EXISTENCE OF SOLUTIONS FOR DEGENERATE PARABOLIC EQUATIONS WITH SINGULAR TERMS

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Abstract. In this paper we deal with parabolic problems whose simplest model is
\[
\begin{cases}
  u' - \Delta_p u + B \frac{\nabla u}{u}^p = 0 & \text{in } (0, T) \times \Omega, \\
  u(0, x) = u_0(x) & \text{in } \Omega, \\
  u(t, x) = 0 & \text{on } (0, T) \times \partial \Omega,
\end{cases}
\]
where \( T > 0, N \geq 2, p > 1, B > 0, \) and \( u_0 \) is a positive function in \( L^\infty(\Omega) \) bounded away from zero.

1. Introduction

In this paper we are concerned with homogeneous nonlinear singular initial-boundary value problems whose model is
\[
u' - \Delta_p u + B \frac{\nabla u}{u}^\gamma = 0,
\]
where \( \Omega \) is a bounded open set of \( \mathbb{R}^N, N \geq 2, p > 1, B \in \mathbb{R}^+ \) and \( \Delta_p u = \text{div}(\nabla u^{p-2}\nabla u) \) is the usual \( p \)-laplacian. Here and below we use the simplified notation \( u' \) in order to indicate the time derivative of \( u \) with respect to \( t \).

Singular problems of this type have been largely studied in the past also for their connection with the theory of non-Newtonian fluids and heat conduction in electrically active materials (see for instance [21, 14] and references therein).

From the mathematical point of view, the non-homogeneous elliptic case has been considered in a series of papers in the last few years. Consider the equation
\[
-\Delta u + B \frac{\nabla u}{u}^\gamma = f,
\]
equipped with homogeneous Dirichlet boundary conditions. Here \( \gamma > 0 \) and \( f \) is a nonnegative (not identically zero) function in \( L^1(\Omega) \). The problem is obviously singular as we ask the solution to vanish at the boundary of \( \Omega \). In [3] the existence of a finite energy (i.e., in \( H_0^1(\Omega) \)) solution to problem (1.1) has been proved if \( \gamma < 2 \) and for data \( f \) locally bounded away from zero.

The case of a possibly degenerate datum \( f \) has been also considered. If \( \gamma < 1 \) the existence of a solution in \( H_0^1(\Omega) \) was proved in [4] for general nonnegative (not identically zero) data, while the case \( \gamma = 1 \) was faced in [18] provided \( B \) was small enough (we also

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mention [13, 23, 2, 1] and the references therein for a quite complete account on the subject. Problems as in (1.1) with possibly changing-sign data have also been considered in [11] in the case $\gamma < 1$ (see also [12] for further considerations concerning the strongly singular case).

In the evolutive case, problems as

$$
\begin{aligned}
&u' - \Delta_p u + B \frac{|\nabla u|^p}{u^\gamma} = f \quad \text{in (0, T) } \times \Omega, \\
&u(0, x) = u_0(x) \quad \text{in } \Omega, \\
&u(t, x) = 0 \quad \text{on } (0, T) \times \partial \Omega,
\end{aligned}
$$

have been considered in the case $p = 2$ and $\gamma < 1$ (see [19]). If $\gamma = 1$ singular problems as (1.2) have been considered in [26, 27] for smooth strictly positive data, while degenerate problems (i.e. $p > 2$) were studied in [28] in the one dimensional case.

We also would like to stress that, in the degenerate case $p > 2$ and if $B > p - 1$, to find a solution for the model problem (1.2) is formally equivalent, through a standard Cole-Hopf transformation, to find a large solution to the doubly nonlinear problem

$$
w' = \Delta_p w^{m}, \quad m < \frac{1}{p - 1},
$$

that is a solution that blows up at the boundary of $\Omega$ (see [9, 17, 8, 20] for further considerations on this fact).

The aim of this paper is to study existence and nonexistence of solutions for a general class of singular homogeneous (i.e. $f \equiv 0$) parabolic problems as (1.2) in the limit case $\gamma = 1$. We will mainly be concerned with the case $p \geq 2$.

The paper is structured as follows: in the next section we set the main assumptions, we state our main result, and we introduce some preliminary tools. Section 3 is devoted to prove existence of a solution in the degenerate case $p > 2$. In Section 4 we prove the existence of a solution in the case $p = 2$ provided the size of the lower order term is small (e.g., $B < 1$ in (1.2)), while in Section 5 we will prove a nonexistence result in the complementary case $B \geq 1$. In the last section of the paper we will also give an account on the singular case $p < 2$ by showing some finite time extinction results, and by discussing some explicit examples of evolution.

**Notations.** From now on, we will set $Q = (0, T) \times \Omega$ and $\Gamma = (0, T) \times \partial \Omega$. If not otherwise specified, we will denote by $C$ several constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data (for instance $C$ can depend on $N, \Omega, T, B, \alpha,$ and $\beta$) but they will never depend on the indexes of the sequences we will often introduce. For the sake of simplicity we will often use the simplified notation

$$
\int_Q f = \int_Q f(t, x) \, dt dx,
$$

when referring to integrals when no ambiguity on the variable of integration is possible.

For fixed $k > 0$ we will made use of the truncation functions $T_k$ and $G_k$ defined as

$$
T_k(s) = \max(-k, \min(s, k)),
$$

$$
G_k(s) = \min(s, k).
$$
and
\[ G_k(s) = s - T_k(s) = (|s| - k)^+ \text{ sign}(s). \]

2. Setting of the problem and preliminary results

We consider a parabolic differential problem of the form
\[
\begin{cases}
  u' - \text{div} a(t, x, \nabla u) + H(t, x, u, \nabla u) = 0 & \text{in } Q, \\
  u(0, x) = u_0(x) & \text{in } \Omega, \\
  u(t, x) = 0 & \text{on } \Gamma,
\end{cases}
\]
(2.3)

where
- \( \Omega \) is a bounded open set in \( \mathbb{R}^N \), with \( N \geq 2 \), \( T \) is a positive number;
- \( a(t, x, \xi) : (0, T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N \) is a Carathéodory vector-valued function such that
  \[
  a(t, x, \xi) \cdot \xi \geq \alpha |\xi|^p,
  \]
(2.4)
  \[
  |a(t, x, \xi)| \leq \beta |\xi|^{p-1},
  \]
(2.5)
  \[
  (a(t, x, \xi) - a(t, x, \eta)) \cdot (\xi - \eta) > 0,
  \]
(2.6)
  for a.e. \((t, x) \in (0, T) \times \Omega\), for every \( \xi, \eta \in \mathbb{R}^N \), with \( \xi \neq \eta \), where \( \alpha, \beta \) are positive constants and \( p > 1 \);
- \( H(t, x, s, \xi) : (0, T) \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R} \) is a Carathéodory function such that
  \[
  0 \leq H(t, x, s, \xi) \leq B |\xi|^p/s,
  \]
(2.7)
  for a.e. \((t, x) \in (0, T) \times \Omega\), for every \( s > 0, \xi \in \mathbb{R}^N \), where \( B \) is a positive constant;
- \( u_0(x) \) is a function in \( L^\infty(\Omega) \) such that \( u_0 \geq c > 0 \) almost everywhere on \( \Omega \).

In (2.3), we denote by \( u' \) the partial derivative with respect to time, while \( \nabla u \) stands for the gradient with respect to the space variable \( x \).

Consider problem (2.3). Here is the meaning of weak solution for such a problem.

**Definition 2.1.** A weak solution to problem (2.3) is a function \( u \) in \( L^p(0, T; W^{1,p}_0(\Omega)) \cap C(0, T; L^1(\Omega)) \) such that for every \( \omega \subset \subset \Omega \) there exists \( c_\omega \) such that \( u \geq c_\omega > 0 \) in \((0, T) \times \omega\); furthermore, we have that
\[
- \int_Q u \varphi' + \int_Q a(t, x, \nabla u) \cdot \nabla \varphi + \int_Q H(t, x, u, \nabla u) \varphi = \int_{\Omega} u_0 \varphi(0),
\]
for every \( \varphi \in C^1_c((0, T) \times \Omega) \), that is, for every \( C^1 \) function which vanishes in a neighborhood of \( \{T\} \times \Omega \) and of \((0, T) \times \partial \Omega\).

Note that under assumption (2.7), the function \( H(t, x, u, \nabla u) \) belongs to \( L^1_{\text{loc}}(Q) \) thanks to the property of local positivity required on \( u \).

Our main result is the following:

**Theorem 2.2.** If \( p > 2 \), there exists a weak solution to problem (2.3). Moreover, if \( p = 2 \), there exists a solution if the constant \( B \) appearing in (2.7) satisfies \( B < \alpha \).
Remark 2.3. Assumption $B < \alpha$ is, in some sense, optimal if $p = 2$: this will be the content of Proposition 5.1 below, in which we show how approximating problems may degenerate if $B \geq \alpha$. Assumption $p \geq 2$ is also needed in this context since, as we will see in Section 6, finite time extinction can occur if $p < 2$, so that Definition 2.1 should be suitably modified.

Our strategy in order to prove Theorem 2.2 will rely on an approximation argument. The next subsection will introduce our approximating problems.

2.1. The approximating problems. We consider the approximating problems

$$
\begin{cases}
  u'_n - \text{div} a(t, x, \nabla u_n) + H(t, x, u_n, \nabla u_n) = 0 & \text{in } Q, \\
  u_n(0, x) = u_0(x) + \frac{1}{n} & \text{in } \Omega, \\
  u_n(t, x) = \frac{1}{n} & \text{on } \Gamma.
\end{cases}
$$

(2.8)

A weak solution to this problem is a function $u_n$ such that $u_n \geq \frac{1}{n}$ a.e. in $Q$, $u_n - 1/n \in L^p(0, T; W^{1,p}_0(\Omega)) \cap C([0, T]; L^1(\Omega))$ and $u'_n \in L^1(Q) + L^p(0, T; W^{-1,p'}(\Omega))$, and such that

$$
\int_0^T \langle u'_n, v \rangle + \int_Q a(t, x, \nabla u_n) \cdot \nabla v + \int_Q H(t, x, u_n, \nabla u_n) v = 0,
$$

(2.9)

for any $v \in L^p(0, T; W^{1,p}_0(\Omega)) \cap L^\infty(\Omega)$. A nonnegative weak solution $u_n$ to problem (2.8) does exist. In fact, problem (2.8) is equivalent to

$$
\begin{cases}
  v'_n - \text{div} a(t, x, \nabla v_n) + H(t, x, v_n + \frac{1}{n} \nabla v_n) = 0 & \text{in } Q, \\
  v_n(0, x) = u_0(x) & \text{in } \Omega, \\
  v_n(t, x) = 0 & \text{on } \Gamma,
\end{cases}
$$

(2.10)

where $v_n = u_n - \frac{1}{n}$.

To prove that a solution of (2.10) exists, we first extend $H(t, x, s, \xi)$ to zero for $s \leq 0$, and, for $\varepsilon > 0$, we consider the problem

$$
\begin{cases}
  v'_{n,\varepsilon} - \text{div} a(t, x, \nabla v_{n,\varepsilon}) + \frac{T_\varepsilon(v_{n,\varepsilon})}{\varepsilon} H(t, x, v_{n,\varepsilon} + \frac{1}{n} \nabla v_{n,\varepsilon}) = 0 & \text{in } Q, \\
  v_{n,\varepsilon}(0, x) = u_0(x) & \text{in } \Omega, \\
  v_{n,\varepsilon}(t, x) = 0 & \text{on } \Gamma.
\end{cases}
$$

(2.11)

A nonnegative solution $v_{n,\varepsilon}$ to problem (2.11) exists by the results proven in [10]. Then, if we take a sequence of values $\varepsilon \downarrow 0$, one can prove that the sequence $\{v_{n,\varepsilon}\}$ converges strongly in $L^p(0, T; W^{1,p}_0(\Omega))$ to some function $v_n$. Then one can pass to the limit for $\varepsilon \downarrow 0$ in the first two terms of (2.11), in the sense of distributions. As far as the third term is concerned, we observe that, on the set $\{v_n > 0\}$, the function $T_\varepsilon(v_{n,\varepsilon})$ converges a.e. to 1, while on the set $\{v_n = 0\}$ (where we cannot identify the limit of $\frac{T_\varepsilon(v_{n,\varepsilon})}{\varepsilon}$, but where $\nabla v_n = 0$ a.e. by Stampacchia’s result contained in [24]) the term $H(t, x, v_{n,\varepsilon} + \frac{1}{n} \nabla v_{n,\varepsilon})$ converges a.e. to $H(t, x, v_n + \frac{1}{n} \nabla v_n)$, which is zero a.e. on this set, since $H(t, x, \frac{1}{n}, 0) = 0$ a.e. by assumption (2.7). Therefore, $v_n$ is a weak solution of (2.10).
2.2. Basic a priori estimates. A standard argument allows us to show that some basic estimates on the approximating solutions hold. We collect them in the following

Lemma 2.4. Let $p \geq 2$, and let $u_n$ be a solution to problem (2.8). Then,
\[ \|u_n\|_{L^p(0,T;W^{1,p}(\Omega))} \leq C, \quad \|u_n\|_{L^\infty(Q)} \leq C, \]
and
\[ (2.12) \int_Q H(t,x,u_n,\nabla u_n) \leq C. \]
Moreover, there exists a function $u$ in $L^p(0,T;W^{1,p}_0(\Omega))$ such that (up to subsequences) $u_n - \frac{1}{n}$ converges to $u$ weakly in $L^p(0,T;W^{1,p}_0(\Omega))$ and a.e. on $Q$. Finally,
\[ \nabla u_n \rightharpoonup \nabla u, \quad \text{a.e. on } Q. \]

Proof. The proof of the first two estimates is quite standard and can be deduced for instance as in [5] (see also [7]) using the fact that the lower order term is positive.

In order to get (2.12) one can use $\frac{1}{\varepsilon} T_\varepsilon(u_n - \frac{1}{n})$ as test function in (2.8). Integrating by parts, dropping nonnegative terms and letting $\varepsilon$ go to zero one gets, by Fatou’s lemma
\[ \int_Q H(t,x,u_n,\nabla u_n) \leq \int_\Omega \left( u_0(x) + \frac{1}{n} \right), \]
which implies (2.12). The almost everywhere convergence of the gradients of $u_n$ is a consequence of (2.12) and of a result in [7].

3. Proof of Theorem 2.2: the case $p > 2$

In this section we give the proof of Theorem 2.2 in the degenerate case $p > 2$. We wish to pass to the limit in the weak formulation of (2.8).

A key tool in order to pass to the limit will be the following one.

Lemma 3.1. Let $p > 2$, and let $u_n$ be a weak solution of problem (2.8). Then, for any $\omega \subset \subset \Omega$, there exists a constant $c_\omega$ such that
\[ u_n \geq c_\omega > 0, \quad \text{in } (0,T) \times \omega, \quad \text{for every } n \in \mathbb{N}. \]

Before the proof we recall some technical tools we will use. The first one is a well known consequence of Gagliardo-Nirenberg inequality which is valid on any cylinder of the type $Q = (0,T) \times \Omega$ with bounded $\Omega$ (see for instance [22], Lecture II).

Lemma 3.2. Let $v \in L^p(0,T;W^{1,p}_0(\Omega)) \cap L^\infty(0,T;L^\beta(\Omega))$, with $p \geq 1$, $\beta \geq 1$. Then $v \in L^\sigma(Q)$ with $\sigma = \frac{pN+\beta}{N\beta}$ and
\[ (3.13) \int_Q |v|^\sigma \leq C \|v\|^{\frac{\beta p}{\beta + p}}_{L^\infty(0,T;L^\beta(\Omega))} \int_Q |\nabla v|^p. \]

Finally, we will need the following local version of a lemma by Stampacchia (see [24]).
Lemma 3.3. Let $\omega(h, r)$ be a function defined on $[0, +\infty) \times [0, 1]$, which is nonincreasing in $h$ and nondecreasing in $r$; suppose that there exist constants $k_0 \geq 0$, $M$, $\rho$, $\sigma > 0$, and $\eta > 1$ such that

$$\omega(h, r) \leq \frac{M \omega(k, R)^\eta}{(h - k)^\rho(R - r)^\sigma},$$

for all $h > k \geq k_0$ and $0 \leq r < R \leq 1$. Then, for every $r$ in $(0, 1)$, there exists $d > 0$, given by

$$d^\rho = \frac{M 2^{\frac{\alpha(p + \sigma)}{\eta - 1}} \omega(k_0, 1)^{\eta - 1}}{(1 - r)^\sigma},$$

such that

$$\omega(d, r) = 0.$$

In order to simplify notations, we henceforth write $a(\nabla u_n)$ instead of $a(t, x, \nabla u_n)$ and $H(u_n, \nabla u_n)$ instead of $H(t, x, u_n, \nabla u_n)$.

**Proof of Lemma 3.1.** We divide the proof into a few steps.

**Step 1.** There is no loss in generality in assuming that the constant $B$ which appears in (2.7) satisfies $B > \max(\alpha, p - 1)$.

We use $v = -u_n^{-B}\psi$ in (2.9), for any nonnegative $\psi(t, x) \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ which is zero in a neighborhood of $(0, T) \times \partial \Omega$, in order to obtain

$$- \int_0^T \langle u_n', u_n^{-B}\psi \rangle + B \int_Q a(\nabla u_n) \cdot \nabla u_n u_n^{-B-1}\psi + \int_Q a(\nabla u_n) \cdot \nabla \psi u_n^{-B} - \int_Q H(u_n, \nabla u_n) u_n^{-B}\psi = 0,$$

from which, taking into account the assumptions (2.4), (2.7), and $B > \alpha$, one obtains

$$- \int_0^T \langle u_n', u_n^{-B}\psi \rangle \leq - \int_Q a(\nabla u_n) \cdot \nabla \psi u_n^{-B} \leq 0.$$

Therefore, if we set $u_n = \frac{B+1}{p-1} w_n^{\frac{p-1}{B+1-p}}$ and $\gamma = \frac{(p-1)(B-1)}{B+1-p}$, we have

$$\int_0^T \langle w_n', w_n^{\gamma-1}\psi \rangle - C \int_Q a(-w_n^{-\frac{B}{B+1-p}} \nabla w_n) \cdot \nabla \psi w_n^{\frac{B(p-1)}{B+1-p}} \leq 0,$$

for every nonnegative $\psi(t, x) \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ which is zero in a neighborhood of $(0, T) \times \partial \Omega$ and for some positive constant $C$ depending only on $B$ and $p$. Observe that $B > p - 1$ and $p > 2$ imply that $\gamma > p - 1 > 1$. Moreover, observe that

$$w_n(t, x) = cn^{\frac{B+1-p}{p-1}} \text{ on } (0, T) \times \partial \Omega, \quad w_n(0, x) = c \left(u_0 + \frac{1}{n}\right)^{-\frac{B+1-p}{p-1}} =: w_{0n},$$

for some positive constant $c$. In particular, since $u_0$ is bounded away from zero, then $w_{0n}$ is bounded in $L^\infty(\Omega)$ and the values of $w_n$ blow up on the boundary as $n$ goes to infinity.

We look for an a priori local bound on the $L^\infty$ norm of $w_n$.

**Step 2.** Local $L^\infty$ bound for $w_n$. 
Without loss of generality we assume that \( 0 \in \Omega \); we will prove that the bound holds true in a ball \( B_\rho \) centered at zero of radius \( \rho \) with \( 0 < \rho < R \leq 1 \), then a standard covering argument will allow us to conclude.

We fix \( k > \| w_0 \|_{L^\infty(B_\rho)} \), and we define the sets
\[
A_{k,\rho}(t) = \{ x \in B_\rho : w_n(t,x) > k \}, \quad A_{k,\rho} = \{ (t,x) \in (0,T) \times B_\rho : w_n(t,x) > k \}.
\]

We consider a cut-off function \( \eta(x) \in C^\infty_c(B_R) \) such that
\[
0 \leq \eta(x) \leq 1, \quad \eta(x) \equiv 1 \text{ in } B_r, \quad |\nabla \eta| \leq \frac{c}{R-r},
\]
and use \( \psi(t,x) = G_k(w_n(t,x))\varphi^\delta(x) \) as test function in (3.14), where \( \delta = \frac{p(\gamma+1)}{\gamma-p+1} \). Integrating between 0 and \( t < T \), and using the assumptions (2.4) and (2.5), we obtain
\[
\int_0^t \int_{A_{k,\rho}(\tau)} |\nabla G_k(w_n)|^p \varphi^\delta \leq \frac{C}{(R-r)^p} \int_{A_{k,\rho}} G_k(w_n)^p \varphi^{\frac{p^2}{\gamma-p+1}}.
\]

We use Young’s inequality in order to absorb the term \( |\nabla G_k(w_n)|^{p-1} \) in the right hand side; using the definition of \( \delta \), we get
\[
\int_0^t \int_{A_{k,\rho}(\tau)} |\nabla G_k(w_n)|^p \varphi^\delta \leq \frac{C}{(R-r)^p} \int_{A_{k,\rho}} G_k(w_n)^p \varphi^{\frac{p^2}{\gamma-p+1}}.
\]

Notice that, since \( 0 \leq \varphi \leq 1 \),
\[
|\nabla (G_k(w_n)\varphi^\delta)|^p \leq C \left( |\nabla G_k(w_n)|^p \varphi^{\delta p} + \frac{G_k(w_n)^p \varphi^{\delta p-p}}{(R-r)^p} \right)
\]
\[
\leq C \left( |\nabla G_k(w_n)|^p \varphi^\delta + \frac{G_k(w_n)^p \varphi^{\delta-p}}{(R-r)^p} \right),
\]
so that, observing that \( \delta - p = \frac{p^2}{\gamma-p+1} \), we finally get
\[
\int_0^t \int_{A_{k,\rho}(\tau)} |\nabla G_k(w_n)|^p \varphi^\delta \leq \frac{C}{(R-r)^p} \int_{A_{k,\rho}} G_k(w_n)^p \varphi^{\frac{p^2}{\gamma-p+1}}.
\]

Since \( \gamma > p-1 \), we can use again Young’s inequality to have
\[
\int_0^t \int_{A_{k,\rho}(\tau)} |\nabla (G_k(w_n)\varphi^\delta)|^p \leq \frac{1}{2(\gamma+1)T} \int_{A_{k,\rho}} G_k(w_n)^{\gamma+1} \varphi^\delta + \frac{C}{(R-r)^{\gamma+1-p}} |A_{k,R}|
\]
\[
\leq \frac{1}{2(\gamma+1)} \sup_t \int_{A_{k,\rho}(t)} G_k(w_n)^{\gamma+1} \varphi^\delta \, dx + \frac{C}{(R-r)^{\gamma+1-p}} |A_{k,R}|.
\]
Now we deal with the time derivative part. Let us define, for $s \geq 0$,
\[ \Psi_k(s) = \int_0^s G_k(\sigma) \sigma^{-1} \, d\sigma. \]
Then it is easy to check that
\[ \Psi_k(s) \geq \frac{1}{\gamma + 1} G_k(s)^{\gamma + 1}. \]
that $k > \|w_0\|_{L^\infty(B_R)}$, we obtain
\[ \int_0^t \langle w'_n, w_n^{\gamma-1} G_k(w_n) \varphi^\delta \rangle = \int_{B_R} \Psi_k(w_n(t,x)) \varphi^\delta \geq \frac{1}{\gamma + 1} \int_{B_R} G_k(w_n)^{\gamma + 1} \varphi^\delta. \]
Therefore we can use (3.15) in order to deduce
\[ \frac{1}{\gamma + 1} \int_{B_R} G_k(w_n)^{\gamma + 1} \varphi^\delta \leq \frac{1}{2(\gamma + 1)} \sup_t \int_{A_k,R} G_k(w_n)^{\gamma + 1} \varphi^\delta + \frac{C}{(R - r)^{\gamma + 1 - p}} |A_k,R| \]
and we can take the supremum over $t \in (0, T)$ on the left in order to get
\[ \sup_t \int_{A_k,R(t)} G_k(w_n)^{\gamma + 1} \varphi^\delta \leq \frac{C}{(R - r)^{\gamma + 1 - p}} |A_k,R|. \]
Gathering together all these facts, and using again that $\varphi^\delta \geq \varphi^{\delta(\gamma + 1)}$, we end up with the following estimate
\[ \sup_t \int_{A_k,R(t)} (G_k(w_n) \varphi^\delta)^{\gamma + 1} + \int_{A_k,R} |\nabla (G_k(w_n) \varphi^\delta)|^p \leq \frac{C}{(R - r)^{\gamma + 1 - p}} |A_k,R|. \]
We are now in the position to apply the Gagliardo-Nirenberg inequality (3.13) to the function $G_k(w_n) \varphi^\delta$, with $\beta = \gamma + 1$; recalling that $\varphi \equiv 1$ on $B_r$, we obtain
\[ \int_{A_k,R} G_k(w_n)^p \frac{N + \gamma + 1}{N} \leq \frac{C}{(R - r)^{(\gamma + 1)pN}} |A_k,R|^{1 + \frac{p}{N}}. \]
Stampacchia’s procedure is now quite standard. For $h > k$, one obtains
\[ \int_{A_k,R} G_k(w_n)^p \frac{N + \gamma + 1}{N} \geq \int_{A_h,R} G_h(w_n)^p \frac{N + \gamma + 1}{N} \geq (h - k)^p \frac{N + \gamma + 1}{N} |A_h,R|, \]
that is,
\[ |A_h,R| \leq \frac{C |A_k,R|^{1 + \frac{p}{N}}}{(h - k)^p \frac{N + \gamma + 1}{N} (R - r)^{(\gamma + 1)pN}}. \]
Therefore, if we choose $\omega(h, r) = |A_h,R|$, we can apply Lemma 3.3 in order to deduce that, for every fixed $\rho \in (r, R)$, $|A_h,r| = 0$ if $h$ is larger than some constant $C_\rho$. It follows that
\[ (3.16) \quad w_n \leq C_\rho, \quad \text{a.e. on } (0, T) \times B_\rho, \quad \text{for every } n. \]

**Step 3.** End of the proof. Recalling that $B > p - 1$ and the definition of $w_n$ we use (3.16) to have, a.e. on $(0, T) \times B_\rho$
\[ u_n = B + 1 - p \frac{w_n^{\rho - 1}}{p - 1} \geq \frac{B + 1 - p}{p - 1} C_\rho^{-\frac{p - 1}{p + 1 - p}} \equiv c_\rho > 0. \]
As we said, by means of a standard covering procedure this estimate can be proven to hold on any set of the form $(0, T) \times \omega$, with $\omega \subset \subset \Omega$.

3.1. **Passing to the limit.** In order to pass to the limit as $n$ tends to infinity, we need the following result.

**Proposition 3.4.** We have

\[ u_n \rightharpoonup u \quad \text{strongly in } L^p(0, T; W^{1,p}(\omega)), \]

for every open set $\omega \subset \subset \Omega$.

**Proof.** The sequence $\{u'_n\}$ is bounded in $L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$. Using the Aubin-Simon compactness argument (see Corollary 4 in [24]) we deduce that, up to a subsequence,

\[ u_n \rightharpoonup u \quad \text{in } L^p(Q), \]

for some $u$ in $L^p(0, T; W^{1,p}(\omega))$. We will prove that, for every open set $\omega \subset \subset \Omega$,

\[ u_n \rightharpoonup u \quad \text{in } L^p(0, T; W^{1,p}(\omega)). \tag{3.17} \]

We now introduce a classical regularization $u_\nu$ of the function $u$ with respect to time (see [16]). For every $\nu \in \mathbb{N}$, we define $u_\nu$ as the solution of the Cauchy problem

\[
\begin{aligned}
\frac{1}{\nu} u'_\nu + u_\nu &= u, \\
 u_\nu(0) &= u_{0,\nu},
\end{aligned}
\]

where $u_{0,\nu}$ belongs to $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ and satisfies

\[ u_{0,\nu} \rightharpoonup u_0 \quad \text{strongly in } L^1(\Omega) \text{ and } \ast\text{-weakly in } L^\infty(\Omega), \quad \lim_{\nu \to +\infty} \frac{1}{\nu} \|u_{0,\nu}\|_{W^{1,p}_0(\Omega)} = 0. \]

Then one has (see [16]):

\[ u_\nu \in L^p(0, T; W^{1,p}_0(\Omega)) \quad u'_\nu \in L^p(0, T; W^{1,p}_0(\Omega)), \]

\[ \|u_\nu\|_{L^\infty(Q)} \leq \|u\|_{L^\infty(Q)}, \]

and, as $\nu$ tends to infinity,

\[ u_\nu \rightharpoonup u \quad \text{strongly in } L^p(0, T; W^{1,p}_0(\Omega)). \tag{3.18} \]

Let $\varphi_\lambda(s) = se^{\lambda s^2}$ (with $\lambda$ to be chosen later). We will denote by $\varepsilon(\nu, n)$ any quantity such that

\[ \lim_{\nu \to +\infty} \limsup_{n \to +\infty} |\varepsilon(\nu, n)| = 0. \]

For $0 \leq \phi \in C^\infty_c(\Omega)$ we have that

\[ \int_0^T \langle u'_\nu, \varphi_\lambda(u_n - u_\nu) \phi \rangle \geq \varepsilon(\nu, n), \tag{3.19} \]

(see [16] [17]).
Now, using (3.19) and \( \varphi_\lambda(u_n - u_\nu) \phi \) as test function in (2.9), we obtain
\[
\int_Q a(\nabla u_n) \cdot \nabla (u_n - u_\nu) \varphi'_\lambda(u_n - u_\nu) \phi + \int_Q a(\nabla u_n) \cdot \nabla \phi \varphi_\lambda(u_n - u_\nu) + \int_Q H(u_n, \nabla u_n) \varphi_\lambda(u_n - u_\nu) \phi \leq -\varepsilon(\nu, n).
\]

Moreover, if \( \omega \subseteq \Omega \) is such that \( \text{supp} \phi \subseteq \omega \), since \( u_n \rightarrow u \) weakly in \( L^p(0,T; W^{1,p}_0(\Omega)) \) and \( \varphi_\lambda(u_n - u_\nu) \) converges to \( \varphi_\lambda(u - u_\nu) \) weakly in \( L^\infty(Q) \), we have
\[
\int_Q a(\nabla u_n) \cdot \nabla \phi \varphi_\lambda(u_n - u_\nu) = \varepsilon(\nu, n).
\]

If \( c_\omega \) is the constant given by Lemma 3.1, we have, recalling that \( \text{supp} \phi \subseteq \omega \),
\[
\left| \int_Q H(u_n, \nabla u_n) \varphi_\lambda(u_n - u_\nu) \phi \right| \leq B \int_{\omega \times (0,T)} |\nabla u_n|^p |\varphi_\lambda(u_n - u_\nu)| \phi \leq \frac{B}{c_\omega} \int_Q |\nabla u_n|^p |\varphi_\lambda(u_n - u_\nu)| \phi.
\]

Thus,
\[
(3.20) \quad \int_Q a(\nabla u_n) \cdot \nabla (u_n - u_\nu) \varphi'_\lambda(u_n - u_\nu) \phi - \frac{B}{c_\omega} \int_Q |\nabla u_n|^p |\varphi_\lambda(u_n - u_\nu)| \phi \leq \varepsilon(\nu, n).
\]

Then we can write
\[
\int_Q a(\nabla u_n) \cdot \nabla (u_n - u_\nu) \varphi'_\lambda(u_n - u_\nu) \phi = \int_Q [a(\nabla u_n) - a(\nabla u_\nu)] \cdot \nabla (u_n - u_\nu) \varphi'_\lambda(u_n - u_\nu) \phi + \int_Q a(\nabla u_\nu) \cdot \nabla (u_n - u_\nu) \varphi'_\lambda(u_n - u_\nu) \phi
\]
\[
= \int_Q [a(\nabla u_n) - a(\nabla u_\nu)] \cdot \nabla (u_n - u_\nu) \varphi'_\lambda(u_n - u_\nu) \phi + \varepsilon(\nu, n).
\]

Similarly,
\[
\int_Q |\nabla u_n|^p |\varphi_\lambda(u_n - u_\nu)| \phi \leq \alpha^{-1} \int_Q a(\nabla u_n) \cdot \nabla u_n |\varphi_\lambda(u_n - u_\nu)| \phi
\]
\[
= \alpha^{-1} \int_Q [a(\nabla u_n) - a(\nabla u_\nu)] \cdot \nabla (u_n - u_\nu) |\varphi_\lambda(u_n - u_\nu)| \phi + \int_Q a(\nabla u_\nu) \cdot \nabla (u_n - u_\nu) |\varphi_\lambda(u_n - u_\nu)| \phi + \int_Q a(\nabla u_n) \cdot \nabla u_\nu |\varphi_\lambda(u_n - u_\nu)| \phi
\]
\[
= \alpha^{-1} \int_Q [a(\nabla u_n) - a(\nabla u_\nu)] \cdot \nabla (u_n - u_\nu) |\varphi_\lambda(u_n - u_\nu)| \phi + \varepsilon(\nu, n).
\]

Therefore, from (3.20) we obtain
\[
\int_Q [a(\nabla u_n) - a(\nabla u_\nu)] \cdot \nabla (u_n - u_\nu) [\varphi'_\lambda(u_n - u_\nu) - \frac{B}{\alpha c_\omega} |\varphi_\lambda(u_n - u_\nu)|] \phi \leq \varepsilon(\nu, n).
\]
Choosing \( \lambda \) large enough so that \( \varphi'_\lambda(s) - \frac{B}{\alpha \omega} |\varphi_\lambda(s)| \geq \frac{1}{2} \) for every \( s \in \mathbb{R} \), we deduce that
\[
\int_Q \left[ a(\nabla u_n) - a(\nabla u_\nu) \right] \cdot \nabla (u_n - u_\nu) \phi \leq \varepsilon(n, \nu).
\]
From here it is standard (see for example [6]) to prove that \( u_n - u_\nu \) tends to zero strongly in \( L^p(0, T; W^{1, p}(\omega)) \). Recalling (3.18), we thus have that (3.17) holds.

Using Proposition 3.4 we can prove the (local) strong convergence of the lower order terms.

**Lemma 3.5.** We have
\[
H(u_n, \nabla u_n) \to H(u, \nabla u), \quad \text{locally strongly in } L^1(Q).
\]

**Proof.** Gathering together the results of Theorem 3.4, Lemma 2.4, and Lemma 3.1, we can apply Lebesgue’s dominated convergence theorem to prove that
\[
\frac{|\nabla u_n|^p}{u_n} \to \frac{|\nabla u|^p}{u}, \quad \text{locally strongly in } L^1(Q).
\]
It is then straightforward to conclude using (2.7) and Vitali’s theorem.

Thanks to all the results proved so far we can pass to the limit in (2.8), to have that \( u \) is a solution of (2.3) in the sense of Definition 2.1, thus concluding the proof of Theorem 2.2 in the degenerate case \( p > 2 \).

**Remark 3.6.** We want to point out that since the same argument of the proof of Lemma 3.1 applies to supersolutions of problem (2.8) Thus, in particular, one could prove the same result of Theorem 2.2 also for nonhomogeneous problems like
\[
\begin{aligned}
&u' - \text{div} \, a(t, x, \nabla u) + H(t, x, u, \nabla u) = f &\text{in } Q, \\
&u(0, x) = u_0(x) &\text{in } \Omega, \\
&u(t, x) = 0 &\text{on } \Gamma,
\end{aligned}
\]
with \( 0 \leq f \) in \( L^r(0, T; L^q(\Omega)) \), with
\[
\frac{p}{r} + \frac{N}{q} < p, \quad r \geq p', \quad q > 1.
\]

4. **Proof of Theorem 2.2: the case \( p = 2 \)**

As before, we start from the approximate problems (2.8), and the key result will be the strong positivity of \( u_n \) on \((0, T) \times \omega, \) with \( \omega \subset \subset \Omega. \)

**Lemma 4.1.** Assume that (2.4), (2.5), (2.6) and (2.7) hold with \( p = 2 \) and \( B < \alpha \), and let \( u_n \) be a weak solution of problem (2.8). Then, for any \( \omega \subset \subset \Omega, \) there exists a constant \( c_\omega \) such that
\[
u 
\int_0^T \int_\omega |\nabla u_n|^2 \phi \leq c_\omega \int_0^T \int_\omega f \phi.
\]

\[\int_Q \left[ a(\nabla u_n) - a(\nabla u_\nu) \right] \cdot \nabla (u_n - u_\nu) \phi \leq \varepsilon(n, \nu).\]
Proof. We multiply the equation in (2.8) by $u_n^{-\theta} \psi$, where
$$\theta = \frac{B}{\alpha} \in (0, 1)$$
and $\psi(t, x) \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ is a nonnegative function which is zero in a neighborhood of $(0, T) \times \partial \Omega$. As in Section 3, a straightforward calculation shows that $u_n$ satisfies
$$\int_0^T \langle u'_n, u_n^{-\theta} \psi \rangle + \int_Q a(\nabla u_n) \cdot \nabla \psi u_n^{-\theta} \geq 0.$$ 
Then, if we define
$$v_n = \left( \frac{u_n}{1-\theta} \right)^{1-\theta},$$
the previous inequality can be read as
$$(1-\theta)^{\theta+2} \int_0^T \langle v'_n, \psi \rangle + \int_Q \tilde{a}(t, x, v_n, \nabla v_n) \cdot \nabla \psi \geq 0,$$
where we have set
$$\tilde{a}(t, x, s, \xi) = s^{-\frac{\theta}{1-\theta}} a(t, x, s^{\frac{\theta}{1-\theta}} \xi).$$
In other words, $v_n$ is a weak solution of the variational inequality
$$\begin{cases}
(1-\theta)^{\theta+2} v'_n - \text{div} \, \tilde{a}(t, x, v_n, \nabla v_n) \geq 0 & \text{in } Q, \\
v_n(0, x) = \left( \frac{u_0 + 1}{1-\theta} \right)^{1-\theta} & \text{in } \Omega, \\
v_n(t, x) = \left( \frac{1}{1-\theta} \right)^{1-\theta} & \text{on } \Gamma.
\end{cases}$$
Since the vector-valued function $\tilde{a}$ satisfies the standard inequalities
$$\tilde{a}(t, x, s, \xi) \cdot \xi \geq \alpha |\xi|^2,$$
$$|\tilde{a}(t, x, s, \xi)| \leq \beta |\xi|,$$
$$(\tilde{a}(t, x, s, \xi) - \tilde{a}(t, x, s, \eta)) \cdot (\xi - \eta) > 0,$$
by the standard maximum principle (see [15]) we easily deduce that $v_n \geq C_\omega > 0$, in any $\omega \subset \subset \Omega$; this implies the result. □

Once we have proved Lemma 4.1, the proof of Theorem 2.2 can be concluded as in Section 3, using Proposition 3.4 and Lemma 3.5, which continue to hold also for $p = 2$, with the same proof.

5. The case $p = 2$ and $B \geq \alpha$

In this section we try to explain how nonexistence of solutions (in the sense of approximating sequences) may actually occur for problem (2.3) with $p = 2$ if the absorption
parameter $B$ is too large. For the sake of explanation, we will restrict our attention to the model problem

$$
\begin{cases}
  u' - \Delta u + B \frac{\nabla u}{u} = 0 & \text{in } Q, \\
  u(0, x) = u_0(x) & \text{in } \Omega, \\
  u(t, x) = 0 & \text{on } \Gamma,
\end{cases}
$$

for $B \geq 1$. An easy rescaling argument will then allow to deal with the case $B \geq \alpha$ for

$$
u' - \alpha \Delta u + B \frac{\nabla u}{u} = 0.
$$

In order to provide a simpler proof, without loss of generality, we will consider a constant initial datum (e.g., $u_0 = 1$); the case of a general initial datum can be obtained in a straightforward way by comparison (recall that $u_0 \geq c > 0$) and elementary changes in the proof.

Our result is the following.

**Proposition 5.1.** Let $B \geq 1$, and let $u_n$ be a solution of problem

$$
\begin{cases}
  u_n' - \Delta u_n + B \frac{\nabla u_n}{u_n} = 0 & \text{in } Q, \\
  u_n(0, x) = 1 & \text{in } \Omega, \\
  u_n(t, x) = \frac{1}{n} & \text{on } \Gamma.
\end{cases}
$$

Then

$$u_n \to 0 \text{ uniformly on compact subsets of } Q.$$

**Proof.** We first analyze the case $B = 1$, readapting an idea of [20]. Let $u_n$ be a solution to (5.21) and define $v_n = -\log(u_n)$. Then $v_n$ solves

$$
\begin{cases}
  v_n' - \Delta v_n = 0 & \text{in } Q, \\
  v_n(0, x) = 0 & \text{in } \Omega, \\
  v_n(t, x) = \log(n) & \text{on } \Gamma.
\end{cases}
$$

On the other hand it is clear that $v_n = (1 - w) \log(n)$, where $w$ is the unique solution to problem

$$
\begin{cases}
  w' = \Delta w & \text{in } Q, \\
  w(0, x) = 1 & \text{in } \Omega, \\
  w(t, x) = 0 & \text{on } \Gamma.
\end{cases}
$$

By the strong maximum principle we have

$$0 < w(t, x) < 1 \text{ for any } (t, x) \text{ in } Q,$$

so that $v_n \to +\infty$ uniformly on compact subsets of $Q$. Hence,

$$u_n = e^{-v_n} \to 0 \text{ uniformly on compact subsets of } Q,$$

as desired.
If $B > 1$, a similar argument applies; here the right change of variable is

$$v_n = -\frac{u_n^{1-B}}{1-B},$$

which transforms problem (5.21) into

$$\begin{cases}
    v'_n - \Delta v_n = 0 & \text{in } Q, \\
    v_n(0, x) = \frac{1}{B-1} & \text{in } \Omega, \\
    v_n(t, x) = \frac{n^{B-1}}{B-1} & \text{on } \Gamma.
\end{cases}$$

The conclusion then follows as before, since $v_n$ locally uniformly on compact subsets of $Q$, and $u_n = [(B - 1)v_n]^{-\frac{1}{p-1}}$. □

6. The case $p < 2$. Some partial results.

In this last section we will briefly discuss the singular case $p < 2$. We will show that no solutions in the sense of Definition 2.1 can be expected as finite time extinction may occur. Moreover, we will also show some explicit nondegenerate examples of evolution.

6.1. Extinction in finite time. Consider

$$\begin{cases}
    u' - \Delta_p u + B \frac{\lvert \nabla u \rvert^p}{u} = 0 & \text{in } Q, \\
    u(0, x) = u_0(x) & \text{in } \Omega, \\
    u(t, x) = 0 & \text{on } \Gamma,
\end{cases}$$

(6.22)

where $T > 0$, $N \geq 2$. Concerning the initial datum $u_0$ we assume that $u_0$ is a nonnegative bounded function such that $u_0 \geq c > 0$ almost everywhere in $\Omega$.

We have the following result that should be compared with finite time extinction property for the, formally equivalent, doubly nonlinear case (see for instance [25] and references therein).

**Lemma 6.1.** Let $p < 2$, $B < p - 1$, $u_0$ be as above, and let $u$ be a solution of (6.22); then there exists $T^* > 0$ such that $u(T^*, x) \equiv 0$.

**Proof.** Let

$$w = \frac{p - 1}{p - 1 - B} u^{\frac{p-1-B}{p-1}},$$

so that $w$ solves

$$\begin{cases}
    w^p w' - \Delta_p w = 0 & \text{in } Q, \\
    w(0, x) = \frac{p-1}{p-1-B} u_0(x)^{\frac{p-1-B}{p-1}} & \text{in } \Omega, \\
    w(t, x) = 0 & \text{on } \Gamma,
\end{cases}$$
with $\beta = \frac{(2-p)B}{p-1-B}$. We consider $w^\gamma$ (where $\gamma$ will be chosen later) as test function in the previous problem, and we integrate on $Q_t = (0,t) \times \Omega$, with $0 < t < T$. We obtain
\[
\int_{\Omega} w(t)^{\gamma+\beta+1} + \int_{Q_t} |\nabla w|^p w^{\gamma-1} \leq C,
\]
that implies
\[
\int_{\Omega} w(t)^{\gamma+\beta+1} + \int_{Q_t} |\nabla w|^\frac{\gamma+p-1}{p} |w| \leq C.
\]
Then using Sobolev inequality
\[
\int_{\Omega} w(t)^{\gamma+\beta+1} + \left( \int_{Q_t} w \frac{(\gamma+p-1)p^*}{p} \right)^\frac{p}{p^*} \leq C.
\]
Now observe that, by Hölder inequality, that can be used as $(\gamma + p - 1)^{p^*} > \gamma + \beta + 1$ (for $\gamma$ large enough),
\[
\int_{Q_t} w^{\gamma+\beta+1} \leq C \left( \int_{Q_t} w \frac{(\gamma+p-1)p^*}{p} \right)^{\frac{p}{p^*}(\gamma+\beta+1)}.
\]
Hence, if we define
\[
\eta(t) = \int_{Q_t} w^{\gamma+\beta+1},
\]
we have, for some positive constants $C_1$ and $C$,
\[
\eta'(t) + C_1 \eta(t)^{\frac{\gamma+p-1}{\gamma+\beta+1}} \leq C.
\]
We now remark that $\frac{\gamma+p-1}{\gamma+\beta+1} < 1$ due to the assumptions $p < 2$ and $B < p - 1$. Therefore, by standard ODE analysis, there exists $T^* > 0$ such that $\eta(T^*) = 0$; this implies the finite time extinction for $w$, hence for $u$. \(\square\)

6.2. An example of nontrivial evolution. As we already said, the analysis in the case $p < 2$ is much more delicate. In the previous section we showed that in some cases finite time extinction may occur and so no solutions (in the sense of Definition 2.1) can be found in general. In this section we provide an explicit example of solution that shows that, in some cases, a solution exists at least for small times.

Example 6.2. Let $\frac{2N}{N+2} < p < 2$ and let $B$ be such that
\[
0 < B < \frac{(p^* - 2)(p - 1)}{p^* - p} < p - 1.
\]
First of all, consider the solution $v$ to problem
\[
\begin{cases}
-\Delta_p v = \gamma^{p-1} v^{\frac{(\gamma-1)(p-1)+1}{\gamma}} & \text{in } \Omega, \\
v = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where,
\[
\gamma = \frac{p-1-B}{p-1} > 0.
\]
Problem (6.23) admits a nontrivial, positive, variational solution obtained by a standard mountain pass procedure as soon as
\[ p - 1 < \frac{(\gamma - 1)(p - 1) + 1}{\gamma} < p^* - 1, \]
and an easy calculation shows that the previous relation is equivalent to
\[ B < \frac{(p^* - 2)(p - 1)}{p^* - p}. \]

Consider now the function \( \phi = v^\frac{1}{p} \), which solves the problem
\[
\begin{aligned}
-\Delta_p \phi + B \frac{\nabla \phi}{\phi} = \phi & \quad \text{in } \Omega, \\
\phi = 0 & \quad \text{on } \partial \Omega,
\end{aligned}
\]
and define
\[
\phi(t,x) = (1 - (2 - p)t)^{\frac{1}{p - 1}} \phi(x).
\]

We can compute the problem solved by \( u \), we have
\[
\begin{aligned}
u' - \Delta_p u + B \frac{\nabla u}{u} = (1 - (2 - p)t)^{\frac{1}{p - 1}} \left( -\Delta_p \phi + B \frac{\nabla \phi}{\phi} - \phi \right) &= 0,
\end{aligned}
\]
so that \( u \) is a solution of problem
\[
\begin{aligned}
u' - \Delta_p u + B \frac{\nabla u}{u} = 0 & \quad \text{in } Q, \\
u(0,x) = \phi(x) & \quad \text{in } \Omega, \\
u(t,x) = 0 & \quad \text{on } \Gamma,
\end{aligned}
\]
that has \( T^* = \frac{1}{2 - p} > 0 \) as extinction time.

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