A GREY RADIATIVE TRANSFER PROCEDURE
FOR $\gamma$-RAY TRANSFER IN SUPERNOVAE

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ABSTRACT

The $\gamma$-ray transfer in supernovae for the purposes of energy deposition in the ejecta can be approximated fairly accurately as frequency-integrated (grey) radiative transfer using a mean opacity as shown by Swartz, Sutherland, & Harkness (SSH). In SSH’s grey radiative transfer procedure (unoptimized) the mean opacity is a pure absorption opacity and it is a constant aside from a usually weak composition dependence. The SSH procedure can be optimized by using a fitted constant mean absorption opacity for the whole supernova at a given time. The optimum value of their mean opacity (which depends on the overall composition and optical depth of the supernova) is obtained by fitting to more accurate Monte Carlo calculations. No fitting is needed in the optically thick limit and (not counting fine adjustments to account for time-dependent and non-static radiative transfer effects) in the optically thin limit.

In this paper, we present a variation on the SSH procedure which uses multiple mean opacities which do not need to be obtained by fitting and which have both absorption and scattering components. There is a mean opacity for each order of Compton scattering. (Compton scattering is the dominant form of $\gamma$-ray opacity in supernovae.) The zeroth order $\gamma$-ray field (i.e., the direct field from the nuclear decay) is calculated numerically as in the SSH procedure. The scattered (i.e., nonzeroth order) $\gamma$-ray fields at a point are calculated by assuming that the scattered $\gamma$-ray source functions at that point (i.e., local to that point) can be used for the whole of the ejecta. This local-state (LS) approximation permits an analytic solution for the $\gamma$-ray transfer of scattered $\gamma$-ray fields. The LS approximation is admittedly crude, but the scattered fields are always of lesser importance to the energy deposition. Since the LS approximation is, however, the distinguishing mark of our procedure, we call our procedure the LS grey radiative transfer procedure or LS procedure for short.

Besides the LS approximation we also need to approximate angle-dependent Compton opacity for the analytic solution of the scattered $\gamma$-ray fields. We call the approximated Compton opacity the iso-Compton opacity.

We give only a limited test of the accuracy of our procedure. For a standard Type Ia supernova (SN Ia) model the uncertainty in $\gamma$-ray energy deposition is estimated to be of order 10% or less. This level of accuracy is often adequate since the deposition is used in spectral synthesis calculations which have uncertainties of the same order from the atomic data used.

Since finding the optimum SSH mean opacity requires doing the detailed (e.g., Monte Carlo) radiative transfer one wants to avoid in using a simplified $\gamma$-ray energy deposition procedure, the LS procedure may be the best choice for that simplified procedure. The extra effort in developing and running an LS procedure code beyond that of an SSH procedure code is small.

The LS procedure code used for this paper can be obtained by request from the author. This code (excluding comment lines, auxiliary subroutines, and data statements) is about 260 lines long.

For completeness and easy reference, we include in this paper a review of the $\gamma$-ray opacities important in supernovae, a discussion of the appropriate mean opacity prescription, and a discussion of the errors arising from neglecting time-dependent and non-static radiative transfer effects.
1. INTRODUCTION

The decay of radioactive elements synthesized in supernova explosions is one of the most important sources of the energy driving observable supernova luminosity. For Type Ia supernovae (SNe Ia) (e.g., Colgate, Petschek, & Kriese 1980; Harkness 1991) and probably Type Ic supernovae (e.g., Young, Baron, & Branch 1995), almost all the observed luminosity comes from radioactive decay. For the other supernova types, radioactive decay drives much, and probably in most cases most, of the observable luminosity in the nebular epoch or even earlier as suggested by Type II supernova SN 1987A (e.g., Woosley 1988) and Type IIb supernova SN 1993J (e.g., Young et al. 1995). The overwhelmingly dominant decay chain for the observable epoch of most supernovae is $^{56}\text{Ni} \rightarrow ^{56}\text{Co} \rightarrow ^{56}\text{Fe}$ with half-lives of 5.9 and 77.27 days for the first and second decays, respectively (Huo 1992). The first decay releases almost all its energy in the form of $\gamma$-rays and the second in $\gamma$-rays and, in 19% of the decays, in positrons (Browne & Firestone 1986; Huo 1992). The $\gamma$-ray and mean positron kinetic energy are in the range $\sim 0.15$–$3.6\text{MeV}$. The longer lived radioactive species $^{57}\text{Co}$ and $^{44}\text{Ti}$ are likely to become important only after about day 800 after explosion as suggested by the observations of SN 1987A (e.g., Nomoto et al. 1994, p. 546ff) and tests we have done for SNe Ia. Very few supernovae are observed so late and by then other energy sources such as circumstellar interaction or pulsar remnants may have become important also (e.g., Fransson 1994, p. 731ff), except probably for SNe Ia.

After earliest and usually unobserved times, the $\gamma$-rays and positrons deposit energy in a relatively cold ($T \lesssim 10^4\text{K}$), low-ionization state or nearly neutral supernova medium. This deposited energy is effectively mainly in the form of fast electrons from Compton scattering, photoionization, and positron collisions. The fast electrons ionize and excite atoms and heat the gas through electron collisions. The fast electron energy gets transformed into other forms of energy by a complicated cascade process that is described by, e.g., Fransson (1994, p. 688ff) and Liu & Victor (1994).

The dispersion of the decay energy through the supernova is by $\gamma$-ray radiative transfer and possibly by the positrons and the fastest fast electrons which both lose kinetic energy at an increasing rate as they are slowed down. The simplest assumption is that the positrons and fast electrons do not disperse their kinetic energy, but only deposit it locally. This would certainly be the case if there are tangled magnetic fields of even weak strength in the supernovae (e.g., Colgate et al. 1980; Chan & Lingenfelter 1993). Unfortunately, little is certain about the magnetic fields in supernovae and it may be that they are radially combed out in which case considerable energy transport via positrons and some of the fast electrons may occur. It is at least certain that the positrons must be much more trapped than the $\gamma$-rays since theoretical SN Ia light curves and absolute spectra at late times require positron kinetic energy deposition to dominate $\gamma$-ray energy deposition in order to match observations (e.g., Liu, Jeffery, & Schultz 1997a, b). Positron transport may be an important process and it is being actively investigated (e.g., Colgate et al. 1980; Chan & Lingenfelter 1993; Ruiz-Lapuente 1997; Milne, The, & Leising 1997; Ruiz-Lapuente & Spruit 1998), but it is outside of the scope of the present paper.

We note that the positrons (unless they escape the supernova altogether) will annihilate with electrons either by a three-continuum-photon process from a triplet positronium state or by a two-$m_e c^2$-photon process from a singlet positronium state or directly with free or bound electrons (e.g., Brown & Leventhal 1987). The positrons probably lose most of their kinetic energy before annihilation—it effectively goes into fast electron energy—and their rest mass energy and the rest mass energy of the annihilated electrons become part of the $\gamma$-ray flux. We assume in this paper that the positrons are completely locally trapped so that the annihilation $\gamma$-rays can be treated on the same footing as the $\gamma$-rays coming directly from the radioactive decay.
The $\gamma$-ray transfer, which is the topic of the present paper, can be treated to high accuracy by Monte Carlo calculations in which all important physical processes can be handled in detail. For the calculation of the spectra of $\gamma$-rays that escape the ejecta treatments at least as detailed as Monte Carlos are probably necessary. Such Monte Carlo spectrum calculations have been done by, e.g., Ambwani & Sutherland (1988), Höflich, Khokhlov, & Müller (1992), Kumagai et al. (1993), Ruiz-Lapuente et al. (1993), Gómez-Gomar, Isern, & Jean (1998) and Höflich, Wheeler, & Khokhlov (1998). Another high accuracy approach to $\gamma$-ray transfer would be to use the comoving frame frame formalism including all special relativistic effects (Mihalas 1980a) and angle and frequency partial redistribution effects (Mihalas 1980b). A possible third approach would be to do an observer frame calculation including all special relativistic and angle and frequency partial redistribution effects. In this third approach, the source functions for scattered $\gamma$-ray fields could be obtained by a $\Lambda$-iteration (e.g., Mihalas 1978, p. 147ff). In supernova $\gamma$-ray transfer case, the $\Lambda$-iteration is expected to converge since the zeroth order field source functions are specified and scattering is only locally important until the ejecta is optically thin. Like the Monte Carlo approach, this third approach could be generalized to three-dimensional $\gamma$-ray transfer. Time-dependent effects could probably be handled in the third approach as well.

The detailed accurate $\gamma$-ray transfer procedures (which can be computationally intensive and/or difficult to code) are not, however, needed to obtain fairly accurately the $\gamma$-ray energy deposited in the ejecta in fast electron energy. A simple frequency-integrated (grey) radiative transfer procedure can do this (Colgate et al. 1980; Sutherland & Wheeler 1984; Ambwani & Sutherland 1988; Swartz, Sutherland, & Harkness 1995, hereafter SSH). In such procedures, a mean opacity replaces the frequency-specific opacity needed in detailed (frequency-dependent) radiative transfer procedures. The basic fortran code for the grey procedures is straightforward to write and requires no more than of order 100 lines (Sutherland 1996). A grey radiative transfer calculation typically requires of order $10^{-5}$ of the computational time of a Monte Carlo calculation (SSH) and the result, the amount of energy deposited, is actually very useful since the fast electron cascade process is quite insensitive to the spectrum of the primary fast electrons (e.g., Liu & Victor 1994). Given the $\gamma$-ray energy deposition and the deposition of the positron kinetic energy, the thermal state of the ejecta and its ultraviolet-optical-infrared (UV-optical-IR) emission can be calculated (e.g., Axelrod 1980; Ruiz-Lapuente 1997; Liu et al. 1997a, b, c).

A key point for grey radiative transfer calculations is the choice of the mean opacity. In earlier work (Colgate et al. 1980; Sutherland & Wheeler 1984; Ambwani & Sutherland 1988), the mean opacity was a pure absorption opacity and a single mean opacity value was suggested for all locations in supernova ejecta and all supernova epochs. In the unoptimized version of the grey $\gamma$-ray transfer procedure of SSH (hereafter the SSH procedure), the mean opacity is a pure absorption opacity and it is a constant aside from a usually weak composition dependence. For this opacity SSH suggest $\kappa = 0.06/\mu_e \text{cm}^2 \text{g}^{-1}$ for most cases. The mean atomic mass per electron $\mu_e$ accounts for the main composition dependence (see §2): for hydrogen dominated matter, $\mu_e$ is about 1; for metal dominated matter, it is about 2.

The SSH procedure can be optimized by using a fitted constant mean absorption opacity for the whole supernova at a given time. This mean opacity value (which depends on the overall composition and optical depth of the supernova) is obtained by fitting to more accurate Monte Carlo calculations. No fitting is needed in the optically thick limit and (not counting fine adjustments to account for time-dependent and non-static radiative transfer effects) in the optically thin limit.

In the SSH procedure, the actual $\gamma$-ray transfer is done by a numerical integration solution of the radiative transfer equation. For the supernova model examined by SSH (SN Ia model W7 [Thielemann, Nomoto, & Yokoi 1986]), the optimized SSH procedure obtained an accuracy in energy deposition of a few percent locally and 2% globally.

The level of accuracy obtained by optimized and also the unoptimized SSH procedure is often adequate, particularly for preliminary calculations, since the energy deposition is used in spectral synthesis calculations which have uncertainties of the same order (e.g., $\sim 10\%$) from the atomic data used. Published calculations
using the unoptimized SSH procedure have been done by, e.g., Houck & Fransson (1996) and Liu et al. (1997a, b, c).

It is probable that accuracy in unoptimized SSH procedure calculations close to that from optimized ones can be obtained using mean opacity values that have been estimated based on past Monte Carlo calculations (see § 6). Nevertheless, it would be more satisfactory to have a procedure which would guarantee reasonable results without adjusting a free parameter and which would yield further physical insight. In this paper we present a variation on the SSH procedure that does this and that uses multiple mean opacities with both absorption and scattering components. This procedure adapts to optical depth conditions, and so no fitting is required. (The composition of the supernova model is known a priori, and so no fitting for composition dependence is needed either.)

In our procedure, there is a mean opacity for each order of Compton scattering. (Compton scattering is the dominant form of \( \gamma \)-ray opacity in supernovae: see § 2). The zeroth order \( \gamma \)-ray field (i.e., the direct field from the nuclear decay) is calculated numerically as in the SSH procedure. The scattered (nonzeroth order) \( \gamma \)-ray fields at a point are calculated by assuming that the scattered \( \gamma \)-ray source functions at that point (i.e., local to that point) can be used for the whole of the ejecta. This local-state (LS) approximation permits an analytic solution for the \( \gamma \)-ray transfer of scattered \( \gamma \)-ray fields. The LS approximation is admittedly crude, but the scattered fields are always of lesser importance to the energy deposition. Since the LS approximation, however, is the distinguishing mark of our procedure, we call our procedure the LS grey radiative transfer procedure or LS procedure for short. A brief presentation of the LS procedure is given by Jeffery (1998).

The LS procedure, like the SSH procedure, treats the \( \gamma \)-ray transfer as occurring in a time-independent, static medium. Note, however, that in the SSH procedure time-dependent and non-static effects can be absorbed into the fitting of the mean opacity and SSH, in fact, do this at least partially (Sutherland 1998).

In § 2 of this paper, we describe the \( \gamma \)-ray opacities relevant to supernovae and introduce what we call the iso-Compton opacity approximation. In § 3, we obtain the mean opacity prescription and mean opacities that we use. The LS procedure is presented in § 4. Section 5 reviews some of the material needed for the treatment of the radioactive sources of energy in supernovae. In § 6, we discuss the adequacy of some of the approximations we make and compare the LS procedure to the SSH procedure. Conclusions are given in § 7. In Appendix A, we discuss the errors arising from the neglect in the LS procedure of the effects of time-dependent and non-static radiative transfer. Appendix B proves some of the mathematical properties of the LS approximation series that we introduce in § 4.

2. \( \gamma \)-RAY OPACITIES IN SUPERNOVAE

To begin we should specify our use of the term opacity. There are two common usages. The first usage is for the inverse of the mean free path; this quantity is also called the extinction (e.g., Mihalas 1978, p. 607). The second usage, which we adopt here, is the extinction divided by density. This usage is much more convenient when discussing \( \gamma \)-ray transfer in supernovae since in this case this kind of opacity is almost entirely independent of density and depends almost entirely on composition. In supernovae, density varies with location and time by many orders of magnitude.

For the case of supernova \( \gamma \)-rays, the opacity can be divided into absorption and scattering components. The former treats the transformation of \( \gamma \)-ray energy into some other form which for our case is effectively fast electron kinetic energy. The scattering component treats the transformation of \( \gamma \)-rays into other \( \gamma \)-rays. The sum of the absorption and scattering opacities is the total opacity or simply the opacity. Throughout this paper we will use the superscript \( R \) as a variable that replaces a symbol designating a quantity as related to total (blank), absorption ("a"), or scattering ("s") opacity: e.g., the general symbol for opacity (second usage) \( \kappa^R \) stands for \( \kappa \), \( \kappa^a \), or \( \kappa^s \). The relation between opacity for a particular particle and the particle
cross section $\sigma^R$ is given by

$$\kappa^R = \frac{n}{\rho} \sigma^R,$$

(1)

where $n$ is the particle density and $\rho$ is (mass) density.

In the energy range 0.05–50 MeV, $\gamma$-rays interact with matter principally through three processes: (1) pair production in the Coulomb field of a nucleus or an electron, (2) the photoelectric effect with bound electrons (which is just $\gamma$-ray photoionization of an atom or ion), and (3) Compton scattering off electrons (e.g., Davison 1965, p. 37). Almost all the $\gamma$-rays from the $^{56}\text{Ni} \to ^{56}\text{Co} \to ^{56}\text{Fe}$ decay chain and other decay chains important in supernovae lie in the energy range 0.05–50 MeV and no $\gamma$-rays exceed $\sim 3.6$ MeV in fact (Browne & Firestone 1986; Huo 1992). Thus the three mentioned processes determine $\gamma$-ray opacity in supernovae (see also SSH’s Fig. 1).

For the pair production opacity, one can use

$$\kappa_{\text{pair}} = \frac{\sigma_{\text{pair}}^*}{m_{\text{amu}}} \sum_i X_i \frac{Z_i^2}{A_i},$$

(2)

where the sum is over all the elements, $X_i$ is the mass fraction an element, $A_i$ is the element’s atomic mass, $Z_i$ is the nuclear charge of the element, $m_{\text{amu}}$ is the atomic mass unit, and $\sigma_{\text{pair}}^*$ is the atomic pair production cross section divided by $Z_i^2$. An expression for $\sigma_{\text{pair}}^*$ (adapted from Hubbell 1969) is

$$\sigma_{\text{pair}}^* = 10^{-27} \times \begin{cases} 0, & E < 2m_e c^2; \\ 0.10063 \times (E - 2m_e c^2), & 2m_e c^2 \leq E < 1.5 \text{ MeV}; \\ 0.0481 + 0.301 \times (E - 1.5), & E \geq 1.5 \text{ MeV}, \end{cases}$$

(3)

where $\sigma_{\text{pair}}^*$ is in cm$^2$, $m_e$ is the electron mass, and $E$ is the $\gamma$-ray energy measured in MeV. (Note the constant of the third case of equation (3) could be changed 0.048101329 to ensure better continuity for $\sigma_{\text{pair}}^*$ although there is no change in physical accuracy.) We note that the positron created in pair production will annihilate to form $\gamma$-rays (see also § 1). Thus in an effective sense, especially with assumption of time independence, pair production opacity can be regarded as having a scattering component. We assume that the positron loses all of its kinetic energy before annihilation, and thus the pair production absorption opacity is given by

$$\kappa_{\text{pair}}^a = \kappa_{\text{pair}} \left( \frac{E - 2m_e c^2}{E} \right)$$

(4)

and the pair production scattering opacity by

$$\kappa_{\text{pair}}^s = \kappa_{\text{pair}} \frac{2m_e c^2}{E}.$$ 

(5)

For the development of the LS procedure formalism (see § 3.2), we assume that a pair production scattering always results in two $m_e c^2$ $\gamma$-rays and not in three $\gamma$-rays with a continuum of energies. This assumption introduces negligible error because it turns out that pair production opacity is of very small importance (see § 3.2) The angular redistribution of pair production scattering is probably very isotropic because of the complicated path the positron will take in slowing down if for no other reason. We assume that it is completely isotropic.

We take the photoelectric opacity to be entirely absorption opacity: i.e., we assume any low energy X-rays resulting from a photoelectric effect ionized and excited atom will be locally absorbed eventually
into fast electron energy or the local thermal pool. The photoelectric opacity can be approximated quite accurately in the range 0.01–1 MeV by

$$\kappa_{\text{pe}} = \kappa_{\text{pe}}^* \left( \frac{E}{0.1 \text{ MeV}} \right)^{-3},$$

(6)

where

$$\kappa_{\text{pe}}^* = \frac{1}{m_{\text{amu}}} \left( \sum_i X_i \frac{1}{A_i} \sigma_{\text{pe},i}^{0.1} \right),$$

(7)

(e.g., SSH). The $\sigma_{\text{pe},i}^{0.1}$ values are the photoelectric cross sections of the atoms at 0.1 MeV. To be more accurate, one can construct tables of $\kappa_{\text{pe}}^*$ as a function of energy for different compositions. The cross section data needed to construct $\kappa_{\text{pe}}^*$ values or $\kappa_{\text{pe}}$ tables can be found in, e.g., Veigele (1973).

The Compton opacity is given to a good approximation by

$$\kappa_C = \frac{n_{\text{e, total}}}{\rho} \sigma_C = \frac{\sigma_C^R}{m_{\text{amu}} \mu_e},$$

(8)

where $n_{\text{e, total}}$ is the total electron density counting both free and bound electrons, $\sigma_C^R$ is the Compton cross section, and $\mu_e$ is the mean atomic mass per electron. The expression for $\mu_e$ is

$$\mu_e^{-1} = \sum_i \frac{X_i Z_i}{A_i},$$

(9)

where the sum is again over all elements. The $Z_i$ is again the nuclear charge since we make the assumption that all electrons, free or bound, act as if they were free. This is a good assumption for $\gamma$-rays with energies much larger electron binding energies (e.g., Davisson 1965, p. 49). The fact that bound electrons are spatially concentrated about atoms makes no difference. The effect of any one electron is minute, and so the effects of all the electrons in an atom just add linearly. In the supernova case, the $\gamma$-rays after several scatterings have lost most of their energy, but nevertheless still mostly have energies much larger than electron binding energies. (The electron binding energies for important atoms are $\lesssim 0.01$ MeV [e.g., Veigele 1973]. For mean $\gamma$-ray energies for the first 5 orders of scattering [as our approximate treatment gives them] see § 3.2, Tables I and II. Recall also that the supernova medium we consider is in a low-ionization or nearly-neutral state, and so the problem of tightly bound electrons in highly-charged ions does not arise.) Thus the error in assuming all electrons act as if they were free is small. Additionally, in the metal-rich compositions such as those of SNe Ia and the deep interior of the other supernova types, the photoelectric opacity can dominate for energies below $\sim 0.1$ MeV (see SSH’s Fig. 1), and thus in these cases the error in using Compton opacity for bound electrons is smaller still.

Below we give the Compton opacity formulae which have been adapted from Davisson (1965, p. 51ff). Note that Compton opacity has both absorption and scattering components since Compton scattering is not coherent (i.e., not elastic or energy-conserving). The Compton total cross section is given by

$$\sigma_C = \sigma_e \left( \frac{3}{4} \right) \left\{ \frac{1 + \alpha}{\alpha^2} \right\} \left[ \frac{2(1 + \alpha)}{1 + 2\alpha} - \frac{\ln(1 + 2\alpha)}{2\alpha} \right] + \frac{\ln(1 + 2\alpha)}{2\alpha} - \frac{1 + 3\alpha}{(1 + 2\alpha)^2} \right\},$$

(10)

where

$$\sigma_e = 0.66524616(18) \times 10^{-24} \text{ cm}^2$$

(11)
(Cohen & Taylor 1987) is the Thomson cross section (with uncertainty in the last digits in the brackets) and
\[ \alpha = \frac{E}{m_e c^2} \] (12)
is the γ-ray energy in units of the electron rest energy. The Compton absorption cross section is given by
\[
\sigma_a^C = \sigma_e \left( \frac{3}{8} \right) \left[ \frac{-3 - 2\alpha + \alpha^2}{\alpha^3} \ln(1 + 2\alpha) \right.
\left. + 2 \left( \frac{9 + 51\alpha + 93\alpha^2 + 51\alpha^3 - 10\alpha^4}{3\alpha^2 (1 + 2\alpha)^3} \right) \right]
\] (13)
and the Compton scattering cross section by
\[
\sigma_s^C = \sigma_e \left( \frac{3}{8} \right) \left[ \ln(1 + 2\alpha) - \frac{2 (1 + \alpha) (1 + 2\alpha - 2\alpha^2)}{\alpha^2 (1 + 2\alpha)^2} + \frac{8\alpha^2}{3 (1 + 2\alpha)^3} \right].
\] (14)
Equations (13) and (14) are angle-averaged expressions since Compton scattering is anisotropic and the energy loss on scattering is angle-dependent. To second order in α, the cross section expressions are
\[
\sigma_{C, 2nd} = \sigma_e \left( 1 - 2\alpha + \frac{26}{5}\alpha^2 \right),
\] (15)
\[
\sigma_{a, 2nd}^a = \sigma_e \left( \alpha - \frac{21}{5}\alpha^2 \right),
\] (16)
and
\[
\sigma_{s, 2nd}^s = \sigma_e \left( 1 - 3\alpha + \frac{47}{5}\alpha^2 \right).
\] (17)
We see that in the limit of α going to 0 Compton scattering reduces to coherent Thomson scattering. The asymptotic forms of the cross section expressions as α goes to infinity are
\[
\sigma_{C,asy} = \sigma_e \left( \frac{3}{8\alpha} \right) \left[ \ln(\alpha) + \ln(2) + \frac{1}{2} \right],
\] (18)
\[
\sigma_{a,asy}^a = \sigma_e \left( \frac{3}{8\alpha} \right) \left[ \ln(\alpha) + \ln(2) - \frac{5}{6} \right],
\] (19)
and
\[
\sigma_{s,asy}^s = \sigma_e \left( \frac{3}{8\alpha} \right) \left( \frac{4}{3} \right).
\] (20)
The Compton total cross section decreases with α from \( \sigma_e \) at \( \alpha = 0 \) to \( \sigma_e \) at \( \alpha = \infty \). The only stationary point is the minimum at infinity. The Compton scattering cross section has the same behavior, except that it decreases more rapidly. The Compton absorption cross section rises from 0 at \( \alpha = 0 \) to a maximum of \(~ 0.14838408 \sigma_e \) at \( \alpha \approx 0.98212734 \) (i.e., an energy of \(~ 0.50186615 \text{MeV} \)) and then decreases to 0 at \( \alpha = \infty \). Aside from the maximum, the only stationary point is the minimum at infinity. The fractional scattering
opacity, \( \kappa_C/\kappa_C = \sigma_C/\sigma_C \), decreases for all \( \alpha \). It is 1 at \( \alpha = 0 \) and goes asymptotically to 0 at \( \alpha = \infty \); the only stationary point is the minimum at \( \alpha = \infty \). The fractional absorption opacity, \( \kappa_a/\kappa_C = \sigma_a/\sigma_C \), behaves, of course, in a complementary manner to the fractional scattering opacity: it increases for all \( \alpha \), is 0 at \( \alpha = 0 \), and goes to 1 at \( \alpha = \infty \) which is the only stationary point.

Compton opacity is, as mentioned in § 1, the dominant opacity for supernova \( \gamma \)-rays. In the metal-rich composition of SNe Ia, in which they are strongest, pair production opacity only begins to be important above \( \sim 3 \) MeV and only dominates at \( \sim 10 \) MeV and photoelectric opacity only begins to be important below \( \sim 0.3 \) MeV and only dominates at \( \sim 0.1 \) MeV (see SSH's Fig. 1). Since very roughly speaking the unscattered decay \( \gamma \)-rays in supernovae have energies of order 1 MeV and 1 MeV \( \gamma \)-rays lose about half their energy in Compton scattering, it is clear in SNe Ia that most \( \gamma \)-ray energy must be lost in Compton scattering. In other kinds of supernovae where metallicity is lower, Compton opacity is even more important.

A key point about Compton scattering is that its angular redistribution is forward peaked and the degree of forward peaking increases with increasing \( \gamma \)-ray energy. The ratio of the forward to the backward scattering differential cross sections (for energy, not photon number) is

\[
\frac{(1 + 2\alpha)^3}{(1 + 2\alpha^2 + \frac{1 + 2\alpha}{1 + 2\alpha})}.
\]

For 1 MeV photons, this ratio is \( \sim 46.372391 \). The angle-dependent energy reduction factor on Compton scattering is given by

\[
\frac{1}{1 + \alpha (1 - \cos \theta)},
\]

where \( \theta \) is the scattering angle (e.g., Davisson 1965, p. 50). From equation (22), it follows that forward scattered photons lose no energy at all.

It is clear that a substantial fraction of Compton scattering is nearly-forward and nearly-coherent. This fraction can almost be neglected since it barely affects the \( \gamma \)-ray flux. On the other hand the non-forward, noncoherent scattering fraction of Compton scattering is relatively small. One concludes that the total scattering component of Compton opacity is of relatively low significance for \( \gamma \)-ray transfer. Therefore, one could try to approximate Compton opacity by using only its absorption component and neglecting the scattering component. To see how this would work consider a medium with an opacity with both absorption and scattering components. One now does the radiative transfer through the medium neglecting the scattering component. This absorption-only approximation will tend to underestimate absorption in cases of finite, nonzero optical thickness. The scattering component (the non-forward scattering component to be precise) of the opacity tends to increase the trapping of flux in the medium by randomizing its direction and the trapped flux has more opportunities to be absorbed. Without the scattering component some absorption tends to be missed. Of course, if the medium is in the optically thick limit, scattering will not add to the trapping and absorption, and the absorption-only approximation will work well. On the other hand, the absorption-only approximation also gives exactly the right absorption in the optically thin limit where a \( \gamma \)-ray scatters on average much less than once.

For the LS procedure we wish to exploit the low significance of the scattering component of Compton opacity, but without making the simple absorption-only approximation. We will do this using two approximations: (1) an approximation to Compton opacity that we call the iso-Compton opacity and (2) an approximate treatment (that uses the LS approximation) of the non-forward, noncoherently scattered flux. The second approximation we describe in § 4. To make the iso-Compton opacity approximation we separate the Compton opacity into two approximate components: an isotropic, noncoherently scattering component (the iso-Compton component) and a forward, coherently scattering component (the forward
component). The iso-Compton component is the iso-Compton opacity itself. Since the forward component is pure scattering its total and scattering cross sections are equal and its absorption cross section is zero.

The forward component is effectively a zero opacity and simply does not appear in the radiative transfer calculations. The iso-Compton cross sections are obtained by subtracting the forward component cross sections from the corresponding Compton cross sections: i.e.,

\[ \sigma_{C}(\text{iso}) = \sigma_{C} - \sigma_{C}^{\ast}(\theta_{t}) , \]  
\[ \sigma_{C}^{\ast}(\text{iso}) = \sigma_{C}^{\ast} - 0 = \sigma_{C}^{\ast} , \]  
and

\[ \sigma_{C}^{\ast}(\text{iso}) = \sigma_{C}^{\ast} - \sigma_{C}^{\ast}(\theta_{t}) , \]  
where “iso” stands for iso-Compton opacity and \( \sigma_{C}^{\ast} \) is the forward component. To obtain the iso-Compton component opacities we just replace the Compton cross sections in equation (8) by the iso-Compton cross sections. We treat the energy reduction on iso-Compton scattering to be a constant for all angle. For the energy reduction factor we use \( \sigma_{C}(\text{iso})/\sigma_{C}(\text{iso}) \). The energy reduction factor of the forward component is, of course, 1.

The physical picture of the forward component cross section is that it is the Compton scattering cross section \( \sigma_{C}^{\ast}(\theta_{t}) \) for a cone of scattering directions with opening angle \( \theta_{t} \) centered on the forward direction. The Compton absorption opacity for the cone has been reassigned to the iso-Compton component. The cone’s average energy reduction factor is \( \sigma_{C}(\theta_{t})/\sigma_{C}(\theta_{t}) \), where \( \sigma_{C}(\theta_{t}) \) is the Compton total cross section for the cone. The physical picture of the iso-Compton component is that it is the opacity for other directions plus the absorption component opacity from the cone which have been spread uniformly over the whole scattering sphere. The average energy reduction factor for the other directions is \( [\sigma_{C}^{\ast} - \sigma_{C}^{\ast}(\theta_{t})]/[\sigma_{C} - \sigma_{C}^{\ast}(\theta_{t})] \) and this is larger than energy reduction factor assigned to the iso-Compton opacity: \( \sigma_{C}^{\ast}(\text{iso})/\sigma_{C}(\text{iso}) = [\sigma_{C}^{\ast} - \sigma_{C}^{\ast}(\theta_{t})]/[\sigma_{C} - \sigma_{C}^{\ast}(\theta_{t})] \).

There seems no precise way of optimizing the iso-Compton opacity: the smaller the cone’s opening angle, the more the forward component represents truly forward and coherent scattering, but the less one exploits the forward peaking of Compton opacity. We suggest the following prescription for \( \sigma_{C}^{\ast}(\theta_{t}) \) which allows us to explore the options:

\[ \sigma_{C}^{\ast}(\theta_{t}) = (\sigma_{C,f}^{\ast} - \sigma_{C,b}^{\ast}) \min (g, 1) + 2\sigma_{C,b}^{\ast} \max (g - 1, 0) , \]  
where \( \sigma_{C,f}^{\ast} \) and \( \sigma_{C,b}^{\ast} \) are the partial Compton scattering cross sections for the front and back scattering hemispheres, respectively, and \( g \in [0, 2] \) is an adjustable parameter. The values for \( \sigma_{C,f}^{\ast} \) and \( \sigma_{C,b}^{\ast} \) can be obtained from

\[ \sigma_{C}^{\ast}(\theta) = \sigma_{e} \left( \frac{3}{8} \right) \left\{ \frac{\ln (1 + \alpha - \alpha \cos \theta)}{\alpha^{3}} \right. \right. 
- \left. \left. \left[ 6\alpha^{2} (1 + \alpha - \alpha \cos \theta)^{3} \right]^{-1} \right. \right. 
- \left. \left. \left[ 6 + 15\alpha + 3\alpha^{2} - 12\alpha^{3} - 8\alpha^{4} \right. \right. \right. 
- \left. \left. \left( 6 + 30\alpha + 27\alpha^{2} - 18\alpha^{3} - 24\alpha^{4} \right) \cos \theta \right. \right. \right. 
+ \left. \left. \left( 15\alpha + 33\alpha^{2} - 24\alpha^{4} \right) \cos^{2} \theta \right. \right. \right. 
- \left. \left. \left( 9\alpha^{2} + 6\alpha^{3} - 8\alpha^{4} \right) \cos^{3} \theta \right\} \right\} \]
(e.g., Davisson 1965, p. 55) which is the Compton scattering opacity for the scattering cone of opening angle \( \theta \) centered on the forward direction: \( \sigma_{C,f}^\alpha = \sigma_C^\alpha (\pi/2) \) and \( \sigma_{C,b}^\alpha = \sigma_C^\alpha - \sigma_C^\alpha (\pi/2) \). Note that \( \sigma_{C,f}^\alpha - \sigma_{C,b}^\alpha \) rises from 0 at \( \alpha = 0 \) to a maximum of \( \sim 0.19479908 \sigma_e \) at \( \alpha = 4.9083380 \) and then declines to 0 at \( \alpha = \infty \).

The point of equation (26) is that the special \( g \) values 0, 1, and 2 give \( \sigma_C^\alpha (\theta_1) \) expressions with identifiable physical significance. If \( g = 0 \), then \( \sigma_C^\alpha (\theta_1) = 0 \) and we have the angle-averaged Compton opacity approximation that we describe below. If \( g = 1 \), then \( \sigma_C^\alpha (\theta_1) = \sigma_C^\alpha - \sigma_{C,b}^\alpha \), and the iso-Compton opacity will equal what can plausibly be identified as the real isotropic component of Compton opacity. If \( g = 2 \), then \( \sigma_C^\alpha (\theta_1) = \sigma_{C,f}^\alpha + \sigma_{C,b}^\alpha = \sigma_C^\alpha \) and the iso-Compton opacity approximation reduces to the absorption-only approximation that we discussed above. We note that when a \( \gamma \)-ray’s energy is zero, equation (26) yields a forward component of zero for all \( g \leq 1 \) since \( \sigma_{C,f}^\alpha = \sigma_{C,b}^\alpha \) for \( \alpha = 0 \).

The quantity \( g \) could be considered a free parameter. Given a highly accurate \( \gamma \)-ray deposition calculation for comparison, one could perhaps fine tune \( g \) to make the LS procedure yield a highly accurate result. But we are seeking a procedure without free parameters. Thus we have chosen to use hereafter (except where we explicitly say otherwise) the \( g \) value that \textit{a priori} seems most reasonable: i.e.,

\[
g = 1 \quad \text{giving} \quad \sigma_C^\alpha (\theta_1) = \sigma_{C,f}^\alpha - \sigma_{C,b}^\alpha .
\]

Some numerical experimentation suggests that \( g = 1 \) will quite accurately reproduce the results of the optimized SSH procedure (see §6) and that \( g \) widely different from 1 will not.

One can equate the \( \sigma_C^\alpha (\theta_1) \) (from eq. [28]) to the right-hand side of equation (27) and solve numerically (e.g., by a Newton-Raphson iteration) for \( \theta_1 \). The \( \theta_1 \) value increases monotonically with energy: it is 0° for \( \alpha = 0 \) and goes to an asymptotic value of \( \arccos(1/3) \approx 70.528779^\circ \) for \( \alpha = \infty \). With this \( \theta_1 \), one can compute the energy reduction factors \( \sigma_C^\alpha (\theta_1) / \sigma_C (\theta_1) \) and \( [\sigma_C^\alpha - \sigma_C^\alpha (\theta_1)] / [\sigma_C - \sigma_C (\theta_1)] \). One can then compare \( \theta_1 \) and these energy reduction factors to 0°, 1, and \( \sigma_C^\alpha (\text{iso}) / \sigma_C (\text{iso}) \). The closer the agreement, the better the iso-Compton opacity approximation represents the actual Compton opacity. The agreement is best for \( \alpha = 0 \) and degrades monotonically as \( \alpha \) increases. For \( \alpha = 2 \) (\( \approx 1 \text{ MeV} \)), \( \theta_1 \approx 63.01^\circ \), \( \sigma_C^\alpha (\theta_1) / \sigma_C (\theta_1) \approx 0.7457 \), and \( [\sigma_C - \sigma_C (\theta_1)] / [\sigma_C (\text{iso})] \approx 0.7537 \). For \( \alpha = 8 \) (\( \approx 4 \text{ MeV} \)), the corresponding values are 68.07°, 0.5243, and 0.4929. By these criteria the iso-Compton opacity approximation can only be expected to be moderately successful for the unscattered decay \( \gamma \)-rays in supernovae, but that it should be better for lower energy, scattered \( \gamma \)-rays. The fact that the Compton absorption component increases with \( \gamma \)-ray energy (e.g., its fractional value is 0.4431 for \( \alpha = 2 \) and 0.6091 for \( \alpha = 8 \)), however, limits the error in treating the scattering using the iso-Compton opacity approximation as \( \gamma \)-ray energy increases.

As mentioned above, in the limit of \( \alpha \) going to 0, Compton scattering reduces to coherent Thomson scattering. Thomson scattering, albeit not isotropic, is symmetric about the plane perpendicular to the forward direction (e.g., Mihalas 1978, p. 30), and thus \( \sigma_C^\alpha (\theta_1) \) (for \( g = 1 \) of course) goes to 0 when \( \alpha \) goes to 0. Therefore, iso-Compton opacity also reduces to coherent scattering and its value becomes equal to that of Compton opacity when \( \alpha \) goes to 0. But iso-Compton scattering does not actually reduce to Thomson scattering since it remains isotropic. Thomson scattering, however, in most practical calculations can be approximated as isotropic scattering. Therefore, iso-Compton scattering reduces to a usual good approximation to Thomson scattering in the limit of \( \alpha \) going to 0.

A simpler alternative to the iso-Compton opacity approximation is the angle-averaged Compton opacity approximation (i.e., the iso-Compton opacity approximation for \( g = 0 \)). In this latter approximation, one just assumes Compton opacity acts like its angle average: i.e., the scattering is isotropic and the energy reduction factor on scattering, \( \sigma_C^\alpha / \sigma_C \), is a constant with angle. Since the angle-averaged Compton opacity approximation does not exploit the forward peaking of Compton opacity, it is a poorer approximation from our point of view than the iso-Compton opacity approximation. In principle, it is clear that a two-component approximation to the angle-dependence of Compton opacity, if well chosen, should be a better approximation than a one-component approximation.
3. THE MEAN OPACITIES

A grey atmosphere is an atmosphere with a frequency-independent opacity (e.g., Mihalas 1978, p. 53ff). In a grey atmosphere, only the frequency-integrated radiation field is needed to solve the radiative transfer problem: this is a great simplification. One can imagine finding a mean opacity that will reduce a non-grey problem to a grey one. For practical purposes, however, (with the important exception of radiative transfer in the diffusion limit with local thermodynamic equilibrium [LTE]) no mean opacity permits a non-grey problem to be completely reduced to a grey problem (e.g., Mihalas 1978, p. 56ff). This is mainly because the mean opacity that would perform the reduction exactly cannot usually be calculated until after the problem is solved. If one wants to use grey radiative transfer as an approximation, one tries to find the mean opacity that a priori offers the best chance for an accurate solution for those effects that are of particular interest. This is the kind of mean opacity we try to find in this section.

3.1 Mean Opacity Prescriptions

What we want ultimately is to find the $\gamma$-ray energy deposition. Thus, we want to get as accurately as possible the amount of energy absorbed from the $\gamma$-ray fields. These $\gamma$-ray fields arise directly from the radioactive decay and from scattering. To get the scattered $\gamma$-ray fields we need the emissivity provided by scattering and this is obtained from the amount of flux from a beam removed by scattering (see § 4). Therefore, we need to consider scattering opacity as well as absorption opacity.

To find the appropriate opacities consider the flux removed at a point either by absorption, scattering, or both processes. For clarity, quantities not at the removal point (non-local quantities) will be distinguished by a functional dependence on the beam path length $s$ which is zero at the removal point. Quantities at the removal point will not be given an explicit position dependence. Thus $h(s)$ is the $h$-quantity at location $s$ and not at the removal point, and $h = h(0)$ is the $h$-quantity at the removal point. Making the approximation of time-independent, static radiative transfer, the energy removed at the removal point at a given frequency $\nu$ from a beam of specific intensity $I_\nu$ is given by

$$\chi^R R I_\nu = \chi^R_\nu I_\nu (s_b) \exp [-\tau_\nu (s_b)] + \chi^R_\nu \int_0^{s_b} ds \eta_\nu (s) \exp [-\tau_\nu (s)] , \quad (29)$$

where $\chi^R_\nu$ is the frequency-specific extinction at the removal point and $R$ (as specified in § 2) is blank, “a”, and “s” for total, absorption, and scattering extinction, respectively. The location of a boundary on the beam path is given by $s_b$. The $I_\nu (s_b)$ is the specific intensity incident on the boundary, $\tau_\nu (s_b)$ is the frequency-specific optical depth to the boundary, $\eta_\nu (s)$ is the frequency-specific emissivity along the beam path, and $\tau_\nu (s)$ is the frequency-specific optical depth to $s$. Note that the optical depth is calculated using total extinction. Also note that $\chi^R_\nu = \rho \kappa^R_\nu$ where $\rho$ is the density at the removal point and $\kappa^R_\nu$ is the frequency-specific opacity at the removal point. To get the total energy removed by either absorption, scattering, or both processes, we integrate equation (29) over all frequency and obtain

$$\int_0^\infty d\nu \chi^R I_\nu = \int_0^\infty d\nu \chi^R_\nu I_\nu (s_b) \exp [-\tau_\nu (s_b)]$$

$$+ \int_0^\infty d\nu \chi^R_\nu \int_0^{s_b} ds \eta_\nu (s) \exp [-\tau_\nu (s)] . \quad (30)$$

The grey equation, with which we want to replace equation (30), is

$$\chi I = \chi^R I (s_b) \exp [-\tau (s_b)] + \chi^R \int_0^{s_b} ds \eta (s) \exp [-\tau (s)] , \quad (31)$$
where
\[ I = \int_{0}^{\infty} d\nu I_\nu, \]  
\[ I(s_b) = \int_{0}^{\infty} d\nu I_\nu(s_b), \]  
and
\[ \eta(s) = \int_{0}^{\infty} d\nu \eta_\nu(s). \]

The $\chi^R$ is the mean extinction at the removal point and $\tau(s)$ is the mean optical depth back along the beam path.

Just by equating the left-hand sides of equations (30) and (31) and trying to solve for $\chi^R$ it becomes clear that the mean opacity that will render equations (30) and (31) exactly equivalent will have a complex dependence on non-local quantities through $I_\nu$. Such a mean opacity would be useless practically since we would have to do the frequency-specific radiative transfer to evaluate it. To obtain a practical mean opacity, we need to make some approximations.

First, let us try to find a mean opacity that will force the agreement of the first terms on the right-hand sides of equations (30) and (31). Such an opacity would be obtained from
\[ \kappa^R \exp [-\tau(s_b)] = \frac{\int_{0}^{\infty} d\nu I_\nu(s_b) \kappa^R_\nu \exp [-\tau_\nu(s_b)]}{\int_{0}^{\infty} d\nu I_\nu(s_b)}. \]  

Equation (35) is clearly a highly non-linear, non-local equation for determining $\kappa^R$ at all points in the atmosphere. To make progress let us assume the optical depth is very small at all frequencies and then the exponential factors can be set to 1. Let us further assume that beyond the boundary, the incident radiation field forms in an optically thin layer of spatial width $\Delta s$ and spatially constant emissivity. This assumption gives $I_\nu(s_b) = \eta_\nu(s_b) \Delta s$. Equation (35) now reduces to
\[ \kappa^R = \frac{\int_{0}^{\infty} d\nu \eta_\nu(s_b) \kappa^R_\nu}{\int_{0}^{\infty} d\nu \eta_\nu(s_b)}. \]  

Equation (36) is formally a non-local definition of the mean opacity since the emissivities are to be evaluated on the boundary not at the removal point. In the supernova $\gamma$-ray case, however, the emissivity of the unscattered $\gamma$-ray field (which is provided by radioactive species) is constant in space aside from the frequency-independent factor of radioactive species number density (see eq. [40] below). We will assume that the emissivity of a scattered $\gamma$-ray field can also be approximated constant in space aside from frequency-independent scale factors. From the above considerations, equation (36) can be rewritten
\[ \kappa^R = \frac{\int_{0}^{\infty} d\nu \eta_\nu \kappa^R_\nu}{\int_{0}^{\infty} d\nu \eta_\nu}. \]  

We will call equation (37) the emissivity-weighted mean opacity. From our derivation, the emissivity-weighted mean opacity is strictly valid only in an optically thin medium where the incident beams are formed in an optically thin, non-local emission region and only for an emissivity whose frequency behavior is constant in space.

Let now us try to find a mean opacity that will force the agreement of the second terms on the right-hand sides of equations (30) and (31). We first make the assumption that the density, emissivities, and opacities
are nonzero constants with respect to location in a region of space that encloses the removal point and are zero outside of this region. This means, of course, that the removal point is embedded in a region of $\gamma$-ray emission. Then from the second terms we obtain the following expression:

$$\frac{\kappa_R}{\kappa} \{1 - \exp [-\tau (s_b)]\} = \frac{\int_0^\infty d\nu \eta_\nu \frac{\kappa_R}{\kappa_\nu} \{1 - \exp [-\tau_\nu (s_b)]\}}{\int_0^\infty d\nu \eta_\nu}.$$  

(38)

Note that we had to introduce the total extinctions $\chi$ and $\chi_\nu$ in order to form the differentials $d\tau = ds \chi$ and $d\tau_\nu = ds \chi_\nu$, for the optical depth integrations. This is where the $\kappa_R/\kappa$ and $\kappa_R/\kappa_\nu$ ratios come from. These ratios, of course, become 1 for the total opacity case: i.e., when $R$ is blank.

Equation (38) is almost a local expression for mean opacity since the only non-local dependence is on the distance to the boundaries. It is, however, highly nonlinear. But certain things can be said about it. First, in the optically thin limit, which is an important supernova case, equation (38) reduces to equation (37). Second, in intermediate optically thick cases where the range in variation of the optical depth with frequency is $\lesssim 1$, equation (38) reduces approximately to equation (37). This situation can roughly arise for the unscattered $\gamma$-ray fields of $^{56}$Co and $^{56}$Ni (see SSH's Fig. 1). Third, in the optically thick limit in supernovae, all $\gamma$-rays are locally trapped and eventually absorbed (see § 4), and so the exact values of the mean opacity and its components are not important for deposition calculations provided only that they are sufficiently large. In this case, equation (38) can be replaced adequately by any reasonable mean opacity prescription. Given these facts, the emissivity-weighted mean opacity equation (37) seems a plausible and practical replacement for equation (38).

It is true that there are cases where the emissivity-weighted mean opacity is a poor choice. For instance, consider equation (35) for the mean opacity when the removal point is outside of the $\gamma$-ray emission region. Assume that the optical depths to the emission region (i.e., to the boundary) are small, but that the emission region is optically thick. In this case, from simple radiative transfer $I_\nu (s_b) = \eta_\nu (s_b) / \chi_\nu (s_b)$ assuming the emission region has spatially constant emissivity and extinction. If we now assume that the opacities (but not necessarily density) are constant in space and assume that emissivities are spatially constant aside from frequency-independent factors, then we obtain from equation (35)

$$\kappa_R = \frac{\int_0^\infty d\nu \eta_\nu \frac{\kappa_R}{\kappa_\nu}}{\int_0^\infty d\nu \eta_\nu \frac{1}{\kappa_\nu}}.$$  

(39)

In this special case, we find that emissivity-weighted inverse-mean opacities are the best choice for the mean opacities. The values of these mean opacities tend to be dominated by the lowest frequency-specific total opacities. The reason is that lowest total opacities, permit greatest transfer of flux in the optically thick emission region. The emissivity-weighted inverse-mean opacities may not be too important in supernovae. They apply in optically thin regions when the emission is occurring in optically thick regions. But in such cases, most $\gamma$-ray deposition in the supernova tends to occurs in the optically thick regions and this energy flows out into the optically thin regions in the form of UV-optical-IR radiation which may mainly determine the thermal, ionization, and excitation energy in those optically thin regions. The emissivity weighted inverse-mean opacity prescription is, of course, only one of many special prescriptions that can be invented for special cases.

The preceding analysis has led us to choose the emissivity-weighted mean opacity prescription given by equation (37) as the mean opacity prescription for the LS procedure. The emissivity-weighted mean opacities become exactly right for supernovae in the optically thin limit (when scattered fields are not important) whether the removal point is embedded in a region of emission or not. In the optically thick
limit when the removal point is embedded in an emission region, they capture some of the right behavior and can do no harm if all the $\gamma$-rays are locally trapped and absorbed. The cases where they are a poor approximation may not be so important. The emissivity-weighted mean opacities are also straightforward to calculate.

A factor that may further limit the error in using the emissivity-weighted mean opacities in cases for which they are formally not optimum is that the dominant $\gamma$-ray opacity in supernovae does not vary strongly with frequency across the frequency band where it is most important. Going from 0.1 MeV to 4 MeV, Compton total opacity falls by only a factor of $\sim 5$ and iso-Compton total opacity by only a factor of $\sim 6$. Since $\gamma$-ray spectra from radioactive decays are often dominated by one or a few $\gamma$-ray lines, it is possible that almost all the emissivity weighting will be given to frequency-specific opacities that are rather close in value since they come from a rather narrow frequency band. If this is so, then the difference between the emissivity-weighted mean opacities, emissivity-weighted inverse-mean opacities, or any other kind of reasonable mean opacity with emissivity weighting may often be rather small for the unscattered $\gamma$-ray field. This expectation is, in fact, fulfilled insofar as we have tested it (see § 3.2). The difference between the emissivity-weighted mean and emissivity-weighted inverse-mean opacities turns out to be even smaller for the scattered $\gamma$-ray fields when iso-Compton opacity is used as we will show in § 3.2.

### 3.2 Mean Opacities for the LS Procedure

For the LS procedure we will need emissivity-weighted mean opacities (which we will usually just call mean opacities hereafter) and some other mean quantities for multiple orders of scattered $\gamma$-ray fields. The 0th order field is the field emergent from the radioactive decay itself. The higher order fields, 1st, 2nd, 3rd, etc., have undergone 1, 2, 3, etc., scattering events.

To obtain the mean quantities we will make a number of sweeping assumptions. The general rationale for proceeding despite the deficiencies in these and the other assumptions we have or will make is that our assumptions allow the LS procedure formalism we develop to capture some of the correct physical behavior while remaining fairly simple. The accuracy of the LS procedure must be verified by comparison to procedures of known accuracy. A limited comparison of this sort is done in § 6.

The most sweeping assumption we will make is that for the purposes of the derivations the medium can be considered as infinite (which implies that the medium is in the optically thick limit), homogeneous, isotropic, and time-independent. This implies that we also consider the $\gamma$-ray fields to be homogeneous, isotropic, and time-independent. We emphasize that we will apply the mean opacities we derive from the assumed state of the medium to cases where the medium is not in the assumed state and that the assumed state is not the optimum state for the application of the emissivity-weighted mean opacity prescription (see § 3.1). The aforementioned general rationale allows us to proceed anyway.

The radioactive decay $\gamma$-ray spectrum is virtually entirely a line spectrum and we will assume it is exactly so. Because of our iso-Compton opacity and pair production opacity approximations, the higher order fields will also then consist of line spectra in our treatment. Because we are dealing only with line spectra, we will from now on use line-integrated or line-mean quantities and do sums over these quantities (instead of integrals) to obtain the mean quantities we need for the LS procedure.

The 0th order $\gamma$-ray emissivity of a line $j$ from some radioactive species is given by

$$
\eta_{0,j}^{\gamma} = \frac{1}{4\pi t_e} \frac{n}{f_{0,j}} E_{0,j},
$$

where $n$ is the number density of the species, $t_e$ is the $e$-folding time for the decay, $f_{0,j}$ is the number of $\gamma$-rays (which is usually less than 1) in line $j$ per decay, and $E_{0,j}$ is the $\gamma$-ray line energy. In order to obtain the 0th order mean opacities we in fact need only the $E_{0,j}$ and $f_{0,j}$ quantities since the other emissivity factors cancel out (see eq. [37]).
Based on our assumption of line spectra in all scattering orders, we will have definite $E_{i,j}$'s and $f_{i,j}$'s for orders $i > 0$. Moreover, the frequency dependence of the emissivity in all orders will be given by $f_{i,j}E_{i,j}$ as we will show below. Thus, we can use the $E_{i,j}$'s and $f_{i,j}$'s to calculate the mean opacities for all orders. To obtain $E_{i,j}$'s and $f_{i,j}$'s we will derive recurrence relations. In order to have simple recurrence relations we need to keep constant the number of lines through all scatterings. Therefore we assume that we can replace the three opacities (pair production, photoelectric, and iso-Compton) by a single combined opacity. (Note the combined opacity is a sum of the opacity types with mean properties, not a frequency mean of opacity.) First let us obtain the recurrence relation for the $E_{i,j}$'s.

The energy of a scattered $\gamma$-ray (assuming there is no angle-dependence on the scattered energy) is usually $\kappa^s/\kappa$. This applies to the iso-Compton opacity. In a case like that of the pair production opacity where we have assumed that two scattered $\gamma$-rays share equally the energy of one incident $\gamma$-ray and where there is no angle-dependence for the scattered energy, the energy of each scattered $\gamma$-ray is $(1/2)\kappa^s/\kappa$. Now for our combined opacity the number (not energy) fractions of scattered $\gamma$-rays that are scattered by the iso-Compton and pair production opacities are

$$
\frac{\kappa(\text{iso})}{\kappa(\text{iso}) + \kappa(\text{pair})} \quad \text{and} \quad \frac{\kappa(\text{pair})}{\kappa(\text{iso}) + \kappa(\text{pair})},
$$

respectively, where “iso” stands for the iso-Compton opacity and “pair” for pair production opacity. Therefore the recurrence relation giving the mean energy of $\gamma$-rays in line $j$ for all scattered orders is

$$
E_{i,j} = E_{i-1,j} \left\{ \frac{\kappa_{i-1,j} (\text{iso})}{\kappa_{i-1,j} (\text{iso}) + \kappa_{i-1,j} (\text{pair})} \left[ \begin{array}{c} \kappa_{i-1,j} (\text{iso}) \\ \kappa_{i-1,j} (\text{iso}) \end{array} \right] \right. \\
+ \frac{\kappa_{i-1,j} (\text{pair})}{\kappa_{i-1,j} (\text{iso}) + \kappa_{i-1,j} (\text{pair})} \left[ \begin{array}{c} 1/2 \kappa_{i-1,j} (\text{pair}) \\ 1/2 \kappa_{i-1,j} (\text{pair}) \end{array} \right] \\
= E_{i-1,j} \left[ \frac{\kappa_{i-1,j} (\text{iso}) + (1/2) \kappa_{i-1,j} (\text{pair})}{\kappa_{i-1,j} (\text{iso}) + \kappa_{i-1,j} (\text{pair})} \right].
$$

Note that the photoelectric opacity is assumed to be a pure absorption opacity, and therefore has no effect on the energy of scattered $\gamma$-rays.

Now we need to determine the $f_{i,j}$'s. Consider a single decay and imagine tracking decay-emitted $\gamma$-rays through all scattering events. Because the medium is assumed to be infinite, homogeneous, and isotropic we can treat all the $\gamma$-rays in line $j$ from the single decay together as a single packet. Because the medium is infinite each packet emitted in any order will eventually be scattered: the distance and travel time will depend on the extinction in the order. In other words, if $f_{i,j}$ $\gamma$-rays are emitted into an order, then $f_{i,j}$ $\gamma$-rays will be removed from the order, either scattered or absorbed. Consider the packet with $\gamma$-ray number $f_{i-1,j}$. On the $(i - 1)$th scattering event with the combined opacity the packet’s $\gamma$-ray number will change according to

$$
f_{i,j} = f_{i-1,j} \left[ \frac{\kappa_{i-1,j} (\text{iso}) + 2\kappa_{i-1,j} (\text{pair})}{\kappa_{i-1,j} (\text{iso}) + \kappa_{i-1,j} (\text{pair}) + \kappa_{i-1,j} (\text{pe})} \right],
$$

where “pe” stands for photoelectric opacity and where the denominator in the fraction accounts for the $\gamma$-rays removed from the $(i - 1)$th order field and the numerator for the $\gamma$-rays emitted into the $i$th field.
Equation (43) is the recurrence relation for the $f_{i,j}$'s. Now consider all decays in the medium. Because the medium is infinite, homogeneous, isotropic, and time-independent, the time and distance a packet took in going from the decay emission to $(i-1)$th order scattering event is irrelevant to the emissivity in the $i$th order. Thus, the emissivity for line $j$ in the $i$th order will be proportional to $f_{i,j}E_{i,j}$ and will not have any other frequency dependence, and will be constant in space.

Using equation (37) with the integrals converted to sums and given the $E_{i,j}$ and $f_{i,j}$ values, the expression for $i$th order emissivity-weighted mean opacities is found to be

$$\kappa_i = \frac{\sum_j f_{i,j}E_{i,j}^R}{\sum_j f_{i,j}E_{i,j}},$$

where $\kappa_{i,j}^R$ is evaluated at energy $E_{i,j}$. We have canceled out the frequency-independent factors in the numerator and denominator of equation (44). Note that by the assumption of an infinite, homogeneous, isotropic, and homogeneous medium, the $f_{i,j}$'s describe the $\gamma$-ray removal spectrum as well as emission spectrum. Thus equation (44) could also be described as a removal-weighted mean opacity prescription.

Equation (44) has been used to construct the mean opacities and mean fractional component opacities for the 0th through 5th order for $^{56}\text{Co}$ and $^{56}\text{Ni}$ $\gamma$-ray fields for several cases. The mean fractional component opacities, which are of direct use in § 4, are defined by

$$\xi_i = \frac{\kappa_{i,R}}{\kappa_i},$$

where $R$ is either “a” or “s”. The results are shown in Tables I and II. Table I includes the infinite order mean opacities which are for zero energy $\gamma$-rays. (We demonstrate in Appendix B that infinite order Compton or iso-Compton scattering will degrade $\gamma$-ray energy to zero.) The $\gamma$-ray spectrum data for nuclear decays of $^{56}\text{Co}$ and $^{56}\text{Ni}$ were taken from Huo (1992) and Browne & Firestone (1986). We have included in the $\gamma$-ray spectrum of $^{56}\text{Co}$ $\gamma$-rays to account for the annihilation of the positron that is produced in 19% of $^{56}\text{Co}$ decays (Huo 1992). We assume that the positron is annihilated locally after a negligible time delay. To maintain our line spectrum assumption for the $\gamma$-rays and for consistency with our pair production opacity assumptions, we assume the positron annihilates to create two $m_e c^2$ $\gamma$-rays. A three-continuum-photon process for annihilation is of course actually possible (see § 1). The X-ray spectra of the nuclear decays were not included for the reasons discussed in § 5.

The emissivity-weighted mean $\gamma$-ray energy $\bar{E}_i$ for each order $i$ in Tables I and II has been calculated from the expression

$$\bar{E}_i = \frac{\sum_j f_{i,j}E_{i,j}}{\sum_j f_{i,j}}.$$
opacity and its fractional component opacities have the same general behavior with decreasing energy (and therefore increasing order) as the Compton opacity and its fractional component opacities (see § 2).

Third, we note that the \( \gamma \)-ray mean energy decreases most in the 1st scattering and that by the 5th order it is only a small fraction of its 0th order value. The decline in mean energy with increasing order is in fact rather slow after the 1st scattering because, as noted above, the iso-Compton fractional absorption opacity decreases with decreasing energy. In the limit of infinite scattering order, the \( \gamma \)-rays have lost all their energy and their mean opacity has become pure scattering opacity: angle-average Thomson scattering opacity in fact (see § 2). Iso-Compton scattering conserves photon number and so, in principle, no photons are destroyed or created in scattering to infinite order.

In Table II we present the mean opacities and fractional mean component opacities now including pair production and photoelectric opacities for the 0th through 5th orders for the solar composition (to be precise the solar system composition of Anders & Grevesse 1989) and for the mean composition of SN Ia model W7 (Thielemann et al. 1986). The \( \kappa_{pe} \) values for the photoelectric opacity (see § 2) were constructed from the results given by Veigele (1973); we did not use the scaling approximation equation (6). Veigele’s photoelectric cross sections extend down only to \( 10^{-4} \) MeV. This is not a problem since none of the \( \gamma \)-rays used creating Table II went below \( 10^{-4} \) MeV even by the 10th scattering order.

By comparison of the mean opacities for the \( \mu_e = 1.179 \) case in Table I to those for the solar composition case in Table II we see that both pair production opacity and the photoelectric opacity are nearly negligible. This shows that for solar composition only Compton opacity is important through to the 5th order field. Since higher order fields can be treated very crudely—there is not much energy deposition to be obtained from them—it is clear that Compton opacity alone is adequate for treating the solar composition.

Note that the nonzero order mean energies for the solar composition in Table II are actually slightly higher than the mean energies for the \( \mu_e = 1.179 \) case in Table I despite there being more opacity in the calculations for Table II. Higher mean energies are made possible by the fact that the photoelectric opacity tends to preferentially destroy the lower energy \( \gamma \)-rays and the destroyed \( \gamma \)-rays, of course, make no contribution to the mean \( \gamma \)-ray energy. The effect of the pair production opacity on the mean energies can, however, either lower or raise them since pair production opacity has both absorption and scattering components whose relative size depends on \( \gamma \)-ray energy. (Note that there is a \( ^{56}\text{Ni} \) \( \gamma \)-ray with enough energy for pair production even though the \( ^{56}\text{Ni} \) 0th order mean energy is much less than \( 2m_{e,c} \).) It is the combination of the small photoelectric and pair production opacities that results in the nonzero order mean energies being slightly higher in the Table II solar composition case.

The total scattered energy in each order (which is proportional to \( \sum_{i,j} f_{i,j}E_{i,j} \)) is slightly higher for the 1st through 3rd orders for \( ^{56}\text{Co} \) in the solar composition case of Table II than for the Table I \( ^{56}\text{Co} \) cases. This is because of the relatively large fractional scattering component of the pair production opacity. (Note only in the 0th order are there \( ^{56}\text{Co} \) \( \gamma \)-rays with enough energy to interact with the pair production opacity, but, of course, 0th order interactions affect the higher order results.) In higher orders for \( ^{56}\text{Co} \) and in all nonzero orders for \( ^{56}\text{Ni} \) in the solar composition case, the total scattered energy is slightly lower than in the Table I cases. This is, of course, because of absorption by the photoelectric opacity. The fraction of initial \( \gamma \)-ray energy remaining by the 5th order is 0.06121 for \( ^{56}\text{Co} \) and 0.1301 for \( ^{56}\text{Ni} \) in the Table I cases and 0.06100 for \( ^{56}\text{Co} \) and 0.1291 for \( ^{56}\text{Ni} \) in the Table II solar composition case.

The (mean) model W7 composition is an almost all metal composition and more than 60\% of it is made of iron peak elements. Pair production and photoelectric opacities are typically much larger for metals than for hydrogen and helium (see § 2, eq. [2] and Veigele 1973). Thus we expect them to be more important for the model W7 composition than for the solar composition. On the other hand, \( \mu_e \) is about 1.8 times larger for the model W7 composition than for the solar composition. This simply means the model W7 composition has only about half the electrons per unit mass and thus half the Compton opacity that the solar composition has. Comparing the mean total opacities and mean fractional absorption opacities of the solar and model W7 compositions, it is clear that the increased metallicity of the model W7 composition
cannot compensate for its reduced iso-Compton opacity for the lower orders of scattering, but can more than compensate for the higher orders. This result is explained by the photoelectric opacity which grows strongly with decreasing energy and is pure absorption opacity.

The increased pair production and photoelectric opacities in the model W7 composition case relative to the solar composition case cause the mean γ-ray energies in each nonzero order of scattering to be slightly higher in the model W7 composition case. The total energy, however, in each order from the 2nd order on for $^{56}\text{Co}$ and from the 1st order on for $^{56}\text{Ni}$ is lower in the model W7 composition case. The fraction of initial γ-ray energy remaining by the 5th order in the model W7 composition case is 0.007632 for $^{56}\text{Co}$ and 0.01025 for $^{56}\text{Ni}$.

Pair production, despite the high metallicity, is actually only a small contributor to the mean opacities in the model W7 composition case. For $^{56}\text{Co}$ in the 0th order, pair production opacity contributes only 3%, 2.5%, and 6% to the mean total, absorption, and scattering opacities, respectively. For $^{56}\text{Ni}$ in the 0th order, pair production opacity contributes much less than a percent to the mean opacities. The contribution in all other orders for both $^{56}\text{Co}$ and $^{56}\text{Ni}$ is zero since there are no γ-rays over the threshold energy of $2m_e c^2$.

We have also calculated the emissivity-weighted inverse-mean opacities for the 0th through 5th orders for the cases reported in Tables I and II. The differences between these and the emissivity-weighted mean opacities tend to decrease with order. In the 0th order they range from a few percent up to 25% in the worst case. The differences in the nonzero orders are quite small and, except in the model W7 composition $^{56}\text{Ni}$ case, are less, usually much less, than ~ 3%. Even in the model W7 composition $^{56}\text{Ni}$ case, the differences are no more than 12% in the 1st order and diminish to being less ~ 2% by the 5th order. The reason for the reduction of the differences beyond the 0th order is the nature of the iso-Compton opacity. It turns out that γ-ray energies in the range ~ 0.1–5 MeV are diminished in an iso-Compton scattering to energies in the range ~ 0.08–0.24 MeV. Thus the relative variation in the energies of typical supernova decay γ-rays is greatly reduced by the first iso-Compton scattering. This means that the relative variation in the opacity for these γ-rays is greatly reduced, and thus the two kinds of mean opacities tend to converge. Other kinds of mean opacities should tend to converge as well.

The grey approximation is, obviously, always good when the range in variation of the input opacities is sufficiently small. That the two kinds of mean opacities we have investigated converge so well in the nonzero orders suggests the grey approximation will always be good for these orders. This conclusion, of course, holds only insofar as the iso-Compton opacity is a good approximation to the Compton opacity. The fact that in the 0th order the two kinds of mean opacities are in not in so close agreement as in the nonzero orders does not necessarily mean the grey approximation is worse than in nonzero orders. It just means that the particular mean opacity prescription we have chosen for the grey approximation for supernova γ-ray transfer needs to be adequate. From the arguments given above, we believe that our choice of the emissivity-weighted mean opacity prescription has a good chance of being adequate.

To end this section, we note that whenever Compton scattering opacity is the dominant form of opacity, the mean opacities tend to simply scale with \( \mu^{-1} \). This is because Compton opacity depends directly on the ratio of electron density to mass density (see § 2, eq. [8]). Even for the model W7 composition case where Compton opacity is least important, the mean opacities of the lowest orders of scattering can be obtained very roughly by dividing the \( \mu_e = 1 \) mean opacities in Table I by 2.095.

4. THE LS GREY RADIATIVE TRANSFER PROCEDURE

What we want from a grey radiative transfer procedure is the γ-ray energy deposition as function of position. We will measure this energy deposition by \( \epsilon_d \), the energy deposited per unit time per unit mass. (Energy deposition tends to decrease with decreasing density whether measured per mass or per volume. Measuring per mass, however, gives a smaller range in variation.) To get the energy deposition we assume
the emissivity-weighted mean opacities we developed in § 3 adequately control the transfer of the frequency-integrated radiation fields of all orders of scattering. The energy deposition from one radioactive species is then given by

\[ \epsilon_i = 4\pi \sum_{i=0}^{\infty} \chi_i J_i = 4\pi \sum_{i=0}^{\infty} \kappa_i J_i , \]

(47)

where \( J_i \) is the \( i \)th order (frequency-integrated) mean intensity, \( \chi_i = \kappa_i \rho \) is the \( i \)th order mean absorption extinction, and \( \kappa_i \) is the \( i \)th order mean absorption opacity.

The 0th order mean intensity (or radiation field) is generated by the true energy source, the radioactive species. Formally this field at any point is given by

\[ J_0 = \oint \frac{d\Omega}{4\pi} \int_0^{\tau_0} dx_0 S_0(x_0) \exp(-x_0) , \]

(48)

where \( \Omega \) is solid angle, \( x_0 \) is the optical depth measured from the point backward along a beam path, \( \tau_0 \) is the optical depth along the beam path to the surface of the medium, and \( S_0 \) is the \( \gamma \)-ray source function. The source function is the radioactive species (frequency-integrated) emissivity divided by 0th order mean total extinction: \( S_0 = \eta_0^\gamma / \chi_0 \). The optical depths for the 0th and all other orders are computed using

\[ dx_i = ds \chi_i = ds \kappa_i \rho , \]

(49)

where \( \chi_i \) is, of course, the \( i \)th order mean total extinction, \( \kappa_i \) is the \( i \)th order mean total opacity, and \( s \) is again the beam path length.

Now the emissivity for any scattered radiation field \( i \) is \( \chi_{i-1} J_{i-1} \). Thus the source function for any scattered field \( i \) is \( (\kappa_{i-1} / \kappa_i) J_{i-1} \), where the ratio \( \kappa_{i-1} / \kappa_i \) is a constant in our case. Adapting equation (48), mutatis mutandis, the mean intensity for \( i \geq 1 \) is given by

\[ J_i = \frac{\kappa_{i-1}}{\kappa_i} \oint \frac{d\Omega}{4\pi} \int_0^{\tau_i} dx_i J_{i-1}(x_i) \exp(-x_i) . \]

(50)

The 0th order radiation field must be calculated numerically. It is in fact a straightforward integration since the 0th order source function is known from the properties of the radioactive species, our mean opacities, and the composition of the supernova model being used. We now make the sweeping assumption that all scattered radiation fields can be calculated in the LS approximation (see § 1): i.e., we assume that \( J_{i-1}(x_i) \) in equation (50) (which is the source function for the \( i \)th field) can be replaced by its value at \( x_i = 0 \) which we denote simply by \( J_{i-1} \). We then pull \( J_{i-1} \) out of the integral in equation (50), do the integration over optical depth, and obtain

\[ J_i = J_{i-1} \frac{\kappa_{i-1}}{\kappa_i} \zeta_i , \]

(51)

where

\[ \zeta_i = \oint \frac{d\Omega}{4\pi} \exp[1 - \exp(-\tau_i)] . \]

(52)

It now follows that all scattered fields can be obtained from

\[ J_i = J_0 \prod_{j=1}^{i} \frac{\kappa_{j-1}}{\kappa_j} \zeta_j . \]

(53)

In equation (53) and in all similar cases, we take the product expression to be 1 when the lower limit exceeds the upper limit.
We note here that the LS approximation becomes good when the emission regions contributing significantly to the $\gamma$-ray field at a point vary only linearly with distance in their physical state from the conditions at the point. Only linear variation will occur in sufficiently optically thick conditions because the emission regions will be close to the point. In this case the average physical state of these regions is well approximated by the local state at the point. In the optically thick limit, the emission regions become the local region itself and the LS approximation becomes exact. In non-optically thick conditions the LS approximation becomes a rough approximation.

Using equation (53), the expression for the energy deposition can be written

$$\epsilon_d = 4\pi \kappa_{\text{eff}}^a J_0,$$

where

$$\kappa_{\text{eff}}^a = \sum_{i=0}^{\infty} \kappa_i^a \prod_{j=1}^{i} \frac{\kappa_j^s - 1}{\kappa_j^s} \zeta_j$$

is the effective absorption opacity. The effective absorption opacity can be rewritten

$$\kappa_{\text{eff}}^a = \kappa_0 L,$$

where

$$L = \sum_{i=0}^{\infty} \xi_i^a \prod_{j=0}^{i-1} \xi_j^s \zeta_{j+1}$$

is what we call the LS approximation series and the $\xi_i^R$ quantities are the mean fractional component opacities defined by equation (45) in § 3.2. The series $L$ is the ratio of the energy that is ultimately absorbed from fields of all orders to the energy removed (but not necessarily absorbed) from the 0th order field.

There are three statements to be made about the series $L$ in equation (56). First, on physical grounds alone it must converge. Second, in the optically thin limit for nonzero scattering orders (i.e., when $\zeta_i = 0$ for all $i \geq 1$) $L$ goes to $\xi_0^a$, and thus

$$\kappa_{\text{eff}}^a = \kappa_0.$$  \hspace{1cm} (57)

The local-state approximation becomes exact in this case for the trivial reason that there is no energy deposited in the nonzero orders. Third, if all nonzero scattering orders are in the optically thick limit and we have an appropriate set of order mean opacities (see Appendix B), $L$ converges to 1. This means that

$$\kappa_{\text{eff}}^a = \kappa_0$$  \hspace{1cm} (58)

and all the energy processed by the 0th order total opacity (i.e., $4\pi \kappa_0 J_0$) is ultimately absorbed. In Appendix B we prove that iso-Compton opacity and Compton opacity itself do give an appropriate set of order mean opacities for this to happen. The addition of a pure absorption opacity (e.g., the photoelectric opacity) or to finite order any other opacity (e.g., the pair production opacity) would not alter the convergence of $L$ to 1. Thus, for the supernova case $\kappa_{\text{eff}}^a = \kappa_0$ in the optically thick limit of the nonzero scattering orders. Recall also that LS approximation becomes exact in the optically thick limit.

Now we could compute the terms in the $L$ series in equation (56) to any order we wish. But computing to a high order will not necessarily yield high accuracy because of the approximations we have made. Instead we propose to approximate the terms in the $L$ series so that we can evaluate it to infinite order. In this way we guarantee that the $L$ series will go to the exact nonzero order optically thick, as well as thin, limit. In fact, in the supernova case if the nonzero orders are in the optically thin or thick limits, then the 0th order will be in those limits or nearly as well because of the nature of Compton opacity. If all the orders are in the optically thin limit, then grey radiative transfer with the emissivity-weighted mean opacities is exact for supernovae (assuming time-independent, static radiative transfer) since it is exact for the optically thin limit.
0th order γ-ray transfer (assuming time-independent, static radiative transfer) as we showed in § 3.1. If all orders are in the optically thick limit, then for supernova γ-ray transfer all γ-rays are locally trapped in all orders including the 0th and thus (following from the discussion in the last paragraph) are locally absorbed. Thus any treatment, however crude, that ensures complete local absorption is exact. Therefore practically speaking for supernova γ-ray transfer, the approximated \( L \) series we propose will yield exact results (within numerical limitations and not counting errors due to neglecting time-dependent, non-static radiative transfer effects) in the (all-order) optically thin and thick limits.

To obtain the approximated \( L \) series we split the \( L \) series into low and high order terms with the \( k \)th term being the first high order term:

\[
L = \left( \sum_{i=0}^{k-1} \xi_i^a \prod_{j=0}^{i-1} \xi_j^a \zeta_j \right) + \left( \prod_{j=0}^{k-1} \xi_j^a \zeta_j \right) \sum_{i=k}^{\infty} \xi_i^a \prod_{m=k}^{i-1} \xi_m^a \zeta_m + 1 . \tag{59}
\]

We now approximate all the high order quantities by their \( k \)th values. Then the high order terms can be summed analytically and we obtain

\[
L = \left( \sum_{i=0}^{k-1} \xi_i^a \prod_{j=0}^{i-1} \xi_j^a \zeta_j \right) + \left( \prod_{j=0}^{k-1} \xi_j^a \zeta_j \right) \frac{\xi_k^a}{1 - \xi_k^a \zeta_k} . \tag{60}
\]

(Note one should never choose \( k \) such that \( \xi_k^a = 1 \) and \( \xi_k^s = 0 \). For supernovae this is not a concern since in practice \( \xi_i^a = 1 \) never happens for any order \( i \).) In the nonzero order optically thin limit (or in the 0th order optically thin limit if \( k = 0 \)), \( L \) reduces to \( \xi_0^a \) of course. In the nonzero order optically thick limit (or in the 0th order optically thick limit if \( k = 0 \)), we find, as desired,

\[
L = 1 - \prod_{j=0}^{k-1} \xi_j^a + \prod_{j=0}^{k-1} \xi_j^s = 1 , \tag{61}
\]

where we have used the fact that \( \xi_i^a + \xi_i^s = 1 \) and equation (B5) from Appendix B.

The mean opacity values needed for determining the \( \zeta_i \)'s and \( \xi_i^R \)'s in equation (60) up to 5th order for some opacity cases can be obtained or scaled from the results shown in § 3.2, Tables I and II. The evaluation of equation (60) will probably not be too sensitive to the choice of \( k \) since the amount of energy absorbed in a scattering order decreases with scattering order as can be seen from Tables I and II. We have computed the net energy deposition for model W7 from early times to day 1000 after the explosion using \( k = 2 \) and \( k = 5 \). The difference between the two cases was always less than \( \sim 0.6\% \) and it vanishes, of course, at early optically thick and late optically thin times. For the other opacity cases shown in Tables I and II we expect similar results. We will take \( k = 5 \) to be our fiducial \( k \) value in order to avoid regarding \( k \) as a free parameter. The \( k = 5 \) choice should be adequately high for all cases.

The integration for the \( \zeta_i \)'s cannot be done exactly analytically. For calculations in spherical symmetry, we propose using the simple two-stream approximation

\[
\zeta_i \approx \frac{1}{2} \left\{ [1 - \exp(-\tau_{i,\text{out}})] + [1 - \exp(-\tau_{i,\text{in}})] \right\} , \tag{62}
\]

where \( \tau_{i,\text{out}} \) and \( \tau_{i,\text{in}} \) are the outward and inward radial \( \tau \) values. The calculation of the \( \zeta_i \) with the two-stream approximation is still non-local, but it is numerically trivial. For atmospheres without spherical symmetry some other prescription for \( \zeta_i \) is needed.
Having done the numerical integration for the 0th order radiation field (which uses the 0th order total opacity $\kappa_0$) and the evaluation of effective absorption opacity using equation (56) and the $L$ series in equation (60), we can then obtain the energy deposition from equation (54). These operations together constitute the LS grey radiative transfer procedure for obtaining the energy deposition.

The only real elaboration of LS procedure beyond the SSH procedure is in the calculation of effective absorption opacity. We have discussed this calculation in detail, but the actual coding for it is straightforward. The basic SSH procedure fortran code of circa 1996 (Sutherland 1996) is about 60 lines excluding comment lines and auxiliary subroutines. The LS procedure fortran code that we have written (starting from the SSH code) is about 260 lines excluding comment lines, auxiliary subroutines, and data statements. The comparison is not completely fair, however, since the LS code is more general than the SSH code and contains some purely diagnostic lines. The actual difference in complexity between the SSH and LS codes is small. Both codes run practically instantly by human perception for ordinary supernova models.

Unlike the optimized SSH procedure, there are no free parameters in the LS procedure. There are, however, many significant approximations. In §6, we will discuss the adequacy of some of these approximations.

5. THE RADIOACTIVE SOURCES

In this section we review some material needed for the treatment of the radioactive sources in the energy deposition in supernovae.

The frequency-integrated emissivity in energy form $i$ for a radioactive isotope is given by

$$\eta^i_0 = \frac{1}{4\pi} \frac{n_i}{t_e} Q_i,$$

(63)

where $Q_i$ is the energy emitted per decay in form $i$, $n$ is the number density of the isotope, and $t_e$ is the $e$-folding time for the decay. Since the volume of any mass element in supernova ejecta is changing constantly, it is more convenient to express the energy from radioactive decay in the form of energy generation per unit time per unit mass

$$\epsilon_i = \frac{4\pi\eta^i_0}{\rho}.$$

(64)

For the case of a radioactive isotope synthesized in a supernova explosion, $\epsilon_i$ can be expressed by

$$\epsilon_i = C_i X(0) \exp(-t/t_e),$$

(65)

where $X(0)$ is the isotope’s initial mass fraction and $t$ is the time since explosion. The $C$ coefficient is defined by

$$C_i = \frac{Q_i}{m_{amu} A t_e},$$

(66)

where $A$ is the isotope’s atomic mass and $m_{amu}$ is again the atomic mass unit. For a radioactive isotope which is the child of an isotope synthesized in the explosion, $\epsilon_i$ can be expressed by

$$\epsilon_i = D_i X_1(0) \left[ \exp(-t/t_{e,2}) - \exp(-t/t_{e,1}) \right],$$

(67)

where 1 and 2 designate parent and child nucleus quantities, respectively. The $D$ coefficient is defined by

$$D_i = \frac{Q_{i,2}}{m_{amu} A_1} \frac{1}{(t_{e,2} - t_{e,1})}.$$  

(68)
(Note the use of $A_1$ in the denominator of equation [68] rather than $A_2$ is formally correct although the expression is for the decay of the child nucleus. The reason is that $X_1(0)\rho/(m_{amu}A_1)$, which comes into the derivation of equation [68], is the initial particle density and it is conserved in weak nuclear decays. Since the mass change is very small in weak decays, in practice $A_2$ would do as well as $A_1$ of course.)

In Table III we present the parameters needed for treatment of the deposition of the various forms of energy. The main forms are $\gamma$-rays, X-rays, and the kinetic energy of $\beta$-particles and atomic electrons (from internal conversions and the Auger process). The $\gamma$-ray deposition is, of course, the main subject of this paper. The 0th order source function for $\gamma$-rays in terms of $\epsilon_\gamma$ is given by

$$S_0 = \frac{\epsilon_\gamma}{4\pi\kappa_0}.$$  \hspace{1cm} (69)

Note we include in the Table III $Q_\gamma$’s the $\gamma$-ray energy from the annihilation of any positron produced in the decay.

The radiative transfer and energy deposition of the X-rays can be treated in a similar manner to those of the $\gamma$-rays. X-ray energy, however, is much lower than $\gamma$-ray energy, and so the photoelectric opacity becomes much more important for the X-rays. Because of the great difference between the $\gamma$-ray and X-ray energy scales and opacities, it is not plausible to lump these radiations together in a single grey radiative transfer procedure. Therefore we do not treat X-ray radiative transfer and energy deposition in this paper. The X-rays can, in fact, be neglected at early times when their energy contribution is negligible. The total energy in the X-rays per decay is typically a few kilo-electron-volts. The X-ray contribution, however, does increase with time because the ejecta becomes transparent to $\gamma$-rays sooner than to X-rays. For model W7 we have calculated that at 300 days the $^{56}\text{Co}$ X-rays contribute about 2% of the $\gamma$-ray deposition and at 500 days, about 7%: $^{56}\text{Ni}$ X-rays are, of course, negligible by these late times. Thus, it is possible that the X-ray contribution to the energy deposition will be significant at late times. If this is not the case for the $^{56}\text{Co}$ decay, it may be true for the decay of longer lived radioactive species (see § 1).

The energy from $\beta$-particles, positrons to be specific, is quite important in supernovae. The $^{56}\text{Ni}$ decay produces effectively no positrons: the upper limit on the fraction of decays leading to a positron emission is $< 0.0013\%$ (Huo 1992). The $^{56}\text{Co}$ decay produces a positron 19% of the time. The $\gamma$-ray energy produced on the annihilation of the positron should be accounted for in the $^{56}\text{Co}$ $Q_\gamma$, as we have done in Table III and in the mean opacities as we have done in Tables I and II in § 3.2. There is, however, positron kinetic energy which is mostly lost prior to annihilation: it is about 3% of the $^{56}\text{Co}$ $Q_\gamma$. As discussed in § 1, we assume that the positrons are completely locally trapped and so this kinetic energy is deposited locally. With completely local deposition the positron kinetic energy becomes the dominant source of energy deposition in SNe Ia sometime in the interval 200–300 days after the explosion because of the increasing transparency of the ejecta to $\gamma$-rays. For example, for model W7 we find that the positron kinetic energy deposition surpasses the $\gamma$-ray energy deposition at about day 227 after the explosion. Even without complete local trapping the positron kinetic energy must become dominant in SNe Ia eventually as we know from the analysis of SN Ia light curves and absolute spectra (see § 1). Other kinds of supernovae are much denser than SNe Ia, and so are much more opaque to $\gamma$-rays. The positron kinetic energy is less important for them.

The atomic electrons released in a radioactive decay typically have a total kinetic energy of order a few kilo-electron-volts. Because the atomic electrons almost certainly deposit their kinetic energy entirely locally, the atomic electrons could make a significant contribution to energy deposition at least at late times and from the decay of longer lived species than $^{56}\text{Co}$ (see § 1).

6. APPROXIMATIONS AND COMPARISON TO SSH

In this section, we discuss the adequacy of some of the approximations we have made and compare the LS procedure to the SSH procedure.
The most obvious approximation is the use of the grey radiative transfer itself. SSH have shown, however, grey radiative transfer can potentially be done very accurately for supernovae: global deposition errors within $\sim 2\%$ percent and local deposition errors of only a few percent can be achieved with the optimized SSH procedure. Consequently, it is how the grey radiative transfer is done that determines the actual accuracy.

The most significant approximation is the replacement of Compton opacity (which is the dominant $\gamma$-ray opacity in supernovae) by what we have called the iso-Compton opacity (see § 2). The iso-Compton opacity approximation divides Compton opacity into two approximate components. The forward component causes the nearly-forward, nearly-coherently scattered $\gamma$-rays to be exactly forward and coherently scattered, and thus allows them to be treated simply as part of the 0th order $\gamma$-ray field. The iso-Compton component (which is the iso-Compton opacity itself) creates the approximate scattered $\gamma$-rays fields with which we need to deal. The flux that needs to be treated as scattered and thus its importance are greatly reduced by using the iso-Compton opacity approximation. The iso-Compton opacity can be adjusted through its $g$ parameter.

We argued in § 2 that a priori $g = 1$ seems best and the results described below confirm that $g = 1$ gives reasonable accuracy.

Because the 0th order field is treated numerically in the LS procedure and we have chosen a mean opacity prescription (i.e., the emissivity-weight mean opacity) favorable for the supernova case and we have shown that the value of the mean opacity is not very sensitive to the exact mean opacity prescription (see § 3.2), we believe that the 0th order deposition in itself is probably treated rather well. Now much more than half the $\gamma$-ray energy is absorbed in the 0th order (see § 3.2, Tables I and II). Thus any error contribution from the nonzero order field treatment will be quite limited insofar as the iso-Compton opacity approximation is valid. To be specific, error contributions from the nonzero order treatments of $^{56}$Co and $^{56}$Ni will be less, and probably much less, than 20% and 40%, respectively. In fact, $^{56}$Ni has such a short half-life (5.9 days) that it largely decays while supernova ejecta is optically thick and deposition is nearly entirely local. Thus even a crude procedure can be used to obtain nearly exact results for $^{56}$Ni in supernovae. One can, for example, use the $^{56}$Co mean opacity values for $^{56}$Ni.

For the nonzero order field treatment we will have error from the LS approximation in addition to that from the iso-Compton and grey approximations. The error in using the grey approximation in the nonzero scattering orders will be small again insofar as the iso-Compton opacity approximation is valid (see § 3.2). Recall that the LS approximation accounts for the nonzero order field deposition analytically through the effective absorption opacity. The LS approximation causes the nonzero order $\gamma$-ray transfer to be only crudely treated, except in the optically thin and thick limits where the LS approximation becomes exact for supernovae as discussed in § 4. One particular problem with the LS approximation is that minima and maxima of the 0th order $\gamma$-ray field will tend to cause under- and overestimates of the higher order $\gamma$-rays fields, and thus under- and overestimates of the energy deposition.

There is also error in the LS procedure from the neglect of time-dependent and non-static radiative transfer effects. We discuss the size of this error at length in Appendix A.

How does the LS procedure compare to the (optimized) SSH procedure? As mentioned in § 1, the SSH procedure uses a pure absorption mean opacity (which we will call the SSH mean opacity and designate by $\kappa_{\text{SSH}}$) in a numerical radiative transfer. The SSH procedure uses a single value for the SSH mean opacity for a given epoch and for all radioactive species which are treated as a single energy source. (SSH themselves consider only $^{56}$Co and $^{56}$Ni.) To compare the SSH procedure to the LS procedure, let us consider a system with only a single radioactive species for the moment. In this case, the SSH mean opacity is in effect the 0th order mean absorption opacity of the LS procedure (for the that radioactive species) plus an extra amount of opacity that accounts for the absorption of the scattered fields plus perhaps time-dependent and non-static effects. Recall from § 2 that simply neglecting the scattering component of opacity will tend to underestimate net absorption. The extra absorption opacity to account for the scattered fields is what the LS procedure effectively obtains by treating the scattered fields in the LS approximation. The SSH procedure obtains
the extra absorption opacity by fitting to accurate Monte Carlo calculations. Since those Monte Carlo calculations can also include time-dependent and non-static effects, those effects can be incorporated into the SSH procedure. The optimum SSH mean opacity must be relatively large at early times when optical depth is high and scattered fields are most important. As time increases and optical depth decreases, the optimum SSH mean opacity decreases. In the optically thin limit, the optimum SSH mean opacity would reduce to the 0th order absorption opacity of the LS procedure if no time-dependent and non-static effects were incorporated in the SSH procedure or if they were negligible.

When multiple radioactive species are included in the system, then in the LS procedure they must be treated individually since emissivity weighted mean opacities cannot be calculated a priori for multiple species with their different time-varying abundances. But in the SSH procedure the single SSH mean opacity is obtained by a fitting procedure and does not need to be calculated a priori. This is what allows the radioactive species to be treated together as a single energy source.

That the SSH procedure cannot in general be optimized without comparison to more accurate calculations is probably not a severe problem in fact. For the model W7 cases SSH examined, there is a range of SSH mean opacity values which yield reasonable accuracy for all epochs. This range is $\sim 0.025 - 0.03 \text{ cm g}^{-1}$ for $\mu_e = 2$. For other $\mu_e$ cases, the range should be scaled with $\mu_e^{-1}$ (see § 3.2.) For the model W7 cases examined by SSH, the error in global deposition in using a non-optimum value that is from the reasonable accuracy range will be less, and often much less, than $\sim 20 \%$ (SSH). Nevertheless the reliance of the SSH procedure on a free parameter is somewhat unsatisfactory. There may be cases where the SSH procedure with a non-optimized SSH mean opacity yields very bad results locally or globally. Our procedure without free parameters has the advantage that it adapts automatically to the optical depth conditions.

The accuracy of the LS procedure is probably best tested by comparison to an accurate Monte Carlo procedure. This comparison is beyond the scope of this paper. We have, however, compared the results for model W7 of the LS procedure to those obtained using the SSH procedure with the optimized mean opacities determined by SSH. Those optimized mean opacities did incorporate some time-dependent and non-static effects (Sutherland 1998). The only radioactive sources considered are $^{56}\text{Co}$ and $^{56}\text{Ni}$. For the comparison we have not used exactly the W7 mean opacities from Table II in § 3.2. Because of differences from SSH in $\gamma$-ray opacity data and in the versions of model W7 used, those values are not quite consistent with the counterpart values used by SSH. Therefore we slightly adjusted our mean opacities for model W7 to force consistency with SSH. For example, instead of using the $^{56}\text{Co} \kappa_0^a = 0.0249 \text{ cm}^2 \text{g}^{-1}$ (which is derivable from Table II), we use $\kappa_0^a = 0.0255 \text{ cm}^2 \text{g}^{-1}$ for $^{56}\text{Co}$ in order to agree with the optically thin limit optimum $\kappa_{\text{SSH}}^a = 0.0255 \text{ cm}^2 \text{g}^{-1}$ found by SSH. (Note the SSH $\kappa_{\text{SSH}}^a = 0.0255 \text{ cm}^2 \text{g}^{-1}$ value seems to have been negligibly affected by any time-dependent, non-static effects.) It should be emphasized that exact consistency with the SSH parameters has not been obtained and recalled that the SSH results themselves are not globally accurate to better than $\sim 2 \%$ relative to their Monte Carlo results. We judge that global discrepancies of less than $2 \%$ from the SSH results to be negligible.

In Figure 1 we show the $\gamma$-ray energy deposition functions for model W7 calculated for day 110 after explosion using the LS procedure and the SSH procedure with three different values of $\kappa_{\text{SSH}}^a$. By day 110 the $^{56}\text{Ni}$ is virtually all gone and the energy source is almost entirely $^{56}\text{Co}$. The deposition function (which differs from the deposition function used by SSH) is energy deposited per unit mass divided by the mean radioactive energy generated per unit mass for the whole model. The integral of the deposition function with respect to mass fraction (which we call the net deposition) is the ratio of total energy deposited to total energy generated.

The table on the figure identifies the calculation and gives the calculation net deposition, $\gamma$-ray optical depth (the $^{56}\text{Co}$ effective absorption opacity optical depth in the LS case and the SSH mean opacity optical depth in the SSH cases), and, for the SSH calculations, the $\kappa_{\text{SSH}}^a$ value. For the LS procedure the effective absorption opacities (the $\kappa_{\text{eff}}^a$’s) are variables of course, and so we only show the $^{56}\text{Co} \kappa_0^a$ which is the lower...
limit of the $^{56}\text{Co }\kappa_{\text{eff}}$. The Thomson optical depth shown in the figure counts all electrons, free and bound. Although the Thomson optical depth is not a directly relevant physical quantity, it is a useful characteristic of the model.

The optimum $\kappa_{\text{SSH}}^a$ value for day 110 is 0.0264 cm$^2$ g$^{-1}$. The deposition function for this value is in fairly close agreement with the LS procedure deposition function. The LS procedure net deposition 0.1300 is about 4% larger than the optimum SSH procedure net deposition 0.1247. This 4% discrepancy is in fact the maximum discrepancy from the optimized SSH procedure for all epochs. If we add this 4% discrepancy to the maximum error in the SSH results of $\sim 2\%$, we obtain an estimate of the maximum error in the LS procedure net deposition of $\sim 6\%$. This result is derived only from model W7 results, but we will take it as a general estimate of the maximum error in the LS procedure. It may well be an underestimate: we are not able to make the LS procedure parameters exactly consistent with the counterpart SSH procedure parameters and the SSH optimization does not seem to have incorporated all time-dependent, non-static effects. From the discussion in Appendix A, a high estimate of the maximum error in the LS procedure is of order 20%.

At days 50, 150, 200, and 300 the deviations of the LS procedure results from the SSH procedure results are $-0.2\%$, $3\%$, $2\%$, and $-0.3\%$, respectively. At times earlier than day 50 and later than day 300, the discrepancies (i.e., the absolute values of the deviations) are always less than 2%, and thus we judge them to be negligible. This is not unexpected given our arguments that the LS procedure should have negligible error in the optically thin and thick limits (see §4). The fairly good agreement with the optimized SSH procedure setting $g = 1$ for the iso-Compton opacity (see §2) suggests that $g = 1$ may be generally good. Values for $g$ significantly different from 1 did less well. We did not search for an optimum $g$ value, however, since that value could only be system-specific.

To test the range sensitivity of the SSH procedure to $\kappa_{\text{SSH}}^a$, we have also calculated for day 110 the SSH deposition function for $\kappa_{\text{SSH}}^a = 0.0255$ cm$^2$ g$^{-1}$ which is the optically thin limit value and for $\kappa_{\text{SSH}}^a = 0.0277$ cm$^2$ g$^{-1}$ (more precisely 0.02769 cm$^2$ g$^{-1}$) which yields the same net deposition as the LS procedure. We can see that the 0.0255 cm$^2$ g$^{-1}$ value yields a deposition function that is rather close to the result for the optimum 0.0264 cm$^2$ g$^{-1}$ value. The 0.0277 cm$^2$ g$^{-1}$ value yields a deposition function that is in close, but not perfect, agreement with the LS procedure deposition function. The 0.0277 cm$^2$ g$^{-1}$ value is optimum for about day 75.

Both the 0.0255 cm$^2$ g$^{-1}$ and the 0.0277 cm$^2$ g$^{-1}$ values are within the reasonable accuracy range of values for the SSH procedure for model W7. We have calculated the net deposition with the SSH procedure for the 0.0255 cm$^2$ g$^{-1}$ and 0.0277 cm$^2$ g$^{-1}$ values for the first 1000 days after explosion and compared the results to the results obtained with the optimized SSH mean opacities. The 0.0255 cm$^2$ g$^{-1}$ value gives a maximum discrepancy of $\sim 9\%$ (from a deviation of $\sim -9\%$) at about day 35; at late times (i.e., after day 500) when the ejecta is optically thin the discrepancy for the 0.0255 cm$^2$ g$^{-1}$ value vanishes. The 0.0255 cm$^2$ g$^{-1}$ value, of course, underestimates the net deposition, except at late times. The 0.0277 cm$^2$ g$^{-1}$ value’s maximum deviation is $\sim 9\%$ at about day 1000; this is the asymptotic limit deviation in fact as time goes to infinity. Its deviation vanishes at about day 75 and is negative at earlier times with a minimum of $\sim -6\%$ at about day 30.

Given the foregoing discussion, it is likely that the LS procedure offers at least a modest improvement in accuracy over the unoptimized SSH procedure. Since finding the optimum SSH mean opacity for any particular system requires doing the computationally intensive $\gamma$-transfer (usually by means of a Monte Carlo calculation) that one wishes to avoid by doing a simplified $\gamma$-ray deposition calculation, the LS procedure may be the best choice for that simplified $\gamma$-ray deposition calculation.
7. CONCLUSIONS

We have developed a simplified, grey radiative transfer procedure sans free parameters for energy deposition in supernovae. This procedure does numerical radiative transfer to handle the 0th order $\gamma$-ray field (i.e., the unscattered $\gamma$-ray field direct from the nuclear decay) and uses the local-state (LS) approximation for treating the higher order (i.e., scattered and multiply scattered) $\gamma$-ray fields. Because we rely on the LS approximation we call the procedure the LS grey radiative transfer procedure or LS procedure for short. In determining the scattered fields we also rely on an approximation of the Compton opacity which we call the iso-Compton opacity. The parameters needed for an LS procedure calculation for radioactive $^{56}$Co and $^{56}$Ni for a range of composition cases are given in Tables I and II ($\S$ 3.2), and Table III ($\S$ 5).

Probably the best test for the LS procedure would be a comparison to an accurate Monte Carlo procedure. Such a test, however, is beyond the scope of this paper. We have done a comparison to the simplified, grey radiative transfer procedure of Swartz, Sutherland, & Harkness (1995) (i.e., the SSH procedure) which requires the adjustment of a free parameter to obtain optimum results. This comparison suggests that the LS procedure will be modestly more reliable overall than the unoptimized SSH procedure. The comparison also suggests that the maximum error in an LS procedure result for net deposition could be as low $\sim 6\%$. An examination of the time-dependent, non-static effects on radiative transfer, however, suggests the maximum error could be as high as of order 20\% (see Appendix A). For the present, we estimate the maximum error in an LS procedure calculation to be of order 10\%. Only an extensive comparison to a truly high accuracy $\gamma$-ray transfer procedure can definitively determine the actual maximum error in using the LS procedure. Such a comparison is left for future work.

Since finding the optimum SSH mean opacity requires doing the detailed (e.g., Monte Carlo) radiative transfer one wants to avoid in using a simplified $\gamma$-ray energy deposition procedure, the LS procedure may be the best choice for that simplified procedure. The extra effort in developing and running an LS procedure code beyond that of an SSH procedure code is small.

The LS procedure code used for this paper can be obtained by request from the author.

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APPENDIX A

TIME-DEPENDENT, NON-STATIC RADIATIVE TRANSFER

In this appendix, we discuss the errors arising from the neglect in the LS procedure of the effects of time-dependent, non-static radiative transfer. Before doing so we need to specify the supernova velocity field. This field a day at most after explosion is ordinarily homologous expansion where the matter elements move with a range of constant velocities and were effectively at a point at time $t = 0$, the time of explosion. The radius of a matter element at any time $t$ after $t = 0$ is given by

$$r = vt,$$

where $v$ is the matter element’s velocity. Thus velocity can be used as a comoving coordinate for homologous
expansion.

The characteristic velocity “radius”, \( v_{\text{ch}} \), of significant \( \gamma \)-ray energy deposition in SNe Ia is \( \sim 10^9 \) cm s\(^{-1}\). Other types of supernovae probably have smaller deposition radii by a factor of 2 or 3. We will assume \( v_{\text{ch}} = 10^9 \) cm s\(^{-1}\) below.

Let us first consider time dependence. The time-dependent effects we will consider do not seem to have been included in SSH’s Monte Carlo calculations, and thus the errors we discuss are in addition errors relative to those in the optimized SSH procedure.

Since most of the \( \gamma \)-ray energy is lost in the first non-forward Compton scattering (i.e., the first iso-Compton “scattering”), the characteristic distance for energy loss is the smaller of the mean free path for the first iso-Compton “scattering” or the characteristic deposition spatial radius (i.e., \( v_{\text{ch}} t \)). Thus the characteristic lifetime of a \( \gamma \)-ray, \( \Delta t \), will satisfy inequality

\[
\Delta t \lesssim \frac{v_{\text{ch}} t}{c} \approx 0.03 \times t .
\]  

(A2)

Thus,

\[
\frac{\Delta t}{t} \lesssim \frac{v_{\text{ch}}}{c} \approx 0.03 .
\]  

(A3)

In homologous expansion the density at any velocity is proportional to \( t^{-3} \), and thus optical depth is approximately proportional to \( t^{-2} \). Therefore the change in the characteristic \( \gamma \)-ray optical depth \( \Delta \tau \) during a \( \gamma \)-ray lifetime is given by

\[
\frac{\Delta \tau}{\tau} \approx 2 \frac{\Delta t}{t} .
\]  

(A4)

Now global deposition goes roughly as \( 1 - \exp(-\tau) \). The relative change in global deposition due to a change in \( \tau \) of \( \Delta \tau \) due in turn to a change in \( t \) of \( \Delta t \) is of order

\[
\frac{\exp(-\tau) \Delta \tau}{1 - \exp(-\tau)} = \frac{\Delta \tau}{\exp(\tau) - 1} \leq \frac{\Delta \tau}{\tau} ,
\]  

(A5)

where the equality holds only asymptotically as \( \tau \) is goes to 0. Consequently, the characteristic relative error in energy deposition \( \Delta \epsilon_d / \epsilon_d \) due to neglecting expansion during a characteristic \( \gamma \)-ray lifetime will satisfy

\[
\frac{\Delta \epsilon_d}{\epsilon_d} \lesssim \frac{\Delta \tau}{\tau} \approx 2 \frac{\Delta t}{t} \lesssim \frac{2v_{\text{ch}}}{c} \approx 0.06 .
\]  

(A6)

We see that this error is limited and is less than or approximately equal to the estimated maximum error \( \sim 6 \% \) (see § 6) in the net deposition in the LS procedure relative to the optimized SSH procedure.

Besides the time since explosion \( t \), there is another important time scale relevant to time dependence: the \( e \)-folding time \( t_e \) of the radioactive decay. If \( \Delta t \) becomes comparable to \( t_e \), then deposition at some time corresponds the radioactive decay energy generation at a significantly earlier time. Since the rate of energy generation goes as \( \exp(-t/t_e) \) at least approximately (see § 5, eqs. 65 and 67), it follows, using equation (A2), that the characteristic relative error in deposition from neglecting the changing rate of energy generation will satisfy

\[
\frac{\Delta \epsilon_d}{\epsilon_d} \approx \frac{\Delta t}{t} \lesssim \frac{v_{\text{ch}} t}{c t_e} .
\]  

(A7)

For the case of \(^{56}\text{Ni}\) with \( t_e = 8.5 \) days (see § 5, Table III), we find

\[
\frac{\Delta \epsilon_d}{\epsilon_d} \lesssim 0.004 \times t_d ,
\]  

(A8)
where $t_d$ is the time since explosion $t$ measured in units of days. By day 20 when almost all the $^{56}\text{Ni}$ is gone, $\Delta \epsilon_d/\epsilon_d \lesssim 0.1$, and thus time independence should be a good approximation for the $^{56}\text{Ni}$ $\gamma$-ray deposition. In fact since the ejecta are still optically thick to $\gamma$-rays on day 20, the $^{56}\text{Ni}$ $\Delta \epsilon_d/\epsilon_d$ will be much less than 0.1. For model W7, the $^{56}\text{Ni}$ $\Delta \epsilon_d/\epsilon_d$ is of order $10^{-5}$ on day 20. For supernovae other than SNe Ia, $\Delta \epsilon_d/\epsilon_d$ will be smaller still on day 20 since these supernovae are much denser, have larger optical depths, and smaller $\gamma$-ray lifetimes. We see that for the period when $^{56}\text{Ni}$ is a significant energy source, the error in energy deposition due to neglect of the time variation of the $^{56}\text{Ni}$ energy generation rate will be quite small.

For the case of $^{56}\text{Co}$ with $t_e = 111.48$ days (see § 5, Table III), we find

$$\frac{\Delta \epsilon_d}{\epsilon_d} \lesssim 3 \times 10^{-4} \times t_d .$$  \hfill (A9)

The equality version of equation (A9) holds for SNe Ia by about 100 days or less after explosion and for other kinds of supernovae (which are denser and have larger optical depths at comparable epochs) by perhaps 1000 days after explosion. Thus by order 1000 days the ratio $\Delta \epsilon_d/\epsilon_d$ is starting to become large for any kind of supernova. We cannot expect time-independent approximation for deposition from $^{56}\text{Co}$ decay to hold as late as day 1000. In the case of SNe Ia, at 300 days after the explosion the error due to neglecting the time variation in energy generation could be as large as $\sim 10\%$.

Now we turn to non-static effects. These effects were at least partially accounted for in the optimized SSH procedure (Sutherland 1998). Consider the specific intensity absorbed at some point from a beam originating some distance away. Let 1 designate the frame of absorption and 0 the frame of emission. The beam in the frame of emission has specific intensity $I_0(v_0)$ and band width $dv_0$. First assume a static case. The energy absorbed (per unit volume per unit time per unit solid angle) from the beam is

$$\chi_1(v_1)I_1(v_1) dv_1 = \chi_1(v_1)I_0(v_0) dv_0 ,$$  \hfill (A10)

where the specific intensity, frequency, and band width in frame 1 are the equal to those in frame 0 because of the static condition. If we now assume that the originating point of the beam is moving away from the absorption point with velocity $\beta$ (measured in units of $c$), then using the relativistic transformations (e.g., Mihalas 1978, p. 495) the energy absorbed is

$$\chi_1(v'_1)I_1(v'_1) dv'_1 = \chi_1(v'_1)I_0(v_0) dv_0\psi^4 ,$$  \hfill (A11)

where

$$v'_1 = v_0\psi , \quad dv'_1 = dv_0\psi , \quad \text{and} \quad \psi = \gamma (1 - \beta) .$$  \hfill (A12)

The relativistic transformations account for the energy changes due to the Doppler shift, advection, and aberration. If we now assume that extinction $\chi_1$ is frequency-independent, then the energy absorbed in the moving case is reduced by the relative amount $|\psi^4 - 1|$ from the energy absorbed in the static case.

For supernovae in the optically thin limit, the appropriate characteristic $\beta$ value is just obtained from the characteristic velocity radius of deposition: thus $\beta = v_{ab}/c \approx 0.033$. Therefore the characteristic relative error in absorbed energy in the optically thin limit due to neglecting expansion is

$$|\psi^4 - 1| \approx 4\beta \approx 0.13 .$$  \hfill (A13)

In optically thick cases the energy loss is smaller because the $\gamma$-rays traverse smaller velocity shifts before absorption.

Now in general the extinction will not be frequency-independent (i.e., $\chi_1(v'_1) \neq \chi_1(v_1)$ in general), and thus there is another possible error in energy absorbed due to neglecting expansion. However, the effective absorption opacity for the first scattering of the decay $\gamma$-rays in supernovae is actually fairly constant.
We estimate that the Doppler shift for $\beta = 0.033$ increases effective absorption opacity and hence energy absorption by only of order 1%. Therefore the error in energy deposition due to the frame transformation (see eq. [A13]) is much more important than the error due to the Doppler shift’s effect on extinction.

From the foregoing analysis we can see there that could be errors in energy deposition of up to of order tens of percent from neglecting the effects of time-dependent, non-static radiative transfer. The errors tend to be largest in the optically thin epoch. Some of these errors may cancel. If we add in a root-mean-square sense characteristic high values of the errors we have discussed for cases where the LS procedure can reasonably be used, then the result is an overall error of order 20%. In § 6 we concluded for model W7 that the maximum error in the LS procedure net deposition relative to the Monte Carlo results of SSH was $\sim 6\%$. This, however, was a limited test and it is not clear that all the effects of time-dependent, non-static radiative transfer were included in the SSH calculations. Thus, errors larger than $\sim 6\%$ are possible in the LS procedure. For now we will adopt 6% and 20% as the low and high estimates, respectively, of the maximum error of the LS procedure. More extensive testing of the LS procedure is needed to determine its actual accuracy.

**APPENDIX B**

**MATHEMATICAL BEHAVIOR OF THE LS APPROXIMATION SERIES**

In § 4, we derived the following series (used in evaluating the effective absorption opacity) from the LS approximation:

$$L = \sum_{i=0}^{\infty} \xi^a_{i} \prod_{j=0}^{i-1} \xi^e_{j} \zeta_{i+1} \quad (B1)$$

(see eq. [56]). The series $L$ is the ratio of the energy that is ultimately absorbed from fields of all orders to the energy removed (but not necessarily absorbed) from the 0th order field. In the optically thick limit for all orders $i \geq 1$, $\zeta_{i \geq 1} = 1$ and the series reduces to

$$L_{\text{thick}} = \sum_{i=0}^{\infty} \xi^a_{i} \prod_{j=0}^{i-1} \xi^e_{j} \quad (B2)$$

Here we will derive some of the properties of equation (B2).

First, we define the finite series

$$L_{\text{thick},n} = \sum_{i=0}^{n} \xi^a_{i} \prod_{j=0}^{i-1} \xi^e_{j} \quad (B3)$$

Using the relation

$$\xi^a_{i} = 1 - \xi^e_{i} \quad (B4)$$

it is straightforward to show that equation (B3) can be rewritten

$$L_{\text{thick},n} = 1 - \prod_{j=0}^{n} \xi^e_{j} \quad (B5)$$

Now, of course,

$$L_{\text{thick}} = 1 - \prod_{j=0}^{\infty} \xi^e_{j} \quad (B6)$$
If all $\xi^s_j = 1$, then
\[ L_{\text{thick}} = 0 \text{.} \quad \text{(B7)} \]
Physically, this means that there is no absorption since the opacity in all orders is pure scattering opacity. Thus no flux is absorbed at all. If there is an order $\ell$ for which $\xi^s_\ell = 0$ (implying $\xi^a_\ell = 1$), then the equation (B2) series truncates at term $\ell$ and equation (B6) shows that
\[ L_{\text{thick}} = 1 \text{.} \quad \text{(B8)} \]
Physically, this means no $(\ell + 1)$th or higher order field can be created because no flux is scattered by the $\ell$th order opacity and that because of the optical thickness all the flux removed from the 0th order field by the 0th order opacity is absorbed in a finite number of orders (i.e., in the 0th through $\ell$th orders).

Now consider the case that $\xi^s_j$ satisfies $0 < \xi^s_j \leq 1$. If
\[ \lim_{j \to \infty} \xi^s_j < 1 \text{,} \quad \text{(B9)} \]
then
\[ \prod_{j=0}^{\infty} \xi^s_j = 0 \text{,} \quad \text{(B10)} \]
(e.g., Arfken 1970, p. 286),
\[ L_{\text{thick}} = 1 \text{,} \quad \text{(B11)} \]
and all the flux is absorbed in the limit of infinite scattering. On the other hand, if
\[ \lim_{j \to \infty} \xi^s_j = 1 \text{,} \quad \text{(B12)} \]
then
\[ \prod_{j=0}^{\infty} \xi^s_j \geq 0 \text{,} \quad \text{(B13)} \]
(e.g., Arfken 1970, p. 286),
\[ L_{\text{thick}} \leq 1 \text{,} \quad \text{(B14)} \]
and all the flux may or may not be absorbed.

The optically thick limit LS approximation series for pure Compton scattering is of particular interest to us. Is $L_{\text{thick}}$ equal to 1 or not? Let us assume that a Compton scattering results in a unique energy photon (rather than in a continuum of energies dependent on the scattering angle) and that $\xi^s_j = 1$ only for zero photon energy. These assumptions are satisfied both by the our iso-Compton opacity and the angle-averaged Compton opacity (see § 2). We will examine the degradation of a single photon in the limit of infinite scattering.

First consider the case that in the limit of infinite scattering the photon energy $\alpha$ (in units of $m_e c^2$) is degraded to $\alpha_\infty > 0$. We then have $\lim_{j \to \infty} \xi^s_j < 1$ since $\xi^s_j = 1$ only for $\alpha = 0$ for both the iso-Compton opacity and the angle-averaged Compton opacity (§ 2). Thus $\prod_{j=0}^{\infty} \xi^s_j = 0$ and $L_{\text{thick}} = 1$. Now the product $\prod_{j=0}^{\infty} \xi^s_j$ is actually the amount of energy remaining with the photon in the limit of infinite scattering. If $\prod_{j=0}^{\infty} \xi^s_j = 0$, then $\alpha_\infty = 0$. Therefore there is an inconsistency showing that our premise that $\alpha_\infty > 0$ is incorrect.

Taking $\alpha_\infty = 0$ implies that $\lim_{j \to \infty} \xi^s_j = 1$ which in turn implies $\prod_{j=0}^{\infty} \xi^s_j \geq 0$. However, $\alpha_\infty = 0$ directly implies that $\prod_{j=0}^{\infty} \xi^s_j = 0$. Thus there is consistency and we conclude that optically thick limit iso-Compton opacity and angle-averaged Compton opacity do yield $\alpha_\infty = 0$ and $L_{\text{thick}} = 1$. Since these approximations to the true angle-dependent Compton opacity are consistent with the “average” Compton opacity behavior, we further conclude that in the optically thick limit the true angle-dependent Compton opacity yields $\alpha_\infty = 0$ and $L_{\text{thick}} = 1$. 
TABLES

TABLE I
Mean opacities and mean fractional component opacities
including only iso-Compton opacity

| Order | \( \bar{E}_i \) | \( \kappa_i \) | \( \xi_i^a \) | \( \xi_i^s \) | \( \bar{E}_i \) | \( \kappa_i \) | \( \xi_i^a \) | \( \xi_i^s \) |
|-------|----------------|-------------|-------------|-------------|----------------|-------------|-------------|-------------|
|       | (MeV)          | (cm\(^2\) g\(^{-1}\)) | (MeV)       | (cm\(^2\) g\(^{-1}\)) | | | | |
| 0     | 1.24226        | 0.0643      | 0.7873      | 0.2127      | 1.24226        | 0.0546      | 0.7873      | 0.2127      |
| 1     | 0.23453        | 0.1542      | 0.3565      | 0.6435      | 0.23453        | 0.1308      | 0.3565      | 0.6435      |
| 2     | 0.15085        | 0.1930      | 0.2518      | 0.7482      | 0.15085        | 0.1637      | 0.2518      | 0.7482      |
| 3     | 0.11286        | 0.2206      | 0.1964      | 0.8036      | 0.11286        | 0.1870      | 0.1964      | 0.8036      |
| 4     | 0.09070        | 0.2414      | 0.1616      | 0.8384      | 0.09070        | 0.2047      | 0.1616      | 0.8384      |
| 5     | 0.07604        | 0.2577      | 0.1376      | 0.8624      | 0.07604        | 0.2185      | 0.1376      | 0.8624      |
| \( \infty \) | 0             | 0.4006      | 0           | 1           | 0              | 0.3397      | 0           | 1           |

\( ^{56}\text{Co} \)

\( ^{56}\text{Ni} \)

NOTE.—The solar \( \mu_e \) value was constructed using the solar system abundances of Anders & Grevesse 1989.
| Order | $\bar{E}_i$ (MeV) | $\kappa_i$ (cm$^2$ g$^{-1}$) | $\xi_i^a$ | $\xi_i^a$ | $\bar{E}_i$ (MeV) | $\kappa_i$ | $\xi_i^a$ | $\xi_i^a$ |
|-------|------------------|------------------|---------|---------|------------------|---------|---------|---------|
| 0     | 1.24226          | 0.0547           | 0.7870  | 0.2130  | 1.24226          | 0.0318  | 0.7829  | 0.2171  |
| 1     | 0.23486          | 0.1307           | 0.3570  | 0.6430  | 0.23995          | 0.0827  | 0.4393  | 0.5607  |
| 2     | 0.15099          | 0.1637           | 0.2524  | 0.7476  | 0.15294          | 0.1306  | 0.4775  | 0.5225  |
| 3     | 0.11294          | 0.1873           | 0.1976  | 0.8024  | 0.11403          | 0.2006  | 0.5810  | 0.4190  |
| 4     | 0.09074          | 0.2052           | 0.1639  | 0.8361  | 0.09147          | 0.3028  | 0.6826  | 0.3174  |
| 5     | 0.07607          | 0.2194           | 0.1413  | 0.8587  | 0.07660          | 0.4459  | 0.7630  | 0.2370  |

$^{56}_{\text{Co}}$

| Order | $\bar{E}_i$ (MeV) | $\kappa_i$ (cm$^2$ g$^{-1}$) | $\xi_i^a$ | $\xi_i^a$ | $\bar{E}_i$ (MeV) | $\kappa_i$ | $\xi_i^a$ | $\xi_i^a$ |
|-------|------------------|------------------|---------|---------|------------------|---------|---------|---------|
| 0     | 0.53479          | 0.0807           | 0.5911  | 0.4089  | 0.53479          | 0.0494  | 0.6230  | 0.3770  |
| 1     | 0.18845          | 0.1443           | 0.3077  | 0.6923  | 0.19557          | 0.1043  | 0.4762  | 0.5238  |
| 2     | 0.12888          | 0.1749           | 0.2245  | 0.7755  | 0.13679          | 0.1573  | 0.5330  | 0.4670  |
| 3     | 0.09966          | 0.1964           | 0.1797  | 0.8203  | 0.10694          | 0.2298  | 0.6205  | 0.3795  |
| 4     | 0.08180          | 0.2128           | 0.1515  | 0.8485  | 0.08797          | 0.3328  | 0.7054  | 0.2946  |
| 5     | 0.06961          | 0.2259           | 0.1326  | 0.8674  | 0.07468          | 0.4764  | 0.7756  | 0.2244  |

$^{56}_{\text{Ni}}$

NOTE.—The solar composition is the solar system composition of Anders & Grevesse 1989. The mean model W7 composition is given by Thielemann et al. 1986.

The mean model W7 composition used is the final composition after all radioactive species have decayed. Earlier time compositions give somewhat different mean opacities from those in the table with the differences increasing with scattering order. The extreme case is the day 0 composition
which gives opacities that differ by \( \sim 1\% \) in the 0th order and by up to \( \sim 20\% \) in the 5th order. Experimentation, however, shows that the final composition opacities can be used for all times without adding any significant error in the energy deposition.
### TABLE III
Parameters for the radioactive decays

| Parameter                  | $^{56}$Co     | $^{56}$Ni     |
|----------------------------|---------------|---------------|
| $t_{1/2}$ (days)           | 77.27         | 5.9           |
| $t_e = t_{1/2}/\ln 2$ (days) | 111.48       | 8.5           |
| $Q_{\text{total}}$ (MeV)  | 4.5661        | 2.136         |
| $Q_\gamma$ (MeV)          | 3.62(4)       | 1.72(2)       |
| $C_\gamma$ (ergs s$^{-1}$ g$^{-1}$) | 6.48(7)+9   | 4.03(14)+10   |
| $D_\gamma$ (ergs s$^{-1}$ g$^{-1}$) | 7.02(8)+9    | —             |
| $Q_{\text{X-ray}}$ (MeV)  | 1.57(5)$-3$   | 2.35(8)$-3$   |
| $C_{\text{X-ray}}$ (ergs s$^{-1}$ g$^{-1}$) | 2.82(9)+6  | 5.5(2)+7      |
| $D_{\text{X-ray}}$ (ergs s$^{-1}$ g$^{-1}$) | 3.05(10)+6 | —             |
| $Q_{\text{KE}}^{\beta^+}$ (MeV) | 0.116(6)   | $\sim 0$      |
| $Q_{\text{KE}}^{\beta^+}/Q_\gamma$ | 0.032(2)    | $\sim 0$      |
| $\beta^+$ fraction (%)    | 19.0(9)       | $< 0.0013$    |
| $E_{\text{KE}}^{\beta^+}$ | 0.61(3)       | $\sim 0$      |
| $Q_{\text{atomic el.}}$ (MeV) | 3.6(3)$-3$  | 6.9(3)$-3$    |
| $C_{\text{lepton}}$ (ergs s$^{-1}$ g$^{-1}$) | 2.14(11)+8   | 1.62(9)+8     |
| $D_{\text{lepton}}$ (ergs s$^{-1}$ g$^{-1}$) | 2.32(12)+8 | —             |

**NOTE.**—The meanings of most of the symbols follow from the text. The $t_{1/2}$ quantity is half-life. The $E_{\beta^+}^{\text{KE}}$ quantity is the mean kinetic energy of a positron. We have put the uncertainties in the last digits of the quantities in brackets and have written $\times 10^{\pm k}$ as $\pm k$.

The values have been taken or derived from Browne & Firestone 1986 and Huo 1992. The uncertainties given in the references have been treated as standard deviations in obtaining the uncertainties in the derived quantities. One of the quantities we derived was the $^{56}$Co mean
positron kinetic energy. We calculated this assuming an allowed $\beta$-decay spectrum for all three positron channels using the Fermi functions given by Rose 1955. The overwhelmingly dominant positron channel has a spectrum that is consistent with an allowed $\beta$ decay (Pettersson, Bergman, & Bergman 1965).

The energy generation $C$ and $D$ coefficients for the positron and atomic electron kinetic energies have been summed and subscripted by lepton. We assume local deposition for both decay products, and so they can be treated together. The individual $C$ and $D$ coefficients for positron and atomic electron kinetic energies can be obtained by scaling the given $C$ and $D$ coefficients: e.g.,

$$C_{\beta^+} = C_{\text{lepton}} Q_{\beta^+} / (Q_{\beta^+} + Q_{\text{atomic el}}).$$
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FIGURE LEGENDS

FIG. 1.—The $\gamma$-ray energy deposition function for model W7 on day 110 calculated using the LS procedure and the SSH procedure with three different values of the SSH mean opacity $\kappa_{\text{SSH}}$. The quantities in the table are described in the text. The units of opacities in the table are cm$^2$ g$^{-1}$. The other quantities are dimensionless.
Fig. 1

- LS procedure: $\text{net dep.} = 0.1300; \tau_{\text{eff}} = 0.445; \kappa_0^a = 0.0255$
- SSH procedure: $\text{net dep.} = 0.1247; \tau_{\text{SSH}} = 0.401; \kappa_{\text{SSH}}^a = 0.0264$
- SSH procedure: $\text{net dep.} = 0.1210; \tau_{\text{SSH}} = 0.388; \kappa_{\text{SSH}}^a = 0.0255$
- SSH procedure: $\text{net dep.} = 0.1300; \tau_{\text{SSH}} = 0.421; \kappa_{\text{SSH}}^a = 0.0277$

$\tau(\text{Thomson}) = 2.89$