Asymptotic analysis in multivariate average case approximation with Gaussian kernels

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January 19, 2021

Abstract

We consider tensor product random fields \( Y_d, d \in \mathbb{N} \), whose covariance functions are Gaussian kernels. The average case approximation complexity \( n^{Y_d}(\varepsilon) \) is defined as the minimal number of evaluations of arbitrary linear functionals needed to approximate \( Y_d \), with relative 2-average error not exceeding a given threshold \( \varepsilon \in (0, 1) \). We investigate the growth of \( n^{Y_d}(\varepsilon) \) for arbitrary fixed \( \varepsilon \in (0, 1) \) and \( d \to \infty \). Namely, we find criteria of boundedness for \( n^{Y_d}(\varepsilon) \) on \( d \) and of tending \( n^{Y_d}(\varepsilon) \to \infty, d \to \infty \), for any fixed \( \varepsilon \in (0, 1) \). In the latter case we obtain necessary and sufficient conditions for the following logarithmic asymptotics

\[
\ln n^{Y_d}(\varepsilon) = a_d + q(\varepsilon)b_d + o(b_d), \quad d \to \infty,
\]

with any \( \varepsilon \in (0, 1) \). Here \( q: (0, 1) \to \mathbb{R} \) is a non-decreasing function, \((a_d)_{d \in \mathbb{N}}\) is a sequence and \((b_d)_{d \in \mathbb{N}}\) is a positive sequence such that \( b_d \to \infty, d \to \infty \). We show that only special quantiles of self-decomposable distribution functions appear as functions \( q \) in a given asymptotics.

Keywords and phrases: average case approximation, multivariate problems, random fields, Gaussian kernels, asymptotic analysis, tractability.

1 Introduction and problem setting

We consider a multivariate approximation problem in average case setting for special random fields with arbitrary large parametric dimension.

Let \( X = \{X(t), t \in \mathbb{R}\} \) be a random process defined on some probability space. Here and below \( \mathbb{R} \) denotes the set of real numbers. Suppose that the process has zero mean and the following covariance function

\[
K(\sigma)(t, s) = \exp\left\{-\frac{(t-s)^2}{2\sigma^2}\right\}, \quad t, s \in \mathbb{R},
\]

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where \( \sigma > 0 \) is a length scale parameter. The process is usually considered as a random element of the space \( L_2(\mathbb{R}, \mu) \), where \( \mu \) is the standard Gaussian measure on \( \mathbb{R} \). Covariance operator acts as follows

\[
K_\sigma f(t) = \int K_\sigma(t, s) f(s) \mu(ds) = \int K_\sigma(t, s) f(s) \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds, \quad t \in \mathbb{R}.
\]  

(1)

We consider \( d \)-variate version of \( X \) with arbitrary large \( d \in \mathbb{N} \) (set of positive integers). Namely, we consider a zero-mean random field \( Y_d = \{Y_d(t), t \in \mathbb{R}^d\} \) with the following covariance function

\[
K_{Y_d}^d(t, s) = \prod_{j=1}^{d} K_{\sigma_j}(t_j, s_j) = \exp\left\{ -\sum_{j=1}^{d} \frac{(t_j - s_j)^2}{2\sigma_j^2} \right\},
\]

(2)

where \( t = (t_1, \ldots, t_d) \) and \( s = (s_1, \ldots, s_d) \) are from \( \mathbb{R}^d \) with arbitrary large \( d \in \mathbb{N} \). Here \( (\sigma_j)_{j \in \mathbb{N}} \) is a given sequence of length scale parameters, which are generally have different values. If every \( K_{\sigma_j} \) corresponds to a zero-mean process \( X_j = \{X_j(t), t \in \mathbb{R}\} \) (defined on some probability space), \( j \in \mathbb{N} \), then \( Y_d \) is called tensor product of \( X_1, \ldots, X_d \) (see [8]). Function (2) is well known as Gaussian kernel, which is often used in numerical computation and statistical learning (see [2], [7], [15], [17], [20]).

For every \( d \in \mathbb{N} \) the random field \( Y_d \) is considered as random element of the space \( L_2(\mathbb{R}^d, \mu_d) \), where \( \mu_d \) is the standard Gaussian measure on \( \mathbb{R}^d \). So the space is equipped with the inner product

\[
\langle f, g \rangle_{2,d} = \int_{\mathbb{R}^d} f(x)g(x) \mu_d(dx) = \int_{\mathbb{R}^d} f(x)g(x) \frac{1}{(2\pi)^{d/2}} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{d} x_j^2 \right\} dx,
\]

and the norm

\[
\lVert f \rVert_{2,d} = \left( \int_{\mathbb{R}^d} f(x)^2 \mu_d(dx) \right)^{1/2} = \left( \int_{\mathbb{R}^d} f(x)^2 \frac{1}{(2\pi)^{d/2}} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{d} x_j^2 \right\} dx \right)^{1/2},
\]

where \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) in the integrals. The covariance operator \( K_{Y_d}^d \) of \( Y_d \) acts as follows

\[
K_{Y_d}^d f(t) = \int_{\mathbb{R}^d} K_{Y_d}^d(t, s)f(s) \mu_d(ds) = \int_{\mathbb{R}^d} K_{Y_d}^d(t, s)f(s) \frac{1}{(2\pi)^{d/2}} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{d} s_j^2 \right\} ds,
\]

where \( t = (t_1, \ldots, t_d) \) and \( s = (s_1, \ldots, s_d) \) are from \( \mathbb{R}^d \).

We consider the average case approximation complexity (approximation complexity for short) of \( Y_d, d \in \mathbb{N} \):

\[
n_{Y_d}^d(\varepsilon) := \min\{n \in \mathbb{N} : e_{Y_d}^d(n) \leq \varepsilon e_{Y_d}^d(0)\},
\]

(3)

where \( \varepsilon \in (0, 1) \) is a given error threshold, and

\[
e_{Y_d}^d(n) := \inf\left\{ \left( E \lVert Y_d - Y_d^{(n)} \rVert_{2,d}^2 \right)^{1/2} : Y_d^{(n)} \in \mathcal{A}_n \right\}
\]

2
is the smallest 2-average error among all linear approximations of \( Y_d \) having rank \( n \in \mathbb{N} \). The corresponding classes of linear algorithms are

\[
\mathcal{A}_n^{Y_d} := \left\{ \sum_{m=1}^{n} \langle Y_d, \psi_m \rangle_{2,d} \psi_m : \psi_m \in L_2(\mathbb{R}^d, \mu_d) \right\}.
\]

We will deal with the normalized error, i.e. we take into account the quantity:

\[
e^{Y_d}(0) := \left( \mathbb{E} \| Y_d \|_{2,d}^2 \right)^{1/2} < \infty,
\]

which is the approximation error of \( Y_d \) by zero element.

For a given sequence \( (\sigma_j)_{j \in \mathbb{N}} \) of length scale parameters in \( \mathbb{R} \) the quantity \( n^{Y_d}(\varepsilon) \) is considered as a function depending on two variables \( d \in \mathbb{N} \) and \( \varepsilon \in (0,1) \). There are a lot of results in this direction concerning the tractability. They provide necessary and sufficient conditions on \( (\sigma_j)_{j \in \mathbb{N}} \) to have upper bounds of given forms for the approximation complexity. The results within the described average case setting can be find in the papers \([3, 4, 10]\). The other setting of the worst case have upper bounds of given forms for the approximation complexity. The results within the described average case setting can be find in the papers \([3, 4, 10]\). The other setting of the worst case was considered in \([5, 13, 18]\). We will investigate \( n^{Y_d}(\varepsilon) \) in the different way. Namely, we are interested the asymptotic behaviour of \( n^{Y_d}(\varepsilon) \) for arbitrarily small fixed \( \varepsilon \) and \( d \to \infty \). We are not aware of any asymptotic results in this way specially for random fields with covariance functions \([2]\). Application of general results from \([9]\) requires an additional analysis and we do it in this paper.

We will use the following notation. We write \( a_n \sim b_n \) if \( a_n/b_n \to 1, \ n \to \infty \). The indicator \( 1(A) \) equals one if \( A \) is true and zero if \( A \) is false. For any function \( f \) we will denote by \( C(f) \) the set of all its continuity points and by \( f^{-1} \) the generalized inverse function \( f^{-1}(y) := \inf \{ x \in \mathbb{R} : f(x) \geq y \} \), where \( y \) is from the range of \( f \). By distribution function \( F \) we mean a non-decreasing function \( F \) on \( \mathbb{R} \) that is right-continuous on \( \mathbb{R} \), \( \lim_{x \to -\infty} F(x) = 0 \), and \( \lim_{x \to \infty} F(x) = 1 \).

2 Preliminaries

The quantity \( n^{Y_d}(\varepsilon) \) can be described in terms of the eigenvalues of the covariance operator \( K^{Y_d} \). Let \((\lambda_m^{Y_d})_{m \in \mathbb{N}}\) denote the sequence of eigenvalues and \((\psi_m^{Y_d})_{m \in \mathbb{N}}\) the corresponding sequence of eigenvectors of \( K^{Y_d} \). The family \((\lambda_m^{Y_d})_{m \in \mathbb{N}}\) is assumed to be ranked in non-increasing order. We have therefore \( K^{Y_d}\psi_m^{Y_d}(t) = \lambda_m^{Y_d}\psi_m^{Y_d}(t) \), \( m \in \mathbb{N} \), \( t \in \mathbb{R}^d \). We denote by \( \Lambda^{Y_d} \) the trace of \( K^{Y_d} \), i.e. \( \Lambda^{Y_d} := \sum_{m=1}^{\infty} \lambda_m^{Y_d} \).

It is well known (see \([1, 16, 19]\)) that for any \( n \in \mathbb{N} \) the following \( n \)-rank random field

\[
\tilde{Y}_d^{(n)}(t) := \sum_{k=1}^{n} \langle Y_d, \psi_k^{Y_d} \rangle_{2,d} \psi_k^{Y_d}(t), \quad t \in \mathbb{R}^d,
\]

minimizes the 2-average case error. Hence formula \((3)\) is reduced to

\[
n^{Y_d}(\varepsilon) = \min \left\{ n \in \mathbb{N} : \mathbb{E} \| Y_d - \tilde{Y}_d^{(n)} \|_{2,d}^2 \leq \varepsilon^2 \mathbb{E} \| Y_d \|_{2,d}^2 \right\}, \quad d \in \mathbb{N}, \ \varepsilon \in (0,1).
\]

Due to \((4)\) and \( \mathbb{E} \langle Y_d, \psi_m^{Y_d} \rangle_{2,d}^2 = \langle \psi_m^{Y_d}, K^{Y_d}\psi_m^{Y_d} \rangle_{2,d} = \lambda_m^{Y_d}, \ m \in \mathbb{N} \), we have the needed representation:

\[
n^{Y_d}(\varepsilon) = \min \left\{ n \in \mathbb{N} : \sum_{m=n+1}^{\infty} \lambda_m^{Y_d} \leq \varepsilon^2 \Lambda^{Y_d} \right\}, \quad d \in \mathbb{N}, \ \varepsilon \in (0,1).
\]
We now consider the sequence \((\lambda^Y_m)_m \in \mathbb{N}\). It has the following description. Let \((\lambda_{\sigma,k})_{k \in \mathbb{N}}\) denote the sequence of eigenvalues (ranked in non-increasing order) of the covariance operator \(K_\sigma\) defined by (1). This sequence is known (see [13], [15], and [21]):

\[
\lambda_{\sigma,k} = (1 - \omega) \omega^{k-1}, \quad k \in \mathbb{N}, \quad \omega := \left(1 + \frac{\sigma^2}{2} \left(1 + \sqrt{1 + \frac{4}{\sigma^2}}\right)\right)^{-1}.
\] (6)

In particular, we have \(\lambda_{\sigma,1} = 1 - \omega\) and \(\sum_{k \in \mathbb{N}} \lambda_{\sigma,k} = 1\). It is well known (see [11] and [13]), that, due to the tensor product structure (2) with given \(\sigma_j, j \in \mathbb{N}\), \((\lambda^Y_m)_m \in \mathbb{N}\) is the sequence of numbers

\[
\prod_{j=1}^d \lambda_{\sigma_j,k_j} = \prod_{j=1}^d (1 - \omega_j) \omega_j^{k_j-1}, \quad k_1, k_2, \ldots, k_d \in \mathbb{N},
\]

ranked in non-increasing order (see [12]). Here, according to (6), we set

\[
\omega_j := \left(1 + \frac{\sigma^2_j}{2} \left(1 + \sqrt{1 + \frac{4}{\sigma^2_j}}\right)\right)^{-1}, \quad j \in \mathbb{N}.
\]

Observe that

\[
\Lambda^Y = \prod_{j=1}^d \sum_{k \in \mathbb{N}} \lambda_{\sigma_j,k} = 1, \quad d \in \mathbb{N}.
\]

Thus each of the sequences \((\sigma_j)_{j \in \mathbb{N}}\) and \((\omega_j)_{j \in \mathbb{N}}\) fully determines \((\lambda^Y_m)_m \in \mathbb{N}\) and hence the quantity \(n^Y_d(\varepsilon)\) for any \(d \in \mathbb{N}\) and \(\varepsilon \in (0, 1)\).

3 Results

Before proceeding to the asymptotic analysis of the quantity \(n^Y_d(\varepsilon)\), we find criteria of its boundedness and unboundedness on \(d\) for any fixed \(\varepsilon \in (0, 1)\). The following propositions show that for any fixed \(\varepsilon \in (0, 1)\) either the quantity \(n^Y_d(\varepsilon)\) is a bounded function on \(d \in \mathbb{N}\) or it tends to infinity as \(d \to \infty\).

Proposition 1 The following conditions are equivalent:

(i) \(\sup_{d \in \mathbb{N}} n^Y_d(\varepsilon) < \infty\) for all \(\varepsilon \in (0, 1)\);

(ii) \(\sum_{j=1}^{\infty} \omega_j < \infty\);

(iii) \(\sum_{j=1}^{\infty} \sigma_j^{-2} < \infty\).

Proof of Proposition 1. By Proposition 5 from [9], the relation \(\sup_{d \in \mathbb{N}} n^Y_d(\varepsilon) < \infty, \varepsilon \in (0, 1)\), is equivalent to convergence of the following series

\[
\sum_{j=1}^{\infty} \omega_j = \sum_{j=1}^{\infty} \frac{\omega_j}{1 - \omega_j} = \sum_{j=1}^{\infty} \frac{\sigma_j^2 + \sqrt{\sigma_j^4 + 4\sigma_j^2}}{\sigma_j^2}.
\] (7)
Since $\omega_j \in (0, 1)$, the convergence of $\sum_{j=1}^\infty \frac{\omega_j}{1-\omega_j}$ implies the convergence of $\sum_{j=1}^\infty \omega_j$. Next, if $\sum_{j=1}^\infty \omega_j < \infty$, then $\omega_j \to 0$ and hence $\frac{\omega_j}{1-\omega_j} \sim \omega_j$, $j \to \infty$. So we have $\sum_{j=1}^\infty \frac{\omega_j}{1-\omega_j} < \infty$.

It is easily seen that the convergence of $\sum_{j=1}^\infty \sigma_j^{-2}$ implies the convergence of (7). Next, if (7) converges, then $\sigma_j \to \infty$ and $\sigma_j^2 + \sqrt{\sigma_j^4 + 4\sigma_j^2} \sim 2\sigma_j^2$, $j \to \infty$. Hence we get the convergence of $\sum_{j=1}^\infty \sigma_j^{-2}$.

Proposition 2 The following conditions are equivalent:

(i) $\lim_{d \to \infty} n^Y_d(\varepsilon) = \infty$ for all $\varepsilon \in (0, 1)$;

(ii) $\sum_{j=1}^\infty \omega_j = \infty$;

(iii) $\sum_{j=1}^\infty \sigma_j^{-2} = \infty$.

Proof of Proposition 2 According to Proposition 4 and 5 from [9], for any $\varepsilon \in (0, 1)$ either $\sup_{d \in \mathbb{N}} n^Y_d(\varepsilon) < \infty$ or $\lim_{d \to \infty} n^Y_d(\varepsilon) = \infty$. Hence the latter is equivalent to divergence of each of the series $\sum_{j=1}^\infty \omega_j$ and $\sum_{j=1}^\infty \sigma_j^{-2}$ by Proposition 1.

It is known (see [9]) that for wide class of tensor product random fields the quantity $n^Y_d(\varepsilon)$ has the logarithmic asymptotics of the following form (8) below. Our next theorem shows that in such asymptotics the function $q$ can be only a special quantile of self-decomposable distribution function (see [6], [14], or [9], Appendix).

Theorem 1 Let $(a_d)_{d \in \mathbb{N}}$ be a sequence, $(b_d)_{d \in \mathbb{N}}$ be a positive sequence such that $b_d \to \infty$, $d \to \infty$. Let a non-increasing function $q: (0, 1) \to \mathbb{R}$ and a distribution function $G$ satisfy the equation $q(\varepsilon) = G^{-1}(1 - \varepsilon^2)$ for all $\varepsilon \in C(q)$. Suppose that the following asymptotics holds

$$\ln n^Y_d(\varepsilon) = a_d + q(\varepsilon)b_d + o(b_d), \quad d \to \infty, \quad \varepsilon \in C(q). \quad (8)$$

Then $G$ is self-decomposable with zero Lévy spectral function on $(−\infty, 0)$.

Proof of Theorem 1 Due to Theorem 1 from [9], the condition (8) is equivalent to the convergence

$$\lim_{d \to \infty} G_d(x) = G(x), \quad x \in C(G), \quad (9)$$

where

$$G_d(x) := \sum_{m \in \mathbb{N}} \lambda^Y_m \mathbf{1}(\lambda^Y_m \geq e^{-a_d - b_dx}), \quad x \in \mathbb{R}, \quad d \in \mathbb{N}. \quad (10)$$

It was shown in [9] that $G_d$, $d \in \mathbb{N}$, can be considered as the following (probability) distribution functions

$$G_d(x) = \mathbb{P}\left(\frac{\sum_{j=1}^d U_j - a_d}{b_d} \leq x\right), \quad x \in \mathbb{R}, \quad d \in \mathbb{N},$$
where $U_j, j \in \mathbb{N}$, are independent random variables on some probability space with the measure $P$. Here $U_j, j \in \mathbb{N}$, have the following distribution

$$P(U_j = |\ln \lambda_{\sigma,j,k}|) = \lambda_{\sigma,j,k}, \quad k \in \mathbb{N}, \quad j \in \mathbb{N}.$$ 

Now we center these variables in the following way: $\hat{U}_j := U_j - |\ln \lambda_{\sigma,j,1}|$, $j \in \mathbb{N}$. So we have

$$P(\hat{U}_j = k|\ln \omega_j|) = (1 - \omega_j)\omega_j^k, \quad k \in \mathbb{N}_0, \quad j \in \mathbb{N}.$$ 

Here $\hat{U}_j \geq 0, j \in \mathbb{N}$. Next, we set

$$\hat{a}_d := a_d - \sum_{j=1}^d |\ln \lambda_{\sigma,j,1}| = a_d - \sum_{j=1}^d |\ln(1 - \omega_j)|, \quad d \in \mathbb{N}.$$ 

Then

$$G_d(x) = P\left(\frac{\sum_{j=1}^d \hat{U}_j - \hat{a}_d}{b_d} \leq x\right), \quad x \in \mathbb{R}, \quad d \in \mathbb{N},\quad (11)$$

For any $d \in \mathbb{N}, j \in \{1, \ldots, d\}$ and $x > 0$ we consider the following tails:

$$P(|\hat{U}_j| > xb_d) = P(\hat{U}_j > xb_d) = \sum_{k \in \mathbb{N}_0; k|\ln \omega_j| > xb_d} (1 - \omega_j)\omega_j^k = \omega_j^{k_{j,d}(x)},$$

where $k_{j,d}(x) := \min\{k \in \mathbb{N}_0 : k|\ln \omega_j| > xb_d\}$. Since $k_{j,d}(x)|\ln \omega_j| > xb_d$, we have

$$P(|\hat{U}_j| > xb_d) < e^{-xb_d}, \quad d \in \mathbb{N}, \quad j \in \{1, \ldots, d\}, \quad x > 0.$$ 

Due to $b_d \to \infty, d \to \infty$, we obtain

$$\max_{j \in \{1, \ldots, d\}} P(|\hat{U}_j| > xb_d) \to 0, \quad d \to \infty.\quad (12)$$

This is the condition of uniform negligibility of $\hat{U}_j/b_d$ in the sums $(\sum_{j=1}^n \hat{U}_j - \hat{a}_d)/b_d$. It is known, that under this condition and (11) the limit distribution in (9) is self-decomposable (see [14], p. 101, or [9], Theorem 10). Let $L$ denote the Lévy spectral function of $G$. Due to non-negativity of $\hat{U}_j$, we have $L(x) = 0, x < 0$ (see [6], p. 124, or [9], Theorem 11). □

The next theorem provides a criterion for the asymptotics (8), where $q$ is a quantile of a given self-decomposable distribution function. This is the main result of the paper.

**Theorem 2** Let $(a_d)_{d \in \mathbb{N}}$ be a sequence, $(b_d)_{d \in \mathbb{N}}$ be a positive sequence such that $b_d \to +\infty, d \to \infty$. Let $G$ be self-decomposable distribution function with triplet $(c,v,L)$ such that $L(x) = 0, x < 0$. Let
a non-increasing function \( q: (0, 1) \to \mathbb{R} \) satisfy the equation \( q(\varepsilon) = G^{-1}(1 - \varepsilon^2) \) for all \( \varepsilon \in (0, 1) \). For the asymptotics

\[
\ln n Y_d(\varepsilon) = a_d + q(\varepsilon)b_d + o(b_d), \quad d \to \infty, \quad \varepsilon \in (0, 1),
\]

the following ensemble of conditions is necessary and sufficient:

\[
\begin{align*}
\text{(A)} & \quad \lim_{d \to \infty} \sum_{j=1}^{d} \omega_j = -L(\tau), \quad \tau > 0, \\
\text{(B)} & \quad \lim_{d \to \infty} \frac{1}{b_d} \left( \sum_{j=1}^{d} \frac{|\ln \omega_j| \omega_j}{1 - \omega_j} - a_d \right) = c + \gamma_\tau, \quad \tau > 0, \\
\text{(C)} & \quad \lim_{\tau \to 0} \lim_{d \to \infty} \frac{1}{b_d^2} \sum_{j=1}^{d} \frac{|\ln \omega_j|^2 \omega_j}{(1 - \omega_j)^2} = \lim_{\tau \to 0} \lim_{d \to \infty} \frac{1}{b_d^2} \sum_{j=1}^{d} \frac{|\ln \omega_j|^2 \omega_j}{(1 - \omega_j)^2} = v^2.
\end{align*}
\]

where

\[
\gamma_\tau := \int_{0}^{\tau} \frac{y^3 dL(y)}{1 + y^2} - \int_{\tau}^{+\infty} \frac{y dL(y)}{1 + y^2}, \quad \tau > 0.
\]

The proof of this theorem is essentially based on the following lemma.

\textbf{Lemma 1} For any \( x > 0 \) and \( d \in \mathbb{N} \) the following identities hold:

\[
\begin{align*}
& \sum_{j=1}^{d} \sum_{k \in \mathbb{N}: k|\ln \omega_j| > x} (1 - \omega_j) \omega_j^k = \sum_{j=1}^{d} \omega_j + R_0(d, x), \\
& \sum_{j=1}^{d} \sum_{k \in \mathbb{N}: k|\ln \omega_j| \leq x} k|\ln \omega_j|(1 - \omega_j) \omega_j^k = \sum_{j=1}^{d} \frac{|\ln \omega_j| \omega_j}{1 - \omega_j} - R_1(d, x), \\
& \sum_{j=1}^{d} \left[ \sum_{k \in \mathbb{N}: k|\ln \omega_j| \leq x} k^2 |\ln \omega_j|^2 (1 - \omega_j) \omega_j^k - \left( \sum_{k \in \mathbb{N}: k|\ln \omega_j| \leq x} k|\ln \omega_j|(1 - \omega_j) \omega_j^k \right)^2 \right] \\
& \quad = \sum_{j=1}^{d} \frac{|\ln \omega_j|^2 \omega_j}{(1 - \omega_j)^2} - R_2(d, x),
\end{align*}
\]

where

\[
k_j(x) = \min \{ k \in \mathbb{N} : k \geq 2, k|\ln \omega_j| > x \}
\]
and

\[ R_0(d, x) := \sum_{j=1, \ldots, d: |\ln \omega_j| \leq x} \omega_j^{k_j(x)}, \]

\[ R_1(d, x) := \sum_{j=1, \ldots, d: |\ln \omega_j| \leq x} \frac{\ln \omega_j}{1 - \omega_j} k_j(x)(1 - \omega_j + \omega_j), \]

\[ R_2(d, x) := \sum_{j=1, \ldots, d: |\ln \omega_j| \leq x} \frac{\ln \omega_j}{(1 - \omega_j)^2} \left( k_j(x)^2 (1 - \omega_j)^2 (1 + \omega_j) + 2k_j(x)(1 - \omega_j)\omega_j^{k_j(x) + 1} + (1 - \omega_j)(\omega_j^{k_j(x) + 2}) \right). \]

**Proof of Lemma 1**. We fix \( x > 0 \) and \( d \in \mathbb{N} \).

1. Let us prove the first identity. Observe that

\[ \sum_{j=1 \ldots d} \sum_{k \in \mathbb{N}: k|\ln \omega_j| > x} (1 - \omega_j)\omega_j^{k} = \sum_{j=1 \ldots d} \sum_{k \in \mathbb{N}: k|\ln \omega_j| > x} (1 - \omega_j)\omega_j^{k} + \sum_{j=1 \ldots d} \sum_{k \in \mathbb{N}: k \leq 2} \sum_{k \in \mathbb{N}: k|\ln \omega_j| > x} (1 - \omega_j)\omega_j^{k}. \]

Here we have

\[ \sum_{j=1 \ldots d} \sum_{k \in \mathbb{N}: k|\ln \omega_j| > x} (1 - \omega_j)\omega_j^{k} = \sum_{j=1 \ldots d} \sum_{k \in \mathbb{N}: k \leq 2} \sum_{k \in \mathbb{N}: k|\ln \omega_j| > x} (1 - \omega_j)\omega_j^{k} = \sum_{j=1 \ldots d} \omega_j, \]

and

\[ \sum_{j=1 \ldots d} \sum_{k \in \mathbb{N}: k|\ln \omega_j| \leq x} \sum_{k \in \mathbb{N}: k|\ln \omega_j| > x} (1 - \omega_j)\omega_j^{k} = \sum_{j=1 \ldots d \leq 2} \sum_{k \in \mathbb{N}: k|\ln \omega_j| > x} (1 - \omega_j)\omega_j^{k} \]

\[ = \sum_{j=1 \ldots d} \sum_{k \equiv k_j(x)} \sum_{k \in \mathbb{N}: k|\ln \omega_j| > x} (1 - \omega_j)\omega_j^{k} \]

\[ = \sum_{j=1 \ldots d} \omega_j^{k_j(x)}. \]

2. Let us prove the second identity. It is obvious that if \( |\ln \omega_j| > x \) then there are no any \( k \in \mathbb{N} \)
such that \( k|\ln \omega_j| \leq x \). Therefore

\[
\sum_{j=1}^{d} \sum_{\substack{k \in \mathbb{N}: \\
 k|\ln \omega_j| \leq x}} k|\ln \omega_j|(1 - \omega_j)\omega_j^k = \sum_{j=1, \ldots, d:} \sum_{k \in \mathbb{N}: k|\ln \omega_j| \leq x} k|\ln \omega_j|(1 - \omega_j)\omega_j^k
\]

\[
= \sum_{j=1, \ldots, d:} |\ln \omega_j|(1 - \omega_j) \sum_{k \in \mathbb{N}: k|\ln \omega_j| \leq x} k\omega_j^k
\]

\[
= \sum_{j=1, \ldots, d:} |\ln \omega_j|(1 - \omega_j) \left( \sum_{k=1}^{\infty} k\omega_j^k - \sum_{k=k_j(x)}^{\infty} k\omega_j^k \right).
\]

It is well known that

\[
\sum_{k=1}^{\infty} k\omega_j^k = \omega_j \sum_{k=1}^{\infty} k\omega_j^{k-1} = \frac{\omega_j}{(1 - \omega_j)^2}.
\]

Next, using this fact we get

\[
\sum_{k=k_j(x)}^{\infty} k\omega_j^k = \omega_j^{k_j(x)} \sum_{k=k_j(x)}^{\infty} k\omega_j^{k-k_j(x)}
\]

\[
= \omega_j^{k_j(x)} \sum_{k=k_j(x)+1}^{\infty} (k - k_j(x))\omega_j^{k-k_j(x)} + k_j(x)\omega_j^{k_j(x)} \sum_{k=k_j(x)}^{\infty} \omega_j^{k-k_j(x)}
\]

\[
= \frac{\omega_j^{k_j(x)+1} - k_j(x)\omega_j^{k_j(x)}}{(1 - \omega_j)^2} + \frac{k_j(x)\omega_j^{k_j(x)}}{1 - \omega_j}.
\]

Then

\[
\sum_{j=1}^{d} \sum_{\substack{k \in \mathbb{N}: \\
 k|\ln \omega_j| \leq x}} k|\ln \omega_j|(1 - \omega_j)\omega_j^k = \sum_{j=1, \ldots, d:} |\ln \omega_j|(1 - \omega_j) \left( \frac{\omega_j}{(1 - \omega_j)^2} - \frac{\omega_j^{k_j(x)+1} - k_j(x)\omega_j^{k_j(x)}}{(1 - \omega_j)^2} - \frac{k_j(x)\omega_j^{k_j(x)}}{1 - \omega_j} \right)
\]

\[
= \sum_{j=1, \ldots, d:} \frac{|\ln \omega_j|\omega_j}{1 - \omega_j} - \sum_{j=1, \ldots, d:} \frac{(|\ln \omega_j|\omega_j^{k_j(x)+1} + k_j(x)|\ln \omega_j|\omega_j^{k_j(x)})}{1 - \omega_j}
\]

\[
= \sum_{j=1, \ldots, d:} \frac{|\ln \omega_j|\omega_j}{1 - \omega_j} - \sum_{j=1, \ldots, d:} \frac{|\ln \omega_j|\omega_j^{k_j(x)}}{1 - \omega_j} (\omega_j + k_j(x)(1 - \omega_j)).
\]
3. We now prove the third identity. Let us consider the sum

$$S_1 := \sum_{j=1}^{d} \sum_{k \in \mathbb{N}} k^2 | \ln \omega_j|^2 (1 - \omega_j) \omega_j^k$$

$$= \sum_{j=1, \ldots, d} \sum_{k \in \mathbb{N}; k \mid \ln \omega_j \leq x} k^2 | \ln \omega_j|^2 (1 - \omega_j) \omega_j^k$$

$$= \sum_{j=1, \ldots, d} | \ln \omega_j|^2 (1 - \omega_j) \sum_{k \in \mathbb{N}; k \mid \ln \omega_j \leq x} k^2 \omega_j^k$$

$$= \sum_{j=1, \ldots, d; | \ln \omega_j | \leq x} | \ln \omega_j|^2 (1 - \omega_j) \left( \sum_{k=1}^{\infty} k^2 \omega_j^k - \sum_{k=k_j(x)}^{\infty} k^2 \omega_j^k \right).$$

It is easily seen that

$$\sum_{k=1}^{\infty} k^2 \omega_j^k = \sum_{k=1}^{\infty} k(k-1) \omega_j^k + \sum_{k=1}^{\infty} k \omega_j^k$$

$$= \omega_j^2 \sum_{k=2}^{\infty} k(k-1) \omega_j^{k-2} + \sum_{k=1}^{\infty} k \omega_j^k$$

$$= \frac{2 \omega_j^2}{(1 - \omega_j)^3} + \frac{\omega_j}{(1 - \omega_j)^2}$$

$$= \frac{\omega_j + \omega_j^2}{(1 - \omega_j)^3}.$$
Thus we obtain

\[
S_1 = \sum_{j=1, \ldots, d; \ln \omega_j \leq x} |\ln \omega_j|^2 (1 - \omega_j) \left( \frac{\omega_j + \omega_j^2}{(1 - \omega_j)^2} - \frac{2 \omega_j^{k_j(x)+2}}{(1 - \omega_j)^3} - \frac{(2k_j(x) + 1)\omega_j^{k_j(x)+1}}{(1 - \omega_j)^2} - \frac{k_j(x)^2 \omega_j^{k_j(x)}}{1 - \omega_j} \right)
\]

\[
= \sum_{j=1, \ldots, d; \ln \omega_j \leq x} |\ln \omega_j|^2 \left( \frac{\omega_j + \omega_j^2}{(1 - \omega_j)^2} - \frac{2 \omega_j^{k_j(x)+2}}{(1 - \omega_j)^3} - \frac{(2k_j(x) + 1)\omega_j^{k_j(x)+1}}{1 - \omega_j} - k_j(x)^2 \omega_j^{k_j(x)} \right).
\]

We now consider the sums

\[
S_2 := \sum_{j=1}^d \left( \sum_{k \in \mathbb{N}; k \ln \omega_j \leq x} k |\ln \omega_j| (1 - \omega_j) \omega_j^k \right)^2
\]

\[
= \sum_{j=1, \ldots, d; \ln \omega_j \leq x} \left( \sum_{k \in \mathbb{N}; k \ln \omega_j \leq x} k |\ln \omega_j| (1 - \omega_j) \omega_j^k \right)^2
\]

\[
= \sum_{j=1, \ldots, d; \ln \omega_j \leq x} |\ln \omega_j|^2 (1 - \omega_j)^2 \left( \sum_{k=1}^\infty k \omega_j^k - \sum_{k=k_j(x)}^\infty k \omega_j^k \right)^2.
\]

By the above remarks, we have

\[
\left( \sum_{k=1}^\infty k \omega_j^k - \sum_{k=k_j(x)}^\infty k \omega_j^k \right)^2 = \left( \frac{\omega_j}{(1 - \omega_j)^2} - \frac{\omega_j^{k_j(x)+1}}{(1 - \omega_j)^2} - \frac{k_j(x) \omega_j^{k_j(x)}}{1 - \omega_j} \right)^2
\]

\[
= \frac{\omega_j^2}{(1 - \omega_j)^4} + \frac{\omega_j^{2k_j(x)+2}}{(1 - \omega_j)^4} + \frac{k_j(x)^2 \omega_j^{2k_j(x)}}{(1 - \omega_j)^2} - \frac{2 \omega_j^{k_j(x)+2}}{(1 - \omega_j)^4} - \frac{2k_j(x) \omega_j^{k_j(x)+1}}{(1 - \omega_j)^3} + \frac{2k_j(x) \omega_j^{2k_j(x)+1}}{(1 - \omega_j)^3}.
\]

Hence we get

\[
S_2 = \sum_{j=1, \ldots, d; \ln \omega_j \leq x} |\ln \omega_j|^2 \left( \frac{\omega_j^2}{(1 - \omega_j)^2} + \frac{\omega_j^{2k_j(x)+2}}{(1 - \omega_j)^2} + k_j(x)^2 \omega_j^{2k_j(x)} - \frac{2 \omega_j^{k_j(x)+2}}{(1 - \omega_j)^2} - \frac{2k_j(x) \omega_j^{k_j(x)+1}}{1 - \omega_j} + \frac{2k_j(x) \omega_j^{2k_j(x)+1}}{1 - \omega_j} \right).
\]
Thus we have

\[ S_1 - S_2 = \sum_{j=1, \ldots, d: |\ln \omega_j| \leq x} |\ln \omega_j|^2 \left( \frac{\omega_j^{k_j(x)+2}}{(1 - \omega_j)^2} - k_j(x)^2 \omega_j^{k_j(x)} \right) \]

\[ = \sum_{j=1, \ldots, d: |\ln \omega_j| \leq x} |\ln \omega_j|^2 \omega_j \left( \frac{1}{1 - \omega_j} k_j(x)^2 \omega_j^{k_j(x)} \right) \]

\[ + \omega_j^{k_j(x)+2} + k_j(x)^2 \omega_j^{2k_j(x)} + 2k_j(x)(1 - \omega_j) \omega_j^{k_j(x)+1} \]

\[ = \sum_{j=1, \ldots, d: |\ln \omega_j| \leq x} |\ln \omega_j|^2 \frac{\omega_j}{(1 - \omega_j)^2} \left( k_j(x)^2 (1 - \omega_j)^2 (1 + \omega_j^{k_j(x)}) \right) \]

\[ + 2k_j(x)(1 - \omega_j) \omega_j^{k_j(x)+1} + (1 - \omega_j) \omega_j^{k_j(x)+2} \].

So we obtain the third identity. \( \square \)

We now turn to proof of the theorem.

**Proof of Theorem 2.** We first recall that \( C(q) = (0,1) \), which follows from properties of self-decomposable distribution functions (see [9], Remarks 2 and 8). Thus, by Theorem 1 from [9], the conditions (A), (B), (C) of Theorem 11 (see [9], Appendix) are necessary and sufficient (with \( Y_j := \hat{U}_j, j \in \mathbb{N} \), \( A_\gamma := a_\gamma, B_\gamma := b_\gamma, \gamma := c \), and \( \sigma^2 := v \)). Thus (13) is equivalent to the following ensemble of conditions:

\[ \lim_{\tau \to 0} \lim_{d \to \infty} \frac{1}{b_d^2} \sum_{j=1}^d \text{Var} \left[ \hat{U}_j | \hat{U}_j \leq \tau b_d \right] = \lim_{\tau \to 0} \lim_{d \to \infty} \frac{1}{b_d^2} \sum_{j=1}^d \text{Var} \left[ \hat{U}_j | \hat{U}_j \leq \tau b_d \right] = v^2. \]

Here \( \gamma_\tau \) is defined by (14).
Let us write the sums in these conditions in terms of \( \omega_j \). First, note that
\[
\Pr(\hat{U}_j > \tau b_d) = \sum_{k \in \mathbb{N}: \ln \omega_j > \tau b_d} (1 - \omega_j) \omega_j^k,
\]
\[
E\left[\hat{U}_j \mathbb{1}(|\hat{U}_j| \leq \tau b_d)\right] = \sum_{k \in \mathbb{N}: \ln \omega_j \leq \tau b_d} k \ln \omega_j (1 - \omega_j) \omega_j^k,
\]
\[
\text{Var}\left[\hat{U}_j \mathbb{1}(|\hat{U}_j| \leq \tau b_d)\right] = E\left[\hat{U}_j^2 \mathbb{1}(|\hat{U}_j| \leq \tau b_d)\right] - \left(E\left[\hat{U}_j \mathbb{1}(|\hat{U}_j| \leq \tau b_d)\right]\right)^2
\]
\[= \sum_{k \in \mathbb{N}: \ln \omega_j \leq \tau b_d} k^2 \ln \omega_j (1 - \omega_j) \omega_j^k - \left(\sum_{k \in \mathbb{N}: \ln \omega_j \leq \tau b_d} k \ln \omega_j (1 - \omega_j) \omega_j^k\right)^2.
\]

Next, applying Lemma \ref{lemma:basic_lemma} we have
\[
\sum_{j=1}^{d} \Pr(\hat{U}_j > \tau b_d) = \sum_{j=1, \ldots, d: \ln \omega_j > \tau b_d} \omega_j + R_0(d, \tau b_d),
\]
\[
\sum_{j=1}^{d} E\left[\hat{U}_j \mathbb{1}(|\hat{U}_j| \leq \tau b_d)\right] = \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} |\ln \omega_j| \omega_j - R_1(d, \tau b_d),
\]
\[
\sum_{j=1}^{d} \text{Var}\left[\hat{U}_j \mathbb{1}(|\hat{U}_j| \leq \tau b_d)\right] = \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \frac{|\ln \omega_j|^2 \omega_j}{(1 - \omega_j)^2} - R_2(d, \tau b_d).
\]

where the functions \( R_0, R_1, \) and \( R_2 \) are defined as in Lemma \ref{lemma:basic_lemma} Therefore (13) is equivalent to the following three conditions

\[(A') \quad \lim_{d \to \infty} \sum_{j=1}^{d} \left(\sum_{j=1, \ldots, d: \ln \omega_j > \tau b_d} \omega_j + R_0(d, \tau b_d)\right) = -L(\tau) \quad \text{for all} \quad \tau > 0;\]

\[(B') \quad \lim_{d \to \infty} \frac{1}{b_d} \left(\sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \frac{|\ln \omega_j| \omega_j}{1 - \omega_j} - R_1(d, \tau b_d) - \hat{a}_d\right) = c + \gamma_\tau \quad \text{for all} \quad \tau > 0;\]

\[(C') \quad \lim_{d \to \infty} \lim_{\tau \to 0} \frac{1}{b_d^2} \left(\sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \frac{|\ln \omega_j|^2 \omega_j}{(1 - \omega_j)^2} - R_2(d, \tau b_d)\right) = \lim_{\tau \to 0} \lim_{d \to \infty} \frac{1}{b_d^2} \left(\sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \frac{|\ln \omega_j|^2 \omega_j}{(1 - \omega_j)^2} - R_2(d, \tau b_d)\right) = v^2.\]

We first show that (A), (B), (C) imply (A'), (B'), (C'). Due to the condition (C), there exist
$\tau_0 > 0$ and $C_0 > 0$ such that for all $d \in \mathbb{N}$ we have

$$\frac{1}{b_d^2} \sum_{j=1}^{d} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} \leq C_0. \tag{16}$$

Since $\omega_j \in (0, 1)$, the inequality $|\ln \omega_j| > 1 - \omega_j$ always holds. Hence we conclude that

$$\frac{1}{b_d^2} \sum_{j=1}^{d} \omega_j \leq C_0, \quad d \in \mathbb{N}. \tag{17}$$

We first show that $R_0(d, \tau b_d) \to 0$, $d \to \infty$ for every $\tau > 0$. We fix $\tau > 0$ and $\tau_* < \min\{\tau_0, \tau\}$. Observe that

$$R_0(d, \tau b_d) = \sum_{j=1}^{d} \omega_j^{k_j(\tau b_d)} \leq \sum_{j=1}^{d} \omega_j^{k_j(\tau b_d)} + \sum_{j=1}^{d} \omega_j^{k_j(\tau b_d)}.$$ 

By the definition of $k_j(\cdot)$ (see Lemma [1]), we have the inequalities $k_j(\tau b_d) \geq 2$ and $\omega_j^{k_j(\tau b_d)} \leq e^{-\tau b_d}$, which give

$$R_0(d, \tau b_d) \leq e^{-\tau b_d} \sum_{j=1}^{d} 1 + \sum_{j=1}^{d} \omega_j^2.$$ 

Using conditions in the sums, we obtain

$$R_0(d, \tau b_d) \leq e^{-\tau \tau_* b_d} \sum_{j=1}^{d} \omega_j + e^{-\tau_* b_d} \sum_{j=1}^{d} \omega_j \leq e^{-\tau \tau_* b_d} \sum_{j=1}^{d} \omega_j + e^{-\tau_* b_d} \sum_{j=1}^{d} \omega_j.$$ 

Here the first sum is less than $C_0 b_d^2$ by (17) and the second sum is bounded by some constant $C_1$ due to (A). Thus

$$R_0(d, \tau b_d) \leq C_0 b_d^2 e^{-\tau \tau_* b_d} + C_1 e^{-\tau_* b_d}.$$ 

Since $\tau > \tau_* > 0$ and $b_d \to \infty$, we obtain that $R_0(d, \tau b_d) \to 0$, $d \to \infty$. This, together with (A), yields (A').

We now consider

$$R_1(d, \tau b_d) = \sum_{j=1}^{d} \frac{|\ln \omega_j| \omega_j^{k_j(\tau b_d)}}{1 - \omega_j} \left(k_j(\tau b_d)(1 - \omega_j) + \omega_j\right).$$

According to the definition of $k_j(\cdot)$, we have

$$k_j(\tau b_d) \leq 2(k_j(\tau b_d) - 1) \leq \frac{2\tau b_d}{|\ln \omega_j|} \tag{18}$$
Hence
\[
R_1(d, \tau d) \leq 2\tau b_d \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \omega_j^{k_j(\tau b_d)} + \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \frac{|\ln \omega_j| \omega_j^{k_j(\tau b_d)}}{1 - \omega_j} \cdot \omega_j^{k_j(\tau b_d)}.
\]

Note that the function \(\omega_j \mapsto \frac{|\ln \omega_j| \omega_j^{k_j(\tau b_d)}}{1 - \omega_j}\) is bounded by some constant \(C_3\) for all values \(\omega_j \in (0, 1)\). Therefore
\[
R_1(d, \tau d) \leq (2\tau b_d + C_3) \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \omega_j^{k_j(\tau b_d)} = (2\tau b_d + C_3) \cdot R_0(d, \tau b_d).
\]

(19)

Since \(R_0(d, \tau b_d) \to 0, d \to \infty\), we have \(R_1(d, \tau d) = o(b_d), d \to \infty\). Due to (B), this yields (B’).

We now consider
\[
R_2(d, \tau b_d) = \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \frac{|\ln \omega_j|^2 \omega_j^{k_j(\tau b_d)}}{(1 - \omega_j)^2} \left(k_j(\tau b_d)^2(1 - \omega_j)^2(1 + \omega_j^{k_j(\tau b_d)}) + 2k_j(\tau b_d)(1 - \omega_j)^2(1 + \omega_j^{k_j(\tau b_d)}) + (1 - \omega_j)\omega_j^{k_j(\tau b_d)} + \omega_j^{2k_j(\tau b_d) + 2} + (1 - \omega_j)\omega_j^{k_j(\tau b_d) + 1}\right).
\]

Using the inequality (18),
\[
R_2(d, \tau b_d) = (2\tau b_d)^2 \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \omega_j^{k_j(\tau b_d)} \left(1 + \omega_j^{k_j(\tau b_d)} + 4\tau b_d \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \frac{|\ln \omega_j|^2 \omega_j^{2k_j(\tau b_d) + 2}}{(1 - \omega_j)^2} + \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \frac{|\ln \omega_j|^2 \omega_j^{k_j(\tau b_d) + 1}}{1 - \omega_j}\right).
\]

The function \(\omega_j \mapsto \frac{|\ln \omega_j|^2 \omega_j^{k_j(\tau b_d)}}{1 - \omega_j}\) is bounded by some constant \(C_4\) for all values \(\omega_j \in (0, 1)\). Using this and the same fact about \(\omega_j \mapsto \frac{|\ln \omega_j| \omega_j^{k_j(\tau b_d)}}{1 - \omega_j}\), we obtain
\[
R_2(d, \tau b_d) = 4\tau^2 b_d^2 \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \omega_j^{k_j(\tau b_d)} \left(1 + \omega_j^{k_j(\tau b_d)} + 4\tau b_d \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \omega_j^{2k_j(\tau b_d)} + C_3 \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \omega_j^{k_j(\tau b_d)} + C_4 \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \omega_j^{2k_j(\tau b_d)}\right).
\]

Next, since \(\omega_j^{k_j(\tau b_d)} \leq 1\), we have
\[
R_2(d, \tau b_d) \leq (8\tau^2 b_d^2 + 4\tau b_d + C_3^2 + C_4) \sum_{j=1, \ldots, d: \ln \omega_j \leq \tau b_d} \omega_j^{k_j(\tau b_d)} \leq (8\tau^2 b_d^2 + 4\tau b_d + C_3^2 + C_4) R_0(d, \tau b_d).
\]

(20)
Here $R_0(d, \tau b_d) \to 0$, $d \to \infty$. Hence $R_1(d, \tau d^2) = o(b_d)$, $d \to \infty$. Due to (C), this yields (C').

We now show that (A'), (B'), (C') imply (A), (B), (C). From (A') it follows that $\sup_{d \in \mathbb{N}} R_0(d, \tau b_d) < \infty$ for every $\tau > 0$. By the above inequality, we have

$$\sup_{d \in \mathbb{N}} \frac{1}{b_d^2} R_2(d, \tau b_d) < \infty, \text{ for every } \tau > 0.$$ 

This and (C') yield (16) for every $d \in \mathbb{N}$ and some $\tau_0 > 0$ and $C_0 > 0$. Hence, as we showed above, it follows that $R_0(d, \tau b_d) \to 0$, $d \to \infty$, for every $\tau > 0$. Here we use

$$\sup_{d \in \mathbb{N}} \sum_{j=1, \ldots, d} \omega_j < \infty \text{ for every } \tau > 0,$$

which follows from (A'). Next, from (19) and (20) we correspondingly obtain that $R_1(d, \tau b_d) = o(b_d)$ and $R_2(d, \tau b_d) = o(b_d^2)$, $d \to \infty$, for every $\tau > 0$. These relations for $R_k(d, \tau b_d)$, $k = 1, 2, 3$ and conditions (A'), (B'), (C') give (A), (B), (C).  

\[\square\]

4 Acknowledgments

The work of A. A. Khartov was supported by RFBR–DFG grant 20-51-1204.

References

[1] J. L. Brown, Mean Square truncation error in series expansions of random functions, J. Soc. Indust. Appl. Math., 8 (1960), 1, 28–32.

[2] M. D. Buhmann, Radial Basis Functions: Theory and Implementations, Cambridge University press, Cambridge, 2003.

[3] J. Chen, H. Wang, Average case tractability of multivariate approximation with Gaussian kernels, J. Approx. Theory, 239 (2019), 51–71.

[4] G. E. Fasshauer, F. J. Hickernell, H. Woźniakowski, Average case approximation: convergence and tractability of Gaussian kernels, in L. Plaskota, H. Woźniakowski (Eds.), Monte Carlo and Quasi-Monte Carlo 2010, Springer Verlag, 2012, 329–345.

[5] G. E. Fasshauer, F. J. Hickernell, H. Woźniakowski, On dimension-independent rates of convergence for function approximation with Gaussian kernels, SIAM J. Numer. Analysis, 50 (2012), 247–271.

[6] B. V. Gnedenko, A. N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables, Addison-Wesley, Cambridge, 1954.

[7] T. Hastie, R. Tibshirani, J. Friedman, Elements of Statistical Learning: Data Mining, Inference, and Prediction, second ed., in: Springer Series in Statistics, Springer, New York, 2009.

[8] A. Karol, A. Nazarov, Ya. Nikitin, Small ball probabilities for Gaussian random fields and tensor products of compact operators, Trans. Amer. Math. Soc., 360 (2008), no. 3, 1443–1474.
[9] A. A. Khartov, *Asymptotic analysis of average case approximation complexity of Hilbert space valued random elements*, J. Complexity, **31** (2015), 835–866.

[10] A. A. Khartov, *A simplified criterion for quasi-polynomial tractability of approximation of random elements and its applications*, J. Complexity, **34** (2016), 30–41.

[11] M. A. Lifshits, A. Papageorgiou, H. Woźniakowski, *Average case tractability of non-homogeneous tensor product problems*, J. Complexity, **28** (2012), 539–561.

[12] E. Novak, H. Woźniakowski, *Tractability of Multivariate Problems. Volume I: Linear Information*, EMS Tracts Math. 6, EMS, Zürich, 2008.

[13] E. Novak, H. Woźniakowski, *Tractability of Multivariate Problems. Volume III: Standard Information for Operators*, in: EMS Tracts Math., vol. 18, EMS, Zürich, 2012.

[14] V. V. Petrov, *Limit Theorems of Probability Theory: Sequences of Independent Random Variables*, Oxford Stud. Prob., vol. 4, Clarendon Press, Oxford, 1995.

[15] C. E. Rasmussen, C. Williams, *Gaussian Processes for Machine Learning*, MIT Press, 2006.

[16] K. Ritter, *Average-case Analysis of Numerical Problems*, Lecture Notes in Math. No. 1733, Springer, Berlin, 2000.

[17] B. Schölkopf, A. J. Smola, *Learning with Kernels: Support Vector Machines, Regularization, Optimization, and Beyond*, MIT Press, Cambridge, Massachusetts, 2002.

[18] I. H. Sloan, H. Woźniakowski, *Multivariate approximation for analytic functions with Gaussian kernels*, J. Complexity, **45** (2018), 1–21.

[19] G. W. Wasilkowski, H. Woźniakowski, *Average case optimal algorithms in Hilbert spaces*, J. Approx. Theory, **47** (1986), 17–25.

[20] H. Wendland, *Scattered Data Approximation*, in: Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 2005.

[21] H. Zhu, C. K. I. Williams, R. J. Rohwer, M. Morciniec, *Gaussian Regression and Optimal Finite Dimensional Linear Models*, in C. M. Bishop (Edt.), Neural Networks and Machine Learning, Springer-Verlag, Berlin, 1998, 1–20.