Almost representations and asymptotic representations of discrete groups

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Abstract

We define for discrete finitely presented groups a new property related to their asymptotic representations. Namely we say that a group has the property AGA if every almost representation generates an asymptotic representation. We give examples of groups with and without this property. For our example of a group Γ without AGA the group $K^0(BΓ)$ cannot be covered by asymptotic representations of Γ.

One of the reasons of attention to almost and asymptotic representations of discrete groups [5, 2] is their relation to $K$-theory of classifying spaces [2, 9]. It was shown in [9] that in the case of finite-dimensional classifying space $BΓ$ in order to construct a vector bundle over it out of an asymptotic representation of Γ it is sufficient to have an $ε$-almost representation of Γ with small enough $ε$. Of course an $ε$-almost representation contains less information than the whole asymptotic representation, but it turns out that often the information contained in an $ε$-almost representation makes it possible to construct the corresponding asymptotic representation. In the present paper we give the definition of this property, prove this property for some classes of groups and finally give an example of a group without this property. We discuss also this example in relation with its $K$-theory.

1 Basic definitions

Let $Γ$ be a finitely presented discrete group, and let $Γ = ⟨F|R⟩ = ⟨g_1, . . . , g_n| r_1, . . . , r_k⟩$ be its presentation with $g_i$ being generators and $r_j = r_j(g_1, . . . , g_n)$ being relations. We assume that the set $F = \{g_1, . . . , g_n\}$ is symmetric, i.e. for every $g_i$ it contains $g_i^{-1}$ too, and the set $R$ of relations contains relations of the form $g_i g_i^{-1}$, though we usually will skip these additional generators and relations.

By $U_∞$ we denote the direct limit of the groups $U_n$ with respect to the natural inclusion $U_n → U_{n+1}$ supplied with the standard operator norm. The unit matrix we denote by $I \in U_∞$.

**Definition 1** A set of unitaries $σ = \{u_1, . . . , u_n\} ⊂ U_∞$ is called an $ε$-almost representation of the group $Γ$ if after substitution of $u_i$ instead of $g_i$, $i = 1, . . . , n$, into $r_j$ one has

$$\|r_j(u_1, . . . , u_n) - I\| \leq ε$$

for all $j = 1, . . . , k$. 
In this case we write $\sigma(g_i) = u_i$. Remark that this definition depends on a choice of presentation of the group $\Gamma$, but we will see that this dependence is not important. Let $\langle h_1, \ldots, h_m | s_1, \ldots, s_l \rangle$ be another presentation of $\Gamma$. For an $\varepsilon$-almost representation $\sigma$ with respect to the first presentation we can define the set of unitaries $v_1, \ldots, v_m \in U_\infty$, $v_i = \sigma(h_i)$ putting $\sigma(h_i) = \sigma(g_{j_1}) \cdots \sigma(g_{j_{n_i}})$, where $h_i = g_{j_1} \cdots g_{j_{n_i}}$. By the same way starting from the set $\sigma(h_i)$ we can construct the set $\sigma(g_i)$.

**Lemma 2** There exist constants $C$ and $D$ (depending on the two presentations) such that $\sigma$ is a $C \varepsilon$-almost representation with respect to the second presentation of $\Gamma$ and for all $g_i$, $i = 1, \ldots, n$, one has $\|\sigma(g_i) - \sigma(g_i)\| \leq D \varepsilon$.

**Proof.** We have to estimate the norms $\|s_q(v_1, \ldots, v_m) - I\|$, $q = 1, \ldots, l$. To do so notice that every relation $s_q$ can be written in the form

$$s_q = a_1^{-1} r_{j_1} \epsilon_1 a_1 \cdots a_m^{-1} r_{j_m} \epsilon_m a_m q$$

(1)

for some $a_1 \in \Gamma$, where $\epsilon_i = \pm 1$. Let $M$ be the maximal length of the words $a_i = a_i(g_1, \ldots, g_n)$. Put $b_i' = a_i^{-1}(g_1, \ldots, g_n) \in U_\infty$, $b_i = a_i(g_1, \ldots, g_n) \in U_\infty$. Then one has

$\|b_i' b_i - I\| \leq M \varepsilon$.

It follows from (1) that

$$s_q(v_1, \ldots, v_m) = b_1' r_{j_1}(u_1, \ldots, u_n) b_1 \cdots b_m' r_{j_m}(u_1, \ldots, u_n) b_m,$$

but as for every $i$ one has

$$\|b_i' r_{j_i}(u_1, \ldots, u_n) b_i - I\| \leq \|b_i' b_i - I\| + \|r_{j_i}(u_1, \ldots, u_n) - I\| \leq (M + 1) \varepsilon,$$

so

$$\|s_q(v_1, \ldots, v_m) - I\| \leq m_q(M + 1) \varepsilon,$$

which proves the first statement of the Lemma. The second statement is proved in a similar way. $\square$

As the number of generators is finite, so the image of every almost representation lies in finite matrices, $\sigma \in U_n$ for some $n$. The minimal such $n$ is called a dimension of $\sigma$. Usually we will ignore the remaining infinite unital tail of the matrices $\sigma(g_i)$ and write $\sigma(g_i) \in U_n$ instead of $U_\infty$.

The set of all $\varepsilon$-almost representations of the group $\Gamma$ we denote by $R_\varepsilon(\Gamma)$. The deviation of an almost representation can be measured by the value

$$\|\sigma\| = \max_j \|r_j(u_1, \ldots, u_n) - I\|.$$

Notice that both these definitions also depend on the choice of a presentation of $\Gamma$.

**Definition 3** (cf. [2]) A set of norm-continuous unitary paths $\sigma_t = \{u_1(t), \ldots, u_n(t)\} \subset U_\infty$, $t \in [0, \infty)$, is called an asymptotic representation of the group $\Gamma$ if $\|\sigma_t\|$ tends to zero when $t \to \infty$. 

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Due to the Lemma 2, this definition does not depend on the choice of a presentation of $\Gamma$. The set of all asymptotic representations of the group $\Gamma$ we denote by $R_{\text{asym}}(\Gamma)$.

Now we are ready to define a new property of finitely generated groups which we call AGA (Almost representations Generate Asymptotic representations).

**Definition 4** A group $\Gamma$ possesses the property AGA if for every $\varepsilon > 0$ one can find a number $\delta(\varepsilon)$ (with the property $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$) such that for every almost representation $\sigma \in R_\varepsilon(\Gamma)$ there exists an asymptotic representation $\sigma_t \in R_{\text{asym}}(\Gamma)$ such that $\sigma_0 = \sigma$ and $\|\sigma_t\| \leq \delta(\varepsilon)$ for all $t \in [0, \infty)$.

**Lemma 5** The property AGA does not depend on the choice of a presentation of the group.

**Proof** immediately follows from the Lemma 2. \qed

**Theorem 6** The following groups have the property AGA:

i) free groups,

ii) free products of groups having the property AGA,

iii) subgroups of finite index in groups having the property AGA,

iv) finite groups,

v) free abelian groups,

vi) fundamental groups of two-dimensional oriented manifolds.

**Proof.** The first item is obvious — free groups have no relations, so every almost representation is a genuine representation. The same argument works for the second item too. The third item can be proved in the same way as the Lemma 2 as subgroups of finite index are defined by a finite number of relations originated from relations of the group. The fourth item was proved in [3] — for finite groups there exists a genuine representation close to every almost representation. The fifth item is non-trivial too and was proved in [7]. It follows also from the Super Homotopy Lemma of [1]. The sixth item we prove in the next section.

2 Case of fundamental groups of oriented surfaces

Let $\Gamma = \langle a_1, b_1, \ldots, a_m, b_m | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_m b_m a_m^{-1} b_m^{-1} \rangle$. Let $\sigma$ be an $\varepsilon$-almost representation of $\Gamma$, $u_i = \sigma(a_i)$, $v_i = \sigma(b_i)$, $u_i, v_i \in U_n$, $i = 1, \ldots, m$. Denote $\gamma(u, v) = uvu^{-1}v^{-1}$, then we have

$$\|\gamma(u_1, v_1) \cdots \gamma(u_m, v_m) - I\| \leq \varepsilon.$$ 

Consider the map

$$\gamma : U_n \times U_n \to SU_n.$$  

(2)

To prove the property AGA for fundamental groups of oriented two-dimensional manifolds we have to use the following elementary statement about the map (2).
Lemma 7 Let \((u_0, v_0) \in U_n \times U_n\) and let \(c(t) \in SU_n\), \(t \in [0, 1]\), be a path such that \(\gamma(u_0, v_0) = c(0)\). Then for any \(\delta > 0\) there exists a path \((u_t, v_t) \in U_n \times U_n\) such that 
\[\|\gamma(u_t, v_t) - c(t)\| < \delta.\]

Proof. Remember that a pair \((u, v) \in U_n \times U_n\) is called irreducible if there is no common invariant subspace for \(u\) and \(v\). It was shown in [4] that the set of regular points for the map \(\gamma\) coincides with the set of irreducible pairs. Denote the set of reducible pairs \((u, v) \in U_n \times U_n\) by \(S\). For any \(k = 1, \ldots, n - 1\) by \(\Sigma_k \subset SU_n\) denote the set of block-diagonal matrices \(\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}\) with respect to some invariant subspace \(V\), \(\dim V = k\), such that \(c_1 \in SU_k\), \(c_2 \in SU_{n-k}\). Put \(\Sigma = \cup_k \Sigma_k \subset SU_n\). Then obviously \(\gamma(S) \subset \Sigma\). Notice that every \(\Sigma_k\) is a submanifold in \(SU_n\) with codimension one. So \(\Sigma\) divides \(SU_n\) into a finite set of closed path components \(M_j\), \(\cup_j M_j = SU_n\) and for every point \(c \in M_j\) the set \(\gamma^{-1}(c)\) consists only of regular points. Hence every path in \(M_j\) transversal to its boundary can be lifted up to a path in \(U_n \times U_n\) with a fixed starting point.

Without loss of generality we can assume that the path \(c(t)\) is transversal to every \(\Sigma_k\). Let \(t_0 \in \{c(t)\} \cap \Sigma_k\). It remains to show that we can lift the path \(c(t)\) in some neighborhood of the point \(c_0 = c(t_0)\). Let \((u_0, v_0) \in U_n \times U_n\) be such point that \(\gamma(u_0, v_0) = c_0\). If the point \((u_0, v_0)\) is a regular point then the statement is obvious. Otherwise we can write

\[u_0 = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \quad v_0 = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}\]

with respect to some basis and we can assume that matrices \(v_1\) and \(v_2\) are diagonal. Let \(e^{2\pi i \varphi_1}, \ldots, e^{2\pi i \varphi_k}\) and \(e^{2\pi i \varphi_{k+1}}, \ldots, e^{2\pi i \varphi_n}\) be the eigenvalues of \(v_1\) and \(v_2\) respectively. Slightly changing \(v_0\) we can assume that for all \(i = 2, \ldots, k, j = k+2, \ldots, n\) the values \(\varphi_i - \varphi_1, \varphi_j - \varphi_{k+1}\) differ from each other. Multiplying \(v_1\) by \(e^{-2\pi i \epsilon_1 t}\) and \(v_2\) by \(e^{-2\pi i \epsilon_{k+1} t}\), \(t \in [0, 1]\), we connect the matrix \(v_0\) with the matrix

\[v'_0 = \begin{pmatrix} v'_1 & 0 \\ 0 & v'_2 \end{pmatrix} = \begin{pmatrix} e^{-2\pi i \epsilon_1 v_1} & 0 \\ 0 & e^{-2\pi i \epsilon_{k+1} v_2} \end{pmatrix}\]

which has two eigenvalues equal to one and all other eigenvalues being different from each other. Obviously the value \(\gamma(u_0, v_0) = \gamma(u_0, v'_0) = c_0\) does not change along this path. Denote by \(e_1, \ldots, e_n\) the basis consisting of the eigenvalues of \(v'_0\) and let \(w(t) \in U_n\), \(t \in [0, 1]\), be a rotation of the vectors \(e_1\) and \(e_{k+1}\):

\[w(t)e_1 = \cos t e_1 - \sin t e_{k+1}, \quad w(t)e_{k+1} = \sin t e_1 + \cos t e_{k+1}, \quad w(t)e_j = e_j \quad \text{for} \quad j \neq 1, k+1.\]

Obviously \(w(t)\) commutes with \(v'_0\). Put \(u_t = u_0w(t)\). Then

\[\gamma(u_t, v'_0) = u_0w(t)v'_0w^{-1}(t)u_t^{-1}(v'_0)^{-1} = \gamma(u_0, v'_0) = c_0\]

and for \(\sin t \neq 0\) the pair \((u_t, v'_0)\) is irreducible (since \(v'_0\) is diagonal with only two coinciding eigenvalues, so its invariant subspaces are easy to describe, then it is easy to check that they are not invariant under the action of \(u_t\)) with the same value of \(\gamma\). Then it is possible to extend the path \((u_t, v_t)\) through the point \(c_0\). \(\Box\)
Proposition 8 Any $\varepsilon$-almost representation of $\Gamma$ is homotopically equivalent in $R_{2\varepsilon}(\Gamma)$ to an $2\varepsilon$-almost representation with $u_2 = v_2 = \ldots = u_m = v_m = I$.

Proof. Connect the matrix $\gamma(u_m, v_m)$ with $I$ by a path $c_m(t)$. Then by the Lemma we can find a path $(u_m(t), v_m(t)) \in U_n \times U_n$ such that
\[\|\gamma(u_m(t), v_m(t)) - c_m(t)\| \leq \frac{\varepsilon}{2m}.\]

Notice that the set $\gamma^{-1}(I) = \{(u, v) : uv = vu\}$ is path-connected, so we can assume that the end point of the path $(u_m(t), v_m(t))$ is $(I, I)$. Put
\[c_{m-1}(t) = \gamma(u_{m-1}, v_{m-1})c_m^{-1}(t).\]

Again by the Lemma we can find a path $(u_{m-1}(t), v_{m-1}(t)) \in U_n \times U_n$ such that
\[\|\gamma(u_{m-1}(t), v_{m-1}(t)) - c_{m-1}(t)\| \leq \frac{\varepsilon}{2m}.\]

Then
\[\|\gamma(u_1, v_1) \cdot \ldots \cdot \gamma(u_{m-1}(t), v_{m-1}(t)) \cdot \gamma(u_m(t), v_m(t)) - I\| \leq \varepsilon + \frac{\varepsilon}{m}\]
and at the end point we have $(u_m(t), v_m(t)) = (I, I)$. Proceeding by induction we finish the proof. \qed

It now follows from the proposition that the property AGA for the group $\Gamma$ follows from the same property for the group $\mathbb{Z}^2$ with generators $u_1, v_1$. \qed

3 Example of a group without AGA

Consider the group $\Gamma = \langle a, b, c | acc^{-1}b^{-1}, (ab)^2 \rangle$.

Theorem 9 The group $\Gamma$ does not possess the property AGA.

Proof. Put $\omega = e^{2\pi i/n}$ and define a family of almost representations $\sigma_n$ taking values in $U_n$ by
\[
\sigma_n(a) = \begin{pmatrix} \omega & 0 & \ldots & 0 \\ \omega^2 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \omega^n & \ldots & 1 & \omega \\ \end{pmatrix},
\sigma_n(c) = \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \ldots & 0 & 1 \\ \end{pmatrix},
\sigma_n(b) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & 1 \\ \end{pmatrix}.
\]

Here the matrices $\sigma_n(a)$ and $\sigma_n(c)$ are the Voiculescu matrices with the winding number equal to one, and one has
\[\sigma_n(a)\sigma_n(c)\sigma_n(a)^{-1}\sigma_n(c)^{-1} = \omega \cdot I, \quad \sigma_n(b)^2 = I, \quad (\sigma_n(a)\sigma_n(b))^2 = \omega \cdot I,\]
\[ \varepsilon_n = \| \sigma_n \| = | \omega - 1 | \to 0 \text{ when } n \to \infty, \]

hence for every \( \varepsilon > 0 \) there exists an \( \varepsilon \)-almost representation \( \sigma \) of \( \Gamma \) such that the winding number of the pair \( (\sigma(a), \sigma(c)) \) equals one.

Suppose the opposite, i.e. that the group \( \Gamma \) has AGA. Then there should be such \( \varepsilon \) that the number \( \delta(\varepsilon) \leq 1 \). Take such \( \varepsilon \) and an \( \varepsilon \)-almost representation \( \sigma_0 \) with a non-zero winding number of the pair \( (\sigma_0(a), \sigma_0(c)) \).

By supposition there exists an asymptotic representation \( \sigma_t \in R_{\text{asyg}}(\Gamma) \) extending \( \sigma_0 \) such that \( \| \sigma_t \| \leq 1 \) for all \( t \in [0, \infty) \) and for any \( \varepsilon' > 0 \) there exists \( t_0 \) such that \( \| \sigma_{t_0} \| \leq \varepsilon' \). Fix this \( t_0 \) and denote \( \sigma_{t_0} \) by \( \sigma \). Let \( n \) and \( n + m \) be the dimension of the almost representation \( \sigma_0 \) and \( \sigma \) respectively. Then one has

\[ \| \sigma(a) \sigma(c) - \sigma(c) \sigma(a) \| \leq \varepsilon', \quad \| \sigma(b) - I \| \leq \varepsilon', \quad \| \sigma(a) \sigma(b) \sigma(a) \sigma(b) - I \| \leq \varepsilon'. \]

Notice that as along the whole path \( \sigma_t \) one has

\[ \| \sigma_t(b)^2 - I \| \leq \| \sigma_t \| \leq \delta(\varepsilon) \leq 1, \]

so the eigenvalues of \( \sigma_t(b) \) satisfy the estimate \( | \lambda^2 - 1 | \leq 1 \), hence the number of eigenvalues \( \lambda \in \text{Sp} \sigma_t(b) \) with \( \text{Re} \lambda < 0 \) does not change along the whole path \( \sigma_t \) in \( U_\infty \), therefore the number of eigenvalues \( \lambda \in \text{Sp} \sigma(b) \) with \( | \lambda + 1 | \leq \varepsilon' \) cannot exceed \( n \) (the maximal number of eigenvalues with \( \text{Re} \lambda < 0 \) of \( \sigma_0(b) \)) and the number of eigenvalues with \( | \lambda - 1 | \leq \varepsilon' \) is not less than \( m \). Then there exists a matrix \( \sigma(b)' \in U_{n+m} \) such that \( \| \sigma(b) - \sigma(b)' \| \leq \varepsilon' \) and that the matrix \( \sigma(b)' \) has not more than \( n \) eigenvalues equal to \(-1\) and not less than \( m \) eigenvalues equal to 1. Then we have

\[ \| \sigma(a) \sigma(b)' \sigma(a) \sigma(b)' - I \| \leq 3\varepsilon'. \] (3)

Notice that \( (\sigma(b)')^2 = I \) and

\[ | \text{tr}(\sigma(b)') | \geq m - n. \] (4)

Let

\[ \sigma(a) = \begin{pmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_{n+m} \end{pmatrix} \]

be the matrix of the operator \( \sigma(a) \) in the basis consisting of its eigenvectors. It was shown in [3] that if the winding number of the pair \( (\sigma(a), \sigma(c)) \) is non-zero then for any \( \varepsilon' > 0 \) there exists \( \delta'(\varepsilon') \) such that \( \delta'(\varepsilon') \to 0 \) when \( \varepsilon' \to 0 \) and that all lacunae in \( \text{Sp} \sigma(a) \) do not exceed \( \delta'(\varepsilon') \). Denote the number of eigenvalues of \( \sigma(a) \) with \( | \text{Im} \omega_j | > 2\varepsilon' \) by \( N \). Then we have

\[ N \geq \frac{2\pi - 10\varepsilon'}{\delta'(\varepsilon')} \] (5)

As \( (\sigma(b)')^2 = I \), so it follows from [3] that

\[ \| \sigma(a) \sigma(b)' - \sigma(b)' \sigma(a) \| \leq 3\varepsilon'. \] (6)
Denote by $b_{ij}$ the matrix elements of the matrix $\sigma(b)'$. It follows from (3) that all matrix elements of $\sigma(a)\sigma(b)' - \sigma(b)'\sigma(a)^*$ do not exceed $3\varepsilon'$, i.e.

$$|b_{ii}(\omega_i - \overline{\omega}_i)| \leq 3\varepsilon', \quad i = 1, \ldots, n + m. \quad (7)$$

Let us estimate $\text{tr}(\sigma(b)')$. We have

$$|\text{tr}(\sigma(b)')| = \left| \sum_{i=1}^{n+m} b_{ii} \right| \leq \sum_{i=1}^{n+m} |b_{ii}| = \sum' |b_{ii}| + \sum'' |b_{ii}|,$$

where $\sum'$ denotes the sum for those numbers $i$ for which one has $|\text{Im} \omega_i| > 2\varepsilon'$ and $\sum''$ is the sum for the remaining numbers. As for all $i$ one has $|b_{ii}| \leq 1$, so the last sum do not exceed the number of summands,

$$\sum'' |b_{ii}| \leq n + m - N.$$

It follows from (3) that for those $i$ which are included into the first sum we have $|\omega_i - \overline{\omega}_i| > 4\varepsilon'$, hence those $b_{ii}$ satisfy

$$|b_{ii}| < \frac{3}{4},$$

so

$$\sum' |b_{ii}| < \frac{3}{4}N,$$

hence we have

$$|\text{tr}(\sigma(b)')| < \frac{3}{4}N + n + m - N = n + m - \frac{N}{4}$$

and it follows from (3) that

$$|\text{tr}(\sigma(b)')| < n + m - \frac{\pi - 5\varepsilon'}{2\delta'(\varepsilon')}. \quad (8)$$

If we take $\varepsilon'$ small enough then $\delta'(\varepsilon')$ is small enough too and we get $\frac{\pi - 5\varepsilon'}{2\delta'(\varepsilon')} > 2n$, then (3) and (5) give a contradiction. \Box

**Corollary 10** Let $\sigma_t \in R_{\text{asym}}(\Gamma)$ be an asymptotic representation. Then the winding number of the pair $(\sigma_t(a), \sigma_t(c))$ is zero for big enough $t$. In particular, it means that $\sigma_t$ is homotopic to an asymptotic representation $\rho_t \in R_{\text{asym}}(\Gamma)$ with $\rho_t(c) = I$ in the class of asymptotic representations. \Box

Denote the Grothendieck group of homotopy classes of asymptotic representations of the group $\Gamma$ by $R_{\text{asym}}(\Gamma)$. Let $H$ denote the subgroup $\langle a, b | b^2, (ab)^2 \rangle \cong \mathbb{Z}_2 \ast \mathbb{Z}_2$. Then $\Gamma \cong \mathbb{Z}^2 \ast \mathbb{Z} H$, so we have $B\Gamma = T^2 \cup BH$, $S^1 = T^2 \cap BH$, where $B\Gamma$, $BH$, $S^1$ and $T^2$ are the classifying spaces of the groups $\Gamma$, $H$, $\mathbb{Z}$ and $\mathbb{Z}^2$ respectively, and the inclusion $S^1 \subset T^2$ is the standard inclusion onto the first coordinate. Then one has an exact sequence

$$K^0(B\Gamma) \longrightarrow K^0(BH) \oplus K^0(T^2) \longrightarrow K^0(S^1) \quad (9)$$

$$K^1(S^1) \leftarrow K^1(BH) \oplus K^1(T^2) \leftarrow K^1(B\Gamma)$$
and as the maps $K^*(T^2)\to K^*(S^1)$ are onto, so the vertical maps in (3) are zero and the group $K^0(B\Gamma)$ contains an element $\beta$ which is mapped onto the Bott generator of $K^0(T^2)$. Remember that in (3) a map
\[ R_{\text{asym}}(\Gamma) \to K^0(B\Gamma) \]
was constructed, which factorizes (8) through the assembly map
\[ R_Q(\Gamma \times \mathbb{Z}) \to K^0(B\Gamma), \]
where $R_Q(\Gamma)$ denotes the Grothendieck group of representations of $\Gamma$ into the Calkin algebra $Q$.

It follows from the Corollary 10 that $R_{\text{asym}}(\Gamma) = R_{\text{asym}}(H)$, therefore the element $\beta \in K^0(B\Gamma)$ does not lie in the image of the map (10), hence we obtain

**Corollary 11** The map (10) is not a rational epimorphism for the group $\Gamma$.  

On the other hand we should remark that the element $\beta \in K^0(B\Gamma)$ can be obtained as an image of a representation of $\Gamma \times \mathbb{Z}$ into the Calkin algebra $Q$. To get such representation we can take $\sigma(b)$ to be the infinite direct sum of matrices of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and to rearrange the basis for the matrices $\sigma_n(a)$ and $\sigma_n(c)$ in such a way that $\sigma_n(a)$ and $\sigma(b)$ would almost commute. We should also insert a number of intermediate matrices between $\sigma_n$ and $\sigma_{n+1}$ as it was done in (3). Denote by $H_n$ the Hilbert space where the matrices $\sigma_n(a)$, $\sigma_n(c)$ and $\sigma(b)$ act and put $H = \oplus_n H_n$ (for negative $n$ put $\sigma_n(a) = \sigma_n(c) = I$). Let $F$ be a shift on $H$. Then the matrices $\oplus_n \sigma_n(a)$, $\oplus_n \sigma_n(c)$, $\oplus_n \sigma(b)$ and $F$ generate a representation of $\Gamma \times \mathbb{Z}$ into $Q$ with necessary property.

Remark that in our example the absence of the AGA property is related to torsion. It would be interesting to know whether torsion-free groups always have AGA.

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