CONSTRUCTION OF SOME ALGEBRAS ASSOCIATED TO DIRECTED GRAPHS AND RELATED TO FACTORIZATIONS OF NONCOMMUTATIVE POLYNOMIALS

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ABSTRACT. This is a survey of recently published results. We introduce and study a wide class of algebras associated to directed graphs and related to factorizations of noncommutative polynomials. In particular, we show that for many well-known graphs such algebras are Koszul and compute their Hilbert series.

Let $R$ be an associative ring with unit and $P(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$ be a polynomial over $R$. Here $t$ is an independent central variable. We consider factorizations of $P(t)$ into a product

\[(0.1) \quad P(t) = a_0(t - y_n)(t - y_{n-1}) \cdots (t - y_1)\]

if such factorizations exist.

When $R$ is a (commutative) field, there is at most one such factorization up to a permutation of factors. When $R$ is not commutative, the polynomial $P(t)$ may have several essentially different factorizations.

The set of factorizations of a polynomial over a noncommutative ring can be rather complicated and studying them is a challenging and useful problem (see, for example, [N, GLR, GR1, GR2, GRW, GGRW, LL, O, B, V, W]).

In this paper we present an approach relating such factorizations to algebras associated with directed graphs and study properties of such algebras.

In the factorization (0.1) the element $y_1$ is called a right root of $P(t)$ and element $y_n$ is called a left root of $P(t)$. This terminology can be justified by the following equalities (see, for example, [L]):

\[
\begin{align*}
a_0 y_1^n + a_1 y_1^{n-1} + \cdots + a_{n-1} y_1 + a_n &= 0, \\
y_n a_0 + y_n^{n-1} a_1 + \cdots + y_n a_{n-1} + a_n &= 0.
\end{align*}
\]

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It is natural (see [GGRSW]) to call the elements \( y_i, i = 1, 2, \ldots, y_n \), in (0.1) \textit{pseudo-roots} of \( P(t) \). Note that any root is a pseudo-root but not every pseudo-root is a root.

There is a connection between factorizations of noncommutative polynomials and finding their right (left) roots and pseudo-roots. Consider the simplest non-trivial example when \( P(t) = t^2 + a_1 t + a_2 \) and \( x_1, x_2 \) are its right roots such that the difference \( x_1 - x_2 \) is invertible in \( R \).

Set
\[
x_{1,2} = (x_2 - x_1) x_2 (x_2 - x_1)^{-1},
\]
\[
x_{2,1} = (x_1 - x_2) x_1 (x_1 - x_2)^{-1}.
\]

One can show that

(0.2) \hspace{1cm} P(t) = (t - x_{1,2})(t - x_1) = (t - x_{2,1})(t - x_2).

Thus we have two different factorizations of \( P(t) \).

In studying factorizations of noncommutative polynomials, one should answer at least two questions:

1) How to obtain factorizations of type (0.1)?
2) How to relate different factorizations?

A partial answer to the first question was given in [GR1, GR2] when a polynomial \( P(t) = t^n + a_1 t^{n-1} + \cdots + a_n \) has \( n \) right roots \( x_1, x_2, \ldots, x_n \) in a \textit{generic position}. Here elements \( x_1, x_2, \ldots, x_n \) are in a generic position if all Vandermonde matrices \( (x_i^{m-1}), \ell, m = 1, 2, \ldots, k \) are invertible over \( R \) when \( k \geq 2 \) and \( i_1, i_2, \ldots, i_k \) are distinct. In [GR1, GR2] explicit formulas for \( n! \) factorizations of \( P(t) \) were found. For \( n = 2 \) these formulas give the factorizations (0.2).

To answer the second question, one has to study relations among pseudo-roots of \( P(t) \) and, more generally, study properties of subalgebras generated by pseudo-roots of \( P(t) \).

For example, the factorizations (0.2) imply the following identities between the corresponding pseudo-roots:

(0.3a) \hspace{1cm} x_{1,2} + x_1 = x_{2,1} + x_2,

(0.3b) \hspace{1cm} x_{1,2} x_1 = x_{2,1} x_2.

Note that one can consider expressions in (0.3) as noncommutative elementary symmetric functions of order 1 and 2 in \( x_1, x_2 \) (see [GR1, GR2, GRW, GGRW]). It is natural to study the algebra \( Q_2 \) with generators \( x_{1,2}, x_{2,1}, x_1, x_2 \) satisfying relations (0.3).

To carry out this idea, a \textit{universal algebra of \( n \) pseudo-roots} in a generic position, called \( Q_n \), was introduced in [GRW] and its properties were studied in detail in [GRW, GGR, GGRSW, SW, Pi]. The algebras \( Q_n \) are defined by linear and quadratic relations similar
to relations (0.3). These algebras are Koszul and have nice Hilbert series. Overall, one can say that $Q_n$ is a rather “tame” algebra despite its exponential growth.

A more general approach to a study of algebras of pseudo-roots starts with $\Gamma_P$, a directed graph of right divisors of a polynomial $P(t)$ similar to the graph of divisors of a natural number. In this graph vertices correspond to right divisors of $P(t)$ and edges to pseudo-roots of $P(t)$. Factorizations (0.1) correspond to paths from a maximal vertex $P(t)$ to a minimal vertex 1.

![Diagram](image)

**Figure 1**

In this setting relations (0.3) for the algebra $Q_2$ can be described with a help of the diamond graph with vertices indexed by subsets of set $\{1, 2\}$ (see Fig 1). Elements $x_{1,2}$ and $x_{2,1}$ correspond to edges $e_1$ and $f_1$, and elements $x_1$ and $x_2$ correspond to edges $e_2$ and $f_2$ respectively. Factorizations (0.2) correspond to two paths with the same origin and the end.

Our main objects are the “universal algebras” $A(\Gamma)$ associated to a directed graph $\Gamma$ and “universal polynomials” over these algebras. These algebras are generated by edges of $\Gamma$ and relations are defined by pairs of directed paths in $\Gamma$ with the same origin and end. If $\Gamma$ is the Hasse graph of the Boolean lattice of subsets of the set $\{1, 2, \ldots, n\}$ then $A(\Gamma) = Q_n$.

It turns out that the algebras $A(\Gamma)$ have a lot of interesting properties similar to the algebras $Q_n$. The “geometric nature” of algebras $A(\Gamma)$ helps us to understand their structure and to simplify the proofs of main results for algebras $Q_n$ compared to the proofs given in [GRW, GGRSW, SW, Pi]. A geometric approach expresses coefficients of the “universal polynomials” via certain pseudo-roots of these polynomials, giving a geometric version of the noncommutative Viète theorem from [GR1, GR2, see also [GGRW].

This paper contains definitions and results from [GRSW, RSW, RSW1, GGRW1]. It is organized in the following way. In Section 1 we recall the noncommutative Viète theorem and the construction of the algebras $Q_n$. Section 2 contains a definition of the algebras
Theorem 1.1.1. \( A(\Gamma) \) and similar algebras. In Section 3 we describe a linear basis in \( A(\Gamma) \) and state that for a large class of directed graphs, the algebras \( A(\Gamma) \) are Koszul. In Section 4 we compute Hilbert series for algebras \( A(\Gamma) \) and discuss some examples. In Section 5 we describe a geometric version of the noncommutative Viète theorem.

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1. The Viète Theorem and algebras of pseudo-roots of noncommutative polynomials

1.1. Factorizations of noncommutative polynomials. A generalization of the factorization (0.2) can be presented as follows (see [GR1, GR2, GRW]). Let \( R \) be an associative ring with unit and \( P(t) = t^n + a_1 t^{n-1} + \cdots + a_n \) be polynomial in \( R[t] \). Let \( x_1, x_2, \ldots, x_n \in R \) be right roots of the polynomial, i.e. \( x_i^n + a_1 x_i^{n-1} + \cdots + a_n = 0 \) for all \( i \).

We say that the roots \( x_1, x_2, \ldots, x_n \) are in generic position if all Vandermonde matrices \( V(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = (x_{i_l}^{m-1}), l, m = 1, 2, \ldots, k \) are invertible over \( R \) when \( k \geq 2 \) and all \( i_1, i_2, \ldots, i_k \) are distinct.

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In this case, one can define the Vandermonde quasideterminant (for the general theory of quasideterminants see [GR, GR2, GGRW])

\[ v(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = x_{i_k}^{k-1} - r \cdot V(x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}})^{-1} \cdot c \]

where \( r \) is a row vector of length \( k-1 \), \( r = (x_{i_1}^{k-1}, x_{i_2}^{k-1}, \ldots, x_{i_{k-1}}^{k-1}) \) and \( c \) is a column vector of the same length, \( c = (1, x_{i_k}, \ldots, x_{i_{k-2}}^T) \).

One can see that \( v(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \) is an element in the ring \( R \). This element is invertible in \( R \) because elements \( x_1, x_2, \ldots, x_n \) are in a generic position.

Example. \( v(x_1, x_2) = x_2 - x_1 \).

Note that \( v(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \) is a rational expression in \( x_{i_1}, x_{i_2}, \ldots, x_{i_k} \) and that it does not depend on the ordering of \( x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}} \) (see [GR, GR2, GGRW]).

Set

\[ x_{A,i_k} = v(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \cdot x_{i_k}^{-1} \cdot v(x_{i_1}, x_{i_2}, \ldots, x_{i_k})^{-1}, \]

where \( A = \{ x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}} \} \). Our notation is justified because \( x_{A,i} \) also does not depend on the ordering of \( A \). It is convenient to set \( x_{\emptyset,i} = x_i, i = 1, \ldots, n \).

Any ordering \( i_1, i_2, \ldots, i_n \) of indices \( 1, 2, \ldots, n \) defines a factorization of \( P(t) \). Namely, let \( A_k = \{ i_1, i_2, \ldots, i_k \}, k = 1, \ldots, n-1 \) and \( A_0 = \emptyset \). The following result was obtained in [GR2].

Theorem 1.1.1.

\[ P(t) = (t - x_{A_{n-1},i_n})(t - x_{A_{n-2},i_{n-1}}) \cdots (t - x_{A_0,i_1}). \]
The elements $x_{A,i}$ (called pseudo-roots of $P(t)$ in Section 1.2) are defined by pairs $A \subseteq \{1, 2, \ldots, n\}$ and $i \in \{1, 2, \ldots, n\} \setminus A$. They satisfy the relations

$$x_{A \cup \{i\}, j} + x_{A,i} = x_{A \cup \{j\}, i} + x_{A,j} \quad (1.1a)$$

$$x_{A \cup \{i\}, j} x_{A,i} = x_{A \cup \{j\}, i} x_{A,j} \quad (1.1b)$$

When $A = \emptyset$, the relations (1.1) give relations (0.3).

To understand these relations better, we consider the Hasse graph of the Boolean lattice of subsets of $\{1, 2, \ldots, n\}$. Thus, we consider the directed graph whose vertices are subsets of $\{1, 2, \ldots, n\}$ and an edge goes from vertex $B$ to vertex $A$ if and only if $B = A \cup \{i\}$ where $i \in \{1, 2, \ldots, n\} \setminus A$. Then the relations (1.1) can be described nicely by a diamond in the graph with vertices $A \cup \{i, j\}, A \cup \{i\}, A \cup \{j\}, A$ (see Fig 1 for $A = \emptyset, i = 1, j = 2$).

The elements $x_{A \cup \{i\}, j}, x_{A \cup \{j\}, i}, x_{A,j}, x_{A,i}$ correspond to the edges from $A \cup \{i\}$ to $A \cup \{j\}$, from $A \cup \{i\}$ to $A \cup \{j\}$ to $A$, and from $A \cup \{i\}$ to $A$.

1.2. Pseudo-roots. The elements $x_{A,i}$ introduced in the previous section are natural examples of pseudo-roots of noncommutative polynomials.

Let $R$ be an associative ring with unit and $P(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$ be a polynomial in $R[t]$. According to [GRSW] an element $x \in R$ is a pseudo-root of $P(t)$ if there exist polynomials $Q_1(t), Q_2(t) \in R[t]$ such that

$$P(t) = Q_1(t)(t-x)Q_2(t).$$

The element $x$ is a right root of $P(t)$ if $Q_2(t) = 1$ and is a left root of $P(t)$ if $Q_1(t) = 1$. As we mentioned in the introduction, it is easy to check that $x$ is a right root of $P(t)$ if and only if $a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$. Similarly, $x$ is a left root if and only if $x^n a_0 + x^{n-1} a_1 + \cdots + a_n = 0$.

For a noncommutative ring $R$, a theory of polynomials over $R$ should be based not only on properties of right (left) roots but on pseudo-roots as well (see Section 1.1). In particular, it is natural to study subrings $R_P$ of $R$ generated by pseudo-roots of a given polynomial $P(t) \in R[t]$. We will construct now a universal algebra of pseudo-roots $Q_n$ introduced in [GRW]. In certain situations there exists a canonical homomorphism of $Q_n$ into $R_P$.

1.3. The algebra $Q_n$. Let $F$ be a field. The algebra $Q_n$ over $F$ is generated by elements $x_{A,i}$ where $A \subseteq \{1, 2, \ldots, n\}$ and $i \in \{1, 2, \ldots, n\} \setminus A$ satisfying relations (1.1).

Let $R$ be an algebra over a field $F$ and let $P(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ be a polynomial over $R$ such that $P(t)$ has $n$ right roots $x_1, \ldots, x_n$ in a generic position. The subalgebra of $R$ generated by all pseudo-roots of $P(t)$ is denoted by $R_P$. The following theorem was proved in [GRW].
Theorem 1.3.1. There exists a homomorphism

\[ \alpha : Q_n \to R_P \]

such that \( \alpha(z_{A,i}) = x_{A,i} \).

It turns out that the algebra \( Q_n \) has several nice properties studied in [GRW, GGR, GGRSW, SW, Pi]. It is easy to see that this algebra has a set of generators satisfying quadratic relations because linear and quadratic relations in (1.1) are separated, i.e. \( Q_n \) is a quadratic algebra. In [GRW] we constructed a linear basis in \( Q_n \), in [GGRSW] we computed the Hilbert series for \( Q_n \) and its quadratic dual, and in [SW] we showed that \( Q_n \) is a Koszul algebra.

Other properties of \( Q_n \), including its relations to the algebra of noncommutative symmetric functions introduced in [GKLLRT], were discussed in [GRW]. Overall, one can say that \( Q_n \) is a rather “tame” algebra (due to the properties listed above) despite its exponential growth.

2. The algebras \( A(i, \Gamma) \) and \( A(\Gamma) \)

2.1. Directed graphs. Let \( \Gamma = (V, E) \) be a directed graph. That is, \( V \) is a set (of vertices), \( E \) is a set (of edges), and \( t : E \to V \) and \( h : E \to V \) are functions. (\( t(e) \) is the tail of \( e \) and \( h(e) \) is the head of \( e \).)

A vertex \( u \in V \) is called maximal if there is no \( e \in E \) such that \( h(e) = u \). A vertex \( v \in V \) is called minimal if there is no \( e \in E \) such that \( t(e) = v \).

We say that \( \Gamma \) is layered if \( V \) is the disjoint union of \( V_i \), \( 0 \leq i \leq n \), \( E \) is the disjoint union of \( E_i \), \( 0 \leq i \leq n \), \( t : E_i \to V_i \), \( h : E_i \to V_{i-1} \). We will write \( |v| = i \) if \( v \in V_i \). In this case the number \( i \) is called the level of \( v \). Note that a layered graph has no loops.

We will assume throughout the remainder of the paper that \( \Gamma = (V, E) \) is a layered graph with \( V = \bigcup_{i=0}^{n} V_i \), and \( V_0 = \{*\} \) where \( * \) is the unique minimal vertex of \( \Gamma \).

To any partially ordered set \( I \) there corresponds the directed graph \( \Gamma_I \) called the Hasse graph of \( I \). Its vertices are elements \( x \in I \) and its edges are pairs \( e = (x, y) \in I \times I \) such that \( y \) is an immediate predecessor of \( x \) (in other words, \( y < x \) and there is no \( z \in I \) such that \( y < z < x \)). The element \( x \) is the tail of \( e \) and the element \( y \) is the head of \( e \).

Recall that a partially ordered set \( I \) with a function \( r : I \to \mathbb{Z}_+ \) is called a ranked poset with the ranking function if \( r(x) > r(y) \) for any \( x > y \). Then \( r \) turns the corresponding Hasse graph \( \Gamma_I \) into a layered graph with \( |x| = r(x) \). To any maximal (minimal) element in \( I \) there corresponds a maximal (minimal) vertex in \( \Gamma_I \).

Note that every layered graph with no multiple edges arrives in in this manner. Although, the theory we develop below applies to graphs with multiple edges (see Example 1 of Section 2.3), all the examples we consider will be Hasse graphs.

Here are some examples of ranked partially ordered sets.
Examples.

1. To any set $S$ corresponds the partially ordered set $\mathcal{P}(S)$ of all subsets of $S$. The order relation is given by inclusion and $r(A)$ equals the cardinality of $A$. The minimal element in $\mathcal{P}(S)$ is the empty set and the maximal element is $S$. We will always assume that $S$ is a finite set.

2. To any finite-dimensional vector space $E$ over a field $F$ corresponds the partially ordered set $\mathcal{W}(E)$ of all vector subspaces of $E$. The order relation is given by inclusion and $r(W)$ is the dimension of $W$ for a subspace $W \subseteq E$.

   The minimal element in $\mathcal{W}(E)$ is the zero subspace $(0)$ and the maximal element is $E$.

3. We say that a layered graph $\Gamma = (V, E)$ with $V = \bigcup_{i=0}^{n_i} V_i$ is complete if for every $i, 1 \leq i \leq n$, and every $v \in V_i, w \in V_{i-1}$, there is a unique edge $e$ with $t(e) = v, h(e) = w$. A complete layered graph is determined (up to isomorphism) by the cardinalities of the $V_i$. We denote the complete layered graph with $V = \bigcup_{i=0}^{n_i} V_i, |V_i| = m_i$ for $0 \leq i \leq n$, by $C[m_n, m_{n-1}, \ldots, m_1, m_0]$. Note that the graph $C[1, m_{n-1}, \ldots, m_1, m_0]$ has a unique minimal vertex of level 0 and the graph $C[1, m_{n-1}, \ldots, m_1, m_0]$ has a unique maximal vertex of level $n$.

4. Recall that the partially ordered set $\mathcal{Y}$ of Young diagrams can be identified with the set of weakly decreasing sequences $\lambda = (\lambda_k)_{k \geq 1}$ such that $\lambda_k = 0$ for $k >> 0$. By definition, $\lambda_k \geq \mu_k$ if and only if $\lambda_k \geq \mu_k$ for all $k$. In fact, $\mathcal{Y}$ is a ranked partially ordered set. The rank is defined as $r((\lambda_k)) = \sum_{k \geq 1} \lambda_k$.

5. Abstract regular polytopes (see, for example, [MS]) also are natural examples of ranked partially ordered sets.

6. A family $\mathcal{F} \subseteq \mathcal{P}(S)$ is called a complex if $B \in \mathcal{F}$ and $A \subseteq B$ imply $A \in \mathcal{F}$. The order and the ranking function on $\mathcal{P}(S)$ induce an order and a ranking function on $\mathcal{F}$.

Another important example of a layered graph is the graph of right divisors of a monic polynomial described in the next subsection.

2.2. The graph of right divisors. Let $P(t)$ be a monic polynomial over an associative algebra $R$ and $S$ be a set of pseudo-roots of $P(t)$. Denote by $R_S$ the subalgebra in $R$ generated by pseudo-roots $x \in S$.

Construct a layered graph $\Gamma(P, S) = (V, E)$,

$$V = V_n \cup V_{n-1} \cup \ldots V_1 \cup V_0$$

as follows. The vertices of $V_k = \{v \in V : r(v) = k\}$ are monic polynomials $B(t) \in R[t]$ such that $\deg B(t) = k$ and

$$P(t) = Q(t)B(t)$$

in $R[t]$.

We say that there is an edge from vertex $B_1(t)$ to $B_2(t)$ in $\Gamma$ if

$$B_1(t) = (t - x)B_2(t)$$
for some $x \in S$.

Note that graph $\Gamma (P, S)$ has only one maximal vertex $v = P(t)$ and only one minimal vertex $w = 1$.

### 2.3. Algebras $A(i, \Gamma)$ and $A(\Gamma)$.

In this section we discuss a class of algebras introduced in [GRSW] and [RSW]. Let $\Gamma = (V, E)$ be a layered directed graph with a finite number of layers. That is, $V = \bigcup_{i=0}^{n} V_i$, $E = \bigcup_{i=1}^{n} E_i$, $t : E_i \to V_i$, $h : E_i \to V_{i-1}$.

Recall that we are assuming throughout the remainder of the paper that $V_0 = \{ \ast \}$ where $\ast$ is the unique minimal vertex of $\Gamma$.

For each $v \in \bigcup_{i=1}^{n} V_i$ we will fix, arbitrarily, some $e_v \in E$, with $t(e) = v$. Recall, that if $v \in V_i$ we write $|v| = i$ and say that $v$ has level $i$. Similarly, if $e \in E_i$ we write $|e| = i$ and say that $e$ has level $i$.

If $v, w \in V$, a path from $v$ to $w$ is a sequence of edges $\pi = \{e_1, e_2, ..., e_k\}$ with $t(e_1) = v$, $h(e_k) = w$ and $t(e_{i+1}) = h(e_i)$ for $1 \leq i < k$. We write $v = t(\pi)$, $w = h(\pi)$ and call $v$ the tail of the path and $w$ the head of the path. We also write $v > w$ if there is a path from $v$ to $w$.

With $\pi$ defined as above, let $l(\pi) = k$ be the length of $\pi$ and let $|\pi| = |e_1| + ... + |e_k|$ be the level of $\pi$.

If $\pi_1 = \{e_1, ..., e_k\}, \pi_2 = \{f_1, ..., f_l\}$ are paths with $h(\pi_1) = t(\pi_2)$ then $\{e_1, ..., e_k, f_1, ..., f_l\}$ is a path; we denote it by $\pi_1 \pi_2$.

For $v \in V$, write $v^{(0)} = v$ and define $v^{(i+1)} = h(e_{v(i)})$ for $0 \leq i < |v|$. Then $v^{(|v|)} = \ast$ and $\pi_v = \{e_{v(0)}, ..., e_{v(|v|-1)}\}$ is a path from $v$ to $\ast$.

Let $T(E)$ denote the free associative algebra on $E$ over a field $F$. We are going to introduce a quotient algebra of $T(E)$ modulo relations generalizing relations (0.2) and (0.3). We will do this by equating coefficients of polynomials associated with pairs of paths with the same origin and the same end.

For a path $\pi = \{e_1, e_2, ..., e_m\}$ define

$$P_\pi(\tau) = (1 - \tau e_1)...(1 - \tau e_m) \in T(E)[\tau]/(\tau^{n+1}).$$

Note that $P_{\pi_1 \pi_2}(\tau) = P_{\pi_1}(\tau)P_{\pi_2}(\tau)$ if $h(\pi_1) = t(\pi_2)$. Write

$$P_\pi(\tau) = \sum_{j=0}^{n+1} (-1)^j e(\pi, j) \tau^j.$$

**Definition 2.3.1.** Let $R$ be the ideal in $T(E)$ generated by

$$\{e(\pi_1, k) - e(\pi_2, k) \mid k \geq 1, t(\pi_1) = t(\pi_2), \pi(\pi_1) = h(\pi_2)\}$$
Set
\[ A(\Gamma) = T(E)/R. \]

In other words, \( A(\Gamma) \) is defined by generators \( e \in E \) and relations
\[
\sum_{j=1}^{k} e_j = \sum_{j=1}^{k} f_j,
\]
\[
\sum_{i<j} e_i e_j = \sum_{i<j} f_i f_j,
\]
\[
\ldots
\]
\[
e_1 e_2 \ldots e_k = f_1 f_2 \ldots f_k.
\]

**Examples.**
1. Suppose \( \Gamma = (V,E), e, f \in E \) and \( t(e) = t(f), h(e) = h(f) \). Then the images of \( e \) and \( f \) in \( A(\Gamma) \) are equal. Thus \( A(\Gamma) \) is isomorphic to \( A(\Gamma') \) where \( \Gamma' \) is the graph without multiple edges obtained from \( \Gamma \) by identifying all edges with the same tail and head.

2. If \( \Gamma \) is a tree-like graph, i.e. there are no distinct paths \( \pi_1, \pi_2 \) such that \( t(\pi_1) = t(\pi_2), h(\pi_1) = h(\pi_2), \) then \( A(\Gamma) = T(E) \) is the free associative algebra generated by \( e \in E \).

3. Let \( \Gamma \) be a diamond graph from Fig 1. Then \( A(\Gamma) \) is defined by the relations
\[
e_1 + e_2 = f_1 + f_2,
\]
\[
e_1 e_2 = f_1 f_2
\]
and \( A(\Gamma) \) is isomorphic to the algebra \( Q_2 \) defined by the relations (1.1).

It also useful to consider larger algebras \( A(i, \Gamma) \) associated to directed graphs. To introduce these classes of algebras, define \( P_{i,\pi}(\tau) \) to be the image of \( P_\pi(\tau) \) in the quotient of \( T(E)[\tau]/(\tau^{i+1}) \) and write
\[
P_{i,\pi}(\tau) = \sum_{k=0}^{\min(l(\pi), i)} (-1)^k e(i, \pi, k) \tau^k
\]
for \( i \geq 1 \).

Note that \( P_{i,\pi_1\pi_2}(\tau) = P_{i,\pi_1}(\tau)P_{i,\pi_2}(\tau) \) if \( h(\pi_1) = t(\pi_2) \).

Set \( e(i, \pi, k) = 0 \) if \( k > \min(l(\pi), i) \). For \( v \in \bigcup_{i=1}^n V_i \), set \( P_{i,v}(t) = P_{i,\pi_v}(t) \) and \( e(i, v, k) = e(i, \pi_v, k) \). Also, set \( P_{i,*}(t) = 1 \) and \( e(i, *, k) = 0 \) if \( k > 0 \).

**Definition 2.3.2.** Let \( R(i) \) be the ideal in \( T(E) \) generated by
\[
\{ e(i, \pi_1, k) - e(i, \pi_2, k) \mid t(\pi_1) = t(\pi_2), h(\pi_1) = h(\pi_2), \; 1 \leq k \leq \min(l(\pi_1), i) \}.
\]
Note that
\[ R(1) \subseteq R(2) \subseteq ... \subseteq R(n) = R(n+1) = ... \]

Let
\[ A(i, \Gamma) = T(E)/R(i). \]

Therefore we have
\[ A(1, \Gamma) \to A(2, \Gamma) \to ... \to A(n-1, \Gamma) \to A(n, \Gamma). \]

Note that \( A(n, \Gamma) = A(\Gamma) \) (as in Definition 2.3.1).

In other words, \( A(i, \Gamma) \) is defined by generators \( e \in E \) and relations
\[ \sum_{j_1 < j_2 < ... < j_k \atop 1 \leq k \leq i} (e_{j_1}e_{j_2}\ldots e_{j_k} - f_{j_1}f_{j_2}\ldots f_{j_k}) \]
for all pairs of paths \( \pi_1 = (e_1, e_2, \ldots, e_s) \), \( \pi_2 = (f_1, f_2, \ldots, f_s) \) such that \( t(\pi_1) = t(\pi_2) \), \( h(\pi_1) = h(\pi_2) \).

In Section 3.3 we will show that for most of the graphs described in Section 2.1, algebras \( A(2, \Gamma) \) and \( A(\Gamma) \) coincide. In this case \( A(\Gamma) \) is described by linear and quadratic relations. In particular, for the Hasse graph \( \Gamma_n \) of subsets of \( \{1, 2, \ldots, n\} \) the algebra \( A(\Gamma_n) \) is isomorphic to the algebra \( Q_n \) described in Section 1.3.

2.4. Universality of \( A(\Gamma) \). In notations of Section 2.3 assume that graph \( \Gamma \) has a unique maximal vertex \( M \) and a unique minimal vertex \( * \). Define a polynomial \( \mathcal{P}_\Gamma(t) \) over algebra \( A(\Gamma) \) corresponding to any path \( \pi_0 = (e_1, e_2, \ldots, e_n) \) from \( M \) to \( * \):
\[ \mathcal{P}_\Gamma(t) = (t - e_1)(t - e_2)\ldots(t - e_n). \]

It follows from the definition of \( A(\Gamma) \) that \( \mathcal{P}_\Gamma(t) \) does not depend of a choice of path \( \pi_0 \).

The polynomial \( \mathcal{P}_\Gamma(t) \) is a monic polynomial. We call it the universal polynomial over \( A(\Gamma) \).

Let \( R \) be an algebra, \( P(t) \) a monic polynomial of degree \( n \) over \( R \), and \( S \) a set of pseudo-roots of \( P(t) \). Let \( \Gamma(P, S) \) be the layered graph constructed in Section 2.2.

Assume that the set \( S \) contains pseudo-roots \( a_1, a_2, \ldots, a_n \) such that
\[ P(t) = (t - a_1)(t - a_2)\ldots(t - a_n). \]

Then graph \( \Gamma(P, S) \) contains a directed path from maximal vertex \( M = P(t) \) to minimal vertex \( * = 1 \).

Following Section 2.3 construct the algebra \( A(\Gamma(P, S)) \). Let \( \mathcal{P}(t) = P_\Gamma(t) \) be the universal polynomial over this algebra.
Theorem 2.4.1. There is a canonical homomorphism

$$\kappa : A(\Gamma(P,S)) \to R$$

such that the induced homomorphism of polynomial algebras

$$\hat{\kappa} : A(\Gamma(P,S))[t] \to R[t]$$

maps $P(t)$ to $P(t)$.

Theorem 2.4.1 was proved in [GGRW1]. To construct the homomorphism $\kappa$, note that to any edge $e \in \Gamma(P,S)$ corresponds a pair of polynomials $B_1(t), B_2(t)$ in $R[t]$ such that $B_1(t), B_2(t)$ divide $P(t)$ from the right and $B_1(t) = (t - a)b_2(t)$. Set $\kappa(e) = a$. One can see that $\kappa$ can be uniquely extended to the homomorphism $A(\Gamma(P,S)) \to R$ and that $\hat{\kappa}(P)(t) = P(t)$.

3. Properties of the algebras $A(i, \Gamma)$ and $A(\Gamma)$

Throughout this section that we continue to assume $\Gamma = (V,E)$ is a layered graph with $V = \cup_{i=0}^{n} V_i$, that $V_0 = \{\ast\}$, and that, for every $v \in V_+ = \cup_{i=1}^{n} V_i$, $\{e \in E \mid t(e) = v\} \neq \emptyset$. For each $v \in V_+$ fix, arbitrarily, some $e_v \in E$ with $t(e_v) = v$. This defines a distinguished path, denoted by $\pi_v$, from $v$ to $\ast$. Namely, for $v \in V^+$ we define $v^{(0)} = v$ and $v^{(i+1)} = h(e_{v(i)})$ for $0 \leq i < |v|$ and we set $\pi_v = \{e_v(0), e_v(1), \ldots, e_v(|v|-1)\}$.

3.1. Linear basis in $A(\Gamma)$. For $v \in V_+$ and $1 \leq k \leq |v|$ we define $\hat{e}(v,k)$ to be the image in $A(\Gamma)$ of the product $e_1 \ldots e_k$ in $T(E)$ where $\pi_v = \{e_1, \ldots, e_{|v|}\}$.

If $(v,k), (u,\ell) \in V \times \mathbb{N}$ we say $(v,k)$ covers $(u,\ell)$ if $v > u$ and $k = |v| - |u|$. In this case we write $(v,k) \triangleright (u,\ell)$. (In [GRSW] we used different terminology and notation: if $(v,k) \triangleright (u,\ell)$ we said $(v,k)$ can be composed with $(u,\ell)$ and wrote $(v,k) \equiv (u,\ell)$.)

Example. In Fig 1 pair $\{(12), 1\}$ covers the pairs $\{(1), 1\}$ and $\{(2), 1\}$.

The following theorem is proved in [GRSW] (see Corollary 4.5).

Theorem 3.1.1. Let $\Gamma = (V,E)$ be a layered graph, $V = \cup_{i=0}^{n} V_i$, and $V_0 = \{\ast\}$ where $\ast$ is the unique minimal vertex of $\Gamma$. Then

$$\{\hat{e}(v_1,k_1) \ldots \hat{e}(v_l,k_l) \mid l \geq 0, v_1, \ldots, v_l \in V_+, 1 \leq k_i \leq |v_i|, (v_i,k_i) \not\equiv (v_{i+1},k_{i+1})\}$$

is a basis for $A(\Gamma)$.

Theorem 3.1.1 implies the construction of a linear basis in the algebra $Q_n$ obtained in [RSW], but the current description is much nicer and the proof is much shorter.
3.2. New description of the algebras $A(i, \Gamma)$ and $A(\Gamma)$. In Section 2.3 we described algebras $A(i, \Gamma)$ as algebras whose generators are edges of $\Gamma$ subject to homogeneous relations of degree $1, 2, \ldots, i$. In this Section, following [RSW], we describe those algebras as algebras whose generators are vertices of $\Gamma$ subject to homogeneous relations of degree $2, \ldots, i$. In fact, we will represent each vertex $v$ in $\Gamma$ by a path from $v$ to $\ast$. By Definition 2.3.1 the sum of edges in each of such paths has the same image in $A(\Gamma)$.

Define $e(v, 1) = e_v(0) + e_v(1) + \ldots + e_v(|v| - 1)$. Thus

$$e_v = e(v, 1) - e(v(1), 1) = e(t(e_v), 1) - e(h(e_v), 1).$$

Let $E' = \{e_v|v \in V^+\}$. Let $F$ be our ground field. For any set $X$ denote by $FX$ the vector space over $F$ with basis elements $x \in X$. Define $\tau : FE \to FE'$ by

$$\tau(f) = e(t(f), 1) - e(h(t), 1).$$

Then $\tau$ is a projection of $FE$ onto $FE'$ with kernel generated by $e(i, \pi_1, 1) - e(i, \pi_2, 1)$ where $\pi_1, \pi_2$ are paths with the same tail and head.

Now define $\eta : FE' \to FV^+$ by $\eta : e_v \mapsto v$.

Then $\eta$ is an isomorphism of vector spaces and $\eta \tau$ induces a surjective homomorphism of graded algebras

$$\theta : T(E) \to T(V^+).$$

Proposition 3.2.1. $\theta$ induces an isomorphism $A(i, \Gamma) \cong T(V^+)/\theta(R(i))$.

It is important to write generators for the ideal $\theta(R(i))$ explicitly. The ideal is generated by elements of the form $\theta(e(i, \pi_1, k) - e(i, \pi_2, k))$ where $2 \leq k \leq i$ (see Section 2.3 for the definition of $e(i, \pi_1, k)$). Therefore it will be sufficient to write a formula for $\theta(e(i, \pi, k))$. Let $\pi = \{e_1, e_2, \ldots, e_m\}$ be a path, let $t(e_j) = v_{j-1}$ for $1 \leq j \leq m$ and let $h(e_m) = v_m$.

**Proposition 3.2.2.**

$$\theta(e(i, \pi, k)) = (-1)^k \sum_{1 \leq j_1 < \ldots < j_k \leq s} (v_{j_1 - 1} - v_{j_1}) \ldots (v_{j_k - 1} - v_{j_k})$$

where $s = \min(i, m)$.

Proposition 3.2.1 immediately implies

**Corollary 3.2.3.** $A(2, \Gamma)$ is a quadratic algebra.

We will show below that for many interesting graphs algebra $A(2, \Gamma)$ coincides with algebra $A(\Gamma)$. 

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3.3. Algebras associated with uniform graphs.

Let \( \Gamma \) be a layered graph. For \( v \in V \) define \( S_-(v) \) to be the set of all vertices \( w \in V \) covered by \( v \), i.e. such that there exists an edge with the tail \( v \) and the head \( w \). Similarly, define \( S_+(v) \) to be the set of all vertices covering \( v \).

For \( v \in V_j, j \geq 2 \), let \( \sim_v \) denote the equivalence relation on \( S_-(v) \) generated by \( u \sim_v w \) if \( S_-(u) \cap S_-(w) \neq \emptyset \).

**Definition 3.3.1.** The layered graph \( V \) is said to be **uniform** if, for every \( v \in V_j, j \geq 2 \), all elements of \( S_-(v) \) are equivalent under \( \sim_v \).

**Example.** The diamond graph in Fig 1 is a uniform graph.

**Proposition 3.3.2.** The Hasse graphs of partially ordered sets listed in Examples 1-5 in Section 2.1 are uniform.

Algebras associated to uniform graphs have especially nice structure.

**Proposition 3.3.3.** Let \( \Gamma \) be a uniform layered graph. Then \( A(\Gamma) = A(2, \Gamma) \cong T(V^+)/R_V \) is a quadratic algebra and \( R_V \) is generated by

\[
\{v(u - w) - u^2 + w^2 + (u - w)x \mid v \in \bigcup_{i=2}^n V_i, u, w \in S_-(v), x \in S_-(u) \cap S_-(w)\}.
\]

**Corollary 3.3.4.** The algebra associated to the Hasse graph of all subsets of \( \{1, 2, \ldots, n\} \) is isomorphic to algebra \( Q_n \) described in Section 1.3.

3.4. **\( A(\Gamma) \) is a Koszul algebra.** Koszul algebras constitute an important class of quadratic algebras (see, for example, [PP, U] for the definitions and properties of Koszul algebras). Most of the known examples are algebras of a polynomial growth. A wide variety of Koszul algebras of exponential growth is given by the following theorem proved in [RSW].

**Theorem 3.4.1.** Let \( \Gamma \) be a uniform layered graph with a unique minimal element. Then \( A(\Gamma) \) is a Koszul algebra.

3.5. **Algebras dual to \( A(2, \Gamma) \).**

Let \( A \) be a quadratic algebra over a field \( F \). Thus \( A \) is isomorphic to a quotient algebra \( T(W)/<L> \) where \( W \) is a vector space over \( F \) and \( L \) is a subspace in \( W \otimes W \). Assume that \( W \) is finite dimensional. Denote by \( W^* \) the dual vector space and set \( L^+ = \{f \in W^* \otimes W^* \mid f|_L = 0 \} \). By definition, the algebra \( A^! = T(W^*)/<L^+> \) is the quadratic dual to algebra \( A \).

Both algebras \( A \) and \( A^! \) are graded and their Hilbert series \( H(A, \tau) \) and \( H(A^!, \tau) \) are well defined. A graded algebra \( A \) is Koszul if and only if \( A^! \) is Koszul and in this case the Hilbert series of \( A \) and \( A^! \) are related by

\[
H(A, \tau)H(A^!, -\tau) = 1
\]
According to Corollary 3.2.3 the algebra $A(2, \Gamma)$ is quadratic. Denote its quadratic dual algebra by $B(\Gamma)$. If the graph $\Gamma$ is uniform than $A(2, \Gamma) = A(\Gamma)$ and $B(\Gamma) = A(\Gamma)^!$.

The following theorem gives a description of $B(\Gamma)$. When $\Gamma$ is the Hasse graph of the partially ordered set of subsets of $\{1, 2, \ldots, n\}$ such description was essentially given in [GGRSW].

Let $\Gamma = (V, E)$ be a layered graph with a unique minimal vertex. Assume that $V$ is a finite set.

**Theorem 3.5.1.** The algebra $B(\Gamma)$ is generated by vertices $v \in V$ subject to the following relations:

1. $uv = 0$ if $u \neq v$ and there is no edge $e \in E$ such $t(e) = u$, $h(e) = v$;
2. $v^2 + v \sum w = 0$, where the sum is taken over all $w \in S_-(v)$;
3. $v^2 + (\sum u)v = 0$, where the sum is taken over all $u \in S_+(v)$.

In Section 4 we will compute Hilbert series for some algebras $B(\Gamma)$ and see that those Hilbert series are polynomial in $\tau$. It follows that in this case, the algebras $B(\Gamma)$ are finite dimensional and the ideal (of codimension one) generated by $V$ is nilpotent. It is not surprising since Theorem 3.5.1 implies that $v^3 = 0$ for any $v \in V$.

4. The Hilbert series of $A(\Gamma)$

In this section we compute the Hilbert series of algebras $A(\Gamma)$ introduced in Section 2.3 and specialize the result for some examples of layered directed graphs. All results formulated in this section were obtained in [RSW1].

**4.1. Main theorem.**

Let $h(\tau)$ denote the Hilbert series $H(A(\Gamma), \tau)$, where $\Gamma = (V, E)$ is a layered graph with unique minimal element $*$ of level 0. Arrange the elements of $V$ in nonincreasing order and index the elements of vectors and matrices by this ordered set. Let $1$ denote the column vector all of whose entries are 1, and let $\zeta(\tau)$ denote the matrix with entries $\zeta_{v, w}(\tau)$ for $v, w \in V$ where $\zeta_{v, w}(\tau) = \tau^{|v| - |w|}$ if $v \geq w$ and 0 otherwise. Then we have

**Proposition 4.1.1.**

$$h(\tau) = \frac{1 - \tau}{1 - \tau^T \zeta(\tau)^{-1} 1}.$$ 

Note that $\zeta(\tau) = 1 + N$ where $N$ is strongly upper-triangular. Consequently, the $(v, w)$-entry of $\zeta(\tau)^{-1}$ can be written as

$$\sum_{v = v_1 > \cdots > v_\ell = w \geq *} (-1)^{\ell+1} \tau^{|v| - |w|}$$

and we have the following result.
Theorem 4.1.2. Let $\Gamma$ be a layered graph with unique minimal element $*$ of level 0 and $h(\tau)$ denote the Hilbert series of $A(\Gamma)$. Then

$$h(\tau) = \frac{1 - \tau}{1 + \sum_{v_1 > v_2 \ldots > v_\ell \geq *}(\tau^{|v_1|} - \tau^{|v_\ell|})^\ell}.$$ 

The proof of Proposition 4.1.1 and Theorem 4.1.2 is based on Theorem 3.1.1 describing a linear basis in $A(\Gamma)$.

We remark that the matrices $\zeta(1)$ and $\zeta(1)^{-1}$ are well-known as the zeta-matrix and the Möbius-matrix of $V$ (cf. [R]).

In the remaining part of this section we will use Theorem 4.1.2 to compute the Hilbert series of the algebras $A(\Gamma)$ associated with certain layered graphs.

4.2. The Hilbert series of the algebra associated with the Hasse graph of the lattice of subsets of $\{1, \ldots, n\}$.

Let $\Gamma_n$ denote the Hasse graph of the lattice of all subsets of $\{1, \ldots, n\}$. Thus the vertices of $\Gamma_n$ are subsets of $\{1, \ldots, n\}$, the order relation $>$ is set inclusion $\supset$, the level $|v|$ of a set $v$ is its cardinality, and the unique minimal vertex $*$ is the empty set $\emptyset$. Then the algebra $A(\Gamma_n)$ is the algebra $Q_n$ defined in [GRW]. Theorem 4.1.2 implies the following theorem (from [GGRSW]). The proof obtained in [RSW] is much shorter and more conceptual than that in [GGRSW].

Theorem 4.2.1.

$$H(Q_n, \tau) = \frac{1 - \tau}{1 - \tau(2 - \tau)^n}.$$ 

4.3. The Hilbert series of algebras associated with the Hasse graph of the lattice of subspaces of a finite-dimensional vector space over a finite field.

We will denote by $L(n, q)$ the Hasse graph of the lattice of subspaces of an $n$-dimensional space over the field $\mathbb{F}_q$ of $q$ elements. Thus the vertices of $L(n, q)$ are subspaces of $\mathbb{F}_q^n$, the order relation $>$ is inclusion of subspaces $\supset$, the level $|U|$ of a subspace $U$ is its dimension, and the unique minimal vertex $*$ is the zero subspace $(0)$. Recall that $\binom{n}{m}_q$ is a $q$-binomial coefficient.

Theorem 4.3.1.

$$\frac{1 - \tau}{H(A(L(n, q)), \tau)} = 1 - \tau \sum_{m=0}^{n} \binom{n}{m}_q (1 - \tau)(1 - \tau q) \ldots (1 - \tau q^{n-m-1}).$$

Note that setting $q = 1$ in the expression in Theorem 4.3.1 gives $1 - \tau(2 - \tau)^n$. By Theorem 4.2.1, this is $(1 - \tau)H(Q_n, \tau)^{-1}$.

Since by Theorem 3.4.1 (see also [RSW]) $A(L(n, q))$ is a Koszul algebra, we have the following corollary.
Corollary 4.3.2.

\[ H(A(L(n, q))^1, \tau) = 1 + \sum_{m=0}^{n-1} \binom{n}{m} (1 + \tau q) \cdots (1 + \tau q^{n-m-1}). \]

4.4. The Hilbert series of algebras associated with complete layered graphs.

Recall (see Example 3, Section 2.1) that a layered graph \( \Gamma = (V, E) \) with \( V = \bigcup_{i=0}^{n} V_i \) is complete if for every \( i, 1 \leq i \leq n \), and every \( v \in V_i, w \in V_{i-1} \), there is a unique edge \( e \) with \( t(e) = v, h(e) = w \). A complete layered graph is determined (up to isomorphism) by the cardinalities of the \( V_i \). We denote the complete layered graph with \( V = \bigcup_{i=0}^{n} V_i, |V_i| = m_i \) for \( 0 \leq i \leq n \), by \( C[m_n, m_{n-1}, \ldots, m_1, 1] \). Note that the graph \( C[m_n, m_{n-1}, \ldots, m_1, 1] \) has a unique minimal vertex of level 0 and so Theorem 4.1.2 applies to \( A(C[m_n, m_{n-1}, \ldots, m_1, 1]) \). This leads to the following theorem.

Theorem 4.4.1.

\[
\frac{1 - \tau}{H(A(C[m_n, m_{n-1}, \ldots, m_1, 1], \tau)} = 1 - \sum_{k=0}^{n} \sum_{a=k}^{n} (-1)^k m_a(m_{a-1} - 1)(m_{a-2} - 1) \cdots (m_{a-k+1} - 1)m_{a-k}\tau^{k+1}.
\]

When \( k = 1 \) and \( k = 2 \), the product \((m_{a-1} - 1)(m_{a-2} - 1) \cdots (m_{a-k+1} - 1)\) in this expression (and also in the expression in Corollary 4.4.2) represents the empty product and so has value 1.

This result applies, in particular, to the case \( m_0 = m_1 = \ldots = m_n = 1 \). The resulting algebra \( A(C[1, \ldots, 1]) \) has \( n \) generators and no relations. Theorem 4.4.1 shows that

\[
\frac{1 - \tau}{H(A(C[1, \ldots, 1], \tau)} = 1 - \sum_{a=0}^{n} \tau + \sum_{a=1}^{n} \tau^2 = (1 - \tau)(1 - n\tau).
\]

Thus

\[
H(A(C[1, \ldots, 1], \tau) = \frac{1}{1 - n\tau}
\]

and we have recovered the well-known expression for the Hilbert series of the free associative algebra on \( n \) generators.

Since by Theorem 3.4.1 (see also [RSW]) the algebras associated to complete directed graphs are Koszul algebras, we have the following corollary.

Corollary 4.4.2.

\[
H(A(C[m_n, m_{n-1}, \ldots, m_1, 1])^1, \tau) = 1 + \sum_{k=1}^{n} \sum_{a=k}^{n} m_a(m_{a-1} - 1)(m_{a-2} - 1) \cdots (m_{a-k+1} - 1) \tau^k.
\]
5. Sufficient sets of pseudo-roots and directed graphs

We return to questions 1) and 2) from the introduction: Given a polynomial \( P(t) \) over a noncommutative algebra, how to obtain its factorizations and how to relate two different factorizations? To answer these questions we will work in general context of algebras \( A(\Gamma) \).

5.1. Defining sets of pseudo-roots. In this section we briefly describe some results from [GGRW1]. Let \( R \) be an associative ring with unit, \( P(t) = t^n + a_1 t^{n-1} + \cdots + a_n \) be a polynomial over \( R \), and \( t \) be a central variable.

Recall (see Theorem 1.1.1) that if \( P(t) \) has right roots \( x_1, x_2, \ldots, x_n \) in a generic position then \( P(t) \) admits factorizations

\[
P(t) = (t - x_{A_{n-1},i_n})(t - x_{A_{n-2},i_{n-1}}) \cdots (t - x_{\emptyset,i_1})
\]

indexed by orderings of \( \{1, 2, \ldots, n\} \). Also recall that pseudo-roots \( x_{A,i} \)'s are rational expressions in \( x_1, x_2, \ldots, x_n \).

According to [GRW] the pseudo-roots \( x_{A,i} \)'s can be obtained from a generic set of \( n \) right roots by a sequence of operations

\begin{equation}
(5.1a) \quad a, b \mapsto (a - b)a(a - b)^{-1}, \quad (b - a)b(b - a)^{-1}
\end{equation}

and from a generic set of \( n \) left roots by a sequence of operations

\begin{equation}
(5.1b) \quad a, b \mapsto (a - b)^{-1}a(a - b), \quad (b - a)^{-1}b(b - a).
\end{equation}

This leads us to the following natural question. We call a set of pseudo-roots \( Y = \{y_1, y_2, \ldots, y_n\} \) of \( P(t) \) is a defining set if

\[
P(t) = (t - y_n)(t - y_{n-1}) \cdots (t - y_1).
\]

Our question is then: given a set of pseudo-roots \( Z \), when it is possible to construct a defining set of pseudo-roots from elements of \( Z \) by a successive application of operations of (5.1)-type?

Our answer to this question is based on a geometrical “diamond” interpretation of operations of (5.1)-type.

5.2. Sufficient sets of pseudo-roots and the algebra \( Q_n \).

To avoid taking inverses, we will slightly change the definition of operations of (5.1)-type. Recall that algebra \( Q_n \) corresponds to the Hasse graph \( \Gamma_n \) of the Boolean lattice of \( \{1, 2, \ldots, \} \) and \( Q_n[t] \) contains a unique universal polynomial, denoted \( P(t) \) (as defined in Section 2.4.)
Definition 5.2.1. We say that a pseudo-root $\xi \in Q_n$ is obtained from an ordered pair of pseudo-roots $x_{A,i}, x_{B,j}$ by the $u$-operation if $A \cup \{i\} = B \cup \{j\}$ and $(x_{A,i} - x_{B,j})x_{A,i} = \xi(x_{A,i} - x_{B,j})$.

A pseudo-root $\eta \in Q_n$ is obtained from an ordered pair of pseudo-roots $x_{A,i}, x_{B,j}$ by the $d$-operation if $A \cup \{i\} = B \cup \{j\}$ have and $(x_{A,i} - x_{B,j})\eta = x_{A,i}(x_{A,i} - x_{B,j})$.

Proposition 5.2.2. The element $x_{A \cup \{i\}, j}$ is obtained by the $u$-operation from the pair $x_{A,j}, x_{A,i}$.

The element $x_{A,i}$ is obtained by the $d$-operation from the pair $x_{A \cup \{j\}, i}, x_{A \cup \{i\}, j}$.

Definition 5.2.3. The set of elements in $Q_n$ that can be obtained from elements of $Z$ by a successive applications of $d$- and $u$-operations is called the $du$-envelope of $Z$.

Definition 5.2.4. A set $Z \subseteq Q_n$ is called sufficient if the $du$-envelope of $Z$ contains a defining set of pseudo-roots of $P(t)$.

Any defining set of elements is a sufficient set. Other examples of sufficient sets in $Q_n$ are given by the following statement.

Proposition 5.2.5. The sets $\{x_{0,k} \mid 1 \leq k \leq n\}$ and $\{x_{\{12\ldots k\ldots n\}, k} \mid 1 \leq k \leq n\}$ are sufficient in $Q_n$ for $P(t)$.

A necessary condition for a subset in $Q_n$ to be sufficient for $P(t)$ is given by the following theorem.

Theorem 5.2.6. If $Z = \{x_{A,i_1}, x_{A_2,i_2}, \ldots, x_{A_n,i_n}\}$ is a sufficient subset of $Q_n$ then $i_1, i_2, \ldots, i_n$ are distinct.

A set of edges in a directed graph is connected if it is connected in the associated non-directed graph (see Section 5.3 below for details).

Theorem 5.2.7. Let $Z = \{x_{A_1,i_1}, x_{A_2,i_2}, \ldots, x_{A_n,i_n}\}$ be a subset of $Q_n$ such that $i_1, i_2, \ldots, i_n$ are distinct. If the set of edges $\{(A_1,i_1),(A_2,i_2),\ldots,(A_n,i_n)\}$ in $\Gamma_n$ is connected then the set $Z$ is sufficient.

Let $f : Q_n \to D$ be a homomorphism of $Q_n$ into a division ring $D$, $\hat{f} : Q_n[t] \to D[t]$ be the induced homomorphism of the polynomial rings and $P(t) = f(P(t))$.

Corollary 5.2.8. Let $Z = \{x_{A_1,i_1}, x_{A_2,i_2}, \ldots, x_{A_n,i_n}\}$ be a subset of $Q_n$ such that $i_1, i_2, \ldots, i_n$ are distinct and the set of edges $\{(A_1,i_1),(A_2,i_2),\ldots,(A_n,i_n)\}$ in $\Gamma_n$ is connected. Then all coefficients of $P(t) \in D[t]$ can be obtained from elements $f(z)$, $z \in Z$, by operations of addition, subtractions, multiplication, and left and right conjugation.

Examples. 1. The set $X = \{x_{0,1}, x_{0,2}, \ldots, x_{0,n}\}$ is connected (the corresponding edges have a common head $\emptyset$). Therefore, $X$ is a sufficient set.

2. For $n = 2$ the sufficient sets are $\{x_{1,j}, x_{0,i}\}$, $\{x_{0,j}, x_{0,i}\}$, $\{x_{1,j}, x_{1,i}\}$. The sets $\{x_{1,j}, x_{0,j}\}$ are not sufficient. Here $i, j = 1, 2$, $i \neq j$. 18
3. Let \( n = 3 \). The set \( \{ x_{(1),2}, x_{(2),1}, x_{(1),3} \} \) is sufficient because
\[
(x_{(2),1} - x_{(1),2})x_{(1),1} = x_{(1),2}(x_{(2),1} - x_{(1),2}),
\]
\[
x_{(1),3}(x_{(1),3} - x_{(1),2}) = (x_{(1),3} - x_{(1),2})x_{(1),3}
\]
and \( \{ x_{(12),3}, x_{(1),2}, x_{(1),3}, x_{(1),2}, x_{(1),3} \} \) is a defining set of pseudo-roots.

The sets \( \{ x_{(1),2}, x_{(2),1}, x_{(1),3} \} \) and \( \{ x_{(1),3}, x_{(1),2}, x_{(1),3} \} \) also are sufficient but not defining sets in \( Q_3 \).

4. The set \( W = \{ x_{(12),3}, x_{(3),2}, x_{(1),3} \} \) is not sufficient because \( d \)- and \( u \)-operations are not defined on elements of \( W \).

Theorem 5.2.7 follows from a more general theorem for algebras associated with directed graphs.

5.3. Sufficient sets of edges for directed graphs.

In this section we will define and study sufficient sets of edges in directed graphs \( \Gamma = (V, E) \). These sets will provide us with a construction of sufficient sets of pseudo-roots of polynomials \( P(t) \) over algebras \( A(\Gamma) \). All graphs considered in this section are simple (i.e., if \( t(e) = t(f) \) and \( h(e) = h(f) \) then \( e = f \)) and acyclic (i.e., there are no directed paths \( P \) such that \( t(P) = h(P) \)).

Let \( \Gamma = (V, E) \) be a directed graph.

**Definition 5.3.1.**

(1) A pair of edges \( f_1, f_2 \) with a common head is obtained from the pair \( e_1, e_2 \) with a common tail by \( D \)-operation if \( h(e_i) = t(f_i) \) for \( i = 1, 2 \);

(2) A pair of edges \( e_1', e_2' \) with a common tail is obtained from the pair \( f_1', f_2' \) with a common head by \( U \)-operation if \( h(e_i') = t(f_i') \) for \( i = 1, 2 \).

**Example.** In Fig 1 edges \( f_1, f_2 \) can be obtained from edges \( e_1, e_2 \) by a \( D \)-operation. Conversely, edges \( e_1, e_2 \) can be obtained from edges \( f_1, f_2 \) by a \( U \)-operation.

**Remark.** We do not require the uniqueness of \( D \)- and \( U \)-operations.

**Definition 5.3.2.** A subset \( E_0 \subseteq E \) is called \( DU \)-complete (or simply complete) if the results of any \( D \)-operation or any \( U \)-operation applied to edges from \( E_0 \) belong to \( E_0 \).

**Proposition 5.3.3.** For any subset \( F \subseteq E \) there exists a minimal \( DU \)-complete set \( \hat{F} \subseteq E \) containing \( F \).

We call \( \hat{F} \) the completion of \( F \).

Let \( \Gamma = (V, E) \) be a directed graph.

**Definition 5.3.4.** A set of edges \( G \) in \( \Gamma \) is called sufficient if its completion \( \hat{G} \) contains a path from a maximal vertex (source) to a minimal vertex (sink).
Definition 5.3.5. A set of vertices \( W \subseteq V \) is called ample if

1. For any non-minimal vertex \( v \in V \) there exists a vertex \( u \in W \) such that there is no directed path in \( \Gamma \) from \( u \) to \( v \);
2. For any non-maximal vertex \( v \in V \) there exists a vertex \( w \in W \) such that there is no directed path in \( \Gamma \) from \( v \) to \( w \).

A set of edges is called ample if the set of its tails and heads is ample.

As an example, consider the graph \( \Gamma_n \) of all subsets of \( \{1, \ldots, n\} \). It has one source \( \{1, \ldots, n\} \) and one sink \( \emptyset \).

Proposition 5.3.6. A set of edges \((A_1, i_1), (A_2, i_2), \ldots, (A_n, i_n)\) in \( \Gamma_n \) is ample if \( i_1, i_2, \ldots, i_n \) are distinct.

Definition 5.3.7. A directed graph is called a modular graph if:

1. For any two edges \( e_1, e_2 \) with a common tail there exist edges \( f_1, f_2 \) with a common head such that \( h(e_i) = t(f_i) \) for \( i = 1, 2 \);
2. For any two edges \( h_1, h_2 \) with a common head there exist edges \( g_1, g_2 \) with a common tail such that \( h(g_i) = t(h_i) \) for \( i = 1, 2 \).

We do not require the uniqueness of \( f_1, f_2 \) and \( h_1, h_2 \).

Theorem 5.3.8. Any ample connected set of edges of a finite modular directed graph is a sufficient set.

Now let \( \Gamma = (V, E) \) be a directed graph such that

1. \( \Gamma \) contains a unique source \( M \) and a unique sink \( m \);
2. For each vertex \( v \in V \) there exist a directed path from \( M \) to \( v \) and a directed path from \( v \) to \( M \).

Recall that we associate to \( \Gamma \) an algebra \( A(\Gamma) \) and the universal polynomial \( P(t) \in A(\Gamma)[t] \). The polynomial \( P(t) \) is constructed using a path \( e_1, e_2, \ldots, e_n \) from \( M \) to \( m \) in \( \Gamma \), but it does not depend on the path. To any edge \( e \in E \) there corresponds to a pseudo-root \( e \in A(\Gamma) \) of \( P(t) \), and to any path \( (e_1, e_2, \ldots, e_n) \) from \( M \) to \( m \) in \( \Gamma \) there corresponds the factorization

\[
P(t) = (t - e_1)(t - e_2)\cdots(t - e_n)
\]

of \( P(t) \) over \( A(\Gamma) \).

Theorem 5.3.8 implies

Theorem 5.3.9. Let \( S \subseteq E \) be an ample connected set of edges in a modular directed graph \( \Gamma = (V, E) \). Then there exists a factorization (3.1) of \( P(t) \) such that the DU-completion of \( S \) contains elements \( e_1, e_2, \ldots, e_n \) and, therefore, coefficients of \( P(t) \).
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