DIFFERENTIABILITY OF THE ARGMIN FUNCTION AND A MINIMUM PRINCIPLE FOR SEMICONCAVE SUBSOLUTIONS

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Abstract. Suppose $f(x, y) + \frac{\kappa}{2} \|x\|^2 - \frac{\sigma}{2} \|y\|^2$ is convex where $\sigma > 0$, and the argmin function $\gamma(x) = \{ \gamma : \inf_y f(x, y) = f(x, \gamma) \}$ exists and is single valued. We will prove $\gamma$ is differentiable almost everywhere. As an application we deduce a minimum principle for certain semiconcave subsolutions.

1. Introduction

The first part of this paper is a proof of the following elementary statement about regularity of certain argmin functions.

Theorem 1. Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is such that

1. There are $\kappa \geq 0$ and $\sigma > 0$ so $f(x, y) + \frac{\kappa}{2} \|x\|^2 - \frac{\sigma}{2} \|y\|^2$ is convex.

2. For each $x \in \mathbb{R}^n$ there is a unique $\gamma(x)$ such that $\inf_y f(x, y) = f(x, \gamma(x))$.

Then the function $\gamma$ is differentiable almost everywhere.

Our motivation for this is the following. After Harvey-Lawson [8, 9], by a (primitive) subequation on an open $X \subset \mathbb{R}^n$ we mean a subset $F \subset J^2(X)$ of the space of 2-jets on $X$ with certain properties. Given such an $F$ and a $C^2$ function $f$, we say that $f$ is $F$-subharmonic if every 2-jet of $f$ lies in $F$. Moreover, using the so-called viscosity technique it is possible to extend the notion of $F$-subharmonicity to any upper-semicontinuous function (details and precise definitions will be given in §3).

In our previous work [16] we introduced a notion of “product subequation” $F\#P$ on $X \times \mathbb{R}^m$ and show (under suitable hypothesis) that if $F$ is convex and $f$ is $F\#P$-subharmonic then its marginal function $g(x) := \inf_y f(x, y)$ is $F$-subharmonic. This statement generalises the classical statement that the marginal function of a convex function is again convex. We will use Theorem 1 to prove a similar minimum principle that does not require $F$ to be convex:

Theorem 2. Let $X \subset \mathbb{R}^n$ be open and $F \subset J^2(X)$ be a constant-coefficient primitive subequation that depends only on the Hessian part. Suppose $f : X \times \mathbb{R} \to \mathbb{R}$
is locally semiconcave, bounded from below, and $F\#P$-subharmonic. Then the marginal function

$$g(x) = \inf_y f(x, y)$$

is $F$-subharmonic on $X$.

A few remarks are in order.

(1) The semiconcavity assumption on $f$ is rather unnatural, since one would expect a subsolution to have some kind of convexity rather than concavity, but it captures what we are able to prove. Observe that $f$ is certainly locally semiconcave if it is $C^{1,1}_{loc}$.

(2) The assumption that $f$ is $F\#P$-subharmonic implies that for each $x$ the function $y \mapsto f(x, y)$ is convex. This along with the semiconcavity assumption implies that $y \mapsto f(x, y)$ is $C^{1,1}_{loc}$.

(3) Theorem 2 can be proved rather easily when $f$ is $C^2$ (see [16, Prop. 7.5] for a stronger statement). To do so, we first approximate $f$ by adding a small multiple of the function $(x, y) \mapsto \|y\|^2$, so there is no loss in assuming $f$ is strictly convex in $y$ and that for each fixed $x$ the function $y \mapsto f(x, y)$ attains its unique minimum at some point $\gamma(x)$. Said another way, $\gamma(x)$ is the unique point such that

$$\frac{\partial f}{\partial y}(x, \gamma(x)) = 0.$$ 

If we assume $f$ is $C^2$ we can then:

(a) Use the implicit function theorem to deduce that $\gamma$ is $C^1$.

(b) Use the chain rule to compute the Hessian of $g$ at a point $x$ in terms of the Hessian of $f$ at the point $(x, \gamma(x))$ and the derivative of $\gamma$ at $x$.

The combination of (b) and assumption that $f$ is $F\#P$-subharmonic yields that $g$ is $F$-subharmonic as claimed.

(4) If we assume furthermore that $F$ is convex, then using smooth mollification to approximate any upper-semicontinuous $F\#P$-subharmonic function by those that are $C^2$, we can deduce a much more general minimum principle – this is the approach taken in [10].

(5) If instead we assume that $f$ is merely $C^{1,1}_{loc}$ then it is of course twice differentiable almost everywhere. However it may well be that $f$ is not twice differentiable at any point of the form $(x, \gamma(x))$ so part (b) of the above argument does not apply.

To prove Theorem 2 we will first use a partial-sup convolution to approximate $f$ by $F\#P$-subharmonic functions $f_\epsilon$, such that

$$f_\epsilon(x, y) + \frac{1}{\epsilon} \|x\|^2 - \frac{\epsilon}{2} \|y\|^2$$

is convex.

In particular for fixed $x$ the function $y \mapsto f_\epsilon(x, y)$ is strongly convex, and we will further arrange so the argmin of $f_\epsilon$ is a well-defined single-valued function $\gamma$. Having done so we can apply Theorem 1 to deduce that $\gamma$ is differentiable almost everywhere, which will act in lieu of the implicit function argument used in (a).

From this one can prove, essentially from the definition, that at almost every point $x$ the Hessian of $g$ is contained in $F$. As $g$ is semiconvex, this is known by the
Almost-Everywhere Theorem of Harvey-Lawson \[7\] to be enough to conclude that \(g\) is \(F\)-subharmonic.

**Comparison with other work:** The authors do not have sufficient expertise to properly survey all previously known regularity results that are related to Theorem \[1\]. It appears that there has been much interest in studying regularity of marginal functions (by which we mean functions of the form \(\inf_y f(x, y)\) or \(\sup_y f(x, y)\) for some function \(f\)) due to its relevance for optimization problems, but less has been said about regularity of the argmin function itself. For example various regularity properties of marginal functions are known both when \(f\) has some convexity property (see for example \[14\], Theorems 23.4 and 24.5) and without this convexity hypothesis (e.g. Clarke \[5\]). The only previous statements about regularity of the argmin function itself we have found are related to continuity rather than differentiability (for example \[18\], Theorem 2.10), which is taken from \[15\], Theorems 1.17 and 7.41, gives conditions under which the argmin function is outer semicontinuous).

Regarding the minimum principle, the fact that the marginal function of a convex function is again convex is a basic property in convex analysis. In the complex case this has an analog for plurisubharmonic functions due to Kiselman \[12\] 13. Both convexity and plurisubharmonicity are massively generalized through the notion of \(F\)-subharmonic functions which uses the viscosity technique that arose in the study of fully non-linear degenerate second-order differential equations (in particular the work of Caffarelli–Nirenberg–Spruck \[4\] and Lions–Crandall–Ishii \[6\], who often refer to such functions as *subsolutions*).

Our motivation for introducing the product \(F\#P\) came from a desire to generalise this minimum principle to general subequations, which we do in \[16\] under the assumption that \(F\) is convex. As discussed above, this assumption is needed only to be able to approximate \(F\#P\)-subharmonic functions by smooth ones, and thus suggests that it is a facet of the proof rather than an essential requirement. Theorem \[2\] is, as far as we know, the first such minimum principle that does not require any convexity hypothesis on the subequation in question. For further background in this area the reader is referred to \[16\].

### 2. Differentiability of the Argmin Function

**2.1. Statement.** In this section we prove that the argmin function of certain semiconvex functions is differentiable (resp. calm) almost everywhere. Suppose \(\Omega \subset \mathbb{R}^{n+m}\) is open and let \(\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n\) be the projection, and write \(\Omega_x = \{y \in \mathbb{R}^m : (x, y) \in \Omega\}\). We will assume throughout that \(\Omega\) is convex, so each \(\Omega_x\) is convex (in particular connected). Now suppose

\[
f : \Omega \to \mathbb{R}
\]

and set

\[
g(x) := \inf_{y \in \Omega_x} f(x, y) \text{ for } x \in \pi(\Omega).
\]

**Definition 3 (Argmin).** The *argmin function* is the set-valued function

\[
\text{argmin}_f(x) := \{\gamma \in \Omega_x : \inf_{y \in \Omega_x} f(x, y) = f(x, \gamma)\}
\]

where we allow the possibility that \(\text{argmin}_f(x)\) is empty.
Below we shall make assumptions on $f$ that ensure that $\text{argmin}(x)$ is defined everywhere on $\pi(\Omega)$ and single-valued. In such cases we shall write

$$\gamma(x) = \text{argmin}_f(x)$$

so

$$f(x, \gamma(x)) = \inf_{y \in \Omega_x} f(x, y) = g(x) \text{ for all } x \in \pi(\Omega).$$

The precise statement we will prove requires some terminology concerning subdifferentials. Let $X \subset \mathbb{R}^n$ be open.

**Definition 4.** Suppose $g : X \to \mathbb{R}$. For each $x_0 \in X$ define

$$\nabla x_0 g = \{ u \in \mathbb{R}^n : g(x) - g(x_0) \geq u.(x-x_0) \text{ for all } x \text{ sufficiently near } x_0 \}$$

which may be empty. We call any $u \in \nabla x_0 g$ a lower support vector for $g$ at $x_0$. Similarly if $\kappa \in \mathbb{R}$ we let

$$\nabla x_0^\kappa g = \{ u \in \mathbb{R}^n : g(x) - g(x_0) \geq u.(x-x_0) - \frac{\kappa}{2}\|x-x_0\|^2 \text{ for all } x \text{ sufficiently near } x_0 \}.$$

**Definition 5 (Semiconvexity and Semiconcavity).** Let $\kappa \geq 0$. We say $g : X \to \mathbb{R}$ is $\kappa$-semiconvex (resp. $\kappa$-semiconcave) if $g(x) + \frac{\kappa}{2}\|x\|^2$ is convex (resp. $g(x) - \frac{\kappa}{2}\|x\|^2$ is concave). If $g$ is $\kappa$-semiconvex/semiconcave for some $\kappa \geq 0$ then we say simply $g$ is semiconvex/semiconcave.

One can check that $g : X \to \mathbb{R}$ is locally $\kappa$-semiconvex if and only if $\nabla x_0^\kappa g$ is non-empty for all $x_0$. Moreover $g$ is differentiable at $x_0$ if and only if $\nabla x_0 g$ is a singleton, in which case its unique element is the derivative of $g$ at $x_0$. Finally if $g$ is a convex function on a convex set $X$ then

$$\nabla x_0^\kappa g = \{ u \in \mathbb{R}^n : g(x) - g(x_0) \geq u.(x-x_0) - \frac{\kappa}{2}\|x-x_0\|^2 \text{ for all } x \}.$$

**Theorem 6 (Argmin is calm almost everywhere).** Let $\Omega \subset \mathbb{R}^{n+m}$ be open and convex. Also let $f : \Omega \to \mathbb{R}$ and suppose there are $\kappa \geq 0$ and $\sigma > 0$ so that

$$f(x, y) + \frac{\kappa}{2}\|x\|^2 - \sigma\|y\|^2 \text{ is convex and}$$

$$\text{argmin}_f(x) \text{ is non-empty for all } x \in \pi(\Omega).$$

Then

(i) The function

$$g(x) := \inf_y f(x, y) = f(x, \gamma(x)) \text{ for } x \in \pi(\Omega)$$

is $\kappa$-semiconvex and $\gamma(x) := \text{argmin}_f(x)$ is single valued for all $x \in \pi(\Omega)$.

(ii) Given any $x_0 \in \pi(\Omega)$ and $u_0 \in \nabla x_0^\kappa g$ there exists a Lipschitz function

$$\phi : V \to \mathbb{R}$$

defined on a neighbourhood $V$ of $(x_0, u_0)$ in $\mathbb{R}^{2n}$ such that

$$\gamma(x) = \phi(x, u) \text{ for all } (x, u) \in V \text{ with } u \in \nabla x g.$$

(iii) The function $\gamma$ is calm almost everywhere. That is, for almost all $x_0 \in \pi(\Omega)$ there are $C$ and $\delta > 0$ such that

$$\|\gamma(x) - \gamma(x_0)\| \leq C\|x-x_0\| \text{ for } \|x-x_0\| < \delta.$$

**Corollary 7 (Argmin is differentiable almost everywhere).** Under the hypothesis of the Theorem the argmin function $\gamma$ is differentiable almost everywhere.
Proof. This follows from (2.3) and Stepanov’s Theorem [17] (see also [10] Theorem 3.4). □

The proof will appear in §2.3; the strategy is to construct a functional equation satisfied by the argmin function, and then apply the implicit function theorem for Lipschitz functions.

2.2. Properties of semiconvex functions. We collect a few basic statements about convex and semiconvex functions. As above Ω ⊂ R^{n+m} is open and convex.

Lemma 8. Suppose f : Ω → R and \( \tilde{f}(x, y) = f(x, y) + \kappa \|x\|^2 \). Then
\[
\nabla_{(x_0, y_0)} f = \nabla_{(x_0, y_0)} \tilde{f} + \kappa x_0
\]
as sets.

Proof. This is immediate from the definition, and left to the reader. □

Lemma 9. Suppose f : Ω → R and h : R^m → R and set
\[
\tilde{f}(x, y) = f(x, y) + h(x) \text{ for } (x, y) \in \Omega.
\]
Then
\[
\argmin f(x) = \argmin f(x) + h(x). \tag{2.4}
\]
In particular argmin \( f \) is single-valued if and only if argmin \( \tilde{f} \) is single-valued.

Proof. Clearly
\[
\argmin f(x) = \{ \gamma : \gamma = \inf_y \tilde{f}(x, y) \} = \{ \gamma : \gamma = \inf_y f(x, y) + h(x) \} = \argmin f(x) + h(x)
\]
giving (2.4). The last statement follows immediately. □

Lemma 10. Let f : Ω → R and suppose f(x, y) + \( \frac{\kappa}{2} \|x\|^2 \) is convex. Then g(x) = inf_y f(x, y) is \( \kappa \)-semiconvex.

Proof. Write \( \tilde{f}(x, y) = f(x, y) + \frac{\kappa}{2} \|x\|^2 \) so
\[
g(x) + \frac{\kappa}{2} \|x\|^2 = \inf_y \tilde{f}(x, y)
\]
which is the marginal function of convex function defined on Ω (which is assumed to be convex and \( \Omega_x \) is connected for each x). Thus \( g(x) + \frac{\kappa}{2} \|x\|^2 \) is convex. □

Lemma 11 (Gradient at argmin). Suppose that f : Ω → R is convex and set g(x) = inf_y f(x, y). Then for all x ∈ π(Ω) and γ ∈ argmin f(x)
\[
u \in \nabla_x g \Rightarrow (u, 0) \in \nabla_{(x, \gamma)} f.
\]

Proof. Suppose γ ∈ argmin f(x) so g(x) = f(x, γ). Let \( u \in \nabla_x g \). Then for any (x', y') ∈ R^n × R^m,
\[
f(x', y') - f(x, \gamma) \geq g(x') - g(x) \geq u.(x' - x) = (u, 0).((x', y') - (x, \gamma)) \tag{2.5}
\]
so \( (u, 0) \in \nabla_{(x, \gamma)} f \) as claimed. □

The next statement is a slight modification of [11] Proposition 6.4.
**Proposition 12.** Let $\sigma > 0$ and suppose $f : \mathbb{R}^{n+m} \to \mathbb{R}$ is such that $f(x, y) - \frac{\sigma}{2} \|y\|^2$ is convex. Define the set-valued function

$$G(p) = p + \nabla_p f$$

for $p = (x, y) \in \mathbb{R}^{n+m}$.

Then

(i) $G$ is non-contractive. That is, if $\zeta_i \in G(p_i)$ for $i = 1, 2$ then

$$\|\zeta_1 - \zeta_2\| \geq \|p_1 - p_2\|. \tag{2.7}$$

(ii) There exist a single-valued function $H : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ that is inverse to $G$, by which we mean

$$H(\zeta) = p \iff \zeta \in G(p). \tag{2.8}$$

(iii) The function $H$ is Lipschitz with Lipschitz constant 1. Moreover there is a $\mu < 1$ such that letting $\pi_2 : \mathbb{R}^{n+m} \to \mathbb{R}^m$ denote the second projection,

$$\|\pi_2 H(\zeta_1) - \pi_2 H(\zeta_2)\| \leq \mu \|\zeta_1 - \zeta_2\| \text{ for all } \zeta_1, \zeta_2 \in \mathbb{R}^{n+m}. \tag{2.9}$$

**Proof.** Let $p_i := (x_i, y_i) \in \mathbb{R}^{n+m}$ for $i = 1, 2$. We first claim

$$(\nabla_{p_2} \tilde{f} - \nabla_{p_1} \tilde{f}).(p_2 - p_1) \geq \sigma\|y_2 - y_1\|^2 \text{ for all } (x_i, y_i) \in \mathbb{R}^{n+m}. \tag{2.10}$$

To see this, let $\tilde{f}(x, y) = f(x, y) - \frac{\sigma}{2} \|y\|^2$ which by assumption is convex and $\nabla_{(x,y)} \tilde{f} = \nabla_{(x,y)} f - (0, \sigma y)$. Then

$$\tilde{f}(p_1) - \tilde{f}(p_2) \geq \nabla_{p_2} \tilde{f}.(p_1 - p_2) = \nabla_{p_2} f.(p_1 - p_2) - \sigma y_2.(y_1 - y_2). \tag{2.11}$$

Swapping the indices we also have

$$\tilde{f}(p_2) - \tilde{f}(p_1) \geq \nabla_{p_1} \tilde{f}.(p_2 - p_1) = \nabla_{p_1} f.(p_2 - p_1) - \sigma y_1.(y_2 - y_1). \tag{2.12}$$

Adding (2.11) and (2.12) and rearranging gives (2.10).

Now from Cauchy-Schwarz and (2.10)

$$\|G(p_1) - G(p_2)\|\|p_1 - p_2\| \geq (G(p_1) - G(p_2)).(p_1 - p_2)$$

$$= (p_1 - p_2 + \nabla_{p_1} f - \nabla_{p_2} f).(p_1 - p_2)$$

$$\geq \|p_1 - p_2\|^2 + \sigma\|y_1 - y_2\|^2 \tag{2.13}$$

which in particular implies (i).

We claim next that $G$ is surjective, by which we mean for all $\zeta \in \mathbb{R}^{n+m}$ there is an $p \in \mathbb{R}^{n+m}$ such that $\zeta \in G(p)$. To see this let

$$\phi(p) := \frac{1}{2} \|p\|^2 + f(p) - p.\zeta.$$

The function $p \mapsto \frac{1}{4} \|p\|^2 - p.\zeta$ is convex, and hence so is $\phi$ and

$$\nabla_{p_0} \phi = \nabla_{p_0} f + p_0 - \zeta = G(p_0) - \zeta.$$

Similarly the function

$$\psi(p) := \frac{1}{4} \|p\|^2 + f(p) - p.\zeta = \phi(p) - \frac{1}{4} \|p\|^2$$

is convex. Pick $b \in \nabla_0 \psi$ so

$$\psi(p) - \psi(0) \geq b.p$$

giving

$$\phi(p) \geq \phi(0) + \frac{1}{4} \|p\|^2.$$
As $\phi$ is continuous this implies $\phi$ has a global minimum at some $p_0 \in \mathbb{R}^{n+m}$, and so $0$ is a lower support vector for $\phi$ at $0$. Thus $0 \in \nabla_{p_0} \phi = G(p_0) - \zeta$ implying that $\zeta \in G(p_0)$. Thus $G$ is surjective as claimed.

In particular the inverse $H$ to $G$ defined by

$$H(\zeta) = \{ p \in \mathbb{R}^{n+m} : \zeta \in G(p) \}$$

is non-empty, and $G$ being non-contractive implies that it is single-valued. That $H$ has Lipschitz constant 1 follows from (i).

Finally given $\zeta_1, \zeta_2$ set $p_i := (x_i, y_i) := H(\zeta_i)$ so by definition $\zeta_i \in G(p_i)$ and $y_i = \pi_2 H(\zeta_i)$. To ease notation let $\alpha := \|x_1 - x_2\|$ and $\beta := \|y_1 - y_2\| = \|\pi_2 H(\zeta_1) - \pi_2 H(\zeta_2)\|$. Then dividing (2.13) by $\|p_1 - p_2\|$ gives

$$\|\zeta_1 - \zeta_2\| \geq (\alpha^2 + \beta^2)^{1/2} + \frac{\beta^2}{(\alpha^2 + \beta^2)^{1/2}}.$$

If $\alpha \geq \sigma \beta$ then $\|\zeta_1 - \zeta_2\| \geq (1 + \sigma^2)^{1/2} \beta$. If $\alpha \leq \sigma \beta$ then

$$\|\zeta_1 - \zeta_2\| \geq \beta + \sigma \frac{\beta^2}{(\sigma^2 \beta^2 + \beta^2)^{1/2}}$$

$$\quad = (1 + \sigma \frac{(1 + \sigma^2)^{1/2}}{\beta}).$$

Hence (2.13) holds with $\mu := \min\{(1 + \sigma^2)^{1/2}, (1 + \sigma \frac{(1 + \sigma^2)^{1/2}}{\beta})\}^{-1} < 1$.  

We will also need the following simpler corollary (which is proved in the same way, or follows formally from Proposition 12 upon taking $m = 0$).

**Corollary 13.** Suppose $g : \mathbb{R}^n \to \mathbb{R}$ is convex and define the set-valued function

$$G_1(x) = x + \nabla_x g$$

for $x \in \mathbb{R}^n$.

Then

(1) $G_1$ is non-contractive, that is

$$\|G_1(x_1) - G_1(x_2)\| \geq \|x_1 - x_2\| \text{ for all } x_1, x_2 \in X.$$  

(2) There exist a single-valued function $H_1 : \mathbb{R}^n \to \mathbb{R}^n$ that is inverse to $G_1$, and $H_1$ is Lipschitz with Lipschitz constant 1.

### 2.3. Functional Equation for argmin.

Suppose now that $f : \mathbb{R}^{n+m} \to \mathbb{R}$ is convex and as usual let $g(x) = \inf_y f(x, y)$ which is also convex. Consider the set-valued functions

$$G_1(x) = x + \nabla_x g,$$

$$G(x, y) = (x, y) + \nabla_{(x, y)} f.$$

By Proposition 12 and Corollary 13 these have single-valued inverses $H_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $H : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$. That is

$$H_1(u) = x \Leftrightarrow u \in G_1(x) \text{ for } x, u \in \mathbb{R}^n$$

$$H(u, v) = (x, y) \Leftrightarrow (u, v) \in G(x, y) \text{ for } (x, y), (u, v) \in \mathbb{R}^{n+m}.$$  

We use these to define a functional equation for argmin. Let

$$J : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$$

$$J(x, u, y) := y - \pi_2 H(H_1(x + u) + u, y).$$
Proposition 14 (Functional Equation for argmin). Suppose that $f(x, y)$ is convex and let $g(x) = \inf_y f(x, y)$. Then
\[ J(x, \nabla_x g, \arg\min_f(x)) = 0 \text{ for all } x \in \mathbb{R}^n. \]
That is,
\[ J(x, u, \gamma) = 0 \text{ for all } x \in \mathbb{R}^n \text{ and } \gamma \in \arg\min_f(x) \text{ and } u \in \nabla_x g. \]

Proof. Let $x \in \mathbb{R}^n$, $\gamma \in \arg\min_f(x)$ and $u \in \nabla_x g$. Then $x + u \in G_1(x)$ so \(^{2.15}\) gives $H_1(x + u) = x$. On the other hand since $\gamma \in \arg\min_f(x)$ we have by Lemma \(^{11}\)
\[ (u, 0) \in \nabla(x, \gamma)f. \]
Thus
\[ (x, \gamma) + (u, 0) = (u + x, \gamma) \in G(x, \gamma) \]
so \(^{2.16}\) gives $H(u + x, \gamma) = (x, \gamma)$. So
\[ J(x, u, \gamma) = \gamma - \pi_2 H_1(x + u) + u, \gamma) \]
= $\gamma - \pi_2 H(x + u, \gamma)$
\[ = \gamma - \pi_2 H(x + u, \gamma) \]
\[ = 0 \]
as claimed. \hfill \square

We next collect two basic properties of $J$:

Lemma 15 (Properties of $J$). The function $J$ is Lipschitz in $(x, u, y)$. Moreover if $f(x, y) - \frac{\sigma}{2}||y||^2$ is convex for some $\sigma > 0$ then there is a $\lambda > 0$ such that for fixed $x, u$
\[ \|J(x, u, y_1) - J(x, u, y_2)\| \geq \lambda\|y_1 - y_2\| \text{ for all } y_1, y_2. \]

Proof. Clearly $J$ is Lipschitz in all variables since both $H$ and $H_1$ are. For the second statement, suppose $f(x, y) - \frac{\sigma}{2}||y||^2$ is convex and let $\pi_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the second projection. We know from Proposition \(^{12}\)(iii) that there is a $\mu < 1$ such that
\[ \|\pi_2 H(v, y_1) - \pi_2 H(v, y_2)\| \leq \mu\|y_1 - y_2\| \text{ for all } v, y_1. \] (2.18)
Now fix $x, u$ and let $v := H_1(x + u) + u$. Then if $y_1, y_2 \in \mathbb{R}^m$,
\[ \|J(x, u, y_2) - J(x, u, y_1)\| = \|y_2 - y_1 - \pi_2 H(v, y_2) + \pi_2 H(v, y_1)\| \]
\[ \geq \|y_2 - y_1\| - \|\pi_2 H(v, y_2) - \pi_2 H(v, y_1)\| \]
\[ \geq (1 - \mu)\|y_2 - y_1\|. \]
\hfill \square

2.4. Statement of Alexandrov’s Theorem. Let $X \subset \mathbb{R}^n$ be open. The following is a precise version of Alexandrov’s Theorem:

Theorem 16 (Alexandrov’s Theorem). Let $g : X \to \mathbb{R}$ be locally convex. Then the set-valued function
\[ x \mapsto \nabla_x g \]
is differentiable at $x_0$ for almost all $x_0$ in $X$. That is, for almost all $x_0$ there is an $L \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ such that for all $\epsilon > 0$ there is a $\delta > 0$ such that for $\|x - x_0\| < \delta$ we have
\[ \|u - u_0 - L (x - x_0)\| \leq \epsilon\|x - x_0\| \text{ for all } u \in \nabla_x g \text{ and } u_0 \in \nabla_{x_0} g. \] (2.19)
Moreover for almost all $x_0$ the function $g$ is twice differentiable at $x_0$ and $\text{Hess}_{x_0}(g) = L$. That is, for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|g(x) - g(x_0) - \nabla g|_{x_0} (x-x_0) - \frac{1}{2}(x-x_0)^t \text{Hess}_{x_0}(g)(x-x_0)| \leq \epsilon \|x-x_0\|^2 \quad (2.20)$$

for all $\|x-x_0\| < \delta$.

**Proof.** This originates in [2] and for an exposition the reader is referred to [11] Theorems 6.1, 7.1. (We remark that the latter cited work requires the function to be convex and defined on all of $\mathbb{R}^n$; but the statement we want is local, and being locally convex, $g$ is also locally Lipschitz [1], and so using [20, Theorem 4.1] we know that $X$ is covered by small open sets $U$ such that $g|_U$ extends to a convex function on $\mathbb{R}^n$ so the cited work applies.)

### 2.5. Proof of Calmness.

**Lemma 17** (Continuity of argmin). Let $\Omega \subset X \times \mathbb{R}$ be convex and such that $\Omega_x$ is connected for each $x \in X$. Let $f : \Omega \to \mathbb{R}$ be continuous, and suppose that for each $x \in X$ the function $y \mapsto f(x, y)$ is strongly convex and attains its minimum at some point. Then $\gamma(x) = \text{argmin}_y f(x, y)$ is single valued and continuous.

**Proof.** For fixed $x$ the hypothesis imply that $y \mapsto f(x, y)$ is a strongly convex function on the connected set $\Omega_x$ that attains its minimum, and thus this minimum $\gamma(x)$ must be a unique. We first claim that $\gamma$ is locally bounded. Fix $x_0 \in X$ and let $a := \gamma(x_0)$. Then by strong convexity there is an $\epsilon > 0$ and $c > 0$ such that $f(x_0, y) > a + \epsilon$ if $\|y - \gamma(x_0)\| \geq c$. By continuity we may take $\delta > 0$ small so if $\|x-x_0\| < \delta$ and $\|y - \gamma(x_0)\| = c$ then $f(x,y) > a + \epsilon$ and, furthermore that $f(x,\gamma(x_0)) < a +\epsilon$. But by strict convexity of $y \mapsto f(x, y)$ this implies $\gamma(x) \in [\gamma(x_0) - c, \gamma(x_0) + c]$ for all $\|x-x_0\| < \delta$, and thus $\gamma$ is locally bounded.

Now suppose $(x_n)$ is a sequence in $X$ converging to $x$ as $n \to \infty$. By the above we may assume $S := \{\gamma(x_n)\}$ is bounded. Let $b$ be a cluster point of $S$, so there is a subsequence $x_{n_r}$ with $\gamma(x_{n_r}) \to b$ as $r \to \infty$. By continuity of $f$ for any $y \in \mathbb{R}^m$,

$$f(x,b) = \lim_{r \to \infty} f(x_{n_r},\gamma(x_{n_r})) \leq \lim_{r \to \infty} f(x_{n_r},y) = f(x,y).$$

Hence $b = \gamma(x)$. As this holds for all cluster points of $S$ we deduce $\gamma(x_n) \to \gamma(x)$ as $n \to \infty$, proving continuity of $\gamma$. □

**Proof of Theorem 2.** We first claim that there is no loss in generality in assuming that $\Omega = \mathbb{R}^{n+m}$. To see this, suppose $f : \Omega \to \mathbb{R}$ has properties (2.1) and (2.2). Then $\gamma = \text{argmin}_F$ is single-valued and continuous (Lemma 17). So given $x_0 \in \pi(\Omega)$ there are small balls $x_0 \in U \subset \pi(\Omega)$ and $\gamma(x_0) \in V \subset \mathbb{R}^m$ so that $U \times V \subset \Omega$ and $\gamma(U) \subset V$. Moreover as $f$ is semiconvex, by shrinking $U, V$ we may assume that $f|_{U \times V}$ is Lipschitz (all convex functions are Lipschitz, see e.g. [1]). Let $\tilde{f}(x,y) := f(x,y) + \frac{\sigma}{2} \|x\|^2 + \frac{\kappa}{2} \|y\|^2$ which we are assuming is convex on $\Omega$. Then [20, Theorem 4.1] we know $\tilde{f}|_{U \times V}$ extends to a convex function $\tilde{h}$ on all of $\mathbb{R}^{n+m}$. Now let

$$h(x,y) := \tilde{h}(x,y) - \frac{\kappa}{2} \|x\|^2 - \frac{\sigma}{2} \|y\|^2.$$

For fixed $x$ the convex function $y \mapsto h(x,y)$ agrees with the function $y \mapsto f(x,y)$ when $y \in V$. Since $V$ contains $\gamma(x) = \text{argmin}_y f(x,y)$, this implies $\text{argmin}_y h(x,y) = \text{argmin}_y f(x,y) = \gamma(x)$. Hence $h$ satisfies the hypothesis of the Theorem with $\Omega =$
\( \mathbb{R}^{n+m} \), and so \( \gamma|_U \) has the properties in the conclusion of the theorem (which are all local), which proves the claim.

So from now on assume \( f : \mathbb{R}^{n+m} \to \mathbb{R} \) satisfies (2.1) and (2.2). Consider first the case \( \kappa = 0 \), so \( (x, y) \mapsto f(x, y) - \frac{\sigma}{2}||y||^2 \) is convex. Then in particular \( f \) is convex, and so \( g(x) = \inf_y f(x, y) \) is also convex. Moreover for fixed \( x \) the function \( y \mapsto f(x, y) \) is strictly convex, and so \( \arg\min_f \) (which is assumed to be non-empty) must be single valued. Consider the functional \( J \) from (2.17) so by Proposition 13:

\[
J(x, u, \gamma(x)) = 0 \quad \text{for all } x \text{ and } u \in \nabla_x g.
\]

(2.21)

Fix \( x_0 \in \mathbb{R}^n \) and \( u_0 \in \nabla_{x_0} g \) so \( J(x_0, u_0, \gamma(x_0)) = 0 \). The properties of \( J \) proved in Lemma 15 mean we can apply the Inverse-Function-Theorem for Lipschitz maps (for convenience of the reader we give a proof of this in Appendix A and apply it here with \( r \) replaced with \( 2n \) and \( s \) replaced with \( m \)). This yields a Lipschitz function \( \phi : V \to \mathbb{R}^m \) defined on a neighbourhood \( V \) of \((x_0, u_0)\) such that

\[
J(x, u, y) = 0 \iff y = \phi(x, u).
\]

This combined with (2.21) gives

\[
\gamma(x) = \phi(x, u) \quad \text{for all } (x, u) \in V \text{ with } u \in \nabla_x g.
\]

We next prove \( \gamma \) is calm almost everywhere. As \( g \) is convex we have by Alexandrov’s Theorem (2.19) that for almost all \( x_0 \) there are \( \delta_1 > 0 \) and linear \( L : \mathbb{R}^n \to \mathbb{R}^n \) such that for \( ||x - x_0|| < \delta_1 \)

\[
||u - u_0|| \leq (1 + ||L||)||x - x_0|| \quad \text{for all } u \in \nabla_x g \text{ and } u_0 \in \nabla_{x_0} g.
\]

(2.22)

Pick \( u_0 \in \nabla_{x_0} g \), and let \( \phi : V \to \mathbb{R} \) be the Lipschitz function constructed above. For concreteness say that \( V \) contains the set \( ||x - x_0|| < \delta_2 \) and \( ||u - u_0|| < \delta_2 \) and that \( \phi \) has Lipschitz constant \( C' \) there. Thus

\[
\gamma(x) = \phi(x, u) \quad \text{for } ||x - x_0|| < \delta_2, ||u - u_0|| < \delta_2 \text{ and } u \in \nabla_x g.
\]

Set

\[
\delta := \min\{\delta_1, \frac{\delta_2}{1 + ||L||}\}
\]

and suppose \( ||x - x_0|| < \delta \). Picking any \( u \in \nabla_x g \), by (2.22) \( ||u - u_0|| < \delta_2 \) and so

\[
||\gamma(x) - \gamma(x_0)|| = ||\phi(x, u) - \phi(x_0, u_0)|| \leq C'||x - x_0|| + ||u - u_0|| \leq C'(2 + ||L||)||x - x_0||.
\]

Thus \( \gamma \) is calm \( x_0 \).

The case of general \( \kappa \) is easily reduced to the case \( \kappa = 0 \). For suppose \( f(x, y) + \frac{\kappa}{2}||x||^2 - \frac{\sigma}{2}||y||^2 \) is convex and \( \arg\min_f \) is single-valued. Set

\[
\tilde{f}(x, y) = f(x, y) + \frac{\kappa}{2}||x||^2
\]

Then \( \tilde{f}(x, y) - \frac{\sigma}{2}||y||^2 \) is convex, and by (2.3)

\[
\arg\min\tilde{f}(x) = \arg\min_f(x).
\]

Thus \( \arg\min_f \) is also single-valued, so by the above the Theorem can be applied to \( \tilde{f} \). Let

\[
\gamma(x) := \arg\min\tilde{f}(x) = \arg\min_f(x)
\]

Setting \( \tilde{g}(x) := \inf_y \tilde{f}(x, y) \), given \( x_0 \) and \( u_0 \in \nabla_{x_0} \tilde{g} \) we know that there is a locally Lipschitz function \( \tilde{\phi} : \tilde{V} \to \mathbb{R} \) defined on a neighbourhood \( \tilde{V} \) of \((x_0, u_0)\) such that

\[
\gamma(x) = \tilde{\phi}(x, u) \quad \text{for } (x, u) \in \tilde{V} \text{ with } u \in \nabla_{x} \tilde{g}.
\]
Set $\phi(x,u) = \tilde{\phi}(x, u + \kappa x)$ which is locally Lipschitz around $(x_0, u_0 + \kappa x_0)$. And if $u \in \nabla^c g$ then $u - \kappa x_0 \in \nabla_x \tilde{g}$ so $\gamma(x) = \tilde{\phi}(x, u - \kappa x_0) = \phi(x, u)$. Thus the conclusion of the Theorem also holds for $f$ and we are done. \hfill \Box

3. F-SUBHARMONIC FUNCTIONS

3.1. Basic definitions. We summarise some basic properties of F-subharmonic functions from the work of Harvey-Lawson. We refer the reader to [16] for a more detailed summary, or the original papers [8, 9]. Let $X \subset \mathbb{R}^n$ be open and

$$J^2(X) := X \times \mathbb{R} \times \mathbb{R}^n \times \text{Sym}_n^2 = X \times J^2_n$$

be the jet-bundle over $X$. For $F \subset J^2(X)$ and $x \in X$ we write

$$F_x = \{(r,p,A) \in J^2_n : (x,r,p,A) \in F\}.$$

\textbf{Definition 18} (Primitive Subequations). We say that $F \subset J^2(X)$ is a \textit{primitive subequation} if

(1) (Closedness) $F$ is closed.
(2) (Positivity)

$$(r,p,A) \in F_x \text{ and } P \in \text{Pos}_n \Rightarrow (r,p,A + P) \in F_x.$$ (3.1)

We say that $F \subset J^2(X)$ has the \textit{Negativity Property} if

(3) (Negativity)

$$(r,p,A) \in F_x \text{ and } r' \leq r \Rightarrow (r',p,A) \in F_x.$$ (3.2)

\textbf{Definition 19} (Upper contact points, Upper contact jets). Let

$$f : X \to \mathbb{R} \cup \{-\infty\}.$$

We say that $x \in X$ is an \textit{upper contact point} of $f$ if $f(x) \neq -\infty$ and there exists $(p,A) \in \mathbb{R}^n \times \text{Sym}_n^2$ such that

$$f(y) \leq f(x) + p.(y - x) + \frac{1}{2}(y - x)^t A(y - x) \text{ for all } y \text{ sufficiently near } x.$$

When this holds we refer to both $(f(x),p,A)$ and $(p,A)$ as an \textit{upper contact jet} of $f$ at $x$.

\textbf{Definition 20} (F-subharmonic function). Suppose $F \subset J^2(X)$. We say that an upper-semicontinuous function $f : X \to \mathbb{R} \cup \{-\infty\}$ is \textit{F-subharmonic} if

$$(f(x),p,A) \in F_x \text{ for all upper contact jets } (p,A) \text{ of } f \text{ at } x.$$

We let $F(X)$ denote the set of F-subharmonic functions on $X$.

Clearly being F-subharmonic is a local condition on $X$.

\textbf{Proposition 21}. Let $F \subset J^2(X)$ be closed. Then

(1) (Maximum Property) If $f, g \in F(X)$ then $\max\{f,g\} \in F(X)$.
(2) (Decreasing Sequences) If $f_j$ is decreasing sequence of functions in $F(X)$ (so $f_{j+1} \leq f_j$ over $X$) then $f := \lim_j f_j$ is in $F(X)$.
(3) (Uniform limits) If $f_j$ is a sequence of functions on $F(X)$ that converge locally uniformly to $f$ then $f \in F(X)$. 
(4) (Families locally bounded above) Suppose \( \mathcal{F} \subset F(X) \) is a family of \( F \)-subharmonic functions locally uniformly bounded from above. Then the upper-semicontinuous regularisation of the supremum
\[
    f := \sup_{x \in X} f
\]
is in \( F(X) \).

(5) If \( F \) is constant coefficient and \( f \) is \( F \)-subharmonic on \( X \) and \( x_0 \in \mathbb{R}^n \) is fixed, then the function \( x \mapsto f(x - x_0) \) is \( F \)-subharmonic on \( X - x_0 \).

**Proof.** See [9, Theorem 2.6] for (1-4). Item (5) is immediate. \( \square \)

**Definition 22.** Let \( F \subset J^2(X) \).

1. We say \( F \) is constant coefficient if \( F_x \) is independent of \( x \), i.e.
   \[
   (x, r, p, A) \in F_x ⇔ (x', r, p, A) \in F_{x'} \text{ for all } x, x', r, p, A.
   \]
2. We say \( F \) depends only on the Hessian part if each \( F_x \) is independent of \( (r, p) \), i.e.
   \[
   (r, p, A) \in F_x ⇔ (r', p', A) \in F_x \text{ for all } x, r, r', p, p', A.
   \]

An important example is
\[
    \mathcal{P} := \{(x, r, p, A) \in J^2(X) : A \text{ is semipositive}\}
\]
which is a constant-coefficient primitive subequation that depends only on the Hessian part. Then \( \mathcal{P} \)-subharmonic functions are precisely those that are locally convex [9, Example 14.2].

**Lemma 23** (Sums of \( F \)-subharmonic and convex functions). Suppose \( \mathcal{F} \subset J^2(X) \) is a constant coefficient primitive subequation that depends only on the Hessian part. If \( f \) is \( F \)-subharmonic on \( X \) and \( g \) is a convex quadratic function on \( X \), then \( f + g \) is \( F \)-subharmonic.

**Proof.** The hypothesis is that \( g(x) = a + b \cdot x + \frac{1}{2} x^t C x \) for some \( a, b \in \mathbb{R}^n \) and some semipositive symmetric matrix \( C \). One can check that if \( (p, A) \) is an upper-contact point of \( f + g \) at \( x \) then \( (x^t C + p - b, A - C) \) is an upper-contact jet for \( f \) at \( x \). As \( f \) is \( F \)-subharmonic this implies \( (f(x), x^t C + p - b, A - C) \in F \). Since \( F \) depends only on the Hessian part, and satisfies the Positivity property, this in turn implies \( (f(x) + g(x), p, A) \in F \) proving that \( f + g \) is \( F \)-subharmonic as required. \( \square \)

### 3.2. Product Subequations.

For \( \Gamma \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) = M_{m \times n}(\mathbb{R}) \) consider
\[
    i_\Gamma : \mathbb{R}^n \to \mathbb{R}^{n+m} \quad i_\Gamma(x) = (x, \Gamma x) \quad (3.3)
\]
\[
    j : \mathbb{R}^m \to \mathbb{R}^{n+m} \quad j(y) = (0, y). \quad (3.4)
\]

which induce natural pullback maps
\[
    i_\Gamma^* : J^2_{n+m} \to J^2_n \text{ and } j^* : J^2_{n+m} \to J^2_m. \quad (3.5)
\]

We can write these explicitly. Suppose
\[
    p := \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^{n+m} \text{ and } A := \begin{pmatrix} B & C \\ C^t & D \end{pmatrix} \in \text{Sym}^2_{n+m}
\]
where the latter is in block form, so \( B \in \text{Sym}^2_n \) and \( D \in \text{Sym}^2_m \). Then
\[
    i_\Gamma^*(r, p, A) = (r, p_1 + \Gamma^t p_2, B + CT + \Gamma^t C^t + \Gamma^t D \Gamma)
    \quad (3.6)
\]
\[
    j^*(r, p, A) = (r, p_2, D). \quad (3.7)
\]
\textbf{Definition 24 (Products).} Let \( X \subset \mathbb{R}^n \) and \( Y \subset \mathbb{R}^m \) be open, and \( F \subset J^2(X) \) and \( G \subset J^2(Y) \). Define
\[
F \# G \subset J^2(X \times Y)
\]
by
\[
(F \# G)(x,y) = \left\{ \alpha \in J^2_{n+m} : i^*_x \alpha \in F_x \quad \text{and} \quad j^*_y \alpha \in G_y \quad \text{for all} \quad \Gamma \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \right\}.
\]

\textbf{Lemma 25.} \quad (1) If \( F \) and \( G \) are primitive subequations then so is \( F \# G \). Moreover if \( F \) and \( G \) both have the Negativity Property then so does \( F \# G \).

(2) Let \( F \) be a constant-coefficient primitive subequation on \( X \). Suppose and \( f \) is \( F \# \mathcal{P} \)-subharmonic on some open \( \Omega \subset X \times Y \). The for each \( x \in X \) the function \( y \mapsto f(x,y) \) is locally convex.

\textit{Proof.} The reader will easily prove these straight from the definition, or otherwise find the proofs in [16]. \( \square \)

3.3. \textbf{The almost everywhere theorem.} We will rely on a very useful theorem of Harvey-Lawson that characterizes \( F \)-subharmonic semiconvex functions in terms of second order jets almost everywhere.

\textbf{Definition 26 (Twice differentiability at a point).} We say that a function \( f : X \to \mathbb{R} \) is \textit{twice differentiable} at \( x_0 \in X \) if there exists a \( p \in \mathbb{R}^n \) and an \( L \in \text{Sym}_n \) such that for all \( \epsilon > 0 \) there is a \( \delta > 0 \) such that for \( \|x - x_0\| < \delta \) we get
\[
|f(x) - f(x_0) - p.(x-x_0) - \frac{1}{2}(x-x_0)^t L(x-x_0)| \leq \epsilon \|x-x_0\|^2.
\]

When \( f \) is twice differentiable at \( x_0 \) then the \( p, L \) in (3.8) are unique, and moreover in this case \( f \) is differentiable at \( x_0 \) and
\[
p = \nabla f|_{x_0} = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)|_{x_0} \in \mathbb{R}^n.
\]

When \( f \) is twice differentiable at \( x_0 \) we shall refer to \( L \) as the \textit{Hessian} of \( f \) at \( x_0 \) and denote it by \( \text{Hess}(f)|_{x_0} \). Of course when \( f \) is \( C^2 \) in a neighbourhood of \( x_0 \) then \( \text{Hess}_2(f) \) is the matrix with entries
\[
(\text{Hess}(f)|_{x_0})_{ij} := \frac{\partial^2 f}{\partial x_i \partial x_j}|_{x_0}.
\]

\textbf{Definition 27 (Second order jet).} Suppose that \( f : X \to \mathbb{R} \) is twice differentiable at \( x_0 \). We denote the \textit{second order jet} of \( f \) at \( x_0 \) by
\[
J^2_{x_0}(f) := (f(x_0), \nabla f|_{x_0}, \text{Hess}(f)|_{x_0}) \in J^2_n = \mathbb{R} \times \mathbb{R}^n \times \text{Sym}_n^2.
\]

We have seen in Alexandrov’s Theorem (Theorem [16]) that if \( f \) is locally semi-convex then \( J^2_{x}(f) \) exists for almost all \( x \).

\textbf{Theorem 28 (The Almost Everywhere Theorem).} Assume that \( F \subset J^2(X) \) is a primitive subequation and let \( f : X \to \mathbb{R} \) be locally semiconvex. Then
\[
f \in F(X) \iff J^2_{x}(f) \in F_x \quad \text{for almost all} \quad x \in X.
\]

\textit{Proof.} See [7] Theorem 4.1. \( \square \)
4. Partial sup-convolutions

Fix open \( U \subset \mathbb{R}^n \) and \( V \subset \mathbb{R}^m \), and suppose \( f : U \times V \to \mathbb{R} \) is upper-semicontinuous and bounded.

**Definition 29** (Partial-Sup-Convolutions). For \( \epsilon > 0 \) the partial sup-convolution of \( f \) is
\[
f_{\epsilon}^{\cdot,p}(x,y) := \sup_{z \in U} \left\{ f(z,y) - \frac{1}{2\epsilon} \| z - x \|^2 \right\} \text{ for } (x,y) \in U \times V. \tag{4.1}
\]

For \( \delta > 0 \) let
\[
U(\delta) = \{ x \in \mathbb{R}^n : B_\delta(x) \subset U \}.
\]

**Lemma 30** (Basic Properties of Partial-Sup-Convolutions).

(i) (Strong Semiconvexity) Assume that for each fixed \( x \) the function \( y \mapsto f(x,y) \) is convex. Then
\[
(x,y) \mapsto f_{\epsilon}^{\cdot,p}(x,y) + \frac{1}{2\epsilon} \| x \|^2
\]
is convex.

(ii) (Monotonicity) For \( 0 < \epsilon' \leq \epsilon \) we have
\[
f \leq f_{\epsilon'}^{\cdot,p} \leq f_{\epsilon}^{\cdot,p}. \tag{4.2}
\]

(iii) Let \( \delta := 2(\epsilon\| f \|_\infty)^{1/2} \). Then
\[
f_{\epsilon}^{\cdot,p}(x,y) = \sup_{\| \tau \| < \delta} \left\{ f(x+\tau,y) - \frac{1}{2\epsilon} \| \tau \|^2 \right\} \text{ for } (x,y) \in U(\delta) \times V.
\]

(iv) (Pointwise convergence)
\[
\lim_{\epsilon \to 0^+} f_{\epsilon}^{\cdot,p}(x,y) = f(x,y) \text{ for } (x,y) \in U \times V.
\]

(v) (Magic-Property) Suppose that \( F \) is a constant-coefficient primitive subequation on \( U \) that has the Negativity Property and \( f \) is \( F \# \mathcal{P} \)-subharmonic. Then \( f_{\epsilon}^{\cdot,p} \) is \( F \# \mathcal{P} \)-subharmonic on \( U(\delta) \times V \).

**Proof.**
\[
f_{\epsilon}^{\cdot,p}(x,y) + \frac{1}{2\epsilon} \| x \|^2 = \sup_{z \in U} \left\{ f(z,y) - \frac{1}{2\epsilon} \| z - x \|^2 + \frac{1}{2\epsilon} \| x \|^2 \right\}
\]
\[
= \sup_{z \in U} \left\{ f(z,y) + \frac{1}{\epsilon} x.z - \frac{1}{2\epsilon} \| z \|^2 \right\}.
\]

Now for fixed \( z \) the function \( y \mapsto f(z,y) \) is assumed to be convex in \( y \), and so the function \( (x,y) \mapsto f(z,y) \) is convex in \( (x,y) \). Thus, again for \( z \) fixed, \( (x,y) \mapsto f(z,y) + \frac{1}{\epsilon} x.z + \frac{1}{2\epsilon} \| z \|^2 \) is convex in \( (x,y) \), and hence so is \( f_{\epsilon}^{\cdot,p}(x,y) + \frac{1}{\epsilon} \| x \|^2 \) proving (i).

Item (ii) is immediate. For (iii) we claim that
\[
f_{\epsilon}^{\cdot,p}(x,y) = \sup_{z \in U : \| z - x \| < \delta} \left\{ f(z,y) - \frac{1}{2\epsilon} \| z - x \|^2 \right\} \text{ for } (x,y) \in U \times V. \tag{4.3}
\]
To see this let \( M := \| f \|_\infty \). Then for \( z \in U \) with \( \| z - x \| \geq \delta = \sqrt{4\epsilon M} \),
\[
f(z,y) - \frac{1}{2\epsilon} \| z - x \|^2 \leq M - \frac{1}{2\epsilon} \delta^2 = -M \leq f(x,y) \leq f_{\epsilon}^{\cdot,p}(x,y)
\]
which proves (4.3). Then (iii) follows upon making the change of variables \( \tau := z - x \). For the pointwise convergence fix \( (x,y) \in U \times V \) and let \( a > f(x,y) \). Then \( f < a \).
on some open neighbourhood of \((x, y)\). Let \(\epsilon\) be small enough so that \(B_\delta(x)\) is contained in this neighbourhood. Then (4.3) implies \(f^{\epsilon, p}(x, y) \leq a\), proving (iv).

For the final statement, since \(F\) is constant coefficient for any fixed \(\tau\) the function \(f(x + \tau, y)\) is \(F\#P\)-subharmonic (where defined), and hence (iii) shows \(f^{\epsilon, p}\) as a supremum of \(F\#P\)-subharmonic functions. Now being \(F\#P\)-subharmonic implies that \(y \mapsto f(x, y)\) is convex, and so by (i) \(f^{\epsilon, p}\) is certainly continuous and hence equal to its upper semicontinuous regularisation. Thus \(f^{\epsilon, p}\) is \(F\#P\)-subharmonic on \(U(\delta) \times V\) as claimed in (v).

The next lemma reveals a surprising property of the above construction, namely that the partial sup-convolution of a semiconcave function is fibre wise semiconcave.

**Lemma 31.** Suppose that \(f\) is \(\kappa\)-semiconcave for some \(\kappa > 0\). Then for \(\epsilon < \kappa^{-1}\) and fixed \(x \in U\) the function

\[
y \mapsto f^{\epsilon, p}(x, y) - \frac{\kappa}{2}\|y\|^2
\]

is concave.

**Proof.** Let \(x\) be fixed. Then

\[
f^{\epsilon, p}(x, y) - \frac{\kappa}{2}\|y\|^2 = \sup_{z \in U} \{ f(z, y) - \frac{\kappa}{2}\|y\|^2 - \frac{1}{2\epsilon}\|z - x\|^2 \}
\]

\[
= \sup_{z \in U} \{ f(z, y) - \frac{\kappa}{2}\|z\|^2 - \frac{\kappa}{2}\|y\|^2 + \frac{\kappa - \epsilon^{-1}}{2}\|z\|^2 + \frac{1}{\epsilon}\|z\| \cdot (z - x) - \frac{1}{2\epsilon}\|z\|^2 \}.
\]

Observe that (since \(x\) is fixed and \(\kappa - \epsilon^{-1} < 0\)) the function \((z, y) \mapsto \frac{\kappa - \epsilon^{-1}}{2}\|z\|^2 + \frac{1}{\epsilon}\|z\| \cdot (z - x) - \frac{1}{2\epsilon}\|z\|^2\) is concave in \((z, y)\). Hence \(y \mapsto f^{\epsilon, p}(x, y) - \frac{\kappa}{2}\|y\|^2\) is a supremum of functions concave in two variables, and thus is concave. \(\square\)

### 5. \(F\)-subharmonicity of marginal functions

Let \(\Omega \subset \mathbb{R}^{n+m}\) be open, convex and such that \(\Omega_x\) is connected for all \(x\).

**Proposition 32.** Let \(F \subset J^2(\mathbb{R}^n)\) be a primitive subequation. Let \(f : \Omega \rightarrow \mathbb{R}\) be \(F\#P\)-subharmonic, and suppose that for some \(\sigma, \kappa_1, \kappa_2 > 0\) the function

\[
f(x, y) + \frac{\kappa_1}{2}\|x\|^2 - \frac{\sigma}{2}\|y\|^2
\]

is convex and

\[f(x, y) + \frac{\kappa_2}{2}\|y\|^2
\]

is concave

and that \(\gamma(x) = \arg\min f(x)\) is single valued. Then

\[
g(x) := \inf_{y \in \Omega_x} f(x, y)
\]

is \(F\)-subharmonic.

**Proof.** By hypothesis

\[
g(x) = f(x, \gamma(x)).
\]

Now \(g\) is \(\kappa\)-semiconvex (Lemma 10) so by Alexandrov’s Theorem (Theorem 16) \(g\) is twice differentiable almost everywhere. Furthermore (5.1) allows us to invoke our results on the argmin function, so by Corollary 7 \(\gamma\) is differentiable almost
everywhere. Let $x_0$ be a point where $g$ is twice differentiable and $\gamma$ is differentiable, and we will show
\[ J^2_{x_0} g = (g(x_0), \nabla g|_{x_0}, \text{Hess}_{x_0}(g)) \in F_{x_0}. \tag{5.3} \]
By the Almost Everywhere Theorem (Theorem 28) this implies that $g$ is $F$-subharmonic.

Actually we will show that for any $\epsilon > 0$ it holds that
\[ (g(x_0), \nabla g|_{x_0}, \text{Hess}_{x_0}(g) + \epsilon I_n) \in F_{x_0}. \tag{5.4} \]
Letting $\epsilon \to 0$ and using that $F_{x_0}$ is closed yields \((5.3)\).

To this end set $y_0 := \gamma(x_0)$ and
\[ \Gamma := D\gamma|_{x_0} \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \]
and $d(x, y)$ be the vertical distance between $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and the tangent to the graph of $\gamma$ at $(x_0, y_0)$, so
\[ d(x, y) := \|y - y_0 - \Gamma(x - x_0)\| \text{ for } (x, y) \in \mathbb{R}^n \times \mathbb{R}^m. \]
Consider the quadratic
\[ q(x, y) = g(x_0) + \nabla g|_{x_0}. (x-x_0) + \frac{1}{2}(x-x_0)^t \text{Hess}_{x_0}(g)(x-x_0) + \frac{\epsilon}{2}\|x-x_0\|^2 + \kappa_2 d(x, y)^2 \]
for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. By construction
\[ q(x_0, y_0) = g(x_0) = f(x_0, \gamma(x_0)) = f(x_0, y_0), \]
and in Lemma 33 below we show that $q \geq f$ sufficiently near $(x_0, y_0)$. Hence $(x_0, y_0)$ is an upper contact point for $f$ and
\[ J^2_{(x_0, y_0)}(q) = (q(x_0, y_0), \nabla q|_{(x_0, y_0)}, \text{Hess}_{(x_0, y_0)}(q)) \tag{5.5} \]
\[ = \begin{pmatrix} f(x_0, y_0); \nabla g|_{x_0} \\ 0 \end{pmatrix}, \begin{pmatrix} \text{Hess}_{x_0}(g) + \epsilon I_n + 2\kappa_2 \Gamma^t \Gamma & -2\kappa_2 \Gamma^t \\ -2\kappa_2 \Gamma & 2\kappa_2 I_m \end{pmatrix} \tag{5.6} \]
is an upper-contact jet of $f$ at $(x_0, y_0)$. So as $f$ is $F\#P$-subharmonic we have
\[ J^2_{(x_0, y_0)}(q) \in (F\#P)_{(x_0, y_0)}. \]
And from the definition of $i^+_1$,
\[ i^+_1(J^2_{(x_0, y_0)}(q)) = \text{Hess}_{x_0}(g) + \epsilon I_n \]
which must lie in $F_{x_0}$. This gives \((5.3)\) and completes the proof. \qed

Lemma 33. With the notation as in the proof of Theorem 22 we have
\[ q(x, y) \geq f(x, y) \text{ for } (x, y) \text{ sufficiently near } (x_0, y_0). \tag{5.7} \]
Proof. Fix $\epsilon' > 0$ small enough so $\epsilon' + \kappa_2 \epsilon'^2 < \epsilon/2$. That $\Gamma = D\gamma|_{x_0}$ means there is a $\delta > 0$ such that for all $\|x - x_0\| < \delta$
\[ \|\gamma(x) - y_0 - \Gamma(x - x_0)\| \leq \epsilon'\|x - x_0\|. \]
Shrinking $\delta$ is necessary, the definition of $g$ being twice differentiable at $x_0$ means \((5.8)\) that for $\|x - x_0\| < \delta$ we also have
\[ |g(x) - g(x_0) - \nabla g|_{x_0}. (x-x_0) - \frac{1}{2}(x-x_0)^t \text{Hess}_{x_0}g(x-x_0)| \leq \epsilon'\|x-x_0\|^2. \]
Consider now a point \((x, y)\) with \(\|x - x_0\| < \delta\) and \(\|y - y_0\| < \delta\). Then
\[
\|y - \gamma(x)\| \leq \|y - y_0 - \Gamma(x - x_0)\| + \|y_0 + \Gamma(x - x_0) - \gamma(x)\| \tag{5.8}
\]
\[
\leq d(x, y) + \epsilon'\|x - x_0\|. \tag{5.9}
\]
So
\[
\|y - \gamma(x)\|^2 \leq 2\epsilon'^2\|x - x_0\|^2 + 2d(x, y)^2.
\]
Now we use (in an essential way) hypothesis \((5.2)\). Since \(\gamma(x)\) is the minimum of the function \(y' \mapsto f(x, y')\) \((5.2)\) implies
\[
f(x, y) \leq f(x, \gamma(x)) + \frac{\kappa_2}{2}\|y - \gamma(x)\|^2.
\]
Thus
\[
f(x, y) \leq g(x) + \kappa_2(\epsilon'^2\|x - x_0\|^2 + d(x, y)^2)
\]
\[
\leq g(x_0) + \nabla g|_{x_0}(x - x_0) + \frac{1}{2}(x - x_0)^tHess_{x_0} g(x - x_0)
\]
\[
+ (\kappa_2\epsilon'^2 + \epsilon')\|x - x_0\|^2 + \kappa_2d(x, y)^2
\]
\[
\leq q(x, y)
\]
as required.

Proof of Theorem 1.3. Let \(f : X \times \mathbb{R} \to \mathbb{R}\) be locally semiconcave, bounded from below and \(F\#P\)-subharmonic. We are to show that \(g(x) := \inf_y f(x, y)\) is \(F\)-subharmonic.

We first claim that without loss of generality we may assume in addition that for each \(x\) it holds that \(\text{argmin}_f(x)\) is non-empty and single valued. To prove this, for \(j \geq 1\) let
\[
f_j(x, y) = f(x, y) + \frac{1}{j}\|y\|^2.
\]
As \(F\) depends only on the Hessian part, \(f_j\) is still \(F\#P\)-subharmonic, and is still bounded from below and semiconcave. Moreover since \(f\) is bounded from below, for each fixed \(x\) the function \(y \mapsto f(x, y)\) is strictly convex and tends to infinity as \(\|y\|\) tends to infinity, implying that it has a unique global minimum. By assumption the theorem applies to \(f_j\) meaning that letting \(g_j(x) := \inf_y f_j(x, y)\) the function \(g_j\) is \(F\)-subharmonic. But \(g_j \searrow g\) pointwise as \(j \to \infty\), and thus \(g\) will be \(F\)-subharmonic as well, proving the claim.

So from now on assume that \(\gamma(x) = \text{argmin}_f(x)\) is single valued. Fix \(x_0 \in \mathbb{R}^n\). As \(\gamma\) is continuous, there exist small balls \(x_0 \in U \subset X\) and \(\gamma(x_0) \in V \subset \mathbb{R}^m\) such that \(\gamma(U) \subset V\) and \(f\) is semiconcave on \(U \times V\). For \(\epsilon > 0\) consider the function
\[
f_\epsilon(x, y) := f^{\epsilon,P}(x, y) + \frac{\epsilon}{2}\|y\|^2 = \sup_{z \in U} \left\{ f(z, y) - \frac{1}{2\epsilon}\|z - x\|^2 \right\} + \frac{\epsilon}{2}\|y\|^2.
\]
We claim that for \(\epsilon\) sufficiently small the following all hold:
\begin{enumerate}
  \item \(f_\epsilon(x, y) + \frac{1}{2}\|x\|^2 - \frac{\epsilon}{2}\|y\|^2\) is convex.
  \item \(f_\epsilon\) is \(F\#P\)-subharmonic on \(U' \times V\) for some smaller ball \(x_0 \in U' \subset U\).
  \item \(f_\epsilon \searrow f\) pointwise on \(U \times V\) as \(\epsilon \to 0^+\).
  \item There is a \(\kappa_2 > 0\) such that for each \(x \in U\) the function \(y \mapsto f_\epsilon(x, y) - \frac{\kappa_2}{2}\|y\|^2\) is concave.
\end{enumerate}
Items (i,ii,iii) follow from Lemma 30 (we have used here the hypothesis that $F$ depends only on the Hessian part so adding a multiple of $\|y\|^2$ preserves the property of being $F\#\mathcal{P}$-subharmonic by Lemma 29). The statement (iv) follows from Lemma 31 (observing that the addition of $\frac{\epsilon}{2}\|y\|^2$ to the partial sup-convolution only means we may need to increase the value of $\kappa_2$).

Thus we are in a position to apply Proposition 32 to $f_\epsilon$ to conclude that if $g_\epsilon(x) := \inf_{y \in V} f_\epsilon(x,y)$ then $g_\epsilon$ is $F$-subharmonic on $U'$. But by (iii) if $x \in U'$ then
$$g_\epsilon(x) \leq \inf_{y \in V} f(x, \gamma(x)) = g(x) \text{ as } \epsilon \to 0^+$$
and hence $g$ is also $F$-subharmonic on $U'$. Since $x_0$ was arbitrary we conclude $g$ is $F$-subharmonic on all of $\mathbb{R}^n$ as required. □

Appendix

Appendix A. The Implicit Function Theorem for Lipschitz Functions

The following version of the Implicit function theorem is taken from [19, Theorem 5.1], and we include a proof for convenience.

**Theorem 34 (Lipschitz Implicit Function Theorem).** Let $U_1 \subset \mathbb{R}^r$ and $U_2 \subset \mathbb{R}^s$ be open and
$$J : U_1 \times U_2 \to \mathbb{R}^s$$
be Lipschitz with the property that there is a $K > 0$ such that
$$\|J(p, y_1) - J(p, y_2)\| \geq K\|y_1 - y_2\| \text{ for all } (p, y_1), (p, y_2) \in U_1 \times U_2.$$  

Suppose $a \in U_1, b \in U_2$ is such that
$$J(a, b) = 0.$$

There there exists an open $a \in V \subset U_1$ and a Lipschitz map
$$\phi : V \to U_2$$
such that $\phi(a) = b$ and
$$J(p, \phi(p)) = 0 \text{ for all } p \in V. \quad (A.1)$$

**Proof.** For small $\epsilon > 0$ (to be determined) let
$$\tilde{J} : U_1 \times U_2 \to \mathbb{R}^{r+s} \text{ be } \tilde{J}(p, y) = (p, \epsilon J(p, y))$$
which is Lipschitz as $J$ is assumed to be Lipschitz. We claim that as long as $\epsilon$ is sufficiently small, $\tilde{J}$ is bi-Lipschitz, i.e. there is a $C > 0$ such that
$$||\tilde{J}(p_1, y_1) - \tilde{J}(p_2, y_2)|| \geq C\|(p_1, y_1) - (p_2, y_2)\| \quad (A.2)$$
fors all $(p_1, y_1) \in U_1 \times U_2$.

To see this, say $J$ has Lipschitz constant $M$ and let $(p_1, y_1) \in U_1 \times U_2$. Then
$$K^2\|y_1 - y_2\|^2 \leq ||J(p_1, y_1) - J(p_1, y_2)||^2$$
$$\leq 2(||J(p_1, y_1) - J(p_2, y_2)||^2 + ||J(p_2, y_2) - J(p_1, y_2)||^2)$$
$$\leq 2||J(p_1, y_1) - J(p_2, y_2)||^2 + 2M^2\|p_2 - p_1\|^2.$$
So if we take \( \epsilon \) small enough so \( 1 - \epsilon^2 M^2 \geq \frac{K^2 \epsilon^2}{2} \), then
\[
\frac{K^2 \epsilon^2}{2} \|y_1 - y_2\|^2 + (1 - \epsilon^2 M^2) \|p_1 - p_2\|^2 \leq \epsilon^2 \|J(p_1, y_1) - J(p_2, y_2)\|^2 + \|p_1 - p_2\|^2
\]
\[
= \|\hat{J}(p_1, y_1) - \hat{J}(p_2, y_2)\|^2.
\]

So if we take \( \epsilon \) small enough so \( 1 - \epsilon^2 M^2 \geq \frac{K^2 \epsilon^2}{2} \), then
\[
\|\hat{J}(p_1, y_1) - \hat{J}(p_2, y_2)\|^2 \geq C^2 (\|y_1 - y_2\|^2 + \|p_1 - p_2\|^2) = C^2 \|p_1, y_1) - (p_2, y_2)\|^2
\]
as claimed in (A.2).

In particular \( \hat{J} \) is continuous and injective. Thus by Brouwer’s Invariance of Domain Theorem [3, Corollary 19.8], \( \hat{J} \) is an open map. So \( V := \hat{J}(U_1 \times U_2) \subset \mathbb{R}^{r+s} \)
is open and \( \hat{J} : U_1 \times U_2 \to V \) is a continuous bijection with continuous inverse \( \hat{J}^{-1} : V \to U_1 \times U_2 \). In fact as \( \hat{J} \) is bi-Lipschitz, we get that \( \hat{J}^{-1} \) is Lipschitz.

Denote by \( \pi_1 : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^r \) and \( \pi_2 : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^s \) the projections, and let \( B \) be a small ball around \( a \) so that \( B \subset U_1 \) and \( B \times \{0\} \subset \hat{J}(\pi_2^{-1}(U_2)) \). Define \( \phi : B \to U_1 \subset \mathbb{R}^r \) by
\[
\phi(p) = \pi_2 \hat{J}^{-1}(p, 0).
\]

Then \( \phi \) is Lipschitz and \( \hat{J}^{-1}(a, 0) = (a, b) \) gives \( \phi(a) = b \). Moreover if \( p \in V \) then
\[
(p, 0) = \hat{J} \hat{J}^{-1}(p, 0) = J(\pi_1 \hat{J}^{-1}(p, 0), \pi_2 \hat{J}^{-1}(p, 0))
\]
\[
= J(\pi_1 \hat{J}^{-1}(p, 0), \phi(p)) = (\pi_1 \hat{J}^{-1}(p, 0), \epsilon J(\pi_1 \hat{J}^{-1}(p, 0), \phi(p)).
\]

Thus
\[
p = \pi_1 \hat{J}^{-1}(p, 0)
\]
and
\[
0 = \epsilon J(\pi_1 \hat{J}^{-1}(p, 0), \phi(p)) = \epsilon J(p, \phi(p))
\]
proving (A.1).

\[\square\]

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