A note on the local theta correspondence for unitary similitude dual pairs

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1. Introduction

The local theta correspondence has been well studied for isometry groups. Meanwhile the theta correspondence for similitude groups is also important and useful. Roberts [Ro] (and others) studied the local theta correspondence (over a non-Archimedean local field) in the orthogonal-symplectic case. In this short note, under the assumption (split): one of the Hermitian and skew-Hermitian spaces is split, we show that the method of [Ro] is also valid for the unitary case. We remark that part of our results may be known to experts, for example, see some works of M. Harris (e.g. [Ha1]). Since we could not find a literature for the statements and proofs of these results, we insist to write this note.

After this note was written, Gan informed the author that [GT, Proposition 3.2] could show that Howe duality conjecture for isometry case implies Howe duality conjecture for similitude case for general dual pairs and the argument of multiplicity-freeness in [Ro] is not needed. Though only the case of quaternionic unitary groups is treated in [GT, Proposition 3.2], it is clear that the argument works generally for other dual pairs.

In Section 2, we recall some notations and definitions that will be used later. In Section 3, we study the problem of splitting metaplectic covers of unitary similitude dual pairs. We show in Proposition 3.2 that this is not always available contrary to the isometry case. We also show that the
preimages of the similitude dual pairs in the metaplectic cover are not always commute (cf. Proposition 3.3). Actually, the assumption (split) is only essentially used in this section. In Section 4, we study the Howe duality for the similitude groups. Our main result is Theorem 4.3.

2. Notations and assumptions

Let $F$ be a $p$-adic field (i.e. a finite extension field of $\mathbb{Q}_p$), and $E$ be a quadratic extension field of $F$. We will always assume $p \neq 2$ in Section 4. Let $\epsilon_{E/F}$ be the quadratic character of $F^\times$ associated to $E$ by the local class field theory. Denote by $\chi \mapsto \delta$ the non-trivial Galois automorphism of $E/F$. Choose an element $\delta \in E^\times$ such that $\delta^2 = \epsilon_{E/F}(\delta)$. Then $\Delta := \delta^2 \in F^\times$ and $\epsilon_{E/F}(\chi) = (\chi, \Delta)_F$, where $(\cdot)_F$ is the Hilbert symbol for $F$. We write $N_{E/F}$ for the norm map from $E^\times$ to $F^\times$. Fix a non-trivial additive character $\psi : F \to \mathbb{C}^\times$, and denote $\eta = \frac{1}{2} \psi$.

Let $(W, (\cdot, \cdot))$ (resp. $(V, (\cdot, \cdot))$) be a non-degenerate skew-Hermitian (resp. Hermitian) space over $E$ with $\dim_E W = n$ (resp. $\dim_E V = m$). Let $H = GU(W)$ and $G = GU(V)$ be the corresponding unitary similitude groups. Recall that $h \in H$ if and only if $h \in GL_E(W)$ and there exists $v \in F^\times$ such that $\langle xh, y \rangle = v \langle x, y \rangle$ for all $x, y \in W$. If $h \in H$, such $v$ is unique and will be denoted by $v(h)$. Thus we have the scale map $\nu : H \to F^\times$. The group $G = GU(V)$ is defined similarly, and we still use $v$ to denote the scale map $G \to F^\times$. Denote $H_1 = U(W)$ and $G_1 = U(V)$, which are the kernels of $\nu$.

Since the distinguished between Hermitian space and skew-Hermitian space is not essential, we will assume that (split): $W$ is split. In other words, there is a complete polarization $W = X + Y$, where $X$ and $Y$ are maximal isotropic subspaces of $W$.

Attached to $(V, W)$, there is a symplectic space $\mathcal{W} = V \otimes_E W$ over $F$, equipped with the symplectic form

$$\langle (\cdot, \cdot) \rangle = \frac{1}{2} \text{Tr}_{E/F}(\langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle).$$

There is a natural embedding

$$\iota : G \times H \to \text{GSp}(\mathcal{W}), \quad (v \otimes w)\iota(g, h) = (g^{-1} v \otimes w h),$$

where $\text{GSp}(\mathcal{W})$ is the symplectic similitude group attached to $\mathcal{W}$ (note that we write the action of $\text{GSp}(\mathcal{W})$ or $\text{Sp}(\mathcal{W})$ on right). We denote by $\iota_V : H \to \text{GSp}(\mathcal{W})$ and $\iota_W : G \to \text{GSp}(\mathcal{W})$ the restrictions of $\iota$ on $H$ and $G$.

Let $\text{Mp}(\mathcal{W})$ be the metaplectic cover of $\text{Sp}(\mathcal{W})$, which is a non-trivial $\mathbb{C}^1$-central extension of $\text{Sp}(\mathcal{W})$. For the fixed additive character $\psi$, let $\omega_\psi$ be the smooth Weil representation of $\text{Mp}(\mathcal{W})$.

For convenience, we recall some notations used for the (strong) Howe duality (cf. [Ro1]). Let $A$ and $B$ be two groups of $td$-type with countable bases, and $(\rho, S)$ be a smooth representation of $A \times B$. For a group $J$ of $td$-type with countable bases, denote by $\text{Irr}(J)$ the set of equivalence classes of irreducible admissible representations of $J$. For $\pi \in \text{Irr}(A)$, put $S(\pi) = S / \bigcap_{f \in \text{Hom}_A(\rho, \pi)} \text{Ker}(f)$. It is a smooth representation of $A \times B$, and there exists a smooth representation $\Theta(\pi)$ of $B$, unique up to isomorphism, such that $S(\pi) \simeq \pi \otimes \Theta(\pi)$ (cf. [MVW]). Denote by $\mathcal{R}(A)$ the set of equivalence classes of $\pi \in \text{Irr}(A)$ such that $S(\pi) \neq 0$, and denote $\mathcal{R}(B)$ analogously. We say that strong Howe duality holds for $\rho$ if for every $\pi \in \mathcal{R}(A)$ the representation $\Theta(\pi)$ has a unique nonzero irreducible quotient $\theta(\pi)$, and analogous statement holds for every $\tau \in \mathcal{R}(B)$. We say that Howe duality holds for $\rho$ if the set $\{ (\pi, \tau) \in \mathcal{R}(A) \times \mathcal{R}(B) : \text{Hom}_{A \times B}(\rho, \pi \otimes \tau) \neq 0 \}$ is the graph of a bijection between $\mathcal{R}(A)$ and $\mathcal{R}(B)$. It is known that strong Howe duality implies Howe duality.

3. The splitting

Notice that we assume $W$ is split with $\dim_E W = n = 2r$, that is there is a complete polarization $W = X + Y$. Let $X = V \otimes_E X$ and $Y = V \otimes_E Y$. Then $X + Y$ is a complete polarization of $\mathcal{W}$. If necessary we write the elements of $H$ or $\text{GSp}(\mathcal{W})$ as matrices with respect to these polarizations.
Let $\text{GMP}(\mathbb{W})$ be the metaplectic cover of $\text{GSp}(\mathbb{W})$ (cf. [Ba]), which is an extension of $\text{GSp}(\mathbb{W})$ by $\mathbb{C}^1$:

$$1 \to \mathbb{C}^1 \to \text{GMP}(\mathbb{W}) \xrightarrow{\text{pr}} \text{GSp}(\mathbb{W}) \to 1.$$ 

Then $\text{GMP}(\mathbb{W}) \simeq \text{GSp}(\mathbb{W}) \times \mathbb{C}^1$ as sets, and the group law can be written as

$$(g, z) \cdot (g', z') = (gg', C(g, g')zz'),$$

where $g, g' \in \text{GSp}(\mathbb{W}), z, z' \in \mathbb{C}^1$, and $C: \text{GSp}(\mathbb{W}) \times \text{GSp}(\mathbb{W}) \to \mathbb{C}^1$ is the cocycle function. The cocycle $C$ can be computed explicitly as follows (cf. [Ba, §1.2]). There is an action of $\mathbb{F}^\times$ on $\text{Sp}(\mathbb{W})$ defined by $s^y = d(y)^{-1}sd(y)$, where $s \in \text{Sp}(\mathbb{W}), y \in \mathbb{F}^\times$ and $d(y) = \begin{pmatrix} 1 & 0 \\ 0 & y1_m \end{pmatrix}$. For $g \in \text{GSp}(\mathbb{W})$, denote $g_1 = d(v(g))^{-1}g \in \text{Sp}(\mathbb{W})$. Then we have

$$C(g, g') = c_{\mathbb{V}}(g^{-1}(g'), g_1)\mu(v(g'), g_1), \quad (3.1)$$

where $c_{\mathbb{V}}(g_1, g') = \gamma_F(\eta \circ L(\mathbb{V}, g_1^{-1}, g_1))$, and $\mu: \mathbb{F}^\times \times \text{Sp}(\mathbb{W}) \to \mathbb{C}^1$ is the functions defined by

$$\mu(y, s) = \left(x(s), y\right)_F \gamma_F(y, \eta)j(s). \quad (3.2)$$

For convenience we recall some of the notations appeared above. For the triple of Lagrangians $(\mathbb{V}, \mathbb{V}g_1^{-1}, \mathbb{V}g_1)$, $L(\mathbb{V}, \mathbb{V}g_1^{-1}, \mathbb{V}g_1)$ is its Leray invariant, which is a quadratic space over $\mathbb{F}$ (cf. [Ra] for the definitions). For a quadratic space $q$ over $\mathbb{F}$ and an additive character $\eta, \gamma_F(\eta \circ q)$ is the Weil index associated to $q$ and $\eta$. For $y \in \mathbb{F}^\times$, $\gamma_F(y, \eta)$ is defined to be $\frac{\gamma_F(y\eta x^2)}{\gamma_F(yx^2)}$, where $y\eta(x) := \eta(yx)$. Let $\{e_1, \ldots, e_{mn}, e_1^*, \ldots, e_{mn}^*\}$ be a basis of $\mathbb{W}$ such that $\{e_1, \ldots, e_{mn}\} \subset X, \{e_1^* \ldots, e_{mn}^*\} \subset \mathbb{V}, \{e_i, e_j\} = \{e_i^*, e_j^*\} = 0$ and $\{e_i, e_j^*\} = \delta_{ij}$ for $1 \leq i, j \leq mn$. Let $P_{\mathbb{V}}$ be the parabolic subgroup of $\text{Sp}(\mathbb{W})$ stabilizing $\mathbb{V}$. Then

$$\text{Sp}(\mathbb{W}) = \bigsqcup_{j=0}^{mn} P_{\mathbb{V}}\tau_j P_{\mathbb{V}},$$

where $\tau_j$ is defined to be

$$e_i \cdot \tau_j = -e_i^*, \quad \text{and} \quad e_i^* \cdot \tau_j = e_i, \quad \text{if} \quad i \in \{1, \ldots, j\}, \quad e_i \cdot \tau_j = e_i - e_i^*, \quad \text{and} \quad e_i^* \cdot \tau_j = e_i^*, \quad \text{otherwise}, \quad (3.3)$$

and $\tau_0 := \text{id}$. Thus for $s = p_1 \tau_j p_2 \in \text{Sp}(\mathbb{W})$ with $p_1, p_2 \in P_{\mathbb{V}}$, define $x(s) := \det(p_1p_2|_{\mathbb{V}})$ in $\mathbb{F}^\times/\mathbb{F}^{\times 2}$ and $j(s) := j$.

Now we study the preimages $\text{pr}^{-1}(G)$ and $\text{pr}^{-1}(H)$ in $\text{GMP}(\mathbb{W})$. For this, we need some preparations. Let $\{e_1, \ldots, e_r, e_1^*, \ldots, e_r^*\}$ be a basis of $W$ such that $\{e_1, \ldots, e_r\} \subset X, \{e_1^* \ldots, e_r^*\} \subset \mathbb{V}, \{e_i, e_j\} = \{e_i^*, e_j^*\} = 0$ and $\{e_i, e_j^*\} = \delta_{ij}$ for $1 \leq i, j \leq r$. Let $P_{\mathbb{V}}$ be the parabolic subgroup of $H_1$ stabilizing $\mathbb{V}$. Then we have Bruhat decomposition $H_1 = \bigsqcup_{j=0}^{r} P_{\mathbb{V}}\tau_j P_{\mathbb{V}}$, where $\tau_j$ is defined in the same way as (3.3). For $h = p_1 \tau_j p_2 \in H_1$ with $p_1, p_2 \in P_{\mathbb{V}}$, define $x(h) := \det(p_1p_2|_{\mathbb{V}})$ in $\mathbb{E}^\times/\mathbb{E}^{\times 2}$ and $j(h) := j$. Recall that, fixing a character $\chi: \mathbb{E}^\times \to \mathbb{C}^\times$ such that $\chi|_{\mathbb{E}^\times} = e_{\mathbb{E}/\mathbb{F}}^m$, there is a splitting homomorphism (cf. [Ku, §3])

$$\iota_{\mathbb{V}, \chi}: H_1 \to \text{Mp}(\mathbb{W}) \simeq \text{Sp}(\mathbb{W}) \times \mathbb{C}^1, \quad h \mapsto \iota_{\mathbb{V}, \chi}(h) = (\iota_{\mathbb{V}}(h), \beta_{\mathbb{V}, \chi}(h)).$$
where $\beta_{V,X}(h) = \chi(x(h))\gamma_F(\eta \circ RV)^{-j(h)}$ and $RV$ is the underlying $2m$-dimensional $F$ vector space with quadratic form $\frac{1}{2}\text{Tr}_{E/F}(.)$. Recall that we have the relation:

$$c_Y\left(i_V(h), i_V(h')\right) = \beta_{V,X}(h)^{-1}\beta_{V,X}(h')^{-1}\beta_{V,X}(hh'), \quad h, h' \in H_1. \quad (3.4)$$

Meanwhile, there is a natural splitting

$$\tilde{\iota}_W : G_1 \rightarrow \text{Mp}(\mathbb{W}) \simeq \text{Sp}(\mathbb{W}) \times \mathbb{C}^1, \quad g \mapsto (\iota_W(g), 1).$$

Similarly as $G\text{Sp}(\mathbb{W})$, for $y \in F^\times$ and $h \in H_1$, define

$$d(y) = \left(\begin{array}{cc} 1_r & 0 \\ 0 & y \cdot 1_r \end{array}\right), \quad \text{and} \quad h^y = d(y)^{-1}hd(y) \in H_1.$$

For $h \in H$, denote $h_1 = d(v(h))^{-1}h \in H_1$. The following lemma will be used in the proof of Proposition 3.2. It can be checked by linear algebra, and we omit the proof.

**Lemma 3.1.** For $h \in H_1$ and $y \in F^\times$, we have

1. $x(h^y) = x(h)^{y^{j(h)}}$ and $j(h^y) = j(h)$;
2. $x(\iota_V(h)) = N_{E/F}(x(h))m \cdot (-\Delta)^{mj(h)}$ and $j(\iota_V(h)) = 2mj(h)$.

**Proposition 3.2.** As an extension of $H$, $pr^{-1}(H)$ is trivial if $m$ is even. As an extension of $G$, $pr^{-1}(G)$ is trivial.

**Proof.** For $h \in H$, it is easy to see that $\iota_V(h)_1 = \iota_V(h_1)$ and $v(h) = v(\iota_V(h))$. For $h, h' \in H$, by (3.1) and (3.4), we have

$$C(\iota_V(h), \iota_V(h')) = c_Y(\iota_V(h_1)^{v(h')}, \iota_V(h'_1)) \cdot \mu(v(h'), \iota_V(h_1))$$

$$= c_Y(\iota_V(h_1^{v(h')}), \iota_V(h'_1)) \cdot \mu(v(h'), \iota_V(h_1))$$

$$= \beta_{V,X}(h_1^{v(h')})^{-1}\beta_{V,X}(h'_1)^{-1}\beta_{V,X}(h_1^{v(h')}) \cdot \mu(v(h'), \iota_V(h_1)).$$

Since

$$h_1^{v(h')}h'_1 = d(v(h'))^{-1}d(v(h'))^{-1}hd(v(h')) \cdot d(v(h'))^{-1}h' = (hh')_1,$$

we have $\beta_{V,X}(h_1^{v(h')}) = \beta_{V,X}((hh')_1)$. On the other hand, by (3.2) and Lemma 3.1, we have

$$\mu(v(h'), \iota_V(h_1)) = (x(\iota_V(h_1)), v(h'))_{\gamma_F}(v(h'), \eta)^{j(\iota_V(h_1))}$$

$$= (N_{E/F}(x(h_1)), v(h'))_{\gamma_F}(\Delta, v(h'))_{\gamma_F}(v(h'), \eta)^{2mj(h_1)}$$

$$= (N_{E/F}(x(h_1)), v(h'))_{\gamma_F}(\Delta, v(h'))_{\gamma_F}(v(h'), \eta)^{mj(h_1)},$$

and
\[ \beta_{V, \chi}(h_1^{(h')}) = \chi(x(h_1^{(h')})) \gamma_F(\eta \circ RV)^{-j(h_1^{(h')})} \]

\[ = \chi(x(h_1)) \gamma_F(\eta \circ RV)^{-j(h_1)} \]

\[ = \beta_{V, \chi}(h_1)(v(h'), \Delta)^{ mj(h_1)} \cdot \varepsilon_{E/F}(v(h')) \]

Thus we have

\[ \beta_{V, \chi}(h_1^{(h')})^{-1} \mu(v(h'), \iota_V(h_1)) = \beta_{V, \chi}(h_1)^{-1}(N_{E/F}(x(h_1)), v(h'))^m. \]

It follows that

\[ C(\iota_V(h), \iota_V(h')) = \beta_{V, \chi}(h_1)^{-1} \beta_{V, \chi}(h_1')^{-1} \beta_{V, \chi}((hh')_1) \cdot (N_{E/F}(x(h_1)), v(h'))^m. \]

Therefore \( pr^{-1}(H) \) is a trivial \( \mathbb{C}^1 \)-extension if \( m \) is even.

For \( g \in G \), it is easy to see that \( v(\iota_W(g)) = v(g)^{-1} \) and \( x(\iota_W(g)_1) = N_{E/F}(\det g)^r \). For \( g, g' \in G \), we have

\[ C(\iota_W(g), \iota_W(g')) = c_{\mathbb{Z}}(\iota_W(g)_1^{(g')^{-1}}, \iota_W(g')_1) \cdot \mu(v(g')^{-1}, \iota_W(g)_1) \]

\[ = \mu(v(g')^{-1}, \iota_W(g)_1) = (v(g')^{-1}, N_{E/F}(\det g)^r)_F \]

\[ = (v(g), v(g'))^{mr}_F, \]

since \( \iota_W(g)_1 \in \mathbb{P}_Y \) and \( N_{E/F}(\det g) = v(g)^m \). Then the conclusion follows from the relation

\[ (v(g), v(g'))_F = \gamma_F(v(g), \eta)^{-1} \gamma_F(v(g'), \eta)^{-1} \gamma_F(v(gg'), \eta). \]

\[ \square \]

**Proposition 3.3.** \( pr^{-1}(H) \) and \( pr^{-1}(G) \) commute in \( \text{GMP}(\mathbb{W}) \) if and only if \( m \) is even.

**Proof.** For \( g \in G \) and \( h \in H \), if \( \text{pr} \tilde{g} = g \) and \( \text{pr} \tilde{h} = h \), we can write \( \tilde{g} = (\iota_W(g), z) \) and \( \tilde{h} = (\iota_V(h), z') \). It is easy to see that the commutator of \( \tilde{g} \) and \( h \) is

\[ [\tilde{g}, \tilde{h}] = \frac{C(\iota_W(g), \iota_V(h))}{C(\iota_V(h), \iota_W(g))}. \]

Then we have

\[ C(\iota_W(g), \iota_V(h)) = c_{\mathbb{Z}}(\iota_W(g)_1^{(h)}, \iota_V(h)_1) \mu(v(h), \iota_W(g)_1) \]

\[ = (N_{E/F}(\det(g)), v(h))^{r}_F \]

\[ = (v(g), v(h))^{mr}_F, \]

and
Propositions 3.2 and 3.3, as in the previous section, we have the Weil representation construction (cf. [Ro, §3]).

above method of constructing representation is not available. For general $

\begin{align*}
C(t_\psi(h), t_\omega(g)) &= c_\psi(t_\psi(h_1)^{\nu(g)^{-1}}, t_\omega(g_1))\mu(\nu(g)^{-1}, t_\psi(h_1)) \\
&= \mu(\nu(g)^{-1}, t_\psi(h_1)) \\
&= (N_{E/F}(x(h_1)), \nu(g)^{-1})_F^m(\Delta, \nu(g)^{-1})_{g(h_1)}^m.
\end{align*}

Hence $pr^{-1}(H)$ and $pr^{-1}(G)$ commute in $\text{GMP}(W)$ if and only if $m$ is even. □

4. Howe duality

Let $\omega_\psi$ be the smooth Weil representation of $\text{Mp}(W)$. Using the fixed splitting homomorphisms as in the previous section, we have the Weil representation $\omega_{\psi, \chi}$ of $G_1 \times H_1$. We simply denote by $\omega$ for $\omega_\psi$ or $\omega_{\psi, \chi}$. Let $\Omega = \text{ind}_{\text{GMP}(W)}\text{G} \omega$ be the compactly induced representation, and recall that there is an one extra variable Schrödinger model $\mathcal{S}(\mathbb{F} \times \mathbb{F}^\times)$ of $\Omega$ (cf. [Ba, §1.3]). If $m = \dim_{\mathbb{F}} V$ is even, by Propositions 3.2 and 3.3, $\Omega$ can be viewed as a representation of $G \times H$. However, when $m$ is odd, the above method of constructing representation is not available. For general $m$, we have the following construction (cf. [Ro, §3]).

Let

$$R = \{(g, h) \in G \times H \mid \nu(g) = \nu(h)\}$$

be a subgroup of $G \times H$ and also a subgroup of $\text{Sp}(W)$. We can extend the Weil representation $\omega|_{G_1 \times H_1}$ to $R$, and still denote it by $\omega$. Let

$$H^+ = \{h \in H \mid \nu(h) \in \nu(G)\}.$$ 

Then $H^+ = H$ when $m$ is even; $[H : H^+] = 2$ when $m$ is odd. Let

$$\Omega^+ = \text{ind}^{G \times H^+}_R \omega$$

be the compactly induced representation, which is a smooth representation of $G \times H^+$.

**Remark 4.1.** If $m$ is even, we thus have two representations $\Omega^+$ and $\Omega$ of $G \times H$. However it can be shown that $\Omega^+ \simeq \Omega$, by the same arguments of [Ro, Proposition 3.5].

**Remark 4.2.** If $W$ (resp. $V$) is an arbitrary skew-Hermitian (resp. Hermitian) space, we can define $H^+$ with respect to $V$ (resp. $G^+$ with respect to $W$), the subgroup $R$ of $G^+ \times H^+$, and the representation $\Omega^+ = \text{ind}_{R}^{G+ \times H^+} \omega$ in the same way.

By such a construction, we can consider the (strong) Howe duality for the representation $\Omega^+$ of $G \times H^+$.

**Theorem 4.3.** If $p > 2$, the strong Howe duality holds for $\Omega^+$.

**Proof.** If $p > 2$, Waldspurger [Wa] proved that the strong Howe duality holds for $\omega|_{G_1 \times H_1}$. The centers of $G$, $H^+$ are both isomorphic to $E^\times$, and there are isomorphisms:

$$G/E^\times G_1 \simeq R/E^\times (G_1 \times H_1) \simeq H^+/E^\times H_1,$$

where we use $E^\times$ to denote the centers of $G$ and $H^+$, and embed $E^\times$ into $R$ diagonally (then $E^\times$ lies in the center of $R$). Notice that the cardinality of the above quotient groups is no more than 2. Thus,
for $\pi \in \text{Irr}(H^+)$, we have either $\pi|_{H_1} \simeq \rho_0$ being irreducible or $\pi|_{H_1} \simeq \rho_1 \bigoplus \rho_2$ where $\rho_i \in \text{Irr}(H_1)$ are pairwise inequivalent. Then it can be checked that the arguments in Sections 4 and 5 of [Ro] can also be applied to our situation, and the conclusion follows. More precisely, for $\pi \in \text{Irr}(H^+)$, if $\pi|_{H_1} \simeq \bigoplus \rho_i$ so that $\rho_i \in \text{Irr}(H_1)$, then $\theta(\pi) = \bigoplus \theta(\rho_i)$ where $\theta(\rho_i) \in \text{Irr}(G_1)$. □

**Remark 4.4.** If $W$ is arbitrary (i.e. not necessary split), applying the above arguments again and noting the key relation $G^+/E^*G_1 \simeq R/E^*(G_1 \times H_1) \simeq H^+/E^*H_1$, we can show that if $p > 2$ then the (strong) Howe duality holds for $\Omega^+$.  

As in [Ro, §6], when $m$ is odd, we can also consider the (strong) Howe duality for the compactly induced representation

$$\tilde{\Omega} = \text{ind}_R^{G \times H} \omega.$$  

As same as the orthogonal-symplectic case, the (strong) Howe duality for $\tilde{\Omega}$ is equivalent to the theta dichotomy for unitary isometry dual pairs. We explain this phenomenon briefly. Up to isometry, there are two different Hermitian spaces over $E$ of dimension $m \geq 1$: $V^\pm$ defined by

$$\epsilon(V^\pm) = \epsilon_{E/F}((-1)^{\frac{m(m-1)}{2}} \det V^\pm) = \pm 1.$$  

To avoid confusion, we write $R(H_1, V^\pm)$ to emphasize its dependence on $V^\pm$. Applying the same arguments of [Ro, Lemma 6.1], we have the following principle for the (strong) Howe duality for $\tilde{\Omega}$.

**Proposition 4.5.** Assume $p > 2$. Then the (strong) Howe duality holds for $\tilde{\Omega}$ if and only if $R(H_1, V^+) \cap R(H_1, V^-) = \emptyset$.

**Remark 4.6.** It is conjectured in [HKS] that for $m \leq n$,

$$R(H_1, V^+) \cap R(H_1, V^-) = \emptyset,$$

which is called theta dichotomy. Theta dichotomy has been proved (cf. [Ha2, Theorem 2.1.7]) under the assumption of the so-called “weak conservation relation”. Recently, Sun and Zhu [SZ] proved the so-called “conservation relation” for all type I irreducible dual pairs and all local fields of characteristic zero. In summary, theta dichotomy has been proved. It is also known that for $m \geq 2n$,

$$R(H_1, V^+) \cap R(H_1, V^-) \neq \emptyset.$$  

Thus the Howe duality does not hold for $\tilde{\Omega}$ when $m \geq 2n$.

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