DEFORMATIONS OF HOMOGENEOUS ASSOCIATIVE SUBMANIFOLDS IN NEARLY PARALLEL $G_2$-MANIFOLDS

KOTARO KAWAI

Abstract. A nearly parallel $G_2$-manifold $Y$ is a Riemannian 7-manifold whose cone $C(Y) = \mathbb{R}_{>0} \times Y$ has the holonomy group contained in Spin(7). In other words, it is a spin 7-manifold with a real Killing spinor.

We have a special class of calibrated submanifolds called Cayley submanifolds in $C(Y)$. An associative submanifold in $Y$ is a minimal 3-submanifold whose cone is Cayley. We study its deformations, namely, Cayley cone deformations, explicitly when it is homogeneous in the 7-sphere $S^7$.

Key words. Associative submanifolds, nearly parallel $G_2$-manifolds, Cayley cones.

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1. Introduction. For any Riemannian manifold $(Y, g)$, consider its Riemannian cone $(C(Y), \overline{g}) = (\mathbb{R}_{>0} \times Y, dr^2 + r^2 g)$. A Riemannian 7-manifold $(Y, g)$ is called a nearly parallel $G_2$-manifold if the holonomy group of $\overline{g}$ is contained in Spin(7). The existence of a nearly parallel $G_2$-structure is equivalent to that of a spin structure with a real Killing spinor ([2]), which is also used in supergravity and superstring theory in physics.

We have a canonical closed 4-form $\Phi$ on $C(Y)$, which defines a calibration. A 3-submanifold $M$ in $Y$ is called associative if its cone $C(M)$ is calibrated by $\Phi$. In other words, $C(M)$ is a Cayley submanifold in $C(Y)$. For example, special Legendrian submanifolds in Sasaki-Einstein manifolds are associative (Lemma 2.19), and Lagrangian submanifolds in the sine cones of nearly Kähler 6-manifolds are associative (Lemma 2.30). Here, Lagrangian submanifolds are defined in terms of the vanishing of a non-closed 2-form which characterizes nearly Kähler geometry. These are also called totally real submanifolds.

The deformation of compact calibrated submanifolds was studied by Mclean [19]. Joyce [11, 12, 13, 14, 15] introduced the notion of the stability index of a special Lagrangian cone to study deformations of a special Lagrangian submanifold with a conical singularity. Lotay [17] generalized it to the coassociative case. Associative and Cayley submanifolds behave differently from special Lagrangian and coassociative submanifolds, and hence it is difficult to generalize it directly to the associative or Cayley case. Thus in this paper, we focus on the Cayley case and study the deformations of homogeneous Cayley cones explicitly. It may help to develop the general deformation theory of a Cayley submanifold with a conical singularity. Our approach is based on the representation theory. This is an analogue of Ohnita’s approach to special Legendrian submanifolds in [24].

The homogeneous associative submanifolds in $S^7$ are classified by Lotay [16] into 8 types: $A_1, A_2$ and $A_3$ not lying in a totally geodesic nearly Kähler $S^6$, Lagrangian submanifolds $L_1, L_2, L_3$, and $L_4$ in $S^6$, and the totally geodesic $S^5$ (Proposition 6.1). Infinitesimal Lagrangian deformations in $S^6$ are studied in [17], and hence we study the infinitesimal deformations of the others and obtain the following.

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†Department of Mathematics, Gakushuin University, 1-5-1, Mejiro, Toshima, Tokyo, 171-8588, Japan (kkawai@math.gakushuin.ac.jp). The author is supported by Grant-in-Aid for JSPS fellows (26-7067).
Theorem 1.1. As an associative submanifold, $A_1$ is rigid, while $A_2$ and $A_3$ are not rigid. The deformation space of $A_2$ is unobstructed, and all non-trivial associative deformations of $A_2$ are induced by the $\text{PGL}(4, \mathbb{C})$-action on $\mathbb{C}P^3$ via the Hopf lift.

Theorem 1.2. All the associative and non-Lagrangian deformations of the totally geodesic $S^3$, $L_1, L_2, L_3$, and $L_4$ are trivial. In other words, such deformations are induced from $\text{Spin}(7) \setminus G_2$.

This paper is organized as follows. In Section 2, we review the fundamental facts of $G_2$, $\text{Spin}(7)$, Sasakian, and nearly Kähler geometry.

In Section 3, we characterize the space of all infinitesimal associative deformations as an eigenspace of a twisted Dirac operator $D$ (Proposition 3.2).

In Section 4 (5), we compute the difference of the dimension between infinitesimal associative and special Legendrian (Lagrangian) deformations. These computations are useful to prove Theorem 1.1 and 1.2 and give the geometrical meanings of some eigenspaces of some differential operators such as the Laplacian.

In Section 6, according to Lotay’s classification, we calculate the dimensions of eigenspaces of homogeneous associative submanifolds by the representation theoretical method in Appendix B, and prove Theorem 1.1 and 1.2.

Notation. Let $M$ be a manifold and $E$ be a vector bundle over $M$. We denote by $C(M, E)$ the space of all continuous sections of $E \to M$, and by $C^\infty(M, E)$ the space of all smooth sections of $E \to M$. Especially, we write $X(\mathbb{M}) = C^\infty(\mathbb{M}, T\mathbb{M})$.

If a Lie group $G$ acts on $M$, we denote by $X^*$ the vector field generated by $X \in \mathfrak{g} = \text{Lie}(G)$.

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2. Preliminaries.

2.1. $G_2$ and $\text{Spin}(7)$ geometry.

Definition 2.1. Define a 3-form $\varphi_0$ on $\mathbb{R}^7$ by

$$\varphi_0 = dx_{123} + dx_1(dx_{45} + dx_{67}) + dx_2(dx_{46} - dx_{57}) - dx_3(dx_{47} + dx_{56}),$$

where $(x_1, \cdots, x_7)$ is the standard coordinate system on $\mathbb{R}^7$ and wedge signs are omitted. The Hodge dual of $\varphi_0$ is given by

$$*\varphi_0 = dx_{4567} + dx_{23}(dx_{67} + dx_{45}) + dx_{13}(dx_{57} - dx_{46}) - dx_{12}(dx_{56} + dx_{47}).$$

Decompose $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$ and denote by $x_0$ the coordinate on $\mathbb{R}$. Define a self-dual 4-form $\Phi_0$ on $\mathbb{R}^8$ by

$$\Phi_0 = dx_0 \wedge \varphi_0 + *\varphi_0.$$ 

If we identify $\mathbb{R}^8 \cong \mathbb{C}^4$ via $\mathbb{R}^8 \ni (x_0, \cdots, x_7) \mapsto (x_0 + ix_1, x_2 + ix_3, x_4 + ix_5, x_6 + ix_7) = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, then $\Phi_0$ is described as

$$\Phi_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \text{Re}\Omega_0,$$

where $\omega_0 = \frac{i}{2} \sum_{j=1}^4 dz_j \overline{\omega_j}$ and $\Omega_0 = dz_1z_2z_3z_4$ are the standard Kähler form and the holomorphic volume form on $\mathbb{C}^4$, respectively.
The stabilizers of $\varphi_0$ and $\Phi_0$ are the exceptional Lie group $G_2$ and Spin$(7)$, respectively:

$$G_2 = \{ g \in GL(7, \mathbb{R}) ; g^* \varphi_0 = \varphi_0 \}, \quad \text{Spin}(7) = \{ g \in GL(8, \mathbb{R}) ; g^* \Phi_0 = \Phi_0 \}.$$ 

The Lie group $G_2$ fixes the standard metric $g_0 = \sum_{i=1}^7 (dx_i)^2$ and the orientation on $\mathbb{R}^7$. They are uniquely determined by $\varphi_0$ via

$$6g_0(v_1, v_2)\text{vol}_{g_0} = i(v_1)\varphi_0 \wedge i(v_2)\varphi_0 \wedge \varphi_0,$$

where $\text{vol}_{g_0}$ is a volume form of $g_0$, $i(\cdot)$ is the interior product, and $v_i \in T(\mathbb{R}^7)$.

Similarly, Spin$(7)$ fixes the standard metric $h_0 = \sum_{i=0}^7 (dx_i)^2$ and the orientation on $\mathbb{R}^8$. We have the following identities:

$$\Phi_0^2 = 14\text{vol}_{h_0}, \quad (i(w_2)i(w_1)\Phi_0)^2 \wedge \Phi_0 = 6\|w_1 \wedge w_2\|^2_{h_0} \text{vol}_{h_0},$$

where $\text{vol}_{h_0}$ is a volume form of $h_0$ and $w_i \in T(\mathbb{R}^8)$.

**Definition 2.2.** Let $Y$ be an oriented 7-manifold and $\varphi$ a 3-form on $Y$. A 3-form $\varphi$ is called a $G_2$-structure on $Y$ if for each $y \in Y$, there exists an oriented isomorphism between $T_y Y$ and $\mathbb{R}^7$ identifying $\varphi_y$ with $\varphi_0$. From (2.1), $\varphi$ induces the metric $g$ and the volume form on $Y$. A $G_2$-structure $\varphi$ is said to be nearly parallel if $d\varphi = 4 \ast \varphi$. We call a manifold with a nearly parallel $G_2$-structure a nearly parallel $G_2$-manifold for short. A $G_2$-structure $\varphi$ is called torsion-free if $d\varphi = d \ast \varphi = 0$.

Let $X$ be an oriented 8-manifold and $\Phi$ a 4-form on $X$. A 4-form $\Phi$ is called a Spin$(7)$-structure on $X$ if for each $x \in X$, there exists an oriented isomorphism between $T_x X$ and $\mathbb{R}^8$ identifying $\Phi_x$ with $\Phi_0$. From (2.2), $\Phi$ induces the metric $h$ and the volume form on $X$. A Spin$(7)$-structure $\Phi$ is called torsion-free if $d\Phi = 0$.

**Lemma 2.3.** [25] A $G_2$-structure $\varphi$ is torsion-free if and only if Hol$(g) \subset G_2$. A Spin$(7)$-structure $\Phi$ is torsion-free if and only if Hol$(h) \subset \text{Spin}(7)$.

**Lemma 2.4.** [1] The following are equivalent:

1. $d\varphi = 4 \ast \varphi$ (i.e. The 3-form $\varphi$ is a nearly parallel $G_2$-structure.),
2. $\nabla \varphi = \frac{1}{4}d\varphi$, where $\nabla$ is the Levi-Civita connection of $g$,
3. $\nabla \varphi = \ast \varphi$,
4. $\nabla_X(\ast \varphi) = -g(X, \cdot) \wedge \varphi$ for any $X \in TY$,
5. $i(X)\nabla_X \varphi = 0$ for any $X \in TY$,
6. The Riemannian cone $C(Y) = \mathbb{R}_{>0} \times Y$ admits a torsion-free Spin$(7)$-structure $\Phi = r^3 dr \wedge \varphi + r^4 \ast \varphi$ with the induced cone metric $\overline{g} = dr^2 + r^2 g$.

Next, we give a summary of the facts about submanifolds. Let $Y$ be a manifold with a $G_2$-structure $\varphi$ and the induced metric $g$.

**Lemma 2.5.** [8] For every oriented $k$-dimensional subspace $V^k \subset T_p Y$ where $p \in Y$ and $k = 3, 4$, we have $\varphi|_{V^k} \leq \text{vol}_{V^k}$, $\ast \varphi|_{V^k} \leq \text{vol}_{V^k}$. An oriented 3-submanifold $L^3 \subset Y$ is called associative if $\varphi|_{TL^3} = \text{vol}_{L^3}$. An oriented 4-submanifold $L^4$ is called coassociative if $\ast \varphi|_{TL^4} = \text{vol}_{L^4}$.

**Lemma 2.6.** [8] Define a tangent bundle valued 3-form $\chi \in C^\infty(Y, \wedge^3 T^* Y \otimes TY)$ by

$$g(v_1, \chi(v_2, v_3, v_4)) = \ast \varphi(v_1, v_2, v_3, v_4)$$
for \( v_i \in TY \). If \( L^k \subset Y \) is an oriented \( k \)-submanifold where \( k = 3, 4 \), then

\[
L^3: \text{associative} \iff \chi|_{TL^3} = 0 \text{ and } \varphi|_{TL^3} > 0,
\]
\[
L^4: \text{coassociative} \iff \varphi|_{TL^4} = 0 \text{ and } * \varphi|_{TL^4} > 0.
\]

**Definition 2.7.** Define the cross product \( \times : TY \times TY \to TY \) by

\[
g(u \times v, w) = \varphi(u, v, w)
\]
for \( u, v, w \in TY \). This satisfies the following relation:

\[
\chi(u, v, w) = u \times (v \times w) + g(u, v)w - g(u, w)v. \tag{2.3}
\]

**Remark 2.8.** When \( L^3 \) is associative, there exists an orthonormal basis \( \{e_1, e_2, e_3\} \) satisfying \( e_3 = e_1 \times e_2 \) at any point in \( L^3 \).

**Definition 2.9.** Let \( X \) be a manifold with a Spin(7)-structure \( \Phi \). Then for every oriented 4-dimensional subspace \( W \subset T_x X \) where \( x \in X \), we have \( \Phi|_W \leq \text{vol}_W \).

An oriented 4-submanifold \( N \subset X \) is called **Cayley** if \( \Phi|_{TN} = \text{vol}_N \).

**Lemma 2.10.** Let \((Y, \varphi, g)\) be a nearly parallel \( G_2 \)-manifold and \( L \subset Y \) be an oriented 3-submanifold. By Lemma 2.4, \( C(Y) \) is a manifold with a torsion-free Spin(7)-structure \( \Phi \). Then \( L \subset Y \) is associative if and only if \( C(L) \subset C(Y) \) is Cayley.

**Lemma 2.11.** \([16]\) There are no coassociative submanifolds of a nearly parallel \( G_2 \)-manifold \((Y, \varphi, g)\).

**Proof.** If \( L \) is a coassociative submanifold, we have \( \varphi|_{TL} = 0 \), which implies that \( 4 \text{vol}_L = 4 * \varphi|_{TL} = d \varphi|_{TL} = 0 \). This is a contradiction. \( \square \)

### 2.2. Sasakian geometry.

**Definition 2.12.** An odd dimensional Riemannian manifold \((S, g)\) is a **Sasakian manifold** if its Riemannian cone \((C(S), \overline{g}) = (\mathbb{R}_{>0} \times S, dr^2 + r^2 g)\) is a Kähler manifold with respect to some integrable complex structure \( J \) over \( C(S) \).

Here, \( r \) is a standard coordinate of \( \mathbb{R}_{>0} \) and we regard \( r \) as the function on \( C(S) \). We identify \( S \) with the submanifold \( \{1\} \times S \subset C(S) \).

**Lemma 2.13.** Let \((S, g)\) be a Sasakian \((2m+1)\)-manifold. If \( g \) is Einstein, the cone \((C(S), \overline{g})\) is Ricci-flat. In addition, if there exists a holomorphic volume form \( \Omega \in \Omega^{(m+1,0)}(C(S)) \) such that

\[
\omega^{m+1}/(m+1)! = (-1)^{m(m+1)/2}(i/2)^{m+1}\Omega \wedge \overline{\Omega}, \tag{2.4}
\]
where \( \omega = \overline{g}(J \cdot, \cdot) \) is the associated Kähler form on \( C(S) \), we call \((C(S), \overline{g}, J, \omega, \Omega)\) a **Calabi-Yau manifold**.

**Lemma 2.14 ([4, Corollary 11.1.8]).** If \( S \) is a compact simply-connected Sasakian-Einstein manifold, \( C(S) \) is a Calabi-Yau manifold.

**Remark 2.15.** The holomorphic volume form \( \Omega \) is not unique. For any \( \theta \in \mathbb{R} \), \( e^{i\theta} \Omega \) also satisfies (2.4).
Let \((S, g)\) be a Sasaki-Einstein 7-manifold with a Calabi-Yau structure on \(C(S)\).

**Lemma 2.16.** There exists a 3-form \(\varphi \in \Omega^3(S)\) such that \((S, \varphi, g)\) is a nearly parallel \(G_2\)-manifold.

**Proof.** Fix a holomorphic volume form \(\Omega\). Then a 4-form
\[
\Phi = \frac{1}{2} \omega \wedge \omega + \text{Re} \Omega \in \Omega^4(C(S))
\] (2.5)
gives a torsion-free Spin(7)-structure on \(C(C(S))\). A 3-form \(\varphi \in \Omega^3(S)\) defined by
\[
\Phi_{(r,p)} = r^3 dr \wedge \varphi_p + r^4 \ast \varphi_p,
\] where \((r, p) \in \mathbb{R}_{>0} \times S,
\] gives the nearly parallel \(G_2\)-structure on \(S\). \(\square\)

Next, we summarize the facts about submanifolds in Sasakian manifolds.

**Definition 2.17.** An \(m\)-submanifold \(L \subset S\) is called **Legendrian** if \(C(L) \subset C(S)\) is Lagrangian: \(\omega|_{TC(L)} = 0\). Fix a holomorphic volume form \(\Omega\) on \(C(S)\). An \(m\)-submanifold \(L \subset S\) is called **special Legendrian** if \(C(L) \subset C(S)\) is special Lagrangian: \(\text{Re} \Omega|_{TC(L)} = \text{vol}_{C(L)}\) \(\iff\) \(\omega|_{TC(L)} = 0\) and \(\text{Re} \Omega|_{TC(L)} > 0\).

The following is a well-known fact. For example, see [21, Proposition 4.5].

**Lemma 2.18.** Let \(L \subset S\) be a Legendrian submanifold. Then \(L\) is minimal if and only if \(\text{Im}(e^{i\theta} \Omega) = 0\) for some \(\theta \in \mathbb{R}\).

By definition, we obtain the following result.

**Lemma 2.19.** Let \(L \subset S\) be an oriented 3-submanifold. If \(L\) is special Legendrian or if the cone \(C(L)\) is a complex submanifold in \(C(S)\), \(L\) is associative.

**Proof.** If \(L\) is special Legendrian, we have \(\frac{1}{2} \omega \wedge \omega|_{TC(L)} = 0\) and \(\text{Re} \Omega|_{TC(L)} = \text{vol}_{C(L)}\). If \(C(L)\) is a complex submanifold, we have \(\frac{1}{2} \omega \wedge \omega|_{TC(L)} = \text{vol}_{C(L)}\) and \(\text{Re} \Omega|_{TC(L)} = 0\). By (2.5) and Lemma 2.10, we see that \(L\) is associative in both cases. \(\square\)

**2.3. Infinitesimal deformation of special Legendrian submanifolds.** Let \((S, g)\) be a Sasaki-Einstein \((2m + 1)\)-manifold with a Calabi-Yau structure on \(C(S)\). Fix a holomorphic volume form \(\Omega\) and let \(L \subset S\) be a special Legendrian submanifold.

**Lemma 2.20 ([24]).** The vector space of all infinitesimal special Legendrian deformations of \(L\) is identified with
\[
\{ f \in C^\infty(L); \Delta_+ f = (2m + 2) f \},
\] (2.6)
where \(\Delta_+\) is the Hodge Laplacian for functions on \(L\).

We write the subscript + of \(\Delta_+\) since every eigenvalue of this Laplacian is non-negative if \(L\) is compact.

**Proof.** Let \(\nu\) be the normal bundle of \(L\) in \(S\). Since \(L\) is Legendrian, there is a canonical isomorphism \(\nu \ni v \mapsto (g(v, J(r \frac{\partial}{\partial r})|_{r=1}), -g(Jv, \cdot)) \in \mathbb{R} \oplus T^*L\). Via this
identification, suppose that
\[ V \in C^\infty(L,\nu) \] corresponds to \((f,\alpha) \in C^\infty(L) \oplus \Omega^1(L).\] Then we have
\[ 0 = L_V \left( i \left( r \frac{\partial}{\partial r} \right) \omega \right) \bigg|_{TL} = -2\alpha + df, \] (2.7)
\[ 0 = L_V \left( i \left( r \frac{\partial}{\partial r} \right) \text{Im}\Omega \right) \bigg|_{TL} = d \alpha + (m+1)fvol_L, \] (2.8)
which implies the proof. \( \Box \)

The same result is obtained in [6] by using the fact that a cone \( C(L) \) of \( L \) is special Lagrangian in \( C(S) \) and applying the deformation theory of special Lagrangian submanifolds in [19].

### 2.4. Nearly Kähler geometry.

**Definition 2.21.** Let \((N,k,J,\sigma)\) be a real 6-dimensional almost Hermitian manifold with a Hermitian metric \(k\), an almost complex structure \(J\) and an associated Kähler form \(\sigma\). Let \(\psi^\pm \in \Omega^3(N)\) be 3-forms on \(N\). A quintuple \((k,J,\sigma,\psi^\pm)\) is called an \(SU(3)\)-structure if we have \(\|\psi^\pm\| = 2\) and \(\Psi := \psi^+ + \sqrt{-1}\psi^-\) is a \((3,0)\)-form with respect to \(J\).

**Remark 2.22.** The \(SU(3)\)-structure with a Kähler structure and a holomorphic \((3,0)\)-form \(\Psi\) is a Calabi-Yau structure. In fact, we can prove
\[ \sigma \wedge \psi^\pm = 0, \quad \sigma^3/3! = (-1)^{\frac{3(3-1)}{2}}(i/2)^3\Psi \wedge \bar{\Psi}. \]

**Definition 2.23.** An \(SU(3)\)-structure satisfying \(d\sigma = 3\psi^+\) and \(d\psi^+ = -2\sigma^2\) is called nearly Kähler.

**Lemma 2.24 ([5]).** Let \((N,k,J,\sigma)\) be a real 6-dimensional almost Hermitian manifold. It admits a nearly Kähler structure if and only if \((\nabla_X J)X = 0\) for every vector field \(X\) on \(N\) and \(\nabla_X J \neq 0\) for every \(0 \neq X \in TN\), where \(\nabla\) is the Levi-Civita connection of \(k\).

**Lemma 2.25.** Let \((N,k,J,\sigma,\psi^\pm)\) be a nearly Kähler manifold. Then \(C_s(N) = \mathbb{R}_{>0} \times N\) admits a torsion-free \(G_2\)-structure \((\varphi,\tilde{k})\) with
\[ \tilde{k} = dr^2 + r^2k, \]
\[ \varphi = r^2dr \wedge \sigma + r^3\psi^+ = \frac{1}{2}d(r^3\sigma), \]
\[ *\varphi = r^3\psi^- \wedge dr + \frac{1}{2}r^4\sigma^2 = -\frac{1}{4}d(r^4\psi^-). \]

**Lemma 2.26 ([4]).** Let \((N,k,J,\sigma,\psi^\pm)\) be a nearly Kähler manifold. Then \(C_s(N) = (0,\pi) \times N\) (a sine cone of \(N\)) admits a nearly parallel \(G_2\)-structure \((\tilde{\varphi},\tilde{k})\) with
\[ \tilde{k} = dt^2 + (\sin^2 t)k, \]
\[ \tilde{\varphi} = (\sin^2 t)dt \wedge \sigma + (\cos t\sin^3 t)\psi^+ - (\sin^4 t)\psi^-, \]
\[ *\tilde{\varphi} = \frac{1}{2}(\sin^4 t)\sigma^2 + (\sin^3 t\cos t)\psi^- \wedge dt - (\sin^4 t)dt \wedge \psi^+. \]
We canonically identify $N$ with the submanifold $N \times \{\frac{\pi}{2}\} \subset C_s(N)$.

Remark 2.27. Since $C(N)$ admits a torsion-free $G_2$-structure, $\mathbb{R} \times C(N)$ admits a torsion-free Spin(7)-structure. The nearly parallel $G_2$-structure on $C_s(N)$ is induced via the identification $C(C_s(N)) = \mathbb{R}_{>0} \times (0, \pi) \times N \ni (r, t, x) \mapsto (r \cos t, r \sin t, x) \in \mathbb{R} \times \mathbb{R}_{>0} \times N = \mathbb{R} \times C(N)$.

Lemma 2.28 ([20]). Let $(N, k, J, \sigma, \psi^\pm)$ be a nearly Kähler manifold. Define a map $G : TN \times TN \to TN$ by $k(G(u, v), w) = \psi^+(u, v, w)$ for $u, v, w \in TN$. Then we have

$$(\nabla_X J)(Y) = G(X, Y), \quad \nabla_X \psi^+ = -k(X, \cdot) \wedge \sigma,$$

where $\nabla$ is the Levi-Civita connection of $k$ and $X, Y \in \mathfrak{X}(N)$.

Lemma 2.29. Let $(N, k, J, \sigma, \psi^\pm)$ be a nearly Kähler manifold. From Lemma 2.25, the cone $C(N) = \mathbb{R}_{>0} \times N$ admits a torsion-free $G_2$ structure. Let $\Sigma \subset N \setminus (L \subset N)$ be an oriented 2(3)-submanifold. Then we have

- $C(\Sigma) \subset C(N)$ is associative if and only if $\Sigma$ is a $J$-holomorphic curve.
- $C(L^3) \subset C(N)$ is a coassociative 4-fold if and only if $L$ is Lagrangian: $\sigma|_{TL} = 0$.

Lemma 2.30. Let $(N, k, J, \sigma, \psi^\pm)$ be a nearly Kähler manifold. From Lemma 2.26, the sine cone $C_s(N) = N \times (0, \pi)$ admits a nearly parallel $G_2$ structure. Let $\Sigma \subset N \setminus (L \subset N)$ be an oriented 2(3)-submanifold. Then it follows that

- $C_s(\Sigma) \subset C_s(N)$ is associative if and only if $\Sigma$ is a $J$-holomorphic curve.
- $L \times \{\frac{\pi}{2}\} \subset C_s(N)$ is associative if and only if $L$ is Lagrangian: $\sigma|_{TL} = 0$.

Remark 2.31. On a nearly Kähler manifold, we know that $d\sigma = 3\psi^+$, which implies that a Lagrangian submanifold $L$ satisfies $\psi^+|_{TL} = 0$. Thus Lagrangian submanifolds in a nearly Kähler manifold are regarded as “special Lagrangian” (with phase $-i$).

We know the following as Lemma 2.20.

Lemma 2.32. The vector space of all infinitesimal Lagrangian deformations of $L$ in a nearly Kähler manifold is identified with

$$\{v \in \mathfrak{X}(L); \text{rot}(v) = 3v\},$$

where $\text{rot}(v) = \sum_{i=1}^{3} e_i \times \nabla^T \nabla v$, $\nabla^T$ is the Levi-Civita connection of the metric $k_L$ on $L$ induced from $(N, k)$ and $\{e_i\}_{i=1, 2, 3}$ is the local orthonormal frame of $TL$.

Proof. Since $L$ is Lagrangian, there is a canonical isomorphism between the tangent bundle $TL$ and the normal bundle of $L$ in $N$ via $v \mapsto Jv$. Then a vector field $v \in \mathfrak{X}(L)$ on $L$ corresponds to an infinitesimal Lagrangian deformation of $L$ if and only if

$$0 = L_{Jv} \sigma|_{TL} = 3i(v)\text{vol}_L - d(k_L(v, \cdot)) \text{vol}_L.$$

Note that $\psi^-|_{TL} = -\text{vol}_L$. Then the equations $\ast(i(v)\text{vol}_L) = k_L(v, \cdot)$ and $\ast d(k_L(v, \cdot)) = k_L(\text{rot}(v), \cdot)$ imply the proof. □
3. Associative deformations in nearly parallel $G_2$-manifolds. Let $(Y, \varphi, g)$ be a nearly parallel $G_2$-manifold, $\iota : M^3 \hookrightarrow Y$ be an associative immersion, and $\{\iota_t : M \rightarrow Y\}_{t \in (-\varepsilon, \varepsilon)}$ be a smooth family of immersions with $\iota_0 = \iota$.

**Definition 3.1.** A family $\{\iota_t\}$ is called an associative deformation of $\iota$ if $\iota_t$ is an associative immersion for each $t$. An associative deformation $\{\iota_t\}$ is called trivial if $\{\iota_t\}$ is induced by a one-parameter family of automorphisms of $(Y, \varphi, g)$. If all infinitesimal associative deformations of $M$ come from trivial deformations, $M$ is called rigid.

First, we characterize the space of infinitesimal associative deformations of $M$.

**Proposition 3.2.** Let $(Y, \varphi, g)$ be a nearly parallel $G_2$-manifold, and $M^3 \subset Y$ be an associative submanifold. Denote by $\nu$ the normal bundle of $M$ in $Y$ and by $\nabla^\perp$ the connection on $\nu$ induced by the Levi-Civita connection $\nabla$ of $(Y, g)$.

Taking any local orthonormal frame $\{e_1, e_2, e_3\}$ of $TM$, define the operator $D : \mathcal{C}_\infty(M, \nu) \rightarrow \mathcal{C}_\infty(M, \nu)$ by

$$D\psi := \sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i} \psi.$$  

Then the vector space of all infinitesimal associative deformations of $M^3 \hookrightarrow Y$ is identified with

$$\{\psi \in \mathcal{C}_\infty(M, \nu); D\psi = -\psi\}.$$

**Remark 3.3.** [19] There exists a rank 4 vector bundle $E \rightarrow M$ satisfying $\nu \cong S \otimes_{\mathbb{H}} E$, where $S \rightarrow M$ is a spinor bundle. Then $D$ is a twisted Dirac operator.

The proof of Proposition 3.2 comes from the following general theory of associative deformations.

**Proposition 3.4 ([7, 19]).** Let $(Y, \varphi, g)$ be a manifold with a $G_2$-structure and $M^3 \subset Y$ be an associative submanifold. Then the vector space of all infinitesimal associative deformations of $M^3 \hookrightarrow Y$ is identified with $\ker \tilde{D}$, where $\tilde{D} : \mathcal{C}_\infty(M, \nu) \rightarrow \mathcal{C}_\infty(M, \nu)$ is defined by

$$\tilde{D}\psi := -\sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i} \psi + \sum_{k=1}^{4} (\nabla_\psi \ast \varphi)(\eta_k, \omega) \eta_k.$$  

Here $\{e_1, e_2, e_3\}$ is an oriented local orthonormal frame of $TM$, $\omega = e_1 \wedge e_2 \wedge e_3$, and $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ is a local orthonormal frame of $\nu$.

**Proof.** We give an outline of the proof. Define a map $F : \mathcal{C}_\infty(M, \nu) \rightarrow \mathcal{C}_\infty(M, TY |_M)$ as $F(\sigma) = \exp_\sigma^* \chi(\omega)$, where $\chi$ is defined in Lemma 2.6. We know that $\exp_\sigma(M)$ is associative if and only if $F(\sigma)$ vanishes. For any $\psi \in \mathcal{C}_\infty(M, \nu)$, we may consider

$$(dF)_0(\psi) = 0.$$  

By a direct computation, the left hand side is equal to $-\sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i} \psi + \sum_{k=1}^{4} (\nabla_\psi \ast \varphi)(\eta_k, \omega) \eta_k$, and hence the statement is proved. \(\square\)
By Lemma 2.4, we see the following lemma, which implies Proposition 3.2.

**Lemma 3.5.** If \((Y, \varphi, g)\) is nearly parallel, then \(\sum_{k=1}^{4}(\nabla_{\varphi} \ast \varphi)(\eta_k, \omega)\eta_k = -\psi\).

**Remark 3.6.** We can prove Proposition 3.2 by using the fact that a cone \(C(M)\) of \(M\) is a Cayley submanifold in \(C(Y)\) with a torsion-free Spin(7)-structure. Applying the deformation theory of Cayley submanifolds in [19], we consider the Cayley cone deformation of \(C(M)\). This is an analogue of the proof of Lemma 2.20 given by [6].

The operator \(D\) has the following properties.

**Lemma 3.7.** The operator \(D\) is elliptic. There exists a vector field \(X \in \mathfrak{X}(M)\) on \(M\) satisfying
\[
g(D\psi, \psi') = -\text{div}(X) + g(\psi, D\psi')
\]
for any \(\psi, \psi' \in C^\infty(M, \nu)\). In particular, when \(M\) is compact, \(D\) is self-adjoint.

**Proof.** The ellipticity of \(D\) is shown in [7]. For any \(\psi, \psi' \in C^\infty(M, \nu)\), we compute
\[
g(D\psi, \psi') = g\left(\sum_{i=1}^{3} e_i \times \nabla_{e_i} \psi, \psi'\right)
\]
\[
= -\sum_{i=1}^{3} g(\nabla_{e_i} \psi, e_i \times \psi')
\]
\[
= \sum_{i=1}^{3} (-e_i(g(\psi, e_i \times \psi')) + g(\psi, \nabla_{e_i} e_i \times \psi') + g(\psi, D\psi')).
\]

Define the vector field \(X \in \mathfrak{X}(M)\) on \(M\) by
\[
g(X, v) = g(\psi, v \times \psi')\]
for \(v \in TM\). Then we obtain (3.1). \(\Box\)

Since \(D\) is a twisted Dirac operator, there is a close relation between \(D^2\) and the Laplacian. Choose a local orthonormal frame \(\{e_1, e_2, e_3\}\) of \(TM\) and define the operators \(\nabla^{\perp} \ast \nabla^{\perp}, \mathcal{R}, A : C^\infty(M, \nu) \to C^\infty(M, \nu)\) by
\[
\nabla^{\perp} \ast \nabla^{\perp} = \sum_{i=1}^{3} (-\nabla_{e_i} \nabla_{e_i}^{\perp} + \nabla_{\nabla_{e_i}^{\perp} e_i}^{\perp}), \quad \mathcal{R} = \pi_{\nu}(\sum_{i=1}^{3} R(e_i, \cdot) e_i), A = {}^tA \circ A,
\]
where \(\nabla^{\perp}\) is the connection on the normal bundle \(\nu\) induced by the Levi-Civita connection \(\nabla\) of \((Y, g)\), \(\nabla^{\perp}\) is the orthogonal projection of \(\nabla\) to \(TM\), \(R\) is the curvature tensor of \(g\), \(\pi_{\nu}\) is the orthogonal projection to \(\nu\), \(A : \nu \ni \psi \mapsto (u \mapsto -\nabla_{u}^{\perp} \psi) \in SM := \{T : TM \to TM; T = T\}\) (the second fundamental form), and \({}^tA\) is the transpose of \(A\).

**Proposition 3.8.** Let \((Y, \varphi, g)\) be a nearly parallel \(G_2\)-manifold and \(M^3 \subset Y\) be an associative submanifold. Then we have
\[
(D - 3i\nu)(D + i\nu) = \nabla^{\perp} \ast \nabla^{\perp} + \mathcal{R} - A.
\]

The proof is given in the appendix. The right hand side \(\mathcal{J} := \nabla^{\perp} \ast \nabla^{\perp} + \mathcal{R} - A\) is called a Jacobi operator, and \(\ker \mathcal{J}\) is known to be the space of infinitesimal minimal
deformations ([26]). By this formula, \( D\psi = -\psi \) implies \( J\psi = 0 \), which ensures that associative deformations are minimal deformations.

**Remark 3.9.** When \( M \) is compact, the space of all infinitesimal minimal and non-associative deformations of \( M \) is identified with \( \{ \psi \in C^\infty(M,\nu); D\psi = 3\psi \} \).

**Proof.** Since \( D \) is elliptic self-adjoint, there is an orthonormal basis \( \{ \psi_i \}_{i=1}^\infty \subset C^\infty(M,\nu) \) of \( L^2(M,\nu) \) consisting of eigensections of \( D \). The set of eigenvalues is discrete and the each eigenspace is finite dimensional. We may assume that \( D\psi_i = \lambda_i \psi_i \) for \( \lambda_i \in \mathbb{R} \). For any \( \psi = \sum_{i=1}^\infty (\psi,\psi_i)_{L^2} \psi_i \in C^\infty(M,\nu) \) where \( (\cdot,\cdot)_{L^2} \) is the \( L^2 \) inner product, we have

\[
(D - 3id_\nu)(D + id_\nu)\psi = \sum_{i=1}^\infty ((D - 3id_\nu)(D + id_\nu)\psi_i)_{L^2} \psi_i \\
= \sum_{i=1}^\infty (\psi_i (D - 3id_\nu)(D + id_\nu)\psi_i)_{L^2} \\
= \sum_{i=1}^\infty (\lambda_i - 3)(\lambda_i + 1)(\psi,\psi_i)_{L^2} \psi_i.
\]

Since \( \{ \psi_i \}_{i=1}^\infty \) is an orthonormal basis, we see that \( (D - 3id_\nu)(D + id_\nu)\psi = 0 \) if and only if \( (\lambda_i - 3)(\lambda_i + 1)(\psi,\psi_i)_{L^2} = 0 \) for each \( i \). Thus elements of \( \ker(D - 3id_\nu)(D + id_\nu) \) are linear combinations of elements of \( \ker(D - 3id_\nu) \) and \( \ker(D + id_\nu) \). \( \Box \)

4. **Associative deformations of special Legendrian submanifolds in Sasaki-Einstein manifolds.** Let \( (S,g) \) be a Sasaki-Einstein 7-manifold with a Calabi-Yau structure \( (\overline{g},J,\omega,\Omega) \) on \( C(S) \). Let \( M \subset S \) be a special Legendrian submanifold. By Lemmas 2.16 and 2.19, \( (S,\varphi,g) \) admits a nearly parallel \( G_2 \)-structure for some \( \varphi \in \Omega^3(S) \) and \( M \) is associative. We study the infinitesimal associative deformations of \( M \).

4.1. **Associative deformations of special Legendrians.** Let \( \nu \rightarrow M \) be the normal bundle of \( M \). First, we rewrite the operator \( D : C^\infty(M,\nu) \rightarrow C^\infty(M,\nu) \) in Proposition 3.2 in the special Legendrian case. Since \( M \) is special Legendrian, there exists canonical isomorphism \( TM \oplus \mathbb{R} \ni (v,x) \mapsto Jv + xJ(\frac{\partial}{\partial r}) \Big|_{r=1} \in \nu \). Via this identification, we obtain the following.

**Proposition 4.1.** The corresponding operator \( D : \mathfrak{X}(M) \oplus C^\infty(M) \rightarrow \mathfrak{X}(M) \oplus C^\infty(M) \) is described as

\[
D(v,f) = (-\text{grad}(f) + \text{rot}(v) + v, \text{div}(v) + 3f),
\]

where \( g_M(\text{grad}(f),\cdot) = df \), \( \text{div}(v) = \text{tr}(\nabla^T v) \), and \( \text{rot}(v) = \sum_{i=1}^3 e_i \times \nabla^T e_i v \).

Here, we denote by \( g_M \) the metric on \( M \) induced from \( (Y,g) \), by \( \nabla^T \) the Levi-Civita connection of \( g_M \), by \( \{e_i\}_{i=1,2,3} \) the local orthonormal frame of \( TM \), and \( g_M(v \times w,\cdot) = \varphi(v,w,\cdot) = \text{vol}_M(v,w,\cdot) \) \( (v,w \in TM) \).

We first give all the statements in this section and then prove them.

**Corollary 4.2.** We have

\[
\dim\{\text{the infinitesimal associative deformations of } M\} = \dim\{ f \in C^\infty(M); \Delta_+ f = 8f \} + \dim\{ v \in \mathfrak{X}(M); \text{rot}(v) = -2v \}.
\]
REMbek 4.3. From Lemma 2.20, \( \dim \{ v \in \mathfrak{X}(M); \text{rot}(v) = -2v \} \) gives the dimension of infinitesimal associative and non-special Legendrian deformations.

We have the same equations as in the vector analysis.

**Lemma 4.4.** For any \( f \in C^{\infty}(M) \) and \( v \in \mathfrak{X}(M) \), we have

\[
\text{rot}(\text{grad}(f)) = 0, \quad \text{div}(\text{rot}(v)) = 0,
\]

\[
\text{rot}(\text{rot}(v)) = \nabla^T \nabla^T v + \text{grad}(\text{div}(v)) + \sum_{i=1}^{3} R(v, e_i) e_i,
\]

where \( \{e_i\}_{i=1,2,3} \) is the local orthonormal frame of \( TM \), \( R \) is the curvature tensor, and \( \nabla^T \nabla^T = \sum_{i=1}^{3} (-\nabla_{e_i} \nabla_{e_i} + \nabla_{\nabla_{e_i} e_i}) \) is the rough Laplacian.

This lemma implies the following, which corresponds to Proposition 3.8.

**Corollary 4.5.**

\[
D^2(v, f) = \left( -4\text{grad}(f) + v + \text{rot}(v) + \nabla^T \nabla^T v + \sum_{i=1}^{3} R(v, e_i) e_i, \Delta_f + 4\text{div}(v) + 9f \right).
\]

Now, we give proofs.

**Proof of Proposition 4.1.** Let \( \{e_1, e_2, e_3\} \subset TM \) be a local oriented orthonormal frame. Set \( e_4 := \frac{\partial}{\partial r} \bigg|_{r=1} \) and \( \eta_j := J(e_j) \) for \( 1 \leq j \leq 4 \). Then \( \{\eta_j\}_{1 \leq j \leq 4} \) is a local oriented orthonormal frame of \( \nu \). Let \( \{e^1, \cdots, e^4, \eta^1, \cdots, \eta^4\} \) be the dual coframe, then we have

\[
\omega = \sum_{i=1}^{4} e^i \wedge \eta^i, \quad \Omega = (e^1 + i\eta^1) \wedge \cdots \wedge (e^4 + i\eta^4).
\]

Denoting \( \nabla_{e_i} e_j = \sum_{k=1}^{3} \Gamma_{ij}^k e_k \) and \( \nabla_{e_i} \eta_a = \sum_{b=1}^{4} \tilde{\Gamma}_{ia}^b \eta_b \) for \( 1 \leq i, j \leq 3 \) and \( 1 \leq a \leq 4 \), we see the following by a direct computation.

**Lemma 4.6.** We have

\[
(e_i \times \eta_a) = \begin{pmatrix}
\eta_4 & \eta_3 & -\eta_2 & -\eta_1 \\
-\eta_3 & \eta_4 & \eta_1 & -\eta_2 \\
\eta_2 & -\eta_1 & \eta_4 & -\eta_3
\end{pmatrix},
\]

\[
\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k, \quad \tilde{\Gamma}_{ij}^4 = -\delta_{ij}, \quad \tilde{\Gamma}_{i4}^k = \delta_{ik}, \quad \tilde{\Gamma}_{i4}^4 = 0,
\]

for \( 1 \leq i, j, k \leq 3, 1 \leq a \leq 4 \).

Then via the identification \( \mathfrak{X}(M) \oplus C^{\infty}(M) \ni (\sum_{j=1}^{3} v_j e_j, f) \mapsto \sum_{j=1}^{3} v_j \eta_j + f \eta_4 \in \)
\[ C^\infty(M, \nu) \text{ where } v_j, f \in C^\infty(M), \text{ we have} \]
\[ D \left( \sum_{j=1}^{3} v_j \eta_j + f \eta_4 \right) \]
\[ = \sum_{i=1}^{3} e_i \times \nabla_{e_i}^T \left( \sum_{j=1}^{3} v_j \eta_j + f \eta_4 \right) \]
\[ = \sum_{i,j=1}^{3} e_i(v_j) e_i \times \eta_j + \sum_{i,j=1}^{3} v_j e_i \times \left( \sum_{k=1}^{3} \Gamma_{ij}^{k} \eta_k - \delta_{ij} \eta_4 \right) + \sum_{i=1}^{3} e_i(f) e_i \times \eta_4 + \sum_{i=1}^{3} f e_i \times \eta_i \]
\[ = \left\{ e_2(v_3) - e_3(v_2) + \sum_{j=1}^{3} v_j (\Gamma_{ij}^{3} - \Gamma_{ij}^{2}) \right\} \eta_1 + \left\{ e_3(v_1) - e_1(v_3) + \sum_{j=1}^{3} v_j (\Gamma_{ij}^{1} - \Gamma_{ij}^{3}) \right\} \eta_2 \]
\[ + \left\{ e_1(v_2) - e_2(v_1) + \sum_{j=1}^{3} v_j (\Gamma_{ij}^{2} - \Gamma_{ij}^{1}) \right\} \eta_3 + \left\{ \sum_{i=1}^{3} e_i(v_1) + \sum_{i,j=1}^{3} v_j \Gamma_{ij}^{1} \right\} \eta_4 \]
\[ + \sum_{i=1}^{3} v_i \eta_i - \sum_{i=1}^{3} e_i(f) \eta_i + 3f \eta_4, \]
which gives the proof. \[ \square \]

**Proof of Lemma 4.4.** The first two equations are easy to prove. We only prove the third equation. By Lemma A.1 and the fact that \( M \) is associative, it follows that
\[ \text{rot(\text{rot}(v))} = \sum_{i,j=1}^{3} e_i \times (\nabla_{e_i}^T e_j \times \nabla_{e_j}^T v + e_j \times \nabla_{e_i}^T \nabla_{e_j}^T v). \]

From (2.3), we have \( u \times (v \times w) = -g(u, v)w + g(u, w)v \) for \( u, v, w \in TM \) on an associative submanifold \( M \). Hence we have
\[ \text{rot(\text{rot}(v))} \]
\[ = \sum_{i,j=1}^{3} \left( \Gamma_{ij}^{k} \nabla_{e_j}^T v + g(e_i, \nabla_{e_j}^T v) \nabla_{e_i}^T e_j - \delta_{ij} \nabla_{e_i}^T v + g(e_i, \nabla_{e_i}^T \nabla_{e_j}^T v) e_j \right) \]
\[ = \nabla^T \nabla^T v + \sum_{i,j=1}^{3} g((\nabla_{e_i}^T \nabla_{e_j}^T - \nabla_{e_j}^T \nabla_{e_i}^T) v, e_j) e_i, \]
where we use the fact that \( \Gamma_{ij}^{k} = -\Gamma_{ij}^{k} \) since \( \{e_i\}_{i=1,2,3} \) is the local orthonormal frame of \( TM \). On the other hand, we have
\[ \text{grad(\text{div}(v))} = \sum_{i,j=1}^{3} e_i (g(\nabla_{e_j}^T v, e_j)) e_i = \sum_{i,j=1}^{3} g((\nabla_{e_i}^T \nabla_{e_j}^T - \nabla_{e_j}^T \nabla_{e_i}^T) v, e_j) e_i, \]
which implies the proof. \( \square \)

**Proof of Corollary 4.2.** By Proposition 4.1, \( D(v, f) = -(v, f) \) is equivalent to \( \text{rot}(v) + 2v = \text{grad}(f), \text{div}(v) = -4f \). Considering the divergence of the first equation, we have \( \Delta f = -2\text{div}(v) \), which implies that \( D(v, f) = -(v, f) \) is equivalent to
\[ \left\{ \begin{array}{l}
\text{rot}(v) + 2v = \text{grad}(f), \\
\Delta f = 8f.
\end{array} \right. \] (4.1)
The second equation is given in (2.6). For any \( f \in C^\infty(M) \) satisfying \( \Delta_+ f = 8f \), \((v, f) = (\frac{1}{2} \text{grad}(f), f)\) is the solution of (4.1), which corresponds to the infinitesimal special Legendrian deformations of \( M \). (See (2.7)). \( \square \)

### 4.2. Associative deformation of homogeneous special Legendrians

The method and the notation in this subsection are summarized in the appendix. We give an explicit description of the operator \( \text{rot} \) when \( M \) is the reductive homogeneous space \( G/K \), where \( G \subset \text{Aut}(S, \varphi, g) \) and \( K \subset G \) is a closed subgroup. Take an \( \text{Ad}(K) \)-invariant vector subspace of \( \mathfrak{p} \subset \mathfrak{g} \) satisfying \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \).

It is well-known that there is an one-to-one correspondence between \( \text{Ad}(K) \)-invariant inner products on \( \mathfrak{p} \) and \( G \)-invariant metrics on \( M = G/K \). Since \( G \subset \text{Aut}(S, \varphi, g) \), there exists a \( G \)-invariant metric \( g_M \) on \( M \) induced from \( (S, g) \). Denote by \( \langle \cdot, \cdot \rangle \) the corresponding \( \text{Ad}(K) \)-invariant inner product and by \( \{ e_1, e_2, e_3 \} \subset \mathfrak{p} \) an oriented orthonormal basis of \( \mathfrak{p} \). Then we have the following.

**Lemma 4.7.** The map \( G \times \text{Ad} \mathfrak{p} \ni ([g, X] \mapsto \frac{d}{dt}g \cdot \exp(tX)K)|_{t=0} \in TM \) is an isomorphism. Thus the tangent bundle \( TM \) of \( M \) is a homogeneous vector bundle.

**Proposition 4.8.** The operator \( \text{rot} : \mathfrak{X}(M) \to \mathfrak{X}(M) \) is a homogeneous differential operator and induces the map \( \widetilde{\text{rot}} : C^\infty(G, \mathfrak{p})^{(K, \text{Ad})} \to C^\infty(G, \mathfrak{p})^{(K, \text{Ad})} \). If we define \( \overline{\text{rot}} \in (\text{End}(\mathfrak{p}) \otimes U(\mathfrak{g}))^K \) by

\[
\overline{\text{rot}} = \sum_{i \in \mathbb{Z}/3} e_i^* \otimes (e_{i+1} \wedge e_{i+2}) - \sum_{i \in \mathbb{Z}/3} \langle [e_{i+1}, e_{i+2}]_{\mathfrak{p}}, \cdot \rangle e_i \otimes 1,
\]

where \( \{ e_i^* \}_{i=1, 2, 3} \) is the dual basis of \( \{ e_i \}_{i=1, 2, 3} \), we have

\[
\overline{\text{rot}}|_{C^\infty(G, \mathfrak{p})^{(K, \text{Ad})}} = \widetilde{\text{rot}}. \tag{4.2}
\]

**Remark 4.9.** Set \( [e_i, e_j]_{\mathfrak{p}} = \sum_{k=1}^3 c_{ij}^k e_k \). Then with respect to \( \{ e_1, e_2, e_3 \} \), \( \overline{\text{rot}} \) is described as the following \( U(\mathfrak{g}) \)-valued matrix:

\[
\overline{\text{rot}} = \begin{pmatrix}
0 & -e_3 & e_2 \\
e_3 & 0 & -e_1 \\
-e_2 & e_1 & 0
\end{pmatrix} - \begin{pmatrix}
c_1^{13} & c_1^{23} & c_1^{33} \\
c_2^{13} & c_2^{23} & c_2^{33} \\
c_3^{13} & c_3^{23} & c_3^{33}
\end{pmatrix}.
\]

**Proof of Proposition 4.8.** It is straightforward to show that \( \text{rot} \) is a homogeneous differential operator. Since \( \overline{\text{rot}} \) is independent of the choice of \( \{ e_i \}_{i=1, 2, 3} \) and \( \text{Ad}(K) \) preserves the orientation and the metric, we see that \( \overline{\text{rot}} \) is \( K \)-invariant.

From Remark B.8, a homogeneous differential operator is completely determined by its value at a point, and so we only have to compute \( \text{rot} \) at \( eK \in G/K = M \).

For any \( \tilde{v} = \sum_{i=1}^3 v_i e_i \in C^\infty(G, \mathfrak{p})^{(K, \text{Ad})} \) where \( v_i \in C^\infty(G) \), denote by \( v \in \mathfrak{X}(M) \) the induced vector field:

\[
v(gK) = \left. \frac{d}{dt}g \cdot \exp \left( t \sum_{i=1}^3 v_i(g) e_i \right) \cdot K \right|_{t=0}.
\]

Take local coordinates \( (y_1, y_2, y_3) \) around \( eK \) defined by \( (y_1, y_2, y_3) \mapsto \exp \left( \sum_{i=1}^3 y_i e_i \right) \). Let \( \pi : G \to G/K = M \) be the projection and \( \tau_g : M \to M \)
for \( g \in G \) be the left translation. Denoting \( \nabla^{T}_{\partial y_i} \frac{\partial}{\partial y_j} = \sum_{k=1}^{3} \Gamma_{ij}^{k} \frac{\partial}{\partial y_k} \), we see the following.

**Lemma 4.10** ([9, 22]). For a sufficiently small \( X \in \mathfrak{p} \), we have

\[
\left( \frac{\partial}{\partial y_i} \right)_{\exp(X) \cdot K} = \left( (\tau_{\exp(X)} \ast) e^{K} (\pi \ast) e \left( \sum_{m=0}^{\infty} \frac{(-\text{ad}(X))^{m}}{(m+1)!} e_{i} \right) \right)
\]

\[
= \left( (\tau_{\exp(X)} \ast) e^{K} \left( \sum_{m=0}^{\infty} \frac{(-\text{ad}(X))^{m}}{(m+1)!} \left( \frac{\partial}{\partial y_i} \right)_{e^{K}} \right) \right)
\]

\[
g_{M} \left( \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} \right)_{e^{K}} = \delta_{ij}, \quad \Gamma_{ij}^{k} (e^{K}) = \frac{1}{2} (c_{ki}^{j} + c_{kij}^{j}).
\]

Now we compute \( \text{rot}(v) \) at \( e^{K} \in G / K = M \). First, we compute \( \left( \nabla^{T}_{\partial y_i} v \right)_{e^{K}} \). Since the metric \( g_{M} \) is \( G \)-invariant, we have

\[
\langle e_{i}, e_{j} \rangle = g_{M} \left( \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right)_{e^{K}} = g_{M} \left( \left( (\tau_{\exp(X)} \ast) e^{K} \left( \frac{\partial}{\partial y_i} \right)_{e^{K}} \right), \left( (\tau_{\exp(X)} \ast) e^{K} \left( \frac{\partial}{\partial y_j} \right)_{e^{K}} \right) \right)
\]

for any \( X \in \mathfrak{p} \). Then for sufficiently small \( t \in \mathbb{R} \), it follows that

\[
g_{M} \left( v, \frac{\partial}{\partial y_j} \right)_{\exp(te_{i}) \cdot K} = g_{M} \left( \sum_{k=1}^{3} v_{k} (\exp(te_{i})) (\tau_{\exp(te_{i})} \ast) \left( \frac{\partial}{\partial y_k} \right)_{e^{K}}, \left( \frac{\partial}{\partial y_j} \right)_{\exp(te_{i}) \cdot K} \right)
\]

\[
= \sum_{k=1}^{3} v_{k} (\exp(te_{i})) \left( e_{k}, \sum_{m=0}^{\infty} \frac{(-t \cdot \text{ad}(e_{i}))^{m}}{(m+1)!} e_{j} \right),
\]

\[
g_{M} \left( \nabla^{T}_{\partial y_i} v, \frac{\partial}{\partial y_j} \right)_{e^{K}} = \left( \frac{\partial}{\partial y_j} \right)_{e^{K}} g_{M} \left( v, \frac{\partial}{\partial y_j} \right) - g_{M} \left( v, \nabla^{T}_{\partial y_i} \frac{\partial}{\partial y_j} \right)_{e^{K}}
\]

\[
= \sum_{k=1}^{3} \left( e_{i} (v_{k}) \delta_{kj} + v_{k} \left( e_{k}, -\frac{1}{2} [e_{i}, e_{j}] \right) \right) - g_{M} \left( v, \sum_{k=1}^{3} \Gamma_{kj}^{i} (e^{K}) \frac{\partial}{\partial y_k} \right)
\]

\[
= e_{i} (v_{j}) - \frac{1}{2} \sum_{k=1}^{3} v_{k} (c_{kj}^{i} + c_{kij}^{i} + c_{kji}^{i}).
\]

Hence we obtain

\[
\left( \nabla^{T}_{\partial y_i} v \right)_{e^{K}} = \sum_{j=1}^{3} e_{i} (v_{j}) \frac{\partial}{\partial y_j} - \frac{1}{2} \sum_{j,k=1}^{3} v_{k} (c_{kj}^{i} + c_{kij}^{i} + c_{kji}^{i}) \frac{\partial}{\partial y_j}.
\]

Thus we compute

\[
\left( \text{rot}(v) \right)_{e^{K}} = \left( \frac{\partial}{\partial y_i} \right)_{e^{K}} \times \left( \nabla^{T}_{\partial y_i} v \right)_{e^{K}}
\]

\[
= \left( -e_{3} (v_{2}) + e_{2} (v_{3}) - \sum_{j=1}^{3} c_{2j}^{3} \right) \frac{\partial}{\partial y_1} + \left( e_{3} (v_{1}) - e_{1} (v_{3}) - \sum_{j=1}^{3} c_{3j}^{i} \right) \frac{\partial}{\partial y_2}
\]

\[
+ \left( -e_{2} (v_{1}) + e_{1} (v_{2}) - \sum_{j=1}^{3} c_{1j}^{i} \right) \frac{\partial}{\partial y_3},
\]

which implies the proof. \( \square \)
5. Associative deformations in the sine cone of nearly Kähler manifolds. Let \((N,k,J,\sigma,\psi^\pm)\) be a nearly Kähler manifold and \(L \subset N\) be a Lagrangian submanifold. From Lemma 2.26 and 2.30, the sine cone \(C_s(N) = (0, \pi) \times N\) admits nearly parallel \(G_2\)-structure \((\tilde{\varphi}, \tilde{k})\) and \(\{\pi/2\} \times L \subset C_s(N)\) is associative. We study the infinitesimal associative deformations of \(\{\pi/2\} \times L\).

Let \(\nu \mapsto \{\pi/2\} \times L\) be the normal bundle of \(\{\pi/2\} \times L \subset C_s(N)\). First, we rewrite the operator \(D : C^\infty(\{\pi/2\} \times L, \nu) \to C^\infty(\{\pi/2\} \times L, \nu)\) in Proposition 3.2 in this case. Since \(L\) is Lagrangian, there exists canonical isomorphism \(TL \oplus \mathbb{R} \ni (v, x) \mapsto Jv + x \frac{\partial}{\partial t}|_{t=\pi/2} \in \nu\). Via this identification, we obtain the following.

**Proposition 5.1.** The corresponding operator \(\nu \mapsto \{\pi/2\} \times L\) of \(\tilde{\varphi}\) is described as

\[
D(v, f) = (-\nabla f - \text{rot}(v) + 2v, \text{div}(v)),
\]

where we use the notation in Proposition 4.1.

By Proposition 5.1, we prove Corollary 5.2 as in the case of Corollary 4.2.

**Corollary 5.2.** We have

\[
\dim\{\text{the infinitesimal associative deformations of } \{\pi/2\} \times L\} = \dim\{f \in C^\infty(L); \Delta_+ f = 3f\} + \dim\{v \in \mathfrak{X}(L); \text{rot}(v) = 3v\}.
\]

**Remark 5.3.** From Lemma 2.32, \(\dim\{f \in C^\infty(L); \Delta_+ f = 3f\}\) gives the dimension of infinitesimal associative and non-Lagrangian deformations.

**Proof of Proposition 5.1.** Let \(\{e_1, e_2, e_3\} \subset TL\) be a local oriented orthonormal frame such that \(\psi^- (e_1, e_2, e_3) = -1\). Set \(\eta_j := J(e_j)\) for \(1 \leq j \leq 3\) and \(\eta_4 := \frac{\partial}{\partial t}|_{t=\pi/2}\). Then \(\{\eta_j\}_{1 \leq j \leq 4}\) is a local oriented orthonormal frame of \(\nu\). Let \(\{e^1, \ldots, e^3, \eta^1, \ldots, \eta^4\}\) be the dual coframe, then we have

\[
\sigma = \sum_{i=1}^3 e^i \wedge \eta^i, \quad \Psi = \psi^+ + i\psi^- = -i(e^1 + i\eta^1) \wedge (e^2 + i\eta^2) \wedge (e^3 + i\eta^3).
\]

Hence at a point of \(L \times \{\pi/2\}\), we have

\[
\tilde{\varphi} = \eta^4 \wedge \sum_{i=1}^3 e^i \wedge \eta^i + e^1(e^{23} - \eta^{23}) - \eta^1(e^2 \wedge \eta^3 + \eta^2 \wedge e^3).
\]

As in the Sasakian case, the definition of the Levi-Civita connection gives the following.

**Lemma 5.4.** For any \(X, Y \in \mathfrak{X}(N \times \{\pi/2\})\), we have

\[
\nabla^C_s(N)_{\tilde{\varphi}} Y|_{N \times \{\pi/2\}} = \nabla^N X, \quad \nabla^C_s(N)_{\tilde{\varphi}} \frac{\partial}{\partial t}|_{N \times \{\pi/2\}} = 0, \quad \nabla^C_s(N)_{\tilde{\varphi}} X|_{N \times \{\pi/2\}} = 0,
\]

where \(\nabla^C_s(N)\) and \(\nabla^N\) are the Levi-Civita connections of \(\tilde{k}\) on \(C_s(N)\) and \(k\) on \(N\), respectively.

Denoting \(\nabla^C_{ij} e_j = \sum_{k=1}^3 \Gamma^k_{ij} e_k\) for \(1 \leq i, j \leq 3\), we see the following from the computations above and Lemma 2.28.
Lemma 5.5.

\[(e_i \times \eta_a) = \begin{pmatrix} \eta_4 & -\eta_3 & \eta_2 & -\eta_1 \\ \eta_3 & \eta_4 & -\eta_1 & -\eta_2 \\ -\eta_2 & \eta_1 & \eta_4 & -\eta_3 \end{pmatrix}, \quad \nabla_{e_i}^v \eta_j = \sum_{k=1}^3 (\epsilon_{ijk} + \Gamma_{ij}^k) \eta_k,\]

where \(\epsilon_{ijk}\) is the permutation symbol and \(1 \leq i, j \leq 3\).

Via the identification \(X(L) \oplus C^\infty(L) \ni (\sum_{j=1}^3 v_j e_j, f) \mapsto \sum_{j=1}^3 v_j \eta_j + f \eta_4 \in C^\infty(L \times \{\pi\}, \nu)\) where \(v_j, f \in C^\infty(L)\), we can calculate as in the proof of Proposition 4.1. \(\square\)

6. Computation in the standard sphere \(S^7\). By Definition 2.1, \(\mathbb{C}^4\) admits a torsion-free Spin(7)-structure \((\Phi_0, h_0)\), which induces the nearly parallel \(G_2\)-structure \((\varphi, g)\) on \(S^7\) by Lemma 2.16. In this section, we study the deformation spaces of homogeneous associative submanifolds in \(S^7\), and prove Theorem 1.1 and 1.2.

6.1. Classification of homogeneous associative submanifolds in \(S^7\). Mashimo [18] classified homogeneous Lagrangian submanifolds in \(S^6\). Applying this classification, Lotay [16] classified homogeneous associative submanifolds in \(S^7\).

Proposition 6.1 ([16, 18]). Let \(A\) be a connected associative 3-fold in \(S^7 \subset \mathbb{C}^4\) which is the orbit of a closed 3-dimensional Lie subgroup of Spin(7). If \(A\) does not lie in a totally geodesic \(S^6\), then, up to the Spin(7)-action, \(A\) is either

1. \(A_1 \cong T^3\) given by Example 6.2,
2. \(A_2 \cong SU(2)/Z_3\), or \(A_3 \cong SU(2)\) given by Example 6.3,
3. \(L_1 \cong SU(2)\) given by Example 6.5,
4. \(L_2 \cong SO(3) \cong SU(2)/Z_2\) given by Example 6.6, or
5. \(L_3 \cong SU(2)/A^*_4\), or \(L_4 \cong SU(2)/D^*_3\) given by Example 6.7.

Note that the automorphism group of nearly parallel \(S^7\) is Spin(7) and that of nearly Kähler \(S^6\) is \(G_2\).

Example 6.2. Define the action of \(T^3 \cong (\mathbb{R}/2\pi\mathbb{Z})^3\) on \(\mathbb{C}^4\) by

\[(\theta_1, \theta_2, \theta_3) \cdot (z_1, z_2, z_3, z_4) = \eta(e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3, e^{-i(\theta_1 + \theta_2 + \theta_3)} z_4),\]

where \(\theta_i \in \mathbb{R}/2\pi\mathbb{Z}\) and \(z_i \in \mathbb{C}\). Then

\[A_1 := T^3 \cdot \frac{1}{2} (1, 1, 1, i) \cong T^3\]

is special Legendrian given in [8].

Example 6.3. Define the SU(2)-action on \(\mathbb{C}^4\) by

\[
\begin{pmatrix}
  a & -\bar{b} \\
  b & \bar{a}
\end{pmatrix}
\begin{pmatrix}
  z_1 \\
  z_2 \\
  z_3 \\
  z_4
\end{pmatrix} =
\begin{pmatrix}
  a^3 & -\sqrt{3}a^2\bar{b} & \sqrt{3}ab^2 & -b^3 \\
  \sqrt{3}ab^2 & a(|a|^2 - 2|b|^2) & -b(2|a|^2 - |b|^2) & \sqrt{3}a^2\bar{b} \\
  b^3 & b(2|a|^2 - |b|^2) & \bar{a}(|a|^2 - 2|b|^2) & -\sqrt{3}ab^2 \\
  \sqrt{3}a^2\bar{b} & b^3 & \sqrt{3}\pi^2b & \sqrt{3}\pi^2b
\end{pmatrix}
\begin{pmatrix}
  z_1 \\
  z_2 \\
  z_3 \\
  z_4
\end{pmatrix},
\]

(6.1)
where $z_1 \in \mathbb{C}$ and $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. Set

$$A_2 = SU(2) \cdot t'(1, 0, 0, 0) \cong SU(2)/\mathbb{Z}_3, \quad A_3 = SU(2) \cdot \frac{1}{\sqrt{2}} t'(0, 1, i, 0) \cong SU(2),$$

where $\mathbb{Z}_3 = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \in SU(2); \zeta^3 = 1 \right\}$. Then $A_2$ is the Hopf lift of the Veronese curve in $\mathbb{CP}^3$:

$$\{ [x^3 : \sqrt{3}x^2 y : \sqrt{3}xy^2 : y^3] \in \mathbb{CP}^3; [x : y] \in \mathbb{CP}^1 \},$$

and hence associative. However, $A_3$ is an associative submanifold which does not arise from other known geometries.

**Remark 6.4.** Set $A_2(\theta) = SU(2) \cdot t'(\cos \theta, 0, 0, \sin \theta)$ for $\theta \in [0, \frac{\pi}{3}]$. It is known that all the $A_2(\theta)$ are congruent up to the Spin(7)-action to $A_2 = A_2(0)$, which is $U(2)$-invariant. In [10], $A_2(\frac{\pi}{4})$ is shown to be special Legendrian.

Next, we give examples of homogeneous Lagrangian submanifolds in $S^6$.

**Example 6.5.** Define the $SU(2)$-action on $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ by

$$\left( \begin{array}{ccc} a & -\bar{b} \\ b & \bar{a} \end{array} \right) \cdot \left( \begin{array}{c} x_1 \\ z_1 \\ z_2 \\ z_3 \end{array} \right) = \left( \begin{array}{c} (|a|^2 - |b|^2)x_1 - 2\text{Im}(\bar{ab}z_1) \\ 2\text{Im}(\bar{ab}x_1) + \bar{a}^2 z_1 + b^2 \bar{z}_1 \\ az_2 - b\bar{z}_3 \\ \bar{b}z_2 + a\bar{z}_3 \end{array} \right),$$

(6.2)

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. Then

$$L_1 := SU(2) \cdot t'(\sqrt{\frac{5}{3}}, 0, 0, 0) \cong SU(2),$$

(6.3)

where $t'(\sqrt{\frac{5}{3}}, 0, 0, 0) \in \mathbb{R} \oplus \mathbb{C}^3$, is Lagrangian in $S^6$ given in [8]. Moreover, $L_1$ is invariant under a $U(2)\langle \subset G_2 \rangle$ action.

**Example 6.6.** Let $L_2 \subset S^6$ be given by

$$L_2 = \left\{ (0, z_1, z_2, z_3) \in \mathbb{R} \oplus \mathbb{C}^3; |z_1|^2 + |z_2|^2 + |z_3|^2 = 1, z_1^2 + z_2^2 + z_3^2 = 0 \right\}.$$  (6.4)

Since $L_2$ is the link of an complex cone, it is Lagrangian in $S^6$. Define the $SO(3)$-action on $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ by the trivial action of $\mathbb{R}$ and the standard (real) action on $\mathbb{C}^3$. Let $\varpi : SU(2) \to SO(3)$ be a standard double covering:

$$\varpi : \left( \begin{array}{ccc} a & -\bar{b} \\ b & \bar{a} \end{array} \right) \mapsto \left( \begin{array}{ccc} |a|^2 - |b|^2 & 2\text{Im}(ab) & -2\text{Re}(ab) \\ -2\text{Im}(\bar{ab}) & \text{Re}(a^2 + b^2) & \text{Im}(a^2 + b^2) \\ 2\text{Re}(\bar{ab}) & \text{Im}(-a^2 + b^2) & \text{Re}(a^2 - b^2) \end{array} \right),$$

(6.5)

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. By composing these actions, $SU(2)$ acts on $\mathbb{R}^7$, and we have

$$L_2 = SU(2) \cdot \frac{1}{\sqrt{2}} t'(0, 0, 1, i) \cong SU(2)/\mathbb{Z}_2 = SO(3).$$
Example 6.7. Let \( \{ \epsilon_1, \cdots, \epsilon_7 \} \) be a standard basis for \( \mathbb{R}^7 \). Identify \( \mathbb{R}^7 \) with the homogeneous harmonic cubics \( \mathcal{H}^3(\mathbb{R}^3) \) on \( \mathbb{R}^3 \) by:

\[
\begin{align*}
\epsilon_1 & \mapsto \frac{\sqrt{10}}{10}x(2x^2 - 3y^2 - 3z^2); \\
\epsilon_2 & \mapsto -\sqrt{6}xyz; \\
\epsilon_3 & \mapsto \frac{\sqrt{6}}{2}x(y^2 - z^2); \\
\epsilon_4 & \mapsto -\frac{\sqrt{15}}{10}y(4x^2 - y^2 - z^2); \\
\epsilon_5 & \mapsto -\frac{\sqrt{15}}{10}z(4x^2 - y^2 - z^2); \\
\epsilon_6 & \mapsto \frac{1}{2}y(y^2 - 3z^2); \\
\epsilon_7 & \mapsto \frac{1}{2}z(z^2 - 3y^2).
\end{align*}
\]

Let \( SU(2) \) act on \( \mathcal{H}^3(\mathbb{R}^3) \cong \mathbb{R}^7 \) as \( A \cdot f(x, y, z) = f((x, y, z)p(A)) \), where \( A \in SU(2) \) and \( f \in \mathcal{H}^3(\mathbb{R}^3) \cong \mathbb{R}^7 \). Set

\[
L_3 := SU(2) \cdot \epsilon_2, \quad L_4 := SU(2) \cdot \epsilon_6.
\]

Then \( L_3 \cong SU(2)/A^*_4 \) and \( L_4 \cong SU(2)/D^*_3 \) are Lagrangian, where \( A^*_4 \) is a binary tetrahedral group of order 24 generated by

\[
k_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad k_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\pi i/4} & -e^{-\pi i/4} \\ e^{\pi i/4} & e^{-\pi i/4} \end{pmatrix}, \quad (6.6)
\]

and \( D^*_3 \) is a binary dihedral group of order 12 generated by

\[
k_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k_5 = \begin{pmatrix} e^{\pi i/3} & 0 \\ 0 & e^{-\pi i/3} \end{pmatrix}. \quad (6.7)
\]

6.2. Computations on \( SU(2) \). For the convenience of the following computations, we summarize formulas on \( SU(2) \). Define the basis of the Lie algebra \( su(2) \) of \( SU(2) \) by

\[
E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (6.8)
\]

which satisfies the relation \( [E_i, E_{i+1}] = 2E_{i+2} \) for \( i \in \mathbb{Z}/3 \). We see the following by Proposition B.5 and Lemma B.6.

Lemma 6.8. Let \( V_n \) be a \( C \)-vector space of all complex homogeneous polynomials with two variables \( z_1, z_2 \) of degree \( n(n \geq 0) \) and define the representation \( \rho_n : SU(2) \to GL(V_n) \) as

\[
\rho_n \left( \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \right) (z_1, z_2) = f \left( (z_1, z_2) \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \right).
\]

Define the Hermitian inner product \( \langle \cdot, \cdot \rangle \) of \( V_n \) such that

\[
\left\{ v_k^{(n)} = \frac{1}{\sqrt{k!(n-k)!}} z_1^{n-k} z_2^k \right\}_{0 \leq k \leq n}
\]

is a unitary basis of \( V_n \). If we denote by \( \hat{SU}(2) \) the set of all equivalence classes of finite dimensional irreducible representations of \( SU(2) \), we know that \( \hat{SU}(2) = \)
\{(V_n, \rho_n); n \geq 0\}. Then every \(\mathbb{C}\)-valued continuous function on \(SU(2)\) is uniformly approximated by the \(\mathbb{C}\)-linear combination of the following functions:

\[
\left\{ \langle \rho_n(\cdot)v_i^{(n)}, v_j^{(n)} \rangle; n \geq 0, 0 \leq i, j \leq n \right\},
\]

which are mutually orthogonal with respect to the \(L_2\) inner product.

By a direct computation, we see the following.

**Lemma 6.9.** Identify \(X \in \mathfrak{su}(2) \subset U(\mathfrak{su}(2))\) with the left invariant differential operator on \(SU(2)\). For \(u = \sum_{l=0}^{n} C_l v_l^{(n)} \in V_n\), set

\[
u^* = \sum_{l=0}^{n} (-1)^{n-l} C_{n-l} v_l^{(n)} \in V_n.
\]

Then for any \(n \geq 0, 0 \leq k, l \leq n, v \in V_n, X \in \mathfrak{su}(2)\), we have

\[
X \langle \rho_n(\cdot)v, u \rangle = \langle \rho_n(\cdot)d\rho_n(X)v, u \rangle,
\]

\[
(d\rho_n(X)v)(z_1, z_2) = \left( \frac{\partial v}{\partial z_1}, \frac{\partial v}{\partial z_2} \right) X \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right),
\]

\[
\langle \rho_n(\cdot)v_k^{(n)}, u \rangle = (-1)^k \langle \rho_n(\cdot)v_{n-k}, u^* \rangle,
\]

\[
(-iE_1 + E_2)\langle \rho_n(\cdot)v_k^{(n)}, u \rangle = \begin{cases} 2i\sqrt{(k+1)(n-k)} \langle \rho_n(\cdot)v_{k+1}, u \rangle, & (k < n) \\ 0, & (k = n) \end{cases}
\]

\[
(iE_1 + E_2)\langle \rho_n(\cdot)v_k^{(n)}, u \rangle = \begin{cases} 2i\sqrt{k(n-k+1)} \langle \rho_n(\cdot)v_{k-1}, u \rangle, & (k > 0) \\ 0, & (k = 0) \end{cases}
\]

\[
iE_3 \langle \rho_n(\cdot)v_k^{(n)}, u \rangle = (-n + 2k) \langle \rho_n(\cdot)v_{k}, u \rangle.
\]

The next lemma is useful for the later computations.

**Lemma 6.10.** Suppose that \(\{e_1, e_2, e_3\} = \{pE_1, pE_2, qE_3\}\) where \(0 \neq p, q \in \mathbb{R}\) is an oriented orthonormal basis of \(\mathfrak{su}(2)\) for some metric and orientation. For \(v = \sum_{i=1}^{3} v_i e_i \in C^\infty(SU(2), \mathfrak{su}(2))\), \(\text{rot}(v) = \alpha v\) for \(0 \neq \alpha \in \mathbb{R}\) is equivalent to

\[
(i e_3 - (2q + \alpha)) (v_1 + iv_2) + (-i e_1 + e_2) v_3 = 0,
\]

\[
(i e_1 + e_2)(v_1 + iv_2) + \left( \alpha + \frac{2p^2}{q} + i e_3 \right) v_3 = 0.
\]

These equations imply that

\[
\Delta_+ + \left( \frac{4p^2}{q} - 4q \right) i e_3 + \left( -\alpha - \frac{2p^2}{q} + 2q \right) (2q + \alpha) (v_1 + iv_2) = 0,
\]

\[
\Delta_+ - \alpha \left( \alpha + \frac{2p^2}{q} \right) v_3 = 0,
\]

where \(\Delta_+ = -\sum_{i=1}^{3} e_i^2\) is a Laplacian. Especially, for any \(n \geq 0, 0 \leq k \leq n, u \in V_n\),
we have
\[\Delta_+ \langle \rho_n(\cdot) v_k^{(n)}, u \rangle = \left\{( -p^2 + q^2)(n - 2k)^2 + p^2(n^2 + 2n) \right\} \langle \rho_n(\cdot) v_k^{(n)}, u \rangle, \quad (6.13)\]
\[
\Delta_+ \left( \frac{4p^2}{q} - 4q \right) ie_3 + \left( -\alpha - \frac{2p^2}{q} + 2q \right) (2q + \alpha) \right\} \langle \rho_n(\cdot) v_k^{(n)}, u \rangle
\]
\[
= \left\{ ( -p^2 + q^2)(n - 2k + 2)^2 + p^2(n^2 + 2n) - \alpha \left( \alpha + \frac{2p^2}{q} \right) \right\} \langle \rho_n(\cdot) v_k^{(n)}, u \rangle. \quad (6.14)\]

**Remark 6.11.** In the case of SU(2)/Γ for some finite subgroup Γ, we have to consider the Γ equivariant solutions of (6.9) and (6.10).

**Proof.** Note that \([e_1, e_2] = \frac{2p^2}{q} e_3, [e_1, e_3] = -2qe_2, [e_2, e_3] = 2qe_1\). Then from Remark 4.9, \(\text{rot}(v) = \alpha v\) is equivalent to
\[
ie_3(v_1 + iv_2) + (-ie_1 + e_2)(v_3) = (2q + \alpha)(v_1 + iv_2), \quad (6.15)\]
\[
\text{Re}((ie_1 + e_2)(v_1 + iv_2)) = -\left( \alpha + \frac{2p^2}{q} \right) v_3. \quad (6.16)\]
It is clear that (6.9) and (6.10) imply (6.15) and (6.16). Conversely, suppose that (6.15) and (6.16) hold. Applying \((ie_1 + e_2)\) to (6.15), we obtain
\[
(i e_3 - \alpha)(i e_1 + e_2)(v_1 + iv_2) + \left( e_1^2 + e_2^2 + \frac{2p^2}{q} i e_3 \right) v_3 = 0.
\]
Considering the real and imaginary parts, we obtain from (6.16)
\[
-\alpha e_3 (v_3) - \alpha \text{Im}((i e_1 + e_2)(v_1 + iv_2)) = 0. \quad (6.17)
\]
The equations (6.16) and (6.18) imply (6.10), and hence we obtain the first statement.

Substituting (6.18) into (6.17), we have (6.12). Applying \((-ie_1 + e_2)\) to (6.10), we obtain from (6.9)
\[
\left( e_1^2 + e_2^2 + \frac{2p^2}{q} i e_3 \right) (v_1 + iv_2)
\]
\[
= \left( -\alpha - \frac{2p^2}{q} + 2q - ie_3 \right) (-ie_1 + e_2)v_3
\]
\[
= \left\{ -\alpha e_3 (v_3) - \alpha \text{Im}((i e_1 + e_2)(v_1 + iv_2)) \right\} (v_1 + iv_2),
\]
which imply (6.11). Then from Lemma 6.9, we obtain (6.13) and (6.14). \(\square\)

**6.3. The case \(A_1, A_2,\) and \(A_3\).** First, we study the deformation of homogeneous associative submanifolds which do not lie in a totally geodesic \(S^6\).

**6.3.1. The case \(A_1 \cong T^3\).** Define the basis of the Lie algebra \(t^3\) of \(T^3\) by
\[
e_1 = (\sqrt{2}, 0, 0), \quad e_2 = (0, \sqrt{2}, -\sqrt{2}), \quad e_3 = (-1, 1, 1) \in \mathbb{R}^3 \cong t^3.
\]
which is an oriented orthonormal basis of $t^3$ with respect to the orientation and the metric induced from $A_1$.

Define the smooth function $f_\gamma \in C^\infty(T^3, \mathbb{C})$ for $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}^3$ on $T^3 \cong (\mathbb{R}/2\pi\mathbb{Z})^3$ by $f_\gamma(\theta_1, \theta_2, \theta_3) = \exp(i \sum_{j=1}^{3} \gamma_j \theta_j)$. By a Fourier series expansion, every $\mathbb{C}$-valued continuous function on $T^3$ is uniformly approximated by the $\mathbb{C}$-linear combination of $f_\gamma$’s. By a direct computation, we obtain the following.

**Lemma 6.12.** Identifying $e_i \in t^3$ with the left invariant differential operator on $T^3$, we have

\[
e_1(f_\gamma) = \sqrt{2} \gamma_1 if_\gamma, \quad e_2(f_\gamma) = \sqrt{2} (\gamma_2 - \gamma_3) if_\gamma, \quad e_3(f_\gamma) = (\gamma_1 + \gamma_2 + \gamma_3) if_\gamma,
\]

$\Delta_+(f_\gamma) = \{2\gamma_1^2 + 2(\gamma_2 - \gamma_3)^2 + (\gamma_1 + \gamma_2 + \gamma_3)^2 \}$

Then we deduce the following.

**Proposition 6.13.** $\dim_{\mathbb{R}} \{f \in C^\infty(T^3); \Delta_+ f = 8f\} = 12$.

**Proposition 6.14.** $\dim_{\mathbb{R}} \{v \in C^\infty(T^3, t^3); \text{rot}(v) = -2v\} = 6$.

By Corollary 4.2, these imply that associative deformations of $A_1$ are trivial since $\text{Spin}(7)$ induces 18($= \dim_{\mathbb{R}}(\text{Spin}(7)/T^3)$)-dimensional associative deformations of $A_1$. Now, we give proofs.

**Proof of Proposition 6.13.** For $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}^3$, we know that

\[
2\gamma_1^2 + 2(\gamma_2 - \gamma_3)^2 + (\gamma_1 + \gamma_2 + \gamma_3)^2 = 8
\]

$\Leftrightarrow (\gamma_1, \gamma_2, \gamma_3) = \pm (2, 1, 1), \pm (0, 1, -1), \pm (1, 2, 1), \pm (1, 1, 2), \pm (1, -1, 0), \pm (1, 0, -1),$

which gives the proof by Lemma 6.12. \[\Box\]

**Proof of Proposition 6.14.** Take any $v = \sum_{i=1}^{3} v_ie_i \in C^\infty(T^3, t^3)$ where $v_i \in C^\infty(T^3)$. Then from Remark 4.9, $\text{rot}(v) = \alpha v$ for $\alpha \in \mathbb{R}$ is equivalent to

\[
(ie_3 - \alpha)(v_1 + iv_2) + (-ie_1 + e_2)(v_3) = 0,
\]

\[
\Re((ie_1 + e_2)(v_1 + iv_2)) = -\alpha v_3.
\]

Eliminating $v_3$, we have

\[
-\alpha(ie_3 - \alpha)(v_1 + iv_2) + (-ie_1 + e_2)\Re((ie_1 + e_2)(v_1 + iv_2)) = 0.
\]

Set $v_1 + iv_2 = \sum_{\gamma \in \mathbb{Z}^3} C_{\gamma}f_{\gamma}$ where $C_{\gamma} \in \mathbb{C}$. Since $\overline{f}_{\gamma} = f_{-\gamma}$, (6.21) is equivalent to

\[
C_{\gamma} \left(-\gamma_1^2 - (\gamma_2 - \gamma_3)^2 + \alpha (\gamma_1 + \gamma_2 + \gamma_3 + \alpha)\right) + \overline{C}_{-\gamma}(\gamma_1 + (\gamma_2 - \gamma_3)i)^2 = 0.
\]

Take the complex conjugation of (6.22) and replace $\gamma$ by $-\gamma$, then we obtain

\[
C_{\gamma}(\gamma_1 + (\gamma_2 - \gamma_3)i)^2 + \overline{C}_{-\gamma}(-\gamma_1^2 - (\gamma_2 - \gamma_3)^2 + \alpha (\gamma_1 - \gamma_2 - \gamma_3 + \alpha)) = 0.
\]

Eliminating $\overline{C}_{\gamma}$ from (6.22) and (6.23), we have

\[
\alpha^2 \left(-2\gamma_1^2 + (\gamma_2 - \gamma_3)^2 - \alpha^2 (\gamma_1 + \gamma_2 + \gamma_3)^2 + \alpha^2\right) C_{\gamma} = 0.
\]

Set $\alpha = -2$. Since we know that $-2\gamma_1^2 + (\gamma_2 - \gamma_3)^2 - (-\gamma_1 + \gamma_2 + \gamma_3)^2 + 4 = 0 \Leftrightarrow (\gamma_1, \gamma_2, \gamma_3) = \pm (1, 1, 0), \pm (1, 0, 1), \pm (0, 1, 1)$, we deduce by (6.22) that

\[
v_1 + iv_2 = C_{(1,1,0)}f_{(1,1,0)} - i\overline{C}_{(1,1,0)}f_{(-1,-1,0)} + C_{(1,0,1)}f_{(1,0,1)} + i\overline{C}_{(1,0,1)}f_{(-1,0,-1)} + C_{(0,-1,-1)}f_{(0,-1,-1)}.
\]

Thus $v_1 + iv_2$ depends on 3 complex parameters $C_{(1,1,0)}, C_{(1,0,1)}, C_{(0,-1,-1)}$, which implies Proposition 6.14. \[\Box\]
6.3.2. The case $A_2 \cong SU(2)/\mathbb{Z}_3$. By Remark 6.4, $A_2 = A_2(0)$ is congruent to $A_2(\frac{\pi}{2})$, which is special Legendrian. We may compute the dimension of the infinitesimal associative deformations of $A_2(\frac{\pi}{2})$ by Corollary 4.2. The action (6.1) induces an inclusion $su(2) \hookrightarrow su(4)$, where $E_1, E_2, E_3$ in (6.8) correspond to
\[
\begin{pmatrix}
0 & \sqrt{3} & 0 & 0 \\
-\sqrt{3} & 0 & 2 & 0 \\
0 & -2 & 0 & \sqrt{3} \\
0 & 0 & -\sqrt{3} & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & \sqrt{3}i & 0 & 0 \\
\sqrt{3}i & 0 & 2i & 0 \\
0 & 2i & 0 & \sqrt{3}i \\
0 & 0 & \sqrt{3}i & 0
\end{pmatrix}, \quad \begin{pmatrix}
3i & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & -i & 0 \\
0 & 0 & 0 & -3i
\end{pmatrix},
\]
respectively. Set $p_0 = \frac{1}{\sqrt{2}}(1, 0, 0, 1) \in \mathbb{C}^4$. Then we have
\[(E_1^*)_p = \sqrt{\frac{3}{2}}(0, -1, 1, 0), \quad (E_2^*)_p = \sqrt{\frac{3}{2}}(0, i, i, 0), \quad (E_3^*)_p = \frac{3i}{\sqrt{2}}(1, 0, 0, -1),\]
Hence if we set $e_1 := E_1/\sqrt{3}, e_2 := E_2/\sqrt{3}, e_3 := E_3/3$, $\{e_i\}_{1 \leq i \leq 3}$ is an oriented orthonormal basis of $su(2)$ with respect to the orientation and the metric induced from $A_2$.

**PROPOSITION 6.15.** $\dim \mathbb{R}\{f \in C^\infty(A_2); \Delta_+ f = 8f\} = 19$.

**PROPOSITION 6.16.** $\dim \mathbb{R}\{v \in \mathfrak{X}(A_2); \text{rot}(v) = -2v\} = 11$.

On the other hand, Spin(7) induces 17 (= $\dim \mathbb{R}(\text{Spin}(7)/U(2))$)-dimensional associative deformations of $A_2$. By Corollary 4.2 and Remark 6.4, we have a 30-dimensional infinitesimal associative deformation space of $A_2$, and hence $A_2$ can have non-trivial associative deformations. In fact, we obtain the following.

**PROPOSITION 6.17.** All non-trivial associative deformations of $A_2$ are induced by the PGL(4, $\mathbb{C}$)-action on $\mathbb{CP}^3$ via the Hopf lift.

**REMARK 6.18.** ([24, 3]) As a special Legendrian submanifold, $A_2(\frac{\pi}{4})$ is not rigid, either. By a non-standard projection $p_2 : S^7 \to \mathbb{CP}^3$, $p_2(A_2(\frac{\pi}{4}))$ is a horizontal holomorphic curve in $\mathbb{CP}^3$, and for any horizontal holomorphic curve $\Sigma$, $p_2^{-1}(\Sigma) \subset S^7$ is a special Legendrian submanifold.

Since the group of biholomorphic maps which preserve the horizontal distribution is PSp(2, $\mathbb{C}$), all non-trivial special Legendrian deformations of $A_2(\frac{\pi}{4})$ are given by the induced action of PSp(2, $\mathbb{C}$) on $\mathbb{CP}^3$.

Now, we give proofs. First, we prove the following lemma.

**LEMMA 6.19.** Let $\{(v^{(n)}_i)^*; \langle \cdot, v^{(n)}_i \rangle \}$ be the dual basis of $\{v^{(n)}_i\}$. Then we have
\[
\text{Hom}_{\mathbb{Z}_3}(V_n, su(2) \otimes_{\mathbb{R}} \mathbb{C}) = \{L \in \text{Hom}_\mathbb{C}(V_n, su(2) \otimes_{\mathbb{R}} \mathbb{C}); L(\rho_n(k)v) = \text{Ad}(k)L(v) \text{ for any } k \in \mathbb{Z}_3, v \in V_n\} = \text{span}_\mathbb{C}\left\{(v^{(n)}_i)^* \otimes X; \right.X = \begin{pmatrix} e_3 & (n - 2i \in 3\mathbb{Z}) \\ e_1 + ie_2 & (n - 2i \in 3\mathbb{Z} + 1) \\ e_1 - ie_2 & (n - 2i \in 3\mathbb{Z} + 2) \end{pmatrix}, \right. \text{span}_\mathbb{C}\left\{(v^{(n)}_i)^*; n - 2i \in 3\mathbb{Z}\right\}.\]
Proof. Take any $v \in V_n$ and $k = \begin{pmatrix} \zeta & 0 \\ 0 & \frac{1}{\zeta} \end{pmatrix} \in \mathbb{Z}_3$ where $\zeta^3 = 1$. By definition, we see that

\[
\begin{aligned}
\text{Ad}(k)e_1 &= \text{Re}(\zeta)e_1 - \text{Im}(\zeta)e_2, \\
\text{Ad}(k)e_2 &= \text{Im}(\zeta)e_1 + \text{Re}(\zeta)e_2, \\
\text{Ad}(k)e_3 &= e_3,
\end{aligned}
\]

\[
\rho_n(k)v^{(n)}_i = \zeta^{n-2l}v^{(n)}_i.
\]

Setting $L = \sum_{i=0}^{n-1}\sum_{i=1}^{3}(C_{l_1}(v^{(n)}_i)^* \otimes e_i) \in \text{Hom}_{\mathbb{C}}(V_n, \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C})$ where $C_{l_1} \in \mathbb{C}$, we know that $L \in \text{Hom}_{\mathbb{Z}_3}(V_n, \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C})$ if and only if

\[
\zeta^{n-2l} \sum_{i=1}^{3} C_{l_1}e_i = C_{l_1}(\text{Re}(\zeta)e_1 - \text{Im}(\zeta)e_2) + C_{l_2}(\text{Im}(\zeta)e_1 + \text{Re}(\zeta)e_2) + C_{l_3}e_3
\]

for any $0 \leq l \leq n$ and $\zeta^3 = 1$. This is equivalent to

\[
\begin{aligned}
(\zeta^{n-2l} - 1)C_{l_3} &= 0, \\
(\zeta^{n-2l+1} - 1)(C_{l_2} - iC_{l_1}) &= 0, \\
(\zeta^{n-2l+2} - 1)(C_{l_2} + iC_{l_1}) &= 0,
\end{aligned}
\]

which implies the first statement. The second is proven in the same way. \qed

Proof of Proposition 6.15. From (6.13), the solution $f$ of $\Delta_+ f = 8f$ is contained in $\text{span}_{\mathbb{C}} \{ \langle \rho_n(\cdot)v^{(n)}_k, v^{(n)}_l \rangle; (n, k) = (6, 0), (6, 6), (4, 2), 0 \leq l \leq n \}$, which are $\mathbb{Z}_3$ invariant. Hence we obtain Proposition 6.15. \square

Proof of Proposition 6.16. First, we consider $\dim_{\mathbb{R}} \{ v \in \mathfrak{X}(A_2); \text{rot}(v) = -2v \}$. Set $(p, q, \alpha) = (\frac{2}{\sqrt{3}}, \frac{1}{3}, -2)$ in Lemma 6.10. Since we know that

\[
-\frac{2}{9}(n - 2k)^2 + \frac{1}{3}(n^2 + 2n) = 0 \iff (n, k) = (4, 0),
\]

\[
-\frac{2}{9}(n - 2k)^2 + \frac{1}{3}(n^2 + 2n) = 0 \iff (n, k) = (0, 0),
\]

we have $v_1 + iv_2 = \langle \rho_4(\cdot)v^{(4)}_0, u \rangle$ for $u \in V_4$ and $v_3$ is constant. We see that $v = \sum_{i=1}^{3} v_i e_i$ satisfies (6.9), (6.10), and is $\mathbb{Z}_3$ equivariant. Hence we obtain $\dim_{\mathbb{R}} \{ v \in \mathfrak{X}(A_2); \text{rot}(v) = -2v \} = 11$. \square

Proof of Proposition 6.17. We find $13(= 30 - 17)$-dimensional family of non-trivial associative deformations.

Let $p_1 : S^7 \to \mathbb{C}P^3$ be the Hopf fibration. By Lemma 2.19, for any holomorphic curve $\Sigma \subset \mathbb{C}P^3$, the Hopf lift $p_1^{-1}(\Sigma) \subset S^7$ of $\Sigma$ is an associative submanifold. Since $p_1(A_2)$ is a holomorphic curve in $\mathbb{C}P^3$, the group of biholomorphic map of $\mathbb{C}P^3$, which is known to be PGL$(4, \mathbb{C})$, induces the associative deformations of $A_2$ via the Hopf lift.

The PGL$(4, \mathbb{C})$-action included in the Spin$(7)$-action is the standard SU$(4)$-action on $S^7$. Thus the dimension of non-trivial associative deformations of $A_2$ induced by
PGL(4, C) is given by
\[ \dim_{\mathbb{R}} \text{PGL}(4, \mathbb{C}) - \dim_{\mathbb{R}} \{ g \in \text{PGL}(4, \mathbb{C}) ; g \cdot p_1(A_2) \subset p_1(A_2) \} \]
\[ - (\dim_{\mathbb{R}} \text{SU}(4) - \dim_{\mathbb{R}} \{ h \in \text{SU}(4) ; h \cdot A_2 \subset A_2 \}) \]
\[ = \dim_{\mathbb{R}} \text{PGL}(4, \mathbb{C}) - \dim_{\mathbb{R}} \text{PGL}(2, \mathbb{C}) - \dim_{\mathbb{R}} \text{SU}(4) + \dim_{\mathbb{R}} \text{U}(2) \]
\[ = 30 - 6 - 15 + 4 = 13, \]
which gives the proof. □

6.3.3. The case \( A_3 \cong \text{SU}(2) \). Since \( A_3 \) is not special Legendrian, we cannot apply Corollary 4.2 to this case. First, we describe the operator \( D \) explicitly. Define \( E_i \in \text{su}(2) \) as (6.8). We denote by \( e_1, e_2, e_3 \) the left invariant vector fields on \( \text{SU}(2) \cong A_3 \) induced by \( \frac{1}{\sqrt{2}} E_1, \frac{1}{\sqrt{2}} E_2, E_3 \), respectively. If we define the vectors \( \eta_k \) for \( 1 \leq k \leq 4 \) as
\[ \eta_1 = \sqrt{\frac{7}{3}} \left( J e_1 + \frac{2}{\sqrt{7}} e_4 \right), \quad \eta_2 = \sqrt{\frac{7}{3}} \left( J e_2 + \frac{2}{\sqrt{7}} e_3 \right), \]
\[ \eta_3 = \sqrt{\frac{7}{3}} \left( J e_3 - \frac{2}{\sqrt{7}} e_2 \right), \quad \eta_4 = \sqrt{\frac{7}{3}} \left( J e_4 - \frac{2}{\sqrt{7}} e_1 \right), \]
where \( J \) is the standard complex structure on \( \mathbb{C}^4 \) and \( e_4 \) is the position vector, then \( \{ e_1, \cdots, e_3 \} \) is the orthonormal frame of \( TA_3 \) and \( \{ \eta_1, \cdots, \eta_4 \} \) is the orthonormal frame of \( \nu \). At \( p_0 = \frac{1}{\sqrt{2}}(0,1,i,0) \), we have
\[ e_1 = \frac{1}{\sqrt{14}} \left( \frac{\sqrt{3}}{2i} \right) , e_2 = \frac{1}{\sqrt{14}} \left( \frac{\sqrt{3}i}{-2} \right) , e_3 = \frac{1}{\sqrt{2}} \left( 0 \ i \right) , e_4 = \frac{1}{\sqrt{2}} \left( 0 \ i \right) , \]
\[ \eta_1 = \frac{1}{\sqrt{2}} \left( i \right) , \eta_2 = \frac{1}{\sqrt{2}} \left( -1 \ 0 \ -i \right) , \eta_3 = \frac{1}{\sqrt{42}} \left( -2\sqrt{3}i \ -3 \ 3i \ 2\sqrt{3} \right) , \eta_4 = \frac{1}{\sqrt{42}} \left( -2\sqrt{3} \ -3 \ 3i \ 2\sqrt{3} \right) . \]

Lemma 6.20. We have
\[ \nabla_{e_i} e_i = 0 \text{ for } i = 1, 2, 3, \quad [e_1, e_2] = \frac{2}{7} e_3, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = 2e_1, \]
\[ (\nabla_{e_i}^\top \eta_j) = \frac{3}{7} \left( \begin{array}{cccc} -\eta_3 & -\eta_3 & \eta_2 & \eta_1 \\ \eta_2 & -\eta_4 & -\eta_1 & \eta_2 \\ 7\eta_2 & -7\eta_1 & -5\eta_4 & 5\eta_3 \\ \eta_3 & -\eta_4 & -\eta_1 & -5\eta_4 & 5\eta_3 \end{array} \right), \quad (e_i \times \eta_j) = \left( \begin{array}{cccc} \eta_4 & \eta_3 & -\eta_2 & -\eta_1 \\ -\eta_3 & \eta_4 & \eta_1 & -\eta_2 \\ \eta_2 & -\eta_1 & \eta_4 & -\eta_3 \end{array} \right). \]

Proof. Since the \( \text{SU}(2) \)-action preserves the \( G_2 \)-structure on \( S^7 \), we only have to consider at \( p_0 \). The equations of \( \nabla_{e_i}^\top e_i \) and \( [e_i, e_j] \) is shown easily. By a direct computation, we have
\[ (\nabla_{e_i}^\top \eta_j) = \frac{3}{7} \left( \begin{array}{cccc} -\eta_3 & -\eta_3 & \eta_2 & \eta_1 \\ \eta_2 & -\eta_4 & -\eta_1 & \eta_2 \\ 7\eta_2 & -7\eta_1 & -5\eta_4 & 5\eta_3 \\ \eta_3 & -\eta_4 & -\eta_1 & -5\eta_4 & 5\eta_3 \end{array} \right) + \frac{2\sqrt{3}}{7} \left( \begin{array}{cccc} -e_1 & -e_2 & 2e_3 & 0 \\ e_2 & -e_1 & 0 & -2e_3 \\ 0 & 0 & 2e_1 & -2e_2 \end{array} \right), \]
and hence we obtain $\nabla^{\perp}_{e_i} \eta_j$. To prove the equations of $e_i \times \eta_j$, let $h_0$ be the standard metric on $\mathbb{C}^4$, $\omega_0$ be the standard Kähler form on $\mathbb{C}^4$, and $\Omega_0$ be the standard holomorphic volume form on $\mathbb{C}^4$. Define $e^i = h_0(e_i, \cdot), \eta^i = h_0(\eta_j, \cdot)$. Then \{$e^1, \ldots, e^4, \eta^1, \ldots, \eta^4$\} is the dual coframe of \{\$e_1, \ldots, e_4, \eta_1, \ldots, \eta_4$\}. We compute

$$
\begin{pmatrix}
e^1(J_i) \\
e^2(J_i) \\
e^3(J_i) \\
e^4(J_i)
\end{pmatrix} = \frac{2}{\sqrt{7}} \begin{pmatrix} e^1 & e^2 & e^3 & e^4 \end{pmatrix} - \sqrt{\frac{3}{7}} \begin{pmatrix} \eta^1 & \eta^2 & \eta^3 & \eta^4 \end{pmatrix} = \sqrt{\frac{3}{7}} \begin{pmatrix} e^1 & e^2 & e^3 & e^4 \end{pmatrix} + \frac{2}{\sqrt{7}} \begin{pmatrix} -\eta^1 \\
\eta^2 \\
\eta^3 \\
\eta^4 \end{pmatrix}.
$$

Since we know $h_0 = \sum_{i=1}^{4}((e^i)^2 + (\eta^i)^2)$, we obtain

$$
\omega_0 = h_0(J_i, \cdot) = \sqrt{\frac{3}{7}} \sum_{i=1}^{4} e^i \wedge \eta^i + \frac{2}{\sqrt{7}}(-e^{14} - e^{23} + \eta^{14} + \eta^{23}).
$$

The holomorphic volume form $\Omega_0$ is of the form $C \cdot (e^1 + ig(e_1, \cdot)) \cdots (e^4 + ig(e_4, \cdot)) = C \cdot (e^1 - ie^4(J_i)) \cdots (e^4 - ie^4(J_i))$ for $C > 0$, and from the relation $\omega_0^4/4! = (i/2)^4 \Omega_0$, we have $C = 7/3$. Hence the $G_2$-structure $\varphi \in \Omega^3(S^7)$ on $S^7$ is described as

$$
\varphi = i(e_4) \left( \frac{1}{2} \omega_0^2 + \text{Re} \Omega_0 \right)
= -e^{123} + e^1 \wedge (\lambda^{14} + \eta^{23}) + e^3 \wedge (-\eta^{13} + \eta^{24}) + e^3 \wedge (\eta^{12} + \eta^{34}),
$$

which implies the lemma. □
On the other hand, $\text{Spin}(7)$ induces $18(= \dim_\mathbb{R} \text{Spin}(7)/\text{SU}(2))$-dimensional associative deformations of $A_3$, and hence $A_3$ could potentially have 16-dimensional nontrivial associative deformations. However, we do not know whether there exists actual 16-dimensional nontrivial deformations.

**Proof of Proposition 6.22.** By Proposition 6.21, $D\psi = \alpha \psi$ for $\alpha \in \mathbb{R}$ is equivalent to

\[
\left( ie_3 - \left( \frac{15}{7} + \alpha \right) \right) \Psi_1 + (-ie_1 + e_2)\Psi_2 = 0, \tag{6.24}
\]

\[
-(ie_1 + e_2)\Psi_1 + (-ie_3 + (3-\alpha))\Psi_2 = 0. \tag{6.25}
\]

Applying $(ie_1 + e_2)$ to (6.24), we obtain

\[
\left( ie_3 - \left( \frac{1}{7} + \alpha \right) \right) (ie_1 + e_2)\Psi_1 + \left( e_1^2 + e_2^2 + \frac{2}{7}ie_3 \right) \Psi_2 = 0. \tag{6.26}
\]

Substituting (6.25) into (6.26), we have $(-7\Delta_+ + 24ie_3 + (7\alpha + 1)(\alpha - 3))\Psi_2 = 0$. By using the notation in Lemma 6.8 and Lemma 6.9, we obtain

\[
\left( -7\Delta_+ + 24ie_3 + (7\alpha + 1)(\alpha - 3) \right) \left( \rho_n(\cdot)v^{(n)}_k, u \right) = \left\{ -6(n - 2k + 2)^2 - n^2 - 2n + 24 + (7\alpha + 1)(\alpha - 3) \right\} \left( \rho_n(\cdot)v^{(n)}_k, u \right),
\]

for $n \geq 0, 0 \leq k \leq n, u \in V_n$.

Set $\alpha = -1$. Since we know that $-6(n - 2k + 2)^2 - n^2 - 2n + 48 = 0 \Leftrightarrow (n, k) = (6, 4), (4, 2), (4, 4)$, we deduce that

\[
\Psi_2 = \left( \rho_6(\cdot)v^{(6)}_3, u_1 \right) + \left( \rho_4(\cdot)v^{(4)}_2, u_2 \right) + \left( \rho_4(\cdot)v^{(4)}_3, u_3 \right),
\]

for $u_1 \in V_6, u_2, u_3 \in V_4$. From (6.24), we see that

\[
\Psi_1 = -i\sqrt{\frac{7}{10}} \left( \rho_6(\cdot)v^{(6)}_3, u_1 \right) - 2i\sqrt{\frac{7}{6}} \left( \rho_4(\cdot)v^{(4)}_3, u_2 \right).
\]

Hence we obtain $\dim_\mathbb{R} \{ \psi \in C^\infty(\text{SU}(2), \mathbb{R}^4); D\psi = -\psi \} = 14 + 2 \cdot 10 = 34$. $\square$

**6.4. The case $S^3, L_1, L_2, L_3$ and $L_4$.** Next, we study the deformations of homogeneous associative submanifolds which lie in a totally geodesic $S^6$. These Lagrangian deformation spaces are studied in [17]. Hence we only consider associative and non-Lagrangian deformations by Remark 5.3.

**6.4.1. The totally geodesic $S^3 \cong \text{SU}(2)$.** In this case, $\{e_1, e_2, e_3\} = \{E_1, E_2, E_3\}$ gives an orthonormal basis of $\text{su}(2)$ with respect to the induced metric from the totally geodesic $S^3$. We easily see the following by (6.13).

**Proposition 6.23.** $\dim_\mathbb{R} \{ f \in C^\infty(S^3); \Delta_+ f = 3f \} = 4$.

This implies that associative and non-Lagrangian deformations of the totally geodesic $S^3$ are trivial since $G_2$ induces $8(= \dim_\mathbb{R} G_2/\text{SO}(4))$-dimensional Lagrangian deformations of $S^3$ and $\text{Spin}(7)$ induces 12-dimensional associative deformations of $S^3$, whose space is known to be $\text{Spin}(7)/K$, where $K \cong \text{SU}(2)^3/\mathbb{Z}_2$ is a Lie subgroup of $\text{Spin}(7)$ ([8, Theorem IV.1.38]).
6.4.2. The case $L_1 \cong SU(2)$. Set $p_0 = \frac{\sqrt{5}}{2} \epsilon_1 + \frac{2}{3} \epsilon_4 = \langle \frac{\sqrt{5}}{3}, 0, \frac{2}{3}, 0 \rangle \in \mathbb{R} \oplus \mathbb{C}^3$. Then we have

$$(E_1^*)_{p_0} = -2 \frac{2\sqrt{5}}{3} \epsilon_3 - \frac{2}{3} \epsilon_6, \quad (E_2^*)_{p_0} = -\frac{2\sqrt{5}}{3} \epsilon_2 - \frac{2}{3} \epsilon_7, \quad (E_3^*)_{p_0} = \frac{2}{3} \epsilon_5.$$ 

Thus $\{e_1, e_2, e_3\} = \{\frac{\sqrt{5}}{4} E_1, \frac{\sqrt{5}}{4} E_2, \frac{3}{2} E_3\}$ gives an orthonormal basis of $su(2)$. We easily see the following by (6.13).

**Proposition 6.24.** $\dim \{f \in C^\infty(S^3); \Delta f = 3f\} = 7$.

This implies that associative and non-Lagrangian deformations of $L_1$ are trivial since $Spin(7) \setminus G_2$ induces 7-dimensional associative deformations of $L_1$.

6.4.3. The case $L_2 \cong SU(2)/\mathbb{Z}_2$. Set $p_0 = \frac{1}{\sqrt{2}}(\epsilon_4 + \epsilon_7) = \frac{1}{\sqrt{2}}(0, 0, 1, i) \in \mathbb{R} \oplus \mathbb{C}^3$. Then we have

$$(E_1^*)_{p_0} = \sqrt{2} \epsilon_3, \quad (E_2^*)_{p_0} = \sqrt{2} \epsilon_2, \quad (E_3^*)_{p_0} = \sqrt{2} (\epsilon_5 - \epsilon_6).$$

Thus $\{e_1, e_2, e_3\} = \{\frac{1}{\sqrt{2}} E_1, \frac{1}{\sqrt{2}} E_2, \frac{3}{2} E_3\}$ gives an orthonormal basis of $su(2)$.

**Lemma 6.25.** If $n$ is even, we have

$\text{Hom}_{\mathbb{Z}_2}(V_n, su(2) \otimes_{\mathbb{R}} \mathbb{C}) = \text{Hom}_{\mathbb{C}}(V_n, su(2) \otimes_{\mathbb{R}} \mathbb{C})$, \quad $\text{Hom}_{\mathbb{Z}_2}(V_n, \mathbb{C}) = \text{Hom}_{\mathbb{C}}(V_n, \mathbb{C})$.

If $n$ is odd, both spaces are $\{0\}$.

From this Lemma, we see the following by (6.13).

**Proposition 6.26.** $\dim \{f \in C^\infty(L_2); \Delta f = 3f\} = 6$.

This implies that associative and non-Lagrangian deformations of $L_2$ are trivial since $Spin(7) \setminus G_2$ induces 6-dimensional associative deformations of $L_2$. Note that $L_2$ is invariant under the action of $\{\text{diag}(e^{-3it}, e^{it}, e^{it}, e^{it}); t \in \mathbb{R}\} \subset Spin(7) \setminus G_2$.

6.4.4. The case $L_3 \cong SU(2)/A_4^*$. We have

$$(E_1^*)_{e_2} = \sqrt{10} \epsilon_4 - \sqrt{6} \epsilon_6, \quad (E_2^*)_{e_2} = \sqrt{10} \epsilon_5 - \sqrt{6} \epsilon_7, \quad (E_3^*)_{e_2} = -4 \epsilon_3.$$ 

Thus $\{e_1, e_2, e_3\} = \{E_1/4, E_2/4, E_3/4\}$ gives an orthonormal basis of $su(2)$.

**Lemma 6.27.**

$\text{Hom}_{A_4^*}(V_6, \mathbb{C}) = \mathbb{C} \left( (v_1^{(6)})^* - (v_5^{(6)})^* \right)$.

**Proof.** Recall that $A_4^*$ is generated by $k_1, k_2, k_3$ in (6.6). Take $L = \sum_{t=0}^{10} C_t (v_t^{(10)})^* \in \text{Hom}_{\mathbb{C}}(V_{10}, \mathbb{C})$ where $C_{it} \in \mathbb{C}$ and consider the condition

$$L(p_{10}(k)v) = L(v), \quad (6.27)$$

for $k \in A_4^*$ and $v \in V_{10}$. As for $k = k_1, k_2$, (6.27) is equivalent to

$$(-1)^t i^6 C_t = C_t, \quad (-1)^t i^{-t} C_{6-t} = C_{6-t}.$$ 

Thus $L$ is of the form $C \left( (v_1^{(6)})^* - (v_5^{(6)})^* \right)$ for $C \in \mathbb{C}$, and we see that $(v_1^{(6)})^* - (v_5^{(6)})^*$ is invariant by $k_3$. □
**Proposition 6.28.** \( \dim \{ f \in C^\infty(L_3) ; \Delta_+ f = 3f \} = 7. \)

This implies that associative and non-Lagrangian deformations of \( L_3 \) are trivial since \( \text{Spin}(7) \setminus G_2 \) induces 7-dimensional associative deformations of \( L_3. \)

**Proof.** The solution \( f \) of \( \Delta_+ f = 3f \) is contained in \( \text{span}_C \{ \langle \rho_6(\cdot)v_a^{(j)}, v_b^{(6)} \rangle ; 0 \leq a, b \leq 6 \} \) from (6.13). From Lemma 6.27, \( A^*_5 \) invariant solutions of \( \Delta_+ f = 3f \) are of the form \( f = \langle \rho_6(\cdot)(v_1^{(6)} - v_5^{(6)}) , u \rangle \) for \( u \in V_6. \) Imposing that \( f \) is \( \mathbb{R} \)-valued, we have \( \dim \{ f \in C^\infty(L_3) ; \Delta_+ f = 3f \} = 7. \)

**6.4.5. The case \( L_4 \cong SU(2)/D_3^* \).** We have

\[
(E_1^*)_{e_6} = \sqrt{6}e_2, \quad (E_2^*)_{e_6} = \sqrt{6}e_3, \quad (E_3^*)_{e_6} = 6e_7.
\]

Thus \( \{ e_1, e_2, e_3 \} = \{ E_1/\sqrt{6}, E_2/\sqrt{6}, E_3/6 \} \) gives an orthonormal basis of \( \mathfrak{su}(2). \)

**Lemma 6.29.** The space \( \text{Hom}_{D_3^*}(V_n, \mathbb{C}) \) is spanned by the following functions:

1. In case \( n = 6m \) where \( m \in \mathbb{Z}_{\geq 0}, \)

\[
(v^{(n)}_{3j})^* + (-1)^j(v^{(n)}_{n-3j})^* \quad \text{for} \quad 0 \leq j \leq m.
\]

2. In case \( n = 6m + 2, \)

\[
(v^{(n)}_{3j+1})^* + (-1)^{j+1}(v^{(n)}_{n-(3j+1)})^* \quad \text{for} \quad 0 \leq j \leq m.
\]

3. In case \( n = 6m + 4, \)

\[
(v^{(n)}_{3j+2})^* + (-1)^j(v^{(n)}_{n-(3j+2)})^* \quad \text{for} \quad 0 \leq j \leq m.
\]

**In case \( n \in 2\mathbb{Z} + 1, \) we have** \( \text{Hom}_{D_3^*}(V_n, \mathbb{C}) = \{0\}. \)

**Proof.** Recall that \( D_3^* \) is generated by \( k_4, k_5 \) in (6.7). Take \( L = \sum_{l=0}^{n} C_l(v^{(n)}_l)^* \otimes e_i \in \text{Hom}_C(V_n, \mathbb{C}) \) where \( C_l \in \mathbb{C}. \) Consider the condition (6.27) for \( k = k_4, k_5, \) it is equivalent to

\[
(-1)^{n-l}C_{n-l} = C_l, \quad (e^{\frac{2\pi}{3}})^{n-2l}C_l = C_l.
\]

Then we easily see Lemma 6.29. \( \square \)

**Proposition 6.30.** \( \dim \{ f \in C^\infty(L_4) ; \Delta_+ f = 3f \} = 7. \)

**Proof.** From (6.13), the solution \( f \) of \( \Delta_+ f = 3f \) is contained in the space spanned by \( \langle \rho_6(\cdot)v_j^{(6)}, v_a^{(6)} \rangle \) where \( j = 0, 6 \) and \( 0 \leq a \leq 6. \) From Lemma 6.29, \( D_3^* \) invariant solutions of \( \Delta_+ f = 3f \) are of the form \( f = \langle \rho_6(\cdot)(v_0^{(6)} + v_6^{(6)}), u \rangle \) for \( u \in V_6. \) Imposing that \( f \) is \( \mathbb{R} \)-valued, we have \( \dim \{ f \in C^\infty(L_3) ; \Delta_+ f = 3f \} = 7. \)

**Appendix A. Proof of Proposition 3.8.** We follow the proof of [7]. First, we show the following lemma.

**Lemma A.1.** For any vector fields \( u, v, w, z, X \in \mathfrak{X}(Y), \) we have

\[
\nabla_X(u \times v) = (\nabla_X u) \times v + u \times (\nabla_X v) - \chi(X, u, v),
\]

\[
R(w, z)(u \times v) = (R(w, z)u) \times v + u \times (R(w, z)v) + \varphi(z, u, v)w - \varphi(w, u, v)z
\]

\[
- g(w, v)u \times z - g(w, v)z \times u + g(z, u)v \times w + g(z, v)w \times u.
\]
When $M^3 \subset Y$ is associative, we have $TM \times TM \subset TM$, $TM \times \nu \subset \nu$, and $\nu \times \nu \subset TM$. Thus for any $X, u, v \in C^\infty(M, TM), \eta \in C^\infty(M, \nu)$, we have
\[
\nabla_X^\top(u \times v) = (\nabla_X^\top u) \times v + u \times (\nabla_X^\top v) - (\nabla_X^\phi u, v) - \nabla_X^\psi w, z \otimes w, z - \nabla_X^\psi w, z \\
\nabla_X^\perp(u \times \eta) = (\nabla_X^\top u) \times \eta + u \times (\nabla_X^\top \eta) - (\nabla_X^\phi u, \eta) \perp.
\]

Proof. Let $\{f_k\}_{k=1}^7$ be any local orthonormal frame of $TY$. Then
\[
\nabla_X(u \times v) = \sum_{i=1}^7 \{ (\nabla_X^\phi)(u, v, f_i) / f_i + \nabla_X^\psi f_i \}
\]
\[
= -\chi(X, u, v) + (\nabla_X^\phi u) \times v + u \times (\nabla_X^\phi v)
\]
since $\nabla g = 0$ and $\nabla \phi = \ast \phi$. For $R(w, z) = \nabla_w \nabla_z - \nabla_z \nabla_w - \nabla_{[w, z]}$, we see the following by a direct computation.
\[
R(w, z) (u \times v) = (R(w, z) u) \times v + u \times (R(w, z) v) - (\nabla_w \chi)(z, u, v) + (\nabla_z \chi)(w, u, v).
\]
Then, the equation $\nabla_w \chi = \sum_k i(f_k)(\nabla_w \phi) \otimes f_k = -\sum_k i(f_k)(g(w, \cdot) \perp \phi) \otimes f_k = -\phi \otimes w + \sum_k (g(w, \cdot) \perp \psi f_k)$ proves the lemma. □

Next, we compute $D^2$. Let $\{e_i\}_{i=1}^3$ be any local orthonormal frame satisfying $e_3 = e_1 \times e_2$ and $\{\eta_k\}_{k=1}^4$ be any local orthonormal frame of $\nu$. Then by Lemma A.1, it follows that
\[
D^2 \psi = \sum_{i,j=1}^3 e_i \times \nabla_{e_i}^\perp(e_j \times \nabla_{e_j}^\perp \psi) = I_1 + I_2,
\]
where
\[
I_1 = \sum_{i,j=1}^3 e_i \times (\nabla_{e_i}^\top e_j \times \nabla_{e_j}^\top \psi + e_j \times \nabla_{e_i}^\top \nabla_{e_j}^\perp \psi),
\]
\[
I_2 = -\sum_{i,j=1}^3 e_i \times \chi(e_i, e_j, \nabla_{e_j}^\perp \psi) \perp.
\]

From (2.3), the following holds:
\[
I_2 = \sum_{i,j} e_i \times ((e_i \times e_j) \times \nabla_{e_j}^\perp \psi)
\]
\[
= -\sum_{i,j} (e_i \times (e_i \times e_j)) \times \nabla_{e_j}^\perp \psi = 2 \sum_j e_j \times \nabla_{e_j}^\perp \psi = 2 D \psi.
\]

By the computation in [7], we have $I_1 = \nabla^\perp \nabla^\perp \psi + \pi_{\nu}(I_3) + I_4$, where
\[
I_3 = -\frac{1}{2} \sum_{i,j} (e_i \times e_j) \times R(e_i, e_j) \psi, \quad I_4 = \sum_{i,j,k} g(A_{(e_i \times e_j)} \otimes \eta_k, A_{\psi (e_j)} \eta_k).
\]

From the next lemma, we obtain Proposition 3.8.
Lemma A.2.

\[ I_3 = \sum_{i=1}^{3} R(e_i, \psi)e_i + 3\psi, \quad I_4 = -A\psi. \]

Proof. By using the relation \( e_i \times e_{i+1} = e_{i+2} \) for \( i \in \mathbb{Z}/3 \) and the Bianchi identity, we have

\[ I_3 = - \sum_{i \in \mathbb{Z}/3} e_i \times R(e_{i+1}, e_{i+2})\psi \]

\[ = \sum_{i \in \mathbb{Z}/3} e_i \times (R(\psi, e_{i+1})e_{i+2} + R(e_{i+2}, \psi)e_{i+1}), \]

\[ e_{i+2} \times R(e_{i+1}, \psi)e_i = R(e_{i+1}, \psi)e_{i+1} - (R(e_{i+1}, \psi)e_{i+2})e_i \]

\[ - \varphi(\psi, e_{i+2}, e_i)e_{i+1} + \varphi(e_{i+1}, e_{i+2}, e_i)\psi \]

\[ = R(e_{i+1}, \psi)e_{i+1} + e_i \times (R(e_{i+1}, \psi)e_{i+2}) + \psi, \]

since \( e_i \times e_{i+1} = e_{i+2} \) for \( i \in \mathbb{Z}/3 \), \( g(e_i, \psi) = 0 \), and \( \varphi(e_i, e_{i+1}, e_{i+2}) = 1 \). Hence we obtain \( I_3 = \sum_{i=1}^{3} R(e_i, \psi)e_i + 3\psi \). For \( I_4 \), we have by Lemma A.1

\[ A(e_i \times e_j) \times \eta_k e_i = -\nabla_{e_i}^\perp ((e_i \times e_j) \times \eta_k) \]

\[ = -\nabla_{e_i}^\perp (e_i \times e_j) \times \eta_k - (e_i \times e_j) \times (\nabla_{e_i}^\perp \eta_k) + \chi(e_i, e_i \times e_j, \eta_k)^\top \]

\[ = -((\nabla_{e_i}^\perp e_i) \times e_j + e_i \times (\nabla_{e_i}^\perp e_j)) \times \eta_k + (e_i \times e_j) \times A_{\eta_k} e_i \]

\[ + \chi(e_i, e_i \times e_j, \eta_k)^\top. \]

Since an associative submanifold is minimal, it follows that \( \sum_i \nabla_{e_i}^\perp e_i = 0 \). Moreover, we see \( \sum_i e_i \times \nabla_{e_i}^\perp e_j = 0 \) for \( j = 1, 2, 3 \) by the relation \( e_3 = e_1 \times e_2 \). Hence we obtain \( I_4 = I_5 + I_6 \), where

\[ I_5 = \sum_{i,j,k} g((e_i \times e_j) \times A_{\eta_k} e_i, A_{\psi} e_j) \eta_k, \quad I_6 = \sum_{i,j,k} g(\chi(e_i, e_i \times e_j, \eta_k)^\top, A_{\psi} e_j) \eta_k. \]

It is shown that \( I_5 = -A\psi \) in [7]. As for \( I_6 \), we compute \( \chi(e_i, e_i \times e_j, \eta_k) = \eta_k \times (e_i \times (e_i \times e_j)) = \eta_k \times (-e_j + \delta_{ij} e_i) \), and obtain \( \sum_i \eta_k \times (-e_j + \delta_{ij} e_i) = -2\eta_k \times e_j \in C^\infty(M, \nu) \), which implies that \( I_6 = 0 \).

Appendix B. Harmonic analysis on a homogeneous vector bundle. We give a summary of harmonic analysis on a homogeneous vector bundle from [27].

B.1. Homogeneous vector bundles.

Definition B.1. Let \( G \) be a Lie group and let \( K \) be a closed subgroup of \( G \). Set \( M := G/K \). A vector bundle \( E \to M \) is called a homogeneous vector bundle if \( G \) acts on \( E \) on the left and the \( G \)-action satisfies:

1. \( g \cdot E_x = E_{g \cdot x} \) for \( g \in G, x \in M \),
2. \( g \cdot : E_x \to E_{g \cdot x} \) is linear for \( g \in G, x \in M \),

where \( E_x \) is the fiber of \( E \) at \( x \in M \).

Lemma B.2. Let \((\tau, E_0)\) be a finite dimensional representation of \( K \). Then the associated vector bundle \( E := G \times \tau E_0 = G \times E_0/\sim \), where \((g, v) \sim (g \cdot k, \tau(k)^{-1} v)\), is a homogeneous vector bundle over \( M \).
All homogeneous vector bundles are described as above by the following lemma.

**Lemma B.3.** Let $E \to M$ be a homogeneous vector bundle. Let $E_0 = E_{eK}$ and $\tau : K \to \text{End}(E_0)$ be the induced action from 2 of Definition B.1. Then we have $E \cong G \times_{\tau} E_0$.

**B.2. Fourier series expansion.** Let $G$ be a compact Lie group, $K$ be a closed subgroup of $G$, $(\tau, E_0)$ be a finite dimensional unitary representation of $K$, and $E \to M$ be the homogeneous vector bundle associated with $(\tau, E_0)$. Assume that $M = G/K$ is orientable. Setting

$$C(G, E_0)^{(K, \tau)} := \{ f \in C(G, E_0); f(g \cdot k) = \tau(k)^{-1} f(g) \text{ for any } g \in G, k \in K \},$$

we have the following.

**Lemma B.4.** For $f \in C(M, E)$, define $\tilde{f} \in C(G, E_0)^{(K, \tau)}$ by $\tilde{f}(g) = g^{-1} f(gK) \in E_{eK} \cong E_0$. Then the map $f \mapsto \tilde{f}$ gives an isomorphism $C(M, E) \cong C(G, E_0)^{(K, \tau)}$. The map $f \mapsto \tilde{f}$ extends to the isomorphism $A : L^2(M, E) \cong L^2(G, E_0)^{(K, \tau)}$.

Let $\hat{G}$ be the set of all equivalence classes of finite dimensional irreducible unitary representations of $G$. For each $\gamma = [(\pi_\gamma, V_\gamma)] \in \hat{G}$, we assign a map $A_\gamma : V_\gamma \otimes \text{Hom}_K(V_\gamma, E_0) \to C(G, E_0)^{(K, \tau)}$, where $\text{Hom}_K(V_\gamma, E_0) = \{ L \in \text{Hom}(V_\gamma, E_0); L(k \cdot v) = \tau(k) L(v) \text{ for any } k \in K, v \in V_\gamma \}$, by $A_\gamma(v \otimes L)(g) = L(g^{-1} \cdot v)$.

**Proposition B.5** (Fourier expansion). The algebraic direct sum

$$\sum_{\gamma \in \hat{G}} A_\gamma(V_\gamma \otimes \text{Hom}_K(V_\gamma, E_0))$$

is uniformly dense in $C(G, E_0)^{(K, \tau)}$ relative to the uniform topology.

**Lemma B.6** (Schur orthogonality relations). Let $(\pi, V)$ and $(\pi', V')$ be irreducible unitary representations of a compact group $G$. Let $(\cdot, \cdot)$ and $(\cdot, \cdot)'$ be inner products on $V$ and $V'$, respectively. Then for $u, v \in V$ and $u', v' \in V'$, we have

$$\int_G (\pi(g)u, v) \overline{(\pi'(g)u', v')} dg = \begin{cases} 0 & (\pi \not\simeq \pi') \\ (u, u')(v, v') / \dim V & (\pi \simeq \pi'). \end{cases}$$

**B.3. Homogeneous differential operators.**

**Definition B.7.** Let $G$ be a Lie group and let $K$ be a closed subgroup of $G$. Set $M = G/K$. Let $E \to M$ and $F \to M$ be homogeneous vector bundles, and $(\tau, E_0)$ and $(\sigma, F_0)$ be the representations of $K$ associated with $E$ and $F$, respectively.

A differential operator $D : C^\infty(M, E) \to C^\infty(M, F)$ is called a **homogeneous differential operator** if $g \cdot Df = D(g \cdot f)$ for $g \in G, f \in C^\infty(M, E)$. Here, $(g \cdot f)(x) = gf(g^{-1} x)$ for $x \in M, g \in G, f \in C^\infty(M, E)$ or $C^\infty(M, F)$.

**Remark B.8.** The map $D$ is completely determined by its value at a point, i.e., given $(Df)_{eK}$ for any $f \in C^\infty(M, E)$, we can determine $(Df)_{gK}$ for each $g \in G, f \in C^\infty(M, E)$.

We give an explicit description of the homogeneous differential operators.
Let \( U(\mathfrak{g}) = \otimes^* \mathfrak{g}/I(\mathfrak{g}) \), where \( I(\mathfrak{g}) \) is the two-sided ideal in \( \otimes^* \mathfrak{g} \) generated by \( \{ X \otimes Y - Y \otimes X \mid X, Y \in \mathfrak{g} \} \). (In other words, \( U(\mathfrak{g}) \) is the universal enveloping algebra of \( \mathfrak{g} \).) Let \( \xi : \otimes^* \mathfrak{g} \to U(\mathfrak{g}) \) be the canonical projection and \( U^k(\mathfrak{g}) := \xi(\bigoplus_{\ell \leq k} \otimes^\ell \mathfrak{g}) \).

Set \( D(\mathcal{G}) \) be the space of all left invariant differential operators on \( \mathcal{G} \). For any \( X \in \mathfrak{g} \) and \( f \in C^\infty(\mathcal{G}) \), define \( Xf \in C^\infty(\mathcal{G}) \) by \( Xf(g) = (d/dt)f(g \cdot \exp(tX))|_{t=0} \).

The map \( X \mapsto (f \mapsto Xf) \) gives the inclusion \( \mathfrak{g} \hookrightarrow D(\mathcal{G}) \), from which an isomorphism \( U(\mathfrak{g}) \xrightarrow{\cong} D(\mathcal{G}) \) is induced.

**Lemma B.9.** The algebra \( U(\mathfrak{g}) \) is isomorphic to \( D(\mathcal{G}) \). If \( \{X_1, \ldots, X_n\} \) is a basis of \( \mathfrak{g} \), then \( \{X_{1}^{m_{1}} \cdots X_{n}^{m_{n}} \mid m_{j} \geq 0\} \) forms a basis of \( U(\mathfrak{g}) \).

Similarly, for \( L \otimes X \in \text{Hom}(E_0, F_0) \otimes U(\mathfrak{g}) \) and \( f \in C^\infty(G, E_0) \), set \( (L \otimes X)f = L.Xf \). Thus the element of \( \text{Hom}(E_0, F_0) \otimes U(\mathfrak{g}) \) is considered as a differential operator \( C^\infty(G, E_0) \to C^\infty(G, F_0) \).

Let \( K \) act on \( \text{Hom}(E_0, F_0) \otimes U(\mathfrak{g}) \) as \( \mu(k)(L \otimes X) = \sigma(k)L\tau(k)^{-1} \otimes \text{Ad}(k)X \) for \( k \in \text{Hom}(E_0, F_0) \) and \( X \in U(\mathfrak{g}) \). Then \( (\mu, \text{Hom}(E_0, F_0) \otimes U^j(\mathfrak{g})) \) is a representation of \( K \) for each \( j \). Setting \( \text{Hom}(E_0, F_0) \otimes U(\mathfrak{g}) \)

\[ \left( \text{Hom}(E_0, F_0) \otimes U(\mathfrak{g}) \right)^K = \{ D \in \text{Hom}(E_0, F_0) \otimes U(\mathfrak{g}) \mid \mu(k)D = D \text{ for any } k \in K \}, \]

we have the following.

**Lemma B.10.** For any \( D \in \text{Hom}(E_0, F_0) \otimes U(\mathfrak{g}) \)^K, we have \( DC^\infty(G, E_0)^{(K, \tau)} \subset C^\infty(G, F_0)^{(K, \sigma)} \). Conversely, if \( D \in \text{Hom}(E_0, F_0) \otimes U(\mathfrak{g}) \) satisfies \( DC^\infty(G, E_0)^{(K, \tau)} \subset C^\infty(G, F_0)^{(K, \sigma)} \), then \( \mu(k)D \in C^\infty(G, E_0)^{(K, \tau)} \).

**Definition B.11.** Let \( \mathfrak{k} \) be the Lie algebra of \( K \). A homogeneous space \( G/K \) is called reductive if there exists an \( \text{Ad}(K) \)-invariant vector subspace \( \mathfrak{p} \subset \mathfrak{g} \) satisfying \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \).

**Proposition B.12.** Suppose that \( M = G/K \) is a reductive homogeneous space. Let \( D : C^\infty(M, E) \to C^\infty(M, F) \) be a homogeneous differential operator of order \( j \) and let \( \overline{D} \) be the corresponding map from \( C^\infty(G, E_0)^{(K, \tau)} \) to \( C^\infty(G, F_0)^{(K, \sigma)} \).

Then there exists \( \overline{D} \in \text{Hom}(E_0, F_0) \otimes U(\mathfrak{g}) \) so that \( \overline{D} \in \text{Hom}(E_0, F_0) \otimes U(\mathfrak{g}) \)^K. If \( K \) is compact, \( \overline{D} \) may be taken to be in \( \text{Hom}(E_0, F_0) \otimes U(\mathfrak{g}) \)^K.

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