Resurgence in QFT: Unitons, Fractons and Renormalons in the Principal Chiral Model

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We explain the physical role of non-perturbative saddle points of path integrals in theories without instantons, using the example of the asymptotically free two-dimensional principal chiral model (PCM). Standard topological arguments based on homotopy considerations suggest no role for non-perturbative saddles in such theories. However, resurgence theory, which unifies perturbative and non-perturbative physics, predicts the existence of several types of non-perturbative saddles associated with features of the large-order structure of perturbation theory. These points are illustrated in the PCM, where we find new non-perturbative ‘fracton’ saddle point field configurations, and give a quantum interpretation of previously discovered ‘uniton’ unstable classical solutions. The fractons lead to a semi-classical realization of IR renormalons in the circle-compactified theory, and yield the microscopic mechanism of the mass gap of the PCM.

Introduction: In general, observables in quantum field theories (QFTs) receive perturbative and non-perturbative contributions. The perturbative contributions summarize information about quantum fluctuations around the trivial perturbative saddle point (vacuum) of the path integral, while the non-perturbative contributions come from quantum fluctuations around the non-trivial non-perturbative (NP) saddle points. In this paper we develop a deeper understanding of what the structure of perturbation theory implies for the nature and existence of NP saddle points of path integrals.

We illustrate these ideas with the two-dimensional (2d) SU(N) principal chiral model (PCM). The PCM is an asymptotically free matrix field theory, and is believed to have a dynamically generated mass gap determined by the strong scale \( \Lambda = \mu e^{-2\pi t/g_s(N)} \), with \( \mu \) the renormalization scale, see e.g. \[1–3\]. The PCM models many features of 4d Yang-Mills (YM) theory, but historically it has received less attention \[4\] than its vector-model cousin, the \( CP^N \) model, because:

(i) since \( \pi_2(SU(N)) = 0 \), there are no topologically stable instanton configurations which may lead to NP factors such as \( e^{-i/t/g^2} \); (ii) its large-N limit is not analytically tractable \[5\]. The \( CP^{N-1} \) model has neither of these issues, but \( \langle i \rangle \) is also shared with many other 2d QFTs, such as \( O(N > 3) \) and \( Sp(N) \) models, which are relevant to condensed matter physics \[6\].

However, the divergent structure of perturbation theory in the PCM is very similar to YM or \( CP^{N-1} \). After regularization and renormalization, the perturbative series has at least two types of factorial divergences. One is due to the combinatorics of the Feynman diagrams, while the other is known as the IR and UV renormalon divergences \[2, 7\] and comes from the low and high momenta in phase space integrals. Resummation of the perturbative series using the standard technique of Borel summation leads to singularities in the Borel plane. Infrared (IR) renormalons render the Borel sum ambiguous and ill-defined, because there is a subset of factorially divergent terms that do not alternate in sign. These problems are ubiquitous in asymptotically free QFTs, including YM and QCD \[8\], as well as in string theory \[9\].

It is generally believed that in quantum mechanics (QM) and QFT, the ambiguities in the summation of perturbative series due to the growth in the number of Feynman diagrams cancel against ambiguities in the contributions from NP instanton-anti-instanton saddle points, \( \{Z_i, a_i \sim nl/(S_{Z_i})^n \} \) \[7\]. On the other hand, the semi-classical meaning of IR renormalons has been unclear until recently, when it was shown that renormalons may also be continuously connected to new semi-classical NP saddle points \[10–13\]. In the weak coupling regime of circle compactified deformed YM and QCD(adj) in 4d, and the \( CP^{N-1} \) model in 2d, it was shown that BPST instantons fractionalize into \( N \) monopole-instantons \( M_i \) \[14, 15\], and \( N \) kink-instantons \( K_i \) \[16–19\], respectively. Correlated \( \{K_i, \bar{K}_i \} \) and \( \{M_i, \bar{M}_i \} \) events control the IR-renormalon singularities in these theories, and render physical observables unambiguous through the mechanism of resurgence \[11, 12, 20\].

However, the PCM has neither instantons nor fractional instantons. In fact, the PCM has no known stable NP saddles which could lead to NP factors such as \( e^{-i/t/g^2} \). This produces a deep puzzle. Since perturbation theory is divergent and non-Borel summable, an attempt to do Borel resummation results in ambiguities of the form \( \pm ie^{-t_i/g^2} \). If the theory is to be semi-classically meaningful and well-defined according to the criterion of \[11, 12\], such NP ambiguities must cancel, i.e., there must exist NP saddles whose amplitude is proportional to \( \pm ie^{-t_i/g^2} \). This is a highly non-trivial prediction of resurgence theory applied to QFT. But since there are no instantons, what are these NP saddles?

Thus the perturbative similarity (in particular the non-Borel-summability due to IR renormalons) between the
PCM and other asymptotically free theories such as YM and $\mathbb{CP}^{N-1}$, appears to be in conflict with their NP difference: YM and $\mathbb{CP}^{N-1}$ have non-trivial homotopy groups, and hence instantons, while the PCM has trivial homotopy, $\pi_2(SU(N)) = 0$, and no instantons. This suggests that topology alone is insufficient to fully characterize NP saddles, and misses a large class of important NP saddle points. In this work, we combine resurgence theory with a physical principle of continuity, and show the existence of new NP saddles in the path integral of the PCM, which we refer to as ‘fractons’ following the groundbreaking early work [21]; see also [22]. Our analysis easily generalizes to other theories, such as $O(N > 3)$ or $Sp(N)$ models, which also have no instantons.

**Unitons:** The PCM has classical action

$$S_b = \frac{N}{2\lambda} \int_M d^2x \text{tr} \partial_\mu U \partial^\mu U^\dagger,$$

where $\lambda = g^2 N$ is a dimensionless coupling constant, and we work in Euclidean space with $M = \mathbb{R}^2$ and $\mathbb{R} \times S^1$. The PCM has the symmetry $SU(N)_L \times SU(N)_R$ acting as $U \rightarrow g_L U g_R^\dagger$. There are no instantons, but there exist ‘unitons’, finite-action solutions to the second order Euclidean equations of motion, discovered in the seminal work of Uhlenbeck [23]. Further properties are discussed in [24–30]. These uniton solutions, which are harmonic maps from $S^2$ into $SU(N)$, did not receive much attention in the QFT literature, mainly because they are unstable; small fluctuations lead to a decrease in the action [31].

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However, the notion that all finite action saddles contributing to path integrals are classified by $\pi_2[T]$ is incorrect, even if $\pi_2[T]$ is non-trivial, as pointed out in [19] for the $\mathbb{CP}^{N-1}$ model, which has exact finite action non-BPS solutions, and where the connection to resurgence was also emphasized. In fact, these points can already be seen in quantum mechanics (QM). Consider an instanton in QM with a periodic potential and coupling $g$, where instantons are classified by their winding number $W \in \pi_1(S^1) = \mathbb{Z}$, and the basic instanton solution has $W = 1$. This is a solution to the first order BPS equation, and possesses an exact zero mode, its position. The amplitude for this event is $I \sim e^{-t_{\gamma}}$, where $\gamma$ is the topological angle [6]. Now, consider a correlated instanton-anti-instanton event $\{\mathcal{I}\}$. This is topologically indistinguishable from the perturbative vacuum, since $W = 1 + (-1) = 0$, but its action is $S = 1 + 1 = 2$, in units of the instanton action. Yet the separation between the two instantons is a negative quasi-zero mode, and the action of this saddle decreases with decreasing separation. To write the two-event amplitude one must integrate over the quasi-zero mode. Naively, when $\arg(g^2) = 0$ this integration is dominated by short distances, and is ill-defined. However, doing the quasi-zero mode integration at $\arg(g^2) = 0^\pm$, we find $[\mathcal{I}]_{\pm} \sim (\log \frac{1}{|g|} - \gamma \pm i\pi) e^{-2S_{\gamma}}$, a two-fold ambiguous result. This is a manifestation of the fact that $\arg(g^2) = 0$ is a Stokes line. Resurgence theory explains that the purely imaginary ambiguous part of the non-perturbative amplitude cures the ambiguity associated with the non-Borel summability of perturbation theory, i.e., $\text{Im}(\mathcal{B}_0 g^2 = 0^\pm + [\mathcal{I}]_{\pm} \mathcal{B}_2 g^2 = 0^\pm) = 0$, where $\mathcal{B}_0 g^2 = 0^\pm$ and $\mathcal{B}_2 g^2 = 0^\pm$ are left/right Borel sums of the formal perturbative series describing quantum fluctuations around the perturbative and non-perturbative $[\mathcal{I}]_{\pm}$ saddle points of the path integral, respectively [32]. For a fuller discussion of this cancellation mechanism, see [12]. Thus, the ‘instability’ of the $[\mathcal{I}]$ saddle point, i.e., a negative mode in the fluctuation operator, is in fact a positive feature, not a deficiency. Without it, the theory would be ill-defined.

**FIG. 1. Action densities $S$ for small (left) and large (right) $SU(2)$ unitons in the setting described in the text. The large uniton splits into two fractons.**

**FIG. 2. Action densities $S$ for large $SU(3)$ and $SU(4)$ unitons, which split into three and four fractons respectively.**

Unitons, as finite action non-BPS field configurations (just like $[\mathcal{I}]$ events), must be summed over in a semi-classical analysis of the path integral. The uniton action is quantized in units of $S_d \equiv \frac{8\pi}{g^2}$ [24]. The minimal uniton solution is easy to obtain. Let $v(z) \in \mathbb{C}^N$, with $z = x_1 + i x_2$, $x_\mu \in M$, be a single instanton solution in $\mathbb{CP}^{N-1}$ [33]. Then the minimal uniton in the $SU(N)$ PCM is given by $U(z, \tau) = e^{i\pi/(N(1 - 2\mathbb{P})}$, where $\mathbb{P}$ is the projector: $\mathbb{P}_{ij} = \frac{v_i v_j^\dagger}{\|v_i\|^4}$. Fig. 1 (left) depicts a small uniton.
in SU(2).

The uniton amplitude provides a substitute for instantons in theories with a trivial homotopy group, as will be demonstrated below. We write the amplitude associated with the basic uniton event and observe its relation to the strong scale

\[ U(\mu) \sim \text{det}[O_{\mu}(\mu)] e^{-\frac{\pi}{\Lambda^2(\mu)}} , \quad \Lambda^{2\theta_0} = \mu^{2\theta_0} U , \quad (2) \]

where \( \text{det}O_{\mu} \) encodes the Gaussian fluctuations around the uniton saddle point. In contrast, for theories with instantons, we would have \( \Lambda^{2\theta} = \mu^{2\theta} I \), and \( \theta_0 = N \) is the one-loop \( \beta \)-function of the theory.

**Fractions.** We cannot directly study the dynamics of the theory on \( \mathbb{R}^2 \), because the PCM becomes strongly coupled at large distances, just like YM and \( \mathbb{C}P^{N-1} \). However, there exists a way to continuously connect the strongly coupled PCM on \( \mathbb{R}^2 \) to a weakly-coupled calculable regime, analogous to the double-trace deformation of YM theory [34]. Then, we can address many NP questions in weak coupling, and by adiabatic continuity, extract universal results valid even at strong coupling. This set-up also permits us to see the fractionalization of unitons into their constituents, the fractons. The construction involves introducing the twisted boundary conditions \( U(x_1, x_2 + L) = e^{iH} U(x_1, x_2) e^{-iH} \), \( e^{iH} = \text{Diag}[e^{2\pi i\mu_1}, e^{2\pi i\mu_2}, \ldots, e^{2\pi i\mu_N}] \), or equivalently turning on an \( x_2 \) component of a background gauge field for the \( SU(N)_V \) symmetry so that \( \partial_{x_2} U \rightarrow D_{\mu} U = \partial_{\mu} U + i\delta_{\mu \lambda}L^{-1}[H, U] \). Then we study the dynamics of the action \( S_0 = \frac{1}{2\Lambda} \int_M d^2x \text{tr}[D_\mu U]^2 \). It turns out that there is a unique choice for the \( \mu_i \), determined by the condition of unbroken \( Z_N \) center symmetry, such that the small-L theory is continuously connected to its large \( L \) limit, without phase transitions or rapid cross-overs as a function of \( L \). For more details see [35].

Working with the special small-L theory defined above, the easiest way to show the splitting of a uniton into fractons is the following. Let \( v_{tw}(z) \) be a single instanton solution in \( \mathbb{C}P^{N-1} \) on \( \mathbb{R} \times S^1 \) in the presence of twisted boundary conditions, which exhibits fractionalization of an instanton into kink-instantons as the size moduli of the instanton is changed from small to large [16–18], [13]. Then, the uniton in the \( SU(N) \) PCM is given by \( U_{tw}(z, \bar{z}) = e^{i\pi/N}(1-2|P_{tw}|) \), where \( P_{tw} = \frac{\text{tr}_w(v_{tw})}{\text{tr}_w} \). Fig. 1 (right) and Fig. 2 depict the fracton constituents of a uniton for \( N = 2, 3, 4 \).

It is straightforward to construct explicit solutions corresponding to isolated fractons in the \( SU(N) \) PCM. For instance, for \( SU(2) \), using Hopf coordinates \( \theta, \phi_1, \phi_2 \) the action is

\[ S = \frac{1}{g^2} \int_M \left[ (\partial_\mu \theta)^2 + \cos^2 \theta (\partial_\mu \phi_1)^2 + \sin^2 \theta (\partial_\mu \phi_2 + \xi \delta_{\mu \lambda})^2 \right] \]

where \( \xi = 2\pi(\mu_2 - \mu_1) \). In the small-L regime, forgetting about the high Kaluza-Klein modes, we land on QM with a non-trivial potential on the \( SU(2) \) manifold:

\[ S = \frac{L}{g^2} \int_\mathbb{R} \left[ \dot{\theta}^2 + \cos^2 \theta \dot{\phi}_1^2 + \sin^2 \theta \dot{\phi}_2^2 + \xi^2 \sin^2 \theta \right] , \]

where the crucial existence of the potential term is due to the non-trivial background holonomy. The equations of motion associated with this action admit the solution \( \phi_{1,2} = \phi_{1,2}^0(t) \) and \( \theta(x_1; x_1^0) = 2\arccot \left[ e^{-\xi(x_1-x_1^0)} \right] \). The constants of integrations \( \{\phi_0^0, \phi_2^0, x_1^0\} \) are the three zero modes associated with a fracton.

As in gauge theory on \( \mathbb{R}^3 \times S^1 \) and the \( \mathbb{C}P^{N-1} \) model on \( \mathbb{R} \times S^1 \), where there exist Kaluza-Klein (KK) monopole-instantons[14, 15] and KK kink-instantons[16–18] respectively, which are associated with the affine root of the \( SU(N) \) algebra, there is also a KK-fracton in the PCM. Taking this into account, there are \( N \) basic types of fractons in the \( SU(N) \) PCM in a \( Z_N \) symmetric background, each of which carries \( \frac{1}{N} \) of the action of a uniton. Namely,

\[ F_i \sim e^{-\frac{2\pi i \mu_{i+1} - \mu_i}{\Lambda^2}} \sim e^{-\frac{2\pi i}{\Lambda^2}}, \quad U = \prod_{i=1}^N F_i \quad (3) \]

The surprise here with respect to earlier work [14–19, 36] is that we are now considering a theory which does not have instantons. Since each fracton carries three zero modes, and each uniton is composed from \( N \) fractons, the number of the combined zero and quasi-zero mode of a uniton must be \( 3N \). This is analogous to what happens in the \( \mathbb{C}P^{N-1} \) model, where each instanton has \( 2N \) exact zero modes, and each kink-instanton has two zero modes.

**Renormalon and uniton ambiguities on \( \mathbb{R}^2 \):** The IR renormalon divergence and ambiguities on \( \mathbb{R}^2 \) can be determined in two different ways. One is by studying a sub-class of planar Feynman diagrams. The number of planar diagrams grows only exponentially [37, 38], but a subset of such diagrams contribute factorially due to momentum integration at large orders, hence the effect is present at large-\( N \) as well [8]. Another way is using the S-matrix and Bethe-Ansatz equations (starting with the standard assumptions thereof, such as that the theory is gapped). The approaches must give the same answer, but for the PCM the second approach has been the main one pursued, with the result that the non-alternating late terms in perturbation theory diverge as \( n! \left( \frac{1}{\Lambda^2} \right)^n \), meaning that perturbation theory is non-Borel resummable [2]. This produces an ambiguity of the form \( \pm i e^{-\frac{2\pi i}{\Lambda^2}} \). The IR renormalon singularities found in [2] lie on the positive real Borel axis at

\[ \mathbb{R}^2 : \quad t_k^+ = 8\pi k/N = k[g^2 S_0]/\beta_0, \quad k \in \mathbb{Z}^+ \quad (4) \]

The appearance of the ’t Hooft coupling in the IR renormalon ambiguity means that, unlike instanton-anti-instanton and uniton ambiguities, it does not go away in the large-\( N \) limit. Also note that the leading IR-renormalon singularity is proportional to the square of...
the strong scale, \( e^{-\pi g_2 N} \sim (\Lambda/Q)^2 \) where \( Q \) is the Euclidean momentum.

In theories with instantons and a non-trivial homotopy group \( \pi_d \), the leading IR-renormalon ambiguity is approximately \( e^{-2\pi g_2/\beta_0} = e^{-2g_2 S_U/\beta_0} \) and is exponentially larger than the \( \mathcal{I} \pm \) ambiguity, as emphasized by ’t Hooft [7]. In the PCM, the relation between the leading renormalon and uniton ambiguity is \( e^{-\pi g_2 N} \sim e^{-S_U/\beta_0} \).

Since \( \pi_2[7] \) is trivial for the PCM, there is nothing preventing a uniton from appearing as a singularity in the Borel plane associated with the perturbative sector. This is to be contrasted with instantons, which carry a topological charge, and cannot appear as a singularity in the Borel plane. Indeed, on \( \mathbb{R}^2 \), we expect a pole associated with \( \mathcal{U} \pm \) upon integration over quasi-zero modes, related to the combinatorics of Feynman diagrams. The cancellation mechanism for the IR renormalon ambiguities on \( \mathbb{R}^2 \) is unknown, but after spatial compactification to \( \mathbb{R} \times S^1 \) the theory is under control. Below, we provide a microscopic mechanism of cancellation on \( \mathbb{R} \times S^1 \) in the regime of the theory continuously connected to \( \mathbb{R}^2 \).

**Continuity and cancellation of semi-classical renormalon ambiguities on \( \mathbb{R} \times S^1 \):** At small-\( L \), the theory reduces to QM, which is continuously connected to the 2d QFT. Consider the ground state energy \( \mathcal{E} \). The late terms of perturbation theory for \( \mathcal{E} \) involve a non-alternating divergent subseries. Upon left/right Borel resummation \( \mathcal{S}_{\mathcal{E}} = \mathcal{E}(g^2 + i0^\pm) \), we find a two-fold ambiguous result [35]:

\[
\mathcal{S}_{\mathcal{E}} = \mathcal{E}(g^2) = \text{Re} \mathcal{E} \mp 32\pi g_2 N e^{-\pi g_2 N} \\text{Re} \mathcal{E} \mp 16\pi g_2 N e^{-\pi g_2 N} \\text{Im} \mathcal{E} \mp 32\pi g_2 N e^{-\pi g_2 N} \\text{Im} \mathcal{E} \tag{5}
\]

reflecting the non-Borel summability of the theory on the \( \arg(g^2) = 0 \) Stokes line. This is the semi-classical realization of the renormalon ambiguity. The associated semi-classical singularities in the Borel plane are located at

\[
\mathbb{R} \times S^1 \!: \ t_k^{+,s.c.} = \frac{16\pi k}{N} = 2 \times k \times \frac{g_2 S_U}{\beta_0}, \quad k \in \mathbb{Z}^+ \quad \tag{6}
\]
diluted by a factor of two with respect to \( \mathbb{R}^2 \), but parametrically in the same neighborhood as the IR renormalon singularities of ’t Hooft seen in (4).

Remarkably, as predicted by the resurgence theory of Écalle [20], this ambiguity cancels against the fraction-anti-fracton correlated events, for which the leading amplitude at \( g^2 + i0^\pm \) are given by [35]

\[
[\mathcal{F} \mathcal{F}^\dagger]_{\theta=0^\pm} = \text{Re}[\mathcal{F} \mathcal{F}^\dagger] + i \text{Im}[\mathcal{F} \mathcal{F}^\dagger]_{\theta=0^\pm} = \left[ \log \left( \frac{\lambda}{16\pi} \right) - \gamma \right] \frac{16\pi}{\lambda} e^{-\frac{16\pi}{\lambda}} \pm \frac{32\pi}{\lambda} e^{-\frac{32\pi}{\lambda}} \tag{7}
\]

This leads to the cancellation of the non-perturbative ambiguities coming from perturbation theory against the ambiguity that arises from the NP saddle. That is:

\[
\text{Im} \mathcal{E}_{\theta=0^\pm} + \text{Im}[\mathcal{F} \mathcal{F}^\dagger]_{\theta=0^\pm} = 0. \quad \tag{8}
\]

This is a QFT example of Borel-Écalle resummation, a generalization of Borel resummation to account for the Stokes phenomenon.

**Mass gap on \( \mathbb{R} \times S^1 \) and \( \mathbb{R}^2 \):** A speculation by ’t Hooft that IR renormalons may be related to the mass gap and confinement in QCD [7] finds a concrete realization in our approach. The leading ambiguity on \( \mathbb{R}^2 \) is proportional to \( e^{-\pi g_2 N} \sim \Lambda^2/Q^2 \), and recent works [10–12] have shown that, in the semi-classical domain, it is always “half” of the renormalon which leads to a mass gap. If we assume that this semi-classical fact extrapolates to the strongly coupled domain, we observe that indeed, \( e^{-\pi g_2 N} \sim \Lambda/Q \), proportional to the first power of the strong scale. In our current example, in the semi-classical domain, the mass gap is a one-fracton (half-renormalon) effect and is given by \( m_g \sim \frac{1}{\alpha_N} e^{-\frac{\pi}{\lambda}} \sim \Lambda(ALN) \) for \( L \Lambda \lesssim 2\pi \). In future work, it would be important to understand fully the origin of the dilution factor highlighted in Eq. (5) as the theory moves continuously from the semi-classical domain to the strongly coupled domain.

**Conclusions:** Resurgence theory shows that in the principal chiral model (an asymptotically free theory without instantons) standard homotopy considerations are insufficient to classify saddle points in the path integral. Requiring the model to be well defined in the sense of Borel-Écalle summability [10–13], together with the physical principle of continuity and spatial compactification [34], leads to the existence of new non-BPS ‘fracton’ solutions giving a semiclassical realization of IR renormalons, and also provides a quantum interpretation to the classical uniton solutions. The fracton contributions to the path integral of the PCM give the microscopic origin of the mass gap of the theory.

**Acknowledgements.** We acknowledge support from U.S. DOE grants FG02-94ER40823 (A.C.), DE-FG02-92ER40716 (G.D.), DE-FG02-12ER41806 (M. ¨U.) and European Research Council Advanced Grant No. 247252 “Properties and Applications of the Gauge/Gravity Correspondence” (D.D.).

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