ON THE DEFORMATION THEORY OF PAIR \((X, E)\)

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Abstract. Huybrechts and Thomas recently constructed relative obstruction theory of objects of the derived category of coherent sheaves over smooth projective family. In this paper, we use this construction to obtain the absolute deformation-obstruction theory of the pair \((X, E)\), with \(X\) smooth projective scheme and \(E\) perfect complex, and show that the obstruction theories for \(E\), \((X, E)\), and \(X\) fit into exact triangle as derived objects on the moduli space.

1. Introduction

The deformation theory of objects of the derived category of coherent sheaves on smooth projective variety has been studied in [5, 6] and developed recently in [2]. The latter uses Illusie’s cotangent complex and Atiyah class [3] to show that the obstruction class is the product of Atiyah and Kodaira-Spencer classes, and describe the relative obstruction theory (in the sense of [1]) for moduli space of perfect simple complexes on smooth projective families of threefold, which is used to obtain virtual cycle generalizing the virtual counting in [7] and [9].

In this paper, we show how the setting of relative obstruction theory in [2] can be used to obtain the absolute obstruction theory of the pair \((X, E)\), with \(X\) smooth projective scheme and \(E\) perfect complex of coherent sheaves. Specifically, given a perfect complex \(E\) on \(X\), Illusie’s Atiyah class gives an element

\[ A(E) \in \text{Ext}^1_X(E, E \otimes L_X^\bullet) \]

where \(L_X^\bullet\) is Illusie’s cotangent complex, which is quasi-isomorphic to the cotangent bundle \(\Omega_X\) in our case when \(X\) is smooth. View it as a map in the derived category

\[ A(E) : R\text{Hom}(E, E)[−1] \to \Omega_X \]

and let \(G\) be the mapping cone. Then the tangent space of deforming the pair is

\[ \text{Ext}^1_X(G, \mathcal{O}_X) \]

and the obstruction space lies in

\[ \text{Ext}^2_X(G, \mathcal{O}_X) \]

The exact triangle

\[ R\text{Hom}(E, E)[−1] \to \Omega_X \to G \to R\text{Hom}(E, E) \]

naturally puts the tangent-obstruction spaces of deforming the complex \(E\) fixing \(X\), deforming the pair \((X, E)\) and deforming the scheme \(X\) into a long exact sequence.

As an application, we specialize to the case that \(E\) is a vector bundle on \(X\). The complex \(G\) can be explicitly described in this case, and we recover the fact that the tangent and obstruction space for deforming the pair \((X, E)\) is obtained via the first and second cohomology of the sheaf of differential operators of order \(\leq 1\) with diagonal symbol (4.2).

Notation. \(k\) is fixed to be algebraic closed field of characteristic zero. We use standard notations for derived functors, for instance, \(L\pi^*\) is the derived pull-back by \(\pi\), \(R\pi_*\) is the derived push-forward by \(\pi\), \((\cdot)^v\) is the derived dual, and \(R\mathcal{H}om\) is the derived \(\mathcal{H}om\).
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2. Moduli of Pair and Relative obstruction theory

In this section, we consider a flat family of smooth projective varieties of dimension $n$ which is universal at every point of $S$. Denote by $i_s : X_s \hookrightarrow X$ the fiber of $X$ over a closed point $s \in S$ and similar notation for other families. Let $\mathcal{M}/S$ be a relative fine moduli space of perfect complexes over $X/S$ (see [2] for more detail). There is a perfect complex (the universal complex for the moduli space)

$$E \in D^b(\mathcal{M} \times_S X)$$

such that $\mathcal{M}/S$ represents the functor which associates any scheme $T$ over $S$ the set of equivalence classes of perfect complexes $E$ over $T \times_S X$ whose restriction to any fiber to $T \times_S X$ is isomorphic to the restriction of $E$ to some fiber of $\mathcal{M} \times_S X/\mathcal{M}$.

Consider the cartesian diagram

$$
\begin{array}{ccc}
\mathcal{M} \times_S X & \xrightarrow{p_X} & X \\
\downarrow \pi & & \downarrow \pi_S \\
\mathcal{M} & \xrightarrow{p_S} & S
\end{array}
$$

where we denote by $\pi_M, \pi_S, p_X, p_S$ the corresponding morphisms as in the diagram. Let $m \in \mathcal{M}$ be a closed point of $\mathcal{M}$, $s = p_S(m)$, $E_m = i^*_m E$ the restriction of $E$ over $m$, then we get a pair $(X_s, E_m)$

Since we assume that $S$ is universal at $s$, the moduli space $\mathcal{M}$ at $m$ actually parameterizes the local deformation space of the pair $(X_s, E_m)$.

A relative obstruction theory for $\mathcal{M}/S$ is constructed via Atiyah class in [2]. We review their construction which will be generalized in the next section to the absolute case. Let $A(E) \in \text{Ext}_M^1(E, E \otimes L^*_{\mathcal{M} \times_S X})$ be Illusie’s Atiyah class. Here $L^*_{\mathcal{M} \times_S X}$ is the cotangent complex. We denote by

$$A_{\pi_M}(E) \in \text{Ext}_M^1(E, E \otimes L^*_{\mathcal{M} \times_S X}/\mathcal{M})$$

$$A_{p_X}(E) \in \text{Ext}_M^1(E, E \otimes L^*_{\mathcal{M} \times_S X}/X)$$

the image of $A(E)$ via the two maps of cotangent complexes

$$L^*_{\mathcal{M} \times_S X} \to L^*_{\mathcal{M} \times_S X}/\mathcal{M}$$

$$L^*_{\mathcal{M} \times_S X} \to L^*_{\mathcal{M} \times_S X}/X$$

Since the map $\pi_S$ is flat, we actually have canonical isomorphisms

$$L^*_{\mathcal{M} \times_S X}/\mathcal{M} \simeq L^p_X(\mathcal{L}_{X/S})$$

$$L^*_{\mathcal{M} \times_S X}/X \simeq L\pi^*_M(L^*_{\mathcal{M}/S})$$
The class $A_{p_X}(E)$ gives a map in the derived category
\[ R\text{Hom}(E, E)[-1] \rightarrow L_{\pi_M}^*L_{M/S}^* \]
which by Verdier duality along the projective morphism $\pi_M$ gives a map
\[ R\pi_M^*(R\text{Hom}(E, E) \otimes \omega_{\pi_M})[n-1] \rightarrow L_{M/S}^* \]
where $\omega_{\pi_M}$ is the relative dualizing sheaf along $\pi_M$.

**Theorem 2.1** ([2]). *The map (2.2) is a relative obstruction theory for $M/S$.*

This means the map (2.2) has the property that $h^{-1}$ is epimorphism, $h^0$ is isomorphism [1].

**Remark 2.2.** *Note that only part of the full Atiyah class $A(E)$, i.e. $A_{p_X}(E)$ is used to obtain the relative obstruction theory. The other part $A_{\pi_M}(E)$ will also be used to obtain the absolute obstruction theory as we will show in the next section.*

### 3. Atiyah Class and Obstruction Theory of the Pair

We will keep the same notation in this section as above. Since $X/S$ is a smooth family, the relative cotangent complex $L_{X/S}^*$ is in fact isomorphic to the one-term locally free sheaf of relative differentials
\[ L_{X/S}^* \cong \Omega_{X/S} \]

The Atiyah class $A_{\pi_M}(E)$ can be written explicitly as an exact sequence of complexes
\[ A_{\pi_M}(E) : 0 \rightarrow L_{M \times S X/M}^* \otimes E \rightarrow E_{\pi_M} \rightarrow E \rightarrow 0 \]

where $E_{\pi_M}$ is isomorphic to $E \oplus \Omega_{M \times S X/M} \otimes E \cong E \oplus L_{M \times S X/M}^* \otimes E$ as $k$-linear spaces, but with $\mathcal{O}_{M \times S X}$-module structure given by
\[ a \cdot (e_1 \oplus e_2 \otimes db) = (a e_1 \oplus a e_2 \otimes db + e_1 \otimes da) \]

for $a, b \in \mathcal{O}_{M \times S X}, e_1, e_2 \in E$. Use the canonical isomorphism
\[ \text{Ext}^1_{M \times S X}(E, E \otimes L_{M \times S X}^*) \cong \text{Ext}^1_{M \times S X}(R\text{Hom}(E, E), L_{M \times S X}^*) \]

we can write $A_{\pi_M}(E)$ as a map
\[ A_{\pi_M}(E) : \text{RHom}(E, E)[-1] \rightarrow L_{M \times S X/M}^* \]

We define the complex $G$ to be the mapping cone of the above map. We get exact triangle
\[ R\text{Hom}(E, E)[-1] \rightarrow L_{M \times S X/M}^* \rightarrow G \rightarrow R\text{Hom}(E, E) \]

Note that we have commutative diagram of cotangent complexes
\[ \begin{array}{c}
L_{M \times S X}^* \\
\downarrow \\
L_{M \times S X/M}^* \\
\downarrow \\
L_{M \times S X/M}^* L_{p_S}^* L_{S}^*[1] \cong L_{p_X}^* L_{p_S}^* L_{S}^*[1]
\end{array} \]

Combined with the Atiyah class $R\text{Hom}(E, E)[-1] \rightarrow L_{M \times S X}^*$, we get commutative diagram
\[ \begin{array}{c}
R\text{Hom}(E, E) \\
\downarrow \\
L_{M \times S X/M}^*[1] \\
\downarrow \\
L_{M \times S X/M}^*[1] \cong L_{p_X}^* L_{p_S}^* L_{S}^*[2] \cong L_{p_X}^* L_{p_S}^* L_{S}^*[2]
\end{array} \]

Note that we also have exact triangle
\[ L_{p_S}^* L_{S}^* \rightarrow L_{M}^* \rightarrow L_{M/S}^* \rightarrow L_{p_S}^* L_{S}^*[1] \]
Pull it back to $\mathcal{M} \times_S X$ via $\pi_{M}$ we get exact triangle
\[ L\pi_M^*Lp_S^*L_S^* \to L\pi_M^*L_M^* \to L\pi_M^*L_{M/S}^* \simeq L_{M \times_S X/X}^* \to L\pi_M^*Lp_S^*L_S^*[1] \]
and hence the above commutative diagram can be fit into maps of exact triangles
\[
\begin{array}{c}
L_{M \times_S X/M}^* \xrightarrow{G} \xrightarrow{R\text{Hom}(\mathbb{E}, \mathbb{E})} A_{\pi_M}(\mathbb{E}) \xrightarrow{A_{\pi_M}(\mathbb{E})} L_{M \times_S X/M}[1] \\
L\pi_M^*Lp_S^*L_S^*[1] \xrightarrow{L\pi_M^*L_M^*[1]} \xrightarrow{L\pi_M^*L_{M/S}[1]} \simeq L_{M \times_S X/X}[1] \xrightarrow{L\pi_M^*Lp_S^*L_S^*[2]} \\
Lp_S^*L_S^* \xrightarrow{L_M^*} \xrightarrow{L_{M/S}^*} \end{array}
\]  
By Verdier duality along the projective morphism $\pi_M$, we get
\[ R\pi_M^*(L_{M \times_S X/M}^* \otimes \omega_{\pi_M})[n-1] \to R\pi_M^*(L(M^* \otimes \omega_{\pi_M})[n-1] \to R\pi_M^*(R\text{Hom}(\mathbb{E}, \mathbb{E} \otimes \omega_{\pi_M})[n-1] \to L_{M/S}^* \\
Lp_S^*L_S^* \xrightarrow{L_M^*} \xrightarrow{L_{M/S}^*} \]
where $\omega_{\pi_M}$ is the relative dualizing sheaf along $\pi_M$.

**Theorem 3.1.** The map
\[ (3.4) \quad R\pi_M^*(\mathbb{G} \otimes \omega_{\pi_M})[n-1] \to L_M^* \]
gives an obstruction theory for $\mathcal{M}$.

**Proof.** Let’s first consider the map
\[ (3.5) \quad R\pi_M^*(L_{M \times_S X/M}^* \otimes \omega_{\pi_M})[n-1] \to Lp_S^*L_S^* \\
\]
Let $\omega_{\pi_S}$ be the relative dualizing sheaf of $X/S$, then
\[ \omega_{\pi_M} = Lp_X^*\omega_{\pi_S} \]
we have
\[ L_{M \times_S X/M}^* \otimes \omega_{\pi_M} = Lp_X^*(L_{X/S}^* \otimes \omega_{\pi_S}) \]
since $\pi_S$ is flat, we have base change property
\[ R\pi_M^*(L_{M \times_S X/M}^* \otimes \omega_{\pi_M})[n-1] \simeq Lp_S^*(R\pi_S^*(L_{X/S}^* \otimes \omega_{\pi_S}))[n-1] \]
and it’s easy to see that the map (3.5) is the pull-back by $p_S$ of the Kodaira-Spencer map (1)
\[ (3.6) \quad R\pi_S^*(L_{X/S}^* \otimes \omega_{\pi_S})[n-1] \to L_S^* \\
\]
By Prop 6.2 in [1], the Kodaira-Spencer map (3.6) gives an obstruction theory on $S$, i.e., $h^{-1}$ is epimorphism and $h^0$ is isomorphism. It follows that the map (3.5) is also epimorphism for $h^{-1}$ and isomorphism for $h^0$.

By Theorem 4.1 in [2], the map
\[ R\pi_M^*(R\text{Hom}(\mathbb{E}, \mathbb{E} \otimes \omega_{\pi_M})[n-1] \to L_{M/S}^* \]
gives a relative obstruction theory for $\mathcal{M}/S$, hence epimorphism for $h^{-1}$ and isomorphism for $h^0$. Using the long exact sequence of cohomology associated to the exact triangle, and by simple diagram chasing, we see that the map
\[ R\pi_M^*(\mathbb{G} \otimes \omega_{\pi_M})[n-1] \to L_M^* \]
is also epimorphism for $h^{-1}$ and isomorphism for $h^0$, hence giving an obstruction theory for $\mathcal{M}$. □
Corollary 3.2. Let $X$ be smooth projective variety, $E \in D^b(X)$ be a perfect complex. Let $G^\bullet$ be the mapping cone of the Atiyah class

$$R\text{Hom}(E, E)[-1] \to \Omega_X$$

Then the deformation functor $\text{Def}_{(X, E)}$ of the pair $(X, E)$ has tangent space

$$\text{Ext}_X^1(G^\bullet, \mathcal{O}_X)$$

and obstruction space can be chosen to be

$$\text{Ext}_X^2(G^\bullet, \mathcal{O}_X)$$

and for any small extension $0 \to k \to A' \to A \to 0$, where $A', A$ are Artin local rings, the restriction map

$$\text{Def}_{(X, E)}(A') \to \text{Def}_{(X, E)}(A)$$

is a torsor under

$$\text{Ext}_X^0(G^\bullet, \mathcal{O}_X)$$

Proof. It follows from Theorem 4.5 in [1], Theorem 3.1 and Serre Duality. \hfill \Box

4. Application: Deformation theory of Vector Bundle on Smooth Projective Variety

In this section, we specialize the above discussion to the case of pair $(X, E)$, where $X$ is projective smooth variety, and $E$ is a vector bundle on $X$. Let

$$0 \to \Omega_X \otimes E \to E_A \to E \to 0$$

be the Atiyah class. Apply $\text{Hom}(\cdot, E)$, we get

(4.1) $$0 \to \text{Hom}(E, E) \xrightarrow{i} \text{Hom}(E_A, E) \xrightarrow{j} \text{Hom}(E \otimes \Omega_X, E) \to 0$$

We denote the following canonical diagonal map by $k$

$$k : \text{Hom}(\Omega_X, \mathcal{O}_X) \to \text{Hom}(E \otimes \Omega_X, E)$$

The dual of Atiyah class as an element in $\text{Ext}_X^1(\text{Hom}(\Omega_X, \mathcal{O}_X), \text{Hom}(E, E))$ is obtained via the pull-back of the exact sequence (4.1) by the diagonal map $k$

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{i} & \text{Hom}(E, E) & \xrightarrow{j} & \text{Hom}(E \otimes \Omega_X, E) & \xrightarrow{k} & \text{Hom}(\Omega_X, \mathcal{O}_X) & \xrightarrow{} & 0 \\
& & & \uparrow & & & \uparrow & & \\
0 & \xrightarrow{} & \text{Hom}(E, E) & \xrightarrow{} & D(E) & \xrightarrow{} & \text{Hom}(\Omega_X, \mathcal{O}_X) & \xrightarrow{} & 0 \\
\end{array}
\]

where

(4.2) $$D(E) = \text{Ker}(\text{Hom}(\Omega_X, \mathcal{O}_X) \oplus \text{Hom}(E_A, E) \xrightarrow{-k} \text{Hom}(\Omega_X \otimes E, E))$$

Using the explicit structure of $E_A$ as in (3.2), it’s easy to see that $D(E)$ is the sheaf of pairs on an open subset $U$

$$(t_U, \phi_U), \ t_U \in \text{Hom}_{\mathcal{O}_U}(\Omega_U, \mathcal{O}_U), \ \phi_U \in \text{Hom}_{k}(E|_U, E|_U)$$

such that

$$\phi_U(a, e) = a\phi_U(e) + t_U(da)e, \ a \in \mathcal{O}_U, \ e \in E|_U$$

$D(E)$ is known as the sheaf of differential operators of order $\leq 1$ with diagonal symbol.

Lemma 4.1. Let $G^\bullet$ be the mapping cone of the Atiyah class

$$\text{Hom}(E, E)[-1] \to \Omega_X$$

then we have quasi-isomorphism

$$(G^\bullet)^v \simeq D(E)$$

where $D(E)$ is considered as an one-term complex of locally free sheaf.
Proof. Let $\alpha(E)$ denote the Atiyah class
\[ \alpha(E) : \mathcal{H}om(E, E)[-1] \to \Omega_X \]
then we have
\[ \mathcal{H}om(E, E)[-1] \xrightarrow{\alpha(E)} \Omega_X \to G^\bullet \to \mathcal{H}om(E, E) \]
take the dual, we get
\[ \mathcal{H}om(E, E) \to (G^\bullet)^\vee \to \mathcal{H}om(\Omega_X, \mathcal{O}_X) \to \mathcal{H}om(E, E)[1] \]
On the other hand, $(\alpha(E))^\vee : \mathcal{H}om(\Omega_X, \mathcal{O}_X) \to \mathcal{H}om(E, E)[1]$ is given by the exact sequence
\[ 0 \to \mathcal{H}om(E, E) \to D(E) \to \mathcal{H}om(\Omega_X, \mathcal{O}_X) \to 0 \]
which fits into the exact triangle
\[ \mathcal{H}om(E, E) \to D(E) \to \mathcal{H}om(\Omega_X, \mathcal{O}_X) \to \mathcal{H}om(E, E)[1] \]
Comparing the two exact triangles, the lemma follows. \qed

Lemma 4.2. Let $A$ be an Artinian local ring with residue field $k$, $X_A/\text{Spec } A$ is a flat deformation of $X/\text{Spec } k$,
\[ X \downarrow \downarrow \]
\[ \text{Spec } k \quad \text{Spec } A \]
let $\tilde{E}^\bullet$ be a finite complex of locally free sheaves on $X_A$ whose derived restriction to $X$ is quasi-isomorphic to $E$, then $\tilde{E}^\bullet$ is quasi-isomorphic to an one-term complex of locally free sheaf.

Proof. Let $\tilde{E}^n$ be the last non-zero term of $\tilde{E}^\bullet$. If $n > 0$, then
\[ \tilde{E}^{n-1}|_X \to \tilde{E}^n|_X \]
is surjective by assumption. Hence $\tilde{E}^{n-1} \to \tilde{E}^n$ is also surjective by Nakayama Lemma, and the kernel of $\tilde{E}^{n-1} \to \tilde{E}^n$ is then locally free. So we can assume that $\tilde{E}^0$ is the last non-zero term.

Now let $\tilde{E}^n$ be the first non-zero term. If $n < 0$, consider the map
\[ \tilde{E}^n \to \tilde{E}^{n+1} \]
Let $K$ be the kernel, $I$ be the image, and $Q$ be the cokernel. By assumption, the map
\[ \tilde{E}^n|_X = \tilde{E}^n \otimes_A k \to \tilde{E}^{n+1}|_X = \tilde{E}^{n+1} \otimes_A k \]
is injective. It implies that the map
\[ \tilde{E}^n \otimes_A k \to I \otimes_A k \]
is isomorphism and
\[ 0 \to I \otimes_A k \to \tilde{E}^{n+1} \otimes_A k \to Q \otimes_A k \to 0 \]
is exact. Since $\tilde{E}^{n+1}$ is locally free and $X_A$ is flat over $A$, we get
\[ Tor^A_1(Q, k) = 0 \]
We see that $Q$ is flat over $A$, hence $I$ is flat over $A$ also. Therefore the sequence
\[ 0 \to K \otimes_A k \to \tilde{E}^n \otimes_A k \to I \otimes_A k \to 0 \]
is exact. We see that $K \otimes_A k = 0$. By Nakayama Lemma,
\[ K = 0 \]
Therefore the map $\tilde{E}^n \to \tilde{E}^{n+1}$ is both injective as a map of sheaves and injective on fibers. So $\tilde{E}^{n+1}/\tilde{E}^n$ is also locally free and $\tilde{E}^\bullet$ can be trimmed. We can keep this operation until we get one-term complex of locally free sheaf, which is quasi-isomorphic to $E^\bullet$. \qed
Corollary 4.3. Let $X$ be projective smooth variety and $E$ a vector bundle on $X$. Then the local deformation functor of the pair $(X, E)$ viewing $E$ as derived objects on $X$ is isomorphic to the local deformation functor of the pair $(X, E)$ viewing $E$ as vector bundle on $X$.

Theorem 4.4. The tangent space for the deformation of the pair $(X, E)$ is given by

$$H^1(X, D(E))$$

and obstruction space can be chosen to be

$$H^2(X, D(E))$$

Proof. Let $G^*$ be the mapping cone of the Atiyah class as above. By corollary 3.2 and corollary 4.3, the tangent space for the deformation of the pair $(X, E)$ is given by

$$\text{Ext}^1_X(G^*, \mathcal{O}_X) = \text{Ext}^1_X(\mathcal{O}_X, (G^*)^v)$$

By lemma 4.1

$$\text{Ext}^1_X(\mathcal{O}_X, (G^*)^v) = \text{Ext}^1_X(\mathcal{O}_X, D(E)) = H^1(X, D(E))$$

similarly, the obstruction space is given by

$$\text{Ext}^2_X(\mathcal{O}_X, (G^*)^v) = H^2(X, D(E))$$

□

Remark 4.5. In the case that $E$ is vector bundle on $X$, this theorem is well-known and can also be obtained in the standard way by Cecc Cohomology (see for example [8] for the line bundle case). Theorem 3.1 actually generalizes the bundle case above to derived objects of coherent sheaves over smooth projective variety.

Let $\text{Def}_E, \text{Def}_{(X, E)}, \text{Def}_X$ be the deformation functor of $E$, the pair $(X, E)$, and $X$ respectively. Then we have maps

$$\text{Def}_E \rightarrow \text{Def}_{(X, E)} \rightarrow \text{Def}_X$$

where the first map is the deformation of $E$ fixing $X$, and the second map is the forgetful map. Taking the cohomology of the exact sequence

$$0 \rightarrow \mathcal{H}om(E, E) \rightarrow D(E) \rightarrow \mathcal{H}om(\Omega_X, \mathcal{O}_X) \rightarrow 0$$

we get long exact sequence

$$0 \rightarrow \text{Ext}^0(E, E) \rightarrow H^0(X, D(E)) \rightarrow H^0(X, T_X) \rightarrow \text{Ext}^1(E, E) \rightarrow H^1(X, D(E)) \rightarrow H^1(X, T_X) \rightarrow \text{Ext}^2(E, E) \rightarrow H^2(X, D(E)) \rightarrow H^2(X, T_X)$$

which can be viewed as

$$0 \rightarrow \text{Aut}_E \rightarrow \text{Aut}_{(X, E)} \rightarrow \text{Aut}_X \rightarrow \text{Def}_E \rightarrow \text{Def}_{(X, E)} \rightarrow \text{Def}_X \rightarrow \text{Ob}_E \rightarrow \text{Ob}_{(X, E)} \rightarrow \text{Ob}_X$$

Therefore we see that the tangent-obstruction theory for deforming bundle, deforming pairs of bundle and scheme, and deforming scheme are naturally combined into long exact sequence coming from exact
triangle in the derived category of coherent sheaves on $X$ via the construction of Atiyah class. We have the same structure if $E$ is a perfect complex of coherent sheaves on $X$ by using the exact triangle (3.3) instead.

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