Domain walls and flow equations in supergravity

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Abstract

Domain wall solutions have attracted much attention due to their relevance for brane world scenarios and the holographic RG flow. In this talk I discuss the following aspects for these applications: (i) derivation of the first order flow equations as Bogomol’nyi bound; (ii) different types of critical points of the superpotential; (iii) the superpotential needed to localize gravity; (iv) the constraints imposed by supersymmetry including an example for an $N=1$ flow and finally (v) sources and exponential trapping of gravity.

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1 Introduction

In a spacetime of $d$ dimensions domain walls appear as $(d-1)$-dimensional objects, that separate the spacetime in two regions corresponding to different vacua. A well-known example is the D8-brane solution of massive type IIA supergravity. Like any other brane solution in supergravity, also domain walls require for stability a gauge potential. It is a $(d-1)$-form potential, which couples naturally to the worldvolume of the domain wall and its $d$-form field strength is dual to a (negative) cosmological constant. In other words, a domain wall is a source for a cosmological constant. In a more general setting with non-trivial couplings to scalar fields, it is not only a constant, but a potential $V(\phi)$. For the D8-brane e.g., it is the dilaton potential: $V = e^{-2\phi}m^2$, with $m$ as the mass parameter of massive type IIA supergravity.

There are two kinds of domain walls. For the “good” ones the potential has extrema yielding “good” vacua of the theory. Typically, the extrema are not isolated and have still flat directions corresponding to remaining moduli (unfixed scalar values). Unfortunately, many potentials have a run-away behavior as for the D8-brane and therefore the corresponding scalars do not settle down at a critical point, but “run-away”. These run-away potentials appear typically in string compactifications, where the run-away scalar is the dilaton and one refers often to them as “dilatonic walls”, for a review of domain walls see [1]. As consequence, these “bad” domain walls do not allow for a flat space or anti-deSitter (AdS) vacuum and exhibit a singularity. Due to the experimental evidence of a small positive cosmological constant, which may further decrease in the cosmological evolution, potentials of this type are widely discussed in cosmology, e.g. under the name quintessence.

We refer to domain walls as kink solutions that interpolates between two extrema. It may happen that the extrema are $\mathbb{Z}_2$-symmetric, but in general the extrema are different and the domain wall describes the transition from one vacuum to another one. In a cosmological setting, these domain walls may describe a cascade of transitions of false vacua towards the true vacuum with zero or very small cosmological constant. These solutions are the so-called “thick walls”.

As consequence of the AdS/CFT correspondence [2, 3, 4] domain walls are expected to encode many information about field theory. There are especially two application that attracted much attention recently: (i) the (holographic) supergravity picture for the renormalization group (RG) flow and (ii) the brane world scenario yielding an “alternative to compactification” by trapping gravity on the wall (brane) [5, 6].

In this lecture we will summarize different aspects relevant for these applications. We start with a discussion of domain walls as solutions in (super) gravity and derive the first order flow equations of the scalars. These flow equations are expected to encode the renormalization group (RG) flow in the dual field theory. We comment on the importance of IR attractive critical points of the superpotential for the brane world scenarios and as an illustrative example we will consider the Sine-Gordan model. In section 4 we discuss
the constraints imposed by supersymmetry and its implication for no-go theorems. Many brane world scenarios start with an $AdS_5$ vacuum and introduce sources to cut-off “unwanted” pieces of the space. This procedure is justified if the scalar fields in the asymptotic vacua are fixed by “correct” critical points of the superpotential. We will end with a discussion on this subject.

2 Scalar flow equations in supergravity

We refer to domain walls as kink solutions interpolating between extrema of a potential $V$ giving different vacua of the theory: it is a flat spacetime if $V_{extr} = 0$, a deSitter vacuum for $V_{extr} > 0$ whereas for $V_{extr} < 0$ an anti-deSitter (AdS) vacuum. The latter case arises naturally in supergravity and implies a negative vacuum energy. In the bosonic case, it immediately rises the question for stability of the vacuum, which is ensured if the potential is expressed in terms of a superpotential $W$ as

$$V = 6 \left( \frac{3}{4} g^{AB} \partial_A W \partial_B W - W^2 \right)$$

and the vacuum is given by an extremum of $W$, i.e. $dW = 0$. This also implies an extremum of $V$, but the opposite is not true. Typically $V$ has more extrema which are rarely stable. If $W_{extr.} = 0$ we obtain a flat space, otherwise an AdS spacetime. For specific superpotentials, as we will discuss below, these models can be embedded into supergravity and the scalars saturate the Breitenlohner/Freedman bound. To be more specific, let us expand the potential around a given extremum and let us denote the eigenvalues of the Hessian of the superpotential with $\Delta^{(A)}$, i.e.

$$(g^{AC} \partial_C \partial_B W)_0 = \frac{1}{3} \Delta^{(A)} W_0^B \delta^A_B$$

with $W_0^2 \sim -\Lambda$ as the negative cosmological constant. On the other hand the scalar masses are eigenvalues of the mass matrix $M_B^A = (\partial^A \partial_B V)_0 = m_{(A)}^2 W_0^B \delta^A_B$, where we again absorbed the mass dimension by the cosmological constant. Both dimensionless parameters, $\Delta^{(A)}$ and $m_{(A)}$, are related by

$$m_{(A)}^2 = \Delta^{(A)} (\Delta^{(A)} - 4) \quad \text{or} \quad \Delta^{(A)} = 2 \pm \sqrt{4 + m_{(A)}^2}$$

So, the Breitenlohner/Freedman bound $m_{(A)}^2 \geq -4$ is ensured for any real $\Delta^{(A)}$. Note, due to the negative curvature the lightest scalar fields in AdS spaces have naturally negative masses. Naively one may argue that this causes naively an instability, because a mass of a particle can be seen as the minimal energy necessary to create it out of the vacuum and a negative mass seems to allow to create an arbitrary number of particles without spending any energy. This is true for a flat space vacuum, but for an AdS space one has to be careful in the definition of the energy and this instability does not occur if
the Breitenlohner/Freedman bound is fulfilled. In the deSitter case the vacuum energy is positive, which however implies a non-vanishing temperature and deSitter solution suffer thus from the usual thermodynamical instabilities. As other solutions with non-vanishing temperature, deSitter solutions break supersymmetry and we will ignore them furthermore.

Vacua in supergravity are often associated with a negative maximum of the potential $V$, but this is in general not the case. Although extrema of the superpotential yield $V_{\text{extr}} < 0$ it does not need to be necessarily a maximum. A maximum of $V$ implies that the Hessian is negative definite or $m(A) < 0$, which is the case only if $0 < \Delta(A) < 4$. Therefore, the potential has a maximum only for these values of $\Delta(A)$ and especially it has a negative minimum whenever $\Delta(A) < 0$, which we will discuss below as IR attractive fixed points.

To be more concrete, we are interested in flat domain walls with a metric

$$ds^2 = e^{2U} \left(-dt^2 + dx^2\right) + dy^2.$$  

This ansatz preserves the Poincaré symmetry if the fields depend only on the transverse coordinate, i.e. $U = U(y)$. For these flat wall solutions, the gauge fields are trivial and the action reads

$$S = \int_M \left[\frac{R}{2} - \frac{1}{2} g_{AB} \partial_a \phi^A \partial^a \phi^B - V\right] - \int_{\partial M} K.$$  

We included also a surface term with the outer curvature $K$ and the scalar fields $\phi^A = \phi^A(y)$ parameterizing a space $M$ with a metric $g_{AB}$.

The Poincaré invariance of the ansatz (4) implies that all worldvolume directions are Abelian isometries, so that we can integrate them out. For our ansatz the Ricci scalar takes the form $R = -20(\dot{U})^2 - 8\ddot{U}$ and after a Wick rotation to an Euclidean time we find the resulting 1-dimensional action

$$S \sim \int dy e^{4U} \left[-6 \dot{U}^2 + \frac{1}{2} g_{AB} \dot{\phi}^A \dot{\phi}^B + V\right].$$  

In deriving this expression, the surface term in (5) was canceled by the total derivative term. The equations of motion of this action describe trajectories $\dot{\phi}^A = \phi^A(y)$ of particles in the target space $M$ with the metric $g_{AB}$. This is closely related to an analogous discussion of black holes [11, 12, 13]. As a consequence of the 5-d Einstein equations, these trajectories are subject to the constraint

$$-6 \dot{U}^2 + \frac{1}{2} |\dot{\phi}^A|^2 - V = 0$$  

with $|\dot{\phi}^A|^2 = g_{AB} \partial_y \phi^A \partial_y \phi^B$. In order to derive the Bogomol’nyi bound we can insert the potential and write the action as

$$S \sim \int dy e^{4U} \left[-6 (\dot{U} \mp W)^2 + \frac{1}{2} |\dot{\phi}^A \pm 3 \partial^A W|^2\right] \mp 3 \int dy \frac{d}{dy} e^{4U} W$$  

\footnote{In our notation, dotted quantities always refer to $y$-derivatives.}
leading to the BPS equations for the function $U = U(y)$ and $\phi^A = \phi^A(y)$:

\[
\dot{U} = \pm W, \quad \dot{\phi}^A = \mp 3 g^{AB} \frac{\partial W}{\partial \phi^B}.
\]

(9)

An analogous derivation of these equations can be found in \[14\] and if they are satisfied, the bulk part of the action vanishes and only the surface term contributes. In the asymptotic $AdS_5$ vacuum the surface term diverges near the AdS boundary ($U \sim y \rightarrow \infty$) and after subtracting the divergent vacuum energy one obtains the expected result that the energy (tension) of the wall is proportional to the difference of the cosmological constants (topological charge):

\[
\sigma \sim \Delta W_0 = W_{+\infty} - W_{-\infty}.
\]

(10)

If one embeds this model into N=2 supergravity \[15, 16\] the first BPS equation becomes equivalent to the variations of the gravitino and the gaugino/hyperino; the tension is given by the gravitino charge (mass) and can be obtained by Nesters procedure \[17, 9, 18\].

Our metric ansatz was motivated by Poincaré invariance which is not spoiled by a reparameterization of the radial coordinate. We have set $g_{yy} = 1$, which is one possibility to fix this residual symmetry, but we can also use this symmetry to solve the first BPS equation: $W \, dy = \pm dU$, i.e. to take $U$ as the new radial coordinate. In this coordinate system the metric reads

\[
ds^2 = e^{2U} \left(-dt^2 + d\vec{x}^2\right) + \frac{dU^2}{W^2}.
\]

(11)

Repeating the same steps as before we obtain the Bogomol’nyi equations for the scalars

\[
- \dot{\phi}^A = g^{AB} \partial_B \log |W|^3 = g^{AB} \partial_B h
\]

(12)

which follow from the one-dimensional action

\[
S \sim \int dy \left[|\dot{\phi}^A|^2 + g^{AB} \partial_A h \partial_B h\right] = \int dy \left[|\dot{\phi}^A|^2 + g^{AB} \partial_B h|^2\right] + \text{(surface term)}
\]

(13)

where $h = 3 \log |W|$. As before, the field equations are subject to the constraint $\frac{1}{2} |\dot{\phi}^A|^2 - g^{AB} \partial_B h \partial_B h = 0$ and the surface term yields the central charge. Supersymmetric vacua are given by extrema of $h$ and the number and type of such vacua can be determined by using Morse theory where $h$ is called the height function, see \[19\]. To be consistent, $h$ has to be a “good” height function, which means especially that $h$ and all its derivatives are well-defined on $\mathcal{M}$ and the Morse inequalities state that the number of critical points is larger or equal to the sum over all Betti numbers of $\mathcal{M}$ \[20, 21\].

Moreover, the height function $h$ and therefore also the superpotential are monotonic along the flow. Namely, multiplying eq. (12) with $g_{ik} \dot{\phi}^k$ one obtains

\[
- \dot{h} = - \dot{\phi}^A \partial_A h = g^{AB} \dot{\phi}^A \dot{\phi}^B \geq 0
\]

(14)

implying that the height function $h$ is a monotonic decreasing function towards larger values of $U$. This is the proposed supergravity analog of the $c$-theorem \[22\], but note it is well defined as long as $h$ is finite, i.e. as long as the superpotential $W$ does not pass a zero or pole.
3 RG flow and localization of gravity

Let us now turn to physical applications that made domain wall solutions of supergravity so popular. The first application concerns the renormalization group (RG) flow, see \[23\] and refs therein. The other application is the localization of gravity on or near the wall, which is discussed nowadays as the Randall-Sundrum scenario \[6\], although the basic idea goes back more than 15 years ago \[5\].

3.1 Holographic RG flow

The holographic RG flow based on the AdS/CFT correspondence \[2, 3, 4\], which conjectures that AdS gravity is dual to a conformal field theory (CFT) and the scalar fields in (super) gravity corresponds to couplings of perturbation in the field theory. Originally this conjecture has been made for \(S_5\)-compactification of type IIB string theory and the dual field theory on the worldvolume of a D3-brane resides on the boundary of the \(AdS_5\) space. This original idea has been extended in a way that any hypersurface of constant radius should have a field theory dual and the radial coordinate was identified as the energy scale in the field theory. In this interpretation, the AdS boundary corresponds to the UV-limit of the field theory and the radial motion translates into the RG flow towards the IR.

Let us start with some general remarks about the RG flow in field theory and repeat some well-known facts. Consider a field theory described by an action

\[ S = S[\mathcal{O}_A, g^B] \] (15)

with a set of operators \(\mathcal{O}_A\) with couplings \(g^B\). In classical field theory these couplings are constant, but due to the renormalization they become scale dependent \(g^B = g^B(\mu)\) (running couplings). This scale dependence is fixed by the \(\beta\)-functions

\[ \mu \frac{d}{d\mu} g^B = \beta^B(g) \] (16)

which can be derived from the renormalization: \(g^B \to g^B_R - \beta^B \log \frac{a}{\mu} + O(\log^2 \frac{a}{\mu})\), where \(a\) is a cut-off and \(\mu\) is the RG-scale. Note, the couplings \(g^B\) are not necessarily gauge couplings, but the couplings to any perturbations of the Lagrangian and the \(\beta\)-functions do not necessarily refer to gauge field \(\beta\)-functions.

The zeros of the \(\beta^B\)-functions are especially interesting, because at these points the theory becomes scale invariant and therefore finite. Near this point the operators \(\mathcal{O}_A\) have a well-defined scaling behavior and the scaling dimensions \(\hat{\Delta}_A\) are given by the eigenvalues of \(\gamma^A_B\) appearing in the expansion around a zero \(\beta^A(g_0) = 0\)

\[ \beta^A = \gamma^A_B \delta g^B + O(\delta g^2) \] (17)
In general the $\beta$-function may have different zeros, which are UV- or IR-attractive. In addition, there may be different poles separating different phases of the field theory. From the supergravity point of view, the singular line I correspond to a pole in $W$ whereas II is a zero in $W$ and in both cases we expect a singularity in field theory.

The equations (16) are first order differential equations describing a flow towards a zero of $\beta^A$, which are fixed points of the flow. Solving these equations near a fixed point we find $\delta g^A \sim e^{\hat{\Delta^A} \log \mu}$ and stability requires that $\hat{\Delta^A} \log \mu \rightarrow -\infty$ while we perform scale transformations $\mu \rightarrow e^{\lambda} \mu$. In an UV scaling: $\lambda \rightarrow +\infty$ this implies $\hat{\Delta^A} < 0$ whereas in an IR scaling $\lambda \rightarrow -\infty$: $\hat{\Delta} > 0$. Hence, we get the picture as shown in figure 1, that in different scaling limits the coupling runs to different zeros of the $\beta$-function. Unfortunately in many cases the $\beta$-functions are known only perturbatively for small couplings and they are often only well-defined in the UV regime. The IR behavior of many field theories are out of reach and it would be interesting to have a dual description which is still valid at points where field theory methods break down. Having this motivation, there has been a significant effort to use the AdS/CFT correspondence to get new information in field theory. The translation table is straightforward: recall the scalar fields $\phi^A$ in gravity correspond to the field theory couplings $g^A$ and the warp factor in eq. (11) $e^U$ corresponds to the RG parameter $\mu$. Obviously, the case $U \rightarrow +\infty$ corresponds to large supergravity length scale and therefore, due to the AdS/CFT correspondence, describes the UV regime of the dual field theory. The opposite happens for $U \rightarrow -\infty$, which is related to small supergravity length scales and thus encodes the IR behavior of the dual field theory. Moreover, the $\beta$-function entering the first order differential equations (16) can be translated into the flow equations (12). Extrema of $W$ are fixed points of the scalar flow equations and translate into fixed points of the RG flow. As long as $W \neq 0$ at the extremum, we obtain an AdS vacuum, which is reached either near the boundary ($U \rightarrow +\infty$) or near the Killing horizon ($U \rightarrow -\infty$).

In order to identify the different fixed points we do not need to solve the equations explicitly; they are determined by the eigenvalues of the Hessian of the height function $h$. Let us go back to the BPS equations and expand these equations around a given fixed
Figure 2: An example for a superpotential $W$. At UV extrema the supergravity solution approaches the boundary of the AdS space, whereas IR extrema correspond to the Killing horizon. The supergravity solution is singular at poles of $W$ but regular at zeros.

point with $\partial_A W|_0 = 0$ at $\phi^A = \phi^A_0$. The superpotential becomes

$$W = W_0 + \frac{1}{2} (\partial_A \partial_B W)_0 \delta \phi^A \delta \phi^B \pm \ldots$$

(18)

with $\delta \phi^A = \phi^A - \phi^A_0$, and the cosmological constant (inverse AdS radius) is given by $\Lambda = -W_0^2 = -1/R^2_{AdS}$. Consequently, near the AdS vacuum we find a solution of the flow equations

$$U = (y - y_0)W_0 , \quad \delta \phi^A = e^{-\frac{1}{2} \Delta(A) W_0 (y - y_0)} = e^{-\frac{1}{2} \Delta(A) U}$$

(19)

with the scaling dimensions introduced in (2). This approximate solution is valid only if $\delta \phi^A = \phi^A - \phi^A_0 \to 0$ in the AdS vacuum with $U \to \pm \infty$ and therefore all eigenvalues $\Delta^{(i)}$ have to have the same sign: $\Delta^{(A)} > 0$ for UV fixed points ($U \to +\infty$), or $\Delta^{(A)} < 0$ for IR fixed points ($U \to -\infty$). Equivalently, UV fixed points are minima of the height function $h = \log |W|^3$ whereas IR fixed points are maxima, see figure 2. For this conclusion we assumed that the scalar metric has Euclidean signature and $W_0 > 0$. It is important to notice that in the definition of the scaling dimensions the matrix $\Omega^A_B$ has one upper index and one lower index. It is straightforward to consider also the possibilities $W_0 < 0$ and/or timelike components of the scalar field metric. Note, the sign ambiguity in the BPS equations (2) interchanges both sides of the wall, i.e. it is related to the parity transformation $y \leftrightarrow -y$, which also flips the fermionic projector onto the opposite chirality.

If the eigenvalues $\Delta^{(A)}$ have different signs for different scalars, this extremum is IR-attractive for some scalars and UV-attractive for others and, therefore, is not stable (a saddle point of $h$). A small fluctuation will initiate a further flow towards a local maximum.
or minimum for the scalars which enter the superpotential. Recall, in our sign convention larger values of the radial parameter $U$ corresponds to the UV region and are minima of the height function $h$. If we start with the UV point ($U = +\infty$) and go towards lower values of $U$, the $c$-theorem states that $h$ has to increase, either towards an IR fixed point (maximum) or towards a positive pole in $h$ ($W^2 \to \infty$), which is singular in supergravity and corresponds to $c_{CFT} = 0$. On the other hand, if we start from an IR fixed point ($U = -\infty$) and go towards larger values of $U$, due to the $c$-theorem $h$ has to decrease, either towards a minimum (UV fixed point) or towards a negative pole ($W^2 \to 0$), which is not singular in supergravity. An example is the asymptotically flat 3-brane, where the height function parameterizes the radius of the sphere, which diverges asymptotically (indicating decompactification) and runs towards a finite value near the horizon which is IR attractive in our language.

In summary, there are the following distinct types of supergravity flows, which are classified by the type of the extremum of the height function or superpotential. Depending on the eigenvalues of the Hessian of $h$, the extrema can be IR attractive (negative eigenvalues), UV attractive (positive eigenvalues) or flat space (singular eigenvalues). Generalizing the above discussion and allowing also possible sign changes in $W$, the following kink solutions are possible:

(i) flat ↔ IR
(ii) IR ↔ IR
(iii) IR ↔ UV
(iv) UV ↔ UV (singular wall)
(v) UV ↔ singularity ($W^2 = \infty$).

Note, there is no kink solution between a UV fixed point and flat space, because the $c$-theorem requires a monotonic $h$-function and the UV point corresponds to a minimum of $h$ whereas the flat space case is a negative pole. Moreover, if there are two fixed points of the same type on each side of the wall, $W$ necessarily has to change its sign implying that the wall is either singular (pole in $W$) or one has to pass a zero of $W$. In addition, between equal fixed points no flow is possible (would violate the $c$-theorem) and therefore this describes a static configuration, where the scalars do not flow. This is also what we would expect in field theory, where the RG-flows go always between different fixed points. Recall, although a zero of $W$ means a singularity in $h$, the domain wall solution can nevertheless be smooth. Type (v) walls appear generically for models which can be embedded into maximal supersymmetric models, for examples see [24, 27], whereas models allowing type (iv) walls typically can not be embedded into maximal supersymmetric model[9], for an explicit example of this singular flow see [11].

Interesting are of course flows towards a confining gauge theory, for which the Wilson

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3Maximal supersymmetric models typically have only one UV extremum.
\[ ds^2 = e^{-W(y)}(-dt^2 + d\vec{x}^2) + dy \]

**Figure 3:** In order to localize gravity near the branes one needs an exponential suppression of the warp factor on both sides. This means that the scalars have to be fixed by an IR attractive fixed point of the potential and the superpotential has to change its sign.

loop shows area law, i.e. it scales like

\[ \langle \frac{1}{N} \text{Tr} P e^{\frac{i}{2} A_u dx^u} \rangle \sim e^{-cA} \]  

with some constant \( c \) and \( A \) is the area enclosed by the Wilson loop. Writing the 5-d metric as

\[ ds^2 = f(y)[-dt^2 + d\vec{x}^2] + g(y)dy^2 \]  

the Wilson loop confinement criterion is fulfilled if the warp factor has a lower bound

\[ f \to f_{\text{min}} > 0, \quad g \sim \frac{1}{y^2} + \text{finite}, \quad \text{for } y \to 0 \]  

which has been nicely summarized in [26], but see also [27] for a review. Notice, the 5-d metric becomes flat for a flow towards a confining gauge theory.

### 3.2 Localization of gravity

There is another application that attracted much attention recently, namely the possibility to localize gravity on a domain wall. This old idea [5] has recently been discussed as an alternative to compactification [6]. The central idea is to employ the exponential warp factor to suppress any dynamics perpendicular to the wall, see figure 3. In the language of the RG flow this means that on both sides of the wall one approaches asymptotically an IR fixed point, i.e. the Hessian of \( h \sim \log |W| \) has to be negative definite at the critical point! So, we need a potential with two IR points as shown in figure 2. Moreover, in order to have an exponential suppression on both sides, \( W \) has to change its sign and it will have the opposite tension than a (singular) wall separating two UV critical points.

For the idealized situation, that the space is always AdS (thin wall approximation), it could be shown that the exponential warp factor traps the massless graviton mode near the wall [5, 6] and if the cosmological constant is large enough the fifth direction becomes invisible – at least at our low energies. In this picture, our 4-d world can be seen as a 3-brane embedded in a higher-dimensional space. However, in the setup that we discussed
so far, the scale of the gravitational and the gauge interaction would be of the same magnitude, which is not the case in our world. In contrast, in our low-energy world the gravitational scale is suppressed by many magnitude and this discrepancy is also known as the gauge hierarchy problem. In a modified version, the brane world scenario has been proposed to yield an elegant resolution of this problem \[28, 29\]. Namely one can introduce a second brane, the so-called Standard model brane, and brings it close to the IR critical point. Obviously, the gravitational scale on the Standard model brane is exponentially suppressed with respect to the former, the so-called Planck brane. In the spirit of the RG-flow, one may assume that in the UV regime both brane are close to each other and only in the limit of low energies our Standard model brane moves closer and closer to the IR critical point and gravity becomes weaker and weaker.

Let us note, that these solutions necessarily violate the proposed c-theorem \([14]\) and there are no-go theorems for constructing smooth domain walls of this type \([30, 31]\). This seems to be in contradiction to the fact that there are smooth solution as in N=1,D=4 supergravity \([32]\), but also in 5 dimensions as we will see in the next section. To resolve this puzzle one may not regard these solutions as flows, but instead as a static configuration where both AdS vacua coexist and not as a decay of one vacuum into another.

### 3.3 As an example: Sine-Gordon model

Unfortunately, supergravity as it comes from compactified string or M-theory exhibits an abundance of UV critical points but almost no IR critical points. This is of course related to difficulties with negative tension branes, which would arise in a thin wall approximation. But the existence of IR critical points is essential for the brane world scenarios discussed before, only IR critical points provide the exponential suppression.

As a illustrative example let us consider the Sine-Gordan model where the superpotential is given by

\[ W = a + b \cos \theta \]  \hspace{1cm} (23)

and the kinetic term of the angular scalar field \( \theta \) is normalized as \( g_{\theta\theta} = 0 \). The critical points are at \( \cos \theta = \pm 1 \) and the type is related to the derivative of the \( \beta \)-function calculated at the different critical points

\[-\partial_b \beta|_\pm = \Delta = 3\partial_b \partial_b \log W|_\pm = -\frac{3b}{b \pm a} = \begin{cases} < 0 & \text{IR} \\ > 0 & \text{UV} \end{cases}. \]  \hspace{1cm} (24)

For \( a > b > 0 \) we have alternately UV and IR critical points, for \( a = b \) we get a flow towards flat space \( (dW = W = 0) \) and for \( a = 0 \) there are only IR critical points. For this model the flow equations \([8]\) can be solved explicitly \([33]\), but let us discuss only two examples. First, for the flat space flow \( (a = b) \) one finds \([33]\)

\[ ds^2 = \left(1 + e^{-12az}\right)^{-\frac{3}{2}} \left(-dt^2 + dx^2\right) + dz^2, \quad \cos \theta = -\tanh 6az. \]  \hspace{1cm} (25)
Figure 4: This figure shows the potential $V$ for the superpotential (23). For positive $a, b$ the supersymmetric extrema at $\theta = 0$ are always infra-red attractive, whereas the nature of the critical point at $\theta = \pi$ depends on the value of $a/b$. If $4a = 7b$ one finds $\Delta = 4$ and the corresponding scalar becomes massless.

This solution describes now a flow from an IR critical point ($\Delta = -\frac{3}{2}$) at $z \to -\infty$ and flat space time at $z \to +\infty$, where the Wilson loop criteria for confinement (22) is fulfilled with $y = e^z$. The IR point is a minimum for the potential $V$ and the flat space $V = 0$ is approached from above ($V'' > 0$), see figure 4. Therefore, near this flat space vacuum the effective cosmological constant is positive and becomes smaller and smaller providing an example for the quintessence scenario.

The other solution, that we want to mention is the case $a = 0$, which has only IR critical points ($\Delta = -3$). The solution can be written as

$$ds^2 = \left( \cosh 6bz \right)^{-\frac{2}{3}} \left( -dt^2 + d\mathbf{x}^2 \right) + dz^2, \quad \cos \theta = -\tanh 6bz$$

and provides an explicit realization of the Randall-Sundrum scenario, which we introduced in the previous subsection. Since the scalar $\theta$ is an angle this solution is periodic and we can identify Plank branes at $\sin \theta = \pm 1$ and the Standard model branes near the fixed points $\sin \theta = 0$. This interpretation becomes obvious in the thin wall approximation $b \to \infty$.

Let us also mention the case, where $\Delta = 4$ where the scalar becomes massless. It happens for $a/b = \frac{7}{4}$ and corresponds to the plateau in figure 4. This is exactly the point, where the extremum of $V$ converts from a minimum ($\Delta > 4$) into a maxima ($0 < \Delta < 4$); see eq. (3). From the field theory point of view this is an UV critical point whereas $\theta = 0$ corresponds to the IR regime. Notice, whenever $a > b$ the solution has both types of critical points and when $a < b$ there are only IR critical points with $\Delta < 0$.

The superpotential (23) yields a supergravity potential of the form $V = a_1 + a_2 \cos \theta + a_3 \cos 2\theta$, where the coefficients depend on $a, b$. Potentials of this type are generated naturally by an instanton/monopole condensation and have an long history in 4-d gauge
theory in the discussion of confinement [34], but where also discussed for 5d domain walls, e.g. recently in [35]. Let us concentrate on the discussion of a single cosine potential and we refer to [10] for more details. In 4-d gauge theory this potential can be derived by summing over instantons and anti instantons in a dilute gas approximation, i.e. widely separated non-interacting instanton and anti instantons. In fact, from the topological term \( \int \theta tr (F \wedge F) \) we obtain after summing over instantons and anti-instantons a cosine potential for the axion

\[
\sum_{n,\bar{n}} \frac{1}{n! \bar{n}!} e^{i \theta (n - \bar{n})} = e^{2 \cos \theta}.
\]

But can something similar also happen in 5-d supergravity? The answer is yes, if we replace the Yang-Mills instantons by M5-brane instantons. To be more concrete, 5-d supergravity can be obtained from M-theory compactification and an M5-brane instanton background translates into a gas of point-like sources in 5 dimensions with \( dG = n \delta^{(5)} \), i.e. the 5-branes wrap the 6-d internal space. The compactification of the topological Chern-Simons term \( \int C \wedge G \wedge G \) yields than a topological term \( \int \theta dG \). So if \( dG \) is non-trivial this term is the 5-d analogue of the familiar universal axionic coupling discussed above and as in 4 dimensions we expect that the sum over an M5-brane instanton gas will reproduce the cosine potential.

4 Supersymmetry: What is possible and what not?

Imposing supersymmetry, the BPS equations [9] become the fermionic supersymmetry variation for the gravitino and gaugino/hyperino depending on the type of scalars. In addition, supersymmetry puts severe constraints on the superpotential \( W \). If we have four unbroken supercharges as for \( N=1,D=4 \) supergravity, any holomorphic \( W \) is allowed. On the other hand, if we have eight supercharges, as for \( N=2 \) supergravity in four and five dimensions, the superpotential has to come from a gauging of global isometries. Restricting to 5-d \( N=2 \) supergravity, the scalar fields can be in three different multiplets: vector-, tensor- or hypermultiplets. In ungauged supergravity vector- and tensormultiplets are equivalent; they are dual to each other, which is not the case in gauged sugra, see [37] for a discussion of non-trivial tensormultiplets. The scalars parameterize a direct product space

\[
\mathcal{M}_{V/T} \times \mathcal{M}_H
\]

where \( \mathcal{M}_{V/T} \) is defined by the cubic equation

\[
F = \frac{1}{6} C_{IJK} X^I X^J X^K = 1
\]

and \( I \) counts the number of vector- and tensorfields. The four scalars of a hypermultiplet can be combined to a quaternion and the scalar manifolds \( \mathcal{M}_H \) has to be quaternionic. In gauged supergravity, one gauges isometries of these scalar manifolds. One can show
that the gauging of isometries of $\mathcal{M}_{V/T}$ does not influence the flow equations for the scalar, but introduces only a further constraint.

More interesting is the gauging of isometries of $\mathcal{M}_H$ which can be gauged with different vector fields $A^I$ and are given by Killing vectors $k^a_I$: $q^a \rightarrow q^a + k^a_I \epsilon^I$ and $dq^a \rightarrow dq^a + k^a_I A^I$, where $q^a$ denote the scalars in the hypermultiplets. Supersymmetry requires now that the Killing vectors have to be tri-holomorphic, which means that they can be expressed by Killing prepotentials

$$\Omega^x u^a_I = -\nabla_u P^x_I$$

where $\Omega^x$ is the triplet of Kähler forms characterizing the quaternionic manifold ($x = 1..3$) and $\nabla$ is the covariant derivative with respect to the $SU(2)$-part of the $SU(2) \times Sp(2n_H)$ holonomy of the quaternionic space, for more details we refer to [40, 39, 41]. The real-valued superpotential which enters the flow equations becomes

$$W^2 = \sum_{x=1}^{3} \left( P^x_I X^I \right)^2$$

and it is especially simple if the $SU(2)$-valued Killing prepotential has only one component, say $P^3_I$. In this case, the Killing prepotential can be shifted by constants

$$P^3_I \rightarrow P^3_I + \alpha_I$$

which are the analogs of the FI terms known in field theory. If only these terms are turned on, one has the special case of a gauged $SU(2)$-R-symmetry [12] and the superpotential reads

$$W = \alpha_I X^I$$

and it depends on the vector scalars $\phi^A$ via the constraint (29). Let us consider this special case in more detail. Critical points of $W$ are given by

$$\partial_A W = \alpha_I \partial_A X^I (\phi^A) = 0$$

and since $\partial_A X^I$ are tangent vectors on $\mathcal{M}_V$, these are points where the vector $\alpha_I$ is normal to $\mathcal{M}_V$ [13, 14]. The normal vector is given by $\partial_I F$ and has therefore be proportional to $\alpha_I$, which becomes the attractor equation [18]

$$\frac{1}{W_0} \alpha_I = \partial_I F.$$  

The proportionality factor has been fixed by contracting the equation with $X^I$. In many simple cases these equations can solved explicitly, as for so-called $STU$-model ($F = STU$) [15, 16] (see [17] for the 4d case). By performing the second derivative we can also determine the type these extrema. Using the formula [18, 49]

$$\partial_A \partial_B W = \alpha_I \partial_A \partial_B X^I = \frac{2}{3} g_{AB} W + \mathcal{O}(\partial_A W)$$

14
one finds that all scaling dimensions as introduced in (2) are

\[ \Delta_{(4)} = +2. \]  

(37)

The “+” sign indicates that we have an UV fixed point and the “2” that they are related to mass deformation in field theory. But, this means especially that there are no IR critical points and therefore the flow has necessarily go towards the singularity \[ F = 0, \] at which the \( \beta \)-function has a regular zero \[ [51], \] but the spacetime metric is singular.

Therefore, if we have a superpotential depending only on vector scalars, no smooth flows are possible and especially the lack of IR critical points means, that brane world scenarios are excluded \[ [38, 52]. \] But notice, this conclusion based on the relation \[ (36) \] which holds only for vector scalars. In fact, turning on hyper scalars and allowing for general Killing prepotentials \( P^x_I = P^x_I(q) \), smooth flows connecting an UV and an IR attractive fixed point may be possible; see discussion in \[ [53] \] where a simple model with one hypermultiplet is considered, which may be related to the configurations described in \[ [54, 55, 56, 57, 58]. \]

5 Sources and stability of brane world scenarios

It is very typical in supergravity, that solutions are only consistent if one adds appropriate sources. This is especially necessary for all (charged) solutions of co-dimensions greater than two, i.e. in 10 dimensions all p-brane solutions with \( p < 7 \). All these solutions are given in terms of a harmonic function \( H = H(\vec{y}) \) on the transverse coordinates \( \vec{y} \)

\[ \partial^2_{\vec{y}} H(\vec{y}) = Q \delta(\vec{y}). \]  

(38)

If the dimension of the transverse \( y \)-space is greater than two we need a source for a non-trivial solution, but not necessarily for the 2- or 1-dimensional case. For codimension-2, as the D7-brane, it can be any holomorphic function whereas for codimension-1 situation, as for a domain wall, any linear function is harmonic. On the other hand, there are good reasons to introduce sources also for codimension-1 and codimension-2 object. One reason is that T-duality should map all D-brane solutions onto one another and this implies that sources on the rhs are present for all of them. Another reason is, that a linear harmonic function has necessarily a zero, which causes in many cases a curvature singularity. This singularity can be avoided if we consider the harmonic function: \( H = a + b \ | y - y_0 | \), with \( a \) and \( b \) as some positive constants. The absolute value obviously corresponds to a source at the position \( y = y_0 \) and both sides are \( Z_2 \) symmetric. Physically, this means, that we cut-off the singular part and glue together two regular pieces at \( y = y_0 \). In this setup the domain wall can be identified with this singular source, see \[ [59] \] for more details.

For the brane world scenarios there are some subtle points, which one has to take into account. To solve the hierarchy problem and for the localization of gravity it is essential that the warp factor of the metric has exponential fall-off – at least asymptotically. This is
in fact the case for the pure $AdS_5$ case, but a cosmological constant is always a very special limit and in general the low-energy supergravity has a potential depending on the various scalar fields. If this potential has an extrema, there exist of course a solution of the flow equations with constant scalars yielding $dW = 0$ and in this case the spacetime is $AdS_5$ everywhere. Next, one may introduce sources and realize a brane world scenario with an exponential warp factor. This setup is robust, if we can allow for small fluctuations of the scalar fields and the exponential suppression survives – at least asymptotically. This however is only the case, if the asymptotic vacua is an IR attractive fixed point with $\Delta < 0$; cp. the asymptotic solution (19). If however, the scalars are fixed at an UV attractive fixed point any small fluctuation will destroy the exponential warp factor. One still has an exponential increase in one direction, but in the other direction the warp factor goes to zero only with a certain power of the radial distance. Actually having models as coming from the superpotential (33) without any IR fixed point, any small fluctuations in the scalars will cast the exponential suppression into a power suppression. On the other hand having models with an IR fixed point, the exponential suppression is stable under small fluctuations of the scalars and in these models we can set the scalars to their critical value. Therefore, the challenge in 5-d supergravity is to find “good” superpotentials and having this, one may cut-off the regions close to the UV and IR critical points and continue in a periodic way. This means that one has to introduce sources and one of them has to have a negative tension and such objects are not understood.

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