STOCHASTIC NAVIER-STOKES EQUATION ON A 2D ROTATING SPHERE WITH STABLE LÉVY NOISE: EXISTENCE AND UNIQUENESS OF WEAK AND STRONG SOLUTIONS

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ABSTRACT. In this note we prove the existence and uniqueness of weak and strong solution (in PDE sense) to the stochastic Navier-Stokes equations on the rotating 2-dimensional unit sphere perturbed by stable Lévy noise.

1. Introduction

The deterministic Navier-Stokes system (NSE) on the rotating sphere serves as a basic model in large scale Ocean dynamics. Many authors have studied the NSE on the unit spheres. Notably, Il’in and Filatov [16,14] tackled the well-posedness to these equations and identified the Hausdorff dimension of their global attractors [15]. Teman and Wang investigated the inertial forms of NSE on the sphere while Teman and Ziane show that the NSE on a 2D sphere is a limit of NSE defined a spherical cell [26]. This paper is concerned with the following stochastic Navier-Stokes equations (SNSE) on a 2D rotating sphere:

$$\partial_t u + \nabla u \cdot \nu L u + \omega \times u + \nabla p = f + \eta(x, t), \quad \text{div } u = 0, \quad u(0) = u_0$$  \hspace{1cm} (1.1)

where $L$ is the stress tensor, $\omega$ is the Coriolis acceleration, $f$ is the external force and $\eta$ is the noise process that can be informally described as the derivative of an $H$-valued Lévy process. Rigorous definitions of all relevant quantities in this equation will be given in section 2 and 3. The question of well-posedness for equation (1.1) with additive Gaussian noise has been studied in [1]. The new features in this paper are the following. First, we prove that given $L^4$-valued noise, $V'$-valued forcing $f$ and small $H$-valued initial data, there exists an uniqueness global weak (variational) solution which depends continuously on initial data. Moreover, with increased regularity of forcing and initial data, we prove an unique strong (PDE) solution for the abstract stochastic Navier-Stokes equations on the 2D unit sphere perturbed by stable Lévy noise. The existence time interval depends on the regularity of force and the assumption of the noise.

The paper is organised as follows. In section 2 we review the fundamental mathematical theory for the deterministic Navier-Stokes equations (NSE) on the sphere. We state some keep results without proofs. In section 3 we define the SNSE on the spheres. We start with some analytic facts; we introduce the driving noise process, which is a stable Lévy noise via subordination. The SNSE is then decomposed into an Ornstein Uhlenbeck (OU) process (associated with the linear part of the SNSE) and a nonlinear PDE. In section 4 we prove there exists global weak solution using the usual Galerkin approximation based on vector spherical harmonic series expansion. (see the proof of Theorem 3.2.5) Moreover, uniqueness is proven using the classical argument in the spirit of Lion and Prodi [19]. Furthermore, the solution is shown to depend continuously on initial data. (see the proof of Theorem 3.2.6) In section 5 we prove strong classical solution (see the proof of Theorem 3.3.7) for smooth initial data, sufficient regular noise following the classical lines in the proof of Theorem 3.1 [4].
2. NAVIER-STOKES EQUATIONS ON A ROTATING 2D UNIT SPHERE

The sphere is the simplest example of a compact Riemannian manifold without boundary hence one may employ the well-developed tools from Riemannian geometry to study objects on such manifold. Nevertheless, all objects of interests in this thesis are defined explicitly under the spherical coordinate. The presentation here follows closely from Goldys et al. [11] and reference therein.

2.1. Preliminaries. Let $S^2$ be the 2D unit sphere in $\mathbb{R}^3$, that is $S^2 = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| = 1 \}$. An arbitrary point $x$ on $S^2$ can be parametrized by the spherical coordinates

$$x = \hat{x}(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi.$$ 

The corresponding angle $\theta$ and $\phi$ will be denoted by $\theta(x)$ and $\phi(x)$, or simply by $\theta$ and $\phi$.

Let $e_\theta = e_\theta(\theta, \phi)$ and $e_\phi = e_\phi(\theta, \phi)$ be the standard unit tangent vectors of $S^2$ at point $\hat{x}(\theta, \phi) \in S^2$ in the spherical coordinate, that is,

$$e_\theta = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta), \quad e_\phi = (-\sin \phi, \cos \phi, 0).$$

Remark that

$$e_\theta = \frac{\partial \hat{x}(\theta, \phi)}{\partial \theta}, \quad e_\phi = \frac{1}{\sin \theta} \frac{\partial \hat{x}(\theta, \phi)}{\partial \phi},$$

where the second identity holds whenever $\sin \theta \neq 0$.

Our first aim is to give a meaning to all the terms in the deterministic Navier-Stokes equation for the velocity field $u(\hat{x}, t) = (u_\theta(\hat{x}, t), u_\phi(\hat{x}, t))$ of a geophysical fluid flow on a 2D rotating unit sphere $S^2$ under the external force $f = (f_\theta, f_\phi) = f_\theta e_\theta + f_\phi e_\phi$. Motion of the fluid is governed by the equation

$$\partial_t u + \nabla u - vL u + \omega \times u + \frac{1}{\rho} \nabla p = f, \quad \nabla \cdot u = 0, \quad u(x, 0) = u_0. \quad (2.1)$$

Here $v$ and $\rho$ are two positive constants denote the viscosity and the density of the fluid, the normal vector field

$$\omega = 2\Omega \cos(\theta(x)) x,$$

where $x = \hat{x}(\theta(x), \phi(x))$, $\Omega$ is the angular velocity of the earth and $\theta$ is the parameter represent the colatitude. Note that $\theta(x) = \cos^{-1}(x_3)$. In what follows we will identify $\omega$ with the corresponding scalar function $\omega$ defined by $\omega(x) = 2\Omega \cos(\theta(x))$.

We will introduce now other terms that appear in the equation. The surface gradient for a scalar function $f$ on $S^2$ is given by

$$\nabla f = \frac{\partial f}{\partial \theta} e_\theta + \frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} e_\phi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi.$$ 

Unless specified otherwise, by a vector field on $S^2$ we mean a tangential vector field, that is, a section of the tangent vector bundle of $S^2$.

On the other hand, for a vector field $u = (u_\theta, u_\phi)$ on $S^2$, that is $u = u_\theta e_\theta + u_\phi e_\phi$, one puts

$$\nabla \cdot u = \frac{1}{\sin \theta} \left( \frac{\partial}{\partial \theta} (u_\theta \sin \theta) + \frac{\partial}{\partial \phi} u_\phi \right). \quad (2.2)$$

Given two vector fields $u$ and $v$ on $S^2$, there exist vector fields $\tilde{u}$ and $\tilde{v}$ defined in some neighbourhood of the surface $S^2$ and such that their restriction to $S^2$ are equal to $u$ and $v$. More precisely, see Definition 3.31 in [10],

$$\tilde{u}|_{S^2} = u : S^2 \to TS^2, \quad \text{and} \quad \tilde{v}|_{S^2} = v : S^2 \to TS^2.$$

For \( x \in \mathbb{R}^3 \), we define the orthogonal projection \( \pi_x : \mathbb{R}^3 \to T_x \mathbb{S}^2 \) of \( x \) onto \( T_x \mathbb{S}^2 \), that is

\[
\pi_x : \mathbb{R}^3 \ni y \mapsto y - (x \cdot y)x = -x \times (x \times y) \in T_x \mathbb{S}^2. \tag{2.3}
\]

**Lemma 2.1.** Suppose \( \tilde{u} \) and \( \tilde{v} \) are \( \mathbb{R}^3 \)-valued vector fields on \( \mathbb{S}^2 \), and \( u, v \) are tangent vector field on \( \mathbb{S}^2 \), defined by \( u(x) = \pi_x(\tilde{u}(x)) \) and \( v(x) = \pi_x(\tilde{v}(x)) \), \( x \in \mathbb{S}^2 \). Then the following identity holds

\[
\pi_x(\tilde{u}(x) \times \tilde{v}(x)) = u(x) \times ((x \cdot v(x))x) + ((x \cdot u(x))x \times v(x), \quad x \in \mathbb{S}^2. \tag{2.4}
\]

Let us fix \( x \in \mathbb{S}^2 \). Then one may decompose vector \( \tilde{u} \) and \( \tilde{v} \) into tangential and normal components as follows

\[
\tilde{u} = u + u^\perp \quad \text{with} \quad u \in T_x \mathbb{S}^2, \quad u^\perp = (u \cdot x)x,
\]

\[
\tilde{v} = v + v^\perp \quad \text{with} \quad v \in T_x \mathbb{S}^2, \quad v^\perp = (v \cdot x)x.
\]

Since \( u \times v \) is normal to \( T_x \mathbb{S}^2 \), \( \pi_x(u \times v) = 0 \). Likewise, \( u^\perp \times v^\perp = 0 \) since the cross product of two parallel vectors yields the 0 vector. Hence, it follows that

\[
\pi_x(\tilde{u} \times \tilde{v}) = \pi_x(u \times v + u \times v^\perp + u^\perp \times v) = u \times v^\perp + u^\perp \times v \tag{2.5}
\]

We will denote by \( \tilde{\nabla} \) the usual gradient in \( \mathbb{R}^3 \) and then we have

\[
(\tilde{\nabla} f)(x) = \pi_x(\nabla f(x)). \tag{2.6}
\]

The operator \( \text{curl} \) is defined by the formula

\[
(\text{curl} u)(x) = (I - \pi_x)((\tilde{\nabla} \times \tilde{u}))(x) = (x \cdot (\tilde{\nabla} \times \tilde{u}))(x). \tag{2.7}
\]

Let \( u \) be a tangent vector field on \( \mathbb{S}^2 \). Applying formula (2.5) to the vector fields \( \tilde{u} \) and \( \tilde{v} = \tilde{\nabla} \times \tilde{u} \), one gets

\[
\pi_x(\tilde{u} \times (\tilde{\nabla} \times \tilde{u})) = \tilde{u} \times (\tilde{\nabla} \times (u^\perp + u))
\]

\[
= u \times ((\tilde{\nabla} \times u)^\perp) + u^\perp \times (\tilde{\nabla} \times u)
\]

\[
= u \times ((x \cdot (\tilde{\nabla} \times \tilde{u}))x)
\]

\[
= (x \cdot (\tilde{\nabla} \times \tilde{u}))(u \times x), \quad x \in \mathbb{S}^2. \tag{2.8}
\]

So, we can now define the curl of the vector field \( u \) on \( \mathbb{S}^2 \), namely

\[
\text{curl} u := \tilde{x} \cdot (\tilde{\nabla} \times \tilde{u})|_{\mathbb{S}^2}. \tag{2.9}
\]

equations (2.9) and (2.4) together yield

\[
\pi_x[\tilde{u} \times (\tilde{\nabla} \times \tilde{u})](x) = [u(x) \times x] \text{curl} u(x), \quad x \in \mathbb{S}^2
\]

Therefore, we have the following

**Definition 2.2.** Let \( u \) be a tangent vector field on \( \mathbb{S}^2 \), and let the vector field \( \psi \) be normal to \( \mathbb{S}^2 \). We set

\[
\text{curl} u = (\tilde{x} \cdot (\tilde{\nabla} \times \tilde{u}))|_{\mathbb{S}^2}, \quad \text{Curl} \psi = (\tilde{\nabla} \times \psi)|_{\mathbb{S}^2}. \tag{2.10}
\]

The first equation above indicates a projection of \( \nabla \times \tilde{u} \) onto the normal direction, while the 2nd equation means a restriction of \( \nabla \times \psi \) to the tangent field on \( \mathbb{S}^2 \). The definitions presented above do not depend on the extensions \( \tilde{u} \) and \( \tilde{\psi} \). A vector field \( \psi \) normal to \( \mathbb{S}^2 \) will often be identified with a scalar function on \( \mathbb{S}^2 \) when it is convenient to do so. The following describe the
relationships among Curl of a scalar function \( \psi \), Curl of a normal vector field \( w = w \hat{\varepsilon} \), and curl of a vector field \( v \) on \( S^2 \) and the surface div and \( \Delta \) operators are given as

\[
\text{Curl} \psi = -\hat{\varepsilon} \times \nabla \psi, \quad \text{Curl} w = -\hat{\varepsilon} \times \nabla w, \quad \text{curl} v = -\text{div}(\hat{\varepsilon} \times v). \tag{2.11}
\]

Let

\[
(\nabla_x u)(x) = \pi_x \left( \sum_{i=1}^{3} \bar{v}_i(x) \partial_i \bar{u}(x) \right) = \pi_x \left( (\bar{v}(x) \cdot \hat{\nabla}) \bar{u}(x) \right), \quad x \in S^2. \tag{2.12}
\]

Invoking (2.4) and the formula

\[
(\bar{u} \cdot \hat{\nabla}) \bar{u} = \hat{\nabla} \left[ \frac{|\bar{u}|^2}{2} \right] - \bar{u} \times (\hat{\nabla} \times \bar{u}),
\]

we find that the covariant derivative \( \nabla_x u \) takes the form

\[
\nabla_x u = \hat{\nabla} \left[ \frac{|u|^2}{2} \right] - \pi_x (\bar{u} \times (\hat{\nabla} \times \bar{u})).
\]

In particular, using (2.4) we obtain

\[
\nabla_x u = \hat{\nabla} \left[ \frac{|u|^2}{2} \right] - \pi_x (\bar{u} \times (\hat{\nabla} \times \bar{u})).
\]

The surface diffusion operator acting on vector fields on \( S^2 \) is denoted by \( \Delta \) (known as the Laplace de Rham operator) and is defined as

\[
\Delta v = \nabla \text{div} v - \text{Curl} \text{curl} v. \tag{2.13}
\]

Using (2.11), one can derive the following relations connecting the above operators:

\[
\text{div} \text{Curl} v = 0, \quad \text{curl} \text{Curl} v = -\hat{\varepsilon} \Delta v, \quad \Delta \text{Curl} v = \text{Curl} \Delta v. \tag{2.14}
\]

Next, we recall the definition of the Ricci tensor \( \text{Ric} \) of the 2D sphere \( S^2 \). Since

\[
\text{Ric} = \begin{pmatrix} E & F \\ F & C \end{pmatrix}
\]

where the coefficients \( E, F, G \) of the first fundamental form are given by

\[
E = x_\theta \cdot x_\theta = 1, \\
F = x_\theta \cdot x_\phi = x_\phi \cdot x_\theta = 0, \\
C = x_\phi \cdot x_\phi = \sin^2 \theta
\]

we find that

\[
\text{Ric} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}. \tag{2.15}
\]

Finally we define the stress tensor \( L \): it is given by

\[
L = \Delta + 2\text{Ric}
\]

where \( \Delta \) is the Laplace-de Rham operator.
2.2. Function spaces on the sphere. In what follows we denote by $dS$ the surface measure on $S^2$. In the spherical coordinate one has locally, $dS = \sin \theta \, d\theta d\phi$. For $p \in [1, \infty)$ we denote by $L^p(S^2, \mathbb{R})$ the space of $p$ integrable scalar function on $S^2$ endowed with the norm
\[ |v|_{L^p} = \left( \int_{S^2} |v(x)|^p \, dS(x) \right)^{1/p}. \]
For $p = 2$ the corresponding inner product is denoted by
\[ (v_1, v_2) = (v_1, v_2)_{L^2(S^2)} = \int_{S^2} v_1 v_2 \, dS. \]
On the other hand, we denote $L^p(S^2, \mathbb{R})$ the space $L^p(S^2, T S^2)$ of vector fields $v : S^2 \to TS^2$ endowed with the norm
\[ |v|_{L^p} = \left( \int_{S^2} |v(x)|^p \, dS(x) \right)^{1/p}, \]
where, for $x \in S^2$, $|v(x)|$ denotes the length of $v(x)$ in the tangent space $T_x S^2$. For $p = 2$ the corresponding inner product is denoted by
\[ (v_1, v_2) = (v_1, v_2)_{L^2} = \int_{S^2} v_1 \cdot v_2 \, dS. \]
Throughout this thesis, the induced norm on $L^2(S^2)$ is denoted by $| \cdot |$. For other inner product spaces, say $V$ with inner product $(\cdot, \cdot)_V$, the associated norm is denoted by $| \cdot |_V$.

The following identities hold for appropriate real valued scalar functions and vector fields on $S^2$, see (2.4)-(2.6) [14]:
\[
\begin{align*}
(\nabla \psi, v) &= - (\psi, \text{div} v), \\
(Curl \psi, v) &= (\psi, \text{curl } v), \\
(Curlcurl w, z) &= (\text{curl } w, \text{curl } z).
\end{align*}
\]
In (2.17), the $L^2(S^2)$ inner product is used on the left hand side while the $L^2(S^2)$ is used on the right hand side. Throughout this thesis, we identify a normal vector field $w$ with a scalar field $w$ and by $w = \hat{x} w$ and hence we put
\[ (\psi, w) := (\psi, w)_{L^2(S^2)}, \quad \text{if } w = \hat{x} w, \quad \psi, w \in L^2(S^2). \quad (2.19) \]
Let us now introduce the Sobolev spaces $H^1(S^2)$ and $H^1(S^2)$ of scalar functions and vector fields on $S^2$. Let $\psi$ be a scalar function and let $u$ be a vector field on $S^2$, respectively. For $s \geq 0$ we define
\[ |\psi|_{H^s(S^2)}^2 \quad (2.20) \]
and
\[ |u|_{H^s(S^2)}^2 \quad (2.21) \]
One has the following Poincaré inequality
\[ \lambda_4 |u| \leq |\text{div} u| + |\text{Curl } u|, \quad u \in H^1(S^2), \quad (2.22) \]
where $\lambda_4 > 0$ is the first positive eigenvalue of the Laplace-Hodge operator, see below. By the Hodge decomposition theorem in Riemannian geometry [9], the space of $C^\infty$ smooth vector field on $S^2$ can be decomposed into three components:
\[ C^\infty(T S^2) = \mathcal{G} \oplus \mathcal{V} \oplus \mathcal{V}, \]
where
\[ S = \{ \nabla \psi \in C^\infty(S^2) : \mu \psi \in C^\infty(S^2) \}, \]
and \( \mathcal{H} \) is the finite-dimensional space of harmonic vector fields. Since the sphere is simply connected, that is, the map \( S^2 \to S^2 \) is a diffeomorphism and so \( \mathcal{H} = \{ 0 \} \). The condition of orthogonality to \( \mathcal{H} \) is dropped out. We introduce the following spaces
\[ H := \{ u \in L^2(S^2) : \nabla \cdot u = 0 \}, \]
\[ V := H \cap H^1(S^2). \]
In other words, \( H \) is the closure of the
\[ \{ u \in C^\infty(TS^2) : \nabla \cdot u = 0 \} \]
in the \( L^2 \) norm \( |u| = (u, u)^{1/2} \), where \( u = (u_\theta, u_\phi) \) and
\[ (u, v) = \int_{S^2} u_\theta(x) v_\theta(x) dx, \quad (2.23) \]
and the space \( V \) is the closure of
\[ \{ u \in C^\infty(TS^2) : \nabla \cdot u = 0 \} \]
in the norm of \( H^1(S^2) \). Since \( V \) is densely and continuously embedded into \( H \) and \( H \) can be identified with its dual \( H' \), one has the following Gelfand triple:
\[ V \subset H \subset H' \subset V'. \quad (2.24) \]

2.3. **Stokes operator.** We will recall first that the Laplace-Beltrami operator on \( S^2 \) can be defined in terms of spherical harmonics \( Y_{l,m} \) as follows. For \( \theta \in [0, \pi], \phi \in [0, 2\pi), \) we define
\[ Y_{l,m}(\theta, \phi) = \left( \frac{(2l + 1)(l - |m|)l}{4\pi(l + |m|)!} \right)^{1/2} P_l^m(\cos \theta)e^{im\phi}, \quad m = -l, \ldots, l, \quad (2.25) \]
with \( P_l^m \) being the associated Legendre polynomials. The family \( \{ Y_{l,m} : l = 0, 1, \ldots, m = -l, \ldots, l \} \) form an orthonormal basis in \( L^2(S^2) \) and then we can define the Laplace-Beltrami operator putting
\[ \Delta Y_{l,m} = -l(l + 1)Y_{l,m}, \]
and then extending by linearity to all functions \( f : L^2(S^2) \) such that
\[ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} l(l+1)^2 (f, Y_{l,m})^2_{L^2(S^2)} < \infty. \]
We consider the following linear Stokes problem, that is given \( f \in V' \), find \( v \in V \) such that
\[ \nu \text{CurlCurl} u - 2\nu \text{Ric}(u) + \nabla p = f, \quad \nabla \cdot u = 0. \quad (2.26) \]
By taking the inner product of the first equation above with a test field \( v \in V \) and then use (2.18), the pressure term drops and we obtain
\[ \nu (\text{CurlCurl} u, \text{CurlCurl} v) - 2\nu (\text{Ric} u, v) = (f, v) \quad \forall \, v \in V. \]
Next, define a bilinear form \( \alpha : V \times V \to \mathbb{R} \) by
\[ \alpha(u, v) := (\text{Curl} u, \text{Curl} v) - 2(\text{Ric} u, v), \quad u, v \in V. \quad (2.27) \]
In view of (2.21) and the formula (2.15) for the Ricci tensor on \( S^2 \), the bilinear form \( \alpha \) satisfies
\[ \alpha(u, v) \leq |u|_{H^1} |v|_{H^1}. \quad (2.28) \]
and so it is continuous on $V$. So, by the Riesz representation theorem, there exists a unique operator $\mathcal{A} : V \to V'$ where $V'$ is the dual of $V$, such that $\alpha(u, v) = \langle \mathcal{A}u, v \rangle$, for $u, v \in V$. Invoking the Poincaré inequality [2.22] we find that $\alpha(u, u) \geq \alpha |u|^2_0$ for a certain $\alpha > 0$, which implies that $\alpha$ is coercive in $V$. Hence by the Lax-Milgram theorem the operator $\mathcal{A} : V \to V'$ is an isomorphism. Let $A$ be a restriction of $\mathcal{A}$ to $H$:

$$
\begin{align*}
\mathcal{D}(A) & := \{u \in V : \mathcal{A}u \in H\}, \\
Au & := \mathcal{A}u, \quad u \in \mathcal{D}(A).
\end{align*}
$$

It is well known (see for instance [25, Theorem 2.2.3]) that $A$ is positive definite, self-adjoint in $H$ and $\mathcal{D}(A^{1/2}) = V$ with equivalent norms. Furthermore, for some positive constants $c_1, c_2$ we have

$$
\langle Au, u \rangle = \langle (u, u) \rangle = |u|^2_V = |\nabla u|^2 = |Du|^2, \quad u \in \mathcal{D}(A).
$$

The spectrum of $A$ consists of an infinite sequence of eigenvalues $\lambda_l$. Using the stream function $\psi_l$ for which $Z_{l, m} = \text{Curl} \psi_{l, m}$ and identities [2.14], one can show that each $\lambda_l$ are in fact the eigenvalues of the Laplace-Beltrami operator $\Delta$, that is $\lambda_l = l(l + 1)$, and there exists an orthonormal basis $(Z_{l, m})_{l \geq 1}$ of $H$ consisting of eigenvector of $A$, where

$$
Z_{l, m} = \lambda_l^{-1/2} \text{Curl} V_{l, m}, \quad l = 1, \ldots, m = -l, \ldots, l.
$$

Therefore, for any $v \in H$, one has,

$$
v = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \hat{v}_{l, m} Z_{l, m}, \quad \hat{v}_{l, m} = \int_{S^2} v \cdot Z_{l, m} dS = \langle v, Z_{l, m} \rangle.
$$

An equivalent definition of the operator $A$ can be given using the so-called Leray-Helmholtz projection $P$ that is defined as an orthogonal projection from $L^2(S^2)$ onto $H$, called Leray-Helmholtz projection. Let $H^2(S^2)$ denote the domain of the Laplace-Hodge operator in $H$ endowed with the graph norm. It can be shown in [12] that $D(A) = H^2(S^2) \cap V$ and $A = -P(\Delta + 2\text{Ric})$. Therefore, we obtain an equivalent definition of the Stokes operator on the sphere.

**Definition 2.3.** The Stokes operator $A$ on the sphere is defined as

$$
A : D(A) \subset H \to H, \quad A = -P(\Delta + 2\text{Ric}), \quad D(A) = H^2(S^2) \cap V,
$$

where $\Delta$ is the Laplace-De Rham operator.

It can be shown that $V = D(A^{1/2})$ when endowed with the norm $|x|_V = |A^{1/2}x|$ and the inner product $\langle (x, y) \rangle = \langle Ax, y \rangle$. After identification of $H$ with its dual space we have $V \subset H \subset V'$ with continuous dense injection. The dual pairing between $V$ and $V'$ is denoted by $\langle \cdot, \cdot \rangle_{V' \times V}$. Moreover, there exist positive constants $c_1, c_2$ such that

$$
c_1|u|^2_V \leq \langle Au, u \rangle \leq c_2|u|^2_V, \quad u \in \mathcal{D}(A).
$$

Let us now introduce the Sobolev spaces $H^s(S^2)$ and $H^2(S^2)$ of scalar functions and vector fields on $S^2$. Let $\psi$ be a scalar function and let $u$ be a vector field on $S^2$, respectively. For $s \geq 0$ we define

$$
|\psi|^2_{H^s(S^2)} = |\psi|^2_{L^2(S^2)} + |(-\Delta)^{s/2}\psi|^2_{L^2(S^2)},
$$

and

$$
|u|^2_{H^s(S^2)} = |u|^2 + |(-\Delta)^{s/2}u|^2,
$$

where $\Delta$ is the Laplace-Beltrami operator and $\Delta$ is the Laplace-de Rham operator on the sphere. Note that, for $k = 0, 1, 2, \cdots$ and $\theta \in (0, 1)$ the space $H^{k+\theta}(S^2)$ can be defined as the interpolation
space between $H^k(S^2)$ and $H^{k+1}(S^2)$. One can apply the same procedure for $H^{k+\theta}(S^2)$. The fractional power $A^{s/2}$ of the Stokes operator $A$ in $H$ for any $s \geq 0$ is given by

$$ D(A^{s/2}) = \left\{ v \in H : v = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \tilde{v}_{l,m} Z_{l,m}, \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \lambda_l^s |\tilde{v}_{l,m}|^2 < \infty \right\}, $$

$$ A^{s/2}v := \sum_{m=1}^{\infty} \sum_{m=-l}^{l} \lambda_l^{s/2} \tilde{v}_{l,m} Z_{l,m} \in H. $$

The Coriolis operator $C_1 : L^2(S^2) \to L^2(S^2)$ is defined by the formula\footnote{The angular velocity vector of earth is denoted as $\Omega$ in consistent to geophysical fluid dynamics literature. It shall not be confused with the notation for probability space $\Omega$ used in this thesis.}

$$ (C_1v)(x) = 2\Omega(x \times v(x))\cos \theta, \quad x \in S^2. \quad (2.36) $$

It is clear from the above definition that $C_1$ is a bounded linear operator defined on $L^2(S^2)$. In the sequel we will need the operator $C = PC_1$ which is well defined and bounded in $H$. Furthermore, for $u \in H$,

$$ (Cu, u) = (C_1u, Pu) = \int_{S^2} 2\Omega \cos \theta ((x \times u) \cdot u(x))dS(x) = 0. \quad (2.37) $$

In addition,

**Lemma 2.4.** For any smooth function $u$ and $s \geq 0$

$$ (Cu, A^{s}u) = 0. \quad (2.38) $$

**Proof.** The case $s = 0$ is obvious as in the line above, due to the fact $(\omega \times u) \cdot u = 0$. For $s > 0$ we refer readers to Lemma 5 in [24].

Let $X = H \cap \mathbb{L}^4(S^2)$ be endowed with the norm

$$ \|v\|_X = \|v\|_H + \|v\|_{\mathbb{L}^4(S^2)}. $$

Then $X$ is a Banach space. It is known that the Stokes operator $A$ generates an analytic $C_0$-semigroup $\{e^{-tA}\}_{t \geq 0}$ in $X$ (see Theorem A.1 in [1]). Since the Coriolis operator $C$ is bounded on $X$ we can define in $X$ an operator

$$ \hat{A} = vA + C, \quad D(\hat{A}) = D(A), $$

with $v > 0$.

**Lemma 2.5.** Suppose that $V \subset H \subset H' \subset V'$ is a Gelfand triple of Hilbert spaces. If a function $u$ being $L^2(0,T;V)$ and $\partial_t u$ belongs to $L^2(0,T;V)$ in weak sense, then $u$ is a.e. equal to a continuous function from $[0,T]$ to $H$, the real function $|u|^2$ is absolutely continuous and, in the weak sense one has

$$ \partial_t |u(t)|^2 = 2\langle \partial_t u(t), u(t) \rangle \quad (2.39) $$

**Proposition 2.6.** The operator $\hat{A}$ with the domain $D(\hat{A}) = D(A)$ generates a strongly continuous and analytic semigroup $\{e^{-t\hat{A}}\}_{t \geq 0}$ in $X$. In particular, there exist $M \geq 1$ and $\mu > 0$ such that

$$ |e^{-t\hat{A}}|_{C(X,X)} \leq Me^{-\mu t}, \quad t \geq 0, \quad (2.40) $$

and for any $\delta > 0$ there exists $M_\delta \geq 1$ such that

$$ |\hat{A}^\delta e^{-t\hat{A}}|_{C(X,X)} \leq M_\delta t^{-\delta}e^{-\mu t}, \quad t > 0. \quad (2.41) $$
Proof. See the proof of Proposition 5.3 in [1].

Now consider the trilinear form \( b \) on \( V \times V \times V \), defined as
\[
b(v, w, z) = (\nabla_v w, z) = \int_{\mathbb{S}^2} \nabla_v w \cdot z \, dS = \pi_x \sum_{i,j=1}^{3} v_i \cdot D_i w_j \, d\mathcal{H}^1,
\] where \( v, w, z \in V \). (2.42)

Via the identity [1],
\[
2\nabla_v w = -\text{curl}(w \times v) + \nabla(w \cdot v) - v \text{div} w + w \text{div} v - v \times \text{curl} w - w \times \text{curl} v,
\]
and equation (2.13), one can write the divergence free fields \( v, w, z \), the trilinear form can be written as
\[
b(v, w, z) = \frac{1}{2} \int_{\mathbb{S}^2} [-v \times w \cdot \text{curl} z + \text{curl} v \times w \cdot z - v \times \text{curl} w \cdot z] \, dS.
\] (2.43)

Moreover,
\[
b(v, w, w) = 0, \quad b(v, z, w) = -b(v, w, z), \quad v \in V, w, z \in H^1(\mathbb{S}^2),
\] (2.44)

and such that
\[
|B(u, v), w| = |b(u, v, w)| \leq c|u||w|(|\text{curl} v|_{L^\infty(\mathbb{S}^2)} + |v|_{L^\infty(\mathbb{S}^2)}), \quad u \in H, v \in V, w \in H,
\] (2.45)

\[
|B(u, v), w| = |b(u, v, w)| \leq c|u|^{1/2}|u|^{1/2}|v|^{1/2}|v|^{1/2}|w|, \quad u, v, w \in V,
\] (2.46)

\[
|B(u, v), w| = |b(u, v, w)| \leq c|u|^{1/2}|u|^{1/2}|v|^{1/2}|v|^{1/2}|Au|^{1/2}|w|, \quad \forall u \in V, v \in D(A), w \in H, \quad n = 2,
\] (2.47)

\[
|b(u, v, w)| \leq c|u|_{L^1(\mathbb{S}^2)}|v|_{L^2(\mathbb{S}^2)}|w|_{L^2(\mathbb{S}^2)}, \quad v \in V, u, w \in H^1(\mathbb{S}^2).
\] (2.48)

In view of (2.46),
\[
\sup_{z \in V, |z| \neq 0} \frac{|B(u, v), z|}{|z|^V} = |B(u, v)|^V \leq c|u|^{1/2}|u|^{1/2}|v|^{1/2}|v|^{1/2} \leq c|u||u||v| \quad \implies \quad |B(u, u)|^V \leq c|u||u||v|,
\] (2.49)

\[
\sup_{z \in H, |z| \neq 0} \frac{|B(u, v), z|}{|z|^H} = |B(u, v)|^H \leq c|u|^{1/2}|u|^{1/2}|v|^{1/2}|v|^{1/2} \leq c|u||u||v| \quad \implies \quad |B(u, u)|^H \leq c|u||u||v|.
\] (2.50)

In view of (2.47),
\[
\sup_{z \in H, |z| \neq 0} \frac{|B(u, v), z|}{|z|^H} = |B(u, v)|^H \leq c|u|^{1/2}|u|^{1/2}|v|^{1/2}|v|^{1/2} \leq c|u||u||v||Au|^{1/2} \quad \forall u \in D(A).
\] (2.51)

In view of (2.48), \( b \) is a bounded trilinear map from \( L^4(S^2) \times V \times L^4(S^2) \) to \( \mathbb{R} \).

**Lemma 2.7.** The trilinear map \( b \) can be uniquely extended from \( V \times V \times V \) to a bounded three-linear map
\[
b : (L^4(S^2) \cap H) \times L^4(S^2) \times V \to \mathbb{R}.
\]
Finally, we recall the interpolation inequality (See [16], p.12),
\[ |u|_{L^4(S^2)} \leq C|u|_{L^2(S^2)}^{1/2}|u|_V^{1/2}. \]  
(2.52)
Inequality 2.46 is deduced from the following Sobolev embedding
\[ H^{1/2} = W^{1/2,2}(S^2) \hookrightarrow L^4(S^2). \]

Then using (2.13), (2.16), (2.29) and (2.43), we arrive with the weak solution of the Navier-Stokes equations 2.22 which is a vector field \( u \in L^2([0, T]; V) \) with \( u(0) = u_0 \) that satisfies the weak form of (2.22):
\[ (\partial_t u, v) + b(u, u, v) + \nu(\text{curl} u, \text{curl} v) - 2\nu(\text{Ric} u, v) + (C u, v) = (f, v), \quad v \in V, \]  
where the bilinear form \( B : V \times V \rightarrow V' \) is defined by
\[ (B(u, v), w) = b(u, v, w) = \sum_{i,j=1}^3 \int_{S^2} \partial_i (v_k) \partial_j u_j \, dx, \quad w \in V. \]  
(2.54)
With a slight abuse of notation, we denote \( B(u) = B(u, u) \) and \( B(u) = \pi(u, \nabla u) \).

3. **Stochastic Navier-Stokes equations on the 2D unit sphere**

By adding a Lévy white noise to (2.1), we obtain the main equation in this thesis.
\[ \partial_t u + \nabla_x u - \nu \Delta u + \omega \times u + \nabla_x p = f + \eta(x, t), \]  
\[ \text{div} \, u = 0, \quad u(x, 0) = u_0, \quad x \in S^2. \]
We assume that, \( u_0 \in H, \ f \in V' \) and \( \eta(x, t) \) is the so-called Lévy white noise, that is a noise process which can be informally described as the derivative of an \( H \)-valued Lévy process, that is rigorously defined in Lemma 5.7. Applying the Leray-Helmholtz projection we can interpret equation (3.1) as an abstract stochastic equation in \( H \)
\[ du(t) + Au(t) + B(u(t), u(t)) + Cu = fdt + GdL(t), \quad u(0) = u_0, \]  
(3.2)
where \( L \) is an \( H \)-valued stable Lévy process and \( G : H \rightarrow H \) is a bounded operator. In order to study this equation we need to consider first some properties of the stochastic convolution.

3.1. **Stochastic convolution of \( \beta \)-stable noise.** In this section we will study a linear version of equation (3.2)
\[ dz(t) + Az(t) + Cz = GdL(t), \quad z(0) = 0. \]  
(3.3)
Under appropriate assumptions formulated below its solution takes the form
\[ z_t = \int_0^t e^{-A(t-s)}GdL(s), \]  
(3.4)
where \( \tilde{A} = A + C \). Let \( W \) be a cylindrical Wiener process on a Hilbert space \( K \) continuously imbedded into \( H \) and let \( X \) be a \( \beta/2 \)-stable subordinator. Then the process \( L = W(X) \) is a symmetric cylindrical \( \beta \)-stable process in \( H \). Assume that \( G : H \rightarrow H \) is \( \gamma \)-radingonifying. Then the process \( GL \) is a well defined Lévy process taking values in \( H \). Under these assumptions the process \( z \) defined by (3.4) is a well defined \( H \)-valued process and moreover, it can be considered as a solution to the following integral equation
\[ z(t) = -\int_0^t e^{-(t-s)A}Cz(s) \, ds + \int_0^t e^{-(t-s)A}GdL(s). \]  
(3.5)
With some abuse of notation, we will denote now by \( \lambda \) the eigenvalues of the Stokes operator \( A \) taking into account their multiplicities that is \( \lambda_1 \leq \lambda_2 \leq \cdots \), and by \( e_t \) the corresponding
eigenvectors that form an orthonormal basis in $H$. We will impose a stronger condition on the operator $G$:

$$Ge_l = \sigma_l e_l, \quad l = 1, 2, \ldots.$$  

We will consider the process

$$z_0^t = \int_0^t e^{-(t-s)A}GdL(s) = \sum_{l=1}^{\infty} z_l^0(t)e_l,$$

where

$$z_l^0(t) = \int_0^t e^{-\lambda_l(t-s)}\sigma_l dL_l(s). \quad (3.6)$$

**Lemma 3.1.** Suppose that there exists some $\delta > 0$ such that $\sum_{l \geq 1} |\sigma_l|^{p\beta_l^{\delta}} < \infty$. Then for all $p \in (0, \beta)$,

$$\mathbb{E}|A^\delta L(t)|^p \leq C(\beta, p) \left( \sum_{l \geq 1} |\sigma_l|^{p\beta_l^{\delta}} \right)^{\frac{p}{\beta}} < \infty. \quad (3.7)$$

**Proof.** Let $L(t) = \sum_{l \geq 1} L_l e_l$, $t \geq 0$ be the cylindrical $\beta$-stable process on $H$, where $e_l$ is the complete orthonormal system of eigenfunctions on $H$ and $L_1, L_2, \ldots, L_l$ are i.i.d. $\mathbb{R}$-valued, symmetric $\beta$-stable process on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Now take a bounded sequence of real number $\sigma = (\sigma_l)_{l \in \mathbb{N}}$, let us define

$$G_\sigma : H \rightarrow H; \quad G_\sigma u := \sum_{l=1}^{\infty} \sigma_l (u, e_l) e_l,$$

and $\sigma_l$ are chosen such that

$$G_\sigma L(t) = \sum_{l=1}^{\infty} \sigma_l (L_l(t), e_l) e_l = \sum_{l=1}^{\infty} \sigma_l L_l(t) e_l.$$

To show (3.7), we follow the argument in the proof of Lemma 3.1 in [29] and Theorem 4.4 in [23]. Take a Rademacher sequence $\{r_k\}_{k \geq 1}$ in a new probability space $(\Omega', \mathcal{F}', \mathbb{P})$, that is, $\{r_k\}_{k \geq 1}$ are i.i.d. with $\mathbb{P}\{r_k = 1\} = \mathbb{P}\{r_k = -1\} = \frac{1}{2}$. By the following Khintchine inequality: for any $p > 0$, there exists some $C(p) > 0$ such that for arbitrary real sequence $\{h_l\}_{l \geq 1}$,

$$\left( \sum_{l \geq 1} h_l^2 \right)^{1/2} \leq C(p) \left( \mathbb{E} \left| \sum_{l \geq 1} r_l h_l \right|^p \right)^{1/p}.$$

Via this inequality, we get

$$\mathbb{E}|A^\delta L(t)|^q = \mathbb{E} \left( \sum_{l \geq 1} \lambda_l^{2\delta} |\sigma_l|^2 |L_l(t)|^2 \right)^{p/2} \leq C\mathbb{E} \left( \sum_{l \geq 1} r_l \lambda_l^{\delta} |\sigma_l||L_l(t)| \right)^p = C\mathbb{E} \left( \sum_{l \geq 1} r_l \lambda_l^{\delta} |\sigma_l||L_l(t)| \right)^p.$$


Lemma 3.2 \([p.3714, [29]]\). Suppose that there exists \(\delta > 0\) such that 
\[
\sum_{l=1}^{\infty} |\alpha_l|^{\beta_l^\delta} < \infty , \quad (3.7)
\]
Then for all \(p \in (0, \beta)\) and \(T > 0\)
\[
E \sup_{0 \leq t \leq T} |A^\delta z_t|^p \leq C \left( 1 + T^{p(1-\delta)} \right) T^{p/\beta} \quad (3.8)
\]
Proof. It is proved in \([29]\) that for \(p > 1\)
\[
E \sup_{0 \leq t \leq T} |A^\delta z_t|^p \leq CT^{p/\beta} . \quad (3.9)
\]
In order to prove the lemma for the process \(z\), we use formula \([55]\). Let \(Z = z - z^0\). Then \([55]\) yields
\[
\frac{dZ}{dt} = -AZ - C \left( Z + z^0 \right) = -\hat{A}Z - Cz^0 , \quad Z(0) = 0 .
\]
Therefore,
\[
Z(t) = - \int_0^T e^{-(t-s)\hat{A}}Cz^0(s) \, ds , \quad t \geq 0 .
\]
Then, by the properties of analytic semigroups we find that
\[
|\hat{A}^\delta Z(t)| \leq \int_0^t \left| e^{-(t-s)\hat{A}} \right| Cz^0(s) \, ds 
\leq \sup_{s \leq t} \left| Cz^0(s) \right| \int_0^t \frac{c}{(t-s)^\delta} \, ds 
\leq c_1 t^{1-\delta} \sup_{s \leq t} \left| Cz^0(s) \right| 
\leq c_1 |C| t^{1-\delta} \sup_{s \leq t} \left| z^0(s) \right| .
\]
Since \(C\) is bounded, we have \(D(\hat{A}) = D(A)\) by Theorem 2.11 in \([22]\). Since \(A \geq 0\) is selfadjoint, the domains of fractional powers can be identified as the complex interpolation spaces, see Section 1.15.3 of \([28]\). Therefore, \(D(\hat{A}^\delta) = D(\hat{A}^\gamma)\) for every \(\gamma \in (0, 1)\), which yields the existence of constants, \(r_1, r_2\) depending on \(\delta\) only, such that
\[
r_1 |\hat{A}^\delta x| \leq |\hat{A}^\delta x| \leq r_2 |\hat{A}^\gamma x| , \quad x \in D(A^\gamma) .
\]
where \(C = C^p(p)\). For any \(\lambda \in \mathbb{R}\), by the fact of \(|r_k| = 1\) and formula (4.7) of \([23]\),
\[
E \exp \left\{ i\eta \sum_{l=1}^{\infty} \left| \alpha_l \right| |L_l(t)| \right\} = \exp \left( -|\eta|^{\delta} \sum_{l=1}^{\infty} |\alpha_l|^{\beta_l^\delta} t \right) .
\]
Using (3.9) we find that
\[
\mathbb{E} \sup_{t \leq T} \left| A^\delta Z(t) \right|^p \leq c^p r^p \mathbb{E} \sup_{s \leq T} \left| \mathcal{Z}^0(s) \right|^p < \infty.
\]

Now the lemma follows since \( z(t) = Z(t) + z^0(t) \).

Finally, for completeness we prove the case \( p \in (0, 1) \) for the process \( z^0 \). As (3.8) is proved for \( q \in (1, \beta) \) we fix \( q \in (1, \beta) \) and then
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left| A^\delta z^0|t) \right|^q \right) \leq C T^{q/\beta}.
\]

Using the Hölder inequality (see for instance [15], p.191) one has that, that is
\[
\mathbb{E}(|X|^p \cdot 1) \leq (\mathbb{E}|X|^q)^{1/q}.
\]

We then have
\[
\begin{align*}
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left| A^\delta z^0|t) \right|^p \right) \\
= \mathbb{E} \left( \left\{ \sup_{0 \leq t \leq T} \left| A^\delta z^0|t) \right|^p \right\}^{\frac{p}{q}} \right)^{\frac{q}{p}} \\
\leq \mathbb{E} \left( \left\{ \sup_{0 \leq t \leq T} \left| A^\delta z^0|t) \right|^q \right\}^{\frac{p}{q}} \right) \\
\leq (C_1 T^{q/\beta})^{p/q} \\
= C T^{p/\beta}.
\end{align*}
\]

\[\square\]

**Proposition 3.3.** [p110][23] Suppose \( \sum_{l \geq 1} \frac{\beta}{\beta + \alpha} < \infty \), then for any \( 0 < p < \beta \), \( t \geq 0 \),
\[
E|z^0|t|p \leq \tilde{c}_p \left( \sum_{l=1}^{\infty} |\sigma_l|^\beta \frac{1 - e^{-\beta(\lambda_l + \alpha)}t}{\beta(\lambda_l + \alpha)} \right)^{p/\beta},
\]
where \( c_p \) depends on \( p \) and \( \beta \). Moreover, as \( \alpha \to \infty \),
\[
E|z^0|t|p \to 0.
\]

**Proof.** Under same theme of the proof of Lemma [3.1] we follow the argument in the proof of Theorem 4.4 in [23] to complete the proof. Let \( z^0 \) be the solution of
\[
dz^0 + (A + \alpha I)z^0 = GdL(t), \quad z^0(0) = 0
\]
which has the expression
\[
\begin{align*}
z^0(t) &= \int_0^t S(t - s)GdL(s) \\
&= \sum_{l=1}^{\infty} \left( \int_0^t e^{-(\lambda_l + \alpha)(t-s)}\sigma_l dL^l_s \right) e_l.
\end{align*}
\]
where we used the notation \( S(t) = e^{-t(A + \alpha I)} \). Take a Rademacher sequence \( \{ r_k \}_{k \geq 1} \) in a new probability space \((\Omega', \mathcal{F}', \mathbb{P}')\), that is \((\Omega', \mathcal{F}', \mathbb{P}')\), that is \( \{ r_k \}_{k \geq 1} \) are i.i.d. with \( \mathbb{P}(r_i = 1) = \mathbb{P}(r_i = -1) = \frac{1}{2} \). By the following Khintchine inequality: for any \( p > 0 \), there exists some \( c_p > 0 \) such that for any arbitrary real sequence \( \{ c_i \}_{i \in \mathbb{N}} \),

\[
\left( \sum_{i \geq 1} c_i^2 \right)^{1/2} \leq c_p \left( \mathbb{E} \left| \sum_{i \geq 1} r_i c_i \right|^p \right)^{1/p},
\]

where \( c_p \) depends only on \( p \).

Now fix \( \omega \in \Omega \), \( t \geq 0 \), write

\[
\left( \sum_{i \geq 1} |r_i(t, \omega)|^2 \right)^{1/2} \leq c_p (\mathbb{E} \left| \sum_{i \geq 1} r_i z_i^0(t, \omega) \right|^p)^{1/p}.
\]

Then

\[
\mathbb{E}|z_i^0|^p = \left( \sum_{i=1}^\infty \left| \int_0^t e^{-|\lambda_i + \alpha||t-s|} \theta \, dL^i_s \right|^2 \right)^{1/2} \leq c_p \mathbb{E} \left( \mathbb{E} \left| \sum_{i \geq 1} r_i z_i^0(t) \right|^p \right) = c_p \mathbb{E} \left( \mathbb{E} \left| \sum_{i \geq 1} r_i \int_0^t e^{-|\lambda_i + \alpha||t-s|} \theta \, dL^i_s \right|^p \right).
\]

For any \( t \geq 0 \), \( \kappa \in \mathbb{R} \) using the fact \( |r_i| = 1 \), \( l \geq 1 \) and formula (4.7) in [23],

\[
\mathbb{E} e^{i \kappa \sum_{i \geq 1} r_i z_i^0(t)} = e^{-|\kappa|^p \sum_{i \geq 1} |r_i|^p} \int_0^t e^{-|\beta \lambda_i + \alpha| t} ds.
\]

Now we use (3.2) in [23]: If \( X \) is a symmetric \( \beta \)-stable r.v. with distribution \( S(\beta, \gamma, 0) \) satisfying

\[
\mathbb{E} e^{i \kappa X} = e^{-\gamma |\kappa|^\beta}
\]

for some \( \beta \in (0, 2) \) and any \( \kappa \in \mathbb{R} \), then for any \( p \in (0, \beta) \), one has

\[
\mathbb{E} X^p = C(\beta, p) \gamma^p.
\]

Since \( \sum_{l \geq 1} \frac{\beta}{\alpha} \frac{1}{l^{\alpha+\alpha}} < \infty \), the assertion follows. Furthermore, \( \mathbb{E}|z_i^0|_p \to 0 \) as \( \alpha \to \infty \). \( \square \)

Now we present a Lemma that allows us to claim that the solution of SNE has càdlàg trajectories. The proof follows closely with Lemma 3.3 in [20].

**Lemma 3.4.** Assume that for a certain \( \delta \in [0, 1) \)

\[
\sum_{l=1}^\infty |\theta_l|^{\beta_l/\alpha_l} < \infty.
\]

Then the process \( z \) defined by (3.6) has a version in \( D([0, \infty); D(A^\delta)] \).

**Proof.** By Lemma 3.2 we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} |A^\delta z(t)|^p < \infty
\]

for any \( p \in (0, \beta) \). Now, by Theorem 2.2 in [20] \( z^0 \) has a càdlàg modification\(^2\) in \( V \). By representation (3.5) the process \( z \) is càdlàg as well and the proof of Lemma is completed. \( \square \)

---

\(^2\)Modification with càdlàg path.
Let $B : H \to H$ be a selfadjoint operator with the complete orthonormal system of eigenfunctions $(e_l) \subset L^p(S^2)$ and the corresponding set of eigenvalues $(\lambda_l)$. It follows from Theorem 2.3 [7] that if further $B$ has compact inverse $B^{-1}$ then the operator $U^{-s} : H \to L^p(S^2)$ is well defined and $\gamma$-radonifying iff

$$
\int_{S^2} \left( \sum_l \lambda_l^{-2s} |e_l(x)|^2 \right)^{p/2} dS(x) < \infty
$$

\[(3.10)\]

We will study the $\gamma$-radonifying property.

**Lemma 3.5.** Let $\Delta$ denotes the Laplace-de Rham operator on $S^2$ and $q \in (1, \infty)$. Then the operator

$$
(-\Delta + 1)^{-s} : H \to L^q(S^2)
$$

is $\gamma$-radonifying iff $s > 1/2$.

**Proof.** See proof of Lemma 3.1 in [2]. \[\square\]

Let $X = L^4(S^2) \cap H$ be the Banach space endowed with the norm

$$|x|_X = |x|_H + |x|_{L^4(S^2)}.$$ 

It follows from Lemma 3.5 that the operator

$$A^{-s} : H \to X$$

is $\gamma$-radonifying iff $s > 1/2$. \[(3.11)\]

We need the OU process to take value in $X$, to this end, we need the following assumption.

**Definition 3.6.** Let $K$ and $X$ be separable Banach spaces and let $\gamma_K$ be the canonical cylindrical (finitely additive) Gaussian measure on $K$. A bounded linear operator $U : K \to X$ is said to be $\gamma$-radonifying iff $U(\gamma_K)$ is a Borel Gaussian measure on $X$.

One has to choose $X$ wisely, so that $U : K \to X$ is $\gamma$-radonifying (in checking validity of subordinator condition as in p.156, [3]). The following is our standing assumption.

**Assumption 1** A continuously embedded Hilbert space $K \subset H \cap L^4$ is such that for any $\delta \in (0, 1/2)$,

$$A^{-\delta} : K \to H \cap L^4$$

is $\gamma$-radonifying. \[(3.12)\]

It follows from (3.11) that $K = D(A^s)$ for some $s > 0$, then assumption 1 is satisfied.

**Remark.** Under the above assumption, we have the facts $K \subset H$ and Banach space $X$ is taken as $H \cap L^4$. In fact, space $K := Q^{1/2}(W)$ is the RKHS of noise $W(t)$ on $H \cap L^4$ with inner product $\langle \cdot, \cdot \rangle_K = \langle Q^{-1/2}x, Q^{-1/2}y \rangle_W$, $x, y \in K$. The notation $Q$ denotes the covariance of the noise $W$.

Note: The parameters used in Lemma 3.5 and Assumption 1 are independent. In the first case, we start with the whole space, a smaller exponent is required to map onto $H \cap L^4(S^2)$, so the assumption $s > 1/2$ justifies. While in Assumption 1, we start with a smaller space, a bigger exponent is required to map onto $H \cap L^4(S^2)$, so $\delta \in (0, 1/2)$.

**Corollary.** In the framework of Proposition 2.6 let us additionally assume that there exists a separable Hilbert space $K \subset X$ such that the operator $A^{-\delta} : K \to X$ is $\gamma$-radonifying for some $\delta \in (0, 1/2)$. Then

$$\int_0^\infty |e^{-tA^2}_{R(K,X)}| dt < \infty.$$
Proof. Since $e^{-tA} = A^δe^{-tA}A^{-δ}$, it follows by Neidhardt\cite{21} that
\[|e^{-tA}|_{R(K,X)} \leq |A^δe^{-sA}|_{L^2(X,X)}|A^{-δ}|_{R(K,X)},\]
and then Proposition\cite{22,26} yields finiteness of the integral. \hfill \Box

Let us recall what one means by $M$-type $p$ Banach space\cite{3}. Suppose $p \in [1, 2]$ is fixed, the Banach space $E$ is called as type $p$, iff there exists a constant $K_p(E) > 0$ such that for any finite sequence of symmetric independent identically distributed r.v. $ξ_1, \cdots, ξ_n : Ω → [-1, 1], n ∈ N,$ and any finite sequence $x_1, \cdots, x_n$ from $E$, satisfying
\[\mathbb{E}\left|\sum_{i=1}^{n} ξ_ix_i\right|^p \leq K_p(E)\sum_{i=1}^{n} |x_i|^p.\]
Moreover, a Banach space $E$ is of martingale type $p$ iff there exists $L_p(E) > 0$ such that for any $E$-valued martingale $\{M_n\}_{n=0}^{N}$ the following holds
\[\sup_{n \leq N} \mathbb{E}|M_n|^p \leq L_p(E)\sum_{n=0}^{N} \mathbb{E}|M_n - M_{n-1}|^p.\]

Lemma 3.7 (Corollary 8.1,\cite{8}). Assume that $p \in (1, 2]$, $X$ is a subordinator Lévy process from the class Sub$(p)$, $E$ is a separable type $p$ Banach space, $U$ is a separable Hilbert space, $E \subset U$ and $W = (W(t), t \geq 0)$ is an $U$-valued Wiener process.

Define a $U$-valued Lévy process as
\[L(t) = W(X(t)), \quad t \geq 0.\]
Then the $E$-valued process
\[z(t) = \int_{0}^{t} e^{-(t-s)[A+αI]}dL(s)\]
is well defined. Moreover, with probability 1, for all $T > 0$,
\[\int_{0}^{T} |z(t)|^2_{L^2}dt < \infty,\]
\[\int_{0}^{T} |z(t)|^4_{L^4}dt < \infty.\]

The following existence and regularity result is a version of the result in\cite{8}.

Theorem 3.8. Let the process $L$ be defined in the same way as in Lemma 3.7. Assume that one of the following conditions is satisfied:

(i) $p ∈ (0, 1]$ or
(ii) the Banach space $E$ is separable of martingale type $p$ for a certain $p ∈ (1, 2]$.

Then the process
\[z_{α}(t) = \int_{-∞}^{t} e^{-(t-s)[A+αI]}dL(s)\quad (3.13)\]
is well defined in $E$ for all $t > 0$. Moreover, if $p \in (1, 2]$, then the process $z$ of (3.13) is càdlàg.

Proof. As $S = (S(t), t \geq 0)$ is a $C_0$ semigroup in the separable martingale type $p$-Banach space $E$, there exists a Hilbert space $H$ as the reproducing Kernel Hilbert space of $W(1)$ such that the embedding $i : H \hookrightarrow E$ is $γ$-radonifying. The proof of this theorem is a straight application of Theorem 4.1 and 4.4 in\cite{8}. \hfill \Box
In order to obtain well-posedness of the \([3.1]\), one need some regularity on the noise term. Fortunately, this becomes attainable using Lemma \([3.7]\). In view of this, we construct the driving Lévy noise \(L = L(t)\) by subordinating a cylindrical Wiener process \(W\) on a Hilbert space \(H\). Let \(\{W^l_t, t \geq 0\}\) be a sequence of independent standard one-dimensional Wiener process on some given probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The cylindrical Wiener process on \(H\) is defined by

\[
W(t) := \sum_l W^l_t e_l,
\]

where \(e_l\) is the complete orthonormal system of eigenfunctions on \(H\).

For \(\beta \in (0, 2)\), let \(X(t)\) be an independent symmetric \(\beta/2\)-stable subordinator, that is, an increasing one dimensional Lévy process with Laplace Transform

\[
\mathbb{E}e^{-rX(t)} = e^{-t|\beta|/2}, \quad r > 0
\]

The subordinated cylindrical Wiener process \(\{L(t), t \geq 0\}\) on \(H\) is defined by

\[
L(t) := W(X(t)), \quad t \geq 0.
\]

Note in general that \(L(t)\) does not belongs to \(H\). More precisely, \(L(t)\) lives on some larger Hilbert space \(U\) with the \(\gamma\)-radonifying embedding \(H \hookrightarrow U\). Now, let us consider the abstract Itô equation in \([3.2]\) (which we restate here) in \(H = L^2(\mathbb{S}^2)\):

\[
\begin{aligned}
&du(t) + \nuAu(t)dt + B(u(t), u(t))dt + Cu = fdt + GdL(t), \quad u(0) = u_0.
\end{aligned}
\tag{3.14}
\]

Write \([3.2]\) into the usual mild form one has

\[
\begin{aligned}
u(t) &= S(t)u_0 - \int_0^t S(t-s)B(u(s))ds + \int_0^t S(t-s)f ds + \int_0^t S(t-s)GdL(s).
\end{aligned}
\tag{3.15}
\]

where \(S(t)\) is an analytic \(C_0\) semigroup \(\{e^{-t\hat{A}}\}\) generated by \(\hat{A} = \nu A + C\), where \(A\) is the Stokes operator in \(H\). Note that \(\hat{A}\) is a strictly positive selfadjoint operator in \(H\) (that is \(A : D(A) \subset H \to H\), \(\hat{A} = \hat{A}^* > 0\), \(\langle Av, v \rangle \geq \gamma|v|^2\) for any \(v \in D(A)\) for some \(\gamma > 0\) and \(v \neq 0\)). The operator \(G : H \to H\) is a bounded linear operator. For a fixed \(\alpha > 0\) we introduce the process

\[
\begin{aligned}
z_a(t) := \int_0^t e^{-(t-s)(\alpha + \hat{A})}GdL(s)
\end{aligned}
\]

that solves the OU equation

\[
\begin{aligned}
dz_a + (\nu A + C + \alpha)z_a dt = GdL(t), \quad t \geq 0.
\end{aligned}
\tag{3.16}
\]

Now let \(v(t) = u(t) - z_a(t)\). Then

\[
\begin{aligned}
&\begin{cases}
\frac{dv(t) + \nu A(u(t) - z_a(t))dt + B(u(t))dt + Cu(u - z_a(t))dt - \alpha z_a(t)dt = f dt, \\
v(0) = v_0.
\end{cases}
\end{aligned}
\]

The problem becomes

\[
\begin{aligned}
&\begin{cases}
\frac{dv(t) + \nu Av(t)dt + B(v(t) + z_a(t))dt + Cv(t)dt - \alpha z_a(t)dt = f dt, \\
v(0) = v_0.
\end{cases}
\end{aligned}
\]

Convert into standard form,

\[
\begin{aligned}
&\begin{cases}
\frac{dv(t) + \nu A(v(t))dt = f - B(v(t) + z_a(t)) + \alpha z_a(t), \\
v(0) = v_0.
\end{cases}
\end{aligned}
\tag{3.17}
\]
where $\frac{d^+v}{dt}$ is the right-hand derivative of $v(t)$ at $t$. Solution to equation (3.17) will be understood in the mild sense, that is as a solution to the integral equation

$$v(t) = S(t)v(0) + \int_0^t S(t-s)(f - B(v(s) + z_\alpha(s)) + az_\alpha(s))ds,$$  

(3.18)

with $v_0 = u_0 - z_\alpha(0)$. One can easily show that (3.17) and (3.18) are equivalent for $v \in C([0, \infty); V) \cap L^2_{\text{loc}}([0, \infty); D(A))$. More precisely, (3.18) follows from (3.17) via integration. Then (3.17) follows from (3.18) via the usual continuity argument (see Lebesgue Dominated Convergence Theorem in Appendix), namely, differentiation the integral when integrand is continuous.

For brevity, we write $z_\alpha$ as $z$. Let us now explain what is meant by a solution of (3.2).

**Definition 3.9.** Suppose that $z \in L^4_{\text{loc}}([0, T); L^4(\mathbb{S}^2) \cap H)$, $v_0 \in H$, $f \in V'$. A weak solution to (3.2) is a function $v \in C([0, T); H) \cap L^2_{\text{loc}}([0, T); V)$ satisfies (3.17) in weak sense for any $\phi \in V$, $T > 0$,

$$\partial_t(v, \phi) = (v_0, \phi) - v(v, A\phi) - b(v + z, v + z, \phi) - (Cv, \phi) + (az + f, \phi).$$  

(3.19)

Equivalently, (3.17) holds as an equality in $V'$ for a.e. $t \in [0, T]$.

Now if $f \in H$, and the following regularity is satisfied,

$$v \in L^\infty([0, T); V) \cap L^2([0, T); D(A)),$$

(3.20)

then the solution becomes strong. More precisely,

**Definition 3.10 (Strong solution).** Suppose that $z \in L^4_{\text{loc}}([0, T); L^4(\mathbb{S}^2) \cap H)$, $v_0 \in V$, $f \in H$. We say that $u$ is a strong solution of the stochastic Navier-Stokes equations (3.2) on the time interval $[0, T]$ if $u$ is a weak solution of (3.2) and in addition

$$u \in L^\infty([0, T); V) \cap L^2([0, T); D(A)).$$

(3.21)

The main theorems proved in this paper are the following.

**Theorem 3.11.** Suppose that $\alpha \geq 0$, $z \in L^4_{\text{loc}}([0, \infty); L^4(\mathbb{S}^2) \cap H)$, $v_0 \in H$ and $f \in V'$. Then there exists a unique solution $v$ of equation (3.17). In particular, if

$$\sum_{\ell = 1}^\infty |a_\ell|^\beta |A_\ell|^{\beta/2} < \infty,$$

then the theorem holds.

Next, we show the weak solution depends continuously on initial data, noise and forcing terms.

**Theorem 3.12.** Assume that,

$$u_n^0 \to u \quad \text{in} \quad H,$$

and for some $T > 0$,

$$z_n \to z \quad \text{in} \quad L^4([0, T]; L^4(\mathbb{S}^2) \cap H) \quad f_n \to f \quad \text{in} \quad L^2([0, T); V').$$

(3.22)

Let us denote by $v(t, z)u_0$ the solution of (3.17) and by $v(t, z_n)u_n^0$ the solution of (3.17) with $z, f, u_0$ being replaced by $z_n, f_n, u_n^0$. Then

$$v(\cdot, z_n)u_n^0 \to v(\cdot, z)u_0 \quad \text{in} \quad C([0, T); H) \cap L^2([0, T); V).$$
Theorem 3.13. Suppose that $\alpha \geq 0$, $z \in L^4_t([0, \infty); \mathbb{L}^4(S^2) \cap H)$, $v_0 \in H$ and $f \in V'$. Then there exists $P$-a.s. a unique solution $u \in D([0, \infty); H) \cap L^2_{loc}([0, \infty); V)$ of equation (3.17). In particular, if
\[
\sum_{l=1}^{\infty} |q_l|^2 \lambda_{l}\beta_l < \infty ,
\]
then the theorem holds.

Analogously to Theorem 3.12, the càdlàg in time) solution to the SNSE depends continuously on initial data, noise and forcing terms.

Theorem 3.14. Assume that,
\[
u^n_0 \to u \quad \text{in} \quad H
\]
and for some $T > 0$,
\[
 z_n \to z \quad \text{in} \quad L^4_t([0, T]; \mathbb{L}^4(S^2) \cap H) \quad f_n \to f \quad \text{in} \quad L^2(0, T; V').
\]  
(3.23)
Let us denote by $u(t, z)u_0$ the solution of (3.14) and by $u(t, z_n)u^n_0$ the solution of (3.17) with $z, f, u_0$ being replaced by $z_n, f_n, u^n_0$. Then
\[
 u(\cdot, z_n)u^n_0 \to u(\cdot, z)u_0 \quad \text{in} \quad D([0, T]; H) \cap L^2(0, T; V).
\]
In particular, $u(T, z_n)u^n_0 \to u(T, z_n)u_0$ in $H$.

Moreover, the weak solution is found to be strong indeed.

Theorem 3.15. Assume that $\alpha \geq 0$, $z \in L^4_t([0, \infty); \mathbb{L}^4(S^2) \cap H)$, $f \in H$ and $v_0 \in H$. Then, there exists unique solution of (3.18) in the space $C(0, T; H) \cap L^2(0, T; V)$ which belongs to $C(h, T; V) \cap L^2_{loc}(h, T; D(A))$ for all $h > 0$ and $T > 0$. Moreover, if $v_0 \in V$, then $v \in C(0, T; H) \cap L^2_{loc}(0, T; D(A))$ for all $T > 0$. In particular, $v(T, z_n)u^n_0 \to v(T, z_n)u_0$ in $H$. Moreover, if
\[
\sum_{l=1}^{\infty} |q_l|^2 \lambda_{l}\beta_l < \infty ,
\]
then the theorem holds.

Theorem 3.16. Assume that $\alpha \geq 0$, $z \in L^4_t([0, \infty); \mathbb{L}^4(S^2) \cap H)$, $f \in H$ and $v_0 \in H$. Then, there exists $P$-a.s. unique solution of (3.22) in the space $D(\epsilon, T; H) \cap L^2(\epsilon, T; V)$ which belongs to $D(\epsilon, T; V) \cap L^2_{loc}(\epsilon, T; D(A))$ for all $\epsilon > 0$ and $T > 0$. Moreover, if $v_0 \in V$, then $u \in D(0, T; V) \cap L^2_{loc}(0, T; D(A))$ for all $T > 0, \omega \in \Omega$. Moreover, if
\[
\sum_{l=1}^{\infty} |q_l|^2 \lambda_{l}\beta_l < \infty ,
\]
then the theorem holds.

4. Weak solutions

In this section, we prove the existence and uniqueness of weak solution.
4.1. Existence of Weak solutions via Galerkin approximation. Our aim in this subsection is to prove the existence part of Theorem 3.11. First, we construct approximate solutions and deduce local existence and uniqueness of the solutions of the Galerkin equations of SNSE. For a comprehensive overview of Galerkin methods on spheres, we refer readers to [17]. Next, we obtain uniform a priori estimates on the solutions \( v_L \) and hence show that they exist globally in time. Last but not least, we extract a convergent subsequence and pass to the limit in the equation.

First, we need some preliminary definitions.

Loosely speaking, a solution to problem (3.2) is a process \( u(t), t \geq 0 \), which can be written in form \( u(t) = v(t) + z(t) \), where \( z(t), t \in \mathbb{R} \), is a stationary OU process with drift \(-vA - C - \alpha I\), i.e. a stationary solution of (3.3) and \( v(t), t \geq 0 \), is the solution of the following problem (with \( v_0 = u_0 - z(0) \)):

\[
\begin{aligned}
\partial_t v(t) &= -vAv(t) + B(v) - B(v, z) - B(z, v) - Cv(t) + F, \\
v(0) &= v_0.
\end{aligned}
\] (4.1)

**Definition 4.1.** Suppose that \( z \in L^2_{\text{loc}}([0, \infty); L^2(S^2)) \), \( f \in V' \) and \( v_0 \in H \). A function \( v \in C([0, \infty); H) \cap L^2_{\text{loc}}([0, \infty); V) \) is a solution to problem (4.1) if and only if \( v(0) = v_0 \) and (4.1) holds in the weak sense, i.e. for any \( \phi \in V \),

\[
\partial_t \langle v, \phi \rangle = -\langle v, A\phi \rangle - b(v, \phi) - b(v, z, \phi) - b(z, v, \phi) - \langle Cv, \phi \rangle + \langle F, \phi \rangle
\] (4.2)

We remark that for (4.2) to make sense, it suffices to assume that \( v \in L^2([0, T]; V) \cap L^\infty(0, T; H) \).

4.1.1. Local existence and uniqueness of solutions. For any \( L \in \mathbb{N} \) denote

\[
H_L = \text{linspan}\{Z_{l,m} : l = 1, \ldots, L; |m| \leq l\}
\]
as the linear space spanned by the first \( L \) eigenfunctions in an orthonormal basis \( \{Z_{l,m} : l = 1, \ldots, L; |m| \leq l\} \) of \( H \), which may be assumed to be the orthogonal in \( V \). In other words, \( H_L \) is the \( L \)-dimensional subspace of \( V \) and \( P_L \) is the orthogonal projection from \( H \) onto \( H_L \). We consider the following Galerkin approximation for (4.1) on the finite dimensional space \( H_L \):

\[
\begin{aligned}
\partial_t v_L(t) &= P_L[-vAv_L - B(v) - B(v, z) - B(z, v) - Cv(t) + F], \\
v(0) &= P_L v_0,
\end{aligned}
\] (4.3)

where \( F = -B(z) + az + f \). In view of (4.3), \( B(z) \) belongs to the dual space \( V' \) and so \( F \in L^2(0, T; V') \).

We notice that (4.3) is an Ordinary Differential Equations (ODE) in \( H_L \), hence the existence and uniqueness of solution \( v_L \) of (4.3) defined on \([0, T_L]\) follows from standard theory of ODE. Since the right-hand side has a bilinear form, it is not clear if \( v_L \) can be defined globally or it could blow up at some time \( T_L < \infty \). We will show in the next subsection that the \( H \) norm of the solution stay finite as \( t \to T_L \), which implies the solution indeed exists globally in time.

4.1.2. Uniform a-priori estimates on the solutions \( v_L \). From the last subsection we already know that \( v_L \) exists on some time interval \([0, T_L]\). Now we want to send \( L \) to infinity and to show a subsequence of the solution \( v_L \) of the approximate problem converges to a weak solution to (4.1). For this, we need some uniform estimates. Take the inner product of (4.3) in \( H \) with \( v_L(t) \) we obtain

\[
\langle \partial_t v_L(t), v_L(t) \rangle = -\langle P_L Av_L, v_L \rangle - \langle P_L B(v_L), v_L \rangle - \langle P_L B(v_L, z), v_L \rangle - \langle P_L B(z, v_L), v_L \rangle - \langle P_L Cv_L, v_L \rangle + \langle F, v_L \rangle.
\]
We notice that
\[
(\partial_t v_L(t), v_L(t)) = \frac{1}{2} \frac{d^+}{dt} |v_L(t)|^2,
\]
\[-v(P_L A v_L, v_L) = -v(A v_L, v_L) = -v|v_L|^2_v,
\]
and by (2.44),
\[
(P_L B(v_L), v_L) = (B(v_L), v_L) = b(v_L, v_L, v_L) = 0, \quad (P_L B(z, v_L), v_L) = b(z, v_L, v_L) = 0,
\]
and by (2.37),
\[
(P_L C v_L, v_L) = (C v_L, v_L) = 0.
\]
Therefore, for any \( t > 0 \) we have
\[
\frac{1}{2} \frac{d^+}{dt} |v_L(t)|^2 = -v|v_L(t)|^2_v - b(v_L(t), v_L(t), v_L(t)) + \langle F(t), v_L(t) \rangle \quad t \in [0, T_L).
\]
Using (2.44) and (2.52) and the Young inequality \((a b \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{with} \quad p = 4, \quad q = 4/3)\), we have
\[
|b(v_L, v_L, z)| \leq C|v_L|_{L^4(S^2)}|v||z|_{L^4(S^2)} \leq C|v_L|^{1/2}|v_L|^{3/2}|z|_{L^2(S^2)} \leq C|v_L|^{1/2}|v_L|^{3/2}|v| \leq \frac{C}{\nu^3}|v_L|^2|z|^\nu + \frac{\nu}{4}|v_L|^2_v. \quad (4.4)
\]
We also have
\[
\langle F(t), v_L \rangle \leq |F(t)|_v|v_L|_v \leq \frac{1}{\nu}|F(t)|_v^2_v + \frac{\nu}{4}|v_L|^2_v.
\]
Hence we obtain
\[
\partial_t |v_L(t)|^2 + v|v_L|^2_v \leq \frac{C}{\nu^3}|v_L|^2|z|^\nu + \frac{2}{\nu}|F(t)|_v^2_v, \quad t \in [0, T_L). \quad (4.5)
\]
Invoking Gronwall Lemma, one has
\[
|v_L(t)|^2 \leq |v(0)|^2 \exp \left( \frac{C}{\nu^3} \int_0^t |z(\tau)|_\nu^\nu d\tau \right) + \int_0^t \frac{2}{\nu}|F(s)|_v^2_v \exp \left( \frac{C}{\nu^3} \int_s^t |z(\tau)|_\nu^\nu d\tau \right) ds, \quad t \in [0, T_L).
\]
It follows that \( v_L \) does not blow up in finite time and so \( T_L = \infty \).
Let us fix \( T > 0 \). Denoting
\[
\psi_T(z) = \exp \left( \frac{C}{\nu^3} \int_0^T |z(\tau)|_\nu^\nu d\tau \right) < \infty, \quad C_F = \int_0^T \frac{2}{\nu}|F(t)|_v^2_v \exp \left( \frac{C}{\nu^3} \int_s^T \frac{1}{2\nu}|z(\tau)|^2_\nu d\tau \right) dt
\]
We find that
\[
\sup_{t \in [0, T]} |v_L(t)|^2 \leq |v(0)|^2 \psi_T(z) + C_F \leq |v(0)|^2 \psi_T(z) + C_F < \infty \quad t \in [0, T) \quad (4.6)
\]
which implies that \( \{v_L : L \in \mathbb{N}\} \) is bounded uniformly (in \( L \)) in the norm of \( L^\infty(0, T; H) \).
Next we integrate in time \( (4.5) \) from 0 to \( T \) and then using \( (4.6) \) to obtain
\[
|v_L(T)|^2 + v \int_0^T |v_L(t)|^2_v dt + \frac{C}{\nu^3} \int_0^T |z(t)|^2_v |v_L(t)|^2_v dt + \frac{2}{\nu} \int_0^T |F(t)|_v^2_v dt.
\]
We will now pass to the limits by sending $L$ to infinity, to build a weak solution of our original problem \((4.1)\). For this we need some convergence results. Notice that the above inequality implies that
\[
\text{the sequence } \{v_L : L \in \mathbb{N}\} \text{ is bounded uniformly in } L^2(0, T; V) \tag{4.7}
\]
Therefore we have shown that $v_L$ is uniformly bounded in $L$ in the norm of $L^\infty(0, T; H) \cap L^2(0, T; V)$. These uniform bounds imply that $\{v_L\}$ has a subsequence that converges weakly in $L^2(0, T; V)$ and weakly* in $L^\infty(0, T; H)$. Then by the Banach-Alaoglu theorem, one can extract a subsequence $\{v_{L_k} \subset v_L\}$ and some limit function $v \in L^2(0, T; V)$ such that
\[
\begin{aligned}
v_L &\rightharpoonup v, \quad \text{weakly in } L^2(0, T; V), \\
v_L &\rightharpoonup v, \quad \text{weakly* in } L^\infty(0, T; H).
\end{aligned} \tag{4.8}
\]
Now we need to show
\[
v_L \to v \quad \text{strongly in } L^2(0, T; H), \tag{4.9}
\]
and this strong convergence result allows us to choose $v_L$ such that $v_L \to v$ in $L^2(S^2)$ for all $t \geq 0$.

The crux to prove \((4.9)\) is a compactness theorem which involves fractional derivatives.

Now, let us assume that $X_0 \subset X \subset X_1$ are Hilbert spaces with the injection being continuous and the injection of $X_0$ into $X$ is compact. If $v$ is a function from $\mathbb{R}$ to $X_1$, let us denote $\hat{v}$ the Fourier Transform as
\[
\hat{v}(\tau) = \int e^{-2\pi i \tau t} v(t) \, dt, \quad \tau \in \mathbb{R}. \tag{4.10}
\]
The fractional derivative in $t$ of order $\gamma$ of $v$ is the Fourier transform of the $X_1$-valued function $\{\mathbb{R} \ni \tau \mapsto (2i\pi \tau)^\gamma \hat{v}(\tau)\}$:
\[
\widehat{D^\gamma_t} v(\tau) = (2i\pi \tau)^\gamma \hat{v}(\tau), \quad \tau \in \mathbb{R}.
\]
The definition makes sense, observe that the first derivative of \((4.10)\) via integration by parts is obtained as,
\[
\widehat{D^\gamma_t} v(\tau) = \int_\mathbb{R} e^{-2i\pi \tau t} v'(t) \, dt \\
= e^{-2i\pi \tau t} v(t) \bigg|_\infty^\infty - \left(-2\pi i \tau \int_\mathbb{R} v(t) e^{-2i\pi \tau t} \, dt\right)
\]
Since $|v(t)| \to 0$ as $|t| \to \infty$, the first term vanishes, and so
\[
\widehat{D^\gamma_t} v(\tau) = 2\pi i \tau \hat{v}_L(t).
\]
For a given $\gamma > 0$, we define the space
\[
\mathcal{H}^\gamma(\mathbb{R}; X_0, X_1) = \{v \in L^2(\mathbb{R}; X_0) : D^\gamma_t v \in L^2(\mathbb{R}; X_1)\}, \tag{4.11}
\]
as a Hilbert space equipped with the norm
\[
\|v\|_{\mathcal{H}^\gamma(\mathbb{R}; X_0, X_1)} = (\|v\|^2_{L^2(\mathbb{R}; X_0)} + \|\tau\|^\gamma \|\hat{v}\|^2_{L^2(\mathbb{R}; X_1)})^{1/2}.
\]
For a given set $K \subset \mathbb{R}$, the subspace $\mathcal{H}^\gamma_{K,Z}$ of $\mathcal{H}^\gamma = \mathcal{H}^\gamma_{\mathbb{R}; X_0, X_1}$ is defined by
\[
\mathcal{H}^\gamma_{K,Z}(\mathbb{R}; X_0, X_1) = \{u \in \mathcal{H}^\gamma(\mathbb{R}; X_0, X_1), \text{spt } u \subset K\}. \tag{4.12}
\]

**Theorem 4.2** (Chapter III, Theorem 2.2 [27]). Suppose that $X_0 \subset X \subset X_1$ is a Gelfand triple of Hilbert spaces and the injection of $X_0$ into $X$ is compact. Then for any bounded set $K \subset \mathbb{R}$ and $\gamma > 0$, the injection of $\mathcal{H}^\gamma_{K,Z}(\mathbb{R}; X_0, X_1)$ into $L^2(\mathbb{R}; X)$ is compact.
To apply this compactness theorem, one first need to identify bounded set. For this let 
\[ \bar{v}_L = 1_{(0, T)} v_L, \]
and Let Fourier Transform in the time variable of \( \bar{v}_L \) denotes by \( \hat{v}_L \). We would like to show that
\[ \int_{\mathbb{R}} |\tau|^{2\gamma} |\hat{v}_L(\tau)|^2 d\tau < \infty. \]  
(4.13)

Observe that (4.3) can be written as
\[ \frac{d^+}{dt} v_L = \hat{f}_L + v_L(0) \delta_0 - v_L(T) \delta_T, \]
(4.14)
where \( \delta_0 \) and \( \delta_T \) are respectively the Dirac distributions at 0 and \( T \) and
\[ f_L = F - v A v_L - B v_L - B(v_L, z) - B(z, v_L) - C v_L. \]

Apply the Fourier Transform to (4.14) (with respect to the time variable \( t \)) we obtain
\[ \hat{D}_t v(\tau) = (2i \pi \tau) \hat{v}(\tau) = \hat{f}_L(\tau) + v_L(0) \delta_0 - v_L(T) \delta_T, \]
(4.15)
where \( \hat{v}_L \) and \( \hat{f}_L \) are the Fourier Transform of \( v_L \) and \( f_L \) respectively. Multiply this equation with the Fourier Transform of \( v_L \) one obtain for each \( \tau \in \mathbb{R} \) that
\[ 2i \pi \tau |\hat{v}_L(\tau)|^2 = (\hat{f}_L(\tau), \hat{v}_L(\tau)) + (v_L(0), \hat{v}_L(\tau)) - (v_L(T), \hat{v}_L(\tau)) \exp(-2i \pi \tau T). \]
(4.16)
From the Parseval equality, (2.37) and (2.44), one has
\[ (\hat{f}_L, \hat{v}_L) = (f_L, v_L) = (F, v_L) - v(A v_L, v_L) - b(v_L, z, v_L). \]  
(4.17)
Therefore, via Cauchy Schwartz and (4.4), we have
\[ |(f_L, v_L)| \leq |F| |v_L| + v |v_L| + \frac{C}{v^3} |v_L|^2 |z|_{L^4(S^2)}^4 + \frac{C}{v^3} |v_L|^2 |z|_{L^4(S^2)}^4 + \frac{v}{4} |v_L|^2. \]  
(4.18)
Now due to (4.7), \( |v_L| \leq C_L \), then integrate over time we conclude
\[ \int_0^T |f_L| |v_L| \, dt \leq \int_0^T \left( |F| + \frac{5v}{4} |v_L| + \frac{C_L}{v^3} |z|_{L^4(S^2)}^4 \right) \, dt \]
(4.19)
and this stays bounded (w.r.t. \( L \)) as \( F \in L^2(0, T; V), z \in L^4_{\text{loc}}([0, \infty); L^4(S^2)) \) remains in a bounded set of \( L^2(0, T; V) \). Hence, there exists a \( C > 0 \) such that
\[ \sup_{L \in \mathbb{N}} \sup_{\tau \in \mathbb{R}} |\hat{f}_L(\tau)| v_L \leq C. \]
(4.20)
Now observe from (4.16) that
\[ 2i \pi \tau |\hat{v}_L(\tau)|^2 = (\hat{f}_L(\tau), \hat{v}_L(\tau)) + (v_L(0), \hat{v}_L(\tau)) - (v_L(T), \hat{v}_L(\tau)) \exp(-2i \pi \tau T) \]
\[ \leq (\hat{f}_L(\tau), \hat{v}_L(\tau)) + v_L(0)|\hat{v}_L(\tau)| + v_L(T)|\hat{v}_L(\tau)| e^{-2i \pi \tau T}. \]
Then from (4.7), we see
\[ |v_L(0)| \leq c_1, |v_L(T)| \leq c_1. \]
(4.21)
Combined with (4.20), one deduces that
\[ |\tau| |\hat{v}_L|^2 \leq c_2 |v_L| + c_3 |v_L| \leq c_4 |v_L| \cdot |v_L| \cdot |v| \]
(4.22)
Let us fix $\gamma \in (0, 1/4)$. Observe that
\[ |\tau|^{2\gamma} \leq C(\gamma)(1 + |\gamma|)/(1 + |\tau|^{1-2\gamma}) \quad \forall \tau \in \mathbb{R}, \tag{4.23} \]
we infer that
\[
\int_{\mathbb{R}} |\tau|^{2\gamma} \leq C(\gamma) \int_{\mathbb{R}} \frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}} |\hat{v}_L(\tau)|^2 d\tau \\
\leq c_5 \int_{\mathbb{R}} |\tau| |\hat{v}_L(\tau)|^2 d\tau + c_6 \int_{\mathbb{R}} |\hat{v}_L(\tau)|^2 d\tau.
\]
In the last step, the first integral is finite since $\gamma < 1/4$\(^3\). Then base on Parseval inequality, one has $|\hat{v}_L| = |v_L|$ and $|\hat{v}_L|_V = |v_L|_V$ which is bounded according to (4.8). Hence we have shown
\[ \{v_L\} \text{ is bounded in } \mathcal{C}^\gamma(\mathbb{R}; V, H). \tag{4.24} \]
This allows us to apply the compactness theorem involves fractional derivatives.

Since the sphere $S^2$ is bounded, the embedding $H^1(S^2) \hookrightarrow L^2(S^2)$ is compact and by (4.24), the sequence $\{v_L : L \in \mathbb{N}\}$ is bounded in $H^\gamma(0, T; H^1(S^2), L^2(S^2))$. Due to (4.24) and Theorem 4.2 we deduce that there exists a subsequence $\{v_{L_k}\}$ such that $\{v_{L_k}\} \to v$ strongly in $L^2(0, T; L^2(S^2))$
\[ v_L \to v \quad \text{strongly} \quad L^2(0, T; H). \tag{4.25} \]
The convergence result (4.3) and (4.9) enable us to pass to the limit. Now we need to show the limit function indeed satisfies (3.17). Take a $C^1([0, T]; \mathbb{R})$ function $\psi$ with $\psi(T) = 0$. Multiply (4.3) with $\psi(T)\phi$ where $\phi \in H_1$ for some $l \in \mathbb{N}^*$, then integrate by parts, one gets
\[
- \int_0^T (v_L(t), \psi(t)\phi) dt = -\nu \int_0^T (P_L A v_L(t), \psi(t)\phi) dt \\
- \int_0^T (P_L B v_L(t), \psi(t)\phi) dt - \int_0^T P_L B v_L(t), z, \psi(t)\phi) dt \\
- \int_0^T P_L B z, v_L(t), \psi(t)\phi) dt - \int_0^T \langle P_L F(t), \psi(t)\phi \rangle dt + \langle v_L(0), \psi(0)\phi \rangle. \tag{4.26}
\]
Now we aim to pass to the limit of (4.26) when $L \to \infty$. Since $\psi(\cdot)\phi \in L^2(0, T; H), \psi \in C^1(0, T; \mathbb{R})$, then $\psi(\cdot)\phi \in L^2(0, T; V)$, combine with the first part of (4.8), we have
\[ \int_0^T (v_L(t), \psi(t)\phi) dt \to \int_0^T (v(t), \psi(t)\phi) dt \quad \text{as} \quad L \to \infty. \tag{4.27} \]
Hence the left hand side of (4.26) converges to $- \int_0^T (v(t), \psi(t)\phi) dt$.

Next, for the linear term, let us take $l \in L$ so that $H_l \subset H_1$ and $P_L \phi = \phi$. For the first term on the right hand side of (4.8), observe that
\[
\int_0^T (P_L A v_L(t), \psi(t)\phi) dt = \int_0^T (A v_L(t), \psi(t)\phi) dt \\
= \int_0^T (A v_L(t), \psi(t)\phi) dt \\
= \int_0^T (v_L(t), \psi(t)\phi) dt.
\]
\(^3\)This integral converges iff. $\int_1^\infty x^{2\gamma - 1} dx < \infty$. This holds iff. $2(2\gamma - 1) < -1$. 24
Again, since $\psi(\cdot)\phi \in L^2(0, T; V)$, it follows from (4.8) that, as $L \to \infty$,

$$
\int_0^T (P_L v_L(t), \psi(t)\phi) dt \to \int_0^T (v(t), \psi(t)\phi)_V dt,
$$

(4.28)

or

$$
\int_0^T (v_L(t) - v(t), \psi(t)\phi)_V dt \to 0.
$$

(4.29)

**Lemma 4.3.** If $v_m \to v$ in $L^2(0, T; V)$ and strongly in $L^2(0, T; H)$, then for any vector function $u : [0, T] \times \mathbb{S}^2 \to \mathbb{R}^2$ with components in $C^1(\mathbb{S}^2 \times [0, T])$,

$$
\int_0^T b(v_m(t), v_m(t), u(t)) dt \to \int_0^T b(v(t), v(t), u(t)) dt.
$$

(4.30)

**Proof.**

$$
\int_0^T b(v_m, v_m, u) dt = -\int_0^T b(v_m, u, v_m) dt
$$

$$
= -\sum_{i,j=1}^3 \int_0^T \int_{\mathbb{S}^2} (v_m)_i (D_i u_j)(v_m)_j dx dt.
$$

Now our two assumptions on $v_m$ imply that

$$
v_m \to v \text{ in } H,
$$

(4.31)

$$
Dv_m \to Dv \text{ weakly in } H.
$$

(4.32)

and which further implies that

$$
|v_m(t)|^2 + \sup_{0 \leq t \leq T} \int_0^T |v_m(t)|_V^2 dt \leq C.
$$

Hence, there exists a function $g(t)$ for which the term $F_m = (v_m)_i (D_i u_j)(v_m)_j$ is dominated for all $t \in [0, T]$. Now by (4.31),

$$
|v_m| \leq c_1, \quad \text{uniformly in } m,
$$

$$
|Dv| \leq c_2.
$$

Hence $F_m(t) \leq g(t) = c_1^2 c_2$. Then by usual continuity argument one has

$$
\lim_{m \to \infty} \int_0^T \sum_{i,j=1}^3 \int_{\mathbb{S}^2} (v_m)_i (D_i u_j)(v_m)_j dx dt = \int_0^T \lim_{m \to \infty} \sum_{i,j=1}^3 \int_{\mathbb{S}^2} (v_m)_i (D_i u_j)(v_m)_j dx dt
$$

$$
= \int_0^T \sum_{i,j=1}^3 \int_{\mathbb{S}^2} v_i (D_i u_j) v_j dx dt.
$$
\[
\int_0^T b(v_m, v_m, u) dt = -\int_0^T b(v_m, u, v_m) dt \\
= -\sum_{i,j=1}^3 \int_0^T \int_{\mathbb{S}^2} [v_m](D_i u_j)(v_m) dx dt \\
= -\sum_{i,j=1}^3 \int_0^T \int_{\mathbb{S}^2} v_i(D_j u_j) v_j dx dt \\
= -\int_0^T b(v, u, v) dt \\
= \int_0^T b(v, v, u) dt.
\]

\[\square\]

An alternative proof is the following \[\text{[1]}\].

\textbf{Lemma 4.4.} Suppose \{v_m\} is bounded in \(L^\infty(0, T; H)\), \(v \in L^\infty(0, T; H)\), \(v_m \to v\) in \(L^2(0, T; V)\) and strongly in \(L^2(0, T; L^2_\text{loc}(\mathbb{S}^2))\). Then for any \(w \in L^4(0, T; L^4(\mathbb{S}^2))\),

\[\int_0^T b(v_m(t), w(t), v_m(t) - v(t)) dt \to 0.\]

\textbf{Proof.} In view of \[\text{[2.44]}\], one has \(b(v_m, v_m, w) = -b(v_m, w, v_m)\). One also has \(b(v_m, w, v_m) - b(v, w, v) = b(v_m, w, v_m - v) + b(v_m - v, w, v)\).

Using \[\text{[2.48]}\], combine with the assumption \(v_m \to v\) strongly in \(L^2(0, T; H)\), one also has

\[|b(v_m, w, v_m - v)| = |b(v_m, v_m - v, w)| \leq C|v_m|_{L^4(0, T; L^4(\mathbb{S}^2))}|v_m - v|_{L^4(0, T; L^4(\mathbb{S}^2))}. \quad (4.33)\]

Moreover, invoke the assumption \(v_m \to v\) strongly in \(L^2(0, T; L^2_\text{loc}(\mathbb{S}^2))\) once again, we conclude that

\[\int_0^T b(v_m(t), w(t), v_m(t) - v(t)) dt \to 0. \quad (4.34)\]

Similarly,

\[\int_0^T b(v_m(t) - v(t), v_m(t), v(t)) dt \to 0. \quad (4.35)\]

\[\square\]

Alternatively, one may prove the above Lemma following the proof as in \[\text{[6]}\].

For the second term of the right hand side of \[\text{[4.26]}\] that is,

\[\int_0^T (P_L B(v_L(t)), \psi(t)) dt.\]

We apply Lemma \[\text{[4.4]}\] with \(w(t, \hat{x}) = \psi(t) \phi(\hat{x})\) for \(t \in [0, T], \hat{x} \in S^2\). Since \(P_L\) is self-adjoint in \(H\) and \((P_L B(v_L), \psi(t) \phi) = (B(v_L), P_L \psi(t) \phi) = (B(v_L), \psi(t) \phi) = b(v_L, v_L, \psi(t) \phi)\), one obtains the following convergence:

\[\int_0^T (P_L B(v_L(t)), \psi(t)) dt = \int_0^T b(v_L, v_L, \psi(t) \phi) dt \to \int_0^T b(v(t), v(t), \psi(t) \phi) dt.\]

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Consider the third term on the right hand side of (4.26) since
\[ \int_0^T P_L B(v_L(t), z, \phi(t) \phi) dt = \int_0^T (B(v_L, z), \psi(t) P_L \phi) dt \]
\[ = \int_0^T (B(v_L, z), \psi(t) \phi) dt = \int_0^T b(v_L, z, \psi(t) \phi) dt. \]

Using (2.44) and (2.45) we obtain,
\[ \int_0^T (P_L B(v_L, z), \psi(t) \phi) dt - \int_0^T b(v, z, \psi(t) \phi) dt \]
\[ = \int_0^T b(v_L(t) - v(t), z(t), \psi(t) \phi) dt = \int_0^T b(v_L(t) - v(t), \psi(t) \phi, z) dt \]
\[ \leq \int_0^T |b(v_L(t) - v(t), \psi(t) \phi, z)| dt \]
\[ \leq C \int_0^T |v_L(t) - v(t)||z||\psi(t)\text{curl}\phi|_{L^\infty(S^2)} + |\psi(t)\phi|_{L^\infty(S^2)}. \]

Since \( v_L \to v \) strongly in \( L^2(0, T; H) \) and \( z \in L^4([0, T]; L^4(S^2) \cap H) \) we infer that the last integral converges to 0 as \( L \to \infty \). Hence,
\[ \int_0^T (P_L B(v_L, z), \psi(t) \phi) dt - \int_0^T b(v, z, \psi(t) \phi) dt \to 0. \]

Similarly,
\[ \int_0^T (P_L B(z, v_L), \psi(t) \phi) dt - \int_0^T b(z, v(t), \psi(t) \phi) dt \to 0. \]

For the fifth term on the rhs of (4.26), we have
\[ \int_0^T (P_L F, \psi(t) \phi) dt = \int_0^T \langle F, \psi(t) P_L \phi \rangle dt = \int_0^T \langle F, \psi(t) \phi \rangle dt. \]

Now we recall (4.8) to find upon passing to the weak limit of (4.26) that
\[ - \int_0^T (v(t), \psi'(t) \phi) dt = -v \int_0^T (Av(t), \psi(t) \phi) dt - \int_0^T (B(v(t)), \psi(t) \phi) dt - \int_0^T (B(v(t), z, \psi(t) \phi) \]
\[ - \int_0^T (B(z, v(t)), \psi(t) \phi) dt - \int_0^T \langle F(t), \psi(t) \phi \rangle dt + (v_0, \psi(0) \phi). \]

This equality holds for any \( \phi \in V \) and any \( \psi \in C_0^1([0, T]) \). Hence, \( v \) solves problem (3.19) and so it satisfies (3.17).

To infer \( v \) indeed satisfies (3.17) one also need to show \( v(0) = v_0 \). For this, let us take an arbitrary function \( \phi \in V \) and \( \psi \in C_0^1([0, T]) \). Multiply (3.17) by \( \psi(t) \phi \) then integrate by parts, one gets
\[ - \int_0^T (v(t), \psi'(t) \phi) dt = -v \int_0^T (Av(t), \psi(t) \phi) dt - \int_0^T (B(v(t)), \psi(t) \phi) dt - \int_0^T (B(v(t), z, \psi(t) \phi) \]
\[ - \int_0^T (B(z, v(t)), \psi(t) \phi) dt - \int_0^T \langle F(t), \psi(t) \phi \rangle dt + (v(0), \psi(0) \phi). \]

(4.37)
by comparing with (4.35), one infers that
\[(v(0) - v_0, \phi)\psi(0) = 0.\]
If we choose \(\psi\) with \(\psi(0) = 1\), then necessarily,
\[\langle v(0) - v_0, \phi \rangle = 0, \quad \forall \ \phi \in V.\]
Then since \(V\) is dense in \(H\), the above holds for any \(\phi \in H\). Since \(v(0) - v_0 \in H\), one has \(\langle v(0) - v_0, v(0) - v_0 \rangle\) and so \(v(0) = v_0\).

The final step is to show \(v \in C([0, T]; H)\). Let us first recall the following weak continuity result from Temam [27].

Observe from the ODE
\[
\frac{dv}{dt}(t) + (vA + C)v(t) = \dot{f} - B(v(t) + z_0(t)) + az_0(t)
\]
and lemma 2.7, since each term of the right hand side belongs to \(L^2(0; T; V')\) and so \(\frac{dv}{dt}(t)\) also belongs to \(L^2(0; T; V')\), hence it follows from Lemma 2.5 that \(v\) is a.e. a function continuous from \([0, T]\) into \(H\). Thus
\[
v \in C([0, T]; H).
\]

Combine with the earlier result (4.25) we conclude that \(v \in L^2(0; T; V) \cap C([0, T]; H)\). Note, the solution is in \(L^2(0, T; L^2(S^2))\) as well. To see this,
\[
\int_0^T \|v(t)\|_{L^2(S^2)}^2 \leq \|v\|_{L^2(S^2)}^2 < \infty,
\]
due to the interpolation inequality in p.12 [10].

So, the proof of existence of global weak solutions is completed. To complete the proof of Theorem 3.11 we now prove uniqueness using the classical argument of Lion and Prodi [19].

4.1.3. Uniqueness of solutions. Suppose \(v_1, v_2\) are two solutions of (3.17) with the same initial condition. Let \(w = v_1 - v_2\), then \(w\) satisfies
\[
\begin{align*}
\partial_t w + vA w &= -B(w, z) - B(z, w) - B(w, v_1) - B(v_2, z) - Cw, \\
w(0) &= 0.
\end{align*}
\]
Multiply (4.40) both sides with \(w\) and integrate against \(w\), using Lemma 2.5, equations 2.44, we get
\[
\partial_t \|w\|^2 + 2|w|^2 \leq -2b(w, z, w) - 2b(w, v_n, w),
\]
Since \(|b(w, z, w)| \leq C|w||w|z|v|\) and \(|b(w, w, v)| \leq C|w||w||v|\), the right hand side
\[
\leq C|w||w|(|z|v + |v_1|v).
\]
Then via usual Young inequality with \(a = \sqrt{v}|w|v\) and \(b = \frac{C}{\sqrt{v}}|w|(|z|v + |v_n|v)\), one has
\[
|b(w, w, v)| \leq \frac{|v||w|^2}{2} + \frac{C}{2|v|}|w|^2(|z|v + |v_1|v)^2
\]
Therefore, by Gronwall lemma one obtains
\[
\partial_t |w|^2 \leq \frac{C}{2|v|}(|z|^2v + |v_1|^2v)|w|^2.
\]
and combine with \( w_0 = v_{1,0} - v_{2,0} = 0 \), it is easy to show

\[
|w(t)|^2 \leq |w(0)|^2 \exp \left( \frac{C}{2V} \int_0^T |z(t)|^2 + |v_1(t)|^2 \right) \, dt < \infty,
\]
as \( \int_0^T |z(t)|^2 + |v_1(t)|^2 \, dt < \infty \). Now, since \( w(0) = 0 \), necessarily \( w(t) \) must be 0.

Therefore Theorem 4.1.1 is proved.

4.1.4. Continuous dependence on initial data, noise and force. This subsubsection is devoted to the proof of Theorem 4.1.2. Write

\[
v_n(t) = v(t, z_n), v(t) = v(t, z), y_n(t) = v(t, z_n) - v(t, z), \quad t \in [0, T],
\]

Then it is clear that \( y_n \) solves

\[
\begin{align*}
\partial_t y_n &= -vAy_n - B(v_n(t) + z_n(t)) + B(v(t) + z(t)) - Cy_n + \alpha z_n + \tilde{f}_n, \\
y_n(0) &= u_n^0 - u_0,
\end{align*}
\]

Since \( y_n \in L^2(0, T; V) \) and \( \partial_t y_n \in L^2(0, T; V') \), it follows from lemma 2.25 that the function \( |y_n|^2 \) is absolutely continuous on \( (0, T) \) and \( \frac{1}{2} \partial_t |y_n(t)|^2 = \langle \partial_t y_n(t), y_n(t) \rangle \) holds in the weak sense. Moreover, by equation (2.30) we have \( \langle Ay_n(t), y_n(t) \rangle = |\nabla y_n(t)|^2 \) a.e. on \( (0, T) \) and \( Cy_n, y_n = 0 \) and so we arrive with,

\[
\begin{align*}
\frac{1}{2} \partial_t |y_n(t)|^2 + v|\nabla y_n(t)|^2 &= -b(y_n, v_n, y_n) - b(v, y_n, y_n) - b(\mathcal{A}z_n, v_n, y_n) - b(z, y_n, y_n) - b(v_n, \mathcal{A}z_n, y_n) \\
&\quad - b(y_n, z, y_n) - b(z, \mathcal{A}z_n, y_n) - b(\mathcal{A}z_n, z, y_n) + \alpha(\mathcal{A}z_n, y_n) + (\tilde{f}_n, y_n), \quad t \geq 0.
\end{align*}
\]

Using the Young inequality, we have

\[
\begin{align*}
b(y_n, v_n, y_n) &\leq |y_n|_{L^4}^2 |v_n|_V \quad \text{via inequality (2.48)} \\
&\leq |y_n||v_n||v_n|_V \quad \text{via (2.52)}
\end{align*}
\]

With \( ab = \sqrt{\frac{10}{20}|y_n|_V \sqrt{\frac{10}{20}|v_n|_V |y_n|_V} \), \( p = 2 \)

\[
\begin{align*}
&\leq \frac{1}{20} |y_n|^2_V + \frac{5}{20} |v_n|^2 |y_n|^2.
\end{align*}
\]

Similarly,

\[
\begin{align*}
b(v, y_n, y_n) &\leq |v|_{L^4}^2 |y_n|_V |y_n|_{L^4} \quad \text{via inequality (2.52)} \\
&\leq |v|_{L^4}^2 |y_n|^2 |y_n|^2.
\end{align*}
\]

Now using Young inequality with \( p = 4/3 \) and \( a = (\frac{15}{2})^{-3/4} |y_n|^3 / 2 \) and \( b = (\frac{15}{2})^{3/4} |y_n|^{1/2} |v|_{L^4}^2 \),

\[
\begin{align*}
&\leq \frac{5}{20} |y_n|^2_V + \frac{155}{44} |y_n|^2 |v|_{L^4}^2.
\end{align*}
\]

\[
\begin{align*}
b(\mathcal{A}z_n, v_n, y_n) &\leq |\mathcal{A}z_n|_{L^4}^2 |y_n|_V |v_n|_{L^4} \quad \text{via inequality (2.52)} \\
&\leq \frac{5}{20} |y_n|^2_V + \frac{5}{20} |\mathcal{A}z_n|_{L^4}^2 |v_n|_V |v_n|^2.
\end{align*}
\]
Now using Young inequality with $p = 4/3$ and $a = (\frac{15}{4v})^{-\frac{3}{4}}|v_n|^\frac{3}{2}$ and $b = (\frac{15}{4v})^\frac{3}{4}|y_n|^\frac{1}{2}|z|_{L^1(\mathbb{S}^2)}$,

$$b(z, y_n, y_n) \leq |z|_{L^1(\mathbb{S}^2)}|y_n|\sqrt{|y_n|L^1(\mathbb{S}^2)}$$

$$\leq |z|_{L^1(\mathbb{S}^2)}|y_n|\sqrt{|y_n|}L^1(\mathbb{S}^2)$$

Hence we have, \(b(z, y_n, y_n) \leq \frac{v}{20}|y_n|^2 + \frac{15^2}{4v^3}|y_n|^2|z|_{L^1(\mathbb{S}^2)}L^1(\mathbb{S}^2).\)

$$b(v_n, Z_n, y_n) \leq |v_n|L^1(\mathbb{S}^2)|v_n||y_n|Z_n|_{L^1(\mathbb{S}^2)}$$

$$\leq \frac{v}{20}|y_n|^2 + \frac{5}{v}|z_n|^2|y_n|^2,$$

$$b(y_n, z, y_n) \leq |y_n|^2|z_n|L^1(\mathbb{S}^2)|y_n|Z_n|_{L^1(\mathbb{S}^2)}$$

$$\leq \frac{v}{20}|y_n|^2 + \frac{5}{v}|z_n|^2|y_n|^2,$$

$$b(z_n, \tilde{z}_n, y_n) \leq |z_n|L^1(\mathbb{S}^2)|y_n|Z_n|_{L^1(\mathbb{S}^2)}$$

$$\leq \frac{v}{20}|y_n|^2 + \frac{5}{v}|z_n|^2|y_n|^2,$$

$$\alpha(\tilde{z}_n, y_n) \leq \alpha|y_n||\tilde{z}_n||y_n|$$

$$\leq \frac{v}{20}|y_n|^2 + \frac{5\alpha^2}{v}|z_n|^2|y_n|^2.$$
Moreover, since

\[ \gamma_n = \frac{10}{v} |v_n|^2 V + \frac{15^3}{2v^3} |v|_{L^4(S^2)}^4 + \frac{15^3}{2v^3} |z|_{L^4(S^2)}^4 + \frac{10}{v} |z|_{V}^2. \]

Then Gronwall yields

\[ |y_n(t)|^2 \leq \left( |y_0|^2 + \frac{10}{v} \int_0^t \beta_n(s) ds \right) \exp \left( \int_0^t \gamma_n(s) ds \right). \]

Note that

\[ \int_0^T \beta_n(s) ds = \int_0^T |[\mathcal{L}_n(s)|_{L^4(S^2)}^4 |v_n(s)| |v_n(s)| |v| + |v_n(s)| |v_n(s)| V] \mathcal{L}_n(s)_{L^4(S^2)} \]

\[ + |z_n(s)|_{L^4(S^2)}^2 \mathcal{L}_n(s)_{L^4(S^2)} + |z_n(s)|_{L^4(S^2)}^2 \mathcal{L}_n(s)_{L^4(S^2)} + \alpha_2 |\mathcal{L}_n(s)|_{L^2}^2 + |\mathcal{F}_n|_{L^2}^2 ds \]

\[ \leq [2 |v_n| L^2(0,T;V) + |z_n|_{L^4(0,T;L^4)}^2 + |z_n|_{L^4(0,T;L^4)}^2] \mathcal{L}_n(s)_{L^4(0,T;L^4)} \]

\[ + \alpha_2 |\mathcal{L}_n(s)|_{L^2}^2 + |\mathcal{F}_n|_{L^2}^2. \]

Hence, by usual continuity argument, pass the limit through the integral one gets

\[ \int_0^T \beta_n(s) ds \to 0 \quad \text{as} \quad n \to \infty. \]

Moreover, since \( |y_n(0)| \to 0 \) as \( n \to \infty \) and for some finite constant \( C \) one has

\[ \int_0^T \gamma_n(s) ds = \int_0^T \left( \frac{10}{v} |v_n|^2 V + \frac{15^3}{2v^3} |v|^4_{L^4(S^2)} + \frac{15^3}{2v^3} |z|^4_{L^4(S^2)} + \frac{10}{v} |z|_{V}^2 \right) ds \]

\[ \leq C. \]

Hence \( y_n(t) \to 0 \) in \( H \) as \( n \to \infty \) uniformly in \( t \in [0, T] \). In other words,

\[ v(\cdot, z_n) u_0^0 \to v(\cdot, z) u_0 \quad \text{in} \quad C(0, T; H). \]

From inequality \( \text{(4.43)} \), we also have

\[ v \int_0^T |y_n(s)|_{V}^2 ds \leq |y_n(0)|^2 + \frac{10}{v} \int_0^T \beta_n(s) ds + \int_0^T \gamma_n(s) |y_n(s)|^2 ds \]

\[ \leq |y_n(0)|^2 + \frac{10}{v} \int_0^T \beta_n(s) ds + \sup_{s \in [0, T]} |v_n(s)|^2 \int_0^T \gamma_n(s) ds. \]

Therefore,

\[ \int_0^T |y_n(s)|_{V}^2 ds \to 0 \quad \text{as} \quad n \to \infty. \quad (4.44) \]

Hence

\[ v(\cdot, z_n) u_0^0 \to v(\cdot, z) u_0 \quad \text{in} \quad L^2([0, T]; V). \]

and Theorem \( \text{3.12} \) is proved.
5. Proof of Theorem 5.15: Strong solutions

Suppose now $f \in H$, in what proceeds we will show that if $u_0 \in V$ then we obtain a more regular kind of solution and deduce that if $v_0 \in H$ then $v(t) \in V$ for every $t > 0$. In this paper, we will construct a unique global strong solution (in PDE sense).

The proof of Theorem 5.15 follows closely to Theorem 3.1 in [4]. However in the proof in [4] there is no Coriolis force and additive noise whereas here there are. In particular our constants in the proof now depend on $|F(t)|$ and $|z(t)|$ and $|z(t)|_V$, but not on the Coriolis term due to the antisymmetric condition $(Cv, Av) = 0$.

Remark. One can alternatively prove Theorem 5.15 via the usual Galerkin approximation which we used in the proof of weak variational solution.

5.1. Existence and uniqueness of strong solution with $v_0 \in V$. The following function spaces are introduced for convenience.

Definition 5.1. The spaces

$$X_T := C(0, T; H) \cap L^2(0, T; V),$$

$$Y_T = C(0, T; V) \cap L^2(0, T; D(A))$$

are endowed with the norm

$$|\cdot|_{X_T} := |\cdot|_{C(0,T;H)} + |\cdot|_{L^2(0,T;V)},$$

$$|\cdot|_{Y_T} := |\cdot|_{C(0,T;V)} + |\cdot|_{L^2(0,T;D(A))}.$$

Or explicitly,

$$|f|^2_{X_T} = \sup_{0 \leq t \leq T} |f(t)|^2 + \int_0^T |f(s)|^2_V^2 ds,$$

$$|f|^2_{Y_T} = \sup_{0 \leq t \leq T} |f(t)|^2_V + \int_0^T |Af(s)|^2 ds.$$

Let $\mathcal{K}$ be the map in $Y_T$ defined by

$$\mathcal{K}(u)(t) = \int_0^t S(t-s)B(u(s))ds, \quad t \in [0, T], \ u \in Y_T.$$

The following is a crucial lemma for the proof of existence and uniqueness.

Lemma 5.2. There exists $c > 0$ such that for every $u, v \in Y_T$,

$$|\mathcal{K}(u)|_{Y_T}^2 \leq c|u|_{Y_T}^2 \sqrt{T},$$

$$|\mathcal{K}(u) - \mathcal{K}(v)|_{Y_T}^2 \leq c|u - v|_{Y_T}^2 \sqrt{T}. $$

Proof. Recall classical facts due to Lions [18],

- for any $f \in L^2(0, T; H)$, the function $t \mapsto x(t) = \int_0^t S(t-s)f(s)ds$ belongs to $Y_T$ and
- the map $f \mapsto x$ is continuous from $L^2(0, T; H)$ to $Y_T$. 

We remark that the second fact implies $\int_0^t |f(s)|_{H_f}^2 ds < \infty$ Now because $B(u) \in L^2(0, T; H)$, that is $\int_0^t |B(u(s))|_{H_f}^2 ds$, using the previous classical facts, combine with (2.51) one has,

$$|\mathcal{K}(u)|_{V_T}^2 \leq c_1 \int_0^T |B(u(s))|_{H_f}^2 ds$$

$$\leq c_2 \int_0^T |u(t)|_{Y_T}^2 |u|_{V} Au|dt$$

$$\leq c_2 \sup_{0 \leq t \leq T} |u(t)|_{Y_T}^2 \int_0^T |u(t)|_{V} |Au(t)|dt$$

$$\leq \sqrt{2c_2} \sup_{0 \leq t \leq T} |u(t)|_{Y_T}^2 \left( \int_0^T |u(t)|_{Y_T}^2 + |Au(t)|_{H_f}^2 dt \right)$$

$$\leq c_8 |u|_{Y_T}^2 \sqrt{T}. $$

Similarly, combine Lions’ results and (2.51), one has

$$|\mathcal{K}(u) - \mathcal{K}(v)|_{V_T}^2 \leq c_4 \int_0^T |B(u - v, u) + B(v, u - v)|_{H_f}^2 dt$$

$$\leq c_5 \int_0^T |B(u - v, u)|_{H_f}^2 + |B(v, u - v)|_{H_f}^2 dt$$

$$\leq c_5 \int_0^T c_7 |u - v|_{Y_T}^2 |u|_{V} |Au| + c_8 |u - v|_{Y_T}^2 |v|_{V} |Av| dt$$

$$\leq c |u - v|_{Y_T}^2 (|u|_{V_T}^2 + |v|_{V_T}^2) \sqrt{T}. $$

**Lemma 5.3.** Assume that $\alpha \geq 0, z \in L^4_{\infty}(0, \infty); L^4(S^2) \cap H) , f \in H$ and $v_0 \in V$. Then, there exists unique solution of (5.15) in the space $C(0, T; V) \cap L^2(0, T; D(A)).$ for all $T > 0.$

**Proof.** First let us prove local existence and uniqueness. Let $Y_T = C(0, \tau; V) \cap L^2(0, \tau; D(A))$ be equipped with the norm

$$|f|_{Y_T}^2 = \sup_{t \leq \tau} |f(t)|^2 + \int_0^T |Af(s)|^2 ds,$$

and Let $\Gamma$ be a nonlinear mapping in $Y_T$ as

$$(\Gamma v)(t) = S(t)v_0 + \int_0^t S(t - s)(f - B(v(s) + z(s) + az(s)))ds.$$ 

Now recall the following classical result due to Lion.

$$A1 \quad S(\cdot)v_0 \in Y_T, \forall v_0 \in H, \tau > 0;$$

$$A2 \quad \text{The map } t \mapsto x(t) = \int_0^t S(t - s)f(s)ds \text{ belongs to } Y_T \text{ for all } L^2(0, T; H);$$

$$A3 \quad \text{The mapping } f \mapsto x \text{ is continuous from } L^2(0, T; H) \text{ to } Y_T.$$ 

Note, our assumption $z(t) \in L^4([0, \infty); L^4(S^2) \cap H)$ implies $z(t) \in Y_T$ as $z(t)$ is square integrable and $V$ can be continuously embedded into $L^4(S^2).$

The first step is to show $\Gamma$ is well defined. Using assumptions A1 and A2 and the assumption for $z(t)$, together with Young inequality, one can show

$$|\Gamma|_{Y_T}^2 \leq c |S(t)v_0|_{Y_T}^2 + c \int_0^t S(t - s)B(v(s) + z(s))ds |_{Y_T}^2 + c \int_0^t S(t - s)fds |_{Y_T}^2 + c\alpha \int_0^t S(t - s)z(s) |_{Y_T}^2.$$ 

For some different constant $c$. Now due to $A_1$ and $A_2$, the first and third terms are finite, due to $A_2$ and the trilinear inequality \ref{2.8}, the second term is finite, and the last term also finite due to the assumption on $z(t)$

$$|Γ|^2_{Y_τ} \leq c_1 + c_2|v|^4_{Y_τ} + c_3 + c_4.$$ \hfill (5.3)

Whence the map $Γv$ is well defined in $Y_τ$, and $Γ$ maps $Y_τ$ into $Y_τ$ itself.

Now we have

$$|Γ(v_1) − Γ(v_2)|^2_{Y_τ} \leq | \int_0^τ S(t − s)(B(v_1(s) + z(s)) − B(v_2(s) + z)|ds|^2_{Y_τ} \leq c_6|v_1 − v_2|^2_{Y_τ}(|v_1 + z|^2_{Y_τ} + |v_2 + z|^2_{Y_τ})\sqrt{τ},$$

for all $v_1, v_2$ and $z$ in $Y_τ$. Therefore, for sufficiently small $τ > 0$, $Γ$ is a contraction in a closed ball of $Y_τ$, yielding existence and uniqueness of a local solution of \ref{5.3} in $Y_τ$. That is, the solutions are bounded in $V$ on some short time $[0, τ]$.

**Remark.** If the following map

$$(Γu)(t) = S(t)u_0 − \int_0^t S(t − s)B(u(s))ds + \int_0^t S(t − s)fds + \int_0^t S(t − s)GdL(s)$$

is used to prove contraction. Then one would have to assume

$$\int_0^τ |Az(t)|^2 dt < \infty.$$

The local existence and uniqueness results indicates that the solution can be extended up to the maximal lifetime $T_{f,x}$ and then is well defined on the right open interval $[0, T_{f,x})$. Next, we will prove the local solution may be continued to the global solution valid for all $t > 0$, in the class of weak solutions satisfying a certain energy inequality. (This is comparable with the satisfactory deterministic literature in 2D that strong solution exist globally in time and is unique, see for instance Theorem 7.4 Foias and Temam \cite{bib1}.

It suffices to find an uniform a priori estimate for the solution $v$ in the space $V_{T_0}$ such that for any $T_0 ∈ [0, T_{f,x})$;

$$|v|^2_{V_{T_0}} \leq C \quad \text{for all } T_0 ∈ [0, T_{f,x}),$$ \hfill (5.4)

where $C$ is independent on $T_0$. This uniform a priori estimate along with the local existence uniqueness proved earlier yields the unique global solution $u$ in $V_{T,x}$ Indeed exist globally in time. Hence one can deduce that the solution is well defined up to time $t = T_{f,x}$, at this point in time the iterated process could be repeated and the solution can be found in $[T_{f,x}, 2T_{f,x}]$ and so forth, hence in $C(0, \infty; V) \cap L_{loc}^2(0, \infty; D(A))$. To prove \ref{5.4}, we first need to show

$$|v|_{X_{T_0}} \leq c_0.$$ 

Toward the above end, we work with a modified version of \ref{3.17}

$$\begin{cases}
\partial_t v + νAv = −B(v) − B(v, z) − B(z, v) − Cv + F, \\
v(0) = v_0,
\end{cases}$$ \hfill (5.5)
where $F = -B(z) + az + f$ is an element of $H$ since the $H$ norm of all its three terms is bounded.
Now multiply both sides with $v$, integrate over $S^2$, one gets
\[
\partial_t |v|^2 + v|v|^2 = -b(v, v, v) - b(v, z, v) - b(z, v, v) - \langle Cv, v \rangle + \langle F, v \rangle = b(v, v, z) + \langle F, v \rangle.
\]

Now by (2.46), one has
\[
|b(v, v, z)| \leq c |v||v||v|
\]
then apply Young inequality with $ab = \sqrt{\frac{2}{\epsilon}} |v||v| \sqrt{\frac{2}{\epsilon}} |z|v$ it follows that
\[
\leq \frac{|v|^2}{4} + \frac{1}{\epsilon} |v|^2 |z|^2.
\]

On the other hand,
\[
\langle F(t), v \rangle = |F(t)||v| \leq \frac{1}{\epsilon} |F(t)|^2 + \frac{\epsilon}{4} |v|^2.
\]

So
\[
\partial_t |v|^2 + (2v - \frac{\epsilon}{2}) |v|^2 \leq \frac{2}{\epsilon} |v|^2 |z|^2 + \frac{2}{\epsilon} |F(t)|^2 + \frac{\epsilon}{2} |v|^2
\]
for all $\epsilon > 0$.

By integrating in $t$ from 0 to $T$, after dropping out unnecessary terms,
\[
\int_0^T |v(t)|^2 \leq \frac{1}{2 \epsilon} \left( |v(0)|^2 + \frac{2}{\epsilon} \int_0^T |v(t)|^2 |z(t)|^2 dt + \frac{2}{\epsilon} \int_0^T |F(t)|^2 dt + \frac{\epsilon}{2} \int_0^T |v(t)|^2 dt \right) \leq K_1,
\]
since $v(0) = u_0$
\[
K_1 = K_1(u_0, F, v, T, z).
\]

On the other hand, by integrating in $t$ of (5.6) from 0 to $s$, $0 < s < T$, we obtain
\[
|v(s)|^2 \leq |u_0|^2 + \frac{2}{\epsilon} \int_0^s |v(t)|^2 |z(t)|^2 dt + \frac{2}{\epsilon} \int_0^s |F(t)|^2 dt + \frac{\epsilon}{2} \int_0^s |v(t)|^2 dt,
\]
\[
\sup_{s \in [0, T_s]} |v(s)|^2 \leq K_2,
\]
\[
K_2 = K_2(u_0, F, v, T, z) = (2v - \frac{\epsilon}{2})K_1.
\]

Hence, for any $\epsilon$ such that $\frac{\epsilon}{2} < 2v$, apply Gronwall lemma to
\[
\partial_t |v|^2 \leq \left( \frac{2}{\epsilon} |z|^2 + \frac{\epsilon}{2} \right) |v|^2 + \frac{2}{\epsilon} |F(t)|^2,
\]
one obtains
\[
|v(t)|^2 \leq |v(0)|^2 \exp \left( \int_0^t \frac{2}{\epsilon} |z|^2 \frac{d\tau}{2} + \frac{\epsilon}{2} d\tau \right) |v|^2 + \int_0^t \frac{2}{\epsilon} |F(s)|^2 \exp \left( \int_s^t \frac{2}{\epsilon} |z|^2 \frac{d\tau}{2} + \frac{\epsilon}{2} d\tau \right) d\tau| ds,
\]
and so
\[
\sup_{t \in [0, T_s]} |v(t)|^2 \leq |v(0)|^2 \exp \left( \int_0^{T_s} \frac{2}{\epsilon} |z|^2 \frac{d\tau}{2} + \frac{\epsilon}{2} d\tau \right) + \int_0^{T_s} \frac{2}{\epsilon} |F(s)|^2 \exp \left( \int_s^{T_s} \frac{2}{\epsilon} |z|^2 \frac{d\tau}{2} + \frac{\epsilon}{2} d\tau \right) d\tau| ds.
\]
To avoid clumsiness, write momentarily \( T_{f,x} = T \). Let
\[
\psi_T(z) = \exp \left( \int_0^T \frac{2}{\epsilon} |z(\tau)|_V^2 + \frac{\epsilon}{2} \, d\tau \right) < \infty, \quad c_F = \int_0^T \frac{2}{\epsilon} |F(s)|^2 \exp \left( \int_s^T \left( \frac{2}{\epsilon} |z(\tau)|_V^2 + \frac{\epsilon}{2} \right) \, d\tau \right) \, ds.
\]
(5.8)
So
\[
\sup_{t \in [0,T]} |v(t)|^2 \leq |v(0)|^2 \psi_T(z) + c_F,
\]
(5.9)
which implies
\[
v \in L^\infty([0,T]; H).
\]
(5.10)
Now integrate
\[
\partial_t |v|^2 + v|v|^2 \leq \left( \frac{2}{\epsilon} |z|^2 + \frac{\epsilon}{2} \right) |v|^2 + \frac{2}{\epsilon} |F(t)|^2,
\]
(5.11)
from 0 to \( T \) one gets
\[
|v(T)|^2 + v \int_0^T |v(t)|_V^2 \, dt \leq \left( \psi_T(z)|v(0)|^2 + c_F \right) \int_0^T \left( \frac{2}{\epsilon} |z(t)|^2 + \frac{\epsilon}{2} \right) \, dt + \frac{2}{\epsilon} \int_0^T |F(t)|^2 \, dt + |v(0)|^2,
\]
(5.12)
which implies
\[
v \in L^2([0,T]; V),
\]
(5.13)
and \( v \) is indeed a weak solution. To show \( v \in C([0,T]; H) \), note that \( A : V \rightarrow V' \) is bounded and \( Av \in L^2([0,T]; V') \). Then \( F \in L^2([0,T]; V') \) since \( z \in L^4([0,T]; L^4(S^2) \cap H) \) which can be continuously embedded into \( V' \), and the terms \( B(z) \), \( B(v, z) \), \( B(z, v) \) all in \( L^2([0,T]; V') \). Combine these facts and invoke lemma 4.1 in [1] we conclude that \( v \in C([0,T]; H) \).

The uniform apriori estimate (5.12) implies that the solution is well defined up to time \( t = T_{f,x} \). The iterative process may be repeated start from \( t = T_{f,x} \) with the initial condition \( z(t) \) and the solution is uniquely extended to \([0, 2T_{f,x}]\) and so on to arbitrary large time.

Now, multiply both sides with \( Av \), noting again the classical fact \( \frac{1}{2} \partial_t |v(t)|^2 = \langle \partial_t v(t), v(t) \rangle \) and \( \langle Cv, Av \rangle = 0 \), integrate over \( S^2 \), one gets
\[
\langle \partial_t v, Av \rangle + v(Av, Av) = -b(v, v, Av) - b(v, z, Av) - b(z, v, Av) + \langle F(t), Av(t) \rangle
\]
\[
\implies \frac{1}{2} \frac{d^+}{dt} |v|^2 + |Av|^2 = -b(v(t), v(t), Av(t)) - b(v(t), z(t), Av(t)) - b(z(t), v(t), Av(t)) + \langle F(t), Av(t) \rangle.
\]
(5.14)
Now,
\[
|b(v, v, Av)| \leq C |v|^2 |v||Av| \quad \forall v \in V, v \in D(A),
\]
\[
|b(v, z, Av)| \leq C |v|^2 |v| |z||Av| \quad \forall v \in V, v \in D(A),
\]
\[
|b(z, v, Av)| \leq C |z|^2 |z| |v||Av| \quad \forall z \in V, v \in D(A).
\]
Also,
\[
\langle F(t), Av \rangle \leq \frac{\epsilon}{4} |Av(t)|^2 + \frac{1}{\epsilon} |F(t)|^2.
\]
Furthermore, using Young inequality with the choice $p = \frac{4}{3}$ and $ab = (\epsilon p)^{\frac{1}{2}} |A\nu|^{\frac{1}{2}} |\nu|^{\frac{1}{2}}$, the above estimates of the three bilinear terms become

\[
|b(v, v, A\nu)| \leq C|v|^2 |v| |A\nu|^3 \leq \epsilon |A\nu|^2 + C(\epsilon) |v|^2 |v|^4,
\]

\[
|b(v, z, A\nu)| \leq C|v|^2 |v|^2 |z|^2 |A\nu|^3 \leq \epsilon |A\nu|^2 + C(\epsilon) |v|^2 |v|^2 |z|^2 |v|^4,
\]

\[
|b(z, v, A\nu)| \leq C|z|^2 |z|^2 |v|^2 |A\nu|^3 \leq \epsilon |A\nu|^2 + C(\epsilon) |z|^2 |v|^2 |z|^2 |v|^4.
\]

Therefore,

\[
d^+ \frac{d}{dt} |v|^2 + (2v - 3\epsilon - \frac{C}{4}) |A\nu|^2 \leq C(\epsilon) (|v|^2 |v|^4 + |v|^2 |v|^2 |z|^2 |v|^2 + |z|^2 |z|^2 |v|^2) + \frac{1}{\epsilon} |F(t)|^2. \tag{5.15}
\]

Momentarily dropping the term $|A\nu(t)|^2$, we have the differential inequality,

\[
y' \leq a + \theta y,
\]

\[
y(t) = |v|^2, \quad a(t) = \frac{1}{\nu} |F(t)|^2, \quad \theta(t) = C(\epsilon) (|v|^2 |v|^4 + |v|^2 |z|^2 + |z|^2 |z|^2 |v|^2).
\]

Then for any $\epsilon$ such that $\epsilon < \frac{8}{13} \nu$, using Gronwall lemma, one has

\[
\sup_{t \in [0, T]} |v(t)|^2 \leq K_3, \tag{5.16}
\]

\[
K_3 = K_3(u_0, F, \nu, T, z) = \left( |v(0)|^2 + \frac{1}{\nu} \int_0^T |F(s)|^2 \, ds \right) \exp(C(\epsilon) K_2 K_1),
\]

which implies

\[
v \in L^\infty(0, T; V). \tag{5.17}
\]

Let us now come back to (5.15), which we integrate from 0 to $T$, after dropping some unnecessary terms, we have

\[
\int_0^T |A\nu(t)|^2 \, dt \leq K_4,
\]
and
\[
K_4 = K_4(u_0, F, v, z, T) = \frac{1}{2v - 3\epsilon - \frac{\nu}{2}} (|u_0|^2 + C(\epsilon) \sup_{t \in [0, T]} |v(t)|^2 |v(t)|_V^4 + C(\epsilon) \sup_{t \in [0, T]} |v(t)|^2 |z(t)|_V^2 + C(\epsilon) \sup_{t \in [0, T]} |z(t)|^2 |v(t)|_V^2 + \frac{1}{\epsilon} \int_0^T |F(t)|^2 dt).
\]
As
\[
\sup_{t \in [0, T]} |v(t)|^2 \leq K_2, \quad \sup_{t \in [0, T]} |v(t)|_V^4 \leq K_2^3, \quad |z(t)|_V^2 \leq C_1, \quad \sup_{t \in [0, T]} |z(t)|^2 \leq C_2.
\]
So,
\[
K_4 = \frac{1}{2v - 3\epsilon - \frac{\nu}{2}} (|u_0|^2 + C(\epsilon)K_2K_2^3 + C(\epsilon)K_2K_2C_1 + C(\epsilon)C_2C_1K_3 + \frac{1}{\epsilon} \int_0^T |F(t)|^2 dt).
\]
This implies
\[
v \in L^2(0, T_f, \mathbb{R}; D(A)). \tag{5.18}
\]
It remains to show \(v \in C([0, T]; V)\). Note, the fact that the solution with \(v_0 \in V\) is in \(L^2([0, T]; V)\) implies that a.e. in \([0, T]\), \(v(t) \in V\). Moreover, since \(v(t) \in C([0, T]; H)\) as previously deduced, and is unique as proved in step 1. It follows that \(u \in C([0, T]; V)\)

With the uniform a priori estimate along with the local existence uniqueness in step 1, we conclude that there exists unique \(u \in C(0, \infty; H) \cap L^2(0, \infty; V) \subset C(0, \infty; V) \cap L^2(0, \infty; D(A))\), for any given \(u_0 \in V, f \in H, z(t) \in L^4_{\text{loc}}([0, \infty); L^4(S^2) \cap H)\). Moreover, our promising a priori bound \(5.16\) yields \(T = \infty\). \(\square\)

5.2. Existence and uniqueness of strong solution with \(v_0 \in H\).

**Corollary.** If \(f \in H, v_0 \in H, z(t) \in L^4_{\text{loc}}([0, \infty); L^4(S^2) \cap H)\), then \(v(t) \in V\) for all \(t > 0\).

We follow the proof in [4]. The idea stems from standard approximation method commonly used in PDE theory. In view of the a priori estimate \(5.15\) one takes approximated solution to \(5.15\) in \(Y_T\) show that approximates converge. Then show the limit function indeed satisfies \(5.15\).

Let \((v_{0n}) \subset V\) be a sequence converging to \(v_0\) in \(H\). For all \(n \in \mathbb{N}\), let \(v_n\) be solution of equation \(6.15\) in \(Y_T\) corresponding to initial data \(v_{0n}\). Similar to the case when \(v_0 \in V\), one can find constant such that \(|v_n|_{X_T} \leq c, \quad \forall n \in \mathbb{N}\). Follow the same lines as in the proofs of \(5.10\) and \(5.13\), \(v_n\) can be proved to be a weak solution.

Moreover, for \(n, m \in \mathbb{N}\), take \(v_{n,m} = v_n - v_m\) with \(v_{n,m} = v_{0} - v_{0}\). Then \(v_{n,m}\) is the solution of
\[
\begin{align*}
\partial_t v_{n,m} + v Av_{n,m} &= -B(v_{n,m}, z) - B(z, v_{n,m}) - B(v_{n,m}, v_n) - B(v_{n,m}, v_m) - C v_{n,m}, \\
v_{n,m}(0) &= v_{0} - v_{0}.
\end{align*}
\tag{5.19}
\]
Multiply \(5.19\) both sides with \(v_{n,m}\) and integrate against \(v_{n,m}\), using again Lemma \(2.45\) and \(2.44\) and noting \(2.27\) one gets
\[
\partial_t |v_{n,m}|^2 + 2v|v_{n,m}|_V^2 = -2b(v_{n,m}, z, v_{n,m}) - 2b(v_{n,m}, v_n, v_m).
\]

Since $|b(w, w, z)| \leq C|w||w|_V|z|_V$ and $|b(w, w, v)| \leq C|w||w|_V|v|_V$

$$\leq C|v_{n,m}||v_n\cdot|_V |z|_V + |v_n|_V$$

Then via usual Young inequality with $\alpha = \epsilon |v_{n,m}|_V$ and $b = \frac{C}{\sqrt{\epsilon}} |v_{n,m}| |z|_V + |v_n|_V$,

$$\leq \frac{\epsilon |v_{n,m}|_V}{2} + \frac{C}{2\epsilon} |v_{n,m}|^2 |z|_V^2 + |v_n|_V^2. \quad (5.20)$$

Therefore, for any $\epsilon > 0$ such that $\frac{\epsilon}{2} < 2\nu$, one applies Gronwall lemma to

$$|\partial_t v_{n,m}|^2 \leq \frac{C}{2\epsilon} (|z|_V^2 + |v_n|_V^2) |v_{n,m}|^2,$$

and combine with $v_{n,m}^0 = v_n^0 - v_m^0$, it is easy to show

$$|v_{n,m}(t)|^2 \leq |v_{n,m}(0)|^2 \exp \left( \frac{C}{2\epsilon} \int_0^T |z(t)|_V^2 + |v_n(t)|_V^2 |v_{n,m}(t)|^2 dt \right) < \infty,$$

as $\int_0^T |z(t)|_V^2 + |v_n(t)|_V^2 < \infty$. Hence $v_{n,m}$ converges in $T$ and so is Cauchy in $T$. That is, for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|v_n - v_m| < \epsilon$ whenever $n, m \geq N$.

Let the limit of $v_n$ be $v$. It remains to show $v$ indeed satisfies (5.15).

Let $v_n$ be the solution to

$$v_n(t) = S(t)v_0 - \int_0^t S(t-s)B(u_n(s))ds + \alpha \int_0^t z_n(s)ds, \quad (5.21)$$

where $z_n = \int_0^t S(t-s)GdL_n(t)$. We would like to show

$$\lim_{n \to \infty} u_n(t) = S(t)u_0 - \int_0^t S(t-s)B(u(s))ds + \int_0^t S(t-s)f ds + \alpha \int_0^t z(s)ds. \quad (5.22)$$

Assume $f_n \to f$ in $L^2(0, T; H)$, $z_n = \int_0^t S(t-s)GdL_n(t) \to z$. In $L^4([0, T]; L^4(S^3) \cap H)$, we would like to check if

$$\lim_{n \to \infty} B(u_n) = B(u) \quad \text{in} \quad H. \quad (5.23)$$

For this, note first that

$$\left| |u_n|_V^2 - |u|_V^2 \right| = \left| (u_n, u_n) - (u, u) \right| = \left| (u_n, u_n) - (u, u_n) + (u, u_n) - (u, u) \right| = \left| (u_n, u_n)_V - (u, u_n)_V + |(u, u_n)_V - (u, u)_V| \right| = |u_n - u|_V u_n + |u|_V u - (u, u)_V \leq |u_n - u|_V |u_n|_V + |u|_V |u_n - u|_V.$$

Now $|u_n|_V$ is Cauchy and so is bounded. So $u_n$ converges to $u$ in $V$ as $n \to \infty$. Then using (5.24) one deduces that

$$|B(u_n) - B(u)| = |B(u_n, u_n) - B(u_n, u) + B(u_n, u) - B(u, u)| \leq C(|u_n^2|_V^2 + |u_n|_V^2 |u| + |u|_V^2) \to C|u|_V^2.$$
Now similar to the earlier work on proving contraction we have,
\[
\|B(u_n(s)) - B(u(s))\|_{V_T}^2 \\
\leq \left| \int_0^t S(t-s)(B(u_n(s)) - B(u(s)))ds \right|_{V_T}^2 \\
\leq c \int_0^T |B(u_n(s)) - B(u(s))|^2 ds \\
\leq c |u|^2 \sqrt{T}.
\]
Therefore, \(B(u_n) - B(u)\) is in \(L^2(0, T; H)\). Now by continuity argument again, one has
\[
\lim_{n \to \infty} \int_0^T S(t-s)B(u_n(s))ds = \int_0^T S(t-s)B(u(s))ds,
\]
and
\[
\lim_{n \to \infty} \int_0^T S(t-s)f_n(s)ds = \int_0^T S(t-s)f(s)ds.
\]
Combine with the assumptions
\[
\lim_{n \to \infty} S(t)u_{0,n} = S(t)u_0, \\
\lim_{n \to \infty} z_n(t) = z(t),
\]
oneduces that
\[
\lim_{n \to \infty} v_n(t) = v(t).
\]
and there exists a solution to (3.15). However, the solution constructed as the limits of \(u_n\) leaves open the possibility that there is more than one limit. So we will now prove \(u\) is unique. The idea is analogous to proving (5.20). Nevertheless we detail as following. Suppose \(v_1, v_2\) are two solutions of (3.17) with the same initial condition. Let \(w = v_1 - v_2\), then \(w\) satisfies
\[
\begin{align*}
\partial_t w + vAw &= -B(w, z) - B(z, w) - B(w, v_1) - B(v_2, w), \\
w(0) &= 0.
\end{align*}
(5.24)
\]
Multiply (5.24) both sides with \(w\) and integrate against \(w\), using the identities \(\partial_t |v(t)|^2 = 2\langle \partial_t v(t), v(t) \rangle\) again in Temam and (2.44) one gets
\[
\partial_t |w|^2 + 2v|w|_V^2 = -2b(w, z, w) - 2b(w, v_1, w).
\]
Since \(|b(w, w, z)| \leq C|w||w|_V|z|_V\) and \(|b(w, w, v)| \leq C|w||w||v|_V\)
\[
\leq C|w||w|(|z|_V + |v_1|_V)
\]
Then via usual Young inequality with \(a = \sqrt{\varepsilon}|w|_V\) and \(b = \frac{C}{\sqrt{\varepsilon}}|w|(|z|_V + |v_1|_V)\)
\[
\leq \frac{\varepsilon|w|_V^2}{2} + \frac{C}{2\varepsilon}|w|_V^2(|z|_V^2 + |v_1|^2_\mathcal{V}).
(5.25)
\]
Therefore, for any \(\varepsilon > 0\) such that \(\frac{C}{2\varepsilon} < 2v\), one applies Gronwall lemma to
\[
\partial_t |w|^2 \leq \frac{C}{2\varepsilon}(|z|^2_\mathcal{V} + |v_1|^2_\mathcal{V})|w|^2,
\]
and combine with \( w_0 = v_{1,0} - v_{2,0} = 0 \), it follows from Gronwall inequality that
\[
|w(t)|^2 \leq |w(0)|^2 \exp \left( \frac{C}{2e} \int_0^T |z(t)|^2 + |v_1(t)|^2 |w(t)|^2 dt \right) < \infty
\]
as \( \int_0^T |z(t)|^2 + |v_1(t)|^2 dt < \infty \). Now, since \( w(0) = 0 \), necessarily \( w(t) \) must be 0.

It remains to show \( v \in C([0, T]; V) \), as observe from the above energy inequality (5.20), the solution starts with with an initial condition \( v_0 \in H \) belongs to \( L^2(0, T; \mathcal{V}) \). This implies that almost everywhere in \( [0, T] \), there must exist a time point \( \varepsilon \) (and \( \varepsilon < T \)) such that \( u(\varepsilon) \in V \). Then one may repeat step 2 to another interval \( [\varepsilon, 2\varepsilon] \), \( [2\varepsilon, 3\varepsilon] \) \( \cdots \) and over the whole \( [\varepsilon, \infty] \). Finally we obtain that \( u \in C([\varepsilon, T]; V) \cap L^2([\varepsilon, T]; D(A))) \) for all \( \varepsilon > 0 \). Note that \( T = \infty \) as implied from the a priori estimate.

In summary, in this section, we have proved

**Lemma 5.4.** Assume that \( \alpha \geq 0 \), \( z \in L^4_{\text{loc}}([0, \infty); \mathbb{L}^4(S^2) \cap H) \), \( f \in H \) and \( v_0 \in H \). Then, there exists unique solution of (3.18) in the space \( C(0, T; H) \cap L^2(0, T; V) \). which belongs to \( C(\varepsilon, T; V) \cap L^2_{\text{loc}}(\varepsilon, T; D(A)) \) for all \( \varepsilon > 0 \) and \( T > 0 \).

Combine Lemma 5.4 with 5.5 we have proved theorem 3.15

**Remark.** Continuous dependence on \( v_0, z \) and \( f \) is implied from the point where local existence and uniqueness is attained and hence holds also for global solutions.

Finally, we give an intuitive meaning of Theorem 3.15

**Remark.** The proof of Theorem 3.15 shows that the solution \( v \), starting from \( v_0 \in H \), belongs to \( V \) for a.e. \( t \geq t_0 \); If we take any \( \tilde{t} \geq t_0 \) such that \( v(\tilde{t}) \in V \), the solution is extended over the interval \( [t_0, \tilde{t} + \varepsilon] \) and is found to be in \( D(A) \) as well. One may repeat this step over another interval \( [t_0 + \varepsilon, \tilde{t} + \varepsilon] \), \( [t_0 + 2\varepsilon, \tilde{t} + 3\varepsilon] \) \( \cdots \). Thus, we obtain that \( v \in C([t_0 + \varepsilon, \infty); V) \cap L^2_{\text{loc}}(t_0 + \varepsilon, D(A)) \).

Furthermore, provided the noise does not degenerate, base on the condition given in the following, we obtained the existence and uniqueness results for the solution to the original equation 5.22. We detail this in the next subsection.

If
\[
\sum_{l} \lambda_l^2 |\alpha_l|^8 < \infty,
\]
then by Lemma 5.6 the process \( z \) has version which has left limits and is right continuous in \( V \). Recall that \( u_t := v_t + z_t \) and for each \( T > 0 \), define
\[
Z_T(\omega) := \sup_{0 \leq s \leq T} |z_t(\omega)|_V, \quad \omega \in \Omega.
\]

(5.27)

If (5.26) holds then by Lemma 5.6 we have
\[
\mathbb{E}Z_T < \infty,
\]
hence there exists a measurable set \( \Omega_0 \subset \Omega \) such that \( \mathbb{P}(\Omega_0) = 1 \) and
\[
Z_T(\omega) < \infty, \quad \omega \in \Omega_0.
\]

Finally, let us study 5.22 for \( \omega \in \Omega_0 \). Since \( z(\cdot, \omega) \in D([0, \infty); V) \), it is of course \( z(\cdot, \omega) \in D([0, \infty); H) \). Therefore, by Theorem 3.12, \( u(\cdot, \omega) = v(\cdot, \omega) + z(\cdot, \omega) \) is the unique càdlàg solution to 5.22. So, we extend the existence theorem of strong solution for \( u \). Moreover, for \( \omega \in \Omega_0 \), we find that \( u(\cdot, \omega) = v(\cdot, \omega) + z(\cdot, \omega) \) is the unique solution to 5.22 in \( D([0, \infty); H) \cap D([0, \infty); V) \) which belongs
to $D([h, \infty); V] \cap L^2_{\text{loc}}([h, \infty); D(A))]$ for all $h > 0$. If $u_0 \in V$, then $u \in D([h, \infty); V] \cap L^2_{\text{loc}}([h, \infty); D(A))]$ for all $h > 0$, $T > 0$.

This completes the proof of Theorem 3.16.

Since the solution is constructed using Banach Fixed Point Theorem, the continuous dependence on initial data is implied from the existence-uniqueness proof of strong solution in above line. Moreover, our existence uniqueness results work naturally when initial time $t_0 \in \mathbb{R}$ other than 0.

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