LOOPING DIRECTIONS AND INTEGRALS OF EIGENFUNCTIONS OVER SUBMANIFOLDS

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Abstract. Let \((M,g)\) be a compact \(n\)-dimensional Riemannian manifold without boundary and \(e_\lambda\) be an \(L^2\)-normalized eigenfunction of the Laplace-Beltrami operator with respect to the metric \(g\), i.e.
\[-\Delta_g e_\lambda = \lambda^2 e_\lambda \quad \text{and} \quad \|e_\lambda\|_{L^2(M)} = 1.\]

Let \(\Sigma\) be a \(d\)-dimensional submanifold and \(d\mu\) a smooth, compactly supported measure on \(\Sigma\). It is well-known (e.g. proved by Zelditch in [15] in far greater generality) that
\[
\int_{\Sigma} e_\lambda \, d\mu = O(\lambda^{n-d-\frac{1}{2}}).
\]

We show this bound improves to \(o(\lambda^{n-d-\frac{1}{2}})\) provided the set of looping directions,
\[
\mathcal{L}_{\Sigma} = \{ (x,\xi) \in S^*\Sigma : \Phi_t(x,\xi) \in S^*\Sigma \text{ for some } t > 0 \}
\]
has measure zero as a subset of \(S^*\Sigma\), where here \(\Phi_t\) is the geodesic flow on the cosphere bundle \(S^*M\) and \(S^*\Sigma\) is the unit conormal bundle over \(\Sigma\).

1. Introduction.

In what follows, \((M,g)\) will denote a compact, boundaryless, \(n\)-dimensional Riemannian manifold. Let \(\Delta_g\) denote the Laplace-Beltrami operator and \(e_\lambda\) an \(L^2\)-normalized eigenfunction of \(\Delta_g\) on \(M\), i.e.
\[-\Delta_g e_\lambda = \lambda^2 e_\lambda \quad \text{and} \quad \|e_\lambda\|_{L^2(M)} = 1.\]

In [13], Sogge and Zelditch investigate which manifolds have a sequence of eigenfunctions \(e_\lambda\) with \(\lambda \to \infty\) which saturate the bound
\[
\|e_\lambda\|_{L^\infty(M)} = O(\lambda^{\frac{n-1}{2}}).
\]

They show that the bound above is necessarily \(o(\lambda^{\frac{n-1}{2}})\) if at each each \(x\), the set of looping directions through \(x\),
\[
\mathcal{L}_x = \{ \xi \in S^*_xM : \Phi_t(x,\xi) \in S^*_xM \text{ for some } t > 0 \}
\]
has measure zero\(^1\) as a subset of \(S^*_xM\) for each \(x \in M\). Here, \(\Phi_t\) denotes the geodesic flow on the unit cosphere bundle \(S^*M\) after time \(t\). The hypotheses were later weakened by Sogge, Toth, and Zelditch in [10], where they showed
\[
\|e_\lambda\|_{L^\infty(M)} = o(\lambda^{\frac{n-1}{2}})
\]
provided the set of recurrent directions at \(x\) has measure zero for each \(x \in M\).

\(^1\)Let \(\psi_j : U_j \subset \mathbb{R}^n \to M\) be coordinate charts of a general manifold \(M\). We say a set \(E \subset M\) has measure zero if the preimage \(\psi_j^{-1}(E)\) has Lebesgue measure 0 in \(\mathbb{R}^n\) for each chart \(\psi_j\). Sets of Lebesgue measure zero are preserved under transition maps, ensuring this definition is intrinsic to the \(C^\infty\) structure of \(M\).
We are interested in extending the result in [13] to integrals of eigenfunctions over submanifolds. Let \( \Sigma \) be a submanifold of dimension \( d \) with \( d < n \) and a measure \( d\mu(x) = h(x)\sigma(x) \) where \( \sigma \) is the surface measure on \( \Sigma \) and \( h \) is a smooth function supported on a compact subset of \( \Sigma \). In his 1992 paper [15], Zelditch proves, among other things, a Weyl law-type bound

\[
\sum_{\lambda_j \leq \lambda} \left| \int_{\Sigma} e_j \, d\mu \right|^2 \sim \lambda^{n-d} + O(\lambda^{n-d-1})
\]

from which follows

\[
\int_{\Sigma} e_{\lambda} \, d\mu = O(\lambda^{\frac{n-d-1}{2}}).
\]

Though (1.2) is already well known, we will give a direct proof which will be illustrative for our main argument.

**Theorem 1.1.** Let \( \Sigma \) be a \( d \)-dimensional submanifold with \( 0 \leq d < n \), and \( d\mu(x) = h(x)\sigma(x) \) where \( h \) is a smooth, real valued function supported on a compact neighborhood in \( \Sigma \). Then, (1.2) holds.

We let \( SN^*\Sigma \) denote the unit conormal bundle over \( \Sigma \). We define the set of looping directions through \( \Sigma \) by

\[
L_{\Sigma} = \{(x,\xi) \in SN^*\Sigma : \Phi_t(x,\xi) \in SN^*\Sigma \text{ for some } t > 0 \}.
\]

Our main result shows the bound (1.2) cannot be saturated whenever the set of looping directions through \( \Sigma \) has measure zero.

**Theorem 1.2.** Assume the hypotheses of Theorem 1.1 and additionally that \( L_{\Sigma} \) has measure zero as a subset of \( SN^*\Sigma \). Then,

\[
\int_{\Sigma} e_{\lambda} \, d\mu = o(\lambda^{\frac{n-d-1}{2}}).
\]

The argument for Theorem 1.2 is modeled after Sogge and Zelditch’s arguments in [13]. In fact if \( d = 0 \) we obtain the first part of [13] Theorem 1.2.

We expect the bound (1.2) to be saturated in the case \( M = S^n \), since \( L_{\Sigma} = SN^*\Sigma \) always. The spectrum of \( -\Delta_g \) on \( S^n \) consists of \( \lambda_j^2 \) where

\[
\lambda_j = \sqrt{j(j+n-1)} \quad \text{for } j = 0, 1, 2, \ldots
\]

(see [9]). For each \( \lambda_j \) we select an eigenfunction \( e_j \) maximizing \( \left| \int_{\Sigma} e_j \, d\mu \right| \). By Zelditch’s Weyl law type bound (1.1), there exists an increasing sequence of \( \lambda \) with \( \lambda \to \infty \) for which

\[
\sum_{\lambda_j \in [\lambda,\lambda+1]} \left| \int_{\Sigma} e_j \, d\mu \right|^2 \gtrsim \lambda^{n-d-1}.
\]

Since the gaps \( \lambda_j - \lambda_{j-1} \) approach a constant width of 1 as \( j \to \infty \), we may pick a subsequence of \( \lambda \)’s so that only one \( \lambda_j \) falls in each band \([\lambda,\lambda+1]\). Hence,

\[
\left| \int_{\Sigma} e_j \, d\mu \right|^2 \gtrsim \lambda_j^{\frac{n-d-1}{2}}
\]

for some subsequence of \( \lambda_j \).

It is worth remarking that there are some cases where the hypotheses of Theorem 1.2 are naturally fulfilled and we obtain an improvement over (1.2). Chen and
Sogge [2] proved that if \( M \) is 2-dimensional and has negative sectional curvature, and \( \Sigma \) is a geodesic in \( M \),

\[
\int_{\Sigma} e^\lambda \, d\mu = o(1).
\]

They consider a lift \( \tilde{\Sigma} \) of \( \Sigma \) to the universal cover of \( M \). Using the Gauss-Bonnet theorem, they show for each non-identity deck transformation \( \alpha \), there is at most one geodesic which intersects both \( \tilde{\Sigma} \) and \( \alpha(\tilde{\Sigma}) \) perpendicularly. Since there are only countably many deck transformations, \( L_\Sigma \) is at most a countable subset of \( SN^*\Sigma \) and so satisfies the hypotheses of Theorem 1.2. This result was extended to a larger class of curves in [14] which similarly have countable \( L_\Sigma \).

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2. Proof of Theorem 1.1

Theorem 1.1 is a consequence of this stronger result.

Proposition 2.1. Given the hypotheses of Theorem 1.1, we have

\[
\sum_{\lambda_j \in [\lambda, \lambda + 1]} \left| \int_{\Sigma} e^\lambda \, d\mu \right|^2 \leq C\lambda^{n-d-1}.
\]

We lay out some local coordinates which we will use repeatedly. Fix \( p \in \Sigma \), and consider local coordinates \( x = (x_1, \ldots, x_n) = (x', \bar{x}) \) centered about \( p \), where \( x' \) denotes the first \( d \) coordinates and \( \bar{x} \) the remaining \( n - d \) coordinates. We let \( (x', 0) \) parameterize \( \Sigma \) on a neighborhood of \( p \) in such a way that \( dx' \) agrees with the surface measure on \( \Sigma \). Let \( g \) denote the metric tensor with respect to our local coordinates. We require

\[
g = \begin{bmatrix} * & 0 \\ 0 & I \end{bmatrix}
\]

wherever \( \bar{x} = 0 \),

where \( I \) here is the \( (n-d) \times (n-d) \) identity matrix. This is ensured after inductively picking smooth sections \( v_j(x') \) of \( SN\Sigma \) for \( j = d + 1, \ldots, n \) with \( \langle v_i, v_j \rangle = \delta_{ij} \), and then using

\[
(x_1, \ldots, x_n) \mapsto \exp(x_{d+1}v_{d+1}(x') + \cdots + x_nv_n(x'))
\]

as our coordinate map.

Now we prove Proposition 2.1. For simplicity, we assume without loss of generality that \( d\mu \) is a real measure. We set \( \tilde{\chi} \in C^\infty(\mathbb{R}) \) with \( \chi \geq 0 \) and \( \tilde{\chi} \) supported on a small neighborhood of 0. It suffices to show

\[
\sum_j \chi(\lambda_j - \lambda) \left| \int_{\Sigma} e^\lambda \, d\mu \right|^2 \leq C\lambda^{n-d-1}.
\]

2This reduction is standard and appears in [13], [2], proofs of the sharp Weyl law as presented in [9] and [8], and in many other similar problems.
By Fourier inversion, we write the left hand side as
\[ \sum_j \int_{\Delta} \int_{\Delta} \chi(\lambda_j - \lambda) e_j(x) \overline{e_j(y)} \, d\mu(x) \, d\mu(y) \]
\[ = \frac{1}{2\pi} \sum_j \int_{-\infty}^{\infty} \int_{\Delta} \int_{\Delta} \hat{\chi}(t) e^{-it\lambda} e^{it\lambda_j} e_j(x) \overline{e_j(y)} \, d\mu(x) \, d\mu(y) \, dt \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\Delta} \int_{\Delta} \hat{\chi}(t) e^{-it\lambda} e^{it\sqrt{-\Delta_g}}(x, y) \, d\mu(x) \, d\mu(y) \, dt \]
\[ (2.2) \]
where \( e^{it\sqrt{-\Delta_g}} \) is the half wave operator with kernel
\[ e^{it\sqrt{-\Delta_g}}(x, y) = \sum_j e^{it\lambda_j} e_j(x) \overline{e_j(y)}. \]

Using the coordinates \( x = (x', \bar{x}) \) as in (2.1), the last line of (2.2) is written
\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} \hat{\chi}(t) e^{-it\lambda} e^{it\sqrt{-\Delta_g}}(x', y') h(x') h(y') \, dx' \, dy' \, dt \]
\[ (2.3) \]
where \( h \) is a smooth function on \( \mathbb{R}^d \) such that \( d\mu(x) = h(x') dx' \), and where by abuse of notation \( x' \) is taken to mean \((x', 0)\) where appropriate. We now use Hörmander’s parametrix as presented in [8], i.e.
\[ e^{it\sqrt{-\Delta_g}}(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^n} e^{i(t\varphi(x, y, \xi) + t\varphi(y, \xi))} q(t, x, y, \xi) \, d\xi \]
modulo a smooth kernel, where
\[ p(y, \xi) = \sqrt{\sum_{j, k} g^{jk}(y) \xi_j \xi_k} \]
is the principal symbol of \( \sqrt{-\Delta_g} \) and \( \varphi \) is smooth for \(|\xi| > 0\), homogeneous of degree 1 in \( \xi \), and satisfies
\[ (2.4) \]
\[ |\partial^\alpha_\xi \varphi(x, y, \xi) - \langle x - y, \xi \rangle| \leq C_\alpha |x - y|^1 + |\xi|^{-|\alpha|} \]
for multiindices \( \alpha \geq 0 \) and for \( x \) and \( y \) sufficiently close. Moreover, \( q \) satisfies bounds
\[ (2.5) \]
\[ |\partial^\beta_\xi \partial^\alpha_\xi \varphi(x, y, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{-|\alpha|}, \]
and where for \( t \in \text{supp} \hat{\chi}, q \) is supported on a small neighborhood of \( x = y \). Hence, we write (2.3) as
\[ = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} e^{i(t\varphi(x', y', \xi) + t\varphi(y', \xi) - t\lambda)} \hat{\chi}(t) q(t, x', y', \xi) h(x') h(y') \, d\xi \, dx' \, dy' \, dt \]
\[ = \frac{\lambda^n}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} e^{i\lambda(t\varphi(x', y', \xi) + t\varphi(y', \xi) - t\lambda)} \hat{\chi}(t) q(t, x', y', \lambda \xi) h(x') h(y') \, d\xi \, dx' \, dy' \, dt. \]

We introduce a function \( \beta \in C_0^\infty(\mathbb{R}) \) with \( \beta \equiv 1 \) near 0 and support contained in a small neighborhood of 0, and cut the integral into \( \beta(\log p(y', \xi)) \) and 1 –
where we have set the amplitude variables \( t_N \) for each \( N \). Note for fixed \( y \)
\( \partial_r \phi = \nabla_x \phi(x', y', \xi) + t \nabla_x p(y', \xi) \)
\( \partial_t \phi = \partial_r \varphi(x', y', \xi) + t \partial_r p(y', \xi). \)

Note for fixed \( y' \) and \( \omega \), \((t, x', \xi', r) = (0, y', 0, 1)\) is a critical point of \( \Phi \). Now we compute the second derivatives at this point. We immediately see that \( \partial^2_r \Phi, \partial_t \nabla_x \Phi, \nabla^2_{\xi r} \Phi, \partial_t \nabla_{\xi r} \Phi, \partial^2_{\xi r} \Phi \) all vanish. Moreover, \( \partial_r \partial_t \Phi = 1 \) since \( p(y', \xi) = r \).
where $\xi' = 0$. By our coordinates (2.1) and the fact that $[g^{ij}]_{i,j \leq d}$ is necessarily positive definite,

$$p(y', \xi) = \sqrt{\sum_{j,k} g^{jk} \xi_j \xi_k} = \sqrt{r^2 + \sum_{j,k \leq d} g^{jk} \xi'_j \xi'_k} \geq r = p(y', r\omega).$$

Hence, $\partial_t \nabla\xi' \Phi = \nabla\xi' p(y', \xi) = 0$. Since $\phi$ is homogeneous of degree 1 in $\xi$, at $\xi' = 0$ and $t = 0$,

$$\nabla_{x'} \partial_r \Phi = \nabla_{x'} \partial_r \phi(x', y', \xi) = \nabla_{x'} \phi(x', y', \omega) = 0$$

since $\phi(x', y', \omega) = O(|x' - y'|^2)$ by (2.4) and the fact that $\langle x' - y', \omega \rangle = 0$. Finally by (2.4),

$$\nabla_{x'} \phi(x', y', \xi' + \omega) = x' + O(|x' - y'|^2)$$

whence at the critical point

$$\nabla_{x'} \nabla_{\xi'} \Phi = I,$$

the $d \times d$ identity matrix. In summary, the Hessian matrix of $\Phi$ at the critical point $(t, x', \xi', r) = (0, y', 0, 1)$ is

$$\nabla^2_{t,x', \xi', r} \Phi = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & * & I & 0 \\ 0 & I & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which has full rank.

3. Microlocal tools.

The hypotheses on the looping directions in Theorem 1.2 ensure that the wavefront sets of $\mu$ and $e^{it\sqrt{-\Delta_g}} \mu$ have minimal intersection for any given $t$. We can then use pseudodifferential operators to break $\mu$ into two parts, the first whose wavefront set is disjoint from that of $e^{it\sqrt{-\Delta_g}} \mu$ and the second which contributes a small, controllable term to the bound. The following propositions will allow us to handle these cases, respectively.

**Proposition 3.1.** Let $u$ and $v$ be distributions on $M$ for which

$$\text{WF}(u) \cap \text{WF}(v) = \emptyset.$$

Then

$$t \mapsto \int_M e^{it\sqrt{-\Delta_g}} u(x)v(x) \, dx$$

is a smooth function of $t$ on some neighborhood of 0.

**Proof.** Using a partition of unity, we write

$$I = \sum_j A_j$$

modulo a smoothing operator where $A_j \in \Psi^0(M)$ with essential supports in small conic neighborhoods. We then write, formally,

$$\int e^{it\sqrt{-\Delta_g}} u(x)v(x) \, dx = \sum_{j,k} \int A_j e^{it\sqrt{-\Delta_g}} u(x) A_k v(x) \, dx.$$
We are done if for each $i$ and $j$,

\[(3.1) \quad \int_M A_j e^{it\sqrt{-\Delta_g} u(x)} A_k v(x) \, dx \quad \text{is smooth for } |t| \ll 1.\]

If the essential supports of $A_j$ and $A_k$ are disjoint, then $A_j^* A_k$ is a smoothing operator, and so $A_j^* A_k v$ is a smooth function and the contributing term

\[\int u(x) e^{it\sqrt{-\Delta_g} A_j^* A_k v(x)} \, dx\]

is smooth in $t$. Assume the essential support of $A_j$ are small enough so that for each $j$ there exists a small conic neighborhood $\Gamma_j$ which fully contains the essential support of $A_k$ if it intersects the essential support of $A_j$. We in turn take $\Gamma_j$ small enough so that for each $j$, $\Gamma_j \cap \text{WF}(e^{it\sqrt{-\Delta_g} u}) = \emptyset$ for $|t| \ll 1$ since both sets above are closed and the geodesic flow is continuous. Then $A_j e^{it\sqrt{-\Delta_g} u(x)}$ is smooth as a function of $t$ and $x$, and we have (3.1).

The second piece of our argument requires the following generalization of Proposition 2.1, modeled after [9, Lemma 5.2.2]. In the proof we will come to a point where it seems like we may have to perform a stationary phase argument involving an eight-by-eight Hessian matrix. Instead, we appeal to Proposition 5.1 in the appendix to break the argument into two steps involving two four-by-four Hessian matrices.

**Proposition 3.2.** Let $b(x, \xi)$ be smooth for $\xi \neq 0$ and homogeneous of degree 0 in the $\xi$ variable. We define $b \in \Psi^0_0(M)$ by

\[b(x, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y, \xi)} b(x, \xi) \, dy \, d\xi \]

for $x$, $y$, and $\xi$ expressed locally according to our coordinates (2.1). Then,

\[\sum_{\lambda_j \in [\lambda, \lambda+1]} \left| \int_{\Sigma} \int_{\Sigma} b e_{\lambda_j}(x) \, d\mu(x) \right|^2 \leq C \left( \int_{\mathbb{R}^d} \int_{S^{n-d-1}} |b(x', \omega)|^2 h(x')^2 \, d\omega \, dx' \right) \lambda^{n-d-1} + C_b \lambda^{n-d-2}\]

where $C$ is a constant independent of $b$ and $\lambda$ and $C_b$ is a constant independent of $\lambda$ but which depends on $b$.

**Proof.** We may by a partition of unity assume that $b(x, D)$ has small $x$-support. Let $\chi$ be as in the proof of Proposition 2.1. It suffices to show

\[\sum_{j} \int_{\Sigma} \int_{\Sigma} \chi(\lambda_j - \lambda) b(x, D) e_{\lambda_j}(x) b(y, D) e_{\lambda_j}(y) \, d\mu(x) \, d\mu(y) \sim \left( \int_{\mathbb{R}^d} \int_{S^{n-d-1}} |b(y', \omega)|^2 h(y')^2 \, d\omega \, dy' \right) \lambda^{n-d-1} + O_b(\lambda^{n-d-2}).\]
Using the same reduction as in Proposition 2.1, the left hand side is

\[ (3.2) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\Sigma} \chi(t) e^{-it\lambda} e^{it\sqrt{-\Delta_s}} b^*(x, y) \, d\mu(x) \, d\mu(y) \, dt. \]

Set \( \beta \in C_0^\infty(\mathbb{R}) \) with small support and where \( \beta \equiv 1 \) near 0. Then,

\[
\int_{\mathbb{R}^d} b(x', D) f(x') h(x') \, dx' = \frac{\lambda^n}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} e^{i\lambda(x' - \eta \cdot w)} b(x', \eta) f(w) h(x') \, dx' \, dw \, d\eta
\]

\[ (3.3) \quad = \frac{\lambda^n}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} e^{i\lambda(x' - \eta \cdot w)} \beta(\log |\eta|) b(x', \eta) f(w) h(x') \, dx' \, dw \, d\eta
\]

\[ + O(\lambda^{-N}), \]

where the second line is obtained by a change of variables \( \eta \mapsto \lambda \eta \), and the third line is obtained after multiplying in the cutoff \( \beta(\log |\eta|) \) and bounding the discrepancy by \( O(\lambda^{-N}) \) by integrating by parts in \( x' \). Additionally,

\[
\int_{\mathbb{R}^d} b^*(z, D) d\mu(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} e^{i(z - y' \cdot \zeta)} b(y', \zeta) h(y') \, dy' \, d\zeta
\]

\[ (3.4) \quad = \frac{\lambda^n}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} e^{i\lambda(z - y' \cdot \zeta)} \beta(\log |\zeta|) b(y', \zeta) h(y') \, dy' \, d\zeta + O(\lambda^{-N})
\]

where the second and third lines are obtained similarly as before and the fourth line is obtained after multiplying by \( \beta(\log |z - y'|) \) and integrating the remainder by parts in \( \zeta \). Using Hörmander’s parametrix,

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(t) e^{-it\lambda} e^{it\sqrt{-\Delta_s}} (w, z) \, dt
\]

\[ (3.5) \quad = \frac{\lambda^n}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} e^{i\lambda(\varphi(w, z, \xi) + t(p(z, \xi) - 1))} \chi(t) q(t, w, z, \xi) \, d\xi \, dt
\]

Here the third line comes from a change of coordinates \( \xi \mapsto \lambda \xi \). The fourth line follows after applying the cutoff \( \beta(\log p(z, \xi)) \) and integrating the discrepancy by parts in \( t \). Combining (3.3), (3.4), and (3.5), we write (3.2) as

\[
(3.6) \quad \lambda^{3n} \int \cdots \int e^{i\Phi(t, x', y', w, z; \eta, \zeta)} b(\lambda; t, x', y', w, z, \eta, \zeta) \, dx' \, dy' \, dw \, d\eta \, d\zeta + O(\lambda^{-N})
\]
with amplitude
\[ a(\lambda; t, x', y', w, z, \eta, \zeta, \xi) = \frac{1}{(2\pi)^{n+1}} e^{i\lambda\Phi(t,x',y',\xi)} a(\lambda; t, x', y', \xi) dx' dy' d\xi \]
and phase
\[ \Phi(t, x', y', w, z, \eta, \zeta, \xi) = \langle x' - w, \eta \rangle + \varphi(w, z, \xi) + t(p(z, \xi) - 1) + \langle z - y', \zeta \rangle. \]
We pause here to make a couple observations. First, \( a \) has compact support in all variables, support which we may adjust to be smaller by controlling the supports of \( \hat{\chi}, \beta, b, \) and the support of \( q \) near the diagonal. Second, the derivatives of \( a \) are bounded independently of \( \lambda \geq 1 \). We are now in a position to use the method of stationary phase – not in all variables at once, though. First, we fix \( t, x', y' \) and \( \xi \), and use stationary phase in \( w, z, \eta, \zeta, \xi \) subject to the constraints \( (3.7) \), and where \( \nabla_\xi \Phi(\lambda, t, x', y', \xi) = 1 \)

At such a critical point we have the Hessian matrix

\[ \nabla^2_{w, z, \eta, \zeta} \Phi = \begin{bmatrix} * & * & -I & 0 \\ * & * & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}, \]

which has determinant \(-1\). By Proposition 5.1 in the appendix, \( 3.6 \) is equal to complex constant times

\[ \lambda^n \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} e^{i\lambda\Phi(t, x', y', \xi)} a(\lambda; t, x', y', \xi) dx' dy' d\xi \ dt + \lambda^{n-1} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} e^{i\lambda\Phi(t, x', y', \xi)} R(\lambda; t, x', y', \xi) dx' dy' d\xi \ dt + O(\lambda^{-N}) \]
where we have phase
\[ \Phi(t, x', y', \xi) = \varphi(x', y', \xi) + t(p(y', \xi) - 1), \]
amplitude
\[ a(\lambda; t, x', y', \xi) = a(\lambda; t, x', y', w, z, \eta, \zeta, \xi) \]
with \( w, z, \eta, \) and \( \zeta \) subject to the constraints \( 3.7 \), and where \( R \) is a compactly supported smooth function in \( t, x', y' \), and \( \xi \) whose derivatives are bounded uniformly with respect to \( \lambda \). Our phase function matches that in the proof of Proposition 2.1 and so we repeat that argument – we write \( \xi = r\omega \) and fix \( y' \) and \( \omega \). We obtain unique nondegenerate stationary points
\[ (t, x', \xi', r) = (0, y', 0, 1). \]
Now,
\[ a(\lambda; 0, y', y', \omega) \sim |b(y', \omega)|^2 h(y')^2. \]
Hence, we have
\[ \lambda^n \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i \lambda \Phi(t, x', y', \xi)} a(\lambda; t, x', y', \xi) \, dx' \, dy' \, d\xi' \, dt \]
\[ \sim \lambda^{n-d-1} \left( \int_{\mathbb{R}^d} \int_{S^{n-1}} b(y', \omega)^2 h(y')^2 \, d\omega \, dy' \right) + O(\lambda^{n-d-2}) \]
by Proposition 5.1 as desired. The same argument applied to the remainder term gives
\[ \lambda^{n-1} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i \lambda \Phi(t, x', y', \xi)} R(\lambda; t, x', y', \xi) \, dx' \, dy' \, d\xi' \, dt = O(\lambda^{n-d-2}) \]
as desired.

4. Proof of Theorem 1.2

Theorem 1.2 follows from the following stronger statement.

Proposition 4.1. Given the hypotheses of Theorem 1.2, we have
\[ \sum_{\lambda_j \in [\lambda, \lambda + \epsilon]} \left| \int_{\Sigma} e^{i \lambda_j} \, d\mu \right|^2 \leq C \varepsilon \lambda^{n-d-1} + C \varepsilon \lambda^{n-d-2}, \]
where \( C \) is a constant independent of \( \varepsilon \) and \( \lambda \), and \( C \varepsilon \) is a constant depending on \( \varepsilon \) but not \( \lambda \).

We make a few convenient assumptions. First, we take the injectivity radius of \( M \) to be at least 1 by scaling the metric \( g \). Second, we assume the support of \( d\mu \) has diameter less than 1/2 by a partition of unity. We reserve the right to further scale the metric \( g \) and restrict the support of \( d\mu \) as needed, finitely many times.

As before, we set \( \chi \in C^\infty(\mathbb{R}) \) with \( \chi(0) = 1 \), \( \chi \geq 0 \), and supp \( \hat{\chi} \subset [-1, 1] \). It suffices to show
\[ \sum_{j} \chi(\ell(\lambda_j - \lambda)) \left| \int_{\Sigma} e^{i \lambda_j} \, d\mu \right|^2 \leq C T^{-1} \lambda^{n-d-1} + C T \lambda^{n-d-2} \]
for \( T > 1 \). Similar to the reduction in the proof of Proposition 2.1, we have
\[ \sum_{j} \int_{\Sigma} \int_{\Sigma} \chi(\ell(\lambda_j - \lambda)) e_j(x) \overline{e_j(y)} \, d\mu(x) \, d\mu(y) \]
\[ \quad = \frac{1}{2 \pi} \sum_{j} \int_{-\infty}^{\infty} \int_{\Sigma} \hat{\chi}(t) e^{it(\lambda_j - \lambda)} e_j(x) \overline{e_j(y)} \, d\mu(x) \, d\mu(y) \, dt \]
\[ \quad = \frac{1}{2 \pi T} \sum_{j} \int_{-\infty}^{\infty} \int_{\Sigma} \hat{\chi}(t/T) e^{-it\lambda} e^{it\lambda_j} e_j(x) \overline{e_j(y)} \, d\mu(x) \, d\mu(y) \, dt \]
\[ \quad = \frac{1}{2 \pi T} \int_{-\infty}^{\infty} \int_{\Sigma} \hat{\chi}(t/T) e^{-it\lambda} e^{it\lambda_j} \Delta_{\varepsilon}(x, y) \, d\mu(x) \, d\mu(y) \, dt. \]

Hence, it suffices to show
\[ \int_{-\infty}^{\infty} \int_{\Sigma} \hat{\chi}(t/T) e^{-it\lambda} e^{it\lambda_j} \Delta_{\varepsilon}(x, y) \, d\mu(x) \, d\mu(y) \, dt \]
\[ \leq C \lambda^{n-d-1} + C T \lambda^{n-d-2}. \]
Set $\beta \in C^\infty_0(\mathbb{R})$ with $\beta(t) \equiv 1$ near 0 and $\beta$. We cut the integral in (1.1) into $\beta(t)$ and $1 - \beta(t)$ parts. Since $\beta(t)\hat{\chi}(t/T)$ and its derivatives are all bounded independently of $T \geq 1$,

$$\left| \int_{-\infty}^{\infty} \int_{\Sigma} \beta(t)\hat{\chi}(t/T)e^{-it\lambda}e^{it\sqrt{-\Delta_s}}(x, y) \, d\mu(x) \, d\mu(y) \, dt \right| \leq C\lambda^{n-d-1}$$

by the proof of Proposition 2.1. Hence, it suffices to show

$$\left| \int_{-\infty}^{\infty} \int_{\Sigma} (1 - \beta(t))\hat{\chi}(t/T)e^{-it\lambda}e^{it\sqrt{-\Delta_s}}(x, y) \, d\mu(x) \, d\mu(y) \, dt \right| \leq C\lambda^{n-d-1} + C_\epsilon \lambda^{n-d-2}.$$  

Here we shrink the support of $\mu$ so that $\beta(d_\theta(x, y)) = 1$ for $x, y \in \text{supp } \mu$. We now state and prove a useful decomposition based off of those in [13], [10], and Chapter 5 of [9]. We let $L_\Sigma(\text{supp } \mu, T)$ denote the subset of $L_\Sigma$ relevant to the support of $\mu$ and the timespan $[1, T]$, specifically

$$L_\Sigma(\text{supp } \mu, T) = \{(x, \xi) \in \text{SN}^*\Sigma : \Phi_t(x, \xi) = (y, \eta) \in \text{SN}^*\Sigma \text{ for some } t \in [1, T] \text{ and where } x, y \in \text{supp } \mu\}.$$  

**Lemma 4.2.** Fix $T > 1$ and $\epsilon > 0$. There exist $b, B \in \Psi_0^0(M)$ supported on a neighborhood of $\text{supp } \mu$ with the following properties.

1. $b(x, D) + B(x, D) = I$ modulo a smoothing operator on $\text{supp } \mu$.
2. Using coordinates (2.1),

$$\int_{\mathbb{R}^d} \int_{S^{n-1}} |b(x', \omega)|^2 \, d\omega \, dx' < \epsilon,$$

where $b(x, \xi)$ is the principal symbol of $b(x, D)$.
3. The essential support of $B(x, D)$ contains no elements of $L_\Sigma(\text{supp } \mu, T)$.

**Proof.** As shorthand, we write

$$\text{SN}_{\text{supp } \mu}^*\Sigma = \{(x, \xi) \in \text{SN}^*\Sigma : x \in \text{supp } \mu\}.$$  

We first argue that $L_\Sigma(\text{supp } \mu, T)$ is closed for each $T > 1$. However, $L_\Sigma(\text{supp } \mu, T)$ is the projection of the set

$$(4.3) \quad \{(t, x, \xi) \in [1, T] \times \text{SN}_{\text{supp } \mu}^*\Sigma : \Phi_t(x, \xi) \in \text{SN}_{\text{supp } \mu}^*\Sigma\}$$

onto $\text{SN}_{\text{supp } \mu}^*\Sigma$, and since $[1, T]$ is compact it suffices to show that (4.3) is closed. However, (4.3) is the intersection of $[1, T] \times \text{SN}_{\text{supp } \mu}^*\Sigma$ with the preimage of $\text{SN}_{\text{supp } \mu}^*\Sigma$ under the continuous map

$$(t, x, \xi) \mapsto \Phi_t(x, \xi).$$

Since $\text{SN}_{\text{supp } \mu}^*\Sigma$ is closed, (4.3) is closed.

Since $L_\Sigma(\text{supp } \mu, T)$ is closed and has measure zero, there is $\tilde{b} \in C^\infty(\text{SN}^*\Sigma)$ supported on a neighborhood of $\text{SN}_{\text{supp } \mu}^*\Sigma$ with $0 \leq \tilde{b}(x, \xi) \leq 1$, $\tilde{b}(x, \xi) \equiv 1$ on an open neighborhood of $L_\Sigma(\text{supp } \mu, T)$, and

$$\int_{\mathbb{R}^d} \int_{S^{n-1}} |b(x', \omega)|^2 \, d\omega \, dx' < \epsilon.$$  

We use the coordinates in (2.1) and define

$$b(x, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y, \xi)}\tilde{b}(x, \xi/|\xi|) f(y) \, dy \, d\xi,$$
hence (2). We set \( \psi \in C_0^\infty(\Sigma) \) to be a cutoff function supported on a neighborhood of \( \text{supp} \mu \) with \( \psi \equiv 1 \) on \( \text{supp} \mu \). Defining

\[
B(x, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y, \xi)} \psi(x)(1 - \tilde{b}(x, |\xi|)) f(y) \, dy \, d\xi
\]

yields (1). We have (3) since the support of \( 1 - \tilde{b}(x, \xi) \) contains no elements of \( \mathcal{L}_\Sigma(\text{supp} \mu, T) \).

Returning to the proof of Proposition 4.1 let \( X_T \) denote the function with

\[
X_T(t) = (1 - \beta(t)) \tilde{\chi}(t/T),
\]

and let \( X_{T, \lambda} \) denote the operator with kernel

\[
X_{T, \lambda}(x, y) = \frac{1}{2\pi} \int_{-\infty}^\infty X_T(t)e^{-it\lambda}e^{it\sqrt{-\Delta_g}}(x, y) \, dt.
\]

We use part (1) of Lemma 4.2 to write the integral in (4.2) as

\[
\int_\Sigma \int_\Sigma X_{T, \lambda}(x, y) \, d\mu(y) \, d\mu(x) = \int_\Sigma \int_\Sigma BX_{T, \lambda}B^*(x, y) \, d\mu(y) \, d\mu(x)
+ \int_\Sigma \int_\Sigma BX_{T, \lambda}b^*(x, y) \, d\mu(y) \, d\mu(x)
+ \int_\Sigma \int_\Sigma bX_{T, \lambda}B^*(x, y) \, d\mu(y) \, d\mu(x)
+ \int_\Sigma \int_\Sigma bX_{T, \lambda}b^*(x, y) \, d\mu(y) \, d\mu(x).
\]

We claim the first three terms on the right are \( OT(\lambda^{-N}) \) for \( N = 1, 2, \ldots \). We will only prove this for the first term – the argument is the same for the second term and the bound for the third term follows since \( X_{T, \lambda} \) is self-adjoint. Interpreting \( \mu \) as a distribution on \( M \), we write formally

\[
\int_\Sigma \int_\Sigma BX_{T, \lambda}B^*(x, y) \, d\mu(y) \, d\mu(x)
= \int_M \int_M X_{T, \lambda}(x, y)B^*\mu(y)\overline{B^*\mu(x)} \, dx \, dy
= \frac{1}{2\pi} \int_{-\infty}^\infty X_T(t)e^{-it\lambda} \int_M e^{it\sqrt{-\Delta_g}}(B^*\mu)(x)\overline{B^*\mu(x)} \, dx \, dt.
\]

Once we show

\[
WF(e^{it\sqrt{-\Delta_g}}B^*\mu) \cap B^*\mu = \emptyset \quad \text{for all} \quad t \in \text{supp} \tilde{X}_T,
\]

the integral over \( M \) will be smooth in \( t \) by Proposition 3.1. Integration by parts in \( t \) then gives the desired bound of \( OT(\lambda^{-N}) \). To prove (4.5), suppose \( (x, \xi) \in WF(B^*\mu) \). By part (3) of Lemma 4.2 \( \Phi_1(x, \xi) \) is not in \( SN_{\text{supp} \mu, \Sigma}^* \) for any \( 1 \leq |t| \leq T \). By propagation or singularities,

\[
WF(e^{it\sqrt{-\Delta_g}}B^*\mu) = \Phi_1 \cdot WF(B^*\mu),
\]

hence

\[
WF(e^{it\sqrt{-\Delta_g}}B^*\mu) \cap WF(B^*\mu) = \emptyset \quad \text{for} \quad 1 \leq |t| \leq T.
\]

Since the support of \( \mu \) has been made small, if there is \( (x, \xi) \in SN_{\text{supp} \mu, \Sigma}^* \) and some \( t > 0 \) in the support of \( (1 - \beta(t))\tilde{\chi}(t/T) \) for which \( \Phi_1(x, \xi) \in SN_{\text{supp} \mu, \Sigma}^* \), then \( t \geq 1 \).
Taking $\varepsilon$ concludes the proof of Proposition 4.1.

What remains is to bound

$$
\left| \int_{\Sigma} \int_{\Sigma} bX_{T,\lambda} b^*(x, y) \, d\mu(x) \, d\mu(y) \right| \leq \lambda^{n-d-1} + C_{T,\lambda} \lambda^{n-d-2}.
$$

We have

$$bX_{T,\lambda} b^*(x, y) = \sum_j X_T(\lambda_j - \lambda) b e_j(x) \overline{b e_j(y)},$$

and so we write the integral in (4.7) as

$$
\sum_j X_T(\lambda_j - \lambda) \left| \int_{\Sigma} b(x, D) e_j(x) \, d\mu(x) \right|^2
$$

By the bounds

$$|X_T(\tau)| \leq C_{T,N}(1 + |\tau|)^{-N} \quad \text{for } N = 1, 2, \ldots$$

and Proposition 3.2

$$
\left| \sum_j X_T(\lambda_j - \lambda) \left| \int_{\Sigma} b(x, D) e_j(x) \, d\mu(x) \right|^2 \right|
\leq C_T \lambda^{n-d-1} \int_{\mathbb{R}^d} \int_{S^{n-d-1}} |b(x', \omega)|^2 h(x')^2 \, d\omega \, dx' + C_{T,\lambda} \lambda^{n-d-2}.
$$

Taking $\varepsilon$ in part (2) of Lemma 4.2 small enough so that $\varepsilon C_T \leq 1$ yields (4.7). This concludes the proof of Proposition 4.1.

5. Appendix: Stationary phase tool.

The following tool is a combination of Corollary 1.1.8 with the discussion at the end of Section 1.1 in [8]. Let $\phi(x, y)$ be a smooth phase function on $\mathbb{R}^m \times \mathbb{R}^n$ with

$$\nabla_y \phi(0, 0) = 0 \quad \text{and} \quad \det \nabla_y^2 \phi(0, 0) \neq 0,$$

and let $a(\lambda; x, y)$ be a smooth amplitude with small, adjustable support satisfying

$$|\partial_j^\alpha \partial_{y}^\beta a(\lambda; x, y)| \leq C_{j,\alpha,\beta} \lambda^{-j} \quad \text{for } \lambda \geq 1$$

for $j = 0, 1, 2, \ldots$ and multiindices $\alpha$ and $\beta$. $\nabla_y^2 \phi \neq 0$ on a neighborhood of 0 by continuity. There exists locally a smooth map $x \mapsto y(x)$ whose graph in $\mathbb{R}^m \times \mathbb{R}^n$ contains all points in a neighborhood of 0 such that $\nabla_y \phi = 0$, by the implicit function theorem. Let $p$ and $q$ be integers denoting the number of positive and negative eigenvalues of $\nabla_y^2 \phi$, respectively, counting multiplicity. By continuity, $p$ and $q$ are constant on a neighborhood of 0. We adjust the support of $a$ to lie in the intersection of these neighborhoods.

**Proposition 5.1.** Let

$$I(\lambda; x) = \int_{\mathbb{R}^n} e^{i\lambda \phi(x, y)} a(\lambda; x, y) \, dy$$

with $\phi$ and $a$ as above. Then,

$$I(\lambda; x) = (\lambda/2\pi)^{-n/2} |\det \nabla_y^2 \phi(x, y(x))|^{-1/2} e^{\pi i (p-q)/4} e^{i\lambda \phi(x, y(x))} a(\lambda; x, y(x))$$

$$\lambda^{-n/2-1} e^{i\lambda \phi(x, y(x))} R(\lambda; x) + O(\lambda^{-N})$$
for \( N = 1, 2, \ldots \), where \( R \) has compact support,
\[
|\partial^j_x \partial^a_y R(\lambda; x)| \leq C_{j,a} \lambda^{-i} \quad \text{for } \lambda \geq 1,
\]
and the \( \mathcal{O}(\lambda^{-N}) \) term is constant in \( x \).

**Proof.** We have
\[
\text{(5.1)} \quad e^{-i\lambda \phi(x,y(x))} I(\lambda; x) = \int_{\mathbb{R}^n} e^{i\lambda \Phi(x,y)} a(\lambda; x, y) \, dy
\]
where we have set
\[
\Phi(x, y) = \phi(x, y) - \phi(x, y(x)).
\]
The proof of the Morse-Bott lemma in [1] lets us construct a smooth map \( F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \) such that \( y \mapsto F(x, y) \) is a diffeomorphism between neighborhoods of \( 0 \) in \( \mathbb{R}^n \) for each \( x \), and for which
\[
F(x, 0) = y(x)
\]
and
\[
\Phi(x, F(x, y)) = \frac{1}{2} (y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2) = Q(y).
\]
Applying a change of variables in \( y \) to (5.1) yields
\[
\begin{align*}
&= \int_{\mathbb{R}^n} e^{i\lambda Q(y)} a(\lambda; x, F(x, y)) |\det D_y F(x, y)| \, dy \\
&= \int_{\mathbb{R}^n} e^{i\lambda Q(y)} a(\lambda; x, F(x, 0)) |\det D_y F(x, 0)| \, dy + \int_{\mathbb{R}^n} e^{i\lambda Q(y)} r(\lambda; x, y) \, dy
\end{align*}
\]
where
\[
r(\lambda; x, y) = a(\lambda; x, F(x, y)) |\det D_y F(x, y)| - a(\lambda; x, F(x, 0)) |\det D_y F(x, 0)|.
\]
The first term evaluates to
\[
\left(\frac{\lambda}{2\pi}\right)^{-n/2} |\det \nabla_y^2 \phi(x, y(x))|^{-1/2} e^{|\pi i (p-q)/4|} a(\lambda; x, y(x))
\]
since
\[
\int_{-\infty}^{\infty} e^{i\lambda t^2/2} \, dt = \left(\frac{\lambda}{2\pi}\right)^{-1/2} e^{\pi i /4}
\]
and
\[
|\det \nabla_y^2 \phi(x, y(x))| = |\det D_y F(x, 0)|^{-2}.
\]
To estimate the second term, we let \( \chi \) be a smooth compactly supported cutoff function with \( \chi(|y|) = 1 \) for all \( y \in \text{supp}_y a \). Then,
\[
\int_{\mathbb{R}^n} e^{i\lambda Q(y)} r(\lambda; x, y) \, dy
\]
\[
= \int_{\mathbb{R}^n} e^{i\lambda Q(y)} \chi(|y|) r(\lambda; x, y) \, dy + \int_{\mathbb{R}^n} e^{i\lambda Q(y)} (1 - \chi(|y|)) r(\lambda; x, y) \, dy
\]
\[
= R_1(\lambda; x) + R_2(\lambda; x),
\]
respectively. Since \( r(\lambda; x, y) \) vanishes for \( y = 0 \),
\[
|\partial^j_x \partial^a_y R_1(\lambda; x)| \leq C_{j,a} \lambda^{-n/2-1-\delta}
\]
by [8] Lemma 1.1.6] applied to the \( x \)-derivatives of \( R_1 \). Finally,
\[
R_2(\lambda; x) = ce \int_{\mathbb{R}^n} e^{i\lambda Q(y)} (1 - \chi(|y|)) \, dy = \mathcal{O}(\lambda^{-N})
\]
by integration by parts.

References

[1] A. Banyaga and D. E. Hurtubise, *A proof of the Morse-Bott Lemma*, Expo. Math., Volume 22, Issue 4, 2004, 365-373.

[2] X. Chen and C. D. Sogge, *On integrals of eigenfunctions over geodesics*, Proceedings of the American Mathematical Society 143.1 (2015), 151-161.

[3] M. do Carmo, *Riemannian geometry*, Birkhäuser, Basel, Boston, Berlin, 1992.

[4] A. Good, *Local analysis of Selberg’s trace formula*, Lecture Notes in Mathematics, vol. 1040, Springer-Verlag, Berlin, 1983.

[5] D. A. Hejhal, *Sur certaines séries de Dirichlet associées aux géodésiques fermées d’une surface de Riemann compacte*, C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), no. 8, 273-276.

[6] A. Reznikov, *A uniform bound for geodesic periods of eigenfunctions on hyperbolic surfaces*, Forum Math. 27 (2015), no. 3, 1569-1590.

[7] R. Schoen and S. Yau, *Lectures on differential geometry*, International Press, 1994.

[8] C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics 105, Cambridge University Press, Cambridge, 1993.

[9] C. D. Sogge, *Hangzhou lectures on eigenfunctions of the Laplacian*, to appear in Annals of Math Studies, Princeton Univ. Press.

[10] C. D. Sogge, J. A. Toth, S. Zelditch, *About the blowup of quasimodes on Riemannian manifolds*, J. Geom. Anal. 21 (2011), no. 1, 150-173.

[11] C. D. Sogge, Y. Xi, C. Zhang, *Geodesic period integrals of eigenfunctions on Riemann surfaces and the Gauss Bonnet theorem*, [arXiv:1604.03189](https://arxiv.org/abs/1604.03189).

[12] C. D. Sogge and S. Zelditch, *On eigenfunction restriction estimates and $L^4$-bounds for compact surfaces with nonpositive curvature*, Advances in analysis: the legacy of Elias M. Stein, 447-461, Princeton Math. Ser., 50, Princeton Univ. Press, Princeton, NJ, 2014.

[13] C. D. Sogge and S. Zelditch, *Riemannian manifolds with maximal eigenfunction growth*, Duke Math. J. 114 (2002), no. 3, 387-437.

[14] E. L. Wyman, *Integrals of eigenfunctions over curves in surfaces of nonpositive curvature*, [arXiv:1709.03552](https://arxiv.org/abs/1709.03552).

[15] S. Zelditch, *Kuznecov sum formulae and Szegő limit formulae on manifolds*, Comm. Partial Differential Equations 17 (1992), no. 1-2, 221–260.

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