Fermion mass hierarchy from nonuniversal abelian extensions of the Standard Model

Carlos E. Díaz, S.F. Mautilla, R. Martinez.

cediaj@unal.edu.co, sfmantillas@unal.edu.co, remartinezm@unal.edu.co

Departamento de Física, Universidad Nacional de Colombia,
Ciudad Universitaria, K. 45 No. 26-85, Bogotá D.C., Colombia

(Dated: June 13, 2018)

A nonuniversal abelian extension $U(1)_X$ free from chiral anomalies is introduced into the Standard Model (SM), in order to evaluate its suitability in addressing the fermion mass hierarchy (FMH) by using seesaw mechanisms (SSM). In order to break the electroweak symmetry, three Higgs doublets are introduced, which give mass at tree-level to the top and bottom quarks, and the muon lepton. With an electroweak singlet scalar field, the $U(1)_X$ symmetry is broken and the exotic particles acquire masses. The light particles in the SM obtain their masses via SSM and Yukawa couplings differences. Active neutrino masses are generated through inverse seesaw mechanisms (ISM). Additionally, the algebraic expressions for the mixing angles for quarks and leptons are also shown in the article.

Keywords: Flavor Problem, Neutrino Physics, Extended Scalar Sectors, Beyond Standard Model, Fermion masses, Inverse seesaw Mechanism.

I. INTRODUCTION

The current phenomenological data in high-energy physics is consistent with the existence of twelve fundamental fermions divided into quarks and leptons with their masses ranging from units of MeV to hundreds of GeV [1], and a large gap until thousandths of eV according to neutrino oscillation data [2, 3]. Such particles are grouped in three chiral anomaly-free families under the gauge symmetries of the Standard Model (SM) $G_{SM} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ [3]. However, although the electroweak spontaneous symmetry breaking (SSB) accounts for the mass acquisition of fermions, it is not understood how fermions masses covers so many orders of magnitude despite there is only one vacuum expectation value (VEV) in the current SM. Moreover, the hierarchy among mixing angles of quark and lepton obtained in the Cabibbo-Kobayashi-Maskawa (CKM) [4] and Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrices [5], respectively, has not been well understood. This issue, called fermion mass hierarchy (FMH) [6], is a motivation to extend the SM by adding new particles or symmetries.

The FMH problem has been addressed from different points of view in order to achieve the simplest model beyond the SM. One of the first and most important schemes is the left-right model proposed by H. Fritzsch whose mass matrices have suited textures to understand the existence of three mass scales in the fermionic spectrum [8]. C. Froggat and H. Nielsen presented a model in which the heaviest fermions acquire mass through the VEV of the Higgs field, while the lightest ones get massive through radiative corrections by employing degrees of freedom heavier than the SM particles [9]. Another remarkable way to understand the hierarchy mechanism is to assume that light neutrinos acquire their masses through radiative corrections [10]. Similar methodologies are shown in Refs. [11] using multiple scalar fields to achieve FMH. As a special case, Y. Koide addressed the model in a geometrical shape, yielding the well-known Koide formula to obtain the $\tau$ lepton mass $[12]$, while Z. Xing analyzed the quark spectrum involving masses and mixing angles of the CKM matrix [13].

Therefore, the confirmation of neutrino oscillations open new possibilities BSM. The smallness of neutrino masses is traditionally explained by the seesaw mechanism (SSM) [14], which adds Majorana fermions $N_R$ as right-handed neutrinos whose masses are at $\mu_N = 10^{14}$ GeV such that SM neutrinos get light masses in accordance with experimental upper limits and square mass differences, $\Delta m^2_{21}$ and $\Delta m^2_{31}$, from oscillation data. However, such a shocking mass can be lower by including a second set of right-handed neutrinos $\nu_R$ which acquire mass at units of TeV so as the inverse SSM (ISS) [15] can be implemented, and the Majorana mass turns out to be at units of $\text{keV} \sim 1\text{keV}$. Similarly, the large lepton mixings have been addressed with new methods. N. Haba and H. Murayama presented a model on neutrino masses through anarchic mass textures, i.e., without any particular structure [16], and discrete symmetries such as $A_4$ were also employed to achieve fermion masses and mixings [17].

Lastly, the detection of the Higgs boson has encouraged new schemes with extended scalar sectors and gauge groups because of the existence of fundamental scalar fields in Nature. In this way, the FMH can be understood in two Higgs doublet models (2HDM) [18] and Next-to-Minimal 2HDMs (N2HDM) [19] and N3HDM [20]. Thus, this article is oriented in nonuniversal abelian extensions of the SM $G_{SM} \otimes U(1)_X$ whose charges are different among families, and its symmetry breaking is ensured by a Higgs singlet $\chi$ at TeV scale. As a consequence, chiral anomalies from triangle diagrams could emerge, so it is important search for solutions of chiral anomaly equations that cancel them.
such a requirement, together with nonuniversality, is satisfied by adding new isospin singlet exotic fermions $\mathcal{T}$, $\mathcal{J}$ and $\mathcal{E}$ to the model. Furthermore, these new fermions might contribute to mass acquisition so as FMH can be obtained by avoiding unpleasant fine-tunings. Additionally, the nonuniversality in the set of $X$ charges which cancels the chiral anomalies could imply flavor violation processes such as $B^+ \rightarrow K^+ \ell^+ \ell^-$ or $h \rightarrow \mu \tau$.

The present article shows a nonuniversal abelian extension which address FMH. First of all, section II presents the seesaw mechanism which employs the vacuum hierarchy (VH) among scalars to yield algebraic expressions of fermion masses which suggest the existence of lighter and heavier fermions than the original VEVs. After that, section III introduces a nonuniversal model corresponding to one solution of the chiral anomaly equations (I) with the corresponding analysis on the different hidden flavor symmetries and its consequences in mass acquisition mechanisms. Then, section IV employs special textures to get a suited fermionic spectrum which might account for FMH.

Lastly, the important features of each model are presented and compared in section V with some conclusions outlined at the end of the article.

**II. SEESAW MECHANISM**

The majority of textures propose finite and null components of the mass matrices in order to get the suited mass eigenvalues and mixing angles. The model achieves in a natural way the fermionic mass hierarchy through *seesaw mechanisms* (SSM), with the existence of elements at two different orders of magnitude in a very special location inside the mass matrix.

The simplest example of the SSM comprises two fermions $\mathcal{F}$ and $\mathcal{J}$ coupled by two Higgs scalars $\phi_{1,2}$. The Yukawa Lagrangian is

$$-\mathcal{L}_Y = h_{1f} \phi_1 \mathcal{F}_R + h_{1f} \phi_1 \mathcal{F}_R + h_{2f} \mathcal{F}_L \phi_2 \mathcal{F}_R,$$

and the corresponding mass matrix after evaluating at the VEVs is

$$M_{SSM} = \left(h_{1f} \phi_1 v_1, h_{1f} \phi_1 v_1, h_{2f} \phi_2 \mathcal{F}_R, h_{2f} \phi_2 \mathcal{F}_R\right).$$

The diagonalization may be done on either $MM^\dagger$ or $M^1M$. Both matrices give the mass eigenvalues

$$m_f^2 \approx \frac{(h_{1f}h_{2f} - h_{1f}h_{1f})^2 v_1^2 v_2^2}{(h_{1f})^2 + (h_{2f})^2} v_1^2 + (h_{2f})^2 + (h_{1f})^2 v_2^2,$$

$$m_F^2 \approx \frac{(h_{1f})^2 + (h_{1f})^2 v_1^2 + (h_{2f})^2 + (h_{2f})^2 v_2^2}{2}.$$
does not, as it is shown below by choosing one of the two possible VHS: when \( v_1 < v_2 \)

\[
\tan \theta_L \approx \frac{\left[ h_1 f_{3f} h_1 f_{2f} \right]}{h_{2f}^2 + (h_{2f})^2} \frac{v_1}{v_2}, \quad \tan \theta_R \approx \frac{h_{2f}}{h_{2f}}.
\]

(7)

while when \( v_2 < v_1 \) the mixing angles turn out to be

\[
\tan \theta_L \approx \frac{\left[ h_1 f_{3f} h_1 f_{2f} \right]}{h_{1f}^2 + (h_{1f})^2} \frac{v_2}{v_1}, \quad \tan \theta_R \approx \frac{h_{1f}}{h_{1f}}.
\]

(8)

In both scenarios the left-handed mixing angle gets suppressed by the specific VHI between the Higgs doublets. On the contrary, the right-handed mixing angle results as the ratio among the Yukawa coupling constants of the dominant VEV.

The following sections show the abelian extensions \( U(1)_X \) with nonuniversal sets of charges and the fermion mass acquisition in the quark and lepton sectors.

### III. NONUNIVERSAL ABELIAN EXTENSIONS, PARTICLE CONTENT AND FLAVOR SYMMETRIES

The model presented in this article is a nonuniversal abelian extension of the SM, in which a new gauge symmetry \( U(1)_X \) is added to the SM gauge group \( G_{SM} \). Such an extension is broken to the SM by an additional Higgs scalar singlet \( \chi \) whose VEV \( v_\chi \) lies at TeV. Then, the electroweak SB is done by three Higgs scalar doublets \( \Phi_{1,2,3} \) whose VEVs fulfill \( v^2 = v_1^2 + v_2^2 + v_3^2 = 246 \text{ GeV} \) in order to get the correct masses for electroweak gauge bosons \( W^\pm \) and \( Z^\mu \). The complete SB chain is

\[
SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \otimes U(1)_X \rightarrow SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \rightarrow SU(3)_C \otimes U(1)_Q
\]

The scalar Higgs potential of the model is given by:

\[
V = \mu_1^2 (\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2 + \Phi_3^\dagger \Phi_3 + \mu_2^2 \chi^* \chi) \\
+ \mu_2 (\Phi_1^\dagger \Phi_2 + \Phi_1^\dagger \Phi_3 + \Phi_2^\dagger \Phi_1 + \mu_3 (\Phi_3^\dagger \Phi_3 + \Phi_1^\dagger \Phi_1) \\
+ \mu_2^2 (\Phi_2^\dagger \Phi_3 + \Phi_3^\dagger \Phi_2) + \frac{f}{\sqrt{2}} (\Phi_1^\dagger \Phi_3 \chi + h.c) \\
+ \lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_3^\dagger \Phi_3)^2 + \lambda_\chi (\chi^* \chi)^2 \\
+ 2 \lambda_1 (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) + 2 \lambda_2 (\Phi_1^\dagger \Phi_3) (\Phi_3^\dagger \Phi_1) + 2 \lambda_3 (\Phi_2^\dagger \Phi_3) (\Phi_3^\dagger \Phi_2) \\
+ 2 \lambda_1 (\Phi_1^\dagger \Phi_1 + \lambda_2 (\Phi_2^\dagger \Phi_2 + \lambda_3 (\Phi_3^\dagger \Phi_3) (\chi^* \chi),
\]

(9)

where the terms with \( \mu_3^2 \) break softly the discrete symmetry \( Z_2 \) and the ones with \( \mu_2^2, \mu_3^2 \) the \( U(1)_X \) gauge symmetry softly [23]. After symmetry breaking the minimal condition for the VEV of the scalar field \( \chi \) is given by

\[
\mu_\chi^2 + \lambda_\chi \chi^2 + \lambda_1 \chi v_1^2 + \lambda_2 \chi v_2^2 + \lambda_3 \chi v_3^2 + f \frac{v_1 v_3}{v_\chi} = 0
\]

(10)

The VEV \( v_\chi \) breaks the symmetry beyond the SM, giving masses to the exotic fermions. Therefore, since \( v_\chi \gg v_1, v_2, v_3 \),

\[
v_\chi^2 \approx -\frac{\mu_2^2}{\lambda_\chi}
\]

(11)

The other conditions for the electroweak VEV are:

\[
v_1 \left\{ \mu_1^2 + \lambda_{11} v_1^2 + \tilde{\lambda}_{12} v_1 v_2 + \tilde{\lambda}_{13} v_3^2 + \lambda_{14} v_\chi^2 + f \frac{v_1 v_3}{v_\chi} \right\} + \mu_2 v_2 + \mu_3 v_3 = 0
\]

\[
v_2 \left\{ \mu_2^2 + \lambda_{22} v_2^2 + \tilde{\lambda}_{23} v_1 v_2 + \tilde{\lambda}_{23} v_3^2 + 2 \lambda_{24} v_\chi^2 + f \frac{v_1 v_3}{v_\chi} \right\} + \mu_2 v_2 + \mu_3 v_3 = 0
\]

\[
v_3 \left\{ \mu_3^2 + \lambda_{33} v_3^2 + \tilde{\lambda}_{32} v_1 v_2 + \tilde{\lambda}_{32} v_3^2 + 2 \lambda_{34} v_\chi^2 + f \frac{v_1 v_3}{v_\chi} \right\} + \mu_2 v_2 + \mu_3 v_3 = 0
\]

(12)

For \( v_1, v_2, v_3 \) the constraints are given by:

\[
v_1^2 \approx -\frac{\mu_1^2 + \tilde{\lambda}_{12} v_2 + \tilde{\lambda}_{13} v_3^2 + \lambda_{14} v_\chi^2 + f \frac{v_1 v_3}{v_\chi}}{\lambda_{11}}
\]

\[
v_2^2 \approx \frac{\mu_2^2 v_1^2}{\left( \mu_2^2 + \tilde{\lambda}_{12} v_2^2 + \tilde{\lambda}_{13} v_3^2 + 2 \lambda_{24} v_\chi^2 \right)^2}
\]

\[
v_3^2 \approx \frac{\mu_3^2 v_1^2}{\left( \mu_3^2 + \tilde{\lambda}_{12} v_2^2 + \tilde{\lambda}_{13} v_3^2 + 3 \lambda_{34} v_\chi^2 + f \frac{v_1 v_3}{v_\chi} \right)^2}
\]

(13)

where \( \tilde{\lambda}_{ij} \equiv \lambda_{ij} + \lambda_{ji} \). The condition for the soft symmetry breaking reads \( |\mu_2|^2, |\mu_3|^2, |\mu_3|^2 \gg |\mu_1|^2, |\mu_2|^2, |\mu_3|^2 \), therefore \( v_1 \gg v_2, v_3 \). Comparing the second and third equations from (13), it can be seen that there is an additional factor \( f \frac{v_1 v_3}{v_\chi} \) in the denominator for \( v_3^2 \), which does not appear in the expression for \( v_2^2 \). Consequently, the space of parameters permits naturally the assumption of the following VH

\[
v_1^2 \gg v_2^2 \gg v_3^2.
\]

Lastly, the scalar sector, together with the Majorana mass scale \( \mu_N \), exhibits the VH

\[
v_1 = 245.7 \text{ GeV} \sim m_t \sqrt{2}, \quad v_\chi = 2.5 \text{ TeV},
\]

\[
v_2 = 12.14 \text{ GeV} \sim m_{h_2} \sqrt{2}, \quad \mu_N \sim 1 \text{ keV},
\]

\[
v_3 = 250 \text{ MeV} \sim m_\nu \sqrt{2}.
\]

This choice of VEVs of the Higgs doublets \( \Phi_{1,2,3} \) plays a fundamental role on the model, because it sets the energy scale for the charm and top quarks and the muon. All other SM particles acquire masses through suppression mechanisms along with such introduced energy scales.

On the other hand, the fermionic sector comprises the SM fermions (including right-handed neutrinos \( \nu_R \)) and an exotic sector composed by up-like quarks \( \tau \), down-like quarks
TABLE I. Scalar content of the model, non-universal $X$ quantum number and $Z_2$ parity.

| Scalar Doublets | Scalar Singlets | $X^\pm$ | $X^\pm$ |
|-----------------|-----------------|---------|---------|
| $\Phi_1$        | $(h_{h,1} + h_{h,2} + h_{h,3})$ | $+1/3^+$ | $X = \frac{\xi_1 + \xi_2 + \xi_3}{\sqrt{2}} - 1/3^+$ |
| $\Phi_2$        | $(h_{h,1} + h_{h,2} + h_{h,3})$ | $+2/3^+$ | |
| $\Phi_3$        | $(h_{h,1} + h_{h,2} + h_{h,3})$ | $+2/3^+$ | |

TABLE II. Particle content of the abelian extensions, non-universal $X$ quantum number and $Z_2$ parity for the model.

|                      | Left-Handed $X^\pm$ | Right-Handed $X^\pm$ |
|----------------------|----------------------|----------------------|
| **SM Quarks**        |                      |                      |
| $q^L_1$ | $u^L_1$ | $0^+$ | $u_R^+$ | $+2/3^+$ | $+1/3^+$ |
| $q^L_2$ | $d^L_2$ | $+1/3$ | $d_R^+$ | $+2/3$ | $+2/3$ |
| $q^L_3$ | $u^L_3$ | $+1/3$ | $u_R^+$ | $+2/3$ | $+2/3$ |
|                      |                      |                      |
| **SM Leptons**       |                      |                      |
| $\ell^L_1$ | $\nu^L_1$ | $-2/3^+$ | $\nu_R^+$ | $-1/3$ | $-1/3$ |
| $\ell^L_2$ | $e^L_2$ | $-1/3$ | $e_R^+$ | $-1/3$ | $-1/3$ |
| $\ell^L_3$ | $e^L_3$ | $-1$ | $e_R^+$ | $-1$ | $-1$ |
|                      |                      |                      |
| **Non-SM Quarks**    |                      |                      |
| $J_1^L$            | $+1/3$ | $J_1^R$ | $+1/3$ |
| $J_2^L$            | $+1/3$ | $J_2^R$ | $+1/3$ |
| $J_3^L$            | $+1/3$ | $J_3^R$ | $+1/3$ |
|                      |                      |                      |
| **Non-SM Leptons**  |                      |                      |
| $\xi_1^L$ | $\xi_1^R$ | $+1/3$ | $\xi_R^+$ | $+1/3$ | $+1/3$ |
| $\xi_2^L$ | $\xi_2^R$ | $-1/3$ | $\xi_R^+$ | $-1/3$ | $-1/3$ |
| $\xi_3^L$ | $\xi_3^R$ | $+5/3$ | $\xi_R^+$ | $+5/3$ | $+5/3$ |
|                      |                      |                      |
| **Majorana Fermions** | $N_{N}^{3/2}$ | $N_{N}^{3/2}$ | $0^+$ | $0^+$ |

$\mathcal{J}$, charged leptons $\mathcal{E}$ and Majorana fermions $\mathcal{N}_R$. The exotic sector of model has two up-like quarks $T^{1,2}$, two down-like quarks $J^{1,2}$, three charged leptons $\mathcal{E}^{1,2,3}$ and three Majorana masses $\mathcal{N}_R^{1,2,3}$. All the exotic fermions are isospin singlets, so they acquire mass through the Higgs singlet $\nu_\chi$. The addition of the exotic sector not only ensures the cancellation of chiral anomalies, which would not be canceled by only SM fermions, but also contributes to the mass acquisition mechanisms of fermions. The set of $U(1)_X$ charges is shown in Tab. III.

Before acting $Z_2$, the quark sector exhibits the global symmetry

$$G^Q_{\text{Flavor}} = SU(2)_{E_{L,3}} \otimes SU(3)_{u_R^{1,2,3}} \otimes SU(2)_{d_R^{1,2,3}},$$

while the lepton sector has the flavor symmetry

$$G^L_{\text{Flavor}} = SU(2)_{\nu_R^{1,2,3}} \otimes SU(2)_{d_R^{1,2,3}},$$

which does not show any universal symmetry, so it does not need breaking. After the action of the discrete symmetry, the quark global symmetry turns out to be

$$G^Q_{\text{Flavor}} \sim SU(2)^{2/3} \otimes SU(2)^{1/3} \otimes SU(2)^{1/3},$$

breaking the universality in the up quark right-handed sector and ensuring the complete acquisition of masses in the fermionic sector.

The section IV focuses on the fermion mass acquisition, mixing angles and the different mechanisms involved to obtain the FMH.

### IV. MASS HIERARCHY ACQUISITION

As it was mentioned in section III, the model lacks of flavor global symmetries. This feature implies that all fermions acquire mass at tree level.

The hadronic sector of the model contains the SM fields with four exotic chiral quarks: two up-like quarks $T^1, T^2$ and two down-like quarks $J^1, J^2$. The leptonic sector of the model contains the SM fields with three exotic chiral charged leptons $\mathcal{E}^1, \mathcal{E}^2, \mathcal{E}^3$ and three Majorana fermions $\mathcal{N}_R = (N^1, N^2, N^3_R)$. The non-universal quantum numbers and parities are shown in Tab. II.

The Yukawa Lagrangians under the symmetry $U(1)_X \otimes Z_2$ in the quark sector are

$$-\mathcal{L}_U = h_{1,1}^L \bar{q}^{L}_1 \Phi_2 u^L_1 + h_{1,2}^L \bar{q}^{L}_2 \Phi_2 u^L_2 + h_{1,3}^L \bar{q}^{L}_3 \Phi_2 u^L_3$$

$$+ h_{1,4}^L \bar{q}^{L}_1 \Phi_2 U^L_1 + h_{1,5}^L \bar{q}^{L}_2 \Phi_2 U^L_2 + h_{1,6}^L \bar{q}^{L}_3 \Phi_2 U^L_3$$

$$+ h_{1,7}^L \bar{q}^{L}_1 \Phi_1 U^L_1 + h_{1,8}^L \bar{q}^{L}_2 \Phi_1 U^L_2 + h_{1,9}^L \bar{q}^{L}_3 \Phi_1 U^L_3$$

$$+ h_{1,10}^L \bar{q}^{L}_1 \Phi_3 U^L_1 + h_{1,11}^L \bar{q}^{L}_2 \Phi_3 U^L_2 + h_{1,12}^L \bar{q}^{L}_3 \Phi_3 U^L_3$$

(18)

and, under the symmetry $U(1)_X \otimes Z_2$, the Yukawa Lagrangians in the lepton sector are

$$-\mathcal{L}_D = h_{1,13}^L \bar{q}^{L}_1 \Phi_3 d^L_1 + h_{1,14}^L \bar{q}^{L}_2 \Phi_3 d^L_2 + h_{1,15}^L \bar{q}^{L}_3 \Phi_3 d^L_3$$

$$+ h_{1,16}^L \bar{q}^{L}_1 \Phi_2 d^L_1 + h_{1,17}^L \bar{q}^{L}_2 \Phi_2 d^L_2 + h_{1,18}^L \bar{q}^{L}_3 \Phi_2 d^L_3$$

$$+ h_{1,19}^L \bar{q}^{L}_1 \Phi_1 d^L_1 + h_{1,20}^L \bar{q}^{L}_2 \Phi_1 d^L_2 + h_{1,21}^L \bar{q}^{L}_3 \Phi_1 d^L_3$$

$$+ h_{1,22}^L \bar{q}^{L}_1 \Phi_3 d^L_1 + h_{1,23}^L \bar{q}^{L}_2 \Phi_3 d^L_2 + h_{1,24}^L \bar{q}^{L}_3 \Phi_3 d^L_3$$

(19)

and, under the symmetry $U(1)_X \otimes Z_2$, the Yukawa Lagrangians in the lepton sector are

$$-\mathcal{L}_N = h_{2,1}^{\nu} \bar{\nu}_{\nu} \Phi_2 \nu^L_R + h_{2,2}^{\nu} \bar{\nu}_{\nu} \Phi_1 \nu^L_R + h_{2,3}^{\nu} \bar{\nu}_{\nu} \Phi_3 \nu^L_R$$

$$+ h_{2,4}^{\nu} \bar{\nu}_{\nu} \Phi_2 \nu^L_R + h_{2,5}^{\nu} \bar{\nu}_{\nu} \Phi_1 \nu^L_R + h_{2,6}^{\nu} \bar{\nu}_{\nu} \Phi_3 \nu^L_R$$

$$+ h_{2,7}^{\nu} \bar{\nu}_{\nu} \Phi_3 \Phi_2 \nu^L_R + h_{2,8}^{\nu} \bar{\nu}_{\nu} \Phi_3 \Phi_1 \nu^L_R + h_{2,9}^{\nu} \bar{\nu}_{\nu} \Phi_3 \Phi_3 \nu^L_R$$

$$+ h_{2,10}^{\nu} \bar{\nu}_{\nu} \Phi_3 \Phi_2 \nu^L_R + h_{2,11}^{\nu} \bar{\nu}_{\nu} \Phi_3 \Phi_1 \nu^L_R + h_{2,12}^{\nu} \bar{\nu}_{\nu} \Phi_3 \Phi_3 \nu^L_R$$

(20)

and, under the symmetry $U(1)_X \otimes Z_2$, the Yukawa Lagrangians in the lepton sector are
The mass term in the flavor basis turns out to be
\[ -\mathcal{L}_U = U_7^\dagger M_U U_R + m_{T2}^2 T_R^2 + \text{h.c.,} \]
where $M_U$ is the up-like quarks mass matrix
\[ M_U = \frac{1}{\sqrt{2}} \begin{pmatrix} h_{3u}^{11} v_3 & h_{3u}^{12} v_2 & h_{3u}^{13} v_3 & h_{12}^{11} v_2 \\ h_{1u}^{11} v_1 & 0 & h_{1u}^{13} v_1 & 0 \\ 0 & h_{1u}^{11} v_1 & 0 & h_{11}^{11} v_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
(24)

Since the determinant of $M_U$ is non-vanishing, all up-like quarks acquire mass. Then, the mass eigenvalues corresponding with the SM quark masses are
\[ m_u^2 \approx \frac{(h_{3u}^{13})^2 - (h_{1u}^{13})^2}{(h_{1u}^{11})^2} v_3^2, \]
\[ m_c^2 \approx \frac{(h_{3u}^{12})^2 - (h_{1u}^{12})^2}{(g_{12}^{11})^2} v_2^2, \]
\[ m_t^2 \approx \frac{(h_{3u}^{11})^2 - (h_{1u}^{11})^2}{(g_{11}^{11})^2} v_1^2, \]
(25)
while the masses of the exotic up-like quarks are
\[ m_{T1}^2 \approx \frac{(g_{12}^{11})^2 + (g_{11}^{11})^2}{2} v_{12}^2, \]
\[ m_{T2}^2 \approx \frac{(g_{12}^{11})^2}{2} v_{12}^2. \]
(26)
The corresponding left-handed rotation matrix can be expressed by
\[ \Psi_{U}^L = \Psi_{L,SSM} \Phi_{L,B} \]
(27)
where the seesaw angle is
\[ \Theta_{U}^L = \begin{pmatrix} (h_{3u}^{12} g_{11}^{12} + h_{12}^{12} g_{11}^{11}) v_2 e^{i(bu - eu)} \\ (g_{12}^{11})^2 + (g_{11}^{11})^2 v_1 e^{i(cu - eu)} \\ (h_{3u}^{13} h_{1u}^{12} + h_{1u}^{11} h_{3u}^{13}) v_1 e^{i(cu - eu)} \\ 0 \end{pmatrix} \]
(28)
while $\Phi_{L,B}$ diagonalizes only the SM-up quarks. Its angles are given by
\[ \tan \theta_{U_{13}}^{U,L} \approx \frac{h_{3u}^{13} h_{1u}^{31} + h_{1u}^{13} h_{3u}^{13} v_3}{h_{1u}^{11} h_{1u}^{12} + (h_{3u}^{13})^2} v_1 \]
\[ \tan \theta_{U_{23}}^{U,L} \approx 0 \]
\[ \tan \theta_{U_{12}}^{U,L} \approx \frac{(h_{3u}^{12} g_{13}^{12} - h_{1u}^{11} h_{3u}^{12}) v_2 e^{i(bu - eu)}}{(h_{1u}^{12} g_{12}^{11} - h_{1u}^{11} g_{12}^{12}) v_1} \]
(29)

The exotic species $T_1$ and $T_2$ got masses through $v_\chi$ at units of TeV. The SM $t$ quark has acquired mass with $v_1$ without any suppression, so its mass remains at the scale of $v_1$, hundreds of GeV. On the contrary, the $c$ quark have acquired mass with $v_1$ and $v_2$ but yielding the suppressed mass of the $c$ quark because of the existence of a SSM together with the $T_1$ exotic quark. Finally, the $u$ quark has acquired mass through $v_3$ with a similar SSM as for $c$ quark, but with the $t$ quark instead of $T_1$. Consequently, the mass of the $u$ quark gets suppressed by the top quark $t$.
B. Down-like quarks

The down-like quarks are described in the bases $\mathbf{D}$ and $\mathbf{d}$, where the former is the flavor basis while the latter is the mass basis

$$
\mathbf{D} = (d^1, d^2, d^3, J^1),
\mathbf{d} = (d, s, b, J^1).
$$

(30)

The mass term in the flavor basis is

$$
- \mathcal{L}_D = \mathbf{D}^T M_D \mathbf{D} + m_j \mathcal{J}_F \mathcal{J}^2 + \text{h.c.},
$$

(31)

where $M_D$ turns out to be

$$
M_D = \frac{1}{\sqrt{2}} \begin{pmatrix}
\chi^{11}_{\mu \nu} v_3 & 0 & 0 & h^{11}_{3, J} v_3 \\
0 & h^{22}_{3, d} v_3 & h^{33}_{2, d} v_3 & 0 \\
h^{11}_{3, J} v_3 & 0 & 0 & g^{11}_{\chi J} v_3 \\
h^{22}_{3, d} v_3 & 0 & 0 & g^{22}_{\chi J} v_3 \\
\end{pmatrix}
$$

(32)

Thus, the mass eigenvalues of the SM quarks are

$$
m^2_1 \approx \frac{(h^{11}_{3, J} g^{11}_{\mu \nu} - h^{11}_{3, d} g^{11}_{\mu \nu})^2 v_3^2}{(g^{11}_{\chi J})^2 + (g^{11}_{\chi J})^2},
$$

$$
m^2_2 \approx \frac{h^{22}_{3, d} h^{33}_{2, d} - h^{22}_{3, d} h^{33}_{2, d} v_3^2}{(g^{22}_{\chi J})^2 + (g^{33}_{\chi J})^2},
$$

$$
m^2_3 \approx \frac{(h^{22}_{3, d})^2 + (h^{33}_{2, d})^2 v_3^2}{2},
$$

(33)

and the masses of the exotic species are given by

$$
m^2_{11} \approx \frac{(g^{11}_{\chi J})^2 + (g^{11}_{\chi J})^2 v_3^2}{2},
$$

$$
m^2_{12} \approx \frac{(g^{22}_{\chi J})^2 v_3^2}{2}.
$$

(34)

The corresponding left-handed rotation matrix is

$$
\Psi^L_D = \Psi^D_{LS} \Psi^D_{LB},
$$

(35)

where the seesaw angle which rotates out species $J^{1,2}$ is

$$
\Theta^{D \dagger}_{L} = \begin{pmatrix}
h^{11}_{3, d} g^{11}_{\mu \nu} + h^{11}_{3, J} g^{11}_{\mu \nu} v_3 \\
(g^{11}_{\chi J})^2 + (g^{11}_{\chi J})^2 v_3 \\
0 \\
0 \\
\end{pmatrix},
$$

(36)

and the SM angles of $\Psi^D_{LB}$ are given by

$$
tan \theta^{D, L}_{13} \approx 0,
$$

$$
tan \theta^{D, L}_{23} \approx \frac{(h^{22}_{3, d} h^{33}_{2, d} + h^{23}_{3, d} h^{32}_{2, d}) v_3}{(h^{23}_{3, d})^2 + (h^{32}_{2, d})^2 v_2},
$$

$$
tan \theta^{D, L}_{12} \approx 0.
$$

(37)

The heaviest quarks $J^1$ and $J^2$ acquired mass at TeV scale due to $v_3$, while the $b$ quark obtained its mass through $v_2$ at units of GeV. The $s$ quark has acquired its mass through $v_3$ at hundreds of MeV with the corresponding SSM with the bottom quark $b$.

Similarly, the quark $d$ got its mass through the SSM with the exotic species $J^1$.

C. Neutral leptons

Neutrinos involve Dirac and Majorana masses in their Yukawa Lagrangian. Since $N_R^\nu$ are Majorana fermions, the bases are chiral and the mass basis describes Majorana neutrinos. The flavor and mass bases are, respectively,

$$
\mathbf{N}_L = (\nu^\tau_{L,R}, (\nu^\tau_{L,R})^C, (N^\tau_{L,R})^C),
\mathbf{n}_L = (\nu^1_{L,R}, (N^1_{L,R})^C, (\tilde{N}^1_{L,R})^C).
$$

(38)

The mass term expressed in the flavor basis is

$$
- \mathcal{L}_N = \frac{1}{2} \mathbf{N}^T_L M_N \mathbf{N}_L,
$$

(39)

where the mass matrix has the following block structure

$$
M_N = \begin{pmatrix}
0 & M^T_\nu & 0 \\
M_\nu & 0 & M^T_N \\
0 & M_N & M_N
\end{pmatrix},
$$

(40)

with $M_\nu$ as the Dirac mass matrix between $\nu_L$ and $\nu_R$

$$
M_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix}
h^{\mu \nu}_{2e} v_2 & h^{\mu \nu}_{2\tau} v_1 & h^{\mu \nu}_{2\tau} v_1 \\
0 & h^{\mu \nu}_{\tau \tau} v_1 & 0 \\
0 & 0 & h^{\mu \nu}_{\tau \tau} v_3
\end{pmatrix},
$$

(41)

$M_N$ the Dirac mass matrix between $\nu^C_R$ and $N_R$

$$
M_N = \frac{v_3}{\sqrt{2}} \begin{pmatrix}
g^{\mu \nu}_{11} v_1 & g^{\mu \nu}_{22} v_2 & g^{\mu \nu}_{33} v_3 \\
g^{\mu \nu}_{11} v_1 & g^{\mu \nu}_{22} v_2 & g^{\mu \nu}_{33} v_3 \\
g^{\mu \nu}_{11} v_1 & g^{\mu \nu}_{22} v_2 & g^{\mu \nu}_{33} v_3
\end{pmatrix},
$$

(42)

where $g^{\mu \nu}_N = \sqrt{2}M_N^\mu / v_3$, and $M_N = G_N M_N$ is the Majorana mass of $N_R$.

By employing the inverse SSM because of the VH in eq. (12), it is found that

$$
(\Psi^N_{LS})^\dagger M_N \Psi^N_{LS} = \begin{pmatrix}
m_\nu & 0 & 0 \\
0 & m_N & 0 \\
0 & 0 & m_N
\end{pmatrix},
$$

(43)

where the new $3 \times 3$ blocks are

$$
m_\nu = M_\nu (M^T_N)^{-1} M_N (M^T_N)^{-1} M_\nu,
$$

$$
M_N \approx M_N - M_N, \quad \tilde{M} \approx M_N + M_N.
$$

(44)

It was assumed $M_N$ diagonal and

$$
G_N = \begin{pmatrix}
G_{N1} & G_{N4} & 0 \\
G_{N4} & G_{N2} & 0 \\
0 & 0 & G_{N3}
\end{pmatrix},
$$

(45)

so as it can yield the adequate mixing angles to fit PMNS matrix. By rejecting terms proportional to $v_3$ in $m_\nu$, the neutrino $\nu^1_L$ turns out to be massless, the masses of the other two neutrinos are

$$
m_{12} \approx \frac{(h^{\mu \nu}_{2e})^2 G_{N2} \mu_N v_2^2}{(g^{\mu \nu}_{N1})^2 v_3^2},
$$

$$
m_{13} \approx \frac{(h^{\mu \nu}_{2e})^2 G_{N1} \mu_N v_2^2}{(g^{\mu \nu}_{N1})^2 v_3^2}.
$$

(46)
and the masses of the exotic species are

\[
m_{N_k}^L = \frac{g_{\chi N}^1 v_{\chi}}{\sqrt{2}} \pm \frac{G_{N1} \mu_N}{2},
\]

\[
m_{N_2}^L = \frac{g_{\chi N}^2 v_{\chi}}{\sqrt{2}} \pm \frac{G_{N2} \mu_N}{2},
\]

\[
m_{N_3}^L = \frac{g_{\chi N}^3 v_{\chi}}{\sqrt{2}} \pm \frac{G_{N3} \mu_N}{2}.
\]

The left-handed rotation matrix can be expressed by

\[
V_{N}^L = V_{L,SS}^N V_{L,B}^N,
\]

where the seesaw angle is

\[
\theta_{13}^{N,L} = \left( \begin{array}{ccc}
\frac{h_{\mu e}^L g_{\chi \mu}^N}{h_{\tau e}^L g_{\chi \tau}^N} & \frac{h_{\mu e}^L g_{\chi \mu}^N}{h_{\tau e}^L g_{\chi \tau}^N} & \frac{h_{\mu e}^L g_{\chi \mu}^N}{h_{\tau e}^L g_{\chi \tau}^N} \\
\frac{h_{\tau e}^L g_{\chi \tau}^N}{h_{\mu e}^L g_{\chi \tau}^N} & \frac{h_{\tau e}^L g_{\chi \tau}^N}{h_{\mu e}^L g_{\chi \tau}^N} & \frac{h_{\tau e}^L g_{\chi \tau}^N}{h_{\mu e}^L g_{\chi \tau}^N} \\
0 & 0 & 0 \\
\end{array} \right)
\]

and \( V_{L,SM}^E \), contained in the block-diagonal mixing matrix \( V_{L,B}^E \) after rotating out the heavy species has the angles

\[
\tan \theta_{13}^{N,L} \approx \frac{h_{\mu e}^L g_{\chi \mu}^N}{h_{\tau e}^L g_{\chi \tau}^N},
\]

\[
\tan \theta_{23}^{N,L} \approx \frac{h_{\tau e}^L g_{\chi \tau}^N}{(h_{\mu e}^L g_{\chi \tau}^N)^2G_{N1} - (h_{\mu e}^L g_{\chi \mu}^N)^2G_{N2}},
\]

\[
\tan \theta_{12}^{N,L} \approx \frac{h_{\mu e}^L g_{\chi \mu}^N}{h_{\tau e}^L g_{\chi \tau}^N},
\]

and

\[
M_E = \frac{1}{\sqrt{2}} \begin{pmatrix}
h_{\mu e}^e v_3 & 0 & h_{\mu e}^\tau v_3 & 0 & h_{\mu e}^\mu 3 v_3 & 0 \\
0 & h_{\mu e}^e v_3 & 0 & 0 & 0 & 0 \\
h_{\tau e}^e v_1 & 0 & h_{\tau e}^\tau v_1 & 0 & h_{\tau e}^\mu 3 v_1 & 0 \\
0 & 0 & 0 & g_{\chi e}^e v_1 & 0 & g_{\chi e}^\mu 3 v_1 \\
g_{\chi e}^e v_3 & 0 & g_{\chi e}^\tau v_3 & 0 & g_{\chi e}^\mu 3 v_3 & 0 \\
0 & 0 & 0 & g_{\chi e}^e v_3 & 0 & g_{\chi e}^\mu 3 v_3
\end{pmatrix}.
\]

The determinant of \( M_E \) is non-vanishing ensuring that all charged leptons acquire mass. Thus, the eigenvalues of the mass matrix yields the masses of the SM leptons

\[
m_e^2 \approx \frac{[(h_{\mu e}^e v_3^2 - h_{\mu e}^e g_{\chi e}^x v_3^2 - h_{\mu e}^\tau g_{\chi e}^x v_3^2 - h_{\mu e}^\mu g_{\chi e}^x v_3^2 - h_{\mu e}^\mu g_{\chi e}^x v_3^2)^2 + (h_{\mu e}^e h_{\tau e}^{\mu e} - h_{\mu e}^\tau g_{\chi e}^{\mu e} g_{\chi e}^{\tau e})^2 + (h_{\mu e}^e h_{\mu e}^{\tau e} - h_{\mu e}^\tau g_{\chi e}^{\tau e} g_{\chi e}^{\mu e})^2 + (h_{\mu e}^e h_{\mu e}^{\mu e} - h_{\mu e}^\mu g_{\chi e}^{\mu e} g_{\chi e}^{\mu e})^2]}{2},
\]

\[
m_\mu^2 \approx (h_{\mu e}^e)^2 \frac{v_3^2}{2},
\]

\[
m_\tau^2 \approx \frac{(h_{\mu e}^e h_{\mu e}^{\tau e} - h_{\mu e}^\tau g_{\chi e}^{\mu e} g_{\chi e}^{\tau e})^2 + (h_{\mu e}^e h_{\mu e}^{\tau e} - h_{\mu e}^\tau g_{\chi e}^{\mu e} g_{\chi e}^{\tau e})^2 + (h_{\mu e}^e h_{\mu e}^{\mu e} - h_{\mu e}^\mu g_{\chi e}^{\mu e} g_{\chi e}^{\mu e})^2}{(g_{\chi e}^{\mu e})^2 + (g_{\chi e}^{\tau e})^2 + (g_{\chi e}^{\mu e})^2},
\]

and the masses of the new exotic charged leptons

\[
m_{E1}^2 \approx \frac{[(g_{\chi e}^{11} v_3^2 + g_{\chi e}^{13} v_3^2) \frac{v_3^2}{2}}{2},
\]

\[
m_{E2}^2 \approx \frac{[(g_{\chi e}^{2 e} v_3^2 + (g_{\chi e}^{22} v_3^2) \frac{v_3^2}{2}}{2},
\]

\[
m_{E3}^2 \approx \frac{[(g_{\chi e}^{3 e} v_3^2 + (g_{\chi e}^{32} v_3^2) v_3^2}{2}.}
\]

D. Charged leptons

The charged leptons are described in the bases \( E \) and \( e \), where the former is the flavor basis while the latter is the mass basis

\[
E = (e^e, e^\mu, e^\tau, \epsilon^1, \epsilon^2, \epsilon^3),
\]

\[
e = (e, \mu, \tau, E^1, E^2, E^3).
\]
The left-handed rotation matrix can be expressed by

$$\mathcal{V}_L^E = \mathcal{V}_{L,SS}^E \mathcal{V}_{L,B}^E,$$

where the seesaw angle is

$$\Theta_{L}^{E+} = \begin{pmatrix} 0 & \frac{h_{e^c}^2 v_p + h_{e^c}^2 g_{e^c}^2 g_{e^c}^2 v_3 e^{i(\alpha_E - \epsilon_E)}}{(g_{e^c}^2)^2 + (g_{e^c}^2)^2 + (g_{e^c}^2)^2 v_p} & 0 \\ 0 & \frac{h_{e^c}^2 g_{e^c}^2 v_p + h_{e^c}^2 g_{e^c}^2 g_{e^c}^2 v_3 e^{i(\alpha_E - \epsilon_E)}}{(g_{e^c}^2)^2 + (g_{e^c}^2)^2 + (g_{e^c}^2)^2 v_p} & 0 \\ 0 & \frac{h_{e^c}^2 g_{e^c}^2 v_p + h_{e^c}^2 g_{e^c}^2 g_{e^c}^2 v_3 e^{i(\alpha_E - \epsilon_E)}}{(g_{e^c}^2)^2 + (g_{e^c}^2)^2 + (g_{e^c}^2)^2 v_p} & 0 \end{pmatrix},$$

and $\mathcal{V}_{L,SM}^E$, contained in $\mathcal{V}_{L,B}^E$ has the mixing angles

$$\tan \theta_{13}^{E,L} \approx \frac{(h_{e^c}^2 h_{e^c}^2 + h_{e^c}^2 h_{e^c}^2 + h_{e^c}^2 h_{e^c}^2) v_3}{(h_{e^c}^2 g_{e^c}^2 + h_{e^c}^2 g_{e^c}^2 + h_{e^c}^2 g_{e^c}^2) v_1} e^{i(\alpha_E - \epsilon_E)}$$
$$\tan \theta_{23}^{E,L} = 0$$
$$\tan \theta_{12}^{E,L} = 0$$

which yields the suppression of the $c$ quark mass

$$m_c^2 \approx \left( \frac{h_{1u}^2 h_{1u}^2 - h_{1u}^2 h_{1u}^2}{(g_{1u}^2)^2 + (g_{1u}^2)^2} \right)^2 v_1^2,$$

from hundreds to units of GeV since it acquires mass via $v_1$, in accordance with experimental observations. Similarly, the $u$ quark mass is suppressed by the SSM involving $u$ and $t$ in the following sub-block of the up-like quarks mass matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} (h_{1u}^{11} v_3 & h_{1u}^{13} v_3) \\ h_{1u}^{31} v_1 & h_{1u}^{31} v_1 \end{pmatrix},$$

obtaining the mass of the $u$ quark through $v_3$ with the subtraction of Yukawa coupling constants

$$m_u^2 \approx \left( \frac{h_{1u}^{11} h_{1u}^{11} - h_{1u}^{13} h_{1u}^{13}}{(h_{1u}^{11})^2 + (h_{1u}^{13})^2} \right)^2 v_3^2,$$

lowering the mass from hundreds of units of MeV. Moreover, the top quark obtains its mass at the scale of hundreds of GeV directly via $v_1$ with no suppression mechanism

$$m_t^2 \approx \left( \frac{h_{1u}^{11} h_{1u}^{11} - h_{1u}^{13} h_{1u}^{13}}{(h_{1u}^{11})^2 + (h_{1u}^{13})^2} \right)^2 v_1^2.$$

The $d$ quark obtains its mass through $v_3$ and an SSM with the exotic species $J^T$,

$$\begin{pmatrix} \frac{1}{\sqrt{2}} (h_{1u}^{11} v_3 & h_{1u}^{11} v_3) \\ g_{1u}^{11} v_1 & g_{1u}^{11} v_1 \end{pmatrix},$$

which gives the suppressed mass of the $d$ quark

$$m_d^2 \approx \left( \frac{h_{1u}^{11} h_{1u}^{11} - h_{1u}^{13} h_{1u}^{13}}{(h_{1u}^{11})^2 + (h_{1u}^{13})^2} \right)^2 v_3^2.$$
TABLE III. Summary of fermion masses.

| Family   | SM Up-like Quarks | F ernion | SM Down-like Quarks |
|----------|-------------------|----------|---------------------|
| 1        | u                 | \( \frac{h_u - h_u^0 v_3}{\sqrt{2}} \) | d                    | \( \frac{h_d - h_d^0 v_3}{\sqrt{2}} \) |
|          | c                 | \( \frac{h_t - h_t^0 v_3}{\sqrt{2}} \) | s                    | \( \frac{h_b - h_b^0 v_3}{\sqrt{2}} \) |
|          | t                 | \( \frac{h_t^0 v_3}{\sqrt{2}} \)       | b                    | \( \frac{h_b^0 v_3}{\sqrt{2}} \) |

SM Neutral Leptons | SM Charged Leptons

| Exotic Up-like Quarks | Exotic Down-like Quarks |
|----------------------|-------------------------|
| 1                    | \( \frac{h_f}{\sqrt{2}} \) | \( J^1 \) |
| 2                    | \( \frac{h_f v_3}{\sqrt{2}} \) | \( \frac{h_f v_3}{\sqrt{2}} \) |

Exotic Neutral Leptons | Exotic Charged Leptons

| 1 | \( \frac{h_N}{\sqrt{2}} \) | \( E^1 \) | \( \frac{h_N}{\sqrt{2}} \) |
| 2 | \( \frac{h_N}{\sqrt{2}} \) | \( E^2 \) | \( \frac{h_N}{\sqrt{2}} \) |
| 3 | \( \frac{h_N}{\sqrt{2}} \) | \( E^3 \) | \( \frac{h_N}{\sqrt{2}} \) |

so as the d quark does not acquire mass at hundreds, but at units of MeV in accordance with phenomenological data. The model suppresses the mass of the s quark with the b quark because of the SSM

\[
\frac{1}{\sqrt{2}} \left( \frac{h_u^0 v_3}{\sqrt{2}} \frac{h_b^0 v_3}{\sqrt{2}} \right)
\]

yielding the mass eigenvalue of a light s quark

\[
m_s^2 = \frac{(h_u^{23} h_d^{32} - h_u^{32} h_d^{23})^2 v_3^2}{2}.
\]

Within this SSM, the bottom quark acquires its mass directly through \( v_2 \)

\[
m_b^2 \approx \frac{(h_d^{32})^2 + (h_d^{32})^2}{2} v_2^2.
\]

For the neutral sector, light active neutrinos and two-folded sterile heavy neutrinos at TeV scale are obtained with the employment of ISS. Moreover, the model selects the normal ordering, with the lightest active neutrino \( \nu_L \) turning out massless when the smallest VEV \( v_3 \) is neglected.

The model predicts the non-suppressed mass of the \( \mu \) given by \( m_\mu = \frac{h_u^{\mu 0} v_3}{\sqrt{2}} \), which turns out at hundreds of MeV because of \( v_3 \). Furthermore, the charged lepton sector shows the largest SSM in the article involving e, \( \tau \) and the exotic species \( E^2 \),

\[
\frac{1}{\sqrt{2}} \left( \frac{h_e^{23} v_3}{h_e^{32} v_3} \frac{h_\tau^{23} v_3}{h_\tau^{32} v_3} \right)
\]

which gives the simultaneous suppression of the e and \( \tau \) masses, from hundreds to units of MeV and GeV, respectively. By setting \( g_{e\tau}^X \) null to simplify algebraic expressions, the masses of the e and \( \tau \) turn out to be

\[
m_e^2 \approx \frac{[h_e^{23} h_\tau^{23} - h_e^{32} h_\tau^{32}]^2 v_3^2}{2} g_{e\tau}^X v_3 g_{e\tau}^X v_3 g_{e\tau}^X v_3
\]

It is remarkable how the exotic \( E^2 \) suppresses the mass from hundreds to units of GeV, and in turn it suppresses the mass of the e from hundreds to units of MeV, as it is shown in the above expressions.

A summary of the fermion masses obtained from the model is shown in Tab. III. The nonuniversal abelian extensions can be considered one of the simplest schemes beyond SM because it only comprises one abelian gauge group \( U(1)_X \). However, they give rich frameworks where fundamental issues such as fermion mass hierarchy can be addressed with the suit particle content and couplings.
Moreover, this article shows how previous schemes can be improved to avoid radiative corrections or fine-tunings and obtain in a natural way light and heavy fermions in accordance with experimental data.

Regarding the global symmetries which appear from the assignation of $U(1)_{X}$ quantum numbers in Table III the discrete $\mathbb{Z}_2$ breaks them in the following scheme,

$$SU(2)_{d_{L}^{u,3}} \otimes SU(3)_{u_{L}^{+,1,2,3}} \otimes SU(2)_{d_{R}^{e,3}} \otimes SU(2)_{e_{L}^{+,3}} \otimes SU(2)_{d_{R}^{\pm,3}}$$

which prevents the existence of massless fermions after the SSB.

On the other hand, it is important to reinforce that the set of chiral anomaly-free $U(1)_{X}$ quantum numbers constrains the set of $U(1)_{X}$ quantum numbers in the scalar sector in order to understand the masses and mixing angles observed in the fermionic spectrum of the SM.

**ACKNOWLEDGMENT**

This work was supported by El Patrimonio Autonomo Fondo Nacional de Financiamiento para la Ciencia, la Tecnología y la Innovación Francisco José de Caldas programme of COLCIENCIAS in Colombia.

**Appendix A: General scheme for diagonalization of mass matrices**

The fermions of each sector are described in two bases: The flavor basis $\mathbf{F}$ and the mass basis $\mathbf{f}$. The mass matrix $M_{F}$ written in the basis $\mathbf{f}$ is found in the diagonal form, having as diagonal elements the masses of the fermions. By the implementation of spontaneous rupture of symmetry, the Yukawa Lagrangian $L_{F}$ can be expressed as:

$$- L_{F} = \mathbf{F} M_{F} \mathbf{f} R + h.c.$$  \hspace{1cm} (A1)

The diagonalization of the mass matrix is made by a unitary transformation, which reads

$$(V_{F}^{T})^\dagger M_{F} V_{F}^{T}.$$  \hspace{1cm} (A2)

The unitary matrices $V_{F}^{T}$ and $V_{F}^{R}$ relate the mass and the flavor bases for both chiralities

$$\mathbf{F}_{L} = V_{F}^{L} \mathbf{f}_{L} \quad \quad \mathbf{F}_{R} = V_{F}^{R} \mathbf{f}_{R}.$$  \hspace{1cm} (A3)

However, $M_{F}$ is not a symmetric matrix and two unitary matrices have to be found. An alternative way is to diagonalize the symmetric constructions from $M_{F}$, which are:

$$M_{F} \cdot M_{F}^{\dagger} \rightarrow (V_{F}^{T})^\dagger M_{F} \cdot M_{F}^{\dagger} V_{F}^{T}, \quad \quad (V_{F}^{R})^\dagger M_{F} \cdot M_{F}^{\dagger} V_{F}^{R},$$  \hspace{1cm} (A4)

where the diagonal form is found only by the mixing of left-handed fermions, in the case of $M_{F} \cdot M_{F}^{\dagger}$, and by the mixing of the right-handed fermions, for the matrix $M_{F}^{\dagger} \cdot M_{F}$ (for neutrinos the mass matrix $M_{N}$ is already symmetric.).

Both quadratic mass matrices lead to the same eigenvalues, that correspond with the square of the masses. Since the main goal is to obtain the values of the fermion masses, only the matrix $M_{F}^{\dagger} \cdot M_{F}$ was diagonalized, giving also the Yukawa mixings of the left-handed fermions. The first step is to notice that the matrix $M_{F}^{\dagger} \cdot M_{F}$ can be written in the following block form:

$$M_{F}^{\dagger} \cdot M_{F} = \begin{pmatrix} M_{F}^{\dagger} \cdot M_{F} & 0 \\ 0 & M_{F}^{\dagger} \cdot M_{F} \end{pmatrix},$$  \hspace{1cm} (A6)

where $n$ is the number of exotic fermions and $(M_{F}^{\dagger} \cdot M_{F})^{T} = M_{F}^{\dagger} \cdot M_{F}$. Thus, a seesaw rotation was applied to separate the SM from the exotic sector.

Then, a diagonalization for each sector was performed, summarized in the following matrix:

$$V_{F} = \begin{pmatrix} V_{F}^{SM} & 0_{3 \times n} \\ 0_{n \times 3} & V_{F}^{Exot} \end{pmatrix},$$  \hspace{1cm} (A9)

where $V_{F}^{SM}$ is expressed by

$$V_{SM}^{F} = R_{13}(\theta_{F}^{\ell}, \delta_{F}^{\ell}) R_{23}(\theta_{F}^{r}, \delta_{F}^{r}) R_{12}(\theta_{F}^{\ell}, \delta_{F}^{\ell}).$$  \hspace{1cm} (A10)

The matrices $R_{ij}$ are complex rotations, that read

$$R_{12}(\theta_{F}^{\ell}, \delta_{F}^{\ell}) = \begin{pmatrix} c_{F}^{12} & s_{F}^{12} \\ -s_{F}^{12} & c_{F}^{12} \end{pmatrix},$$  \hspace{1cm} (A11)

$$R_{13}(\theta_{F}^{\ell}, \delta_{F}^{\ell}) = \begin{pmatrix} c_{F}^{13} & 0 & s_{F}^{13} \\ 0 & 1 & 0 \\ -s_{F}^{13} & 0 & c_{F}^{13} \end{pmatrix},$$  \hspace{1cm} (A11)

$$R_{23}(\theta_{F}^{r}, \delta_{F}^{r}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{F}^{23} & s_{F}^{23} \\ 0 & -s_{F}^{23} & c_{F}^{23} \end{pmatrix},$$  \hspace{1cm} (A11)

where $c_{ij}^{F} = \cos \theta_{ij}^{F}$ and $s_{ij}^{F} = \sin \theta_{ij}^{F} \exp(i \delta_{ij}^{F})$. The angles $\theta_{ij}^{F}$ are determined, in the calculus, from their tangents in an approximate way. Thus, the unitary transformation that leaves the symmetric mass matrix for the left-handed fermions in a diagonal form is

$$V_{L}^{F} = V_{L,SS}^{F} V_{L,B}^{F},$$  \hspace{1cm} (A12)
