The generalized mathematical model of the failure of the cutting tool

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Abstract. We offer a mathematical model which takes into account the following factors: the spread of the cutting properties of the tool, parameters spread of gear blanks and consideration of the factor of a possible fracture of the cutting wedge tool. The reliability function, taking into account the above-mentioned factors, has five parameters for which assessment we propose a method according to our experience. A numerical illustration of the method is shown in the article. We suggest using the model in the optimization mode of the cutting tool preventive measures.

1. Introduction
Efficiency of tooling on metal cutting machines is largely dependent on the reliability (stability) of cutting tools. This efficiency can be increased at the cost of the rational use of the cutting tools life. In addition, by predicting the durability of cutting tools, the cost of tool maintenance of machine-building enterprises can be reduced. These enterprises are characterized by a very wide range of consumable cutting tools [1]. The task of forecasting of cutting tools durability and efficiency improvement of the tool life are complicated by the fact that the cutting tools life (Mean-Time-Between-Failures (MTBF) according to the terminology of the reliability theory) is a random variable that depends on many factors: cutting parameters, cutting properties of tools, the type of cutting, hardness of work pieces, the value of allowances for processing, pre-stress and strain state, vibration, geometric errors of the machine and other factors [2-7].

Cutting tools properties even in one batch are subject to variety. Hardness of work pieces and machining allowances are subject to variety too. All these factors, and the stochastic nature of the process of wear and tear of the cutting tool active part results in the gain of the tool life, and, however, the tool life of even one batch of the cutting tools varies fairly considerably (15-35\%) [8].

For the purpose of the efficiency increase of the resource use of the cutting tool, it is necessary to organize the monitoring and replacement system of the cutting tool with account of the specified dispersion factors. This requires the development of a mathematical model of the resource use of the cutting tool, adequately considering the mentioned factors.

2. Materials and methods
When elaborating the mathematical model of the process of the resource use of the cutting tool with account of the specified dispersion factors, the following models of the tool wear have been applied: a
fibered model, an accumulation model, a complex model and a fracture model. For the solution to the given task, methods of the probability theory and the mathematical statistics have been used.

3. The generalized mathematical model of failure of the cutting tool

The failure of the cutting tool is usually associated with the deterioration or destruction of the cutting wedge (chipping, sliding fracture, breakage). According to the researchers’ data, the proportion of failures because of destruction during the roughing and finishing operation reaches 50% or more. Since the consequences of these failures can be different, while determining the parameters of tools reliability, the type of failure should be considered. The peculiarity of the wear process of the cutting tool also consists in the fact that in case of failure, a wear condition of the tool can be judged by the degree of wear, but it is impossible in case of tool destruction.

If $T'_1$ – the time to failure because of wear and tear, $T'_2$ – the time to failure because of destruction; the actual time to failure in general (durability) is $T = \min(T'_1, T'_2)$, and there is a probability of no-failure during operating time $t$ (reliability function), i.e. probability that $T \geq t$ $P(t) = P'_1(t) \cdot P'_2(t)$, where $P'_1(t)$ is the reliability function by wear, and $P'_2(t)$ is the reliability function by the destruction of the cutting wedge. It is impossible to evaluate the reliability of private functions $P'_1(t)$ and $P'_2(t)$ directly by experience, as the wear process and the development process of the destruction take place in parallel and cannot be divided. Sorting out life tests results by the types of failures leads to other conditional functions of reliability $\pi_i(t)$ and $\pi_2(t)$, connected with $P'_1(t)$ and $P'_2(t)$ by mutually inverse transformations:

$$\pi_i(t) = \frac{1}{p_i} \int_0^\infty P(\tau) \cdot \lambda_i(\tau) \, d\tau, \quad P'_i(t) = \exp\left( \int_0^t \frac{\pi'_i(\tau)}{P(\tau)} \, d\tau \right), \quad i = 1, 2, \quad (1)$$

and $P(t) = p_1 \cdot \pi_1(t) + p_2 \cdot \pi_2(t)$. Here, $p_1$ and $p_2$ are failure shares of the first and second types:

$$p_i = \int_0^\infty P(\tau) \cdot \lambda_i(\tau) \, d\tau,$$

and failure intensity by $i$ factor $(i = 1, 2)$ $\lambda_i(t) = -P_i'(t)/P_i(t) = -p_i \cdot \pi'_i(t)/P(t)$.

General distribution density of mean time to failure is $f(t) = -P'(t) = f_1(t) \cdot P'_2(t) + f_2(t) \cdot P'_1(t)$, where $f_1(t) = -P'_1(t)$, $f_2(t) = -P'_2(t)$ – distribution densities $T'_1$ and $T'_2$, relatively.

Theoretical considerations based on the types of failure modes lead to the following reliability functions for failures related to wear and tear:

$$P_i(t) = \int_0^\infty \psi(a) \Phi^*( \frac{L - at}{\sigma \sqrt{t}} ) \, da, \quad \psi(a) = \frac{1}{\sqrt{2\pi} \delta a} \exp\left[ -\frac{(\ln a - \ln \hat{a})^2}{2\delta^2} \right]. \quad (2)$$

Here $\psi(a)$ is the density of the logarithmically normal distribution of the average rate of wear by batch tools. $\Phi^*( \frac{L - at}{\sigma \sqrt{t}} )$ is likelihood that tool wear $Y(t)$ after operating time $t$ at medium intensity $a$ and quadratic wear per life length unit $\sigma$ does not exceed limit value $L$, which means it does not fail. $\Phi^*(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \, dx$ is the standard normal distribution function, which table is in the reference books.

For failures related to the destruction, we suggest using Weibull’s distribution, which has found wide use in failure predicting of technological equipment [9]. When determining this law of durability
distribution, we will proceed from this model. Initially, in the body of the cutting wedge, there are many micro-defects, which are gradually developing in the process of cutting because of fluctuations of the cutting force in the fatigue cracks with subsequent destruction of the cutting wedge.

Let us assume that \( t_1, t_2, ..., t_N \) is the time of development of \( i \) microdefects into a critical size of cracks, leading to the destruction of the tool cutting wedge. The actual wedge breakdown will occur when \( T = \min(t_1, t_2, ..., t_N) \).

In such model for \( T \), Weibull’s distribution is limited. The distribution density and the reliability function in this case is expressed as follows:

\[
f_z(t) = \frac{\beta}{r} \left( \frac{t}{r} \right)^{\beta-1} \exp\left[-\left(\frac{t}{r}\right)^\beta\right], \quad P_z(t) = \exp\left[-\left(\frac{t}{r}\right)^\beta\right],
\]

where \( r \) and \( \beta \) are distribution parameters.

Mathematical expectation and the coefficient of variation of the lifetime during destruction are expressed through parameters \( r \) and \( \beta \) as follows:

\[
\bar{T}_2 = r \cdot \Gamma\left(1 + \frac{1}{\beta}\right), \quad K_{T2} = \frac{\Gamma\left(1 + 2/\beta\right)}{\Gamma^2\left(1 + 1/\beta\right)} - 1.
\]

Here \( \Gamma(x) \) is the gamma-function.

With regard to quotations (1), (2) and (3), we receive

\[
P(t) = \frac{1}{\sqrt{2\pi\delta^2}} \int_a^\infty \exp\left[-\frac{(\ln a - \ln \hat{a})^2}{2\delta^2}\right] \cdot \Phi^*\left(\frac{L - at}{\sigma \sqrt{t}}\right) \cdot da \cdot \exp\left[-\left(\frac{t}{r}\right)^\beta\right],
\]

This reliability function has five parameters: \( \hat{a}, \sigma, \delta, r \) and \( \beta \).

When \( \sigma \to 0 \) and \( 1/r \to 0 \), within this range, we get a fibered model [10], whereby the reliability function follows the logarithmically normal law, which means that

\[
P(t) = \frac{1}{\sqrt{2\pi\delta^2}} \exp\left[-\frac{(\ln t - \ln \hat{T})^2}{2\delta^2}\right], \quad \bar{T} = \frac{L}{\hat{a}}.
\]

When \( \delta \to 0 \) and \( 1/r \to 0 \), within this range, we get an accumulation model according to which the reliability function takes the following form:

\[
P(t) \approx \Phi^\ast\left(\frac{L - \hat{a}}{\sigma \sqrt{t}}\right).
\]

When \( \hat{a} \to 0 \) and \( \sigma \to 0 \), within this range, we get a real fracture model, according to which \( P(t) = P_z(t) \).

When \( 1/r \to 0 \), within this range, we get a complex wear model, according to which \( P(t) = P_1(t) \).

4. Parameters assessment of a generic mathematical model of the cutting tool failure

Estimation of parameters \( \hat{a}, \sigma, \delta, r \) and \( \beta \) by experience is not an easy task. If we assess using the method of maximum likelihood, based on the statistics of operating time to failure \( t_i, \quad i = 1, ..., N \), where \( N \) – the sample size, then the likelihood function will look like

\[
I(\hat{a}, \sigma, \delta, r, \beta) = \prod_{i=1}^{N} f(t_i),
\]

where \( f(t) = -P^\prime(t) = f_z(t)P_z(t) + f_z(t)P_z(t) \).
It is not easy to find parameters $\hat{a}$, $\sigma$, $\delta$, $r$ and $\beta$ that lead to the maximum such as the complex function. A solving method for the system of equations, obtained by equating the partial derivatives of parameters $\hat{a}$, $\sigma$, $\delta$, $r$ and $\beta$ from function (4), seems here too lengthy. It is simpler to look for the function maximum (4) directly, for example by sorting or random search. Although the time of machine calculation increases, the time required for programming and programs debugging reduces significantly.

If we take into account the type of failure during parameters estimation, in other words, proceeding statistics of operating time to failure due to wear $t_{1,1}, ..., t_{1,N_1}$ and statistics of operating time to failure due to destruction $t_{2,1}, ..., t_{2,N_2}$ under $N = N_1 + N_2$, parameters $\hat{a}$, $\sigma$, $\delta$, $r$ and $\beta$ can be estimated more accurately by means of inclusion of this additional information. Let us consider this case in more detail, and use the method of moments. In this case the estimates of the mentioned parameters are obtained by solving the following system of five equations:

$$T_1^* = T_1, \quad T_2^* = T_2, \quad D_1^* = D_1, \quad D_2^* = D_2, \quad p_1^* = p_1,$$

(5)

where $T_j = \sum_{i=1}^{N_j} t_{j,i}$, $D_j^* = \sum_{i=1}^{N_j} t_{j,i}^2 - (T_j^*)^2$, $p_1 = \frac{N_1}{N}$, and

$$T_j = \int_0^\infty \pi_j(t)dt, \quad D_j = 2 \int_0^\infty \pi_j(t)dt - (T_j^*)^2, \quad p_j = \int_0^\infty f_j(t)P_j(t)dt, \quad (j = 1, 2).$$

System (5) with respect to $\hat{a}$, $\sigma$, $\delta$, $r$ and $\beta$ is solved numerically. For this purpose, we propose the following algorithm of a random search, consisting of the following basic steps.

1) According to formulas (4), the statistical characteristics of the empiric lifetime choice are calculated.

2) From the a priori consideration, we determine the limits of the possible values of required parameters $a'$, $a''$, $\sigma'$, $\sigma''$, $\delta'$, $\delta''$, $r'$, $r''$, $\beta'$ and $\beta''$.

3) We generate current parameter values $\hat{a}$, $\sigma$, $\delta$, $r$ and $\beta$ by formulas

$$\hat{a} = \gamma_1(a'' - a'), \quad \sigma = \gamma_2(\sigma'' - \sigma'), \quad ..., \quad \beta = \gamma_5(\beta'' - \beta'),$$

where $\gamma_1, ..., \gamma_5$ are pseudo-random numbers distributed uniformly between 0 and 1. These numbers are generated, for example, a Pascal algorithmic language using random functions.

4) By formulas (5) generated by the values of parameters $\hat{a}$, $\sigma$, $\delta$, $r$ and $\beta$, theoretical characteristics as well as the absolute value of the difference between them are calculated. Let us call them residuals.

$$\Delta_1 = \frac{T_1 - T_1^*}{T_1}, \quad \Delta_2 = \frac{T_2 - T_2^*}{T_2}, \quad \Delta_3 = \sqrt{D_1 - D_1^*}, \quad \Delta_4 = \sqrt{D_2 - D_2^*}, \quad \Delta_5 = \frac{p_1 - p_1^*}{p_1}.$$

5) Maximum of the residuals is calculated as $\Delta = \max(\Delta_1, ..., \Delta_5)$.

6) If received value $\Delta$ is less than the value of the previous iteration, then the value and the corresponding parameters are stored and new parameters values are generated and we return to step 3. Otherwise, just go to step 3. According to the first iteration, $\Delta$ is obviously taken as a large value (theoretical infinity).

7) The search process continues for a defined number of search iterations or until the given residual level corresponding to the desired accuracy of parameter estimation is gained. At the end of search iterations, the best values of the required parameters will remain in the computer memory.
5. Illustration of the method of estimating model parameters

We will demonstrate the method of parameter estimation by the example of the source data shown in figure 1.

| Source data for the program ‘Ocenk_Ob.exe’ | 
|------------------------------------------|
| Search ranges of parameter estimates     | 
| Average wear intensity per tool, μm/pcs. | 1.0 3.0 |
| CV of wear intensity per tool             | 0.200 0.400 |
| RMSD of wear per operation time, μm/pcs.  | 0.1 3.0 |
| Mean time to failure, pcs.                | 50.0 200.0 |
| CV of time to failure                     | 0.100 1.000 |
| Wear limit, mm                            | 0.30 |
| Search iterations count                   | 3000 |
| Failures due to wear count                | 28 |
| Failures due to the destruction count     | 12 |
| Lifetime choice by wear, pcs.             | 158 155 135 107 178 173 118 132 149 162 158 156 153 147 |
| Lifetime choice by the destruction, pcs.  | 124 186 163 154 145 104 148 145 87 129 124 162 147 161 |

**Figure 1.** Initial data for evaluating the parameters of reliability function $P(t)$.

The calculation results are shown in figure 2.

| Loop | $\hat{a}$ | $\delta$ | $\sigma$ | $r$ | $\beta$ | $\Delta$ |
|------|-----------|----------|----------|----|---------|----------|
| 1    | 0.001     | 0.204    | 0.003    | 82.110 | 9.062 | 65.000 |
| 2    | 0.002     | 0.258    | 0.001    | 107.192 | 7.368 | 55.552 |
| 3    | 0.001     | 0.287    | 0.000    | 176.472 | 11.425 | 22.101 |
| 57   | 0.001     | 0.253    | 0.000    | 156.476 | 8.425 | 11.697 |
| 198  | 0.001     | 0.294    | 0.002    | 156.841 | 9.623 | 10.857 |
| 732  | 0.001     | 0.327    | 0.000    | 157.794 | 11.137 | 10.623 |
| 2544 | 0.001     | 0.274    | 0.001    | 152.051 | 7.105 | 9.101  |

| Statistical parameters of the sample |
|--------------------------------------|
| $\bar{T}_1$ = 145.0  $\bar{T}_2$ = 143.8  $K_{T1}^*$ = 0.15  $K_{T2}^*$ = 0.21  $\bar{T}$ = 144.6  $K_T^*$ = 0.172 |

| Durability distribution parameters   |
|--------------------------------------|
| $\hat{a}$ = 0.0013  $\delta$ = 0.274  $\sigma$ = 0.0008  $r$ = 152.1  $\beta$ = 7.11 |

| Residual $\Delta$                  |
|-------------------------------------|
| 9.1                                 |

| Average wear intensity per tool, μm/pcs. |
|------------------------------------------|
| 1.4                                      |

| CV of wear intensity per tool            |
|------------------------------------------|
| 0.279                                    |

| RMSD of wear per operation time, μm/pcs. |
|------------------------------------------|
| 0.8                                      |

| Mean time to failure, pcs.               |
|------------------------------------------|
| 142.3                                    |

| CV of time to failure                    |
|------------------------------------------|
| 0.166                                    |

**Figure 2.** The intermediate and final results of the assessment of tools reliability parameters.
The calculation was carried out according to the program, developed in the algorithmic language Pascal. The count of search iterations was 3000, and a better solution was found when the iteration number was 2544. Great accuracy estimates can be obtained by setting a greater number of iterations.

6. Conclusion
Thus, we have proposed a generalized mathematical model of failure of cutting tools, which takes into account the wear and the wear rate of its spread because of tools quality dispersion and the parameters variety, taking into account the fact of the destruction of the instrument. We offer a method for model parameters as well, which is based on our experience. It is assumed that among other factors, the use of the mathematical model allows optimizing the model of cutting tool preventive measures.

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