Multitype branching processes with immigration in random environment and polling systems

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Abstract

For multitype branching processes with immigration evolving in a random environment and producing a final product we find the tail distribution of the size of the final product accumulated in the system for a life period. Using this result we investigate the tail distribution of the busy periods of the branching type polling systems with random service disciplines and random positive switch-over times.

1 Introduction

Branching processes with and without immigration are powerful tools in studying various models of queueing systems (see, for instance, [25],[26],[27],[28],[35], [36], [41] and [43]). In this paper we use multitype branching processes with accumulation of a final product and immigration which evolve in random environment (MBPFPIRE) to study the tail distribution of busy periods of a class of polling systems in which input parameters, service disciplines and the distributions of switch-over times vary in a random manner. This article complements the results of paper [16] established for polling systems with zero switch-over times.

The paper is organized as follows. In Sections 2-6 we established various results related to MBPFPIRE. These results are applied in Section 7 to investigate the tail distribution and moments of the branching type polling systems with positive switch-over times.

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2 Branching processes in random environment with final product

Let \( s := (s_1, \ldots, s_m) \in [0, 1]^m \) be a \( m \)-dimensional variable,
\[
s^k := s_1^{k_1} \cdots s_m^{k_m}, k_i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},
\]
and \((\xi; \varphi)\) be a \((m + 1)\)-dimensional vector, where the vector \( \xi := (\xi_1, \ldots, \xi_m) \) has integer-valued nonnegative random variables as components and \( \varphi \) is a nonnegative random variable, and let
\[
F(s; \lambda) := \mathbb{E} \left[ s^\xi e^{-\lambda \varphi} \right] = \mathbb{E} \left[ s_1^{\xi_1} s_2^{\xi_2} \cdots s_m^{\xi_m} e^{-\lambda \varphi} \right], s \in [0, 1]^m, \lambda \geq 0,
\]
be the respective mixed probability generating function (m.p.g.f.). Denote \( \mathcal{F} := \{F(s; \lambda)\} \) the set of all such m.p.g.f.'s and let
\[
\mathcal{F}^m := \left\{ F(s; \lambda) = \left( F^{(1)}(s; \lambda), \ldots, F^{(m)}(s; \lambda) \right) : F^{(i)}(s; \lambda) \in \mathcal{F}, i = 1, \ldots, m \right\}
\]
be the \( m \)-times direct product of \( \mathcal{F} \). Let, further,
\[
\mathcal{F}_0 := \left\{ f(s) = F(s; 0) = \mathbb{E} \left[ s_1^{\xi_1} s_2^{\xi_2} \cdots s_m^{\xi_m} \right] : F(s; \lambda) \in \mathcal{F} \right\}
\]
be the set of all ordinary probability generating functions (p.g.f.'s) and
\[
\mathcal{F}_0^m := \left\{ f(s) = \left( f^{(1)}(s), \ldots, f^{(m)}(s) \right) : f^{(i)}(s) \in \mathcal{F}_0, i = 1, \ldots, m \right\}
\]
be the set of all \( m \)-dimensional (vector-valued) p.g.f.'s.

Let, further, \((\eta; \psi)\) be a \((m + 1)\)-dimensional tuple where the vector \( \eta := (\eta_1, \ldots, \eta_m) \) has integer-valued nonnegative random components and \( \psi \) is a nonnegative random variable and let
\[
G(s; \lambda) := \mathbb{E} \left[ s^\eta e^{-\lambda \psi} \right] = \mathbb{E} \left[ s_1^{\eta_1} s_2^{\eta_2} \cdots s_m^{\eta_m} e^{-\lambda \psi} \right], s \in [0, 1]^m, \lambda \geq 0,
\]
be the respective m.p.g.f. Denote \( \mathcal{G} := \{G(s; \lambda)\} \) the set of all such m.p.g.f.'s (which, of course, is equivalent to \( \mathcal{F} \)). Let, further,
\[
\mathcal{G}_0 := \{g(s) = G(s; 0) = \mathbb{E} \left[ s_1^{\eta_1} s_2^{\eta_2} \cdots s_m^{\eta_m} \right], G(s; \lambda) \in \mathcal{G} \}
\]
be the set of all ordinary probability generating functions (p.g.f.'s).

Assume that a probability measure \( \mathbb{Q} \) is specified on the natural \( \sigma \)-algebra \( \mathcal{A} \) generated by the subsets of \( \mathcal{F}^m \times \mathcal{G} \). Let \( \mathbf{H}(s; \lambda), \mathbf{H}_0(s; \lambda), \mathbf{H}_1(s; \lambda), \ldots, \mathbf{H}_k(s; \lambda), \ldots \) where
\[
\mathbf{H}(s; \lambda) := (F(s; \lambda); G(s; \lambda)) = \left( F^{(1)}(s; \lambda), \ldots, F^{(m)}(s; \lambda); G(s; \lambda) \right),
\]
and
\[
\mathbf{H}_n(s; \lambda) := (F_n(s; \lambda); G_{n+1}(s; \lambda)) = \left( F_n^{(1)}(s; \lambda), \ldots, F_n^{(m)}(s; \lambda); G_{n+1}(s; \lambda) \right),
\]
be the set of all ordinary probability generating functions (p.g.f.'s).
$n = 0, 1, 2, \ldots$ be a sequence of vector-valued m.p.g.f.'s selected from $\mathcal{F}^m \times \mathcal{G}$ in an iid manner in accordance with measure $\mathcal{Q}$. The sequence $\{H_n, n \in \mathbb{N}_0\}$ is called a random environment. With each m.p.g.f. $F_n^{(i)}(s;\lambda)$ we associate a random vector of offsprings $\xi_i(n) := (\xi_{i1}(n), \xi_{i2}(n), \ldots, \xi_{in}(n))$ and a random variable $\varphi_i(n)$ such that

$$F_n^{(i)}(s;\lambda) := \mathbb{E} \left[ s^{\xi_i(n)} e^{-\lambda \varphi_i(n)} \right] = \mathbb{E} \left[ s_{1}^{\xi_{i1}(n)} s_{2}^{\xi_{i2}(n)} \ldots s_{m}^{\xi_{im}(n)} e^{-\lambda \varphi_i(n)} \right] \overset{d}{=} F^{(i)}(s;\lambda)$$

and with each m.p.g.f. $G_n(s;\lambda)$ we associate a random vector of immigrants $\eta(n) := (\eta_1(n), \eta_2(n), \ldots, \eta_m(n))$ and a random variable $\psi(n)$ such that

$$G_n(s;\lambda) := \mathbb{E} \left[ s^{\eta(n)} e^{-\lambda \psi(n)} \right] = \mathbb{E} \left[ s_{1}^{\eta_{1}(n)} s_{2}^{\eta_{2}(n)} \ldots s_{m}^{\eta_{m}(n)} e^{-\lambda \psi(n)} \right] \overset{d}{=} G(s;\lambda).$$

Now we may give an informal description of a multitype branching processes with accumulation of a final product and immigration which evolve in random environment (MBPFPIRE)

$$T(n) = (V(n); \Theta(n)), n \in \mathbb{N}_0,$$

which may be treated as a process describing the evolution of a $m-$type population of particles with immigration and accumulation of a final product.

Given an environment $\{H_n, n \in \mathbb{N}_0\}$ the starting conditions for the process are: a vector of particles $V(0) = (V_1(0), \ldots, V_m(0))$ (may be random or equal zero), where $V_i(0)$ denotes the number of particles of type $i \in \{1, \ldots, m\}$ in the process at moment 0, and an amount $\Theta(0)$ (may be random or equal zero) of a final product. All the initial particles have the unit life-length and just before the death produce children and some amount of the final product independently of each other. A particle, say, of type $i$, produces at the end of her life particles of different types and adds some amount of the final product to the existing amount of the final product in accordance with m.p.g.f. $F_0^{(i)}(s;\lambda)$. In addition, a random tuple of immigrants specified by the vector $\eta(1) =: (\eta_1(1), \eta_2(1), \ldots, \eta_m(1))$ arrives to the system at moment 1, where $\eta_j(1)$ is the number of type $j$ particles immigrating in the first generation of the population, and the final product of size $\psi(1)$ is added to the system with the joint distribution specified by the m.p.g.f. $G_1(s;\lambda)$. Thus, the amount of the final product $\Theta(1)$ accumulated in the system to this moment is

$$\Theta(1) = \Theta(0) + \psi(1) + \sum_{i=1}^{m} \sum_{k=1}^{V_i(0)} \varphi_i(0, k),$$

where $\varphi_i(0, k)$ – is the amount of the final product added to the system at the death moment of the $k$-particle of type $i$ among those which existed at moment 0.

The newborn particles and immigrants entering the system at moment $n = 1$ constitute the first generation of the MBPFPIRE, have the unit life-length and
dying produce, independently of each other and of the prehistory of the process, offsprings and final product in accordance with their types and subject to the m.p.g.f. \( F^{(i)}_k(s;\lambda), i = 1, 2, \ldots, m \). In addition, at moment \( n = 2 \) immigrants and some amount of final product described by a tuple \((\eta(2); \psi(2)) =: (\eta_1(2), \eta_2(2), \ldots, \eta_m(2); \psi(2))\) is contribute to the process, which is specified by the m.p.g.f. \( G_2(s;\lambda) \). And so on.

Note that in our model \( \psi(n) \) may be positive even if \( \eta(n) = 0 \).

**Definition 1** A \( m \)-type Galton-Watson branching process

\[
T_{\varphi, \psi}(n) = T(n) := (V(n); \Theta(n)) = (V_1(n), \ldots, V_m(n); \Theta(n)), \quad n \in \mathbb{N}_0
\]

with immigration and final product \((\varphi, \psi)\) in a fixed (but picked at random) environment \( \{H_n, n \in \mathbb{N}_0\} \) is a time-inhomogeneous Markov process with the state space

\[
\mathbb{N}_0^m \times \mathbb{R}_+ := \{z = (z_1, \ldots, z_m; w), \ z_i \in \mathbb{N}_0; w \in [0, \infty)\}
\]

defined as

\[
E\left[ s^{V(n+1)} e^{-\lambda \Theta(n+1)} | H_0, \ldots, H_n; T(0), \ldots, T(n) \right] = e^{-\lambda \Theta(n)} (F_n(s;\lambda))^{V(n)} G_{n+1}(s;\lambda).
\]

Thus,

\[
T(n+1) = (V(n+1); \Theta(n+1))
\]

\[
= (0; \Theta(n)) + (\eta(n+1); \psi(n+1)) + \sum_{i=1}^m \sum_{k=1}^{V_i(n)} (\xi_i(n; k); \varphi_i(n; k)),
\]

where \((\xi_i(n; k); \varphi_i(n; k))\) is a random vector representing the offspring vector and the size of the final product contributed to the process at the death moment of the \( k \)-th particle of type \( i \) of the \( n \)-th generation. Given the environment and \( V(n) \) the tuple

\[
(\xi_i(n; k); \varphi_i(n; k)), \ k = 1, 2, \ldots, V_i(n), i \in \{1, \ldots, m\}, \ (\eta(n+1), \psi(n+2)), \ n \in \mathbb{N}_0;
\]

consists of independent vectors and, moreover, for each \( n \in \mathbb{N}_0 \) and \( i \in \{1, \ldots, m\} \) the vectors

\[
(\xi_i(n; k); \varphi_i(n; k)), \ k = 1, 2, \ldots, V_i(n)
\]

are identically distributed: \((\xi_i(n; k); \varphi_i(n; k)) \sim (\xi_i(n); \varphi_i(n))\).

In the sequel we write for brevity for fixed \( n \in \mathbb{N}_0 \)

\[
P_{H_n}(\cdot) := P_{H_n}(\cdot | H_n), \ E_{H_n}[\cdot] := E[\cdot | H_n],
\]

\[
P_H(\cdot) := P_H(\cdot | H_0, \ldots, H_n), \ E_H[\cdot] := E[\cdot | H_0, \ldots, H_n]
\]

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or (where it will cause no confusion)

\[ P_H (\cdot) := P_H (\cdot | H_0, \ldots, H_n, \ldots), \quad E_H [\cdot] := E [\cdot | H_0, \ldots, H_n, \ldots] \]

Similar meaning will have the notation \( P_{F_n}, E_{F_n}, P_F, E_F \) and so on.

Observe that if \( \Theta(0) = 0 \) and \( \varphi_i(n; k) \equiv 1, \psi(n) \equiv 0, \ n \in \mathbb{N}_0 \), then \( \Theta(N) - 1 \) is the total number of particles born and immigrating in the process within generations \( 0, 1, \ldots, N - 1 \); if

\[
\varphi_i(n; k) = I \left\{ \sum_{j=1}^m \xi_{ij}(n; k) \geq t \right\}, \ \psi(n + 1) \equiv 0, \ n \in \mathbb{N}_0 
\]

for some positive integer \( t \) (here and in what follows \( I \{ A \} \) stands for the indicator of the event \( A \)) and \( \Theta(0) = 0 \), then \( \Theta(N) \) is the total number of particles of all types in generations \( 0, 1, \ldots, N - 1 \) each of which has at least \( t \) children, and so on.

Letting \( \lambda = 0 \) in Definition 1 we arrive to the definition of the multitype branching process with immigration evolving in random environment (MBPIRE) which we call the underlying MBPIRE for the initial MBPIFPRE.

**Definition 2** A \( m \)-type Galton-Watson branching process with immigration \( V(n) \) in a fixed (but picked at random) environment \( \{(f_n, g_{n+1}), n \in \mathbb{N}_0\} \) is a time-inhomogeneous Markov chain with the state space

\[ \mathbb{N}_0^m := \{z = (z_1, \ldots, z_m), z_i \in \mathbb{N}_0\} \]

defined as \( V(0) = z \) and for \( n \in \mathbb{N}_0 \)

\[
E \left[ s^{V(n+1)} | f_0, g_1, \ldots, f_n, g_{n+1}; V(0), \ldots, V(n) \right] = (f_n(s))^{V(n)} g_{n+1}(s). \quad (1)
\]

Thus,

\[
V(n + 1) = \sum_{i=1}^m \sum_{k=1} V_i(n) \xi_i(n; k) + \eta(n + 1),
\]

where, given the environment and fixed \( V(n) \), the tuple

\[ \xi_i(n; k), k = 1, 2, \ldots V_i(n); i = 1, 2, \ldots, m; \eta(n + 1) \]

consists of independent random variables.

In the sequel we need the definition of the ordinary multitype branching process with final product which evolves in random environment (MBPFPRE).

**Definition 3** A \( m \)-type Galton-Watson branching process

\[ R_{\varphi}(n) = R(n) := (Z(n); \Phi(n)) = (Z_1(n), \ldots, Z_m(n); \Phi(n)), n \in \mathbb{N}_0, \]

\[ P^{\Phi}_0, E^{\Phi}_0, P^-_\varphi, E^-_\varphi \]
with final product $\varphi$ in a fixed (but picked at random) environment $\{F_n, n \in \mathbb{N}_0\}$ is a time-inhomogeneous Markov process with the state space

$$\mathbb{N}_0^m \times \mathbb{R}_0 := \{z = (z_1, \ldots, z_m; w), z_i \in \mathbb{N}_0; w \in [0, \infty)\}$$

defined as

$$R(0) = (Z(0); \Phi(0)) = (z; \varphi_0),$$

$$\mathbb{E} \left[ s^{Z(n+1)} e^{-\lambda \Phi(n+1)} | F_0, \ldots, F_n; R(0), \ldots, R(n) \right] = e^{-\lambda \Phi(n)} (F_n(s; \lambda))^{Z(n)}.$$

Note that the initial value $R(0)$ may be random. Thus,

$$\Phi(n + 1) = \Phi(0) + \sum_{l=0}^{n} \sum_{i=1}^{m} \sum_{k=1}^{Z_i(l)} \varphi_i(l; k)$$

where $(\xi_i(n; k); \varphi_i(n; k))$ is a random vector representing the offspring vector and the size of the final product of the $k$-th particle of type $i$ of the $n$-th generation of the process.

Finally, excluding immigration and final product we obtain the definition of the ordinary underlying multitype branching process in random environment.

**Definition 4** A $m$-type Galton-Watson process

$$Z(n) = (Z_1(n), \ldots, Z_m(n)), n \in \mathbb{N}_0$$
in a fixed (but selected at random) environment $\{f_n, n \in \mathbb{N}_0\}$ is a time-inhomogeneous Markov chain with the state space

$$\mathbb{N}_0^m := \{z = (z_1, \ldots, z_m), z_i \in \mathbb{N}_0\}$$
defined as

$$Z(0) = z, \mathbb{E} \left[ s^{Z(n+1)} | f_0, \ldots, f_n; Z(0), \ldots, Z(n) \right] = (f_n(s))^{Z(n)}.$$
Let \( e_i := (0, \ldots, 1, \ldots, 0) \) be a \( m \)–dimensional vector with zero components except the \( i \)-th equal to 1, 0 and 1 are \( m \)–dimensional vectors all whose components are equal to 0 and 1, respectively.

Denote

\[
A_n = (a_{ij}(n))_{i,j=1}^m := \left( \left. \frac{\partial f_n^{(i)}(s)}{\partial s_j} \right|_{s=1} \right)_{i,j=1}^m
\]

the mean matrix of the vector-valued p.g.f. \( f_n \),

\[
C_n := (E_{F_n} \varphi_1(n), \ldots, E_{F_n} \varphi_1(n))' = \left( \left. \frac{dF_n^{(1)}(1, \lambda)}{d\lambda} \right|_{\lambda=0}, \ldots, \left. \frac{dF_n^{(m)}(1, \lambda)}{d\lambda} \right|_{\lambda=0} \right)'
\]

the mean vector of the amount of the final product of particles of the \( n \)-th generation and

\[
B_n := (B_1(n), \ldots, B_m(n))' := \left( \left. \frac{\partial g_n(s)}{\partial s_1} \right|_{s=1}, \ldots, \left. \frac{\partial g_n(s)}{\partial s_m} \right|_{s=1} \right)
\]

\[
D_n := E_{G_n} \psi(n) = \frac{dG_n(1, \lambda)}{d\lambda} \bigg|_{\lambda=0}
\]

the respective characteristics for immigrants.

By our assumptions the tuples \((A_n, B_{n+1}, C_n, D_{n+1}), n \in \mathbb{N}_0\) are iid:

\[
(A_n, B_{n+1}, C_n, D_{n+1}) \overset{d}{=} (A, B, C, D).
\]

For vectors \( u = (u_1, \ldots, u_m) \) and \( v = (v_1, \ldots, v_m)' \in \mathbb{R}^m \) denote

\[
\langle u, v \rangle := \sum_{k=1}^m u_k v_k
\]

their inner product.

For a \( m \times m \) matrix \( A = (a_{ij})_{i,j=1}^m \) and a \( m \)–dimensional vector \( u = (u_1, \ldots, u_m) \) introduce the norms

\[
\|A\| := \sum_{i,j=1}^m |a_{ij}|, \quad \|u\| := \sum_{i=1}^m |u_i|
\]

and

\[
\|A\|_2 := \sqrt{\sum_{i,j=1}^m |a_{ij}|^2}, \quad \|u\|_2 := \sqrt{\sum_{i=1}^m |u_i|^2}.
\]

Let, further,

\[
\Pi_{l,n} := \prod_{i=l}^{n-1} A_i, \quad 1 \leq l \leq n,
\]

with the agreement that \( \Pi_{n,n} := E \) is the unit \( m \times m \) matrix.
3 Statement of main results

Denote $D := \{ x > 0 : \mathbb{E} \| A \|^x < \infty \}$ and, for a given $x \geq 0$, set
\[
s(x) := \lim_{n \to \infty} (\mathbb{E} \| A_{n-1} \cdots A_0 \|^x)^{1/n} = \lim_{n \to \infty} (\mathbb{E} \| \Pi_{0,n} \|^x)^{1/n}
\]
and let
\[
\alpha := s'(0) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \log \| A_{n-1} \cdots A_0 \| = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \log \| \Pi_{0,n} \|
\]
be the top Lyapunov exponent for this sequence of matrices.

It is known that the limits in (8) and (9) exist and, moreover, $s(x)$ is a log-convex continuous function in $D$ (see, for instance, [34]). Put
\[
\kappa := \inf \{ x > 0 : s(x) > 1 \}
\]
and $\kappa = \infty$ if $s(x) \leq 1$ for all $x > 0$. Observe that $s(0) = 1$ and, therefore, $\kappa = 0$ if $\alpha > 0$ and $\kappa \in (0, \infty]$ if $\alpha < 0$.

In the last case (which will be our main concern) the series $\sum_{n=0}^{\infty} \mathbb{E} \| \Pi_{0,n} \|^x$ converges if $0 < x < \kappa$ and diverges if $x > \kappa$.

In what follows we call the underlying MBPRE subcritical if $\alpha < 0$ and supercritical if $\alpha > 0$. The same terminology we keep for the respective MBPIRE.

Now we formulate important statements concerning properties of MBPRE and MBPIRE.

Let
\[
q_i(f) := \lim_{n \to \infty} \mathbb{P}_f (\| Z(n) \| = 0 | Z(0) = e_i)
\]
be the extinction probability of a MBPRE initiated at time 0 by a single individual of type $i$ and
\[
q(f) := (q_1(f), \ldots, q_m(f)).
\]

**Theorem 5 ([40])** If the mean matrices of a MBPRE meets the condition
\[
\mathbb{E} \log^+ \| A \| < \infty
\]
and there exists a positive integer $L$ such that
\[
\mathbb{P} \left( \min_{1 \leq i,j \leq m} (A_{L-1}A_{L-2} \cdots A_0)_{ij} > 0 \right) = 1
\]
and $1 \leq l \leq m$ such that
\[
\mathbb{E} \log (1 - \mathbb{P}_f (Z_l(L) = 0 | Z(0) = e_i)) < \infty,
\]
then, for $\alpha$ specified by (9)
1) $\alpha < 0$ implies $\mathbb{P}(q(f) = 1) = 1$;
2) $\alpha > 0$ implies $\mathbb{P}(q(f) < 1) = 1$ and
\[
\mathbb{P}_f \left( \lim_{n \to \infty} n^{-1} \log \| Z(n) \| = \alpha | Z(0) = e_i \right) = 1 - q_i(f)
\]
with probability 1 for $1 \leq i \leq m$. 

8
The next statement deals with MBPIRE.

**Theorem 6** ([32] and [37]) Let a MBPIRE satisfy condition (11), $\alpha < 0$ and $E \log^+ \|B\| < \infty$.

Then, for any $v \in \mathbb{N}_0^m$ the limit

$$
\lim_{n \to \infty} P(V(n) = v) =: D(v)
$$

exists and defines a probability distribution on $\mathbb{N}_0^m$. If, in addition, there exists $L \geq 1$ such that

$$
P(P_g(V(L) = 0|V(0) = 1) > 0 > 0
$$

then $D(0) > 0$ and, therefore,

$$
P(V(n) = 0 \ i.o. \ ) = 1.
$$

Introduce the set

$$
U_+ = \{u = (u_1, \ldots, u_m) \in \mathbb{R}^m : u_i \geq 0, 1 \leq i \leq m, \|u\|_2 = 1\}
$$

and associate with the tuple $(A_n, C_n), n \in \mathbb{N}_0$ of iid pairs the series of vectors

$$
\Xi_l := \sum_{k=l}^{\infty} A_l A_{l+1} \cdots A_{k-1} C_k = \sum_{k=l}^{\infty} \Pi_{l,k} C_k, \ l \in \mathbb{N}_0; \ \Xi := \Xi_0. \quad (13)
$$

Our main results are established under the following hypothesis.

**Condition T.** There exist positive constants $\kappa$ and $K_0$ and a continuous strictly positive function $l(u)$ on $U_+$ such that for all $u \in U_+$

$$
\lim_{y \to \infty} y^\kappa P((u,\Xi) > y) = K_0 l(u). \quad (14)
$$

In Section 4 we list sufficient conditions imposed on the distributions of the pairs $(A_n, C_n)$ which provide the validity of Condition T. These conditions are extracted from paper [30] where the behavior of the tail distribution of sums and products of random matrices were investigated.

Let

$$
\Phi := \lim_{n \to \infty} \Phi(n) = \Phi(0) + \sum_{n=0}^{\infty} \sum_{i=1}^{m} \sum_{k=1}^{Z(n)} \phi_i(n; k) \quad (15)
$$

be the total size of the final product produced by the particles of the MBPFPRE up to the extinction moment (if any).

The following theorem was proved in [16].

**Theorem 7** Let a MBPFPRE satisfy the following hypotheses:

1) the underlying MBPRE is subcritical and meets the conditions of Theorem 5;
2) for $\kappa$ specified by (10) the following assumptions fulfill:

- if $\kappa > 1$ then
  \[
  \max_{1 \leq i \leq m} E \left| \sum_{j=1}^{\kappa} (\xi_{ij} - E_{F}\xi_{ij}) \right| ^{\kappa} < \infty \quad \text{and} \quad E \left( \sum_{i=1}^{\kappa} (\varphi_{i} - E_{F}\varphi_{i}) \right) ^{\kappa} < \infty,
  \]

- if $\kappa \leq 1$ then
  \[
  \max_{1 \leq i \leq m} E \left( \sum_{j=1}^{\kappa} \text{Var}_{F}\xi_{ij} \right) ^{\kappa} < \infty \quad \text{and} \quad E \left( \sum_{i=1}^{\kappa} \text{Var}_{F}\varphi_{i} \right) ^{\kappa} < \infty;
  \]

4) there exists $\delta > 0$ such that $0 < E_{F}\kappa^{\delta} < \infty$, $i = 1, \ldots, m$;

5) the mean matrix (4) and the vector (5) are such that Condition $T$ is valid.

Then, as $y \to \infty$

\[
P \left( \Phi > y \middle| Z(0) = z \right) \sim C(z)y^{-\kappa}, \quad C(z) \in (0, \infty).
\]

The theorem has been used in [16] to investigate the tail distributions of the busy periods and some other characteristics for a wide class of polling systems with zero switch-over time. In the present paper we consider MBPIFPRE and apply in Section 7 the respective results to deduce similar statements for busy periods and certain other characteristics of polling systems with positive (possibly random) switch-over times. To this aim we introduce the notion of life periods of a MBPIFPRE.

**Definition 8** We say that a branching process $T(n), n \in \mathbb{N}_0$ with $m$ types of particles, immigration and final product has a life period of length $\Upsilon := \Upsilon_{1}$, and study the tail distribution of the quantity

\[
\Theta = \Theta(\Upsilon) := \theta + \sum_{n=0}^{\Upsilon-1} \left( \psi(n+1) + \sum_{i=1}^{m} \sum_{k=1}^{V_i(n)} \varphi_i(n; k) \right)
\]

- the total amount of the final product accumulated in the MBPIFPRE during the life-period which starts by $\eta(1) = z \neq 0$ particles at moment $N = 1$. Since our proofs (but not constants!) do not depend on the particular value of $\eta(1)$ we do not specify it explicitly in the subsequent arguments.

Note that if $\theta = 0$ and $\varphi_i(n; k) \equiv 1$, $\psi(n) \equiv 0$ for all $n \in \mathbb{N}_0$ then $\Theta$ is the total number of individuals existing in the MBPIFPRE during the respective life-period.

Now we are ready to formulate the main result of the present paper.
Theorem 9 Let the conditions of Theorem 7 be valid,
\[ \max_{1 \leq i \leq m} E \eta_i^{1+\kappa} < \infty, \quad E |\psi - E_G \psi|^\kappa < \infty \]
and, in addition,
\[ E \|B\|^q < \infty \quad \text{for some } q > 0 \quad \text{and} \quad E \|A\|^{\kappa} < \infty. \]

Then, as \( y \to \infty \)
\[ P(\Theta > y) \sim C_I y^{-\kappa}, \quad C_I \in (0, \infty). \] (19)

In particular,
\[ E \Theta^x < \infty \] (20)

if and only if \( x < \kappa \).

In Section 7 we use Theorem 9 to show that the tail distributions of the busy periods of a wide class of branching type polling systems whose input parameters and service disciplines vary in a random manner and switch-over times are positive, decay at infinity as \( y^{-\kappa} \) for some \( \kappa > 0 \).

4 Auxiliary results

The proof of Theorem 9 uses Condition T whose validity is not easy to check. We list here a set of assumptions given in [30] which imply Condition T.

Let \( \Lambda(A) \) be the spectral radius of the matrix \( A \). The following statement is a refinement of a Kesten theorem from [30].

Theorem 10 (see [22]) Let \( \{A_n, n \geq 0\} \) be a sequence of iid matrices generated by a measure \( P_A \) with support concentrated on nonnegative matrices and \( A = (a_{ij})_{i,j=1}^d = A_n \). Assume that the following conditions are valid:
1) there exists \( q > 0 \) such that \( E \|A\|^q < \infty \);
2) \( A \) has no zero rows a.s.;
3) the group generated by
\[ \{\log \Lambda(a_n \cdots a_0) : a_n \cdots a_0 > 0 \text{ for some } n \text{ and } a_i \in \text{supp}(P_A)\} \]
is dense in \( \mathbb{R} \);
4) there exists \( \kappa_0 > 0 \) for which
\[ E \left[ \min_{1 \leq i \leq m} \left( \sum_{j=1}^m a_{ij} \right)^{\kappa_0} \right] \geq m^{\kappa_0/2} \]
and
\[ E \|A\|^\kappa_0 \log^+ \|A\| < \infty. \]
Then there exists a $\kappa \in (0, \kappa_0]$ such that

$$s'(\kappa) = \lim_{n \to \infty} \frac{1}{n} \log E \|A_{n-1} \cdots A_0\|^\kappa = 0.$$  

If, in addition, the tuple of $m$-dimensional vectors $\{C_n, n \geq 0\}$ is such that the pairs $(A_n, C_n), n \in \mathbb{N}_0$ are iid: $(A_n, C_n) \overset{d}{=} (A, C)$ and such that

$$P(C = 0) < 1, \quad P(C \geq 0) = 1, \quad E \|C\|^\kappa < \infty,$$

then there exist a constant $K_0 \in (0, \infty)$ and a continuous strictly positive function $l(u)$ on $\mathbb{U}_+$ such that relation (14) holds.

The next lemma provides estimates of moments of a random walk (see, for instance, [29], Theorem 1.5.1):

**Lemma 11** If $X_i, i = 1, 2, \ldots$ is a sequence of iid random variables such that $E|X_i|^p < \infty$ and $E X_i = 0$ if $p \geq 1$ and $M$ is a stopping time for the sequence $\Gamma_n := X_1 + \ldots + X_n$ then there exists a constant $R_p \in (0, \infty)$ such that

$$E |\Gamma_M|^p \leq R_p E |X_i|^p E M^{p/2 \vee 1}.$$  

From now on we agree to denote by $K, K_1, K_2, \ldots$ positive constants which may be different from formula to formula.

**Lemma 12** ([16]) If a subcritical MBPRE starts by $Z(0) = z$ individuals and parameter $\kappa$ in (10) exceeds 1 then for each $x \in [1, \kappa)$ there exist $\rho = \rho(x) \in (0, 1)$ and $K = K(x) < \infty$ such that

$$E \|Z(n)\|^x \leq K \rho^n \|z\|^x$$  \hspace{1cm} (21)

for all $n \in \mathbb{N}_0$.

The next statement is borrowed from [32], page 351.

**Lemma 13** If conditions of Theorem 6 are valid and, in addition,

$$E \|B\|^q < \infty \quad \text{and} \quad E \|A\|^q < \infty$$

for some $q > 0$, then there exist positive constants $K_1$ and $K_2$ such that for all $t > 0$

$$P(\Upsilon > t) \leq K_1 e^{-K_2 t}.$$  

Let

$$Z^I(i, j; n, k) := (Z^I_1(i, j; n, k), \ldots, Z^I_m(i, j; n, k))$$

be the $m$-dimensional vector of the total progeny alive at moment $k > n$ of the $j$-th immigrant of type $i$ entering the process at time $n$,

$$Z^I(n, k) := \sum_{i=1}^m \sum_{j=1}^{n_i(n)} Z^I(i, j; n, k)$$

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be the $m$–dimensional vector of the total progeny alive at moment $k > n$ of all the immigrants entering the process at time $n$, and

$$Y(n) := \eta(n) + \sum_{k=n+1}^{\infty} Z^I(n, k)$$

be the $m$–dimensional vector representing the total progeny steaming from the immigrants entering the process at time $n$.

Lemma 14 If the conditions of Theorem 7 are valid and, in addition,

$$\max_{1 \leq i \leq m} E|\eta_i|^{\kappa+1} < \infty$$

then there exists a constant $K_1 \in (0, \infty)$ such that for all $x > 0$

$$P(\|Y(1)\| \geq x) \leq K_1 x^{-\kappa}.$$  

Proof. We have

$$Y(1) = \eta(1) + \sum_{k=1}^{\infty} \sum_{i=1}^{m} \eta_i(1) \sum_{j=1}^{\infty} Z^I(i, j; 1, k + 1).$$

Hence, observing that, for any fixed $i \in \{1, \ldots, m\}$ and any $j = 1, 2, \ldots, \eta_i(1)$

$$\{Z^I(i, j; 1, k + 1), k = 1, 2, \ldots \} \overset{d}{=} \{Z(k), k = 1, 2, \ldots | Z(0) = e_i\}$$

and using Theorem 7 with $\varphi_i \equiv 1$, $i \in \{1, \ldots, m\}$ we see that

$$P(\|Y(1)\| \geq x) \leq P(\|\eta\| \geq \frac{x}{2}) + \sum_{i=1}^{m} P\left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{m} \eta_i(1) Z^I(i, j; 1, k + 1)\right) \geq \frac{x}{2m}\right) \right)$$

$$\leq \frac{2^\kappa E|\eta|^{\kappa}}{x^{\kappa}} + \sum_{i=1}^{m} E\left[\sum_{j=1}^{\eta_i(1)} P\left(\sum_{k=1}^{\infty} |Z(k)| \geq \frac{x}{m\eta_i(1)} | Z(0) = e_i; \eta_i(1)\right)\right]$$

$$= \frac{2^\kappa E|\eta|^{\kappa}}{x^{\kappa}} + \sum_{i=1}^{m} E\left[\eta_i(1) P\left(\sum_{k=1}^{\infty} |Z(k)| \geq \frac{x}{m\eta_i(1)} | Z(0) = e_i; \eta_i(1)\right)\right]$$

$$\leq \frac{2^\kappa E|\eta|^{\kappa}}{x^{\kappa}} + K_1 \sum_{i=1}^{m} E\left[\eta_i(1) \left(\frac{m\eta_i(1)}{x^{\kappa}}\right)^{\kappa}\right] \leq \frac{K_2}{x^{\kappa}} \sum_{i=1}^{m} E\eta^{\kappa+1}_i = \frac{K_1}{x^{\kappa}}.$$

The lemma is proved.

Let

$$\mu = \mu(r) = \min \{n : \|V(n)\| > r\}$$

be the first moment when the size of the population of the MBPIRE exceeds level $r$ with the natural agreement that $\mu = \infty$ if $\max_n \|V(n)\| \leq r$. 

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Lemma 15. Under the conditions of Theorem 9 for any $\varepsilon > 0$ there exists $r_1 = r_1(\varepsilon) > 0$ such that

$$P\left(\sum_{n=\mu+1}^{\Upsilon-1} \|Y(n)\| \geq \varepsilon y, \mu < \Upsilon\right) \leq \varepsilon y^{-K}$$

for all $r \geq r_1$.

Proof. In view of the estimate

$$\sum_{n=1}^{\infty} n^{-2} = \pi^2/6 \leq 2$$

we have

$$P\left(\sum_{n=\mu+1}^{\Upsilon-1} \|Y(n)\| \geq \varepsilon y, \mu < \Upsilon\right) = P\left(\sum_{n=\mu+1}^{\infty} \|Y(n)\| I\{\mu < n < \Upsilon\} \geq 6\pi^{-2}\varepsilon y \sum_{n=1}^{\infty} \frac{1}{n^2}\right) \leq \sum_{n=1}^{\infty} P\left(\|Y(n)\| \geq \frac{\varepsilon y}{2n^2}\right) \leq \sum_{n=1}^{\infty} P(\mu < n < \Upsilon) P\left(\|Y(n)\| \geq \frac{\varepsilon y}{2n^2}\right),$$

where the last equality is justified by the fact that the event $\{\mu < n < \Upsilon\}$ is specified by $V(0), V(1), ..., V(n-1)$ while $Y(n)$ depends on the environment and immigration after the moment $n-1$. Besides, the manipulations with sums are correct since the sums run only till $\Upsilon$ which is finite with probability 1 by Lemma 13.

Clearly,

$$P\left(\|Y(n)\| \geq \frac{\varepsilon y}{2n^2}\right) = P\left(\|Y(1)\| \geq \frac{\varepsilon y}{2n^2}\right).$$

Now using Lemma 14 we have

$$P\left(\|Y(1)\| \geq \frac{\varepsilon y}{2n^2}\right) \leq \frac{2^K Kn^{2K}}{\varepsilon^n y^K}. $$

Thus,

$$P\left(\sum_{n=\mu+1}^{\Upsilon-1} \|Y(n)\| \geq \varepsilon y, \mu < \Upsilon\right) \leq \frac{K_1}{\varepsilon^n y^K} \sum_{n=1}^{\infty} n^{2K} P(\mu < n < \Upsilon) \leq \frac{K_2}{\varepsilon^n y^K} E[\Upsilon^{2K+1} I\{\mu < \Upsilon\}] \leq \frac{\varepsilon}{y^K}$$

for $r \geq r_1(\varepsilon)$ since $E\Upsilon^{2K+1} < \infty$ in view of Lemma 13 and $\mu = \mu(r) \uparrow \infty$ in probability as $r \to \infty$.

The lemma is proved.
Lemma 16 If $E (\|A\| + E \|\eta\|)^x < \infty$

then for any fixed $r$

$E [\|V(\mu)\|^r I \{\mu < Y\}] < \infty.$

Proof. It is easy to check that for any $y, w \geq 0$, any $\varepsilon \in (0, 1)$ and $x > 0$

$y^x \leq (1 + \varepsilon)w^x + K |y - w|^x,$

where $K = K(x, \varepsilon) := \left(1 - (1 + \varepsilon)^{-1/x}\right)^{-x}$. Hence introducing a temporary notation $V[0, n) := (V(0), \ldots, V(n-1))$ we have

$E [\|V(n)\|^x |V(0), \ldots, V(n-1)|]$

$= E [\|V(n-1)A_{n-1} + V(n) - V(n-1)A_{n-1}\|^x |V[0, n)]$

$\leq (1 + \varepsilon)\|V(n-1)A_{n-1}\|^x + KE [\|V(n) - V(n-1)A_{n-1}\|^x |V[0, n)]$

$\leq (1 + \varepsilon)\|V(n-1)A_{n-1}\|^x + 2^xKE \|\eta(n)\|^x$

$+ 2^x K \sum_{i=1}^{m} E \left[\left|\sum_{k=1}^{\infty} \sum_{j=1}^{m} |\xi_{ij}(n-1; k) - a_{ij}(n-1)|\right|^x \right] |V[0, n)] . (25)$

Recalling the definition $a_{ij}(n) = E \xi_{ij}(n)$, set

$\beta_i(n) := \sum_{j=1}^{m} (\xi_{ij}(n) - a_{ij}(n))$

and let

$M_x(n; i) := E |\beta_i(n)|^x, \quad M_x := \max_{1 \leq i \leq m} E |\beta_i(n)|^x = \max_{1 \leq i \leq m} E \left[\sum_{j=1}^{m} (\xi_{ij} - a_{ij}) \right]^x.$

Observe that $M_x < \infty$ in view of condition (16).

By Lemma 11 for $x > 1$ we have

$E \left[\left|\sum_{k=1}^{\infty} \sum_{j=1}^{m} |\xi_{ij}(n-1; k) - a_{ij}(n-1)|\right|^x \right] |V[0, n)]$

$\leq m^2 \sum_{j=1}^{m} E \left[\left|\sum_{k=1}^{\infty} |\xi_{ij}(n-1; k) - a_{ij}(n-1)|\right|^x \right] |V(n-1)|$

$\leq R_x m^2 \sum_{i=1}^{m} M_x(n-1, i) V_i^{x/2\vee 1}(n-1).$
Thus,
\[
\mathbb{E} \left[ \| V(n) \|^{x} \mid V[0,n] \right] \leq (1 + \varepsilon) \| V(n-1)A_{n-1} \|^{x} + 2^{x} K \mathbb{E} \| \eta(n) \|^{x} \\
+ 2^{x} K R \kappa m^{x} \sum_{i=1}^{m} M_{x}(n-1,i)V_{x}^{x/2\nu 1}(n-1) =: \Lambda_{x}(n-1).
\] (27)

On the other hand, for \( x \leq 1 \)
\[
\mathbb{E} \left[ \| V(n) \|^{x} \mid V[0,n] \right] \leq \mathbb{E} \left[ \| V(n) - \eta(n) \|^{x} + \| \eta(n) \|^{x} \mid V[0,n] \right] \leq \mathbb{E} \left[ \| V(n-1)A_{n-1} \|^{x} \right] + \mathbb{E} \| \eta \|^{x} \leq \Lambda_{x}(n-1).
\] (28)

Clearly,
\[
\Lambda_{\kappa}(n-1)I \{ n-1 < \mu < \Upsilon \} \leq Q_{n-1}(r) := (1 + \varepsilon)r^{\kappa} \| A_{n-1} \|^{\kappa} + 2^{\kappa} K \mathbb{E} \| \eta \|^{\kappa} \\
+ 2^{\kappa} K R \kappa m^{\kappa} r^{\kappa/2\nu 1} \sum_{i=1}^{m} M_{\kappa}(n-1,i).
\]

Observe that by our conditions the expectation \( \mathbb{E} [Q_{n-1}(r)] \) is independent of \( n \) and, moreover,
\[
\mathbb{E} [Q_{n-1}(r)] = (1 + \varepsilon)r^{\kappa} \mathbb{E} \| A \|^{\kappa} + 2^{\kappa} K \mathbb{E} \| \eta \|^{\kappa} \\
+ 2^{\kappa} K R \kappa m^{\kappa} r^{\kappa/2\nu 1} \left[ \sum_{i=1}^{m} M_{\kappa}(n-1,i) \right] \leq (1 + \varepsilon)r^{\kappa} \mathbb{E} \| A \|^{\kappa} + 2^{\kappa} K \mathbb{E} \| \eta \|^{\kappa} + 2^{\kappa} K R \kappa m^{\kappa+1} r^{\kappa/2\nu 1} M_{\kappa} := K_{1} < \infty.
\]

Besides,
\[
\| V(\mu) \|^{\kappa} I \{ \mu < \Upsilon \} = \Lambda_{\kappa}(\mu-1) \frac{\| V(\mu) \|^{\kappa}}{\Lambda_{\kappa}(\mu-1)} I \{ \mu < \Upsilon \} \leq Q_{n-1}(r) \frac{\| V(\mu) \|^{\kappa}}{\Lambda_{\kappa}(\mu-1)} \leq \sum_{\mu \leq n < \Upsilon} Q_{n-1}(r) \frac{\| V(n) \|^{\kappa}}{\Lambda_{\kappa}(n-1)}.
\]
Now by (27) and (28) we see that
\[
E \|\mathbf{V}(\mu)\|^{\kappa} I \{\mu < \Upsilon\} \leq E \left[ \sum_{\mu \leq n < \Upsilon} Q_{n-1}(r) \frac{\|\mathbf{V}(n)\|^{\kappa}}{\Lambda_n(n-1)} \right]
\]
\[
\leq E \left[ \sum_{n=1}^{\infty} \frac{Q_{n-1}(r) \|\mathbf{V}(n)\|^{\kappa}}{\Lambda_n(n-1)} I \{\Upsilon \geq n\} \right]
\]
\[
= \sum_{n=1}^{\infty} E \left[ \frac{Q_{n-1}(r)}{\Lambda_n(n-1)} I \{\Upsilon \geq n\} E \|\mathbf{V}(n)\|^{\kappa} |\mathbf{V}[0,n)\right]
\]
\[
\leq \sum_{n=1}^{\infty} E [Q_{n-1}(r) I \{\Upsilon \geq n\}] = \sum_{n=1}^{\infty} E [Q_{n-1}(r)] P (\Upsilon \geq n)
\]
\[
\leq K_1 \sum_{n=1}^{\infty} P (\Upsilon \geq n) < \infty.
\]

The lemma is proved.

Let
\[
\mathbf{Z}(i,j;l,n) := (Z_1(i,j;l,n),...,Z_l(i,j;l,n))
\]
be the \(m\)-dimensional vector of the total progeny alive at moment \(n > l\) of the \(j\)-th particle of type \(i\) existing in the process at time \(l\). Let, further,
\[
\mathbf{Z}(l,l) := \mathbf{V}(l) \quad \text{and} \quad \mathbf{Z}(l,n) = (Z_1(l,n),...,Z_m(l,n)) := \sum_{i=1}^{m} \sum_{j=1}^{V_i(l)} \mathbf{Z}(i,j;l,n)
\]
be the total progeny at time \(n\) of all particles existing in the process at moment \(l < n\). Recalling definition (5) we set
\[
\Gamma (\mu,\mu) := \langle \mathbf{V}(\mu), \mathbf{C}_\mu \rangle = \langle \mathbf{Z}(\mu,\mu), \mathbf{C}_\mu \rangle
\]
and for \(n > \mu\) let
\[
\Gamma (\mu, n) := \sum_{i=1}^{m} \sum_{j=1}^{V_i(\mu)} \langle \mathbf{Z}(i,j;\mu,n), \mathbf{C}_n \rangle = \langle \mathbf{Z}(\mu,n), \mathbf{C}_n \rangle
\]
(29)
be the inner product of \(\mathbf{C}_n\) and the vector of the total progeny at time \(n\) of the \(\mathbf{V}(\mu)\) particles presented in the process at time \(\mu\), and
\[
\Gamma (\mu) := \sum_{n=\mu}^{\Upsilon-1} \Gamma (\mu, n) = \sum_{n=\mu}^{\Upsilon-1} \langle \mathbf{Z}(\mu,n), \mathbf{C}_n \rangle.
\]
(30)

Denote by \(\mathbf{C}_n\) \((n = 1, 2, \ldots)\) be the \(\sigma\)-algebra generated by the tuple
\[
\mathbf{H}_{0}(s,\lambda), \mathbf{H}_{1}(s,\lambda), \ldots, \mathbf{H}_{n-1}(s,\lambda), \mathbf{V}(0), \ldots, \mathbf{V}(n).
\]
The next lemma shows that for large $r$ the random variable $\Gamma (\mu) = \Gamma (\mu(r))$ is, in a sense, close to the conditional expectation $E [\Gamma (\mu) | C_\mu] = V(\mu)\Xi_\mu$ (recall definition (13)).

**Lemma 17** Under the conditions of Theorem 9 for any $\varepsilon > 0$ there exists $r_1 = r_1(\varepsilon)$ such that for all $y > 0$

$$P \left( |\Gamma (\mu) - \langle V(\mu), \Xi_\mu \rangle| > \varepsilon y; \mu < \Upsilon \right) \leq \frac{\varepsilon}{y^\epsilon} E \left[ \|V(\mu)\|^\kappa I \{ \mu < \Upsilon \} \right]. \quad (31)$$

**Proof.** Evidently, for $n \geq \mu + 1$

$$\langle Z(\mu, n) - V(\mu) \Pi_{\mu, n}, C_n \rangle = \sum_{l=\mu+1}^{n} \langle Z(\mu, l) \Pi_{l, n} - Z(\mu, l - 1) \Pi_{l-1, n}, C_n \rangle$$

which implies

$$|\Gamma (\mu) - \langle V(\mu), \Xi_\mu \rangle| = \left| \sum_{n=\mu+1}^{\infty} \langle Z(\mu, n) - V(\mu) \Pi_{\mu, n}, C_n \rangle \right|$$

$$\leq \sum_{n=\mu+1}^{\infty} |\langle Z(\mu, n) - V(\mu) \Pi_{\mu, n}, C_n \rangle|$$

$$= \sum_{n=\mu+1}^{\infty} \left| \sum_{l=\mu+1}^{n} \langle Z(\mu, l) - Z(\mu, l - 1) A_{l-1}, \Pi_{l, n} C_n \rangle \right|$$

$$\leq \sum_{l=\mu+1}^{\infty} \sum_{n=l}^{\infty} |\langle Z(\mu, l) - Z(\mu, l - 1) A_{l-1}, \Pi_{l, n} C_n \rangle|$$

$$\leq \sum_{l=\mu+1}^{\infty} \|Z(\mu, l) - Z(\mu, l - 1) A_{l-1}\| \left\| \sum_{n=l}^{\infty} \Pi_{l, n} C_n \right\|$$

$$= \sum_{l=\mu+1}^{\infty} \|Z(\mu, l) - Z(\mu, l - 1) A_{l-1}\| \|\Xi_l\|. \quad (32)$$

Here the interchange of summation order is justified since the process having $V(\mu)$ particles at moment $\mu < \Upsilon < \infty$ dies out with probability 1.

Thus,

$$P(\{|\Gamma (\mu) - \langle V(\mu), \Xi_\mu \rangle| > \varepsilon y; \mu < \Upsilon\})$$

$$\leq P \left( \sum_{l=\mu+1}^{\infty} \|Z(\mu, l) - Z(\mu, l - 1) A_{l-1}\| \|\Xi_l\| > \varepsilon y; \mu < \Upsilon \right).$$
Hence, using (22) we conclude
\[
\begin{align*}
P\left(\|\Gamma(\mu) - (V(\mu), \Xi_\mu)\| > \varepsilon y; \mu < \Upsilon \mid \mathcal{C}_\mu\right) \\
\leq P\left(\sum_{n=1}^\infty \|Z(\mu, n + \mu) - Z(\mu, n + \mu - 1)A_{n+\mu-1}\| \|\Xi_{n+\mu}\| > \frac{\varepsilon y}{2} \sum_{n=1}^\infty \frac{1}{n^2}; \mu < \Upsilon \mid \mathcal{C}_\mu\right) \\
\leq \sum_{n=1}^\infty P\left(\|Z(\mu, n + \mu) - Z(\mu, n + \mu - 1)A_{n+\mu-1}\| \|\Xi_{n+\mu}\| > \frac{\varepsilon y}{2} \frac{1}{n^2}; \mu < \Upsilon \mid \mathcal{C}_\mu\right).
\end{align*}
\]
Since \(Z(\mu, n + \mu) - Z(\mu, n + \mu - 1)A_{n+\mu-1}\) and \(\Xi_{n+\mu}\) are independent random objects on the event \(\mu = k < \Upsilon < \infty\) and \(\Xi_{n+\mu} \overset{d}{=} \Xi\) (see (13)), we get
\[
P\left(\|Z(\mu, n + \mu) - Z(\mu, n + \mu - 1)A_{n+\mu-1}\| \|\Xi_{n+\mu}\| > \frac{\varepsilon y}{2} \frac{1}{n^2}; \mu < \Upsilon \mid \mathcal{C}_\mu\right) \\
= \int_0^\infty P\left(\|Z(\mu, n + \mu) - Z(\mu, n + \mu - 1)A_{n+\mu-1}\| \|\Xi_{n+\mu}\| \in dt; \mu < \Upsilon \mid \mathcal{C}_\mu\right) P\left(\|\Xi\| > \frac{\varepsilon y}{2t^2n^2}\right).
\]
According to Condition T there exists a constant \(K_1 \in (0, \infty)\) such that for all \(n > 0\)
\[
P\left(\|Z(\mu, n + \mu) - Z(\mu, n + \mu - 1)A_{n+\mu-1}\| \|\Xi_{n+\mu}\| > \frac{\varepsilon y}{2} \frac{1}{n^2}; \mu < \Upsilon \mid \mathcal{C}_\mu\right) \\
\leq \int_0^\infty P\left(\|Z(\mu, n + \mu) - Z(\mu, n + \mu - 1)A_{n+\mu-1}\| \|\Xi_{n+\mu}\| \in dt; \mu < \Upsilon \mid \mathcal{C}_\mu\right) \frac{K_1 t^\kappa}{\varepsilon y^\kappa} n^{2\kappa} \\
\leq \frac{K_1}{\varepsilon y^\kappa} n^{2\kappa} E\left[\|Z(\mu, n + \mu) - Z(\mu, n + \mu - 1)A_{n+\mu-1}\| \|\Xi_{n+\mu}\| \right]^{\kappa/2}. (33)
\]
Now we consider the cases \(\kappa \leq 1\) and \(\kappa > 1\) separately.
For the first case we use for \(l > \mu\) the estimate
\[
E\left[\|Z(\mu, l) - Z(\mu, l - 1)A_{l-1}\|^\kappa I\{\mu < \Upsilon\} \mid \mathcal{C}_\mu\right] \\
\leq \left(E\left[\|Z(\mu, l) - Z(\mu, l - 1)A_{l-1}\|^2 I\{\mu < \Upsilon\} \mid \mathcal{C}_\mu\right]\right)^{\kappa/2}. (34)
\]
Further, we have (recall (26))
\[
E\left[\|Z(\mu, l) - Z(\mu, l - 1)A_{l-1}\|^2 I\{\mu < \Upsilon\} \mid \mathcal{C}_\mu\right] \\
= E\left[\left(m \sum_{i=1}^m Z_i(\mu, l-1) I_{\{\mu < \Upsilon\} \mid \mathcal{C}_\mu} \right)^2 \right] \\
= m \sum_{i=1}^m E_\mu \beta_i^2(l-1) E[Z_i(\mu, l-1) I\{\mu < \Upsilon\} \mid \mathcal{C}_\mu] \\
= m \sum_{i=1}^m E_\mu \beta_i^2(l-1) (V(\mu)\Pi_{\mu,l-1})_i \leq \|V(\mu)\Pi_{\mu,l-1}\| \sum_{i=1}^m E_\mu \beta_i^2(l-1). (35)
\]
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Thus, for $\kappa \leq 1$ and $l = n + \mu > \mu$

$$
\left( \mathbb{E} \left[ \| \mathbf{Z}(\mu, n + \mu) - \mathbf{Z}(\mu, n + \mu - 1) \mathbf{A}_{n+\mu-1} \|^2 \, I \{ \mu < \Upsilon \} \left| \mathcal{C}_\mu \right. \right] \right)^{\kappa/2}
\leq \| \mathbf{V}(\mu) \mathbf{P}_{\mu,n+\mu-1} \|^\kappa/2 \left( \sum_{i=1}^{m} \mathbb{E} \beta_i^2 (n + \mu - 1) \right)^{\kappa/2}. \quad (36)
$$

This, in view of the inequality

$$
s(\kappa/2) = \lim_{n \to \infty} \left( \mathbb{E} \| \mathbf{P}_{0,n} \|^\kappa/2 \right)^{1/n} < 1,
$$

the first part of condition (17), and relations (35)-(36) leads to the estimate

$$
P \left( |\Gamma (\mu) - \langle \mathbf{V}(\mu), \Xi_\mu \rangle | > \varepsilon y; \mu < \Upsilon \right)
\leq \frac{K}{\varepsilon^ny^\kappa} \mathbb{E} \left[ \sum_{n=1}^{\infty} n^{2\kappa} \| \mathbf{V}(\mu) \|^\kappa/2 \| \mathbf{P}_{\mu,n+\mu-1} \|^\kappa/2 \, I \{ \mu < \Upsilon < \infty \} \right]
\leq \frac{K_1}{\varepsilon^ny^\kappa} \mathbb{E} \left[ \| \mathbf{V}(\mu) \|^\kappa \, I \{ \mu < \Upsilon < \infty \} \right] \leq \frac{\varepsilon}{y^\kappa} \mathbb{E} \left[ \| \mathbf{V}(\mu) \|^\kappa \, I \{ \mu < \Upsilon < \infty \} \right]
$$

for all $r \geq r_0(\varepsilon)$, proving the lemma for $\kappa \leq 1$.

For the case $\kappa > 1$ we use Lemma 11 to conclude that for any $l > \mu$

$$
\mathbb{E} \left[ \| \mathbf{Z}(\mu, l) - \mathbf{Z}(\mu, l - 1) \mathbf{A}_{l-1} \|^\kappa \, I \{ \mu < \Upsilon < \infty \} \left| \mathcal{C}_\mu \right. \right]
\leq R_\kappa m^\kappa \sum_{i=1}^{m} M_\kappa(l; i) \mathbb{E} \left[ \| \mathbf{Z}(\mu, l - 1) \|^\kappa/2 \, I \{ \mu < \Upsilon < \infty \} \left| \mathcal{C}_\mu \right. \right]
\leq R_\kappa m^\kappa \mathbb{E} \left[ \| \mathbf{Z}(\mu, l - 1) \|^\kappa/2 \, I \{ \mu < \Upsilon < \infty \} \left| \mathcal{C}_\mu \right. \right] \sum_{i=1}^{m} M_\kappa(l - 1; i).
$$

By Lemma 12 there exist constants $K = K (\kappa/2 \vee 1) < \infty$ and $\rho = \rho(\kappa/2 \vee 1) \in (0, 1)$ such that

$$
\mathbb{E} \left[ \| \mathbf{Z}(\mu, l - 1) \|^\kappa/2 \, I \{ \mu < \Upsilon < \infty \} \left| \mathcal{C}_\mu \right. \right] \leq K \rho^{l-\mu - 1} \| \mathbf{V}(\mu) \|^\kappa/2 \, I \{ \mu < \Upsilon < \infty \} \left| \mathcal{C}_\mu \right. \right.
$$
This inequality combined with condition (16) yields for \( l = n + \mu \) the estimates

\[
\mathbb{P} (\| \Gamma (\mu) - (V(\mu), \Xi_\mu) \| > \varepsilon y; \mu < \Upsilon)
\leq \frac{K_1 R_\kappa m^n}{\varepsilon y^\kappa} \mathbb{E} \left[ \sum_{n=1}^{\infty} n^{2\kappa} \mathbb{E} \left[ \| Z(\mu, n + \mu - 1) \|^{\kappa/21} \sum_{i=1}^{m} M_\kappa(n + \mu - 1; i) | C_\mu \right] I \{ \mu < \Upsilon < \infty \} \right]
\leq \frac{K_1 R_\kappa m^{n+1} M_\kappa}{\varepsilon y^\kappa} \mathbb{E} \left[ \sum_{n=1}^{\infty} n^{2\kappa} \mathbb{E} \left[ \| V(\mu) \|^{\kappa/21} \| C_\mu \right] I \{ \mu < \Upsilon < \infty \} \right]
\leq \frac{K_2}{\varepsilon y^\kappa} \mathbb{E} \left[ \| V(\mu) \|^{\kappa/21} I \{ \mu < \Upsilon \} \sum_{n=1}^{\infty} n^{2\kappa} \mathbb{E} \left[ \| V(\mu) \|^{\kappa/21} \| C_\mu \right] I \{ \mu < \Upsilon \} \right]
\leq \frac{K_3}{\varepsilon y^{\kappa+\kappa - \kappa/21}} \mathbb{E} \left[ \| V(\mu) \|^{\kappa} I \{ \mu < \Upsilon \} \right] \leq \frac{\varepsilon y^\kappa}{\varepsilon y^\kappa} \mathbb{E} \left[ \| V(\mu) \|^{\kappa} I \{ \mu < \Upsilon \} \right]
\]

(the last is valid by selecting \( r \) sufficiently large) which justifies the statement of the lemma for \( \kappa > 1 \).

The lemma is proved.

5 The accumulated amount of the final product

In this section we deduce some estimates related with the total size of the final product accumulated in a subcritical MBPIFPRE within a life-period.

Let

\[
\Delta_V(0) := \Theta(0) + \psi(1) I \{ \| \eta(1) \| > 0 \}
\]

and let

\[
\Delta_V(n) := \sum_{i=1}^{m} \sum_{k=1}^{V_i(n)} \varphi_i(n; k), n \geq 1,
\]

be the total amount of the final product produced by the particles of the \( n \)-th generation of a MBPIFPRE,

\[
\Delta_I(n) = \Delta_V(n) + \psi(n + 1)
\]

be the total amount of the final product produced by the individuals of the \( n \)-th generation of a MBPIFPRE plus the final product contributed by the particles immigrating at moment \( n + 1 \), and

\[
Y_I(n) := \psi(n + 1) + \sum_{i=1}^{m} \sum_{j=1}^{\eta_i(n+1)} \varphi_i(n + 1, j) + \sum_{k=n+2}^{\Upsilon-1} \sum_{t=1}^{\sum_{w=1}^{\sum_{i=1}^{m} \sum_{j=1}^{\eta_i(n+1)}} \varphi_t(k, w)}
\]

be the total size of the final product contributed to the system by all particles immigrating in the process at moment \( n + 1 \) and by their progeny.
Of interest will be the quantity

\[ \Theta_Z(\mu, n) := \sum_{t=1}^{m} \sum_{k=1}^{\varphi_t(n; k)} \varphi_t(n; k), \]

i.e., the total amount of the final product produced by the particles of the \( n \)-th generation, \( n > \mu \), of the MBPIFPRE which belong to the progeny of the \( V(\mu) \) particles of generation \( \mu \) and the random variable

\[ \Theta_Z(\mu) := \sum_{n=\mu}^{Y-1} \Theta_Z(\mu, n) \]

which is equal to the total amount of the final product produced by the \( V(\mu) \) particles of generation \( \mu \) and their progeny up to the moment of extinction of the MBPRE generated by the \( V(\mu) \) particles of the \( \mu \)-th generation.

Clearly, the total amount

\[ \Theta := \sum_{n=0}^{Y-1} \Delta_I(n) \]

of the final product accumulated in the system during the life period which starts at moment \( n = 1 \) admits on the set \( \mu < Y \) the representation

\[ \Theta = \sum_{n=0}^{\mu-1} \Delta_I(n) + \Theta_Z(\mu) + \sum_{n=\mu}^{Y-1} Y_I(n). \]

Our aim is to investigate the asymptotic behavior of the tail distribution of \( \Theta \).

As the first step in solving this problem we compare \( \Theta_Z(\mu) \) with \( \Gamma(\mu) \). This will be done by the arguments similar to those used to demonstrate Lemma 17.

**Lemma 18** Under the conditions of Theorem 9 for any \( \varepsilon > 0 \) there exists \( r = r(\varepsilon) \) such that for all \( y > 0 \)

\[ P(|\Theta_Z(\mu) - \Gamma(\mu)| > \varepsilon y; \mu < Y) \leq \frac{\varepsilon}{y^\kappa} E[\|V(\mu)\|^\kappa \ I\{\mu < Y\}]. \quad (37) \]

**Proof.** For \( l > \mu \) we have

\[ \Theta_Z(\mu, l) - (Z(\mu, l), C_l) = \sum_{t=1}^{m} \sum_{k=1}^{\varphi_t(l; k)} (\varphi_t(l; k) - E\varphi_t(l)). \]

Now we consider separately the cases \( \kappa \leq 1 \) and \( \kappa > 1 \).
For $\kappa \geq 1$ we use Lemmas 11 and 12 with $\rho = \rho(\kappa/2 \vee 1)$ and $K = K(\kappa/2 \vee 1)$ to get

$$
P \left( |\Theta_Z(\mu, l) - \langle Z(\mu, l), C_l \rangle| > \frac{\varepsilon y}{2n^2} | C_\mu \right) \leq 4^\kappa n^{2\kappa} E \left[ \sum_{i=1}^m \sum_{k=1}^{Z_i(\mu, l)} \langle \varphi_l(k) - E_l \varphi_l(l) \rangle | C_\mu \right] \leq (4m)^\kappa R_\kappa \frac{n^{2\kappa}}{(\varepsilon y)} E \left[ \|Z(\mu, l)\|^{\kappa/2 \vee 1} | C_\mu \right] \sum_{i=1}^m E_l | \varphi_l(l) - E_l \varphi_l(l) |^\kappa \leq (4m)^\kappa R_\kappa \frac{n^{2\kappa}}{(\varepsilon y)} \|V(\mu)\|^{\kappa/2 \vee 1} \rho^l \sum_{i=1}^m E_l | \varphi_l(l) - E_l \varphi_l(l) |^\kappa.
$$

Hence, in view of condition (16) and with $l = \mu + n$ we have

$$
P \left( |\Theta_Z(\mu) - \Gamma(\mu)| > \varepsilon y; \mu < \mathcal{Y} \right) \leq E \sum_{n=1}^\infty P_F \left( |\Theta_Z(\mu, \mu + n) - \langle Z(\mu, \mu + n), C_{\mu + n} \rangle| > \frac{\varepsilon y}{2n^2} | C_\mu \right) I(\mu < \mathcal{Y}) \leq \frac{(4m)^\kappa R_\kappa K_1}{(\varepsilon y)^\kappa} E \left[ \|V(\mu)\|^{\kappa/2 \vee 1} \rho^l \sum_{n=1}^\infty n^{2\kappa} \rho^n I(\mu < \mathcal{Y}) \right] \leq \frac{K_2}{(\varepsilon y)^\kappa} E \left[ \|V(\mu)\|^{\kappa} I(\mu < \mathcal{Y}) \right] \leq \frac{K_3}{(\varepsilon y)^\kappa r^{-(\kappa/2 \vee 1)}} E \left[ \|V(\mu)\|^{\kappa} I(\mu < \mathcal{Y}) \right] \leq \frac{\varepsilon}{y^n} E \left[ \|V(\mu)\|^{\kappa} I(\mu < \mathcal{Y}) \right]
$$

for all $r \geq r(\varepsilon)$.

To analyze the case $\kappa \leq 1$ we apply for $l > \mu$ the inequality

$$
E \left[ |\Theta_Z(\mu, l) - \langle Z(\mu, l), C_l \rangle|^{\kappa} | C_\mu \right] \leq \left( E \left[ |\Theta_Z(\mu, l) - \langle Z(\mu, l), C_l \rangle|^2 | C_\mu \right] \right)^{\kappa/2}
$$

Further, we have

$$
E \left[ |\Theta_Z(\mu, l) - \langle Z(\mu, l), C_l \rangle|^2 | C_\mu \right] \leq m^2 \sum_{i=1}^m E_l \beta_i^2(l) E \left[ Z_i(\mu, l) | C_\mu \right]
$$

Thus, for $\kappa \leq 1$

$$
\left( E \left[ |\Theta_Z(\mu, l) - \langle Z(\mu, l), C_l \rangle|^2 | C_\mu \right] \right)^{\kappa/2} \leq m^\kappa \|V(\mu)\|^{\kappa/2} \left( \sum_{i=1}^m E_l \beta_i^2(l) \right)^{\kappa/2}.
$$
This combined with the assumption (17) shows that for \( l = \mu + n \)

\[
P(\{|\Theta_Z(\mu) - \Gamma(\mu)| > \varepsilon y; \mu < \Upsilon\})
\leq E\left[\sum_{n=1}^{\infty} P\left(\Theta_Z(\mu, \mu + n) - (Z(\mu, \mu + n), C_{\mu+n}) > \frac{\varepsilon y}{2n^2} | C_{\mu}^\top I(\mu < \Upsilon)\right)\right]
\leq \frac{m\kappa K_1}{\varepsilon^\kappa y^\kappa} E\left[\sum_{n=1}^{\infty} n^{2\kappa} \|V(\mu)\|^{\kappa/2} \|\Pi_{\mu,\mu+n}\|^{\kappa/2} I(\mu < \Upsilon)\right]
= \frac{m\kappa K_1}{\varepsilon^\kappa y^\kappa} E\left[\|V(\mu)\|^{\kappa/2} I(\mu < \Upsilon)\right] \sum_{n=1}^{\infty} n^{2\kappa} E\|\Pi_{0,n}\|^{\kappa/2}
\leq \frac{K_2}{\varepsilon^\kappa y^\kappa \varepsilon^{\kappa/2}} E[\|V(\mu)\|^{\kappa} I(\mu < \Upsilon)] \leq \frac{\varepsilon}{y^\kappa} E[\|V(\mu)\|^{\kappa} I(\mu < \Upsilon)]
\]

for all \( r \geq r(\varepsilon) \).

The lemma is proved.

Up to now we have assumed that the initial number of particles and the initial amount of the final product are nonrandom. The next two lemmas are free of this restriction.

Let

\[
\Theta(N) := \sum_{n=0}^{N} \Delta I(n)
\]

**Lemma 19** Let a MBPIFRE be subcritical, and \( n = 1 \) be the starting point of a life period of the process. If there exists \( \delta > 0 \) such that

\[
E\Theta^{\kappa+\delta}(0) < \infty, \quad \max_{1 \leq i \leq m} E\varphi_i^{\kappa+\delta} < \infty, \quad E\psi_i^{\kappa+\delta} < \infty,
\]

then for any \( \varepsilon > 0 \) there exists \( r = r(\varepsilon) \) such that for all \( y \geq y_0 \)

\[
P(\Theta(\mu - 1) > \varepsilon y; \mu < \Upsilon) < \frac{1}{y^{\kappa+\delta/2}}.
\]

**Proof.** Let, as earlier, \( \Upsilon \) be the length of the life period in question. Recalling (2) put

\[
\Theta(N, r) := \Theta(0) + \sum_{l=0}^{N} \left(\psi(l+1) + \sum_{i=1}^{m} \sum_{k=1}^{r} \varphi_i(l;k)\right).
\]
Clearly,
\[
P(\Theta(N-1, r) > \varepsilon y; N < \Upsilon) 
\leq P(\Theta(0) > \frac{\varepsilon y}{3}) P(N < \Upsilon) + P\left(\sum_{i=0}^{N} \psi(i+1) > \frac{\varepsilon y}{3}\right) 
+ P\left(\sum_{i=0}^{N} \sum_{l=1}^{m} l; k > \frac{\varepsilon y}{3}\right) 
\leq \left(\frac{3}{\varepsilon y}\right)^{\kappa+\delta} E\Theta^{\kappa+\delta}(0) P(N < \Upsilon) + (N + 1) P\left(\psi > \frac{\varepsilon y}{3(N + 1)}\right) 
+ (N + 1) r P\left(\sum_{i=1}^{m} \varphi_i > \frac{\varepsilon y}{3(N + 1)r}\right) 
\leq \left(\frac{3}{\varepsilon y}\right)^{\kappa+\delta} E\Theta^{\kappa+\delta}(0) P(N < \Upsilon) + (N + 1) \left(\frac{3(N + 1)}{\varepsilon y}\right)^{\kappa+\delta} E\Theta^{\kappa+\delta} 
+ (N + 1) r \left(\frac{3(N + 1)r}{\varepsilon y}\right)^{\kappa+\delta} E\left(\sum_{i=1}^{m} \varphi_i\right)^{\kappa+\delta}. 
\]

In view of Lemma 13 there exist constants \(K_3\) and \(K_4\) such that for \(y > 0\)
\[
\sum_{N > K_4 \ln y} P(N < \Upsilon) \leq \frac{K_3}{y^{\kappa+\delta}}.
\]

Now using conditions (38) we have
\[
P(\Theta(\mu - 1) > \varepsilon y; \mu < \Upsilon) = \sum_{N=1}^{\infty} P(\Theta(N-1) > \varepsilon y; \mu = N < \Upsilon) 
\leq \sum_{1 \leq N \leq K_4 \ln y} P(\Theta(N-1, r) > \varepsilon y; \mu = N < \Upsilon) 
+ \sum_{N > K_4 \ln y} P(N < \Upsilon) 
\leq \frac{K_5}{(\varepsilon y)^{\kappa+\delta}} \sum_{1 \leq N \leq K_4 \ln y} (N + 1)^{\kappa+\delta+1} 
\leq \frac{K_6 \ln^{\kappa+\delta+2} y}{(\varepsilon y)^{\kappa+\delta}} \leq \frac{1}{y^{\kappa+\delta/2}}.
\]

The lemma is proved.

**Lemma 20** Under the conditions of Lemma 19 there exists \(r\) such that for all \(y \geq y_0\)
\[
P(\Theta > y; \mu > \Upsilon) \leq \frac{1}{y^{\kappa+\delta/2}}. \quad (40)
\]
Proof. Letting $K_3$ and $K_4$ be the same as in the previous lemma we have for $y \geq y_0$

$$P(\Theta > y; \mu > \Upsilon) \leq P(\Theta(\Upsilon, r) > y)$$

$$\leq P(\Theta([K_4 \ln y], r) > y) + P(\Upsilon > K_4 \ln y)$$

$$\leq P(\Theta(0) > \frac{y}{3}) + \sum_{l=0}^{[K_4 \ln y]} P(\psi(l + 1) > \frac{y}{3[K_4 \ln y]})$$

$$+ \sum_{l=0}^{[K_4 \ln y]} \sum_{i=1}^{r} \sum_{k=1}^{r} P(\phi_i(l; k) > \frac{y}{3mr[K_4 \ln y]}) + P(\Upsilon > K_4 \ln y)$$

$$\leq K \frac{\ln \kappa + \delta + 1}{y^{\kappa + \delta}} + \frac{1}{y^{\kappa + \delta}}.$$

The lemma is proved.

6 Proof of Theorem 9

Now we are ready to prove the main result of the paper, Theorem 9. First observe that by the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_2$ and estimates (31),(37), (39) and (40), for any $\varepsilon \in (0, 1/3)$ one can find $r = r(\varepsilon)$ such that for all $y \geq y_0(r, \varepsilon)$

$$P(\Theta > y) \leq P(\langle V(\mu), \Xi_\mu \rangle > y(1 - 3\varepsilon); \mu < \Upsilon)$$

$$+ P(|\Gamma(\mu) - \langle V(\mu), \Xi_\mu \rangle| > \varepsilon y; \mu < \Upsilon) + P(|\Theta_Z(\mu) - \Gamma(\mu)| > \varepsilon y; \mu < \Upsilon)$$

$$+ P(\Theta > y; \mu > \Upsilon) + P(\Theta(\mu - 1) > \varepsilon y; \mu < \Upsilon)$$

$$\leq P(\langle V(\mu), \Xi_\mu \rangle > y(1 - 3\varepsilon); \mu < \Upsilon) + \frac{2\varepsilon}{y^{\kappa}} E[\|V(\mu)\|_2^\kappa I\{\mu < \Upsilon\}] + \frac{2}{y^{\kappa + \delta + 2}}. \tag{41}$$

Let $l(u) > 0$ be the function involved in Condition $T$. By this condition and the independency of $V(\mu)$ and $\Xi_\mu$ given $\mu < \Upsilon$ we conclude

$$\limsup_{y \to \infty} y^\kappa P(\langle V(\mu), \Xi_\mu \rangle > y(1 - 3\varepsilon); \mu < \Upsilon)$$

$$\leq \limsup_{y \to \infty} y^\kappa \int_{\|u\|_2 = r}^{\infty} P(V(\mu) \in du; \mu < \Upsilon) P\left(\left\langle \frac{u}{\|u\|_2}, \Xi_\mu \right\rangle \geq (1 - 3\varepsilon)y\right)$$

$$= K_0(1 - 3\varepsilon)^{-\kappa} \int_{\|u\|_2 = r}^{\infty} P(V(\mu) \in du; \mu < \Upsilon) \|u\|_2^\kappa l\left(\frac{u}{\|u\|_2}\right)$$

$$= K_0(1 - 3\varepsilon)^{-\kappa} E\left[\|V(\mu)\|_2^\kappa I\{\mu < \Upsilon\}\left(\frac{V(\mu)}{\|V(\mu)\|_2}\right) I\{\mu < \Upsilon\}\right] < \infty.$$
Since \( y^\epsilon \mathbb{P} (\Theta > y) \) does not depend on \( r \) and \( \varepsilon \), the previous estimate and (41) yield
\[
\limsup_{y \to \infty} y^\epsilon \mathbb{P} (\Theta > y) < \infty \quad (42)
\]
and, moreover,
\[
\limsup_{y \to \infty} y^\epsilon \mathbb{P} (\Theta > y) \leq K_0 \liminf_{r \to \infty} \mathbb{E} \left[ \|V(\mu)\|^2 \mathbb{I} \left\{ \mu < \Upsilon \right\} \right].
\]
(43)

To get a similar estimate from below we use for \( \varepsilon > 0 \) the inequality
\[
\mathbb{P} (\Theta > y) \geq \mathbb{P} (\Theta_{\geq} (\mu); \mu < \Upsilon)
\]
\[
\geq \mathbb{P} ((V(\mu), \Xi_{\mu}) > y (1 + 2\varepsilon); \mu < \Upsilon) - \mathbb{P} (|\Theta_{\geq} (\mu) - \Gamma (\mu)| > \varepsilon y; \mu < \Upsilon)
\]
\[
- \mathbb{P} (|\Gamma (\mu) - (V(\mu), \Xi_{\mu})| > \varepsilon y; \mu < \Upsilon).
\]

Now we select \( r \) as large to meet estimates (31), (37) and (39). This gives for sufficiently large \( y > r \) the inequality
\[
\mathbb{P} (\Theta > y) \geq \mathbb{P} ((V(\mu), \Xi_{\mu}) > y (1 + 2\varepsilon); \mu < \Upsilon) - \frac{2\varepsilon}{y^\epsilon} \mathbb{E} [\|V(\mu)\|^2 \mathbb{I} \{\mu < \Upsilon\}].
\]

Letting \( y \) tend to infinity we obtain
\[
\liminf_{y \to \infty} y^\epsilon \mathbb{P} ((V(\mu), \Xi_{\mu}) > y (1 + 2\varepsilon); \mu < \Upsilon)
\]
\[
= \liminf_{y \to \infty} y^\epsilon \int_{\|u\|_2 = r}^\infty \mathbb{P} (V(\mu) \in du; \mu < \Upsilon) \mathbb{P} \left( \left\| \frac{u}{\|u\|_2} \times \Xi \right\| \geq (1 + 2\varepsilon)y \right)
\]
\[
= K_0 (1 + 2\varepsilon)^{-\epsilon} \mathbb{E} [\|V(\mu)\|^2 \mathbb{I} \left\{ \mu < \Upsilon \right\} - 2\varepsilon \mathbb{E} [\|V(\mu)\|^2 \mathbb{I} \{\mu < \Upsilon\}]]
\]

Let \( K(l) := \inf_{u \in U_1} l(u) > 0 \). We know that for sufficiently small \( \varepsilon > 0 \) and an appropriate \( r \)
\[
K_0 (1 + 2\varepsilon)^{-\epsilon} \mathbb{E} [\|V(\mu)\|^2 \mathbb{I} \left\{ \mu < \Upsilon \right\} - 2\varepsilon \mathbb{E} [\|V(\mu)\|^2 \mathbb{I} \{\mu < \Upsilon\}]]
\]
\[
\geq (K_0 K(l) (1 + 2\varepsilon)^{-\epsilon} - 2\varepsilon) \mathbb{E} [\|V(\mu)\|^2 \mathbb{I} \{\mu < \Upsilon\}] > 0
\]
which implies
\[
\liminf_{y \to \infty} y^\epsilon \mathbb{P} (\Theta > y) > 0
\]
leading in turn to
\[
\liminf_{y \to \infty} y^\epsilon \mathbb{P} (\Theta > y) \geq K_0 \limsup_{r \to \infty} \mathbb{E} [\|V(\mu)\|^2 \mathbb{I} \left\{ \mu < \Upsilon \right\}].
\]
This combined with (43) shows existence of the limit
\[
\lim_{r \to \infty} \mathbb{E} [\|V(\mu)\|^2 \mathbb{I} \left\{ \mu < \Upsilon \right\}] \in (0, \infty)
\]
and gives
\[
\lim_{y \to \infty} y^\kappa \mathbf{P}(\Theta > y) = K_0 \lim_{r \to \infty} \mathbb{E} \left[ \|\mathbf{V}(\mu)\|_2^\kappa \right] I\{\mu < \Upsilon\}
\]
proving (19). The validity of (20) is now evident.

The theorem is proved.

7 Polling systems

We consider a polling system consisting of a single server and \( m \) stations with infinite-buffer queues indexed by \( i \in \{1, \ldots, m\} \). Initially there are no customers in the system and the server waits their arrival at a parking place \( R \). Customers arrive to the queues in accordance with a point process whose parameters are changing in a random manner each time when the server switches from station to station. When a customer arrives to the system, say to station \( i \), the server immediately begins the service by visiting the stations in cyclic order (\( i \to i+1 \to \cdots \to m \to 1 \to \cdots \)) starting at station \( i \) according to a selected service policy (to be described later on) and with random positive switch-over times between the queues. Later on the initial stage of services (\( i \to i+1 \to \cdots \to m \to \cdots \to i-1 \)) will be called the zero cycle. The subsequent routes of the server will be called the first cycle, the second cycle and so on. The server immediately moves to its parking place if there are no customers in the system to be served.

Our goal is to investigate various characteristics related with busy periods of the system whose work starts by the arrival of a single customer to a station \( i \in \{1, \ldots, m\} \) at moment 0.

To give a rigorous description of the arrival and service processes for the system in question we need some notions. Let
\[
\chi^{(i)}(s;\lambda) := \mathbb{E} \left[ s_1^{\theta_{i1}} s_2^{\theta_{i2}} \cdots s_m^{\theta_{im}} e^{-\lambda \phi_i} \right], i = 1, \ldots, m
\]
be the mixed probability generating functions (m.p.g.f.’s) of \((m+1)-\)dimensional vectors \((\theta_{i1}, \ldots, \theta_{im}; \phi_i)\), where \(\theta_{ij}\) are nonnegative integer-valued random variables and \(\phi_i\) is a nonnegative random variable. Denote
\[
\chi(s;\lambda) := \left( \chi^{(1)}(s;\lambda), \ldots, \chi^{(m)}(s;\lambda) \right)
\]
the respective vector-valued m.p.g.f.

Let, further,
\[
\rho^{(i)}(s;\lambda) := \mathbb{E} \left[ s_1^{\zeta_{i1}} s_2^{\zeta_{i2}} \cdots s_m^{\zeta_{im}} e^{-\lambda \sigma_i} \right], i = 1, \ldots, m
\]
be the m.p.g.f.’s of \((m+1)-\)dimensional vectors \((\zeta_{i1}, \ldots, \zeta_{im}; \sigma_i)\), where \(\zeta_{ij}\) are nonnegative integer-valued random variables and \(\sigma_i\) is a nonnegative random variable. Denote
\[
\rho(s;\lambda) := \left( \rho^{(1)}(s;\lambda), \ldots, \rho^{(m)}(s;\lambda) \right)
\]
the respective vector-valued m.p.g.f. and let $\mathcal{H} = \{ (\chi(s;\lambda), \rho(s;\lambda)) \}$ be the set of pairs of the described m.p.g.f.’s. Assume that a probability measure $\mathbb{P}$ is specified on the natural $\sigma$-algebra of $\mathcal{H}$ and let

$$(\chi_0(s;\lambda), \rho_0(s;\lambda)), (\chi_1(s;\lambda), \rho_1(s;\lambda)), (\chi_2(s;\lambda), \rho_2(s;\lambda)), \ldots,$$

be a sequence of pairs $(\chi_n(s;\lambda), \rho_n(s;\lambda))$ selected in iid manner from $\mathcal{H}$ according to measure $\mathbb{P}$, where $\chi_n(s;\lambda) := \left( \chi_n^{(1)}(s;\lambda), \ldots, \chi_n^{(m)}(s;\lambda) \right)$ and $\rho_n(s;\lambda) := \left( \rho_n^{(1)}(s;\lambda), \ldots, \rho_n^{(m)}(s;\lambda) \right)$ have the components

$$\chi_n^{(i)}(s;\lambda) := \mathbb{E} \left[ \frac{\theta_1(n)}{s_1} \frac{\theta_2(n)}{s_2} \cdots \frac{\theta_m(n)}{s_m} e^{-\lambda \phi_i(n)} \right],$$

$$\rho_n^{(i)}(s;\lambda) := \mathbb{E} \left[ \frac{\zeta_1(n)}{s_1} \frac{\zeta_2(n)}{s_2} \cdots \frac{\zeta_m(n)}{s_m} e^{-\lambda \sigma_i(n)} \right].$$

In the present paper we investigate such polling systems whose arrival and service procedures of customers meet the following two conditions.

**Branching property.** When the server arrives to station $i$ for the $n$-th time and find there, say, $k_i$ customers labelled $1, 2, \ldots, k_i$, then, during the course of the server’s visit, the arrival of customers is arranged in such a way that after the end of each stage of service of customer $j$ (the number of such stages may be more than one if the service discipline admits feedback) the queues at the system will be increased by a random population of customers $(\theta_{i1}(n, j), \ldots, \theta_{im}(n, j))$, where $\theta_{il}(n, j)$ is the number of customers added to the $l$-th station, and, in addition, a final product of size $\phi_i(n, j) \geq 0$ will be added to the system. It is assumed that the vectors $(\theta_{i1}(n, j), \ldots, \theta_{im}(n, j); \phi_i(n, j)), j = 1, 2, \ldots, k_i$ are iid and such that

$$\mathbb{E} \left[ \frac{\theta_{i1}(n, j)}{s_1} \frac{\theta_{i2}(n, j)}{s_2} \cdots \frac{\theta_{im}(n, j)}{s_m} e^{-\lambda \phi_i(n, j)} \right] = \chi_n^{(i)}(s;\lambda).$$

Note that $\chi_n^{(i)}(s;\lambda)$ is a random m.p.g.f. and, therefore, the parameters of the polling system are changed from cycle to cycle in a random manner.

**Immigration property.** Upon departing from the $i$-th station for the $n$-th time the server switches to the queue existing at station $(i + 1)(\text{mod} \ m)$. The process of switching takes a random time at the end of which the queues at the system will be increased by a random population of customers $(\zeta_{i1}(n), \ldots, \zeta_{im}(n))$, where $\zeta_{il}(n)$ is the number of customers added to the $l$-th station, and, in addition, a final product of size $\sigma_i(n) \geq 0$ will be added to the system. It is assumed that the vectors $(\zeta_{i1}(n), \ldots, \zeta_{im}(n); \sigma_i(n)), n = 1, 2, \ldots$ are iid and such that

$$\mathbb{E} \left[ \frac{\zeta_{i1}(n)}{s_1} \frac{\zeta_{i2}(n)}{s_2} \cdots \frac{\zeta_{im}(n)}{s_m} e^{-\lambda \sigma_i(n)} \right] = \rho_n^{(i)}(s;\lambda).$$

In the sequel we call the described polling systems as the branching type polling systems with final product and random environment (BTPSFPRE).

Polling systems are rather popular subject of investigations in queueing theory (see surveys [25], [35] and [43] for definitions and more details). For instance,
polling systems possessing the branching property in which the probability generating functions \( h_n^{(i)}(s) \), \( n = 1, 2, \ldots \) are nonrandom and identical for all cycles were considered in \([12],[36],[41]\) and quite recently in \([4],[5]\). These models cover many classical service policies, including the exhaustive, gated, binomial-gated and their feedback modifications.

Polling systems with input parameters and service disciplines changing in a random manner and (or) depending on the states of systems were investigated by the fluid method in \([9]-[11]\) and by the method based on the construction of appropriate Lyapunov functions in \([13]-[15]\). The present paper may be viewed as a compliment to article \([16]\) in which the BTPSFPRE with zero switchover times were analyzed by means of MBPRE without immigration.

Before we pass to general results, consider two examples of BTPSFPRE.

Let \( \mathcal{T}_+ = \{T\} \) be the set of all probability distributions of nonnegative random variables,

\[
\mathcal{T}_+^m \times \mathcal{I}_+^m := \{(T_1, \ldots, T_m) \times (I_1, \ldots, I_m) : T_i \in \mathcal{T}_+, I_j \in \mathcal{I}_+, i, j = 1, \ldots, m\}
\]

be the set of all \( 2m \)-dimensional tuples of such distributions, \( \mathcal{M}_\varepsilon = \{\mathcal{E}\} \) be the set of all \( m \times m \) matrices \( \mathcal{E} = (\varepsilon_{ij})_{i,j=1}^m \) with nonnegative elements, and \( \mathcal{M}_\gamma = \{\Gamma\} \) be the set of all \( m \times (m+1) \) matrices \( \Gamma = (\gamma_{ij})_{i=1,j=0}^m \) with nonnegative elements such that

\[
\sum_{j=0}^m \gamma_{ij} = 1, \quad i = 1, \ldots, m, \quad \max_{1 \leq i \leq m} \gamma_{i0} > 0.
\]

Let \( \mathcal{P} \) be a measure on the Borel \( \sigma \)-algebra of the space

\[
\mathcal{S} := \mathcal{M}_\varepsilon \times \mathcal{M}_\varepsilon \times \mathcal{M}_\gamma \times \mathcal{T}_+^m \times \mathcal{I}_+^m.
\]

**Example 21** (motivated by \([15]\)). Consider a polling system with \( m \) stations and a single server performing cyclic service of the customers at the stations. Assume that initially there are no customers in the system and the server is located at a parking place \( R \). Assume that given the idle system the flow of customers arriving to station \( l \) is Poisson with, say, a deterministic rate \( \varepsilon_l \). When the first customer appears in the system the server selects a random element \( (\mathcal{E}_0, \mathcal{E}_0^I, \Gamma_0, T_0, I_0) \) from \( \mathcal{S} \) with

\[
\mathcal{E}_0 = (\varepsilon_{ij}(0))_{i,j=1}^m, \quad \mathcal{E}_0^I = (\varepsilon_{ij}^I(0))_{i,j=1}^m, \quad \Gamma_0 = (\gamma_{ij}(0))_{i=1,j=0}^m, \quad T_0 = (T_{10}, \ldots, T_{m0}), \quad I_0 = (I_{10}, \ldots, I_{m0})
\]

and immediately starts its zero service cycle \((i \rightarrow i + 1(\text{mod} \ m) \rightarrow \cdots \rightarrow i - 1(\text{mod} \ m))\) adopting the gated server policy with random switch-over times. Namely, the server serves all the customers that were queueing, say at station \( i \) when the server arrived and then proceeds to the next (in cyclic order) station with switch-over times \( \sigma_i(0) \) distributed according to \( I_{i0}(x) := \mathcal{P}(\sigma_i(0) \leq x) \). During the switch-over period \( \sigma_i(0) \) new customers arrive to
the system according to independent Poisson flows with intensities given by the 
vector \((\varepsilon_{i1}(0), \varepsilon_{i2}(0),...,\varepsilon_{im}(0))\) (some of the components may be equal to zero).

For the period while the server performs the batch of services at station \(i\) new customers arrive to the system according to independent Poisson flows with 
intensities given by the vector \((\varepsilon_{i1}(0), \varepsilon_{i2}(0),...,\varepsilon_{im}(0))\) (some of the components 
may be equal to zero) and the service times of customers are iid and distributed 
according to \(T_{00}(x):=\P(\tau_i(0) \leq x)\). Each served customer either goes to station 
\(j\) with probability \(\gamma_{ij}(0)\) or leaves the system with probability \(\gamma_{i0}(0)\) independently of other events. Besides, after the end of each service stage of a 
customer the customer contributes to the system the service time at this stage 
as the final product.

It is assumed that given \((\mathcal{E}_0, \mathcal{E}_0^J, \Gamma_0, T_0 \times I_0)\) the service times, switch-over 
times and the arrival process of new customers are independent.

The subsequent routes \(n=1,2,\ldots\) have the same probabilistic structure speci-
fied by the tuples

\[
\begin{align*}
\mathcal{E}_n &= (\varepsilon_{ij}(n))_{i,j=1}^m, \quad \mathcal{E}_n^J = (\varepsilon_{ij}^J(n))_{i,j=1}^m, \\
T_n &= (T_{1n}, ..., T_{mn}), \\
I_n &= (I_{1n}, ..., I_{mn})
\end{align*}
\]

with only difference that at the beginning of cycle \(n \geq 1\) there is a possibility to 
have more than one customer in the system.

One can show (see [16]) that this system possesses a branching property in 
which the m.p.g.f. \(\chi_n(s;\lambda)\) has the components (in our setting and the service 
time of customers as the final product)

\[
\chi_n^{(i)}(s;\lambda) := \E \left[ s_1^{\theta_{i1}(n)} s_2^{\theta_{i2}(n)} \cdots s_m^{\theta_{im}(n)} e^{-\lambda \tau_i(n)} \right] \\
= \left( \gamma_{i0}(n) + \sum_{j=1}^m \gamma_{ij}(n) s_j \right) t_{in} \left( \lambda + \sum_{j=1}^m \varepsilon_{ij}(n)(1-s_j) \right), (44)
\]

where

\[
t_{in}(\lambda) := \int_0^{\infty} e^{-\lambda x} dT_{in}(x),
\]

and an immigration property with \(\rho_n(s;\lambda)\) having components

\[
\rho_n^{(i)}(s;\lambda) := \E \left[ s_1^{\zeta_{i1}(n)} s_2^{\zeta_{i2}(n)} \cdots s_m^{\zeta_{im}(n)} e^{-\lambda \tau_i(n)} \right] \\
= \int_0^{\infty} \exp \left\{ - \left( \lambda + \sum_{j=1}^m \varepsilon_{ij}^J(n)(1-s_j) \right) x \right\} dI_{in}(x). \quad (45)
\]

**Example 22** (compare with [15]). Consider the same polling system as earlier 
but assume now that at each station the server adopts the exhaustive server 
policy: it serves all the customers that were queueing at the station when the 
server arrived together with all subsequent arrivals up until the queue at this 
station becomes empty and then switches over to the next station.
Let
\[ w_{in}(\lambda) := \frac{1 - \gamma_{ii}(n)}{1 - \gamma_{ii}(n)t_{in}(\lambda)} t_{in}(\lambda) \] (46)
and
\[ y_{in}(s) := \frac{\gamma_{ii}(n) + \sum_{j \neq i} \gamma_{ij}(n) s_j}{1 - \gamma_{ii}(n)}. \]

It is known (see, for instance, [16]) that the system above possesses a branching property with \( \phi_n(s; \lambda) \) whose components are unique solutions of the equations
\[ \chi_n^{(i)}(s; \lambda) = \frac{\gamma_{ii}(n) + \sum_{j \neq i} \gamma_{ij}(n) s_j}{1 - \gamma_{ii}(n)} \]
\[ + \lambda + \sum_{j \neq i} \varepsilon_{ij}(n)(1 - s_j) + \varepsilon_{ii}(n)(1 - \chi_n^{(i)}(s; \lambda)). \]

The immigration property for this system fulfills with \( \rho_n(s; \lambda) \) specified by (45).

Now we come back to the general case. We assume that the server may start the service of a customer of any type arrived to an idle system immediately and would like to study the distribution of the total size of the final product accumulated in the system during a busy period of the server. For this reason the law of arrival of customers to an idle system plays no role for the subsequent arguments. The only assumption we need that the probability of arrival two or more customer to an idle system simultaneously is zero.

Set
\[ \eta_n^{(i)}(s) := \chi_n^{(i)}(s; 0), i = 1, 2, \ldots, m \]
and for \( n = 0, 1, 2, \ldots \) introduce m.p.g.f.'s
\[ F_n^{(i)}(s; \lambda) = \mathbb{E} \left[ \xi_{i1}(n) \xi_{i2}(n) \ldots \xi_{im}(n)e^{-\lambda \varepsilon_i(n)} \right] \]
and p.g.f.'s
\[ f_n^{(i)}(s) = \mathbb{E} \left[ s_1 \xi_{i1}(n) \xi_{i2}(n) \ldots \xi_{im}(n) \right] \]
by the equalities
\[ F_n^{(m)}(s; \lambda) := \chi_n^{(m)}(s; \lambda), \]
\[ F_n^{(i)}(s; \lambda) := \chi_n^{(i)} \left( s_1, \ldots, s_i, F_n^{(i+1)}(s; \lambda), \ldots, F_n^{(m)}(s; \lambda); \lambda \right), i < m, \] (47)
and
\[ f_n^{(m)}(s) := \eta_n^{(m)}(s), \]
\[ f_n^{(i)}(s) := \eta_n^{(i)} \left( s_1, \ldots, s_i, f_n^{(i+1)}(s), \ldots, f_n^{(m)}(s) \right), i < m. \] (48)

Further, set
\[ l_n^{(i)}(s) := \rho_n^{(i)}(s; 0), i = 1, 2, \ldots, m \]
and for \( n = 0, 1, 2, \ldots \) introduce m.p.g.f.'s
\[ G_n(s; \lambda) := \mathbb{E} \left[ s_1^{m(n)} s_2^{m(n)} \ldots s_m^{m(n)} e^{-\lambda \nu(n)} \right] \]
\[ = \prod_{i=1}^{m} \rho_n^{(i)} \left( s_1, \ldots, s_i, F_n^{(i+1)}(s; \lambda), \ldots, F_n^{(m)}(s; \lambda); \lambda \right) \] (49)
and p.g.f.’s

\[ g_n(s) := E \left[ s_1^{n_1(n)} s_2^{n_2(n)} \cdots s_m^{n_m(n)} \right] \]

\[ = \prod_{i=1}^{m} l_n^{(i)} \left( s_1, \ldots, s_i, f_n^{(i+1)}(s), \ldots, f_n^{(m)}(s) \right). \]

We would like to describe conditions on the branching type polling system under which power moments of the amount of the final product accumulated in the system during its busy period are finite or infinite. Note that letting the final product \( \phi_i(n,j) \) be the service time of the \( j \)-th customer served during the \( n \)-th visit of the server to station \( i \) and \( \sigma_i(n) \) be the switch-over time from station \( i \) to station \( (i+1) \mod m \) after the \( n \)-th visit of the server to station \( i \) we provide conditions under which the tail distribution of the length of the busy period of a polling system decays, as \( y \to \infty \), like \( y^{-\kappa} \) for some \( \kappa > 0 \).

Our results are based on an important statement revealing connections between the behavior of certain characteristics of the busy periods of BTPSFPRE and the related characteristics of life-length periods of a MBPIFPRE. To formulate this statement we introduce the notion of generalized busy period as follows.

If a busy period of a BTPSFPRE starts by the arrival of a single customer at station \( J \) then, after the end of the busy period generated by this customer we continue to follow the performance of the system (with the described laws of arrival of customers within switch-over times, the service disciplines and accumulating of final product) up to the first visit of the server to station \( J \) when there are no customers in the system.

**Definition 23** *The time interval between the start of service of the first customer and the moment when, for the first time, the system is idle after the end of the switch-over time from station \( J-1 \) to station \( J \) is called a generalized busy period.*

Denote by \( \mathcal{M}_J \) the length of the respective generalized busy period.

Since a generalized busy period includes a possibility to have an idle system at the moment when the server arrives to station \( J-1 \mod m \) but have customers at the moment when the server arrives to station \( J \) (which arrive to the system during the switch-over time \( J-1 \to J \)), the length of a generalized busy period is not less than the length of the standard busy period.

In the sequel we assume without loss of generality that \( J = 1 \).

**Theorem 24** *If a generalized busy period of a BTPSFPRE starts by the arrival of a single customer at station 1 then the distribution of the amount of the final product accumulated in the system to the end of the generalized period coincides with the distribution of the total amount of the final product produced in a MBPIFPRE within a life period which starts at moment 0 by the birth of a single particle of type 1 and where the joint distribution of the number of direct descendants and immigrants and the amount of the final product produced by*
particles of different types of the $k$-th generation is given by the vector-valued m.p.g.f.’s

$$H_k(s; \lambda) := \left( F_k^{(1)}(s; \lambda), \ldots, F_k^{(m)}(s; \lambda); G_{k+1}(s; \lambda) \right), \; k = 1, 2, \ldots, n$$

specified by (47) and (49).

To prove this theorem one should modify in a natural and evident way the proof of Theorem 3 in [36] and we omit the respective arguments.

The MBPIFPRE described in Theorem 24 will be called the associated for the BTPSFPRE.

Now we apply the obtained earlier results for MBPIFPRE to our polling system.

Let $H_n := (h_{ij}(n))_{i,j=1}^m$ be the matrix with elements

$$h_{ij}(n) := \frac{\partial h_n^{(i)}(s)}{\partial s_j} \bigg|_{s=1} = E_{h_n} \theta_{ij}(n),$$

and $A_n := (a_{ij}(n))_{i,j=1}^m$ be the matrix with elements

$$a_{ij}(n) := \frac{\partial f_n^{(i)}(s)}{\partial s_j} \bigg|_{s=1} = E_{f_n} \xi_{ij}(n). \quad (51)$$

Then in view of (48) $a_{mj}(n) = h_{mj}(n), j = 1, 2, \ldots, m$ and for $i < m$

$$a_{ij}(n) = h_{ij}(n) I \{ j \leq i \} + \sum_{k=i+1}^m h_{ik}(n) a_{kj}(n). \quad (52)$$

For $i = 1, \ldots, m$ introduce auxiliary matrices

$$H_n^{(i)} := \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & \cdots \\
h_{i1}(n) & h_{i2}(n) & \cdots & h_{i(i-1)}(n) & h_{i,i}(n) & h_{i(i+1)}(n) & \cdots & \cdots & h_{im}(n) \\
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & 0 & 0 & 1 & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix},$$

where for each $i$ the elements of the matrix $H_n$ are located only in row $i$ of the matrix $H_n^{(i)}$. It is not difficult to check by (52) that

$$A_n = H_n^{(1)} H_n^{(2)} \cdots H_n^{(m)}. \quad (53)$$
Further, let $C_n := (C_1(n), \ldots, C_m(n))'$ be a random vector with components
\[ C_i(n) := \frac{dF^{(i)}_n(\lambda)}{d\lambda} \bigg|_{\lambda=0} = E_{F_n} \phi_i(n), \ i = 1, \ldots, m \]
and let $c_n := (c_1(n), \ldots, c_m(n))'$ be a random vector with components
\[ c_i(n) := \frac{d\phi^{(i)}_n(\lambda)}{d\lambda} \bigg|_{\lambda=0} = E_{\phi_n} \phi_i(n), \ i = 1, \ldots, m. \]

Then, by (47) $C_m(n) = c_m(n)$ and, for $i < m$
\[ C_i(n) = c_i(n) + \sum_{k=i+1}^{m} h_{ik}(n) C_k(n). \]

Hence we get $C_n = (E - H_n^\Delta)^{-1} c_n$ where
\[ H_n^\Delta := (h_{ij}(n) I (i < j))_{i,j=1}^{m} \]
is the upper triangular matrix generated by $H_n$.

Let, further, $L_n := (l_{ij}(n))_{i,j=1}^{m}$ be a random matrix with elements
\[ l_{ij}(n) = \frac{\partial g_n(s)}{\partial s_j} \bigg|_{s=1} = E_{l_n} \zeta_{ij}(n), \]
and let
\[ p_i(n) := \frac{d\rho_n^{(i)}(\lambda)}{d\lambda} \bigg|_{\lambda=0} = E_{\rho_n} \sigma_i(n), \ i = 1, \ldots, m. \]

Then, by (49) and (50)
\[ B_j(n) = \frac{\partial g_n(s)}{\partial s_j} \bigg|_{s=1} = \sum_{i=1}^{j-1} l_{ij}(n) + \sum_{i=1}^{m} \sum_{k=i+1}^{m} l_{ik}(n) a_{kj}(n) \]
and
\[ D_n = E_{G_n} \psi(n) = \frac{dG_n(1, \lambda)}{d\lambda} \bigg|_{\lambda=0} = \left( \sum_{i=1}^{m} p_i(n) + \sum_{k=i+1}^{m} l_{ik}(n) C_k(n) \right). \]

The next two statements are easy consequences of Theorem 9.

**Theorem 25** Assume that the MBPIFPRE associated with a BTPSFPRE is subcritical and satisfies conditions of Theorem 9. Then there exist constants $C_1, C_2 \in (0, \infty)$ such that
\[ C_1 y^{-\kappa} \leq P(\Theta_P > y) \leq C_2 y^{-\kappa}, \ y \geq y_0. \]

In particular, if the final product of any customer is its service time then the tail distribution of the length $\Theta_P$ of a busy period of the system satisfies (54).

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Corollary 26 Under the conditions of Theorem 25 $E\Theta_P < \infty$ if and only if $x \in (0, \kappa)$.

Proof of Theorem 25. We associate with the initial polling system an auxiliary polling system obtained from the initial one by excluding certain customers and the final product contributed by them. We split the exclusion process into several stages.

At the first stage we exclude from the initial system all the customers (and the final product contributed by them) which arrive to the system during the switch-over times of the server. We call these customers the customers of the first level.

At the second stage we exclude from the initial system all the customers (and the final product contributed by them) which arrive to the system during the service times of the first level customers. We call these customers the customers of the second level.

At the $k$-th stage we exclude from the initial system all the customers (and the final product contributed by them) which arrive to the system during the service times of the $(k-1)$-th level customers. And so on.

Denote by $\Phi$ – the total amount of the final product accumulated in the auxiliary branching type polling system. It is clear that $\Theta_P \geq \Phi$. As shown in [16], under the conditions of Theorem 25 $P(\Phi > y) \sim Ky^{-\kappa}$ as $y \to \infty$. Hence

$$P(\Theta_P > y) \geq P(\Phi > y) \geq C_1 y^{-\kappa}, y \geq y_0.$$ 

This proves the needed bound from below.

To get the desired estimate from above note that $\Theta_P \leq \Theta_{ext}$ where $\Theta_{ext}$ is the total amount of the final product accumulated in the analyzed branching type polling system within the generalized busy period. By Theorem 24 $\Theta_{ext} \overset{d}{=} \Theta$ where $\Theta$ is the total amount of the final product produced in the associated MBPIFPRE within a life period which starts at moment 0 by the birth of a type 1 particle. Using now Theorem 9 we obtain

$$P(\Theta_P > y) \leq P(\Theta > y) \sim C_1 y^{-\kappa}, y \to \infty,$$

which gives the desired estimate from above.

For completeness we formulate the following statement concerning polling systems with $\alpha > 0$.

Theorem 27 Assume that the MBPIFPRE associated with a BTPSFPRE is such that its underlying MBPRE satisfies conditions of Theorem 5 with $\alpha > 0$ and, in addition,

$$P\left(\min_{1 \leq i \leq m} E_{i} \varphi_i > 0\right) > 0.$$ 

If $\Theta_P$ is the total size of the final product accumulated in the BTPSFPRE during a busy period then $P(\Theta_p = \infty) > 0$. In particular, if the service time of any customer at any station is positive with probability 1 then the busy period of the BTPSFPRE is infinite with positive probability.
**Proof.** As shown in [16] under the conditions of Theorem 27 $P (\Phi = \infty) > 0$ where $\Phi$ is the same as in the proof of Theorem 25. Hence the desired statement follows.

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