INTERIOR ESTIMATES IN CAMPAANATO SPACES RELATED TO QUADRATIC FUNCTIONALS

Maria Alessandra Ragusa & Atsushi Tachikawa

Abstract
In this paper we obtain interior estimates in Campanato spaces for the derivatives of the minimizers of quadratic functionals.

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1 Introduction and Preliminary Tools

The papers concerned with the regularity problem almost always have as a common starting point the Euler’s equation related to a generic functional $I$. In the paper by Giaquinta and Giusti [11], the authors investigate the hölder continuity of the minima working directly with the functional $I$ instead of Euler’s equation.

In the present paper, following the method in [11], we have studied regularity properties of the minima of variational integrals of the type:

$$A(u, \Omega) = \int \Omega f(x, u, Du)dx,$$

$\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded opens set $u : \Omega \to \mathbb{R}^N$, $u(x) = (u^1(x), \ldots, u^N(x))$, $Du = D_\alpha u^i$, $D_\alpha = \frac{\partial}{\partial x^\alpha}$, $i = 1, \ldots, N$, $\alpha = 1, \ldots, n$,

$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$
defined by
\[ f(x, u, Du) = A_{ij}^{\alpha \beta}(x, u)D_\alpha u^i D_\beta u^j + g(x, u, Du). \]

Let us now give the following definitions useful in the sequel.

**Definition 1.1.** (see [16], [20]). Let \( 1 \leq p < \infty, 0 \leq \lambda < n. \)

By \( L^{p,\lambda}(\Omega) \) we denote the linear space of functions \( f \in L^p(\Omega) \) such that
\[
\|f\|_{L^{p,\lambda}(\Omega)} = \left\{ \sup_{0 < \rho < \text{diam } \Omega} \rho^{-\lambda} \int_{\Omega \cap B(x, \rho)} |f(y)|^p dy \right\} < +\infty
\]
where \( B(x, \rho) \) ranges in the class of the balls of \( \mathbb{R}^n \) of radius \( \rho \) around \( x. \)

We have that \( \|f\|_{L^{p,\lambda}(\Omega)} \) is a norm respect to which \( L^{p,\lambda}(\Omega) \) is a Banach space, and also that
\[
\|f\|_{L^p(\Omega)} \leq (\text{diam } \Omega)^\frac{\lambda}{p} \|f\|_{L^{p,\lambda}(\Omega)}.
\]

Before the definition of the Campanato spaces let us define \( f_{B(x, \rho)} \) as the integral average
\[
f_{B(x, \rho)} = \frac{1}{|B(x, \rho)|} \int_{B(x, \rho)} f(y) dy
\]
of the function \( f(x) \) over the balls \( B(x, \rho) \) of \( \mathbb{R}^n. \) When no confusion may arise, we will write \( f_{B, \rho} \) or \( f_{\rho} \) instead of \( f_{B(x, \rho)} \).

**Definition 1.2.** (see e.g. [3], [9]). Let \( 1 \leq p < \infty \) and \( \lambda \geq 0. \)

By Campanato spaces \( \mathcal{L}^{p,\lambda}(\Omega) \) we denote the linear space of functions \( u \in L^p(\Omega) \) such that
\[
[f]_{p,\lambda} = \left\{ \sup_{x \in \Omega, 0 < \rho < \text{diam } \Omega} \rho^{-\lambda} \int_{\Omega \cap B(x, \rho)} |f(y) - f_{\rho}|^p dy \right\}^{\frac{1}{p}} < +\infty.
\]

\( \mathcal{L}^{p,\lambda}(\Omega) \) are Banach spaces with the following norm
\[
\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{p,\lambda}
\]
which simply demonstrate that \( u \in \mathcal{L}^{p,\lambda}(\Omega) \) if and only if
\[
\sup_{x \in \Omega, 0 < \rho < \text{diam} \Omega} \rho^{-\lambda} \inf_{c \in \mathbb{R}} \int_{\Omega \cap B(x, \rho)} |f - c|^p dy < \infty.
\]

Using Hölder inequality we have that
\[
\mathcal{L}^{p_1,\lambda_1}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega)
\]
where
\[
p \leq p_1, \quad \frac{n - \lambda}{p} \geq \frac{n - \lambda_1}{p_1}.
\]

Let us observe that
\[
\int_{\Omega \cap B(x, \rho)} |f - f_\rho|^p dy \leq C \int_{\Omega \cap B(x, \rho)} \left( |f|^p + |\Omega \cap B(x, \rho)| \cdot |f_\rho|^p \right) dy
\]
and also
\[
|f_\rho|^p \leq \frac{1}{|\Omega \cap B(x, \rho)|} \cdot \int_\Omega |f|^p dy.
\]

Below, we consider \( 0 \leq \lambda < n \).

We use
\[
[f]_{p,\lambda} \leq C \|f\|_{\mathcal{L}^{p,\lambda}(\Omega)}.
\]
to obtain the following relation between Morrey and Campanato spaces
\[
L^{p,\lambda}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega).
\]

Let us now recall the definitions of the BMO and VMO classes.

**Definition 1.3.** (see [13]). We say that a function \( f \) belongs to the John-Nirenberg space BMO, or that \( f \) has "bounded mean oscillation", if
\[
\|f\|_* \equiv \sup_{B \subseteq \mathbb{R}^n} \frac{1}{|B|} \int_{B} |f(y) - f_\rho| dy < \infty
\]
where \( f_\rho \) is the integral average of the function \( f \) over the balls \( B_\rho \).

Let us define, for a function \( f \in \text{BMO} \),
\[
\eta(r) = \sup_{x \in \mathbb{R}^n, \rho \leq r} \frac{1}{|B_\rho|} \int_{B_\rho} |f(x) - f_\rho| dx.
\]
Definition 1.4. (see [22]). A function \( f \in BMO \) belongs to the class \( VMO \), or \( f \) has "vanishing mean oscillation" if

\[
\lim_{r \to 0^+} \eta(r) = 0.
\]

We are now ready to formulate the hypothesis on the terms \( A^\alpha_{ij}(x, u) \) and \( g(\cdot, u, Du) \).

We suppose that \( A^\alpha_{ij}(x, u) \) are bounded functions in \( \Omega \times \mathbb{R}^N \), such that:

(A1) \( A^\alpha_{ij} = A^\beta_{ji} \).

(A2) For every \( u \in \mathbb{R}^N \), \( A^\alpha_{ij}(\cdot, u) \in VMO(\Omega) \).

(A3) For every \( x \in \Omega \) and \( u, v \in \mathbb{R}^N \),

\[
|A_{ij}^\alpha(x, u) - A_{ij}^\alpha(x, v)| \leq \omega(|u - v|^2)
\]

for some monotone increasing concave function \( \omega \) with \( \omega(0) = 0 \).

(A4) There exists a positive constant \( \nu \) such that

\[
\nu|\xi|^2 \leq A_{ij}^\alpha(x, u)\xi_i^\alpha \xi_j^\beta
\]

for a.e. \( x \in \Omega \), all \( u \in \mathbb{R}^N \) and \( \xi \in \mathbb{R}^nN \).

We suppose that the function \( g \) is a Charathéodory function, that is:

(g1) \( g(\cdot, u, Du) \) is measurable in \( x \) \( \forall u \in \mathbb{R}^N, \forall z \in \mathbb{R}^nN \);

(g2) \( g(x, \cdot, \cdot) \) is continuous in \( (u, z) \) a. e. \( x \in \Omega \);

moreover we consider \( g \) satisfying the condition:

(g3)

\[
|g(x, u, z)| \leq g_1(x) + H|z|^{\gamma},
\]

\( g_1 \geq 0 \), a. e. in \( \Omega \), \( g_1 \in L^p(\Omega), \ 2 < p \leq \infty, \ H \geq 0, \ 0 \leq \gamma < 2 \).

We point out that since \( C^0 \) is a proper subset of \( VMO \), the continuity of \( A^\alpha_{ij}(x, u) \) with respect to \( x \) is not assumed.

Let us make some remarks on \( VMO \) class. It was at first defined by Sarason in 1975 and later it was considered by many others. For instance we recall the papers by Chiarenza, Frasca and Longo [4] where the authors
answer a question raised thirty years before by C. Miranda in [18]. In his paper he considers a linear elliptic equation where the coefficients $a_{ij}$ of the higher order derivatives are in the class $W^{1,n}(\Omega)$ and asks whether the gradient of the solution is bounded, if $p > n$. Chiarenza, Frasca and Longo suppose $a_{ij} \in VMO$ and prove that $Du$ is Hölder continuous for all $p \in ]1, +\infty[$. We point out that $W^{1,n} \subset VMO$ because, using Poincare’s inequality

$$\int_B |f(x) - f_B| \leq c(n) \left( \int_B |\nabla u| dx \right)^{\frac{1}{n}}$$

and the term on the right-hand side tending to zero as $|B| \to 0$. Later the interior estimates obtained by Chiarenza, Frasca and Longo were extended to boundary estimates in [5]. From these papers on, many authors have used this space VMO to obtain regularity results for partial differential equations and systems with discontinuous coefficients. We recall for example Bramanti and Cerutti [2] for parabolic equations and many others.

With this useful assumption we investigate the regularity of the minimizers for the quadratic functional. Its existence is guaranteed, being the functional $A$ sequentially lower semicontinuous with respect to the $H^{1,2}$-weak topology (see [10]).

2 Main Results

**Theorem 2.1.** Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a minimum of the functional $A(u, \Omega)$ defined above. Suppose that assumptions (A-1), (A-2), (A-3), (A-4), $1 < q \leq 2$ and (g-1), (g-2) and (g-3) are satisfied. Then for $\lambda = n(1 - \frac{2}{p})$ we have

$$Du \in \mathcal{L}^{q,\lambda}_{\text{loc}}(\Omega_0, \mathbb{R}^n)$$

where

$$\Omega_0 = \{ x \in \Omega : \liminf_{R \to 0} \frac{1}{R^{n-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0 \}.$$ 

The set $\Omega_0$ is obligatory, in fact when we pass from the regularity theory for scalar minimizers of solutions of elliptic equations to the regularity theory for vector-valued minimizers of solutions of elliptic systems, the situation changes completely: regularity is an exceptional occurrence everywhere, excluding the two dimensional case. In 1968 De Giorgi in [7] showed that his
regularity result for solutions of second order elliptic equations with measurable bounded coefficients cannot be extended to solutions of elliptic systems. He presented the quadratic functional

\[ S = \int_{\Omega} A_{ij}^{\alpha \beta}(x) D_\alpha u^i D_\beta u^j dx \]

with \( A_{ij}^{\alpha \beta} \in L^\infty(\Omega) \), such that

\[ \exists \nu > 0: A_{ij}^{\alpha \beta} \chi^i \chi^j \geq \nu |\chi|^2 \text{ a.e. } x \in \Omega, \chi \in \mathbb{R}^n. \]

De Giorgi proves that \( S \) has a minimizer that is a function having a point of discontinuity in the origin. Later, Souček in [23] showed that minimizers of functionals of the type \( S \) can be discontinuous, not only in a point, but also on a dense subset of \( \Omega \). Modifying De Giorgi’s example, Giusti and Miranda in [13] showed that solutions of elliptic quasilinear systems of the type

\[ \int_{\Omega} A_{ij}^{\alpha \beta}(u) D_\alpha u^i D_\beta \varphi^j dx = 0, \forall \varphi \in C^\infty_0(\Omega, \mathbb{R}^N) \]

with analytic elliptic coefficients \( A_{ij}^{\alpha \beta} \) have singularities in dimension \( n \geq 3 \).

We observe that we can get global regularity for some special cases, see for example [24].

Similar examples were presented in the meantime independently by Maz’ya in [17]. Even Giaquinta in 1993 in [10], Morrey in [19] and others were interested in these problems of solutions of elliptic systems, solutions in general non regular. Then we can prove regularity except on a set, hopefully not too large.

For linear systems, regularity results assuming \( A_{ij}^{\alpha \beta} \) constant or in \( C^0(\Omega) \), have been obtained by Campanato (see [3]). Without assuming continuity of coefficients, we mention the study by Acquistapace where Campanato’s results are refined considering that \( A_{ij}^{\alpha \beta} \) belongs to a class that neither contains nor is being contained by \( C^0(\Omega) \), hence in general discontinuous. Moreover, we recall the study made by Huang in [14] where he shows the regularity of weak solutions of linear elliptic systems with coefficients \( A_{ij}^{\alpha \beta}(x) \in VMO \). So, it seems to be natural to expect partial regularity results under the condition that the coefficients of the principal terms \( A_{ij}^{\alpha \beta} \in VMO \), even for nonlinear cases. Daneček and Viszus in [6] consider the regularity of minimizer for the functional

\[ \int_{\Omega} \left\{ A_{ij}^{\alpha \beta}(x) D_\alpha u^i D_\beta u^j + g(x, u, Du) \right\} dx \]
where the term \( g(x, u, Du) \) is such that

\[
|g(x, u, z)| \leq f(x) + |z|^{\gamma}
\]

where \( f \in L^p(\Omega), 2 < p \leq \infty, f \geq 0 \) a.e. on \( \Omega, L \) is a non-negative constant and \( 0 \leq \gamma < 2 \).

They obtained Hölder regularity of minimizer assuming that \( A_{ij}^{\alpha \beta}(x) \in VMO \).

We also recall the paper [8] where the authors obtain regularity results for minimizers of the quasilinear functionals

\[
\int_{\Omega} A_{ij}^{\alpha \beta}(x, u) D_{\alpha} u^i D_{\beta} u^j dx,
\]

where the coefficients \( A_{ij}^{\alpha \beta}(x, u) \) have VMO dependence on the variable \( x \) and continuous dependence on \( u \).

In the paper [21] we improve the last mentioned result in Morrey spaces because we have considered inside the integral the term \( g(x, u, Du) \) and the result by Daneček and Viszus because we consider \( A_{ij}^{\alpha \beta} \) dependent not only on \( x \) but also on \( u \). In the present note we extend our above cited regularity results because we consider the more general class Campanato spaces.

Before an outline of the proof we state a preliminary Lemma by Campanato.

**Lemma 2.2.** Let \( B(x_0, R) \) be a fixed ball and \( u \in W^{1,2}(B(x_0, R), \mathbb{R}^N) \) be a weak solution of the system

\[
D_{\alpha}(A_{ij}^{\alpha \beta} D_{\beta} u^j) = 0, \quad i = 1, \ldots, N
\]

where \( A_{ij}^{\alpha \beta} \) are constant and satisfy the ellipticity condition. Then \( \forall t \in (0, 1] \)

\[
\int_{B(x_0, tR)} |Du|^2 dx \leq c \cdot t^n \int_{B(x_0, R)} |Du|^2 dx.
\]

**Proof of the Theorem**

For simplicity we’ll consider the case \( g = 0 \). Let \( R > 0 \) and \( x_0 \in \Omega \) such that \( B(x_0, R) \subset \subset \Omega \).

Let \( v \) be the minimum of the “freezing” functional \( \mathcal{A} \), that is

\[
\mathcal{A}_0(v, B(x_0, R/2)) = \int_{B(x_0, R/2)} A_{ij}^{\alpha \beta}(x_0, v_{B}) D_{\alpha} u^i D_{\beta} u^j dx
\]
with \( v \equiv u \) on \( \partial B(x_0, \frac{R}{2}) \). The idea of freezing is the same used by Chiarenza, Frasca and Longo in [14].

For \( 0 \leq \lambda < n \) and \( q \leq 2 \) we have that

\[
\| Du \|_{L^q,\lambda(\Omega)} = \| Du \|_{L^q(\Omega)} + [Du]_{q,\lambda} \leq K \| Du \|_{L^q,\lambda(\Omega)}
\]

where the constant \( K \) is independent of \( u \).

We observe that \( A_{ij}^{\alpha \beta}(x, u, R) \) are constant coefficients, then from the above Lemma, \( \forall t \in (0, 1] \),

\[
\int_{B(x_0, \frac{tR}{2})} |Du|^2 dx \leq c \cdot t^n \int_{B(x_0, \frac{R}{2})} |Du|^2 dx.
\]

Let \( w = u - v \), then \( w \in W^{1,2}_0(B(x_0, \frac{R}{2})) \)

\[
\int_{B(x_0, \frac{tR}{2})} |Du|^2 dx \leq c \cdot \left\{ t^n \int_{B(x_0, \frac{R}{2})} |Du|^2 dx + \int_{B(x_0, \frac{R}{2})} |Dw|^2 dx \right\}.
\]

Let us estimate:

\[
\int_{B(x_0, \frac{R}{2})} |Dw|^2 dx.
\]

From the hypothesis and a Lemma in [12] we have

\[
\nu \int_{B(x_0, \frac{R}{2})} |Dw|^2 dx \leq \left\{ A_0^\alpha(u, B(x_0, \frac{R}{2})) - A_0^\alpha(v, B(x_0, \frac{R}{2})) \right\}
\]

adding and subtracting:

\[
A_{ij}^{\alpha \beta}(x, u, \frac{R}{2}) D_\alpha u^i D_\beta u^j, \quad A_{ij}^{\alpha \beta}(x, u, \frac{R}{2}) D_\alpha v^i D_\beta v^j
\]

\[
A_{ij}^{\alpha \beta}(x, u) D_\alpha u^i D_\beta u^j, \quad A_{ij}^{\alpha \beta}(x, v) D_\alpha v^i D_\beta v^j
\]

we obtain some different kinds of integrals that we now examine.
Using $L^p$ estimate, we can estimate the terms with $|Du|^2$ as follows:

$$
\int_{B(x_0, \frac{R}{2})} |A_{ij}^{\alpha\beta}(x_0, u_{\frac{R}{2}}) - A_{ij}^{\alpha\beta}(x, u_{\frac{R}{2}})| \cdot |Du|^2 \, dx \leq
$$

$$
\leq c \left\{ \eta(A(\cdot, u_{\frac{R}{2}}); R) \right\}^{1-\frac{2}{p}} \int_{B(x_0, R)} |Du|^2 \, dx.
$$

We also observe that:

$$
\int_{B(x_0, \frac{R}{2})} |A_{ij}^{\alpha\beta}(x, u_{\frac{R}{2}}) - A_{ij}^{\alpha\beta}(x, u)| \cdot |D\alpha u^i D\beta u^j| \, dx \leq
$$

$$
\leq c \left( \int_{B(x_0, R)} |Du|^2 \, dx \right) \left( \int_{B(x_0, \frac{R}{2})} \omega(|u_{\frac{R}{2}} - u|^2) \, dx \right)^{1-\frac{2}{p}} \leq
$$

$$
\leq c \left( \int_{B(x_0, R)} |Du|^2 \, dx \right) \left( \omega \left( R^{2-n} \int_{B(x_0, \frac{R}{2})} |Du|^2 \, dx \right) \right)^{1-\frac{2}{p}}.
$$

Moreover, we can estimate the terms having $|Dv|^2$ similarly using $L^p$ estimates for $Dv$.

Then if $\rho = tR$

$$
\int_{B(x_0, \rho)} |Du|^2 \, dx \leq
$$

$$
\leq c \left\{ \left( \frac{\rho}{R} \right)^n + \left( \omega \left( R^{2-n} \int_{B(x_0, \frac{R}{2})} |Du|^2 \, dx \right) \right)^{1-\frac{2}{p}} \right\}
$$

Using a Lemma contained in [9] and selecting $\rho$ sufficiently small, specifically $\rho < \frac{R}{2}$, we have

$$
\int_{B(x_0, \rho)} |Du|^2 \, dx \leq c \cdot \rho^\lambda.
$$
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M.A. Ragusa, Dipartimento di Matematica, Università di Catania, Viale A. Doria, 6, 95125 Catania, Italia
E-mail address: maragusa@dipmat.unict.it

A. Tachikawa, Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, Noda, Chiba, 278-8510, Japan
E-mail address: tachikawa_atsushi@ma.noda.tus.ac.jp