PERIODIC SOLUTIONS FOR NONLINEAR NONMONOTONE EVOLUTION INCLUSIONS

LESZEK GASIŃSKI *
Jagiellonian University
Faculty of Mathematics and Computer Science
ul. Łojasiewicza 6, 30-348 Kraków, Poland

NIKOLAOS S. PAPAGEORGIU
National Technical University
Department of Mathematics
Zografou Campus, Athens 15780, Greece

Abstract. We study periodic problems for nonlinear evolution inclusions defined in the framework of an evolution triple \((X, H, X^*)\) of spaces. The operator \(A(t, x)\) representing the spatial differential operator is not in general monotone. The reaction (source) term \(F(t, x)\) is defined on \([0, b]\times X\) with values in \(2^{X^*}\setminus\{\emptyset\}\). Using elliptic regularization, we approximate the problem, solve the approximation problem and pass to the limit. We also present some applications to periodic parabolic inclusions.

1. Introduction. In this paper we study periodic problems for a class of evolution inclusions defined in the framework of an evolution triple of spaces.

Let \(T = [0, b]\) and let \((X, H, X^*)\) be an evolution triple (see Section 2). We assume that the embedding of \(X\) into \(H\) is compact (hence so is the embedding of \(H^* = H\) into \(X^*\)). The problem under consideration is the following:

\[
\begin{aligned}
&u'(t) + A(t, u(t)) \in F(t, u(t)) \quad \text{for a.a. } t \in T, \\
&u(0) = u(b).
\end{aligned}
\]

Here \(A: T \times X \to X^*\) and \(F: T \times X \to 2^{X^*}\setminus\{\emptyset\}\). Problem (1.1) is the abstract formulation of parabolic boundary value problem and so it has attracted considerable interest in the past. Initially they considered the case where \(A(t, \cdot)\) is linear and \(F(t, x)\) is a single valued Carathéodory function. In this direction, we mention the works of Amann [1], Becker [3], Browder [4], Prüss [28]. The approach in the aforementioned works is similar. Namely they consider the Poincaré map of the problem and they look for a fixed point of this map. To produce such a fixed point, Amann [1] uses the theory of fixed points in ordered Banach spaces. Becker [3] assumes a time-independent map \(A(x)\) defined on a separable Hilbert space which satisfies that \(A - aI\) is maximal monotone for some \(a > 0\) (strong monotonicity hypothesis). The reaction term \(F(t, x)\) is single valued, continuous and...
and bounded. Browder [4] has a family \( \{ A(t) \} \) of closed monotone operators defined on a Hilbert space and a single valued continuous reaction which is monotone too. Prüss [28] has a time-independent, linear, densely defined operator \( A(x) \) and a reaction term \( F(t, x) \) which is single-valued, continuous and satisfies a Nagumo-type tangential condition with respect to the tangent cone of a closed, convex, bounded set with nonempty interior. The first nonlinear result was proved by Vrabie [31], who assumed that \( A : X \supseteq D(A) \to 2^X \) is a time-independent \( m \)-accretive operator defined on a Banach space \( X \), which generates a compact nonlinear semigroup of contractions. Similar to Becker [3], Vrabie [31] assumes that for some \( a > 0 \), \( A - aI \) is \( m \)-accretive too and the reaction term \( F(t, x) \) is single-valued, continuous and satisfies the following asymptotic condition

\[
\lim_{r \to +\infty} \frac{1}{r} \sup \{ \| F(t, x) \| : t \in \mathbb{R}, x \in D(A), \| x \| \leq r \} < a.
\]

The approach of Vrabie [31] is based on the existence of a fixed point for an appropriately defined nonlinear map. The result of Vrabie [31] can be viewed as the nonlinear extension of that of Becker [3]. Soon thereafter, Hirano [17] partially improved the work of Vrabie [31]. Assuming that the underlying space is a Hilbert space and that \( A \) is of the subdifferential type, he was able to remove the requirements that \( A \) generates a compact semigroup and instead requires only that the resolvents of \( A \) are compact maps. The reaction \( F(t, x) \) is a single-valued Carathéodory map exhibiting a sublinear growth in \( x \) and \( A - F \) satisfies a coercivity condition. The approach of Hirano [17] uses a kind of elliptic regularization of the equation. Another extension of the nonlinear work of Vrabie [31] was produced by Caşcaval-Vrabie [5]. Additional nonlinear results can be found in the works of Shioji [30] and Sattayatham-Tangmanee-Wei [29].

Extensions to evolution inclusions were obtained by Bader-Papageorgiou [2], Hu-Papageorgiou [18], Kasyanov-Mel’nik-Toscano [21], Lakshmikantham-Papageorgiou [22], Paicu [25], Xue-Cheng [32]. All imposed a monotonicity condition on the map \( A \).

Here we go beyond all the aforementioned works. We consider evolution inclusions without any monotonicity condition on \( A(t, \cdot) \) and with a multivalued reaction term defined on \( T \times X \) with values in \( 2^X \setminus \{ \emptyset \} \). It is an interesting open problem, if we can have \( A(t, x) \) to be set-valued too (as, for example, in Defranceschi [8]). Our approach is inspired by the elliptic regularization method developed by Lions [24] (see Section 3.1). Finally for other kinds of generalizations of the evolution inclusions we refer to Gasiński [9, 10, 11, 12] and Gasiński-Smolka [15, 16].

In the next section for the convenience of the reader, we recall the main mathematical tools which we will use in this work.

2. Mathematical background. Let \( (\Omega, \Sigma) \) be a measurable space and let \( X \) be a separable Banach space. We will use the following notation:

\[
P_f(X) = \{ C \subseteq X : C \text{ is nonempty, closed, (convex)} \},
\]

\[
P_{w(K)}(X) = \{ C \subseteq X : C \text{ is nonempty, (weakly-) compact, (convex)} \}.
\]

For a multifunction \( F : \Omega \to 2^X \setminus \{ \emptyset \} \) the graph of \( F \) is the set

\[
\text{Gr } F = \{ (\omega, x) \in \Omega \times X : x \in F(\omega) \}.
\]

We say that \( F \) is graph measurable, if \( \text{Gr } F \in \Sigma \times \mathcal{B}(X) \), with \( \mathcal{B}(X) \) being the Borel \( \sigma \)-field of \( X \). According to the Yankov-von Neumann-Aumann selection theorem (see
Hu-Papageorgiou [19, pp. 158–159]), if µ is a σ-finite measure on Σ and F: Ω → 2^X \ {∅} is graph measurable, then F admits a measurable selection, that is there exists a Σ-measurable function f: Ω → X such that
\[ f(\omega) \in F(\omega) \text{ µ-a.e. in } \Omega. \]
In fact the result is true if X is only Souslin and we can have a whole sequence \( \{f_n: \Omega \to X\}_{n \geq 1} \) of Σ-measurable functions such that
\[ F(\omega) \leq \bigcup_{n \geq 1} f_n(\omega) \text{ µ-a.e. in } \Omega. \]
We say that F: Ω → P_2(X) is measurable, if for all x ∈ X the map
\[ \omega \mapsto d(x, F(\omega)) = \inf_{v \in F(\omega)} \|x - v\| \]
is Σ-measurable. A P_2(X)-valued multifunction which is measurable is also graph measurable. The converse is not in general true (see Hu-Papageorgiou [19]).

Now assume that (Ω, Σ, µ) is a σ-finite measure space and for 1 ≤ p ≤ +∞ consider the Lebesgue-Bochner space \( L^p(\Omega; X) \). Given a multifunction F: Ω → 2^X \ {∅}, we define
\[ S^p_p = \{ f \in L^p(\Omega; X) : f(\omega) \in F(\omega) \text{ µ-a.e. in } \Omega \}. \]
The set \( S^p_p \) is decomposable, in the sense that if \( (A, f_1, f_2) \in \Sigma \times S^p_p \times S^p_p \), then
\[ \chi_A f_1 + \chi_{\Omega \setminus A} f_2 \in S^p_p. \]
Recall that \( \chi_A \) is the characteristic function of \( C \in \Sigma \).

Let H be a separable Hilbert space with norm \( \| \cdot \| \). Also, let X be a separable reflexive Banach space with norm \( \| \cdot \| \) which is embedded continuously and densely into H. Identifying H with its dual (pivot space), we have \( X \hookrightarrow H \hookrightarrow X^* \) with all embeddings being continuous and dense. Such a triple of spaces is usually known in the literature as evolution triple (or Gelfand triple or spaces in normal position).

Here we will also assume that the embedding \( X \hookrightarrow H \) is compact, hence so is \( H \hookrightarrow X^* \) (Schauder theorem; see Gasiński-Papageorgiou [13, p. 275]). By \( \| \cdot \| \), we denote the norm of \( X^* \). Also by \( \langle \cdot, \cdot \rangle \) we denote the duality brackets for the pair \( (X^*, X) \) and by \( \langle \cdot, \cdot \rangle \) the inner product of H. We know that
\[ \langle \cdot, \cdot \rangle_{H^* \times X} = \langle \cdot, \cdot \rangle. \]
Let \( T = [0, b], 1 < p, p' < +\infty \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \). We define
\[ W_p(0, b) = \{ u \in L^p(T; X) : u' \in L^{p'}(T; X^*) \}. \]
In this definition, the derivative of \( u \) is understood in the sense of vector valued distributions. Furnished with the norm
\[ \| u \|_{W_p} = \| u \|_{L^p(T; X)} + \| u' \|_{L^{p'}(T; X^*)}, \]
the space \( W_p(0, b) \) becomes a Banach space which is separable and reflexive. We know that
\[ W_p(0, b) \hookrightarrow C(T; H) \text{ continuously } \quad (2.1) \]
and
\[ W_p(0, b) \hookrightarrow L^p(T; H) \text{ compactly. } \quad (2.2) \]
Moreover, we have the following integration by parts formula. If \( u, v \in W_p(0, b) \), then \( t \mapsto \langle u(t), v(t) \rangle \) is absolutely continuous and
\[ \frac{d}{dt} \langle u(t), v(t) \rangle = \langle u'(t), v(t) \rangle + \langle u(t), v'(t) \rangle \text{ for a.a. } t \in T. \quad (2.3) \]
For more details on evolution triples and related topics, we refer to the books of Gasiński-Papageorgiou [13] and Hu-Papageorgiou [20].

Let $Y$ be a reflexive Banach space, $Y^*$ its topological dual and by $\langle \cdot, \cdot \rangle_Y$ the duality brackets for the pair $(Y^*, Y)$. Let $A : Y \to 2^{Y^*}$ be a multivalued map.

(a) We say that $A$ is pseudomonotone if

(i): for all $u \in Y$, $A(u) \in P_{wkc}(Y^*)$;

(ii): $A$ is bounded (that is, maps bounded sets to bounded sets);

(iii): if $u_n \xrightarrow{w} u \in Y$, $u_n^* \in A(u_n)$ for $n \geq 1$ and $\limsup_{n \to +\infty} \langle u_n^*, u_n - u \rangle_Y \leq 0$, then for all $v \in Y$, there exists $x^*(v) \in A(u)$ such that

$$\langle x^*(v), u - v \rangle_Y \leq \liminf_{n \to +\infty} \langle u_n^*, u_n - v \rangle_Y.$$

(b) We say that $A$ is generalized pseudomonotone, if the following holds:

"if $u_n \xrightarrow{w} u$ in $Y$, $u_n^* \xrightarrow{w} u^*$ in $Y^*$ with $u_n^* \in A(u_n)$ for $n \geq 1$ and $\limsup_{n \to +\infty} \langle u_n^*, u_n - u \rangle_Y \leq 0$, then $u^* \in A(u)$ and $\langle u_n^*, u_n \rangle_Y \to \langle u^*, u \rangle_Y".$

In the next two propositions we state some useful results about these operators which we will need later. Details can be found in Gasiński-Papageorgiou [13, pp. 330–337].

**Proposition 2.1.** (a) Pseudomonotonicity implies generalized pseudomonotonicity;

(b) If $A : Y \to P_{f.c}(Y^*)$ is bounded (maps bounded sets to bounded sets) and generalized pseudomonotone, then $A$ is pseudomonotone.

Pseudomonotone maps exhibit remarkable surjectivity properties and for this reason play an important role in the study of evolution equations.

**Proposition 2.2.** If $A : Y \to 2^{Y^*}$ is pseudomonotone and coercive, that is,

$$\inf \{ \langle u^*, u \rangle : u^* \in A(u) \} \to +\infty \quad \text{as } \|u\|_Y \to +\infty,$$

then $A$ is surjective.

Given a sequence $\{C_n\}_{n \geq 1} \subseteq 2^Y \setminus \emptyset$, we define

$$w^{-}\lim_{n \to +\infty} C_n = \{ u \in Y : u = \lim_{k \to +\infty} u_{n_k}, \quad u_{n_k} \in C_{n_k}, n_k < n_{k+1} \text{ for all } k \geq 1 \}.$$

Here $w$ denotes the weak topology in $Y$.

Returning to the setting of an evolution triple $(X, H, X^*)$ described earlier by $X_w$ (respectively $X^*_w$) we denote the space $X$ (respectively $X^*$) equipped with the weak topology. In what follows using the Troyanski renorming theorem, we assume without any loss of generality that both $X$ and $X^*$ are locally uniformly convex.

Then the duality map $\mathcal{F} : X \to X^*$ is a homeomorphism and maximal monotone (see Gasiński-Papageorgiou [13, p. 316] and Zeidler [33, p. 860]). Recall that

$$L^p(T; X) = L^p(T; X^*)$$

(see Gasiński-Papageorgiou [13, p. 129]) and both spaces are locally uniformly convex. Hence the duality map $\widehat{\mathcal{F}} : L^p(T; X) \to L^{p'}(T; X^*)$ defined by

$$\widehat{\mathcal{F}}(u)(t) = \frac{1}{\|u\|_{L^p(T; X)}^{p-2}} \|u(t)\|^{p-2} \mathcal{F}(u(t)) \quad \forall u \in L^p(T; X), \ u \neq 0,$$
is a homeomorphism and maximal monotone.

Let

\[ W_p^{\text{per}}(0, b) = \{ u \in W_p(0, b) : u(0) = u(b) \}. \]

Because of (2.1) the evaluations at \( t = 0 \) and \( t = b \) make sense. Of course \( W_p^{\text{per}}(0, b) \) is a closed subspace of \( W_p(0, b) \), hence it is a separable reflexive Banach space with the \( W_p(0, b) \) norm.

Consider the operator \( L : L^p(T; X) \supseteq D(L) = W_p^{\text{per}}(0, b) \longrightarrow L^{p'}(T; X^*) \) defined by

\[ L(u) = u' \quad \forall u \in D(L) = W_p^{\text{per}}(0, b) \]

(see Zeidler [33, p. 855]) This is a linear densely defined operator which is maximal monotone (see Hu-Papageorgiou [19, Proposition 3.9, p. 419] and Zeidler [33, Proposition 32.10, p.855]). Note that \( D(L) \) furnished with the graph norm becomes the separable reflexive Banach space \( W_p^{\text{per}}(0, b) \). Since \( L \) is linear and densely defined, we can define its adjoint \( L^* \) which is also linear, densely defined, maximal monotone and \( D(L^*) = D(L) = W_p^{\text{per}}(0, b) \).

By \( \langle \cdot, \cdot \rangle \) we denote the duality brackets for the pair \( (L^p(T; X^*); L^p(T; X)) \). So, we have

\[ \langle (h^*, h) \rangle = \int_0^b \langle h^*(t), h(t) \rangle \, dt \quad \forall h^* \in L^{p'}(T; X^*), \; h \in L^p(T; X). \]

Using integration by parts formula, for \( u, h \in W^{1,\text{per}}(0, b) \) we have

\[ \langle (L(u), h) \rangle = \int_0^b \langle u'(t), h(t) \rangle \, dt \]

\[ = \langle u(b), h(b) \rangle - \langle u(0), h(0) \rangle - \int_0^b \langle u(t), h'(t) \rangle \, dt \]

\[ = -\int_0^b \langle u(t), h'(t) \rangle \, dt = \langle (u, L^*(h)) \rangle. \]

So, we see that \( L^* = -L \) and \( D(L) = D(L^*) \).

Now we are ready to introduce the hypotheses on the data of (1.1).

**H(A):** \( A : T \times X \longrightarrow X^* \) is a map such that:

(i) for all \( x \in X \), the map \( t \mapsto A(t, x) \) is measurable;

(ii) for almost all \( t \in T \), the map \( x \mapsto A(t, x) \) is pseudomonotone;

(iii) \( \| A(t, x) \|_* \leq a_1(t) + c_1 \| x \|^{p-1} \) for almost all \( t \in T \), all \( x \in X \) with \( a_1 \in L^p(T) \), \( c_1 > 0 \), \( 2 \leq p < +\infty \);

(iv) \( A(t, x) \geq c_2 \| x \|^{p-2} - a_2(t) \) for almost all \( t \in T \), all \( x \in X \) with \( c_2 > 0 \), \( a_2 \in L^1(T)_+ = \{ a \in L^1(T) : a(t) \geq 0 \text{ a.e. on } T \} \).

**H(F):** \( F : T \times X \longrightarrow P_{c}(X^*) \) is a multifunction such that:

(i) for all \( x \in X \), the multifunction \( t \mapsto F(t, x) \) is graph measurable;

(ii) for almost all \( t \in T \), \( \text{Gr} F(t, \cdot) \) is sequentially closed in \( X_w \times X_w^* \);

(iii) for almost all \( t \in T \), all \( (x, x^*) \in \text{Gr} F(t, \cdot) \), we have

\[ \| x^* \|_* \leq a_3(t) + c_3 \| x \|^{p-1}, \]

with \( a_3 \in L^p(T), \; c_3 > 0 \);

(iv) for almost all \( t \in T \), all \( (x, x^*) \in \text{Gr} F(t, \cdot) \), we have

\[ \langle x^*, x \rangle \leq \beta(t) + \tilde{c} \| x \|^\tau, \]

where \( \tilde{c} > 0 \) and \( \beta(t) \in L^\tau(T) \).
with $\beta \in L^1(T)$, $\hat{c} > 0$ and $1 < r \leq 2$; if $\tau = 2$ we assume additionally that $\hat{c}b^{p+\frac{1}{p-2}} < c_2$.

(\textbf{v}) if $\{u_n\}_{n \geq 1} \subseteq W^{\alpha, p}_{p} (0, b)$ is a sequence such that $u_n(t) \rightharpoonup u(t)$ weakly in $X$ for almost all $t \in T$ and $f_n \in S^\beta_{F(i, u_n(\cdot))}$, then
\[
\limsup_{n \to +\infty} (f_n, u_n - u) \leq 0.
\]

Let $a: L^p(T; X) \to L^p(T; X^*)$ be the nonlinear map defined by
\[
a(u)(\cdot) = A(\cdot, u(\cdot)) \quad \forall u \in L^p(T; X).
\]

**3. Existence of periodic solutions.** First we consider the multivalued map $N: L^p(T; X) \to 2^{L^p(T; X^*)}$ defined by
\[
N(u) = \{ f \in L^p(T; X^*) : f(t) \in F(t, u(t)) \text{ for a.e. } t \in T \} \quad \forall u \in L^p(T; X).
\]

**Proposition 3.1.** If hypotheses $H(F)$ hold, then
\[
N(u) \in P_{wkc}(L^p(T; X^*)) \quad \forall u \in L^p(T; X).
\]

**Proof.** Evidently the values of $N$ are closed, convex and bounded subsets of $L^p(T; X^*)$, hence weakly compact and convex. So, the only thing that we need to show, is that $N$ has nonempty values. To this end, let $\{s_n\}_{n \geq 1} \subseteq L^p(T; X)$ be simple functions such that
\[
s_n(t) \to u(t) \quad \text{in } X, \text{ for a.a. } t \in T \quad (3.1)
\]
and
\[
\|s_n(t)\| \leq \|u(t)\| \quad \text{for a.a. } t \in T, \text{ all } n \geq 1. \quad (3.2)
\]

Hypothesis $H(f)(i)$ implies that for every $n \geq 1$, the multifunction $t \mapsto F(t, s_n(t))$ is graph measurable. So, we can use the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [19, pp. 158–159]) and find a measurable map $f_n: T \to X^*$ such that $f_n(t) \in F(t, s_n(t))$ for a.a. $t \in T$, all $n \geq 1$,

so the sequence $\{f_n\}_{n \geq 1} \subseteq L^p(T; X^*)$ is bounded (see hypothesis $H(F)(iii)$ and (3.1)-(3.2)). By passing to a suitable subsequence if necessary, we may assume that
\[
f_n \rightharpoonup f \quad \text{in } L^p(T; X^*).
\]

Proposition 6.6.33 of Papageorgiou-Kyritsi [27, p. 521] implies that
\[
f(t) \in \overline{\conv} (\limsup_{n \to +\infty} f_n(t)) \subseteq \overline{\conv} (\limsup_{n \to +\infty} F(t, s_n(t)) \subseteq \overline{\conv} F(t, u(t) = F(t, u(t))
\]
(see (3.1), (3.2) and hypothesis $H(F)(ii)$), so $f \in N(u)$ and thus
\[
N(u) \neq \emptyset \quad \forall u \in L^p(T; X).
\]

This proves that
\[
N(u) \in P_{wkc}(L^p(T; X^*)) \quad \forall u \in L^p(T; X).
\]

\[\square\]
Let \(((\cdot, \cdot))_0\) denote the duality brackets for the pair \((W^{\text{per}}_p(0,b)^*, W^{\text{per}}_p(0,b))\) and for \(\varepsilon > 0\) consider the map \(K_{\varepsilon}: W^{\text{per}}_p(0,b) \rightarrow W^{\text{per}}_p(0,b)^*\) defined by

\[
((K_{\varepsilon}(u), h))_0 = ((\varepsilon\tilde{F}^{-1}(u'), h')) + ((u', h)) + ((a(u), h)) \quad \forall h \in W^{\text{per}}_p(0,b).
\]

(3.3)

Recall that \(((\cdot, \cdot))\) denotes the duality brackets for \((L^p(T; X^*), L^p(T; X))\).

The next proposition establishes the main property of this map.

**Proposition 3.2.** If hypotheses \(H(A)\) hold, then for every \(\varepsilon > 0\), the operator \(K_{\varepsilon}: W^{\text{per}}_p(0,b) \rightarrow W^{\text{per}}_p(0,b)^*\) defined by (3.3) is pseudomonotone.

**Proof.** Evidently \(K_{\varepsilon}\) is bounded. So, according to Proposition 2.1, in order to prove the pseudomonotonicity of \(K_{\varepsilon}\), it suffices to show that it is generalized pseudomonotone. To this end, let \(\{u_n\}_{n \geq 1} \subseteq W^{\text{per}}_p(0,b)\) be a sequence such that

\[
\left\{
\begin{aligned}
u_n &\stackrel{w}{\rightarrow} u \quad \text{in} \quad W^{\text{per}}_p(0,b), \\
K_{\varepsilon}(u_n) &\stackrel{w}{\rightarrow} u^* \quad \text{in} \quad W^{\text{per}}_p(0,b)^*, \\
\limsup_{n \rightarrow +\infty}((K_{\varepsilon}(u_n), u_n - u))_0 &\leq 0.
\end{aligned}
\right.
\]

(3.4)

The monotonicity of \(\tilde{F}^{-1}\) implies that

\[
((\varepsilon\tilde{F}^{-1}(u'_n), u'_n - u')) \geq ((\varepsilon\tilde{F}^{-1}(u), u' - u')) \quad \forall n \geq 1.
\]

(3.5)

Then from (3.3) and (3.5), for all \(n \geq 1\) we have

\[
((K_{\varepsilon}(u_n), u_n - u))_0 \geq ((\varepsilon\tilde{F}^{-1}(u), u'_n - u')) + ((u'_n, u_n - u)) + ((a(u_n), u_n - u)).
\]

(3.6)

From (2.3) and the periodic boundary condition, we have

\[
((u'_n, u_n - u)) = ((u', u_n - u)) \longrightarrow 0
\]

(3.7)

(see (3.4)). Similarly, from (3.4) we infer that

\[
((\varepsilon\tilde{F}^{-1}(u), u'_n - u')) \longrightarrow 0.
\]

(3.8)

So, if in (3.6) we pass to the limit as \(n \rightarrow +\infty\) and use (3.4), (3.7) and (3.8), we obtain

\[
\limsup_{n \rightarrow +\infty}((a(u_n), u_n - u)) \leq 0,
\]

so

\[
a(u_n) \stackrel{w}{\rightarrow} a(u) \quad \text{in} \quad L^p(T; X^*) \quad \text{and} \quad ((a(u_n), u_n)) \longrightarrow ((a(u), u))
\]

(3.9)

(see Hu-Papageorgiou [20, Theorem 2.35, p.41]). Also, from (2.3), we have

\[
((u'_n, u_n)) = ((u', u)) = 0 \quad \forall n \geq 1.
\]

(3.10)

Finally note that (3.4), (3.7) and (3.9) imply that

\[
\limsup_{n \rightarrow +\infty}((\varepsilon\tilde{F}^{-1}(u'_n), u'_n - u')) \leq 0.
\]

(3.11)

But recall that \(\tilde{F}^{-1}\) is maximal monotone, hence generalized pseudomonotone (see Gasiński-Papageorgiou [13, Corollary 3.2.50, p. 334]). So, from (3.11) it follows that

\[
\left\{
\begin{aligned}
\tilde{F}^{-1}(u'_n) &\stackrel{w}{\rightarrow} \tilde{F}^{-1}(u') \quad \text{in} \quad L^p(T; X), \\
((\varepsilon\tilde{F}^{-1}(u'_n), u'_n)) &\longrightarrow ((\varepsilon\tilde{F}^{-1}(u'), u')).
\end{aligned}
\right.
\]

(3.12)

From (3.3), (3.4), (3.9) and (3.12) we conclude that

\[
K_{\varepsilon}(u_n) \stackrel{w}{\rightarrow} K_{\varepsilon}(u) \quad \text{in} \quad W^{\text{per}}_p(0,b)^* \quad \text{and} \quad ((K_{\varepsilon}(u_n), u_n))_0 \longrightarrow ((K_{\varepsilon}(u), u))_0.
\]
Proposition 3.3. If hypotheses $H(A)$ and $H(F)$ hold, then $Q_\varepsilon: W^p_0(0,b) \rightarrow 2^{W^p_0(0,b)^*}$ be defined by
\[ Q_\varepsilon(u) = K_\varepsilon(u) - N(u) \quad \forall u \in W^p_0(0,b). \] (3.13)

**Proposition 3.3.** If hypotheses $H(A)$ and $H(F)$ hold, then $Q_\varepsilon: W^p_0(0,b) \rightarrow 2^{W^p_0(0,b)^*}$ defined by (3.13) is pseudomonotone and coercive.

**Proof.** First we show the pseudomonotonicity of $Q_\varepsilon$.

Since $Q_\varepsilon$ is bounded, it suffices to show that $Q_\varepsilon$ is generalized pseudomonotone (see Proposition 2.1). To this end, let \( \{u_n\}_{n \geq 1} \subseteq W^p_0(0,b) \), \( \{u^*_n\}_{n \geq 1} \subseteq W^p_0(0,b)^* \) be such that
\[
\begin{align*}
    u_n &\rightarrow u \text{ in } W^p_0(0,b), \\
    u^*_n &\rightarrow u^*_0 \text{ in } W^p_0(0,b), \quad u^*_n \in Q_\varepsilon(u_n) \quad \forall n \geq 1, \\
    \limsup_{n \to +\infty} &((u^*_n, u_n - u))_0 \leq 0.
\end{align*}
\] (3.14)

We need to show that
\[ u^* \in Q_\varepsilon(u) \text{ and } ((u^*_0, u_n))_0 \rightarrow ((u^*, u))_0. \] (3.15)

From (3.13) and (3.14) we have
\[ u^*_n = K_\varepsilon(u_n) - f_n \text{ with } f_n \in N(u_n) \quad \forall n \geq 1. \] (3.16)

So, we have
\[
\begin{align*}
    \limsup_{n \to +\infty} &((K_\varepsilon(u_n), u_n - u))_0 - \limsup_{n \to +\infty}((f_n, u_n - u))_0 \\
    &\leq \limsup_{n \to +\infty}(((K_\varepsilon(u_n), u_n - u))_0 - ((f_n, u_n - u))_0) \\
    &= \limsup_{n \to +\infty}((u^*_n, u_n - u))_0 \leq 0
\end{align*}
\] (3.17)
(see (3.14) and (3.16)).

Let \( \eta_n(t) = (A(t, u_n(t)), u_n(t) - u(t)) \) for \( n \geq 1 \) and let \( E \subseteq T \) be the Lebesgue-null set outside of which hypotheses $H(A)(ii)$, $(iii)$ and $(iv)$ hold. Then for \( t \in T \setminus E \), we have
\[ \eta_n(t) \geq c_2\|u_n(t)\|^{p-1} - a_2(t) - (a_1(t) + c_1\|u_n(t)\|^{p-1})\|u(t)\| \] (3.18)
(see hypotheses $H(A)(iii)$ and $(iv)$).

Set
\[ D = \{ t \in T : \liminf_{n \to +\infty} \eta_n(t) < 0 \}. \]

Evidently $D \subseteq T$ is Lebesgue measurable. Denoting by $|\cdot|$ the Lebesgue measure on $\mathbb{R}$, suppose that $|D| > 0$. From (3.18) it is clear that the sequence \( \{u_n(t)\}_{n \geq 1} \subseteq X \) is bounded for all \( t \in D \cap (T \setminus E) \neq \emptyset \).

Then from (3.14) and the Urysohn criterion for the convergence of sequences (see Gasiński-Papageorgiou [14, p. 331]), we have
\[ u_n(t) \xrightarrow{w} u(t) \quad \text{in } X \quad \forall t \in D \cap (T \setminus E). \] (3.19)

Fix \( t \in D \cap (T \setminus E) \) and choose a subsequence \( \{n_k\} \) of \( \{n\} \) (the subsequence in general will depend on \( t \)) such that
\[ \lim_{k \to +\infty} \eta_{n_k}(t) = \liminf_{n \to +\infty} \eta_n(t). \]
Since \( t \in D \cap (T \setminus E) \), exploiting the pseudomonotonicity of \( A(t, \cdot) \) (see hypothesis \( H(A)(ii) \)) and (3.19), we have
\[
\lim_{k \to +\infty} \langle A(t, u_{n_k}(t)), u_{n_k}(t) - u(t) \rangle \geq 0,
\]
a contradiction. Therefore \( |D|_1 = 0 \) and so
\[
0 \leq \liminf_{n \to +\infty} \eta_n(t) \quad \text{for a.a. } t \in T. \tag{3.20}
\]
Using Fatou’s lemma (see (3.18)), we have
\[
0 \leq \int_0^b \liminf_{n \to +\infty} \eta_n(t) \, dt \leq \liminf_{n \to +\infty} \int_0^b \eta_n(t) \, dt \leq 0
\]
(see (3.17) and the proof of Proposition 3.2), so
\[
\int_0^b \eta_n(t) \, dt \to 0. \tag{3.21}
\]
We have
\[
|\eta_n(t)| = \eta_n^+(t) + \eta_n^-(t) = \eta_n(t) + 2\eta_n^-(t) \quad \forall n \geq 1. \tag{3.22}
\]
Note that
\[
\eta_n^-(t) \to 0 \quad \text{for a.a. } t \in T \tag{3.23}
\]
(see (3.20)). Moreover, from (3.18) we see that
\[
\eta_n(t) \geq \gamma_n(t) \quad \text{for a.a. } t \in T, \text{ all } n \geq 1,
\]
with \( \{\gamma_n\}_{n \geq 1} \subseteq L^1(T) \) uniformly integrable. Then
\[
\eta_n^-(t) \leq \gamma_n^-(t) \quad \text{for a.a. } t \in T, \text{ all } n \geq 1 \tag{3.24}
\]
and \( \{\gamma_n^-\}_{n \geq 1} \subseteq L^1(T) \) is uniformly integrable. From (3.23), (3.24) and Vitali’s theorem (see Gasiński-Papageorgiou [14, p. 443]), we have
\[
\int_0^b \eta_n^-(t) \, dt \to 0,
\]
so
\[
\int_0^b |\eta_n(t)| \, dt \to 0
\]
(see (3.21) and (3.22)), thus
\[
\eta_n \to 0 \quad \text{in } L^1(T).
\]
Hence by passing to a suitable subsequence if necessary, we may assume that
\[
\eta_n(t) \to 0 \quad \text{for a.a. } t \in T. \tag{3.25}
\]
Claim. For almost all \( t \in T \), we have
\[
\sup_{n \geq 1} \|u_n(t)\| < +\infty.
\]
Arguing by contradiction, suppose that the Claim is not true. Then we can find \( D^* \subseteq T \) measurable with \( |D^*|_1 > 0 \) such that
\[
\sup_{n \geq 1} \|u_n(t)\| = +\infty \quad \forall t \in D^*.
\]
Let $\varepsilon = \frac{1}{2}|D^*|_1 > 0$. Invoking the Egorov theorem, we can find $D_1^* \subseteq T$ measurable with $|D_1^*|_1 < \varepsilon$ such that

$$\eta_n(t) \longrightarrow 0 \text{ uniformly on } T \setminus D_1^*$$

(see (3.25)). Let $D_2^* = (T \setminus D_1^*) \cap D^*$. Evidently $D_2^* \subseteq T$ is measurable and we claim that $|D_2^*|_1 > 0$. Indeed, if $|D_2^*|_1 = 0$, then

$$b = |T|_1 \geq |(T \setminus D_1^*) \cup D^*|_1 = |(T \setminus D_1^*) \cup D^*|_1 + |D_2^*|_1$$

$$= |(T \setminus D_1^*) \cup D^*|_1 + |(T \setminus D_1^*) \cap D^*|_1$$

$$= |T \setminus D_1^*|_1 + |D^*|_1$$

$$\geq b - \frac{1}{2}|D^*|_1 + |D^*|_1 = b + \frac{1}{2}|D^*|_1 > b,$$

a contradiction. This proves that $|D_2^*|_1 > 0$.

Let $E_1 \subseteq T$ be the Lebesgue-null set outside of which (3.18), (3.25) and hypotheses $H(A)(ii)$, (iii) and (iv) hold. Then we have

$$\sup_{n \geq 1} |\eta_n(t)| \leq M_1 \quad \forall t \in (T \setminus E_1) \cap D_2^*, \quad (3.26)$$

for some $M_1 > 0$. As before, using hypotheses $H(A)(iii)$, (iv) and (3.26), we obtain

$$\sup_{n \geq 1} \|u_n(t)\| \leq M_2 \quad \forall t \in (T \setminus E_1) \cap D_2^* \subseteq D^*, \quad (3.27)$$

for some $M_2 > 0$, a contradiction to the definition of $D^*$. This proves the Claim.

Because of the Claim and by passing to a suitable subsequence if necessary (the subsequence in general depends on $t$), we can have

$$u_n(t) \overset{w}{\longrightarrow} \hat{u}(t) \quad \text{in } X, \text{ for a.a. } t \in T. \quad (3.28)$$

On the other hand from (3.14) and (2.1), we have

$$u_n \overset{w}{\longrightarrow} u \quad \text{in } C(T; H),$$

so

$$u_n(t) \overset{w}{\longrightarrow} u(t) \quad \text{in } H.$$ 

Then from (3.27) it follows that $\hat{u} = u$ and in fact by Urysohn’s criterion for the convergence of sequences, we conclude that for the initial sequence we have

$$u_n(t) \overset{w}{\longrightarrow} u(t) \quad \text{in } X, \text{ for a.a. } t \in T. \quad (3.29)$$

Recall that

$$f_n(t) \in F(t, u_n(t)) \quad \text{for a.a. } t \in T, \text{ all } n \geq 1$$

(see (3.16)), so the sequence $\{f_n\}_{n \geq 1} \subseteq L^p(T; X^*)$ is bounded (see hypothesis $H(F)(ii)$ and (3.14)). Passing to a subsequence if necessary, we may assume that

$$f_n \longrightarrow f \quad \text{in } L^p(T; X^*). \quad (3.29)$$

As before, using Proposition 6.6.33 of Papageorgiou-Kyritsi [27, p. 521], we have

$$f(t) \in \overset{\text{conv}}{\lim} \sup_{n \rightarrow +\infty} \{f_n(t)\}$$

$$\subseteq \overset{\text{conv}}{\lim} \sup_{n \rightarrow +\infty} F(t, u_n(t))$$

$$\subseteq F(t, u(t)) \quad \text{for a.a. } t \in T$$

(see (3.28) and hypothesis $H(F)(ii)$), so $f \in N(u)$.
Also, hypothesis $H(F)(v)$ implies that
\[
\limsup_{n \to +\infty}((f_n, u_n - u)_0 = \limsup_{n \to +\infty}((f_n, u_n - u)) \leq 0. \quad (3.30)
\]
Then from (3.17) and (3.27) it follows that
\[
\limsup_{n \to +\infty}((K_\varepsilon(u_n), u_n - u)_0 \leq 0. \quad (3.31)
\]
From (3.31) and Proposition 3.2 (see also Proposition 2.1), we infer that
\[
K_\varepsilon(u_n) \rightharpoonup K_\varepsilon(u) \ \text{in} \ W^p_{\text{per}}(0, b)^* \ \text{and} \ \((K_\varepsilon(u_n), u_n)_0 \to ((K_\varepsilon(u), u)). \quad (3.32)
\]
Recall that
\[
u_n^* = K_\varepsilon(u_n) - f_n \ \forall n \geq 1 \quad (3.33)
\]
(see (3.16)) and
\[
u_n^* \to u^* \ \text{in} \ W^p_{\text{per}}(0, b)^* \quad (3.34)
\]
(see (3.14)). From (3.32), (3.33) and (3.34), it follows that for the initial sequence $\{f_n\}_{n \geq 1} \subseteq L^p(T; X^*)$, we have
\[
f_n \rightharpoonup f \ \text{in} \ L^p(T; X^*) \ \text{and} \ f \in N(u) \quad (3.35)
\]
(see (3.29)). Moreover, we have
\[
((f_n, u_n)) = ((f_n, u_n)_0 \to ((f, u))_0 = ((f, u)) \quad (3.36)
\]
(see (3.17), (3.32) and hypothesis $H(f)(v)$). Therefore finally we can say that
\[
((u_n^*, u_n)_0 \to ((u^*, u)_0
\]
and since $f = N(u)$, we infer that
\[
u^* \in Q_\varepsilon(u).
\]
So, we have proved that (3.15) holds and this implies the pseudomonotonicity of $Q_\varepsilon$.

Next we show the coercivity of $Q_\varepsilon$. For every $u \in W^p_{\text{per}}(0, b)$ and every $f \in N(u)$, we have
\[
((K_\varepsilon(u) - f, u)_0 = ((K_\varepsilon(u), u)_0 - ((f, u))
\]
\[
= ((\varepsilon \tilde{F}^{-1}(u'), u')) + ((u', u)) + ((a(u), u)) - ((f, u)).
\]
Note that
\[
((\varepsilon \tilde{F}^{-1}(u'), u')) = \varepsilon \|u'\|_{L^{p'}(T; X^*)} \ \text{and} \ ((u', u)) = 0
\]
(see (2.3)). Then using also hypotheses $H(A)(iv)$ and $H(F)(iv)$, we have
\[
((K_\varepsilon(u) - f, u)_0 \geq \varepsilon \|u'\|_{L^{p'}(T; X^*)}^2 + c_2 \|u\|_{L^p(T; X)}^p - \varepsilon \|u\|_{L^p(T; X)} - c_4, \quad (3.35)
\]
for some $c_4 > 0$.

If $\tau \in (1, 2)$, then from (3.35), we have
\[
((K_\varepsilon(u) - f, u)_0 \geq c_5 \|u\|_{W^{p,\tau}}^2 - c_6 \|u\|_{W^{p,\tau}} - c_4
\]
for some $c_5, c_6 > 0$ and we conclude the coercivity of $Q_\varepsilon$.

If $\tau = 2$, then using Hölder inequality in (3.35), we have
\[
((K_\varepsilon(u) - f, u)_0 \geq \varepsilon \|u'\|_{L^{p'}(T; X^*)}^2 + c_2 \|u\|_{L^p(T; X)}^p - \varepsilon \|u\|_{L^p(T; X)}^2 - c_4. \quad (3.36)
\]
We may assume that $\|u\|_{L^p(T; X)} \geq 1$. Then from (3.36) and since $p \geq 2$, $\varepsilon \frac{\|u\|^2_{L^p(T; X)}}{\|u\|_{L^p(T; X)}} < c_2$, we conclude the coercivity of $Q_\varepsilon$. \hfill \square
Proposition 3.3 implies that \( Q_{\varepsilon} \) is surjective (see Proposition 2.2). So, for every \( \varepsilon > 0 \) we can find \( u_{\varepsilon} \in W^{per}_p(0,b) \) such that

\[ 0 \in Q_{\varepsilon}(u_{\varepsilon}), \]

thus

\[ K_{\varepsilon}(u_{\varepsilon}) = f_{\varepsilon}, \quad (3.37) \]

with \( f_{\varepsilon} \in N(u_{\varepsilon}) \). Letting \( \varepsilon \searrow 0 \), we will produce a solution for problem (1.1).

**Theorem 3.4.** If hypotheses \( H(A) \) and \( H(F) \) hold, then problem (1.1) admits a solution \( u \in W^{per}_p(0,b) \).

**Proof.** Let \( \varepsilon_n \searrow 0 \) and let \( u_n = u_{\varepsilon_n} \in W^{per}_p(0,b) \), \( f_n = f_{\varepsilon_n} \in N(u_n) \) be such that

\[ K_{\varepsilon_n}(u_n) = f_n \quad \forall n \geq 1 \]

(see (3.37)). We have

\[ 0 = ( (K_{\varepsilon_n}(u_n), u_n) )_0 - ( (f_n, u_n) )_0 \]
\[ = ( (K_{\varepsilon_n}(u_n), u_n) )_0 - ( (f_n, u_n) )_0 \]
\[ = ( (\varepsilon_n \hat{F}^{-1}(u'_n), h') ) + ( (u'_n, u_n) ) + ( (a(u_n) - f_n, u_n) ) \]
\[ \geq \varepsilon_n \| u'_n \|^2_{L^{p'}(T;X^*)} + c_2 \| u_n \|^2_{L^{p'}(T;X^*)} - c_7 \| u_n \|_{L^p(T;X)} - c_8 \]
\[ \geq \varepsilon_n \| u'_n \|^2_{L^{p'}(T;X^*)} + c_9 \| u_n \|_{L^p(T;X)} - c_{10} \]

(3.38)

for some \( c_7, c_8, c_9, c_{10} > 0 \) (recall that \( \tau < 2 \leq p \)). From (3.38) it follows that

\[ \{u_n\}_{n \geq 1} \subseteq \text{L}^p(T;X) \text{ is bounded}. \]

(3.39)

Recall that

\[ ( (\varepsilon_n \hat{F}^{-1}(u'_n), h') ) = ( (f_n - u'_n - a(u_n), h) ) \quad \forall h \in W^{per}_p(0,b), \ n \geq 1. \]

(3.40)

So, for every \( n \geq 1 \) the map

\[ h \mapsto ( (\varepsilon_n \hat{F}^{-1}(u'_n), h') ) \]

is continuous on \( W^{per}_p(0,b) \) furnished with the relative \( L^p(T;X) \)-topology. Therefore we infer that

\[ \hat{F}^{-1}(u'_n) \in D(L^*) = W^{per}_p(0,b) \quad \forall n \geq 1. \]

Then from (3.40) and the density of \( W^{per}_p(0,b) \) in \( \text{L}^p(T;X) \), we have

\[ ( (u'_n, h) ) = ( (f_n, h) ) - ( (a(u_n), h) ) - ( (\varepsilon_n L^* \hat{F}^{-1}(u'_n), h) ) \quad \forall h \in \text{L}^p(T;X), \ n \geq 1. \]

Let \( h = \hat{F}^{-1}(u'_n) \). Then

\[ \| u'_n \|^2_{L^{p'}(T;X^*)} = ( (f_n, \hat{F}^{-1}(u'_n)) ) - ( (a(u_n), \hat{F}^{-1}(u'_n)) ) \]
\[ - ( (\varepsilon_n L^* \hat{F}^{-1}(u'_n), \hat{F}^{-1}(u'_n)) ) \]

(3.41)

for all \( n \geq 1 \). From (3.39) and hypotheses \( H(F)(iii) \) and \( H(A)(iii) \) we have

\[ | ( (f_n, \hat{F}^{-1}(u'_n)) ) |, | ( (a(u_n), \hat{F}^{-1}(u'_n)) ) | \leq c_{11} \| u'_n \|_{L^{p'}(T;X^*)} \quad \forall n \geq 1, \]

(3.42)

for some \( c_{11} > 0 \). Also recall that \( L^* \) is linear monotone. So, we have

\[ ( (\varepsilon_n L^* \hat{F}^{-1}(u'_n), \hat{F}^{-1}(u'_n)) ) \geq 0 \quad \forall n \geq 1. \]

(3.43)

Returning to (3.41) and using (3.42) and (3.43), we refer that

\[ \| u'_n \|^2_{L^{p'}(T;X^*)} \leq c_{10} \| u'_n \|_{L^{p'}(T;X^*)} \]

so

\[ \| u'_n \|_{L^p(T;X)} \leq c_{11} \quad \forall n \geq 1. \]

(3.44)
From (3.39) and (3.44) it follows that
\[ \{u_n\}_{n \geq 1} \subseteq W_p^{\text{per}}(0, b) \text{ is bounded.} \] (3.45)

So, passing to a subsequence if necessary, we may assume that
\[ u_n \xrightarrow{w} u \quad \text{in } W_p^{\text{per}}(0, b). \] (3.46)

We have
\[ ((\varepsilon_n \hat{F}^{-1}(u_n'), u_n' - u')) + ((u_n', u_n - u)) + ((a(u_n), u_n - u)) = ((f_n, u_n - u)) \] (3.47)
for all \( n \geq 1 \). Note that from (3.45) we have that
\[ \{\hat{F}^{-1}(u_n')\}_{n \geq 1} \subseteq L^p(T, X) \text{ is bounded.} \]
so
\[ ((\varepsilon_n \hat{F}^{-1}(u_n'), u_n' - u')) \rightarrow 0. \] (3.48)

Also from (3.46), (2.3) and hypothesis \( H(F)(iii) \), we have
\[ ((u_n', u_n - u)) \rightarrow 0 \quad \text{and} \quad ((f_n, u_n - u)) \rightarrow 0. \] (3.49)

Therefore from (3.47) and using (3.48) and (3.49), we obtain
\[ \lim_{n \to +\infty} ((a(u_n), u_n - u)) = 0, \]
so
\[ a(u_n) \xrightarrow{w} a(u) \quad \text{in } L^{p'}(T; X^*) \] (3.50)
(see Hu-Papageorgiou [20, Theorem 2.35, p. 41]). Also, we may assume that
\[ f_n \xrightarrow{w} f \quad \text{in } L^{p'}(T; X^*) \] (3.51)
(see hypothesis \( H(F)(iii) \)). Moreover, reasoning as in the proof of Proposition 3.3, we can show that
\[ u_n(t) \xrightarrow{w} u(t) \quad \text{in } X \quad \text{for a.a. } t \in T. \] (3.52)

Then from (3.51), (3.52) and hypothesis \( H(F)(ii) \), as in the proof of Proposition 3.1, we show that
\[ f \in N(u). \] (3.53)

Note that
\[ ((\varepsilon_n \hat{F}^{-1}(u_n'), h')) + ((u_n', h)) + ((a(u_n), h)) = ((f_n, h)) \quad \forall h \in L^p(T, X), \quad n \geq 1, \]
so
\[ ((u', h)) + ((a(u), h)) = ((f, h)) \quad \forall h \in L^p(T, X) \]
(see (3.50), (3.51) and recall that \( \varepsilon_n \searrow 0 \), thus \( u \in W_p^{\text{per}}(0, b) \) is a solution of (1.1) (see (3.52)).
4. Applications. In this section, we illustrate the existence theorem by considering
parabolic inclusions.

Let \( T = [0, b] \) and \( \Omega \subseteq \mathbb{R}^N \) be a bounded domain with Lipschitz boundary
\( \partial \Omega \). We examine the following nonlinear parabolic boundary value problem with
multivalued terms and periodic conditions:

\[
\begin{cases}
\frac{\partial u}{\partial t} - \text{div} a(t, z, u, Du) + \bar{c} \sin u \sum_{k=1}^{N} D_k u \in G(t, z, u) & \text{in } T \times \Omega, \\
u|_{T \times \partial \Omega} = 0, \ u(0, z) = u(b, z) & \text{for a.a. } z \in \Omega.
\end{cases}
\]

(4.1)

The hypotheses on the data of problem (4.1) are the following:

\( H(a): a: T \times \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a function such that:

(i) for all \( (x, y) \in \mathbb{R} \times \mathbb{R}^N \), the map \((t, z) \mapsto a(t, z, x, y)\) is measurable;

(ii) for almost all \((t, z) \in T \times \Omega\), the map \((x, y) \mapsto a(t, z, x, y)\) is continuous and
for almost all \((t, z) \in T \times \Omega\) and all \( x \in \mathbb{R} \), the map \( y \mapsto a(t, z, x, y)\) is strictly
monotone;

(iii) for almost all \((t, z) \in T \times \Omega\) and all \((x, y) \in \mathbb{R} \times \mathbb{R}^N\), we have
\[ |a(t, z, x, y)| \leq \tilde{a}_1(t, z) + \tilde{c}_1(|x|^{p-1} + |y|^{p-1}) \]
with \( \tilde{a}_1 \in L^p(T \times \Omega), \; \tilde{c}_1 > 0, \; 2 \leq p < +\infty \);

(iv) for almost all \((t, z) \in T \times \Omega\) and all \((x, y) \in \mathbb{R} \times \mathbb{R}^N\), we have
\[ (a(t, z, x, y), y)_{\mathbb{R}^N} \geq \tilde{c}_2 |y|^p - \tilde{a}_2(t, z) \]
with \( \tilde{a}_2 \in L^1(T \times \Omega), \; \tilde{c}_2 > 0 \).

Remark 4.1. A particular case of these conditions is the map
\[ a(t, z, x, y) = \vartheta(t, z, x)|y|^{p-2}y, \]
with \( \vartheta(t, z, x) \) being a Carathéodory function (that is, for all \( x \in \mathbb{R}, \; (t, z) \mapsto \vartheta(t, z, x) \) is measurable and for almost all \((t, z) \in T \times \Omega, \; z \mapsto \vartheta(t, z, x) \) is continuous) which satisfies
\[ |\vartheta(t, z, x)| \leq M_3 \; \text{for a.a. } (t, z) \in T \times \Omega, \; \text{all } x \in \mathbb{R}, \]
for some \( M_3 > 0 \). Then the elliptic differential operator in (4.1) is a weighted
\( p \)-Laplacian with the weight depending on \( u \).

\( H(G): G: T \times \Omega \times \mathbb{R} \rightarrow P_{fc}(\mathbb{R}) \) is a multifunction such that:

(i) for all \( x \in \mathbb{R} \), the map \((t, z) \mapsto G(t, z, x)\) is measurable;

(ii) for almost all \((t, z) \in T \times \Omega\), the map \( x \mapsto G(t, z, x) \) has closed graph;

(iii) we have
\[ |G(t, z, x)| = \sup_{v \in G(t, z, x)} |v| \leq \tilde{a}_3(t, z) + \tilde{c}_3|x|^{p-1} \]
for almost all \((t, z) \in T \times \Omega, \; \text{all } x \in \mathbb{R}, \) with \( \tilde{a}_3 \in L^p(T \times \Omega), \; \tilde{c}_3 > 0 \);

(iv) for almost all \((t, z) \in T \times \Omega\) and all \((x, v) \in \text{Gr } G(t, z, \cdot)\), we have
\[ vx \leq 0 \]
(sign condition).

Remark 4.2. From Hu-Papageorgiou [19, Example 2.8, p. 3], we know that
\[ G(t, z, x) = [g_1(t, z, x), g_2(t, z, x)], \]
for all \( t \in [0, b], \; z \in \Omega, \; x \in \mathbb{R} \).
with \( g_1, g_2 : T \times \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) two functions such that

for all \( x \in \mathbb{R}, (t, z) \mapsto g_1(t, z, x), g_2(t, z, x) \) are measurable

for a.a. \( (t, z) \in T \times \Omega, x \mapsto g_1(t, z, v) \) is lower semicontinuous

for a.a. \( (t, z) \in T \times \Omega, x \mapsto g_2(t, z, v) \) is upper semicontinuous.

So, problem (4.1) incorporates equations with discontinuous reaction term. To deal with such problems, we pass to an inclusion by filling in the gaps at the discontinuity points (see Chang [6] and Hu-Papageorgiou [20]).

\[ H_0 : |\tilde{c}| b^{\frac{p-2}{p}} < \tilde{c}_2. \]

For problem (4.1) the evolution triple is:

\[ X = W_0^{1,p}(\Omega), \quad H = L^2(\Omega), \quad X^* = W^{-1,p'}(\Omega). \]

From the Sobolev embedding theorem (recall that \( p \geq 2 \)), we know that \( X \hookrightarrow H \) compactly. Hence, so does \( H^* = H \) into \( X^* \).

Let \( A : T \times X \rightarrow X^* \) be defined by

\[ \langle A(t, u), h \rangle = \int_{\Omega} (a(t, z, u, Du), Dh)_{\mathbb{R}^N} \, dz \quad \forall (t, u, h) \in T \times X \times X. \]

From the Pettis measurability theorem (see Gasiński-Papageorgiou [13, Theorem 2.1.3, p. 109]), we have that for all \( x \in X \), the map \( t \mapsto A(t, x) \) is measurable. Moreover, from Leray-Lions [23], we know that for almost all \( t \in T \), the map \( x \mapsto A(t, x) \) is pseudomonotone. Finally hypotheses \( H(a)(iii) \) and (iv) imply that hypotheses \( H(A)(iii) \) and (iv) hold.

Next consider the multifunction \( \tilde{G} : T \times X \rightarrow 2^{X^*} \) defined by

\[ \tilde{G}(t, u) = S_{\tilde{G}(t, u)}^p \quad \forall (t, u) \in T \times X. \]

Evidently \( \tilde{G} \) has nonempty, weakly compact, convex values in \( L^p(\Omega) \hookrightarrow X^* \). Moreover, hypotheses \( H(G)(i), (ii) \) and (iii) imply that

for all \( x \in X \), the map \( t \mapsto \tilde{G}(t, x) \) is graph measurable; \hspace{1cm} (4.2)

for a.a. \( t \in T \), the map \( x \mapsto \tilde{G}(t, x) \) has a closed graph in \( X \times L^p(\Omega)_w \). \hspace{1cm} (4.3)

Let \( V : X \rightarrow X^* \) be the nonlinear map defined by

\[ \langle V(u), h \rangle = \int_{\Omega} \bar{c} \sum_{k=1}^{N} (\sin u)(D_k u)h \, dz \quad \forall u, h \in X \]

and we set

\[ F(t, u) = \tilde{G}(t, u) - V(u) \quad \forall (t, u) \in T \times X. \]

Consider a sequence \( \{(u_n, f_n)\}_{n \geq 1} \subseteq X \times X^* \) such that

\[ f_n \in F(t, u_n) \quad \forall n \geq 1 \]

and

\[ u_n \overset{w}{\rightarrow} u \quad \text{in} \ X, \quad f_n \overset{w}{\rightarrow} f \quad \text{in} \ X^*. \]

From Zeidler [33, p. 593], we know that \( V \) is completely continuous, so

\[ V(u_n) \overset{}{\longrightarrow} V(u) \quad \text{in} \ W^{-1,p'}(\Omega) = X^*. \]

We have

\[ f_n = \tilde{g}_n + V(u_n) \quad \forall n \geq 1, \]
with \( \hat{g}_n \in \hat{G}(t, u_n) \). It is clear from hypothesis \( H(G)(iii) \) that the sequence \( \{\hat{g}_n\}_{n \geq 1} \subseteq L^{p'}(\Omega) \) is bounded. So, passing to a subsequence if necessary, we may assume that

\[
\hat{g}_n \xrightarrow{w} \hat{g} \quad \text{in } L^{p'}(\Omega),
\]

thus

\[ \hat{g} \in \hat{G}(t, u) \quad \text{(4.7)} \]

(see (4.3)). From (4.4), (4.5), (4.6) and (4.7) it follows that

\[ f = \hat{g} - V(u), \]

with \( \hat{g} \in \hat{G}(t, u) \).

Thus hypothesis \( H(F)(ii) \) is satisfied.

Also from (4.2) we see that the map \( t \mapsto F(t, u) \) is graph measurable (see hypothesis \( H(F)(i) \)).

Hypothesis \( H(F)(iii) \) is a consequence of hypotheses \( H(G)(iii) \) and the definition of \( V \).

Next, let \( f = \hat{g} - V(u) \) with \( \hat{g} \in \hat{G}(t, u) \). We have

\[ \langle f, u \rangle = \int_\Omega \hat{g} u \, dz - \langle V(u), u \rangle \leq \|\hat{g}\|_X^2. \]

(see hypotheses \( H(G)(iv) \)). Thus hypothesis \( H(F)(iv) \) is satisfied with \( \beta \equiv 0, \tau = 2 \) and \( \hat{c} = |\hat{c}| \) (see hypothesis \( H_0 \)).

Finally suppose \( \{u_n\}_{n \geq 1} \subseteq W^{1,p}_{\text{per}}(0, b) \) is a sequence such that

\[ u_n(t) \xrightarrow{w} u(t) \quad \text{in } X \quad \text{(4.8)} \]

and

\[ f_n \in L^{p'}(T; X^*), \quad f_n(t) \in F(t, u_n(t)) \quad \text{for a.a. } t \in T, \quad \forall n \geq 1. \quad \text{(4.9)} \]

We have

\[ f_n(t) = \hat{g}_n(t) + V(u_n) \quad \forall n \geq 1, \quad \text{(4.10)} \]

with \( \hat{g}_n \in L^{p'}(T; H) \hookrightarrow L^{p'}(T; X^*) \) such that

\[ \hat{g}_n(t) \in \hat{G}(t, u_n(t)) \quad \text{for a.a. } t \in T, \quad n \geq 1. \]

From (4.8)-(4.9) we have

\[ u_n \longrightarrow u \quad \text{in } L^p(T; L^2(\Omega)) \subseteq L^2(T \times \Omega) \quad \text{(4.11)} \]

(recall that \( 2 \leq p \)). Note that from hypothesis \( H(G)(iii) \) we have that the sequence \( \{\hat{g}_n\}_{n \geq 1} \subseteq L^{p'}(T \times \Omega) \) is bounded. So, passing to a subsequence if necessary, we may assume that

\[ \hat{g}_n \xrightarrow{w} \hat{g} \quad \text{in } L^{p'}(T \times \Omega), \quad \hat{g}(t) \in \hat{G}(t, u(t)) \quad \text{for a.a. } t \in T. \quad \text{(4.12)} \]

Then we have

\[ ((\hat{g}_n, u_n - u)) = \int_0^b \int_\Omega \hat{g}_n(t, z)(u_n - u)(t, z) \, dz \, dt \longrightarrow 0 \quad \text{(4.13)} \]

(see (4.11), (4.12)). Also, from Zeidler [33, p. 593], we know that \( V \) is completely continuous, thus

\[ V(u_n) \longrightarrow V(u) \quad \text{in } W^{-1, p'}(\Omega) \]

and so

\[ ((V(u_n), u_n - u)) = \int_0^b \langle V(u_n), u_n - u \rangle \, dt \longrightarrow 0. \quad \text{(4.14)} \]
From (4.13) and (4.14), we see that hypothesis \( H(F)(v) \) is satisfied.

We rewrite problem (4.1) as the following equivalent abstract evolution inclusion:
\[
\begin{cases}
u'(t) + A(t, u(t)) \in F(t, u(t)) & \text{for a.a. } t \in T = [0, b], \\
u(0) = u(b).
\end{cases}
\]

For this problem we can use Theorem 3.4 and state the following existence result for problem (4.1).

**Theorem 4.3.** If hypotheses \( H(a), H(G), H_0 \) hold, then problem (4.1) admits a solution \( u \in L^p(T; W_0^{1,p}(\Omega)) \cap C(T; L^2(\Omega)) \) such that
\[
\frac{\partial u}{\partial t} \in L^{p'}(T; W^{-1,p'}(\Omega)).
\]

We can also consider the following parabolic inclusion:
\[
\begin{cases}
\frac{\partial u}{\partial t} - \operatorname{div}(a(t, z, u, Du)) \in G(t, z, u) + \vartheta(z)Du & \text{in } T \times \Omega, \\
|\partial u|_{\Omega} = 0, u(0, z) = u(b, z) & \text{for a.a. } z \in \Omega.
\end{cases}
\]

We assume the following on the coefficient \( \vartheta(z) \):

\[
H(\vartheta): \vartheta \in C^1(\Omega), \vartheta(z) \geq c > 0 \text{ for all } z \in \overline{\Omega} \text{ and } \|\vartheta\|_{\infty} \leq \lambda^2 c.
\]

In this case \( V: X \rightarrow X^* \) is defined by
\[
\langle V(u), h \rangle = \langle \vartheta(z)Du, h \rangle \quad \forall(u, h) \in X \times X.
\]

We know that
\[
\operatorname{div}(\vartheta(z)Du) = \vartheta(z)Du + (D\vartheta, Du)_{\mathbb{R}N} \quad \forall u \in X,
\]
so for all \( u, h \in X = W_0^{1,p}(\Omega) \), we have
\[
\langle V(u), h \rangle = \langle \vartheta(z)Du, h \rangle = -\int_{\Omega} \vartheta(z)(Du, Dh)_{\mathbb{R}N} dz - \int_{\Omega} (D\vartheta, Du)_{\mathbb{R}N} h dz.
\]

Clearly \( \text{Gr} V \) is sequentially closed in \( X \times X^* \).

Also, if \( u_n \rightharpoonup u \) in \( X \), then
\[
\limsup_{n \to \infty} \langle V(u_n), u_n - u \rangle \leq 0.
\]

By Fatou’s lemma, hypothesis \( H(F)(v) \) is satisfied.

Also, using the variational characterization of \( \lambda_1 \) and the choice of \( c \) (see hypotheses \( H(\vartheta) \)), for \( u \in X \), we have
\[
\langle V(u), u \rangle = \langle \vartheta(z)Du, u \rangle \\
= -\int_{\Omega} \vartheta(z)|Du(z)|^2 dz - \int_{\Omega} (D\vartheta, Du)_{\mathbb{R}N} dz \\
\leq -c\|Du\|_2^2 + \|D\vartheta\|_{\infty}\|Du\|_2\|u\|_2 \\
\leq -c\|Du\|_2^2 + \|D\vartheta\|_{\infty} \frac{1}{\lambda_1^2} \|Du\|_2^2
\]
so hypothesis \( H(F)(iv) \) is satisfied. Thus, we can apply Theorem 3.4 and have the following result.

**Theorem 4.4.** If hypotheses \( H(a), H(G), H(\vartheta) \) hold, then problem (4.15) admits a solution \( u \in L^p(T; W^{1,p}_0(\Omega)) \cap C(T; L^2(\Omega)) \) such that

\[
\frac{\partial u}{\partial t} \in L^p\left(T; W^{-1,p'}(\Omega)\right).
\]

Finally, we may have an example, where the multivalued term \( G \) depends on the gradient \( Du \). So consider the following parabolic inclusion:

\[
\begin{cases}
\frac{\partial u}{\partial t} - \text{div} a(t, z, u, Du) \in G(t, z, u, Du) & \text{in } T \times \Omega, \\
u|_{T \times \partial \Omega} = 0, & u(0, z) = u(b, z) \text{ for a.a. } z \in \Omega.
\end{cases}
\]

(4.16)

We put the following assumptions on the multivalued part \( G \):

\( H(G) \): \( G: T \times \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow P_{fc}(\mathbb{R}) \) is a multifunction such that:

(i) for all \( x \in \mathbb{R} \), the map \( (t, y) \mapsto G(t, z, x, y) \) is measurable;

(ii) for almost all \( (t, z) \in T \times \Omega \), the map \( (x, y) \mapsto G(t, z, x, y) \) is upper semicontinuous;

(iii) we have

\[
|G(t, z, x, y)| = \sup_{v \in G(t, z, x, y)} |v| \leq \hat{\alpha}_4(t, z) + \tilde{\alpha}_4(|x| + |y|)
\]

for almost all \( (t, z) \in T \times \Omega \), all \( (x, y) \in \mathbb{R} \times \mathbb{R}^N \), with \( \hat{\alpha}_4 \in L^p(T \times \Omega) \), \( \tilde{\alpha}_4 > 0 \), \( \hat{\alpha}_4 b^{\frac{N+2}{p}} < c_2 \);

(iv) for almost all \( (t, z) \in T \times \Omega \), all \( x \in \mathbb{R} \), \( G(t, z, x, \cdot) \) is concave (that is \( \text{Gr} G(t, z, x, \cdot) \) is convex).

Consider the multifunction \( F: T \times X \rightarrow P_{wkc}(L^p(X)) \subseteq P_{wkc}(X^*) \) defined by

\[
F(t, u) = S^p_{G(t, z, x, y, Du)}
\]

and the support function

\[
\sigma_{F(t, u)}(h) = \int_{\Omega} \sigma_{G(t, z, x, y, Du)}(h(z)) \, dz \quad \forall \ t \in T, \ u \in X, \ h \in L^p(\Omega)
\]

(see Gasiński-Papageorgiou [13, p. 514] and Papageorgiou-Kyritsi [27, p. 492]). Then the concavity of \( G(t, z, x, \cdot) \) implies that \( u \mapsto \sigma_{F(t, u)} \), is upper semicontinuous from \( X_w \) into \( L^p(\Omega) \) (see Papageorgiou-Kyritsi [27, p. 70]). So, hypothesis \( H(F)(ii) \) is satisfied.

Also from the growth condition on \( G(t, z, x, \cdot) \) and the condition on \( c \) (see hypothesis \( H(G)(iii) \)), we see that hypothesis \( H(F)(iv) \) holds.

Finally, assuming that \( u_n \overset{w}{\rightarrow} u \) in \( X \), we have that \( u_n \rightarrow u \) in \( L^p(\Omega) \), so

\[
(f_n, u_n - u) = \int_{\Omega} f_n(u_n - u) \, dz \rightarrow 0,
\]

hence hypothesis \( H(F)(v) \) holds.

**Theorem 4.5.** If hypotheses \( H(a), H(G)' \) hold, then problem (4.16) admits a solution \( u \in L^p(T; W^{1,p}_0(\Omega)) \cap C(T; L^2(\Omega)) \) such that

\[
\frac{\partial u}{\partial t} \in L^p\left(T; W^{-1,p'}(\Omega)\right).
\]
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E-mail address: Leszek.Gasinski@ii.uj.edu.pl

E-mail address: npapg@math.ntua.gr