Regular Graphs with Minimum Spectral Gap

M. Abdi\textsuperscript{a,b} E. Ghorbani\textsuperscript{a,b} W. Imrich\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, K. N. Toosi University of Technology, 
P. O. Box 16765-3381, Tehran, Iran

\textsuperscript{b}School of Mathematics, Institute for Research in Fundamental Sciences (IPM), 
P. O. Box 19395-5746, Tehran, Iran

\textsuperscript{c}Montanuniversität Leoben, Leoben, Austria

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Abstract

Aldous and Fill conjectured that the maximum relaxation time for the random walk on a connected regular graph with $n$ vertices is $(1 + o(1))^{3n^2}$. This conjecture can be rephrased in terms of the spectral gap as follows: the spectral gap (algebraic connectivity) of a connected $k$-regular graph on $n$ vertices is at least $(1 + o(1))^{2k^2}$, and the bound is attained for at least one value of $k$. Based upon previous work of Brand, Guiduli, and Imrich, we prove this conjecture for cubic graphs. We also investigate the structure of quartic (i.e. 4-regular) graphs with the minimum spectral gap among all connected quartic graphs. We show that they must have a path-like structure built from specific blocks.

Keywords: Spectral gap, Algebraic connectivity, Relaxation time, Cubic graph, Quartic graph

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E-mail Addresses: m.abdi@email.kntu.ac.ir (M. Abdi), e.ghorbani@ipm.ir (E. Ghorbani), wilfried.imrich@unileoben.ac.at (W. Imrich)
1 Introduction

All graphs we consider are simple that is undirected graphs without loops or multiple edges. The difference between the two largest eigenvalues of the adjacency matrix of a graph $G$ is called the spectral gap of $G$. If $G$ is a regular graph, then its spectral gap is equal to the second smallest eigenvalue of its Laplacian matrix and known as algebraic connectivity.

In 1976, Bussemaker, Čobelić, Cvetković, and Seidel ([4], see also [5]), by means of a computer search, found all non-isomorphic connected cubic graphs with $n \leq 14$ vertices. They observed that when the algebraic connectivity is small the graph is long. Indeed, as the algebraic connectivity decreases, both connectivity and girth decrease and diameter increases. Based on these results, L. Babai (see [9]) made a conjecture that described the structure of the connected cubic graph with minimum algebraic connectivity. Guiduli [9] (see also [8]) proved that the cubic graph with minimum algebraic connectivity must look like a path, built from specific blocks. The result of Guiduli was improved as follows confirming the Babai’s conjecture.

**Theorem 1.1.** (Brand, Guiduli, and Imrich [3]) Among all connected cubic graphs on $n$ vertices, $n \geq 10$, the graph $G_n$ (given in Figure 1) is the unique graph with minimum algebraic connectivity.

![Figure 1](image)

Figure 1: The cubic graph $G_n$, $n \geq 10$, with minimum spectral gap on $n \equiv 2 \pmod{4}$ and $n \equiv 0 \pmod{4}$ vertices, respectively

The relaxation time of the random walk on a graph $G$ is defined by $\tau = 1/(1-\eta_2)$, where $\eta_2$ is the second largest eigenvalue of the transition matrix of $G$, that is the matrix $D^{-1}A$ in which $D$ and $A$ are the diagonal matrix of vertex degrees and the adjacency matrix of $G$, respectively. A central problem in the study of random walks is to determine the mixing time, a measure of how fast the random walk converges to the stationary distribution. As seen throughout the literature [2, 6], the relaxation time is the primary term controlling mixing time. Therefore, relaxation time is directly associated with the rate of convergence of the random walk.

Our main motivation in this work is the following conjecture on the maximum relaxation time of the random walk in regular graphs.
Conjecture 1.2. (Aldous and Fill [2, p. 217]) Over all connected regular graphs on \(n\) vertices, \(\max \tau = (1 + o(1))\frac{3\pi^2}{2n^2}\).

In terms of the eigenvalues of the normalized Laplacian matrix, that is the matrix \(I - D^{-1/2}AD^{-1/2}\), the Aldous–Fill conjecture says that the minimum second smallest eigenvalue of the normalized Laplacian matrices of all connected regular graphs on \(n\) vertices is \((1 + o(1))\frac{2\pi^2}{3n^2}\). This can be rephrased in terms of the spectral gap as follows, giving another equivalent statement of the Aldous–Fill conjecture.

Conjecture 1.3. The spectral gap (algebraic connectivity) of a connected \(k\)-regular graph on \(n\) vertices is at least \((1 + o(1))\frac{2k\pi^2}{3n^2}\), and the bound is attained at least for one value of \(k\).

It is worth mentioning that in [3], it is proved that the maximum relaxation time for the random walk on a connected graph on \(n\) vertices is \((1 + o(1))\frac{3n^3}{54}\) settling another conjecture by Aldous and Fill ([2, p. 216]).

In [3], it is mentioned without proof that the algebraic connectivity of the graphs \(G_n\) (of Theorem 1.1) is \((1 + o(1))\frac{2\pi^2}{n^2}\), where its proof is postponed to another paper which has not appeared. We prove this equality, thus, showing that the minimum spectral gap (algebraic connectivity) of connected cubic graphs on \(n\) vertices is \((1 + o(1))\frac{2\pi^2}{n^2}\), which implies the Aldous–Fill conjecture for \(k = 3\). As the next case of the Aldous–Fill conjecture and as a continuation of Babai’s conjecture, we investigate the connected quartic, i.e. 4-regular, graphs with minimum spectral gap (algebraic connectivity). We show that similar to the cubic case, these graphs must have a path-like structure with specified blocks (see Theorem 3.1 below). Finally, we put forward a conjecture about the unique structure of the connected quartic graph of any order with minimum spectral gap.

### 2 Minimum spectral gap of cubic graphs

In this section, we prove that the minimum spectral gap (algebraic connectivity) of connected cubic graphs on \(n\) vertices is \((1 + o(1))\frac{2\pi^2}{n^2}\).

Let \(G\) be a graph on \(n\) vertices and \(L(G)\) be its Laplacian matrix. For any \(x \in \mathbb{R}^n\), the value \(\frac{x^\top L(G)x}{x^\top x}\) is called a Rayleigh quotient. We denote the second smallest eigenvalue of \(L(G)\) known as the algebraic connectivity of \(G\) by \(\mu(G)\). It is well known that

\[
\mu(G) = \min_{x \neq 0, x \perp 1} \frac{x^\top L(G)x}{x^\top x},
\]

(1)
where \( \mathbf{1} \) is the all-1 vector. An eigenvector corresponding to \( \mu(G) \) is known as a **Fiedler vector** of \( G \). In passing we note that if \( \mathbf{x} = (x_1, \ldots, x_n)^	op \), then

\[
\mathbf{x}^	op L(G)\mathbf{x} = \sum_{ij \in E(G)} (x_i - x_j)^2,
\]

where \( E(G) \) is the edge set of \( G \).

Considering the graphs \( G_n \) of Theorem 1.1, we let \( \Pi = \{C_1, C_2, \ldots, C_k\} \) (numbered consecutively from left to right) be a partition of the vertex set \( V(G_n) \) such that each cell \( C_i \) has size 1 or 2, consisting of the vertices drawn vertically above each other as depicted in Figure 1. We note in passing that partition \( \Pi \) is a so-called ‘equitable partition’ of \( G_n \).

**Lemma 2.1.** (\([3]\)) Let \( \mathbf{x} \) be a Fiedler vector of \( G_n \).

(i) Then the components of \( \mathbf{x} \) on each cell \( C_i \) of the partition \( \Pi \) are equal.

(ii) Let \( x_1, \ldots, x_k \) be the values of \( \mathbf{x} \) on the cells of \( \Pi \). Then the \( x_i \) form a strictly monotone sequence changing sign once.

Recall that a block of a graph is a maximal connected subgraph with no cut vertex—a subgraph with as many edges as possible and no cut vertex. So a block is either \( K_2 \) (a trivial block) or is a graph which contains a cycle. If a graph \( G \) has no cut vertex, then \( G \) itself is also called a block. The blocks of a connected graph fit together in a tree-like structure, called the block tree of \( G \). The block tree of the graphs \( G_n \) are paths which justifies the description ‘path-like structure.’

We now present the the main result of this section.

**Theorem 2.2.** The minimum algebraic connectivity of cubic graphs on \( n \) vertices is \( (1 + o(1))\frac{2\pi^2}{n^2} \).

**Proof.** In view of Theorem 1.1 it suffices to show that \( \mu(G_n) = (1 + o(1))\frac{2\pi^2}{n^2} \). To prove this, we consider two cases based on the value of \( n \mod 4 \).

**Case 1.** \( n \equiv 2 \mod 4 \)

In this case \( G_n \) is the upper graph of Figure 1. Let \( m + 2 \) be the number of non-trivial blocks of \( G_n \). So we have \( n = 4m + 10 \).

We first prove that \( (1 + o(1))\frac{2\pi^2}{n^2} \) is an upper bound for \( \mu(G_n) \).

We define the vector \( \mathbf{x} = (x_1, \ldots, x_{2m})^	op \) with

\[
x_i = \cos \left( \frac{(2i-1)\pi}{4m} \right), \quad i = 1, \ldots, 2m.
\]
Note that $\mathbf{x}$ is a skew symmetric vector, i.e. $x_{2m-i+1} = -x_i$, for $i = 1, \ldots, m$, and so $\mathbf{x} \perp \mathbf{1}$. We extend $\mathbf{x}$ to define the vector $\mathbf{x}'$ on $G_n$ as follows:

![Diagram](image)

The vector $\mathbf{x}'$ (like $\mathbf{x}$) is a skew symmetric. It follows that $\mathbf{x}' \perp \mathbf{1}$. Therefore, by (1) we have

$$
\mu(G_n) \leq \frac{\mathbf{x}'^T L(G_n) \mathbf{x}'}{\mathbf{x}'^T \mathbf{x}'} \\
\leq \frac{\sum_{i=1}^{2m-1} (x_i - x_{i+1})^2}{\sum_{i=1}^{2m} x_i^2 + 2 \sum_{i=1}^m \frac{1}{4}(x_{2i-1} + x_{2i})^2 + 10x_1^2} \\
\leq \frac{4 \sin^2 \left( \frac{\pi}{4m} \right) \sum_{i=1}^{2m-1} \sin^2 \left( \frac{\pi i}{2m} \right)}{4m \sin^2 \left( \frac{\pi}{4m} \right)} \\
= \frac{\sum_{i=1}^{2m-1} \sin^2 \left( \frac{\pi i}{2m} \right) + 2 \cos^2 \left( \frac{\pi}{4m} \right) \sum_{i=1}^m \cos^2 \left( \frac{(2i-1)\pi}{2m} \right)}{m + m \cos^2 \left( \frac{\pi}{4m} \right)} \\
= (1 + o(1)) \frac{2\pi^2}{n^2}. \quad (2)
$$

Note that (2) is obtained using the identities $\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$ and $\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$. For (3) we use the identities

$$
\sum_{i=1}^{2m-1} \sin^2 \left( \frac{\pi i}{2m} \right) = \sum_{i=1}^m \cos^2 \left( \frac{(2i-1)\pi}{2m} \right) = m, \quad \sum_{i=1}^m \cos^2 \left( \frac{\pi i}{2m} \right) = \frac{m}{2}
$$

which are a consequence of the fact that $\sin^2(\alpha) + \sin^2 \left( \frac{\pi}{2} - \alpha \right) = \cos^2(\alpha) + \cos^2 \left( \frac{\pi}{2} - \alpha \right) = 1$.

We now prove that $(1 + o(1)) \frac{2\pi^2}{n^2}$ is a lower bound for $\mu(G_n)$.

Let $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ be a Fiedler vector of $G_n$. Let $B_1, \ldots, B_{m+2}$ be the non-trivial blocks of $G_n$, and $E_1$ be the set of edges of $B_1, \ldots, B_{m+2}$ and $E_2$ be the set of all bridges of $G_n$. Then we have

$$
\mu(G_n) = \frac{\mathbf{y}^T L(G_n) \mathbf{y}}{\mathbf{y}^T \mathbf{y}} \\
= \frac{\sum_{ij \in E(G_n)} (y_i - y_j)^2}{\sum_{i=1}^n y_i^2} \\
= \frac{\sum_{ij \in E_1} (y_i - y_j)^2 + \sum_{ij \in E_2} (y_i - y_j)^2}{\sum_{i=1}^n y_i^2}. \quad (5)
$$
The graph $G_n$ has $2m+2$ cut vertices. Consider the components of $\mathbf{y}$ on the cut vertices of $G_n$ together with the four components $y_1, y_3, y_{n-2}, y_{n}$; we define $\mathbf{z}$ as the vector consisting of these $2m+6$ components, as depicted below:

Note that $\mathbf{y}$ is skew symmetric. To verify this, observe that by the symmetry of $G_n$, $\mathbf{y}' = (y_n, y_{n-1}, \ldots, y_1)$ is also an eigenvector for $\mu(G_n)$. It follows that $\mathbf{y} - \mathbf{y}'$ itself is a skew symmetric eigenvector for $\mu(G_n)$ (note that from Lemma 2.1 it is seen that $\mathbf{y} - \mathbf{y}' \neq \mathbf{0}$), so that we may replace $\mathbf{y} - \mathbf{y}'$ for $\mathbf{y}$. Now, from Lemma 2.1 it follows that $\mathbf{y} - \mathbf{y}'$ itself is a skew symmetric eigenvector for $\mu(G_n)$ (note that from Lemma 2.1, it is seen that $\mathbf{y} - \mathbf{y}' \neq \mathbf{0}$), so that we may replace $\mathbf{y} - \mathbf{y}'$ for $\mathbf{y}$. Now, from Lemma 2.1 it follows that $\mathbf{z} = (z_1, z_2, \ldots, z_{2m+6}) \neq \mathbf{0}$. As $\mathbf{y}$ is skew symmetric, it follows that $\mathbf{z}$ is also skew symmetric and thus $\mathbf{z} \perp \mathbf{1}$. Let $B_k$ be one of the middle blocks of $G_n$, i.e. $2 \leq k \leq m+1$. The components of $\mathbf{y}$ on the left vertex and the right vertex of $B_k$ are $z_{2k}$ and $z_{2k+1}$, respectively. Let $t$ be the component of $\mathbf{y}$ on the two middle vertices of $B_k$ (which are equal by Lemma 2.1) as shown below:

Then

$$\sum_{ij \in E(B_k)} (y_i - y_j)^2 = 2(z_{2k} - t)^2 + 2(t - z_{2k+1})^2.$$

The right hand side, considered as a function of $t$, is minimized at $t = \frac{1}{2}(z_{2k} + z_{2k+1})$. This implies that

$$\sum_{ij \in E(B_k)} (y_i - y_j)^2 \geq (z_{2k} - z_{2k+1})^2.$$

It follows that

$$\sum_{ij \in E_1} (y_i - y_j)^2 = \sum_{ij \in E(B_1)} (y_i - y_j)^2 + \sum_{k=2}^{m+1} \sum_{ij \in E(B_k)} (y_i - y_j)^2 + \sum_{ij \in E(B_{m+2})} (y_i - y_j)^2$$

$$\geq 4(z_1 - z_2)^2 + 2(z_2 - z_3)^2 + \sum_{k=2}^{m+1} (z_{2k} - z_{2k+1})^2$$

$$+ 2(z_{2m+4} - z_{2m+5})^2 + 4(z_{2m+5} - z_{2m+6})^2$$

$$\geq (z_1 - z_2)^2 + \sum_{k=1}^{m+2} (z_{2k} - z_{2k+1})^2 + (z_{2m+5} - z_{2m+6})^2,$$

6
which in turn implies that
\[
\sum_{ij \in E_1} (y_i - y_j)^2 + \sum_{ij \in E_2} (y_i - y_j)^2 \geq \sum_{r=1}^{2m+5} (z_r - z_{r+1})^2. \tag{6}
\]

We also have
\[
\sum_{i=1}^{n} y_i^2 \leq 2 \sum_{i=1}^{2m+6} z_i^2 \tag{7}
\]
(Indeed, \(y_1^2 + y_2^2 = 2z_1^2, y_3^2 + y_4^2 = 2z_3^2, y_5^2 + y_6^2 \leq 2z_5^2, y_7^2 + y_8^2 \leq 2z_7^2, \ldots, y_{n-4}^2 + y_{n-7}^2 \leq 2z_{2m+4}^2, \ldots\)). Now, from (5), (6) and (7) we infer that
\[
\mu(G_n) \geq \frac{\sum_{i=1}^{2m+5} (z_i - z_{i+1})^2}{2 \sum_{i=1}^{2m+6} z_i^2}. \tag{8}
\]

Note that the right hand side of (8) is the Rayleigh quotient of \(z\) for the path \(P_{2m+6}\). Thus, by the fact that \(\mu(P_h) = 2(1 - \cos \frac{\pi}{h})\) (see [7]), it follows that
\[
\frac{\sum_{i=1}^{2m+5} (z_i - z_{i+1})^2}{\sum_{i=1}^{2m+6} z_i^2} \geq \mu(P_{2m+6}) = (1 + o(1)) \frac{\pi^2}{4m^2}.
\]

Therefore,
\[
\mu(G_n) \geq (1 + o(1)) \frac{2\pi^2}{n^2}.
\]

**Case 2.** \(n \equiv 0 \pmod{4}\)

In this case, \(G_n\) is the lower graph of Figure 1. We define the graph \(H_{n+2}\) as follows:

![Graph H_{n+2}](https://example.com/graph.png)

**Figure 2:** The graph \(H_{n+2}\)

The symmetries of \(H_{n+2}\) are similar to those of the graph \(G_{n-2}\). So the arguments of the previous case also work for \(H_{n+2}\), in particular \(H_{n+2}\) has a skew symmetric Fiedler vector. Therefore, we have \(\mu(H_{n+2}) = (1 + o(1)) \frac{2\pi^2}{(n+2)^2}\). Let \(x = (x_1, \ldots, x_n)^\top\) be the Fiedler vector of \(G_n\) with \(\|x\| = 1\). We define the vector \(y\) of length \(n + 2\) by
\[
y_i = \begin{cases} 
  x_i - \delta & i = 1, 2, 3, 4, \\
  x_5 - \delta & i = 5, 6, \\
  x_{i-2} - \delta & i = 7, \ldots, n + 2,
\end{cases}
\]
where $\delta = \frac{2x_5}{n+2}$. It is seen that $y_1$ is orthogonal to $1$. We label the vertices of $H_{n+2}$ by the components of $y$ as follows:

By the definition of $y$, we have $\sum_{ij \in E(G_n)} (x_i - x_j)^2 = \sum_{ij \in E(H_{n+2})} (y_i - y_j)^2$. On the other hand,

$$||y||^2 = \sum_{i=1}^{n+2} y_i^2 = \sum_{i=1}^{n} (x_i - \delta)^2 + 2(x_5 - \delta)^2 = \sum_{i=1}^{n} x_i^2 - 2\delta \sum_{i=1}^{n} x_i + n\delta^2 + 2(x_5 - \delta)^2 = 1 + 2x_5^2 \left(1 - \frac{2}{n+2}\right).$$

So $||y|| > 1$, which means that the Rayleigh quotient for $y$ on $H_{n+2}$ is smaller than $\mu(G_n)$. It follows that $(1 + o(1))\frac{2n^2}{(n+2)^2} = \mu(H_{n+2}) \leq \mu(G_n)$. By a similar argument, we see that $\mu(G_n) \leq \mu(G_{n-2}) = (1 + o(1))\frac{2(n-2)^2}{n^2}$. Therefore, $\mu(G_n) = (1 + o(1))\frac{2n^2}{n^2}$. 

\section{Structure of quartic graphs with minimum spectral gap}

Motivated by the Aldous–Fill Conjecture and also as an analogue to Babai’s conjecture on connected cubic graphs with minimum spectral gap, we consider the problem of determining the structure of connected quartic graphs with minimum spectral gap. We prove that the connected quartic graphs with minimum spectral gap have a path-like structure and specify their blocks. Finally, we pose a conjecture which precisely describes the connected quartic graphs with minimum spectral gap.

Here is the main result of this section.

\textbf{Theorem 3.1.} Let $G$ be a graph with the minimum spectral gap in the family of connected quartic graphs on $n$ vertices. If $G$ is a block then either $n \leq 9$ and $G$ is one of the graphs
of Figure 3, or \( n \geq 10 \) and \( G \) is of B-type:

\[
\begin{array}{c}
\langle B_1 \rangle \\
\langle B_2 \rangle \\
\vdots \\
\langle B_s \rangle \\
\end{array}
\]

for some \( s \geq 2 \), where \( B_1 \) is either \( D_3 \) or \( D_4 \) (see Figure 4), \( B_i \in \{M_1, M_2\} \) for \( i = 2, \ldots, s - 1 \), and \( B_s \) is the mirror image of either \( D_3 \) or \( D_4 \).

If \( G \) itself is not a block, then it has a path-like structure:

\[
\begin{array}{c}
\langle B_1 \rangle \\
\langle B_2 \rangle \\
\vdots \\
\langle B_s \rangle \\
\end{array}
\]

in which the structure of each block is as follows.

(i) Each left end block is either \( D_1 \), \( D_2 \), or it is of B-type for some \( s \geq 1 \), where \( B_1 \in \{D_3, D_4\} \), \( B_i \in \{M_1, M_2\} \) for \( i = 2, \ldots, s - 1 \), and \( B_s \in \{M_1, M_2, M_4\} \). Each right end block is the mirror image of some left end block described above.

(ii) Each middle block is either \( M \) or it is of B-type for some \( s \geq 1 \), where \( B_1 \in \{M_1, M_2, M_3\} \), \( B_i \in \{M_1, M_2\} \) for \( i = 2, \ldots, s - 1 \), and \( B_s \in \{M_1, M_2, M_4\} \).

![Figure 3: The graphs of Theorem 3.1 on \( n \leq 9 \) vertices](image)

![Figure 4: The building parts of the quartic graphs of Theorem 3.1](image)

Subsection 3.2 is devoted to the proof of Theorem 3.1. In fact, Theorem 3.1 follows from Theorems 3.11 and 3.15 below.
3.1 Elementary moves and their effect on algebraic connectivity

In this subsection we present the main tool of the proof of Theorem 3.1, that is, a local operation on edges of a graph which preserves the degree sequence of the graph.

Let $G$ be a graph. By ‘∼’ and ‘≁’ we denote, respectively, adjacency and non-adjacency in $G$. An elementary move or switching in $G$ is a switching of parallel edges: let $a ∼ b, c ∼ d$ and $a ≁ c, b ≁ d$, then the elementary move denoted by $\text{sw}(a, b, c, d)$ removes the edges $ab$ and $cd$ and replaces them by the edges $ac$ and $bd$.

**Definition 3.2.** Let $G$ be a graph and $\rho : V(G) \rightarrow \mathbb{R}$ be a Fiedler vector of $G$, considered as a weighting on the vertices; for $v \in V(G)$ we write $\rho_v = \rho(v)$. For convenience, we may assume the vertex set is $[n] = \{1, 2, \ldots, n\}$ and that the vertices are numbered so that $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. We call this a proper labeling of the vertices (with respect to the eigenvector $\rho$).

The following two lemmas were initially used by Guiduli [9] (see also [8]) for cubic graphs but they also hold for quartic graphs.

**Lemma 3.3.** Let $G$ be a connected graph. Let $\rho : V(G) \rightarrow \mathbb{R}$ be a Fiedler vector of $G$. If there are vertices $\{a, b, c, d\}$ in $G$ such that $a ∼ b, c ∼ d, a ≁ c, b ≁ d$, with $\rho_a \geq \rho_d$, and $\rho_c \geq \rho_b$, then $\text{sw}(a, b, c, d)$ does not increase the algebraic connectivity.

**Definition 3.4.** A switch or elementary move is said to be proper if it satisfies the conditions of Lemma 3.3.

We will use proper switchings to transfer the graphs into the path-like structure without increasing the algebraic connectivity. The following lemma keeps the graph connected during this procedure.

**Lemma 3.5.** Let $G$ be a connected graph on $[n]$, properly labeled with respect to a Fiedler vector of $G$. Assume that $G \setminus [r]$ is disconnected and that each of its components has an edge which is not a bridge. Then we may reconnect the graph using proper elementary moves to make $G \setminus [r]$ connected, not increasing the algebraic connectivity.

In the arguments which follow, we use proper elementary moves to connect two specific vertices $x$ and $y$. The following remark demonstrates when such a switch does, or does not, exist.

**Remark 3.6.** Let $G$ be a graph whose vertices $[n]$ are properly labeled and $x, y$ be two vertices of a graph $G$ with $x < y$. Suppose we are looking for a proper switch to connect $x$ and $y$ without altering the induced subgraph on $[x]$. From Lemma 3.3 it is evident that such a switch does not exist if and only if any neighbor of $x$ in $[n] \setminus [y]$ is adjacent to any neighbor of $y$ in $[n] \setminus [x]$.
3.2 Proof of Theorem 3.1

Theorem 3.1 follows from Theorems 3.11 and 3.15 which will be proved in this subsection.

Hereafter, we assume that $\Gamma$ is a connected quartic graph with $n$ vertices, whose vertices are labeled properly as described in Definition 3.2, with $\rho$ being the Fiedler vector used for the labeling. Our goal is to utilize proper elementary moves to transfer $\Gamma$ to one of the graphs described in Theorem 3.1.

3.2.1 The subgraph on the first few vertices

Our first goal is to prove that we can reconnect (by proper elementary moves) the first few vertices of $\Gamma$ to get one of the four subgraphs $D_1$, $D_2$, $D_3$, $D_4$. We first need two lemmas.

By $\Gamma[r]$ we denote the induced subgraph of $\Gamma$ on the vertices $[r]$, by $d_{\Gamma[r]}(v)$ the number of neighbors of $v$ in $\Gamma[r]$, and by $\text{dist}_{\Gamma}(u, v)$, the distance between the vertices $u$ and $v$ in $\Gamma$.

**Lemma 3.7.** Assume that $\Gamma \setminus [r]$ is connected and $d_{\Gamma[r]}(r + 1) = d_{\Gamma[r]}(r + 2) = 1$. If $n > r + 8$, then by proper switchings, $r + 1 \sim r + 2$. Moreover, if $n = r + 5$ or $n = r + 8$, then the last five (resp. eight) vertices of $\Gamma$, by proper switchings, can be reconnected to form the mirror image of $D_4$ (resp. $D_3$).

**Proof.** We consider the following three cases based on $\text{dist}_{\Gamma \setminus [r]}(r + 1, r + 2)$.

(i) $\text{dist}_{\Gamma \setminus [r]}(r + 1, r + 2) > 3$. Let $P$ be a shortest path between $r + 1$ and $r + 2$ in $\Gamma \setminus [r]$ and $x$ and $y$ be the neighbors of $r + 1$ and $r + 2$, respectively, on $P$. We have $x \sim y$. Now $\text{sw}(r + 1, x, r + 2, y)$ connects $r + 1$ to $r + 2$.

(ii) $\text{dist}_{\Gamma \setminus [r]}(r + 1, r + 2) = 3$. If some neighbor of $r + 1$ is not adjacent to some neighbor of $r + 2$, the desired switch would exist by Remark 3.6. Otherwise, $\Gamma[r + 8]$ is already 4-regular which implies $n = r + 8$. Now, let $x = r + 3$. By the symmetry, we may assume that $x \sim r + 1$. So we are in the following situation:

![Diagram](image.png)
We first \( \text{sw}(r+2, y, x, w) \) and then \( \text{sw}(r+2, x, r+1, z) \), which result in the following:

The above block is an end block of \( \Gamma \) and we show that we can end up with the mirror image of \( D_3 \) by performing the switches described below. If \( z = r + 4 \), then \( \text{sw}(r + 1, m, z, x), \text{sw}(r + 2, w, x, l), \) and \( \text{sw}(r + 2, l, z, y) \), result in \( D_3 \) as follows:

Similarly, \( D_3 \) can be obtained in the remaining cases: if \( m = r + 4 \), then \( \text{sw}(r + 2, w, x, l) \) and \( \text{sw}(r + 2, l, m, y) \); if \( y = r + 4 \), then \( \text{sw}(r + 1, m, y, z), \text{sw}(r + 2, w, x, l) \), and \( \text{sw}(r + 2, l, y, x) \); if \( w = r + 4 \), then \( \text{sw}(r + 1, m, w, z) \) and \( \text{sw}(r + 2, l, y, x) \); if \( l = r + 4 \), then \( \text{sw}(r + 1, m, l, z) \) and \( \text{sw}(r + 2, w, x, l) \).

(iii) \( \text{dist}_{\Gamma \setminus [r]}(r + 1, r + 2) = 2 \). If \( r + 1 \) and \( r + 2 \) share three neighbors and all of these are adjacent to each other, then \( n = r + 5 \) and we have the mirror image of \( D_4 \). Now let \( r + 1 \) and \( r + 2 \) share two neighbors, say \( x \) and \( y \). If \( x \approx y \), then \( \text{sw}(r + 1, x, r + 2, y) \) connects \( r + 1 \) to \( r + 2 \). Otherwise, let \( z \) and \( w \) be the other neighbors of \( r + 1 \) and \( r + 2 \), respectively. We have either \( z \approx x \) or \( w \approx x \), then \( \text{sw}(r + 1, z, r + 2, x) \) or \( \text{sw}(r + 1, x, r + 2, w) \), respectively. Finally, let \( r + 1 \) and \( r + 2 \) share one neighbor, say \( x \). Then there is a neighbor \( w \) of \( r + 1 \) or \( r + 2 \) which is not adjacent to \( x \). Then \( \text{sw}(r + 1, w, r + 2, x) \) or \( \text{sw}(r + 1, x, r + 2, w) \).

\( \square \)

**Lemma 3.8.** Assume that \( \{r + 1, r + 2\} \) is a vertex cut of \( \Gamma \), \( \Gamma \setminus [r + 2] \) is connected, \( d_{\Gamma \setminus [r]}(r + 1) = d_{\Gamma \setminus [r]}(r + 2) = 2 \), and \( r + 1 \) is adjacent to both \( r + 3 \) and \( r + 4 \). If \( n \neq r + 6 \), then by proper switchings, \( r + 2 \sim r + 3 \). Moreover, if \( n = r + 6 \), then by proper switchings, the last six vertices of \( \Gamma \) can be reconnected to form the mirror image of the first six vertices of \( D_1 \).

**Proof.** We consider four cases:
(a) \( r + 2 \sim r + 4 \) and \( r + 3 \sim r + 4 \). We are in the situation of Lemma 3.7. (Here \( r + 3 \) and \( r + 4 \) have the same role as \( r + 1 \) and \( r + 2 \) in Lemma 3.7, and further note that the second case of Lemma 3.7 on end blocks does not occur here as \( \Gamma \setminus [r + 1] \) is disconnected now.) Thus \( r + 3 \sim r + 4 \). This reduces \( \Gamma \) to Case (b).

(b) \( r + 2 \sim r + 4 \) and \( r + 3 \sim r + 4 \).

(i) \( \text{distr}_{[r+1]}(r + 2, r + 3) > 3 \). The desired switch will clearly exist, as in Case (i) of the proof of Lemma 3.7.

(ii) \( \text{distr}_{[r+1]}(r + 2, r + 3) = 3 \). In view of Remark 3.6, the desired switch is available, except in the following situation:

![](https://example.com/diagram.png)

Then \( \text{sw}(r + 2, y, r + 4, x) \) reduces the graph to Case (c).

(iii) \( \text{distr}_{[r+1]}(r + 2, r + 3) = 2 \). If \( r + 2 \) and \( r + 3 \) share one neighbor, the desired switch is available. Now, let \( r + 2 \) and \( r + 3 \) share two neighbors, say \( x, y \). If \( x \sim y \), then \( \text{sw}(r + 2, x, r + 3, y) \). Now let \( x \sim y \). If \( r + 4 \) is not adjacent to one of \( x \) or \( y \), then \( \text{sw}(r + 2, x, r + 3, r + 4) \) or \( \text{sw}(r + 2, y, r + 3, r + 4) \). Otherwise \( r + 4 \) is adjacent to both \( x, y \), and so \( n = r + 6 \) and the last six vertices of \( \Gamma \), form the mirror image of the first six vertices of \( D_1 \).

(c) \( r + 2 \sim r + 4 \) and \( r + 3 \sim r + 4 \). Let \( x, y \notin \{r + 1, r + 4\} \) be two other neighbors of \( r + 3 \). At least one of these two vertices, say \( x \), is non-adjacent to \( r + 4 \). Then \( \text{sw}(r + 2, r + 4, r + 3, x) \).

(d) \( r + 2 \sim r + 4 \) and \( r + 3 \sim r + 4 \). Let \( x, y, z \neq r + 1 \) be three other neighbors of \( r + 3 \). At least one of these three vertices, say \( x \), is non-adjacent to \( r + 4 \). Then \( \text{sw}(r + 2, r + 4, r + 3, x) \).

Lemma 3.9. By proper switchings, the induced subgraphs on the first few vertices in \( \Gamma \) can be transferred by elementary moves into one of the four subgraphs \( D_1, D_2, D_3, D_4 \). Furthermore, if \( n \leq 9 \), then \( \Gamma \) can be transferred into one of the graphs \( G_5, G_6, G_7, G_8, G_8', \) or \( G_9 \).
Proof. In Steps 1–7 below, we show that the induced subgraph on first five to seven vertices of Γ can be transferred into $D_4$ or to one of the subgraphs $H_1, H_2$ given in Figure 5 or Γ has at most 9 vertices and it is one of the graphs $G_5, G_6, G_7, G_8, G_8', G_9$. In the final Step 8, from $H_1, H_2$ we obtain one of $D_1, D_2, D_3, D_4$.

![Figure 5: Two subgraphs on the first few vertices](image)

**Step 1.** Connecting 1 to 2. If $1 \not\sim 2$, then consider a shortest path $(1, i_1, \ldots, i_r, 2)$ from 1 to 2. Let $x$ be a neighbor of 1 such that $x \not= i_1$ and $x \sim i_r$, then we may apply the proper $\text{sw}(1, x, 2, i_r)$, leaving 1 adjacent to 2 and Γ connected.

**Step 2.** Connecting 1 to 3, 4, and 5. If $1 \sim 3$, then let $x \not= 2$ be a neighbor of 1. Note that each connected component of $\Gamma \setminus [1]$ contains a cycle. We may therefore use Lemma 3.5 to assume that $\Gamma \setminus [1]$ is connected. Let $(x, i_1, \ldots, i_r, 3)$ be a shortest path from $x$ to 3 not passing through 1. Let $y$ be a neighbor of 3 so that $y \not= i_r$ and $y \sim i_r$. Then $\text{sw}(1, x, 3, y)$. In the same way, we can connect 1 to each of 4 and 5.

**Step 3.** Connecting 2 to 3. We may assume that $\Gamma \setminus [1]$ is connected. If $\text{dist}_{\Gamma \setminus [1]}(2, 3) = 2$, then the desired switch is there, except when 2 and 3 share three neighbors in $\Gamma \setminus [1]$ and all three neighbors are adjacent to each other. But this contradicts the fact that $\Gamma \setminus [1]$ is connected. Similarly, we are done if $\text{dist}_{\Gamma \setminus [1]}(2, 3) = 3$.

**Step 4.** Connecting 2 to 4. Again we may assume by Lemma 3.5 that $\Gamma \setminus [3]$ is connected. If no proper switch to have $2 \sim 4$ exists, then, similarly to Step 3, we see that $4 \sim 5$. Let $x \not= 1, 3$ be the other neighbor of 2. We consider the following two cases:

(4a) $2 \sim 5$.

(i) $\text{dist}_{\Gamma \setminus [3]}(x, 4) = 1$. If $x \sim 5$, then $\text{sw}(2, x, 4, 5)$. Otherwise, 1, 2, 4, $x$ are all the neighbors of 5. Let $y \not= 1, 5, x$ be the fourth neighbor of 4. Then $\text{sw}(2, 5, 4, y)$.

(ii) $\text{dist}_{\Gamma \setminus [3]}(x, 4) = 2$. This follows by the same argument as in the previous item.

(4b) $2 \sim 5$.

(i) $\text{dist}_{\Gamma \setminus [3]}(x, 4) = 1$. By Remark 3.6 the desired switch exists, except in the two
following situations:

For the left one, we use $sw(2, x, 4, 5)$. The right one is impossible as it contradicts the fact that $\Gamma \setminus [3]$ is connected.

(ii) $\text{dist}_{\Gamma \setminus [3]}(x, 4) = 2$. The desired switch exists, except in the following situation:

Then $sw(2, x, 5, y)$ reduces the graph to (4a).

**Step 5. Connecting 2 to 5.** We may assume that $\Gamma \setminus [3]$ is connected. Let $x \neq 1, 3, 4$ be the fourth neighbor of 2. We consider the following two cases based on $\text{dist}_{\Gamma \setminus [3]}(x, 5)$.

(i) $\text{dist}_{\Gamma \setminus [3]}(x, 5) = 1$. Let $y$ and $z$ be the other two neighbors of $x$. If $y \not\sim 5$ or $z \not\sim 5$, then the desired switch is available. Otherwise we have the following situation:

We first show that $3 \sim 5$ or $4 \sim 5$. If $y \sim z$, then by examining the neighbors of 3 and 4, proper switches to $3 \sim 5$ or $4 \sim 5$ will clearly exist. If $y \sim z$, then the desired switch will exist, except when $3 \sim 4$, $3 \sim y$, and $4 \sim z$ in which case $n = 8$ and $\Gamma = G_8$. So we are in either of the following situations:
(Note that if there is no edge 4x in the left and 3x in the right situation, then it is easy to find a switch that connects 2 to 5.) For the left one, 3 has a neighbor \( y \neq 4 \) and \( y \sim x \). Then \( \text{sw}(3, y, 5, x) \). For the right one, 4 has a neighbor \( y \neq 3 \) and \( y \sim x \). Then \( \text{sw}(4, y, 5, x) \). Now we have the following subgraph:

![Graph](image)

If both 3 and 4 be adjacent to \( x \), then \( n = 6 \) and we get \( G_6 \). Therefore we suppose that both 3 and 4 cannot be adjacent to \( x \). Then either \( 3 \sim x \) or \( 4 \sim x \) for which we apply \( \text{sw}(2, x, 5, 3) \) or \( \text{sw}(2, x, 5, 4) \), respectively.

(ii) \( \text{dist}_{\Gamma \setminus \{5\}}(x, 5) = 2 \). If the remaining neighbors of \( x \) and 5 are not the same, then the desired switch is available. Otherwise, similarly to (i), we have \( 3 \sim 5 \) or \( 4 \sim 5 \), so in view of Remark 3.6, we are in either of the following situations:

![Graphs](images)

For the left one, let \( y \sim z \). If \( x \leq z \), then \( \text{sw}(x, y, 5, z) \), and if \( z \leq x \), then \( \text{sw}(z, 5, 4, x) \) connects \( x \) to 5, which reduces the graph to Case (i). Now let \( y \sim z \). If \( 3 \sim y \) and \( 3 \sim z \), then \( n = 8 \), and by \( \text{sw}(3, y, 5, 4) \) and then \( \text{sw}(3, 5, 4, x) \) we transfer \( \Gamma \) to \( G_8 \). If \( 3 \sim y \) or \( 3 \sim z \), then there is a neighbor \( w \) of 3 such that either \( w \sim y \) and \( w \neq y \), and then \( \text{sw}(3, w, 5, y) \), or \( w \sim z \) and \( w \neq z \), and then \( \text{sw}(3, w, 5, z) \). We do the same for the right one to connect 4 to 5. So we have the following:

![Graph](image)

Now \( \text{sw}(3, x, 4, 5) \) connects \( x \) to 5, which reduces the graph to Case (i).

**Step 6. Connecting 3 to 4.** Let \( x \neq 1, 2 \) be a neighbor of 3. If \( 4 \sim 5 \), we may choose \( x \) so that \( x \sim 5 \), and then \( \text{sw}(3, x, 4, 5) \). So assume that \( 4 \sim 5 \). From Remark 3.6, it is seen
that the desired switch is available, except in the following cases:

For each of them, we first show that $4 \sim 5$. Then, with this edge, the desired switches can be found. In the left one, if 5 is adjacent to both of $y$ and $z$, then $n = 8$ and $\Gamma = G'_8$. Otherwise 5 has a neighbor $w \neq y, z$ and $w \sim x$. We first $\text{sw}(4, x, 5, w)$ and then $\text{sw}(3, x, 4, 5)$. The other two cases are similar. Note that in the second case if $5 \sim x$ and $5 \sim y$, then $n = 7$ and $\Gamma = G'_7$; and in the third case if $z \sim w$, $5 \sim x$, and $5 \sim y$, then $n = 9$, and by $\text{sw}(5, x, 3, y)$ and then $\text{sw}(3, 5, 4, z)$, we transfer $\Gamma$ to $G'_9$.

**Step 7.** So far we obtain one of the following subgraphs on the first five vertices:

If the left one is the case, letting $x$ to be the fourth neighbor of 3, then $\text{sw}(3, x, 5, 4)$ connects 3 to 5, and so we obtain $D_4$. Now, assume that the right one is the case. If we can find a switch to connect 3 to 5, we again reach $D_4$. Otherwise, it is easily seen that by proper switching we can connect 3 to 6 as follows:

Furthermore, if we can find a suitable switch to connect 4 to 6, we reach the graph $H_1$ of Figure 3. Otherwise, it is easily seen by switching that $4 \sim 7$ and that we can reach the following graph:

If there is no switch to connect 4 to 6, then we can find a proper switch to connect 5 to 6, except when all the three vertices 5, 6, and 7 are adjacent to 8 and 9 and $8 \sim 9$, in which
case \( n = 9 \) and \( \Gamma = G_9 \). Now, if \( 5 \sim 7 \), then \( \text{sw}(4, 7, 6, 5) \) connects 4 to 6. Otherwise \( 5 \sim 7 \) and we reach the graph \( H_2 \) of Figure 5.

**Step 8.** So far we have obtained one of the subgraphs \( D_4 \), or \( H_1, H_2 \) of Figure 5, unless \( n \leq 9 \), in which case we obtained the graphs \( G_i \) of Figure 3. We show that continuous reconnecting, starting from \( H_1 \) and \( H_2 \), leads to \( D_1, D_2, D_3, \) or \( D_4 \).

First, consider \( H_1 \). We have either \( 5 \sim 6 \) or \( 5 \not\sim 6 \). Let \( 5 \sim 6 \). It is easy to find a switch that connects 5 to 7. If further \( 6 \sim 7 \), then we have the block \( D_1 \). If \( 6 \not\sim 7 \), it is easily seen, by switching, that \( 6 \sim 8 \). Then \( \text{sw}(3, 6, 5, 7) \) reduces the subgraph on \( [5] \) to \( D_4 \). Now, let \( 5 \sim 6 \). By switching it is seen that \( 5 \sim 7 \) and \( 5 \sim 8 \). Thus we are in the situation of Lemma 3.8 for \( r = 4 \), which leads to either of the graphs of Figure 6.

![Figure 6: Two subgraphs which can be obtained by proper switchings starting from \( H_1 \)](image)

Now, \( \text{sw}(3, 6, 5, x) \) reduces the graph of Figure 6(b) to the graph of Figure 7, which is the unique graph of Theorem 3.1 on 10 vertices.

![Figure 7: The unique graph of Theorem 3.1 on 10 vertices](image)

In the graph of Figure 6(a), let \( 6 \sim 8 \). If further \( 7 \sim 8 \), then we get \( D_3 \), otherwise \( \text{sw}(5, 8, 6, 7) \) and then \( \text{sw}(3, 6, 5, 7) \) reduce the subgraph on \( [5] \) to \( D_4 \). Now, let \( 6 \sim 8 \). Then \( \text{sw}(3, 6, 5, 8) \) reduces the subgraph on \( [5] \) to \( D_4 \).

Secondly, consider \( H_2 \). We have either \( 6 \sim 7 \) or \( 6 \not\sim 7 \). Let \( 6 \sim 7 \). It is easy to find a switch that connects 6 to 8. If further \( 7 \sim 8 \), then we obtain the block \( D_2 \). If \( 7 \not\sim 8 \), then \( \text{sw}(4, 7, 6, 8) \) reduces the graph to \( D_1 \). Now, let \( 6 \sim 7 \). Then \( \text{sw}(3, 6, 5, 7) \) reduces the subgraph on \( [5] \) to \( D_4 \).

**3.2.2 General Steps**

In this section, we continue reconnecting \( \Gamma \) by proper switchings to construct the middle and end blocks with the structure described in Theorem 3.1.
Lemma 3.10. Let $r \in [n]$ be a cut vertex of $\Gamma$. Then, either by proper switchings we can transform the induced subgraph of $\Gamma$ on the next four or five vertices into one of the subgraphs given in Figure 8, or the vertex $r$ is the last cut vertex of $\Gamma$, where the last block can be transformed into one of the blocks $D_1, D_2, D_3, D_4$.

![Figure 8: Possible subgraphs on vertices following a cut vertex obtained by proper switchings](image)

Proof. First note that as quartic graphs have no bridges, the vertex $r$ has two neighbors in each component of $\Gamma - r$.

**Step 1.** Connecting $r$ to $r + 1$. Let $x$ be the neighbor of $r$ closest to $r + 1$ and let $y$ be the neighbor of $r + 1$ furthest from $r$. Then $sw(r, x, r + 1, y)$.

**Step 2.** Connecting $r$ to $r + 2$. As $r$ is a cut vertex of $\Gamma$, it has another neighbor, say $x$, in the block containing $r + 1$. By Lemma 3.5, we may assume that $\Gamma \setminus [r]$ is connected. Let $y$ be a neighbor of $r + 2$ furthest from $x$ in $\Gamma \setminus [r]$. Then $sw(r, x, r + 2, y)$.

**Step 3.** Connecting $r + 1$ to $r + 2$. Again we may assume that $\Gamma \setminus [r]$ is connected. This follows from Lemma 3.7, and we can get the mirror image of $D_3$ or $D_4$ at this step.

**Step 4.** Connecting $r + 1$ to $r + 3$ and $r + 4$. We can use the same arguments as in Step 2.

**Step 5.** Connecting $r + 2$ to $r + 3$. We may assume that $\Gamma \setminus [r + 2]$ is connected. This follows from Lemma 3.8, and we can get the mirror image of the block $D_1$ at this step.

**Step 6.** We distinguish four cases:

1. **(6a)** $r + 2 \sim r + 4, r + 3 \sim r + 4$. As before, we obtain the subgraph given in Figure 8(a).

2. **(6b)** $r + 2 \sim r + 4, r + 3 \sim r + 4$. In this case, we obtain the subgraph given in Figure 8(b).

3. **(6c)** $r + 2 \sim r + 4, r + 3 \sim r + 4$. We may assume that $\Gamma \setminus [r + 3]$ is connected. Our goal is to show that $r + 2 \sim r + 5$ and $r + 3 \sim r + 5$. Let $x \neq r, r + 1, r + 3$ be the fourth neighbor of $r + 2$. There are two possibilities:
(i) $\text{distr}_{\Gamma[r+3]}(x, r + 4) = 1$. A switch to connect $r + 2$ to $r + 4$ exists, except in the following situation:

![Diagram](image)

If $x = r + 5$, then we are done by reaching the subgraph given in Figure 8(c). If $y = r + 5$, let $z$ and $w$ be the other neighbors of $r + 5$. Then $\text{sw}(r + 2, x, r + 5, z)$ and $\text{sw}(r + 3, x, r + 5, w)$ give rise to Figure 8(c) again. If $y \neq r + 5$ and $x \neq r + 5$, then $r + 5$ has two neighbors $z$ and $w$ that are non-adjacent to $x$. Then $\text{sw}(r + 2, x, r + 5, z)$ and $\text{sw}(r + 3, x, r + 5, w)$ give rise to Figure 8(d).

(ii) $\text{distr}_{\Gamma[r+3]}(x, r + 4) = 2$. A switch to connect $r + 2$ to $r + 4$ exists, except in the following situation:

![Diagram](image)

If $x = r + 5$, then we are done by reaching the subgraph of Figure 8(d). Otherwise, in a similar manner, the switches which give rise to Figure 8(d) can be found easily by examining the adjacencies between neighbors of $r + 5$ and $r + 2$ (or $r + 3$).

(6d) $r + 2 \not\sim r + 4, r + 3 \not\sim r + 4$. By Lemma 3.5, we may assume that $\Gamma \setminus [r + 2]$ is connected. We show that an appropriate switch can be found to make $r + 3 \sim r + 4$, and so the graph is reduced to Case (6c).

Let $x \neq r, r + 1, r + 3$ be the fourth neighbor of $r + 2$. We consider the following two cases based on the distance between $x$ and $r + 4$ in $\Gamma \setminus [r + 2]$.

(i) $\text{distr}_{\Gamma[r+2]}(x, r + 4) = 1$. The desired switch is available, except in the following situation:
So $n = r + 7$ and we obtain the mirror image of the block $D_2$.

(ii) $\text{dist}_{r+2}(x, r + 4) = 2$. In view of Remark 3.6 the desired switch is available in any situation other than the following:

Then either $y \sim z$ and so $\text{sw}(r + 3, y, r + 4, z)$, or $y \sim w$ and so $\text{sw}(r + 3, y, r + 4, w)$.

We are now in a position to prove the ‘first half’ of the proof Theorem 3.1, that is to conclude that $\Gamma$ can be transferred to one of the graphs of Theorem 3.1.

**Theorem 3.11.** By proper switchings, any connected quartic graph can be turned into one of the graphs described in Theorem 3.1.

**Proof.** For $n \leq 9$ the assertion is proved in Lemma 3.9. So we may assume that $n \geq 10$. We start rebuilding $\Gamma$ on its first few vertices as in Lemma 3.9. As we saw there, the first few vertices of $\Gamma$ can be transformed into one of the subgraphs $D_1, D_2, D_3, D_4$. Moreover, whatever we obtained, we ended up either with a cut vertex, or with one of the situations (i) or (ii) of Table 1. Also, after reconnecting following a cut vertex, employing Lemma 3.10, we again reach at one of the situations (i), (ii), or (iii). We now demonstrate what can be constructed afterwards. As verified below, by proper switchings, the situation of the next few vertices can be determined from the situation of $v, v + 1$ according to Table 1.

In Case (i) it is easily seen, by switching, that $v \sim v + 2$. If further $v + 1 \sim v + 2$, then we obtain the first possible outcome. If $v + 1 \sim v + 2$, it is easily seen, by switching, that $v + 1 \sim v + 3$. Now, we can employ Lemma 3.7 (with $r = v + 1$), which implies that we have $v + 2 \sim v + 3$ or that we have reached the mirror images of either $D_3$ or $D_4$ as an end block.

In (ii), we assume that $v \sim v + 1$, otherwise we return to Case (i). It is easily seen, by switching, that $v \sim v + 2$ and $v \sim v + 3$. Then, by Lemma 3.8 we see that $v + 1 \sim v + 2$. 

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Table 1: The situation of two vertical vertices and the possible structures following them

or we obtain the third possible outcome, in which case we either get $D_3$ or we are left
with one of the following two situations:

For the left one, by $\text{sw}(y, v + 1, x, v)$ and then $\text{sw}(x, v + 1, v, z)$, and for the right one, by $\text{sw}(x, v + 1, v, z)$, we obtain $D_4$. If we further have $v + 1 \sim v + 3$, we come up with the
first possible outcome. So assume that $v + 1 \not\sim v + 3$. Then it easy to find a switch that
ensures $v + 1 \sim v + 4$. If $v + 2 \sim v + 3$ and $v + 2 \sim v + 4$, then we obtain the first possible
outcome again. Otherwise, we have either $v + 2 \not\sim v + 3$ or $v + 2 \not\sim v + 4$, and then
$\text{sw}(v, v + 3, v + 1, v + 2)$ or $\text{sw}(v, v + 2, v + 1, v + 4)$, respectively, ensures that $v \sim v + 1$,
which return us to Case (i).

In (iii), it is easily seen, by switching, that $v \sim v + 2$ and $v \sim v + 3$, as follows:

If $v + 1 \sim v + 2$, by $\text{sw}(x, v + 1, v, v + 2)$, $v$ is turned to a cut vertex $v$. Now, let $v + 1 \sim v + 2$
If further $v + 1 \sim v + 3$, then we obtain the first outcome, otherwise by $\text{sw}(x, v + 1, v, v + 3)$,
$v$ is turned into a cut vertex $v$. 

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The outcome of Table 1 is either an end block or, after proper reconnecting, we are again in one of the situations (i), (ii), (iii). Therefore, we may keep repeating this until we end up with an end block. We need further switchings to transform the blocks into the structure desired by Theorem 3.1.

First, note that we may have the following structure in our graph:

\[
\begin{array}{cccc}
\text{a} & \text{b} \\
\text{c} & \text{d}
\end{array}
\]

which can be turned by \( \text{sw}(a, b, c, d) \) to

\[
\begin{array}{cccc}
\text{a} & \text{b} \\
\text{c} & \text{d}
\end{array}
\]

This guarantees that our blocks have the general B-type structure. We need to take care of the location of the structures of the shape \( \bigotimes \). Such a shape may appear in one of the following structures:

\[
\begin{array}{cccc}
\text{x} & \text{c} & \text{d} \\
\text{a} & \text{b} & \text{z}
\end{array}
\quad \begin{array}{cccc}
\text{x} & \text{c} & \text{d} \\
\text{a} & \text{b} & \text{z}
\end{array}
\quad \begin{array}{cccc}
\text{x} & \text{c} & \text{d} \\
\text{a} & \text{b} & \text{w}
\end{array}
\]

all of which can be transformed by \( \text{sw}(a, b, c, d) \) to

\[
\begin{array}{cccc}
\text{x} & \text{y} & \text{d} \\
\text{a} & \text{c} & \text{z}
\end{array}
\quad \begin{array}{cccc}
\text{x} & \text{b} & \text{d} \\
\text{a} & \text{y} & \text{z}
\end{array}
\quad \begin{array}{cccc}
\text{x} & \text{b} & \text{d} \\
\text{a} & \text{c} & \text{z}
\end{array}
\]

This in turn implies that \( \bigotimes \) can only occur in the first part of a B-type block, \( \bigotimes \) can only occur in the last part of a B-type block, and \( \bigotimes \) can only occur when a B-type block consists of a single building part, i.e. when it is the block \( M \).

The above arguments show that \( \Gamma \) can be transformed into one of the graphs described in Theorem 3.1. □
3.2.3 Final Step

Let $\mathcal{M}$ denote the family of graphs described in Theorem 3.1. To complete the proof of Theorem 3.1, we need to show that all connected quartic graphs with minimum algebraic connectivity belong to $\mathcal{M}$. In fact, it might be possible that $\Gamma$ is transformed (by means of proper switchings) to a graph $G \in \mathcal{M}$, where we still have $\mu(\Gamma) = \mu(G)$. We show that, under this circumstance, $\Gamma$ must be isomorphic to $G$.

**Remark 3.12.** Considering the structure of the graphs $G \in \mathcal{M}$, we regard the vertices drawn vertically above each other as a cell. The cells of $G$, in fact constitute an ‘equitable partition’ of $G$. Each cell contains one or two vertices (except for the first cells in $D_1, D_4$, or some cells in the $G_i$’s (of Figure 3) that have size 4 and 3, respectively). Further, we know that the weights on the vertices of $G$ given by a Fiedler vector $\rho$ of $G$ are non-increasing from left to right. We may assume that the vertices that are in the same cell have the same weight. Otherwise, let $\rho'$ be a vector obtained from $\rho$ by interchanging the weights of the vertices in all cells (in fact this is carried out by the action of an automorphism of $G$, which also works for the first cells in $D_1, D_4$). Then $\rho'$ and thus $\rho + \rho'$ is an eigenvector corresponding to $\mu(G)$ where $\rho + \rho'$ is constant on each cell. Thus we may assume that $\rho$ is a non-increasing eigenvector for $\mu(G)$ and is constant on each cell. The above argument may not work for $G_8'$; for this small graph this can be done by direct inspection.

**Lemma 3.13.** Let $G \in \mathcal{M}$ and $\rho$ be a non-increasing Fiedler vector of $G$ which is constant on each cell. Then $\rho$ is indeed strictly decreasing on the cells from left to right.

**Proof.** By contradiction, suppose that there are two vertices $a, b$ in two different cells with the same weight under $\rho$. We may assume that $a \sim b$ and that at least one of $a$ or $b$ has a neighbor $c$ with $\rho_c \neq \rho_a = \rho_b$. Let $\alpha$ and $\beta$ be the sum of the weights of the neighbors of $a$ and $b$, respectively. Then, from the structure of the graphs in $\mathcal{M}$, it is evident that $\alpha \geq \beta$. But we have the strict inequality $\alpha > \beta$ by the existence of $c$.

We may suppose that $\|\rho\| = 1$. Let $\lambda$ be the second largest eigenvalue of the adjacency matrix $A$ of $G$. Then $\mu(G) = 4 - \lambda$ and $\lambda = \rho^T A \rho$. We choose a real $\epsilon$ with $0 < \epsilon < (\alpha - \beta)/(1 + \lambda)$. Now, in the vector $\rho$ we replace the weights of $a$ and $b$ by $\rho_a + \epsilon$ and $\rho_b - \epsilon$, respectively, to obtain a new vector $\rho'$. As $\rho \perp 1$, we have $\rho' \perp 1$. We have

$$\lambda = \max_{x \neq 0, x \perp 1} \frac{x^T A x}{x^T x} \geq \frac{\rho'^T A \rho'}{\rho'^T \rho'} = \frac{\lambda + 2 \epsilon (\alpha - \beta - \epsilon)}{1 + 2 \epsilon^2},$$

where the right hand side is larger than $\lambda$ by the choice of $\epsilon$, a contradiction. \qed

**Lemma 3.14.** Any proper elementary move on a graph in $\mathcal{M}$, leaves a graph isomorphic to the original.
Proof. For the graphs in $\mathcal{M}$, with a Fiedler vector which satisfies Lemma 3.13, proper switchings cannot be found except when $a, b$ are in the same cell, and $c, d$ are in the same cell, $a \sim c$, $b \sim d$, $a \sim d$, and $b \sim c$. In this case, $sw(a, c, d, b)$ leaves a graph isomorphic to the original. Also, any proper elementary move on $G_5, G_6, G_7, G_8, G_9, G_{10}$, and $D_1, D_2, D_3$, and $D_4$ gives a structure isomorphic to themselves. \qed

Now we can settle the ‘second half’ of Theorem 3.1. The following theorem, combined with Theorem 3.11, completes the proof of Theorem 3.1.

**Theorem 3.15.** Let $\Gamma$ be a connected quartic graph such that after a sequence of proper switchings, it is turned to $G \in \mathcal{M}$. If $\mu(\Gamma) = \mu(G)$, then $\Gamma$ is isomorphic to $G$.

*Proof.* Let $sw_1, \ldots, sw_t$ be a sequence of proper switchings which turn $\Gamma$ into $G$. Consider the graphs $\Gamma = G_0, G_1, \ldots, G_t = G$ in which $G_i$ is obtained from $G_{i-1}$ by applying $sw_i$. Since $\mu(\Gamma) = \mu(G)$, we have $\mu(G_i) = \mu(G)$, for $i = 1, \ldots, t$. Let $sw_t = sw(a, b, c, d)$. Then

$$0 = \mu(G_{t-1}) - \mu(G) \leq \rho^\top L(G_{t-1})\rho - \rho^\top L(G)\rho = 2(\rho_a - \rho_d)(\rho_c - \rho_b) \leq 0.$$ 

It follows that $\rho_a = \rho_d$ or $\rho_c = \rho_b$. Without loss of generality, suppose that $\rho_a = \rho_d$. From Lemma 3.13 it then follows that $a, d$ are in the same cell of $G$. Note that $sw(d, b, c, a)$ is the reverse of $sw(a, b, c, d)$, and so, when applied on $G$, yields $G_{t-1}$. However, $sw(d, b, c, a)$ is indeed a proper switching, and so by Lemma 3.14, $G_{t-1}$ must be isomorphic to $G$. Similarly, it follows that all $G_i$, for $i = 0, \ldots, t - 2$, are isomorphic to $G$. \qed

3.3 Concluding Remarks

By Theorem 3.1, it can be seen that the connected quartic graphs on $n \leq 10$ vertices with minimum spectral gap are $G_5, G_6, G_7, G_8, G_9$, and the graph of Figure 7 respectively. For $n \geq 11$, we pose the following conjecture on the puniness and the precise structure of the connected quartic graphs with minimum spectral gap. The conjecture suggests that for any given order the quartic graph with minimum spectral gap is unique in which end blocks consist only of one part (see Figure 4) and the middle blocks also consist only of one part namely $M_1$.

**Conjecture 3.16.** The connected quartic graph on $n \geq 11$ vertices with minimum spectral gap is the unique graph $G$ described below. Let $q$ and $r < 5$ be non-negative integers such that $n - 11 = 5q + r$. Then $G$ consists of $q$ middle blocks $M_1$ and each end block is one of $D_1$–$D_4$ of Figure 9. If $r = 0$, then both end blocks are $D_1$. If $r = 1$, then the end blocks are $D_1$ and $D_2$. If $r = 2$, then both end blocks are $D_2$. If $r = 3$, then the end blocks are $D_2$ and $D_3$. Finally, if $r = 4$, then the end blocks are $D_1$ and $D_4$. (The blocks at the right ends are the mirror images of $D_i$’s).
In [3] it was shown that the graphs $G_n$ of Theorem 1.1 are graphs of maximum diameter among all trivalent graphs on $n$ vertices. However, there are other cubic graphs of the same diameter. When it comes to quartic graphs we wish to mention in support of our conjecture that the quartic graphs of Conjecture 3.16 have the largest diameter among the graphs of the same order in $\mathcal{M}$. This is not hard to see, we leave the details to the reader.

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