EXPLICIT FORMULAS FROM THE CONTINUOUS SPECTRUM

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The purpose of this note is to announce the results of our investigation into the role played by the continuous spectrum in the development of the Selberg trace formula vis-à-vis a pair \((G, \Gamma)\). For the sake of simplicity, we shall restrict ourselves to a “rank-2” situation, a case in point being when

\[
\begin{align*}
G &= \text{SL}(3, \mathbb{R}) \\
\Gamma &= \text{SL}(3, \mathbb{Z})
\end{align*}
\]

Full details (in all generality) will appear elsewhere.

Let \(G\) be a reductive Lie group, \(\Gamma\) a lattice in \(G\), both subject to the usual conditions (cf. [6, p. 62]). As is well-known, there is a decomposition of \(L^2(G/\Gamma)\) into the orthogonal direct sum of

\[L^2_{\text{dis}}(G/\Gamma) : \text{the discrete spectrum}\]

and

\[L^2_{\text{con}}(G/\Gamma) : \text{the continuous spectrum}\]

Consider the following statement:

**Main Conjecture (MC).** The operator \(L^\text{dis}_{G/\Gamma}(\alpha)\) is trace class for every \(K\)-finite \(\alpha\) in \(C^\infty_c(G)\).

This conjecture is a theorem when \(\text{rank}(\Gamma) = 0\) (cf. [6, p. 355]) or when \(\text{rank}(\Gamma) = 1\) (cf. Donnelly [3, p. 349]) and is actually true in general (cf. Müller [4, p. 473]).

Owing to the theory of the parametrix (cf. [6, p. 21]), it then automatically holds for all \(K\)-finite \(\alpha\) in \(C^1(G)\).
These points made, the fundamental problem of the theory is to compute

$$\text{tr} \left( \text{I}^\text{dis}_{G/\Gamma} (\alpha) \right)$$

in explicit terms. Thanks to the considerations to be found in [7-(c), §8], the problem can be divided into two parts:

1. Determine the contribution to the trace arising from the conjugacy classes.
2. Determine the contribution to the trace arising from the continuous spectrum.

[Note: Naturally, when rank (\(\Gamma\)) = 0, (2) is irrelevant, so only (1) is of interest, an elementary matter.]

Our approach dictates that the second issue be addressed first. The essence of the method of attack can be found already in [7-(a)], the key being the cancellation principle. There, of course, rank (\(\Gamma\)) = 1 and all the Arthur polynomials are linear, so everything, by comparison is fairly simple. The situation when rank (\(\Gamma\)) > 1 is far more complicated. Nevertheless, it is still possible to arrive at an explicit determination, the basis for the cancellation being a certain remarkable “addition” property enjoyed by the Arthur polynomials, combined with a multidimensional Dini calculus. The way it works is this. Each proper \(G\)-conjugacy class \(C\) of \(\Gamma\)-cuspidal split parabolic subgroups of \(G\) makes a contribution

$$\text{Con}(\alpha : \Gamma : C)$$

to the trace, the total contribution to the trace furnished by the continuous spectrum being the sum

$$\text{Con} - \text{Sp}(\alpha : \Gamma) = \sum_C \text{Con}(\alpha : \Gamma : C).$$

Accordingly, fix a \(C\) containing \(P = M \cdot A \cdot N\), say — then

$$\text{Con}(\alpha : \Gamma : C) = \sum_{\mathcal{O}} \sum_{w \in W(A)} \text{Con}(\alpha : \Gamma : \mathcal{C} : \mathcal{O} : w),$$

the actual form of the contribution

$$\text{Con}(\alpha : \Gamma : C : \mathcal{O} : w)$$
depending on $w$ through
\[ \text{rank}(1 - w), \]
the orbit type $\mathcal{O}$ having a passive part in the overall procedure.

To provide some motivation for [7-(d)], we shall explicate here the position when \( \text{rank}(\Gamma) = 2 \). Before doing this, though, it will be a good idea to recall how things go when \( \text{rank}(\Gamma) = 1 \). For use below, denote by \(*((\mathcal{C}))\) the number of chambers in $A$ (cf. [6, p. 104]).

Fixing $\mathcal{O}$, let us now suppose that \( \text{rank}(\Gamma) = 1 \). Then \( \#(W(A)) = 2 \). Thus, there are two terms appearing in the contribution from the continuous spectrum.

**[w = 1] In this case,**

\[
\text{Con}(\alpha : \Gamma : \mathcal{C} : \mathcal{O} : 1)
\]

is equal to
\[
-\frac{1}{2\pi} \cdot \frac{1}{*((\mathcal{C}))} \cdot \sum_{w \in W(A)} \times \int_{\mathcal{M}(\Lambda) = 0} \text{tr} \left( \text{Ind}_{P}^{G}((\mathcal{O}, \Lambda))(\alpha) \right.
\]
\[
\left. \bullet c(P|A : P|A : w : \Lambda) \cdot \frac{d}{d\Lambda} c(P|A : P|A : w : \Lambda) \right) |d\Lambda|. \]

[Note: Since the $c$-function attached to the trivial element of $W(A)$ is a constant, the contribution is concentrated entirely in the $c$-function of the nontrivial element of $W(A)$. Still, this mode of expression possesses an inherent symmetry that can be generalized.]

**[w \neq 1] In this case,**

\[
\text{Con}(\alpha : \Gamma : \mathcal{C} : \mathcal{O} : w)
\]

is equal to
\[
-\frac{1}{2\pi} \cdot 2\pi \cdot \frac{1}{*((\mathcal{C}))} \cdot \frac{1}{|\det(1 - w)|}
\]
\[
\times \text{tr} \left( \text{Ind}_{P}^{G}((\mathcal{O}, 0))(\alpha) \bullet c(P|A : P|A : w : 0) \right). \]

[Note: Since
\[ w \neq 1 \implies |\det(1 - w)| = 2, \]
the prefacing constant is 1/4. The “1/2π” is inherent in the Fourier inversion formula; the “2π is inherent in the Dini calculus. Because 1 – w is nonsingular, they cancel.

Keeping the orbit type fixed, assume now that rank (Γ) = 2. There are then two G-conjugacy classes C' and C'' of maximal Γ-cuspidal split parabolic subgroups of G and one G-conjugacy class C of minimal Γ-cuspidal split parabolic subgroups of G. It will be best to discuss each level separately.

\[ C', C'' \] Two cases can occur.

(I) Suppose that \( P' \in C' \), \( P'' \in C'' \) are associate (e.g. \( A_2 \)) then

\[
\begin{align*}
W(A'') &= \{ w' \} \\
W(A'', A') &= \{ w'' \}
\end{align*}
\]

In this case,

\[
\text{Con}(\alpha : \Gamma : C' : O' : 1)
\]

is equal to

\[
-\frac{1}{2\pi} \bullet \frac{1}{*(C')}
\]

\[
\times \int_{\Re(\Lambda)=0} \text{tr} \left( \text{Ind}_{GP'}^{G}((O', \Lambda'))(\alpha) \right) \\
\bullet c(P''|A'' : P'|A' : w' : \Lambda') * \frac{d}{d\Lambda} c(P''|A'' : P'|A' : w' : \Lambda') \right) |d\Lambda'|
\]

and

\[
\text{Con}(\alpha : \Gamma : C'' : O'' : 1)
\]

is equal to

\[
-\frac{1}{2\pi} \bullet \frac{1}{*(C'')}
\]

\[
\times \int_{\Re(\Lambda'')=0} \text{tr} \left( \text{Ind}_{GP''}^{G}((O'', \Lambda''))(\alpha) \right) \\
\bullet c(P'|A' : P''|A'' : w'' : \Lambda'') * \frac{d}{d\Lambda''} c(P'|A' : P''|A'' : w'' : \Lambda'') \right) |d\Lambda''|.
\]
(II) Suppose that $P' \in C', P'' \in C''$ are not associate (e.g. $A_1 \times A_1, B_2, G_2$) — then

\[
\begin{align*}
\begin{cases}
W(A') &= \{1, w'\} \\
W(A'') &= \{1, w''\}
\end{cases}
\end{align*}
\]

In this case,

\[\text{Con}(\alpha : \Gamma : C' : O' : 1)\]

is equal to

\[
-\frac{1}{2\pi} \cdot \frac{1}{\ast(C')} \cdot \sum_{w' \in W(A')} \\
\times \int_{\mathbb{R}(\Lambda') = 0} \text{tr} \left( \text{Ind}^G_{P'}((O', \Lambda'))(\alpha) \right) \cdot c(P'|A' : P'|A' : w' : \Lambda') \cdot \left| \frac{d}{d\Lambda'} \right| \right| d\Lambda' \\
\]

and

\[\text{Con}(\alpha : \Gamma : C' : O' : w')\]

is equal to

\[
-\frac{1}{2\pi} \cdot \frac{1}{\ast(C')} \cdot \frac{1}{|\text{det}(1 - w')|} \\
\times \text{tr} \left( \text{Ind}^G_{P'}((O', 0))(\alpha) \cdot c(P'|A' : P'|A' : w' : 0) \right),
\]

while

\[\text{Con}(\alpha : \Gamma : C'' : O'' : 1)\]

is equal to

\[
-\frac{1}{2\pi} \cdot \frac{1}{\ast(C'')} \cdot \sum_{w'' \in W(A'')} \\
\times \int_{\mathbb{R}(\Lambda'') = 0} \text{tr} \left( \text{Ind}^G_{P''}((O'', \Lambda''))(\alpha) \right)
\]
\[ \bullet c(P''|A'' : P''|A'' : w'' : \Lambda'') \cdot \frac{d}{d\Lambda''} c(P''|A'' : P''|A'' : w'' : \Lambda'') \bigg| d\Lambda'' \]

and

\[ \text{Con}(\alpha : \Gamma : C'' : \mathcal{O}'') \]

is equal to

\[ -\frac{1}{2\pi} \cdot 2\pi \cdot \frac{1}{\chi(C'') \cdot \det(1 - w'')} \]

\[ \times \text{tr} \left( \text{Ind}_{P''}^G((\mathcal{O}'', 0))(\alpha) \cdot c(P''|A'' : P''|A'' : w'' : 0) \right). \]

[Note: At the maximal level, therefore, the contribution to the trace is entirely analogous to what obtains when rank(\Gamma) = 1, including the interpretation of the constants.]

\[ \text{C} \]

Given \( w \in W(A) \), there are three possibilities:

\[ \begin{cases} \text{rank}(1 - w) = 0 \\
\text{rank}(1 - w) = 1 \\
\text{rank}(1 - w) = 2 \end{cases} \]

The two extreme cases are the easiest to treat and will be dealt with first.

Let \( \lambda_1 \) and \( \lambda_2 \) be the simple roots: let \( \lambda^1 \) and \( \lambda^2 \) be their duals. Generically, write

\[ \hat{\lambda} = \frac{\lambda}{\|\lambda\|}. \]

\[ \text{rank}(1 - w) = 0 \] This requirement implies that \( w = 1 \). Introduce

\[ P^G_P(H) = \frac{1}{2} \{(H, \hat{\lambda}_1)(H, \hat{\lambda}_2) + (H, \hat{\lambda}_1)(H, \hat{\lambda}_2)\}. \]

Then \( P^G_P \) is an Arthur polynomial. As such, it is homogeneous of degree 2. Denote by \( D^G_P \)
the associated differential operator. In this case,

$$\text{Con}(\alpha : \Gamma : C : O : 1)$$

is equal to

$$- \frac{1}{(2\pi)^2} \cdot \frac{1}{*C} \cdot \sum_{w \in W(A)}$$

$$\times \int_{\Re(\Lambda) = 0} \text{tr} \left( \text{Ind}^G_P((O, \Lambda))(\alpha) \right)$$

$$\bullet c(P|A : P|A : w : \Lambda) \ast D^G_P c(P|A : P|A : w : \Lambda) |d\Lambda|.$$ 

[Note: The similarity with the “$w = 1$” contribution when $\text{rank}(\Gamma) = 1$ is quite striking. In particular, the constants have the “right” interpretation and the derivative is “logarithmic” in character. Needless to say, in the sum over $w \in W(A)$, the term corresponding to $w = 1$ is a priori, zero.]

$$\text{rank}(1 - w) = 2$$ The requirement implies that $1 - w$ is nonsingular. In this case,

$$\text{Con}(\alpha : \Gamma : C : O : w)$$

is equal to

$$- \frac{1}{(2\pi)^2} \cdot (2\pi)^2 \cdot \frac{1}{*C} \cdot \frac{1}{|\det(1 - w)|}$$

$$\times \text{tr} \left( \text{Ind}^G_P((O, 0))(\alpha) \bullet c(P|A : P|A : w : 0) \right).$$

[Note: Again, the resemblance to the “$w \neq 1$” contribution when $\text{rank}(\Gamma) = 1$ is immediately apparent. Once more, the “$1/(2\pi)^2$” is inherent in the Fourier inversion formula; the $2(2\pi)^2$ in the Dini calculus. Because $1 - w$ is nonsingular, they cancel.]

$$\text{rank}(1 - w) = 1$$ This requirement implies that $w$ is a reflection, say $w = w_\lambda$, where, without loss of generality, $\lambda$ is a positive short root. The extra “$1/2$” that arises in what follows has its origins in a change of variables, which can be traced back to the fact
that

$$(1 - w_\lambda)(\lambda) = 2\lambda.$$  

Put

$$*(\mathcal{C}(\lambda)) = 2$$

and let $D_\lambda$ be the differential operator corresponding to $-\tilde{\lambda}$. We distinguish two cases.

(I) $\lambda \not\perp \lambda_1$ and $\lambda \not\perp \lambda_2$. Write

$$\begin{cases}
\theta_1 & \text{for the angle between } \lambda \text{ and } \lambda_1 \\
\theta_2 & \text{for the angle between } \lambda \text{ and } \lambda_2
\end{cases}.$$  

In this case,

$$\text{Con}(\alpha : \Gamma : \mathcal{C} : \mathcal{O} : w_\lambda)$$

is equal to

$$-\frac{1}{(2\pi)^2} \cdot 2\pi \cdot \frac{1}{*(\mathcal{C}(\lambda))} \cdot \frac{1}{\det((1 - w_\lambda)[\text{Ker}(1 - w_\lambda)]^\perp)} \cdot \frac{1}{2} \cdot \frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1)\cos(\theta_2)} \times D_\lambda|_{\Lambda' = 0} \int_{\text{Ker}(1 - w_\lambda)} \text{tr} \left( \text{Ind}^G_P((\mathcal{O}, \Lambda + \Lambda'))(\alpha) \right) \cdot c(P|A : P|A : w_\lambda : \Lambda + \Lambda') |d\Lambda|.$$  

[Note: The constant

$$\frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1)\cos(\theta_2)}$$

is strictly positive or strictly negative.]

(II) $\lambda \perp \lambda_1$ or $\lambda \perp \lambda_2$. Let $i = 1$ or 2 and suppose that $\lambda \perp \lambda_i$. In this case

$$\text{Con}(\alpha : \Gamma : \mathcal{C} : \mathcal{O} : w_\lambda)$$

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is equal to the sum of a pair of terms, namely:

\[(\text{II}_1)\] Call \(w_i\) the simple reflection in \(\lambda_i\) — then the first term is

\[-\frac{1}{(2\pi)^2} \cdot 2\pi \cdot \frac{1}{*(C(\lambda))} \cdot \frac{1}{|\text{det}((1 - w_\lambda)|\text{Ker}(1 - w_\lambda)^\perp)|} \cdot \frac{1}{2} \]

\[\times \int_{\text{Ker}(1 - w_\lambda)} \text{tr} \left( \text{Ind}_G^G((\mathcal{O}, \Lambda))(\alpha) \right) \]

\[\cdot c(P|A : P|A : w_i w_\lambda : \Lambda)^* \frac{d}{d\Lambda} c(P|A : P|A : w_i : \Lambda) \right\} |d\Lambda|.

[Note: Here, the \(c\)-function enters as a “hybrid” logarithmic derivative.]

\[(\text{II}_2)\] Call \(\theta_{12}\) the angle between \(\lambda_1\) and \(\lambda_2\) — then the second term is

\[-\frac{1}{(2\pi)^2} \cdot 2\pi \cdot \frac{1}{*(C(\lambda))} \cdot \frac{1}{|\text{det}((1 - w_\lambda)|\text{Ker}(1 - w_\lambda)^\perp)|} \cdot \frac{1}{2} \cdot \cot(\pi - \theta_{12}) \]

\[\times D_{\lambda|\Lambda'=0} \int_{\text{Ker}(1 - w_\lambda)} \text{tr} \left( \text{Ind}_G^G((\mathcal{O}, \Lambda + \Lambda'))(\alpha) \right) \]

\[\cdot c(P|A : P|A : w_\lambda : \Lambda + \Lambda') \right\} |d\Lambda|.

[Note: Since \((\lambda_1, \lambda_2)\) is \(\leq 0\), the cotangent of \(\pi - \theta_{12}\) is \(\geq 0\) and can = 0 (e.g. in \(\mathbf{A}_1 \times \mathbf{A}_2\)).]

We remark that the “2\(\pi\)” supra is the Dirac constant, hence does not cancel the “1/(2\(\pi\))^2”, the Fourier constant. Also, \(\forall \lambda\),

\[\left|\text{det}((1 - w_\lambda)|\text{Ker}(1 - w_\lambda)^\perp)\right| = 2.

To have a specific illustration of all this, take

\[
\begin{cases}
G = \text{SL}(3, \mathbb{R}) \\
\Gamma = \text{SL}(3, \mathbb{Z})
\end{cases}
\]
Then \( \#(W(A)) = 6 \). Apart from \( w = 1 \), there are two rotations, \( w' \) and \( w'' \), and three reflections, \( w_1, w_2, \) and \( w_3 \). Regarding the latter, only case I applies and we accordingly pick up a sum

\[
\sum_{i=1}^{3} \text{Con}(\alpha : \Gamma : \mathcal{C} : \mathcal{O} : w_i)
\]

of three “orthogonal derivatives”.

The appearance of

\[
D_\lambda|_{\Lambda = 0} \int_{\text{Ker}(1-w_\lambda)} \text{tr} \left( \text{Ind}^G_P((\mathcal{O}, \Lambda + \Lambda'))(\alpha) \cdot c(P|A : P|A : w_\lambda : \Lambda + \Lambda') \right) |d\Lambda|
\]

is not a total surprise, if only because in higher rank derivatives of Dirac distributions are produced by the Dini calculus in the presence of quadratic denominators (via the two roots). Indeed, if

\[
\delta(\text{Ker}(1-w_\lambda))
\]

is the Dirac distribution concentrated on \( \text{Ker}(1-w_\lambda) \), then our “orthogonal derivative” is, up to a constant, the result of applying

\[
\delta'(\text{Ker}(1-w_\lambda))
\]

to

\[
\text{tr} \left( \text{Ind}^G_P((\mathcal{O}, ?))(\alpha) \cdot c(P|A : P|A : w_\lambda : ?) \right).
\]
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