Research Article

Stability Analysis of the Rhomboidal Restricted Six-Body Problem

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1.Introduction

In the restricted n-body problem, a body of infinitesimal mass moves under the gravitational influence of n − 1 massive bodies called primaries. The force exerted by the infinitesimal mass on the primaries can be ignored as it has a negligible mass, whereas primaries revolve perpetually in concentric circles around their common center of mass by retaining a particular central configurations (CCs). The CC play a vital role in the understanding of the n-body problem in celestial mechanics. It can be used to find simple or special solutions of the n-body problem since the geometry formed by the arrangement of the primaries remains constant for all time.

The CC for n > 3 has been investigated extensively after being identified as the problem of the 21st century by Arnold [1]. Xia [2] used the method of analytical continuation to find exact number of central configurations for n positive masses. Hampton and Moeckel [3] studied the finiteness of relative equilibria of the four-body problem and showed that the number of relative equilibria is always between 32 and 8472 for the Newtonian four-body problem. The homographic solutions to the rhombi four-body problem are the variational minimizers of the Lagrangian action confined on a holonomically constrained rhombus loop space, according to Mansur and Offin [4]. The approach used by Mansur et al. [5] to demonstrate spectral instability for the entire parameter ranges of mass ratio and eccentricity e ≥ 0 for the homographic family of rhombus solutions within the four degree of freedom parallelogram four-body problem is based on a topological invariant called the Maslov index. Shi and Xie [6], using analytical approach, have shown that there is exactly one family of concave and one family of convex central configurations in addition to the family of equilateral triangle configurations. Llibre and Mello [7] classified the central configurations of the four-body problem. Liu and Zhou [8] investigated the four-body problem with three masses forming a Lagrangian triangle, used the bifurcation diagram of linearly stable and unstable regions, and found two linearly stable subregions with respect to \(\alpha, \beta,\) and \(e\). Deng et al. [9] investigated the CC of the four-body problem with equal masses and showed that, for the planar
Newtonian four-body problem having adjacent equal masses i.e., \( m_1 = m_2 \neq m_3 = m_4 \) and equal lengths for the two diagonals, any convex noncollinear CC must have a symmetry and must be an isosceles trapezoid. They also showed that when the length between \( m_1 \) and \( m_2 \) equals the length between \( m_3 \) and \( m_4 \), the CC is also an isosceles trapezoid. Marchesin [10] investigated the rhomboidal configuration stability with a central mass and two pairs of equal masses. The mass \( m_5 \) is at the center of the configuration, and the equilibria obtained in this case were all shown to be unstable. Shoaib et al. [11] established the central configuration for the rhomboidal 5-body problem and highlighted the regions in the phase plane where it is possible to have central configuration. On the axis of symmetry, Shoaib et al. [12] considered a symmetric five-body problem with three unequal collinear masses. The remaining two masses were symmetrically arranged on both sides of the axis of symmetry, and areas of feasible central configurations were identified analytically and numerically for the rhomboidal and triangular four- and five-bodies. Using Levi-Civita type transformations, the equations of motion were regularized, and the phase space for chaotic and periodic orbits was explored using the Poincaré surface of sections. Zotos [13] numerically explored the restricted four-body problem with three equal masses with a dynamically stable triangular configuration and found that the linearly stable Lagrange points only exist when one of the three masses has a considerably larger mass. Dewangan et al. [14] investigated the elliptic restricted four-body problem by considering radiation and oblateness effects; they considered a bigger primary as a radiation source and the other primaries to be of equal masses as oblate spheroid. They found that the equilibrium points to be linearly stable. Liu et al. [15] studied the four-body problem and found that the boundaries of possible motions obey the change in parameter \( c^2E \), that is, if the value of \( c^2E \) is less than or equal to a critical value \( c^2E_{cr} \), then the system is stable. Ismail et al. [16] studied the four-body problem by considering the effects of radiation pressure and oblateness and used the Lyapunov function to show the stability of equilibrium points. Wang and Gao [17] did a numerical study of the restricted five-body problem regarding the zero velocity surface and transfer trajectory by considering four equal masses (primaries) forming a regular tetrahedron configuration and the fifth (infinitesimal) mass moving under the gravitational influence of the four primaries. They numerically simulated the zero velocity surface of the infinitesimal mass in the three-dimensional space and designed the transfer trajectory of the infinitesimal mass. Suraj et al. [18] studied the five-body problem to investigate the effects of perturbation parameter on the positions, motion, and stability of the libration points due to the variable mass of the fifth mass. Li and Liao [19] obtained 695 families of Newtonian periodic planar collisionless orbits of three-body systems with equal mass and zero angular momentum numerically. In the planar restricted three-body problem, Sosnitskii [20] investigated Lagrange stability and proved a theorem on the Lagrange stability of the infinitesimal particle particularly for the circular restricted three-body problem. Ding et al. [21] proved that there exists another Eulerian collinear central configuration at any instant in the three-body problem. Using the port-Hamiltonian approach, Liu and Dong [22] reformulated the Circular Restricted Three-Body Problem (CRTPB) and obtained the closed loop Hamiltonian by designing a control strategy (based on energy shaping and dissipation injection) as a candidate of the Lyapunov function that assures asymptotic stability. This control technique also demonstrated global stability within the CRTPB model's application region. Corbera et al. [23] established that every four-body convex center configuration with perpendicular diagonals will have a kite configuration. Lara and Bengochea [24] investigated the symmetric periodic orbits of the four-body system both theoretically and numerically. Alvarez-Ramírez and Medina [25] studied the planar restricted five-body problem in which the four primaries form a axisymmetric four-body central configuration and described the equilibrium points that depend on the mass parameters of the primaries.

In the six-body problem, Mello et al. [26] demonstrated the existence of three new families of stacked spatial central configurations. Alsadi et al. [27] used variational methods and computational algorithm to investigate the six-body problem and its new style periodic solutions. Idrisia and Ullah [28] studied the CC of the restricted six-body problem with the central body and showed that all the libration (equilibrium) points exist on the concentric circles \( C_1 \), \( C_2 \), and \( C_3 \) having center at the origin. The libration points that lie on circles \( C_1 \) and \( C_3 \) are unstable, while there are some stable libration points on circle \( C_2 \).

In the present paper, we consider a restricted six-body problem, where four of the primaries are at the vertices of a rhombus and the fifth mass is at the intersection of the two diagonals. Three of the primaries have equal masses of \( m_1 = m_2 = m_3 = m \) and are located on the horizontal axis, and two other equal mass primaries are located on the vertical axis with masses \( m_4 = m \). Before dividing the equations of motion for the restricted 6-body in Section 3, we find continuous families of central configurations for the rhomboidal 5-body problem. The rest of the paper is organized as follows: we investigate the Hill region and possible region of motion of \( m_5 \) according to the Jacobi constant in Section 4. In Sections 5 and 6, we show the existence, uniqueness, and stability of equilibrium points, respectively. Conclusions are given in Section 7.

### 2. Rhomboidal Central Configurations

In this section, we prove the existence and uniqueness of central configuration of a rhomboidal 5-body problem for positive masses. The mass ratio is written as a function of “a” and “b” which can be used to find regions of central configuration for the rhomboidal 5-body problem. The classical equation of motion for the \( n \)-body problem has the following form:

\[
m_i \mathbf{r}_i = \sum_{j=0, j \neq i}^{n} m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^3},
\]
where the units are chosen so that the gravitational constant is equal to one. A central configuration is a particular configuration of the \( n \) bodies where the acceleration vector of each body is proportional to its position vector, and the constant of proportionality is the same for the \( n \) bodies. Therefore, a CC is a configuration that satisfies the following equation:

\[
-\omega^2 (\mathbf{r}_i - \mathbf{c}) = \sum_{j=0, j \neq i}^{n} \frac{m_j m_i (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3},
\]

where \( \omega \) is angular speed and

\[
\mathbf{c} = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{\sum_{i=1}^{n} m_i},
\]

which is the center of mass of the \( n \) bodies. We take the position vector (see Figure 1) of the five primaries \( m_j \), where \( j = 0, 1, \ldots, 4 \) as

\[
\mathbf{r}_0 = 0,
\]

\[
\mathbf{r}_1 = ae^{i\omega t},
\]

\[
\mathbf{r}_2 = -ae^{i\omega t},
\]

\[
\mathbf{r}_3 = bei^{i\omega t},
\]

\[
\mathbf{r}_4 = -bei^{i\omega t}.
\]

For \( i = 1 \) and \( 3 \) in equation (2), the CC equation for masses \( m_1 \) and \( m_3 \) is

\[
-\omega^2 (a, 0) = \frac{m(-a, 0)}{a^3} + \frac{m(-2a, 0)}{8a^3} + \frac{\tilde{m}(-a, b)}{(a^2 + b^2)^{3/2}} + \frac{\tilde{m}(-a, -b)}{(a^2 + b^2)^{3/2}}
\]

\[
-\omega^2 (0, b) = \frac{m(0, -b)}{b^3} + \frac{m(a, -b)}{(a^2 + b^2)^{3/2}} + \frac{m(-a, -b)}{(a^2 + b^2)^{3/2}} + \frac{m(0, -2b)}{8b^3}
\]

where \( m_1 = m_2 = m_0 = m \) and \( m_3 = m_4 \). The positions of the primaries are taken from equation (4). Writing equations (5) and (6) in components’ form, we get

\[
\omega^2 = \frac{5m}{4a^3} + \frac{2\tilde{m}}{(a^2 + b^2)^{3/2}}
\]

\[
\omega^2 = m \left( \frac{1}{b^3} + \frac{2}{(a^2 + b^2)^{3/2}} \right) + \frac{\tilde{m}}{4b^3}.
\]

Because of the symmetry of the problem, the equations for \( m_2 \) and \( m_4 \) are identical to equations (7) and (8). The equation for \( m_0 \) is identically zero. We have extended the astrophysical model given in [29] by keeping the mass \( m_0 \) at the center. The results of the model given in [29] can be reproduced by taking \( m_0 = 0 \) in our model. Without loss of generality, we take \( \omega = 1 \) and solve equations (7) and (8) simultaneously for \( m(a, b) \) and \( \tilde{m}(a, b) \):

\[
m(a, b) = \frac{N_m(a, b)}{D_m(a, b)},
\]

\[
\tilde{m}(a, b) = \frac{N_{\tilde{m}}(a, b)}{D_m(a, b)},
\]

where

\[
N_m(a, b) = 4a^3 \left( a^2 + b^2 \right)^{3/2} \left( (a^2 + b^2)^{3/2} - 8b^3 \right),
\]

\[
N_{\tilde{m}}(a, b) = 4(a^2 + b^2)^{3/2} \left( -4a^2 - 8a^2 b^2 + 5a^4 b^3 - 6a^3 b^4 + 10a^2 b^5 - 8a^3 b^3 \sqrt{a^2 + b^2} + 5b^2 \right),
\]

\[
D_m(a, b) = -32a^7 - 64a^5 b^2 - 32a^3 b^4 + 5a^6 b^2 + 15a^4 b^4 + 15a^2 b^4 + 5b^6 \left( \sqrt{a^2 + b^2} \right).
\]

Defining \( \tau = (\tilde{m}/m) \), then equations (9)–(11) give

\[
\tau(a, b) = \frac{(4 - (5b^3/a^3))(1 + (b^2/a^2))^{3/2} + 8(b^4/a^3)}{8(b^3/a^3) - (1 + (b^3/a^2))^{3/2}}.
\]

Again, let \( \mu = (b/a) \) in equation (13); we get an alternate form of equation (13):

\[
\tau(\mu) = \frac{8\mu^3 + (4 - 5\mu^3)(1 + \mu^2)^{3/2}}{8\mu^3 - (1 + \mu^2)^{3/2}}.
\]

**Lemma 1.** \( \tau(\mu) \) given by (13) is a continuous and strictly decreasing positive function in the interval \((1/\sqrt{3}, 1.1394282249562009)\), \( \lim_{\mu \rightarrow 1/\sqrt{3}} \tau(\mu) = \infty \), and \( \tau(1.1394282249562009) = 0 \).
Proof. One can easily check the continuity and \( \lim_{\tau \to 0} = 1 \). From equation (14), \( \mu(1.394282249562009) = 0 \) directly from equation (14). It means the value of \( \tau(\mu) \) which is the ratio of the distance parameters varies between \( \left( \frac{1}{\sqrt{3}} \right), 1.394282249562009 \). Now we need to prove that the function is decreasing and positive for the given interval. Taking the derivative of \( \tau(\mu) \),

\[
\tau'(\mu) = 5\kappa(\mu),
\]

\[
\kappa'(\mu) = \frac{15\mu^2 \sqrt{\mu^2 + 1} \left( (\mu^2 + 1)^{3/2} - 8(\mu^2 + 1) \right)}{(\mu^2 + 1)^{3/2} - 8\mu^3},
\]

where \( \tau'(\mu) \) is a constant multiple of \( \kappa'(\lambda) \) given in [29] by Kulesza et al. The proof will therefore be similar and can be seen in [29].

Here, we present some particular cases for different values of \( \tau \) and \( \mu \). We show these different shapes of rhombus for different values of \( \tau \) and \( \mu \) in Figure 2.

(i) If \( \tau = 1 \), then \( \mu = 1 \); from equation (13), we get the shape of true square with \( m = \bar{m} = 1 \) and \( a = b = 1 \)

(ii) If \( \tau = 0.67 \), then \( \mu = 1.04232 \); from equation (13), we get \( \bar{m} = 0.67m \) and \( b = 1.04232a \)

(iii) If \( \tau = 0.97 \), then \( \mu = 1.00367 \); from equation (13), we get \( \bar{m} = 0.97m \) and \( b = 1.00367a \)

There are two cases in which the RR6BP central configuration degenerates.

(i) If \( \mu = 1.394282249562009 \), then \( \tau = 0 \); this implies \( \bar{m} = 0 \) and \( b = 1.394282249562009a \)

(ii) If \( \mu = 1/\sqrt{3} \), then \( \tau = \infty \); this implies \( m = 0 \) and \( b = (1/\sqrt{3})a \)

3. Equations of Motion

In this section, we describe the motion of the infinitesimal body, \( m_5 \), under the gravitational attraction of the five primaries. We assume that the sixth body has a significantly smaller mass compared to the masses of the primary \( (m_5 \ll m_0, m_1, m_2, m_3, m_4) \). On this basis, the sixth body acts as an infinitesimal test particle, and therefore, it does not influence the motion of the five primaries. In the RR6BP, the equations of motion of \( m_5 \) are

\[
r_5 = \sum_{j=0}^{4} m_j r_j - r_5, \quad \text{dot} \quad r_5 - r_j
\]

where dot represents the derivative with respect to time. From here onward, without loss of generality, we take the value of \( a = 1 \). The equations of motion of \( m_5 \) in the synodical coordinates \( x \) and \( y \) are

\[
\dot{x} - 2y = U_x, \quad \text{and} \quad y + 2x = U_y,
\]

where

\[
U(x, y) = \frac{(x^2 + y^2)}{2} + m \left( \frac{1}{r_{50}} + \frac{1}{r_{51}} + \frac{1}{r_{52}} + \frac{1}{r_{53}} + \frac{1}{r_{54}} \right),
\]

which is the effective potential. The mutual distances of \( m_5 \) from the primaries in the corotating frame are

\[
\begin{align*}
    r_{50} &= \sqrt{x^2 + y^2}, \\
    r_{51} &= \sqrt{(x-1)^2 + y^2}, \\
    r_{52} &= \sqrt{(x+1)^2 + y^2}, \\
    r_{53} &= \sqrt{x^2 + (y-b)^2}, \\
    r_{54} &= \sqrt{x^2 + (y+b)^2}.
\end{align*}
\]

The Jacobian constant is given by Curtis [30]:

\[
C + U = \frac{1}{2} \left( x^2 + y^2 \right) = v^2.
\]

For a given value of the Jacobi constant, \( v^2 \) is only a function of position in the rotating frame. Since \( v^2 \) cannot be negative, it must be true that

\[
C + U \geq 0.
\]

The boundaries between forbidden and allowed regions of motion are found by setting \( v^2 = 0 \), i.e.,

\[
C + U = 0.
\]
It is now trivial to show that $C(x, y)$ is the first integral of motion of system (11) by proving that $\dot{C}(x, y) = 0$.

4. The Hill Regions

The region of permitted motion is also known as the Hill region, and the curves found by equation (16) for various values of $C$ are known as the zero velocity curves.

The zero velocity curves when $b = 0.67$, $m = 0.07717$, and $\bar{m} = 0.7879$ and when $b = 0.97$, $m = 0.4588$, and $\bar{m} = 0.5766$ are given in Figure 4, and the corresponding Hill regions are given in Figure 3. We also give the region of possible motion of $m_5$ for six different values of Jacobi constants $C$ in Figures 5–7 for mass parameters $b = 0.67, 0.97$, and 1.13. The shaded regions represent the forbidden regions of motion for the infinitesimal mass $m_5$. It is numerically confirmed that the permitted regions are completely disconnected for $C \geq -2.20, -2.85$, and $-2.56$, as shown in Figures 5–7, for the above values of $b$. For the increasing values of $C$, the allowed region of motion (white region) becomes partially connecting at $C = -2.574, -3.328$, and $-3.342$ and completely connected at $C = -3.2, -3.9$, and $-3.5$. It can be seen from Figures 5–7 that the transition of motion from totally disconnected to completely connected occurs in six stages for $b = 0.67, 0.97$, and 1.13. For these values of $b, m_5$ can freely move in the gravitational field of the CC region for $C \geq -2.2, -2.85$, and $-2.56$ and cannot reach any of the primaries for $C \leq -3.2, -3.9$, and $-3.5$.

5. Equilibrium Solutions

Equilibrium solutions of the RR6BP are the solutions of $U_x(x, y) = 0$ and $U_y(x, y) = 0$. The derivative $U_x$ and $U_y$ of the effective potential, given in equation (19), are found as follows:

$$U_x = x - \frac{mx}{(x^2 + y^2)^{3/2}} - m \left( \frac{x - a}{(x - a)^2 + y^2}^{3/2} + \frac{x + 1}{(x + 1)^{3/2} + y^2} \right)$$

$$- \bar{m} x \left( \frac{1}{(y - b)^2 + x^2}^{3/2} + \frac{1}{(y + b)^2 + x^2}^{3/2} \right)$$

$$U_y = y - \frac{my}{(x^2 + y^2)^{3/2}} - my \left( \frac{1}{(x - 1)^{3/2} + y^2}^{3/2} + \frac{1}{(x + 1)^2 + y^2} \right)$$

$$- \bar{m} \left( \frac{y - b}{(y - b)^2 + x^2}^{3/2} + \frac{b + y}{(y + b)^2 + x^2}^{3/2} \right).$$

5.1. Equilibrium Solutions on the Coordinates’ Axes. Since the potential given in equation (19) is invariant under the symmetry $(x, -y), (-x, y)$, and $(-x, -y)$, we will restrict our computation to the first quadrant: $x \geq 0$ and $y \geq 0$. Initially, we study the existence and number of equilibrium solutions on the axes and then off the coordinate axes. To study the equilibrium solutions on the $y$-axis, let $x = 0$; then, equations (24) and (25) are given as

$$U_x = 0,$$

$$U_y = y - \frac{my}{(y^2)^{3/2}} - \frac{2my}{(1 + y^2)^{3/2}} + \bar{m} \left( \frac{1}{(y - b)^2 + x^2}^{3/2} + \frac{b + y}{(y + b)^2 + x^2} \right).$$

To solve $U_x = 0$, divide $y$ into subintervals $0 < y < b$ and $y > b$. 

5.2. $0 < y < b; \ (1/\sqrt{3}) < b < 1.1394282249562009$. Rewrite the right hand side of equation (25) by taking into account that $y \in (0, b)$:

$$f_1(y) = y - \frac{m}{y^2} - \frac{2my}{(1 + y^2)^{3/2}} + \bar{m} \left( \frac{1}{(y - b)^2} - \frac{1}{(y + b)^2} \right).$$

At $y = 0$, $f_1(y) < 0$; and, at $y = b$, $f_1(y) > 0$; therefore, by the mean value theorem, there is at least one zero of $f_1(y)$ when $y \in (0, b)$. The derivative of $f_1(y)$ is given by

$$\frac{df_1(y)}{dy} = 1 + 2m \left( \frac{1}{y^2} + \frac{3y^2}{(y^2 + 1)^{3/2}} \right) + 2\bar{m} \frac{1}{(y + b)^2} - \frac{1}{(y - b)^2}.$$
The only term \((-3y^2/(y^2 + 1)^{3/2})\) that can make \(df_1(y)/dy\) negative for \(y \in (0, b)\) is dominated by the rest of the term in equation (26); therefore, \((df_1(y)/dy) > 0\). This proves the existence of the unique equilibrium solution inside rhombus on the \(y\)-axis.

5.3. \(y > b\). Now, consider the case \(y > b\). Use equation (26) and rewrite as

\[
f_2(y) = y - \frac{m}{y^2} - \frac{2my}{(1 + y^2)^{3/2}} - \bar{m}\left(\frac{1}{(y-b)^2} + \frac{1}{(y+b)^2}\right).
\]

(29)

At \(y = b\), \(f_2(y) < 0\); and, at \(y = \infty\), \(f_2(y) > 0\); therefore, by mean value theorem, there is at least one zero of \(f_2(y)\) when \(y \in (b, \infty)\). The derivative of \(f_2(y)\) is given by

\[
\frac{df_2(y)}{dy} = 1 + 2m\left(\frac{1}{y^3} + \frac{1}{(y^2 + 1)^{3/2}} - \frac{3y^2}{(y^2 + 1)^{3/2}}\right)
\]

\[+ 2\bar{m}\left(\frac{1}{(y+b)^3} + \frac{1}{(y-b)^3}\right).
\]

(30)

Following the same procedure as given for \(0 < y < b\), one can easily prove the uniqueness of the equilibrium solution for \(y > b\).

We discuss here two special cases of CC for \(b \in ((1/\sqrt{3}), 1.1394282249562009)\). When \(b = (1/\sqrt{3})\), then the masses on the horizontal axis are zero, i.e., \(m = 0\). In this case, we only get the two equilibrium points along \(y\)-axis. When \(b = 1.1394282249562009\), then \(\bar{m} = 0\). In this case, we get four equilibrium points along \(x\)-axis. The positions of the masses and the corresponding equilibrium points for these two cases are shown in Figure 8. The stability of these cases will be discussed in Section 6.

5.4. Equilibrium Solutions off the Coordinates Axes. It is numerically confirmed that, for \(b = 0.67, 0.97,\) and \(1.13\), there are always a total of 12 equilibrium points. As shown in Figure 9, four of the equilibrium points are on the \(x\)-axis, four on the \(y\)-axis, and remaining four of the equilibrium points are off the axes. Since the gravitational field is a function of mass parameters \(m(b)\), therefore the equilibrium points change their positions around the primaries for changing values of \(b\). It is numerically confirmed that majority of equilibrium points are around the primaries along the horizontal axis if \(b\) is around \((1/\sqrt{3})\) as the masses on the horizontal axis are dominant (see Figures 9(a)). For \(b \geq 1\), the equilibrium points concentrated around the primaries on the vertical axis (see Figure 9(b)).

6. Stability Analysis

To study the stability of the equilibrium points obtained in the previous section, we will follow the standard linearization procedure by linearizing the equation of motion of the infinitesimal mass. Let the location of an equilibrium point in the RR6BP be denoted by \((x, y)\), and consider a small displacement \((X, Y)\) from the point such that \(x + X\) and \(y + Y\) are the new position of the infinitesimal. Using Taylor’s series expansion in equations (17) and (18), we obtain a new set of second-order linear differential equations:

\[
\begin{align*}
\ddot{X} - 2Y &= UX_{xx} + YU_{xy}, \\
\ddot{Y} + 2\dot{X} &= UX_{xy} + YU_{yy}.
\end{align*}
\]

(31)

The matrix form of the linearized equations is

\[
\begin{pmatrix}
\ddot{X} \\
\ddot{Y}
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
\dot{X} \\
\dot{Y}
\end{pmatrix}.
\]

(32)
Figure 4: The evolution of zero velocity curves. (a) $b = 0.67; m = 0.07717; \bar{m} = 0.7879$. (b) $b = 0.97; m = 0.4588; \bar{m} = 0.5766$.

Figure 5: The regions of motion of $m_3$ (white region) for energy interval $C \in (-3.2, -2.20)$ when $b = 0.67$ from the top-left corner to the bottom-right corner. (a) $C = -3.2$. (b) $C = -3.157$. (c) $C = -2.574$. (d) $C = -2.2832$. (e) $C = -2.269$. (f) $C = -2.20$. 
Figure 6: The regions of motion of $m_5$ (white region) for energy interval $C \in (-3.5, -2.85)$ when $b = 0.97$ from the top-left corner to the bottom-right corner. (a) $C = -3.5$. (b) $C = -3.47$. (c) $C = -3.328$. (d) $C = -3.1868$. (e) $C = -3.121$. (f) $C = -2.85$.

Figure 7: Continued.
These equations can also be written in the following matrix form:

\[ \Psi = \mathcal{A} \Psi, \quad (33) \]

where

\[
\Psi = \begin{pmatrix} \dot{\chi} \\ \dot{\gamma} \\ \dot{x} \\ \dot{y} \end{pmatrix},
\]

\[
\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & 0 & -2 \end{pmatrix}.
\]

The characteristic polynomial for \( \mathcal{A} \) is

\[ \Lambda^4 + \alpha \Lambda^2 + \beta = 0, \quad (35) \]

where \( \alpha = 4 - U_{xx} - U_{yy} \) and \( \beta = U_{xx} U_{yy} - U_{xy}^2 \). Let \( \Lambda^2 = \lambda \); then, equation (35) reduces to

\[ \lambda^2 + \alpha \lambda + \beta = 0. \quad (36) \]

Now, in order for a Lagrange point to be linearly stable to a small perturbation, all four roots, \( \Lambda \), of equation (35) must be purely imaginary. Thus, in turn, it implies that the two roots of equation (36),

\[ \lambda_{\pm} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}, \quad (37) \]
must be real and negative. For $\lambda \neq 0$, we must have
\begin{equation}
(i) \alpha > 0 \text{ and } 0 < \beta \leq \frac{\alpha^2}{4}, \\
(ii) \alpha > 0 \text{ and } \alpha^2 - 4\beta = 0.
\end{equation}

Figure 9: (a–c) 12 equilibrium points (red colors) for different values of $b = 0.67, 0.97$, and 1.13, respectively.

We will numerically identify regions when either condition (i) or condition (ii) is satisfied. Figures 10(a) and 10(b) (left to right) give the stability region in case (i) and case (ii), and their projections for $b = 0.67, b = 1$, and $b = 1.13$ are shown in Figures 11 and 12, respectively.

We have tested a large number of equilibrium points for many values of $b$ and found all of them unstable. In other words, the intersection of $U_x = 0$ and $U_y = 0$ within
\begin{equation}
\{\alpha > 0 \text{ and } 0 < \beta \leq (\alpha^2/4)\} \text{ and } \{\alpha > 0 \text{ and } \alpha^2 - 4\beta = 0\}
\end{equation}
is an empty set. Representative examples are given in Tables 1–3.
Figure 10: (a-b) Shaded regions represent stability regions for cases (i) and (ii), respectively.

Figure 11: Continued.
Figure 11: Projection of stability regions for fixed values of the parameter $b = 0.67$, 1, and 1.13.

Figure 12: Continued.
Figure 12: Case (ii): projection of stability regions for fixed values of the parameter \( b = 0.67 \), 1, and 1.13.

Table 1: Equilibrium points and stability analysis for \( b = 0.67 \).

| Equilibrium point \( L_{1,2} \) | Eigenvalues \( \pm 1.60658, \pm 1.52567i \) | Stability Unstable |
|---------------------------------|-----------------------------------|-------------------|
| \( L_{3,4} \) = ( \( \pm 0.67893,0 \) ) | \( \pm 1.94310, \pm 1.16850i \) | Unstable |
| \( L_{5,6} \) = (0, \( \pm 1.472145 \)) | \( \pm 1.21869, \pm 1.35167i \) | Unstable |
| \( L_{7,8} \) = (0, \( \pm 0.180864 \)) | \( \pm 6.34778, \pm 4.592599i \) | Unstable |
| \( L_{9,10,11,12} \) = ( \( \pm 1.02395, \pm 0.479505 \)) | | |

Table 2: Equilibrium points and stability analysis for \( b = 0.97 \).

| Equilibrium point \( L_{1,2} \) | Eigenvalues \( \pm 1.480313, \pm 1.458732i \) | Stability Unstable |
|---------------------------------|-----------------------------------|-------------------|
| \( L_{3,4} \) = ( \( \pm 0.509436,0 \) ) | \( \pm 3.583445, \pm 2.544469i \) | Unstable |
| \( L_{5,6} \) = (0, \( \pm 1.660744 \)) | \( \pm 1.454216, \pm 1.451670i \) | Unstable |
| \( L_{7,8} \) = (0, \( \pm 0.464669 \)) | \( \pm 4.047077, \pm 2.908229i \) | Unstable |
| \( L_{9,10,11,12} \) = ( \( \pm 0.950136, \pm 0.874620 \)) | \( \pm 0.909938 \pm 0.998607i \) | Unstable |

Table 3: Equilibrium points and stability analysis for \( b = 1.13 \).

| Equilibrium point \( L_{1,2} \) | Eigenvalues \( \pm 1.508096, \pm 1.499923i \) | Stability Unstable |
|---------------------------------|-----------------------------------|-------------------|
| \( L_{3,4} \) = ( \( \pm 0.495101,0 \) ) | \( \pm 4.863781, \pm 3.594754i \) | Unstable |
| \( L_{5,6} \) = (0, \( \pm 1.413901 \)) | \( \pm 1.797668, \pm 1.600502i \) | Unstable |
| \( L_{7,8} \) = (0, \( \pm 0.874617 \)) | \( \pm 2.535781, \pm 1.929722i \) | Unstable |
| \( L_{9,10,11,12} \) = ( \( \pm 0.423616, \pm 1.103657 \)) | \( \pm 0.903618 \pm 0.9990127i \) | Unstable |
7. Conclusions

We have modeled and studied the Rhomboidal Restricted Six-Body Problem which has four masses at the vertices of the rhombus, and the fifth mass is at the intersection of the two diagonals placed at the origin. It is assumed that $m_1 = m_2 = m_0 = m$ and $m_3 = m_4 = \tilde{m}$. The primaries always move in the rhomboidal configuration. The sixth mass $m_5 \ll m_0$, where $i = 0, \ldots, 4$, is moving in the gravitational field of the five primaries. To get rid of the time dependency, the equation of motion of $m_3$ is written in the rotating coordinate system. It is shown that, for the mass parameter $b \in (\{1/\sqrt{3}\}, 1.1394282249562009)$, the total number of equilibrium points is always 12. Out of the 12 equilibrium points, 4 are on the $x$-axis and 4 are on the $y$-axis. As the mass parameters vary, the equilibrium points also change their positions around the five primaries, and the number of equilibrium points remains same. The linear stability analysis revealed that none of the equilibrium points are stable. The permissible region of motion is also discussed according to the variation of Jacobian constant of $C$. We have identified the values of $C$ at which the permissible regions of motion are partially disconnected and fully disconnected.

Data Availability

No data were used in the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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