Statistical interparticle potential of an ideal gas of non-Abelian anyons

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Abstract
We determine and study the statistical interparticle potential of an ideal system of non-Abelian Chern–Simons (NACS) particles, comparing our results with the corresponding results of an ideal gas of Abelian anyons. In the Abelian case, the statistical potential depends on the statistical parameter $\alpha$; it has ‘quasi-bosonic’ behavior for $0 < \alpha < 1/2$ (non-monotonic with a minimum) and ‘quasi-fermionic’ behavior for $1/2 \leq \alpha < 1$ (monotonically decreasing without a minimum). In the non-Abelian case, the behavior of the statistical potential depends on the Chern–Simons coupling and the isospin quantum number: as a function of these two parameters, a phase diagram with quasi-bosonic, quasi-fermionic and bosonic-like regions is obtained and investigated. Finally, using the obtained expression for the statistical potential, we compute the second virial coefficient of the NACS gas, which correctly reproduces the results available in the literature.

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(Some figures may appear in colour only in the online journal)

1. Introduction
A key property of the statistics of quantum systems in two space dimensions is provided by the possibility of displaying intermediate fractional statistics interpolating between bosons and fermions: the properties of anyons, the two-dimensional (2D) identical particles obeying...
fractional braiding statistics and carrying fractional charge, have been the subject of intense and continuing interest [1–5]. Both Abelian and non-Abelian anyons (associated with respectively one-dimensional and irreducible higher-dimensional representations of the braid group) have been extensively studied both for their intrinsic interest and their connection with quantum Hall systems [5–7]. In particular, there is increasing interest in the investigation of the properties of non-Abelian anyons for their application to topologically fault-tolerant quantum information processing [5].

The ideal gas of anyons is also interesting for the phenomenon of statistical transmutation [2–4], i.e. the fact that one can treat non-interacting anyons as interacting bosons or fermions. The idea behind statistical transmutation is that one can alternatively consider the system as made of interacting particles with canonical statistics or of non-interacting particles, but obeying non-canonical statistics. In general, this makes it difficult to study the equilibrium thermodynamic properties of ideal anyon gases [4] and therefore any result that gives even qualitative information on such ideal gases is valuable.

Dating back to the classical work by Uhlenbeck and Gropper [8], a standard way of characterizing the effects of quantum statistics on the properties of ideal gases is provided by the determination of the so-called statistical potential. As detailed in textbooks [9, 10], one can show that the partition function (PF) of a gas of particles approaches—for sufficiently high temperatures—the PF of the classical gas (with the correct Boltzmann counting). When this computation is done for an ideal quantum gas (under the condition that the thermal wavelength is much smaller than the interparticle distance) one finds that the quantum PF becomes the PF of the classical ideal gas. Evaluating the first quantum correction, one can appreciate that the quantum PF can be formally written as the PF of a classical gas in which a fictitious two-body interaction term (the statistical potential) is added [9, 10]. The statistical potential gives a simple characterization of the effects of the quantum statistics of the ideal gases: the statistical potential is ‘attractive’ for bosons and ‘repulsive’ for fermions; it is respectively monotonically increasing (decreasing) for bosons (fermions). Since the ideal Abelian anyon gas represents in all respects an interpolating case between Bose and Fermi systems [4], the study of its statistical potential gives the first qualitative tool to characterize the effect of non-canonical quantum statistics and to ascertain similarities and analogies with the Bose and Fermi limits: repulsive or attractive behavior of the statistical potential respectively provides fingerprints of ‘fermionic-like’ or ‘bosonic-like’ behavior of the anyon gas. It would then be useful to obtain the corresponding information for the non-Abelian anyon gas.

Another important property of the statistical potential is that a suitable integral of it directly gives the second coefficient of the virial expansion, which is a key quantity in the study of the equation of state in the dilute limit [9, 10]. For the Abelian gas, the corresponding second virial coefficient varies between the values for the limiting fermionic/bosonic cases, which are $\pm \lambda_T^2/4$, where $\lambda_T$ is the thermal de Broglie wavelength [4].

However, the statistical potential $v$ does not only give information integrated over all the distances: indeed, $e^{-\beta v(r)}$ can be viewed as the quantum propagator of a two anyon system—or, equivalently, of a system of two composite flux-charged bosons—at distance $r$ [11]. Being defined as the interaction potential energy between two classical particles, which need to have equilibrium properties of their bosonic, fermionic or Abelian anyonic counterpart, the statistical potential then provides clear information on the qualitative behavior of the statistical interaction between the particles as a function of their distance $r$. A similar consideration applies for the systems fulfilling non-Abelian braiding statistics, which is the subject of our study in section 3.

The statistical potential of a gas of ideal Abelian anyons has been studied in [12–14]; it depends on the statistical parameter $\alpha$ (we recall that $\alpha = 0$ and $\alpha = 1$ correspond respectively
to free 2D spinless bosons and fermions, while \( \alpha = 1/2 \) corresponds to semions \([4]\). It is found that for \( 1/2 \leq \alpha \leq 1 \) the statistical potential \( v_\alpha(r) \) is monotonically decreasing, while for \( 0 < \alpha < 1/2 \) it has a minimum at a finite value of \( r \) and it is increasing for larger values of \( r \) (while for \( \alpha = 0 \) it is monotonically increasing). We can refer to these types of behavior respectively as ‘quasi-fermionic’ (\( 1/2 \leq \alpha < 1 \)) and ‘quasi-bosonic’ (\( 0 < \alpha < 1/2 \)). The purpose of this paper is to compute the statistical potential of the ideal gas of non-Abelian anyons: since it is possible to show that the statistical potential for the non-Abelian gas may be written in terms of sums of Abelian statistical potentials (having different statistical parameters depending on the projection of the isospin quantum number), we recap in section 2 the main properties of the statistical potential in the Abelian case. For the non-Abelian case, we find that the behavior of the statistical potential depends on the Chern–Simons coupling and the isospin quantum number. As a function of these two parameters, quasi-bosonic and quasi-fermionic regions emerge; they are part of a phase diagram which will be presented below. Furthermore, a third type of behavior (‘bosonic-like’) appears in this phase diagram, corresponding to a monotonically increasing statistical potential.

The plan of the paper is the following: in section 2 we review the steps that lead to the computation of the statistics potential \( v_\alpha \) of an ideal gas of Abelian anyons with statistical parameter \( \alpha \). In section 2 we also provide a detailed study of \( v_\alpha \): this will turn out to be useful in section 3, since the statistical potential for the non-Abelian gas is written as a sum of Abelian statistical potentials—for this reason, we present a compact and useful integral representation for it, showing that it can be written in terms of bivariate Lommel functions \([15]\). Furthermore, using the Sumudu transform of the statistical potential \([16]\), we also give a closed expression of \( v_\alpha(r) \) in terms of the inverse Laplace transform of an algebraic function of \( r \) and \( \alpha \) (we observe that similar integral representations have already been worked out in the context of planar Brownian paths \([11]\)). Using these results, we are able to give a simple expression for the statistical potential of the ideal gas of semions; further, we show that for a general value \( \alpha \) it is possible to retrieve the well-known result for the second virial coefficient of an ideal anyon gas found in \([17]\) (with a hard-core boundary condition for the two-body wavefunction at zero distance). For the sake of a comparison with the non-Abelian case that follows, the limit behavior at both small and large distances of the statistics potentials is presented. In section 3 we introduce the non-Abelian Chern–Simons (NACS) model studied in the rest of the paper: we compute the statistics potential (within the hard-core boundary condition frame) as a function of the Chern–Simons coupling \( \kappa \) and the isospin quantum number \( l \); we build a phase diagram summarizing the behavior of the statistical potential in terms of \( \kappa \) and \( l \). We then show that the second virial coefficient, previously studied in \([18–21]\), is correctly retrieved. The asymptotic expressions for the small and large distances of the statistics potentials are also given. Finally, our conclusions are drawn in section 4, while supplementary material is presented in the appendices.

2. Statistical potential for Abelian anyons

In this section we introduce the model for an ideal gas of Abelian anyons; we then derive its statistical potential \( v_\alpha(r) \) as a function of the statistical parameter \( \alpha \), obtaining the expression for \( v_\alpha(r) \) given in \([12–14]\); we also provide an explicit formula for the semions (half-integer values of \( \alpha \)). The results for \( v_\alpha(r) \) and the asymptotic expressions for small and large distances will be used in the next section, where the statistical interparticle potential of an ideal gas of non-Abelian anyons is derived and studied. We point out that our treatment of the Abelian anyonic case is primarily intended to make the paper self-contained and to introduce notations and results used in the next sections devoted to non-Abelian anyons.
Abelian anyons admit a concrete representation by the flux-charge composite model [4]; the statistics of these objects can be understood in terms of Aharonov–Bohm type interference [22, 23]. The Hamiltonian for the quantum dynamics of an ideal system of anyons reads [3, 4]

\[ H_N = \sum_{n=1}^{N} \frac{1}{2M} (\vec{p}_n - \alpha \vec{a}_n)^2, \]  

(1)

where \( \vec{p}_n \) is the momentum of the \( n \)th particle \( (n = 1, \ldots, N) \). Similarly, we denote the position of the \( n \)th particle by \( \vec{r}_n \equiv (x_n^1, x_n^2) \). In equation (1), \( \alpha \) is the statistical parameter: notice that the physical quantities, e.g. the virial coefficients, are periodic with period 2 [4]: the bosonic points are defined by \( \alpha = 2j \) and the fermionic ones by \( \alpha = 2j + 1 \), \( j \) integer. For this reason, we consider in the following \( \alpha \in [0, 2] \).

In equation (1), \( \vec{a}_n \) is the vector potential carrying the flux attached to the bosons; indeed, the Hamiltonian (1) is written in the so-called bosonic representation, i.e. it is a Hamiltonian associated in the following \( \alpha \in [0, 2] \).

Let us recall that this model of Abelian anyons also admits a field-theoretic description: in fact (non-relativistic) anyons can be described by bosonic Schrödinger fields \( \psi \) and \( \psi^\dagger \) coupled to a Chern–Simons gauge field \( a_\mu \) living in (2+1)-D [24, 25] (then \( \mu = 0, 1, 2 \)). The Lagrangian density of such a system reads

\[ \mathcal{L} = \frac{c}{2} e^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \psi^\dagger \left( i D_t + \frac{1}{2M} D^2 \right) \psi, \]

where \( c \) gives the measure of the interaction among particles mediated by the \( U(1) \) gauge potential \( a_\mu \), with the covariant derivatives given by \( D_t = \partial_t + iqa_0, D = \vec{\nabla} - iq\vec{a} \); the anyon statistical parameter \( \alpha \) is identified as \( \alpha \equiv q^2/(2\pi c) \).

In the study of a quantum-mechanical ideal gas, the effect of the symmetry properties of the wavefunction can be interpreted, from a classical point of view, as the consequence of a fictitious classical potential introduced by Uhlenbeck and Gropper [8], referred to as an effective statistical potential, which represents the first quantum correction for the classical PF [9, 10]. For our purposes we have to consider the two-body case, which is relevant for the subsequent computation of the statistical interparticle potential [9, 10]. The statistical potential completely determines the second virial coefficient \( B_2 \), which gives the first order correction, in the expansion in powers of the density, for the pressure in the equation of state: therefore, from knowledge of \( B_2 \), one can determine the limit \( \rho \lambda^2 \ll 1 \) all the thermodynamic quantities by evaluating their shift from those of the Boltzmann ideal gas [9, 10]. The virial coefficient \( B_2 \) is particularly interesting for Abelian anyon gases, likewise for the family of non-Abelian anyon gases discussed in section 3: e.g., as discussed in [21] one can show the validity at the first order in \( \rho \lambda^2 \) of the energy–pressure relation \( E = PA \) for any temperature for both Abelian and non-Abelian anyon gases, unlike general interacting 2D systems. Remarkably, \( B_2 \) and the related quantities result in being non-monotonic in the statistical coupling constant, in both Abelian and non-Abelian cases [21].

For the two-anyon system, after separating in (1) with \( N = 2 \) the center-of-mass dynamics (i.e. that of a particle having mass \( 2M \)), one is left with the dynamics of the relative wavefunction:

\[ -\frac{1}{M} \left[ \vec{\nabla}_r - i \alpha \frac{\partial}{\partial \vec{r}} \right] \psi(\vec{r}) = E \psi(\vec{r}), \]

(3)
where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the relative coordinate and $a^l(r) = \epsilon^{ij} x^j r^l$. Therefore, the relative dynamics is equivalent to that of a single particle in the presence of a point vortex $\mathbf{a}(\mathbf{r})$ placed at the origin.

An analysis of the statistical interparticle potential for an ideal gas of Abelian anyons is presented in [12, 13]. The eigenfunctions of (3) read

$$\Psi_\alpha = e^{iK \cdot R} J_{|l-\alpha|}(kr) = e^{iK \cdot R} \psi_\alpha,$$

where the capital (italic) letters respectively refer to center-of-mass (relative) coordinates. The bosonic description used in (1) imposes the condition $l = \text{even}$; furthermore, $J_m(x)$ denote the Bessel functions of the first kind [15] (their definition is recalled in appendix A). Notice that the wavefunctions (4) are the eigenfunctions of (3), provided that the hard-core boundary conditions $\Psi_\alpha(0) = 0$ are imposed.

The two-body PF is

$$Z = \text{Tr} e^{-\beta H_2} = 2A \lambda_T^{-2} Z',$$

where $Z'$ is the single-particle PF in the relative coordinates, $\beta = 1/k_B T$, $\lambda_T = (\beta \hbar^2/2\pi M)^{1/2}$ is the thermal wavelength and $A$ is the area of the system. The relative PF $Z'$ is given by

$$Z' = \frac{1}{2} \int_0^\infty \int_{-\infty}^\infty d^2 p \int d^2 r e^{-\beta p^2/M} |\psi_\alpha|^2$$

(with $p = h k$). It is possible to conveniently rewrite equation (5) using the following integral property [26] of the Bessel functions:

$$\int_0^\infty e^{-\alpha x} J_\nu(2\beta \sqrt{x}) J_\nu(2\gamma \sqrt{x}) \, dx = \frac{1}{\alpha} I_\nu \left( \frac{2\beta \gamma}{\alpha} \right) e^{-(\beta^2+\gamma^2)/\alpha},$$

where $I_\nu$ is the modified Bessel function of the first kind [15] (see also appendix A). The relative PF then takes the form

$$Z' = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int_0^\infty dx e^{-x I_{|2n-\alpha|}(x)},$$

where

$$x = \frac{M r^2}{2\beta \hbar^2} = \frac{\pi r^2}{\lambda_T^2}.$$
The interparticle statistical potential admits a closed expression in terms of the bivariate Lommel functions (alias, Lommel functions of two variables). Indeed, as is evident from appendix A, 

\[ \mathcal{M}_\alpha(x) = i^{-\alpha} U_{\alpha}(ix, ix) - i^\alpha U_{2-\alpha}(ix, ix), \]  

where \( U_{\alpha} \) denote the Lommel functions of two variables [15]. Notice that the symmetry property \( v_{\alpha}(r) = v_{2-\alpha}(r) \) (\( \forall \alpha \in \mathbb{R} \)) holds for the statistical potential.

In appendix B we prove the following integral representation for \( v_{\alpha}(r) \), which will result in being useful in subsection 2.1 for discussing the large-distance limit behavior:

\[ e^{-\beta v_{\alpha}(r)} = 1 + e^{-2x} \cos \alpha \pi - 2 \frac{\sin \alpha \pi}{\pi} e^{-x} \int_0^\infty dt \frac{\sinh [(1 - \alpha)t]}{\sinh t} e^{-x \cosh t}. \]  

Similar integral representations have already been worked out in the literature [11]. Other integral representations for \( e^{-\beta v_{\alpha}(r)} \) are presented in appendix B. In the remainder of this section we discuss the asymptotic behavior of \( v_{\alpha}(r) \) for small and large values of the dimensionless parameter \( x \); we give a closed expression for the statistical potential \( v_{\alpha}(r) \) in terms of the inverse Laplace transform of an algebraic function of \( r \) and \( \alpha \); this manipulation allows us to straightforwardly regain the known expressions for \( v_{\alpha}(r) \) in the bosonic/fermionic cases, to obtain its expression in the case of semions and to finally recover the value of the second virial coefficient for a generic \( \alpha \) presented in [17].

2.1. Limit behavior

In order to quantitatively understand the tendency of anyons to bunch together or vice versa to repel each other in different limit regimes of density, let us recall and further discuss the asymptotic behavior of their effective statistical potential, for small and large distances [13, 14].

For small \( r \) (that is \( x \ll 1 \)), we can approximate the summation term in (10) as

\[ \sum_{n=-\infty}^{\infty} I_{2n-\alpha}(x) \approx \sum_{n=0}^{\infty} \left( \frac{1}{\Gamma(2n+\alpha+1)} (x/2)^{2n+\alpha} + \frac{1}{\Gamma(2n+3-\alpha)} (x/2)^{2n+2-\alpha} \right), \]

\[ \approx [\Gamma(\alpha+1)]^{-1}(x/2)^{\alpha} + [\Gamma(3-\alpha)]^{-1}(x/2)^{3-\alpha}. \]  

Since

\[ \beta v_{\alpha}(r) = -\ln \left[ 2 e^{-x} \sum_{n=-\infty}^{\infty} I_{2n-\alpha}(x) \right] \]

\[ \approx -\ln[2 e^{-x}([\Gamma(\alpha+1)]^{-1}(x/2)^{\alpha} + [\Gamma(3-\alpha)]^{-1}(x/2)^{3-\alpha})], \]

\[ e^{-\beta v_{\alpha}(r)} \]

\[ \]
it follows
\[
\beta v_{\alpha}(r) \approx \begin{cases} 
- \ln \left[ 2 - 2\pi r^2 / \lambda_1^2 \right] & \approx - \ln 2 + \frac{\pi}{2} (r / \lambda_T)^2, \quad \alpha = 0, 2 \\
- \ln \left[ 2 \left( \pi r^2 / 2 \lambda_1^2 \right)^\alpha / \Gamma(\alpha + 1) \right], & 0 < \alpha < 1 \\
- \ln \left[ 2\pi r^2 / \lambda_2^2 \right], & \alpha = 1 \\
- \ln \left[ 2 \left( \pi r^2 / 2 \lambda_2^2 \right)^{2-\alpha} / \Gamma(3 - \alpha) \right], & 1 < \alpha < 2. 
\end{cases} \tag{17}
\]

We may summarize the small-distance behavior as follows: \(v_{\alpha}(r)\) is repulsive and logarithmically divergent to \(+\infty\) for any \(\alpha \in (0, 2)\), whereas for \(\alpha = 0, 2\) it is attractive and quadratically increasing in \(r\) starting from the finite value \(v_0(r = 0) = -\ln 2\). The small-\(r\) asymptotic function for \(v_{\alpha}(r)\) is discontinuous in \(\alpha = 1\), being twice the limit of asymptotic functions for \(\alpha \to 1^\pm\) (in fact two equal Bessel terms contribute to the asymptotic behavior for \(\alpha = 1\), whereas only one of them dominates when \(\alpha \neq 1\)). Notice that the above expressions are symmetric about \(\alpha = 1\).

We observe that the short range behavior of the anyonic statistical potential \(v_{\alpha}(r)\) shows the same logarithmically divergent trend known for the case of 2D fermions and qualitatively corresponds to a mutual exclusion. This makes the ideal anyon gas much more closely related to the Fermi gas than to the Bose one. This is a consequence of having assumed the standard hard-core wavefunction boundary condition for the anyons, as previously remarked after (4).

The whole set of eigenfunctions of the relative Hamiltonian (3) is the continuous spectrum (4), whose states are indexed by the momentum parameter \(k\) and have positive energy \(k^2 / M\). The absence of bound states for (3) can be discussed using the statistical potential: as can be qualitatively seen in the limit case of bosons, corresponding to the steepest possible attractive behavior within the Abelian anyon family, in the effective classical picture provided by the statistical potential, the competition between temperature and attractive potential does not allow trapping of the (fictitious) particle in the potential \(v(r)\) since the maximal depth \(| - k_B T \ln 2 |\) of \(v\) is smaller than the thermal energy \(k_B T\).

To study the behavior of the statistical potential for large distance \(r\) (i.e. \(x \gg 1\)), we employ the integral representation (14). The method of steepest descent allows one to evaluate, to an arbitrary order, the last term of this integral representation. At the first significant order, we get
\[
-2 \frac{\sin \alpha \pi}{\pi} e^{-z} \int_0^\infty dr \frac{\sinh [(1 - \alpha)t]}{\sinh t} e^{-z \cosh t} \approx \frac{2(\alpha - 1) \sin \alpha \pi}{\sqrt{2\pi z}} e^{-2z}.
\tag{18}
\]
Therefore the large distance behavior of the statistical potential is given by
\[
\beta v_{\alpha}(r) \approx \left[ -\cos \alpha \pi + \frac{\sqrt{2}(1 - \alpha) \sin \alpha \pi}{\pi r / \lambda_T} \right] e^{-2\pi r^2 / \lambda_2^2}
\tag{19}
\]
for any \(\alpha \in [0, 2]\). Let us note here that this result differs from the corresponding one in [13] and that the asymptotic behaviors for \(\alpha = 0, 1, 2\) (see formulas (26)–(28) in the following) are correctly retrieved. The statistical potential for large distances is vanishing for \(r \to \infty\) and the interval \(\alpha \in [0, 1]\) (as well as the interval \(\alpha \in [1, 2]\), due to the symmetry property \(v_{\alpha} = v_{2-\alpha}\) is divided in two regions: for a large distance, \(v_{\alpha}(r)\) is attractive for \(0 \leq \alpha < 1/2\) and repulsive for \(1/2 \leq \alpha \leq 1\). Apart from that, we observe that the decay of the potential for large \(r\) is similar for any element of the anyon family (being invariably \(\propto e^{-2\pi r^2 / \lambda_2^2}\)), but for the case of semions, whose potential asymptotically decays as \(\frac{1}{r} \propto e^{-2\pi r^2 / \lambda_2^2}\).

The large- and short-distance behavior, considered together, imply that \(v_{\alpha}(r)\) must admit a minimum point at finite distance \(r_{\alpha}(\alpha)\), for any \(0 < \alpha < 1/2\) (see figure 2). We denote the corresponding dimensionless quantity by \(x_{\alpha}(\alpha) = \pi r_{\alpha}^2(\alpha) / \lambda_2^2\). The minimum point \(x_{\alpha}(\alpha)\) tends to \(+\infty\) for \(\alpha \to \frac{1}{2}^+\), as shown in figure 3.
Figure 2. $\beta v_\alpha(r)$ versus $r/\lambda_T$ for different values of $\alpha$: from top to bottom $\alpha = 1, 0.7, 0.5, 0.3, 0.2, 0.1, 0$. The bosonic curve ($\alpha = 0$) is monotonically increasing while the fermionic curve ($\alpha = 1$) is monotonically decreasing and divergent for $r \to 0$. All the curves for $0 < \alpha < 1/2$ diverge for $r \to 0$ and have a minimum point at finite $r$.

Figure 3. The dimensionless value of the minimum point $x_{cr}(\alpha)$ of the statistical potential $v_\alpha$ of Abelian anyons as a function of the statistical parameter $\alpha$ near $\alpha = 1/2$.

2.2. Laplace transform of the statistical potential and the second-virial coefficient

In this section we write an explicit formula for $e^{-\beta v_\alpha(x)}$ as the inverse Laplace transform of a function of $x$ and $\alpha$. This result will then allow us to write a simple formula for the statistical potential for the semionic gas and to compute the second virial coefficient, which of course coincides with the result reported in the seminal reference [17].

Let us start by writing down the Sumudu transform [16] of the function $M_\alpha$, defined in equation (11). The Sumudu transform of $M_\alpha$ is defined as

$$[M_\alpha(x)]_S \equiv \int_0^\infty e^{-\theta x} M_\alpha(\theta x) \, d\theta = \int_0^\infty e^{-t/x} M_\alpha(t) \frac{dt}{x} = s \int_0^\infty e^{-st} M_\alpha(t) \, dt,$$

where $s \equiv \frac{1}{x}$, whence

$$\left[ M_\alpha \left( \frac{1}{x} \right) \right]_S = x \mathcal{L}[M_\alpha](x),$$

where $\mathcal{L}[M_\alpha](x)$ is the Laplace transform of $M_\alpha$. This allows us to compute the second virial coefficient for the semionic gas, which coincides with the result reported in [17].
\( \mathcal{L} \) being the ordinary (one-sided) Laplace transform. The function \( \mathcal{M}_\alpha(x) \) is given by

\[
\mathcal{M}_\alpha(x) = \sum_{n=-\infty}^{\infty} I_{2\alpha-1}(x) = e^{-\gamma \pi / 2i} \sum_{n=0}^{\infty} (-1)^n J_{2n+\gamma}(ix) + e^{-\alpha \pi / 2i} \sum_{n=0}^{\infty} (-1)^n J_{2n+\alpha}(ix) \tag{22}
\]

where \( \gamma \equiv 2 - \alpha \). Substituting (22) in (20), one gets that the Sumudu transform of \( \mathcal{M}_\alpha \) becomes

\[
\left[ \mathcal{M}_\alpha(x) \right]_S = e^{-\gamma \pi / 2i} \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} e^{-\gamma t} J_{2n+\gamma}(ix) \, dt + e^{-\alpha \pi / 2i} \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} e^{-\alpha t} J_{2n+\alpha}(ix) \, dt.
\]

The use of the integral properties of the Bessel functions of the first kind \[15\] (see page 386) gives the following expression:

\[
\left[ \mathcal{M}_\alpha(x) \right]_S = \frac{e^{-\gamma \pi / 2i}}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} (-1)^n \left( \frac{\sqrt{1-x^2} - 1}{ix} \right)^{2n+\gamma} + \frac{e^{-\alpha \pi / 2i}}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} (-1)^n \left( \frac{\sqrt{1-x^2} - 1}{ix} \right)^{2n+\alpha}
\]

\[
= \frac{1}{\sqrt{1-x^2}} \left\{ \left( \frac{1 - \sqrt{1-x^2}}{x} \right)^\gamma + \left( \frac{1 - \sqrt{1-x^2}}{x} \right)^\alpha \right\} \frac{1}{1 - (\frac{1-\sqrt{1-x^2}}{x})^2}
\]

\[
= \frac{1}{2} \left( \frac{1}{1-x^2} + \frac{1}{\sqrt{1-x^2}} \right) \left[ \left( \frac{1 - \sqrt{1-x^2}}{x} \right)^\gamma + \left( \frac{1 - \sqrt{1-x^2}}{x} \right)^\alpha \right]. \tag{23}
\]

By sending \( x \to 1/x \) in (23) and applying (21), we obtain

\[
\mathcal{M}_\alpha \left( \frac{1}{x} \right)_S = \frac{1}{2} \left( \frac{x^2}{x^2 - 1} + \frac{x}{\sqrt{x^2 - 1}} \right) \left[ (x - \sqrt{x^2 - 1})^\gamma + (x - \sqrt{x^2 - 1})^\alpha \right]
\]

and

\[
\mathcal{L}[\mathcal{M}_\alpha](x) = \frac{(x - \sqrt{x^2 - 1})^{1-\alpha} + (x - \sqrt{x^2 - 1})^{\alpha-1}}{2(x^2 - 1)}. \tag{24}
\]

Hence the interparticle statistical potential admits the following form, for many purposes easier to handle than (10), since it does not contain infinite sums:

\[
e^{-\beta x_\alpha(r)} = e^{-x} L^{-1} \left[ \frac{(x - \sqrt{x^2 - 1})^{1-\alpha} + (x - \sqrt{x^2 - 1})^{\alpha-1}}{x^2 - 1} \right]. \tag{25}
\]

The correct potentials for the bosonic and fermionic cases are straightforwardly reproduced: using known results for the Laplace transforms \[27\], we get

\[
e^{-\beta x_{\alpha,\text{bos}}(r)} = e^{-x} L^{-1} \left[ \frac{2x}{x^2 - 1} \right] = e^{-x} 2 \cosh x = 1 + e^{-2x}; \tag{26}
\]

\[
e^{-\beta x_{\alpha,\text{ferm}}(r)} = e^{-x} L^{-1} \left[ \frac{2}{x^2 - 1} \right] = e^{-x} 2 \sinh x = 1 - e^{-2x}. \tag{27}
\]

Equation (25) gives a closed formula for the potential in the case of semions (\( \alpha = 1/2 \) or \( \alpha = 3/2 \)):

\[
e^{-\beta x_{\text{sem}}(r)} = e^{-x} L^{-1} \left[ \frac{(x - \sqrt{x^2 - 1})^{1/2} + (x - \sqrt{x^2 - 1})^{-1/2}}{x^2 - 1} \right] = \text{erf}(\sqrt{2x}) \tag{28}
\]

where erf is the error function \[28\].
Equation (25) allows us to recover the second virial coefficient of a gas made of identical Abelian $\alpha$-anyons, which is given by [17]:

$$B_2(\alpha, T) = \frac{1}{4} \lambda_T^2 (-1 + 4\alpha - 2\alpha^2).$$

The link between the second-virial coefficient and the statistical potential can be expressed in the form

$$B_2(\alpha, T) = \frac{\lambda_T^2}{2} \int_0^\infty dx [1 - e^{-\beta v(x)}].$$

Using equations (26)–(28), for the three special cases $\alpha = 0$, 1 and 1/2 (corresponding respectively to bosons, fermions and semions), one immediately finds

$$B_2(\alpha = 0, T) = \frac{\lambda_T^2}{2} \int_0^\infty dx [1 - (1 + e^{-2x})] = -\frac{1}{4} \lambda_T^2;$$

$$B_2(\alpha = 1, T) = \frac{\lambda_T^2}{2} \int_0^\infty dx [1 - (1 - e^{-2x})] = +\frac{1}{4} \lambda_T^2;$$

$$B_2 \left( \alpha = \frac{1}{2}, T \right) = \frac{\lambda_T^2}{2} \int_0^\infty dx \left[1 - \text{erf}(\sqrt{2}x)\right] = \frac{\lambda_T^2}{2} \int_0^\infty dy \left[1 - \text{erf} y\right] = \frac{1}{8} \lambda_T^2$$

as it should be.

Finally, the effective two-body statistical potential written as in equation (25) allows us to easily recover the expression of the second-virial coefficient, even for a general statistical parameter $\alpha$. Indeed, by virtue of (25) and the dominated convergence theorem, one has

$$\frac{B_2(\alpha, T)}{\lambda_T^2} = \frac{1}{2} \int_0^\infty dx [1 - e^{-\beta v(x)}]$$

$$= \frac{1}{2} \lim_{\epsilon \to 0} \int_0^\infty dx \left[e^{-\epsilon x} - e^{-(1+\epsilon)x}\right] \left[L^{-1} \left(\frac{(x - \sqrt{x^2 - 1})^{1-a}}{x^2 - 1}\right)\right]$$

$$= \frac{1}{2} \lim_{\epsilon \to 0} \left[1 - \frac{1}{\epsilon} L_{|x|=1+\epsilon} \left[L^{-1} \left(\frac{(x - \sqrt{x^2 - 1})^{1-a}}{x^2 - 1}\right)\right] - L^{-1} \left[\frac{(x - \sqrt{x^2 - 1})^{1-a}}{x^2 - 1}\right]\right]$$

$$= \frac{1}{2} \lim_{\epsilon \to 0} \left[1 - \frac{1}{\epsilon} \left(1 + \epsilon - \sqrt{2\epsilon + \epsilon^2}\right)^{1-a} + \frac{1 + \epsilon - \sqrt{2\epsilon + \epsilon^2}}{2\epsilon + \epsilon^2}\right]$$

$$= \frac{1}{4} (-1 + 4\alpha - 2\alpha^2),$$

that is just (29).

As a byproduct of equations (12), (13), (30) and (34), we find an interesting integral property relevant to the bivariate Lommel functions:

$$\int_0^\infty dx [1 - 2 e^{-i\alpha x} U_0 (ix, ix) - i\alpha U_{2-a} (ix, ix)] = -\frac{1}{2} + 2\alpha - \alpha^2.$$  

We point out that equivalent formulas, involving definite integrals of series of modified Bessel functions, appear in the context of calculating the area spanned by a closed Brownian planar path, of a given length starting from and ending at a given point [29].
3. Statistical potential for non-Abelian anyons

In this section we discuss the statistical interparticle potential for a 2D system of $SU(2)$ NACS spinless particles. The NACS particles are point-like sources mutually interacting via the non-Abelian gauge field attached to them [30]. As a consequence of their interaction, equivalent to a non-Abelian statistical interaction for a system of bosons, they are endowed with fractional spins and obey braid statistics as non-Abelian anyons.

Let us briefly introduce the NACS quantum mechanics [31–34]. The Hamiltonian describing the dynamics of the $N$-body system of free NACS particles can be derived by a Lagrangian with a Chern–Simons term and a matter field coupled with the Chern–Simons gauge term [34]: the resulting Hamiltonian reads

$$H_N = - \sum_{i=1}^{N} \frac{1}{M_i} \left( \nabla_{z_i} \nabla_{\bar{z}_i} + \nabla_{\bar{z}_i} \nabla_{z_i} \right)$$

(36)

where $M_i$ is the mass of the $i$th particles, $\nabla_{z_i} = \frac{\partial}{\partial z_i}$ and

$$\nabla_{z_i} = \frac{\partial}{\partial z_i} + \frac{1}{2\pi \kappa} \sum_{j \neq i} \hat{Q}^a_j \frac{1}{z_i - z_j}.$$  

(37)

In formula (36) $i = 1, \ldots, N$ labels the particles, $(x_i, y_i) = (z_i + \bar{z}_i, -i(z_i - \bar{z}_i))/2$ are their spatial coordinates and $\hat{Q}^a_j$'s are the isovector operators which can be represented by some generators $T^a_j$ in a representation of isospin $l$ [33]. The quantum number $l$ labels the irreducible representations of the group of rotations induced by the coupling of the NACS particle matter field with the non-Abelian gauge field; as a consequence, the values of $l$ are of course quantized and vary over all the integer and the half-integer numbers, with $l = 1/2$ being the smallest possible non-trivial value ($l = 0$ corresponds to a system of free bosons). As usual, the basis of isospin eigenstates can be labeled by $l$ and the magnetic quantum number $m$ (varying in the range $-l, -l+1, \ldots, l-1, l$). We observe that substituting (37) in the Hamiltonian (36) one explicitly sees that the Hamiltonian features a kinetic term in the presence of a vector potential, which provides an example of the statistical transmutation procedure, whose limiting cases will be discussed in the following.

For NACS particles, the statistical potential depends in general on the value of the isospin quantum number $l$ and on the coupling $\kappa$ (and of course on the distance $r$ and the temperature $T$). The quantity $\kappa$ present in the covariant derivative is a parameter of the theory. The condition $4\pi \kappa = \text{integer}$ has to be satisfied for consistency reasons [35, 31]. In the following we denote for simplicity by $k$ the integer $4\pi \kappa$.

For non-Abelian anyons, in an analogy with (9), the effective statistical potential can be related to the relative PF according to the following expression:

$$Z_2' (\kappa, l, T) - Z_2^{(n,l)} (l, T) = \frac{1}{2\hbar^2} \int d^3 p \ e^{-\beta \hat{p}^2 / M}$$

$$\times \int d^3 r \left[ \exp[-\beta v(\kappa, l, r)] - \exp[-\beta v^{(n,l)}(l, r)] \right].$$  

(38)

where $v^{(n,l)}(l, r)$ refers to the system of particles with isospin $l$ and without statistical interaction ($\kappa \rightarrow \infty$). The potential $v^{(n,l)}(l, r)$ can be expressed in terms of the potentials $v_{\alpha=0}(r)$ and $v_{\alpha=1}(r)$ for the free Bose and Fermi systems (endowed with the chosen wavefunction boundary conditions). $Z_2' (\kappa, l, T) - Z_2^{(n,l)} (l, T)$ is the (convergent) variation of the divergent PF for the two-body relative Hamiltonian, between the interacting case being examined and the non-interacting limit $\kappa \rightarrow \infty$. 

\[11\]
The computation of $Z_2 = \text{Tr} e^{-\beta H_2}$ is discussed in [19], where the results for the hard-core case are presented. It is convenient to separate the center-of-mass and relative coordinates: defining $Z = (z_1 + z_2)/2$ and $\bar{z} = z_1 - z_2$ one can write

$$H_2 = H_{cm} + H_{rel} = -\frac{1}{2\mu} \partial_\bar{z} \partial_\bar{z} - \frac{1}{\mu} (\nabla_\bar{z} \nabla_\bar{z} + \nabla_\bar{z} \nabla_\bar{z}),$$

where $\mu \equiv M/2$ is the two-body reduced mass, $\nabla_\bar{z} = \partial_\bar{z}$ and

$$\Omega_2 = \partial_\bar{z} + \frac{\Omega}{\bar{z}}.$$

$\Omega$ is a block-diagonal matrix given by

$$\Omega = \hat{Q}_l^0 \hat{Q}_l^0 / (2\pi \kappa) = \sum_{j=0}^{2l} \omega_j \otimes I_j,$$

with $\omega_j \equiv \frac{1}{4\pi} [j(j+1) - 2(l+1)]$. Furthermore, $Z_2$ can be then written as

$$Z_2 = 2 \lambda \Omega^{-2} Z_2,'$$

(40)

where $Z_2,' = \text{Tr}_{rel} e^{-\beta H_{rel}}$. The similarity transformation $G(\bar{z}, \bar{z}) = \exp \{-\frac{\Omega}{2} \ln(\bar{z}z)\}$, acting as

$$H_{rel} \rightarrow H_{rel}' = G^{-1} H_{rel} G,$$

$$\Psi(\bar{z}, \bar{z}) \rightarrow \Psi'(\bar{z}, \bar{z}) = G^{-1} \Psi(\bar{z}, \bar{z}),$$

(41)

gives rise to a Hamiltonian $H_{rel}'$ manifestly Hermitian and leaves invariant $Z_2,' [19]$. The explicit expression for $H_{rel}'$ is

$$H_{rel}' = -\frac{1}{\mu} (\nabla_\bar{z} \nabla_\bar{z} + \nabla_\bar{z} \nabla_\bar{z}),$$

(42)

where $\nabla_\bar{z} = \partial_\bar{z} + \Omega/2\bar{z}$ and $\nabla_\bar{z} = \partial_\bar{z} - \Omega/2\bar{z}$.

By rewriting $H_{rel}'$ in polar coordinates and projecting it onto the subspace of total isospin $j$, one is left [19] with the Hamiltonian for (Abelian) anyons in the Coulomb gauge, having the statistical parameter given by $\alpha_s = \omega_j$:

$$H_{j} = -\frac{1}{2\mu} \left[ \frac{\partial^2}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} + \frac{1}{\bar{r}^2} \left( \frac{\partial}{\partial \bar{\theta}} + i \omega_j \right)^2 \right].$$

(43)

The starting point of the subsequent computation is the non-interacting case, i.e. the determination of the quantity $\psi^{(j)}(l, r)$ entering equation (38): therefore $\omega_j = 0$ in equation (43). We may proceed in an analogy with the procedure presented in [20, 19, 21]: in the space of symmetric wavefunctions one projects the Hamiltonian (43) on the sectors labeled by the total isospin number $j = 0, \ldots, 2l$. For even (odd) values of $j$, the spatial part of the wavefunction has to be symmetric (antisymmetric): it follows that the contribution of the $j$th sector to $e^{-\beta \psi^{(j)}}(l, r)$ is proportional to the bosonic one $e^{-\beta \psi_{bos}(r)}$ (for even $j$) and the fermionic one $e^{-\beta \psi_{ferm}(r)}$ (for odd $j$). Each of these contributions has to be of course weighted with the degeneracy factor $2j + 1$: one then obtains

$$e^{-\beta \psi^{(j)}(l, r)} = \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \left[ \frac{1 + (-1)^j 2l}{2} e^{-\beta \psi_{bos}(r)} + \frac{1 - (-1)^j 2l}{2} e^{-\beta \psi_{ferm}(r)} \right].$$

(44)

where the normalization constant $1/(2l+1)^2$ follows from the expression (5) for the PF. Reference [21] contains a discussion and a comparison with previously available results about the manner of performing this projection onto the sectors, distinguished one from another by different total isospin principal numbers in the two-particle space of internal degrees of
Both (47) and (48) are periodic quantities under the shift

\[ \exp[-\beta v^{(n.l.)}(l, r)] = 1 + \frac{e^{-2x}}{2l+1}. \]  

(45)

Notice that this non-interacting quantity exactly corresponds to \((-1)^{2l}\) times the statistical potential for a system of identical \((2l)\)-spin ordinary particles (fulfilling the spin-statistics constraint) at the same temperature, similarly to what is argued in [21] about the second-virial coefficient for the same system.

In the interacting case (i.e. finite \(k\)), we can likewise express the statistical potential in terms of the statistical potentials of Abelian anyons:

\[ e^{-\beta v_{\nu_{\omega_j}}(r)} = \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \left[ \frac{1}{2} e^{-\beta v_{\omega_j}^B(x)} + \frac{1}{2} e^{-\beta v_{\omega_j}^F(x)} \right] \]

(46)

where \(\omega_j \equiv [j(j+1)-2l(l+1)]/k\); \(v_{\omega_j}^B(r), v_{\omega_j}^F(r)\) are the potentials for the Abelian \(\omega_j\)-anyon gases respectively in the bosonic and fermionic bases, given by

\[ e^{-\beta v_{\omega_j}^B(r)} = 2 e^{-x} M_{\omega_j}(x) \]

(47)

and

\[ e^{-\beta v_{\omega_j}^F(r)} = 2 e^{-x} M_{\omega_j+1}(x). \]

(48)

Both (47) and (48) are periodic quantities under the shift \(\omega_j \rightarrow \omega_j + 2\); it follows

\[ e^{-\beta v_{\nu_{\omega_j}}(r,T)} = \frac{2 e^{-x}}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \left[ \frac{1}{2} M_{\omega_j}(x) + \frac{1}{2} M_{\omega_j+1}(x) \right] \]

(49)

Equation (49) gives the statistical potential for a gas of NACS particles.

### 3.1. Second-virial coefficient

A useful application (and check, at the same time) of equation (49) consists in computing the second virial coefficient. The analogous equation to (30) reads

\[ B_2(k, l, T) = \frac{\lambda_T^2}{2} \int_0^\infty dx \left[ 1 - e^{-\beta v_{\nu_{\omega_j}}(r,T)} \right]. \]

(50)

Substituting in its integrand both equation (49) and the following decomposition of the unity

\[ 1 = \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \left[ \frac{1}{2} + \frac{1}{2} \right], \]

(51)

one obtains for \(B_2(k, l, T)\)

\[ \frac{\lambda_T^2}{2(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \int_0^\infty dx \left[ \frac{1}{2} \left( 1 - 2e^{-x} M_{\omega_j}(x) \right) \right. \]

\[ + \frac{1}{2} \left( 1 - 2e^{-x} M_{\omega_j+1}(x) \right) \].

(52)
By virtue of (35) one then has

\[ B_2(k, l, T) = \frac{\lambda_r^2}{(2l + 1)^2} \sum_{j=0}^{2l} (2j + 1) \left[ \frac{1 + (-1)^{j+2l}}{2} \left( -\frac{1}{4} + y_j - \frac{1}{2}\eta_j \right) + \frac{1 - (-1)^{j+2l}}{2} \left( -\frac{1}{4} + \eta_j - \frac{1}{2}\eta_j \right) \right] \]

where \( y_j \equiv \omega_j \mod 2 \) and \( \eta_j \equiv (\omega_j + 1) \mod 2 \). This result matches with previous results reported in the literature [20, 21], see in particular equations (38) and (41) of [21]. In this respect, we mention that in the context of non-Abelian anyons, a study of the thermodynamic properties in the lowest Landau level of a strong magnetic field has been performed in [36], showing that the virial coefficients are in that case independent of the statistics.

### 3.2. Minimum points

Using equation (49), we can study whether the gas of NACS has a ‘bosonic-like’, ‘quasi-bosonic’ or ‘quasi-fermionic’ behavior, according to its characterization in terms of the statistical potential. In correspondence with the analysis carried out in subsection 2.1, we can address the problem of determining: which points of the discrete parameter space \([k, l]\) are associated with the presence of an (interior) minimum point \( r_{\text{crit}}(k, l, T) \) for the statistical potential \( v(k, l, r, T) \) (which will be referred to as ‘bosonic’), which points correspond to a monotonically increasing \( v(k, l, r, T) \) (‘bosonic-like’ points) and which ones instead correspond to a \( v(k, l, r, T) \) monotonically decreasing in \( r \) (‘quasi-fermionic’ ones). With this aim, let us exploit the limit behavior of \( \exp[-\beta v(k, l, l, T)] \) for small distance \((x \ll 1)\) and large distance \((x \gg 1)\), which straightforwardly arises from equations (15) and (18). Notice that at \( x = 0 \), one has

\[ e^{-\beta v(k, l, r=0)} = \frac{1}{(2l + 1)^2} \sum_{j=0}^{2l} (2j + 1) \left[ (1 + (-1)^{j+2l}) \delta(y_j, 0) + (1 - (-1)^{j+2l}) \delta(y_j, 1) \right], \]

where \( \delta \) denotes the Kronecker delta function and \( y_j \equiv \omega_j \mod 2 \). For large distance it is

\[ e^{-\beta v(k, l, r=\infty)} \approx \begin{cases} 1 + \frac{e^{-2x}}{(2l + 1)^2} s(j, k, l), & \text{if } s(j, k, l) \neq 0 \\ 1 - \frac{e^{-2x}}{(2l + 1)^2 2\pi x} t(j, k, l), & \text{otherwise} \end{cases}, \]

where

\[ s(j, k, l) = \sum_{j=0}^{2l} (2j + 1)(-1)^{j+2l} \cos(\omega_j\pi) \]

and

\[ t(j, k, l) = \sum_{j=0}^{2l} (2j + 1) \sin(y_j\pi) \left[ (1 + (-1)^{j+2l})(y_j - 1) - (1 - (-1)^{j+2l})(\eta_j - 1) \right] \]

with \( y_j \equiv \omega_j \mod 2 \), \( \eta_j \equiv (\omega_j + 1) \mod 2 \).

Our results can be summarized in the ‘phase diagram’ shown in figure 4, in which we distinguish pairs of parameters \([k, l]\) for which \( v(k, l, r, T) \) has a minimum point at finite \( r \) (in black), pairs for which the statistical potential is monotonically increasing in \( r \) (in magenta) and the remaining pairs (which are left blank), for which it is monotonically decreasing. In this
Figure 4. Phase diagram in the parameter space \( [k, l] \) for the family of non-Abelian anyon models under consideration. Points \((k, l)\) in the black regions correspond to the presence of an interior minimum point for the statistical potential \(v(k, l, r, T)\) and are associated with quasi-bosonic behavior. Points in magenta correspond to bosonic-like, monotonically increasing behavior of the statistical potential. Finally, the remaining points are left blank and correspond to quasi-fermionic behavior.

way the black points denote ‘quasi-bosonic’ behavior and the magenta ones denote ‘bosonic-like’ behavior, according to the classification operated in section 2 to extract information from the statistical potential for Abelian anyons. Bosonic-like behavior occurs only for pairs \((k, l)\) having one of the forms: \((k \text{ generic}, l = 0), (k = 1, l \text{ integer})\) or \((k = 2, l \text{ even})\). One sees that for non-Abelian anyons there are mixed regions in which quasi-bosonic and quasi-fermionic behavior alternates, separated by regions dominated by quasi-bosonic behavior.

4. Conclusions

In this paper we have studied the two-body effective statistical potential (which models, in the dilute regime, the dominant term of the statistical interaction between the particles) of ideal systems of Abelian and non-Abelian anyons, described within the picture of flux-charge composites. In both cases we have derived a closed expression for the statistical potential (in terms of known special functions) and we have studied its behavior. Asymptotic expansions have been provided and the second virial coefficients of both systems have been found using a
compact expression of the statistical potential in terms of Laplace transforms. A phase diagram for the non-Abelian gas as a function of the Chern–Simons coupling and the isospin quantum number has been derived.

In our study we have considered hard-core boundary conditions for the relative anyonic wavefunctions; however, it would be interesting to use the results obtained so far in order to analyze the statistical potential in the more general case of soft-core conditions (both for the Abelian and non-Abelian ideal anyon gases).

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Appendix A. Definition of the used Bessel functions

In the main text we used the Bessel functions of the first kind \( J_\alpha \) and the modified Bessel function of the first kind \( I_\alpha \): their definition is respectively given by

\[
J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left( \frac{x}{2} \right)^{2m+\alpha}
\]

and

\[
I_\alpha(x) = i^{-\alpha}J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + \alpha + 1)} \left( \frac{x}{2} \right)^{2m+\alpha}.
\]  

The Lommel functions of two variables are defined in equation (5) of [15, p 537] and read

\[
U_\nu(w, z) = \sum_{m=0}^{\infty} (-1)^m \left( \frac{w}{z} \right)^{\nu+2m} J_{\nu+2m}(z)
\]

\[
V_\nu(w, z) = \cos \left( \frac{w^2 + z^2}{2w} + \frac{\nu\pi}{2} \right) + U_{2-\nu}(w, z).
\]

Appendix B. Properties of the statistical potential \( v_\alpha \)

In this appendix we provide details and further information on the statistical potential \( v_\alpha \) for Abelian anyons.

We first give a derivation of equation (10): the relative PF (7) is given by

\[
Z' = \frac{1}{2} \sum_{l=-\infty}^{\infty} \int_{0}^{\infty} dx e^{-x} I_{|l-\alpha|}(x), \quad x = Mr^2/2\beta\hbar^2
\]  

(B.1)

and it can be rewritten as

\[
Z' = \frac{1}{2\hbar^2} \int d^3p e^{-p^2/M} \int d^2r \sum_{l=-\infty}^{\infty} 2e^{-Mr^2/2\beta\hbar^2} I_{|l-\alpha|} \left( \frac{Mr^2}{2\beta\hbar^2} \right).
\]

(B.2)

as pointed out in [12]. That allows for its comparison with the PF (9) for classical systems associated with a generic potential \( v(r) \), whence the result (10).
We also observe that
\[
e^{-\beta_{J}(r)} = 2 e^{-x} \left[ i^{-\alpha} \sum_{n=0}^{\infty} (-1)^n J_{2n+\alpha}(ix) + i^{\alpha-2} \sum_{n=0}^{\infty} (-1)^n J_{2n+2-\alpha}(ix) \right].
\] (B.3)

The expression that follows here below is a possible closed form for the statistical potential, but at the cost of using an integral representation given in formula (7) [37, p 652]. It stands for

\[
J_{\nu}(g) = \int_{0}^{\infty} J_{\nu}(gt) J_{\nu-1}(t) \, dt,
\]

for \( \nu \in (0, 2) \), can be produced by using the following property ([15, p 540]) of the bivariate Lommel function \( U \):

\[
U_{\nu}(w, z) = \frac{w^\nu}{e^z - 1} \int_{0}^{1} J_{\nu-1}(z t) \cos \left\{ \frac{1}{2} w (1 - t^2) \right\} t^\nu \, dt, \quad \text{Re}(\nu) > 0
\]

together with expressions (B.3), (13) and (A.1). The resulting integral representation is

\[
e^{-\beta_{J}(r)} = 2 x e^{-x} \int_{0}^{1} \cosh \left( \frac{x}{2} (1 - t^2) \right) \left( I_{\nu-1}(x t) t^{\alpha} + I_{1-\alpha}(x t) t^{2-\alpha} \right) \, dt.
\] (B.5)

In the final part of this appendix we provide the derivation of the integral representation (14). To this end, we use the following representation [26] for the modified Bessel function of the first kind:

\[
I_{\nu} = \frac{1}{\pi} \int_{0}^{\pi} e^{z \cos \theta} \cos \nu \theta \, d\theta - \frac{\sin \nu \pi}{\pi} \int_{0}^{\infty} e^{-z \cosh t - \nu t} \, dt, \quad \text{arg} |z| \leq \frac{\pi}{2}, \quad \text{Re} \nu > 0.
\] (B.6)

Then the summation term in (10) is:

\[
\sum_{n=-\infty}^{\infty} I_{(2n+1)}(z) = \sum_{n=0}^{\infty} I_{2n+2-\alpha}(z) + \sum_{n=0}^{\infty} I_{2n+\alpha}(z)
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} d\phi \, e^{\alpha \cos \phi} \sum_{n=0}^{\infty} \cos (2n + \alpha) \phi + \sum_{n=1}^{\infty} \cos (2n - \alpha) \phi
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} d\phi \, e^{-\alpha \cos \phi} f(t, \alpha),
\] (B.7)

where

\[
f(t, \alpha) = \sum_{n=0}^{\infty} e^{-(2n+1)t} \sin (2n + \alpha) \pi + \sum_{n=1}^{\infty} e^{-(2n-1)t} \sin (2n - \alpha) \pi = \sin \alpha \pi \frac{\sin[(1 - \alpha) t]}{\sin t}
\]

for \( t \neq 0 \)

\[
f(0, \alpha) \equiv \lim_{t \to 0^+} f(t, \alpha) = (1 - \alpha) \sin \alpha \pi.
\]
The first addend of the last integral representation is
\[
\frac{1}{\pi} \int_0^{\pi} d\phi \ e^{z \cos \phi} \left[ \sum_{n=0}^{\infty} \cos (2n + \alpha)\phi + \sum_{n=1}^{\infty} \cos (2n - \alpha)\phi \right]
\]
\[
= \frac{1}{\pi} \int_0^{\pi} d\phi \ e^{z \cos \phi} \left[ \sum_{n=0}^{\infty} \cos (2n + \alpha)\phi \sum_{n=-\infty}^{1} \cos (2n + \alpha)\phi \right]
\]
\[
= \frac{1}{\pi} \int_0^{\pi} d\phi \ e^{z \cos \phi} \sum_{n=-\infty}^{\infty} \cos (2n + \alpha)\phi \ Re \ \left[ \sum_{n=-\infty}^{\infty} (e^{i\alpha\phi} e^{2\alpha\phi}) \right]
\]
\[
= \frac{1}{\pi} \int_0^{\pi} d\phi \ e^{z \cos \phi} \ Re \ \left[ \sum_{n=-\infty}^{\infty} (e^{i\alpha\phi} e^{2\alpha\phi}) \right]
\]
\[
= \frac{1}{2} \left( e^{z} + e^{-z} \cos \alpha \pi \right). \quad (B.8)
\]

As a result of (10), (B.7) and (B.8), one has then
\[
e^{-\beta v(\alpha)} = 1 + e^{-2z} \cos \alpha \pi - 2 \frac{\sin \alpha \pi}{\pi} e^{-z} \int_0^\infty dt \ \frac{\sinh [(1 - \alpha)t]}{\sinh t} e^{-z \cosh t}. \quad (B.9)
\]

By direct inspection this result, notwithstanding the hypothesis of validity for (B.6), is valid also for \( \alpha \) at the extremes of the interval \([0, 2]\), so that the derivation of (14) is completed.

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