Family of Hermitian Low-Momentum Nucleon Interactions with Phase Shift Equivalence

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Using a Schmidt orthogonalization transformation, a family of Hermitian low-momentum NN interactions is derived from the non-Hermitian Lee-Suzuki (LS) low-momentum NN interaction. As special cases, our transformation reproduces the Hermitian interactions of Okubo and Andreozzi. Aside from their common preservation of the deuteron binding energy, these Hermitian interactions are shown to be phase shift equivalent, all preserving the empirical phase shifts up to decimation scale $\Lambda$. Employing a solvable matrix model, the Hermitian interactions given by different orthogonalization transformations are studied; the interactions can be very different from each other particularly when there is a strong intruder state influence. However, because the parent LS low-momentum NN interaction is only slightly non-Hermitian, the Hermitian low momentum nucleon interactions given by our transformations, including the Okubo and Andreozzi ones, are all rather similar to each other. Shell model matrix elements given by the LS and several Hermitian low momentum interactions are compared.

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I. INTRODUCTION

A fundamental problem in nuclear physics has been the determination of the effective nucleon-nucleon ($NN$) interaction appropriate for complex nuclei. Typically, one starts from a NN interaction, $V_{NN}$, constrained by the deuteron properties and the empirical low-energy NN scattering phase shifts. Several realistic meson models [1, 2, 3, 4] for $V_{NN}$ have been obtained in this way, and while they all share the same one pion tail, they differ significantly in how they treat the shorter distance components. Despite this difference, these models all give approximately the same low-energy phase shifts and deuteron binding energy. This result clearly manifests the main theme of the renormalization group (RG) and effective field theory (EFT) approach, namely physics in the infrared region is insensitive to the details of the short-distance dynamics [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. It is thus possible to have infinitely many theories that differ substantially at small distances, but still give the same low-energy physics, as long as they possess the same symmetries and the “correct” long-wavelength structure. Since low-energy physics isn’t concerned with these high-energy details, one should just use an effective theory with the short-wavelength modes integrated out.

Following this RG-EFT idea, a low-momentum NN effective interaction, $V_{\text{low}-k}$, was recently developed [15, 16, 17, 18, 19, 20]. While similar in spirit to traditional EFT, $V_{\text{low}-k}$ is not derived via the usual RG-EFT methods; rather, it combines the standard nuclear physics approach (SNPA) with EFT making it a more effective EFT (MEEFT) [21, 22, 23], as discussed in [24]. As such, a main step in its derivation is the integrating out of the high momentum components of some realistic NN potential model $V_{NN}$ such as those of Ref. [1, 2, 3, 4]. Even though these $V_{NN}$ models are quite different from each other, it is remarkable that the $V_{\text{low}-k}$’s derived from them are nearly identical to each other, suggesting a nearly unique $V_{\text{low}-k}$ [25]. Furthermore, shell model calculations using $V_{\text{low}-k}$ have given very encouraging results over a wide range of nuclei [16, 17, 18, 20]. Applications of $V_{\text{low}-k}$ to quasi-particle interaction, superfluid gaps and equation of state for neutron matter have also been highly successful [18].

An important problem in deriving $V_{\text{low}-k}$ is how to obtain a low-momentum NN interaction which is Hermitian. The $V_{\text{low}-k}$ given by the T-matrix equivalence approach [15, 16] is not Hermitian, and some additional transformation is needed to make it Hermitian. There are a number of methods used to obtain a Hermitian effective interaction, such as those of Okubo [26], Suzuki and Okamoto [27], and Andreozzi [28]. Which of these methods should one use? How different are the Hermitian $V_{\text{low}-k}$’s given by them? These questions seem to have not been investigated. In the present work, we shall study the $V_{\text{low}-k}$’s given by these methods as well as develop a unified method by which a family of phase-shift equivalent Hermitian low-momentum NN interactions can be obtained. It is, of course, important that $V_{\text{low}-k}$ preserve phase shifts, and while the non-Hermitian $V_{\text{low}-k}$ given by the T-matrix equivalence [15, 16] can be shown to preserve phase shifts in a straightforward way, it seems to be more involved to prove the phase shift preservation by Hermitian $V_{\text{low}-k}$. Epelbaum et al. [8] have pointed out that phase shifts are preserved for the Okubo low-momentum interaction [5], but phase shift preservation...
for other Hermitian $V_{\text{low-}k}$'s seems to have not been investigated - an issue we will also examine.

To conclude our introduction, we shall briefly review the non-Hermitian low-momentum interaction given by the T-matrix equivalence approach. Then in Section II we present a general method, based on Schmidt orthogonalization transformation, for generating a family of Hermitian effective interactions, and show that the Hermitian interactions of Okubo, Suzuki and Okamoto and Andreozzi all belong to this family. In Section III, we study our method using a simple solvable model of the Hoffmann type \[24], focusing on the influence of intruder states and the difference of the Hermitian interactions given by the various methods. In Section IV we present a proof that phase shifts are preserved by the Hermitian interactions generated by our method. This preservation will also be checked by phase shift calculations using different Hermitian interactions. Finally in Section V we present our results for the $V_{\text{low-}k}$'s corresponding to the various Hermitian interactions, where we show that our approach allows us to construct a $V_{\text{low-}k}$ which preserves the deuteron wave function, in addition to the preservation of deuteron binding energy and low-energy phase shifts.

In the following let us first briefly review the $V_{\text{low-}k}$ given by T-matrix equivalence \[12,16\]. One starts from the half-on-shell T matrix

\[
T(k', k, k^2) = V_{NN}(k', k) + P \int_0^{\infty} dq V_{NN}(k', q) \frac{1}{k^2 - q^2 + i0^+} T(q, k, k^2). \tag{1}
\]

An effective low-momentum T matrix is then defined by

\[
T_{\text{low-}k}(p', p, p^2) = V_{\text{low-}k}(p', p) + P \int_0^{\Lambda} dq V_{\text{low-}k}(p', q) \frac{1}{q^2 - p^2 + i0^+} T_{\text{low-}k}(q, p, p^2) \tag{2}
\]

where the intermediate state momentum is integrated up to $\Lambda$. In the above two equations we employ the principal value boundary conditions, as indicated by the symbol $P$ in front of the integral sign. We require the T matrices satisfy the condition

\[
T(p', p, p^2) = T_{\text{low-}k}(p', p, p^2); \quad (p', p) \leq \Lambda. \tag{3}
\]

The above equations define the effective low-momentum interaction $V_{\text{low-}k}$. Using a $Q$-box folded-diagram method \[32,33\], it has been shown \[12,16,17\] that the above equations are satisfied by the solution

\[
V_{\text{low-}k} = \hat{Q} - \hat{Q}' \int \hat{Q} + \hat{Q}' \int \hat{Q} \int \hat{Q} - \hat{Q}' \int \hat{Q} \int \hat{Q} + \cdots, \tag{4}
\]

where $\hat{Q}$-box denotes the irreducible vertex function whose intermediate states are all beyond $\Lambda$, and $\hat{Q}'$ is the same vertex function except that it starts with terms second order in the interaction. The low-momentum effective NN interaction of Eq.(4) can be calculated using iteration methods such as the Lee-Suzuki \[30\], Andreozzi \[28\] or Krenciglowa-Kuo \[31\] methods.

The above $V_{\text{low-}k}$ preserves both the deuteron binding energy and the half-on-shell T-matrix of $V_{NN}$ (which implies the preservation of the phase shifts up to $E_{\text{lab}} = 2h^2\Lambda^2/M$, $M$ being the nucleon mass). This $V_{\text{low-}k}$ is not Hermitian, as indicated by Eq.(4). As we will show soon, starting from this $V_{\text{low-}k}$ a family of phase-shift equivalent Hermitian low-momentum NN interactions can be obtained.

## II. FORMALISM

Before presenting our general Hermitization procedure, let us first review some basic formulations about the model space effective interaction. We start from the Schroedinger equation

\[
(H_0 + V)\Psi_n = E_n\Psi_n, \tag{5}
\]

where $H_0$ is the unperturbed Hamiltonian and $V$ the interaction. The eigenstates of $H_0$ are $\phi_n$ with eigenvalues $\epsilon_n$. For example, $H_0$ can be the kinetic energy operator and $V$ the NN interaction $V_{NN}$. A model-space projection operator $P$ is defined as $\sum_d | \phi_d \rangle \langle \phi_d |$, where $d$ is the dimension of the model space. The projection operator complement to $P$ is denoted as $Q$, and as usual, one has $P^2 = P, Q^2 = Q$ and $PQ = 0$. In the present work, $P$ represents all the momentum states with momentum less than the cut-off scale $\Lambda$.

A model-space effective interaction $V_{\text{eff}}$ is introduced with the requirement that the effective Hamiltonian $P(H_0 + V_{\text{eff}})P$ reproduces some of the eigenvalues and certain information about the eigenfunctions of the original Hamiltonian $(H_0 + V)$. There are a number of ways to derive $V_{\text{eff}}$, but, as indicated by Eq.(4), our effective interaction is obtained by the folded diagram method \[32,33\] and can be calculated conveniently using the Lee-Suzuki-Andreozzi \[28\] or Krenciglowa-Kuo \[31\] iteration methods. We denote this effective interaction as $V_{\text{LS}}$, with the corresponding model space Schroedinger equation

\[
P(H_0 + V_{\text{LS}})P\chi_m = E_m\chi_m, \tag{6}
\]

where $\{E_m\}$ is a subset of $\{E_n\}$ of Eq.(5) and $\chi_m = P\Psi_m$.

It is convenient to rewrite the above effective interaction in terms of the wave operator $\omega$, namely

\[
P V_{\text{LS}} P = P e^{-\omega} (H_0 + V) e^{\omega} P - P H_0 P, \tag{7}
\]
where $\omega$ possesses the usual properties

\[ \omega = Q\omega P; \]
\[ \chi_m = e^{-\omega}\Psi_m; \]
\[ \omega \chi_m = Q\Psi_m. \]  

(8)

While the eigenvectors $\Psi_m$ of Eq.(5) are orthogonal to each other, it is clear that the eigenvectors $\chi_m$ of Eq.(6) are not so and the effective interaction $V_{LS}$ is not Hermitian. We now make a $Z$ transformation such that

\[ Z\chi_m = v_m; \]
\[ \langle v_m | v_{m'} \rangle = \delta_{mm'}; \ m, m' = 1, d, \]  

(9)

where $d$ is the dimension of the model space. This transformation reorients the vectors $\chi_m$ such that they become orthonormal to each other. We assume that $\chi_m$'s ($m = 1, d$) are linearly independent so that $Z^{-1}$ exists, otherwise the above transformation is not possible. Since $\chi_m$ and $Z$ exist entirely within the model space, we can write $v_m = Pv_m$ and $Z = PZP$.

Using Eq.(9), we transform Eq.(6) into

\[ Z(H_0 + V_{LS})Z^{-1}v_m = E_m v_m, \]  

(10)

which implies

\[ Z(H_0 + V_{LS})Z^{-1} = \sum_{m \in P} E_m | v_m \rangle \langle v_m |. \]  

(11)

Since $E_m$ is real (it is an eigenvalue of Eq.(5)) and the vectors $v_m$ are orthonormal to each other, $Z(H_0 + V_{LS})Z^{-1}$ must be Hermitian. The original problem is now reduced to a Hermitian model-space eigenvalue problem

\[ P(H_0 + V_{herm})Pv_m = E_m v_m \]  

(12)

with the Hermitian effective interaction

\[ V_{herm} = Z(H_0 + V_{LS})Z^{-1} - PH_0P, \]  

(13)

or equivalently

\[ V_{herm} = Ze^{-\omega}(H_0 + V)e^{-\omega}Z^{-1} - PH_0P. \]  

(14)

To calculate $V_{herm}$, we must first have the $Z$ transformation. Since there are certainly many ways to construct $Z$, this generates a family of Hermitian effective interactions, all originating from $V_{LS}$. For example, we can construct $Z$ using the familiar Schmidt orthogonalization procedure, namely:

\[ v_1 = Z_{11}\chi_1 \]
\[ v_2 = Z_{21}\chi_1 + Z_{22}\chi_2 \]
\[ v_3 = Z_{31}\chi_1 + Z_{32}\chi_2 + Z_{33}\chi_3 \]
\[ v_4 = \ldots, \]  

(15)

with the matrix elements $Z_{ij}$ determined from Eq.(9). We denote the Hermitian effective interaction using this $Z$ transformation as $V_{chm}$. Clearly there are more than one such Schmidt procedures. For instance, we can use $v_2$ as the starting point, which gives $v_2 = Z_{22}\chi_2$, $v_3 = Z_{32}\chi_2 + Z_{33}\chi_3$, and so forth. This freedom in how the orthogonalization is actually achieved, gives us infinitely many ways to generate a Hermitian interaction, and this is our family of Hermitian interactions produced from $V_{LS}$.

We now show how some well-known Hermitization transformations relate to (and in fact, are special cases of) ours. We first look at the Okubo transformation \[28]\. From Eq.(8) we have

\[ \langle \chi_m | (1 + \omega^+\omega) | \chi_{m'} \rangle = \delta_{mm'}. \]  

(16)

It follows that an analytic choice for the $Z$ transformation is

\[ Z = P(1 + \omega^+\omega)^{1/2}P. \]  

(17)

This leads to the Hermitian effective interaction

\[ V_{okb1} = P(1 + \omega^+\omega)^{1/2}P(H_0 + V_{LS})P(1 + \omega^+\omega)^{-1/2}P - PH_0P. \]  

(18)

From Eqs. (8), (9), (16) and (17), it is easily seen that the above is equal to the Okubo Hermitian effective interaction

\[ V_{okb} = P(1 + \omega^+\omega)^{-1/2}(1 + \omega^+\omega)(H_0 + V) \times (1 + \omega)(1 + \omega^+\omega)^{-1/2}P - PH_0P \]

(19)

giving us an alternate expression, Eq.(18), for the Okubo interaction.

There is another interesting choice for the transformation $Z$. As pointed out by Andreozzi \[28]\, the positive definite operator $P(1 + \omega^+\omega)P$ can be decomposed into two Cholesky matrices, namely

\[ P(1 + \omega^+\omega)P = PLL^T P, \]  

(20)

where $L$ is a lower triangle Cholesky matrix, $L^T$ being its transpose. Since $L$ is real and it is within the $P$-space, we have from Eq.(16) that

\[ Z = L^T \]  

(21)

and the corresponding Hermitian effective interaction from Eq.(13) is

\[ V_{cho} = PLL^T P(H_0 + V_{LS})P(L^{-1})^T P - PH_0P. \]  

(22)

This is the Hermitian effective interaction of Andreozzi \[28\].
The final Hermitian effective interaction we consider is that of Suzuki and Okamota \[27, 34\], which is of the form

\[ V_{\text{suzu}} = Pe^{-G}(H_0 + V)e^G P - PH_0P \]  \hspace{1cm} (23)

with \( G = \text{tanh}^{-1}(\omega - \omega^\dagger) \) and \( G^\dagger = -G \). It has been shown that this interaction is the same as the Okubo interaction \[27\]. In terms of the \( Z \) transformation, it is readily seen that the operator \( e^{-G} \) in Eq. (23) is equal to \( Ze^{-\omega} \) with \( Z \) given by Eq. (17). Thus, three well-known and particularly useful Hermitian effective interactions indeed belong to our family.

### III. MODEL CALCULATIONS

The results from the previous section show that a family of Hermitian effective interactions can be derived from a Schmidt-type transformation of the non-Hermitian interaction \( V_{LS} \). We now check to see if these interactions do reproduce some of the eigenvalues of the original Hamiltonian, and how the effective interactions given by the various methods differ. In this section, we shall use a solvable matrix model to study these questions. Hoffmann et al. \[29\] have employed a matrix model to study the influence of intruder states on effective interactions. Since we are also interested to see how intruder states might affect our Hermitian potentials, we use a matrix model of this type to study \( V_{\text{schm}}, V_{\text{okb}} \) and \( V_{\text{cho}} \), together with their parent non-Hermitian interaction \( V_{LS} \).

We employ a 5-by-5 matrix model \( H = H_0 + xV \), \( x \) being a strength parameter. We take \( H_0 = \{1, 1, 1, 5, 9\} \) and

\[ V = \begin{pmatrix} 1 & 5 & 5 & 0 & 5 \\ 5 & 25 & 5 & 5 & 0 \\ 5 & 5 & 15 & 2 & 2 \\ 0 & 5 & 2 & -5 & 1 \\ 5 & 0 & 2 & 1 & -5 \end{pmatrix}. \]  \hspace{1cm} (24)

Our \( P \)-space is chosen as the space spanned by the three lowest eigenstates of \( H_0 \), namely \( PH_0P = \{1, 1, 1\} \). The rather large diagonal matrix elements are used for \( V \) so that intruder states will enter as the strength parameter \( x \) increases. How this happens is shown in Fig. 1, where the eigenvalues of \( H \) are plotted as a function of the strength parameter. In this figure the states are labelled by \( P \) and \( Q \) states, where the \( P \)-space components of the wave functions are dominated by their \( P \)-space components, i.e. \( \langle \Psi | P | \Psi \rangle \). In contrast, the \( Q \) states are dominated by their \( Q \)-space components. For a weak interaction (small \( x \)), the lowest three states are all \( P \) states. As the interaction strength increases, states \( Q_1 \) and \( Q_2 \) decrease in value and states \( P_1 \) and \( P_2 \) increase, and we would expect these eigenvalues to intersect. Of course, they don’t actually cross, but at certain interaction strength a \( Q \)-state “intrudes” into the \( P \)-space, becoming lower than the rising \( P \)-state. At \( x=0.5 \), for example, the lowest three states are \( P_3, Q_2 \) and \( Q_1 \), so \( Q_2 \) and \( Q_1 \) are intruder states in the sense that they have entered the \( P \)-space when the interaction is strong. We want our model-space effective Hamiltonian \( PH_{eff}P \) to reproduce the lowest three states of \( H \). Thus at large \( x \), we are requiring our \( PH_{eff}P \) to reproduce two intruder states. In

| \( V_{LS} \) | \( V_{\text{okb}} \) | \( V_{\text{cho}} \) | \( V_{\text{schm}} \) |
|---|---|---|---|
| -0.0368 0.3657 0.3976 | -0.0358 0.3732 0.4368 | -0.0041 0.4063 0.5215 | -0.0251 0.3823 0.4698 |
| 0.3793 0.8811 0.4482 | 0.3732 0.8814 0.4690 | 0.4063 0.9020 0.5143 | 0.3823 0.8846 0.4695 |
| 0.4722 1.3999 0.4695 | 0.4368 1.3986 0.4728 | 0.5215 1.3461 0.5143 | 0.4698 1.3846 0.4695 |

Table I: Comparison of 3x3 Hermitian (\( okb, cho, schm \)) and non-Hermitian (\( LS \)) effective interactions. The matrix model of Eq. (24) is employed with interaction strength \( x=0.1 \).
by Andreozzi [28]; we denote this method as the ALS method, and by Suzuki [29].

The effective interactions are calculated using the following procedures. First we calculate the wave operator \( \omega \) using the Lee-Suzuki iteration method developed by Andreozzi [29]; we denote this method as the ALS method, and \( V_{LS} \) is then given by Eq.(7). \( V_{schm} \) is obtained from Eqs.(9), (13) and (15), and \( V_{cho} \) is calculated using Eqs.(20) and (22). For the Okubo interaction, it is convenient to calculate it using the method of Suzuki et al. [21, 31, 32]. With this method, we first find the eigenvalues and eigenfunctions defined by

\[
(1 + \omega^2) \langle \alpha | \alpha \rangle = \mu^2_{\alpha} \langle \alpha | \alpha \rangle.
\]

Then the Okubo Hermitian effective interaction is given by

\[
\langle \alpha | V_{okb} | \beta \rangle = D(\alpha, \beta) \times \left( \sqrt{\mu^2_{\alpha} + 1} \langle \alpha | V_{LS} | \beta \rangle + \sqrt{\mu^2_{\beta} + 1} \langle \alpha | V_{LS}^d | \beta \rangle \right)
\]

with

\[
D(\alpha, \beta) = \left[ \sqrt{\mu^2_{\alpha} + 1} + \sqrt{\mu^2_{\beta} + 1} \right]^{-1}.
\]

Using these methods we can calculate \( V_{LS}, V_{okb}, V_{cho}, \) and \( V_{schm} \) for the model potential of Eq.(24). Two different strength parameters, \( x = 0.1 \) and \( x = 0.55 \), have been used to see how the intruder states influence the effective interactions. Our results for these two parameters are shown in Tables I and II. First let us inspect the Hermiticity of our effective interactions; clearly \( V_{okb}, V_{cho}, \) and \( V_{schm} \) are all Hermitian, irrespective of the strength parameter, as they should be. The degree of non-Hermiticity of \( V_{LS} \), however, is highly dependent on \( x \). In Table I, we see that \( V_{LS} \) is only slightly non-Hermitian, as the largest difference in symmetric matrix elements is only of the order of 20%. The impact of the strength parameter on the non-Hermiticity of \( V_{LS} \) can be seen when comparing this with table II. Here \( V_{LS} \) is strongly non-Hermitian with some symmetric matrix elements differing by more than a factor of 4. Thus we see that when no intruder states are present (low strength parameter), our parent interaction is approximately Hermitian, but when the intruder states enter (high strength parameter), our parent interaction, namely \( V_{LS} \), loses that Hermiticity in a striking manner.

The next point to note concerning Tables I and II is the differences between the Hermitian effective interactions given by the various methods. In Table I, we see that the Hermitization procedures produce potentials which do not differ greatly from the parent potential. This, however, is expected since the parent potential is already approximately Hermitian. For a high strength parameter, we see that the resultant Hermitized potentials \( (V_{okb}, V_{cho}, V_{schm}) \) are all indeed quite different. Thus, if we want Hermitian potentials which are similar, it is crucial that the influence of intruder states be minimal.

Finally, we examine Table III where we show the eigenenergies and wavefunctions for the parent and Hermitized potentials. As expected, we see that the Hermitization procedures preserve eigenenergies. Note, however,
the eigenfunctions of the various interactions are very different, although they correspond to the same eigenvalues. We note also that the ground state wave function of both $V_{LS}$ and $V_{schm}$ is equal to the P-space projection of the ground state wave function of the full-space Hamiltonian.

IV. PHASE SHIFT EQUIVALENCE

The non-Hermitian $V_{LS}$ given by the Lee-Suzuki (or folded diagram) method is specifically constructed to preserve the half-on-shell T-matrix $T(p', p, p^2)$; this interaction of course preserves the phase shift which is given by the fully-on-shell T-matrix $T(p, p, p^2)$. It would be of interest to study if phase shifts are also generally preserved by the Hermitian interactions generated using the transformations described in Section II.

Let us consider two T-matrices $T_1(\omega_1) = V_1 + V_1g_1(\omega_1)T_1(\omega_1)$ and $T_2(\omega_2) = V_2 + V_2g_2(\omega_2)T_2(\omega_2)$, with the propagators $g_1(\omega_1) = \frac{1}{m-H_0}$ and similarly for $g_2(\omega_2)$. The unperturbed state is defined by $H_0$, namely $H_0|q\rangle = q^2|q\rangle$. The symbol $P$ denotes the principal value boundary condition. These T-matrices are related by the well known two-potential formula

$$T_2 = T_1 + T_2^1(g_2 - g_1)T_1 + \Omega_2^1(V_2 - V_1)\Omega_1,$$  \hspace{1cm} (28)

where the wave operator $\Omega$ is defined by $T_1(\omega_1) = V_1\Omega_1(\omega_1)$ and similarly for $\Omega_2$. Applying the above relation to the half-on-shell T-matrices in momentum space, we have

$$\langle p'|T_2^1(p^2)|p\rangle = \langle p'|T_1(p^2)|p\rangle + \langle p'|T_2^1(p^2)(g_2(p^2) - g_1(p^2))T_1(p^2)|p\rangle + \langle p'|T_2^1(p^2)|v_2(p)|v_2(p)|\rangle.$$

Here the true and unperturbed wave functions are related by $|\psi_1(p)\rangle = \Omega_1(p^2)|\psi_1(p)\rangle$ and similarly for $\psi_2$.

Using the above relation, we shall now show that the phase shifts of the full-space interaction $V$ are preserved by the Hermitian interaction $V_{herm}$, for momentum $\leq \Lambda$. Let us denote the last term of Eq.(29) as $D(p', p)$. We use $V_{herm}$ for $V_2$ and $V$ for $V_1$. Recall that the eigenfunction of $(H_0 + V_{herm})$ is $\psi_2$ (see Eq.(12)) and that for $H \equiv (H_0 + V)$ is $\Psi_m$. We define a wave operator

$$U_P = \sum_{m\in P} |v_m\rangle\langle \Psi_m|.$$

Then $|v_m\rangle = U_P|\Psi_m\rangle$ and $PV_{herm}P = U_P(H_0 + V)U_P^\dagger - PH_0P$.

$$D(p', p) = \langle \Psi_p|[(H - p)^\dagger, U_P^\dagger, H]\Psi_p\rangle = \langle p'|(p^2 - p'^2)|v_p|\Psi_p\rangle.$$

Clearly $D(p, p) = 0$. The second term on the right hand side of Eq.(29) vanishes when $p' = p$. Hence

$$\langle p|T_{herm}(p^2)|p\rangle = \langle p|T(p^2)|p\rangle, \hspace{1cm} p \leq \Lambda,$$

where $T_{herm}$ is the T-matrix for $(H_0 + V_{herm})$ and $T$ for $(H_0 + V)$. Consequently the phase shifts of $V$ are preserved by $V_{herm}$. Recall that our T-matrices are real, because of the principal-value boundary conditions employed.
To double check this preservation, we have calculated the phase shifts of the various Hermitian potentials ($V_{okb}$, $V_{cho}$ and $V_{schm}$) together with $V_{LS}$. As shown in Fig. 2 the $^1S_0$ phase shifts obtained from them all agree with each other very well, as is the case for other partial waves. The calculations were performed with the CD-Bonn potential and $\Lambda = 2.0 fm^{-1}$. Since the phase shifts of $V_{LS}$ are, by construction, the same as those of the full-space potential $V$, the Hermitian potentials preserve the phase shifts of $V$. It has been pointed out that the Okubo Hermitian potential preserves the phase shifts, but we have found that there is a family of Hermitian potentials, including Okubo, which all preserve the phase shifts up to the decimation scale $\Lambda$.

V. HERMITIAN LOW-MOMENTUM INTERACTIONS

A main purpose of having a low-momentum nucleon interaction $V_{low-k}$ is to use it in nuclear many body problems such as the shell model nuclear structure calculations. As we have seen, however, there is a family of phase-shift equivalent Hermitian interactions. How different are they? Which one should one use for nuclear structure calculations? Will these Hermitian effective interactions give rise to the same physical properties? It is these questions we now seek to answer.

We have calculated the Hermitian $V_{low-k}$’s corresponding to $V_{okb}$, $V_{cho}$ and $V_{schm}$ using several NN potentials. Calculations for the non-Hermitian $V_{low-k}$ corresponding to $V_{LS}$ were also performed. In Fig. 3 we compare the results for the $^1S_0$ channel, obtained with the CD-Bonn potential and $\Lambda = 2.0 fm^{-1}$. Clearly the Hermitian interactions are all quite similar to each other and to the parent non-Hermitian potential $V_{LS}$. In Fig. 4, we show a similar plot for the $^3S_1$ channel. Again, with the exception of very low momentum, the Hermitian potentials are all nearly identical to the parent $V_{LS}$. It is of interest that at very low momentum, the $V_{schm}$ matrix elements are slightly more attractive than the others. We note that the Hermitian effective interactions all preserve the deuteron binding energy (2.225 MeV). In addition, they are all phase shift equivalent up to the decimation scale $\Lambda$, as illustrated in Fig. 2.

Despite the similarities, there is, however, an additional degree of preservation that $V_{schm}$ satisfies but the other two Hermitian interactions don’t. By construction, the deuteron ground-state wave function given by $V_{schm}$ is exactly equal to the P-space projection of the wave function of $V$, which is not true for $V_{okb}$ and $V_{cho}$. This additional preservation is worth studying further. We refer to Figures 5 and 6, where we plot the S and D-state deuteron wavefunctions for a cutoff of $\Lambda = 2.0 fm^{-1}$. Overall, the agreement is very good between all potentials, not just $V_{LS}$ and $V_{schm}$, which is to be expected considering that the interactions themselves are approximately the same. The fact that $V_{schm}$ gives exactly the same wavefunction as $V_{LS}$ can be seen from the D-state probability of the deuteron for each interaction, which we list for convenience in Fig. 6. Whereas the $P_D$’s for the effective interactions are close, they are exactly equal for $V_{LS}$ and $V_{schm}$. This exact preservation of the ground state wavefunction presents us with an extra constraint.
that might be useful in deciding which Hermitian potential to use.

To ensure our Hermitian potentials will be useful for nuclear structure calculations, we examine in Table IV some shell model matrix elements calculated with $V_{\text{low-}k}$ corresponding to $V_{\text{okb}}, V_{\text{cho}}, V_{\text{schm}}$ and $V_{LS}$. As seen they are all virtually identical, as would be expected owing to the similarity between the potentials themselves. This good agreement is a desirable result, as it implies that shell model calculations will not be sensitive to which interaction one employs.

We have seen that the Hermitian potentials generated above are all approximately the same, and we would like to offer an explanation as to why this is so. Although the $V_{\text{low-}k}$ corresponding to $V_{LS}$ is non-Hermitian, we emphasize that it is only slightly so. In reference to our model study of Section 2, this corresponds to the situation with a small strength parameter, and thus it would not be surprising to see the Hermitian $V_{\text{low-}k}$ potentials are so similar to the parent $V_{\text{low-}k}$. For example, this is especially transparent in the case of the Okubo interaction of Eq. (18) or (19), as it can be written as:

$$\langle \alpha | V_{\text{okb}} | \beta \rangle = \langle \alpha | \frac{1}{2} (V_{LS} + V_{L_S}^T) | \beta \rangle$$

This tells us that $V_{\text{okb}}$ can be well approximated by $V_{LS}$, if $V_{LS}$ is only slightly non-Hermitian. In fact in this case $V_{\text{okb}}$ is very accurately reproduced by the simple average ($V_{LS} + V_{L_S}^T$)/2.

We note that the D-state probabilities given in Fig.6 for the various effective interactions are significantly different from that of the bare potential. This is due to the fact that the effective interactions are subjected to a momentum cutoff at 2.0 fm$^{-1}$, while the bare CD Bonn potential extends far beyond that. This can clearly be seen from the figure where a large portion of the wavefunction is simply cut off. While $P_D$ is not an observable, it has theoretical relevance-it is an important characteristic of modern nucleon-nucleon potentials.

According to the MEEFT prescription, it is necessary to impose the cutoff at the limit of experimental data (in this case the limit of NN scattering experiments at 2.0 fm$^{-1}$), and it is at this cutoff that the effective interactions from all the bare potentials become nearly identical. Raising the cutoff would erode this approximate

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**TABLE IV:** A comparison of shell model relative matrix element calculated using $V_{LS}$ with those calculated using the Hermitian interactions. The oscillator length parameter $b$ is given by $b = \sqrt{\frac{\mu}{m}}$, in units of $fm$. The matrix elements are in units of $MeV$.

| $|n\alpha_0\rangle \langle V | n\alpha_0\rangle$ | $n = 0$ | $n = 1$ | $n = 0$ | $n = 1$ |
|---------------------------------------------|--------|--------|--------|--------|
| $V_{LS}$ | $V_{\text{okb}}$ | $V_{\text{cho}}$ | $V_{\text{schm}}$ | $V_{LS}$ | $V_{\text{okb}}$ | $V_{\text{cho}}$ | $V_{\text{schm}}$
| 1.4 | -9.96 | -9.96 | -9.96 | -5.43 | -5.43 | -5.43 | -5.43 |
| 2.0 | -4.85 | -4.85 | -4.85 | -4.40 | -4.40 | -4.40 | -4.40 |
| 2.6 | -2.59 | -2.59 | -2.60 | -2.80 | -2.80 | -2.81 | -2.81 |
| 1.4 | -6.98 | -6.98 | -6.98 | -3.06 | -3.06 | -3.06 | -3.06 |
| 2.0 | -4.48 | -4.48 | -4.48 | -3.83 | -3.83 | -3.83 | -3.83 |
| 2.6 | -2.64 | -2.64 | -2.65 | -2.69 | -2.69 | -2.70 | -2.70 |
| 1.4 | -4.04 | -4.04 | -4.04 | -1.06 | -1.06 | -1.06 | -1.06 |
| 2.0 | -3.80 | -3.80 | -3.80 | -3.13 | -3.13 | -3.13 | -3.13 |
| 2.6 | -2.49 | -2.49 | -2.49 | -2.49 | -2.49 | -2.49 | -2.49 |
| 1.4 | -12.29 | -12.29 | -12.29 | -9.11 | -9.11 | -9.11 | -9.11 |
| 2.0 | -5.51 | -5.51 | -5.51 | -5.70 | -5.70 | -5.70 | -5.70 |
| 2.6 | -2.85 | -2.84 | -2.85 | -3.32 | -3.31 | -3.33 | -3.33 |
| 1.4 | -10.18 | -10.18 | -10.18 | -6.65 | -6.65 | -6.65 | -6.65 |
| 2.0 | -5.46 | -5.46 | -5.46 | -5.33 | -5.33 | -5.33 | -5.33 |
| 2.6 | -3.02 | -3.01 | -3.02 | -3.32 | -3.31 | -3.30 | -3.32 |
uniqueness and model independence. But at the above cutoff, we cannot preserve $P_D$. To illustrate, we refer to Figure 7 where we plot the D-state wavefunctions with a cutoff of $\Lambda = 3.0$ fm$^{-1}$. It shows that much more of the wavefunction is contained in the region below the cutoff, and as a result, the $P_D$’s for the effective interactions are much closer to that of the bare potential. If we increase the cutoff to 4.0 fm$^{-1}$, they are almost exact. How to resolve this disparity in how to choose the cutoff warrants further study.

VI. SUMMARY AND CONCLUSION

We have studied a general method for deriving low-momentum NN interactions which are phase shift equivalent. By integrating out the high momentum components of a realistic NN potential such as the CD-Bonn potential, the Lee-Suzuki or folded-diagram method is employed to derive a parent low-momentum NN potential. This potential preserves the deuteron binding energy and phase shifts up to cutoff scale $\Lambda$. In addition it preserves the half-on-shell T-matrix up to the same scale. This Lee-Suzuki low-momentum NN interaction is not Hermitian, and further transformation is needed to obtain low-momentum interactions which are Hermitian. We have shown how to construct a family of such an interaction using the Schmidt orthogonalization procedure, and we have seen that two existing Hermitization schemes, namely Okubo and Andreozzi, are in fact special cases of our general process. We have shown that all the Hermitian interactions so generated are phase shift equivalent, all reproducing empirical phase shifts up to scale $\Lambda$. These potentials also preserve the deuteron binding energy.

Through an analysis of this procedure using a solvable matrix model we have seen some interesting properties of Schmidt transformation method. In particular, with the entrance of intruder states, the parent potential can become highly non-Hermitian, and that the Hermitian potentials can deviate largely from each other and from the parent potential. It is fortunate that such deviations are not present for low-momentum NN potentials, for cutoff momentum $\Lambda \sim 2.0$ fm$^{-1}$. This is mainly because our parent $V_{low-k}$ is only slightly non-Hermitian, and as a result, the Hermitian low-momentum nucleon interactions generated from our orthogonalization procedure are all close to each other and close to the parent $V_{low-k}$. Shell model matrix elements of the low-momentum nucleon interactions are found to be approximately independent of the starting Hermitian potential, indicating the usefulness of our procedure in nuclear many body calculations.

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