Asymptotic stability of synchronous orbits for a gravitating viscoelastic sphere

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Abstract We study the dynamics of a viscoelastic body whose shape and position evolve due to the gravitational forces exerted by a pointlike planet. We work in the quadrupole approximation. We consider the solution in which the center of mass of the body moves on a circular orbit, and the body rotates in a synchronous way about its axis, so that it always shows the same face to the planet as the Moon does with the Earth. We prove that if any internal deformation of the body dissipates some energy, then such an orbit is locally asymptotically stable. The proof is based on the construction of a suitable system of coordinates and on the use of LaSalle’s principle. A large part of the paper is devoted to the analysis of the kinematics of an elastic body interacting with a gravitational field. We think this could have some interest in itself.

Keywords Spin orbit resonance · Viscoelastic satellite · Dynamical systems · Lyapunov stability · Lagrangian mechanics · Dissipative dynamics

1 Introduction

In this paper we study the dynamics of a deformable celestial body interacting with a planet. In particular we are interested in understanding the interaction of the internal degrees of freedom with the orbital and spin ones.

The study of such a problem goes back to Darwin (1879, 1880) (see also Ferraz-Mello 2008; Efroimsky and Williams 2009) and the theory he initiated has been developed by many
Such a theory consists of a procedure which allows to separate the internal degrees of freedom (DOFs) from the orbital and spin ones. This is obtained by showing that, in some approximation, the effect of the internal DOFs is just that of producing an effective force acting on the orbital and spin DOFs. In particular one of the main issues of the theory is that a dissipation acting on the internal DOFs induces an effective dissipation on the orbital and spin DOFs (for a recent reference see Efroimsky 2012). We emphasize that Darwin’s procedure is heuristic, and from a mathematical point of view, its range of validity is far from being clear.

Our purpose in this paper is to prove the phenomenon of stabilization of orbital and spin DOFs in a mathematically rigorous way, at least in one simple model. Our model is the following one: we first approximate the planet by a pure point. Then we model the satellite by an elastic sphere, whose shape will change under the action of the gravitational and dynamical forces.

From a dynamical system point of view, the system must be described by a system of coupled differential equations governing both the evolution of the orbital and spin DOFs and the internal (elastic) DOFs of the satellite. In particular the equations of motion for the elastic DOFs are of course partial differential equations.

The first problem we address is that of writing in a resonable way the equations of motion of the system. It turns out that this is possible, although nontrivial, by making very little assumptions, at least in the quadrupole approximation. The introduction of suitable coordinates occupies a large part of the paper.

Then we prove that, as expected, the so obtained equations have a particular stationary solution in which the center of mass of the satellite moves uniformly on a circular orbit and the satellite is stretched in the direction of the planet (of course the shape of the deformed body corresponds to the standard Love equilibria). Then we prove that, if any internal deformation dissipates energy, then such a stationary solution is asymptotically stable. In particular the orbital DOFs relax to the circular ones. We emphasize that our theory is local, so in particular it is valid only for small initial values of the eccentricity.

We also would like to mention that our approach also applies to satellites whose unper-turbed shape is triaxial (like a rock), but in order to conclude the proof one needs to substitute the argument of Sect. 5 with a different argument which will be the object of a separate paper.

We now briefly describe the proof and the structure of the paper.

The starting point of the proof is the remark that, in quadrupole approximation, the gravitational potential of an extended body in an external gravitational field is a function only of the principal moments of inertia and of the directions of the principal axes of inertia. So it is natural to use such quantities as coordinates in the configuration space of the elastic satellite. We first prove that it is possible to complete such quantities to a system of coordinates. Furthermore, due to the symmetries of the system, the kinetic energy and the potential energy of the body have quite a simple form [see Eq. (4.2)]. For simplicity, we restrict ourselves to the “planar situation” in which the center of mass lies in a plane, the spin axis is orthogonal to such a plane and the deformations are such that one of the principal axes of inertia of the body is always orthogonal to the plane of the orbit. We first study the dynamical system with such a Lagrangian proving the existence of the above mentioned orbits, then we add dissipation and use LaSalle’s principle in order to get the main result. Concerning dissipation, we only assume that any internal deformation of the body produces some nonzero dissipation, and that the stress at a given time is function of the strain and of its time derivative at that fixed time, so that there are no memory effects.