Particle density mobility edge

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Many-body localization provides a mechanism to avoid thermalization in isolated interacting quantum systems. The breakdown of thermalization may be complete, when all eigenstates in the many-body spectrum become localized, or partial, when the so-called many-body mobility edge separates localized and delocalized parts of the spectrum. Previously, De Roeck et al. [Phys. Rev. B 93, 014203 (2016)] suggested a possible instability of the many-body mobility edge in energy density. The local ergodic regions — so called “bubbles” — resonantly spread throughout the system, leading to delocalization. In order to study such instability mechanism, in this work we design a model featuring many-body mobility edge in particle density: the states at small particle density are localized, while increasing the density of particles leads to delocalization. Using numerical simulations with matrix product states we demonstrate the stability of MBL with respect to small bubbles in large dilute systems for experimentally relevant timescales. In addition, we demonstrate that processes where the bubble spreads are favored over processes that lead to resonant tunneling, suggesting a possible mechanism behind the observed stability of many-body mobility edge. We conclude by proposing experiments to probe particle density mobility edge in Bose-Hubbard model.

Introduction. — Many-body localization (MBL) provides a mechanism to avoid thermalization in isolated quantum interacting systems [1, 2]. Despite intensive theoretical [3, 4] and experimental [5–11] studies, only the so-called fully-MBL phase in one spatial dimension is relatively well understood. In particular, the fate of MBL in higher dimensions [12–16] and the possibility of the coexistence of localized and delocalized eigenstates in the same many-body spectrum [17] remain debated.

Similarly to the case of Anderson localization [18], the MBL and delocalized eigenstates cannot coexist at the same energy. Analogously, in many-body systems where not all eigenstates are localized, there exists a many-body mobility edge (MBME) — a certain energy in the spectrum that separates localized and delocalized eigenstates [2]. In contrast to the non-interacting case, the energy of MBME scales extensively with system size. In the absence of a coupling to the bath, such scaling of MBME would lead to an exactly vanishing conductivity (in contrast to an exponentially small but finite value in Anderson insulator) until a certain critical temperature [2].

Recently De Roeck et al. [17] suggested a possible mechanism that may destroy MBME in large systems: a finite region with local energy density above the mobility edge — a “bubble” — may resonantly spread throughout the system thereby destroying localization everywhere. However, recent experiments [11] gave evidence of MBME, at least on intermediate timescales. In addition, a number of numerical studies observed a mobility edge [19–22] using exact diagonalization (ED). Unfortunately, the ED puts severe limitations on system size accessible for numerical studies; experiments with MBME in energy density are also challenging since they require energy resolution.

In order to overcome the above challenges, in this work we propose to study a MBME in particle density. This allows us to directly probe the mechanism of instability suggested in Ref. [17], which equally applies to MBME in any extensive conserved quantity. First, using numerical simulation with matrix product states (MPS), we demonstrate that uniform dilute states remain localized even at system sizes of $L = 40$ sites up to 250 tunneling times (i.e. more than two orders of magnitudes larger than the inverse local hopping). Next, we use a region with a large particle density as a bubble and track its influence on the dilute remainder of the system in a quantum quench. We do not find any evidence of resonant tunneling of the bubble, at least on experimentally relevant timescales.

In summary, the study of the particle density MBME facilitates the state preparation and analysis and allows us to access the dynamics of much larger systems using time evolution with MPS. We report the stability of the particle density mobility edge on long timescales and suggest that similar physics may be experimentally probed using Bose-Hubbard model.

Correlated hopping model. — We consider hard-core bosons on an open chain of size $L$, with dynamics governed by the following Hamiltonian,

$$
\hat{H} = t_1 \sum_{i=1}^{L-1} (c_{i+1}^\dagger c_i + \text{h.c.}) + \sum_{i=1}^{L} \epsilon_i \hat{n}_i \\
+ t_2 \sum_{i=2}^{L-1} (c_{i-1}^\dagger \hat{n}_i c_{i+1} + \text{h.c.}).
$$

The first two terms correspond to the non-interacting Anderson’s model [23], where random on-site potential $\epsilon_i \in [-W, W]$ is drawn from a uniform distribution. The last term introduces the interaction in the form of a facilitated hopping, which allows motion of a pair of particles with hopping amplitude $t_2$, $\bullet \bullet \leftrightarrow \bullet \bullet$. In this way, we introduce two channels for dynamics, one related to single
particle hopping that is dominant in the dilute case, and the second related to the pair hopping term that plays a dominant role at large particle density.

We note that a somewhat similar model was discussed in Ref. [17] in two dimensions, although there the authors considered only two particles. The enhancement of localization length in the case of two interacting particles also received significant attention [24, 25]. In a different direction, the fate of the single particle mobility edge in the presence of interactions was studied [10, 26]. In contrast, we study model (1) that does not have a single particle mobility edge and focus on the regime of finite density of particles.

In order to choose the value of the hopping parameters \(t_{1,2}\), we numerically calculate the localization lengths of a single particle, \(\xi_{SP}\), and a pair of particles, \(\xi_{P}\), see [27]. We fix \(t_1 = 0.5\) and \(t_2 = 2\) so that \(\xi_{SP} \lesssim 1\) at the same time \(\xi_{P} \gtrsim 2.5\). For such a choice, we expect dense states, which typically contain many pairs separated by just a few lattice sites, to be delocalized. In contrast, at lower particle density existing pairs would be separated by a distance significantly larger than \(\xi_{P}\) and thus will be localized. In [27] we demonstrate that our model does not suffer from finite size effects [28] and has a monotonic density of states.

Eigenstate probes of localization.—We use exact diagonalization (ED) and shift-invert (SI) numerical techniques to study the spectrum and eigenstate properties of the Hamiltonian (1). First we analyze the average ratio of level spacings in the middle of the spectrum, \(r_{av} = \langle \min(\delta_i, \delta_{i+1}) / \max(\delta_i, \delta_{i+1}) \rangle\), where \(\delta_i = E_{i+1} - E_i\) is the level spacing. This is a commonly used probe of the MBL transition [20, 29] that attains the value \(r_P \simeq 0.39\) for the Poisson level statistics, characteristic of the MBL phase, and \(r_{GOE} \simeq 0.53\) for the case of random Gaussian orthogonal ensemble (GOE), typical for chaotic Hamiltonians with time-reversal symmetry.

Figure 1 displays that at half-filling, \(\nu = N/L = 1/2\), where \(N\) is the total number of particles and \(L\) is the chain length, the level statistics approaches GOE with an increasing system size, which is consistent with the delocalized phase. In contrast, at \(\nu = 1/2\) filling \(r_{av}\) flows towards \(r_P\) at strong disorder. In what follows we fix the disorder strength to be \(W = 6.5\), since at this value the dilute limit is localized while the dense limit clearly flows towards delocalization. The scaling of entanglement entropy also reveals the coexistence of localized and delocalized phases at one-fifth and half-filling, respectively, for disorder \(W = 6.5\) [27].

Quench dynamics.—Having provided numerical evidence for the coexistence of localized and delocalized phases in small systems, we turn to quantum quench dynamics that distinguishes MBL from ergodic phase [5, 30]. We consider quenches where the system is initially prepared in a product state and then evolved with the Hamiltonian (1). Starting with a density wave of period \(1/\nu\), we calculate the density profile at late times. For the dilute case, \(\nu = 1/5\), we use the time-evolved block decimation (TEBD) with MPS [31, 32] (see [27] for additional details and benchmarks). This allows to monitor dynamics of systems as large as \(L = 40\) sites with 8 particles up to times \(T_{max} = 500\). In the dense case \(\nu = 1/2\) we use ED and Krylov subspace time evolution method. While ED allows to access the infinite-time density profile, with the Krylov method, we simulate quantum dynamics up to time \(T_{max} = 1000\).

The density profiles at late times look very different in the dense and dilute cases. While in the dilute case the system retains memory of the initial state, see Fig. 2(a), at \(\nu = 1/2\) quantum dynamics leads to a progressively more uniform density profile with increasing system size, Fig. 2(b). In order to quantify the difference in the form of the density profile at late times, in Fig. 2(c) we plot the average deviation of the density from an equilibrium thermal value, \(\Delta n = \langle n \rangle - \langle n \rangle_{eq}\), where \(\langle n \rangle_{eq}\) is the particle density. The deviation of late-time density from the thermal value, \(\Delta n\), in the dense regime decays exponentially with the system size as \(\Delta n \sim e^{-L/\xi_T}\), where \(\xi_T \simeq 6.27\). In contrast, for the dilute case \(\Delta n\) shows no dependence on the system size, as is apparent in the density profiles. The characteristic length \(\xi_T\) extracted in the dense case is related to the size of the typical thermal region that is sufficient to destroy the MBME [17].

Having confirmed the coexistence of localized and delocalized states at different values of particle density \(\nu\) for the same disorder strength, we proceed with a more detailed study of the effect of a bubble. Figure 2(d) illustrates the evolution of a non-uniform initial state, where a dense region represents the bubble. The bubble region

![Figure 1](image-url)
Figure 2. (a)-(c) The quantum quench from the uniform density wave with period $1/\nu$ reveals memory of the initial state at $\nu = 1/5$ in (a), whereas in the dense case $\nu = 1/2$ (b) the charge pattern relaxes to zero exponentially in the system size as is shown in (c). (d)-(f) Stability of the dilute system against the bubble consisting of a half-filled region with 4 particles is illustrated in panel (d) by the density profile at $T_{\text{max}}$. (e) The time dynamics of density in the coarse grained regions (see the legend at the top) shows the absence of significant relaxation in regions away from the bubble. The density in the region at the boundary with the bubble increases logarithmically in time. (f) The onset of logarithmic entanglement dynamics after a transient is visible for all cuts (see the legend at the top) away from the bubble. (g)-(j) In contrast, the bubble delocalizes the system when the overall density $\nu = 1/2$. The residual density profile at $T_{\text{max}} = 1000$ in panel (g) has only weak memory of the initial state. In addition, densities coarse grained over 3-site regions in (f) all tend to the equilibrium value of 1/2 and entanglement entropy in (j) displays faster than logarithmic growth for all cuts. The data are generated using TEBD and Krylov (ED) dynamics for dilute and dense limits. The averaging is performed over $5 \times 10^4, 2 \times 10^4, 8 \times 10^3, 10^3$ disorder realizations for the dilute systems and $3 \times 10^4, 1.5 \times 10^4, 8 \times 10^3, 10^3$ disorder realizations for the dense systems (from smallest to largest).

consists of 8 sites with two pairs of particles and has a local density of $\nu = 1/2$. The bubble is followed by a period-5 density wave that occupies $L - 10$ sites and two additional empty sites at the end of the chain. The bubble leaks into the dilute region only weakly at late times, see Fig. 2(d), with particles far away from the bubble not being affected. In contrast, in the dense case, Fig. 2(g), the bubble with average density of $\nu = 2/3$ successfully melts the period-3 density wave state throughout the system.

Next, in panels Fig. 2(e) and (h) we further illustrate the differences between the density dynamics in the dense and dilute cases in presence of a bubble. In both cases we plot the density of particles within subregions of a small size $k$, $\bar{\nu}_i = (1/k) \sum_{j=i}^{i+k-1} \langle n_j \rangle$, that are shown at the top of the plot. In the dilute case, Fig. 2(e), we observe that $\bar{\nu}$ remains far from its thermal value even at late times. The densities of regions in the bubble and adjacent to the bubble seem to saturate, while the regions far away from the bubble show very slow dynamics. In contrast, the dense case, Fig. 2(h), shows that all expectation values tend to an equilibrium value, although the regions far away from the center of the chain display slow, logarithmic in time, growth of density.

Finally, we study the dynamics of the bipartite entanglement entropy, $S_{\text{VN}}$, see Fig. 2(f) and (j). The entanglement is defined as $S_{\text{VN}} = -\text{tr} \rho \ln \rho$, where $\rho$ is the density matrix of the left subregion calculated from $\rho(t) = e^{-iHt} |\psi_0\rangle$. Different entanglement cuts shown at the top of Fig. 2(f) and (j) are encoded by their color. Consistent with MBL, the increase of entanglement in the region close to the bubble is logarithmic in time in Fig. 2(f) [33–36]. The entanglement across the cuts further away from the bubble begins to grow at significantly later times. For these more distant cuts, the initial uprise in entanglement corresponds to a slow logarithmic change of density [see Fig. 2(e)], and after saturation of density dynamics, we expect an onset of the logarithmic growth of entanglement. In contrast, the entanglement dynamics in Fig. 2(j) is always faster than logarithmic. In [27] we provide more details on the contribution of the particle transport to entanglement [36, 37], demonstrating that particle transport is responsible for logarithmic entanglement increase, whereas the remaining part of entanglement grows faster than logarithmic.

Bubble tunneling vs. decay processes.—The quench dynamics discussed above suggests that a bubble is not able to spread through the entire localized chain and remains in the vicinity of its initial position. At the same time, most of our quench simulations are restricted to finite,
albeit long, times. In order to give a complementary evidence for the bubble localization, we return to eigenstate properties that effectively probe the infinite time limit. We start with an initial product state in the half-filled case

$$|\psi_1\rangle = \bigotimes_{i=1}^{L/2} |\psi_{\text{half}}\rangle$$

that contains a bubble of $k = L/2$ sites with $\nu = 2/3$ filling (boxed region), followed by a sparser region with the same number of sites and density $\nu = 1/3$. For illustrative purposes we fixed $L = 12$. We are interested in comparing the probability of the bubble tunneling to the opposite end of the system and the probability of the bubble spreading throughout the system. We use a spatial reflection of $|\psi_1\rangle$ as a representative of the state with bubble tunneling to the opposite end of the chain:

$$|\psi_2^\dagger\rangle = \bigotimes_{i=1}^{L/2} |\psi_{\text{half}}\rangle$$

again illustrated here for $L = 12$ and $\nu = 1/2$ filling. At the same time, the state where the bubble has spread and decayed can be taken to be a uniform density wave,

$$|\psi_2\rangle = \bigotimes_{i=1}^{L/2} |\psi_{\text{uniform}}\rangle$$

For dilute configurations at $\nu = 1/3$ we define the bubble as a region of size $2(N - 1)$ with density $\nu = 1/2$, joined with a dilute remainder. Here $N = L/5$ is the number of particles. For $L = 20$ such a state with the bubble is

$$|\psi_1\rangle = \bigotimes_{i=1}^{L/2} |\psi_{\text{dilute}}\rangle$$

It is straightforward to show that the infinite time average probability of finding the system with the wave function $e^{-i\hat{H}t} |\psi_1\rangle$ in the product state $|\psi_2\rangle$ is given by the sum $\sum_{\alpha} |\langle\psi_2|E_\alpha\rangle|^2$, where $|E_\alpha\rangle$ are the complete set of eigenstates of $\hat{H}$. Taking the inverse of this probability, we define the mutual inverse participation ratio (mIPR) that quantifies the similarity in the expansion of two different states $|\psi_1,2\rangle$ over the basis $|E_\alpha\rangle$:

$$\mathcal{I}(|\psi_1\rangle, |\psi_2\rangle) = \left[ \sum_{\alpha=1}^{N} |\langle E_\alpha |\psi_1\rangle \langle E_\alpha |\psi_2\rangle|^2 \right]^{-1}$$

The mIPR reduces to the conventional inverse participation ratio when $|\psi_1\rangle = |\psi_2\rangle$.

In order to understand the relation between bubble decay and tunneling processes, we study the ratio of two mIPRs. Figure 3 shows ratio of $\mathcal{I}_L / \mathcal{I}_s$ in Fig. 3 is rapidly increasing with both disorder, and system size. This proves that the bubble tunneling processes are strongly suppressed compared to the processes where the bubble spreads throughout the volume of the system.

**Experimental realization.**—Finally, we discuss a possible way to observe the physics related to MBME in experiments with ultracold atoms. Within the disordered Aubry-André bosonic Hamiltonian,

$$\hat{H} = \sum_i \left[ t(a_i^\dagger a_{i+1} + \text{h.c.}) + \epsilon_i n_{i,\sigma} + U n_i (n_i - 1) \right]$$

that is actively used to study the MBL physics [36, 38], the bubbles can be represented by regions with $\langle a_i^\dagger a_i \rangle = \rho > 1$ bosons per site. A particle within such region has a hopping matrix element enhanced by the Bose-factor of $\langle \rho \rangle$, thus playing a role of hopping $t_2$ in model (1). In the regime of densities and disorder strengths such that the enhanced hopping $\langle \rho \rangle t$ corresponds to localization lengths significantly larger than lattice spacing, $\xi_{\text{dense}} > a$, whereas a single boson localization length is $\xi = a$, this model will implement similar physics to our toy model. Note that at the same time it is important to keep interaction $U$ low enough, $U < t$, to avoid the formation of minibands related to long-lived doublons.

By initializing the system in a product state with a dense region of bosons in the center of the trap along with low density of bosons away from such a region, the dynamics under Hamiltonian (7) will probe the ability of the bubble to melt the imbalance [5] away from its original position. From our simulations of the toy model we expect the absence of imbalance relaxation far away from the bubble.

**Discussion.**—We presented a model with MBME in particle density and investigated its properties numerically using ED and time evolution with MPS. We find
strong evidence of the persistence of localization at infinite times for small systems and also observe memory of initial configuration until times of $T_{\text{max}} = 500$ for systems with up to $L = 40$ sites. These times are at least two orders of magnitude larger compared to the inverse local hopping, $h/t_1$, and are achievable with cold atoms experiments. While we cannot rule out a residual very slow delocalization at much later times, the constructed model allows us to bound the timescale up to which the localization remains stable in very large systems that are beyond the reach for ED.

The model with MBME in particle density presented in this work allows for direct tests of the arguments of Ref. [17]. The argument about the instability of MBME relies on the processes where the bubble moves throughout the system. For these processes it is important that the bubble does not disappear by spreading and that configurations with bubbles situated at different locations are effectively coupled to each other. Our simulations reveal that dilute systems have no trace of bubble reemerging at a different location within the system. Moreover, even the expectation value of the pair density $\langle n_in_{i+1}\rangle$ (recall, that pairs are most mobile in our model and they are building blocks of the bubble) is exponentially suppressed away from the original location of the dense bubble [27]. In an alternative approach, we directly test the probability of the bubble to emerge at the opposite end of the system at infinite time and find it to be strongly suppressed.

To conclude, we expect that the proposed model will enable further investigations of particle density MBME. Studies of the structure of matrix elements that govern the resonant processes is a promising avenue for future work. Also, it would be interesting to extend the theory of LIOMs [39, 40] to systems with MBME in particle density analogously to the results for energy MBME [21]. In a different direction, our model provides an alternative way to study the effect of a small bath on a localized system [41–44].

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Supplementary material for “Particle density mobility edge”

In this supplementary material we present additional data and details of the methods used in the main text. First, we show data for additional probes of ETH breakdown such as entanglement entropy. After this we present benchmarks of our time-evolving block decimation simulation of dynamics. Finally, we explore the behavior of the mutual IPR defined in the main text.

I. LOCALIZATION LENGTH AND PARAMETER CHOICE

The dynamics generated by the constrained Hamiltonian, Eq. (1), strongly depends on the choice of the hopping parameters $t_{1,2}$. In order to choose the most suitable parameters for the study of MBME in particle density, we explore localization lengths for a single particle $\xi_{SP}$ and for one pair of particles $\xi_p$. These localization lengths are evaluated using ED. We calculate the infinite-time averaged of the occupation number at each site for an initial state where either a single particle or a single pair are initialized at the first site of the chain. We extract the localization lengths $\xi_{SP}$ ($\xi_p$) from an exponential fit of the density curve $\langle n_i \rangle$.

Resulting values of $\xi_{SP}$ for fixed $t_1 = 0.5$ and different disorder values and different values of hopping $t_2$ are shown in Fig. S1. The single particle hopping localization length (dashed line in Fig. S1) does not depend on $t_2$, and becomes smaller than one lattice spacing for $W > 4$. The pair localization length is monotonously increasing with $t_2$ at fixed value of disorder strength, $W$. Our aim is to have $\xi_p$ in the range between 2 and 5. In this regime, the half-filling case is expected to be delocalized.

Figure S1. The localization length decreases, as expected, with the disorder strength for all the values of $t_2$. For every constrained hopping amplitude $t_2$ it is possible to locate the region of disorder where we expect to see a MBME in particle density as the area among the two dashed lines. As the curve crosses the first dashed line, systems with typical particle spacing 5 will be localized. Nevertheless denser states will still be delocalized, having smaller distance among particles. Data were obtained on a lattice of length $L = 50$ and averaged over 5000 disorder realizations.

II. ED PROBES OF LOCALIZATION

While in the main text we focused on the two values of filling, $\nu = 1/2$ and $1/5$, here we demonstrate the density dependence of critical disorder. For this purpose we calculate the average ratio of level spacings $r_{av}$ for a single system size $L = 18$ at varying values of density. Fig-

Figure S2. The DOS from single disorder realizations show a relatively smooth behavior and a Gaussian shape, thus confirming the absence of strong finite size effects. DOS refers to a chain with $L = 20$ and $\nu = 1/4$. Disorder strength is $W = 5.0$. Green, blue and orange curves correspond to different disorder realizations, while the black dashed line shows disorder-averaged DOS.
Figure S3. The sharp difference of $r_{av}$ obtained for different $\nu$ at the same disorder $W$ clearly shows the MBME in our model. Interestingly, the mobility edge curve $W_c(\nu)$ is not symmetric, but is peaked around $\nu = 2/3$, implying that the states with the maximum number of pairs for fixed size are the hardest to localize. The data is obtained for a system of size $L = 18$, using shift-invert method with $10 - 10^3$ states from the middle of the spectrum and $5 \times 10^4 - 10^5$ disorder realizations.

Figure S4. The behavior of half-chain entanglement entropy shows very distinct behavior for dilute (blue-shaded curves) and dense (red-shaded curves) states. The crossing in the dilute states implies that they entered the MBL phase, and thus have area-law entanglement entropy. On the other hand, dense states do not show a similar crossing in this range of disorder, suggesting that they are still in the ergodic phase. The data are obtained with shift-invert method for $10 - 10^3$ eigenstates in the middle of the spectrum and averaged over $5 \times 10^4 - 5 \times 10^3$ disorder realizations.

III. MPS SIMULATIONS OF QUENCH DYNAMICS

In our MPS simulation, we time evolve dilute states in large systems $L \geq 30$ up to time $T_{\text{max}} = 500$. For this we use the time-evolving block decimation (TEBD) algorithm with a fourth-order Trotter evolution based on the ITensor library [32]. The main parameter involved in the time evolution algorithm is the time step $\delta t$ used to split the unitary evolution into a sequence of gates. The error related to the finite size of the time step in the $p$-th order Trotter expansion grows as $\delta t^p$. The other source of error is the finite cutoff, $\epsilon$, that governs the truncation of singular values in the singular value decomposition (SVD).

While the instantaneous errors related to the truncation and finite time step are known, understanding the propagation of these errors with time and their possible interference is challenging. First we tested TEBD algorithm by evolving the ground state of the same model. Provided that the time evolution is numerically exact, the overlap between the TEBD-evolved ground state, $|\psi_0(t)\rangle = U^{\text{TEBD}}(t)|\text{GS}\rangle$ and the exact time evolution of the ground state, $|\text{GS}(t)\rangle = e^{-i\mathcal{H}t}|\text{GS}\rangle$, is supposed to give the identity $\langle\psi_0(t)|\text{GS}(t)\rangle = 1$ at all times. For the fourth-order Trotterization the behavior of $F = 1 - |\langle\psi_0(t)|\text{GS}(t)\rangle|$ is known to be proportional to $(\delta t)^8$. The numerical results plotted in Fig. S5(a), confirm these expectations.

Next, we performed a benchmarking of TEBD algorithm against ED time evolution for several disorder realizations and simulation parameters. An illustration of such benchmarking is shown in Fig. S5(c) and (d). In particular, we observed that time step $\delta t = 0.05$ and cut-
of our long-time numerical simulations. Nevertheless, ∆E bubble in the lattice. Nevertheless, ∆E(t) remains very small in both configurations. In spite of that, a clear difference can be observed among the two quenches, noticing that the non-uniform state has larger error. This is probably due to the enhanced entanglement caused by the presence of the bubble in the lattice. Nevertheless, ∆E(t) remains very small even at long times, thus confirming the reliability of our long-time numerical simulations.

In all the simulations performed using ITensor [32], we used the U(1) symmetry implementation. In particular, to obtain the numerical results presented in Fig. 2(a) and (d) we set the maximum bond dimension to be 500 and 3000 respectively. As the histograms in Fig. S7 show, all the disorder realizations remained well below the maximum threshold. This fact ensures that we have a control on the error encountered in the evolution, in contrast to time evolution with TDVP with fixed bond dimension, where error estimation is more challenging [45].

IV. ADDITIONAL RESULTS FROM QUENCH DYNAMICS

In the main text we discussed dynamics in quenches that begin from a uniform state or a bubble joined to a more dilute remainder. Below we present details for the quenches in presence of bubble. In addition, we discuss the dynamics resulting from the initial state containing density wave of particle pairs.
Figure S6. The normalized absolute value of the energy difference from the initial energy $\Delta E(t) = |\langle H(t) \rangle - E(0)|/|E(0)|$ remains very small for both the density wave, blue curve, and the non-uniform, red curve, configurations, confirming the good accuracy of our numerical simulations beyond the ED benchmark. The larger deviation displayed by the non-uniform configuration is understood as a result of the presence of the bubble in the lattice, that increases entanglement growth. The results here shown are obtained averaging over 100 disorder realizations, for the system sizes, $L = 30$, and initial states described in the main text.

A. Pair density and entanglement dynamics in presence of a bubble

Since particle pairs are the most mobile objects, we consider the pair density in quenches that are initialized with the bubble (see Fig. 2(d),(g) in the main text). The pair density is of special interest in these quenches as in Ref. [17] suggested that the instability of the system is ascribed to the ability of the bubble to move. In our model the bubble consists of several pairs, thus motion of the bubble throughout the system would imply the spreading of pairs.

The pair density defined as $\langle n_i n_{i+1} \rangle$ measured at late or infinite times is shown in Fig. S8. In the dense case the late time pair density profile supports delocalization: at late times the density of pairs becomes homogeneous throughout the formerly more dilute region of the system. We note, that the pair density is not a conserved quantity, and it can increase in the process of unitary dynamics.

In contrast, for the dilute case the pair density profile has a pronounced exponential tail away from the initial ergodic region. This shows that pairs spreading away from the initial bubble do not delocalize when encountering additional particles on their way. Indeed, while the late time pair density profile has small peaks around the initial position of particles, these peaks are not very pronounced. In addition, the study of the pair density profile in the uniform density wave at $\nu = 1/5$ reveals an almost constant behavior, centered around $\langle n_i n_{i+1} \rangle \sim 10^{-3}$, which corresponds to the values reached at the end of the exponential tail in the system with $L = 30$ in Fig. S8.

Next, we focus on understanding different contributions to entanglement growth. Exploiting the $U(1)$ symmetry of our model and following Refs. [36, 37], we split the von Neumann entanglement entropy into a configuration and a particle transport contributions. Indeed, due to conservation of the total number of bosons the full reduced density matrix $\rho$ must have a block-diagonal

Figure S7. The histogram representing the maximum bond dimension of different disorder realizations show that the threshold values of 500 and 3000 for the uniform density wave (left) and bubble states (right) were never saturated in our simulations.

Figure S8. The finite size scaling of the pair density $\langle n_i n_{i+1} \rangle$ shows opposite trend for the dense and dilute cases. The red-shaded curves represent $\nu = 1/2$ configurations: increasing the system size (from yellow to dark red) the pair density becomes more uniform and approaches the thermal value, hence in the thermodynamic limit the probability of finding a pair far from the bubble is almost the same as finding it in the bubble. On the contrary, blue curves ($\nu = 1/5$) show exponential vanishing of the pair density and, furthermore, increasing system size (from light blue to dark blue) the density decreases, suggesting that at the thermodynamic limit there will be no pair outside the thermal region. Data were obtained with ED, Krylov ($T_{\text{max}} = 1000$) and TEBD ($T_{\text{max}} = 500$) algorithms averaging over 100 disorder samples for the largest MPS simulations ($L = 20, 30$), $3 \times 10^4, 10^4, 5 \times 10^3$ and $10^3$ for ED (from $L = 10$ to $L = 16$) and over $10^3$ for Krylov algorithm ($L = 18$).
form. Individual blocks within $\rho$ can be written as $p_n \rho^{(n)}$, where $p_n$ gives the probability to have $n$ particles in the subsystem $A$ and $\rho^{(n)}$ is normalized as $\text{tr} \rho^{(n)} = 1$. Using such representation of the reduced density matrix we can split the full entropy into $S_{vN} = S_C + S_n$ as:

$$
S_{vN} = - \text{tr} \rho \log \rho = - \sum_n p_n \text{tr} \rho^{(n)} \log(p_n \rho^{(n)})
$$

$$
= - \sum_n p_n \log p_n - \sum_n p_n \text{tr} \rho^{(n)} \log \rho^{(n)} \quad \text{(S1)}
$$

$$
= S_n + S_C.
$$

In this way the entanglement growth is split into two contributions: one coming from the particle transport, and another originating from dephasing between different configurations with the same particle number. Interestingly, Fig. S9 shows that while the overall entanglement entropy grows faster than logarithmic, this is due only to the configuration part (yellow curve) and the entanglement due to particle transport has logarithmic growth. The logarithmic growth of $S_n$ is consistent with the logarithmic particle transport presented in Fig. 2(h) in the main text. We identify this behavior as a hallmark of MBME, and note that it happens on long, yet experimentally accessible timescales $t \sim 50(\hbar/t_1)$.

### B. Quench dynamics from a pair density wave state

Below we consider quench from a pair-density wave of period $2/\nu$. These configurations accommodate the maximal possible number of pairs in the uniform state. Figure S10 confirms that such state is localized at $\nu = 1/5$ and is relaxing in the dense case. Dense systems display strong dependence on the system size and increased tendency towards relaxation at larger system sizes, $L$. In contrast, at $\nu = 1/5$ the late time density profile has almost no dependence on the size of the system. In particular, even at very large lengths the curves do not approach the average density represented by the dashed black line.
Figure S11. The mutual IPR, $I_d$, that quantifies the inverse probability of bubble tunneling $d$ sites, increases exponentially with $d$ at strong disorder. At weak disorder the mIPR approaches the Hilbert space dimension, $N$, shown by a dashed line. In the dilute system in (b), $W = 4.5$ marks the onset of the exponential growth, suggesting that the thermal bubble is frozen at its initial position. On the other hand, for $\nu = 1/2$, in (a), the clear exponential behavior emerges only at larger disorder. $I_d$ was calculated for system sizes $L = 15$ and $L = 12$ in dilute and dense case respectively and averaged over $10^4$ disorder realizations.

V. MUTUAL IPR AS A FUNCTION OF BUBBLE DISTANCE

In the text we introduced a generalized version of the inverse participation ratio, that we dubbed mutual IPR. This quantity takes as argument two vectors in the Hilbert space and quantifies the similarity in expansion of these vectors over a complete set of eigenstates: values of mIPR $O(N)$ correspond to two vectors that have similar expansion over eigenstates, while very large values of mIPR imply that the expansion is very different. In the main text we analyzed the mIPR between two product states where the bubble is located at the left and right end of the system respectively, see Eqs (2)-(3). Such pair of states corresponds to the maximum possible displacement of the bubble in the chain. Below we illustrate the behavior of mIPR between pair of states which correspond to a smaller bubble displacement.

In our analysis we measure the mIPR, $I_d = I(\psi_L, \psi_d)$, between the following states in the dense limit (half-filling, $L = 12$),

$$|\psi_L\rangle = \bullet \bullet \bullet \bullet \circ \circ \circ \circ \circ,$$  \hspace{1cm} (S2)

$$|\psi_d\rangle = \circ \circ \circ \circ \bullet \bullet \bullet \bullet \circ \circ.$$ \hspace{1cm} (S3)

Here we use the bubble that contains all particles to maximize the range of achievable displacements. For the dilute case, $L = 15$, we use similar pair of states with bubble containing 3 particles ($\nu = 1/5$).

In the thermal phase, eigenstates are approximately given by random vectors in the Hilbert space and their average overlap with other normalized vectors approaches the value predicted by random matrix theory, irrespective of the state or the eigenstate. In the weak disorder limit, we then expect $I_d$ to be independent on the distance between the two bubbles and to have the same behavior as the conventional IPR: $I_d \sim N$. This expectation is confirmed by the results presented in fig. S11(a) and (b) for $W = 0.5$.

On the other hand, in the MBL phase eigenstates are not similar to random vectors, but instead are characterized by a set of local integrals of motion that have a finite overlap with the local particle density. Thus, two product states with globally different arrangement of particles are expected to have drastically different expansion over eigenstates. Therefore, we expect $I_d \propto \exp [d/\xi]$. As presented in figure S11(a) and (b), at strong disorder our results support this hypothesis for both dilute (a) and dense (b) states.

At intermediate disorder strength, we observe a qualitative difference between dense and dilute cases. Dilute configurations, Fig. S11(a), show exponential behavior already at $W = 4.5$, whereas dense states in Fig. S11(b) need much stronger disorder to clearly present the same trend. This result confirms the presence of mobility edge and is consistent with the observed absence of pair spreading reported in Figure S8 and also with the finite size scaling of mIPRs shown in the main text, Fig. 3.

VI. DYNAMICAL PROBE OF THE ABSENCE OF RESONANCES

The discussion on mutual IPR showed how tunneling processes are strongly suppressed in the dilute case of our model. In addition to eigenstates analysis, we also studied long time dynamics of states with a thermal bubble. In this way, it was possible to verify whether a bubble initialized at a certain position can dynamically give rise to a dense region somewhere else in the chain. In order to study this process we defined a projector onto the subset
of Hilbert space that has large density in a certain region. More specifically, we define

\[ \hat{P}_{\nu_c}(L_0, i) = \sum_{|\phi_{\alpha}\rangle \in C} |\phi_{\alpha}\rangle \langle \phi_{\alpha}|, \]  

(S4)

where states \(|\phi\rangle\) are all possible product states that satisfy the condition \(\nu \geq \nu_c\) in the region \([i, i + L_0]\). This projector selects all configurations where the system is locally above the mobility edge. We notice that \(\hat{P}_{\nu_c}(L_0, i)\) takes into account all possible configurations, thus considering also the entropic factor.

In order to understand what is the minimal required size of the region \(L_0\), we use the lengthscale extracted from the decay of \(\Delta n\). Fit in Fig. 2(c) in the main text yields \(L_0 \simeq 6 \div 7\), while fit in Fig. S10(c) gives a somewhat larger scale. We define an initial state \(|\psi_0\rangle\) that has an entangled dense region of size approximately \(L_0\) (described by a linear superposition of product states \(|\phi_i\rangle\)) followed by a product state:

\[ |\psi_0\rangle = \frac{1}{\sqrt{N_C}} \sum_{i=1}^{N_C} |\phi_i\rangle \otimes |\circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ + \]  

(S5)

Below, we fix the overall density to \(\nu = 1/4\) and \(W = 6.5\), which still corresponds to a localized system. The dense region is obtained as a superposition of different configurations with \(N - 1\) particles in \(L_0 = 2(N - 1)\) sites. The remaining particle is initialized in the middle of the last segment of the chain. For instance, for \(L = 16\) this results into following initial state:

\[ |\psi_0\rangle = \frac{1}{\sqrt{N_C}} \begin{bmatrix} \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ + \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ + \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ + \cdots \end{bmatrix}, \]  

(S6)

where the boxed area contains a dense entangled bubble and the remainder is in the dilute state.

The initial state \(|\psi_0\rangle\) is then evolved through the Hamiltonian (1) in a quench protocol. After time evolution up to a maximum time \(T_{\text{max}} = 1000\), we measure \(\langle P_{\nu_c}(L_0, d) \rangle = \langle \psi(t) | \hat{P}_{\nu_c}(L_0, d) | \psi(t) \rangle\), which quantifies the probability of encountering a bubble shifted by \(d\) sites from the initial position of the bubble.

Finally, averaging over all different product states in the dilute part of the chain and over disorder we obtain the data in Fig. S12. This plot reveals that the probability of having a dense \((\nu > \nu_c)\) region decays exponentially with the distance \(d\) from its initial location. This is in agreement with our long-time TEBD dynamics, Fig. S8, that reveals localization of individual pairs. Thus, we conclude that bubble does not spread resonantly but rather tunnels throughout the system. Moreover, the finite size scaling analysis shows that increasing the system size the decay of \(\langle P_{\nu_c}(L_0, d) \rangle\) with distance \(d\) is enhanced. Therefore in the dilute regime of our model the bubble remains localized around its initial position.