Relating the neutrino mixing angles to a lepton mass hierarchy

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Abstract

We propose two phenomenological scenarios of lepton mass matrices and show that either of them can exactly give rise to $\tan^2 \theta_{13} = m_e/(m_e + 2m_\mu)$, $\tan^2 \theta_{23} = m_\mu/(m_e + m_\mu)$ and $\tan^2 \theta_{12} = (m_e m_2 + 2m_\mu m_1)/(m_e m_1 + 2m_\mu m_2)$ in the standard parametrization of lepton flavor mixing. The third relation, together with current experimental data, predicts a normal but weak hierarchy for the neutrino mass spectrum. We also obtain $\theta_{13} \approx 2.8^\circ$ for the smallest neutrino mixing angle and $\mathcal{J} \approx 1.1\%$ for the Jarlskog invariant of leptonic CP violation, which will soon be tested in the long-baseline reactor and accelerator neutrino oscillation experiments. A seesaw realization of both scenarios is briefly discussed.

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I. INTRODUCTION

In the standard parametrization of the Cabibbo-Kobayashi-Maskawa (CKM) quark flavor mixing matrix \( V \) [1] a hierarchy exists among its three mixing angles: \( \vartheta_{12} \sim \lambda, \vartheta_{23} \sim \lambda^2 \), and \( \vartheta_{13} \sim \lambda^4 \) with \( \lambda \approx 0.22 \). This hierarchy might intrinsically be related to the hierarchy of the quark masses: \( m_u/m_c \sim m_c/m_t \sim \lambda^4 \) and \( m_d/m_s \sim m_s/m_b \sim \lambda^2 \), since the CKM matrix \( V \) actually measures a mismatch between the mass and flavor eigenstates of both up- and down-type quarks. For the Maki-Nakagawa-Sakata-Ponte corvo (MNSP) lepton flavor mixing matrix \( U \) [2], two of its three mixing angles are found to be unexpectedly large: \( \vartheta_{12} \sim 34^\circ \) and \( \vartheta_{23} \sim 45^\circ \). This phenomenon seems incompatible with the mass hierarchy of the three charged leptons, \( m_e/m_\mu \sim \lambda^4/2 \) and \( m_\mu/m_\tau \sim 4\lambda^2/3 \) [1]. One might ascribe the largeness of \( \vartheta_{12} \) and \( \vartheta_{23} \) to a very weak hierarchy of the three neutrino masses (e.g., \( m_1 \sim m_2 \sim m_3 \) [3]). Another possibility of understanding the observed pattern of neutrino mixing is to invoke a certain flavor symmetry such that all three mixing angles are pure numbers in the symmetry limit and their weak dependence on \( m_\alpha \) (for \( \alpha = e, \mu, \tau \)) and \( m_i \) (for \( i = 1, 2, 3 \)) could result from the details of the symmetry breaking [4]. One may also speculate that one mixing angle is purely determined by an underlying flavor symmetry, while the other two are more or less associated with \( m_\alpha \) and \( m_i \), or vice versa.

In Refs. [3,5], we recommended a useful parametrization of the MNSP matrix \( U \):

\[
U = \begin{pmatrix}
  c_l & s_l & 0 \\
  -s_l & c_l & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  e^{-i\phi} & 0 & 0 \\
  0 & c & s \\
  0 & -s & c
\end{pmatrix}
\begin{pmatrix}
  c_\nu & -s_\nu & 0 \\
  s_\nu & c_\nu & 0 \\
  0 & 0 & 1
\end{pmatrix} P
\]

\[
= \begin{pmatrix}
  c_\nu s_\nu c + c_l c_\mu e^{-i\phi} & s_\nu c_\nu c - c_l s_\mu e^{-i\phi} & s_l s \\
  c_\nu s_\nu c - s_l c_\mu e^{-i\phi} & c_\nu c_\nu c + s_l s_\mu e^{-i\phi} & c \\
  -s_\nu s & c_\nu s & c
\end{pmatrix} P
\]

(1)

with \( c_{\ell\nu} \equiv \cos \theta_{\ell\nu}, s_{\ell\nu} \equiv \sin \theta_{\ell\nu}, c \equiv \cos \theta \) and \( s \equiv \sin \theta \). \( P = \text{Diag} \{ e^{i\rho}, e^{i\sigma}, 1 \} \) is the Majorana phase matrix. We conjectured that two lepton mixing angles could be related to two lepton mass ratios in the following way [5]:

\[
\tan \theta_l = \sqrt{\frac{m_e}{m_\mu}}, \quad \tan \theta_\nu = \sqrt{\frac{m_1}{m_2}}.
\]

(2)

This conjecture is reasonable because \( \theta_l \) and \( \theta_\nu \) describe the \( e-\mu \) mixing in the charged-lepton sector and the \( \nu_e-\nu_\mu \) mixing in the neutrino sector respectively. Hence they can naturally vanish in the limit \( m_e \to 0 \) and \( m_1 \to 0 \), consistent with the experimental fact that \( m_e \ll m_\mu \) and \( m_1 < m_2 \) hold. Note that the three mixing angles of \( U \) in the standard parametrization (i.e., \( \vartheta_{12}, \vartheta_{13} \) and \( \vartheta_{23} \) [1]) are approximately related to those in Eq. (1):

\[
\vartheta_{12} \approx \theta_\nu, \quad \vartheta_{23} \approx \theta, \quad \vartheta_{13} \approx \theta \sin \theta,
\]

(3)

since the solar and atmospheric neutrino oscillations can approximately be decoupled from each other, Eq. (2) indicates that the smallness of \( \vartheta_{13} \) is attributed to a very strong charged-lepton mass hierarchy (i.e., \( m_e \ll m_\mu \)). The large value of \( \vartheta_{12} \) is related to a relatively weak neutrino mass hierarchy (i.e., \( m_1 < m_2 \)).
In this paper we propose two simple scenarios of the charged-lepton and neutrino mass matrices from which the phenomenological conjecture made in Eq. (2) can exactly be derived. We show that both scenarios can accommodate $\theta = \pi/4$ and $\phi = \pi/2$ in our parametrization of $U$, leading to the predictions $\tan^2 \theta_{13} = m_e/(m_e + 2m_\mu)$, $\tan^2 \theta_{23} = m_\mu/(m_e + m_\mu)$ and $\tan^2 \theta_{12} = (m_e m_2 + 2m_\mu m_1)/(m_e m_1 + 2m_\mu m_2)$ in the standard parametrization of $U$. We find that the third relation, together with current experimental data, predicts a normal but weak hierarchy for the neutrino mass spectrum: $m_1 \approx 4.28 \times 10^{-3}$ eV, $m_2 \approx 9.74 \times 10^{-3}$ eV and $m_3 \approx 4.92 \times 10^{-2}$ eV. Furthermore, we obtain $\theta_{13} \approx 2.8^\circ$ for the smallest neutrino mixing angle and $J \approx 1.1\%$ for the Jarlskog invariant of leptonic CP violation, which will soon be tested in the long-baseline reactor and accelerator neutrino oscillation experiments. We discuss a simple seesaw realization of these two scenarios.

II. TWO SCENARIOS

Our main goal is to reproduce the relations conjectured in Eq. (2) and to predict $\theta = \pi/4$ for the MNSP matrix $U$. We find that there are two simple possibilities of realizing this goal. Their consequences on neutrino mixing and CP violation are exactly the same.

Scenario (A)

Let us consider the following textures of charged-lepton and neutrino mass matrices at the electroweak scale (i.e., $\Lambda = M_Z$):

\[
M_l = \begin{pmatrix}
0 & A & 0 \\
A^* & B & 0 \\
0 & 0 & C
\end{pmatrix}, \quad M_\nu = \begin{pmatrix}
0 & X & -X \\
X & Y & Z \\
-X & Z & Y
\end{pmatrix}.
\]  

One may easily diagonalize the Hermitian matrix $M_l$ by using the unitary transformation $O_l^\dagger M_l O_l^{\ast} = \text{Diag}\{m_e, m_\mu, m_\tau\}$, where

\[
O_l = \begin{pmatrix}
e^{i\phi_l} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
c_l & -s_l & 0 \\
s_l & c_l & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
O_l' = \begin{pmatrix}
e^{i\phi'_l} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
c_l & s_l & 0 \\
-s_l & c_l & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

with $\phi_l \equiv \text{arg}(A) - \pi$, $\phi'_l \equiv \text{arg}(A)$ and $\tan \theta_l = \sqrt{m_e/m_\mu}$. We have $|A| = \sqrt{m_e m_\mu}$, $B = m_\mu - m_e$ and $C = m_\tau$. Because $M_\nu$ is symmetric, it can be diagonalized by the transformation $O_\nu^\dagger M_\nu O_\nu^{\ast} = \text{Diag}\{m_1, m_2, m_3\}$, where

\[
O_\nu = \begin{pmatrix}
e^{i\phi_\nu} & 0 & 0 \\
0 & c & s \\
0 & -s & c
\end{pmatrix} \begin{pmatrix}
c_\nu & -s_\nu & 0 \\
s_\nu & c_\nu & 0 \\
0 & 0 & 1
\end{pmatrix} P,
\]  

3
with $\phi_\nu \equiv \arg(X) - \pi$, $\theta = \pi/4$, $\tan \theta_\nu = \sqrt{m_1/m_2}$ and $P = \text{Diag}\{i, 1, 1\}$. We have $|X| = \sqrt{m_1 m_2/2}$, $Y = (m_3 + m_2 - m_1)/2$ and $Z = (m_3 - m_2 + m_1)/2$.

With the help of Eqs. (5) and (6), we are able to calculate the MNSP matrix $U = O_l^l O_\nu$ with the definition of $\phi \equiv \phi_l - \phi_\nu = \arg(A) - \arg(X)$. Then the expression of $U$, which is compatible with our parametrization in Eq. (1), can explicitly be given as

$$U = \begin{pmatrix} \sqrt{m_\mu/m_\mu + m_\mu} & \sqrt{m_\mu/m_\mu + m_\mu} & 0 \\ -\sqrt{m_\mu/m_\mu + m_\mu} & \sqrt{m_\mu/m_\mu + m_\mu} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & e^{i\pi/2} & e^{i\pi/2} \\ 0 & -e^{i\pi/2} & e^{i\pi/2} \end{pmatrix} \begin{pmatrix} \sqrt{m_\mu/m_\mu + m_\mu} & -\sqrt{m_\mu/m_\mu + m_\mu} & 0 \\ \sqrt{m_\mu/m_\mu + m_\mu} & \sqrt{m_\mu/m_\mu + m_\mu} & 0 \\ 0 & 0 & 1 \end{pmatrix} P \ (7)$$

We assume $\phi = \pi/2$ for simplicity. This assumption is consistent with the one made in the quark sector, where the counterpart of $\phi$ is denoted as $\varphi$ [6]. Indeed, $\varphi = \pi/2$ is strongly favored by current experimental data on the CKM unitarity triangle simply because $\varphi - \alpha \approx 1.1^\circ$ holds [7,8] and $\alpha = 89.0^{+4.4}_{-4.2}^\circ$ has been measured [9]). Comparing Eq. (7) with the standard parametrization of the MNSP matrix [1],

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} P \ (8)$$

with $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$ (for $ij = 12, 13, 23$) and $P = \text{Diag}\{e^{i\rho}, e^{i\sigma}, 1\}$, we immediately arrive at the predictions

$$\tan^2 \theta_{12} = \frac{m_\mu m_2 + 2m_\mu m_1}{m_\mu m_1 + 2m_\mu m_2}, \quad \tan^2 \theta_{13} = \frac{m_\mu}{m_\mu + 2m_\mu}, \quad \tan^2 \theta_{23} = \frac{m_\mu}{m_\mu + m_\mu}, \quad (9)$$

together with $\rho = \pi/2, \sigma = 0$ and

$$\sin^2 \delta = \frac{m_1 m_2 (m_\mu + 2m_\mu)^2}{(m_\mu m_1 + 2m_\mu m_2)(m_\mu m_2 + 2m_\mu m_1)}. \quad (10)$$

We have used $\phi = \pi/2$ in obtaining the results of $\tan^2 \theta_{12}$ and $\sin^2 \delta$. Given the strong mass hierarchy $m_\mu \ll m_\mu$, Eqs. (9) and (10) can be written as

$$\tan \theta_{12} \approx \sqrt{\frac{m_1}{m_2}} \left(1 + \frac{m_\mu m_2^2 - m_\mu^2}{4m_\mu m_2} \right), \quad \tan \theta_{23} \approx 1 - \frac{1}{2} \frac{m_\mu}{m_\mu}, \quad (11)$$

Some comments on these results are in order.
(1) The neutrino mass ratio $m_1/m_2$ can be determined from the experimental values of $m_e/m_\mu$ and $\tan^2 \theta_{12}$. Eq. (9) yields

$$m_1 = \frac{2m_\mu \tan^2 \theta_{12} - m_e}{2m_\mu - m_e \tan^2 \theta_{12}} \approx \tan^2 \theta_{12} \left( 1 - \frac{2m_e \cos 2\theta_{12}}{m_\mu \sin^2 2\theta_{12}} \right).$$

(12)

Because of $m_e = 0.486570161 \pm 0.000000042$ MeV and $m_\mu = 102.7181359 \pm 0.0000092$ MeV at the scale $\Lambda = M_Z$ [10,11], we get $m_e/m_\mu \approx 0.004737$, which is far below the error bar of $\tan^2 \theta_{12}$ extracted from the present experimental data [12]. Hence it is good enough to use the leading-order relation $m_1/m_2 \approx \tan^2 \theta_{12}$ in our numerical calculation. Taking the best-fit value $\sin^2 \theta_{12} = 0.304$ [12] for example, we obtain $m_1/m_2 \approx 0.44$. This result, together with the best-fit values $\Delta m^2_2 = m^2_2 - m^2_1 = 7.65 \times 10^{-5}$ eV$^2$ and $|\Delta m^2_3| \equiv |m^2_3 - m^2_1| = 2.40 \times 10^{-3}$ eV$^2$ obtained from a global analysis of current neutrino oscillation data [12], allows us to determine the neutrino mass spectrum:

$$m_1 \approx 4.28 \times 10^{-3} \text{ eV} , \quad m_2 \approx 9.74 \times 10^{-3} \text{ eV} , \quad m_3 \approx 4.92 \times 10^{-2} \text{ eV} ,$$

(13)

which exhibits a normal but weak hierarchy.

(2) The value of $\theta_{13}$ and the deviation of $\theta_{23}$ from $\theta = \pi/4$ are both determined by the ratio $m_e/m_\mu$, while the deviation of $\delta$ from $\phi = \pi/2$ depends both on $m_e/m_\mu$ and $m_1/m_2$ in this ansatz:

$$\theta_{13} \approx 2.8^\circ , \quad \theta_{23} \approx 44.9^\circ , \quad \delta \approx 87.6^\circ .$$

(14)

In obtaining the value of $\delta$, we have used the values of $m_1$ and $m_2$ in Eq. (13).

(3) The Jarlskog invariant of leptonic CP violation [13] turns out to be

$$\mathcal{J} = c_3 s_1 c_\nu s_\nu c^2 \sin \phi = c_{12} s_{12} c_{13} s_{13} c_{23} s_{23} \sin \delta = \frac{1}{2\sqrt{2}} \frac{\sqrt{m_e m_\mu m_1 m_2}}{(m_e + m_\mu)(m_1 + m_2)} .$$

(15)

Taking the values of $m_1$ and $m_2$ given in Eq. (13), we obtain $\mathcal{J} \approx 1.1\%$. This amount of CP violation can be observed in the future long-baseline neutrino oscillation experiments [14], provided the terrestrial matter effects do not mimic it [15].

With the help of Eq. (13) together with $m_\tau = 1746.24^{+0.20}_{-0.19}$ MeV at $\Lambda = M_Z$ [10], we find the following textures of $M_t$ and $M_\nu$:

$$M_t \approx 1.00 m_\tau \begin{pmatrix} 0 & -0.0040 e^{i\phi_t} & 0 \\ -0.0040 e^{-i\phi_t} & 0.058 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(16)

$$M_\nu \approx 0.56 m_3 \begin{pmatrix} 0 & -0.17 e^{i\phi_\nu} & 0.17 e^{i\phi_\nu} \\ -0.17 e^{i\phi_\nu} & 1 & 0.80 \\ 0.17 e^{i\phi_\nu} & 0.80 & 1 \end{pmatrix},$$

where the phases $\phi_t$ and $\phi_\nu$ have been defined in Eqs. (5) and (6), respectively. Eq. (16) clearly shows the weak hierarchy of $M_\nu$ in contrast with the strong hierarchy of $M_t$. It is therefore natural that two large flavor mixing angles $\theta_{12}$ and $\theta_{23}$ stem from $M_\nu$, while the smallest flavor mixing angle $\theta_{13}$ comes from $M_t$. 

5
Scenario (B)

Now we consider the following ansatz of lepton mass matrices, whose textures are essentially the interchanged ones of $M_l$ and $M_\nu$ given in Eq. (4), at the electroweak scale:

$$\tilde{M}_l = \begin{pmatrix} 0 & \tilde{A} & -\tilde{A} \\ \tilde{A}^* & \tilde{B} & \tilde{C} \\ -\tilde{A}^* & \tilde{C} & \tilde{B} \end{pmatrix}, \quad \tilde{M}_\nu = \begin{pmatrix} 0 & \tilde{X} & 0 \\ \tilde{X} & \tilde{Y} & 0 \\ 0 & 0 & \tilde{Z} \end{pmatrix}. \quad (17)$$

The Hermitian matrix $\tilde{M}_l$ is diagonalized by using the unitary transformation $\hat{O}_l^\dagger \tilde{M}_l \hat{O}_l = \text{Diag}\{m_e, m_\mu, m_\tau\}$, where

$$\hat{O}_l = \begin{pmatrix} e^{i\phi_l} & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}, \quad \hat{O}_\nu = \begin{pmatrix} e^{i\phi_\nu} & 0 & 0 \\ 0 & c & s \\ 0 & s & c \end{pmatrix},$$

with $\phi_l \equiv \arg(\tilde{A}) - \pi$, $\phi_\nu \equiv \arg(\tilde{X})$, $\theta = -\pi/4$ and $\tan \theta_l = \sqrt{m_e/m_\mu}$. In addition, we have $|\tilde{A}| = \sqrt{m_\tau m_\mu/2}$, $\tilde{B} = (m_\tau + m_\mu - m_e)/2$ and $\tilde{C} = (m_\tau - m_\mu + m_e)/2$. Since $\tilde{M}_\nu$ is symmetric, it can be diagonalized by means of the transformation $\hat{O}_\nu^\dagger \tilde{M}_\nu \hat{O}_\nu^* = \text{Diag}\{m_1, m_2, m_3\}$, where

$$\hat{O}_\nu = \begin{pmatrix} e^{i\phi_\nu} & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}, \quad P,$$

with $\phi_\nu \equiv \arg(\tilde{X}) - \pi$, $\tan \theta_\nu = \sqrt{m_1/m_2}$ and $P = \text{Diag}\{i, 1, 1\}$. Furthermore, we have $|\tilde{X}| = \sqrt{m_1 m_2}$, $\tilde{Y} = m_2 - m_1$ and $\tilde{Z} = m_3$.

We are then able to calculate the MNSP matrix $\hat{U} = \hat{O}_l^\dagger \hat{O}_\nu$ with the definition of $\phi \equiv \phi_l - \phi_\nu = \arg(\tilde{A}) - \arg(\tilde{X})$. The result of $\hat{U}$, after a redefinition of the phases of the $\tau$ and $\nu_3$ fields (i.e., $\tau \to -\tau$ and $\nu_3 \to -\nu_3$ in order to absorb the minus sign coming from $\theta = -\pi/4$), is $\hat{U} = U \hat{P}$, where $U$ has been given in Eq. (7) and $\hat{P} = \text{Diag}\{i, 1, -1\}$ holds. Given $\phi = \pi/2$, the predictions of the present scenario for $\theta_{12}, \theta_{13}, \theta_{23}$ and $\delta$ in the standard parametrization are exactly the same as those obtained in Eqs. (9) and (10). Hence the scenarios of lepton mass matrices proposed in Eqs. (4) and (17) are phenomenologically identical and indistinguishable.

To illustrate the textures of $\tilde{M}_l$ and $\tilde{M}_\nu$, we adopt the values of three neutrino masses in Eq. (13) and the central values of three charged-lepton masses at $\Lambda = M_Z$. We obtain

$$\tilde{M}_l \approx 0.53 m_\tau \begin{pmatrix} 0 & -0.0054 e^{i\phi_l} & 0.0054 e^{i\phi_l} \\ -0.0054 e^{-i\phi_l} & 1 & 0.89 \\ 0.0054 e^{-i\phi_l} & 0.89 & 1 \end{pmatrix},$$

$$\tilde{M}_\nu \approx 1.00 m_3 \begin{pmatrix} 0 & -0.13 e^{i\phi_\nu} \\ -0.13 e^{i\phi_\nu} & 0.11 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (20)$$
where the phases $\phi_l$ and $\phi_\nu$ have been defined in Eqs. (5) and (6), respectively. A comparison between Eqs. (16) and (20) shows the slight changes of the structural hierarchies in both the charged-lepton and neutrino sectors.

III. SEESAW MECHANISM

The simple neutrino mass texture discussed above implies that it can be derived from the canonical seesaw mechanism [16], which attributes the small masses of the three known neutrinos to the existence of some heavy degrees of freedom far above the electroweak scale $\Lambda = M_Z$. Introducing three right-handed neutrinos into the standard model, one can write down the gauge-invariant Lagrangian associated with the lepton masses. After spontaneous gauge symmetry breaking the resultant lepton mass terms are

$$-\mathcal{L}_{\text{lepton}} = \overline{l}_L M_l l_R + \overline{\nu}_L M_D N_R + \frac{1}{2} \overline{N}_R M_R N_R + \text{h.c.}, \quad (21)$$

where $M_l$ and $M_D$ stand respectively for the mass matrices of charged leptons and Dirac neutrinos, while $M_R$ is a symmetric Majorana mass matrix of right-handed neutrinos. As the Majorana mass term is not subject to the electroweak symmetry breaking, the scale of $M_R$ can be much higher than those of $M_l$ and $M_D$ characterized by $M_Z$. Hence the effective mass matrix of the three active neutrinos is given by the well-known seesaw relation [16]

$$M_\nu \approx M_D M_R^{-1} M_D^T \quad (22)$$

to a good degree of accuracy. Although the small mass scale of $M_\nu$ is qualitatively ascribed to the smallness of $M_D/M_R$ in Eq. (22), this seesaw picture itself has no quantitative predictability because the flavor structures of $M_D$ and $M_R$ are entirely unspecified. To generate large flavor mixing angles in the neutrino sector, extra assumptions must be made so as to specify the textures of $M_D$ and $M_R$ in the seesaw framework. Instead of going into any details of model building, here we give some brief discussions about the seesaw-invariant property of neutrino mass matrices for our scenarios.

We first discuss a seesaw realization of the texture of $M_\nu$ proposed in Eq. (4). We assume that $M_D$ and $M_R$ have a similar texture which might result from a certain (underlying) flavor symmetry:

$$M_D = \begin{pmatrix} 0 & x & -x \\ x & y & z \\ -x & z & y \end{pmatrix}, \quad M_R = \begin{pmatrix} 0 & X & -X \\ X & Y & Z \\ -X & Z & Y \end{pmatrix}, \quad (23)$$

With the help of Eq. (22), one can easily show that $M_\nu$ takes the same texture as $M_D$ and $M_R$ [17]. Thus it is consistent with scenario (A). The matrix elements of $M_\nu$ turn out to be

$$X = \frac{x^2}{X},$$

$$Y = \frac{(y + z)^2}{2(Y + Z)} + \frac{x(y - z)}{X} - \frac{x^2(Y - Z)}{2X^2},$$

$$Z = \frac{(y + z)^2}{2(Y + Z)} - \frac{x(y - z)}{X} + \frac{x^2(Y - Z)}{2X^2}. \quad (24)$$
Such a seesaw-invariant texture is interesting since its matrix elements can be expressed in terms of the corresponding mass eigenvalues.

Now we examine whether the following texture of $M_D$ and $M_R$ is seesaw-invariant for scenario (B):

$$
\tilde{M}_D = \begin{pmatrix}
0 & \tilde{x} & 0 \\
\tilde{x} & \tilde{y} & 0 \\
0 & 0 & \tilde{z}
\end{pmatrix},
\quad
\tilde{M}_R = \begin{pmatrix}
0 & \tilde{X} & 0 \\
\tilde{X} & \tilde{Y} & 0 \\
0 & 0 & \tilde{Z}
\end{pmatrix}.
$$

By using the seesaw formula $\tilde{M}_\nu \approx \tilde{M}_D \tilde{M}_R^{-1} \tilde{M}_D^T$, we find that $\tilde{M}_\nu$ has the same texture which is consistent with scenario (B) proposed in Eq. (17) [18]. The non-vanishing matrix elements of $\tilde{M}_\nu$ are given as

$$
\tilde{X} = \frac{\tilde{x}^2}{X},
\quad
\tilde{Y} = \frac{2\tilde{x}\tilde{y}}{X} - \frac{\tilde{x}^2\tilde{Y}}{X^2},
\quad
\tilde{Z} = \frac{\tilde{z}^2}{Z}.
$$

One may express the matrix elements of $\tilde{M}_D$, $\tilde{M}_R$ or $\tilde{M}_\nu$ in terms of the corresponding mass eigenvalues, but there remain some unknown degrees of freedom (e.g., the phases in each mass matrix).

We conclude with two brief comments on scenarios (A) and (B).

1) The texture of $M_D$, $M_R$ and $M_\nu$ in scenario (A) has the following $Z_2$-like symmetry:

$$
TM_D T = M_D,
\quad
TM_R T = M_R,
\quad
TM_\nu T = M_\nu,
$$

where

$$
T = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
$$

It is therefore possible to obtain such a texture by introducing several Higgs doublets into the Lagrangian and imposing a global symmetry like the one in Eq. (27) on the neutrino mass terms [19,20]. An implementation of the global $U(1) \times Z_2$ symmetry may allow one to obtain the texture of $M_D$, $M_R$ and $M_\nu$ in scenario (B) [21].

2) The seesaw scale $\Lambda_{SS}$ is usually much higher than the electroweak scale $M_Z$, but the texture of $M_\nu$ (or $\tilde{M}_\nu$) is expected to be essentially stable against radiative corrections from $\Lambda_{SS}$ down to $M_Z$ [22]. Because the neutrino mass spectrum has a normal hierarchy in either scenario (A) or scenario (B), radiative corrections to flavor mixing angles and CP-violating phases are actually negligible in most cases [22].

**IV. SUMMARY**

We have proposed two simple scenarios of the lepton mass matrices from which the interesting relations $\tan \theta_l = \sqrt{m_e/m_\mu}$ and $\tan \theta_\nu = \sqrt{m_1/m_2}$ can be derived \(^1\). Both

\(^{1}\text{Note that the approximate relations of this type can also be derived from the four-zero textures of lepton (or quark) mass matrices [23].}\)
scenarios can be realized in a seesaw framework. They predict $\tan^2 \theta_{13} = m_e/(m_e + 2m_\mu)$, $\tan^2 \theta_{23} = m_\mu/(m_e + m_\mu)$ and $\tan^2 \theta_{12} = (m_e m_2 + 2m_\mu m_1)/(m_e m_1 + 2m_\mu m_2)$ in the standard parametrization of the MNSP matrix. We find that the third relation, together with current experimental data, allows us to obtain a normal but weak hierarchy for the neutrino mass spectrum: $m_1 \approx 4.28 \times 10^{-3}$ eV, $m_2 \approx 9.74 \times 10^{-3}$ eV and $m_3 \approx 4.92 \times 10^{-2}$ eV. Our predictions $\theta_{13} \approx 2.8^\circ$ for the smallest neutrino mixing angle and $J \approx 1.1\%$ for the Jarlskog invariant of CP violation will soon be tested in the long-baseline accelerator and reactor neutrino oscillation experiments [14], such as the Daya Bay experiment [24].

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