Synchrotron emission from anisotropic disordered magnetic fields

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Received

ABSTRACT
We derive expressions for the total and linearly-polarized synchrotron emissivity of an element of plasma containing relativistic particles and disordered magnetic field which has been sheared or compressed along three independent directions. Our treatment follows that given by Matthews & Scheuer (1990) in the special case of a power-law electron energy spectrum. We show that the emissivity integrals depend on a single parameter, making it straightforward to generate one-dimensional look-up tables. We also demonstrate that our formulae give identical results to those in the literature in special cases.

Key words: magnetic fields – radiation mechanisms: non-thermal – radio continuum: general

1 INTRODUCTION
The inference of the three-dimensional structure of the magnetic field in a source of synchrotron radiation from the observed emission at a single frequency is not unique, but considerable progress can be made if well-resolved images of total intensity and linear polarization are available. We have developed models of jets in low-luminosity radio galaxies which predict the observed emission for comparison with observations (Laing et al. 1999; Laing & Bridle, in preparation). Our fundamental assumption about the field structure is that it is random on small scales, but made anisotropic by shear and compression. Given the large number of observed data points, the need to integrate along the line of sight and the iteration required to optimize the fit between model and data, we found that existing algorithms for the calculation of synchrotron emissivity from disordered fields were too cumbersome. In order to minimize the resulting computation, we have developed an efficient method of evaluating the Stokes parameters $I, Q$ and $U$ for synchrotron radiation from electrons radiating in a three-dimensional, anisotropic magnetic field. This technique should be applicable to a variety of astrophysical synchrotron sources and is the subject of the present paper.

The basic theory of synchrotron emission from an optically-thin power-law energy distribution of relativistic electrons $n(E)dE \propto E^{-(2\alpha+1)}dE$ is well-documented (e.g. Rybicki & Lightman 1979, Hughes & Miller 1991). The spectrum of the emitted radiation also has a power-law form with spectral index $\alpha$ ($S \propto \nu^{-\alpha}$). For a uniform field, $B$, at an angle $\theta$ to the line of sight, the emissivity is $\propto (B \sin \theta)^{1+\alpha}$ and the degree of polarization has its maximum value $p_0 = (3\alpha+3)/(3\alpha+5)$. The observed $E$-vector is perpendicular to the projected field direction on the sky in the absence of propagation effects. Scheuer (1967) pointed out that the same results hold for a one-dimensional disordered field with many reversals: $I, Q$ and $U$ depend only on the orientation of the field, not on its vector direction.

Expressions for synchrotron emission from a two-dimensional disordered field with equal component variances along two orthogonal directions and no component in the third were derived by Laing (1980), and discussed further by Laing (1981). The more general case of an initially isotropic field compressed by an arbitrary factor along one direction was considered by Hughes et al. (1985). These authors wrote down expressions for elements of field in terms of spherical polar angles and averaged over the angles to derive the emitted Stokes parameters. For a field structure described by one compression factor, this approach is well suited to numerical calculation, but it becomes extremely cumbersome if velocity shear is included. Matthews & Scheuer (1990, hereafter MS) showed that a much simpler treatment is possible if it is assumed that the initial field is isotropic with a Gaussian distribution of field strengths. Their formulae also include the effects of inverse Compton and synchrotron losses, which cause a deviation of the emitted spectrum from a power-law form. In this paper, we describe a simplified version of their method, which applies to the three-dimensional case, but is restricted to power-law spectra. The advantage is that com-
putation of the emissivity is reduced to a one-dimensional integration or table look-up.

The new formulae are derived in Section \(2\). In Section \(3\), we illustrate the use of the method in practice by rederving expressions in the literature for a number of special field configurations. Section \(2\) briefly summarizes our results.

2 COMPUTATION OF THE STOKES PARAMETERS

2.1 Assumptions

The jet emission is taken to be optically-thin synchrotron radiation with a power law frequency spectrum of index \(\alpha\) as defined earlier. We make the conventional assumptions that the pitch-angle distribution of radiating electrons is isotropic. The observing frequency is taken to be high enough that Faraday rotation may be neglected, or corrected accurately.

The magnetic field is assumed to be disordered on small scales with zero mean. We and others have argued that this is likely to be a good approximation for extragalactic radio sources, except perhaps on the smallest linear scales (Laing 1983, Begelman, Blandford & Rees 1984, Laing & Bridle, in preparation). We note, however, that if one of the three field components is vector-ordered, then the results of our emissivity calculations are unaltered.

2.2 Geometry

Following MS, the field structure is assumed to be generated from an element of plasma containing radiating particles and an initially isotropic field distribution by shear and compression. We start with a small cube of plasma whose sides are defined by three unit vectors in a coordinate system \(x, y, z\) moving with the emitting plasma (the distinction between emitted and observed frames is important for relativistic flow). The coordinate axes are usually chosen along obvious natural directions. This element of particles and field is then compressed and sheared, forming a parallelepiped whose sides are defined by the vectors \(a, b\) and \(c\).

We then calculate the components of these vectors in a coordinate system \(X, Y, Z\) fixed in space, with \(Z\) along the line of sight.

MS show that the position angle \(\chi_0\) of the apparent magnetic field (perpendicular to the \(E\)-vector for synchrotron radiation) is given by:

\[
\tan 2\chi_0 = \frac{2(ax ay + bx by + cx cy)}{a_x^2 - a_y^2 + b_x^2 - b_y^2 + c_x^2 - c_y^2}
\]

for any spectral index. \(a_X\) is defined by:

\[
a_X = a_X/\mathbf{a} \times \mathbf{c}
\]

where \(\mathbf{X}\) is a unit vector along the \(X\)-axis, and so on. We will refer to \(a_X, a_Y, b_X, b_Y, c_X, c_Y\) collectively as the edge vector components. We rotate about the line of sight by an angle \(\chi_0\) into a coordinate system \(X, Y, Z (Z = Z)\) in which Stokes \(U = 0\). The rms field components along the \(X\) and \(Y\) axes are \(B_X\) and \(B_Y\), where:

\[
B_X^2 = B_0^2 ([a_X^2 + b_X^2 + c_X^2] \cos^2 \chi_0 + (a_Y^2 + b_Y^2 + c_Y^2) \sin^2 \chi_0)
\]

Here, \(B_0\) is the rms field along any direction for the initial isotropic field. Note that MS, in their equation (3.8), use the mean square value of the total field (3\(B_0^2\) in our notation).

2.3 Gaussian field distributions

For a power-law electron energy distribution, the observed emissivity from a element of field of strength \(B\) is always of the form \(B^{1+\alpha} \propto \) a geometrical factor which is independent of \(B\). The simplest initial field distribution is one where the field has constant intensity and random direction. Whilst this gives simple expressions for the emissivity if \(\alpha = 1\), the more general case is extremely messy. MS showed that a much simpler treatment is possible if the initial distribution of field strengths is isotropic and Gaussian, with probability distribution function (PDF):

\[
F(B_x, B_y, B_z) dB_x dB_y dB_z = (2\pi B_0^2)^{-3/2} \exp\left(-\frac{B^2}{2B_0^2}\right) dB_x dB_y dB_z
\]

where \(B_0\) is again the rms field in one coordinate, \(B_x, B_y\) and \(B_z\) are the fields along \(x, y\) and \(z\) and \(B^2 = B_x^2 + B_y^2 + B_z^2\). In this case, the PDF of the field after shear or compression can still be described by a multivariate Gaussian function. The PDF of the components of magnetic field perpendicular to the line of sight (responsible for synchrotron emission) then has its major and minor axes along \(X\) and \(Y\), with the rms fields \(B_X\) and \(B_Y\) given earlier. Unless \(B_\perp = 0\), the joint PDF for \(B_\perp\) and the position angle \(\chi\) is as given by MS:

\[
F_{\chi, \perp}(B_\perp, \chi) dB_\perp d\chi = \frac{B_\perp}{2\pi B_X B_Y} \exp(-uB_\perp^2 - vB_\perp^2 \cos 2\chi) dB_\perp d\chi
\]

where

\[
u = \frac{1}{4} \left(\frac{1}{B_X^2} + \frac{1}{B_Y^2}\right)
\]

\[
u = \frac{1}{4} \left(\frac{1}{B_X^2} - \frac{1}{B_Y^2}\right)
\]

The PDF for \(B_\perp\) is given by integration of equation (5) over \(\chi\):

\[
F_\perp (B_\perp) = \frac{B_\perp}{2\pi B_X B_Y} \int_0^{2\pi} \exp(-uB_\perp^2 \cos 2\chi) d\chi
\]

\[
= \frac{B_\perp}{B_X B_Y} \exp(-uB_\perp^2 I_0(-vB_\perp^2))
\]

\(I_0\) is the modified Bessel function of order 0 and we have used the integral representation given by Abramovitz & Stegun (1970), equation (9.6.16). This is identical to equation (3.12) of MS, since \(I_0\) is an even function.
2.4 Synchrotron emissivities for a Gaussian field distribution

We work with total and polarized intensity functions \( j_I \) and \( j_P \) defined such that the rest-frame emissivities are given by \( e_I = k(n, \alpha) j_I \) and \( e_P = k(n, \alpha) j_P \) where \( k(n, \alpha) \) is proportional to the particle density \( n \) and the functions are normalized to have unit value for a uniform field of unit strength perpendicular to the line of sight. \( j_I \) and \( j_P \) are then functions of the field strength, geometry and spectral index:

\[
j_I = \int_0^\infty B_\perp^{1+\alpha} F_\perp(B_\perp) dB_\perp
\]

\[
j_P = \frac{1}{B_X B_Y} \int_0^\infty B_\perp^{1+\alpha} \exp(-u B_\perp^2) I_0(-v B_\perp^2) dB_\perp
\]  

\[= \frac{1}{B_X B_Y} \int_0^\infty B_\perp^{1+\alpha} \frac{\exp(-u B_\perp^2)}{(1+\alpha/2)} dB_\perp
\]

We have again used the integral representation for \( I_1 \) given by Abramovitz & Stegun (1970), equation (9.6.19). It is convenient to write these formulae in terms of the dimensionless field PDF and to define the functions \( G_\alpha(q) = \exp(-q) I_\alpha(q) \) and \( G_1(q) = \exp(-q) I_1(q) \). For \( q \gg n \),

\[
I_n(q) \approx \frac{1}{\sqrt{2\pi q}} \exp(q)
\]

so \( G_\alpha(q) \propto q^{-1/2} \). The total and polarized intensity functions then become:

\[
j_I = \frac{B_\perp^{1+\alpha}}{\eta} \int_0^\infty q^{2+\alpha} \exp(-q^2/2) G_0 \left[ \frac{q^2}{4} \left( \frac{1}{\eta^2} - 1 \right) \right] dq
\]

\[
j_P = \frac{p_0 B_\perp^{1+\alpha}}{\eta} \int_0^\infty q^{2+\alpha} \exp(-q^2/2) G_1 \left[ \frac{q^2}{4} \left( \frac{1}{\eta^2} - 1 \right) \right] dq
\]

In the special case \( B_\perp = 0 \) (\( \eta = 0 \)), the PDF for \( B_\perp \) is a one-dimensional Gaussian function:

\[
F_\perp(B_\perp) dB_\perp = \left( \frac{2}{\pi B_X^2} \right)^{1/2} \exp \left( -\frac{B_\perp^2}{2B^2_X} \right) dB_\perp
\]

and the integral for \( j_I \) can be solved explicitly:

\[
j_I = B_\perp^{1+\alpha} \frac{1}{\pi} \frac{1}{2^{1+\alpha}} \Gamma(1+\alpha/2)
\]  

\[
j_P = p_0 j_I
\]

using the relations \( I_0(w) = J_0(iw) \) and \( I_1(w) = -iJ_1(iw) \) and the substitution \( w = q^2 \). The results are:

\[
j_I = B_\perp^2 (1 + \eta^2)
\]  

\[
j_P = p_0 B_\perp^2 (1 - \eta^2)
\]

and remain valid if \( \eta = 0 \). The degree of polarization is simply:

\[
p = j_P/j_I = p_0 \left( 1 - \eta^2 \right)
\]

Finally, the Stokes parameters in the fixed \( X, Y \) coordinate system are given by:

\[
j_Q = j_P \cos 2\chi_0
\]

\[
j_U = j_P \sin 2\chi_0
\]

Note that these definitions of \( Q \) and \( U \) refer to the direction of the apparent magnetic field, and that the position angle of the synchrotron \( E \)-vector is \( \chi_0 + \pi/2 \).

2.5 Emissivities for an initial field of constant magnitude

An initially isotropic Gaussian field with component rms \( B_0 \) can be regarded as the superposition of a set of isotropic fields of constant magnitude \( B \) with a distribution:

\[
F_{3D}(B) dB = \frac{2}{(2\pi)^{1/2} B_0^3} B^2 \exp(-B^2/2B_0^2) dB
\]

Consider two cubes, both containing initially isotropic fields, but with different distributions. The first has a Gaussian distribution, with component rms \( B_0 \), as before. The second has a field of constant strength \( B_0 \). They are stretched and sheared in identical families. Since the geometrical factors in the emissivity calculation are independent of field strength, the ratio of emissivities (Gaussian/constant) is:

\[
r = \frac{1}{B_0^{1+\alpha}} \int_0^\infty B_\perp^{1+\alpha} F_{3D}(B) dB
\]

\[
= \frac{2^{(5+\alpha)/2} \pi^{-1/2}}{\Gamma(2+\alpha/2)} \int_0^\infty \exp(-t^2) t^{3+\alpha} dt
\]

again using GR 8.310.

The total and polarized emissivities for the case of a constant initial field are \( j_I/r \) and \( j_P/r \), respectively, \( r = 3 \) if \( \alpha = 1 \), reflecting the fact that the emissivity is proportional to the square of the field and that the mean square fields along a given direction are \( B_0^2 \) and \( B_0^2/3 \) for the Gaussian and constant-field cases, respectively.

2.6 Implementation

The integrals can be evaluated using standard routines, for example QROMO and MIDEXP from Press et al. (1992). These implement Romberg integration for an integrand which decreases exponentially and an infinite upper integration limit.
The functions $G_0$ and $G_1$ can be computed by slightly modified versions of the subroutines BESJ0 and BESJ1, also from Press et al. (1992). The emissivity functions depend only on the ratio of the field rms's, $\eta$, for a given spectral index. Look-up tables can be generated as functions of $\eta$ at the beginning of a computation and thereafter polynomial interpolation can be used to derive values as required.

The emissivity functions $j_I$ and $j_P$ are plotted against axial ratio $\eta$ in Fig. 1 for a range of spectral indices typical of optically-thin synchrotron radiation. The corresponding degrees of polarization, $p = j_P/j_I$, are shown in Fig. 2. Although the variation of total intensity with spectral index is substantial, that of the degree of polarization is not, and the analytical formula for $\alpha = 1$ (equation 19) provides a good approximation for most sources, as previously noted in less general cases by Laing (1981) and Hughes et al. (1985).

3 EXAMPLE CALCULATIONS

In this section, we outline the use of the method in practice and demonstrate that the formulae given in the previous section are equivalent to expressions in the literature for a number of simple field distributions. These distributions are non-Gaussian, and their emissivity functions are denoted by $f_I$ and $f_P$, to distinguish them from the Gaussian equivalents.

3.1 General approach

In one class of problem, we are given the rms field components along three orthogonal directions. Such a configuration can always be generated from a unit cube containing an isotropic field by stretch or compression along the three axes. We can choose orthogonal vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ along the axes so that the volume $V$ of the resulting cuboid has any specified value. The initial field $B_0$ and the magnitudes of the vectors are determined by:

\begin{align}
(B_0^2)^{1/2} &= aB_0/V \\
(B_0^2)^{1/2} &= bB_0/V \\
(B_0^2)^{1/2} &= cB_0/V \\
V &= abc
\end{align}

so we can solve for $B_0$, $a$, $b$ and $c$.

We can extend this approach to follow the emission from an element of fluid in a prescribed flow pattern (including the effects of shear) by tracking the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as in MS, using the values given above as initial conditions.

3.2 One-dimensional fields

To get a one-dimensional field, we start with a cube of initially isotropic, Gaussian field of rms $B_0$ in each coordinate and expand it by a factor $c \to \infty$ in one direction ($z$), leaving the other two dimensions constant. The rms field in the $z$ direction remains $B_0$, whilst those in the other two coordinates $\to 0$. Suppose that the $z$-axis makes an angle $\theta$ with the line of sight. In the formulation of Section 2, $\eta = 0$ and the only non-zero edge vector component is $c_X = \sin \theta.$

The emissivities are given by equations (15) and (16) with $B_X = B_0 \sin \theta$.

Alternatively, we can use the standard emissivity functions for a one-dimensional field of strength $B$:

\begin{align}
J_I &= (B \sin \theta)^{1+\alpha} \\
J_P &= p_0 f_I
\end{align}

Figure 1. Plots of the emissivity functions for total intensity (lower panel) and linear polarization (upper panel) against the axial ratio of the magnetic field PDF, $\eta$ for different spectral indices $\alpha$. The values of $\alpha$ are: 0.0 (dotted), 0.5 (short dashes), 1.0 (full line), 1.5 (long dashes) and 2.0 (dash-dot).
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3.3 Two-dimensional fields

Analytical expressions for the total and polarized emissivities for a sheet of disordered field with no component in one direction but equal components in the other two coordinates were given by Laing (1980), who considered a field of constant strength $B$ and random direction, confined to a plane (note that this distribution is not produced by simply squashing a three-dimensional field of constant magnitude and random direction).

In order to compare these formulae with their equivalents from Section 2, we consider a cube of initially isotropic, Gaussian field of rms $B_0$ in each coordinate which is expanded by factors $a$, $a$ and $1/a$ along the $x$, $y$ and $z$ axes in the limit $a \to \infty$, so that there is no field in the $z$ direction, but the rms values in the $x$ and $y$ directions remain equal to $B_0$. Since the $z$-axis makes an angle $\theta$ with the line of sight, the edge vector components are:

\[
an_x = 1, \quad b_x = 0, \quad c_x = 0
\]
\[
a_y = 0, \quad b_y = \cos \theta, \quad c_y = 0
\]

so $B_X = B_0$ (since the field component along the major axis is not projected) and $\eta = \cos \theta$. The emissivity functions are given by inserting these values into equations (12), (13), (15) and (16). For $\theta \neq \pi/2$, we make the substitutions:

\[
t = \frac{q^2}{4} \left( \frac{1}{\eta^2} - 1 \right)
\]
\[
w = 1 + \cos^2 \theta \quad \frac{1}{2 \cos \theta}
\]

into the expressions for $j_t$ and $j_P$, so

\[
j_t = B_0^{1+\alpha} 2^{2+\alpha} \frac{\cos^{2+\alpha} \theta}{\sin^{3+\alpha} \theta}
\]
\[
\times \int_0^\infty \frac{t^{1+\alpha}}{2^{2+\alpha}} \exp[-tw(w^2 - 1)^{-1/2}]I_0(t) dt
\]
\[
j_P = p_0 B_0^{1+\alpha} 2^{2+\alpha} \frac{\cos^{2+\alpha} \theta}{\sin^{3+\alpha} \theta}
\]
\[
\times \int_0^\infty \frac{t^{1+\alpha}}{2^{2+\alpha}} \exp[-tw(w^2 - 1)^{-1/2}]I_1(t) dt
\]

The integrals can be written in terms of Associated Legendre and Gamma functions using GR 6.624 and 8.731:

\[
j_t = B_0^{1+\alpha} 2^{4+\alpha} \frac{3+\alpha}{2} \left( 1 + \cos^2 \theta \right)^{\frac{2+\alpha}{1+\alpha}}
\]
\[
\times P_{\frac{2+\alpha}{1+\alpha}} \left( \frac{1 + \cos^2 \theta}{2 \cos \theta} \right)
\]
\[
j_P = p_0 B_0^{1+\alpha} 2^{4+\alpha} \frac{3+\alpha}{2} \left( 1 + \cos^2 \theta \right)^{\frac{2}{1+\alpha}}
\]
\[
\times P_{1+\alpha} \left( \frac{1 + \cos^2 \theta}{2 \cos \theta} \right)
\]

unless $\theta = \pi/2$, in which case equations (12) and (16) apply, with $B_X = B_0$.

For a spectral index $\alpha = 1$, equations (13) and (18) give:

\[
j_t = B_0^2 (1 + \cos^2 \theta)
\]
\[
j_P = p_0 B_0^2 \sin^2 \theta
\]

Alternatively, we can start from the expressions given by Laing (1980) and average over the PDF for the total field. We make three modifications to the notation:

(i) We normalize over the PDF for azimuthal angle, introducing an extra factor of $2\pi$.
(ii) We use the angle $\theta$ between the normal to the field sheet and the line of sight instead of $\beta = \pi/2 - \theta$.
(iii) We make use of GR 8.731 to simplify the expression for polarized emissivity.

The emissivity functions become:

\[
f_t(\theta) = B_0^{1+\alpha} (\cos \theta)^{\frac{1}{2}+\alpha} \frac{1 + \cos^2 \theta}{2 \cos \theta}
\]
\[
f_P(\theta) = p_0 B_0^{1+\alpha} (\cos \theta)^{\frac{1}{2}+\alpha}
\]
\[
\times \left( \frac{2}{1+\alpha} \right) \frac{p_1^{1+\alpha}}{2 \cos \theta}
\]

For an edge-on field sheet ($\theta = \pi/2$), the argument of the...
Legendre functions $\rightarrow \infty$ and the expressions are replaced by:

$$f_I(\pi/2) = B^{3+\alpha} \pi^{-1/2} \Gamma(1+\alpha/2) \Gamma(3+\alpha/2)$$

$$f_P = p_0 f_I$$

If $\alpha = 1$,

$$f_I = B^2 (1 + \cos^2 \theta)/2$$

$$f_P = p_0 B^2 \sin^2 \theta/2$$

The total field this time has a two-dimensional Gaussian PDF:

$$F_{2D}(B)dB = B B_0^2 \exp\left(-\frac{B^2}{2B_0^2}\right) dB$$

We can calculate the emissivities for this Gaussian field distribution by integrating $B^{3+\alpha}$ over the PDF:

$$\int_0^{\infty} B^{3+\alpha} F_{2D}(B)dB = B_0^{3+\alpha} 2^{1+\alpha} \Gamma\left(\frac{3+\alpha}{2}\right)$$

The last expression is as given by Hughes et al. (1985). We have also verified that our numerical results (equations [41] and [42]) are identical to those of Hughes et al. (1985) for $\alpha \neq 1$.

### 3.4 Fields compressed along one axis

A slightly more complex field configuration was studied by Hughes et al. (1985) in the context of shocks propagating through a jet. We calculate its emissivity in this section in order to illustrate the use of the formalism developed earlier for a three-dimensional case. Hughes et al. (1985) considered a cube containing an initially isotropic field which is compressed to a factor $k$ times its original length along one axis ($z$) and observed in a frame where the plane of compression makes an angle $\epsilon$ with respect to the line of sight. We take the $X$ axis to be in the plane of compression, in which case the vector components are:

$$a_X = 1/k, \quad b_X = 0, \quad c_X = 0$$

$$a_Y = 0, \quad b_Y = \sin \epsilon/k, \quad c_Y = \cos \epsilon$$

The apparent field position angle $\chi_0 = 0$, so $X = X$ and $Y = Y$. The field components and axial ratio are:

$$B_X^2 = B_0^2/k^2$$

$$B_Y^2 = B_0^2(\sin^2 \epsilon/k^2 + \cos^2 \epsilon)$$

$$\eta = (\sin^2 \epsilon + k^2 \cos^2 \epsilon)^{1/2}$$

The emissivity functions for $\alpha = 1$ and the degree of polarization, $p$, are given by equations [43] and [44], with $r = 3$ (equation [23]):

$$f_I = \frac{B_0^2}{3k^2} (1 - k^2 \cos^2 \epsilon)$$

$$f_P = p_0 B_0^2 (1 - k^2) \cos^2 \epsilon$$

We have shown that the formulae reduce to the standard expressions in the literature for one- and two-dimensional fields and for an initially isotropic field compressed along one axis.

### Table 1. Equations for total and polarized emissivity functions for the cases considered in the text.

| $\alpha$ | $\eta$ | $\eta$ |
|--------|------|------|
| $\neq 1$ | $0$ | $0$ |
| $= 1$ | any $\eta$ |

The steps in a practical calculation are as follows:

(i) Start with a cube containing isotropic field, either of constant magnitude $B_0$, or with a Gaussian distribution of component rms $B_0$.

(ii) Compress and/or shear the cube to get the prescribed field configuration (Section 3.1; equations 24 – 27).

(iii) Evaluate the components of the 3 vectors defining the edges of the resulting parallelepiped in the observed coordinate system and normalize them by the volume to give the edge vector components $a_X, a_Y, b_X, b_Y, c_X, c_Y$.

(iv) Work out the apparent field position angle $\chi_0$ from equation [10].

(v) Derive the field components along the principal axes, and their ratio $\eta$ (equations [3] and [4]).

(vi) Calculate the total and polarized emissivities. The relevant pairs of equations for a Gaussian initial field strength distribution are summarized in Table 1.

(vii) For a constant initial field strength, divide by the factor $r$ given in equation [23].

(viii) Rotate by $\chi_0$ to obtain observed Stokes $Q$ and $U$ (equations [20] and [21]).

We have derived expressions for the synchrotron emissivity in Stokes parameters $I$, $Q$ and $U$ for a three-dimensional, disordered, anisotropic field. These are computationally straightforward, requiring only two one-dimensional integrations, which can in turn be used to generate look-up tables. In the special case $\alpha = 1$, the expressions reduce to simple analytical formulae.
ACKNOWLEDGMENTS

This paper has made extensive use of routines from Numerical Recipes (Press et al. 1992), which is gratefully acknowledged.

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