RELATIONS AMONG $\mathbb{P}$-TWISTS

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Abstract. Given two $\mathbb{P}$-objects in some algebraic triangulated category, we investigate the possible relations among the associated $\mathbb{P}$-twists. The main result is that, under certain technical assumptions, the $\mathbb{P}$-twists commute if and only if the $\mathbb{P}$-objects are orthogonal. Otherwise, there are no relations at all. In particular, this applies to most of the known pairs of $\mathbb{P}$-objects on hyperkähler varieties. In order to show this, we relate $\mathbb{P}$-twists to spherical twists and apply known results about the absence of relations between pairs of spherical twists.

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1. INTRODUCTION

Spherical objects and the associated spherical twists are well-studied, in particular, one knows a lot about the relations between spherical twists. Much is already present in [ST01] by P. Seidel and R. Thomas, where spherical objects were introduced in algebraic geometry. The easiest geometric examples of such objects are line bundles in the derived category of a K3 surface, or, more generally, of a strict Calabi–Yau variety of arbitrary dimension.

Spherical objects and their twists generalise quite naturally to $\mathbb{P}$-objects and their $\mathbb{P}$-twists, see [HT06] by D. Huybrechts and R. Thomas. Here, the easiest examples are line bundles in the derived category of a hyperkähler variety. Most of the story about spherical objects carries over to $\mathbb{P}$-objects, somewhat in analogy to the passage from K3 surfaces to higher dimensional hyperkähler varieties.

We want to highlight a connection between these two notions, which is central for this article. Given a hyperkähler variety $X$, there is the inclusion $j : X \to \mathcal{X}$ into its twistor space. The twistor space $\mathcal{X}$ is the total space of all possible complex structures that can be put on $X$, which is an analytic space over $\mathbb{P}^1$. 
By [HT06], the (derived) pushforward \( j_* : \mathcal{D}(X) \to \mathcal{D}(X) \) turns any \( \mathbb{P} \)-object \( P \in \mathcal{D}(X) \) into a spherical object in \( \mathcal{D}(X) \), under an assumption, which can be interpreted as \( P \) does not deform from \( X \) to \( X \). Moreover, the associated twists fit together in the sense that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{D}(X) & \xrightarrow{j_*} & \mathcal{D}(X) \\
P_P \downarrow & & \downarrow T_{j_*P} \\
\mathcal{D}(X) & \xrightarrow{j_*} & \mathcal{D}(X)
\end{array}
\]

where \( P_P \) is the \( \mathbb{P} \)-twist and \( T_{j_*P} \) is the spherical twist associated to \( j_*P \).

This functor \( j_* : \mathcal{D}(X) \to \mathcal{D}(X) \) is the prototype of what we call a spherification functor, a functor that turns \( \mathbb{P} \)-objects into spherical objects; see Subsection 3.1 for details on our definition. We establish the existence of such functors that turn \( \mathbb{P} \)-objects into spherical objects also more abstractly, using our previous work on formality of \( \mathbb{P} \)-objects [HK19]:

**Theorem A (Corollary 3.13).** Let \( P_1, \ldots, P_m \) be \( \mathbb{P}^n[k] \)-objects in some algebraic triangulated category, such that \( k \geq 2 \) is even, \( n \geq 2 \), \( \gcd(k, nk/2) > 1 \) and \( \text{Hom}^*(P_i, P_j) \) is concentrated in degree \( nk/2 \) for \( i \neq j \) (or zero). Then there is a spherification functor \( F : \langle P_1, \ldots, P_m \rangle \to \mathcal{B} \).

We conjecture that the assumptions of Theorem A are mainly of technical nature, that is, we suppose that abstract spherification functors exist in greater generality.

Anyway, with a spherification functor at hand (of geometric origin or an abstract one), we can use the compatibility of \( \mathbb{P} \)-twists and spherical twists. The relations between the spherical twists of two spherical objects are known by work of Y. Volkov [Vol19], which we pull back to \( \mathbb{P} \)-twists along a spherification functor to obtain the following theorem. We say that two objects \( P_1 \) and \( P_2 \) of a triangulated category are isomorphic up to shift if there exists some \( m \in \mathbb{Z} \) such that \( P_1 \cong P_2[m] \). Note that, if two \( \mathbb{P} \)-objects are isomorphic up to shift, their associated twists are the same; see e.g. [Kru15, Lem. 2.4(ii)].

**Theorem B (Theorem 4.3).** Let \( P_1, P_2 \) be two \( \mathbb{P}^n[k] \)-objects with \( n \geq 2 \) which are not isomorphic up to shift in some algebraic triangulated category \( \mathcal{F} \) with \( \text{Hom}^*(P_1, P_2) \neq 0 \). If there is a spherification functor \( F : \langle P_1, P_2 \rangle \to \mathcal{B} \), then the associated \( \mathbb{P} \)-twists \( P_1 \) and \( P_2 \) generate the free group \( F_2 \).

Note that the assumption \( n \geq 2 \) is logically not necessary, as for any pair of non-orthogonal \( \mathbb{P}^1[k] \)-objects (i.e. \( k \)-spherical objects) the associated \( \mathbb{P} \)-twists (which are then the squares of the spherical twists) have no relation – independently of the existence of a spherification functor. This follows directly from the results of Volkov on the (absence of) relations between pairs of spherical twists; see Remark 4.4 for details.

We remark, that if \( \text{Hom}^*(P_1, P_2) = 0 \), then \( \langle P_1, P_2 \rangle \cong \mathbb{Z}^2 \). This was already known in essence, for completeness we discuss this case in detail in Section 5.
Symplectic geometry. Finally, we want to mention that spherical objects and \(\mathbb{P}\)-objects appear not only in algebraic geometry. Actually, spherical twists appeared first in symplectic geometry as Dehn twists about Lagrangian spheres. It was the homological mirror symmetry conjecture by M. Kontsevich [Kon95] that motivated the successful search for spherical objects in algebraic geometry [ST01].

Similarly, \(\mathbb{P}\)-twists are (generalised) Dehn twists about Lagrangian projective spaces \(\mathbb{P}^n\) (this statement seems to be in parts still only conjectural; see [MW18] by C.Y. Mak and W. Wu for details).

In the symplectic set-up, results analogous to Theorem B on the absence of relations between Dehn twists along Lagrangian projective spaces can be found in [Tor20] by B.C. Torricelli. The authors were not aware of loc. cit. before putting the first version of the present paper on the arXiv. Not only the results, but also the proofs in [Tor20] are somewhat analogous to ours. We use spherification functors to deduce our results from Volkov’s result on the absence of relations between spherical twists, while Torricelli uses the Hopf correspondence to deduce her result from known results about relations among Dehn twists about Lagrangian spheres by A. Keating [Kea14].

Conventions. By \(k\) we denote some arbitrary base field. All categories in this article will be additive and \(k\)-linear.

All triangles are distinguished, and we write \(A \to B \to C\) for them, suppressing the degree increasing morphism \(C \to A[1]\).

All functors are exact. In particular, in case of derived functors, we will drop the decoration with \(L\) and \(R\).

All functors between algebraic triangulated categories are assumed to be liftable to functors between the (dg-)enhancements. In a geometric context, this means functors of Fourier–Mukai type.

In a triangulated category, we write \(\text{Hom}^*(A, B)\) for the graded vector space of derived homomorphisms \(\bigoplus_i \text{Hom}(A, B[i])[−i]\). In the setting of dg-categories, we will use \(\text{Hom}^*(A, B)\) to denote the complex of homomorphisms.

As our main examples come from hyperkähler varieties, we assume some regularity and properness of the algebraic triangulated categories \(\mathcal{T}\). More precisely, we want that the graded vector space \(\text{Hom}^*(A, B)\) is finite-dimensional for all \(A, B \in \mathcal{T}\), and moreover, that \(\mathcal{T}\) consists only of perfect objects. In particular, \(\text{Hom}^*(P, \_\_)\) defines a functor to \(\mathcal{D}(k) = \mathcal{D}^b(k - \text{mod})\).

The typical example is the bounded derived category of coherent sheaves on a smooth projective variety \(X\) over \(k\), which we will denote by \(\mathcal{D}(X) = \mathcal{D}^b(\text{coh}(X))\). In this context, we denote by \(\langle P \rangle\) the smallest thick triangulated category of \(\mathcal{T}\) containing \(P \in \mathcal{T}\). We have an equivalence \(\langle P \rangle \cong \mathcal{D}(A)\) where \(\mathcal{D}(A)\) is the subcategory of compact objects in the derived category of modules over the dg-algebra \(A = \text{End}^*(P)\); see [Kel94, §4.2], [LS16, Prop. B.1], or [HK19, Thm. 1.10].

If the taste of the reader is for the infinite, one can also assume all triangulated categories to be cocomplete, in which case no regularity or properness is necessary. Here, the typical example is the unbounded derived category of quasi-coherent sheaves \(\mathcal{D}(\text{QCoh}(X))\) of some variety \(X\) over \(k\). In this case, \(\langle P \rangle\) should be read as the smallest cocomplete triangulated
category of \( \mathcal{F} \) containing \( P \in \mathcal{F} \). Again, we have an equivalence \( \langle P \rangle \cong \mathcal{D}(A) \), but now \( \mathcal{D}(A) \) denotes the whole derived category of modules over the dg-algebra \( A = \text{End}^\ast(P) \).

We stress that all statements of this article hold in both contexts: either regular and proper, or cocomplete.

**Acknowledgements.** We thank Ivan Smith for making us aware that, in the first version of this paper, we missed the relevant reference [Tor20] as well as the assumption that the objects in Theorem B must not be isomorphic up to shift.

2. Enter spherical and \( \mathbb{P} \)-objects

We recall some basic facts from [ST01] and [HT06], using the slightly generalised notations as introduced in [HKP16], [Kru18] and [HK19].

**Definition 2.1.** Let \( P \) be an object in some \((k\)-linear\) triangulated category, and let \( n, k, d \) be integers with \( n > 0 \).

We say that \( P \) is \( \mathbb{P}^n[k] \)-like if \( \text{End}^\ast(P) \cong k[t]/t^{n+1} \) with \( \deg(t) = k \). If \( P[d] \) is a Serre dual of \( P \), that is,

\[
\text{Hom}^\ast(P, \underline{\_}) \cong \text{Hom}^\ast(\underline{\_}, P[d])^\vee
\]

then we say that \( P \) is a \( d \)-Calabi–Yau object. We will drop \( d \) as a prefix, if it is clear from the context.

If \( P \) is \( \mathbb{P}^n[k] \)-like and Calabi–Yau (in which case necessarily \( d = nk \)), then we say that \( P \) is a \( \mathbb{P}^n[k] \)-object.

If \( n = 1 \), then a \( \mathbb{P}^1[k] \)-like object, or \( \mathbb{P}^1[k] \)-object, is better known as a spherical or spherical object, respectively.

If \( k = 2 \), then a \( \mathbb{P}^n[2] \)-object is better known as a \( \mathbb{P}^n \)-object. We will drop the exponent \( n \), if it is clear from the context.

**Proposition 2.2 ([ST01]).** Let \( S \) be a spherical object in some algebraic triangulated category \( \mathcal{T} \). Then the spherical twist \( T_S \) defined by the triangle

\[
\text{Hom}^\ast(S, \underline{\_}) \otimes S \xrightarrow{\text{ev}} \text{id} \rightarrow T_S
\]

is an autoequivalence of \( \mathcal{T} \).

**Proposition 2.3 ([HT06]).** Let \( P \) be a \( \mathbb{P}^n[k] \)-object in some algebraic triangulated category \( \mathcal{T} \). Then the \( \mathbb{P} \)-twist \( T_P \) defined in following diagram, where the tilted rows are exact triangles

\[
\text{Hom}^\ast(P, \underline{\_}) \otimes P[-k] \xrightarrow{H = t \otimes \text{id} - \text{id} \otimes t} \text{Hom}^\ast(P, \underline{\_}) \otimes P \xrightarrow{\text{ev}} \text{Cone}_H(\underline{\_}) \xrightarrow{\text{id}} T_P \text{P}_{P}
\]

is an autoequivalence of \( \mathcal{T} \).
Remark 2.4. Associated to a spherical object $S$, we have the spherical twist $T_S$ and the $\mathbb{P}$-twist $P_S$ which are related by
\[ T_S^2 \cong P_S, \]
see [HT06, Prop. 2.9].

The notion of spherelike and spherical objects can be generalised to functors. Before giving the definitions, we need some canonical triangles associated to a functor $F: \mathcal{A} \to \mathcal{B}$ between algebraic triangulated categories, which admits both adjoints $L$ and $R$.

The twist $T$ and dual twist $T'$ associated to $F$ are defined by the triangles (using unit and counit of the adjunction):
\[ FR \to \text{id}_B \to T \quad \text{and} \quad T' \to \text{id}_B \to FL. \]

Note that $T'$ is the left adjoint of $T$.

Similarly, the cotwist $C$ and dual cotwist $C'$ associated to $F$ are defined by the triangles
\[ C \to \text{id}_A \to RF \quad \text{and} \quad LF \to \text{id}_A \to C'. \]

Again, $C'$ is the left adjoint of $C$.

Additionally note that we have natural morphisms
\[ \phi: R \to RFL \to CL[1] \quad \text{and} \quad \psi: LT[-1] \to LFR \to R. \]

Definition 2.5 ([AL17], [HM20]). Let $F: \mathcal{A} \to \mathcal{B}$ be a functor between algebraic triangulated categories, which admits both adjoints $L$ and $R$.

If $C$ is an autoequivalence, then we call $F$ a spherelike functor.

If additionally $\phi: R \to CL[1]$ is an isomorphism, then we call $F$ a spherical functor.

Proposition 2.6 ([AL17, Thm. 1.1]). Let $F: \mathcal{A} \to \mathcal{B}$ be a functor between algebraic triangulated categories, which admits both adjoints $L$ and $R$.

If $F$ satisfies two of the following four conditions, then $F$ satisfies all four of them:
\begin{itemize}
  \item $C$ is an autoequivalence;
  \item $T$ is an autoequivalence;
  \item $\phi: R \to CL[1]$ is an isomorphism;
  \item $\psi: LT[-1] \to R$ is an isomorphism.
\end{itemize}
In particular, such an $F$ is a spherical functor.

The notion is motivated by the following example.

Example 2.7. Let $S$ be some object in an algebraic triangulated category $\mathcal{B}$, which admits a Serre dual. Then the functor $F_S = (\_ \otimes S): D(k - \text{mod}) \to \mathcal{B}$ admits both adjoints.

Moreover, we have that $F_S$ is a spherelike/spherical functor, if and only if, $S$ is a spherelike/spherical object, respectively.

Remark 2.8. There is also the generalisation of $\mathbb{P}$-objects to $\mathbb{P}$-functors. We do not give the definition here, as we will not use it. We refer to [Add16] where the notion of a (split) $\mathbb{P}$-functor was first introduced, this was generalised to (possibly non-split) $\mathbb{P}$-functors in [AL19].
3. From \( \mathcal{P} \)-objects to spherical objects

3.1. Introducing spherification functors. Consider a functor \( \mathcal{F} : \mathcal{A} \to \mathcal{B} \) of algebraic triangulated categories, which admits a left adjoint \( \mathcal{L} \). Recall that there is the triangle

\[
\mathcal{L} \mathcal{F} \xrightarrow{\alpha} \text{id} \xrightarrow{\circ} \mathcal{C}'
\]

where \( \mathcal{C}' \) is the dual cotwist of \( \mathcal{F} \). In the following, the morphism \( \alpha : \text{id} \to \mathcal{C}' \) will be of importance.

**Definition 3.1.** Let \( \mathcal{F} : \mathcal{A} \to \mathcal{B} \) be a functor of algebraic triangulated categories, which admits a left adjoint \( \mathcal{L} \). We say that \( \mathcal{F} \) is a weak spherification functor for \( F \) if

- \( \mathcal{C}' \mathcal{F} \cong \mathcal{P}[k] \)
- the natural morphism \( \alpha_\mathcal{F} : \mathcal{P} \to \mathcal{P}[k] \) is non-zero.

Note that the second property implies that \( \text{Hom}^*(\mathcal{P}, \mathcal{P}[k]) = k[\alpha_\mathcal{F}]/(\alpha_\mathcal{F}^{n+1}) \).

We justify the name by the following lemma.

**Lemma 3.2.** Let \( \mathcal{F} : \mathcal{A} \to \mathcal{B} \) be a functor of algebraic triangulated categories, which admits a left adjoint \( \mathcal{L} \). Let \( \mathcal{P} \) be a \( \mathbb{P}^n[k] \)-like object in \( \mathcal{A} \).

If \( \mathcal{F} : \mathcal{A} \to \mathcal{B} \) is a weak spherification functor for \( \mathcal{P} \), then \( \mathcal{F} \mathcal{P} \) is a \( (nk + k - 1) \)-spherelike object.

Conversely, if \( \mathcal{F} \mathcal{P} \) is a \( (nk + k - 1) \)-spherelike object and \( \mathcal{C}' \mathcal{P} \cong \mathcal{P}[k] \), then \( \mathcal{F} \) is a weak spherification functor for \( \mathcal{P} \).

**Proof.** Let \( \mathcal{F} \) be a weak spherification functor for \( \mathcal{P} \). Apply \( \text{Hom}^*(\_, \mathcal{P}) \) to the triangle \( \mathcal{L} \mathcal{F} \mathcal{P} \to \mathcal{P} \to \mathcal{C}' \mathcal{P} \cong \mathcal{P}[k] \) to get

\[
\text{Hom}^*(\mathcal{F} \mathcal{P}, \mathcal{F} \mathcal{P}) \leftarrow \text{Hom}^*(\mathcal{P}, \mathcal{P}) \leftarrow \alpha_{\mathcal{F} \mathcal{P}}^* \text{Hom}^*(\mathcal{P}[k], \mathcal{P}).
\]

Under the isomorphism \( \text{Hom}^*(\mathcal{P}, \mathcal{P}) \cong k[\alpha_\mathcal{P}]/(\alpha_\mathcal{P}^{n+1}) \), the map \( \alpha_{\mathcal{F} \mathcal{P}}^* \) induces isomorphisms on the components \( k \cdot \alpha_\mathcal{P}^i[-k] \cong k \cdot \alpha_\mathcal{P}^{i+1} \) for \( 0 \leq i < n \). It follows that \( \text{Hom}^*(\mathcal{F} \mathcal{P}, \mathcal{F} \mathcal{P}) \cong k[s]/s^2 \) with \( \deg(s) = nk + k - 1 \). In other words, \( \mathcal{F} \mathcal{P} \) is a \( (nk + k - 1) \)-spherelike object.

Now suppose that \( \mathcal{F} \mathcal{P} \) is a \( (nk + k - 1) \)-spherelike object and \( \mathcal{C}' \mathcal{P} \cong \mathcal{P}[k] \). By the triangle above we can now conclude that \( \mathcal{P} \to \mathcal{P}[k] \) is non-zero, hence \( \mathcal{F} \) is a weak spherification functor for \( \mathcal{P} \).

**Remark 3.3.** The only reason, why we prefer the left adjoint of \( \mathcal{F} \), is that the triangle \( \mathcal{L} \to \text{id} \to \mathcal{C}' \) appears in the geometric situation of Proposition 3.8 which is our point of departure. There is an analogous definition using the right adjoint of \( \mathcal{R} \), which yields similar statements.

**Definition 3.4.** Let \( \mathcal{F} : \mathcal{A} \to \mathcal{B} \) be a functor of algebraic triangulated categories, which admits a left adjoint \( \mathcal{L} \). We say that \( \mathcal{F} \) is a spherification functor for a \( \mathbb{P}^n[k] \)-like object \( \mathcal{P} \) if \( \mathcal{F} \) is a weak spherification functor for \( \mathcal{P} \) and \( \mathcal{F} \mathcal{P} \) is a \( (nk + k - 1) \)-Calabi–Yau object in \( \mathcal{B} \).

**Remark 3.5.** Let \( \mathcal{F} : \mathcal{A} \to \mathcal{B} \) be a spherification functor for a \( \mathbb{P}^n[k] \)-like object \( \mathcal{P} \). Then by Lemma 3.2, we have that \( \mathcal{F} \mathcal{P} \) is \( (nk + k - 1) \)-spherelike, and by definition of spherification functor that \( \mathcal{F} \mathcal{P} \) is \( (nk+k-1) \)-Calabi–Yau. Therefore \( \mathcal{F} \mathcal{P} \) is a \( (nk + k - 1) \)-spherical object in \( \mathcal{B} \).
Remark 3.6. We will assume from now on, that for all $P^n[k]$-like objects holds $n \geq 2$. In other words, we exclude weak spherification of $k$-spherelike objects. This is coherent with our two main results Theorem A and Theorem B.

We do not make any general assumptions on the value of $k$, however. In derived categories of smooth projective varieties, due to Serre duality, only $P^n[k]$-objects with $k > 0$ can appear. In representation theory, however, $P^n[k]$-objects with $k \leq 0$ can occur; see Example 3.18. The proof of Proposition 3.19, which is an important ingredient of our proof of Theorem B, would have been shorter (but not really conceptually easier) if we restricted ourselves to the case $k > 0$, but we decided to go with the greater generality.

We note in passing that most of the following results still apply to the case $n = 1$, that is, $k$-spherelike objects (but in some cases the proofs must be slightly adapted) as long as $k > 1$, but some break in the special case $k = 1$. We decided it was not worth the complications that would arise when including always the special case $n = 1$. The reason is that the question of relations of pairs of twists for $n = 1$ is already settled by the results of [Vol19]; see Remark 4.4.

It is an idiosyncrasy of our definition that a weak spherification functor turns a $k$-spherelike object into a $(2k - 1)$-spherelike object. Note that, contrary to what one might expect from the name, the identity functor is not a weak spherification functor of spherelike objects as it has $C' = 0$.

Lemma 3.7. Let $F : \mathcal{A} \to \mathcal{B}$ be a spherelike functor with $C' \cong [k]$. Then $F$ is a weak spherification functor for all $P^n[k]$-like objects $P$ such that the natural morphism $\alpha_P : P \to C'P$ is non-zero.

If $F$ is even a spherical functor with $C' \cong [k]$, then $F$ becomes even a spherification functor for all $P^n[k]$-objects $P$ with $\alpha_P \neq 0$.

Proof. The first part holds by the definition of weak spherification functor. Note that we are asking much more than necessary: $C'$ needs to be a shift only for the $P^n[k]$-like objects in question, a spherelike functor admits also a right adjoint which is not needed here.

Assume now that $F$ is a spherical functor, that is, $C$ is an equivalence with $C^{-1} = C'$ and $R \cong \mathcal{C}[1]$. We compute

$$\Hom^*(FP, -) \cong \Hom^*(P, R-) \cong \Hom^*(R-, P[nk])^\vee \cong \Hom^*(C[1], P[nk])^\vee \cong \Hom^*([L-, C^{-1}P[nk-1]])^\vee \cong \Hom^*([-, FP[nk+k-1]])^\vee$$

where we used adjunction, that $P$ is a $nk$-Calabi–Yau object, and $C'P \cong P[k]$.

Hence $FP$ is a $(nk+k-1)$-Calabi–Yau object, and therefore $F$ is a spherification functor.

3.2. The guiding example. The following proposition is the motivation for our definition of a spherification functor, coming from [HT06].
Let $X$ be a smooth projective variety. Suppose there is a smooth family $\mathcal{X} \to C$ over a smooth curve $C$ with distinguished fibre $j : X = X_0 \hookrightarrow \mathcal{X}$, where $0 \in C$ is a closed point. Note that the inclusion $j : X \to \mathcal{X}$ gives rise to the push-forward functor $j_* : D(X) \to D(\mathcal{X})$ which admits both adjoints.

The family gives rise to the Atiyah class $A(E)$ for $E \in D(X)$ and the Kodaira-Spencer class $\kappa(\mathcal{X})$. For further information on these classes, see [HT10].

We reformulate the following proposition of [HT06] using the language of spherification functors.

**Proposition 3.8** ([HT06, Prop. 1.4]). Let $\mathcal{X} \to C$ be a smooth family over a smooth curve $C$, and let $j : X \to \mathcal{X}$ be the inclusion of a fiber $X$. Then $j_*$ is a spherification functor for all $\mathbb{P}$-objects $P$ with $A(P) \cdot \kappa(\mathcal{X}) \neq 0$.

**Remark 3.9.** Note that the condition $A(P) \cdot \kappa(\mathcal{X}) \neq 0$ is exactly the one asked for in the definition of a weak spherification functor, as the triangle of the dual cotwist to $j^*$ is

$$j^* j_* \to \text{id} \xrightarrow{[2]} A(\mathcal{X}) \cdot \kappa(\mathcal{X})$$

Also note that $j_*$ is a spherical functor, as $j : X \to \mathcal{X}$ is the inclusion of a divisor. So by Lemma 3.7, $j_*$ is indeed a spherification functor for all $\mathbb{P}$-objects $P$ with $A(P) \cdot \kappa(\mathcal{X}) \neq 0$.

Note that associated to a $\mathbb{P}$-object $P \in D(X)$, we have a $\mathbb{P}$-functor $F_P = (\_ \otimes P : D(k{-}\text{mod}) \to D(X))$. Hence the proposition can be reformulated, that the composition $j_* \circ F_P$ becomes a spherical functor. So one might wonder, whether there is a condition generalising $A(\_ \cdot \kappa(\mathcal{X}) \neq 0$ (or the one of Lemma 3.7), which guarantees that the composition of a $\mathbb{P}$-functor and a spherical functor becomes spherical. See [MR19], where this question is discussed for the spherical functor $j_*$.

**Example 3.10** ([HT06, Ex. 1.5]). Let $X$ be a hyperkähler variety of dimension $2n$. Then $X$ allows a $\mathbb{P}^1$-family of complex structures, which fit together into an analytic manifold $\mathcal{X}$, the *twistor space* of $X$, see, for example, [Huy99] for a background. So $X$ provides both a wealth of $\mathbb{P}$-objects and a smooth (analytic) family $\mathcal{X} \to \mathbb{P}^1$.

Any non-trivial line bundle $L$ is a $\mathbb{P}^n$-object in $D(X)$ with $A(L) \cdot \kappa(\mathcal{X}) \neq 0$. So $j_*$ is a spherification functor for any non-trivial line bundle $L$, that is, $j_* L$ is a spherical object in $D(\mathcal{X})$.

Note that $\mathcal{O}_X$ cannot be spherificated this way: It deforms to $\mathcal{O}_X$. Equivalently, $A(\mathcal{O}_X) \cdot \kappa(\mathcal{X})$ vanishes, because of the splitting

$$j^* j_* \mathcal{O}_X \cong \mathcal{O}_X \oplus \mathcal{O}_X[-1].$$

Hence, $j_*$ is not a spherification functor for $\mathcal{O}_X$.

Also for any $\mathbb{P}^n \subset X$, we get that $\mathcal{O}_{\mathbb{P}^n}$ is a $\mathbb{P}^n$-object in $D(X)$ with $A(\mathcal{O}_{\mathbb{P}^n}) \cdot \kappa(\mathcal{X}) \neq 0$. Again, $j_*$ is a spherification functor for such $\mathcal{O}_{\mathbb{P}^n}$, so $j_* \mathcal{O}_{\mathbb{P}^n}$ is a spherical object in $D(\mathcal{X})$.

### 3.3. Existence of spherification functors.

The geometric situation of Proposition 3.8 establishes already the existence of a spherification functor for many $\mathbb{P}^n$-objects. In this section, we construct spherification functors for more general $\mathbb{P}^n[k]$-objects.
Lemma 3.11. Let $A$ be a dg-algebra over $k$ and let $h$ be a homogeneous element of even degree $k$ which is central. Then the cone of the multiplication by $h$

$$A[-k] \overset{h}{\to} A 	o B$$

can be turned into a dg-algebra over $A$.

Proof. By the definition of the cone, as an $A$-module, $B$ can be written as $A \oplus A[1-k]$ with differential

$$d_B = \begin{pmatrix} d & h \\ 0 & a \end{pmatrix}$$

where $d$ is the differential of $A$ (or the differential of $A[1-k]$ which differs by the one of $A$ by $(-1)^{1-k} = -1$).

We write $B = A[\varepsilon]/\varepsilon^2$ where $\varepsilon$ is an element of degree $k-1$, which implicitly defines a graded multiplication

$$(a_1 + \varepsilon a_2) \cdot (a_1' + \varepsilon a_2') = a_1 a_1' + \varepsilon((-1)^{k-1}a_1 a_2' + a_2 a_1') = a_1 a_1' + \varepsilon((-1)^{|a_1|}a_1 a_2' + a_2 a_1')$$

for homogeneous elements $a_i$ of degree $|a_i|$, where we use that $k$ is even.

The differential of $B$ is then

$$d_B(a_1 + \varepsilon a_2) = d(a_1) + ha_2 + (-1)^{k-1}\varepsilon d(a_2) = d(a_1) + ha_2 - \varepsilon d(a_2)$$

again using that $k$ is even.

We check that with this differential $B$ becomes a dg-algebra, that is, that the Leibniz rule holds. On the one hand, we have for homogeneous elements

$$d_B((a_1 + \varepsilon a_2)(a_1' + \varepsilon a_2')) =$$

$$= d_B(a_1 a_1' + \varepsilon((-1)^{|a_1|}a_1 a_2' + a_2 a_1')) =$$

$$= d(a_1 a_1') + h((-1)^{|a_1|}a_1 a_2' + a_2 a_1') - \varepsilon d((-1)^{|a_1|}a_1 a_2' + a_2 a_1') =$$

$$= d(a_1) a_1' + (-1)^{|a_1|}a_1 d(a_1') + (-1)^{|a_1|}ha_1 a_2' + ha_2 a_1' -$$

$$- (-1)^{|a_1|}\varepsilon d(a_1) a_2' - \varepsilon a_1 d(a_2') - \varepsilon d(a_2) a_1' - (-1)^{|a_2|}\varepsilon a_2 d(a_1').$$

On the other hand, we have:

$$d_B(a_1 + \varepsilon a_2)(a_1' + \varepsilon a_2') + ((-1)^{|a_1|}a_1 - (-1)^{|a_2|}\varepsilon a_2)d_B(a_1' + \varepsilon a_2') =$$

$$= (d(a_1) + ha_2 - \varepsilon d(a_2))(a_1' + \varepsilon a_2') +$$

$$+ ((-1)^{|a_1|}a_1 - (-1)^{|a_2|}\varepsilon a_2)(d(a_1') + ha_2' - \varepsilon d(a_2')) =$$

$$= d(a_1) a_1' + d(a_1)\varepsilon a_2' + ha_2 a_1' + ha_2\varepsilon a_2' - \varepsilon d(a_2) a_1' +$$

$$+ (-1)^{|a_1|}a_1 d(a_1') + (-1)^{|a_1|}a_1 ha_2' - (-1)^{|a_1|}a_1\varepsilon d(a_2') -$$

$$- (-1)^{|a_2|}\varepsilon a_2 d(a_1') + (-1)^{|a_2|}\varepsilon a_2 ha_2'$$

which coincides with the previous computation, using that $h$ is central and the graded commutativity of $\varepsilon$. \hfill \square

Given a $\mathbb{P}^n[k]$-object $P_i$, we denote in the following by $t_i$ a non-zero element of $\text{End}^k(P_i)$. 

Proposition 3.12. Let $P_1, \ldots, P_m$ be $\mathbb{P}^n[k]$-objects in some algebraic triangulated category with $k$ even. Consider the dg-algebra $A = \text{End}^*(\bigoplus_i P_i)$, and suppose that there is a central element $h = h_1 + \cdots + h_m \in A$ such that

$$H^0(h_i) = t_i \in H^0(A) = \text{End}^*(\bigoplus_i P_i).$$

Then there is a spherification functor $F : \langle P_1, \ldots, P_m \rangle \to \mathcal{B}$.

Proof. Note that $\langle P_1, \ldots, P_m \rangle \cong \mathcal{D}(A)$. We can apply Lemma 3.11 to obtain the following triangle

$$(1) \quad A[-k] \xrightarrow{h} A \to B$$

where $B = A[\varepsilon]/\varepsilon^2$ is a dg-algebra with differential $d_B(a_1 + \varepsilon a_2) = d(a_1) + ha_2 - \varepsilon d(a_2)$.

The morphism $A \to B$ gives rise to

$$F = B \otimes_A (\_): \mathcal{D}(A) \to \mathcal{D}(B).$$

Note that, by (1), $B$ is semi-free as a dg-module over $A$. Hence, we do not need to replace $B$ by a resolution in order to compute the derived tensor product $F$. This $F$ has a right adjoint $R = A(\_)$, as

$$\text{Hom}_B^*(B \otimes_A M, N) \xrightarrow{\sim} \text{Hom}_A^*(M, \underbrace{\text{Hom}_B^*(B, N)}_{\cong_A N}).$$

We have also a left adjoint $L = \text{Hom}_A^*(B, A) \otimes_B (\_)$.

We compute first the cotwist of $F$ using the triangle

$$C \to \text{id} \to RF = A(B \otimes_A (\_)).$$

Plugging $A$ into this triangle, we get

$$CA \to A \to B,$$

so $CA \cong A[-k]$. This implies $C \cong [-k]$ as $A$ is a generator of $\mathcal{D}(A)$. Moreover, the morphism $C \to \text{id}$ is non-zero for each $P_i$, as it is just the multiplication with $h_i$. Dually, this holds also for the natural morphism $\text{id} \to C'$. In particular, by the first part of Lemma 3.7, $F$ is a weak spherification functor for the $P_i$.

Next we check that $R \cong \text{CL}[1]$. To make $L$ more explicit, we can calculate $\text{Hom}_A^*(B, A)$ by applying $\text{Hom}_A^*(\_)$ to (1):

$$\text{Hom}_A^*(A[-k], A) \xleftarrow{\text{Hom}_A^*(A, A)} \text{Hom}_A^*(B, A).$$

From this we get that $\text{Hom}_A^*(B, A) \cong B[1-k]$, so

$$L = \text{Hom}_A^*(B, A) \otimes_B (\_) \cong AB \otimes_B (\_)[1-k] \cong A(\_)[1-k] \cong R[1-k].$$

Hence $F$ is a spherical functor. In particular, by the second part of Lemma 3.7, $F$ is a spherification functor of $P_1, \ldots, P_m$. □

The following is Theorem A of the introduction.

Corollary 3.13. Let $P_1, \ldots, P_m$ be $\mathbb{P}^n[k]$-objects in some algebraic triangulated category, such that $k \geq 2$ is even, $\gcd(k, nk/2) > 1$ and $\text{Hom}^*(P_i, P_j)$ is concentrated in degree $nk/2$ for $i \neq j$ (or zero). Then there is a spherification functor $F : \langle P_1, \ldots, P_m \rangle \to \mathcal{B}$. 

Proof. By slightly adapting the proof of [HK19, Prop. 4.3], we get that \( A = \text{End}^*(\bigoplus_i P_i) \) is intrinsically formal.

Moreover, \( h = t_1 + \cdots + t_m \) is a central element (note that, for \( i \neq j \), the product of \( h \) with elements in \( \text{Hom}^*(P_i, P_j) \) is zero for degree reasons). So we can apply Proposition 3.12 to the dg-algebra \( A \) with trivial differential, in order to conclude the proof. \( \square \)

Remark 3.14. In particular, we have that there is always a spherification functor for a single \( \mathbb{P}^n[k] \)-object \( P \) such that \( k \) is even.

Remark 3.15. Let \( P_1 \) and \( P_2 \) be two \( \mathbb{P}^n[k] \)-objects, such that \( k \) even and \( \gcd(k, nk/2) > 1 \) (for example, if \( n \) is even). In order to obtain a spherification functor for \( P_1 \) and \( P_2 \), it is enough to assume additionally that \( \text{Hom}^*(P_1, P_2) \) is concentrated in a single degree. To see this, replace one of the two objects by a suitable shift, so we can ensure that \( \text{Hom}^*(P_1, P_2) \) (and by Serre duality \( \text{Hom}^*(P_2, P_1) \)) is concentrated in degree \( nk/2 \).

The following example starts from results taken from [PS14], see also [HK19, §6].

Example 3.16. Let \( Y \) be some smooth projective variety of even dimension \( k \) and \( S \in \mathcal{D}(Y) \) a spherical object.

Then for \( n > 0 \) we can consider the equivariant derived category \( \mathcal{D}_{\mathfrak{S}_n}(Y^n) \), where \( S \) gives rise to two \( \mathbb{P}^n[k] \)-objects \( S^+(n) \) and \( S^-(n) \) by linearising \( S^\mathbb{G}_m \) with the trivial or the sign representation of \( \mathfrak{S}_n \), respectively.

If there are two spherical objects \( S_1, S_2 \) with \( \text{Hom}^*(S_1, S_2) \) in a single degree \( d \), then we have that \( \text{Hom}^*(S_1^\pm(n), S_2^\pm(n)) \) is again concentrated in a single degree, namely \( nd \). In particular, there is a spherification functor for any of the pairs of \( \mathbb{P} \)-objects of the form \( (S_1^\pm(n), S_2^\pm(n)) \) by Corollary 3.13.

In the situation of \( \mathbb{P}^n \)-objects on hyperkähler varieties, we can often spherificate using \( j_* : \mathcal{D}(X) \to \mathcal{D}(X) \) of Proposition 3.8, or we can use the more abstract spherification of Corollary 3.13. There is a non-empty intersection, as we see in the following example.

Example 3.17. We specialise Example 3.16. Recall that in case of \( Y \) a surface, we get \( \Phi : \mathcal{D}_{\mathfrak{S}_n}(Y^n) \to \mathcal{D}(\text{Hilb}^n(Y)) \). In particular, if \( Y \) is a K3 surface, \( X = \text{Hilb}^n(Y) \) is a hyperkähler variety. Note that if \( S \) is a (non-trivial) line bundle on a K3 surface \( Y \), then so is \( \Phi(S^\pm(n)) \) on \( X \), hence can be spherified using \( j_* : \mathcal{D}(X) \to \mathcal{D}(X) \) or abstractly by Corollary 3.13.

We noted in Example 3.10 that \( j_* : \mathcal{D}(X) \to \mathcal{D}(X) \) is not a spherification functor for \( \mathcal{O}_X = \Phi(\mathcal{O}_{Y}^\pm(n)) \). This problem can be circumvented easily: just precompose \( j_* \) with a suitable autoequivalence, for example, tensoring with a non-trivial line bundle. But by Remark 3.14, we can also spherificate \( \mathcal{O}_X \) directly, using the abstract spherification of Corollary 3.13.

If \( S = \mathcal{O}_C \) for some rational curve on a K3 surface \( Y \), then \( S^+(n) \) becomes isomorphic to \( \mathcal{O}_{\mathbb{P}^n} \) under the equivalence \( \Phi \), see [HK19, Prop. 6.6]. Hence, up to this equivalence, it can again be spherified either way. But \( S^-(n) \) will yield a more complicated object in \( \mathcal{D}(\text{Hilb}^n(Y)) \), see [HK19, Prop. 6.7] for \( n = 2 \). For that reason it is not immediately clear, whether spherification using \( j_* \) is possible. Still, we can spherificate abstractly by Corollary 3.13.
Example 3.18. As far as we know, there are no examples of \( \mathbb{P}^n[k] \)-objects with \( k < 0 \) in the literature. But there are \( k \)-spherical objects with \( k < 0 \) studied in representation theory. For example, [CS17] treats the triangulated category \( \mathcal{T} \) generated by a single \( k \)-spherical object. Considering the symmetric power of this category \( \mathcal{S}^n \mathcal{T} \) as introduced in [GK14], one obtains two \( \mathbb{P}^n[k] \)-objects there (like in the geometric situation of Example 3.16). Note that Corollary 3.13 assumes \( k \geq 2 \), hence does not yield a spherification functor for these negative \( \mathbb{P} \)-objects.

3.4. Properties of spherification functors. In the following, we write \( \text{hom}^*(A, B) \) for the dimension of the graded \( k \)-vector space \( \text{Hom}^*(A, B) \).

Proposition 3.19. Let \( P_1, P_2 \) be two \( \mathbb{P}^n[k] \)-objects in some algebraic triangulated category \( \mathcal{A} \). Let \( F : \mathcal{A} \to \mathcal{B} \) be a weak spherification functor for \( P_1 \) and \( P_2 \).

(i) If \( \text{hom}^*(P_1, P_2) \geq 1 \) then \( \text{hom}^*(FP_1, FP_2) \geq 2 \).

(ii) If \( FP_1 \cong FP_2 \) implies \( P_1 \cong P_2 \).

Proof. Apply \( \text{Hom}^*(-, P_2) \) to the triangle \( LFP_1 \to P_1 \to P_1[k] \) to get

\[
\text{Hom}^*(FP_1, FP_2) \xleftarrow{\alpha_{P_2}} \text{Hom}^*(P_1, P_2) \xrightarrow{\alpha_{P_1}} \text{Hom}^*(P_1, P_1)[−k].
\]

Let us first prove (i) in the case \( k < 0 \). By (3), we see that there is an injection of the minimal degree part of \( \text{Hom}^*(P_1, P_2) \) into \( \text{Hom}^*(FP_1, FP_2) \), and a surjection from \( \text{Hom}^*(FP_1, FP_2) \) to the maximal degree part of \( \text{Hom}^*(P_1, P_2)[−k] \). This gives the claim.

For \( k < 0 \), we instead have an injection of the maximal degree part of \( \text{Hom}^*(P_1, P_2) \) into \( \text{Hom}^*(FP_1, FP_2) \), and a surjection from \( \text{Hom}^*(FP_1, FP_2) \) to the minimal degree part of \( \text{Hom}^*(P_1, P_2)[−k] \).

For \( k = 0 \), we have that \( \alpha_{P_i} : P_i \to P_i \) is a nilpotent endomorphism of degree zero for \( i = 1, 2 \), as \( \text{Hom}^*(P_i, P_i) \cong k[\alpha_i]/(\alpha_i^{n+1}) \). Hence, in the long exact cohomology sequence associated to (2), the morphisms \( \alpha_{P_i}^* : \text{Hom}^i(P_i, P_2) \to \text{Hom}^i(P_1, P_2) \) are nilpotent endomorphisms. In particular, they cannot be isomorphisms for any \( j \in \mathbb{Z} \). From this follows (i) for the last remaining case, \( k = 0 \).

We now prove (ii). Since \( FP_1 \cong FP_2 \), we have

\[
\text{Hom}^*(FP_1, FP_2) \cong \text{Hom}^*(FP_1, FP_1) \cong k \oplus k[−nk − k + 1]
\]

where the last isomorphism is due to Lemma 3.2.

Let us first assume that \( k \neq 0 \). Reinspecting the arguments above involving (2) shows that

\[
\text{mindeg} \text{Hom}^*(P_1, P_2) = \text{mindeg} \text{Hom}^*(FP_1, FP_2) \quad \text{for } k > 0,
\]

\[
\text{maxdeg} \text{Hom}^*(P_1, P_2) = \text{maxdeg} \text{Hom}^*(FP_1, FP_2) \quad \text{for } k < 0,
\]

where \text{mindeg} and \text{maxdeg} denote the minimal and maximal degrees, respectively, in which the graded vector spaces are non-zero. Furthermore, we have \( \text{mindeg} \text{Hom}^*(FP_1, FP_2) = 0 \) for \( k > 0 \) and \( \text{maxdeg} \text{Hom}^*(FP_1, FP_2) = 0 \) for \( k < 0 \); see (3). Hence, in any case, we get an isomorphism of one-dimensional vector spaces \( F : \text{Hom}(P_1, P_2) \iso \text{Hom}(FP_1, FP_2) \) in degree 0. In complete analogy, we have an isomorphism \( F : \text{Hom}(P_2, P_1) \iso \text{Hom}(FP_2, FP_1) \).
Furthermore, inspecting the proof of Lemma 3.2 shows that there are isomorphisms $F: \text{Hom}(P_i, P_i) \xrightarrow{\sim} \text{Hom}(FP_i, FP_i)$ for $i = 1, 2$. Now, let $0 \neq \phi: P_1 \to P_2$ be some morphism. Then $F(\phi) \neq 0$, hence it is an isomorphism. Let $\psi \in \text{Hom}(P_2, P_1)$ be the unique preimage of $F(\phi)^{-1}$ under $F: \text{Hom}(P_2, P_1) \xrightarrow{\sim} \text{Hom}(FP_2, FP_1)$. Then

$$F(\psi \circ \phi) = F(\psi) \circ F(\phi) = \text{id}_{FP_1}.$$  

By the injectivity of $F: \text{Hom}(P_1, P_1) \to \text{Hom}(FP_1, FP_1)$, it follows that $\psi \circ \phi = \text{id}_{P_1}$. The same way, we obtain $\phi \circ \psi = \text{id}_{P_2}$. Hence, $\phi$ and $\psi$ are mutually inverse isomorphisms.

Recall that, for $k = 0$, the morphisms $\alpha_{P_1}^* : \text{Hom}^j(P_1, P_2) \to \text{Hom}^j(P_1, P_2)$ occurring in the long exact sequence associated to (2) cannot be isomorphisms for any $j \in \mathbb{Z}$. Furthermore, $\text{Hom}^*(FP_1, FP_2) \cong k \oplus k[1]$ by (3). It follows that $\text{Hom}^*(P_1, P_2)$ is concentrated in degree 0 and $F: \text{Hom}(P_1, P_2) \to \text{Hom}(FP_1, FP_2)$ is surjective. Analogously, also $F: \text{Hom}(P_2, P_1) \to \text{Hom}(FP_2, FP_1)$ is surjective. Similarly to the case $k \neq 0$, we now pick some $\phi \in \text{Hom}(P_1, P_2)$ such that $F(\phi)$ is an isomorphism, and some $\psi \in \text{Hom}(P_2, P_1)$ with $F(\psi) = F(\phi)^{-1}$. Then $F(\psi \circ \phi) = \text{id}_{FP_1}$. Inspecting the proof of Lemma 3.2, we see that the kernel of

$$F: \text{Hom}(P_1, P_1) \to \text{Hom}(FP_1, FP_1)$$

is, under the isomorphism $\text{Hom}(P_1, P_1) \cong k[\alpha_{P_1}] / (\alpha_{P_1}^{n+1})$, the maximal ideal $(\alpha_{P_1})$. It follows that $\psi \circ \phi$, not being contained in this kernel, is an automorphism of $P_1$. Analogously, $\phi \circ \psi$ is an automorphism of $P_2$. Hence, $\phi$ and $\psi$ are isomorphisms.  

Example 3.20. Let $F: \mathcal{A} \to \mathcal{B}$ be a weak spherification functor for two $\mathbb{P}^n[k]$-objects $P_1$ and $P_2$. If $\text{Hom}^*(P_1, P_2) \cong k^m[d]$, then $\text{Hom}^*(FP_1, FP_2) \cong k^m[d] \oplus k^m[d-k+1]$.

To see this, we look at the triangle of the proof of Proposition 3.19:

$$\text{Hom}^*(FP_1, FP_2) \xleftarrow{\psi} \text{Hom}^*(P_1, P_2) \xrightarrow{\phi} \text{Hom}^*(P_1, P_2)[-k].$$

As there is no cancellation possible between the middle and the right hand term, we arrive at $\text{Hom}^*(FP_1, FP_2) \cong k[d] \oplus k[d-k+1]$.

The following statement generalises [HT06, Prop. 4.7] to arbitrary spherification functors.

Proposition 3.21. Let $P$ be a $\mathbb{P}^n[k]$-objects in some algebraic triangulated category $\mathcal{A}$. Let $F: \mathcal{A} \to \mathcal{B}$ be a weak spherification functor for $P$. Then the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
\downarrow_{P_\mathcal{A}} & & \downarrow_{T_{FP}} \\
\mathcal{A} & \xrightarrow{F} & \mathcal{B}
\end{array}$$
Proof. Apply $F$ to the triangles defining the $P$-twist $P$ to get

$$\text{Hom}^*(P, \_ \otimes FP[-k]) \xrightarrow{FH} \text{Hom}^*(P, \_ \otimes FP) \xrightarrow{\text{ev}} \text{Hom}^*(P, \_ \otimes FP) \xrightarrow{F \circ \text{ev}} \text{F Cone}_H(\_ \otimes FP)$$

(4)

where we write $H = t \otimes \text{id} - \text{id} \otimes t$.

On the other hand $T = T_{FP}$ fits into the triangle

$$\text{Hom}^*(FP, \_ \otimes FP) \xrightarrow{\text{ev}} \text{id} \rightarrow T.$$

Precomposing with $F$ we obtain

$$\text{Hom}^*(FP, \_ \otimes FP) \rightarrow \text{ev} \rightarrow \text{F Cone}_H(\_ \otimes FP).$$

(5)

For the left hand term we have that

$$\text{Hom}^*(FP, \_ \otimes FP) \cong \text{Hom}^*(LFP, \_ \otimes FP).$$

We can compute the latter from the triangle of the dual cotwist for $P$:

$$LFP \rightarrow P \xrightarrow{t} P[k]$$

from which we get

$$\text{Hom}^*(LFP, \_ \otimes FP) \leftarrow \text{Hom}^*(P, \_ \otimes FP) \xleftarrow{t \otimes \text{id}} \text{Hom}^*(P, \_ \otimes FP)[-k].$$

Note that $FH = F(t \otimes \text{id} - \text{id} \otimes t) = t \otimes \text{id}$, as the second term cancels for degree reasons (we recall that we always assume that $n > 1$). So comparing this last triangle with the upper triangle of (4), we obtain that

$$\text{F Cone}_H(\_ \otimes FP) \cong \text{Hom}^*(LFP, \_ \otimes FP).$$

This in turn implies that the lower triangle in (4) coincides with the triangle (5), so we obtain $FP \equiv T_{FP}F$ as claimed. □

Remark 3.22. If $F: A \rightarrow B$ is a spherification functor for $P$, then we get also

$$T_{FP}^{-1}F \cong TP^{-1}$$

by precomposing $FP \cong T_{FP}F$ with $P^{-1}$ and postcomposing it with $T_{FP}^{-1}$.

4. Returning from spherical objects to $P$-objects

The following proposition is part of a far more general statement about spherical twists associated to spherical sequences. To our knowledge, spherical sequences were first defined and studied in [Efi07], but they were rediscovered independently as exceptional cycles in [BPP17]. Anyway, we will stick to the special case of spherical objects.

Proposition 4.1 ([Vol19, Thm. 2.7]). Let $S_1, S_2$ be two $k$-spherical objects in some algebraic triangulated category which are not isomorphic up to shift. If $\text{hom}^*(S_1, S_2) \geq 2$, then there are no relations among the associated spherical twists.
Remark 4.2. In [Kea14, Thm. 1.2], a similar statement to Proposition 4.1 is shown in the context of Lagrangian spheres inside a symplectic manifold (giving rise to spherical objects in a derived Fukaya category).

In [Ans13, Thm. 1.2], the statement was shown in the special case of a \( A_1 \)-configurations of spherical objects in K3 categories.

The following gives Theorem B of the introduction.

**Theorem 4.3.** Let \( P_1, P_2 \) be two \( \mathbb{P}^n[k] \)-objects in some algebraic triangulated category \( \mathcal{A} \) which are not isomorphic up to shift. Let \( P_1, P_2 \) be the associated \( \mathbb{P} \)-twists. Suppose that there is a spherification functor \( F : \mathcal{A} \to \mathcal{B} \) for \( P_1 \) and \( P_2 \). If \( \text{Hom}^*(P_1, P_2) \neq 0 \), then there are no relations between the \( \mathbb{P} \)-twists, that is, \( \langle P_1, P_2 \rangle \cong F_2 \).

**Proof.** The objects \( S_1 = FP_1 \) and \( S_2 = FP_2 \) are spherical by definition, and by Proposition 3.19, we have that \( \text{hom}^*(S_1, S_2) \geq 2 \), and that \( S_1, S_2 \) are not isomorphic up to shift. So we find that \( \langle T_1, T_2 \rangle \cong F_2 \) by Proposition 4.1 where \( T_1 := T_{S_1} \) and \( T_2 := T_{S_2} \) denote the associated spherical twists.

We recall the key idea of the proof of Proposition 4.1: see the proof of [Vol19, Cor. 3.4]. We denote by \( \mathcal{O}_T \) the union of the orbits of \( S_1 \) and \( S_2 \) under the action of \( \langle T_1, T_2 \rangle \) on \( \mathcal{B} \), that is,

\[ \mathcal{O}_T = \langle T_1, T_2 \rangle \cdot \{ S_1, S_2 \}. \]

We define two subsets of \( \mathcal{O}_T \):

\[ \mathcal{X} = \{ X \in \mathcal{O}_T \mid \text{hom}^*(S_2, X) > \frac{\text{hom}^*(S_1, S_2)}{2} \cdot \text{hom}^*(S_1, X) \} , \]

\[ \mathcal{X}' = \{ X \in \mathcal{O}_T \mid \text{hom}^*(S_1, X) > \frac{\text{hom}^*(S_1, S_2)}{2} \cdot \text{hom}^*(S_2, X) \} . \]

As \( \text{hom}^*(S_1, S_2) \geq 2 \), the two sets do not intersect.

Now Y. Volkov shows that \( T_1 S_2 \in \mathcal{X} \) (and analogously, \( T_2 S_1 \in \mathcal{X}' \)), so both sets are non-empty. Finally, he shows that \( T_1^n \cdot \mathcal{X}' \subseteq \mathcal{X} \) and \( T_2^n \cdot \mathcal{X} \subseteq \mathcal{X}' \) for \( n \neq 0 \). Hence, one can conclude by the Ping-Pong Lemma, that \( T_1 \) and \( T_2 \) generate the free group \( F_2 \).

To obtain the statement about \( \mathbb{P}^n[k] \)-objects, we bring Proposition 3.21 into play. Define for \( \mathcal{O}_p = \langle P_1, P_2 \rangle \cdot \{ P_1, P_2 \} \) the subsets

\[ \mathcal{Y} = \{ Y \in \mathcal{O}_p \mid FY \in \mathcal{X} \} , \]

\[ \mathcal{Y}' = \{ Y \in \mathcal{O}_p \mid FY \in \mathcal{X}' \} . \]

We note that both are non-empty, as \( P_1 P_2 \in \mathcal{Y} \) and \( P_2 P_1 \in \mathcal{Y}' \), using Proposition 3.21. By the same proposition and Remark 3.22, \( P_1^n \cdot \mathcal{Y}' \subset \mathcal{Y} \) and \( P_2^n \cdot \mathcal{Y} \subset \mathcal{Y}' \) for \( n \neq 0 \). Hence, we can now conclude that the Ping-Pong Lemma applies here, so \( P_1 \) and \( P_2 \) generate the free group \( F_2 \).

**Remark 4.4.** As mentioned in the introduction, our assumption that \( n \geq 2 \) is not logically necessary for Theorem 4.3. Indeed, for \( n = 1 \), there are never relations between \( P_1 \) and \( P_2 \) if \( \text{Hom}^*(P_1, P_2) \neq 0 \), independently of the existence of a spherification functor. The reason is that by [Vol19] the spherical twists \( T_{P_1} \) and \( T_{P_2} \) either generate the free group \( F_2 \) or the braid group \( A_2 \). In either case, its squares \( T_{P_i}^2 \) do not satisfy any relations.
5. Twists along orthogonal $\mathbb{P}$-objects

**Proposition 5.1.** Let $P_1, P_2$ be two $\mathbb{P}^n[k]$-objects in some algebraic triangulated category with $\text{Hom}^*(P_1, P_2) = 0$. Then the following holds:

- The associated $\mathbb{P}$-twists $P_1$ and $P_2$ commute.
- If $(n, k) \neq (1, 1)$, the two $\mathbb{P}$-twists span a free abelian group:
  
  $$(P_1, P_2) \cong \mathbb{Z}^2.$$

**Proof.** This is proved in [Kru15, Cor. 2.5], but there it is formulated only for $\mathbb{P}^n$-objects in (equivariant) derived categories of smooth projective varieties. The general proof is exactly the same. However, we reproduce it here, as it is not long, and we want to make clear where the assumption $(n, k) \neq (1, 1)$ is needed.

By [Kru15, Prop. 2.1], which summarises some results from [Add16], we have

(6) $P_i(P_j) = P_j$ for $i \neq j$,

(7) $P_i(P_i) = P_i[−(n + 1)k + 2].$

By (6) together with [Kru15, Lem. 2.4(ii)], we have

$$P_1 = P_{P_2(P_1)} = P_2P_1P_2^{-1}$$

which proves the first part of the assertion.

For the second part, note that, due to the commutativity, we can write every $g \in \langle P_1, P_2 \rangle$ in the form $g = P_2^i P_1^2$. By (6) and (7), we have

$$g(P_i) = P_i[\nu_i(-(n + 1)k + 2)]$$

for $i = 1, 2$.

In particular, only for $\nu_1 = 0 = \nu_2$ it happens that $g = \text{id}$, as for $(n, k) \neq (1, 1)$ we find that $-(n + 1)k + 2 \neq 0$. □

**Remark 5.2.** The assumption $(n, k) \neq (1, 1)$ is not only needed for the proof, but actually there are counterexamples to the statement for $(n, k) = (1, 1)$. In other words, in the case when we are speaking about 1-spherical objects.

The following can be found in [ST01, Sect. 3.4] and [Huy06, Ex. 8.25] as part of an example of a $A_3$-sequence of 1-spherical objects where the associated braid group action is not faithful. Let $C$ be an elliptic curve, and let $x, y \in C$ be two points such that $\mathcal{O}_C(x - y)$ is a 2-torsion line bundle. Let $P_1 = \mathcal{O}_x$ and $P_2 = \mathcal{O}_y$ be the associated skyscraper sheaves. They are orthogonal 1-spherical objects with associated spherical twists

$$T_1 = - \otimes \mathcal{O}_C(x)$$

and

$$T_2 = - \otimes \mathcal{O}_C(y).$$

As the $\mathbb{P}$-twists are the squares of the spherical twists, $P_i = T_i^2$, we get the relation

$$P_1P_2^{-1} = - \otimes \mathcal{O}_C(2x - 2y) = - \otimes \mathcal{O}_C = \text{id}$$

which means that the two $\mathbb{P}$-twists are equal in this case.
6. Open questions and speculation

Let us finish by asking some open questions related to our results. Recall that we proved the absence of relations between \(\mathbb{P}\)-twists associated to non-orthogonal pairs of \(\mathbb{P}\)-objects under the assumption that a spherification functor for both objects exists, and provided the existence of spherification functors for many, but not all, pairs of \(\mathbb{P}\)-objects.

**Question 6.1.** Given a pair \(P_1, P_2\) of \(\mathbb{P}^n[k]\)-objects in some algebraic triangulated category \(\mathcal{A}\), is there always a spherification functor \(F: \langle P_1, P_2 \rangle \to \mathcal{B}\) for \(P_1\) and \(P_2\)? More generally, one may ask the same question for a collection \(P_1, \ldots, P_m\) of \(\mathbb{P}^n[k]\)-objects.

It seems to us that without the assumption of Proposition 3.12 that there is a central element of \(A = \text{End}^\bullet(P_1 \oplus P_2)\) lifting \(t_1 + t_2 \in \text{End}^\bullet(P_1 \oplus P_2)\), then one has to come up with a construction really different from the one used there (provided it is even true that there is a spherification functor in this case). To explain the problem, let us rephrase the idea of our construction of a spherification functor \(F\) of Proposition 3.12. We want to have a functor \(F\) whose associated comonad \(LF\) is the cone of the morphism of functors \(h: [-k] \to \text{id}\) in \(\mathcal{D}(A)\). However, if \(h\) is not central, there is not even a morphism of functors \(h: [-k] \to \text{id}\) as, for a general dg-module \(M\) over \(A\), multiplication by \(h\) will not be an \(A\)-linear map \(M[-k] \to M\).

**Question 6.2.** Given a pair \(P_1, P_2\) of non-orthogonal \(\mathbb{P}^n[k]\)-objects in some algebraic triangulated category \(\mathcal{A}\) which are not isomorphic up to shift, is it always true that there are no relations between the associated twists?

By Theorem 4.3, a positive answer to Question 6.1 would imply a positive answer to Question 6.2. If the answer to Question 6.1 should be negative, one could still try to prove a positive answer to Question 6.2 by hand. This seems feasible, but much more work than just pulling-back Volkov’s result about the absence of relations between spherical twists (or rather its proof via the ping-pong lemma) along a spherification functor.

On the other hand, the authors would not be completely surprised if there should be examples of \(\mathbb{P}^n[k]\)-objects \(P_1, P_2\) where the associated \(\mathbb{P}\)-twists \(P_1, P_2\) satisfy some relation when \(t_1 + t_2\) is not a central element of \(\text{End}^\bullet(P_1 \oplus P_2)\), in which case the answer to Question 6.2 would be negative. The reason is that the action of \(t_1\) and \(t_2\) on \(\text{Hom}^\bullet(P_1, P_2)\) and \(\text{Hom}^\bullet(P_2, P_1)\) comes into play when we compute \(P_1[P_2]_1\) and \(P_2[P_1]_1\); see Proposition 2.3.

Another natural question is about relations (or the lack thereof) between more than just two \(\mathbb{P}\)-twists. However, there the answer is not even known for spherical twists, so it would be natural to first try to extend Volkov’s results to more than just two spherical twists.

A last question is if and how our results generalise from \(\mathbb{P}\)-objects to \(\mathbb{P}\)-functors. Again, it seems like a sensible approach to first try to generalise Volkov’s results from spherical objects to spherical functors.

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