AN ALGEBRAIC SOLUTION OF EINSTEIN’S FIELD EQUATIONS IN $X_4$

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Abstract. The main goal in the present paper is to obtain a particular solution $g_{\lambda \mu}$, $\Gamma^\nu_{\lambda \mu}$ and an algebraic solution $\bar{g}_{\lambda \mu}$, $\bar{\Gamma}^\nu_{\lambda \mu}$ by means of $g_{\lambda \mu}$, $\Gamma^\nu_{\lambda \mu}$ in UFT $X_4$.

1. Introduction

Einstein([1]) proposed a new unified field theory that would include both gravitation and electromagnetism. Hlavatý([6]) gave the mathematical foundation of the Einstein’s unified field theory in a 4-dimensional generalized Riemannian space $X_4$ (i.e., space-time) for the first time. Since then this theory had been generalized in a generalized Riemannian manifold $X_n$, the so-called Einstein’s n-dimensional unified field theory (UFT $X_n$), and many consequences of this theory has been obtained by a number of mathematicians. However, it has been unable yet to represent a general n-dimensional Einstein’s connection in a surveyable tensorial form, probably due to the complexity of the higher dimensions. The purpose of the present paper is to obtain a particular solution $g_{\lambda \mu}$, $\Gamma^\nu_{\lambda \mu}$ of Einstein’s field equation in UFT $X_4$. In the next, we shall obtain an algebraic solution $\bar{g}_{\lambda \mu}$, $\bar{\Gamma}^\nu_{\lambda \mu}$ by means of $g_{\lambda \mu}$, $\Gamma^\nu_{\lambda \mu}$ in UFT $X_4$.

2. Preliminary

Let $X_n$ be an n-dimensional generalized Riemannian manifold covered by a system of real coordinate neighborhoods $\{U; x^\nu\}$, where, here and in the sequel, Greek indices run over the range $\{1, 2, \cdots , n\}$ and follow

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the summation convention. The algebraic structure on $X_n$ is imposed by a basic real non-symmetric tensor $g_{\lambda\mu}$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},$$

where we assume that

(2.2) \((a)\) $\det((g_{\lambda\mu})) < 0$, \( (b)\) $\det((h_{\lambda\mu})) < 0$, \( (c)\) $\det((k_{\lambda\mu})) \geq 0$.\n
Since $\det((h_{\lambda\mu})) \neq 0$, we may define a unique tensor $h^{\lambda\nu} (= h^{\nu\lambda})$ by

$$h^{\lambda\mu}h_{\lambda\nu} = \delta_{\nu}^{\mu}.$$ \hspace{8cm} (2.3)\n
We use the tensors $h^{\lambda\nu}$ and $h^{\lambda\mu}$ as tensors for raising and/or lowering indices for all tensors defined on $X_n$ in the usual manner. The manifold $X_n$ is assumed to be connected by a general real connection $\Gamma_{\lambda\nu}^{\mu}$ which may also be split into its symmetric part $\Lambda_{\lambda\nu}^{\mu}$ and skew-symmetric part $S_{\lambda\mu}^{\nu}$, called the torsion tensor of $\Gamma_{\lambda\nu}^{\mu}$.

The Einstein’s $n$-dimensional unified field theory in $X_n(UFT_{X_n})$ is governed by the following set of equations:

$$\partial_{\nu}g_{\lambda\mu} - g_{\alpha\mu}\Gamma_{\lambda\omega}^{\alpha} - g_{\lambda\alpha}\Gamma_{\omega\mu}^{\alpha} = 0 \quad (\partial_{\nu} = \frac{\partial}{\partial x^{\nu}}),$$

and

(2.5) \((a)\) $S_{\lambda} = S_{\lambda\alpha}^{\alpha} = 0$, \( (b)\) $R_{\lambda\mu\nu} = \partial_{\nu}P_{\lambda\mu} - \partial_{\nu}P_{\lambda_{\mu}}, \quad (c)\) $R_{\lambda\mu} = 0$,

where $P_{\mu}$ is an arbitrary vector, called the Einstein’s vector, and $R_{\lambda\mu}$ is the contracted curvature tensor $R_{\lambda\mu\alpha}^{\alpha}$ of the curvature tensor $R_{\lambda\mu\nu}^{\alpha}$:

$$R_{\lambda\mu\nu}^{\alpha} = \partial_{\mu}\Gamma_{\lambda\nu}^{\alpha} - \partial_{\nu}\Gamma_{\lambda\mu}^{\alpha} + \Gamma_{\lambda\nu}^{\alpha}\Gamma_{\omega\mu}^{\alpha} - \Gamma_{\lambda\mu}^{\alpha}\Gamma_{\omega\mu}^{\alpha}.$$ \hspace{8cm} (2.6)\n
The equation (2.4) is called the Einstein’s equation, and the solution $\Gamma_{\lambda\mu}^{\nu}$ of the Einstein’s equation is called an Einstein’s connection. And the vector $S_{\lambda}$, defined by (2.5)(a), is the called the torsion vector.

The following two theorems were proved by Lee([3]).

**Theorem 2.1.** In $UFT_{X_n}$, if the system (2.4) admits a solution $\Gamma_{\lambda\mu}^{\nu}$ such that its torsion tensor is, for some nonzero vector $Y_{\lambda}$,

$$S_{\lambda\mu}^{\nu} = \frac{2}{n-1}\delta_{\nu}^{\nu}k_{\lambda\mu}^{\nu} + k_{\lambda\mu}^{\nu},$$

then it must be of the form

$$\Gamma_{\lambda\mu}^{\nu} = \left\{\lambda^{\nu}_{\mu}\right\} + \frac{2(2 - n)}{n - 1}k_{\lambda\mu}^{\nu}k_{\mu\lambda}^{\nu} + \frac{2}{n - 1}\delta_{\nu}^{\nu}k_{\lambda\mu}^{\nu} + k_{\lambda\mu}^{\nu},$$

where $\left\{\lambda^{\nu}_{\mu}\right\}$ are the Christoffel symbols defined by $h_{\lambda\mu}$.
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**Theorem 2.2.** In UFT $X_n$, the connection (2.8) is an Einstein’s connection if and only if the vector $Y_\lambda$ defining (2.8) satisfies the following condition

$$\nabla_\nu k_\lambda^\mu = \frac{2}{n-1} h_{\nu(\lambda} k_{\mu)\alpha} Y^\alpha - 2 k_{\nu|\lambda} Y_\mu + \frac{2(n-2)}{n-1} k_{\nu|\lambda} k_{\mu)\alpha} Y^\alpha,$$

where $\nabla_\omega$ is the symbolic vector of the covariant derivative with respect to $\{\lambda^\nu_\mu\}$.

3. A particular solution of field equations in UFT $X_4$

In this section we shall display a particular solution of (2.4) and (2.5) in UFT $X_4$. Let a tensor $g_{\lambda\mu}$ be given by the following matrix:

$$\begin{pmatrix} 1 & 0 & -e^t & e^t \\ 0 & 1 & 0 & 0 \\ e^t & 0 & 1 & 0 \\ -e^t & 0 & 0 & -1 \end{pmatrix},$$

where $t = x^3 - x^4$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$ given by the following matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -e^t & e^t \\ 0 & 0 & 0 & 0 \\ e^t & 0 & 0 & 0 \\ -e^t & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$\begin{align*} (a) \ det((g_{\lambda\mu})) &= -1, & (b) \ det((h_{\lambda\mu})) &= -1, & (c) \ det((k_{\lambda\mu})) &= 0, \end{align*}$$

we can choose the tensor $g_{\lambda\mu}$ given by (3.1) as a basic tensor in UFT $X_4$, by the assumption (2.2). On the other hand, in virtue of (3.2), all the Christoffel symbols $\{\lambda^\nu_\mu\}$ vanish. Hence the components of the first covariant derivatives with respect to $\{\lambda^\nu_\mu\}$ are ordinary derivatives, and $H^\nu_{\lambda\mu} = 0$. Define two vectors by

$$\begin{align*} (a) \ A_\lambda : (0, 0, 1, -1), & \quad (b) \ B_\lambda : (e^t, 0, 0, 0). \end{align*}$$
Then the skew-symmetric part \( k_{\lambda \mu} \) satisfies the following relation:

\[ k_{\lambda \mu} = 2A[\lambda B_{\mu}], \tag{3.6} \]

Furthermore, making use of (3.2) and (3.5), we obtain

\[ \begin{align*}
(a) \ A^\lambda &= h^{\lambda \nu} A_\nu \colon (0, 0, 1, 1), \\
(b) \ B^\lambda &= h^{\lambda \nu} B_\nu \colon (e^t, 0, 0, 0),
\end{align*} \tag{3.7} \]

and

\[ \begin{align*}
(a) \ A_\alpha A^\alpha &= 0, \\
(b) \ A_\alpha B^\alpha &= 0, \\
(c) \ k_{\lambda \alpha} A^\alpha &= 0, \\
(d) \ \nabla_\lambda A_\mu &= 0, \\
(e) \ \nabla_\omega B_\mu &= A_\omega B_\mu. \tag{3.8} \end{align*} \]

**Theorem 3.1.** In UFT \( X_4 \), the vector \( A_\lambda \) given by (3.5)(a) is a solution of the condition (2.9). In this case, the corresponding Einstein connection may be given by

\[ \Gamma^\nu_{\lambda \mu} = 2A[\lambda B_{\mu}]A^\nu. \tag{3.9} \]

And its curvature tensor \( R^\omega_{\lambda \mu \nu} \) may be given by

\[ R^\omega_{\lambda \mu \nu} = 2A_\lambda A_{[\mu} B_{\nu]} A^\omega. \tag{3.10} \]

**Proof.** Substituting the vector \( A_\lambda \) into the condition (2.9), the vector \( A_\lambda \) is a solution of the condition (2.9) if, in virtue of (3.8)(c),

\[ \nabla_\nu k_{\lambda \mu} = -2k_{\nu[\lambda A_{\mu]}}. \tag{3.11} \]

But, making use of (3.6), (3.8)(d) and (3.8)(e),

\[ \nabla_\nu k_{\lambda \mu} = A_\lambda A_\nu B_\mu - A_\mu A_\nu B_\lambda = -2k_{\nu[\lambda A_{\mu]}}, \tag{3.12} \]

which implies that the vector \( A_\lambda \) is a solution of the condition (2.9). Substituting the vector \( A_\lambda \) into (2.8), making use (3.6) and (3.8)(c), and remembering \( \{\lambda^\nu_{\mu}\} = 0 \), we obtain an Einstein connection (3.9). Substituting the Einstein connection (3.9) into (2.6), we obtain (3.10) by a straightforward computation. \( \square \)

**Conclusion.** In virtue of Theorem 3.1, if UFT \( X_4 \) is endowed with the basic tensor \( g_{\lambda \mu} \) given by (3.1), then an Einstein connection \( \Gamma^\nu_{\lambda \mu} \) is given by (3.9), which satisfy (2.5)(a). In the next, since the contracted curvature tensor \( R_{\lambda \mu} \) with respect to the connection (3.9) is given by \( R_{\lambda \mu} = 0 \), in virtue of (3.10), the field equation (2.5)(c) is satisfied automatically. And since the field equation (2.5)(b) is equivalent to \( \partial_{[\lambda} P_{\mu]} = 0 \), the field equation (2.5)(b) is satisfied by a vector \( P_\mu = \partial_\mu P \), that is, the vector \( P_\mu = \partial_\mu P \) is an Einstein’s vector.
4. An algebraic solution of field equations in UFT $X_4$

Assume that we have a particular solution $g_{\lambda\mu}$, $\Gamma^\nu_{\lambda \mu}$ of (2.4) and (2.5). The question arises whether there exist a tensor $\bar{g}_{\lambda\mu}$ which together with $\bar{\Gamma}^\nu_{\lambda \mu}$ is a solution of (2.4) and (2.5). In order to answer this question we put $\bar{g}_{\lambda\mu}$ in the form

(4.1) \[ \bar{g}_{\lambda\mu} = g_{\lambda\mu} + X_{\lambda\mu} \]

where the tensor $X_{\lambda\mu}$ has to be founded. From now on, we shall hold to the following agreement: If $T$ is a function of $g_{\lambda\mu}$, then we denote by $\bar{T}$ the same function of $\bar{g}_{\lambda\mu}$. If, in particular, $T$ is a tensor, so is $\bar{T}$. From (4.1), we obtain

(4.2) \[ (a) \quad \bar{h}_{\lambda\mu} = h_{\lambda\mu} + p_{\lambda\mu}, \quad (b) \quad \bar{k}_{\lambda\mu} = k_{\lambda\mu} + q_{\lambda\mu}, \]

where $p_{\lambda\mu}$ and $q_{\lambda\mu}$ are the symmetric part and the skew-symmetric part of the tensor $X_{\lambda\mu}$, respectively. And we assume that $\text{det}(\bar{h}_{\lambda\mu}) \neq 0$. We may define a unique tensor $\bar{h}_{\lambda\nu}(= \bar{h}_{\nu\lambda})$ by

(4.3) \[ \bar{h}_{\lambda\mu} \bar{h}^{\lambda\nu} = \delta^\nu_\mu. \]

**Theorem 4.1.** If we put

(4.4) \[ \boxed{\lambda_{\nu}^{\omega}} = \boxed{\lambda_{\nu}^{\omega}} + P_{\lambda\nu}^{\omega}, \]

then $P_{\lambda\nu}^{\omega}$ is a tensor symmetric in the indices $\lambda$ and $\nu$, and it is given by

(4.5) \[ P_{\lambda\nu}^{\omega} = \frac{1}{2} \bar{h}^{\nu\alpha}(\nabla_\lambda p_{\mu\alpha} + \nabla_\mu p_{\alpha\lambda} - \nabla_\alpha p_{\lambda\mu}). \]

**Proof.** By the law of transformation of the Christoffel symbols, $P_{\lambda\nu}^{\omega} = \frac{1}{2} \bar{h}^{\nu\alpha}(\nabla_\lambda p_{\mu\alpha} + \nabla_\mu p_{\alpha\lambda} - \nabla_\alpha p_{\lambda\mu})$. Multiplying by $\bar{h}_{\omega\mu}$ and summing for $\omega$ on both sides of (4.4), and using the expression (4.2)(a) in the right-hand member of (4.4), we obtain

(4.6) \[ \boxed{\lambda_{\nu}^{\mu}} = \boxed{\lambda_{\nu}^{\mu}} + \boxed{\lambda_{\nu}^{\mu}} P_{\lambda\nu}^{\alpha}, \]

In accordance with the definition of the Christoffel symbols we have, from (4.2)(a),

(4.7) \[ \boxed{\lambda_{\nu}^{\mu}} = \boxed{\lambda_{\nu}^{\mu}} + \boxed{\lambda_{\nu}^{\mu}} p_{\mu\nu}, \]

where $\boxed{\lambda_{\nu}^{\mu}}$ are the Christoffel symbols of the first kind formed with respect to $p_{\lambda\nu}$. Substituting (4.7) into (4.6), we obtain

(4.8) \[ \boxed{\lambda_{\nu}^{\mu}} = \boxed{\lambda_{\nu}^{\mu}} + \boxed{\lambda_{\nu}^{\mu}} P_{\lambda\nu}^{\alpha}. \]
If we add to this equation the one obtained by interchanging \( \lambda \) and \( \mu \), then the result may be written

\[
\nabla_\nu p_{\lambda \mu} = \bar{h}_{\mu \alpha} P_\alpha^{\nu} + \bar{h}_{\lambda \alpha} P_\alpha^{\mu}.
\]

Subtracting this equation from the sum of the two others which are obtained from it by cyclic permutation of the indices \( \lambda, \mu \) and \( \nu \), we obtain (4.5).

In order to obtain an algebraic solution \( \bar{g}_{\lambda \mu}, \Gamma_\nu^{\lambda \mu} \) by means of a particular solution \( g_{\lambda \mu}, \Gamma_\nu^{\lambda \mu} \) of (2.4) and (2.5), where \( \bar{g}_{\lambda \mu} \) and \( \Gamma_\nu^{\lambda \mu} \) are given by (3.1) and (3.8), let us consider a tensor \( \bar{g}_{\lambda \mu} \) given by the following matrix :

\[
(\bar{g}_{\lambda \mu}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
2e^t & 0 & 1 & 0 \\
-2e^t & 0 & 0 & -1
\end{pmatrix},
\]

where \( t = x^3 - x^4 \). The tensor \( \bar{g}_{\lambda \mu} \) may be split into its symmetric part \( \bar{h}_{\lambda \mu} \) and skew-symmetric part \( \bar{k}_{\lambda \mu} \) given by the following matrices :

\[
(\bar{h}_{\lambda \mu}) = \begin{pmatrix}
1 & 0 & e^t & -e^t \\
0 & 1 & 0 & 0 \\
e^t & 0 & 1 & 0 \\
-e^t & 0 & 0 & -1
\end{pmatrix},
\]

\[
(\bar{k}_{\lambda \mu}) = \begin{pmatrix}
0 & 0 & -e^t & e^t \\
0 & 0 & 0 & 0 \\
e^t & 0 & 0 & 0 \\
-e^t & 0 & 0 & 0
\end{pmatrix}.
\]

Hence the tensor \( \bar{h}_{\lambda \mu} \) may be split into \( h_{\lambda \mu} \) and \( p_{\lambda \mu} \) given by the following matrices :

\[
(h_{\lambda \mu}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

\[
(p_{\lambda \mu}) = \begin{pmatrix}
0 & 0 & e^t & -e^t \\
0 & 0 & 0 & 0 \\
e^t & 0 & 0 & 0 \\
-e^t & 0 & 0 & 0
\end{pmatrix}.
\]
Hence we obtain
\begin{align}
(4.15) & \quad (a) \quad \bar{h}_{\lambda\mu} = h_{\lambda\mu} + p_{\lambda\mu}, \quad (b) \quad q_{\lambda\mu} = 0, \quad (c) \quad \tilde{k}_{\lambda\mu} = k_{\lambda\mu}(= 2A_{\lambda[B_\mu]}). \\
& \text{Since}
\end{align}

\begin{align}
(4.16) & \quad (a) \quad \det((\bar{g}_{\lambda\mu})) = -1, \quad (b) \quad \det((\bar{h}_{\lambda\mu})) = -1, \quad (c) \quad \det((\tilde{k}_{\lambda\mu})) = 0,
\end{align}
we can choose the tensor $\bar{g}_{\lambda\mu}$ given by (4.10) as a basic tensor in UFT $X_4$, by the assumption (2.2).

**Theorem 4.2.** In UFT $X_4$, for the vectors $A_{\lambda}$ and $B_{\lambda}$ given by (3.5), the following relations hold.
\begin{align}
(4.17) & \quad (a) \quad p_{\lambda\mu} = 2A_{(\lambda}B_{\mu)} , \quad (b) \quad p_{\lambda\alpha}A^{\alpha} = 0, \quad (c) \quad p_{\lambda\alpha}B^{\alpha} = A_{\lambda}B_{\alpha}B^{\alpha}, \\
& \quad (d) \quad A_{\alpha}\bar{h}^{\alpha\nu} = A^{\nu}, \quad (e) \quad B_{\alpha}\bar{h}^{\alpha\nu} = B^{\nu} - A^{\nu}B_{\beta}B^{\beta} \\
& \quad (f) \quad P^{\nu}_{\lambda\mu} = A_{\lambda}A_{\mu}(B^{\nu} - A^{\nu}B_{\beta}B^{\beta})
\end{align}

**Proof.** A simple inspection based on (3.6), (3.8) and (4.15) shows (4.17)(a)~(d). From (4.15) and (4.17)(c), we obtain
\begin{align}
(4.18) & \quad \bar{h}_{\lambda\alpha}B^{\alpha} = B_{\lambda} + A_{\lambda}B_{\alpha}B^{\alpha}
\end{align}
Multiplying $\bar{h}^{\lambda\beta}$ on both sides of (4.18) and summing for $\lambda$, we obtain
\begin{align}
(4.19) & \quad B^{\beta} = \bar{h}_{\lambda\beta}B_{\lambda} + A_{\beta}B_{\alpha}B^{\alpha},
\end{align}
which implies (4.17)(e). Next, substituting (4.17)(a) in (4.5), and making use of (3.8) and (4.17)(e), obtain
\begin{align}
(4.20) & \quad P^{\nu}_{\lambda\mu} = \bar{h}^{\nu\alpha}A_{\mu}A_{\lambda}B_{\alpha} = A_{\lambda}A_{\mu}(B^{\nu} - A^{\nu}B_{\beta}B^{\beta}).
\end{align}

**Theorem 4.3.** In UFT $X_4$, let $g_{\lambda\mu}$ and $\Gamma^{\nu}_{\lambda\mu}$ be given by (3.1) and (3.9), respectively. For the basic tensor $\bar{g}_{\lambda\mu}$ given by (4.10), let $\bar{\Gamma}^{\nu}_{\lambda\mu}$ be a connection with the same torsion tensor as $\Gamma^{\nu}_{\lambda\mu}$. Then $\bar{\Gamma}^{\nu}_{\lambda\mu}$ is an Einstein connection which is given by
\begin{align}
(4.21) & \quad \bar{\Gamma}^{\nu}_{\lambda\mu} = 2A_{[\lambda}B_{\mu]}A^{\nu} + A_{\lambda}A_{\mu}(B^{\nu} - A^{\nu}B_{\beta}B^{\beta}).
\end{align}
And its curvature tensor $\bar{R}^{\nu}_{\lambda\mu\nu}$ may be given by
\begin{align}
(4.22) & \quad \bar{R}^{\nu}_{\lambda\mu\nu} = 2A_{\lambda}A_{[\mu}B_{\nu]}A^{\nu}.
\end{align}

**Proof.** Since $\bar{\Gamma}^{\nu}_{\lambda\mu}$ is a connection with the same torsion tensor as $\Gamma^{\nu}_{\lambda\mu}$,
\begin{align}
(4.23) & \quad \bar{S}^{\nu}_{\lambda\mu} = k_{\lambda\mu}A^{\nu} = 2A_{[\lambda}B_{\mu]}A^{\nu}.
\end{align}
Hence, in virtue of Theorem 2.1, and making use of (3.8)(c), (4.4), (4.5) and (4.20), and remembering \( \{ \lambda_\nu^\mu \} = 0 \), the connection \( \Gamma^\nu_{\lambda\mu} \) may be given by

\[
(4.24) \quad \Gamma^\nu_{\lambda\mu} = \{ \lambda_\nu^\mu \} + k_{\lambda\mu} A^\nu = A_\lambda A_\mu (B^\nu - A^\nu B_\beta B^\beta) + 2A_{[\lambda} B_{\mu]} A^\nu.
\]

And, in virtue of Theorem 2.2, this connection (4.24) is an Einstein connection if and only if, making use of (3.8)(c),

\[
(4.25) \quad \nabla_\nu k_{\lambda\mu} = -2k_{[\nu} [A_{\mu]}]
\]

But since, making use of (4.17)(f), (3.6) and (3.8),

\[
(4.26) \quad \nabla_\nu k_{\lambda\mu} = \partial_\nu k_{\lambda\mu} - k_{\alpha\mu} \{ \lambda_\alpha^\nu \} - k_{\lambda\alpha} \{ \nu_\lambda^\alpha \} = \partial_\nu k_{\lambda\mu} - k_{\alpha\mu} P_\lambda^\alpha - k_{\lambda\alpha} P^\alpha_{\mu\nu}
\]

the connection (4.21) is an Einstein connection which satisfies (2.4).

Next Substituting (4.21) into the curvature tensor :

\[
(4.27) \quad R^\omega_{\lambda\mu\nu} = \partial_\mu \Gamma^\omega_{\lambda\nu} - \partial_\nu \Gamma^\omega_{\lambda\mu} + \Gamma^\alpha_{\lambda\nu} \Gamma^\omega_{\alpha\mu} - \Gamma^\alpha_{\lambda\mu} \Gamma^\omega_{\alpha\nu},
\]

we obtain (4.22), by a straightforward computation.

**Conclusion.** In virtue of Theorem 4.3, if UFT \( X_4 \) is endowed with the basic tensor \( g_{\lambda\mu} \) given by (4.10), then an Einstein connection \( \Gamma^\nu_{\lambda\mu} \) is given by (4.21), which satisfy (2.5)(a). Furthermore, since from (4.22), the contracted curvature tensor \( R_{\lambda\mu\nu\rho} \) with respect to the connection (4.21) is given by \( R_{\lambda\mu\nu\rho} = 0 \), the field equation (2.5)(c) is satisfied automatically. On the other hand, since the field equation (2.5)(b) is equivalent to \( \partial_\nu P_{\mu} = 0 \), the field equation (2.5)(b) is satisfied by a vector \( P_{\mu} = \partial_\mu P \), that is, the vector \( P_{\mu} = \partial_\mu P \) is an Einstein’s vector. Consequently, for a particular solution \( g_{\lambda\mu}, \Gamma^\nu_{\lambda\mu} \) of (2.4) and (2.5), \( g_{\lambda\mu} \) is an algebraic solution which together with the \( \Gamma^\nu_{\lambda\mu} \) is a solution of (2.4) and (2.5).

**References**

[1] A. Einstein, *The meaning of relativity*, Princeton University Press, Princeton, New Jersey, 1950

[2] J. W. Lee, *Field equations of SE(k)-manifold \( X_n \)*, International Journal of Theoretical Physics 30 (1991), 1343-1353

[3] J. W. Lee, *An Einstein’s connection with zero torsion vector in even-dimensional UFT \( X_n \)*, Jour. of the Chungcheong Math. Soc. 24 (2011), no. 4, 869-881

[4] J. W. Lee, *A solution of Einstein’s field equations for the third class in \( X_4 \)*, Jour. of the Chungcheong Math. Soc. 26 (2013), no. 1, 167-174
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[5] J. W. Lee and K. T. Chung, *A solution of Einstein’s unified field equations*, Comm. Korean Math. Soc. **11** (1996), no. 4, 1047-1053.

[6] V. Hlavatý, *Geometry of Einstein’s unified field theory*, P. Noordhoff Ltd. New York, 1957.

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