ON THE COMPOSITION IDEALS OF LIPSCHITZ MAPPINGS

KHALIL SAADI

ABSTRACT. In this paper, we study some property of Lipschitz mappings which admit factorization through an operator ideal. Lipschitz cross-norms has been established from known tensor norms in order to represent certain classes of Lipschitz mappings. Inspired by the definition of $p$-summing linear operators, we derive a new class of Lipschitz mappings that is called strictly Lipschitz $p$-summing.

1. Introduction and preliminaries

Let $X$ be a metric space and $E$ be a Banach space. Every Lipschitz mapping $T : X \to E$ admits a factorization of the form

$$T = \hat{T} \circ \delta_X,$$

where $\hat{T}$ is the linearization of $T$ and $\delta_X$ is the canonical embedding. Let $\mathcal{I}$ be an operator ideal, there is a constructive method for defining new classes of Lipschitz mappings which it consists in composing of linear operators of $\mathcal{I}$ and Lipschitz mappings, the resulting space is denoted by $\mathcal{I} \circ Lip_0$. This technique is usually used to generate some ideals of multilinear mappings and homogeneous polynomials (see [2, 11]). The study of the space $\mathcal{I} \circ Lip_0$ is well-motivated, many interesting spaces which are resulting by this technique belong to famous classes of Lipschitz mappings. This is the case, for example, of the spaces of Lipschitz-Cohen strongly $p$-summing, Lipschitz compact, Lipschitz weakly compact, strongly Lipschitz $p$-integral and strongly Lipschitz $p$-nuclear operators. Moreover, the appearance of a linear operator and a Lipschitz mapping in the formula (1.1) motivates us to investigate the connection between the Lipschitz operator $T$ and its linearization. Given an operator ideal $\mathcal{I}$, by considering the correspondence $T \leftrightarrow \hat{T}$ we can obtain the following identification

$$\mathcal{I} \circ Lip_0 (X; E) = \mathcal{I} (\mathcal{F} (X); E).$$

In this paper, our main objective is to derive and study new classes of Lipschitz mappings which satisfy (1.2). We are interested to represent these classes by using Lipschitz cross-norms, such norms have been recently studied by Cabrera-Padilla et al., in [3]. First, we established some relations between tensor norms which

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defined on tensor product and Lipschitz cross-norms and we prove that every tensor norm generates a Lipschitz cross-norm. So, if \( \mathcal{I} \) is an operator ideal which admits a representation through a tensor norm \( \alpha \), i.e.,

\[
\mathcal{I}(E; F^*) = (E \tilde{\otimes}_\alpha F)^* ,
\]

for every Banach spaces \( E, F \), then there is a Lipschitz cross-norm \( \alpha^L \) for which the space \( \mathcal{I} \circ Lip_0 \) admits a Lipschitz tensor representation, i.e.,

\[
\mathcal{I} \circ Lip_0 (X; E^*) = (X \tilde{\boxtimes}_{\alpha^L} E)^* ,
\]

for every metric space \( X \) and every Banach space \( E \). Among our results, we will investigate the Lipschitz cross-norms corresponding to Chevet-Saphar norms. We will define a new concept in the category of Lipschitz operators, that is Lipschitz strictly \( p \)-summing. The operators of this class have a strong relationship with their linearizations for the concept of \( p \)-summing. Certain results and properties of this new class are obtained.

This paper is organized as follows. First, we recall some standard notations which will be used throughout. In section 2, we define for a given operator ideal \( \mathcal{I} \), the class \( \mathcal{I} \circ Lip_0 \) of Lipschitz mappings satisfying that their linearizations belong to \( \mathcal{I} \). Some examples of classes of Lipschitz mappings which are represented by this procedure are given. Section 3 contains the main results, we start by studying Lipschitz cross-norms generated by tensor norms. Then, we consider the Chevet-Saphar norms and we study the corresponding Lipschitz cross-norms. Inspired by the definition of \( p \)-summing we introduce the concept of Lipschitz strictly \( p \)-summing for which we prove that the Lipschitz mapping \( T \) is strictly \( p \)-summing if and only if its linearization \( \hat{T} \) is \( p \)-summing. This notion coincides with the notions of \( p \)-summing and Lipschitz \( p \)-summing operators when we are considering only linear operators.

Now, we recall briefly some basic notations and terminology. Throughout this paper, the letters \( E, F \) will denote Banach spaces and \( X, Y \) will denote metric spaces with a distinguished point (pointed metric spaces) which we denote by 0. Let \( E \) be a Banach space and \( n \in \mathbb{N} \). We denote by \( l^n_p (E) \), \( 1 \leq p \leq \infty \), the space of all sequences \((x_i)_{1 \leq i \leq n}\) in \( E \) equipped with the norm

\[
\|(x_i)_{1 \leq i \leq n}\|_{l^n_p (E)} = \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} ,
\]

and by \( l^n_\omega (E) \) the space of all sequences \((x_i)_{1 \leq i \leq n}\) in \( E \) equipped with the norm

\[
\|(x_i)_{1 \leq i \leq n}\|_{l^n_\omega (E)} = \sup_{\|x^*\|_{E^*} = 1} \left( \sum_{i=1}^{n} |x^* (x_i)|^p \right)^{\frac{1}{p}}
\]

Let \( X \) be a pointed metric space. We denote by \( X^\# \) the Banach space of all Lipschitz functions \( f : X \to \mathbb{R} \) which vanish at 0 under the Lipschitz norm.
given by
\[
\operatorname{Lip}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}.
\]

We denote by \( \mathcal{F}(X) \) the free Banach space over \( X \), i.e., \( \mathcal{F}(X) \) is the completion of the space
\[
AE = \left\{ \sum_{i=1}^{n} \lambda_i \delta(x_i, y_i), (\lambda_i)_{i=1}^{n} \subset \mathbb{R}, (x_i)_{i=1}^{n}, (y_i)_{i=1}^{n} \subset X \right\},
\]
with the norm
\[
\|m\|_{\mathcal{F}(X)} = \inf \left\{ \sum_{i=1}^{n} |\lambda_i| d(x_i, y_i) : m = \sum_{i=1}^{n} \lambda_i \delta(x_i, y_i) \right\},
\]
where the function \( \delta(x, y) : X^\# \to \mathbb{R} \) is defined as follows
\[
\delta(x, y) (f) = f(x) - f(y).
\]

We have \( \mathcal{F}(X)^* = X^\# \). For a general theory of free Banach space see \([10, 11, 16]\). Let \( X \) be a metric space and \( E \) be a Banach space, we denote by \( \operatorname{Lip}_0(X; E) \) the Banach space of all Lipschitz functions \( T : X \to E \) such that \( T(0) = 0 \) with pointwise addition and Lipschitz norm. Note that for any \( T \in \operatorname{Lip}_0(X; E) \) there exists a unique linear map (linearization of \( T \)) \( \hat{T} : \mathcal{F}(X) \to E \) such that \( \hat{T} \circ \delta_X = T \) and \( \|\hat{T}\| = \operatorname{Lip}(T) \), i.e., the following diagram commutes
\[
\begin{array}{ccc}
X & \overset{T}{\to} & E \\
\delta_X \downarrow & & \nearrow \hat{T} \\
\mathcal{F}(X)
\end{array}
\]
where \( \delta_X \) is the canonical embedding so that \( \langle \delta_X(x), f \rangle = \langle \delta(x, 0), f \rangle = f(x) \) for \( f \in X^\# \). The Lipschitz transpose map of a Lipschitz operator \( T : X \to E \) is a linear operator \( T^* : E^* \to X^\# \) which is defined by
\[
T^*(e^*)(x) = e^*(T(x)).
\]
We have
\[
T^* = Q_X^{-1} \circ \hat{T}^*,
\]
where \( Q_X \) is the isomorphism isometric between \( X^\# \) and \( \mathcal{F}(X)^* \) such that
\[
Q_X(f)(m) = m(f), \quad \text{for every } f \in X^\# \text{ and } m \in \mathcal{F}(X).
\]
If \( X \) is a Banach space and \( T : X \to E \) is a linear operator, then the corresponding linear operator \( \hat{T} \) is given by
\[
\hat{T} = T \circ \beta_X,
\]
where \( \beta_X : \mathcal{F}(X) \to X \) is linear quotient map which verifies \( \beta_X \circ \delta_X = \operatorname{id}_X \) and \( \|\beta_X\| \leq 1 \), see \([13, \text{p 124}]\) for more details about the operator \( \beta_X \). Let \( X \) be a metric space and \( E \) be a Banach space, by \( X \boxtimes E \) we denote the Lipschitz tensor
product of $X$ and $E$. This is the vector space spanned by the linear functional
\[ \delta_{(x,y)} \otimes e \] on $\text{Lip}_0(X; E^*)$ defined by
\[ \delta_{(x,y)} \otimes e (f) = \langle f(x) - f(y), e \rangle. \]
Let $\alpha$ be a norm on $X \otimes E$; $\alpha$ is called Lipschitz cross-norm if it satisfies the condition
\[ \alpha(\delta_{(x,y)} \otimes e) = d(x, y) \|e\|. \]
A Lipschitz cross-norm $\alpha$ is called dualizable if given $f \in X^*$ and $e^* \in E^*$, for all $\sum_{i=1}^n \delta_{(x_i, y_i)} \otimes e_i \in X \otimes E$, we have
\[ \left| \sum_{i=1}^n (f(x_i) - f(y_i)) \langle e^*, e_i \rangle \right| \leq \text{Lip}(f) \|e^*\| \alpha \left( \sum_{i=1}^n \delta_{(x_i, y_i)} \otimes e_i \right). \]
Every Lipschitz mapping $T : X \to E^*$ admits a linear functional $\varphi_T$ defined on the Lipschitz tensor product $X \otimes E$ by
\[ \varphi_T \left( \sum_{i=1}^n \delta_{(x_i, y_i)} \otimes e_i \right) = \sum_{i=1}^n (f(x_i) - f(y_i), e_i). \]
As in [1], a subclass $\mathcal{A}$ of $\text{Lip}_0$ is to said a normed (Banach) Lipschitz ideal if for every pointed metric space $X$ and every Banach space $E$, the pair $(\mathcal{A}(X; E), \|\|_{\mathcal{A}})$ is a normed (Banach) space and
(a) For every $f \in X^*$ and $e \in E$ the Lipschitz operator $f \otimes e : X \to E$ defined by $f \otimes e(x) = f(x)e$ is in $\mathcal{A}(X; E)$ and
\[ \|f \otimes e\|_{\mathcal{A}} \leq \text{Lip}(f) \|e\|. \]
(b) For all $T \in \mathcal{A}(X; E)$, we have
\[ \text{Lip}(T) \leq \|T\|_{\mathcal{A}}. \]
(c) Ideal property: Let $Z$ be a metric space and $F$ a Banach space. The composed operator $u \circ T \circ g$ is in $\mathcal{A}(Z; F)$ and
\[ \|u \circ T \circ g\|_{\mathcal{A}} \leq \|u\| \|T\|_{\mathcal{A}} \text{Lip}(g) \]
for every $g \in \text{Lip}_0(Z; X)$, $T \in \mathcal{A}(X; E)$ and $u \in \mathcal{B}(E; F)$, ($\mathcal{B}(E; F)$ is the Banach space of all linear operators from $E$ into $F$).

2. Lipschitz spaces generated by the composition method

In this section, we apply composition ideals technique to generate new classes of Lipschitz mappings. Given an operator ideal $\mathcal{I}$, let $X$ be a pointed metric space and $E$ be a Banach space. A Lipschitz operator $T \in \text{Lip}_0(X; E)$ is said to be of type $\mathcal{I} \circ \text{Lip}_0$ if there exist a Banach space $Z$, a Lipschitz operator $L \in \text{Lip}_0(X; Z)$ and a linear operator $u \in \mathcal{I}(Z; E)$ such that the following diagram commutes
\[ \begin{array}{ccc}
X & \xrightarrow{T} & E \\
\downarrow{L} & & \uparrow{u} \\
Z & & \end{array} \]
On the other hand, \( T = L \circ u \). If \((\mathcal{I}, \| \cdot \|_{\mathcal{I}})\) is a normed (Banach) ideal, the space \( \mathcal{I} \circ \text{Lip}_0 (X; E) \) is a normed (Banach) Lipschitz ideal with respect to the following norm

\[
\| T \|_{\mathcal{I} \circ \text{Lip}_0} = \inf \text{Lip}(L) \| u \|_{\mathcal{I}}.
\]

In [1], the connection between the Lipschitz operators of \( \mathcal{I} \circ \text{Lip}_0 \) and their linearizations is established.

**Theorem 2.1.** [1] Let \( \mathcal{I} \circ \text{Lip}_0 \) be the space of Lipschitz mappings generated by the normed operator ideal \( \mathcal{I} \). The following properties are equivalent.

1. The Lipschitz operator \( T \) belongs to \( \mathcal{I} \circ \text{Lip}_0 (X; E) \).
2. The linearization \( \hat{T} \) belongs to \( \mathcal{I} (\mathcal{F}(X); E) \).

In this case we have \( \| T \|_{\mathcal{I} \circ \text{Lip}_0} = \| \hat{T} \|_{\mathcal{I}} \) and then

\[
\mathcal{I} \circ \text{Lip}_0 (X; E) = \mathcal{I} (\mathcal{F}(X); E)
\]

holds isometrically.

**Proposition 2.2.** Let \( \mathcal{I}_1, \mathcal{I}_2 \) be two operator ideals. Then,

1. If \( \mathcal{I}_1 \circ \text{Lip}_0 (X; E) \subset \mathcal{I}_2 \circ \text{Lip}_0 (X; E) \), then, \( \mathcal{I}_1 (\mathcal{F}(X); E) \subset \mathcal{I}_2 (\mathcal{F}(X); E) \).
2. If \( \mathcal{I} \circ \text{Lip}_0 (X; E) = \text{Lip}_0 (X; E) \) then, \( \mathcal{I} (\mathcal{F}(X); E) = \mathcal{B}(\mathcal{F}(X); E) \).

**Proof.** (1) Let \( u \in \mathcal{I}_1 (\mathcal{F}(X); E) \), then the Lipschitz operator \( T = u \circ \delta_X : X \to E \) verifies

\[
\hat{T} = u.
\]

By Theorem 2.1, \( T \in \mathcal{I}_1 \circ \text{Lip}_0 (X; E) \) and hence \( T \in \mathcal{I}_2 \circ \text{Lip}_0 (X; E) \). Consequently \( \hat{T} = u \in \mathcal{I}_2 (\mathcal{F}(X); E) \).

(2) Let \( u \in \mathcal{B}(\mathcal{F}(X); E) \), then \( T = u \circ \delta_X \in \text{Lip}_0 (X; E) \). Hence

\[
\hat{T} = u \in \mathcal{I} (\mathcal{F}(X); E).
\]

\( \square \)

The next Proposition follows directly from the previous one.

**Proposition 2.3.** Let \( E \) be a Banach space. The following properties are equivalent.

1. \( \text{id}_E \in \mathcal{I} (E; E) \).
2. \( \mathcal{I} \circ \text{Lip}_0 (X; E) = \text{Lip}_0 (X; E) \) for every pointed metric space \( X \).

As in the linear case studied in [1] we give the definition of Lipschitz dual of a given operator ideal.

**Definition 2.4.** [1] The Lipschitz dual of a given operator ideal \( \mathcal{I} \) is defined by

\[
\mathcal{I}^{\text{Lip}_0-\text{dual}} (X; E) = \{ T \in \text{Lip}_0 (X; E) : T^t \in \mathcal{I} (E^*; X^\#) \}.
\]

If \((\mathcal{I}, \| \cdot \|_{\mathcal{I}})\) is a normed (Banach) ideal, define

\[
\| T \|_{\mathcal{I}^{\text{Lip}_0-\text{dual}}} = \| T^t \|_{\mathcal{I}},
\]

then, the space \( \mathcal{I}^{\text{Lip}_0-\text{dual}} (X; E) \) becomes a normed (Banach) Lipschitz ideal.
An operator ideal $\mathcal{I}$ is symmetric if
\[ I = I^\text{dual} = \{ u \in I(F; G) : u^* \in I(G^*; F^*) \}. \]
If $\mathcal{I}$ is symmetric, we have the following coincidence between a Lipschitz ideal and its dual.

**Proposition 2.5.** The following properties are equivalent.

1. $I$ is symmetric.
2. $I\circ \text{Lip}_0(X; E) = \Pi_{p}^{\text{dual}} \circ \text{Lip}_0(X; E)$ for every pointed metric space $X$ and Banach space $E$.

Now, in the rest of this section we present some examples of classes of Lipschitz mappings which generated by known operator ideals using the composition method. In [15], we have introduced the class $\mathcal{D}_p^L$ of Lipschitz-Cohen strongly $p$-summing operators. Proposition 3.1 in [15] asserts that $T : X \to E$ is Lipschitz-Cohen strongly $p$-summing if and only if its linearization $\hat{T}$ is strongly $p$-summing. Combining with Theorem 2.1, the class $\mathcal{D}_p^L$ can be interpreted in terms of the composition method as follows.

**Theorem 2.6.** Let $X$ be a pointed metric space and $E$ be a Banach space. Let $1 < p \leq \infty$ and let $p^*$ be its conjugate ($\frac{1}{p} + \frac{1}{p^*} = 1$). We have
\[ \mathcal{D}_p^L(X; E) = \mathcal{D}_p \circ \text{Lip}_0(X; E) = \Pi_{p^*}^{\text{dual}} \circ \text{Lip}_0(X; E) = \Pi_{p}^{Lip_0-\text{dual}}(X; E), \]
where $\Pi_p$ and $\mathcal{D}_p$ are the classes of $p$-summing and strongly $p$-summing linear operators, respectively.

The classes of Lipschitz compact and weakly compact operators have been introduced in [14]. By $\text{Lip}_{0K}$ and $\text{Lip}_{0W}$ we denote the Banach Lipschitz ideals of Lipschitz compact and Lipschitz weakly compact operators, respectively. In [14, Proposition 2.1 and 2.2], a similar relation as in Theorem 2.1 has been established.

**Theorem 2.7.** Let $X$ be a pointed metric space and $E$ be a Banach space. Then
1. $\text{Lip}_{0K}(X; E) = \mathcal{K} \circ \text{Lip}_0(X; E) = \mathcal{K}^{\text{dual}} \circ \text{Lip}_0(X; E)$.
2. $\text{Lip}_{0W}(X; E) = \mathcal{W} \circ \text{Lip}_0(X; E) = \mathcal{W}^{\text{dual}} \circ \text{Lip}_0(X; E)$.

A simple consequence of the linear result given in [8] asserts that the Banach space $F$ is reflexive if and only if, for every Banach space $G$ and linear operator $v : F \to G$, $v$ is weakly compact. We have the next characterization.

**Theorem 2.8.** Let $X$ be a pointed metric space. The following properties are equivalent.

1. The metric space $X$ is finite.
2. For all Banach space $E$, we have $\text{Lip}_{0W}(X; E) = \text{Lip}_0(X; E)$.

*Proof.* (1) $\Rightarrow$ (2): Immediate.

(2) $\Rightarrow$ (1): Let $E$ be a Banach space and $v : \mathcal{F}(X) \to E$ be a linear operator. We will show that $v$ is weakly compact. By (2), the Lipschitz operator $T = v \circ \delta_X : X \to E$ is Lipschitz weakly compact. Hence, $\hat{T} = v$ is weakly compact. As consequence, the space $\mathcal{F}(X)$ is reflexive. But by [7, Theorem 1] the space $\mathcal{F}(X)$ is never reflexive if $X$ is an infinite metric space. \qed
Now, we recall the definition of strongly $p$-nuclear operators introduced in \cite{5}.

**Definition 2.9.** Let $1 \leq p < \infty$. A Lipschitz operator $T \in \text{Lip}_0(X; E)$ is called a strongly Lipschitz $p$-nuclear ($1 \leq p < \infty$) if there exist operators $A \in B(l_p; E)$, $b \in \text{Lip}_0(X; l_\infty)$ and a diagonal operator $M_\lambda \in B(l_p; l_p)$ induced by a sequence $\lambda \in l_p$ such that the following diagram commutes

$$
\begin{array}{ccc}
X & \xrightarrow{T} & E \\
\downarrow b & & \uparrow A \\
l_\infty & \xrightarrow{M_\lambda} & l_p
\end{array}
$$

The triple $(A, b, \lambda)$ is called a strongly Lipschitz $p$-nuclear factorization of $T$. We denote $\mathcal{N}^{\text{SL}_p}(X; E)$ the Banach space of all strongly Lipschitz $p$-nuclear operators from $X$ into $E$ with the norm

$$sv^L_p (T) = \inf \|A\| \|M_\lambda\| \text{Lip} (b),$$

where the infimum is taken over all the above factorizations. For Banach spaces $F, G$, we denote by $\mathcal{N}_p (F; G)$ the space of all $p$-nuclear linear operators which admit a factorization as in the Lipschitz case with the operator $b$ being linear.

**Proposition 2.10.** Let $1 \leq p < \infty$. The Lipschitz operator $T : X \to E$ is strongly Lipschitz $p$-nuclear if and only if its linearization $\hat{T}$ is $p$-nuclear. Consequently

$$\mathcal{N}^{\text{SL}_p}(X; E) = \mathcal{N}_p \circ \text{Lip}_0 (X; E).$$

**Proof.** Let $T$ be a strongly Lipschitz $p$-nuclear operator, we have

$$T = A \circ M_\lambda \circ b.$$

We use the Lipschitz factorization of $T$ and $b$

$$\hat{T} \circ \delta_X = A \circ M_\lambda \circ \hat{b} \circ \delta_X,$$

by the uniqueness of linearization we obtain

$$\hat{T} = A \circ M_\lambda \circ \hat{b},$$

hence, $\hat{T}$ is $p$-nuclear. The converse is immediate. $\square$

In analogy with the definition of strongly Lipschitz $p$-nuclear operator, the authors in \cite{14} have introduced the definition of strongly Lipschitz $p$-integral operator. In fact, the same definition has been introduced in \cite{5}. In the first definition, the authors have considered a factorization in which the left operator is linear and the right one is Lipschitz. In the second definition, the role of these operators has been changed.

**Definition 2.11.** Let $1 \leq p < \infty$. A Lipschitz operator $T \in \text{Lip}_0(X; E)$ is called strongly Lipschitz $p$-integral if there exist a finite measure space $(\Omega, \Sigma, \mu)$, a bounded linear operator $A \in \mathcal{B}(L_p(\mu); E^{**})$ and a Lipschitz operator $b \in \text{Lip}_0(X; L_\infty(\mu))$ such that the following diagram commutes

$$
\begin{array}{ccc}
X & \xrightarrow{T} & E \\
\downarrow b & & \uparrow A \\
L_\infty(\mu) & \xrightarrow{L_p(\mu)} & L_p(\mu)
\end{array}
$$

\\[\text{Diagram}\]
where \( I_{\infty, p} : L_\infty(\mu) \to L_p(\mu) \) is the formal inclusion operator. The triple \((A, b, \mu)\) is called a strongly Lipschitz \(p\)-integral factorization of \(T\). We denote by \( J_p^{SL}(X; E) \) the Banach space of all strongly Lipschitz \(p\)-integral operators from \(X\) into \(E\) with the norm

\[
s_i^L(T) = \inf \text{Lip}(b) \|A\|.
\]

For Banach spaces \(F, G\) we denote by \( J_p(F; G) \) the space of all \(p\)-integral linear operators. Using the same argument in the proof of Proposition 2.10, we can prove the following.

**Proposition 2.12.** Let \(1 \leq p < \infty\). The Lipschitz operator \(T : X \to E\) is strongly Lipschitz \(p\)-integral if and only if its linearization \(\hat{T}\) is \(p\)-integral. Consequently

\[
J_p^{SL}(X; E) = J_p \circ \text{Lip}_0(X; E)
\]

holds isometrically. If \(p = 1\) we have

\[
J_1^{SL}(X; E) = J_1 \circ \text{Lip}_0(X; E) = J_1^\text{dual} \circ \text{Lip}_0(X; E).
\]

As in the linear case, we give a factorization result for strongly Lipschitz \(p\)-nuclear operators. For the proof, we use the linearization operators and the result [9, Theorem 5.27].

**Theorem 2.13.** Let \(1 \leq p < \infty\). A Lipschitz operator \(T : X \to E\) is strongly Lipschitz \(p\)-nuclear if and only if there exist a Banach space \(Z\), a compact linear operator \(v : Z \to E\) and a strongly Lipschitz \(p\)-integral operator \(L : X \to Z\) such that

\[
T = v \circ L.
\]

In this case

\[
s_i^L_p(T) = \inf \|v\| s_i^L_p(L).
\]

**Proof.** Let \(T : X \to E\) be a strongly Lipschitz \(p\)-nuclear operator, then \(\hat{T} : \mathcal{F}(X) \to E\) is \(p\)-nuclear. Theorem 2.27 in [9] asserts that there exist a Banach space \(Z\), a compact operator linear \(v : Z \to E\) and a \(p\)-integral operator \(w : \mathcal{F}(X) \to Z\) such that \(\hat{T} = v \circ w\). Then

\[
\hat{T} \circ \delta_X = v \circ w \circ \delta_X \Rightarrow T = v \circ L
\]

where \(L = w \circ \delta_X\) which is strongly Lipschitz \(p\)-nuclear by Proposition ??.

Conversely, suppose that

\[
T = v \circ L
\]

where \(v\) is compact operator and \(L\) is strongly Lipschitz \(p\)-integral, then

\[
\begin{array}{ccc}
X & \xrightarrow{T} & E \\
\delta_X & \downarrow & \uparrow v \\
\mathcal{F}(X) & \xrightarrow{\hat{L}} & Z
\end{array}
\]

i.e., \(\hat{T} = v \circ \hat{L}\), with \(\hat{L}\) is \(p\)-integral. So, \(\hat{T} = v \circ \hat{L}\) is \(p\)-nuclear and then \(T\) is strongly Lipschitz \(p\)-nuclear. \(\square\)
3. Main results

3.1. Results on Lipschitz tensor product. Let $F, G$ be two Banach spaces. We denote by $F \otimes G$ its algebraic tensor product. There are various tensor norms that may be defined on the tensor product $F \otimes G$. If we consider a pointed metric space $X$ and a Banach space $E$, there is an attempt to generalize the definition of tensor product to the category of metric spaces. The authors in \[3\] have studied the space $X \otimes E$ which called Lipschitz tensor product. Some Lipschitz cross-norms have been defined on this space. In this section we provide to give some relations between Lipschitz cross-norms and tensor norms. In the sequel, we will use the terminology of Lipschitz cross-norms for norms defined on $X \otimes E$ and tensor norms for norms defined on $F \otimes G$.

**Theorem 3.1.** Every tensor norm $\alpha$ generates a dualizable Lipschitz cross-norm $\alpha^L$ such that for all pointed metric space $X$ and Banach space $E$ we have
\[
\alpha^L(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i) = \alpha(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i),
\]
where $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \otimes E$. In this case, the linear map $\Phi : X \otimes_{\alpha^L} E \to \mathcal{F}(X) \otimes_{\alpha} E$ defined by
\[
\Phi\left(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i\right) = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i,
\]
is well-defined and is an isometry.

**Proof.** Let $\alpha$ be a tensor norm. The properties of the norm $\alpha^L$ have been inherited from that of $\alpha$. Let $X$ be a pointed metric space and $E$ a Banach space. Let $x, y \in X$ and $e \in E$. Then
\[
\alpha^L(\delta_{(x,y)} \otimes e) = \alpha(\delta_{(x,y)} \otimes e) = \|\delta_{(x,y)}\| \|e\| = d(x,y) \|e\|.
\]

So, $\alpha^L$ is Lipschitz cross-norm. Let $f \in X^\# (= \mathcal{F}(X)^*)$ and $e^* \in E^*$. We have
\[
\left| \sum_{i=1}^{n} (f(x_i) - f(y_i)) (e^*, e_i) \right| = \left| f \otimes e^*(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i) \right| \\
\leq \text{Lip}(f) \|e^*\| \alpha(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i) \\
\leq \text{Lip}(f) \|e^*\| \alpha^L(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i).
\]

Then $\alpha^L$ is dualizable. Now, it is easy to show that $\Phi$ is linear. Let $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i = 0$, we will show that $\Phi(u) = 0$. Indeed, let $f \in \mathcal{F}(X)^*$ and $e^* \in E^*$. Since $\mathcal{F}(X)^* = X^\#$ we have by \[3, \text{Proposition 1.6}\]
\[
\sum_{i=1}^{n} (f(x_i) - f(y_i)) e^*(e_i) = 0.
\]
Then
\[ \sum_{i=1}^{n} f(\delta_{(x_i,y_i)}) e^*(e_i) = 0. \]

So, \( \Phi(u) = 0 \) thus tell us that \( \Phi \) is well defined. Let \( u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \bar{\otimes}_\alpha E \). By (3.1) we have
\[
\alpha(\Phi(u)) = \alpha\left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right) = \alpha^L(u).
\]
then \( \Phi \) is an isometry. \( \Box \)

Since \( \Phi \) is a linear isometry, its range \( \Phi(X \bar{\otimes}_\alpha E) \) is closed. On the other hand, the tensors of the form \( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \) are dense in \( \mathcal{F}(X) \bar{\otimes}_\alpha E \). This shows that the range \( \Phi(X \bar{\otimes}_\alpha E) \) is dense in \( \mathcal{F}(X) \bar{\otimes}_\alpha E \) and thus \( X \bar{\otimes}_\alpha E \) is isometrically isomorphic to \( \mathcal{F}(X) \bar{\otimes}_\alpha E \).

**Corollary 3.2.** For every pointed metric space \( X \) and Banach space \( E \) we have,
\[ X \bar{\otimes}_\alpha E = \mathcal{F}(X) \bar{\otimes}_\alpha E \quad (3.2) \]
holds isometrically.

As a consequence of Theorem 2.1 and Corollary 3.2, we get the following result.

**Corollary 3.3.** Let \( \mathcal{I} \circ L \) be a Lipschitz ideal generated by the operator ideal \( \mathcal{I} \). Suppose that \( \mathcal{I} \) can be interpreted through a tensor product, i.e., there is a tensor norm \( \alpha \) such that for every Banach spaces \( F, G \) we have
\[ \mathcal{I}(F; G^*) = (F \bar{\otimes}_\alpha G)^*. \]
Then, there is a Lipschitz cross-norm \( \alpha^L \) defined as in (3.1) such that
\[ \mathcal{I} \circ \text{Lip}_0(X; E^*) = (X \bar{\otimes}_\alpha E)^*. \]
If we consider the projective tensor norm \( \pi \) and injective tensor norm \( \varepsilon \), by using (3.2) and the last corollary, it is not hard to see that
\[ \text{Lip}_0(X; E^*) = (X \bar{\otimes}_\pi E)^* = (\mathcal{F}(X) \bar{\otimes}_\pi E)^*; \]
and
\[ \mathcal{J}_1(\mathcal{F}(X); E^*) = (X \bar{\otimes}_\varepsilon E)^* = (\mathcal{F}(X) \bar{\otimes}_\varepsilon E)^* = \mathcal{J}_1^SL(X; E^*). \]

### 3.2. Lipschitz Chevet-Saphar norms.

We will consider the Chevet-Saphar tensor norms and we will discuss their corresponding Lipschitz cross-norms. Let \( F, G \) be two Banach spaces. The Chevet-Saphar norms \( g_p \) and \( d_p \) are defined on a tensor product \( F \otimes G \) for \( 1 \leq p \leq \infty \) as
\[
d_p(u) = \inf \left\{ \| (x_i) \|_{l^p(F)} \| (g_i) \|_{l^p(G)} : u = \sum_{i=1}^{n} x_i \otimes g_i \right\}
\]

and
\[ g_p(u) = \inf \left\{ \| (x_i_i)_{i=1}^n \|_{F_p^p(F)} \| (g_i_i)_{i=1}^n \|_{G_p^p(G)} : u = \sum_{i=1}^n x_i \otimes g_i \right\}, \]

These norms are mainly introduced to study the classes of \( p \)-summing and strongly \( p \)-summing linear operators. The dual spaces of the corresponding tensor products coincide with these last spaces, i.e.,
\[ D_p(F; G^*) = (F \hat{\otimes} g_k) \] and \( \Pi_p(F; G^*) = (F \hat{\otimes} d_p^*, G)^* \).

We recall that a linear operator \( T : F \rightarrow G \) is \( p \)-summing if there exists a positive constant \( C \) such that for every \( x_1, ..., x_n \in F \) and \( g_1^*, ..., g_n^* \in G^* \) we have
\[ \left| \sum_{i=1}^n \langle T(x_i), g_i^* \rangle \right| \leq C d_p(u), \tag{3.3} \]
where \( u = \sum_{i=1}^n x_i \otimes g_i^* \). The space \( \Pi_p(F, G) \) stands the Banach space of all \( p \)-summing linear operators and
\[ \| T \|_{\Pi_p} = \inf \{ C, \text{verifying the equality (3.3)} \}. \]

Moreover, for the definition of strongly \( p \)-summing linear operators, we substitute in (3.3) \( d_p(u) \) by \( g_p(u) \). Again, \( D_p(F; G) \) stands the Banach space of all strongly \( p \)-summing linear operators with the norm \( \| T \|_{D_p} \). For more details about these notions see [6, 9]. Now, let \( X \) be a pointed metric space and \( E \) be a Banach space. We define \( d_p^L \), the corresponding norm of \( d_p \), as follows: for every \( u = X \otimes E \) we have
\[ d_p^L(u) = d_p(\Phi(u)) = \inf \left\{ \| m_i \|_{F_p^p(F(X))} \| (e_i_i)_{i=1}^n \|_{G_p^p(E)} : u = \sum_{i=1}^n m_i \otimes e_i \right\}, \]
where the infimum is taken over all representations of the form \( \sum_{i=1}^n m_i \otimes e_i \in F(X) \otimes E \) such that \( \Phi(u) = \sum_{i=1}^n m_i \otimes e_i \). By (3.2), we obtain the following identification for \( 1 \leq p < \infty \)
\[ X \hat{\otimes} d_p^L E = F(X) \hat{\otimes} d_p^L E. \]

Let \( T \in Lip_0(X; E) \) be a Lipschitz operator, the operator \( T \) can be see as a linear functional on \( X \otimes E^* \) which its action on a tensor \( u = \sum_{i=1}^n \delta_{(x_i, y_i)} \otimes e_i^* \) is given by
\[ \langle T, u \rangle = \sum_{i=1}^n \langle T(x_i) - T(y_i), e_i^* \rangle. \]

Inspired by the definition of \( p \)-summing linear operators (3.3), we introduce a new definition in the category of Lipschitz mappings.

**Definition 3.4.** Let \( 1 \leq p < \infty \). Let \( X \) be a pointed metric space and \( E \) a Banach space. A Lipschitz operator \( T : X \rightarrow E \) is said to be **strictly Lipschitz**
A metric space and $E$ be a Banach space. The following properties are equivalent.

(1) The Lipschitz operator $T$ belongs to $\Pi_p^{\text{SL}}(X; E)$.

(2) The linearization $\hat{T}$ belongs to $\Pi_p(F(X); E)$.

Proof. (2) $\Rightarrow$ (1): Suppose that $\hat{T} \in \Pi_p(F(X); E)$. Let $x_i, y_i \in X$ and $e_i^* \in E^*$ $(1 \leq i \leq n)$. We put $u = \sum_{i=1}^{n} \delta(x_i , y_i) \otimes e_i^*$, then

$$\left| \sum_{i=1}^{n} \langle T(x_i) - T(y_i), e_i^* \rangle \right| \leq \left| \sum_{i=1}^{n} \langle \hat{T}(\delta(x_i, y_i)), e_i^* \rangle \right| \leq \left\| \hat{T} \right\|_{\Pi_p} d_p(\Phi(u)) = \left\| \hat{T} \right\|_{\Pi_p} d_p^L(u),$$

where $u = \sum_{i=1}^{n} \delta(x_i, y_i) \otimes e_i^*$. We denote by $\Pi_p^{\text{SL}}(X; E)$ the Banach space of all strictly Lipschitz $p$-summing operators from $X$ into $E$ which its norm $\left\| T \right\|_{\Pi_p^{\text{SL}}}$ is the smallest constant $\mathcal{C}$ verifying \eqref{eq:3.4}.

In \cite{4}, the author has studied the class $\Pi_p^L(X; E)$ of Lipschitz $p$-summing operators. He has defined a norm on the space of molecules $\mathcal{F}(X; E)$ of which we have the next duality

$$\Pi_p^L(X; E^*) = \mathcal{F}_{cs_p}(X; E)^*,$$

where $cs_p$ is defined by

$$cs_p(u) = \inf \left\{ \left\| \delta(x,y) \right\|^p_{\Pi_p^L(F(X))} \left\| (e_i)_{i \in I} \right\|_{p(E)} \right\},$$

where the infimum is taken over all representations of $u$ of the form $u = \sum_{i=1}^{n} \delta(x_i, y_i) \otimes e_i \in \mathcal{F}(X; E)$. Note that the space of molecules $\mathcal{F}(X; E)$ plays the same role of Lipschitz tensor product $X \otimes E$ which its norms can be defined on both spaces. Definitions $cs_p$ and $d_p^L$ look very similar. However, they do not coincide. In the definition of $cs_p$, we are only using elements in $\mathcal{F}(X)$ of the form $\delta(x, y)$, but in the second case we have to consider all elements of $\mathcal{F}(X)$. Therefore, the infimum in $d_p^L$ will in general be smaller. It means that

$$\Pi_p^{\text{SL}}(X; E) \subset \Pi_p^L(X; E).$$

In \cite{15}, we have seen that if the linearization $\hat{T}$ of $T$ is $p$-summing then $T$ is Lipschitz $p$-summing, but the converse is not true in general. In our case, we show that it is true for the concept of strictly Lipschitz $p$-summing.

**Theorem 3.5.** Let $1 \leq p < \infty$. Let $X$ be a metric space and $E$ be a Banach space. The following properties are equivalent.

(1) The Lipschitz operator $T$ belongs to $\Pi_p^{\text{SL}}(X; E)$.

(2) The linearization $\hat{T}$ belongs to $\Pi_p(F(X); E)$.

Proof. (2) $\Rightarrow$ (1): Suppose that $\hat{T} \in \Pi_p(F(X); E)$. Let $x_i, y_i \in X$ and $e_i^* \in E^*$ $(1 \leq i \leq n)$. We put $u = \sum_{i=1}^{n} \delta(x_i, y_i) \otimes e_i^*$, then

$$\left| \sum_{i=1}^{n} \langle T(x_i) - T(y_i), e_i^* \rangle \right| \leq \left| \sum_{i=1}^{n} \langle \hat{T}(\delta(x_i, y_i)), e_i^* \rangle \right| \leq \left\| \hat{T} \right\|_{\Pi_p} d_p(\Phi(u)) = \left\| \hat{T} \right\|_{\Pi_p} d_p^L(u),$$
hence $T$ is strictly Lipschitz $p$-summing and
\[ \| T \|_{\Pi_p^{SL}} \leq \| \hat{T} \|_{\Pi_p} . \]

(1) $\Rightarrow$ (2): Suppose that $T \in \Pi_p^{SL} (X; E)$. Let $m_i \in F (X) (m_i = \sum_{j=1}^{k_i} \lambda^i_j \delta_{(x^i_j, y^i_j)})$ and $e^*_i \in E (1 \leq i \leq n)$
\[ \left| \sum_{i=1}^n \left< \hat{T} (m_i) , e^*_i \right> \right| \leq \| T \|_{\Pi_p^{SL}} \| d_p^L (u) \| = \| T \|_{\Pi_p^{SL}} \| \Phi (u) \| , \]
where
\[ u = \sum_{i=1}^n \sum_{j=1}^{k_i} \lambda^i_j \delta_{(x^i_j, y^i_j)} \otimes e^*_i , \]
then $\Phi (u) = \sum_{i=1}^n m_i \otimes e^*_i$. So, $\hat{T}$ is $p$-summing and $\| \hat{T} \|_{\Pi_p} \leq \| T \|_{\Pi_p^{SL}} . \]

As immediate consequences, we have the following results.

**Corollary 3.6.** For every pointed metric space $X$ and Banach space $E$ we have
\[ \Pi_p^{SL} (X; E) = (X \boxtimes d_p^E)^* = (F (X) \boxtimes d_p E)^* = \Pi_p (F (X) ; E) . \]

**Corollary 3.7.** The next inclusion is strict
\[ \Pi_p^{SL} (\mathbb{R}; l_1 (\mathbb{R})) \subset \Pi_p^L (\mathbb{R}; l_1 (\mathbb{R})) . \]

*Proof.* We know that $\delta_\mathbb{R} : \mathbb{R} \rightarrow F (\mathbb{R})$ ($F (\mathbb{R}) = l_1 (\mathbb{R})$ in fact) is Lipschitz $p$-summing. Its linearization is the identity on $F (\mathbb{R})$ which cannot be $p$-summing because $F (\mathbb{R})$ is infinite-dimensional. Hence, $\delta_\mathbb{R}$ is not strictly Lipschitz $p$-summing.

If $X$ is a Banach space and $T : X \rightarrow E$ is a linear operator, we have the following result.

**Proposition 3.8.** Let $X, E$ be two Banach spaces and $T : X \rightarrow E$ be a linear operator. The following properties are equivalent.
1. $T$ is Lipschitz $p$-summing.
2. $T$ is $p$-summing.
3. $T$ is strictly Lipschitz $p$-summing.

In this case we have
\[ \| T \|_{\Pi_p} = \| T \|_{\Pi_p^L} = \| T \|_{\Pi_p^{SL}} . \]

*Proof.* The equivalence (1) $\Leftrightarrow$ (2) has been proved by Farmer and Johnson in [11] and we have $\| T \|_{\Pi_p} = \| T \|_{\Pi_p^L}$. Suppose that $T$ is $p$-summing, by (1.4) $\hat{T} = T \circ \delta_X$ which is $p$-summing by the ideal property, consequently $T$ is strictly Lipschitz $p$-summing and we have
\[ \| T \|_{\Pi_p^{SL}} = \| T \circ \delta_X \|_{\Pi_p} \leq \| T \|_{\Pi_p} . \]

The last implication is immediate with $\| T \|_{\Pi_p^{L}} \leq \| T \|_{\Pi_p^{SL}} . \]
In the linear case, every $p$-integral operator linear is $p$-summing. Then, by Proposition 2.12 and Theorem 3.5 we conclude that every strongly Lipschitz $p$-integral operator is strictly Lipschitz $p$-summing. In the next results, we give some coincidence situations as in the linear case, for the proof we use the linearization of both classes and the linear results given in [9, p 99].

**Corollary 3.9.** Let $1 \leq p < \infty$. Let $X$ be a pointed metric space and $E$ be an injective Banach space. Then

$$\Pi_p^{SL}(X; E) = \mathcal{J}_p^{SL}(X; E),$$

with equality of norms.

**Corollary 3.10.** Let $X$ be a pointed metric space and $E$ be a Banach space. Then

$$\Pi_2^{SL}(X; E) = \mathcal{J}_2^{SL}(X; E),$$

with equality of norms.

**Corollary 3.11.** If $E$ is a subspace of an $L_p$-space, $1 \leq p \leq 2$. Then for every pointed metric space $X$

$$\Pi_q^{SL}(X; E) = \mathcal{J}_q^{SL}(X; E) = \mathcal{J}_2^{SL}(X; E),$$

for all $2 \leq q < \infty$.

We next show a Lipschitz version of well know characterization of an $L_\infty$-space which said that a Banach space $X$ is an $L_\infty$-space if and only if for every Banach space $E$ and 1-summing linear operator $T : X \to E$, $T$ is 1-integral, see [9, Corollary 6.24] for more details about this characterization.

**Theorem 3.12.** Let $X$ be a pointed metric space. The following properties are equivalent.

1. The space $\mathcal{F}(X)$ is an $L_\infty$-space.
2. For all Banach space $E$ we have $\Pi_1^{SL}(X; E) = \mathcal{J}_1^{SL}(X; E)$.

Proof. (1) $\Rightarrow$ (2): Suppose that $\mathcal{F}(X)$ is an $L_\infty$-space. Let $E$ be a Banach space and $T \in \Pi_1^{SL}(X; E)$, then $\hat{T} : \mathcal{F}(X) \to E$ is 1-summing. By the characterization of an $L_\infty$-space, $\hat{T}$ is 1-integral. Consequently, $T$ is in $\mathcal{J}_1^{SL}(X; E)$.

(2) $\Rightarrow$ (1): Let $v : \mathcal{F}(X) \to E$ be 1-summing linear operator. It suffices to see that $v$ is 1-integral. Put $T = v \circ \delta_X$. Then $T$ is strictly Lipschitz 1-summing and $\hat{T} = v$. By (2), $T$ is strongly Lipschitz 1-integral and then its linearization is 1-integral.

We finish this section by discussing the Lipschitz tensor norm associated to Chevet-Saphar norm $g_p$. For every $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \bigotimes e_i \in X \bigotimes E$ we have

$$g_p^L(u) = g_p(\Phi(u)).$$

Then

$$(X \bigotimes_{g_p} E)^* = (\mathcal{F}(X) \bigotimes_{g_p} E)^*.$$
Proposition 3.1 in [15] gives that there is an equivalence between a Lipschitz mapping $T$ and its linearization $\hat{T}$ for the concept of strongly $p$-summing. Moreover, we have
\[
\mathcal{F}_{\mu_p}(X; E)^* = \mathcal{D}_p^L(X; E^*),
\]
where the norm $\mu_p$ is defined as follows
\[
\mu_p(u) = \inf \left\{ \|\delta_{(x_i,y_i)}\|_{l_p^w(\mathcal{F}(X))} \| (e_i)_i \|_{l_p^w(E)} : u = \sum_{i=1}^n \delta_{(x_i,y_i)} e_i \right\}.
\]
Combining with Theorem 2.1, we obtain the following identification.

**Theorem 3.13.** Let $X$ be a metric space and $E$ be a Banach space. We have
\[
(X \hat{\otimes}_{g_p} E)^* = (\mathcal{F}(X) \hat{\otimes}_{g_p} E)^* = \mathcal{D}_p^L(X; E^*) = \mathcal{D}_p(\mathcal{F}(X); E^*) .
\]

**Corollary 3.14.** The norms $g_p^L$ and $\mu_p$ are the same.

**Proof.** Let $u \in X \otimes E$. By the definition of $g_p^L$ we have
\[
g_p^L(u) \leq \mu_p(u).
\]
On the other hand,
\[
\mu_p(u) = \sup_{T \in B_{\mathcal{F}_{\mu_p}(X; E)^*}} \langle T, u \rangle = \sup_{\hat{T} \in B_{\mathcal{D}_p(\mathcal{F}(X); E^*)}} \langle \hat{T}, u \rangle
\]
\[
= \sup_{\hat{T} \in B_{\mathcal{D}_p(\mathcal{F}(X); E^*)}} \left| \sum_{i=1}^n \langle \hat{T} \left( \delta_{(x_i,y_i)} \right), e_i \rangle \right|
\]
\[
\leq \sup_{\hat{T} \in B_{\mathcal{D}_p(\mathcal{F}(X); E^*)}} \| \hat{T} \|_{\mathcal{D}_p} g_p \left( \sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i \right)
\]
\[
\leq g_p(\Phi(u)) = g_p^L(u).
\]
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University of M’sila, Laboratoire d’Analyse Fonctionnelle et Géométrie des Espaces, 28000 M’sila, Algeria.
E-mail address: kh_saadi@yahoo.fr