ON THE BOUNDARY COMPONENTS OF CENTRAL STREAMS IN THE TWO SLOPES CASE

Nobuhiro Higuchi

Abstract

In 2004 Oort studied the foliation on the space of $p$-divisible groups. In his theory, special leaves called central streams play an important role. It is still meaningful to investigate central streams, for example, there remain a lot of unknown things on the boundaries of central streams. In this paper, we classify the boundary components of the central stream for a Newton polygon consisting of two segments, where one slope is less than $1/2$ and the other slope is greater than $1/2$. Moreover we determine the generic Newton polygon of each boundary component using this classification.

1 Introduction

In [11, p. 1023], Oort introduced the notion of minimal $p$-divisible groups, where $p$-divisible groups are often called Barsotti-Tate groups. Let $k$ be an algebraically closed field of characteristic $p$. Oort showed in [11, 1.2] that they have the following special property: Let $X$ be a minimal $p$-divisible group over $k$. Let $Y$ be a $p$-divisible group over $k$. If $X[p] \sim Y[p]$, then $X \sim Y$, where $X[p]$ is the kernel of $p$-multiplication $p : X \rightarrow X$. For a Newton polygon $\xi$, we have a minimal $p$-divisible group $H(\xi)$. See Section 2.1 (2) for the definition of $H(\xi)$. A minimal $p$-divisible group is a $p$-divisible group which is isomorphic to $H(\xi)$ for some $\xi$ over an algebraically closed field.

Let $X_0$ be a $p$-divisible group over $k$. Let $\text{Def}(X_0) = \text{Spf}(\Gamma)$ be the deformation space of $X_0$. Here, the deformation space is the formal scheme pro-representing the functor $\text{Art}_k \rightarrow \text{Set}$ sending $R$ to the set of isomorphism classes of $p$-divisible groups $X$ over $R$ satisfying $X_k \cong X_0$, where $\text{Art}_k$ is the category of local Artinian rings with residue field $k$. It was proved by de Jong in [4, 2.4.4] that the category of $p$-divisible groups over $\text{Spf}(\Gamma)$ is equivalent to the category of $p$-divisible groups over $\Delta := \text{Spec}(\Gamma)$. Let $\mathcal{X}' \rightarrow \text{Spf}(\Gamma)$ be the universal $p$-divisible groups, and let $\mathcal{X}$ be the $p$-divisible groups over $\Delta$ obtained from $\mathcal{X}'$ by the equivalence above.

In [10, 2.1], for a $p$-divisible group $Y$ over $k$ and a $k$-scheme $S$, Oort introduced a locally closed subset $\mathcal{C}_Y(S)$ for a $p$-divisible group $Y$ over $S$ characterized by $s \in \mathcal{C}_Y(S)$ if and only if $Y_s$ is isomorphic to $Y$ over an algebraically closed field containing $k(s)$ and $k$. He called $\mathcal{C}_Y(S)$ the leaf associated with $Y$ in $S$; see (3) in Section 211 for the details. We are interested in the case that

\begin{itemize}
  \item 2010 Mathematics Subject Classification : Primary:14L15 Group schemes; Secondary:14L05 formal groups, $p$-divisible group; 14K10 algebraic moduli, classification.
  \item Key words and phrases. $p$-divisible group; deformation space; Newton polygons.
\end{itemize}
Let $X$ and $Y$ be $p$-divisible groups over $k$. We say that $X$ appears as a specialization of $Y$ if there exists a family of $p$-divisible group $X \to \text{Spec}(R)$ with discrete valuation ring $(R, m)$ in characteristic $p$ such that $X_L$ is isomorphic to $Y$ over an algebraically closed field containing $L$ and $k$, and $X_k$ is isomorphic to $X$ over an algebraically closed field containing $\kappa$ and $k$, where $L = \text{frac} R$ is the field of fractions of $R$, and $\kappa = R/m$ is the residue field of $R$. We say that a specialization $X$ of $Y$ is generic if $\ell(X[p]) = \ell(Y[p]) - 1$ holds, where $\ell(X[p])$ is the length of the element of the Weyl group corresponding to $X[p]$; see the paragraph below Corollary 4.12 for this correspondence between elements of the Weyl group and $p$-kernels of $p$-divisible groups.

Let $\xi$ and $\zeta$ be Newton polygons. See the paragraph containing (2) for the definition of Newton polygons. We say $\zeta \prec \xi$ if each point of $\zeta$ is above or on $\xi$. Moreover, we say that $\zeta \prec \xi$ is saturated if there exists no Newton polygon $\eta$ such that $\zeta \preceq \eta \preceq \zeta$.

In this paper, we will give two main results. The first result (Theorem 1.1) classifies the boundary components of certain central streams. See Section 4 for the statement as we need some notation given in Section 3 to state the first result. From this result, we expect that Conjecture 4.13 stated in Section 4 is true. This conjecture says that it suffices to deal with central streams associated to boundary components of arbitrary central streams. The second result is

**Theorem 1.1.** Let $\xi$ be a Newton polygon consisting of two segments with slopes $\lambda$ and $\lambda'$ satisfying $\lambda < 1/2 < \lambda'$. Let $X$ be an arbitrary generic specialization of $H(\xi)$. Then there exists a Newton polygon $\zeta$ satisfying that

(i) $\zeta \prec \xi$ is saturated, and

(ii) $H(\zeta)$ appears as a specialization of $X$.

With the same notation as Theorem 1.1 by Grothendieck-Katz [5] 2.3.1, we have

$$\zeta \prec \text{NP}(X) \preceq \zeta,$$

where $\text{NP}(X)$ is the Newton polygon of $X$. Here we note that $\text{NP}(X) \neq \xi$ will be proved in Lemma 2.2 below. Then the saturatedness of $\zeta \prec \xi$ implies $\text{NP}(X) = \zeta$ and therefore

**Corollary 1.2.** Let $\xi$ and $X$ be as in Theorem 1.1. Then $\text{NP}(X) \prec \xi$ is saturated.

For a finite flat commutative group scheme $G$ over some base scheme, we say that $G$ is a truncated Barsotti-Tate group of level one (abbreviated as BT$_1$) if Frobenius $F$ and Verschiebung $V$ satisfy that $\text{Ker} F = \text{Im} V$ and $\text{Ker} V = \text{Im} F$. The dimension of BT$_1$ $G$ is defined by $\log_p(\text{Ker} F)$. Let $W$ be the Weyl group of the general linear group $GL_h$. By Kraft [6], Oort [9] and Moonen-Wedhorn [8], for each algebraically closed field $k$, there exists a canonical one-to-one correspondence

$$\{\text{BT}_1’s \text{ over } k \text{ of height } h \text{ and dimension } d\} \leftrightarrow J W$$

for a subset $J W$ of $W$ depending on $h$ and $d$; see the paragraph below Example 2.6 for the definition of $J W$ and this correspondence. Let $w$ and $w'$ be elements of $J W$. We say $w' \subset w$ if there exists a discrete valuation ring $R$ of characteristic $p$ such that there exists a finite flat commutative group scheme $G \to \text{Spec}(R)$ satisfying that $G_{\mathfrak{m}}$ is a BT$_1$ of type $w'$, and $G_{\mathfrak{m}}$ is a BT$_1$ of type $w$, where $L$
is the fractional field of $R$, and $\kappa$ is the residue field $R/\mathfrak{m}$ with the maximal ideal $\mathfrak{m}$ of $R$. We say that $w'$ is a generic specialization of $w$ if $w' \subset w$ and $\ell(w') = \ell(w) - 1$ holds.

The reason to use the word “generic” comes from the following fact. For a $p$-divisible group $X_0$ of type $w$, i.e., for a $p$-divisible group such that its $p$-kernel corresponds to $w \in JW$, let $S_w(\Delta)$ be the reduced subscheme of $\Delta = \text{Spec}(\Gamma)$ consisting of $s$ satisfying that $\mathfrak{X}_s[p]$ is of type $w$. Then it is known that $\dim S_w(\Delta) = \ell(w)$ and $S_w(\Delta)$ is non-empty; see [13] 6.10 and [7] 3.1.6. This justifies the terminology “generic” for elements $w$ of $JW$.

For a Newton polygon $\xi$, let $w_\xi$ be the element of $JW$ corresponding to $H(\xi)[p]$. Using notation of the Weyl group, by [2] 4.1, Theorem 1.1 is paraphrased as

**Theorem 1.3.** Let $\xi$ be a Newton polygon consisting of two segments, where one slope is less than $1/2$ and the other is larger than $1/2$. For an arbitrary generic specialization $w \in JW$ of $w_\xi$, there exists a Newton polygon $\zeta$ such that

(i) $\zeta \prec \xi$ is saturated, and

(ii) $w_\zeta \subset w$.

The results of this paper lead us to expect that Theorem 1.3 can be generalized to the case that $\xi$ is an arbitrary Newton polygon:

**Conjecture 1.4.** For an arbitrary Newton polygon $\xi$, let $w \in JW$ be a generic specialization of $w_\xi$. Then there exists a Newton polygon $\zeta$ such that

(i) $\zeta \prec \xi$ is saturated, and

(ii) $w_\zeta \subset w$.

If Conjecture 1.3 stated in Section 4 holds, then for Newton polygons $\xi$, the two segments case is essential to show Conjecture 1.4.

This paper is organized as follows: In Section 2 we recall definitions of $p$-divisible groups, truncated Dieudonné modules of level one and related matters. Moreover, we introduce a notion of “arrowed binary sequences”. The proofs of our main results are described using this notion. In Section 3 to show the first result, we give explanations about tools which are used to construct specializations combinatorially, and show some properties of these tools. In Section 4 we give a proof of Theorem 4.1. This theorem classifies boundary components of central streams. In Section 5 we see the key proposition (Proposition 5.1) which is used to prove the second result, and we show Theorem 1.3.

## 2 Preliminary

In this section, we recall definitions of $p$-divisible groups, central leaves, minimal $p$-divisible groups, central streams and DM$_1$’s. Finally, we introduce the notion of arrowed binary sequences; see Definition 2.8 used in the proofs of main theorems.
2.1 \( p \)-divisible groups and Dieudonné modules

First, let us recall the definition of \( p \)-divisible groups. Let \( p \) be a prime number. Let \( h \) be a non-negative integer. Let \( S \) be a scheme in characteristic \( p \). We say that \( X \) is a \( p \)-divisible group (Barsotti-Tate group) of height \( h \) over \( S \) if \( X \) is an inductive system \( X = (G_v, i_v)_{v \geq 1} \) for natural numbers \( v \), where \( G_v \) is a finite locally free commutative group scheme over \( S \) of order \( p^v \), and for each \( v \), there exists the exact sequence of commutative group schemes

\[
0 \to G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{p^v} G_{v+1},
\]

where \( i_v \) is a canonical inclusion. Let \( X = (G_v, i_v)_{v \geq 1} \) be a \( p \)-divisible group over \( S \). For an arbitrary scheme \( T \) over \( S \), we have the \( p \)-divisible group \( X_T \) over \( T \) which is defined by \( (G_v \times_S T, i_v \times \text{id})_{v \geq 1} \). In particular, if \( T \) is a closed point \( s \) over \( S \), then the \( p \)-divisible group \( X_s \) is called fiber of \( X \) over \( s \).

Let \( K \) be a perfect field of characteristic \( p \). We denote by \( W(K) \) the ring of Witt-vectors with coefficients in \( K \). Let \( \sigma \) be the Frobenius over \( K \). We denote by the same symbol \( \sigma \) the Frobenius over \( W(K) \) if no confusion can occur. We say that \( M \) is a Dieudonné module over \( K \) if \( M \) is a finite \( W(K) \)-module equipped with \( \sigma \)-linear homomorphism \( F : M \to M \) and \( \sigma^{-1} \)-linear homomorphism \( V : M \to M \) satisfying that \( F \circ V \) and \( V \circ F \) is equal to the multiplication by \( p \). We use the covariant Dieudonné theory, which says that there exists a canonical categorical equivalence \( \mathcal{D} \) from the category of \( p \)-divisible groups (resp. finite commutative group schemes) over \( K \) to that of Dieudonné modules over \( K \) which are free as \( W(K) \)-modules (resp. are of finite length).

Here, let us recall the notion of minimal \( p \)-divisible groups. We define a \( p \)-divisible group \( H_{m,n} \) over \( \mathbb{F}_p \) as follows: \( H_{m,n} \) is of dimension \( n \), and its Serre-dual is of dimension \( m \). Moreover, the Dieudonné module is obtained by

\[
\mathcal{D}(H_{m,n}) = \bigoplus_{i=1}^{h} \mathbb{Z}_p e_i,
\]

where \( h = m + n \), and \( \mathbb{Z}_p \) is the ring of \( p \)-adic integers. For the basis \( e_i \), operations \( F \) and \( V \) satisfy that \( F e_i = e_{i-h} \), \( V e_i = e_{i-n} \) and \( e_{i-h} = p e_i \).

Let \( \{ (m_i, n_i) \}_{i=1, \ldots, z} \) be a finite number of pairs of coprime non-negative integers endowed with a non-increasing order of \( \lambda_i := n_i/h_i \) with \( h_i = m_i + n_i \), i.e., we have \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_z \). A Newton polygon \( \xi = \sum_{i=1}^{z} (m_i, n_i) \) is a lower convex polygon in \( \mathbb{R}^2 \), breaking on integral coordinates, consisting of slopes \( \lambda_i \). We call each coprime pair \( (m_i, n_i) \) segment.

For a Newton polygon \( \xi = \sum_i (m_i, n_i) \), we set a \( p \)-divisible group

\[
H(\xi) = \bigoplus_i H_{m_i, n_i}.
\]

We say that a \( p \)-divisible group \( X \) is minimal if there exists a Newton polygon \( \xi \) such that \( X \) is isomorphic to \( H(\xi) \) over an algebraically closed field. For a \( p \)-divisible group \( Y \), there exists an isogeny from \( Y \) to \( H(\xi) \) over an algebraically closed field for some Newton polygon \( \xi \). This \( \xi \) is called the Newton polygon of \( Y \), which is denoted by \( \text{NP}(Y) \).

In [10 2.1], for a \( p \)-divisible group \( Y \) of height \( h \) over a scheme \( S \) in characteristic \( p \) and for a \( p \)-divisible group \( Y \) over a field of characteristic \( p \), Oort gave the definition of a leaf by

\[
\mathcal{C}_Y(S) = \{ s \in S \mid \mathcal{Y}_s \text{ is isomorphic to } Y \text{ over an algebraically closed field}\},
\]
which is considered as a locally closed subscheme of $S$ by giving $C_Y(S)$ the induced reduced structure.

Let $K$ be a field of characteristic $p$, and let $X$ be a $p$-divisible group over $K$. Let $Y \to S$ be a $p$-divisible group of height $h$ and dimension $d$ over a noetherian scheme $S$ over $K$. Let $\xi$ be a Newton polygon starting at $(0,0)$, ending at $(h,d)$. We write

\[ W_\xi^0(S) = \{ s \in S \mid \text{NP}(Y_s) = \xi \}. \]

By Grothendieck-Katz [5], we know that $W_\xi^0(S) \subset S$ is locally closed. We recall Oort’s result on leaves:

**Theorem 2.1** ([10], Theorem 2.2). Let $K$ be a field, and let $X_0 \to \text{Spec}(K)$ be a $p$-divisible group over $K$. Set $\xi = \text{NP}(X_0)$. Let $S \to \text{Spec}(K)$ be an excellent scheme over $K$. For a $p$-divisible group $Y \to S$,

\[ C_{X_0}(S) \subset W_\xi^0(S) \]

is a closed subset.

Using Theorem 2.1 we show

**Lemma 2.2.** Let $X$ be a specialization of the minimal $p$-divisible group $H(\xi)$ for a Newton polygon $\xi$. Assume that $X$ is not isomorphic to $H(\xi)$ over an algebraically closed field. Then $\text{NP}(X) \preceq \xi$.

**Proof.** It suffices to show the case that $X$ is a $p$-divisible group over an algebraically closed field $k$ of characteristic $p > 0$. Let $R$ be a discrete valuation ring over $k$. Let $L$ denote the fractional field of $R$. Let $X \to \text{Spec}(R)$ be a $p$-divisible group satisfying that $X_k \simeq X$ and $X_L \simeq H(\xi)_L$. Suppose that $\text{NP}(X) = \xi$ were true. By applying Theorem 2.1 to $(Y \to S) = (X \to \text{Spec}(R))$ and $X_0 = H(\xi)_k$, we have $X \simeq H(\xi)_k$. This is a contradiction. \qed

For a $p$-divisible group $X$, the kernel of $p$-multiplication $p : X \to X$ is called the $p$-kernel of $X$, denoted by $X[p]$. The Dieudonné module of $X[p]$ makes a truncated Dieudonné module of level one, defined below:

**Definition 2.3.** Let $N$ be a $K$-vector space of finite dimension. Let $F$ and $V$ be a $\sigma$-linear map and a $\sigma^{-1}$-linear map respectively from $N$ to itself. A triple $(N, F, V)$ is a truncated Dieudonné module of level one over $K$ (abbreviated as $\text{DM}_1$) if the above $F$ and $V$ satisfy that $\text{Ker} F = \text{Im} V$ and $\text{Im} F = \text{Ker} V$. We say that a $\text{DM}_1$ $(N, F, V)$ is of height $h$ and dimension $d$ if $\dim_K N = h$ and $\dim_K N/VN = d$.

Let us recall the notion of specializations. Let $R$ be a commutative ring of characteristic $p > 0$. Let $\sigma : R \to R$ be the Frobenius endomorphism defined by $\sigma(a) = a^p$. Then the definition of $\text{DM}_1$’s over the ring $R$ is given as follows:

**Definition 2.4.** A $\text{DM}_1$ over $R$ of height $h$ is a quintuple $N = (N, C, D, F, V^{-1})$ satisfying

(i) $N$ is a locally free $R$-module of rank $h$,

(ii) $C$ and $D$ are submodules of $N$ which are locally direct summands of $N$,

(iii) $F : (N/C) \otimes_{R,\sigma} R \to D$ and $V^{-1} : C \otimes_{R,\sigma} R \to N/D$ are $R$-linear isomorphisms.
Put $R = K[t]$. Let $\mathcal{N}$ be an arbitrary $\text{DM}_1$ over $R$. We set $\mathcal{N}_K = \mathcal{N} \otimes_R K$, and we have a $\text{DM}_1$ over $K$. From this we obtain a map, called a specialization,

$$\{\text{DM}_1 \text{ over } R\} \rightarrow \{\text{DM}_1 \text{ over } K\}$$

which maps $\mathcal{N}$ to $\mathcal{N}_K$.

Let $\xi = \sum_i (m_i, n_i)$ be a Newton polygon. We denote by $N_\xi$ the $\text{DM}_1$ associated to the $p$-kernel of $H(\xi)$. Equivalently, $N_\xi$ is described as

$$N_\xi = \bigoplus N_{m_i, n_i},$$

where $N_{m,n}$ is the $\text{DM}_1$ associated to the $p$-kernel of $H_{m,n}$. In this paper, we mainly treat $\text{DM}_1$’s $N_\xi$ with $\xi = (m_1, n_1) + (m_2, n_2)$ satisfying $\lambda_2 < 1/2 < \lambda_1$.

Let $k$ be an algebraically closed field of characteristic $p$. Then the following classification of $\text{DM}_1$’s over $k$ is given by Kraft [6], Oort [9] and Moonen-Wedhorn [8].

**Theorem 2.5.** There exists a bijection:

$$\{0, 1\}^h \leftrightarrow \{\text{DM}_1 \text{ over } k \text{ of height } h\}/\sim = .$$

Let us review the construction of the bijection above. We often identify an element $S$ of $\{0, 1\}^h$ with the pair of a totally ordered set $\tilde{S} = \{t_1 < \cdots < t_h\}$ and a map $\delta : \tilde{S} \rightarrow \{0, 1\}$, so that the $i$-th coordinate of $S$ is $\delta(t_i)$. We write this identification as an equality $S = (\tilde{S}, \delta)$. The bijection of Theorem 2.5 is obtained in the following way. Let $S = (\tilde{S}, \delta)$ be as above. To $S$, we associate a $\text{DM}_1 (N, F, V)$ as follows. Let $N = ke_1 \oplus \cdots \oplus ke_h$. We define a map $F$ by

$$F_{e_i} = \begin{cases} e_j, & j = \# \{l \mid \delta(t_l) = 0, l \leq i\} \text{ for } \delta(t_i) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $t_{j_1}, \ldots, t_{j_c}$, with $j_1 < \cdots < j_c$, be the elements of $\tilde{S}$ satisfying $\delta(t_{j_i}) = 1$. Put $d = h - c$. Then a map $V$ is defined by

$$V_{e_i} = \begin{cases} e_{j_l}, & l = i - d \text{ for } i > d, \\ 0 & \text{otherwise.} \end{cases}$$

We call $\{e_1, \ldots, e_h\}$ a standard basis of $(N, F, V)$.

**Example 2.6.** Let us see an example of $\text{DM}_1$’s. Let $S = (1, 1, 0, 1, 0)$ be an element of $\{0, 1\}^5$. Then the $\text{DM}_1 (N, F, V)$, with $N = ke_1 \oplus \cdots \oplus ke_5$, corresponding to $S$ is given by the following diagram.

For the above diagram, if there exists no vector of $F$ (resp. $V$) from $e_i$, then we regard $F$ (resp. $V$) maps $e_i$ to zero. One can check that the above satisfies the condition of $\text{DM}_1$’s.
Let $h$ and $c$ be non-negative integers. Put $d = h - c$. Let $W$ be the Weyl group of the general linear group $GL_h$. We identify $W$ with the symmetric group $S_h$. Here, we associate $DM_1$’s over $k$ with elements of the Weyl group. Let $s_i \in W$ be the simple reflection $(i, i + 1)$ for $i = 1, \ldots, h - 1$. Let $\Omega$ be the standard generator of $S_h$, namely $\Omega = \{s_1, \ldots, s_{h-1}\}$. Set $J_\ell = \Omega \setminus \{s_c\}$. Let $W_J$ be the subgroup of $W$ generated by $J = J_\ell$. Let $JW$ be the set consisting of elements of minimal length in $W_J \setminus W$, i.e., the shortest representatives of $W_J \setminus W$. Let $w$ be an element of $JW$. We associate $S = (\tilde{S}, \delta)$ to $w$ by the pair of a totally ordered set $\tilde{S} = \{t_1 < \cdots < t_h\}$ and the map $\delta : \tilde{S} \to \{0, 1\}$ defined by $\delta(t_i) = 0$ if and only if $w(i) > c$. Here the property of the minimal length of $w$ is used. We regard this pair $(\tilde{S}, \delta)$ as the element of $\{0, 1\}^h$. One can check that this gives a one-to-one correspondence between $JW$ and the set of elements $S$ of $\{0, 1\}^h$ satisfying $\#\{t \in \tilde{S} \mid \delta(t) = 0\} = d$. Thus there exists a bijection between $JW$ and the set of isomorphism classes of $DM_1$’s over $k$ of height $h$ and dimension $d$. We denote by $w_\xi$ the element of $JW$ associated to $N_\xi$.

In the rest of this subsection, we show a lemma used for the construction of generic specializations. Let $W$ and $J = J_\ell$ be as above. Set $d = h - c$. We define $x \in W$ to be $x(i) = i + d$ if $i \leq c$ and $x(i) = i - c$ otherwise. We define $\theta : W \to W$ by $u \mapsto xu\text{x}^{-1}$. It follows from [12 4.10] by Viehmann-Wedhorn that $w' \subset w$ if and only if there exists an element $u$ of $W_J$ such that $u^{-1}w'\theta(u) \leq w$, where $\leq$ denotes the Bruhat order.

**Lemma 2.7.** Let $w'$ and $w$ be elements of $JW$ with $w' \subset w$. If $\ell(w') = \ell(w) - 1$, then there exist $v \in W$ and $u \in W_J$ such that

(i) $v = ws$ for a transposition $s$,

(ii) $\ell(v) = \ell(w) - 1$,

(iii) $w' = uv\theta(u^{-1})$.

**Proof.** Let $w \in JW$. Let $w' \in JW$ satisfying that $w' \subset w$ and $\ell(w') = \ell(w) - 1$. Choose an element $u$ of $W_J$ satisfying that $u^{-1}w'\theta(u) < w$. Set $v = u^{-1}w'\theta(u)$. We show (ii). Since $w'$ is an element of $JW$, we have $\ell(v) \geq \ell(u^{-1}w') - \ell(\theta(u)^{-1}) = \ell(u) + \ell(w') - \ell(\theta(u))$. Moreover, $\ell(u) + \ell(w') - \ell(\theta(u)) = \ell(w')$ since for all element $u'$ of $W_J$ we have $\ell(u') = \ell(\theta(u'))$ by the definition of $\theta$. As $v < w$, we have $\ell(v) < \ell(w)$. These prove (ii). Let $w = s_1s_2 \cdots s_i$ be a reduced expression of $w$ with $v = s_1 \cdots s_{i-1}s_{i+1} \cdots s_1$. Set $s = (s_1 \cdots s_{i-1})s_{i+1} \cdots s_1$. Then $s$ is a transposition, and this $s$ satisfies $v = ws$. \hfill $\square$

### 2.2 Definition of arrowed binary sequences

To show our main results, we introduce arrowed binary sequences. This object can be regarded as a generalization of $DM_1$’s.

**Definition 2.8.** An arrowed binary sequence (we often abbreviate as ABS) is the triple $(\tilde{S}, \delta, \pi)$ consisting of a totally ordered set $\tilde{S} = \{t_1 < \cdots < t_h\}$, a map $\delta : \tilde{S} \to \{0, 1\}$ and a bijection $\pi : \tilde{S} \to \tilde{S}$. We denote by $\mathcal{H}$ the set of all arrowed binary sequences.

**Definition 2.9.** Let $N = (N, F, V)$ be a $DM_1$ of height $h$. Let $e_1, \ldots, e_h$ be a standard basis of $N$. Let $S = (\tilde{S}, \delta)$ be the element of $\{0, 1\}^h$ corresponding to $N$ with $\tilde{S} = \{t_1, \ldots, t_h\}$. We define a bijective map $\pi : \tilde{S} \to \tilde{S}$ by $\pi(t_i) = t_j$, where $j$ is uniquely determined by

\[
\begin{align*}
F(e_i) &= e_j & \text{if } \delta(t_i) = 0, \\
V(e_j) &= e_i & \text{otherwise}.
\end{align*}
\]
Note that \((\tilde{S}, \delta, \pi)\) is an ABS. We say that an ABS \(S = (\tilde{S}, \delta, \pi)\) is admissible if there exists a DM \(N = (N, F, V)\) such that \((\tilde{S}, \delta)\) corresponds to \(N\) by Theorem 2.5 and \(\pi\) is constructed by (7) from \(N\). The admissible ABS \(S\) obtained from a DM \(N\) is called the ABS associated to \(N\). We denote by \(\mathcal{H}'\) the set of all admissible ABS’s.

For an arrowed binary sequence \(S = (\tilde{S}, \delta, \pi)\), as seen in Example 2.10, we obtain a diagram of the ABS using elements of \(\tilde{S}\) and arrows

\[
\begin{array}{ccc}
\bullet & \xrightarrow{\pi} & \bullet \\
\end{array}
\]

Example 2.10. The diagram of the ABS corresponding to \((1, 1, 0, 1, 0)\) is

\[
\begin{array}{ccc}
\circ & \xrightarrow{\pi} & \circ \\
\circ & \xrightarrow{\pi} & \circ \\
\end{array}
\]

From Example 2.6 and this diagram, one can check that the admissible ABS \((\tilde{S}, \delta, \pi)\) is obtained by a DM.

Now we show a property of admissible ABS’s. Let \(S = (\tilde{S}, \delta, \pi)\) be an ABS. For each element \(t\) of \(\tilde{S}\), we define the binary expansion \(b(t)\) by \(b(t) = 0.b_1b_2\cdots\), where \(b_i = 0\) if \(\delta(\pi^{-i}(t)) = 0\), and \(b_i = 1\) otherwise.

Lemma 2.11. Let \(S = (\tilde{S}, \delta, \pi)\) be an admissible ABS. Let \(t\) and \(t'\) be elements of \(\tilde{S}\). Then the following holds.

(i) Suppose \(\delta(t) = \delta(t')\). Then \(t < t'\) if and only if \(\pi(t) < \pi(t')\),

(ii) Suppose \(b(t) \neq b(t')\). Then \(t < t'\) if and only if \(b(t) < b(t')\).

Proof. (i) follows from the construction of \(\pi\) defined in Definition 2.9

Let us see (ii). To show “only if” part, we suppose \(b(t) > b(t')\). It implies that there exists a natural number \(u\) such that \(\delta(\pi^{-v}(t)) = \delta(\pi^{-v}(t'))\) for all \(u < v\) and \(\delta(\pi^{-v}(t)) > \delta(\pi^{-v}(t'))\). Then we clearly have \(\delta(\pi^{-v}(t)) = 1\) and \(\delta(\pi^{-v}(t')) = 0\). By the construction of \(\pi\), we have \(\pi^{-v+1}(t) > \pi^{-v+1}(t')\), and this contradicts with (i). Let us see “if” part. If \(b(t) < b(t')\), then there exists a natural number \(v\) such that \(\delta(\pi^{-u}(t)) = \delta(\pi^{-u}(t'))\) for \(u < v\) and \(\delta(\pi^{-v}(t)) < \delta(\pi^{-v}(t'))\). Then we have \(\delta(\pi^{-v}(t)) = 0\) and \(\delta(\pi^{-v}(t')) = 1\), and \(\pi^{-v+1}(t) < \pi^{-v+1}(t')\) holds. Using (i) repeatedly, we see \(t < t'\).

Let us define special DM\(1\)’s which correspond to special \(p\)-divisible groups \(H_{m,n}\).

Definition 2.12. We say a DM \(N\) is \(DM_1\)-simple if there exist coprime natural numbers \(h\) and \(m\) such that \(N\) is associated to the ABS \(\{(t_1, \ldots, t_h), \delta, \pi\}\), where \(\delta(t_i) = 1\) if \(i \leq m\) and \(\delta(t_i) = 0\) otherwise, with the map \(\pi\) defined as Definition 2.9. In other words, we have \(\pi(t_i) = t_{i-m \ mod \ h}\). That kind of DM\(1\) corresponds to \(N_{m,n}\) with \(n = h - m\), and we call this DM\(1\) simple DM\(1\) if no confusion can occur.
Lemma 2.14. Here, let us recall the direct sum of admissible ABS’s, which corresponds to the direct sum of associated Dieudonné modules. We define the direct sum $S = (\tilde{S}, \delta, \pi)$ of elements $A = (\tilde{A}, \delta_A, \pi_A)$ and $B = (\tilde{B}, \delta_B, \pi_B)$ of $\mathcal{H}'$ as follows. We define a set $\tilde{S}$ by $\tilde{S} = \tilde{A} \sqcup \tilde{B}$. We define a map $\delta : \tilde{S} \to \{0, 1\}$ to be $\delta|_{\tilde{A}} = \delta_A$ and $\delta|_{\tilde{B}} = \delta_B$, and let $\pi : \tilde{S} \to \tilde{S}$ be a map satisfying that $\pi|_{\tilde{A}} = \pi_A$ and $\pi|_{\tilde{B}} = \pi_B$. We define an order $<$ in $\tilde{S}$ so that

(i) for elements $t, t' \in \tilde{S}$, if $b(t) \leq b(t')$, then $t < t'$, and

(ii) there is no elements $t, t'$ of $\tilde{S}$ such that $t < t'$ and $\pi(t') < \pi(t)$,

where $b(t)$ is the binary expansion of $t \in \tilde{S}$ determined by $\pi$. For instance, if $A = B$ with $\tilde{A} = \{t_1, \ldots, t_h\}$ and $\tilde{B} = \{t'_1, \ldots, t'_h\}$, then $\tilde{S} = \{t_1, t'_1, \ldots, t_h, t'_h\}$. Thus we get the ABS $A \oplus B = (\tilde{S}, \delta, \pi)$ which also belongs to $\mathcal{H}'$. Let $M$ and $N$ be DM’s, and let $A$ (resp. $B$) be the ABS corresponding to $M$ (resp. $N$). Then the ABS $A \oplus B$ corresponds to the direct sum of DM’s $M \oplus N$. In this paper, we consider the case that $M = N_{m_1,n_1}$ and $N = N_{m_2,n_2}$, with pairs of coprime non-negative integers $(m_1, n_1)$ and $(m_2, n_2)$.

Example 2.13. Let $M = N_{2,7}$ and $N = N_{5,3}$. Let $A$ and $B$ be ABS’s corresponding to $M$ and $N$ respectively. Then diagrams of these ABS’s are given by the following:

\[
A = \begin{array}{cccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}, \quad B = \begin{array}{cccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{array}.
\]

We write $t_1$ for the element of $\tilde{A}$ corresponding to the underlined element in the diagram. Then the binary expansion $b(t_1)$ is given by $b(t_1) = 0.000010001 \cdots$. In the same way, we obtain the binary expansions of all elements of $\tilde{A}$ and $\tilde{B}$. We sort all elements by the binary expansions, and the direct sum $A \oplus B$ is given by the following:

\[
1^A_1 1^A_2 0^A_3 0^A_4 0^A_5 0^A_6 1^B_7 1^B_8 0^B_9 0^B_4 1^B_5 1^B_6 0^B_0 0^B_7 0^B_8,
\]

where $\tau^A_i$ (resp. $\tau^B_i$), with $\tau = 0$ or 1, is the $i$-th element of $A$ (resp. $B$).

For certain DM’s, we have the following:

Lemma 2.14. Let $\xi = (m_1, n_1) + (m_2, n_2)$ be a Newton polygon satisfying $\lambda_2 < 1/2 < \lambda_1$. Set $h_1 = m_1 + n_1$ and $h_2 = m_2 + n_2$. Let $N_\xi$ be the minimal DM of $\xi$. For the above notation, the ABS $S$ associated to $N_\xi$ is obtained by the following:

\[
1^A_1 \cdots 1^A_{m_1} 0^A_{m_1+1} \cdots 0^A_{n_1} 1^B_1 \cdots 1^B_{n_1} 0^A_{n_1+1} \cdots 0^A_{h_1} 1^B_{n_1+1} \cdots 1^B_{h_2 + n_2} 0^B_{m_2+1} \cdots 0^B_{h_2}.
\]

Proof. See [1], Proposition 4.20.
3 Constructing a specialization using arrowed binary sequences

In this section, we give a method to construct a specialization of a DM \(_1\) using arrowed binary sequences. We introduce some sets which help us to investigate properties of the specialization obtained by this method. These properties are useful for classification of boundary components of central streams, given in Section 4.

3.1 Some notation for specializations

Here, we prepare some notation for describing a specialization of a DM \(_1\) in arrowed binary sequences. First, we define the lengths of ABS’s. We use some notation of Section 2.2.

Definition 3.1. Let \(S = (\tilde{S}, \delta, \pi)\) be an ABS. We define the length of \(S\) by

\[
\ell(S) = \# \{ (t', t) \mid \delta(t') = 0 \text{ and } \delta(t) = 1 \text{ with } t' < t \}.
\]

Remark 3.2. If an ABS \(S\) corresponds to a DM \(_1\) \(N\), then the value \(\ell(S)\) is equal to the length \(\ell(w)\) for the element \(w\) of \(JW\), which corresponds to \(N\). In other words, the length of \(w\) can be calculated by the above \(\ell(S)\).

Example 3.3. For the ABS \(A \oplus B\) associated to \(N_{2,7} \oplus N_{5,3}\), which is constructed in Example 2.13, we have \(\ell(A \oplus B) = 29\). In general, for a Newton polygon \(\xi = (m_1, n_1) + (m_2, n_2)\) of two segments with \(A_2 < 1/2 < \lambda_1\), the length \(\ell(S)\) of the ABS \(S\) corresponding to \(N_{\xi}\) is equal to \(m_2n_1 - m_1n_2\).

Let \(H'(h, d)\) denote the set of admissible ABS’s whose corresponding DM’s are of height \(h\) and dimension \(d\). By translating the ordering \(\subset\) on \(JW\) via the bijection from \(JW\) to \(H'(h, d)\), we have an ordering on \(H'(h, d)\) as well as the notion of specialization of admissible ABS’s.

Here we give a method to construct a type of the specializations of ABS’s. Those will turn out to correspond to specializations of the form \(w' \subset w\) with \(v = ws < w\) and \(w' = uv\theta(u^{-1})\), where \(s\) is a transposition and \(u \in W_J\). See the paragraph before Lemma 2.7 for this specialization. Starting with \(S = (\tilde{S}, \delta, \pi_0) \in H'(h, d)\), we construct a new admissible ABS \(S' = (\tilde{S}', \delta, \pi)\). First we consider a “small modification” of \(\pi_0\).

Definition 3.4. Let \(S = (\tilde{S}, \delta, \pi_0)\) be an ABS with \(\tilde{S} = \{ t_1 < \cdots < t_h \}\). Choose elements \(t_i\) and \(t_j\) of \(\tilde{S}\). We define a map \(\pi\) on \(\tilde{S}\) by \(\pi(\pi_0^{-1}(t_i)) = t_j, \pi(\pi_0^{-1}(t_j)) = t_i\) and \(\pi(t) = \pi_0(t)\) for the other elements \(t\) of \(\tilde{S}\). We call this map \(\pi\) a small modification by \(t_i\) and \(t_j\) of \(\pi_0\).

We require \(\delta(t_i) = 0, \delta(t_j) = 1\) and \(t_i < t_j\), when we consider specializations. Let \(\pi : \tilde{S} \to \tilde{S}\) be the small modification by elements \(t_i\) and \(t_j\) of \(\pi_0\). Let \(\tilde{S}' = \tilde{S}\) as sets. There uniquely exists an ordering \(\prec'\) on \(\tilde{S}'\) so that

(i) If \(t \prec' t'\), then \(b(t) \leq b(t')\).

(ii) Suppose \(\delta(t) = \delta(t')\). Then \(t \prec' t'\) if and only if \(\pi(t) \prec' \pi(t')\),

where \(b(t)\) denotes the binary expansion determined by \(\delta\) and \(\pi\). Then we get an admissible ABS \(S' = (\tilde{S}', \delta, \pi)\); see Remark 3.5. We call this \(S'\) the specialization of \(S\) obtained by exchanging \(t_i\) and \(t_j\). We say an exchange of \(t_i\) and \(t_j\) is good if the specialization \(S'\) obtained by exchanging \(t_i\) and \(t_j\) satisfies \(\ell(S') = \ell(S) - 1\). We often write \(S^{-}\) for \(S'\) when \(S'\) is the specialization obtained by a good exchange. We call \(S^{-}\) a generic specialization of \(S\). Any good exchange produces a generic specialization, and conversely every generic specialization is obtained by a good exchange. This follows from Lemma 2.7 and Remark 3.5 below:
Remark 3.5. We use the same notation as Section 2.1. Let \( S = (\tilde{S}, \delta, \pi_0) \in \mathcal{H}'(h, d) \) with \( \tilde{S} = \{t_1 < \cdots < t_h\} \). Let \( S' = (\tilde{S}', \delta, \pi) \) be the specialization of \( S \) obtained by exchanging \( t_i \) and \( t_j \). Let \( w \) be the element of \( J^W \) corresponding to \( S \). Set \( s = (i, j) \) the transposition. We regard \( \pi_0 \) and \( \pi \) as elements of \( W \). Then \( \pi_0 = xw \). Put \( \tilde{S}' = \{t_1' < \cdots < t_h'\} \). Set \( \tilde{S}'(0) = \{t_1'(0) < \cdots < t_h'(0)\} \) to be \( t_s(\varepsilon) = t_z(0) \). We define an element \( \varepsilon \) of \( W \) to be \( t_s'(0) = t_z' \). Note that \( \varepsilon \) stabilizes \( \{1, 2, \ldots, d\} \) (and therefore \( \{d + 1, \ldots, h\} \)), since \( b(t_z'(0)) < 0.1 \) for \( z \leq d \) and \( b(t_z'(0)) > 0.1 \) for \( z > d \), where binary expansions are determined by \( \delta \) and \( \pi \). Put \( v = ws \). Then \( u' = uv \theta(u^{-1}) \) corresponds to \( S' \) with \( u = x^{-1}e^{-1}x \in W_J \). The map \( \pi \) is obtained by \( \varepsilon^{-1} \pi_0 s \varepsilon \). In terminologies of ABS’s, multiplying \( \pi_0 \) by \( s \) corresponds to constructing the small modification. By the coordinate transformation by \( \varepsilon \), we obtain the map \( \pi \) on \( \tilde{S}' \). The main purpose of Section 3.2 is to give the construction of this \( \varepsilon \) combinatorially.

Example 3.6. Let us see an example of constructing a specialization of an ABS. Let \( S = (\tilde{S}, \delta, \pi_0) \) be the admissible ABS associated to \( (1,0,0,1,0,1,0,0) \). This \( S \) is described as the following diagram.

Let \( \pi \) be the small modification map by \( 0_2 \) and \( 1_4 \). Then the ABS \( S' = (\tilde{S}', \delta, \pi) \) is described as

One can check that this specialization is generic by calculating lengths of \( S \) and \( S' \).

Let \( \xi = (m_1, n_1) + (m_2, n_2) \) be a Newton polygon satisfying that \( \lambda_2 < 1/2 < \lambda_1 \). Let \( S \) denote the ABS of \( N_\xi \). To classify generic specializations of \( S \), in Section 3.2 we introduce a method to construct specializations \( S' \) combinatorially. The method gives an explicit construction of permutations \( \varepsilon \) as in Remark 3.5. Moreover, we show some properties of generic specializations; see Section 4.2. These properties are useful for to show Proposition 5.1, which is a key step of the proof of Theorem 1.3. Let \( A \) and \( B \) be ABS’s corresponding to \( N_{m_1, n_1} \) and \( N_{m_2, n_2} \) respectively. Then \( S \) is obtained by \( A \oplus B \). We often denote by \( \tau_i^A \) (resp. \( \tau_i^B \)), with \( \tau = 0 \) or \( 1 \), the \( i \)-th symbol of \( A \) (resp. \( B \)). By Lemma 2.14, there exist only three cases as the choice of exchange of \( 0_i^A \) and \( 1_j^B \):

1. \( m_1 < i \leq n_1 \) and \( 1 \leq j \leq n_2 \);
2. \( m_1 < i \leq n_1 \) and \( n_2 < j \leq m_2 \);
3. \( n_1 < i \leq h_1 \) and \( n_2 < j \leq m_2 \).

We assume that \( m_1 + 1 < n_1 \) or \( n_2 + 1 < m_2 \) in the case (2). For \( \xi = (m_1, m_1 + 1) + (n_2 + 1, n_2) \) with \( m_1 > 0 \) or \( n_2 > 0 \), the case (2) is treated separately; see below the proof of Proposition 3.20. Since the case (3) can be regarded as the dual of (1), it suffices to deal with the case (1) and (2). From now on, we assume \( m_1 < i \leq n_1 \). We often treat the cases (1) and (2) simultaneously, but for example, the proof of Proposition 3.20 is divided into the cases (1) and (2).
3.2 Constructing a specialization

Let $\xi = (m_1, n_1) + (m_2, n_2)$ be a Newton polygon of two segments satisfying $\lambda_2 < 1/2 < \lambda_1$. Let $S = (\tilde{S}, \delta, \pi_0)$ be the ABS associated to the minimal DM $N_\xi$. In this section, we introduce a combinatorial method to construct a specialization $S'$ of $S$ in order to classify generic specializations. Concretely, for the small modification $\pi$ by $t_i$ and $t_j$, we shall construct the ordered set $\tilde{S}'$, which coincides with $\tilde{S}$ as sets, satisfying that $t < t'$ if and only if $\pi(t) < \pi(t')$ for elements $t$ and $t'$ of $\tilde{S}'$ with $\delta(t) = \delta(t')$. For this ordered set, using Theorem 2.25, we get the specialization $N'_{\xi}$ of $N_\xi$. The method gives a useful characterization to ABS’s $S'$ satisfying $\ell(S') = \ell(S) - 1$. The main purpose of this section is to show that this combinatorial operation is well-defined.

Let $A$ (resp. $B$) be the ABS corresponding to $N_{m_1,n_1}$ (resp. $N_{m_2,n_2}$). Then $S = A \oplus B$. We denote by $\tau^A_i$ (resp. $\tau^B_j$) the $i$-th element of $A$ (resp. the $j$-th element of $B$) with $\tau = 0$ or $1$. Fix natural numbers $i$ and $j$ satisfying that $m_1 < i \leq n_1$ and $1 \leq j \leq m_2$. Let $\pi$ be the small modification by $0^A_i$ and $1^B_j$. For the ordered set

$$\tilde{S} = \{1^A_1 < \cdots < 0^A_i < 0^A_{i+1} < \cdots < 1^B_{j-1} < 1^B_j < 1^B_{j+1} < \cdots < 0^B_{n_2}\},$$

we define an ordered set $S(0)$ by $S(0) = \tilde{S}$ as sets with the ordering

$$S(0) = \{1^A_1 < \cdots < 0^A_i < 1^B_j < 0^A_{i+1} < \cdots < 1^B_{j-1} < 0^A_i < 1^B_j < 1^B_{j+1} < \cdots < 0^B_{n_2}\}.$$ 

In Definition 3.7 and Definition 3.8 we construct a sequence $(S(0), \delta, \pi), (S(1), \delta, \pi), \ldots$ of ABS’s. We will show that there exists a non-negative integer $n$ such that the ABS $(S(n), \delta, \pi)$ coincides with the specialization of $S$ obtained by exchanging $0^A_i$ and $1^B_j$.

**Definition 3.7.** For the ABS $(S(0), \delta, \pi)$, we define a set $A(0)$ by

$$A(0) = \{ t \in S_A^{(0)} \mid t < 0^A_i \text{ and } \pi(0^A_i) < \pi(t) \text{ in } S^{(0)} \text{ with } \delta(t) = 0 \},$$

where $S_A^{(0)}$ is a subset of $S^{(0)}$ consisting of all elements of $A$. For non-negative integers $n$, put $\alpha_n = \pi^n(0^A_i)$. For a natural number $n$, we construct an ordered set $S(n)$ and a set $A(n)$ from the ABS $(S^{(n-1)}, \delta, \pi)$ and the set $A^{(n-1)}$ if $A^{(n-1)}$ is not empty. Let $S(n) = S^{(n-1)}$ as sets. We define the order on $S(n)$ to be for $t < t'$ in $S^{(n-1)}$, we have $t > t'$ in $S(n)$ if and only if $\alpha_n < t' \leq \pi(t_{\text{max}})$ in $S^{(n-1)}$ and $t = \alpha_n$. We can regard $\pi$ as the map on $S(n)$. Thus we obtain an ABS $(S(n), \delta, \pi)$. In other words, we define the ABS $(S(n), \delta, \pi)$ by moving the element $\alpha_n$ to the right of $\pi(t_{\text{max}})$ in $S^{(n-1)}$. We define

$$A^{(n)} = \{ t \in S_A^{(n)} \mid t < \alpha_n \text{ and } \alpha_{n+1} < \pi(t) \text{ in } S^{(n)} \text{ with } \delta(t) = \delta(\alpha_n) \},$$

where $S_A^{(n)}$ is the subset of $S^{(n)}$ consisting of all elements of $A$.

We will see that there exists a non-negative integer $a$ satisfying $A^{(a)} = \emptyset$ in Proposition 3.14.

**Definition 3.8.** We define the set

$$B^{(0)} = \{ t \in S^{(0)} \mid 1^A_j < t \text{ and } \pi(t) < \pi(1^B_j) \text{ in } S^{(0)} \text{ with } \delta(t) = 1 \},$$

and let $I$ be the subset of $B^{(0)}$ consisting of elements $t$ which are of the form $t = 1^A_j$ with a natural number $x$. Note that for an element $t$ of $I$, there exists a non-negative integer $n$ with $n < a$ such
that $t = \alpha_n$. We have $1^B_j < 1^B_z$ and $\pi(1^B_j) < \pi(1^B_z)$ for natural numbers $z$ with $z < j$ in $S(\alpha)$. Thus the set $B^{(0)}$ is determined as $I \cup \{1^B_1, \ldots, 1^B_j\}$. Set $\beta_n = \pi^n(1^B_j)$ for non-negative integers $n$. We construct the ordered set $S(\alpha+n)$ for natural numbers $n$ inductively. For the ABS $(S(\alpha+n-1), \delta, \pi)$ and the set $B^{(n-1)}$, if $B^{(n-1)}$ is not empty, then let $S(\alpha+n) = S(\alpha+n-1)$ as sets. The ordering of $S(\alpha+n)$ is given so that for $t < t'$ in $S(\alpha+n-1)$, we have $t > t'$ in $S(\alpha+n)$ if and only if $\pi(t) \leq t < \beta_n$ in $S(\alpha+n-1)$ and $t' = \beta_n$. We can regard $\pi$ as the map on $S(\alpha+n)$. Thus we obtain an ABS $(S(\alpha+n), \delta, \pi)$. In other words, we obtain the ABS $(S(\alpha+n), \delta, \pi)$ by moving the element $\beta_n$ to the left of $\pi(t)$ in $S(\alpha+n-1)$. We define

$$B^{(n)} = \{ t \in S(\alpha+n) \mid \beta_n < t \text{ and } \pi(t) < \beta_{n+1} \text{ in } S(\alpha+n) \text{ with } \delta(t) = \delta(\beta_n) \}.$$ 

If there exists a non-negative integer $b$ such that $B^{(b)} = \emptyset$, then we call the ABS $(S(\alpha+b), \delta, \pi)$ the full modification by $0^A_1$ and $1^B_2$. If there exists the full modification by $0^A_1$ and $1^B_2$, then the specialization of $S$ obtained by exchanging $0^A_1$ and $1^B_2$ is given by this full modification. We show that there exists a non-negative integer $b$ such that $B^{(b)} = \emptyset$ in Proposition 3.20.

**Example 3.9.** Let us see an example of constructing $S'$ from $S$. Let $N = N_{2,7} \oplus N_{5,3}$ be the DM, and let $S = (S, \delta, \pi)$ denote the ABS associated to $N$. In Example 2.13, we obtain the diagram of $S$. Construct the small modification by $0^A_6$ and $1^B_3$. Then we obtain the ABS $(S^{(0)}, \delta, \pi)$, and the diagram of this ABS is described as

$$(S^{(0)}, \delta, \pi) : \begin{array}{cccccccccccccccccc}
1^A_1 & 1^A_2 & 0^A_3 & 0^A_4 & 0^A_5 & 1^B_3 & 0^B_6 & 0^B_7 & 0^B_8 & 0^B_9 & 0^B_{10} & 0^B_{11} & 0^B_{12} & 0^B_{13} & 0^B_{14} & 0^B_{15} & 0^B_{16} & 0^B_{17} & 0^B_{18} & 0^B_{19} & 0^B_{20}.
\end{array}$$

First, let us construct sets $A^{(n)}$. We have the set $A^{(0)} = \{0^A_1\}$. To construct the ordered set $S^{(1)}$, we move the element $0^A_1$ to the right of $0^A_3$, and we obtain the following diagram of $(S^{(1)}, \delta, \pi)$:

$$(S^{(1)}, \delta, \pi) : \begin{array}{cccccccccccccccccc}
1^A_1 & 1^A_2 & 0^A_3 & 0^A_4 & 0^A_5 & 1^B_3 & 0^B_6 & 0^B_7 & 0^B_8 & 0^B_9 & 0^B_{10} & 0^B_{11} & 0^B_{12} & 0^B_{13} & 0^B_{14} & 0^B_{15} & 0^B_{16} & 0^B_{17} & 0^B_{18} & 0^B_{19} & 0^B_{20}.
\end{array}$$

We have the set $A^{(1)} = \{0^A_1\}$. In the same way, we get $S^{(2)}$ by moving the element $1^A_2$ to the right of $0^A_3$, and we see that $A^{(2)}$ is empty. Hence we obtain $a = 2$. Next, let us construct sets $B^{(n)}$. We have $B^{(0)} = \{1^B_1, 1^B_2\}$. We move the element $0^B_6$ to the left of $1^B_4$ which is the minimum element of $\pi(B^{(0)})$ to construct $S^{(3)}$. Then $B^{(1)} = \emptyset$. Hence we get an ABS $S'$ by $(S^{(3)}, \delta, \pi)$. Thus the diagram of the specialization $S'$ is described as follows.

$$(S') : \begin{array}{cccccccccccccccccc}
1^A_1 & 0^A_3 & 1^A_4 & 0^A_5 & 1^B_3 & 1^B_4 & 0^B_6 & 0^B_7 & 0^B_8 & 0^B_9 & 0^B_{10} & 0^B_{11} & 0^B_{12} & 0^B_{13} & 0^B_{14} & 0^B_{15} & 0^B_{16} & 0^B_{17} & 0^B_{18} & 0^B_{19} & 0^B_{20}.
\end{array}$$

One can check that these admissible ABS’s $S'$ and $S$ satisfy that $\ell(S') = \ell(S) - 1$. Moreover, one can
see that for the same notation as Remark \textbf{3.5} the permutation $\varepsilon$ satisfies $\varepsilon^{-1} = (13, 14, 15)(2, 3)(4, 5)$ in this case.

To construct full modifications of $S$, we show some properties of ordered sets $S^{(n)}$ and sets $A^{(n)}$ in Lemma \textbf{3.10} Proposition \textbf{3.11} Corollary \textbf{3.12} and Proposition \textbf{3.13}. For a Newton polygon $\xi = (m_1, n_1) + (m_2, n_2)$ consisting of two segments satisfying $\lambda_2 < 1/2 < \lambda_1$, we denote by $S = (\tilde{S}, \delta, \pi_0)$ the ABS associated to the DM1 $N_\xi$. Let $A$ (resp. $B$) be the ABS corresponding to the DM1 $N_{m_1, n_1}$ (resp. $N_{m_2, n_2}$). We denote by $\tilde{A}$ (resp. $\tilde{B}$) the ordered set of $A$ (resp. $B$). Let $\pi$ denote the small modification by elements $0_i^A$ and $1_j^B$ of $\tilde{S}$ with $m_1 < i \leq n_1$. Then we obtain sets $\mathcal{A}^{(n)}$ for non-negative integers $n$. We will show that there exists a non-negative integer $n$ such that $\mathcal{A}^{(n)} = \emptyset$ in Proposition \textbf{3.14}.

**Lemma 3.10.** For every non-negative integer $n$, let $\Sigma(n)$ be the set consisting of elements $t$ of $\tilde{A}$ satisfying that $t' < t$ and $\pi(t) < \pi(t')$ in $S^{(n)}$ for some elements $t'$ of $S^{(n)}$ with $\delta(t) = \delta(t')$. If $\mathcal{A}^{(n)}$ is not empty, then $\Sigma(n) = \{\alpha_n\}$, where $\alpha_n = \pi^n(0_i^A)$. Moreover, if $\mathcal{A}^{(n)} = \emptyset$, then $\Sigma(n) = \emptyset$.

**Proof.** Let us show the assertion by induction on $n$. The case $n = 0$ is obvious. For a natural number $n$, suppose that $\Sigma(n-1) = \{\alpha_{n-1}\}$. The statement follows from the construction of the set $S^{(n)}$. In fact, by the construction of $S^{(n)}$, we have $\pi(t_{\text{max}}) < \alpha_n$ in $S^{(n)}$, where $t_{\text{max}}$ is the maximum element of $\mathcal{A}^{(n-1)}$. It implies that $\pi(t) < \alpha_n$ holds for all elements $t$ of $\mathcal{A}^{(n-1)}$ in $S^{(n)}$. Then $\alpha_{n-1}$ does not belong to $\Sigma(n)$. Thus we see that elements $t'$ of $\tilde{A} \setminus \{\alpha_n\}$ do not belong to $\Sigma(n)$, and we obtain $\Sigma(n) = \{\alpha_n\}$.

**Proposition 3.11.** For all non-negative integers $m$, set $\alpha_m = \pi^m(0_i^A)$. Let $n$ be a natural number. Then $\mathcal{A}^{(n)}$ is obtained by

$$\mathcal{A}^{(n)} = \{\pi(t) \in \tilde{A} \mid t \in \mathcal{A}^{(n-1)} \text{ and } \delta(\pi(t)) = \delta(\alpha_n)\}.$$  

**Proof.** We fix a natural number $n$. Choose an element $t$ of $\mathcal{A}^{(n-1)}$ satisfying that $\delta(\pi(t)) = \delta(\alpha_n)$. Let us see that if $\pi(t)$ belongs to $\tilde{A}$, then the set $\mathcal{A}^{(n)}$ contains $\pi(t)$. Since $\alpha_n < \pi(t)$ in $S^{(n-1)}$, we have $\alpha_{n+1} < \pi(\pi(t))$ in $S^{(n-1)}$ and $S^{(n)}$. By the construction of $S^{(n)}$, we have $\pi(t) < \alpha_n$ in $S^{(n)}$. Hence the set $\mathcal{A}^{(n)}$ contains $\pi(t)$. Conversely, let $t'$ be an element of $\mathcal{A}^{(n)}$. We denote by $t$ the element of $S^{(n)}$ satisfying that $\pi(t) = t'$. We have to see that $t$ belongs to $\mathcal{A}^{(n-1)}$. In the sets $S^{(n)}$ and $S^{(n-1)}$, we have $t < \alpha_{n-1}$. Moreover, since $\alpha_{n+1} < \pi(\pi(t))$ holds in $S^{(n)}$ and $S^{(n-1)}$, we get $\alpha_n < \pi(t)$ in $S^{(n-1)}$. Hence $t$ belongs to $\mathcal{A}^{(n-1)}$.

Applying Proposition \textbf{3.11} repeatedly, we obtain the following:

**Corollary 3.12.** Let $n$ be a natural number. Then $\mathcal{A}^{(n)}$ is described as follows:

$$\mathcal{A}^{(n)} = \{\pi^m(t) \mid t \in \mathcal{A}^{(0)} \text{ and } \delta(\pi^m(t)) = \delta(\alpha_m) \text{ and } \pi^m(t) \in \tilde{A} \text{ for all } m \text{ with } 0 \leq m \leq n\}.$$  

**Proposition 3.13.** Let $n$ be a non-negative integer. Then the set $\mathcal{A}^{(n)}$ does not contain an element $t$ which is of the form $t = \pi^m(0_i^A)$ with a non-negative integer $m$ for $m \leq n$.

**Proof.** For every non-negative integer $n$, the set $\mathcal{A}^{(n)}$ consists of some elements of $\tilde{A}$. Hence sets $\mathcal{A}^{(n)}$ do not contain the element $\pi^{-1}(0_i^A)$ which is an element of $\tilde{B}$. We show the assertion by induction on $n$. The case $n = 0$ is obvious. For a natural number $n$, assume that the set $\mathcal{A}^{(n-1)}$ contains no element $\pi^m(0_i^A)$ with $m \leq n - 1$. We consider the set $\mathcal{A}^{(n)}$. Suppose that $\mathcal{A}^{(n)}$ contains an element $\pi^m(0_i^A)$ for $m \leq n$. Then by Proposition \textbf{3.11} $\pi^{m-1}(0_i^A)$ belongs to $\mathcal{A}^{(n-1)}$. This contradicts with the hypothesis of induction.
Proposition 3.14. For every small modification by $0_i^A$ and $1_j^B$ with $m_1 < i < n_1$, there exists a non-negative integer $a = a(i, j)$ such that $A^{(a)} = \emptyset$.

Proof. In the case of $i = n_1$ and $1 \leq j \leq n_2$, we immediately have $A^{(0)} = \emptyset$. Let us see the other cases. By the hypothesis $m_1 < i < n_1$, the map $\pi$ sends $\theta_{i+m_1}^A$ to $1_j^B$ in $S^{(n)}$ for all non-negative integers $n$. Let $m$ be the minimum natural number satisfying $\pi^m(0_i^A) = 1_{m_1}^A$. If there exists a natural number $n$ such that $A^{(n-1)} = \{\theta_{i+m_1}^A\}$ for $n < m$, we have then $A^{(n)} = \emptyset$ by definition of $A^{(n)}$. Assume that sets $A^{(0)}, \ldots, A^{(m-1)}$ are not empty. Let us consider the set $A^{(m)}$. Note that $1_{m_1}^A$ is the maximum element of the set $\{t \in A \mid \delta(t) = 1\}$. Proposition 3.13 induces that every element $t$ of $A^{(m-1)}$ satisfies that $\delta(\pi(t)) = 0$ or $\pi(t) = 1_j^B$, whence we see $A^{(m)} = \emptyset$. \hfill \Box

Thus by a small modification $\pi$ by $0_i^A$ and $1_j^B$, we obtain the non-negative integer $a = a(i, j)$ satisfying that $A^{(a)} = \emptyset$. By the ABS $(S^{(\alpha)}, \delta, \pi)$, we obtain the set $B^{(0)} = \{1_j^B, \ldots, 1_j^{B-1}\} \cup I$, where $I$ is the subset of $S^{(\alpha)}$ consisting of elements $\alpha_n = \pi^n(0_i^A)$ satisfying that $1_j^B < \alpha_n$ and $\alpha_{n+1} < \pi(1_j^B)$. Here, we show some properties of sets $B^{(0)}, B^{(1)}, \ldots$ in Lemma 3.15 Proposition 3.16 Proposition 3.14.Lemma 3.15 and Lemma 3.19 to see that there exists a non-negative integer $n$ such that $B^{(n)} = \emptyset$ in Proposition 3.20. We suppose that if $n_1 = m_1 + 1$ and $n_2 = n_2 + 1$, then $(i, j) \neq (n_1, m_2)$.

Lemma 3.15. Let $n'$ be a non-negative integer with $n' \geq a$. Set $n = n' - a$. Let $\Sigma^{(n)}$ be the set consisting of elements $t$ of $\hat{S}$ satisfying that $t < t'$ and $\pi(t') < \pi(t)$ in $S^{(n')}$ for some elements $t'$ of $S^{(\alpha)}$ with $\delta(t) = \delta(t')$. If $B^{(n)}$ is not empty, we have then $\Sigma^{(n)} = \{\beta_n\}$, where $\beta_n = \pi^n(1_j^B)$. Moreover, if $B^{(n)} = \emptyset$, then $\Sigma^{(n)} = \emptyset$.

Proof. A proof is given by the same way as Lemma 3.10. \hfill \Box

Proposition 3.16. Set $\beta_m = \pi^m(1_j^B)$ for all non-negative integers $m$. Let $n$ be a natural number. The set $B^{(n)}$ is obtained by

$$B^{(n)} = \{\pi(t) \in S^{(\alpha+n)} \mid t \in B^{(n-1)} \text{ and } \delta(\pi(t)) = \delta(\beta_n)\}.$$  

Moreover, this set is described as

$$B^{(n)} = \{\pi^n(t) \mid t \in B^{(0)} \text{ and } \delta(\pi^m(t)) = \delta(\beta_m) \text{ for all } m \text{ with } 0 \leq m \leq n\}.$$  

Proof. A proof is given by the same way as Proposition 3.11. \hfill \Box

Proposition 3.17. Let $n$ be a non-negative integer. If the set $B^{(n)}$ is a subset of $B$, then $B^{(n)}$ does not contain an element $t$ which is of the form $t = \pi^m(1_j^B)$ with a non-negative integer $m$ for $m \leq n$.

Proof. A proof is given by the same way as Proposition 3.13. \hfill \Box

Lemma 3.18. If $1 \leq j \leq n_2$, then all elements $t$ of $I$ satisfy that $\delta(\pi^2(t)) = 0$.

Proof. If $m_1$ and $n_1$ satisfy $n_1 - m_1 > m_1$, then $\delta(\pi^2(t)) = 0$ holds for all elements $t$ of $A$ with $\delta(t) = 1$. It suffices to see the case of $n_1 - m_1 < m_1$. Assume that there exists an element $t$ of $I$ satisfying $\delta(\pi^2(t)) = 1$. We recall that there exists a natural number $n$ such that $t = \pi^n(0_i^A)$. On the other hand, we can denote by $t = 1_i^A$ this element $t$ with a natural number $r$. Put $r' = r + n_1 - m_1$. We have then $\pi^2(1_i^A) = 1_j^A$. By the definition of $I$, the set $A^{(n-1)}$ contains the inverse image of $1_j^B$. By construction, we have $|A^{(m)}| < n_1 - m_1$ for all $m$. In $S^{(n-1)}$, the number of elements between $1_r^A$ and $1_j^B$ is greater than $|A^{(n-1)}|$, whence we have $1_r^A < 1_j^B$ in $S^{(n)}$. This contradict with $1_i^A \in I$.
Lemma 3.19. Let \( n \) be a natural number with \( n > 1 \). Suppose \( 1 \leq j \leq n_2 \). Then \( B^{(n)} \) is a subset of \( B \).

Proof. First, let us see that sets \( B^{(n)} \) do not contain \( 0_i^1 \) for all \( n \). Assume that a set \( B^{(n-1)} \) contains the inverse image of \( 0_i^1 \). Note that the inverse image of \( 0_i^1 \) is obtained by \( 0_{j+m_2}^B \) in this hypothesis. We have then \( \delta(\pi^{n-1}(1_j^B)) = 0 \) by the definition of \( B^{(n-1)} \). The condition \( \lambda_2 < 1/2 \) induces that \( \delta(\pi^n(1_j^B)) = 1 \). It implies that the element \( 0_i^1 \) does not belong to \( B^{(n)} \).

To show the statement of this lemma, let us consider two cases depending on the value of \( \delta(\pi(1_j^B)) \). On the one hand, suppose that \( \delta(\pi(1_j^B)) = 1 \), and let us see that \( B^{(n)} \) consists of some elements of \( B \) for \( n \geq 1 \). We have \( B^{(1)} = \{\pi(t) \mid t \in B^{(0)} \text{ with } \delta(\pi(t)) = 1\} \) in this assumption. For elements \( t \) of \( I \), all elements \( \pi(t) \) do not belong to \( B^{(1)} \) since \( \delta(\pi(t)) = 0 \) holds for all elements \( t \) of \( A \) with \( \delta(t) = 1 \). It induces that \( B^{(1)} \) is a subset of \( B \), whence we see that \( B^{(n)} \subset B \) for all natural numbers in this case.

On the other hand, suppose \( \delta(\pi(1_j^B)) = 0 \). Let \( \Lambda \) be the subset of \( B^{(1)} \) consisting of elements \( t \) which belong to \( B \). Then \( B^{(1)} \) is obtained by the union of \( \Lambda \) and \( \pi(I) \). By the condition of \( m_2 \) and \( n_2 \), for all elements \( t \) of \( \Lambda \), we have \( \delta(\pi(t)) = 1 \). Moreover, by Lemma 3.19, we have \( \delta(\pi(t)) = 0 \) for all elements \( t \) of \( \pi(I) \). Thus we see that \( B^{(2)} \) is a subset of \( B \) since \( \delta(\pi^2(1_j^B)) = 1 \) in this hypothesis. Hence \( B^{(n)} \) is a subset of \( B \) for every natural number \( n \) with \( n > 1 \).

By the above properties, we show Proposition 3.20. By this proposition, we see that specializations \( S'\) of \( S \) are obtained by the combinatorial method.

Proposition 3.20. For the small modification by \( 0_i^1 \) and \( 1_j^B \) with \( m_1 < i \leq n_1 \), there exists a non-negative integer \( b = b(i, j) \) such that \( B^{(b)} = \emptyset \).

Proof. First, let us consider the case \( 1 \leq j \leq n_2 \). If \( j = 1 \), then we have \( B^{(0)} = I \) and \( B^{(1)} = \emptyset \). For \( j > 1 \), we have \( B^{(0)} = \{1_i^B, \ldots, 1_{j-1}^B\} \cup I \). Let \( m \) be the minimum natural number satisfying \( \pi^m(1_j^B) = 0_{m_2+1}^B \). If \( m = 1 \), then \( B^{(1)} = \pi(I) \). Since \( \delta(\pi^2(1_j^B)) = 1 \) in this case and Lemma 3.19, we have \( B^{(2)} = \emptyset \). Let us see the case \( m > 1 \). Suppose that there exists a natural number \( n \), with \( n < m \), such that \( B^{(n)} = \{0_{j+m_2}^B\} \), i.e., the set \( B^{(n)} \) consists of an element the inverse image of \( 0_i^1 \). By the proof of Lemma 3.19, the element \( 0_i^1 \) does not belong to \( B^{(n+1)} \), whence we have \( B^{(n+1)} = \emptyset \). Assume that \( B^{(0)}, \ldots, B^{(m-1)} \) are non-empty sets. Note that \( 0_{m_2+1}^B \) is the minimum element of all elements \( t \) of \( \bar{B} \) satisfying \( \delta(t) = 0 \). For an element \( t \) of \( B^{(m-1)} \), if \( \pi(t) \) belongs to \( \bar{B} \), then \( \delta(\pi(t)) = 1 \) holds. Moreover, it follows from Lemma 3.19 that \( B^{(m)} \) does not contain an element of \( A \). Therefore we have \( B^{(m)} = \emptyset \).

Next, let us see the case \( n_2 < j \leq m_2 \). We divide the proof into two cases depending on whether \( I \) is empty. First, suppose \( I = \emptyset \). For the minimum non-negative integer \( n \) satisfying that \( \pi^n(1_j^B) = 1_i^B \), we have \( B^{(n)} = \emptyset \). In fact, if the set \( B^{(n-1)} \) is not empty, then \( B^{(n-1)} = \{0_i^1\} \) holds since \( \pi^{n-1}(1_j^B) = 0_{m_2+1}^B \). If \( \delta(\pi(0_i^1)) = 1 \), then \( I \) contains \( \pi(0_i^1) \) since \( A^{(0)} = \{0_{i+1}^1, \ldots, 0_{m_1+n_3}^B\} \) contains the inverse image of \( 1_j^B \). This contradicts the assumption. Hence we have \( \delta(\pi(0_i^1)) = 0 \), and then \( B^{(n)} = \emptyset \) holds.

Next, suppose \( I \neq \emptyset \). If \( j > n_2 + 1 \), then the non-negative integer \( b \) is obtained by the number \( n \) satisfying that \( \pi^n(1_j^B) = 1_i^B \). In fact, \( B^{(n-1)} \) consists of some elements \( t \) of \( A \) since \( \pi^{n-1}(1_j^B) = 1_i^B \), is the minimum element of \( \bar{B} \). It is clear that \( \delta(\pi(t)) = 0 \) for all elements \( t \) of \( B^{(n-1)} \), and we have then \( B^{(n)} = \emptyset \). If \( j = n_2 + 1 \), then the non-negative integer \( b \) is obtained by the number \( n \) satisfying \( \pi^n(1_j^B) = 1_{n_2+2}^B \).
For the Newton polygon $\xi$, we now suppose $n_1 = m_1 + 1$ and $m_2 = n_2 + 1$ with $m_1 > 0$ or $n_2 > 0$. If $m_1 < i \leq n_1$ and $n_2 < j \leq m_2$, then $i$ and $j$ must be $i = m_1 + 1$ and $j = m_2$. We construct the specialization by a well-known method using binary expansions for such ABS’s. For the ABS $S = (\tilde{S}, \delta, \pi_0)$ corresponding to $N_\xi$, we have then the following diagram of $S$:

\[
\begin{array}{cccccccc}
1^A & \cdots & 1^A & 0^A_{m_1+1} & 1^B & \cdots & 1^B & 0^A & \cdots & 0^A & 1^B_{m_2} & 0^B & \cdots & 0^B. \\
\end{array}
\]

Note that the diagram satisfies that $\delta(t) \neq \delta(\pi_0^{-1}(t))$ for all elements $t$ except for these two elements. Let $\pi$ be the small modification by $0^A_{m_1+1}$ and $1^B_{m_2}$, and we obtain the admissible ABS $S' = (\tilde{S}', \delta, \pi)$. By the construction of $S'$, we have $\delta(t) \neq \delta(\pi(t))$ for all elements of $\tilde{S}'$. Hence we obtain binary expansions of elements by $b(t) = 0.1010\cdots$ if $\delta(t) = 0$ and $b(t) = 0.0101\cdots$ otherwise. Therefore, the ABS $S'$ is associated to a DM$_1 mN_{1,1}$ with $m = m_1 + m_2$. Note that this ABS satisfies that $\ell(S') < \ell(S) - 1$. In this case, by the small modification, we have $B^{(0)} = \{1^A_1, \ldots, 1^A_{m_1}, 1^B_1, \ldots, 1^B_{m_2}\}$. For two elements $t$ and $t'$ of $B^{(0)}$ and for every non-negative integer $n$, we have $\delta(\pi^n(t)) = \delta(\pi^n(t'))$. Hence there exists no non-negative integer $n$ satisfying $B^{(n)} = \emptyset$.

Let $S$ be the ABS associated to $N_\xi$ with a Newton polygon $\xi = (m_1, n_1) + (m_2, n_2)$ satisfying that $\lambda_2 < 1/2 < \lambda_1$. For the case $\xi = (m_1, m_1 + 1) + (m_2, m_2 - 1)$, we have seen that the specialization of $S$ obtained by exchanging $0^A_{m_1+1}$ and $1^B_{m_2}$ is associated to the DM$_1 (m_1 + m_2)N_{1,1}$. Moreover, for the other cases, we obtain specializations by full modifications. From now on, for a small modification $\pi$ by $0^A_i$ and $1^B_j$, we denote by $a$ and $b$ the minimum non-negative integers satisfying that $A^{(a)} = \emptyset$ and $B^{(b)} = \emptyset$. The specialization $S'$ obtained by exchanging $0^A_i$ and $1^B_j$ is equal to the ABS $(S^{(a+b)}, \delta, \pi)$.

### 3.3 Some examples

In Example 3.9 we introduced an example of full modifications by $0^A_i$ and $1^B_j$ with $m_1 < i \leq n_1$ and $1 \leq j \leq n_2$. Here, let us see examples of the case $n_2 < j \leq m_2$, which are examples of the latter part of the proof of Proposition 3.20.

**Example 3.21.** Let $N = N_{2,7} \oplus N_{5,3}$. For the ABS $S = (\tilde{S}, \delta, \pi_0)$ associated to $N$, we construct the small modification $\pi$ by $0^A_0$ and $1^B_5$. Then we get $A^{(0)} = \{0^A_7, 0^A_8, 0^A_9\}$ and $a = 2$. The ABS $(S^{(2)}, \delta, \pi)$ is described as

\[
(S^{(2)}, \delta, \pi) : 1^A_1 0^A_3 0^A_5 1^B_2 0^A_7 0^A_4 1^B_1 1^B_3 1^B_5 0^A_3 0^A_7 0^A_5 0^A_1 0^A_6 0^A_2 0^B_6 0^B_8. 
\]

Then we have $I = \emptyset$, and sets $B^{(n)}$ are given by

\[
B^{(0)} = \{1^B_1, 1^B_2, 1^B_3, 1^B_4\}, \ B^{(1)} = \{0^A_6, 0^B_6, 0^B_7\}, \ B^{(2)} = \{1^B_1, 1^B_2\}, \ B^{(3)} = \{0^A_6\}, \ B^{(4)} = \emptyset.
\]

The ABS $S'$ is obtained by the following:

\[
S' = 1^A_1 0^A_3 0^A_5 0^A_7 1^B_2 0^A_3 1^B_4 1^B_6 0^A_8 0^A_9 0^B_6 1^B_4 0^A_6 0^B_7.
\]
Example 3.22. Let \( N = N_{2.7} \oplus N_{5.3} \). For the \( \text{ABS} S = (\tilde{S}, \delta, \pi_0) \) associated to \( N \), we construct the small modification \( \pi \) by \( 0^4 \) and \( 1^B \). Then we get \( \mathcal{A}^{(0)} = \{0^4, 0^A, 0^A, 0^A, 0^A \} \) and \( a = 1 \). The \( \text{ABS} (S^{(1)}, \delta, \pi) \) is described as follows:

\[
(S^{(1)}, \delta, \pi) : 1^A_{1} \ 0^A_{3} \ 1^B_{4} \ 0^A_{5} \ 0^A_{6} \ 0^A_{7} \ 1^B_{8} \ 0^A_{9} \ 1^B_{10} \ 1^B_{11} \ 0^A_{12} \ 1^B_{13} \ 0^A_{14} \ 1^B_{15} \ 0^B_{16} \ 0^B_{17} \ 0^B_{18} .
\]

Then \( I = \{1^A_2\} \), and we have sets

\[
\mathcal{B}^{(0)} = \{1^A_2, 1^B_1, 1^B_2, 1^B_3\}, \quad \mathcal{B}^{(1)} = \{0^A_9, 0^A_4, 0^B_6\}, \quad \mathcal{B}^{(2)} = \{1^A_2, 1^B_1\}, \quad \mathcal{B}^{(3)} = \emptyset.
\]

The \( \text{ABS} S' \) is obtained by the following:

\[
S' = 1^A_{1} \ 0^A_{3} \ 1^B_{4} \ 0^A_{5} \ 0^A_{6} \ 1^B_{7} \ 0^A_{8} \ 1^B_{9} \ 0^A_{10} \ 1^B_{11} \ 0^A_{12} \ 1^B_{13} \ 0^A_{14} \ 1^B_{15} \ 0^B_{16} \ 0^B_{17} \ 0^B_{18}.
\]

4 Classification of boundary components

Let \( \xi \) be a Newton polygon consisting of two segments satisfying \( \lambda_2 < 1/2 < \lambda_1 \). In this section, for the arrowed binary sequence \( S \) associated to the minimal DM \( N_\xi \), we characterize specializations \( S' \) of \( S \) satisfying \( \ell(S') = \ell(S) - 1 \), i.e., we classify boundary components of central streams. Moreover, in Section 4.2, we show some properties of generic specializations. We will give a proof of Theorem 4.3 in Section 5 using these properties.

4.1 Criterion of boundary components

Now, we state the first result (Theorem 4.1). Let \( \xi = (m_1, n_1) + (m_2, n_2) \) be a Newton polygon with \( \lambda_2 < 1/2 < \lambda_1 \), and let \( N_\xi \) be the minimal DM associated to \( \xi \). Let \( S = A \oplus B \) be the \( \text{ABS} \) corresponding to \( N_\xi \), where \( A \) (resp. \( B \)) is the \( \text{ABS} \) corresponding to \( N_{m_1, n_1} \) (resp. \( N_{m_2, n_2} \)). Put \( (\tilde{S}, \delta, \pi_0) = S \). Set \( h_1 = m_1 + n_1 \) and \( h_2 = m_2 + n_2 \). We construct the small modification \( \pi \) by \( 0^A \) with \( m_1 < i \leq n_1 \) and \( 1^B \) with \( 1 \leq j < m_2 \). Then for the notation of Section 3 we obtain sets \( \mathcal{A}^{(0)} \) for \( 0 \leq n \leq a \), sets \( \mathcal{B}^{(n)} \) for \( 0 \leq n \leq b \) and \( \text{ABS}'s \) \( (S^{(n)}, \delta, \pi) \) for \( 0 \leq n \leq a + b \), where \( a \) and \( b \) are the smallest non-negative integers satisfying that \( \mathcal{A}^{(a)} = \emptyset \) and \( \mathcal{B}^{(b)} = \emptyset \). The main result of this section is

**Theorem 4.1.** Let \( \xi = (m_1, n_1) + (m_2, n_2) \) be a Newton polygon with \( \lambda_2 < 1/2 < \lambda_1 \). Let \( S = (\tilde{S}, \delta, \pi_0) \) be the \( \text{ABS} \) associated to the DM \( N_\xi \). Let \( S' = (\tilde{S}', \delta, \pi) \) be the \( \text{ABS} \) obtained by exchanging \( 0^4 \) and \( 1^B \). Then \( \ell(S') = \ell(S) - 1 \) holds if and only if there exists no non-negative integer \( n \) such that \( \mathcal{A}^{(n)} \) contains the inverse image of \( 1^B \) or \( \mathcal{B}^{(n)} \) contains the inverse image of \( 0^4 \) for the small modification \( \pi \).

This theorem gives a classification of generic specializations of \( H(\xi) \). To show the above, we divide the problem into three cases depending on conditions of \( i \) and \( j \) as follows.

**Definition 4.2.** For the \( \text{ABS} S = A \oplus B \) associated to the minimal DM \( N_\xi \) with the Newton polygon \( \xi = (m_1, n_1) + (m_2, n_2) \), we denote by \( S' = S'(i, j) \) the specialization obtained by exchanging
$0_i^A$ and $1_j^B$. We define sets

$$
\mathcal{H}_1(S) = \{ S'(i,j) \mid m_1 < i \leq n_1 \text{ and } 1 \leq j \leq n_2 \},
$$

$$
\mathcal{H}_2(S) = \{ S'(i,j) \mid m_1 < i \leq n_1 \text{ and } n_2 < j \leq m_2 \},
$$

$$
\mathcal{H}_3(S) = \{ S'(i,j) \mid n_1 < i \leq h_1 \text{ and } n_2 < j \leq m_2 \}.
$$

Let $T$ be the ABS associated to a DM $N$, and let $T_D$ be the ABS associated to the dual $N^D$ of $N$. We call this ABS dual ABS of $T$. For the above notation, we will give a concrete condition of $i$ and $j$ satisfying that the exchange of $0_i^A$ and $1_j^B$ is a good. Theorem 4.1 follows from this proposition:

**Proposition 4.3.** The following holds:

1. For $S' \in \mathcal{H}_1(S)$, the formula $\ell(S') = \ell(S) - 1$ holds if and only if there exists no non-negative integer $n$ satisfying that $\mathcal{A}^{(n)}$ contains $0_i^{A_{i+m_1}}$ or $\mathcal{B}^{(n)}$ contains $0_j^{B_{j+m_2}}$.

2. For $S' \in \mathcal{H}_2(S)$, we have $\ell(S') < \ell(S) - 1$.

3. For $S' \in \mathcal{H}_3(S)$, the formula $\ell(S') = \ell(S) - 1$ holds if and only if the dual ABS’s $S_D$ and $S_D'$ satisfy $\ell(S_D') = \ell(S_D) - 1$.

As an example, for an element $S'$ of $\mathcal{H}_1(S)$, we immediately see that $0_i^{A_{i+m_1}}$ and $0_j^{B_{j+m_2}}$ are inverse images of $1_j^B$ and $0_i^A$ respectively.

If the Newton polygon $\xi = (m_1, n_1) + (m_2, n_2)$ satisfies that $n_1 = m_1 + 1$ and $m_2 = n_2 + 1$ with $m_1 > 0$ or $n_2 > 0$, then the specialization $S'$ obtained by exchanging $0_i^{A_{i+m_1}}$ and $1_j^{B_{j+m_2}}$ corresponds to the DM $mN \tilde{N}_{1,1}$ with $m = m_1 + m_2$. Clearly this $S'$ satisfies $\ell(S') < \ell(S) - 1$. In this case, the set $\mathcal{A}^{(0)}$ contains the inverse image of $1_j^B$ for the small modification $\pi$ by $0_i^{A_{i+m_1}}$ and $1_j^{B_{j+m_2}}$. We may assume that every specialization $S'$ is obtained by the full modification $(S'(a+b), \delta, \pi)$.

First, we will show (i) and (ii) of Proposition 4.3. For the ABS $(\tilde{S}, \delta, \pi_0)$ associated to $N$, fix elements $0_i^A$ and $1_j^B$ with $m_1 < i \leq n_1$. Let $\pi$ be the small modification by $0_i^A$ and $1_j^B$. Let $a$ (resp. $b$) denote the smallest non-negative integer such that $\mathcal{A}^{(a)} = \emptyset$ (resp. $\mathcal{B}^{(b)} = \emptyset$). We introduce some definitions to calculate the length of the specialization $S' = (S'(a+b), \delta, \pi)$. For simplicity, we often write $\ell(S^{(n)})$ for the length of the ABS $(S^{(n)}, \delta, \pi)$.

**Notation 4.4.** For non-negative integers $n$ with $n < a$, we define $d_A(n)$ by

$$
d_A(n) = |\mathcal{A}^{(n)}| - |\mathcal{A}^{(n+1)}|.
$$

Moreover, we define $\Delta \ell_A(n)$ by

$$
\Delta \ell_A(n) = \ell(S^{(n+1)}) - \ell(S^{(n)}).
$$

Put $\Delta \ell_A = \sum_n \Delta \ell_A(n)$.

**Notation 4.5.** Let $n'$ be a non-negative integer with $n' \geq a$. Put $n = n' - a$. We define $d_B(n)$ by

$$
d_B(n) = |\mathcal{B}^{(n)}| - |\mathcal{B}^{(n+1)}|.
$$

Moreover, we define $\Delta \ell_B(n)$ by

$$
\Delta \ell_B(n) = \ell(S^{(n'+1)}) - \ell(S^{(n')}).
$$

Put $\Delta \ell_B = \sum_n \Delta \ell_B(n)$.
Example 4.6. Let \( N = N_{3,7} \oplus N_{5,3} \). In Example 3.9, we constructed the small modification by elements \( 0_6^A \) and \( 1_3^B \). By this small modification, the length of the ABS’s decrease by four from \( S \) to \( S^{(0)} \). We have \( \Delta \ell_A = 1 \) and \( \Delta \ell_B = 2 \). Thus the length is increased by three from \( S^{(0)} \) to \( S' \), and eventually we see \( \ell(S') = \ell(S) - 1 \).

For the above definition of \( d_A(n) \) and \( d_B(n) \), we obtain Lemma 4.7 and Lemma 4.8 which are used for evaluating values \( \Delta \ell_A \) and \( \Delta \ell_B \). Recall that \( I \) is the subset of \( B^{(0)} \) consisting of elements \( t \) of \( \tilde{A} \).

Lemma 4.7. Let \( S' \in \mathcal{H}_1(S) \). The following are true:

1. \( \sum_n d_A(n) = n_1 - i \),
2. \( \sum_n d_B(n) = |I| + j - 1 \).

Proof. Clearly \( \sum_n d_A(n) \) is given by \( |A^{(0)}| - |A^{(a)}| \). As \( |A^{(0)}| = n_1 - i \) and \( |A^{(a)}| = 0 \), we obtain the desired value. Similarly, we obtain (2) since \( |B^{(0)}| = |I| + j - 1 \) and \( |B^{(b)}| = 0 \).

Lemma 4.8. Let \( S' \in \mathcal{H}_2(S) \). The following are true:

1. \( \sum_n d_A(n) = h_1 - i \),
2. \( \sum_n d_B(n) = |I| + j - 1 \).

Proof. Note that we have \( A^{(0)} = \{0_1^A, \ldots, 0_{h_1}^A \} \) and \( B^{(0)} = I \cup \{1_1^B, \ldots, 1_{j-1}^B \} \) in this case. A proof is given by the same way as Lemma 4.7.

Notation 4.9. For an element \( t \) of \( S^{(n)} \) with \( \delta(t) = 1 \), we define \( \ell(t, n) \) by the number of elements \( t' \) satisfying that \( t' < t \) and \( \delta(t') = 0 \) in \( S^{(n)} \). For instance, the sum \( \sum_t \ell(t, n) \) is equal to the length of the ABS \( (S^{(n)}, \delta, \pi) \).

We will give a criterion of generic specializations in Proposition 4.11 and Theorem 4.1 by comparing values of \( d_A(n) \) and \( \Delta \ell_A(n) \), or \( d_B(n) \) and \( \Delta \ell_B(n) \) using Proposition 4.11 and Proposition 4.12 below.

Lemma 4.10. Let \( n \) be a non-negative integer. Put \( \alpha = \pi^{n+1}(0_1^A) \) and \( \beta = \pi^{n+1}(1_1^B) \). The following holds:

\[
\begin{align*}
|\Delta \ell_A(n)| &= \# \{ t \in A^{(n)} \mid \delta(\pi(t)) \neq \delta(\alpha) \}, \\
|\Delta \ell_B(n)| &= \# \{ t \in B^{(n)} \mid \delta(\pi(t)) \neq \delta(\beta) \}.
\end{align*}
\]

Concretely, if \( \delta(\alpha) = 0 \) (resp. \( \delta(\alpha) = 1 \)), we have then \( \Delta \ell_A(n) \leq 0 \) (resp. \( \Delta \ell_A(n) \geq 0 \)), and if \( \delta(\beta) = 0 \) (resp. \( \delta(\beta) = 1 \)), we have then \( \Delta \ell_B(n) \geq 0 \) (resp. \( \Delta \ell_B(n) \leq 0 \)).

Proof. We fix a non-negative integer \( n \), and let us show that the equation (5) holds. In the same way, we can obtain the equation (6). We divide the proof into two cases depending on the value of \( \delta(\alpha) \). First, suppose \( \delta(\alpha) = 0 \). If \( \pi(A^{(n)}) \) does not contain \( 1_1^B \), then it follows from Proposition 3.13 that all elements \( t \) of \( \pi(A^{(n)}) \) satisfy \( \delta(t) = 0 \), and hence we have \( d_A(n) = 0 \). Moreover, in this case \( \Delta \ell_A(n) = 0 \) holds. If \( \pi(A^{(n)}) \) contains \( 1_1^B \), then all elements \( t \) of \( \pi(A^{(n)}) \) satisfy \( \delta(t) = 0 \) except for \( 1_1^B \), whence we have \( d_A(n) = 1 \). Then \( \ell(1_1^B, n + 1) = \ell(1_1^B, n) - 1 \) holds. Moreover, \( \ell(t, n + 1) = \ell(t, n) \) holds for the other elements, whence we have \( \Delta \ell_A(n) = -1 \). Next, let us see
the case of $\delta(\alpha) = 1$. By the construction of $S^{(n+1)}$, it is clear that $\ell(\alpha, n + 1) = \ell(\alpha, n) + r$, where $r = \# \{ t \in A^{(n)} \mid \delta(\pi(t)) \neq \delta(\alpha) \}$. Moreover, $\ell(t, n + 1) = \ell(t, n)$ holds for the other elements $t$. Clearly we have $d_B(n) = r$. Hence we get desired equality for the case of $\delta(\alpha) = 1$. This completes the proof.

\[ \square \]

**Proposition 4.11.** For all non-negative integers $n$, an inequality $\Delta \ell_A(n) \leq d_A(n)$ holds. The equality holds for all $n$ if and only if $A^{(n)}$ do not contain the inverse image of $1^n_j$ for all $n$.

**Proof.** By the condition $m_1 < i \leq n_1$, in ABS’s $(S^{(n)}, \delta, \pi)$, the inverse image of $1^n_j$ is obtained by $0^{A}_{i+m_1}$. We fix a non-negative integer $n$ with $n < a$. Put $\alpha = \pi^{n+1}(0^A_i)$. In the ordered set $S^{(n+1)}$, the element $\alpha$ is located in the right of elements of $\pi(A^{(n)})$. Let us suppose that $A^{(n)}$ does not contain $0^{A}_{i+m_1}$. This assumption implies that $\pi(A^{(n)})$ contains no element of $B$. If $\delta(\alpha) = 0$, then all elements $t$ of $A^{(n)}$ satisfy $\delta(\pi(t)) = 0$. Hence we have $\Delta \ell_A(n) = 0$, and then $d_A(n) = 0$ holds. If $\delta(\alpha) = 1$, by Proposition 3.11 the set $A^{(n+1)}$ is obtained by $\pi(A^{(n)}) \setminus \Xi$, where

$$
\Xi = \{ \pi(t) \mid t \in A^{(n)} \text{ and } \delta(\pi(t)) \neq \delta(\alpha) \}.
$$

We immediately obtain $|\Xi| = d_A(n)$. Lemma 4.11 concludes that $\Delta \ell_A(n) = d_A(n)$.

Assume that there exists a non-negative integer $n$ such that $A^{(n)}$ contains $0^{A}_{i+m_1}$. We divide the proof into two cases depending on the value of $\delta(\alpha)$. First, if $\delta(\alpha) = 0$, it follows from $\ell(1^n_j, n + 1) = \ell(1^n_j, n) - 1$ that $\Delta \ell_A(n) = -1$. As $A^{(n+1)}$ is obtained by $A^{(n)} = \pi(A^{(n)}) \setminus \{1^n_j\}$, we have $d_A(n) = 1$. Hence we get $\Delta \ell_A(n) < d_A(n)$. Next, if $\delta(\alpha) = 1$, we obtain the set $A^{(n+1)}$ by $\pi(A^{(n)}) \setminus \Xi'$, where

$$
\Xi' = \{ \pi(t) \mid t \in A^{(n)} \text{, with } \delta(\pi(t)) = 0 \text{ or } \pi(t) \in B \}.
$$

It is clear that $\Xi'$ contains $1^n_j$ in this hypothesis. We have $d_A(n) = |\Xi'|$. On the other hand, since $\delta(1^n_j) = \delta(\alpha)$, we have $\ell(\alpha, n + 1) = \ell(\alpha, n) + (|\Xi'| - 1)$, and it implies that $\Delta \ell_A(n) = |\Xi'| - 1$. Hence we get $\Delta \ell_A(n) < d_A(n)$.

\[ \square \]

**Proposition 4.12.** For all non-negative integers $n$, an inequality $\Delta \ell_B(n) \leq d_B(n)$ holds. Moreover, for $1 \leq j \leq n_2$, the equality holds for all $n$ if and only if

1. $B^{(n)}$ do not contain the inverse image of $0^n_i$ for all $n$, and

2. $I = \emptyset$.

**Proof.** For all $j$, the inequality follows from Lemma 4.11. To see the latter part, we treat the case of $1 \leq j \leq n_2$. In this hypothesis, in ABS’s $(S^{(n)}, \delta, \pi)$, the inverse image of $0^n_i$ is obtained by $0^{B}_{j+m_2}$. If sets $B^{(n)}$ do not contain $0^{B}_{j+m_2}$ for all $n$ and $I = \emptyset$, then we can show that the equality $\Delta \ell_B(n) = d_B(n)$ holds in the same way as Proposition 4.11.

Let us see the converse. Put $\beta_n = \pi^n(1^n_j)$ for non-negative integers $n$. We assume that $B^{(n)}$ contains $0^{B}_{j+m_2}$ for a non-negative integer $n$. By the condition $n_2/h_2 < 1/2$ with $h_2 = m_2 + n_2$, we have $\delta(t) = 0$ and $\delta(\pi(t)) = 1$ for all elements $t$ of $B^{(n)}$ except for $0^{B}_{j+m_2}$. Moreover, we have $\delta(\beta_n) = 0$ and $\delta(\beta_{n+1}) = 1$. In the ABS $(S^{(n+a+1)}, \delta, \pi)$ we have $\beta_{n+1} < \pi(t_{\min})$, where $t_{\min}$ is the minimum element of $B^{(n)}$. We have then $\Delta \ell_B(n) = -1$ since $\ell(\beta_{n+1}, n + 1) = \ell(\beta_{n+1}, n) - 1$ and $\ell(t, n + 1) = \ell(t, n)$ for the other elements $t$ of $S^{(n+a+1)}$. On the other hand, we have $d_B(n) = 1$. In fact, $B^{(n+1)}$ is given by $B^{(n+1)} = \pi(B^{(n)}) \setminus \{0^n_i\}$. 


Next, assume $I \neq \emptyset$. We divide the proof into two cases depending on values of $\delta(\beta_1)$. If $\delta(\beta_1) = 0$, then the set $B^{(1)}$ is the union of $\Lambda$ and $\pi(I)$, where $\Lambda$ is the subset of $B^{(1)}$ consisting of elements $t$ of $B^{(1)}$ satisfying $t \in B$. Note that $\delta(\beta_2) = 1$ in this hypothesis. We have $\delta(\pi(t)) = 1$ for every element $t$ of $\Lambda$. Moreover, Lemma 3.15 implies that $\delta(\pi(t)) = 0$ for every element $t$ of $\pi(I)$. Hence we have $\ell(\beta_2, 2) = \ell(\beta_2, 1) - |I|$, and it implies that $\Delta \ell_B(1) = -|I|$. On the other hand, we have $d_B(1) = |I|$, whence $\Delta \ell_B(1) < d_B(1)$ holds. Let us suppose $\delta(\beta_1) = 1$. In this case, we obtain $\Delta \ell_B(0) = -|I|$. Since $d_B(0) = |I|$, we have $\Delta \ell_B(0) < d_B(0)$.

Thanks to the above propositions, we can prove Proposition 4.3.

Proof of Proposition 4.3 (i). We have $\ell((S^{(0)}, \delta, \pi)) - \ell(S) = -(n_1 - i + j)$. Note that if there exists no non-negative integer $n$ such that $A^{(n)}$ contains the inverse image of $1^B_j$, then $I = \emptyset$ holds. Furthermore, if there exists no non-negative integer $n$ such that $B^{(n)}$ contains the inverse image of $0^A_i$, then Lemma 4.7, Proposition 4.11 and Proposition 4.12 imply that $\Delta \ell_A = n_1 - i$ and $\Delta \ell_B = j - 1$. Hence we have $\ell(S') - \ell((S^{(0)}, \delta, \pi)) = n_1 - i + j - 1$, and $\ell(S') = \ell(S) - 1$ holds.

Suppose that there exists a non-negative integer $n$ such that $A^{(n)}$ contains the inverse image of $1^B_j$ or $B^{(n)}$ contains the inverse image of $0^A_i$. If $I = \emptyset$, then we have $\Delta \ell_A < n_1 - i$ or $\Delta \ell_B < j - 1$. Then we have $\ell(S') < \ell(S) - 1$. On the other hand, if $I \neq \emptyset$, then we have $\Delta \ell_A \leq n_1 - i - |I|$. Moreover, by the proof of Proposition 4.12 as there exists a non-negative integer $m$ such that $\Delta \ell_B(m) = -|I|$, we have $\Delta \ell_B < j - 1$. Hence we have $\ell(S') < \ell(S) - 1$.

Proof of Proposition 4.3 (ii). In this case, the set $A^{(0)}$ is given by $A^{(0)} = \{0^A_{i_1+1}, \ldots, 0^A_{h_1}\}$. We have $\ell((S^{(0)}, \delta, \pi)) - \ell(S) = -(h_1 - i + j)$. By the condition of $i$, the set $A^{(0)}$ contains $0^A_{i_1+m_3}$ which is the inverse image of $1^B_{j_1}$. Hence we have $\Delta \ell_A < h_1 - i$ by Lemma 4.8 and Proposition 4.11. Moreover, we have $\Delta \ell_B \leq j - 1$ since if $B^{(n)}$ contains the element $\pi^n(t)$ for $t \in I$, then $\Delta \ell_B(n) < 0$ and hence $\ell(S') < \ell(S) - 1$ holds. In the case $n_1 = m_1 + 1$ and $m_2 = n_2 + 1$, for the small modification $\pi$ by $0^A_{i_1}$ and $1^B_{m_2}$, the set $A^{(0)}$ contains the inverse image of $1^B_{m_2}$.

Let us classify specializations satisfying $\ell(S') = \ell(S) - 1$ for $S' \in H_3(S)$. We use the duality to consider this case. Let $N = N^\xi$ be the DM$_1$ with a Newton polygon $\xi = (m_1, n_1) + (m_2, n_2)$ satisfying $\lambda_2 < 1/2 < \lambda_1$. Let $N^D$ be the dual of $N$. Then we have $N^D = N_{n_2, m_2} \oplus N_{n_1, m_1}$. Let $S_D$ be the ABS corresponding to $N^D$. Note that $\ell(S) = \ell(S_D)$ follows from the definition of the lengths of ABS’s.

Proof of Proposition 4.3 (iii). Fix an ABS $S' \in H_3(S)$. For the same notation as above, $S_D = B_D \oplus A_D$, where $B_D$ and $A_D$ correspond to $N_{n_2, m_2}$ and $N_{n_1, m_1}$ respectively. The elements $0^A_i$ and $1^B_j$ correspond to $1_j'$ and $0_j'$, respectively. The duality, with $i' = h_2 - j + 1$ and $j' = h_1 - i + 1$. Then $S'_D = (S_D)'$ is the ABS obtained by the small modification by $0_j'$ and $1_j'$ in $S_D$. Since $S'_D$ belongs to $H_1(S_D)$, the exchange of $0_j'$ and $1_j'$ in $S_D$ is well-defined if and only if the small modification by $0_j'$ and $1_j'$ satisfies the necessary and sufficient condition of Proposition 4.3 (i). As the small modification by $0^A_i$ and $1^B_j$ in $S$ corresponds to the small modification by $0_j'$ and $1_j'$ in $S_D$, the equality $\ell(S') = \ell(S) - 1$ holds if and only if $\ell(S'_D) = \ell(S_D) - 1$ holds.

Finally, Proposition 4.3 induces the main theorem of this section (Theorem 4.1).

Proof of Theorem 4.1. Proposition 4.3 (i) and (ii) imply that the statement of Theorem 4.1 holds for ABS’s $S'$ of $H_1(S)$ and $H_2(S)$. Moreover, by the duality, Proposition 4.3 (iii) concludes that
\( \ell(S') = \ell(S) - 1 \) holds if and only if there exists no non-negative integer \( n \) such that \( A^{(n)} \) (resp. \( B^{(n)} \)) does not contain the inverse image of \( 1_j^B \) (resp. \( 0_i^A \)) for \( S' \in \mathcal{H}_3(S) \). \( \square \)

We want to determine boundary components of \( H(\xi) \) for arbitrary Newton polygons \( \xi \). If Conjecture 4.13 is true, then the two segments case is essential for classification of boundary components of all central streams.

**Conjecture 4.13.** Let \( \xi \) be a Newton polygon of \( z \) segments. Let \( N_\xi \) be the minimal DM\(_1\) of \( \xi \). A specialization of \( N_\xi \) is generic if and only if it is the direct sum of a generic specialization of \( N_r \oplus N_{r+1} \) and the minimal \( p \)-divisible group \( N_1 \oplus \cdots \oplus N_{r-1} \oplus N_{r+2} \oplus \cdots \oplus N_z \) for a natural number \( r \). Here, \( N_i \) is the simple DM\(_1\) associated with the \( i \)-th segment of \( \xi \).

### 4.2 Some properties of generic specializations

Here, we introduce some notation and properties of generic specializations, which are useful for showing Theorem 1.3.

Theorem 4.1 and Proposition 4.3 imply that it suffices to deal with ABS’s of \( \mathcal{H}_1(S) \) to study boundary components of central streams. From now on, in this paper, for elements \( 0_i^A \) and \( 1_j^B \) used in the construction of the small modification \( \pi \), we make the assumption \( m_1 < i \leq n_1 \) and \( 1 \leq j \leq n_2 \). Note that for the good exchange of \( 0_i^A \) and \( 1_j^B \), if \( a \) and \( b \) are positive, then non-negative integers \( a \) and \( b \) satisfy that \( \pi^a(0_i^A) = 1_{m_1}^A \) and \( \pi^b(1_j^B) = 0_{m_2+1}^B \).

By Theorem 4.3 if the exchange of \( 0_i^A \) and \( 1_j^B \) is good, then sets \( A^{(n)} \) (resp. \( B^{(n)} \)) are subsets of \( \tilde{A} \) (resp. \( \tilde{B} \)) for all non-negative integers \( n \), whence for the small modification by \( 0_i^A \) and \( 1_j^B \), sets \( A^{(n)} \) (resp. \( B^{(n)} \)) and the value \( \Delta \ell_A \) (resp. \( \Delta \ell_B \)) do not depend on \( j \) (resp. \( i \)). Thus we define the following sets.

**Definition 4.14.** Let \( A = (\tilde{A}, \delta_A, \pi_A) \) and \( B = (\tilde{B}, \delta_B, \pi_B) \) be the ABS of \( N_m_{1, n_1} \) and \( N_m_{2, n_2} \) respectively. For the ABS \( S = A \oplus B \), which is associated to \( N_\xi \), let \( G \) be the subset of \( \tilde{A} \times \tilde{B} \) consisting of pairs \( (0_i^A, 1_j^B) \) such that exchanges of \( 0_i^A \) and \( 1_j^B \) are good. By the above, this set is described as \( G = C' \times D' \) with \( C' \subset \tilde{A} \) and \( D' \subset \tilde{B} \). Equivalently, we obtain a generic specialization of \( S \) if and only if we construct a small modification by an element of \( C' \) and an element of \( D' \). Moreover, let \( C \) (resp. \( D \)) be the subset of \( C' \) (resp. \( D' \)) consisting of elements satisfying that \( \mathcal{A}(0) \) (resp. \( \mathcal{B}(0) \)) is not empty.

From now on, we fix the notation of \( C' \), \( C \), \( D' \) and \( D \). We have \( C' \setminus C = \{0_i^A\} \) and \( D' \setminus D = \{1_j^B\} \). We call the construction of a full modification using an element of \( C' \) and an element of \( D' \) good exchange.

**Example 4.15.** Let \( \xi = N_{2,7} \oplus N_{5,3} \). Then the set \( G = C' \times D' \) is given by

\[
C' \times D' = \{(0_i^A, 1_j^B), (0_i^A, 1_j^B), (0_i^A, 1_j^B), (0_i^A, 1_j^B)\}.
\]

We have \( C' = \{0_i^A, 0_i^A\} \) and \( D' = \{1_j^B, 1_j^B\} \).

In Lemma 4.16 and Proposition 4.17 we give some properties of \( \{A^{(n)}\}_{n=0, ..., a} \) and \( \{B^{(n)}\}_{n=0, ..., b} \) for generic specializations. For the ABS \( S \) associated to \( N_\xi \), let \( (S^{(n)}, \delta, \pi) \) be ABS’s obtained by constructing the full modification by \( 0_i^A \in C' \) and \( 1_j^B \in D' \) for \( n = 0, \ldots, a + b \).
Lemma 4.16. Let $n$ be a non-negative integer. For a good exchange, if $d_A(n) > 0$ (resp. $d_B(n) > 0$) and $A^{(n+1)}$ (resp. $B^{(n+1)}$) is not empty, then $A^{(n+1)}$ (resp. $B^{(n+1)}$) has the maximum element $1_{m_1}^A$ (resp. the minimum element $0_{m_2+1}^B$).

Proof. Let us see the case $d_A(n) > 0$. We can show the case $d_B(n) > 0$ in the same way. Fix a non-negative integer $n$. Put $\alpha = \pi^n(0_A^1)$. Clearly, if $\delta(\alpha) = \delta(\pi(\alpha)) = 0$, then $\delta(\pi(t)) = \delta(\pi(\alpha))$ holds for all elements $t$ of $A^{(n)}$. Moreover, if $\delta(\alpha) = 1$, we have then $\delta(\pi(\alpha)) = \delta(\pi(t)) = 0$ for all elements $t$ of $A^{(n)}$. It implies that $d_A(n) = 0$ holds in these cases. Hence it suffices to see the case that $\delta(\alpha) = 0$ and $\delta(\pi(\alpha)) = 1$. Then we have $\pi(A^{(n)}) = \{1^A_{x}, \ldots, 1^A_{m_1}, 0_{m_1+1}^A, \ldots, 0_{y}^A\}$. It induces that the maximum element of $A^{(n+1)}$ is $1_{m_1}^A$. \hfill \qed

Proposition 4.17. For a good exchange, let $n$ be a non-negative integer satisfying $n < a$, and let $c$ be the natural number satisfying that $\pi^n(0_A^1) = \tau_c^A$ with $\tau = 0$ or 1. The set $A^{(n)}$ is obtained by

$$A^{(n)} = \{\tau_{c+1}^A, \tau_{c+2}^A, \ldots, \tau_{c+u}^A\},$$

where $u = |A^{(n)}|$. Similarly, for a non-negative integer $n$ with $n < b$ and the natural number $c$ satisfying $\pi^n(0_B^1) = \tau_c^B$, the set $B^{(n)}$ is obtained by

$$B^{(n)} = \{\tau_{c-v}^B, \tau_{c-v+1}^B, \ldots, \tau_{c-1}^B\},$$

where $v = |B^{(n)}|$. 

Proof. Let us see sets $A^{(n)}$. Note that we have $I = \emptyset$, and we can see the latter part in the same way. We use induction on $n$. Clearly the statement holds for $n = 0$. For a natural number $n$, suppose that $A^{(n-1)} = \{\tau_{c+1}^A, \ldots, \tau_{c+u}^A\}$ with $u = |A^{(n-1)}|$. We have $A^{(n)} = \pi(A^{(n-1)}) \setminus T$, with

$$T = \{\pi(t) \mid \delta(\pi(t)) \neq \delta(\pi^n(0_A^1)) \text{ for } t \in A^{(n-1)}\}. $$

If $T = \emptyset$, we clearly have the property for $A^{(n)}$. Suppose that $T \neq \emptyset$. By Lemma 4.16, we see $A^{(n)} = \{1^A_{c+1}, 1^A_{c+2}, \ldots, 1^A_{m_1-1}, 1^A_{m_1}\}$. This completes the proof. \hfill \qed

5 Determining Newton polygons of generic specializations

The main purpose of this section is to show Theorem 1.3. For a minimal DM $N_{\xi}$ and the ABS $S$ associated to $N_{\xi}$, we say a specialization $N'$ of $N_{\xi}$ is generic if the specialization $S'$ associated to $N'$ satisfies $\ell(S') = \ell(S) - 1$, and we denote by $N_{\xi'}$ this generic specialization of $N_{\xi}$. We will show Proposition 5.1, which is a key step of constructing specializations from a generic specialization of $N_{\xi}$ to a minimal DM $N_{\zeta}$ with $\zeta < \xi$ is saturated. The main result of this section is

Proposition 5.1. Let $\xi = (m_1, n_1) + (m_2, n_2)$ be a Newton polygon satisfying $\lambda_2 < 1/2 < \lambda_1$, where $\lambda_1 = n_1/(m_1 + n_1)$ and $\lambda_2 = n_2/(m_2 + n_2)$. Assume that $\xi$ is not $(0,1) + (1,0)$. Let $N_{\xi}$ be the DM of $\xi$. For every generic specialization $N_{\xi'}$, there exist $N_{\xi}^{\sim\sim}$ and $N_{\xi'}^{\sim\sim}$ satisfying

$$N_{\xi}^{\sim\sim} = N_{\xi'}^{\sim\sim} \oplus N_{\rho},$$

where $\rho = (f,g)$ is a Newton polygon and $\rho$ is uniquely determined by $\xi$ so that the area of the region surrounded by $\xi$, $\xi'$ and $\rho$ is one. The diagram of these Newton polygons is described as
either of the following:

\[ \xi' \]

\[ \xi \]

For the above statement, we will determine the diagram of Newton polygons in Proposition 5.2 dividing the cases into the conditions of \( \xi \). We fix notations as follows: Let \( \xi = (m_1, n_1) + (m_2, n_2) \) be a Newton polygon satisfying that \( \lambda_2 < 1/2 < \lambda_1 \). Set \( h_1 = m_1 + n_1 \) and \( h_2 = m_2 + n_2 \). For the ABS \( S = A \oplus B \) associated to the DM \( N^e_\xi = N_{m_1,n_1} \oplus N_{m_2,n_2} \), let \( S^- \) denote the ABS obtained by a good exchange by \( 0^A_i \) and \( 1^B_j \), and let \( \xi^- \) be the DM corresponding to \( S^- \). Recall that by the small modification by \( 0^A_i \) and \( 1^B_j \), we obtain sets \( \{A^{(n)}\}_{n=0,...,a} \) and \( \{B^{(n)}\}_{n=0,...,b} \), where \( a \) (resp. \( b \)) is the smallest non-negative integer satisfying that \( A^{(a)} = \emptyset \) (resp. \( B^{(b)} = \emptyset \)). Note that if \( a \) and \( b \) are positive, then \( \pi^a(0^A_i) = 1^A_{m_1} \) and \( \pi^b(1^B_j) = 0^B_{m_2+1} \). For the ABS \( S = (S, \delta, \pi_0) \), the generic specialization \( S^- \) obtained by exchanging \( 0^A_i \) and \( 1^B_j \) is constructed by ABS’s \( \{(S^{(n)}, \delta, \pi)^{-}\}_{n=0,...,a+b} \). Since we can use the duality of DM’s for \( S^- \in \mathcal{H}_3(S) \), it suffices to see the case that \( S^- \) belongs to \( \mathcal{H}_3(S) \). Hence we assume that \( m_1 < i \leq n_1 \) and \( 1 \leq j < n_2 \).

Theorem 1.3 is obtained by applying Proposition 5.1 inductively. In Proposition 5.2 we concretely give an operation to obtain \( \xi^- \) satisfying the equality (10).

**Proposition 5.2.** Let \( \xi = (m_1, n_1) + (m_2, n_2) \) be a Newton polygon satisfying \( 0 < \lambda_2 < 1/2 < \lambda_1 < 1 \). Let \( \xi^- \) be any generic specialization of \( \xi^- \). Let \( S^- \) be the ABS associated to \( \xi^- \).

(I) If \( n_1 > m_1 + 1 \), then we obtain the equality (10) for \( N^-_{\xi^-} \) corresponding to \( S^- \), where \( S^- \) is either of the following:

(a) \( S^- \) is the specialization obtained by exchanging \( 0^A_{m_1+1} \) and \( 1^A_{m_1} \) for \( S^- \), or

(b) \( S^- \) is the specialization obtained by exchanging \( 0^A_i \) and \( 1^B_j \) for \( S^- \).

For these cases, the Newton polygon \( \xi' \) of (10) is of the form \( \xi' = (m_1 - f, n_1 - g) + (m_2, n_2) \).

(II) If \( n_1 = m_1 + 1 \), then we obtain the equality (10) for \( N^-_{\xi^-} \) corresponding to \( S^- \), where \( S^- \) is any of the following:

(c) \( S^- \) is the specialization obtained by exchanging \( 0^B_{m_2+1} \) and \( 1^B_{m_2} \) for \( S^- \),

(d) \( S^- \) is the specialization obtained by exchanging \( 0^A_i \) and \( 1^B_{j+1} \) for \( S^- \),

(e) \( S^- \) is the specialization obtained by exchanging \( 0^A_{n_1} \) and \( 1^B_{n_2+1} \) for \( S^- \).

For (c) and (d), the Newton polygon \( \xi' \) of (10) is of the form \( \xi' = (m_1, n_1) + (m_2 - f, n_2 - g) \), and for the case (e), we have \( \xi' = (m_1 - 1, n_1 - 1) + (m_2, n_2) \).
5.1 Proof of Proposition 5.2 (I)

Let us show Proposition 5.2 (I). Assume \( n_1 > m_1 + 1 \). By the construction of \( S^- \) which is obtained by a good exchange of \( 0^A_i \in C' \) and \( 1^B_j \in D' \), we have \( 0^A_{i-1} < 1^B_j \) in this ABS; see Definition 4.14 for the definition of sets \( C' \) and \( D' \). In fact, it is clear that \( 0^A_{i-1} < 1^B_j \) in \( S(0) \). Moreover, if \( 1^B_j < 0^A_{i-1} \) is true in \( S^- \), then there exists a natural number \( n \) with \( n < a \) such that \( \pi^n(0^A_i) = 0^A_{i-1} \). Then \( A^{(n-1)} \) contains the inverse image of \( 1^B_j \). This contradicts with \( \ell(S^-) = \ell(S) - 1 \). Here, to treat the case (a) of Proposition 5.2, in Proposition 5.4, we will see that \( 0^A_{m_1+1} < 1^A_m \) holds in \( S^- \). The following notation is useful for showing some properties of ABS’s.

**Notation 5.3.** Let \((T, \delta, \pi)\) be an ABS. For an element \( t \) of \( T \), we often express a subset \( \{ \pi^m(t) \mid 0 \leq m \leq n \} \) of \( T \) as

\[ t \to \pi(t) \to \cdots \to \pi^n(t), \]

and we call this diagram a path.

**Proposition 5.4.** We have \( 0^A_{m_1+1} < 1^A_m \) in the ABS \( S^- \) obtained by a good exchange of \( 0^A_i \in C \) and \( 1^B_j \in D' \). Moreover, there exists no non-negative integer \( n \) such that \( \pi^n(0^A_i) = 0^A_{m_1+1} \) with \( n \leq a \).

**Proof.** By the condition \( n_1 - m_1 > 1 \), we have \( h_1 > 2 \). First, to show the former statement, let us see that the non-negative integer \( a \) is not greater than \( h_1 - 2 \). For a good exchange, the non-negative integer \( a \) satisfies that \( \pi^a(0^A_i) = 1^A_{m_1} \). We clearly have \( a < h_1 \). If \( a = h_1 - 1 \), we have then \( i = h_1 \). This contradicts with the condition of the natural number \( i \). Hence we have \( a \leq h_1 - 2 \).

In the ABS \( A \) which is associated to the simple DM \( N_{m_1, n_1} \), binary expansions of \( 1^A_{m_1} \) and \( 0^A_{m_1+1} \) are given by

\[
\begin{align*}
(11) & \quad b(1^A_{m_1}) = 0.b_1 b_2 \ldots b_{h_1-2} 01 \ldots, \\
(12) & \quad b(0^A_{m_1+1}) = 0.b_1 b_2 \ldots b_{h_1-2} 10 \ldots.
\end{align*}
\]

For elements \( 0^A_i \) and \( 0^A_{i+1} \) of \( S^- \), we have two paths as follows:

\[
\begin{align*}
(13) & \quad 0^A_i \xrightarrow{\pi} \cdots \xrightarrow{\pi} 1^A_{m_1}, \\
(14) & \quad 0^A_{i+1} \xrightarrow{\pi} \cdots \xrightarrow{\pi} 0^A_{m_1+1}.
\end{align*}
\]

It is clear that \( 0^A_{i+1} \) belongs to \( A(0) \). Moreover, the above binary expansions and Corollary 3.12 induce that \( \pi^n(0^A_{i+1}) \) belongs to the set \( \pi(A^{(n-1)}) \) for every natural number \( n \) with \( n \leq a \). In particular, we apply this property for \( n = a \) and we have \( 0^A_{m_1+1} \in \pi(A^{(a-1)}) \). By the construction of \( S^{(a)} \), we obtain \( 0^A_{m_1+1} < 1^A_{m_1} \) in \( (S^{(a)}, \delta, \pi) \) and \( S^- \).

Let us see the latter statement. Suppose that there exists a non-negative integer \( n \) with \( n \leq a \) satisfying that \( \pi^n(0^A_i) = 0^A_{m_1+1} \). If this hypothesis leads \( 1^A_{m_1} < 0^A_{m_1+1} \), then we have a contradiction with the former statement. Suppose that \( 0^A_{m_1+1} < 1^A_{m_1} \) in \( S^- \). Then the set \( \pi(A^{(a-1)}) \) contains the element \( \pi^n(0^A_i) \). It implies that the set \( A^{(a-1)} \) contains the element \( \pi^{n-1}(0^A_i) \). Proposition 3.13 concludes that this is a contradiction, and hence \( 1^A_{m_1} < 0^A_{m_1+1} \) holds.

In the same way, we obtain the following assertion which is used in the proof of Proposition 5.2 (II).
Proposition 5.5. For a good exchange of $0^A_i \in C'$ and $1^B_j \in D$, we have $0^B_{m_1+1} < 1^B_{m_2}$ in the ABS $S^-$. Moreover, there exists no non-negative integer $n$ such that $\pi^n(1^B_j) = 1^B_{m_2}$ with $n \leq b$.

Notation 5.6. For the ABS $A$ associated to $N_{m_1,n_1}$, we define sub-paths $P$ and $Q$ of $A$ by the following:

\[
P : 1^A_{m_1} \to 0^A_{i_1} \to \cdots \to 0^A_{2m_1+1},
Q : 0^A_{m_1+1} \to 1^A_{i_1} \to \cdots \to 0^A_{2m_1}.
\]

Clearly $A$ is a disjoint union of $P$ and $Q$ as sets.

The above paths $P$ and $Q$ are useful. For instance, in the case (a) of Proposition 5.2 for the ABS $S^\rho = (\tilde{S}_\rho, \delta_\rho, \pi_\rho)$ which is associated to $N_\rho$ of Proposition 5.1, the set $\tilde{S}_\rho$ consists of all elements of $P$. Moreover, the path $Q$ has the following property:

Lemma 5.7. The set $C$ is contained in $Q$.

Proof. Suppose that there exists an element $0^A_i$ of $C$ which belongs to $P$. Then there exists a natural number $n$ such that $\pi^n(0^A_i) = 0^A_{m_1+1}$ with $n < a$. This contradicts with the latter statement of Proposition 5.4. \qed

Here, let us consider the construction of the ABS obtained by (a) or (b) of Proposition 5.2. The ABS $S^-$ obtained by the small modification by $0^A_i$ and $1^B_j$ is described as the following diagram:

```
• · · · 1^A_{m_1} 0^A_{2m_1} · · · 0^A_i 0^B_{j+m_2} · · ·
\downarrow
• · · · 0^A_{2m_1+1} 0^A_{m_1+1} · · · 0^A_{i+m_1} 1^B_j · · ·
```

First, let us consider the case (a). By constructing the small modification by $0^A_{m_1+1}$ and $1^A_{m_1}$, images of $0^A_{2m_1+1}$ and $0^A_{2m_1}$ are switched, and we obtain the ABS which consists of two components as follows:

```
• · · · 1^A_{m_1} 0^A_{2m_1} · · · 0^A_i 0^B_{j+m_2} · · ·
\uparrow
\downarrow
• · · · 0^A_{2m_1+1} 0^A_{m_1+1} · · · 0^A_{i+m_1} 1^B_j · · ·
```

The former component consists of all elements of $P$. The DM_{1} corresponding to this component is described as $N_\rho$ with a Newton polygon $\rho = (f,g)$. Since this component coincides with the component obtained from $A$ by applying [1] Lemma 5.6 to the adjacent $1^A_{m_1}$ and $0^A_{m_1+1}$, we have $fn_1 - gm_1 = 1$. Next, let us see the case (b). By constructing the small modification by $0^A_{i-1}$ and $1^B_j$, images of $0^A_{i-1+m_1}$ and $0^A_{i+m_1}$ are switched, and we obtain the ABS which consists of two components:

```
• · · · 0^A_{i-1} 0^A_{i-1+m_1} · · ·
\uparrow
\downarrow
• · · · 0^A_{i+m_1} 1^B_j · · · 0^A_i
eq 0
```

Finally, let us see the case (a). By constructing the small modification by $0^A_{m_1+1}$ and $1^A_{m_1}$, images of $0^A_{2m_1+1}$ and $0^A_{2m_1}$ are switched, and we obtain the ABS which consists of two components:
The latter component contains elements \( \pi^n(0)^i \) for all non-negative integers \( n \) with \( n \leq a \). The former component corresponds to the DM1 \( N_\rho \) with a Newton polygon \( \rho = (f, g) \). Since this component coincides with the component obtained from \( A \) by applying [1, Lemma 5.6] to the adjacent \( 0i_{-1} \) \( 0^i \), we have \( fn_1 - gm_1 = 1 \).

Hence for the cases (a) and (b), the ABS \( S^- \) has two components, where one is associated to the DM1 \( N_\rho \). We will show that for the other component, there exists a Newton polygon \( \xi' \) such that this component corresponds to the DM1 \( N^-_{\xi'} \) satisfying (10) of Proposition 5.2.

**Definition 5.8.** We define \( C_1 \) (resp. \( C_2 \)) to be the subset of \( C' \) consisting of elements \( 0^i \) satisfying that for \( N^-_{\xi'} \) obtained by a small modification by \( 0^i \) and \( 1^B \in D' \), we have the equality (10) by the case (a) (resp. (b)) of Proposition 5.2.

We denote by \( S'' \) the ABS obtained by the small modification by \( 0^i_{m+1} \) and \( 1^A_{m_1} \) or the small modification by \( 0^i_{-1} \) and \( 1^B_j \) for \( S^- \). By the above, we see that the ABS \( S'' \) consists of two components, and a component of \( S'' \) is associated to \( N^{-}_{\rho} \). Let \( \Psi \) be the other component of \( S'' \). To show \( C_1 \cup C_2 = C' \), we give a condition that an element \( t \) of \( C' \) belongs to \( C_1 \) or \( C_2 \) in Proposition 5.9.

In the proof of this proposition, we give a method to determine the structure of the ABS \( \Psi \).

**Proposition 5.9.** Let \( S^- \) be the generic specialization obtained by a small modification by \( 0^i \) and \( 1^B \). If \( \Psi \) contains no element \( t \) satisfying that \( 0^i_{m+1} < t < 1^A_{m_1} \) (resp. \( 0^i_{-1} < t < 1^B_j \)) in \( S^- \), then \( 0^i \) belongs to \( C_1 \) (resp. \( C_2 \)).

**Proof.** Let us see the case that we construct \( S'' \) by the small modification by \( 0^i_{m+1} \) and \( 1^A_{m_1} \). The other case \( (S'') \) is constructed by the small modification by \( 0^i_{-1} \) and \( 1^B_j \) is shown by the same way. For the ABS \( S^- = (S', \delta, \pi) \), we construct the small modification \( S'' \) by \( 0^i_{m+1} \) and \( 1^A_{m_1} \), and we obtain the admissible ABS \( S'' = (S'', \delta, \pi'') \) consisting of two components. A component corresponds to \( N^{-}_{\rho} \) with \( \rho = (f, g) \) satisfying \( fn_1 - gm_1 = 1 \). Let \( N \) be the DM1 associated to the other component \( \Psi \). To see that there exists a Newton polygon \( \xi' \) such that \( N = N^-_{\xi'} \), we consider the small modification by \( 1^B_j \) and \( 0^i \) in \( S'' \). Let \( \chi \) be the small modification by \( 1^B_j \) and \( 0^i \), and we obtain the map \( \chi \) on \( \Psi' \). For the ordered set \( \{t_1 < \cdots < t_{h'}\} \) of \( \Psi' \), put \( \Psi''(0) = \{t'_1 < \cdots < t'_{h'}\} \), where if \( t_i \) and \( 1^j \) are \( i \)-th element and \( j \)-th element of \( \Psi' \) respectively, then for \( s = (i', j') \) transposition, we set \( t'_s = t_{s(i')} \). We have the ABS \( (\Psi'(0), \delta, \chi) \). Here, we define sets

\[
D^{(0)} = \{t \in \Psi(0) \mid a^A < t \text{ and } \chi(t) < \chi(a^A) \text{ in } \Psi(0) \text{ with } \delta(t) = 0\}.
\]

\[
\mathcal{E}^{(0)} = \{t \in \Psi(0) \mid t < 1^B \text{ and } \chi(t) < \chi(t) \text{ in } \Psi(0) \text{ with } \delta(t) = 1\}.
\]

Put \( \alpha_n = \chi^n(0^A) \) and \( \beta_n = \chi^n(1^B) \) for all non-negative integers \( n \). For sets \( \Psi(0), \ldots, \Psi(n-1) \) and sets \( D^{(0)}, \ldots, D^{(n-1)} \) with a natural number \( n \), let \( \Psi(n) = \Psi(n-1) \) as sets. We define the order on \( \Psi(n) \) to be for \( t < t' \) in \( \Psi(n-1) \), we have \( t > t' \) if and only if \( \pi(t_{\min}) \leq t < \alpha_n \) in \( \Psi(n-1) \) and \( t' = \alpha_n \), where \( t_{\min} \) is the minimum element of \( D^{(n-1)} \). In other words, in the ABS \( \Psi(n-1) \), we move the element \( \alpha_n \) to the left of \( \chi(t_{\min}) \) to construct \( \Psi(n) \). We regard \( \chi \) as a map on \( \Psi(n) \). Then we obtain the ABS \( (\Psi(n), \delta, \chi) \). We define a set

\[
D^{(n)} = \{t \in \Psi(n) \mid \alpha_n < t \text{ and } \chi(t) < \alpha_{n+1} \text{ in } \Psi(n) \text{ with } \delta(t) = \delta(\alpha_n)\}.
\]

By hypothesis, we have \( D^{(n)} = A^{(n)} \setminus \tilde{S}_\rho \) for all \( n \). Here, \( S_\rho = (\tilde{S}_\rho, \delta, \pi') \) is the ABS associated to \( N_\rho \), where \( \delta \) (resp. \( \pi' \)) denotes the restriction of \( \delta \) (resp. \( \pi'' \)) to \( \tilde{S}_\rho \). Hence there exists the
smallest integer \( a' \) such that \( D(a') = \emptyset \). For sets \( \Psi(a'), \ldots, \Psi(a' + n - 1) \) and sets \( \mathcal{E}(0), \ldots, \mathcal{E}(n-1) \), let \( \Psi(a'+n) = \Psi(a'+n-1) \) as sets. The ordering of \( \Psi(a'+n) \) is given so that for \( t < t' \) in \( \Psi(a'+n-1) \), we have \( t > t' \) if and only if \( \beta_n < t' \leq \pi(t_{\text{max}}) \) in \( \Psi(a'+n-1) \) and \( t = \beta_n \), where \( t_{\text{max}} \) is the maximum element of \( \mathcal{E}(n-1) \). In other words, to obtain \( \Psi(a'+n) \), we move the element \( \beta_n \) to the right of \( \chi(t_{\text{max}}) \). We regard \( \chi \) as a map on \( \Psi(a'+n) \). Thus we obtain the ABS \( (\Psi(a'+n), \delta, \chi) \). We define a set

\[
\mathcal{E}(n) = \{ t \in \Psi(n) \mid t < \beta_n \text{ and } \beta_{n+1} < \chi(t) \in \Psi(a'+n), \text{ with } \delta(t) = \delta(\beta_n) \}.
\]

By hypothesis, we have \( \mathcal{E}(n) = \mathcal{B}(n) \) for all \( n \). Hence there exists the smallest integer \( b' \) such that \( \mathcal{E}(b') = \emptyset \). We obtain the admissible ABS \( \Psi' = (\Psi(a'+b'), \delta, \chi) \) which is associated to \( N_{\xi} \) with \( \xi' = (m_1 - f, n_1 - g) + (m_2, n_2) \). We immediately obtain \( \ell(\Psi') = \ell(\Psi) + 1 \). Hence the ABS \( \Psi \) corresponds to \( N_{\xi}^{-} \).

**Example 5.10.** Let us see an example of constructing \( N_{\xi}^{-} \) and \( N_{\xi} \) from \( N_{\xi}^{-} \). Let \( N_{\xi} = N_{2,7} \oplus N_{5,3} \), and we construct the full modification by \( 0_6^A \) and \( 1_3^B \). By Example 5.9 we have \( N_{\xi}^{-} \) as follows:

\[
N_{\xi}^{-} : 1_1^A \rightarrow 0_3^A \rightarrow 1_2^A \rightarrow 0_6^A \rightarrow 0_4^A \rightarrow 1_3^B \rightarrow 0_7^A \rightarrow 1_4^B \rightarrow 0_8^A \rightarrow 0_4^B \rightarrow 1_5^B \rightarrow 0_9^A \rightarrow 0_6^B .
\]

We construct the small modification by elements \( 0_3^A \) and \( 1_2^A \) for the ABS associated to \( N_{\xi}^{-} \), and we obtain \( N_{\xi}^{-} \) by the following diagram which is decomposed into two cycles:

\[
N_{\xi}^{-} : 1_1^A \rightarrow 0_3^A \rightarrow 0_4^A \rightarrow 1_3^B \rightarrow 1_1^B \rightarrow 1_2^B \rightarrow 0_6^A \rightarrow 0_8^A \rightarrow 1_4^B \rightarrow 1_5^B \rightarrow 0_6^B \oplus 1_2^A \rightarrow 0_5^A \rightarrow 0_7^A \rightarrow 0_9^A .
\]

It is clear that the latter component is associated to the simple DM1 \( N_{1,3} \). Let us see that the former component \( N \) is a specialization of a minimal DM1. We construct the small modification by elements \( 1_3^B \) and \( 0_6^A \), and we obtain the following diagram:

\[
1_1^A \rightarrow 0_3^A \rightarrow 0_4^A \rightarrow 0_6^A \rightarrow 0_8^A \oplus 1_1^B \rightarrow 1_2^B \rightarrow 1_5^B \rightarrow 0_6^B \rightarrow 1_4^B \rightarrow 1_5^B \rightarrow 0_7^B \rightarrow 0_8^B .
\]

Clearly the former summand is associated to \( N_{1,4} \). For the notation of Proposition 5.9, we have \( D(0) = \emptyset \) and \( \mathcal{E}(0) = \{ 1_1^B, 1_2^B \} \). We move the element \( 0_6^B \) to between \( 1_5^B \) and \( 0_7^B \). Thus we see that the latter summand is associated to the simple DM1 \( N_{5,3} \). Therefore we see that this diagram corresponds to \( N_{1,4} \oplus N_{5,3} \), and we have \( N_{\xi}^{-} = N_{1,4} \oplus N_{5,3} \).

Here, for an element \( t \) of \( C \), we give a condition of \( t \) belongs to \( C_2 \) as follows:

**Proposition 5.11.** For a good exchange of \( 0_i^A \in C \) and \( 1_j^B \in D' \), we have an element \( t \) of \( S' \) satisfying \( 0_i^{A-1} < t < 1_j^B \) in \( S^- \) if and only if there exists a non-negative integer \( n \) such that the set \( \pi(A(n)) \) has the maximum element \( 0_{i-1}^A \).
Proof. In the ABS \((S^{(0)}, \delta, \pi)\), clearly there exists no element \(t\) satisfying \(0^A_{i-1} < t < 1^B_j\). Hence every element \(t\) between \(0^A_{i-1}\) and \(1^B_j\) in \(S^-\) is of the form \(\pi^m(0^A_i)\) for a natural number \(m\) with \(m \leq a\). Fix a natural number \(n\). By definitions of \(A^{(n)}\) and \(S^{(n)}\), there exists an element \(t\) between \(0^A_{i-1}\) and \(1^B_j\) in \(S^{(n+1)}\) if and only if \(\pi(A^{(n)})\) has the maximum element \(0^A_{i-1}\). Indeed, the element \(t\) is obtained by \(t = \pi^{n+1}(0^A_i)\).

We consider the case that \(C\) is not empty. We set an order on the set \(C\) which plays an important role.

**Notation 5.12.** Put \(\nu = |C|\). For \(x = 1, \ldots, \nu\), let \(i_x\) be the natural number with \(m_1 < i_x \leq h_1\) such that \(0^A_{i_x}\) is the element of \(C\) appearing in the \(x\)-th in the path \(Q\). In the other words, we put \(C = \{0^A_{i_1}, 0^A_{i_2}, \ldots, 0^A_{i_{\nu}}\}\), where for elements \(0^A_{i_x} = \pi^l_x(0^A_{m_1+1})\) and \(0^A_{i_y} = \pi^l_y(0^A_{m_1+1})\) of \(C\), we have \(x < y\) if and only if \(l_x < l_y\). The elements \(0^A_{i_x}\) for \(x = 1, \ldots, \nu\), of \(C\) appear in the path \(Q\) as follows.

\[
Q : 0^A_{m_1+1} \rightarrow \cdots \rightarrow 0^A_{i_1} \rightarrow \cdots \rightarrow 0^A_{i_2} \rightarrow \cdots \rightarrow 0^A_{i_\nu} \rightarrow \cdots \rightarrow 0^A_{2m_1}.
\]

Here, we give a characterization of the “first” element \(0^A_{i_1}\) of \(C\) in Lemma 5.13.

**Lemma 5.13.** If there exists the minimum number \(y\) such that the element \(t = \pi^y(0^A_{m_1+1})\) of \(Q\) satisfies \(0^A_{m_1+1} < t < 0^A_{i_1}\) in \(A\), then \(t = 0^A_{i_1}\).

**Proof.** First, let us see that for an small modification by \(0^A_{i_1}\) and \(1^B_j\), there exists no non-negative integer \(n\) such that \(A^{(n)}\) contains the element \(0^A_{m_1+1}\). If \(A^{(n)}\) contains \(0^A_{m_1+1}\) for a non-negative integer \(n\), then \(\pi^{n+1}(0^A_{i_1}) < 1^A\) holds. This is a contradiction.

To see that \(t\) belongs to \(C\), for the ABS \(S\) of \(N_\varepsilon\), we consider an small modification \(\pi\) by \(t\) and \(1^B_j\) with \(1^B_j \in D'\). Let \(S^{(n)}\) and \(A^{(n)}\) denote sets obtained by the small modification by \(t\) and \(1^B_j\). Assume that for a non-negative integer \(n\), the set \(A^{(n)}\) contains the inverse image \(t^B_j\) for \(\pi\). In the ABS \(A\), this \(t^B_j\) is the inverse image of \(t\). It is clear that \(t^B_j\) belongs to \(Q\). By Corollary 3.12, there exists an element \(t''\) of \(A^{(0)}\) such that \(\pi^n(t'') = t'\). By definition of \(t\), and since elements \(1^A_{m_1}\) and \(0^A_{i_1}\) belong to \(P\), elements \(\tau^A_{\delta_i}\), with \(\tau = 0\) or \(1\), of the path \(Q\) between \(0^A_{m_1+1}\) and \(t\) satisfies \(x < m_1\) or \(n_1 < x\). It implies that these elements do not belong to \(A^{(0)} \subset \{0^A_{m_1+1}, \ldots, 0^A_{i_1}\}\). Hence \(t''\) belongs to \(P\). Then the path from \(t''\) to \(t'\) through \(0^A_{m_1+1}\), i.e., there exists a natural number \(m\) such that \(\pi^m(t'') = 0^A_{m_1+1}\) with \(m < a\). This contradicts with the above property. □

We consider the case \(C \neq \emptyset\). Then we have the element \(0^A_{i_1}\) of \(C\). To apply Proposition 5.9, it is important to study elements between \(0^A_{m_1+1}\) and \(1^A_{m_1}\) in \(S^-\). The following set given in Notation 5.14 and the element of \(C\) given in Proposition 5.15 are useful for studying the elements between \(0^A_{m_1+1}\) and \(1^A_{m_1}\) in \(S^-\).

**Notation 5.14.** For an element \(0^A_{i_1}\) of \(C\), we often write \(A^{(n)}_{i_1}\) for sets \(A^{(n)}\) obtained by the small modification by \(0^A_{i_1}\) and \(1^B_j\) to avoid confusion. Moreover, we often write \(a_i\) for the smallest non-negative integer \(a\) satisfying \(A^{(n)}_{i_1} = \emptyset\). For sets \(\{A^{(n)}_{i_1}\}_{n=0, \ldots, a_i}\) with \(0^A_{i_1} \in C\), we set

\[
T_i = \pi(A^{(n-1)}_{i_1}).
\]

This set consists of all elements \(t\) satisfying \(0^A_{m_1+1} \leq t < 1^A_{m_1}\) in \(S^-\).
Proposition 5.15. Put $i = n_1 - \gamma$, with $\gamma = |T_{i_1}|$. Then $0^A_i$ belongs to $C$. Moreover, $T_{i_1} = T_i$ holds.

Proof. Since $T_{i_x}$ consists of some elements of \{0^A_{m_1+1}, \ldots, 0^A_{n_1}\}, we have $|T_{i_x}| < n_1 - m_1$ for all $0^A_x \in C$. Hence we have $m_1 < i < n_1$. To show that $0^A_i$ belongs to $C$, it suffices to see that there exists a natural number $m$ such that $A^{(0)}_{i_1} = A^{(m)}_{i_1}$. In fact, if this statement is true, then $A^{(n)}_{i_1} = A^{(m+n)}_{i_1}$ holds for all $n$ with $0 \leq n \leq a_i$. Put $\alpha = \pi^{m-1}(0^A_{i_1})$. Note that $\alpha$ is the inverse image of $0^A_i$ in the ABS’s $(S^{(n)}, \delta, \pi)$, where $\pi$ is the small modification by $0^A_i$ and $1^B_j$. By Proposition 3.13 sets $A^{(m+n)}_{i_1}$ do not contain $\alpha$. Hence sets $A^{(n)}_{i_1}$ do not contain the inverse image of $0^A_i$ for all $n$, and this completes the proof. Let us see that there exists such a number $m$. By the definition of $\gamma$ and Lemma 4.16 there exists a natural number $m'$ such that $A^{(m')}_{i_1} = \{1^A_{m_1-\gamma+1}, \ldots, 1^A_{n_1}\}$. We have then $A^{(m'+2)}_{i_1} = \{0^A_{m_1-\gamma+1}, \ldots, 0^A_{n_1}\}$, and this set is equal to $A^{(0)}_{i_1}$. Therefore the number $m$ is obtained by $m = m' + 2$. The latter part $|T_{i_1}| = |T_i|$ follows from the former part. \hfill \Box

Let $d$ be a natural number satisfying $i_d = n_1 - \gamma$. From now on, we fix notations of the non-negative integers $d$ and $\gamma$. As can be seen in Proposition 5.16 this natural number $d$ plays an important role to show the former part of Proposition 5.2.

Proposition 5.16. Let $0^A_{i_x}$ be an element of $C$. For the above natural number $d$, we have

1. If $x \leq d$, then $0^A_{i_x}$ belongs to $C_1$,

2. If $x > d$, then $0^A_{i_x}$ belongs to $C_2$.

Thanks to this proposition, we can show the case of $n_1 > m_1 + 1$ for Proposition 5.2.

Proof of Proposition 5.2 (I). If Proposition 5.16 is true, then the equality (10) of Proposition 5.2 holds for DM1’s $N_{\xi}$ which are obtained by small modifications by elements $0^A_i$ and $1^B_j$ of $C = C' \backslash \{0^A_{i_1}\}$ and $D'$ respectively. Let us see the case of $i = n_1$. In this case, we have $A^{(0)} = \emptyset$, and there exists no element $t$ of $S^{-}$ satisfying $0^A_{i-1} < t < 1^B_j$ in $S^{-}$. Hence we obtain desired $N_{\xi}^{-} = N_{\xi^*} \oplus N_{\rho}$ by the case (b). \hfill \Box

Let us show some properties of sets $T_i$, the natural numbers $d$ and $\gamma$. These properties are used for showing Proposition 5.16.

Proposition 5.17. The following are true:

1. If $x < y$, then $T_{i_x} \subset T_{i_y}$ holds;

2. For all $n$ with $n < a_{i_d}$, we have $|A^{(n)}_{i_d}| = \gamma$;

3. For all $x$ and $n$ with $n < a_{i_x}$, we have $|A^{(n)}_{i_x}| \geq \gamma$;

4. $T_{i_d} \subset T_{i_x}$ holds for all $x$ with $x > d$;

5. $T_{i_x} = T_{i_d}$ is true if and only if $x \leq d$. 
Proof. Let us see (i). By assumption, there exists a natural number \( n \) such that \( \pi^n(0^A_{i_d}) = 0^A_{i_d} \) and \( n < a_{i_d} \). We have then \( \mathcal{A}^{(n)}_{i_d} = \{0^A_{i_d+1}, \ldots, 0^A_z\} \), with \( z \leq n_1 \). Hence \( \mathcal{A}^{(n)}_{i_d} \subset \mathcal{A}^{(m+n)}_{i_d} \) holds for all non-negative integers \( m \), and it induces the desired relation.

It is clear that \( |\mathcal{A}^{(n)}| \geq |\mathcal{A}^{(n+1)}| \) for all \( n \) and all elements \( 0^A_i \) of \( C \). By the definition of \( i_d \) and Proposition 5.15, we have \( |\mathcal{A}^{(0)}_{i_d}| = |\mathcal{A}^{(a-1)}_{i_d}| = \gamma \) for \( a = a_{i_d} \). Thus (ii) holds.

By (i) and the definition of \( \gamma \), we have \( |T_{i_d}| \geq \gamma \) for all \( x \). Moreover, it is clear that \( |\mathcal{A}^{(n)}_{i_d}| \geq |T_{i_d}| \) for all \( n \) and \( x \). Hence we see (iii).

To see (iv), we fix a natural number \( x \) satisfying \( x > d \). It suffices to see that \( |T_{i_d}| \) is greater than \( \gamma \). Suppose that \( |T_{i_d}| = \gamma \). Then there exists the minimum number \( u \) such that \( |\mathcal{A}^{(u)}_{i_d}| = \gamma \). We have \( \mathcal{A}^{(u)}_{i_d} = \{1^A_{m_1-\gamma+1}, \ldots, 1^A_{m_1}\} \) and \( \pi^u(0^A_{i_d}) = 1^A_{m_1-\gamma} \). Then we have a sub-path of \( Q \):

\[
0^A_{i_d} \to \cdots \to 0^A_{i_d} \to \cdots \to 1^A_{m_1-\gamma} \to 0^A_{m_1-\gamma} \to 0^A_{m_1-\gamma}
\]

which implies that there exists a natural number \( m \) such that \( \pi^m(0^A_{i_d}) = 0^A_{i_d} \) for \( m < m_1 + n_1 \). This is a contradiction.

The statement (v) follows from Proposition 5.15 (i) and (iv). \( \square \)

The natural number \( d' \) introduced in Notation 5.18 has a relation with the natural number \( d \). We give the proof of Proposition 5.16 using this relation.

Notation 5.18. For the admissible ABS \( S \) of \( N_\xi \) and the set \( C \), we define a set \( \mathcal{L} \) by

\[
\mathcal{L} = \{ x \in \bar{C} \mid \text{For a non-negative integer } n = n(x), \text{ the maximum element of } \mathcal{A}^{(n)}_{i_d} \text{ is } 0^A_{i_d-1} \}.
\]

where \( \bar{C} = \{ x \in \mathbb{N} \mid 1 \leq x \leq |C| \} \). Let \( d' \) denote the maximum element of \( \mathcal{L} \). If \( \mathcal{L} \) is empty, then we set \( d' = 0 \).

Proposition 5.19. For the set \( \mathcal{L} \) and the natural number \( d' \) of Notation 5.18 if \( \mathcal{L} \) is not empty, then \( \mathcal{L} = \{1, 2, \ldots, d'\} \).

Proof. Fix a natural number \( x \) satisfying \( x \leq d' \). We show that \( 0^A_{i_d-1} \) belongs to \( \mathcal{A}^{(u)}_{i_d} \) for a non-negative integer \( u \). Let \( n \) be the non-negative integer satisfying that the maximum element of \( \mathcal{A}^{(n)}_{i_d} \) is \( 0^A_{i_d-1} \). Let us consider the path consisting of maximum elements of \( \mathcal{A}^{(m)}_{i_d} \) for \( 0 \leq m \leq n \)

\[
0^A_{i_d} \to \cdots \to 0^A_{i_d-1}.
\]

(15)

For the sub-path of \( Q \)

\[
0^A_{m_1+1} \to 0^A_{m_1+1} \to \cdots \to 0^A_{i_d} \to \cdots \to 0^A_{i_d},
\]

since this path contains \( 0^A_{i_d} \), the path (15) contains \( 0^A_{i_d-1} \). This completes the proof. \( \square \)

Proposition 5.20. For the natural number \( d' \) of Notation 5.18 and the natural number \( d \) obtained by Proposition 5.15 we have \( d' \leq d \).
Proof. First, let us see that there exists no non-negative integer \( n \) such that the maximum element of \( A_{i_d}^{(n)} \) is \( 1_{m_1-1}^A \). Assume that the set \( A_{i_d}^{(n)} \) has the maximum element \( 1_{m_1-1}^A \) for a non-negative integer \( n \). Then this set has the minimum element \( 1_{m_1-\gamma}^A \), and we have \( \pi(1_{m_1-\gamma}^A) = 0_{i_d+m_1}^A \). It implies that the set \( A_{i_d}^{(n+1)} \) contains the inverse image of \( 0_{i_d}^A \), and this is a contradiction.

Suppose \( d < d' \), and we fix a natural number \( x \) satisfying \( d < x \leq d' \). Let us consider the ABS \((S', \delta, \pi)\) obtained by the small modification by \( 0_{i_d}^A \) and \( 1_{B}^j \in D' \). By Proposition 5.19 we obtain the path which consists of maximum elements of \( A_{i_d}^{(n)} \) for all \( n \) and \( T_{i_d} \) as follows:

\[
0_{m_1}^A \to \cdots \to 0_{i_x-1}^A \to \pi(0_{i_x-1}^A) \to \cdots \to 0_{m_1+\gamma}^A.
\]

Let us consider the sub-path of \( A = (\tilde{A}, \delta_A, \rho_A) \):

\[
0_{i_x}^A \to \pi_A(0_{i_x}^A) \to \cdots \to 0_{m_1+\gamma+1}^A.
\]

Since the path (16) does not contain \( 1_{m_1-1}^A \), the path (17) does not contain \( 1_{m_1}^A \). Hence for the ABS \((S', \delta, \pi_x)\) obtained by the small modification by \( 0_{i_x}^A \) and \( 1_{j}^B \) for \( S \), we have a natural number \( n \) satisfying that \( \pi_x^n(0_{i_x}^A) = 0_{m_1+\gamma+1}^A \) with \( n < a_{i_x} \). By Proposition 5.17 (iv), the set \( T_{i_x} \) contains \( 0_{m_1+\gamma+1}^A \). This contradicts with Proposition 3.13. \( \square \)

We fix notation of the non-negative integer \( d' \), and we give a proof of Proposition 5.16 as follows.

Proof of Proposition 5.16. For the ABS \( S^- \), let \( \Psi \) and \( S_p \) be components of \( S^- \) given by (a) or (b) of Proposition 5.2 where \( S_p \) is the ABS associated to the DM \( N_p \). Let us see the case (1) of Proposition 5.16. Note that \( S_p \) obtained by (a) of Proposition 5.2 consists of all elements of \( P \). Recall that \( T_{i_x} \) is the set which consists of elements \( t \) satisfying that \( 0_{m_1+\gamma+1}^A \leq t < 1_{m_1}^A \) in \( S^- \). To apply Proposition 5.9, we will show that if \( x \leq d \), then all elements of \( T_{i_x} \) except for \( 0_{m_1+\gamma+1}^A \) belong to \( \tilde{S}_p \) for the small modification by \( 0_{m_1+1}^A \) and \( 1_{m_1}^A \). By Proposition 5.17 (v), it suffices to show that each element of \( Q \) except for \( 0_{m_1+\gamma+1}^A \) does not belong to \( T_{i_1} \). In fact, this claim induces that all elements of \( T_{i_1} \) belong to \( P \) and \( \tilde{S}_p \). We put two sub-paths \( \omega_1 \) and \( \omega_2 \) of \( Q \) as follows:

\[
Q : 0_{m_1+\gamma+1}^A \to \cdots \to 0_{m_1+m_1}^A \to 0_{m_1}^A \to \cdots \to 0_{2m_1}^A.
\]

It follows from Proposition 3.13 that every element, which is of the form \( \pi^m(0_{i_1}^A) \) with \( m < a_{i_1} \), of \( \omega_2 \) does not belong to \( T_{i_1} \). The property, which is given in Lemma 5.13 of \( i_1 \) implies that all elements of \( \omega_1 \) except for \( 0_{m_1+\gamma+1}^A \) do not belong to \( T_{i_1} \) which is a subset of \( \{0_{m_1+1}^A, \ldots, 0_{m_1}^A\} \).

Next, let us see the case (2). We consider the small modification by \( 0_{i_x-1}^A \) and \( 1_{j}^B \) in \( S^- \). By Proposition 5.9 it suffices to show that if \( x > d \), then there exists no element \( t \) between \( 0_{i_x-1}^A \) and \( 1_{j}^B \) in \( S^- \). For the natural number \( x \) with \( x > d \), by Proposition 5.20 clearly \( x > d' \) holds. To lead a contradiction, let us suppose that there exists an element between \( 0_{i_x-1}^A \) and \( 1_{j}^B \) in the ABS \( S^- \). By Proposition 5.11, there exists a non-negative integer \( v \) such that \( 0_{i_x-1}^A \) is the maximum element of \( \pi(A_{i_x}^{(v)}) =: T \). For sets \( \{A_{i_d}^{(n)}\}_{n=0, \ldots, a-1} \) and \( T_{i_d} \), let \( \Phi \) be the path consisting of maximum elements of these sets:

\[
\Phi : 0_{m_1}^A \to \cdots \to 0_{m_1+\gamma}^A.
\]
We define a non-negative integer \( m' \) to be \( A_{\xi}^{(m')} = \{1^A_{m_1-n+1}, \ldots, 1^A_{m_1} \} \), with \( u = |T| \). Since \( m' \) is the minimum number satisfying that \( |A_{\xi}^{(m')}| = |T| \), we have \( m' < v \). Put \( m = m' + 2 \). Then \( A_{\xi}^{(m)} \) has the maximum element \( 0^A_{m_1} \). We consider the path consisting of the maximum elements of sets \( A_{\xi}^{(m)}, A_{\xi}^{(m+1)}, \ldots, T \), and we obtain the path \( O \) from \( 0^A_{m_1} \) to \( 0^A_{m_1-1} \). Here, let us show that \( O \) can be regarded as a subset of \( \Phi \). It suffices to see that \( O \) does not contain \( 0^A_{m_1+\gamma} \). If \( 0^A_{m_1+\gamma} \) is contained in the path \( O \), then \( |T_{\xi_2}| \leq \gamma \) follows from Proposition 4.17. This contradicts with the hypothesis \( x > d \) and Proposition 5.17 (iv). Hence \( O \) is a subset of \( \Phi \), and it implies that for a non-negative integer \( n \), the set \( A_{\xi_2}^{(n)} \) contains the maximum element \( 0^A_{n-x-1} \) with \( x > d' \). This contradicts with the definition of \( d' \).

5.2 Proof of Proposition 5.2 (II)

Here, let us see the remaining case \( n_1 = m_1 + 1 \) for \( \xi = (m_1, n_1) + (m_2, n_2) \), and we will give a proof of Proposition 5.2 (II). The discussion of this proof is given by the same way as the proof of Proposition 5.2 (I). As we have seen in Proposition 5.2, for every good exchange of \( 0^A_i \) and \( 1^B_j \) with \( 1^B_j \in D \), we have \( 0^B_{m+1} < 1^B_{m+1} \) in \( S^- \). Assume \( n_1 = m_1 + 1 \) for the Newton polygon \( \xi = (m_1, n_1) + (m_2, n_2) \). In this case, we have \( C' = \{0^A_{m_1} \} \). Let us see the case \( D \neq \emptyset \).

**Notation 5.21.** For the simple DM \( N_{m_2,n_2} \) and its ABS \( B \), we define sub-paths \( U \) and \( V \) of \( B \) by the following:

\[
U : \quad 1^B_{m_2} \rightarrow 0^B_{m_2} \rightarrow 1^B_{n_2} \rightarrow \cdots \rightarrow 1^B_{m_2-n_2+1}, \\
V : \quad 0^B_{m_2+1} \rightarrow 1^B_1 \rightarrow 1^B_{n_2+1} \rightarrow \cdots \rightarrow 1^B_{m_2-n_2}.
\]

We clearly have \( U \sqcup V = B \) as sets.

The above components \( U \) and \( V \) of \( B \) are useful. Concretely, as can be seen in Lemma 5.22, all elements of \( D \) belongs to the component \( U \). Moreover, in the case (c) of Proposition 5.2, the ABS corresponding to \( N_0 \) consists of all elements of \( V \).

**Lemma 5.22.** For the above notation, \( D \subset U \) holds. Set \( \nu = |D| \). Let \( j_1, \ldots, j_\nu \) be natural numbers such that \( 1^B_{j_\nu} \) is the element of \( D \) appearing in the \( x \)-th in the path \( U \):

\[
1^B_{m_2} \rightarrow \cdots \rightarrow 1^B_{j_1} \rightarrow \cdots \rightarrow 1^B_{j_\nu} \rightarrow 1^B_{j_\nu+1} \rightarrow \cdots \rightarrow 1^B_{m_2-n_2+1}.
\]

We have then \( j_1 = n_2 \).

**Proof.** For the former part, we see that there exists no non-negative integer \( n \) with \( n \leq b \) such that \( \pi^n(1^B_j) = 1^B_{m_2} \) for every element \( 1^B_j \) of \( D \) by the same reasoning as Proposition 5.4. A proof is given by the same way as Lemma 5.7.

Let us see the latter part. By the former part and the hypothesis \( 1 \leq j \leq n_2 \), if \( 1^B_{n_2} \) belongs to \( D \), then immediately we see \( j_1 = n_2 \). Suppose that the set \( B^{(n)} \) contains \( 0^B_{n_2} \), which is the inverse image of \( 0^A_i \) in \( S^{(n)} \) for a non-negative integer \( n \). We have then \( 0^B_{n_2} < \pi^n(1^B_j) \) in \( S^{(a+n-1)} \). Since \( 0^B_{n_2} \) is the maximum element of the ABS \( S \), this is a contradiction. □

**Definition 5.23.** Let \( D_1 \) (resp. \( D_2 \)) be the subset of \( D' \) consisting of \( 1^B_j \) satisfying that for a generic specialization \( S^- \) obtained by \( 0^A_i \in C' \) and \( 1^B_j \), we have the equality \( (11) \) by the case (c) (resp. (d)) of Proposition 5.2.
We give a key element of $D$ in Proposition 5.25 to show Proposition 5.2 (II). For the above notation, this element characterize the sets $D_1$ and $D_2$ as seen in Proposition 5.26. Sets given in Notation 5.24 are used for introducing the key element and describing the proof of Proposition 5.26.

**Notation 5.24.** For an element $1^B_j$ of $D'$, we often write $B_j^{(n)}$ for sets $B^{(n)}$ to avoid confusion. Moreover, we often write $b_j$ for the smallest non-negative integer $b$ satisfying $B_j^{(b)} = \emptyset$. For sets $\{B_j^{(n)}\}_{n=0,\ldots,b_j}$, we define

$$Z_j = \pi(B_j^{(b-1)}).$$

This set consists of all elements $t$ satisfying $0^B_{m_2+1} < t \leq 1^B_{m_2}$ in $S^-$. 

**Proposition 5.25.** Put $j = 1 + \mu$, with $\mu = |Z_{j_1}|$. Then $1^B_j$ belongs to $D$. Let $e$ be the natural number satisfying $j = e$. Then $Z_{j_1} = Z_{j_e}$ holds.

**Proof.** By Lemma 4.16, there exists a natural number $n$ such that $B_j^{(n-1)} = \{0^B_{m_2+1}, \ldots, 0^B_{m_2+\mu}\}$. We have then $B_j^{(n)} = \{1^B_1, \ldots, 1^B_\mu\}$. This set is equal to $B_j^{(0)}$. A proof is given by the same way as Proposition 5.15. □

Proposition 5.26 is shown in the same way as the proof of Proposition 5.16. The proposition is used for the proof of Proposition 5.2 (II).

**Proposition 5.26.** Let $j_x$ be an element of $D$. We have then

1. If $x \leq e$, then $1^B_{j_x}$ belongs to $D_1$,
2. If $x > e$, then $1^B_{j_x}$ belongs to $D_2$.

**Proof.** Let $S^-$ be the generic specialization obtained by the exchange of $0^A_i$ and $1^B_i$ in $S$. For the ABS $S^-$, by (c) or (d) of Proposition 5.2, we obtain two components $\Psi$ and $S_\rho$, where $S_\rho$ is associated with $N_\rho$ for $\rho = (f,g)$. Since $S_\rho$ coincides with the component obtained from $B$ by applying [1] Lemma 5.6 to the adjacent $1^B_{m_2}$ or $1^B_{m_2+1}$, we have $gm_2 - fn_2 = 1$. In the same way as Proposition 5.9, we have the property: If there exists no element $t$ of $\Psi$ satisfying that $0^B_{m_2+1} < t < 1^B_{m_2}$ (resp. $0^A_i < t < 1^B_{j+1}$) in $S^-$, then $1^B_j$ belongs to $D_1$ (resp. $D_2$).

First, let us show the statement (1). We have properties

1. If $x < y$, then $Z_{j_x} \subset Z_{j_y}$ holds;
2. For all $n$ with $n \leq b_x$, we have $|B_{j_x}^{(n)}| = \mu$;
3. For all $j_x$ and all $n$ with $n < b_x$, we have $|B_{j_x}^{(n)}| \geq \mu$;
4. $Z_{j_x} \subset Z_{j_x}$ holds for all $x$ with $x > e$;
5. $Z_{j_x}$ is true if and only if $x \leq e$.

These properties are shown by the same way as the proof of Proposition 5.17. By (v), it suffices to consider the case $x = 1$ to show the statement (1). Note that $S_\rho$ obtained by the small modification
by $0^B_{m_2+1}$ and $1^B_{m_2}$ consists of all elements of $V$. There exists no element $t$ of $U$ satisfying that $0^B_{m_2+1} < t < 1^B_{m_2}$ in $S^-$. In fact, for the path $U$:

$$
0^B_{m_2} \rightarrow 0^B_{h_2} \rightarrow 1^B_{j_1} \rightarrow \cdots \rightarrow 1^B_{m_2-n_2+1},
$$

clearly if $t = 1^B_{m_2}$ or $t = 0^B_{h_2}$, then $t$ does not satisfy $0^B_{m_2+1} < t < 1^B_{m_2}$ in $S^-$. By Proposition 3.17, every element of the latter sub-path, which is of the form $\pi^m(1^B_j)$ with $m < b$, does not belong to $Z_{j_1}$. It induces that all elements in between $0^B_{m_2+1}$ and $1^B_{m_2}$ in $S^-$ do not belong to $\Psi$.

Next, let us show the statement (2). We define a non-negative integer $e'$ to be the maximum number of the set \{ $x \in \bar{D} \mid 1^B_{j_2+1}$ is the maximum element of $B^{(n)}_j$ for $n = n(x)$\}, where $\bar{D} = \{x \in \mathbb{N} \mid 1 \leq x \leq t\}$. If this set is empty, then we define $e' = 0$. By the same way as Proposition 5.20, we have $e' \leq e$. For these notation, a proof is obtained by the same way as the proof of Proposition 5.16 (2).

Proof of Proposition 5.2 (II). By Proposition 5.20, for every element of $D = D' \setminus \{1^B_1\}$, we construct the equality (10) by (c) or (d). Let us see the remaining case $j = 1$. If $n_2 > 1$, then we have $0^A_{i+1} < 1^B_{j+1}$ and there exists no element $t$ in between $0^A_i$ and $1^B_{j+1}$. Hence we obtain the equality (10) by (d).

Suppose $n_2 = 1$. In this case, we construct small modification by elements $0^A_{h_1}$ and $1^B_{n_2+1}$ in $S^-$, and we obtain two components $\Psi$ and $S_\rho$ of $S^-$, where $\rho = (1, 1)$. Concretely, we have $S_\rho = 1^B_0 0^A_{h_1}$. Let $N$ be the DM$_1$ associated with $\Psi$. We have then $N = N^-_\xi$ with $\xi' = (m_1 - 1, n_1 - 1) + (m_2, n_2)$. Hence we obtain the equality (10) by (e).

5.3 Proofs of Proposition 5.1 and Theorem 1.3

In this section, we show Proposition 5.1. Theorem 1.3 follows from this proposition.

Proof of Proposition 5.1. By Proposition 5.2, it remains to show the case $\lambda_1 = 1$ or $\lambda_2 = 0$. If $\lambda_1 = 1$, then for $S^-$ obtained by a good exchange of $0^A_i$ and $1^B_j$, we get ABS’s corresponding to $N^-_\xi$ and $N_\rho$ by (c) or (d) of Proposition 5.2. If $\lambda_2 = 0$, then for $S^-$ obtained by a good exchange $0^A_i$ and $1^B_j$, we get ABS's corresponding to $N^-_\xi$ and $N_\rho$ by (a) or (b) of Proposition 5.2.

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. The assertion is paraphrased as follows: For any generic specialization $N^-_\xi$ of the DM$_1$ $N_\xi$ with $\xi = (m_1, n_1) + (m_2, n_2)$, there exists a Newton polygon $\zeta$ such that $N^-_\zeta$ appears as a specialization of $N^-_\xi$, and $\zeta < \xi$ is saturated. We show this by induction on height of $\xi$.

If the height of $\xi$ is two (the case that the height is minimal), then $\xi = (0, 1) + (1, 0)$. In this case $N^-_\xi = N_{1,1}$ holds, whence there is nothing to prove.

Assume that the height of $\xi$ is greater than two. By Proposition 5.1, we obtain Newton polygons $\xi'$ and $\rho$ such that $N^-_\xi = N^-_{\xi'} \oplus N_{\rho}$, where the area of the region surrounded by $\xi$, $\xi'$ and $\rho$ is one. By the hypothesis of induction, there exists a Newton polygon $\zeta'$ such that $\zeta' < \xi'$ is saturated and $N^-_{\zeta'}$ is a specialization of $N^-_{\xi'}$. If we put $\zeta = \zeta' + \rho$, then $\zeta < \xi$ is saturated, and $N^-_{\zeta}$ is a specialization of $N^-_{\xi'} \oplus N_{\rho}(= N^-_{\xi''})$, and therefore is a specialization of $N^-_{\xi}$ (cf. [3 Proposition 3.5]).
Acknowledgments

I thank the referee for careful reading and helpful comments. This paper is written when the author is a Ph.D student. I thank the supervisor professor Harashita for the constant support from the early stage of this paper.

References

[1] S. Harashita, Configuration of the central streams in the moduli of abelian varieties, Asian J. Math. 13 (2009), no. 2, 215–250.

[2] S. Harashita, The supremum of Newton polygons of p-divisible groups with a given p-kernel type, Geometry and Analysis of Automorphic Forms of Several Variables, Proceedings of the international symposium in honor of Takayuki Oda on the occasion of his 60th birthday, Series on Number Theory and Its Applications, Vol. 7, (2011), pp. 41-55.

[3] N. Higuchi and S. Harashita, On specializations of minimal p-divisible groups, Yokohama Math. J., 64 (2018), 1-20.

[4] A. J. de Jong, Crystalline Dieudonne module theory via formal and rigid geometry, Inst. Hautes Études Sci. Publ. Math. No. 82 (1995), 5–96 (1996).

[5] N. M. Katz, Slope filtrations of F-crystals, Journ. Géom. Algébr. de Rennes, Vol. I, Astérisque, 63 (1979), Soc. Math. France, pp. 113–164

[6] H. Kraft, Kommutative algebraische p-Gruppen (mit Anwendungen auf p-divisible Gruppen und abelsche Varietäten), Sonderforschungsbereich. Bonn, September 1975. Ms. 86 pp.

[7] B. Moonen, A dimension formula for Ekedahl-Oort strata, Ann. Inst. Fourier 54 (2004), 666–698.

[8] B. Moonen and T. Wedhorn, Discrete invariants of varieties in positive characteristic, Int. Math. Res. Not. 2004, no. 72, 3855–3903.

[9] F. Oort, A stratification of a moduli space of abelian varieties, Moduli of Abelian Varieties (Ed. C. Faber, G. van der Geer, F. Oort), Progr. Math., 195, Birkhäuser, Basel, 2001; pp. 345-416.

[10] F. Oort, Foliations in moduli spaces of abelian varieties, J. Amer. Math. Soc. 17 (2004), no.2, 267-296.

[11] F. Oort, Minimal p-divisible groups, Ann. of Math. (2) 161 (2005), 1021–1036.

[12] E. Viehmann and T. Wedhorn, Ekedahl-Oort and Newton strata for Shimura varieties of PEL type, Math. Ann. 356 (2013), 1493–1550.

[13] T. Wedhorn, The dimension of Oort strata of Shimura varieties of PEL-type, Moduli of abelian varieties (Ed. C. Faber, G. van der Geer, F. Oort), Progress in Math. 195, Birkhäuser Verlag 2001; pp. 441–471.
GRADUATE SCHOOL OF ENVIRONMENT AND INFORMATION SCIENCES,
YOKOHAMA NATIONAL UNIVERSITY,
79-1 TOKIWADAI, HODOGAYA-KU,
YOKOHAMA 240-8501
JAPAN

E-mail address: higuchi-nobuhiro-sy@ynu.jp