Elements of Finite Order in the Group of Formal Power Series Under Composition

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Abstract. We consider formal power series \( f(z) = \omega z + a_2 z^2 + \cdots \) (\( \omega \neq 0 \)), with coefficients in a field of characteristic 0. These form a group under the operation of composition (= substitution). We prove (Theorem 1) that every element \( f(z) \) of finite order is conjugate to its linear term \( \ell_\omega(z) = \omega z \), and we characterize those elements which conjugate \( f(z) \) to \( \omega z \). Then we investigate the construction of elements of order \( n \) and prove (Theorem 2) that, given a primitive \( n \)th root of unity \( \omega \) and an arbitrary sequence \( \{a_k\}_{k\neq nj+1} \) there is a unique sequence \( \{a_{nj+1}\}_{j=1}^{\infty} \) such that the series \( f(z) = \omega z + a_2 z^2 + a_3 z^3 + \cdots \) has order \( n \). Sections 1 - 5 give an exposition of this classical subject, written for the 2005 - 2006 Morgan State University Combinatorics Seminar. We do not claim priority for these results in this classical field, though perhaps the proof of Theorem 2 is new. We have now (2018) added Section 6 which gives references to valuable articles in the literature and historical comments which, however incomplete, we hope will give proper credit to those who have preceded this note and be helpful and of interest to the reader.

1. Introduction

Suppose that \( F \) is a field of characteristic 0. We denote by \( F[[z]] \) the set of all formal power series with coefficients in \( F \). Composition in \( F[[z]] \) is defined for elements of the form \( f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \ g(z) = b_1 z + b_2 z^2 + \cdots \) by:

\[
(f \circ g)(z) \equiv f(g(z)) = a_0 + a_1 g(z) + a_2 g(z)^2 + \cdots = \sum_{n \geq 0} a_n g(z)^n.
\]

Define: \( G = G[[z]] \equiv \{ f \in F[[z]] \mid f(z) = \sum_{n} a_n z^n, \ a_0 = 0, \ a_1 \neq 0 \} \).

It is a classical fact (See Section 6.1) that \( G \) is a group under composition. The identity element of \( G \) is the series \( id \) given by \( id(z) = z \). We denote the inverse of \( f \) by \( \overline{f} \) and the composition \( f \circ f \circ f \cdots \circ f \) (\( n \) times) by \( f^{(n)} \). We define \( \ell_\omega \in G \) by \( \ell_\omega(z) = \omega z \).
The purpose of this note is to prove the following two theorems concerning formal power series of finite compositional order. In Section 5, we comment on series of order two and series of infinite order.

**Theorem 1 (Classification of finite order elements up to conjugation).**

Suppose that \( f(z) = \omega z + \sum_{k=2}^{\infty} a_k z^k \) is an element of \( G \) of order \( n \). Define \( f^* \) by

\[
f^* = \frac{1}{n} \sum_{j=1}^{n} \omega^{n-j} f^{(j)}.
\]

Then:

A. \( f^* \in G \), \( n \) is the multiplicative order of \( \omega \in F - \{0\} \) and

\[
f^* \circ f \circ f^* (z) = \omega z.
\]

Two elements of order \( n \) in \( G \) are conjugate iff their lead coefficients are the same primitive \( n \)'th root of unity.

B. (a) If \( g \in G \) then

\[
g \circ f \circ \overline{g} = \ell_\omega \iff \exists h(z) = \sum_{j=0}^{\infty} h_{nj+1} z^{nj+1} \in G \text{ such that } g = h \circ f^*.
\]

(b) For every sequence \( \{ g_{nj+1} \}_{j=0}^{\infty} \) of elements of \( F \) with \( g_1 \neq 0 \) there exists a unique sequence \( \{ g_k \}_{1<k \neq nj+1}^{\infty} \) such that the series \( g(z) = \sum_{k=1}^{\infty} g_k z^k \) satisfies \( g \circ f \circ \overline{g} = \ell_\omega \).

**Note:** The same proof of Theorem 1A, in which the conjugating element \( f^* \) is used, is given by Cheon and Kim [5] and by O’Farrell and Short [10]. (See Section 6.) When \( F = \mathbb{C} \), the field of complex numbers, Theorem 1A. is a very special case of Theorem 8 (p. 19) of [1]. The proof given here, using \( f^* \), is our interpretation of page 23 of [1] in the case of a finite cyclic group of formal power series of a single variable. The definition of \( f^* \) is a finite version of the formal power series \( S \) in equation (85) on page 23 of [1]. See also the final Remark in Section 5 of this paper.

If \( F = \mathbb{C} \) and \( f \) is analytic (has a positive radius of convergence) of compositional order \( n \) then clearly \( f^* \) is also analytic. Thus \( f \) is conjugate by an analytic function in a neighborhood of the origin to a rotation of order \( n \). (See Theorem 5 (p. 55) of [1], for a more general result.)

In the light of Theorem 1 one there are uncountably many different series of compositional order \( n \) which have a given primitive \( n \)'th root of unity \( \omega \) as lead coefficient — namely, the conjugates of \( \ell_\omega \). How much freedom is there in choosing the other coefficients?
Theorem 2 (The coefficients of finite order elements of $G[[z]]$).
If $n$ is a positive integer and $\omega$ is a primitive $n'$th root of unity in the field $F$ then for every infinite sequence $\{a_k\}_{1<k\neq n_j+1}$ of elements of $F$ there exists a unique sequence $\{a_{nj+1}\}_{j=1}^{\infty}$ such that the formal power series

$$f(z) = \omega z + \sum_{k=2}^{\infty} a_k z^k$$

has order $n$ in $G[[z]]$.

Corollary. If $F = \mathbb{C}$ and $2 \leq n \in \mathbb{N}$ then there exist uncountably many elements $f(z) \in G[[z]]$ of order $n$ such that the power series $f(z)$ does not converge for any non-zero $z \in \mathbb{C}$.

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2. The $n$th power of a series under composition

We do a number of elementary calculations based on the fact that if $g(z) = b_1 z + b_2 z^2 + \cdots$ then the multiplicative (as opposed to compositional) power $g(z)^k$ has as its lead term $b_1^k z^k$:

$$g(z)^k = (b_1 z + b_2 z^2 + \cdots)(b_1 z + b_2 z^2 + \cdots) \cdots (b_1 z + b_2 z^2 + \cdots) = b_1^k z^k + \text{higher powers.}$$

To start with, this combines with the definition of composition to give

**Lemma 2.1.** If $f(z) = a_1 z + a_2 z^2 + \cdots$, and $g(z) = b_1 z + b_2 z^2 + \cdots$ then $(f \circ g)(z) = (a_1 b_1) z + (a_1 b_2 + a_2 b_1^2) z^2 + (a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3) z^3 + \text{higher powers.} \quad \square$

**Notation:** We write $f^{(n)}(z) = \sum_{k=1}^{\infty} f_k^{(n)} z^k$. The following well known lemma can be proved by induction. We omit the proof since it is a corollary to Lemma 2.4 below. The fact that $F$ is a field of characteristic 0 is used.

**Lemma 2.2.** If $f(z) = z + a_k z^k + a_{k+1} z^{k+1} + \cdots$, with $a_k \neq 0$ and $n$ a positive integer then

$$f^{(n)}(z) = (z + a_k n z^k + \text{higher powers})$$

and $f$ has infinite order in $G$. \quad \square.

**Lemma 2.3.** If $f(z) = \omega z + a_2 z^2 + \cdots \in G$ has finite order under composition then the order of $f$ equals the order of $\omega \in F - \{0\}$. 


By Lemma 2.2, we see that $f^{(a)}(z)=\omega^a z + \text{higher powers} = z + \text{higher powers}.

\[ f^{(n)}(z) = z = (f^{(n)})^{(b)}(z). \]

By Lemma 2.2, we see that $f^{(a)}(z)=z$. Therefore $a = \text{order}(\omega) = n = \text{order}(f)$. □

Suppose that $f(z) = a_1 z + a_2 z^2 + \ldots \in G$. From the definition of $f^{(n)}$ and Lemma 2.1 we see that

\[ f_k^{(1)} = a_k, \quad f_1^{(n)} = a_1^n, \quad f_2^{(2)} = a_2 a_1 (1 + a_1) \quad f_3^{(2)} = a_3 a_1 (1 + a_1^2) + 2a_1 a_2^2 \]

More generally we have the following key lemma, which will be used in proving the uniqueness in Theorem 2.

**Lemma 2.4.** If $n, k$ are positive integers then there exists a polynomial $P^{(n)}_k(x_1, \ldots, x_{k-1})$ in $k$-1 variables, with integer coefficients, such that $P^{(n)}_k(x_1, 0, 0, \ldots, 0) = 0$ (i.e., each summand contains an $x_j$ with $j \neq 1$), and such that $P^{(n)}_k$ satisfies the following:

If $f(z) = (\omega z + a_2 z^2 + \cdots a_k z^k + \cdots) \in G[[z]]$ then

\[ f^{(n)}_k = a_k \omega^{n-1} \left( 1 + \omega^{k-1} + \cdots \omega^{(k-1)(n-1)} \right) + P^{(n)}_k(\omega, a_2, \ldots, a_{k-1}) \]

**Proof:** We proceed by induction on $n$

$n = 1 : \quad f^{(1)}_k = a_k = a_k \omega^0(1) \quad \text{and} \quad P^{(1)}_k = 0$. Suppose $n > 1$. For simplicity we introduce the further notation

\[ b_k = f^{(n-1)}_k \quad \text{(including} \quad b_1 = \omega^{n-1} \text{)} \]

\[ [Q(z)]_k = \text{coefficient of} \ z^k \text{in the polynomial} \ Q(z) \]

Note that $a_k \left[ (b_1 z)^k \right]_k = a_k \omega^{(n-1)k}$ but that, by induction, $b_i = f^{(n-1)}_i$ has no $a_k$’s in it if $1 \leq i < k$. Thus we have

\[ f^{(n)}(z) = f(f^{(n-1)}(z)) = f \left( \sum_{k=1}^{\infty} f^{(n-1)}_k z^k \right) = f \left( \sum_{k=1}^{\infty} b_k z^k \right) \]

\[ = \omega(b_1 z + b_2 z^2 + \cdots) + \cdots + a_j(b_1 z + b_2 z^2 + \cdots)^j + \cdots a_k(b_1 z + b_2 z^2 + \cdots)^k + \cdots \]
Considering how $z^k$ can occur in the power $(b_1z + b_2z^2 + \cdots)^j$, we get

$$f_k^{(n)} = \omega b_k + \cdots + a_j \left[ (b_1z + \cdots + b_{k-j+1}z^{k-j+1})^j \right] + \cdots + a_k \left[ (b_1z)^k \right]$$

where $P_k^{(n)}(a_1, \ldots, a_{k-1})$ is the result of summing all the terms which do not contain $a_k$. By induction on $P_k^{(n-1)}$ and definition of $P_k^{(n)}$, it follows that if $a_j = 0$ for all $j$ with $1 < j < k$ then $P_k^{(n)}(a_1, 0, \ldots, 0) = 0$. Moreover, since all computations involve taking integral powers, the coefficients of $P_k^{(n)}$ are integers. Since the computational process does not depend on the values of the $a_j$, the polynomial is independent of $f(z)$. □

3. Proof of The Conjugation Theorem

**Theorem 1A.** [Classification of finite order elements up to conjugation ]

Suppose that $f(z) = \omega z + \sum_{k=2}^{\infty} a_k z^k$ is an element of $G$ of order $n$. Define $f^*$ by

$$f^* = \frac{1}{n} \sum_{j=1}^{n} \omega^{n-j} f(j).$$

Then $f^* \in G$, $n$ is the multiplicative order of $\omega \in F - \{0\}$ and

$$f^* \circ f \circ f^* (z) = \omega z.$$

Two elements of order $n$ in $G$ are conjugate iff their lead coefficients are the same primitive $n'$th root of unity.

**Proof:** By Lemma 2.3 the order of $\omega$ is $n$. From the definition of composition we have, for all $g, h, k \in G$:

- $(g + h) \circ k = g \circ k + h \circ k$
- $\ell_\omega \circ (g + h) = \ell_\omega \circ g + \ell_\omega \circ h$
Then we have
\[
\ell_\omega \circ f^* = \frac{1}{n} \sum_{j=1}^{n} \omega^{n-j+1} f(j), \quad \text{and}
\]
\[
f^* \circ f = \frac{1}{n} \sum_{j=1}^{n} \omega^{n-j} f(j+1)
\]
Since \(\omega^n = 1\) and \(f^{(n)} = \text{id}\), these sums run through the same terms of \(G\). Hence
\[
f^* \circ f = f_\omega \circ f^* \quad \text{so that} \quad f^* \circ f \circ f^* = f_\omega.
\]
Finally we note from Lemma 2.1 that any conjugate \(g \circ f_\omega \circ g^{-1}(z) = \omega z + \text{higher terms}\). Hence \(f_\omega\) is conjugate in \(G_1\) to \(f_\omega\) if and only if \(\omega = \omega'\). So elements of finite order in \(G\) are conjugate iff their lead terms are the same. \(\square\)

**THEOREM 1B.** [Determination of the Conjugating Elements]

**NOTATION 3.1.** \(Z_\omega \overset{\text{def}}{=} \{ h \in G \mid h \circ \ell_\omega = \ell_\omega \circ h \} = \text{the centralizer of } \ell_\omega \text{ in } G\).

H. Furstenberg has pointed out the following fact:

**LEMMA 3.2.** If \(\omega\) is a primitive \(n\)'th root of unity and \(h = h(z) \in G[[z]]\) then
\[
h \in Z_\omega \iff h(z) = \sum_{j=0}^\infty h_{nj+1}z^{nj+1}.
\]
**Proof.**
\[
h \circ \ell_\omega(z) = \sum_{k=1}^\infty h_k(\omega z)^k \quad \text{and} \quad \ell_\omega \circ h(z) = \sum_{k=1}^\infty \omega h_k z^n.
\]
Therefore these are equal \(\iff \omega h_k = \omega^k h_k\) for all \(k \iff h_k = 0\) for all \(k \neq 1\ (n)\). \(\square\).

We have the following Corollary to Theorem 1.

**COROLLARY 3.3.** Suppose that \(f(z) = (\omega z + a_2 z^2 + a_3 z^3 \cdots)\) has finite order \(n\).

1. If \(g \in G\) then
\[
g \circ f \circ g^{-1} = \ell_\omega \iff \exists h(z) = \sum_{j=0}^\infty h_{nj+1}z^{nj+1} \in G \text{ such that } g = h \circ f^*.
\]

2. For every sequence \(\{g_{nj+1}\}_{j=0}^\infty\) of elements of \(F\) with \(g_1 \neq 0\) there exists a unique sequence \(\{g_k\}_{1<k\neq nj+1}\) such that the series \(g(z) = \sum_{k=1}^\infty g_k z^k\) satisfies
\[
g \circ f \circ g^{-1} = \ell_\omega.
\]
Proof: From Theorem 1,
\[ g \circ f \circ g = \ell \iff g \circ (f \circ \ell \circ f^*) \circ g = \ell \iff g \circ f^* \in Z. \]

Statement (1) now follows from Lemma 3.2.

We use (1) to prove (2): A sequence \( \{g_{n+1}\}_{j=0}^{\infty} \), with \( g_1 \neq 0 \), extends to a sequence \( \{g_k\}_{k=1}^{\infty} \) for which \( g(z) = \sum g_k z^k \) conjugates \( f \) to \( \ell \) iff \( g = h \circ f^* \) as in (1). We can solve for the missing coefficients of \( g \) and the coefficients of \( h \) recursively. If \( f^*(z) = b_1 z + b_2 z^2 + b_3 z^3 + \cdots \) (\( b_1 \) necessarily equals 1) then one equates the coefficients of the \( z^k \) in
\[ g_1 z + g_2 z^2 + \cdots + g_n z^{n+1} + \cdots = h_1(z + b_2 z^2 + b_3 z^3 + \cdots) + h_{n+1}(z + b_2 z^2 + b_3 z^3 + \cdots)^n + \cdots, \]
We leave the details to the reader. \( \square \)

4. Proof of the Construction Theorem for Elements of Order \( n \)

Theorem 2: [Determination of the coefficients of a formal power series of finite compositional order]

If \( n \) is a positive integer and \( \omega \) is a primitive \( n \)'th root of unity in the field \( F \) then for every infinite sequence \( \{a_k\}_{1 \leq k \leq n+1} \) of elements of \( F \) there exists a unique sequence \( \{a_{nj+1}\}_{j=1}^{\infty} \) such that the formal power series

\[ f(z) = \omega z + \sum_{k=2}^{\infty} a_k z^k \]

has order \( n \) in \( G \).

Proof of existence: Since \( \ell_\omega \) has order \( n \) in \( G \), any conjugate \( \overline{h} \ell_\omega h \) has order \( n \). Given a sequence \( \{a_k\}_{1 \leq k \leq n+1} \), we shall recursively construct a series \( h(z) = \sum_{k=1}^{\infty} h_k z^k \) and a sequence \( \{a_{nj+1}\}_{j=1}^{\infty} \) such that \( \overline{h} \ell_\omega h \) has the form of the desired \( f(z) \). Indeed we construct \( h \) and \( f \) simultaneously so that
\[ (h \circ f)(z) = (\ell \circ h)(z). \]
This is true precisely when
\[ (*) \quad h_1 \left( \omega z + a_2 z^2 + \cdots \right) + h_2 \left( \omega z + a_2 z^2 + \cdots \right)^2 + \cdots + h_k \left( \omega z + a_2 z^2 + \cdots \right)^k + \cdots \]
\[ = \quad \omega h_1 z + \omega h_2 z^2 + \cdots + \omega h_k z^k + \cdots \]
Set \( h_1 = 1 \). Suppose that \( h_m (m < k) \) and \( a_{nj+1} (nj + 1 < k) \) have been chosen so that the coefficients of \( z^m (m < k) \) on the two sides of equation \( (*) \) are equal. Consider the coefficient of \( z^k \) on both sides of the above equation:
\[ (**) \quad h_1 a_k + Q_k(h_i, a_l | 2 \leq i, \ell < k) + h_k \omega^k \]
where \( h_1 = 1 \) and \( Q_k \) is a polynomial in the given variables.
CASE 1: $k \not\equiv 1(n)$. Note that $a_k$ has been prespecified and $\omega^k \neq \omega$. Then we choose according to (***) (and have no choice in so doing):

$$h_k = \frac{a_k + Q_k}{\omega - \omega^k}.$$ 

CASE 2: $k \equiv 1(n)$. Note that $\omega^k = \omega$, $h_1 = 1$ and to satisfy (***) we choose

$$a_k = -Q_k = -Q_k(h_2, \ldots, h_{k-1}, a_1, \ldots a_{k-1})$$

We may choose $h_k$ arbitrarily. □

Remark:

- In the above proof, the value of $a_k$ which we are forced to choose when $k \equiv 1(n)$ seems to depend on the preceding freely chosen $h_{nj+1}$. In fact, we show that Lemma 2.4 implies that each $a_{nj+1}$ is uniquely determined by the preceding $a_k$'s:

Proof of uniqueness: If an infinite sequence $\{a_k\}_{1 < k \neq nj+1}$ is given and $f(z) = \omega z + \sum_{k=2}^{\infty} a_k z^k$ has order $n$ in $G$ then

- $f_k^{(a)} = 0$ for $k > 1$.
- $k \equiv 1(\text{mod } n) \implies \omega^{k-1} = 1$.

Hence, for $k > 1$ and $k \equiv 1(\text{mod } n)$, Lemma 2.4 gives

$$0 = a_k \omega^{n-1}n + P_k^{(n)}(a_1, \ldots, a_{k-1}).$$

Therefore

$$a_k = -\frac{\omega}{n} P_k^{(n)}(a_1, \ldots, a_{k-1}).$$ □

5. Further Comments

5.1. On formal power series of order two and of infinite order.

From Lemma 2.3 we see that, when $F = \mathbb{R}$, a real power series of finite order can only have order one or two and that, for any field of characteristic 0, a series in $G$ of order two is of the form $f(z) = -z + a_2 z^2 + a_3 z^3 + \cdots$. Note that if $f(z)$ has order two then $f^*(z) = \frac{1}{2} (z - f(z))$.

L. Shapiro has pointed out that the following is an example of Theorem 1.

Example 5.1 (Stanley [14], page 50, problem 41a)

Suppose that $f(z) \in G$.

Then $f(-f(-z)) = z \iff \exists g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ such that $f(z) = f(-g(z))$.

In our terms this just says that $f \circ \ell_{-1}$ has order two iff it is conjugate to $\ell_{-1}$.
5.2. Muckenhoupt’s results on formal series of infinite order.

From Lemma 2.3 we see that \( f(z) = (\omega z + a_2 z^2 + \cdots) \) has infinite order in \( G \) if \( \omega \) has infinite multiplicative order in \( F \). And from Theorem 2 we see that if \( \omega \) has finite order, it is still possible that \( f(z) \) has infinite order – one can choose any one of the \( a_{nj+1} \) in such a way that \( f(z) \) won’t have finite order. In these two situations, Muckenhoupt [8] has explained what happens (while implying that the following results were classically known):

**Theorem** (Muckenhoupt). Suppose that \( f(z) = (\omega z + a_2 z^2 + a_3 z^3 + \cdots) \in G \)

1. If \( \omega \) has infinite order then \( f \) is conjugate to \( \ell_\omega \).
2. If \( \omega \) has finite order \( n \) then \( f \) is conjugate to \( g \in G \), where \( g \) is of the form
   \[
   g(z) = \omega z + \sum_{j=1}^{\infty} g_{nj+1} z^{nj+1}.
   \]

**Remark:** Our first proof that formal power series of finite order under composition are conjugate to their linear parts used an inductive argument from Lemmas 2.3 and 2.4 above and Part (2) of Muckenhoupt’s theorem, without explaining the role of \( f^* \).

6. Useful References and Comments on the History of Formal Power Series

The subject of Formal Power Series is a deep classical subject of importance in Analysis, Combinatorics and Algebra. I hope the comments below will be helpful and will add some perspective, but there is no claim to completeness and I apologize to those contributors to the subject whom I omit.

6.1. Introductory surveys. Excellent introductions to formal power series and surveys of the field are given by

1. Ivan Niven [9], (1969) – this article won the MAA’s Lester R. Ford Award for expository excellence.
2. Henri Cartan [4], Section I.1 – originally published 1961.
3. Stephen Scheinberg [12], (1970)
4. The volumes [15], [16] of Richard P. Stanley (1997, 1999).
5. Thomas S. Brewer [2], (2014).
6. Anthony G. Farrell and Ian Short [10], Ch. 10 (2015).

6.2. E. Schröder’s foundational 1871 paper.

Ernst Schröder’s foundational 1871 paper [13], “Über Iterate Functionen” (“On Iterated Functions”) set the stage for much to come. In his opening he states (translated)

“I present here investigations into a field in which I have seen very little previous work.”
This rich paper formally introduces the concept of the $r^{th}$ iterate of a function $F(z)$ and, in particular, contains the following two items which relate directly to the work above; namely, to Lemma 2.4 on the determination of $f_k^{(n)}$ and to the conjugation problem in Theorem 1.

1. §6, page 310: **Coefficients of $F^{(r)}(z)$ when $F(z) = z + a_2 z^2 + \ldots$ is a MacLaurin series.**

   Between pages 310 and 315, this gives a remarkably deep and detailed calculation of the $k^{th}$ coefficient of $F_k^{(r)}$ (which Schröder denotes $F_k^r(z)$). Since he does not consider convergence questions until after he develops these formulas, the development of his formulas really takes place within the realm of formal power series over $\mathbb{C}$. I conjecture that there is much to be gained by those who can wield his formulas in particular cases.

2. §3(D), page 303: Schröder assumes given a function $F(\zeta)$ of the variable $\zeta$. He points out how useful it would be in evaluating the $r^{th}$ iterate $F(F(\cdots(\zeta)\cdots))$, if there existed a constant $m$ and a change-of-coordinates function $\psi$ such that

   $$\psi(m\zeta) = F\psi(\zeta)$$

   ("Schröder’s Equation")

   In our notation, if $\psi$ is an invertible function, we would then have $F = \psi \circ \ell_m \circ \psi^{-1}$ and it becomes possible to easily compute

   $$F^{(r)}(\zeta) = \psi(m^r \cdot \psi^{-1}(\zeta)).$$

   The solution of Schröder’s equation has been sought in many different settings and formulation of the Conjugacy Problem is often phrased in terms of “solving Schröder’s equation”.

6.3. Conjugacy of Formal Power Series.

1. **Great Historical Papers Can Be Used Incorrectly.**

   The algebraic subject of Formal Power Series over $\mathbb{C}$ is so deeply intertwined with the subject of MacLaurin series in Analysis, that one can easily misapply a historical result from one of these subjects incorrectly in the other subject.

   For example, the following mistakes are possible:

   a. Suppose given a MacLaurin series $F(z) = a_1 z + a_2 z^2 + \ldots$ with positive radius of convergence. Seeking an analytic function which linearizes a curvilinear angle, one might formally find a conjugator $\psi(\zeta)$ satisfying Schröder’s equation which is not analytic (does not converge in any neighborhood of the origin) and mistakenly think that they’ve found an analytic solution. In 1917, Pfeiffer [11] pointed out this problem by giving an analytic $F(z)$, where $a_1$ has infinite order, such that there exists no analytic solution to Schröder’s equation.
It is noteworthy that Pfeiffer does prove (page 186 of [11]) that, when a formal power series $F(z)$ has $a_1$ of infinite order, a unique formal solution to Schröder’s equation always exists.

(b) A person wishing to prove Theorem 1A, given a formal power series of finite order, $F(z) = a_1z + a_2z^2 + \ldots$, might wish to use a paper in analysis – such as Siegel’s famous paper [14] – to show that this can be conjugated to a linear function. However, Siegel’s beautiful argument on the opening page of [14] assumes that the function being conjugated is analytic in a neighborhood of the origin and is stable (a valid assumption for analytic functions of finite compositional order). He then finds the conjugator with a mixture of topology (the universal covering space) and analysis (Schwartz’ lemma). However, this analyticity assumption is invalid in proving Theorem 1A, since we have proven in the Corollary to Theorem 2 that not every formal power series of finite order is analytic.

But note: Despite these warnings about the possible misuse of the historical documents,

**the beauty and insights of the early papers is not to be bypassed.**

And often it happens that, inside an argument in one field there will be a gem which can be of great use in the other. For example, on page 42, line 3, of [3], in the midst of an analysis argument which assumes that “the sequence of iterates $\{f^n\}$ is uniformly bounded in some neighborhood of the origin” the formula for the sum which we call $f^*$ in Theorem 1A appears. If $f$ is an analytic function of finite order, the conjugating function $\varphi$ which that proof gives is in fact $\varphi = f^*$. Seeing this one could conjecture that $f^*$ is also the conjugating element in the group of formal power series, and come up with an algebraic proof.

One cannot help but conjecture that the great mathematicians of the past, having the proof of the conjugacy-to-linear-term in the analytic setting, knew from this a proof in the formal power series setting. Indeed, the 1948 proof of Bochner and Martin [1] mentioned above notably avoids compactness considerations in the complex plane by putting “the ordinary weak topology based on convergence for each coefficient separately” on the set of formal power series directly, and proving a theorem (Theorem 8) about bounded groups of formal power series. Clearly any formal power series of finite order generates a finite, hence bounded, group of formal power series.

(2) **Scheinberg’s Table of Normal Forms of Conjugation**

Scheinberg [12] gives a very rich discussion of formal power series as of 1970, with interesting historical detail. He includes at the end of §3 with Table 1, a table of canonical forms of formal power series under conjugation, including, for example, Muckenhoupt’s Theorem above.
(3) Conjugating Formal Power Series of Finite Order

Summarizing much of what has been said above, the fact (Theorem 1 above) that a formal power series
\[ f(z) = \omega z + \sum_{k=2}^{\infty} a_k z^k \]
of compositional order \( n \) is conjugate in the group \( G \) of formal power series to \( \ell_\omega(z) = \omega z \) is very well known. It has appeared at least in [1], [2], [5], [10] and [12].

The fact that \( f^* \) of Theorem 1 may be used as the conjugator appears in [1], [5] and [10], and has been known at least since Bochner and Martin’s 1948 book [1].

Historical challenge:

Check out all of the references given in Peiffer’s 1917 paper [11] and see whether the knowledge of this result goes back to the early twentieth or even the nineteenth century.

6.4. The Construction of Formal Power Series of Finite Order.

Independent proofs of Theorem 2 above were given by the author and Thomas S. Brewer [2]. We do not know (May, 2018) of other proofs, but in this venerable field, we would hesitate to make brash claims.

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