On Crossing Symmetry and Modular Invariance in Conformal Field Theory and S-Duality in Gauge Theory

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In this note, we explore the relation between crossing symmetry and modular invariance in conformal field theory and S-duality in gauge theory. It is shown that partition functions of different S dual theories of $N=2$ $SU(2)$ gauge theory with four fundamentals can be derived from the crossing symmetry of the Liouville four point function. We also show that the partition function of $N=4$ $SU(2)$ gauge theory can be derived from the Liouville partition function on torus.

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I. INTRODUCTION

Historically, string theory is originated as an attempt to provide a theory of strong interactions. The famous Veneziano amplitude is shown to be derivable from the vibrating string which sweeps out a two dimensional world sheet in space time. The scattering amplitude is calculated from the correlator function on the Riemann surface which can be regarded geometrically as punctured Riemann surface. Two dimensional conformal field theory (CFT) plays a central role in the development of the string theory. The most important dynamical principle of this approach is the associativity of the operator product expansion (OPE) which in turn leads to the crossing symmetry of the four particle scattering amplitude. This crossing symmetry is the underlying reason to eliminate the high energy divergences of tree level diagrams. On the other hand, modular invariance of the one-loop partition function of conformal CFT leads a miraculous way to eliminate high energy divergences of one loop diagrams in string theory.

Furthermore, people were trying to understand the strong coupled dynamics of four dimensional quantum field theory from the Strong-Weak duality (S duality) of the quantum field theory itself. During the past two decades, we gained a lot of understandings about four dimensional gauge theories from the S dualities. S dualities are also related to Geometric Langlands Duality.

These two seemingly unrelated subjects are put together by Gaiotto’s observation on $N=2$ superconformal field theory (SCFT)(see recent developments in [1, 2, 3, 4, 5, 6, 7, 8, 9]). It is shown that four dimensional $N=2$ SCFT can be realized as six dimensional $(0,2)$ theory compactified on a punctured Riemann surface. The coupling constants are determined by the complex structure of this punctured Riemann surface while different S dual frames are determined by various degeneration limits of this surface. The flavor groups are determined by the puncture types which are labeled by Young Tableaux. Everything about the gauge theory is encoded into this punctured Riemann surface. Recalling our discussion about the conformal field theory, it is natural to wonder if there is any connection between those two subjects.

AGT [10] proposed a remarkable conjecture that for $SU(2)$ theory the partition function of the gauge theory is equivalent to the correlation function of the Liouville theory. Liouville theory falls into the general framework of CFT [11]. One may wonder why a four dimensional theory can be equivalent to a two dimensional theory. This might be related to a remarkable property of the Liouville theory: the central charge is adjustable. Then the degrees of freedom of Liouville theory is rather mysterious and makes a four dimensional correspondence possible.

It is interesting to further explore the correspondence between gauge theory and Liouville theory. In this note, we make the observation that we can determine the transformation law under S-duality of the partition function of $N=2$ gauge theory with four fundamentals from the crossing symmetry of the Liouville four point function. The transformation law for $N=4$ $SU(2)$ theory under S-duality can be determined similarly from the one-loop partition function of Liouville theory.

II. CROSSING SYMMETRY AND MODULAR INVARIANCE IN CONFORMAL FIELD THEORY

Let’s first review how the dual string model was proposed. Consider an elastic scattering amplitude with incoming spinless particles of momentum $p_1, p_2$ and outgoing spinless particles of momentum of $p_3, p_4$ (see Figure 1). The conventional Mandelstam variables are

$$s = -(p_1 + p_2)^2, \ t = -(p_2 + p_3)^2, \ u = -(p_1 + p_4)^2. \ (1)$$

Consider first the $t$ channel contribution. There are various particles with mass $M_J$ and spin $J$ which might...
FIG. 1: An elastic scattering with incoming particles of momentum $p_1$, $p_2$ and outgoing particles of momentum $p_3$, $p_4$. We indicate the contribution from $s$ channel and $t$ channel. The field theory amplitude is constructed from the sum of these contributions.

be exchanged:

$$A(s, t) = - \sum_J \frac{g_J^2(-s)^J}{t - M_J^2}. \quad (2)$$

We have the following similar amplitude if we consider $s$ channel:

$$A'(s, t) = - \sum_J \frac{g_J^2(-s)^J}{s - M_J^2}. \quad (3)$$

Two remarkable properties of the scattering amplitude are that the above sums are infinite and these two amplitudes are equal to each other $A(s, t) = A'(s, t)$. The last property which is called $s - t$ duality motivates the proposed Veneziano amplitude.

It is well known that the Veneziano amplitude can be derived from two dimensional (world sheet) string theory. The infinite sum is due to the infinite number of states in mass spectrum. The $s - t$ duality is simply the crossing symmetry of the four point function of conformal field theory. This crossing symmetry is also equivalent to the associativity of the OPE on the world sheet:

$$A_i(\zeta)A_j(0) = \sum_k C_{ij}^k(\zeta)A_k(0). \quad (4)$$

We briefly summarize some properties of CFT. For the Virasoro algebra is

$$[L_n, L_m] = L_{m+n} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}. \quad (5)$$

here $c$ is the central charge and $L_m$ are generators of conformal symmetry. The representations of this algebra are labeled by primary states which satisfy:

$$L_0|V_\alpha > = \Delta_\alpha |V_\alpha >, \quad L_n|V_\alpha > = 0, \quad n > 0, \quad (6)$$

$\Delta_\alpha$ is the conformal dimension of this primary state. The other states of this representation are represented as:

$$L_{-k_n}L_{-k_{n-1}}...L_{-k_1}|V_\alpha >, \quad (7)$$

here $k_n \geq k_{n-1}... \geq k_1$. These secondary states have conformal weights $\bar{\Delta} = \Delta_\alpha + |Y|$; here $|Y|$ is the total boxes of the Young Tableaux with rows $k_1, ..., k_n$. The correlation functions involving the energy momentum tensor and secondary states are expressed by the correlation functions of the primary states.

The OPE of two primary states are given as $[11]$;

$$\phi_m(z, \bar{z})\phi_n(0, 0) = \sum_p c^p_{nm} z^{\Delta_p - \Delta_n - \Delta_m} \bar{z}^{\bar{\Delta}_p - \bar{\Delta}_n - \bar{\Delta}_m} \psi_p(z, \bar{z}, 0, 0). \quad (8)$$

The most important dynamical information is $c^p_{nm}$ and the conformal dimensions.

The four point function has the form

$$G^{lk}_{nm}(x, \bar{x}) = \langle k|\phi_l(1, 1)\phi_n(x, \bar{x})|m >, \quad (9)$$

here we fixed the positions of three vertex operators as $0, 1, \infty$ and $x$ is the projective invariant variable. The crossing symmetry is

$$G^{kl}_{nm}(x, \bar{x}) = G^{ml}_{nl}(1-x, 1-\bar{x}) = x^{-2\Delta_n + 2\bar{\Delta}_n}G^{lk}_{nm}(1-x, 1-\bar{x}) \quad (10)$$

Using OPE of $\phi_n, \phi_m$, the four point function is

$$G^{lk}_{nm}(x, \bar{x}) = \sum_p c^p_{nm}c_{lp}F_{nm}(p|x)\bar{F}_{lp}(\bar{p}\bar{|x}), \quad (11)$$

here $F$ is the conformal block which is entirely determined by the conformal symmetry (we give the $s$ channel contribution). The crossing symmetry relates the contributions of different channels (see for instance Figure 2):

Next let’s consider the one-loop partition function defined on a torus:

$$Z(\tau) = \sum_i q^{h_i - c/24}\bar{q}^{\bar{h}_i - c/24}(-1)^{F_i}, \quad (12)$$

here $i$ runs over all the states in CFT and $F_i$ is the fermion number, $q = \exp(2\pi i \tau)$. It is well known that this partition function is needed to be invariant under the $SL(2, Z)$ modular group transformation of the torus. The high energy density states of this theory is determined by the central charge for compact unitary conformal field theory. For Liouville theory, there is a modification to this result, see [18].
are given by \( N \) original theory. This proposal is naturally realized in which the elementary particles are the monoples of the \( SL(1) \) to a \( N = 2 \) this corresponds to different weakly coupled S-dual theory of this theory is \( \epsilon_1 = \epsilon_2 = \frac{1}{b} \).

Next, let’s study the \( N = 4 \) \( SU(2) \) theory. It is given by the six dimensional \( A_1 \) theory compactified on a smooth torus. The \( SL(2, Z) \) duality of the gauge theory is interpreted as the \( SL(2, Z) \) modular invariance of this torus. The full partition function of this theory is of the same form as formula (10). It is interesting to note that for \( N = 4 \) \( U(M) \) gauge theory the one-loop part is trivial \( Z_{1-\text{loop}} = 1 \) and the full partition function for \( U(M) \) gauge theory is:

\[
Z = C \frac{1}{|\eta(\tau)|^{2M}(2\pi\sqrt{2})^M}. \tag{18}
\]

where \( \eta(\tau) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k) \), \( q = e^{2\pi i \tau} \).

IV. S DUALITY FROM THE CONFORMAL FIELD THEORY

AGT \cite{15} made a conjecture that the full partition function of the above \( N = 2 \) \( SU(2) \) SCFT is equivalent to the correlation function of Liouville theory. It is shown in \cite{15} that the instanton part of the gauge theory partition function is equal to the conformal block of the correlation function and the one-loop part and classical part correspond to the structure constant part of the correlation function. It is also argued that the energy momentum tensor of Liouville theory is related to the operator \cite{13} (this can be seen from the classical uniformization problem with the punctured sphere).

The relation between the deformation parameters and the parameters in Liouville field theory is

\[
e_1 = b, \quad e_2 = \frac{1}{b}, \tag{19}
\]

here \( e_1 \) and \( e_2 \) are the deformation parameters in Nekrasov’s instanton partition function. Notice that in order to use the partition function on \( S^4 \), we need to set \( b = 1 \).

We associate an exponential vertex operator \( e^{a_\phi} \) to each puncture. We also associate a intermediate state \( e^{a_\phi} \) to weakly coupled \( SU(2) \) group with Coulomb parameter \( a \). The exact relations between the parameters in gauge theory and Liouville theory are

\[
\alpha_1 = m_a + \frac{Q}{2}, \quad \alpha_2 = m_b, \quad \alpha_3 = m_c + \frac{Q}{2}, \quad \alpha_4 = m_d.
\]

Here \( Q \) is the conventional parameter for the Liouville theory \( Q = b + \frac{1}{b} \). See Figure 4 for the correspondence.

Now the crossing symmetry \cite{10} of CFT states that the correlation function of different channels are related.
When we consider the gauge theory, the different channels mean different S dual frames (see Figure 3). With the identification between the partition function of gauge theory and correlation function of CFT, we conclude that:

The partition function of this four dimensional SCFT in different S dual frames are related as in formula (10). Notice that the gauge coupling is identified with the position of the unfixed coordinate of the vertex operator, and the second identity in (10) relates theories with gauge couplings \( q = \frac{1}{q'} \). So this identity implies that the partition function of one strongly coupled theory is determined by another weakly coupled theory as predicted by S-duality! We have determined the exact relations between them from Liouville theory.

Next let’s consider \( N = 4 \) SU(2) theory, then we are tempted to identify the gauge theory partition function with the partition function of Liouville theory, the Liouville partition function can be calculated from (12):

\[
Z(\tau) = V_\phi \frac{1}{2\pi \sqrt{\tau_2} |\eta(\tau)|^2}  
\]  

(20)

here \( V_\phi \) is the zero mode contribution and is independent of \( \tau \), this is identified with the gauge theory partition function; we also need to identify \( \alpha = 1 + a \) for intermediate state and the gauge coupling is \( \frac{1}{g_{YM}^2} = \tau_2 \) which is consistent with our previous identification. Comparing to (18) with \( M = 2 \), we can see that the \( U(1) \) part contribution to gauge theory partition function is

\[
Z^{U(1)} = C' \frac{1}{2\pi \sqrt{\tau_2} |\eta(\tau)|^2}. 
\]  

(21)

This result can be generalized to SU\( (N) \) case in which the \( A_{N-1} \) conformal Toda field theory is involved. The one-loop partition function of \( A_{N-1} \) theory is equivalent to \( N - 1 \) free scalars so the partition function has a factor \( \frac{1}{2\pi \sqrt{\tau_2} |\eta(\tau)|^2} \), comparing with (18), the \( U(1) \) part contributes a factor \( \frac{1}{2\pi \sqrt{\tau_2} |\eta(\tau)|\tau} \).

V. CONCLUSION

In this note, we derive the transformation law for the partition function of certain SU(2) SCFT under the S duality transformation from Liouville theory point of view. It is easy to generalize to higher rank gauge theory (for higher rank gauge theory, the CFT side is the conformal Toda field theory [22]).

Consider a six dimensional \((0,2)\) theory compactified on a punctured Riemann surface \( \Sigma \), we get a four dimensional gauge theory on a four manifold \( M \); on the other hand, we can first compactify six dimensional theory on a four manifold \( M \) (with possible singularities), we can get a CFT on \( \Sigma \). It is interesting to identify \( \Sigma \) and \( M \) and the corresponding gauge theory and conformal field theory. We may also have the beautiful correspondence between gauge theory and CFT.

Finally, Liouville theory plays a fundamental role in non-critical string theory, and it has interesting application to cosmology[23]. Any understanding of the dynamics of the theory is interesting. The dynamical information of the theory is encoded in the structure constant of the theory. This can be calculated from the one-loop part of the partition function. It is interesting to see what we can learn from gauge theory about the property of Liouville theory, for instance: the spectrum, the Seiberg bound...

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