FINITE ENERGY WEAK SOLUTIONS OF 2D BOUSSINESQ EQUATIONS WITH DIFFUSIVE TEMPERATURE

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Abstract. We show the existence of finite kinetic energy solution with prescribed kinetic energy to the 2d Boussinesq equations with diffusive temperature on torus.

1. Introduction. The Boussinesq equation was introduced for understanding the effect of potentially large conversions between internal energy and mechanical energy in fluids, and simulates many geophysical flows, such as atmospheric fronts and ocean circulations (see, for example, [31], [35]). Moreover, it was used in recent theoretical discussion of the energetics of horizontal convection and the energetics of turbulent mixing in stratified fluids.

In this paper, we consider the following 2-dimensional Boussinesq system

\[
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p + (-\Delta)^\alpha v &= \theta e_2, & \text{in} & \quad [0,1] \times T^2 \\
\text{div} v &= 0, & \text{in} & \quad [0,1] \times T^2 \\
\partial_t \theta + v \cdot \nabla \theta - \Delta \theta &= 0, & \text{in} & \quad [0,1] \times T^2
\end{aligned}
\] (1)

where \( \alpha < 1 \) is a positive number, \( e_2 = (0,1)^T \) and \( T^2 \) is 2d torus. Here, \( v \) is the velocity vector, \( p \) is the pressure, \( \theta \) denotes the temperature which is a scalar function.

The global well-posedness have been established by many authors for the Cauchy problem of (1) in 2d for regularity data (see, for example, [9], [20]). For the 3-dimensional case, the global existence of smooth solution of (1) remains open. To understand the turbulence phenomena in hydrodynamics, one needs to go beyond classical solutions, and in this paper we are interested in constructing weak solutions

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of (1) with bounded kinetic energy. The triple \((v, p, \theta)\) on \([0, 1] \times \mathbb{T}^2\) is called a weak solution of (1) if they belong to \(L^\infty(0, 1; L^2(\mathbb{T}^2))\) and solve (1) in the following sense:

\[
\int_0^1 \int_{\mathbb{T}^2} (v \cdot \partial_t \varphi + v \otimes v : \nabla \varphi - (-\Delta)\alpha \varphi \cdot u + \theta e_2 \cdot \varphi)dxdt = 0
\]

for all \(\varphi \in C^\infty_c((0, 1) \times \mathbb{T}^2; \mathbb{R}^2)\) with \(\text{div} \varphi(t, x) = 0\).

\[
\int_0^1 \int_{\mathbb{T}^2} (\partial_t \phi\theta + v \cdot \nabla \phi\theta + \Delta \phi\theta)dxdt = 0
\]

for all \(\phi \in C^\infty_c((0, 1) \times \mathbb{T}^2; \mathbb{R})\) and

\[
\int_{\mathbb{T}^2} v(t, x) \cdot \nabla \psi(x)dx = 0
\]

for all \(\psi \in C^\infty(\mathbb{T}^2; \mathbb{R})\) and any \(t \in [0, 1]\).

The study of constructing non-unique or dissipative weak solution to fluid system is very fashionable in recent years, and the construction is based on convex integration method which pioneered by De Lellis-Székelyhidi Jr in [15, 17], where the author tackle the Onsager conjecture for the incompressible Euler equation. So far, there are many important work about weak solution of the incompressible Euler equation, see [10, 12, 16, 19, 21, 23, 34, 36, 37, 38, 40, 41]. The Onsager conjecture was proved by P.Isett in [24], based on a series of process on this problem in [1, 3, 4, 5, 11, 13, 14, 18, 22], see also [6] for the construction of admissible weak solution. Moreover, the idea and method can be used to construct dissipative weak solution for other model, see [7, 26, 29, 39, 42, 43, 44].

Recently, Buckmaster and Vicol establish the non-uniqueness of weak solution to the 3D incompressible Navier-Stokes in [8] by introducing some new ideas. The main idea is to use “intermittent” building blocks in the convex integration scheme to control the dissipative term \(\nabla v\), called “intermittent Beltrami flow”, which are space inhomogeneous version of the classical Beltrami flow, also see [2]. Compared with the homogenous case, the “intermittent Beltrami flow” has different scaling in different \(L^p\) norms. In particular, one can ensure small \(L^p\) norm for small \(p > 1\) which is key to control the dissipative term. By choosing the parameter suitably, T.Luo and E.S.Titi in [27] construct weak solution with compact support in time for hyperviscous Navier-Stokes equation. For high dimension \(d \geq 4\) stationary Navier-Stokes equation, X. Luo in [30] show the non-uniqueness by constructing the concentrated Mikado flows introduced in [32, 33]. Moreover, S.Modena and Székelyhidi established the non-uniqueness for the linear transport equation (and transport-diffusion equation) with divergence-free vector in some Sobolev space, see [32, 33], where they use Mikado density and Mikado fields which is highly concentrated such that the \(L^p\) norm of Mikado field is small for \(p > 1\) small.

Motivated by the above earlier works, we consider the 2d Boussinesq equations (1) and want to know if the similar phenomena can also happen when adding the temperature effects. Following the general scheme in the construction of non-uniqueness to Navier-Stokes equation in [8], we obtain the following existence result.

**Theorem 1.1.** For any smooth function \(e(t) : [0, 1] \rightarrow [1, +\infty)\) and \(\frac{1}{2} \leq \alpha < 1\), there exist \(v \in C([0, 1], L^2(\mathbb{T}^2)), \theta \in \bigcap_{2 \leq p < \infty} C([0, 1], L^p(\mathbb{T}^2)) \cap L^2([0, 1], H^1(\mathbb{T}^2))\),
which is weak solution of Boussinesq equations (1) with
\[ \int_{T^2} |v(x,t)|^2 dx = e(t), \quad \forall t \in [0,1], \]
and for any \( t \in [0,1] \)
\[ \frac{1}{2} \| \theta(t,\cdot) \|_{L^2}^2 + \int_0^t \| \nabla \theta(s,\cdot) \|_{L^2}^2 ds = \frac{1}{2} \| \theta(0,\cdot) \|_{L^2}^2. \]

Remark 1. For \( 0 \leq \alpha < \frac{1}{2} \), we also can construct weak solution with prescribed energy curve in the class \( C([0,1],L^2(T^2)) \) by the same method. However, in a separate paper [28], we will construct Hölder continuous solution for this case.

Remark 2. The \( \theta \) we construct in this paper is rather regular and satisfies energy equality. When \( \theta = 0 \), the equation (1) is 2d Navier-Stokes equation with fractional diffusion, and our construction also works for this case.

Remark 3. In the paper [8], Buckmaster and Vicol introduced the “intermittent Beltrami flow” in convex integration which is key to construct weak solution to Navier-Stokes equation, by using the effect of concentration to control the errors induced by viscosity. The extra difficulties of the Boussinesq equation are the interaction between the velocity \( v \) and the temperature \( \theta \), and the fact that the concentration effect is not strong enough to control the errors induced by diffusion in 2D, in contrast to the case in 3D. The key observation in this paper is that we can use the technique of convex integration to construct the velocity \( v \), and construct the temperature \( \theta \) by solving the transport-diffusion equation directly. The regularity of the transport-diffusion equation enables us to control the interaction between the velocity \( v \) and the temperature \( \theta \) (See Section 7.3.2).

The rest of the paper is organized as follows. In Section 2, we state the main proposition and give a proof of Theorem 1.1. In Section 3, we collect some technical tool which will be used frequently. In Section 4, we introduce the intermittent plane wave which is the building block in our perturbation. In Section 5 and 6, we construct velocity perturbations and temperature perturbation, respectively. After the construction, we establish the related estimates. In Section 7, we construct the Reynold-Stress and establish the related estimates. Finally, in Section 8, we give a proof of Proposition 2.1.

2. Main proposition and proof of main theorem.

2.1. Iterative proposition and proof of main theorem. In this section, we state our main iterative proposition and give a proof of theorem 1.1 by the help of main proposition.

Definition 2.1. Assume that \( (v_0,p_0,\theta_0, \hat{R}_0) \in C^\infty([0,1] \times T^2, R^2 \times R \times R \times S_0^{2 \times 2}) \). We say that they solve the Boussinesq-Reynold equation if
\[
\begin{aligned}
\partial_t v_0 + \text{div}(v_0 \otimes v_0) + \nabla p_0 + (-\Delta)^\alpha v_0 &= \theta e_2 + \text{div} \hat{R}_0, &\text{in } [0,1] \times T^2 \\
\text{div} v_0 &= 0, &\text{in } [0,1] \times T^2 \\
\partial_t \theta_0 + \text{div}(v_0 \theta_0) - \Delta \theta_0 &= 0, &\text{in } [0,1] \times T^2.
\end{aligned}
\]

(2)

Here and throughout the paper, \( S_0^{2 \times 2} \) is the set of trace-free symmetric 2 \( \times \) 2 matrices.

We now state our main proposition, and Theorem 1.1 is a corollary.
Proposition 1. Let $e(t), \alpha$ be as in Theorem 1.1 and $\varepsilon_0$ be a universal constant from the Geometric Lemma 4.1. Then there exist universal constant $M$ and $\tilde{M}$ such that the following hold.

Let $\delta \leq 1$ be any positive number, $\theta^0 \in C^\infty(\mathbb{T}^2)$ be any function satisfies $\int_{\mathbb{T}^2} \theta^0(x) dx = 0$, and $(v_0, p_0, \theta_0, R_0)$ is a solution of Boussinesq-Reynold equation (2) with

$$\| \dot{R}_0 \|_{L^\infty_t L^1_x} \leq \frac{\varepsilon_0 \delta}{10000}, \quad \int_{\mathbb{T}^2} \theta_0(t, x) dx = 0, \quad \forall t \in [0, 1]$$

and

$$\frac{3\delta}{4} e(t) \leq e(t) - \int_{\mathbb{T}^2} |v_0(t, x)|^2 dx \leq \frac{5\delta}{4} e(t), \quad \forall t \in [0, 1].$$

Then there exist another smooth functions $(v_1, p_1, \theta_1, \dot{R}_1)$ which is also a solution of Boussinesq-Reynold equation (2), and for every $t \in [0, 1],$

$$\| \dot{R}_1(t, \cdot) \|_{L^1_x} \leq \frac{\varepsilon_0 \delta}{20000}, \quad \int_{\mathbb{T}^2} \theta_1(t, x) dx = 0,$$

$$\| v_1(t, \cdot) - v_0(t, \cdot) \|_{L^2} \leq M \sqrt{\delta},$$

$$\theta_1(0, x) = \theta^0(x), \quad \| \theta_1(t, \cdot) \|_{L^\infty} \leq \tilde{M},$$

$$\| \theta_1(t, \cdot) - \theta_0(t, \cdot) \|_{L^2}^2 + 2 \int_0^t \| \nabla \theta_1(s, \cdot) - \nabla \theta_0(s, \cdot) \|_{L^2}^2 \leq M \delta,$$

and

$$\| \theta_1(t, \cdot) \|_{L^2}^2 + 2 \int_0^t \| \nabla \theta_1(s, \cdot) \|_{L^2}^2 = \| \theta_1(0, \cdot) \|_{L^2}^2,$$\hspace{1cm} (5)

and

$$\frac{3\delta}{8} e(t) \leq e(t) - \int_{\mathbb{T}^2} |v_1(t, x)|^2 dx \leq \frac{5\delta}{8} e(t).$$\hspace{1cm} (6)

We will prove the above proposition in the next several sections. Here we first give a proof of Theorem 1.1.

Proof. We first fix $\delta = 1$ and set

$$v_0 := 0, \quad \theta_0 := 0, \quad p_0 := 0, \quad \dot{R}_0 := 0.$$  

Obviously, they solve Boussinesq-Reynolds system (2) and

$$\frac{3\delta}{4} e(t) \leq e(t) - \int_{\mathbb{T}^2} |v_0|^2(x, t) dx \leq \frac{5\delta}{4} e(t), \quad \forall t \in [0, 1]$$

$$\sup_{t \in [0, 1]} \| \dot{R}_0(t, \cdot) \|_{L^1_x} = 0 \leq \frac{\varepsilon_0 \delta}{10000}.$$  

Then choosing $\theta^0 \in C^\infty(\mathbb{T}^2)$ satisfies $\int_{\mathbb{T}^2} \theta^0(x) dx = 0$, and using Proposition 2.1 iteratively, we can construct a sequence $(v_n, p_n, \theta_n, \dot{R}_n)$, which solve (2) and
satisfy, for every $t \in [0, 1]$
\[
\frac{3}{4} e(t) \leq e(t) - \int_{T^2} |v_n|^2(x, t) \, dx \leq \frac{5}{4} e(t), \quad (7)
\]
\[
\|\tilde{R}_n(t, \cdot)\|_{L^1_T} \leq \frac{\varepsilon_0}{2^n \times 10000},
\]
\[
\|v_{n+1}(t, \cdot) - v_n(t, \cdot)\|_{L^2} \leq M \sqrt{\frac{1}{2^n}},
\]
\[
\|\theta_n(t, \cdot)\|_{L^\infty} \leq M, \quad \theta_n(0, x) = \theta^0(x),
\]
\[
\|\theta_{n+1}(t, \cdot) - \theta_n(t, \cdot)\|_{L^2}^2 + \int_0^t \|\nabla(\theta_{n+1}(s, \cdot) - \theta_n(s, \cdot))\|_{L^2}^2 \leq \frac{M}{2^n},
\]
\[
\|\theta_{n+1}(t, \cdot)\|_{L^2}^2 + 2 \int_0^t \|\nabla\theta_{n+1}(s, \cdot)\|_{L^2}^2 = \|\theta_{n+1}(0, \cdot)\|_{L^2}^2. \quad (12)
\]
From (8)-(12), we know that $(v_n, \theta_n, \tilde{R}_n)$ are, respectively, Cauchy sequence in $C([0, 1], L^2(T^2)) \cap L^2([0, 1], H^1(T^2))$ and $C([0, 1], L^1(T^2))$(12), therefore there exist
\[
v \in C([0, 1], L^2(T^2)), \quad \theta \in \bigcap_{2 \leq p < \infty} C([0, 1], L^p(T^2)) \cap L^2([0, 1], H^1(T^2))
\]
such that
\[
v_n \to v \quad \text{in} \quad C([0, 1], L^2(T^2)),
\]
\[
\theta_n \to \theta \quad \text{in} \quad \bigcap_{2 \leq p < \infty} C([0, 1], L^p(T^2)) \cap L^2([0, 1], H^1(T^2)),
\]
\[
\tilde{R}_n \to 0 \quad \text{in} \quad C([0, 1], L^1(T^2))
\]
as $n \to \infty$.

Passing into the limit in (2), we conclude that $(v, \theta)$ solve (1) in the sense of distribution. Moreover, by (7),
\[
e(t) = \int_{T^2} |v|^2(x, t) \, dx, \quad \forall t \in [0, 1].
\]
Moreover, by (12), we deduce that the temperature $\theta$ satisfies the energy equality: for every $t \in [0, 1]$
\[
\|\theta(t, \cdot)\|_{L^2}^2 + 2 \int_0^t \|\nabla\theta(s, \cdot)\|_{L^2}^2 = \|\theta(0, \cdot)\|_{L^2}^2.
\]
This complete the proof of Theorem 1.1.

2.2. Strategy of the proof of Proposition 2.1. Our strategy is as follows: the construction of $v_1$ is based on the technique of convex integration inspired by the work [8] of Buckmaster and Vicol, and $\theta_1$ is obtained by solving the transport-diffusion equation.

More specifically, we first construct $v_1$ by adding perturbation to $v_0$ as follows:
\[
v_1 = v_0 + w_0^p + w_1^c + w_1^t := v_0 + w_1,
\]
where $w_0^p$ is the main part which consists of intermittent plane waves, $w_1^c$ is the incompressibility corrector, $w_1^t$ is temporal corrector, and all of them are smooth functions given by explicit formulas.
Then we construct new temperature $\theta_1$ by solving the following transport-diffusion equation
\[
\begin{cases}
\partial_t \theta_1 + v_1 \cdot \nabla \theta_1 - \Delta \theta_1 = 0, \\
\theta_1|_{t=0} = \theta^0,
\end{cases}
\tag{13}
\]
where $\theta^0$ is the function appeared in Proposition 2.1.

Finally, we focus on the construction of $\hat{R}_1, p_1$. By careful computation of the interaction term $\text{div}(w_1^p \otimes w_1^p)$, we can construct $p_1$ and $\hat{R}_1$ which satisfied the system (2) and the desired estimates.

3. Technical tool. In this section, we collect some technical tools which will be frequently used in the following.

3.1. Properties of fast oscillatory. In this subsection, we discuss some properties of fast oscillatory and the proof can be found in [32], [33], which was inspired by [8]. More precisely, we give an improved Hölder inequality which concern the $L^p$ norm of the product of a slow oscillating function with a fast oscillating function, and a mean value estimate which concern the mean value of the product of a slow oscillating function with a fast oscillating function.

For a given function $f: T^2 \to \mathbb{R}$, $\lambda \in \mathbb{N}$, we set $f_\lambda(x) := f(\lambda x)$.

**Lemma 3.1.** Let $f, g: T^2 \to \mathbb{R}$ be smooth functions, $\lambda \in \mathbb{N}$. Then for every $p \in [1, +\infty)$, we have
\[
\|fg_\lambda\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^p} + \frac{C}{\lambda^p}\|f\|_{C^1}\|g\|_{L^p}.
\]

**Lemma 3.2.** Let $f, g: T^2 \to \mathbb{R}$ be smooth function with $\int_{T^2} g(x)dx = 0$, and $\lambda \in \mathbb{N}$. Then there hold
\[
\left| \int_{T^2} fg_\lambda dx \right| \leq \frac{\sqrt{2}\|f\|_{C^1}\|g\|_{L^1}}{\lambda}.
\]

3.2. Commutator for fast oscillation.

**Lemma 3.3.** Fix $\kappa \geq 1$. Let $a \in C^2(T^3)$. For $1 < p < \infty$, and any $f \in L^p(T^3)$, we have
\[
\|\|\nabla\|_{L^p}^{-1}P_{\neq 0}(aP_{\geq k}f)\|_{L^p} \leq CK^{-1}(\|a\|_{L^\infty} + \|\nabla^2a\|_{L^\infty})\|f\|_{L^p}.
\]

The proof of this Lemma can be found in [8].

4. Intermittent plane waves. In this section, we describe in detail the construction of the *intermittent plane waves* which will form the building blocks of the convex integration scheme.

We first recall the following stationary solution for the 2d Euler equation. Our building block in this paper is the inhomogeneous version of it.
4.1. Stationary flows in 2D and Geometric Lemma.

**Proposition 2.** Let $\Lambda$ be a given finite symmetric subset of $S^1 \cap Q^2$ with $\lambda \Lambda \in Z^2$. Then for any choice of coefficients $a_\xi \in C$ with $\overline{a_\xi} = a_{-\xi}$, the vector field

\[
W(x) = \sum_{\xi \in \Lambda} a_\xi \xi^T e^{\xi \cdot x}, \quad \Psi(x) = \sum_{\xi \in \Lambda} a_\xi e^{\xi \cdot x}
\]

is real-valued and satisfies

\[
\text{div}(W \otimes W) = \nabla \left( \frac{|W|^2}{2} + \frac{\Psi^2}{2} \right), \quad W(x) = \nabla^\perp \Psi(x).
\]

Here and throughout the paper, we denote $\bar{\xi} \equiv (-\xi_2, \xi_1)$ if $\xi = (\xi_1, \xi_2)$, and denote $\nabla^\perp = (-\partial_2, \partial_1)$. Furthermore,

\[
\langle W \otimes W \rangle := \frac{1}{(2\pi)^2} \int_{T^2} W \otimes W(x) dx = \sum_{\xi \in \Lambda} |a_\xi|^2 (\text{Id} - \xi \otimes \bar{\xi}).
\]

The proof of this Proposition can be found in [11], and we omit it here.

Let

\[
\Lambda_0^+ = \left\{ e_1, \frac{3}{5}e_1 + \frac{4}{5}e_2, \frac{3}{5}e_1 - \frac{4}{5}e_2 \right\}, \quad \Lambda_0^- = -\Lambda_0^+, \quad \Lambda_0 = \Lambda_0^+ \cup \Lambda_0^-,
\]

and $\Lambda_1$ be given by the rotation of $\Lambda_0$ counter clock-wise by $\pi/2$:

\[
\Lambda_1^+ = \left\{ e_2, \frac{3}{5}e_2 + \frac{4}{5}e_1, \frac{3}{5}e_2 - \frac{4}{5}e_1 \right\}, \quad \Lambda_1^- = -\Lambda_1^+, \quad \Lambda_1 = \Lambda_1^+ \cup \Lambda_1^-.
\]

Clearly $\Lambda_0, \Lambda_1 \subseteq Q^2 \cap S^1$ and we have the representation

\[
\frac{25}{32} \left( \left( \frac{3}{5}e_1 + \frac{4}{5}e_2 \right) \otimes \left( \frac{3}{5}e_1 + \frac{4}{5}e_2 \right) + \left( \frac{3}{5}e_1 - \frac{4}{5}e_2 \right) \otimes \left( \frac{3}{5}e_1 - \frac{4}{5}e_2 \right) \right) + \frac{7}{16} e_1 \otimes e_1 = \text{Id}.
\]

In fact, such representation holds for $2 \times 2$ symmetric matrices near $\text{Id}$.

**Lemma 4.1** (Geometric Lemma). There exists $\varepsilon_0 > 0$, and smooth positive functions $\gamma_\xi$:

\[
\gamma_\xi \in C^\infty(B_{\varepsilon_0}(\text{Id}))
\]

such that for every $2 \times 2$ symmetric matrix $R \in B_{\varepsilon_0}(\text{Id})$, we have

\[
R = \sum_{\xi \in \Lambda_0^+} \gamma_\xi^2(R) \xi \otimes \xi.
\]

**Remark 4.** By rotational symmetry, Geometric Lemma 4.1 also holds for $\xi \in \Lambda_1^+$. It is convenient to introduce a small geometric constant $c_0 \in (0, 1)$ such that

\[
|\xi + \xi'| \geq 2c_0
\]

for all $\xi, \xi' \in \Lambda_0 \cap \Lambda_1, \xi \neq -\xi'$. Moreover, for $\xi \in \Lambda_0^-$ with $k = 0, 1$, we set $\gamma_\xi := \gamma_{-\xi}$.

**Remark 5.** When $\text{Id} - \hat{R} \in B_{\varepsilon_0}(\text{Id})$, by Geometric Lemma, we have

\[
\text{Id} - \hat{R} = \sum_{\xi \in \Lambda_0^+} \gamma_\xi^2(\text{Id} - \hat{R}) \xi \otimes \xi.
\]

Thus, taking trace in both side, we obtain

\[
\sum_{\xi \in \Lambda_0^+} \gamma_\xi^2(\text{Id} - \hat{R}) = 2.
\]
4.2. **Intermittent plane flow.** The Dirichlet kernel $\tilde{D}_r$ is defined as

$$\tilde{D}_r(x) := \sum_{\xi = -r}^{r} e^{ix \cdot \xi} = \frac{\sin((r + \frac{1}{2})x)}{\sin(\frac{x}{2})},$$

and it obeys the estimates: for any $p > 1$,

$$\|\tilde{D}_r\|_{L^p} \sim r^{1 - \frac{1}{p}},$$

where the implicit constant only depend only on $p$. Define a 2d square

$$\Omega_r := \{(j, k) : j, k \in \{-r, \ldots, r\}\}$$

and normalizing to unit size in $L^2$, we obtain a kernel

$$D_r(x) := \frac{1}{2r + 1} \sum_{\xi \in \Omega_r} e^{ix \cdot \xi} = \frac{1}{2r + 1} \sum_{(j, k) \in \Omega_r} e^{ijx_1 + kx_2}$$

which has the property: for $1 < p \leq \infty$

$$\|D_r\|_{L^p} \leq C r^{1 - \frac{2}{p}}, \quad \|D_r\|_{L^2} = C,$$

where the implicit constant only depend on $p$. This computation is very easy due to the fact:

$$\sum_{\xi \in \Omega_r} e^{ix \cdot \xi} = \left( \sum_{i = -r}^{r} e^{ijx_1} \right) \left( \sum_{i = -r}^{r} e^{ikx_2} \right).$$

We first fixed a large parameter $\lambda \in \mathbb{N}$, then introduce a parameter $\sigma$ such that $\lambda \sigma \in \mathbb{N}$ which parameterizes the spacing between frequencies. We assume that

$$\sigma r \leq c_0 \frac{50}{N},$$

where $c_0$ is the constant in Remark 4, and $N$ is a fixed integer (for example, we can take $N = 5$). Furthermore, we introduce a parameter $\mu \in (0, \lambda)$, which describes temporal oscillation in the building blocks.

As in [8], for $\xi \in \Lambda^j_+$, we define a directed and rescaled $(\frac{2\pi}{\lambda \sigma})^2$-periodic Dirichlet kernel by

$$\eta_{\xi}(x, t) := \eta_{\xi, \lambda, \sigma, r, \mu}(x, t) = D_r\left(\lambda \sigma N(\xi \cdot x + \mu t), \lambda \sigma N \xi^\perp \cdot x\right)$$

and set $\eta_{\xi}(x, t) = \eta_{(-\xi)}(x, t)$ for $\xi \in \Lambda^j_-$, where $N$ is a integer (we can set $N = 5$ due to our construction of $\Lambda_0, \Lambda_1$). Observe that $\eta_{\xi}(x, t)$ satisfies the following important identity:

$$\frac{1}{\mu} \partial_t \eta_{\xi}(x, t) = \xi \cdot \nabla \eta_{\xi}(x, t)$$

for every $\xi \in \Lambda^j_+$, $j = 0, 1$.

A change of variable gives that

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \eta_{\xi}^2(x, t) dx = 1, \quad \|\eta_{\xi}(t)\|_{L^p} \leq C r^{1 - \frac{2}{p}}$$

for all $1 < p \leq \infty$, for any $t$. 
Let $W(\xi)(x)$ be the stationary wave at frequency $\lambda$, namely

$$W(\xi)(x) = W_{\xi,\lambda} = i\xi e^{i\lambda \xi \cdot x}.$$

We define the intermittent plane waves $W(\xi)$ as

$$W(\xi)(x,t) = \eta(\xi,\lambda,\sigma,r,\mu)(x,t)W_{\xi,\lambda}(x) = \eta(\xi,\lambda,\sigma,r,\mu)(x,t)W_{\xi,\lambda}(x).$$

**Remark 6.** The explicit representation of $W(\xi)(x,t)$ is as following:

$$W(\xi)(x,t) = \left( \sum_{j=-r}^{r} e^{ij\sigma N(\xi \cdot x + \mu t)} \right) \left( \sum_{k=-r}^{r} e^{ik\sigma N\xi \cdot x} \right) i\xi e^{i\lambda \xi \cdot x}.$$

Some facts about the frequency support of $\eta(\xi)$ and $W(\xi)$:

**Lemma 4.2.** We have the following frequency support property:

$$P_{\leq 2\lambda \sigma r N} \eta(\xi) = \eta(\xi), \quad P_{\leq 2\lambda} W(\xi) = W(\xi).$$

For $\xi + \xi' \neq 0$, by the definition of $c_0$, we have

$$P_{\leq 4\lambda} P_{\geq c_0} \left( W(\xi) \otimes W(\xi') \right) = W(\xi) \otimes W(\xi').$$

These facts can be obtained directly from the definition. In fact, the frequency support of $\eta(\xi)$ is obvious. Then, using the fact that $2\lambda \sigma r N \leq \frac{c_0 \lambda}{5}$, it’s easy to obtain the frequency support of $W(\xi)$. Finally, a direct computation gives that

$$W(\xi) \otimes W(\xi') = -\eta(\xi) \eta(\xi') \xi \otimes \xi' e^{i\lambda(\xi + \xi') \cdot x}.$$

Due to

$$P_{\leq 4\lambda} \eta(\xi) \eta(\xi') = \eta(\xi) \eta(\xi'), \quad |\xi + \xi'| \geq 2c_0$$

and the fact

$$4\lambda \sigma r N \leq \frac{2\lambda c_0}{5},$$

we obtain the frequency support of $W(\xi) \otimes W(\xi')$ for $\xi + \xi' \neq 0$.

From these frequency support properties (from which we can use the Berstein inequality) and the estimates for Dirichlet kernel, we have the following estimates.

**Proposition 3.** Let $W(\xi)$ be defined as above. Then

$$\|\nabla^N \partial^K_t W(\xi)\|_{L^p} \leq C(N, K, p) \lambda^N (\lambda \sigma r)^{K, r} p^{-\frac{N}{2}},$$

$$\|\nabla^N \partial^K_t \eta(\xi)\|_{L^p} \leq C(N, K, p) (\lambda \sigma r)^N (\lambda \sigma r)^{K, r} p^{-\frac{N}{2}},$$

for any $1 < p \leq \infty$, and $N, K \geq 0$ are integer.

These estimates are direct, and we omit the proof here.

5. The velocity perturbation: Construction and estimates. In this section, we construct the perturbation of velocity and give some estimates for it.

5.1. Construction of the velocity perturbation. In this subsection, we give the detailed construction of velocity perturbation.
5.1.1. **Definition of amplitude.** Choose two smooth cutoff functions \( \tilde{\chi}_0, \tilde{\chi}_1 \) such that

\[
\text{supp} \tilde{\chi}_0 \subseteq [0, 4], \quad \text{supp} \tilde{\chi}_1 \subseteq \left[ \frac{1}{4}, 4 \right]
\]

and

\[
\tilde{\chi}_0^2(y) + \sum_{j \geq 1} \tilde{\chi}_j^2(y) = 1
\]

for any \( y > 0 \), where \( \tilde{\chi}_j(y) = \tilde{\chi}_1(4^{-j}y) \). We then define

\[
\chi_j(t, x) := \tilde{\chi}_j \left( \frac{50\tilde{R}_0(t, x)}{\varepsilon_0 \delta} \right)
\]

for all \( j \geq 0 \). Here and throughout the paper we use the notation \( \langle A \rangle = (1 + |A|^2)^{\frac{1}{2}} \) where \( |A| \) denotes the standard norm of the matrix \( A \). By the definition of the cutoff functions, we have

\[
\sum_{j \geq 0} \chi_j^2(t, x) = 1, \quad \chi_j, \chi_{j'}(t, x) = 0 \quad \text{if} \quad |j_1 - j_2| \geq 2.
\]

Moreover, it’s obvious that there exists an index \( j_{\text{max}} = j_{\text{max}}(\tilde{R}_0, \delta) \) such that \( \chi_{j}(t, x) = 0 \) for all \( j \geq j_{\text{max}} \).

In fact,

\[
\chi_0(t, x) = 0 \iff \left\langle \frac{50\tilde{R}_0(t, x)}{\varepsilon_0 \delta} \right\rangle \geq 4.
\]

Thus, for fixed \( t \in [0, 1] \), we have

\[
\chi_0(t, x) = 0 \Rightarrow 50|\tilde{R}_0(t, x)| \geq 3\varepsilon_0 \delta.
\]

Hence, for fixed \( t \in [0, 1] \), we have

\[
\int_{T^2} |\tilde{R}_0(t, x)| dx \geq \frac{3}{50} \varepsilon_0 \delta.
\]

Thus, the assumption (3) tells us that for every \( t \in [0, 1] \), \( \int_{T^2} \chi_0^2(t, x) dx > 0 \).

For \( j \in \mathbb{N} \), denote

\[
\Lambda_{(j)} = \Lambda_j \mod 2, \quad \Lambda^\pm_{(j)} = \Lambda^\pm_j \mod 2.
\]

For \( \xi \in \Lambda^\pm_{(j)} \), define the coefficient function \( a_{(\xi, j)} \) by

\[
a_{(\xi, j)}(t, x) := \sqrt{\rho_j} \chi_j(t, x) \gamma_2 \left( \text{Id} - \frac{\tilde{R}_0}{\rho_j} \right)
\]

where \( \rho_j, j \geq 1 \), are defined by

\[
\rho_j := 4^j \delta,
\]

and \( \rho_0 \) is defined later.

We first claim that \( a_{(\xi, j)}(t, x) \) is well-defined for \( j \geq 1 \). We only need to show that \( \text{Id} - \frac{\tilde{R}_0}{\rho_j} \in B_{\varepsilon_0} (\text{Id}) \). In fact, when \( \chi_j \neq 0 \), there holds

\[
\left\langle \frac{50\tilde{R}_0(t, x)}{\varepsilon_0 \delta} \right\rangle \leq 4^{j+1},
\]
which implies
\[
\frac{50|\hat{R}_0(t, x)|}{\varepsilon_0 \delta} \leq 4^{j+1}.
\]
Thus,
\[
\frac{|\hat{R}_0(t, x)|}{\rho_j} = \frac{|\hat{R}_0(t, x)|}{4^j \delta} \leq \varepsilon_0,
\]
thus \(a(\xi, j)(t, x)\) is well-defined for \(j \geq 1\). We define \(\rho_0(t)\) as following:
\[
\rho_0(t) := \frac{1}{2} \left( \int_{T^2} \chi_0^2(t, x) dx \right)^{-1} \left[ e(t) \left( 1 - \frac{\delta}{2} \right) - \int_{T^2} |v_0(t, x)|^2 dx \right].
\] (22)

Due to (19), we deduce that \(\rho_0(t)\) is well-defined. Next, we show that
\[
\left\| \frac{\hat{R}_0}{\rho_0} \right\|_{L^\infty(\text{supp}\chi_0)} \leq \varepsilon_0.
\]
In fact, using the assumption (4), we know that
\[
\frac{\delta}{4} e(t) \leq e(t) \left( 1 - \frac{\delta}{2} \right) - \int_{T^2} |v_0(t, x)|^2 dx \leq \frac{3\delta}{4} e(t), \quad \forall t \in [0, 1].
\] (23)

On the support of \(\chi_0\), there holds
\[
50|\hat{R}_0(t, x)| \leq 4\varepsilon_0 \delta.
\]
Thus, on the support of \(\chi_0\), for any \(t \in [0, 1]\), there holds
\[
\left| \frac{\hat{R}_0(t, x)}{\rho_0(t)} \right| \leq \frac{4\varepsilon_0 \delta}{50} \frac{8 \int_{T^2} \chi_0^2(t, x) dx}{\delta e(t)} \leq \varepsilon_0.
\]
Thus, \(a(\xi, 0)\) is well-defined.

5.1.2. Construction of velocity perturbation. Let us fix \(\lambda_1, \sigma_1, r_1, \mu_1\) such that \(\lambda_1 \sigma_1 \in \mathbb{N}\) and the integer \(r_1\), the parameter \(\sigma_1\) and \(\mu_1\) are defined by
\[
r_1 = [\lambda_1], \quad \sigma_1 = \lambda_1^{-\frac{1+\mu_1}{\lambda_1}}, \quad \mu_1 = [\lambda_1^{-\frac{1}{\lambda_1}}].
\] (24)

The principle part of perturbation \(w_1^p\) will be defined as
\[
w_1^p = \frac{1}{\sqrt{2}} \sum_j \sum_{\xi \in \Lambda_{j}^1} a_{(\xi, j)} \eta_\xi (W_\xi + W_{-\xi}) = \frac{1}{\sqrt{2}} \sum_j \sum_{\xi \in \Lambda_{j}^1} a_{(\xi, j)} \eta_\xi W_\xi
\] (25)

where \(0 \leq j \leq j_{\text{max}}\). Here and throughout the paper, \(\eta_\xi, W_\xi\) denote, respectively,
\[
\eta_\xi = \eta_{\xi, \lambda_1, \sigma_1, r_1, \mu_1}, \quad W_\xi = W_{\xi, \lambda_1},
\]
where \(\xi \in \Lambda_{j}^1\).

Then we define an incompressibility corrector
\[
w_1^c := -\frac{1}{\sqrt{2}} \sum_j \sum_{\xi \in \Lambda_{j}^1} \nabla^\perp \left( a_{(\xi, j)} \eta_\xi \right) e^{i\lambda_1 \xi^+ \cdot x} + e^{-i\lambda_1 \xi^+ \cdot x}. \]
(26)

Here and throughout the paper, we denote \(\nabla^\perp\) as \(\nabla^\perp = (-\partial_2, \partial_1)\). Thus, we have
\[
w_1^p + w_1^c = \frac{1}{\sqrt{2}} \sum_j \sum_{\xi \in \Lambda_{j}^1} \nabla^\perp \left( a_{(\xi, j)} \eta_\xi \frac{e^{i\lambda_1 \xi^+ \cdot x} + e^{-i\lambda_1 \xi^+ \cdot x}}{\lambda} \right)
\]
and
\[ \text{div}(w_p^t + w_c^t) = 0. \]

As in paper [8], in addition to the incompressibility corrector \( w_c^t \), we introduce a temporal corrector \( w_t^t \), which is defined by

\[ w_t^t := -\frac{1}{\mu} \sum_j \sum_{\xi \in \Lambda_{(j)}^+} P_H P \neq 0 (a_{(\xi,j)}^2 \eta^2_{(\xi)\xi}). \tag{27} \]

Here \( P \neq 0 f = f - \frac{1}{(2\pi)^2} \int_{T^2} f dx \) and \( P_H f = f - \nabla \Delta^{-1} \text{div} f \). Finally, we define the velocity increment \( w_1 \) by

\[ w_1 = w_p^t + w_c^t + w_t^t. \]

It’s obvious that
\[ \text{div} w_1 = 0, \quad \int_{T^2} w_1(t,x) dx = 0. \]

After the construction of \( w_1 \), we define the new velocity field \( v_1 \) as
\[ v_1 := v_0 + w_1. \]

5.2. Estimate of the perturbation. In this subsection, we establish some estimates for the velocity perturbation.

Firstly, we collect some estimates concerning the cutoffs function \( \chi_j(t,x) \).

**Lemma 5.1.** There exists a \( j_{\max} = j_{\max}(\hat{R}_0, \delta) \) such that
\[ \chi_j(t,x) = 0, \quad \text{for all } j > j_{\max}. \]

Moreover, for all \( 0 \leq j \leq j_{\max} \), there holds
\[ \rho_j \leq 4^j. \]

**Proof.** For \( j \geq 1 \),
\[ \chi_j(t,x) \neq 0 \Leftrightarrow 4^{j-1} \leq \frac{50\hat{R}_0(t,x)}{\varepsilon_0 \delta} \leq 4^{j+1}. \]

Thus, \( \chi_j(t,x) \neq 0 \) implies
\[ 4^{j-2} \leq \frac{50|\hat{R}_0|}{\varepsilon_0 \delta}. \]

Thus, there exists \( j_{\max} = j_{\max}(\hat{R}_0, \delta) \) such that
\[ \chi_j(t,x) = 0, \quad \text{for all } j > j_{\max}. \]

More precisely, we have
\[ 4^{j_{\max}} \leq \frac{800|\hat{R}_0|}{\varepsilon_0 \delta}. \tag{28} \]

**Lemma 5.2.** Let \( 0 \leq j \leq j_{\max} \). There holds
\[ \| \chi_j \|_{C_{L,\delta}^t} \leq C(\hat{R}_0, \delta, L), \]
where \( L \) is integer, and the constant \( C \) also depend on \( \varepsilon_0 \), but \( \varepsilon_0 \) is a universal constant and we omit it.
Proof. Direct computation gives that
\[
\partial_t(\langle A \rangle) = \frac{A \cdot \partial_t A}{\langle A \rangle}, \quad \partial_t(\langle A \rangle) = \frac{\partial_t A : \partial_t A + A : \partial_t A}{\langle A \rangle} - \frac{(A : \partial_t A)^2}{\langle A \rangle^2}
\]

hence
\[
|\partial_t(\langle A \rangle)| \leq |\partial_t A|.
\]

Since
\[
\partial_t(\chi_j(t,x)) = \dot{\chi}'\left(\frac{1}{4}\langle \hat{R}_0 \rangle \right) \frac{1}{4\varepsilon_0^\delta} \partial_t \langle \hat{R}_0 \rangle,
\]

thus
\[
|\partial_t(\chi_j(t,x))| \leq C(\varepsilon_0, \hat{R}_0, \delta), \quad |\partial_{ii}(\chi_j(t,x))| \leq C(\varepsilon_0, \hat{R}_0, \delta).
\]

By the inequality
\[
\|f \circ g\|_{C^L} \leq C_L(\|f\|_{C^L} \|g\|_{L^\infty} + \|f\|_{C^1} \|g\|_{C^L}),
\]

we know that
\[
\|\chi_j\|_{C_t^L} \leq C(\varepsilon_0, \hat{R}_0, \delta, L).
\]

Lemma 5.3 (Estimate on the amplitude). For \(0 \leq j \leq j_{\text{max}}\), we have
\[
\|a(\xi,j)\|_{L^\infty} \leq \sqrt{p_j} \leq 2^j \sqrt{\delta}, \quad \|a(\xi,j)\|_{C_t^L} \leq C(\hat{R}_0, \delta, L).
\]

Proof. Recall that
\[
a(\xi,j)(t,x) = \sqrt{p_j} \chi_j(t,x) \gamma_\xi \left(\text{Id} - \frac{\hat{R}_0}{\rho_j}\right).
\]

By Lemma 5.2, the estimate on \(\|a(\xi,j)\|_{L^\infty}\) and \(\|a(\xi,j)\|_{C_t^L}\) is obvious. In fact, noticing (28), we have
\[
\|a(\xi,j)\|_{C_t^L} = \sqrt{p_j} \|\chi_j \gamma_\xi \left(\text{Id} - \frac{\hat{R}_0}{\rho_j}\right)\|_{C_t^L} \leq \sqrt{p_j} \|\chi_j \gamma_\xi \|_{C_t^L} \|\left(\text{Id} - \frac{\hat{R}_0}{\rho_j}\right)\|_{C_t^L} \leq C(\hat{R}_0, \delta, L) \sqrt{\delta} 2^j \delta^{j_{\text{max}}} \leq C(\hat{R}_0, \delta, L).
\]

Proposition 4 (Estimate on the perturbation). For the velocity perturbation, we have the following bound: for every \(t \in [0, 1]\)
\(1. L^2\) estimate:
\[
\|w_1^p(t)\|_{L^2} \leq \frac{M \sqrt{\delta}}{16} + C(\hat{R}_0, \delta)(\lambda_1 \sigma_1)^{-\frac{1}{2}}, \quad \|w_1^c(t)\|_{L^2} \leq C(\hat{R}_0, \delta) \sigma_1 r_{1}, \quad \|w_1^l(t)\|_{L^2} \leq C(\hat{R}_0, \delta) r_1 \mu_1^{-1}.
\]
where $M$ is a universal constant.

2. $L^p$ estimate: for $p > 1$, there holds

$$
\|w_1^p(t)\|_{L^p} \leq C(\hat{R}_0, \delta, p)r_1^{1-\frac{2}{p}},
$$

$$
\|w_2^p(t)\|_{L^p} \leq C(\hat{R}_0, \delta, p)\sigma_1 r_1^{2-\frac{2}{p}},
$$

$$
\|w_3^p(t)\|_{L^p} \leq C(\hat{R}_0, \delta, p)\mu_1^{-1}\sigma_1 r_1^{3-\frac{2}{p}}.
$$

3. $W^{1,p}$ estimate: for $p > 1$, there holds

$$
\|w_1^p(t)\|_{W^{1,p}} \leq C(\hat{R}_0, \delta, p)\lambda_1 r_1^{1-\frac{2}{p}},
$$

$$
\|w_2^p(t)\|_{W^{1,p}} \leq C(\hat{R}_0, \delta, p)\lambda_1 \sigma_1 r_1^{2-\frac{2}{p}},
$$

$$
\|w_3^p(t)\|_{W^{1,p}} \leq C(\hat{R}_0, \delta, p)\mu_1^{-1}\lambda_1 \sigma_1 r_1^{3-\frac{2}{p}}.
$$

In particular, we have

$$
\|w_1^p(t)\|_{C_L} \leq C(\hat{R}_0, \delta, L)\lambda_1^L r_1,
$$

$$
\|w_2^p(t)\|_{C_L} \leq C(\hat{R}_0, \delta, L)\lambda_1^L \sigma_1 r_1^2,
$$

$$
\|w_3^p(t)\|_{C_L} \leq C(\hat{R}_0, \delta, L)\mu_1^{-1}(\lambda_1 \sigma_1 r_1)^L r_1^2.
$$

4. Time derivative estimate: for $p > 1$, there holds

$$
\|\partial_t w_1^p(t)\|_{L^p} \leq C(\hat{R}_0, \delta, p)\lambda_1 \mu_1 r_1^{2-\frac{2}{p}}.
$$

**Proof.** Step 1. $L^2$ estimate. Recall the definition (25) of $w_1^p$, using the support property (18) of cutoff function $\chi_i$, Proposition 3, Lemma 5.3 and Lemma 3.1, we have

$$
\frac{1}{2} \sum_{j} \sum_{\xi \in \Lambda_0^+} \int_{T^2} a_0^2(\xi, j)\eta_{\xi}^2 \left|W(\xi) + W(-\xi)\right|^2 dx
$$

$$
\leq 2 \sum_{j} \sum_{\xi \in \Lambda_0^+} \int_{T^2} a_0^2(\xi, j)\eta_{\xi}^2 dx
$$

$$
\leq 2 \sum_{j} \sum_{\xi \in \Lambda_0^+} \left(\|a(\xi, j)\|_{L^2}^2 \|\eta_{\xi}\|_{L^2}^2 + \frac{C}{\lambda_1 \sigma_1} \|a(\xi, j)\|_{C^1} \|\eta_{\xi}\|_{L^2}^2\right)
$$

$$
\leq 6 \sum_{j} \left(4^j \delta \int_{T^2} \chi_j^2(t, x) dx + \frac{C(\hat{R}_0, \delta)}{\lambda_1 \sigma_1}\right).
$$

Moreover, for any $t \in [0, 1]$, there hold

$$
\sum_{j \geq 1} 4^j \delta \int_{T^2} \chi_j^2(t, x) dx \leq \delta \sum_{j \geq 1} 4^j \int_{\left\{ x \in T^2 : 4^{j-1} \varepsilon_0 \delta \leq |\hat{R}_0(t, x)| \leq 4^{j+1} \varepsilon_0 \delta \right\}} dx
$$

$$
\leq \delta \sum_{j \geq 1} 4^j \int_{\left\{ x \in T^2 : 4^{j-1} \varepsilon_0 \delta \leq |\hat{R}_0(t, x)| \leq 4^{j+1} \varepsilon_0 \delta \right\}} \frac{\left|\hat{R}_0(t, x)\right|}{4^{j-1} \varepsilon_0 \delta} dx.
$$
Thus, we obtain

$$\leq 4\varepsilon_0^{-1} \sum_{j \geq 1} \int_{|x| \leq 4^{j+1} \delta \leq |\tilde{R}_0| \leq 4^{j+1} \delta} |\tilde{R}(t, x)| dx$$

$$\leq 4\varepsilon_0^{-1} \int_{T^2} |\tilde{R}_0(t, x)| dx \leq \frac{\delta}{500}. $$

Finally, due to the definition of $\rho_0$ and estimate (23), we deduce that

$$\frac{1}{2} \sum_{j \geq 1} \frac{\sum_{\xi \in \Lambda_{t(j)}}}{\sum_{\xi \in \Lambda_{t(j)}}} \int_{T^2} a_{(ξ, j)}^2 (\eta(ξ))^2 |W(ξ) + W(−ξ)|^2 dx \leq \frac{C(\tilde{R}_0, \delta)}{\lambda_1 \sigma_1}. \quad (29)$$

where $C_0$ is a universal constant.

Taking $M$ to be a universal constant, we obtain

$$\|w_1\|^2_{L^2(t)} \leq \frac{M\sqrt{\delta}}{16} + C(\tilde{R}_0, \delta) \frac{1}{\sqrt{\lambda_1 \sigma_1}}.$$

From the definition (26) of $w_1$, Proposition 3 and Lemma 5.3, and noticing the fact (28)(we use this fact frequently below), we deduce that

$$\|w_1(t)\|^2_{L^2} \leq \frac{1}{\lambda_1} \sum_{j \geq 1} \sum_{\xi \in \Lambda_{t(j)}} (\|a(\xi, j)\|_{C^1} \|\eta(\xi)\|_{L^2} + \|a(\xi, j)\|_{L^\infty} \|\nabla \eta(\xi)\|_{L^2})$$

$$\leq \frac{1}{\lambda_1} \sum_{j \geq 1} \sum_{\xi \in \Lambda_{t(j)}} (C(\tilde{R}_0, \delta) + 2^{j} \sqrt{\delta} \lambda_1 \sigma_1 r_1)$$

$$\leq C(\tilde{R}_0, \delta) \sigma_1 r_1.$$

By (27), Proposition 3 and Lemma 5.3, we obtain

$$\|w_1(t)\|^2_{L^2} \leq \frac{1}{\mu_1} \sum_{j \geq 1} \sum_{\xi \in \Lambda_{t(j)}} \|a_{(ξ, j)}^2 (\eta(ξ))^2\|_{L^2}$$

$$\leq \frac{1}{\mu_1} \sum_{j \geq 1} \sum_{\xi \in \Lambda_{t(j)}} \|a(\xi, j)\|_{L^\infty} \|\eta(\xi)\|_{L^4}^4$$

$$\leq \frac{1}{\mu_1} \sum_{j \geq 1} \sum_{\xi \in \Lambda_{t(j)}} \|a_{(ξ, j)}(t)\|_{L^p} \|\eta(\xi)\|_{L^p}^4$$

$$\leq C(\tilde{R}_0, \delta) \frac{r_1}{\mu_1}.$$

**Step 2.** $L^p$ estimate. By Lemma 3.1, proposition 3 and Lemma 5.3, we obtain

$$\|w_1(t)\|_{L^p} \leq C(\tilde{R}_0, \delta, p) \sum_{j \geq 1} \sum_{\xi \in \Lambda_{t(j)}} 2^j r_1^{1-\frac{2}{p}} \leq C(\tilde{R}_0, \delta, p) r_1^{1-\frac{2}{p}}.$$

For \( \|w_1^f(t)\|_{L^p} \) and \( \|w_1^f\|_{L^p} \), there hold
\[
\|w_1^f(t)\|_{L^p} \leq \frac{C}{\mu_1} \sum_j \sum_{\xi \in \Lambda_j} \left( \|a_{(\xi,j)}\|_{C_2} \|\eta_\xi\|_{L^p} + \|a_{(\xi,j)}\|_{L^\infty} \|\nabla \eta_\xi\|_{L^p} \right)
\leq C(\hat{R}_0, \delta, p) \sigma_1 r_1^{2 - \frac{2}{p}}
\]
and
\[
\|w_1^f(t)\|_{L^p} \leq \frac{C}{\mu_1} \sum_j \sum_{\xi \in \Lambda_j} \|a_{(\xi,j)}\|_{L^\infty}^2 \|\eta_\xi\|_{L^p}^2
\leq C(\hat{R}_0, \delta, p) r_1^{2 - \frac{2}{p} \mu_1^{-1}}.
\]

**Step 3.** \( W^{1,p} \) estimate. Recalling (25), a direct computation gives that
\[
\partial_t w_1^p = \frac{1}{\sqrt{2}} \sum_j \sum_{\xi \in \Lambda_j} \left( \partial_t (a_{(\xi,j)} \eta_\xi)(W_\xi + W_{-\xi}) + a_{(\xi,j)} \eta_\xi (\partial_t W_\xi + \partial_t W_{-\xi}) \right),
\]
thus by Lemma 3.1, Proposition 3 and Lemma 5.3, we have
\[
\|\nabla w_1^p\|_{L^p} \leq \sum_j \sum_{\xi \in \Lambda_j} \left( \|\nabla a_{(\xi,j)}\|_{L^p} \|\eta_\xi\|_{L^p} + \|a_{(\xi,j)}\|_{L^p} \|\nabla \eta_\xi\|_{L^p} \right)
+ C_p(\lambda_1 \sigma_1)^{-\frac{1}{p}} \|a_{(\xi,j)}\|_{C_2} \|\eta_\xi\|_{W^{1,p}}
+ \lambda_1 \sum_j \sum_{\xi \in \Lambda_j} \left( \|a_{(\xi,j)}\|_{L^p} \|\eta_\xi\|_{L^p} + C_p(\lambda_1 \sigma_1)^{-\frac{1}{p}} \|a_{(\xi,j)}\|_{C_2} \|\eta_\xi\|_{L^p} \right)
\leq C(\hat{R}_0, \delta, p) \lambda_1 r_1^{1 - \frac{2}{p}},
\]
and
\[
\|\nabla w_1^p\|_{L^\infty} \leq C(\hat{R}_0, \delta) \lambda_1 r_1.
\]
Recalling (26), there holds
\[
\partial_t w_1^f := -\frac{1}{\sqrt{2}} \sum_j \sum_{\xi \in \Lambda_j} \nabla \partial_t (a_{(\xi,j)} \eta_\xi) \frac{e^{i \lambda_1 \xi^+ \cdot x} + e^{-i \lambda_1 \xi^+ \cdot x}}{\lambda_1}
- \frac{1}{\sqrt{2}} \sum_j \sum_{\xi \in \Lambda_j} \nabla \partial_t (a_{(\xi,j)} \eta_\xi) \partial_t \left( \frac{e^{i \lambda_1 \xi^+ \cdot x} + e^{-i \lambda_1 \xi^+ \cdot x}}{\lambda_1} \right).
\]
Thus, by Lemma 3.1, Proposition 3 and Lemma 5.3, we get
\[
\|\nabla w_1^f\|_{L^p} \leq C(\hat{R}_0, \delta, p) \lambda_1 \sigma_1 r_1^{2 - \frac{2}{p}}.
\]
Recalling (27), we have
\[
\partial_t w_1^f := -\frac{1}{\mu_1} \sum_j \sum_{\xi \in \Lambda_{\xi,j}} P_t (\xi, \eta_\xi) \partial_t \left( a_{(\xi,j)} \eta_\xi^2 \xi \right).
\]
Thus, by Lemma 3.1, Proposition 3 and Lemma 5.3, we get
\[
\|\nabla w_1^f\|_{L^p} \leq C(\hat{R}_0, \delta, p) \mu_1^{-1} \lambda_1 \sigma_1 r_1^{3 - \frac{2}{p}}.
\]
The same argument gives that
\[
\|w_1^f(t)\|_{C_2} \leq C(\hat{R}_0, \delta) \lambda^L r_1.
\]
The estimate for \( \|w_1^1(t)\|_{C^1}\), \( \|w_1^1(t)\|_{C^2}\) is similar, and we omit the detail here.

**Step 4. Time derivative estimate.** A direct computation gives that
\[
\partial_t w_1^1 = \frac{1}{\sqrt{2}} \sum_j \sum_{\xi \in \Lambda_j} \partial_t (a_{(\xi,j)} \eta_{(\xi)})(W_{(\xi)} + W_{(-\xi)}),
\]
thus by Lemma 3.1, Proposition 3 and Lemma 5.3, we deduce that
\[
\|\partial_t w_1^1\|_{L^p} \leq \sum_j \sum_{\xi \in \Lambda_j} \left( \|\partial_t a_{(\xi,j)}\|_{L^p} \|\eta_{(\xi)}\|_{L^p} + \|a_{(\xi,j)}\|_{L^p} \|\partial_t \eta_{(\xi)}\|_{L^p} \right)
+ C_p(\lambda_1 \sigma_1)^{-\frac{1}{p}} \|a_{(\xi,j)}\|_{C^2_{\xi,j}} \|\partial_t \eta_{(\xi)}\|_{L^p}
\leq C(\tilde{R}_0, \delta, p) \lambda_1 \sigma_1 \mu_1 r_1^{2-\frac{2}{p}}.
\]

**Corollary 1.** For all \(1 < p < 2\), by taking \(\lambda_1\) large enough, we have
\[
\|w_1\|_{L^2} \leq \frac{M \sqrt{\delta}}{10}, \quad \|w_1\|_{L^p} \leq C(\tilde{R}_0, \delta) r_1^{1-\frac{2}{p}}.
\]

**Proof.** From the definition of \(w_1\), by the \(L^2\) estimate in Proposition 4, we deduce that
\[
\|w_1\|_{L^2} \leq \frac{M \sqrt{\delta}}{10} + C(\tilde{R}_0, \delta) \left( \lambda_1 \sigma_1^{-\frac{1}{2}} + \sigma_1 r_1 + r_1 \mu_1^{-1} \right).
\]
Using the relationship of parameter (24) and taking \(\lambda_1\) large enough, we obtain
\[
\|w_1\|_{L^2} \leq \frac{M \sqrt{\delta}}{10}.
\]

Similarly, we get
\[
\|w_1(t)\|_{L^p} \leq C(\tilde{R}_0, \delta, p) \left( r_1^{1-\frac{2}{p}} + \sigma_1 r_1^{2-\frac{2}{p}} + r_1^{2-\frac{2}{p}} \mu_1^{-1} \right).
\]
Then, the parameter relationship gives
\[
\|w_1(t)\|_{L^p} \leq C(\tilde{R}_0, \delta, p) r_1^{1-\frac{2}{p}}.
\]

**6. Construction and estimate on temperature perturbation.** After the construction of new velocity \(v_1\), we construct new temperature \(\theta_1\) as following. Consider the transport-diffusion equation:
\[
\begin{cases}
\partial_t \theta_1 + v_1 \cdot \nabla \theta_1 - \Delta \theta_1 = 0, \\
\theta_1|_{t=0} = \theta_0, \quad (30)
\end{cases}
\]
where \(\theta_0(x)\) is the function in Proposition 2.1. From the standard theory, we know that there exists a unique solution \(\theta_1 \in C^\infty([0, 1] \times T^2)\) and it obeys the following estimates:
\[
\|\theta_1\|_{L^\infty_{x,t}} \leq \|\theta_0\|_{L^\infty}, \quad \|\theta_1(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla \theta_1(s)\|_{L^2}^2 ds = \|\theta_0\|_{L^2}^2
\]
and
\[
\begin{cases}
\partial_t (\theta_1 - \theta_0) + v_1 \cdot \nabla (\theta_1 - \theta_0) + (v_1 - v_0) \cdot \nabla \theta_0 - \Delta (\theta_1 - \theta_0) = 0, \\
(\theta_1 - \theta_0)|_{t=0} = 0.
\end{cases} \quad (31)
\]
Direct energy estimate gives that
\[
\frac{d}{dt} \| \theta_1 - \theta_0 \|_{L^2}^2 + 2 \| \nabla (\theta_1 - \theta_0) \|_{L^2}^2 = 2 \int_{T^2} \theta_0 (v_1 - v_0) \cdot \nabla (\theta_1 - \theta_0) dx \\
\leq \| \nabla (\theta_1 - \theta_0) \|_{L^2}^2 + \| \theta_0 \|_\infty \| v_1 - v_0 \|_{L^2}^2.
\]
which implies
\[
\sup_{t \in [0,1]} \| \theta_1 - \theta_0 \|_{L^2}^2 (t) + \int_0^1 \| \nabla (\theta_1 - \theta_0) \|_{L^2}^2 (t) dt \leq \| \theta^0 \|_\infty M^2 \delta.
\]

7. Reynold stress: Construction and estimate.

7.1. Anti-divergence operator. We first recall the anti-divergence operator:

**Lemma 7.1.** There exists an operator $\mathcal{R}$ satisfying the following property:

- For any $v \in C^\infty (T^2; \mathbb{R}^2)$, $\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x \in T^2$ and $\text{div} \mathcal{R}v(x) = v(x) - \frac{1}{(2\pi)^2} \int_{T^2} v(x) dx$.
- The following estimates hold: $\| \nabla |\mathcal{R}| \|_{L^p \rightarrow L^p} \leq C_p$, $\| \mathcal{R} \|_{L^p \rightarrow L^p} \leq C_p$, $\| \mathcal{R} \|_{C^0 \rightarrow C^0} \leq C$.

**Proof.** Let $u \in C^\infty_0 (T^2)$ be a solution to
\[
\Delta u = v - \frac{1}{(2\pi)^2} \int_{T^2} v(x) dx,
\]
where $C^\infty_0 (T^2) = \{ f \in C^\infty (T^2) : \int f(x) dx = 0 \}$. Then set
\[
\mathcal{R}v(x) := \nabla u + (\nabla u)^T - (\text{div} u) \text{Id}.
\]
Then $\mathcal{R}$ satisfies the above property. \hfill $\Box$

7.2. Construction of new error $\hat{R}_1$. In this subsection, we define the new error $\hat{R}_1$. We first compute the interaction $w^p_1 \otimes w^p_1$ of principle perturbation. Recalling the definition (25) of $w^p_1$, we have
\[
w^p_1 \otimes w^p_1 = T_{self} + T_{inter},
\]
where
\[
T_{self} := \sum_j \sum_{\xi \in \Lambda^j_1} a_{i,j}^2 (\xi \otimes \xi) (\xi \otimes \xi - \frac{1}{2} \xi \otimes \xi (e^{2i\lambda_1 \xi^+ : x} + e^{-2i\lambda_1 \xi^+ : x})),
\]
\[
= \sum_j \sum_{\xi \in \Lambda^j_1} a_{i,j}^2 \xi \otimes \xi + \sum_j \sum_{\xi \in \Lambda^j_1} a_{i,j}^2 (\eta_{i,j}^2 - 1) \xi \otimes \xi
\]
\[
- \frac{1}{2} \sum_j \sum_{\xi \in \Lambda^j_1} a_{i,j}^2 \eta_{i,j}^2 \xi \otimes \xi (e^{2i\lambda_1 \xi^+ : x} + e^{-2i\lambda_1 \xi^+ : x}),
\]
and $T_{inter} = w^p_1 \otimes w^p_1 - T_{self}$. Recalling (20), we deduce
\[
\sum_j \sum_{\xi \in \Lambda^j_1} a_{i,j}^2 \xi \otimes \xi = \sum_j \sum_{\xi \in \Lambda^j_1} \rho_j \lambda_j^2 \gamma_j^2 (\text{Id} - \frac{\hat{R}_0}{\rho_j}) \xi \otimes \xi.
\]
However, by Geometric Lemma 4.1,
\[ \dot{R}_0 = \sum_j \lambda_j^2 \dot{R}_0 = \left( \sum_j \rho_j \lambda_j^2 \right) \text{Id} - \sum_j \rho_j \lambda_j^2 \left( \text{Id} - \frac{\dot{R}_0}{\rho_j} \right) \]
\[ = \left( \sum_j \rho_j \lambda_j^2 \right) \text{Id} - \sum_j \rho_j \lambda_j^2 \left[ \sum_{\xi \in \Lambda^+_j} \gamma_2 \left( \text{Id} - \frac{\dot{R}_0}{\rho_j} \right) \xi \otimes \xi \right]. \]

Thus, there holds
\[ \dot{R}_0 + \sum_j \sum_{\xi \in \Lambda^+_j} a^2_{\xi,j} \xi \otimes \xi = \left( \sum_j \rho_j \lambda_j^2 \right) \text{Id}. \]

Furthermore, by (15) and using the identity \( \text{div} f = P_{\neq 0} \text{div} f \), there hold
\[ \text{div} \left( \sum_j \sum_{\xi \in \Lambda^+_j} a^2_{\xi,j} \left( \eta^2_{\xi,j} - 1 \right) \xi \otimes \xi \right) \]
\[ = P_{\neq 0} \left( \sum_j \sum_{\xi \in \Lambda^+_j} \left( \eta^2_{\xi,j} - 1 \right) \xi \otimes \xi \nabla \left( a^2_{\xi,j} \right) \right) + P_{\neq 0} \left( \sum_j \sum_{\xi \in \Lambda^+_j} a^2_{\xi,j} \xi \otimes \xi \nabla \left( \eta^2_{\xi,j} \right) \right) \]
\[ = P_{\neq 0} \left( \sum_j \sum_{\xi \in \Lambda^+_j} \left( \eta^2_{\xi,j} - 1 \right) \xi \otimes \xi \nabla \left( a^2_{\xi,j} \right) \right) + P_{\neq 0} \left( \sum_j \sum_{\xi \in \Lambda^+_j} a^2_{\xi,j} \frac{\xi}{\mu_1} \partial_t \left( \eta^2_{\xi,j} \right) \right) \]
\[ = P_{\neq 0} \left( \sum_j \sum_{\xi \in \Lambda^+_j} \left( \eta^2_{\xi,j} - 1 \right) \xi \otimes \xi \nabla \left( a^2_{\xi,j} \right) \right) + P_{\neq 0} \left( \frac{1}{\mu_1} \sum_j \sum_{\xi \in \Lambda^+_j} \partial_t \left( a^2_{\xi,j} \eta^2_{\xi,j} \right) \right) \]
\[ - P_{\neq 0} \left( \frac{1}{\mu_1} \sum_j \sum_{\xi \in \Lambda^+_j} \partial_t \left( a^2_{\xi,j} \eta^2_{\xi,j} \right) \right) \]

and
\[ \text{div} \left( \frac{1}{2} \sum_j \sum_{\xi \in \Lambda^+_j} a^2_{\xi,j} \eta^2_{\xi,j} \xi \otimes \xi \left( e^{2i\lambda_1 \xi^+ \cdot x} + e^{-2i\lambda_1 \xi^+ \cdot x} \right) \right) \]
\[ = \frac{1}{2} \sum_j \sum_{\xi \in \Lambda^+_j} \xi \otimes \xi \nabla \left( a^2_{\xi,j} \eta^2_{\xi,j} \right) \left( e^{2i\lambda_1 \xi^+ \cdot x} + e^{-2i\lambda_1 \xi^+ \cdot x} \right). \]

Using (18), we have
\[ T_{\text{inter}} = \sum_{j \neq j'} \sum_{\xi \in \Lambda^+_j, \xi' \in \Lambda^+_j} a_{\xi,j} a_{\xi',j'} \eta_{\xi,j} \eta_{\xi',j'} \left( W_{\xi,j} + W_{\xi,j'} \right) \otimes \left( W_{\xi,j} + W_{\xi,j'} \right) \]
\[ + \sum_{j} \sum_{\xi, \xi' \in \Lambda^+_j, \xi \neq \xi'} a_{\xi,j} a_{\xi',j'} \eta_{\xi,j} \eta_{\xi',j'} \left( W_{\xi,j} + W_{\xi,j'} \right) \otimes \left( W_{\xi,j} + W_{\xi,j'} \right). \]

It follows from Proposition 2 that
\[ \text{div} \left( W_{\xi,j} \otimes W_{\xi,j'} \right) = 2 \nabla \tilde{P}_{\xi,j}. \]
Thus, we obtain

\[
\tilde{P}_{\xi', \xi} = 2 \cos(\lambda \xi^+ \cdot x) \cos(\lambda \xi'^+ \cdot x) + 2 \xi \cdot \xi' \sin(\lambda \xi^+ \cdot x) \sin(\lambda \xi'^+ \cdot x).
\]

Set

\[
P_{\xi', \xi} = 2 \cos(\lambda \xi^+ \cdot x) \cos(\lambda \xi'^+ \cdot x) + 2 \xi \cdot \xi' \sin(\lambda \xi^+ \cdot x) \sin(\lambda \xi'^+ \cdot x),
\]

then

\[
\text{div}(W_{\xi', \xi} \otimes W_{\xi', \xi'}) = 2 \nabla P_{\xi', \xi'}.
\]

Thus, we obtain

\[
\text{div} T_{\text{inter}} = \sum_{|j-j'|=1} \sum_{\xi \in \Lambda^+_{(j)}} \sum_{\xi' \in \Lambda^+_{(j')}} \left((W_{\xi}) + W_{(-\xi)} \otimes (W_{\xi'}) + W_{(-\xi')} \right) \nabla (a_{\xi, j} a_{\xi', j'}) \eta(\xi) \eta(\xi')
\]

\[
+ \nabla (a_{\xi, j} a_{\xi', j'} \eta(\xi) \eta(\xi') P_{\xi', \xi'}) - P_{\xi', \xi'} \nabla (a_{\xi, j} a_{\xi', j'} \eta(\xi) \eta(\xi'))
\]

\[
+ \nabla (a_{\xi, j} a_{\xi', j'} \eta(\xi) \eta(\xi') P_{\xi', \xi'}) - P_{\xi', \xi'} \nabla (a_{\xi, j} a_{\xi', j'} \eta(\xi) \eta(\xi'))
\]

Finally, by combining the definition (27) of \( w_1 \), we obtain

\[
\text{div} T_{\text{self}} + \text{div} T_{\text{inter}} + \text{div} \tilde{R}_0 + \partial_t w_1 = -\nabla (p_1 - p_0) + T_{1,\text{osc}},
\]

where we define the new pressure \( p_1 \) such that

\[
p_1 - p_0 = -\left( \sum_j \rho_j \lambda_j^2 + \frac{1}{\mu_1} \sum_j \sum_{\xi \in \Lambda^+_{(j)}} \Delta^{-1} \partial_t (a_{\xi, j} \eta(\xi) \eta(\xi'))
\]

\[
+ \sum_{|j-j'|=1} \sum_{\xi \in \Lambda^+_{(j)}} \sum_{\xi' \in \Lambda^+_{(j')}} a_{\xi, j} a_{\xi', j'} \eta(\xi) \eta(\xi') P_{\xi', \xi'}
\]

\[
+ \sum_j \sum_{\xi, \xi' \in \Lambda^+_{(j)}, \xi \neq \xi'} a_{\xi, j} a_{\xi', j} \eta(\xi) \eta(\xi') P_{\xi', \xi'},
\]

and oscillatory term

\[
T_{1,\text{osc}} := P \left( \sum_j \sum_{\xi \in \Lambda^+_{(j)}} \eta^2(\eta) - 1 \right) \xi \otimes \xi \nabla (a_{\xi, j}^2)
\]

\[
- P \left( \frac{1}{\mu_1} \sum_j \sum_{\xi \in \Lambda^+_{(j)}} \partial_t (a_{\xi, j}^2) \eta(\eta) \xi \right)
\]

\[
- \frac{1}{2} \sum_j \sum_{\xi \in \Lambda^+_{(j)}} \xi \otimes \xi \nabla (a_{\xi, j}^2) \eta^2(\eta) (e^{2i \lambda_1 \xi^+} x + e^{-2i \lambda_1 \xi^+} x)
\]
Since computation gives that

Thus, the new function \((v_1)_{1,\text{osc}}\) satisfies Boussinesq-Reynold equation (2).

Next, we prove that the error \(\|R_{1}\|_{L^\infty L^p_t}\) is very small.

7.3. Estimate on \(R_{1}\). In this subsection, we estimate \(R_{1}\). We deal with it term by term.

7.3.1. Estimate on linear term \(R_{\text{linear}}\): For \(1 < p < 2\),

\[
\|R(\partial_t (w^1_t + w^0_t))\|_{L^p_t} \leq C(\tilde{R}_0, \delta, p)\sigma_1\mu_1 r_1^{2-s}.
\]
\[ |\mathcal{R}((-\Delta)^a w_1)| \leq C(\tilde{R}_0, \delta, p)\lambda_1^{2a-1} r_1^{1-\frac{2}{p}}, \]
\[ |\mathcal{R}(\text{div}(v_0 \otimes w_1 + w_1 \otimes v_0))| \leq C(\tilde{R}_0, v_0, \delta, p)r_1^{1-\frac{2}{p}}. \]

**Proof.** Due to the $L^p$ estimate, $W^{1,p}$ estimate and time derivative estimate in Proposition 4, we deduce
\[
|\mathcal{R}(\partial_t (w_1^p + w_1^2))|_{L^p} = \frac{1}{\lambda_1^2} |\mathcal{R}\nabla^\perp (\partial_t w_1^p)|_{L^p} \leq C |\partial_t w_1^p|_{L^p} \leq C(\tilde{R}_0, \delta, p)\sigma_1 \mu_1 r_1^{2-\frac{2}{p}},
\]
\[
|\mathcal{R}((-\Delta)^a w_1)|_{L^p} \leq C|\mathcal{R}\nabla [2a (w_1)]|_{L^p} \leq C(\tilde{R}_0, \delta, p)\lambda_1^{2a-1} r_1^{1-\frac{2}{p}},
\]
\[
|\mathcal{R}(\text{div}(v_0 \otimes w_1 + w_1 \otimes v_0))|_{L^p} \leq C(p)|v_0 \otimes w_1 + w_1 \otimes v_0|_{L^p} \leq C(\tilde{R}_0, \delta, p)r_1^{1-\frac{2}{p}}.
\]

\[
7.3.2. \text{Estimate on corrector term } R_{\text{cor}}: \text{ For } 1 < p < 2,
\]
\[
|\mathcal{R}(\text{div}(w_0^p \otimes (w_1^2 + w_1^1) + (w_0^2 + w_0^1) \otimes w_1))|_{L^p} \leq C(\tilde{R}_0, \delta, p)(\sigma_1 + \mu_1^{-1}) \lambda_1^{3(1-\frac{2}{p})}.
\]

**Proof.** By Proposition 4 and parameter relationship (24), we get
\[
|\mathcal{R}(\text{div}(w_0^p \otimes (w_1^2 + w_1^1) + (w_0^2 + w_0^1) \otimes w_1))|_{L^p} \leq C_p \|w_0^p \otimes (w_1^2 + w_1^1) + (w_0^2 + w_0^1) \otimes w_1\|_{L^p} \leq C_p \left( \|w_0^p \otimes (w_1^2 + w_1^1)\|_{L^p}^{\frac{1}{2}} \|w_0^p \otimes (w_1^2 + w_1^1)\|_{L^\infty}^{1-\frac{1}{2}} + \|w_0^2 + w_0^1\|_{L^1} \|w_1^2 + w_1^1\|_{L^\infty}^{1-\frac{1}{2}} \right) \leq C(\tilde{R}_0, \delta, p)(\sigma_1 + \mu_1^{-1}) \lambda_1^{3(1-\frac{2}{p})}.
\]

\[
7.3.3. \text{Estimate on temperature term } R_{\text{tem}}: \text{ For } 1 < p < 2,
\]
\[
|\mathcal{R}((\theta_1 - \theta_0)v_2)|_{L^p} \leq C(\theta_0, \tilde{R}_0, \delta, p)\sigma_1 r_1^{1-\frac{2}{p}}.
\]

**Proof.** For $1 < p < 2$, we have
\[
|\mathcal{R}((\theta_1 - \theta_0)v_2)|_{L^p} \leq C_p \|\theta_1 - \theta_0\|_{L^p}.
\]
From the equation (31), we try $L^p$ estimates: Multiplying $|\theta_1 - \theta_0|^{p-2}(\theta_1 - \theta_0)$, we arrive at
\[
\frac{1}{p} \frac{d}{dt} \|\theta_1 - \theta_0\|^p_p + \int_{T^2} \nabla(\theta_1 - \theta_0) \cdot \nabla(|\theta_1 - \theta_0|^{p-2}(\theta_1 - \theta_0)) dx \leq \|\theta_1 - \theta_0\|_{L^p} \|\nabla \theta_0\|_{L^\infty} \|\theta_1 - \theta_0\|_{L^p}^{p-1}.
\]
A direct computation gives
\[
\int_{T^2} \nabla(\theta_1 - \theta_0) \cdot \nabla(|\theta_1 - \theta_0|^{p-2}(\theta_1 - \theta_0)) dx = (p-1) \int_{T^2} |\nabla(\theta_1 - \theta_0)|^2 |\theta_1 - \theta_0|^{p-2}.
\]
Thus, we obtain
\[
\|\theta_1 - \theta_0\|_p(t) \leq \int_0^t \|\nabla \theta_0\|_\infty(s) \|v_1 - v_0\|_p(s) ds
\]
\[
\leq \int_0^1 \|\nabla \theta_0\|_\infty(s) \|v_1 - v_0\|_p(s) ds,
\]
then by Corollary 1
\[
\sup_{t \in [0,1]} \|\theta_1 - \theta_0\|_p \leq \sup_{t \in [0,1]} \|\nabla \theta_0\|_{L^p} \sup_{t \in [0,1]} \|v_1 - v_0\|_p \leq C(\theta_0, R_0, \delta, p) r_1^{1-\frac{3}{p}}.
\]

7.3.4. *Estimate on the oscillatory term \( R_{osc} \):* For every \( 1 < p < 2 \),
\[
\|R_{osc}\|_{L^p} \leq C(R_0, \delta, p)(\lambda_1 \sigma_1)^{-1} r_1^{2-\frac{2}{p}}.
\]

**Proof.** Recall (34). By Lemma 3.3 and Proposition 3
\[
\left\| R P_{\neq 0} \left( \sum_{j} \sum_{\xi \in \Lambda_j^+} (\eta_{\xi j}^2 - 1) \xi \otimes \xi \nabla (a_{\xi j}^2) \right) \right\|_{L^p}
\]
\[
\leq \sum_{j} \sum_{\xi \in \Lambda_j^+} \left\| \nabla^{-1} P_{\neq 0} \left( (\eta_{\xi j}^2 - 1) \xi \otimes \xi \nabla (a_{\xi j}^2) \right) \right\|_{L^p}
\]
\[
\leq C(R_0, \delta, p) \left( 1 + \|a_{\xi j}^2\|_{C^3} \right) \left( \|\eta_{\xi j}^2 - 1\|_{L^p} \right) \leq C(R_0, \delta, p) \frac{1}{\lambda_1 \sigma_1}.
\]

Similarly, we have
\[
\left\| R \left( \sum_{j} \sum_{\xi \in \Lambda_j^+} (a_{\xi j}^2 \eta_{\xi}) \nabla (a_{\xi j}^2 \eta_{\xi}) (e^{2i\lambda_1 \xi^+} + e^{-2i\lambda_1 \xi^+}) \right) \right\|_{L^p}
\]
\[
\leq \|\nabla^{-1} (\sum_{j} \sum_{\xi \in \Lambda_j^+} \xi \otimes \xi \nabla (a_{\xi j}^2 \eta_{\xi}) \nabla (a_{\xi j}^2 \eta_{\xi}) (e^{2i\lambda_1 \xi^+} + e^{-2i\lambda_1 \xi^+})) \|_{L^p}
\]
\[
+ \|\nabla^{-1} (\sum_{j} \sum_{\xi \in \Lambda_j^+} \xi \otimes \xi a_{\xi j}^2 \nabla (\eta_{\xi}) (e^{2i\lambda_1 \xi^+} + e^{-2i\lambda_1 \xi^+})) \|_{L^p}
\]
\[
\leq C(R_0, \delta, p) \left( 1 + \|a_{\xi j}^2\|_{C^3} \right) \left( \frac{\|\eta_{\xi j}^2\|_{L^p}}{\lambda_1} \right) + C(R_0, \delta, p) \left( 1 + \|a_{\xi j}^2\|_{C^3} \right) \frac{\|\nabla (\eta_{\xi})\|_{L^p}}{\lambda_1}
\]
\[
\leq C(R_0, \delta, p) \sigma_1 r_1^{3-\frac{2}{p}}.
\]

Recalling Remark 4 and (32), we have
\[
\left\| R P_{\neq 0} \left( \sum_{j} \sum_{\xi \in \Lambda_j^+} P_{\xi, \xi'} \nabla (a_{\xi j}^2 a_{\xi'} (\xi') \eta_{\xi} \eta_{\xi'}) \right) \right\|_{L^p}
\]
\[
\leq C(R_0, \delta, p) \sigma_1 r_1^{3-\frac{2}{p}},
\]
\[
\left\| R P_{\neq 0} \left( \sum_{j} \sum_{\xi \in \Lambda_j^+} (W(\xi) + W(-\xi)) \otimes (W(\xi') + W(-\xi')) \right) \right\|_{L^p}
\]
\[
\leq C(R_0, \delta, p) \sigma_1 r_1^{3-\frac{2}{p}}.
\]
\[
+ W(-\varepsilon') \nabla \left( a(\xi, j) a(\xi', j') \eta(\xi') \right) \right) \bigg\|_{L^p} \leq C(R_0, \delta, p) \sigma_1 r_1^{3 - \frac{2}{p}}.
\]

Similarly,
\[
\left\| \mathcal{R}P \left( \sum_j \sum_{\xi, \xi' \in \Lambda_j^0} P(\xi, \xi') \nabla \left( a(\xi, j) a(\xi', j') \eta(\xi) \eta(\xi') \right) \right) \right\|_{L^p} \leq C(R_0, \delta, p) \sigma_1 r_1^{3 - \frac{2}{p}}
\]
\[
\left\| \mathcal{R}P \left( \sum_j \sum_{\xi, \xi' \in \Lambda_j^0, \xi \neq \xi'} (W(\xi) + W(-\varepsilon)) \otimes (W(\xi')
\]
\[
+ W(-\varepsilon') \nabla \left( a(\xi, j) a(\xi', j') \eta(\xi) \eta(\xi') \right) \right) \bigg\|_{L^p} \leq C(R_0, \delta, p) \sigma_1 r_1^{3 - \frac{2}{p}}.
\]

By Proposition 3 and Lemma 5.3, there hold
\[
\frac{1}{\mu_1} \left\| \mathcal{R}P \left( \sum_j \sum_{\xi \in \Lambda_j^0} \partial_t \left( a(\xi, j) \eta^2(\xi) \right) \right) \right\|_{L^p} \leq \frac{C_p}{\mu_1} \sum_j \sum_{\xi \in \Lambda_j^0} \| \partial_t \left( a(\xi, j) \right) \|_{L^\infty} \| \eta^2(\xi) \|_{L^p} \leq C(R_0, \delta, p) r_1^{2 - \frac{2}{p}} \mu_1^{-1}.
\]

Summing the parts and using the parameter relationship (24), we complete the proof. \hfill \square

Finally, collecting these term together and noticing that \( \alpha < 1 \), we obtain the estimate on the error term \( \tilde{R}_1 \):
\[
\| \tilde{R}_1 \|_{L^p} \leq C(R_0, \delta, v_0, \theta_0, p) \left( (\lambda_1 \sigma_1)^{-1} r_1^{2 - \frac{2}{p}} + r_1^{1 - \frac{2}{p}} + \sigma_1 r_1^{2 - \frac{2}{p}} \right.
\]
\[
+ \sigma_1 r_1^{3 - \frac{2}{p}} + \mu_1^{-1} r_1^{3\left(1 - \frac{1}{p}\right)} + \lambda_1^{2\alpha - 1} r_1^{1 - \frac{2}{p}} \right).
\]

Using the parameter relationship (24), taking \( 1 < p < \frac{2\alpha}{2\alpha - 1} \) and noticing \( \frac{1}{2} \leq \alpha < 1 \), we obtain
\[
\| \tilde{R}_1 \|_{L^p} \leq C(R_0, \delta, v_0, \theta_0, p) \lambda_1^{3\alpha - 1 - \frac{2\alpha}{p}}.
\]

**In summary:** we have constructed smooth function \((v_1, p_1, \theta_1, R_1) \in C^\infty([0, 1] \times T^2, R^2 \times R \times R \times S^{2\alpha - 2})\), they satisfies Boussinesq-Reynold equation (2). By taking \( \lambda_1 \) large enough, the following estimate hold: for any \( t \in [0, 1] \)
\[
\| v_1 - v_0 \|_{L^2(t)} \leq \frac{M \sqrt{\delta}}{10}, \quad \| \theta_1 \|_{L^\infty(t)} \leq \| \theta_0 \|_{L^\infty},
\]
\[
\| \theta_1 \|_{L^2(t)} + 2 \int_0^t \| \nabla \theta_1 \|_{L^2(s)}^2 \| \theta_0 \|_{L^2(t)},
\]
\[
\| \theta_1 - \theta_0 \|_{L^2(t)} + \int_0^t \| \nabla (\theta_1 - \theta_0) \|_{L^2(s)}^2 \| \theta_0 \|_{L^\infty} (\delta, \mu),
\]
\[
\| \tilde{R}_1 \|_{L^\infty L^1(t)} \leq C(R_0, \delta, v_0, \theta_0, p) \lambda_1^{3\alpha - 1 - \frac{2\alpha}{p}}.
\]
8. Proof of main proposition. In this section, we give a proof of Proposition 2.1 by combining the above construction and estimate.

Noticing that $3\alpha - 1 - \frac{2\alpha}{p} < 1$ for $1 < p < \frac{2\alpha}{\alpha - 1}$, thus we first take $\lambda_1$ to be an integer, large enough such that

$$C(\tilde{R}_0, \delta, v_0, \theta_0, p)\lambda_1^{3\alpha - 1 - \frac{2\alpha}{p}} \leq \frac{\varepsilon_0\delta}{6000}.$$

Thus, there holds

$$\|\tilde{R}_1\|_{L^p} \leq \frac{\varepsilon_0\delta}{20000}.$$

To complete the proof of Proposition 2.1, we only need to estimate the energy difference between $e(t)$ and $\int_{T^2} |v_1(t, x)|^2dx$.

Direct computation gives that

$$\int_{T^2} |v_1(t, x)|^2dx = \int_{T^2} |v_0(t, x) + w_1(t, x)|^2dx
\quad = \int_{T^2} \left(|v_0(t, x)|^2 + |w_1(t, x)|^2\right)dx + 2\int_{T^2} v_0(t, x) \cdot w_1(t, x)dx.$$

From the definition of $w_1$ we deduce

$$\int_{T^2} |w_1^p(t, x)|^2dx = \frac{1}{2}\sum_{\xi \in \Lambda_0^+} \sum_{j=1}^{N_0} a_{\xi,j}^2 |W_\xi + W_{(-\xi)}|^2 dx
\quad + \frac{1}{2} \sum_{|j-j'|=1} \sum_{\xi \in \Lambda_0^+} a_{\xi,j}^2 a_{\xi',j'}^2 (W_{\xi} + W_{(-\xi)}) \cdot (W_{\xi'} + W_{(-\xi')}) dx
\quad + \frac{1}{2} \sum_{\xi, \xi' \in \Lambda_0^+, \xi \neq \xi'} a_{\xi,j} a_{\xi',j'} (W_{\xi} + W_{(-\xi)}) \cdot (W_{\xi'} + W_{(-\xi')}) dx.$$

Recalling (25) and (18), it’s easy to deduce that

$$I = \sum_{\xi \in \Lambda_0^+} \int_{T^2} \rho_0 \lambda_0^2 \gamma_\xi^2 \left(\text{Id} - \frac{\tilde{R}_0}{\rho_0}\right) dx + \sum_{\xi \in \Lambda_0^+} \int_{T^2} \rho_0 \lambda_0^2 \gamma_\xi^2 \left(\text{Id} - \frac{\tilde{R}_0}{\rho_0}\right) (1 - \varepsilon_0) dx
\quad + \frac{1}{2} \sum_{\xi \in \Lambda_0^+} \int_{T^2} \rho_0 \lambda_0^2 \gamma_\xi^2 (1 - \frac{\tilde{R}_0}{\rho_0}) (1 - \varepsilon_0) dx
\quad + \frac{1}{2} \sum_{\xi \in \Lambda_0^+} \sum_{j \geq 1} a_{\xi,j}^2 (W_{\xi} + W_{(-\xi)})^2 dx.$$

Recalling (14), there holds

$$\sum_{\xi \in \Lambda_0^+} \int_{T^2} \rho_0 \lambda_0^2 \gamma_\xi^2 \left(\text{Id} - \frac{\tilde{R}_0}{\rho_0}\right) dx = 2\rho_0 \int_{T^2} \chi_0^2 dx.$$
Set
\[ E(t)_{\text{err}} := 2 \int_{T^2} v_0(t, x) \cdot w_1 dx + \int_{T^2} \left( 2w_1^p \cdot (w_1^c + w_1^i) + |w_1^c + w_1^i|^2 \right) dx + II \]

\[ + \sum_{\xi \in \Lambda_0^+} \int_{T^2} \rho_0 \rho_0^2 \gamma_1^2 \left( \text{Id} - \frac{\hat{R}_0}{\rho_0} \right) (\eta_0^2 - 1) dx \]

\[ + \frac{1}{2} \sum_{\xi \in \Lambda_0^+} \int_{T^2} \rho_0 \rho_0^2 \gamma_1^2 \left( \text{Id} - \frac{\hat{R}_0}{\rho_0} \right) \eta_0^2 \left( -e^{2\lambda_1 \xi^+ x} - e^{-2\lambda_1 \xi^+ x} \right) dx \]

\[ + \frac{1}{2} \int_{T^2} \sum_{j \geq 1} \sum_{\xi \in \Lambda_0^+} \rho_0^2 \gamma_1^2 \eta_0^2 W_{(\xi)} + W_{(-\xi)}^2 dx \]

Thus, combining the definition \((22)\) of \(\rho_0\), we obtain
\[ \int_{T^2} |v_1(t, x)|^2 dx = \int_{T^2} |v_0(t, x)|^2 dx + 2 \int_{T^2} \rho_0^2 \chi_{\delta} dx + E(t)_{\text{err}} \]
\[ = e(t) \left( 1 - \frac{\delta}{2} \right) + E(t)_{\text{err}}. \]

Next, we will show that by choosing the parameter \(\lambda_1\) sufficiently large, there holds
\[ |E(t)_{\text{err}}| \leq \frac{\delta e(t)}{8}, \quad \forall t \in [0, 1], \]

thus, we obtain
\[ |e(t) \left( 1 - \frac{\delta}{2} \right) - \int_{T^2} |v_1(t, x)|^2 dx| \leq \frac{\delta e(t)}{8}, \quad \forall t \in [0, 1], \]

which gives \((6)\).

**Estimate on** \(E(t)_{\text{err}}\): We estimate \(E(t)_{\text{err}}\) term by term.

**Estimate on** \((1)\): By Lemma 3.1 and the \(L^p\) estimate in Proposition 4, we deduce that for every \(t \in [0, 1]\) and \(p > 1\)
\[ |(1)| \leq C(v_0)\|w_1(t)\|_{L^1} \leq C(\hat{R}_0, v_0, \delta, p) r_1^{1 - \frac{2}{p}}. \]

**Estimate on** \((2)\): By Proposition 4, it’s direct to get
\[ \int_{T^2} |w_1^c + w_1^i|^2 dx \leq C(\hat{R}_0, \delta) \left( (r_1 \sigma_1)^2 + (r_1 \mu_1^{-1})^2 \right), \]
\[ \left| \int_{T^2} w_1^p \cdot (w_1^c + w_1^i) dx \right| \leq C(\hat{R}_0, \delta) (r_1 \sigma_1 + r_1 \mu_1^{-1}). \]

Then, by Lemma 3.2 and Proposition 4, we deduce that for any \(t \in [0, 1]\) and \(p > 1\)
\[ |II| \leq C(\hat{R}_0, \delta, p) \lambda_1^{-1} \|w_1\|_{L^2}^2 \leq C(\hat{R}_0, \delta, p) \lambda_1^{-1} r_1^{2(1 - \frac{1}{p})}. \]

Summing these term, we obtain
\[ |(2)| \leq C(\hat{R}_0, \delta, p) \left( r_1 \sigma_1 + r_1 \mu_1^{-1} + \lambda_1^{-1} r_1^{2(1 - \frac{1}{p})} \right). \]
Estimate on (3): Notice that
\[ \int_{T^2} (\eta_\xi^2 - 1) = 0, \]
thus,
\[ P \geq \frac{\lambda_1}{\| \eta_\xi \|} (\eta_{\xi}^2 - 1) = (\eta_{\xi}^2 - 1). \]

Lemma 3.2, Lemma 5.1 and Lemma 5.2 implies
\[ |(3)| \leq C(\tilde{R}_0, \delta) \frac{1}{\lambda_1 \sigma_1}. \]

Estimate on (4): Notice the fact
\[ P \geq \frac{\lambda_1}{\| \eta_\xi \|} \left( -e^{2i\lambda_1 \xi^+ \cdot x} - e^{-2i\lambda_1 \xi^+ \cdot x} \right) = \eta_\xi^2 \left( -e^{2i\lambda_1 \xi^+ \cdot x} - e^{-2i\lambda_1 \xi^+ \cdot x} \right). \]
Thus, Lemma 3.2, Lemma 5.1 and Lemma 5.2 gives
\[ |(4)| \leq C(\tilde{R}_0, \delta) \frac{1}{\lambda_1}. \]

Estimate on (5): Recalling (29), we obtain
\[ |(5)| \leq \frac{\delta}{20} + C(\tilde{R}_0, \delta) \frac{1}{\lambda_1 \sigma_1} \leq \frac{\delta e(t)}{20} + C(\tilde{R}_0, \delta) \frac{1}{\lambda_1 \sigma_1}. \]

Finally, collecting estimate (1)-(5), noticing the parameter relationship (24), taking \( p \) sufficiently close to 1 and parameter \( \lambda_1 \) sufficiently large, we arrive at
\[ E_{err}(t) \leq \frac{\delta e(t)}{8}. \]

This completes the proof.

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