finally Section 8 contains the conclusions. In two Appendices, for the sake of completeness, the tunneling method and Wald entropy method are briefly discussed.

## 2 Lovelock Black Hole Solutions

In this section, as warm up, we review Lovelock theory with the related static and spherically symmetric black hole solutions. This theory is a very interesting higher dimensional generalization of Einstein gravity. In general, by making use of higher order geometrical invariants in the action, in the metric formalism for the field equations one obtains fourth order partial differential equations. However, as Lovelock had shown, one can obtain second order differential equation by making use
of higher dimensional extended Euler densities, the so called $m$-th order Lovelock terms defined by

$$L_m = \frac{1}{2^m} \delta^{\lambda_1 \sigma_1 \cdots \lambda_m \sigma_m}_{\rho_1 \kappa_1 \cdots \rho_m \kappa_m} R_{\lambda_1 \sigma_1} \cdots R_{\lambda_m \sigma_m} \rho_m \kappa_m,$$  \hspace{1cm} m = 1, 2, 3, ... \hspace{1cm} (23)

where $R_{\lambda \sigma}$ is the Riemann tensor in arbitrary $D$-dimensions and $\delta^{\lambda_1 \sigma_1 \cdots \lambda_m \sigma_m}_{\rho_1 \kappa_1 \cdots \rho_m \kappa_m}$ is the generalized totally antisymmetric Kronecker delta defined by

$$\delta^{\mu_1 \mu_2 \cdots \mu_p}_{\nu_1 \nu_2 \cdots \nu_p} = \det \begin{pmatrix} \delta^{\mu_1}_{\nu_1} & \delta^{\mu_1}_{\nu_2} & \cdots & \delta^{\mu_1}_{\nu_p} \\ \delta^{\mu_2}_{\nu_1} & \delta^{\mu_2}_{\nu_2} & \cdots & \delta^{\mu_2}_{\nu_p} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{\mu_p}_{\nu_1} & \delta^{\mu_p}_{\nu_2} & \cdots & \delta^{\mu_p}_{\nu_p} \end{pmatrix}.$$

The action for Lovelock gravitational theory reads

$$I = \int d^Dx \sqrt{-g} \left[ -2\Lambda + \sum_{m=1}^{k} \left\{ \frac{a_m}{m} L_m \right\} \right],$$  \hspace{1cm} (24)

where we defined the maximum order $k = [(D - 1)/2]$ and $a_m$ are arbitrary constants. Here $\lfloor z \rfloor$ represents the maximum integer satisfying $\lfloor z \rfloor \leq z$. Hereafter we set $a_1 = 1$.

For such a kind of theory, the equations of motion in vacuum are second order quasi-linear partial differential equations in the metric tensor and read

$$\mathcal{G}_{\mu \nu} = 0,$$  \hspace{1cm} (25)

the Lovelock tensor $\mathcal{G}_{\mu \nu}$ being given by

$$\mathcal{G}_{\mu \nu} = \Lambda \delta_{\mu \nu} - \sum_{m=1}^{k} \frac{1}{2^{m+1}} \frac{a_m}{m} \delta^{\mu_1 \sigma_1 \cdots \mu_m \sigma_m}_{\nu_1 \kappa_1 \cdots \nu_m \kappa_m} R_{\mu_1 \sigma_1} \cdots R_{\mu_m \sigma_m} \rho_m \kappa_m.$$  \hspace{1cm} (26)

As we said in previous Section, we shall focus our attention on static, spherically symmetric solutions, thus we look for metric of the form

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_n^2,$$  \hspace{1cm} (27)

where $d\Omega_n^2$ is the metric of a $n$-dimensional sphere $S^n$ ($n = D - 2$). Such kind of theories become quite interesting for $D > 4$, the four-dimensional case being equivalent to Schwarzschild-de Sitter, since $\mathcal{L}_1 = R$ and $\mathcal{L}_2$ is equal to the Gauss-Bonnet quadratic term, which in four-dimensions is a topological invariant.

A direct evaluation of field equations gives $\cite{32}$

$$\mathcal{G}_{t t}^i = \mathcal{G}_{t r}^i = -\frac{n}{2r^{n-1}} \frac{d}{dr} \left[ r^{n+1} W(r) \right],$$  \hspace{1cm} (28)

$$\mathcal{G}_{r r}^i = -\frac{1}{2r^{n-1}} \frac{d^2}{dr^2} \left[ r^{n+1} W(r) \right],$$  \hspace{1cm} (29)

where $W$ is given by

$$W(r) = \sum_{m=2}^{k} \frac{\alpha_m}{m} [1 - B(r)]^m r^{-2m} + [1 - B(r)] r^{-2} - \frac{2\Lambda}{n(n+1)},$$  \hspace{1cm} (30)

with $\alpha_m = a_m \prod_{p=1}^{2m-2} (n-p)$. 

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For example, for $D = 4$, $k = 1$, and so one has the Schwarzschild-de Sitter solution, while for $D = 5$, $k = 2$, there is one Lovelock non trivial term (the Gauss-Bonnet, which in five-dimensions is not a topological invariant) and one has the Boulware-Deser solution. For higher dimensions one has an algebraic equation of increasing complexity, but, as we shall see in the following, for our purposes it will be not necessary to know explicitly the expression for the solution $B(r)$.

For the static metric in (27) one has the Killing vector $V^\mu = (1, \vec{0})$ and since

$$\nabla_\nu G^\nu_\mu = 0, \quad G_{\mu\nu} = G_{\nu\mu},$$

(31)

the vector $J_\mu = G_{\mu\nu} V^\nu$ is covariantly conserved and gives rise to a Killing conserved charge. This corresponds to the quasi-local generalized Misner-Sharp mass which reads

$$E(r) = -\frac{1}{8\pi G} \int d^3 r J^\mu = \frac{nV(\Omega_n)}{16\pi G} \int_0^r d\rho \frac{d(\rho^{n+1} W)}{d\rho} = \frac{nV(\Omega_n)}{16\pi G} r^{n+1} W(r),$$

(32)

where $\Sigma$ is a spatial volume at fixed time, $d\Sigma_\mu = (d\Sigma, \vec{0})$, and assuming spherical horizons, $V(\Omega_n) = \frac{2\pi^{n/2+1/2}}{\Gamma(n/2+1/2)}$.

In the absence of matter Eq. (28) can be integrated and one has

$$r^{n+1} W(r) = C,$$

(33)

$C$ being a constant of integration which we will show to be related to the mass of the black hole. On shell, that is at the horizon $r = r_H$, $B(r_H) = 0$, Eqs. (32) and (33) leads to

$$E_K = \frac{nV(\Omega_n)}{16\pi G} C,$$

(34)

Now let us show that a First Law of black hole thermodynamics holds true, with the “energy” of the black hole solution, namely the Killing charge obtained below, proportional to constant of integration $C$. In the case of Lovelock gravity the validity of the First Law of black hole thermodynamics has been investigated in many places (see for example [34, 35, 36, 37]). For the static case we present a direct and simple proof.

First of all we introduce the horizon defined by the existence of the largest positive root $r_H$ of

$$B(r_H) = 0, \quad \frac{dB(r_H)}{dr} \neq 0.$$

(35)

Then from Eq. (33) we have the identity

$$C = r_H^{n+1} W_H = \sum_{m=2}^k \frac{\alpha_m}{m} r_H^{n+1-2m} + r_H^{n-1} - \frac{2\Lambda r_H^{n+1}}{n(n+1)}.$$  

(36)

On the other hand, taking the derivative with respect to $r$ of Eq. (33) and putting $r = r_H$, and making use again of Eq. (33), we obtain

$$\sum_{m=2}^k \frac{\alpha_m(n+1-2m)}{m} r_H^{n+1-2m} + (n-1)r_H^{n-1} - \frac{2\Lambda r_H^{n+1}}{n} = \frac{dB_H}{dr} \left( \sum_{m=2}^k \frac{\alpha_m r_H^{n+2-2m} + r_H^n}{m} \right).$$  

(37)

Now, let us compute the “thermodynamical” change of $C$ with respect to a small change of $r_H$. From Eq. (33) one has

$$dC = \left( \sum_{m=2}^k \frac{\alpha_m(n+1-2m)}{m} r_H^{n-2m} + (n-1)r_H^{n-2} - \frac{2\Lambda r_H^n}{n} \right) dr_H.$$  

(38)

Making use of Eq. (37) this expression may be rewritten in the form

$$dC = \frac{dB_H}{dr} \left( \sum_{m=2}^k \frac{\alpha_m r_H^{n+1-2m} + r_H^{n-1}}{m} \right) dr_H.$$  

(39)
Let us interpret the r.h.s of the latter identity. Here we are dealing with a static, spherically
symmetric metric admitting a Killing vector. If there is an event horizon located at \( r_H \), then the
Hawking temperature of the related black hole is given by Eq. (15).

Now, all thermodynamical quantities associated with these black holes solutions can be computed
by standard methods. In particular, the entropy can be calculated by the Wald method
\cite{27, 38, 39} or other methods if you like, and one has (see for example \textit{\cite{35, 36, 40}})

\[
S_W = \frac{2\pi V(\Omega_n)}{8\pi G} r_H^n \left( 1 + n \sum_{m=2}^{k} \frac{\alpha_m}{n + 2 - 2m} r_H^{2m-2} \right). \tag{40}
\]

As a result, from Eqs. (34), (38), and (40), one has the First Law of black hole thermodynamics
for Lovelock gravity, that is

\[
T_K dS_W = dE_K. \tag{41}
\]

We have shown that for a generic Lovelock gravity, the First Law of black hole thermodynamics
holds and one can identify the energy of a static, spherically symmetric black hole with the constant
of integration and Killing conserved charge.

The generalization to topological Lovelock black holes has been investigated in \cite{41}, and again
the First Law of black hole thermodynamics has been shown to hold.

3 \( F(R) \) four-dimensional modified gravity

In this Section we will come back to \( D = 4 \). To begin with, we recall that the action of modified
\( F(R) \)-theories reads

\[
I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} F(R), \tag{42}
\]

where \( g \) is the determinant of metric tensor \( g_{\mu\nu} \), and \( F(R) \) is a generic function of the Ricci scalar
\( R \). For dimensional reason, \( F(R) \) may contain a multiplicative functional dependence on \( G \), the
Newton constant.

The equations of motion in vacuum for a general \( F(R) \) model read

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu}^{MG}. \tag{43}
\]

Here, \( R_{\mu\nu} \) is the Ricci tensor and the part of ‘modified gravity’ (\( MG \)) is formally included into
the tensor \( G_{\mu\nu}^{MG} \), which is given by

\[
G_{\mu\nu}^{MG} = \frac{1}{F'(R)} \left\{ \frac{1}{2} g_{\mu\nu} [F(R) - RF'(R)] + (\nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \Box) F'(R) \right\}. \tag{44}
\]

The prime denotes derivative with respect to the curvature \( R \), \( \nabla_{\mu} \) is the covariant derivative
operator associated with \( g_{\mu\nu} \) and \( \Box \phi \equiv g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \phi \) is the D’Alembertian of a scalar field \( \phi \). The
trace of Eq. (43) gives

\[
3 \Box F'(R) + RF'(R) - 2F(R) = 0, \tag{45}
\]

which shows that there exists an additive scalar dynamical degree of freedom represented by \( F'(R) \).

In the metric (12) the scalar curvature reads

\[
R = -3 \left[ \frac{d}{dr} B(r) \right] \frac{d}{dr} \alpha(r) - 2B(r) \left[ \frac{d}{dr} \alpha(r) \right]^2 - \frac{d^2}{dr^2} B(r) - 2 B(r) \frac{d^2}{dr^2} \alpha(r)
-4 \frac{d B(r)}{r} - 4 \frac{B(r) d^2 \alpha(r)}{r} - 2 \frac{B(r)}{r^2} + \frac{2}{r^2}. \tag{46}
\]
In Ref. [13] the following equations of motion have been found by Lagrangian methods [42, 43, 6]:

\[
e^\alpha(r) \left( RF'(R) - F(R) - 2F'(R) \frac{(1 - B(r) - r(dB(r)/dr))}{r^2} \right) + 2B(r)F''(R) \left( \frac{d^2 R}{dr^2} + \left( \frac{2}{r} + \frac{dB(r)/dr}{2B(r)} \right) \frac{dR}{dr} + \frac{F''(R)}{F'(R)} \left( \frac{dR}{dr} \right)^2 \right) = 0,
\]

\[
e^\alpha(r) \left[ \frac{d\alpha(r)}{dr} \left( \frac{2}{r} + \frac{F''(R) dR}{F'(R) dr} \right) - \frac{F''(R) d^2 R}{F'(R) dr^2} - \frac{F''(R)}{F'(R)} \left( \frac{dR}{dr} \right)^2 \right] = 0.
\]

Once \( F(R) \) is given, together with Eq. (49), the above equations form a system of three differential equations in the quantities \( \alpha(r) \), \( B(r) \) and \( R(r) \).

As already said, the static solutions describe a black hole if there exists a real positive solution \( r_H \) of \( B(r_H) = 0, B'(r_H) \neq 0 \). If this happens \( r_H \) is the radius of the event horizon. The Killing surface gravity reads

\[
\kappa_K \equiv \frac{1}{2} \sqrt{\frac{|d(e^{2\alpha(r)}B(r)/dr) dB(r)/dr|}{dr}} \bigg|_{r=r_H} = \frac{1}{2} e^\alpha(r_H) \frac{dB(r)}{dr} \bigg|_{r=r_H}.
\]

Non trivial examples of such \( F(R) \) gravity black hole solutions have been discussed in [13] and we shall deal with them in the next Sections. In the following, we shall show that the First Law of black hole thermodynamics holds, making use of the equations of motion and of the expressions for the Hawking temperature and entropy obtained by independent methods. First, the tunneling method gives for the Hawking temperature

\[
T_K = \frac{e^\alpha(r_H) dB(r_H)}{4\pi}.
\]

The entropy associated to these black holes solutions can be calculated by the Wald method (see Appendix II). One has

\[
S_W = \frac{A_H}{4G} F'(R_H).
\]

For simplicity we will consider only spherical horizons, thus the area is \( A_H = 4\pi r_H^2 \) and the volume \( V_H = \frac{4}{3}\pi r_H^3 \). By evaluating the equation of motion on the event horizon, and multiplying both sides of equation by \( dr_H \), we have

\[
T_K dS_W = e^\alpha(r_H) \left( \frac{F_H'}{2G} - \frac{R_H F_H' - F_H r_H^2}{4G} \right) dr_H.
\]

Thus, we have derived for a generic \( F(R) \) gravitational model the First Law of black hole thermodynamics as soon as the identification

\[
E_K = \int e^\alpha(r_H) \left( \frac{F_H'}{2G} - \frac{R_H F_H' - F_H r_H^2}{4G} \right) dr_H,
\]

can be made. Within these \( F(R) \) modified gravity theories, this is one of the main result of this paper. In the next Sections, by making use of several exact solutions, we will provide a support for this identification.

Our proposal, expressed by Eq. (53), should be compared with a similar proposal contained in Ref. [44]. In Ref. [45] an attempt to define a local Misner-Sharp mass has been presented. There, however, the proposed formula is not really satisfactory, because the quasi-local form is only present in some particular cases, one of which will be discussed in the next Section.
3.1 The constant curvature case

As a simple but important example, let us consider the class of static solutions with constant curvature $R_0$. In this case one has the solution with $\alpha = 0$ (in [12]), and the first equation of motion (47) reduces to

$$\frac{d}{dr} \left( r - rB(r) + \frac{\Lambda_0}{3} r^3 \right) = 0,$$

where

$$\Lambda_0 = \frac{R_0 F_0' - F_0}{2 F_0}.$$  

(55)

Thus, one arrives at Schwarzschild-de Sitter solution

$$B(r) = \left( 1 - \frac{C}{r} - \Lambda_0 \frac{r^2}{3} \right),$$

and $R_0 = 4\Lambda_0$. Here $C$ is a constant of integration. The horizon is located at $r = r_H$, where

$$1 = \frac{C}{r_H} + \Lambda_0 \frac{r_H^2}{3}.$$  

(57)

Making use of Eq. (53) one has

$$E_K = \frac{1}{2G} \left( F_0'(r_H) - \frac{\Lambda_0 r_H^2}{3} \right),$$

and by Eq.(57) one finally gets

$$E_K = \frac{F_0' C}{2G},$$

(59)

which is our identification of mass-energy expression for this class of black hole, in agreement with Ref.[44].

4 The Clifton-Barrow solution

Let us apply the same procedure for the highly non-trivial Clifton-Barrow solution[12], for which $\alpha$ is not a constant. The starting point is the following $F(R)$-modified gravity model:

$$F(R) = R^{3+\delta} G^\delta.$$  

(60)

For dimensional reasons we have also included the Newton constant $G^\delta$, $\delta$ being a numerical parameter. When $\delta = 0$ the Hilbert-Einstein action of GR is recovered. Note that in this case the modification with respect GR is not additive.

Looking for static, spherically symmetric metric of the type (12), we find the Clifton-Barrow solution of Eqs. (46)-(48), that it

$$e^\alpha(r) = \left( \frac{r}{r_0} \right)^{\delta(1+2\delta)/(1-\delta)} \left( \frac{1 - 2\delta + 4\delta^2}{1 - \delta} \right)^{1/2},$$

(61)

$$B(r) = \frac{(1-\delta)^2}{(1-2\delta + 4\delta^2)(1-\delta)} \left( 1 - \frac{C}{r^{1-2\delta + 4\delta^2}} \right),$$

(62)

and

$$R = \frac{c_\delta}{r^2}, \quad c_\delta = \frac{6\delta(1+\delta)}{(2\delta^2 + 2\delta - 1)}.$$  

(63)

Above, $r_0 > 0$ is an arbitrary constant while $C > 0$ is the integration constant of the model. We assume $\delta \neq 1$. 

9
The horizon radius, defined by \( B(r_H) = 0 \) and \( \partial_r B(r_H) \neq 0 \) reads
\[
r_H = C^{(1-\delta)/(1-2\delta+4\delta^2)},
\]
(64)
and since \( C > 0 \), the Clifton-Barrow metric is a black hole solution.

According to Equation (49) the Killing-horizon surface gravity reads
\[
\kappa_K = \frac{1}{2}\sqrt{\frac{(1-2\delta+4\delta^2)}{(1-2\delta-2\delta^2)}} \frac{r^{(2\delta+2\delta^2-1)/(1-\delta)}}{r_H^{(1+2\delta)/(1-\delta)}},
\]
(65)
which can be used to find the Killing-Hawking temperature \( T_K = \kappa_K/2\pi \).

With regard to the black hole entropy associated with the event horizon of the Clifton-Barrow solution, from the Wald formula in Equation (51) we find \[46\]:
\[
S_W = \frac{A_H}{4G^{1-\delta}}(1+\delta) \left[ \frac{6\delta(1+\delta)}{(2\delta^2 + 2\delta - 1)r_H^2} \right]^{\delta}.
\]
(66)
In order to have the positive sign of entropy, we must require \( \delta > (\sqrt{3} - 1)/2 \) or \( -1 < \delta < 0 \). The solutions with \( 0 < \delta < (\sqrt{3} - 1)/2 \) or \( \delta < -1 \) are unphysical, whereas for \( \delta = 0 \) we find the result of General Relativity. On the other hand, only the solutions of \( -1 < \delta < 0 \) give a real value for the Killing surface gravity \( \kappa_H \). If \( \delta > (\sqrt{3} - 1)/2 \) the Hawking Temperature becomes imaginary.

Making use of Eqs. (52) one has
\[
dE_K = A_3 r_H^{(4\delta^2-\delta)/(1-\delta)} dr_H,
\]
(67)
As a result, the energy turns out to be
\[
E_K = \frac{A_3(1-\delta)}{1+4\delta^2-\delta} r_H^{(1+4\delta^2-\delta)/(1-\delta)}.
\]
(68)
Finally, from Eq. (51) one gets again that the energy is proportional to the constant of integration of the BH solution since
\[
E_K = \frac{\Psi_\delta G^{\delta-1}}{r_0^{(1+2\delta)/(1-\delta)}} C,
\]
(69)
where we have introduced the dimensionless constant depending on \( \delta \)
\[
\Psi_\delta = \left( \frac{2^{\delta-1}3^\delta (\delta - 1)^2(\delta + 1)^{\delta+1}}{\sqrt{1-2\delta - 2\delta^2} \sqrt{1 - 2\delta + 4\delta^2 (2\delta^2 + 2\delta - 1)^\delta}} \right).
\]
(70)
We conclude this Section with some remarks. In the above expression, the range of parameter \( \delta \) has to be restricted to the ranges already discussed in order to have a positive temperature and entropy. As a check, it is easy to show that in the limit \( \delta \to 0 \), one gets the GR value \( C = 2EG \). Furthermore, the Killing energy \( E_K \) and the Killing temperature depend on the dimensional constant \( r_0 \), and we may take it proportional to Planck length \( \sqrt{G} \).

5 1/R Model

As a further non trivial example, let us consider the following \( F(R) \)-model:
\[
F(R) = -\gamma \left( \frac{1}{R} - \frac{\hbar^2}{6} \right).
\]
(71)
where $h$ and $\gamma$ are positive, dimensional, arbitrary constants (we may choose, for example $\gamma = 1/G^2$). In Ref.\[13\] it has been shown that this model admits a static, spherically symmetric solution of the type \[12\]

$$e^{\alpha(r)} = \left(\frac{r}{r_0}\right)^{1/2},$$

$$B(r) = \frac{4}{7} \left(1 - \frac{7}{6h} r + \frac{C}{r^{7/2}}\right),$$

and $R = 6/(hr)$. Here $r_0$ is an arbitrary constant, which is present for dimensional reasons, and $C$ is the integration constant. Let us consider the solution of $B(r_H) = 0$, namely

$$\left(1 - \frac{7}{6h} r + \frac{C}{r^{7/2}}\right) = 0.$$  \[74\]

If we assume $C > 0$, it is easy to show that there exists always a simple zero $r_H > 0$, which defines the event horizon, and so the above solution represents a black hole. With regard to the related entropy, Eq.\[61\] gives

$$S_W = \frac{\pi \gamma h^2 r_H^3}{36G},$$

the entropy being positive, since $\gamma > 0$. The Killing temperature associates with the horizon reads

$$T_K = \frac{|\kappa_K|}{2\pi} = 4 \left(\frac{1}{6h} \sqrt{r_H} + \frac{C}{2r_H^{1/2}}\right) \left(\frac{1}{r_0}\right)^{1/2}.$$  \[76\]

By computing the Killing energy from \[53\] we have

$$E_K = \int \frac{h\gamma}{54G} \left(\frac{1}{r_0}\right)^{1/2} \left(r_H^{9/2} - \frac{6}{7h} r_H^{7/2}\right) = \frac{h\gamma}{54G} \left(\frac{1}{r_0}\right)^{1/2} \left(r_H^{9/2} - \frac{6}{7h} r_H^{7/2}\right).$$

Thus, making use of Eq.\[74\] one arrives at

$$E_K = \frac{h^2 \gamma}{63G} \left(\frac{1}{r_0}\right)^{1/2} C.$$  \[78\]

Also in this case we can identify the integration constant of the model as a quantity proportional to the black hole Killing energy.

### 6 The Deser-Sarioglu-Tekin topological black hole solutions

In this Section, first we generalize the modified gravity black hole solution of Deser et al.\[11\], and then we shall show that also for these solutions the First Law of black hole thermodynamics is valid and the constant of integration is proportional to the Killing energy.

For the sake of simplicity we shall restrict ourselves to the four-dimensional case, but, since we are interested in black hole with generalized topological horizon, we have to include a non vanishing cosmological constant (see for example the GR case \[47, 48, 49\]). The $D$-dimensional case as well as the inclusion of Electromagnetism presents no difficulties.

To begin with, we write down the action of the model

$$I = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left(R - 2\Lambda + \sqrt{3}\sigma \sqrt{F}\right).$$  \[79\]

where $\sigma$ is a real dimensionless parameter and $F = C_{\mu\nu\rho\delta}C^{\mu\nu\rho\delta}$ is the square of the Weyl tensor. For $\sigma = 0$ the Weyl contribution turns off and GR result is recovered. This model is a very interesting additive modification of GR with cosmological constant.
For more generality we look for static, (pseudo)-spherically symmetric solutions with various topology and so we write the metric in the form

\[ ds^2 = -a^2(r)B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 \left( \frac{d\rho^2}{1-kr^2} + \rho^2 d\phi^2 \right), \tag{80} \]

where the horizon manifold will be a sphere \( S_2 \), a torus \( T_2 \) or a compact hyperbolic manifold \( Y_2 \), according to whether \( k = 1, 0, -1 \).

A direct computation shows that the noteworthy properties of the Weyl scalar \( F \) discussed in Ref.\[11\] for \( k = 1 \), are still valid for \( k = 0, -1 \). Thus the unknown functions \( a(r) \) and \( B(r) \) can be obtained by imposing the stationary condition \( \delta I = 0 \), where, \( I \) is the original action evaluated on the metric (80) (up to integration by parts and on the “topological” variable \( \rho, \phi \)). It reads

\[ I = \int dr \left\{ (1-\sigma)[ra'(r)B(r) + ka(r)] + 3\sigma a(r)B(r) - \Lambda r^2a(r) \right\}, \tag{81} \]

from which it follows

\[ (1-\sigma)ra'(r) + 3\sigma a(r) = 0, \tag{82} \]

\[ rB'(r) + \frac{(1-4\sigma)}{1-\sigma} B(r) = k - \Lambda \frac{r^2}{1-\sigma}. \tag{83} \]

Here we are assuming \( \sigma \neq 1, \frac{1}{2}, \frac{1}{4} \). The general solutions are

\[ a(r) = \left( \frac{r}{r_0} \right)^{3\sigma}, \tag{84} \]

\[ B(r) = k \frac{(1-\sigma)}{(1-4\sigma)} - Cr^{-\frac{1+4\sigma}{1-\sigma}} - \Lambda \frac{r^2}{3(1-2\sigma)}, \tag{85} \]

\( C \) and \( r_0 \) being integration constants.

One can see that black hole solutions exists only for negative cosmological constant, but in the case \( k = 1 \), already discussed in \[46\], where \( \Lambda \) can assume any arbitrary value. As usual, the horizon is given by the positive root \( r_H \) of \( B(r) = 0 \) with \( B'(r_H) \neq 0 \). The algebraic equation can be easily solved for the integration constant \( C \) and gives

\[ C = \left( k \frac{1-\sigma}{1-4\sigma} - \Lambda \frac{r_H^2}{3(1-2\sigma)} \right) r_H^{-\frac{1+4\sigma}{1-\sigma}}. \tag{86} \]

The equation (83) evaluated on the horizon leads to

\[ r_H B'(r_H) = k - \Lambda \frac{r_H^2}{1-\sigma}. \tag{87} \]

Thus, since the Killing-Hawking temperature \( T_K = a_H B'(r_H)/4\pi \), taking into account Eq. (84) we get

\[ 4\pi r_H T_K = \left( k - \Lambda \frac{r_H^2}{(1-\sigma)} \right) \left( \frac{r_H}{r_0} \right)^{\frac{3\sigma}{1-\sigma}}. \tag{88} \]

On the other hand, a direct computation along the line discussed in \[46\] for the case \( k = 1 \) leads to the BH entropy

\[ S_W = \frac{A_H}{4}(1+\sigma), \tag{89} \]
where, in order to deal with a positive entropy, we have to restrict to the interval \( \sigma \in (-1, 1) \).

Above, \( A_H = V V_1^2 \), in which \( V_1 = 4\pi \) (the sphere), \( V_0 = |3\pi| \), with \( \gamma \) the Teichmüller parameter for the torus, and finally \( V_{-1} = 4\pi g \), \( g > 2 \), for the compact hyperbolic manifold with genus \( g \) [47].

As a result we have

\[
T_K dS_W = \frac{V_k (1 + \sigma)}{8\pi G} \left( k - \Lambda \frac{r_H^2}{(1 - \sigma)} \right) \left( \frac{r_H}{r_0} \right)^{\frac{2}{1 + \sigma}} dr_H .
\]

(90)

Furthermore, Eq. (56) gives

\[
dC = \left( k - \Lambda \frac{r_H^2}{1 - \sigma} \right) \left( \frac{r_H}{r_0} \right)^{\frac{2}{1 + \sigma}} dr_H .
\]

(91)

As a consequence the first Law holds and

\[
E_K = \frac{V_k (1 + \sigma)}{8\pi G} C .
\]

(92)

In this class of modified gravitational models the energy of black hole is particularly simple, since the modification is describe by the dimensionless parameter \( \sigma \).

7 Topological Conformal Weyl Gravity

In this Section, first we revisit the higher gravity black hole solution of Riegert and others [50, 51], and its topological version [52].

To begin with, we write down the action of the model in the form

\[
I = \int d^4x \sqrt{-g} \left[ \gamma (R - 2\Lambda) + 3\omega F \right] ,
\]

where \( \gamma \) is an arbitrary parameter, which may be proportional to the square of Plank mass, \( \omega \) is a dimensionless parameter and \( F = C_{\mu\nu\rho\delta}C^{\mu\nu\rho\delta} \) is the square of the Weyl tensor. The pure conformal invariant model \( \gamma = 0 \) is very interesting and its phenomenology has been investigated in Ref. [53].

As in previous Section, also here we shall consider various topology and this means that the metric will have the form [50], and the arbitrary functions \( a(r), B(r) \) will be obtained from the reduced action

\[
\tilde{I} = \int dr \left[ \gamma \left( rB(r)a'(r) + ka(r) - 2\Lambda r^2 a(r) \right) + \omega \frac{A^2(r)}{r^2 a(r)} \right] ,
\]

(94)

where we have put

\[
A(r) = r^2 a(r)B'(r) + 3r^2 a'(r)B'(r) - 2ra(r)B'(r) + 2r^2 a''(r)B(r) - 2ra'(r)B(r) + 2a(r)B(r) - 2ka(r) .
\]

(95)

As a result, we are dealing with a higher order Lagrangian system, the Lagrangian depending on the first and second derivative of the unknown functions \( a(r) \) and \( B(r) \).

The equations of motion read

\[
4\frac{d^2}{dr^2} \left( \frac{AB(r)}{a(r)} \right) - 2 \frac{d}{dr} \left( \frac{A}{ra(r)} \left[ 3rB'(r) - 2B(r) \right] \right) + 2A \frac{\gamma}{r^2 a(r)} \left[ r^2 B''(r) - 2rB'(r) + 2B(r) - 2k \right] - \frac{A^2}{r^2 a(r)} + \frac{\gamma}{\omega} \left[ k - B(r) - rB'(r) - 2\Lambda r^2 \right] = 0 ,
\]

(96)
For simplicity let us look for exact solutions with $a(r) = 1$. With this Ansatz Eq. (97) can be integrated and one obtains

$$B(r) = \frac{b_1}{r} + c_0 + c_1 r + c_2 r^2,$$

(98)

$b_1$ and $c_k$ being integration constants. In order to satisfy Eq. (96) we have to distinguish the two cases $\gamma \neq 0$ (a modified Einstein gravity) and $\gamma = 0$, (pure conformal gravity), since they provide completely different solutions.

In the case $\gamma \neq 0$ Eq. (96) is satisfied only if

$$c_0 = k, \quad c_1 = c_2 = 0 \quad c_3 = -\frac{1}{3} \Lambda,$$

(99)

while $b_1$ remains a free parameter. We see that this is a topological Schwarzschild-de Sitter(AdS) black hole like solution, since

$$B(r) = k - \frac{C}{r} - \frac{1}{3} \Lambda r^2,$$

(100)

where here $b_1$ has been replaced by $C$. It has to be noted that this is the solution which one would have obtained from the Hilbert-Einstein action with cosmological constant, that is with $\omega = 0$. 

As we already said, if $\gamma = 0$ the solution is completely different and in fact, in such a case Eq. (96) is satisfied only if

$$c_1 = \frac{c_0^2 - k^2}{3b_1}.$$

(101)

Now the solution depends on the three arbitrary parameters $c_0$, $c_2$ and $b_1$. By a redefinition of them by $c_0 \to k + 3c_0$, $c_2 \to \lambda$, $b_1 \to -C$, we write it in the form

$$B(r) = k + 3c_0 - \frac{c_0}{C} (2k + 3c_0) r + \lambda r^2 - \frac{C}{r},$$

(102)

in agreement with the topological black hole solution already found by Klemm in [52].

The event horizon exists as soon as there is positive solution $r_H$ of $B(r) = 0$. For example, if $C > 0$ and $\lambda = 1/L^2 > 0$, it is easy to show that there exists always a positive root independently on the values of $c_0$ and of $L$, while, in the opposite case $\lambda < 0$, a positive root of $B(r) = 0$ exists only if $c_0 > 0$ and the value of $|\lambda|$ is sufficiently small. The special $\lambda = 0$ case will be discussed at the end of this Section.

With regard to the computation of Entropy, assuming that there exists an event horizon $B(r_H) = 0$, with $r_H > 0$ and $B'(r_H) \neq 0$, for the pure Weyl gravity case the Wald method gives

$$S_W = 2\omega V_K \left( \frac{C}{r_H} - c_0 \right) = 2\omega V_K (x - c_0) \quad \Rightarrow \quad dS_W = 2\omega V_K \, dx,$$

(103)

where for convenience we have introduced the variable $x = C/r_H$. Here $A_H = V_k r_H^2$ ($k = 1, 0, -1$), with $V_k = 4\pi r^3$, for the sphere, $V_0 = |3\tau|$, $\tau$ being the Teichmüller parameter for the torus, and $V_{-1} = 4\pi g$, $g > 2$, for the compact hyperbolic manifold with genus $g$ [47]. The integration constant $C$ in Eq. (103) can be seen as a function of $r_H$ obtained by solving the equation $B(r_H) = 0$, which, as it follows from (102), it is a second-order algebraic equation in $C$. Of course, in order to have a positive entropy we have to choose $c_0 < C/r_H = x$ and moreover $C$ has to be positive being proportional to the energy.
Now we restrict ourselves to the $\lambda = 1/L^2 > 0$ case. In this way, by solving the equation $B(r_H) = 0$ with respect to $C$ we get

$$2x = \frac{2C}{r_H} - \frac{r_H^2}{L^2} + k + 3c_0 + \sqrt{W}, \quad W = \left(\frac{r_H^2}{L^2} + k + 3c_0\right)^2 - 4c_0(2k + 3c_0) > 0,$$

and from the latter equation it follows

$$dx = \frac{r_H}{L^2} \left(1 + \frac{r_H^2}{L^2} + k + 3c_0\right) dr_H, \quad dC = r_H dx + x dr_H.$$  \hspace{1cm} (105)

On the other hand the Hawking temperature can be written in the convenient form

$$T_K = \frac{B'(r_H)}{4\pi} = \frac{1}{4\pi r_H} \left(\frac{2r_H^2}{L^2} + \sqrt{W}\right),$$

and using Eqs. (103) and (105) we obtain

$$T_K dS_W = \frac{\omega V_k}{2\pi L^2} \left(\frac{3r_H^2}{L^2} + k + 3c_0 + \sqrt{W} + \frac{2r_H^2}{L^2} \frac{r_H^2}{L^2} + k + 3c_0\right) dr_H = \frac{\omega V_k}{\pi L^2} (r_H dx + x dr_H).$$  \hspace{1cm} (107)

We finally see that the First Law of black hole thermodynamics reads

$$T_K dS_W = \frac{\omega V_k}{\pi L^2} dC.$$  \hspace{1cm} (108)

As a result, we may again identify the energy as

$$E_K = \frac{\omega V_k}{\pi L^2} C.$$  \hspace{1cm} (109)

We conclude this Section with some remarks. The pure Weyl conformal gravity does not contain dimensional parameters. Thus, one could think that there exists a trivial entropy and a vanishing energy, but, as we have shown above, the solution gives rise to a length scale $L$ related to the integration constant $\lambda$. In such a case the First Law of black hole thermodynamics holds and the energy of black hole solution is proportional to the other dimensional constant of integration $C$.

The situation is different when $\lambda = 0$, since in such a case the scale does not emerge and for the horizon one gets

$$\frac{r_H}{C} = \frac{k + 3c_0 + \sqrt{(c_0 + k)(k - 3c_0)}}{2c_0(2k + 3c_0)}.$$  \hspace{1cm} (110)

The latter equation gives a positive $r_H$ for $k \neq 0$ and a suitable value for $c_0$. In any case we see that $x = C/r_H$ is a pure number and so $dx = 0$ and the entropy is trivially constant. The First Law of black hole thermodynamics is trivially valid with a vanishing energy. This is the particular case discussed in [52].

8 Conclusions

In this paper the issue of defining the energy associated with a static, spherically symmetric black hole solution in higher order modified gravitational models has been tackled. We have proposed to identify the black hole energy as a quantity proportional to the constant of integration, which appears in the explicit black hole solution. The identification is substantiated by the fact that
in all explicit and known examples, we have been able to show that the First Law of black hole thermodynamics (Clausius relation) holds true as a consequence of equations of motion, and evaluating in an independent way the related entropy via Wald method and the Killing-Hawking temperature via quantum mechanics techniques in curved space time. In the case of $F(R)$ modified gravity some non trivial exact black hole solutions have been considered. In the case of modified gravity in which the quadratic Weyl scalar is additively present, first we have found the corresponding new topological black hole solution, and then we have verified for it our proposal. Finally our proposal has been shown to work also in another non trivial higher order gravity theory, namely the topological conformal Weyl gravity. It is easy to show that the proposal is also working for constant curvature black holes solutions in the usual Einstein gravity with cosmological constant modified by generic curvature-squared terms (see, for example [54]).

On general grounds, we may say that our explicit results are in agreement with the general result obtained recently in Ref. [28], and together other results appeared in literature seem to indicate that the thermodynamic origin of a generalized modified gravity, when horizons are present, has a broad validity.

9 Appendix I: the Tunneling method

In this Appendix, for the sake of completeness, we present a short review of the tunneling method in its Hamilton-Jacobi variant [25]. The method is based on the computation of the classical action $I$ along a trajectory starting slightly behind the trapping horizon but ending in the bulk, and the associated WKB approximation ($\hbar = 1$)

$$\text{Amplitude} \propto e^{i\frac{I}{\hbar}}. \quad (111)$$

The related semi-classical emission rate reads

$$\Gamma \propto |\text{Amplitude}|^2 \propto e^{-\frac{2\hbar I}{\omega}}. \quad (112)$$

The imaginary part of the classical action is due to deformation of the integration path according to the Feynman prescription, in order to avoid the divergence present on the horizon. As a result, one asymptotically gets a Boltzmann factor, in which an energy $\omega$ appears, i.e.

$$\Gamma \propto e^{-\frac{2\hbar}{\omega}}, \quad (113)$$

and the Hawking temperature is $T = \frac{1}{\beta}$.

To evaluate the action $I$, let us start with a generic static, spherically symmetric solution in $D$-dimension, written in Eddington-Finkelstein gauge, which, as it is well known, is regular gauge on the horizon

$$ds^2 = -B(r)e^{2\alpha(r)}dv^2 + 2e^{\alpha(r)}dr dv + r^2d\Omega^2_{D-2} = \gamma^{ij}(x^i)dx^i dv + r^2d\Omega^2_{D-2}. \quad (114)$$

Here $x^i = (v, r)$, where $v$ is the advanced time. Since we are dealing with static, spherically symmetric solution space-times, one may restrict to radial trajectories, and only the two-dimensional normal metric is relevant, and the Hamilton-Jacobi equation for a (massless) particle is

$$\gamma^{ij}\partial_i I \partial_j I = +2e^{\alpha(r)}\partial_r I \partial_r r + e^{2\alpha(r)}B(r)(\partial_r I)^2 = 0. \quad (115)$$

Thus

$$\partial_r I = \frac{2\omega}{e^{\alpha(r)}B(r)}. \quad (116)$$

in which $\omega = -\partial_v I$ is the Killing energy of the emitted particle. In the near horizon approximation, $B(r) \simeq B_H'(r - r_H)$. As a consequence, making use of Feynman prescription for the simple pole in $r - r_H$, one has

$$I = \int dr \partial_r I = \int dr \frac{2\omega}{e^{\alpha(r)}B_H'(r - r_H - i\varepsilon)}. \quad (117)$$
where the range of integration over $r$ contains the location of the horizon $r_H$. Thus

$$\Im I = \frac{2\pi\omega}{e^{\alpha R} B_H'},$$

and the Hawking-Killing temperature is

$$T_K = \frac{e^{\alpha R} B_H' H}{4\pi}.$$

If one had introduced the Kodama energy $\omega_H = e^{-\alpha R} \omega$, one would have obtained the Hayward temperature $T_H = \frac{B_H'}{4\pi}$.

10 Appendix II: the Wald Entropy

In this Appendix, we recall the basic formula for computing black hole entropy for a generalized covariant theory of gravity. Following [27, 38, 39], the explicit calculation of the black hole entropy $S_W$ is provided by the formula

$$S_W = -2\pi \int_S \left( \frac{\delta \mathcal{L}}{\delta R_{\alpha\beta\gamma\delta}} \right)_H e_{\alpha\beta} e_{\gamma\delta} dS,$$

(120)

where $\mathcal{L} = \mathcal{L}(R_{\alpha\beta\gamma\delta})$ is the Lagrangian density of any general theory of gravity, $e_{\alpha\beta} = -e_{\beta\alpha}$, is the binormal vector to the (bifurcate) horizon. It is normalized so that $e_{\alpha\beta} e^{\alpha\beta} = -2$.

For the metric (12), the binormal turns out to be

$$e_{\alpha\beta} = e^{\alpha(r)} (\delta_0^\alpha \delta_1^\beta - \delta_1^\alpha \delta_0^\beta).$$

(121)

The induced area form, on the bifurcate surface $r = r_H$, $t = constant$, is represented by $dS$.

Finally, the subscript $(H)$ indicates that the partial derivative is evaluated on the horizon, and the variation of the Lagrangian density with respect to $R_{\alpha\beta\gamma\delta}$ is performed as if $R_{\alpha\beta\gamma\delta}$ and the metric $g_{\alpha\beta}$ are independent. For example,

$$\frac{\delta R}{\delta R_{\mu\nu\alpha\beta}} = \frac{1}{2} \left( g^{\alpha\mu} g^{\nu\beta} - g^{\alpha\beta} g^{\mu\nu} \right).$$

(122)

As a result, for the modified gravity model of the $F(R)$ class, one obtains

$$S_W = \frac{A_H F_H'}{4G}.$$

(123)

This is the formula used in Section III and IV.

Acknowledgments

We thank L. Vanzo for discussions. We also would like to thanks the referee for several useful remarks that have permitted to improve the final version of the paper.

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