3-cobordisms with their rational homology on the boundary∗†‡
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Abstract
The object of this paper is to define a subcategory of the category of 3-cobordisms to which invariants of rational homology 3-spheres should generalize. We specify the notion of Topological Quantum Field Theory (in the sense of Atiyah [1]) to this case, and prove two interesting properties that these TQFTs always have. In the case of the LMO invariant these properties amount to saying that the TQFT is anomaly-free.

This paper is organized as follows. In Section 1 we define the topological categories of 3-cobordisms $Q$, $Z$, and $L$ that we want to consider, where the morphisms are certain connected oriented 3-cobordisms (with parametrizations of the boundary components). Their domain and range are also connected oriented closed surfaces. All the proofs in this section are elementary when given rigorous definition. In Section 2 we give the axioms of TQFT for these categories, and prove two interesting properties. There are two essential differences between this type of TQFTs and the classical TQFTs, such as Reshetikhin-Turaev’s for quantum invariants. Firstly, we tailor the construction for integer and rational homology spheres. Therefore we restrict to connected cobordisms between connected surfaces, and hence have no tensor product $\otimes$ structure induced by disjoint union. Secondly, when gluing, we discriminate between the domain and the range of a cobordism. In particular, while we regard the standard surface $\Sigma_g$ of genus $g$ in the domain as the boundary of the standard handlebody $N_g$, we regard $\Sigma_g$ in the range as the boundary of the complement of $N_g$ in $S^3$. Thus, gluing identically in our TQFT produces $S^3$ as opposed to $\#_g(S^2 \times S^1)$ in the Reshetikhin-Turaev TQFT. This framework still allows to derive associated representations of the Torelli group, in fact of a larger Lagrangian subgroup of the Mapping Class Group.

These definitions and properties are motivated by the construction of a TQFT for the Le-Murakami-Ohtsuki invariant $Z^{LMO}$ [3,14], because this invariant is strong for rational homology three-spheres, and is weaker if the rank of homology is bigger. However the two simple properties that we prove in section 2 can be useful for any TQFT on $\Omega$.

1 The categories $\Omega \supset \mathcal{Z} \supset \mathcal{L}$ of semi-Lagrangian cobordisms

All maps and homeomorphisms in this paper are piecewise-linear, hence we will generally drop the term “PL”. Denote by $\Gamma^g$ and call chain graph (terminology borrowed from [14]) the abstract trivalent graph $\bigcirc \overrightarrow{\bigcirc} \cdots \overrightarrow{\bigcirc}$ with oriented edges as indicated. Label its subgraphs $\bigcirc$ from 1 to $g$ from left to right. For $g = 1$, set $\Gamma^1 = \bigcirc$, one oriented edge, no vertices. For $g = 0$, set $\Gamma^0 = \text{one point}$.

Definition 1) A pair $(\Gamma, R)$, consisting of an ordered disjoint union of chain graphs and an oriented surface with boundary, will be called a ribbon pair if it is the union of finitely many copies of the two pairs depicted in the figure 1a, such that each point of $\Gamma$ has a neighbourhood PL-homeomorphic to one
Definition

Let $M$ be a compact oriented 3-manifold with boundary $\partial M = (-S_1) \cup S_2$, suppose also that parametrizations $f_1, f_2$ of each $S_1, S_2$ are fixed. We will call such $(M, f_1, f_2)$ a (parametrized) $(2+1)$-cobordism. $S_1$ will be called the bottom, $S_2$ – the top of the cobordism. The cobordisms $(M, f_1, f_2)$ and $(N, h_1, h_2)$ will be called equivalent (homeomorphic) if there is a PL-homeomorphism $F : M \to N$ sending bottom to bottom and top to top preserving the parametrizations, i.e. $F \circ f_i = h_i$, $i = 1, 2$. We will use $\cong$ to denote equivalent cobordisms.

Using the parametrizations, we can glue the standard handlebody $N_{g_1}$ to the bottom and the standard
anti-handlebody $\overline{N_{g_2}}$ to the top of $M$. Denote the result $M \cup_{f_1} N_{g_1} \cup_{-f_2} (\overline{-N_{g_2}})$ by $\tilde{M}$ and call it the filling of $(M, f_1, f_2)$.

### 1.1 Surgery description of gluing cobordisms

Let $\mathfrak{G}$ denote set of equivalence classes of triplets $(L, G_1, G_2)$ in $S^3$. Let $\mathfrak{C}$ denote the set of equivalence classes of 3-cobordisms, with non-empty bottom and top.

**Proposition 1.** 1) There is a natural well-defined map $\kappa : \mathfrak{G} \to \mathfrak{C}$ that associates to every equivalence class of triplets $(L, G_1, G_2)$ the equivalence class of cobordisms $(M, f_1, f_2)$, obtained by doing surgery on $L \subset S^3$, removing tubular neighbourhoods $N_1, N_2$ of each $G_1, G_2$, and recording the parametrizations of the two obtained boundary components. If one glues according to these parametrizations a standard handlebody to $-\partial N_1$ and a standard anti-handlebody to $\partial N_2$, then one obtains $S^3_{\kappa}$.

2) $\kappa$ is surjective, and hence there exist maps $v : \mathfrak{C} \to \mathfrak{G}$ such that on 3-cobordisms with two non-empty boundary components $\kappa \circ v = id$.

3) Let a **first Kirby move** on a triplt be the cancellation / insertion of a $O^{\pm 1}$ separated by an $S^2$ from anything else, and an **extended (generalized)** second Kirby move be a slide over a link component of an arc, either from another link component or from a chain graph. Then if one factors $\mathfrak{G}$ by the extended Kirby moves and changes of orientations of link components, the induced map $\pi$ is a bijection.

**Proof.** 1) The parametrizations are determined as follows. For $i = 1, 2$, let $N_i$ be the closure of a tubular neighbourhood of $G_i$ such that $\partial R_i \subset \partial N_i$. (Since $R_i$ is compact, there is a neighbourhood of $G_i$ of the form $R_i \times [-\varepsilon, \varepsilon]$. Take this as $N_i$.) Let $M = S^2_L - (N_1 \cup N_2)$, $S_1 = \partial N_1, S_2 = -\partial N_2$. In particular, this construction yields $S_i \approx S^2$ if $G_i \approx \Gamma^0$. In the last case the possible parametrization is unique up to isotopy.

Fix a preferred point $x$ on each (open) upper half-circle of each $\Gamma^0$. This determines a preferred point $x'$ on each circle component of $G$. The construction of $N_1$ produces a preferred disk in $N_1$, centered at $x'$, with boundary in $\partial N_1$. Orient the boundary curves so that they twist right-handedly with respect to the circle components of $G$. Similarly, fix a preferred point $y$ on each (open) lower half-circle of each $\Gamma^0$ and construct an ordered system $b$ of oriented curves for the handlebody $N_2$. (We assume $N_2$ contains the point at infinity $\infty \in S^3$, hence we will refer to $N_2$ as the anti-handlebody.) Push $G_1$ along a framing transversal to $R_1$ to a graph on $S_1$, call it the b-graph. Analogously, push $G_2$ along a framing transversal $R_2$ to a graph on $S_2$, call it the a-graph.

If we cut-open $S_1$ along the b-graph and the system $a$, we get an oriented surface homeomorphic to $D^2$. Define the parametrization of $S_1$ by sending the standard b-graph on $S_{g_1}$ to the b-graph on $S_1$, and the system of loops $a_i$ on $S_{g_1}$ to the system $a$. Similarly for $S_2$: both $\Sigma - (\{ \text{standard a-graph} \} \cup \{ \text{standard system b} \})$ and $S_2 - (\{ a - \text{graph} \} \cup \{ \text{system b} \})$ are homeomorphic as oriented surfaces to the oriented $D^2$. Any two orientation-preserving homeomorphisms of $D^2$ are isotopic. Therefore the parametrization of $S_1$ is uniquely determined up to isotopy by $G_1 = (\Gamma_i, R_i)$. (Observe that different choices of transversal framings to the ribbons lead to isotopic b-graphs / a-graphs on $S_i$; different choices of the points $x$ lead to isotopic systems $a / b$.)

In conclusion, $\kappa$ of each triplt is well-defined as an equivalence class of 3-cobordisms. It is obvious that via the above construction equivalent triplets yield the same equivalence class of 3-cobordisms.

Moreover, in the above construction of parametrisations, the homeomorphism between the standard surface $\Sigma$ and $S$ extends to a homeomorphism between the standard handlebody / anti-handlebody and $N_i$. Hence $\tilde{M} \approx S^3_{\kappa}$.  

2) Let $\tilde{M}$ be the filling of a 3-cobordism $(M, f_1, f_2)$, $\Gamma_1, \Gamma_2$ - the images in $\tilde{M}$ of the cores and $R_1, R_2$ - the images in $\tilde{M}$ of the preferred ribbon graph neighbourhoods of the cores of the handlebody, respectively anti-handlebody of $\tilde{M}$. Since $\tilde{M}$ is a closed 3-manifold, there is a banded (unoriented) link $L \subset S^3$, such that $\tilde{M} \approx S^3_L$, the result of surgery on $L$. Choose one such link $L$. Then there exist two disjoint framed graphs $G_i, i = 1, 2$ in $S^3$, also disjoint from the link $L$, such that their remains after
surgery coincide (up to ambient isotopy) with the pairs \((\Gamma_i, R_i), i = 1, 2\) in \(\tilde{M}\),\(^2\) 
3) see \[12\] proposition 2.1

![Diagram](image)

**Figure 2:** The preferred choice of ribbons \(R_i, i = 1, 2\) for \((\Sigma_g \times I, p_1, p_2)\).

To visualize the above proof it is helpful to imagine the standard b-graph and system \(b\) (and respectively the standard \(a\)-graph and system \(a\)) cutting \(\Sigma_g\) to a 2-disk. Strictly speaking the definitions of the b-graph and system \(a\) make sense if both \(g_1, g_2 > 0\). However we can add the remaining cases by making the following convention: if \(\Gamma_i\) is a point, the b-graph/a-graph and the system \(a/b\) are to be considered the empty set. To draw a framed graph \(G_i = (\Gamma_i, R_i) \subset S^3\), we only need to draw the projection of \(\Gamma_i\) on \(\mathbb{R}^2\), which can be done in such a way that the preferred blackboard framing determines \(R_i\) up to isotopy.

Let us consider \(\Sigma_g \times I \subset S^3\). \(\Sigma_g \times \{0\}\) and \(\Sigma_g \times \{1\}\) are identified with two very near (isotopic) copies of the standard embedding \(\Sigma_g \subset S^3\). \((\Sigma_g \times I, p_1, p_2)\) is a 3-cobordism, and its filling is homeomorphic to \(S^3\). The parametrizations of the bottom \(p_1\) and top \(p_2\) are the ones induced via the isotopies in \(S^3\) between the standard \(\Sigma_g\) and \(\Sigma_g \times \{i\}\) from the identity \(id : \Sigma_g \to \Sigma_g \subset S^3\). To represent this 3-cobordism we choose framed graphs \(R_1, R_2\) as depicted in figure 2 (projections on \(\mathbb{R}^2 \subset \mathbb{R}^3 \subset S^3\)). Observe the consistency with Figure 1d. Denote by \([D_g]_{L_0}\) the result of surgery on the link \(L_0\) in \(D_g\), the three-dimensional manifold depicted in figure 3. Note the disks \(E_k\) and \(E_k'\) in \(D_g\), that intersect \(L_0\) once. If \(g = 0\), set \(D_g \approx S^2 \times I\) and \(L_0 = \emptyset\). \([D_g]_{L_0, p_1, p_2}\) is a 3-cobordism equivalent to \((\Sigma_g \times I, p_1, p_2)\).

**Proposition 2** Let \((M, f_1, f_2)\) and \((N, h_1, h_2)\) be arbitrary 3-cobordisms with connected bottoms and tops. The following 3-cobordisms are equivalent:

\[
(N \cup_{h_1 \circ f_2^{-1}} M, f_1, h_2) \cong (N \cup_{h_1 \circ p_2^{-1}} [D_g]_{L_0} \cup_{p_1 \circ f_2^{-1}} M, f_1, h_2)
\] (1.1)

**Proof.** Start with the right-hand-side. Using Kirby calculus, slide the handles of the surface \(S_1\) along the upper components of the link \(L_0\). Then the lower components of \(L_0\) bound disks, so the link can be canceled altogether. The two surfaces that remain are clearly isotopic, and the parametrizations are equivalent since both are images under isotopies of the identity parametrization of \(\Sigma_g \subset S^3\). The equivalence thus follows.

\[\square\]

Let \(\Gamma\), respectively \(\Gamma'\) generically denote the bottom, respectively the top of a triplet. Call the union of the lower half-circles and the horizontal segments of \(\Gamma\), the horizontal line of \(\Gamma\). Similarly, call the union of the upper half-circles and the horizontal segments of \(\Gamma'\), the horizontal line of \(\Gamma'\). See figure 4a.

For a given genus \(g\), let us decompose the manifold \(D_g\), together with the link \(L_0\) inside, into a union of upper handles \(D^U\), lower handles \(D^L\), and the rest of it \(D^M\): \(D_g = D^U \cup D^M \cup D^L\). In figure 3 instead of the whole handles, we have represented only disks \(E_k\) and \(E_k'\), whose neighbourhoods the handles are. Accordingly, \(L_0 \subset D_g\) can be decomposed into three framed oriented tangles: an oriented framed tangle in the upper handles, the oriented tangle \(T_g\) with the blackboard framing (see figure 4b), and an oriented framed tangle in the lower handles. For \(g = 0\), \(T_0 = \emptyset\).\(\footnote{Indeed, isotope the image of \((\Gamma_i, R_i)\) via \(\tilde{M} \leftarrow S^4_L\) to avoid the union of surgery tori, a bounded nonseparating subset of \(S^4_L\).}^2\)
Proposition 3 Let \((M_1, f_1, f'_1)\) and \((M_2, f_2, f'_2)\) be two 3-cobordisms with connected non-empty bottoms and tops. Let \(v(M_1, f_1, f'_1) = (L_1, G_1, G'_1)\), \(v(M_2, f_2, f'_2) = (L_2, G_2, G'_2)\) where \(v\) is some section of \(\kappa\). Remove a 3-ball neighbourhood of the horizontal line of \(G'_1 \subset S^3\), and identify the remain with \(B(0,1)\). Remove a 3-ball neighbourhood of the horizontal line of \(G_2 \subset S^3\), and identify the remain with \(S^3 - B(0,2)\). Glue the framed tangle \(T_g \subset B(0,2) - B(0,1)\) shown in figure 4 to the ends of the remains of \(G'_1\) in \(B(0,1)\) and \(G_2\) in \(S^3 - B(0,2)\), strictly preserving the order of the points, so that the composition of these framed tangles is a smooth framed oriented link \(L_0\) in \(S^3 = (S^3 - B(0,2)) \cup B(0,2) - B(0,1) \cup (B(0,1))\). Then

\[
\kappa(L_1 \cup L_0 \cup L_2, G_1, G'_2) = (M_2 \cup f_2 \circ (f'_1)^{-1} M_1, f_1, f'_2) \tag{1.2}
\]

where the ribbon neighbourhoods \(R_1, R'_2\) of \(G_1, G'_2\) in this formula are determined by the original \(R_1, R'_2\) in the two copies of \(S^3\). Hence, any triplet representing \((M_2 \cup f_2 \circ (f'_1)^{-1} M_1, f_1, f'_2)\) is equivalent to \((L_1 \cup L_0 \cup L_2, G_1, G'_2)\) by extended (generalized) Kirby moves and changes of orientations of link components.

**Proof.** Let us first note that the result is obvious if the two cobordisms are glued along a 2-sphere. For the general case, first use proposition 1.3 to "insert" \([D_g]_0\) "between" the two cobordisms (for appropriate \(g\)). This changes \((M_2 \cup f_2 \circ (f'_1)^{-1} M_1, f_1, f'_2)\) to an equivalent 3-cobordism.

Observe that gluing the standard handlebody \(N_g\) to \(D_g\) along \(p_1\) produces a manifold homeomorphic to \(N_g\). Moreover, gluing \(D_g \cup p_2\) along \(f_2 \circ p_2^{-1}\) to the bottom of the cobordism \(M_2\) produces a manifold.
homeomorphic to the one obtained by gluing \( N_g \) directly (along \( f_2 \)). In fact this homeomorphism is identity outside a collar neighbourhood of the bottom of \( M_2 \).

Now, using the decomposition \( D_g = D^L \cup D^M \cup D^U \), we note that gluing \( D_g \cup p_1 N_g \) to \( M_2 \) is the same as first gluing \( D^L \) on part of its boundary along the corresponding “restriction” of the map \( f_2 \circ \tilde p_2^{-1} \), then gluing \( (D^M \cup D^L) \cup p_1 N_g \approx D^3 \) along a 2-sphere.

Let us look at this glued 3-ball in the presentation of the filling \( \hat M_2 \) as \( S^3 \). By enlarging \( D^U \) and \( D^L \) if necessary, we may assume that inside this 3-ball there is only a neighbourhood of the horizontal line of \( \Gamma_2 \) with \( R_2 \) of blackboard framing.

Apply a similar procedure to the top of the 3-cobordism \( M_1 \): we may thus assume that \( M_1 \cup f_1 \circ p_1^{-1} \overline{N_g} \) is equivalent (and the respective homeomorphism is identity except in a collar neighbourhood of the top of \( M_1 \)) to a cobordism decomposed along a 2-sphere into \( M_1 \cup \text{restiction of } f_1 \circ p_1^{-1} D^L \) and \( D^M \cup D^U \cup p_1^{-1} \overline{N_g} \), the latter homeomorphic to a 3-ball; and that the corresponding triplet in \( S^3 \) has inside that 3-ball only a neighbourhood of the horizontal line of \( \Gamma_1 \) with \( R_1 \) of blackboard framing.

Hence \( (M_2 \cup f_2 \circ p_2^{-1} [D_g] \cup_{p_1 \circ (f_1)}^{-1} M_1, f_1, f_2) \) can be decomposed into three pieces: \( M_2 \cup \text{restiction of } f_2 \circ p_2^{-1} D^U \), \( D^M \), and \( D^L \cup \text{restriction of } p_1 \circ (f_1)^{-1} M_1 \), that have to be glued along two 2-spheres. Observe that \( D^M \) with the remaining tangle inside it is homeomorphic with \( B(0,2) - B(0,1) \) with tangle \( T_g \) inside. Therefore \( \text{12a} \), where all objects are as described in the statement, holds.

\[ \square \]

1.2 \( \mathbb{Q} \)- and \( \mathbb{Z} \)-cobordisms

For a chain graph \( \Gamma^g \) embedded in \( S^3 \), denote \( \mu_1, \ldots, \mu_g \) the meridians of the upper half-circles, defined by the right-hand rule, just as the meridians of oriented link components are defined.

**Proposition 4** Let \( \Gamma^g \) be embedded in an arbitrary way in \( S^3 \), then \( H_1(S^3 - \Gamma^g, \mathbb{Z}) \cong \mathbb{Z}^g \), with free generators \( \mu_1, \ldots, \mu_g \).

**Proof.** Either write down the exact homology sequence of the pair \( (S^3, S^3 - \Gamma^g) \) between \( H_2(S^3, \mathbb{Z}) = 0 \) and \( H_1(S^3, \mathbb{Z}) = 0 \); or use Alexander duality; or write down the Mayer-Vietoris sequence (for the reduced homology) of the decomposition \( S^3 - \Gamma^g = A \cup B \), where \( A \) is the complement in \( D^3 \) of the graph:

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]

and \( B \) is some embedding of \( \sqcup \mu I \) in \( S^3 - D^3 \), to get \( H_1(S^3 - \Gamma^g, \mathbb{Z}) \) as a quotient of \( H_1(A, \mathbb{Z}) \oplus H_1(B, \mathbb{Z}) \cong \mathbb{Z}^{2g-1} \oplus \mathbb{Z}^g \) through the image of \( H_1(A \cap B, \mathbb{Z}) \cong \mathbb{Z}^{2g-1} \) via the map given by the matrix above. \( \square \)

This proposition admits an obvious generalization to the case of several connected components \( \Gamma^{g'} \). Loosely speaking, it says that homology does not detect the horizontal lines.
Proposition 5 Suppose $M$ is a connected compact oriented 3-manifolds with two distinguished (not necessarily connected) boundary components $\partial M = (-S_1) \cup S_2$, let $f_1$, $f_2$ be parametrizations of these surfaces, let $N_1$ and $N_2$ be corresponding-to-the-genera disjoint unions of standard handlebodies, respectively anti-handlebodies, and let $i = (i_1, i_2) : \partial M \rightarrow M$ be the inclusion. The following conditions are equivalent:

1. $H_i(\tilde{M}, \mathbb{Z}) = 0$
2. $H_i(M, \mathbb{Z}) = i_*(H_1(\partial M, \mathbb{Z}))/ (f_1, -f_2)_* H_1(N_1 \cup -N_2, \mathbb{Z}))$

They imply:

3. $2 \cdot \text{rank } H_1(M; \mathbb{Z}) = \text{rank } H_1(\partial M; \mathbb{Z})$

Proof. We will prove this proposition for $S_1 \approx \Sigma_{g_1}$ and $S_2 \approx \Sigma_{g_2}$, $g_1, g_2 \geq 0$. The general case is absolutely analogous.\footnote{The case when $S_j = \emptyset$ follows readily from the case $S_j \approx S^2$. Our convention then is that $\emptyset_* = 0$, where $\emptyset$ is the topological map and $0$ is the algebraic one. Then $\text{Im}((i_j \circ f_j)_*) = 0$. In the case $S_1 = S_2 = \emptyset$, the condition (2) reads $H_1(M, \mathbb{Z}) = 0$.}

Applying Mayer-Vietoris to $\tilde{M} = (M \cup f_1, N_{g_1}) \cup -f_2(-N_{g_2})$, using the fact that $(M \cup f_1, N_{g_1}) \cap N_{g_2}$ is connected, and then the second isomorphism theorem, we obtain $H_1(\tilde{M}, \mathbb{Z}) \cong H_i(M \cup f_1, N_{g_1}, \mathbb{Z}) \oplus H_i(N_{g_2}, \mathbb{Z})$. In a similar fashion $M \cap N_{g_2} = \partial N_{g_2}$ is connected, and $H_1(M \cup f_1, N_{g_1}, \mathbb{Z}) \cong H_1(M, \mathbb{Z})$. Therefore $H_1(\tilde{M}, \mathbb{Z}) \cong \frac{H_1(M, \mathbb{Z})}{i_1*(H_1(S_1, \mathbb{Z})/ f_1, H_1(N_{g_1}, \mathbb{Z})) + i_2*(H_1(S_2, \mathbb{Z})/ (-f_2), H_1(N_{g_2}, \mathbb{Z}))}$, which proves (1) $\iff$ (2).

(1) $\implies$ (3) We will give a geometric proof, naturally extending linking relations from the case of closed 3-manifolds.\footnote{The case when $S_j = \emptyset$ follows readily from the case $S_j \approx S^2$.}

There is a link $L \subset S^3$ such that $\tilde{M} \approx S^3_L$. Moreover, this link can be taken disjoint from the embedding $\Gamma^{g_1} \sqcup \Gamma^{g_2} \rightarrow S^3$, such that (identifying as before $\Gamma^{g_1} \sqcup \Gamma^{g_2}$ with the remain after surgery on $L$) $\tilde{M} - N(\Gamma^{g_1} \sqcup \Gamma^{g_2}) \approx M$ for a tubular neighbourhood $N(\Gamma^{g_1} \sqcup \Gamma^{g_2})$. Hence, there is a neighbourhood $N(L \sqcup \Gamma^{g_1} \sqcup \Gamma^{g_2})$ such that $M$ is obtained from $S^3 - N(L \sqcup \Gamma^{g_1} \sqcup \Gamma^{g_2})$ by adding a 2-handle and a 3-handle for each component $L$. By proposition 4 (and the remark afterwards) $H_1(S^3 - N(L \sqcup \Gamma^{g_1} \sqcup \Gamma^{g_2}), \mathbb{Z}) \cong \mathbb{Z}^{\left|L\right|+g_1+g_2}$ with generators $\mu_1, \ldots, \mu_{\left|L\right|+g_1+g_2}$, the right-handed oriented meridians of the link components $K_j$, the upper-half-circles $U_j$ of $\Gamma^{g_1}$, and the lower half-circles $V_j$ of $\Gamma^{g_2}$. Therefore $H_1(M, \mathbb{Z})$ is a quotient of the former by $|L|$ relations, one for each 2-handle. If the component $K_i$ of $L$ has surgery coefficient (framing) $l_i$ and the preferred longitude $l_i$ (i.e. $l_i$ has framing 0 in $S^3$), then the corresponding relation is $l_i \mu_i + l_i^{*} = 0$. But $l_i$ is the boundary of a Seifert surface $F_i$ in $S^3$, punctured by the other components $K_j$, $j \neq i$, $U_j$ and $V_j$ (clearly $F_i$ can be taken disjoint from 3-balls, hence the intersection with $\Gamma^{g_1} - U_j$ and $\Gamma^{g_2} - V_j$ can be avoided). These intersections result in a surface $F'_i$ with additional boundary components homologous to $\pm \mu_j$. Therefore $F_i$ determines a relation $l_i = \sum_j l_k(K_j, K_i) \mu_j + \sum_j l_k(U_j, K_i) \mu_{\left|L\right|+j} + \sum_j l_k(V_j, K_i) \mu_{\left|L\right|+g_1+g_2}$ for the homology of $M$.

Hence $H_1(M, \mathbb{Z})$ is a quotient of $\mathbb{Z}^{\left|L\right|+g_1+g_2}$ through the image of $A : \mathbb{Z}^{\left|L\right|} \rightarrow \mathbb{Z}^{\left|L\right|+g_1+g_2}$ given by the $(|L|+g_1+g_2) \times |L|$-matrix $\left( \begin{array}{c} l_{ij} \\ l_{kij} \end{array} \right)$. (Here $l_{kij} = l_{ij}$.)

On the other side, adding a 2-handle (along the corresponding $\mu_j$) for each component $U_j$ and $V_j$, as well as a 3-handle for $\Gamma^{g_1}$ and 3-handle for $\Gamma^{g_2}$, one obtains $\tilde{M}$. At the level of homology this adds precisely the relations $l_j = 0$ for $j = |L| + 1, \ldots, |L| + g_1 + g_2$. This means that $H_1(M, \mathbb{Z})$ is a quotient of $\mathbb{Z}^{|L|}$ through the image of the linear homomorphism $B$ given in the $\mu$-basis by the $|L| \times |L|$-matrix $(l_{kij})$, the linking matrix of $L$. Since $H_1(M, \mathbb{Z}) = 0$, $(l_{kij})$ is unimodular, hence invertible (over $\mathbb{Z}$). Therefore there is a basis $(v_1, \ldots, v_{|L|})$ of the module freely generated by $\mu_1, \ldots, \mu_{|L|}$, such that the matrix representing $B$ in the new basis has the form $\left( \begin{array}{cc} \pm 1 & 0 \\ 0 & \pm 1 \end{array} \right)$. Correspondingly $A = B \otimes l_{kij}$ has in
the new basis the form \[
\begin{pmatrix}
\pm 1 & 0 & \cdots & 0 \\
0 & \ddots & \cdots & 0 \\
0 & \cdots & \pm 1
\end{pmatrix}
\]. Therefore, when writing down the relations in \( H_1(M, \mathbb{Z}) \)
for the system of generators \( \nu_1, \ldots, \nu_{|L|}, \mu_{|L|+1}, \ldots, \mu_{|L|+g_1+g_2} \), the generators \( \nu_j, j = 1, \ldots, |L| \) can be
eliminated, together with all relations, without adding any new relations. Hence \( H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{g_1+g_2} \).
freely generated by \( \mu_{|L|+1}, \ldots, \mu_{|L|+g_1+g_2} \). The statement (3) now follows.

**Proposition 6** Suppose \( M \) is a connected compact oriented 3-manifolds with two distinguished (not
necessarily connected) boundary components \( \partial M = (-S_1) \cup S_2 \), let \( f_1, f_2 \) be parametrizations of these
surfaces, let \( N_1 \) and \( \overline{N}_2 \) be corresponding-to-the-genera disjoint unions of standard handlebodies, respec-
tively anti-handlebodies, and let \( i = (i_1, i_2) : \partial M \hookrightarrow M \) be the inclusion. The following conditions are
equivalent:

1. \( H_1(\widehat{M}, \mathbb{Q}) = 0 \)
2. \( H_1(M, \mathbb{Q}) = i_* (H_1(\partial M, \mathbb{Q})/\langle f_1, -f_2 \rangle, H_1(N_1 \cup -\overline{N}_2, \mathbb{Q})) \)

They imply:

3. \( 2 \cdot \text{rank } H_1(M; \mathbb{Q}) = \text{rank } H_1(\partial M; \mathbb{Q}) \)

**Proof.** is identical to the proof of the previous proposition in every aspect, except that in the course of
proving (1) \( \implies \) (3), \( H_1(\widehat{M}, \mathbb{Q}) = 0 \) implies only that the matrix of \( B \) is invertible (over \( \mathbb{Q} \)). The basis \( \nu_1, \ldots, \nu_{|L|} \) is then over \( \mathbb{Q} \), i.e. it is a linear combination of \( \mu_1, \ldots, \mu_{|L|} \), but in general only with rational
coefficients. Again, these \( \nu \)-generators can be eliminated together with all relations. \( \Box \)

Of cause, the conditions (3) in propositions 5 and 6 are equivalent.

A 3-cobordism satisfying the equivalent conditions (1), (2) of Proposition 5 (respectively 6) will be
called a \( \mathbb{Z} \)-cobordism (respectively a \( \mathbb{Q} \)-cobordism). In both deinitions we have allowed one or both \( S_i \)
to be empty, although from the point of TQFT the case of empty top and/or bottom is indistinguished
from the case when that component is \( S^2 \).

**Corollary 7** If \( H_1(\widehat{M}, \mathbb{Z}) = 0 \), then \( H_1(M, \mathbb{Z}) \) is free of rank sum of the genera of its boundary compo-
ments. More generally, \( H_1(M, \mathbb{Z}) \) can have only the kind of torsion \( H_1(\widehat{M}, \mathbb{Z}) \) has.

**Proof.** The proof of proposition 6 can be repeated for \( \mathbb{Z}_p \). Hence \( \text{rank } H_1(M, \mathbb{Z}_p) = \text{rank } H_1(M, \mathbb{Q}) \)
for all \( p \) for which \( H_1(\widehat{M}, \mathbb{Z}_p) = 0 \), which implies that then \( p \)-torsion can not occur. In particular if \( M \) is a
\( \mathbb{Z} \)-cobordism, this is true for all \( p \). Apply the structure theorem for finitely generated abelian groups. \( \Box \)

In general (3) does not imply (1) in the statement of the above propositions. For example, let \( M \) be
the manifold obtained from \( S^3 \) by excising one component of the Hopf link in \( S^3 \) and performing surgery
on the other. Then (3) is true, while (1) is not. But, if we restrict to 3-cobordisms those equivalence
class is in the category \( \mathcal{L} \) below, then such phenomena are excluded a priori (Proposition 9).

For connected \( \partial M \) condition (2) clearly implies \( H_4(M; \partial M) = 0 \). However, for example \( \kappa(\emptyset, G, G') \)
where \( G \) and \( G' \) are the components of the two-component unlink in \( S^3 \), obviously satisfies (1), and hence
(2), but fails to satisfy \( H_4(M; S_i) = 0 \), the second condition from the definition of an \( h \)-cobordism \( \Box \).

### 1.3 Description of the categories

We will be interested in three categories, \( \mathcal{Q}, \mathcal{B} \subset \mathcal{L} \), which we now describe. Objects in each of these are
natural numbers. The morphisms between \( g_1 \) and \( g_2 \) are certain equivalence (homeomorphism) classes of
connected 3-cobordisms with connected bottom \( S_1 \) of genus \( g_1 \) and connected top \( S_2 \) of genus \( g_2 \). The
composition-morphism is the equivalence class of the 3-cobordisms in \( \mathcal{L} \). The equivalence classes of
the cobordisms \( ([D]_{a_1} + [P_1 + P_2]), g \geq 0 \) play the role of identity in these categories.

Let us first describe some additional objects. Let \( L^a, L^b \) be the submodules of \( H_1(\Sigma_g, \mathbb{Z}) \) generated
by \( a_i \)'s and \( b_i \)'s respectively. Each is a Lagrangian submodule with respect to the algebraic intersection
form \( \omega \) on \( H_1(\Sigma_g, \mathbb{Z}) \). Suppose \( M \) is a \( \mathbb{Q} \)-cobordism, with boundary \( (-S_1) \cup S_2 \). Let \( L_i, i = 1, 2 \) be
submodules of $H_1(S_i, \mathbb{Z})$, viewed inside $H_1(M, \mathbb{Z})$. We say that $L_1 \geq L_2$ in $M$, if every element in $L_2$ is $\mathbb{Z}$-homologically equivalent to an element in $L_1$. (Or, intuitively, $L_2$, which is in the homology of the component $S_2$, can be “moved” to the part $L_1$ of the homology of the component $S_1$.) Similarly we define $L_1 \leq L_2$. If $L_1 \geq L_2$ and $L_1 \leq L_2$, we have $L_1 = L_2$. Using the parametrizations of the boundary components, we can speak about the submodules $L_i^a, L_i^b$ of $H_1(S_i)$, $i = 1, 2$, which correspond to $L^a, L^b$.

The morphisms in $\mathcal{Z}$ from $g_1$ to $g_2$ are equivalence classes of $\mathbb{Z}$-cobordisms with boundary $(-S_1) \cup S_2$ such that $g(S_i) = g_i$ and

$$L_i^a \geq L_i^b, \quad \text{and} \quad L_1^b \leq L_2^b \quad (1.3)$$

Such $\mathbb{Z}$-cobordisms will be called semi-Lagrangian. An absolutely similar construction works for $\mathcal{Q}$, just replace $\mathbb{Z}$ by $\mathcal{Q}$.

**Example** In general condition (1.3) over $\mathbb{Z}$ is stronger than (1.3) over $\mathcal{Q}$. Let $(M, f_1, f_2)$ be a representative of the equivalence class of 3-cobordisms obtained by applying $\kappa$ to the triplet $(K = K_1 \cup K_2, L, U)$ shown in figure 6. Let $a, b, \mu_1, \mu_2$ be the corresponding meridians. Then $H_1(M, \mathbb{Z})$ is the abelian group with the classes of these as generators, and relations $\mu_1 + \mu_2 + a = 0$ and $-3\mu_2 + \mu_1 - a = 0$. They imply that $2(\mu_1 - \mu_2) = 0$, i.e. $\mu_1 - \mu_2$ is a torsion element. $L_1^a, L_1^b, L_2^a, L_2^b$ are generated respectively by 1 element each, the classes of respectively $a, L, U, b$. Note that $L$ is homologous to $\mu_1 - \mu_2 - b$, hence over $\mathbb{Z}$, $L_1^b \not\in L_2^b$, but over $\mathcal{Q}$, $\mu_1 - \mu_2 = 0$, hence $L_1^a = L_2^a$. In this example $H_1(M, \mathbb{Z}) \cong \mathbb{Z}_4$, i.e. $(M, f_1, f_2)$ is a $\mathcal{Q}$-cobordism, but not a $\mathbb{Z}$-cobordism. If one assumes, however, that $(M, f_1, f_2)$ is a $\mathbb{Z}$-cobordism, then conditions (1.3) over $\mathbb{Z}$ and $\mathcal{Q}$ are equivalent. This follows from (2) in proposition 5, and from the fact that for $\mathbb{Z}$-cobordisms, $L_2^a$ and $L_1^b$ are free (by Proposition 7).

Note that if condition (1.3) is satisfied with equalities, then necessarily $g_1 = g_2$. (1.3) is defined with the assumption that $M$ is already a $\mathbb{Q}$-cobordism.)

**Example** Condition (1.3) may hold with strict inclusion. Consider $\kappa(\emptyset, G, G')$, where $G$ is the braid-closure of a generator of the braid group $B_2$, and $G'$ is the braid axis (in $S^3$). In the case of integer homology both condition in (1.3) are strict, as it is easy to check. This example also shows that 3-cobordisms representing elements of the category $\mathcal{Z}$ do not necessarily satisfy $H_*(M; S_1) = 0$. (They are both $\cong \mathbb{Z}_2$ in this example.) It is not hard to see that if a homology-cobordism triad $(M, S_1, S_2)$ is enhanced with parametrizations such that we get a $\mathbb{Q}$-cobordism, then the second condition from the definition of an h-cobordism (1.3) implies (1.3), and in fact with equalities. Hence, if we restrict $\mathbb{Q}$-cobordism, requiring (1.3) contains all homology-cobordisms, and more.

**Proposition 8** The composition of two morphisms (say, class of $M$ and class of $N$) in category $\mathcal{Q}$ (resp. $\mathcal{Z}$) is again a morphism in the category $\mathcal{Q}$ (resp. $\mathcal{Z}$), i.e. $\mathcal{Q}$ and $\mathcal{Z}$ are categories.

**Proof.** The fact that both $M, N$ represent $\mathbb{Q}$-cobordism (respectively $\mathbb{Z}$-cobordism) means (by propositions 5 and 6) that all 1-dimensional homology (over $\mathbb{Q}$, respectively over $\mathbb{Z}$) in $M$ can be considered as
in the boundary. So when we glue \(M\) with \(N\) along a surface \(S\), the 1-dimensional homology is either in the top component of \(N\), in the bottom component of \(M\), or in the “middle” surface \(S\). But now, by condition [\textcolor{red}{9}], all the cycles of type \(L^a\) in \(S\) can be moved down to the bottom, and all the cycles of type \(L^b\) can be moved up to the top. So the 1-dimensional homology is still sitting on the boundary. Since for any 3-cobordism \((M, f_1, f_2)\), \(f_1 \circ f_1, H_1(N_{g_1}, \mathbb{Z}) = L^1_1\), and \((i_2 \circ f_2, H_1(N_{g_2}, \mathbb{Z}) = L^2_2\), using [\textcolor{red}{3}], one can verify condition (2) in Proposition 6 (respectively 5). Therefore \(N \cup_{f_2 \circ (f_1)^{-1}} M\) is a \(\mathbb{Q}\)-cobordism (respectively a \(\mathbb{Z}\)-cobordism). For \(N \cup_{f_2 \circ (f_1)^{-1}} M\) one can easily check [\textcolor{red}{3}]. □

The morphisms in the category \(\mathcal{L}\) we define to be the equivalence classes of 3-cobordisms of the form \((M = \Sigma_\mathcal{g} \times I, f \times 0, f' \times 1)\), where \(f, f' \in \text{Aut}(\Sigma_\mathcal{g})\), which are in \(\mathcal{Q}\). The following proposition shows that \(\mathcal{Q} \cap \mathcal{L} = 3 \cap \mathcal{L}\), hence to require the equivalence class of \((\Sigma_\mathcal{g} \times [0, 1], f, f')\) to be a morphism in category \(\mathcal{Q}\) or \(\mathcal{L}\) is equivalent. Clearly, in \(\mathcal{L}\) there are no morphisms between non-equal natural numbers. Recall that we use the following notation for the indices 1, 2 and 3:

**Proposition 9** Consider a 3-cobordism \(M = (\Sigma_\mathcal{g} \times [0, 1], f \times 0, f' \times 1)\), where the parameterization of the top differs by that of the bottom by the automorphism \(w = (f')^{-1} \circ f\). Then its equivalence class depends only on the isotopy class of \(w\) (i.e., we don’t need to specify both \(f, f'\)), and the following are equivalent:

1. the equivalence class of \((\Sigma_\mathcal{g} \times [0, 1], f \times 0, f' \times 1)\) is a morphism in the category \(\mathcal{L}\).
2. \(L^a = w_a(L^a)\) and \(L^b = w_a(L^b)\).

In particular, \(\widehat{M}\) is a \(\mathbb{Z}\)-homology sphere.

**Proof.** \((\Sigma_\mathcal{g} \times I, f \times 0, f' \times 1)\) is equivalent as a 3-cobordism to \((\Sigma_\mathcal{g} \times I, ((f')^{-1} \circ f) \times 0 = w \times 0, id \times 1)\), therefore \(\widehat{M} \cong (\Sigma_\mathcal{g} \times I, w \times 0, id \times 1)\). As a 3-manifold, \(\widehat{M}\) is homeomorphic to \(\overline{\Sigma_\mathcal{g}} \cup_w N_g\). Using Mayer-Vietoris theorem for the decomposition \(\overline{\Sigma_\mathcal{g}} \cup_w N_g\), we can see that condition (2) already ensures that \(\widehat{M}\) is a \(\mathbb{Z}\)-cobordism. Hence (2) \(\implies\) (1).

(1) \(\implies\) (2). Since \(w\) is an automorphism, \(w_a\) is an automorphism over \(\mathbb{Q}\) of each \(L^a, L^b\). On the other side \(w_a(L^a) \subset H_1(M, \mathbb{Z}) = L^a \oplus L^b\), and as a \(\mathbb{Z}\)-submodule of a free module, \(w_a(L^a)\) must be free. Using this and the fact that \(w_a(L^a) = L^a \oplus \mathbb{Q}\), we conclude \(w_a(L^a) = L^a \oplus \mathbb{Q}\).

Suppose a closed 3-manifold is the result of gluing a standard handlebody \(N_g\) to the standard anti-handlebody \(\overline{N}_g\) along a homeomorphism \(w\) of the standard surface \(\Sigma_g\), whose action on the homology (in the \(a_1, \ldots, a_g, b_1, \ldots, b_g\) basis) is given by a symplectic matrix \(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\). By Mayer-Vietoris theorem this 3-manifold is a ZHS if and only if \(A\) is invertible. But the set of symplectic matrices \(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\) with \(A\) invertible is not closed under multiplication, as it is easy to notice from the following example: \(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) \(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\). The result of gluing along each of the first two homeomorphisms (of \(S^1 \times S^1\)) is \(S^3\), while the result of gluing along their composition is \(S^1 \times S^2\).

**Corollary 10** The composition of two cobordisms \((\Sigma_\mathcal{g} \times I, f_1 \times 0, f'_1 \times 1) \cong (\Sigma_\mathcal{g} \times I, w_1 \times 0, id \times 1)\) and \((\Sigma_\mathcal{g} \times I, f_2 \times 0, f'_2 \times 1) \cong (\Sigma_\mathcal{g} \times I, w_2 \times 0, id \times 1)\) along \((f_2 \times 0) \circ ((f'_1)^{-1} \times 1)\) (respectively \((w_2 \times 0) \circ (id \times 1)^{-1}\)) is the 3-cobordism \((\Sigma_\mathcal{g} \times I, f_f \circ (f'_1)^{-1} \circ f_1 \times 0, f'_2 \times 1) \cong (\Sigma_\mathcal{g} \times I, f_2 \circ (f'_1)^{-1} \circ f_1 \times 0, id \times 1) \cong (\Sigma_\mathcal{g} \times I, (w_2 \circ w_1) \times 0, id \times 1)\). In particular, the composition of two morphisms of category \(\mathcal{L}\) is again a morphism in the same category. □

**Definition** Denote by \(\mathcal{L}_g\) the subgroup of the Mapping Class Group, consisting of isotopy classes of elements \(w \in \text{Aut}(\Sigma_\mathcal{g})\) such that \(w_1(L^a) = L^a\) and \(w_1(L^b) = L^b\) (over \(\mathbb{Q}\) or over \(\mathbb{Z}\), is equivalent by the previous proposition), and call it the Lagrangian subgroup of the MCG.

TQFTs based on \(\mathcal{Q}\) induce representations of \(\mathcal{L}_g\). This subgroup of \(\text{MC}(g)\) is big enough to be interesting, it contains the Torelli group. In fact its image under the action on homology is the group of matrices of the form \(\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}\), where \(A \in GL(g, \mathbb{Z})\). This subgroup of \(Sp(2g, \mathbb{Z})\) is not normal.
Proposition 9 above also shows that within this particular type \((M = \Sigma_g \times I\) and \(w \in L_g)\), statement (3) from propositions 5 and 6 implies statement (1).

**Remark.** Let \(\lambda\) denote the Casson invariant of homology 3-spheres. By fixing the standard handlebody of genus \(g\) in \(\mathbb{R}^3 \subset S^3\) we fixed a Heegaard homeomorphism that Morita \cite{Morita} calls \(\iota_g\), and by taking the filling \((\Sigma_g \times I, \phi, id)\) we obtain a manifold denoted by Morita \(W_\phi\). Every \(\varphi\) is a composition of Dehn twists \(\tau_{\pm 1}^\gamma\). Using Proposition 2, we can “insert” \([D_g, I_0]\) between every two twists, or put it another way, express \((\Sigma_g \times I, \phi, id)\) as a composition cobordisms \((\Sigma_g \times I, \tau_{\pm 1}^\gamma, id)\). Every twist can be replaced with \(\pm 1\)-surgery on a knot \(K_i\). Hence we obtain \(W_\varphi\) as surgery on the link \(L = K_1 \cup \ldots \cup K_n \cup L_0 \cup \ldots \cup L_{n-1}\), such that if one removes \(L_0\)’s, then the remaining \(K_1 \cup \ldots \cup K_n\) is split. If \(\varphi \in K_g\), the kernel of the Johnson homomorphisms, then it is a composition of Dehn twists \(\tau_{\pm 1}^\gamma\) on separating simple closed curves,\(^4\) i.e. \(\gamma\) has zero linking number with every circle component of \(L_0\)’s.

![Diagram](image.png)

**Figure 7:** \((L, G)\) less a tubular neighbourhood of the horizontal line of \(G\).

**Remark.** Let \(\mathcal{C}_\emptyset\) denote the set of connected 3-cobordisms with empty bottom and connected top. For \((M_1, \emptyset, f_1), (M_2, \emptyset, f_2) \in \mathcal{C}_\emptyset\) and \((L_1, G_1) \subset S^3\), \((L_2, G_2) \subset S^3\) such that \(\kappa(L_1, G_1) = (M_1, \emptyset, f_1), \kappa(L_2, G_2) = (M_2, \emptyset, f_2)\), we can remove a tubular neighbourhood of the horizontal line of \(G_1\) (= a ball in \(S^3\)), respectively of \(G_2\), and glue the two “boxes” as in figure 7, from left to right, afterwards filling back in the standard way a horizontal line. Denote the result by \((L_1 \cup L_2, G_1 \bullet G_2)\), and define:

\[
(M_1, \emptyset, f_1) \bullet (M_2, \emptyset, f_2) = \kappa(L_1 \cup L_2, G_1 \bullet G_2) \tag{1.4}
\]

Observe that the new 3-cobordism does not depend on the choice of pairs \((L_i, G_i)\). In the case of \(g = 0\), \(\bullet\) is the connected sum, i.e. this operation is another way (alternative to composition of cobordisms) of generalizing the connected sum. Note that \((M_1, \emptyset, f_1) \bullet (M_2, \emptyset, f_2) = (M_1, \emptyset, f_1) \# (M_2, \emptyset, f_2)\). In particular the sets \(\mathcal{C}_\emptyset \cap \{Z\text{-cobordism}\}\) and \(\mathcal{C}_\emptyset \cap \{Q\text{-cobordism}\}\) are closed under \(\bullet\). However, the Kontsevich-LMO invariant of 3-cobordisms \(\mathcal{B}L\mathcal{E}\) is not multiplicative with respect to \(\bullet\), except in some particular cases.

**Remark.** One can consider a slightly modified version of the category \(\Omega\), where the objects are connected parametrized surfaces (instead of natural numbers) and morphisms are certain classical (i.e. not parametrized) 3-cobordisms. Our choice here was motivated by our main example of TQFT (LMO), where \(\Omega\) gives the simplest formulations. In the next section one can replace it by a "\(\Omega\text{-like category}"\) instead.

\(^4\)The problem of finding an algorithm to express an arbitrary \(\varphi \in K_g\) as a product of \(\tau_{\pm 1}^\gamma\)’s is unsolved.

\(^5\)A pair \((L, G)\) is similar to a triplet \((\emptyset, L, G)\).
2 TQFTs for the categories \( \mathfrak{Q} \supset 3 \supset \mathfrak{L} \)

Define a TQFT \((\mathcal{T}, \tau)\) based on the cobordism category \(\mathfrak{Q}\) (or a subcategory of it, or a \(\mathfrak{Q}\)-like category) to be 1) a covariant functor \(\mathcal{T}\) from the category those objects are the objects of \(\mathfrak{Q}\) (i.e. natural numbers) and morphisms are the homeomorphisms of parametrized surfaces to a subcategory \(\mathfrak{K}\) of the category of \(K\)-modules, such that \(\mathcal{T}(\emptyset) = K\), where \(K\) is a commutative module with a conjugation operation; and 2) a map \(\tau\) that associates to each 3-cobordism \((M, f_1, f_2)\) a \(K\)-homomorphism \(\tau(M) : \mathcal{T}(\Sigma_1) \to \mathcal{T}(\Sigma_2)\), satisfying the following axioms:

\[\text{(A1) (Naturality)} \quad \text{If } (M_1, \Sigma_1, \Sigma_1'), (M_2, \Sigma_2, \Sigma_2') \text{ are two 3-cobordisms, and } f : M_1 \to M_2 \text{ is a homeomorphism of 3-cobordisms, preserving the parametrizations, then the following diagram is commutative:} \]
\[\begin{array}{ccc}
\mathcal{T}(\Sigma_1) & \xrightarrow{\tau(M_1)} & \mathcal{T}(\Sigma_1') \\
\mathcal{T}(f|_{\Sigma_1}) & \downarrow & \mathcal{T}(f|_{\Sigma_2}) \\
\mathcal{T}(\Sigma_2) & \xrightarrow{\tau(M_2)} & \mathcal{T}(\Sigma_2')
\end{array}\]

\[\text{(A2) (Functoriality)} \quad \text{If } M_1, M_2 \text{ are 3-cobordisms, } f = f_2 \circ (f_1')^{-1} : \partial_{\text{top}}(M_1) \to \partial_{\text{bottom}}(M_2) \text{ is the gluing homeomorphism, and denote } M = M_2 \cup_f M_1, \text{ then } \tau(M) = k \cdot \tau(M_2) \circ \tau(M_1). \]

\[\text{if } k \in K \text{ is called the anomaly.} \]

\[\text{(A3) (Normalization)} \quad \text{Let } (\Sigma \times [0, 1], (\Sigma \times 0, p_1), (\Sigma \times 1, p_2)) \text{ be the 3-cobordism mentioned on page \ref{page} then} \]
\[\tau(\Sigma \times [0, 1], (\Sigma \times 0, p_1), (\Sigma \times 1, p_2)) = \text{id}_{\mathcal{T}(\Sigma)} \]

\[\text{(A4) (pseudo-Hermitian structure)} \quad \text{There is a superstructure on each element } V \text{ of } \mathfrak{B}_K, \text{ i.e. it admits an antimorphism } \tau : V \to V^{*} \text{ (a map linear in } 0\text{-supergrading and antilinear in } 1\text{-supergrading), that commutes with } (= \text{ is natural with respect to) surface homeomorphisms. There is a canonical map } V \to V^{*}.\text{, which composed with the above antimorphism extends from the particular case when } \mathcal{T}(\Sigma_1) = K \text{ to an antimorphism } \tau : \text{Mor}(\mathcal{T}(\Sigma_1), \mathcal{T}(\Sigma_2)) \to \text{Mor}(\mathcal{T}(-\Sigma_2), \mathcal{T}(-\Sigma_1)), \text{ that commutes with homeomorphisms of } 3\text{-cobordisms, such that} \]
\[\tau(-M) = \tau(M) \]

We can not require multiplicativity or self-duality since in the category \(\mathfrak{Q}\) all cobordisms are connected. Conditions (A1-A3) say that \(\tau : \mathfrak{Q} \to \mathfrak{A}\) is a pseudo-functor. \(\tau\) would be a true functor when there is no anomaly. If the set of \(\tau(M, S^2, \Sigma)\)'s, spans (in the closure for infinite-dimensional modules) \(\mathcal{T}(\Sigma)\), the TQFT is called non-degenerate.

2.1 The behavior of the signature and the determinant

Proposition 11 Let \((M_1, f_1, f_1')\) and \((M_2, f_2, f_2')\) be two 3-cobordisms. Suppose \((M_1, f_1, f_1') = \kappa(L_1, G_1, G'_1), (M_2, f_2, f_2') = \kappa(L_2, G_2, G'_2), \text{ and } (M_2 \cup f_2 \circ (f_1')^{-1}, M_1, f_1, f_2') = \kappa(L_1 \cup S(L_1) \cup L_2, G_1, G'_2), \text{ the later triplet obtained from the previous two by the construction described in Proposition 3. Denote } \sigma_+ = \text{sign}_+(\text{l}(L_1)), \]
\[\sigma_+ = \text{sign}_+(\text{lk}(L_2)), \sigma_+ = \text{sign}_+(\text{lk}(L_1 \cup S(L_1) \cup L_2)), \text{ and let } q \text{ be the genus of the connected closed surface along which is this splitting. Then the integer } s(M, M_1, M_2) := \sigma_+ + \sigma'_+ + g - \sigma_{2+} \text{ is an invariant of the decomposition } M = M_2 \cup f_2 \circ (f_1')^{-1} M_1, \text{ i.e. it does not depend on the choice of triplets representing the } 3\text{-cobordisms } M_1 \text{ and } M_2. \]

Proof. Suppose we have another such choice of triplets. The new unoriented links \(L_1'\) and \(L_2'\) are related to \(L_1\) and \(L_2\) by a finite sequence of Kirby moves and changes of orientations of link components. Each such move can be thought of as also a move from \(L_1' \cup S(L_1) \cup L_2' \) to \(L_1 \cup S(L_1) \cup L_2\). If a K-1 move changes \(\sigma_{2+}\) by \(\pm 1\), then so does it to \(\sigma_{2+}\), and hence \(\sigma_+ + \sigma'_+ + g - \sigma_{2+}\) remains unchanged. Similarly happens if a K-1 move changes \(\sigma_{2+}\). K-2 moves do not affect the signature. Indeed, recall that the operations of the type “add a \(j^{th}\) row and column to an \(i^{th}\) row an column” (which corresponds to sliding \(i^{th}\) component over
the \(^{jth}\) component of the link), do not change the signature of a symmetric matrix. They correspond to re-writing a symmetric bilinear form in a different basis, and Sylvester’s inertia theorem applies. If the orientation of a link component of \(L_1\) or \(L_2\) changes, it corresponds to multiplying both linking matrices \(\text{lk}(L_i)\) and \(\text{lk}(L_1 \cup L_0 \cup L_2)\) on the left and on the right by a diagonal matrix with all entries 1 except one entry \(-1\). Hence again by Sylvester’s theorem all signatures involved are invariant. \(\square\)

Note that \(s(M, M_1, M_2)\) is not an invariant of a triad (in the sense of Milnor [12], i.e. if one ignores parametrizations.

Let \((L, G, G')\) be a triad and \((M, f, f') = \kappa(L, G, G')\). Recall that (see Proposition 5) we can talk about linking number between a link component \(K\) and a circle \(U\) of a chain graph, as well as between two circles \(U\) and \(V\) of chain graphs: \(\text{lk}(K, U) = \text{lk}(U, K)\) is the linking number between \(K\) and the knot obtained from the graph by deleting all but the circle component \(U\); and similarly for \(\text{lk}(U, V)\). We can then define the linking matrix of a triad:

\[
\text{lk}(L, G, G') = \begin{pmatrix}
\text{lk}(L, G) & \text{lk}(L, G') \\
\text{lk}(G, L) & \text{lk}(G, G') \\
\text{lk}(G', L) & \text{lk}(G', G')
\end{pmatrix} = \begin{pmatrix}
A & B^T & C^T \\
B & D & E^T \\
C & E & F
\end{pmatrix}
\]

(2.1)

where \(A, D, F\) are symmetric matrices.

Let \(\mu\) be the column-vector consisting of the meridians of \(L, m\) the column-vector of the meridians of \(G, \) and \(m'\) be the column-vector of the meridians of \(G'\). Then \(H_1(M, Z)\) is the \(Z\)-module generated by the elements of \(\mu, m, m'\) with \(|L|\) relations – the elements of the column-vector \(A\mu + B^T m + C^T m'\). The semi-Lagrangian conditions \(L^b_1 \leq L^b_2, L^g_2 \leq L^g_1\) can be expressed \(Z < B\mu + Dm + E^T m' > Z < m'\), respectively \(Z < C\mu + Em + Fm' > Z < m >\). If \(\tilde{M}\) is a \(Q\)- or \(Z\)-homology sphere, \(A\) is invertible over \(Q\). Moreover, \(A^{-1} \in \mathcal{M}_{|L| \times |L|}(\tilde{Z})[\frac{1}{\text{det} A}]\). Therefore \(\mu = -a^{-1}B^T m - A^{-1}C^T m'\). (This equality of column-vectors with entries in \(\tilde{Z}\) has to be read over \(Z\), i.e. that multiplying an entry on the left with the denominator of the corresponding entry on the right gives the numerator on the right side.) Hence the semi-Lagrangian conditions can be expressed \(Z < (D - BA^{-1}B^T)m + (E^T - BA^{-1}C^T)m' > Z < m'\), respectively \(Z < (E - CA^{-1}B^T)m + (F - CA^{-1}C^T)m' > Z < m >\), i.e.

\[
\begin{align*}
D &= BA^{-1}B^T \\
F &= CA^{-1}C^T
\end{align*}
\]

(2.2)

(for \(Q\)-cobordisms this in particular means that the entries on the left-hand side, a priori in \(\tilde{Z}\), must be in \(Z\)), and for \(Q\)-cobordisms additionally:

\[
BA^{-1}C^T \in \mathcal{M}_{g_1 \times g_2}(Z)
\]

(2.3)

We will need the following elementary

**Lemma 12** The signature of a symmetric \(2g \times 2g\)-matrix \(\begin{pmatrix} A & -I \\
-I & 0 \end{pmatrix}\) with integer, respectively real entries is \((g, g)\). The determinant of such a matrix is \((-1)^g\). \(\square\)

With these notations, Proposition 3 and [22] imply that the linking matrix \(\text{lk}(L_1 \cup L_0 \cup L_2)\) is:

\[
\text{lk}(L_1 \cup L_0 \cup L_2) = \begin{pmatrix}
A & B^T & 0 & 0 \\
B & BA^{-1}B^T & -I & 0 \\
0 & -I & DC^{-1}D^T & D \\
0 & 0 & D^T & C
\end{pmatrix}
\]

(2.4)

where \(A = \text{lk}(L_1) \in \mathcal{M}_{|L_1| \times |L_1|}(Z), C = \text{lk}(L_2) \in \mathcal{M}_{|L_2| \times |L_2|}(Z), B = \text{lk}(G', L_1) \in \mathcal{M}_{g \times |L_1|}(Z), D = \text{lk}(G, L_2) \in \mathcal{M}_{g \times |L_2|}(Z), BA^{-1}B^T, DC^{-1}D^T \in \mathcal{M}_{g \times g}(Z).\) With the same notations:

**Proposition 13** The signature of the matrix [22] is \((\sigma_1 + \sigma_2^2 + g, \sigma_1^2 + \sigma_2^1 + g)\), where \((\sigma_1, \sigma_2)\), respectively \((\sigma_1^2, \sigma_2^2)\) is the signature of \(\text{lk}(L_1)\), respectively \(\text{lk}(L_2)\). Also the following holds:

\[
\det(\text{lk}(L_1 \cup L_0 \cup L_2)) = (-1)^g \cdot \det(\text{lk}(L_1)) \cdot \det(\text{lk}(L_2))
\]

(2.5)
Proof. By the classification of quadratic forms with real coefficients there exist matrices \( X \in SL([L_1], \mathbb{R}) \), \( Y \in SL([L_2], \mathbb{R}) \) such that \( XAX^T \) and \( YCY^T \) are diagonal. Since \( A \) and \( C \) are symmetric, so are \( BA^{-1}B^T \) and \( DC^{-1}D^T \), hence there exist \( P, Q \in SL(g, \mathbb{R}) \) such that \( PBA^{-1}B^TP^T \) and \( QDC^{-1}D^TQ^T \) are diagonal. Then:

\[
\begin{pmatrix}
X & 0 & 0 & 0 \\
0 & P & 0 & 0 \\
0 & 0 & Q & 0 \\
0 & 0 & 0 & Y \\
\end{pmatrix}
\begin{pmatrix}
A & B^T & 0 & 0 \\
B & BA^{-1}B^T & -I & 0 \\
0 & -I & DC^{-1}D^T & D \\
0 & 0 & 0 & C \\
\end{pmatrix}
\begin{pmatrix}
X^T & 0 & 0 & 0 \\
0 & P^T & 0 & 0 \\
0 & 0 & Q^T & 0 \\
0 & 0 & 0 & Y^T \\
\end{pmatrix}
= 
\begin{pmatrix}
XAX^T & XB^TP^T & 0 & 0 \\
PBX^T & PBA^{-1}B^TP^T & 0 & 0 \\
0 & 0 & -Q^TP & 0 \\
0 & 0 & 0 & QDC^{-1}D^TQ^T \\
\end{pmatrix}
\begin{pmatrix}
D_1 & F_1^T & 0 & 0 \\
F_1 & D_2 & \mathcal{E}^T & 0 \\
0 & \mathcal{E} & D_3 & F_2 \\
0 & 0 & 0 & D_4 \\
\end{pmatrix}
\begin{pmatrix}
I & -D_1^{-1}T & F_1T & 0 \\
0 & D_1 & 0 & 0 \\
0 & 0 & \mathcal{E} & D_3 \\
0 & 0 & 0 & D_2 \\
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 & 0 \\
0 & \mathcal{E} & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
\end{pmatrix}
\begin{pmatrix}
D_1 & 0 & 0 & 0 \\
0 & D_2 - F_1D_1^{-1}F_1^T & \mathcal{E}^T & 0 \\
0 & \mathcal{E} & D_3 - F_2D_4^{-1}F_2^T & 0 \\
0 & 0 & 0 & D_4 \\
\end{pmatrix}
\]

where \( D_i \) are diagonal matrices, \( D_1 \) and \( D_4 \) necessarily invertible, and \( \mathcal{E} \in SL(g, \mathbb{R}) \). Therefore:

\[
\begin{pmatrix}
I & 0 & 0 & 0 \\
0 & \mathcal{E} & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
\end{pmatrix}
\begin{pmatrix}
0 & \mathcal{E} & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
0 & 0 & 0 & I \\
\end{pmatrix}
= 
\begin{pmatrix}
D_1 & 0 & 0 & 0 \\
0 & D_2 - F_1D_1^{-1}F_1^T & \mathcal{E}^T & 0 \\
0 & \mathcal{E} & D_3 - F_2D_4^{-1}F_2^T & 0 \\
0 & 0 & 0 & D_4 \\
\end{pmatrix}
\]

But \( D_2 - F_1D_1^{-1}F_1^T = PBA^{-1}B^TP^T - PBX^T \cdot (X^T)^{-1}A^{-1}X^{-1} \cdot XB^TP^T = 0 \), and similarly \( D_3 - F_2D_4^{-1}F_2^T = 0 \). Also observe that

\[
\begin{pmatrix}
I & 0 \\
0 & -\mathcal{E} \\
\end{pmatrix}
\begin{pmatrix}
0 & \mathcal{E}^T \\
\mathcal{E} & 0 \\
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & -\mathcal{E}^T \\
\end{pmatrix}
= 
\begin{pmatrix}
0 & -I \\
-I & 0 \\
\end{pmatrix}
\]

The later matrix, by Lemma 12, has signature \((g, g)\) and determinant \((-1)^g\). Therefore the signature of the matrix \([23]\) is \((\sigma_+^1 + \sigma_+^2 + g, \sigma_-^1 + \sigma_-^2 + g)\). Since all conjugations above where by matrices of determinant \(\pm 1\), the determinant of the original matrix \([23]\) remained unchanged, i.e. we also have the relation \([23]\).

As immediate consequences we obtain the two central results of this paper:

**Corollary 14** For semi-Lagrangian \(Q\)-cobordisms the integer \(s(M, M_1, M_2)\) is always equal to 0.

**Corollary 15** For semi-Lagrangian \(Q\)-cobordisms \((M, f_1, f_2)\) the cardinality of \(H_1(\widehat{M}, \mathbb{Z})\) is multiplicative with respect to the composition of cobordisms.

**Remark.** For the category \((C_0, \cdot)\) the same properties, with the integer \(s := \sigma_+^1 + \sigma_+^2 - \sigma_-\), are obvious.

### 2.2 Consequences

Set \(T(f|\Sigma) = id_{T(g)}\) for any homeomorphism \(f\) of the parametrized surfaces. Then \(T\) is a covariant functor, and the naturality axiom \((A1)\) is obvious.

**Basic example.** Let \(N\) be an arbitrary integer, and \(\tau(L) \in K^*\) an invariant of framed links, also invariant under the second Kirby move, and under changing orientation of link components. It is obvious that, if the linking matrix of \(L\) is non-singular and has signature \((\sigma_+, \sigma_-)\), and \(M \equiv S^3_L\), then \(\tau(M) = |H_1(M, \mathbb{Z})| \cdot \frac{\tau(L)}{\tau(D^0)\tau(D^1)}\) is an invariant of rational homology 3-spheres.
Assume the link invariant has two additional properties:

\[
\tau(T) = \overline{\tau(L)}
\]

\[
\tau(L_1 \cup L_0 \cup L_2) = \tau(L_1) \tau(L_2) \tau(O^{+1})^g \tau(O^{-1})^g
\]  

(2.6)

(2.7)

for any link \(L, g \in \mathbb{N}\), and link \(L_1 \cup L_0 \cup L_2\) obtained from \(L_1\) and \(L_2\) as follows: add arbitrary \(g\) components to \(L_i\), and join the two links along the \(g\) additional components, preliminary inserting the tangle \(T_g\) (figure 4b) in between. The second property generalizes multiplicativity under connected sum. The first property implies that \(\tau(M) = \overline{\tau(M)}\).

Let \(T(g)\), \(g \geq 1\) be the \(K\)-vector space freely generated by homeomorphism classes of semi-Lagrangian \(\mathbb{Q}\)-cobordisms of the form \((P, s, h), s \cup h : \Sigma_0 \cup \Sigma_g \xrightarrow{\sim} \partial M\). The relation \([P, s, h] = [P, s, h]\) defines a natural antiharmonic on \(T(g)\). Defines \(\tau(M, f, f')\) be the \(K\)-homomorphism sending \([P, s, h] \in T(g)\)

\(\tau(M, f, f')\) to \(\overline{\tau(M)} \cdot [M \cup_{\text{framed}} P, s, f'] \in T(g')\).

(A2) Observe that \(\tau(\sigma_g \times [0, 1], p_1, p_2)\) sends \([P, s, h]\) to \(\tau(S^3) \cdot [P, s, h] = [P, s, h]\);

(A3) If \(M_1\) and \(M_2\) are two cobordisms from \(\Omega\), then \(\tau(M_2 \cup M_1)\) sends \([P, s, h]\) to \([H_1(M_2 \cup M_1)]^N \cdot \tau(L_1) \tau(L_2) \tau(O^{+1})^g \tau(O^{-1})^g \cdot [M_2 \cup M_1 \cup P, s, f'_2]\), while \(\tau(M_2) \circ \tau(M_1)\) sends it to \([H_1(M_2)]^N \cdot [H_1(M_1)]^N \cdot \tau(L_1) \tau(L_2) \tau(O^{+1})^g \tau(O^{-1})^g \cdot [M_2 \cup M_1 \cup P, s, f'_2]\). These are equal by \(\hat{\mathbb{E}}\) and Corollaries 14 and 15;

(A4) For \([Q, r, j]^* \in T(g)\) define the dual map \([Q, r, j]^*\) to send \([P, s, h]\) to \(\tau(\hat{Q}) \cdot \tau(-Q \cup_{\text{framed}} P, s, -r)\).

Note that if one represents \((M, f, f')\) by a triplet (Proposition 1), takes a generic projection on \(\mathbb{R}^2\), change all crossing to their opposites, and apply \(\kappa\), one obtains the cobordism \((-M, -f', -f)\). Therefore defining \(\tau(M, f, f') = \tau(-M, -f', -f)\) we obtain for the case when the domain of \(f\) is \(\Sigma_0\) the composition \(\text{proj}(\hat{\tau}(\tau))\), where \(\text{proj} : T(\partial) \to K, \text{proj}[P, s, r] = \tau(\hat{P})\).

If we replace \(T(\partial)\) by its image through \(\text{proj}\), the above data defines a non-degenerate anomaly-free TQFT on \(\Omega\).

Of cause, given a specific \(\tau\), one should define \(T(g)\), \(\tau(M, f, f')\), the composition and conjugation in a way directly related to \(\tau\) of links. Non-degeneracy should also be addressed in \(\tau\)-specific terms. In \(\mathbb{E}\) we use the results of this paper to construct a TQFT for the Le-Murakami-Ohnuki invariant of \(\mathbb{Q}\), as well as for its degree truncations.

Remark. Throughout this paper we have taken oriented chain graphs. This is motivated by the fact that for our main example (LMO) absence of orientation would induce unjustified complications. However for the correspondence in Proposition 1 here it suffices to consider banded (rather than framed) graphs with a specification of g meridional disks for its circle components (which would replace our notion of "horizontal line").

The TQFTs based on \(\Omega\), when restricted to \(\mathfrak{L}\) produce linear representations \(\mathcal{L}_g \to \text{GL}_K(T(g))\). The group \(\mathcal{L}_g\) has not been studied before, no set of relations\(^6\) is known. Even, if we restrict to Torelli group, although a finite set of generators for \(T_g\) is well-known \(\mathcal{H}\), existence of a finite presentation is an open problem.

References

[1] M.Atiyah, Topological Quantum Field Theories, Publications Mathématiques IHES 68, 175-186 (1988)
[2] C.Blanchet, H.Habegger, G.Masbaum, P.Vogel, Topological quantum field theories derived from the Kauffman bracket, Topology 34, no.4, 883-927 (1995)
[3] D.Cheptea, T.Le, A TQFT associated to LMO invariant of three-dimensional manifolds, math.GT.0508220 v2
[4] F.Deloup, An explicit construction of an abelian topological quantum field theory in dimension 3, Topology and Its Applications 127, no 1-2, 199-211 (2003)

\(^{6}\)A set of generators can be easily obtained by taking products of generators of \(T_g\) and of \(\text{GL}(2, \mathbb{Z})\)
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[5] A.Fomenko, S.Matveev, Algorithmic and computer methods for three-manifolds, Kluwer Academic Publishers (1997)
[6] S.Gervais, A finite presentation of the mapping class group of a punctured surface, Topology 40, no.4, 703-725 (2001)
[7] R.E.Gompf, A.I.Stipsicz, 4-manifolds and Kirby calculus, Graduate Studies in Mathematics 20, AMS (1999)
[8] K.Habiro, Claspers and finite-type invariants of links, Geometry and Topology 4, 1-83 (2000)
[9] D.Johnson, The structure of the Torelli group I: A finite set of generators for \( \mathcal{T} \), Annals of Mathematics 118, 423-442 (1983)
[10] T.T.Q.Le, J.Murakami, T.Ohtsuki, On a universal perturbative invariant of 3-manifolds, Topology 37, no.3, 539-574 (1998)
[11] S.V.Matveev, Generalized surgery of three-dimensional manifolds and representations of homology spheres, Matematicheskie Zametki 42, no.2, 268-278 (1986)
[12] J.Milnor, Lectures on the h-cobordism theorem, Princeton University Press (1965)
[13] S.Morita, Casson's invariant for homology 3-spheres and characteristic classes of surface bundles I, Topology 28, no.3, 305-323 (1989)
[14] J.Murakami, T.Ohtsuki, Topological Quantum Field Theory for the Universal Quantum Invariant, Commun. Math. Phys. 188, 501-520 (1997)
[15] V.Turaev, Quantum Invariants of Knots and 3-Manifolds, Walter de Gruyter (1994)
[16] P.Vogel, Invariants de type fini, en “Nouveaux Invariants en Géométrie et en Topologie”, publié par D. Bennequin, M. Audin, J. Morgan, P. Vogel, Panoramas et Synthèses 11, Société Mathématique de France, 99-128 (2001)

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