ADMISSIBLE CONTROLS AND CONTROLLABLE SETS FOR A LINEAR TIME-VARYING ORDINARY DIFFERENTIAL EQUATION

LIJUAN WANG
School of Mathematics and Statistics, Wuhan University
Wuhan, MO 430072, China

YASHAN XU*
School of Mathematical Sciences, Fudan University, KLMNS
Shanghai, MO 200433, China

Abstract. In this paper, for a time optimal control problem governed by a linear time-varying ordinary differential equation, we give a description to check whether the set of admissible controls is nonempty or not by finite times.

1. Introduction. The existence of admissible controls is an important subject in the studies of time optimal control problems. This subject seems to be independent of other research issues, such as the Pontryagin Maximum Principle, the bang-bang property, and so on. However, it is a prerequisite to study time optimal control problems. A control is called an admissible one if it takes value in a set of constrained controls and the corresponding solution of the controlled equation reaches a given target at some time from a starting set at the initial time.

Generally, people always connect the existence of admissible controls with controllability. In the study of the existence of admissible controls, there are two misunderstandings. The first misunderstanding is: controllability should imply the existence of admissible controls. Here is a counterexample. Consider the controlled equation:

\[ \dot{y}(t) = y(t) + u(t), \quad t > 0. \]  

It is obvious that this equation is controllable. We introduce a time optimal control problem:

\[ \inf\{T > 0 : y(T; 0, 2, u) = 0 \text{ and } u(\cdot) \in \mathcal{V}\}, \]

where

\[ \mathcal{V} \triangleq \{ u : (0, +\infty) \to \mathbb{R} \text{ is measurable | } |u(t)| \leq 1 \text{ for a.e. } t > 0\}, \]

and \( y(\cdot; 0, 2, u) \) denotes the solution of (1) with the control \( u(\cdot) \in \mathcal{V} \) and the initial condition \( y(0) = 2 \). We can check that there is no admissible control for the problem (2). The second misunderstanding is: if a controlled system is uncontrollable, then

\[ 2010 \text{ Mathematics Subject Classification. Primary: 49J15, 93C15; Secondary: 93B05.} \]

\[ \text{Key words and phrases.} \] Ordinary differential equation, time optimal control, admissible control.

The first author is supported by the National Natural Science Foundation under grants 11771344 and 11371285; the second author is supported by the National Natural Science Foundation under grants 11471080 and 11631004.

* Corresponding author: Yashan Xu.
its corresponding time optimal control problem has no admissible control. Here is a counterexample. Consider the controlled system:
\[
\begin{align*}
\dot{y}_1(t) &= u(t), \\
\dot{y}_2(t) &= 0,
\end{align*}
\] (3)
It is clear that this system is uncontrollable. We introduce a time optimal control problem:
\[
\inf\{T > 0 : y_1(T; 0, (1, 0)^T, u) = y_2(T; 0, (1, 0)^T, u) = 0 \text{ and } u(\cdot) \in \mathcal{V}\},
\] (4)
where \((y_1(\cdot; 0, (1, 0)^T, u), y_2(\cdot; 0, (1, 0)^T, u))^T\) denotes the solution of (3) with the control \(u(\cdot) \in \mathcal{V}\) and the initial condition \((y_1(0), y_2(0))^T = (1, 0)^T\). We can check that the control defined by:
\[
u(t) \triangleq \begin{cases}
-1, & t \in (0, 1), \\
0, & t \geq 1,
\end{cases}
\]
is an admissible control for the problem (4).

Generally, differences between an admissible control problem and a controllability problem are as follows. First, an admissible control problem has some constraints on the control set, the starting set and the target set, while a controllability problem has no such constraint. Second, controllability requires that for any two elements in state space, there is a control which drives the state of the system from one element to another element, while for the existence of admissible controls, there are two elements (which are in the starting set and the target set, respectively) in state space, there is a control which drives the state of the system from the element (in the starting set) to another element (in the target set). From these viewpoints, the existence of admissible controls is totally different from the controllability.

Though the existence of admissible controls is a base for the study of time optimal control problems, there is little work on this topic (see [1], [2], [3], [5], [8]-[11], [14] and [16]). In general, there are two angles to study the existence of admissible controls governed by differential equations. The first is: under the assumption that a set of constrained controls, a starting set and a target set are given, what conditions should a differential equation satisfy in order to guarantee the existence of admissible controls? (see [4] and [13].) The second is: under the assumption that the controlled system, the set of constrained controls and the starting set (or the target set, respectively) are given, what conditions should the target set (or the starting set, respectively) satisfy in order to guarantee the existence of admissible controls? For the second case, it turns out to be an attainability problem (or a constrained controllability problem). There are some existing works about it (see [15], [6] and [12]). In the study of the second case, a very interesting question arises: for a given linear differential equation with the null target set and a fixed set of constrained controls (closed ball with radius \(r\)), can we give a criterion for initial data, by which we can judge whether admissible control exists or not? For this question, there have been some sufficient and necessary conditions for the time-invariant ordinary differential equations (see [4] and [13]). When the ending time is fixed, some results are derived for the controlled heat equations (see [15], [6] and [12]). However, to the best of our knowledge, for the time-varying controlled ordinary differential equations and free ending time, there is no result. In this paper, we will study this problem.

In this paper, we consider the following controlled system:
\[
\dot{y}(t) = A(t)y(t) + B(t)u(t), \quad t \in \mathbb{R}_+,
\] (5)
The time optimal control problem studied in this paper is as:
The set of constrained control is:

\[ U \triangleq \{ u : \mathbb{R}_+ \to \mathbb{R}^m \text{ is measurable} \mid u(t) \in B_{r}(0) \text{ for a.e. } t \in \mathbb{R}_+ \}. \]

The time optimal control problem studied in this paper is as:

\[ (P_{y_0}) \quad \inf \{ T : y(T; 0, y_0, u) = 0 \text{ and } u \in U \}. \]

Here, \( y_0 \in \mathbb{R}^n \setminus \{0\} \) is arbitrarily fixed, and \( y(\cdot; 0, y_0, u) \) is the unique solution of (5) with the initial condition \( y(0) = y_0 \). We set

\[ U_{y_0} \triangleq \{ u \in U : y(T; 0, y_0, u) = 0 \text{ for some } T \in \mathbb{R}_+ \}. \]

It is called the admissible control set of \((P_{y_0})\); every element in \( U_{y_0} \) is called an admissible control of \((P_{y_0})\). We now give the definition about the solvability of the problem \((P_{y_0})\).

**Definition 1.1.** Let \( y_0 \in \mathbb{R}^n \setminus \{0\} \). The problem \((P_{y_0})\) is said to be solvable if we can check whether \((P_{y_0})\) has an admissible control or not by finite times.

The main result of this paper is as follows.

**Theorem 1.2.** For each \( y_0 \in \mathbb{R}^n \setminus \{0\} \), the problem \((P_{y_0})\) is solvable.

In order to prove Theorem 1.2, we need some notations. Let \( \hat{A}(\cdot) \in L_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^{n \times n}) \) and \( \hat{B}(\cdot) \in L_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^{n \times m}) \) be given. Denote by \( W_{\hat{A}, \hat{B}} \) and \( W^r_{\hat{A}, \hat{B}} \) controllable subspace without control constraint and controllable set with constrained controls for the controlled system:

\[ \hat{y}(t) = \hat{A}(t)\hat{y}(t) + \hat{B}(t)u(t), \quad t \in \mathbb{R}_+, \quad (6) \]

more precisely,

\[ W_{\hat{A}, \hat{B}} \triangleq \{ \hat{y}_0 \in \mathbb{R}^n \mid \exists T \in \mathbb{R}_+ \text{ and } u \in L^2(0, T; \mathbb{R}^m) \text{ so that } \hat{y}(T; 0, \hat{y}_0, u) = 0 \} \quad (7) \]

and

\[ W^r_{\hat{A}, \hat{B}} \triangleq \bigcup_{T \in \mathbb{R}_+} W^r_{\hat{A}, \hat{B}}(T), \quad (8) \]

where \( \hat{y}(\cdot; 0, \hat{y}_0, u) \) is the unique solution of (6) with the initial condition \( \hat{y}(0) = \hat{y}_0 \), and

\[ W^r_{\hat{A}, \hat{B}}(T) \triangleq \{ \hat{y}_0 \in \mathbb{R}^n \mid \exists u \in L^\infty(0, T; B_r(0)) \text{ so that } \hat{y}(T; 0, \hat{y}_0, u) = 0 \}. \quad (9) \]

Further, we set

\[ N_{\hat{A}, \hat{B}} \triangleq \left\{ \psi \in \mathbb{R}^n \mid \hat{B}(\cdot)^*\hat{\Phi}^{-1}(\cdot, 0)^*\psi = 0 \text{ in } \mathbb{R}_+ \right\} \quad (10) \]

and

\[ D_{\hat{A}, \hat{B}} \triangleq N^\perp_{\hat{A}, \hat{B}} \bigcap \partial B_1(0). \quad (11) \]

Here, \( \hat{\Phi}(\cdot, \cdot) \) is the evolution operator generated by \( \hat{A}(\cdot) \). From (8) and (9), one can directly verify the following proposition.

**Proposition 1.** For arbitrarily fixed \( y_0 \in \mathbb{R}^n \setminus \{0\} \), the problem \((P_{y_0})\) has an admissible control if and only if \( y_0 \in W^r_{\hat{A}, \hat{B}}. \)
Lemma 2.1. Let $2.

Preliminaries.

and preliminaries in Section 2, we give the proof of Theorem 1.2. In order to overcome this difficulty, we introduce candidate controllable set $CW_{\hat{A},\hat{B}}$ (see (18)), which plays an important role in the proof of Theorem 1.2.

This paper is organized as follows. In Section 2, we present some preliminaries, which will be used in the proof of Theorem 1.2. In Section 3, based on Proposition 1 and preliminaries in Section 2, we give the proof of Theorem 1.2.

2. Preliminaries. We start this section with the following result.

Lemma 2.1. Let $N_{\hat{A},\hat{B}}$ and $W_{\hat{A},\hat{B}}$ be defined by (10) and (7), respectively. Then

$$W_{\hat{A},\hat{B}} = N_{\hat{A},\hat{B}}^1.$$ 

Proof. We first show that $W_{\hat{A},\hat{B}} \subseteq N_{\hat{A},\hat{B}}^1$. To this end, we let $\hat{y}_0 \in W_{\hat{A},\hat{B}}$. According to (7), there exists a constant $T \in \mathbb{R}_+$ and $u \in L^2(0,T;\mathbb{R}^n)$ so that $y(T;0,\hat{y}_0,u) = 0$, which indicates that

$$\hat{y}_0 = -\int_0^T \hat{\Phi}^{-1}(s,0)\hat{B}(s)u(s)ds.$$ 

From the above and (10), it follows that

$$\langle \psi, \hat{y}_0 \rangle_{\mathbb{R}^n} = -\int_0^T \langle \hat{B}(s)^\dagger \hat{\Phi}^{-1}(s,0)^\ast \psi, u(s)\rangle_{\mathbb{R}^m}ds = 0, \quad \forall \psi \in N_{\hat{A},\hat{B}}.$$ 

This implies that $\hat{y}_0 \in N_{\hat{A},\hat{B}}^1$. Hence, $W_{\hat{A},\hat{B}} \subseteq N_{\hat{A},\hat{B}}^1$.

We next show that $N_{\hat{A},\hat{B}}^1 \subseteq W_{\hat{A},\hat{B}}$. By contradiction, there would exist a $z \in \mathbb{R}^n$ so that $z \in N_{\hat{A},\hat{B}}^1 \setminus W_{\hat{A},\hat{B}}$. By the convexity and closedness of $W_{\hat{A},\hat{B}}$ (in $\mathbb{R}^n$), we use the separation theorem of convex sets to find a vector $\varphi \in \mathbb{R}^n$ so that

$$\langle \varphi, z \rangle_{\mathbb{R}^n} > \langle \varphi, w \rangle_{\mathbb{R}^n}, \quad \forall \ w \in W_{\hat{A},\hat{B}}. \quad (12)$$ 

Since $0 \in W_{\hat{A},\hat{B}}$, it follows from (12) that

$$\langle \varphi, z \rangle_{\mathbb{R}^n} > 0.$$ 

Noting that $z \in N_{\hat{A},\hat{B}}^1$ by the above inequality, we see that $\varphi \notin N_{\hat{A},\hat{B}}$, i.e.,

$$\hat{B}(\cdot)^\dagger \hat{\Phi}^{-1}(\cdot,0)^\ast \varphi \neq 0.$$ 

Hence, there is a constant $\hat{T} \in \mathbb{R}_+$ so that

$$\chi_{(0,\hat{T})}(\cdot)\hat{B}(\cdot)^\dagger \hat{\Phi}^{-1}(\cdot,0)^\ast \varphi \neq 0. \quad (13)$$

For any $k > 0$, we set

$$u_k(\cdot) \triangleq k\chi_{(0,\hat{T})}(\cdot)\hat{B}(\cdot)^\dagger \hat{\Phi}^{-1}(\cdot,0)^\ast \varphi \quad \text{and} \quad w_k \triangleq -\int_0^\hat{T} \hat{\Phi}^{-1}(s,0)\hat{B}(s)u_k(s)ds.$$ 

We can easily check that

$$y(\hat{T};0,-w_k,-u_k) = 0.$$ 

This, together with (7), implies that $-w_k \in W_{\hat{A},\hat{B}}$. Thus by (12), we have that

$$\langle \varphi, z \rangle_{\mathbb{R}^n} > \langle \varphi, -w_k \rangle_{\mathbb{R}^n} = k \int_0^\hat{T} \|\hat{B}(s)^\dagger \hat{\Phi}^{-1}(s,0)^\ast \varphi\|^2_{\mathbb{R}^m}ds. \quad (14)$$
It follows from (13) and (14) that
\[ \langle \varphi, z \rangle_{\mathbb{R}^n} = +\infty, \]
which leads to a contradiction.

In summary, we finish the proof of Lemma 2.1. \qed

Now, for any \( \psi \in \mathbb{R}^n \), we denote
\[ C_{\hat{A}, \hat{B}, \psi} \triangleq \sup_{t \in [0, +\infty)} \int_0^t \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \psi \|_{\mathbb{R}^m} ds, \]  
(15)
a closed set
\[ (CHS)_{\hat{A}, \hat{B}, \psi}^r \triangleq \{ \hat{z} \in \mathbb{R}^n \mid \langle \hat{z}, \psi \rangle_{\mathbb{R}^n} \leq r C_{\hat{A}, \hat{B}, \psi} \} \]  
(16)
and an open set
\[ (OHS)_{\hat{A}, \hat{B}, \psi}^r \triangleq \{ \hat{z} \in \mathbb{R}^n \mid \langle \hat{z}, \psi \rangle_{\mathbb{R}^n} < r C_{\hat{A}, \hat{B}, \psi} \}. \]  
(17)
Furthermore, we define a candidate controllable set
\[ CW_{\hat{A}, \hat{B}}^r \triangleq \left( \bigcap_{\psi \in D_{\hat{A}, \hat{B}}} (HS)_{\hat{A}, \hat{B}, \psi}^r \right) \bigcap W_{\hat{A}, \hat{B}}. \]  
(18)
Here, when the supreme in (15) is achieved,
\[ (HS)_{\hat{A}, \hat{B}, \psi}^r \triangleq (CHS)_{\hat{A}, \hat{B}, \psi}^r; \]  
(19)
otherwise,
\[ (HS)_{\hat{A}, \hat{B}, \psi}^r \triangleq (OHS)_{\hat{A}, \hat{B}, \psi}^r. \]  
(20)
Then we have the following result.

Lemma 2.2. \( W_{\hat{A}, \hat{B}}^r \subseteq CW_{\hat{A}, \hat{B}}^r. \)

Proof. We arbitrarily fix \( \hat{y}_0 \in W_{\hat{A}, \hat{B}}^r. \) According to (8) and (9), there exists a constant \( T \in \mathbb{R}_+ \) and \( u \in L^\infty(0, T; B_r(0)) \) so that
\[ \hat{y}_0 = -\int_0^T \hat{\Phi}^{-1}(s, 0) \hat{B}(s) u(s) ds. \]
From the latter it follows that for any \( \psi \in \mathbb{R}^n, \)
\[ \langle \psi, \hat{y}_0 \rangle_{\mathbb{R}^n} = -\int_0^T \langle \psi, \hat{\Phi}^{-1}(s, 0) \hat{B}(s) u(s) \rangle_{\mathbb{R}^n} ds \leq r \int_0^T \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \psi \|_{\mathbb{R}^m} ds. \]
This, along with (15)-(17), (19) and (20), implies that
\[ \hat{y}_0 \in (HS)_{\hat{A}, \hat{B}, \psi}^r, \quad \forall \psi \in D_{\hat{A}, \hat{B}}. \]  
(21)
Moreover, by (7), (8) and (9), we have that
\[ W_{\hat{A}, \hat{B}}^r \subseteq W_{\hat{A}, \hat{B}}^r, \]
which, combined with (21) and (18), indicates that
\[ \hat{y}_0 \in CW_{\hat{A}, \hat{B}}^r. \]
Hence, the result follows at once.

This completes the proof. \qed
Now two questions arise: are these two sets the same? If not, what conditions can we need to ensure that they are the same? For the first question, we consider the following example, which shows that generally, these two sets are not the same.

**Example 2.3.** Consider the controlled system:

\[
\frac{d}{dt} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \chi_{[1, +\infty)}(t) \begin{pmatrix} 0 & 0 \\ 0 & 1/t^2 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad t \in \mathbb{R}_+.
\]

Let \( r = 1 \). We will show that \((1, 1)^T \notin CW_{\tilde{A}, \tilde{B}} \setminus W_{\tilde{A}, \tilde{B}}^1\). Its proof will be carried out by the following three steps.

**Step 1.** We claim that

\[
W_{\tilde{A}, \tilde{B}}^1 = \left\{ \begin{pmatrix} y_1 \\ y_2 + z \end{pmatrix} \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1, \ |z| < 1 \right\} \text{ and } (1, 1)^T \notin W_{\tilde{A}, \tilde{B}}^1. \tag{22}
\]

Indeed, on one hand, by (8) and (9), we see that

\[
W_{\tilde{A}, \tilde{B}}^1 = \left( \bigcup_{t \in (0, 1]} S_1(t) \right) \bigcup \left( \bigcup_{t > 1} (S_1(1) + S_2(t)) \right), \tag{23}
\]

where

\[
S_1(t) \triangleq \left\{ \int_0^t \begin{pmatrix} u(s) \\ v(s) \end{pmatrix} ds \mid u(\cdot), v(\cdot) \in L^\infty(0, t; \mathbb{R}) \text{ and } u^2(s) + v^2(s) \leq 1 \text{ for a.e. } s \in (0, t) \right\}, \quad t \in (0, 1], \tag{24}
\]

and

\[
S_2(t) \triangleq \left\{ \int_1^t \begin{pmatrix} 0 \\ u(s)/s^2 \end{pmatrix} ds \mid u(\cdot) \in L^\infty(1, t; \mathbb{R}) \text{ and } |u(s)| \leq 1 \text{ for a.e. } s \in (1, t) \right\}, \quad t > 1. \tag{25}
\]

On the other hand, by (24) and (25), we can directly check that

\[
S_1(t_1) \subseteq S_1(t_2) \text{ when } 0 < t_1 \leq t_2 \leq 1, \quad S_1(1) = B_1(0),
\]

\[
S_2(t_1) \subseteq S_2(t_2) \text{ when } 1 < t_1 \leq t_2, \quad \text{ and } \bigcup_{t > 1} S_2(t) = \left\{ \begin{pmatrix} 0 \\ z \end{pmatrix} \mid |z| < 1 \right\}.
\]

These, along with (23), imply (22).

**Step 2.** We prove that \((1, 1)^T \in CW_{\tilde{A}, \tilde{B}}^1\).

For this purpose, on one hand, by (10) and Lemma 2.1, we have that

\[
N_{\tilde{A}, \tilde{B}} = \{0\} \text{ and } W_{\tilde{A}, \tilde{B}} = \mathbb{R}^2. \tag{26}
\]

It follows from the first equality in (26) and (11) that

\[
D_{\tilde{A}, \tilde{B}} = \partial B_1(0). \tag{27}
\]

On the other hand, by (27) and (15), for each \( \psi = (\psi_1, \psi_2)^T \in D_{\tilde{A}, \tilde{B}} \), we get that

\[
C_{\tilde{A}, \tilde{B}, \psi} = 1 + |\psi_2|
\]

and

\[
C_{\tilde{A}, \tilde{B}, \psi} \text{ can be achieved when } \psi = (1, 0)^T \text{ and } \psi = (-1, 0)^T. \tag{28}
\]
The above two conclusions, together with (19), (20), (16) and (17), yield that
\[(HS)_{\hat{A},\hat{B},(1,0)} = \{(z_1, z_2)^\top \in \mathbb{R}^2 \mid z_1 \leq 1\}, \tag{29}\]
\[(HS)_{\hat{A},\hat{B},(-1,0)} = \{(z_1, z_2)^\top \in \mathbb{R}^2 \mid z_1 \geq -1\}, \tag{30}\]
and for each \(\psi = (\psi_1, \psi_2)^\top \in D_{\hat{A},\hat{B}} \setminus \{(1,0)^\top, (-1,0)^\top\},\)
\[(HS)_{\hat{A},\hat{B},\psi} = \{(z_1, z_2)^\top \in \mathbb{R}^2 \mid z_1 \psi_1 + z_2 \psi_2 < 1 + |\psi_2|\}. \tag{31}\]
According to (18), (29)-(31) and the second conclusion in (26), \((1, 1)^\top \in CW_{\hat{A},\hat{B}}^1.\)

**Step 3.** End of the proof.

The result follows from Steps 1 and 2 immediately.

In order to answer the second question, we introduce the following notations.

Let
\[V_{\hat{A},\hat{B}} \equiv \{\psi \in N_{\hat{A},\hat{B}} \mid \text{the supreme in (15) can be achieved}\}\tag{32}\]
and
\[\hat{D}_{\hat{A},\hat{B}} \equiv V_{\hat{A},\hat{B}} \bigcap \partial B_1(0).\tag{33}\]
It is obvious that by (33), (32) and (11), we obtain that
\[\hat{D}_{\hat{A},\hat{B}} \subseteq D_{\hat{A},\hat{B}}; \tag{34}\]
Moreover, we need the following Lemma.

**Lemma 2.4.** ([7]) Let \(Z\) be a closed and convex set in \(\mathbb{R}^n\). Then
\[Z = \bigcap_{\psi \in \partial B_1(0)} (CHS)_{Z,\psi}.\]
Here, when \(\sup_{z \in Z} \langle z, \psi \rangle_{\mathbb{R}^n} = +\infty, (CHS)_{Z,\psi} \equiv \mathbb{R}^n; \) When \(\sup_{z \in Z} \langle z, \psi \rangle_{\mathbb{R}^n} < +\infty, \)
\[(CHS)_{Z,\psi} \equiv \{y \in \mathbb{R}^n \mid \langle y, \psi \rangle_{\mathbb{R}^n} \leq \sup_{z \in Z} \langle z, \psi \rangle_{\mathbb{R}^n}\}.\]

Based on Lemma 2.1 and Lemma 2.4, we have the following result.

**Theorem 2.5.** Let \(z \in CW_{\hat{A},\hat{B}}^r.\) If
\[z \notin \bigcup_{\psi \in \hat{D}_{\hat{A},\hat{B}}} \partial (HS)_{\hat{A},\hat{B},\psi}; \tag{35}\]
then there is a constant \(T \in \mathbb{R}_+\) so that \(z \in W_{\hat{A},\hat{B}}^r(T).\)

**Proof.** Since \(z \in CW_{\hat{A},\hat{B}}^r,\) by (18)-(20), (16), (17), (33) and (32), we get that
\[\langle z, \psi \rangle_{\mathbb{R}^n} \leq rC_{\hat{A},\hat{B},\psi}, \forall \psi \in \hat{D}_{\hat{A},\hat{B}}, \tag{36}\]
and
\[\langle z, \psi \rangle_{\mathbb{R}^n} < rC_{\hat{A},\hat{B},\psi}, \forall \psi \in D_{\hat{A},\hat{B}} \setminus \hat{D}_{\hat{A},\hat{B}}. \tag{37}\]
We claim that there is a constant \(T \in \mathbb{R}_+\) so that
\[\langle z, \psi \rangle_{\mathbb{R}^n} \leq r \int_0^T \|\hat{B}(s)^*\hat{\Phi}^{-1}(s,0)^*\psi\|_{\mathbb{R}^n} ds, \forall \psi \in D_{\hat{A},\hat{B}}. \tag{38}\]
By contradiction, there would be a sequence of \( \{ \psi_k \}_{k \geq 1} \subseteq D_{\tilde{A}, \tilde{B}} \) so that
\[
\langle z, \psi_k \rangle_{\mathbb{R}^n} > r \int_0^k \| \tilde{B}(s)^* \tilde{\Phi}^{-1}(s, 0)^* \psi_k \|_{\mathbb{R}^m} ds.
\] (39)
Moreover, by (11), we have that \( \{ \psi_k \}_{k \geq 1} \subseteq N_{\tilde{A}, \tilde{B}} \cap \partial B_1(0) \). Hence, there exists a subsequence of \( \{ \psi_k \}_{k \geq 1} \), still denoted by itself, and \( \psi^* \in D_{\tilde{A}, \tilde{B}} \) so that
\[
\lim_{k \to +\infty} \psi_k = \psi^*.
\]
Thus, for any \( j \geq 1 \), we obtain from (39) that
\[
\langle z, \psi^* \rangle_{\mathbb{R}^n} = \lim_{k \to +\infty} \langle z, \psi_k \rangle_{\mathbb{R}^n} \geq r \lim_{k \to +\infty} \int_0^j \| \tilde{B}(s)^* \tilde{\Phi}^{-1}(s, 0)^* \psi_k \|_{\mathbb{R}^m} ds
\]
which, combined with (15), indicates that
\[
\langle z, \psi^* \rangle_{\mathbb{R}^n} \geq rC_{\tilde{A}, \tilde{B}, \psi^*}.
\] (40)
Moreover, since \( \psi^* \in D_{\tilde{A}, \tilde{B}} \), by (36), (37) and (40), we have that
\[
\psi^* \in \tilde{D}_{\tilde{A}, \tilde{B}} \quad \text{and} \quad \langle z, \psi^* \rangle_{\mathbb{R}^n} = rC_{\tilde{A}, \tilde{B}, \psi^*}.
\] (41)
Noting that \( z \in CW_{\tilde{A}, \tilde{B}}^T \) and \( \psi^* \in D_{\tilde{A}, \tilde{B}} \), by (18), (41), (34), (33), (32), (19) and (16), we obtain that
\[
z \in \partial (HS)^T_{\tilde{A}, \tilde{B}, \psi^*},
\]
which contradicts (35). Hence, (38) follows.

Now, for any \( \phi \in \partial B_1(0) \), there are two functions \( \phi_1 \in N_{\tilde{A}, \tilde{B}} \) and \( \phi_2 \in N_{\tilde{A}, \tilde{B}}^+ \) so that
\[
\phi = \phi_1 + \phi_2.
\] (42)
Since \( z \in CW_{\tilde{A}, \tilde{B}}^T \), it follows from (42), (18), Lemma 2.1, (11), (38) and (10) that
\[
\langle z, \phi \rangle_{\mathbb{R}^n} = \langle z, \phi_2 \rangle_{\mathbb{R}^n} \leq r \int_0^T \| \tilde{B}(s)^* \tilde{\Phi}^{-1}(s, 0)^* \phi_2 \|_{\mathbb{R}^m} ds
\]
which contradicts (35). Hence, (38) follows.

Moreover, by (9), we can easily check that
\[
\sup_{\tilde{g}_0 \in W_{\tilde{A}, \tilde{B}}^T(\mathbb{T})} \langle \tilde{g}_0, \psi \rangle_{\mathbb{R}^n} = r \int_0^T \| \tilde{B}(s)^* \tilde{\Phi}^{-1}(s, 0)^* \psi \|_{\mathbb{R}^m} ds, \quad \forall \psi \in \mathbb{R}^n,
\] (44)
and
\[
W_{\tilde{A}, \tilde{B}}^T(\mathbb{T}) \quad \text{is a convex and closed subset in} \quad \mathbb{R}^n.
\] (45)
Then, by Lemma 2.4, (45) and (44), we get that
\[
W_{\tilde{A}, \tilde{B}}^T(\mathbb{T}) = \left\{ \tilde{z} \in \mathbb{R}^n \mid \langle \tilde{z}, \psi \rangle_{\mathbb{R}^n} \leq r \int_0^T \| \tilde{B}(s)^* \tilde{\Phi}^{-1}(s, 0)^* \psi \|_{\mathbb{R}^m} ds, \quad \forall \psi \in \partial B_1(0) \right\}.
\] (46)
This, together with (43), implies that \( z \in W_{\tilde{A}, \tilde{B}}^T(\mathbb{T}) \).

In summary, we finish the proof of Theorem 2.5.
The difficulty lies in dealing with the following case:

$$z \in CW_{\hat{A}, \hat{B}}^R \text{ and } z \in \bigcup_{\psi \in D_{\hat{A}, \hat{B}}} \partial(HS)_{\hat{A}, \hat{B}, \psi}^R. \quad (47)$$

In order to deal with this case, we need some preliminary results.

First we present the following result.

**Lemma 2.6.** $V_{\hat{A}, \hat{B}}$ is a subspace of $W_{\hat{A}, \hat{B}}$ and

$$\{ \text{supp}(\hat{B}(\cdot)\hat{\Phi}^{-1}(\cdot, 0)^*\psi) \}_{\psi \in V_{\hat{A}, \hat{B}}}$$

is uniformly bounded.

**Proof.** On one hand, for any $\psi_1, \psi_2 \in V_{\hat{A}, \hat{B}}$, by (32) and (15), we have that

$$\psi_1, \psi_2 \in N_{\hat{A}, \hat{B}}, \quad (48)$$

and there are two constants $T_1, T_2 \in \mathbb{R}_+$ so that

$$\int_{T_j}^{T_j^*} \| \hat{B}(s)^*\hat{\Phi}^{-1}(s, 0)^*\psi \|_{\mathbb{R}_+} ds = \int_{T_j}^{+\infty} \| \hat{B}(s)^*\hat{\Phi}^{-1}(s, 0)^*\psi \|_{\mathbb{R}_+} ds, \quad j = 1, 2. \quad (49)$$

It is obvious that (49) implies that

$$\hat{B}(s)^*\hat{\Phi}^{-1}(s, 0)^*\psi_j = 0 \text{ for a.e. } s \in (\max\{T_1, T_2\}, +\infty), \quad j = 1, 2.$$ 

Then, for any $\lambda_1, \lambda_2 \in \mathbb{R}$, we get that

$$\hat{B}(s)^*\hat{\Phi}^{-1}(s, 0)^*(\lambda_1 \psi_1 + \lambda_2 \psi_2) = 0 \text{ for a.e. } s \in (\max\{T_1, T_2\}, +\infty).$$

From the above conclusion and (48) it follows that

$$\lambda_1 \psi_1 + \lambda_2 \psi_2 \in N_{\hat{A}, \hat{B}}^\perp \quad (50)$$

and

$$\int_0^{\max\{T_1, T_2\}} \| \hat{B}(s)^*\hat{\Phi}^{-1}(s, 0)^*(\lambda_1 \psi_1 + \lambda_2 \psi_2) \|_{\mathbb{R}_+} ds = \int_0^{+\infty} \| \hat{B}(s)^*\hat{\Phi}^{-1}(s, 0)^*(\lambda_1 \psi_1 + \lambda_2 \psi_2) \|_{\mathbb{R}_+} ds. \quad (51)$$

By (50), (51), (15) and (32), we deduce that

$$\lambda_1 \psi_1 + \lambda_2 \psi_2 \in V_{\hat{A}, \hat{B}}. \quad (52)$$

Moreover, by (32) and Lemma 2.1, we have that

$$V_{\hat{A}, \hat{B}} \subseteq N_{\hat{A}, \hat{B}}^\perp = W_{\hat{A}, \hat{B}}. \quad (53)$$

This, along with (52), yields that $V_{\hat{A}, \hat{B}}$ is a subspace of $W_{\hat{A}, \hat{B}}$.

On the other hand, select a basis of $V_{\hat{A}, \hat{B}}$, denoted by $\xi_1, \xi_2, \ldots, \xi_k$. By (32) and (15), we can take $\{T_j\}_{j=1}^k \subseteq \mathbb{R}_+$ so that

$$\hat{B}(s)^*\hat{\Phi}^{-1}(s, 0)^*\xi_j = 0 \text{ for a.e. } s \in (T_j, +\infty), \quad j = 1, 2, \ldots, k. \quad (54)$$

Set

$$\hat{T} \triangleq \max\{T_1, T_2, \ldots, T_k\}. \quad (54)$$

It follows from (53) and (54) that

$$\hat{B}(s)^*\hat{\Phi}^{-1}(s, 0)^*\psi = 0 \text{ for a.e. } s \in (\hat{T}, +\infty), \ \forall \psi \in V_{\hat{A}, \hat{B}},$$

which indicates the result.
Hence, we finish the proof of Lemma 2.6.

Based on Lemma 2.6 and (33), we introduce some notations. For any \( \psi \in \hat{D}_{\hat{A}, \hat{B}} \), we set
\[
E_{\hat{A}, \hat{B}, \psi}^c \triangleq \text{supp}(\hat{B}(\cdot)^*)^*\overline{\hat{\Phi}}^{-1}(-,0)^*\psi), \quad E_{\hat{A}, \hat{B}, \psi}^c \triangleq \mathbb{R}^+ \setminus E_{\hat{A}, \hat{B}, \psi},
\]
and
\[
B_{\hat{A}, \hat{B}, \psi}(\cdot) \triangleq \chi_{E_{\hat{A}, \hat{B}, \psi}^c} \hat{B}(\cdot).
\]

Here and throughout this paper, if \( E_{\hat{A}, \hat{B}, \psi} = \mathbb{R}^+ \), we denote \( \chi_{E_{\hat{A}, \hat{B}, \psi}}(\cdot) \triangleq 0 \). Moreover, we write
\[
V_{\hat{A}, \hat{B}, \psi} \triangleq \left\{ \hat{\psi} \in \mathbb{R}^n \mid B_{\hat{A}, \hat{B}, \psi}(\cdot)^*\overline{\hat{\Phi}}^{-1}(-,0)^*\hat{\psi} = 0 \text{ in } \mathbb{R}^+ \right\}
\]
and
\[
\theta_{\hat{A}, \hat{B}, \psi} \triangleq r \int_{E_{\hat{A}, \hat{B}, \psi}} \overline{\hat{\Phi}}^{-1}(s,0) \hat{B}(s) \frac{\hat{B}(s)^*\overline{\hat{\Phi}}^{-1}(s,0)^*\psi}{\|\hat{B}(s)^*\overline{\hat{\Phi}}^{-1}(s,0)^*\psi\|_{\mathbb{R}^n}} ds.
\]

**Remark 1.** We can easily check that
\[
W_{\hat{A}, \hat{B}, A, B, \psi} \subseteq W_{\hat{A}, \hat{B}}, \forall \psi \in \hat{D}_{\hat{A}, \hat{B}},
\]
and
\[
\dim W_{\hat{A}, \hat{B}, A, B, \psi} < \dim W_{\hat{A}, \hat{B}}, \forall \psi \in \hat{D}_{\hat{A}, \hat{B}}.
\]
Indeed, (59) follows immediately from Lemma 2.1, (10) and (56). In order to prove (60), we use a contradiction argument. By contradiction and (59), we would have that
\[
W_{\hat{A}, \hat{B}, \hat{A}, \hat{B}, \psi} = W_{\hat{A}, \hat{B}} \text{ for some } \hat{\psi} \in \hat{D}_{\hat{A}, \hat{B}}.
\]
This, together with Lemma 2.1, implies that
\[
N_{\hat{A}, \hat{B}, \psi} = N_{\hat{A}, \hat{B}}.
\]
On one hand, since \( \hat{\psi} \in \hat{D}_{\hat{A}, \hat{B}} \subseteq N_{\hat{A}, \hat{B}} \cap \partial B_1(0) \) (see (33) and (32)), by (61), we get that
\[
\hat{\psi} \in N_{\hat{A}, \hat{B}, \hat{A}, \hat{B}, \psi} \text{ and } \|\hat{\psi}\|_{\mathbb{R}^n} = 1.
\]
On the other hand, by (55) and (56), we have that
\[
B_{\hat{A}, \hat{B}, \psi}(\cdot)^*\overline{\hat{\Phi}}^{-1}(-,0)^*\hat{\psi} = \chi_{E_{\hat{A}, \hat{B}, \psi}} \hat{B}(\cdot)^*\overline{\hat{\Phi}}^{-1}(-,0)^*\hat{\psi} = 0 \text{ in } \mathbb{R}^+,
\]
which, combined with (10), indicates that
\[
\hat{\psi} \in N_{\hat{A}, \hat{B}, \hat{A}, \hat{B}, \psi}.
\]
By (62) and (63), we arrive at a contradiction and (60) follows.

Now we present the following result.

**Lemma 2.7.** Let \( z \in W_{\hat{A}, \hat{B}}(T) \) for some \( T \in \mathbb{R}^+ \). If there is \( \psi \in \hat{D}_{\hat{A}, \hat{B}} \) so that
\[
z \in \partial(\text{HS})_{\hat{A}, \hat{B}, \psi},
\]
then
\[
T \geq \text{esssup} \ E_{\hat{A}, \hat{B}, \psi}.
\]
Besides, for any \( u \in L^\infty(0,T;B_r(0)) \) satisfying \( y(T;0,z,u) = 0 \), it holds that
\[
u(s) = - \frac{r \hat{B}(s)^* \mathbf{\phi}^{-1}(s,0)^* \psi}{\| \hat{B}(s)^* \mathbf{\phi}^{-1}(s,0)^* \psi \|_{\mathbb{R}^m}} \text{ for a.e. } s \in E_{\mathbf{\hat{A}},\mathbf{\hat{B}},\psi}.
\]

Furthermore,
\[
z - \theta_{\mathbf{\hat{A}},\mathbf{\hat{B}},\psi} \in W^r_{\mathbf{\hat{A}},\mathbf{\hat{B}},\psi}(T).
\]

Proof. Since \( z \in W^r_{\mathbf{\hat{A}},\mathbf{\hat{B}}}(T) \), according to (9), there exists a control \( u \in L^\infty(0,T;B_r(0)) \) so that
\[
z = - \int_0^T \hat{\mathbf{\phi}}^{-1}(s,0) \hat{B}(s)u(s)ds.
\]
This implies that
\[
\langle z, \psi \rangle_{\mathbb{R}^n} = - \int_0^T \langle \hat{B}(s)^* \hat{\mathbf{\phi}}^{-1}(s,0)^* \psi, u(s) \rangle_{\mathbb{R}^m} ds.
\]

By (33), (32), (64), (19), (16) and (15), we get that
\[
\langle z, \psi \rangle_{\mathbb{R}^n} = r \int_0^{+\infty} \| \hat{B}(s)^* \hat{\mathbf{\phi}}^{-1}(s,0)^* \psi \|_{\mathbb{R}^m} ds.
\]

From (70) and (69) it follows that
\[
r \int_0^{+\infty} \| \hat{B}(s)^* \hat{\mathbf{\phi}}^{-1}(s,0)^* \psi \|_{\mathbb{R}^m} ds = - \int_0^T \langle \hat{B}(s)^* \hat{\mathbf{\phi}}^{-1}(s,0)^* \psi, u(s) \rangle_{\mathbb{R}^m} ds
\]
\[
\leq r \int_0^T \| \hat{B}(s)^* \hat{\mathbf{\phi}}^{-1}(s,0)^* \psi \|_{\mathbb{R}^m} ds.
\]

Hence, according to (71) and (55), (65) and (66) hold.

Finally, by (68), (58), (65), (66) and (56), we obtain that
\[
z - \theta_{\mathbf{\hat{A}},\mathbf{\hat{B}},\psi} = - \int_{(0,T)\setminus E_{\mathbf{\hat{A}},\mathbf{\hat{B}},\psi}} \hat{\mathbf{\phi}}^{-1}(s,0) \hat{B}(s)u(s)ds = - \int_0^T \hat{\mathbf{\phi}}^{-1}(s,0)B_{\mathbf{\hat{A}},\mathbf{\hat{B}},\psi}(s)u(s)ds,
\]
which, combined with (9), indicates (67).

In summary, we finish the proof of Lemma 2.7. \(\square\)

Remark 2. Assume that \( z \in CW^r_{\mathbf{\hat{A}},\mathbf{\hat{B}}}, \psi \in D_{\mathbf{\hat{A}},\mathbf{\hat{B}}} \) and (64) holds, i.e., (47) holds. Does the relation
\[
z - \theta_{\mathbf{\hat{A}},\mathbf{\hat{B}},\psi} \in CW^r_{\mathbf{\hat{A}},\mathbf{\hat{B}},\psi}
\]
hold like Lemma 2.7? The answer is false. Example 2.3 presents a counterexample.

For the controlled system in Example 2.3, by (33), (32), the first conclusion in (26), (27) and (28), we have that
\[
D_{\mathbf{\hat{A}},\mathbf{\hat{B}}} = \{(1,0)^\top,(-1,0)^\top\}.
\]
Choose \( z = (1,1)^\top \) and \( \psi = (0,1)^\top \). It follows from Step 2 in Example 2.3 and (29) that (47) holds. Besides, it follows from (55), (56) and (58) that
\[
E_{\mathbf{\hat{A}},\mathbf{\hat{B}},(1,0)^\top} = [0,1], \quad E_{\mathbf{\hat{A}},\mathbf{\hat{B}},(1,0)^\top} = (1,+-\infty),
\]
and
\[
\tilde{B}(t) \triangleq B_{\mathbf{\hat{A}},\mathbf{\hat{B}},(1,0)^\top}(t) = \chi_{(1,+-\infty)}(t) \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad \theta_{\mathbf{\hat{A}},\mathbf{\hat{B}},(1,0)^\top} = (1,0)^\top.
\]
Thus $z - \theta_{\hat{A},\hat{B},(1,0)^\top} = (0,1)^\top$. Consider the controlled system:

$$
\frac{d}{dt} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \chi_{(1,\infty)}(t) \begin{pmatrix} 0 & 0 \\ 0 & 1/t^2 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad t \in \mathbb{R}_+.
$$

By similar arguments as those in Example 2.3, we can easily check that

$$W_{\hat{A},\hat{B}} = \{0\} \times \mathbb{R}, \quad D_{\hat{A},\hat{B}} = \{(0,1)^\top, (0,-1)^\top\}
$$

(72)

and

$$(HS)^1_{\hat{A},\hat{B},(0,1)^\top} = \{(z_1, z_2)^\top \in \mathbb{R}^2 \mid z_2 < 1\}.
$$

(73)

From (18), (72) and (73) it follows that $(0,1)^\top \notin CW^1_{\hat{A},\hat{B}}$, i.e.,

$$z - \theta_{\hat{A},\hat{B},\psi} \notin CW^1_{\hat{A},\hat{B},\psi}.
$$

The following result is concerned with the case (47).

**Theorem 2.8.** Let $z \in CW^r_{\hat{A},\hat{B}}$ and there is $\psi \in \tilde{D}_{\hat{A},\hat{B}}$ so that

$$z \in \partial(HS)^r_{\hat{A},\hat{B},\psi}.
$$

(74)

Let $\theta_{\hat{A},\hat{B},\psi}$ and $B_{\hat{A},\hat{B},\psi} \cdot (\cdot)$ be defined as (58) and (56), respectively. Then

$$z - \theta_{\hat{A},\hat{B},\psi} \in \left( \bigcap_{\psi \in D_{\hat{A},\hat{B}}} (CHS)^r_{\hat{A},\hat{B},\psi} \right) \cap W_{\hat{A},\hat{B}},
$$

(75)

where

$$\tilde{B}(\cdot) \triangleq B_{\hat{A},\hat{B},\psi}(\cdot).
$$

(76)

**Proof.** On one hand, for any $\hat{\psi} \in D_{\hat{A},\hat{B}}$, by (34) and (11), we have that

$$\psi^\varepsilon \triangleq \psi + \varepsilon \hat{\psi} \in N_{\hat{A},\hat{B}} \setminus \{0\}, \quad \forall \varepsilon \in (0,1),
$$

which, combined with (11), indicates that

$$\psi^\varepsilon / \|\psi^\varepsilon\|_{\mathbb{R}^n} \in D_{\hat{A},\hat{B}}.
$$

(77)

Since $z \in CW^r_{\hat{A},\hat{B}}$, by (18) and (77), we get that

$$z \in (HS)^r_{\hat{A},\hat{B},\psi_{\mathbb{R}^n}}.
$$

This, together with (19), (20), (16), (17) and (15), implies that

$$\langle z, \psi^\varepsilon \rangle_{\mathbb{R}^n} \leq \int_{0}^{+\infty} \|\tilde{B}(s)^* \tilde{\Phi}^{-1}(s,0)^*\psi^\varepsilon\|_{\mathbb{R}^n} \, ds.
$$

(78)

It follows from (74), (33), (32), (19), (16) and (55) that

$$\langle z, \psi \rangle_{\mathbb{R}^n} = \int_{E_{\hat{A},\hat{B},\psi}} \|\tilde{B}(s)^* \tilde{\Phi}^{-1}(s,0)^*\psi\|_{\mathbb{R}^n} \, ds.
$$

(79)

By (78) and (79), we obtain that

$$\langle z, \hat{\psi} \rangle_{\mathbb{R}^n} = \lim_{\varepsilon \to 0^+} \frac{\langle z, \psi + \varepsilon \hat{\psi} \rangle_{\mathbb{R}^n} - \langle z, \psi \rangle_{\mathbb{R}^n}}{\varepsilon}
$$

\leq \lim_{\varepsilon \to 0^+} \left[ \int_{0}^{+\infty} \|\tilde{B}(s)^* \tilde{\Phi}^{-1}(s,0)^*(\psi + \varepsilon \hat{\psi})\|_{\mathbb{R}^n} \, ds
$$

$$- \int_{E_{\hat{A},\hat{B},\psi}} \|\tilde{B}(s)^* \tilde{\Phi}^{-1}(s,0)^*\psi\|_{\mathbb{R}^n} \, ds \right].$$
which indicates that
\[
\langle z, \varphi \rangle_{\mathbb{R}^n} \leq \lim_{\varepsilon \to 0^+} r \int_{E_{A,B,\varphi}} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^*(\psi + \varepsilon \hat{\psi})\|_{\mathbb{R}^m} ds + \frac{r}{\varepsilon} \int_{E_{A,B,\varphi}} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^*(\psi + \varepsilon \hat{\psi})\|_{\mathbb{R}^m} ds \quad (80)
\]

Moreover, by (55), we get that
\[
\frac{1}{\varepsilon} \int_{E_{A,B,\varphi}} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^*(\psi + \varepsilon \hat{\psi})\|_{\mathbb{R}^m} ds = \int_{E_{A,B,\varphi}} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \hat{\psi}\|_{\mathbb{R}^m} ds.
\]

This, along with (80) and (81), implies that
\[
\langle z, \varphi \rangle_{\mathbb{R}^n} \leq r \int_{E_{A,B,\varphi}} \frac{\langle \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \psi, \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \hat{\psi} \rangle_{\mathbb{R}^m}}{\| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \hat{\psi}\|_{\mathbb{R}^m}} ds + \frac{r}{\varepsilon} \int_{E_{A,B,\varphi}} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \hat{\psi}\|_{\mathbb{R}^m} ds, \quad \forall \, \psi \in D_{\tilde{A}, \tilde{B}}.
\]

From the latter and (58) it follows that
\[
\langle z - \theta_{\tilde{A}, \tilde{B}, \psi}, \hat{\psi} \rangle_{\mathbb{R}^n} \leq r \int_{E_{A,B,\varphi}} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \hat{\psi}\|_{\mathbb{R}^m} ds, \quad \forall \, \psi \in D_{\tilde{A}, \tilde{B}}. \quad (82)
\]

By (76), (56), (10) and (11), we observe that \(N_{\tilde{A}, \tilde{B}} \subseteq N_{\hat{A}, \hat{B}}\) and
\[
D_{\tilde{A}, \tilde{B}} \subseteq D_{\hat{A}, \hat{B}}. \quad (83)
\]

It follows from (16), (15), (76), (56), (82) and (83) that
\[
z - \theta_{\tilde{A}, \tilde{B}, \psi} \in (CHS)_{\tilde{A}, \tilde{B}}^r \hat{\psi}, \quad \forall \, \psi \in D_{\tilde{A}, \tilde{B}}. \quad (84)
\]

On the other hand, by (82), we infer that
\[
\langle z - \theta_{\tilde{A}, \tilde{B}, \psi}, \hat{\psi} \rangle_{\mathbb{R}^n} = r \int_{0}^{+\infty} \chi_{E_{A,B,\varphi}} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \hat{\psi}\|_{\mathbb{R}^m} ds, \quad \forall \, \hat{\psi} \in D_{\tilde{A}, \tilde{B}}.
\]

This, together with (11), implies that
\[
\langle z - \theta_{\tilde{A}, \tilde{B}, \psi}, \hat{\psi} \rangle_{\mathbb{R}^n} = r \int_{0}^{+\infty} \chi_{E_{A,B,\varphi}} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \hat{\psi}\|_{\mathbb{R}^m} ds, \quad \forall \, \hat{\psi} \in N_{\tilde{A}, \tilde{B}}. \quad (85)
\]

Now, for any \(\tilde{\psi} \in V_{\tilde{A}, \tilde{B}, \psi}\) (see (57)), we write
\[
\tilde{\psi} = \bar{\psi}_1 + \bar{\psi}_2 \text{ with } \bar{\psi}_1 \in N_{\tilde{A}, \tilde{B}}, \bar{\psi}_2 \in N_{\tilde{A}, \tilde{B}}^\perp. \quad (86)
\]
Then, it follows from (85), (86), (56), (57) and (10) that
\[ \langle z - \theta_{\hat{A},\hat{B},\psi}, \tilde{\psi}_2 \rangle_{\mathbb{R}^n} = r \int_0^{+\infty} \chi_{W_\hat{A},\hat{B},\psi} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* (\tilde{\psi} - \tilde{\psi}_1) \|_{\mathbb{R}^m} ds = 0. \] (87)
Moreover, by (58), (55), Lemma 2.6 and (7), we have that
\[ \theta_{\hat{A},\hat{B},\psi} \in W_{\hat{A},\hat{B}}. \] (88)
Noting that \( z \in CW_{\hat{A},\hat{B}}^r \subseteq W_{\hat{A},\hat{B}} \) (see (18)) and \( \tilde{\psi}_1 \in \tilde{N}_{\hat{A},\hat{B}} \), by Lemma 2.1 and (88), we get that
\[ \langle z - \theta_{\hat{A},\hat{B},\psi}, \tilde{\psi}_1 \rangle_{\mathbb{R}^n} = 0, \] which, combined with (87) and (86), indicates that
\[ \langle z - \theta_{\hat{A},\hat{B},\psi}, \tilde{\psi} \rangle_{\mathbb{R}^n} = 0, \quad \forall \tilde{\psi} \in V_{\hat{A},\hat{B},\psi}. \]
Hence,
\[ z - \theta_{\hat{A},\hat{B},\psi} \in V_{\hat{A},\hat{B},\psi}^\perp. \] (89)
By (10), (76) and (57), we can easily check that
\[ N_{\hat{A},\hat{B}} = V_{\hat{A},\hat{B},\psi}^\perp. \] This, along with Lemma 2.1, implies that
\[ W_{\hat{A},\hat{B}} = N_{\hat{A},\hat{B}}^\perp = V_{\hat{A},\hat{B},\psi}^\perp. \]
From the latter and (89) it follows that
\[ z - \theta_{\hat{A},\hat{B},\psi} \in W_{\hat{A},\hat{B}}. \] (90)
The result (75) follows from (84) and (90).

In summary, we finish the proof of Theorem 2.8.

**Theorem 2.9.** Let \( z \in CW_{\hat{A},\hat{B}}^r \) and \( \dim W_{\hat{A},\hat{B}}^r = 1 \). Then \( z \in W_{\hat{A},\hat{B}}^r \).

**Proof.** Since \( z \in CW_{\hat{A},\hat{B}}^r \), according to (18), (19) and (20), for any \( \psi \in D_{\hat{A},\hat{B}} \), there exists a constant \( T(z, \psi) > 0 \) so that
\[ \langle z, \psi \rangle_{\mathbb{R}^n} \leq r \int_0^{T(z, \psi)} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \psi \|_{\mathbb{R}^m} ds. \] (91)
Here and throughout the proof of this theorem, \( T(\cdot, \cdot) \) denotes a generic positive constant which depends on what are enclosed in the bracket. Since \( \dim W_{\hat{A},\hat{B}} = 1 \), we assume that \( W_{\hat{A},\hat{B}} = \text{span}\{\phi\} \) and \( \| \phi \|_{\mathbb{R}^n} = 1 \). Then it follows from Lemma 2.1, (11) and (91) that
\[ \langle z, \phi \rangle_{\mathbb{R}^n} \leq r \int_0^{T(z, \phi)} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \phi \|_{\mathbb{R}^m} ds \]
and
\[ \langle z, -\phi \rangle_{\mathbb{R}^n} \leq r \int_0^{T(z, \phi)} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \phi \|_{\mathbb{R}^m} ds. \]
The above two inequalities imply that
\[ |\langle z, \phi \rangle_{\mathbb{R}^n}| \leq r \int_0^{T(z, \phi)} \| \hat{B}(s)^* \hat{\Phi}^{-1}(s, 0)^* \phi \|_{\mathbb{R}^m} ds. \] (92)
Noting that \( W_{\hat{A},\hat{B}} = \text{span}\{\phi\} \), by (11), Lemma 2.1 and (92), we obtain that
\[
|\langle z, \psi \rangle_{\mathbb{R}^n}| \leq r \int_0^{T(z,\phi)} \| \hat{B}(s) \hat{\Phi}^{-1}(s,0)^* \psi \|_{\mathbb{R}^m} ds, \quad \forall \psi \in D_{\hat{A},\hat{B}}. \tag{93}
\]
Now, for any \( w \in \partial B_1(0) \), we write \( w = w_1 + w_2 \) with \( w_1 \in N_{\hat{A},\hat{B}} \) and \( w_2 \in N_{\hat{A},\hat{B}}^\perp \). By (18), Lemma 2.1, (11), (93) and (10), we have that
\[
\| w \|_{\mathbb{R}^n} \leq r \int_0^{T(z,\phi)} \| \hat{B}(s) \hat{\Phi}^{-1}(s,0)^* w_2 \|_{\mathbb{R}^m} ds.
\]
This completes the proof. \( \square \)

3. Proof of main result. We now are on the position to prove Theorem 1.2.

Proof of Theorem 1.2. For each \( y_0 \in \mathbb{R}^n \setminus \{0\} \), we fix it. We can judge whether it belongs to \( W_{\hat{A},\hat{B}}^r \) or not by finite times as follows.

Step 1. Let \( k = 0 \), \( B_k(\cdot) = B(\cdot) \) and \( y_k = y_0 \).

Step 2. Is \( y_k \in CW_{\hat{A},\hat{B}}^r \)? If the answer is false, then \( y_k \notin W_{\hat{A},\hat{B}}^r \); If the answer is yes, then we continue Step 3.

Indeed, if \( y_k \notin CW_{\hat{A},\hat{B}}^r \), by Lemma 2.2, we have that \( y_k \notin W_{\hat{A},\hat{B}}^r \).

Step 3. Is there \( \psi_k \in \hat{D}_{\hat{A},\hat{B}} \) so that \( y_k \in \partial (HS)_{\hat{A},\hat{B},\psi_k}^r \)? If the answer is false, then \( y_k \in W_{\hat{A},\hat{B}}^r \); If the answer is yes, then we continue Step 4.

In fact, by Step 2, we see that \( y_k \in CW_{\hat{A},\hat{B}}^r \). If \( y_k \notin \bigcup_{\psi \in D_{\hat{A},\hat{B}}} \partial (HS)_{\hat{A},\hat{B},\psi}^r \), by Theorem 2.5, we get that \( y_k \in W_{\hat{A},\hat{B}}^r(T) \) for some \( T \in \mathbb{R}_+ \). This implies that \( y_k \in W_{\hat{A},\hat{B}}^r \).

Step 4. Define \( B_{k+1}(\cdot) \triangleq B_{A,B_k,\psi_k}(\cdot) \) and \( y_{k+1} \triangleq y_k - \theta_{A,B_k,\psi_k} \).

Step 5. Is there \( \psi \in D_{A,B_k+1} \) so that \( \langle y_{k+1}, \psi \rangle_{\mathbb{R}^n} = r C_{A,B_k+1,\psi} \) when
\[
\sup_{t \in [0,\infty)} \int_0^t \| B_{k+1}(s)^* \Phi^{-1}(s,0)^* \psi \|_{\mathbb{R}^m} ds
\]
can not be achieved? If the answer is yes, then \( y_k \notin W_{\hat{A},\hat{B}}^r \); If the answer is false, then turn back to Step 3.

Indeed, by Step 2 and Step 3, we have that \( y_k \in CW_{\hat{A},\hat{B}}^r \) and there is \( \psi_k \in \hat{D}_{\hat{A},\hat{B}} \) so that \( y_k \in \partial (HS)_{\hat{A},\hat{B},\psi_k}^r \). Then by Step 4 and Theorem 2.8, we get that
\[
y_{k+1} \in \left( \bigcap_{\psi \in D_{A,B_k+1}} (CHS)_{A,B_k+1,\psi}^r \right) \cap W_{A,B_k+1}
\] \quad \( \tag{94} \)

Now two cases may occur:

Case 3-1. There is \( \psi \in D_{A,B_k+1} \) so that \( \langle y_{k+1}, \psi \rangle_{\mathbb{R}^n} = r C_{A,B_k+1,\psi} \) when
\[
\sup_{t \in [0,\infty)} \int_0^t \| B_{k+1}(s)^* \Phi^{-1}(s,0)^* \psi \|_{\mathbb{R}^m} ds
\]
can not be achieved.

In this case, by (18), (20), (17) and (15), we get that

\[ y_{k+1} \notin CW_{A,B_{k+1}}^r. \]

This, along with Lemma 2.2, implies that

\[ y_{k+1} \notin W_{A,B_{k+1}}^r. \]  \hfill (95)

From (95), we claim that

\[ y_k \notin W_{A,B_k}^r. \]  \hfill (96)

Otherwise, according to (8), there would exist a constant \( T \in \mathbb{R}_+ \) such that \( y_k \in W_{A,B_k}^r(T) \). By Lemma 2.7 and (8) again, we have that

\[ y_{k+1} \in W_{A,B_k}^r(T) \subseteq W_{A,B_{k+1}}^r, \]

which contradicts (95).

**Case 3-2.** For any \( \psi \in D_{A,B_{k+1}} \), when

\[ \sup_{t \in [0,\infty)} \int_0^t \|B_{k+1}(s)\Phi^{-1}(s,0)^*\psi\|_{\mathbb{R}^m} ds \]

can not be achieved, \( \langle y_{k+1}, \psi \rangle_{\mathbb{R}^n} \neq rC_{A,B_{k+1},\psi} \).

In this case, by (94) and (16), we get that

\[ \langle y_{k+1}, \psi \rangle_{\mathbb{R}^n} < rC_{A,B_{k+1},\psi}. \]

Hence, it follows from the latter, (94) and (18) that

\[ y_{k+1} \in CW_{A,B_{k+1}}^r. \]  \hfill (97)

We now turn to Step 3.

(I) If \( y_{k+1} \notin \bigcup_{\psi \in \hat{D}_{A,B_{k+1}}} \partial(HS)^r_{A,B_{k+1},\psi} \), by (97) and Theorem 2.5, we get that

\[ y_{k+1} \in W_{A,B_{k+1}}^r(T) \text{ for some } T \in \mathbb{R}_+. \]

Since \( B_{k+1}^r(\cdot) = B_{A,B_k,\psi_k}^r(\cdot) = \chi_{E_{A,B_k,\psi_k}^r}B_k(\cdot) \) (see (56)), there exists a control \( u \in L^\infty(0,T;B_r(0)) \) so that

\[
y_{k+1} = -\int_0^T \Phi^{-1}(s,0)B_{k+1}(s)u(s)ds
= -\int_{(0,T) \cap E_{A,B_k,\psi_k}^r} \Phi^{-1}(s,0)B_k(s)u(s)ds. \]  \hfill (98)

Moreover, by (58), we have that

\[
\theta_{A,B_k,\psi_k} = \int_{E_{A,B_k,\psi_k}^r} \Phi^{-1}(s,0)B_k(s) \frac{rB_k(s)^*\Phi^{-1}(s,0)^*\psi_k}{\|B_k(s)^*\Phi^{-1}(s,0)^*\psi_k\|_{\mathbb{R}^m}} ds. \]  \hfill (99)

It follows from (98) and (99) that

\[ y_k = y_{k+1} + \theta_{A,B_k,\psi_k} \in W_{A,B_k}^r. \]

(II) If \( \psi_{k+1} \in \hat{D}_{A,B_{k+1}} \) and \( y_{k+1} \in \partial(HS)^r_{A,B_{k+1},\psi_{k+1}} \), we define \( B_{k+2}(\cdot) = B_{A,B_k,\psi_{k+1}}(\cdot) \) and \( y_{k+2} = y_{k+1} - \theta_{A,B_{k+1},\psi_{k+1}} \). By Theorem 2.8 and (97), we get that

\[ y_{k+2} \in \bigcap_{\psi \in \hat{D}_{A,B_{k+2}}} (CHS)^r_{A,B_{k+2},\psi} \bigcap W_{A,B_{k+2}}^r. \]

Then two cases may occur:
(II-1) There is \( \psi \in D_{A,B_k+2} \) so that \( \langle y_{k+2}, \psi \rangle_{\mathbb{R}^n} = rC_{A,B_k+2} \psi \) when
\[
\sup_{t \in [0, +\infty)} \int_0^t \| B_{k+2}(s)^* \Phi^{-1}(s, 0)^* \psi \|_{\mathbb{R}^m} \, ds
\]
can not be achieved.

In this case, by similar arguments as (96), we can deduce that
\[
y_{k+1} \not\in \overline{W}_{A,B_{k+1}}^r.
\]  
(100)
From which it follows that
\[
y_k \not\in \overline{W}_{A,B_k}^r.
\]  
Otherwise, according to (8) and (9), there would exist a constant \( T \in \mathbb{R}_+ \) and \( u \in L^\infty(0, T; B_r(0)) \) so that
\[
y_k = - \int_0^T \Phi^{-1}(s, 0) B_k(s) u(s) \, ds,
\]  
(101)
which indicates that
\[
\langle y_k, \psi_k \rangle_{\mathbb{R}^n} = \int_0^T \langle B_k(s)^* \Phi^{-1}(s, 0)^* \psi_k, -u(s) \rangle_{\mathbb{R}^m} \, ds.
\]  
(102)
Moreover, since \( y_k \in \partial(HS)^*_{A,B_k,\psi_k} \) and \( \psi_k \in \overline{D}_{A,B_k} \) (see the beginning of Step 3), by (34), (33), (32), (19) and (16), we have that
\[
\langle y_k, \psi_k \rangle_{\mathbb{R}^n} = r \int_0^{+\infty} \| B_k(s)^* \Phi^{-1}(s, 0)^* \psi_k \|_{\mathbb{R}^m} \, ds.
\]  
(103)
It follows from (102) and (103) that
\[
B_k(s)^* \Phi^{-1}(s, 0)^* \psi_k = 0 \quad \text{for a.e. } s \in (T, +\infty),
\]  
(104)
and
\[
u(s) = - \frac{B_k(s)^* \Phi^{-1}(s, 0)^* \psi_k}{\| B_k(s)^* \Phi^{-1}(s, 0)^* \psi_k \|_{\mathbb{R}^m}} \quad \text{for a.e. } s \in \text{supp}(B_k(\cdot)^* \Phi^{-1}(\cdot, 0)^* \psi_k).
\]  
(105)
Hence, by (101), (58), (104) and (105), we get that
\[
y_{k+1} = y_k - \theta_{A,B_k,\psi_k} = - \int_0^T \Phi^{-1}(s, 0) B_k(s) \chi_{E_{A,B_k,\psi_k}} u(s) \, ds
\]
\[
= - \int_0^T \Phi^{-1}(s, 0) B_{k+1}(s) u(s) \, ds.
\]
This, together with (8) and (9), implies that \( y_{k+1} \in \overline{W}_{A,B_{k+1}}^r(T) \subseteq \overline{W}_{A,B_{k+1}}^r \). It contradicts (100).

(II-2) For any \( \psi \in D_{A,B_k+2} \), when
\[
\sup_{t \in [0, +\infty)} \int_0^t \| B_{k+2}(s)^* \Phi^{-1}(s, 0)^* \psi \|_{\mathbb{R}^m} \, ds
\]
can not be achieved, \( \langle y_{k+2}, \psi \rangle \neq rC_{A,B_k+2,\psi} \).

In this case, by similar arguments to get (97), we have that
\[
y_{k+2} \in \overline{CW}_{A,B_{k+2}}^r.
\]
Then we turn to Step 3.

Finally, it follows from (18), (59) and (60) that for \( i \geq 1 \),
\[
\overline{CW}_{A,B_{k+i}}^r \subseteq \overline{W}_{A,B_{k+i}}, \quad \overline{W}_{A,B_{k+i}} \subseteq \overline{W}_{A,B_{k+i-1}} \quad \text{and} \quad \dim \overline{W}_{A,B_{k+i}} < \dim \overline{W}_{A,B_{k+i-1}}.
\]
These, along with Theorem 2.9, imply that the above judging process ends in finite times, i.e., we can check whether \((P_{y_0})\) has an admissible control or not by finite times. Thus, the problem \((P_{y_0})\) is solvable (see Definition 1.1).

In summary, we finish the proof of Theorem 1.2.

The following result is concerned with the case that \(A(\cdot)\) and \(B(\cdot)\) are time-invariant.

**Theorem 3.1.** Let \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\). Then \(y_0 \in W^r_{A,B}\) if and only if
\[
y_0 \in CW^r_{A,B}. \tag{106}
\]
Moreover, (106) is equivalent to the following conditions:
\[
y_0 \in W_{A,B} \text{ and } \langle y_0, \psi \rangle_{\mathbb{R}^n} < r \int_0^{+\infty} \|B^* e^{-A^* s} \psi\|_{\mathbb{R}^m} ds, \quad \forall \psi \in D_{A,B}. \tag{107}
\]

**Proof.** First, we claim that
\[
(HS)^r_{A,B,\psi} = (OHS)^r_{A,B,\psi}, \quad \forall \psi \in D_{A,B}. \tag{108}
\]
By contradiction, there would exist \(\hat{\psi} \in D_{A,B}\) so that
\[
\sup_{t \in [0, +\infty)} \int_0^t \|B^* e^{-A^* s} \hat{\psi}\|_{\mathbb{R}^m} ds \text{ is achieved.}
\]
Then there exists a constant \(T \in \mathbb{R}_+\) so that
\[
B^* e^{-A^* t} \hat{\psi} = 0, \quad \forall t \geq T. \tag{109}
\]
Since \(B^* e^{-A^* t} \hat{\psi}, t \geq 0,\) is an analytic function, it follows from (109) that
\[
B^* e^{-A^* t} \hat{\psi} = 0, \quad \forall t \geq 0.
\]
This, along with (10), implies that
\[
\hat{\psi} \in N_{A,B},
\]
which contradicts \(\hat{\psi} \in D_{A,B} = N^+_{A,B} \cap \partial B_1(0)\) (see (11)).

Second, on one hand, if \(y_0 \in W^r_{A,B}\), it follows from Lemma 2.2 that (106) holds. On the other hand, if \(y_0 \in CW^r_{A,B}\), by (18), we have that
\[
y_0 \in W_{A,B} \text{ and } y_0 \in (HS)^r_{A,B,\psi}, \quad \forall \psi \in D_{A,B}. \tag{110}
\]
Noting that \(\bar{D}_{A,B} \subseteq D_{A,B}\) (see (34)), by (33), (32), (110) and (108), we get that
\[
y_0 \notin \partial (HS)^r_{A,B,\psi}, \quad \forall \psi \in \bar{D}_{A,B}.
\]
This, together with Theorem 2.5 and (8), yields that \(y_0 \in W^r_{A,B}\).

Finally, by (18) and (108), we can easily check that (106) is equivalent to (107). This completes the proof.

**REFERENCES**

[1] M. E. Achbab, F. M. Callier and V. Wertz, Admissible controls and attainable states for a class of nonlinear systems with general constraints, *Internat. J. Robust Nonlinear Control*, 4 (1994), 267–288.

[2] S. A. Alisagaliev and M. K. Ospanova, Existence of admissible controls for ordinary differential equations with fixed end-points of trajectories in the presence of phase and integral constraints, (Russian) *Vestn. Minist. Obraz. Nauki Nats. Akad. Nauk Resp. Kaz.*, (2003), 16–26.
[3] V. Barbu, *Optimal Control of Variational Inequalities*, Research Notes in Mathematics, 100, Pitman, Boston, MA, 1984.

[4] R. Conti, *Teoria del Controllo e del Controllo Ottimo*, UTET, Torino, Italy, 1974.

[5] A. L. Dontchev, On the admissible controls of constrained linear systems, *C. R. Acad. Bulgare Sci.*, 42 (1989), 33–36.

[6] H. Hermes, On the closure and convexity of attainable sets in finite and infinite dimensions, *SIAM J. Control*, 5 (1967), 409–417.

[7] J. B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of Convex Analysis*, Springer-Verlag, Berlin, 2001.

[8] V. A. Komarov, Estimates for the accessibility set and the construction of admissible controls for linear systems, (Russian) *Dokl. Akad. Nauk SSSR*, 268 (1983), 537–541.

[9] S. R. Musaev, A certain sufficient condition for the existence of admissible controls for a multidimensional optimal control problem, (Russian) *Akad. Nauk Azerbaïdjan, SSR Dokl.*, 32 (1976), 3–7.

[10] S. R. Musaev and T. M. Efendiev, Construction of scalar admissible controls by the Picard-Rakovshchik method, (Russian) *Questions of Mathematical Cybernetics and Applied Mathematics*, “Elm”, Baku, 1980, 134–145.

[11] L. D. Pustyl’nikov, On a method for finding admissible controls in a linear system with phase constraints, (Russian) *Differentsial’nye Uravneniya*, 17 (1981), 2176–2184, 2300.

[12] E. O. Roxin, The attainable set in control systems, in *Mathematical Theory Of Control* (Bombay, 1990), 307–319, Lecture Notes in Pure and Appl. Math., 142, Dekker, New York, 1993.

[13] W. E. Schmitendorf and B. R. Barmish, Null controllability of linear systems with constrained controls, *SIAM J. Control and Optim.*, 18 (1980), 327–345.

[14] G. Wang, The existence of time optimal control of semilinear parabolic equations, *Systems Control Lett.*, 53 (2004), 171–175.

[15] G. Wang, Y. Xu and Y. Zhang, Attainable subspaces and the bang-bang property of time optimal controls for heat equations, *SIAM J. Control Optim.*, 53 (2015), 592–621.

[16] L. Wang and Q. Yan, Bang-bang property of time optimal null controls for some semilinear heat equation, *SIAM J. Control Optim.*, 54 (2016), 2949–2964.

Received August 2017; revised June 2018.

E-mail address: ljwang.math@whu.edu.cn
E-mail address: yashanxu@fudan.edu.cn