AN APPLICATION OF LIMITING INTERPOLATION TO THE FOURIER SERIES THEORY

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Abstract. Limiting real interpolation method is applied to describe the behaviour of the Fourier coefficients of functions that belong to spaces which are “very close” to \( L^2 \).

Keywords: orthonormal system; Fourier coefficients; real interpolation method; limiting reiteration theorems.

1. Introduction

We consider (equivalent classes of) complex-valued measurable functions on \([0,1]\). Let \( \{\varphi_n\} \) \((n \in \mathbb{N} \text{ or } n \in \mathbb{Z})\) be an orthonormal system in \( L^2 \) bounded in \( L^\infty \):

\[
\sup \|\varphi_n\|_{L^\infty} = M (<\infty).
\]

(1.1)

Everywhere below we denote by \( \{c_n(f)\} \) the Fourier coefficients of a function \( f \) with respect to the system \( \{\varphi_n\} \):

\[
c_n \equiv c_n(f) := \int_0^1 f(x)\overline{\varphi_n(x)} dx.
\]

We write \( \mathcal{F} \) for the Fourier series map assigning the sequence of Fourier coefficients to any function \( f \), i.e. \( \mathcal{F}(f) = \{c_n(f)\} \). It is known that \( \mathcal{F} \) is a linear bounded operator from \( L^2 \) to \( l^2 \) and from \( L^1 \) to \( l^\infty \):

\[
\|\mathcal{F}\|_{L^2 \rightarrow l^2} = 1,
\]

(1.2)

\[
\|\mathcal{F}\|_{L^1 \rightarrow l^\infty} \leq M.
\]

(1.3)

The results of Hausdorff, Young and Paley describe the behaviour of the Fourier coefficients of functions that belong to Lebesgue spaces \( L_p \) \((1 < p < 2)\). For further information about classical results dealing with Fourier series map we refer to e.g. \([2, 3, 27, 28, 39]\). Interpolation between (1.2) and (1.3) by the classical real interpolation method \( (\ast, \ast)_{\theta,q} \) \((0 < \theta < 1)\) provides such description for the Lorentz spaces \( L_{p,q} \) \([1, 35]\):

\[
\|\mathcal{F}\|_{L_{p,q} \rightarrow l_{r,q}} \prec M^\frac{2-\theta}{p}.
\]

Application of the real interpolation functor \( (\ast, \ast)_{\theta,q,a} \) \((0 < \theta < 1, \ a \in \mathbb{R})\) involving logarithmic factor shows that for the Lorentz–Zygmund spaces \( L_{p,q}(logL)_a \) it holds \([2, 14, 15]\)

\[
\mathcal{F}: L_{p,q}(logL)_a \rightarrow l_{r,q}(logl)_a.
\]

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In both formulae $1 < p < 2$, $\frac{1}{r} = 1 - \frac{1}{p}$, $0 < q \leq \infty$. Note that in the scale $L_{p,q}(\log L)_a$ this result is optimal [34, Theorem 5.3]. Similar results were obtained also by other approaches (see [26, 33, 34] and references therein).

But for the spaces which are “very close” to $L_2$ all these methods do not work and two other approaches can be applied. One of them is a “direct way” based on estimates of a $K$-functional [5, 6, 29, 30, 32]. The other one is based on limiting real interpolation methods (see e. g. [7, 14, 16, 36]).

Comparable results are also known for the inverse transformation

$$\mathcal{F}^{-1}: \{c_n\} \rightarrow f(x) := \sum_{n=1}^{\infty} c_n \varphi_n(x).$$

Here $\{c_n\}$ are sequences of complex numbers. It is known [35, 39] that

$$\| \mathcal{F}^{-1} \| _{L_2} \rightarrow L_2 \| = 1,$$

$$\| \mathcal{F}^{-1} \| _{L_\infty} \rightarrow L_\infty \| = M.$$  \hfill (1.5)

Interpolation between (1.4) and (1.5) by the real interpolation method $(*, *)_{q,a}$ or $(*,*)_{q,a \alpha}$ \hfill (0<\theta<1) leads also to results for Lorentz and Lorentz–Zygmund sequence spaces with main parameter laying between 1 and 2 [14, 15, 35]. But for the spaces which are “very close” to $l_2$ all these methods do not work either.

The main objective of this work is to study the Fourier series map and its inverse for the spaces which are “very close” to $L_2$ or $l_2$ resp. via limiting interpolation methods. More precisely, we prove the following assertions.

**Theorem 1.1.** Let $0 < q \leq \infty$ and $\alpha < -1/q$. Then

$$\left\{ \sum_{k=1}^{\infty} \left( 1 + |\log k | \right)^{\alpha} \left( \sum_{i=1}^{k} \left( c_i^* (f) \right)^2 \right)^{\frac{q}{2}} \right\}^{\frac{1}{q}} k^{-1} \leq$$

$$\leq C \min (M, (1+\log M)^{\frac{1}{q}}) \left\{ \int_0^1 \left( 1 + |\log t | \right)^{\alpha} \left( \int_0^t \left( f^*(u) \right)^2 \, du \right)^{\frac{q}{2}} t^{-1} \, dt \right\}^{\frac{1}{q}}$$

for some constant $C$ which depends only on $q$ and $\alpha$. (As usual, the integral and the sum should be replaced by the supremum when $q=\infty$.)

**Theorem 1.2.** Let $0 < q \leq \infty$ and $\alpha > -1/q$. Then for any $\varepsilon > 0$

$$\left\{ \int_0^1 \left( 1 + |\log t | \right)^{\alpha} \left( \int_0^t \left( f^*(u) \right)^2 \, du \right)^{\frac{q}{2}} t^{-1} \, dt \right\}^{\frac{1}{q}} \leq$$
\[
\leq C \min(M, (1+\log M)^{[q]+1/q}) \left( \sum_{k=1}^{\infty} (1 + |\log k|)^{q} \left( \sum_{i=k}^{\infty} (c_{i}^{*}(f))^{2} \right)^{\frac{q}{2}} \right)^{1/q} k^{-1} \]

for some constant \( C \) which depends only on \( q, a, \) and \( \varepsilon \). (As usual, the integral and the sum should be replaced by the supremum when \( q=\infty \).) In particular, this means that if the expression on the right-hand side of the estimate above exists then the row \( \sum_{n=1}^{\infty} c_{n} \varphi_{n}(x) \) converges in the (quasi-) norm determined by the expression on the left-hand side.

**Remark 1.2.**

(i) The orthonormality of the system \( \{ \varphi_{n} \} \) and (1.1) implies \( M \geq 1 \).

(ii) The system \( \{ \varphi_{n} \} \) may also be bounded in \( L_{Q} \) for some \( Q \in (2, \infty] \) [28]. The estimates of Theorems 1.1 and 1.2 do not depend on \( Q \).

The second objective of the present paper is to compare the estimate given in Theorem 1.1 with known results. It turns out that the results established with the help of the “direct way” approach are weaker than those of Theorem 1.1.

This paper is organized as follows. Section 2 contains necessary notations, definitions and auxiliary results. Theorems 1.1 and 1.2 will be proven and discussed in Sections 3 and 4 resp.

2. **Notation, definitions and auxiliary results**

If \( X \) is a (quasi-) Banach space and \( x \in X \) then its (quasi-) norm is denoted as \( \| x \|_X \). \( X \approx Y \) means that the Banach spaces \( X \) and \( Y \) are isomorphic. Throughout the paper \( L_{q}(a,b) \) \((0<q<\infty, -\infty \leq a < b \leq \infty) \) is the usual quasinormed Lebesgue space \( L_{q} \) on the interval \((a,b)\). For \( q \geq 1 \) it is a Banach space. \( L_{q} \) implies \( L_{q}(0,1) \). By \( C \) we designate different positive constants which are independent of all significant arguments. If \( f \) and \( g \) are positive functions, we will write \( f \preceq g \) if \( f \leq C \cdot g \) and \( f \simeq g \) if \( f \preceq g \) and \( g \preceq f \).

2.1. **Interpolation spaces**

Let \( \overset{\rightarrow}{X} = (X_{0}, X_{t}) \) be a compatible couple of (quasi-) Banach spaces and let

\[
K(t,x) \equiv K(t,x, \overset{\rightarrow}{X}) := \inf_{x = x_{0} + t x_{1}, x_{0} \in X_{0}, x_{1} \in X_{t}} (\| x_{0} \|_{X_{0}} + t \| x_{1} \|_{X_{t}}) .
\]

be Peetre’s \( K \)-functional. For further information about properties of the \( K \)-functional and the real interpolation method, we refer to [3, 4, 27, 38]. For our purposes, it is enough to consider only ordered couples \( X_{0} \rhd X_{1} \) with the norm of embedding equal to 1. This will be denoted as \( X_{0} \rhd X_{1} \). In this case \( K(t,x) \approx \| x \|_{X_{0}} \) for \( t > 1 \) [4].
**Definition 2.1.** Let \( X_0 \supset X_1 \), \( 0 \leq \theta \leq 1 \), \( 0 < q \leq \infty \), and \( \alpha \in \mathbb{R} \). We set
\[
\tilde{X}_{\theta,q,a} = (X_0, X_1)_{\theta,q,a} := \{ x \in X_0 + X_1 \mid \| x \|_{\tilde{X}_{\theta,q,a}} := \| x \|_{\theta} < \infty \}.
\]

It only makes sense to consider the spaces \( \tilde{X}_{\theta,q,a} \) on the set
\[
\{ (\theta, q, a) \in [0, 1] x (0, \infty] x \mathbb{R} \mid 0 < \theta < 1, \text{ or } \theta = 0, q \leq \infty, \alpha \geq -1/q, \text{ or } \theta = 1, q \leq \infty, \alpha < -1/q, \text{ or } \theta = 1, q = \infty, \alpha = 0 \}.\]

Note that the functors \((X_0, X_1)_{\theta,q,a} \) and \((X_0, X_1)_{1,q,a} \) produce spaces which are “very close” to \( X_0 \) and to \( X_1 \) respectively. This definition can be found in a lot of papers (see e. g. \([10, 12, 15, 16, 17, 19, 21, 36]\)).

Different analogues of the next lemma can be found in literature. See \([12, \text{ Theorem 2.5}], [8, \text{ Theorem 4.9}], [9, (2.3) and (2.4)], [18], \) and \([22, \text{ Theorem 3.5}]\).

**Lemma 2.2.** Let \( X_0 \supset X_1 \) and \( Y_0 \supset Y_1 \) be (quasi-) Banach spaces, and let \( T \) be a (quasi-) linear bounded operator, \( T : X_j \rightarrow Y_j \) with the norms \( M_j := \| T \mid X_j \rightarrow Y_j \| \) \((j = 0, 1)\). Additionally suppose that \( 0 < q \leq \infty \).

(a) If \( M_0 \geq M_1 \) and \( \alpha < -1/q \), then \( T \) is bounded from \( \tilde{X}_{1,q,a} \) to \( \tilde{Y}_{1,q,a} \) and
\[
\| T \mid \tilde{X}_{1,q,a} \rightarrow \tilde{Y}_{1,q,a} \| \leq \min(M_0, (1 + \log(M_0/M_1))^{[\alpha]} M_1).
\]

(b) If \( M_0 \leq M_1 \) and \( \alpha \geq -1/q \), then \( T \) is bounded from \( \tilde{X}_{0,q,a} \) to \( \tilde{Y}_{0,q,a} \) and for any \( \varepsilon > 0\)
\[
\| T \mid \tilde{X}_{0,q,a} \rightarrow \tilde{Y}_{0,q,a} \| < \min(M_1, (1 + \log(M_1/M_0))^{[\alpha]} \varepsilon^{1/q} M_0).
\]

**Proof.** First, notice that if \( x \in X_0 + X_1 \) then
\[
K(t, Tx; \tilde{Y}) \leq \max(M_0, M_1) K(t, x; \tilde{X}).
\]
(2.1)

Moreover,
\[
K(t, Tx; \tilde{Y}) \leq M_0 K(tM_1/M_0, x; \tilde{X}).
\]
(2.2)

We begin with the assertion (a). (2.1) implies
\[
\| T \mid \tilde{X}_{1,q,a} \rightarrow \tilde{Y}_{1,q,a} \| \leq M_0.
\]
(2.3)

It is not difficult to show that
\[
(1 + |\log(uv)|)^{\alpha} \leq (1 + |\log u|)^{\alpha} (1 + |\log v|)^{\beta} \quad (u, v > 0).
\]

By means of this inequality and of (2.2), because \( M_1/M_0 \leq 1 \), and using the change of variable \( u = tM_1/M_0 \), we obtain
\[ \| T \mid X_{0,q,a} \rightarrow Y_{0,q,a} \| \leq M_1. \] (2.4)

For real numbers \( \omega \) und \( \alpha \) we use as usual
\[ l^{(\omega,\alpha)}(t) := \begin{cases} (1 - \log t)^\omega, & \text{if } 0 < t \leq 1, \\ (1 + \log t)^\alpha, & \text{if } 1 \leq t < \infty. \end{cases} \]

Now we show that for any \( \varepsilon > 0 \)
\[ \| t^{-1/\alpha}(1 + \log t)^\alpha K(t, Tx; Y) \| \leq \| t^{-1/\alpha}(1 + \log t)^\alpha K(t, Tx; Y) \| \leq \| t^{-1/\alpha}(1 + \log t)^\alpha K(t, Tx; Y) \| \leq \| t^{-1/\alpha}(1 + \log t)^\alpha K(t, Tx; Y) \| \leq \| t^{-1/\alpha}(1 + \log t)^\alpha K(t, Tx; Y) \|. \] (2.5)

It is enough to check that
\[ \| t^{-1/\alpha}(1 + \log t)^\alpha K(t, Tx; Y) \| \leq \| t^{-1/\alpha}(1 + \log t)^\alpha K(t, Tx; Y) \|. \]

Observe that
\[ \| t^{-1/\alpha}(1 + \log t)^\alpha \| \leq \infty \]
and if \( \varepsilon > 0 \)
\[ \| t^{-1/\alpha}(1 + \log t)^\alpha \| \leq \infty. \]

Using that \( t^{-1/\alpha}(1 + \log t)^\alpha \) is non-increasing and \( K(t, Tx; Y) \approx K(1, Tx; Y) \) for \( t > 1 \), we obtain
\[ \| t^{-1/\alpha}(1 + \log t)^\alpha K(t, Tx; Y) \| \leq \| t^{-1/\alpha}(1 + \log t)^\alpha K(t, Tx; Y) \| \leq \| t^{-1/\alpha}(1 + \log t)^\alpha K(t, Tx; Y) \| \leq \| t^{-1/\alpha}(1 + \log t)^\alpha K(t, Tx; Y) \|. \]

So, (2.5) is proven. It can be shown that (cf. [9], p. 169.)
\[ l^{(\omega,\alpha)}(uv) \leq l^{(\omega,\alpha)}(u) (1 + |\log u|)^{\omega |\log v|} \]

Therefore,
\[ l^{(\omega,\alpha)}(uv) \leq l^{(\omega,\alpha)}(u) (1 + |\log u|)^{\omega |\log v|}. \]
Since $M_0 \leq M_1$, using (2.2) and (2.5), and by means of the change of variable $u = tM_1/M_0$, we obtain:

$$
\| T x \|_{Y_{0,q,\alpha}} \approx \left\| t^{-\frac{1}{q}} \left( 1 + \ln t \right)^{\alpha} K(t, T x; Y) \| L_q(0,1) \right\| \approx \left\| t^{-\frac{1}{q}} I(t)^{\alpha} K(t, T x; Y) \| L_q(0,1) \right\| \leq M_0 \left\| t^{-\frac{1}{q}} I(t)^{\alpha} K(t, x; X) \| L_q(0,1) \right\| = M_0 \left\| t^{-\frac{1}{q}} I(uM_0/M_1)^{\alpha} K(u, x; X) \| L_q(0,1) \right\| \leq M_0(1 + \log(M_1/M_0))^{|\alpha| + \frac{1}{q}} \left\| t^{-\frac{1}{q}} I(u)^{\alpha} K(u, x; X) \| L_q(0,1) \right\| \approx M_0(1 + \log(M_1/M_0))^{|\alpha| + \frac{1}{q}} \left\| \frac{1}{M_0} \left( 1 + \log u \right)^{\alpha} K(u, x; X) \| L_q(0,1) \right\| = M_0(1 + \log(M_1/M_0))^{|\alpha| + \frac{1}{q}} \| x \|_{X_{0,q,\alpha}} \right\|.
$$

Combining this with (2.4) we complete the proof. \hfill \Box

**Remark 2.3.** Due to Lemma 2.2 we have in Theorems 1.1 and 1.2 expression of the form

$$
G(M,\gamma) = \min\left( M, (1 + \log M)^{\gamma} \right)
$$

with $M \geq 1$ and $\gamma \geq 0$. It is clear that $G(1,\gamma) = 1$ and

$$
G(M,\gamma) = \begin{cases} 
(1 + \log M)^{\gamma}, & \text{if } \gamma \leq \log M / \log(1 + \log M), \\
M, & \text{otherwise}. 
\end{cases}
$$

In particular, $G(M,\gamma) = (1 + \log M)^{\gamma}$ if $\gamma \leq 1$.

Next we consider the following limiting interpolation spaces. These spaces allow to formulate reiteration theorems in the limiting cases $\theta = 0$ and $\theta = 1$. In more general form these spaces were introduced and investigated in [14, 16, 19, 23, 25].

**Definition 2.4.** Let $X_0 \equiv X_1$, $0 < q, r \leq \infty$, $\alpha \in \mathbb{R}$. We denote by $X_{\theta,q,\alpha,\mathbb{R}}$ ($0 \leq \theta < 1$) and $X_{\theta,q,\alpha,\mathbb{E}}$ ($0 \leq \theta \leq 1$) the sets of elements $x \in X_0$ for which the expressions

$$
\left\| x \right\|_{X_{\theta,q,\alpha,\mathbb{R}}} := \left\| t^{-\frac{1}{q}} \left( 1 + \ln t \right)^{\alpha} K(t, x) \| L_r(0,t) \| L_q(0,1) \right\|, \\
\left\| x \right\|_{X_{\theta,q,\alpha,\mathbb{E}}} := \left\| t^{-\frac{1}{q}} \left( 1 + \ln t \right)^{\alpha} K(t, x) \| L_r(t,1) \| L_q(0,1) \right\|
$$

resp. are finite.

The next lemma follows from [14, Lemma 6.2], [16, Lemma 4], see also [19].

**Lemma 2.5.** Let $0 < q, r \leq \infty$, $0 \leq \theta < 1$ and $\alpha > -1/q$ (or $0 < \theta \leq 1$ and $\alpha < -1/q$); then
\[
X_{\theta,q,a,r} \supseteq X_{\theta,q,a,r} (\text{or } X_{\theta,q,a,r} \text{ resp.) } \supseteq X_{\theta,r,a,r} \cap X_{\theta,\max(r,q),a,r}.
\]

2.2. **Function spaces**

In this section we give necessary definitions of function and sequence spaces. We consider (equivalent classes of) complex-valued measurable functions on \([0,1]\) and bounded complex-valued sequences \(\{c_k\}\). As usual, \(f^*\) is the non-increasing rearrangement of \(|f|\), and \(\{c_k^*\} \ (k \in \mathbb{N})\) is the non-increasing rearrangement of the sequence \(|c_k|\). Lorentz–Zygmund spaces can be defined as follows.

**Definition 2.6.** Let \(0<p,q \leq \infty\) and \(\alpha \in \mathbb{R}\). Put

\[
L_{p,q}(\log L, \alpha) := \{f \mid \|f\|_{L_{p,q}(\log L, \alpha)} := \|u^{1/p-1/q} (1 + |\log u|)^\alpha |f^*(u)| L_q(0,1)\| < \infty\}.
\]

Analogously

\[
l_{p,q}(\log l, \alpha) := \{c_k \mid \|c_k\|_{l_{p,q}(\log l, \alpha)} := \|k^{1/p-1/q} (1 + \log k)^\alpha c_k^* \mid l_q \| < \infty\}.
\]

These spaces are studied in [2, 13, 15, 16, 17, 19, 20, 21, 31]. See also [3] and [25]. Note that \(L_{p,q} = L_{p,q}(\log L, 0)\) and \(l_{p,q} = l_{p,q}(\log l, 0)\). Concerning next definition we refer to [14, 19, 25].

**Definition 2.7.** Let \(0<p,q \leq \infty\) and \(\alpha \in \mathbb{R}\). Put

\[
\begin{align*}
L_{p,q,a,r}^E & := \{f \mid \|f\|_{L_{p,q,a,r}^E} := \|u^{1/p-1/q} (1 + |\ln u|)^\alpha \|f^*(u)| L_r(0,t)\| \| L_q(0,1)\| \}, \\
L_{p,q,a,r}^R & := \{f \mid \|f\|_{L_{p,q,a,r}^R} := \|u^{1/p-1/r} (1 + |\ln u|)^\alpha \|f^*(u)| L_r(t,1)\| \| L_q(0,1)\| \}.
\end{align*}
\]

Analogously

\[
\begin{align*}
l_{p,q,a,r}^E & := \{c_k \mid \|c_k\|_{l_{p,q,a,r}^E} := \left\| \sum_{k=1}^{\infty} \left[ \frac{1}{k} \right]^{1/q} (1 + \ln k)^\alpha \left( \frac{1}{k} \sum_{i=1}^{k} \frac{c_i^*}{i^{1/r}} \right)^{1/q} \right\|^{1/q}, \\
l_{p,q,a,r}^R & := \{c_k \mid \|c_k\|_{l_{p,q,a,r}^R} := \left\| \sum_{k=1}^{\infty} \left[ \frac{1}{k} \right]^{1/q} (1 + \ln k)^\alpha \left( \frac{1}{k} \sum_{i=1}^{k} \frac{c_i^*}{i^{1/r}} \right)^{1/q} \right\|^{1/q} \}.
\end{align*}
\]

(As usual, the sum should be replaced by the supremum when \(q = \infty\) or \(r = \infty\).)

We need only the combination \(p=r=2\) for these spaces:

\[
\|f\|_{L_{2,q,a,2}^E} = \|u^{1/q} (1 + |\ln u|)^\alpha \|f^*(u)| L_2(0,1)\| \| L_q(0,1)\|.
\]
\[ \| f \|_{L^p_{2,q,\alpha,2}} = \| r^{1/q} (1+|\ln t|)^\alpha \| f^*(u) \|_{L^2(t,1)} \| L_0(0,1)}, \]

\[ \| \{ c_k \} \|_{L^p_{2,q,\alpha,2}} = \left( \sum_{k=1}^{\infty} \frac{1}{q} (1+\ln k)^\alpha \left( \sum_{i=k}^{\infty} (c_i^*)^2 \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}. \]

\[ \| \{ c_k \} \|_{L^p_{2,q,\alpha,2}} = \left( \sum_{k=1}^{\infty} \frac{1}{q} (1+\ln k)^\alpha \left( \sum_{i=k}^{\infty} (c_i^*)^2 \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}. \]

Note that in the terminology of [24] \( L^p_{2,q,\alpha,2} \) is the generalized grand Lorentz space \( L^p_{2,q,\alpha,2} \) and \( L^p_{2,q,\alpha,2} \) is the generalized grand Lorentz space of sequences \( L^p_{2,q,\alpha,2} \).

The following result is a consequence of [14, Corollaries 7.3 and 7.9], [19, Theorem 8.9], and [25, Theorem 5.7].

**Lemma 2.8.** Let \( 0 < q \leq \infty \). If \( \alpha < -1/q \), then
\( (L_1, L_2)_{1,q,\alpha} \cong L^p_{2,q,\alpha,2} \) and \( (l_\infty, l_2)_{1,q,\alpha} \cong l^p_{2,q,\alpha,2} \).

If \( \alpha > -1/q \), then
\( (L_2, L_\infty)_{0,q,\alpha} \cong L^p_{2,q,\alpha,2} \) and \( (l_2, l_\infty)_{0,q,\alpha} \cong l^p_{2,q,\alpha,2} \).

By Lemmas 2.5 and 2.8 we get following embeddings (sf. [14, Corollaries 7.4 and 7.8]).

**Lemma 2.9.** Let \( 0 < q \leq \infty \). If \( \alpha < -1/q \) (or \( \alpha > -1/q \)), then
\( L_{2,q}(\log L)_{\alpha+1/\min(q,2)} \subset L^p_{2,q,\alpha,2} \) (or \( L^p_{2,q,\alpha,2} \), resp.) \( \subset L_{2,q}(\log L)_{\alpha+1/\max(q,2)} \cap L_{2,\max(q,2)}(\log L)_{\alpha+1/q} \),

\( l_{2,q}(\log l)_{\alpha+1/\min(q,2)} \subset l^p_{2,q,\alpha,2} \) (or \( l^p_{2,q,\alpha,2} \), resp.) \( \subset l_{2,q}(\log l)_{\alpha+1/\max(q,2)} \cap l_{2,\max(q,2)}(\log l)_{\alpha+1/q} \).

**Remark 2.10.** For \( 0 < q < 2 \) the spaces \( L_{2,q}(\log L)_{\alpha+1/\max(q,2)} \) and \( L_{2,\max(q,2)}(\log L)_{\alpha+1/q} \) are incomparable [37].

**Corollary 2.11.** If \( \alpha < -1/2 \) (or \( \alpha > -1/2 \)) we have isomorphisms:
\( L^p_{2,q,\alpha,2} \) (or \( L^p_{2,\alpha,2} \), resp.) \( \cong L_{2,q}(\log L)_{\alpha+1/2} \), \( l^p_{2,q,\alpha,2} \) (or \( l^p_{2,\alpha,2} \), resp.) \( \cong l_{2,q}(\log l)_{\alpha+1/2} \).

For the scale \( l^p_{2,q,\alpha,2} \) we also need the following inclusion.

**Lemma 2.12.** If \( 0 < q < \infty \) and \( \alpha < -1/q \), then \( l^p_{2,q,\alpha,2} \subset l^p_{2,\alpha,2,\frac{1}{q}} \).
Proof. Let \( \{ c_k \} \in l^{\ell}_{q,\alpha,2} \). This means that
\[
\| \{ c_k \} \|_{l^{\ell}_{q,\alpha,2}} = \left( \sum_{k=1}^{\infty} \left[ (1 + \log k)^{\alpha} \left( \sum_{i=1}^{k} |c_i|^2 \right)^{\frac{q}{2}} k^{-1} \right]^{\frac{1}{q}} \right)^{\frac{1}{q}} < \infty.
\]
The sequence \( \{ b_k \} := \left( \sum_{i=1}^{k} |c_i|^2 \right)^{\frac{1}{2}} \) is non-negative and non-decreasing. Because \( q < \infty \) and \( q \alpha < -1 \), for \( m \geq 1 \) we obtain
\[
(1 + \log m)^{1 + q \alpha} \approx \int_{m}^{\infty} (1 + \log x)^{q \alpha} \frac{dx}{x} \approx \sum_{k=m}^{\infty} (1 + \log k)^{q \alpha} \frac{1}{k}.
\]
Therefore
\[
(1 + \log m)^{1 + q \alpha} b_m \approx b_m \sum_{k=m}^{\infty} (1 + \log k)^{q \alpha} \frac{1}{k} \leq \sum_{k=m}^{\infty} (1 + \log k)^{q \alpha} b_k \frac{1}{k}.
\]
So, for all \( m \geq 1 \)
\[
(1 + \log m)^{\alpha + \frac{1}{q}} \left( \sum_{i=1}^{m} |c_i|^2 \right)^{\frac{1}{2}} \approx \left( \sum_{k=m}^{\infty} (1 + \log k)^{\alpha} \left( \sum_{i=1}^{k} |c_i|^2 \right)^{\frac{1}{2}} k^{-1} \right)^{\frac{1}{q}}.
\]
and hence
\[
\| \{ c_k \} \|_{l^{\ell}_{q,\alpha,2}} = \sup_{m \geq 1} (1 + \log m)^{\alpha + \frac{1}{q}} \left( \sum_{i=1}^{m} |c_i|^2 \right)^{\frac{1}{2}} \approx \left( \sum_{k=1}^{\infty} (1 + \log k)^{\alpha} \left( \sum_{i=1}^{k} |c_i|^2 \right)^{\frac{1}{2}} k^{-1} \right)^{\frac{1}{q}} = \| \{ c_k \} \|_{l^{\ell}_{q,\alpha,2}}.
\]

3. Proof of Theorem 1.1, corollaries, and remarks

Proof of Theorem 1.1. The assertion of Theorem 1.1 can be reformulated as follows: if \( 0 < q \leq \infty \) and \( \alpha < -1/q \) then the Fourier series map \( F \) is bounded from \( L^{R}_{2,q,\alpha} \) to \( l^{\ell}_{2,q,\alpha,2} \) and
\[
\| F \|_{L^{R}_{2,q,\alpha} \rightarrow l^{\ell}_{2,q,\alpha,2}} \approx \min (M, (1 + \log M)^{\frac{1}{q}}).
\]
By Lemma 2.8 we have \( (L_1, L_2)_{1,q,\alpha} \equiv L^{R}_{2,q,\alpha} \) and \( (L_{\infty}, L_{2})_{1,q,\alpha} \equiv l^{\ell}_{2,q,\alpha,2} \). Now due to (1.2) and (1.3) it only remains to apply Lemma 2.2. \(\]

Remark 3.1. For the system \( \{ e^{izk} \} \) Theorem 1.1 recovers [14, Theorem 8.2 (b)].
Using isomorphisms (2.8) we get the following corollary.

**Corollary 3.2.** If $\alpha < -1/2$, then
\[
\| \mathcal{F} | L_{2,2} \log L \alpha + 1/2 \rightarrow I_{2,2} \log L \alpha + 1/2 \| < \min (M, (1 + \log M)^{|\beta|}).
\]

**Remark 3.3.** The special case $\alpha = -1$ and the system $\{ e^{l_{2,ms}} \}$ of Corollary 3.2 recovers [11, Theorem 8.5].

The following result is a consequence of (3.1) and the left side of the inclusions (2.6).

**Corollary 3.4.** If $2 \leq q \leq \infty$ and $\alpha < 1/2 - 1/q$, then
\[
\| \mathcal{F} | L_{2,q} \log L \alpha \rightarrow I_{2,q,\alpha - 1/2,2} \| < \min (M, (1 + \log M)^{|\alpha - 1/2|}).
\]
In particular, due to Remark 2.3, if $2 < q \leq \infty$, then
\[
\| \mathcal{F} | L_{2,q} \rightarrow I_{2,q,\alpha - 1/2,2} \| \propto (1 + \log M)^{1/2}.
\]

**Remark 3.5.** In [5 and 6] (see also [29]) Bochkarev has proven the following estimate
\[
\| \mathcal{F} | L_{2,q} \rightarrow I_{2,q,\alpha - 1/2,2} \| \propto M \quad (2 < q \leq \infty). \tag{3.2}
\]
Note that in the case $q = \infty$ formula (3.2) was proven in [32]. In [30] Bochkarev’s inequality was improved in Lorentz–Zygmund spaces:
\[
\| \mathcal{F} | L_{2,q} \log L \alpha \rightarrow I_{2,q,\alpha - 1/2,2} \| \propto M \quad (2 < q \leq \infty, \alpha < 1/2 - 1/q). \tag{3.3}
\]
Due to Lemma 2.12 if $2 < q \leq \infty$ and $\alpha < 1/2 - 1/q$ we have $I_{2,q,\alpha - 1/2,2} \subset I_{2,q,\alpha - 1/2,2}$. So, Corollary 3.4 improves both inequalities (3.2) and (3.3).

We finish this section by applying Lemma 2.9 to both spaces in (3.1).

**Corollary 3.6.** Let $0 < q \leq \infty$ and $\alpha < 1/q$; then
\[
\| \mathcal{F} | L_{2,q} \log L | \alpha + 1/\max(q,2) \rightarrow I_{2,q} \log L | \alpha + 1/\max(q,2) \| \propto \min (M, (1 + \log M)^{|\alpha|}).
\]

**Remark 3.7.** For the system $\{ e^{l_{2,ms}} \}$ Corollary 3.6 recovers [7, Theorem 5.3].

### 4. Proof of Theorem 1.2, corollaries and remarks

**Proof of Theorem 1.2.** The assertion of Theorem 1.2 can be reformulated as follows: if $0 < q \leq \infty$ and $\alpha > -1/q$, then the operator $\mathcal{F}^{-1}$ is bounded from $I_{2,q,\alpha,2}^{\Re}$ to $L_{2,q,\alpha,2}^{\Re}$ and for any $\varepsilon > 0$
\[
\| \mathcal{F}^{-1} | I_{2,q,\alpha,2}^{\Re} \rightarrow L_{2,q,\alpha,2}^{\Re} \| \propto \min (M, (1 + \log M)^{|\alpha + 1/\varepsilon|}). \tag{4.1}
\]
By Lemma 2.8 we have $(l_{2}, l_{1})_{0,q,\alpha} \equiv I_{2,q,\alpha,2}^{\Re}$ and $(L_{2}, L_{\infty})_{0,q,\alpha} \equiv L_{2,q,\alpha,2}^{\Re}$. Now due to (1.4) and (1.5) it only remains to apply Lemma 2.2. \qed
The following result is a consequence of (4.1) and the left side of the inclusions (2.7).

**Corollary 4.1.** Let \(0 < q \leq \infty\) and \(\alpha > -1/q\); then for any \(\varepsilon > 0\)
\[
\| \mathcal{F}^{-1} |_{L_{2,q}(\log L)_{\alpha+1/\min(q,2)}} \rightarrow L_{2,q,\alpha,2}^L \| < \min(M, (1+\log M)^{|\alpha|+\varepsilon+1/q}).
\]

Applying Lemma 2.9 to the both spaces in (4.1) we get the next corollary.

**Corollary 4.2.** Let \(0 < q \leq \infty\) and \(\alpha > -1/q\); then for any \(\varepsilon > 0\)
\[
\| \mathcal{F} |_{L_{2,q}(\log L)_{\alpha+1/\min(q,2)}} \rightarrow L_{2,q}(\log L)_{\alpha+1/\max(q,2)} \cap L_{2,\max(q,2)}(\log L)_{\alpha+1/q} \| \lessapprox \min(M, (1+\log M)^{|\alpha|+\varepsilon+1/q}).
\]

In particular, if \(\alpha > -1/2\), then for any \(\varepsilon > 0\)
\[
\| \mathcal{F}^{-1} |_{L_{2,2}(\log L)_{\alpha+1/2}} \rightarrow L_{2,2}(\log L)_{\alpha+1/2} \| \lessapprox \min(M, (1+\log M)^{|\alpha|+\varepsilon+1/2}).
\]

**Remark 4.3.** For the system \(\{ e^{i2\pi x} \} \) Theorem 1.2 and Corollary 4.2 recovers [14, Theorem 8.2 (g) and Lemma 8.4] respectively.

Using Remark 2.3 and Corollaries 4.1 and 4.2 we can formulate following assertions.

**Corollary 4.4.** Let \(1 < q \leq \infty\) and \(\max(-1/q, 1/q-1) < \alpha < 1-1/q\); then for any sufficiently small \(\varepsilon > 0\)
\[
\| \mathcal{F}^{-1} |_{L_{2,q}(\log L)_{\alpha+1/\min(q,2)}} \rightarrow L_{2,q,\alpha,2}^L \| \lessapprox (1+\log M)^{|\alpha|+\varepsilon+1/q}
\]
and
\[
\| \mathcal{F}^{-1} |_{L_{2,q}(\log L)_{\alpha+1/\min(q,2)}} \rightarrow L_{2,q}(\log L)_{\alpha+1/\max(q,2)} \cap L_{2,\max(q,2)}(\log L)_{\alpha+1/q} \| \lessapprox (1+\log M)^{|\alpha|+\varepsilon+1/q}.
\]

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