ON CONJUGATE PSEUDO-HARMONIC FUNCTIONS.

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Abstract. We prove the following theorem. Let $U$ be a pseudo-harmonic function on a surface $M^2$. For a real valued continuous function $V : M^2 \to \mathbb{R}$ to be a conjugate pseudo-harmonic function of $U$ on $M^2$ it is necessary and sufficient that $V$ is open on level sets of $U$.

Keywords: a pseudo-harmonic function, a conjugate, a surface, an interior transformation

Let $M^2$ be a surface, i.e. a 2-dimensional and separable manifold, $U : M^2 \to \mathbb{R}$ be a real-valued function on $M^2$. Denote also by $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ the open unit disk in the plane.

Definition 1 (see [1, 2]). A function $U$ is called pseudo-harmonic in a point $p \in M^2$ if there exist a neighbourhood $N$ of $p$ on $M^2$ and a homeomorphism $T : D \to N$ such that $T(0, 0) = p$ and a function

$$u = U \circ T : D \to \mathbb{R}^2$$

is harmonic and not identically constant.

A neighbourhood $N$ is called simple neighbourhood of $p$.

We can even choose $N$ and $T$ from previous definition to comply with the equality

$$u(z) = U \circ T(z) = \text{Re} z^n + U(p), \quad z = x + iy \in D,$$

for a certain $n = n(p) \in \mathbb{N}$ (see [2]).

Definition 2 (see [1, 2]). A function $U$ is called pseudo-harmonic on $M^2$ if it is pseudo-harmonic in each point $p \in M^2$.

Let $U : M^2 \to \mathbb{R}$ be a pseudo-harmonic function on $M^2$ and $V : M^2 \to \mathbb{R}$ be a real valued function.

Definition 3 (see [1]). A function $V$ is called a conjugate pseudo-harmonic function of $U$ in a point $p \in M^2$ if there exist a neighbourhood $N$ of $p$ on $M^2$ and a homeomorphism $T : D \to N$ such that $T(0, 0) = p$ and

$$u = U \circ T : D \to \mathbb{R}^2 \quad \text{and} \quad v = V \circ T : D \to \mathbb{R}^2$$

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are conjugate harmonic functions.

We can choose $N$ and $T$ from previous definition in such way that
\[ u(z) = U \circ T(z) = \text{Re} \, z^n + U(p), \]
\[ v(z) = V \circ T(z) = \text{Im} \, z^n + V(p), \quad z = x + iy \in D, \]
for a certain $n = n(p) \in \mathbb{N}$ (see [2]).

**Definition 4** (see [1]). A function $V$ is called a conjugate pseudo-harmonic function of $U$ on $M^2$ if it is a conjugate pseudo-harmonic function of $U$ in every $p \in M^2$.

**Definition 5.** Let $U$ and $V$ be continuous real valued functions on a surface $M^2$. We say that $V$ is open on level sets of $U$ if for every $c \in U(M^2)$ a mapping
\[ V|_{U^{-1}(c)} : U^{-1}(c) \to \mathbb{R} \]
is open on the space $U^{-1}(c)$ in the topology induced from $M^2$.

**Theorem 1.** Let $U$ be a pseudo-harmonic function on $M^2$. For a real valued continuous function $V : M^2 \to \mathbb{R}$ to be a conjugate pseudo-harmonic function of $U$ on $M^2$ it is necessary and sufficient that $V$ is open on level sets of $U$.

Let us remind following

**Definition 6** (see [3]). A mapping $G : M^2_1 \to M^2_2$ of a surface $M^2_1$ to a surface $M^2_2$ is called interior if it complies with conditions:

1) $G$ is open, i. e. an image of any open subset of $M^2_1$ is open in $M^2_2$;

2) for every $p \in M^2_2$ its full preimage $G^{-1}(p)$ does not contain any nondegenerate continuum (closed connected subset of $M^2_1$).

In order to prove theorem 1 we need following

**Lemma 1.** Let $U$ be a pseudo-harmonic function on $M^2$. Let a real valued continuous function $V$ be open on level sets of $U$.

Then the mapping $F : M^2 \to \mathbb{C}$,
\[ F(p) = U(p) + iV(p), \quad p \in M^2 \]
is interior.

First we will verify one auxiliary statement. Denote $I = [0,1]$, $\overset{\circ}{I} = (0,1) = I \setminus \{0,1\}$.

**Proposition 1.** In the condition of Lemma 1 the following statement holds true.

Let $\gamma : I \to M^2$ be a simple continuous curve and $\gamma(I) \subseteq U^{-1}(c)$ for a certain $c \in \mathbb{R}$. If the set $\gamma(I)$ is open in $U^{-1}(c)$ in the topology induced from $M^2$, then the function $V \circ \gamma : I \to \mathbb{R}$ is strictly monotone.
Proof. Suppose that contrary to the statement of Proposition the equality \( V \circ \gamma(\tau_1) = V \circ \gamma(\tau_2) \) is valid for certain \( \tau_1, \tau_2 \in I, \tau_1 < \tau_2 \).

Since the function \( V \circ \gamma \) is continuous and a set \([\tau_1, \tau_2]\) is compact, then following values

\[
d_1 = \min_{t \in [\tau_1, \tau_2]} V \circ \gamma(t),
\]

\[
d_2 = \max_{t \in [\tau_1, \tau_2]} V \circ \gamma(t),
\]

are well defined. Let us fix \( s_1, s_2 \in [\tau_1, \tau_2] \) such that \( d_i = V \circ \gamma(s_i), \)

\( i = 1, 2 \).

We designate \( W = (\tau_1, \tau_2) \). It is obviously the open subset of \( I \).

Let us consider first the case \( d_1 = d_2 \). It is clear that \([\tau_1, \tau_2] \subseteq (V \circ \gamma)^{-1}(d_1)\) in this case. So the open subset \( \gamma(W) \) of the level set \( U^{-1}(c) \) is mapped by \( V \) onto a one-point set \( \{d_1\} \) which is not open in \( \mathbb{R} \) and \( V \) is not open on level sets of \( U \).

Assume now that \( d_1 \neq d_2 \). Since \( V \circ \gamma(\tau_1) = V \circ \gamma(\tau_2) \) due to our previous supposition, then either \( s_1 \) or \( s_2 \) is contained in \( W \).

Let \( s_1 \in W \) (the case \( s_2 \in W \) is considered similarly). Then \( V \circ \gamma(W) \subseteq [d_1, +\infty) \) and the open subset \( \gamma(W) \) of the level set \( U^{-1}(c) \) can not be mapped by \( V \) to an open subset of \( \mathbb{R} \) since its image contains the frontier point \( d_1 = V \circ \gamma(s_1) \). So, in this case \( V \) is not open on level sets of \( U \).

The contradiction obtained shows that our initial supposition is false and the function \( V \circ \gamma \) is strictly monotone on \( I \). \( \square \)

Proof of Lemma 1. Let \( p \in M^2 \) and \( Q \) be an open neighbourhood of \( p \).

We are going to show that the set \( F(Q) \) contains a neighbourhood of \( F(p) \). At the same time we shall show that \( p \) is an isolated point of a level set \( F^{-1}(F(p)) \).

Without loss of generality we can assume that \( U(p) = V(p) = 0 \).

Let \( N \) be a simple neighbourhood of \( p \) and \( T : D \rightarrow N \) be a homeomorphism such that for a certain \( n \in \mathbb{N} \) the following equality holds true \( u(z) = U \circ T(z) = \text{Re} \ z^n, z \in D \) (see Definition 1 and the subsequent remark). It is clear that without losing generality we can regard that \( N \) is small enough to be contained in \( Q \).

Observe that for an arbitrary level set \( \Gamma \) of \( U \) an intersection \( \Gamma \cap T(D) = \Gamma \cap N \) is open in \( \Gamma \). Consequently, since \( T \) is homeomorphism then a mapping \( v = V \circ T : D \rightarrow \mathbb{R} \) is open on level sets of \( u = U \circ T : D \rightarrow \mathbb{R} \) (see Definition 5).

Let us consider two possibilities.

Case 1. Zero is a regular point of the smooth function \( u = U \circ T \), i. e. \( n = 1 \) and \( u(z) = \text{Re} \ z, z \in D \).

In this case \( u^{-1}(u(0)) = u^{-1}(U(p)) = T^{-1}(U^{-1}(U(p))) = \{0\} \times (-1, 1) \). According to Proposition 1 the function \( v \) is strictly monotone on every segment which is contained in this interval, so it is strictly
monotone on \{0\} \times (-1, 1). Consequently, for points \(z_1 = 0 - i/2\) and
\(z_2 = 0 + i/2\) the following inequality holds true \(v(z_1) \cdot v(z_2) < 0\).

Let us note that from previous it follows that \(V\) is monotone on the
arc \(\beta = T(\{0\} \times (-1, 1)) = U^{-1}(U(p)) \cap N\). And since \(F^{-1}(F(p)) \cap N \subset \beta\) then \(F^{-1}(F(p)) \cap N = \{p\}\) and \(p\) is an isolated point of its level set
\(F^{-1}(F(p))\).

Let \(d_1 = v(z_1) < 0\) and \(d_2 = v(z_2) > 0\) (The case \(d_1 > 0\) and \(d_2 < 0\)
is considered similarly). Denote
\[\varepsilon = \frac{1}{2} \min(|d_1|, |d_2|) > 0.\]

Function \(v\) is continuous, so there exists \(\delta > 0\) such that following
implications are fulfilled
\[|z - z_1| < \delta \Rightarrow |v(z) - d_1| < \varepsilon,\]
\[|z - z_2| < \delta \Rightarrow |v(z) - d_2| < \varepsilon.\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure}

Let us examine a neighbourhood \(W = (-\delta, \delta) \times (-1/2, 1/2)\) of 0,
which is depicted on Figure 1. It can be easily seen that for every
\(x \in (-\delta, \delta)\) following relations are valid
\[u(x + iy) = x, \quad y \in (-\varepsilon, \varepsilon),\]
\[v(x - i/2) < v(z_1) + \varepsilon < -2\varepsilon + \varepsilon = -\varepsilon,\]
\[v(x + i/2) > v(z_2) - \varepsilon > 2\varepsilon - \varepsilon = \varepsilon.\]

From two last lines and from the continuity of \(v\) on a segment \(\{x\} \times
[-1/2, 1/2]\) it follows that \(v(\{x\} \times [-1/2, 1/2]) \supseteq (-\varepsilon, \varepsilon)\). Therefore
\[F \circ T(\{x\} \times [-1/2, 1/2]) \supseteq \{x\} \times (-\varepsilon, \varepsilon), \quad x \in (-\delta, \delta).\]

Since \(T(W) \subseteq N \subseteq Q\) by the choice of \(N\), then
\[0 = F(p) \in (-\delta, \delta) \times (-\varepsilon, \varepsilon) \subseteq F \circ T(W) \subseteq F(Q).\]
Case 2. Zero is a saddle point of \( u = U \circ T \), i.e. \( u(z) = \text{Re } z^n \), \( z \in D \) for a certain \( n > 1 \).

In this case
\[
  u^{-1}(u(0)) = T^{-1}(U^{-1}(U(p))) = \{0\} \cup \bigcup_{k=0}^{2n-1} \gamma_k,
\]
where \( \gamma_k = \{z \in D \mid z = a \cdot \exp(\pi i (k - 1/2)/n), \ a \in (0,1)\}, \ k = 1, \ldots, 2n - 1 \).

As above, applying Proposition \( \overline{\Pi} \) we conclude that function \( v = V \circ T \) is strictly monotone on each arc \( \gamma_k \), \( k = 1, \ldots, 2n - 1 \). Since \( v \) is continuous and 0 is a boundary point for each \( \gamma_k \), then \( v(z) \neq v(0) \) for all \( z \in \bigcup_k \gamma_k \). Therefore, \( 0 = (F \circ T)^{-1}(F \circ T(0)) \) and \( F^{-1}(F(p)) \cap N = \{p\} \), i.e. \( p \) is the isolated point if its level set \( F^{-1}(F(p)) \).

Let us designate by
\[
  R_k = \left\{ z \in D \mid z = ae^{i\varphi}, \ a \in [0,1), \ \varphi \in \left[\frac{\pi (k-1/2)}{2}, \frac{\pi (k+1/2)}{2}\right]\right\},
\]
\[
  k = 0, \ldots, 2n - 1
\]
sectors on which disk \( D \) is divided by the level set \( u^{-1}(u(0)) \).

We also denote
\[
  D_l = \{ z \in D \mid \text{Re } z \leq 0 \},
\]
\[
  D_r = \{ z \in D \mid \text{Re } z \geq 0 \}.
\]

Consider map \( \Phi : D \rightarrow D \) given by the formula \( \Phi(z) = z^n, \ z \in D \). It is easy to see that for every \( k \in \{0, \ldots, 2n - 1\} \) depending on its parity sector \( R_k \) is mapped homeomorphically by \( \Phi \) either onto \( D_l \) or onto \( D_r \). Let a mapping \( \Phi_k : R_k \rightarrow D_r \) is given by relation
\[
  \Phi_k = \begin{cases} 
  \Phi|_{R_k}, & \text{if } k = 2m, \\
  \text{Inv} \circ \Phi|_{R_k}, & \text{if } k = 2m + 1, 
  \end{cases}
\]
\[
  k = 0, \ldots, 2n - 1
\]
where \( \text{Inv} : D \rightarrow D \) is defined by formula \( \text{Inv}(z) = -z, \ z \in D \). Evidently, all \( \Phi_k \) are homeomorphisms.

We consider now inverse mappings \( \varphi_k = \Phi_k^{-1} : D_r \rightarrow D, \ k = 0, \ldots, 2n - 1 \). By construction all of these mappings are embeddings. Moreover, it is easy to see that
\[
  u_k(z) = u \circ \varphi_k(z) = \begin{cases} 
  \text{Re } z, & \text{when } k = 2m, \\
  -\text{Re } z, & \text{when } k = 2m + 1.
  \end{cases}
\]

Let us fix \( k \in \{0, \ldots, 2n - 1\} \). It is clear that \( \varphi_k \) homeomorphically maps a domain
\[
  \tilde{D}_r = \{ z \in D \mid \text{Re } z > 0 \}
\]
ono into a domain
\[
  \tilde{R}_k = \left\{ z \in D \mid z = ae^{i\varphi}, \ a \in (0,1), \ \varphi \in \left(\frac{\pi (k-1/2)}{2}, \frac{\pi (k+1/2)}{2}\right)\right\},
\]
so with the help of argument similar to the observation preceding to case 1 we conclude that the mapping \( \hat{v}_k = v \circ \varphi_k |_{D_r} : \hat{D}_r \to \mathbb{R} \) is open on level sets of the function \( \hat{u}_k = u \circ \varphi_k |_{D_r} : \hat{D}_r \to \mathbb{R} \). As above, applying Proposition 11 we conclude that function \( \hat{v}_k \) is strictly monotone on each arc
\[
\alpha_c = \hat{u}_k^{-1}(\hat{u}_k(c + 0i)) = \{z \in \hat{D}_r \mid \Re z = c\}, \quad c \in (0, 1).
\]
We already know that the function \( v \) is strictly monotone on the arcs \( \gamma_k \) and \( \gamma_s \), where \( s \equiv k + 1 \) (mod \( 2n \)). Therefore the function \( v_k = v \circ \varphi_k : D_r \to \mathbb{R} \) is strictly monotone on the arcs
\[
\alpha_- = \varphi_k^{-1}(\gamma_k) = \{z \in D_r \mid \Re z = 0 \text{ and } \Im z < 0\},
\]
\[
\alpha_+ = \varphi_k^{-1}(\gamma_s) = \{z \in D_r \mid \Re z = 0 \text{ and } \Im z > 0\}.
\]
Let us verify that \( v_k \) is strictly monotone on the arc
\[
\alpha_0 = \alpha_- \cup \{0\} \cup \alpha_+ = u_k^{-1}(u_k(0)) = \{z \in D_r \mid \Re z = 0\}.
\]
Since \( v_k(0) = v(0) = V(p) = 0 \) according to our initial assumptions and 0 is the boundary point both for \( \alpha_- \) and \( \alpha_+ \), then \( v_k \) is of fixed sign on each of these two arcs.
So we have two possibilities:
- either \( v_k \) has the same sign on \( \alpha_- \) and \( \alpha_+ \), then \( v_k|_{\alpha_0} \) has a local extremum in 0;
- or \( v_k \) has different signs on \( \alpha_- \) and \( \alpha_+ \), then \( v_k \) is strictly monotone on \( \alpha_0 \).

Suppose that \( v_k \) has the same sign on \( \alpha_- \) and \( \alpha_+ \).
We will assume that \( v_k \) is negative both on \( \alpha_- \) and \( \alpha_+ \). The case when \( v_k \) is positive on \( \alpha_- \) and \( \alpha_+ \) is considered similarly.
Denote \( z_1 = 0 - i/2 \in \alpha_- \), \( z_2 = 0 + i/2 \in \alpha_+ \). Let
\[
\hat{\varepsilon} = \frac{1}{2} \min(|v_k(z_1)|, |v_k(z_2)|) > 0.
\]
From the continuity of \( v_k \) it follows that there exists \( \hat{\delta} > 0 \) to comply with the following implications
\[
|z - z_1| < \hat{\delta} \Rightarrow |v_k(z) - v_k(z_1)| < \hat{\varepsilon},
\]
\[
(1) \quad |z - z_2| < \hat{\delta} \Rightarrow |v_k(z) - v_k(z_2)| < \hat{\varepsilon},
\]
\[
|z| = |z - 0| < \hat{\delta} \Rightarrow |v_k(z) - v_k(0)| = |v_k(z)| < \hat{\varepsilon}.
\]
Let \( c \in (0, \hat{\delta}) \). Then the point \( w_0 = c + i0 \) is situated on the curve \( \alpha_c \) between points \( w_1 = c - i/2 \) and \( w_2 = c + i/2 \). It follows from \( \text{(i)} \) that \( v_k(w_1) < -\hat{\varepsilon}, v_k(w_2) < -\hat{\varepsilon} \) and \( v_k(w_0) \in (-\hat{\varepsilon}, 0) \). But these three correlations can not hold true simultaneously since \( v_k \) is strictly monotone on \( \alpha_c \) as we already know.
The contradiction obtained shows us that \( v_k \) has different signs on \( \alpha_- \) and \( \alpha_+ \). So, \( v_k \) is strictly monotone on \( \alpha_0 \).
Now, repeating argument from case 1 we find such \( \varepsilon_k > 0 \) and \( \delta_k > 0 \) that the set
\[
\hat{W}_k = [0, \delta_k) \times (-\frac{1}{2}, \frac{1}{2})
\]
meets the relations
\[
\begin{align*}
F \circ T \circ \varphi_k(\hat{W}_k) & \supseteq [0, \delta_k) \times (-\varepsilon_k, \varepsilon_k), & \text{if } k = 2m, \\
F \circ T \circ \varphi_k(\hat{W}_k) & \supseteq (-\delta_k, 0] \times (-\varepsilon_k, \varepsilon_k), & \text{if } k = 2m + 1.
\end{align*}
\]

Let us denote \( W_k = \varphi_k(\hat{W}_k) \),
\[
W = \bigcup_{k=0}^{2n-1} W_k, \quad \delta = \min_{k=0,\ldots,2n-1} \delta_k > 0, \quad \varepsilon = \min_{k=0,\ldots,2n-1} \varepsilon_k > 0.
\]

\[\text{Figure 2.}\]

It is easy to show that \( W \) is an open neighbourhood of 0 in \( D \). From (2) and from our initial assumptions it follows that
\[
F(Q) \supseteq F(N) \supseteq F \circ T(W) \supseteq (-\delta, \delta) \times (-\varepsilon, \varepsilon).
\]

So, we have proved that for an arbitrary point \( p \in M^2 \) and its open neighbourhood \( Q \) a set \( F(Q) \) contains a neighbourhood of \( F(p) \). Hence the mapping \( F : M^2 \to \mathbb{C} \) is open.

At the same time we have shown that an arbitrary \( p \in M^2 \) is an isolated point of its level set \( F^{-1}(F(p)) \). It is easy to see now that any level set \( F^{-1}(F(p)) \) can not contain a nondegenerate continuum.

Consequently, the map \( F \) is interior. \( \square \)

**Proof of Theorem 1 Necessity.** Let \( U, V : M^2 \to \mathbb{R} \) be conjugate pseudoharmonic functions on \( M^2 \) (see Definitions 3 and 4).

Obviously, \( V \) is continuous on \( M^2 \). Suppose that contrary to the statement of Theorem there exists such \( c \in \mathbb{R} \) that \( V \) is not open on the level set \( \Gamma_c = U^{-1}(c) \subset M^2 \), i.e. a map \( V_c = V|_{\Gamma_c} : \Gamma_c \to \mathbb{R} \) is not open on \( \Gamma_c \) in the topology induced from \( M^2 \).
Let us verify that $V_c$ has a local extremum in some $p \in \Gamma_c$.

Note that the space $\Gamma_c$ is locally arcwise connected, i.e. for every point $a \in \Gamma_c$ and its open neighbourhood $Q$ there exists a neighbourhood $\hat{Q} \subseteq Q$ of $a$ such that every two points $b_1, b_2 \in \hat{Q}$ can be connected by a continuous curve in $Q$. This is a straightforward corollary of the remark subsequent to Definition 4.

Since the map $V_c$ is not open by our supposition, then there exists an open subset $O$ of $\Gamma_c$ such that its image $R = V_c(O)$ is not open in $\mathbb{R}$. Therefore there is a point $d \in R \setminus \text{Int} R$. Fix $p \in V_c^{-1}(d) \cap O$.

Let us show that $p$ is a point of local extremum of $V_c$. Fix a neighbourhood $\hat{O} \subseteq O$ of $p$ such that every two points $b_1, b_2 \in \hat{O}$ can be connected by a continuous curve $\beta_{b_1,b_2} : I \to \Gamma_c$ which meets relations $\beta(0) = b_1, \beta(1) = b_2$ and $\beta(I) \subseteq O$. It is clear that an image of a path-connected set under a continuous mapping is path-connected, therefore following inclusions are valid

$$(V_c(b_1), V_c(b_2)) \subseteq V_c(I) \quad \text{if} \quad V_c(b_1) < V_c(b_2),$$
$$(V_c(b_2), V_c(b_1)) \subseteq V_c(I) \quad \text{if} \quad V_c(b_2) > V_c(b_1).$$

Evidently, $p$ is not an interior point of $V_c(\hat{O})$ since it is not the interior point of $V_c(O)$ by construction and $V_c(\hat{O}) \subseteq V_c(O)$. Then there does not exist a pair of points $b_1, b_2 \in \hat{O}$ such that $V_c(b_1) < V_c(p) < V_c(b_2)$ and either $V(b) \leq V(p)$ for all $b \in \hat{O}$ or $V(b) \geq V(p)$ for all $b \in \hat{O}$, i.e. $p$ is the point of local extremum of $V_c$.

Now, since $V$ is the conjugate pseudo-harmonic function of $U$ in the point $p$ (see Definition 3), we can take by definition a neighbourhood $N$ of $p$ in $\mathbb{M}^2$ and a homeomorphism $T : D \to N$ such that a map $f : D \to \mathbb{C}$

$$f(z) = u(z) + iv(z), \quad z \in D$$

is holomorphic on $D$. Here $u = U \circ T : D \to \mathbb{R}$ and $v = V \circ T : D \to \mathbb{R}$.

It is clear that without loss of generality we can choose $N$ so small that either $V(b) = V_c(b) \leq V_c(p) = V(p)$ for every $b \in N \cap \Gamma_c$ or $V(b) \geq V(p)$ for all $b \in N \cap \Gamma_c$.

Let for definiteness $p$ is the local maximum of $V_c$ and $V(b) \leq V(p)$ for every $b \in N \cap \Gamma_c$. The case when $p$ is the local minimum of $V_c$ is considered similarly.

On one hand it follows from what we said above that

$$\left(\{U(p)\} \times (V(p), +\infty)\right) \cap f(D) = \emptyset$$

since $u^{-1}(U(p)) = T^{-1}(\Gamma_c \cap N)$ and $v(z) = V(T(z)) \leq V(p)$ for all $z \in T^{-1}(\Gamma_c \cap N)$ by construction. Therefore a point $U(p) + iV(p) = f(T^{-1}(p))$ is not the interior point of a set $f(D)$.

On the other hand it is known that the holomorphic map $f$ is open, so the point $f(T^{-1}(p))$ must be the interior point of the domain $f(D)$. 

The contradiction obtained shows that our initial assumption is false and \( V \) is open on level sets of \( U \).

**Sufficiency.** Let \( U \) be a pseudo-harmonic function on \( M^2 \) and a continuous function \( V : M^2 \to \mathbb{R} \) be open on level sets of \( U \).

From Lemma 1 it follows that the mapping \( F : M^2 \to \mathbb{C}, F(p) = U(p) + iV(p), p \in M^2 \) is interior.

Let \( p \in M^2 \) and \( N \) is a simple neighbourhood of \( p \) in \( M^2 \). Then there exists a homeomorphism \( T : D \to N \). It is straightforward that for the open set \( N \) a mapping \( F_N = F|_N : N \to \mathbb{C} \) is interior and its composition \( F_N \circ T = F \circ T : D \to \mathbb{C} \) with the homeomorphism \( T \) is also an interior mapping.

Now from Stoilov theorem it follows that there exists a complex structure on \( D \) such that the mapping \( F \circ T \) is holomorphic in this complex structure (see [3]). But from the uniformization theorem (see [4]) it follows that a simply-connected domain has a unique complex structure. So the mapping \( F \circ T \) is holomorphic on \( D \) in the standard complex structure. Thus the functions \( u = \text{Re}(F \circ T) = U \circ T \) and \( v = \text{Im}(F \circ T) = V \circ T \) are conjugate harmonic functions on \( D \). Consequently, \( V \) is a conjugate pseudo-harmonic function of \( U \) in the point \( p \).

From arbitrariness in the choice of \( p \in M^2 \) it follows that \( V \) is a conjugate pseudo-harmonic function of \( U \) on \( M^2 \). \( \square \)

**Corollary 1.** Let \( U, V : M^2 \to \mathbb{R} \) be conjugate pseudoharmonic functions on \( M^2 \).

Then there exists a complex structure on \( M^2 \) with respect to which \( U \) and \( V \) are conjugate harmonic functions on \( M^2 \).

**Proof.** This statement follows from Theorem 1 Lemma 1 and the Stoilov theorem which says that there exists a complex structure on \( M^2 \) such that the interior mapping \( F(p) = U(p) + iV(p), p \in M^2 \) is holomorphic in this complex structure (see [3]). \( \square \)

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