The CMB neutrino mass / vacuum energy degeneracy: a simple derivation of the degeneracy slopes.

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Submitted to MNRAS, 06 Sep 2017; revised 20 Feb 2018

ABSTRACT

It is well known that estimating cosmological parameters from cosmic microwave background (CMB) data alone results in a significant degeneracy between the total neutrino mass and several other cosmological parameters, especially the Hubble constant $H_0$ and the matter density parameter $\Omega_m$. Adding low-redshift measurements such as baryon acoustic oscillations (BAOs) breaks this degeneracy and greatly improves the constraints on neutrino mass. The sensitivity is surprisingly high, e.g. adding the $\sim 1$ percent measurement of the BAO ratio $r_s/D_V$ from the BOSS survey leads to a limit $\Sigma m_\nu < 0.19\ eV$, equivalent to $\Omega_\nu < 0.0045$ at 95% confidence. For the case of $\Sigma m_\nu < 0.6\ eV$, the CMB degeneracy with neutrino mass almost follows a track of constant sound horizon angle (Howlett et al 2012). For a $\Lambda$CDM + $m_\nu$ model, we use simple but quite accurate analytic approximations to derive the slope of this track, giving dimensionless multipliers between the neutrino to matter ratio ($x_\nu \equiv \omega_\nu/\omega_m$) and the shifts in other cosmological parameters. The resulting multipliers are substantially larger than 1: conserving the CMB sound horizon angle requires parameter shifts $\delta \ln H_0 \approx -2 \delta x_\nu$, $\delta \ln \Omega_m \approx +5 \delta x_\nu$, $\delta \ln \omega_\Lambda \approx -6.2 \delta x_\nu$, and most notably $\delta \Omega_\nu \approx -14 \delta x_\nu$. These multipliers give an intuitive derivation of the degeneracy direction, which agrees well with the numerical likelihood results from Planck team.

Key words: cosmic background radiation – cosmological parameters – dark energy – cosmology:miscellaneous

1 INTRODUCTION

There is a long history of cosmological constraints on neutrino masses; from the 1970s, the simple requirement that the cosmic neutrino background should not over-close the Universe required $\Sigma m_\nu \lesssim 50\ eV$ (Cowick & McClelland 1972). This limit steadily improved with new data and simulations of large-scale structure during the 1990s, with a notable improvement to $1.8\ eV$ from the galaxy power spectrum in the 2dFGRS survey (Elgaroy et al 2002), and the limit continued to improve through WMAP in 2003–2012 (Hinshaw et al 2013).

Since the discovery of atmospheric neutrino oscillations by Super-Kamiokande (Fukuda et al 1998) showed that neutrinos have non-zero mass, and the decisive solution of the solar neutrino problem by the Solar Neutrino Observatory (Ahmad et al 2002), many oscillation experiments with solar, nuclear reactor and accelerator neutrinos have given precise measurements of neutrino mass-squared differences (Olive et al 2014); these imply $\Sigma m_\nu \gtrsim 0.060\ eV$, but do not set an absolute mass scale. Current laboratory measurements give a model-independent upper limit $m_\nu < 2\ eV$ for the electron neutrino (Olive et al 2014), while upper limits from cosmological observations are now much stronger than this (though with some model-dependence).

Many previous works have studied the effects of neutrino mass on CMB anisotropy, see e.g. Ma & Bertschinger (1995); Jungman et al (1996); Kaplinghat, Knox & Song (2003); Bashinsky & Seljak (2004); Lesgourgues et al (2006); Hamann et al (2011); Howlett et al (2012); Hou et al (2013); Riemer-Sorensen, Parkinson & Davis (2014); see also the reviews by Lesgourgues & Pastor (2006) and Wong (2011) and the modern textbook by Lesgourgues et al (2013), and references therein.

Currently, constraints from the Planck CMB data alone (Ade et al 2016) provide an upper limit $\Sigma m_\nu < 0.6\ eV$ at 95% confidence; this essentially requires that neutrinos

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1 More formally, the 95% c.l. lower limits on neutrino mass-squared differences imply a lower limit $\Sigma m_\nu \gtrsim 0.057\ eV$, while the best-fit mass-squared differences imply $\Sigma m_\nu \gtrsim 0.060\ eV$; these are so close that 0.06 eV is generally adopted as the fiducial minimum.
remained quasi-relativistic until after the epoch of CMB last scattering at \(z_s \approx 1090\), and in this case the primary CMB anisotropies cannot improve much on this upper limit. Secondary anisotropies, notably the gravitational lensing of the CMB, may substantially improve the bound in the future (Kaplinghat, Knox & Song 2003; Allison et al 2015; Archidiacono et al 2017; Challinor et al 2017); however in the Planck case adding lensing information only slightly changes the upper limit. (In more detail, adding lensing information does narrow the posterior, but also moves the likelihood peak from zero to positive neutrino mass; the result is that the Planck-only upper limit changes only slightly with the addition of lensing data). For future CMB experiments, the best sensitivity to neutrino mass is anticipated from small-angle (\(t > 1000\)) polarization data, in a regime where the Planck data is noise-limited (Adel et al 2016).

However, the CMB power spectrum alone gives a known degeneracy between \(\Sigma_m\) and low-redshift parameters such as \(H_0, \Omega_m\), which can be broken by addition of low-redshift data such as baryon acoustic oscillations (BAOs) sensitive to these parameters. Combining Planck with several BAO datasets including BOSS (Alam et al 2017), 6dFGS (Beutler et al 2011) and WiggleZ (Blake et al 2011) gives an upper limit \(\Sigma_m < 0.17\) eV at 95\% confidence for a \(\Lambda CD\) + \(m_\nu\) model (Eq. 54d of Ade et al 2016), or 0.23 eV if polarization is not included (Eq. 57 of ibid). The mid-point of these, 0.20 eV, is equivalent to a present-day neutrino/matter ratio \(\lesssim 1.5\) percent or \(\Omega_\nu < 0.005\) at 95\% c.l., an impressively small limit given the \(\approx 1\) percent precision of the most precise BAO measurement. Even stronger limits \(\Sigma_m < 0.13\) eV have been derived using Ly-\(\alpha\) forest data (Palanque-Delabrouille et al 2015; Yeche et al 2017), but these are slightly more model-dependent.

In this paper we give a simplified but fairly accurate semi-analytic derivation of the slope of this degeneracy track: in Section 2 we note an interesting but not well-known feature that the CMB sound horizon angle is approximately \(2.5\times\) more sensitive to small changes in neutrino density compared to CDM+baryon density; we then estimate various dimensionless multipliers relating parameter variations along the CMB-only degeneracy track. In Section 3 we compare these analytic approximations with numerical results, including the public Planck likelihood results. In Section 4 we consider effects on the matter power spectrum, and note that the secondary effects from varying \(H_0\) turn out of similar size to the primary effects of neutrino mass. In Section 5 we briefly discuss extended models, and we conclude in Section 6.

2 THE LEVER-ARM BETWEEN NEUTRINO MASS AND LOW-REDSHIFT PARAMETERS

In this section we give a simple derivation of the lever-arm between the present-day neutrino/matter density ratio to low-redshift parameters such as \(\Omega_m, \Omega_\Lambda, H_0, \omega_\Lambda\), defined below.

2.1 Notation

Our default model is flat \(\Lambda CD\) extended with arbitrary neutrino mass, unless specified otherwise. We use the standard notation that \(h \equiv H_0/(100\) km s\(^{-1}\) Mpc\(^{-1}\)), and \(\Omega_i\) is the present-day density of species \(i\) in units of the critical density, where \(i = c, b, \nu\) respectively for CDM, baryons, neutrinos and the cosmological constant. The physical densities \(\omega_i\) are defined by \(\omega_i \equiv \Omega_i h^2\). We assume zero curvature \(\Omega_k = 0\), dark energy equation of state \(w = -1\), and effective number of neutrino species \(N_{\text{eff}} = 3.046\), except in Section 5 where we briefly explore deviations from these.

We use \(\Omega_{cb} \equiv \Omega_c + \Omega_b\) to denote the dark + baryonic matter density (excluding neutrinos), \(\Omega_m \equiv \Omega_{cb} + \Omega_\nu\) includes neutrinos, \(D_s \equiv (1 + z_s)D_4(z_s) \approx 13.9\) Gpc is the comoving angular diameter distance to photon decoupling at redshift \(z_s \approx 1090\), \(\theta_s \equiv r_s(z_s)/D_s\) is the CMB sound horizon angle, and \(z_{eq} \approx 3375\) is the redshift of matter-radiation equality.

It is helpful below to work mostly with dimensionless parameters, so we define the present-day neutrino / other matter ratio as

\[
x_\nu \equiv \omega_\nu/\omega_{cb};
\]

note that a more common parameter choice is \(f_\nu \equiv \omega_{b\nu}/(\omega_\nu + \omega_b) = x_\nu/(1 + x_\nu)\) where \(f_\nu\) includes neutrinos in the denominator; these are clearly very similar for \(x_\nu, f_\nu \ll 1\), but it is convenient later to choose a parameter which is strictly linear in \(\Sigma_m\) for fixed \(\omega_{cb}\). For the concordance value \(\omega_{cb} \approx 0.141\), this gives \(x_\nu = \Sigma_m/0.13\) eV, and a default value (for \(\Sigma_m = 0.06\) eV) of \(x_\nu \approx 4.6 \times 10^{-3}\). Since we are mostly interested in "differences in observables relative to the 6-parameter model with neutrino masses fixed to the default, we also define \(\delta x_\nu \equiv x_\nu - 0.0046\) to be the shift in \(x_\nu\) above this minimal value.

2.2 Neutrino effects on the sound horizon length

If the total neutrino mass is \(\Sigma m_\nu \lesssim 0.6\) eV (a conservative limit from Planck data alone), then the oscillation experiments require all three single neutrino masses \(\lesssim 0.22\) eV. At high redshift the neutrino temperature is \(T_\nu \approx (4/11)^{1/3} T_\gamma\) where \(T_\gamma\) is the photon temperature, hence at photon decoupling we have \(T_\nu(z_s) = 2122\) K and \(kT_\nu(z_s) = 0.183\) eV. From the accurate fitting functions in Sect. 3.3 of Komatsu et al (2011), each single neutrino with \(m_\nu = 0.183\) eV would contribute 6.5 percent higher energy density at decoupling than one negligible-mass neutrino, which is a quite substantial shift. However, the effects of neutrino mass on the sound horizon length are suppressed by several factors as follows: since minimal-mass neutrinos contribute 10.0 percent of the total matter+radiation density at \(z_s \approx 1090\), changing to \(\Sigma m_\nu = 0.55\) eV (\(\delta x_\nu = 0.037\), i.e. three neutrinos with masses close to 0.183 eV each) gives only a 0.65 percent increase in total energy density at \(z_s\), thus 0.32 percent increase in expansion rate \(H(z_s)\). Finally, the sound horizon length \(r_{s4}(z_s)\) contains an integral over \(\infty > z > z_s\), and the fractional shift in \(H(z)\) decreases towards higher redshift, so the change in sound horizon length is smaller again at \(-0.15\) percent. Also for \(\Sigma m_\nu < 0.55\) eV the fractional effect falls faster than linearly, becoming almost negligible at \(\Sigma m_\nu \lesssim 0.3\) eV. (See also Section 3 for a numerical verification of the above).

However, neutrino mass does have important effects on \(D_s\) and hence \(\theta_s\) as we see below.
2.3 Do massive neutrinos affect matter-radiation equality?

The short answer is “very little”, in the case of fitting the Planck data. Traditionally, many early works studied the consequences of varying $\Sigma m_\nu$ at fixed $\Omega_m h$, in which case the CDM density is implicitly reduced 1:1 as neutrino density increases; however this affects the CMB by altering the epoch of matter-radiation equality, $z_{eq}$ and also shifts $\theta_*$ as we see below; so the observable degeneracy track is substantially different to fixing $\Omega_m h$.

For neutrino masses below $\Sigma m_\nu < 0.6$ eV or single neutrino masses $< 0.2$ eV, the shift in neutrino energy density (relative to minimal-mass neutrinos) around $z_{eq} \sim 3375$ is no more than 1 percent, and the shift in radiation (photon + neutrino) density is $\approx 0.41 x$ this hence $\leq 0.41$ percent, which is substantially smaller than the Planck precision on $z_{eq}$. Thus the “direct” effect of neutrino mass around $z \sim 3000$ has very little impact on $z_{eq}$, and any change in $z_{eq}$ is driven mainly by any consequential shift in $\omega_{ch}$, which turns out to be small in the Planck case (see § 3).

If we adopt the common choice of $\omega_{ch}$ and $f_\nu$ among the base cosmological parameters, clearly $\omega_{ch} \equiv \omega_{m}(1 - f_\nu)$, so raising $f_\nu$ at constant $\omega_{m}$ trades CDM for neutrino density today in equal ratio; in that case increasing $f_\nu$ clearly does reduce $z_{eq}$ nearly in proportion. This has the apparent benefit of keeping $D_*$ (and the age of the universe, $t_0$) almost constant at $f_\nu$ varies, but this benefit is largely illusory, since the change in $\omega_{ch}$ also changes the sound horizon length and hence $\theta_*$ (see below); and it is $\theta_*$ which is constrained most precisely by Planck data, rather than $D_*$ or $t_0$.

Thus we argue that $\omega_{ch}$ and $f_\nu$ are not an optimal choice for basic parameters, and a more natural choice is to use $\omega_{ch}$ and $x_\nu$; so $\omega_{m} \equiv \omega_{ch}(1 + x_\nu)$ becomes a derived parameter. This is preferable since varying $x_\nu$ up to $\sim 0.04$ at constant $\omega_{ch}$ has a nearly negligible effect on $z_{eq}$.

In any of these parameter choices the sound horizon angle $\theta_*$ does vary with $f_\nu$ or $x_\nu$: it turns out that this can only be compensated by a change in vacuum energy density and hence $h$, for reasons given below.

2.4 Sensitivity of $\theta_*$ to neutrino and CDM density

Simple intuition suggests that increasing neutrino mass should be compensated by a reduction in CDM density to conserve consistency with CMB data. This intuition turns out to be incorrect, for the following reasons.

The observed sound horizon angle $\theta_* \equiv r_S(z_*)/D_*$ (where $r_S(z_*)$ is the comoving sound horizon length at last scattering) is the most precise cosmological observable (apart from the absolute temperature $T_0$): $\theta_*$ is constrained to 0.06 percent precision by Planck (Ade et al. 2016), and the corresponding length $r_S(z_*)$ is also well constrained at 0.25 percent precision, since the latter follows from the measurements of $\omega_c$ and $\omega_b$ from the acoustic peaks. (Given the high precision on $\omega_m$ from Planck, variations in $\omega_b$ have very little effect on $r_S(z_*)$, so in practice it is the combined value $\omega_{ch}$ which is relevant below). Thus, if we vary $x_\nu$, to remain consistent with the Planck data it is necessary to vary other parameter(s) to preserve a near-constant angle $\theta_*$. This degeneracy is studied in detail numerically by Howlett et al (2012), and is found to be well represented by constant $\omega_{ch}$ and $D_*$ as above. As seen above, for the interesting range $0.06 < \Sigma m_\nu < 0.6$ eV ($x_\nu \lesssim 0.046$), varying the neutrino mass has nearly negligible effect on the sound horizon length and the heights of acoustic peaks; but it does have a significant effect on the distance $D_*$ to last scattering, since the heaviest neutrino(s) must have mass $> 0.05$ eV and became non-relativistic at $z \gtrsim 250$, thus increasing the expansion rate during most of the post-recombination era.

To get a semi-analytic estimate of this degeneracy track, a good approximation to the present-day horizon size in flat-L models was given by Vittorio & Silk (1985) as

$$r_H \approx \frac{2 c}{H_0 \Omega_m^{0.4}}$$  \hspace{1cm} (2)

The value of $D_*$ is about 1.8 percent smaller than the above due to the finite redshift of last scattering, which leads to

$$D_* \approx \frac{5888 \text{ Mpc}}{h \Omega_m^{0.4}}$$  \hspace{1cm} (3)

this is accurate to < 0.1 percent for the range of $\Lambda CDM + m_\nu$ models allowed by Planck. This small error is fairly unimportant in the following, since it is smaller than the $\sim 0.25$ percent observational uncertainty in $r_S(z_*)$ and hence $D_*$. It is convenient to rewrite this as

$$D_* \approx \frac{5888 \text{ Mpc}}{h \Omega_m^{0.4}} \left(1 + x_\nu\right)^{0.4} \left(\omega_{ch} + \omega_\nu + \omega_\Lambda\right)^{0.07} ;$$  \hspace{1cm} (4)

Thus, if $x_\nu$ increases from its minimal value $x_\nu \approx 4.6 \times 10^{-3}$, we must adjust other parameter(s) to restore $\theta_*$ to the precisely-measured Planck value. At first sight it appears we could reduce $\omega_{ch}$ to compensate, but we now illustrate qualitatively that this does not lead to an acceptable solution. Concerning variations in $\omega_{ch}$, although the distance $D_*$ does scale as $\omega_{ch}^{-0.4}$ (for fixed $h$), varying $\omega_{ch}$ also produces a shift in the sound horizon length as $r_S(z_*) \propto \omega_{ch}^{-0.25}$ which partly compensates, so the net sensitivity of $\theta_*$ to $\omega_{ch}$ becomes

$$\left(\frac{\partial \ln \theta_*}{\partial \ln \omega_{ch}}\right)_{x_\nu,h} \simeq +0.15 ;$$  \hspace{1cm} (5)

where the subscripted parameters are held fixed.

However, varying neutrino mass has (almost) negligible compensation from $r_S(z_*)$; small neutrino masses ($\Sigma m_\nu < 0.6$ eV) affect $D_* \propto (1 + x_\nu)^{-0.4}$ but have nearly negligible effect on sound horizon length, hence

$$\left(\frac{\partial \ln \theta_*}{\partial x_\nu}\right)_{\omega_{ch},h} \simeq +0.4 .$$  \hspace{1cm} (6)

(Numerical differentiation with CAMB actually gives $+0.34$ rather than 0.40, see Section 3 below for more details). Note that if we fix $\omega_m$ and vary $f_\nu$, then we get the difference of these, i.e.

$$\left(\frac{\partial \ln \theta_*}{\partial f_\nu}\right)_{\omega_m,h} \simeq +0.25 .$$  \hspace{1cm} (7)

From Eq. 4 we also have the sensitivity to $h$ as

$$\left(\frac{\partial \ln \theta_*}{\partial \ln h}\right)_{\omega_{ch},x_\nu} \simeq +0.2 .$$  \hspace{1cm} (8)

Although all of the $\theta_*$ sensitivity coefficients above are

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fairly small compared with 1, the *Planck* estimate of $\theta_c$ is much more precise than any other parameter, so it is the relative sizes of these coefficients which mainly determine the direction of the CMB degeneracy track. A notable point above, comparing (6) and (7), is that $\theta_c$ is more than twice as sensitive to a small change in neutrino density than an equal shift in CDM+baryon density; the effects on $D_*$ are similar, but the CDM effect on $\theta_c$ is substantially compensated by variation in sound horizon length, while the effect from $x_\nu$ is almost uncompensated.

This turns out to be a major reason (see below) why increasing neutrino mass cannot in practice be compensated by reducing dark matter density $\omega_m$, but instead requires a (considerably larger) reduction in dark energy density.

If we take an example case of $\delta x_\nu = +0.01$ (i.e. increasing $\Sigma m_\nu$ from 0.06 eV to 0.191 eV, thus near the current CMB+BAO upper limit) we can consider three illustrative cases for varying $\omega_{\text{cdm}}$:

(i) If $\omega_{\text{cdm}}$ and $h$ were both held fixed then the above shows that $\theta_c$ would increase by $\approx 0.4$ percent, which is over 6× outside the *Planck* precision.

(ii) We may compensate the change in $x_\nu$ with an equal (1 percent) reduction in physical matter density $\omega_{\text{cdm}}$, thereby keeping constant $\omega_m$, $h$ and almost constant $D_*$; such a shift in $\omega_{\text{cdm}}$ and $\omega_m$ would be tolerable at around the 1 $\sigma$ *Planck* precision. However, due to the differing sensitivities above, this would give a $\sim +0.25$ percent increase in $\theta_c$, which is still $\sim 4 \times$ larger than the *Planck* precision and therefore ruled out.

(iii) Finally, we could reduce $\omega_{\text{cdm}}$ by a larger percentage in order to conserve $\theta_c$ at its fiducial value. From above, this would require $\sim 2.5$ percent reduction in $\omega_{\text{cdm}}$ and $\omega_m$; however, a shift this large in $\omega_m$ would lead to substantial tension with the acoustic peak heights.

The above example shows that if $\delta x_\nu \gtrsim +0.01$ (and $\omega_\Lambda$ or $h$ were held fixed), an arbitrary adjustment to $\omega_{\text{cdm}}$ could conserve either $\theta_c$ or $\omega_m$ within the *Planck* bounds, but not both simultaneously. Thus, the only way to preserve consistency with the *Planck* data is to conserve $\omega_\Lambda$ and $\omega_m$, but to reduce the physical dark energy density $\omega_\Lambda$ (thereby reducing $h$) to preserve the concordance value of $D_*$; we estimate the resulting degeneracy direction in the next subsection.

### 2.5 The lever-arm from neutrino mass to dark energy

If $\omega_{\text{cdm}}$ is fixed by the CMB acoustic peak heights and a value for $x_\nu$ is assumed, then (for a flat-$\Lambda$ model) specifying any one of $\omega_\Lambda$, $h$, $\Omega_{\text{cdm}}$, $\Omega_m$ determines the other three; so any of those four may be adopted as the independent “low-redshift” variable, and the degeneracy track is approximately a line through a five-dimensional $x_\nu$, $h$, $\omega_\Lambda$, $\Omega_{\text{cdm}}$, $\Omega_m$ space. We now estimate the direction of this track.

It is easily seen from Eqs. 4 and 5 that a small increase $\delta x_\nu \equiv x_\nu - 0.0046$ above the minimal value requires relative changes

$$\delta \ln (\omega_{\text{cdm}} + \omega_\nu + \omega_\Lambda) \simeq -4 \delta x_\nu \quad \text{and} \quad (10)$$

$$\delta \ln h \simeq -2 \delta x_\nu \quad \text{and} \quad (11)$$

to conserve $D_*$ at its fiducial value.

From these the consequential shifts in other parameters readily follow as

$$\delta \ln \Omega_{\text{cdm}} \simeq +4 \delta x_\nu \quad ; \quad \delta \Omega_{\text{cdm}} \simeq +4 \delta \Omega_\nu \quad ; \quad (12)$$

$$\delta \Omega_m \simeq +5 \delta \Omega_\nu \quad ; \quad \delta \Omega_\Lambda \simeq -5 \delta \Omega_\nu \quad (13)$$

here we have chosen “matching units” i.e. $\ln(\Omega_{\text{cdm}})$ vs dimensionless neutrino fraction $x_\nu$, or $\Omega_\nu$ on both sides.

It is notable that the dimensionless multiplier is dramatically larger in physical density units, $\omega_\Lambda$ vs $\omega_\nu$: rearranging Equation (10) we have

$$\delta (\omega_{\text{cdm}} + \omega_\nu + \omega_\Lambda) \simeq -4 \frac{\delta \omega_\nu}{\omega_{\text{cdm}}} (\omega_{\text{cdm}} + \omega_\nu + \omega_\Lambda)$$

$$\simeq -4 \frac{\delta \omega_\nu}{\Omega_{\text{cdm}}} \delta \omega_\nu \simeq -14 \delta \omega_\nu \quad (14)$$

where the last line assumes $\delta \omega_{\text{cdm}} \simeq 0$.

For the relative change in $\omega_\Lambda$, we can divide the above by $\delta \omega_\Lambda$ and rearrange to

$$\delta \ln \omega_\Lambda \simeq \left(\frac{-4 - \Omega_{\text{cdm}}}{\Omega_\Lambda}\right) \delta x_\nu \simeq -6.2 \delta x_\nu.$$  

The various dimensionless multipliers above are notably larger than 1, with many between 4 to 6, and the surprisingly large factor $-14$ in physical densities $\delta \omega_\Lambda \simeq -14 \delta \omega_\nu$. This is arguably the “root cause” of the degeneracy in physical density units, i.e. increasing neutrino density requires a $\sim 14 \times$ larger reduction in vacuum energy density to minimise the changes in the observed CMB power spectrum.

Qualitatively, this is explained because an increase in neutrino mass increases the expansion rate across almost the entire matter-dominated era (causing a decrease in $D_*$); to restore $D_*$, we must then reduce the expansion rate $H(z)$ in the $\Lambda$-dominated (accelerating) era $z \lesssim 0.67$ by reducing $\omega_\Lambda$. The latter era contributes nearly half of cosmic time, but only about 19 percent of the comoving distance $D_*$, to last scattering, roughly explaining the “factor of 4” in Equation (10). Another factor of $\Omega_{\text{cdm}}^{-1}$ appears in Equation (14), since we defined $x_\nu$ relative to the CDM+baryon density while $\delta \ln (\omega_{\text{cdm}} + \omega_\nu + \omega_\Lambda)$ is relative to the total mass-energy density today.

Although the physical CDM+baryon density $\omega_{\text{cdm}}$, changes very little in response to varying neutrino mass, the density parameter $\Omega_m$ is affected substantially: assuming flatness we can write

$$\Omega_m = \frac{\omega_{\text{cdm}} + \omega_\nu + \omega_\Lambda}{\omega_{\text{cdm}} + \omega_\nu + \omega_\Lambda} \quad , \quad (16)$$

so it is clear that as $\omega_\nu$ increases, it is the steep decrease in dark energy density in the denominator which is reponsible for around four-fifths of the increase in $\Omega_m$, while the “direct” contribution of $\omega_\nu$ in the numerator accounts for only the remaining one-fifth.

In the next subsection we compare these approximate estimates with selected numerical results from the *Planck* collaboration likelihood chains and the literature, and find rather good agreement.
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3 COMPARISON WITH NUMERICAL RESULTS

The derivations above are approximate, but turn out to be quite close to the numerical degeneracy direction in current and near-future CMB experiments, as follows.

Numerical calculations with CAMB indicate that if $\omega_c, \omega_b$ are held fixed while $\Sigma m_\nu$ is varied, then the condition for constant $\theta_*$ is actually $\delta \ln h \simeq -1.75 \delta x_\nu$: this slope is comparable but slightly shallower than the value $-2$ from approximation 11 above. The difference between $-2$ and $-1.75$ arises mainly from two small effects: neutrinos are not fully matter-like at $300 \lesssim z < 1090$ which makes the sensitivity of $D_s$ to $x_\nu$ slightly weaker than the $-0.4$ power in Eq. 5 above, actually $\simeq -0.36$; also increasing neutrino mass very slightly reduces the sound horizon length $r_S(z_\nu)$. (The main point remains, that $\theta_*$ is substantially more sensitive to $x_\nu$ than $\omega_{ch}$ and $h$).

However, in full 7-parameter CMB fits, varying $x_\nu$ also gives additional small correlated changes in other parameters, with $\omega_{ch}$ being the next most important contributor to changes in $\theta_*$: changes in $\omega_\nu$ are much less important. In the full parameter space, the condition for constant $\theta_*$ is well approximated by the relationship

$$
\delta \ln h \simeq -1.75 \delta x_\nu - 0.8 \delta \ln \omega_{ch} .
$$

In the Planck case, the likelihood ridge shows a small but positive correlation$^2$ of $\omega_{ch}$ with $x_\nu$, in the direction $\delta \ln \omega_{ch} \approx +0.2 \delta x_\nu$: so marginalising over $\omega_{ch}$ leads to an overall degeneracy direction for Planck of $\delta \ln h \approx -1.9 \delta x_\nu$, hence fortuitously moving closer to the simple $-2$ approximation from the previous Section.

Fig. 1 shows a scatter plot of $\omega_A$ vs $\omega_\nu$ in the public Planck Monte-Carlo Markov chains, here the 7-parameter chain base_mnu/planck_TTTEEE_lowTEB. Fitting a linear relation for $\omega_A$ vs $\omega_\nu$ to the Planck chain gives a slope of $-12.9$, while a quadratic fit has a slope of $-14.4$ at the fiducial $\Sigma m_\nu = 0.06$ eV; the latter in particular agrees well with the approximate slope $-14$ from Eq. 14. Here the quadratic is a better fit, since the observed degeneracy track starts to curve for $\Sigma m_\nu \gtrsim 0.4$ eV where neutrinos are not fully relativistic at recombination; this pulls the linear fit to shallower slope.

Looking to the future, a detailed set of predictions for the proposed CoRE CMB spacecraft (Delabrouille et al 2017) are given by Archidiacono et al (2017) and Challinor et al (2017). In the case of simulated CoRE

$^2$ The source of this correlation appears to be that when increasing neutrino mass at constant $\omega_{ch}, \theta_*$, the largest fractional change of the theoretical CMB spectrum is a small reduction at low multipoles, since the higher $\Omega_m$ and thus lower $\Omega_A$ reduces the late-time ISW effect. The shift is well within the cosmic variance, but in the MCMC fits this can be partially compensated by a combination of small red tilt (lowering $n_s$) and a small increase in $\omega_{ch}$.
data, the improved sensitivity to high-ℓ polarisation and CMB lensing leads to a substantially more positive correlation of \( \omega_{\text{ch}} \) versus \( x_{\nu} \) for CoRE than for Planck, with a predicted CoRE likelihood ridge given by \( \delta \ln \omega_{\text{ch}} \approx +1 \delta x_{\nu} \). Also, Eq. 2.4 of Archidiacono et al (2017) converts to

\[
\delta \ln h \simeq -2.5 \delta x_{\nu} \tag{18}
\]

in our notation, which is consistent with substituting \( \delta \ln \omega_{\text{ch}} \approx +1 \delta x_{\nu} \) into approximation (17) above. This predicted degeneracy slope -2.5 for CoRE is somewhat steeper than the Planck case and our approximate slope -2 above. However, it is not dramatically steeper, because the larger coefficient of \( \delta x_{\nu} \) in approximation (17) implies that the \( \delta x_{\nu} \) term still dominates over the \( \omega_{\text{ch}} \) term.

We can also compare with the numerical estimates of Pan & Knox (2015), who show \( H(z) \) for CMB-fitted models with several selected values of \( \Sigma m_{\nu} = (0.05, 0.1, 0.2) \, \text{eV} \) imposed as a prior. At \( z \lesssim 100 \) we can write

\[
H(z) = 100 \, \text{km s}^{-1} \, \text{Mpc}^{-1} \sqrt{[\omega_{\text{ch}} + \omega_{\nu}](1 + z)^3 + \omega_{\Lambda}}. \tag{19}
\]

From the condition in Equation (14), \( \delta \omega_{\nu} \approx -14 \delta x_{\nu} \), it is clear that this predicts a ‘crossover’ redshift given by \( 1 + z_{\text{cr}} \approx \frac{1}{14} \) i.e. \( z_{\text{cr}} \approx 1.4 \), at which the CMB-preferred value of \( H(z_{\text{cr}}) \) becomes independent of neutrino mass. The numerically-fitted crossover point seen in Figure 1 of Pan & Knox (2015) agrees very well with this simple estimate.

To verify that the direct dependence of the sound horizon length on neutrino mass is nearly negligible, we did a fit of \( r_{\text{s}}(z) \left[ \omega_{\text{ch}}/0.1410 \right]^{0.25} \) as a quadratic function of \( \Sigma m_{\nu} \), using the above Planck chain. The scaling with \( \omega_{\text{ch}} \) is included in order to cancel the secondary effect from correlated shifts of \( \omega_{\text{ch}} \), \( \Sigma m_{\nu} \), which otherwise dominate the variation in \( r_{\text{s}}(z) \) alone. The result of this fit is

\[
r_{\text{s}}(z) \left( \frac{\omega_{\text{ch}}}{0.1410} \right)^{0.25} = 144.84 + 0.078 \left( \frac{\Sigma m_{\nu}}{1 \, \text{eV}} - 0.06 \right) -0.511 \left( \frac{\Sigma m_{\nu}}{1 \, \text{eV}} - 0.06 \right)^2, \tag{20}
\]

with the rms of the Planck chain only 0.076 Mpc (0.05 percent) relative to the above fitting function. The fit gives a mean shift of \( \approx 0 \) percent for \( \delta x_{\nu} = 0.01 \), and \( -0.08 \) percent for \( \Sigma m_{\nu} = 0.6 \, \text{eV} \) (\( \delta x_{\nu} = 0.04 \)). This validates the argument in Section 2.2 that neutrino mass has nearly negligible direct effect on sound horizon length.

To summarise this section, we expect that numerical degeneracy tracks from CMB experiments alone will in general give a track with slope \( \delta \ln h/\delta x_{\nu} \) between \(-1.75 \) and \(-2.5 \) in order to conserve the sound horizon angle \( \theta_{\text{s}} \). Here the simple slope estimate \(-2 \) from approximation (10) is the leading-order term, while smaller effects from errors in that approximation and correlations between \( \omega_{\text{ch}} \) and \( x_{\nu} \) give moderate corrections to the simple \(-2 \). In physical density units, the slope of \( \delta \omega_{\Lambda} \) vs \( \delta x_{\nu} \) is approximately \( 7 \times \) steeper than the above.

![Figure 2](https://example.com/figure2.png)

**Figure 2:** Linear-theory matter power spectrum for three models, all with \( \delta x_{\nu} = +0.01 (\Sigma m_{\nu} = 0.191 \, \text{eV}) \), relative to the baseline model with \( \delta x_{\nu} = 0 (\Sigma m_{\nu} = 0.06 \, \text{eV}) \). The lines show three choices for which other parameters are fixed: the short-dashed line has fixed \( \omega_{\nu} \), \( h \) (i.e. \( \omega_{\text{ch}} = -1 \delta x_{\nu} \)); the long-dashed line has fixed \( \omega_{\text{ch}}, h \) (i.e. \( \omega_{\Lambda} = -1 \delta x_{\nu} \)); and the solid line has fixed \( \omega_{\text{ch}}, \theta_{s} \) (i.e. \( \omega_{\Lambda} \) and \( \theta_{s} \) reduced to conserve \( \theta_{s} \) ) approximating the Planck degeneracy track.

### 4. CONSEQUENCES FOR THE MATTER POWER SPECTRUM

The degeneracy between \( \Sigma m_{\nu} \) and \( h \) turns out to have interesting consequences for matter power spectrum observables, as follows. There is a well-known effect that massive neutrinos reduce the matter power spectrum on small scales (large wavenumber \( k \)) due to neutrino free-streaming; there is negligible effect at \( k < k_{\text{fs}} \), where \( k_{\text{fs}} \) is the free-streaming scale, while the suppression shows a downward ramp at \( k > k_{\text{fs}} \), then asymptotically approaching \( \delta P/P \approx -8 f_{s} \) on small scales (Hu, Eisenstein & Tegmark 1998), as in the short-dashed line in Figure 2. However, most early studies (e.g. Eisenstein & Hu (1997); Lesgourgues & Pastor (2006)) compared models with different \( \Sigma m_{\nu} \) but identical \( \Omega_{\Lambda}, h \) to derive this simple rule-of-thumb: if instead we compare models with different \( \Sigma m_{\nu} \) moving along the CMB degeneracy track, then the resulting variations in \( h \) and \( \Omega_{\Lambda} \) also become comparably important for the low-redshift power spectrum, as follows:

(i) If we keep fixed \( h \), but now fix \( \omega_{\text{ch}} \) instead of \( \omega_{\nu} \), as shown by the long-dashed line in Figure 2, then the small-scale power suppression is somewhat reduced to \( \delta P/P \approx -6 \delta x_{\nu} \), and we see a slight large-scale suppression \( \delta P/P \approx -1 \delta x_{\nu} \).

(ii) If we consider (for simplicity) a pair of models with identical \( \omega_{\text{ch}}, \omega_{\nu}, x_{\nu} \) and identical early-time matter power spectra in physical Mpc units, but slightly different values of \( h \), the observables from a low-redshift galaxy survey are \( P \) in units of \( h^{-3} \, \text{Mpc}^{-3} \), and \( k \) in units of \( h \, \text{Mpc}^{-1} \). Defining \( q = k/h \) and \( P = P/h^{-3} \), the power-spectrum actually observed corresponds to \( P(q) = h^{3} P(k = hq) \) in units of \( \text{Mpc}^{-3} \) and \( \text{Mpc}^{3} \). Then, comparing two models with identical \( P(k) \) but a small difference in \( h \), two observers measuring the low-z power spectrum in these models would observe an offset in
\( P(q) \) given by

\[
\delta \ln P(q) \approx \left[ 3 + \left( \frac{d \ln P}{d \ln k} \right)_{k=k_{c}} \right] \delta \ln h. \tag{21}
\]

Given the \( \Lambda \)CDM power spectrum shape, the square-bracket term above is almost +1 on very large scales \((k \lesssim k_{c})\), then declines to \( \sim +1\) at \( q \sim 0.5 \text{ Mpc}^{-1}\), and asymptotes to zero on very small scales (where linear theory breaks down).

Since we saw above that the Planck-only degeneracy track is well approximated by \( \delta \ln h \approx -2 \delta x_{\nu} \), the resulting offset in \( h \) contributes a fractional power spectrum shift \(^3\) which is \( \delta \ln P \approx -8 \delta x_{\nu} \) at small \( q \), and ramps smoothly to \( \sim -2 \) at small scales \( q \sim 0.5 \text{ Mpc}^{-1}\). By a rather remarkable apparent coincidence, this ramp from the \( h \)-offset has similar magnitude but on the opposite slope to the direct neutrino-mass effect above; and the respective crossover scales \( k_{c} \) and \( k_{x} \) also have a different origin but are roughly similar for interesting neutrino masses; so effects (i) and (ii) above combine to produce a roughly uniform power suppression \( \delta \ln P(q) \approx -8 \delta x_{\nu} \), now nearly independent of scale \( q \).

(iii) Finally, there is another shift in low-redshift power due to the differing growth factor from the CMB era to today, which depends mainly on \( \Omega_{m} \); the linear-theory \( z = 0 \) power spectrum contains a factor of \( g^{2} \) where \( g \propto \Omega_{m}^{0.24} \) is the linear-theory growth function. Since we saw above that the CMB degeneracy track follows \( \delta \ln \Omega_{m} \approx +5 \delta x_{\nu} \), this effect contributes a fractional shift in low-redshift \( P(q) \) by \( \delta \ln g^{2} \approx 0.48 \delta \ln \Omega_{m} \approx +2.4 \delta x_{\nu} \), contributing an increase in power with \( x_{\nu} \); this is in the opposite direction but smaller than effects (i) and (ii) above.

(For a slightly different but comparable effect based on super-sample density fluctuations, see Li et al (2014)).

Thus, the total effect of varying neutrino mass on the low-redshift matter power spectrum is substantially dependent on which parameter(s) are held fixed: in Figure 2 we show linear-theory power spectra for three example cases, all with a common value \( \delta x_{\nu} = +0.01 \) but different choices for fixing other parameters.

To a fairly good approximation for \( x_{\nu} \lesssim 0.03 \), since \( \omega_{b} \) and \( \omega_{\gamma} \) are almost unchanged we expect the high-redshift power spectra in physical \( k \) units to vary only with effect (i) above; but at low redshift, effects (ii) and (iii) also contribute.

In the approximation that effects (i) - (iii) combine additively in \( \ln P \), simply adding them predicts that varying neutrino mass (along the CMB degeneracy track) results in an approximately scale-independent suppression of the broad-band low-redshift power spectrum by a fractional shift \( \delta \ln P(q) \sim -5.6 \delta x_{\nu} \). This approximate estimate is similar to the solid line in Figure 2, except for the wiggles. Since the key parameter \( \sigma_{8} \) is measured in an \( 8 \text{ h}^{-1} \) Mpc sphere, this also depends on \( P(q) \) and thus \( \delta \ln \sigma_{8} \sim -2.8 \delta x_{\nu} \). This broadband overall power offset is largely degenerate with the galaxy bias parameter in galaxy power spectra measurements, but the \( \sigma_{8} \) effect is distinctive.

However, the BAO peaks do shift: along the Planck degeneracy track the values of \( \omega_{b} \) and \( \omega_{\gamma} \) are almost independent of \( x_{\nu} \), so the BAO scale is almost independent of neutrino mass in physical \( k \) units, but it does shift in \( q \) units due to the consequential shift in \( h \); this shift is responsible for the pronounced wiggles in the solid line in Figure 2.

Finally, we note another feature derived from the above: weak-lensing measurements at moderate redshift are especially sensitive to the parameter combination \( S_{8} \), usually defined as \( S_{8} \equiv \sigma_{8}(\Omega_{m}/0.30)^{0.5} \). Along the CMB degeneracy direction, the combination of the \( \sigma_{8} \) reduction as above with the positive degeneracy \( \delta \ln \Omega_{m} \sim +5 \delta x_{\nu} \) results in a near cancellation of the two effects on \( S_{8} \); again this is largely coincidental. This helps to explain why the CMB constraints on \( S_{8} \) are counter-intuitively rather insensitive to varying neutrino mass (e.g. MacCrann et al 2015).

To summarise this section, we have seen that when varying neutrino mass along with other parameters following the CMB degeneracy track, the “secondary” effects on the matter power spectrum at large scales caused by the consequential shifts in \( \Omega_{m}, h \) and \( \Omega_{m} \) are (mainly coincidentally) of a similar magnitude to the “primary” effect of neutrinos suppressing small-scale power. This explains qualitatively why BAO measurements and also \( \sigma_{8} \) measurements are in practice considerably more effective than broad-band galaxy power spectrum measurements for breaking the CMB-only neutrino mass degeneracy (see e.g. Cuesta et al (2016)).

5 Eight-Parameter Models

The estimates in previous Sections assumed the six+one parameter flat \( \Lambda \)CDM + \( m_{\nu} \) model. However, it is interesting to consider the effect of allowing an eighth free parameter such as dark energy equation of state \( w \), curvature \( \Omega_{k} \) or additional relativistic species \( (N_{\text{eff}} > 3.046) \), since an extra free parameter may generally relax the upper limits on \( \Sigma m_{\nu} \). Here we give just a short qualitative discussion of these three possible extra parameters in turn.

In the case of allowing \( w \neq -1 \), it is well known that CMB fits give an anticorrelation between \( w \) and \( H_{0} \) (Weinberg et al 2013): assuming flatness, increasing \( w > -1 \) requires lower \( h \) and higher \( \Omega_{m} \) to fit the CMB data, i.e. the same direction as increasing neutrino mass. Adding BAO measurements gives primarily a constraint on \( \Omega_{k} \), hence implying an anticorrelation between neutrino mass and \( w \) in a combined CMB+BAO fit. This suggests that allowing “phantom” dark energy with \( w < -1 \) can relax upper limits on neutrino mass, but allowing time-variable dark energy with a choice of a “no-phantom” prior \( w(z) \gtrsim -1 \), in general should not much weaken the upper limits on neutrino mass (though if future CMB+BAO fits show a deviation from 6-parameter \( \Lambda \)CDM, there may well be potential ambiguity between the cases \( \Sigma m_{\nu} > 0.06 \text{ eV} \) or \( w > -1 \)).

For the case of small non-zero curvature \( \Omega_{k} \neq 0 \), the largest change in observables is a significant shift in the sound horizon angle, with a high sensitivity \( \partial \ln \theta_{s}/\partial \Omega_{k} \approx -1.6 \). Since we have seen above that the sound horizon angle is a key factor giving rise to the neutrino mass/dark energy degeneracy, we expect that allowing non-flat models will significantly weaken the current constraints on neutrino mass,
compared to assuming flatness. This is consistent with the results of Chen et al. (2016).

In the case of allowing \( N_{\text{eff}} > 3.046 \), we recall the argument of Eisenstein & White (2004) and Sutherland (2012): allowing free \( N_{\text{eff}} \) leads to a degeneracy direction whereby to minimise changes in (dimensionless) CMB+BAO+SNe observables, the physical densities of matter and vacuum energy increase almost pro-rata with early-time radiation density. Along this degeneracy track, truly dimensionless parameters such as \( \Omega_{\Lambda}, \theta_*, z_{eq} \) (which depend on density ratios) have best-fit values almost independent of \( N_{\text{eff}} \), while the pseudo-dimensionless parameters \( h \) and \( \omega_{\text{cdm}} \) (which include an arbitrary normalisation to \( H_0 = 100 \text{ km s}^{-1} \text{ Mpc}^{-1} \)) do show a substantial degeneracy with \( N_{\text{eff}} \). Since neutrino mass has a substantial degeneracy with \( \Omega_{\Lambda} \), but \( N_{\text{eff}} \) has little degeneracy with \( \Omega_{\nu} \), this suggests that allowing non-standard \( N_{\text{eff}} \) will not substantially weaken neutrino mass limits from dimensionless data combinations such as CMB+BAO+SNe; this is broadly consistent with the results of Ade et al. (2016).

As a numerical check, we have repeated the procedure from Section 3 of fitting a quadratic to \( \omega_\Lambda \) vs. \( \omega_\nu \) in a Planck MCMC chain, this time to the 8-parameter chain with variable \( N_{\text{eff}} \) and \( m_\nu \) and selecting the subset of the chain with \( 3.4 < N_{\text{eff}} < 3.6 \), near the Planck upper limit. This fit gave a slope of -17.4, only slightly steeper than the -14.4 found previously for standard \( N_{\text{eff}} \).

To summarise this section, we estimate qualitatively that allowing phantom dark energy \( (w < -1) \) or non-zero curvature can substantially weaken the constraints on neutrino mass compared to the 7-parameter \( \Lambda \)CDM + \( \Sigma m_\nu \) case; but allowing \( w > -1 \) or non-standard \( N_{\text{eff}} \) will tend to give only marginal weakening of neutrino mass upper limits from combined CMB+BAO+SNe datasets.

6 CONCLUSIONS

We have given a simple and intuitive semi-analytic explanation for the observed CMB degeneracy direction in flat \( \Lambda \)CDM models extended with non-minimal neutrino mass. A notable point is that a non-cosmological estimate of vacuum energy density \( \omega_\Lambda \), either from a future laboratory detection or an \textit{ab initio} theoretical calculation, could give strong constraints on neutrino mass; unfortunately at present there is no well-agreed route to such an estimate, though there are some speculative proposals (e.g. Hogan 2012; Padmanabhan 2016).

We also gave an approximate explanation how the degeneracy between \( \Sigma m_\nu \) and \( h \) produces a suppression in large-scale power in observable \( h \)-dependent units; when combined with the small-scale effect of neutrino mass, this explains the nearly scale-independent suppression of broad-band power at low redshift, combined with a sideways shift in the BAO features.

These multipliers above are helpful to intuitively explain the strong constraints on total neutrino mass obtained from adding low-redshift cosmological observations such as BAOs to the Planck data.

ACKNOWLEDGEMENTS

We thank the anonymous referee for helpful comments which significantly improved the clarity of the paper. We acknowledge the use of data from the Planck Legacy Archive supported by ESA at planck.esa.int.

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