Attitude and Angular Velocity Tracking for a Rigid Body using Geometric Methods on the Two-Sphere (Stability Proof)

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Abstract—Stability proof of the controller proposed in [1]. This proof must be studied together with Sec. III-C in [1].

I. POINTING DIRECTION AND ANGULAR VELOCITY TRACKING

A control system is defined in L2 under the assumption that we have a fairly accurate estimate of the model parameters showing exponential convergence in an envelope around the zero equilibrium of $b_{eq}$, $b_{ew}$ using Lyapunov analysis.

Proposition 1: For $\Lambda, \gamma, \eta \in \mathbb{R}^+$ and a desired pointing direction curve $q_d(t) \in S^2$ and a desired angular velocity profile $b_{d}(t), b_{d}(t)$, around the $q_d(t)$ axis, we define the control moment $b_u$ as,

\begin{align}
    b_u &= \eta^{-1} \tilde{J}(-\eta \hat{f} + d) - (\Lambda + \Psi) b_{eq} - \Psi b_{eq} + \gamma s \tag{18a} \\
    d &= (\hat{b} B) J^{-1} B^T Q_d b_{d} - Q_d^T b_{d} \hat{b} \tilde{J} \tag{18b} \\
    f &= \tilde{J}(\hat{b} B) J^{-1} B^T (\hat{b} B - \hat{b} \omega + \tau) \tag{18c} \\
    s &= (\Lambda + B) b_{eq} + \gamma b_{ew} \tag{18d} \\
    \eta &= A_{1}\Lambda / (\gamma A_{1} - A_{2}) \tag{18e} \\
    \gamma > A_{1}
\end{align}

where $A_{1} = \lambda_{min}(J^{-1} \tilde{J})$ while $\lambda_{min}$ signifies estimated parameters due to parameter identification errors. The bounds $A_{1}, A_{2}$ are found via the expressions after eq. (22) and are given in [23]. It will be shown that the above control law stabilizes and maintains $b_{eq}, b_{ew}$ in a bounded set around the zero equilibrium. Furthermore for perfect control, the system parameters above the law stabilizes $b_{eq}, b_{ew}$ to zero exponentially.

Proof: We utilize a sliding structure in L2 by defining the surface in terms of the configuration error vectors (12), (15) and the attitude error function (8) so that they appear explicitly in the Lyapunov candidate function. Then the control design is similar to nonlinear control design in Euclidean spaces [2].

\[ s^{T} s \leq -k \| s \|^2, \quad k > 0 \]

Substituting (18a) and (13) to (21), after some manipulations,

\[ \dot{V} = s^{T} A_{1} - \eta A_{2} \]

Employing (15), after several manipulations,

\[ \dot{V} = s^{T} A_{1} - \eta A_{2} \]

where $A \in \mathbb{R}^{3 \times 3}, B \in \mathbb{R}^{3 \times 1}$, are given by,

\begin{align}
    A_1 &= \Delta J \left( b_{eq} b_{eq}^{T} + (\Lambda + \Psi) b_{eq} b_{eq}^{T} \right) + \eta (f - \tilde{J} \tilde{f}) + \Delta J (Q_{d} Q_{d}^{T} b_{d}) \times \\
    B &= \Delta J \left( (\Lambda + \Psi) E b_{d} - Q_{d} Q_{d}^{T} b_{d} \right) + \eta (f - \tilde{J} \tilde{f}) \end{align}

Under the assumption mentioned in the beginning of this section, i.e. we have a fairly accurate estimate of the model parameters, the following holds,

\[ \| f - \tilde{J} \tilde{f} \| \leq f_{max} < \infty, f_{max} \in \mathbb{R}^{+} \]
Additionally since $b\omega_d, b\dot{\omega}_d$ are bounded, the following holds,

$$\exists A_{1, max}, A_{2, max}, B_{max} \in \mathbb{R}^+ - \{\infty\}$$

$$\|A_1\| \leq A_{1, max}, \|A_2\| \leq A_{2, max}, \|B\| \leq B_{max} \tag{23}$$

Expanding (22) and rearranging we have,

$$\dot{V} \leq -\gamma\lambda J S^T s + Y + \dot{\Lambda} \|e_w\|^2 + (\gamma^2 A_{2, max} + \eta A_{1, max})\|e_w\|^2 + k\dot{\gamma} s^T e_w$$

$$\leq -\gamma\lambda J S^T s + Y + \dot{\Lambda} \|e_w\|^2 + (\gamma^2 A_{2, max} + \eta A_{1, max})\|e_w\|^2 + k\dot{\gamma} s^T e_w$$

$$\Upsilon = (\Lambda + \Psi)\|B\|$$

$$\dot{\Lambda} = \gamma\|B\| + (\Lambda + \Psi)\|A\|$$

Using (18a) after several manipulations,

$$\dot{V} \leq -\gamma\lambda J S^T s - \zeta^T W_3 \zeta + Y - \gamma\lambda J (\Lambda + \Psi)^2 \|e_w\|^2$$

$$-\gamma\lambda J 2\Psi \eta \|e_w\|^2 + \gamma\|e_w\|^2 + (\gamma^2 A_{2, max} + \eta A_{1, max})\|e_w\|^2$$

$$\dot{W}_3 = \left[ \begin{array}{c} \gamma\lambda J (\Lambda + \Psi)^2 - \gamma\lambda J \Psi \eta \\ -\gamma \lambda J \Psi \eta - \gamma^2 \lambda J \end{array} \right]$$

Employing $\gamma_3 = \sum_{i=1}^5 \gamma_i, \gamma_i \in \mathbb{R}^+$ after some manipulations,

$$\dot{V} \leq -\gamma\lambda J S^T s - \lambda_{min}(W_3)\|z_q\|^2 - \lambda_{min}(W_4)\|z_q\|^2 + Y$$

Finally for,

$$\|z_q\| > \sqrt{\frac{\Upsilon}{\lambda_{min}(W_3)}} \tag{25}$$

then

$$\dot{V} \leq -\gamma\lambda J S^T s - \lambda_{min}(W_4)\|z_q\|^2 \leq -\lambda_{min}(W_4)\|z_q\|^2$$

Notice that the first term above ensures sliding behavior.

Boundedness: Employing conditions (24) and (25) the following sets are defined,

$$M_1 = \{b e_q, b e_w\} \in \mathbb{R}^3 \times \mathbb{R}^3|\text{Eq. (25)}, \text{Eq. (24)}\}$$

To ensure that $\|b e_q\| < 1$, i.e. the states are in $L_2$, the following set is defined,

$$M_2 = \{b e_q, b e_w\} \in \mathbb{R}^3 \times \mathbb{R}^3|\|z_q\| < 1, \text{Eq. (24)}\}$$

Finally for proper $\gamma, b\omega_d, b\dot{\omega}_d$ the following hold,

$$\dot{\gamma} < \lambda_{min}(W_3) \tag{25a}$$

$$\dot{\Lambda} < (\gamma^2 - \eta A_{1, max})$$

Conditions (25a), (25b), ensure that $M_1 \subset M_2$.

Thus $b e_q, b e_w$ are exponentially stabilized on an envelope of radius $\sqrt{\lambda_{min}(W_3)}$ around the zero equilibrium, with the radius decreasing as $\gamma$ increases. Additionally for perfect knowledge of the system parameters, we have perfect cancellation, which yields,

$$\dot{V} \leq -\gamma\lambda J S^T s - \lambda_{min}(W_5)\|z_q\|^2 \leq -\lambda_{min}(W_5)$$

$$\dot{W}_5 = \left[ \begin{array}{c} (\gamma_2 + \gamma_3)(\Lambda + \Psi)^2 - (\gamma_2 + \gamma_3)\Psi \eta \\ - (\gamma_2 + \gamma_3)\Psi \eta - (\gamma_2 + \gamma_3)^2 \end{array} \right]$$

and by the comparison lemma, [4],

$$V(t) \leq V(0)e^{-\lambda_{min}(W_5)}$$

Proving that the zero equilibrium of the attitude and angular velocity tracking errors $b e_q, b e_w$, is exponentially stable. ■

REFERENCES

[1] Ramp, M. and Papadopoulos, E., “Attitude and Angular Velocity Tracking for a Rigid Body using Geometric Methods on the Two-Sphere,” Proc. of the European Control Conference 2015, Johannes Kepler University, Linz, Austria, July 15-17, 2015, DOI: 10.1109/ECC.2015.7331033.

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APPENDIX

Vector space isomorphism where $r \in \mathbb{R}^3$,

$$(r)^\times = [0, -r_3, r_2; r_3, 0, -r_1; r_2, r_1, 0], ((r)^\times)^\vee = r \tag{A1}$$

Exponential map using the Rodrigues formulation [4],

$$\exp(\epsilon \xi^\times) = I + \xi^\times \sin \epsilon + (\xi^\times)^2(1 - \cos \epsilon) \tag{A2}$$

Derivative of the configuration error vector (12),

$$b e_q = E b e_w + \Xi b \omega_d \tag{A3}$$

$$E = \frac{(b e_q q_T)(q_T)^T Q}{2(1 + q_T^T q_T)} + \frac{Q^T(q_T^T q_T - q_T^T Q)}{\sqrt{1 + q_T^T q_T}} q_d \tag{A4}$$

$$\Xi = \frac{(b e_q q_T)(q_T)^x + (b e_q T)(q_d)^x}{2(1 + q_T^T q_T)} Q_d \tag{A5}$$

Alternative expression for (18b),

$$d = -(Q_T^T q_d b \omega_d)^x b e_w - Q_T^T Q_d b \dot{\omega}_d \tag{A6}$$