Anomalous time correlation in two-dimensional driven diffusive systems

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We study the time correlation function of a density field in two-dimensional driven diffusive systems within the framework of fluctuating hydrodynamics. It is found that the time correlation exhibits power-law behavior in an intermediate time regime in the case that the fluctuation-dissipation relation is violated and that the power-law exponent depends on the extent of this violation. We obtain this result by employing a renormalization group method to treat a logarithmic divergence in time.

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I. INTRODUCTION

The anomalous time correlation of hydrodynamic modes has been studied for a long period. For an equilibrium, it is understood that this anomaly arises from nonlinear mode coupling effects. By contrast, there is no systematic understanding of the time correlation in nonequilibrium steady states (NESSs) far from local equilibrium. In particular, it is not known how violation of the fluctuation-dissipation relation (FDR) influences the time correlation.

As the simplest example realizing a NESS far from local equilibrium, we consider a two-dimensional driven diffusive system, in which a fluctuating density field is driven locally by an external force, diffusion and random noise. Such a system can be realized in laboratory experiments. Perhaps the simplest model for a theoretical study of the long time behavior in the driven diffusive system is a stochastic differential equation consisting of terms representing a drift due to the external force, diffusion and random noise.

The time correlation function for such a stochastic model has been calculated by employing mode coupling theory. However, the model analyzed in Ref. does not exhibit the long-range spatial correlation that is a generic feature of NESSs in driven diffusive systems of $d \geq 2$ dimensions. The reason that long-range correlation does not appear in that model is that violation of the FDR is not taken into account. Indeed, it is known that, in general, long-range correlation cannot exist when the FDR holds. By contrast, it has been found that the long-range correlation of driven diffusive systems can be described by a linear model with the violation of the FDR.

With the above considerations, in the present paper, we study a nonlinear model in which the FDR can be violated. We demonstrate that as a result of this violation, the time correlation is qualitatively altered. Specifically, by employing a perturbative renormalization group (RG) method that treats a logarithmic divergence in time, we obtain an expression for the time correlation function. From this expression, we find that power-law behavior appears in the time correlation if and only if the FDR is violated and that the power-law exponent depends on the extent of the violation.

II. MODEL

We consider the time evolution of a fluctuating density field $\rho(x, t)$ in a two-dimensional space, under the influence of an external driving force in one direction, say the $x_1$ direction, where $x = (x_1, x_2)$. Note that we study NESSs in the high temperature regime, far from the critical point. We now describe the model we study. First, the conserved quantity $\rho$ obeys the continuity equation

$$\frac{\partial \rho(x, t)}{\partial t} + \sum_{i=1}^{2} \frac{\partial J_i(x, t)}{\partial x_i} = 0. \quad (1)$$

We assume that the $i$-th component of the density current, $J_i(x, t)$, is given by

$$J_i(x, t) = -D_i \delta \rho(x, t) + \delta \xi_i(x, t). \quad (2)$$

Here, the functional form of $J$ is such that, with $\bar{\rho}$ the average density, $\bar{J}(\bar{\rho})$ is the average current along the $x_1$ direction in the steady state. We then approximate $\bar{J}(\rho(x, t))$ in the form

$$\bar{J}(\rho(x, t)) \approx \bar{J}(\bar{\rho}) + c(\bar{\rho})\delta \rho(x, t) + \lambda(\bar{\rho})(\delta \rho(x, t))^2, \quad (3)$$

where $\rho(x, t) = \bar{\rho} + \delta \rho(x, t)$. The term $\xi_i(x, t)$ in (2) represents a random current constituting zero mean Gaussian white noise, with

$$\langle \xi_i(x, t) \xi_j(x', t') \rangle = 2B_i \delta(x - x') \delta(t - t'). \quad (4)$$

Note that because anisotropy in both the diffusion and noise intensity is expected to arise through effects of the external driving, the diffusion constant, $D_i$, and the noise intensity, $B_i$, are assumed to be anisotropic, in general.

Let us simplify the model given above. First, note that the first term on the right-hand side of (3) does
not contribute to the time evolution of the density, and the second term can be eliminated when we study density fluctuations in a frame moving with the velocity \(e_1\) given in [3]. To make this explicit, we define the density \(\phi(x, t) \equiv \rho(x + ce_1t, t)\), where \(e_1\) is the unit vector in the \(x_1\) direction. Furthermore, introducing the parameters \(\chi\) and \(\Delta\), we rewrite \(B_1\) and \(B_2\) as

\[
B_i = D_i \chi(1 - (-1)^i \Delta). 
\]

Thus, \(\Delta\) corresponds to the extent of the violation of the FDR of the second kind [1]. Then, replacing \(x_i\) by \(\sqrt{D_i}x_i\), \(\phi\) by \(\sqrt{D_iD_2}^{1/4}\phi\) and \(\xi_i\) by \(\sqrt{D_iD_1D_2}^{-1/4}\xi_i\), we obtain the following dimensionless form of the equation for \(\phi\):

\[
\frac{\partial \phi(x, t)}{\partial t} = \sum_{i=1}^{2} \left[ \nabla^2 \phi(x, t) - \frac{\partial \xi_i(x, t)}{\partial x_i} \right] - \lambda \phi(x, t)^2. 
\]

(6)

Here,

\[
\langle \xi_i(x, t)\xi_j(x', t') \rangle = 2\delta_{ij}(1 - (-1)^i \Delta)\delta(x - x')\delta(t - t'), 
\]

(7)

and \(\bar{\lambda}\) is a dimensionless constant given by

\[
\bar{\lambda} = \lambda(D_1D_2)^{-1/4}\chi^{1/2}. 
\]

(8)

The renormalization group flow of \((\bar{\lambda}, \Delta)\) for the model [9] with \(\bar{\chi} = 0\) is studied in Ref. [10]. Also, the time correlation function has been calculated in the special cases that \(\Delta = 0\) (using the mode coupling equation) [3] and \(\bar{\lambda} = 0\) [4]. However, as far as we know, the time correlation function for the nonlinear model [6] with the anisotropic noise intensity [9] has never been investigated.

In the analysis below, employing a perturbative expansion with respect to \(\bar{\lambda}\) and \(\Delta\), we calculate the time correlation function \(\hat{C}(k, t)\) defined by

\[
(2\pi)^2 \hat{C}(k, t)\delta(k + k') = \left\langle \hat{\phi}(k, 0)\hat{\phi}(k', t) \right\rangle. 
\]

(9)

Here and below, for an arbitrary function \(f(x, t)\), we define \(\hat{f}(k, t)\) by

\[
\hat{f}(k, t) \equiv \int d^2x e^{-ik\cdot x} f(x, t). 
\]

(10)

From the definition [9] and the symmetry of the steady state with respect to translation in time, the equality \(\hat{C}(k, t) = \hat{C}(k, -t)\) holds. Therefore, we consider only \(\hat{C}(k, t)\) with \(t \geq 0\).

III. ANALYSIS

First, we fix \(\Delta\) and consider the expansion of \(\hat{\phi}(k, t)\) in \(\bar{\lambda}\):

\[
\hat{\phi}(k, t) = \hat{\phi}^{(0)}(k, t) + \bar{\lambda}\hat{\phi}^{(1)}(k, t) + \bar{\lambda}^2\hat{\phi}^{(2)}(k, t) + \cdots. 
\]

(11)

Substituting (11) into (9) with (10) and extracting all terms proportional to \(\bar{\lambda}\), we obtain a linear differential equation for \(\hat{\phi}^{(n)}\) containing all lower order \(\hat{\phi}^{(k)}\) and \(\hat{\xi}^{(k)}(t, t')\). Solving these differential equations under initial conditions set at \(t = -\infty\), we can iteratively derive expressions for \(\hat{\phi}^{(0)}, \hat{\phi}^{(1)}, \cdots\). We then substitute these results into (9). In this way, the correlation function \(C(k, t)\) is calculated in the form

\[
\hat{C}(k, t) = \hat{C}^{(0)}(k, t) + \bar{\lambda}\hat{C}^{(1)}(k, t) + \bar{\lambda}^2\hat{C}^{(2)}(k, t) + \cdots. 
\]

(12)

It turns out that it is simplest to obtain the terms \(\hat{C}^{(n)}(k, t)\) in the above expansion of \(\hat{C}(k, t)\) by first deriving the terms \(\hat{C}^{(n)}(k, \omega)\) in the analogous expansion of \(\hat{C}(k, \omega)\), the Fourier transform with respect to time of \(\hat{C}(k, t)\), and then taking the inverse Fourier transform of these.

The lowest-order contribution to \(\hat{C}(k, t)\) can be easily calculated as

\[
\hat{C}^{(0)}(k, t) = \left(1 + \Delta \frac{k_1^2 - k_2^2}{|k|^2} \right) e^{-|k|^2 t}. 
\]

(13)

Note that the spatial correlation function, obtained through the Fourier transformation of \(\hat{C}^{(0)}(k, 0)\), exhibits power-law decay of the type \(1/r^2\), unless \(\Delta = 0\). This illustrates the long-range correlation of driven diffusive systems. To this order, we find that there is no interesting behavior of the time dependence of \(\hat{C}^{(0)}(k, t)\), which merely exhibits an exponentially decaying form.

The next contribution to \(\hat{C}(k, t)\) appears at second order in \(\bar{\lambda}\). Through a straightforward calculation, we obtain

\[
\hat{C}^{(1)}(k, t) = \hat{C}(k, -t) = -2\int_{-\infty}^{\infty} dt' \int \frac{d^2k'}{(2\pi)^2} \sum_{j=1}^{2} \left(1 - (-1)^j \Delta \right)\frac{k_j^2}{|k|^2} e^{-|k|^2|t-t'|} e^{-|k|^2|t-t'| - |k-k'|^2|t-t'|} \hat{C}^{(0)}(k_j, t) \hat{C}^{(0)}(k_j', t) 
\]

\[
\left[ \sum_{j=1}^{2} \left(1 - (-1)^j \Delta \right) \frac{k_j^2}{|k|^2} k_j^2 (k_j - k_j') \right] \left( (t-t') \frac{t}{|t|} + |t-t'| + \frac{1}{|k|^2} \right) - \frac{1}{2} \frac{k_1^2}{|k|^2} \sum_{j=1}^{2} \left(1 - (-1)^j \Delta \right) (k_j - k_j')^2 \right]. 
\]

(14)

\[
\int_{-\infty}^{\infty} dt' \int \frac{d^2k'}{(2\pi)^2} \sum_{j=1}^{2} \left(1 - (-1)^j \Delta \right)\frac{k_j^2}{|k|^2} e^{-|k|^2|t-t'|} e^{-|k|^2|t-t'| - |k-k'|^2|t-t'|} \hat{C}^{(0)}(k_j, t) \hat{C}^{(0)}(k_j', t) 
\]
derivation.) Note that the equal-time correlation \( \hat{C}(k,t) \) must be obtained as \( \lim_{t \to 0} \hat{C}(k,t) \), because the expression \( 17 \) is physically sound only for \( t \gg \tau_m \). Next, we introduce a time scale \( \tau_M \) which can be chosen arbitrarily and define a dimensionless parameter \( \mu = \tau_M / \tau_m \). Then, using

\[
\log \frac{t}{\tau_m} = \log \frac{t}{\tau_M} + \log \frac{\tau_M}{\tau_m},
\]

we rewrite \( 17 \) as

\[
\hat{C}(k,t) = Z(\mu) \hat{C}^{(0)}(k,t)
\]

\[
\left[ 1 - (c_0(k)\Delta + c_1(\Delta)k^2 t) \hat{\lambda}^2 \log \frac{t}{\tau_M} \right] + \hat{\lambda}^2 \hat{C}^{(2)}(k,t) + o(\hat{\lambda}^2, \Delta),
\]

where we have introduced the renormalization constant \( Z(\mu) \). Here, the bare perturbation result \( 17 \) is equivalent to \( 21 \) with

\[
Z(\mu) = 1 - (c_0(k)\Delta + c_1(\Delta)k^2 t) \hat{\lambda}^2 \log \mu + o(\hat{\lambda}^2, \Delta).
\]

Now, we regard \( 22 \) as the bare perturbation result for \( Z(\mu) \) and calculate the improved perturbation result by using the fact that \( \hat{C}(k,t) \) does not depend on \( \tau_M \). That is, differentiating \( 21 \) with respect to \( \tau_M \), we obtain the equation

\[
\frac{d \log Z(\mu)}{d \log \mu} + (c_0(k)\Delta + c_1(\Delta)k^2 t) \hat{\lambda}^2 + o(\hat{\lambda}^2, \Delta) = 0,
\]

which is referred to as the “renormalization group equation”. Solving \( 23 \) under the condition

\[
Z(\mu = 0) = 1,
\]

we derive

\[
Z(\mu) = \mu^{- (c_0(k)\Delta + c_1(\Delta)k^2 t) \hat{\lambda}^2 + o(\hat{\lambda}^2, \Delta)},
\]

which provides the improved result of \( 22 \). Finally, substituting \( 24 \) into \( 21 \) and setting \( \tau_M = t \) (recall that \( \tau_M \) is arbitrary), we obtain the expression

\[
\hat{C}(k,t) = \hat{C}^{(0)}(k,t) \left( \frac{t}{\tau_M} \right)^{-(c_0(k)\Delta + c_1(\Delta)k^2 t) \hat{\lambda}^2 + o(\hat{\lambda}^2, \Delta)} + \hat{\lambda}^2 \hat{C}^{(2)}(k,t) + o(\hat{\lambda}^2, \Delta),
\]

which may be reliable for all \( t \geq \tau_m \).

IV. RESULTS AND REMARKS

From the expression \( 25 \), we have the following physically interesting results. First, we note that there is a crossover time \( \tau_c(k) \) given by

\[
\Delta |c_0(k)| \hat{\lambda}^2 = |k|^{2 \tau_c(k)}.
\]

We focus on the small wavenumber regime satisfying \( \tau_c(k) \gg \tau_m \). Then, for \( t \) satisfying \( t / \log(t/\tau_m) \ll \tau_c(k) \),
with the method to study the RG flow of system parameter.

There are several related works [9, 10] in which such a diffusion for a deterministic nonlinear diffusion equation. The RG method to derive a solution representing anomalous term was treated in Ref. [5] within the framework of the bare perturbation result is the key to obtaining the fluctuation-dissipation relation is violated [see (5)]. We believe that such power-law behavior can be observed in experiments.

In addition to the above result, from [20], we find that the decay rate of the correlation for \( t \) satisfying \( t/\log(t/\tau_m) \gg \tau_m(k) \) is expressed by

\[
- \frac{1}{t} \log \hat{C}(k, t) \simeq |k|^2 + c_1(\Delta)k_1^2\lambda^2 \log t. \tag{29}
\]

This shows that the decay rate of the correlation increases slowly as a function of time. Such enhancement of the decay rate exists even in the case \( \Delta = 0 \). A similar result was obtained from analysis of the mode coupling equation [3].

The appearance of the singular term \( \log t/\tau_m \) in the bare perturbation result is the key to obtaining the power-law behavior of the correlation. A similar singular term was treated in Ref. [3] within the framework of the RG method to derive a solution representing anomalous diffusion for a deterministic nonlinear diffusion equation. There are several related works [3, 11] in which such a divergence is treated in a similar way.

The RG analysis given here should not be confused with the method to study the RG flow of system parameters that occurs with the change of the wavenumber scale. For the model under consideration, this type of the RG flow of \( (\lambda, \Delta) \) is investigated in Ref. [3]. With that method, for example, the relevancy of the parameters can be studied, but explicit calculation of the time correlation is not possible.

In conclusion, we calculated the time correlation function [20] for the driven diffusive model [3] with [3]. The expression we obtained indicates that a power-law regime appears in the time correlation function if the FDR is violated. In addition to predicting this new type of physical phenomenon, our analysis provides an instructive example for the application of the perturbative RG method.

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APPENDIX A: DERIVATION OF (14)

For an arbitrary function \( f(x, t) \), we define \( \tilde{f}(k, \omega) \) as

\[
\tilde{f}(k, \omega) = \int d^2x \exp[-i\omega t - ik \cdot x] f(x, t). \tag{A1}
\]

Then, the quantity \( \hat{C}(k, \omega) \) satisfies

\[
(2\pi)^3 \delta(z + z') \hat{C}(z) = \left\langle \tilde{\phi}(z) \tilde{\phi}(z') \right\rangle, \tag{A2}
\]

where \( z = (k, \omega) \). Here, the Fourier transformation of (3) yields

\[
\hat{\phi}(z) = G(z) \left[ -2 i k_1 \tilde{x}_1(z) - \lambda i k_1 (\hat{\phi} \circ \tilde{\phi})(z) \right], \tag{A3}
\]

with

\[
G(z) = \frac{1}{i \omega + \sum_{i=1}^2 k_i^2}, \tag{A4}
\]

where \( (\hat{f} \circ \hat{g})(z) \) denotes the convolution of \( \hat{f}(z) \) and \( \hat{g}(z) \). From (10), for \( \hat{\phi}^{(n)}(z), (n = 0, 1, 2, \ldots) \), defined by (11) and (A1), we obtain

\[
\hat{\phi}^{(0)}(z) = G(z) \left[ -2 i k_1 \tilde{x}_1(z) \right], \tag{A5}
\]

\[
\hat{\phi}^{(1)}(z) = G(z) \left[ -i k_1 (\hat{\phi}^{(0)} \circ \tilde{\phi}^{(0)})(z) \right], \tag{A6}
\]

\[
\hat{\phi}^{(2)}(z) = G(z) \left[ -2 i k_1 (\hat{\phi}^{(0)} \circ \tilde{\phi}^{(1)})(z) \right]. \tag{A7}
\]

We expand \( \hat{C}(z) \) in the form

\[
\hat{C}(z) = \hat{C}^{(0)}(z) + \lambda \hat{C}^{(1)}(z) + \lambda^2 \hat{C}^{(2)}(z) + \ldots. \tag{A8}
\]

The lowest order contribution of (A8) is calculated as

\[
\hat{C}^{(0)}(z) = 2 |G(z)|^2 \sum_{i=1}^2 k_i^2 (1 + (-1)^{(i-1)} \Delta). \tag{A9}
\]

Using the inverse Fourier transformation in \( \omega \), we obtain (14). It can be easily checked \( \hat{C}^{(1)}(z) = 0 \), and \( \hat{C}^{(2)}(z) \) is expressed in the form

\[
\hat{C}^{(2)}(z) = \hat{C}^{(2)}_1(z) + \hat{C}^{(2)}_\parallel(z) + \hat{C}^{(2)}_\perp(z), \tag{A10}
\]

where
\[ \hat{C}_1^{(2)}(z) = 8|G(z)|^2 k_1^2 \int d^3 z' |G(z - z')|^2 \left( \sum_{i=1}^{2} (k_i - k'_i)^2 (1 - (-1)^i \Delta) |G(z')|^2 \right) \left( \sum_{j=1}^{2} k_j^2 (1 - (-1)^j \Delta) \right), \]

\[ \hat{C}_2^{(2)}(z) = 32|G(z)|^4 \omega k_1 \sum_{i=1}^{2} k_i^2 (1 - (-1)^i \Delta) \int d^3 z' |G(z - z')|^2 \left( \sum_{j=1}^{2} (k_j - k'_j)^2 (1 - (-1)^j \Delta) |G(z')|^2 \omega k'_1, \right) \]

\[ \hat{C}_3^{(2)}(z) = -32|G(z)|^4 |k|^2 k_1 \sum_{i=1}^{2} k_i^2 (1 - (-1)^i \Delta) \int d^3 z' |G(z - z')|^2 \left( \sum_{j=1}^{2} (k_j - k'_j)^2 (1 - (-1)^j \Delta) |G(z')|^2 |k'|^2 k'_1. \right) \]

APPENDIX B: DERIVATION OF (17)

We expand \( \hat{C}^{(2)}(k, t) \) in the form

\[ \hat{C}^{(2)}(k, t) = \hat{C}^{(2,0)}(k, t) + \Delta \hat{C}^{(2,1)}(k, t) + o(\Delta). \]  

Through a straightforward calculation, we obtain

\[ \hat{C}^{(2,0)}(k, t) = \frac{1}{4\pi k_1^2} \left[ \int_0^t dt' e^{\frac{|k|^2 t'}{2}} - t \int_0^t dt' \frac{1}{|k|^2} e^{\frac{|k|^2 t'}{2}} \right]. \]

In order to calculate \( \hat{C}^{(2,1)}(k, t) \), we extract terms proportional to \( \Delta \) from (14). The obtained expression becomes

\[ \hat{C}^{(2,1)}(k, t) = -4 \int_0^t dt' k_1 (t - t') e^{-|k|^2 (t - t')} \int \frac{d^2 k'}{(2\pi)^2} \left[ \frac{k_1^2 - k_2'^2}{|k'|^2} + \frac{k_1^2 - k_2'^2}{|k|^2} \right] (k_1 - k_1') e^{-|k_k - k'|^2 + |k'|^2 |t'|} \]

\[ + 2 \frac{k_1^2}{|k|^2} \int dt' e^{-|k|^2 |t' - t|} \int \frac{d^2 k'}{(2\pi)^2} \frac{k_1^2 - k_2'^2}{|k'|^2} e^{-|k_k - k'|^2 + |k'|^2 |t'|} \]

\[ - 2 \frac{k_1^2}{|k|^2} \int dt' e^{-|k|^2 |t' - t|} \int \frac{d^2 k'}{(2\pi)^2} \left[ \frac{k_1^2 - k_2'^2}{|k'|^2} + \frac{k_1^2 - k_2'^2}{|k|^2} \right] (k_1 - k_1') e^{-|k_k - k'|^2 + |k'|^2 |t'|}. \]

We first evaluate the Gauss integrals in \( |k'| \) and perform the \( t' \) integrals with picking up singular terms. Then, \( \hat{C}^{(2,1)}(k, t) \) is obtained as

\[ \hat{C}^{(2,1)}(k, t) = \frac{1}{8\pi} e^{-|k|^2 t} k_1^2 t \ln \frac{t}{\tau_m} \]

\[ - \frac{1}{8\pi} e^{-|k|^2 t} k_1^2 \frac{t}{\tau_m} \ln \frac{t}{\tau_m} \]

\[ + \text{(non-singular term)}. \]

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