Three-Dimensional SU(3) gauge theory and the
Spatial String Tension of the
(3+1)-Dimensional Finite Temperature SU(3) Gauge Theory

F. Karsch, E. Laermann and M. Lütgemeier

Fakultät für Physik
Universität Bielefeld
P.O.Box 100131
D-33501 Bielefeld
Germany.

Abstract

We establish a close relation between the spatial string tension of the (3+1)-dimensional SU(3) gauge theory at finite temperature ($\sigma_s$) and the string tension of the 3-dimensional SU(3) gauge theory ($\sigma_3$) which is similar to what has been found previously for SU(2). We obtain $\sqrt{\sigma_3} = (0.554 \pm 0.004) g_3^2$ and $\sqrt{\sigma_s} = (0.586 \pm 0.045) g^2(T) T$, respectively. For temperatures larger than twice the critical temperature results are consistent with a temperature dependent coupling running according to the two-loop $\beta$-function with $\Lambda_T = 0.118(36) T_c$. 
1 INTRODUCTION

$SU(N)$ gauge theories at high temperature are known to undergo a phase transition to a deconfined phase. At the phase transition point basic thermodynamic observables show a drastic qualitative change, which suggests that the high temperature phase consists of weakly interacting, *asymptotically free* partons. The heavy quark potential, for instance, changes from a confining potential in the low temperature phase to a Debye screened potential at high temperature. At the same time there exist, however, correlation functions, whose structure does not change qualitatively at $T_c$, although there may be significant changes in their temperature dependence. For instance, the pseudo-potential, extracted from space-like Wilson loops, is known to be confining for all temperatures $[1, 2]$. This has been taken as indication for the survival of certain *confining* properties in the high temperature phase. The different high temperature behaviour of observables like the heavy quark potential and the spatial string tension can be understood at least qualitatively in terms of the structure of the effective, three-dimensional theory which describes the high temperature phase of QCD. This effective theory, obtained from *dimensional reduction* $[3]$, is a 3-$d$ gauge-Higgs model with Higgs-fields in the adjoint representation. The gauge sector of the model is confining and at the same time it leads to a screened Coulomb potential for the adjoint Higgs fields, which represent the static 0-components of the gauge potential in (3+1) dimensions.

It recently has been shown $[4, 5]$ that in the case of an $SU(2)$ gauge theory the resulting spatial string tension remains temperature independent up to $T_c$ and then starts rising rapidly. A similar behaviour has been found in lower dimensions and also in $Z(2)$ gauge theories $[3, 7]$. The qualitative behaviour of the spatial string tension thus seems to be a generic feature of the finite temperature phase transition in confining gauge theories. The temperature dependence may be qualitatively understood in terms of string fluctuations, which get suppressed due to the reduction of one space-time dimension with increasing temperature. This leads to a more rigid string, i.e. a larger string tension $[7]$.

In a recent high statistics Monte Carlo study of the $SU(2)$ gauge theory $[4]$ it could be shown that the linear slope of the pseudo-potential extracted from spatial Wilson loops - the
spatial string tension $\sigma_s$ - is closely related to the string tension of the gauge sector of the 3-d effective theory, $\sigma_3$, alone. The Higgs part does not seem to contribute substantially to this quantity. In fact, for temperatures larger than twice $T_c$ the spatial string tension has been found to rise like $g^4(T)T^2$ with a proportionality constant, which is only 10% larger than that for the string tension in a 3-d gauge theory. The running coupling, however, turned out to be quite large, $g^2(2T_c) \simeq 2.7$.

It is the purpose of the present paper to study the relation between $\sigma_s$ and $\sigma_3$ in the case of the SU(3) gauge theory in (3+1) and 3 dimensions, respectively. Unlike in the case of SU(2), where the string tension in 3-d had been studied quite accurately [8], we have to start here with a determination of $\sigma_3$ for the 3-d SU(3) gauge theory. We will discuss this calculation in section 2. In section 3 we describe the calculation of the spatial string tension of the finite temperature SU(3) gauge theory in (3+1) dimensions for three values of the temperature. Both results will be discussed in section 4.

2 STRING TENSION OF THE 3-D SU(3) GAUGE THEORY

We have analyzed the heavy quark potential in a 3-d SU(3) gauge theory on lattices of size $32^3$. The 3-d Wilson action is given by

$$S_3 = \beta_3 \sum_{0 \leq \mu < \nu \leq 2} \left(1 - \frac{1}{3} Re \text{Tr} U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu} U_{x,\nu} \right).$$

(1)

Here $\beta_3 = 6/ag_3^2$ denotes the dimensionless coupling, which in the continuum limit is related to the lattice spacing "a" and the dimensionful gauge coupling, $g_3^2$. The latter sets the scale for all dimensionful quantities in this limit. Lattice observables thus will scale with appropriate powers of the coupling $\beta_3$.

The heavy quark potential has been calculated from ratios of smeared Wilson loops, $W(R, S)$, [4] as

$$V(R)a = - \lim_{S \to \infty} \ln \frac{W(R, S + 1)}{W(R, S)} .$$

(2)

We find that generally $S > S_{\text{min}} \simeq 4$ is sufficient to reach a plateau in the ratio of smeared Wilson loops. We then extract $V(R)a$ from a fit to ratios with $S > S_{\text{min}}$. 

2
The potential has been calculated at six values of $\beta_3$ in the interval $18 \leq \beta_3 \leq 24$. The results are shown in Fig. 1. At each value of the coupling we have performed 37,500 overrelaxed/heat-bath updates and have analyzed 1500 configurations separated by 5 iterations (1 iter. $\equiv$ 4OR+1HB). Errors have been obtained from a jackknife-analysis.

![Figure 1: The heavy quark potential in 3-d SU(3) gauge theory obtained from calculations on a 32\(^3\) lattice. The errors are smaller than the size of the symbols.](image)

At short distances the 3-d potentials clearly differ from their 4-d counterparts. They show a much weaker $R$-dependence\(^1\) in this regime as may be expected from a logarithmic Coulomb potential. The long-distance part of the heavy quark potential is dominated by the linearly rising confinement part, $V_{\text{conf}} \sim \hat{\sigma}_3 R$. In addition, however, one expects to find also a universal contribution from string fluctuations, $\pi(d - 2)/24R$, which might be visible at intermediate distances \([11]\).

We extract the string tension from two and three parameter fits to the long distance part of the potential using the general ansatz

$$V(R) a = v - \frac{c}{R} + \hat{\sigma}_3 R \quad \text{for} \quad R > R_{\text{min}} .$$  \hspace{1cm} (3)

\(^1\)Throughout this paper $R$ is given in lattice units.
Table 1: The three-dimensional string tension calculated on lattices of size $32^3$ at various values of $\beta_3$

In the case of two parameter fits we fix the coefficient of the $1/R$-term to the value of the string model, $c = \pi/24$, and vary $R_{\text{min}}$ until a plateau is found for the coefficient of the linear term, $\hat{\sigma}_3(R_{\text{min}})$. We find that this coefficient rises with increasing $R_{\text{min}}$. We need $R_{\text{min}} \simeq 3$ for $\beta_3 = 18$ and $R_{\text{min}} = 6$ for $\beta_3 = 24$ until a plateau is reached within statistical errors. In the case of three parameter fits we proceed in the same way. Here we find stable string tension values for $R_{\text{min}} \geq 4$. We have verified that the results of both types of fits agree within statistical errors [12]. Our final results for $\hat{\sigma}_3$, obtained from two parameter fits as described above, are summarized in Table 1. In the continuum limit the string tension scales like

$$\sqrt{\hat{\sigma}_3} = \frac{b_1}{\beta_3} + \frac{b_2}{\beta_3^2} .$$

The results for $\hat{\sigma}_3\beta_3$ obtained from our two parameter fits are shown in Fig. 2. These data can be fitted to the form given by eq. (4). We find,

$$\sqrt{\hat{\sigma}_3\beta_3} = 3.326(22) + 7.2(4)/\beta_3 ,$$

which gives for the string tension in the continuum limit, $\sigma_3 a^2 \equiv \hat{\sigma}_3$,

$$\sqrt{\sigma_3} = (0.554 \pm 0.004)g_3^2 .$$
Figure 2: Scaling behaviour of the string tension in 3-d $SU(3)$ gauge theory. Errors on the data points are of the size of the symbols. The solid line is the fit given by Eq. 5, the star at $1/\beta_3 = 0$ shows the constant appearing in this fit.

3 THE SPATIAL STRING TENSION

Let us now come to a discussion of the spatial string tension of the $SU(3)$ gauge theory at finite temperature. In analogy to the analysis performed in the $SU(2)$ case [4] we study the spatial pseudo-potentials resulting from Wilson loops in hyperplanes orthogonal to the temporal direction of the lattice. At short distances these pseudo-potentials have been studied previously and have been shown to agree quite well with results obtained within the dimensionally reduced effective theory at high temperatures [9]. Here we aim at an analysis of the large distance behaviour of these pseudo-potentials using smeared Wilson loops as discussed in the previous section. Our calculations have been performed at a fixed value of the gauge coupling, $\beta = 6/g^2 = 6.0$, i.e. at fixed lattice cut-off. The temperature is varied by varying the temporal extent, $N_\tau$, of a lattice of size $32^3 \times N_\tau$. The deconfinement transition on a lattice with temporal extent $N_\tau = 8$ is known to occur close to this value of the gauge coupling [10]. For the purpose of the present calculation it is sufficient to identify $\beta = 6.0$ with the critical coupling on lattices with temporal extent $N_\tau = 8$. We
Table 2: The spatial string tension calculated on lattices of size $32^3 \times N_T$ for $N_T = 2, 3, 4$ (this paper) and 32 ([14]) at $\beta = 6.0$.

| $N_T$ | $T/T_c$ | $\hat{\sigma}_s$ |
|-------|---------|------------------|
| 32    | 0.25    | 0.0513 $\pm$ 0.0025 |
| 4     | 2       | 0.0874 $\pm$ 0.0014 |
| 3     | 2.67    | 0.1224 $\pm$ 0.0052 |
| 2     | 4       | 0.2384 $\pm$ 0.0090 |

have performed simulations for $N_T = 2, 3$ and 4. These values thus cover the temperature interval, $2 \lesssim T/T_c \lesssim 4$. On these lattices we have collected between 200 and 400 configurations separated by 50 iterations (1 iter. $\equiv$ 7OR+1HB).

The spatial pseudo-potentials, extracted from smeared spatial Wilson loops are shown in Fig. 3. In $(3+1)$ dimensions at zero temperature the Coulomb term in the heavy quark potential is a $1/R$-term. This is expected to transform into a logarithmic behaviour at very high temperatures. In order to avoid a prejudice on the subleading $R$ dependence at large distances, we have extracted the string tension from a linear fit

$$V(R)a = v + \hat{\sigma}_s R \quad \text{for} \quad R > R_{\text{min}},$$

(7)

where we have varied the lower limit of the fitting range, $R_{\text{min}}$, until the string tension has been found to be independent of it within errors. For all three temperatures we find this to be the case for $R_{\text{min}} > 4$ [12]. The resulting values for the spatial string tension are summarized in Table 2.

The detailed investigation performed in the case of $SU(2)$ has shown that the temperature dependence of the spatial string tension above $2T_c$ is consistent with what one would expect from dimensional reduction if the Higgs-sector does not significantly contribute to the string tension, $\sqrt{\sigma_s} \sim g^2(T)T$, with $g^2(T)$ given by the 2-loop $\beta$-function. The same does seem to hold true in the case of $SU(3)$ as can be seen from Fig. 4 where we show the ratio $T/\sqrt{\sigma_s}$.
from our present analysis of the $SU(3)$ gauge theory as well as the data for $SU(2)$ [4]. Also shown in this figure is $T_c/\sqrt{\sigma}$, where $\sigma$ is the zero temperature ($T/T_c \simeq 0.25$) string tension obtained on a $32^4$ lattice [13], and a first result from an ongoing more detailed investigation of the spatial string tension on lattices with temporal extent $N_\tau = 6$ ($T/T_c \simeq 1.33$) [14]. In both cases the string tension has also been calculated at $\beta = 6.0$.

Figure 3: The heavy quark potential in (3+1)-dimensional $SU(3)$ gauge theory obtained from calculations on $32^3 \times N_\tau$ lattices, with $N_\tau = 2$ (circles), 3 (triangles) and 4 (squares) at $\beta = 6.0$.

Assuming the validity of 2-loop scaling for the running coupling constant,

$$g^{-2}(T) = \frac{11}{8\pi^2} \ln(T/\Lambda_T) + \frac{51}{88\pi^2} \ln(2 \ln(T/\Lambda_T)) ,$$

for temperatures larger than twice $T_c$ we find for the spatial string tension of (3+1)-dimensional $SU(3)$,

$$\sqrt{\sigma_s(T)} = (0.586 \pm 0.055)g^2(T)T , \quad T \geq 2T_c .$$

Here the additional free parameter in the $\beta$-function is determined as $\Lambda_T = 0.118(36)T_c$. 

7
4 DISCUSSION

We have studied the string tension in three-dimensional $SU(3)$ gauge theory and performed a first exploratory investigation of the spatial string tension in $(3+1)$-dimensional $SU(3)$ at finite temperature. We find a behaviour which closely resembles results found in the case of an $SU(2)$ gauge theory: For temperatures above $2T_c$ the spatial string tension is well described by the string tension of a three-dimensional gauge theory, $\sqrt{\sigma_s} \simeq \sqrt{\sigma_3}$, with $g_3^2 \equiv g^2(T)T$. This gives further support to the observation that the Higgs-sector in dimensionally reduced QCD does not contribute significantly to the spatial string tension.

From the temperature dependence of $\sqrt{\sigma_s}$ above $2T_c$ we find evidence for a running coupling constant. The coupling $g^2(T)$ turns out to be somewhat smaller than in the case of $SU(2)$ at comparable temperatures. For instance we find $g^2(2T_c) \simeq 2.0$ for the $SU(3)$ gauge theory compared to 2.7 in the case of $SU(2)$. Similarly the scale parameter in the perturbative 2-loop $\beta$-function is slightly larger. It is, however, interesting to note that in both cases the value of $\Lambda_T$ turns out to be of the order of 10% of the scale $\Lambda_{\overline{MS}}$ determined in
pure gauge (zero flavour) simulations at zero temperature for instance from the heavy quark potential \cite{13}. In fact, it is consistent with the relation, $\Lambda_T = \Lambda_{\overline{\text{MS}}} / 4\pi$, a combination which naturally occurs in the renormalization scale dependent terms in higher order perturbative calculations\cite{13,16}.

Acknowledgements:
We thank J. Fingberg and U. Heller for helpful discussions. The computations have been performed on the Cray Y-MP at HLRZ and the Q1 Quadrics at the University of Bielefeld. The work has been supported in part by the Deutsche Forschungsgemeinschaft under contracts Pe 340/3-2 and 340/6-1.

References

[1] C. Borgs, Nucl. Phys. B261 (1985) 455.

[2] E. Manousakis and J. Polonyi, Phys. Rev. Lett. 58 (1987) 847.

[3] T. Reisz, Z. Phys. C - Particles and Fields 53 (1992) 169 and references therein.

[4] G.S. Bali, J. Fingberg, U.M. Heller, F. Karsch and K. Schilling, Phys. Rev. Lett. 71 (1993) 3059.

[5] L. Kärkkäinen, P. Lacock, D.E. Miller, B. Petersson and T. Reisz, Phys. Lett. B312 (1993) 173.

[6] M. Caselle, R. Fiore, F. Gliozzi, P. Guaita and S. Vinti, Nucl. Phys. B422 (1994) 397.

[7] M. Caselle and A. D’Adda, The Spatial String Tension in High Temperature Lattice Gauge Theories, DFTT 8/94, March 1994.

[8] M. Teper, Phys. Lett. B311 (1993) 223.

\footnote{In Ref.\cite{13} the relation, $\Lambda_T = \Lambda_{\overline{\text{MS}}} / 4\pi \exp\{-\gamma_E\}$, has been suggested for the $\Lambda$-parameter entering the running coupling in the effective 3-$d$ theory obtained by dimensional reduction.}
[9] L. Kärkkäinen, P. Lacock, D.E. Miller, B. Petersson and T. Reisz, Phys. Lett. B282 (1992) 121.

[10] S.A. Gottlieb, J. Kuti, D. Toussaint, A.D. Kennedy, S. Meyer, B.J. Pendleton and R.L. Sugar, Phys. Rev. Lett. 55 (1985) 1958; H. Ding and N. Christ, Phys. Rev. Lett. 60 (1988) 1367.

[11] M. Lüscher, K. Symanzik and P. Weisz, Nucl. Phys. B173 (1980) 365; O. Alvarez, Phys. Rev. D24 (1981) 440.

[12] M. Lütgemeier, contribution to Lattice 94, to appear in Nucl. Phys. B (Proc.Suppl.).

[13] G.S. Bali and K. Schilling, Phys. Rev. D47 (1993) 661.

[14] G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lütgemeier and B. Petersson, work in progress.

[15] K. Farakos, K. Kajantie, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. B425 (1994) 67.

[16] P. Arnold and C. Zhai, The three-loop free energy for pure gauge QCD, University of Washington preprint, UW/PT-94-03.