STRONG MORITA EQUIVALENCE FOR INCLUSIONS OF
C*-ALGEBRAS INDUCED BY TWISTED ACTIONS OF A
COUNTABLE DISCRETE GROUP

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Abstract. Let \((\alpha, w_\alpha)\) and \((\beta, w_\beta)\) be twisted actions of a countable discrete group \(G\) on \(\sigma\)-unital \(C^*\)-algebras \(A\) and \(B\) and \(A \rtimes_{\alpha, w_\alpha} G\) and \(B \rtimes_{\beta, w_\beta} G\) the twisted reduced crossed products of \(A\) and \(B\) by \((\alpha, w_\alpha)\) and \((\beta, w_\beta)\), respectively. Then we obtain the inclusions of \(C^*\)-algebras \(A \subset A \rtimes_{\alpha, w_\alpha} G\) and \(B \subset B \rtimes_{\beta, w_\beta} G\). We suppose that \(A' \cap M(A \rtimes_{\alpha, w_\alpha} G) = C_1\). In this paper, we show that if \(A \subset A \rtimes_{\alpha, w_\alpha} G\) and \(B \subset B \rtimes_{\beta, w_\beta} G\) are strongly Morita equivalent, then there is an automorphism \(\phi\) of \(G\) such that \((\alpha^\phi, w_\alpha^\phi)\) and \((\beta, w_\beta)\) are strongly Morita equivalent, where \((\alpha^\phi, w_\alpha^\phi)\) is the twisted action of \(G\) on \(A\) defined by \(a^\phi_t(a) = a_{\phi(t)}(a)\) and \(w_\alpha^\phi_t(s) = w_\alpha(\phi(t), \phi(s))\) for any \(t, s \in G, a \in A\).

1. Introduction

In the previous papers [10], [11], we discussed strong Morita equivalence for twisted coactions of a finite dimensional \(C^*\)-Hopf algebra on unital \(C^*\)-algebras and unital inclusions of unital \(C^*\)-algebras. Also, in [12], we clarified the relation between strong Morita equivalence for twisted coactions of a finite dimensional \(C^*\)-Hopf algebra and strong Morita equivalence for their unital inclusions of the unital \(C^*\)-algebras induced by the twisted coactions of the finite dimensional \(C^*\)-Hopf algebra on the unital \(C^*\)-algebras. In the present paper, we shall discuss the same subject as above in the case of twisted actions of a countable discrete group on \(C^*\)-algebras.

Let \((\alpha, w_\alpha)\) and \((\beta, w_\beta)\) be twisted actions of a countable discrete group \(G\) on \(\sigma\)-unital \(C^*\)-algebras \(A\) and \(B\) and \(A \rtimes_{\alpha, w_\alpha} G\) and \(B \rtimes_{\beta, w_\beta} G\) the twisted reduced crossed products of \(A\) and \(B\) by \((\alpha, w_\alpha)\) and \((\beta, w_\beta)\), respectively. Then we obtain the inclusions of \(C^*\)-algebras \(A \subset A \rtimes_{\alpha, w_\alpha} G\) and \(B \subset B \rtimes_{\beta, w_\beta} G\). We note that \(A(\alpha, w_\alpha, r G) = A \rtimes_{\alpha, w_\alpha} G\) and \(B(B \rtimes_{\beta, w_\beta} G) = B \rtimes_{\beta, w_\beta} G\) by routine computations. In this paper, we review that if \((\alpha, w_\alpha)\) and \((\beta, w_\beta)\) are strongly Morita equivalent, then in the same way as in Combes [5] or Curto, Murphy and Williams [6], the inclusions \(A \subset A \rtimes_{\alpha, w_\alpha} G\) and \(B \subset B \rtimes_{\beta, w_\beta} G\) are strongly Morita equivalent by easy arguments. However, its inverse is not clear. We shall show the following: We suppose that \(A\) and \(B\) are \(\sigma\)-unital and that \(A' \cap M(A \rtimes_{\alpha, w_\alpha} G) = C_1\). If the inclusions \(A \subset A \rtimes_{\alpha, w_\alpha} G\) and \(B \subset B \rtimes_{\beta, w_\beta} G\) are strongly Morita equivalent, then there is an automorphism \(\phi\) of \(G\) such that \((\alpha^\phi, w_\alpha^\phi)\) and \((\beta, w_\beta)\) are strongly Morita equivalent, where \((\alpha^\phi, w_\alpha^\phi)\) is the twisted action of \(G\) on \(A\) defined by \(a^\phi_t(a) = a_{\phi(t)}(a)\) and \(w_\alpha^\phi_t(s) = w_\alpha(\phi(t), \phi(s))\) for any \(t, s \in G, a \in A\). This result is similar one to [12, Theorem 6.2].

Let \(K\) be the \(C^*\)-algebra of all compact operators on a countably infinite dimensional Hilbert space and \(\{c_{ij}\}_{i,j \in \mathbb{N}}\) its system of matrix units. For each \(C^*\)-algebra

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A, we denote by $M(A)$ the multiplier $C^*$-algebra of $A$. Let $\pi$ be an isomorphism of $A$ onto a $C^*$-algebra $B$. Then there is the unique strictly continuous isomorphism of $M(A)$ onto $M(B)$ extending $\pi$ by Jensen and Thomsen [5 Corollary 1.1.15]. We denote it by $\widehat{\pi}$. For an algebra $A$, we denote $1_A$ and $\text{id}_A$ the unit element in $A$ and the identity map on $A$, respectively. If no confusion arises, we denote them by $1$ and $\text{id}$, respectively. Throughout this paper, we denote by $G$ a countable discrete group with the unit element $e$.

2. Preliminaries

First, we give some definitions. Let $A$ be a $C^*$-algebra and $G$ a countable discrete group with the unit element $e$. Let $\text{Aut}(A)$ be the group of all automorphisms of $A$ and $U(M(A))$ the group of all unitary elements in $M(A)$.

Definition 2.1. By a twisted action $(\alpha,w_\alpha)$ of $G$ on $A$, we mean a map $\alpha$ from $G$ to $\text{Aut}(A)$ and a map $w_\alpha$ from $G \times G$ to $U(M(A))$ satisfying following:

1. $\alpha_t \circ \alpha_s = \text{Ad}(w_\alpha(t,s)) \circ \alpha_{ts}$,
2. $w_{\alpha}(t,s)w_{\alpha}(s,r) = \alpha_t(w_\alpha(s,r))w_\alpha(t,sr)$,
3. $w_\alpha(t,e) = w_\alpha(e,t) = 1_{M(A)}$ for any $t, s, r \in G$.

By easy computations, for any $t \in G$,

$$\alpha_e = \text{id}, \quad w_\alpha(t,t^{-1}) = \alpha_t(w_\alpha(t^{-1},t)),$$

$$\alpha_t^{-1} = \alpha_{t^{-1}} \circ \text{Ad}(w_\alpha(t^{-1},t)^*) = \text{Ad}(w_\alpha(t^{-1},t)^*) \circ \alpha_{t^{-1}}$$

Let $(\alpha,w_\alpha)$ be a twisted action of $G$ on a $C^*$-algebra $A$. Then we have a twisted action $(\lambda, w_\lambda)$ of $G$ on $M(A)$ such that $\lambda_t$ is the unique strictly continuous automorphism of $M(A)$ extending $\alpha_t$ to $M(\widehat{A})$ for any $t \in G$.

Let $A$ and $B$ be $C^*$-algebras. Let $X$ be an $A-B$-equivalence bimodule. For any $a, b \in A$, $x \in X$, we denote by $a \cdot x$ the left $A$-action on $X$ and by $x \cdot b$ the right $B$-action on $X$. Let $\tilde{X}$ be the dual $B-A$-equivalence bimodule of $X$ and $\tilde{x}$ denotes the element in $\tilde{X}$ induced by an element $x \in X$. Also, we regard $X$ as a Hilbert $M(A) - M(B)$-bimodule in the sense of Brown, Mingo and Shen [3] as follows: Let $B_B(X)$ be the $C^*$-algebra of all adjointable right $B$-linear operators on $X$. We note that an adjointable right $B$-linear operator on $X$ is bounded. Then $B_B(X)$ can be identified with $M(A)$. Similarly let $A_B(X)$ be the $C^*$-algebra of all adjointable left $A$-linear operators on $X$ and $(A_B(X))'$ is identified with $M(B)$. In this way, we regard $X$ as a Hilbert $M(A) - M(B)$-bimodule. Let $\text{Aut}(X)$ be the group of all bijective linear maps on $X$.

Definition 2.2. Let $(\alpha,w_\alpha)$ and $(\beta,w_\beta)$ be twisted actions of $G$ on a $C^*$-algebra $A$. We say that $(\alpha,w_\alpha)$ and $(\beta,w_\beta)$ are exterior equivalent if there are unitary elements $\{w_1\}_{t \in G} \subset M(A)$ satisfying the following:

1. $\beta_t = \text{Ad}(w_1) \circ \alpha_t$,
2. $w_{ts} = w_\beta(t,s)^*w_\alpha \alpha_t(w_\alpha(t,s))w_\alpha(t,ts)$

for any $t, s \in G$.

Definition 2.3. Let $(\alpha,w_\alpha)$ and $(\beta,w_\beta)$ be twisted actions of $G$ on $C^*$-algebra $A$ and $B$, respectively. We say that $(\alpha,w_\alpha)$ and $(\beta,w_\beta)$ are strongly Morita equivalent if there are an $A-B$-equivalence bimodule $X$ and a map $\lambda$ from $G$ to $\text{Aut}(X)$ satisfying the following:

1. $\alpha_t(\lambda(x,y)) = \lambda(\alpha_t(x), \lambda_t(y))$,
2. $\beta_t(\langle x,y \rangle_B) = \langle \lambda_t(x), \lambda_t(y) \rangle_B$,
3. $\langle \lambda_t \circ \lambda_s(x), y \rangle_B = w_\beta(t,s) \cdot \lambda_{ts}(x) \cdot w_\beta(t,s)^*$

for any $t, s \in G$, $x, y \in X$, where we regard $X$ as a Hilbert $M(A) - M(B)$-bimodule in the above way.
We note that if twisted actions \((\alpha, w_\alpha)\) and \((\beta, w_\beta)\) of \(G\) on \(A\) are exterior equivalent, then they are strongly Morita equivalent. Now we show it.

Let \((\alpha, w_\alpha)\) and \((\beta, w_\beta)\) be twisted actions of \(G\) on \(A\) and \(B\), respectively. We suppose that they are strongly Morita equivalent. Also, we suppose that there are an \(A - B\)-equivalence bimodule \(X\) and a map \(A\) to \(\operatorname{Aut}(X)\) satisfying Conditions (1)-(3) in Definition 2.3. We show that the inclusions of \(C^\ast\)-algebras \(A \subset A \rtimes_{\alpha, w_\alpha, r} G\) and \(B \subset B \rtimes_{\beta, w_\beta, r} G\) are strongly Morita equivalent in the same way as in Combes [5] or Curto, Muhly and Williams [6]. Let \(L_X\) be the linking \(C^\ast\)-algebra for \(X\) defined by

\[
L_X = \begin{bmatrix} A & X \\ X & B \end{bmatrix} = \begin{bmatrix} a & x \\ y & b \end{bmatrix} | a \in A, b \in B, x, y \in X).
\]

Let \(t, s \in G\). Let \(\gamma_t\) be the map on \(L_X\) defined by

\[
\gamma_t\begin{bmatrix} a & x \\ y & b \end{bmatrix} = \begin{bmatrix} \alpha_t(a) & \lambda_t(x) \\ \lambda_t(y) & \beta_t(b) \end{bmatrix}
\]

for any \begin{bmatrix} a & x \\ y & b \end{bmatrix} \in L_X. Also, let \(w_\gamma(t, s)\) be the unitary element in \(M(L_X)\) defined by

\[
w_\gamma(t, s) = \begin{bmatrix} w_\alpha(t, s) & 0 \\ 0 & w_\beta(t, s) \end{bmatrix}.
\]

Then by routine computations, we can see that \((\gamma, w_\gamma)\) is a twisted action of \(G\) on \(L_X\). Let \(L_X \rtimes_{\gamma, w_\gamma, r} G\) be the twisted reduced crossed product of \(L_X\) by \((\gamma, w_\gamma)\).

Let \(p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\) and \(q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\). Then \(p\) and \(q\) are projections in \(M(L_X) \subset M(L_X \rtimes_{\gamma, w_\gamma, r} G)\) and they are full in \(L_X \rtimes_{\gamma, w_\gamma, r} G\). Let \(Y = p(L_X \rtimes_{\gamma, w_\gamma, r} G)q\). Then \(Y\) is an \(A \rtimes_{\alpha, w_\alpha, r} G - B \rtimes_{\beta, w_\beta, r} G\)-equivalence bimodule and by routine computations, the inclusions of \(C^\ast\)-algebras \(A \subset A \rtimes_{\alpha, w_\alpha, r} G\) and \(B \subset B \rtimes_{\beta, w_\beta, r} G\) are strongly Morita equivalent with respect \(Y\) and its closed subspace \(X\). Hence we obtain the following proposition.

**Proposition 2.1.** Let \((\alpha, w_\alpha)\) and \((\beta, w_\beta)\) be twisted actions of a countable discrete group \(G\) on \(C^\ast\)-algebras \(A\) and \(B\), respectively. If \((\alpha, w_\alpha)\) and \((\beta, w_\beta)\) are strongly Morita equivalent, then the inclusions of \(C^\ast\)-algebras \(A \subset A \rtimes_{\alpha, w_\alpha, r} G\) and \(B \subset B \rtimes_{\beta, w_\beta, r} G\) are strongly Morita equivalent.

Let \((\alpha, w_\alpha)\) be a twisted action of \(G\) on \(A\). Then we have \((\alpha \otimes \operatorname{id}_K, w_\alpha \otimes 1_{M(K)})\), the twisted action of \(G\) on \(A \otimes K\). Hence we obtain the inclusions of \(C^\ast\)-algebras

\[
A \subset A \rtimes_{\alpha, w_\alpha, r} G,
\]

\[
A \otimes K \subset (A \otimes K) \rtimes_{\alpha \otimes \operatorname{id}, w_\alpha \otimes 1, r} G.
\]

Also, we have the inclusion of \(C^\ast\)-algebras

\[
A \otimes K \subset (A \rtimes_{\alpha, w_\alpha, r} G) \otimes K.
\]

Then there is an isomorphism of \((A \otimes K) \rtimes_{\alpha \otimes \operatorname{id}, w_\alpha \otimes 1, r} G\) onto \((A \rtimes_{\alpha, w_\alpha, r} G) \otimes K\) such that its restriction to \(A \otimes K\) is the identity map on \(A \otimes K\). Thus

\[
A \otimes K \subset (A \otimes K) \rtimes_{\alpha \otimes \operatorname{id}, w_\alpha \otimes 1, r} G
\]

and

\[
A \otimes K \subset (A \rtimes_{\alpha, w_\alpha, r} G) \otimes K
\]

are isomorphic as inclusions of \(C^\ast\)-algebras. We identify them as inclusions of \(C^\ast\)-algebras in this paper.
We regard $A$ and $M(A)$ as $C^*$-subalgebras of $A \rtimes_{\alpha,w_A,r} G$ and $M(A) \rtimes_{\alpha,w_A,r} G$ in the usual way, respectively. We note that we have the following inclusions of $C^*$-algebras:

$$A \subset A \rtimes_{\alpha,w_A,r} G \subset M(A) \rtimes_{\alpha,w_A,r} G \subset M(A) \rtimes_{\alpha,w_A,r} G.$$  

We construct a conditional expectation from $A\rtimes_{\alpha,w_A,r}G$ onto $A$. Before we do it, we give a definition of a conditional expectation from $C$ onto $A$, where $A$ and $C$ are $C^*$-algebras with $A \subset C$.

Definition 2.4. Let $E^A$ be a positive linear map from $C$ onto $A$. We say that $E^A$ is a conditional expectation from $C$ onto $A$ if $E^A$ satisfies the following conditions:

1. $||E^A|| \leq 1$,
2. $E^A \circ E^A = E^A$,
3. $E^A(ac) = aE^A(c), E^A(a) = E^A(e)a$ for any $a, c \in C$.

Let $E^{M(A)}$ be the faithful canonical conditional expectation from $M(A)\rtimes_{\alpha,w_A,r}G$ onto $M(A)$ defined in Bélos and Conti [11 Section 3]. Let $E^A = E^{M(A)}|_{A\rtimes_{\alpha,w_A,r}G}$. Then it is a faithful conditional expectation from $A\rtimes_{\alpha,w_A,r}G$ onto $A$.

For each $t \in G$, let $\delta_t$ be the function on $G$ defined by

$$\delta_t(s) = \begin{cases} 
1 & \text{if } s = t \\
0 & \text{if } s \neq t.
\end{cases}$$

We regard $\delta_t$ as an element in $M(A \rtimes_{\alpha,w_A,r} G)$ (See Pedersen [13]). Let $\{u_i\}_{i \in I}$ be an approximate unit of $A$ with $||u_i|| \leq 1$ for any $i \in I$. We fix the approximate unit $\{u_i\}_{i \in I}$ of $A$ in this paper. Let $x \in M(A \rtimes_{\alpha,w_A,r} G)$. Then $\{E^A(x(u_i)\delta_t)\}_{i \in I}$ is a Cauchy net in $A$ under the strict topology for any $t \in G$ by easy computations since $\alpha_t(a) = \delta_t a \delta_t^*$ for any $t \in G$ and $a \in A$. Let $x_t = \lim_i E^A(x(u_i)\delta_t)$ for any $x \in A \rtimes_{\alpha,w_A,r} G$ and $t \in G$, where the limit is taken under the strict topology in $M(A)$. Then $x_t \in M(A)$ for any $t \in G$.

Definition 2.5. Let $x \in M(A \rtimes_{\alpha,w_A,r} G)$. For any $t \in G$, let $x_t$ be an element in $M(A)$ defined in the above. We call $x_t$ the Fourier coefficient of $x \in M(A \rtimes_{\alpha,w_A,r} G)$ at $t \in G$ with respect to the approximate unit $\{u_i\}_{i \in I}$. And we call $\{x_t\}_{t \in G}$ the Fourier coefficients of $x \in M(A \rtimes_{\alpha,w_A,r} G)$ with respect to the approximate unit $\{u_i\}_{i \in I}$. Furthermore, a formal series $\sum_{t \in G} x_t \delta_t$ is called the Fourier series of $x$ with respect to the approximate unit $\{u_i\}_{i \in I}$.

Lemma 2.2. With the above notation, let $x \in A \rtimes_{\alpha,w_A,r} G$. If $x_t = E^A(x\delta_t) = 0$ for any $t \in G$, then $x = 0$.

Proof. For any $a \in A$, $t \in G$,

$$0 = E^A(x\delta_t)\alpha_t^{-1}(a) = E^A(x\delta_t\alpha_t^{-1}(a)) = E^A(xa\delta_t)$$

since $\alpha_t(a) = \delta_t a \delta_t^*$. Since $A \rtimes_{\alpha,w_A,r} G$ is a closed linear span of $\{a\delta_t | a \in A, t \in G\}$, $E^A(xy) = 0$ for any $y \in A \rtimes_{\alpha,w_A,r} G$. Since $E^A$ is faithful, $x = 0$. \hfill \square

Lemma 2.3. With the above notation, let $x \in M(A \rtimes_{\alpha,w_A,r} G)$. If $x_t$, the Fourier coefficient of $x$ at any $t \in G$ is zero, then $x = 0$.

Proof. For any $t \in G$, $a \in A$,

$$0 = x_{t^{-1}} \alpha_t^{-1}(a) = \lim_i E^A(x(u_i)\delta_t)\alpha_t^{-1}(a) = \lim_i E^A(x(u_i)a)\delta_t = E^A(xa\delta_t),$$

since $\alpha_t(a) = \delta_t a \delta_t^*$. Since $E^A(xa\delta_t)$ is the Fourier coefficient of the element $xa$ at $t^{-1} \in G$, by Lemma 2.2 $xa = 0$ for any $a \in A$. Hence $x = 0$. \hfill \square
3. Relative commutants of inclusions of $\sigma$-unital $C^*$-algebras

Let $A$ be a $C^*$-algebra and $p$ a projection in $M(A \otimes K)$. We note that by [2] Proposition 3.1, there is an isomorphism of $M(p(A \otimes K)p)$ onto $pM(A \otimes K)p$ such that its restriction to $p(A \otimes K)p$ is the identity map on $p(A \otimes K)p$. We identify $M(p(A \otimes K)p)$ with $pM(A \otimes K)p$ by the above isomorphism. We begin this section with the following lemma:

**Lemma 3.1.** Let $A \subseteq C$ be an inclusions of $C^*$-algebras with $\overline{AC} = C$. Then the following conditions are equivalent:

1. $A' \cap M(C) = C1$,
2. $(A \otimes K)' \cap M(C \otimes K) = C1$.

**Proof.** (1) $\Rightarrow$ (2): Let $x \in (A \otimes K)' \cap M(C \otimes K)$. Then for any $a \in A$, $k \in K$,

$x(a \otimes k) = (a \otimes k)x.$

Since $A$ is dense in $M(A)$ under the strict topology, $x(1 \otimes k) = (1 \otimes k)x$ for any $k \in K$. We note that

$$(1 \otimes e_{11})M(C \otimes K)(1 \otimes e_{11}) = M(((1 \otimes e_{11})(C \otimes K))(1 \otimes e_{11})) = M(C \otimes e_{11}) = M(C) \otimes e_{11}.$$  

For any $a \in A$,

$$(1 \otimes e_{11})x(1 \otimes e_{11})(a \otimes e_{11}) = (a \otimes e_{11})x(1 \otimes e_{11}).$$

Since $A' \cap M(C) = C1$, there is a $c \in C$ such that $(1 \otimes e_{11})x(1 \otimes e_{11}) = c(1 \otimes e_{11})$. Thus $x(1 \otimes e_{11}) = c(1 \otimes e_{11})$. For any $n \in \mathbb{N}$,

$$x(1 \otimes e_{nn}) = x(1 \otimes e_{n1})(1 \otimes e_{11})(1 \otimes e_{11}) = (1 \otimes e_{n1})x(1 \otimes e_{11})(1 \otimes e_{11}) = (1 \otimes e_{n1})c(1 \otimes e_{11})(1 \otimes e_{11}) = c(1 \otimes e_{nn}).$$

Therefore, $x = c1$. Thus $(A \otimes K)' \cap M(C \otimes K)$. (2) $\Rightarrow$ (1): Let $x \in A' \cap M(C)$. Then $x \otimes 1 \in M(C \otimes K)$. For any $a \in A$, $k \in K$,

$$(a \otimes k)(x \otimes 1) = ax \otimes k = xa \otimes k = (x \otimes 1)(a \otimes k).$$

Thus $x \otimes 1 \in (A \otimes K)' \cap M(C \otimes K) = C1$. Therefore, there is a $c \in C$ such that $x \otimes 1 = c(1 \otimes 1)$. Hence since $x = c1 \in C$ for any $x \in A' \cap M(C)$, $A' \cap M(C) = C1$. $\square$

Let $A$ and $B$ be $\sigma$-unital $C^*$-algebras and $A \subseteq C$ and $B \subseteq D$ inclusions of $C^*$-algebras with $\overline{AC} = C$ and $\overline{BD} = D$. Then $C$ and $D$ are also $\sigma$-unital. We suppose that $A \subseteq C$ and $B \subseteq D$ are strongly Morita equivalent. Then in the same way as in the proof of [2] Proposition 3.5] or Brown, Green and Rieffel [2 Proposition 3.1], there is an isomorphism $\theta$ of $D \otimes K$ onto $C \otimes K$ such that $\theta|_{B \otimes K}$ is an isomorphism of $B \otimes K$ onto $A \otimes K$. Let $p = \theta(1_{M(D)} \otimes e_{11})$. Then $p$ is a projection in $M(A \otimes K) \subseteq M(C \otimes K)$ and $p$ is full in $A \otimes K$ and $C \otimes K$, that is, $(A \otimes K)p(A \otimes K) = A \otimes K$ and $(C \otimes K)p(C \otimes K) = C \otimes K$. By the definitions of $\theta$ and $p$, the unital inclusion of unital $C^*$-algebras $B \subseteq D$ is isomorphic to the unital inclusion of unital $C^*$-algebras $p(A \otimes K)p \subseteq p(C \otimes K)p$ as unital inclusions of unital $C^*$-algebras. We show that if $B' \cap M(D) = C1$, then $A' \cap M(C) = C1$.

**Lemma 3.2.** With the above notation,

$$(A \otimes K)' \cap M(C \otimes K) \cong ((A \otimes K)' \cap M(C \otimes K))p.$$
Proof. Let $\pi$ be the map from $(A \otimes K)\cap M(C \otimes K)$ to $((A \otimes K)\cap M(C \otimes K))p$ defined by $\pi(x) = xp$ for any $x \in (A \otimes K)\cap M(C \otimes K)$. Let $x \in (A \otimes K)\cap M(C \otimes K)$. Then for any $a \in A \otimes K$, $xa = ax$. Hence $xa = ax$ for any $a \in M(A \otimes K)$. This implies that $\pi$ is a homomorphism of $(A \otimes K)\cap M(C \otimes K)$ to $((A \otimes K)\cap M(C \otimes K))p$. It is clear that $\pi$ is surjective. We show that $\pi(x) = 0$. Then $xp = 0$. For any $y, z \in A \otimes K$, $yxpz = 0$. Since $(A \otimes K)p(A \otimes K) = A \otimes K$, $xa = 0$ for any $a \in A \otimes K$. Therefore $x = 0$. Since $\pi$ is injective, we obtain the conclusion. 

Lemma 3.3. With the above notation, 

$$(A \otimes K)\cap M(C \otimes K)p \subset (p(A \otimes K)p')\cap pM(C \otimes K)p.$$ 

Proof. Let $x \in ((A \otimes K)\cap M(C \otimes K))p$. We note that $xp = px$. Then $xp = px$ for any $a \in A \otimes K$. Hence $xpap = paxp$ for any $a \in A \otimes K$. Hence $x \in (p(A \otimes K)p')\cap pM(C \otimes K)p$. 

Lemma 3.4. With the above notation and assumptions, if $B'\cap M(D) = C1$, then $A' \cap M(C) = C1$. 

Proof. Since $B'\cap M(D) = C1$, by the discussions before Lemma 3.2, $(p(A \otimes K)p)\cap pM(C \otimes K) = Cp$. Hence by Lemma 3.3, $((A \otimes K)\cap M(C \otimes K))p = Cp$. Also, by Lemma 3.2, $(A \otimes K)\cap M(C \otimes K) = C1$. Thus by Lemma 3.1, $A' \cap M(C) = C1$. 

4. Free twisted actions of a countable discrete group

Following Zarikian [15] or Choda and Kosaki [4], we give the following definitions. Let $\alpha$ be an automorphism of a $C^*$-algebra $A$.

Definition 4.1. We say that $\alpha$ is inner if there is a unitary element $w \in M(A)$ such that $\alpha = Ad(w)$. We say that $\alpha$ is outer if $\alpha$ is not inner.

Definition 4.2. We say that $\alpha$ is free if $\alpha$ satisfies the following conditions:
If $x \in M(A)$ satisfies that $xa = \alpha(a)x$ for any $a \in A$, then $x = 0$.

Let $(\alpha, w_\alpha)$ be a twisted action of $G$ on a $C^*$-algebra $A$.

Definition 4.3. We say that $(\alpha, w_\alpha)$ is free if the automorphism $\alpha_t$ is free for any $t \in G \setminus \{e\}$.

Definition 4.4. We say that $(\alpha, w_\alpha)$ is outer if $\alpha_t$ is outer for any $t \in G \setminus \{e\}$.

Let $(\alpha, w_\alpha)$ be a twisted action of $G$ on a unital $C^*$-algebra $A$. Let $E^A$ be the faithful canonical conditional expectation from $A \rtimes_{\alpha, w_\alpha, r} G$ onto $A$ defined in Section 2. In the same way as Zarikian [15 Theorem 3.1.2], we obtain the following proposition.

Proposition 4.1. With the above notation, the following conditions are equivalent:
(1) The conditional expectation $E^A$ is unique,
(2) $A' \cap (A \rtimes_{\alpha, w_\alpha, r} G) = A' \cap A$,
(3) the twisted action $(\alpha, w_\alpha)$ is free.

Proof. (1) $\Rightarrow$ (2): Since $E^A$ is an $A - A$-bimodule map, $E^A(A' \cap (A \rtimes_{\alpha, w_\alpha, r} G)) = A' \cap A$. Let $x$ be a selfadjoint element in $A' \cap (A \rtimes_{\alpha, w_\alpha, r} G)$ with $||x|| < 1$. Then $1 - x$ is a positive invertible element in $A' \cap (A \rtimes_{\alpha, w_\alpha, r} G)$ and $1 - E^A(x)$ is a positive invertible element in $A' \cap A$. Let $F$ be the map from $A \rtimes_{\alpha, w_\alpha, r} G$ onto $A$ defined by $F(a) = E^A((1 - x)x^*a(1 - x)x^*) (1 - E^A(x))^{-1}$. 

\[ F(a) = E^A((1 - x)x^*a(1 - x)x^*) (1 - E^A(x))^{-1}. \]
Then by easy computation, $F$ is a conditional expectation from $A \rtimes_{\alpha,w_A,r} G$ onto $A$. Hence by the assumption, $F = E^A$. Thus

$$E^A(x) = F(x) = E^A((1 - x)x)(1 - E^A(x))^{-1}.$$  

Hence $E^A(x)^2 = E^A(x^2)$. It follows that $x$ is in the multiplicative domain of $E^A$. Thus $E^A|_{A' \cap (A \rtimes_{\alpha,w_A,r} G)}$ is a homomorphism of $A' \cap (A \rtimes_{\alpha,w_A,r} G)$ onto $A' \cap A$ by Størmer [14, Proposition 2.1.5]. Since $E^A$ is faithful, $E^A|_{A' \cap (A \rtimes_{\alpha,w_A,r} G)}$ is injective. Hence $x = E^A(x)$ for any $x \in A' \cap (A \rtimes_{\alpha,w_A,r} G)$ since $E^A(x - E^A(x)) = 0$. Therefore $A' \cap (A \rtimes_{\alpha,w_A,r} G) = A' \cap A$.

(2) $\Rightarrow$ (3): Let $t \in G \setminus \{e\}$ and $x \in A$. We suppose that $xa = \alpha_t(a)x$ for any $a \in A$. Then since $\alpha_t(a) = \delta_t \alpha_t \delta_t^*$, $\delta_t^* xa = a \delta_t^* x$ for any $a \in A$. Hence $\delta_t^* x \in A' \cap (A \rtimes_{\alpha,w_A,r} G)$. By the definition of $E^A$, $E^A(\delta_t^* x) = 0$. On the other hand, since $A' \cap (A \rtimes_{\alpha,w_A,r} G) = A' \cap A$, $E^A(\delta_t^* x) = \delta_t^* x$. Hence $x = 0$.

(3) $\Rightarrow$ (1): We suppose that $(\alpha, w_A)$ is free. Let $F$ be a conditional expectation from $A \rtimes_{\alpha,w_A,r} G$ onto $A$. Let $G \setminus \{e\}$. For any $a \in A$, $F(\delta_t a) = F(\delta_t a) = F(\alpha_t(a) \delta_t) = \alpha_t(a) F(\delta_t)$ since $\alpha_t(a) = \delta_t \alpha_t \delta_t^*$. Since $(\alpha, w_A)$ is free, $F(\delta_t) = 0$. Hence $F = E^A$ since $E^A(\delta_t a) = 0$ for any $a \in A$, $t \in G \setminus \{e\}$.

Let $(\alpha, w_A)$ be a twisted action of $G$ on a $C^*$-algebra $A$ and let $(\alpha, w_A)$ the twisted action of $G$ on the $C^*$-algebra $M(A)$ induced by $(\alpha, w_A)$. Let $E^{M(A)}$ be the faithful canonical conditional expectation from $M(A) \rtimes_{\alpha,w_A,r} G$ onto $M(A)$ defined in Section 2. We note that $E^A = E^{M(A)}|_{A \rtimes_{\alpha,w_A,r} G}$, a conditional expectation from $A \rtimes_{\alpha,w_A,r} G$ onto $A$.

Corollary 4.2. With the above notation, the following conditions are equivalent:

1. $A' \cap M(A \rtimes_{\alpha,w_A,r} G) = C_1$,
2. $A' \cap M(A) = C_1$ and $(\alpha, w_A)$ is free.

Proof. (1) $\Rightarrow$ (2): It is clear that $A' \cap M(A) = C_1$. We show that $(\alpha, w_A)$ is free. Since $M(A) \rtimes_{\alpha,w_A,r} G \subset M(A \rtimes_{\alpha,w_A,r} G)$,

$M(A)' \cap (M(A) \rtimes_{\alpha,w_A,r} G) \subset A' \cap M(A \rtimes_{\alpha,w_A,r} G) = C_1$.

Hence by Proposition 4.1, $(\alpha, w_A)$ is free. Let $t \in G \setminus \{e\}$ and $x \in M(A)$. We suppose that $xa = \alpha_t(a)x$ for any $a \in A$. Then since $\alpha$ is strictly continuous, $xa = \alpha_t(a)x$ for any $a \in M(A)$. Hence $x = 0$ since $(\alpha, w_A)$ is free. Thus $(\alpha, w_A)$ is free. (2) $\Rightarrow$ (1): Let $x \in A' \cap M(A \rtimes_{\alpha,w_A,r} G)$. For any $t \in G$ let $x_t$ be the Fourier coefficient of $x$ at $t \in G \setminus \{e\}$ defined in Section 2. Then for any $a \in A$,

\begin{align*}
x_{t^{-1} a} &= \lim_{i} E^A(x(u_i \delta_t))a = \lim_{i} E^A(x(u_i \delta_t)a) = \lim_{i} E^A(x(u_i \alpha_t(a)) \delta_t) \\
&= E^A(x \alpha_t(a) \delta_t) = E^A(x \alpha_t(a) x \delta_t) = \lim_{i} E^A(\alpha_t(a) u_i x \delta_t) \\
&= \alpha_t(a) \lim_{i} E^A(u_i x \delta_t) = \alpha_t(a) x_{t^{-1}}.
\end{align*}

Since $(\alpha, w_A)$ is free $x_e = 0$ for any $t \in G \setminus \{e\}$. Hence $x = x_e$ by Lemma 2.3 where $x_e$ is regarded as an element in $M(A \rtimes_{\alpha,w_A,r} G)$ and the Fourier coefficient of $x_e$ at any $t \in G \setminus \{e\}$ is zero. Since $A' \cap M(A) = C_1$, $x = x_e \in C_1$. Therefore $A' \cap M(A \rtimes_{\alpha,w_A,r} G) = C_1$.

Lemma 4.3. Let $(\alpha, w_A)$ be a free twisted action of $G$ on a $C^*$-algebra $A$. Then $(\alpha \otimes \text{id}, w_A \otimes 1)$ is a free twisted action of $G$ on $A \otimes K$.

Proof. Let $t \in G \setminus \{e\}$. Let $x$ be an element in $M(A \otimes K)$ satisfying that

\begin{align*}
xy &= (\alpha_t \otimes \text{id})(y)x
\end{align*}
for any \( y \in A \otimes K \). Then we see that
\[
xy = (\alpha_t \otimes \text{id})(y)
\]
for any \( y \in M(A) \otimes K \). Hence we see that
\[
x(1 \otimes e_i) = (1 \otimes e_i)x
\]
for any \( i \in N \). Also, we can see that
\[
x(\alpha_t(a) \otimes e_i) = (\alpha_t(a) \otimes e_i)x
\]
for any \( a \in A, i \in N \). Hence
\[
(1 \otimes e_i)x(1 \otimes e_i)(a \otimes e_i) = (\alpha_t(a) \otimes e_i)(1 \otimes e_i)x(1 \otimes e_i)
\]
for any \( a \in A, i \in N \). Since \((\alpha, w_a)\) is a free twisted action on \( A \) and we can identify \( A \) with \((1 \otimes e_i)(A \otimes K)(1 \otimes e_i)\),
\[
(1 \otimes e_i)x(1 \otimes e_i) = 0
\]
for any \( i \in N \). Thus since \( x(1 \otimes e_i) = 0 \) for any \( i \in N \), \( x = 0 \).

**Lemma 4.4.** Let \((\alpha, w_a)\) be a twisted action of \( G \) on a \( C^*\)-algebra \( A \). If \((\alpha, w_a)\) is free, then \((\alpha, w_a)\) is outer. Furthermore, if \( A' \cap M(A) = C1 \), then the inverse holds.

**Proof.** We suppose that \((\alpha, w_a)\) is free. We suppose that \((\alpha, w_a)\) is not outer. Then there are an element \( t \in G \setminus \{e\} \) and a unitary element \( w \in M(A) \) such that \( \alpha_t = \text{Ad}(w) \). Thus for any \( a \in A \), \( \alpha_t(a) = waw^* \). Hence \( waw = \alpha_t(a)w \) for any \( a \in A \). Since \((\alpha, w_a)\) is free, \( w = 0 \). This is a contradiction. Therefore, \((\alpha, w_a)\) is free. Next, we suppose that \( A' \cap M(A) = C1 \) and that \((\alpha, w_a)\) is outer. We suppose that \((\alpha, w_a)\) is not free. Then there are an element \( t \in G \setminus \{e\} \) and a non-zero element \( x \in M(A) \) such that \( xw = \alpha(a)x \) for any \( a \in A \). Hence \( ax^* = x^*\alpha_t(a) \) for any \( a \in A \). Thus \( x^*x = x^*\alpha_t(a)x = x^*ax^* \) for any \( a \in A \). Hence \( x^*x \in C1 \). Thus \( x^*x \in C1 \). It follows that there is a \( c \in R \) with \( c > 0 \) such that \( x^*x = xx^* = c1 \). Let \( w = \frac{1}{\sqrt{c}}x \). Then \( w \) is a unitary element in \( M(A) \) such that \( \alpha_t(a) = waw^* \) for any \( a \in A \). This is a contradiction. Therefore, \((\alpha, w_a)\) is free. \( \square \)

5. **Strong Morita equivalence for inclusions of \( C^*\)-algebras**

Let \((\beta, w_b)\) be twisted actions of \( G \) on \( \sigma \)-unital \( C^*\)-algebras \( A \) and \( B \) and \( A \rtimes_{\alpha, w_a, r} G \) and \( B \rtimes_{\beta, w_b, r} G \) the twisted reduced crossed products of \( A \) and \( B \) by \((\alpha, w_a)\) and \((\beta, w_b)\), respectively. Then we obtain the inclusions of \( C^*\)-algebras \( A \subset A \rtimes_{\alpha, w_a, r} G \) and \( B \subset B \rtimes_{\beta, w_b, r} G \). We suppose that \( A \subset A \rtimes_{\alpha, w_a, r} G \) and \( B \subset B \rtimes_{\beta, w_b, r} G \) are strongly Morita equivalent. We regard \( K \) as the trivial \( K \)-\( K \)-equivalence bimodule. Then the inclusions of \( C^*\)-algebras \( A \otimes K \subset (A \rtimes_{\alpha, w_a, r} G) \otimes K \) and \( B \otimes K \subset (B \rtimes_{\beta, w_b, r} G) \otimes K \) are also strongly Morita equivalent. Hence in the same way as in the proof of \[2\] Proposition 3.5] or Brown, Green and Rieffel \[2\] Proposition 3.1], there is an isomorphism \( \theta : (A \rtimes_{\alpha, w_a, r} G) \otimes K \) onto \((B \rtimes_{\beta, w_b, r} G) \otimes K \) such that \( \theta|_{A \otimes K} = \text{an isomorphism of } A \otimes K \) onto \( B \otimes K \). Also, by the discussions of Preliminaries, the inclusions of \( C^*\)-algebras \( A \otimes K \subset (A \rtimes_{\alpha, w_a, r} G) \otimes K \) and \( B \otimes K \subset (B \rtimes_{\beta, w_b, r} G) \otimes K \) are isomorphic to \( A \otimes K \subset (A \otimes K) \rtimes_{\alpha \otimes \text{id}, w_a \otimes 1, r} G \) and \( B \otimes K \subset (B \otimes K) \rtimes_{\beta \otimes \text{id}, w_b \otimes 1, r} G \) as inclusions of \( C^*\)-algebras, respectively. Thus \( \theta \) can be regarded as an isomorphism of \((A \otimes K) \rtimes_{\alpha \otimes \text{id}, w_a \otimes 1, r} G \) onto \((B \otimes K) \rtimes_{\beta \otimes \text{id}, w_b \otimes 1, r} G \) such that \( \theta|_{A \otimes K} \) is an isomorphism of \( A \otimes K \) onto \( B \otimes K \). Let \( \theta_s = \theta|_{A \otimes K} \). Let \( \gamma_t = \theta_t \circ (\alpha_t \otimes \text{id}) \circ \theta_t^{-1} \) and \( w_{x}(t, s) = \theta_{x}(w_{x}(t, s) \otimes 1) \) for any \( t, s \in G \).
**Lemma 5.1.** With the above notation, $(\alpha \otimes \id, w_\alpha \otimes 1)$ and $(\gamma, w_\gamma)$ are strongly Morita equivalent.

**Proof.** Let $\theta_t$ be the isomorphism of $A \otimes K$ onto $B \otimes K$ defined as above. Let $X_{\theta_t}$ be the $A \otimes K - B \otimes K$-equivalence bimodule defined in the following way: Let $X_{\theta_t} = A \otimes K$ as vector spaces over $C$. For any $a \in A \otimes K, b \in B \otimes K, x, y \in X_{\theta_t}$, 

$$a \cdot x = ax, \quad x \cdot b = x\theta_t^{-1}(b),$$

$$A \otimes K \langle x, y \rangle = xy^*, \quad \langle x, y \rangle_{B \otimes K} = \theta_t(x^*y).$$

By easy computations, $X_{\theta_t}$ is an $A \otimes K - B \otimes K$-equivalence bimodule. For any $t \in G$, let $\lambda_t$ be the linear automorphism of $X_{\theta_t}$ defined by 

$$\lambda_t(x) = (\alpha_t \otimes \id)(x)$$

for any $x \in X_{\theta_t}$. Then 

$$A \otimes K \langle \lambda_t(x), \lambda_t(y) \rangle = (\alpha_t \otimes \id)(xy^*) = (\alpha_t \otimes \id)(A \otimes K \langle x, y \rangle),$$

$$\langle \lambda_t(x), \lambda_t(y) \rangle_{B \otimes K} = \theta_t((\alpha_t \otimes \id)(x^*y)) = \gamma_t((x^*y)) = \gamma_t((x, y)_{B \otimes K})$$

for any $x, y \in X_{\theta_t}$, $t \in G$. Therefore, we obtain the conclusion. \qed

Also, the twisted $C^*$-dynamical systems $(A \otimes K, G, \alpha \otimes \id, w_\alpha \otimes 1)$ and $(B \otimes K, G, \gamma, w_\gamma)$ are covariantly isomorphic. Thus there is an isomorphism $\pi_\theta$ of $(A \otimes K) \rtimes_{\alpha \otimes \id, w_\alpha \otimes 1, r} G$ onto $(B \otimes K) \rtimes_{\gamma, w_\gamma, r} G$ such that $\pi_\theta|_{A \otimes K} = \theta_t$. Let $\rho$ be the isomorphism of $(B \otimes K) \rtimes_{\beta \otimes \id, w_\beta \otimes 1, r} G$ onto $(B \otimes K) \rtimes_{\gamma, w_\gamma, r} G$ defined by $\rho = \pi_\theta \circ \theta_t^{-1}$. Then for any $b \in B \otimes K$

$$\rho(b) = (\pi_\theta \circ \theta_t^{-1})(b) = \theta_t(\theta_t^{-1}(b)) = b.$$

Thus $\rho|_{B \otimes K} = \id$ on $B \otimes K$. For any $t \in G$, let $u_t^{\beta \otimes \id}$ be a unitary element in $M((B \otimes K) \rtimes_{\gamma, w_\gamma, r} G)$ implementing $\beta_t \otimes \id$. Let $v_t = \rho(u_t^{\beta \otimes \id})$ for any $t \in G$. Then for any $b \in B \otimes K$,

$$v_t b v_t^* = \rho(u_t^{\beta \otimes \id}) \rho(b) \rho(u_t^{\beta \otimes \id*}) = \rho((\beta_t \otimes \id)(b)) = (\beta_t \otimes \id)(b).$$

Hence $v_t b = (\beta_t \otimes \id)(b)v_t$ for any $b \in B \otimes K, t \in G$. Let $\{U_i\}_{i \in I}$ be an approximate unit of $B \otimes K$. Let $\{b^\gamma_i\}_{i \in G}$ be the Fourier coefficients of $v_t$ in $M((B \otimes K) \rtimes_{\gamma, w_\gamma, r} G)$ with respect to the approximate unit $\{U_i\}_{i \in I}$ of $B \otimes K$.

**Lemma 5.2.** With the above notation, for any $t, s \in G, b \in B \otimes K$,

$$b^\gamma_t b = (((\beta_t \otimes \id) \circ \gamma_s^{-1})(b)) b^\gamma_s. \quad (\ast)$$

**Proof.** Let $u^\gamma_s$ be a unitary element in $M((B \otimes K) \rtimes_{\gamma_s, w_\gamma, r} G)$ implementing $\gamma_s$ for any $s \in G$. Let $t \in G$ and $b \in B \otimes K$. Then the Fourier series of $v_t b$ with respect to the approximate unit $\{U_i\}_{i \in I}$ of $B \otimes K$ is:

$$\sum_{s \in G} b^\gamma_s u^\gamma_s b = \sum_{s \in G} b^\gamma_s \gamma_s(b) u^\gamma_s.$$

And the Fourier series of $(\beta_t \otimes \id)(b)v_t$ with respect to the approximate unit $\{U_i\}_{i \in I}$ of $B \otimes K$ is:

$$\sum_{s \in G} (\beta_t \otimes \id)(b) b^\gamma_s u^\gamma_s.$$

Since $v_t b = (\beta_t \otimes \id)(b)v_t$,

$$b^\gamma_t \gamma_s(b) = (\beta_t \otimes \id)(b) b^\gamma_s$$

for any $t, s \in G, b \in B \otimes K$. Since $b$ is an arbitrary element in $B \otimes K$, replacing $b$ by $\gamma_s^{-1}(b)$,

$$b^\gamma_t b = (\beta_t \otimes \id)(\gamma_s^{-1}(b)) b^\gamma_s$$

for any $t, s \in G, b \in B \otimes K$. \qed
We suppose that the actions \((\alpha \otimes \text{id}, w_\alpha \otimes 1)\) and \((\beta \otimes \text{id}, w_\beta \otimes 1)\) are free on \(A \otimes K\) and \(B \otimes K\), respectively and that \((B \otimes K)^t \cap M(B \otimes K) = C_1(b \otimes K)\). Let \(t\) be any element in \(G\). Since \(v_t \neq 0\), there is an element \(s_0 \in G\) such that \(b_{s_0}^t \neq 0\). Since \(s_0 \in G\) is depending on \(t \in G\), we denote it by \(\phi(t)\). Hence \(b_{\phi(t)}^t \neq 0\) and by Lemma [5.2] for any \(b \in B \otimes K\),

\[
b_{\phi(t)}^t b = ((\beta_t \otimes \text{id}) \circ \gamma_{\phi(t)}^{-1})(b)b_{\phi(t)}^t.
\]

Since \((B \otimes K)^t \cap M(B \otimes K) = C_1\), by the proof of Lemma [4.4] there is a unitary element \(w_t \in M(B \otimes K)\) such that

\[
((\beta_t \otimes \text{id}) \circ \gamma_{\phi(t)}^{-1})(b) = w_t b w_t^*
\]

for any \(b \in B \otimes K\). That is,

\[
(\beta_t \otimes \text{id})(b) = w_t \gamma_{\phi(t)}(b) w_t^*
\]

for any \(b \in B \otimes K\). Thus for any \(s \in G\), \(b \in B \otimes K\),

\[
(\beta_t \otimes \text{id})(\gamma_{s^{-1}}(b)) = w_t \gamma_{\phi(t)}(\gamma_{s^{-1}}(b)) w_t^*.
\]

By Equation (**) in Lemma [5.2] and the above equation,

\[
b_{\phi(t)}^t b = (\beta_t \otimes \text{id})(\gamma_{\phi(t)}^{-1}(b))b_s^t
\]

\[
= ((\beta_t \otimes \text{id}) \circ \text{Ad}(w_\gamma(s^{-1}, s)^*) \circ \gamma_{s^{-1}}(b))b_s^t
\]

\[
= [\text{Ad}((\beta_t \otimes \text{id})(w_\gamma(s^{-1}, s)^*)) \circ (\beta_t \otimes \text{id}) \circ \gamma_{s^{-1}}(b)] b_s^t
\]

\[
= [\text{Ad}((\beta_t \otimes \text{id})(w_\gamma(s^{-1}, s)^*)) \circ \text{Ad}(w_t) \circ \gamma_{\phi(t)}(\gamma_{s^{-1}}(b)) b_s^t
\]

\[
= [\text{Ad}((\beta_t \otimes \text{id})(w_\gamma(s^{-1}, s)^*)) w_t \circ \text{Ad}(w_\gamma(\phi(t), s^{-1}) \circ \gamma_{\phi(t)s^{-1}}(b)) b_s^t
\]

\[
= [\text{Ad}((\beta_t \otimes \text{id})(w_\gamma(s^{-1}, s)^*)) w_t \gamma_\phi(t, s^{-1}) \circ \gamma_{\phi(t)s^{-1}}(b)) b_s^t.
\]

Hence

\[
w_\gamma(\phi(t), s^{-1})^* w_t^* (\beta_t \otimes \text{id})(w_\gamma(s^{-1}, s)) b_s^t b
\]

\[
= \gamma_{\phi(t)s^{-1}}(b) w_\gamma(\phi(t), s^{-1})^* w_t^* (\beta_t \otimes \text{id})(w_\gamma(s^{-1}, s)) b_s^t
\]

for any \(s \in G\), \(b \in B \otimes K\). Since \((\alpha \otimes \text{id}, w_\alpha \otimes 1)\) is free, so is \((\gamma, w_\gamma)\). Hence the automorphism \(\gamma_{\phi(t)s^{-1}}\) is free if \(s \neq \phi(t)\). Thus \(b_{\phi(t)}^t \neq 0\) for any \(s \in G\) with \(s \neq \phi(t)\).

Therefore, for any \(t \in G\), there is the unique \(\phi(t) \in G\) such that

\[
b_{\phi(t)}^t \neq 0, \quad \rho(u_t^{\phi(t)} \otimes \text{id}) = u_t = b_{\phi(t)}^t u_{\phi(t)}^t.
\]

By the above discussions, we can regard \(\phi\) as a map on \(G\). We show that \(\phi\) is an automorphism of \(G\).

**Lemma 5.3.** With above notation, \(\phi\) is an automorphism of \(G\).

**Proof.** Let \(t, s \in G\). Then

\[
v_t v_s = \rho(u_t^{\phi(t)} \otimes u_s^{\phi(s)}) = \rho((w_\beta(t, s) \otimes 1) u_t^{\phi(t)} \otimes \text{id}) = \rho(w_\beta(t, s) \otimes 1) v_{ts}.
\]

Since \(\rho(b) = b\) for any \(b \in B \otimes K\), \(\rho(w_\beta(t, s) \otimes 1) = w_\beta(t, s) \otimes 1\). Hence \(v_t v_s = (w_\beta(t, s) \otimes 1) v_{ts}\).

Thus

\[
b_{\phi(t)}^t u_{\phi(t)}^t b_{\phi(s)}^t u_{\phi(s)}^t = (w_\beta(t, s) \otimes 1) b_{\phi(ts)}^s u_{\phi(ts)}^s.
\]

Here

\[
b_{\phi(t)}^t u_{\phi(t)}^t b_{\phi(s)}^t u_{\phi(s)}^t = b_{\phi(t)}^t \gamma_{\phi(t)}(b_{\phi(s)}^s u_{\phi(s)}^s u_{\phi(ts)}^s)
\]

\[
= b_{\phi(t)}^t \gamma_{\phi(t)}(b_{\phi(s)}^s u_{\phi(s)}^s u_{\phi(ts)}^s)
\]

Thus

\[
b_{\phi(t)}^t(\gamma_{\phi(ts)}(b_{\phi(s)}^s u_{\phi(s)}^s u_{\phi(ts)}^s)) w_{\gamma}(\phi(t), \phi(s)) u_{\phi(ts)}^s = (w_\beta(t, s) \otimes 1) b_{\phi(ts)}^s u_{\phi(ts)}^s.
\]
We note that the both sided elements in the above equation are the Fourier series of $v_t$, with respect to the approximate unit $\{U_i\}_{i \in I}$ of $B \otimes K$ and that the non-zero Fourier coefficient is uniquely determined by the above discussions. Thus we obtain that

$$u_{\phi(t)}^{\gamma}(s) = u_{\phi(t)}^{\gamma}(s),$$

$$b_{\phi(t)}^{t}(\sum_{s} (b_{\phi(s)}^{s} w_{\gamma}(\phi(t), \phi(s))) = (w_{\beta}(t, s) \otimes 1)b_{\phi(ts)}^{t}$$

for any $t, s \in G$. Hence $\phi(t)\phi(s) = \phi(ts)$ for any $t, s \in G$ and $\phi$ is a homomorphism of $G$ to $G$. We show that $\phi$ is bijective. Let $t \in G \setminus \{e\}$. We suppose that $\phi(t) = e$. By the definition of $\phi$, $\rho(\phi_\beta^{\beta, id}) = v_t = b_c^t u_c^t = b_c^t$ since $u_c^t = 1$. By Equation $(\star)$ in Lemma [5.2]

$$b_c^t b = (\beta_t \otimes id)(b)b_c^t$$

for any $b \in B \otimes K$. Since $\beta_t \otimes id$ is a free automorphism of $B \otimes K$, $b_c^t = 0$. Hence $v_t = 0$. This is a contradiction. Thus $\phi(t) \neq e$, that is, $\phi$ is injective. Also, we note that $(B \otimes K) \rtimes \gamma_{t, w_c} \ast r \ast G$ is the closed linear span of the set

$$\{v_t \mid b \in B \otimes K, t \in G\} = \{b b_{\phi(t)}^{t} u_{\phi(t)}^{\gamma} \mid b \in B \otimes K, t \in G\}$$

This implies that $\phi$ is surjective. Therefore, we can see that $\phi$ is an automorphism of $G$.

**Lemma 5.4.** With the above notation, $(\beta \otimes id, w_\gamma)$ and $(\gamma^0, w_\beta^0)$ are exterior equivalent, where $(\gamma^0, w_\beta^0)$ is the twisted action of $G$ on $B \otimes K$ defined by $\gamma_t^0(b) = \gamma_{\phi(t)}(b)$ and $w_{\gamma}^{t}(t, s) = w_{\gamma}(\phi(t), \phi(s))$ for any $t, s \in G$, $b \in B \otimes K$.

**Proof.** By the discussions before Lemma [5.3], $v_t = b_{\phi(t)}^{t} u_{\phi(t)}^{\gamma}$ for any $t \in G$. Since $v_t$ and $u_{\phi(t)}^{\gamma}$ are unitary elements in $M(B \otimes K)$, $b_{\phi(t)}^{t}$ is a unitary element in $M(B \otimes K)$ for any $t \in G$. Also, for any $b \in B \otimes K$, $t \in G$

$$(\beta_t \otimes id)(b) = \gamma(b) u_{\phi(t)}^{r} = b_{\phi(t)}^{t} u_{\phi(t)}^{\gamma} b_{\phi(t)}^{s} w_{\gamma}(\phi(t), \phi(s)) = (Ad(b_{\phi(t)}^{t}) \gamma_{\phi(t)}(b))b_{\phi(t)}^{r}$$

And by Equation $(\star\star)$ in the proof of Lemma [5.3]

$$b_{\phi(t)}^{t}(\sum_{s} (b_{\phi(s)}^{s} w_{\gamma}(\phi(t), \phi(s))) = (w_{\beta}(t, s) \otimes 1)b_{\phi(ts)}^{t}$$

for any $t, s \in G$. Therefore, $(\beta, w_\beta)$ and $(\gamma^0, w_\gamma^0)$ are exterior equivalent.

We give the main theorem.

**Theorem 5.5.** Let $(\alpha, w_\alpha)$ and $(\beta, w_\beta)$ be twisted actions of a countable discrete group $G$ on $\sigma$-unital $C^*$-algebras $A$ and $B$ and $A \rtimes_{\alpha, w_\alpha} G$ and $B \rtimes_{\beta, w_\beta} G$ the twisted reduced crossed products of $A$ and $B$ by $(\alpha, w_\alpha)$ and $(\beta, w_\beta)$, respectively. We suppose that $A' \cap M(A) \rtimes_{\alpha, w_\alpha} G$ is $C_1$. If the inclusions of $C^*$-algebras $A \subset A \rtimes_{\alpha, w_\alpha} G$ and $B \subset B \rtimes_{\beta, w_\beta} G$ are strongly Morita equivalent, then there is an automorphism $\phi$ of $G$ such that $(\alpha^0, w_\alpha^0)$ and $(\beta, w_\beta)$ are strongly Morita equivalent, where $(\alpha^0, w_\alpha^0)$ is the twisted action of $G$ on $A$ defined by $\alpha^0_{\phi(t)}(a) = \alpha_{\phi(t)}(a)$ and $w_{\alpha}^{t}(t, s) = w_{\alpha}(\phi(t), \phi(s))$ for any $t, s \in G$, $a \in A$.

**Proof.** Since $A' \cap M(A) \rtimes_{\alpha, w_\alpha} G = C_1$, by Lemma [5.4], $B' \cap M(B \rtimes_{\beta, w_\beta} G) = C_1$. Also, by Corollary [5.2], $A' \cap M(A) = C_1$, $B' \cap M(B) = C_1$ and $(\alpha, w_\alpha)$, $(\beta, w_\beta)$ are free. Thus by Lemma [5.4], $B \otimes K \rtimes_{\beta, w_\beta} G$ is $C_1$. Furthermore, by Lemma [5.3], $(\beta \otimes id, w_\beta)$ and $(\beta \otimes id, w_\beta)$ are free. On the other hand, since $A \subset A \rtimes_{\alpha, w_\alpha} G$ and $B \subset B \rtimes_{\beta, w_\beta} G$ are strongly Morita equivalent, by the discussions before Lemma [5.2] there is an isomorphism $\theta$ of $(A \otimes K) \rtimes_{\alpha, w_\alpha \otimes 1, r} G$ onto $(B \otimes K) \rtimes_{\alpha, w_\alpha \otimes 1, r} G$ such that $\theta|_{A \otimes K}$ is an isomorphism of $A \otimes K$ onto...
Let $\theta_r = \theta_r^t \vert_{A \otimes K}$ and $\gamma_l = \theta_r \circ (\alpha \otimes \text{id}) \circ \theta_r^{-1}$, $w_{\gamma}(t, s) = \theta_r \nu_{\alpha}(t, s) \otimes 1$ for any $t, s \in G$. Then by Lemmas 5.3 and 5.4, there is an automorphism $\phi$ of $G$ such that $(\beta \otimes \text{id}, w_\phi)$ is the twisted action of $G$ on $B \otimes K$ defined by $\gamma_{\phi}(b) = \gamma_{\alpha}(b)$ and $w_\phi^\gamma(t, s) = w_{\gamma}(\phi(t), \phi(s))$ for any $t, s \in G$ and $b \in B \otimes K$. We note that $(\gamma, w_\gamma)$ and $(\alpha \otimes \text{id}, w_{\alpha} \otimes 1)$ are strongly Morita equivalent by Lemma 5.1. Hence $(\alpha^\phi \otimes id, w_{\phi}^\gamma)$ and $(\beta \otimes id, w_{\phi} \otimes 1)$ are strongly Morita equivalent. Furthermore, $(\alpha^\phi, w_{\alpha}^\phi)$ and $(\beta, w_{\beta})$ are strongly Morita equivalent to $(\alpha^\phi \otimes id, w_{\phi}^\gamma \otimes 1)$ and $(\beta \otimes id, w_{\beta} \otimes 1)$, respectively. Therefore, $(\alpha^\phi, w_{\alpha}^\phi)$ and $(\beta, w_{\beta})$ are strongly Morita equivalent.

In the similar way to the proof of Theorem 5.5, we obtain the following.

**Corollary 5.6.** Let $(\alpha, w_{\alpha})$ and $(\beta, w_{\beta})$ be free twisted actions of a countable discrete group $G$ on $\sigma$-unital $C^*$-algebras $A$ and $B$ and $A \rtimes_{\alpha, w_{\alpha}} G$ and $B \rtimes_{\beta, w_{\beta}} G$ the twisted full crossed products of $A$ and $B$ by $(\alpha, w_{\alpha})$ and $(\beta, w_{\beta})$, respectively. We suppose that $A' \cap M(A) = C1$ and $B' \cap M(B) = C1$. If $A \subset A \rtimes_{\alpha, w_{\alpha}} G$ and $B \subset B \rtimes_{\beta, w_{\beta}} G$ are strongly Morita equivalent, then there is an automorphism of $\phi$ of $G$ such that $(\alpha^\phi, w_{\alpha}^\phi)$ and $(\beta, w_{\beta})$ are strongly Morita equivalent.

**References**

[1] E. Bédos and R. Conti, *On discrete twisted C*-dynamical systems, Hilbert C*-modules and regularity*, preprint, arXiv: 1104.1738v1.

[2] L. G. Brown, P. Green and M. A. Rieffel, *Stable isomorphism and strong Morita equivalence of C*-algebras*, Pacific J. Math., 71 (1977), 349–363.

[3] L. G. Brown, J. Mingo and N-T. Shen, *Quasi-multipliers and embeddings of Hilbert C*-bimodules*, Can. J. Math., 46 (1994), 1150–1174.

[4] M. Choda and H. Kosaki, *Strongly outer actions for an inclusion of factors*, J. Func. Anal., 122 (1994), 315-332.

[5] F. Combes, *Crossed products and Morita equivalence*, Proc. London Math. Soc., 49 (1984), 289-306.

[6] R. E. Curto, P. S. Muhly and D. P. Williams, *Cross products of strong Morita equivalent C*-algebras*, Proc. Amer. Math. Soc., 90 (1984), 528–530.

[7] K. K. Jensen and K. Thomsen, *Elements of KK-theory*, Birkhäuser, 1991.

[8] M. Izumi, *Inclusions of simple C*-algebras*, J. reine angew. Math., 547 (2002), 97–138.

[9] K. Kodaka, *The Picard groups for unital inclusions of unital C*-algebras*, preprint, arXiv:1712.09499v1, Acta Math., to appear.

[10] K. Kodaka and T. Teruya, *The strong Morita equivalence for coactions of a finite dimensional C*-Hopf algebra on unital C*-algebras*, Studia Math., 228 (2015), 259–294.

[11] K. Kodaka and T. Teruya, *The strong Morita equivalence for inclusions of C*-algebras and conditional expectations for equivalence bimodules*, J. Aust. Math. Soc., 105 (2018), 103–144.

[12] K. Kodaka and T. Teruya, *Coactions of a finite dimensional C*-Hopf algebra on unital C*-algebras, unital inclusions of unital C*-algebras and the strong Morita equivalence*, preprint, arXiv:1706.00430.

[13] G. K. Pedersen, *C*-algebras and their automorphism groups*, Academic Press, London, New York, Sn Francisco, 1979.

[14] E. Størmer, *Positive linear maps of operator algebras*, Springer-Verlag, Berlin Heidelberg, 2013.

[15] V. Zarikian, *Unique expectations for discrete crossed products*, preprint, arXiv:1707.09339v1.

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