Determining the characteristics of securities price fluctuations correctly is basic to scientifically grasp the law of securities market operation, and provides a guarantee for the implementation of asset optimization allocation and systematic risk management. The efficient market hypothesis (EMH) was developed based on the premise that securities price movement follows Brownian motion and random walk, and yields subject to normal distribution (Fama, 1965, 1970). Therefore, the securities market is characterized by linearity, continuity, static, and independence; through which, the investment risk can be estimated and controlled.

With the continuous integration and development of technologies, such as computer science, communication engineering, and big data, financial physics and some other emerging interdisciplinary disciplines in the academic community have been deepening their understanding on the characteristics of securities price fluctuation and the laws of securities market operation. In fact, non-normal characteristics such as the leptokurtic and fat-tailed of securities yield distribution, finance phenomena such as the P/E effect and calendar effect, and high frequency and cluster financial risk all have shown significant evidence that securities price fluctuation is not always moderate and the securities market may have some characteristics of discontinuity, nonlinearity, and complexity (Mandelbrot, 1971, 1997; Mandelbrot & Hudson, 2009; Peters, 1991). Thus, in a broader and more integrated perspective, it is necessary to adopt a new complex paradigm to analyze complex issues of social sciences and examine the development situation between various disciplines (Jiang, 2018). The fractal market hypothesis (FMH), as one of the most important theoretical innovations established from Fractal Geometry, has a profound impact on the development of modern securities investment theories and practices (Peters, 1994).

As soon as its emergence, the FMH began to reconstruct the analysis framework for the investment theory. Compared with a number of foreign innovation achievements, these domestic studies are mostly still at the stage of citing or paraphrasing the existing achievements abroad, which not only lacks enough breakthrough, but also lacks in-depth combing and clear analysis of some fundamental issues. For example, what the differences and similarities between the FMH and the EMH, and how FMH supplement or enhance EMH are still unknown. However, many studies in recent years such as Ghadiri Moghadam et al. (2014) and Moradi et al. (2019) both proved that fractal market hypothesis was accepted for Tehran Stock Exchange and rejected for London Stock Exchange. Further, the present and future situations of the research on the FMH are yet to be
determined. This paper, therefore, aims to respond and achieve answers to these issues.

**Theoretical Analysis**

**Origin and Development of the EMH**

It is generally accepted that Bachelier (1900) was the forerunner of the EMH. He first discovered and observed that securities price follows Brownian motion and “irregular” random walk in a speculative financial market, so investors cannot get any excess return by “detecting” the securities price fluctuations. Subsequently, Cowles (1933), Kendall (1953), and Cootner (1964) successively conducted further empirical studies to expand knowledge on securities price fluctuation and returns distribution. Finally, Samuelson (1965) invented the most general form of mathematical analysis on this subject for the first time, laying the foundation of “scientificity” on the securities market researches.

According to the academic contribution of information classification theorized by Roberts (1967) and Fama (1965, 1970), an awardee of the 2013 Nobel Prize in Economics, formally proposed the EMH and divided it into three distinguishing levels of securities market namely: (1) weak form efficiency, (2) semi-strong form efficiency, and (3) strong form efficiency. Fama (1965, 1970) observed that the securities price reflects all existing information in a strong efficient market, and that there is no statistical short-term or long-term “memory” in the sequence of securities yields. Therefore, there is no “free lunch” in the securities market, so investors cannot obtain excess return by using any historical, public, or inside information.

The EMH describes the standard ideal state of the securities market, and provides a reference for a deep understanding of the fluctuations in securities price and the operating rules of the securities market. Since its conception, the EMH has become the cornerstone of neoclassical financial theory. The modern portfolio theory (MPT; Markowitz, 1952), the capital asset pricing model (CAPM; Sharpe, 1964), the option pricing model (OPM; Merton, 1973), and the arbitrage pricing theory (APT; Ross, 1976) were all established based on the EMH.

However, the EMH is still far from being perfect and universal. For instance, several scholars doubt its assumption that a series of return rate distribution changes in a linear fashion in the securities market (Mandelbrot, 1970; Peters, 1991). Because of this, its scientificity is theoretically challenged by some persistent financial anomalies (such as P/E effect and calendar effect mentioned above), and the long-standing irreconcilable differences and contradictions between fundamental analysis and technical analysis (such as the history of price movements can be repeated) have made it difficult to guide investment practice (Mandelbrot, 1970). These problems have inspired the academic circles to correct and perfect the classical theory continuously, promote the theory closest to reality, and get a stronger interpretation value and practical guidance ability (Mandelbrot, 1970, 1997; Mandelbrot & Taylor, 1967; Peters, 1991).

Many studies have shown that any efforts on finding solutions to these problems that are still in the context of linear science are almost in vain (Mandelbrot & Hudson, 2009; Peters, 1991). As a result, scholars tried to establish a broader and more general analytical framework at a higher level of horizon, and the FMH emerged in response to this (Mandelbrot, 1997; Mandelbrot & Van Ness, 1968; Peters, 1991, 1994).

**Origin and Development of the FMH**

The FMH emerged because of two historical events: (1) the development of interdisciplinary sciences, such as Econophysics; and (2) the pioneering and creative studies of Mandelbrot on Fractal Geometry and Multifractal Geometry (Mandelbrot, 1967, 1999; Mandelbrot & Hudson, 2009; Mantegna & Stanley, 2006).

The FMH was first proposed by Peters (1991, 1994), which hypothesizes that the securities price is subject to the fractal Brownian movement, and its yield follows fractal distribution characterized by self-similarity and long-term memory. The FMH is established on the roughness and irregularity of the reality, and pays more attention to the auto-correlation and self-similarity of securities price, which are closer to the objective law. Since the FMH authorizes and respects the nonlinear characteristics of the securities market, it does not only connect with other theories better, but also reconciles the differences between different financial theories.

The FMH has two modes namely monofractal analysis and multifractal analysis. Local researches on FMH began in the late 1990s (Fan & Zhang, 2002; Xu, 1999). Some systematically reviewed the emergence, development, and current research status of the FMH, and prepared for its application to the securities market (Liu & Song, 2013). In addition, both domestic and foreign studies have verified the fractal dynamics characteristics of stocks, bonds, currencies, foreign exchange, commodity futures, stock index futures, P2P, cryptocurrencies, and other markets, as well as the asymmetrical disturbances between the different markets (Cao & Shi, 2007; Li et al., 2020; Mantegna & Stanley, 1995; Shi et al., 2014; Yuan et al., 2016; Zhu et al., 2021). The features of these local studies are discussed in detail in the succeeding paragraphs.

The complementarities of the relationship between the EMH and the FMH have already been proven qualitatively (Liu & Song, 2013; Mantegna & Stanley, 2006; Peters, 1994), but the microscopic formation mechanism of their relationship has not been discussed deeply and mathematically. Because of this, misunderstandings and mistakes in their basic concepts and methodologies can be easily made.

In addition, Mandelbrot (1967) noted that the fractal structure of the securities market is relatively associated with the violent and discontinuous fluctuation of the securities.
price; thus, the volatility of securities price is considered not mild, but has the Noah effect caused by volatility clustering, and the Joseph effect caused by long term persistence and aperiodic circulation simultaneously. He defined that the Noah effect refers to the occasional discontinuous and sudden big jump of stock price, where a big jump is often followed by a big jump, and a small jump is often followed by a small jump, which forms a phenomenon of volatility clustering. The Joseph effect refers to the phenomenon of long-term continuous (long memory) and non-cyclical movement of stock price. Unfortunately, the analysis on the deeper causes of these effects was abandoned at that time. The subsequent studies are still mostly at this stage and lacked theoretical breakthrough. Obviously, although decades have passed, the law and characteristics of price volatility have still not been understood profoundly and comprehensively.

Fractal geometry, being a vital nonlinear science discipline and methodology, is not only employed to determine the volatility law of securities prices, but also used to reconstruct optimal portfolios and manage investment risk (Mandelbrot & Hudson, 2009). Some international studies have illustrated the importance of fractal geometry (Mandelbrot & Van Ness, 1968; Mantegna & Stanley, 2006), but local studies are still in their infancy and imitation research stage (Chen, 2014; Fan & Zhang, 2002; Liu & Song, 2013).

Finally, although the FMH could relatively describe the reality of the securities market better than the EMH, it is still limited in terms of the psychological characteristics and the behavioral decision-making process of the market participants, placing constraints on its utilization. In the future, the FMH also needs to learn from other emerging disciplines and better develop itself by integrating with them continuously.

**Cross Development of the EMH and the FMH**

When comparing neoclassical and emerging finance, the analysis of the similarities and differences between the EMH and the FMH is almost always a part of the discussion (Chen, 2014; Tian et al., 2016). The linearity of the securities market serves as the foundation of neoclassical finance; however, the non-linearity and complexity of the securities market have never been disregarded. Previous researches on the cross-development of financial theories have shown that the EMH and the FMH are rarely distinguished from each other. In contrast, both have always been observed to grow through mutual permeation and cross merging (see Figure 1). For instance, Mandelbrot, who is the forerunner of the FMH, is the student of the mathematician Von Neumann and the probability theory expert Lévy. While another famous scholar Fama, who is the founder of the EMH, is the student of Mandelbrot in turn.

**Integration Between the EMH and the FMH**

**Uniformity of distribution functions.** Based on the EMH, Samuelson (1965) was the first to precisely point out that securities price movement follows Brownian motion and random walk, and its yields follow the normal distribution. Its logarithmic characteristic function is illustrated in equation (1).

\[
\ln \phi(k) = i\mu k - \gamma |k|^\gamma; \tag{1}
\]

Where the \( k \) represents any real number and the \( i \) represents any imaginary unit. The \( \gamma \) is a scale parameter, reflecting the dimensional change of random variables. The \( \mu \) is a
position parameter, reflecting the specific position of the distribution curve.

The Cauchy distribution shown in equation (2) describes the logarithmic characteristic function when the securities price volatility is discontinuous and violent.

\[
\ln \phi(k) = i \mu k - \gamma |k|, \quad (2)
\]

Where the \( k, i, \gamma \), and \( \mu \) have the same meanings as in equation (1).

Both logarithmic characteristic functions (equations (1) and (2)) are valid under finite variance.

For those more general movement types of securities prices, Lévy (1937), who laid the theoretical foundation for FMH based on Pareto’s (1897) research on income distribution, originally proposed a relative comprehensive logarithmic characteristic function which can link various distribution curves together. The stable Lévy distribution is shown in the following equation (3).

\[
\ln \varphi(k) = \begin{cases} 
  i \mu k - \gamma |k|^\alpha & \text{if } \alpha \neq 1, \\
  i \mu k - \gamma |k|^{\beta} / (k / |k|)^{\gamma} \ln(\gamma / k) & \text{if } \alpha = 1,
\end{cases} \quad (3)
\]

Where the \( k \) and \( i \) have the same meanings as equation (1).

For any stationary process \( X(\alpha, \beta, \gamma, \mu) \), its distribution curve is determined by the four parameters \( \alpha, \beta, \gamma \), and \( \mu \), where \( \alpha \) and \( \beta \) are the two key parameters. The meanings of the four parameters are as follows.

The \( \alpha \) is the characteristic parameter of stationary distribution, and its value range is \( \alpha \in (0, 2] \). The \( \alpha \) not only characterizes the peak and fat tail of the distribution curve, that is, the probability of extreme values distributed at the tail and the height of the density function, but also determines the moment of distribution, as well as the distribution of the sum and its standardization characteristics of random variable.

The \( \beta \) is the partial parameter, reflecting the direction in which the peak value of the distribution function deviates from the mean value (or position parameter). Its value range is \( \beta \in [-1, 1] \). If \( \beta > 0 \), it means that the peak value is right deviation (positive deviation), and the closer the value of the \( \beta \) is to 1, the greater the right deviation. If \( \beta < 0 \), it means that the peak value is left deviation (negative deviation), and the closer the value of the \( \beta \) is to \(-1\), the greater the left deviation. If \( \beta = 0 \), it means that the peak value is no deviation, and the distribution function is symmetric about \( \mu \).

The \( \gamma \) is the scale parameter, reflecting the width of the distribution curve. Its value range is \( \gamma \in (0, +\infty) \). The larger the value of \( \gamma \), the wider the distribution curve is. On the contrary, the smaller the value of \( \gamma \), the narrower the distribution curve is.

The \( \mu \) is the position parameter, reflecting the center position of the distribution curve. Its value range is \( \mu \in (-\infty, +\infty) \). When \( \alpha > 1 \), the \( \mu \) is the mean of the distribution function. When \( \alpha \leq 1 \), the distribution function has no mean.

According to equation (3), when the values of the four parameters are different, the following conclusions can be obtained.

(a) If \( \alpha = 2 \), then equation (3) condenses to equation (1), and the stable Lévy distribution is equivalent to the normal distribution, where the mean of yields is represented by \( \mu \), and the variance of yields by \( 2 \gamma \). If \( \alpha \neq 2 \), then equation (3) is equal to the stable Non-Gaussian distribution. If \( 0 < \alpha < 2 \), the curve of the Levy distribution function has a thicker tail. The smaller the value of \( \alpha \), the greater the probability that the tail takes the extreme value.

(b) If \( \alpha = 1 \) and \( \beta = 0 \), then equation (3) condenses to equation (2), and the stable Lévy distribution is equal to the Cauchy distribution.

(c) If \( \alpha \in (1, 2) \), then equation (3) is equal to the Pareto-Lévy distribution.

It can be seen from the above functions that the normal distribution of yield is one of the special cases of the stable Lévy distribution, that is, it is the bias rather than the normality of the yield distribution. So, the EMH and the FMH have internal consistency, and the former is the special case of the latter.

**Uniformity of characteristic exponents.** Mandelbrot and Van Ness (1968) originally extended the price fluctuations of securities from the standard Brownian motion to the fractal Brownian motion. Subsequently, Mandelbrot (1970, 1971) instituted the rescaled range analysis \((R/S)\) into the securities market. According to his theory, for any fractal Brownian motion, \( B_H(t) \), the correlation \( CN \) between the yields of securities can be expressed by the equation (4).

\[
CN = 2^{H-1} - 1, \quad (4)
\]

where: \( H \) denotes the Hurst exponent.

(a) If \( H = 1/2 \), then \( CN = 0 \), and \( B_H(t) \) is equal to the standard Brownian motion, which indicates that the yields of securities are random and unrelated.

(b) If \( 1/2 < H < 1 \), then \( CN > 0 \), and \( B_H(t) \) is equal to fractal Brownian motion. That is to say, the yields of securities have persistence and long-term memory, that is, securities price fluctuation has inertia or momentum effect.

(c) If \( 0 < H < 1/2 \), then \( CN < 0 \), and \( B_H(t) \) is also equal to the fractal Brownian motion. However, the yields of securities have anti-persistence and no long-term memory here, that is, securities price fluctuation has a contrarian effect.

The equation (4) shows that the standard Brownian motion is one of the special cases of the fractal Brownian motion. It proves that in terms of characteristic exponent, the
random uncorrelated of yields is still not the general but the special case. The standard Brownian motion is just a specific state of price fluctuation. The EMH and the FMH have internal consistency, and the former is the special case of the latter.

Aside from the $H$ exponent, other exponents including the fractal dimension $D$, stable index $\alpha$, fractional difference parameter $\xi$, and power spectral density $\eta$ can be used to identify the fractal characteristics of the securities market. According to Zhao et al. (1999), the relationships of the exponents mentioned above are as follows:

\[ D = 2 - H, \quad \alpha = 1 / H, \quad \xi = H - 0.5, \quad \text{and} \quad \eta = 2H - 1. \]

These exponents are equivalent in revealing the uniformity of the EMH and the FMH; thus, further discussion about them is not needed.

### Differences and Similarities Between the EMH and the FMH

Mantegna and Stanley (2006) made a mathematical description of the relationship between the FMH and the EMH. They expounded their viewpoints from the two dimensions of finiteness and stability of the stochastic process. Their description tends to be theoretical, and cannot be understood easily or quoted widely. In order to demonstrate more intuitively the differences and similarities between the EMH and the FMH in terms of theoretical connotations, Table 1 lists their characteristics on the basis of existing researches. Though there are many differences between them, this study highlights that intrinsic uniformity is the essential characteristic of the EMH and the FMH.

In addition to conducting comparative analysis by mathematical and qualitative methods as above, we are inspired by the fractal geometry theory and try developing a third method to express the relationship between the two hypotheses more distinctly within the perspective of nonlinear science. The main steps of this method are described as follows.

Firstly, we draw a semicircle $C_0$ with a diameter of any unit length, and choose it as the initial semicircle. Next, another two parallel semicircles, $C_{0-1}$ and $C_{0-2}$, are drawn in the $C_0$. The diameter of $C_{0-1}$ is $1/3$ of the diameter of $C_0$, and the diameter of $C_{0-2}$ is $2/3$ of the diameter of $C_0$. Then, following this recursive rule, another two parallel semicircles, the $C_{0-2-1}$ and $C_{0-2-2}$, are drawn in the $C_{0-2}$. And, another two parallel semicircles, the $C_{0-2-2-1}$ and $C_{0-2-2-2}$, are drawn in the $C_{0-2-2}$. Finally, a simple fractal image that is similar to a “Pharaoh’s Breastplate” (Figure 2) is obtained. In this image, the semicircle $C_0$, $C_{0-2}$, $C_{0-2-2}$, and $C_{0-2-2-2}$, respectively, represent the form of nonlinear financial market.
fractal financial market, efficient market, and strong efficient market. Meanwhile, the semicircle $C_0 - 1$, $C_0 - 2 - 1$, and $C_0 - 2 - 2 - 1$, respectively, represent the market described by the chaos theory, the Cauchy distribution, and the weak- form or the semi-strong efficient hypothesis.

It can be clearly seen that the semicircle $C_0 - 2 - 2$ is only one part of the semicircle $C_0 - 2$. This indicates that the market defined by FMH consists of the market defined by EMH, and the both belong to the category of the nonlinear market.

**Evolutions of the FMH**

The FMH is one of the most notable and rapidly developing theories in the field of emerging finance. The fluctuation of securities price is assumed to be affected by more factors under the analysis framework of the FMH than that of the EMH; thus, the connotation of price fluctuation is more extensive. On the one hand, the fractal characteristics of the distribution of securities yields cannot be completely attributed to the randomness of price fluctuations; meanwhile, some previous exogenous variables under the analysis framework of the EMH, such as time, is internalized into the pricing model for the first time (Mandelbrot & Taylor, 1967). Therefore, time is no longer uniform and meaningless, but is included in the price behavior research instead. The fractal characteristic of securities price fluctuation is state continuity, that is, there is a time scale in yields difference. In addition, the covariance between securities price fluctuation and market index fluctuation is also influenced by time and has a time scale. That is to say, securities price fluctuation is closely related to time. Time determines the sensitivity and path dependence of price behavior to initial conditions, which in turn forms long-term memory and self-similarity of yields. Because investors with different investment terms coexist, the efficiency of the securities market is guaranteed. Compared to the EMH, the FMH acknowledges that the investment terms are inconsistent, which is a huge improvement.

Furthermore, Cont and Bouchaud (2000) showed a static percolation model and suggested that reciprocal imitation between investors could create herd behavior. Furthermore, herd behavior causes the power-law tail distribution of securities yields. In addition to time and behavioral variables, it can be predicted that more exogenous variables, such as investor sentiment, may be merged into the analysis framework of the FMH, which would make FMH more closely integrated with other nonlinear theories in the future.

Since its establishment, the FMH has made great progress in identifying the fractal characteristics of the securities market. Although early researches mainly identified the monofractal characteristics of the securities market, Mandelbrot (1967) emphasized in his later years that the multifractal theory is a superior improvement to the monofractal theory, because the former can be more suitably employed in risk estimation, investment analysis, and bankruptcy avoidance, even though the multifractal securities market is more challenging to characterize. However, both Ola et al. (2014) and Abbaszadeh et al. (2020) demonstrated that the existence of multifractality process in the evolution of time series stock price. There are only a few quantitative measurement methods used to solve this problem at present. They are mostly built on the method of detrended fluctuation analysis (DFA; Peng et al., 1994), which mainly includes the multifractal volatility measurement (MFV), multifractal detrended fluctuation analysis (MF-DFA; Kantelhardt et al., 2002), multifractal detrended cross-correlation analysis (MF-DCCA; Zhou, 2008), multifractal detrended cross-correlation analysis (MFXDFA), multifractal detrending moving average cross-analysis (MFXDMA; Jiang & Zhou, 2011), multifractal cross-correlation analysis-correlation analysis on statistical moments (MFSMXA; Wang et al., 2012), and multifractal detrended cross-correlation analysis (MF-ADCCA; Cao et al., 2014). Although some already underwent trial applications in practice, they have not yet been unanimously recognized theoretically in the academic community.

After the theoretical basis and method system were completed, proponents of the FMH tried to improve the EMH through micro-modeling, which was attempted earlier by Mandelbrot (1997). Subsequently, in the aspect of capital asset pricing, the classical CAPM tends to make over-differencing problem in securities prices when used to estimate systemic risk, which causes the identification and interpretation of systemic risk is usually underestimated, and the risk reward ratio of securities is usually overestimated. Therefore, some studies attempted to use the ratio of the fractal differential to replace the traditional rate of return, plugged it into the CAPM equation, obtained the fractal $\beta$ by fitting, and then acquired the corresponding model of fractal CAPM (FCAPM; Ding et al., 2012; Raei & Mohammadi, 2008). Meanwhile, some studies used the generalized rate of return to fit the $\beta$, which developed the model of scaled variable CAPM (SV-CAPM; Wei et al., 2014).

In the aspect of portfolio optimization, some studies optimizd portfolio by the mean-DCCA model (Mean-DCCA) and the mean-MF-DCCA model (Tang & Zhu, 2018; Yuan et al., 2020); while others identified the trend of the securities market or distinguished the forecasting ability of the securities market to the macro-economy by modeling (Liu et al., 2020; Tan et al., 2018).

In the aspect of risk management, some studies revealed the endogenous dynamics mechanism of the financial storm based on the nonlinearities and complexities of the securities market or monitored the systemic risk with methods such as the “earthquake magnitude index”; and others used the method of wavelet leader multifractal analysis to detect the risk of the securities market (Zhang et al., 2014) or manage the multifractal risk of the securities market and portfolio with methods such as the twin-support vector machine model (Twin-SVM), and the multifractal volatility in week-VaR model (MFVW-VaR; Wang & Huang, 2018; Xu & Wang, 2019).
In addition, some researches on FMH, such as improving the B-S model for financial derivatives pricing and tracking the time-variability characteristic of $\beta$ are highly active branches of emerging finance (Tang & Chen, 2011). The mentioned researches first expanded the traditional financial theories and tentatively grafted the behavioral finance into the FMH, which provided better explanations to emerging financial markets, and offered good references and enlightenments on the follow-up researches.

Conclusions

The EMH and the FMH are the representative theories of neoclassical finance and emerging finance, which depict the linearity and nonlinearity of the securities market, respectively. It can be observed from the two aspects of the yield distribution functions and characteristic exponents that the fractal market is the general form and steady state of the securities market, while the effective market is the special form and bias state of the securities market. Therefore, the two theories have intrinsic uniformity. The EMH reveals the ideal and special state of the securities market and provides a target and anchor for a deep understanding of the volatility of securities price and the laws of market operation. In a more general sense, the FMH depicts the volatility of securities price and the laws of market operation and offers a higher-level abstraction and description of the securities market. Basically, the FMH is not a negation or subversion of the EMH, but is the latter’s improvement and sublimation.

The FMH complements and improves the EMH, while the FMH reconstructs the financial investment analysis framework. It is important to correctly understand and handle the relationship between the EMH and the FMH, and explore the theoretical and the application prospects of the FMH. This study’s efforts are meant to further improve the development of nonlinear scientific theories and provide guidance to the practice of financial investment.

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