The matching polytope does not admit fully-polynomial size relaxation schemes

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April 3, 2014

Abstract

The groundbreaking work of Rothvoß [2014] established that every linear program expressing the matching polytope has an exponential number of inequalities (formally, the matching polytope has exponential extension complexity). We generalize this result by deriving strong bounds on the polyhedral inapproximability of the matching polytope: for fixed $0 < \varepsilon < 1$, every polyhedral $(1 + \varepsilon/n)$-approximation requires an exponential number of inequalities, where $n$ is the number of vertices. This is sharp given the well-known $\rho$-approximation of size $O((\rho/n^{\rho/(\rho-1)})$ provided by the odd-sets of size up to $\rho/(\rho - 1)$. Thus matching is the first problem in P, whose natural linear encoding does not admit a fully polynomial-size relaxation scheme (the polyhedral equivalent of an FPTAS), which provides a sharp separation from the polynomial-size relaxation scheme obtained e.g., via constant-sized odd-sets mentioned above.

Our approach reuses ideas from Rothvoß [2014], however the main lower bounding technique is different. While the original proof is based on the hyperplane separation bound (also called the rectangle corruption bound), we employ the information-theoretic notion of common information as introduced in Braun and Pokutta [2013], which allows to analyze perturbations of slack matrices. It turns out that the high extension complexity for the matching polytope stem from the same source of hardness as for the correlation polytope: a direct sum structure.

1 Introduction

In recent years, the expressive power of linear programs came under scrutiny. The core (motivating) question is which polytopes or combinatorial optimization problems can be expressed as linear programs with a small number of inequalities. In contrast to other complexity approaches, we disregard the encoding length of the coefficients and we only count the number of inequalities. The resulting notion of complexity, called extension complexity, provides an alternative notion of complexity independent of P vs. NP. Formally, the extension complexity $\text{xc}(P)$ of a polytope $P$ is the minimum number of inequalities needed in any polyhedral description $Q$, so that there exists an affine linear map $\pi$ with $P = \pi(Q)$. Such a $(Q, \pi)$ is called an extended formulation of $P$.

Most of the recent progress is ultimately rooted in Yannakakis [1988, 1991] seminal paper ruling out symmetric linear programs with a polynomial number of inequalities for the TSP polytope and the matching polytope. It was the innocent question whether the same still holds true when the symmetry assumption is removed that spurred a significant body of work (see Section 1.1 below). Yannakakis’s question was answered in the affirmative in Fiorini et al. [2012b] for the TSP polytope.
via the correlation polytope, and in the groundbreaking work [Rothvoss 2014] for the matching polytope.

The fact that the matching problem has high extension complexity raises some fundamental questions. After all, matching can be solved in polynomial time. Two key questions are (1) how well we can approximate the matching polytope from a polyhedral perspective, and (2) what commonality the correlation polytope and the matching polytope possess so that both require an exponential number of inequalities. In this work, we answer both of these questions.

We generalize the results in [Rothvoss 2014] to show in Theorem 3.1 that the matching polytope cannot be approximated within a factor of $1 + \Theta(1/n)$ with a linear program with a polynomial number of inequalities. Thus, the matching polytope does not admit a fully polynomial-size relaxation scheme (FPSRS), the polyhedral equivalent of an FPTAS, i.e., it cannot be $(1 + \varepsilon)$-approximated with a linear program of size $\text{poly}(n, 1/\varepsilon)$. This complements the folklore polynomial-sized $(1 + 1/k)$-approximation for the matching (considering odd sets up to size $k$, see Example 2.10) and provides a full analysis of the extension complexity of the matching polytope across all approximation regimes.

Further, our approach is based on a direct information-theoretic argument for the exponential lower bound on the polyhedral approximations of the matching polytope, extending the techniques developed in [Braun and Pokutta 2013]. This proof exhibits key commonalities with the one for the correlation polytope, revealing that the reason for high extension complexity of both, the matching polytope and the correlation polytope, is an (almost identical) direct sum structure contained in both problems.

1.1 History of extended formulations

As mentioned above, the interest in extended formulation was initiated by the question of Yannakakis whether the TSP polytope and the matching polytope have (asymmetric) linear programs with a polynomial number of inequalities. In [Kaibel et al. 2010] it was shown that the symmetry actually can make a significant difference for the size of extended formulations. The authors studied the $\ell$-matching polytope (polytope of all matchings with exactly $\ell$ edges), closely related to Yannakakis’s work prompting the question whether every 0/1 polytope has an efficient polyhedral lift (note that this is independent of P vs. NP as we disregard encoding length). This question was answered in the negative in [Rothvoss 2012], establishing the existence of 0/1 polytopes requiring an exponential number of inequalities in any of their formulations, and it was recently extended to the case of SDP-based extended formulations in [Briët et al. 2013]. The same technique was used in [Fiorini et al. 2012d] to derive lower bounds on the extension complexity of polygons in the plane, which are of prototypical importance for many related applications; the analogous results for the SDP case have been obtained in [Briët et al. 2013].

Shortly after Rothvoß’s result it was shown in [Fiorini et al. 2012c] that the first half of Yannakakis’s question is in the affirmative: the TSP polytope requires an exponential number of linear inequalities in any linear programming formulation, irrespective of P vs. NP. In fact, it was shown that the correlation polytope (or equivalently the cut polytope) has extension complexity $2^n$ where $n$ is the length of the bit strings and that the stable set polytope (over a certain family of graphs) as well as the TSP polytope have extension complexity $2^{\Omega(n^{1/2})}$ where $n$ is the number of nodes in the graph. In [Braun et al. 2012] the result for the correlation polytope was generalized to the approximate case, showing that the correlation polytope and (a linear encoding of) the CLIQUE problem cannot be approximated with a polynomial-size linear program better than $n^{1/2-\varepsilon}$, which was subsequently improved in [Braverman and Moitra 2012] to $n^{1-\varepsilon}$ matching Håstad’s celebrated inapproximability result for the CLIQUE problem (see Håstad [1999]). In [Braun and Pokutta 2013]...
the results for the correlation polytope and the CLIQUE problem were further improved to average-case type results via a general information-theoretic framework to lower bound the nonnegative rank of matrices. In *Braun et al. [2013]* these average-case arguments were used to show that the average-case (as well as high-probability) extension complexity of the stable-set problem is high. Additional bounds for various polytopes including the knapsack polytope have been established in *Pokutta and Van Vyve [2013]* and *Avis and Tiwary [2013]* using the reduction mechanism outlined in *Fiorini et al. [2012b]*.

The main tool for lower bounds is Yannakakis’s result that the extension complexity of a polytope is equal to the nonnegative rank of any of its slack matrices. Combinatorial as well as communication based lower bounds for the nonnegative rank have been explored in *Faenza et al. [2012]*, *Fiorini et al. [2012a]*. At the core of all of the above super-polynomial lower bounds on the extension complexity is the all-important UDISJ problem whose partial matrix appears as a pattern in the slack matrix of these problems. The current best lower bound on the nonnegative rank of the UDISJ matrix is established in *Kaibel and Weltge [2013]* by a remarkably short combinatorial argument.

One notable exception not using UDISJ is *Chan et al. [2013]*, where lower bounds on the size of linear programs of constraint satisfaction problems have been obtained by showing that an arbitrary linear programming formulation cannot do much better than the linear program obtained from the Sherali-Adams hierarchy (of compatible size). Another exception is the very recent result of *Rothvoss [2014]*, which finally answers the second half of Yannakakis’s question, showing that the matching polytope has exponential extension complexity. The proof is based on Razborov’s technique (see *Razborov [1992]*), and provides the second base matrix with linear rank but exponential nonnegative rank, namely, the slack matrix of the matching problem.

Similar to the approach in *Karchmer et al. [1995]*, where via LP-duality it is shown that the fractional rectangle covering number can be computed from the largest rectangle under adversarial distributions, the lower bounding technique for the results in *Fiorini et al. [2012b]* as well as *Rothvoss [2014]* analyze a single rectangle that serves as a duality certificate obtained from the hyperplane separation lemma (also known as rectangle corruption bound); see *Rothvoss [2014]* for an explicit statement of this technique. The analysis of the rectangles is highly geared towards the respective problem, potentially hiding the key commonalities between the matching polytope and the correlation polytope (which is linearly isomorphic to the cut polytope).

There are only a few results on approximate (linear) extended formulations. For any fixed $\varepsilon > 0$, a polynomial-sized linear program giving a $(1 + \varepsilon)$-approximation of the knapsack problem has been provided in *Bienstock [2008]*. For the CLIQUE polytope, exponential lower bounds on approximate extension complexity were established in see *Braun et al. [2012]*, *Braun and Pokutta [2013]*, *Braun et al. [2013b]*. Recently, an exponential lower on the approximate polyhedral complexity of the metric capacitated facility location problem was established in *Kolliopoulos and Moysoglou [2013]*, however only in the original space; extended formulations were not considered.

### 1.2 Related work

We recall the works whose methodology is closely related to ours. The most closely related one is *Braun and Pokutta [2013]* providing a general framework for lower bounding the extension complexity of polytopes in terms of information, motivated by *Braverman et al. [2012]* as well as *Jain et al. [2013]*. A dual information-theoretic approach, similar to the one for the fractional rectangle covering number has been explored in *Braun et al. [2013b]*, however, we will stick to the primal approach here, dealing directly with distribution over potential rank-1 matrices. The corner stone of our framework is the notion of common information introduced by *Wyner [1975]* for (a completely
unrelated) use in information theory. In order to cross-over from the information-theoretic argument into actual quantities and probabilities [Braun and Pokutta, 2013] used the Hellinger distance inspired by Bar-Yossef et al. [2004], however here it is more effective to employ Pinsker’s inequality for the estimation. The inapproximability framework for the matching polytope is based on Braun et al. [2012].

1.3 Contribution

Our main result is Theorem 3.1 proving inapproximability of the matching polytope. Its contribution is three-fold:

1. **Polyhedral inapproximability of the matching problem:** As mentioned above, the matching problem can be solved in polynomial time, even though its extension complexity is exponential. However, it can be \((1 + \varepsilon)\)-approximated by a linear program of polynomial size, i.e., it admits a *polyhedral-size relaxation scheme (PSRS)*, which is the polyhedral equivalent of a PTAS. A fundamental question is whether we can even go a step further and find a linear program of size \(\text{poly}(n, 1/\varepsilon)\) approximating the matching polytope within a factor of \(1 + \varepsilon\), i.e., whether it admits a *fully polynomial-size relaxation scheme (FPSRS)*; the polyhedral equivalent of an FPTAS. We resolve this question in the negative, showing that the (linear) extension complexity and (classical) computational complexity are very different notions. The non-existence of an FPSRS for matching can be also interpreted as an indicator that matching might be especially hard from a polyhedral point of view, similar to the notion of strong NP-hardness which also precludes the existence of an FPTAS (in most cases) in the computational complexity setup.

2. **Information-theoretic proof:** We provide an information-theoretic proof for the lower bound on the extension complexity of the matching polytope and its polyhedral approximations. The proof is based on an extension of the framework in Braun and Pokutta [2013]. By arguing directly via the distribution of potential factorizations, rather than considering single rank-1 factors or rectangles as typically required by Razborov’s method, the setup can be considerably simplified and the information-theoretic framework naturally lends itself to polyhedral approximations. We obtain a simple and short proof that provides additional insight into the structure of matchings and its polyhedral hardness.

3. **High extension complexity implied by direct sum structure:** The information-theoretic approach is well suited for proving exponential lower bounds by partitioning up the structure \(P\) into copies of a fixed-size substructures \(P_0\), and applying a direct sum argument to show

\[
\log \text{xc}(P) \geq \Theta(n) \cdot \log \text{xc}(P_0),
\]

where \(\text{xc}(Q)\) denotes the extension complexity of \(Q\). This was already used for the correlation polytope in Braun and Pokutta [2013], and we show that the matching polytope exhibits a similar direct sum structure. The partition we use is a simplified variant of the one in Rothvoß [2014], however it is still slightly more involved than the one for the correlation polytope due to additional structure we need to consider.

In the above direct sum framework, the difficulty arises by actually having an information-theoretic quantity in place of \(\log \text{xc}(P_0)\), which is used to make the direct sum argument work in first place. Thus a positive constant lower bound is not immediate, in particular when considering approximations the constant might tend to 0 for \(n\) growing. Information theory further provides
a nice analysis of this lower bound: via Yannakakis’s Theorem (see Theorem 2.6) an extension is equivalent to a nonnegative factorization of the slack matrix, which we interpret as distribution. We then analyze its information structure, observing that rank-1 factors far away from a certain standard factor contribute significantly to our lower bounding quantity.

The authors believe that the same approach can be used to obtain high extension complexity (and polyhedral inapproximability) for a variety of other problems.

1.4 Outline

We start with preliminaries in Section 2, introducing necessary notions and notations from information theory in Section 2.1 as well as providing a recap of the theory of (approximate) extended formulations in Section 2.2. In Section 2.3 we provide the connection between common information and the nonnegative rank which is subsequently extended to provide bounds for the matching polytope in Section 3. We choose notations mostly consistent with [Rothvoß 2014] for easy relation of both approaches.

2 Preliminaries

We use \([k] := \{1, \ldots, k\}\) as a short-hand. Let \(\log(.)\) denote the logarithm to base 2. We will denote random variables by bold capital letters such as \(\mathbf{A}\) to avoid confusion with sets that we will also denoted by capital letters. Events will be denoted by capital script letters such as \(\mathcal{E}\) when not written out and combinations of random variables and events (possible being solely an event or a random variable), which we will refer to as conditions or conditionals, will be denoted by script bold letters such as \(\mathbf{Z}\). We use \(\perp\) to indicate independence: e.g., \(\mathbf{A} \perp \mathbf{B}, \mathcal{E}\).

2.1 Information-theoretic basics

We briefly recall standard basic notions from information theory in this section. We refer the reader to [Cover and Thomas 2006] for more details, and as an excellent introduction. We measure information in bits, as is standard for discrete random variables.

We will now introduce the notion of mutual information which is at the core of our arguments. The mutual information of two discrete random variables \(\mathbf{A}, \mathbf{B}\) is defined as

\[
\mathbb{I}[\mathbf{A}; \mathbf{B}] := \sum_{a \in \text{range}(\mathbf{A})} \sum_{b \in \text{range}(\mathbf{B})} P[\mathbf{A} = a, \mathbf{B} = b] \log \frac{P[\mathbf{A} = a, \mathbf{B} = b]}{P[\mathbf{A} = a] \cdot P[\mathbf{B} = b]}.
\]

It captures how much information about \(\mathbf{A}\) is leaked by considering \(\mathbf{B}\); and vice versa: mutual information is symmetric. We will often have \(\mathbf{A}\) and \(\mathbf{B}\) being a collection of random variables. We use a comma to separate the components of \(\mathbf{A}\) or \(\mathbf{B}\), and a semicolon to separate \(\mathbf{A}\) and \(\mathbf{B}\) themselves, e.g., \(\mathbb{I}[\mathbf{A}_1, \mathbf{A}_2; \mathbf{B}] = \mathbb{I}[(\mathbf{A}_1, \mathbf{A}_2); \mathbf{B}]\). We can naturally extend mutual information to conditional mutual information

\[
\mathbb{I}[\mathbf{A}; \mathbf{B} | \mathbf{C}, \mathcal{E}] := \mathbb{E}_\mathbf{C}[\mathbb{I}[\mathbf{A} | \mathbf{C}, \mathcal{E}) ; (\mathbf{B} | \mathbf{C}, \mathcal{E})]]
\]

by using the respective conditional distributions, where \(\mathbf{C}\) is a random variable and \(\mathcal{E}\) is an event. Note that the expectation is implicitly taken over random variables in the condition.

We shall use the following bounds on mutual information. An obvious upper bound is

\[
\mathbb{I}[\mathbf{A}; \mathbf{B} | \mathbf{Z}] \leq \log |\text{range}(\mathbf{A})|,
\]
which provides the lower bound on the logarithm of the nonnegative rank in Lemma 2.13.

Exponential lower bounds of the nonnegative rank will be obtained via the direct sum property, which states that for mutually independent pairs \((A_1, B_1), \ldots, (A_n, B_n)\) given a condition \(Z\) we have

\[
\mathbb{I} [A_1, \ldots, A_n; B_1, \ldots, B_n | Z] \geq \sum_{i \in [n]} \mathbb{I} [A_i; B_i | Z].
\]

In order to lower bound each summand we will use a divergence measure. The relative entropy or Kullback–Leibler divergence measures the difference of two probability distributions. It is always non-negative, but it is neither symmetric, nor does it satisfy the triangle inequality. For simplicity we only define it for random variables.

**Definition 2.1** (Relative entropy). Let \(X, Y\) be discrete random variables on the same domain. The relative entropy of \(X\) and \(Y\) is

\[
D(X \parallel Y) = \sum_{x \in \text{range}(X)} \sum_{y \in \text{range}(Y)} P[X = x] \cdot \log \frac{P[X = x]}{P[Y = x]}.
\]

The relative entropy is related to mutual information via the following identity:

\[
\mathbb{I} [X; Y] = \mathbb{E}_Y [D(X \parallel Y)],
\]

i.e., the mutual information is the expectation of the deviation over \(Y\). Pinsker’s inequality provides a convenient lower bound on the relative entropy (see e.g., [Cover and Thomas 2006, Lemma 11.6.1]).

**Lemma 2.2** (Pinsker’s inequality). Let \(X, Y\) be discrete random variables with identical domains. Then

\[
D(X \parallel Y) \geq 2(\log e) \left( \max_{\mathbb{E} : \text{event}} |P_X [\mathbb{E}] - P_Y [\mathbb{E}]| \right)^2.
\]

The quantity \(\max_{\mathbb{E} : \text{event}} |P_X [\mathbb{E}] - P_Y [\mathbb{E}]|\) is called the total variation distance between \(X\) and \(Y\), which is the maximal difference of probabilities the distributions of \(X\) and \(Y\) assign to the same event. For mutual information, via Pinsker’s inequality we obtain the lower bound

\[
\mathbb{I} [X; Y] \geq 2(\log e) \cdot \mathbb{E}_Y \left[ \max_{\mathbb{E} : \text{event}} |P_X [\mathbb{E}] - P [\mathbb{E} | Y]|^2 \right].
\]

### 2.2 Polyhedral preliminaries

Our main object of interest will be the (perfect) matching polytope (over the complete graph \(K_n\)).

**Definition 2.3** (Matching polytope). Let \(n \in \mathbb{N}\). Then the matching polytope \(P_M(n)\) is defined as

\[
P_M(n) := \text{conv} \left( \left\{ \chi_M \in \mathbb{R}^{(\binom{n}{2})} \mid M \text{ is a matching in } K_n \right\} \right),
\]

and the perfect matching polytope \(P_{PM}(n)\) is given by

\[
P_{PM}(n) := \text{conv} \left( \left\{ \chi_M \in \mathbb{R}^{(\binom{n}{2})} \mid M \text{ is a perfect matching in } K_n \right\} \right).
\]
Let \( E[U] \) denote the edges of the graph \( K_n \) being contained in \( U \subseteq V \) and let \( \delta(U) \) for \( U \subseteq V \) denote the set of edges with one endpoint in \( U \) and one in its complement; we use \( \delta(v) := \delta(\{v\}) \) for \( v \in V \). Edmonds [1965] showed that \( P_M(n) \) has the inequality description:

\[
P_M(n) = \left\{ x \in \mathbb{R}^{|\mathcal{E}|} \left| \begin{array}{l} x(\delta(v)) \leq 1 \quad \forall v \in V, x(E[U]) \leq \frac{|U| - 1}{2} \quad \forall U \subseteq V : |U| \text{ odd}, x \geq 0 \end{array} \right. \right\}.
\]

For even \( n \), the perfect matching polytope \( P_{PM}(n) \) has the description

\[
P_{PM}(n) = \left\{ x \in \mathbb{R}^{|\mathcal{E}|} \left| \begin{array}{l} x(\delta(v)) = 1 \quad \forall v \in V, x(E[U]) \leq \frac{|U| - 1}{2} \quad \forall U \subseteq V : |U| \text{ odd}, x \geq 0 \end{array} \right. \right\}.
\]

Observe that in both cases, we have an exponential number of odd-set inequalities, whereas all other inequalities are of a polynomial number. It will therefore suffice in the following to consider only the odd-set inequalities as we will see soon.

We recall necessary notions and results from (approximate) extended formulations. For an overview the reader is referred to the excellent surveys Conforti et al. [2010] and Kaibel [2011] as well as Pashkovich [2012], Braun et al. [2012] for the approximate extended formulation framework.

The motivating example is having a set of feasible solutions and a set \( C \) of linear objective functions we would like to maximize over the set of feasible solutions. We choose \( P \) to be the convex hull of feasible solutions, and \( Q := \{ x \mid c x \leq \max_{y \in P} c y \quad \forall c \in \mathcal{C} \} \) defined by the objective functions, so that maximizing over \( P \) and \( Q \) are equivalent. Thus \( P \subseteq Q \), but equality need not hold in general. The \( \rho \)-dilate of a polyhedron \( Q := \{ x \mid A x \leq b \} \neq \emptyset \) is \( \rho Q = \{ x \mid A x \leq \rho b \} \).

Now given an approximation factor \( \rho \), for every objective \( c \), the maximum over the \( \rho \)-dilate \( \rho Q \) of \( Q \) is clearly \( \rho \) times the maximum over \( Q \), so optimizing over a polyhedron \( K \) with \( P \subseteq K \subseteq \rho Q \), the maximum is off by a factor of at most \( \rho \). In order to avoid technical complications we assume that 0 is an interior point of \( Q \). This model captures many linear programming setups used in approximation algorithms.

For the matching we shall employ the uniform model, where the feasible solutions are independent of the instances, and the instances are encoded only into the objective functions. Therefore for the maximum matching problem, it is natural to choose \( P = P_{PM}(n) \) to be the convex hull of all perfect matchings on \( n \) vertices. For a graph \( G \), by restriction, every perfect matching provides a matching, and all matchings of \( G \) can be obtained this way. Therefore for \( G \) we consider the following objective function: the number of edges of the matching lying inside \( G \). For our purposes, it will suffice to consider only complete graphs \( G \) on odd-sized subsets, providing the polytope \( Q = Q(n) \) defined by some of the defining inequalities of \( P_{PM}(n) \):

\[
Q(n) := \left\{ x \in \mathbb{R}^{|\mathcal{E}|} \left| \begin{array}{l} x(E[U]) \leq \frac{|U| - 1}{2} \quad \forall U \subseteq V : |U| \text{ odd} \end{array} \right. \right\}.
\]

Now we recall the general setup. In the approximate extended formulation framework we consider the question, how many inequalities we need in any polyhedral description \( K \) with \( P \subseteq \pi(K) \subseteq \rho Q \) for a pair of polyhedra \( P \subseteq Q \) and affine linear map \( \pi \). In particular, for \( P = Q \) and \( \rho = 1 \) we get back the standard extended formulation model. Informally, we ask

How hard is it to approximate \( P \) with respect to the objective functions coming from \( Q \) within a factor of \( \rho \) via a linear programming formulation?
To capture this complexity we will use the notion of extension complexity of a pair. For convenience we define the size of a polytope $Q$ to be the number of its facets (disregarding encoding size of the coefficients).

**Definition 2.4** (Extension complexity of pair). The extension complexity $\text{xc}(P, Q)$ of the pair $P, Q$ is the minimum number of facets of a polyhedron $K$ so that there exists an affine map $\pi$ with $P \subseteq \pi(K) \subseteq Q$.

The theory of extended formulations naturally extends to the (approximate) setting with pairs and we can study the extension complexity of a pair via its associated slack matrix.

**Definition 2.5** (Slack matrix of a pair of polyhedra). Given a polytope $P = \text{conv}(v_1, \ldots, v_n)$ and a polyhedron $Q = \{ x : Ax \leq b \}$ with $b \in \mathbb{R}^m$, the slack matrix $S$ of the pair $P, Q$ is the $m \times n$ matrix with entries $S_{ij} = b_i - A_i v_j$ for all $i, j$.

Also Yannakakis’s factorization theorem (see Yannakakis [1988, 1991]) extends naturally:

**Theorem 2.6** ([Pashkovich 2012, Braun et al. 2012]). Let $P, Q$ be a polyhedral pair and let $S$ be any of its slack matrices. Then $\text{rk}_+ S - 1 \leq \text{xc}(P, Q) \leq \text{rk}_+ S$. If $P, Q$ are both polytopes, then $\text{xc}(P, Q) = \text{rk}_+ S$.

In order to express approximation factors, we use the notion of the $\rho$-approximate extension complexity of the pair $P, Q$.

**Definition 2.7** ($\rho$-approximate extension complexity). Let $P, Q$ be a polyhedral pair and let $\rho \geq 1$. The $\rho$-approximate extension complexity of $P, Q$ is defined as $\text{xc}(P, \rho Q)$, where $\rho Q$ is the $\rho$-dilate of $Q$ (and we assume $P \subseteq \rho Q$).

The notion of a $\rho$-approximate extended formulations corresponds precisely to the minimum number of facets in any polyhedral relaxation of $P$ so that optimizing objective functions generated from $Q$ is within at most a $\rho$-factor compared to the optimum value over $P$.

We can easily derive the slack matrix of the pair $(P, \rho Q)$ from $P, Q$: let $S$ be a slack matrix for the pair $P = \text{conv}(v_1, \ldots, v_n)$ and $Q = \{ x : Ax \leq b \}$, then a slack matrix $S^\rho$ for the pair $(P, \rho Q)$ is obtained simply as

$$S_{ij}^\rho = S_{ij} + (\rho - 1)b_i,$$

i.e., we shift the slack matrix by adding positive entries. For brevity we write $S^\rho = S + (\rho - 1)B$, where $B$ is the matrix with row $i$ being $b_i \cdot (1, \ldots, 1)$. We obtain

**Corollary 2.8.** Let $P = \text{conv}(v_1, \ldots, v_n)$, $Q = \{ x : Ax \leq b \}$ be a polyhedral pair and let $S$ be the associated slack matrix. Then

$$\text{rk}_+(S + (\rho - 1)B) - 1 \leq \text{xc}(P, \rho Q) \leq \text{rk}_+(S + (\rho - 1)B),$$

where $B_{ij} = b_i$ for all $i, j$. If $P$ and $Q$ are both polytopes, then $\text{xc}(P, \rho Q) = \text{rk}_+(S + (\rho - 1)B)$.

We will also examine the trade-off between approximation factor and size. The following two notions are the polyhedral equivalents of PTAS and FPTAS.

**Definition 2.9** ((Fully) polynomial-size relaxation scheme). Let $(P_n, Q_n) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ with $n \in \mathbb{N}$ be a family of polyhedral pairs. We say that $\{(P_n, Q_n)\}_n$ admits a

1. polynomial-size relaxation scheme (PSRS) if for every fixed $\epsilon > 0$ there exists a family of polyhedra $\{K_n\}_n$ with $\text{xc}(K_n) = \text{poly}(n)$ and $P_n \subseteq K_n \subseteq (1 + \epsilon)Q_n$.

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2. fully polynomial-size relaxation scheme (FPSRS) if for every $\epsilon > 0$ there exists a family of polyhedra $\{K_n\}_n$ with $xc(K_n) = \text{poly}(n, 1/\epsilon)$ and 

$$P_n \subseteq K_n \subseteq (1 + \epsilon)Q_n.$$ 

Thus both PSRS and FPSRS require a polynomial $p$ with $xc(K_n) \leq p(n)$. The difference between PSRSs and FPSRSs is that an FPSRS requires $p$ to depend polynomially on $1/\epsilon$ as well, whereas for a PSRS the polynomial $p$ is allowed to depend arbitrarily on $1/\epsilon$.

It is known that the knapsack polytope admits a PSRS (in the maximization sense) as shown in [Bienstock 2008] however it is not known whether it also admits an FPSRS. The matching polytope has the following folklore PSRS:

**Example 2.10 (PSRS for matching).** For $\rho > 1$ consider the polytope

$$K_n = \left\{ x \in \mathbb{R}^{\binom{n}{2}} \mid x(\delta(v)) \leq 1 \ \forall v \in V, x(E[U]) \leq \rho \frac{|U| - 1}{2} \ \forall U \subseteq V : |U| \text{ odd}, x \geq 0 \right\} \subseteq \rho P_M(n).$$

Clearly $P_{PM}(n) \subseteq K_n \subseteq \rho Q(n)$.

We will now argue that $K_n$ is defined by roughly $O(n^{p/(p-1)})$ inequalities. For this observe if $U \subseteq [n]$ with $|U| > \frac{\rho}{p-1}$ and $|U|$ odd, we have $\rho \frac{|U| - 1}{2} = \frac{|U| + |U| - \rho}{2} \geq \frac{\rho}{2}$. Thus the inequality $x(E[U]) \leq \rho \frac{|U| - 1}{2}$ is redundant and dominated by $x(E[U]) \leq \frac{|U|}{2}$ which arises from positive combinations of the inequalities $x(\delta(v)) \leq 1$ for $v \in U$. Thus, this leaves only $O(n^{p/(p-1)})$ undominated inequalities in the description of $K_n$.

Thus, in order to establish the non-existence of an FPSRS for the matching problem, we will prove a lower bound for $xc(P_{PM}(n), \rho Q(n))$. We will need the slack matrix of this pair. The slack matrix $S$ for the (perfect) matching polytope has its rows indexed by perfect matchings $M$ and its columns indexed by odd-sized sets $U \subseteq [n]$, where

$$S_{M,U} := |\delta(U) \cap M| - 1.$$ 

(For convenience, here and below we omit a constant factor $1/2$, which obviously does not change the nonnegative rank.) The slack matrix of the pair $(P_{PM}(n), \rho Q(n))$ is obtained via a $\rho$-shift, i.e., the entries are of the form

$$S^\rho_{M,U} := |\delta(U) \cap M| - 1 + (\rho - 1)(|U| - 1).$$

Observe that for $\rho > 1$, the value of an entry will also depend on the size of the odd-set $U$ not just the number of crossing edges. However, we prefer to let the slack depend only on the number on crossing edges, and therefore we use the following relaxation of $\rho Q(n)$ instead:

$$Q^{+\epsilon}(n) := \left\{ x \in \mathbb{R}^{\binom{n}{2}} \mid x(E[U]) \leq \frac{|U| - 1 + \epsilon}{2} \ \forall U \subseteq V : |U| \text{ odd} \right\}.$$

This is indeed a relaxation, since $(1 + \epsilon/(n - 1))Q(n) \subseteq Q^{+\epsilon}(n)$. The pair $(P_{PM}(n), Q^{+\epsilon}(n))$ has slack matrix:

$$S_{M,U}^{+\epsilon} := |\delta(U) \cap M| - 1 + \epsilon.$$
2.3 Lower bounds on the nonnegative rank via common information

We further extend the common information-based framework that was introduced in [Braun and Pokutta 2013] and expanded in [Braun et al. 2013b]. The underlying approach is based on the sampling framework introduced in [Braverman and Moitra 2012] and implicitly related to previous lower bounding techniques given in [Bar-Yossef et al. 2004]. We bound the log nonnegative rank from below by common information, an information-theoretic quantity that was introduced in [Wyner 1975]. We recall the basic framework here and refine it in Section 3.

Definition 2.11 (Common information). Let $A$, $B$ be random variables, and $Z$ be a conditional. The common information of $A, B$ given $Z$ is the quantity

$$C[A; B | Z] := \inf_{\Pi: A \perp B | \Pi \perp Z | A, B} \Pi [A, B; \Pi | Z],$$

where the infimum is taken over all random variables $\Pi$ in all extensions of the probability space, so that

1. $A$ and $B$ are conditionally independent given $\Pi$, and
2. $Z$ and $\Pi$ are conditionally independent given $A$ and $B$.

We refer to $\Pi$ as seed whenever it satisfies the above properties.

The conditional independence of $Z$ and $\Pi$ is to ensure that no information about $\Pi$ should be leaked from $Z$. We will link common information to nonnegative matrices, by reinterpreting the latter as probability distributions over two random variables $A$ and $B$.

Definition 2.12 (Induced distribution). Let $M \in \mathbb{R}^{m \times n}_+$ be a nonnegative matrix. The induced distribution on $A, B$ with range$(A) = [m]$ and range$(B) = [n]$ via $M$ is given by

$$P[A = a, B = b] = \frac{M(a, b)}{\sum_{x,y} M(x, y)}$$

for every row $a$ and column $b$. Slightly abusing notation, we write $(A, B) \sim M$.

For convenience we define the common information of $M$ conditioned on $Z$ as

$$C[M | Z] := C[A; B | Z],$$

where $(A, B) \sim M$. It can be shown that common information is a lower bound on the log of the nonnegative rank.

Lemma 2.13 ([Braun and Pokutta 2013]). Every factorization of a nonnegative matrix $M$ induces a seed with range of size of the number of summands in the factorization. In particular, $\log \text{rk}_+ M \geq C[M | Z]$ for any condition $Z$.

3 Extension complexity of the matching polytope

We will now show how to bound the common information of the ($\varepsilon$-shift of the) slack matrix of the matching polytope, which leads to a lower bound on the nonnegative rank of the ($\varepsilon$-shift of the) slack matrix and hence the approximate extension complexity of the matching polytope via Theorem 2.6 in the case $P = P_{PM}(n)$ and $Q = Q^{+\varepsilon}(n)$. 

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Theorem 3.1. Let \( 0 < \varepsilon < 1 \) be fixed and \( n \) even. Then \( xc(P_{PM}(n), Q^{+\varepsilon}(n)) = 2^{\Theta(n)} \). In particular, the extension complexity of the \( \rho \)-approximation of the perfect matching polytope is \( xc(P_{PM}(n), \rho Q) = 2^{\Theta(n)} \) for \( \rho \leq 1 + \varepsilon / n \), and also \( xc(P_{PM}(n)) = 2^{\Theta(n)} \). Moreover, the perfect matching polytope does not admit an FPSRS.

The upper bound \( 2^{\Theta(n)} \) on the extension complexity follows from the number of facets of the matching polytope. For the lower bounds, note that the first bound implies the remaining ones, as \((\rho-1)(|U|-1) \leq (\rho-1)n \leq \varepsilon \). The non-existence of an FPSRS follows as \( xc(P_{PM}(n), (1+\varepsilon/n)Q) = 2^{\Theta(n)} \) for fixed \( \varepsilon \); we pick the approximation factor \( \alpha = \varepsilon / n \), then \( xc(P_{PM}(n), (1+\alpha)Q) \) is super-polynomial in \( n \) and \( 1/\alpha = n/\varepsilon \).

The lower bound \( xc(P_{PM}(n), Q^{+\varepsilon}(n)) = 2^{\Omega(n)} \) is obtained via Theorem 2.6 by lower bounding the nonnegative rank of the slack matrix \( S^{+\varepsilon} \) by Lemma 2.13. All we need to prove is \( |C[M; U | \mathcal{Z}] = \Omega(n) \) for \( (M, U) \sim S^{+\varepsilon} \) under a suitable condition \( \mathcal{Z} \).

We need some preparations to introduce \( \mathcal{Z} \). We partition the vertices of \( K_n \) in a manner similar to but simpler than in [Rothvoß (2014)], see Figure 1. The purpose is to break up the graph into small chunks, in order to amplify the complexity of the chunks.

![Figure 1: The condition \( \mathcal{E} \) together with a partition of \( K_n \) to reduce extension complexity to fixed-size small chunks.](image)

First, we choose a 3-matching \( \mathbf{H} \) between two disjoint 3-element subsets \( \mathbf{C}_H \) and \( \mathbf{D}_H \). The intention is to consider only pairs \((\mathbf{M}, \mathbf{U})\) in \( \mathcal{Z} \) with \( \delta(\mathbf{U}) \cap \mathbf{M} = \mathbf{H} \), especially with \( \mathbf{C}_H \subseteq \mathbf{U} \) and \( \mathbf{U} \cap \mathbf{D}_H = \emptyset \). We partition the vertices not covered by \( \mathbf{H} \) into equal-sized chunks \( T_1, \ldots, T_m \) of even size \( 2(k - 3) \). This might not be possible, and some vertices might be left out, therefore we add a remainder chunk \( \mathbf{L} \) (which might be empty) of size \( l \leq 2(k - 3) \). In particular, \( n - 3 = 2(k - 3)m + l \).

We partition every \( T_i \) into a pair of \((k - 3)\)-element sets \( \mathbf{C}_i, \mathbf{D}_i \). We will denote by \( \mathbf{T} \) the collection of \( \mathbf{C}_1, \mathbf{D}_1, \ldots, \mathbf{C}_m, \mathbf{D}_m, \mathbf{C}_H, \mathbf{D}_H, \mathbf{L} \). Let \( \mathbf{T} \) and \( \mathbf{H} \) be jointly uniformly distributed independently of \( \mathbf{M}, \mathbf{U} \).

Second, we need a collection \( \mathbf{N} \) of \( m \) mutually independent random fair coins \( \mathbf{N}_1, \ldots, \mathbf{N}_m \), which are also independent of the random variables introduced before. Finally, we introduce the
In particular, given \( E \) bound on common information:

Let \( \delta(U) \cap M = H \). This actually also fixes \( \delta(U) \cap M = H \).

Now we are in the position to define \( \mathcal{Z} \) as \( T, H, N, \mathcal{E} \), and hence to formulate the exact lower bound on common information:

**Proposition 3.2.** Let \( k \geq 7 \) be a fixed odd number, \( 0 \leq \varepsilon < 1 - 4/k \), and \( n = 2(k - 3)m + 1 + 3 \) for some \( l < k \). Consider the complete graph \( K_n \) on \( n \) vertices. Furthermore, let \( M \) be a random perfect matching and \( U \) a random subset of vertices of odd size, with \( (M, U) \sim S^{+\varepsilon} \). Then there exists a constant \( c_{k,\varepsilon} > 0 \) depending only on \( k \) and \( \varepsilon \), so that

\[
\mathbb{C}[M; U | T, N, H, \mathcal{E}] \geq c_{k,\varepsilon}m.
\]

We will now provide the proof of Proposition 3.2. We follow the framework of Braun and Pokutta [2013]:

1. **Reduction to local case:** We first reduce the general case to the case \( m = 1 \) and \( l = 0 \). This follows via a direct sum argument and the partitions from above serve the purpose of creating independent subproblems.

2. **Bound from local case:** We then analyze the case \( m = 1, l = 0 \), and \( k \geq 5 \) fix and show that

\[
\mathbb{I}[M, U; \Pi | T, N, H, \mathcal{E}] \geq c_{k,\varepsilon}
\]

via its relative entropy equivalent and applying Pinsker’s inequality (see Lemma 2.2).

**Proof of Proposition 3.2.** Reduction to the local case with \( m = 1 \) and \( l = 0 \). Suppose that the statement of the proposition holds for \( m = 1 \) and \( l = 0 \).

First, observe that the event \( \mathcal{E} \) ensures that \( \delta(U) \cap M = H \). Thus, as the probability of a pair \( (M, U) \) depends only on the number of crossing edges, \( (M, U) \) is uniformly distributed given \( \mathcal{E} \).

The matching \( M \) decomposes into \( M_i := M \cap E[T_i] \) for \( i \in [m] \), together with \( M_L := M \cap E[L] \) and \( H \). Similarly, the set \( U \) decomposes as \( U = C_H \cup \bigcup_{i \in [m]} U_i \) with \( U_1 := U \cap T_1 \). The pairs \( (M_i, U_i) \) together with \( (M_L, \emptyset) \) are mutually independent, therefore by the direct sum property

\[
\mathbb{I}[M, U; \Pi | T, N, H, \mathcal{E}] \geq \sum_{i \in [m]} \mathbb{I}[M_i, U_i; \Pi | T, N, H, \mathcal{E}] \geq c_{k,\varepsilon}m,
\]

where the last inequality is concluded from the local case as follows.

We prove that every summand is at least \( c_{k,\varepsilon} \) via reduction to the local case. Let us consider a fixed \( i \in [m] \), and let us also fix \( L \) and \( T_j, N_j \) for all \( j \neq i \). This actually also fixes \( T_i \), and hence the \( 2k \)-element subset \( V_i = T_i \cup C_H \cup D_H \). Only the partition of \( V_i \) into \( C_i, D_i, C_H, D_H \) remains random. Let \( \mathcal{E}_{-i} \) be the event \( \mathcal{E} \) with the restrictions for \( U_i \) and \( M_i \) omitted, it ensures that all crossing edges lie inside \( V_i \). Therefore, given \( T_j, N_j \) for \( j \neq i \) and \( \mathcal{E}_{-i} \), the distribution of
\(M_i, U_i, \Pi, H, N_i, C_i, D_i, C_H, D_H\) on the complete graph on \(V_i\) is exactly the one given in the proposition for the case \(m = 1\) and \(l = 0\). The events \(\mathcal{E}_0\) and \(\mathcal{E}\) also have the same interpretation in the local and global case.

We invoke the proposition for the local case, which provides
\[
\mathbb{I} [M_i, U_i; \Pi | T_i, N_i, H, \mathcal{E}] \geq c_{k, \varepsilon}.
\]
Unfixing the \(T_j, N_j\) and \(L\), and taking expectation over those leads to
\[
\mathbb{E} [M_i, U_i; \Pi | T, N, H, \mathcal{E}] \geq c_{k, \varepsilon},
\]
as claimed.

### 3.1 Proof of Proposition 3.2: The local case

In the local case \(m = 1\), \(l = 0\), we first adjust the setup, as illustrated in Figure 2. We introduce

![Figure 2: Condition in the local case. The blocks \(C, D\) and a complete matching \(F\) between them are fixed.](image)

some auxiliary random variables. Let \(C := C_1 \cup C_H\) and \(D := D_1 \cup D_H\). Note that \(C, D\) and \(H\) are uniformly distributed (independently of \(M, U, \Pi, N\)), and together determine the \(C_1, D_1, C_H, D_H\). Furthermore, we introduce \(F\) as a uniformly random extension of \(H\) into a full matching between \(C\) and \(D\), depending only on \(C, D\) and \(H\). This independence ensures that adding it as condition to the mutual information has no effect:
\[
\mathbb{I} [M, U; \Pi | T, H, N, \mathcal{E}] = \mathbb{I} [M, U; \Pi | T, H, F, N, \mathcal{E}] = \mathbb{I} [M, U; \Pi | C, D, F, H, N, \mathcal{E}]
\]
\[
= \mathbb{E}_{C \sim C_1 \cup C_H, D \sim D_1 \cup D_H, \mathcal{E}} \left[ \mathbb{I} [M, U; \Pi | C = C, D = D, F = F, H, N, \mathcal{E}] \right].
\]
We show that the inner term is always at least \(c_{k, \varepsilon}\). To this end, we fix \(C, D, F\), and drop them from the condition to simplify notation. For convenience, we rewrite the events \(\mathcal{E}_0\) and \(\mathcal{E}\) in a simple form:
\[
\mathcal{E}_0 := \{U \in \{C, C(H)\}, H \subseteq M\} \quad \mathcal{E} := \begin{cases} U = C(H), & \text{if } N = 0 \\ \delta(C) \cap M = H, & \text{if } N = 1. \end{cases}
\]
Here and below for a 3-matching \(h \subseteq F\), let \(C(h)\) denote the endpoints of the edges of \(h\) lying in \(C\). (I.e., \(C(H) = C_H\) but we shall use the notation for other \(h\), too.) As \(N\) consists of only one coin, we simply identify it with \(N\).

To proceed, we will distinguish two types of pairs \((\pi, h)\): the good pairs are where the conditional distribution \((M, U) | \Pi = \pi, H = h\) is close to the distribution \((M, U) | H = h\) with the condition on \(\Pi\) left out. Therefore the contribution of good pairs to mutual information is negligible. The bad pairs are where the two distributions differ significantly, and hence contribute much to mutual information. As we will see in Section 3.1.1 there are not too many good pairs, and this is the key to the proof.

We now state the exact definitions of goodness and badness, taking also the conditions in the mutual information into account:
**Definition 3.3.** A pair \((\pi, h)\) is **M-good** if for all matchings \(m \supseteq h\)

\[
1 - \delta \leq \frac{\Pr \left[ M = m \mid \Pi = \pi, H = h, N = 0, \mathcal{E} \right]}{\Pr \left[ M = m \mid H = h, N = 0, \mathcal{E} \right]} \leq 1 + \delta.
\]

Otherwise the pair is **M-bad**, denoted as **M-BAD**(\(\pi, h\)). Similarly, a pair \((\pi, h)\) is **U-good** if for \(u = C(h)\) and \(u = C\)

\[
1 - \delta \leq \frac{\Pr \left[ U = u \mid \Pi = \pi, H = h, N = 1, \mathcal{E} \right]}{\Pr \left[ U = u \mid H = h, N = 1, \mathcal{E} \right]} \leq 1 + \delta.
\]

Otherwise the pair is **U-bad**, denoted as **U-BAD**(\(\pi, h\)).

The pair \((\pi, h)\) is **good** if it is both **M-good** and **U-good**. It is denoted by **GOOD**(\(\pi, h\)). The pair is **bad**, denoted as **BAD**(\(\pi, h\)) if it is not good.

Now we reduce the proposition to estimating the probability of bad pairs.

**Proof of Proposition 3.2** the case \(m = 1\) and \(l = 0\). We will use Pinsker’s inequality (Lemma 2.2) to lower bound the mutual information induced by those \((\pi, h)\) which are **M-bad** or **U-bad** (depending on the outcome of the coin \(N\)). Therefore we rewrite the mutual information using relative entropy:

\[
\mathbb{I} \left[ M, U ; H, N, \mathcal{E} \right] = \mathbb{E}_{\Pi, H, N \in \mathcal{E}} \left[ \mathbb{D} \left( M, U \mid \Pi, H, N, \mathcal{E} \parallel M, U \mid H, N, \mathcal{E} \right) \right] = \sum_{i \in \{0, 1\}} \Pr \left[ N = i \mid \mathcal{E} \right] \cdot \mathbb{E}_{\Pi, H, N = i, \mathcal{E}} \left[ \mathbb{D} \left( M, U \mid \Pi, H, N, \mathcal{E} \parallel M, U \mid H, N, \mathcal{E} \right) \right].
\]

By definition, for **M-bad** pairs there is an \(m\) where the probabilities differ by at least

\[
\delta \Pr \left[ M = m \mid H = h, N = 0, \mathcal{E} \right] = \delta \alpha,
\]

where \(\alpha = 1/(2k - 7)!!\). Similarly, for **U-bad** pairs, the probabilities differ by at least

\[
\delta \Pr \left[ U = u \mid H = h, N = 1, \mathcal{E} \right] = \delta \frac{1}{2}.
\]

Thus via Lemma 2.2 we have

\[
\mathbb{E}_{\Pi, H | N = 0, \mathcal{E}} \left[ \mathbb{D} \left( M, U \mid \Pi, H, N, \mathcal{E} \parallel M, U \mid H, N, \mathcal{E} \right) \right] \geq \Pr \left[ \text{M-BAD} \left( \Pi, H \right) \mid N = 0, \mathcal{E} \right] 2(\log e)(\delta \alpha)^2,
\]

\[
\mathbb{E}_{\Pi, H | N = 1, \mathcal{E}} \left[ \mathbb{D} \left( M, U \mid \Pi, H, N, \mathcal{E} \parallel M, U \mid H, N, \mathcal{E} \right) \right] \geq \Pr \left[ \text{U-BAD} \left( \Pi, H \right) \mid N = 1, \mathcal{E} \right] 2(\log e) \left( \frac{\delta}{2} \right)^2.
\]

The main part of the proof is to lower bound the probability of being bad. To simplify computations, we rewrite the probabilities appearing above in a more manageable form:

\[
\Pr \left[ \text{M-BAD} \left( \Pi, H \right) \mid N = 0, \mathcal{E} \right] = \frac{\Pr \left[ \text{M-BAD} \left( \Pi, H \right) \mid U = C(H), \mathcal{E}_0 \right]}{\Pr \left[ U = C(H) \mid \mathcal{E}_0 \right]},
\]

\[
\Pr \left[ \text{U-BAD} \left( \Pi, H \right) \mid N = 1, \mathcal{E} \right] = \frac{\Pr \left[ \text{U-BAD} \left( \Pi, H \right) \mid H = \delta(C) \cap M, \mathcal{E}_0 \right]}{\Pr \left[ H = \delta(C) \cap M \mid \mathcal{E}_0 \right]}.
\]
We shall obtain a constant $B_{k, \varepsilon} > 0$ lower bounding the sum of the probabilities of being $M$-bad or $U$-bad

$$P[M \text{-BAD}(\Pi, H), U = C(H) \mid \varepsilon_0] + P[U \text{-BAD}(\Pi, H), H = \delta(C) \cap M \mid \varepsilon_0] \geq B_{k, \varepsilon}.$$ 

Note that the probabilities cannot be bounded separately as individually they can be 0. This bound then leads to

$$P[M, U; \Pi \mid H, N, \varepsilon] \geq P[N = 0 \mid \varepsilon] \cdot P[M \text{-BAD}(\Pi, H) \mid N = 0, \varepsilon] 2(\log e)(\delta \alpha)^2$$

$$+ P[N = 1 \mid \varepsilon] \cdot P[U \text{-BAD}(\Pi, H) \mid N = 1, \varepsilon] 2(\log e) \left(\frac{\delta}{2}\right)^2$$

$$\geq B_{k, \varepsilon}(2 \log e)\delta^2 \min \left\{ \frac{P[N = 0 \mid \varepsilon] \alpha^2}{P[U = C(H) \mid \varepsilon_0]}, \frac{P[N = 1 \mid \varepsilon] / 4}{P[\delta(C) \cap M = H \mid \varepsilon_0]} \right\} =: c_{k, \varepsilon}.$$ 

This is a positive constant depending only on $k$ and $\varepsilon$ provided $B_{k, \varepsilon} > 0$, which we prove in the next section.

### 3.1.1 Bounding probability of being good

To obtain the claimed lower bound $B_{k, \varepsilon} > 0$ on the probability of being bad, we investigate how much the good pairs contribute to the distribution of $M, U$. We start by rewriting of being good into a form with less conditions on probabilities. We remove $H, N$ using their independence of $M, \Pi, U$. For any 3-matching between $C$ and $D$, any perfect matching $m \supseteq h$ and for $u \in \{C(h), C\}$ we have

$$P[M = m \mid \Pi = \pi, H = h, N = 0, \varepsilon] = P[M = m \mid \Pi = \pi, H = h, N = 0, U = C(h), h \subseteq M]$$

$$= P[M = m \mid \Pi = \pi, h \subseteq M] = \frac{P[M = m \mid \Pi = \pi]}{P[h \subseteq M \mid \Pi = \pi]},$$

$$P[U = u \mid \Pi = \pi, H = h, N = 1, \varepsilon] = P[U = u \mid \Pi = \pi, H = h, N = 1, h = \delta(C) \cap M, U \in \{C(h), C\}]$$

$$= \frac{P[U = u \mid \Pi = \pi]}{P[U \in \{C(h), C\} \mid \Pi = \pi]}.$$ 

This is mostly useful for comparing probabilities for the various values of $m, u$. Let $\beta := \left(\frac{1 - \varepsilon}{1 + \varepsilon}\right)^2$. When $(\pi, h)$ is good, we obtain

$$\frac{1}{\sqrt{\beta}} P[U = C \mid \Pi = \pi] \leq P[U = C(h) \mid \Pi = \pi] \leq \sqrt{\beta} P[U = C \mid \Pi = \pi],$$

and

$$\frac{1}{\sqrt{\beta}} P[M = F \mid \Pi = \pi] \leq P[M = m \mid \Pi = \pi] \leq \sqrt{\beta} P[M = F \mid \Pi = \pi],$$

for $h \subseteq m$.

Let us fix $\pi$, and let $h_1, \ldots, h_N$ be all the 3-submatchings of $F$ with $(\pi, h_i)$ good. Let $u_i = C(h_i)$ denote the vertices of $h_i$ in $C$. We divide the $h_i$ into two classes: the first class consists of the $h_i$ having two common edges with every other $h_j$, i.e., the intersection $u_i \cap u_j$ has 2 elements for all $j \neq i$. Without loss of generality, the first class is $h_1, \ldots, h_{\ell}$. The second class consists of the $h_i$ which do not have two common edges with all other $h_j$.

The Erdős–Ko–Rado Theorem [Erdős et al., 1961, Theorem 2] bounds the size of the first class by $\ell \leq k - 2$ for $k \geq 6$, (slightly improving over the 3$k$ in the combinatorial part of [Rothvoss, 2014].
Lemma 8). We consider matchings \( m \) satisfying \( h_i = \delta(C) \cap m \) for an \( i \). For \( i \leq \ell \), i.e., \( h_i \) is in the first class, we use the estimation
\[
\mathbb{P} [M = F, U = C \mid \Pi = \pi] = \mathbb{P} [M = F \mid \Pi = \pi] \mathbb{P} [U = C \mid \Pi = \pi] \\
\geq \sqrt{\beta} \mathbb{P} [M = m \mid \Pi = \pi] \mathbb{P} [U = u_i \mid \Pi = \pi] \\
= \beta \mathbb{P} [M = m, U = u_i \mid \Pi = \pi].
\]
Taking the average over all \( m \), we obtain
\[
\mathbb{P} [M = F, U = C \mid \Pi = \pi] \geq \frac{\beta}{A} \mathbb{P} [h_i = \delta(C) \cap M, U = u_i \mid \Pi = \pi], \quad i \leq \ell, 
\]
with \( A := (k - 3)!^2 \) being the number of matchings \( m \) with \( \delta(C) \cap m = h_i \).

For \( i > \ell \) there is an \( i' \) such that \( |u_i \cap u_{i'}| \leq 1 \). Thus there is an edge \( e \in E[u_i] \) not incident with any vertices of \( u_i \). We replace the two edges \( e_1, e_2 \) in \( F \) at the endpoints of \( e \) by the two edges \( e \) and \( f \) connecting their endpoints in \( C \) and \( D \), respectively, see Figure 3. Let \( m' := F \setminus \{e_1, e_2\} \cup \{e, f\} \)

![Figure 3: Constructing the matching \( m' \) by replacing \( e_1, e_2 \) in \( F \) by the dashed edges: \( e \) in \( u_i \) and its counter part \( f \) in \( D \).](image)

be the new matching, it contains \( e \). Note that \( |\delta(m') \cap u_i| = 1 \) and
\[
\mathbb{P} [M = m', U = u_i \mid \Pi = \pi] = \mathbb{P} [M = m' \mid \Pi = \pi] \mathbb{P} [U = u_i \mid \Pi = \pi] \\
\geq \sqrt{\beta} \mathbb{P} [M = F \mid \Pi = \pi] \mathbb{P} [U = u_i \mid \Pi = \pi] \\
\geq \beta \mathbb{P} [M = m \mid \Pi = \pi] \mathbb{P} [U = u_i \mid \Pi = \pi] \\
= \beta \mathbb{P} [M = m, U = u_i \mid \Pi = \pi].
\]
Taking the average over all \( m \) provides
\[
\mathbb{P} [\delta(U) \cap M] = 1, \, |M \setminus F| = 2, \, U = u_i \mid \Pi = \pi] \geq \mathbb{P} [M = m', U = u_i \mid \Pi = \pi] \\
\geq \frac{\beta}{A} \mathbb{P} [h_i = \delta(C) \cap M, U = u_i \mid \Pi = \pi]. 
\]
Now we add \( H \) to (1) and (2), using its independence of the variables involved there, we have
\[
\frac{\beta}{A} \mathbb{P} [H = \delta(C) \cap M, U = C(H), H = h_i \mid \Pi = \pi] \\
\leq \begin{cases} 
\mathbb{P} [M = F, U = C \mid \Pi = \pi] / (k^3), & \text{for } i \leq \ell, \\
\mathbb{P} [\delta(U) \cap M] = 1, \, |M \setminus F| = 2, \, H = h_i, U = C(H) \mid \Pi = \pi], & \text{for } i > \ell.
\end{cases}
\]
We sum up over all \( i \) to obtain
\[
\frac{\beta}{A} P[H = \delta(C) \cap M, U = C(H), \text{GOOD}(\Pi, H) \mid \Pi = \pi] \\
\leq \frac{\ell}{(\frac{1}{5})} P[M = F, U = C \mid \Pi = \pi] + P[\delta(U) \cap M] = 1, |M \setminus F| = 2, U = C(H) \mid \Pi = \pi].
\]
We take expectation over \( \pi \sim \Pi \). Recall that \( \ell \leq k - 2 \), hence
\[
\frac{\beta}{A} \left( P[H = \delta(C) \cap M, U = C(H)] - P[H = \delta(C) \cap M, U = C(H), \text{BAD}(\Pi, H)] \right) \\
= \frac{\beta}{A} P[H = \delta(C) \cap M, U = C(H), \text{GOOD}(\Pi, H)] \\
\leq \frac{k - 2}{(\frac{1}{5})} P[M = F, U = C] + P[\delta(U) \cap M] = 1, |M \setminus F| = 2, U = C(H)].
\]
We compute the values of the various probabilities from \( S^{+\epsilon} \). The probability of a fixed pair \((M, U)\) is proportional to their slack value, e.g.,
\[
P[M = F, U = C] = (k - 1 + \epsilon)\gamma.
\]
(Actually, \( \gamma = [(k/2 - 1 + \epsilon) \cdot (2k - 1)!! \cdot 2^{k-2}]^{-1} \), but we do not need the exact value.)
For the other events, it is easier to compute the probability conditioned on \( H \):
\[
P[H = \delta(C) \cap M, U = C(H)] = E_H \left[ P[H = \delta(C) \cap M, U = C(H) \mid H] \right] = (2 + \epsilon)\gamma A.
\]
Recall that \( A \) is the number of pairs satisfying the event for \( H \) fixed. Here we have used that \((M, U)\) is independent of \( H \).
The probability of the third event is determined similarly. Note that for any fixed 3-element \( U \), there are 3 matchings \( m \) with \( |\delta(U) \cap M| = 1 \) and \( |m \setminus F| = 2 \), actually arising by an analogous construction depicted in Figure 3
\[
P[\delta(U) \cap M = 1, M \setminus F = 2, U = C(H)] = \epsilon\gamma \cdot 3.
\]
Substituting the probabilities back into our formula, we obtain
\[
P[H = \delta(C) \cap M, U = C(H), \text{BAD}(\Pi, H)] \geq (2 + \epsilon)\gamma A - \left( \frac{6(k - 1 + \epsilon)}{k(k-1)} - \gamma + \epsilon \cdot 3 \right) \frac{A}{\beta^2}.
\]
Finally, we add back the event \( E_0 \) as a condition. Recall that \( \epsilon \leq 1 - 4/k \), hence
\[
P[H = \delta(C) \cap M, U = C(H), \text{BAD}(\Pi, H) \mid E_0] \\
\geq \left( 2 + \epsilon - \left( \frac{6(k - 1 + \epsilon)}{k(k-1)} + \epsilon \cdot 3 \right) \frac{1}{\beta^2} \right) \frac{\gamma A}{P[E_0]} \\
= \left( 2 - \left( \frac{6(k - 1 + \epsilon)}{k(k-1)} + (3 - \beta)\epsilon \right) \frac{1}{\beta^2} \right) \frac{\gamma A}{P[E_0]} \\
\geq \left( 2 - \left( \frac{6(k - 1 + (1 - 4/k))}{k(k-1)} + (3 - \beta)(1 - 4/k) \right) \frac{1}{\beta^2} \right) \frac{\gamma A}{P[E_0]} \\
= \left( 2 - (3k - 4)(1 - \beta) - \frac{6(k - 4)}{k(k-1)} \right) \frac{\gamma A}{k\beta P[E_0]}.
\]
The last expression has a positive lower bound $B_{k, \varepsilon}$ depending only on $k$ and $\varepsilon$. (Note that $\mathbb{P} [ \varepsilon_0] / \gamma$ involves $\varepsilon$.) To this end, we choose $\delta$ sufficiently close to 0, ensuring that $\beta$ is sufficiently close to 1, making the last parenthesized expression positive. The number $A$ depends just on $k$.

All in all, $\mathbb{P} [ H = \delta ( C) \cap M, U = C ( H), \text{BAD}(\Pi, H) | \varepsilon_0] \geq B_{k, \varepsilon} > 0$. Finally,

$$\mathbb{P} [ U = C ( H), M-\text{BAD}(\Pi, H) | \varepsilon_0] + \mathbb{P} [ H = \delta ( C) \cap M, U-\text{BAD}(\Pi, H) | \varepsilon_0] \geq \mathbb{P} [ H = \delta ( C) \cap M, U = C ( H), \text{BAD}(\Pi, H) | \varepsilon_0] \geq B_{k, \varepsilon}. \square$$

Acknowledgements

The authors would like to thank Thomas Rothvoß for the helpful comments. Research was partially supported by NSF grants CMMI-1300144 and CCF-1415496.

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