Higher Genus Correlators from the Hermitian One-Matrix Model

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Abstract

We develop an iterative algorithm for the genus expansion of the hermitian $N \times N$ one-matrix model (= the Penner model in an external field). By introducing moments of the external field, we prove that the genus $g$ contribution to the $m$-loop correlator depends only on $3g - 2 + m$ lower moments ($3g - 2$ for the partition function). We present the explicit results for the partition function and the one-loop correlator in genus one. We compare the correlators for the hermitian one-matrix model with those at zero momenta for $c = 1$ CFT and show an agreement of the one-loop correlators for genus zero.

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1 Introduction

Recently there has renewed interest in the hermitian one-matrix model which is defined by the partition function

$$Z_N[t.] = \int DX \ e^{-\sum_{k=0}^{\infty} t_k \text{tr} X^k}$$

where the integration goes over $N \times N$ hermitian matrices. As is proven in Ref.[1], this model is equivalent as $N \to \infty$ to the following model in an external field:

$$Z_N[\eta; \alpha] = e^{-\frac{N}{2} \text{tr} \eta^2} \int DX \ e^{N \text{tr} \left(-\frac{1}{2} X^2 + \eta X + \alpha \log X\right)}$$

with $\eta$ and $\{t_k\}$ being related by the Miwa transformation

$$t_k = \frac{1}{k} \text{tr} \eta^{-k} - \frac{N}{2} \delta_{k2} \quad \text{for } k \geq 1, \quad t_0 = \text{tr} \log \eta^{-1}.$$  

The partition function (1.2), in turn, is associated [2] with an external field problem for the Penner model [3]:

$$Z_N[\Lambda; \alpha] = e^{-\frac{\alpha^2 N}{2} \text{tr} \Lambda^{-2}} (\det \Lambda)^{N(\alpha+1)} \int DX \ e^{N \text{tr} \left(-\frac{1}{2} \Lambda X X + \alpha [\log (1+X) - X]\right)}$$

providing $\eta = \Lambda - \alpha \Lambda^{-1}$. The extra coefficients are introduced to provide

$$Z_{\alpha N}[t.] = Z_N[\eta; \alpha] = Z_N[\Lambda; \alpha]$$

(1.5)

to any order of the genus expansion.

A surprising property of the hermitian one-matrix model, which is advocated in Refs.[1, 2], is that it reveals some features of $c = 1$ CFT interacting with 2D gravity. For the case of the Penner model, this property was discovered by Distler and Vafa [4] in the double-scaling limit and has been extended to more general models by Tan [5], by Chaudhuri, Dykstra and Lykken [6] and by Gilbert and Perry [7], in particular to the model (1.4) with $\Lambda$ being proportional to a unit matrix. However, these features of $c = 1$ have been observed in Refs.[4, 6] in genus zero and genus one, identifying $\alpha$ with the cosmological constant, without taking the double scaling limit. Therefore, it was conjectured that the hermitian one-matrix model has something to do with the continuum $c = 1$ case similar to the Kontsevich model [8] which is associated with the continuum pure 2D gravity ($c = 0$) or to its generalizations [9] which are associated with $c < 1$.

A direct way to verify this conjecture is to study correlation functions of the loop operators. For the hermitian one-matrix model with an arbitrary (not necessarily symmetric) potential, the one-loop correlator has been known in genus zero for a long time [10], while the two- and three-loop correlators were obtained in Ref.[11]. The genus one contribution to the one-loop correlator was explicitly calculated in Ref.[12] for a quartic symmetric potential and in Ref.[4] for an arbitrary symmetric potential. On the other
hand, much is known now about \( c = 1 \) correlators since pioneering works by Kostov [13] and Boulatov [14]. An incomplete list of references includes [15]–[24].

In the present paper we develop an iterative algorithm for calculating the genus expansion of the hermitian \( N \times N \) one-matrix model (= the Penner model in an external field). We introduce the moments, \( I_p \) and \( J_p \), of the external field as well as the ‘basis vectors’, \( \chi^{(n)} \) and \( \psi^{(n)} \), which are determined by a recursion relation and diagonalize the iterative procedure. We prove that the genus \( g \) contribution to the partition function depends only on \( I_p \) and \( J_p \) with \( p \leq 3g - 2 \) while the genus \( g \) contribution to the \( m \)-loop correlator depends on \( I_p \) and \( J_p \) with \( p \leq 3g - 2 + m \). We present the explicit calculation of the partition function and the one-loop correlator for an arbitrary potential in genus one. We compare the correlators for the hermitian one-matrix model with those at zero momenta for \( c = 1 \) CFT and show an agreement of the one-loop correlators (this correlator in \( c = 1 \) is the only one which vanishes at non-zero momenta) in genus zero. We did not find such an agreement for the two-loop correlator.

2 The iterative scheme

We propose in this section a general iterative procedure for calculating higher genus contributions to the hermitian one-matrix model. We shall be solving, iteratively in \( 1/N^2 \), the loop equation

\[
\int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\lambda - \omega)} W(\omega) = (W(\lambda))^2 + W(\lambda; \lambda),
\]

where

\[
V(\lambda) = \sum_{k=0}^{\infty} t_k \lambda^k, \quad \frac{\delta}{\delta V(\lambda)} = -\sum_{k=0}^{\infty} \lambda^{-k-1} \frac{\partial}{\partial t_k}
\]

and

\[
W(\lambda) = \frac{\delta}{\delta V(\lambda)} \log Z[t], \quad W(\lambda; \lambda) = \frac{\delta}{\delta V(\lambda)} W(\lambda)
\]

with the partition function \( Z[t] \) given by Eq.(1.1). The one-loop correlator \( W(\lambda) \) obeys the normalization condition

\[
\lambda W(\lambda) \to \alpha N \quad \text{as} \quad \lambda \to \infty.
\]

While we consider in this paper the loop equation (2.1), it should be noticed that a similar iterative procedure can be formulated for the equivalent Schwinger–Dyson equation

\[
\left\{ \frac{\partial^2}{\partial \eta_i^2} + \sum_{j \neq i} \frac{1}{\eta_i - \eta_j} \left( \frac{\partial}{\partial \eta_i} - \frac{\partial}{\partial \eta_j} \right) + N \eta_i \frac{\partial}{\partial \eta_i} - \alpha N^2 \right\} Z[\eta; \alpha] = 0
\]

where the partition function \( Z[\eta; \alpha] \) is given by Eq.(1.2) and \( \{t_k\} \) are related to \( \eta_i \) — the eigenvalues of \( \eta \) — by Eq.(1.3). The genus one solution of Ref.[1] was obtained for a symmetric distribution of the eigenvalues by solving Eq.(2.5).
Our goal is to solve Eq. (2.1) iteratively order by order in $1/N^2$ starting with the one-cut genus zero solution \[ W_0(\lambda) = \int_{C_1} \frac{d\omega}{4\pi i} \frac{V'(<\omega)}{(\lambda - \omega)^{1/2}} \sqrt{\omega^2 + b\omega + c}, \] (2.6)

with $b$ and $c$ given by

\[ \int_{C_1} \frac{d\omega}{2\pi i} \sqrt{\omega^2 + b\omega + c} = 0, \quad \int_{C_1} \frac{d\omega}{2\pi i} \omega V'(<\omega) = 2\alpha N. \] (2.7)

In this way we will obtain the genus expansion of the (logarithm of the) partition function

\[ \log Z[t.] = \sum_{g=0}^{\infty} F_g[t.] \] (2.8)

and that of the associated multi-loop correlators.

Following Ref. [1], we introduce the new variables

\[ I_p = \int_{C_1} \frac{d\omega}{2\pi i} \frac{\omega V'(<\omega)}{(\omega^2 + b\omega + c)^{p+1/2}} - (2\alpha + 1)N\delta_{p0}, \]

\[ J_p = \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(<\omega)}{(\omega^2 + b\omega + c)^{p+1/2}}. \] (2.9)

Their relation to the moments of the external field $\eta$, which is described by the density of eigenvalues $\rho(x)$, can easily be established by substituting

\[ \frac{1}{N}V'(<\lambda) = \int dx \rho(x) x - \lambda \] (2.10)

into Eq. (2.9):

\[ \frac{1}{N}I_p = \int dx \frac{x\rho(x)}{(x^2 + bx + c)^{p+1/2}} - (\frac{3}{8}b^2 - \frac{1}{2}c)\delta_{p0} - \frac{1}{2}b\delta_{p1}, \]

\[ \frac{1}{N}J_p = \int dx \frac{\rho(x)}{(x^2 + bx + c)^{p+1/2}} - \frac{1}{2}b\delta_{p0}. \] (2.11)

The terms with $\delta_{p0}$ and $\delta_{p1}$ come from the second term on the r.h.s. of Eq. (2.10) taking the residual at infinity. It is easy to see, expanding the integrands in Eq. (2.11) in $1/x$ that $I_p$ depends on $t_k$ with $k \geq 2p$ while $J_p$ depends on $t_k$ with $k \geq 2p + 1$.

As has been proposed in Ref. [1], $F_g$ depends at $1 \leq g < \infty$ only on $I_p$ and $J_p$ for $p \leq P_g - 1$ (where $P_g = 3g - 1$ as is proven below). This is in contrast to the $t$-dependence of $F_g$ which always depends on the whole set $\{t_k\}$.

For an iterative solution of the loop equation, it is convenient to introduce the operator $\hat{K}$ by

\[ \hat{K}\Phi(<\lambda) = \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(<\omega)}{(\lambda - \omega)} \Phi(<\omega). \] (2.12)

Inserting here Eq. (2.11), one finds the relation with the operator $K$ from Ref. [1]: $-\frac{1}{N}\hat{K} = K + \lambda$, when applied to a function $\Phi(<\lambda)$ which decays at infinity faster than $1/\lambda$ (if $\Phi(<\lambda) \sim 1/\lambda$ as $\lambda \to \infty$, $-1$ should be added to the r.h.s. which comes from the residual at infinity).
It is easy to calculate of how the operator $\hat{\mathcal{K}}$ acts on the set of functions
\begin{equation}
\Phi^{(n)}(\lambda) = \frac{1}{(\lambda^2 + b\lambda + c)^{n+1/2}}
\end{equation}
as well as on $\lambda \Phi^{(n)}(\lambda)$. Let us start with $n = 0$. Comparing with Eq. (2.6), one gets immediately
\begin{equation}
\hat{\mathcal{K}} \Phi^{(0)}(\lambda) = 2W_0(\lambda)\Phi^{(0)}(\lambda).
\end{equation}
To calculate of how $\hat{\mathcal{K}}$ acts on $\Phi^{(n)}(\lambda)$ with $n \geq 1$, let us insert the following expansion of the denominator
\begin{equation}
\frac{1}{\lambda - \omega} = (\lambda + \omega + b) \sum_{k=1}^{n} \frac{\omega^2 + b\omega + c}{(\lambda^2 + b\lambda + c)^k} + \frac{\omega^2 + b\omega + c}{(\lambda^2 + b\lambda + c)^n} \frac{1}{\lambda - \omega} \quad \text{for } n \geq 1.
\end{equation}
Using Eq. (2.9) we get finally
\begin{equation}
\hat{\mathcal{K}} \Phi^{(n)}(\lambda) = \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\lambda - \omega)} \Phi^{(n)}(\omega) = \sum_{k=1}^{n} \frac{I_{n+1-k} + (\lambda + b)J_{n+1-k}}{(\lambda^2 + b\lambda + c)^k} + 2W_0(\lambda)\Phi^{(n)}(\lambda).
\end{equation}
Quite similarly, one calculates of how $\hat{\mathcal{K}}$ acts on $\lambda \Phi^{(n)}(\lambda)$:
\begin{equation}
\hat{\mathcal{K}} \lambda \Phi^{(n)}(\lambda) = -J_n + \lambda \hat{\mathcal{K}} \Phi^{(n)}(\lambda) = \sum_{k=1}^{n} \frac{\lambda J_{n+1-k} - cJ_{n+1-k}}{(\lambda^2 + b\lambda + c)^k} \sum_{k=1}^{n-1} \frac{J_{n-k}}{(\lambda^2 + b\lambda + c)^k} + 2W_0(\lambda)\lambda \Phi^{(n)}(\lambda).
\end{equation}
Some comments concerning these formulas are in order. The expansion is in integer powers of $1/(\lambda^2 + b\lambda + c)$ due to Eq. (2.17). The sum over $k$ is bounded from above by $n$ by construction. For the symmetric potential $V(\lambda) = V(-\lambda)$, which corresponds to a symmetric $\rho(\lambda) = \rho(-l)$ in Eq. (2.10), when $b = 0$ and $J_0 = 0$, Eq. (2.16) recovers Eq. (5.13) of Ref. [1].

Let us now turn to the iterative solution of Eq. (2.1). Substituting the genus expansion for $W(\lambda)$ and $W(\lambda; \lambda)$:
\begin{equation}
W(\lambda) = \sum_{g=0}^{\infty} W_g(\lambda); \quad W(\lambda; \lambda) = \sum_{g=0}^{\infty} W_g(\lambda; \lambda),
\end{equation}
and comparing the terms of the same order in $1/N^2$, one rewrites Eq. (2.1) as
\begin{equation}
(\hat{\mathcal{K}} - 2W_0(\lambda)) W_g(\lambda) = \sum_{g' = 1}^{g-1} W_{g'}(\lambda) W_{g-g'}(\lambda) + W_{g-1}(\lambda; \lambda)
\end{equation}
where $g \geq 1$. This equation allows us to calculate $W(\lambda)$ order by order of genus expansion.

As is proven below by induction, the r.h.s. of Eq. (2.19), which is known from previous orders of genus expansion, has a structure of the sum over $n$ up to
\begin{equation}
n_g = 3g - 1
\end{equation}
of the terms of the type \( A_g^{(n)}/(\lambda^2 + b\lambda + c)^n \) and \( D_g^{(n)}\lambda/(\lambda^2 + b\lambda + c)^n \) with the coefficients \( A_g^{(n)} \) and \( D_g^{(n)} \) being some functions of the moments \( I_p \) and \( J_p \) to be discussed latter. For this reason, it is convenient to introduce two sets of ‘basis vectors’ \( \chi^{(n)}(\lambda) \) and \( \psi^{(n)}(\lambda) \) which obey

\[
(\hat{K} - 2W_0(\lambda)) \chi^{(n)}(\lambda) = \frac{1}{(\lambda^2 + b\lambda + c)^n}, \quad (\hat{K} - 2W_0(\lambda)) \psi^{(n)}(\lambda) = \frac{\lambda}{(\lambda^2 + b\lambda + c)^n}.
\]

(2.21)

This idea is analogous to the one suggested by Gross and Newman \([25]\) for the unitary matrix model and for the hermitian model with a cubic potential.

We can derive now recursion relations which defines \( \chi^{(n)}(\lambda) \) and \( \psi^{(n)}(\lambda) \) explicitly. Using Eqs.(2.16) and (2.17), we get

\[
\chi^{(n)}(\lambda) = \frac{I_1 - \lambda J_1}{\Delta} \Phi^{(n)}(\lambda) - \frac{1}{\Delta} \sum_{k=1}^{n} \chi^{(k)}(\lambda) \{ I_1[I_{n+1-k} + bJ_{n+1-k}] + J_1[cJ_{n+1-k} - J_{n-k}] \} - \frac{1}{\Delta} \sum_{k=1}^{n} \psi^{(k)}(\lambda) \{ I_1J_{n+1-k} - J_1I_{n+1-k} \}
\]

(2.22)

and

\[
\psi^{(n)}(\lambda) = \frac{\lambda(I_1 + bJ_1) + cJ_1}{\Delta} \Phi^{(n)}(\lambda) - \frac{1}{\Delta} \sum_{k=1}^{n} \chi^{(k)}(\lambda) \{ cJ_1[I_{n+1-k} + bJ_{n+1-k}] + (I_1 + bJ_1)[J_{n-k} - cJ_{n+1-k}] \} - \frac{1}{\Delta} \sum_{k=1}^{n} \psi^{(k)}(\lambda) \{ cJ_1J_{n+1-k} + (I_1 + bJ_1)I_{n+1-k} \}
\]

(2.23)

where

\[
\Delta = I_1^2 + bI_1J_1 + cJ_1^2.
\]

(2.24)

Eqs.(2.21) and (2.22), (2.23) allow us to restore \( W_g(\lambda) \) provided the r.h.s. of Eq.(2.19) has the advertised form. The result reads

\[
W_g(\lambda) = \sum_{n=1}^{n_g} \left[ A_g^{(n)}\chi^{(n)}(\lambda) + D_g^{(n)}\lambda\psi^{(n)}(\lambda) \right].
\]

(2.25)

Notice that we do not add the terms with \( n = 0 \) which are annihilated by the operator \( \hat{K} - 2W_0(\lambda) \). These terms would contradict the boundary condition (2.4). The expression on the r.h.s. of Eq.(2.25) is analytic everywhere in the complex \( \lambda \)-plane except for the cut coinciding with the one of \( W_0(\lambda) \). This is in accordance with the different iterative procedure advocated by Migdal \([10]\) and elaborated by David \([26]\).

It remains to prove that the r.h.s. of Eq.(2.19) indeed has such a form. We see from Eq.(2.25) that the first term on the r.h.s. of Eq.(2.19) has exactly this form order by order.
in the genus expansion, while the second term needs a more careful analysis since we have to calculate \( \delta W_{g-1}(\lambda)/\delta V(\lambda) \). To do this we need some more formulas.

From Eq. (2.9) one gets

\[
\frac{\partial I_p}{\partial b} = -(p + \frac{1}{2})[bI_{p+1} + cJ_{p+1} + J_p], \quad \frac{\partial I_p}{\partial c} = -(p + \frac{1}{2})I_{p+1};
\]

\[
\frac{\partial J_p}{\partial b} = -(p + \frac{1}{2})I_{p+1}, \quad \frac{\partial J_p}{\partial c} = -(p + \frac{1}{2})J_{p+1}
\]

(2.26)

while Eq. (2.7), which determines \( b \) and \( c \), can be rewritten as

\[
I_0 = -N, \quad J_0 = 0.
\]

(2.27)

One needs as well the following formulas for calculating \( \delta / \delta V(\lambda) \):

\[
\frac{\delta b}{\delta V(\lambda)} = \frac{\partial}{\partial \lambda} \frac{2}{\sqrt{\lambda^2 + b\lambda + c}} \frac{I_1 - \lambda J_1}{\Delta},
\]

\[
\frac{\delta c}{\delta V(\lambda)} = \frac{\partial}{\partial \lambda} \frac{2}{\sqrt{\lambda^2 + b\lambda + c}} \frac{(\lambda + b)I_1 + cJ_1}{\Delta},
\]

(2.28)

with \( \Delta \) given by Eq. (2.24), and

\[
\frac{\delta I_p}{\delta V(\lambda)} = \frac{\partial}{\partial \lambda} \Phi^{(p)}(\lambda) - \frac{\delta b}{\delta V(\lambda)}(p + \frac{1}{2})[bI_{p+1} + cJ_{p+1} + J_p] - \frac{\delta c}{\delta V(\lambda)}(p + \frac{1}{2})I_{p+1},
\]

\[
\frac{\delta J_p}{\delta V(\lambda)} = \frac{\partial}{\partial \lambda} \Phi^{(p)}(\lambda) - \frac{\delta b}{\delta V(\lambda)}(p + \frac{1}{2})I_{p+1} - \frac{\delta c}{\delta V(\lambda)}(p + \frac{1}{2})J_{p+1}.
\]

(2.29)

These formulas can be obtained from Eqs. (2.9) and (2.27) using Eq. (2.26).

It is easy to see that the result of acting of \( \delta / \delta V(\lambda) \) on \( W_{g-1} \) given by Eq. (2.25) has exactly the form discussed above. Moreover, we have presented an explicit algorithm for calculating \( W_g(\lambda) \), say by symbolic computer calculations with the only input parameter being \( g \). The explicit results for genus one are presented in the next section.

We perform now a power-counting analysis to express \( n_g \) and \( P_g \), which are introduced above, via \( g \). Let us note first that the highest moments \( I_{P_g} \) and \( J_{P_g} \) emerge on the r.h.s. of Eq. (2.25) from the highest term \( 1/(\lambda^2 + b\lambda + c)^n_g \) on the r.h.s. of Eq. (2.13) according to Eqs. (2.22), (2.23) (they are associated with the \( k = 1 \) terms). Therefore, one gets \( P_g = n_g \).

To pass to the next order of the genus expansion, one analyze the structure of the r.h.s. of Eq. (2.13). It is easy to estimate the highest power of each term \( W_g(\lambda)W_{g-g}(\lambda) \) for the solution (2.25), which is known from the previous order, to be \( n_g + n_{g-g} + 1 \) while the two-loop correlator gives the power \( P_g + 3 \) by virtue of Eqs. (2.28) and (2.29). Therefore, one gets \( n_{g+1} = n_g + 3 \) and finally

\[
P_g = 3g - 1
\]

(2.30)

We have not calculated explicitly \( \partial / \partial \lambda \) in order that \( \delta / \delta \rho(\lambda) \), which is related to \( \delta / \delta V(\lambda) \) by

\[
\frac{\delta}{\delta V(\lambda)} = \frac{1}{N} \frac{\partial}{\partial \lambda} \frac{\delta}{\delta \rho(\lambda)},
\]

could be extracted easily as well.
since \( n_1 = 2 \). Notice that both types of terms contribute to the maximal power since \( n_{g'} + n_{g-g'} + 1 = 3g - 1 \).

According to Eqs. (2.8), (2.3) and (2.28), (2.29), the highest moments which contribute to \( F_g \) are \( I_{3g-2} \) and \( J_{3g-2} \). An analogous result for the Kontsevich model has been obtained by Itzykson and Zuber [27]. Applying Eqs. (2.28), (2.29) \( m \) times to obtain the \( m \)-loop correlator, one finds highest moments to be \( I_{3g-2+m} \) and \( J_{3g-2+m} \). Thus, we have proven the following

**Theorem** The genus \( g \) contribution to the partition function depends on \( I_p \) and \( J_p \) with \( p \leq 3g - 2 \). The genus \( g \) contribution to the \( m \)-loop correlator depends on \( I_p \) and \( J_p \) with \( p \leq 3g - 2 + m \).

### 3 The partition function and correlators in genus one

We present in this section the explicit results for the genus one one-loop correlator \( W_1(\lambda) \) and partition function \( F_1 \) for the case of an arbitrary potential \( V(\lambda) \) which are obtained according to the algorithm of the previous section.

In genus one there is no sum on the r.h.s. of Eq. (2.19) while

\[
W_0(\lambda; \mu) = \frac{b^2 - 4c}{16(\lambda^2 + b\lambda + c)^2}, \quad W_0(\lambda; \mu) = \frac{1}{2(\lambda - \mu)^2} \left[ \frac{\lambda\mu - \frac{1}{2}b(\lambda + \mu) + c}{\sqrt{\lambda^2 + b\lambda + c} \sqrt{\mu^2 + b\mu + c}} - 1 \right].
\]

Eq. (2.21) yields immediately

\[
W_1(\lambda) = \frac{b^2 - 4c}{16} \chi^{(2)}(\lambda) = \frac{b^2 - 4c}{16} \left\{ \frac{\Phi^{(2)}(\lambda)}{\Delta} [I_1 - \lambda J_1] - \frac{\Phi^{(1)}(\lambda)}{\Delta^2} [J_1^3 + I_1^2 J_2 - 2I_1 J_1 I_2 - bJ_1^2 I_2 - cJ_1^2 J_2] \right\}
\]

where we have substituted the explicit form of \( \chi^{(2)}(\lambda) \) given by Eqs. (2.22) and (2.23). For the symmetric case when \( b = 0 \) and \( J_p = 0 \), Eq. (3.2) recovers Eq. (5.18) of Ref. [1].

We are now in a position to integrate Eq. (3.2) (w.r.t. \( \rho \)) and find the functional \( F_1 \) whose variation generates this \( W_1(\lambda) \). First, we integrate over \( \lambda \) using the formulas

\[ \text{It would be suggestive to relate these numbers to the (complex) dimension of the moduli space } M_{g,m}, \text{ which equals } 3g - 3 + m. \]
\[
\frac{1}{4} \int \frac{dx}{(x^2 + bx + c)^{3/2}} = -\frac{x + \frac{1}{2}b}{(b^2 - 4c)^{1/4}(x^2 + bx + c)} + \text{const}, \\
\frac{1}{4} \int \frac{dx}{(x^2 + bx + c)^{5/2}} = -\frac{x + \frac{1}{2}b}{3(b^2 - 4c)(x^2 + bx + c)^{3/2}} + \frac{8(x + \frac{1}{2}b)}{3(b^2 - 4c)^{3/2}} + \text{const}. 
\]  

(3.3)

It is worth to note the appearance of the factor \(b^2 - 4c\) in the denominators.

Using Eqs.\((2.29)\) and \((2.28)\), we can now integrate \(W_1(\lambda)\) explicitly. Namely, it is more or less clear that one can produce coefficients like in Eq.\((3.2)\) by differentiating terms like log \(\Delta\) (the terms with \(\Delta^2\) in the denominators originate from the variations \((2.29), (2.28)\)). Moreover, due to the appearance of \(\frac{1}{4\Delta(b^2-4c)}\) term after the integration of \(W_1(\lambda)\) over \(\lambda\), it is reasonable to assume the presence of log \((4c-b^2)\) term in \(F_1\). After all these preliminaries, a straightforward, though lengthy, calculation gives the answer

\[F_1 = -\frac{1}{12} \log(4c - b^2) - \frac{1}{24} \log \Delta + \text{const},\]  

(3.4)

which, in particular, reproduces Eq.\((5.18)\) of Ref.\[1\] for the reduced model (the symmetric potential).

## 4 Comparison with \(c = 1\) correlators

We compare in this section the explicit formulas for the loop correlators in the hermitian one-matrix model with those in \(c = 1\) CFT. A problem immediately arises that the loop operator in \(c = 1\) depends both on the length of the loop \(l\) and on momentum \(p\) which is associated with a matter field. Since in the hermitian one-matrix model there is no dependence on \(p\), we should put \(p\) equal to some value (or sum up somehow over special values of \(p\)). The procedure is, however, unambiguous for the one-loop correlator which is non-vanishing only at \(p = 0\) due to the momentum conservation. Therefore, our idea is to compare first the one-loop correlators.

The genus zero one-loop correlator, given by Eqs.\((2.6)\) and \((2.7)\), is greatly simplified after differentiating w.r.t. the cosmological constant \(\alpha\):

\[
\frac{1}{N} \frac{dW_0(\lambda)}{d\alpha} = \frac{1}{\sqrt{\lambda^2 + b\lambda + c}}. 
\]  

(4.1)

Let us put for simplicity \(b = 0\) keeping in mind that the general case can be restored by the shift \(\lambda \rightarrow \lambda + b/2; c \rightarrow c - b^2/4\). The equation \((2.7)\) which determines \(c\) can be written in the form of the genus zero string equation

\[
\sum_m mt_{2m} \frac{(2m - 1)!!}{(2m)!!} (-c)^m = \alpha. 
\]  

(4.2)

If we restrict ourselves to the simplest case \(t_2 \neq 0, t_{2m} = 0\) for \(m \geq 2\), Eq.\((4.2)\) gives \(c = -\alpha/t_2\) and \(c > 0\) corresponds to the ‘upside-down’ potential which is familiar from
the quantum mechanical description of \( c=1 \) correlators. Moreover, the r.h.s. of Eq.(4.2) is as well the known expression for the (diagonal) resolvent of the Schrödinger operator with the potential \( c(\alpha) \) in genus zero. Therefore, we conclude that the genus zero one-loop correlators coincide.

The genus zero two-loop correlator in \( c=1 \) theory at zero momentum can easily be extracted, applying the Gegenbauer’s addition theorem, from the results by Moore and Seiberg [22]:

\[
W_0(l_1, p = 0; l_2, p = 0) = K_0 \left( \sqrt{c(l_1 + l_2)} \right)
\] (4.3)

where \( K_0 \) is the modified Bessel function of the third kind.

It is easy to see that the Laplace transform of Eq.(4.3) does not coincide with Eq.(3.1). A possible way out (a pure speculative one) might be to sum up the general formula of Moore and Seiberg [22] over the momenta which corresponds to the discrete special states of \( c = 1 \) CFT [18, 28, 29]. This problem deserves further investigation.

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