Phase fluctuations in low-dimensional Gross-Neveu models

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We consider the Gross-Neveu model with a continuous chiral symmetry in two and three space-time dimensions at zero and finite temperature. In order to study long-range phase coherence, we construct an effective low-energy Lagrangian for the phase $\theta$. This effective Lagrangian is used to show that the fermionic two-particle correlation function at finite temperature decays algebraically in 2+1 dimensions and exponentially in 1+1 dimensions.

**Introduction.** Low-dimensional field theories have been studied extensively as toy models for QCD at zero and finite temperature. Many of these theories are remarkably rich, sharing a number of properties with four-dimensional QCD. For example, both the Gross-Neveu model and the $O(N)$ nonlinear sigma model in 1+1 dimensions are asymptotically free and have nonperturbatively generated mass gaps. In addition, the latter has instanton solutions for $N = 3$. In contrast with QCD in 3+1 dimensions, these models are renormalizable in the $1/N$ expansion both in 1+1 and 2+1 dimensions. Thus one can systematically study these models in a nonperturbative setting [1, 2]. There is, however, an important difference between QCD and low-dimensional field theories, namely the possibility of spontaneous breaking of continuous global symmetries. In QCD with $N_f$ massless fermions, chiral symmetry is spontaneously broken at zero temperature and this gives rise to $N_f^2 - 1$ massless Goldstone particles. For $N_f = 2$, these are the well known pions. Chiral symmetry is expected to be restored at a critical temperature of approximately 150 MeV depending on the number of flavors. In 1+1 and 2+1 dimensions, the Mermin-Wagner theorem [3, 4] forbids the spontaneous breakdown of continuous symmetries at any finite temperature [1]. The reason is that the phase fluctuations are so strong that they destroy the presence of a condensate. The role of phase fluctuations was studied in the seminal papers on the Kosterlitz-Thouless phase transition of vortex unbinding [3, 5]. The low-temperature phase where the vortices are bound in pairs is characterized by an algebraic decay of the one-particle density matrix and superfluidity. The high-temperature phase of unbound vortices is characterized by loss of superfluidity and exponential fall-off of correlations. More recently, phase fluctuations have been studied in connection with Bose-Einstein condensation of atomic gases in harmonic traps in one - and two-dimensional Bose gases [6, 7, 8, 9, 10]. In the following, we consider the Gross-Neveu model with a continuous chiral symmetry [11]. Many of its properties have been examined in detail [12, 13, 14, 15]. In the present paper we discuss the possible phases of the theory. By using density and phase variables, one can conveniently address the issues of long-range order and phase structure as a function of temperature. These issues were discussed in Refs. [12, 13]; here we present a somewhat more general treatment at zero and finite temperature.

**Gross-Neveu model.** The Euclidean Lagrangian of the Gross-Neveu model with a $U(1)$ symmetry is [11]

$$\mathcal{L} = \bar{\psi}_j \partial_\tau \psi_j - \frac{g^2}{2N} \left( (\bar{\psi}_j \psi_j)^2 - (\bar{\psi}_j \gamma_5 \psi_j)^2 \right), \quad (1)$$

where $j = 1, 2, ..., N$. In 2+1 dimensions, the $\gamma$-matrices are $4 \times 4$ matrices and in 1+1 dimensions they are $2 \times 2$ matrices [2]. They satisfy $[\gamma_i, \gamma_j] = 2\delta_{ij}$ and $\gamma_5 = -i\gamma_0\gamma_1$. The Lagrangian [11] has a $U(1)$ chiral symmetry:

$$\psi_j \rightarrow e^{i\phi} \psi_j,$$  \hspace{1cm} (2)

where $\phi$ is a constant phase. Following [11], we introduce the auxiliary fields $\sigma = \frac{2}{N} \bar{\psi}_j \gamma_5 \psi_j$ and $\pi = \frac{2}{N} \bar{\psi}_j \gamma_0 \gamma_1 \psi_j$. The Lagrangian can then be written as

$$\mathcal{L} = \bar{\psi}_j \partial_\tau \psi_j + \frac{N}{2g^2} \left( \sigma^2 + \pi^2 \right) - \bar{\psi}_j \left( \sigma + i\pi \gamma_5 \right) \psi_j. \quad (3)$$

In terms of the fields $\sigma$ and $\pi$, chiral symmetry can be written as

$$(\sigma + i\pi) \rightarrow (\sigma + i\pi) e^{2i\phi}. \quad (4)$$

Defining the density $\rho$ and phase $\theta$ variables by

$$\sigma + i\pi = \rho e^{i\theta}, \quad (5)$$

and the new fermion field $\chi$ by

$$\chi_L = e^{-i\theta/2} \psi_L, \hspace{1cm} \chi_R = e^{i\theta/2} \psi_R, \quad (6)$$

we can write the Lagrangian [11] as

$$\mathcal{L} = \bar{\chi} \left( \partial + \frac{1}{2} \gamma_5 \partial_\theta + \rho \right) \chi + \frac{N}{2g^2} \rho^2. \quad (7)$$

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1 In 2+1 dimensions, spontaneous breaking of continuous symmetries may occur at $T = 0$. Consequently, the critical temperature for the restoration of a continuous symmetry is $T_c = 0$.**
In terms of $\rho$ and $\theta$, the chiral symmetry is simply $\theta \to \theta + c$, where $c$ is a constant. In these variables the chiral symmetry is manifest. Moreover, the operator $\partial_\theta$ acting on $\theta$ in $\Pi_{\rho}(Q)$ ensures that the field is massless and that it is derivatively coupled. The auxiliary field $\rho$ is now written as a sum of a space-time independent background $\rho_0$ and a quantum fluctuating field $\tilde{\rho}$; $\rho = \rho_0 + \tilde{\rho}$. Note that a nonzero expectation value for $\rho$ does not violate chiral symmetry since $\rho$ is invariant under global $U(1)$ transformations. Since the Lagrangian is quadratic in the fermion fields, we can integrate over $\chi$, and we obtain the effective action for $\tilde{\rho}$ and $\theta$:

$$S = -N \text{Tr} \ln \left[ \Theta_0 + i \gamma_5 \partial_\theta + \rho_0 + \tilde{\rho} \right] + \frac{N}{2g^2} \int_0^\beta d\tau \int d^{d-1}x \left( (\rho_0 + \tilde{\rho})^2 \right), \quad (8)$$

where $\beta = 1/T$ is the temperature (in units where $\hbar = 1$) and $d - 1$ is the dimension of space. The trace is over both Dirac indices and spacetime. The next step is to expand the functional determinant around the classical solution of the equation of motion $\theta = 0$ and $\tilde{\rho} = 0$. This expansion generates propagators for $\tilde{\rho}$ and $\theta$ as well as interaction vertices among them.

At leading order in $1/N$, the auxiliary fields $\rho$ and $\theta$ do not propagate. However, at next-to-leading order, quantum fluctuations induce the inverse propagators. These are obtained from expanding the functional determinant in $\Pi_{\rho}(Q)$ to second order in the fluctuations:

$$\Pi_{\rho}(Q) = \frac{N}{g^2} - \text{Tr}(I) \sum_{\{P\}} \frac{1}{P^2 + \rho_0^2} + \frac{1}{2} \text{Tr}(I) \sum_{\{P\}} \frac{4\rho_0^2 + Q^2}{P^2 + \rho_0^2} \frac{1}{(P + Q)^2 + \rho_0^2}, \quad (9)$$

$$\Pi_{\theta}(Q) = \frac{1}{2} \text{Tr}(I) \sum_{\{P\}} \frac{Q^2}{P^2 + \rho_0^2} \frac{1}{(P + Q)^2 + \rho_0^2}, \quad (10)$$

where $\text{Tr}(I)$ is the trace of the identity matrix $I$. We have introduced the notation

$$\sum_{\{P\}} \equiv T \sum_{P_0 = 2(n+1)\pi T} \int \frac{d^{d-1}p}{(2\pi)^{d-1}}, \quad (11)$$

and $P = (P_0, \mathbf{p})$ is the Euclidean momentum. The integral over spatial momenta is regularized by using an ultraviolet cutoff $\Lambda$. Eq. (10) is ultraviolet divergent and the divergence can be eliminated by renormalizing the coupling constant $g$. By analytic continuation to Minkowski space ($P_0 \to i\omega + i\eta$), one can examine the analytic structure of the propagators for $\rho$ and $\theta$. For example, at zero temperature, both propagators have a branch cut starting at $\omega = \sqrt{q^2 + 4\rho_0^2}$. The $\theta$ propagator has in addition a pole at $\omega = q$, which ensures that the field is massless. This pole remains at finite temperature, which follows immediately from Eq. (10).

The effective potential $\mathcal{V}$ through leading order in $1/N$ is

$$\mathcal{V} = \frac{N\rho_0^2}{2g^2} - \frac{1}{2} \text{Tr}(I) \sum_{\{P\}} \ln \left[ P^2 + \rho_0^2 \right]. \quad (12)$$

The value of $\rho_0$ is determined by minimizing the effective potential and to leading order in the $1/N$ expansion this leads to the gap equation

$$\frac{1}{g^2} = \text{Tr}(I) \sum_{\{P\}} \frac{1}{P^2 + \rho_0^2}. \quad (13)$$

This gap equation coincides with the gap equation in Gross-Neveu model with a discrete $Z_2$ symmetry. Eq. (13) is ultraviolet divergent and it is rendered finite by renormalizing the coupling constant $g$. At $T = 0$, the gap equation can be solved explicitly for $\rho_0$. For example, in $1+1$ dimension, we obtain $\rho_0 = \Lambda^2 e^{-2\pi g^2/\pi}$, where $g_{\text{re}}$ is the renormalized coupling and $\Lambda$ is the ultraviolet cutoff. Finally, note that one can also use the gap equation (13) to eliminate the divergent sum-integral in the expression for the $\rho$-propagator.

If we are interested in the long-distance properties of the theory, we can simplify calculations by noticing that for momenta $p$ much smaller than $\rho_0$, the massive field $\rho$ decouples and we are left with an effective long-distance theory for the phase $\theta$. To leading order in derivatives, this can be written as a free field theory:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \alpha (\partial_\theta)^2, \quad (14)$$

where $\alpha$ is the stiffness of the phase fluctuations and is given by

$$\alpha = \text{Tr}(I) \sum_{\{P\}} \frac{1}{P^2 + \rho_0^2}. \quad (15)$$

The sumnation in (15) is over the bosonic Matsubara modes $P_0 = 2\pi n T$. The expression for $\alpha$ follows immediately from (10). Chiral symmetry, $\theta \to \theta + c$, forbids terms in the effective Lagrangian not involving derivatives of $\theta$ such as $\theta^4$ and $\theta^2 (\nabla \theta)^2$. Thus the next operator in the low-energy expansion involves four derivatives and the results concerning the infrared properties of the Gross-Neveu model obtained using (14) are stable against perturbations (12).

In order to discuss the possible phases of the Gross-Neveu model, we must examine the long-distance behav-
ior of the following four-fermion correlation function \[12\]

\[
\rho_2(x, 0) = \langle \bar{\psi}(x)(1 + \gamma_5)\psi(x)\bar{\psi}(0)(1 - \gamma_5)\psi(0) \rangle .
\] (16)

The function \(\rho_2(x, 0)\) can be written as

\[
\rho_2(x, 0) = \left(\rho(x)e^{-i\theta(x)}\rho(0)e^{i\theta(0)}\right) \approx \rho_0^2e^{-\frac{\pi}{2}(\theta(x)-\theta(0))^2},
\] (17)

where we in the second line have replaced \(\rho\) by its expectation value \(\rho_0\) and used Wick’s theorem to rewrite the exponent. This replacement is justified since density fluctuations are negligible for long distances. Using the effective Lagrangian \[14\], the exponent \(18\) can be written as

\[
\langle [\theta(x) - \theta(0)]^2 \rangle = \frac{1}{\alpha} \int \frac{dp^{d-1}}{(2\pi)^{d-1}} \coth \left( \frac{\beta p}{2} \right) \times \left[ 1 - \cos(p \cdot x) \right].
\] (18)

The behavior of \(18\) in the limit \(x \to \infty\) then determines the possible phases of the Gross-Neveu model.

2+1 dimensions. We first notice that the gap equation \[13\] and the stiffness \(15\) can be calculated explicitly:

\[
\frac{1}{g_R^2} = -\frac{\rho_0}{\pi} - \frac{2T}{\pi} \ln \left[ 1 + e^{-\beta \rho_0} \right],
\] (19)

\[
\alpha = \frac{N}{2\pi} \rho_0 \tanh \frac{\beta \rho_0}{2}.
\] (20)

Averaging over the angle between \(p\) and \(x\) in \(15\), we obtain

\[
\langle [\theta(x) - \theta(0)]^2 \rangle = \frac{1}{2\pi \alpha} \int_0^\infty dp \coth \left( \frac{\beta p}{2} \right) \left[ 1 - J_0(px) \right],
\] (21)

where \(J_0(px)\) is a Bessel function of the first kind. At \(T = 0\), the expression \(21\) reduces to

\[
\langle [\theta(x) - \theta(0)]^2 \rangle = -\frac{1}{2\pi \alpha x}.
\] (22)

In the large-\(x\) limit, the exponent vanishes and this shows that there is a real condensate for \(T = 0\). This is the usual phase with the \(U(1)\) symmetry spontaneously broken and long-range order \[2\]. At finite temperature the exponent \(21\) cannot be evaluated in closed form. However, one can expand the integrand and integrate term by term. This gives the following convergent series

\[
\langle [\theta(x) - \theta(0)]^2 \rangle = -\frac{1}{2\pi \alpha x} + \frac{T}{\pi \alpha} \sum_{n=1}^\infty \frac{1}{n} \left[ 1 - \sqrt{\frac{\beta^2 n^2 + x^2}{\beta^2 n^2 + x^2}} \right].
\] (23)

The large-\(x\) behavior can be found by integrating the second term in \(23\) considering \(n\) as a continuous variable. Alternatively, one can apply dimensional-reduction argument that for distance scales \(x\) much larger than \(1/T\) and \(1/\rho_0\), the correlator is dominated by the zeroth Matsubara mode. Either way, in the limit \(x \to \infty\) one obtains

\[
\langle [\theta(x) - \theta(0)]^2 \rangle = \frac{T}{\alpha \pi} \left[ \ln x + C \right],
\] (24)

where \(C\) is a constant. The correlator decreases algebraically with exponent \(\gamma = T/2\alpha\). This shows that at finite temperature there is a quasicondensate whose density is \(\rho_0\). In this phase, there is a local gap, but the phase is incoherent. This is referred to as a pseudogap.

The gap equation \(13\) ceases to have a nontrivial solution for \(\rho_0\) for temperatures \(T\) above \(T_\rho = \rho_0(T = 0)/2\ln 2 \ [2, 14\] (Note that \(T_\rho\) is the critical temperature for the phase transition in the case of the 2+1 dimensional Gross-Neveu model with a \(Z_2\)-symmetry). Above this temperature, the leading-order effective potential then reduces to the ideal-gas value, \(V = -3T^3\zeta(3)/4\pi\).

1+1 dimensions. Averaging over the angle between \(p\) and \(x\) in \(13\), one obtains

\[
\langle [\theta(x) - \theta(0)]^2 \rangle = \frac{1}{2\pi \alpha} \int_0^\infty dp \coth \left( \frac{\beta p}{2} \right) \times \left[ 1 - \cos(px) \right],
\] (25)

At zero temperature and in the large-\(x\) limit, the exponent reduces to

\[
\langle [\theta(x) - \theta(0)]^2 \rangle = \frac{1}{2\pi \alpha} \ln x.
\] (26)

At zero temperature, the stiffness \(15\) reduces to \(\alpha = N/2\pi\) which implies that \(\rho_2(x, 0) = \rho_0^2 x^{-1/N}\). This agrees with Refs. \[2, 12\]. Thus at \(T = 0\), the two-particle correlation function decays algebraically and one has “almost long-range order”. At finite temperature the exponent \(25\) can be calculated exactly and reads

\[
\langle [\theta(x) - \theta(0)]^2 \rangle = \frac{1}{2\pi \alpha} \left[ \ln \sinh(\pi x/\beta) + K \right],
\] (27)

where \(K\) is a constant. We then obtain

\[
\lim_{x \to \infty} \langle [\theta(x) - \theta(0)]^2 \rangle = \frac{1}{2\alpha} T x,
\] (28)

which shows that the correlator decays exponentially, \(\rho_2(x, 0) \sim e^{-\frac{\pi x}{\beta} T x}\). Thus not even a quasicondensate is present at finite temperature. The large-\(x\) \(25\) behavior could also have been derived by calculating the contribution from the static Matsubara mode, which dominates in this limit.

For temperatures above \(T_\rho = \rho_0(T = 0)e^{\gamma_\epsilon/\pi} \ [13\], the gap equation has only the trivial solution \(\rho_0 = 0\) (The
critical temperature for the phase transition in the discrete case is $T_c = 0$). The leading-order effective potential therefore reduces to the ideal-gas value, $\mathcal{V} = -\pi T^2/6$. More generally, due to asymptotic freedom we expect in the high-temperature limit that the $1/N$ expansion reduces to perturbation theory.

Summary. In the present paper, we have studied the phase fluctuations of the $U(1)$-symmetric Gross-Neveu model in 1+1 and 2+1 dimensions. Constructing an effective low-momentum field theory for the phase $\theta$, we have calculated the two-particle correlation function. In 2+1 dimensions, there is a real gap at zero temperature, while the phase fluctuations turn it into a pseudo-gap at finite temperature. In 1+1 dimensions, the quasicondensate which is present at $T = 0$, is destroyed at finite temperature due to phase fluctuations. In particular, this implies that the Kosterlitz-Thouless-like phase present at $T = 0$ [12] disappears at finite temperature.

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