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**BOUNDARY LAYERS IN WEAK SOLUTIONS OF HYPERBOLIC CONSERVATION LAWS**

K.T. Joseph$^1$ and P.G. LeFloch$^2$

**Abstract.** This paper is concerned with the initial-boundary value problem for a nonlinear hyperbolic system of conservation laws. We study the boundary layers that may arise in approximations of entropy discontinuous solutions. We consider both the vanishing viscosity method and finite difference schemes (Lax-Friedrichs type schemes, Godunov scheme). We demonstrate that different regularization methods generate different boundary layers. Hence, the boundary condition can be formulated only if an approximation scheme is selected first. Assuming solely uniform $L^\infty$ bounds on the approximate solutions and so dealing with $L^\infty$ solutions, we derive several entropy inequalities satisfied by the boundary layer in each case under consideration. A Young measure is introduced to describe the boundary trace. When a uniform bound on the total variation is available, the boundary Young measure reduces to a Dirac mass. Form the above analysis, we deduce several formulations for the boundary condition which apply whether the boundary is characteristic or not. Each formulation is based a set of admissible boundary values, following Dubois and LeFloch’s terminology in “Boundary conditions for nonlinear hyperbolic systems of conservation laws”, *J. Diff. Equa.* **71** (1988), 93–122. The local structure of those sets and the well-posedness of the corresponding initial-boundary value problem are investigated. The results are illustrated with convex and nonconvex conservation laws and examples from continuum mechanics.

1. **Introduction.**

This paper considers the initial-boundary value problem for an hyperbolic system of conservation laws

$$
\partial_t u + \partial_x f(u) = 0, \quad u(x, t) \in \mathcal{U} \subset \mathbb{R}^N, \quad x > 0, \quad t > 0,
$$

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1. an initial condition at time $t = 0$

$$u(x,0) = u_I(x), \quad x > 0,$$

(1.2)

2. the entropy inequality

$$\partial_t U(u) + \partial_x F(u) \leq 0,$$

(1.3)

3. and a weak form of the following Dirichlet boundary condition at $x = 0$

$$u(0,t) = u_B(t), \quad t > 0.$$

(1.4)

Indeed the hyperbolic problem (1.1)–(1.4) is usually not well-posed when the boundary data is required to be assumed in the (strong) sense (1.4), even when (1.1) is a linear system (cf. Kreiss [28]). It is the objective of this paper to provide a general framework which leads to (mathematically correct) formulations for the boundary condition. Following Dubois-LeFloch [15], our strategy is to reformulate (1.4) in the (weak) form

$$u(0+,t) \in \mathcal{E}(u_B(t)), \quad t > 0,$$

(1.5)

where $\mathcal{E}(u_B(t)) \subset \mathcal{U}$ is a time-dependent set (the set of admissible boundary values) to be defined from the boundary data, and $u(0+,t)$ is the trace (its existence is discussed in this paper) of the solution $u$ at the boundary. We shall consider several methods of approximation for the problem (1.1)–(1.4), including the artificial vanishing viscosity method and a class of finite difference schemes, for which the boundary condition (1.4) can be easily implemented. As the approximation parameter goes to zero, a sharp transition layer generally develops near the boundary $\{x = 0\}$ and the limiting solution does not satisfy the boundary condition (1.4). Our aim in this paper is to provide some contribution to the following program: perform a rigorous analysis of the boundary layer for weak solutions, then derive several suitable definitions for the set in (1.5), and finally investigate the structure of the latter to decide whether the boundary-value problem is well-posed.

In (1.1), $\mathcal{U}$ is assumed to be a convex and open subset of $\mathbb{R}^N$, the flux-function $f : \mathcal{U} \to \mathbb{R}^N$ to be a smooth mapping, and the initial data $u_I$ to belong to $L^\infty(\mathbb{R}^+, \mathcal{U})$. It will be convenient to assume that the boundary data $u_B$ has bounded total variation on any interval $[0, T]$ for all $T > 0$. It is assumed that (1.1) admits at least one strictly convex entropy pair. By definition, a pair of functions $(U, F) : \mathcal{U} \to \mathbb{R} \times \mathbb{R}$ of class $C^2$ is called a convex (or strictly convex) entropy pair iff $\nabla F^T = \nabla U^T \nabla f$ and the Hessian matrix $\nabla^2 U$ is non-negative (or positive definite). The existence of at least one strictly convex entropy pair implies that (1.1) is hyperbolic. For background on hyperbolic systems, we refer to Lax [29, 30, 31], Dafermos [11] and Smoller [45], concerning the theory of existence of entropy solutions to the pure Cauchy problem, to Glimm [21] and Liu [40] for initial data with small total variation, and DiPerna [12,13] for systems of two equations with $L^\infty$ initial data.

This paper contributes to establishing a framework for the initial-boundary value problem for (1.1). It is intended to pursue the efforts initiated in recent years on this problem (Cf. review
below). In particular we built upon the recent contributions in Gisclon-Serre [20] and Xin [49], who studied the boundary layers associated with the vanishing viscosity approximations assuming the solution to the hyperbolic problem be smooth. A formal asymptotic expansion is introduced in [20, 49] and the convergence including $L^2$ error estimates is proven for the boundary layer in the smooth regime. For linear hyperbolic systems, Joseph [25] constructed boundary layers explicitly and obtained error estimates in $L^2$ Sobolev space.

One of the motivations here is to treat several approximation methods simultaneously and compare the results obtained with each of them. We consider the vanishing viscosity method, a class of Lax-Friedrichs type schemes, and the Godunov scheme.

In Section 2, we rigorously derive conditions satisfied by the boundary layer, which take the form of a family of boundary entropy inequalities and a boundary layer equation. The regularity of the relevant traces at the boundary are discussed. The whole analysis is performed by assuming only a uniform $L^\infty$ bound on the approximate solutions; in particular no assumption is required on the regularity of the limiting solution to (1.1). Since high frequency oscillations in the approximate solutions can not be a priori excluded, the conditions above are formulated in terms of a boundary Young measure associated with the boundary layer. Note that, in the derivation of Section 2, the boundary is possibly characteristic, i.e. the eigenvalues of the matrix $\nabla f(u)$ may vanish for certain values of $u$.

Observe also that, in general, the equations and inequalities we derive depend upon the approximation method in use. Fundamentally the boundary condition can not be formulated from the mere knowledge of the function $u_B$, but depend upon the underlying “physical” regularization. This feature arises in weak solutions to many nonlinear hyperbolic problems. See, for instance, the review paper by LeFloch [33] on regularization-sensitive shock waves.

In Section 3, we introduce several sets of admissible boundary values and investigate their local structure. When the boundary is non-characteristic, we establish that the sets based on the boundary layer equations are manifold with the “correct” dimension. That is, the corresponding initial-boundary value problem is well-posed, at least for constant boundary and initial data (a generalization to the Riemann problem). We also prove a similar (but stronger) result for the set based on the boundary layer equation derived by the Godunov scheme. Strictly speaking this scheme does not produce any boundary layer; however analyzing that scheme leads to a formulation of the boundary condition as it was first pointed out in [15, 16]. We recall that setting the boundary condition via an upwinding difference scheme is a classical idea in the computing literature.

Sections 4 is devoted to studying several examples of particular interest. It is expected that, in general, different approximation method for (1.1) leads to a different set in (1.5). However we prove in Section 4, for both convex and non-convex conservation laws, that this is not the case when $N = 1$. In other words the boundary layer for the scalar conservation laws is independent of the approximation method. The same is true of the linear hyperbolic systems; and we conjecture that this also holds for the nonlinear systems in the class with coinciding shock and rarefaction curves introduced by Temple [48]. We also consider examples from continuum mechanics, i.e. the system of nonlinear elasticity and the system of gas dynamics.

To complete this presentation, we give a short overview of the literature on the boundary
conditions for (1.1). Most of the activity was restricted to scalar equations, i.e. $N = 1$. The pioneering work by Leroux [35] and Bardos-Leroux-Nedelec [4] based on the vanishing viscosity method provides a derivation of "the" correct formulation of the boundary condition for multidimensional scalar conservation laws. Specifially, [4] shows that (1.4) should be replaced by the weaker statement:

$$\left( \text{sgn}(u(0+, t) - k) - \text{sgn}(u_B(t) - k) \right) \left( f(u(0+, t)) - f(k) \right) \geq 0 \quad \text{for all } k \in \mathbb{R}, \quad (1.6)$$

where $\text{sgn}(a) = -1$ if $a < 0$, $\text{sgn}(a) = 0$ if $a = 0$, and $\text{sgn}(a) = 1$ if $a > 0$. The convergence of finite difference schemes, again for scalar equations, is established by Leroux in an unpublished work: it is remarkable that the finite difference scheme approach leads to the same formulation (1.6) of the boundary condition. The condition is used by LeFloch [32] in order to extend Lax’s explicit formula [30] to the initial-boundary value problem. Joseph [24] used the vanishing viscosity method and the Hopf-Cole transformation to extend Lax’s formula for the inviscid Burgers equation. Another derivation is given by Joseph and Veerappa Gowda [27]; see also Gisclon [18] and LeFloch-Nedelec [34]. We also refer to the paper [47] by Szepessy for a very general result of existence and uniqueness.

The statement (1.6) is a special case (when applied to Kruzkov entropies) of a more general inequality:

$$F(u(0, t)) - F(u_B(t)) - \nabla U(u_B(t)) \left( f(u(0, t)) - f(u_B(t)) \right) \leq 0, \quad (1.7)$$

which has to hold for every convex entropy pair $(U, F)$. The latter was derived formally using the vanishing viscosity method in Dubois-LeFloch [15], who pointed out that (1.7) holds even when $N \geq 2$ and introduced the notion of set of admissible boundary values, cf. (1.5). These inequalities were obtained independently by Bourdel-Delorme-Mazet [8] based on an analysis of the characteristics of the system (1.1), and by Benabdallah [5] for a specific system. The first result of existence for the initial-boundary value problem for a system was given by Benabdallah-Serre [6, 7]: the vanishing viscosity method applied to the $p$-system of gas dynamics converges to a solution to (1.1) satisfying the set of inequalities (1.7).

The Glimm scheme with various type of boundary conditions was studied by Liu, for instance [37, 38, 39]. In the case that the boundary is assumed to be non-characteristic and the number of boundary conditions is equal to the number of positive eigenvalues of the matrix $\nabla f$, Goodman proves the convergence of the Glimm scheme in his unpublished thesis [22]; cf. also Dubroca-Gallice [17] and Sablé-Tougeron [42, 43].

More recently Amadori [1, 2] used the formulation in [15] and proved the convergence of a front tracking scheme in the characteristic case. In particular, Amadori establishes that a condition of the form (1.5) can be satisfied pointwise except at countably many times.

Finally we refer to the IMA report [26] by the authors for an extended version of the present article.

2. Boundary Layers in Weak Solutions.

In this section, we consider sequences of approximate solutions to the initial boundary value problem (1.1)–(1.4), and aim at characterizing their limiting behavior near the boundary. Here we rigorously derive entropy inequalities satisfied by the boundary layer. We deal with
a sequence of $L^\infty$ functions with uniformly bounded amplitude. As is well-known, for general systems of conservation laws, proving the strong convergence of a sequence of approximate solutions is an open problem. It seems therefore natural to formulate those entropy inequalities in terms of a Young measure (for instance Ball [3] for this concept) associated with the sequence of approximate solutions. Further analysis can be performed on a case by case basis only.

In the following, certain averages will be shown to belong to the space $BV(\mathbb{R}_+)$ of functions of locally bounded total variation, i.e. measurable and bounded functions $w : \mathbb{R}_+ \to \mathbb{R}$ whose distributional derivative is a bounded Borel measure on every interval $(0, T)$ for all $T > 0$. We denote by $TV_0^T(w)$ the total variation, and by $\|w\|_{BV(0,T)} = \|w\|_{L^\infty(0,T)} + TV_0^T(w)$ the norm, of a BV function $w$ on an interval $(0, T)$. By convention, a BV function will be always normalized by selecting its right continuous representative.

2.1 Vanishing Viscosity Method. Let $u^\epsilon$ be the approximate solutions obtained by solving the following parabolic regularization of (1.1)-(1.4):

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial^2_{xx} u^\epsilon, \quad x > 0, t > 0,$$

$$u^\epsilon(x, 0) = u^\epsilon_t(x), \quad x > 0,$$

$$u^\epsilon(0, t) = u^\epsilon_B(t), \quad t > 0.$$  

The smooth functions $u^\epsilon_I \in L^\infty(\mathbb{R}_+)$ and $u^\epsilon_B \in BV(\mathbb{R}_+)$ are chosen to be uniformly bounded and a.e. convergent approximations of the corresponding data $u_I$ and $u_B$. We assume the existence of a (smooth enough) solution $u^\epsilon$ to the problem (2.1)-(2.3). Note that compatibility conditions at $(x, t) = (0, 0)$, such as $u^\epsilon_I(0) = u^\epsilon_B(0)$, are implicitly required. We shall also assume that

$$u^\epsilon \text{ is uniformly bounded in } L^\infty(\mathbb{R} \times \mathbb{R}_+).$$

We introduce a new function $v^\epsilon$ by setting

$$v^\epsilon(y, t) = u^\epsilon(\epsilon y, t),$$

so that the system of equations (2.1) transforms into

$$\epsilon \partial_t v^\epsilon + \partial_y f(v^\epsilon) = \partial^2_{yy} v^\epsilon.$$  

It is expected that the ($\epsilon \to 0$) limit of the $v^\epsilon$’s will give us a good description of the boundary layer at $x = 0$, at least under additional assumptions, although a different scaling may more adapted in certain circumstances.

By definition (e.g. Ball [3]), a Young measure associated with a sequence $u^\epsilon$ satisfying (2.4) is a weak-star measurable mapping $\nu$ from the $(x, t)$ plane to the space $\text{Prob}(\mathcal{U})$ of all probability measures (i.e. non-negative measures with mass one) with the property that for every continuous function $g : \mathcal{U} \to \mathbb{R}$

$$g(u^\epsilon) \rightharpoonup < \nu, g > \quad \text{weakly-}\star \text{ in } L^\infty(\mathbb{R}_+^2).$$

In view of (2.4), the functions $v^\epsilon$ also are uniformly bounded in $L^\infty(\mathbb{R} \times \mathbb{R}_+)$. We denote by $\mu$ a Young measure associated with the functions $v^\epsilon$.  

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**Theorem 2.1.** The following statements hold for all convex entropy pairs \((U, F)\) associated with the system \((1.1)\), all functions \(\theta \in BV(R_+)\), and any bounded interval \((T_1, T_2)\).

1) When \(\theta(t) \geq 0\), the distribution
\[
y \mapsto \int_{T_1}^{T_2} < \mu_{y,t}, F > \theta(t) \, dt - \frac{d}{dy} \int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t) \, dt
\]
is in fact a function of locally bounded variation and thus is defined pointwise as a right continuous function. There exists a Young measure \(\mu_{0,t}\), such that the following limit exists and is given by \(\mu_{0,t}\):
\[
\lim_{y \to 0^+} \int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t) \, dt = \int_{T_1}^{T_2} < \mu_{0,t}, U > \theta(t) \, dt.
\]

When \(\theta(t) \geq 0\), the function
\[
x \mapsto \int_{T_1}^{T_2} < \nu_{x,t}, F > \theta(t) \, dt
\]
has locally bounded variation. There exists a Young measure \(\nu_{0,t}\), the “trace” of \(\nu_{x,t}\) at \(x = 0\), such that the following limit exists and is given by \(\nu_{0,t}\):
\[
\lim_{x \to 0^+} \int_{T_1}^{T_2} < \nu_{x,t}, F > \theta(t) \, dt = \int_{T_1}^{T_2} < \nu_{0,t}, F > \theta(t) \, dt.
\]

When \((U, F) = (u_j, f_j), 1 \leq j \leq N\), all of the results above still hold when the function \(\theta\) has no specific sign.

2) For all \(0 < y_1 < y_2\) and in the sense of distributions for \(t \in R_+\), one has
\[
F(u_B) + \nabla U(u_B)(< \nu_{0,t}, f > - f(u_B)) \geq < \mu_{y_1,t}, F > - \partial_y < \mu_{y_1,t}, U > \geq < \mu_{y_2,t}, F > - \partial_y < \mu_{y_2,t}, U > \geq < \nu_{0,t}, F >.
\]

3) Moreover one has
\[
\mu_{0,t} = \delta_{u_B(t)} \quad a.e. \ t \in R_+
\]
and, when \(\theta \geq 0\),
\[
\lim_{y \to \infty} \left( \int_{T_1}^{T_2} < \mu_{y,t}, F > \theta(t) \, dt - \frac{d}{dy} \int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t) \, dt \right) \geq \int_{T_1}^{T_2} < \nu_{0,t}, F > \theta(t) \, dt.
\]
\(\square\)
Theorem 2.1 provides a rigorous basis to the formal asymptotic expansion approach. We collect here several important remarks, including the property that the Young measures \( \nu \) and \( \mu \) reduce to Dirac masses when a uniform total variation bound is available.

First of all, the inequalities (2.8) actually hold in the (stronger) sense:

\[
\int_{T_1}^{T_2} \left( F(u_B(t)) + \nabla U(u_B(t))(< \nu_{0,t}, f > - f(u_B(t))) \right) \theta(t) \, dt \\
\geq \int_{T_1}^{T_2} < \mu_{y_1,t}, F > \theta(t) \, dt - \frac{d}{dy} \left( \int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t) \, dt \right) \bigg|_{y=y_1} \\
\geq \int_{T_1}^{T_2} < \mu_{y_2,t}, F > \theta(t) \, dt - \frac{d}{dy} \left( \int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t) \, dt \right) \bigg|_{y=y_2} \\
\geq \int_{T_1}^{T_2} < \nu_{0,t}, F > \theta(t) \, dt
\]

for all non-negative \( \theta \in BV(\mathbb{R}_+) \) and all \( 0 < y_1 < y_2 \). Observe that this is a stronger statement than the convergence in the sense of distributions since \( \theta \) is a function of bounded total variation, not necessarily having compact support in \((T_1, T_2)\), rather than a smooth function with compact support. All the formulas to be derived in this section hold in this sense. Note also that (2.10) is an immediate consequence of (2.8) by taking \( y \to \infty \).

The following inequalities, rigorously derived in Theorem 2.1,

\[
F(u_B) + \nabla U(u_B)(< \nu_0, f > - f(u_B)) \geq < \nu_0, F >
\]

will be referred to as the boundary entropy inequalities. They do not refer explicitly to the boundary layer itself but only to its limiting values.

The inequalities (2.8) also contain constraints for the boundary layer. In particular, using the trivial entropies \((U, F) = \pm(u_j, f_j(u)))\), \(1 \leq j \leq N\), in (2.8) leads us to the equation

\[
< \mu, f > - \partial_y < \mu, \text{id} > = < \nu_{0,t}, f >,
\]

where the right hand side is independent of the variable \( y \) and only depends on \( t \).

For scalar equations and when the method of compensated compactness due to Murat-Tartar applies (i.e., mainly, for systems of two conservation laws), it is known that \( \nu \) is a Dirac mass concentrated at a point \( u(x, t) \) which is an entropy weak solution. In those two situations, it is conceivable that the Young measure \( \mu \) also would be a Dirac mass.

If one assumes that \( \mu \) is a Dirac mass, say

\[
\mu_{y,t} = \delta_{v(y,t)} \quad \text{for almost every } (y, t)
\]

with \( v \in L^\infty \), then the formulas in Theorem 2.1 take a much simpler form. Namely if (2.12) holds, then (2.12) becomes what will be referred to as boundary layer equation:

\[
f(v) - \partial_y v = < \nu_0, f >.
\]

(2.14)
This is nothing but the equation that would be obtained formally by plugging an asymptotic expansion of the form \( u_\epsilon(x, t) = u(x, t) + v(x/\epsilon, t) + O(\epsilon) \) in the equations (2.1). More generally, if (2.12) holds, the inequalities (2.8) become
\[
F(u_B) + \nabla U(u_B)(< \nu_0, f > - f(u_B)) \geq F(v(y_1)) - \partial_y U(v|y=y_1) \\
\geq F(v(y_2)) - \partial_y U(v|y=y_2) \\
\geq < \nu_0, F > .
\]

When \( \nu_0 \) also is a Dirac mass for a.e. \( t \), say \( \nu_{0,t} = \delta_{u_0(t)} \), for instance when \( u \) has bounded variation in \( x \) and so admits a trace at \( x = 0 \) in a classical sense, then the boundary layer equation (2.14) becomes
\[
f(v) - \partial_y v = < \nu_0, f > .
\]
and the boundary entropy inequalities (2.11) take the form
\[
F(u_0) - F(u_B) - \nabla U(u_B)(f(u_0) - f(u_B)) \leq 0,
\]
which was derived in Dubois-LeFloch [14, 15] by assuming a uniform BV bound on the \( u_\epsilon \).

Note finally that the behavior of \( \mu_{y,t} \) as \( y \to \infty \) is controlled by the set of inequalities (2.10), only. If it is assumed that \( v \) has a limit in a classical sense and \( \partial_y v(y, t) \to 0 \) as \( y \to \infty \), then we can set
\[
v_\infty(t) \equiv \lim_{y \to \infty} v(y, t)
\]
and (2.10) becomes
\[
F(v_\infty) \geq F(u_0) \quad \text{for all entropy flux } F
\]
(the flux \( F \) must be associated with a convex entropy). In fact (2.17) need not imply
\[
v_\infty(t) = u_0(t).
\]
However (2.17) does imply
\[
f(v_\infty(t)) = f(u_0(t))
\]
so, in the non-characteristic case i.e. when \( \nabla f \) is invertible, (2.17) implies (2.17'). In the characteristic case, (2.17') may very well be violated. This difficulty is related to the choice of the scaling in the definition of the functions \( v_\epsilon \). Cf. the examples in Sections 4 and 5.

Uniform bounds on the total variation of \( u_\epsilon \) are available for scalar equations, linear systems and systems in the so-called Temple’s class having coinciding shock and rarefaction curves. In the general case we have:

**Corollary 2.2.** Assume that the solutions to the boundary-value problem (2.1)-(2.3) additionally satisfy the bound
\[
TV(u_\epsilon(t)) := \int_0^\infty |\partial_x u_\epsilon(t)| dx \leq C,
\]
where $C$ is independent of $\epsilon$. Then the Young measures $\mu$ and $\nu_0$ in Theorem 2.1 reduce to Dirac masses, i.e.,

$$\mu_{y,t} = \delta_{v(y,t)} \quad \text{and} \quad \nu_{0,t} = \delta_{u_0(t)},$$

and the functions $v = v(y,t)$ and $u_0 = u_0(t)$ satisfy the conditions (2.14) and (2.16) for almost every $y, t$.

\begin{proof}[Proof of Theorem 2.1] We decompose the proof into several steps. For the whole of this proof, we denote by $(U, F)$ a given convex entropy pair.

\textbf{Step 1: Preliminaries.}

We gather here several properties of $\nu$ and $\mu$ that are readily obtained. Let us multiply the equation (2.6) by the gradient of $U$ and obtain

$$\epsilon \partial_t U(v^\epsilon) + \partial_y (U'(v^\epsilon) - \partial_y U(v^\epsilon)) = -\nabla^2 U(v^\epsilon) \cdot (\partial_y v^\epsilon, \partial_y U^\epsilon) \leq 0.$$  \hspace{1cm} (2.18)

Using the definition of the Young measure $\mu$, it is a simple matter to pass to the limit in the inequality (2.18). For any $\theta \in BV$ and uniformly in $y \in \mathbb{R}_+$, we have

$$\epsilon \left| \int_{T_1}^{T_2} \partial_t U(v^\epsilon) \theta \, dt \right| \leq \int_{T_1}^{T_2} \epsilon U(v^\epsilon) \partial_t \theta \, dt + \epsilon \left[ U(v^\epsilon) \theta \right]_{T_1}^{T_2} \leq O(1) \| \theta \|_{BV} \| U(v^\epsilon) \|_{L^\infty} \to 0,$$

so we obtain

$$\partial_y \left( \int_{T_1}^{T_2} < \mu_{y,t}, F > \theta \, dt - \frac{d}{dy} \int_{T_1}^{T_2} < \mu_{y,t}, U > \theta \, dt \right) \leq 0,$$  \hspace{1cm} (2.19)

which provides the second inequality in (2.8). Therefore time-averages of the function $< \mu_{y,t}, F > - \partial_y < \mu_{y,t}, U >$ are non-increasing, and so have bounded variation on any compact set. The limits as $y \to 0+$ or $y \to +\infty$ exist, although at this stage of the proof, we can not exclude that those limits could be $\pm \infty$. We shall see later that actually $< \mu_{y,t}, F > - \partial_y < \mu_{y,t}, U > \in L^\infty$. Moreover the function

$$\int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t) \, dt$$

has a trace at $y = 0$, which defines $< \mu_{0,t}, U >$. Note also that (2.19) with the choices $(U, F) = \pm(u_j, f_j)$, $1 \leq j \leq N$, leads us to

$$< \mu_{y,t}, f > - \partial_y < \mu_{y,t}, \text{id} > = C_*(t),$$  \hspace{1cm} (2.20)

where $C_*(t)$ has to be determined. In fact it will be immediate from the results in Step 5 below that

$$C_*(t) = < \nu_{0,t}, \text{id} > \quad \text{for a.e. } t > 0.$$
Similarly, following DiPerna [13] and using the Young measure $\nu_{x,t}$ associated with $u^\epsilon$, one can pass to the limit in (2.1) and obtain the entropy inequality:

$$\partial_t < \nu_{x,t}, U > + \partial_x < \nu_{x,t}, F > \leq 0.$$  \hfill (2.21)

From (2.21), we deduce first that, for any smooth function $\theta(t) \geq 0$,

$$\frac{d}{dx} \int_{T_1}^{T_2} < \nu_{x,t}, F > \theta(t) dt \leq \int_{T_1}^{T_2} < \nu_{x,t}, U > \partial_t \theta(t) dt \leq O(1) \| \theta \|_{BV}.$$  \hfill (2.22)

For $\theta$ fixed, the right hand side of (2.22) is a constant, thus its left hand side is a locally bounded Borel measure and the function

$$g_\theta(x) \equiv \int_{T_1}^{T_2} < \nu_{x,t}, F > \theta(t) dt$$

has bounded total variation. Therefore the trace $\nu_{0,t}$ introduced in Theorem 2.1 exists, at least on entropy fluxes. This gives a meaning to the last term in the right hand side of (2.8). In fact it is possible to establish the estimate

$$TV(g_\theta) \leq O(1) \| \theta \|_{BV}$$

for arbitrary functions $\theta \in BV$. (For such $\theta$, (2.22) can be obtained directly from (2.1).) Thus the trace $\nu_{0,t}$ exists for $\theta \in BV$ as well.

Observe that the traces $\mu_{0,t}$ and $\nu_{0,t}$ are uniquely determined on entropies and entropy fluxes, respectively. They can be easily extended as Young measures defined on the whole set of continuous functions, in a non-unique way however. Namely, to construct $\mu_{0,t}$, take any sequence $y_k \to 0$ and consider a Young measure associated with the sequence of measures $\{\mu_{y_k,t}\}$.

This completes the proof of the part 1) in Theorem 2.1.

**Step 2: A General Identity.**

It remains to analyze the behavior of $\mu$ at the end point $y = 0$ which shall provide us with the desired boundary entropy inequality. We are going to use a general identity which immediately follows from the Green formula applied to (2.6).

Let $\theta(t)$ and $\varphi(y)$ be smooth functions not necessarily having compact support. We multiply the equation (2.6) by $\nabla U(v^\epsilon) \theta \varphi$ and integrate over the domain $(y_1, y_2) \times (T_1, T_2)$. Integrating by parts and re-ordering the terms, we obtain the identity

$$E_I^\epsilon + E_{II}^\epsilon + E_{III}^\epsilon = E_{IV}^\epsilon$$  \hfill (2.23)

with

$$E_I^\epsilon \equiv -\epsilon \int_{T_1}^{T_2} \int_{y_1}^{y_2} U(v^\epsilon) \partial_t \theta \varphi \, dy \, dt + \epsilon \theta(T_2) \int_{y_1}^{y_2} U(v^\epsilon(T_2)) \varphi \, dy - \epsilon \theta(T_1) \int_{y_1}^{y_2} U(v^\epsilon(T_1)) \varphi \, dy,$$  \hfill (2.24.1)
\[ E_{II}^\epsilon \equiv - \int_{T_1}^{T_2} \int_{y_1}^{y_2} F(v^\epsilon) \theta \partial_y \varphi \, dy \, dt + \varphi(y_2) \int_{T_1}^{T_2} (F(v^\epsilon(y_2)) - \partial_y U(v^\epsilon)_{y=y_2}) \theta \, dt \]
\[ - \varphi(y_1) \int_{T_1}^{T_2} (F(v^\epsilon(y_1)) - \partial_y U(v^\epsilon)_{y=y_1}) \theta \, dt, \]  
(2.24.II)

\[ E_{III}^\epsilon \equiv - \int_{T_1}^{T_2} \int_{y_1}^{y_2} U(v^\epsilon) \theta \partial_y \varphi \, dy \, dt + \partial_y \varphi(y_2) \int_{T_1}^{T_2} U(v^\epsilon(y_2)) \theta \, dt \]
\[ - \partial_y \varphi(y_1) \int_{T_1}^{T_2} U(v^\epsilon(y_1)) \theta \, dt, \]  
(2.24.III)

and

\[ E_{IV}^\epsilon \equiv - \int_{T_1}^{T_2} \int_{y_1}^{y_2} \nabla U(v^\epsilon) \cdot (\partial_y v^\epsilon, \partial_y v^\epsilon) \theta \varphi \, dy \, dt. \]  
(2.24.IV)

In case that \( \theta \geq 0 \) and \( \varphi \geq 0 \) and since \( U \) is assumed to convex, one has

\[ E_{IV}^\epsilon \leq 0, \]  
(2.25)

so we can focus attention on estimating the terms \( E_I^\epsilon, E_{II}^\epsilon \) and \( E_{III}^\epsilon \).

**Step 3:** Viscous Flux at the Boundary.

We prove here that the viscous flux at the boundary, i.e. the function \( \partial_y v^\epsilon(0, t) \), is uniformly bounded in a certain sense and we determine its weak limit as \( \epsilon \to 0 \). We use the identity (2.23)-(2.24) with the following choice of parameters:

\[ \text{supp } \theta \subset [T_1, T_2], \quad \text{supp } \varphi \subset [0, 1), \quad y_1 = 0, \quad y_2 = 1, \quad (U, F) = (u_j, f_j), \quad 1 \leq j \leq N. \]

For \( \varphi \) fixed, we obtain

\[ |E_I^\epsilon| \leq O(\epsilon) \| \theta \|_{BV}, \]
\[ E_{II}^\epsilon = - \int_{T_1}^{T_2} \int_0^1 f(v^\epsilon) \theta \partial_y \varphi \, dy \, dt - \varphi(0) \int_{T_1}^{T_2} (f(u_B^\epsilon) - \partial_y v^\epsilon(0, .)) \theta \, dt \]
\[ = O(1) \| \theta \|_{L^\infty} - \varphi(0) \int_{T_1}^{T_2} (f(u_B^\epsilon) - \partial_y v^\epsilon(0, .)) \theta \, dt, \]

and

\[ E_{III}^\epsilon = - \int_{T_1}^{T_2} \int_0^1 v^\epsilon \theta \partial_y \varphi \, dy \, dt - \int_{T_1}^{T_2} u_B^\epsilon \theta \partial_y \varphi(0) \, dt \]
\[ = O(1) \| \theta \|_{L^\infty}. \]

Since in this case \( E_{IV}^\epsilon = 0 \) and choosing \( \varphi \) so that \( \varphi(0) \neq 0 \), it follows

\[ \left| \int_{T_1}^{T_2} (f(u_B^\epsilon) - \partial_y v^\epsilon(0, .)) \theta \, dt \right| \leq O(1) \| \theta \|_{L^\infty} + O(\epsilon) \| \theta \|_{BV}. \]  
(2.26)
More precisely we can pass to the limit in the identity (2.23) and get

\[ \varphi(0) \lim_{\epsilon \to 0} \int_{T_1}^{T_2} (f(u_B) - \partial_y v^\epsilon(0,t)) \theta \, dt \]

\[ = - \int_{T_1}^{T_2} \int_0^1 \langle \mu, f > \theta \partial_y \varphi \, dy \, dt - \int_{T_1}^{T_2} \int_0^1 \langle \mu, \text{id} > \theta \partial_y \varphi \, dy \, dt - \int_{T_1}^{T_2} u_B \theta \, dt. \]

On the other hand, it has been observed in Step 1 that (2.20) holds and \( \langle \mu, \text{id} > \theta \partial_y \varphi \) has a trace at \( y = 0 \). Thus one has

\[ \int_{T_1}^{T_2} \int_0^1 \langle \mu, f > \theta \partial_y \varphi \, dy \, dt + \int_{T_1}^{T_2} \int_0^1 \langle \mu, \text{id} > \theta \partial_y \varphi \, dy \, dt \]

\[ = \int_{T_1}^{T_2} \int_0^1 C_\ast(t) \theta \partial_y \varphi \, dy \, dt - \int_{T_2}^{T_1} \mu_0, \text{id} > \theta \partial_y \varphi(0) \, dt \]

\[ = - \int_{T_1}^{T_2} C_\ast(t) \theta \varphi(0) \, dt - \int_{T_1}^{T_2} \mu_0, \text{id} > \theta \partial_y \varphi(0) \, dt \]

and therefore

\[ \varphi(0) \lim_{\epsilon \to 0} \int_{T_1}^{T_2} (f(u_B) - \partial_y v^\epsilon(0,t)) \theta \, dt \]

\[ = \varphi(0) \int_{T_1}^{T_2} C_\ast(t) \theta \, dt + \partial_y \varphi(0) \int_{T_1}^{T_2} \mu_0, \text{id} > \theta \, dt - \partial_y \varphi(0) \int_{T_1}^{T_2} u_B \theta \, dt. \]

Choosing two test-functions \( \varphi \), one such that \( \varphi(0) = 0 \) but \( \partial_y \varphi(0) \neq 0 \), and the other such that \( \varphi(0) \neq 0 \) but \( \partial_y \varphi(0) = 0 \), we deduce from the above formula that

\[ \lim_{\epsilon \to 0} \int_{T_1}^{T_2} (f(u_B) - \partial_y v^\epsilon(0,t)) \theta \, dt = \int_{T_1}^{T_2} C_\ast(t) \theta \, dt \]

\[ \int_{T_1}^{T_2} \mu_0, \text{id} > \theta \, dt = \int_{T_1}^{T_2} u_B \theta \, dt. \]  

(2.27)

The first statement in (2.27) is the desired convergence result. The second statement is a first step toward proving (2.9).

**Step 4:** Boundary Entropy Inequalities (I).

Using (2.27), we are now able to obtain the boundary entropy inequalities. We use the identity (2.23)-(2.24) with

\[ \theta \geq 0, \quad \text{supp } \theta \subset [T_1, T_2], \quad \varphi \geq 0, \quad \text{supp } \varphi \subset [0, \infty), \quad y_1 = 0, \quad y_2 > 0, \]
and \((U, F)\) arbitrary. We obtain

\[
|E'_I| \leq O(\epsilon) \|\theta\|_{BV},
\]

\[
E''_I = -\int_{T_1}^{T_2} \int_0^{y_2} F(v') \theta \partial_y \varphi \, dydt - \varphi(0) \int_{T_1}^{T_2} (F(u_B') - \partial_y U(v_y=0)) \theta \, dt
\]

\[
= -\int_{T_1}^{T_2} \int_0^{y_2} F(v') \theta \partial_y \varphi \, dydt - \varphi(0) \int_{T_1}^{T_2} \left( F(u_B') - \nabla U(u_B') \partial_y v(0,.) \right) \theta \, dt
\]

\[
\to -\int_{T_1}^{T_2} \int_0^{y_2} \mu, F > \theta \partial_y \varphi \, dydt - \varphi(0) \int_{T_1}^{T_2} \left( F(u_B) - \nabla U(u_B)(f(u_B) - C_*(.)) \right) \theta \, dt,
\]

where we have used (2.27) and the fact that \(u_B' \in BV\) converges strongly to \(u_B \in BV\), and

\[
E''_I = -\int_{T_1}^{T_2} \int_0^{y_2} U(v') \theta \partial_{yy} \varphi \, dydt - \int_{T_1}^{T_2} U(u_B) \theta \partial_y \varphi(0) \, dt.
\]

Since \(E''_I \leq 0\) we pass to the limit in (2.23) and get

\[
\varphi(0) \int_{T_1}^{T_2} \left( F(u_B) - \nabla U(u_B)(f(u_B) - C_*(t)) \right) \theta \, dt,
\]

\[
\geq -\int_{T_1}^{T_2} \int_0^{y_2} \left( <\mu_{y,t}, F > \partial_y \varphi + <\mu_{y,t}, U > \partial_{yy} \varphi \right) \, dydt
\]

\[
+ \varphi(y_2) \int_{T_1}^{T_2} \left( <\mu_{y_2,t}, F > - \partial_y <\mu_{y,t}, U > y=y_2 \right) \theta \, dt
\]

\[
+ \partial_y \varphi(y_2) \int_{T_1}^{T_2} <\mu_{y_2,t}, U > \theta \, dt - \partial_y \varphi(0) \int_{T_1}^{T_2} U(u_B) \theta \, dt.
\]

On one hand, using the test-function \(\varphi(y) \equiv 1\), we deduce that

\[
\int_{T_1}^{T_2} \left( F(u_B) - \nabla U(u_B)(f(u_B) - C_*(t)) \right) \theta \, dt \geq \int_{T_1}^{T_2} \left( <\mu, F > + \partial_y <\mu, U > y=y_2 \right) \theta \, dt
\]

(2.28)

which proves the first inequality in (2.8).

On the other hand, using the function \(\varphi(y) = y\), we obtain

\[
0 \geq -\int_{T_1}^{T_2} \int_0^{y_2} \mu_{y,t} F > \theta \, dydt + y_2 \int_{T_1}^{T_2} \left( <\mu_{y_2,t}, F > - \partial_y <\mu_{y,t}, U > y=y_2 \right) \theta \, dt
\]

\[
+ \int_{T_1}^{T_2} \mu_{y_2,t} U > \theta \, dt - \int_{T_1}^{T_2} U(u_B) \theta \, dt,
\]

which as \(y_2 \to 0\) yields

\[
\int_{T_1}^{T_2} U(u_B) \theta \, dt \geq \lim_{y_2 \to 0^+} \int_{T_1}^{T_2} \mu_{y,t} U > \theta \, dt.
\]

(2.29)
In particular, plugging \((U, F) = (u_j, f_j), 1 \leq j \leq N\), in (2.29), we recover the second statement in (2.27), which used together with (2.29) for any fixed, strictly convex entropy \(U\) gives:

\[
\int_{T_1}^{T_2} < \mu_{0,t}, U - U(u_B) - \nabla U(u_B)(id - u_B) > \theta \, dt
\]

\[
= \lim_{y \to 0^+} \int_{T_1}^{T_2} < \mu_{y,t}, U - U(u_B) - \nabla U(u_B)(id - u_B) > \theta \, dt
\]

\[
\leq \int_{T_1}^{T_2} U(u_B) \theta \, dt - \int_{T_1}^{T_2} U(u_B) \theta \, dt
\]

\[
= 0.
\]

But the function \(u \to U(u) - U(u_B) - \nabla U(u_B)(u - u_B)\) is positive everywhere except at \(u_B\) where it achieves its global minimum value. It follows that \(\mu_{0,t}\) is a Dirac mass concentrated at \(u_B\). That proves (2.9).

**Step 5:** Boundary Entropy Inequalities (II).

We now establish the third inequalities in (2.8). We use once more the identity (2.23)-(2.24) with now \(\theta \geq 0, \text{ supp } \theta \subset [T_1, T_2], \varphi \geq 0, \text{ supp } \varphi \subset [y_1, \infty), \ y_1 > 0, \ y_2 = \infty,\)

with a function \(\varphi\) depending on \(\epsilon\), that is

\[
\varphi^\epsilon(y, t) \equiv \tilde{\varphi}(\epsilon y, t)
\]

with \(\tilde{\varphi}\) fixed. In that situation one can check that

\[
E^\epsilon_I = - \int_{T_1}^{T_2} \int_{y_1}^{\infty} U(u^\epsilon) \partial_t \varphi \, dx dt
\]

\[
\rightarrow - \int_{T_1}^{T_2} \int_{0}^{\infty} < \nu_{x,t}, U > \partial_t \varphi \, dx dt,
\]

\[
E^\epsilon_{II} = - \int_{T_1}^{T_2} \int_{y_1}^{\infty} F(u^\epsilon) \theta \partial_x \varphi \, dx dt
\]

\[
- \tilde{\varphi}(\epsilon y_1) \int_{T_1}^{T_2} (F(v^\epsilon) - \partial_y U(v^\epsilon)|_{y=y_1}) \theta \, dt
\]

\[
\rightarrow - \int_{T_1}^{T_2} \int_{0}^{\infty} < \nu_{x,t}, F > \theta \partial_x \varphi \, dx dt - \tilde{\varphi}(0) \int_{T_1}^{T_2} (\mu_{y_1,t,F} > -\partial_y < \mu, U >_{y=y_1}) \theta \, dt,
\]

and

\[
E^\epsilon_{III} = - \epsilon \int_{T_1}^{T_2} \int_{y_1}^{\infty} U(u^\epsilon) \theta \partial_{xx} \varphi \, dx dt - \partial_x \tilde{\varphi}(\epsilon y_1) \int_{T_1}^{T_2} U(v^\epsilon)|_{y=y_1} \theta \, dt
\]

\[
\rightarrow 0.
\]
Since $E^t_{IV} \leq 0$ and 

$$- \int_{T_1}^{T_2} \int_0^\infty < \nu_{x,t}, F > \theta \partial_x \tilde{\varphi} \, dx \, dt = \int_{T_1}^{T_2} < \nu_0, F > \theta \, dt + O(1) \| \tilde{\varphi} \|_{L^1},$$

we obtain an inequality of the form

$$\tilde{\varphi}(0) \int_{T_1}^{T_2} \left(< \mu_{y_1,t}, F > - \partial_y < \mu, U > \big|_{y=y_1}\right) \theta \, dt \geq \tilde{\varphi}(0) \int_{T_1}^{T_2} < \nu_0, F > \theta \, dt + O(1) \| \tilde{\varphi} \|_{L^1},$$

which proves the third inequality in (2.8) by choosing $\tilde{\varphi} \geq 0$ such that $\| \tilde{\varphi} \|_{L^1} \to 0$ but $\tilde{\varphi}(0) > 0$.

This completes the proof of Theorem 2.1. $\square$

**Remark.** Additional uniform estimates and regularity can be obtained from the identity in Step 2 of the proof of Theorem 2.1. Let $(U, F)$ be a non-negative entropy pair that is uniformly convex on $U$. Use the identity (2.23)-(2.24) with

$$\theta \equiv 1, \quad T_1 = 0, \quad T_2 = T, \quad \varphi \equiv 1, \quad y_1 = 0, \quad y_2 = \infty.$$

We assume additionally here that, for a fixed state $u_\infty$ and for all $t$,

$$u^\varepsilon(x, t) \to u_\infty, \quad u_x^\varepsilon(x, t) \to 0 \quad \text{as } x \to \infty.$$

The initial data $u_I$ should also decay rapidly at infinity. We obtain the following identity

$$\varepsilon \int_0^T U(v^\varepsilon(y, T)) \, dy - \varepsilon \int_0^\infty U(v^\varepsilon(y, 0)) \, dy + \int_0^T F(u_\infty) \, dt$$

$$- \int_0^T (F(u_B^\varepsilon) - \nabla U(u_B^\varepsilon) \partial_y v^\varepsilon(0, \cdot)) \, dt + \int_0^T \int_0^\infty \nabla^2 U(v^\varepsilon) \cdot (\partial_y v^\varepsilon, \partial_y v^\varepsilon) \, dy \, dt = 0.$$
For every Lipschitz continuous function \( g \), it follows from (2.31) that the sequence \( \partial_y g(v^\varepsilon) \) is bounded in \( L^2 \), so converges weakly to a limit which is nothing but \( \partial_y <\mu, g> \):

\[
\partial_y g(v^\varepsilon) \rightharpoonup \partial_y <\mu, g> \quad \text{weak in } L^2(R \times R_+).
\] (2.32)

\[\square\]

2.2. Finite Difference Schemes. We now extend the above analysis to several classes of finite difference schemes that are known to be consistent with the entropy inequality (1.3). Theorem 2.3 below deals with the entropy flux-splittings introduced by Chen-LeFloch [9], which also includes as a special case the Lax Friedrichs type schemes. We treat the Godunov scheme in Theorem 2.4.

We are given two mesh parameters \( \tau \) and \( h \) with \( \lambda \equiv \tau/h \) kept constant and small enough in order to guarantee the stability of the scheme. We define the approximate solutions \( u^h(x,t) \) by the scheme

\[
\begin{align*}
  u^h(x,t+\tau) &= u^h(x,t) - \lambda g(u^h(x,t), u^h(x+h,t)) + \lambda g(u^h(x-h,t), u^h(x,t)) \\
&= u^h(x,t) - \lambda g(v^h(\varepsilon)(v^\varepsilon, \varepsilon)), \\
  \text{and the initial and boundary conditions:} & \\
  u^h(x,t) &= u_I(x) \quad \text{for all } t < \tau, \\
  u^h(x,t) &= u_B(t) \quad \text{for all } x < h.
\end{align*}
\] (2.33)

By convention, the functions \( u^h \) are right continuous. For the Lax-Friedrichs type schemes, the numerical flux \( g \) is given by

\[
g_{\text{Lax}}(v,w) = \frac{1}{2}(f(v) + f(w)) - \frac{Q}{\lambda}(w - v),
\] (2.35)

where \( Q \in (0, 1) \) is called the numerical coefficient of the scheme. (Symmetric positive definite matrices \( Q \) could also be dealt with.) For the flux-splitting schemes, \( g \) takes the form

\[
g_{\text{split}}(v,w) = f^-(w) + f^+(v),
\] (2.36)

where \( f = f^- + f^+ \) is a given entropy flux-splitting for the system (1.1). By definition [9], the matrix \( \nabla f^\pm \) have real eigenvalues and a basis of eigenvectors and there exists a pair of functions \( F^\pm \) such that \( (U, F^\pm) \) is an entropy pair for the system associated with flux-functions \( f^\pm \). Observe that (2.35) is a special case of (2.36) as was pointed out by Chen-LeFloch.

As in the analysis of Section 2.1, we assume a uniform \( L^\infty \) bound:

\[
\|u^h\|_{L^\infty(R \times R_+)} \leq O(1). \tag{2.37}
\]

We rescale \( u^h \) and define the function \( v^h : R \times R_+ \to U \) by

\[
v^h(y,t) = u^h(yh,t) \quad y \geq 0, t \geq 0.
\]
Let \( \nu \) and \( \mu \) be two Young measures associated with \( u^h \) and \( v^h \), respectively.

The entropy flux-splitting schemes satisfy discrete entropy inequalities of the form

\[
U(u^h(x,t + \tau)) - U(u^h(x,t + \tau)) + \lambda \left( G(u^h(x,t), u^h(x+h,t)) - G(u^h(x-h,t), u^h(x,t)) \right) \leq 0,
\]

where \( G \) is called the numerical entropy flux. With obvious notation, we have

\[
G_{\text{Lax}}(v, w) = \frac{1}{2}(F(v) + F(w)) - \frac{Q}{\lambda} (U(w) - U(v))
\]

(2.35bis)

and

\[
G_{\text{split}}(v, w) = F^-(w) + F^+(v).
\]

(2.36bis)

Note that (2.38) hold for (2.36)-(2.36bis) provided \( u \) takes its value in a sufficiently small neighborhood of a given state in \( \mathcal{U} \). This is in contrast with the vanishing viscosity method where no such assumption was necessary.

Theorem 2.1 admits the following extension to the flux-splitting schemes. We omit the proof which follows the lines of the one of Theorem 2.1.

**Theorem 2.3.** Assume that \( \mathcal{U} \) is a small neighborhood of a constant state in \( \mathbb{R}^N \). The measure \( \mu_{y,t} \) is defined for all \( y \geq 0 \) and almost every \( t \), and is constant for \( y \in [k, k+1) \) for any integer \( k \). For all convex entropy pairs \((U, F)\), all \( y \geq 0 \), and in the sense of distributions in \( t \in \mathbb{R}_+ \), one has

\[
F^+(u_B) + < \mu_{1,t}, F^- > \geq < \mu_{y,t}, F^+ > + < \mu_{y+1,t}, F^- > \\
\geq < \mu_{y+1,t}, F^+ > + < \mu_{y+2,t}, F^- > \\
\geq < \nu_{0,t}, F >,
\]

(2.39)

and

\[
\mu_{0,t} = \delta_{u_B(t)} \quad \text{for a.e. } t > 0,
\]

(2.40)

\[
\lim_{y \to +\infty} < \mu_{y,t}, F^+ > + < \mu_{y+1,t}, F^- > \geq < \nu_{0,t}, F >.
\]

(2.41)

Consider next the Godunov scheme corresponding to the flux \( g \) given by

\[
g_{\text{Godunov}}(v, w) = f(R(v, w)),
\]

(2.42)

where we denote by \( R(v, w) \) the value at \( x/t = 0+ \) of the solution to the Riemann problem with \( v \) and \( w \) as left and right initial data, respectively. The entropy flux is

\[
G_{\text{Godunov}}(v, w) = F(R(v, w)),
\]

(2.42bis)

Here it is more convenient to consider the values \( R(u^h(x,t), u^h(x+h,t)) \) and define a function \( w^h \)

\[
w^h(y,t) = R(u^h(yh,t), u^h(yh+h,t))
\]

(2.43)

for all \( y \geq 0 \). We denote by \( \pi \) a Young measure associated with \( w^h \) and by \( \nu \) a Young measure for \( u^h \). It is not difficult to extend Theorem 2.3 as follows:
Theorem 2.4. The measure $\pi_{y,t}$ is defined for all $y \geq 1/2$ and almost every $t$, and is constant in $y$ for $y \in [k-1/2, k+1/2)$ for any integer $k \geq 1$. For all convex entropy pairs $(U, F)$, all $y \geq 1/2$, and in the sense of distributions in $t \in \mathbb{R}_+$, one has

$$< \pi_{1/2,t}, F > \geq < \pi_{y,t}, F > \geq < \pi_{y+1,t}, F > \geq < \nu_{0,t}, F >,$$

(2.44)

and, at $y = 1/2$ and $y = \infty$, $\pi$ satisfies

$$< \pi_{1/2,t}, F > = \lim_{h \to 0} R(u_B, v^h(1,t)),$$

(2.45)

and

$$\lim_{y \to \infty} < \pi_{y,t}, F > \geq < \nu_{0,t}, F >.$$

(2.46)

We conclude this section by giving the main conditions satisfied by the discrete boundary layer, which will be studied in the rest of this paper.

Assuming in the results of Theorem 2.3 that $\mu$ is a Dirac mass, say $\mu = \delta_v$, the discrete boundary layer equation associated with the scheme (2.33) takes the form:

$$g(v(y-1), v(y)) - g(v(y), v(y+1)) = 0 \quad \text{for all } y \geq 1,$$

$$v(y) = u_B, \quad y \in [0, 1),$$

(2.47)

while the discrete boundary entropy inequality is

$$G(u_B, v_1) \geq F(u_0),$$

(2.48)

where $v_1$ plays the role of a parameter. Formally, Theorem 2.4 leads to the same equations (2.47)-(2.48) with flux and entropy-fluxes given by (2.42).

3. Sets of Admissible Boundary Values.

Based on the results in Section 2, we introduce in this section several sets which can be used to formulate the boundary condition. For every method of approximation considered in Section 2, we introduce two different sets of admissible boundary values:

1. One is based on the entropy inequalities, $\mathcal{E}_{\text{entropy}}(u_B)$ and yields a boundary condition of the form (1.5). This boundary condition is rigorously satisfied by the limiting function generated by a sequence of approximate solution, as was proven in Section 2. For arbitrary systems having few or even just one entropy, the set $\mathcal{E}_{\text{entropy}}(u_B)$ may be too large to lead to a well-posed problem;

2. Another set, $\mathcal{E}_{\text{layer}}(u_B)$, is based on the boundary layer equation, which was obtained formally after the analysis in Section 2. This set is more adapted to deal with general systems and lead to a well-posed problem when the boundary is non characteristic.
In this section, we study the local structure of those sets; under certain assumptions, we can prove that the sets $\mathcal{E}^{\text{layer}}(u_B)$ are manifolds with dimension equal to the number of negative wave speeds of the system (1.1). This ensures that the initial-boundary value problem is well posed if, for instance, the data are constant states (boundary Riemann problem) as can be seen by applying the theory in [36]. We recall that (1.1) is assumed to be strictly hyperbolic throughout this section and we denote by $\lambda_j(u)$ the $N$ real and distinct eigenvalues of the matrix $\nabla f(u)$ and by $\ell_j(u)$ and $r_j(u)$ corresponding basis of left and right eigenvectors.

### 3.1 Vanishing Viscosity Method.

For the sake of generality, we consider

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_x \left( B(u^\varepsilon) \partial_x u^\varepsilon \right), \quad x > 0, \, t > 0. \quad (3.1)$$

Theorem 2.1 could be partially extended to this case. We assume that the viscosity matrix $B(u)$ depends smoothly upon its argument $u$ and is positive. We consider entropies $U$ that are $B$-convex in the sense that $\nabla^2 U(u)B(u) > 0$ for all $u$ under consideration. The boundary layer equation here takes the form

$$\partial_y f(v) = \partial_y (B(v)\partial_y v) \quad (3.2)$$

and the boundary entropy inequalities have the same form (2.16) but now $U$ must be $B$-convex.

Following Dubois-LeFloch [15], we introduce a set based on the boundary entropy inequalities. From now on, the time-dependence may be omitted.

**Definition 3.1.** Given $u_B \in \mathcal{U}$, the set of admissible boundary values based on the entropy inequalities associated with the vanishing viscosity method (3.1) is

$$\mathcal{E}^{\text{entropy}}(u_B) = \{ u_0 \in \mathcal{U}; \text{ for all } B\text{-convex } (U,F), \, F(u_B) + \nabla U(u_B) (f(u_0) - f(u_B)) \geq F(u_0) \}. \quad (3.3)$$

It is obvious that this set may be quite large when the system (1.1) only admits few entropies. For most systems ($N \geq 3$), this set is too large to be used to formulate the boundary condition. In any case, it is difficult to get information on its local structure at $u_B$. For general systems, the following observation is immediate. Fix a state $u_B \in \mathcal{U}$ and suppose that for some $p$ one has

$$\lambda_p(u_B) < 0 < \lambda_{p+1}(u_B) \quad (3.4)$$

and the basis $r_j(u)$ is a family of eigenvectors for $B(u)$. Then the set obtained by formally plugging the expansion

$$f(u_0) \approx f(u_B) + \nabla f(u_B)(u_0 - u_B) + \nabla^2 f(u_B) \cdot (u_0 - u_B, u_0 - u_B),$$

$$F(u_0) \approx F(u_B) + \nabla F(u_B)(u_0 - u_B) + \nabla^2 F(u_B) \cdot (u_0 - u_B, u_0 - u_B) \quad (3.5)$$

in the definition of $\mathcal{E}^{\text{entropy}}(u_B)$ contains $u_B +$ the span of $r_j(u_B), \, j = 1, \ldots, p$ and is contained in a cone with vertex $u_B$. Indeed the inequality under consideration in (3.3) then becomes

$$\nabla^2 U(u_B) \nabla f(u_B)(u_0 - u_B, u_0 - u_B) \leq 0. \quad (3.6)$$
Since $U$ is a convex entropy, the eigenvalues of $\nabla f(u_B)$ satisfy (3.4), and $\nabla^2 U(u_B)$ is a positive definite matrix, our claim follows.

We also consider a second set of admissible boundary values, first introduced by Gisclon and Serre [20].

**Definition 3.2.** Given any $u_B \in \mathcal{U}$, the set of admissible boundary values $\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B)$, based on the boundary layer equation associated with the vanishing viscosity method is the set of all $v_\infty \in \mathcal{U}$ such that the problem

$$
B(v) \partial_y v = f(v) - f(v_\infty),
$$

$$
v(0) = u_B,
$$

$$
\lim_{y \to \infty} v(y) = v_\infty.
$$

admits a (smooth) solution $v(y) \in \mathcal{U}$ for $y \geq 0$. \hfill \square

To study the local structure of $\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B)$, we apply the following theorem concerning the existence of invariant manifolds. Cf. Hartman [23] for a proof.

**Theorem 3.3.** Consider the differential equation

$$
\frac{d\xi}{dy} = E\xi + H(\xi; \xi_0), \quad \xi(y) \in \mathbb{R}^N, \quad y \in \mathbb{R},
$$

where $H : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is of class $C^1$ and for each $\xi_0$

$$
H(0, \xi_0) = \frac{dH}{d\xi}(0, \xi_0) = 0,
$$

and $E$ is a constant square matrix with $d$ eigenvalues having negative real part, $e$ eigenvalues having positive real part, and $N - d - e$ eigenvalues having zero real part. For every (small enough) $\xi_0 \in \mathbb{R}^N$, let $\xi_y = \xi(y; \xi_0)$ be the solution of (3.7) with the initial condition $\xi(0; \xi_0) = \xi_0$. Denote by $T_y$ the mapping $\xi_0 \rightarrow \xi(y; \xi_0)$.

There exists a one-to-one mapping of class $C^1$, $S : \xi \rightarrow S(\xi) = (w^I, w^{II}, w^{III})$, having non-vanishing Jacobian and defined on a neighborhood of $\xi = 0 \in \mathbb{R}^N$ onto a neighborhood of $(w^I, w^{II}, w^{III}) = (0, 0, 0) \in \mathbb{R}^d \times \mathbb{R}^{N-d-e} \times \mathbb{R}^e$, such that the mapping $ST_y S^{-1}$ takes the simple form

$$
ST_y S^{-1} : \quad w^I_y = e^{P^I} y w^I_0 + W^I(y; w^I_0, w^{II}_0, w^{III}_0),
$$

$$
w^{II}_y = e^{P^{II}} y w^{II}_0 + W^{II}(y; w^I_0, w^{II}_0, w^{III}_0),
$$

$$
w^{III}_y = e^{P^{III}} y w^{III}_0 + W^{III}(y; w^I_0, w^{II}_0, w^{III}_0),
$$

where $P^I$, $P^{II}$, and $P^{III}$ are constant real-valued matrices with all eigenvalues having moduli less than one so that the matrix exponentials $e^{P^I}$, $e^{P^{II}}$, and $e^{P^{III}}$ are well-defined, the absolute
value of any eigenvalue of $e^{D't}$ is less than 1, and that for $e^{D''t}$ is greater than 1, and that for $e^{D''t}$ is exactly 1. Moreover the mapping $W', W''$, and $W'''$ are of class $C^1$ and their first order partial derivatives with respect to $(w_0^I, w_0^I, \bar{w}_0^I)$ vanish at $(0, 0, 0)$. Moreover one has

$$W^I = 0 \quad \text{and} \quad W'' = 0 \quad \text{if} \quad w_0^I = 0 \quad \text{and} \quad \bar{w}_0^I = 0,$$

(3.11)

and

$$W'' = 0 \quad \text{and} \quad W''' = 0 \quad \text{if} \quad w_0^I = 0 \quad \text{and} \quad \bar{w}_0^I = 0.$$

(3.12)

The condition (3.11) means that the $e$-dimensional plane $\{w_0^I = 0, \bar{w}_0^I = 0\}$ is a locally invariant manifold. If $S(\xi_0)$ belongs to this plane, then $|\xi(y; \xi_0)| \to \infty$ as $y \to \infty$. The manifold $\{\xi / w_0^I = 0, \bar{w}_0^I = 0\}$ is called the unstable manifold of initial data for the equation (3.8).

The condition (3.12) means that the $d$-dimensional plane $\{w_0^I = 0, \bar{w}_0^I = 0\}$ is a locally invariant manifold. If $S(\xi_0)$ belongs to this plane, then $\xi(y; \xi_0) \to 0$ as $y \to \infty$. The manifold $\{\xi / w_0^I = 0, \bar{w}_0^I = 0\}$ is called the stable manifold.

Using Theorem 3.3 we prove the following result.

**Theorem 3.4.** Let $u_B \in U$ be given and assume that, for all $u$ in a small neighborhood of $u_B$,

the basis $r_j(u)$ is a family of eigenvectors for $B(u)$,

(3.13)

where

$$\lambda_p(u) < 0 \leq \lambda_{p+1}(u)$$

(3.14)

holds for some $p$. Then the set $E_{\text{viscosity}}^\text{layer}(u_B)$ contains the point $u_B$ and, locally nearby $u_B$, contains a manifold with dimension $p$ at least. When $0 < \lambda_{p+1}(u_B)$, $E_{\text{viscosity}}^\text{layer}(u_B)$ is a manifold with dimension exactly $p$ and its tangent space at the point $u_B$ is spanned by the eigenvectors $r_j(u_B)$, $j = 1, 2, \cdots, p$.

Assumption (3.13) holds for the examples considered later in this paper, but could easily be relaxed. A result similar to our Theorem 3.4 is also proved by Gisclon in [19], by another method.

**Proof of Theorem 3.4.** The system in (3.6) can be written in the form

$$\frac{d\tilde{v}}{dy} = B(v_\infty)^{-1}\nabla f(v_\infty)\tilde{v} + G(\tilde{v}, v_\infty),$$

$$\tilde{v}(0) = u_C - v_\infty,$$

$$\tilde{v}(\infty) = 0,$$

(3.15)

where $\tilde{v}(y) = v(y) - v_\infty$ and the mapping $G(\tilde{v}, v_\infty)$ satisfies $G(0, v_\infty) = 0, \frac{\partial G}{\partial \tilde{v}}(0, v_\infty) = 0$. In view of the assumption (3.13), the two matrices $\nabla f(v_\infty)$ and $B(v_\infty)^{-1}\nabla f(v_\infty)$ have the same eigenvectors, and so exactly the same number of positive, zero, and negative eigenvalues. Let

$$\hat{\lambda}_j(v_\infty) = b_j(v_\infty)^{-1}\lambda_j(v_\infty)$$
be the eigenvalues of $B(v_\infty)^{-1}\nabla f(v_\infty)$. Applying Theorem 3.3 with

$$\xi(y; \xi_0) = \tilde{v}(y; u_B - v_\infty),$$

we see that there exists a one-to-one $C^1$ mapping $S$, defined on a neighborhood of $0 \in \mathbb{R}^N$, onto a neighborhood of $(w^I, w^{II}, w^{III}) = (0, 0, 0) \in \mathbb{R}^p \times \mathbb{R}^{N-p-1} \times \mathbb{R}^1$, such that the manifold

$$E \equiv \{ \tilde{v} / w^{II}(\tilde{v}) = 0, \quad w^{III}(\tilde{v}) = 0 \},$$

which is of dimension $p$, is stable. For any point $u_B - v_\infty$ taken in this manifold as an initial data for the differential equation in (3.15), the solution $\tilde{v}(y)$ converges to 0 as $y \to \infty$, which is the third condition required in (3.15).

If $v_\infty$ belongs to this manifold, then (3.15) has a solution and hence $v_\infty$ solves the boundary layer problem. Furthermore the local structure of the set nearby $u_B$ can be described as follows.

Suppose that $0 < \lambda_{p+1}(u_B)$. The following estimate follows from (3.15):

$$\tilde{v}(y) = \sum_{j=1}^N e^{\lambda_j y} \ell_j(v_\infty) \cdot (u_B - v_\infty) r_j(v_\infty) + 0((\tilde{v}(y))^2. \quad (3.16)$$

For the right handside of (3.16) to go to zero, we must have

$$g_j(v_\infty) \equiv \ell_j(v_\infty) \cdot (u_B - v_\infty) = 0, \quad j = p + 1, \cdots N. \quad (3.17)$$

Keeping $u_B$ fixed, consider the map $g : \mathcal{U} \to \mathbb{R}^{N-p}$ with components $g_j$ given by (3.17). We have

$$\frac{dg}{dv_\infty}(u_B) = -(\ell_{p+1}(u_B), \cdots, \ell_N(u_B)), \quad (3.18)$$

whose rank is $N - p$. By the implicit function theorem, (3.17) defines a manifold passing through $u_B$ and of dimension $p$. By construction its tangent space at $u_B$ coincides with the one for the stable manifold $E$. Therefore, in view of (3.18), the tangent space at $u_B$ for $E$ is spanned by the $r_j(u_B), j = 1, 2, \cdots, p$. \hfill \qed

A general inclusion can be proven regarding the sets introduced in the previous sections. It has been first pointed out by Serre [44] (cf. also [19]) that:

**Proposition 3.5.** The two family of sets introduced in Definitions 3.1 and 3.2 satisfy the inclusion

$$E^\text{layer}_{\text{viscosity}}(u_B) \subset E^\text{entropy}_{\text{viscosity}}(u_B) \quad (3.19)$$

for all $u_B \in \mathcal{U}$. \hfill \qed
Proof of Proposition 3.5. Let \( v_\infty \) be a point in \( \mathcal{E}_{\text{layer}}(u_B) \) and denote by \( y \to v(y) \) the associated boundary layer function which satisfies \( v(0) = u_B \) and \( v(\infty) = v_\infty \). Consider the following function of the variable \( y > 0 \):

\[
\Omega(y) \equiv F(v_\infty) - F(v(y)) - \nabla U(v(y)) (f(v_\infty) - f(v(y))).
\]

It is easy to see that

\[
\frac{d\Omega}{dy}(y) = \nabla^2 U(v(y)) \left( f(v_\infty) - f(v(y)), f(v_\infty) - f(v(y)) \right) \geq 0
\]

So the function \( \Omega \) is non-decreasing, and since \( \lim_{y \to \infty} \Omega(y) = 0 \), we deduce that \( \Omega(y) \leq 0 \) for all \( y \), in particular for \( y = 0 \), that is

\[
F(v_\infty) - F(u_B) - \nabla U(u_B)(f(v_\infty) - f(u_B)) \leq 0.
\]

Thus \( v_\infty \) belongs to \( \mathcal{E}_{\text{entropy}}(u_B) \). \( \square \)

3.2 Finite Difference Schemes. We now turn to formulations of the boundary condition that are based on finite difference approximations. We use the notation in Section 2.2. We consider a scheme characterized by its mesh parameters \( \tau \) and \( h \) with \( \lambda = \tau/h \) small enough, and by its numerical flux \( g(.,.) \) and its family of numerical entropy fluxes \( G(.,.) \). It is tacitly assumed that the values \( u \) remain in a small neighborhood of a given state and attention is restricted to those entropies \( U \) such that the discrete entropy inequalities (2.38) are satisfied. In fact attention is mostly restricted to the Lax-Friedrichs type schemes and the Godunov scheme.

Definition 3.6. Given \( u_B \in \mathcal{U} \), the set of admissible boundary values based on the entropy inequalities associated with difference scheme is

\[
\mathcal{E}_{\text{scheme}}(u_B) = \{ u_0 \in \mathcal{U} ; \text{ there exists } v_1 \text{ s.t. for all convex } (U,F), G(u_B, v_1) \geq F(u_0) \}. \tag{3.20}
\]

As for \( \mathcal{E}_{\text{scheme}}(u_B) \), this set may be too large to guarantee that the boundary value problem is well posed. We also use the obvious notation \( \mathcal{E}_{Lax}(u_B) \), \( \mathcal{E}_{\text{splitting}}(u_B) \), and \( \mathcal{E}_{\text{Godunov}}(u_B) \).

For general systems and the diagonalizable splittings, i.e. those such that the vectors \( r_j \) form a basis of eigenvectors for the matrices \( \nabla f^\pm \), we have the following fact. Consider a Lax-Friedrichs type scheme or, more generally a diagonalizable, entropy flux-splitting scheme. Fix a state \( u_B \in \mathcal{U} \) and suppose that (3.4) holds for some \( p \). Then the set obtained by formally linearizing the inequalities in the definition of \( \mathcal{E}_{\text{scheme}}(u_B) \) contains \( u_B + \) the span of \( r_j(u_B), j = 1, \ldots, p \) and is contained in a cone with the vertex at \( u_B \). To see this we formally plug the second order expansion

\[
F^\pm(u_0) \approx F^\pm(u_B) + \nabla F^\pm(u_B)(u_0 - u_B) + \nabla^2 F^\pm(u_B)(u_0 - u_B, u_0 - u_B) \tag{3.21}
\]
and obtain the second order version of the inequalities in (3.20):

$$\nabla F(u_B)(u_0-u_B)+\nabla^2 F(u_B)(u_0-u_B, u_0-u_B) \leq \nabla F^-(u_B)(v_1-u_B)+\nabla^2 F^-(u_B)(v_1-u_B, v_1-u_B).$$

Using the trivial entropies (i.e. choose for $F$ the components of $f$), we get an (second order) expression for $v_1$:

$$\nabla f(u_B)(v_1-u_B)+\nabla^2 f(u_B)(v_1-u_B, v_1-u_B) = \nabla f(u_B)(u_0-u_B)+\nabla^2 f(u_B)(u_0-u_B, u_0-u_B),$$

which can be used to rewrite the above inequality:

$$\nabla^2 U(u_B)\nabla f(u_B)(u_0-u_B, u_0-u_B) \leq \nabla^2 U(u_B)\nabla f^{-}(u_B)(v_1-u_B, v_1-u_B).$$

At the first order, $v_1$ is given by

$$\nabla f^{-}(u_B)(v_1-u_B) = \nabla f(u_B)(u_0-u_B)$$

so we arrive at the inequality

$$-\nabla f^+(u_B)^T \nabla f^-(u_B)^{-T} \nabla^2 U(u_B)\nabla f(u_B)(u_0-u_B, u_0-u_B) \leq 0.$$ 

The desired result follows immediatly since $r_j$ is a basis of eigenvectors for the matrices $\nabla f^+$, $\nabla f^-$, and $\nabla f$, and the function $U$ is convex.

The second family of sets is now defined.

**Definition 3.7.** Given any $u_B \in U$, the set of admissible boundary values $\mathcal{E}^{layer}_{scheme}(u_B)$, based on the boundary layer equation associated with the difference scheme is the set of all $v_\infty \in U$ such that the problem

$$g(v(y), v(y+1)) = f(v_\infty),$$

$$v(y) = u_B \quad \text{for} \; y \in [0, 1),$$

$$\lim_{y \to \infty} v(y) = v_\infty,$$  \hspace{1cm} (3.22)

admits a (piecewise constant) solution $v(y) \in U$ for $y \geq 0$. \hspace{1cm} \Box

To study the local structure of $\mathcal{E}^{layer}_{scheme}(u_B)$, we apply the following theorem concerning the existence of discrete invariant manifolds. (Cf. Hartman [23] for a proof.)

**Theorem 3.8.** Let $T : \mathbb{R}^N \to \mathbb{R}^N$, $\xi_0 \to \xi_1$, be a mapping of the form

$$\xi_1 = \Gamma \xi_0 + E(\xi_0),$$  \hspace{1cm} (3.23)

where $E(\xi_0)$ is of class $C^1$ for small $\xi_0$ and satisfy $E(0) = 0$ and $\frac{DE}{D\xi_0}(0) = 0$, and the matrix $\Gamma$ is constant, non-singular, and has $d \geq 0$, $N-d-e$, $e \geq 0$ eigenvalues of absolute value less than 1, equal to 1, and greater than 1, respectively.
There exists a map $S$ of a neighborhood of $\xi_0 = 0$ onto a neighborhood of the origin in the space of $(w^I_0, w^{II}_0, w^{III}_0) \in \mathbb{R}^d \times \mathbb{R}^{N-d-e} \times \mathbb{R}^e$ such that $S$ is of class $C^1$ with non-vanishing Jacobian and $STS^{-1}$ takes the simple form

\[
STS^{-1} : \quad w^I_1 = A^I w^I_0 + W^I(w^I_0, w^{II}_0, w^{III}_0),
\]
\[
w^{II}_1 = A^{II} w^{II}_0 + W^{II}(w^I_0, w^{II}_0, w^{III}_0),
\]
\[
w^{III}_1 = A^{III} w^{III}_0 + W^{III}(w^I_0, w^{II}_0, w^{III}_0),
\]

(3.24)

where $P^I$, $P^{II}$, and $P^{III}$ are $d \times d$, $(N-d-e) \times (N-d-e)$, and $e \times e$ square matrices with eigenvalues of absolute value less than 1, equal to 1, greater than 1, respectively, and the mapping $W^I$, $W^{II}$, and $W^{III}$ are of class $C^1$ and their first order partial derivatives with respect to $(w^I_0, w^{II}_0, w^{III}_0)$ vanish at $(0, 0, 0)$. Moreover one has

\[
W^I = 0 \quad \text{and} \quad W^{II} = 0 \quad \text{if} \quad w^I_0 = 0 \quad \text{and} \quad w^{II}_0 = 0,
\]

(3.25)

and

\[
W^{II} = 0 \quad \text{and} \quad W^{III} = 0 \quad \text{if} \quad w^{II}_0 = 0 \quad \text{and} \quad w^{III}_0 = 0.
\]

(3.26)

The condition (3.25) means that the plane $v_0 = 0, w_0 = 0$ of dimension $d$ is locally invariant manifold and if $R(\xi_0)$ belongs to this manifold then $T^n \xi_0 \to 0$ as $n \to \infty$.

The condition (3.26) means that the plane $u_0 = 0, w_0 = 0$ is a locally invariant manifold and if $R(\xi_0)$ belongs to this manifold, $|T^n \xi_0| \to \infty$ as $n \to \infty$.

Using this theorem we shall prove:

**Theorem 3.9.** Consider a Lax-Friedrichs type scheme. Let $u_B \in \mathcal{U}$ be given and assume that (3.14) holds for some $p$. Then the set $E^\text{layer}_{\text{Lax}}(u_B)$ contains the point $u_B$ and, locally nearby $u_B$, contains a manifold with dimension $p$. When $0 < \lambda_{p+1}(u_B)$, $E^\text{layer}_{\text{Lax}}(u_B)$ is a manifold with dimension exactly $p$ and its tangent space at the point $u_B$ is spanned by the eigenvectors $r_j(u_B)$, $j = 1, 2, \cdots, p$. \(\square\)

**Proof of Theorem 3.9.** We search for all $v_\infty$ that solve the problem:

\[
H(v(y), v(y+1), v_\infty) = 0
\]
\[
v(0) = 0,
\]
\[
v(\infty) = v_\infty
\]

(3.27)

with

\[
H(v(y), v(y+1), v_\infty) \equiv v(y+1) - v(y) - \frac{\lambda}{2Q} (f(v(y)) + f(v(y+1)) - 2f(v_\infty)).
\]

(3.28)
Using the notation $H = H(v, w, v_{\infty})$, we compute
\[
\frac{\partial H}{\partial v}(v, w, v_{\infty}) = Id + \frac{\lambda}{2Q} \nabla f(v), \\
\frac{\partial H}{\partial w}(v, w, v_{\infty}) = Id - \frac{\lambda}{2Q} \nabla f(w).
\]

(3.29)

For $\lambda/(2Q)$ small enough, the matrix $\partial H/\partial w$ is invertible and its inverse is uniformly bounded w.r.t the variables $v$, $w$, and $v_{\infty}$. By the global implicit function theorem (see J.T. Schwartz [46]) the system (3.27) can be solved for $v(y+1)$. So there exists a smooth mapping $K(v(y), v_{\infty})$ such that
\[
v(y + 1) = K(v(y), v_{\infty})
\]
and $K(v_{\infty}, v_{\infty}) = 0$. Moreover one has
\[
\frac{\partial K}{\partial v}(v(y), v_{\infty}) = (Id - \frac{\lambda}{2Q} \nabla f(v(y+1)))^{-1}(Id + \frac{\lambda}{2Q} \nabla f(v(y))).
\]

(3.30)

The system (3.30) can be linearized around $v_{\infty}$:
\[
v(y + 1) = - (Id - \frac{\lambda}{2Q} \frac{\partial f}{\partial u}(v_{\infty}))^{-1}(Id + \frac{\lambda}{2Q} \frac{\partial f}{\partial u}(v_{\infty}))v(y)
\]
\[
+ K(v(y), v_{\infty}) + (Id - \frac{\lambda}{2Q} \frac{\partial f}{\partial u}(v_{\infty}))^{-1}(Id + \frac{\lambda}{2Q} \frac{\partial f}{\partial u}(v_{\infty}))v(y).
\]

Set $v^*(y + 1) = v(y + 1) - v_{\infty}$. The system can be written as
\[
v^*(y + 1) = - (Id - \frac{\lambda}{2Q} \frac{\partial f}{\partial u}(v_{\infty}))^{-1}(Id + \frac{\lambda}{2Q} \frac{\partial f}{\partial u}(v_{\infty}))v^*(y)
\]
\[
+ G(v^*(y) + v_{\infty}, v_{\infty}) + (Id - \frac{\lambda}{2Q} \frac{\partial f}{\partial u}(v_{\infty}))^{-1}(Id + \frac{\lambda}{2Q} \frac{\partial f}{\partial u}(v_{\infty}))v^*(y).
\]

In other words
\[
v^*(y + 1) = A(v_{\infty})v^*(y) + K^*(v^*(y), v_{\infty}),
\]
where
\[
A(v_{\infty}) \equiv (Id - \frac{\lambda}{2Q} \nabla f(v_{\infty}))^{-1}(Id + \frac{\lambda}{2Q} \frac{\partial f}{\partial u}(v_{\infty}))
\]
(3.33a)

and
\[
K^* \text{ and } \frac{\partial K^*}{\partial v^*(y)} \text{ vanish at } v^*(y) = 0.
\]

(3.33b)

We observe that
\[
\text{the eigenvalues of the matrix } A(v_{\infty}) \text{ are } \frac{1 + \frac{\lambda}{2Q} \lambda_i(v_{\infty})}{1 - \frac{\lambda}{2Q} \lambda_i(v_{\infty})},
\]
(3.34)
where $\lambda_i(v_{\infty})$ are the eigenvalues of $\nabla f(v_{\infty})$.

Namely (3.34) follows from the fact that the following two statements are equivalent:

(1) $a$ is an eigenvalue of $A(v_{\infty})$;

(2) there exists $r \neq 0$ such that $A(v_{\infty})r = ar$.

Using the expression (3.33a) of $A(v_{\infty})$ and simplifying the resulting equation, we get

$$\nabla f(v_{\infty})r = \frac{2Q(a-1)}{\lambda(1+a)} r.$$

So $a$ is an eigenvalue of $A(v_{\infty})$ if and only if $\frac{2Q(a-1)}{\lambda(1+a)}$ is an eigenvalue of $\partial f/\partial u(v_{\infty})$ with right eigenvector $r$; so

$$\frac{2Q(a-1)}{\lambda(a+1)} = \lambda_i(v_{\infty})$$

for some $i$ with left eigenvector $\ell_i(v_{\infty})$ and right eigenvector $r_i(v_{\infty})$. Solving (3.35) for $a$ we get

$$a_i = \frac{1 + \frac{\lambda}{2Q} \lambda_i(v_{\infty})}{1 - \frac{\lambda}{2Q} \lambda_i(v_{\infty})}.$$ (3.36)

Let $T$ be a matrix which diagonalize $\nabla f(v_{\infty})$. Then the same matrix diagonalize $A(v_{\infty})$:

$$TAT^{-1} = \text{diag}(a_1, a_2, \ldots, a_n).$$

Set $w^*(y+1) = Tv^*(y+1)$, we get

$$w^*(y+1) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} w^*(y) + G^*(T^{-1}w^*(y), v_{\infty}) \end{pmatrix},$$

where $G^*$ and $\frac{\partial G^*}{\partial w^*(y)}$ are zero at $w^*(y) = 0$.

Note that

$$a_1 < a_2 < \cdots < a_p < 1 \leq a_{p+1} < \cdots < a_n.$$ (3.37)

and

$$a_{p+1} = 1 \iff \lambda_{p+1}(v_{\infty}) = 0.$$

Since all the hypothesis of Theorem 3.8 are satisfied, there exists a $p$-dimensional invariant manifold defined near 0 such that, if the data $v^*_0$ belongs to this manifold, then $w^*(y+1) \to 0$ as $y \to \infty$. In fact in terms of the original variable $v(y+1)$, we have the expansion

$$v(y+1) - v_{\infty} = \sum_{j=1}^{N} a_j^y \ell_j(u), v_b - v_{\infty} > r_j(v_{\infty}) + 0(|v(y+1) - v_{\infty}|)^2.$$ (3.38)
In order for this to go to zero, as \( y \to 0 \) we must have
\[
< \ell_j(v_\infty), u_B - v_\infty > = 0, \quad j = p + 1, \cdots N. \tag{3.39}
\]
This for fixed \( u_B \) defines a map from \( \mathbb{R}^N \to \mathbb{R}^{N-p} \) and whose Jacobian at \( u_B = v_\infty \) is the matrix whose \( N-p \) rows are \( \ell_j(v_\infty) \). Since \( \ell_j(v_\infty) \) are linearly independent by implicit function theorem we deduce that (3.39) defines a \( p \) dimensional manifold passing through \( u_B \) and if \( v_\infty \) is in this manifold then there exist a solution to (3.29) whose local structure is given by (3.39).

The following general inclusion can be proven:

**Proposition 3.10.** The two family of sets introduced in Definitions 3.6 and 3.7 satisfy, for all \( u_B \in \mathcal{U} \),
\[
\mathcal{E}_{\text{layer}}(u_B) \subset \mathcal{E}_{\text{entropy}}(u_B). \tag{3.40}
\]

**Proof of Proposition 3.10.** We consider as before a difference scheme that satisfies discrete entropy inequalities. For every \( v_\infty \) in the set \( \mathcal{E}_{\text{layer}}(u_B) \), there exists a corresponding boundary layer profile \( v(y) \), solution of
\[
g(v(y), v(y+1)) = f(v_\infty).
\]
The function \( v(y) \) is actually a stationary solution to the scheme since
\[
v(y) - v(y) + \lambda(g(v(y), v(y+1)) - g(v(y-1), v(y))) = 0.
\]
Therefore for every convex entropy pair \((U, F)\), it satisfies the entropy inequality
\[
U(v(y)) - U(v(y)) + \lambda(G(v(y), v(y+1)) - G(v(y-1), v(y))) \leq 0,
\]
which is nothing but
\[
G(v(y), v(y+1)) - G(v(y-1), v(y)) \leq 0
\]
Since \( \lim_{y \to \infty} v(y) = v_\infty \), we get
\[
G(v(y), v(y+1)) \geq F(v_\infty)
\]
and so with \( y = 0 \), since \( v(y) = u_B \) for \( y \in [0, 1) \),
\[
G(u_B, v_1) \geq F(u_0)
\]
with \( v_1 = v(1) \). That establishes that \( v_\infty \) belongs to the set \( \mathcal{E}_{\text{entropy}}(u_B) \).

Finally we treat the Godunov scheme. The sets \( \mathcal{E}_{\text{Godunov}}^{\text{layer}}(u_B) \) and \( \mathcal{E}_{\text{Godunov}}^{\text{entropy}}(u_B) \) are defined by Definitions 3.6 and 3.7. We now prove:
Theorem 3.11. Consider the Godunov scheme and let \( u_B \in \mathcal{U} \) be given. We have
\[
\mathcal{E}_{\text{layer}}^{\text{Godunov}}(u_B) = \mathcal{E}_{\text{entropy}}^{\text{Godunov}}(u_B).
\] (3.41)

This set can also be described as the set
\[
\mathcal{E}_{\text{Riemann}}^{\text{Godunov}}(u_B) = \{ R(u_B, w) \mid w \in \mathcal{U} \},
\]
where \( R(u_B, w) \) denotes the value at \( x/t = 0^+ \) of the solution of the Riemann problem with data \( u_B \) and \( w \) on the left and right, respectively. Moreover when (3.4) holds for some \( p \), the set above contains the point \( u_B \) and, locally nearby \( u_B \), is a manifold with dimension \( p \) and with tangent space at the point \( u_B \) spanned by the eigenvectors \( r_j(u_B) \), \( j = 1, 2, \ldots, p \). \( \square \)

Observe that the Godunov scheme does not produce any boundary layer, in the sense that the layer contains no interior point.

Proof of Theorem 3.11. We recall that the set \( \mathcal{E}_{\text{layer}}^{\text{Godunov}}(u_B) \) is defined by the equation
\[
\begin{align*}
f(u_B) &= f(R(v(y), v(y + 1)), \\
v(y) &= u_B \quad \text{for all } y \in [0, 1), \\
\lim_{y \to \infty} v(y) &= v_\infty,
\end{align*}
\] (3.42)

while the set \( \mathcal{E}_{\text{entropy}}^{\text{Godunov}}(u_B) \) is defined by the inequalities
\[
F(R(u_B, v_1)) \geq F(u_0) \quad \text{for all convex pair } (U, F)
\] (3.43)

and for some \( v_1 \in \mathcal{U} \). So it is not hard to see from the definition that
\[
\mathcal{E}_{\text{Riemann}}^{\text{Godunov}}(u_B) \subset \mathcal{E}_{\text{layer}}^{\text{Godunov}}(u_B).
\]

On the other hand the inclusion
\[
\mathcal{E}_{\text{layer}}^{\text{Godunov}}(u_B) \subset \mathcal{E}_{\text{entropy}}^{\text{Godunov}}(u_B)
\]
also holds in view of Proposition 3.10.

It remains to show that
\[
\mathcal{E}_{\text{entropy}}^{\text{Godunov}}(u_B) \subset \mathcal{E}_{\text{Riemann}}^{\text{Godunov}}(u_B).
\]
Consider a pair \( (u_0, v_1) \) that solves (3.43). Then we need show that there exists \( w \) such that
\[
R(u_B, w) = u_0.
\] (3.44)

Using the trivial entropies, we get
\[
f(R(u_B, v_1)) = f(u_0)
\]
which, combined with the inequality (3.43), shows that the pair of states \( (R(u_B, v_1), u_0) \) is an entropy satisfying, stationary shock wave. On the other hand the Riemann problem with left and right initial data \( u_B \) and \( R(u_B, v_1) \), respectively, contains only waves with non-positive speeds. Therefore the Riemann solution, with \( u_B \) as a left state and \( u_0 \) as a right state, only contains waves with non-positive speeds. This function takes the value \( u_0 \) in the whole half-interval \( x/t > 0 \) and thus \( R(u_B, u_0) = u_0 \), which proves (3.44) with \( w = u_0 \). \( \square \)
4. Selected Examples.

In this section, we consider first the convex scalar conservation laws and establish that all the sets introduced in Section 2 are essentially the same. Some remarks are then given for the linear hyperbolic systems. Next we return to the scalar equation and treat a non-convex flux function, showing again that the sets are the same with the exception of the set based on the boundary layer equations. Finally we treat the elastodynamics system and isentropic Euler system.

4.1. Scalar Conservation laws: Convex Fluxes. We consider a scalar conservation law with strictly convex flux, i.e. \( f''(u) > 0 \) and analyze the boundary layer equation. Let \( u_* \) be the unique point such that \( f'(u_*) = 0 \). To the state \( u_B \), when \( u_B \neq u_* \), we associate the solution \( u_B^* \neq u_B \) of the equation \( f(u_B^*) = f(u_B) \).

We state here a theorem which says that some of the sets introduced in Section 3 coincide in this case. We also recover the formulation of the boundary condition discovered by Bardos-Leroux-Nedelec [4] and Leroux [35].

**Theorem 4.1.** Consider a scalar conservation laws with convex flux.

1) For any \( u_B \in U \equiv \mathbb{R} I \), the sets of admissible boundary values \( E^{\text{entropy}}(u_B) \), \( E^{\text{layer}}(u_B) \), and \( E^{\text{entropy Godunov}}(u_B) \), coincide with

\[
E^{\text{Riemann}}(u_B) = \begin{cases} 
(-\infty, u_B^*] \cup \{u_B\} & \text{if } u_B > u_*, \\
(-\infty, u_*] & \text{if } u_B \leq u_*.
\end{cases}
\]

and

\[
E^{\text{layer viscosity}}(u_B) = E^{\text{Riemann}}(u_B) \setminus \{u_B^*\}
\]

2) Given \( u_B \in U \equiv [-M, M] \) for a fixed value of \( M > 0 \), we set \( \|f'\|_\infty = \sup_{w \in [-8M,8M]} |f'(w)| \) and consider a Lax-Friedrichs type scheme with coefficient \( \lambda \) and \( Q \) satisfying \( \|f'\|_\infty \lambda/Q \leq 1 \), then

\[
E^{\text{layer}}(u_B) \cap [-M, M] = E^{\text{Riemann}}(u_B) \cap [-M, M] \setminus \{u_B^*\}
\]

\[
E^{\text{entropy}}(u_B) \cap [-M, M] = E^{\text{Riemann}}(u_B) \cap [-M, M]
\]

\[\square\]

4.2 Linear Hyperbolic Systems.

It is not hard to prove that for a linear and strictly hyperbolic system, the sets defined in Section 3 are all equivalent when boundary is not characteristic. We only consider here the case of the discrete boundary layer based on the Lax-Friedrichs scheme.

We also focus attention in this section to establish that the restriction (3.12) on the viscosity matrix is essential to our purpose here, as was observed in another context by Majda-Pego [41] in their study of traveling wave solutions to (2.1). The following example shows a situation where the viscosity matrix is a positive diagonal matrix, and does not satisfy (3.12), while the formulation may lead to a “wrong” boundary condition.
We consider the linear system
\[ \partial_t u + \begin{pmatrix} -5 & 5 \\ -3 & 3 \end{pmatrix} \partial_x u = \epsilon \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \partial_{xx} u. \] (4.1)

According to our earlier analysis, the boundary layer equation is
\[ \partial_{yy} v(y) = \begin{pmatrix} 1/5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -5 & 5 \\ -3 & 3 \end{pmatrix} \partial_y v(y), \]
i.e.
\[ \partial_{yy} v(y) = \begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix} \partial_y v(y). \]

Integrating this equation once and using \( v(\infty) = v_\infty \), we get
\[ \partial_y v(y) = \begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix} (v - v_\infty). \] (4.2)

Now the eigenvalues of \( \begin{pmatrix} -5 & 5 \\ -3 & 3 \end{pmatrix} \) are \( \lambda_1 = -2 \) and \( \lambda_2 = 0 \). On the other hand, the eigenvalues of \( \begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix} \) are \( \mu_1 = 0 \) and \( \mu_2 = 2 \). The solution of (4.2) with the initial condition \( v(0) = v_B - v_\infty \) is
\[ v(y) - v_\infty = \bar{\ell}_1 v_B - v_\infty > \bar{r}_1 + \bar{\ell}_2 v_B - v_\infty > \bar{r}_2 e^{2y}, \]
where
\[ \bar{\ell}_1 = \begin{pmatrix} -3/2, 1 \\ 1/2 \end{pmatrix}, \quad \bar{\ell}_2 = \begin{pmatrix} 1 \sqrt{2} & -1 \sqrt{2} \\ 1 \sqrt{2} & 1 \sqrt{2} \end{pmatrix}, \quad \bar{r}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad \bar{r}_2 = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}. \]

In order for \( v(y) \to v_\infty \) as \( y \to \infty \), we must have \( \bar{\ell}_1, v_B - v_\infty >= 0 \) and \( \bar{\ell}_2, v_B - v_\infty >= 0 \) which means that \( v_\infty = v_B \). This requires that we prescribe \( u \) at the boundary. This is wrong boundary condition for the hyperbolic system
\[ \partial_t u + \begin{pmatrix} -5 & 5 \\ -3 & 3 \end{pmatrix} \partial_x u = 0 \]
because none of the characteristics are entering.

Let us now consider the numerical boundary layer for a general linear and strictly hyperbolic system. Set \( f(u) = Au \), where \( A \) is a constant matrix. The boundary layer equation becomes
\[ \frac{\lambda}{2} A v(y + 1) + \frac{\lambda}{2} A v(y) - \frac{1}{2} (v(y + 1) - v(y)) = \lambda A v_\infty, \] (4.3)
For a given \( u_B \), we search for the set of states \( v_\infty \) for which this problem has a solution. Set \( v^\infty(y) = v(y+1) - v_\infty \). The first equation in (4.3) becomes

\[
(\lambda A - I)v^\infty(y) = -(\lambda A + I)v^\infty(y-1).
\]

(4.4)

Let \( \ell_j \) and \( r_j \) be the left- and right- eigenvectors for \( A \) associated with the eigenvalues \( \lambda_j \). Set \( C^j(y) = \langle \ell_j, v(y+1) \rangle \). From (4.4) we get

\[
(1 - \lambda \lambda_j)C^j(y)^\prime = (1 + \lambda \lambda_j)C^j(y-1)
\]

or

\[
C^j(y) = \left( \frac{1 + \lambda \lambda_j}{1 - \lambda \lambda_j} \right) C^j(y-1)
\]

with

\[
C^0_0 = \langle \ell_j, v_B - v_\infty \rangle.
\]

Integrating this, we get

\[
C^j(y) = \langle \ell_j, v_B - v_\infty \rangle \left( \frac{1 + \lambda \lambda_j}{1 - \lambda \lambda_j} \right)^y
\]

or

\[
v(y+1) - v_\infty = v^\infty(y) = \sum_{j=1}^n \left( \frac{1 + \lambda \lambda_j}{1 - \lambda \lambda_j} \right)^y < \ell_j, u_B - v_\infty > r_j.
\]

For \( v(y+1) \rightarrow v_\infty \), we need \( < \ell_j, v_B - v_\infty > = 0, j = p+1, \ldots, n \) because \( \lambda_1 < \lambda_2 < \cdots < \lambda_p < 0 < \lambda_{p+1} < \cdots < \lambda_n \). This gives correct boundary condition when the eigenvalues are not zero; i.e. to prescribe

\[
< \ell_j, u > \quad \text{for } j = p+1, \ldots, N.
\]

4.3 Scalar Conservation Laws: Non-Convex Fluxes.

We return to scalar conservation laws but now with non-convex fluxes. For definiteness we treat the case of the cubic flux given by

\[
f(u) = \frac{1}{2}(u^3 - 3u),
\]

(4.5)

which has one minima and one maxima; indeed

\[
f(1) = -1, \quad f'(1) = 0, \quad f''(1) = 3, \quad f(-1) = 1, \quad f'(-1) = 0, \quad f''(-1) = -3.
\]

For a given \( u_B \in \mathbb{R} \) and the function \( f \) given by (4.5), we shall need the solution of the equation

\[
f(u) = f(u_B), u \neq u_B.
\]

(4.6)
If $u_B < -2$ or $u_B > 2$, there is no solution for (4.6). If $u_B \in (-2, -1) \cup (1, 2)$, then (4.6) has exactly two solutions. In this case we denote by $u^l_B$ and $u^s_B$ the largest and smallest solutions of (4.6), respectively. If $u_B = -2, -1, 1$, or 2, then (4.6) has exactly one solution; namely 1, 2, -2, and -1, respectively.

For the formulation of the results in this subsection, it will be convenient to introduce the following set, which is either the empty set or contains a single element:

$$\begin{cases} 
\emptyset, & \text{if } u_B \in (-\infty, -2) \cup [-1, 1] \cup (2, \infty) \\
\{1\}, & \text{if } u_B = -2 \\
\{u^s_B\}, & \text{if } -2 < u_B < -1 \\
\{u^l_B\}, & \text{if } 1 < u_B < 2. \\
\{-1\}, & \text{if } u_B = 2.
\end{cases}$$

(4.7)

**Theorem 4.2.** Consider the scalar conservation law with the non-convex flux (4.5).

1) For any $u_B \in U = R$, the set of admissible boundary values $\mathcal{E}^{\text{entropy}}_{\text{viscosity}}(u_B), \mathcal{E}^{\text{entropy}}_{\text{Godunov}}(u_B)$, and $\mathcal{E}^{\text{entropy}}_{\text{Godunov}}(u_B)$ coincide with

$$\mathcal{E}^{\text{Riemann}}(u_B) = \begin{cases} 
\{u_B\}, & \text{if } u_B < -2 \\
\{-2, 1\}, & \text{if } u_B = -2 \\
[u^s_B, 1] \cup \{u_B\}, & \text{if } -2 < u_B < -1 \\
[-1, 1], & \text{if } -1 \leq u_B \leq 1 \\
[-1, u^l_B] \cup \{u_B\}, & \text{if } 1 < u_B < 2 \\
\{u_B\}, & \text{if } u_B > 2 \\
\{2, -1\}, & \text{if } u_B = 2
\end{cases}$$

and

$$\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B) = \mathcal{E}^{\text{Riemann}}(u_B) - E(u_B).$$

2) Given any state $u_B \in U = [-M, M]$ for a fixed value $M > 2$, we set $\|f'\|_\infty = \sup_{w \in [-M, M]} |f'(w)|$ and consider a Lax-Friedrichs type scheme with coefficient $\lambda$ and $Q$ satisfying $\|f'\|_\infty \lambda Q \leq 1$. Then

$$\mathcal{E}^{\text{layer}}_{\text{Lax}}(u_B) \cap [-M, M] = \mathcal{E}^{\text{Riemann}}(u_B) \cap [-M, M] \setminus E(u_B),$$

$$\mathcal{E}^{\text{entropy}}_{\text{Lax}}(u_B) \cap [-M, M] = \mathcal{E}^{\text{Riemann}}(u_B) \cap [-M, M].$$
The proof of this is straightforward and is omitted.

4.4 Nonlinear Elastodynamics. The system considered now arises in the modeling of elastic materials [10]:

\[
\begin{align*}
\partial_t v - \partial_x u &= 0, \\
\partial_t u - \partial_x \sigma(v) &= 0.
\end{align*}
\]  

(4.8)

It describes the evolution of a nonlinear material with deformation gradient \(v\) and velocity \(u\). The stress function \(\sigma\) is assumed to be smooth enough and satisfy the following conditions:

\[
\sigma'(v) > 0, \quad v \sigma''(v) > 0.
\]  

(4.9)

Let us discuss the vanishing viscosity approximation for the viscosity matrix \(B(u) = I\). The boundary layer problem to be studied here is

\[
\begin{align*}
- \partial_y u &= \partial_y^2 v, \\
- \partial_y \sigma(v) &= \partial_y^2 u, \\
v(0) &= v_B, \quad v(\infty) = v_\infty, \\
u(0) &= u_B, \quad u(\infty) = u_\infty.
\end{align*}
\]  

(4.10)

We need determine the set of \((v_\infty, u_\infty)\) for which (4.10) has a solution. Integrating once the ODE’s and using the boundary condition at infinity, we get

\[
\partial_y v = u_\infty - u, \quad u_y = \sigma(v_\infty) - \sigma(v).
\]  

(4.11)

Cross multiplying the equations and integrating, we get

\[
\frac{(u - u_\infty)^2}{2} = \int_{v_\infty}^{v} (\sigma(s) - \sigma(v_\infty)) \, ds,
\]

so

\[
(u - u_\infty) = \pm \left( \int_{v_\infty}^{v} 2 (\sigma(s) - \sigma(v_\infty)) \, ds \right)^{1/2}.
\]  

(4.12)

Note that \(\int_{v_\infty}^{v} (\sigma(s) - \sigma(v_\infty)) \, ds \geq 0\) because of the condition \(\sigma'(v) > 0\). From (4.12) it follows that

\[
v(y) = v_\infty \iff u(y) = u_\infty.
\]

Since we are interested in a solution connecting \((v_B, u_B)\) at \(y = 0\) to \((v_\infty, u_\infty)\) at \(y = \infty\), we get from (4.11) that either

\[
v_B < v(y) < v_\infty \quad \text{and} \quad u_B < u(y) < u_\infty
\]

or

\[
v_B > v(y) > v_\infty \quad \text{and} \quad u_B > u(y) > u_\infty.
\]  

(4.13)
This determines the sign in (4.12):

\[
\begin{align*}
\mathbf{u} &= \begin{cases} 
    u_{\infty} + \left( \int_{v_{\infty}}^{v} 2 (\sigma(s) - \sigma(\bar{v})) \, ds \right)^{1/2} & \text{if } v > v_{\infty} \\
    u_{\infty} - \left( \int_{v_{\infty}}^{v} 2 (\sigma(s) - \sigma(\bar{v})) \, ds \right)^{1/2} & \text{if } v < v_{\infty}.
\end{cases}
\end{align*}
\]

Since we need \((v_B, u_B)\) to be on this curve, we obtain that the set of \((v_{\infty}, u_{\infty})\) so that (4.10) has a solution lies on the curve

\[
\begin{align*}
\mathbf{u}_{\infty} &= \begin{cases} 
    u_B - \left( \int_{v_{\infty}}^{v_B} 2 (\sigma(s) - \sigma(\bar{v})) \, ds \right)^{1/2} & \text{if } v_{\infty} < v_B \\
    u_B + \left( \int_{v_{\infty}}^{v_B} 2 (\sigma(s) - \sigma(\bar{v})) \, ds \right)^{1/2} & \text{if } v_{\infty} > v_B,
\end{cases}
\end{align*}
\]

where \((v_B, u_B)\) is fixed.

Let us now turn to the Lax Friedrichs scheme. For the system (4.8), the discrete boundary layer equation is

\[
H(v(y), v(y+1), u_{\infty}, v_{\infty}) \equiv \left( \frac{\lambda(u(y+1) + u(y)) + v(y+1) - v(y) - 2 \lambda u_{\infty}}{\lambda \sigma(v(y+1)) + \sigma(v(y)) + u(y+1) - u(y) - 2 \lambda \sigma(v_{\infty})} \right) = 0,
\]

(4.15)

\[
(v, u)(0) = (u_B, u_0), \quad (v, u)(\infty) = (v_{\infty}, u_{\infty}).
\]

Here the eigenvalues of (4.8) are

\[
\lambda_2(v_{\infty}, u_{\infty}) = -\lambda_1(v_{\infty}, u_{\infty}) = \sigma'(v_{\infty})^{1/2}
\]

and, with the notations of Section 3,

\[
a_1(v_{\infty}, u_{\infty}) = \frac{1 - \lambda \sigma'(v_{\infty})^{1/2}}{1 + \lambda \sigma'(v_{\infty})^{1/2}}, \quad a_2(v_{\infty}, u_{\infty}) = \frac{1 + \lambda \sigma'(v_{\infty})^{1/2}}{1 - \lambda \sigma'(v_{\infty})^{1/2}}.
\]

Thus \(0 < a_1(v_{\infty}, u_{\infty}) < 1, a_2(v_{\infty}, u_{\infty}) > 1\). By the analysis of Section 3, it follows that the set of \((v_{\infty}, u_{\infty})\) near \((v_B, u_B)\) for which (4.15) has a solution lie on a curve passing through \((v_B, u_B)\).

4.5 Eulerian Isentropic Gas Dynamics. We now consider the isentropic approximation to the compressible Euler system. The system is composed of the two conservation laws for the mass and the momentum of a gas [10]:

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2 + p(\rho)) &= 0.
\end{align*}
\]

(4.16)
The main unknowns are the specific density $\rho$ and the velocity $u$. The pressure is a function of the density and, for simplicity, we shall restrict to a polytropic perfect gas:

$$p(\rho) = \rho^\gamma, \quad \gamma \in (1, \infty).$$

(4.17)

We consider the boundary layer equation generated by the vanishing viscosity method with $B(u) = I$:

$$\begin{align*}
\partial_y (\rho u) &= \partial_y^2 \rho \\
\partial_y (\rho u^2 + p(\rho)) &= \partial_y^2 u \\
\rho(0) &= \rho_B, \quad u(0) = u_B, \quad \rho(\infty) = \rho_\infty, \quad u(\infty) = u_\infty.
\end{align*}$$

(4.18)

Integrating the ODE'S and using the boundary condition at infinity, we get

$$\begin{align*}
\partial_y \rho &= \rho u - \rho_\infty u_\infty \\
\partial_y u &= \rho u^2 + p(\rho) - \rho_\infty u_\infty^2 - p(\rho_\infty) \\
\rho(0) &= \rho_B, \quad u(0) = u_B, \quad \rho(\infty) = \rho_\infty, u(\infty) = u_\infty.
\end{align*}$$

(4.19)

The eigenvalues of the matrix obtained by linearizing the R.H.S. of (4.19) around $(\rho_\infty, u_\infty)$ are

$$\lambda_1(\rho_\infty, u_\infty) = u_\infty - c(\rho_\infty), \lambda_2(\rho_\infty, u_\infty) = u_\infty + c(\rho_\infty)$$

(4.20)

where $c^2(\rho_\infty) = p'(\rho)$. We have to distinguish between five different cases. We define the following regions in $(\rho, u)$–plane:

$$\begin{align*}
\Omega_I &= \{(\rho, u) : u - c(\rho) < 0, u + c(\rho) < 0\} \\
\Omega_{II} &= \{(\rho, u) : u - c(\rho) < 0, u + c(\rho) = 0\} \\
\Omega_{III} &= \{(\rho, u) : u - c(\rho) < 0, u + c(\rho) > 0\} \\
\Omega_{IV} &= \{(\rho, u) : u - c(\rho) = 0, u + c(\rho) > 0\} \\
\Omega_V &= \{(\rho, u) : u - c(\rho) > 0, u + c(\rho) > 0\}
\end{align*}$$

(4.21)

Thus in $\Omega_I$ both eigenvalues are negative, whereas in $\Omega_{II}$ one has $\lambda_1 < 0, \lambda_2 = 0$. In $\Omega_{III}$, one has $\lambda_1 < 0, \lambda_2 > 0$, whereas in $\Omega_{IV}$, one has $\lambda_1 = 0, \lambda_2 > 0$ and in $\Omega_V$, $\lambda_1 > 0$ and $\lambda_2 > 0$. Following the analysis that we did for the proof of Theorem 3.2, it is not hard to get the following local result.

**Case 1 :** $(\rho_B, u_B) \in \Omega_I$. In this case the set of $(\rho_\infty, u_\infty)$ close to $(\rho_B, u_B)$ for which (4.19) has a solution is an open neighborhood of $(\rho_B, u_B)$.

**Case 2 :** $(\rho_B, u_B) \in \Omega_{II}$. In this case the set of $(\rho_\infty, u_\infty)$ close to $(\rho_B, u_B)$ for which (4.19) has a solution is a union of a two-dimensional region $U$ in $\Omega_I$ and a curve in $\Omega_{III}$ through $(\rho_B, u_B)$ intersecting $U$. 
Case 3: \((\rho_B, u_B) \in \Omega_{III}\). In this case the set of states \((\rho_\infty, u_\infty)\) close to \((\rho_B, u_B)\) for which (4.19) has a solution is a curve through \((\rho_B, u_B)\).

Case 4: \((\rho_B, u_B) \in \Omega_{IV}\). In this case the set of states \((\rho_\infty, u_\infty)\) near \((\rho_B, u_B)\) for which (4.19) has a solution lies in a curve in \(\Omega_{III}\) through \((\rho_B, u_B)\). This does not extend to \(\Omega_V\).

Case 5: \((\rho_B, u_B) \in \Omega_V\). There cannot be any point \((\rho_\infty, u_\infty)\) in \(\Omega_V\) for which (4.19) has a solution.

Next we consider the Lax-Friedrichs scheme. The discrete boundary layer problem to be solved is

\[
\lambda (\rho(y+1)v(y+1) + \rho(v(y))) - 2\lambda \rho_\infty u_\infty - (\rho(y+1) - \rho(y)) = 0
\]

\[
\lambda (\rho(y+1)u(y+1)^2 + \rho(y)u(y)^2) - 2\lambda \rho_\infty u_\infty^2 - (\rho(y+1)v(y+1) - \rho(y)v(y)) + \lambda(p(\rho(y+1)) + p(\rho(y)))
\]

(4.22)

\((\rho_B, u_B)\) given and \((\rho, u)(\infty) = (\rho_\infty, u_\infty)\).

Given \((\rho_B, u_B)\) we determine \((\rho_\infty, u_\infty)\) close to \((\rho_B, u_B)\) for which (4.22) has a solution. Following the analysis of the proof of Theorem 3.4, we get the eigenvalues of the linearized matrix at \((\rho_\infty, u_\infty)\) are

\[
a_1 = a_1(\rho_\infty, u_\infty) = \frac{1 + \lambda_1(\rho_\infty, u_\infty)}{1 - \lambda_1(\rho_\infty, u_\infty)}, a_2 = a_2(\rho_\infty, u_\infty) = \frac{1 + \lambda_2(\rho_\infty, u_\infty)}{1 - \lambda_2(\rho_\infty, u_\infty)}
\]

where \(\lambda_1\) and \(\lambda_2\) are given by (4.20). If \((\rho_\infty, u_\infty) \in \Omega_I, a_1 < 1, a_2 < 1\), if \((\rho_\infty, u_\infty) \in \Omega_{II}, a_1 < 1, a_2 = 1\), if \((\rho_\infty, u_\infty) \in \Omega_{III}, a_1 < 1, a_2 > 1\), if \((\rho_\infty, u_\infty) \in \Omega_{IV}, a_1 = 1, a_2 > 1\) and if \((\rho, u_\infty) \in \Omega_V, a_1 > 1, a_2 > 1\). It follows from the proof of Theorem (3.4), that if \((\rho_B, u_B) \in \Omega_I\), then the set of states \((\rho_\infty, u_\infty)\) near \((\rho_B, u_B)\) for which (4.22) has a solution connecting \((\rho_B, u_B)\) to \((\rho_\infty, u_\infty)\) is a neighborhood of \((\rho_B, u_B)\). If \((\rho_B, u_B) \in \Omega_{II}\) this set is a union of an open set \(U\) in \(\Omega_I\) and a curve in \(\Omega_{III}\) through \((\rho_B, u_B)\) which interset \(U\). If \((\rho_B, u_B) \in \Omega_{III}\) this set of \((\rho_\infty, u_\infty)\) near \((\rho_B, u_B)\) consists of a curve through \((\rho_B, u_B)\) and if \((\rho_B, u_B) \in \Omega_{IV}\) this set consists of a curve in \(\Omega_{III}\) through \((\rho_B, u_B)\). If \((\rho_B, u_B) \in \Omega_V\) no point \((\rho_\infty, u_\infty)\) in \(\Omega_V\) can be connected by a solution of (4.22) from \((\rho_B, u_B)\).

4.6 Lagrangian Isentropic Gas Dynamics. Finally, we consider the system of gas dynamics in Lagrangian coordinates

\[
\partial_t v_t - \partial_x u = 0,
\]

\[
\partial_t u + \partial_x \left(\frac{1}{v}\right) = 0,
\]

(4.23)

where \(u\) is the velocity and \(v > 0\) is the specific density. The eigenvalues of (4.23) are

\[
\lambda_1 = -\frac{1}{v} < 0, \quad \lambda_2 = \frac{1}{v} > 0;
\]

(4.24)
hence the boundary $x = 0$ is not characteristic.

The purpose of this section is to provide an explicit formula for the boundary layer set associated with the Lax-Friedrichs scheme. The boundary layer equation takes the form

$$\lambda (u(y + 1) + u(y)) - 2 \lambda u_\infty + v(y + 1) - v(y) = 0$$

$$\lambda \left( \frac{1}{v(y + 1)} + \frac{1}{v(y)} \right) - 2 \frac{\lambda}{v_\infty} - u(y + 1) + u(y) = 0$$

with

$$(v(0), u(0)) = (v_B, u_B), \quad (v, u)(\infty) = (v_\infty, u_\infty).$$

We restrict attention to $v_B > \delta > 0$ for fixed $\delta$, and we determine the set of $(v_\infty, u_\infty)$ for which (4.25) has a solution. We set

$$w(y) = \frac{v(y)}{\lambda}$$

so that (4.25) becomes

$$\frac{1}{w(y + 1)} + \frac{1}{w(y)} - u(y + 1) + u(y) = \frac{2}{w_\infty}$$

$$w(y + 1) - w(y) + u(y + 1) + u(y) = 2 u_\infty.$$  

Adding the two equalities, we get

$$w(y + 1) + \frac{1}{w(y + 1)} + \frac{1}{w(y)} - w(y) + 2u(y) = \frac{2}{w_\infty} + 2 u_\infty.$$  

Setting

$$N(y) = -2 u(y) + 2 u_\infty - \frac{1}{w(y)} + \frac{2}{w_\infty} + w(y),$$

we obtain a quadratic equation for $w(y + 1)$:

$$w^2(y + 1) - N(y) w(y + 1) + 1 = 0.$$  

Therefore

$$w(y + 1) = \frac{1}{2} \left( N(y) \pm \left( N(y)^2 - 4 \right)^{1/2} \right)$$

from which we get an expression for $u(y + 1)$ as well:

$$u(y + 1) = \frac{\lambda}{2} N(y) \pm \frac{\lambda}{2} \left( N(y)^2 - 4 \right)^{1/2}.$$  

(4.29)
Observe that $N(\infty) = w_\infty + 1/w_\infty$, where $w_\infty = v_\infty/\lambda$ and $N(\infty)^2 - 4 = (w_\infty - 1/w_\infty)^2$. The product of the two roots of (4.28) is equal to one. Stability requires $w_\infty > 1$ so we choose the larger root in (4.29). We have finally from (4.29) and (4.25).

\[
v'(y + 1) = \frac{\lambda}{2} N(y) + \frac{\lambda}{2} (N(y)^2 - 4)^{1/2}
\]

\[
u'(y + 1) = 2u_\infty - u(y) + \frac{v(y)}{\lambda} - \frac{1}{2} - \frac{1}{2}(N(y)^2 - 4)^{1/2}
\]

The Jacobian of the R.H.S. of (4.30) at $(v_\infty, u_\infty)$ is easily seen to be

\[
A(v_\infty, u_\infty) = \begin{pmatrix}
w^2_\infty + 1 & -2\lambda w^2_\infty \\
w^2_\infty - 1 & w^2_\infty - 1 \\
-2 & w^2_\infty + 1 \\
\lambda(w^2_\infty - 1) & w^2_\infty - 1
\end{pmatrix},
\]

whose eigenvalues are

\[
a_1 = \frac{w_\infty - 1}{w_\infty + 1}, \quad a_2 = \frac{w_\infty + 1}{w_\infty - 1}.
\]

In terms of $v_\infty$, we have

\[
a_1 = \frac{1 - \lambda}{1 + \lambda v_\infty}, \quad a_2 = \frac{1 + \lambda}{1 - \lambda v_\infty}.
\]

If the data for the Lax-Friedrichs scheme are chosen such that the $v$ component is bounded away from zero, then so is the approximate solution. Hence we can restrict attention to $v_\infty > \delta'$ for some $\delta' > 0$. For $\lambda$ small enough, we have

\[
0 < a_1 < 1 \quad \text{and} \quad a_2 > 1,
\]

and Theorem 3.10 applies. We deduce that the set of all states $(v_\infty, u_\infty)$ near $(v_B, u_B)$ for which (4.25)-(4.26) has a solution is a curve passing through $(v_B, u_B)$.

5. Concluding Remarks. Given a family of sets such as those introduced in this paper, we can formulate the boundary condition for the hyperbolic problem. When the solutions $u$ under consideration are functions of bounded variation, the traces exist in a strong sense and one can require that

\[
u(0^+, t) \in E(u_B(t)), \quad t > 0,
\]

holds for all, except countably many, $t$. This type of regularity has been recently rigorously established by Amadori in her thesis [1], using the front tracking scheme and the sets $E_{\text{Godunov}}$ ($= E_{\text{Godunov}}^{\text{layer}} = E_{\text{Godunov}}^{\text{entropy}}$). It would be interesting to extend [1] to the other sets we introduced here.

For general $L^\infty$ solutions constructed by the vanishing viscosity method, the boundary condition

\[
\text{supp } u_{0,t} \subset E_{\text{viscosity}}
\]
has been rigorously derived in Theorem 2.1. When the method of compensated compactness applies [12], an existence theorem for the boundary-value problem (1.1)–(1.3), (5.2) follows immediately from Theorem 2.1. Such a result is satisfactory only when the condition (5.2) yields, for simple enough initial and boundary data at least, a well-posed problem. This is the case for the scalar equations and the linear systems, and, likely, for any system in the so-called Temple’s class (having coinciding shock and rarefaction curves).

In a recent preprint by Grenier and Gues, a scaling of the type \(1/\sqrt{\epsilon}\) is used for linear systems of equations to obtain a more precise description of the boundary features. However, as far as the formulation of a well-posed, limiting boundary-value problem for the hyperbolic equations is sought, the scaling \(1/\epsilon\) we used in (2.5) happens to be sufficiently discriminating. The formulation based on the boundary layer equation may not be appropriate as it is when the boundary is characteristic. On the other hand, the formulation based on entropy inequality capture rigorously some of the features in the solution near the boundary, but is more difficult to work with analytically. Further study of the connection between the two sets for systems is in progress.

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