Analysis of the angular distribution asymmetries and the associated CP asymmetries in bottom baryon decays

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Abstract

We introduce a set of observables representing the angular distribution asymmetries, which can be viewed as a generalization of the Forward-Backward Asymmetry of the angular distributions, and can be used as an effective tool for searching for CP violation in bottom and charmed baryon decays. We propose to search for such kind of CP asymmetries in decays with Λ⁰, Σ±, or Λ⁺ involved, such as Λ_b → Λ⁰D and Λ_b → Λ⁰ρ⁰(770). We also propose to search for such kind of CP asymmetries in three-body decays of bottom baryon with opposite parity intermediate resonances involved. Typical examples include Ξ_b⁻ → pK⁻K⁻, Λ_b⁰ → pK_Sπ⁻, Λ_b⁰ → Λ⁰π⁺π⁻, etc..
I. INTRODUCTION

*CP* violation (*CPV*), which is an important ingredient of the Standard Model (SM) of particle physics [1], has been observed in *K*, *B*, *B_s* and *D* meson decay processes. Theoretical investigations show that there can be relatively large *CP* asymmetries (*CPA*s) in some decay channels of bottom baryons [2]. On the experimental side, the *CPAs* have been investigated in some typical two-body decay channels of *Λ_0^b* by CDF [3, 4] and LHCb [5], in three- or four-body decays of *Λ_0^b*, *Ξ_0^b*, *Ξ_+^b*, and *Ξ_-^b* by LHCb [6–12], and very recently in hyperon decays by BESIII [13]. However, the pursuit of *CPV* in the baryonic sectors, which although is of great importance for the test of SM and for indirect searching of New Physics beyond SM, has not reached in a positive result after years of efforts.

Since the baryons are spined particles, other than the decay width, *CPV* can present in the observables associated with the angular distributions of the final particle. One such observable is the decay parameters in two-body weak decay processes of baryons. Typical examples include *Λ^0* → *pπ^−* and *Λ_0^b* → *DΛ^0*, where the decay parameter and the associated *CPA*s of the former channel has been measured by BESIII through the baryon-anti-baryon pair production *J/ψ* → *Λ^0Λ^0* [13], and those of the latter were proposed for extracting the CKM phases *γ* through the dual weak cascade decays *Λ_0^b* → *DΛ^0* with *Λ^0* → *pπ^−* [14–16]. In both cases, the decay parameters can be related to Forward-Backward Asymmetries (FBA) of the final particle distributions in certain reference frames.

*CPV* can also leave track in the angular distributions of final particles in multi-body decay processes of hadrons. Examples include *CPAs* associated with the triple product asymmetries in baryon decay processes [7, 9, 17, 18] and partial-wave *CPAs* (PW*CPAs*) [19]. In fact, the largest *CPAs* ever been observed are those localized in certain regions of the phase space in three-body decay channels of bottom mesons, such as *B^±* → *π^±π^+π^−*, *B^±* → *K^±K^+K^−*, *B^±* → *K^±π^±π^−*, *B^±* → *π^±K^+K^−* etc, though the integrated *CPAs* are all relatively quite small [20–25]. Take *B^±* → *π^±π^+π^−* as an example, in which very large regional *CPAs* were observed in part of the *f_0(500)−ρ(770)^0* interfering region corresponding to the angle between the two same-sign pions smaller than 90°. This large regional *CPA* can be explained by the interference of the *s* - and *p*-wave amplitudes (corresponding to *f_0(500)* and *ρ(770)^0* respectively) with a natural inclusion of a non-perturbative strong phase difference between the two waves [26], which can be ideally studied through the angular
distribution asymmetry observables FBA [27, 28]. It is naturally expected that there exist similar CPAs associated with the an-isotropy of angular distributions of the final particle in multi-body decays of baryons.

The aim of this paper is to introduce a set of angular distribution asymmetry observables, which can be viewed as a generalization of the aforementioned FBAs in the multi-body decay processes of baryons and mesons. The newly introduced observables can be used in searching for CP violations in the baryon decay processes, especially in the bottom baryon decays.

II. ANGULAR DISTRIBUTION ASYMMETRY

To put the discussions on a more general ground, we consider a three-body weak decay process of a hadron $H$, $H \rightarrow h_1 h_2 h_3$. It can be proven that the spin-averaged decay amplitude squared, which is defined as $|\mathcal{M}^J|^2 \equiv \frac{1}{2J+1} \sum_{m_z, \lambda_i} |\mathcal{M}^{m_z}_{\lambda_1, \lambda_2, \lambda_3}|^2$ for unpolarised $H$, can be expressed as

$$
|\mathcal{M}^J|^2 = \sum_j w^{(j)} P_j \left( c_{\theta'_1} \right),
$$

where $J$ and $m_z$ are the spin and its $z$-axis component, $\lambda_i (i = 1, 2, 3)$ is the helicity of $h_i$, $P_j$ is the $j$-th Legendre polynomial, $c_{\theta'_1} \equiv \cos(\theta'_1)$, with $\theta'_1$ being the angle between the momenta of $h_1$ and $H$ in the rest frame of $R_i$ (see FIG. 1 for illustration), $w^{(j)}$ represents the weight of the $j$-th wave.

The presence of odd-$j$ terms $w^{(j)}$ will result angular distributions asymmetries, i.e., asymmetries between $\theta' \leftrightarrow \pi - \theta'$. To account for this kind of asymmetries, we introduce a set of
observables, which is defined as
\[
A_{FB}^j = \frac{\left( \int_{-1}^{x_{j}^{(1)}} - \int_{x_{j}^{(2)}}^{x_{j}^{(1)}} + \int_{x_{j}^{(3)}}^{x_{j}^{(2)}} \cdots - \int_{x_{j}^{(j+1)}}^{x_{j}^{(j)}} \right) |M|^2 dc_{\theta'}}{\int_{-1}^{+1} |M|^2 dc_{\theta'}}
\] \hspace{1cm} (2)
for odd \( j \), where \( x_{k}^{(j)} \) \((k = 1, 2, \cdots, j)\) is the \( k \)-th zero point of the Legendre polynomial \( P_j(x) \). Note that this can be viewed as a generalization of the Forward-Backward Asymmetry (FBA) for meson decays such as \( B^\pm \rightarrow \pi^\mp \pi^+ \pi^- \), or for baryon decays such as \( \Lambda_0^0 \rightarrow \Lambda \rightarrow p\pi^- \) \(D\).\(^1\) As one can see for the case \( k = 1 \), \( A_{FB}^1 \) reduces to the FBA. Hence we will call \( A_{FB}^j \) as the \( j \)-th FBA. The corresponding \( CP \) violating observables, which will be named as the \( k \)-th FBA induced \( CP \) Asymmetry \((k\)-th FB-CPA\), can then be defined as
\[
A_{FB,CP}^j = \frac{1}{2} \left( A_{FB}^j - \overline{A_{FB}^j} \right),
\]
where \( \overline{A_{FB}^j} \) is the \( j \)-th FBA of the charge conjugate process, and the presence of the minus sign in front of it is because that \( A_{FB,CP}^j \) and \( A_{FB}^j \) are parity-even observables.

One can easily see from the definition that \( A_{FB}^j \) (and hence \( A_{FB,CP}^j \)) can only get contributions from \( w^{(j)} \) with odd \( j' \). All the \( w^{(j')} \)'s with even \( j' \) do not contribute to any of the \( A_{FB}^j \)'s. On the other hand, there is no one-to-one correspondence between \( A_{FB}^j \) and \( w^{(j)} \), meaning that each \( A_{FB}^j \) gets contributions from all the \( w^{(j')} \) for odd \( j' \). Despite of this, for a fixed \( j = j_0 \), the most important contributions to \( A_{FB}^{j_0} \) and \( A_{FB,CP}^{j_0} \) come from \( w^{(j_0)} \).

In what follows, we consider cascade decay \( H \rightarrow R_i (\rightarrow h_1 h_2) h_3 \), where the subscript \( i \) in \( R_i \) indicates that there may be more than one intermediate particles with similar masses. We first leave it open to whether \( R_i \rightarrow h_1 h_2 \) is weak or strong. The terms \( w^{(j)} \) can be expressed in terms of the corresponding decay amplitudes. After some algebra, one has
\[
w^{(j)} = \sum_{ii'} \left\langle \frac{S_{ii'}^{(j)} W_{ii'}^{(j)}}{I_{R_i} I_{R_{i'}}} \right\rangle,
\]
where the notation “\( \langle \cdots \rangle \)” indicates the integral with respect to \( s_{12} \) over a small interval which covers all the interested resonances \( R_i \), \( W_{ii'}^{(j)} \) and \( S_{ii'}^{(j)} \) respectively contain the decay

\(^1\) Note however, the newly introduced observables can not be viewed as a generalization of the decay parameter of the hyperon two-body decay \( \Lambda^0 \rightarrow p\pi^- \) \([29, 30]\). As the FBA, a.k.a. the decay parameter of the hyperon decay \( \alpha \Lambda \rightarrow p\pi^- \), is defined according to the angle between the spin of the hyperon and the momentum of the proton, which is a parity-odd quantity. Hence the corresponding \( CP \)A should be defined as \( A_{CP}^\alpha = \frac{1}{2} \left( \alpha^{\Lambda \rightarrow p\pi^-} + \alpha^{\Lambda \rightarrow p\pi^+} \right) \).
amplitudes of $H \rightarrow R_i h_3$ and $R_i \rightarrow h_1 h_2$, and take the form

$$W^{(j)}_{i'i} = \sum_{\sigma \lambda_3} (-)^{s-s'} \langle s_{R_i} - \sigma s_{R_{i'}} \sigma | s_{R_i} s_{R_{i'}} j0 \rangle F^J_{R_i, \sigma \lambda_3} F^{J*}_{R_{i'}, \sigma \lambda_3}, \tag{5}$$

and

$$S^{(j)}_{i'i} = \sum_{\lambda_1 \lambda_2} (-)^{s'-s} \lambda' \langle s_{R_i} - \lambda s_{R_{i'}} \lambda' | s_{R_i} s_{R_{i'}} j0 \rangle F^{R_i,s_{R_i}}_{\lambda_1} F^{R_{i'},s_{R_{i'}}}_{\lambda_2}, \tag{6}$$

respectively $^2$, where the notation $(\cdots | \cdots)$’s are the Clebsh-Gordan coefficients. Note that the introduction of $s$ in the above two equations is to make $\sigma - s$ and $s - \lambda'$ integer. It could be either $s_{R_i}$ or $s_{R_{i'}}$. The Clebsh-Gordan coefficients in Eqs. (5) and (6) restrict that $j$ in Eq. (1) can only take integer values from 0 to $\max_i(2s_{R_i})$:

$$\frac{|\mathcal{M}^J|^2}{\mathcal{M}^J} = \sum_{j} w^{(j)} P_j \left(c_{\theta_i}\right). \tag{7}$$

### III. APPLICATION

As for the applications of the newly introduced observables $A_{ij}^{FB}$ and $A_{ij,CP}^{FB}$, we want to consider two situations, according to whether the decay $R_i \rightarrow h_1 h_2$ is weak or strong. The first one is that the decay $R_i^{(i)} \rightarrow h_1 h_2$ is a weak process. In this situations, the intermediate state $R_i^{(i)}$ has negligibly narrow decay width. Hence, there is no need to consider the interference of nearby resonances, which means that there is only one intermediate state, $R_i$. Typical processes include: (1) $\Lambda_b^0 \rightarrow \Lambda^0 M$ with $M$ being mesons such as $\pi$, $\rho$, $D$, or $J/\psi$, and $\Lambda$ as $R_i$ and decaying through $N\pi$; (2) $\Lambda_b^0 \rightarrow \Sigma^\pm M$ with $\Sigma^\pm$ as $R_i$ and decaying through $N\pi$; (3) $\Lambda_b^0 \rightarrow \Lambda_c^+ M$, with $\Lambda_c^+$ as $R_i$ and decaying through $pK_s$ or $\Lambda^0\pi^+$. Since the sub-process $R_i^{(i)} \rightarrow h_1 h_2$ is a weak process, there is no extra constraints from the parity conservation. The only worth mentioning constraints come from the Clebsh-Gordan coefficients in Eqs. (5) and (6), which tell us that $j$ can only take integer values 0 and 1 for all these aforementioned examples with spin-parity $(1/2)^+$ baryons as $R_i$. Hence the spin-averaged amplitude squared always takes the form $|\mathcal{M}^J|^2 \propto 1 + A_{ij}^{FB} c_{\theta_i}$. The only practically relevant observable is the 1st-FBA $A_{ij}^{FB}$.

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$^2$ One can also replace the helicity decay amplitudes by the spin-angular ones. For example, $F^J_{R_i, \sigma \lambda_3} = \sum_{ls} \left(\frac{2l+1}{2J+1}\right)^{\frac{1}{2}} \langle l0|s - \lambda_3|ls0\rangle F^{J}_{R_i, \sigma \lambda_3} F^{J*}_{R_i, \sigma \lambda_3}$.

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Another situation is when the decays \( R_i^{(i)} \rightarrow h_1 h_2 \) are strong processes. These strong decay processes respect parity symmetry, which implies from Eq. (6) that

\[
S_{ii'}^{(j)} = \Pi_{R_i} \Pi_{R_i'} (-)^j S_{ii'}^{(j)},
\]

where \( \Pi_{R_i} \) is the parity of \( R_i^{(i)} \). If there is only one resonance \( R_i \) contributes, Eq. (8) reduces to \( S_{ii'}^{(j)} = (-)^j S_{ii'}^{(j)} \). Hence \( S_{ii'}^{(j)} = 0 \) for odd \( j \). That means that there no need for considerations of \( A_{FB}^i \) and \( A_{j,CP}^i \) at all. The only way out of this dilemma is when there are (at least) two resonances, say \( R_{i_1} \) and \( R_{i_2} \), with opposite parities and similar masses. It is this situation that has the most similarities with the aforementioned three-body decays of \( B^\pm \), in which the largest regional CPAs are observed. Now \( S_{i_1 i_2}^{(j)} \) can be nonzero for odd \( j \) according to Eq. 8, hence \( A_{j}^{FB} \) and \( A_{j,CP}^{FB} \) can be nonzero.

The interference of nearby intermediate resonances are pretty common phenomenons in multi-body decay of bottom or charmed hadrons. We list three typical examples of three-body decays of bottom baryons for the second situation.

- The first one is \( \Xi_b^- \rightarrow pK^-K^- \). It has already been observed by LHCb that there are some resonant structures in the low invariant mass region of the \( pK^- \) system, such as \( \Sigma(1775) \) and \( \Sigma(1915) \), whose spin-parities are \( (\frac{5}{2})^+ \) and \( (\frac{5}{2})^- \) respectively [10]. Consequently, \( j \) can take integer values from 0 to 5, according to Eqs. (5) and (6). Hence \( w(1) \), \( w(3) \) and \( w(5) \) will be nonzero. Correspondingly, there can be angular distribution asymmetries, which is suitably be studied through the measurements of \( A_{1}^{FB}, A_{3}^{FB} \) and \( A_{5}^{FB} \). Moreover, the associated CP asymmetries, \( A_{1,CP}^{FB}, A_{3,CP}^{FB} \), and \( A_{5,CP}^{FB} \), can also be measured. It should be pointed that \( A_{3}^{FB} \) and \( A_{5}^{FB} \) (hence \( A_{3,CP}^{FB} \) and \( A_{5,CP}^{FB} \)) have never been studied in any decay channels before.

- Another example is \( \Lambda_b \rightarrow pK_S\pi^- \). Although the nature of the resonant structures observed by LHCb in the low invariant mass region of the \( pK_s \) system is still unclear [31], there still good chance for the presence of \( N^+(1440) \) and \( N^+(1520) \), whose spin-parities are \( (\frac{1}{2})^+ \) and \( (\frac{3}{2})^- \) respectively, \( j \) can take integer values 0, 1, and 2, according to Eqs. (5) and (6). Consequently, \( w(1) \) will be nonzero. The corresponding angular distributions asymmetries \( A_{1}^{FB} \) will be nonzero. There can also be nonzero associated CP asymmetries, \( A_{1,CP}^{FB} \).
• The resonances $R_i$ can also appear as mesons. For example, the decay process $\Lambda_0^b \to \Lambda_0^0 \pi^+ \pi^-$, with $f_0(600)$ and $\rho^0(770)$ as $R_i$ and decaying into $\pi^+ \pi^-$. According to similar analysis as above, the interference of $f_0(500)$ and $\rho^0(770)$ will appear in $w^{(1)}$, resulting in a nonzero $A_{1,CP}^{FB}$, and potentially a nonzero $A_{1,CP}^{FB}$. This is quite similar with the case $B^\pm \to \pi^\pm \pi^+ \pi^-$, where large FBA and FB-CPA caused mainly by the interference of $f_0(500)$ and $\rho^0(770)$ have been observed in the $f_0(600) - \rho^0(770)$ interference region [22, 28], except that the current situation is more complex. There are both two independent decay amplitudes for each of the decays $\Lambda_b \to \Lambda f_0$ and $\Lambda_b \to \Lambda \rho^0$, where the former constitutes the s- and p-wave amplitudes, while the latter constitutes the $p$- and $d$-wave amplitudes.

IV. SUMMARY

In summary, a set of angular distribution asymmetry observables, which are called the $j$-th Forward-Backward Asymmetry for odd $j$, are introduced. They can be used in searching for $CP$ violations in decay channels of bottom baryons. Two typical situations for the application of the newly introduced observables were discussed. The first is the two-body decays of the type $\Lambda_b^0 \to \Lambda^0 M$, $\Lambda_b^0 \to \Sigma^\pm M$, and $\Lambda_b^0 \to \Lambda^+_c M$, with $\Lambda^0$, $\Sigma^\pm$, or $\Lambda^+_c$ decaying weakly to two hadrons. In this situation, the newly introduced observables is in fact equivalence to the decay asymmetry parameters. The second situation corresponds to three-body decays of bottom baryons with the interference of intermediate resonances of similar masses and opposite parities. A typical example for the second situation is the decay channel $\Xi_b^- \to pK^-K^-$, where possible interference between intermediate resonances $\Sigma(1775)$ and $\Sigma(1915)$ presents. We suggest to measure $A_{1,CP}^{FB}$, $A_{3,CP}^{FB}$, and $A_{5,CP}^{FB}$, and the corresponding $CP$ asymmetry observables $A_{1,CP}^{FB}$, $A_{3,CP}^{FB}$, and $A_{5,CP}^{FB}$. Other examples include $\Lambda_b^0 \to pK_S\pi^-$, $\Lambda_b^0 \to \Lambda^0 \pi^+ \pi^-$, etc.

The measurements of the angular distribution asymmetry observables and their corresponding $CP$ violation observables can also be performed in other decay channels of bottom or charmed hadrons.
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Appendix A: decay amplitude for $H \to h_1 h_2 h_3$

To obtain the expression of $w^{(j)}$ in Eq. (4), one needs to write down the decay amplitude for the cascade decay $H \to R_i h_3$, $R_i \to h_1 h_2$. Two reference frames, the rest frame of $H$ (RFH) and that of $R_i$ (RF$R_i$), are needed for the helicity forms of the decay amplitudes. In RFH, the $z$-axis is chosen along the quantization direction of the spin of $H$, and the momenta (helicities) of $R_i$ and $h_k$ (k=1,2,3), will be denoted as $p(\sigma)$ and $q_k(\lambda_k)$, respectively. The decay amplitude of $H \to R_i h_3$ can be expressed in the helicity form as

$$\mathcal{M}^{H,Jm_3}_{\sigma\lambda_3} = \mathcal{F}^{J}_{R_i,\sigma\lambda_3} D^{J*}_{m_z,\sigma-\lambda_3}(\phi_B, \theta_B, 0),$$  \hspace{1cm} (A1)$$

where $D$ is the Wigner-$D$ matrix, $(\phi_B, \theta_B)$ are the polar and azimuthal angles of $\vec{p}$ in RFH, $\mathcal{F}$ is the helicity decay amplitude. In RF$R_i$, the $z'$-axis is chosen along the direction of the 3-momentum of $H$. The point that the $z'$-axis is chosen this way is that the helicity of $R_i$ in the RFH is just the $z'$-component of the spin of $R_i$ in RF$R_i$. In RF$R_i$, the momenta (helicities) of $h_k$ (k=1,2,3), will be denoted as $q'_k(\lambda'_k)$. The decay amplitude for $R_i \to h_1 h_2$ can be expressed as

$$\mathcal{M}^{R_i,sR_i\sigma}_{\lambda'_1\lambda'_2} = \mathcal{F}^{R_i,sR_i}_{\lambda'_1\lambda'_2} D^{s*}_{\sigma,\lambda'_1-\lambda'_2}(\phi'_1, \theta'_1, 0),$$ \hspace{1cm} (A2)$$
where \((\phi_1', \theta_1')\) is the polar and azimuthal angles of \(\vec{q}_1'\) in \(\text{RF}_i\). The decay amplitude for \(H \to h_1 h_2 h_3\) (with \(h_1 h_2\) decaying from \(R_i\)’s) can then be expressed as

\[
\mathcal{M}_{J m_z}^{\lambda_1 \lambda_2 \lambda_3} = \sum_i \sum_{\sigma} \frac{\tilde{\mathcal{M}}_{\lambda_1 \lambda_2}^{R_i, s R_i \sigma} \mathcal{M}_{\sigma \lambda_3}^{H, J m_z}}{\mathcal{I}_{R_i}},
\]

(A3)

where \(\mathcal{I}_{R_i} = s_{12} - m_{R_i}^2 + i m_{R_i} \Gamma_{R_i}\), \(s_{j k} = (q_j + q_k)^2\), \((j, k = 1, 2, 3)\) is the invariant mass squared of \(h_j\) and \(h_k\), \(\Gamma_{R_i}\) is the decay width of \(R_i\), and \(\tilde{\mathcal{M}}_{\lambda_1 \lambda_2}^{R_i, s R_i \sigma}\) is the decay amplitude of \(R_i \to h_1 h_2\) in \(\text{RF}_H\), which can be obtained by a Lorentz transform of the amplitude \(\mathcal{M}_{\lambda_1' \lambda_2'}^{R_i, s R_i \sigma}\) according to

\[
\tilde{\mathcal{M}}_{\lambda_1 \lambda_2}^{R_i, s R_i \sigma} = \sum_{\lambda_1' \lambda_2'} \mathcal{M}_{\lambda_1' \lambda_2'}^{R_i, s R_i \sigma} D_{\lambda_1 \lambda_1'}^{j_1} (\phi_{W_1}, \theta_{W_1}, 0) D_{\lambda_2 \lambda_2'}^{j_2} (\phi_{W_2}, \theta_{W_2}, 0),
\]

(A4)

with \(\Omega_{W_k} = (\phi_{W_k}, \theta_{W_k})\), \((k = 1, 2)\) being the polar and azimuthal angles of the Wigner rotation, \(W_k\) being a pure Lorentz boost that transforms \(q_k\) into \(q_k'\).