STRESS MATRICES AND GLOBAL RIGIDITY OF FRAMEWORKS ON SURFACES

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Abstract. In 2005, Connelly showed that a generic framework in \( \mathbb{R}^d \) is globally rigid if it has a stress matrix of maximum possible rank, and that this sufficient condition for generic global rigidity is preserved by the 1-extension operation. His results gave a key step in the characterisation of generic global rigidity in the plane. We extend these results to frameworks on surfaces in \( \mathbb{R}^3 \). For a framework on a family of concentric cylinders, cones or ellipsoids, we show that there is a natural surface stress matrix arising from assigning edge and vertex weights to the framework, in equilibrium at each vertex, such that the rank of this matrix being maximum is sufficient to guarantee generic global rigidity on the surface. We then show that this sufficient condition for generic global rigidity on these surfaces is preserved under 1-extension and use this to make progress on the problem of characterising generic global rigidity on the cylinder.

1. Introduction

A bar-joint framework \((G, p)\) in \(d\)-dimensional Euclidean space \(\mathbb{R}^d\) is a realisation of a (simple) graph \(G = (V, E)\), via a map \(p : V \rightarrow \mathbb{R}^d\), with vertices considered as universal joints and edges as fixed length bars. A framework \((G, p)\) is rigid if the only continuous motions of the vertices in \(\mathbb{R}^d\) that preserve the edge lengths, arise from isometries of \(\mathbb{R}^d\). More strongly, \((G, p)\) is globally rigid in \(\mathbb{R}^d\) if every realisation \((G, q)\) with the same edge lengths as \((G, p)\) arises from an isometry of \(\mathbb{R}^d\). We refer the reader to \cite{22} for more information on rigidity and its applications.

It is NP-hard to determine if an arbitrary framework is rigid \cite{1} or globally rigid \cite{20}. These problems become more tractable if we restrict to generic frameworks, for which rigidity and global rigidity can be determined in polynomial time when \(d = 1, 2\). It remains a difficult open problem to characterise, in an efficient combinatorial way, when a 3-dimensional generic framework is rigid or globally rigid.

Results have recently been obtained concerning the rigidity and global rigidity of frameworks in \(\mathbb{R}^3\) that are constrained to lie on a fixed surface \cite{18, 19, 15}. In this paper we obtain a natural sufficient condition for such a framework to be globally rigid.

We first recall some fundamental results about the generic (global) rigidity of bar-joint frameworks in Euclidean space. A graph \(G = (V, E)\) is \((2, k)\)-sparse if for every subgraph \(G' = (V', E')\), with at least one edge, the inequality \(|E'| \leq 2|V'| - k\) holds. Moreover \(G\) is \((2, k)\)-tight if \(|E| = 2|V| - k\) and \(G\) is \((2, k)\)-sparse.

Theorem 1.1 (\cite{11}). A generic framework \((G, p)\) in \(\mathbb{R}^2\) is rigid if and only if \(G\) contains a spanning \((2, 3)\)-tight subgraph.

A framework \((G, p)\) is redundantly rigid if \((G - e, p)\) is rigid for all edges \(e\) of \(G\).
Theorem 1.2 ([7] [12]). A generic framework \((G, p)\) in \(\mathbb{R}^2\) is globally rigid if and only if \(G\) is a complete graph on at most three vertices or \((G, p)\) is redundantly rigid and \(G\) is 3-connected.

Hendrickson [10] had previously shown that \((d + 1)\)-connectivity and redundant rigidity are necessary conditions for generic global rigidity in \(\mathbb{R}^d\) for all \(d\). Examples constructed by Connelly [6] show that they are not sufficient to imply generic global rigidity when \(d \geq 3\). Connelly also obtained a different kind of sufficient condition for generic global rigidity in terms of ‘stress matrices’ (which will be defined in Section 5).

Theorem 1.3 ([7]). Let \((G, p)\) be a generic framework in \(\mathbb{R}^d\) with \(n \geq d + 2\) vertices. Suppose that \((G, p)\) has an equilibrium stress \(\omega\) whose associated stress matrix \(\Omega\) has rank \(n - d - 1\). Then \((G, p)\) is globally rigid in \(\mathbb{R}^d\).

Gortler, Healy and Thurston [9] have shown that Connelly’s sufficient condition for generic global rigidity is also a necessary condition. This characterization implies that generic global rigidity depends only on the underlying graph (but does not give rise to a polynomial algorithm for deciding which graphs are generically globally rigid in \(\mathbb{R}^d\)).

In this paper we develop natural analogues of an equilibrium stress and a stress matrix for frameworks constrained to lie on a surface. Our main result is an analogue of Theorem 1.3 giving a sufficient condition for generic frameworks on families of concentric cylinders, cones and ellipsoids to be globally rigid.

We conclude the introduction by giving a short outline of what follows. Section 2 recalls basic definitions and results for frameworks on surfaces. We give some results from algebraic geometry in Section 3 and prove a key technical result of the paper, Proposition 3.3. We describe the rigidity map for surfaces in Section 4. In Section 5 we develop basic properties of stresses, stress matrices and energy functions for frameworks on surfaces. One of the key results here is Theorem 5.1, which shows that an equilibrium stress for a generic framework on a surface is also an equilibrium stress for any equivalent framework. Section 6 contains our main result, Theorem 6.2, an analogue of Theorem 1.3 for generic frameworks on surfaces. We use this result in Section 7 to show that the property of having a maximum rank surface stress matrix is preserved by 1-extensions. We finish by applying our results to make some progress on the problem of characterising generic global rigidity on the cylinder.

2. Frameworks on Surfaces

It was shown in [18] that the rigidity of a generic framework on a surface depends crucially on the number of continuous isometries of \(\mathbb{R}^3\) admitted by the surface, see Theorems 2.1, 2.2 and 4.1 below. Since generic rigidity and global rigidity on the plane and sphere [8], the surfaces with 3-dimensional isometry groups, are now well understood, we consider cylinders, cones and ellipsoids as natural examples of surfaces with 2, 1 and 0-dimensional isometry groups, respectively.

Let \(r = (r_1, r_2, \ldots, r_n)\) be a vector of (not necessarily distinct) positive real numbers. For \(1 \leq i \leq n\), let \(Y_i = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r_i\}\), \(C_i = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r_i z^2\}\) and \(E_i = \{(x, y, z) \in \mathbb{R}^3 : x^2 + \alpha y^2 + \beta z^2 = r_i\}\) for some fixed \(\alpha, \beta \in \mathbb{Q}\) with \(1 < \alpha < \beta\). Let \(Y = \bigcup_{i=1}^{n} Y_i\), \(C = \bigcup_{i=1}^{n} C_i\) and \(E = \bigcup_{i=1}^{n} E_i\). We will use \(S = \bigcup_{i=1}^{n} S_i\) to denote one of the three surfaces \(Y, C, E\), and \(\ell\) for the dimension of its space of infinitesimal isometries (so \(\ell = 2, 1\) or 0 when \(S = Y, C\) or \(E\), respectively. We will occasionally use \(S(r)\) when we wish to specify a particular vector \(r\) and \(S(1)\) for the special case when \(r_1 = r_2 = \cdots = r_n\) (there is no loss in generality in assuming that \(r_i = 1\) for all \(1 \leq i \leq n\) when the \(r_i\) are all equal).
A framework on $S$ is a pair $(G, p)$ where $G = (V, E)$ is a graph with $V = \{v_1, v_2, \ldots, v_n\}$, and $p : V \to \mathbb{R}^3$ with $p(v_i) \in S_i$ for all $1 \leq i \leq n$. Two frameworks $(G, p)$ and $(G, q)$ on $S$ are equivalent if $\|p(v_i) - p(v_j)\| = \|q(v_i) - q(v_j)\|$ for all $v_i, v_j \in E$ and congruent if $\|p(v_i) - p(v_j)\| = \|q(v_i) - q(v_j)\|$ for all $v_i, v_j \in V$. The framework $(G, p)$ is globally rigid on $S$ if every framework $(G, q)$ on $S$ which is equivalent to $(G, p)$ is congruent to $(G, p)$. We say that $(G, p)$ is rigid on $S$ if there exists an $\epsilon > 0$ such that every framework $(G, q)$ on $S$ which is equivalent to $(G, p)$, and has $\|p(v_i) - q(v_i)\| < \epsilon$ for all $1 \leq i \leq n$, is congruent to $(G, p)$. (This is equivalent to saying that every continuous motion of the vertices that stays on $S$ and preserves equivalence also preserves congruence). If $(G, p)$ is not rigid on $S$ then $(G, p)$ is said to be flexible on $S$. The framework $(G, p)$ is minimally rigid on $S$ if it is rigid and, for every edge $e \in E$, $(G - e, p)$ is flexible on $S$. It is redundantly rigid on $S$ if $(G - e, p)$ is rigid on $S$ for all $e \in E$.

An infinitesimal flex $s$ of $(G, p)$ on $S$ is a map $s : V \to \mathbb{R}^3$ such that $s(v)$ is tangential to $S$ at $p(v)$ for all $v \in V$ and $(p(u) - p(v)) \cdot (s(u) - s(v)) = 0$ for all $uv \in E$. The framework $(G, p)$ is infinitesimally rigid on $S$ if every infinitesimal flex of $(G, p)$ is an infinitesimal isometry of $S$.

We consider a framework $(G, p)$ on $S = S(r)$ to be generic if $\text{td} [\mathbb{Q}(r, p) : \mathbb{Q}(r)] = 2|V|$, where $\text{td} [\mathbb{Q}(r, p) : \mathbb{Q}(r)]$ denotes the transcendence degree of the field extension. Thus $(G, p)$ is generic on $S$ if the coordinates of the vertices of $G$ are as algebraically independent as possible. The following results characterise when a generic framework on $\mathbb{Y}$ or $\mathbb{C}(1)$ is minimally rigid.

**Theorem 2.1** ([18]). Let $(G, p)$ be a generic framework on $\mathbb{Y}$. Then $(G, p)$ is minimally rigid if and only if $G$ is $K_n$ for $1 \leq n \leq 3$ or $G$ is $(2, 2)$-tight.

**Theorem 2.2** ([19]). Let $(G, p)$ be a generic framework on $\mathbb{C}(1)$. Then $(G, p)$ is minimally rigid if and only if $G$ is $K_n$ for $1 \leq n \leq 4$ or $G$ is $(2, 1)$-tight.

It remains an open problem to characterise generic minimally rigid frameworks on $\mathbb{E}$. (The natural analogue of the above theorems is known to be false.)

The final result of this section gives necessary conditions for generic global rigidity of frameworks on $S$ which are analogous to Hendrickson’s conditions for $\mathbb{R}^d$.

**Theorem 2.3** ([15]). Suppose $(G, p)$ is a generic globally rigid framework on $S$ with $n \geq 7 - \ell$ vertices. Then $(G, p)$ is redundantly rigid on $S$, and $G$ is $k$-connected, where $k = 2$ if $S \in \{\mathbb{Y}, \mathbb{C}\}$ and $k = 1$ if $S = \mathbb{E}$.

We believe that these necessary conditions for generic global rigidity are also sufficient when $S \in \{\mathbb{Y}, \mathbb{C}\}$, see [14] Conjecture 9.1. One motivation for the current paper is to try to verify this conjecture by using the same proof technique as Theorem 1.2. We will return to this in Section 8.

### 3. Generic Points and Smooth Maps

Let $M$ be a smooth manifold and $f : M \to \mathbb{R}^m$ be a smooth map. Then $x \in M$ is a regular point of $f$ if $df|_x$ has maximum rank, and $f(x)$ is a regular value of $f$ if, for all $y \in f^{-1}(f(x))$, $y$ is a regular point of $f$.

**Lemma 3.1.** For $i = 1, 2$, let $M_i$ be an open subset of $\mathbb{R}^n$, $p_i \in M_i$, and $f_i : M_i \to \mathbb{R}^m$ be a smooth map with rank $df_i|_{p_i} = m$ and $f_1(p_1) = f_2(p_2)$. Then there exist open neighbourhoods $N_1$ of $p_1$, $N_2$ of $p_2$, and a diffeomorphism $g : N_1 \to N_2$ such that $f_2(g(x)) = f_1(x)$ for all $x \in N_1$. 
Proof. We first consider the case when \( m = n \). By the Inverse Function Theorem there exist neighbourhoods \( \tilde{N}_i \subseteq M_i \) of \( p_i \) such that \( f_i \) maps \( \tilde{N}_i \) diffeomorphically onto \( f_i(\tilde{N}_i) \) for \( i = 1, 2 \). Let \( W = f_1(\tilde{N}_1) \cap f_2(\tilde{N}_2) \) and then let \( N_i = f_i^{-1}(W) \) for \( i = 1, 2 \). We have \( f_1(N_1) = W = f_2(N_2) \). Thus we may choose \( g = f_2^{-1} \circ f_1 \) and find \( f_2(g(x)) = f_2(f_2^{-1}(f_1(x))) = f_1(x) \) for all \( x \in N_1 \).

We next consider the case when \( m < n \). Let \( F_i : M_i \to \mathbb{R}^m \times \mathbb{R}^{n-m} \) be defined by \( F_i(x) = (f_i(x), x_{m+1}, x_{m+2}, \ldots, x_n) \). Then rank \( dF_i|_{p_i} = n \). By the Inverse Function Theorem there exist neighbourhoods \( \tilde{N}_i \subseteq M_i \) of \( p_i \) such that \( F_i \) is a diffeomorphism from \( \tilde{N}_i \) to \( F_i(\tilde{N}_i) \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m} \). Let \( F_i(N_i) = U_i \times V_i \) where \( U_i \subseteq \mathbb{R}^m \) and \( V_i \subseteq \mathbb{R}^{n-m} \). Then \( V_i \) is diffeomorphic to \( \mathbb{R}^{n-m} \) for \( i = 1, 2 \) so we can choose a diffeomorphism \( h : V_1 \to V_2 \) such that \( h(\tilde{p}_i) = \tilde{p}_2 \), where \( \tilde{p}_i \) is the projection of \( p_i \) onto its last \( n-m \) coordinates. Let \( \iota \) be the identity map on \( U_1 \) and let \( H = (\iota, h) : U_1 \times V_1 \to U_1 \times V_2 \). Let \( F_i' = H \circ F_i \). Then we have \( F_i'(p_1) = (f_1(p_1), h(\iota \tilde{p}_1)) = (f_2(p_2), \tilde{p}_2) = F_2(p_2) \). By the previous paragraph there exist neighbourhoods \( N_i \subseteq \tilde{N}_i \) of \( p_i \) and a diffeomorphism \( g : N_1 \to N_2 \subseteq \mathbb{R}^n \) such that \( F_2(g(x)) = F_1'(x) \) for all \( x \in N_1 \). Since \( F_1'(x) = (f_1(x), h(\iota \tilde{p})) \) and \( F_2(g(x)) = (f_2(g(x)), g(\tilde{p})) \) we have \( f_1(x) = f_2(g(x)) \) for all \( x \in N_1 \).

Let \( \mathbb{K} \) be a field such that \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} \). A set \( W \subseteq \mathbb{R}^n \) is an algebraic set defined over \( \mathbb{K} \) if \( W = \{ x \in \mathbb{R}^n : F_i(x) = 0 \text{ for all } 1 \leq i \leq m \} \) for each \( P_i \in \mathbb{K}[X_1, \ldots, X_n] \) with \( \deg P_i \leq m \). An algebraic set \( W \) is irreducible if it cannot be expressed as the union of two algebraic proper subsets defined over \( \mathbb{R} \). The dimension of \( W \), \( \dim W \), is the largest integer \( t \) for which \( W \) has an open subset homeomorphic to \( \mathbb{R}^t \). A point \( p \in W \) is generic over \( \mathbb{K} \) if every \( h \in \mathbb{K}(X) \) satisfying \( h(p) = 0 \) has \( h(x) = 0 \) for all \( x \in W \).

**Lemma 3.2 ([15]).** Let \( \mathbb{K} \) be a field with \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} \). \( W \subseteq \mathbb{R}^n \) be an algebraic set defined over \( \mathbb{K} \) and \( p \in W \). Then \( \dim W \geq \text{td} [\mathbb{K}(p) : \mathbb{K}] \). Furthermore, if \( W \) is irreducible and \( \dim W = \text{td} [\mathbb{K}(p) : \mathbb{K}] \), then \( p \) is a generic point of \( W \).

Note that, if \( (G,p) \) is a generic framework on \( S \), then Lemma 3.2 implies that \( p \) is a generic point of the irreducible algebraic set \( S_1 \times S_2 \times \ldots \times S_n \) defined over \( \mathbb{Q}(r) \) in \( \mathbb{R}^{3n} \).

A set \( A \subseteq \mathbb{R}^n \) is a semi-algebraic set defined over \( \mathbb{K} \) if it can be expressed as a finite union of sets of the form

\[ \{ x \in \mathbb{R}^n : P_i(x) = 0 \text{ for } 1 \leq i \leq s \text{ and } Q_j(x) > 0 \text{ for } 1 \leq j \leq t \} , \]

where \( P_i, Q_j \in \mathbb{K}[X_1, \ldots, X_n] \) for \( 1 \leq i \leq s \) and \( 1 \leq j \leq t \). It is easy to see that the family of semi-algebraic sets defined over \( \mathbb{K} \) is closed under union and intersection. A deeper result is that if \( A \subseteq \mathbb{R}^n \) is a semi-algebraic set defined over \( \mathbb{K} \) and \( f : A \to \mathbb{R}^m \) is a map in which each component is a polynomial with coefficients in \( \mathbb{K} \), then \( f(A) \) is a semi-algebraic set defined over \( \mathbb{K} \). Another result we shall need is that a semi-algebraic set \( A \) can be partitioned into a finite number of semi-algebraic subsets \( C_1, C_2, \ldots, C_t \), called cells, such that, for all \( 1 \leq i \leq t \), \( C_i \) is diffeomorphic to \( \mathbb{R}^{m_i} \) for some integer \( m_i \geq 0 \) where \( \mathbb{R}^0 \) is taken to be a single point. The dimension of \( A \) is the largest integer \( t \) for which \( A \) has an open subset homeomorphic to \( \mathbb{R}^t \). The Zariski closure, \( A^* \), of \( A \) is the smallest algebraic set defined over \( \mathbb{R} \) which contains \( A \). It is known that \( A^* \) is an algebraic set defined over \( \mathbb{L} \), for some finite field extension \( \mathbb{L} \) of \( \mathbb{K} \), and that \( \dim A = \dim A^* \). We refer the reader to [2, 6] for more information on semi-algebraic sets.

We can now obtain an analogue of [7, Proposition 3.3].

**Proposition 3.3.** Let \( \mathbb{K} \) be a field with \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} \), \( W \subseteq \mathbb{R}^n \) be an irreducible algebraic set defined over \( \mathbb{K} \) of dimension \( n \), and \( f : W \to \mathbb{R}^p \) be a function where each coordinate is a
polynomial with coefficients in $\mathbb{K}$. Suppose that the maximum rank of the differential of $f$ is $m$, and that $p_0 \in W$ with $td[\mathbb{K}(p_0) : \mathbb{K}] = n$. Then rank $df|_{p_0} = m$. Furthermore, if $q_0 \in W$ and $f(p_0) = f(q_0)$, then there exist open neighbourhoods $N_{p_0}$ of $p_0$ and $N_{q_0}$ of $q_0$ in $W$ and a diffeomorphism $g : N_{q_0} \to N_{p_0}$ such that $g(q_0) = p_0$ and, for all $q \in N_{q_0}$, $f(g(q)) = f(q)$.

Proof. We first show that rank $df|_{p_0} = m = \text{rank} df|_{q_0}$, and that there exist open neighbourhoods $M_{p_0}$ of $p_0$ and $M_{q_0}$ of $q_0$ in $W$ such that $f(M_{p_0}) = f(M_{q_0})$ and $f(M_{p_0})$ is diffeomorphic to $\mathbb{R}^m$. We then complete the proof by applying Lemma 3.1.

By Lemma 3.2, $p_0$ is a generic point of $W$. We can now use [9] Lemma 2.7 and Proposition 2.32 to deduce that $f(p_0)$ is a regular value of $f$. In particular, we have rank $df|_{p_0} = m = \text{rank} df|_{q_0}$. The Constant Rank Theorem (see, for example, [21] Theorem 9) now implies that we can choose disjoint open balls $B(p_1, \epsilon)$ and $B(q_1, \delta)$ in $\mathbb{R}^n$ such that: $p_0 \in B(p_1, \epsilon) \cap W$; $q_0 \in B(q_1, \delta) \cap W$; $p_1, q_1 \in \mathbb{Q}^n$; $\epsilon, \delta \in \mathbb{Q}$; both $B(p_1, \epsilon) \cap W$ and $B(q_1, \delta) \cap W$ are diffeomorphic to $\mathbb{R}^n$; both $f(B(p_1, \epsilon) \cap W)$ and $f(B(q_1, \delta) \cap W)$ are diffeomorphic to $\mathbb{R}^m$.

Let $U_{p_0} = B(p_1, \epsilon) \cap W$ and $U_{q_0} = B(q_1, \delta) \cap W$. Since $U_{p_0}$ and $U_{q_0}$ are both semi-algebraic defined over $\mathbb{K}$, $f(U_{p_0})$ and $f(U_{q_0})$ are both semi-algebraic defined over $\mathbb{K}$, and hence $T = f(U_{p_0}) \cap f(U_{q_0})$ is semi-algebraic defined over $\mathbb{K}$. The facts that $f$ is a polynomial map, $td[\mathbb{K}(p_0) : \mathbb{K}] = n$ and rank $df|_{p_0} = m \leq n$ imply that $td[\mathbb{K}(f(p_0)) : \mathbb{K}] = m$, see for example [13] Lemma 3.1. Let $C_1, C_2, \ldots, C_l$ be a cell decomposition of $T$ with $f(p_0) \in C_1$, and let $C_1^*$ be the Zariski closure of $C_1$. Then $C_1^*$ is an algebraic set defined over some finite field extension $\mathbb{L}$ of $\mathbb{K}$. Since $f(p_0) \in C_1^*$, Lemma 3.2 gives

$$\dim C_1 = \dim C_1^* \geq \text{td}(\mathbb{L}(f(p_0)) : \mathbb{L}) = \text{td}(\mathbb{K}(f(p_0)) : \mathbb{K}) = m.$$ 

Since $C_1 \subseteq f(U_{p_0})$ and $f(U_{p_0})$ is diffeomorphic to $\mathbb{R}^m$, we must have $\dim C_1 = m$. We can now take $M_{p_0} = f^{-1}(C_1) \cap U_{p_0}$ and $M_{q_0} = f^{-1}(C_1) \cap U_{q_0}$. Then $f(M_{p_0}) = C_1 = f(M_{q_0})$ and $C_1$ is diffeomorphic to $\mathbb{R}^m$.

The proposition now follows from Lemma 3.1 by choosing $M_1 = M_{p_0}, M_2 = M_{q_0}$, and $f_i = f|_{M_i}$ for $i = 1, 2$.

4. The Rigidity Map

We assume henceforth that $G = (V, E)$ is a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. The rigidity map $F^G : \mathbb{R}^{3n} \to \mathbb{R}^m$ is defined by $F^G(p) = (||e_1||^2, \ldots, ||e_m||^2)$ where $||e_i||^2 = ||p(v_i) - p(v_k)||^2$ when $e_i = v_jv_k$. Its differential at the point $p$ is the map $dF_p^G : \mathbb{R}^{3n} \to \mathbb{R}^m$ defined by $dF_p^G(q) = 2R(G, p) \cdot q$ where $R(G, p)$ is the $|E| \times 3|V|$ matrix with rows indexed by $E$ and 3-tuples of columns indexed by $V$ in which, for $e = v_iv_j \in E$, the submatrices in rows $e$ and columns $v_i$ and $v_j$ are $p(v_i) - p(v_j)$ and $p(v_j) - p(v_i)$, respectively, and all other entries are zero. We refer to $R(G, p)$ as the rigidity matrix for $(G, p)$.

We next define a rigidity map and matrix for a framework $(G, p)$ constrained to lie on our surface $S$. Let $\Theta^G : \mathbb{R}^{3n} \to \mathbb{R}^n$ be the map defined by $\Theta^G(p) = (h_1(p(v_1)), \ldots, h_n(p(v_n)))$ where, for each $1 \leq i \leq n$,

$$h_i(x, y, z) = \begin{cases} x^2 + y^2 - r_i, & \text{if } S = \Upsilon(r_1, r_2, \ldots, r_n); \\ x^2 + y^2 - r_iz^2, & \text{if } S = \Theta(r_1, r_2, \ldots, r_n); \\ x^2 + \alpha y^2 + \beta z^2 - r_i, & \text{if } S = \Theta(r_1, r_2, \ldots, r_n, \alpha, \beta). \end{cases}$$
Then the differential of $\Theta^G$ at the point $p$ is the map $d\Theta^G_p : \mathbb{R}^{3n} \to \mathbb{R}^m$ defined by $\Theta^G_p(q) = 2S(G, p) \cdot q$ where

$$S(G, p) = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n \end{bmatrix},$$

$s_i = s_i(p(v_i))$ and

$$(4.2) \quad s_i(x, y, z) = \begin{cases} (x, y, 0), & \text{if } S = \mathcal{Y}; \\ (x, y, -rz), & \text{if } S = \mathcal{C}; \\ (x, \alpha y, \beta z), & \text{if } S = \mathcal{E}. \end{cases}$$

It follows that $\text{rank} d\Theta^G_p = n$ if $p \in \mathcal{W} = S_1 \times S_2 \times \ldots \times S_n$ and $p(v_i) \neq (0, 0, 0)$ for all $1 \leq i \leq n$. Hence $p \in \mathcal{W}$ is a regular point of $\Theta^G$ unless $S = \mathcal{C}$ and $p(v_i) = (0, 0, 0)$ for some $1 \leq i \leq n$.

The $S$-rigidity map $F_{G, S} : \mathbb{R}^{3n} \to \mathbb{R}^{m+n}$ is defined by $F_{G, S} = (F^G, \Theta^G)$. The rigidity matrix

$$R_S(G, p) = \begin{bmatrix} R(G, p) \\ S(G, p) \end{bmatrix}$$

for the framework $(G, p)$ on $S$ is (up to scaling) the Jacobian matrix for the differential of $F_{G, S}$ evaluated at the point $p$. It is shown in [18] that the null space of $R_S(G, p)$ is the space of infinitesimal flexes of $(G, p)$ on $S$. This allows us to characterise infinitesimal rigidity in terms of $R_S(G, p)$.

**Theorem 4.1** ([18]). Let $(G, p)$ be a framework on $S$. Then $(G, p)$ is infinitesimally rigid on $S$ if and only if $\text{rank } R_S(G, p) = 3n - \ell$.

Theorem 4.1 implies that the (redundant) rigidity of a generic framework $(G, p)$ on $S$ depends only on the graph $G$. Hence we say that $G$ is (redundantly) rigid on $S$ if some, or equivalently every, generic realisation of $G$ on $S$ is (redundantly) rigid. We close this section by pointing out that rigidity of a graph $G$ on a family of concentric cylinders is independent of the radii of the cylinders.

**Lemma 4.2.** Let $(G, p)$ be a framework on $\mathcal{Y}(r_1, r_2, \ldots, r_n)$ with $p(v_i) = (x_i, y_i, z_i)$ and let $\hat{p}(v_i) = (\frac{x_i}{r_i}, \frac{y_i}{r_i}, \frac{z_i}{r_i})$ for all $1 \leq i \leq n$. Then $(G, p)$ is infinitesimally rigid on $\mathcal{Y}(r_1, r_2, \ldots, r_n)$ if and only if $(G, \hat{p})$ is infinitesimally rigid on $\mathcal{Y}(1)$.

**Proof.** Suppose $q : V \to \mathbb{R}^{3n}$ and $q(v_i) = (a_i, b_i, c_i)$ for all $v_i \in V$. Then $q$ is an infinitesimal flex of $(G, p)$ if and only if $q(v_i) \cdot (x_i, y_i, 0) = 0$ for all $v_i \in V$ and $(p(v_i) - p(v_j)) \cdot (q(v_i) - q(v_j)) = 0$ for all $v_i \in V$. It follows that, if we put $\hat{q}(v_i) = (\frac{a_i}{r_i}, \frac{b_i}{r_i}, c_i)$ for all $v_i \in V$, then $q$ is an infinitesimal flex of $(G, p)$ if and only if $\hat{q}$ is an infinitesimal flex of $(G, \hat{p})$. Thus the map $q \mapsto \hat{q}$ gives a linear bijection between the spaces of infinitesimal flexes of $(G, p)$ and $(G, \hat{p})$. \[\square\]

This result implies that if a graph is (redundantly) rigid on some family of concentric cylinders, then it is (redundantly) rigid on all families of concentric cylinders. (We can also deduce this from Theorem 2.1.) We do not know if analogous results hold for the families of concentric cones or ellipsoids.
5. Stresses and stress matrices

In this section we develop the notion of an equilibrium stress for a framework on our surface $S$ and use Proposition 3.3 to obtain the key result that an equilibrium stress for a generic framework $(G, p)$ on $S$ is an equilibrium stress for any equivalent framework $(G, q)$. We use this to show that if $(G, p)$ has a maximum rank stress matrix then every equivalent framework on $S$ is an affine image of $(G, p)$. We also show that the same conclusion holds for non-generic frameworks which have a maximum rank positive semi-definite stress matrix.

A stress for a framework $(G, p)$ on $S$ is a pair $(\omega, \lambda)$, where $\omega : E \rightarrow \mathbb{R}$ and $\lambda : V \rightarrow \mathbb{R}$. A stress $(\omega, \lambda)$ is an equilibrium stress if it belongs to the cokernel of $R_S(G, p)$. Thus $(\omega, \lambda)$ is an equilibrium stress for $(G, p)$ on $S$ if and only if

\[ \sum_{j=1}^{n} \omega_{ij} (p(v_i) - p(v_j)) + \lambda_i s_i(p(v_i)) = 0 \quad \text{for all } 1 \leq i \leq n, \]

where $s_i(p(v_i))$ is as defined in Equation (1.2), $\omega_{ij}$ is taken to be equal to $\omega_i$ if $e = v_i v_j \in E$ and to be equal to 0 if $v_i v_j \notin E$. We can think of $\omega$ as a weight function on the edges and $\lambda$ as a weight function on the vertices. Note that, if the rows of $R_S(G, p)$ are linearly independent, then the only equilibrium stress for $(G, p)$ is the all-zero equilibrium stress.

We first use Proposition 3.3 to show that an equilibrium stress for a generic framework on $S$ is an equilibrium stress for any equivalent framework on $S$.

**Theorem 5.1.** Let $(G, p_0)$ be a generic framework on $S$ and $(\omega, \lambda)$ be an equilibrium stress for $(G, p_0)$. Let $(G, q_0)$ be equivalent to $(G, p_0)$. Then $(\omega, \lambda)$ is an equilibrium stress for $(G, q_0)$.

*Proof.* Let $F = F^{G,S}$ and put $f = F^{G,S}|_S$. By Proposition 3.3 there exist open neighbourhoods $U$ of $p_0$ and $W$ of $q_0$ in $S$ and a diffeomorphism $g : W \rightarrow U$ such that $g(q_0) = p_0$ and, for all $q \in W$, $f(g(q)) = f(q)$. Taking differentials at $q_0$ we obtain $df_{q_0} = df_{p_0} dg_{q_0}$. Since the Jacobian matrix of $dF_p$ is $2R_S(G, p)$ and $df_p = df_p|_S$ for all $p \in S$, we can rewrite this equation in terms of Jacobian matrices as $R_S(G, q_0) = R_S(G, p_0) D$, where $D$ is the Jacobian of $dg_{q_0}$. Thus $(\omega, \lambda) R_S(G, q_0) = (\omega, \lambda) R_S(G, p_0) D$. Since $(\omega, \lambda)$ is an equilibrium stress for $(G, p_0)$ we have $(\omega, \lambda) R_S(G, q_0) = (\omega, \lambda) R_S(G, p_0) D = 0 D = 0$. \qed

Given a stress $(\omega, \lambda)$ for a framework $(G, p)$ on $S$ we define: $\Omega = \Omega(\omega)$ to be the $n \times n$ symmetric matrix with off-diagonal entries $-\omega_{ij}$ and diagonal entries $\Sigma = \Sigma(\omega)$; $\Lambda = \Lambda(\lambda)$ to be the $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$; and $\Delta = \Delta(\lambda, r)$ to be the $n \times n$ diagonal matrix with diagonal entries $\lambda_1 r_1, \lambda_2 r_2, \ldots, \lambda_n r_n$. The stress matrix associated to $(\omega, \lambda)$ on $S$ is the $3n \times 3n$ symmetric matrix

\[
\Omega_S(\omega, \lambda) = \begin{bmatrix}
\Omega + \Lambda & 0 & 0 \\
0 & \Gamma & 0 \\
0 & 0 & \Sigma
\end{bmatrix}
\]

where: $\Gamma = \Omega + \Lambda$ if $S \in \{Y, E\}$ and $\Gamma = \Omega + \alpha \Lambda$ if $S = \xi; \Sigma = \Omega$ if $S = Y, \Sigma = \Omega - \Delta$ if $S = E$, and $\Sigma = \Omega + \beta \Lambda$ if $S = \xi$. Our next result, which follows immediately from the definition of an equilibrium stress, tells us how we can use $\Omega_S(\omega, \lambda)$ to determine if $(\omega, \lambda)$ is an equilibrium stress for $(G, p)$ on $S$.

**Lemma 5.2.** Let $(G, p)$ be a framework on $S$ with $p(v_i) = (x_i, y_i, z_i)$ and let

\[
\Pi = \begin{bmatrix}
x_1 & \ldots & x_n & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & y_1 & \ldots & y_n & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & z_1 & \ldots & z_n
\end{bmatrix}.
\]
Then \((\omega, \lambda)\) is an equilibrium stress for \((G, p)\) on \(S\) if and only if \(\Pi \Omega_S = 0\).

We next define the configuration matrix \(C_S(G, p)\) for a framework \((G, p)\) on \(S\) by modifying the above matrix \(\Pi\) as follows:

\[
C_S(G, p) = \begin{bmatrix}
x_1 & \ldots & x_n & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & y_1 & \ldots & y_n & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & z_1 & \ldots & z_n \\
y_1 & \ldots & y_n & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & x_1 & \ldots & x_n & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 1 & \ldots & 1
\end{bmatrix}
\]

if \(M = Y\),

\[
C_S(G, p) = \begin{bmatrix}
x_1 & \ldots & x_n & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & y_1 & \ldots & y_n & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & z_1 & \ldots & z_n \\
y_1 & \ldots & y_n & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & x_1 & \ldots & x_n & 0 & \ldots & 0
\end{bmatrix}
\]

if \(M = E\),

and \(C_S(G, p) = \Pi\) if \(M = E\). We can use the configuration matrix to obtain an upper bound on the rank of a stress matrix.

**Lemma 5.3.** Let \((\omega, \lambda)\) be an equilibrium stress for a framework \((G, p)\) on \(S\). Then each row of \(C_S(G, p)\) belongs to the cokernel of \(\Omega_S(\omega, \lambda)\), \(\text{rank } \Omega_S(\omega, \lambda) \leq 3n - \text{rank } C_S(G, p)\) and, if equality holds, then the rows of \(C_S(G, p)\) span the cokernel of \(\Omega_S(\omega, \lambda)\).

**Proof.** Equation (5.1) and the definitions of \(\Omega_S(\omega, \lambda)\) and \(C_S(G, p)\) imply that

\[C_S(G, p) \Omega_S(\omega, \lambda) = 0.\]

Thus each row of \(C_S(G, p)\) belongs to the cokernel of \(\Omega_S(\omega, \lambda)\). Hence \(\dim \text{coker } \Omega_S(\omega, \lambda) \geq \text{rank } C_S(G, p)\) and we have \(\text{rank } \Omega_S(\omega, \lambda) = 3n - \dim \text{coker } \Omega_S(\omega, \lambda) \leq 3n - \text{rank } C_S(G, p)\). Furthermore, if equality holds, then \(\text{coker } \Omega_S(\omega, \lambda)\) is equal to the row space of \(C_S(G, p)\). \(\square\)

We next use Lemma 5.3 to show that, if a framework \((G, p)\) on \(S\) has an equilibrium stress \((\omega, \lambda)\) whose associated stress matrix has maximum rank, then every framework \((G, q)\) on \(S\) which has \((\omega, \lambda)\) as an equilibrium stress can be obtained from \((G, p)\) by an affine transformation.

**Lemma 5.4.** Let \((G, p)\) and \((G, q)\) be frameworks on \(S\) and let \((\omega, \lambda)\) be an equilibrium stress for both \((G, p)\) and \((G, q)\). Suppose that \(\text{rank } \Omega_S(\omega, \lambda) = 3n - \text{rank } C_S(G, p)\). Then, for some fixed \(a, b, c, d, e, f \in \mathbb{R}\), we have

\[
q(v_i) = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix} p(v_i) + \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} \quad \text{for all } 1 \leq i \leq n,
\]

where \(f = 0\) if \(S \in \{E, \mathcal{E}\}\) and \(b = c = 0\) if \(S = \mathcal{E}\).

**Proof.** Lemma 5.3 implies that the rows of \(C_S(G, p)\) span the cokernel of \(\Omega_S(\omega, \lambda)\), and that each row of \(C_S(G, q)\) belongs to the cokernel of \(\Omega_S(\omega, \lambda)\). It follows that each row of \(C_S(G, q)\) is a linear combination of the rows of \(C_S(G, p)\). The lemma now follows from the structure of the matrices \(C_S(G, p)\) and \(C_S(G, q)\). \(\square\)

We will say that \((G, q)\) is an \(S\)-affine image of \((G, p)\) if it satisfies the conclusion of Lemma 5.4. Our next result gives a converse to Lemma 5.4.
**Lemma 5.5.** Let \((G, p)\) and \((G, q)\) be frameworks on \(S\) such that \((G, q)\) is an \(S\)-affine image of \((G, p)\). Then every equilibrium stress \((\omega, \lambda)\) for \((G, p)\) is an equilibrium stress for \((G, q)\).

**Proof.** Since \((G, q)\) is an \(S\)-affine image of \((G, p)\), we have \(q(v_i) = Ap(v_i) + t\) for some fixed \(A, t\) satisfying the conclusion of Lemma 5.4 and all \(1 \leq i \leq n\). Hence

\[
\sum_j \omega_{ij}(q(v_i) - q(v_j)) + \lambda_is_i(q(v_i)) = \sum_j \omega_{ij}A(p(v_i) - p(v_j)) + \lambda_is_i(Ap(v_i) + t)
\]

\[
= A \left( \sum_j \omega_{ij}(p(v_i) - p(v_j)) + \lambda_is_i(p(v_i)) \right),
\]

since \(s_i(Ap(v_i) + t) = As_i(p(v_i))\) by the definitions of \(s_i, A, t\). The lemma now follows by applying Equation (5.1). \(\square\)

A framework \((G, p)\) on \(S\) is **fully realised** on \(S\) if the rows of its configuration matrix are linearly independent i.e. we have rank \(C_S(G, p) = \mu\) where

\[
\mu = \begin{cases} 6 & \text{if } S = Y; \\ 5 & \text{if } S = C; \\ 3 & \text{if } S = E. \end{cases}
\]

It can be seen that \((G, p)\) is fully realised on \(S\) if and only if its points do not all lie on: a plane containing or perpendicular to the \(z\)-axis when \(S = Y\); a plane containing the \(z\)-axis when \(S = C\); one of the planes \(x = 0, y = 0\) or \(z = 0\) when \(S = E\). We can use these observations to deduce that all equivalent realisations of a generic framework \((G, p)\) which has a maximum rank stress-matrix, are affine images of \((G, p)\).

**Theorem 5.6.** Let \((G, p)\) and \((G, q)\) be equivalent frameworks on \(S\). Suppose that \((G, p)\) is generic and that \((\omega, \lambda)\) is an equilibrium stress for \((G, p)\) with rank \(\Omega_S(\omega, \lambda) = 3n - \mu\). Then \((G, q)\) is an \(S\)-affine image of \((G, p)\).

**Proof.** Theorem 5.4 tells us that \((\omega, \lambda)\) is an equilibrium stress for \((G, q)\). Since \((G, p)\) is generic, it is fully realised on \(S\). We can now use Lemma 5.4 and the hypothesis that rank \(\Omega_S(\omega, \lambda) = 3n - \mu\) to deduce that \((G, q)\) is an \(S\)-affine image of \((G, p)\). \(\square\)

We next consider non-generic frameworks. We will use a similar argument to that used by Connelly in \([5]\) to show that the conclusion of Theorem 5.6 remains valid when the hypothesis that \((G, p)\) is generic is replaced by an hypothesis that \(\Omega_S(\omega, \lambda)\) is positive semi-definite.

The **energy function** associated to a stress \((\omega, \lambda)\) for a framework \((G, q)\) and a family of concentric surfaces \(S\) is defined as

\[
E_{\omega, \lambda, S}(q) = \sum_{1 \leq i < j \leq n} \omega_{ij}\|q(v_i) - q(v_j)\|^2 + \sum_{i=1}^n \lambda_i k_i(q(v_i))
\]

where

\[
k_i(x, y, z) = \begin{cases} x^2 + y^2 & \text{if } S = Y; \\ x^2 + y^2 - r_i z^2 & \text{if } S = C; \\ x^2 + ay^2 + bz^2 & \text{if } S = E. \end{cases}
\]

Then the differential of \(E_{\omega, \lambda, S}(q)\) at a point \(q\) with \(q(v_i) = (x_i, y_i, z_i)\) for all \(1 \leq i \leq n\) is given by

\[
dE_{\omega, \lambda, S}|_q = 2(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n)\Omega_S(\omega, \lambda).
\]
Hence, when \((G, q)\) is a framework on \(S\), \(q\) is a critical point of \(E_{\omega,\lambda,S}\) if and only if \((\omega, \lambda)\) is an equilibrium stress for \((G, q)\) on \(S\).

**Lemma 5.7.** Suppose \(q \in \mathbb{R}^{3n}\). If \(q\) is a critical point of \(E_{\omega,\lambda,S}\) then \(E_{\omega,\lambda,S}(q) = 0\). In addition, when \(\Omega_S(\omega, \lambda)\) is positive semi-definite and \((G, q)\) lies on \(S\), we have \(E_{\omega,\lambda,S}(q) = 0\) if and only if \(q\) is a critical point of \(E_{\omega,\lambda,S}\).

**Proof.** Suppose \(q\) is a critical point of \(E_{\omega,\lambda,S}\). Then the differential of \(E_{\omega,\lambda,S}(q)\) in the direction of \(q\) is zero. This implies that \(E_{\omega,\lambda,S}(tq)\) is constant for all \(t \in \mathbb{R}\). We can now take \(t = 0\) to deduce that \(E_{\omega,\lambda,S}(q) = E_{\omega,\lambda,S}(0) = 0\).

Observe that, if \(q(v_i) = (x_i, y_i, z_i)\) for all \(1 \leq i \leq n\), then
\[
E_{\omega,\lambda,S}(q) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n)\Omega_S(\omega, \lambda)(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n)^T.
\]
Thus, when \(\Omega_S(\omega, \lambda)\) is positive semi-definite, we have \(E_{\omega,\lambda,S}(q) \geq 0\) for all \(q \in \mathbb{R}^{3n}\). Hence \(q\) is a critical point of \(E_{\omega,\lambda,S}\) if \(E_{\omega,\lambda,S}(q) = 0\). □

We can now deduce that equivalent frameworks with maximum rank positive semi-definite stress matrices are linked by affine transformations.

**Theorem 5.8.** Let \((G, p)\) be a framework which is fully realised on \(S\) and let \((\omega, \lambda)\) be an equilibrium stress for \((G, p)\). Suppose that \(\Omega_S(\omega, \lambda)\) is positive semi-definite and rank \(\Omega_S(\omega, \lambda) = 3n - \mu\). Let \((G, q)\) be a framework on \(S\) which is equivalent to \((G, p)\). Then \((G, q)\) is an \(S\)-affine image of \((G, p)\).

**Proof.** Since \((\omega, \lambda)\) is an equilibrium stress for \((G, p)\) we have \(E_{\omega,\lambda,S}(p) = 0\). Then
\[
E_{\omega,\lambda,S}(q) = E_{\omega,\lambda,S}(q) - E_{\omega,\lambda,S}(p) = \sum_{i=1}^{n} \lambda_i [k_i(q(v_i)) - k_i(p(v_i))] = 0
\]
since \((G, p)\) and \((G, q)\) are equivalent and both lie on \(S\). Lemma 5.7 now implies that \(q\) is a critical point of \(E_{\omega,\lambda,S}\) and hence \((\omega, \lambda)\) is an equilibrium stress for \((G, q)\). The theorem now follows from Lemma 5.4. □

6. A SUFFICIENT CONDITION FOR GLOBAL RIGIDITY

We say that a framework \((G, p)\) on \(S\) is quasi-generic if it is congruent to a generic framework on \(S\). The framework \((G, p)\) is said to be in standard position on \(S\) if \(p(v_i) = (x_i, y_i, z_i)\) and: \(p(v_1) = (0, y_1, 0)\) when \(S = \mathbb{Y}\); \(p(v_1) = (0, y_1, z_1)\) when \(S = \mathbb{C}\). All frameworks on \(E\) are taken to be in standard position. Two frameworks on \(S\) are \(S\)-congruent if there is an isometry of \(S\) which maps one on to the other. We use \(\mathbb{K}\) to denote the algebraic closure of a field \(\mathbb{K}\).

We will need the following result, [15, Lemma 8].

**Lemma 6.1.** Suppose \((G, p)\) and \((G, p_0)\) are \(S\)-congruent frameworks on \(S\) and \((G, p_0)\) is in standard position on \(S\). Then \((G, p)\) is quasi-generic if and only if \(td[\mathbb{Q}(r, p_0) : \mathbb{Q}(r)] = 2n - \ell\).

We can now prove an analogue of Theorem 1.3.

**Theorem 6.2.** Let \((G, p)\) be a generic framework on \(S\) with \(n \geq 3\). Suppose that \((G, p)\) has an equilibrium stress \((\omega, \lambda)\) with rank \(\Omega_S(\omega, \lambda) = 3n - \mu\). Then \((G, p)\) is globally rigid on \(S\).

**Proof.** Since \(n \geq 3\), \((\omega, \lambda)\) must be non-zero. The fact that \((G, p)\) is generic and has a non-zero equilibrium stress implies that \(n \geq 5\). Suppose \((G, q)\) is a framework on \(S\) which is equivalent to \((G, p)\). Then \((G, q)\) is an \(S\)-affine image of \((G, p)\) by Theorem 5.8. We
may apply two isometries of $S$ to move $(G, p)$ and $(G, q)$ to two frameworks $(G, p_0)$ and $(G, q_0)$ in standard position on $S$. Then $(G, q_0)$ will be an $S$-affine image of $(G, p_0)$. Let $p_0(v_i) = (x_i, y_i, z_i)$ and $q_0(v_i) = (\hat{x}_i, \hat{y}_i, \hat{z}_i)$. We will analyse each choice of $S$ in turn.

Case 1: $S = Y$. We have

\begin{equation}
q_0(v_i) = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix} \cdot p_0(v_i) + \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}
\end{equation}

for all $1 \leq i \leq n$.

Applying Equation (6.1) with \( q_0(v_1) = (0, y_1, 0) = p_0(v_1) \) (\( \hat{y}_1 = y_1 \) since \((G, p)\) and \((G, q)\) are on \( S \)) reveals that \( b = 0 = f \) and \( d = 1 \). For \( i = 2, 3, \ldots, n \), Equation (6.1) now gives

\[
\begin{bmatrix}
\hat{x}_i \\
\hat{y}_i \\
\hat{z}_i
\end{bmatrix} = q_0(v_i) = \begin{bmatrix} a & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & e \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} ax_i \\ cx_i + y_i \\ ez_i \end{bmatrix}.
\]

Using the fact \( q_0(v_i) \) and \( p_0(v_i) \) are on \( Y \), we deduce that

\begin{equation}
(a^2 - 1 + c^2)x_i^2 + 2cx_iy_i = 0.
\end{equation}

Suppose \( c \neq 0 \). Then we have

\[ y_i = \frac{(1 - a^2 - c^2)x_i}{2c}, \]

and

\[ r_i = x_i^2 + y_i^2 = x_i^2 + \frac{(1 - a^2 - c^2)x_i^2}{4c^2}. \]

These equations imply that \( x_i, y_i \in \mathbb{Q}(r, a, c) \). We may now deduce that

\[ \text{td} [\mathbb{Q}(r, p_0) : \mathbb{Q}(r)] \leq \text{td} [\mathbb{Q}(r, z_2, z_3, \ldots, z_n, a, c) : \mathbb{Q}(r)] \leq n + 2. \]

Since \( n \geq 5 \), this contradicts the fact that \( \text{td} [\mathbb{Q}(r, p_0) : \mathbb{Q}(r)] = 2n - 2 \) by Lemma 6.1.

Hence \( c = 0 \). Equation (6.2) and the fact that \( c = 0 \) implies \( a = \pm 1 \). It remains to show that \( e = \pm 1 \).

We may assume, without loss of generality, that \( v_1v_2 \in E \). Then

\[
x_1^2 + (y_1 - y_2)^2 + z_2^2 = \| (0, y_1, 0) - (x_2, y_2, z_2) \|^2 = \| p_0(v_1) - p_0(v_2) \|^2
\]

\[
= \| q_0(v_1) - q_0(v_2) \|^2 = \| (0, y_1, 0) - (\hat{x}_2, \hat{y}_2, \hat{z}_2) \|^2
\]

\[
= \| (0, y_1, 0) - A(x_2, y_2, z_2) \|^2 = \| (0, y_1, 0) - (\pm x_2, y_2, z_2) \|^2
\]

\[
= x_2^2 + (y_1 - y_2)^2 + e^2z_2^2.
\]

Hence \( z_2^2 = e^2z_2^2 \) and \( e = \pm 1 \).

We have shown that, if \( q_0 \neq p_0 \), then \((G, q_0)\) is a reflection of \((G, p_0)\) in a plane which contains \((0, y_1, 0)\) and either contains, or is perpendicular to, the \( z \)-axis or a composition thereof. Hence \((G, p_0)\) and \((G, q_0)\) are congruent. This implies that \((G, p)\) and \((G, q)\) are congruent, so \((G, p)\) is globally rigid.

Case 2: $S = E$. We have

\begin{equation}
q_0(v_i) = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix} \cdot p_0(v_i)
\end{equation}

for all \( 1 \leq i \leq n \).

Since \( p_0(v_1) = (0, y_1, z_1) \), \( q_0(v_1) = (0, \hat{y}_1, \hat{z}_1) \), \( y_1^2 = r_1z_1^2 \) and \( \hat{y}_1^2 = r_1\hat{z}_1^2 \) applying Equation (6.3) shows that \( b = 0 \) and \( d = e \). For \( i = 2, 3, \ldots, n \), we have
\[
\begin{bmatrix}
\dot{x}_i \\
\dot{y}_i \\
\dot{z}_i
\end{bmatrix} = q_0(v_i) = \begin{bmatrix}
a & 0 & 0 \\
c & d & 0 \\
0 & 0 & e
\end{bmatrix} \begin{bmatrix}
x_i \\
y_i \\
z_i
\end{bmatrix} = \begin{bmatrix}
a x_i \\
c x_i + d y_i \\
d z_i
\end{bmatrix}.
\]

Using the fact \( q_0(v_i) \) and \( p_0(v_i) \) are on \( \mathcal{E}_i \) we deduce that

\[
(a^2 - 1 + c^2)x_i^2 + 2cdx_i y_i + (d^2 - 1)y_i^2 - r_i(d^2 - 1)z_i^2 = 0.
\]

Suppose \( d^2 \neq 1 \). Then

\[
z_i^2 = \frac{(a^2 - 1 + c^2)x_i^2 + 2cdx_i y_i + (d^2 - 1)y_i^2}{r_i(d^2 - 1)}.
\]

Since

\[
x_i^2 + y_i^2 = r_i z_i^2 = \frac{(a^2 - 1 + c^2)x_i^2 + 2cdx_i y_i + (d^2 - 1)y_i^2}{d^2 - 1},
\]

this implies \( x_i, z_i \in \mathbb{Q}(r, a, c, d, y_i) \). We may now deduce that

\[
\text{td } [\mathbb{Q}(r, p_0) : \mathbb{Q}(r)] \leq \text{td } [\mathbb{Q}(r, y_1, y_2, \ldots, y_n, a, c, d) : \mathbb{Q}(r)] \leq n + 3.
\]

Since \( n \geq 5 \), this contradicts the fact that \( \text{td } [\mathbb{Q}(r, p_0) : \mathbb{Q}(r)] = 2n - 1 \), by Lemma 6.1.

Hence \( d^2 = 1 \). Substituting \( d^2 = 1 \) into Equation (6.4) gives

\[
(a^2 - 1 + c^2)x_i^2 + 2cdx_i y_i = 0.
\]

Similar arguments to those used in Case 1 can now be applied to deduce \( c = 0 \) and \( a = \pm 1 \).

We have shown that, if \( q_0 \neq p_0 \), then \((G, q_0)\) is a reflection of \((G, p_0)\) in the plane containing \((0, y_1, z_1)\) and the \( z \)-axis, or a rotation by \( \pi \) around the \( x \)-axis, or a composition thereof. Hence \((G, p_0)\) and \((G, q_0)\) are congruent. This implies that \((G, p)\) and \((G, q)\) are congruent, so \((G, p)\) is globally rigid.

**Case 3:** \( S = \mathcal{E} \). We have

\[
q_0(v_i) = \begin{bmatrix}
a & 0 & 0 \\
0 & d & 0 \\
0 & 0 & e
\end{bmatrix} \cdot p_0(v_i) \text{ for all } 1 \leq i \leq n.
\]

Since \( p_0(v_i) \) and \( q_0(v_i) \) both lie on \( \mathcal{E}_i \), we have \( x_i^2 + ay_i^2 + bz_i^2 = r_i \) and \( a^2x_i^2 + cdx_i y_i^2 + ay_i^2 = r_i \). We can eliminate \( x_i^2 \) from these equations to obtain

\[
r_i(a^2 - 1) + ay_i^2(d^2 - a^2) + bz_i^2(e^2 - a^2) = 0.
\]

Hence, if \( d^2 - a^2 \neq 0 \), then \( x_i, y_i \in \mathbb{Q}(r, a, d, e, z_i) \). This would imply that

\[
2n = \text{td } [\mathbb{Q}(r, p_0) : \mathbb{Q}(r)] \leq \text{td } [\mathbb{Q}(r, a, d, e, z_1, z_2, \ldots, z_n) : \mathbb{Q}(r)] \leq n + 3,
\]

a contradiction since \( n \geq 5 \). Hence \( d^2 = a^2 \). We can deduce similarly, from Equation (6.7), that \( a^2 = c^2 \). Equation (6.7) now implies that \( a^2 = 1 \).

We have shown that \((G, q_0)\) is a reflection of \((G, p_0)\) in either the plane \( x = 0, y = 0 \) or \( z = 0 \) or a composition thereof. Hence \((G, p_0)\) and \((G, q_0)\) are congruent. This implies that \((G, p)\) and \((G, q)\) are congruent, so \((G, p)\) is globally rigid. \(\square\)
7. 1-EXTENSIONS AND GLOBAL RIGIDITY

Given a graph $G$, the 1-extension operation constructs a new graph by first deleting an edge $v_1v_2$ and then adding a new vertex $v_0$ and three new edges $v_0v_1, v_0v_2, v_0v_3$ for some vertex $v_3$ distinct from $v_1, v_2$. Our aim is to show that the property of having a maximum rank stress matrix is preserved by the 1-extension operation. To do this we will have to change our viewpoint from the surface $S \subset \mathbb{R}^3$ to a point $p \in \mathbb{R}^3$.

Given a map $p : V \rightarrow \mathbb{R}^3$, there is a unique family of concentric surfaces $S$ with $p(v_i) \in S_i$ for each $S \in \{Y, C, E\}$, as long as $p(v_i)$ does not lie on the $z$-axis for all $1 \leq i \leq n$ when $S \in \{Y, C\}$ and $p(v_i) \neq (0,0,0)$ for all $1 \leq i \leq n$ when $S = E$. We will refer to $S$ as the surface induced by $p$ and denote it by $S^p$.

**Lemma 7.1.** Suppose $(G, p)$ is a generic framework on $S$ with $n \geq 3$. Let $G' = (V', E')$ be a 1-extension of $G$, obtained by deleting an edge $e = v_1v_2$ and adding a new vertex $v_0$ and new edges $v_0v_1, v_0v_2, v_0v_3$. Then there exists a map $q : V' \rightarrow \mathbb{R}^3$ such that $\text{rank } R_{S^p}(G', q) = \text{rank } R_S(G, p) + 3$. Furthermore, if $(\omega, \lambda)$ is an equilibrium stress for $(G, p)$ on $S$ and $\omega \neq 0$, then there exists an equilibrium stress $(\omega', \lambda')$ for $(G', q)$ on $S^q$ such that $\text{rank } \Omega_{S^q}(\omega', \lambda') = \text{rank } k\Omega_S(\omega, \lambda) + 3$.

**Proof.** Define $(G', q)$ by putting $q(v) = p(v)$ for all $v \in V$ and $q(v_0) = \frac{1}{2}(p(v_1) + \frac{1}{2} p(v_2))$. Let $S^q$ be the surface induced by $q$.

We first consider the framework $(G' + v_1v_2 - v_0v_2, q)$ on $S^q$. Its rigidity matrix $R$ can be constructed from $R_S(G, p)$ by adding 3 new columns indexed by $v_0$, and 3 new rows indexed by $v_0, v_0v_1$ and $v_0v_3$. Since $(p(v_1), p(v_2), p(v_3))$ is a generic point on $S(r_1) \times S(r_2) \times S(r_3)$, the $3 \times 3$ submatrix $M$ of $R$ with rows indexed by $v_0, v_0v_1, v_0v_3$ and columns indexed by $v_0$ is non-singular.

The fact that the new columns contain zeros everywhere except in the new rows now gives rank $R = \text{rank } R_{S^q}(G', p) + 3$. Since $q(v_0), q(v_1)$ and $q(v_2)$ are collinear, the rows in $R_{S^q}(G' + v_1v_2, q)$ corresponding to $v_0v_1, v_0v_2, v_1v_2$ are a minimal linearly dependent set. Thus

$$\text{rank } R_{S^q}(G', q) = \text{rank } R_{S^q}(G' + v_1v_2, q) = \text{rank } R = \text{rank } R_S(G, p) + 3.$$

Let $(\omega', \lambda')$ be the stress for $(G', q)$ on $S^q$ defined by putting $\omega'_f = \omega_f$ for all $f \in E - e$, $\omega'(v_1v_0) = 2\omega_e, \omega'(v_2v_0) = 2\omega_e, \lambda'(v) = \lambda(v)$ for all $v \in V$ and $\lambda'(v_0) = 0$. It is straightforward to verify that $(\omega', \lambda')$ is an equilibrium stress for $(G', q)$ on $S^q$. Let $\omega_{ij}$ be the $ij$-th entry of $\Omega(\omega')$ for $i \neq j$ and $\lambda_i$ be the $ii$-th entry of $\Lambda(\lambda')$.

We first consider $\Omega(\omega') + \Lambda(\lambda')$. We have

$$\Omega(\omega') + \Lambda(\lambda') = \begin{bmatrix}
4\omega_{12} & -2\omega_{12} & 0 & \cdots & 0 \\
-2\omega_{12} & \sum_{j=3}^{n} \omega_{1j} + \omega_{12} + \lambda_1 & -2\omega_{12} & 0 & \cdots & 0 \\
-2\omega_{12} & 0 & \sum_{j=3}^{n} \omega_{2j} + \omega_{12} + \lambda_2 & -\omega_{13} & \cdots & -\omega_{1n} \\
0 & -\omega_{13} & -\omega_{23} & \cdots & \cdots & -\omega_{3n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}.$$
By adding 1/2 times the first row to the second and third rows, respectively, this reduces to
\[
\begin{bmatrix}
4\omega_{12} & -2\omega_{12} & -2\omega_{12} & 0 & \ldots & 0 \\
0 & \sum_j \omega_{1j} + \lambda_1 & \sum_j \omega_{1j} + \lambda_1 & \ldots & \ldots & \ldots \\
0 & -\omega_{12} & \sum_j \omega_{2j} + \lambda_2 & -\omega_{23} & \ldots & -\omega_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}.
\]

Now adding 1/2 times the first column to the second and third columns, respectively, gives
\[
\begin{bmatrix}
4\omega_{12} & 0 & 0 & 0 & \ldots & 0 \\
0 & \sum_j \omega_{1j} + \lambda_1 & -\omega_{12} & -\omega_{13} & \ldots & -\omega_{1n} \\
0 & -\omega_{12} & \sum_j \omega_{2j} + \lambda_2 & -\omega_{23} & \ldots & -\omega_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix} = \begin{bmatrix}
4\omega_{12} & 0 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots \\
\vdots & \Omega(\omega) + \Lambda(\lambda) \\
0 \\
\end{bmatrix}.
\]

Since \(\omega_{12} \neq 0\), we have \(\text{rank}(\Omega(\omega') + \Lambda(\lambda')) = \text{rank}(\Omega(\omega) + \Lambda(\lambda)) + 1\).

Since \(\lambda'(v) = 0\), we can repeat the above argument for \(\Omega(\omega')\) when \(S = \mathbb{E}\), for \(\Omega(\omega') - \Delta(\lambda')\) when \(S = \mathcal{E}\), and for both \(\Omega(\omega') + \alpha \Lambda(\lambda')\) and \(\Omega(\omega') + \beta \Lambda(\lambda')\) when \(S = \mathcal{E}\), to deduce that \(\text{rank}(\Omega_{S_q}(\omega', \lambda')) = \text{rank}(\Omega_{S}(\omega, \lambda)) + 3\).

We do not know whether we can find a framework \((G', q)\) which satisfies the conclusions of Lemma 7.1 and in addition has \(S^q = S^p^2\). Lacking such a result, we are forced to consider frameworks on ‘generic surfaces’ i.e. surfaces \(S^q\) induced by some generic \(q \in \mathbb{R}^{3n}\).

Lemma 7.2. Suppose \((G, p)\) is an infinitesimally rigid framework on some surface \(S\). Then \((G, q)\) is infinitesimally rigid on \(S^q\) for all generic \(q \in \mathbb{R}^{3n}\).

Proof. Choose \(q : V \rightarrow \mathbb{R}^3\) such that \(q(v_i)\) does not lie on the z-axis for all \(1 \leq i \leq n\) when \(S \in \{\mathbb{Y}, \mathbb{E}\}\) and \(q(v_i) \neq (0, 0, 0)\) for all \(1 \leq i \leq n\) when \(S = \mathcal{E}\). Since \(q(v_i) \in S^q\) for all \(1 \leq i \leq n\), the \(S^q\)-rigidity matrix for \((G, q)\) has the form \(R_{S^q}(G, q) = \begin{bmatrix} R(G, q) \\ S(G, q) \end{bmatrix}\) where \(R(G, q)\) is the ordinary rigidity matrix of \((G, q)\),

\[
S(G, q) = \begin{bmatrix}
s_1 & 0 & \ldots & 0 \\
0 & s_2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & s_n \\
\end{bmatrix},
\]

and

\[
s_i = \begin{cases} 
(x_i, y_i, 0), & \text{if } S^q = \mathbb{Y}; \\
(x_i, y_i, \frac{x_i^2 + y_i^2}{z_i}), & \text{if } S^q = \mathbb{E}; \\
(x_i, \alpha y_i, \beta z_i), & \text{if } S^q = \mathcal{E}.
\end{cases}
\]

The expression for \(s_i\) when \(S^q = \mathbb{E}\) is obtained by substituting \(r_i = (x_i^2 + y_i^2)/z_i^2\) into Equation (4.2). Since the entries in \(R_{S^q}(G, q)\) are rational functions of \(q\), its rank will be maximised when \(q\) is a generic point in \(\mathbb{R}^{3n}\).

The analogous result for frameworks with a maximum rank stress matrix is not true in general. It becomes true, however, if we restrict our attention to infinitesimally rigid frameworks.

---

2Partial results are known for particular surfaces: there exists a framework \((G, q)\) with rank \(R_{S^q}(G', q) = \text{rank} R_{S}(G, p) + 3\) and \(S^q = S\) when \(S = \mathbb{Y}\) [13], and when \(S = \mathbb{E}(1)\) or \(S = \mathcal{E}(1)\) [19].
Lemma 7.3. Suppose \((G, p)\) is an infinitesimally rigid framework on \(S\) and \(\text{rank}\Omega_S(\omega, \lambda) = 3n - \mu\) for some equilibrium stress \((\omega, \lambda)\) of \((G, p)\). Then \((G, q)\) has an equilibrium stress \((\omega', \lambda')\) on \(S^9\) with \(\text{rank}\Omega_{S^9}(\omega', \lambda') = 3n - \mu\) for all generic \(q \in \mathbb{R}^{3n}\).

Proof. We adapt the proof technique of Connelly and Whiteley [8, Theorem 5]. Choose \(q : V \to \mathbb{R}^{3n}\) such that \((G, q)\) is infinitesimally rigid on \(S^9\). We saw in the proof of Lemma 7.2 that the entries in \(R_G(G, q)\) are rational functions of \(q\). Since the space of equilibrium stresses of \((G, q)\) is the cokernel of \(R_G(G, q)\), each equilibrium stress of \((G, q)\) can be expressed as a pair of rational functions \((\omega(q, t), \lambda(q, t))\) of \(q\) and \(t\), where \(t\) is a vector of \(m - 2n + \ell\) indeterminates. This implies that the entries in the corresponding stress matrix \(\Omega_S(\omega(q, t), \lambda(q, t))\) will also be rational functions of \(q\) and \(t\). Hence the rank of \(\Omega_S(\omega(q, t), \lambda(q, t))\) will be maximised whenever \(q, t\) is algebraically independent over \(\mathbb{Q}\). In particular, for any generic \(q \in \mathbb{R}^{3n}\), \((G, q)\) is infinitesimally rigid on \(S^9\) by Lemma 7.2 and we can choose \(t \in \mathbb{R}^{m-2n+\ell}\) such that \(\text{rank}\Omega_{S^9}(\omega(q, t), \lambda(q, t)) = 3n - \mu\). □

We need one final lemma before we can obtain our result on generic 1-extensions. An equilibrium stress \((\omega, \lambda)\) for a framework \((G, p)\) on \(S\) is said to be nowhere zero if \(\omega_e \neq 0\) for all \(e \in E\).

Lemma 7.4. Suppose \((G, p)\) is a generic framework on \(S\) and \(\text{rank}\Omega_S(\omega, \lambda) = 3n - \mu\) for some equilibrium stress \((\omega, \lambda)\) of \((G, p)\). Then \(\text{rank}\Omega_S(\omega', \lambda') = 3n - \mu\) for some nowhere zero equilibrium stress \((\omega', \lambda')\) of \((G, p)\) on \(S\).

Proof. We may assume that \((\omega, \lambda)\) has been chosen such that the number of edges \(e \in E\) with \(w_e = 0\) is as small as possible. Suppose that \(w_e = 0\) for some \(e \in E\). Then \((\omega|_{E-e}, \lambda)\) is an equilibrium stress for \((G-e, p)\) on \(S\) and \(\text{rank}\Omega_S(\omega|_{E-e}, \lambda) = 3n - \mu\). By Theorem 7.2 \((G-e, p)\) is globally rigid on \(S\). In particular \((G-e, p)\) is rigid on \(S\). Since \(p\) is generic, \((G-e, p)\) is infinitesimally rigid on \(S\). This implies that the row of \(R_S(G, p)\) indexed by \(e\) is contained in a minimal linearly dependent set of rows of \(R_S(G, p)\). This gives us an equilibrium stress \((\hat{\omega}, \hat{\lambda})\) for \((G, p)\) on \(S\) with \(\hat{\omega}_e \neq 0\). Then \((\omega', \lambda') = (\omega, \lambda) + c(\hat{\omega}, \hat{\lambda})\) is an equilibrium stress for \((G, p)\) on \(S\) for any \(c \in \mathbb{R}\). We can now choose a small \(c > 0\) so that \(\text{rank}\Omega_S(\omega', \lambda') = 3n - \mu\), and \(\omega' \neq 0\) for all \(f \in E\) with \(\omega_f \neq 0\). This contradicts the choice of \((\omega, \lambda)\). □

Theorem 7.5. Suppose \((G, p)\) is an infinitesimally rigid framework on \(S\) and \((\omega, \lambda)\) is an equilibrium stress for \((G, p)\) with \(\text{rank}\Omega_S(\omega, \lambda) = 3n - \mu\). Let \(G' = (V', E')\) be a 1-extension of \(G\) and \(q : V' \to \mathbb{R}^3\) such that \(q\) is generic in \(\mathbb{R}^{3(n+1)}\). Then \((G', q)\) is infinitesimally rigid on \(S^9\) and has an equilibrium stress \((\omega', \lambda')\) with \(\text{rank}\Omega_{S^9}(\omega', \lambda') = 3(n+1) - \mu\).

Proof. We may assume that \(p\) is a generic point in \(\mathbb{R}^{3n}\) by Lemmas 7.2 and 7.3 and that \((\omega, \lambda)\) is nowhere zero by Lemma 7.4. We can now use Lemma 7.1 to deduce that there exists a map \(p^* : V' \to \mathbb{R}^3\) such that \((G, p^*)\) is infinitesimally rigid on \(S^9\) with an equilibrium stress \((\omega^*, \lambda^*)\) for \((G', p^*)\) on \(S^9\) such that \(\text{rank}\Omega_{S^9}(\omega^*, \lambda^*) = 3(n+1) + 3\). The theorem now follows by another application of Lemmas 7.2 and 7.3. □

8. Global rigidity on concentric cylinders

In this section we apply our results to make progress on the conjectured characterisation of global rigidity on concentric cylinders given in [13, Conjecture 9.1], see also [17, Conjecture 5.7].
Conjecture 8.1. Suppose \((G, p)\) is a generic framework on a family of concentric cylinders \(Y\). Then \((G, p)\) is globally rigid if and only if \(G\) is a complete graph on at most four vertices, or \(G\) is 2-connected and redundantly rigid on \(Y\).

We have seen that the redundant rigidity of \(G\) on \(Y\) is independent of the radii of the cylinders in \(Y\). Thus Conjecture 8.1 would imply that the global rigidity of a generic realisation of \(G\) on a family of concentric cylinders is also independent of the radii of the cylinders.

Theorem 2.3 shows that the combinatorial conditions given in Conjecture 8.1 are necessary for global rigidity. We could try to demonstrate sufficiency using a similar proof technique to that of Theorem 1.2. This would involve two steps: (i) a graph theoretic step obtaining a recursive construction for 2-connected, redundantly rigid graphs; (ii) a geometric step showing that each operation used in the recursive construction preserves global rigidity. Part (i) would be resolved by the following conjecture (which uses the base graphs \(K_5 - e, H_1, H_2\) and the operations of 1-, 2- and 3-join illustrated in Figures 1 and 2).

Conjecture 8.2. Suppose \(G\) is a 2-connected graph which is redundantly rigid on some (or equivalently every) family of concentric cylinders. Then \(G\) can be obtained from either \(K_5 - e, H_1, H_2\) by recursively applying the operations of edge addition, 1-extension, and 1-, 2- and 3-join.

The results of [17] verify the special case of this conjecture when \(|E| = 2|V| - 1\) i.e. \(E\) is a circuit in the generic rigidity matroid for the cylinder.

![Graphs K5−e, H1, H2](image)

**Figure 1.** The graphs \(K_5 - e, H_1, H_2\).

We close by showing that all graphs constructed from our base graphs using the edge addition and 1-extension operations are generically globally rigid on concentric cylinders with algebraically independent radii.

Theorem 8.3. Suppose \(G\) is a graph on \(n\) vertices which can be constructed from \(K_5 - e, H_1, H_2\) by a sequence of 1-extensions and edge additions. Then \((G, p)\) is globally rigid on \(Y^p\) for all generic \(p \in \mathbb{R}^3\).

**Proof.** We use induction on \(n\) to show that \((G, p)\) is infinitesimally rigid on \(Y^p\) and has an equilibrium stress \((\omega, \lambda)\) with rank \(\Omega_{Y^p}(\omega, \lambda) = 3n - 6\). The result will then follow from Theorem 6.2. The base case of the induction is when \(G \in \{K_5 - e, H_1, H_2\}\). We construct a particular realisation \((G, p)\) for each such \(G\) which is infinitesimally rigid on \(Y^p\) and has an equilibrium stress with a full rank stress matrix in the Appendix. We may deduce that the same properties hold for all generic \(p\) by applying Lemmas 7.2 and 7.3. To complete the induction we need to show that the 1-extension and edge addition operations preserve
the properties of infinitesimal rigidity and having a maximum rank stress matrix. This is trivially true for edge addition. It holds for 1-extension by Theorem 7.5.

\[ \square \]

We conjecture that Theorem 8.3 can be strengthened to show that, if \( G \) can be constructed as in Theorem 8.3 and \( (G, p) \) is a generic framework on any family of concentric cylinders \( \mathcal{Y} \), then \( (G, p) \) is globally rigid on \( \mathcal{Y} \).

9. Closing Remark

It follows from [7] and [9] that a generic framework in \( \mathbb{R}^d \) with \( n \geq d + 2 \) vertices is globally rigid if and only if it has a stress matrix of rank \( n - d - 1 \). It is conceivable that the stress matrix condition given in Theorem 6.2 provides a necessary, as well as a sufficient, condition for the global rigidity of a generic framework on \( \mathcal{S} \) whenever the framework has at least \( 7 - \ell \) vertices. The following examples indicate why we need this lower bound on \( n \).

The smallest redundantly rigid graph on the cone is \( K_5 \), but no framework \( (K_5, p) \) on \( \mathcal{C} \) can have a stress matrix with the maximum possible rank of \( 3n - \mu = 10 \). To see this consider a generic \( p \in \mathbb{R}^5 \). Since every equilibrium stress for \( (K_5, p) \) in \( \mathbb{R}^3 \) is an equilibrium stress for \( (K_5, p) \) on \( \mathcal{C}^p \), and since the spaces of equilibrium stresses for \( (K_5, p) \) in \( \mathbb{R}^3 \) and on \( \mathcal{C}^p \) are both 1-dimensional, these spaces are the same. This implies that every equilibrium stress \( (\omega, \lambda) \) for \( (K_5, p) \) has \( \lambda = 0 \) and rank \( \Omega(\omega) \leq 3 \). Hence rank \( \Omega_{\mathcal{C}^p}(\omega, \lambda) \leq 9 \). On the other \( (K_5, p) \) is globally rigid on \( \mathcal{C}^p \) for all \( p \).

Similarly, the smallest redundantly rigid graph on the ellipsoid is \( K_6 - \{e, f\} \) for two nonadjacent edges \( e, f \), but no framework \( (K_6 - \{e, f\}, p) \) on \( \mathcal{E} \) can have a stress matrix with the maximum possible rank of \( 3n - \mu = 15 \). (We do not know whether every generic framework \( (K_6 - \{e, f\}, p) \) on \( \mathcal{E}^p \) is globally rigid.)

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We define a framework \((G,p)\) for \(G \in \{K_5 - e, H_1, H_2\}\) which is infinitesimally rigid on \(\mathbb{R}^p\) and has a self-stress \((\omega, \lambda)\) on \(\mathbb{R}^p\) with maximum rank stress matrix. We will use the labeling of the vertices given in Figure 1 and adopt the convention that \(\omega_{ij}\) is the weight on the edge \(v_i v_j\) in \(\omega\) and \(\lambda_i\) is the weight on the vertex \(v_i\) in \(\lambda\).

**Case 1:** \(G = K_5 - e\). Let \((G,p)\) and \((\omega, \lambda)\) be defined by \(p(v_1) = (0,1,0), p(v_2) = (1,1,1), p(v_3) = (-1,-2,-1), p(v_4) = (2,3,4), p(v_5) = (5,1,-1),\)

\[(\omega_{12}, \omega_{13}, \omega_{14}, \omega_{15}, \omega_{23}, \omega_{24}, \omega_{25}, \omega_{35}, \omega_{45}) = (-369, 192, 153, 51, -96, -279, -138, 32, 45)\]

and

\[(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (-270, -270, -192, 54, -6)).\]
It is straightforward to check that rank $R_Y(G, p) = 13$, that $(\omega, \lambda) \cdot R_Y(G, p) = 0$ and that rank $\Omega_Y(\omega, \lambda) = 9$.

**Case 2:** $G = H_1$. Let $(G, p)$ and $(\omega, \lambda)$ be defined by $p(v_1) = (0, 1, 0), p(v_2) = (3, 1, 0), p(v_3) = (1, 4, 1), p(v_4) = (1, 2, 2), p(v_5) = (2, 2, 3), p(v_6) = (6, 0, 2), \quad (\omega_{12}, \omega_{13}, \omega_{15}, \omega_{23}, \omega_{24}, \omega_{25}, \omega_{36}, \omega_{35}, \omega_{37}, \omega_{45}, \omega_{56})$

$= (41, -246, 369, -123, 30, 48, 60, 50, -40, 492, 56)$

and

$\quad (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (-123, -39, 30, 123, -102, 28).$

It is straightforward to check that rank $R_Y(G, p) = 16$, that $(\omega, \lambda) \cdot R_Y(G, p) = 0$ and that rank $\Omega_Y(\omega, \lambda) = 12$.

**Case 3:** $G = H_2$. Let $(G, p)$ and $(\omega, \lambda)$ be defined by $p(v_1) = (0, 1, 0), p(v_2) = (3, 1, 0), p(v_3) = (1, 4, 1), p(v_4) = (1, 2, 2), p(v_5) = (2, 2, 3), p(v_6) = (6, 0, 2), p(v_7) = (3, 4, 3), \quad (\omega_{12}, \omega_{13}, \omega_{15}, \omega_{23}, \omega_{24}, \omega_{25}, \omega_{36}, \omega_{35}, \omega_{37}, \omega_{45}, \omega_{56}, \omega_{57}, \omega_{67})$

$= (-58, 348, -522, -108, -24, -40, 14, 21, -696, 56, 588, -42)$

and

$\quad (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) = (-174, -6, 24, 174, 372, -28, -252).$

It is straightforward to check that rank $R_Y(G, p) = 19$, that $(\omega, \lambda) \cdot R_Y(G, p) = 0$ and that rank $\Omega_Y(\omega, \lambda) = 15$.

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