LONG TIME DYNAMICS FOR THE FOCUSING INHOMOGENEOUS FRACTIONAL SCHRÖDINGER EQUATION

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Abstract. We consider the following fractional NLS with focusing inhomogeneous power-type nonlinearity

\[ i\partial_t u - (\Delta)^s u + |x|^{-b} |u|^{p-1} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \]

where \( N \geq 2, \frac{1}{2} < s < 1, 0 < b < 2s \) and \( 1 + \frac{2(2s-b)}{N} < p < 1 + \frac{2(2s-b)}{N-2s} \).

We prove the ground state threshold of global existence and scattering versus finite time blow-up of energy solutions in the inter-critical regime with spherically symmetric initial data. The scattering is proved by the new approach of Dodson-Murphy (Proc. Am. Math. Soc. 145: 4859–4867, 2017). This method is based on Tao’s scattering criteria and Morawetz estimates. One describes the threshold using some non-conserved quantities in the spirit of the recent paper by Dinh (Discr. Cont. Dyn. Syst. 40: 6441–6471, 2020). The radial assumption avoids a loss of regularity in Strichartz estimates. The challenge here is to overcome two main difficulties. The first one is the presence of the non-local fractional Laplacian operator. The second one is the presence of a singular weight in the non-linearity. The greater part of this paper is devoted to prove the scattering of global solutions in \( H^{s}(\mathbb{R}^N) \).

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1. Introduction

We consider the following fractional NLS with focusing inhomogeneous power non-linearity

\[
\begin{cases}
    i\partial_t u - (-\Delta)^s u = -|x|^{-b}|u|^{p-1}u, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
    u(0, x) = u_0(x),
\end{cases}
\tag{1.1}
\]

where \( N \geq 2, s \in (0, 1), p > 1 \) and \( b > 0 \). The fractional Laplacian operator \((-\Delta)^s\) is defined by \((-\Delta)^s u := F^{-1}(|\xi|^{2s}F(u))\) where \( F \) and \( F^{-1} \) are the Fourier transform and inverse Fourier transform, respectively.

The fractional Schrödinger equation arises for instance as an effective equation in the continuum limit of discrete models with long-range interactions. In [50] Kirkpatrick, Lenzmann and Staffilani refer to models of mathematical biology, specifically for the charge transport in biopolymers like the DNA. Numerous applications of fractional NLS-type equations in the physical sciences could be mentioned, ranging from the description of Boson stars [37] to water wave dynamics. The fractional Laplacian also appears as a natural operator when considering jump processes [61], which makes it valuable for Lévy processes in probability theory with applications in financial mathematics.

From the mathematical point of view, there is a quite large literature devoted to (1.1) and its variants. We emphasize that the case \( s = 1, b = 0 \) corresponds to the nonlinear Schrödinger equation (NLS). There have been tremendous amount of researches on (NLS), and the monographs [12, 36, 57, 59] cover all known results in great mathematical detail.

The case \( s = 1, b \neq 0 \) corresponds to the inhomogeneous nonlinear Schrödinger equation (INLS). The Cauchy problem for (INLS) with \( b > 0 \) has received a lot of interested in the mathematical community; see, among others, [38, 44, 25, 1, 49, 23, 33, 4, 34, 35, 9, 26, 51, 11, 27, 52] and references therein. Note that the case of spatial growing non-linearity, that is \( b < 0 \), has been recently investigated in [30]. See also [13, 14, 15, 16, 39].

We stress that the 2D energy critical counterpart has received more attention in the past decade. See, among others, [21, 46, 5, 6].

Let us turn now to the fractional homogeneous case, that is, \( 0 < s < 1, b = 0 \), which we call for short (FNLS). A first analysis of the well-posedness in Sobolev spaces has been done by Hong and Sire [45]. It is worth noticing that the results obtained in [45] require some extra assumptions due to the loss of \( N(1 - s) \) derivatives in the dispersion [19]. In the radial settings, using Strichartz estimates without loss of regularity [42], the local well-posedness of solutions holds in the energy space [24]. See also [40, 41] for
a cubic source term. The scattering of radial focusing solutions below the ground state threshold was investigated in [56]. In the energy-critical radial regime, the global well-posedness and scattering in the defocusing case, and in the focusing case with energy below the ground state, were obtained in [43]. Due to the lack of a variance identity, the finite time blow-up of solutions was open for a long time. A partial result was given in [8] by use of a localized variance identity.

To the best authors knowledge, there exist few works dealing with the inhomogeneous fractional Schrödinger equation (1.1), with \( b \neq 0 \) and \( 0 < s < 1 \). Sufficient conditions for global existence in \( H^s \) were derived in [53]. In addition, a blow-up criterion of radial solutions for the inter-critical regime was give in [53]. By using a sharp Gagliardo-Nirenberg inequality and potential well method, the second author obtained the well-posedness in the case of spatial growing nonlinearity, that is \( b < 0 \); see [54]. See also [55] for the bi-harmonic Schrödinger equation.

The main goal of this paper is to fill in a gap in the literature. Indeed, the ground state threshold of global existence and scattering versus finite time blow-up of focusing radial solutions seems not to be treated before. One needs to deal with two combined difficulties. The first one is the presence of the non-local fractional Laplacian operator, which gives a loss of derivatives in the dispersive estimate (2.16). The second one is the singular weight \( |x|^{-b} \) in the non-linearity. The main part of this work is devoted to proving the scattering. We avoid here the Kenig-Merle road-map, based on the concentration-compactness method [47]. Instead, we use the Dodson-Murphy method [31] based on Tao’s scattering criteria [60] and Morawetz estimates. Moreover, we express the threshold using some non-conserved quantities in the spirit of the recent paper by Dinh [28]. This gives as a consequence the classical mass-energy threshold obtained first by Roudenko et al. [32].

The non-linear Schrödinger equation (1.1) satisfies the scaling invariance

\[
0 < \lambda \mapsto u_\lambda(t, x) := \lambda^{\frac{2s-b}{p-1}} u(\lambda^{2s} t, \lambda x).
\]

The only one homogeneous Sobolev norm invariant under the above scaling is relevant in this study. Indeed, the next identity

\[
\|u_\lambda(t)\|_{\dot{H}^s} = \lambda^{\mu-(\frac{N}{2} - \frac{2s-b}{p-1})} \|u(\lambda^{2s} t)\|_{\dot{H}^s},
\]

gives rise to the critical Sobolev index \( s_c := \frac{N}{2} - \frac{2s-b}{p-1} \). The energy-critical regime corresponds to \( s_c = s \) or \( p = p^* := 1 + \frac{2(2s-b)}{N-2s} \) and is related to the energy conservation law

\[
E[u(t)] := \int_{\mathbb{R}^N} |D^s u(t)|^2 \, dx - \frac{2}{1+p} \int_{\mathbb{R}^N} |x|^{-b} |u(t)|^{1+p} \, dx = E[u_0],
\]
where $D^\sigma$ stands for the operator with Fourier multiplier

$$\mathcal{F}(D^\sigma \phi)(\xi) = |\xi|^\sigma \mathcal{F}(\phi)(\xi).$$  

(1.3)

In particular, $(-\Delta)^s = D^{2s}$.

The mass-critical regime corresponds to $s_c = 0$, or $p = p_* := 1 + \frac{2(2s-b)}{N}$ and is related to the mass conservation law

$$M[u(t)] := \int_{\mathbb{R}^N} |u(t,x)|^2 \, dx = M[u_0].$$  

(1.4)

Unless otherwise specified, we restrict ourselves to the inter-critical regime $0 < s_c < s$. The last condition can be written in terms of $p$ as $p_* < p < p^*$.

Define the positive real number $\gamma_c := \frac{s - s_c}{s_c}$ and the quantities

$$P[\phi] := \int_{\mathbb{R}^N} |x|^{-b} |\phi(x)|^{p+1} \, dx,$$

$$I[\phi] := \|D^s \phi\|^2 - \frac{B}{1 + p} \int_{\mathbb{R}^N} |x|^{-b} |\phi(x)|^{1+p} \, dx,$$

(1.5)

(1.6)

where $B$ is given by (2.11) below and $\| \cdot \|$ denotes the $L^2(\mathbb{R}^N)$ norm. Define also the scale invariant quantities

$$\mathcal{M}E[\phi] := \left( \frac{M[\phi]}{M[Q]} \right)^\gamma \left( \frac{E[\phi]}{E[Q]} \right),$$

$$\mathcal{M}G[\phi] := \left( \frac{\|\phi\|}{\|Q\|} \right)^\gamma \left( \frac{\|\nabla \phi\|}{\|\nabla Q\|} \right),$$

(1.7)

(1.8)

where $Q \in H^s$ is the unique non-negative radially symmetric decreasing solution of (2.13) given by Lemma 2.5 below.

From now on, we hide the variable $t$ for simplicity, spreading it out only when necessary. Let $H^{s}_{\text{rad}}$ denote the space of functions in $H^s$ which are radially symmetric. Our main contribution reads as follows.

**Theorem 1.1.** Assume that $N \geq 2$, $s \in (\frac{N}{2N-1}, 1)$, $0 < b < 2s$ and $p_* < p < p^*$.

Let $u_0 \in H^{s}_{\text{rad}}$ and $u \in C([0,T^*); H^{s}_{\text{rad}})$ be the corresponding maximal solution of (1.1) given by Proposition 2.8.

(i) Suppose that

$$\sup_{t \in [0,T^*)} P[u(t)][M[u(t)]^\gamma < P[Q][M[Q]]^\gamma.$$  

Then, $u$ is global. Moreover, if $N \geq 3$, $s > \frac{N}{1+N}$, $p > 2(1 - \frac{b}{N})$ ($p < \frac{N-2b}{N-2s}$ if $N = 3$), then $u$ scatters in $H^s$.

(ii) Suppose that

$$\sup_{0 \leq t < T^*} I[u(t)] < 0.$$  

Then, $u$ blows-up in finite or infinite time in the sense that $T^* < \infty$ or $T^* = \infty$ and there is $t_n \to \infty$ such that $\|D^s u(t_n)\| \to \infty$ as $n$ goes to infinity.
In view of the results stated in the above theorem, some comments are in order.

- The scattering in $H^s$ means that there exists $\varphi_{\pm} \in H^s$ such that
  \[ \lim_{t \to \pm \infty} \| u(t) - e^{-itD^{2s}} \varphi_{\pm} \|_{H^s} = 0, \]
  where $e^{-itD^{2s}}$ denotes the free fractional Schrödinger operator given by (2.15).
- In Corollary 1.2, we derive the scattering under the ground state threshold as a consequence of Theorem 1.1.
- As one can see later in the proof of the above theorem, the effect of the fractional operator appears in the scattering criteria in Section 5. Precisely, when using the dispersive estimate (2.16) to handle the term $F_1$ appearing in (5.6). The natural assumption $s \geq N(1 - s)$ here gives the restriction $s \geq \frac{N}{1+2N}$. This is stronger than the Strichartz estimates requirement $s > \frac{N}{2N-1}$.
- It seems that with the method used here one has the scattering for some $p_* < \tilde{p} < p < p^*$. The same restriction was observed in [56].
- The radial assumption is necessary to avoid a loss of regularity in Strichartz estimates [42].
- In the homogeneous case $b = 0$, there is a scattering result under the ground state threshold of the (FNLS), see [56].
- In the non-radial case, based on the local well-posed result [45], the authors will treat the asymptotics of energy solutions in a forthcoming work.
- In the space dimension $N = 2$, the inequality (5.15) below is false. Arguing with a different way, one obtains the scattering for $\max\{1 + \frac{b}{s}, 2\} \leq p \leq 1 + \frac{b}{s} + (1 - \frac{b}{s}) \frac{N}{N-2s}$; see appendix A for the details.
- If one adds the supplementary assumption $p < 1 + 4s$, then arguing as in [8], one can prove the finite time blow-up provided that (1.10) is satisfied.
- In the classical case $s = 1$, we can remove the radial assumption by the use of the decay of the inhomogeneous term [10]. But here, the spherically symmetric condition is also needed in Strichartz estimates.
- In the attractive regime, namely the equation (B.1), the scattering of global solutions with spherically symmetric data is proved in the Appendix B.
- In a paper in progress, the authors investigate the scattering of (1.1) with $b < 0$.

As a consequence of the above result, one has the next dichotomy of global/non global existence of energy solutions under the ground state threshold.

**Corollary 1.2.** Assume that $N \geq 2$, $s \in (\frac{N}{2N-1}, 1)$, $0 < b < 2s$ and $p_* < p < p^*$. Let $u_0 \in H^s_{rad}$ and $u \in C([0, T^*); H^s_{rad})$ be the corresponding maximal solution of (1.1) given by Proposition 2.8. Suppose further that

\[ \mathcal{M}\mathcal{E}[u_0] < 1. \]  

(1.11)

Then,
\( (i) \) the solution \( u \) is global provided that 
\[
\mathcal{M}_G[u_0] < 1. \tag{1.12}
\]
Moreover, it scatters in \( H^s \) if \( N \geq 3, s > \frac{N}{1 + 2N} \), \( 2(1 - \frac{b}{N}) < p \) and \( p < \frac{N - 2b}{N - 2s} \) if \( N = 3; \)
\( (ii) \) the solution \( u \) blows-up in finite or infinite time provided that 
\[
\mathcal{M}_G[u_0] > 1. \tag{1.13}
\]

Remark 1.3. The finite time blow-up was proved in [53] under the supplementary assumption \( p < 1 + 4s \).

We conclude the introduction with an outline of the paper. In Section 2, we recall some useful tools needed in our proofs. In addition, we give some auxiliary results in order to facilitate the reading of the rest of the paper. Section 3 is devoted to some variational analysis. We derive Morawetz-type inequalities in Section 4. The scattering criterion is stated and proved in Section 5. The proof of Theorem 1.1 is given in Section 6. Section 7 contains the proof of the scattering versus blow-up under the ground state threshold, that is, Corollary 1.2. The two-dimensional case and the defocusing regime are treated in Appendix A and Appendix B, respectively.

Finally, the notation \( A \lesssim B \) (resp. \( A \gtrsim B \)) for positive numbers \( A \) and \( B \), means that there exists a positive constant \( C \) such that \( A \leq CB \) (resp. \( A \geq CB \)).

2. Useful tools & Auxiliary results

2.1. Useful tools. For future convenience, we recall some known and useful tools which will play an important role in the proof of our main results.

The fractional radial Sobolev inequality reads [18, Proposition 1]
\[
sup_{x \neq 0} |x|^\frac{N}{2} - \alpha |u(x)| \leq C(N, \alpha) \|D^\alpha u\|, \quad \forall u \in \dot{H}^\alpha_{rad}, \tag{2.1}
\]
provided that \( N \geq 2 \) and \( 1/2 < \alpha < N/2 \).

We also recall the homogeneous Sobolev embedding [3, Theorem 1.38]
\[
\|u\|_{L^\frac{2N}{N-2\alpha}(\mathbb{R}^N)} \leq C(N, \alpha) \|D^\alpha u\|, \quad \forall u \in \dot{H}^\alpha, \tag{2.2}
\]
provided that \( 0 \leq \alpha < N/2 \).

The classical Sobolev embedding [2] states that
\[
H^\alpha(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \quad 2 \leq q \leq \frac{2N}{N - 2\alpha}, \tag{2.3}
\]
provided that \( 0 \leq \alpha < N/2 \).

We know from [3, Theorem 1.62, p. 41] that multiplication by a function of Schwartz space \( \mathcal{S}(\mathbb{R}^N) \) is a continuous map from \( H^s(\mathbb{R}^N) \) into itself for any \( s \in \mathbb{R} \). Precisely we
have
\[ \| \varphi u \|_{H^s} \leq 2^{\frac{|s|}{2}} \left( 1 + \| \cdot \|_{L^1}^{\frac{1}{2}} \right) \| \varphi \|_{L^1} \| u \|_{H^s}, \]
where \( s \in \mathbb{R} \), \( \varphi \in \mathcal{S}(\mathbb{R}^N) \) and \( u \in H^s(\mathbb{R}^N) \).

The next fractional chain rule \[20\] will be useful.

**Lemma 2.1.** Let \( N \geq 1 \), \( 0 < \alpha \leq 1 \), \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_i} \), \( i = 1, 2 \) and \( F \in C^1(\mathbb{C}) \). Then,
\[ \| D^\alpha F(u) \|_{L^p} \lesssim \| D^\alpha u \|_{L^p} \| F'(u) \|_{L^p}, \tag{2.4} \]
and
\[ \| D^\alpha (uv) \|_{L^p} \lesssim \| D^\alpha u \|_{L^p} \| v \|_{L^{p_1}} + \| D^\alpha v \|_{L^p} \| u \|_{L^{q_2}}. \tag{2.5} \]

The next result gives a Leibniz rule for fractional derivatives \[48\].

**Lemma 2.2.** Let \( N \geq 1 \), \( s_1 + s_2 = s \in (0,1) \), such that \( s_1, s_2 \in (0,1) \) and \( 1 < p,p_1,p_2 < \infty \) such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Then,
\[ \| D^s (uv) - u D^{s_1} v - v D^{s_2} u \|_{L^p} \lesssim \| D^{s_1} v \|_{L^p} \| D^{s_2} u \|_{L^p}. \tag{2.6} \]
Moreover, for \( s_1 = 0 \), the value \( p_1 = \infty \) is allowed.

A useful Sobolev’s embedding reads as follows (see \[7\], Theorem 6.1.6).

**Lemma 2.3.** Let \( s_1, s_2 \in \mathbb{R} \) and \( 1 < p_1 \leq p_2 < \infty \) such that
\[ s_1 - \frac{N}{p_1} = s_2 - \frac{N}{p_2}. \]
Then
\[ W^{s_1,p_1}(\mathbb{R}^N) \hookrightarrow W^{s_2,p_2}(\mathbb{R}^N). \]

The following continuity argument (or bootstrap argument) will also be useful for our purpose. See \[29\], Lemma 2.11 for a similar statement.

**Lemma 2.4.** \[58\], Lemma 3.7, p. 437
Let \( I \subset \mathbb{R} \) be a time interval, and \( X : I \to [0, \infty) \) be a continuous function satisfying, for every \( t \in I \),
\[ X(t) \leq a + b(X(t))^\theta, \tag{2.7} \]
where \( a, b > 0 \) and \( \theta > 1 \) are constants. Assume that, for some \( t_0 \in I \),
\[ X(t_0) \leq a, \quad ab^{\frac{1}{\theta-1}} < (\theta - 1) \theta^{\frac{\theta}{\theta-1}}. \tag{2.8} \]
Then, for every \( t \in I \), we have
\[ X(t) < \frac{\theta a}{\theta - 1}. \tag{2.9} \]
Proof. We give here a simpler proof than done in [58]. We argue by contradiction. Suppose that $X(t_1) \geq \frac{\theta a}{\theta - 1}$ for some $t_1 \in I$. Then $X(t_0) \leq a < \frac{\theta a}{\theta - 1} \leq X(t_1)$. By continuity, there exists $t_2 \in I$ such that $X(t_2) = \frac{\theta a}{\theta - 1}$. This contradicts the second assumption in (2.8) since $\frac{\theta a}{\theta - 1} \leq a + b \left( \frac{\theta a}{\theta - 1} \right)^\theta$ imply that $a b^{\frac{1}{\theta - 1}} \geq (\theta - 1)\theta^{\frac{a}{\theta - 1}}$. This leads to (2.9) as desired. □

The next Gagliardo-Nirenberg type inequality [53, Theorem 2.2] is crucial in our proofs.

**Lemma 2.5.** Let $N \geq 2$, $s \in (0, 1)$, $0 < b < 2s$ and $1 < p < \frac{4s - 2b}{N - 2s}$. Then the following sharp Gagliardo-Nirenberg inequality holds

$$\int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1} \, dx \leq \mathcal{K}_{\text{opt}} \|u\|^A \|D^s u\|^B,$$

where

$$B := \frac{1}{2s} \left( N(p - 1) + 2b \right) \quad \text{and} \quad A := p + 1 - B.$$ (2.11)

Moreover, the sharp constant is given by

$$\mathcal{K}_{\text{opt}} = \frac{p + 1}{A} \left( \frac{A}{B} \right)^{B/2} \|Q\|^{1-p},$$ (2.12)

where $Q \in H^s$ is the unique non-negative radially symmetric decreasing solution of

$$- (\Delta)^s Q = Q - |x|^{-b} |Q|^{p-1} Q = 0.$$ (2.13)

Furthermore,

$$\|D^s Q\| = \sqrt{\frac{B}{A}} \|Q\| \quad \text{and} \quad \int_{\mathbb{R}^N} |x|^{-b} |Q(x)|^{p+1} \, dx = \frac{p + 1}{A} \|Q\|^2.$$ (2.14)

Now, let us collect some standard estimates related to the Schrödinger equation. The free operator associated to the fractional Schrödinger equation is given by

$$e^{-itD^{2s} \phi} := \mathcal{F}^{-1}(e^{-it|\xi|^{2s}}) \ast \phi.$$ (2.15)

It is classical that (1.1) has the following integral formulation

$$u(t) = e^{-itD^{2s} u_0} + i \int_0^t e^{-i(t-\tau)D^{2s}} [|x|^{-b} |u(\tau)|^{p-1} u(\tau)] \, d\tau.$$ (2.16)

The following dispersive estimate can be found in [19] for instance.

$$\|e^{-itD^{2s} \phi}\|_{L^r(\mathbb{R}^N)} \leq \frac{C}{|t|^{N(\frac{1}{2} - \frac{1}{r})}} \|D^{2N(1-s)(\frac{1}{2} - \frac{1}{r})} \phi\|_{L^r(\mathbb{R}^N)}, \quad \forall \ r \geq 2, \ \forall \ t \neq 0.$$ (2.16)

**Definition 2.6.**
1) A pair \((q,r)\) is said admissible if \(q,r \geq 2\) and
\[
\frac{4N + 2}{2N - 1} \leq q \leq \infty, \quad \frac{2}{q} + \frac{2N - 1}{r} \leq N - \frac{1}{2},
\]
or
\[
2 \leq q \leq \frac{4N + 2}{2N - 1}, \quad \frac{2}{q} + \frac{2N - 1}{r} < N - \frac{1}{2}.
\]

2) \((q,r) \in \Gamma_\gamma\) if it is an admissible pair such that \((N,q,r) \neq (2,2,\infty)\) and
\[
N\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{2s}{q} + \gamma.
\]

Moreover, \(\Gamma := \Gamma_0\).

3) let \(I \subset \mathbb{R}\) be an interval, one denotes the Strichartz spaces
\[
S^\gamma(I) := \bigcap_{(q,r) \in \Gamma_\gamma} L^q(I,L^r), \quad S(I) := S^0(I).
\]

Let us now state some Strichartz estimates [42, 17].

**Proposition 2.7.** Let \(N \geq 2\), \(\gamma \in \mathbb{R}\) and \(u_0 \in L^2_{\text{rad}}\). Then,
\[
1) \|e^{-itD^{2s}}u_0\|_{S^\gamma(I)} \lesssim \|D^s u_0\|;
2) \|u - e^{-itD^{2s}}u_0\|_{S^\gamma(I)} \lesssim \inf_{(q,r) \in \Gamma_\gamma} \|i\partial_t u - D^{2s} u\|_{L^q(I,L^r)};
3) \|u\|_{S(I)} \lesssim \|u_0\| + \inf_{(q,r) \in \Gamma_\gamma} \|i\partial_t u - D^{2s} u\|_{L^q(I,L^r)},\text{ provided that } \frac{N}{2N-1} < s \leq 1.
\]

### 2.2. Auxiliary results

Using a contraction mapping technique based on Strichartz estimates, the Cauchy problem (1.1) is locally well-posed in the energy space [53, Proposition 3].

**Proposition 2.8.** Let \(N \geq 2\), \(s \in \left(\frac{N}{2N-1}, 1\right)\), \(0 < b < 2s\), \(1 < p < p^*\) and \(u_0 \in H^s_{\text{rad}}\). Then, there exist \(0 < T^* \leq \infty\) and a unique maximal solution \(u \in C([0,T^*), H^s_{\text{rad}})\) to (1.1). Moreover, the solution \(u\) satisfies the conservation of mass and energy (1.4) and (1.2).

In trying to apply the arguments in [8] and obtain a localized Morawetz estimate, one encounters serious difficulties due to the nonlocal operator \((-\Delta)^s = D^{2s}\). To handle this difficulty, we use the following representation known as Balakrishnan’s formula
\[
D^{2s} = (-\Delta)^s = \frac{\sin \pi s}{s} \int_0^\infty m^{s-1} \left( \frac{-\Delta}{-\Delta + m} \right) dm.
\]

We define the auxiliary function \(u_m\), for \(m > 0\), by
\[
u_m := c_s R_m u := c_s (m - \Delta)^{-1} u := c_s \mathcal{F}^{-1}\left( \frac{\hat{u}(\xi)}{m + |\xi|^2} \right),\text{ (2.18)}
\]
where
\[
c_s = \sqrt{\frac{\sin \pi s}{\pi}}.
\]
The following formula can be easily derived by using Plancherel’s and Fubini’s theorem as in [8, (2.12)]

\[ s \|D^s u\|^2 = \int_0^\infty m^s \int_{\mathbb{R}^N} |\nabla u_m|^2 \, dx \, dm. \quad (2.19) \]

Here and hereafter, we denote by \( B(R) := \{ x \in \mathbb{R}^N ; |x| \leq R \} \) the ball of \( \mathbb{R}^N \) centered at the origin and with radius \( R > 0 \). Let \( \psi \in C_0^\infty(\mathbb{R}^N) \) be a radial bump function such that

\[ \psi = 1 \text{ on } B \left( \frac{1}{2} \right), \quad \psi = 0 \text{ for } |x| \geq 1 \text{ and } 0 \leq \psi \leq 1. \quad (2.20) \]

For \( R > 0 \), define

\[ \psi_R(x) = \psi \left( \frac{|x|}{R} \right). \quad (2.21) \]

**Lemma 2.9.** Let \( \varphi \in C_0^\infty(\mathbb{R}^N) \) be a real-valued function and \( v \in H^1(\mathbb{R}^N) \). Then

\[ \int_{\mathbb{R}^N} |\nabla(\varphi v)|^2 \, dx = \int_{\mathbb{R}^N} \varphi^2 |\nabla v|^2 \, dx - \int_{\mathbb{R}^N} \varphi \Delta \varphi |v|^2 \, dx, \quad (2.22) \]

\[ |\nabla (\varphi v)_m|^2 = |\nabla (\varphi v)_m|^2 - \nabla (\varphi v_m) \cdot \nabla (R_m[-\Delta, \varphi] \bar{v}_m) - \nabla (R_m[-\Delta, \varphi] v_m) \cdot \nabla (\varphi \bar{v}_m) \quad (2.23) \]

and

\[ s \|D^s (\varphi v)\|^2 = \int_0^\infty \int_{\mathbb{R}^N} m^s |\nabla (\varphi v)_m|^2 \, dx \, dm - \int_0^\infty \int_{\mathbb{R}^N} m^s |\nabla (\varphi v_m)|^2 \, dx \, dm \\
+ \int_0^\infty \int_{\mathbb{R}^N} m^s \varphi^2 |\nabla v_m|^2 \, dx \, dm - \int_0^\infty \int_{\mathbb{R}^N} m^s \varphi \Delta \varphi |v_m|^2 \, dx \, dm. \quad (2.24) \]

**Proof.** The proof of (2.22) is straightforward and uses integration by parts. In particular, we obtain the useful identity

\[ \int_0^\infty \int_{\mathbb{R}^N} m^s |\nabla (\varphi v_m)|^2 \, dx \, dm = \int_0^\infty \int_{\mathbb{R}^N} m^s \varphi^2 |\nabla v_m|^2 \, dx \, dm \\
- \int_0^\infty \int_{\mathbb{R}^N} m^s \varphi \Delta \varphi |v_m|^2 \, dx \, dm. \quad (2.25) \]

To prove (2.23), let us write using computation done in the proof of [56, Lemma 4.8],

\[ \nabla (\varphi v)_m = \nabla (\varphi v_m) - \nabla [\varphi, R_m] v \\
= \nabla (\varphi v_m) - \nabla (R_m[-\Delta + m, \varphi] R_m v) \\
= \nabla (\varphi v_m) - \nabla (R_m[-\Delta, \varphi] v_m). \]

Hence

\[ |\nabla (\varphi v)_m|^2 = |\nabla (\varphi \bar{v})_m (\nabla (\varphi v_m) - \nabla (R_m[-\Delta, \varphi] v_m)) | \quad (2.26) \]

\[ = \nabla (\varphi v_m) (\nabla (\varphi \bar{v}_m) - \nabla (R_m[-\Delta, \varphi] \bar{v}_m) - \nabla (R_m[-\Delta, \varphi] v_m) \nabla (\varphi \bar{v})_m) \]

\[ = |\nabla (\varphi v_m)|^2 - \nabla (\varphi v_m) \nabla (R_m[-\Delta, \varphi] \bar{v}_m) - \nabla (R_m[-\Delta, \varphi] v_m) \nabla (\varphi \bar{v}). \]
Finally, the proof of (2.24) follows easily from (2.23) and (2.25).

Now, let us give an improvement of [56, Lemma 4.2].

**Lemma 2.10.** Let $N \geq 2$, $\theta \in (0, 1)$ and $u \in L^2(\mathbb{R}^N)$. Then

$$
\left| \int_0^\infty \int_{\mathbb{R}^N} m^\theta \psi_R \Delta \psi_R |u_m|^2 \, dx \, dm \right| \leq C \, R^{-2\theta},
$$

(2.27)

where $u_m$ given by (2.18) and $C = C(N, s, \theta, \psi, ||u||)$ is a positive constant depending only on $N, s, \theta, \psi$ and $||u||$.

**Proof.** Note that, for any $0 \leq \alpha < 2$, we have

$$
||D^\alpha u_m|| \lesssim m^\frac{\alpha}{2} ||u||.
$$

(2.28)

Indeed, by (2.18), we get

$$
||D^\alpha u_m|| = c_s \left\| \frac{\xi^\alpha}{m + |\xi|^2} \hat{u}(\xi) \right\|
\leq C_{N,s} \left\| \frac{t^\alpha}{m + t^2} \right\|_{L^\infty(\mathbb{R}^+)} ||u||
\leq C_{N,s} m^\frac{\alpha}{2} ||u||.
$$

Next, for $M > 0$ we write

$$
\left| \int_0^\infty \int_{\mathbb{R}^N} m^\theta \psi_R \Delta \psi_R |u_m|^2 \, dx \, dm \right| \leq (I) + (II),
$$

where

$$(I) := \left| \int_0^M \int_{\mathbb{R}^N} m^\theta \psi_R \Delta \psi_R |u_m|^2 \, dx \, dm \right|,$$

$$(II) := \left| \int_M^\infty \int_{\mathbb{R}^N} m^\theta \psi_R \Delta \psi_R |u_m|^2 \, dx \, dm \right|.
$$

As in [56, Lemma 4.2], we have

$$(II) \lesssim R^{-2} M^{\theta - 1}.
$$

To handle the first term $(I)$ let us choose $\alpha = 1 - \frac{\theta}{2}$ and write by Hölder’s inequality, (2.2) and (2.28),

$$(I) \leq \int_0^M m^\theta \|\psi_R\|_{\frac{N}{2\alpha}} \|\Delta \psi_R\|_{\infty} \|u_m\|^{2\frac{N}{N-2\alpha}} \, dm
\lesssim R^{2\alpha - 2} \int_0^M m^\theta m^{\alpha - 2} \, dm
\lesssim R^{-\theta} \int_0^M m^{\frac{\theta}{2} - 1} \, dm
\lesssim R^{-\theta} m_{\frac{\theta}{2}}.
$$

We conclude the proof by choosing $M = R^{-2}$.  \qed
We also need the following estimates.

**Lemma 2.11.** Let $u \in H^s(\mathbb{R}^N)$ with $s \in (\frac{1}{2}, 1)$ and $\theta \in (0, s)$. Then,

$$
\int_0^\infty \int_{\mathbb{R}^N} m^\theta |\Delta \psi_R|^2 |u_m|^2 \, dx \, dm \lesssim \frac{1}{R^2},
$$

(2.29)

$$
\|\psi_R \ u\|_{H^\theta} \leq C(s, \psi) \|u\|_{H^s}, \quad R \geq 1,
$$

(2.30)

and

$$
\int_0^\infty \int_{\mathbb{R}^N} m^\theta |\nabla \psi_R|^2 |\nabla u_m|^2 \, dx \, dm \lesssim \frac{1}{R^2}.
$$

(2.31)

**Proof.** First, let us prove (2.29). For $M > 0$ we write

$$
\int_0^\infty \int_{\mathbb{R}^N} m^\theta |\Delta \psi_R|^2 |u_m|^2 \, dx \, dm = \int_0^M \int_{\mathbb{R}^N} m^\theta |\Delta \psi_R|^2 |u_m|^2 \, dx \, dm + \int_M^\infty \int_{\mathbb{R}^N} m^\theta |\Delta \psi_R|^2 |u_m|^2 \, dx \, dm.
$$

The second term in the RHS can be estimated as follows

$$
\int_M^\infty \int_{\mathbb{R}^N} m^\theta |\Delta \psi_R|^2 |u_m|^2 \, dx \, dm \leq \int_M^\infty m^\theta \|\Delta \psi_R\|^2_\infty \|u_m\|^2 \, dm
\lesssim R^{-4} \int_M^\infty m^{-2} \, dm
\lesssim R^{-4} M^{\theta-1}.
$$

To estimate the first term in the RHS, we use Hölder’s inequality together with (2.2) and (2.28) to obtain

$$
\int_0^M \int_{\mathbb{R}^N} m^\theta |\Delta \psi_R|^2 |u_m|^2 \, dx \, dm \leq \int_0^M m^\theta \|\Delta \psi_R\|^2_{N/\alpha} \|\Delta \psi_R\|_{\infty} \|u_m\|^2_{N-2N/2\alpha} \, dm
\lesssim R^{2\alpha-4} \int_0^M m^{\theta-\alpha} \, dm
\lesssim R^{-\theta-2} M^{\theta/2},
$$

where $\alpha = 1 - \theta/2$. Choosing $M = R^{-2}$ yields (2.29).

Next, we turn to (2.30). Since $\theta \in (0, s)$ then $\|\psi_R u\|_{H^\theta} \leq \|\psi_R u\|_{H^s}$. Using (2.1), we conclude the proof of (2.30).

Finally, let us prove (2.31). By (2.19) and the fact that $\theta \in (0, s)$, we have

$$
\int_0^\infty \int_{\mathbb{R}^N} m^\theta |\nabla \psi_R|^2 |\nabla u_m|^2 \, dx \, dm \lesssim \frac{1}{R^2} \|u\|^2_{H^\theta} \lesssim \frac{1}{R^2} \|u\|^2_{H^s}.
$$

□

**Lemma 2.12.** Let $u \in H^s$ with $s \in (1/2, 1)$. Then

$$
\left| \int_0^\infty \int_{\mathbb{R}^N} m^\theta \psi_R \Delta \psi_R |u_m|^2 \, dx \, dm \right| \lesssim \frac{1}{R},
$$

(2.32)
\[
\left| \int_0^\infty \int_{\mathbb{R}^N} m^s \nabla (R_m[-\Delta, \psi_R] u_m) \cdot \nabla (\psi_R u) \, dx \, dm \right| \lesssim \frac{1}{R}, \quad (2.33)
\]
and
\[
\left| \int_0^\infty \int_{\mathbb{R}^N} m^s \nabla (\psi_R u_m) \cdot \nabla (R_m[-\Delta, \psi_R] u_m) \, dx \, dm \right| \lesssim \frac{1}{R}. \quad (2.34)
\]

**Proof.** The estimate (2.32) follows easily from (2.27) and \(2s > 1\).

The proof of (2.33) uses similar arguments as in [56, Lemma 4.3] together with (2.29)-(2.30)-(2.31).

To prove (2.34) let us first define
\[
I(R) := \left| \int_0^\infty \int_{\mathbb{R}^N} m^s \nabla (\psi_R u_m) \cdot \nabla (R_m[-\Delta, \psi_R] u_m) \, dx \, dm \right|
\]
Note that
\[
[-\Delta, \psi_R] u_m = -(\Delta \psi_R) u_m - 2\nabla \psi_R \cdot \nabla u_m,
\]
and
\[
\|\nabla (R_m w)\| \lesssim m^{-1/2} \|w\|.
\]
Taking \(\beta \in \left(\frac{1}{2s}, 1\right)\) and noticing that \(2s(1-\beta) \in (0, 1)\), \(2s\beta - 1 \in (0, s)\), one writes thanks to Cauchy-Schwarz’s inequality
\[
I(R) \lesssim \|m^{s \beta - \frac{1}{2}} (\Delta (\psi_R) u_m + 2\nabla \psi_R \cdot \nabla u_m)\|_{L^2(\mathbb{R}^+, L^2(\mathbb{R}^N))} \|m^{s(1-\beta)} \nabla (\psi_R u_m)\|_{L^2(\mathbb{R}^+, L^2(\mathbb{R}^N))} \lesssim \frac{1}{R} \|m^{s(1-\beta)} \nabla (\psi_R u_m)\|_{L^2(\mathbb{R}^+, L^2(\mathbb{R}^N))},
\]
where we have used (2.29) and (2.30). Now, applying Lemma 2.10, one has
\[
\|m^{s(1-\beta)} \nabla (\psi_R u_m)\|_{L^2(\mathbb{R}^+, L^2(\mathbb{R}^N))}^2 = \int_0^\infty m^{2s(1-\beta)} \int_{\mathbb{R}^N} |\nabla (\psi_R u_m)|^2 \, dx \, dm
\]
\[
= \int_0^\infty m^{2s(1-\beta)} \int_{\mathbb{R}^N} (\psi_R^2 |\nabla u_m|^2 - \psi_R \Delta \psi_R |u_m|^2) \, dx \, dm
\]
\[
\lesssim \|u\|_{H^{s(1-\beta)}}^2 + \frac{1}{R^{4s(1-\beta)}} \lesssim \|u\|_{H^s}^2 + \frac{1}{R}.
\]
This gives (2.34) as desired. \[\square\]

As a consequence, we obtain

**Lemma 2.13.** Let \(u \in H^s(\mathbb{R}^N)\). Then, as \(R \to \infty\), we have
\[
s \|D^s (\psi_R u)\| \leq \int_0^\infty \int_{\mathbb{R}^N} m^s \psi_R^2 |\nabla u_m|^2 \, dx \, dm + O\left(\frac{1}{R}\right)
\]
\[
\leq s \|D^s u\| + O\left(\frac{1}{R}\right). \quad (2.35)
\]
Proof. Using (2.24) and (2.26), we have
\[ s\|D^s(\psi_R u)\|^2 = \int_0^\infty \int_{\mathbb{R}^N} m^s \psi_R^2 |\nabla u_m|^2 \, dx \, dm - \int_0^\infty \int_{\mathbb{R}^N} m^s \psi_R \Delta \psi_R |u_m|^2 \, dx \, dm \]
\[- \int_0^\infty \int_{\mathbb{R}^N} m^s \nabla(\psi_R u_m) \cdot \nabla(R_m[-\Delta, \psi_R]u_m) \, dx \, dm \]
\[- \int_0^\infty \int_{\mathbb{R}^N} m^s \nabla(R_m[-\Delta, \psi_R]u_m) \cdot \nabla(\psi_R \tilde{u})_m \, dx \, dm. \]
It follows from the estimates (2.32)-(2.33)-(2.34) that
\[ s\|D^s(\psi_R u)\|^2 \leq \int_0^\infty \int_{\mathbb{R}^N} m^s \psi_R^2 |\nabla u_m|^2 \, dx \, dm + O\left(\frac{1}{R}\right) \]
\[ \leq s\|D^su\|^2 + O\left(\frac{1}{R}\right), \text{ as } R \to \infty. \]
This finishes the proof of Lemma 2.13. □

3. Variational analysis

Recall that \( Q \) stands for the unique nonnegative radially symmetric decreasing solution to (2.13). The following inequality will be useful in obtaining a coercivity result.

Lemma 3.1. Let \( u \in H^s \). Then
\[ P[u] \leq \frac{p+1}{B} \left( \frac{M[u]^{\gamma_c} P[u]}{M[Q]^{\gamma_c} P[Q]} \right)^{\frac{B}{p-2}} \|D^su\|^2, \tag{3.1} \]
where \( B \) is given by (2.11).

Proof. Thanks to Pohozaev identities (2.14) (see also [53, Theorem 2.2]), one has
\[ P[Q] = \frac{p+1}{A} M[Q] = \frac{p+1}{B} \|D^sQ\|^2. \]
Using the Gagliardo-Nirenberg inequality (2.10), the expression of \( K_{opt} \) given by (2.12) and the identities \((p-1)s_c = s(B-2)\) and \( \gamma_c(B-2) = A\), one writes
\[
\begin{align*}
[P[u]]^\frac{B}{p} & \leq K_{opt} \left(\|u\|^{2\gamma_c} P[u]\right)^{\frac{B}{p-2}} \|D^su\|^B \\
& \leq \frac{p+1}{A} \left( \frac{A}{B} \right)^{\frac{B}{p-2}} \|Q\|^{-(p-1)} \left( M[u]^{\gamma_c} P[u] \right)^{\frac{B}{p-2}} \|D^su\|^B \\
& \leq \frac{1+p}{A} \left( \frac{A}{B} \right)^{\frac{B}{p-2}} \left( M[u]^{\gamma_c} P[u] \right)^{\frac{B}{p-2}} \left( \frac{M[Q]^{\gamma_c} P[Q]}{M[Q]^{\gamma_c} P[Q]} \right)^{\frac{B}{p-2}} \|D^su\|^B \\
& \leq \left( \frac{A [P[Q]]}{B M[Q]} \right)^{\frac{B}{p}} \left( \frac{M[u]^{\gamma_c} P[u]}{M[Q]^{\gamma_c} P[Q]} \right)^{\frac{B}{p-2}} \left( 1+p \|D^su\|^2 \right)^{\frac{B}{p-2}}. \\
\end{align*}
\]
This leads to (3.1) as desired. □
As a consequence of the above lemma, we obtain the following coercivity result.

**Corollary 3.2.** Let \( u \in H^s \) and \( \varepsilon \in (0,1) \) satisfying
\[
P[u][M[u]]^\gamma \leq (1 - \varepsilon)P[Q][M[Q]]^\gamma.
\] (3.2)

Then,
\[
P[u] \leq \frac{p + 1}{B} (1 - \varepsilon) \frac{B^{p - 2}}{p + 1} \|D^s u\|^2 \leq \frac{p + 1}{B} \|D^s u\|^2,
\] (3.3)

and
\[
\|D^s u\|^2 - \frac{B}{p + 1} P[u] \geq c(\varepsilon, B) \|D^s u\|^2,
\] (3.4)

where \( c(\varepsilon, B) := 1 - (1 - \varepsilon)^\frac{B^{p - 2}}{B} > 0 \). Moreover, for \( \varepsilon \) small enough, we have
\[
E[u] \geq \frac{B - 2}{B} \|D^s u\|^2.
\] (3.5)

**Proof.** Inequality (3.3) follows immediately from (3.1) and (3.2). To prove (3.4) we use the first inequality in (3.3) and the fact that \( B > 2 \) and \( \varepsilon \in (0,1) \). Finally, using the first inequality in (3.3), we infer
\[
E[u] = \|D^s u\|^2 - \frac{2}{1 + p} P[u]
\geq \left(1 - \frac{2}{B} \left(1 - \varepsilon\right)^\frac{B^{p - 2}}{B}\right) \|D^s u\|^2.
\]

Since \( 1 - \frac{2}{B} \left(1 - \varepsilon\right)^\frac{B^{p - 2}}{B} \to 1 - \frac{2}{B} \) as \( \varepsilon \to 0 \) and \( B > 2 \), we get (3.5). \( \square \)

**Remark 3.3.** Since
\[
P[\psi_R u] \leq P[u] \quad \text{and} \quad M[\psi_R u] \leq M[u], \quad \forall \quad R > 0,
\]
inequalities (3.3)-(3.4) remain true for \( \psi_R u \) instead of \( u \). Namely, we have
\[
P[\psi_R u] \leq \frac{p + 1}{B} \|D^s(\psi_R u)\|^2,
\] (3.6)

and
\[
\|D^s(\psi_R u)\|^2 - \frac{B}{p + 1} P[\psi_R u] \geq c(\varepsilon, B) \|D^s(\psi_R u)\|^2.
\] (3.7)

**Remark 3.4.** The solution is global by (3.5).

4. Morawetz estimates

In this section, we assume that \( N \geq 2, s \in (\frac{N}{2N-1},1), b \in (0,2s), p \in (p_*,p^*) \) and \( u \in C([0,\infty); H^s(\mathbb{R}^N)) \) is a global solution of (1.1).

Consider a smooth real-valued function \( f \) such that
\[
0 \leq f'' \leq 1 \quad \text{and} \quad f(r) = \begin{cases} 
\frac{r^2}{2}, & \text{if } 0 \leq r \leq 1; \\
1, & \text{if } r \geq 2.
\end{cases}
\] (4.1)
Lemma 4.1. We have
\[ u \text{ where } M \]

The next lemma gives the evolution of \( M \).

Denote the differential operator \( \Gamma \) by
\[ \Gamma \phi = -i \left( \text{div}(\phi \nabla f_R) + \nabla f_R \cdot \nabla \phi \right), \]
which satisfies
\[ < u(t), \Gamma u(t) > = M[u(t)]. \]

Using \( \Gamma \), the time derivative of \( M[u(t)] \) reads
\[ \frac{d}{dt} M[u(t)] = \frac{d}{dt} A(t) + B(t), \]
where the commutator of \( X \) and \( Y \) is \([X,Y] := XY - YX\). Thanks to the computations done in \([8]\), one writes
\[ A(t) = \int_0^\infty m^s \int_{\mathbb{R}^N} \left( 4 \partial_k \bar{u}_m \partial_k^2 f_R \partial_k u_m - \Delta^2 f_R |u_m|^2 \right) dx \ dm \]
\[ = 4 \int_0^\infty m^s \int_{|x|<R} |\nabla u_m|^2 dx \ dm + 4 \int_0^\infty m^s \int_{|x|<2R} f'' \left( \frac{|x|}{R} \right) |\nabla u_m|^2 dx \ dm \]
\[ - \int_0^\infty m^s \int_{|x|angle R} \Delta^2 f_R |u_m|^2 dx \ dm. \]
Let us denote the source term $\mathcal{N} := |x|^{-b}|u|^{p-1}u$ and compute

$$B(t) = - <u, \frac{\mathcal{N}}{u} \nabla f_R \cdot \nabla u> - <u, \frac{\mathcal{N}}{u} \text{div}(u \nabla f_R)> + <u, \nabla f_R \nabla \mathcal{N}> + <u, \text{div}[\mathcal{N} \nabla f_R]>
$$

$$= -2 <u, \frac{\mathcal{N}}{u} \nabla f_R \nabla u> + 2 <u, \nabla f_R \nabla \mathcal{N}>
$$

$$= 2 \int_{\mathbb{R}^N} |u|^2 \nabla f_R \nabla \left[ \frac{\mathcal{N}}{u} \right] dx.$$

Integrating by parts yields

$$B(t) = 2 \int_{\mathbb{R}^N} |u|^2 \nabla f_R \nabla |x|^{-b}|u|^{p-1} dx$$

$$= -2 \int_{\mathbb{R}^N} \nabla(|u|^2) \nabla f_R + |u|^2 \Delta f_R |x|^{-b}|u|^{p-1} dx$$

$$= -2 \int_{\mathbb{R}^N} \nabla(|u|^2) \nabla f_R |x|^{-b}|u|^{p-1} dx - 2 \int_{\mathbb{R}^N} \Delta f_R |x|^{-b}|u|^{p+1} dx$$

$$= -2 \int_{\mathbb{R}^N} \Delta f_R |x|^{-b}|u|^{p+1} dx - \frac{4}{1+p} \int_{\mathbb{R}^N} |x|^{-b} \nabla f_R \nabla (|u|^{1+p}) dx.$$

Hence,

$$B(t) = - \frac{2(p-1)}{1+p} \int_{\mathbb{R}^N} \Delta f_R |x|^{-b}|u|^{p+1} dx - \frac{4b}{1+p} \int_{\mathbb{R}^N} \frac{x \cdot \nabla f_R}{|x|^2} |x|^{-b}|u|^{1+p} dx$$

$$- \frac{4b}{1+p} \int_{\mathbb{R}^N} \frac{x \cdot \nabla f_R}{|x|^2} |x|^{-b}|u|^{1+p} dx$$

$$= - \frac{4b}{1+p} \int_{|x|<R} |x|^{-b}|u|^{p+1} dx + \frac{2(p-1)}{1+p} \int_{|x|>R} (N - \Delta f_R) |x|^{-b}|u|^{p+1} dx$$

$$- \frac{4b}{1+p} \int_{|x|>R} \frac{x \cdot \nabla f_R}{|x|^2} |x|^{-b}|u|^{1+p} dx.$$

Thus, with the above calculus we obtain (4.5) as desired.

**Corollary 4.2.** We have

$$\int_0^T \int_{|x|<R} |x|^{-b}|u(t,x)|^{1+p} dx dt \lesssim R + TR^{-b}, \quad \text{for any} \quad T, R > 0. \quad (4.6)$$

**Proof.** By [8, Lemma A.2], one has

$$\left| \int_0^\infty m^s \int_{\mathbb{R}^N} \Delta^2 f_R |u_m|^2 dx dm \right| \lesssim R^{-2s}.$$

Thus, using the properties of $f_R$ and the fact that $b < 2s$, one writes

$$\frac{d}{dt} M_R[u(t)] \geq 4 \int_0^\infty m^s \int_{|x|<R} |\nabla u_m|^2 dx dm - \frac{Asb}{1+p} \int_{|x|<R} |x|^{-b}|u|^{1+p} dx + O(R^{-b}).$$
Taking into account Corollary 3.2 and Lemma 2.13, with the fact that $0 \leq \psi \leq 1$ and $\psi = 0$ on $B(\frac{1}{2})$, one gets

$$\frac{d}{dt} M_R[u(t)] \geq 4 \int_0^\infty m^s \int_{|x|<R} |\nabla u_m|^2 \, dx \, dm - \frac{4sB}{1+p} \int_{|x|<R} |x|^{-b}|u|^{1+p} \, dx + O(R^{-b})$$

$$\geq 4 \int_0^\infty m^s \int_{\mathbb{R}^N} \psi_2^2 |\nabla u_m|^2 \, dx \, dm + 4 \int_0^\infty m^s \int_{R/2<|x|<R} (1 - \psi_2^2)|\nabla u_m|^2 \, dx \, dm$$

$$- \frac{4sB}{1+p} P[\psi_R u] - \frac{4sB}{1+p} \int_{R/2<|x|<R} |x|^{-b}(1 - \psi_R^{p+1})|u|^{1+p} \, dx + O(R^{-b})$$

$$\geq 4\|D^s(\psi_R u)\|^2 - \frac{4sB}{1+p} P[\psi_R u] + O(R^{-b})$$

$$\geq cP[\psi_R u] + O(R^{-b}).$$

Hence,

$$\sup_{t \in [0,T]} |M_R[u(t)]| \geq \int_0^T P[\psi_R u] \, ds + O(R^{-b})T$$

$$\geq \int_0^T \int_{|x|<\frac{R}{2}} |x|^{-b}|u|^{1+p} \, dx \, ds + O(R^{-b})T.$$

Thus, with previous computation via [8, Lemma A.1] and the assumption $s > \frac{1}{2}$, one gets

$$\int_0^T \int_{|x|<\frac{R}{2}} |x|^{-b}|u|^{1+p} \, dx \, dt \leq C \left( \sup_{[0,T]} |M_R[u(t)]| + TR^{-b} \right)$$

$$\leq C \left( R + TR^{-b} \right).$$

□

**Corollary 4.3.** For any sequence $R_n \to \infty$, there exists a sequence $t_n \to \infty$ such that

$$\lim_{n \to \infty} \int_{|x|<R_n} |x|^{-b}|u(t_n,x)|^{1+p} \, dx = 0. \quad (4.7)$$

**Proof.** Let $(R_n)$ be a sequence of positive numbers tending to infinity. By taking $T_n = R_n^{1+b}$, it follows from (4.6) that

$$\frac{2}{T_n} \int_{T_n/2}^{T_n} \int_{|x|<R_n} |x|^{-b}|u(t,x)|^{1+p} \, dx \, dt \lesssim R_n^{-b} \to 0 \text{ as } n \to \infty.$$ 

The proof follows using the integral mean value theorem. □
5. Scattering criterion

Here and hereafter, one denotes the real numbers
\[ a := \frac{s(1 + p - \theta)}{s - s_c}, \quad d := \frac{s(1 + p - \theta)}{s + (p - \theta)s_c}; \]
\[ q := \frac{2s(1 + p - \theta)}{2(s - s_c) + s_c(1 + p - \theta)}; \]
\[ r := \frac{2N(1 + p - \theta)}{(N - 2s_c)(1 + p - \theta) - 4(s - s_c)}. \]

Clearly, one can choose \( \theta > 0 \) small enough so that
\[(q, r) \in \Gamma, \quad (a, r) \in \Gamma_{s_c}, \quad (d, r) \in \Gamma_{-s_c} \quad \text{and} \quad (p - \theta)d' = a.\]

This section is devoted to the proof of the following scattering criterion.

**Proposition 5.1.** Take the assumptions of Theorem 1.1. Let \( u \in C(\mathbb{R}, H^s_{\text{rad}}) \) be a global solution to (1.1). Assume that
\[ 0 < \sup_{t \geq 0} \| u(t) \|_{H^s} := E < \infty. \tag{5.1} \]
Then, there exist \( R, \varepsilon > 0 \) depending on \( E, N, p, b, s \) such that if
\[ \liminf_{t \to \infty} \int_{|x| < R} |u(t, x)|^2 \, dx < \varepsilon^2, \tag{5.2} \]
then, \( u \) scatters for positive time.

Before proving Proposition 5.1, let us first give a technical result.

**Lemma 5.2.** Let \( I \) be a time slab. There exists \( \theta > 0 \) small enough such that the global solution \( u \) to (1.1) satisfies
1) \[ \| u - e^{-it\Delta^{2s}}u_0 \|_{L^q(I, L^r)} \lesssim \| u \|_{L^\infty(I, H^r)} \| u \|_{L^p(I, L^r)}^{p - \theta}; \]
2) \[ \|(1 + |\nabla|^s)(u - e^{-it\Delta^{2s}}u_0)\|_{L^q(I, L^r)} \lesssim \| u \|_{L^\infty(I, H^r)} \| u \|_{L^p(I, L^r)}^{p - 1 - \theta} \| \nabla^s u \|_{L^q(I, L^r)}. \]

**Proof of Lemma 5.2.**

1) Using Hölder’s inequality, one writes
\[ \||x|^{-b}|u|^{p-1}u|_{L^{r'}(|x| < 1)} = \||x|^{-b}|u|^{p-\theta}|u|^\theta|_{L^{r'}(|x| < 1)} \leq \||x|^{-b}|u|^\theta|_{L^p(|x| < 1)} \| u \|_{L^{N-2s}}^{2N-2s} \| u \|_{L^r}^{p-\theta}, \]
where
\[ \frac{1}{r'} = \frac{1}{\mu} + \frac{\theta(N - 2s)}{2N} + \frac{p - \theta}{r}, \]
promised that
\[ 0 < \theta < \frac{2N}{N - 2s}, \quad 0 < \theta < p, \quad p - r < \theta, \quad \frac{b}{N} < \frac{1}{\mu}. \]
The last integrability condition reads

\[
\frac{b}{N} < \frac{1}{\mu} = 1 - \frac{\theta (N - 2s)}{2N} - \frac{1 + p - \theta}{r} \tag{5.3}
\]

This is equivalent to

\[
2b < 2N - \theta (N - 2s) - (N - 2s_e)(1 + p - \theta) + 4(s - s_e)
\]

This is obviously satisfied and gives via Sobolev embedding

\[
\left\| |x|^{-b} |u|^{p-1} u \right\|_{L^{q'}(I, L^{r'}(|x| < 1))} \lesssim \left\| u \right\|_{L^q(\mathbb{R}^N)}^{\theta} \left\| u(t) \right\|_{L^r(I)}^{p-\theta} \left\| u \right\|_{L^{r'}(I)}.
\]

Let us estimate the same term on the complementary to the unit ball. Using Hölder’s inequality, we have

\[
\left\| |x|^{-b} |u|^{p-1} u \right\|_{L^{q'}(|x| > 1)} \leq \left\| |x|^{-b} \right\|_{L^{q'}(|x| > 1)} \left\| u \right\|_{L^{r_1}} \left\| u \right\|_{L^{r'}}^{p-\theta}.
\]

Here, the integrability condition reads

\[
\frac{b}{N} > \frac{1}{\mu_1} = 1 - \frac{\theta}{r_1} - \frac{1 + p - \theta}{r} \tag{5.4}
\]

This is equivalent to

\[
\frac{\theta}{r_1} > 1 - \frac{b}{N} - \frac{(N - 2s_e)(1 + p - \theta) - 4(s - s_e)}{2N}
\]

Thus, it is sufficient to choose \( r_1 \in [2, \frac{2N}{N - 2s_e}) \). The first point follows with Strichartz estimates arguing as previously.

2) Using the first point with the equality \( \frac{1}{q} = \frac{p - 1 - \theta}{a} + \frac{1}{q'} \), one has by Strichartz estimates

\[
\left\| u - e^{-itD^{2s}} u_0 \right\|_{L^q(I, L^r)} \lesssim \left\| |x|^{-b} |u|^{p-1} u \right\|_{L^{q'}(I, L^{r'})}
\]

\[
\lesssim \left\| u \right\|_{L^q(\mathbb{R}^N)}^{\theta} \left\| u(t) \right\|_{L^r(I)}^{p-\theta} \left\| u \right\|_{L^{r'}(I)}
\]

\[
\lesssim \left\| u \right\|_{L^q(\mathbb{R}^N)}^{\theta} \left\| u \right\|_{L^{r'}(I, L^{r'})} \left\| u \right\|_{L^q(I, L^r)}.
\]
Now, let us estimate the term
\[ \|D^s [x^{-b} |u|^{p-1} u]\|_{L^r(I,L^r')} \]
\[ \lesssim \|x^{-b-s} |u|^{p-1} u\|_{L^r(|x|<1)} + \|x^{-b-s} |u|^{p-1} u\|_{L^r(|x|>1)} \]
\[ + \|x^{-b} D^s [u|^{p-1} u]\|_{L^r'(I,L^r')}(|x|<1) + \|x^{-b} D^s [u|^{p-1} u]\|_{L^r'(I,L^r')}(|x|>1). \]

Using Hölder’s inequality, one writes
\[ \|x^{-b-s} |u|^{p-1} u\|_{L^r'(|x|<1)} \leq \|x^{-b-s} \|_{L^p(|x|<1)} \|u\|_{L^r}^{p-1-\theta} \|u\|_{L^r} \]
\[ \lesssim \|u\|_{H^s}^{\theta} \|u\|_{L^r}^{p-1-\theta} \|D^s u\|_{L^r}. \]

Here, one has the same condition (5.3) above
\[ \frac{s+b}{N} < \frac{1}{\mu} = 1 + \frac{s}{N} + \theta (-\frac{1}{2} + \frac{s}{N}) - \frac{p+1-\theta}{r}. \]

On the complementary of the unit ball, one has
\[ \|x^{-b-s} |u|^{p-1} u\|_{L^r'(|x|>1)} \leq \|x^{-b-s} \|_{L^p(|x|>1)} \|u\|_{L^r}^{p-1-\theta} \|u\|_{L^r} \]
\[ \lesssim \|u\|_{H^s}^{\theta} \|u\|_{L^r}^{p-1-\theta} \|D^s u\|_{L^r}. \]

Here, one has the same condition (5.4) above
\[ \frac{s+b}{N} > \frac{1}{\mu_1} = 1 + \frac{s}{N} + \frac{\theta}{r_1} - \frac{p+1-\theta}{r}. \]

Thus, one can choose \( r_2 \in \left[2, \frac{2N}{N-2s+1}\right) \). Now, using the fractional chain rule in Lemma 2.1, one gets
\[ \|x^{-b} D^s [u|^{p-1} u]\|_{L^r'(|x|<1)} \lesssim \|x^{-b} \|_{L^p(|x|<1)} \|u\|_{L^r_2}^{p-1-\theta} \|D^s u\|_{L^r} \]
\[ \lesssim \|u\|_{H^s}^{\theta} \|u\|_{L^r}^{p-1-\theta} \|D^s u\|_{L^r}. \]

Here
\[ \begin{cases} \frac{1}{N} < \frac{1}{\mu_2} = 1 - \frac{\theta}{r_2} - \frac{p+1-\theta}{r} ; \\ 2 \leq r_2 \leq \frac{2N}{N-2s} \end{cases} \]

This condition is satisfied as before for \( r_2 = \frac{2N}{N-2s} \). The last term reads
\[ \|x^{-b} D^s [u|^{p-1} u]\|_{L^r'(|x|>1)} \lesssim \|x^{-b} \|_{L^p(|x|>1)} \|u\|_{L^r_3}^{p-1-\theta} \|D^s u\|_{L^r} \]
\[ \lesssim \|u\|_{H^s}^{\theta} \|u\|_{L^r}^{p-1-\theta} \|D^s u\|_{L^r}. \]

Here
\[ \begin{cases} \frac{1}{N} > \frac{1}{\mu_3} = 1 - \frac{\theta}{r_3} - \frac{p+1-\theta}{r} ; \\ 2 \leq r_3 \leq \frac{2N}{N-2s} \end{cases} \]
This condition is satisfied as before for $r_3 \in \left[ 2, \frac{2N}{N + 2s} \right)$. Now, plugging all the above estimates together and using the equality $\frac{1}{q'} = \frac{p-1-\frac{1}{a}}{a} + \frac{1}{q}$, we obtain

\[
\| D^s |x|^{-b} |u|^{p-1} u \|_{L_t^q(I, L_x^r)} \lesssim \| u \|_{H^s}^\theta \| u \|_{L_x^r}^{p-1-\theta} \| D^s u \|_{L_t^q(I, L_x^r)} \lesssim \| u \|_{H^s}^\theta \| u \|_{L^q(I, L^r)}^{p-1-\theta} \| D^s u \|_{L^q(I, L^r)}.
\]

The proof is finished via Strichartz estimates.

The key of the proof of the scattering criterion is the next result.

**Proposition 5.3.** Suppose that the assumptions of Proposition 5.1 are fulfilled. Then, for any $\varepsilon > 0$, there exist $T, \mu > 0$ satisfying

\[
\| e^{-i(t-T)D^{2s}} u(T) \|_{L^q(T, \infty), L^r} \lesssim \varepsilon^\mu. \tag{5.5}
\]

**Proof.** By the integral formula

\[
e^{-i(t-T)D^{2s}} u(T) = e^{-iD^{2s}} u_0 + i \int_0^T e^{-i(t-\tau)D^{2s}} |x|^{-b} |u|^{p-1} u \, d\tau = e^{-iD^{2s}} u_0 + i \left( \int_0^{T-\varepsilon^\beta} + \int_{T-\varepsilon^\beta}^T \right) e^{-i(t-\tau)D^{2s}} |x|^{-b} |u|^{p-1} u \, d\tau := e^{-iD^{2s}} u_0 + F_1 + F_2. \tag{5.6}
\]

- **Estimate of the linear term.**
  Since $(a, \frac{2N}{N + 2s}) \in \Gamma$, by Strichartz estimate and Sobolev embedding, one has

\[
\| e^{-iD^{2s}} u_0 \|_{L^q((T, \infty), L^r)} \lesssim \| \nabla |s| e^{-iD^{2s}} u_0 \|_{L^q((T, \infty), L^q)} \lesssim \| u_0 \|_{H^s}.
\]

Thus, one may choose $T_0 > \varepsilon^\beta > 0$, where $\beta > 0$, such that

\[
\| e^{-iD^{2s}} u_0 \|_{L^q((T_0, \infty), L^r)} \leq \varepsilon^2. \tag{5.7}
\]

- **Estimate of the term $F_2$.**
  By the assumption (5.2), one has for $T > \varepsilon^\beta$ large enough,

\[
\int_{\mathbb{R}^N} \psi_R(x) |u(T, x)|^2 \, dx < \varepsilon^2.
\]

Moreover, a computation with the use of (1.1) gives

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \psi_R(x) |u(t, x)|^2 \, dx = 2 \int_{\mathbb{R}^N} \psi_R(x) \mathbb{R}[\dot{u}(t, x) \ddot{u}(t, x)] \, dx = 2 \int_{\mathbb{R}^N} \psi_R(x) \mathbb{I}[D^{2s} u(t, x) \ddot{u}(t, x)] \, dx.
\]
Hence,
\[
\frac{d}{dt} \int_{\mathbb{R}^N} \psi_R |u|^2 \, dx = 2\Im \int_{\mathbb{R}^N} D^s(\psi_R \bar{u}) D^s u \, dx
\]
\[
= 2\Im \int_{\mathbb{R}^N} \bar{u} D^s \psi_R D^s u \, dx
\]
\[
+ 2\Im \int_{\mathbb{R}^N} D^s u \left( D^s(\psi_R \bar{u}) - \bar{u} D^s(\psi_R) - \psi_R D^s(\bar{u}) \right) \, dx.
\]
By Lemma 2.2 with \( \psi_R \) rather than \( v \), one has via Sobolev embedding
\[
\left| \frac{d}{dt} \int_{\mathbb{R}^N} \psi_R(x) |u(t,x)|^2 \, dx \right| \lesssim R^{-s} + \| D^{s_1} \psi_R \|_{L^p} \| D^{s_2} \bar{u} \|_{L^q},
\]
provided that \( s, s_2, p_1, p_2 \) satisfy
\[
s = s_1 + s_2, \quad 2 \leq p_2 \leq \frac{2N}{N - 2(s - s_2)}, \quad \frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}.
\]
Let us choose \( s_1 = \theta s \) and \( p_1 = \frac{2N}{\theta s} \) for some \( \theta \in (0, 1) \). Hence \( p_2 = \frac{2N}{N - \theta s} \) and
\[
\frac{N}{p_1} - s_1 = -\frac{\theta}{2}s, \quad \text{It follows that}
\]
\[
\left| \frac{d}{dt} \int_{\mathbb{R}^N} \psi_R(x) |u(t,x)|^2 \, dx \right| \lesssim R^{-s} + R^{-\frac{\theta}{2}s} \| u \|_{H^s},
\]
\[
\leq C(N, s, \psi, \theta) R^{-\frac{\theta}{2}s},
\]
where \( C(N, s, \psi, \theta) \) comes from the sharp constant in the Sobolev embedding [22, Theorem 1.1, p. 226] and \( \| D^{s_1} \psi \|_{p_1} \). Since
\[
C(N, s, \psi, \theta) \to C(N, s, \psi) > 0 \quad \text{as} \quad \theta \to 1,
\]
we deduce that
\[
\left| \frac{d}{dt} \int_{\mathbb{R}^N} \psi_R(x) |u(t,x)|^2 \, dx \right| \lesssim R^{-\frac{\theta}{2}}.
\]
Therefore, for any \( T - \varepsilon^{-\beta} \leq t \leq T \) and \( R > \varepsilon^{-\frac{2(2+\beta)}{s}} \), we obtain
\[
\| \psi_R u(t) \| \leq \left( \int_{\mathbb{R}^N} \psi_R(x) |u(T,x)|^2 \, dx + C \frac{T - t}{R^2} \right)^{\frac{1}{2}} \leq C \varepsilon.
\]
This gives
\[
\| \psi_R u \|_{L^\infty([T-\varepsilon^{-\beta}, T], L^2)} \leq C \varepsilon.
\]
By Strichartz estimate and Lemma 5.2, one has
\[
\| \psi_R |x|^{-b} |u|^p \|_{L^{s'}((T-\varepsilon^{-\beta},T), L^{s'})} \lesssim \| u \|_{L^\theta((T-\varepsilon^{-\beta}, T), H^s)} \| u \|_{L^{p-1}(T-\varepsilon^{-\beta}, T), L^{p-1}} \| \psi_R u \|_{L^s((T-\varepsilon^{-\beta}, T), L^r)}
\]
\[
\lesssim \| (T - \varepsilon^{-\beta}, T)^{\frac{1}{2} - \frac{\theta}{2}} \| u \|_{L^{\infty}((T-\varepsilon^{-\beta}, T), H^s)} \| \psi_R u \|_{L^s((T-\varepsilon^{-\beta}, T), L^r)}
\]
\[
\lesssim \varepsilon^{-\beta \frac{p-1}{2}} \| \psi_R u \|_{L^\infty((T-\varepsilon^{-\beta}, T), L^{r'})}.
\]
Using Gagliardo-Nirenberg inequality, one has
\[
\|\psi_R^b u\|_{L^{p'}((T-e^{-\beta}T),L^{r'})} \lesssim \varepsilon^{-\beta \frac{p-1}{p}} \|\psi_R^b u\|_{L^\infty(J_1,L^2)} \\
\lesssim \varepsilon^{-\beta \frac{p-1}{p}} \varepsilon^{1-\frac{N}{2}(\frac{1}{b} - \frac{1}{2})} \\
\lesssim \varepsilon^{(1-\beta)(\frac{s-\gamma}{s}(p-1) - \theta)}.
\]

Now, let us estimate
\[
\|\left(1 - \psi_R^b u\right)\|_{L^{p'}(|x|>R/2)} \lesssim \|\left| x \right|^{-b} u^p \|_{L^{p'}}(\|x|>R/2) \\
\lesssim \|\left| x \right|^{-b} u^p \|_{L^p(|x|>R/2)} \| u \|_{L^1} \| u \|_{L^{p'}} \\
\lesssim R^{N-b\mu} \| u \|_{L^1} \| u \|_{L^{p'}}.
\]

Here
\[
b > \frac{N}{\mu} = N - \frac{N\theta}{r_1} - \frac{N(1+p-\theta)}{r} \\
= N - \frac{N\theta}{r_1} - \left(\frac{N}{2} - s_c\right)(1+p-\theta) + 2(s-s_c).
\]

This is equivalent to
\[
r_1 < \frac{N\theta}{-b + N - \left(\frac{N}{2} - s_c\right)(1+p-\theta) + 2(s-s_c)} \\
= \frac{N\theta}{-b + N - \frac{2s-b}{p-1}(p-1+2-\theta) + 2(s-s_c)} \\
= \frac{N(p-1)}{2s-b}.
\]

Since \(p > p_c\), one can choose \(2 \leq r_1 \leq 2^*\). Thus, by Sobolev embedding
\[
\|\left(1 - \psi_R^b u\right)\|_{L^{p'}(|x|>R/2)} \lesssim R^{N-b\mu} \| u \|_{H^1} \| u \|_{L^{p'}} \lesssim R^{N-b\mu} \| u \|_{H^1}^{p-\theta}.
\]

Hence,
\[
\|\left(1 - \psi_R^b u\right)\|_{L^{p'}((T-e^{-\beta}T),L^{r'})} \lesssim R^{N-b\mu} \varepsilon^{\frac{-\theta}{p'}} \\
\lesssim R^{N-b\mu} \varepsilon^{\frac{1+(s-\gamma)(p-1) - \theta}{s(1+p-\theta)}} \\
\lesssim \varepsilon^{\left(1-\beta\right)(\frac{s-\gamma}{s}(p-1) - \theta)}.
\]

Then, choosing \(0 < \beta \ll 1\), there is \(\gamma > 0\) such that
\[
\|F_2\|_{L^p((T,\infty),L^{r'})} \lesssim \varepsilon^{\left(1-\beta\right)(\frac{s-\gamma}{s}(p-1) - \theta)} + \varepsilon^{\left|N-b\mu\right| \frac{2s-b}{s} \frac{\beta(s-s_c)(p-1) - \theta}{s(1+p-\theta)}} \\
\lesssim \varepsilon^\gamma.
\]  

(5.8)
Estimate of the term $F_1$. 
Take $\frac{1}{r} := \frac{1}{2} + \frac{N}{q}$. Then $(a, c) \in \Gamma$ and there is $\lambda \in [0, 1]$ such that $\frac{1}{r} := \frac{\lambda}{c}$. By interpolation via the mass conservation, one writes

$$\|F_1\|_{L^b((T, \infty), L^c)} \lesssim \|F_1\|_{L^b((T, \infty), L^{\frac{\lambda}{c}})} \|F_1\|_{L^b((T, \infty), L^\infty)}^{1-\lambda} \lesssim \|F_1\|_{L^b((T, \infty), L^\infty)}^{1-\lambda}.$$ 

Using the dispersive estimate (2.16), one has for $T \leq t$,

$$\|F_1\|_{L^\infty} \lesssim \int_0^T \frac{1}{(t-s)\frac{a}{2}} \|D^{N(1-s)}(\|x|-b|u|^p-1u)\|_{L^1} ds.$$ 

Thanks to the fractional chain rule Lemma 2.1, we get

$$\|D^{N(1-s)}(\|x|-b|u|^p-1u)\|_{L^1} \lesssim \|\|x|-b-N^s|u|^p\|_{L^1} + \|\|x|-b|u|^p-1D^{N(1-s)}u\|_{L^1}^2 \lesssim \lambda,$$

(5.10)

By Hölder estimate, the term $(I_1) := \|\|x|-b-N^s|u|^p\|_{L^1(\|x|<1)}$ satisfies

$$(I_1) \leq \|\|x|-b-N^s\|_{L^p(\|x|<1)}\|u|^p\|_{L^d} \lesssim \|u\|_{H^s}^p.$$ 

Here, $\mu$ and $d$ satisfy

$$\begin{cases} 
1 = \frac{1}{\mu} + \frac{p}{d}, \\
\mu \leq \frac{N}{6+N(1-s)}, \\
2 \leq d \leq \frac{2N}{N-2s}. 
\end{cases}$$

This requires that

$$d \in \left(\frac{Np}{8N-b}, \infty\right) \cap \left(2, \frac{2N}{N-2s}\right),$$

which is possible if $p < 2\frac{Ns-b}{N-2s}$. Moreover, the condition $p^* \leq 2\frac{Ns-b}{N-2s}$ is equivalent to $s \geq \frac{N}{2(N-1)}$, which is satisfied because $s < 1$ and $N \geq 2$.

By Hölder’s inequality, the term $(I_2) := \|\|x|-b-N^s|u|^p\|_{L^1(\|x|>1)}$ can be estimated as

$$(I_2) \leq \|\|x|-b-N^s\|_{L^\gamma(\|x|>1)}\|u|^e\|_{L^d} \lesssim \|u\|_{H^s}^e.$$ 

Here, $\gamma$ and $e$ satisfy

$$\begin{cases} 
1 = \frac{1}{\gamma} + \frac{p}{e}, \\
\gamma \geq \frac{N}{6+N(1-s)}, \\
2 \leq e \leq \frac{2N}{N-2s}. 
\end{cases}$$

This requires that

$$e \in \left(1, \frac{pN}{Ns-b}\right) \cap \left(2, \frac{2N}{N-2s}\right),$$
which is possible if and only if $p > \frac{2(Ns-b)}{N}$.

Again, by Hölder’s inequality, the term $(II_1) := \| |x|^b |u|^{p-1} D^{N(1-s)} u \|_{L^1(|x|<1)}$ satisfies

\[
(II_1) \leq \| |x|^b \|_{L^\beta(|x|<1)} \| u \|_{L^\beta}^{p-1} \| D^{N(1-s)} u \|
\lesssim \| u \|_{H^s} \| u \|_{L^\beta} \| D^{N(1-s)} u \|
\lesssim \| u \|_{H^s} \| u \|^{\theta} \| D^{s} u \|^{1-\theta}
\lesssim \| u \|_{H^s}^{1-\theta}.
\]

Here, $\theta \in [0, 1]$ and $\beta$ and $f$ satisfy

\[
\begin{aligned}
1 = \frac{1}{\beta} + \frac{p-1}{f} + \frac{1}{2}, \\
\beta < \frac{N}{b}, \\
2 \leq f \leq \frac{2N}{N-2s}, \\
0 \leq N(1-s) \leq s.
\end{aligned}
\]

This is equivalent to

\[
\begin{aligned}
\frac{1}{2} - \frac{p-1}{f} = \frac{1}{f} > \frac{b}{N}, \\
2 \leq f \leq \frac{2N}{N-2s}, \\
s \geq \frac{N}{1+N}.
\end{aligned}
\]

which is possible if $\frac{N-2s}{2N} \leq \frac{1}{f} < \frac{1}{\frac{1}{p-1} \left( \frac{1}{2} - \frac{b}{N} \right) \left( \frac{1}{2} - \frac{b}{N} \right)}$. Thus,

\[
p - 1 < \frac{2N}{N-2s} \left( \frac{1}{2} - \frac{b}{N} \right).
\]

Using again Hölder’s inequality, we estimate $(II_2) := \| |x|^b |u|^{p-1} D^{N(1-s)} u \|_{L^1(|x|>1)}$ as

\[
(II_2) \leq \| |x|^b \|_{L^\alpha(|x|>1)} \| u \|_{L^\alpha}^{p-1} \| D^{N(1-s)} u \|
\lesssim \| u \|_{H^s} \| u \|^{\theta} \| D^{s} u \|^{1-\theta}
\lesssim \| u \|_{H^s}^{1-\theta}.
\]

Here, $\theta \in [0, 1]$, $\alpha$ and $j$ satisfy

\[
\begin{aligned}
1 = \frac{1}{\alpha} + \frac{p-1}{j} + \frac{1}{2}, \\
\alpha > \frac{N}{b}, \\
2 \leq j \leq \frac{2N}{N-2s}, \\
s \geq \frac{N}{1+N}.
\end{aligned}
\]

This leads to

\[
\begin{aligned}
\frac{1}{2} - \frac{b}{N} < \frac{p-1}{2} \leq \frac{p-1}{2}, \\
2 \leq j \leq \frac{2N}{N-2s}, \\
s \geq \frac{N}{1+N}.
\end{aligned}
\]
which is possible if
\[ p - 1 > 1 - \frac{2b}{N}. \]  
(5.12)

Summarize the above estimates, we infer that
\[ 2 - \frac{2b}{N} < p < 1 + \frac{2N}{N - 2s}\left(\frac{1}{2} - \frac{b}{N}\right). \]  
(5.13)

Under the above restrictions (5.13), one has
\[ \|F_1\|_{L^\infty} \lesssim \int_0^{T-\varepsilon^{-\beta}} \frac{1}{(t-\tau)^{\frac{N}{2}}} \|D^{N(1-s)}(|x|^{-b}|u|^{p-1}u)\|_{L^1} d\tau \]
\[ \lesssim \int_0^{T-\varepsilon^{-\beta}} \frac{1}{(t-\tau)^{\frac{N}{2}}} \|u(\tau)\|_{H^s}^p d\tau \]
\[ \lesssim (t-T+\varepsilon^{-\beta})^{1-\frac{N}{2}}. \]

It follows that, if \( N - \frac{1}{2} - \frac{1}{a} > 0 \),
\[ \|F_1\|_{L^p((T,\infty),L^r)} \lesssim \|F_1\|_{L^p((T,\infty),L^\infty)}^{1-\lambda} \left( \int_T^{\infty} (t-T+\varepsilon^{-\beta})^{a[1-\frac{N}{2}]} dt \right)^{\frac{1-\lambda}{a}} \]
\[ \lesssim \varepsilon^{(1-\lambda)\beta\left(\frac{N}{2}-1-\frac{1}{a}\right)}. \]  
(5.14)

Note that the condition \( N - \frac{1}{2} - \frac{1}{b} > 0 \) translate to
\[ \frac{N}{2} - 1 > \frac{s-s_c}{s(1+p)}. \]  
(5.15)

Clearly (5.15) is false for \( N = 2 \) and reads for \( N \geq 3 \), \( Q(p-1) > 0 \), where \( Q \) is the polynomial function
\[ Q(X) := s(N-2)X^2 + ((1+2s)N-6s)X - 2(2s-b). \]

Let us compute
\[ Q(p^*-1) = s(N-2)\left(\frac{2(2s-b)}{N-2s}\right)^2 + ((1+2s)N-6s)\left(\frac{2(2s-b)}{N-2s}\right) - 2(2s-b) \]
\[ = \frac{2(2s-b)}{(N-2s)^2}\left[2s(N-2)(2s-b) + ((1+2s)N-6s)(N-2s) - (N-2s)^2 \right] \]
\[ = \frac{2(2s-b)}{(N-2s)^2}\left[2s(N-2)(2s-b) + (N-2s)((1+2s)N-6s - (N-2s)) \right] \]
\[ = \frac{2(2s-b)}{(N-2s)^2}\left[2s(N-2)(2s-b) + 2s(N-2s)^2 \right] \]
\[ = \frac{4s(2s-b)}{(N-2s)^2}\left[(N-2)(2s-b) + (N-2s)^2 \right] \]
\[ > 0. \]
Moreover, one can easily verify that
\[ Q(p_\ast - 1) = \frac{4s(2s - b)}{N^2} \left[ (N - 2)(2s - b) + N(N - 3) \right] > 0, \]
provided that \( N \geq 3 \). This means that \( Q(p - 1) > 0 \) as long as \( p \in (p_\ast, p_\ast') \).

Now, the desired estimate (5.5) easily follows from (5.7), (5.8) and (5.14). This finishes the proof of Proposition 5.3. \( \square \)

Now, we are ready to prove the scattering criterion.

**Proof of Proposition 5.1.** Take \( 0 < \varepsilon < < 1 \). By Proposition 5.3, there exists \( \mu > 0 \) such that
\[ \| e^{-iT\mathcal{D}^{2s}u(T)} \|_{L^a((0,\infty),L^r)} = \| e^{-i(t-T)\mathcal{D}^{2s}u(T)} \|_{L^a((T,\infty),L^r)} \lesssim \varepsilon^\mu. \]
So, with Lemma 5.2 together with the continuity argument Lemma 2.4, one gets
\[ \| u \|_{L^a((T,\infty),L^r)} \lesssim \varepsilon^\mu. \]
Again, thanks to Lemma 5.2 via the continuity argument Lemma 2.4, we obtain the global bound
\[ u \in L^q(\mathbb{R},W^{s,r}). \]

Now, for \( t > t' > > 1 \), we have by Lemma 5.2 and Strichartz estimates
\[ \| u(t') - u(t) \|_{H^s} \lesssim \| (1 + |\nabla|^s)\| \| x \|^{-b} |u|^{p-1} u \|_{S(t',t)} \lesssim \| u \|_{L^p((t,t'),L^r)} \| u \|_{L^q((t',t),W^{s,r})} \rightarrow 0. \]

The proof is achieved via the Cauchy criterion. \( \square \)

**6. Proof of Theorem 1.1**

Let \( R, \varepsilon > 0 \) given by Proposition 5.1 and \( R_n \rightarrow \infty \). Let \( t_n \rightarrow \infty \) given by Corollary 4.3. For \( n > > 1 \) such that \( R_n > R \), one gets by Hölder’s inequality
\[ \int_{|x| \leq R} |u(t_n, x)|^2 \, dx \leq R^{2b/p} \int_{|x| \leq R} |x|^{-2b/(1+p)} |u(t_n, x)|^2 \, dx \]
\[ \leq R^{2b/p} \left( \int_{|x| \leq R} |x|^{-b} |u(t_n, x)|^{1+p} \, dx \right)^{1/p} \left( \int_{|x| \leq R} |x|^{-b} |u(t_n, x)|^{1+p} \, dx \right)^{1/(1+p)} \rightarrow 0. \]
Hence, the part (i) of Theorem 1.1 follows from Proposition 5.1.
Let us turn now to part (ii) of Theorem 1.1. We have

\[
\frac{d}{dt} M_R[u(t)] = 4 \int_0^\infty m^s \int_{\mathbb{R}^N} |\nabla u_m|^2 \, dx \, dm - 4 \int_0^\infty m^s \int_{|x|<2R} \left( 1 - f'' \left( \frac{|x|}{R} \right) \right) |\nabla u_m|^2 \, dx \, dm \\
- \int_0^\infty m^s \int_{\mathbb{R}^N} \Delta^2 f_R |u_m|^2 \, dx \, dm \\
- \frac{4sB}{1+p} \int_{|x|<R} |x|^{-b} |u|^{p+1} \, dx + \frac{2(p-1)}{1+p} \int_{|x|>R} (N - \Delta f_R) |x|^{-b} |u|^{p+1} \, dx \\
- \frac{4b}{1+p} \int_{|x|>R} \frac{x \cdot \nabla f_R}{|x|^2} |x|^{-b} |u|^{p+1} \, dx \\
\leq 4s|D^s u|^2 + CR^{-2s} \\
- \frac{4sB}{1+p} \int_{|x|<R} |x|^{-b} |u|^{p+1} \, dx + \frac{2(p-1)}{1+p} \int_{|x|>R} (N - \Delta f_R) |x|^{-b} |u|^{p+1} \, dx \\
- \frac{4b}{1+p} \int_{|x|>R} \frac{x \cdot \nabla f_R}{|x|^2} |x|^{-b} |u|^{p+1} \, dx,
\]

where $M_R[u(t)]$ is given by (4.4). Take $0 < \varepsilon < \frac{p-1}{s}(s - \frac{1}{2})$ and $\frac{1}{2} < \alpha := \frac{1}{2} + \frac{s}{p-1} < s < \frac{N}{2}$. Using the radial assumption and the Strauss type estimate in Lemma 2.1, via the interpolation $\|D^\alpha \phi\| \lesssim \|\phi\|^{\frac{\alpha}{s}} \|D^s \phi\|^{\frac{\alpha}{s}}$, one gets

\[
\int_{|x|>R} |x|^{-b} |u(t, x)|^{1+p} \, dx \leq R^{-b} \|u(t)\|^2 \|u(t)\|^{p-1}_{L^\infty(|x|>R)} \\
\lesssim R^{-b} \left( R^{\alpha - \frac{s}{2}} \|D^\alpha u(t)\| \right)^{p-1} \\
\lesssim R^{-b} \left( R^{\frac{(N-1)(p-1)}{2} + \varepsilon} \|D^s u(t)\|^{(p-1)\frac{\alpha}{2}} \right) \\
\lesssim R^{-b} \left( R^{\frac{(N-1)(p-1)}{2} + \varepsilon} \|D^s u(t)\|^{(s + \frac{1}{2})} \right).
\]

Therefore,

\[
\frac{d}{dt} M_R[u(t)] \leq 4s I[u(t)] + CR^{-2s} + CR^{-b - \frac{b-1}{2}(N-1) + \varepsilon} \|D^s u(t)\|^{\varepsilon + \frac{b-1}{2s}}.
\]

If $u$ is global and does not blow-up in infinite time, it follows that $\|D^s u\|_{L^\infty(\mathbb{R}, L^2)} \leq 1$. Thus, taking $R >> 1$ and using (1.10), one gets $\frac{d}{dt} M_R[u(t)] \leq -2\eta < 0$, for some $\eta > 0$. Hence, for $t > 0$ large enough, we get

\[
M_R[u(t)] \leq -\eta t. \tag{6.1}
\]

Combining (6.1) with [8, Lemma A.1], we obtain, for $t$ large enough,

\[
\eta t \leq -M_R[u(t)] = |M_R[u(t)]| \lesssim \left( 1 + \|D^s u(t)\|^{\frac{1}{2}} \right). \tag{6.2}
\]

This gives $\|D^s u(t)\| \gtrsim t^s$ for $t$ large enough, which is a contradiction. This finishes the proof of part (ii) of Theorem 1.1.
7. Proof of Corollary 1.2

The scattering part follows by Theorem 1.1 with the next result. Indeed, the classical scattering condition below the ground state threshold is stronger than (1.9).

Lemma 7.1. Suppose that assumptions (1.11) and (1.12) are fulfilled. Then, there exists $\varepsilon > 0$ such that (3.2) is satisfied.

Proof. Using the identity $A + 2\gamma_c = B\gamma_c$, we have

$$E[u][M[u]]^{\gamma_c} \geq \|\nabla u\|^2\|u\|^{2\gamma_c} - \frac{2K_{opt}}{1+p}\|u\|^{A+2\gamma_c}\|\nabla u\|^B$$

$$= g(\|\nabla u\|\|u\|^{\gamma_c}),$$

where $g(\tau) = \tau^2 - \frac{2K_{opt}}{1+p}\tau^B$.

Now, with Pohozaev identities and the conservation laws, one has for some $0 < \varepsilon < 1$,

$$g(\|\nabla u\|\|u\|^{\gamma_c}) \leq E[u][M[u]]^{\gamma_c} < (1 - \varepsilon)E[Q][M[Q]]^{\gamma_c}$$

$$= (1 - \varepsilon)g(\|\nabla Q\|\|Q\|^{\gamma_c}).$$

Thus, with time continuity, (1.12) is invariant under the flow (1.1) and hence $T^\ast = \infty$.

Moreover, by Pohozaev identities, one writes

$$E[Q][M[Q]]^{\gamma_c} = \frac{B - 2}{B}(\|\nabla Q\|\|Q\|^{\gamma_c})^2 = \frac{K_{opt}(B - 2)}{1+p}(\|\nabla Q\|\|Q\|^{\gamma_c})^B$$

and so

$$1 - \varepsilon \geq \frac{B - 2}{B - 2 - \frac{2}{B-2}\tau^B} - \frac{2}{B - 2 - \frac{2}{B-2}\tau^B}.$$ 

Following the variations of $\tau \mapsto \frac{B - 2}{B - 2 - \frac{2}{B-2}\tau^B} - \frac{2}{B - 2 - \frac{2}{B-2}\tau^B}$ via the assumption (1.12) and a continuity argument, there is a real number denoted also by $0 < \varepsilon < 1$, such that

$$\|\nabla u\|\|u\|^{\gamma_c} \leq (1 - \varepsilon)\|\nabla Q\|\|Q\|^{\gamma_c} \text{ on } \mathbb{R}. \quad (7.1)$$

Now, by (7.1) and Pohozaev identities, there exists $0 < \varepsilon < 1$, such that

$$P[u][M[u]]^{\gamma_c} \leq K_{opt}\|\nabla u\|^B\|u\|^{A+2\gamma_c}$$

$$\leq K_{opt}(1 - \varepsilon)(\|\nabla Q\|\|Q\|^{\gamma_c})^B$$

$$\leq (1 - \varepsilon)P[Q]M[Q]^{\gamma_c}.$$ 

This finishes the proof of Lemma 7.1. \qed

Let us turn now to the blow-up part in Corollary 1.2. Assume that (1.13) is satisfied. Taking into account [53, Section 3], one has $\mathcal{MG}[u(t)] > 1$ on $[0, T^\ast)$. Thus, by the
Pohozaev identity $BE[Q] = (B - 2)\|D^s Q\|^2$. It follows that

$$I[u][M[u]]^{\gamma_c} = \left(\|D^s u\|^2 - \frac{B}{1 + p} P[u]\right) [M[u]]^{\gamma_c} = \frac{B}{2} E[u][M[u]]^{\gamma_c} - \left(\frac{B}{2} - 1\right)\|D^s u\|^2 [M[u]]^{\gamma_c} \leq \frac{B}{2} (1 - \varepsilon) E[Q][M[Q]]^{\gamma_c} - \left(\frac{B}{2} - 1\right)\|D^s Q\|^2 [M[Q]]^{\gamma_c} \leq -\varepsilon\|D^s Q\|^2 [M[Q]]^{\gamma_c}.$$  

The proof follows by the use of Theorem 1.1.

Appendix A. The Two Dimensional Case

One keeps the notations of section 5 about the scattering criteria and the estimate of the term $F_1$ for $N = 2$. Take $\frac{1}{c} := \frac{1}{f} + \frac{2c}{N}$. Then $(a, c) \in \Gamma$ and for $f \geq r$, there is $\lambda \in [0, 1]$ such that

$$\frac{1}{r} := \frac{\lambda}{c} + \frac{1 - \lambda}{f} = \lambda\left(\frac{1}{r} + \frac{sc}{N}\right) + \frac{1 - \lambda}{f}.$$  

Indeed, $\frac{1}{r} - \frac{1}{f} = \lambda\left(\frac{1}{r} + \frac{sc}{N} - \frac{1}{f}\right)$ gives

$$\frac{1}{r} + \frac{sc}{N} - \frac{1}{f} = \lambda \in [0, 1], \quad \text{for} \quad f \geq r.$$  

By interpolation via the mass conservation, one writes

$$\|F_1\|_{L^s((T, \infty), L^r)} \lesssim \|F_1\|_{L^s((T, \infty), L^c)}^{\frac{1 - \lambda}{\lambda}}\|F_1\|_{L^s((T, \infty), L^f)}^{\frac{1}{\lambda}} \lesssim \|e^{-i(t-(T-\varepsilon\beta))D^{2s}} u(T - \varepsilon^{-\beta}) - e^{-itD^{2s}} u_0\|_{L^s((T, \infty), L^c)}^{\lambda} \|F_1\|_{L^s((T, \infty), L^f)}^{1 - \lambda} \lesssim \|F_1\|_{L^s((T, \infty), L^f)}^{1 - \lambda}.$$  

With the free fractional Schrödinger operator dispersive estimate (2.16), one has for $T \leq t$ and $f := \frac{2}{3}$, such that $1 \geq \frac{2}{r} \geq \delta > 0$,

$$\|F_1\|_{\frac{2}{3}} \lesssim \int_0^{T - \varepsilon^{-\beta}} \frac{1}{(t - s)^{\frac{N(1 - \delta)}{2}}} \|D^{N(1-s)(1-\delta)}(\{x|^{-b}\}|u|^{p-1} u)\|_{L^{\frac{2}{p}}} ds.$$  

Using the fractional chain rule in Lemma 2.1, one writes

$$\|D^{N(1-s)(1-\delta)}(\{x|^{-b}\}|u|^{p-1} u)\|_{L^{\frac{2}{p}}} \lesssim \|\{x|^{-b-N(1-s)(1-\delta)}|u|^{p}\|_{L^{\frac{2}{p}}} + \|\{x|^{-b}\}|u|^{p-1}D^{N(1-s)(1-\delta)} u\|_{L^{\frac{2}{p}}} := (I) + (II).$$
The first term (I) can be estimated as previously. It remains to control the second term. By Hölder’s inequality and Sobolev embedding, we have

\[
\begin{align*}
(II) & = \left\| |x|^{-\frac{b}{2}}|u|^{p-1}D^{N(1-s)(1-\delta)}u \right\|_{L^2}^2 \\
& = \left\| |x|^{-\frac{b}{2}}u^{\frac{1}{2}}D^{N(1-s)(1-\delta)}u \right\|_{L^2}^2 \\
& \leq \left\| |x|^{-}\frac{b}{2}u \right\|^\frac{1}{p-1} \left\| u \right\|^\frac{1}{p-1} \left\| D^{N(1-s)(1-\delta)}u \right\|_{L^2} \\
& \lesssim \left\| |x|^{-}\frac{b}{2}u \right\|^\frac{1}{p-1} \left\| u \right\|^\frac{1}{p-1} \left\| D^{N(1-s)(1-\delta)}u \right\|_{L^2} \\
& \lesssim \left\| u \right\|^\frac{1}{p-1} \left\| u \right\|^\frac{1}{p-1} \left\| D^{N(1-s)(1-\delta)}u \right\|_{L^2} \\
& \lesssim \left\| u \right\|^p_H.
\end{align*}
\]

Here, one takes

\[
\begin{align*}
\begin{cases}
\frac{2-\delta}{2} = \frac{b}{2s} + \frac{p-1}{r_1} + \frac{1}{2}, \\
p \geq 1 + \frac{b}{2s}, \\
0 \leq N(1-s)(1-\delta) \leq s, \\
2 \leq r_1 \leq \frac{2N}{N-2s}.
\end{cases}
\end{align*}
\]

So,

\[
\begin{align*}
\begin{cases}
(p-1 - \frac{b}{s}) \frac{N-2s}{2N} \leq \frac{1-\delta}{2} - \frac{b}{2s} = \frac{p-1}{r_1} - \frac{p-1}{2}, \\
p \geq 1 + \frac{b}{2s}, \\
0 \leq \frac{N(1-\delta)}{1+N(1-\delta)} \leq s.
\end{cases}
\end{align*}
\]

Then,

\[
\begin{align*}
\begin{cases}
2\left(\frac{1-\delta}{2} - \frac{b}{2s}\right) \leq p - 1 - \frac{b}{s} \leq \frac{2N}{N-2s}\left(\frac{1-\delta}{2} - \frac{b}{2s}\right); \\
p \geq 1 + \frac{b}{2s}, \\
0 \leq \frac{N(1-\delta)}{1+N(1-\delta)} \leq s.
\end{cases}
\end{align*}
\]

Therefore,

\[
\begin{align*}
\begin{cases}
\max\left\{ 1 + \frac{b}{2s}, 2 - \delta \right\} \leq p \leq 1 + \frac{b}{2s} + \frac{N}{N-2s}\left(1 - \delta - \frac{b}{2s}\right); \\
\frac{N(1-\delta)}{1+N(1-\delta)} \leq s.
\end{cases}
\end{align*}
\]

The proof is achieved by taking \(\delta \to 0\).

**APPENDIX B. A REMARK ON THE DEFOCUSING CASE**

Although the main goal of this paper is concerned with the focusing regime, we want to draw the reader’s attention to the fact that the defocusing case can be studied in a more simpler way. For the sake of completeness, we give the statement and the proof in this appendix.

Consider the defocusing fractional inhomogeneous Schrödinger equation

\[
\begin{align*}
\begin{cases}
i\partial_t u - (-\Delta)^s u & = |x|^{-\frac{b}{2}}|u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
u(0, x) & = u_0(x).
\end{cases}
\end{align*}
\]
One proves the following scattering result.

**Proposition B.1.** Assume that \( N \geq 3, s \in (\frac{N}{2N-1}, 1), 0 < b < 2s \) and \( p_* < p < p^* \). Then, for \( u_0 \in H^s_{rad} \), the maximal energy solution of \( (B.1) \) is global and scatters in \( H^s \).

**Proof.** The global existence follows by the conservation of the energy and the energy sub-critical regime. Let the Morawetz action

\[
M_f[u(t)] := 23 \int_{\mathbb{R}^N} \tilde{u} \nabla f \cdot \nabla u \, dx := 23 \int_{\mathbb{R}^N} \tilde{u} \partial_{\bar{t}} f \partial_t u \, dx.
\]

Taking into account the calculus performed in the proof of Lemma 4.1, we get

\[
\partial_t \left( M_f[u(t)] \right) = \int_0^\infty m^s \int_{\mathbb{R}^N} \left( 4 \partial_k \bar{u}_m \partial_{kl} f \partial_l u_m - \Delta^2 f |u_m|^2 \right) \, dx \, dm
\]

\[
+ \frac{2(p-1)}{1+p} \int_{\mathbb{R}^N} \Delta f |x|^{-b}|u|^{p+1} \, dx + \frac{4b}{1+p} \int_{\mathbb{R}^N} \frac{x \cdot \nabla f}{|x|^2} |x|^{-b}|u|^{1+p} \, dx.
\]

One picks \( f(x) := |x| \). Then,

\[
\nabla f(x) = \frac{x}{|x|}, \quad \Delta f = N - 1, |x|,
\]

and

\[
-\Delta^2 f = \begin{cases} 
4\pi(N-1)\delta_0 & \text{if } N = 3, \\
\frac{(N-1)(N-3)}{|x|^4} & \text{if } N \geq 4.
\end{cases}
\]

Clearly we have

\[
\partial_k \partial_\bar{t} f \Re(\partial_\bar{t} u_m \partial_k \bar{u}_m) \geq 0,
\]

where \( u_m \) is given by \((2.18)\). Indeed, denoting \( r := |x| \), one computes

\[
\partial_\bar{t} \partial_k f(x) = \frac{1}{r} (\delta_k - \frac{x_k x}{r^2});
\]

\[
\partial_\bar{t} \partial_k f \Re(\partial_\bar{t} u_m \partial_k \bar{u}_m) = \frac{1}{r} \left( |\nabla u_m|^2 - \frac{|x \cdot \nabla u_m|^2}{r^2} \right) \geq 0.
\]

Hence,

\[
\partial_t \left( M_f[u(t)] \right) \geq \frac{2(p-1)}{1+p} \int_{\mathbb{R}^N} \Delta f |x|^{-b}|u|^{p+1} \, dx + \frac{4b}{1+p} \int_{\mathbb{R}^N} \frac{x \cdot \nabla f}{|x|^2} |x|^{-b}|u|^{1+p} \, dx
\]

\[
\geq \frac{2(p-1)(N-1)}{1+p} \int_{\mathbb{R}^N} |x|^{-b-1}|u|^{p+1} \, dx + \frac{4b}{1+p} \int_{\mathbb{R}^N} |x|^{-b}|u|^{1+p} \, dx.
\]

Since by [8, Lemma A.1], one has \( |M_f[u(t)]| \leq C(\|D^{\frac{1}{2}} u(t)\|^2 + \|u(t)\|\|D^{\frac{1}{2}} u(t)\|) \), one gets

\[
\int_{\mathbb{R}^N} |x|^{-b-1}|u(t,x)|^{p+1} \, dx + \int_{\mathbb{R}^N} |x|^{-b}|u(t,x)|^{1+p} \, dx \leq 1.
\]
Therefore, by the fractional radial Sobolev inequality \((2.1)\), we get
\[
1 \gtrsim \int_{\mathbb{R} \times \mathbb{R}^N} |x|^{-b-1} |u(t, x)|^{p+1} dx \, dt + \int_{\mathbb{R}^N} |x|^{-b} |u(t, x)|^{1+p} dx \, dt
\]
\[
\gtrsim \int_{\mathbb{R} \times \mathbb{R}^N} |u(t, x)|^{1+\frac{2(1+b)}{N-2s}+p} dx \, dt + \int_{\mathbb{R}^N \times \mathbb{R}} |u(t, x)|^{1+\frac{2b}{N-2s}+p} dx \, dt.
\]
Hence,
\[
 u \in L^{1+\frac{2(1+b)}{N-2s}+p}\left(\mathbb{R}, L^{1+\frac{2(1+b)}{N-2s}+p}(\mathbb{R}^N)\right) \cap L^{1+\frac{2b}{N-2s}+p}\left(\mathbb{R}, L^{1+\frac{2b}{N-2s}+p}(\mathbb{R}^N)\right).
\]
Now, since \(0 < s_c < 1\) implies that \(0 < 1 + \frac{2b}{N-2s} + p < s\), one has \((1 + \frac{2b}{N-2s} + p, 1 + \frac{2b}{N-2s} + p) \in \Gamma_\gamma\) for some \(0 < \gamma < s\). The scattering follows by standard arguments via Stichartz estimates. \(\square\)

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