ON ISOSPECTRAL COMPACTNESS IN CONFORMAL CLASS FOR 4-MANIFOLDS

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Abstract. Let \((M, g_0)\) be a closed 4-manifold with positive Yamabe invariant and with \(L^2\)-small Weyl curvature tensor. Let \(g_1 \in [g_0]\) be any metric in the conformal class of \(g_0\) whose scalar curvature is \(L^2\)-close to a constant. We prove that the set of Riemannian metrics in the conformal class \([g_0]\) that are isospectral to \(g_1\) is compact in the \(C^\infty\) topology.

1. Introduction

Let \(M\) be a compact smooth manifold without boundary and let \(g\) be a smooth Riemannian metric on \(M\). We will denote by \(\Delta_g\) the Laplace-Beltrami operator associated to \(g\). It is well known that the eigenvalues of \(\Delta_g\) form a discrete sequence that tends to infinity:

\[
\text{Spec}(\Delta_g) : 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty.
\]

Although one can’t compute individual eigenvalues explicitly in most cases, it has long been known that the sequence \(\text{Spec}(\Delta_g)\) is quite rigid, at least in the \(k \to \infty\) limit. For example, the Weyl’s law states that the leading asymptotic of \(\lambda_k\)’s is completely determined by the dimension and the volume of \((M, g)\). A very interesting question is to study the exact relation between the geometry of \((M, g)\) and \(\text{Spec}(\Delta_g)\). This turns out to be a very subtle question: lots of theorems (both in positive and negative directions) have been proved, while many major conjectures are still widely open. For some of the conjectures and their current status, we refer to \([Yau], [Zel]\) and references therein.

Two Riemannian metrics \(g\) and \(g'\) on \(M\) are said to be isospectral if \(\text{Spec}(\Delta_g) = \text{Spec}(\Delta_{g'})\). People had found plenty of examples of isospectral pairs (e.g. \([Mil], [Sun]\)), or even families (e.g. \([Gor], [BrG]\)), of Riemannian metrics. However, it is still believed that the set of isospectral metrics on any smooth manifold is not too large. The famous isospectral compactness problem asks: Given any compact smooth manifold \(M\), for any Riemannian
metric \( g \), is the set of Riemannian metrics on \( M \) that is isospectral to \( g \) compact in the \( C^\infty \) topology? In other words, does any sequence of isospectral metrics admits a convergent subsequence in the \( C^\infty \) topology?

One of the first remarkable works on the isospectral compactness problem was done by B. Osgood, R. Phillips and P. Sarnak [OPS]: they proved the compactness of isospectral metrics on any given compact Riemann surface. For manifolds of dimension greater than two, it is still not known whether the isospectral sets of metrics on a given manifold are compact or not. However, if one restricted to the isospectral metrics in the same conformal class, then it was proved by A. Chang and P. Yang ([CY1], [CY2]) that for three dimensional compact manifolds, the isospectral metrics in the same conformal class is compact. People also studied isospectral compactness under other extra assumptions, see e.g. [And], [BPY], [BPP], [Gur] and [Zhou]. We remark that even inside the same conformal class, one can find isospectral families of non-isometric Riemannian metrics ([BrG]).

Before we continue to describe the isospectral compactness results for 4 dimensional manifolds, we would like to say a few words about the ideas of [CY1] for 3-dimensional manifolds. Recall that any Riemannian metric in the conformal class \([g_0]\) of \( g_0 \) is of the form \( g = u^2 g_0 \). So to prove the compactness of isospectral Riemannian metrics, one need to prove the isospectral compactness of the corresponding conformal factors. In other words, suppose \( u_j \in C^\infty(M) \) be a sequence of conformal factors so that \( g_j = u_j^2 g_0 \) are isospectral, one need to prove that the sequence of functions \( \{u_j\} \) admits a convergent subsequence. [This is not quite precise, since one has to modulo the effect of isometries. See the remark after theorem 1.1 below.] In their proof of the isospectral compactness in conformal class for 3-manifolds, A. Chang and P. Yang introduced the following “non-blowup” condition for the sequence of the conformal factors \( u_j \)’s:

\[
\text{There exist positive constants } r_0, l_0 \text{ so that for all } j, \quad \text{Vol}\{x|u_j(x) \geq r_0\} \geq l_0 \text{Vol}(M, g_0).
\]

(1.1)

Their proof are then divided into two parts, the easier part being proving the isospectral compactness for sequences \( u_j \)’s satisfying (1.1), while the harder part is to verify that the condition (1.1) is always true under the isospectral assumption: They showed that if (1.1) fails, then \( (M,g_0) \) is conformal to the standard \( (S^3, g_0) \).

The isospectral compactness in conformal class problem for 4 dimensional manifolds was analyzed in [Xu1], [Xu2] and [ChX]. For example, in [ChX] they proved the isospectral compactness in conformal class under the extra conditions

\[
\int_M S_{g_1} \, dv_{g_1} < \frac{6}{C_s} \left( \int_M dv_{g_1} \right)^{1/2}
\]
and
\[ \int_M S_{g_1}^2 dv_{g_1} - \frac{(\int_M S_{g_1} dv_{g_1})^2}{\int_M dv_{g_1}} \leq \frac{11^2}{52^2 C_s^2}, \]
where \( C_s \) is the Sobolev constant in (1.2) below. Then showed that the first inequality implies that the conformal factors \( u_j \)'s satisfy the condition (1.1). Moreover, from these two inequality they proved that \( \int |R_g|^4 dv \) is bounded, which implies that the conformal factors are bounded from below and above uniformly. Note that as Chen and Xu remarked, their arguments also works for positive scalar curvature case.

One of the main tools in studying isospectral compactness problem is the Sobolev inequality of Aubin (c.f. [Aub]). When restricted to 4-dimensional Riemannian manifold \((M^4, g_0)\), the inequality takes the form
\[ \left( \int_{M^4} |f|^4 dv_0 \right)^{1/2} \leq C_s \int_{M^4} |\nabla f|^2 dv_0 + K_s \int_{M^4} f^2 dv_0 \]
for any function \( f \in H^2_0(M^4) \), where \( C_s \) and \( K_s \) are positive constants. In what follows we will always make the following assumption:

**Assumption:** \( g_0 \) is a metric with constant scalar curvature \( S_0 > 0 \).

Let \( C_s \) and \( K_s \) be the Sobolev constants as in (1.2) for the metric \( g_0 \). We define a constant (which depends only on \( g_0 \))
\[ C_0 := \max(C_s, \frac{6K_s}{S_0}). \]
Note that according to [AuL] (see also [Bcc]), one has \( C_0 \geq \frac{\sqrt{6}}{8\pi} \).

In this paper, we will study isospectral compactness in conformal class for 4-manifolds with positive Yamabe invariant. Instead of bound the scalar curvature (as in the first condition of [ChX] cited above), we will assume that the Weyl curvature tensor has a small \( L^2 \)-norm. Since the norm of the Weyl curvature tensor is a conformal invariant, we only have one restriction (see the condition (1.5) below) on the scalar curvature. Note that the condition (1.5) is in fact a condition on the spectrum.

Our main theorem in this paper is

**Theorem 1.1.** Let \((M, g_0)\) be a compact Riemannian 4-manifold with positive Yamabe invariant. Suppose the Weyl curvature of \((M, g_0)\) satisfies
\[ \int |W|^2_{g_0} dv_{g_0} \leq \frac{1}{625C_0^2}, \]
where \( C_0 \) is as in (1.3). Suppose \( g_1 \in [g_0] \) be any metric in the same conformal class of \( g_0 \) satisfying
\[ \int_M S_{g_1}^2 dv_{g_1} - \frac{(\int_M S_{g_1} dv_{g_1})^2}{\int_M dv_{g_1}} \leq \frac{1}{64C_0^2}. \]
Then the set of Riemannian metrics $g$ in $[g_0]$ which are isospectral to $g_1$ is compact in the $C^\infty$-topology.

**Remark 1.2.** For the case $M = S^4$ with $g_0 = g_{\text{can}}$ the canonical round metric, the compactness is in the following sense: For any sequence $g_j$ in $[g_0]$ that are isospectral to each other, there is a choice of conformal factors $\{u_j\}$’s so that each $g_j$ is isometric to $u_j^2g_0$, and $\{u_j\}$’s has a convergent subsequence.

We shall say a few words of the proof. As in [CY1], we will first prove theorem 1.1 under the extra assumption (1.1). The argument is closely related to that of [Xu1] and [ChX], i.e. we first deduce that such $u_j$’s are uniformly bounded both from below and from above. As noticed by [CY2] and [Xu2], modulo isometries the conformal factors on the standard $S^4$ can be chosen to be satisfying (1.1). So in particular this implies that theorem 1.1 is true for $S^4$. This will be done in section 4. For the rest of this paper, we will show that (1.1) holds under the condition of theorem 1.1. Motivated by [CY1], we will prove the following “conformal sphere theorem”:

**Theorem 1.3.** Let $(M, g_0)$ be a 4-dimensional closed Riemannian manifold with positive Yamabe invariant and satisfies

\begin{equation}
\int_M |W|^2 dv_0 < 16\pi^2. \tag{1.6}
\end{equation}

Let $\{u_j\}$ be a sequence of positive smooth functions $M$ so that

1. The integral $\int_M |R|^4 dv_0$ is bounded for the sequence $g_j = u_j^2g_0$,
2. There exist $x_0 \in M$ and a sequence of constants $C_j > 0$ with $C_j \to \infty$ so that the sequence $\{C_ju_j\}$ converges uniformly on compact subset of $M\setminus\{x_0\}$ to the Green’s function of the conformal Laplacian $L$.

Then $(M, g_0)$ is conformally equivalent to $(S^4, g_{\text{can}})$.

The main ingredients in proving theorem 1.3 are the conformal sphere theorem of [CGY], and the classification of complete connected flat manifolds in [Wolf]. We remark that the conformal sphere theorem of [CGY] assumes

\[ \int_M |W|^2 dv_0 < 16\pi^2\chi(M), \]

which requires $\chi(M) > 0$. There are plenty of closed manifolds with positive Yamabe invariant and non-positive Euler characteristic. For a generalized sphere theorem, c.f. [ChZ]. We will prove theorem 1.3 in the second half of section 6. In section 5 and the first half of section 6 we will show that under the conditions of theorem 1.1, if the non-blowup condition (1.1) fails, then the condition in theorem 1.3 must hold. So the proof of the main theorem is completed.
2. Preliminaries

2.1. Heat invariants. One of the main tools used in studying the isospectral compactness problem is the heat trace expansion. It is well known that as $t \to 0$,
\[ \text{Tr}(e^{-t\Delta}) = \sum_i e^{-t\lambda_i} \sim (4\pi t)^{-\frac{n}{2}} (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots), \]
where $a_0, a_1, a_2, \cdots$ are integrals of derivatives of curvature terms on $M$. For Riemannian manifolds of dimension 4, the first several heat invariants are explicitly given by

\begin{align*}
(2.1a) \quad a_0 &= \int_M dv_g = \text{Vol}(M), \\
(2.1b) \quad a_1 &= \frac{1}{6} \int_M S_g dv_g, \\
(2.1c) \quad a_2 &= \frac{1}{180} \int_M \left( |W|^2 + |B|^2 + \frac{29}{12} S_g^2 \right) dv_g,
\end{align*}

and (c.f. [Xu2], [Sak], [Gi1], [Gi2])
\begin{align*}
(2.2) \quad a_3 &= \frac{1}{7!} \int_M \left( -\frac{7}{3} |\nabla W|^2 - \frac{4}{3} |\nabla B|^2 - \frac{152}{9} |\nabla S_g|^2 \\
&\quad + 5 S_g |W|^2 + \frac{50}{9} S_g |B|^2 + \frac{185}{54} S_g^3 \\
&\quad + \frac{38}{9} W^{ijkl} W_{kl}^{mn} W_{mnij} + 12 B^{ij} W_{ijkl} W_{jklm} \\
&\quad + \frac{40}{9} B^{ij} B^{kl} W_{ijkl} + \frac{56}{9} W^{ijkl} W_{i k l}^{mn} W_{jmln} \right) dv_g,
\end{align*}

where $S_g, \text{Ric}, B, W, R$ and $dv_g$ are the scalar curvature, the Ricci curvature tensor, the traceless Ricci curvature tensor, the Weyl curvature tensor, the full Riemannian curvature tensor, and the volume element associated to the given metric $g$ respectively. For 4-manifolds, $R, W, B$ and $S$ are related by
\[ R_g = W_g + \frac{1}{2} B_g \otimes g + \frac{S}{24} g \otimes g, \]
where $\otimes$ is the Kulkarni-Nomizu product. In particular,
\begin{equation}
|R_g|^2 = |W_g|^2 + 2 |B_g|^2 + \frac{1}{6} S_g^2.
\end{equation}

2.2. Conformal change of metric. Let $M$ be a 4-manifold. Then under the conformal change $g = u^2 g_0$, the volume forms of the metrics $g$ and $g_0$ are related by
\[ dv_g = u^4 dv_0, \]
while the corresponding scalar curvatures are related by the equation
\[ 6\Delta_{g_0}u + S_g u^3 = S_0 u. \]

Another very important fact for us is that the integral
\[ \int_M |W_g|^2 dv_g \]
is invariant under the conformal change.

We also notice that on 4-manifolds, the quantity
\[ \int_M S_g^2 dv_g \]
is a spectral invariant if we assume that the metrics sit in the same conformal class. For a proof, c.f. [ChX].

3. SOME NORM ESTIMATES

We recall that \(|W|^2 = W^{ijkl}W_{ijkl}|.\n
**Lemma 3.1.** Under the assumption (1.4), we have

\[ (3.1a) \quad \left| \int_M W^{ijkl}W_{kl}^{mn}W_{mnij}dv_g \right| \leq \frac{1}{25} C_0 \left( \int_M |W_g|^4 dv_g \right)^{1/2}, \]

\[ (3.1b) \quad \left| \int_M B^{ij}B^{kl}W_{ikjl}dv_g \right| \leq \frac{1}{25} C_0 \left( \int_M |B_g|^4 dv_g \right)^{1/2}, \]

and

\[ (3.1c) \quad \left| \int_M B^{ij}W_{i}^{kln}W_{jklm}dv_g \right| \leq \frac{1}{50} C_0 \left[ \eta \left( \int_M |B_g|^4 dv_g \right)^{1/2} + \frac{1}{\eta} \left( \int_M |W_g|^4 dv_g \right)^{1/2} \right], \]

where \( \eta \) is any positive constant.

**Proof.** (3.1a) follows from

\[ \int_M |W_g|^3 dv_g \leq \left( \int_M |W_g|^2 dv_g \right)^{1/2} \left( \int_M |W_g|^4 dv_g \right)^{1/2} \leq \frac{1}{25 C_0} \left( \int_M |W_g|^4 dv_g \right)^{1/2}. \]

The proof of (3.1b) is similar. To prove (3.1c), one only need to notice

\[ \int_M |B_g|^2 |W_g|^2 dv_g \leq \left( \int_M |B_g|^2 dv_g \right)^{1/2} \left( \int_M |W_g|^2 dv_g \right)^{1/2} \leq \frac{1}{25 C_0} \left( \int_M |B_g|^4 dv_g \right)^{1/4} \left( \int_M |W_g|^4 dv_g \right)^{1/4}, \]

and use the fact that for any positive \( a, b \) and \( \eta, ab \leq \frac{1}{2} (\eta a^2 + \frac{1}{\eta} b^2). \)
Lemma 3.2. Assume \([1.3]\), then one has

\[(3.2a) \quad \left( \int_M S_g^4 dv_g \right)^{1/2} \leq C_0 \int_M |\nabla S_g|^2 dv_g + \frac{C_0}{6} \int_M S_g^3 dv_g,\]

\[(3.2b) \quad \left( \int_M |W_g|^4 dv_g \right)^{1/2} \leq C_0 \int_M |\nabla W_g|^2 dv_g + \frac{C_0}{6} \int_M S_g |W_g|^2 dv_g,\]

\[(3.2c) \quad \left( \int_M |B_g|^4 dv_g \right)^{1/2} \leq C_0 \int_M |\nabla B_g|^2 dv_g + \frac{C_0}{6} \int_M S_g |B_g|^2 dv_g.\]

Proof. The proofs of (3.2a) and (3.2c) are essentially the same as in [ChX]. In fact, according to the Sobolev inequality (1.2) and the definition of \(C_0\),

\[
\left( \int_M S_g^4 dv_g \right)^{1/2} = \left( \int_M S_g^4 u^4 dv_0 \right)^{1/2} \leq C_0 \int_M |\nabla (S_g u)|^2 dv_0 + K_s \int_M S_g^2 u^2 dv_0.
\]

For the first term, we have (c.f. the proof of lemma 4.2 of [ChX])

\[
\int_M |\nabla (S_g u)|^2 dv_0 = \int_M |\nabla_0 S_g|^2 u^2 dv_0 + 2 \int_M \langle \nabla S_g, \nabla u \rangle_0 S_g u dv_0 + \int_M S_g^2 |\nabla u|^2 dv_0
\]

\[
= \int_M |\nabla_0 S_g|^2 u^2 dv_0 + \int_M S_g^2 (S_g u^3 - S_0 u) dv_0.
\]

So we get

\[
\left( \int_M S_g^4 dv_g \right)^{1/2} \leq C_0 \int_M |\nabla S_g|^2 dv_g + \frac{C_0}{6} \int_M S_g^3 dv_g + (K_s - \frac{C_0 S_0}{6}) \int_M S_g^2 u^2 dv_0,
\]

which proves (3.2a). To prove (3.2b), we start with

\[
\left( \int_M |W_g|^4 dv_g \right)^{1/2} = \left( \int_M |W_g|^4 u^4 \right)^{1/2} dv_0 \leq C_0 \int_M |\nabla |W_g|^2| u^2 dv_0 + K_s \int_M |W_g|^2 u^2 dv_0,
\]

and for the first term, we use

\[
\int_M |\nabla |W_g|^2| u^2 dv_0 = \int_M |\nabla |W_g|^2|^2 u^2 dv_0 + \frac{1}{2} \int_M \langle \nabla |W_g|^2, \nabla u^2 \rangle_0 dv_0 + \int_M |W_g|^2 |\nabla u|^2 dv_0
\]

\[
= \int_M |\nabla |W_g|^2|^2 dv_0 - \frac{1}{2} \int_M |W_g|^2 \Delta_0 u^2 dv_0 + \int_M |W_g|^2 |\nabla u|^2 dv_0
\]

\[
= \int_M |\nabla |W_g|^2|^2 dv_0 - \int_M |W_g|^2 u \Delta_0 u dv_0
\]

\[
= \int_M |\nabla |W_g|^2|^2 dv_0 + \int_M |W_g|^2 S_g u^3 - S_0 u dv_0
\]

\[
\leq \int_M |\nabla |W_g|^2|^2 dv_0 + \frac{1}{6} \int_M |W_g|^2 S_g dv_g - \frac{S_0}{6} \int_M |W_g|^2 u^2 dv_0.
\]

The proof of (3.2c) is similar.
Lemma 3.3. Assume (1.5), then

\[ (3.3a) \quad \left| \int_M S_g |W|^2_g dv_g \right| \leq \frac{1}{8C_0} \left( \int_M |W|^4_g u^4 dv_0 \right)^{1/2} + 1080 \frac{a_1a_2}{a_0}, \]

\[ (3.3b) \quad \left| \int_M S_g^2 dv_g \right| \leq \frac{1}{8C_0} \left( \int_M S_g^4 dv_g \right)^{1/2} + \frac{12960}{29} \frac{a_1a_2}{a_0}, \]

\[ (3.3c) \quad \left| \int_M S_g |B|^2_g dv_g \right|^2 \leq \frac{1}{8C_0} \left( \int_M |B|^4_g dv_g \right)^{1/2} + 1080 \frac{a_1a_2}{a_0}. \]

**Proof.** The estimate (3.3a) follows from

\[
\int_M S_g |W|^2_g dv_g = \int_M \left( S_g u^2 - \frac{\int_M S_g u^4 dv_0}{\int_M u^4 dv_0} u^2 \right) |W|^2_g u^2 dv_0 \\
+ \frac{\int_M S_g u^4 dv_0}{\int_M u^4 dv_0} \int_M |W|^2_g u^2 dv_0 \\
\leq \left[ \int_M S_g u^4 dv_0 - \left( \frac{\int_M S_g u^4 dv_0}{\int_M u^4 dv_0} \right)^2 \right]^{1/2} \left( \int_M |W|^4_g u^4 dv_0 \right)^{1/2} + 1080 \frac{a_1a_2}{a_0},
\]

and the proofs of (3.3b) and (3.3c) are similar. \qed

Substituting the estimates (3.1a)-(3.3c) into the heat invariant \( a_3 \), we get

**Lemma 3.4.** Under the assumptions of lemmas 3.1, 3.2 and 3.3, we have

\[
(3.4) \quad \frac{7}{3} \left( \int |W|^4_g dv_g \right)^{1/2} + \frac{4}{3} \left( \int |B|^4_g dv_g \right)^{1/2} + \frac{152}{9} \left( \int_M S_g^4 dv_g \right)^{1/2} \\
\leq \left[ \frac{971}{18} + \frac{1}{25} \left( \frac{94}{9} + \frac{6}{\eta} \right) \right] \left( \int_M |W|^4_g dv_g \right)^{1/2} \\
+ \frac{1}{25} (6\eta + \frac{40}{3}) + \frac{521}{9} \left( \int_M |B|^4_g dv_g \right)^{1/2} \\
+ \frac{337}{54} \frac{1}{8} \left( \int_M S_g^4 dv_g \right)^{1/2} + 18800C_0 \frac{a_1a_2}{a_0} - 7!a_3C_0.
\]

As a consequence, we can prove

**Proposition 3.5.** Let \( g \) be any metric as described in theorem 1.1, then there exist constants \( A_1, A_2 \) such that

\[
(3.5) \quad \int_M |\nabla R|^2_g dv_g \leq A_1
\]
and

\[ \int_M |R|^4 dv_g \leq A_2. \]  

Proof. Take \( \eta = \frac{1}{5} \) in (3.4). It is easy to see that the quantity

\[ \left( \int_M S^4_g dv_g \right)^{1/2} + \left( \int_M |B|^4_g dv_g \right)^{1/2} + \left( \int_M |W|^4_g \right)^{1/2} \]

is bounded. In view of (2.3) we get

\[ \int_M |R|^4 dv_g \leq A_2. \]

To obtain a bound on \( \int_M |\nabla R|^2 dv_g \), we notice that according to the formula (2.2) of \( a_3 \), we can write

\[ \int_M \left[ \frac{7}{3} |\nabla W|^2 + \frac{4}{3} |\nabla B| + \frac{152}{9} |\nabla S_g|^2 \right] dv_g \]

as

\[ \int_M \left[ 5S_g |W|^2 + \frac{50}{9} S_g |B|^2 + \frac{185}{54} S_g^3 + \frac{38}{9} W^{ijkl} W_{ijklmn} \right. \]

\[ + 12 B^{ij} W_{ijkl} W_{ijklmn} + \frac{40}{3} B^{ij} B^{kl} W_{ijkl} + \frac{56}{9} W^{ijkl} W_{ijklmn} \left. \right] dv_g - 7! a_3, \]

which can be controlled by the integral

\[ \int_M |R|^4 dv_g \]

using the Hölder inequality and the estimates above.

4. The proof of theorem 1.1 under condition (1.1)

We will start with proving the following proposition, which claims that under the condition (1.1), the conformal factors \( u_j \)'s are uniformly bounded. We remark that for the negative scalar curvature case, this was proved in [Xu1].

Proposition 4.1. Let \( g_0 \) be a metric on \( M^4 \), and \( g = u^2 g_0 \) a metric satisfy conditions (1.1), (1.4) and (1.5), then there exist constants \( C_\alpha, C_\beta > 0 \) such that \( C_\alpha \leq u \leq C_\beta \).

Proof. We first notice that although the proposition 3.1 in [ChX] was stated under the condition \( S_0 < 0 \), the same argument works for the case \( S_0 > 0 \) without any change. So there exists a constant \( C_1 \) such that

\[ \int_M u^{-4} dv_0 \leq C_1. \]

In particular, this also implies \( \int_M u^{-1} dv_0 \leq C'_1 \) for some constant \( C'_1 \).
Also in the proof of proposition 3.5 above we see that there is a constant $C_2$ so that
\[ \int_M S_g^4 u^4 dv_0 = \int_M S_g^4 dv_0 \leq C_2. \]

So by (2.4), one can find a constant $C$ so that
\[
\int_M (\Delta g_0 u)^4 u^{-8} dv_0 = \frac{1}{64} \int_M |S_g u^3 - S_0 u|^4 u^{-8} dv_0 \\
\leq \frac{16}{64} \int_M (|S_g u^3|^4 + |S_0 u|^4) u^{-8} dv_0 \\
\leq C.
\]

On the other hand, if we let $G$ be the Green’s function on $M$ (with respect to $g_0$) which can be assumed to be positive everywhere, then $G_x(y) = G(x, y)$ is $L^p$ integrable for $p < 2$ since $M$ is 4-dimensional. Since
\[ \Delta \frac{1}{u} = -\frac{1}{u^2} \Delta u + \frac{2}{u^3} |\nabla u|^2, \]
we get from Green’s formula that for any point $x \in M$,
\[
\frac{1}{u(x)} - (\int_M dv_0)^{-1} \int_M \frac{1}{u} dv_0 = - \int_M G(x, y) \left[ -\frac{1}{u^2} \Delta u + \frac{2}{u^3} |\nabla u|^2 \right] dv_0(y) \\
\leq \int_M G(x, y) \left[ \frac{1}{u^2} \Delta u \right] dv_0(y) \\
\leq \|G_x\|_{L^{4/3}} \left\| \frac{1}{u^2} \Delta u \right\|_{L^4}.
\]

It follows that there exists a constant $C_\alpha > 0$ which does not depend on $u$ such that for any $x \in M$,
\[ u(x) \geq C_\alpha > 0. \]

The proof of the upper bound is similar to that of proposition 1 in [Xu1]. So we will omit the details here.

We shall use the $C^k$ version of the Cheeger-Gromov compactness:

**Cheeger-Gromov $C^k$ Convergence Theorem** ([OPS], [ChX], [Ch], [Gro]). For any $k$, the space of $n$-dimensional Riemannian manifolds satisfying the bounds

\begin{align*}
(4.2a) \quad |\nabla^j R|_{C_\alpha} &\leq \Lambda(j), \quad j \leq k, \\
(4.2b) \quad \text{Vol}(M, g) &\geq v > 0, \\
(4.2c) \quad \text{diam}(M, g) &\leq D
\end{align*}

is (pre)compact in the $C^{k+1, \alpha}$ topology on $M$. 

More precisely, given any $\alpha < 1$, any sequence of metrics $\{g_i\}$ on $M$ satisfying (4.2a)-(4.2c) has a subsequence converging in the $C^{k+1, \alpha'}$ topology for $\alpha' < \alpha$ to a limit $C^{k+1, \alpha}$ Riemannian metric $g$ on $M$.

Proof of theorem 1.1 under the condition (1.1). One need to verify (4.2a)-(4.2c) for Riemannian metrics $g_j = u_j^2 g_0$ satisfying the condition (1.1).

The bound (4.2a) follows from proposition 3.5, proposition 4.1 above, and proposition 3 of [Xu1]. The bound (4.2b) follows from the first heat invariant $a_0$. To prove (4.2c), one need to use the fact that $C_\alpha \leq u_j \leq C_\beta$. It follows that $C_\alpha^2 g_0 \leq g_j = u_j^2 g_0 \leq C_\beta^2 g_0$. Now for any $p, q \in M$, let $\gamma$ be the minimal geodesic (with respect to the metric $g_1$) connecting $p$ and $q$. Then

$$\text{dist}_{g_j}(p, q) \leq L_{g_j}(\gamma) \leq C_\beta^2 L_{g_1}(\gamma) = C_\beta^2 \text{dist}_{g_1}(p, q) \leq C_\beta^2 \text{diam}(M, g_1).$$

So (4.2c) follows.

As noticed by [CY1] and [Xu2], on $(S^4, g_{\text{can}})$ where $g_{\text{can}}$ is the canonical round metric on $S^4$, if $g_j = u_j^2 g_{\text{can}}$ is a sequence of conformal metrics satisfying

$$C_0 = \int_{S^4} u_j^4 dv_0$$

and $\lambda_1(g_j) \geq \Lambda > 0$, then there exist a sequence of conformal factors $v_j$'s such that each $v_j^2 g_{\text{can}}$ is isometric to $u_j^2 g_{\text{can}}$, and $v_j$'s satisfy the condition (1.1) with universal $r_0, l_0$ depending only on $C_0$ and $\Lambda$. As a consequence,

**Corollary 4.2 ([Xu2]).** Theorem 1.1 holds for $S^4$ with the canonical round metric $g_{\text{can}}$.

5. Mass Concentration

For the rest of the paper, we will study the isospectral compactness for the sequence $\{g_j = u_j^2 g_0\}$ under the assumption that the condition (1.1) fails for any subsequence of $\{u_j\}$. We will show this can happen only when some subsequence of $u_j$ has its mass “concentrate” at some point $x_0 \in M$.

We will start with a technical lemma that we will need several times later. For simplicity we denote

$$C_g = \int_M S_g^2 dv_g - \left( \frac{\int_M S_g dv_g}{\int_M dv_g} \right)^2.$$

We notice that for $g$ in the same conformal class, $C_g$ is in fact a spectral invariant.
Lemma 5.1. Let \((M, g_0)\) be a 4-dimensional manifold, \(g = u^2 g_0\), and \(\eta\) a positive cut-off function which will be chosen later. Then for \(\beta \neq 0\) and \(\beta \neq -1\) we have

\[
(5.2) \quad \left( \int_M \omega^4 \eta^4 d\nu_0 \right)^{1/2} \leq 2C_s \left( 6 \frac{A_\beta}{|\beta|} + 1 \right) \int_M |\nabla \eta|^2 \omega^2 d\nu_0 + (A_\beta |S_0| + K_s) \int_M \omega^2 \eta^2 d\nu_0 \\
+ C_s C^{1/2} A_\beta \left( \int_M \omega^4 \eta^4 d\nu_0 \right)^{1/2} + A_\beta C_s \left[ \frac{\int_M S_g u^4 d\nu_0}{\int_M u^4 d\nu_0} \right] \int_M u^2 \omega^2 \eta^2 d\nu_0,
\]

where \(\omega = u^{1+\beta}/2\), \(A_\beta = \frac{|1+\beta^2|}{6|\beta|}\), \(C_s\) and \(K_s\) are as in (1.2).

Proof. For simplicity, we will henceforth abbreviate \(\int d\nu_0\) as \(\int\). Applying the Sobolev inequality (1.2) to the function \(f = \omega \eta\), we get

\[
(5.3) \quad \left( \int \eta^4 \omega^4 \right)^{1/2} \leq C_s \int |\nabla (\omega \eta)|^2 + K_s \int \omega^2 \eta^2 \\
\leq 2C_s \int |\nabla \omega|^2 \eta^2 + 2C_s \int \omega^2 |\nabla \eta|^2 + K_s \int \omega^2 \eta^2.
\]

Next we multiply both sides of

\[
6 \Delta u + S_g u^3 = S_0 u
\]

by \(\eta^2 u^\beta\) and integrate, to get

\[
6 \beta \int_M \eta^2 u^{\beta - 1} |\nabla u|^2 + 12 \int_M \nabla u \cdot \nabla \eta u^\beta + S_0 \int_M \eta^2 u^{\beta + 1} = \int_M S_g u^2 \eta^2 u^{\beta + 1}.
\]

We can control the second term via

\[
\left| 2 \int_M \nabla u \nabla \eta u^\beta \right| \leq \frac{1}{t} \int_M |\nabla \eta|^2 u^{\beta + 1} + t \int_M \eta^2 |\nabla u|^2 u^{\beta - 1}.
\]

For any \(t\) with \(0 < t < 1/\beta\),

\[
- \frac{1}{t} \int |\nabla \eta|^2 u^{\beta + 1} - t \int \eta^2 |\nabla u|^2 u^{\beta + 1} \\
\leq 2 \int (\nabla u \cdot \nabla \eta) u^\beta \\
\leq \frac{1}{t} \int |\nabla \eta|^2 u^{\beta + 1} + t \int \eta^2 |\nabla u|^2 u^{\beta - 1}.
\]

It follows that for \(\beta < 0\) one has

\[
(5.3) \quad 6(|\beta| - t) \int \eta^2 u^{\beta - 1} |\nabla u|^2 \leq \frac{6}{t} \int |\nabla \eta|^2 u^{\beta + 1} + |S_0| \int \eta^2 u^{\beta + 1} - \int S_g u^2 \eta^2 u^{\beta + 1},
\]
while for $\beta > 0$, one has
\[(5.4)\]
\[6(|\beta|^{-t})\int \eta^2 u^{\beta - 1} |\nabla u|^2 \leq \frac{6}{t} \int |\nabla \eta|^2 u^{\beta + 1} + |S_0| \int \eta^2 u^{\beta + 1} + \int S_g u^2 \eta^2 u^{\beta + 1}.
\]

Take $t = \frac{|\beta|}{2}$ we get for $\beta < 0$,
\[
\frac{12|\beta|}{|1 + \beta|^2} \int |\nabla \omega|^2 \eta^2 \leq \frac{12}{|\beta|} \int |\nabla \eta|^2 \omega^2 + |S_0| \int \omega^2 \eta^2 - \int S_g u^2 \omega^2 \eta^2
\]
and for $\beta > 0$,
\[
\frac{12|\beta|}{|1 + \beta|^2} \int |\nabla \omega|^2 \eta^2 \leq \frac{12}{|\beta|} \int |\nabla \eta|^2 \omega^2 + |S_0| \int \omega^2 \eta^2 + \int S_g u^2 \omega^2 \eta^2.
\]
So if $\beta < 0$, we get
\[
\left(\int \eta^4 \omega^4\right)^{1/2} \leq 2C_s\left[\frac{1 + \beta}{|\beta|^2}\right] \left(\frac{12}{|\beta|} \int |\nabla \eta|^2 \omega^2 + |S_0| \int \omega^2 \eta^2 - \int S_g u^2 \omega^2 \eta^2\right)
+ 2C_s \int w^2 |\nabla \eta|^2 + K_s \int \omega^2 \eta^2
\]
\[
\leq 2C_s \left(\frac{1 + \beta^2}{|\beta|^2} + 1\right) \int |\nabla \eta|^2 \omega^2 + \left[2C_s \frac{1 + \beta^2}{|\beta|^2} |S_0| + K_s\right] \int \omega^2 \eta^2
+ 2C_s C_g \left(\frac{1 + \beta^2}{12|\beta|^2}\right) \left(\int (\omega \eta)^4\right)^{1/2} - 2C_s \frac{1 + \beta^2}{12|\beta|^2} \int S_g u^4 \int u^2 \omega^2 \eta^2.
\]
Similarly, when $\beta > 0$, we have
\[
\left(\int \eta^4 \omega^4\right)^{1/2} \leq 2C_s \left(\frac{1 + \beta^2}{|\beta|^2} + 1\right) \int |\nabla \eta|^2 \omega^2 + \left[2C_s \frac{1 + \beta^2}{|\beta|^2} |S_0| + K_s\right] \int \omega^2 \eta^2
+ 2C_s \frac{1 + \beta^2}{12|\beta|^2} C_g^{1/2} \left(\int (\omega \eta)^4\right)^{1/2} + 2C_s \frac{1 + \beta^2}{12|\beta|^2} \int S_g u^4 \int u^2 \omega^2 \eta^2.
\]
This completes the proof. \[\blacksquare\]

The following lemma is an analogue of lemma 1 in section 3 of [CY1]:

**Lemma 5.2.** Suppose (1.1) fails for any subsequence of a sequence of positive functions $\{u_j\}$ that satisfy
\[
\int_M u_j^4 dv_0 = C_0,
\]
then $u_j \to 0$ in $L^p$ for any $1 \leq p < 4$.

**Proof.** For each $r > 0$ we set
\[
\Omega_{r,j} \equiv \{x \in M : u_j(x) \geq r\}.
\]
We argue by contradiction. Suppose the lemma fails, i.e. there exists some \( p < 4 \) and \( \delta_0 > 0 \) such that
\[
\int u_j^p dv_0 \geq \delta_0
\]
for some subsequence of \( u_j \), which we still denote by \( u_j \) for simplicity. Then for each \( r > 0 \) we have
\[
\delta_0 \leq \int u_j^p dv_0 = \int_{\Omega_{r,j}} u_j^p dv_0 + \int_{M \setminus \Omega_{r,j}} u_j^p dv_0 \\
\leq \left( \int u_j^4 \right)^{p/4} \text{Vol}(\Omega_{r,j})^{(4-p)/4} + r^p \text{Vol}(\Omega_{r,j}).
\]
Choose \( r_0 \) small so that
\[
r_0^p \text{Vol}(M, g_0) < \frac{\delta_0}{2},
\]
then we get
\[
\frac{\delta_0}{2} \leq C_0^{p/4} \text{Vol}(\Omega_{r_0,j})^{(4-p)/4}.
\]
Thus
\[
\text{Vol}(\Omega_{r_0,j}) \geq \left( \frac{\delta_0}{2C_0^{p/4}} \right)^{4/(4-p)} =: l_0
\]
for each \( u_j \), which contradicts with our assumption that the condition (1.1) fails for the sequence \( \{u_j\} \).}

The proof of the following proposition is similar to the proof of the proposition B in section 3 of [CY1]. The main differences are that we use lemma 5.1 and 5.2 for 4-dimensional manifolds, while they use their formula (9b) and lemma 1 in their paper for 3-manifolds. For completeness, we will give the detail of the proof in the appendix.

**Proposition 5.3** ([CY1], proposition B). Suppose \( \{u_j\} \) is a sequence of positive functions defined on \((M^4, g_0)\) such that \( g_j = u_j^2 g_0 \) satisfy the following conditions
\begin{enumerate}
  \item \( a_0(g_j) = \alpha_0 \),
  \item \( a_1(g_j) \leq \alpha_1 \),
  \item \( \int S_{g_j}^2 u_j^4 dv_0 \leq \alpha_2 \),
  \item \( 0 < \Lambda \leq \lambda_1(g_j) \),
  \item The condition (1.1) fails for any subsequence of \( \{u_j\} \).
\end{enumerate}
Then there exists some subsequence of \( \{u_j\} \) whose mass concentrates at some point \( x_0 \in M \).

The next lemma is served as a replacement of Lemma 2 in [CY1].
Lemma 5.4. Let $u$ be any positive function on $M$. Then for each point $x \in M$, there exists some neighborhood $\Omega(x)$ such that for every point $y \in \Omega(x)$ and geodesic ball $B(y, \rho) \subset \Omega(x)$ we have

$$\int_{B(y, \rho)} |\nabla \log u| dv_0 \leq k \rho^3,$$

where $k$ is a constant depending only on $c_2 = \int S_2 g u^4 dv_0$, where $g = u^2 g_0$.

In particular, there exists a constant $p_0 > 0$ that depends only on $c_2$, such that

$$\int_{B(y, \rho)} u^{p_0} dv_0 \int_{B(y, \rho)} u^{-p_0} dv_0 \leq c \rho^8.$$

Proof. The proof is similar to that of [CY1], so we only describe the difference here. By choosing a cut-off function $\eta$ satisfying $|\nabla \eta| \leq 2 \rho$ on $B(2 \rho)$, taking $\beta = -1$ and $t = \frac{1}{2}$ in (5.3), we get the following replacement of (21) in [CY1]:

$$3 \int_{B(\rho)} u^{-2} |\nabla u|^2 \leq 12 \int_{B(2 \rho)} \frac{|S|}{\rho^2} + \int_{B(2 \rho)} 1 + \int_{B(2 \rho)} |S_g| \rho u^2.$$

Since

$$\int_{B(\rho)} |S_g| \rho u^2 \leq \left( \int S_g^2 u^4 \right)^{1/2} \text{Vol}(B(2 \rho))^{1/2},$$

we immediately see

$$\int \frac{|\nabla u|^2}{u^2} \leq k_1 \rho^2$$

for some $k_1 = k_1(c_2)$. So

$$\int_{B(\rho)} |\nabla \log u| dv_0 = \int_{B(\rho)} \left| \frac{\nabla u}{u} \right| dv_0 \leq \left( \int \frac{|\nabla u|^2}{u^2} \right)^{1/2} \text{Vol}(B(\rho))^{1/2} \leq k \rho^3.$$

The proof of (5.6) goes the same as in [CY1]. Namely, we only need to apply the Jonh-Nirenberg inequality ([JN], see also [GT]) to the function $\log u$.

Finally by applying the Nash-Moser iteration as in [CY1], with their lemma 2 replace by our lemma 5.4, one can prove the following proposition. (The proof will be included in the appendix.)

Proposition 5.5 ([CY1], Proposition C and Remark). Suppose \{u_j\} is a sequence of functions as in Proposition 5.3 with $x_0$ its concentration point. Then for each fixed $r$ that is small enough and each $p \geq 2$, there exists some integer $j(r, p)$ and some universal constant $C = C(p, p_0)$ so that

$$\int_{B(x_0, r) - B(x_0, \frac{r}{2})} u_j^p dv_0 \leq C \int_{B(x_0, 2r) - B(x_0, r)} u^p_j dv_0.$$
for all \( j \geq j(r, p) \).

We end this section by the following proposition that serves as a replacement of proposition D in [CY1]:

**Proposition 5.6.** Let \( \{u_j\} \) be a sequence of positive smooth functions that satisfies the assumptions of Proposition 5.3, and so that the conformal sequence \( g_j = u_j^2 g_0 \) satisfies (3.6). Then there exist constants \( C_j > 0 \) with \( C_j \to \infty \) so that the sequence

\[
v_j = C_j u_j
\]

converges uniformly on compact subset of \( M \setminus \{x_0\} \) to the Green’s function of the conformal Laplacian \( L = -6\Delta_0 + S_0 \).

**Proof.** For simplicity we denote \( B_r = B(x_0, r) \) and \( B^c_r = M \setminus B_r \). In what follows we will fix a small ball \( B_r \) and choose a constant \( C_j \) so that

\[
C_j^4 \int_{B^c_r} u_j^4 dv_0 = 1.
\]

According to Proposition 5.3, it is clear that \( C_j \to \infty \).

By (3.6), one can find constant \( D \) such that

\[
\int_M |S_{g_j}|^4 dv_{g_j} \leq D.
\]

Notice that if we denote \( \tilde{g}_j = v_j^2 g_0 \), then

\[
\int_{B^c_r} \left( \frac{Lv_j}{v_j^2} \right)^4 dv_0 = \int_{B^c_r} S_{g_j}^4 v_j^4 dv_0 = \frac{1}{C_j^4} \int_{B^c_r} S_{g_j}^4 u_j^4 dv_0.
\]

So as \( j \to \infty \),

\[
\int_{B^c_r} (Lv_j)^{4/3} dv_0 \leq \left( \int_{B^c_r} \left( \frac{Lv_j}{v_j^2} \right)^4 dv_0 \right)^{1/3} \left( \int_{B^c_r} v_j^4 dv_0 \right)^{2/3} \leq \left( \frac{D}{C_j^4} \right)^{1/3} \to 0.
\]

It follows that \( \{v_j\} \) has a subsequence converges strongly in \( W^{2,4/3} \) to a solution \( \omega \) of the equation

\[
L \omega = 0
\]

on \( B^c_r \). We need to verify that \( \omega \) is strictly positive. Since \( S_0 > 0 \), according to the minimum principle for elliptic operators it is enough to prove \( \omega \neq 0 \) on \( B^c_r \).

Assume on the contrary that \( \omega \equiv 0 \) on \( B^c_r \). Then

\[
\lim_{j \to \infty} \int_{B^c_r} v_j^2 = \int_{B^c_r} \omega^2 = 0.
\]

This implies

\[
\lim_{j \to \infty} \int_{B^c_{r/2}} v_j^2 = 0,
\]
since by Proposition 5.5
\[ \int_{B_{r/2}^c} v_j^2 = C_j^2 \left( \int_{B_r \setminus B_{r/2}} u_j^2 + \int_{B_r^c} u_j^2 \right) \leq CC_j^2 \int_{B_r^c} u_j^2 = C \int_{B_r^c} v_j^2. \]
Also if we apply Proposition 5.5 with \( p = 4 \), we get
\[ \int_{B_{r/2}^c} v_j^4 = C_j^4 \left( \int_{B_r \setminus B_{r/2}} u_j^4 + \int_{B_r^c} u_j^4 \right) \leq \tilde{C} C_j^4 \int_{B_r^c} u_j^4 = \tilde{C} \int_{B_r^c} v_j^4 = \tilde{C}, \]
where \( \tilde{C} \) is a constant that depends only on \( p_0 \).

On the other hand, by choosing a cut-off function \( \eta \) so that
\[ |\nabla \eta| \leq \frac{c}{r}, \quad \text{and} \quad \eta \equiv 1 \text{ on } B_r^c, \quad \eta \equiv 0 \text{ on } B_r, \]
and applying Lemma 5.1 with \( \beta = 1 \) and \( u = v_j \) we get
\[
\left( \int_M \eta^4 v_j^4 \right)^{1/2} \leq \frac{C}{r^2} \int_{B_{r/2}^c} v_j^2 + \frac{2C_s C_{g_j}^{1/2}}{3} \left( \int_M v_j^4 \eta^4 \right)^{1/2} + \frac{2C_s \int_M S_{g_j} v_j^4}{3} \int_M v_j^2 \eta^2 \\
= \frac{C}{r^2} \int_{B_{r/2}^c} v_j^2 + \frac{2C_s C_{g_j}^{1/2}}{3} \left( \int_M v_j^4 \eta^4 \right)^{1/2} + \frac{2C_s \int_M S_{g_j} u_j^4}{3} \frac{1}{C_j^2} \int_{B_{r/2}^c} v_j^4.
\]
Dividing both sides by \( \left( \int_M \eta^4 v_j^4 \right)^{1/2} \geq 1 \) and letting \( j \to \infty \) we get
\[ 1 \leq \frac{2C_s C_{g_j}^{1/2}}{3}, \]
which contradicts to (1.5).

The rest of the proof goes exactly like [CY1]: One can apply the standard diagonal trick to construct a sequence of functions \( v_j = c_j u_j \), such that on \( M \setminus \{ x_0 \} \), \( v_j \) converges to a positive solution \( \omega \) of \( L \omega = 0 \). Then we apply the isolated singularity theorem of Gilbarg-Serrin [GS] to conclude that \( \omega \sim d(x,x_0)^{-2} \) which is the Green function of the conformal Laplacian.

6. Proof of theorem 1.3

Finally we will prove theorem 1.3 which will also complete our proof of theorem 1.1.

Proof of theorem 1.3. We first prove that
\[ g_\omega := \omega^2 g_0 \]
defines a flat metric on \( M \setminus \{ x_0 \} \). In fact, by (3.6) one can find a constant \( C \) so that
\[ \int_M |W(g_j)|_{g_j}^4 u_j^4 dv_0 \leq C. \]
So as $j \to \infty$ we have
\[
\int_M \left| W(\tilde{g}_j) \right|^4_{\tilde{g}_j} v_j^4 \, dv_0 = \frac{1}{C^4_j} \int_M \left| W(g_j) \right|^4_{g_j} u_j^4 \, dv_0 \to 0,
\]
where $\tilde{g}_j = v_j^2 g_0$ as we used in the proof of proposition 5.6. Now let $K$ be any compact subset of $M \setminus \{x_0\}$. Then
\[
L v_j \to L \omega = 0
\]
in $L^{4/3}$, hence $v_j \to \omega$ in $W^{2,4/3}$. By Fatou lemma,
\[
\int_K \left| W(g_\omega) \right|^4_{g_\omega} \omega^4 \, dv_0 \leq \liminf_{j \to \infty} \int_K \left| W(\tilde{g}_j) \right|^4_{\tilde{g}_j} v_j^4 \, dv_0 = 0.
\]
Hence the Weyl tensor $W(g_\omega) = 0$ on $M \setminus \{x_0\}$. Apply the same argument to $Ric$, we also get $Ric(g_\omega) = 0$ on $M \setminus \{x_0\}$. So $g_\omega$ is a flat metric on $M \setminus \{x_0\}$.

Next we prove that the Euler characteristic $\chi(M) > 0$. Suppose on the contrary that $\chi(M) \leq 0$. Then we take $r > r'$ small and denote $A = B_r(x_0)$, $B = M \setminus B_{r'}(x_0)$ such that $B$ can be retractable to $M \setminus \{x_0\}$. Then $A \cap B$ is homotopic to $S^3$, while $A \cup B = M$. So we have the following Mayer-Vietoris sequence
\[
\cdots \to H_4(S^3) \to H_4(A) \oplus H_4(B) \to H_4(M) \to H_3(S^3) \to H_3(A) \oplus H_3(B) \to H_3(M) \to \cdots
\]
\[
\to H_2(S^3) \to H_2(A) \oplus H_2(B) \to H_2(M) \to H_1(S^3) \to H_1(A) \oplus H_1(B) \to H_1(M) \to H_0(S^3) \to H_0(A) \oplus H_0(B) \to H_0(M) \to 0.
\]
We can rewrite it as
\[
0 \to H_4(B) \xrightarrow{\phi_1} H_4(M) \xrightarrow{\phi_2} \mathbb{Z} \xrightarrow{\phi_3} H_3(B) \xrightarrow{\phi_4} H_3(M) \xrightarrow{\phi_5} H_2(B) \xrightarrow{\phi_6} H_2(M) \xrightarrow{\phi_7} 0
\]
\[
\to 0 \to H_1(B) \to H_1(M) \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0,
\]
from which we get isomorphisms
\[
H_1(M) \cong H_1(B), \quad H_2(B) \cong H_2(M)
\]
and the relation
\[
\text{rank } H_4(B) - \text{rank } H_4(M) + 1 - \text{rank } H_3(M) + \text{rank } H_3(B) = 0.
\]
So our assumption $\chi(M) \leq 0$ implies $\chi(B) \leq -1$ i.e. $\chi(M \setminus \{x_0\}) \leq -1$. This contradicts with corollary 3.3.5 in [Wolf].
So we must have $\chi(M) > 0$. According to [CGY], $M$ is diffeomorphic to either $S^4$ or $\mathbb{R}P^4$. We now show that $M$ cannot be diffeomorphic to $\mathbb{R}P^4$. Our argument is close to the proof of corollary 3.3.5 in [Wolf]. In fact, suppose $M \simeq \mathbb{R}P^4$, then we have

$$\pi_1(M \setminus \{x_0\}) \cong \pi_1(M) \cong \mathbb{Z}_2. \quad (6.1)$$

But by theorem 3.3.3 and theorem 3.3.1 in [Wolf], $M \setminus \{x_0\}$ admits a deformation retraction onto a compact totally geodesic submanifold $N$, and $N$ admits an $r$-fold covering by a torus $T$ for some $r > 0$. If $\dim T > 0$, then $\pi_1(N)$ contains a free abelian subgroup, which contradicts to (6.1). If $\dim T = 0$, then $M \setminus \{x_0\}$ contracts to a point, which contradicts to (6.1) again.

So we finally arrive at the conclusion that $M$ is diffeomorphic to $S^4$. Hence we have

$$\chi(M \setminus \{x_0\}) = 1.$$ 

We can apply corollary 3.3.5 in [Wolf] to conclude that $(M \setminus \{x_0\}, w^2 g_0)$ is isometric to the flat $\mathbb{R}^4$. Hence by Liouville’s theorem, $(M, g)$ is conformally equivalent to the round sphere $(S^4, g_0)$. This completes the proof.

**Appendix: The proofs of proposition 5.3 and 5.5**

For completeness, we will include the detailed proofs of proposition 5.3 and proposition 5.5 here.

**Proof of Proposition 5.3.**

As in [CY1], we will prove the proposition in two steps.

- Step I: The set of points where the mass of some subsequence of $\{u_j\}$ accumulates is nonempty:

$$A = \left\{ x \in M : \lim_{r \to 0} \lim_{j \to \infty} \int_{B(x,r)} u_j^4 \neq 0 \right\} \neq \emptyset.$$ 

- Step II: The set $A$ above consists of exactly one point $x_0$.

**Proof of Step I.** Suppose for each $x \in M$, we have

$$m_x := \lim_{r \to 0} \lim_{j \to \infty} \int_{B(x,r)} u_j^4 = 0.$$ 

Then for any $x \in M$ fixed and any $\varepsilon > 0$, one can find a subsequence $\{u_j\}$ so that as $j \to \infty$ and $r$ sufficiently small,

$$\int_{B(x,r)} u_j^4 < \varepsilon.$$ 

We fix $r$ small and choose a cut-off function $\eta$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B(x, \frac{r}{2}), \quad \eta \equiv 0 \text{ off } B(x, r), \quad \text{and } |\nabla \eta| \leq \frac{c}{r}.$$
Applying lemma 5.1 to $\beta = 1, \omega = u_j$ and the above $\eta$, and notice
\[
\int_M u_j^2 \omega^2 \eta^2 \leq \left( \int_{\text{supp} \eta} \omega^4 \eta^4 \right)^{1/2} \left( \int_{\text{supp} \eta} u_j^4 \right)^{1/2},
\]
we obtain, for some constant $C$,
\[
\left( \int \eta^4 \omega^4 \right)^{1/2} \leq \frac{2C_s C_g^{1/2}}{3} \left( \int \eta^4 \omega^4 \right)^{1/2} + \frac{C}{r^2} \int_{B(x,r)} u_j^2 dv_0 + \frac{2C_s}{3} \left( \int M \sup \eta \omega \eta \right)^{1/2} \left( \int \omega^4 \eta^4 \right)^{1/2} \left( \int_{B(x,r)} u_j^4 \right)^{1/2},
\]
where we used $C_s C_g^{1/2} \leq \frac{1}{8}$. So for $\varepsilon$ sufficiently small and $j$ large we have
\[
\frac{1}{2} \left( \int_{B(x,r/2)} u_j^4 dv_0 \right)^{1/2} \leq \frac{1}{2} \left( \int_{B(x,r)} \eta^4 \omega^4 \right)^{1/2} \leq \frac{C}{r^2} \int_{B(x,r)} u_j^2 dv_0.
\]
Now we cover $M$ by finitely many such balls $B(x_1, \frac{r_1}{2}), \cdots, B(x_N, \frac{r_N}{2})$. Then
\[
a_0 = \int_M u_j^4 dv_0 \leq \sum_{k=1}^{N} \int_{B(x_k, r_k/2)} u_j^4 dv_0 \leq 4 \sum_{k=1}^{N} \left( \frac{C}{r^2} \int_{B(x_k, r_k)} u_j^2 dv_0 \right)^2 \to 0,
\]
where we used lemma 5.2, which is a contradiction.

Proof of Step II. Assume we have at least two points $x_1, x_2 \in A$. By passing to a subsequence of $\{u_j\}$ we may assume
\[
\lim_{r \to 0} \lim_{j \to \infty} \int_{B(x_k, r)} u_j^4 = m_k, \quad k = 1, 2.
\]
As in [CY1] we let $\rho = \text{dist}(x_1, x_2)$ and set
\[
\mu_1 = \limsup_{j \to \infty} \int_{B(x_1, 1) \setminus B(x_1, 1/2)} u_j^4 dv_0,
\]
which, by passing to a subsequence, becomes a limit. We then inductively choose subsequences of subsequences so that
\[
\mu_t = \lim_{j \to \infty} \int_{B(x_1, 2^{-t+1}) \setminus B(x_1, 2^{-t})} u_j^4 dv_0.
\]
Since $\sum_{j=1}^{\infty} \mu_t \leq a_0$, we can find $l_0$ so that for $l \geq l_0$ we have
\[
\mu_l \leq \frac{(a_0)^{1/2}}{4} \left( \frac{1}{m_1} + \frac{1}{m_2} \right)^{-1/2}.
\]
We do the same argument near $x_2$ and choose a common $l_0$. Then we pick $\rho_0 \leq \min\{\rho/2, 2^{-l_0}\}$ small so that for all $r \leq 2\rho_0$ and $j$ sufficiently large,

$$\left| \int_{B(x_k, r)} u_j^4 d\nu_0 - m_k \right| \leq \varepsilon, \quad k = 1, 2.$$ 

Then we choose $\phi$ to be a $C^\infty$-function on $M$ with

$$\phi = \begin{cases} \frac{1}{m_1}, & \text{on } B(x_1, \rho_0), \\ \frac{1}{m_2}, & \text{on } B(x_2, \rho_0), \\ 0, & \text{on } (B(x_1, 2\rho_0) \cup B(x_2, 2\rho_0))^c \end{cases}$$

and we extend $\phi$ “linearly” in the rest of $M$, then

$$\int u_j^4 \phi^2 \geq \frac{m_1 - \varepsilon}{m_1^2} + \frac{m_2 - \varepsilon}{m_2^2} \geq \frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right).$$

and

$$\left| \int \phi u_j^4 \right| \leq \left| \int_{B(x_1, \rho_0) \cup B(x_2, \rho_0)} \phi u_j^4 \right| + \left| \int_{\cup_{k=1}^2 (B(x_k, 2\rho_0) \setminus B(x_k, \rho_0))} \phi u_j^4 \right|

\leq \left( \frac{m_1 + \varepsilon}{m_1} - \frac{m_2 - \varepsilon}{m_2} \right) + a_0^{1/2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right)^{1/2}

\leq \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \varepsilon + a_0^{1/2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right)^{1/2}$$

and

$$\int |\nabla \phi|^2 u_j^2 \leq C \left( \frac{1}{\rho_0^6m_2^2} + \frac{1}{\rho_0^6m_1^2} \right) \int_M u_j^2.$$ 

According to the Rayleigh-Ritz inequality, for any smooth function $\varphi$ we have

$$\int \varphi^2 dv \leq \left( \int dv \right)^{-1} \left( \int \varphi dv \right)^2 + \frac{1}{\lambda_1} \int |\nabla \varphi|^2 dv.$$ 

Apply this to the metric $g = u_j^2 g_0$, we get

$$\lambda_1 \leq \frac{a_0}{a_0} \int |\nabla \phi|^2 u_j^2 \leq \frac{Ca_0}{a_0} \left( \frac{1}{\rho_0^6m_2^2} + \frac{1}{\rho_0^6m_1^2} \right) \int_M u_j^2 \rightarrow 0,$$

where we used lemma 5.2 again in the last step. This contradicts with the isospectrality.

**Proof of Proposition 5.5.**

Given any $p \geq 2$, we choose $\bar{p}$ to be the smallest number of the form $3 \times 2^l$ that is greater than or equal to $p$, where $l$ is a non-negative integer. Let $p_0$ be a constant so that lemma 5.4 holds. Note that by Hölder’s inequality,
\[ p_0 \text{ can be taken to be any sufficiently small number, and we will take } p_0 \text{ to be a number such that } 0 < p_0 < \frac{1}{4} \text{ and such that there exists } m \in \mathbb{N} \text{ with } 2^{m+1} p_0 = \bar{p}. \]

As in [CY1], we denote \( B_r = B(x_0, r) \) and \( B_{r_1, r_2} = B_{r_1} \setminus \overline{B_{r_2}}. \) We will fix \( r \) small enough such that \( B_{4r} \) is contained in a normal coordinate patch at \( x_0. \) Now for \( k = 0, 1, 2, \ldots, m \) we define \( \delta_k \) and \( \beta_k \) by

\[
\delta_k = \frac{r}{2^{k+2}}, \quad 1 + \beta_k = 2^k p_0.
\]

Then for each \( k \leq m \) we have \( |1 + \beta_k| \leq \bar{p}/2. \) Moreover, the minimum of \( |\beta_k| \) is attained at \( |\beta_{m-1}| = 1/4. \) So we get \( \frac{|1+\beta_k|^2}{|\beta_k|} \leq \bar{p}^2 \) and \( \frac{|1+\beta_k|^2}{1/4} \leq 4\bar{p}^2 \) for all \( k. \) Next we let \( \rho_m = r, \sigma_m = \frac{r}{2}, \) and define \( \rho_k, \sigma_k (1 \leq k \leq m-1) \) by

\[
\rho_{k+1} = \rho_k + \delta_k, \quad \sigma_{k+1} = \sigma_k - \delta_k.
\]

Note that for each \( k \) we have \( \sigma_k - \delta_k > r/4, \) so \( \text{supp } \eta_k \subset B^{c}_{4\rho - \delta_k} \subset B^{c}_r. \) Moreover, by definition \( \rho_0 + \delta_0 < 3r. \) As a consequence, for each \( k \) the triples of numbers \( (\rho, \sigma, \delta) = (\rho_k, \sigma_k, \delta_k) \) satisfies \( 2\delta < \sigma < \rho + 3r \) and \( \delta < r. \) We choose a smooth cut-off function \( \eta = \eta_k \) so that

\[
\eta \equiv 1 \text{ on } B_{\rho, \sigma}, \quad \text{supp } \eta \subset B_{\rho + \delta, \sigma - \delta}, \quad |\nabla \eta| \lesssim \frac{1}{\delta} \text{ on its support.}
\]

Now for \( \beta = \beta_k \) and \( u = u_j \) with \( j, \) we let

\[
A_{\beta, u, \eta} = 1 - C_s \frac{|1 + \beta|^2}{6|\beta|} \left( \int_M S_g^4 u^4 \right)^{1/4} \left( \int_{\text{supp } \eta} u^4 \right)^{1/4}.
\]

Then for \( j \) large enough we have \( A_{\beta, u, \eta} > 1/2, \) since

1. by our choice of \( \beta = \beta_k, \) we have \( |1 + \beta_k|^2/|\beta_k| \leq \bar{p}^2, \)
2. according to Proposition 3.5, the integral \( \int_M S_g^4 u^4 \) is bounded,
3. we have supp \( \eta \subset B^c_{4\rho}, \) so for \( j > j(r, p) \) large enough the integral \( \int_{\text{supp } \eta} u^4 \) is sufficiently small.

Applying the last step of the proof of lemma 5.1 to \( \eta \) and \( \omega = u \frac{\omega^4}{\omega^2}, \) we get

\[
\left( \int \eta^4 \omega^4 \right)^{1/2} \leq 2C_s \left( \frac{|1 + \beta|^2}{|\beta|} \int |\nabla \eta| \omega^2 + \frac{|1 + \beta|^2}{|\beta|} |S_0| \int \omega^2 \eta^2 \right)^{1/2} + \frac{|1 + \beta|^2}{12|\beta|} \int S_g u^2 \omega^2 \eta^2 + 2C_s \int w^2 |\nabla \eta|^2 + K_s \int \omega^2 \eta^2
\]

\[
\leq 2C_s \left( \frac{|1 + \beta|^2}{|\beta|} + 1 \right) \int |\nabla \eta| \omega^2 + \left[ 2C_s \frac{|1 + \beta|^2}{12|\beta|} |S_0| + K_s \right] \int \omega^2 \eta^2
\]

\[
+ C_s \frac{|1 + \beta|^2}{6|\beta|} \left( \int_M S_g^4 u^4 \right)^{1/4} \left( \int_{\text{supp } \eta} u^4 \right)^{1/4} \left( \int \eta^4 \omega^4 \right)^{1/2},
\]
which implies
\[ A_{\beta, u, \eta} \left( \int_{B_{p, \sigma}} \omega^4 \right)^{1/2} \leq \frac{B_\beta}{\delta^2} \int_{B_{p+\delta, \sigma-\delta}} \omega^2, \]

where
\[ B_\beta = C \left( \frac{|1 + \beta|^2}{|\beta|^2} + \frac{|1 + \beta|}{|\beta|} |S_0| + 1 \right), \]

and \( C \) is a universal constant that depends only on \( C_s, K_s \). We denote \( \Phi(u, \rho, \sigma) = \left( \int_{\Omega} u^p \right)^{1/p} \).

Since \( A_{\beta, u, \eta} > 1/2 \) and \( \frac{|1+\beta_k|^2}{|\beta_k|^2} \leq 16p^2 \), there is a universal constant \( C'(p) \) such that \( B_\beta/A_{\beta, u, \eta} < C'(p) \). So the formula (A.2) reads

(A.3)
\[ \Phi(u, 2(1 + \beta), B_{\rho, \sigma}) \leq (C'/\delta^2)^{1/|1+\beta|} \Phi(u, 1 + \beta, B_{\rho+\delta, \sigma-\delta}) \text{ if } 1 + \beta > 0 \]

and

(A.4)
\[ \Phi(u, 1 + \beta, B_{\rho+\delta, \sigma-\delta}) \leq (C'/\delta^2)^{1/|1+\beta|} \Phi(u, 2(1 + \beta), B_{\rho, \sigma}) \text{ if } 1 + \beta < 0. \]

Now we iteratively apply (A.3) to get
\[ \Phi(u_j, \bar{p}, B_{r,r/2}) \leq \prod_{k=0}^{m} (C'/\delta_k^2)^{1/|1+\beta_k|} \Phi(u_j, p_0, B_{p_0+\delta_0, \sigma_0-\delta_0}) \]
\[ \leq \prod_{k=0}^{m} (C'/\delta_k^2)^{1/|1+\beta_k|} \Phi(u_j, p_0, B_{3r,r/4}). \]

Let \( c = \sum_{k=0}^{m} \frac{2k+6}{2^{k}} \). Then
\[ \prod_{k=0}^{m} \left( \frac{C'}{\delta_k^2} \right)^{1/|1+\beta_k|} \leq (C')^{\sum_{k=0}^{m} \frac{1}{2^{k}}} \sum_{k=0}^{m} \left( \frac{2^{k}+6}{2^{k}p_0} \frac{2^{k}+6}{2^{k+1}} r^{-2} \sum_{k=0}^{m} \frac{1}{2^{k}} \right) \leq (C')^{\frac{2}{p_0}} \frac{2^{m}c}{2^{m+1}} r^{-4(\frac{1}{p_0} - \frac{1}{\bar{p}})}. \]

So we get

(A.5)
\[ \Phi(u_j, \bar{p}, B_{r,r/2}) \leq (C')^{\frac{2}{p_0}} \frac{2^{m}c}{2^{m+1}} r^{-4(\frac{1}{p_0} - \frac{1}{\bar{p}})} \Phi(u_j, p_0, B_{3r,r/4}). \]

Similarly we let \( \tilde{\beta}_k \) be such that \( 1 + \tilde{\beta}_k = -2^k p_0 \), let \( \tilde{\delta}_k = \frac{r}{2^{k+1}} \) as before, let \( \tilde{\rho}_m = 2r, \tilde{\sigma}_m = r \) and let \( \tilde{\rho}_{k-1} = \tilde{\rho}_k + \tilde{\delta}_k, \tilde{\sigma}_{k-1} = \tilde{\sigma}_k - \tilde{\delta}_k \) for \( 0 \leq k \leq m \).

One can check that all the inequalities we need among these quantities are
satisfied. We then iteratively use (A.4) to get
\[
\Phi(u_j, -p_0, B_{3r,r/4}) \leq \Phi(u_j, 1 + \tilde{\beta}_0, B_{\rho_0 + \delta_0, \sigma_0 - \delta_0})
\]
(A.6)
\[
\leq \prod_{k=0}^m \left( C'/\delta_k^2 \right)^{1/\left(1 + \tilde{\beta}_k\right)} \Phi(u_j, -\bar{p}, B_{2r,r})
\]
\[
\leq (C'/\delta_0)^{2/3} \frac{2}{160} r^{-4 \left( \frac{1}{p} - \frac{1}{p'} \right)} \Phi(u_j, -\bar{p}, B_{2r,r}).
\]

By definition,
\[
\Phi(u_j, p_0, B_{3r,r/4})^{p_0} = \int_{B_{3r,r/4}} u_j^{p_0} \leq \int_{B_{3r}} u_j^{p_0}
\]
and
\[
\Phi(u_j, -p_0, B_{3r,r/4})^{-p_0} = \int_{B_{3r,r/4}} u_j^{-p_0} \leq \int_{B_{3r}} u_j^{-p_0}.
\]

Apply lemma 5.4 to \( u = u_j \) with \( B_\rho = B_{3r} \), we get
\[
\Phi(u_j, p_0, B_{3r,r/4})^{p_0} \Phi(u_j, -p_0, B_{3r,r/4})^{-p_0} \leq \int_{B_{3r}} u_j^{p_0} \int_{B_{3r}} u_j^{-p_0} \leq C(p_0) r^8,
\]
or in other words,
(A.7) \[
\Phi(u_j, p_0, B_{3r,r/4}) \leq C'(p_0)^{r^8/p_0} \Phi(u_j, -p_0, B_{3r,r/4}).
\]
So we get
\[
\Phi(u_j, p, B_{r,r/2}) \leq C_1 \Phi(u_j, \bar{p}, B_{r,r/2})^{\frac{4}{p} - \frac{4}{p'}} \quad \text{(by Hölder)}
\]
\[
\leq C_2 r^{\frac{1}{p} - \frac{1}{p'}} \Phi(u_j, p_0, B_{3r,r/4})^{\frac{4}{p} - \frac{4}{p'}} \quad \text{(by (A.5))}
\]
\[
\leq C_3 r^{\frac{1}{p} - \frac{1}{p'}} \frac{2}{160} r^{-4 \left( \frac{1}{p} - \frac{1}{p'} \right)} \Phi(u_j, -\bar{p}, B_{2r,r})^{\frac{4}{p} - \frac{4}{p'}} \quad \text{(by (A.7))}
\]
\[
\leq C_4 r^{\frac{1}{p} - \frac{1}{p'}} \frac{2}{160} r^{-4 \left( \frac{1}{p} - \frac{1}{p'} \right)} \Phi(u_j, -\bar{p}, B_{2r,r})^{\frac{4}{p} - \frac{4}{p'}} \quad \text{(by (A.6))}
\]
\[
= C_4 r^{\frac{1}{p} - \frac{4}{p'}} \Phi(u_j, -p, B_{2r,r}) \quad \text{(by Hölder)}
\]
\[
\leq C_5 r^{\frac{4}{p'}} \Phi(u_j, -p, B_{2r,r}) \quad \text{(by Cauchy-Schwartz)}
\]
\[
\leq C_6 r^{\frac{4}{p'}} \Phi(u_j, p, B_{2r,r}) \quad \text{(by Cauchy-Schwartz)}
\]
\[
= C_6 \Phi(u_j, p, B_{2r,r}).
\]

where \( C_6 \) is a constant that depends only on \( p_0 \). This completes the proof.

References

[And] M. Anderson, Remarks on the compactness of isospectral sets in lower dimensions, Duke Math. J. 63 (1991), 699-711.

[Aub] T. Aubin, Nonlinear analysis on manifolds.Monge-Ampere equations, Grundlehren Math. Wiss., Springer-Verlag, New York,1982.
[AuL] T. Aubin and Y. Li, On the best Sobolev inequality, J. Math. Pures Appl. 78 (1999), 353-387.

[Bec] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, Ann. Math. (2) 138 (1993), 213-242.

[BrG] R. Brooks and C. Gordon, Isospectral families of conformally equivalent Riemannian metrics, Bull. Amer. Math. Soc. 23 (1990), 433-436.

[BPP] R. Brooks, P. Perry, and P. Petersen, Compactness and finiteness theorems for isospectral manifolds, J. Reine Angew. Math. 426 (1992), 67-89.

[BPY] R. Brooks, P. Perry and P. Yang, Isospectral sets of conformally equivalent metrics, Duke Math. J. 58 (1989), 131-150.

[C] A. Chang, Nonlinear elliptic equations in conformal geometry, Zurich Lectures in Advanced Mathematics, European Mathematical Society, 2004.

[CGY] A. Chang, M. Gursky and P. Yang, A conformally invariant sphere theorem in four dimensions, Publications Mathmatiques de l'Institut des Hautes tudes Scientifiques, Vol 98 (2003), pp 105-143.

[Ch] J. Cheeger, Finiteness theorems for Riemannian manifolds, Amer. J. Math. 92 (1970), 61-74.

[CY1] A. Chang and P. Yang, Isospectral conformal metrics on 3-manifolds, J. Amer. Math. Soc. 3 (1990), 117-145.

[CY2] A. Chang and P. Yang, Compactness of isospectral conformal metrics on $S^4$, Comment. Math. Helv. 64 (1989), 363-374.

[ChZ] B. Chen and X. Zhu, A conformally invariant classification theorem in four dimensions, Comm. Anal. Geom. 22 (2014), 811-831.

[ChX] R. Chen and X. Xu, Compactness of isospectral conformal metrics and isospectral potentials on a 4-manifold, Duke Math. J. 84 (1996), 131-154.

[GS] D. Gilbarg and J. Serrin, On isolated singularities of solutions of second order elliptic differential equation, J. Analyse Math. 4 (1955), 309-340.

[GT] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of Second Order, Berlin-Heidelberg-New York-Tokyo, Springer-Verlag, 1983.

[Gil1] P. Gilkey, The spectral geometry of a Riemannian manifold, J. Differential Geom. 10 (1975).

[Gil2] P. Gilkey, Recursion relations and the asymptotic behavior of eigenvalues of the Laplacian, Compositio Math. 38 (1979), 201-240.

[Gor] C. Gordon, Isospectral Families of Conformally equivalent Riemannian metrics, Bull. of the A.M.S. 23 (1990), 433-436.

[Gro] M. Gromov, Structures Matriques pour les Varieties Riemannnienes, Textes Mathematiques 1, CEDIC, Paris, 1981.

[Gur] M. Gursky, Compactness of conformal metrics with integral bounds on curvature, Duke Math. J. 72 (1993), 339-367.

[JN] F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math. 14 (1961), 415-426.

[Mil] J. Milnor, Eigenvalues of the Laplace operator on certain manifolds, Proc. Natl. Acad. Sci. USA 51 (4): 542.

[OPS] B. Osgood, R. Phillips and P. Sarnak, Compact isospectral sets of surface, J. Funct. Anal. 80 (1988), 212-234.

[Sak] T. Sakai, On eigenvalues of Laplacian and curvature of Riemannian manifolds, Tohoku Math. J. 23 (1971), 589-603.

[Sun] T. Sunada, Riemannian coverings and isospectral manifolds, Annals of Mathematics 121 (1): 169-186.
[Wolf] J. Wolf, Spaces of Constant Curvature, AMS Chelsea Publishing. Sixth Edition, 2011.

[Xu1] X. Xu, On compactness of isospectral conformal metrics of 4-manifolds, Nagoya Math. J. 140 (1995), 77-99.

[Xu2] X. Xu, On compactness of isospectral conformal metrics of 4-sphere, Comm. Anal. Geom. 3 (1995), 335-370.

[Yau] S.T. Yau, Selected Expository Works of Shing-Tung Yau with Commentary, Vols. 28 of the Advanced Lectures in Mathematics series. International Press of Boston (2014).

[Zel] S. Zelditch, Survey on the inverse spectral problem, Notices of the International Congress of Chinese Mathematicians 2 (2014), 1-21.

[Zhou] G. Zhou, Compactness of isospectral compact manifolds with bounded curvatures, Pac. Jour. Math. 181 (1997), 187-200.

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