A mean field game approach modeling pedestrian excursionists’ flow

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Abstract—In this paper, we address the problem of modeling the tourists’ flow in historic city centers whose narrow alleys are represented by a transportation network. We consider a mean filed games approach where the standard forward backward system of equations is also intertwined with a tourists’ path preferences dynamics. The path preferences are influenced by the congestion status on the whole network as well as the possible hassle of being forced to run during the tour.

We prove the existence of a mean field game equilibrium as a fixed point, over a suitable set of time-varying distributions, of a map obtained as a limit of a sequence of approximating functions. Then, a bi-level optimization problem is formulated for an external controller who aims to induce a specific mean field game equilibrium.

Index Terms—Tourist flow optimal control; mean field games; path preference dynamics; dynamical flow networks.

I. INTRODUCTION

In the recent years, the continuous growth of tourists’ flow and the resulting overcrowding have led some heritage cities to seek solutions to manage this phenomenon. The crowd motion modeling and the study of pedestrian flow dynamics have become in the last decades two of the main targets of the transportation research community. Different modeling approaches have been proposed which can generally be classified into two categories: microscopic models and macroscopic models. The former include the cellular automaton model [1]–[2], the social force model [3]–[4], and the lattice gas model [5]–[6], and are particularly well suited for use with small crowds. Macroscopic models, in contrast, focus on the overall behavior of pedestrian flows and are more applicable to investigations of extremely large crowds, especially when examining aspects of motion in which individual differences are less important [7]–[10]. In this paper, starting from the results in [11]–[12], we introduce a Mean Field Games (MFG) approach to modeling and analytically studying the flow of daily pedestrian excursionists (hereinafter referred as agents) along the narrow alleys of the historic center of a heritage city. Mean field games theory goes back to the seminal work by Lasry-Lions [13] (see also [14]). This theory includes methods and techniques to study differential games with a large population of rational players and it is based on the assumption that the population influences individuals’ strategies through mean field parameters. Several application domains such as economics, physics, biology and network engineering accommodate MFG theoretical models (see [15]–[19]). In particular, models to study dynamics on networks and/or pedestrian movement can be found for example in [20]–[23]. In this paper, beside the usual framing of mean field games (which is typically defined by the pair made of Hamilton-Jacobi-Bellman and transport equations), we consider the agent’s path preferences dynamics. In particular, we assume that it evolves following a perturbed best response to global information about the congestion status of the whole network and to the control vector. Moreover, this path preferences dynamics evolves at a slow time scale as compared to the physical dynamics. We apply these arguments to the possible paths considering a network topology as in Figure I.

Our main result shows the existence of a MFG equilibrium for our framework. This equilibrium is a time-varying distribution of agents, \( \tilde{\rho}(s) \) for \( s \in [0,T] \), on the network. Distribution \( \tilde{\rho}(s) \), when plugged in the cost to minimize, generates an optimal control \( \tilde{u}(s) \), for any agent starting at the origin \( o \) which, in turn, yields the path preference \( z(s) \) providing the time-varying distribution \( \tilde{\rho} \). We also introduce a possible bi-level optimization problem for an external controller who aims to force the equilibrium to be as close as possible (in uniform topology) to a reference value \( \check{\rho} \). We suppose that the external controller (the city hall, for example) may act on the latency functions choosing them among a suitable set of admissible functions.

The rest of this paper is organized as follows. In Section II we describe the model and state the hypotheses used in the paper. In Section III we claim the existence of a MFG equilibrium (due to the space limitation, we only include the sketches of the proofs) and, in Section IV we define a bi-level optimization problem. In Section V we draw conclusions and suggest future works.

II. MODEL DESCRIPTION

A. Network characteristics

We model the topology of the network as a directed multigraph \( G = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} \) is a finite set of nodes and \( \mathcal{E} \) is a finite set of directed links. Each link \( e = (\nu_e, \kappa_e) \) in \( \mathcal{E} \) is directed from its tail node \( \nu_e \) to its head node \( \kappa_e \neq \nu_e \). An oriented path from a node \( \nu_{v_0} \) to a node \( \nu_r \) is an ordered set of \( r \) adjacent links \( p = (e_1, e_2, \ldots, e_r) \) such that \( \nu_{e_i} = \nu_{e_{i+1}} \), \( \kappa_{e_i} = \nu_{e_{i+1}} \), \( \nu_{s} = \nu_{e_{i+1}} \) for \( 1 \leq s \leq r-1 \), and no node is visited twice, i.e., \( \nu_{s} \neq \nu_{l} \) for all \( 0 \leq l < s \leq r \), except possibly for \( \nu_{v_0} = \nu_{v_r} \), in which case the path is referred to as a cycle. Moreover, let \( \ell_e \) be the length of the link \( e \in \mathcal{E} \).
and let \( \ell_p = \sum_{e \in p} \ell_e \) be the length of the path \( p \). A node \( v_j \) is said to be reachable from another node \( v_k \) if there exists at least a path from \( v_k \) to \( v_j \).

We hold the following assumptions on the multi-graph \( \mathcal{G} \) and on the agents that move along its links.

**Assumption 1:**

1) \( \mathcal{G} \) contains no cycles.
2) \( \mathcal{G} \) includes an origin node \( o \), from which any node in \( \mathcal{V} \) can be reached, and a destination node \( d \neq o \), which is reachable from any node in \( \mathcal{V} \), \( o \) included.
3) Excursionists arrive at \( o \) in the morning and desire to leave from \( d \) in the afternoon.

We denote \( \Gamma \) as the set of all the paths from \( o \) to \( d \). In particular, in this paper, we consider a graph \( \mathcal{G} \) on which agents have only three possible paths to reach the destination \( d \) starting from the origin \( o \) (see Figure 1).

![Diagram of the graph topology used in the paper.](image)

Fig. 1. The graph topology used in the paper.

For every link \( e \in \mathcal{E} \) and time instant \( t \in [0, T] \), we denote the current mass and flow by \( \rho_e(t) \) and \( f_e(t) \), respectively, defined as

\[
\rho_e : [0, T] \rightarrow [0, \rho_{\text{max}}], \quad f_e : [0, T] \rightarrow [0, C_e]
\]

with \( T > 0 \) the final horizon, i.e., the time by which every agent has to reach the destination \( d \), and \( C_e \) is the maximum flow capacity. Moreover let

\[
\rho(t) := \{ \rho_e(t) : e \in \mathcal{E} \}, \quad f(t) := \{ f_e(t) : e \in \mathcal{E} \}
\]

be the vectors of masses and flows, respectively.

**B. Agents’ dynamics and costs**

\[
\begin{align*}
\dot{\theta}_e(s) &= u^e(s), \quad s \in [t, T] \\
\theta_e(t) &= 0 \quad \forall t \in [0, T],
\end{align*}
\]

where \( \theta_e(s) \in [0, \ell_e] \), being \( \ell_e \) the length of the link. Using this space-coordinates, \( \dot{\theta}_e(s) = 0 \) means that the agent is in \( \nu_e \), while \( \dot{\theta}_e(s) = \ell_e \) means that the agent is in \( \kappa_e \) and hence he is inside the link \( e \) as long as \( 0 \leq \theta(s) \leq \ell_e \). By definition, \( \theta_e(s) \) and \( u^e(s) \) are equal to \( NaN \) when the agent is not on link \( e \). Hereinafter, \( NaN \) stands for not a number.

The control, \( s \mapsto u^e(s) \in \mathbb{R}_+ \), is measurable and locally integrable, namely \( u^e \in L^1_{\nu_e}(T, T) \) \( \forall t \). For ease of notation, from now on, we call \( \mathcal{U}_e \) the set of these kind of controls. There is no loss of optimality in assuming \( u^e \geq 0 \) as we discuss later in Remark 1. The cost to be minimized by every agent crossing a link \( e \), takes into account: i) the possible hassle of running in the link \( e \) to reach \( d \) on time; ii) the pain of being entrapped in highly congested link; iii) the disappointment of not being able to reach \( d \) by the final horizon \( T \). Such a cost can be analytically represented by

\[
J_e(t, u^e) = \int_t^T \chi(0 \leq \theta_e(s) \leq \ell_e) \left( \frac{(u^e(s))^2}{2} + \varphi_e(\rho_e(s)) \right) ds + \chi(0 \leq \theta_e(T) < \ell_e) \alpha \sum_{e \in \mathcal{E}} \ell_j,
\]

(3)

where \( p(e) \) is the shortest path from \( \nu_e \) to \( d \), \( \alpha > 0 \) is a constant parameter representing a cost per unit of length and \( \chi \) is the characteristic function. In (3), the quadratic term inside the integral stands for cost i); while the other term stands for the congestion cost ii); the last addendum stands for cost iii).

We define, with a little abuse of terminology, as value functions the following:

\[
V^e(t) = \inf_{u^e \in \mathcal{U}_e} \left\{ J_e(t, u^e) + \min_{e \in \mathcal{E}, j \in \mathcal{P}(e)} \{ V(j, u^e) \} \right\} \quad \forall e \in \mathcal{E}
\]

(4)

\[
V^0(t) = \min_{e \in \mathcal{E}, e \in \mathcal{P}(e)} \{ V^e(t) \}.
\]

(5)

In (3), \( \tau(t, u^e) \) is the time at which an agent entering link \( e \) at time \( t \) and choosing a control \( u^e \) arrives in \( \nu_j = \kappa_e \).

Note that the recursive definition of \( V^e \) in (4) is not meaningless since the absence of oriented cycles in the network \( \mathcal{G} \) prevents self-referring, that is it cannot occur that we define \( V^e \) in terms of itself. Also \( V^0 \) can be seen as associated to a fictitious link \( c_0 \) with null length, such that \( \nu_{c_0} = \kappa_{c_0} = o \), through which the flow enters into the network. Finally, observe that when the distributional evolution \( t \rightarrow \rho(t) \), i.e. the mass of the agents on the link \( e \), is initially given then \( V^e \) does not depend on \( \rho_e \).

In this paper, we assume that the physical traffic flow \( f \) consist of indistinguishable homogeneous agents which enter in the network through the origin node, travel through it on the different paths and finally exit from the network through the destination node. The relative appeal of the different paths to the agents is modelled by a time-varying nonnegative vector \( z(t) \) in the simplex

\[
\mathcal{S}_z(t) = \left\{ z \in \mathbb{R}^\mathcal{E}_+ : 1'z(t) = \lambda(t) \right\},
\]

(6)

where \( \lambda(t) : [0, T] \rightarrow [0, \rho_{\text{max}}] \) is the throughput, i.e., the total flow that goes through the network. We refer to the vector \( z(t) \) as the current aggregate path preference and let

\[
y^z(t) = Az(t)
\]

(7)
be the flow vector associated to it. The vector $z(t)$ is updated as agents access global information about the current congestion status of the whole network (that is embodied by the mass vector $\rho(t)$) and it is also influenced by the vector $u(t) = \{u^e(t) \neq NaN : e \in E\}$. Hereinafter, $U$ is the set of these vectors. Specifically, the cost perceived by each agent, traversing a link $e \in E$, is given by (3). The cost that an agent expects to incur along a path $p$ is

$$JP(t, u) = \sum_{e \in \mathcal{E}} A_{ep}J_e(t^e_p, u^e)$$

where $t^e_p(t)$, when $e \in p$, is the time instant in which an agent, arriving in $t$ in the origin $o$ and following the path $p$, reaches $v_e$. Clearly, $t^e_p$ depends on the controls $u^l$ for each link $l \in E$ that precedes the link $e$ along $p$. Arbitrarily, we set $t^e_p(t) = t$ when $e \notin p$.

We denote with

$$J(t, u) = \{JP(t, u) : p \in \Gamma\}$$

the vector of costs on all the paths $p \in \Gamma$. Then, we assume that the path preferences are updated at some rate $\eta > 0$, according to a noisy best response dynamics

$$\dot{z}(t) = \eta(F^{(\beta)}(u(t), \rho(t)) - z(t)), \quad z(0) = z_0$$

where for every fixed noise parameter $\beta > 0$ the function

$$F^{(\beta)} : U \times \prod_{e \in \mathcal{E}} [0, \rho_{max}] \rightarrow S_{\lambda(t)}$$

is the perturbed best response defined as follows:

$$F^{(\beta)}(u, \rho) = \frac{\lambda \exp(-\beta(J(. , u)))}{V \exp(-\beta(J(., u)))},$$

and associated to the negative entropy function $\gamma : S_{\lambda} \rightarrow \mathbb{R}$

$$\gamma(z) = \beta^{-1} \sum_{p \in \Gamma} z_p \log z_p$$

using the standard convention that $0 \log 0 = 0$. The perturbed best response (9) provides an idealized description of the behavior of agents whose decisions are based on inexact information about the state of the network. We will need to let $\gamma$ tend to zero through the limit for $\beta \rightarrow \infty$ when interesting in obtaining the exact values of the mass $\rho$, as explained in [24].

We now describe the local decision function $G : S_{\lambda(t)} \rightarrow \mathbb{R}^{\mathcal{E}}$ characterizing the fractions of agents choosing each outgoing link $e$ when traversing a non destination node $v$. It is given by

$$G_e(z) = \begin{cases} \sum_{j \in \mathcal{E} : \theta_j = \theta_e} y_j^e & \text{if } \sum_{j \in \mathcal{E} : \theta_j = \theta_e} y_j^e > 0 \\ \frac{1}{|\{j \in \mathcal{E} : \theta_j = \theta_e\}|} & \text{if } \sum_{j \in \mathcal{E} : \theta_j = \theta_e} y_j^e = 0 \end{cases}$$

for each outgoing link $e$ in $\mathcal{E}$ and it is continuously differentiable. Equations (10) state that, at every node $v \in V$ the outflow is split proportionally to the flow vector $y^v$, if there is flow $y^v$ passing through node $v$, otherwise, the outflow is split uniformly among the immediately downstream links.

Now, for every non-destination node $v$ and outgoing link $e \in E$ conservation of mass implies that

$$\dot{\rho}(t) = H(f(t), z(t)), \quad \rho(0) = \rho_0$$

where $H : \prod_{e \in \mathcal{E}} [0, C_e] \times \mathbb{R} \rightarrow \mathbb{R}^\mathcal{E}$ is

$$H_e(f, z) := G_e(z)(\lambda \delta_{\rho_e} + \sum_{j : \kappa_j = \theta_e} f_j) - f_e, \quad \forall e \in \mathcal{E}$$

and each component $f_e$ of $f$ is $f_e(t) = \rho_e(t)u^e(\rho)/\epsilon_e$, i.e., the flow on the link $e$ estimated by an agent entering in the very link at time $t$.

Next we introduce the hypotheses that we will use to prove the existence of a MFG equilibrium:

(H1) the throughput $\lambda : [0, T] \rightarrow [0, +\infty]$ is a bounded and Lipschitz continuous function;

(H2) $\rho(0) = 0$;

(H3) $\varphi_e : [0, \rho_{max}] \rightarrow [0, +\infty]$ are Lipschitz continuous;

(H4) $\varphi_e$ does not depend explicitly on state variable $\theta$.

Note that $\rho(0) = 0$ in (H2) means that no one is around the city at $t = 0$, while (H4) means that all agents in the same link at the same instant equally suffer the same congestion. In [11, Theorem 1] similar assumptions are used to prove that the value function $V^e$ coincides in every link with the unique (continuous) viscosity solution of the corresponding system of Hamilton-Jacobi-Bellman equations.

Remark 1: Note that (H1) (H3) (H4) also imply that the optimal control $u^e$ is always non-negative.

C. Value Functions

In order to define the MFG equilibrium instead of coupling the two standard MFG equations, i.e., Hamilton-Jacobi-Bellman equations and mass conservation ones (11), with the preferences equations (8) we write conditions equivalent in terms of the value functions. An agent standing at $\nu_e$, and hence at $\theta_e = 0$, for $e \in \{e_4, e_5\}$ at time $t$ has two possible choices: either staying at $\nu_e$ indefinitely or moving to reach $\kappa_e = d$ exactly at time $T$ (it is not optimal to reach $d$ before $T$ and wait there for a positive time length, see, e.g., [11]). Accordingly, the controls are

$$u_1^e = 0, \quad u_2^e = \frac{\ell_e}{T-t}.$$  

Hence, given the cost functional (3), we derive

$$V^e(t) = \min \left\{ \alpha \ell_e, \frac{1}{2} \left(\frac{\epsilon_e}{t} - \frac{\epsilon_e}{T-t}\right)^2 \right\} + \int_t^T \varphi_e ds$$

(note that we do not display the argument $\rho_e$ of $\varphi_e$ for $e \in \{e_4, e_5\}$). We use this convention also for the other $\varphi_e$ in the
formulas below). An agent standing at $\nu_e$ at time $t \in [0, T]$, has two possible choices (meaning, the optimal behaviour may only be one of the following two): staying in $\nu_e$ or moving to reach $\kappa_{\nu_e}$ at $\tau \in [t, T]$. The controls among which the agent chooses are then, respectively

$$u_1^{\nu_e} = 0, \quad u_2^{\nu_e} = \frac{\ell_{\nu_e}}{\tau - t}. \quad (15)$$

Hence,

$$V^{\nu_e}(t) = \min \left\{ \alpha \left( \ell_{\nu_e} + \ell_{u_e} \right), \right. \left. \inf_{\tau \in [t, T]} \left\{ \frac{1}{2} \left( \frac{\ell_{\nu_e}}{\tau - t} \right)^2 + \int_t^\tau \varphi_{\nu_e} \, ds + V^{\nu_e}(\tau) \right\} \right\}. \quad (16)$$

An agent standing at $\nu_{e_1}$ at time $t$ may choose: staying in $\nu_{e_1}$ or to reach $\kappa_{\nu_{e_1}}$ at a certain $\tau \in [t, T]$. Hence, the control is chosen among

$$u_1^{\nu_{e_1}} = 0, \quad u_2^{\nu_{e_1}} = \frac{\ell_{\nu_{e_1}}}{\tau - t}. \quad (17)$$

Consistently, one has

$$V^{e_1}(t) = \min \left\{ \alpha \min \left\{ \ell_{e_1} + \ell_{u_1}, \ell_{e_1} + \ell_{u_1} + \ell_{e_2} \right\}, \right. \left. \inf_{\tau \in [t, T]} \left\{ \frac{1}{2} \left( \frac{\ell_{e_1}}{\tau - t} \right)^2 + \int_t^\tau \varphi_{e_1} \, ds + \min\{V^{e_3}(\tau), V^{e_4}(\tau)\} \right\} \right\}. \quad (18)$$

Analogous arguments hold for computing $V^{e_2}(t)$ when an agent is standing at $\nu_{e_2}$, indeed in that case the controls are

$$u_1^{\nu_{e_2}} = 0, \quad u_2^{\nu_{e_2}} = \frac{\ell_{\nu_{e_2}}}{\tau - t}. \quad (19)$$

Hence the value function is

$$V^{e_2}(t) = \min \left\{ \alpha (\ell_{e_2} + \ell_{e_3}), \right. \left. \inf_{\tau \in [t, T]} \left\{ \frac{1}{2} \left( \frac{\ell_{e_2}}{\tau - t} \right)^2 + \int_t^\tau \varphi_{e_2} \, ds + V^{e_5}(\tau) \right\} \right\}. \quad (20)$$

Finally,

$$V^0(t) = \min \{V^{e_1}(t), V^{e_2}(t)\}. \quad (21)$$

Note that, the optimal controls described in (15), (17), (19) are detected along with the arrival time $\tau$ along the minimization process carried on in (16), (18), (20). Also, when $\rho$ is given, the construction of the optimal controls is performed backwardly, starting from the problem (14). Finally, observe that $V^0(t)$ in (21) is determined without the necessity of computing any optimal control on the fictitious link $\nu_0$.

We summarize the previous discussion as follows.

**Theorem 1:** Suppose that $\rho$ is given and that (H1)–(H4) hold. Then, the value function $V^\epsilon : [0, \ell_e] \times [0, T] \to \mathbb{R}$, at $0$ i.e., at the beginning of each link, and for all $t \in [0, T]$, is determined through (14)–(20). In addition, the set $V$ of functions $V^\epsilon$ is the unique viscosity solution to the set of Hamilton-Jacobi-Bellman equations defined for the links of the network $G$ with suitable boundary conditions. Moreover, $V$ is Lipschitz continuous with respect to time, with its Lipschitz constant independent of $\rho$.

**Proof:** The proof in analogous to the one in [12].

**Remark 2:** Notice that, since the optimal controls are necessarily equi-bounded by a constant depending only on the parameters of the problem and since (H1) holds, by results on the mass conservation equations (see, e.g., [25]), follow that all $\rho_e$ are Lipschitz continuous, with Lipschitz constant $\tilde{L}$ depending only on the parameters of the problem.

### III. EXISTENCE OF A MEAN FIELD EQUILIBRIUM

In this section we give a proof of the existence of a MFG equilibrium. Let $L(w)$ the Lipschitz constant of a function $w$. As space to search for a fixed point, we choose

$$X = \left\{ w : [0, T] \to [0, \rho_{\text{max}}] : L(w) \leq \tilde{L}, \ |w| \leq \rho_{\text{max}} \right\}.$$

(22)

the Cartesian product five times of the space of Lipschitzian functions with Lipschitz constant not greater than $\tilde{L}$ and overall bounded by $\rho_{\text{max}}$, where $\tilde{L}$ is the constant introduced in Remark 2. Note that $X$ is convex and compact with respect to the uniform topology.

We then search for a fixed point of the function $\psi : X \to X$, with $\rho \mapsto \rho' = \psi(\rho)$ where $\rho'$ is obtained as follows: (i) $\rho$ is inserted in (14)–(20), the optimal control $u$ is derived; (ii) $u$ is inserted in (8) and the path preference vector $v$ is obtained; (iii) $\rho'$ is derived from (11) after the computation of (10) and the passage to the limit for $\beta \to \infty$.

**Remark 3:** Note that in general $\psi$ may be a multi-function, that is $\psi(\rho)$ may be a subset of $X$. Indeed the optimal control may not be unique as, for any fixed $t$, the minimization procedure in (14), (16), (18), (20) may return more than one minimizer. In particular, this may happen even along a whole time interval. With uniqueness of the optimal control all agents in the same position at the same instant make the same choice (consistently with MFG models, where agents are homogeneous and indistinguishable). When instead uniqueness fails, and hence a multiplicity situation occurs, $\psi$ can be built in many ways, as many as the different optimal behaviors. To bound the times at which such multiplicities appear, we will obtain $\psi$ and its fixed point $\tilde{\rho}$ through a limiting procedure as in [12]. However, we point out that the procedure used here differs by the one in [12] due to the presence of the path preference dynamics (8).

Let $\{\psi_e\}_{e \geq 0}$ be a sequence of functions approximating $\psi$ in a suitable sense, and the corresponding fixed points $\rho_e$. The single function $\psi_e$ is obtained through (i)–(iii) above, with the difference that in (ii), rather than choosing optimal controls, one chooses $\epsilon$–optimal controls and, along time, an $\epsilon$–optimal stream. Accordingly the path preference vector will be computed once the values of that $\epsilon$–optimal stream are given. We divide the construction into two steps.

**Step 1:** $\epsilon$–optimal streams.

**Definition 1:** Assume for each link $\epsilon = (\nu_e, \kappa_e) \in \mathcal{E}$ to be in $\theta_e = 0$, i.e., at node $\nu_e$, and let $u^{\nu_e}_i, i \in \{1, 2\}$ be the
controls defined through (13) (15) (17) (19). Consider also a partition \( \tau^e = \{ t^n \}_{n} \) of the interval \([0, T]\), and fix \( \varepsilon > 0 \). Then \( u^e_n \) is an \( \varepsilon \)-optimal stream for the link \( e \) associated to the partition \( \tau^e \) if

\[
u^e_n(s) = u^e_n(s), \quad s \in [t^n, t^{n+1}]
\]

where \( u^e_n \) is optimal at \( t^n \) and \( \varepsilon \)-optimal at all \( s \in ]t^n, t^{n+1}[, \) that is, it realizes the minimum cost up to an error not greater than \( \varepsilon \).

Notice that an \( \varepsilon \)-optimal stream associated to a general partition \( \tau \) may or may not exist, but it certainly does when the partition is refined enough. Indeed, considering the functions involved in the minimization process in (14), (16), (18), (20), if the minimum is realized up to the error \( \varepsilon \), the minima are attained within intervals of type \([t + h, T]\), for some suitable \( h > 0 \), so that the functions cited above are Lipschitz continuous. Denote by \( L \) the greater of Lipschitz constants of these functions. Then a control \( u^e \) optimal at \( t \) remain \( \varepsilon \)-optimal at least along \([t, t + \varepsilon/L] \). In addition note that, for every link \( e \), there may exist more than one \( \varepsilon \)-partition, as optimal controls determined in \( \theta_e = 0 \) may be multiple. Anyway, the number of \( \varepsilon \)-partitions is overall bounded by a number \( N_e \), as the optimal controls are at most two. This argument proves the following Lemma.

**Lemma 1:** Fixed \( \varepsilon > 0 \), set \( N = \max\{n \in \mathbb{N} : n < (TL)/\varepsilon \} + 1 \). Consider the partition \( \tau_e \) of \([0, T]\) such that:

1. \( t^0_e = 0; t^N_e = T; \)
2. \( t^n_e = nL/\varepsilon \) for all \( n \in \{1, \ldots, N - 1\} \).

Then the set of \( \varepsilon \)-optimal streams associated to \( \tau_e \) is nonempty and finite.

**Remark 4:** By definition, an agent implementing a control chosen along a \( \varepsilon \)-optimal stream is using an \( \varepsilon \)-optimal control, that is, it is realizing the minimum of the payoff up to a maximum error of \( \varepsilon \). Streams all start at \( t^0_e = 0 \) and are defined along the interval \([0, T]\). Note that an \( \varepsilon \)-optimal stream is not the strategy of a single agent at different times, whereas a strategy that is \( \varepsilon \)-optimal if adopted, along time, by the agents at \( \theta_e = 0 \), independently of the fact that any agent is present at the beginning of link \( e \) at that time.

Now, for a fixed \( \varepsilon > 0 \), consider the partition \( \tau_e \) defined in Lemma 1. Consider then the vector \( u_e = \{ u^e_e : e \in E \} \) whose components are \( \varepsilon \)-optimal streams associated to \( \tau_e \) in \( \theta_e = 0 \). As consequence of Lemma 1, each \( \varepsilon \)-stream includes a finite number of elements. Note that once \( u_e \) is fixed, all agents make the same decision, in time. Indeed, given the value of the \( \varepsilon \)-stream on each subinterval of \( \tau_e \), each agent, in that subinterval, can evaluates the total cost that he expects to incur in each path \( p \in \Gamma \) and consequently it will be possible to compute \( z \) through (18). Hereinafter, we refer to this \( z \) as \( z_e \), since its value depends on \( u_e \). The possibility for agents to split into fractions among different vectors \( u_e \) (that is, splitting among multiple optimal controls on instants of \( \tau_e \), is given by the local decision function (10) after the computation of the vector \( z_e \).

**Step 2:** Construction of \( \psi_e(\rho) \).

Let \( \varepsilon > 0 \) and \( \rho = (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) \in X \) be fixed. Let also \( \tau_e \) be the partition of \([0, T]\) described above. Using the \( \varepsilon \)-optimal vector \( u_e \) together with the paradigm (i)-(iii) we build the multifunction \( \psi_e(\rho) \subseteq X \) with compact and convex images and closed graph, to which later we can apply Kakutani fixed point theorem.

(a) We define \( \psi_e(\rho) \subseteq X \) as the finite set of vectors \( \rho' = (\rho'_1, \rho'_2, \rho'_3, \rho'_4, \rho'_5) \in X \) constructed in the following way. We consider an \( \varepsilon \)-optimal vector \( u_e \) associated to \( \tau_e \) in \( \theta_e = 0 \) for every \( e \in E \). Then, we use \( u_e \) to determine the path preference vector \( z_e \). Finally, given the arrival flow of agents \( \lambda^{(0)} \) and computed the flow \( f \) associated to \( \rho \), we use \( z_e \) to solve \( 11 \) and the limit for \( \beta \) goes to infinity of its solution gives the total mass \( \rho' \). We repeat the construction for all possible (and finite, by Lemma 1) choices \( u_e \) and call the set of all outcomes \( \psi_e(\rho) \) which is a finite set.

(b) We define \( \psi_e(\rho) \subseteq X \) as follows. We consider an arbitrary \( \varepsilon \)-local decision function \( G^e \) (called in this way because it is given by \( z_e \)), and assume that at every point of \( \tau_e \) the outflow of agents \( f_e \) from every incoming link \( j \) gets split in fraction \( G^e_e(z_j) \) among the links \( e \) immediately downstream the link \( j \). After the end of the process and subsequently to the passage to the limit \( \beta \), the output is \( \rho' \). We define \( \psi_e(\rho) \) as the set of all \( \rho' \) which are the limits of all \( \rho'_e \) generated by all possible choices of the \( \varepsilon \)-local decision function \( G^e \). Clearly \( \psi_e(\rho) \supseteq \psi_e(\rho) \).

**Lemma 2:** The set \( \psi_e(\rho) \) is a non-empty convex and compact subset of \( X \), for any \( \rho \in X \). Moreover, the map \( \rho \mapsto \psi_e(\rho) \) has closed graph and it has a fixed point \( \rho_e \in X \).

**Proof:** We preliminarily observe that the set \( \psi_e(\rho) \) is non-empty by construction. In addition, it is also closed, and hence, since \( X \) is compact, it is compact. Moreover, it is possible to prove that \( \psi_e(\rho) \) is the convex hull of \( \psi_e(\rho) \) which is the set of all \( \rho \) whose corresponding \( G^e \) are obtained through all possible \( \varepsilon \)-streams. Observe that the set \( \psi_e(\rho) \) is finite and includes at most \( M_e \) elements. Hence the set \( \psi_e(\rho) \) has a finite number of extremal points. This fact, together with the regularity assumptions (H1) and (H3), implies, reasoning as in (12) but taking into account \( \psi_e(\rho) \) and the limit in \( \beta \), that the multifunction \( \rho \mapsto \psi_e(\rho) \) has closed graph. Hence, by the Kakutani theorem, there exists a fixed point \( \rho_e \in \psi_e(\rho_e) \).

Hereinafter, we denote by \( \rho_e \) a fixed point for \( \psi_e(\rho) \), i.e., a total mass that satisfies \( \rho_e \in \psi_e(\rho_e) \).

Before stating the existence of a MFG equilibrium, we introduce the following definition that help restrict the equilibrium concept to the purpose of our problem.

**Definition 2:** Let \( \psi \) and \( \psi_e \) be the functions described at the beginning of Section 11.1.

- A \( \varepsilon \)-MFG equilibrium is a total mass \( \rho_e \in X \) that satisfies \( \rho_e \in \psi_e(\rho_e) \).
- A MFG equilibrium is a total mass \( \rho \in X \) that satisfies \( \rho \in \psi_e(\rho) \).

Note that \( \rho \in \psi_e(\rho) \) implies that \( \rho \) induces a set of optimal controls as in (13), (15), (17), (19) used to compute the correspondent path preference vector \( z \) trough the perturbed
best response $F^{(\beta)}$. Accordingly, the local decision function is obtained, which splits the flows in every link. Then, using (11) we get a $\rho_\beta$ whose limit for $\beta \to \infty$ is just $\rho$.

**Theorem 2:** Assume (H1)–(H4). Then there exists a MFG equilibrium.

**Proof:** (Rough sketch) The proof is based on the arguments in Theorem 1 of [12]. Here we also to consider the agent’s preference dynamics and the consequent limit in $\beta$.

IV. BI-LEVEL OPTIMIZATION

In this section, we discuss how the results introduced in the previous sections can be used by a central authority (CA), e.g., the city hall, interested in controlling the value of the mass $\rho$. For any given input flow $\lambda(0)$, heritage CA may be consider sustainable for their historical centers that mass of agents is closed to some reference value $\bar{\rho}$. Typically, a CA may slow down (and sometimes also speed up) the agent flows by intervening on the width of the streets with mobile barriers. Formally, we can assume that the CA can consider congestion cost functions of the form

$$\varphi_e(\rho_e) = \alpha'_e \rho_e(t) + \alpha''_e \quad \forall e \in E,$$

where $(\alpha', \alpha'')$ are parameters in a compact $K \subset \mathbb{R}^5 \times \mathbb{R}^5$.

Then, the CA is interested in solving the bi-level problem:

$$\inf_{(\alpha', \alpha'')} \left( \sup_{\rho} \| \rho - \bar{\rho} \|_{\infty} \right)$$

subject to

$$\inf_{u(\rho) \in U_{\alpha', \alpha''}} \check{J}(t, u)$$

with $\check{J}(t, u) = \{J_e(t, u_e) : e \in E\}$ and where the CA chooses the value of pair $(\alpha', \alpha'')$ in the light of the expected response of the agents.

We claim that, if the set $U_{\alpha', \alpha''}$ is a compact set of controls depending by $(\alpha', \alpha'')$, then the existence of a pair $(\alpha', \alpha'')$’ solution of (23) can be proved provided that a unique MFG equilibrium exists for each fixed $(\alpha', \alpha'') \in K$.

V. CONCLUSIONS

In this paper the existence of a mean-field games equilibrium for a pedestrian tourists’ flow model is provided and a bi-level optimization problem is formulated. In future research we plan to refine the optimization problem from both a theoretical and an applicative perspective, and to consider different objectives of central authority. Moreover a comparison between our MFG model and the Wardrop one also through numerical simulations will be provided. A further step could be to consider a more general networks that includes oriented cycles. For this kind of networks the backward approach used for the resolution of equations (14–21) cannot immediately applied.

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