Towards a metamaterial simulation of a spinning cosmic string

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Abstract

Establishing the constitutive parameters of a nonhomogeneous bianisotropic medium that is equivalent to the spacetime metric of a spinning cosmic string, in a noncovariant formalism, we found a metamaterial route to investigate the existence of closed timelike curves.

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1 Introduction

Metamaterials are artificial composite materials which—through judicious design—exhibit remarkable characteristics not displayed by their constituent materials [1]. Most conspicuously, recent research efforts have been directed towards the development of metamaterials which support negative refraction [2] and those which facilitate a degree of concealment [3]. Metamaterials may also provide useful models for the study of general relativistic scenarios which are otherwise impractical or impossible to explore [4, 5, 6, 7]. Furthermore, certain metrics, such as for the Schwarzschild [8], Schwarzschild–de Sitter [9], Kerr [10, 11, 12], Kerr–Newman [13] and Reissner–Nordström [14] spacetimes, are associated with negative-phase-velocity propagation of light—a notable property of certain negatively refracting metamaterials [15]—in a noncovariant formalism.

The formal analogy between light propagation in vacuum subjected to a gravitational field and light propagation in certain nonhomogeneous bianisotropic mediums in flat spacetime has been appreciated for at least 85 years [16]. The realization of such bianisotropic mediums is moving towards practicality, since the emergence of metamaterials. The aim of this letter is to establish the constitutive relations of a nonhomogeneous bianisotropic medium which is formally equivalent to the spacetime metric associated with a spinning cosmic string [17]. The metric describing spinning cosmic strings is particularly interesting because it can support the existence of closed timelike curves (CTCs) [18]. But CTCs are problematic as they violate the principle of causality. Accordingly, a practical model to simulate this spacetime metric would be of considerable value.

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2 Spacetime metric of a spinning cosmic string

Consistently with the signature \((-, +, +, +)\), the line element of a spinning cosmic string has the form \([19]\)

\[
ds^2 = -(F \, dt + M \, d\phi)^2 + A^2 \, d\phi^2 + dz^2 + d\rho^2,
\]

in terms of time \(t\) and cylindrical polar coordinates \((\rho, \phi, z)\). We adopt the “ballpoint pen” model of Jensen and Soleng \([20]\) wherein the metric describing the interior region of the string \(\rho < \rho_s\) is matched smoothly at \(\rho = \rho_s\) to the metric describing the string’s exterior region \(\rho > \rho_s\). Accordingly, we have \(F \equiv 1\) with

\[
A = \begin{cases} 
\frac{1}{\sqrt{\lambda}} \sin \left(\rho \sqrt{\lambda}\right), & \rho \leq \rho_s \\
(1 - 4G\nu) \left[\rho + \rho_s \left(\frac{\tan \left(\rho_s \sqrt{\lambda}\right)}{\rho_s \sqrt{\lambda}} - 1\right)\right], & \rho > \rho_s
\end{cases}
\]

and

\[
M = \begin{cases} 
2\alpha \left[(\rho - \rho_s) \cos \left(\rho \sqrt{\lambda}\right) - \frac{1}{\sqrt{\lambda}} \sin \left(\rho \sqrt{\lambda}\right) + \rho_s\right], & \rho \leq \rho_s \\
4GJ, & \rho > \rho_s
\end{cases}
\]

Herein, \(\nu\) and \(J\) represent the mass and angular momentum per unit length of the string, respectively; \(G\) is the gravitational constant; \(\lambda\) is a positive–valued constant; and \(\alpha \leq 1\). Smooth matching at \(\rho = \rho_s\) imposes the equalities

\[
\nu = \frac{1}{4G} \left[1 - \cos \left(\rho_s \sqrt{\lambda}\right)\right] \quad J = \frac{\alpha}{2G} \left[-\frac{1}{\sqrt{\lambda}} \sin \left(\rho_s \sqrt{\lambda}\right) + \rho_s\right].
\]

After replacing the cylindrical polar coordinates by the Cartesian coordinates \((x, y, z)\), the spacetime metric is represented by \(g_{\alpha\beta}\) per \(^3\)

\[
[g_{\alpha\beta}] = \begin{pmatrix}
-1 & \frac{My}{\rho^2} & -\frac{Mx}{\rho^2} & 0 \\
\frac{Mx}{\rho^2} & \frac{\rho^2 x^2 + (A^2 - M^2) y^2}{\rho^4} & \frac{(\rho^2 - A^2 + M^2) xy}{\rho^4} & 0 \\
-\frac{My}{\rho^2} & \frac{(\rho^2 - A^2 + M^2) xy}{\rho^4} & \frac{\rho^2 y^2 + (A^2 - M^2) x^2}{\rho^4} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\(^3\)Roman indexes take the values 1, 2 and 3, while Greek indexes take the values 0, 1, 2 and 3.
where \( \rho^2 = x^2 + y^2 \). We note the following limits:

\[
\lim_{\rho \to \infty} [g_{\alpha\beta}] = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \cos^2 \phi + \cos^2 \left( \rho_s \sqrt{\lambda} \right) \sin^2 \phi & \sin^2 \left( \rho_s \sqrt{\lambda} \right) \cos \phi \sin \phi & 0 \\
0 & \sin^2 \left( \rho_s \sqrt{\lambda} \right) \cos \phi \sin \phi & \sin^2 \phi + \cos^2 \left( \rho_s \sqrt{\lambda} \right) \cos^2 \phi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

(6)

\[
\lim_{\rho \to 0} [g_{\alpha\beta}] = \text{diag} (-1, 1, 1, 1),
\]

(7)

where \( \cos \phi = x/\rho \) and \( \sin \phi = y/\rho \). In particular, in the limit \( \rho \to \infty \), the eigenvalues of \([g_{\alpha\beta}]\) are \([-1, 1, 1, \cos^2 \left( \rho_s \sqrt{\lambda} \right)]\). Accordingly, \([g_{\alpha\beta}] \to \text{diag} (-1, 1, 1, 1)\) as \( \rho \to \infty \) only when \( \rho_s \sqrt{\lambda} \) is an integer multiple of \( \pi \).

Following the approach of Tamm [16]—which was later developed by others [21, 22, 23, 24]—electromagnetic fields in the curved spacetime associated with a spinning cosmic string may be described by the constitutive relations of an equivalent medium per

\[
D_\ell = \varepsilon_0 \gamma_{\ell m} E_m + \sqrt{\varepsilon_0 \mu_0 \varepsilon_{\ell mn} \Gamma_m H_n},
\]

\[
B_\ell = \mu_0 \gamma_{\ell m} H_m - \sqrt{\varepsilon_0 \mu_0 \varepsilon_{\ell mn} \Gamma_m E_n},
\]

(8)

in SI units. Herein, the scalar constants \( \varepsilon_0 \) and \( \mu_0 \) denote the permittivity and permeability of vacuum in the absence of a gravitational field; \( \varepsilon_{\ell mn} \) is the three-dimensional Levi-Civita symbol; and the components of \( \gamma_{\ell m} \) and \( \Gamma_m \) are defined as

\[
\begin{align*}
\gamma_{\ell m} &= \sqrt{-g} \frac{g_{\ell m}}{g_{00}}, \\
\Gamma_m &= \frac{g_{0 m}}{g_{00}},
\end{align*}
\]

(9)

where the sign of the square root term in the definition of \( \gamma_{\ell m} \) is chosen to ensure that the metric for Minkowskian spacetime is represented by the matrix \( \gamma_{\ell m} = \text{diag} (1, 1, 1) \). Thus, in this noncovariant formalism, the curved spacetime associated with a spinning cosmic string is represented by the fictitious, nonhomogeneous, spatiotemporally local, bianisotropic medium characterized by (8). With respect to the Cartesian basis vectors, the matrix and vector representations of \( \gamma_{\ell m} \) and \( \Gamma_m \) are

\[
[\gamma_{\ell m}] = \frac{1}{\rho} \begin{pmatrix}
A^2 x^2 + \rho^2 y^2 & \frac{(A^2 - \rho^2) x y}{A \rho^2} & 0 \\
\frac{(A^2 - \rho^2) x y}{A \rho^2} & A^2 y^2 + \rho^2 x^2 & 0 \\
0 & 0 & A
\end{pmatrix},
\]

(10)

and

\[
[\Gamma_m] = \frac{M}{\rho^2} (-y, x, 0).
\]

(11)
We note the following limits:

\[
\lim_{\rho \to \infty} [\gamma_{\ell m}] = \begin{pmatrix}
\cos^2 \phi + \frac{\sin^2 \phi}{\cos^2 (\rho_s \sqrt{\lambda})} & -\tan^2 (\rho_s \sqrt{\lambda}) \cos \phi \sin \phi & 0 \\
-\tan^2 (\rho_s \sqrt{\lambda}) \cos \phi \sin \phi & \sin^2 \phi + \frac{\cos^2 \phi}{\cos^2 (\rho_s \sqrt{\lambda})} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
\lim_{\rho \to 0} [\gamma_{\ell m}] = \text{diag} (1, 1, 1),
\]

\[
\lim_{\rho \to \infty} [\Gamma_m] = \lim_{\rho \to 0} [\Gamma_m] = \text{diag} (0, 0, 0).
\]

In particular, we observe that \([\gamma_{\ell m}] \to \text{diag} (1, 1, 1)\) as \(\rho \to \infty\) only when \(\rho_s \sqrt{\lambda}\) is an integer multiple of \(\pi\).

3 Numerical illustration

Following Jensen and Soleng [20], we take \(\alpha = 1\). Let us present numerical results for the limiting case where \(\nu\) attains its maximum value of \(1/(2G)\). Closed timelike curves are supported when the coefficient \((A^2 - M^2)\) of \(d\phi^2\) in the line–element expression (1) is negative–valued [18]. This quantity is plotted against \(\rho/\rho_s\) in Fig. 1. We find that \(A^2 - M^2 < 0\) for \(0.11 < (\rho/\rho_s) < 3\); i.e., both the interior and exterior string regions support CTCs.

![Figure 1: The quantity \(A^2 - M^2\) plotted versus \(\rho/\rho_s\), for \(\alpha = 1\) and \(r_s \sqrt{\lambda} = \pi\).](image)

The fictitious bianisotropic medium characterized by (8) becomes equivalent to vacuum (in flat spacetime) in the limits \(\rho \to 0\) and \(\rho \to \infty\), for all values of \(\phi\). For definiteness, let us fix the angle \(\phi = 0\). The matrix \([\gamma_{\ell m}]\) then is diagonal with \(\gamma_{11} = \gamma_{33}\), and the vector \([\Gamma_m]\) has only one nonzero component: \(\Gamma_2 = M/\rho\). The constitutive parameters \(\gamma_{11}, \gamma_{22}\) and \(\Gamma_2\) are plotted against \(\rho/\rho_s\) in Fig. 2. The constitutive parameters \(\gamma_{11}\) and \(\Gamma_2\) remain bounded for all values of \(\rho/\rho_s\), whereas \(\gamma_{22}\) becomes unbounded at \(\rho = \rho_s\). For a metamaterial simulation of a cosmic string, one would have to use \(\gamma_{22} \gg 1\) at \(\rho = \rho_s\).
The limiting behavior displayed in Fig. 2 as \( \rho \to 0 \) and \( \rho \to \infty \) also arises when \( \nu \) attains its minimum value of zero. Therefore, a realistic metamaterial simulation of a massless string [25] is also possible. Just as for \( \nu = 1 / (2G) \), our calculations (not presented here) reveal that \( \nu = 0 \) is also compatible with the existence of CTCs in both the interior and exterior regions.

4 Closing remarks

The chief purpose of this letter was to present the constitutive matrix (10) and vector (11) which together specify the fictitious bianisotropic medium that simulates the curved spacetime associated with a spinning cosmic string. Thereby, a recipe for a simulating metamaterial has been formulated. Some variation of this recipe could be implemented to resolve fundamental issues pertaining to the existence of CTCs.

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