Some Generalizations of the MacMahon Master Theorem

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Abstract

We consider a number of generalizations of the $\beta$-extended MacMahon Master Theorem for a matrix. The generalizations are based on replacing permutations on multisets formed from matrix indices by partial permutations or derangements over matrix or submatrix indices.

1 Introduction

The Master Theorem due to Percy MacMahon first appeared in 1915 in his classic text Combinatory Analysis [MM]. A generalization known as the $\beta$-extended MacMahon Master Theorem was discovered in more recent times by Foata and Zeilberger [FZ]. This present paper is concerned with several further generalizations of the $\beta$-extension informed by recent results in the theory of vertex operator algebras concerning the partition and correlation functions on a genus zero and higher Riemann surface [MT1, MT2, TZ1, HT, TZ2, TZ3].

One formulation of the MacMahon Master Theorem (MMT) is the identity of $\det(I - A)^{-1}$, for a given matrix $A$, to an infinite weighted sum over all permanents for matrices indexed by multisets formed from the indices of $A$ [W, KP]. The $\beta$-extended MMT relates $\det(I - A)^{-\beta}$ to a similar sum
over so-called $\beta$-extended permanents [FZ, KP]. We consider the following generalizations:

(i) The Submatrix MMT. Here the infinite sum runs over multisets formed from the indices of a given submatrix of $A$.

(ii) The Partial Permutation MMT. In this case the $\beta$-extended permanent is replaced by what we refer to as a $(\beta, \theta, \phi)$-extended partial permanent defined in terms of a sum over all partial permutations of the $A$-indices.

(iii) The Derangement MMT. We replace the $\beta$-extended permanent by what we refer to as a $\beta$-extended deranged partial permanent defined in terms of a sum over the derangements of the $A$-indices.

We begin in Section 2 with a review of the $\beta$-extended MMT [FZ]. We provide a graph theoretic proof based on an enumeration of appropriate weights of non-isomorphic permutation graphs labelled by multisets of the indexing set for $A$. In particular, the connected subgraphs are cycles corresponding to permutation cycles. Section 3 describes our first generalization, the Submatrix MMT (Theorem 3.1), where the set of permutation graphs is modified to account for multisets formed from the indices of an $A$ submatrix. In Section 4 we introduce the $(\beta, \theta, \phi)$-extended partial permanent of a matrix, a variation on the $\beta$-extended permanent involving a sum over the partial permutations of the matrix indices. The corresponding Partial Permutation MMT (Theorem 4.1) is proved by a consideration of partial permutation graphs whose connected subgraphs are cycles and open necklaces. Section 5 combines both of the previous generalizations into one general result in Theorem 5.1. Finally, in Section 6 we introduce another variation, the $\beta$-extended deranged permanent of a matrix, where we sum over the derangements (fixed point free permutations) of the matrix indices. We conclude with a Derangement MMT (Theorem 6.1) and a corresponding Submatrix Derangement MMT (Theorem 6.2) which are proved by applying the graph theory description of Sections 2 and 3 respectively, but where no 1-cycle graphs occur.
2 The $\beta$-Extended MacMahon Master Theorem

Let $A = (A_{ij})$ be an $n \times n$ matrix indexed by $i, j \in \{1, \ldots, n\}$. The $\beta$-extended Permanent of $A$ is defined by [FZ], [KP]

$$\text{perm}_{\beta} A = \sum_{\pi \in \Sigma_n} \beta^{C(\pi)} \prod_{i=1}^{n} A_{i\pi(i)},$$

(1)

where $C(\pi)$ is the number of cycles in $\pi \in \Sigma_n$, the symmetric group. The permanent and determinant are the special cases:

$$\text{perm} A = \text{perm}_{+1} A, \quad \det(-A) = \text{perm}_{-1} A.$$

(2)

Let $r = (r_1, \ldots, r_n)$ denote an $n$-tuple of non-negative integers. Define

$$r! = r_1! \cdots r_n!,$$

(3)

and let

$$n^r = \{1^{r_1}2^{r_2} \cdots n^{r_n}\} = \{1, \ldots, 1_{r_1}, \ldots, n_1, \ldots, n_{r_n}\},$$

(4)

denote the multiset of size $N = \sum_{i=1}^{n} r_i$ formed from the original index set $\{1, \ldots, n\}$ where the index $i$ is repeated $r_i$ times. We sometimes notate a repeated index by $i_a$ for label $a = 1, \ldots, r_i$. For an $n \times n$ matrix $A$, we let $A(n^r, n^r)$ denote the $N \times N$ matrix indexed by the elements of $n^r$ and define $A(n^r, n^r) = 1$ for $r = (0, 0, \ldots, 0)$.

We now describe a generalization, due to Foata and Zeilberger [FZ], of the MacMahon Master Theorem (MMT) of classical combinatorics [MM]. We give a detailed proof based on a graph theory method which is extensively employed throughout this paper. This proof is very similar to that of Theorem 5 of [MT1] where the MMT was essentially rediscovered.

**Theorem 2.1 (The $\beta$-Extended MMT)**

$$\sum_{r_i \geq 0} \frac{1}{r!} \text{perm}_{\beta} A(n^r, n^r) = \frac{1}{\det(I - A)^{\beta}}.$$

(5)
Remark 2.2 For $\beta = 1$, Theorem 2.1 reduces to the MMT. For $\beta = -1$ we use (2) to find that only proper subsets of $\{1, \ldots, n\}$ contribute resulting in the determinant identity for $B = -A$ e.g. [TZ1]

$$\sum_{r_i \in \{0, 1\}} \det B(n^r, n^r) = \det(I + B).$$

Proof of Theorem 2.1. Let $\Sigma(n^r)$ denote the symmetric group of the multiset $n^r$. For $\pi \in \Sigma(n^r)$ we define a permutation graph $\gamma_\pi$ with $N$ vertices labelled by $i \in \{1, \ldots, n\}$, and with directed edges

$$e_{ij} = i \rightarrow j,$$

provided $j = \pi(i)$. The connected subgraphs of $\gamma_\pi \in \Gamma$ are cycles arising from the cycles of $\pi$. For example, for $n = 4$ with $r = (3, 2, 0, 1)$ and permutation $\pi = (1_2 1_2)(1_3 4_1)$ the corresponding graph has two cycles as shown in Fig. 1.

![Graph](image)

Fig. 1 $\gamma_\pi$ for $\pi = (1_2 1_2 2_2)(1_3 4_1)$.

Define a weight for each edge of $\gamma_\pi$ by

$$w(e_{ij}) = A_{ij},$$

and a weight for $\gamma_\pi$ by

$$w(\gamma_\pi) = \beta^{C(\pi)} \prod_{e_{ij} \in \gamma_\pi} w(e_{ij}).$$

(6)

where $C(\pi)$ is the number of cycles in $\pi$. Note that the weight is multiplicative with respect to the cycle decomposition of $\pi$. (6) also implies

$$\text{perm}_\beta A(n^r, n^r) = \sum_{\pi \in \Sigma(n^r)} w(\gamma_\pi).$$

(7)

Let $\Lambda(r) = \Sigma_{r_1} \times \ldots \times \Sigma_{r_n} \subseteq \Sigma(n^r)$ denote the label group of order $|\Lambda(r)| = r!$ which permutes the identical elements of $n^r$. $\Lambda(r)$ generates
isomorphic graphs with $\gamma_\pi \sim \gamma_{\lambda \pi \lambda^{-1}}$ for $\lambda \in \Lambda(r)$ and the automorphism group of $\gamma_\pi$ is the $\pi$ stabilizer $\text{Aut}(\gamma_\pi) = \{ \lambda \in \Lambda(r) | \lambda \pi = \pi \lambda \} \subseteq \Lambda(r)$. Using the Orbit-Stabilizer theorem it follows that the number of isomorphic graphs generated by the action of $\Lambda(r)$ on $\gamma_\pi$ is given by

$$|\Lambda(r)\gamma_\pi| = \frac{|\Lambda(r)|}{|\text{Aut}(\gamma_\pi)|},$$

(8)

(e.g. in Fig. 1, $\Lambda(r) = \Sigma_2 \times \Sigma_3$ and $\text{Aut}(\gamma_\pi) = \Sigma_2$ so that there are 6 permutations in $\Sigma(n^r)$ with graph $\gamma_\pi$). Combining (7) and (8) we find that

$$\sum_r \frac{1}{r!} \text{perm}_\beta A(n^r, n^r) = \sum_{\gamma \in \Gamma} \frac{w(\gamma)}{|\text{Aut}(\gamma)|},$$

(9)

where $\Gamma$ denotes the set of non-isomorphic graphs.

Consider the decomposition of a graph $\gamma$ into cycle graphs

$$\gamma = \gamma_{\sigma_1}^{m_1} \cdots \gamma_{\sigma_K}^{m_K},$$

where $\{\gamma_{\sigma_i}\}$ are non-isomorphic and $\gamma_{\sigma_i}$ occurs $m_i$ times. The automorphism group is

$$\text{Aut}(\gamma) = \prod_{i=1}^M \text{Aut}(\gamma_{\sigma_i}^{m_i}),$$

where $\text{Aut}(\gamma_{\sigma_i}) = \Sigma_m \times \text{Aut}(\gamma_{\sigma_i})^m$ of order $m! |\text{Aut}(\gamma_{\sigma_i})|^m$. Furthermore, since the weight is multiplicative, $w(\gamma) = \prod_{i=1}^M w(\gamma_{\sigma_i})^{m_i}$. Thus we find

$$\sum_{\gamma \in \Gamma} \frac{w(\gamma)}{|\text{Aut}(\gamma)|} = \prod_{\gamma_{\sigma} \in \Gamma_{\sigma}} \sum_{m \geq 0} \frac{1}{m!} \left( \frac{w(\gamma_{\sigma})}{|\text{Aut}(\gamma_{\sigma})|} \right)^m = \exp \left( \sum_{\gamma_{\sigma} \in \Gamma_{\sigma}} \frac{w(\gamma_{\sigma})}{|\text{Aut}(\gamma_{\sigma})|} \right),$$

(10)

where $\Gamma_{\sigma}$ denotes the set of non-isomorphic cycle graphs. For a cycle $\sigma$ of order $|\sigma| = t$ we have $\text{Aut}(\gamma_{\sigma}) = \langle \sigma^s \rangle$ for some $s | t$ with $|\text{Aut}(\gamma_{\sigma})| = \frac{t}{s}$. Using the trace identity

$$\sum_{\gamma_{\sigma}, |\sigma| = t} s \ w(\gamma_{\sigma}) = \beta \text{Tr}(A^t),$$

where
we find
\[
\sum_{\gamma_\sigma \in \Gamma_\sigma} \frac{w(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} = \beta \sum_{t \geq 1} \frac{1}{t} \text{Tr}(A^t) \\
= -\beta \text{Tr} \log(I - A) \\
= -\beta \log \det(I - A).
\]

Thus
\[
\sum_{r} \frac{1}{r!} \text{perm}_\beta A(n^r, n^r) = \det(I - A)^{-\beta}. \quad \square
\]

Let \(w_1(\gamma)\) denote the weight for \(\gamma\) with \(\beta = 1\) in (6). Define a cycle to
be primitive (or rotationless) if \(|\text{Aut}(\gamma_\sigma)| = 1\). For a general cycle \(\sigma\) with
\(|\text{Aut}(\gamma_\sigma)| = k\) we have \(\gamma_\sigma = \gamma_\rho^k\) for a primitive cycle \(\rho\). Let \(\Gamma_\rho\) denote the set
of all primitive cycles. Then
\[
\sum_{\gamma_\sigma \in \Gamma_\sigma} \frac{w_1(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} = \sum_{\gamma_\rho \in \Gamma_\rho} \sum_{k \geq 1} \frac{1}{k} w_1(\gamma_\rho)^k \\
= -\sum_{\gamma_\rho \in \Gamma_\rho} \log \det(1 - w_1(\gamma_\rho)).
\]

Combining this with (10) implies [MT1]

**Proposition 2.3**

\[
\det(I - A) = \prod_{\gamma_\rho \in \Gamma_\rho} (1 - w_1(\gamma_\rho)).
\]

### 3 The Submatrix MMT

Our first generalization of Theorem 2.1 concerns submatrices. Consider an
\((n' + n) \times (n' + n)\) matrix with block structure
\[
\begin{bmatrix}
B & U \\
V & A
\end{bmatrix},
\]
where $A = (A_{ij})$ is an $n \times n$ matrix indexed by $i, j$, $B = (B_{ij'})$ is an $n' \times n'$ matrix indexed by $i', j'$, $U = (U_{ij})$ is an $n' \times n$ matrix and $V = (V_{ij'})$ is an $n \times n'$ matrix. For a multiset $n^r$ of size $N$ define the $(n' + N) \times (n' + N)$ matrix

$$
\begin{bmatrix}
  B & U(n^r) \\
  V(n^r) & A(n^r, n^r)
\end{bmatrix},
$$

where, as before, $A(n^r, n^r)$ denotes the $N \times N$ matrix indexed by $n^r$, $U(n^r)$ is an $n' \times N$ matrix and $V(n^r)$ is an $N \times n'$ matrix. We then find

**Theorem 3.1**

$$
\sum \frac{1}{r!} \text{perm}_\beta \begin{bmatrix} B & U(n^r) \\ V(n^r) & A(n^r, n^r) \end{bmatrix} = \text{perm}_\beta \tilde{B} \det(I - A)^\beta,
$$

for $n' \times n'$ matrix

$$
\tilde{B} = B + U(I - A)^{-1}V,
$$

where $(I - A)^{-1} = \sum_{k \geq 0} A^k$.

This result is related to Theorem 10 of [MT1] when $\beta = 1$ and Theorem 2 of [TZ1] for $\beta = -1$.

**Proof.** Let $n = \{1, \ldots, n\}$ and $n' = \{1', \ldots, n'\}$ and let $n' \cup n^r$ denote the multiset indexing the block matrix (12). Define a permutation graph $\gamma_\pi$ with weight $w(\gamma_\pi)$ for each $\pi \in \Sigma(n' \cup n^r)$ as follows. Each vertex is labelled by an element of $n$ or $n'$ which we refer to as $n$-vertex or $n'$-vertex respectively. For $l = \pi(k)$ with $k, l \in n' \cup n^r$ we define an edge $e_{kl} = k \rightarrow l$ with weight

$$
w(e_{kl}) = \begin{bmatrix} B & U(n^r) \\ V(n^r) & A(n^r, n^r) \end{bmatrix}_{kl}.
$$

Define a weight for $\gamma_\pi$ by

$$
w(\gamma_\pi) = \beta^{C(\pi)} \prod_{e_{kl} \in \gamma_\pi} w(e_{kl}),
$$

where $C(\pi)$ is the number of cycles in $\pi$. As before, we find

$$
\sum \frac{1}{r!} \text{perm}_\beta \begin{bmatrix} B & U(k) \\ V(k) & A(k, k) \end{bmatrix} = \sum \frac{w(\gamma)}{|\text{Aut}(\gamma)|},
$$

7
where \( \hat{\Gamma} \) denotes the set of non-isomorphic graphs. Each \( \gamma \in \hat{\Gamma} \) has a decomposition into cycles \( \gamma_{\sigma} \) which contain \( n \)-vertices only and cycles \( \gamma_{\sigma'} \) which contain at least one \( n' \)-vertex:

\[
\gamma = \gamma_{\sigma_1}^{m_1} \cdots \gamma_{\sigma_K}^{m_K} \gamma_{\sigma'_1} \cdots \gamma_{\sigma'_L},
\]

with weight

\[
w(\gamma) = \prod_a w(\gamma_{\sigma_a})^{m_a} \prod_b w(\gamma_{\sigma'_b}).
\]

The set of non-isomorphic \( \gamma_{\sigma} \) cycle graphs labelled by \( n \) is equivalent to \( \Gamma_{\sigma} \) introduced in the proof of Theorem 2.1. Since each \( n' \)-vertex occurs exactly once in \( \gamma \), each \( \gamma_{\sigma'_b} \) cycle occurs at most once and has trivial automorphism group. Hence

\[
|\text{Aut}(\gamma)| = \prod_a |\text{Aut}(\gamma_{\sigma_a})|^{m_a} m_a!,
\]

as before. Thus the sum over weights of all graphs decomposes into the product

\[
\sum_{\gamma \in \hat{\Gamma}} \frac{w(\gamma)}{|\text{Aut}(\gamma)|} = \sum_{\gamma_{\sigma'}} \sum_{\gamma_{\sigma} \in \Gamma_{\sigma}} \sum_{m \geq 0} \frac{w(\gamma_{\sigma})}{|\text{Aut}(\gamma_{\sigma})|^{m} m!} = \sum_{\gamma_{\sigma'}} \frac{w(\gamma_{\sigma'})}{\det(I - A)^3},
\]

using Theorem 2.1 and where \( \gamma_{\sigma'} \) ranges over non-isomorphic cycles in \( \hat{\Gamma} \) containing at least one \( n' \)-vertex.

It remains to compute \( \sum_{\gamma_{\sigma'}} w(\gamma_{\sigma'}) \). Let \( \sigma' \in \Sigma_{n'} \) denote the permutation cycle corresponding to the cyclic sequence of \( n' \)-vertices in a \( \gamma_{\sigma'} \)-cycle (for arbitrary intermediate \( n \)-vertices). The total edge weight coming from all subgraphs, illustrated in Fig. 2, joining two \( n' \)-vertices, \( i' \) and \( j' \), summed over all intermediate \( n \)-vertices is

\[
B_{i'j'} + (UV)_{i'j'} + (UAV)_{i'j'} + (UA^2V)_{i'j'} + \ldots
\]

\[
= (B + U(I - A)^{-1}V)_{i'j'} = \tilde{B}_{i'j'}.
\]

![Fig. 2](attachment:image_url)
Thus the total weight of all $\gamma_{\sigma'}$ cycles for a given $n'$-vertex cycle $\sigma' = (i'_1 \ldots i'_p)$ is $\beta \prod_i \tilde{B}_{\sigma'_{\pi}(i')}$. Altogether, it follows that

$$
\sum_{\gamma_{\sigma'}} w(\gamma_{\sigma'}) = \sum_{\pi' \in \Sigma_{n'}} \beta^{C(\pi')} \prod_{i'} \tilde{B}_{\pi'(i')} = \text{perm}_\beta \tilde{B}. \quad \square
$$

**Lemma 3.2** For $\beta = -1$ Theorem 3.1 implies

$$
\sum_{r_i \in\{0, 1\}} \det \begin{bmatrix}
B & U(n^r) \\
V(n^r) & A(n^r, n^r)
\end{bmatrix} = \det \begin{bmatrix}
B & -U \\
-V & I + A
\end{bmatrix}.
$$

**Proof.** For $\beta = -1$ the right hand side of (13) gives

$$
\text{perm}_{-1} \tilde{B} \det(I - A) = (-1)^n \det(B + U(I - A)^{-1}V) \det(I - A)
= \det \begin{bmatrix}
-B & U \\
V & I - A
\end{bmatrix},
$$

by means of the matrix identity

$$
\begin{bmatrix}
-B & U \\
V & I - A
\end{bmatrix} = \begin{bmatrix}
-I' & U(I - A)^{-1} \\
0 & I
\end{bmatrix} \begin{bmatrix}
B + U(I - A)^{-1}V & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
I' & 0 \\
0 & I - A
\end{bmatrix},
$$

where $I$ and $I'$ are respectively $n \times n$ and $n' \times n'$ identity matrices. The result follows on replacing $A, B, U, V$ by $-A, -B, -U, -V$. \quad \square

### 4 The Partial Permutation MMT

The next generalization of Theorem 2.1 is concerned with replacing permutations by partial permutations with a suitable generalization of the notions of permanent and $\beta$-extended permanent. Let $\Psi$ denote the set of partial permutations of the set $\{1, \ldots, n\}$ i.e. injective partial mappings from $\{1, \ldots, n\}$ to itself. For $\psi \in \Psi$ we let $\text{dom} \psi$ and $\text{im} \psi$ denote the domain and image respectively and let $\pi_{\psi}$ denote the (possibly empty) permutation of $\text{dom} \psi \cap \text{im} \psi$ determined by $\psi$.  

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9
We introduce the Partial Permanent of an $n \times n$ matrix $A = (A_{ij})$ indexed by $i, j \in \{1, \ldots, n\}$ as follows

\[ \text{pperm}\ A = \sum_{\psi \in \Psi} \prod_{i \in \text{dom}\ \psi} A_{i\psi(i)}, \quad (14) \]

with unit contribution for the empty map. Let $\theta = (\theta_i), \phi = (\phi_i)$ be $n$-vectors and define the $(\beta, \theta, \phi)$-extended Partial Permanent by

\[ \text{pperm}_{\beta\theta\phi} A = \sum_{\psi \in \Psi} \beta^{C(\pi_{\psi})} \prod_{i \in \text{dom}\ \psi} A_{i\psi(i)} \prod_{j \in \text{im}\ \psi} \theta_j \prod_{k \notin \text{dom}\ \psi} \phi_k, \quad (15) \]

where $C(\pi_{\psi})$ is the number of cycles in $\pi_{\psi}$ e.g.,

\[ \text{pperm}_{\beta\theta\phi} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \theta_1 \phi_1 + \beta (A_{11} \theta_2 \phi_2 + A_{22} \theta_1 \phi_1) \]

\[ + A_{12} \theta_1 \phi_2 + A_{21} \theta_2 \phi_1 + \beta^2 A_{11} A_{22} + \beta A_{12} A_{21}. \]

A recent application of an extended partial permanent appears in [HT].

Let $A(n^r, n^r)$ denote the $N \times N$ matrix indexed by a multiset $n^r$ as before. We also let $\text{pperm}_{\beta\theta\phi} A(n^r, n^r)$ denote the corresponding partial permanent with $N$-vectors $(\theta_1, \ldots, \theta_{n^r})$ and $(\phi_1, \ldots, \phi_{n^r})$. We then find

**Theorem 4.1**

\[ \sum_r \frac{1}{r!} \text{pperm}_{\beta\theta\phi} A(n^r, n^r) = \frac{e^{\theta(I-A)^{-1} \phi^T}}{\det(I-A)^{\beta}}, \quad (16) \]

where $\phi^T$ denotes the transpose of the row vector $\phi$.

This result is related to Theorem 11 of [MT1] for $\beta = 1$.

**Proof.** Let $\Psi(n^r)$ denote the partial permutations of $n^r$. Define a partial permutation graph $\gamma_\psi$ labelled by $\{1, \ldots, n\}$ for each $\psi \in \Psi(k)$ with edges

\[ e_{ij} = i \bullet \longrightarrow \bullet j, \]

for $j = \psi(i)$ with $i \in \text{dom}\ \psi$ and $j \in \text{im}\ \psi$. Let $v_i$ denote the vertex of $\gamma_\psi$ with label $i$. If $i \notin \text{dom}\ \psi$ then either $\deg v_i = 0$ or $\deg v_i = \text{indeg} v_k = 1$ whereas if $i \notin \text{im}\ \psi$ then either $\deg v_i = 0$ or $\deg v_i = \text{outdeg} v_i = 1$. In all other cases...
deg \( v_i = 2 \) with indeg \( v_i = \text{outdeg} \ v_i = 1 \). The connected subgraphs in this case consist of cycles and open necklaces i.e. graphs with two end points of degree one. We regard a graph consisting of a single degree zero vertex as a degenerate necklace. For example, for \( n = 4, \ r = (3, 2, 0, 1) \) and partial permutation \( \psi = \begin{pmatrix} 1_1 & 1_2 & 1_3 & 2_1 & 2_2 & 4_1 \\ 2_1 & 4_1 & 1_2 & 1_3 \end{pmatrix} \) then \( \gamma_{\psi} \) is shown in Fig. 3. In this case \( \text{dom} \ \psi = \{1_1, 1_3, 2_1, 4_1\} \) and \( \text{im} \ \psi = \{1_2, 1_3, 2_1, 4_1\} \) and \( \pi_{\psi} = (1_34_1) \).

![Fig. 3](image)

Define an edge weight as before by \( w(e_{ij}) = A_{ij} \) and introduce a vertex weight

\[
\begin{align*}
\omega(v_k) & = \begin{cases} 
1, & \text{deg} \ v_k = 2, \\
\theta_k, & \text{deg} \ v_k = \text{outdeg} \ v_k = 1, \\
\phi_k, & \text{deg} \ v_k = \text{indeg} \ v_k = 1, \\
\theta_k\phi_k, & \text{deg} \ v_k = 0.
\end{cases}
\end{align*}
\]

The weight of a graph \( \gamma_{\psi} \) is defined by

\[
\omega(\gamma_{\psi}) = \beta^{C(\pi_{\psi})} \prod_{e_{ij}} \omega(e_{ij}) \prod_{v_k} \omega(v_k),
\]

where \( C(\pi_{\psi}) \) is the number of cycles in \( \pi_{\psi} \). The weight is multiplicative with respect to the cycle and necklace decomposition. We find again that

\[
\sum_{r} \frac{1}{r!} \text{perm}_{\beta\theta\phi} A(n^r, n^r) = \sum_{\gamma \in \tilde{\Gamma}} \frac{w(\gamma)}{|\text{Aut}(\gamma)|},
\]

where \( \tilde{\Gamma} \) denotes the set of non-isomorphic graphs. Each \( \gamma \in \tilde{\Gamma} \) has a decomposition into connected cycle graphs \( \gamma_{\sigma_a} \) and open necklaces \( \nu_b \):

\[
\gamma = \nu_1^{l_1} \cdots \nu_L^{l_L} \gamma_{\sigma_1}^{m_1} \cdots \gamma_{\sigma_K}^{m_K},
\]

with weight

\[
w(\gamma) = \prod_{b} w(\nu_b)^{l_b} \cdot \prod_{a} w(\gamma_{\sigma_a})^{m_a}.
\]
Each necklace has trivial automorphism group but can have multiple occurrences. Hence we find that

$$|\text{Aut}(\gamma)| = \prod_b b! \cdot \prod_a |\text{Aut}(\gamma_{\sigma_a})|^{ma} |m_a|!.$$

Thus the sum over weights of all graphs decomposes into the product

$$\sum_{\gamma \in \Gamma} \frac{w(\gamma)}{|\text{Aut}(\gamma)|} = \prod_{\nu \in \Gamma_{\nu}} \sum_{l \geq 0} \frac{w(\nu)^l}{l!} \cdot \prod_{\gamma_{\sigma} \in \Gamma_{\sigma}} \sum_{m \geq 0} \frac{|\text{Aut}(\gamma_{\sigma})|^m}{|\text{Aut}(\gamma_{\sigma})|^{m} m!} = \exp \left( \sum_{\nu \in \Gamma_{\nu}} w(\nu) \right) \frac{1}{\det(I - A)^{\beta}},$$

where $\Gamma_{\nu}$ denotes the set of non-isomorphic open necklaces and using Theorem 2.1 again. Finally, the sum over the weights of connected necklaces, such as depicted in Fig. 4, is

$$\sum_{\nu \in \Gamma_{\nu}} w(\nu) = \theta \phi^T + \theta A \phi^T + \theta A^2 \phi^T + \ldots$$

$$= \theta (I - A)^{-1} \phi^T. \quad \square$$

Fig. 4

**Example.** Consider $n = 1$ with $A = z$ and $\theta_1 = \phi_1 = \sqrt{\alpha z}$. Then we find

$$p_{\text{perm}}_{\beta \theta \phi} A(1^r, 1^r) = p_r(\alpha, \beta) z^r,$$

where $p_r(\alpha, \beta) = \sum_{s,t} p_{rst} \alpha^s \beta^t$ is the generating polynomial for $p_{rst}$ the number of graphs with $r$ identically labelled vertices, $s$ open necklaces and $t$ cycles. Theorem 4.1 provides the exponential generating function for $p_r(\alpha, \beta)$ [HT]

$$\sum_{r \geq 0} \frac{p_r(\alpha, \beta)}{r!} z^r = \frac{\exp \left( \frac{\alpha z}{1-z} \right)}{(1-z)^{\beta}}.$$
The Submatrix Partial Permutation MMT

We can combine the two generalizations above into one theorem concerning partial permutations of submatrices of the \((n' + n) \times (n' + n)\) block matrix \(\mathbf{11}\). Let \(\theta' = (\theta'_i)\) and \(\phi' = (\phi'_i)\) be \(n'\)-vectors and \(\theta = (\theta_i)\) and \(\phi = (\phi_i)\) be \(n\)-vectors. For a multiset \(n^r\) of size \(N\) and block matrix \(\mathbf{12}\) labelled by \(n' = \{1', \ldots, n'\}\) and \(n^r\), we let \(\text{pperm}_{\beta \theta \phi} \left[ \begin{array}{cc} B & U(n^r) \\ V(n^r) & A(n^r, n^r) \end{array} \right]\) denote the \((\beta, \theta, \phi)\)-extended partial permanent with \((n' + N)\)-vectors \((\theta'_1, \ldots, \theta'_n', \theta_1, \ldots, \theta_{n^r})\) and \((\phi'_1, \ldots, \phi'_n', \phi_1, \ldots, \phi_{n^r})\) respectively. We then find

Theorem 5.1

\[
\sum_r \frac{1}{r!} \text{pperm}_{\beta \theta \phi} \left[ \begin{array}{cc} B & U(n^r) \\ V(n^r) & A(n^r, n^r) \end{array} \right] = \frac{e^{\theta(I-A)^{-1}\phi^T} \cdot \text{pperm}_{\beta \tilde{\theta} \tilde{\phi}'} \tilde{B}}{\det(I-A)^\beta}, \tag{17}
\]

for

\[
\tilde{B} = B + U(I-A)^{-1}V, \\
\tilde{\theta} = \theta' + \theta(I-A)^{-1}V, \\
\tilde{\phi}^T = \phi^T + U(I-A)^{-1}\phi^T.
\]

This result is related to Theorem 13 of [MT1] for \(\beta = 1\).

Proof. We sketch the proof since it runs along very similar lines to the preceding ones. Define a partial permutation graph \(\gamma_\psi\) for each partial permutation \(\psi\) of \(n' \cup n^r\). In this case, the connected subgraphs consist of cycle graphs \(\Gamma_\sigma\) and open necklaces \(\Gamma_\nu\) containing only \(n\)-vertices, and cycles and open necklaces containing at least one \(n'\)-vertex. Define a graph weight \(w(\gamma_\psi)\) as a product of edge weights, vertex weights and cycle factors as before. This results in

\[
\sum_r \frac{1}{r!} \text{pperm}_{\beta \theta \phi} \left[ \begin{array}{cc} B & U(n^r) \\ V(n^r) & A(n^r, n^r) \end{array} \right] = \frac{e^{\theta(I-A)^{-1}\phi^T} \cdot \text{pperm}_{\beta \tilde{\theta} \tilde{\phi}'} \tilde{B}}{\det(I-A)^\beta} \sum_{\gamma' \in \Gamma'} w(\gamma'),
\]

where the sum is over all graphs \(\Gamma'\) containing at least one \(n'\)-vertex. The remaining terms arise as before.

Each \(\gamma' \in \Gamma'\) canonically determines a partial permutation \(\psi' \in \Psi(n')\) described by the corresponding ordered sequences of \(n'\)-vertices (for any intermediate \(n\)-vertices). As before, the total edge weight coming from all
subgraphs joining two \( n' \)-vertices \( i' \) and \( j' \) with intermediate \( n \)-vertices is \( \widetilde{B}_{i'j'} \). The total weight arising from the subgraphs of all necklaces joining \( n \)-vertices to an \( n' \)-vertex \( i' \) with intermediate \( n \)-vertices as depicted in Fig. 5 is

\[
\theta'_{i'} + (\theta V)_{i'} + (\theta AV)_{i'} + \ldots = \left( \theta' + \theta (I - A)^{-1} V \right)_{i'} = \tilde{\theta}_{i'}.
\]

Likewise, the total weight arising from all subgraphs joining an \( n' \)-vertex \( j' \) to \( n \)-vertices with intermediate \( n \)-vertices is \( \tilde{\phi}_{j'} \). Combining these results we find that

\[
\sum_{\gamma' \in \Gamma'} w(\gamma') = \text{pperm}_{\beta \tilde{\theta} \tilde{\phi}} \widetilde{B}. \quad \square
\]

6 The Derangement MMT

Let \( \Delta_n \subset \Sigma_n \) denote the derangements of the set \( \{1, \ldots, n\} \) i.e. each \( \pi \in \Delta_n \) contains no cycles of length 1. We introduce the \( \beta \)-extended Deranged Permanent of an \( n \times n \) matrix \( A \) by

\[
d \text{perm}_\beta A = \sum_{\pi \in \Delta_n} \beta^{C(\pi)} \prod_i A_{i\pi(i)}.
\]

Using the same multiset notation as before we find

**Theorem 6.1**

\[
\sum_{r} \frac{1}{r!} d \text{perm}_\beta A(n^r, n^r) = \frac{e^{-\beta \text{Tr} A}}{\det(I - A)^{\beta'}}.
\]

14
Proof. Following the proof of Theorem 2.1 we find
\[
\sum_{r} \frac{1}{r!} \text{dperm}_\beta A(n^r, n^r) = \exp \left( \sum_{\gamma_\sigma \in \Gamma_{\sigma}, |\sigma| \geq 2} \frac{w(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} \right),
\]
where cycles of length one are excluded. Using
\[
\sum_{\gamma_\sigma \in \Gamma_{\sigma}, |\sigma| \geq 2} \frac{w(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} = \beta \sum_{s \geq 1} \frac{1}{s} \text{Tr}(A^s) - \beta \text{Tr}A
\]
\[
= -\beta \text{Tr} \log(I - A) - \beta \text{Tr}A,
\]
the result follows. \(\square\)

Example. Consider \(n = 1\) with \(A = z\). Then for multisets \(\{1^r\}\) we find
\[
\text{dperm}_\beta A(1^r, 1^r) = d_r(\beta)z^r,
\]
where \(d_r(\beta) = \sum_s d_{rs} \beta^s\) is the generating polynomial for \(d_{rs}\) the number of derangements of \(r\) labels with \(s\) cycles. From Theorem 6.1 the exponential generating function for \(d_r(\beta)\) is \(\text{HT}\)
\[
\sum_{r \geq 0} \frac{1}{r!} z^r d_r(\beta) = \left( \frac{e^{-z}}{1-z} \right)^\beta.
\]

Finally, we can further generalize Theorem 6.1 to deranged permanents of submatrices as in Theorem 3.1. Using the notation of (11) and (12) we find using similar techniques that

Theorem 6.2
\[
\sum_{r} \frac{1}{r!} \text{dperm}_\beta \begin{bmatrix} B & U(n^r) \\ V(n^r) & A(n^r, n^r) \end{bmatrix} = \frac{e^{-\beta \text{Tr}A} \cdot \text{perm}_\beta \hat{B}}{\det(I - A)^\beta},
\]
for \(n' \times n'\) matrix
\[
\hat{B} = B - \text{diag} B + U(I - A)^{-1}V,
\]
where \(\text{diag} B_{i'j'} = B_{i'j'} \delta_{i'j'}\). \(\square\)
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