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I. INTRODUCTION

It is likely that superstring or M-theory governs the physics of gravity or space-time in higher energy stages. Such theories are naturally formulated in the higher dimensions than four. We expect a plausible scenario that such a higher dimensional space-time somehow evolves to the stable four dimensional space-time according to the history of the Universe. The so-called brane world scenario is the most actively being investigated along to this line. This scenario is motivated by Horava and Witten’s theory which shows that an eleven dimensional supergravity theory on the orbifold $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ is related to the ten-dimensional $E_8 \times E_8$ heterotic string theory. Therein the matters are confined to the ten-dimensional space-time (three-brane) and gravitons are propagating in the full eleven dimensions. The brane world space-time should be stable.

Although the brane world scenario may be plausible at the reduction from eleven to ten dimensions, the space-time will be still compactified to four dimensions in the normal Calabi-Yau’s way. Regarding to these full scenario of the compactification, the stability of the space-time becomes the important issue to be investigated. The positive energy theorem guarantees the stability of the four dimensional asymptotically flat space-time in the framework of general relativity. Surprisingly, the existence of the extra dimensions can drastically change the situation. Witten showed that the five dimensional Minkowski space-time decays into the so-called Kaluza-Klein (KK) bubble space-time unless we assume the existence of the elementary fermion related to supersymmetry, of which existence we can not expect generally. This also may indicate that the ‘bubble’ appears somewhere at the bulk or on the brane in the brane world scenario and disturbed the three-brane where we are living.

The metric of the KK bubble space-time given by Witten is written as

$$ds_5^2 = -r^2 dt^2 + \left(1 - \frac{r_0^2}{r^2}\right) d\chi^2 + \left(1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 \cosh^2 t d\Omega^2,$$

(1.1)

where the $\chi$-direction will be compactified and $r \geq r_0$, and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$. In general case, the metric has a conical singularity at $r_0$. However, if we carefully take a periodicity along the $\chi$-direction, the metric can be regularized. More precisely to see this, we write the metric near $r = r_0$ as

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Fate of Kaluza-Klein Bubble

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We numerically study classical time evolutions of Kaluza-Klein bubble space-time which has negative energy after a decay of vacuum. As the zero energy Witten’s bubble space-time, where the bubble expands infinitely, the subsequent evolutions of Brill and Horowitz’s momentarily static initial data show that the bubble will expand in terms of the area. At first glance, this result may support Corley and Jacobson’s conjecture that the bubble will expand forever as well as the Witten’s bubble. The irregular signatures, however, can be seen in the behavior of the lapse function in the maximal slicing gauge and the divergence of the Kretchman invariant. Since there is no appearance of the apparent horizon, we suspect an appearance of a naked singularity as the final fate of this space-time.

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\[ ds^2_5 \simeq -r_0^2 dt^2 + \frac{2(r - r_0)}{r_0} d\chi^2 + \frac{r_0}{2(r - r_0)} dr^2 + r_0^2 \cosh^2 t d\Omega^2 \]
\[ = -r_0^2 dt^2 + 2r_0 \left[ R^2 d\left( \frac{\chi}{r_0} \right)^2 + dR^2 \right] + r_0^2 \cosh^2 t d\Omega^2, \] (1.2)

where \( R = \sqrt{r - r_0} \). Then we realize that the period should be set to be \( \chi_p = 2\pi r_0 \). As one can see, the ‘boundary of bubble’ located at \( r = r_0 \) expands rapidly like \( \cosh t \) and the space-time does not have naked singularities. Here, we remind you that the total energy is zero. We are imaging the boundary of the space-time, \( r = r_0 \), as the surface of the ‘bubble’.

Interestingly Brill and Pfister \[9\] gave an initial data which has the negative total energy related to the size of the compactified dimension. The space-time with the negative energy may be favorable in the aspect of energetics. One year later, Brill and Horowitz \[10\] gave an initial data in a simple way. (We will briefly review their construction in Sec. III.) Contrasted to the ‘Witten bubble’, their solution has arbitrary negative energy regardless of the size of the compactified space. This is too far from our intuition that the negative energy is proportional to the Casimir energy due to the boundary effect of the compactified space. Therefore it is difficult to imagine the classical evolution after the vacuum decay.

Corley and Jacobson \[11\] discussed the subsequent evolution of Brill-Horowitz’s initial data. They found that the positive acceleration of the bubble’s surface area for the negative mass bubbles, and they conjectured that KK bubble with negative energy cannot collapse. However, their study is not sufficient to conclude the final fate of the bubble, as they already mentioned, because they considered only the initial behavior of the time-symmetric data and they did not do any dynamical studies.

In this paper, we report our numerical analysis on this final fate problem of KK bubble, especially of the negative mass bubble. We start our numerical simulation from the Brill-Horowitz’s initial data, and evolve the space-time using the standard Arnowitt-Deser-Misner formulation (but 4+1 dimensional decomposition). We will show that the space-time initially behaves as Corley-Jacobson’s analysis, and expands forever, although the acceleration will be negative. Despite of the expanding, we will observe the irregular behavior of the curvature invariant.

This paper is organized as follows. In Sec. II we give a brief review of Brill-Horowitz’s construction of their initial data. In Sec. III we describe numerical method and equations. The results of our simulations are shown in in Sec. IV. Finally, we summarize our results in Sec. V.

II. BRILL-HOROWITZ’S INITIAL DATA

In this section we briefly review Brill and Horowitz’s argument \[10\]. Let us consider an initial slice with \( K_{ij} = 0 \) in five dimensional vacuum space-times, where \( K_{ij} \) is the extrinsic curvature of a four dimensional spacelike hypersurface. In this slice the Hamiltonian constraint equation becomes \((4) R = 0\), where \((4) R \) is the four dimensional Ricci scalar. Here one can easily see that the Euclidean Reissner-Nordstrom metric with imaginary ‘charge’ \( iq \) satisfies the Hamiltonian constraint equation, because the ‘energy-momentum’ tensor of the four dimensional Maxwell field is traceless. The metric of the hypersurface is given by

\[ (4) g = U(r) d\chi^2 + \frac{dr^2}{U(r)} + r^2 d\Omega^2, \] (2.1)

where \( U(r) = 1 - 2m/r - q^2/r^2 \) and \( r \geq r_+ := m + \sqrt{m^2 + q^2} \). In the same way as the previous Witten’s example, the metric is approximately written as

\[ (4) g \simeq \frac{4}{U'(r_+)} \left[ R^2 d\left( \frac{U'(r_+) \chi}{2} \right)^2 + dR^2 \right] + r_+^2 d\Omega^2, \] (2.2)

near \( r = r_+ \), where \( R = \sqrt{r - r_+} \). To avoid a conical singularity at \( r = r_+ \), we assume the period \( \chi_p = 4\pi/U'(r_+) = 2\pi r_+^2/(r_+ - m) \) along the \( \chi \)-direction.

The total energy is evaluated as \( E = m/2 \). The \( m \) is arbitrary parameter and \( q \) determines the size of the compactified space. So the total energy can be arbitrary negative.
III. FIELD EQUATIONS AND OUR NUMERICAL METHOD

To know the final fate of the KK bubbles with the negative energy, we study the subsequent time evolution for a long time numerically. We apply 4+1 decomposition of space-time along to the Arnowitt-Deser-Misner formulation for the actual time integrations. We describe equations and basic numerical techniques in this section.

The metric of the full space-time is assumed to be

$$ds^2 = -N(r, t)^2 dt^2 + e^{2a(r, t)} U(r) dx^2 + e^{2b(r, t)} U(r)^{-1} dr^2 + r^2 e^{2c(r, t)} dΩ^2,$$

where $U(r) = 1 - 2m/r - q^2/r^2$, $N$ is the lapse function, and the metric components $a$, $b$ and $c$ are now time dependent. The evolution equations of the four-metric $\gamma_{ij}$ and the extrinsic curvature $K_{ij}$ become

$$\dot{K}_{ij} = N \left( R^i_j + KK^i_j \right) - D_i D_j N,$$

$$\dot{\gamma}_{ij} = -2 N K_{ij},$$
a dot denotes the time derivative, and $(4) R^i_j$ and $(4) D^i$ denote four dimensional Ricci curvature and the covariant derivative, respectively. For the reader’s convenience, we write down several terms in $(3.2)$ for the metric $(3.1)$ as:

$$(4) R^X_X = e^{-2b} \left( -a'^2 - a'' - 2a'c' - \frac{2a'}{r} + ab' \right) U + \left( -\frac{3}{2} a' + 2b' - c' \right) U' - \frac{1}{2} U'',$$

$$(4) R^r_r = e^{-2b} \left( -a'^2 - a'' - 2c'^2 - 2a'c' - \frac{4c'}{r} + ab' + 2b'c' + \frac{2b'}{r} \right) U + \left( -\frac{3}{2} a' + 2b' - c' \right) U' - \frac{1}{2} U'',$$

$$(4) R^\theta_\phi = (4) R^\phi_\theta = e^{-2b} \left[ -2a'^2 - a'' - \frac{1}{r^2} + c'b' + \frac{b'}{r} - a'c' - \frac{a'}{r} \right] U + \left( -c' - \frac{1}{r} \right) U' + \frac{2b - 2c}{r^2},$$

and

$$(4) D_X (4) D^X N = e^{-2b} \left( a' + \frac{1}{2} U' \right) U' N',$$

$$(4) D_r (4) D^r N = e^{-2b} \left( \left( N'' - b' N' \right) U + \frac{1}{2} N' U' \right),$$

$$(4) D_\theta (4) D^\theta N = e^{-2b} \left( c' + \frac{1}{2} \right) N' U,$$

where a dash denotes the derivative on $r$.

We start our simulation from the initial data of Brill-Horowitz’s momentarily static solution, such as

$$a(r, 0) = b(r, 0) = c(r, 0) = 0,$$

$$K^X_X(r, 0) = K^r_r(r, 0) = K^\theta_\phi(r, 0) = 0.$$  

The numerical region is taken as $r_+ \leq r \leq r_*$, where $r_+ := m + \sqrt{m^2 + q^2}$ is the location of the bubble at the initial data and $r_*$ is the numerical outer boundary. We stress from the construction that the Kaluza-Klein bubble space-time is restricted in $r_+ \leq r \leq \infty$. We apply the Robin boundary condition at $r = r_*$ such as all the components fall off as they form an asymptotically flat spacetime. At the inner boundary $r = r_+$, we use the fact that both $a$ and $b$ evolve synchronously as we describe in the Appendix, and use both the evolution equation for $\text{tr} K$,

$$\dot{K} = N K_{ij} K^{ij} - (4) D_i (4) D_i N,$$

where we used the Hamiltonian constraint equation, and the momentum constraint equation,

$$(4) D_j K^j_r - (4) D_r K = 0,$$

1Here, for simplicity, we tacitly supposed the boundary condition so that the location of the bubble is ‘fixed’ under the variation of the action. As a result we obtain the 5-dimensional vacuum Einstein equation and can show the consistent result given in Appendix A. Since the Cauchy development of the initial data cannot cover all region outside the bubble, one may be able to consider another boundary conditions, which might be artificial.
so as the system evolves properly.

In order to specify the lapse function, \( N \), we apply both the geodesic slicing condition, \( N = 1 \), and the maximal slicing condition, \( K = 0 \), which equation becomes (directly from Eq. (3.12))

\[
(4) \Delta N = NK_{ij}K^{ij}. 
\]

This elliptic equation is solved using the incomplete Cholsky conjugate gradient method. The outer boundary for \( N \) is set again as asymptotically flat, and the inner boundary at \( r_+ \) for solving (3.14) we lineally extrapolate 4-metric components.

We apply Brailovskaya integration scheme (a second order predictor-corrector method) \([14]\) for the time evolution.

The accuracy of the calculation is checked by monitoring the violation of the Hamiltonian constraint equation. The numerical code passed convergence tests, and the results shown in this paper are all obtained with acceptable accuracy.

IV. RESULTS

A. Acceleration of the bubble surface

We first check whether our code reveals the initial behavior discussed by Corley and Jacobson \([11]\). We calculate the area of the bubble,

\[
A(t) = 4\pi g_{\theta\theta}(r_+, t),
\]

together with its time derivative \( \dot{A} \), and its acceleration,

\[
\ddot{A} = 4\pi g_{\theta\theta} = 4\pi r^2\left[-2\dot{N}K_{\theta\theta} - 2\dot{N}K_{\theta\theta} + 4N^2(K_{\theta\theta})^2\right]e^{2c}.
\]

We first show this acceleration in Fig. 1(a)(b), since this was the quantity discussed by Corley and Jacobson. The Figs. 1(a) and (b) are of the geodesic slicing condition and of the maximal slicing condition, respectively. We fix the charge \( q \) and varied \( m \) from negative to positive values. Except for the transition at \( m = 0 \), as one can see later, our result is not qualitatively sensitive under changes of \( m/q \). Under both slicing conditions, we see that the negative mass bubble start expanding (positive \( \ddot{A} \)) initially, yet will soon be in de-accelerating phase (negative \( \ddot{A} \)), while the positive mass bubble keep accelerating all the way in Figs. 1(a), and in the region in Figs. 1(b). More precisely, for the positive mass cases in Figs. 1(b), we observe from the numerical results that the acceleration will reach and stay at a positive value in the final stage, even if it goes negative for a short time, which is happen to quite small positive mass cases.

Such an initial behavior (for both positive and negative mass bubbles) does agree with Corley and Jacobson’s analysis (we remark that their analysis was under the geodesic slicing condition). However, the turning behavior into de-accelerating phase could not find in their analysis. The de-accelerating does not mean collapsing feature directly. Actually, up to we stop our time evolution, the numerical data of the area, (4.1), monotonically increases [Figs. 1(c)], while its velocity goes down for negative mass bubbles [Figs. 1(d)]. However, from this facts, we can not say that negative mass bubbles will expand forever, because we can see the blow-ups of the Riemann invariant and collapsing lapse behavior as we show next. (We had to stop time evolution for negative mass bubble case when we face the blow-ups of the Riemann invariant.)

B. Collapse of lapse

Since we found that the time integration using the maximal slicing condition survives long term time evolution than that of the geodesic slicing condition, we will show only the results of the maximal slicing condition hereafter.

The maximal slicing condition is known as a robust gauge condition for singularity avoidance (or, exactly speaking, avoiding the vanishing of the volume elements of the associated Eulerian observers) \([15]\). This is because the lapse will go quite small value in the strong gravitational field. Contrary, we may guess whether the space-time will collapse or not by monitoring the lapse function.

We plot the lapse function, \( N \), in Fig. 2(a). Fig. 2(a) is the lapse function at the bubble surface, \( r = r_+ \), versus time. We see the lapse evolves small value for the case of negative mass bubble space-time. The lines end at the time when the violation of the constraint equation begin growing. From above standard behavior of the maximal sliced lapse functions, we may say that the negative mass bubble space-time is ‘collapsing’ in some senses. Fig. 2(b) is snapshots of \( N \) at several time for the case of negative mass bubble space-time.
C. Riemann invariant

In order to confirm our guess of the ‘collapsing’ behavior of the negative mass bubble, we calculated the Kretschman invariant (Riemann invariant) $R_{ijkl} R^{ijkl}$ of both 4 and 5-dimensional Riemann tensor. We see both blow up in the cases of negative mass bubbles. We plotted a typical behavior of the invariant as a function of $r$ and $t$, in Fig.3. These lines suggest that the possibility of the formation of singularity in the final phase of evolution.

D. Apparent horizons

In order to confirm whether a black hole is formed or not in such a case, we check the appearance of the apparent horizon. The location of the apparent horizon is given by the position that the expansion rate, of the outgoing null geodesic congruence turns into the negative. The appearance of the apparent horizon indicates the existence of the event horizon.

The definition of the apparent horizon might not be unique in our five-dimensional space-time because it depends on the dimension of the space-time which the null geodesic congruence runs. If the null congruence propagating in full five dimensional space-time, the expansion rate, $(4)\theta_+$, is given by

$$ (4)\theta_+ = (4)\nabla_a s^a - (4)K + s^a s^b (4)K_{ab} $$

$$ = (a' + 2c' + \frac{2}{r} + \frac{U'}{2U})\sqrt{U} e^{-b} - (K^\chi_\chi + K^\theta_\theta + K^\varphi_\varphi) $$  \hspace{1cm} (4.3)\)

where $s^a = (0, 1/\sqrt{g_{rr}}, 0, 0)$ is a outer pointing vector in our spatial four metric. On the other hand, if the null is confined to non-compactified four dimensions, the expansion rate, $(3)\theta_+$, is given similarly by

$$ (3)\theta_+ = (3)\nabla_a s^a - (3)K + s^a s^b (3)K_{ab} $$

$$ = 2(c' + \frac{1}{r})\sqrt{U} e^{-b} - (K^\theta_\theta + K^\varphi_\varphi). $$  \hspace{1cm} (4.4)\)

We analyzed both $(3)\theta_+$ and $(4)\theta_+$ in our evolving space-time. Surprisingly, in all cases (positive and negative mass bubbles), both expansions remain positive definite everywhere as we show an example in Fig.4. These suggest us no-appearance of apparent horizons.

V. DISCUSSION

We numerically studied the dynamical evolution of the Brill and Horowitz’s initial data which can have the negative energy. As the zero energy Witten’s bubble space-time, we show that the ‘bubbles’ with negative energy will expand by mean of area upto the time we stop the simulations. At first glance this result supports Corley and Jacobson’s conjecture. However, from the facts that the curvature invariant blows up, and no appearance of the apparent horizon, we suspect that a formation of a naked singularity as the final fate of Kaluza-Klein negative energy bubble. Hence, we may have to consider seriously the decay problem from the Kaluza-Klein vacuum to the Witten-type ‘bubble’ space-time. Possible resolution to this may be given by assuming the supersymmetry which may forbid the decay, or by constructing quantum gravity theory which may smooth out singularities as normally been expected. Although the negative mass bubbles are expanding, we obtained the result that the bubble spacetime terminates at the singularity. At first glance, they are incompatible, because the naive picture, which the expanding keeps regularity. However, the picture may be based on the Raychaudhri-type equation and the equation does not hold in the present case. Moreover, the area cannot properly describes whether the system will collapse or not. Properly speaking, we need the proper radius from the center which is absent in the present case.

2 In usual Kaluza-Klein picture, it is natural that the null geodesic congruence runs in four dimensional part.
3 An anonymous referee of this article pointed out the similarity of the positive and negative mass bubble results. However, from the results we obtained, we believe that there are qualitative differences between positive and negative mass cases in their dynamical behaviors. (We remark that our simulations are only up to a finite time in order to keep the resolution against the expansion of the spacetime.)
Finally, we would like to comment on the so-called brane world scenario\[3\]. The brane world is motivated by the reduction from the M-theory to the $E_8 \times E_8$ heterotic superstring theory. This reduction drastically changes the picture of the reduction\[3\] because ‘matters’ are confined to the ten dimensions and gravitons are propagating in the eleven-dimensions. Here we call the timelike hypersurface, where matters are confined, by ‘brane’\[4\]. A plausible history of the compactification are still quite actively discussed recently and we do not reach the consensus at this moment. Apart from this compactification scenario, the reduction from ten to four-dimensions follows the well-known Kaluza-Klein compactification. More precisely, we may say that the Witten-type Kaluza-Klein ‘bubble’ space-time on the 4-brane will be reduced from at least 6 dimensional space-time. As was recently reported\[3\], the effective Einstein equations on the brane are different from the normal that the Witten-type Kaluza-Klein compactification. More precisely, we may say the picture of the reduction\[2\] because ‘matters’ are confined to the ten dimensions and gravitons are propagating in the reduction from the M-theory to the Einstein equations. Therefore, it might be worth re-asking what is the final fate of Kaluza-Klein bubble if we describe the space-time by such a modified Einstein equation when we take a brane world scenario.

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APPENDIX A: THE DYNAMICAL EQUATIONS AT THE SURFACE OF THE BUBBLE

In this appendix, we show that $\dot{a} = \dot{b}$ at the location of the bubble, $r = r_+$, during the time evolution. The explicit expression of Eqs (3.2) and (3.3) are given by

\[
\begin{align*}
\left(\frac{\dot{a}}{N}\right)' &= -Ne^{-2b}\left[\left(-a'^2 - a'' - 2ac' - \frac{2a'}{r} + a'b\right)U + \left(-\frac{3}{2}a' + \frac{1}{2}b' - c' - \frac{1}{r}\right)U' - \frac{1}{2}U''\right] \\
&\quad - \frac{1}{N}(\dot{a} + \dot{b} + 2\dot{c})a + e^{-2b}\left(a'U + \frac{1}{2}U'\right)N', \\
\left(\frac{\dot{b}}{N}\right)' &= -Ne^{-2b}\left[\left(-a'^2 - a'' - 2c'^2 - 2c'' - \frac{4c'}{r} + a'b' + 2b'c' + \frac{2b'}{r}\right)U + \left(-\frac{3}{2}a' + \frac{1}{2}b' - c' - \frac{1}{r}\right)U' - \frac{1}{2}U''\right] \\
&\quad - \frac{1}{N}(\dot{a} + \dot{b} + 2\dot{c})b + e^{-2b}\left[(N'' - b'N')U + \frac{1}{2}N'U'\right], \\
\left(\frac{\dot{c}}{N}\right)' &= -Ne^{-2b}\left[\left(-c'^2 - c'' - \frac{4c'}{r} - \frac{1}{r} + c'b' + \frac{b'}{r} - a'c' - \frac{a'}{r}\right)U + \left(-c' - \frac{1}{r}\right)U' + \frac{e^{2b - 2c - 2}}{r^2}\right] \\
&\quad - \frac{1}{N}(\dot{a} + \dot{b} + 2\dot{c})c + e^{-2b}\left(c' + \frac{1}{r}\right)N'U.
\end{align*}
\]

At $r = r_+$, we can truncate $U$, since $U = 0$. By adding a suffix $+$ for the variables which is evaluated at $r = r_+$, the above equations become

\[
\begin{align*}
\left(\frac{\dot{a}_+}{N_+}\right)' &= -N_+e^{-2b_+}\left[\left(-\frac{3}{2}a'_+ + \frac{1}{2}b'_+ - c'_+ - \frac{1}{r_+}\right)U'_+ - \frac{1}{2}U''_+\right] \\
&\quad - \frac{1}{N_+}(\dot{a}_+ + \dot{b}_+ + 2\dot{c}_+)\dot{a}_+ + \frac{1}{2}e^{-2b_+}U'_+N'_+,
\end{align*}
\]

\[
\begin{align*}
\left(\frac{\dot{b}_+}{N_+}\right)' &= -N_+e^{-2b_+}\left[\left(-\frac{3}{2}a'_+ + \frac{1}{2}b'_+ - c'_+ - \frac{1}{r_+}\right)U'_+ - \frac{1}{2}U''_+\right] \\
&\quad - \frac{1}{N_+}(\dot{a}_+ + \dot{b}_+ + 2\dot{c}_+)\dot{b}_+ + \frac{1}{2}e^{-2b_+}U'_+N'_+.
\end{align*}
\]

Subtracting Eq. (A5) from Eq. (A4), we obtain

\[
\left(\frac{\dot{a}_+ - \dot{b}_+}{N_+}\right)' = -(\dot{a}_+ + \dot{b}_+ + 2\dot{c}_+)\frac{\dot{a}_+ - \dot{b}_+}{N_+}.
\]
Therefore we get

\[
\frac{\dot{a} - \dot{b}}{N_+} = Ae^{-(a+b+2c_+)},
\]

(A7)

where \(A\) is a constant. Since the initial conditions, (3.10) and (3.11), imply \(a_+ = b_+\), which implies \(A = 0\). Therefore at the boundary, \(r = r_+\), we can set \(a_+ = b_+\) even after the long time integration.
Figure Captions

Fig.1
Acceleration of the bubble surface, $\frac{dA}{dt}$, versus time. The figures (a) and (b) are of the geodesic slicing condition and of the maximal slicing condition, respectively. In both figures, we see that the negative mass bubble will soon be in collapse phase, although they start expanding initially. We set $q = 1$ (hereafter for all figures). For the case of evolutions with the maximum slicing condition, we plot (c) area of the bubble surface $A$, (4.1), versus time, and (d) the velocity of the bubble surface, $dA/dt$.

Fig.2
The lapse function, $N$, are plotted (the solutions of maximal slicing condition). The figure (a) is $N$ at $r = r_+$ versus time. We see the lapse evolves small value for the case of negative mass bubble space-time. The figure (b) is snapshots of $N$ at several times for a case of negative mass bubble ($m = -0.4$) space-time.

Fig.3
(a) Typical snapshots of the Riemann invariant $R_{ijkl}R^{ijkl}$ of 4-dimensional Riemann curvature for the case of negative mass bubble ($m = -0.4$) are plotted. Only the region near the bubble surface is drawn. (b) The Riemann invariant $R_{ijkl}R^{ijkl}$ of 5-dimensional Riemann curvature are plotted as a function of time. We see blow-ups in the cases of negative mass bubbles (we cut the display range at $10^5$). The values are evaluated at a point right from the bubble surface (that is, at $r_+ + \Delta r$).

Fig.4
A typical sample of the outgoing null expansion rate $^{(3)}\theta_+$ and $^{(4)}\theta_+$ are plotted for the case of negative mass ($m = -0.4$) bubbles.
FIG. 1. Acceleration of the bubble surface, $\frac{d^2A}{dt^2}$, versus time. The figures (a) and (b) are of the geodesic slicing condition and of the maximal slicing condition, respectively. In both figures, we see that the negative mass bubble will soon be in collapse phase, although they start expanding initially. We set $q = 1$ (hereafter for all figures).
FIG. 1. (continued)
For the case of evolutions with the maximal slicing condition, we plot (c) area of the bubble surface $A$, versus time, and
(d) the velocity of the bubble surface, $dA/dt$. 
FIG. 2. The lapse function, $N$, are plotted (the solutions of maximal slicing condition). The figure (a) is $N$ at $r = r_+$ versus time. We see the lapse evolves small value for the case of negative mass bubble space-time. The figure (b) is snapshots of $N$ at several times for a case of negative mass bubble ($m = -0.4$) space-time.
FIG. 3. (a) Typical snapshots of the Riemann invariant $R_{ijkl} R^{ijkl}$ of 4-dimensional Riemann curvature for the case of negative mass bubble ($m = -0.4$) are plotted. Only the region near the bubble surface is drawn. (b) The Riemann invariant $R_{ijkl} R^{ijkl}$ of 5-dimensional Riemann curvature are plotted as a function of time. We see blow-ups in the cases of negative mass bubbles (we cut the display range at $10^5$). The values are evaluated at a point right from the bubble surface (that is, at $r_+ + \Delta r$).
FIG. 4. A typical sample of the outgoing null expansion rate $^{(3)}\theta_+$ and $^{(4)}\theta_+$ are plotted for the case of negative mass ($m = -0.4$) bubbles.