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Random Periodic Solutions of Random Dynamical Systems

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Summary. In this paper, we give the definition of the random periodic solutions of random dynamical systems. We prove the existence of such periodic solutions for a $C^1$ perfect cocycle on a cylinder using a random invariant set, the Lyapunov exponents and the pullback of the cocycle.

Keywords: Random periodic solution, perfect cocycle, random dynamical system, invariant set, Lyapunov exponent.

1 Introduction

Similar to the deterministic dynamical systems, in stochastic dynamical systems, the problem of the long time or infinite horizon behaviour is a fundamental problem occupying a central place of research. Studies of dynamic properties of such systems, both deterministic and stochastic, usually involve an appropriate definition of a steady state (viewed as a dynamic equilibrium) and conditions that guarantee its existence and local or global stability.

Fixed points or periodic solutions capture the intuitive idea of a stationary state or an equilibrium of a dynamical system. For a deterministic dynamical system $\Phi_t : H \rightarrow H$ over time $t \in I$, where $H$ is the state space, $I$ is the set of all real numbers, or discrete real numbers, a fixed point is a point $y \in H$ such that

$$\Phi_t(y) = y \quad \text{for all } t \in I, $$

and a periodic solution is a periodic function $\psi : I \rightarrow H$ with period $T \neq 0$ such that

$$\psi(t + T) = \psi(t) \quad \text{and} \quad \Phi_t(\psi(t_0)) = \psi(t + t_0) \quad \text{for all } t, t_0 \in I.$$

However, in many real problems, such kinds of the deterministic equilibria actually don’t exist due to the presence of random factors. For stochastic dynamical systems, it would be reasonable to say that stationary states are not actually steady states in the sense of (1) and periodic solutions in the sense of (2). Due to the fact that the random external force pumps to the system constantly, the relation (1) or (2) breaks down.

Let $\Phi : \Omega \times I \times H \rightarrow H$ be a measurable random dynamical system on a measurable space $(H, \mathcal{B})$ over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \geq 0})$. Then a stationary solution of $\Phi$ is a $\mathcal{F}$ measurable random variable $Y : \Omega \rightarrow H$ such that (c.f. Arnold [2])

$$\Phi(\omega, t, Y(\omega)) = Y(\theta_t \omega) \quad \text{for all } t \in I \text{ a.s.}$$

The concept of the stationary random point of a random dynamical system is a natural extension of the equilibrium or fixed point in deterministic systems. Such a random fixed point for SPDEs is especially interesting. It consists of infinitely many random moving invariant surfaces on the configuration space. It is a more realistic model than many deterministic models...
as it demonstrates some complicated phenomena such as turbulence. Finding such stationary solutions for SPDEs is one of the basic problems. But the study of random cases is much more difficult and subtle, in contrast to deterministic problems. This “one-force, one-solution” setting describes the pathwise invariance of the stationary solution over time along $\theta$ and the pathwise limit of the random dynamical system. Their existence and/or stability for various stochastic partial differential equations have been under active study recently (Duan, Lu and Schmalfuss [6], E, Khanin, Mazel and Sinai [7], Mohammed, Zhang and Zhao [12], Zhang and Zhao [16]).

 Needless to say that the random periodic solution is another fundamental concept in the theory of random dynamical systems. According to our knowledge, such a notion did not exist in literatures. In this paper, we will carefully define such a notion and give a sufficient condition for the existence on a cylinder $S^1 \times \mathbb{R}^d$. To see the motivation for such a definition, let’s first note two obvious but fundamental truth in the definition of periodic solution (2) of the deterministic systems when $I$ is the set of real numbers:-

(i) The function $\psi$ (given in the parametric form here) is a closed curve in the phase space;
(ii) If the dynamical system starts at a point on the closed curve, the orbit will remain on the same closed curve.

But note, in the case of stochastic dynamical systems, although the function $\psi$ may still be a periodic function, one would expect that $\psi$ depend on $\omega$. In other words, we would expect infinitely many periodic functions $\psi^\omega$, $\omega \in \Omega$. Moreover, even the random dynamical system starts at a point on the curve $\psi^\omega$, it will not stay in the same periodic curve when time is running. In fact, the periodic curve actually is not the orbit of the random dynamical system, but the random dynamical system will move from one periodic curve $\psi^\omega$ to another periodic curve $\psi^\theta t^\omega$ at time $t \in I$. Now we give the following definition:

**Definition 1.1** A random periodic solution is an $\mathcal{F}$-measurable periodic function $\psi: \Omega \times I \to \mathcal{H}$ of period $T$ such that

$$\psi^\omega(t + T) = \psi^\omega(t) \quad \text{and} \quad \Phi^\omega_t(\psi^\omega(t_0)) = \psi^\theta t^\omega(t + t_0) \quad \text{for all} \quad t, t_0 \in I. \quad (4)$$

Comparing to the stationary solution, which consists of infinitely many single points on the phase space, the random periodic solution consists of infinite many random moving periodic curves on the phase space. It describes more complex random nonlinear phenomenon than a stationary solution. The closed curve on the phase space is a pathwise invariant set over the time and the $P$-preserving transformation $\theta : I \times \Omega \to \Omega$ along the orbit defined by the random dynamical system. To understand and give the existence of such a solution is an interesting problem. The systematic study of the deterministic periodic solutions of the deterministic dynamical systems began in Poincaré in his seminal work [13]. The Poincaré-Bendixson Theorem has been very useful in the study of the periodic solutions ([4]). Periodic solutions have been studied for many important systems arising in numerous physical problems e.g. van der Pol equations (van der Pol [15]), Liénard equations (Liénard [11], Filipov [8], Zheng [18]). Now, after over a century, this topics is still one of the most interesting nonlinear phenomena to study in the theory of deterministic dynamical systems. The periodic solutions have occupied a central place in dynamical systems such as in the study of the bifurcation theory (V. Arnold [3], Chow and Hale [5], Andronov [1], Li and Yorke [10] to name but a few). The Hilbert’s 16th problem involves determining the number and location of limits cycles for autonomous planar polynomial vector field. This problem still remains unsolved although much progress has been made. See Ilyashenko [9] for a summary. Needless to say that the study of the random periodic solution is more difficult and subtle. The extra essential difficulty comes from the fact that the trajectory of the random dynamical systems starting at a point on the periodic curve does not follow the periodic curve, but moves from one periodic curve to another one corresponding to different $\omega$. If one looks at a family of trajectories (therefore forms a map or a
flow) starting from different points in the closed curve $\psi_\omega$, then the whole family of trajectories at time $t \in I$ will lie on a closed curve corresponding to $\theta_\omega$. This is essentially different from the deterministic case. Of course, the definition (2) in the deterministic case is a special case of the definition (4) of the random case.

In section 2, we will give the definition of the random periodic solutions of a random dynamical system on a cylinder. In section 3, we will prove the existence of the finite number of periodic solution assuming a contraction condition using Lyapunov exponent near the attractor. We use the compactness argument and obtain a finite number of open covering and eventually to prove the existence of the finite number of closed curve. But this does not give the estimate on how many periodic solutions and the winding numbers. They are interesting problems to investigate in the future. Moreover, we will prove that the winding number of each closed orbit, the period of each periodic solution and the number of periodic solutions are invariant under the perturbation of noise.

2 The notion of periodic invariant solutions on cylinders and an example

The extension of the notion of a periodic solution on a cylinder to the random case is given as follows (see Fig.1), where $I$ is either $[0, \infty)$, or $(-\infty, 0]$, or $(-\infty, +\infty)$. Note here there is a natural parameter $s \in S^1$ for a closed curve on $S^1 \times R^d$. Some cases on $R^{d+1}$ can be transformed to the cylinders case. In the next section, we will give the existence of the periodic solutions of the random dynamical systems on a cylinder. The following definition gives more information about the winding number of the closed curves. Except for the winding number, on a cylinder, Definition 2.1 is equivalent to Definition 1.1 if we reparameterize the natural parameter $s$ using time by putting $s = s(t)$. But on a cylinder, it is more convenient to use the natural parameter $s$.

Definition 2.1 Let $\varphi_\omega : R \rightarrow R^d$ be a continuous periodic function of period $\tau \in \mathbb{N}$ for each $\omega \in \Omega$. Define $L^\omega = \text{graph}(\varphi_\omega) = \{(s \mod 1, \varphi_\omega(s)) : s \in R^1\}$. If $L^\omega$ is invariant with respect to the random dynamical system $\Phi : \Omega \times I \times S^1 \times R^d \rightarrow \Omega \times (S^1 \times R^d)$, i.e. $\Phi^\omega(t)L^\omega = L^\theta \omega$, and there exists a minimum $T > 0$ (or maximum $T < 0$) such that for any $s \in [0, \tau)$,

$$\Phi^\omega(T, (s \mod 1, \varphi_\omega(s))) = (s \mod 1, \varphi^0 \tau_\omega(s)), \quad (5)$$

for almost all $\omega$, then it is said that $\Phi$ has a random periodic solution of period $T$ with random periodic curve $L^\omega$ of winding number $\tau$.

It is easy to see that $\delta_{L^\omega}(dx)P(d\omega)$ is an invariant measure of the skew-product $(\theta, \Phi) : I \times \Omega \times (S^1 \times R^d) \rightarrow \Omega \times (S^1 \times R^d)$. Needless to say, an invariant measure may not give a random periodic solution. The support of an invariant measure may be a random fixed point (stationary solution), or random periodic solutions, or a more complicated set.

The periodic invariant orbit is a new concept in the literature. We believe it has some importance in random dynamical systems, for example, it can be studied systematically to establish the Hopf bifurcation theory of stochastic dynamical systems. This is not the objective of this paper, we will study this problem in future publications. But here in order to illustrate the concept, as a simple example, we consider the random dynamical system generated by a perturbation to the following deterministic ordinary differential equation in $R^2$:

$$\begin{align*}
\frac{dx(t)}{dt} &= x(t) - y(t) - x(t)x^2(t) + y^2(t)), \\
\frac{dy(t)}{dt} &= x(t) + y(t) - y(t)x^2(t) + y^2(t)).
\end{align*} \quad (6)$$
It is well-known that above equation has a limit cycle \( x^2(t) + y^2(t) = 1 \).

Consider a random perturbation

\[
\begin{align*}
\frac{dx}{dt} &= (x - y - x(x^2 + y^2))dt + x \circ dW(t), \\
\frac{dy}{dt} &= (x + y - y(x^2 + y^2))dt + y \circ dW(t).
\end{align*}
\]

Here \( W(t) \) is a one-dimensional motion on the canonical probability space \( (\Omega, \mathcal{F}, P) \) with the \( P \)-preserving map \( \theta \) being taken to the shift operator \( (\theta_t)\omega(s) = W(t + s) - W(t) \). Using polar coordinates

\[ x = \rho \cos 2\pi \alpha, \quad y = \rho \sin 2\pi \alpha, \]

then we can transform Eq. (7) on \( R^2 \) to the following equation on the cylinder \([0,1] \times R^1\):

\[
\begin{align*}
\frac{d\rho(t)}{dt} &= (\rho(t) - \rho^3(t))dt + \rho(t) \circ dW(t), \\
\frac{d\alpha}{dt} &= \frac{1}{2\pi} dt.
\end{align*}
\]
This equation has a unique close form solution as follows:

\[ \rho(t, \alpha_0, \rho_0, \omega) = \frac{\rho_0 e^{t + W_t(\omega)}}{(1 + 2\rho_0^2 \int_0^t e^{2(s + W_s(\omega))} ds)^{1/2}}, \quad \alpha(t, \alpha_0, \rho_0, \omega) = \alpha_0 + \frac{t}{2\pi}. \]

It is easy to check that

\[ \rho^*(\omega) = \left(2 \int_0^{-\infty} e^{2s + 2W_s(\omega)} ds\right)^{-\frac{1}{2}} \]

is the stationary solution of the first equation of (8) i.e.

\[ \rho(t, \alpha_0, \rho^*(\omega), \omega) = \rho^*(\theta_t \omega) \]

and

\[ \Phi(t, \omega)(\alpha_0, \rho_0) = (\alpha_0 + \frac{t}{2\pi} \mod 1, \rho(t, \alpha_0, \rho_0, \omega)) \]

defines a random dynamical system on the cylinder \([0, 1] \times R^1\): \(\Phi(t, \omega) = (\Phi_1(t, \omega), \Phi_2(t, \omega)) : [0, 1] \times R^1 \rightarrow [0, 1] \times R^1\). Define

\[ L^\omega = \{ (\alpha, \rho^*(\omega)) : 0 \leq \alpha \leq 1 \}. \]

Then

\[ L^{\theta\omega} = \{ (\alpha, \rho^*(\theta_t \omega)) : 0 \leq \alpha \leq 1 \}. \]

It is noticed that

\[ \Phi(t, \omega)L^\omega = \{ (\alpha + \frac{t}{2\pi} \mod 1, \rho^*(\theta_t \omega)) : 0 \leq \alpha \leq 1 \} = \{ (\alpha, \rho^*(\theta_t \omega)) : 0 \leq \alpha \leq 1 \}. \]

Therefore

\[ \Phi(t, \omega)L^\omega = L^{\theta\omega}, \]

i.e. \(L^\omega\) is invariant under \(\Phi\). Moreover

\[ \Phi(2\pi, \omega)(\alpha, \rho^*(\omega)) = (\alpha, \rho^*(\theta_{2\pi} \omega)). \]

Therefore the random dynamical system has a random solution of period \(2\pi\) with invariant closed curve \(L^\omega\) of winding number 1 on \([0, 1] \times R^1\). Now we can transform the periodic solution back to \(R^2\). For this define for \((x, y) \in R^2\), \(x = \rho \cos 2\pi \alpha, \ y = \rho \sin 2\pi \alpha\)

\[ \Phi(t, \omega)(x, y) = (\Phi_2(t, \omega)(\alpha, \rho) \cos(2\pi \Phi_1(t, \omega)(\alpha, \rho)), \ \Phi_2(t, \omega)(\alpha, \rho) \sin(2\pi \Phi_1(t, \omega)(\alpha, \rho))), \]

and

\[ \psi^\omega(t) = (\rho^*(\omega) \cos(2\pi \alpha + t), \rho^*(\omega) \sin(2\pi \alpha + t)). \]

It is obvious that

\[ \psi^\omega(2\pi + t) = \psi^\omega(t), \]
and
\[\Phi(t,\omega)\psi(0) = \Phi(t,\omega)(\rho^*(\omega) \cos(2\pi\alpha), \rho^*(\omega) \sin(2\pi\alpha)) = (\rho^*(\theta_1\omega) \cos(2\pi\alpha + t), \rho^*(\theta_1\omega) \sin(2\pi\alpha + t)) = \psi^{\theta_1\omega}(t).\]

From this we can tell that the random dynamical system generated by the stochastic differential equation (7) has a random periodic solution \(\psi^\omega : \Omega \times t \to R^2\) defined above. Moreover if \(x(0)^2 + y(0)^2 \neq 0\), then
\[x^2(t,\theta(-t,\omega)) + y^2(t,\theta(-t,\omega)) \to \rho^*(\omega)^2\]
as \(t \to \infty\).

3 The existence of random periodic solutions

Consider a continuous time differentiable random dynamical system \(\gamma\) over a metric dynamical system \((\Omega, F, P, \theta)\) on a cylinder \(S^1 \times R^d\). Though the precise definition given below is standard, it is crucial to the development of this article (c.f. [2], [12]).

**Definition 3.1** A \(C^1\) perfect cocycle is a \(B((-\infty, +\infty)) \otimes F \otimes B(S^1 \times R^d), B(S^1 \times R^d)\) measurable random field \(\gamma : (-\infty, +\infty) \times \Omega \times S^1 \times R^d \to S^1 \times R^d\) satisfying the following conditions:

\((a)\) for each \(\omega \in \Omega\), \(\gamma(0, \omega) = Id,\)
\((b)\) for each \(\omega \in \Omega\), \(\gamma^z(t_1 + t_2) = \gamma^z(t_1)(t_2)\) for all \(z \in S^1 \times R^d\) and \(t_1, t_2 \in R,\)
\((c)\) for each \(\omega \in \Omega\), the mapping \(\gamma^z(\cdot) : (-\infty, +\infty) \times S^1 \times R^d \to S^1 \times R^d\) is continuous,
\((d)\) for each \((t, \omega) \in (-\infty, +\infty) \times \Omega\), the mapping \(\gamma^\omega(t) : S^1 \times R^d \to S^1 \times R^d\) is a \(C^1\) diffeomorphism.

In the following, we will assume there is an invariant set \(X^\omega \subset S^1 \times Y^\omega\), for some compact set \(Y^\omega\), and use a random quasi-periodic winding system on \(S^1 \times Y^\omega\) to characterize the invariant set \(X^\omega\) further and to study the existence of random periodic curves on the cylinder. We will prove under the following three conditions, the invariant set \(X^\omega\) consists of a finite number of periodic curves. The Lyapunov exponent and the pullback are key techniques in our approach. So it is essential to assume the random dynamical systems are \(C^1\) perfect cocycles. It is known from the works of many mathematicians that cocycles can be generated from some random differential equations and stochastic differential equations (c.f. Arnold [2]).

**Condition (i)** Assume that there exists a random compact subset \(Y\) of \(R^d\) and \(t_1 > 0\) such that for any \(s \in S^1\) and \(y = (x_1, x_2, \ldots, x_d) \in Y^\omega\) (denote \(z_0 = (s, x_1, \ldots, x_d)\)), \(\gamma^\omega(z_0)(t_1) = (s', x_1', \ldots, x_d')\) with
\[s' = s + 1,\]
and
\[y' = (x_1', x_2', \ldots, x_d') \in Y^{\theta_1\omega}.\]

We assume there is an invariant compact set \(X^\omega \subset S^1 \times Y^\omega\), that is to say \(X^\omega\) satisfies
\[\gamma^\omega(t)X^\omega = X^{\theta_1\omega},\]
for
for all \( t \in \mathbb{R} \) and the projection of \( X^\omega \) to the subspace \( S^1 \) is \( S^1 \), there exist a constant \( b^* > 0 \) such that \( |X^\omega| < b^* \) where \( |X^\omega| \) denotes the diameter of \( X^\omega \) in the y-direction.

Note that \( t_1 \) is the time that a particle on \( S^1 \) rotate a full circle, no matter where it starts, (10) is just an invariant condition about the random compact subset \( Y^\omega \subset \mathbb{R}^d \).

In the following, we denote \( \tilde{\theta} \omega = \theta_1 \omega \). Under the above assumption, a random winding system can be defined from the continuous time \( C^1 \) random dynamical system \( \gamma \) by letting

\[
h(s) = s'
\]

and

\[
g^\omega(s, y) = y'.
\]

Consider the following discrete time random winding system

\[
\begin{aligned}
s_{n+1} &= h(s_n) \mod 1, s_n \in S^1 \\
y_{n+1} &= g^\omega(s_n, y_n), \ y_n \in \mathbb{R}^d,
\end{aligned}
\]  

(12)

where \( S^1 \) is the unit circle, \( g : \Omega \times S^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is \( \mathcal{F} \otimes \mathcal{B}(S^1) \otimes \mathcal{B}(\mathbb{R}^d) \) measurable and \( g^\omega : S^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is jointly continuous for each \( \omega \) and \( g^\omega(s, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is differentiable for each \( (\omega, s) \in \Omega \times S^1 \). To be convenient, denote

\[
H^\omega(s, y) = (h(s) \mod 1, g^\omega(s, y)),
\]

(13)

for the skew product map on \( S^1 \times \mathbb{R}^d \). We shall also define \( g^{(n)} \) iteratively by \( H^{(n)\omega}(s, y) = (h^{(n)}(s) \mod 1, g^{(n)\omega}(s, y)) \). It is easy to know that \( \tilde{\theta} : \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} \times \Omega \rightarrow \Omega \) is a \( \mathcal{F} \)-preserving map such that for all \( m, n \in \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} \)

\[
H^{(m+n)\omega}(s, y) = H^{(m)\tilde{\theta}^n\omega}(H^{(n)\omega}(s, y)),
\]

(14)

for all \((s, y) \in S^1 \times \mathbb{R}^d \) almost surely.

Let \( \pi : \mathbb{R} \times \mathbb{R}^d \rightarrow S^1 \times \mathbb{R}^d \) be the natural covering \( \pi(a, y) = (a \mod 1, y) \) and \( \varphi^\omega \) a periodic continuous function. If \( \pi(\text{graph } \varphi) \) is invariant under \( (H, \tilde{\theta}) \), that is to say that \( H^{\tilde{\theta}^{-1}\omega}\pi(\text{graph } \varphi^{\tilde{\theta}^{-1}\omega}) = \pi(\text{graph } \varphi^\omega) \) a.s., it is said that \( \varphi \) is an invariant curve for the skew product (13).

To analyse the structure of \( X^\omega \), we pose the following conditions.

**Condition (ii)** Assume that there exists \( \delta_1 > 0 \) and \( L_1 > 0 \) such that for any \((s, y) \in X^\omega \), there exists a Lipschitz continuous function \( f : S^1 \rightarrow \mathbb{R}^d \) with Lipschitz constant \( L_1 \) and \( f(s) = y \) such that \((s^*, f(s^*)) \in X^\omega \) when \( s^* \in [s - \delta_1, s + \delta_1] \).

Define the \((\delta, \varepsilon)\) neighbourhood of \((s, y) \in X^\omega \) and \( X^\omega \) as

\[
B(s, y, \delta, \varepsilon) = \{(s', y') : s' \in [s - \delta, s + \delta], ||y' - f(s')|| \leq \varepsilon\},
\]

and, for any \( s \in S^1 \),

\[
B_s(\delta, \varepsilon) = \bigcup_{(s, y) \in X^\omega} B(s, y, \delta, \varepsilon),
\]

and

\[
B(X^\omega, \varepsilon) = \bigcup_{s \in S^1} B_s(\delta, \varepsilon),
\]

respectively. In fact, \( B_s(\delta, \varepsilon) \) is the \( \varepsilon \)-neighbourhood of \( X^\omega \cap [s - \delta, s + \delta] \times Y^\omega \) in the y-direction and \( B(X^\omega, \varepsilon) \) is the \( \varepsilon \)-neighbourhood of \( X^\omega \) in the y-direction. Note that \( \bigcup_{s \in S^1} B_s(\delta, \varepsilon) \) does not depend on \( \delta \).
**Condition (iii)** Assume there exists an $\lambda < 1$, $\varepsilon_1 > 0$ and an $n_0 \in \mathbb{N}$ such that for almost all $\omega \in \Omega$,

$$||D_y g^{(n_0) \omega}(s,y)|| \leq \lambda \text{ for all } (s,y) \in B(X^\omega, \varepsilon_1),$$  
(15)

and

$$c = \text{ess sup} \sup_{\omega} (s,y) \in B(X^\omega, \varepsilon_1) \left| \frac{\partial g^{(n_0) \omega}}{\partial s}(s,y) \right| < +\infty,$$  
(16)

This assumption can be understood as a condition on the amplitude of Lyapunov exponent of the random map. It is not difficult to prove that for all $m \in \mathbb{N}$

$$H^{mn_0,\omega}(B(X^\omega, \varepsilon_1)) \subset B(X^{\hat{\varphi}^{mn_0} \omega}, \varepsilon_1).$$  
(17)

That is to say that there exists a forward random invariant compact set $B(X^\omega, \varepsilon_1)$. By the chain rule, for all $(s,y) \in B(X^\omega, \varepsilon_1)$ and $m \in \mathbb{N}$,

$$||D_y g^{(mn_0) \omega}(y)|| \leq \lambda^m.$$  
(18)

Moreover, it is easy to see that for any $(s,y) \in X^\omega$,

$$H^{mn_0,\omega}(B(s,y, \delta_1, \varepsilon_1)) \subset B(H^{mn_0,\omega}(s,y), \delta_1, \varepsilon_1),$$  
(19)

and for any $(s',y_1), (s',y_2) \in B(s,y, \delta_1, \varepsilon_1)$,

$$H^{mn_0,\omega}(s',y_1) = (s' + mn_0 \mod 1, g_{s'}^{(mn_0) \omega}(y_1)) \in B(H^{mn_0,\omega}(s,y), \delta_1, \varepsilon_1),$$

$$H^{mn_0,\omega}(s',y_2) = (s' + mn_0 \mod 1, g_{s'}^{(mn_0) \omega}(y_2)) \in B(H^{mn_0,\omega}(s,y), \delta_1, \varepsilon_1)$$

and

$$||g_{s'}^{(mn_0) \omega}(y_1) - g_{s'}^{(mn_0) \omega}(y_2)|| \leq \lambda^m ||y_1 - y_2||.$$  
(20)

From this, it is easy to see that $X^\omega$ actually is a random attractor. The main aim of this section is to prove the following theorem.

**Theorem 3.2** Under Conditions (i), (ii), (iii), the cocycle $\gamma$ has $r^\omega$ random periodic solutions of period $t_i^\omega$ with random periodic curve $L_i^\omega$ of winding number $\tau_i^\omega$, $i = 1, 2, \cdots, r$. Moreover, for any $t \in R$, $r^{\theta-t} = r^\omega$, and $t_i^\theta = t_i^\omega$, $r_i^\theta = r_i^\omega$ for each $i = 1, 2, \cdots, r^\omega$.

We need a series of preparations to prove this theorem. First, for any $s \in S^1$, let’s define

$$X^\omega_s = X^\omega \cap \{s\} \times Y^\omega.$$  

Let $G^\omega_s$ be a connected component of $X^\omega_s$. For any $y_1, y_2 \in R^d$ and $(s,y_1), (s,y_2) \in G^\omega_s$, there exist $y'_1, y'_2 \in R^d$ with $(s',y'_1), (s',y'_2) \in X_{\hat{\varphi}^{mn_0} \omega}$ such that

$$s' + mn_0 \mod 1 = s, \; g_{s'}^{\hat{\varphi}^{mn_0} \omega}(y'_1) = y_1, \; g_{s'}^{\hat{\varphi}^{mn_0} \omega}(y'_2) = y_2,$$

and

$$||y_1 - y_2|| \leq \lambda^m ||y'_1 - y'_2||.$$  
(21)

Let $|G^\omega_s|$ be the largest radius of the set $G^\omega_s$. Define

$$|G^\omega_s| = \sup_{s \in S^1} |G^\omega_s|.$$  

Set for $l > 0$

$$\Omega^l_0 = \{\omega : |G^\omega_s| \geq l\}.$$  

Then we have
Lemma 3.3 For any \( l > 0 \),
\[
P(\Omega^l_0) = 0.
\] (22)

Proof. Assume the claim of the lemma is not true, i.e. there exists \( \alpha_l > 0 \) such that \( P(\Omega^l_0) = \alpha_l \). Also define
\[
\Omega^l_m = \{ \omega : |G^\omega| \geq \frac{l}{\lambda_m} \}.
\]
Then from the invariance of \( X^\omega \), the Conditions (ii) and (iii) and (21), we know that
\[
\hat{\theta}^m ((\Omega^l_m)^c) \subset (\Omega^l_0)^c.
\]
So
\[
\hat{\theta}^m (\Omega^l_m) \supset \Omega^l_0.
\]
Now as \( \hat{\theta} \) is measure preserving we know that for any \( m \geq 0 \),
\[
P (\Omega^l_m) = P (\hat{\theta}^m \Omega^l_m) \geq P (\Omega^l_0) = \alpha_l > 0.
\] (23)
Note that \( \Omega^l_m \downarrow \emptyset \). So by the continuity of probability measure with respect to decreasing sequence of sets, we have as \( m \to +\infty \)
\[
P (\Omega^l_m) \to 0.
\] (24)
Clearly, (23) and (24) contradict each other. The lemma is proved. \( \blacksquare \)

In the following, we will use the pullback of random maps ([2]), the Poincaré map, and the Lyapunov exponent to prove that \( X^\omega \) is a union of finite number of Lipschitz periodic curves. That is to say that there exists \( r^\omega \) continuous periodic functions \( \varphi^\omega_1, \varphi^\omega_2, \ldots, \varphi^\omega_r \) on \( R^1 \) with periods \( \tau^\omega_1, \tau^\omega_2, \ldots, \tau^\omega_r \) \( \in \mathbb{N} \) respectively such that
\[
X^\omega = L_1^\omega \cup L_2^\omega \cup \cdots \cup L_r^\omega,
\] (25)
where
\[
L^\omega_i = \text{graph}(\varphi^\omega_i) = \{(s \mod 1, \varphi^\omega_i(s)) : s \in [0, \tau^\omega_i)\}, \quad i = 1, 2, \ldots, r^\omega,
\] (26)
are invariant under \((H, \hat{\theta})\). Some estimates in the proof of the following lemmas (Lemma 3.4-Proposition 3.9) are the extension of the results in [14] to the stochastic case. This is not trivial and the pullback technique has to be used to make the estimates work. Moreover, it is essential to prove that
\[
\tau^\omega_{i-t} = \tau^\omega_i \quad \text{and} \quad r^\omega_{i-t} = r^\omega_i
\] (27)
for all \( t \in R \). The fact that the periodic curves are not the trajectories of the random dynamical system, and the fact that the periodic curves are different corresponding to different \( \omega \), make it difficult to follow the trajectories of the random dynamical systems. The essential difficulty in the stochastic case arises from the fact that although \( X^\omega \) is a random invariant set satisfying (11), but in general, \( X^\omega \) and \( \gamma^\omega(t)X^\omega \) are different sets. The set \( X^\omega \) is not invariant in the sense as in the deterministic case. This is fundamentally different from the deterministic case. As a consequence, the neighbourhood of \( X^\omega \) is only forward invariant in the sense of (17). The assumption (18) makes the \((\delta_1, \varepsilon_1)\)-neighbourhood of \( X^\omega \) under the map \( H^m_{\omega_0}\omega \) contracting in the \( y \)-direction to the \((\delta_1, \varepsilon_1)\)-neighbourhood of \( X^\omega_{\delta_1, \varepsilon_1} \omega \), rather than the \((\delta_1, \varepsilon_1)\)-neighbourhood of \( X^\omega \) as in the deterministic case. Locally, (19) says the map \( H^m_{\omega_0}\omega \) maps the
Lemma 3.4
Under Conditions (i), (ii), (iii), the function with a Lipschitz constant $L$
Here, unlike the deterministic case, $H^{\overline{\nu},\omega}(s, y)$ is not on the same $X$ as $(s, y)$ is.

To prove the above claim, for $(s, y) \in B(X^\omega, \varepsilon_1) \subset S^1 \times Y^\omega$, denote
\[
H_1^\omega(s, y) = (h_1(s), g_1(s, y)).
\]

For any $(s^*, y^*) \in S_1 \times Y^{\overline{\nu}}$, define
\[
\xi_m^\overline{\nu} : [h_1^m(s^*) - \delta_1, h_1^m(s^*) + \delta_1] \rightarrow Y^\omega
\]
by an induction:
\[
\xi_0^\overline{\nu}(s) = f(s) \in Y^{\overline{\nu}}, \forall s \in [s^* - \delta_1, s^* + \delta_1],
\]
\[
\xi_1^\overline{\nu}(s) = g_1^\overline{\nu}(h_1^{-1}(s), \xi_0^\overline{\nu}(h_1^{-1}(s))) \in Y^{\overline{\nu}},
\]
\[
\forall s \in [h_1(s^*) - \delta_1, h_1(s^*) + \delta_1],
\]
and
\[
\xi_m^\overline{\nu}(s) = g_1^\overline{\nu}(h_1^{-1}(s), \xi_m^{-1}(s) \in Y^\omega,
\]
\[
\forall s \in [h_1(s^*) - \delta_1, h_1(s^*) + \delta_1].
\]

Denote
\[
\delta_2 = \min\{\frac{\delta_1}{2}, \frac{\varepsilon_1}{4L_1}\} \text{ and } L = \frac{c}{1 - \lambda}.
\]

Lemma 3.4
Under Conditions (i), (ii), (iii), the function $\xi_i^\overline{\nu}$ is Lipschitz continuous with a Lipschitz constant $L$ for all $m \in \mathbb{N}$ and $i = 1, 2, \ldots, m$, that is, for any $m \in \mathbb{N}$ and $i = 1, 2, \ldots, m$,
\[
(s, \xi_i^\overline{\nu}(s)), (s', \xi_i^\overline{\nu}(s')) \in B(X^{\overline{\nu} - (m - 1)\overline{\nu}}, \varepsilon_1),
\]
and
\[
\|\xi_i^\overline{\nu}(s) - \xi_i^\overline{\nu}(s')\| \leq L|s - s'|,
\]
for any $s, s' \in [h_1^m(s^*) - \delta_2, h_1^m(s^*) + \delta_2]$.

Proof We prove this lemma by the induction on $i = 1, 2, \ldots, m$ for an arbitrary $m$. When $i = 1$ and $s, s' \in [h_1(s^*) - \delta_2, h_1(s^*) + \delta_2]$, by (16), we have
\[
\|\xi_1^\overline{\nu}(s) - \xi_1^\overline{\nu}(s')\|
\]
\[
= \|g_1^\overline{\nu}(h_1^{-1}(s), f(s)) - g_1^\overline{\nu}(h_1^{-1}(s'), f(s))\|
\]
\[
\leq c|h_1^{-1}(s) - h_1^{-1}(s')|
\]
\[
< L|s - s'|,
\]
and by (17), we have 
\( (s, \xi_1, \xi_1', \xi_1^\omega - \xi_1^\omega) \in B(X \hat{\omega}^{-(m-1)n_0}, \varepsilon_1) \). Now suppose the required result holds for \( i - 1 \in \{1, 2, \cdots, m - 1\} \), then for any \( s, s' \in [h_1^i(s^*) - \delta_2, h_1^i(s^*) + \delta_2] \)

\[
\| \xi_i^\omega - \xi_i^\omega(s') \| \\
= \| g_1^\omega(h_1^i(s), \xi_i^\omega(h_1^i(s))) - g_1^\omega(h_1^i(s'), \xi_i^\omega(h_1^i(s'))) \| \\
\leq \| g_1^\omega(h_1^i(s), \xi_i^\omega(h_1^i(s))) - g_1^\omega(h_1^i(s'), \xi_i^\omega(h_1^i(s))) \| \\
+ \| g_1^\omega(h_1^i(s'), \xi_i^\omega(h_1^i(s))) - g_1^\omega(h_1^i(s'), \xi_i^\omega(h_1^i(s'))) \| \\
\leq (c + \lambda L)|s - s'| \\
\leq L|s' - s|, 
\]

and the claim \( (s, \xi_i^\omega(s), s', \xi_i^\omega(s')) \in B(X \hat{\omega}^{-(m-1)n_0}, \varepsilon_1) \) follows from (17). The lemma is proved. \[\blacksquare\]

For any \( (s, y) \in X_s^\omega \), let \( N(s, y, \delta_2, \varepsilon_1) \) be the interior of \( B^\omega(s, y, \delta_2, \varepsilon_1) \). Then for any \( s^* \in S^1 \), \( \{N(s^*, y, \delta_2, \varepsilon_1) \} \) is an open covering of \( X_s^\omega \). By compactness of \( X_s^\omega \), a finite subcover, \( N(s^*, y^{(1)} \omega, \delta_2, \varepsilon_1) \), \( N(s^*, y^{(2)} \omega, \delta_2, \varepsilon_1) \), \( \ldots \), \( N(s^*, y^{(p)} \omega, \delta_2, \varepsilon_1) \), could be found. Define

\[
N^\omega(s^*, \delta_2, \varepsilon_1) = \bigcup_{i=1}^{p_\omega} N(s^*, y^{(i)} \omega, \delta_2, \varepsilon_1),
\]

\[
B^\omega(s^*, \delta_2, \varepsilon_1) = \bigcup_{i=1}^{p_\omega} B(s^*, y^{(i)} \omega, \delta_2, \varepsilon_1).
\]

Note that \( B^\omega(s^*, \delta_2, \varepsilon_1) \) is the closure of \( N^\omega(s^*, \delta_2, \varepsilon_1) \). It is easy to see that \( X_s^\omega \in N^\omega(s^*, \delta_2, \varepsilon_1) \).

It is possible for \( B(s^*, y^{(1)} \omega, \delta_2, \varepsilon_1) \) to overlap, which leads to the inconvenience in the argument below. It is therefore to merge such boxes and work with the connected components of \( B^\omega(s^*, \delta_2, \varepsilon_1) \). Denote them by \( B_1^\omega(s^*, \delta_2, \varepsilon_1) \), \( B_2^\omega(s^*, \delta_2, \varepsilon_1) \), \( B_3^\omega(s^*, \delta_2, \varepsilon_1) \), \( \ldots \)

Choose \( (h_1^{-m}(s), \hat{y}), (h_1^{-m}(s'), \hat{y}') \in B_j^\omega(s^*, \delta_2, \varepsilon_1) \) such that

\[
H_1^m \hat{\omega}^{-(m-1)n_0}(h_1^{-m}(s), \hat{y}) = (s, y),
\]

\[
H_1^m \hat{\omega}^{-(m-1)n_0}(h_1^{-m}(s'), \hat{y}') = (s', y').
\]

Then it is obvious that

\[
y = g_1(h_1^{-m}(s), \hat{y}),
\]

\[
y' = g_1(h_1^{-m}(s'), \hat{y}').
\]
Let \((s^*, y^*) \in X^{\hat{m}, \omega} \cap B_j^{\hat{m}, \omega}(s^*, \delta_2, \varepsilon_1)\) such that \((h_1^{-m}(s), y^*), (h_1^{-m}(s'), y^*) \in B_j^{\hat{m}, \omega}(s^*, \delta_2, \varepsilon_1)\), then from (18) and Lemma 3.4,

\[
\|y - y'\| = \|g_1^{(m)}(\hat{m}, \omega)(h_1^{-m}(s)), \hat{y} - g_1^{(m)}(\hat{m}, \omega)(h_1^{-m}(s'), \hat{y}')\| \\
\leq \|g_1^{(m)}(\hat{m}, \omega)(h_1^{-m}(s)), \hat{y} - g_1^{(m)}(\hat{m}, \omega)(h_1^{-m}(s), y^*)\| + \|g_1^{(m)}(\hat{m}, \omega)(h_1^{-m}(s), y^*) - g_1^{(m)}(\hat{m}, \omega)(h_1^{-m}(s'), y^*)\| + \|g_1^{(m)}(\hat{m}, \omega)(h_1^{-m}(s'), y^*) - g_1^{(m)}(\hat{m}, \omega)(h_1^{-m}(s'), \hat{y}')\| \\
\leq 2\lambda^m b^* + L|h_1^{-m}(s) - h_1^{-m}(s')| \\
\leq 2\lambda^m b^* + L|s - s'|.
\]

Choose \(N^\omega \in \mathbb{N}\) such that

\[
N^\omega > \frac{\log \Delta^\omega}{2b^*}.
\]

This implies that

\[2\lambda^N b^* < \Delta^\omega.\]

Choose \(\delta_3 \in (0, \delta_2)\) satisfying

\[
\delta_3 < \frac{\Delta^\omega - 2\lambda^N b^*}{L}. \tag{28}
\]

Lemma 3.6 Under Conditions (i), (ii), (iii), for any \(m \in \{N^\omega, N^\omega + 1, N^\omega + 2, \ldots\}\) and any \(j \in \{1, 2, \ldots, r_\omega \}\), there exists a unique \(i \in \{1, 2, \ldots, r_\omega\}\) such that

\[B_i^\omega(s^*, \delta_3, \varepsilon_1) \cap H_1^{m, \omega}(B_j^{\hat{m}, \omega}(s^*, \delta_3, \varepsilon_1)) \neq \emptyset.\]

**Proof** By the definition of \(B_j^{\hat{m}, \omega}(s^*, \delta_3, \varepsilon_1)\), we know that there exists a \((s^*, y^*) \in X^{\hat{m}, \omega} \cap B_j^{\hat{m}, \omega}(s^*, \delta_3, \varepsilon_1)\). Because of the invariance of \(X\) with respect to \(H_1\), we know that \(H_1^{m, \omega}(s^*, y^*) \in X^\omega\). So \(H_1^{m, \omega}(s^*, y^*) \in B^\omega(s^*, \delta_3, \varepsilon_1)\). Hence there exists an \(i \in \{1, 2, \ldots, r_\omega\}\) such that \(H_1^{m, \omega}(s^*, y^*) \in B_i^\omega(s^*, \delta_3, \varepsilon_1)\). So

\[B_i^\omega(s^*, \delta_3, \varepsilon_1) \cap H_1^{m, \omega}(B_j^{\hat{m}, \omega}(s^*, \delta_3, \varepsilon_1)) \neq \emptyset.\]

Now we prove the uniqueness of \(i\). For any \((s, y) \in B_j^{\hat{m}, \omega}(s^*, \delta_3, \varepsilon_1), (h_1^{m}(s), y^*) \in H_1^{m, \omega}(B_j^{\hat{m}, \omega}(s^*, \delta_3, \varepsilon_1))\). From Lemma 3.5 and (28) we know that

\[
\|g_1^{(m)}(\hat{m}, \omega)(s^*, y^*) - g_1^{(m)}(\hat{m}, \omega)(s, y)\| \leq L|s - s^*| + 2\lambda^m b^* \\
< L\delta_3 + 2\lambda^m b^* \\
< \Delta^\omega.
\]

So for any \(i' \in \{1, 2, \ldots, r_\omega\}\) \(\{i\}\), \((h_1^{(m)}(s), g_1^{m, \omega}(s, y)) \notin B_i^\omega(s^*, \delta_3, \varepsilon_1)\). Thus

\[H_1^{m, \omega}(B_j^{\hat{m}, \omega}(s^*, \delta_3, \varepsilon_1)) \cap B_i^\omega(s^*, \delta_3, \varepsilon_1) = \emptyset,
\]

and the uniqueness of \(i\) is proved.
Definition 3.7 Given any \( m \in \{N^\omega, N^\omega + 1, N^\omega + 2, \ldots \} \) and \( j \in \{1,2,\ldots, r^\omega \} \), denote by \( \sigma_m^\omega \) the unique \( i \in \{1,2,\ldots,r^\omega \} \) such that \( B_i^m(s^*, \delta_3, \varepsilon_1) \cap H_{s^*, \delta_3, \varepsilon_1}(s^*, \delta_3, \varepsilon_1) \neq \emptyset \).

Lemma 3.8 Under Conditions (i), (ii) and (iii), for any \( m \in \{N^\omega, N^\omega + 1, N^\omega + 2, \ldots \} \), \( \sigma_m^\omega \) is a permutation and the function \( \sigma_m^\omega : \{1,2,\ldots, r^\omega \} \rightarrow \{1,2,\ldots, r^\omega \} \) is a permutation. In particular, \( \sigma_m^\omega \) is invertible and given any \( m \in \{N^\omega, N^\omega + 1, N^\omega + 2, \ldots \} \), \( i \in \{1,2,\ldots,r^\omega \} \), there exists a unique \( j = (\sigma_m^\omega)^{-1}(i) = \tau_m^\omega(i) \in \{1,2,\ldots,r^\omega \} \) such that \( B_i^m(s^*, \delta_3, \varepsilon_1) \cap H_m^{\hat{\omega}}(B_j^{\hat{\omega}}(s^*, \delta_3, \varepsilon_1)) \neq \emptyset \).

Proof Clearly, for any \( i \in \{1,2,\ldots,r^\omega \} \), \( B_i^m(s^*, \delta_3, \varepsilon_1) \cap X_{s^*}^\omega \neq \emptyset \). Hence, using Lemma 3.6, we have

\[
B_i^m(s^*, \delta_3, \varepsilon_1) \cap \left( \bigcup_{j=1}^{r^\omega} H_m^{\hat{\omega}}(B_j^{\hat{\omega}}(s^*, \delta_3, \varepsilon_1)) \right) \neq \emptyset.
\]

Thus the map \( \sigma_m : \{1,2,\ldots, r^\omega \} \rightarrow \{1,2,\ldots, r^\omega \} \) is onto. We need to prove that \( \sigma_m \) is one-to-one. As the map \( H_m^{\hat{\omega}}(B_j^{\hat{\omega}}(s^*, \delta_3, \varepsilon_1)) \) is a contraction in the \( y \)-direction and a shift in the \( s \) direction, it is evident that for such a \( j \) with \( \sigma_m^\omega(j) = i \),

\[
H_m^{\hat{\omega}}(B_j^{\hat{\omega}}(s^*, \delta_3, \varepsilon_1)) \subset B_i^m(s^*, \delta_3, \varepsilon_1).
\]

But for \( j' \neq j \), there exists \( i' \in \{1,2,\ldots, r^\omega \} \) such that

\[
H_m^{\hat{\omega}}(B_j^{\hat{\omega}}(s^*, \delta_3, \varepsilon_1)) \subset B_{i'}^m(s^*, \delta_3, \varepsilon_1),
\]

and

\[
B_i^m(s^*, \delta_3, \varepsilon_1) \cap B_i^m(s^*, \delta_3, \varepsilon_1) = \emptyset.
\]

It follows that

\[
B_i^m(s^*, \delta_3, \varepsilon_1) \cap H_m^{\hat{\omega}}(B_j^{\hat{\omega}}(s^*, \delta_3, \varepsilon_1)) = \emptyset.
\]

Therefore \( \sigma_m \) is an one-to-one map and \( r^\omega = r^\omega \). In particular, \( \sigma_m^\omega \) is a permutation.

Proposition 3.9 Under Conditions (i), (ii), (iii), there exist \( r^\omega \) Lipschitz functions \( \varphi_i^\omega : [s^* - \delta_3^\omega, s^* + \delta_3^\omega] \rightarrow Y^\omega \) such that \( X^\omega \cap \left( [s^* - \delta_3^\omega, s^* + \delta_3^\omega] \times Y^\omega \right) \subset \bigcup_{i=1}^{r^\omega} \text{graph}(\varphi_i^\omega) \) and for each \( i \in \{1,2,\ldots, r^\omega \} \), we have \( \text{graph}(\varphi_i^\omega) \subset B_i^m(s^*, \delta_3, \varepsilon_1) \).

Proof Let \( \tau_m^\omega \) be the inverse of \( \sigma_m^\omega \) and for each \( i \in \{1,2,\ldots, r^\omega \} \), define

\[
W_i^m = \bigcap_{m=N^\omega}^{\infty} H_m^{\hat{\omega}}(B_m^{\hat{\omega}}(s^*, \delta_3, \varepsilon_1)).
\]

For any \( (s, y), (s', y') \in W_i^m \), by Lemma 3.5, we have

\[
\|y - y'\| \leq L|s - s'| + 2\lambda^m b^*.
\]

for any \( m \in \{N^\omega, N^\omega + 1, N^\omega + 2, \ldots \} \). Let \( m \rightarrow \infty \), we get \( \|y - y'\| \leq L|s - s'| \). That is, each \( W_i^m \) is contained in the graph of a Lipschitz function with a Lipschitz constant \( L \). Let
\[ X_i^\omega = X^\omega \cap B_i^\omega(s^*, \delta_3, \varepsilon_1) = X^\omega \cap \left( [s^* - \delta_3, s^* + \delta_3] \times Y^\omega \right) \cap B_i^\omega(s^*, \delta_3, \varepsilon_1). \] (34)

It is easy to see that
\[ X^\omega \cap \left( [s^* - \delta_3, s^* + \delta_3] \times Y^\omega \right) \subset \bigcap_{m=N^\omega}^{\infty} \bigcap_{j=1}^{r^\omega} H_{1, m, \hat{\delta} - m^\omega \omega} \left( B_{\hat{j}}^{\delta - m^\omega \omega}(s^*, \delta_3, \varepsilon_1) \right). \] (35)

By Lemma 3.8, for any \( m \in \{ N^\omega, N^\omega + 1, N^\omega + 2, \ldots \} \),
\[ B_i^\omega(m, \delta - m^\omega \omega \omega)(B_{\hat{j}}^{\delta - m^\omega \omega}(s^*, \delta_3, \varepsilon_1)) \neq \emptyset, \] (36)
\[ B_i^\omega(s^*, \delta_3, \varepsilon_1) \cap \left( \bigcup_{j \neq \hat{j}}^{r^\omega} H_{1, m, \hat{\delta} - m^\omega \omega \omega}(s^*, \delta_3, \varepsilon_1) \right) = \emptyset. \] (37)

So it follows from (33)-(37) that
\[ \hat{X}_i^\omega \subset W_i^\omega. \]

It is easy to show that
\[ \hat{X}_i^\omega \cap X_i^* \neq \emptyset, \]
for any \( s \in [s^* - \delta_3, s^* + \delta_3] \), \( i = 1, 2, \ldots, r^\omega \). Moreover, for each \( s \in [s^* - \delta_3, s^* + \delta_3] \), \( \hat{X}_i^\omega \cap X_i^* \) contains exactly one point. This can be seen from \( ||y - y'|| \leq L|s - s'| \), for any \( (s, y), (s', y') \in \hat{X}_i^\omega \). Denote this point by \( (s, \varphi^\omega_i(s)) \). But when \( s \) varies in \( [s^* - \delta_3, s^* + \delta_3] \), \( (s, \varphi^\omega_i(s)) \) traces the graph(\( \varphi^\omega_i \)). It is obvious that graph(\( \varphi^\omega_i \)) = \( \hat{X}_i^\omega \subset B_i^\omega(s^*, \delta_3, \varepsilon_1) \), and \( \varphi^\omega_i \) is a Lipschitz function with the Lipschitz constant \( L \).

**Theorem 3.10** Under Conditions (i), (ii), (iii), \( X^\omega \) is a union of a finite number of Lipschitz periodic curves.

**Proof:** First note that \( \delta_3 > 0 \) is independent of \( s^* \in S^1 \). Let \( M \in \mathbb{N} \) such that \( \frac{1}{M} \leq \delta_3 \). Define \( s_m = \frac{m}{M}, m = 1, 2, \ldots, M \). Then \( \{ (s_{m-1}, s_{m+1}) : m = 1, 2, \ldots, M \} \) (in which \( s_{M+1} = s_1, s_0 = s_M \) ) covers \( S^1 \). By Proposition 3.9, we know that \( X^\omega \cap \left( [s_{m-1}, s_{m+1}] \times Y^\omega \right) \) contains a finite number of Lipschitz curves, denote their number by \( r^\omega(m) \). Since \( [s_{m-1}, s_m] \subset [s_{m-2}, s_m] \cap [s_{m-1}, s_{m+1}] \), so we have \( r^\omega(m_1) = r^\omega(m_2) \) when \( m_1 \neq m_2 \). So \( r^\omega \) is independent of \( m \) and define all of them by \( r^\omega \). Thus the Lipschitz curves on \( X^\omega \cap \left( [s_{m-1}, s_{m+1}] \times Y^\omega \right) \) could be expanded to \( S^1 \) and we have the following random Poincare map
\[ H_{n_0, \hat{\delta} - n_0 \omega} : G_{\hat{s}}^\delta - n_0 \omega \rightarrow G_{\hat{s}}^\omega, \]
in which \( G_{\hat{s}}^\delta - n_0 \omega, G_{\hat{s}}^\omega \) are finite sets containing \( r^\omega \) elements:
\[ G_{\hat{s}}^\delta - n_0 \omega = \{(s \text{ mod } 1, \varphi^\delta_i - n_0 \omega(s)) : i = 1, 2, \ldots, r^\omega \}, \]
\[ G_{\hat{s}}^\omega = \{(s \text{ mod } 1, \varphi^\omega_i(s)) : i = 1, 2, \ldots, r^\omega \}, \]
for a fixed \( s \in R^1 \). By the finiteness of \( G_{\hat{s}}^\omega \), we know
Actually above is true for any $s$ due to the continuity of $\varphi_i^\omega$, $i = 1, 2, \cdots, r^\omega$. Therefore there are three cases:

(i). Exact one of $i_1, i_2, \cdots, i_r^\omega$ is equal to $i$. Say $i_{\tau^i} = i$. Then

$$\varphi_i^\omega (s + \tau_i^\omega) = \varphi_i^\omega (s),$$

for any $s \in R$. So $\varphi_i^\omega$ is a periodic function of period $\tau_i^\omega$.

(ii). More than one of $i_1, i_2, \cdots, i_r^\omega$ is equal to $i$. Denote $\tau_i^\omega$ the smallest number $j$ such that $i_j = i$ and $\tilde{\tau}_i^\omega > \tau_i^\omega$ such that $i_{\tilde{\tau}_i^\omega} = i$. Then

$$\varphi_i^\omega (s + \tilde{\tau}_i^\omega) = \varphi_i^\omega (s),$$

$$\varphi_i^\omega (s + \tilde{\tau}_i^\omega) = \varphi_i^\omega (s).$$

But

$$\varphi_i^\omega (s + \tilde{\tau}_i^\omega) = \varphi_i^\omega (s + \tilde{\tau}_i^\omega - \tau_i^\omega + \tau_i^\omega)$$
$$= \varphi_i^\omega (s + \tilde{\tau}_i^\omega - \tau_i^\omega)$$
$$= \cdots$$
$$= \varphi_i^\omega (s + \tilde{\tau}_i^\omega - k\tau_i^\omega),$$

where $k$ is the smallest integer such that $\tilde{\tau}_i^\omega - (k + 1)\tau_i^\omega \leq 0$. Then by definition of $\tau_i^\omega$,

$$\tilde{\tau}_i^\omega - k\tau_i^\omega = \tau_i^\omega,$$

so

$$\tilde{\tau}_i^\omega = (k + 1)\tau_i^\omega.$$

Therefore $\varphi_i$ is a periodic function of period $\tau_i^\omega$.

(iii). None of $i_1, i_2, \cdots, i_r^\omega$ is equal to $i$. In this case, at least two of $i_1, i_2, \cdots, i_r^\omega$ must be equal. Say $\tau_2^\omega > \tau_1^\omega$ are the two such integers such that $i_{\tau_2^\omega} = i_{\tau_1^\omega}$ with smallest difference $\tau_2^\omega - \tau_1^\omega$. Then

$$\varphi_i^\omega (s + \tau_1^\omega) = \varphi_i^\omega (s + \tau_2^\omega).$$

Denote $s + \tau_1^\omega$ by $s_1$, then

$$\varphi_i^\omega (s) = \varphi_i^\omega (s + \tau_2^\omega - \tau_1^\omega), \forall s \in \mathbb{R}^1.$$

Same as (ii) we can see for all other possible $\tilde{\tau}_2^\omega$ and $\tilde{\tau}_1^\omega$, $\tilde{\tau}_2^\omega > \tilde{\tau}_1^\omega$ and $i_{\tilde{\tau}_2^\omega} = i_{\tilde{\tau}_1^\omega}$, $\tilde{\tau}_2^\omega - \tilde{\tau}_1^\omega$ must be an integer multiple of $\tau_2^\omega - \tau_1^\omega$. That is to say $\varphi_i^\omega$ is a periodic curve with period $\tau_2^\omega - \tau_1^\omega$. ■

Theorem 3.10 says there exists a finite number of continuous periodic functions $\varphi_i^\omega, \varphi_2^\omega, \cdots, \varphi_r^\omega$ on $\mathbb{R}^1$. Denote their periods by $\tau_1^\omega, \tau_2^\omega, \cdots, \tau_r^\omega \in \mathbb{N}$ respectively. So

$$X^\omega = L_1^\omega \cup L_2^\omega \cup \cdots \cup L_r^\omega.$$
where

\[ L^\omega_i = \text{graph}(\varphi^\omega_i) = \{(s \mod 1, \varphi^\omega_i(s)) : s \in [0, \tau^\omega_i)\}. \]

But

\[ H^{\hat{\omega}_0}_i \theta^{-\omega} \omega(X^{\hat{\omega}_0} \omega) = X^\omega. \]

So

\[ H^{\hat{\omega}_0}_i \theta^{-\omega} \omega(L^{\hat{\omega}_0}_i \omega) \cup H^{\hat{\omega}_0}_i \theta^{-\omega} \omega(L^\omega_2 \omega) \cup \cdots \cup H^{\hat{\omega}_0}_i \theta^{-\omega} \omega(L^\omega_\ell \omega) = L^\omega_i \cup L^\omega_2 \cup \cdots \cup L^\omega_\ell. \quad (38) \]

It is easy to know that \( H^{\hat{\omega}_0}_i \theta^{-\omega} \omega(L^{\hat{\omega}_0}_i \omega) \) is a closed curve since \( L^{\hat{\omega}_0}_i \omega \) is a closed curve and \( H^{\hat{\omega}_0}_i \theta^{-\omega} \omega \) is a continuous map. Moreover, since \( H_i \) is a homeomorphism, so

\[ H^{\hat{\omega}_0}_i \theta^{-\omega} \omega(L^{\hat{\omega}_0}_i \omega) \cap H^{\hat{\omega}_j}_i \theta^{-\omega} \omega(L^{\hat{\omega}_j}_i \omega) = \emptyset, \quad (39) \]

when \( i \neq j \). Therefore the left hand side of (38) is a union of \( r_i \hat{\omega}_0 \omega \) distinct closed curves and the right hand side of of (38) is a union of \( r_0 \omega \) distinct closed curves. Thus for any \( i \in \{1, 2, \ldots, r_i \hat{\omega}_0 \omega\} \), there exists a unique \( j \in \{1, 2, \ldots, r_0 \omega\} \) such that

\[ H^{\hat{\omega}_0}_i \theta^{-\omega} \omega(L^{\hat{\omega}_j}_i \omega) = L^\omega_j. \quad (40) \]

Denote \( j = K(i) \). It is easy to see now that \( r_i \hat{\omega}_0 \omega = r_0 \omega \). Therefore \( K : \{1, 2, \ldots, r_i \hat{\omega}_0 \omega\} \rightarrow \{1, 2, \ldots, r_0 \omega\} \) and \( K \) is a permutation. Reorder \( \{L^\omega_i : i = 1, 2, \ldots, r_i \hat{\omega}_0 \omega\} \), we can have

\[ H^{\hat{\omega}_0}_i \theta^{-\omega} \omega(L^{\hat{\omega}_j}_i \omega) = L^\omega_j. \quad (41) \]

Moreover, using a similar argument, and note for any \( t \in R \)

\[ X^{\theta^{-\omega} \omega} = L^{\theta^{-\omega} \omega}_1 \cup L^{\theta^{-\omega} \omega}_2 \cup \cdots \cup L^{\theta^{-\omega} \omega}_r \]

and

\[ \gamma^{\theta^{-\omega} \omega}(t)(L^{\theta^{-\omega} \omega}_1) \cup \gamma^{\theta^{-\omega} \omega}(t)(L^{\theta^{-\omega} \omega}_2) \cup \cdots \cup \gamma^{\theta^{-\omega} \omega}(t)(L^{\theta^{-\omega} \omega}_r) = L^{\omega}_1 \cup L^{\omega}_2 \cup \cdots \cup L^{\omega}_r. \]

we have the following proposition.

**Proposition 3.11** Under Conditions (i), (ii) and (iii), for each \( \omega \), we have for any \( t \in R \)

\[ \gamma^{\theta^{-\omega} \omega}(t)L^{\theta^{-\omega} \omega}_i = L^{\omega}_i. \quad (42) \]

**Lemma 3.12** For any \( t \in R \), \( \tau^{\theta^{-\omega} \omega}_i = \tau^{\omega}_i \) for any \( i = 1, 2, \ldots, r_0 \omega \).

**Proof.** Consider first the case when \( t = kt_1 \) \( (k \in \{0, 1, 2, \ldots\}) \). Note for any \( s \in \{0, 1, 2, \ldots, \tau^{\theta^{-\omega} \omega}_i\} \),

\[ s_1 = \left( \gamma^{\theta^{-\omega} \omega}(s \mod 1, \varphi^{\theta^{-\omega} \omega}(s)) \right) = s + k. \quad (43) \]

Here \( (\cdot)_S \) denotes the \( S_1 \) coordinate of the vector. So for \( t = kt_1 \), from (42) and (43), it turns out that
Therefore from (49) and the cocyle property of \( \gamma \), the lemma is proved.
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