THE $\Lambda$CDM COSMOLOGY: FROM INFLATION TO DARK ENERGY THROUGH RUNNING $\Lambda$

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Abstract. Perhaps the deepest mystery of our accelerating Universe in expansion is the existence of a tiny and rigid cosmological constant, $\Lambda$. Its size is many orders of magnitude below the expected one in the standard model (SM) of particle physics. This is a very welcome fact, namely if we care at all about our own existence and fate. However, we do not have a minimally satisfactory explanation for our good fortune and for the failing of the SM at that crucial point. To start with, an expanding Universe is not expected to have a static vacuum energy density. We should rather observe a mildly dynamical behavior $\delta \Lambda(t) \sim R \sim H^2(t)$ with the expansion rate $H$. At the same time, it is natural to think that the huge value of the primeval vacuum energy (presumably connected to some grand unified theory) was responsible for the initial inflationary phase. In the traditional inflaton models such phase is inserted by hand in the early epoch of the cosmic evolution, and it is assumed to match the concordance $\Lambda$CDM regime during the radiation epoch. Here, instead, we consider a class of dynamical vacuum models which incorporate into a single vacuum structure $\Lambda(H)$ the rapid stage of inflation, followed by the radiation and cold matter epochs, until achieving our dark energy Universe. The early behavior of such “running vacuum model” ($\Lambda$CDM) bares resemblance with Starobinsky’s inflation in the early Universe and is very close to the concordance model for the entire post-inflationary history. Most remarkably, the inflationary period in the $\Lambda$CDM terminates with “graceful exit” and the large entropy problem can be solved. The model is compatible with the latest cosmological data on Hubble expansion and structure formation, and at the same time presents distinctive observational features that can be tested in the near future.
1 Introduction

When Einstein first introduced the cosmological term, $\Lambda$, in the gravitational field equations, it was assumed to be constant and positive [1]. Its value was naturally fixed to be of the order of the critical density. The reasons why Einstein introduced that term are well-known [1], and at present are of mere historical interest. However, there is one thing that remains intact: its tiny value and the belief (at least within the standard lore) that $\Lambda$ is a true constant of Nature. Certainly we cannot disagree about its value since it has apparently been measured [3, 4]. It roughly remains of the same order of magnitude as in the time of Einstein, namely it is of order of the critical density. Later on a nonvanishing $\Lambda$ was used to try to cure various astronomical problems, until it became clear that its theoretical status was quite precarious since it was realized that it was far too small by any standards in particle physic units. This was the origin of the so-called “cosmological constant (CC) problem” [2, 5]. At this juncture people felt it was time to desperately find a theoretical explanation for $\Lambda$ being very small, most likely zero. It was thought that some symmetry would do nicely the job. Supersymmetry [6] (SUSY), for instance, has been frequently cited as of the time when Wess and Zumino explored this theoretical possibility – more than forty years ago [7].

But nowadays we know that the CC term, $\Lambda$, is nonzero and that SUSY (and for that matter any other extension of the standard model (SM) of particle physics) is virtually impotent to explain the CC problem in any less troublesome way than the SM itself. Both the SM and the MSSM [8] (the minimal supersymmetric version of the SM) have essentially the same acute CC problem. This is of course very disturbing, specially for the health of our cherished SM of particle physics, a model whose theoretical status became lately significantly augmented after the proclaimed discovery of the Higgs boson [9], the last missing piece to “crown” the particle physics puzzle. The bare truth is that despite the tremendous effort and final triumph of the Higgs boson finding [10], the “excessive success” of the SM makes it “so simple yet so unnatural” [11]. Without extra physics the Higgs boson (and hence the entire backbone structure of the SM) stays radiatively unprotected from the physics of the high scales, namely those extremely large scales associated to the grand unified theories (GUT’s), which – we should perhaps recall at this point – are not just there to provide aesthetical ideas for the unification of the gauge couplings, but are also badly needed to explain the essential facts of inflation and baryogenesis, for example.

Quite surprisingly, it is usually said that there is no trace of physics beyond the SM. However this is not quite true: in fact, for all its particle physic success the SM is in manifest disagreement with the most basic cosmological observations. If we do not find at the moment any obvious disagreement at the microphysical level (through e.g. $5\sigma$ deviations, or more, in some particle physics observable) does not necessarily mean that the SM is a truly watertight theory, since it makes a catastrophic prediction as to the value of the vacuum energy and hence of the cosmological constant (see [2] for a more vivid account), let alone its complete inability to explain or to hint at the nature of the dark matter. The upshot is that some fundamental aspects of the SM of particle physics turn out to be in blunt disagreement with the standard (or “concordance”) model of cosmology – the so-called $\Lambda$CDM model. We could say that by finding the Higgs boson (or even “a Higgs boson”, of whatever nature) we have certified experimentally the reality of the CC problem. The CC problem, therefore, is no longer a formal conundrum!

It is thus more than plausible to say that there is, or there must be, physics beyond the SM, and also beyond the $\Lambda$CDM! What is not obvious at all, at least to us, is where it lies the new physics that should rescue the two standard models, one of particle physics and the other of cosmology, from wreckage. We believe at least on one thing: wherever it hides and whatever it be the new physics, it should provide a sober and sound explanation for the nature of the vacuum energy and its connection with the value of the cosmological constant. And we also believe that

1See e.g. Ref. [2] for a recent review.
this explanation should involve a dynamical cosmological term, namely one capable of tracing the history of the Universe from its origin till the present days in a single unified framework.

While the traditional “explanation” is the existence of a nonvanishing and positive cosmological constant, \( \Lambda \), whose energy density equivalent, \( \rho_\Lambda = \Lambda / 8\pi G \), is of order of the critical density, this cannot be a truly convincing explanation, as in fact an expanding Universe is not expected to have a static vacuum \[2\]. A smoothly evolving vacuum energy density \( \rho_\Lambda(t) = \rho_\Lambda(H(t)) \) that borrows its time-dependence from the Hubble rate \( H = H(t) \) — taken as the natural dynamical variable in the Friedmann-Lemaître-Robertson-Walker (FLRW) background — is not only a qualitatively more plausible and intuitive idea, but is also suggested by fundamental physics, in particular by quantum field theory (QFT) in curved space-time \[12\]. This more formal point of view, based on the renormalization group approach, in which \( \Lambda = \Lambda(H) \) is a running quantity with the cosmic expansion, has been introduced in \[13\] \[14\] \[15\] \[16\] \[17\] \[18\] \[19\], and recently emphasized in \[2\] \[20\] \[21\] \[22\] Furthermore, some recent applications of these ideas for a possible description of the complete cosmic history have been put forward in the literature \[23\]. These vacuum models represent a conceptually new approach that goes beyond the first phenomenological approaches on time evolving cosmological constant, cf. \[24\] \[25\] \[26\] \[27\] \[28\] \[29\]. For alternative formulations and recent developments along these lines, see \[30\] \[31\] \[32\] and references therein. In general, the notion of a variable vacuum energy has also been entertained in the literature from different and interesting points of view, also from the historical and philosophical perspective \[33\] \[34\] \[35\].

In the last years, and well within the tradition of the old Dirac’s ideas, an independent source of puzzling news has generated also a lot of interest. Frequent hints that the electromagnetic fine structure constant \( \alpha_{em} \) and/or the proton mass might be changing with the cosmic time (and locally in space) are reported in the literature \[36\] \[37\] – for reviews, see e.g. \[38\] \[39\]. Theoretical models already exist in the literature trying to explain such phenomena, see e.g. \[40\]. It is tantalizing to conceive that, if \( \alpha_{em} \) can evolve with the cosmic expansion, all of the fundamental “constants” should change in time as well, including the gravity coupling and the masses of all the elementary particles. Recently, these ideas have been linked to a possible time variation of the vacuum energy as well \[41\] \[42\] \[43\], and this opens a line of thought well in the context of the dynamical vacuum energy in an expanding Universe \[2\]. Since the Newtonian coupling, \( G_N \), determines the Planck mass \( M_P = G_N^{-1/2} \), a possible time variation of it would be tantamount to say that \( M_P \) slowly evolves with the cosmic expansion \[42\].

In this presentation of the “running \( \Lambda \)CDM model” (in fact a class of models) we show that the idea of a dynamical vacuum in an expanding Universe can be the sought-for touchstone enabling a substantially improved account of the entire cosmological history as compared to the concordance \( \Lambda \)CDM with rigid \( \Lambda \) term. The content is as follows: In sections 2-3 we summarize the theoretical framework for dynamical vacuum models. In sections 4-5 we discuss inflation in the context of the \( \Lambda \)CDM model and address some important thermodynamical aspects of it, in particular the large entropy problem. Subsequently, in Sect. 6, we compare the early cosmological behavior of the running \( \Lambda \)CDM model with Starobinsky inflation in alternative formulations. The low-energy implications of the \( \Lambda \)CDM class are explored in Sect. 7, where we confront these models with the most recent cosmological data from distant supernovae (SNIa), the cosmic microwave background (CMB) anisotropies, the baryonic acoustic oscillations (BAOs) and the input from structure formation. We show that the notion of a running vacuum can be, in principle, compatible with the current observations, and suggest that traces of their dynamics should be testable in the near future. In the final section we provide some discussion and our conclusions.

\[2\]Recent comprehensive analyses successfully testing a wide class of dynamical vacuum models in the light of the recent observations are also available, see \[22\] \[23\].
2 Dynamical vacuum in an expanding Universe

The idea that in an expanding Universe the cosmological term $\Lambda$ and Newton’s constant $G_N$ could be variable with time is not new and in fact it can be viewed as reasonable and even natural. Dirac’s ideas in the thirties [44, 45] on the so-called “large number hypothesis” were seminal concerning the possible time evolution of the gravitational “constant” $G_N$. They were disputed by E. Teller [46] and further qualified by R.H. Dicke [47]. It also triggered subsequent speculations by G. Gamow [48] on the possible variation of the fine structure constant. Since then the subject has been in continuous evolution, and more and more sophisticated experiments are being designed to monitor the possible time (and space) variation of the fundamental constants – see e.g. [38, 39] for reviews.

While in the old days the cosmological term, $\Lambda$, in Einstein’s equations may not have attracted a lot of attention regarding to its potential time variability (setting aside occasional episodes where some astrophysical observations had suggested this possibility), it is natural to entertain this option in earnest when we cope with the full cosmological context. This is especially so if we take into account that $\Lambda$ defines the energy density parameter $\rho_\Lambda = \Lambda/(8\pi G_N)$, which is interpreted as the vacuum energy density of the expanding Universe $^3$. It should perhaps be surprising if an accelerating Universe were to carry a static vacuum energy density throughout the entire cosmic history [2]. A more natural possibility, which is perfectly compatible with the Cosmological Principle, is that $\Lambda = \Lambda(t)$ and hence $\rho_\Lambda = \rho_\Lambda(t)$.

From that point of view, it is instructive to consider the possible modifications that may undergo the basic conservation laws if one makes allowance for the time variability of the fundamental gravitational parameters $G_N$ and $\Lambda$. The Bianchi identity satisfied by the Einstein tensor on the l.h.s. of Einstein’s equations reads $\nabla^\mu G_{\mu\nu} = 0$, where $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$. It follows that the covariant derivative of the r.h.s. of Einstein’s equations must be zero as well: $\nabla^\mu \left( G_{N} \tilde{T}_{\mu\nu} \right) = 0$, where $\tilde{T}_{\mu\nu} \equiv T_{\mu\nu} + g_{\mu\nu} \rho_\Lambda$ is the full energy-momentum tensor of the cosmic fluid composed of matter and vacuum. Using the explicit form of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, the generalized conservation law emerging from this dynamical framework reads

$$\frac{d}{dt} \left[ G_N (\rho_m + \rho_\Lambda) \right] + 3 G_N H (1 + \omega_m) \rho_m + 3 G_N H (1 + \omega_\Lambda) \rho_\Lambda = 0, \quad (1)$$

where $\omega_m = p_m/\rho_m$ is the equation of state (EoS) for matter and $\omega_\Lambda = p_\Lambda/\rho_\Lambda$ is the EoS for the vacuum. The Hubble rate $H$ dynamics ensues directly from Einstein’s equations. We generalize them for the case of dynamical vacuum (and restrict to the flat FLRW metric):

$$3H^2 = 8\pi G (\rho_m + \rho_\Lambda) \quad (2)$$
$$2\dot{H} + 3H^2 = -8\pi G (\omega_m \rho_m + \omega_\Lambda \rho_\Lambda). \quad (3)$$

The overdot denotes derivative with respect to cosmic time $t$. One can easily check that only two of the equations (1)-(3) are independent. For example, substituting (2) in (1) we arrive at

$$\dot{H} + 4\pi G_N (1 + \omega_m) \rho_m + 4\pi G_N (1 + \omega_\Lambda) \rho_\Lambda = 0. \quad (4)$$

One can check that this equation can also be obtained by combining (2) and (3).

$^3$Ten years later after $\Lambda$ was introduced by Einstein [1] to insure a static non-evolving Universe, G. Lemaître [49] introduced a nonvanishing $\Lambda$ to discuss his dynamical models of the expansion of the Universe, strongly motivated by E. Hubble’s observations prior to their publication in 1929 [50]. A few years later, in 1934, Lemaître discussed for the first time [51] the interpretation of the CC term as vacuum energy and its associated negative pressure – see e.g. [52, 53] for further historical discussions.
Let us note that the previous formulae remain valid if we sum over all cosmic components (matter and vacuum). However, in the particular but rather common situation where there is a dominant matter component (e.g. cold matter or relativistic matter), it is possible to obtain the evolution law for the Hubble function solely in terms of the vacuum term and that dominant material fluid. A simple calculation from the above equations leads to

$$\dot{H} + \frac{3}{2} (1 + \omega_m) H^2 = 4\pi G (\omega_m - \omega_\Lambda) \rho_\Lambda.$$  \hspace{1cm} (5)

As advertized, in this equation we cannot sum over the matter components, as \(\omega_m\) in it stands for the EoS of the dominant one. This equation will be useful in the next sections.

Up to this point we have assumed that the EoS of the vacuum component is completely general. While the previous equations account for the general situation (applicable even if \(\omega_\Lambda\) would correspond to a general DE fluid), we shall henceforth adopt the simplest scenario corresponding to the vacuum state, i.e. \(\omega_\Lambda = -1\), as being a characteristic feature of the EoS of vacuum even for time-evolving \(\rho_\Lambda = \rho_\Lambda(t)\).

In view of the above considerations, we next consider the following cosmological scenarios beyond the concordance model:

**Scenario I**: \(\rho_\Lambda = \rho_\Lambda(t)\) is assumed variable, and \(G_N = \text{const.}\). In this case, Eq. (1) implies

$$\dot{\rho}_m + 3(1 + \omega_m) H \rho_m = -\dot{\rho}_\Lambda.$$  \hspace{1cm} (6)

Since we now have \(\dot{\rho}_\Lambda \neq 0\) it means we permit some energy exchange between matter and vacuum, e.g. through vacuum decay into matter, or vice versa. Obviously if \(\dot{\rho}_\Lambda = 0\) we recover \(\dot{\rho}_m + 3 H (1 + \omega_m) \rho_m = 0\), i.e. the standard covariant matter conservation law. Its solution in terms of the scale factor is well-known:

$$\rho_m(a) = \rho_m^0 a^{-3(1+\omega_m)}.$$  \hspace{1cm} (7)

**Scenario II**: \(\rho_\Lambda = \rho_\Lambda(t)\) is again variable, but \(G_N = G_N(t)\) is also variable. In contrast to the previous case, here we assume matter conservation in the standard form (7). As a result the following conservation law ensues:

$$(\rho_m + \rho_\Lambda) \dot{G}_N + G_N \dot{\rho}_\Lambda = 0.$$  \hspace{1cm} (8)

In this setting the evolution of the vacuum energy density is possible at the expense of a running gravitational coupling: \(\dot{G} \neq 0\).

**Scenario III**: Here we keep \(\rho_\Lambda = \text{const.}\), but \(G_N = G_N(t)\) is again variable. Now we find:

$$\dot{G}_N(\rho_m + \rho_\Lambda) + G_N[\dot{\rho}_m + 3H(1 + \omega_m)\rho_m] = 0.$$  \hspace{1cm} (9)

In this case matter is again non-conserved and the gravitational coupling is running. Despite the vacuum energy is constant in this scenario, such situation can mimic a form of dynamical dark energy since it implies a different expansion rate \[53\].

The above three generalized cosmological scenarios differ from the concordance ΛCDM model, but can stay sufficiently close to it if we consider the recent history of our Universe. Let us finally note that the above dynamical vacuum models can be extended at high energies for a successful explanation of the inflationary Universe \[24\].

### 3 ΛCDM: a vacuum model for the complete cosmic history

Thus far we have sketched some feasible frameworks for the time evolution of the vacuum energy and the gravitational coupling. Let us however note that with the generalized conservation law (6)

and Friedman’s equation (2) is not possible to determine the solution of the cosmological equations. We need some information on the evolution laws \( \rho_A = \rho_A(t) \) or \( G = G(t) \). Such information has been provided in the past based on some phenomenological ansatz (see e.g. [27] [25] [26] [28] and references therein). However, here we wish to focus on a class of models motivated by the theoretical framework of QFT in curved spacetime, see [14] [15] [16] [17] [18] [19] [20] [21] – cf. also the review [2] and the long list of references therein.

Recall that in particle physics we have theories such as QED or QCD where the corresponding gauge coupling constants \( g_i \) run with an energy scale \( \mu \), i.e. \( g_i = g_i(\mu) \). The scale \( \mu \) is usually associated to the typical energy of the process. Following the same line of thought we can think of \( \rho_A \) as an effective coupling sensitive to the quantum effects and thereby running with an energy scale \( \mu \) representative of the cosmological evolution. In the context of QFT in a curved background, where gravity is a classical external field, the quantum effects that shift the value of \( \rho_A \) are exclusively produced by the loops of matter fields [12]. Naturally \( \mu \) should be fixed on physical grounds, but it is quite reasonable to assume that the running of \( \rho_A \) should be associated with the change of the spacetime curvature because if there is no change in the background geometry there is no cosmological evolution at all. In the FLRW metric, this means that \( \mu \) should be related with \( H \) and its time derivatives. We denote this association in the simplified way

\[ \mu \sim H, \dot{H}, \ddot{H}, H^2, \dot{H}^2, \ddot{H}^2 \text{ etc.} \]

In this expression, \( M_i \) are the masses of the particles contributing in the loops, and \( B_i, C_i, \ldots \) are dimensionless parameters. The RG equation (10) provides the rate of change of the quantum effects on the CC as a function of the scale \( \mu \). Provided we are interested only on the dynamics of the current Universe, we may cut off the series at the quadratic contributions, i.e. only the “soft-decoupling” terms of the form \( \sim M_i^2 \mu^2 \) will be of significance. Notice that the \( M_i^4 \) terms are absent, as they would trigger a too fast running of the CC term. As a matter of fact these effects are ruled out by the RG formulation itself since only the fields satisfying \( \mu > M_i \) are to be included as active degrees of freedom. Being \( \mu = \mathcal{O}(H) \) (as indicated above) it is obvious that such condition cannot be currently satisfied by the SM particles – see, however [13]. The leading effects on the running of \( \rho_A \) are, according to Eq. (10), of order \( M_i^4 \mu^2 \sim M_i^2 H^2 \), and hence it is dominated by the heaviest fields. In the context of a typical GUT near the Planck scale, these are the fields with masses \( M_i \sim M_X \lesssim M_P \). For instance, in Ref. [18] a specific scenario is described which is connected to the effective action of QFT in curved spacetime.

Let us recall that because of the general covariance of the effective action, among the list of possible terms emerging from the quantum effects one expects only terms carrying an even number of time derivatives of the scale factor. If expressed in terms of the Hubble rate (which is the most convenient quantity to parameterize the extra contributions), it amounts to terms of the form \( H^2, \dot{H}, H^4, \dot{H}^2, H^2 \dot{H} \) etc. In contrast, the linear terms in \( H \) (and in general any term with an odd number of derivatives of the scale factor, such as \( H^3, \dot{H} H, \ddot{H} \) etc) are not expected since they would be incompatible with the general covariance of the effective action [14] [18] [19] [20] [21]. Let us remark that at low energies only the \( H^2 \) and \( \dot{H} \) terms are relevant. The higher order ones can however be important for the early Universe [24] [51] [55] [59] [67].

As we have agreed, \( \mu \sim H \) is the natural association of the RG-scale in cosmology. However, a more general option is to associate \( \mu^2 \) to a linear combination of \( H^2 \) and \( \dot{H} \) (both terms being dimensionally homogeneous). Adopting this setting and integrating (10) up to the terms of \( \mathcal{O}(\mu^4) \)
it is easy to see that we can express the result as follows:

\[
\rho_{\Lambda}(H, \dot{H}) = a_0 + a_1 \dot{H} + a_2 H^2 + a_3 \dot{H}^2 + a_4 H^4 + a_5 \dot{H} H^2,
\]  

(11)

where the coefficients \(a_i\) have different dimensionalities in natural units. Specifically, \(a_0\) has dimension 4 since this is the dimension of \(\rho_{\Lambda}\); \(a_1\) and \(a_2\) have dimension 2; and, finally, \(a_3, a_4\) and \(a_5\) are dimensionless. The “running vacuum Universe” (\(\Lambda\)CDM) is the extension of the \(\Lambda\)CDM model based on a dynamical vacuum energy density of the form (11), stemming from the basic RG equation (10). While higher order term are still possible, that expression contains the basic terms up to four derivatives of the scale factor, and hence encodes the basic potential of the model both for the low and the high energy Universe.

Let us now consider a particularly simple and illustrative case of the running \(\Lambda\)CDM Universe. Suppose that rather than associating \(\mu^2\) with a linear combination of \(H^2\) and \(\dot{H}\) we would just set \(\mu^2 = H^2\) (in this case the linear combination reduces to just one term and we can just adopt the canonical choice \(\mu = H\).). In this situation we have \(a_1 = a_3 = a_5 = 0\) in (11). The remaining coefficients can be related immediately to those in (10), and one can show that the final result can be cast as follows:

\[
\rho_{\Lambda}(H) = \frac{3}{8\pi G_N} \left( c_0 + \nu H^2 + \frac{H^4}{H_I^2} \right),
\]  

(12)

where \(c_0\) has dimension 2 and we have introduced the dimensionless coefficient \(\nu\) and the dimensionful one \(H_I\). Comparing with (10) it is easy to see that

\[
\nu = \frac{1}{6\pi} \sum_{i=f,b} B_i \frac{M_i^2}{M_P^2}.
\]  

(13)

The dimensionful coefficient \(H_I\) absorbs any other dimensionless factor, but it is unnecessary to further specify its structure since it represents a physical quantity (connected with the mechanism of inflation) and can be determined (or at least bounded) by observations, as we shall see in a moment. But let us first discuss the interpretation of the dimensionless coefficient \(\nu\). The sum in (13) involves both fermions and bosons. Coefficient \(\nu\) plays the role of the \(\beta\)-function coefficient within the structure of the effective action in QFT in curved spacetime. This is confirmed by the fact that \(\nu\) depends on the ratio squared of the masses of the matter particles to the Planck mass, which is indeed the expected result in particular realizations of the RG in curved spacetime – see e.g. [18]. As we shall see, \(H_I\) stands (to within very good approximation) for the Hubble parameter in the inflationary epoch, which is the only epoch where the higher order term \(H^4\) can be of relevance. During the inflationary period \(H =\)const. \((\dot{H} = 0)\), so at least in the pure inflationary stage the terms we have dropped from Eq. (11) should not be determinant. These terms, however, can be important for the different modalities of reheating, just after inflation.

Obviously the dynamical vacuum model (12) aims at providing an unified description of the entire cosmic history, valid from inflation to the present days. We will confirm if this is the case in the subsequent sections. For the current Universe it is enough to consider (12) up to the quadratic term, or type-A1 vacuum model:

\[
\rho_{\Lambda}(H) = \frac{3}{8\pi G_N} \left( c_0 + \nu H^2 \right) \quad \text{(Type A1)}.
\]  

(14)

Note that if \(\rho_\Lambda^0\) and \(H_0\) are the current values of the vacuum energy density and the Hubble parameter, then

\[
c_0 = \frac{8\pi G_N}{3} \rho_\Lambda^0 - \nu H_0^2.
\]  

(15)
Taking into account that at low energy the terms $\sim \dot{H}$ can be equally important as the $\sim H^2$ ones, one may also consider the slightly extended model variant

$$\rho_{\Lambda}(H) = \frac{3}{8\pi G_N} \left( c_0 + \dot{\nu} \dot{H} + \nu H^2 \right) \quad \text{(Type A2)},$$

(16)

where $\dot{\nu}$ is a new dimensionless coefficient; it stems from the original (dimensionful) $a_1$ in Eq. (10) after a convenient redefinition. Although we use the same symbol $c_0$, it is understood that the connection of $c_0$ in (16) with the current cosmological parameters is not exactly the same as in (15) but it can be obtained easily.

We will also introduce briefly two more vacuum types (B1 and B2) for comparison. These are defined as follows:

$$\rho_{\Lambda}(H) = \frac{3}{8\pi G_N} \left( c_0 + \epsilon H_0 H \right) \quad \text{(Type B1)}$$

(17)

$$\rho_{\Lambda}(H) = \frac{3}{8\pi G_N} \left( c_0 + \epsilon H_0 H + \nu H^2 \right) \quad \text{(Type B2)},$$

(18)

where $\epsilon$ is a new dimensionless coefficient and $H_0$ is the current value of the Hubble parameter.

The dynamical vacuum models could provide an alternative explanation for the dynamical DE in the Universe within the context of QFT in curved spacetime, thereby representing an alternative to quintessence and other exotic DE options. In the case of B1 and B2, however, the presence of the linear term $\sim H$ (rather than $\sim \dot{H}$) is more phenomenological because we do not expect linear terms in $H$ in the effective action for the reasons explained above. Still these terms can be admitted (they could represent bulk viscosity effects [58], for instance) provided $c_0 \neq 0$ and/or the standard terms (with an even number of derivatives of the scale factor) are also present. In the case $c_0 = 0$ the above models do not have a well-defined $\Lambda$CDM limit (i.e. none of them has in this case a behavior near that of the concordance model), and one can show that this is problematic [59]. Recent analyses have confronted these models to observations [22, 23] – see also [15, 17, 60, 61, 62, 63] for previous studies on a variety of possible scenarios. In Sect. 7 we provide a summarized presentation of the main results for the most recent analyses.

4 Inflation in the $\Lambda$CDM model and Grand Unified Theories

Let us now concentrate on the high energy implications of the unified dynamical vacuum model. To simplify our discussion we will adopt the canonical form (12) since it already contains the main features. The Hubble function for the dynamical vacuum models under consideration can be derived from solving the basic equations of Sect. 2. However, these equations depend on the particular scenario (I, II and III) we choose. For the present study we will focus on Scenario I of Sect. 2 and hence on Eq. (6).

4.1 Solving the model in the early Universe

Assuming that there is a dominant matter component in the early Universe (typically radiation, $\omega_m = 1/3$), we may combine equations (5) and (12) and we arrive at the following differential equation for $H$:

$$\dot{H} + \frac{3}{2}(1 + \omega)H^2 \left( 1 - \nu - \frac{c_0}{H^2} - \left( \frac{H}{H_I} \right)^2 \right) = 0.$$  

(19)

The analysis of the modifications introduced at high energies by the more general structure (11) will be presented elsewhere.
Now, if Eq. (14) is to describe an approximate CC term in the late Universe, the constant $c_0$ must be of the order of the current value of the CC, or $\Lambda_0 \simeq 3c_0$ to be more precise, and at the same time the term $\sim \nu H^2$ can only represent a small (dynamical) departure from it, thus $|\nu| \ll 1$. This is not only what we expect, but in fact what we find when we compare the low-energy models (14) and (16) with the precision observational data collected from the cosmological observations, we find indeed $|\nu|, |\bar{\nu}| \lesssim 10^{-3}$ (cf. Sect. 7).

In the early Universe, when $H$ is large, we can dismiss the term $c_0/H^2 \ll 1$ in Eq. (19) to a very good approximation. This implies the existence of a constant solution in that epoch, namely

$$H = \sqrt{1 - \nu} H_I \simeq H_I ,$$

which corresponds to a de Sitter phase driven by a huge value of the Hubble parameter $H_I$. We will corroborate that $H_I$ is the characteristic value of the Hubble function of the high energy phase of the early Universe, i.e. the effective value of $H$ in the inflationary period, from our analysis of the energy densities below. In what follows we will neglect $\nu$ in all practical considerations concerning the early Universe. Notice that when $H \simeq H_I$, the $\sim H^2$ term in Eq.(12) is suppressed with respect to the $\sim H^4$ by precisely a factor $|\nu| \ll 1$. In the current Universe, where $H = H_0 \sim 10^{-42}$ GeV (the present value in natural units), the $\sim H^4$ term in Eq. (12) is suppressed with respect to the $\sim H^2$ one by more than 100 orders of magnitude! – cf. Eq. (24) below.

The value of the inflationary scale $H_I$ can be estimated from the anisotropies of the cosmic microwave background as follows. According to the CMB observations, the (primordial) spectrum $P_\zeta$ of the curvature perturbation [64] at typical “pivot” length scales $k_0^{-1}$ (of a few tens to hundreds of Mpc) is $P_{\zeta}^{1/2} \simeq 5 \times 10^{-5}$. On the other hand, the theoretical calculation of $P_\zeta$ is usually performed in the context of the inflaton model with effective potential $V_{\text{eff}}$, and yields

$$P_\zeta(k_0) = \frac{8}{3 M_P^2} \langle V_{\text{eff}} \rangle \bigg|_{k_0} ,$$

where $\langle V_{\text{eff}} \rangle$ is the vacuum expectation value of the potential and $\epsilon$ is the standard slow roll parameter [64]. Recall that $\epsilon$ is related to the tensor-to-scalar fluctuation ratio $r \equiv n_T/n_s$ through $r = 16\epsilon$. Furthermore, the value of the potential during inflation is related to the GUT scale $M_X$ through $\langle V_{\text{eff}} \rangle \sim M_X^4$. For definiteness and simplicity we take them equal. Obviously this also defines the critical density at the GUT epoch, $\rho_I = M_X^4$, which is related to $H_I$ by means of

$$\rho_I = \frac{3H_I^2}{8\pi G_N} = M_X^4 .$$

From the previous equations we immediately find:

$$M_X = \langle V_{\text{eff}} \rangle^{1/4} = \left( \frac{3r}{128} \right)^{1/4} M_P P_{\zeta}^{1/4} = \left( \frac{r}{0.2} \right)^{1/4} 2.2 \times 10^{16} \text{ GeV} ,$$

and

$$H_I = \sqrt{\frac{8\pi}{3} \frac{M_X^2}{M_P}} \simeq \left( \frac{r}{0.2} \right)^{1/2} 1.1 \times 10^{14} \text{ GeV} .$$

Notice that we have normalized the tensor-to-scalar ratio to the approximate value $r \simeq 0.2$ furnished by the BICEP2 collaboration [65] – with the understanding that we are waiting for a future confirmation/reanalysis of this important experimental result. Notwithstanding, it is pretty clear that owing to the fourth and squared roots involved of the factor $r/0.2$ in the previous equations, the final estimates on $M_X$ and $H_I$ are not critically sensitive to the precise value of $r$. 


It is remarkable that the obtained value for $M_X$ in (23) lies in the ballpark of $\sim 10^{16}$ GeV, which is the expected order of magnitude in all viable GUT’s compatible with the current limits on proton decay. At the same time, the value of $H_I$ obtained from (24) is seen to satisfy

$$\frac{H_I}{M_P} < 10^{-5},$$

which is the condition that the fluctuations from the tensor modes do not induce CMB temperature anisotropies larger than the observed ones. Indeed, the vacuum fluctuations become classical a few Hubble times after horizon exit, and have a spectrum $\left(\frac{H_I}{2\pi}\right)^2$ determined by the Gibbons-Hawking temperature $T_{GH} = \frac{H_I}{2\pi}$. To be more precise, the primordial tensor perturbation $h_{ij}$ has the spectrum

$$P_h(k_0) = \frac{64\pi}{M_P^2} \left(\frac{H_I}{2\pi}\right)^2 = \frac{16}{\pi} \left(\frac{H_I}{M_P}\right)^2 |_{k_0}.$$  

Therefore, by requiring that $P_h^{1/2} < 5 \times 10^{-5}$ we retrieve the bound (25) within the correct order of magnitude from $P_h$ alone.

Of course the previous considerations are just an estimate inasmuch as they are partially based on the standard inflaton picture, which is not exactly the one we are subscribing here. They nevertheless lead to a reasonable estimate, for we could have just reversed the sense of the argument and started from Eq. (26) rather than from (21). Following this logic, we could have next applied the bound (25) – imposed by the measured anisotropies of the CMB – and derived an upper limit on the inflationary value of the Hubble parameter directly from (26), which is under the constraint $P_h^{1/2} < 5 \times 10^{-5}$, and then immediately derived the corresponding GUT scale (23). The advantage of this way is that we know that the general form of the tensor spectrum, $P_h \sim \left(\frac{H}{M_P}\right)^2$, is only sensitive to the absolute value of the Hubble parameter squared – measured in Planck units – at the inflation phase, a fairly general fact that holds good irrespective of the underlying details of the inflation dynamics. In this sense, the previous considerations are essentially safe in so far as Eq. (25) is compatible with them.

Far away from the inflationary period (i.e. $H \ll H_I$) we find another de Sitter solution of Eq. (19), namely $H = \left[c_0/(1 - \nu)^2\right]^{1/2}$, whereby $\Lambda \approx 3c_0 \sim \Lambda_0$ (recall that $|\nu| \ll 1$). This is of course the solution we have mentioned before, which leads to the late time (approximate) cosmological constant behavior, i.e. the current DE epoch. Let us emphasize that the pure de Sitter phase of this late time solution is achieved only in the remote future, but the fact that $\nu$ is small but not exactly zero is responsible for a slow vacuum dynamics $\sim H^2$ – cf. e.g. Eq. (14) – in our recent past (still going on at present) which could mimic a dynamical DE scenario. This is precisely the situation when $\nu \neq 0$ matters, and it can affect the present observations. We will address the details later (see Sect. 7), but first we further analyze the inflationary phase.

4.2 Achieving “graceful exit” in the $\Lambda$CDM model

Let us now elucidate the cosmology of the $\Lambda$CDM model in the early de Sitter epoch. We can virtually set $c_0, \nu \to 0$ and assume $\omega = 1/3$ since the subsequent matter epoch is presumably relativistic. It is convenient to trade the cosmic time for the scale factor through $d/dt = aHd/da$. Eq. (19) takes on the form

$$H'(a) + 2aH(a) \left[1 - \frac{H^2}{H_I^2}\right] = 0,$$

where prime denotes differentiation with respect to the scale factor. The solution of the previous equation renders

$$H(a) = \frac{H_I}{\left[1 + Da^4\right]^{1/2}}.$$
Substituting the above result in Eq. (12) we get the explicit form of the vacuum energy density in terms of the scale factor:

$$\rho_\Lambda(a) = \rho_I \frac{1}{[1 + D a^4]^2}. \quad (29)$$

The matter energy density then ensues after inserting the two previous results in Friedmann’s Eq. (2). We obtain:

$$\rho_r(a) = \rho_I D a^4 \frac{[1 + D a^4]^2}{[1 + D a^4]}. \quad (30)$$

In the previous formulas $\rho_I$ is the same quantity defined in (22). We confirm now that it represents the primeval energy density for $a \to 0$, i.e. in the inflationary period, as it is obvious from (29). Furthermore, since the (relativistic) matter energy density $\rho_r \to 0$ as $a \to 0$, it follows that ultimately $\rho_I$ is “pure vacuum energy” stored in the very early Universe. As shown by the above equations, in the course of the fast evolution the vacuum energy transforms into matter, so the Universe becomes eventually dominated by the material component. The maximum density of radiation achieved is $\rho_r^{\text{max}} = \rho_I/4$. Beyond that point the radiation density starts to decrease and at the same time the inflationary period (which is powered only by the vacuum energy) comes naturally to an end. This is tantamount to saying that the $\Lambda$CDM model incorporates natural inflation with “graceful exit”, as we shall further discuss below.

In the initial period, $D a^4 \ll 1$, the solution (28) can be approximated by the constant value (20), i.e. $H \approx H_I$, the vacuum energy density remains essentially constant and the scale factor increases exponentially. This fact can be verified from (28) by integrating from some initial (unspecified) scale factor up to a value $a(t)$ well within the inflationary period, in which $D a^4 \ll 1$ still holds good. The result is just the inflationary solution

$$a(t) = a_i \exp \{H_I t\}. \quad (31)$$

The meaning of $t$ in this equation is the elapsed time within the inflationary period, and $a_i$ is the purported value of the scale factor at the time when inflation started (defined to be at $t = 0$), hence $a(0) = a_i$. What happened before that instant of time, we don’t care. We will probably never know since the pre-inflationary Universe is not accessible.

Once the inflationary phase is left behind, the term $D a^4$ can be comparable to 1 or much greater. Here the integration of Eq. (28) must be done with the full expression, and we find

$$\int_{a_*}^{a} \frac{d\tilde{a}}{\tilde{a}} [1 + D \tilde{a}^4]^{1/2} = H_I t. \quad (32)$$

In this case $t$ is (at variance with the previous case) the time elapsed after approximately the end of the inflationary period, indicated by $t_*$, and we have defined $a_* = a(t_*)$. The integration constant $D$ is fixed from the condition $H(a_*) \equiv H_*$, and it satisfies $Da_*^4 = (H_I/H_*)^2 - 1$.

We can now show that the “graceful exit” from the inflationary phase can be successfully accommodated in this class of models [24]. In fact, considering the limit $D a^4 \gg 1$ of equation (32) we find

$$a \sim t^{1/2}, \quad (33)$$

which confirms our contention that the vacuum phase decays into a radiation-dominated Universe. We can reach the same conclusion from the analysis of the energy densities in the transition period. In effect, from (29) and (30) we see that in the limit $D a^4 \gg 1$ the decay law of matter is the characteristic one of radiation density, $\rho_r \sim a^{-4}$, whereas the vacuum energy density decays much faster, $\rho_\Lambda \sim a^{-8}$, so it becomes suppressed as $\rho_\Lambda/\rho_r \sim a^{-4}$ once the radiation regime has been settled. Such suppression is welcome, of course, as the success of BBN (Big Bang Nucleosynthesis)
Figure 1: The transition from the inflationary period of the running vacuum model (12) into the FLRW radiation epoch. The vacuum and matter energy densities are normalized with respect to primeval density \( \rho_I \), and the scale factor is normalized with respect to the transition one \( a_* \) (see the text). The figure shows an initial (and constant) dominance of the vacuum phase, and its subsequent decay into relativistic particles (whose energy density rapidly increases) until the Universe is dominated by radiation and the vacuum energy becomes negligible. As a result inflation ends with “gracefully exits” into the standard radiation phase. We have used \( 8\pi G = 1 \) units.

Figure 1: The transition from the inflationary period of the running vacuum model (12) into the FLRW radiation epoch. The vacuum and matter energy densities are normalized with respect to primeval density \( \rho_I \), and the scale factor is normalized with respect to the transition one \( a_* \) (see the text). The figure shows an initial (and constant) dominance of the vacuum phase, and its subsequent decay into relativistic particles (whose energy density rapidly increases) until the Universe is dominated by radiation and the vacuum energy becomes negligible. As a result inflation ends with “gracefully exits” into the standard radiation phase. We have used \( 8\pi G = 1 \) units.

is not jeopardized in this model. Recall also that when \( \nu \neq 0 \), we have \( |\nu| \ll 1 \) (cf. Sect. 7) so the \( H^2 \)-term is also harmless at the BBN epoch.

As a concrete illustration, the evolution of the primeval matter and vacuum energy densities from the very early Universe to the present day is highlighted in Fig. 1, where we can clearly see the transition from the de Sitter vacuum-dominated phase into the radiation dominated phase.

5 The horizon and entropy problems in the \( \bar{\Lambda} \)CDM

From the analysis of the previous section it follows that the \( \bar{\Lambda} \)CDM Universe (12) starts with a huge vacuum energy density of the order \( \sim M_X^4 \) – confer Eqs. (22) and (23) – and therefore without an initial singularity, and has no horizon problem. It follows that a light pulse beginning in the remote past at \( t = t_1 \gtrsim t_i \) (i.e. shortly after inflation started at \( t_i \)) will have traveled until the end of inflation, \( t_f \), the physical distance

\[
d_H(t_f) = a(t_f) \int_{t_1}^{t_f} \frac{dt'}{a(t')} = \frac{af}{aiH_I} \left( e^{-H_i t_1} - e^{-H_i t_f} \right) \approx \frac{af e^{-H_i t_1}}{aiH_I},
\]

where \( a(t) \) is given by Eq. (31) during the inflationary stage, and \( H = H_I \) remains constant under inflation. In practice \( H \) need not be strictly constant, it is enough that it remains approximately so for some interval \( t_i < t_1 < t_f \). The integral (31) diverges for \( t_1 \rightarrow t_i \) if the initial time of inflation is a sufficiently early one (viz. if \( a_i \rightarrow 0 \)). This proves the non-existence of particle horizons in the running vacuum Universe. Equivalently, and perhaps more transparently, the above integral can be written as

\[
d_H(a_f) = a(t_f) \int_{a_1}^{a_f} \frac{da'}{a'^2 H} = \frac{af}{H_I} \left( \frac{1}{a_1} - \frac{1}{a_f} \right) \approx \left( \frac{af}{a_1} \right) H_I^{-1}.
\]
Being $a_1$ at time $t_1 \gtrsim t_i$ exponentially smaller than $a_f$ at the end of inflation, we have $a_f/a_1 \gg 1$ and so the above integral (hence the horizon) can be as big as desired. The result (35) is of course the same as (34) because $a_1 = a_i e^{H_i t_i}$.

Notice that $\chi \sim d_H/a_f \simeq H_f^{-1}/a_1$ is the angle subtending the horizon from the point where inflation starts. This angle is a comoving coordinate, and if it is fixed appropriately at time $t_1$ (i.e. if it is made large enough such that it subtends at least the current horizon) it will remain so for the entire cosmic history. The exponential solution triggered by a period of approximately constant vacuum energy (starting from an early point $a_1$) makes this arrangement possible and hence there is effectively no horizon in the $\Lambda$CDM model.

In contrast, in the concordance $\Lambda$CDM model the above integral gives (e.g. in the radiation dominated epoch) $d_H \sim a^2$ (this is again $d_H \sim H^{-1}$, but now $H$ is not constant) and as a result $d_H/a \rightarrow 0$ for $a \rightarrow 0$, which is tantamount to say that the observers become isolated in the past (namely, their horizon becomes infinitely smaller than the size of the Universe for $t \rightarrow 0$ and hence cannot exchange information with it). This is the well-known horizon problem of the concordance model, which as we have seen can be overcome in the $\Lambda$CDM model. On more physical grounds, we can say that in the $\Lambda$CDM model the local interactions (the exchange of information) are able to homogenize the whole Universe since the very beginning.

### 5.1 The large cosmological entropy

We can reconfirm the absence of the horizon problem in the running vacuum model from a thermodynamical perspective, namely from the point of view of the so-called “entropy problem” in cosmology. Succinctly formulated reads as follows: how did the universe manage to start with an extremely low entropy value and develop later a huge one that gave rise to the arrow of time in accordance with the second law of thermodynamics? There is no solution to it within the $\Lambda$CDM.

Let us start by recalling some basic thermodynamical facts of our Universe that are intimately connected with the horizon problem. The current horizon is defined by $H_0^{-1}$ and we normalize the scale factor as $a_0 = 1$. We can take $H_0^{-3}$ as a fiducial volume of the observable Universe (up to an irrelevant numerical factor common to all our estimates). From standard thermodynamical formulae of the thermal history we can estimate the total entropy of our Universe multiplying the entropy density by its fiducial volume. In natural units, we find:

$$S_0 = \frac{2\pi^2}{45} g_{s,0} T_{\gamma 0}^3 \left( H_0^{-1} \right)^3 \simeq 2.3 h^{-3} 10^{87} \sim 10^{88}. \quad (36)$$

Here $T_{\gamma 0} \simeq 2.725 K \simeq 2.35 \times 10^{-13}$ GeV is the present CMB temperature, $H_0 = 2.133 h \times 10^{-42}$ GeV (with $h \simeq 0.67$) is the corresponding value of the Hubble function, and the coefficient

$$g_{s,0} = 2 + 6 \times (7/8) (T_{\nu,0}/T_{\gamma 0})^3 \simeq 3.91$$

is the entropy factor for the light d.o.f. today, which involves the well-known ratio $T_{\nu,0}/T_{\gamma 0} = (4/11)^{1/3}$ of the current neutrino and photon temperatures.

However, in the context of the $\Lambda$CDM concordance model, the huge number (36) cannot be understood without generating a phenomenal causality problem, and hence recreating the horizon problem again. The reason is that in the $\Lambda$CDM the total entropy contained in the horizon at earlier times is much smaller. In fact, we can project the result (36) backwards in time taking into account that it evolves as $(T H^{-1})^3$. Now, for an adiabatic evolution $T \sim a^{-1}$. Moreover during the nonrelativistic epoch $H \sim a^{-3/2}$, so that altogether this renders $S \sim a^{3/2} = (1 + z)^{-3/2}$. We conclude that at recombination ($z_{rec} \simeq 1100$) the total entropy was a factor $\sim z_{rec}^{3/2} \sim 10^{-5}$ smaller, or, to be more precise: $S_{rec} \sim 10^{83}$. It means that the current Hubble volume should contain some hundred thousand causally disconnected regions at recombination! This is completely unacceptable.

\footnote{For this it suffices that $a_f/a_1$ be only $\sim 1\%$ of the total expansion $a_0/a_f \sim 10^{28}$ after inflation. Equivalently, the number of inflationary e-folds ($e^N = a_f/a_1$) must be $N > 26 \ln 10 \approx 60$.}
since the smoothness of the CMB must have a causal explanation in terms of interactions that propagate at subluminal velocities. Such an unsettling situation worsens more and more when we travel deeper and deeper into the past, where the number of causally disconnected regions keeps increasing inordinately. It goes without saying that this a serious drawback of the standard picture, in fact one of the biggest cosmological conundrums of the ΛCDM model!

5.2 Solving the entropy problem with running vacuum in a generic GUT

The situation in the running vacuum Universe is completely different. We shall show that the total entropy contained in the current horizon \( H_0^{-3} \) can be computed from \( S_0 \sim S_\infty H_0^{-3} \), where \( S_\infty \) is an asymptotic value attained by the entropy per comoving volume (or “comoving entropy” for short) in the early Universe. Specifically we find that \( S(a) \sim T^3 a^3 \rightarrow S_\infty \) for \( a \gg a_{eq} \), where \( a_{eq} \) is the vacuum-radiation equality point. This point is placed in the (still incipient) radiation epoch, and in it the energy density equals that of the decaying vacuum. We will show that the comoving (hence intrinsic, volume-independent) radiation entropy increases very fast \( (S \propto a^{6}) \) owing to the primeval vacuum decay until reaching that saturation value \( S_\infty \). It follows that the amount of entropy contained in a physical patch of volume \( V \sim H^{-3} \) (taken at any \( a \gg a_{eq} \)) increases just proportional to the fixed number \( S_\infty \) times the physical volume of the region, i.e. \( S \sim S_\infty V \). In particular, for the region encompassed by the current horizon, \( V_0 \sim H_0^{-3} \), we have \( S_0 \sim S_\infty H_0^{-3} \).

Remarkably, this value is of the correct order of magnitude of the entropy today, Eq. (36). Since the comoving entropy \( S/V \sim S_\infty \) is conserved after we have left well behind the vacuum-radiation equality point, there is no causality problem at all in the running ΛCDM model.

To substantiate these claims, let us first estimate the equality point \( a_{eq} \). It can be obtained from equating the densities \( \rho_r \) and \( \rho_{\Lambda} \), i.e. \( \rho_r(a_{eq}) = \rho_{\Lambda}(a_{eq}) \), hence \( D a_{eq}^4 = 1 \). This determines \( D = a_{eq}^{-4} \) and from \( (28) \) we see that at \( a = a_{eq} \) the Hubble function reads \( H_{eq} = H_1/\sqrt{2} \) (obviously below \( H_1 \)). As we shall see, the main results of this analysis do not depend on the details of the GUT framework that rules the dynamics of the vacuum. However, the precise value of \( a_{eq} \) is of course model dependent. We need not know it accurately, but it is convenient to have a numerical estimate. Inserting \( H_{eq} = H_1/\sqrt{2} \) and \( a = a_{eq} \) in Friedmann’s equation relating the evolution of radiation from the time when \( H = H_{eq} \) up to now, yields

\[
a_{eq} \sim \left( \frac{8\pi^3 g_*}{45} \right)^{1/4} \frac{T_\gamma 0}{\sqrt{M_p H_1}} = \left( \frac{\pi^2 g_*}{15} \right)^{1/4} \frac{T_\gamma 0}{M_X},
\]

where in the last step use has been made of Eq. (24). This result is of course not exact (as one cannot reach the equality point assuming that the evolution is always within the radiation epoch), but it provides the estimate \( a_{eq} \sim 3 \times 10^{-20} \), which is sufficient as an order of magnitude value\(^6\) for a typical GUT defined at the scale \( M_X \sim 10^{16} \) GeV.

To compute the entropy generated from the primeval vacuum decay we have to estimate the temperature of the radiation heat bath that is formed during the conversion of the vacuum energy into relativistic particles\(^7\). This post-vacuum heat bath will mimic the reheating process of the standard inflationary scenarios, but the process to achieve it is actually quite different. Let us call the radiation temperature of the heat bath \( T_r \). Equating the radiation density \( \rho_r \) emerging from vacuum decay in our model to the general form of the radiation energy at an effective temperature \( T_r(a) \) we obtain:

\[
\rho_r(a) = \rho_{\Lambda}(a/a_{eq})^4 \left( 1 + (a/a_{eq})^4 \right) = \frac{\pi^2}{30} g_* T_r^4(a).
\]

\(^6\)The early equality point \( a_{eq} \) is specific of the ΛCDM Universe, and it should not be confused with the standard equality point at which the energy densities of radiation and cold dark matter become equal: \( \rho_{eq} \sim 3 \times 10^{-14} \) (i.e. at redshift \( z_{eq} \approx 3300 \)). The latter occurs much later and is coincident in both models. We have, \( a_{eq} \sim 10^{-20} \).

\(^7\)See \(^{67}\) and \(^{68}\) for recent alternative formulations, and \(^{69}\) for early developments.
Here $g_*$ counts the total number of effectively massless degrees of freedom (d.o.f.) in the heat bath at the given temperature. For the standard model (SM) of particle physics, $g_* = 106.75$, if we include the top quark and the Higgs boson since all of them can be relativistic d.o.f. at high temperatures. In general $g_*$ will be larger than the SM value when we operate at a GUT scale.

From Eq. (38) we find $T_r$ as a function of the scale factor:

$$T_r(a) = T_X \frac{a/a_{eq}}{[1 + (a/a_{eq})^4]^{1/2}}. \quad (39)$$

Here

$$T_X = \left( \frac{30 \rho_I}{\pi^2 g_*} \right)^{1/4} = \left( \frac{30}{\pi^2 g_*} \right)^{1/4} M_X \quad (40)$$

is the value of the temperature associated to the GUT scale $M_X$ through Eq. (22). The temperature at $a = 0$ is $T = 0$. It then rises (linearly with $a$ in the beginning) until a maximum value, which is attained at $a = a_{eq}$. It is natural to think of such maximum value as a kind of effective “reheating temperature” after vacuum decay:

$$T_{RH} \equiv T_X \sqrt{2} = \left( \frac{15}{2\pi^2 g_*} \right)^{1/4} M_X. \quad (41)$$

We point out that, in the conventional inflaton scenario [66], the reheating temperature of the radiation heat bath is not just determined by the initial GUT temperature $T_X$. A new parameter comes into play, the inflaton width $\Gamma_\phi$, and the reheating temperature is

$$T_{RH} \sim g_*^{-1/4} \sqrt{M_P \Gamma_\phi} \sim T_X \sqrt{\frac{\Gamma_\phi}{H_I}},$$

where $H_I^2 \sim g_* T_X^4/M_P^2$.

Notice that neither $T_X$ (nor $T_{RH}$ of course) are very sensitive to the precise value of $g_*$, even if the number of d.o.f. changes by one order of magnitude, because (40)-(41) depend on the quartic root of $g_*$. Therefore we will just take the SM value $g_* = 106.75$ as a fiducial estimate for our calculations. In this case we find e.g. $T_X \simeq 0.41 M_X$, which is smaller than $M_X$ but not far away from it. As the GUT scale $M_X$ is some three orders of magnitude below the Planck mass, $M_P \sim 10^{19}$ GeV, we are well reassured that our working regime is perfectly compatible with a semiclassical description of QFT in curved spacetime (which is the precise context where the running $\Lambda$CDM cosmological framework is formulated [2]).

At the maximum temperature (41) the radiation density is also maximum, so at that point the Universe becomes maximally populated of relativistic particles emerging form vacuum decay. Eventually both the radiation energy and the temperature decay (for $a \gg a_{eq}$), respectively as

$$\rho_r(a) \sim \rho_I \left( \frac{a_{eq}}{a} \right)^4 \sim M_X^4 \left( \frac{a_{eq}}{a} \right)^4 \quad (42)$$

and

$$T_r(a) \sim T_X \frac{a_{eq}}{a} \sim M_X \frac{a_{eq}}{a}. \quad (43)$$

It follows that, in the asymptotic adiabatic regime, the thermodynamic behavior of the running vacuum model boils down to the standard form, namely $\rho_r \sim 1/a^4$ and $T \sim 1/a$. Thus, the conventional BBN picture can proceed normally since the changes affect only the inflationary epoch.

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*Strictly speaking there is no “reheating” in this context since the Universe is just heating up progressively from $a = 0$ till the point $a = a_{eq}$, where it reaches the maximum temperature (11). Later on, when the evolution enters fully the radiation epoch, the temperature falls down standard as in (13).*
Having found an expression for the effective temperature of the primeval radiation heat bath, the comoving radiation entropy of the relativistic particles that now populate the Universe can be estimated. Starting from

\[ S_r(a) = \frac{\rho_r(a) + p_r(a)}{T_r(a)} a^3 = \frac{4}{3} \frac{\rho_r(a) T_r^3(a) a^3}{3 T_r(a)} = \frac{2\pi^2}{45} g_s T_r^3(a) a^3, \]  

and then using (39) we arrive at:

\[ S(r) = \frac{2\pi^2}{45} g_s T_r^3(a_{\text{eq}}) f(r) a^3 \]  

We have defined the following function of the ratio \( r \equiv a/a_{\text{eq}} \):

\[ f(r) = \frac{r^6}{(1 + r^4)^{3/2}}. \]  

Since \( f(0) = 0, f(1) = 2^{-3/2} \simeq 0.35 \) and \( f(r) \to 1 \) for \( r \gg 1 \), it follows that the radiation entropy (45) evolves monotonically very fast (\( S \propto a^6 \) in the beginning) starting from zero, then goes through the equality point \( a = a_{\text{eq}} \) (where is still increasing) and finally reaches the maximum value

\[ S(a) \to S_{\infty} \equiv \frac{2\pi^2}{45} g_s T_X^3 a_{\text{eq}}^3 \]  

for \( a \gg a_{\text{eq}} \), where it saturates. This substantiates our claim on the rapid increase of the comoving entropy until achieving an asymptotic value \( S_{\infty} \).

But of course we still have to prove that we can use \( S_{\infty} \) to predict the total entropy contained in our present horizon, Eq. (39). We are now in position to complete this task. First of all let us rewrite the expression \( T_X^3 a_{\text{eq}}^3 \) appearing in parenthesis in the asymptotic formula (47) in terms of the temperature, \( T_{\text{rad}} \), and the scale factor, \( a_{\text{rad}} \), deep in the radiation epoch, i.e. \( a_{\text{rad}} \gg a_{\text{eq}} \). From Eq. (39) we find:

\[ T_{\text{rad}} = T_X \frac{a_{\text{rad}}/a_{\text{eq}}}{[1 + (a_{\text{rad}}/a_{\text{eq}})^4]^{1/2}} \simeq T_X \frac{a_{\text{eq}}}{a_{\text{rad}}}. \]  

It is important to realize that this approximation is valid for any value of \( a_{\text{rad}} \) well after we have passed the equality point \( a_{\text{eq}} \), i.e. when we have entered the adiabatic regime of the evolution. From (48) it follows that \( T_X a_{\text{eq}} \simeq T_{\text{rad}} a_{\text{rad}} \) in very good approximation in this regime. Inserting the result in Eq. (47), we find that the total entropy in the current physical horizon is predicted to be

\[ S_{\infty} H_0^{-3} = \frac{2\pi^2}{45} (g_s T_{\text{rad}}^3 a_{\text{rad}}^3) H_0^{-3} = \frac{2\pi^2}{45} (g_{s,0} T_{\gamma_0}^3 a_0^3) H_0^{-3} = S_0, \]  

where in the last step we have used the entropy conservation law of the adiabatic regime, which states that \( g_s T_{\text{rad}}^3 a_{\text{rad}}^3 = g_{s,0} T_{\gamma_0}^3 a_0^3 \). Recall that in our normalization \( a_0 = 1 \), and hence we have reproduced the result \( S_0 \) given by (39).

The upshot is significant: it turns out that the theoretical prediction (49) of the running \(^{\Lambda}\)CDM model for the total entropy enclosed in our current horizon is precisely the observed entropy in our Universe, i.e. the huge entropy number (36) – in natural units. This result may be viewed as a potential solution to the cosmological entropy problem in the context of the running \(^{\Lambda}\)CDM. As we have seen, we can assume some generic GUT at the very high scale, which triggers the decay of the huge vacuum energy and generates the large entropy in the radiation phase. Remarkably, the important CMB constraint (25) is preserved and the final result for the entropy is universal, meaning that it does not depend on the details of the GUT. The universality of the prediction is
a reflex of the vacuum decaying dynamics and of the entropy conservation law, which holds good in the Universe once the cosmic evolution has entered the adiabatic regime.

The vacuum model \((12)\) that we have studied here (in which the highest power of the Hubble rate is \(H^4\)) is just the simplest implementation of the running \(\Lambda\)CDM as a candidate model for a complete description of the cosmic history from the inflationary times to the present day. However, the main results are maintained if an arbitrary even power \(H^{2n}\) \((n > 2)\) of the Hubble rate is used for the higher order term (as required by the covariance of the effective action \([2]\)). At the same time, one has to keep the additive term so as to reproduce the standard \(\Lambda\)CDM model at low energies. In Sect. [7] we shall see that the observations are compatible with keeping also the \(H^2\)-term of \((12)\). This is interesting as the latter endows the current cosmological vacuum with some mild dynamics (as a kind of “smoking gun” of the entire vacuum decaying mechanism) and can be interpreted as dynamical dark energy.

6 Starobinsky inflation versus running vacuum model

In this section we wish to elaborate on the fact that the class of vacuum models \((12)\) realize in an effective way the Starobinsky type of inflationary regime \([70]\). This occurs somehow through the dominance of the highest power \(H^4\) in the early Universe, which is of the order of the Starobinsky’s correction term \(R^2 \sim H^4\) in the effective action, see Eq. \((50)\) below. In actual fact the situation is a bit more complicated since at the level of equations of motion the term \(R^2\) in the action gives rise not just to a pure \(H^4\) contribution but to a fairly involved combination of powers of \(H\) and its time derivatives. Still there are some interesting similarities worth noticing. Some of them have been recently pinpointed in \([54, 56]\), and explored in great detail in \([57]\).

6.1 The standard Starobinsky action

Let us recall that the original Starobinsky model \([70]\) is based on the following action:

\[
S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} + bR^2 \right) + S_{\text{matter}},
\]

where \(b\) is the dimensionless coefficient of the higher order derivative term \(R^2\). It is usually written as \(b = M_P^2 / 6M_s^2\), where \(M_s\) is a mass dimension parameter – playing the role of scalaron mass in the original model \([70]\). Recall that \(R\) is, if written in the context of the FLRW metric, a linear combination of \(H^2\) and \(\dot{H}\), and therefore \(R^2\) involves terms of the form \(H^4, \dot{H}^2, H^2 \dot{H}\), all of which are roughly speaking of order \(\sim H^4\), and hence one may envision a kind of close connection of the Starobinsky model and the \(\Lambda\)CDM model. This is true in part, but it is not quite so, at least not for the original form \((50)\) of the Starobinsky action. In the next section we briefly mention how to improve the connection of the running \(\Lambda\)CDM with an alternative Starobinsky-like action \([18]\). At the moment we want to continue with the action \((50)\) and show the form of the inflationary solution that emerges from it, as this will enable us to compare with the inflationary scenario derived from the \(\Lambda\)CDM model in Sect.4.

The equations of motion for the case where the energy-momentum tensor of matter contains a mixture of various fluids \(N = 1, 2...\) with densities and pressures \((\rho_N, p_N)\) can be obtained after varying the action \((50)\) with respect to the metric. The result reads as follows:

\[
G_{\mu\nu} + 32\pi Gb(\nabla_\mu \nabla_\nu R - g_{\mu\nu} \Box R + RR_{\mu\nu} - \frac{g_{\mu\nu} R^2}{4}) = 8\pi G \sum_N \left[ \rho_N g_{\mu\nu} - (\rho_N + p_N)U^N_\mu U^N_\nu \right].
\]

\(^9\)For recent and past detailed studies of Starobinsky inflation, see e.g. \([71, 72]\), and references therein.
Figure 2: The inflationary solution of (55) corresponding to the Starobinsky model (50). On the left it is shown the exponential growth $a \sim e^{H_I t}$ of the scale factor and its stabilization into the radiation regime $a \sim t^{1/2}$. We have taken $H_I = M_P$ in (56), and $b = 10^8$ from the mentioned bound in the text. On the right we depict the corresponding behavior of the Hubble function and (in the inner window) the characteristic oscillations when the Universe leaves the inflationary phase and enters the radiation epoch in the form $a \sim t^{1/2} + \text{oscillations}$. In that window we have set $b = 100$ to make the oscillations more clearly visible. Time has been rescaled as $t = (\sqrt{96/\pi} M_P) \hat{t}$, and $\hat{H} = (1/a) \hat{a} / \hat{t}$ is the correspondingly rescaled Hubble function.

For $b = 0$ we recover, of course, the standard Einstein’s equations. However, for $b \neq 0$ the result is more complicated. Setting $(\mu, \nu) = (0, 0)$ and $(\mu, \nu) = (i, j)$ and using the flat FLRW metric, we find two independent equations:

$$H^2 = \frac{8\pi G}{3} \sum_N \rho_N + 96\pi G b(\dot{H}^2 - 2H \ddot{H} - 6H^2 \dot{H})$$

and

$$H^2 + \frac{2\ddot{H}}{3} + 32\pi G b(2\dddot{H} + 12 \dot{H} H + 18H^2 \dot{H} + 9\dot{H}^2) = -\frac{8\pi G}{3} \sum_N p_N.$$  

If we now just project the result for a single matter component $(\rho_R, p_R)$, and assume that this component is relativistic $(p_R = \rho_R/3)$, we can combine the two previous expressions to obtain a single differential equation for the Hubble rate:

$$2H^2 + \dot{H} + 48\pi G b(2\dddot{H} + 14 \ddot{H} H + 24H^2 \dot{H} + 8\dot{H}^2) = 0.$$  

It is pretty clear that this equation (a third order, nonlinear, differential equation) is considerably more involved than the first order equation (19) for the $\Lambda$CDM model in the early Universe. This shows that the initial analogy between the two models, based on the fact that $R^2 \sim H^4$, was an exceeding simplification since the models have indeed to be compared at the level of the equations of motion.

Expressing (54) in terms of the scale factor it becomes a fourth order differential equation:

$$\frac{\dddot{a}}{a} + 3\frac{\ddot{a}}{a^2} + \frac{\dot{a}^2}{a^2} - 5\frac{\ddot{a}^2}{a^3} + \frac{M^2}{96\pi b} \left( \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \right) = 0.$$  

This equation is to be solved under initial conditions, which we take as follows:

$$a(0) = a_i, \quad \dot{a}(0) = a_i H_I, \quad \ddot{a}(0) = a_i H_I^2, \quad \dddot{a}(0) = a_i H_I^3$$
and assume $H_I \sim M_P$ in this case. An analytic solution of this problem is not possible in general. However, for $b \to 0$ the dominant term is the last one of (55), and the solution becomes $a \sim t^{1/2}$ corresponding to the radiation epoch. In contrast, for large $b$ the last term of (55) becomes negligible and, then, the solution is easily seen to be the exponential one: $a(t) = a_i e^{H_I t}$. The numerical solution of Eq. (55) under the initial conditions (56) is given in Fig. 2. Remarkably an approximate solution of (55), or equivalently of (54), can be found to describe the end of the inflationary period. Since $\dot{H}$ remains essentially constant until we are very near the end of the inflationary phase (see the straight line in the plot on the right in Fig. 2), we can solve (54) by neglecting $\dot{H}/H^2 \ll 1$ and all higher derivative terms. In this way we are left with the equation

$$576 \pi G_N b \dot{H} = -1,$$

whose immediate solution is

$$H(t) = H_I - M_P^2 t / 576 \pi b,$$

and hence the corresponding scale factor reads:

$$a(t) = a_i e^{H_I t} e^{-M_P^2 t / 1152 \pi b}.$$  (57)

Obviously, $b > 0$ in order to have a stable inflationary solution until the inflationary phase terminates at around $t_f \simeq 1152 \pi b / M_P$. The larger is $b$ the longer is the inflationary time. One can show that if we impose the limit (25) from the CMB anisotropies the parameter $b$ has a lower bound of order $10^8$ – see [72].

Needless to say the approximate solution (57) does not perfectly interpolate from the inflationary epoch to the radiation epoch and hence, in contrast to the situation with the $\Lambda$CDM model, we do not have an analytically rigorous description of the graceful exit. The solution (57) is only indicative that the inflationary process comes to an end, and that there is a chance for a correct transition to the $a \sim t^{1/2}$ radiation epoch. But it misses the nontrivial details accounting for the reheating stage. Unfortunately, the issue of graceful exit is still a complicated and unsolved matter in Starobinsky inflation, despite it has been discussed in different places in the literature, see e.g. [71] and [73] in different formulations. A more detailed study, though, is still needed [57].

In point of fact, we see from the comparison of Figures 1 and 2 that the connection of the Starobinsky model and the $\Lambda$CDM model is not that close after all, at least using the original action (50). It turns out that the connection with the $\Lambda$CDM model is much better accomplished if one adopts the conformally invariant formulation of the Starobinsky model, which is only broken by quantum effects; namely, the so-called anomaly-induced effective action (see next section). The reason is that only in this formulation the $\Lambda$CDM model can be motivated from an effective action, which is certainly not the one given by (50). The latter is, in effect, not conformally invariant.

### 6.2 The anomaly-induced effective action

A generalized form of Starobinsky’s inflation, based on the effective action of anomaly-induced inflation was considered long ago in [73, 74, 75] – and references therein – but the first explicit connection of this framework with the running vacuum model $\Lambda$CDM was signaled in [18]. One finds that by taking the masses of the fields into account, using the conformization procedure of the cosmon model [76], the inflationary process automatically slows down and can therefore favor the graceful exit. Let us briefly mention the solution of this alternative form of Starobinsky inflation. We follow Ref. [18], where more details are provided. Rather than starting from the action (50) one takes the classically conformally invariant higher derivative action of the vacuum:

$$S_{\text{vac}} = \int d^4x \sqrt{-g} \left\{ a_1 C^2 + a_2 E + a_3 \nabla^2 R \right\},$$

where $a_{1,2,3}$ are dimensionless coefficients, $C^2$ is the square of the Weyl tensor and $E$ is the Gauss-Bonnet topological invariant in 4 dimensions [12]. This action can then be completed with the
conformally invariant realization of the Einstein-Hilbert (EH) term:

$$S_{EH} = \frac{M_P^2}{16\pi M^2} \int d^4x \sqrt{-g} \left[ R\chi^2 - 6(\partial\chi)^2 \right].$$ (59)

Here $\chi$ is a background scalar field that realizes the conformal symmetry at the high energy scale $\mathcal{M}$, presumably close to the GUT scale $M_X$. The same field and scale are used to conformize the masses of all the matter fields in the action [18] following the prescription of [76]. After $\chi$ acquires a vacuum expectation value of order $M_X$, that symmetry is spontaneously broken and we recover the standard EH term. Now, of course, the conformal symmetry of the total action is also intrinsically broken by quantum effects, not only the traditional ones from the trace anomaly associated to the $S_{\text{vac}}$ part [12] but also by the additional contribution form $S_{\text{EH}}^c$. The anomalous induced part of the action (i.e. that one breaking conformal symmetry from quantum effects) follows from solving the functional differential equation:

$$<T^\mu_\nu> = -\frac{2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta \Gamma}{\delta g^{\mu\nu}} + \frac{1}{\sqrt{-g}} \chi \frac{\delta \Gamma}{\delta \chi}$$ (60)

$$= b_1 C^2 + b_2 E + b_3 \nabla^2 R + \frac{\tilde{f}}{16\pi G_N M^2} [R\chi^2 - 6(\partial\chi)^2].$$

The one-loop values of the $\beta$-functions $b_1, b_2, b_3$ are well established since long time ago [12] and depend on the matter content of the model. The new ingredient is the $\beta$-function of the conformal EH term, which is given by [18]

$$\tilde{f} = \frac{1}{3\pi} \sum_F N_F \frac{\bar{M}_P^2}{M_P^2} + \frac{1}{2\pi} \sum_V N_V \frac{\bar{M}_P^2}{M_P^2}.$$ (61)

It involves contributions from fermions (F) and vector bosons (V) since the scalar part vanishes in the classical conformal limit. The trace-anomaly equation (60) can be solved following the standard method [77], namely from a conformal transformation of the metric $g_{\mu\nu} = e^{2\sigma} \tilde{g}_{\mu\nu}$, extended with the new background field: $\chi = e^{-\sigma} \tilde{\chi}$. In this way one finds the corresponding effective action that includes the effects of $S_{\text{EH}}^c$ [18]. The relevant part of it can be summarized as follows:

$$S_{\text{eff}} = \int d^4x \sqrt{-g} \frac{\bar{M}_P^2(\sigma)}{16\pi M^2} [\bar{R}\chi^2 - 6(\partial\chi)^2] + S_{\text{matter}} + \text{high. deriv. terms},$$ (62)

where

$$\bar{M}_P^2(\sigma) = M_P^2(1 - \tilde{f} \sigma).$$ (63)

is a “running Planck mass” that is sensitive to the dynamics of the conformal factor $\sigma$. We may next fix the conformal gauge as $\chi = \mathcal{M}$ (i.e. $\tilde{\chi} = \mathcal{M}e^{\sigma}$) so as to recover the EH form of the action. The scale $\mathcal{M}$ cancels and the resulting dynamics does not depend on it. Finally, we know that the FLRW metric is conformally flat in conformal time $\eta$, so that with the choice $e^{2\sigma} = a^2(\eta)$ we can take the flat metric $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$. This renders the new curvature terms trivial, $\bar{R} = 0$, but of course the action depends in a nontrivial way on the conformal factor $\sigma = \ln a(\eta)$. Varying this action with respect to it and reverting to the cosmic time $t$ one finds a fourth order differential equation for $a(t)$ similar to (55), although not identical. We shall omit cumbersome details here [73, 18], but we remark that an approximate solution of it can also be found in the form [57] as follows:

$$a(t) = a_i e^{H_i t} e^{-\frac{1}{2}H_f \tilde{f} t^2}.$$ (64)

We see that $\tilde{f}$ plays, up to a factor, the role of $1/b = 6M_P^2/M_P^2$ in the original model, and so the effective scalar mass squared $\bar{M}_P^2$ is substituted here by a combination of squared masses of fermion and boson fields – confer Eq. (61).
With the anomaly-induced formulation we have recovered once more a “tempered” form of inflation, namely possessing an evolution pattern very similar to that indicated in Fig. 2, and thus presumably having a paved way to graceful exit. Not only so, now with the alternative formulation we have gained an additional bonus, to wit: the running Planck mass squared (63) leads to an effective gravitational coupling

\[ G_N(a) = \frac{G_N}{1 - \tilde{f} \ln a}, \quad (65) \]

where \( G_N = 1/M_P^2 \) is the current value \((a = 1)\). Let us assume that this situation is roughly maintained at low energies.\(^\text{10}\) If we next insert (65) into the general low-energy equation for energy conservation in the presence of variable cosmological parameters \( G_N \) and \( \rho_\Lambda \), i.e. Eq. (1), we obtain a differential equation relating \( \rho_\Lambda \) and \( \rho_m \). However, if we further assume matter conservation, then we can derive the evolution law for the vacuum energy density \( \rho_\Lambda \) that is compatible with (65) according to the Bianchi identity (8). The exact result is a bit complicated, but in the limit \(|\tilde{f}| \ll 1\) (which is in fact the natural situation for a \( \beta \)-function coefficient) it turns out that the final result for the matter-dominated epoch reads very simple \(^{18}\):

\[ \rho_\Lambda(H) = \rho_\Lambda^0 + \frac{\tilde{f}}{8\pi G} (H^2 - H_0^2), \quad (66) \]

where \( \rho_\Lambda(H_0) = \rho_\Lambda^0 \) is the current value. This expression is noticeable as it shows that the vacuum energy of the anomaly-induced formulation of the Starobinsky inflation leads, at low energies, to a dynamical vacuum model of type A1, see Eq. (14). We can easily identify by comparison of the two equations that \( \nu = \tilde{f}/3 \). It shows that the coefficient \( \nu \) (which could have been introduced in (14) on mere phenomenological grounds) can be interpreted as a \( \beta \)-function coefficient for the CC running in semiclassical curved spacetime. One can indeed verify that the structure of \( \tilde{f} \) in (61) takes on the expected general form (13). This result reinforces the physical meaning of the general renormalization group equation of the cosmological term, Eq. (10), in QFT in curved spacetime.

At the end of the day, we conclude that the class of vacuum models (12) encodes basic features of different realizations of Starobinsky’s type of inflation, mainly if realized in the anomaly-induced form. It remains to be seen what is the final observational status of the Starobinsky model. At the day of writing this work the situation is inconclusive \(^{65, 78}\) and therefore we must wait for more input.

7 Vacuum dynamics from \( \Lambda \)CDM in the current Universe

In this section we go phenomenological and report on the confrontation of the dynamical vacuum models A1, A2, B1 and B2 defined in Sect. 3 against the latest observational data. All of these models can be solved both at the background and perturbative level. The Type-B ones, however, are more difficult to deal with analytically and require more numerical work, specially in regard to the perturbation equations and the study of structure formation. For details see the recent works \(^{22, 23}\), which are to our knowledge the most comprehensive studies on these matters currently existing in the literature. Here we limit ourselves to provide a quick summarized presentation of the solution of models A1 and A2 only, but we provide also some numerical results of the type-B ones.

\(^{10}\) We cannot exclude that as we run down to the low energy regime, corresponding to the present Universe, there might occur some additional, infrared, renormalization of the coupling \( \tilde{f} \). This could enhance its value as compared to the UV regime.
Figure 3: The evolution of the energy densities (normalized with respect to $H_0^2$) for the type-A1 model \([14]\) in the post-inflationary epoch until our days. The curves shown are: radiation (dashed line), non-relativistic matter (dotted line) and vacuum (solid line, in red). Used inputs: $\nu = 10^{-3}$, $\Omega^0_m = 0.27$, $\Omega^0_r = (1 + 0.227N_v)\Omega^0_m$, $(N_v, \Omega^0_m, h) \simeq (3.04, 2.47 \times 10^{-5}h^{-2}, 0.71)$. We have used $8\pi G = 1$ units.

7.1 The cosmology of type-A models

Let us consider the vacuum model A2 defined in Eq. \([16]\). Using the equations of Sect. 2 we can solve for the energy densities as a function of the scale factor. For cold matter and radiation one finds:

$$\rho_m(a) = \rho^0_m a^{-3\xi}, \quad \rho_r(a) = \rho^0_r a^{-4\xi'},$$

and for the vacuum energy density:

$$\rho\Lambda(a) = \rho^0\Lambda + \rho^0_m (\xi-1 - 1) (a^{-3\xi} - 1) + \rho^0_r (\xi'-1 - 1) (a^{-4\xi'} - 1),$$

where we have defined

$$\xi = \frac{1 - \nu}{1 - 3\nu/2}, \quad \xi' = \frac{1 - \nu}{1 - 2\nu}.$$

Obviously, for $\nu = 0$ the model A2 becomes simply the basic running vacuum model A1 (introduced in \([14]\)). The corresponding Hubble function reads

$$H^2(a) = H_0^2 \left[ 1 + \frac{\Omega^0_m}{\xi} (a^{-3\xi} - 1) + \frac{\Omega^0_r}{\xi'} (a^{-4\xi'} - 1) \right],$$

where the cosmological parameters satisfy the usual sum rule $\Omega^0_m + \Omega^0_r + \Omega^0\Lambda = 1$. As expected, for $\nu = \nu = 0$ (i.e. $\xi = \xi' = 1$) the Hubble function $H(a)$ takes on the form of the concordance model. In this case we recover $\rho_m \sim a^{-3}$, $\rho_r \sim a^{-4}$ and, of course, also $\rho\Lambda = $const. As an illustration, in Fig. 3 the evolution of the matter and vacuum densities are plotted from the radiation epoch to our days. It corresponds to a type-A1 model ($\nu = 0$) and we have assumed $\nu = 10^{-3}$, i.e. a typical value obtained from the fits to the observational data, see below. We can see that the vacuum density performs a substantial evolution until it asymptotes to the current value $\rho\Lambda \rightarrow \rho^0\Lambda$. It certainly looks more natural than the situation in the $\Lambda$CDM model, where it stays $\rho\Lambda = \rho^0\Lambda$ all the time since the radiation epoch.
some degeneracies in the two parameters of models $A_2$ and $B_2$, for the former we have set $\nu = \nu_\text{eff}$ (see the text). For model $B_2$, we have set $\nu = \epsilon$.

To study the evolution of matter perturbations we generalize the standard second order differential equation [79] for the growth factor $D \equiv \delta \rho_\text{m}/\rho_\text{m}$ for the case when the vacuum term is dynamical, $\dot{\rho}_\Lambda \neq 0$. The result (after neglecting contributions which are subleading) is [22]:

$$\ddot{D} + (2H + \Psi) \dot{D} - \left(4\pi G \rho_\text{m} - 2H \Psi - \dot{\Psi}\right) D = 0, \tag{71}$$

with $\Psi \equiv -\dot{\rho}_\Lambda/\rho_\text{m}$. For the $\Lambda$CDM model we have $\rho_\Lambda = \text{const.}$ and hence $\Psi = 0$, so that the above equation correctly reduces to the standard one [79].

### 7.2 Confronting the vacuum models $A$ and $B$ with observations

Equation (71) can be solved for type-$A$ models (for which $\rho_\text{m}$ and $\rho_\Lambda$ have been given above) in terms of hypergeometric functions. In the case of type-$B$ models the solution is more complicated and one has to use direct numerical methods [22, 23]. From the corresponding solution for $D(a)$ we can e.g. study the linear growth rate of clustering [79], namely the logarithmic derivative of the linear growth factor $D(a)$ with respect to $\ln a$:

$$f(a) \equiv \frac{1}{D(a)} \frac{dD(a)}{d\ln a} = \frac{d\ln D(a)}{d\ln a} = -(1 + z) \frac{d\ln D(z)}{dz}. \tag{72}$$

This quantity is important enough since it is measured observationally, so we can use it to investigate the performance of our vacuum models. In Fig.4 we compare the theoretical growth prediction derived from our fits to the data (see below) with the latest growth data (as collected e.g. by [80] and references therein). Specifically we plot the combined observable $f(z)\sigma_8(z)$, viz. the ordinary linear growth rate weighted by the rms mass fluctuation field, which it is claimed to have some advantages in practice [81]. The value of $\sigma_8(z)$ can also be computed for each model through $\sigma_8(z) = \sigma_8 D(z)/D(0)$, where we use $\sigma_8 = 0.829 \pm 0.012$ from Planck+WP [1].

It is pretty clear from Fig.4 that the dynamical vacuum models under consideration are able to adjust the linear growth data in a way comparable to the $\Lambda$CDM. This was expected since the “running” of the vacuum density in our time does not depart too much from the constant $\rho_\Lambda^0$ value. However, if one sets $c_0 = 0$ in the running models the departure from the structure formation data becomes dramatic. This is quite manifest in the anomalous behavior of the $C_1$ curve in Fig.4.

We have performed the fit of the various vacuum models from the combined data on type Ia supernovae (SNIa) [82], the shift parameter of the Cosmic Microwave Background (CNB) [4], and the data on the Baryonic Acoustic Oscillations (BAOs) [83]. In the last case we used the $\text{BAO}_{dz}$ data based on the $d_\text{z}$ estimator defined in that reference – confer Table 3 in it. The basic fitting procedure is explained in detail in Ref. [22]. Our results are collected in Table 1 above. Owing to some degeneracies in the two parameters of models $A_2$ and $B_2$, for the former we have set $\nu = \nu_\text{eff}/2$ (such that $\xi' = 1$) and for the latter $\nu = \epsilon$. With this setting, and taking into account

| Model | $\Omega_\text{m}^0$ | $\nu$ | $\epsilon$ | $\chi^2/\text{dof}$ |
|-------|-----------------|------|--------|--------------|
| $\Lambda$CDM | $0.293 \pm 0.013$ | - | - | 567.8/586 |
| $A_1$ | $0.292 \pm 0.014$ | +0.0013 $\pm$ 0.0018 | - | 566.3/585 |
| $A_2$ | $0.290 \pm 0.011$ | +0.0024 $\pm$ 0.0024 | - | 565.6/585 |
| $B_1$ | $0.297^{+0.015}_{-0.014}$ | - | $-0.014^{+0.016}_{-0.013}$ | 587.2/585 |
| $B_2$ | $0.300^{+0.017}_{-0.003}$ | $-0.0039^{+0.0020}_{-0.0021}$ | $-0.0039^{+0.0020}_{-0.0021}$ | 583.1/585 |

Table 1: The fit values for the various models using SNIa+CMB+BAO$_{dz}$ data, together with their statistical significance according to $\chi^2$ statistical test. For model $A_2$ we provide the fit value of $\nu_\text{eff}$ (see the text). For model $B_2$, we have set $\nu = \epsilon$. 


Figure 4: Comparison of the observed (solid points with vertical error bars) and theoretical evolution of the weighted growth rate $f(z)\sigma_8(z)$ for the various A and B models. The uppermost (red) line corresponds to the $\Lambda$CDM model, used as a reference. The curve C1 (black line) that deviates significantly from the others in the two panels corresponds to model B2 for $c_0 = 0$. The curves have been obtained for the best fit values indicated in Table 1. For more details, see [22].

Parameterizing the linear growth as $f(z) \simeq \Omega_m(z)^{\gamma(z)}$, one can define the linear growth rate index [79] $\gamma$. It can be used to distinguish cosmological models, see e.g. [84]. For the $\Lambda$CDM model such index is approximated by $\gamma_\Lambda \simeq 6/11 \simeq 0.545$. For the dynamical vacuum models we make use of (72) and the knowledge of the corresponding growth factor $D(a)$ for each model, and we get

$$\gamma(z) \simeq \frac{\ln \left[ - (1 + z) \frac{d \ln D}{dz} \right]}{\ln \Omega_m(z)}.$$  

The growth rate index $\gamma$ for each model is defined from the value of the previous expression for $z = 0$.

In Fig. 5 we plot the evolution of $\gamma(z)$ for the A and B type of vacuum models. In the same figures we can also see our determination of $\gamma_\Lambda(z)$ for the $\Lambda$CDM as a function of the redshift, and in particular we find $\gamma_\Lambda(0) \simeq 0.58$. From that figure we see that the growth index of the dynamical vacuum models (especially for type-A models) is well approximated by the $\Lambda$CDM constant value for $z < 1$, while at large redshifts there are deviations. It is worth mentioning that the differences with respect to the $\Lambda$CDM are at the edge of the present experimental limits. For example, in a recent analysis it is found that $\gamma = 0.56 \pm 0.05$ and $\Omega_m^0 = 0.29 \pm 0.01$ [84]. The prediction of $\gamma$ for all our vacuum models lies within 1σ of that range. Being the differences on the verge of being observed, we expect that in the future it should be possible to discriminate among the various $\bar{\Lambda}$CDM models by this procedure.

Finally, we briefly mention an alternative procedure to distinguish the running vacuum models from the standard $\Lambda$CDM. It is based on the cluster number counts method, see [61, 60] for previous applications in this context. The method is based on the Press-Schechter formalism [85] and generalizations thereof [86]. We refer the reader to Refs. [22, 23] for a comprehensive presentation, and in what follows we show only the most important results. The main observable of the method is the fractional difference $\delta N(z)/N(z)$ in the number of counts of clusters between the vacuum models and the concordance $\Lambda$CDM model at any given redshift $z$. The number of counts at
redshift $z$ is given by the following expression:

$$N(z) = -\frac{4\pi r^2(z)\bar{\rho}(z)}{H(z)} \int_{M_1}^{M_2} \frac{1}{M} \left( \frac{1}{\sigma} d\sigma \right) f_{\text{PS}}(\sigma) dM. \tag{74}$$

Here $\bar{\rho}(z)$ is the comoving background mass density. The original Press-Schechter function is [85]

$$f_{\text{PS}}(\sigma) = \sqrt{2/\pi} \frac{\delta_c}{\sigma} \exp\left(-\frac{\delta_c^2}{2\sigma^2}\right).$$

It depends on the parameter $\delta_c$, the linearly extrapolated density threshold above which structures collapse. This parameter must be computed for each vacuum model using the non-linear perturbations equations [22, 23]. A generalized and improved form of the Press-Schechter function $f_{\text{PS}}(\sigma)$, which we adopt here, is the one provided by Reed et al. in [86]. Finally, $r(z)$ in the above equation is the comoving radial distance out to redshift $z$, namely:

$$r(z) = \int_0^z \frac{dz'}{H(z')}; \tag{75}$$

and $\sigma^2(M, z)$ is the mass variance of the smoothed linear density field. It depends on the redshift $z$ at which the halos are identified and is given by

$$\sigma^2(M, z) = \frac{D^2(z)}{2\pi^2} \int_0^\infty k^2 P(k) W^2(kR) dk. \tag{76}$$

In this expression, $P(k)$ is the power-spectrum and $D(z)$ is the linear growth factor of perturbations. It is obtained from solving the perturbations Eq. (71) and is therefore characteristic of each model. Finally, $W$ is the Fourier transform of the standard top-hat function with spherical symmetry – for details, see e.g. [22].

By selecting an appropriate redshift range for the observations, as well as a characteristic range of masses $M_1 < M < M_2$ for the observed clusters, one can provide definite predictions from (74). Using the best fit values of the parameters of our vacuum models, as indicated in Table 1, and the full machinery of the generalized Press-Schechter formalism, we can compute the redshift distribution of clusters predicted in each vacuum model and subtract the corresponding prediction from the ΛCDM, i.e. the quantity $\delta N(z) = N(z) - N(z)_{\Lambda CD M}$. Then we can compute the fractional difference $\delta N(z)/N(z)_{\Lambda CD M}$, which encodes the relative deviations with respect to the concordance model. The numerical results are shown in Fig. 6.

The vacuum models are seen to be clearly separated by the cluster number counts method, with the type-B ones providing an excess in the number of counts, and the type-A models a
Figure 6: Comparison of the fractional difference $\delta N / N$ in the redshift distribution of cluster number counts (in the indicated range of masses) for the vacuum models under consideration with respect to the concordance $\Lambda$CDM model, i.e. $\delta N = N - N_{\Lambda\text{CDM}}$. Inputs as in the previous two figures.

defect, as compared to the $\Lambda$CDM. The deviations can be significant ($+30 - 40\%$) for the former (especially for B2) but are moderate (say $-10\%$ to $-20\%$) for the latter, at around the optimal redshift $z \simeq 1.5$ where it lies (approximately) the maximum total number of counts (cf. [22]). The fractional differences are still bigger for higher redshifts, reaching $+50\%$ for B2 at $z \simeq 2$. Beyond redshift $z = 3$, however, the statistics of the total number of counts depletes significantly and as a result this range is not appropriate for practical measurements.

8 Conclusions

In this work we have discussed various aspects of the “running $\Lambda$CDM cosmology” (a large class of dynamical vacuum models) in which the vacuum energy density can be expressed as a power series of the Hubble function and its cosmic time derivatives. Some of these models can be well motivated within the context of quantum field theory (QFT) in curved spacetime and they can provide an overarching description of the cosmic history, namely starting from the early inflationary times till our dark energy (DE) days. For the study of the current Universe the series naturally terminates at the level of the $H^2$ and $\dot{H}$ terms, but the higher order ones are crucially important for a proper description of the inflationary phase.

The class of $\Lambda$CDM models involving higher powers of the Hubble rate (typically $H^4$) have a non-singular starting point characterized by an initial de Sitter period of rapid inflation, presumably triggered by the high energy dynamics of a Grand Unified Theory at a scale $M_X$ near (but below) the Planck scale. Explicit solution of the model shows that the inflationary period is followed by successful “graceful exit” into the standard radiation epoch. The Universe continues into the cold dark matter era and finally leads to a dark energy epoch with a very small amount of vacuum energy.

It is important to emphasize that the post-inflationary period leads to a cosmological regime
that is very similar to the concordance ΛCDM model, but with a distinctive feature: the vacuum energy is not rigid but mildly dynamical ($\rho_\Lambda \sim c_0 + \nu H^2$). This feature can be tested and may prove a low-energy “smoking gun” of the underlying vacuum dynamics. Let us remark the recent indications on dynamical DE in the current data [87]. The slow vacuum evolution in our recent past, still going on in our days, could be responsible for the dynamical DE and could provide an alternative explanation to it which is completely different from quintessence approaches and the like. Similarly, the same unified vacuum structure explaining the DE is responsible, at the very early times, of the inflationary period and without invoking at all any sort of ad hoc inflaton fields. Therefore, in contrast to the ΛCDM model, the very early times of the cosmic history in the ¯ΛCDM cosmology is characterized by a powerful dynamical vacuum ($\sim H^4 \sim M^4_X$), which triggers inflation and upon decay into matter becomes much more moderate ($\sim H^2 \gtrsim H_0^2$) near our present until effectively behaving as DE.

Quite noticeably, the running ¯ΛCDM model provides a natural explanation for the huge entropy of the current Universe. It also emerges from the primeval vacuum decay in the early Universe. While the details of the vacuum decay depend on the Grand Unified Theory that brings about inflation, the final prediction of the entropy is universal and does not depend on the particular GUT implementation. The mechanism we have described here explains the value of the entropy in the current horizon. No horizon problem exists at all in the ¯ΛCDM model since all of the points of the current Hubble sphere remain causally connected as of the early times when a huge amount of relativistic particles emerged out of the decaying dynamics of the primeval vacuum.

We have signaled some potential connection of the $\sim H^4$ structure of the ¯ΛCDM cosmology with Starobinsky inflation, characterized by $R^2 \sim H^4$. The two sorts of models present some similarities but they actually involve also important differences, which depend on particular implementations.

After the inflationary epoch is left behind, the total amount of vacuum energy decreases very fast, the primordial nucleosynthesis can operate normally within a standard radiation epoch and the Universe can go through the cold dark matter era until the present DE era. The latter appears here as a slowly time-varying vacuum-dominated epoch. The low-energy vacuum dynamics can, however, adopt different forms (which we have called type-A and type-B).

We have shown that at leading order all these forms are admissible structures for the consistent description of our Universe, but they exhibit some differences that can be checked observationally. In particular we have confronted our vacuum models against the Hubble expansion data and structure formation. In addition we have assessed their considerably different capability in populating the Universe with virialized structures at different redshifts as compared to the ΛCDM model.

The current Universe appears in all these models as FLRW-like, except that the vacuum energy is not a rigid quantity but a mildly evolving one. The typical values we have obtained for the coefficients $\nu$, $\alpha$ and $\epsilon$ responsible for the time evolution of $\rho_\Lambda$ lie in the ballpark of $\sim 10^{-3}$. This order of magnitude value is roughly consistent with the theoretical expectations, some of them interpreted in QFT as one-loop $\beta$-functions of the running cosmological constant [2, 18, 19]. It is rather encouraging since it points to a fundamental origin of the theoretical structure of the ΛCDM models in the context of QFT in curved spacetime.

To summarize, the class of ΛCDM cosmologies may offer an appealing and phenomenologically consistent perspective for describing inflation and dynamical dark energy without introducing extraneous dark energy fields. Ultimately they might offer a clue to better understand the origin of the $\Lambda$-term and the cosmological constant problem in the context of fundamental physics. It would be a timely achievement, if we take into consideration that we are currently approaching the centenary of the introduction of the cosmological term in Einstein’s equations in 1917 [1].
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