Brane-Worlds and Cosmology

M. D. Maia
Universidade de Brasília, Instituto de Física
Brasília. D.F. 70919-970
maia@fis.unb.br

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I first met Mario back in 1965 when we were students at the University of Brasília, and I recall one day when he invited me for a chat on the expansion of the universe. We never finished that conversation because the university was shutdown soon after and we went to different places. However, Mario always remained faithful to his cosmological quest. On his 60th birthday, I find it rewarding to have the chance to continue that chat, in a much larger and complex world and under the brighter light of type Ia supernovas. I hope we will never end.

Abstract

The Friedmann equation for a FRW-brane-world in a flat bulk is derived and applied to the accelerated expansion of the universe.

1 Brane-Worlds

It has been recently noted that the predicted scale of $10^{15}$ TeV for quantum gravity is only a conjecture without experimental support, and that the only experimentally verified scale of gauge interactions in four dimensions lies within the TeV scale. Therefore, the assumptions that gravitation becomes strong at the TeV scale, while the standard gauge interactions remain confined to the four dimensional space-time, does not conflict with the
today’s experimental data [1, 2]. These ideas, aimed to solve the hierarchy problem of the fundamental interactions, were originally inspired by the Horawa-Witten M-theory [3].

We may identify four basic principles of the brane-world program: The first one, of phenomenological nature, sets the fundamental scale of interaction, including gravity to the TeV scale. The other three principles are of theoretical nature, stating that the space-time (or, the brane-world) is embedded in a higher dimensional bulk space; that ordinary matter and the standard gauge interaction remain confined to the brane-world; and that the extra dimensions are probed by gravitons.

Under these assumptions, the extra dimensions can be large and sometimes non-compact. This follows from an simple argument using the Einstein-Hilbert action for the brane-world $V_4$, assuming that the bulk has product topology $V_D \sim V_4 \times B_N$, where $B_N$ is the space generated by the extra dimensions

\[ A = \int R \sqrt{-G} dx^{4+N} \approx \frac{1}{M_s^{2+N}}, \quad G_s = \hbar c / M^2_s, \quad \hbar = c = 1 \]

where $M_s$ denotes the equivalent to Planck’s mass in the higher dimensional space.

On the other hand, assuming that ordinary Einstein’s gravity remains confined to the space-time $V_4$, we find that

\[ A = \int R \sqrt{-G} dx^4 x^V \approx \frac{1}{M_{Pl}^2 V}, \quad G = \hbar c / M_{Pl}^2, \quad \hbar = c = 1 \]

where $V$ is the volume of $B_N$. Comparing the two expressions and denoting by $\ell$ the typical length of the extra dimensions, we obtain

\[ \ell \approx \frac{M_{Pl}^{2/N}}{M_s^{1+2/N}} \]

If $M_s \approx 1$TeV e $M_{Pl} \approx 10^{15}$TeV, then $\ell \approx 10^{13}$cm when $N = 1$. This is approximately the diameter of the solar system, quite unsuitable as an internal dimension. On the other hand if $N = 2$, a six dimensional bulk, we obtain the sub millimeter size $\ell \approx 10^{-2}$cm. Larger number of dimensions would produce even smaller values for $\ell$. 
One immediate consequence is the modification of Newton's gravitational theory at small distances. In fact, replacing $M_{Pl}$ in the Newtonian gravitational potential, we find

$$U = \frac{m_1 m_2}{M_{Pl}^2} \frac{1}{r} = \frac{m_1 m_2}{M_{2+N}^2 \ell^N r}$$

For large values of $r$ as compared with $\ell$, we obtain the usual Newtonian theory. However, when $r \approx \ell$ we find

$$U = \frac{m_1 m_2}{M_{2+N}^2} \frac{1}{r^{N+1}}$$

so that for $N = 2$ the gravitational field would decay as $1/r^3$ at distances of the order of $10^{-2}$ cm. Interesting enough, Newton's theory has never been accurately verified at those small distances. A number of experiments to verify this are currently being assembled [4].

Other notable consequences are the possible generation of (high frequency) gravitational waves and the detection of black-holes (and worm-holes) in the space-time at the TeV scale, estimated to occur sometime around 2006 at the LHC [5]. Cosmology would also be a laboratory to observe these phenomena, where the presence of TeV gravitons at the core of galaxies could give a new estimate for the vacuum energy. Even at the classical level, we may be already observing the effect of the extra dimensions in the form of the accelerated expansion of the universe.

### 2 Some Geometrical Aspects

The brane-worlds program has been implemented mostly on particular models where the bulk has a fixed geometry and the space-time has a specific metric ansatz. Nonetheless, it is possible to give a fairly general description based on the four mentioned principles. Here we present only a brief sketch [6].

Consider that we have a $D$-dimensional bulk $V_D$ in which we have a local and isometric embedded brane-world $V_4$ given by the embedding map $Z : V_4 \rightarrow V_D$ whose derivative map satisfies the embedding equations

$$Z^{\mu}_{,i} Z^{\nu}_{,j} G_{\mu \nu} = g_{ij}, \quad Z^{\mu}_{,i} \eta^{\nu}_{A} G_{\mu \nu} = g_{iA}, \quad \eta^{\mu}_{A} \eta^{\nu}_{B} G_{\mu \nu} = g_{AB}$$

(1)
where $\mathcal{G}_{\mu\nu}$ is the metric of $V_D$ in arbitrary coordinates, $g_{AB}$ denotes the metric of the space orthogonal to $V_4$, and $g_{iA} = s^M A_{iAM}$, where

$$A_{iAB} = \eta^B_A \eta^i_B \mathcal{G}_{\mu\nu}$$

(2)

The embedding associates an extrinsic curvature to $V_4$, defined for each direction $\eta_A$ by

$$b_{ijA} = -Z_{\mu, i}^\mu \eta_{\nu j}^\nu \mathcal{G}_{\mu\nu}$$

(3)

Contrarily to the Riemannian curvature, the extrinsic curvature gives a measure of the deviation from the brane-world and its tangent plane at any point, which we call the bending of the geometry. Thus, we may have a bent space even if it is flat in the Riemannian sense. We may define a mean curvature for each normal direction to $V_4$ by

$$h^A = g_{ij} \kappa_{ijA}$$

The total mean curvature is

$$h^2 = g^{AB} h_A h_B$$

The integrability conditions for the embedding equations (1) are the Gauss-Codazzi-Ricci equations, respectively

$$R_{ijkl} = 2 g^{MN} b_{[i[j} b_{k]}^N + \mathcal{R}_{\mu\nu\rho\sigma} Z_{\mu, i}^\mu Z_{\nu, j}^\nu Z_{\rho, k}^\rho Z_{\sigma, l}^\sigma$$

$$b_{[i[A, k]} = g^{MN} A_{[i[M A} b_{k]}^N + \mathcal{R}_{\mu\nu\rho\sigma} Z_{\mu, i}^\mu \eta_{\nu j}^\nu \mathcal{Z}_{\rho, k}^\rho Z_{\sigma, l}^\sigma$$

(4)

$$2 A_{[jAB; k]} + 2 g^{MN} A_{[j[M A k]}^N B] + g^{mn} b_{[i[m A} b_{k]}^n B] + \mathcal{R}_{\mu\nu\rho\sigma} Z_{\mu, j}^\mu Z_{\sigma, k}^\sigma \eta_{iA}^\mu \eta_{jB}^\nu = 0$$

From (1), we obtain

$$g^{ij} Z_{\mu, i}^\mu Z_{\nu, j}^\nu = \mathcal{G}^{\mu\nu} - g^{AB} \eta^\mu_A \eta^\nu_B$$

(5)

Applying this, the contractions of Gauss’ equation gives

$$R = \mathcal{R} - (\omega^2 + h^2) - 2 g^{AB} \frac{\partial h_A}{\partial s^B}$$

where we have denoted $\omega^2 = b_{ijA} b^{ijA}$ [6]. The divergence term can be discarded under a volume integration in the extra dimensions and assuming that the boundary of the integration region is minimal (corresponding to $h_A = 0$).

After adding the Lagrangian for the confined matter in the right hand side, we obtain the brane-worlds Lagrangian

$$L(g) = R \sqrt{g} = 8 \pi G \mathcal{L}_m \sqrt{g} + \mathcal{R} \sqrt{G} + (k^2 + h^2) \sqrt{g}$$

(6)
The corresponding Einstein’s equations for the brane-world can be calculated directly from the contractions of Gauss’ equations

\[ R_{ij} - \frac{1}{2} R g_{ij} = 8\pi G T_{ij}^m + (R_{\nu\rho} - \frac{1}{2} R G_{\nu\rho}) Z^\nu_i Z^\rho_j + Q_{ij} + S_{ij} \]  

(7)

where we have denoted the extrinsic term by

\[ Q_{ij} = g^{MN} (b^m_{iM} b_{jN} - h_{M} b_{ijN}) - \frac{1}{2} (\omega^2 - h^2) g_{ij} \]  

(8)

and an additional term which also depends on the Riemann and Ricci curvatures of the bulk

\[ S_{ij} = g^{AB} R_{\mu\nu} \eta^\mu_A \eta^\nu_B g_{ij} + g^{AB} R_{\mu\rho\sigma} \eta^\mu_A \eta^\rho_B Z^\nu_i Z^\sigma_j - \frac{1}{2} g^{AB} g^{MN} R_{\mu\nu\rho\sigma} \eta^\mu_A \eta^\rho_B \eta^\nu_M \eta^\sigma_N g_{ij} \]  

(9)

Now, the confinement hypothesis implies that ordinary matter, is confined to the brane-world. Therefore, if the bulk is a solution of the high dimensional Einstein’s equations, it is either a vacuum or cosmological constant solution

\[ (R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu}) Z^\mu_i Z^\nu_j = \Lambda g_{ij} \]

Placing this on the left hand side of (7) we obtain

\[ R_{ij} - \frac{1}{2} R g_{ij} - \Lambda g_{ij} = 8\pi G T_{ij}^m + Q_{ij} + S_{ij} \]  

(10)

To these equations we add the conservation law, obtained from the contracted Bianchi identity

\[ 8\pi G T_{ij}^m + S_{ij} = 0, \quad Q_{ij} = 0 \]  

(11)

We focus our attention to \( Q_{ij} \) which depends essentially on the extrinsic curvature and does not depend on the Riemannian curvature.

This is as far as we can go without being more specific. A full account of the brane-worlds equations in a general bulk, including the confinement and some quantum perspectives can be found on [6] and references therein.
3 FRW as a Brane-World in Flat Bulk

The standard FRW cosmological model is known to be embeddable in a flat 5-dimensional flat bulk with metric signature (4, 1) and $g_{55} = 1$ [7]. We use the following parametrization

$$dS^2 = g_{ij}dx^idx^j = -dt^2 + a^2[dr^2 + f(r)(d\theta^2 + sen^2\theta d\varphi^2)]$$

where $f(r) = senr, r, senhr$ corresponding to $k = 1, 0, -1$ respectively. The confined matter is the perfect fluid $T_{ij} = (p + \rho)U_iU_j + pg_{ij}$, where $U_i = \delta_i^0$ in commoving coordinates. The embedding equations (4) become simply

$$R_{ijkl} = 2b_{i[k}b_{l]j}$$

$$b_{[ij;k]} = 0$$

Codazzi’s equation may be easily solved, with solution [8]

$$b_{ab} = \frac{b}{a^2}g_{ab}, \text{ and } b_{44} = -\frac{1}{\dot{a}}\frac{d}{dt}\left(\frac{b}{a}\right)$$

or, after denoting $B = \dot{b}/b$ e $H = \dot{a}/a$, it follows that

$$\omega^2 = b^{ij}b_{ij} = \frac{b^2}{a^4}\left(\frac{2B^2}{H^2} - 2\frac{B}{H} + 4\right), \text{ } h = g^{ij}b_{ij} = \frac{b}{a^2}\left(\frac{B}{H} + 2\right)$$

Therefore

$$\omega^2 - h^2 = -\frac{6b^2}{a^4} \frac{B}{H}$$

and

$$Q_{aa} = \frac{b^2}{a^4}\left(\frac{2B}{H} - 1\right)g_{aa}, \text{ } Q_{44} = \frac{3b^2}{a^4} \text{ } e \text{ } Q = tr(Q_{ij}) = \frac{6b^2}{a^4}\left(\frac{B}{H} - 1\right)$$

In this example we have $S_{ij} = 0$ and the dynamical equations (7) for the FRW metric become

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p + \rho) + \frac{1}{3}(Q_{44} + \frac{1}{2}Q)$$

$$a\ddot{a} + 2\dot{a}^2 + 2k = 4\pi G(p - \rho) - (Q_{ab} - \frac{1}{2}Qg_{ab})$$
After eliminating $\ddot{a}$, we obtain the modified Friedmann’s equation

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 + \frac{b^2}{a^2}$$  \hspace{1cm} (14)$$

Notice that $b(t) = b_{11}$, measures how the universe bends from the tangent space along the radial direction. To this we add the conservation law and the state equation $p = (\gamma - 1)\rho$, which are combined in

$$\frac{\dot{\rho}}{\rho} + 3\gamma \frac{\dot{a}}{a} - \frac{3}{\rho} \frac{d}{dt} \left(\frac{b^2}{a^2}\right) = 0$$

It is possible to add an extra equation relating $b(t)$ to the matter density. This follows from the sandwich conjecture in general relativity using junction conditions such as Israel’s adapted to higher dimensions in the Randall-Sundrum models. However, these junction conditions are not unique and their use lead to instabilities [9]. The application of these conditions to homogeneous and isotropic cosmologies produces a modified Friedmann equation, with an added a term proportional to the square of the density [10].

On the other hand, the accelerated expansion of the universe may be seen as an evidence for equation (14), suggesting that the universe bends in proportion to its expansion:

$$b(t) = k' a(t)$$

In fact, we easily see that the negative constant $k'^2$ on the left side of (14) acts as an accelerator for the expansion, even when $k = 0$.

Another interesting comment is that the signature of the bulk may vary as a response to the quantum fluctuations [6]. This means that the term $k'^2$ in (14) may change sign with the consequent shift from accelerated expansion to contraction.

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