Binary branching processes with Moran type interactions

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Abstract. The aim of this paper is to study the large population limit of a new collection of interacting particle systems (IPS) that encompasses branching models and fixed size Moran type IPS (see [8, 9, 22, 23, 66]). We define an IPS where particles evolve, reproduce and die independently and, with a probability that may depend on the configuration of the whole system, the death of a particle may trigger the reproduction of another particle, while a branching event may trigger the death of another one. We study the occupation measure of the new model, explicitly relating it to the Feynman-Kac semigroup of the underlying Markov evolution and quantifying the \(L^2\) distance between their normalisations. We show that our model outperforms the fixed size Moran type IPS when used to approximate a birth and death process. We discuss several other applications of our model including the neutron transport equation [18, 45] and population size dynamics.

Résumé. L’objectif de cet article est d’étudier la limite en grande population d’une nouvelle famille de systèmes de particules en interaction (IPS) qui englobe les modèles de branchement et les IPS de type Moran de taille fixe (voir [8, 9, 22, 23, 66]). Nous définissons un IPS où les particules évoluent, se reproduisent et meurent indépendamment et, avec une probabilité qui peut dépendre de la configuration du système entier, la mort d’une particule peut déclencher la reproduction d’une autre particule, tandis qu’un événement de branchement peut déclencher la mort d’une autre particule. Nous étudions la mesure d’occupation du nouveau modèle, en la reliant explicitement au semigroupe de Feynman-Kac de l’évolution markovienne sous-jacente et en quantifiant la distance \(L^2\) entre leurs normalisations. Nous montrons que notre modèle est plus performant que l’IPS de Moran de taille fixe lorsqu’il est utilisé pour approximer un processus de naissance et mort. Nous discutons de plusieurs autres applications de notre modèle, notamment l’équation de transport des neutrons [18, 45] et la dynamique de la taille des populations.

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1. Introduction

Branching processes naturally arise as pertinent models in population size dynamics [49, 50, 52], neutron transport [17], genetic dynamics [56], growth-fragmentation processes [5, 6] and cell proliferation kinetics [69], and as theoretical objects in their own right [21, 46–49]. These models are characterised by the independence of the branching and killing events in the system, which leads to a multiplicative behaviour, as well as interest in the scaling behaviour and the characterisation of the system at large times. On the other hand, processes with Moran type interactions are natural models for finite populations with either variety-increasing or variety-reducing effects such as genetic drift, genetic mutations and natural selection. First introduced by Moran [57], the Moran model describes the evolution of \(N\) genes such that at an exponential rate, two particles are chosen uniformly at random from the population, one of which is killed and the other splits in two. Thus, the independence between particles is lost. We refer the reader to [33] and references therein for an overview of this model and of its extensions. This type of resampling has since been employed in a range of particle system models to numerically solve Feynman-Kac models, see [15, 25, 26, 66] and references therein.

The model we consider in this paper provides a combination of these two types of dynamics: (natural) branching and killing, as well as Moran type interactions. More precisely, when the system is initiated from \(N\) particles, each particle evolves according to an independent copy of a given Markov process, \(X\), until either a (binary) branching or killing event happens. Here, binary refers to the fact that the particle is replaced by exactly two independent copies of itself. If such a
branching event occurs, with a probability that may depend on the configuration of the whole system, another particle is removed from the system a selection mechanism. Similarly, if a killing event occurs, with a probability which may also depend on the configuration of the whole system, another particle is duplicated via a resampling mechanism. We refer to this model as the binary branching model with Moran interactions, or BBMMI for short.

Our main result (cf. Theorem 2.6) gives an explicit relation between the average of the empirical distribution of the particle system and the average of the underlying Markov process $X$. Indeed, letting $m_T$ denote the empirical distribution of the particle system at time $T$, we show that for any $T \geq 0$,

$$
\mathbb{E} \left[ \Pi^A_T \Pi^B_T m_T f \right] = \mathbb{E}_{m_0} \left[ f(X_t) \exp \left( \int_0^t b(X_s) - \kappa(X_s) \, \text{d}s \right) 1_{t < \tau_0} \right],
$$

where $b$ is the branching rate, $\kappa$ is the soft killing rate, $\tau_0$ is the absorption time, $\Pi^A_T$ and $\Pi^B_T$ are weights that compensate for the resampling and selection events that occur up to time $T$, which we state explicitly in Theorem 2.6, and $\mathbb{E}$ is the expectation with respect to the law of $X$. As we will later discuss, by choosing particular model parameters, we recover both the classical many-to-one formula [42] and the unbiased estimator proved in [66] for fixed size Moran type genetic algorithms. The second part of Theorem 2.6 then provides a precise bound on the $L^2$ distance between the normalised semigroup on the right-hand side of (1) and the normalised occupation measure of the particle system. In particular, it states that this distance converges to zero as the number of initial particles tends to infinity.

In the third section of this article, we illustrate several applications of this model by focussing on a particular selection/resampling mechanism in our model. More precisely, we consider a binary branching process whose population size is constrained to remain in $\{N_{\min}, \ldots, N_{\max}\}$, where $2 \leq N_{\min} \leq N_{\max} < +\infty$. In order to constrain the size of the process, when the size of the population reaches $N_{\max}$ (resp. $N_{\min}$) and a natural branching (resp. killing) event occurs, we set the probability of selection (resp. resampling) to be 1. We often refer to this model as the $N_{\min}$--$N_{\max}$ model. As will be clear in the next section, the choice $N_{\min} = 0$ and $N_{\max} = +\infty$ is also allowed, so that the collection of models we consider ranges from constant population size models with neither branching nor killing (corresponding to the choice $N_{\min} = N_{\max}$) to branching population models (when $N_{\min} = 0$ and $N_{\max} = +\infty$) with or without a selection/resampling mechanism.

For comparable results concerning the resampling algorithm, we refer the reader to [16, 34, 58, 60] and to [27, 32, 54] for associated central limit theorems. Whether a central limit theorem can be proved in our setting remains open, but could be approached by carefully studying the square increments of the martingale decomposition used in the proof of Theorem 2.6.

Our model and results are also reminiscent of the genetic algorithms introduced by Del Moral (see [22, 23] and references therein), where a particle system with killing is constrained to remain at a constant size via a resampling mechanism. We also refer the reader to [8, 9, 15, 38, 60, 65] and references therein.

Finally, we note that our model fits into the more general class of controlled branching processes introduced by Svestka and Zubkov in [61], where the number of reproductive individuals in each generation depends on the size of the previous generation via a control function. This work was later extended by Yanev [68] to allow for discrete random control functions. We refer the reader to [37, 61, 64, 67] for discrete time versions of this process and to [36] for the continuous time version. In these articles, the authors study the convergence of the survival probability in different regimes, as well as the expected population size. We also note that the control function can be seen as a way to model immigration and emigration of particles, [67]. We refer the reader to [2, 4, 51, 53, 55, 59, 63] for results regarding these latter processes.

The rest of the paper is set out as follows. In the next section, we introduce some notation and assumptions, and formally describe the particle system introduced above. We will then state our main result in Theorem 2.6 and discuss several implications.

In section 3, we focus on the $N_{\min}$--$N_{\max}$ model and consider a variety of applications in order to illustrate the scope of our results, as well as some of the possible difficulties and extensions. In particular, in Section 3.1, we describe and study a stochastic population model with constrained size, based on the $N_{\min}$--$N_{\max}$ model. We give a sufficient condition ensuring that the number of resampling events does not explode in finite time when the killing rate of $X$ is unbounded. We also relate the long-time convergence of the empirical distribution of the population to the existence of a quasi-stationary distribution for an auxiliary sub-Markov process related to $X$ and show the uniform in time convergence of the renormalised empirical distribution of the particle system toward its conditional distribution, at a polynomial speed when $\sqrt{N_{\max}/N_{\min}}$ goes to infinity. We prove these results in the particular setting of this model, but our methods could be extended to a more general setting.

In Section 3.2, we deal with the particular case where the underlying Markov process is a piecewise deterministic Markov process (PDMP). In this case, it is possible for two particles to be killed at the same time, meaning that our main result is not directly applicable without further work. Working in the particular setting of the neutron transport equation
(NTE), see [18, 45], we overcome this difficulty using the notion of $h$-transform introduced in [18] for the NTE. We show that particles whose dynamics are given by an appropriately $h$-transformed process fulfill the assumptions of our main theorem, and thus, we may use the $N_{\text{min}} - N_{\text{max}}$ model with the $h$-transformed process in order to obtain estimates for quantities associated to the original process.

In Section 3.3, we focus on some numerical properties of the resampling/selection process. First, we make comparisons between the $N_{\text{min}}-N_{\text{max}}$ model and the fixed size Moran type interacting particle system (IPS) for the approximation of quantities related to non-conservative semigroups (cf. [8, 9, 22, 23]), which corresponds to the particular case $N_{\text{min}} = N_{\text{max}}$ with no branching. We observe that in the case of a birth-and-death process, $X_t$ and state-dependent Markov transitions, the computational cost, measured as the number of resampling/selection events, is significantly lower when using the $N_{\text{min}}-N_{\text{max}}$ algorithm. We also study the bias and standard deviation of an estimator for the normalised empirical stationary distribution for each of the $N_{\text{min}}-N_{\text{max}}$ model with the $h$-transform introduced in [18] for the NTE. We show that particles whose dynamics are given by an appropriately $h$-transformed process fulfil the assumptions of our main theorem, and thus, we may use the $N_{\text{min}} - N_{\text{max}}$ model with the $h$-transformed process in order to obtain estimates for quantities associated to the original process.

Finally, in Section 4 we provide the proof of Theorem 2.6.

2. Main results

2.1. Description of the model

Let $(\Omega, \mathcal{F}, (X_t)_{t \in [0, +\infty)})$ be a continuous time progressively measurable Markov process with values in a measurable state space $E \cup \partial$, where $\partial$ is an absorbing (measurable) set such that $\partial \cap E = \emptyset$. Denoting by $\tau_\partial = \inf \{t \geq 0, X_t \in \partial\}$ its absorption time, it follows that

$$X_t \in \partial, \forall t \geq \tau_\partial.$$ 

We also introduce the following notation. Denote by $P_x$ its law when initiated at $x \in E$ and by $E_x$ the corresponding expectation operator. We extend, whenever necessary, any measurable function $f : E \to [0, +\infty]$ by $f \equiv 0$ on $\partial$. We also assume that we are given two bounded functions $b : E \to \mathbb{R}_+$ and $\kappa : E \to \mathbb{R}_+$, denoting respectively, the single particle branching rate and the single particle killing rate.

We will shortly define a particle system, where particles move according to copies of $X$ between interactions. Before doing so, we make the following assumption, necessary for the system to be well defined. It entails that two independent copies of $X$ are absorbed simultaneously with probability 0.

**Assumption 1.** For any $x \in E$ and $t \in [0, +\infty)$, $P_x(\tau_\partial = t) = 0$ and $P_x(\tau_\partial > t) > 0$.

We also introduce the following notation. Denote by $P_f(\mathbb{N})$ the collection of finite subsets of $\mathbb{N} := \{1, 2, \ldots\}$. Then, for $i \in \mathbb{N}$, we let

$$b^i : (E \cup \partial)^{P_f(\mathbb{N})} \to [0, +\infty),$$

$$\kappa^i : (E \cup \partial)^{P_f(\mathbb{N})} \to [0, +\infty)$$

be a collection of bounded measurable functions. Also for $i \in \mathbb{N}$, let

$$p^i : (E \cup \partial)^{P_f(\mathbb{N})} \to [0, 1],$$

$$q^i : (E \cup \partial)^{P_f(\mathbb{N})} \to [0, 1]$$

be measurable functions. We will assume that, for any $s \in P_f(\mathbb{N})$, for any collection of points $\{x_i : i \in s, x_i \in E\}$, and any $i_0 \in s$,

$$b^{i_0}(x_i, i \in s) - \kappa^{i_0}(x_i, i \in s) = b(x_{i_0}) - \kappa(x_{i_0})$$  \hspace{1cm} (2)
and that
\[ p^{i_0}(x_i, i \in s) = 0 \text{ whenever } |s| = 1, \]
where $|s|$ denotes the number of elements in the set $s$. See Remarks 2.2 and 2.3 for a discussion of the implications of these conditions.

We now proceed with an algorithmic description of the dynamics of the particle system, which we call the binary branching model with Moran type interactions (BBMMI). The formal construction of the process is a non-trivial task, and is given in the supplementary material [19]. To this end, fix $\bar{N}_0 \geq 2$. We consider the particle system $(S_t, (X^i_t)_{i \in S_t})_{t \in [0, +\infty)}$, where the component $S_t \in \mathcal{P}_f(\{0, 1\})$ is the set enumerating the particles in the system at time $t$.

1. The particles $X^i, i \in \bar{S}_0$, evolve as independent copies of $X$, and we consider the following times:
\[
\tau^{b,i}_1 := \inf\{t \geq 0, \int_0^t b^i(X^i_s, i \in \bar{S}_0) ds \geq e^+_i\},
\]
and
\[
\tau^{\kappa,i}_1 := \inf\{t \geq 0, \int_0^t \kappa^i(X^i_s, i \in \bar{S}_0) ds \geq E^+_i\},
\]
and
\[
\tau^{\partial,i}_1 := \inf\{t \geq 0, X^i_t \in \emptyset\},
\]
where $e^+_i, E^+_i, i = 1, \ldots, \bar{N}_0$ are exponential random variables with parameter 1, and are independent of each other and everything else.

2. Denoting by $i_0$ the index of the (unique) particle such that $\tau^{b,i_0}_1 \wedge \tau^{\kappa,i_0}_1 \wedge \tau^{\partial,i_0}_1 = \tau_1$, where $\tau_1 = \min_{i \in \bar{S}_0} \tau^{b,i}_1 \wedge \tau^{\kappa,i}_1 \wedge \tau^{\partial,i}_1$, we set $\tilde{S}_t = \bar{S}_0$ for all $t < \tau_1$ and
   (a) if $\tau_1 = \tau^{b,i_0}_1$, then a branching event occurs;
   (b) if $\tau_1 = \tau^{\kappa,i_0}_1$, then a soft killing event occurs;
   (c) if $\tau_1 = \tau^{\partial,i_0}_1$, then a hard killing event occurs.

3. Then a resampling or selection event may occur, depending on the following situations:

   **killing:** if a (hard or soft) killing event occurred at the preceding step, then we say that $i_0$ is killed at time $\tau_1$ and
   - with probability $p^{i_0}(X^i_{\tau_1}, i \in \bar{S}_0)$, the particle $i_0$ is removed from the system and a resampling event occurs: one chooses $j_0$ uniformly in $\bar{S}_0 \setminus \{i_0\}$ and sets
     \[ X^{\max\bar{S}_0+1}_{\tau_1} = X^{j_0}_{\tau_1} \text{ and } \bar{S}_{\tau_1} := \bar{S}_0 \setminus \{i_0\} \cup \{\max \bar{S}_0 + 1\}; \]
     observe that the number of particles in the system at time $\tau_1$ is then $\bar{N}_{\tau_1} = \bar{N}_0$; we say that $j_0$ is duplicated at time $\tau_1$.
   - with probability $1 - p^{i_0}(X^i_{\tau_1}, i \in \bar{S}_0)$, the particle $i_0$ is removed from the system; the set of particles at time $\tau_1$ is enumerated by $\bar{S}_{\tau_1} := \bar{S}_0 \setminus \{i_0\}$ and the number of particles in the system is then $\bar{N}_{\tau_1} = \bar{N}_0 - 1$;

   **branching:** if a branching event occurred at the preceding step, then we say that $i_0$ has branched at time $\tau_1$ and
   - with probability $q^{i_0}(X^i_{\tau_1}, i \in \bar{S}_0)$, a new particle is added to the system at position $X^{j_0}_{\tau_1}$ and a selection event occurs: one chooses $j_0$ at random uniformly in $\bar{S}_0 \cup \{\max \bar{S}_0 + 1\}$ and removes the particle $j_0$ from the system:
     \[ X^{\max\bar{S}_0+1}_{\tau_1} = X^{j_0}_{\tau_1} \text{ and } \bar{S}_{\tau_1} := \{\max \bar{S}_0 + 1\} \cup \bar{S}_0 \setminus \{j_0\}; \]
     observe that the number of particles at time $\tau_1$ is thus $\bar{N}_{\tau_1} = \bar{N}_0$; we say that $j_0$ is removed at time $\tau_1$ and that $\max \bar{S}_0 + 1$ is born at time $\tau_1$;
• with probability $1 - q^i(X_{\tau_i}^i, i \in S_0)$, a new particle is added to the system at position $X_{\tau_1}^{i_0}$:

$$X_{\tau_1}^{\max S_0 + 1} = X_{\tau_1}^{i_0} \text{ and } \bar{S}_{\tau_1} = \bar{S}_0 \cup \{\max \bar{S}_0\},$$

and we say that $\max S_0 + 1$ is born at time $\tau_1$.

After time $\tau_1$ the system evolves as independent copies of $X$ until the next killing/branching event, denoted by $\tau_2$, and at time $\tau_2$ it may undergo a resampling/selection event as above. By iteration, we define the sequence $\tau_0 := 0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$.

We will also make use of the following assumption, which ensures that the process described above is well defined at any time $t \geq 0$.

**Assumption 2.** The sequence $(\tau_n)_{n \in \mathbb{N}}$ converges to $+\infty$ almost surely.
In words, the above dynamics describe an interacting particle system such that, between interactions, the individual particles evolve as copies of $X$, undergoing binary branching at rate $b^i$ (where $i$ denotes the index of the particle) and being absorbed either at rate $\kappa^i$ or when the particle hits an absorbing state. On the event that a particle has branched, with probability $p^i$, a uniformly chosen particle is removed from the system. Conversely, on the event that a particle has been absorbed, with probability $q^i$, a uniformly chosen particle is forced to branch.

Let us now make several remarks concerning the above particle system and potential parameter choices.

**Remark 2.1.** The enumeration of the particle system does not keep track of the genealogy of the particles in the system in order to simplify notation; this could be done by replacing $\mathcal{P}_f(\mathbb{N})$ by $\mathcal{P}_f(\Omega)$, the collection of finite subsets of \[ \Omega := \bigcup_{n \geq 0} \mathbb{N}^n. \]

A particle with label $v = v_1v_2\ldots v_n \in \Omega$ means that the particle is the $v_n$-th child of the $v_{n-1}$-th child $\ldots$ of $v_1$. In this case, the functions $b^i, \kappa^i, p^i, q^i$, should then be indexed by elements of $\Omega$ instead of $\mathbb{N}$.

**Remark 2.2.** The functions $b^{i_0}$ and $\kappa^{i_0}$ are the rate at which the particle $i_0$ in the system branches and is killed respectively (in addition to hard killing events occurring when a particle hits the absorbing set $\partial$). We emphasise that the balance condition (2) is crucial to ensure the relation with the semigroup $Q$ defined in the next section, and should not be considered a technical assumption. Extending our results to situations where $b^{i_0}$ and $\kappa^{i_0}$ do not satisfy (2) is the subject of ongoing work.

**Remark 2.3.** The functions $p^{i_0}$ (respectively $q^{i_0}$) denote the probability that a killing event (respectively a branching event) triggers a branching (respectively a killing) among the particles in the system. We call such an event a *resampling event* (respectively a *selection event*). In the context of Moran type models, they correspond to selection by death events (respectively selection by birth events). Condition (3) prevents a resampling event from occurring when the population decreases to 0 particles. We emphasize that, since $\partial$ is a set, $p^{i_0}$ may depend on the killing location.

**Remark 2.4.** The dynamics of the IPS allow one to constrain the size of the process to remain between two bounds $N_{\text{min}} \neq 1$ and $N_{\text{max}}$, with $0 \leq N_{\text{min}} \leq N_{\text{max}}$. In order to do so, one simply chooses

\[ p^{i_0}(x_i, i \in s) = 1_{|s| = N_{\text{min}}} \quad \text{and} \quad q^{i_0}(x_i, i \in s) = 1_{|s| = N_{\text{max}}}. \]

Throughout the rest of this article, we refer to the IPS with these choices for $p^i$ and $q^i$ as the $N_{\text{min}}$-$N_{\text{max}}$ process. Other natural choices include, for instance,

\[ p^{i_0}(x_i, i \in s) = \frac{1}{s+1} 1_{|s| \geq 2} \quad \text{and} \quad q^{i_0}(x_i, i \in s) = 1 - \frac{1}{s+1}, \]

i.e. resamplings become less probable when the number of particles increases and selection becomes stronger when the number of particles increases, respectively. Note also that, since $X$ is only assumed to be Markov, it can carry the time as a component, which would allow $N_{\text{min}}$ and $N_{\text{max}}$ to vary with time (in the former situation) or the selection pressure to vary with time (in the later example), by simply letting $p, q$ depend on time.

Condition (2) allows to consider models where the activity (branching/killing) of the particles increase when they are close or far from each others. For instance, if $(E, d)$ is a metric space with $b$ and $\kappa$ fixed,

\[ b^{i_0}(x_i, i \in s) = b(x_{i_0}) + \left( \sum_{i \in s} d(x_{i_0}, x_i) \right) \wedge 1 \quad \text{and} \quad \kappa^{i_0}(x_i, i \in s) = \kappa(x_{i_0}) + \left( \sum_{i \in s} d(x_{i_0}, x_i) \right) \wedge 1 \]

in which case the activity of the particle $i_0$ is increased when it is far from the other particles. A similar construction replacing $\sum_{i \in s} d(x_{i_0}, x_i)$ by its inverse would lead to an increased activity when $x_{i_0}$ is close the other particles.

**Remark 2.5.** In our model, we consider general killing, which corresponds to a conservative Markov process $X$ killed at a possibly unbounded rate. Finding general criteria ensuring that Assumptions 1 and 2 hold is usually involved (it is trivial when the killing rate is bounded and there is no hard killing). In Section 3.1 we provide a criterion based on a Foster-Lyapunov type assumption ensuring the non-explosion of the system when the killing rate $\kappa$ is unbounded (and when there is no hard killing) and also show in Section 3.2 that Assumption 2 holds true for a specific PDMP model, obtained as the $h$-transform of a Neutron Random Walk killed at the boundary of domain. There is also a substantial literature on these types of problems in the presence of hard killing for the more classical fixed size Moran type systems, and, in particular, the interested reader may find it useful to adapt the methods developed in [9, 38, 65, 66] to our setting.
In the rest of this article, we denote by $\sigma_n$ and $\rho_n$ the times of the $n^{th}$ selection and resampling events, respectively. We let $A_t$ denote the total number of resampling events up to time $t$, $B_t$ denote the total number of selection events up to time $t$ and we set $C_t = \inf \{ n \geq 0 : t < \tau_{n+1} \}$ to be the total number of events up to time $t$. We will also use the following quantities throughout,

\begin{equation}
\Pi^A_t := \prod_{n=1}^{A_t} \left( \frac{N_{\rho_n} - 1}{N_{\rho_n}} \right), \quad \Pi^B_t := \prod_{i=1}^{B_t} \left( \frac{N_{\sigma_n} + 1}{N_{\sigma_n}} \right).
\end{equation}

### 2.2. $L^2$ bounds for the empirical distribution of the process

For all bounded measurable functions $f : E \to \mathbb{R}$, all $t \geq 0$ and all $x \in E$, we set

\[ Q_t f(x) = \mathbb{E}_x \left[ f(X_t) \exp \left( \int_0^t (b(X_s) - \kappa(X_s)) \, ds \right) \mathbf{1}_{t<\tau_0} \right] , \]

where we recall that $\tau_0 = \inf \{ t \geq 0 : X_t \in \partial \}$. This defines a Feynman-Kac semigroup $(Q_t)_{t \geq 0}$, which is related to the binary branching model where particles move as copies of $X$ that are killed at rate $\kappa$ and branch at rate $b$ resulting in the creation of two independent copies of the original particle. The relation between $Q$ and this process is given by the well known many-to-one formula (see for instance [42] and references therein, see also Remark 2.8). In this section, we consider the BBMMI defined above and study its relation with $Q$.

In what follows, we let $m_t$ (resp $\hat{m}_t$) denote the empirical measure (resp. normalised empirical measure) of the particle system,

\begin{equation}
m_t = \sum_{i \in S_t} \delta_{X^i_t}, \quad \text{and} \quad \hat{m}_t = \frac{1}{N_t} m_t, \quad t \geq 0.
\end{equation}

The following result first provides an equality between a weighted version of the empirical measure of the system at any time $T$ and the semigroup $Q$ at time $T$. The second part is dedicated to the control of the $L^2$ distance between renormalised versions of $Q_T$ and $m_T$.

**Theorem 2.6.** Under Assumptions 1 and 2, the BBMMI satisfies, for all time $T \geq 0$ and all bounded measurable functions $f : E \to \mathbb{R}$, the following many-to-one formula

\begin{equation}
m_0 Q_T f = \mathbb{E} \left( \Pi^A_T \Pi^B_T m_T f \right).
\end{equation}

Moreover,

\begin{equation}
\left\| \frac{m_0 Q_T f}{m_0 Q_T \mathbf{1}_E} - \frac{m_T(f)}{m_T(\mathbf{1}_E)} \mathbf{1}_{X_T \neq 0} \right\|_2 \leq C \exp(c \| b \|_\infty T) \frac{\| f \|_\infty}{m_0 Q_T \mathbf{1}_E / N_0} \frac{1}{\sqrt{N_0}},
\end{equation}

where $C, c$ are positive constants.

The proof of this result is given in Section 4, where a variance estimate for the unbiased particle measure $\Pi^A_T \Pi^B_T m_T f$ is obtained (see Step 4 therein). Let us first make some further comments on the model and the above result.

**Remark 2.7.** In Theorem 2.6, $m_0$ is assumed deterministic with total mass $N_0$. The result immediately extends to the case where $m_0$ is random by taking the expectation in (6) and (7) (in the latter, the right hand term may be infinite). Moreover, the expressions may also be written in terms of the normalised measure, $\hat{m}$, in which case (6) becomes $\hat{m}_0 Q_T f = \mathbb{E} \left( \Pi^A_T \Pi^B_T \hat{m}_T f \right)$ and (7) can be written as

\[ \left\| \frac{m_0 Q_T f}{m_0 Q_T \mathbf{1}_E} - \frac{\hat{m}_T(f)}{\hat{m}_T(\mathbf{1}_E)} \mathbf{1}_{X_T \neq 0} \right\|_2 \leq C \exp(c \| b \|_\infty T) \frac{\| f \|_\infty}{m_0 Q_T \mathbf{1}_E} \frac{1}{\sqrt{N_0}}, \]

making explicit the ‘classical’ dependence $N_0^{-1/2}$ on the number of particles which appears on the right-hand side.
Remark 2.8. When \( p = q \equiv 0 \), the particle system has the dynamics of the classical binary branching process where, from their point of creation particles move as independent copies of \( X \) and, when at \( x \in E \), particles are either killed at rate \( \kappa(x) \) or they branch at rate \( b(x) \) at which point the parent particle is replaced by two daughter particles. In this situation, we have \( \Pi_A^\alpha = \Pi_B^b = 1 \) almost surely, and we thus recover the classical many-to-one formula:

\[
\psi_t(g)(x) = E_x \left[ e^{\int_t^0 b(X_s) - \kappa(X_s) ds} g(X_t) 1_{t < \tau_0} \right],
\]

where \( \psi_t(g) \) is the linear semigroup of the binary branching process.

It is also possible to treat branching processes with non-local branching. In this case, when a branching event occurs (at rate \( b(x) \)), the parent is removed from the system and replaced by two particles at (random) positions \( x_1, x_2 \in E \), which may be different to \( x \). Denoting by \( (\psi_t)_{t \geq 0} \) the linear semigroup of this process, in this case, the many-to-one formula is given by

\[
\psi_t(g)(x) = E_x \left[ e^{\int_t^0 b(Y_s) - \kappa(Y_s) ds} g(Y_t) 1_{t < \tau_0} \right],
\]

where \( (Y_t)_{t \geq 0} \) behaves like \( (X_t)_{t \geq 0} \), except that at rate \( 2b(Y_t) \), the process \( Y_t \) jumps to a new location given by \( (\delta_{x_1} + \delta_{x_2})/2 \). We refer the reader to section 3.2 for an example of a branching process with non-local branching.

Remark 2.9. Although we have remained in the setting of binary branching in this article, it should be possible to extend the definition of the BBMMI to allow for more general offspring distributions when a (natural) branching event occurs by modifying the selection mechanism. We intend to address this in future work.

Remark 2.10. The definition of our model allows a population with varying size by including branching, death and Moran type interactions, which are key features in the context of population dynamics. However, when restricted to the situation where \( p = q \equiv 1 \) (so that the size of the process remains constant equal to \( N_0 \)) we recover the standard Moran particle model (see [15, 26, 28, 60] for similar results), where the process is constrained to remain of constant size \( N_0 \) and with uniformly bounded killing rate. Our main contribution to this constant size setting is that our assumptions allow for hard killing events (a feature observed in several models, such as diffusion processes killed at the boundary of a domain), that we consider general Markov processes (allowing implicit dependence with respect to time and with respect to the past of the process), and do not require continuity of the branching/killing rates with respect to the empirical measure of the process.

Remark 2.11. In the fixed size setting with bounded killing rate, it is well known that strong mixing conditions on the semi-group \( Q \) are sufficient to entail uniform convergence with respect to time (see e.g. [22, 28, 29, 60], and more recently [1, 31, 58, 62]). The same proof as in [26], but using the convergence result of Theorem 2.6, also applies. More precisely, if one assumes that the killing rate is bounded (with no hard killing) and that there exists a probability measure \( \nu_Q \) on \( E \) and some constants \( C_Q, \gamma_Q > 0 \) such that

\[
\left\| \frac{\mu Q_t}{\mu Q_1} - \nu' \right\|_{TV} \leq C_Q e^{-\gamma_Q t},
\]

for all \( t \geq 0 \) and all probability measures \( \mu \) on \( E \), one deduces that, for all initial empirical measure \( m_0 \) of the particle system and all bounded measurable function \( f : E \to \mathbb{R}_+ \),

\[
\sup_{T \in [0, +\infty)} \left\| \frac{m_T(f)}{m_T(1_E)} - \frac{m_0 Q_T}{m_0 Q_1 1_E} \right\|_2 \leq (2C_Q + C) N_{\text{min}}^{-\alpha/2} \| f \|_{\infty},
\]

where \( \alpha = \gamma_Q/((c + 1)\|b\|_{\infty} + \|\kappa\|_{\infty} + \gamma_Q) \in (0, 1) \) and \( C, c \) are from Theorem 2.6. In particular,

\[
\left\| \frac{m_T(f)}{m_T(1_E)} - \nu_X(f) \right\|_2 \leq \left(2C_Q + C\right) N_{\text{min}}^{-\alpha/2} + C_Q e^{-\gamma_Q T} \| f \|_{\infty}.
\]

In Section 3.1, we show that such a time-uniform convergence can be proved when \( \kappa \) is not bounded, when \( C_Q \) depends on the initial distribution \( \mu \), and when \( f \) is not necessarily bounded.

Remark 2.12. The main property of the process \((\bar{S}_t, (X^1_t)_{t \geq 0})_{t \geq 0}\) that is used in the proof of Theorem 2.6 is the identity

\[
E^x \left[ Q_T - (t+s) f(X^1_{t+s}) 1_{t+s < \tau_1} \right] = E^x \left[ Q_T - (t+s) f(Y^1_{t+s}) e^{-\int_{t+s}^{t+\gamma} b_u du} 1_{t+s < \tau_0} \right] 1_{t < \tau_1},
\]
where \( E^t \) is the expectation on a probability space \( \Omega' \) such that \( (Y_u^i)_{u \geq t}, \ i \in \bar{S}_0 \), are independent copies of \( X \), starting from \( X^i_t \) at time \( t \), and where

\[
h_u := \sum_{j \in \bar{S}_0} b^j((Y_u^i)_{i \in \bar{S}_0}) + \kappa^j((Y_u^i)_{i \in \bar{S}_0}).
\]

This identity is derived from the Markov property, and the assumption on the dynamics of the process \((\bar{S}_t, (X^i_t)_{i \in \bar{S}_0})_{t \geq 0}\). As such, it is possible to see that Theorem 2.6 still holds under milder conditions on the construction. Specifically, suppose that there exist a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and a progressively measurable process \((\bar{S}_t, (X^i_t)_{i \in \bar{S}_0})_{t \geq 0}\) defined on this space, such that the rates \(b^j, \kappa^j, p^i, q^i\) are no longer functions, but merely progressively measurable processes, satisfying the equivalents\(^1\) of (2), (3):

\[
b^i_t = \kappa^i_t = b(X^i_t) - \kappa(X^i_t) \\
p^i_t = 0 \quad \text{whenever} \ |\bar{S}_t| = 1.
\]

Then the proof of Theorem 2.6 still holds, provided that condition (10) can be verified in the more general case, and where we interpret \( E^t \) as the expectation conditional on the full sigma-algebra \( \mathcal{F}_t \). Of course, such a property is something that may, in certain settings, be obtained through construction.

An example where such a construction might be natural is in a distributed computing environment. In this case, one may choose to impose \( N_{\min} - N_{\max}\)-like criterion on each separate processor. Roughly, each processor might be allocated an initial set of particles, which evolve without communication (by only having killing/branching events), so long as the number of particles remains within \([N_{\min}, N_{\max}]\). Since global properties of the particle system are only needed when resampling or selection events occur, the need for communication events in the system, which cause computational bottlenecks, could be reduced by choosing to allow killing or branching events provided the number of particles on a given processor does not move outside a specified range. In such a case, the number of particles on a given processor would depend on the allocation of particles to given processors (and which may itself be randomised independently of the history of the particle system). In practice, communication between servers will still be necessary, but potentially less frequently, and hence with lower computational burden, than with other methods. In these examples, the processes \(b^j\) and \(\kappa^j\) would not be simply functions of the particle positions (since we also need to know which particles are on the same processor), but would be expected to be \((\mathcal{F}_t)_{t \geq 0}\)-progressively measurable, for an appropriate filtration, which may then be larger than the filtration generated by the particle histories.

3. Applications

3.1. Population size dynamics with constrained population size

In this section, we introduce a branching model for a population where the size is constrained between \(N_{\min} \geq 2\) and \(N_{\max} \geq N_{\min}\) and demonstrate that the results from the previous section may be applied. In this model, we associate with each individual a health state \(x \in E := \mathbb{Z}^d_+\), \(d \geq 1\), which evolves according to the dynamics of a multi-dimensional birth and death process during the life of an individual. The parameters of the pure jump Markovian dynamics are denoted by

\[
\begin{cases}
\beta_i(x) &= \text{transition rate from } x \text{ to } x + e_i, \\
\delta_i(x) &= \text{transition rate from } x \text{ to } x - e_i,
\end{cases}
\]

where \(e_i \in \mathbb{Z}^d_+\) denotes the vector with 1 in the \(i^{th}\) position and 0 everywhere else, \(\beta_i, \delta_i : \mathbb{Z}^d_+ \to [0, +\infty)\) for all \(i \in \{1, \ldots, d\}\), and \(\delta_i(x) = 0\) for all \(x = (x_1, \ldots, x_d)\) such that \(x_i = 0\) (so that the process cannot leave the set \(\mathbb{Z}^d_+\)).

When its health state is \(x \in \mathbb{Z}^d_+\), at rate \(\kappa(x)\) an individual dies, where \(\kappa : \mathbb{Z}^d_+ \to [0, +\infty)\), and at rate \(b(x)\), it gives birth to an identical individual with equal health state \(x\), where \(b : \mathbb{Z}^d_+ \to [0, +\infty)\) is bounded. Note that, in our model, a higher health state may thus be associated with a worse health condition. This corresponds, for instance, to the situation where \(x\) is the size of a (harmful) parasite population or the concentration of a virus.

\(^1\)Technically, we need that the integrals of the respective processes are equal up to modification, that is, if we define \(t^1_{i,1} := \int_0^t (b^i_\cdot - \kappa^i_\cdot) \, ds\) and \(t^1_{i,2} := \int_0^t (b^i_\cdot - \kappa^i_\cdot) \, ds\), then we require the processes \(t^1_{i,1}\) and \(t^1_{i,2}\) to agree up to a modification. Note that as a consequence of the difference \(b(\cdot) - \kappa(\cdot)\) being bounded, this implies that the processes are also indistinguishable.
In addition, an external mechanism controls the total size of the individuals population: the total number of individuals is bounded above by $N_{\text{max}}$ by removing an individual chosen randomly and uniformly in the population when the population size reaches $N_{\text{max}} + 1$, and the total number of individuals is lower bounded by $N_{\text{min}}$ by cloning an individual chosen randomly and uniformly when the population size reaches $N_{\text{min}} - 1$ (in the situation where $N_{\text{min}} = 0$, this last mechanism does not operate).

Formally, the state space of our population model is $F := \bigcup_{N=N_{\text{min}}}^{N_{\text{max}}} [E]^N$, where $[E]^N$ denotes the unordered $N$-tuples of $E$. We represent the elements of $F$ by $x = [x_1, \ldots, x_N]$ and note that repetitions are allowed. The extended infinitesimal generator of the process acting on bounded measurable functions $f : F \rightarrow \mathbb{R}$, is given, for any $x = [x_1, \ldots, x_N] \in F$, by

$$L f(x) = \sum_{k=1}^{N} \sum_{i=1}^{d} \left( \beta_i(x_{k,i}) \Delta_{x,x + e_{k,i}} f + \delta_i(x_{k,i}) \Delta_{x,x - e_{k,i}} f \right)$$

$$+ \sum_{k=1}^{N} b(x_k) \Delta_{x,x | x_k} f + \sum_{k=1}^{N} \kappa(x_k) \Delta_{x,x | x_k} f(x)$$

$$+ 1_{N=N_{\text{min}}} \sum_{k=1}^{N} \kappa(x_k) \frac{1}{N-1} \sum_{\ell=1, \ell \neq k}^{N} \Delta_{x | [x_k], x | [x_\ell]} f$$

$$+ 1_{N=N_{\text{max}}} \sum_{k=1}^{N} b(x_k) \frac{1}{N+1} \sum_{\ell=1}^{N} (1 + 1_{\ell=k}) \Delta_{x | [x_k], x \cup \{x_\ell\} | [x_k]} f,$$

where

- $x_{k,i}$ is the $i^{\text{th}}$ component of $x_k$,
- $x + e_{k,i}$ is $x$ but with $x_{k,i}$ replaced by $x_{k,i} + 1$,
- $x - e_{k,i}$ is $x$ but with $x_{k,i}$ replaced by $x_{k,i} - 1$,
- for all $x, y$, $\Delta_{x,y} f := f(y) 1_{y \in F} - f(x) 1_{x \in F}$,
- for all $y \in E$, $x \cup \{y\} = [x_1, \ldots, x_N, y]$,
- $x \setminus [x_k] = [x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N]$.

In a similar manner to the previous section, we denote by $A_T$ the number of times an individual is duplicated by the size constraining mechanism before time $T$, and by $B_T$ the number of times an individual is removed from the population by the size constraining mechanism. We also set $(X(t))_{t \geq 0}$ to be the population process, $([X(t)])_{t \geq 0}$ its size, and $(X_t)_{t \geq 0}$ a multi-dimensional birth and death process with (state dependent) transition rates $\beta$ and $\delta$. The notation $\mathbb{E}_x$ (respectively $\mathbb{E}_x^\pi$) is used for the law of the population process $X$ starting from $x \in F$ (respectively for the law of $X$ starting from $x \in E$). We denote as usual by $\mathbb{E}_x$ and $\mathbb{E}_x^\pi$ the associated expectations.

The first difficulty is to ensure that the constrained branching model defined above does not explode in finite time, where explosion is interpreted in the sense that infinitely many jumps of the system happen in finite time. Indeed, since $\kappa$ is not assumed to be bounded, the rate at which new individuals are added to the system is not bounded. The following result provides a sufficient criterion for the non-explosion of the process but before stating it, let us first introduce some definitions that will be used in what follows.

We say that two functions $V$ and $\kappa$ are co-monotone if $(\kappa(x) - \kappa(y))(V(x) - V(y)) \geq 0$ for all $x, y \in E$. We say that $V$ and $\kappa$ are almost co-monotone if there exists a function $\kappa'$ co-monotone with $V$ such that $\kappa - \kappa'$ is bounded. We say that $V$ goes to infinity at infinity if the set of $x \in E$ such that $V(x) < c$ is finite for all $c > 0$.

**Proposition 3.1.** Assume that there exists a function $V : E \rightarrow [1, +\infty)$ almost co-monotone with $\kappa$, such that $V$ tends to infinity at infinity, and such that, for all $x \in E$,

$$\sum_{i=1}^{d} [\beta_i(x)(V(x + e_i) - V(x)) + \delta_i(x)(V(x - e_i) - V(x))] \leq CV(x),$$

for some constant $C \in \mathbb{R}$. Then the process with transition rates given by $L$ is non-explosive.

**Remark 3.2.** The proof of this non-explosion result makes use of the extended infinitesimal generator of the particle system and of Dynkin’s formula. As a consequence, its proof can be extended without too much effort to more general continuous time Markov processes, by replacing the left hand term in (11) by the extended infinitesimal generator of the
more general process. One difficulty can appear though when the process is not a simple pure jump process on a discrete state space: to prove in a general context that $\mathcal{L}$ (adapted to fit the general individual dynamics of the particles) is indeed the extended infinitesimal generator of the particle system can be challenging.

**Remark.** More complex resampling/selection mechanisms could be considered with only slight modifications of the proof of Proposition 3.1. For instance, the main line of the proof applies as well if $N_{\text{max}} = +\infty$, or if we assume that, immediately after each branching time and with probability $1 + \frac{N}{N_{\text{max}} + 1}$, a uniformly chosen individual is removed from the population, and/or that, immediately after each killing time and with probability $\frac{N}{N_{\text{max}} - 1}$, a uniformly chosen individual is duplicated (where $N$ is the population size just after the event). Note that these choices of parameters may be used to model an increasing competition between individuals when the size of the population increases.

**Proof of Proposition 3.1.** We define

$$\tilde{V} : F \to [1, +\infty)$$

$$\tilde{V}(x) = \sum_{k=1}^{N} V(x_k),$$

so that $\tilde{V}$ goes to infinity at infinity and for $x = [x_1, \ldots, x_N] \in F$, we have

$$\tilde{L}V(x) = \sum_{k=1}^{N} \sum_{i=1}^{d} \left[ \beta_i(x_k)(V(x_k + e_i) - V(x_k)) + \delta_i(x_k)(V(x_k - e_i) - V(x_k)) \right]$$

$$+ \sum_{k=1}^{N} \left( b(x_k) - \kappa(x_k) \right) V(x_k)$$

$$+ \mathbf{1}_{N=N_{\text{min}}} \sum_{k=1}^{N} \kappa(x_k) \frac{1}{N-1} \sum_{\ell=1, \ell \neq k}^{N} V(x_\ell)$$

$$- \mathbf{1}_{N=N_{\text{max}}} \sum_{k=1}^{N} b(x_k) \frac{1}{N+1} \sum_{\ell=1}^{N} (1 + \mathbf{1}_{\ell=k}) V(x_\ell).$$

Now let $\kappa'$ be co-monotone with $V$ such that $\kappa'' = \kappa - \kappa'$ is bounded. Then, using the fact that $b$ is bounded, we deduce that

$$\tilde{L}V(x) \leq (C + \|b\|_\infty)\tilde{V}(x) + \mathbf{1}_{N=N_{\text{min}}} \left( \frac{1}{N-1} \sum_{k=1}^{N} \kappa(x_k) \sum_{\ell=1, \ell \neq k}^{N} V(x_\ell) - \sum_{k=1}^{N} \kappa(x_k) V(x_k) \right)$$

$$= (C + \|b\|_\infty)\tilde{V}(x) + \mathbf{1}_{N=N_{\text{min}}} \left( \sum_{k=1}^{N} \kappa(x_k) \sum_{\ell=1}^{N} V(x_\ell) - N \sum_{k=1}^{N} \kappa(x_k) V(x_k) \right)$$

$$= (C + \|b\|_\infty)\tilde{V}(x) + \frac{N^2 \mathbf{1}_{N=N_{\text{min}}}}{N-1} \left[ \left( \sum_{k=1}^{N} \kappa(x_k) \right) \left( \sum_{k=1}^{N} V(x_k) \frac{1}{N} \right) - \sum_{k=1}^{N} \kappa(x_k) V(x_k) \right]$$

$$\leq (C + \|b\|_\infty + 4\|\kappa''\|_\infty)\tilde{V}(x)$$

$$+ \frac{N^2 \mathbf{1}_{N=N_{\text{min}}}}{N-1} \left[ \left( \sum_{k=1}^{N} \kappa'(x_k) \right) \left( \sum_{k=1}^{N} V(x_k) \frac{1}{N} \right) - \sum_{k=1}^{N} \kappa'(x_k) V(x_k) \right]$$
where we used the fact that \( \kappa' \) and \( V \) are co-monotone and the FKG inequality for the last inequality.

For all \( n \geq 1 \), let \( U_n = \{ x \in F, \max_{k \in \{1, \ldots, |x|\}} x_k > n \} \) and define the random variable \( \theta_n := \inf\{ t \geq 0, (X_t^1, \ldots, X_t^n) \in U_n \} \). Note that \( \theta_n \) is a stopping time since the constrained branching process is càdlàg. Moreover, denoting by \( C_n \) the set of points in \( F \) that can be reached in one jump from \( F \setminus U_n \), we observe that \( C_n \) is bounded and hence that \( V(X_t \wedge \theta_n) \) is almost surely uniformly bounded in \( t \geq 0 \). In particular, we deduce, using Dynkin’s formula and the inequality (12), that, for all \( n \geq 0 \), all \( t \geq 0 \) and all \( x \in F \),

\[
\mathbb{E}_x(\bar{V}(X_{t \wedge \theta_n})) \leq \bar{V}(x) + \left( C + \|b\|_\infty + 4\|\kappa''\|_\infty \right) \int_0^t \mathbb{E}_x(\bar{V}(X_{s \wedge \theta_n})) \, ds.
\]

Hence, using Grönwall’s inequality, we deduce that

\[
\mathbb{E}_x(\bar{V}(X_{t \wedge \theta_n})) \leq \bar{V}(x) e^{t(C + \|b\|_\infty + 4\|\kappa''\|_\infty)}.
\]

Since \( \{ X_{t \wedge \theta_n} \in U_n, \theta_n \leq t \} = \{ \theta_n \leq t \} \) up to a negligible event, we conclude that

\[
\inf_{y \in U_n} \bar{V}(y) \mathbb{P}_x(\theta_n \leq t) \leq \bar{V}(x) e^{t(C + \|b\|_\infty + 4\|\kappa''\|_\infty)}.
\]

Denote by \( \tau_\infty \) the explosion time of the process. Since the jump rate of the process is bounded up to time \( \theta_n \), we deduce that \( \theta_n \leq \tau_\infty \) almost surely. Hence

\[
\mathbb{P}_x(\tau_\infty \leq t) \leq \frac{\bar{V}(x) e^{t(C + \|b\|_\infty + 4\|\kappa''\|_\infty)}}{\inf_{y \in U_n} \bar{V}(y)} \xrightarrow{n \to +\infty} 0,
\]

so that \( \tau_\infty = +\infty \) almost surely. This concludes the proof.

Since the process defined in this section is a particular instance of the process described in Section 2.1, the following corollary is a direct consequence of Proposition 3.1 and Theorem 2.6.

**Corollary 3.4.** Under the assumptions of Proposition 3.1, we have, for all \( T > 0 \), all bounded measurable functions \( f : \mathbb{Z}^d_+ \to \mathbb{R} \) and all initial position \( X(0) \in F \),

\[
\mathbb{E}_{X(0)} \left[ \Pi_T^{-1} \Pi_T^B \sum_{k=1}^{[X(T)]} f(X(T)_k) \right] = \sum_{k=1}^{[X(0)]} \mathbb{E}_{X(0)_k} \left[ f(X_T) \exp \left( \int_0^T (b(X_s) - \kappa(X_s)) \, ds \right) \right]
\]

where

\[
\Pi_T^A = \left( \frac{N_{\min} - 1}{N_{\min}} \right)^{A_T} \quad \text{and} \quad \Pi_T^B = \left( \frac{N_{\max} + 1}{N_{\max}} \right)^{B_T}.
\]

In addition, for some constant \( C > 0 \),

\[
\left\| \sum_{k=1}^{[X(0)]} \mathbb{E}_{X(0)_k} \left[ f(X_T) \exp \left( \int_0^T (b(X_s) - \kappa(X_s)) \, ds \right) \right] \right\|_2 \leq \frac{C \exp(\|b\|_\infty T) \|f\|_\infty}{\sqrt{[X(0)]}} \frac{1}{\|X(0)\|}
\]

Our aim is now to study the long-time behaviour of the (normalised) empirical distribution of the health parameter across the population. As mentioned in Remark 2.11, adaptation of classical arguments from [22, 26, 28, 29] shows that bounded killing rate and a strong mixing conditions on the semi-group \( Q \) are sufficient to obtain uniform in time estimates. Using for instance the results of [14], one can checks that this is the case if \( \kappa \) is bounded and if there exist
constants $C_1 > 0$, $C_2 > 0$, $\eta_0 > 0$, $n_0 \geq 0$ such that, for all $x \in \mathbb{Z}_+^d$ satisfying $|x| \geq n_0$,

\begin{align}
\sum_{i=1}^{d} (\delta_i(x) - \beta_i(x)) & \geq C_1 |x|^{1+\eta_0} \\
\sum_{i=1}^{d} (\delta_i(x) - \beta_i(x)) & \geq C_2(\|b\|_{\infty} - b(x) + \kappa(x))|x|.
\end{align}

We consider now the more delicate situation where mixing condition is weaker, that is when (9) holds true but with $C_Q$ depending on the initial distribution. More precisely, we consider the situation where there exist two positive functions $\psi_1, \psi_2 : E \to (0, +\infty)$ such that $\psi_2 \leq \psi_1$ and such that, for all $f : E \to \mathbb{R}$ with $|f| \leq \psi_1$, we have

\[ \left| \frac{E_{\mu}(f(X_0) \exp\left(\int_0^t b(X_s) - \kappa(X_s) \, ds\right))}{E_{\mu}(\exp\left(\int_0^t b(X_s) - \kappa(X_s) \, ds\right))} - \nu_X(f) \right| \leq C e^{-\gamma t} \frac{\mu(\psi_1)}{\mu(\psi_2)}, \quad \forall t \geq 0, \]

where $\nu_X$ is a probability measure on $E$. This type of properties has been extensively studied in the past years and several criteria have been provided (see e.g. [3, 13, 24, 35, 41, 43]).

Compared to the strong mixing condition mentioned in Remark 2.11, the novelty of the following result is that we consider unbounded killing rates, that we allow unbounded test functions $f$ and that the convergence of the renormalized semi-group is non-uniform in the initial distribution.

**Theorem 3.5.** Assume that there exists $a_0 > 0$ such that

\[ \theta_N(x) \xrightarrow{|x| \to +\infty} +\infty. \]

where

\[ \theta_N(x) := \sum_{i=1}^{d} \left[ (e^{-a_0} - 1)\delta_i(x) - (e^{a_0} - 1)\beta_i(x) \right] + \frac{N}{N-1} \left( \kappa(x) - \max_{|y| \leq |x|} \kappa(y) \right). \]

Then the process is non-explosive and, for any sequence of initial positions $(X^{(N_{\min})}(0))_{N_{\min} \geq 2}$ such that

\begin{equation}
\sup_{N_{\min} \geq 2} \frac{1}{|X^{(N_{\min})}(0)|} \sum_{k=1}^{N_{\min}} \exp\left( a_0 |X^{(N_{\min})}(0)| \right) < +\infty,
\end{equation}

we have, for all $f : E \to \mathbb{R}$ such that $f(x) \leq \exp(ax)$ for some $a \in (0, a_0)$,

\[ \limsup_{N_{\min} \to +\infty} \sup_{T \in (0, +\infty)} \mathbb{E}_{X^{(N_{\min})}(0)} \left| \frac{m_0 Q_T f}{m_0 Q_T 1_E} - \tilde{m}_T f \right| = 0. \]

**Remark 3.6.** For the sake of readability, we only provide a proof for the particular model of this section. However, the method used therein can be easily adapted to a more general situation, at the expense of additional technicalities which depend on the underlying dynamic of the particles. In the general case, and technical difficulties put aside, $\theta_N(x)$ should be replaced by $\frac{C_W(x)}{W(x)} + \frac{N}{N-1} \left( \kappa(x) - \max_{|y| \leq |x|} \kappa(y) \right)$ for some well chosen Lyapunov function $W$ which goes to infinity at infinity.

**Remark 3.7.** This result can be adapted to include the case where $X$ is a Markov process with hard killing at a boundary, provided one can prove that the particle system does not degenerate toward the boundary. However this can be challenging, see for instance [7, 39, 65].

**Proof.** Let us first observe that our assumptions entail that, for all $a \in (0, a_0)$,

\[ (e^{-a} - 1) \sum_{i=1}^{d} \delta_i(x) - (e^a - 1) \sum_{i=1}^{d} \beta_i(x) \xrightarrow{|x| \to +\infty} +\infty. \]
According to [13] and setting \( \psi_{1,a} = \exp(a|x|) \) for all \( a \in (0, a_0] \) and \( x \in E = \mathbb{Z}^d_+ \), this implies that there exist a bounded positive function \( \psi_{2,a} \) and \( c_{2,a}, \gamma_{2,a} > 0 \) such that \( Q_t \psi_{2,a} \geq c_{2,a} e^{-\gamma_{2,a} t} \), a probability measure \( \nu_{QS} \) on \( E \) such that, for all \( f : E \to \mathbb{R} \) with \( |f| \leq \psi_{1,a} \),

\[
\left| \frac{E_{\mu} \left( f(X_t) \exp \left( \int_0^t b(X_s) - \kappa(X_s) \, ds \right) \right) - \nu_X(f)}{E_{\mu} \left( \exp \left( \int_0^t b(X_s) - \kappa(X_s) \, ds \right) \right) \nu_X(f)} \leq C_a e^{-\gamma_{2,a} t} \frac{\mu(\psi_{1,a})}{\mu(\psi_{2,a})}, \quad \forall t \geq 0,
\]

where \( C_a \) and \( \gamma_{2,a} \) are positive constant that may depend on \( a \in (0, a_0] \). Note that, in order to apply the results of [13], we can choose \( \psi_{2,a} = \psi_{2,a_0} \) for all \( a \in (0, a_0] \), so that we can choose \( \psi_{2,a} \) (resp. \( c_{2,a}, \gamma_{2,a} \)) to not depend on \( a \). For this reason, in what follows, we will use instead the notation \( \psi_2 \) (resp. \( c_2, \gamma_2 \)).

We define the functions \( W = \psi_{1,a_0} \) and \( \tilde{W} : F \to \mathbb{R}_+ \) by

\[
\tilde{W}(x) = \frac{1}{N} \sum_{k=1}^N W(x_k) = \frac{1}{N} \sum_{k=1}^N \exp(a_0 x_k).
\]

One checks that, for all \( x \in F \),

\[
\tilde{L} \tilde{W}(x) = \frac{1}{N} \sum_{k=1}^N \sum_{i=1}^d \left[ \beta_i(x_k)(W(x_k + e_i) - W(x_k)) + \delta_i(x_k)(W(x_k - e_i) - W(x_k)) \right]
+ \sum_{k=1}^N b(x_k) \left( \frac{W(x_k)}{N+1} - \frac{\tilde{W}(x)}{N+1} \right) + \sum_{k=1}^N \kappa(x_k) \left( \frac{W(x_k)}{N-1} - \frac{\tilde{W}(x)}{N-1} \right)
\leq -\frac{1}{N} \sum_{k=1}^N \theta_N(x_k) W(x_k) + \|b\|_\infty \tilde{W}(x) + \frac{N}{N-1} \left( \frac{1}{N} \sum_{k=1}^N \kappa(x_k) \right) \left( \frac{1}{N} \sum_{k=1}^N W(x_k) \right)
- \frac{1}{N-1} \sum_{k=1}^N \left( \max_{|g| \leq a_0} \kappa(g) \right) W(x_k)
\leq -\frac{1}{N} \sum_{k=1}^d \left( \theta_N(x_k) - \|b\|_\infty \right) W(x_k).
\]

Using Dynkin’s formula and the fact that \( \theta_N(x) \to +\infty \) when \( |x| \to +\infty \), this classically entails that the process is non-explosive and that there exists some constant \( M > 0 \) such that, for any \( X(0) \in F \),

\[
\sup_{t \geq 0} \mathbb{E}_{X(0)}(W(X(t))) \leq \tilde{W}(X(0)) + M,
\]

and hence, using (15), we deduce that

\[
\sup_{t \geq 0} \mathbb{P}_{x} \left( W(X_t) \geq A \right) \leq \sup_{t \geq 0} \frac{\mathbb{E}(\tilde{W}(X(t)))}{A} \xrightarrow{A \to +\infty} 0.
\]

Since \( \psi_2 \) is positive (and hence locally bounded away from 0) and \( W(x) \to +\infty \) when \( |x| \to +\infty \), we also deduce that

\[
\sup_{t \geq 0} \mathbb{P}_{x} \left( \tilde{\psi}_2(X(t)) \leq \varepsilon \right) \xrightarrow{\varepsilon \to 0} 0,
\]

where \( \tilde{\psi}_2(x) = \frac{1}{N} \sum_{k=1}^N \psi_2(x_k) \).

From now on, fix \( a \in (0, a_0] \) and let \( f : E \to \mathbb{R} \) such that \( |f| \leq \psi_{1,a} \). Then, for all \( T \geq 0 \), we have, setting \( p = a_0/a > 1 \) and using Jensen’s inequality,

\[
\mathbb{E}_{X(0)} \left( \left| \frac{m_0 Q_T f}{m_0 Q_T 1_E} - \tilde{m}_T f \right| \right) \leq \mathbb{E}_{X(0)} \left( \left| \frac{m_0 Q_T f}{m_0 Q_T 1_E} - \tilde{m}_T f \right| 1_{\tilde{m}_T - \psi_{1,a} \leq A \text{ and } \tilde{m}_T - \psi_2 \geq \varepsilon} \right)
+ \frac{m_0 Q_T \psi_{1,a}}{m_0 Q_T 1_E} \mathbb{P}_{X(0)} \left( \tilde{m}_T - \psi_{1,a} > A \text{ or } \tilde{m}_T - \psi_2 < \varepsilon \right)
\]
we decompose the first term as follows: for any $t$ so that, using the inequality (21)

\[
\begin{align*}
\mathbb{E}_{X(0)} \left( \frac{m_0 Q_T f}{m_0 Q_T 1_E} - \bar{m}_T f \right) 1_{E_{A,\varepsilon}} & \leq \mathbb{E}_{X(0)} \left( \frac{m_0 Q_T f}{m_0 Q_T 1_E} - \bar{m}_T f \right) 1_{E_{A,\varepsilon}} \\
& \quad + \mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t f}{\bar{m}_T Q_t 1_E} - \bar{m}_T f \right) 1_{E_{A,\varepsilon}} \\
& \leq C a e^{-\gamma t} A + \mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t f}{\bar{m}_T Q_t 1_E} - \bar{m}_T f \right) 1_{E_{A,\varepsilon}}.
\end{align*}
\]

where $A$ and $\varepsilon$ go to 0 by (18) and (19). Writing $E_{A,\varepsilon} = \{ \bar{m}_T - \tau \psi_1, A \leq A \text{ and } \bar{m}_T - \tau \psi_2 \geq \varepsilon \}$, we decompose the first term as follows: for any $t \in [0, T]$, we have

\[
\begin{align*}
\mathbb{E}_{X(0)} \left( \frac{m_0 Q_T f}{m_0 Q_T 1_E} - \bar{m}_T f \right) 1_{E_{A,\varepsilon}} & \leq \mathbb{E}_{X(0)} \left( \frac{m_0 Q_T f}{m_0 Q_T 1_E} - \bar{m}_T f \right) 1_{E_{A,\varepsilon}} \\
& \quad + \mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t f}{\bar{m}_T Q_t 1_E} - \bar{m}_T f \right) 1_{E_{A,\varepsilon}} \\
& \leq C a e^{-\gamma t} A + \mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t f}{\bar{m}_T Q_t 1_E} - \bar{m}_T f \right) 1_{E_{A,\varepsilon}}.
\end{align*}
\]

where we used (16). Then, for any $K > 0$, we have

\[
\begin{align*}
\mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t f}{\bar{m}_T Q_t 1_E} - \bar{m}_T f \right) 1_{E_{A,\varepsilon}} & \leq \mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t f}{\bar{m}_T Q_t 1_E} - \bar{m}_T (f \wedge K) \right) 1_{E_{A,\varepsilon}} \\
& \quad + \mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t (\psi_1, A, f \wedge K)}{\bar{m}_T Q_t 1_E} \right) + \mathbb{E}_{X(0)} \left( \bar{m}_T (\psi_1, A, f \wedge K) \right) \\
& \leq E \left( \frac{\|Q_1 1_E\|_{\infty}}{\bar{m}_T Q_t 1_E} \right) K + K^{1-a_0/a} E_{X(0)} \left( \frac{\bar{m}_T Q_t W}{\bar{m}_T Q_t 1_E} \right) \\
& \quad + K^{1-a_0/a} E_{X(0)} \left( \frac{\bar{m}_T Q_t W}{\bar{m}_T Q_t 1_E} \right) + m_T W
\end{align*}
\]

where we used Theorem 2.6. According to [13], there exists some constant $C > 0$ such that

\[
\bar{m}_T Q_t W \leq C + \bar{m}_T - \tau W = C + \bar{W}(X(t)).
\]

Using in addition (17) and the fact that $m_T - \tau 1_E \geq N_{\text{min}}$, we deduce that

\[
\begin{align*}
\mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t f}{\bar{m}_T Q_t 1_E} - \bar{m}_T f \right) 1_{E_{A,\varepsilon}} & \leq \mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t f}{\bar{m}_T Q_t 1_E} - \bar{m}_T (f \wedge K) \right) 1_{E_{A,\varepsilon}} \\
& \quad + \mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t (\psi_1, A, f \wedge K)}{\bar{m}_T Q_t 1_E} \right) + \mathbb{E}_{X(0)} \left( \bar{m}_T (\psi_1, A, f \wedge K) \right) \\
& \leq E \left( \frac{\|Q_1 1_E\|_{\infty}}{\bar{m}_T Q_t 1_E} \right) K + K^{1-a_0/a} (C + 2M + 2M'),
\end{align*}
\]

where $M' > 0$ is the finite supremum in 15. Assuming without loss of generality that $\tau \leq 1$, we observe that

\[
\bar{m}_T Q_t 1_E \geq \bar{m}_T - \tau 1_E \geq \psi_2 \geq \tau e^{-\gamma t} \bar{m}_T - \tau 1_E \geq \tau e^{-\gamma t} \varepsilon \text{ on } E_{A,\varepsilon},
\]

so that, using the inequality $\|Q_1 1_E\|_{\infty} \leq e^{\|b\|_{\infty} t}$,

\[
\begin{align*}
\mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t f}{\bar{m}_T Q_t 1_E} - \bar{m}_T f \right) 1_{E_{A,\varepsilon}} & \leq \mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t f}{\bar{m}_T Q_t 1_E} - \bar{m}_T (f \wedge K) \right) 1_{E_{A,\varepsilon}} \\
& \quad + \mathbb{E}_{X(0)} \left( \frac{\bar{m}_T Q_t (\psi_1, A, f \wedge K)}{\bar{m}_T Q_t 1_E} \right) + \mathbb{E}_{X(0)} \left( \bar{m}_T (\psi_1, A, f \wedge K) \right) \\
& \leq e^{\|b\|_{\infty} t} \frac{K}{e^{2, a_0 e^{-\gamma t} \varepsilon}} + K^{1-a_0/a} (C + 2M + 2M').
\end{align*}
\]

Finally, using this, (20) and (21), we deduce that

\[
\begin{align*}
\mathbb{E}_{X(0)} \left| \frac{m_0 Q_T f}{m_0 Q_T 1_E} - \bar{m}_T f \right| & \leq C a e^{-\gamma t} A + \frac{e^{\|b\|_{\infty} t}}{e^{2, a_0 e^{-\gamma t} \varepsilon}} \frac{K}{e^{N_{\text{min}}}} + K^{1-a_0/a} (C + 2M + 2M') + g(A, \varepsilon),
\end{align*}
\]

where $g(A, \varepsilon) \to 0$ when $A$ or $\varepsilon$ goes to 0. We easily deduce that, for all $\eta > 0$, the exists $T_{\eta} > 0$ such that

\[
\limsup_{N_{\text{min}} \to +\infty} \sup_{T \geq T_{\eta}} \mathbb{E}_{X(0)} \left| \frac{m_0 Q_T f}{m_0 Q_T 1_E} - \bar{m}_T f \right| \leq \eta.
\]

In addition, it is straightforward from Theorem 6 that

\[
\limsup_{N_{\text{min}} \to +\infty} \sup_{T \leq T_{\eta}} \mathbb{E}_{X(0)} \left| \frac{m_0 Q_T f}{m_0 Q_T 1_E} - \bar{m}_T f \right| = 0.
\]
We conclude that
\[
\limsup_{N \to \infty} \sup_{T \in [0, +\infty]} \mathbb{E}_{X(0)} \left| \frac{m_0 Q_T f}{m_0 Q_T 1_E} - \bar{m}_T f \right| = 0.
\]

### 3.2. Piecewise deterministic Markov processes

Due to Assumption 1 in Section 2, the results presented thus far do not hold for the case where \((X_t)_{t \geq 0}\) is a piecewise deterministic Markov process (PDMP) and \(E\) is a bounded domain with absorbing boundary conditions, since two independent copies of the process may hit \(\partial E\) at the same time. In this section, we consider a particular example, namely the neutron transport equation (NTE), in order to discuss a way to avoid this problem.

The NTE is a balance equation that describes the behaviour of neutrons in a fissile medium such as a nuclear reactor. In such systems neutrons move in straight lines with a fixed speed until they either come into contact with the boundary of the reactor, at which point they are absorbed, or they collide with the nucleus of an atom. When the latter occurs, either the neutron undergoes a scattering event where the particle bounces off the nucleus and continues its motion but with a new velocity, or a fission event occurs where the collision causes new neutrons to be produced with identical spatial positions, but potentially different velocities.

In this setting, we have \(E = D \times V\) where \(D \subset \mathbb{R}^3\) is the spatial domain, which we assume to be open and bounded with smooth boundary, \(\partial D\), and \(V = \{v \in \mathbb{R}^3 : v_{min} < |v| < v_{max}\}\) is the velocity domain, where \(0 < v_{min} \leq v_{max} < \infty\). The NTE can then be stated as follows:

\[
\frac{\partial}{\partial t} \psi_t(r, v) = v \cdot \nabla \psi_t(r, v) - (\sigma_s(r, v) + \sigma_f(r, v))\psi_t(r, v)
\]

\[
+ \sigma_s(r, v) \int_V \psi_t(r, v') \pi_s(r, v, v') dv' + \sigma_f(r, v) \int_V \psi_t(r, v') \pi_f(r, v, v') dv',
\]

where

\[
\sigma_s(r, v) : \text{the rate at which scattering occurs from incoming velocity } v \text{ at position } r,
\]

\[
\sigma_f(r, v) : \text{the rate at which fission occurs from incoming velocity } v \text{ at position } r,
\]

\[
\pi_s(r, v, v') : \text{probability density that an incoming velocity } v \text{ at position } r \text{ scatters to an outgoing velocity, with probability } v' \text{ satisfying } \int_V \pi_s(r, v, v') dv' = 1, \text{ and}
\]

\[
\pi_f(r, v, v') : \text{density of expected neutron yield at velocity } v' \text{ from fission with incoming velocity } v \text{ satisfying } \int_V \pi_f(r, v, v') dv' < \infty.
\]

Moreover, we impose the following initial and boundary conditions

\[
\begin{align*}
\psi_0(r, v) &= g(r, v) \quad \text{for } r \in D, v \in V, \\
\psi_t(r, v) &= 0 \quad \text{for } r \in \partial D \text{ if } v \cdot n_r > 0,
\end{align*}
\]

where \(n_r\) is the outward facing normal of \(D\) at \(r \in \partial D\) and \(g : D \times V \to [0, \infty)\) is a bounded, measurable function.

Under the assumptions,

(A1) \(\sigma_s, \sigma_f, \pi_s\) and \(\pi_f\) are uniformly bounded away from infinity.

(A2) \(\inf_{r \in D, v, v' \in V} (\sigma_s(r, v)\pi_s(r, v, v') + \sigma_f(r, v)\pi_f(r, v, v')) > 0,
\]

it was shown in [17, 44, 45] that the NTE can be modelled using a weighted PDMP, also known as the neutron random walk (NRW). More precisely, setting

\[
\alpha(r, v) = \sigma_s(r, v) + \sigma_f(r, v) \int_V \pi_f(r, v, v') dv'
\]

\[
\pi(r, v, v') = (\alpha(r, v))^{-1} [\sigma_s(r, v)\pi_s(r, v, v') + \sigma_f(r, v)\pi_f(r, v, v')],
\]

\[
\beta(r, v) = \alpha(r, v) - \sigma_s(r, v) - \sigma_f(r, v),
\]

...
we define the NRW \(((R_t, \Upsilon_t)_{t \geq 0}, P_{(r,v)})\) as follows. From an initial configuration \((r,v) \in D \times V\), the particle will propagate linearly until it is either absorbed at the boundary of the domain or, at rate \(\alpha\) the process scatters off a nucleus and a new velocity is chosen according to \(\pi\). In order words, the infinitesimal generator associated to \(((R_t, \Upsilon_t)_{t \geq 0}, P)\) is given by

\[
Lf(r,v) = T f + \alpha(r,v) \int_V (f(r,v') - f(r,v)) \pi(r,v,v'), \, dv,
\]

where \(T\) is defined for regular functions by \(T f(r,v) = v \cdot \nabla_r f(r,v)\) and, more generally and when the limit exists, by

\[
T f(r,v) = \lim_{\varepsilon \to 0^+} \frac{f(r+\varepsilon v) - f(r,v)}{\varepsilon}.
\]

Then, for a bounded, measurable function \(g\), and \((r,v) \in D \times V\)

\[
\psi_t[g](r,v) := E_{(r,v)} \left[ \int_0^t \beta(R_s, \Upsilon_s) ds \, g(R_t, \Upsilon_t) 1_{(t < \tau_D)} \right],
\]
solves (22)–(23), where \(\tau_D\) is the first exit time of the NRW from \(D\).

It was also shown in [44] that if (A1) and (A2) are satisfied, and that if there exists an \(\varepsilon > 0\) such that

1. \(D_\varepsilon := \{ r \in D : \inf_{y \in \partial D} |r - y| > \varepsilon \text{max} \}\) is non-empty and connected,
2. there exist \(0 < s_\varepsilon < t_\varepsilon\) and \(\gamma > 0\) such that, for all \(r \in D \setminus D_\varepsilon\), there exists \(K_r \subset V\) measurable such that \(\text{Vol}(K_r) \geq \gamma > 0\) and for all \(v \in K_r\), \(r + vs \in D_\varepsilon\) for every \(s \in [s_\varepsilon, t_\varepsilon]\) and \(r + vs \notin \partial D\) for all \(s \in [0, s_\varepsilon]\),

the semigroup \((\psi_t)_{t \geq 0}\) exhibits the following Perron Frobenius behaviour.

![Fig 2: Illustration of B2 in \(\mathbb{R}^2\). Assumption B2 implies that the set \(\{r + vs : v \in K_r, s \in [s_\varepsilon, t_\varepsilon]\}\) \(\subset D_\varepsilon\), which is illustrated by the area enclosed by \(v_1s_\varepsilon\) and \(v_2t_\varepsilon\), with \(v_1 = \min\{|v| : v \in K_r\}\) and \(v_2\) denoting the maximum.](image)

**Theorem** ([44]). There exists a \(\lambda_+ \in \mathbb{R}\), a positive right eigenfunction \(\varphi \in L_{\infty}^+(D \times V)\) and a left eigenmeasure which is absolutely continuous with respect to Lebesgue measure on \(D \times V\) with density \(\tilde{\varphi} \in L_{\infty}^+(D \times V)\), both having associated eigenvalue \(e^{\lambda_+ t}\), and such that \(\varphi\) (resp. \(\tilde{\varphi}\)) is uniformly (resp. a.e. uniformly) bounded away from zero on each compactly embedded subset of \(D \times V\). In particular, for all \(g \in L_{\infty}^+(D \times V)\),

\[
\langle \tilde{\varphi}, \psi_t[g] \rangle = e^{\lambda_+ t} \langle \tilde{\varphi}, g \rangle \quad \text{(resp.} \psi_t[\varphi] = e^{\lambda_+ t} \varphi), \quad t \geq 0.
\]

Moreover, there exist \(C, \varepsilon > 0\) such that

\[
\sup_{g \in L_{\infty}^+(D \times V) : \|g\|_{\infty} \leq 1} \|e^{-\lambda_+ t} \varphi^{-1} \psi_t[g] - (\tilde{\varphi}, g)\|_{\infty} \leq C e^{-\varepsilon t}, \text{ for all } t \geq 0 \text{ large enough.}
\]
This theorem therefore provides a way to build Monte Carlo simulations to estimate the eigenvalue $\lambda_*$ and the eigenfunctions $\phi$ and $\tilde{\phi}$ via the NRW. We refer the reader to [18] for further details. However, although simulating a single weighted path has advantages over simulating an entire tree of neutrons, the transience of this process means that many of the particles exit the domain relatively quickly and therefore, a large number of simulations are required in order to obtain information about the system. This also leads to problems with the variance of the estimators. In order to deal with this problem, the notion of ‘$h$-transform’ was developed in [18], where one biases the NRW to prevent it from exiting the domain. This idea is similar to that presented in [60], where the author use $h$-transforms to build Monte-Carlo approximations. Therein, the main result requires that the resulting $h$-transformed process has bounded killing rate. In our case, although this can be theoretically obtained, the natural candidates for function $h$ result in a process with unbounded killing rate. We show below that the associated BBMMI is non-explosive, which ensures that Theorem 2.6 applies.

More precisely, let $h$ be a bounded positive function on $D \times V$ such that $h = 0$ on $\partial D$ and define the following change of measure,

$$
\frac{d\mathbb{P}^h_{(r,v)}}{d\mathbb{P}^{r,v}}_{\sigma((R_s,T_s),s\leq t)} := \exp \left( - \int_0^t \frac{\partial h(R_s,T_s)}{h(R_s,T_s)} ds \right) \prod_{i=1}^{N_t} \frac{h(R_{T_i},T_{T_i})}{h(R_{T_i-1},T_{T_i-1})} \mathbf{1}_{(t \leq \tau^D)},
$$

where

$$
\tilde{\alpha}(r,v) = \alpha(r,v) \int_V [g(r,v') - g(r,v)] \pi(r,v,v') dv',
$$

$(T_i)_{i \geq 0}$ are the scatter times of the NRW with $T_0 = 0$, and $N_t = \sup \{i : T_i \leq t\}$.

Then, $((R, \tau), \mathbb{P}^h)$ also defines a NRW but with scattering operator

$$
\tilde{\alpha}(r,v) = \alpha(r,v) \int_V [g(r,v') - g(r,v)] \frac{h(r,v')}{h(r,v)} \pi(r,v,v') dv'.
$$

The idea is that the factor $h(r,v')/h(r,v)$ in the above integral forces the NRW to scatter when it is approaching the boundary, thus preventing it from being killed. Thus, if we can show that this process satisfies Assumption 1, we may use the BBMMI process in conjunction with the $h$-transformed process to estimate the leading eigentriple $(\lambda_*, \phi, \tilde{\phi})$, for example. Indeed, it is straightforward to show that the leading eigentriple of the $h$-transformed process is given by $(\lambda_*, \phi/h, \tilde{\phi})$ and so we may use the BBMMI process to estimate this triple, and hence obtain an estimate for the original quantities $(\lambda_*, \phi, \tilde{\phi})$. The rest of this section is dedicated to verifying Assumption 1 for the transformed process. We refer the reader to Section 3.3 for the numerical aspects.

To this end, fix $\delta = 1/(2\|\alpha\|_{\infty})$ and let $\phi : [0, +\infty) \to [0, \delta]$ be a regular non-decreasing function such that $\phi(x) = x$ for all $x \in [0, \delta/2]$ and $\phi(x) = \delta$ for all $x \geq \delta$. We set

$$
h(r,v) = \phi\left( \kappa_{r,v}^P \right),
$$

where $\kappa_{r,v}^P = \inf \{ t > 0 : r + vt \notin D \}$. Using the following result, one can express expectations with respect to $\mathbb{P}$ as Feynman-Kac formulas related to $\mathbb{P}^h$. In what follows, we say that $(r,v)$ is in a vicinity of $\partial D$ if $h(r,v) < \delta/2$.

**Lemma 3.32.** Suppose (A1), (A2) hold. A process with law $\mathbb{P}^h_{(r,v)}$ does not hit the boundary with probability one. Moreover, setting $b = \left( \frac{Lh}{\kappa} \right)_+$ and $\kappa = \left( \frac{Lh}{\kappa} \right)_-$, where $L$ was defined in (27), the function $b$ is bounded on $D \times V$ and $\kappa$ is locally bounded. Let $\mathbb{P}^h_{r,v}$ be the law of a process with law $\mathbb{P}^h_{r,v}$ with additional soft killing at rate $\kappa$, then we have

$$
\mathbb{E}_{(r,v)} \left[ f(R_t, \tau_D) \mathbf{1}_{(t < \tau_D)} \right] = h(r,v) \mathbb{E}_{(r,v)}^h \left[ \exp \left( \int_0^t b(R_s, \tau_s) ds \right) \frac{f(R_t, \tau_s)}{h(R_t, \tau_s)} \mathbf{1}_{(t < \tau_s)} \right],
$$

where $\tau_s$ denotes the soft killing time of the process with law $\mathbb{P}^h_{r,v}$.

**Proof.** We have

$$
\psi_t[g](r,v) = \mathbb{P}^h_{(r,v)} \left[ \exp \left( \int_0^t \frac{\partial h(R_s, \tau_s)}{h(R_s, \tau_s)} + \beta(R_s, \tau_s) ds \right) \prod_{i=1}^{N_t} \frac{h(R_{T_i}, \tau_{T_i-1})}{h(R_{T_i-1}, \tau_{T_i-1})} g(R_t, \tau_1) \mathbf{1}_{(t < \tau_D)} \right],
$$

where $\tau_D$ denotes the killing time of the process with law $\mathbb{P}^h_{r,v}$.
Note that
\begin{equation}
\inf_{r \in D, v \in V} \lim_{s \to -\kappa_{r,v}^D} \frac{|v| (\kappa_{r,v}^D - s)}{h(r + vs, v)} > 0,
\end{equation}
where $\kappa_{r,v}^D := \inf\{t > 0 : r + vt \notin D\}$. Hence, Theorem 7.1 in [18] implies that, under $P_{(r,v)}^h$, $r \in D, v \in V$, $(R, \Upsilon)$ does not hit the boundary with probability one. In particular, the conditions of Remark 7.1 of [18] are satisfied and hence
\begin{equation}
\frac{dP_{(r,v)}^h}{dP_{(r,v)}}_{\sigma((r_1, \Upsilon_1), s \leq t)} = \exp \left( - \int_0^t \frac{Lh(R_s, \Upsilon_s)}{h(R_s, \Upsilon_s)} ds \right) \frac{h(R_t, \Upsilon_t)}{h(r, v)} 1_{(t < \tau^D)}(r, \Upsilon_t),
\end{equation}
where we recall that $L$ was defined in (27). Finally, we obtain, for all measurable functions $f : D \times V \to [0, +\infty),$
\begin{equation}
E_{(r,v)}[f(R_t, \Upsilon_t) 1_{(t < \tau^D)}] = h(r, v) E_{(r,v)}^{h} \left[ \exp \left( \int_0^t \frac{Lh(R_s, \Upsilon_s)}{h(R_s, \Upsilon_s)} ds \right) \frac{f(R_t, \Upsilon_t)}{h(r, v)} 1_{(t < \tau^D)}(r, \Upsilon_t) \right],
\end{equation}
where have used the fact that $1_{(t < \tau^D)} = 1$, $P_{(r,v)}^h$-almost surely, deduced from (35). Now, introducing $b = \left( \frac{Lh}{\kappa} \right)_-$ and $\kappa = \left( \frac{Lh}{\kappa} \right)_-$, and considering the law $P_{r,v}^\kappa$ of a process with law $P_{r,v}^h$, with soft killing at rate $\kappa$, we deduce that
\begin{equation}
E_{(r,v)}[f(R_t, \Upsilon_t) 1_{(t < \tau^D)}] = h(r, v) E_{(r,v)}^{\kappa} \left[ \exp \left( \int_0^t b(R_s, \Upsilon_s) ds \right) \frac{f(R_t, \Upsilon_t)}{h(r, v)} 1_{(t < \tau^D)}(r, \Upsilon_t) \right],
\end{equation}
where $\tau_\kappa$ denotes the soft killing time of the process (with law $P_{r,v}^\kappa$, with soft killing at rate $\kappa$).

Now, we observe that, in a vicinity of $\partial D$,
\begin{equation}
Lh(r, v) = -1 + \alpha(r, v) \int_{V'} (h(r, v') - h(r, v)) \pi(r, v, v') dv' \\
\leq -1 + \|\alpha\|_{\infty} \delta = -1/2
\end{equation}
and hence that
\[
\sup_{(r,v) \in D \times V} \frac{Lh(r, v)}{h(r, v)} < +\infty.
\]
This implies that $b$ is bounded. The fact that $\kappa$ is locally bounded is an immediate consequence of the regularity of $h$. \hfill \Box

Our aim is now to apply the results of Section 2.2 to the Feynman-Kac expression (33) in order to obtain information about $P$. Assumption 1 is clearly satisfied by a process with law $P_{(r,v)}$, and it only remains to check that Assumption 2 holds true in order to apply Theorem 2.6. This is the purpose of the following result. In what follows, we refer to a bounded BBMMI as any BBMMI whose selection mechanism imposed by the parameters $p^i$ results in a size constrained process, i.e. there is a constant $N_{max}$ such that $N_t \leq N_{max}$ for all $t \geq 0$.

**Proposition 3.9.** Under the assumptions (A1), (A2), (B1) and (B2), any bounded BBMMI driven by $P_{(r,v)}^\kappa$ satisfies Assumption 2.

**Proof.** We denote by $\sigma_1 < \sigma_2 < \ldots$ the sequence of times at which events occur for a given particle in the system (each time is either a scattering, a branching or a soft killing). Note that each time $\sigma_n$ is well defined since, almost surely, the number of event occurring before a time $T$ goes to infinity when $T \to +\infty$. Since one can use the strong Markov property at the birth time of the given particle, we assume without loss of generality that the particle is already alive at time 0.

From the expression of the scattering operator $z_h$ in (32), one observes that the scattering rate toward a direction in a set $V' \subset V$ is given by $\int_{V'} \alpha(r, v') \frac{h(r, v')}{h(r, v)} \pi(r, v, v') dv'$. But, under the assumptions (B1–2), there exists $\epsilon > 0$, $\beta > 0$ and $\eta > 0$ such that, for any point $r \in D$, there exists $V_r$ with $r + [0, \beta]V_r \subset D$ and $\text{Leb}_d((r + [0, \beta]V_r) \cap \{r' \in D, d(r', \partial D) \geq \epsilon\}) > \eta$, where $\text{Leb}_d$ denotes the d-dimensional Lebesgue measure. In particular, using the fact that $h(r, v')$ is lower bounded on $V_r$ (uniformly over $r \in D$). We deduce that there exists a constant $\check{c} > 0$ such that the scattering rate toward a direction in $V_r$ is lower bounded by $\check{c}/h(r, v)$, uniformly in $(r, v) \in D \times V$.

Using (39) to bound $Lh$ from below, there exists a constant $\bar{c}$ such that the total rate at which a particle undergoes an event (either a scattering, a branching or a (soft) killing) is upper bounded by $\bar{c}/h(r, v)$, uniformly in $(r, v) \in D \times V$. 


Hence each event has a probability greater than \( s/\bar{c} \) to be a scattering event toward a direction in \( V_r \), independently of the past of the process, and in particular independently of the time event \( \sigma_n \): formally, for all \( n \geq 0 \),

\[
P^\kappa(\sigma_n \text{ is a scattering event and } (R_{\sigma_n}, Y_{\sigma_n}) \in \{R_{\sigma_n}^-\} \times V_{R_{\sigma_n}^-} \mid \sigma_n, R_{\sigma_n}^-, Y_{\sigma_n}^-) \geq s/\bar{c} > 0,
\]

where \((R, Y)\) denotes the position and direction of the given particle.

Now, using the fact that the killing rate and the scattering rate are uniformly bounded in \( \{r' \in D, d(r', \partial D) \geq \varepsilon\} \) and the fact that the total branching rate of the system is uniformly bounded, we deduce that there exists a constant \( p > 0 \) and a time \( T > 0 \) such that, for all \( r \in D \) and \( v \in V_r, P(\sigma_1 \geq T) \geq p \).

Hence, we deduce from the strong Markov property that

\[
P^\kappa(\sigma_{n+1} < T) \leq E^\kappa \left[ 1_{\sigma_n < T} P^\kappa_{X_{\sigma_n}}(\sigma_1 < T) \right] \\
\leq E^\kappa \left[ 1_{\sigma_n < T} \left( 1 - 1_{Y_{\sigma_n} \in V_{R_{\sigma_n}}} P^\kappa_{X_{\sigma_n}}(\sigma_1 < T) \right) \right] \\
\leq E^\kappa \left[ 1_{\sigma_n < T} \left( 1 - 1_{Y_{\sigma_n} \in V_{R_{\sigma_n}}} P \right) \right] \\
\leq P^\kappa(\sigma_n < T)(1 - s/\bar{c}p).
\]

This shows that the number of events occurring before time \( T \) is stochastically dominated by a geometric random variable with parameter \( s/\bar{c}p > 0 \), so that it is finite almost surely and that its expectation is bounded by a constant \( \bar{a} \). Since this is true for all particles, we deduce that total number of events has an expectation bounded by \( N_{\text{max}} \bar{a} \). Finally, observing that this bound does not depend on the initial position, we deduce using the Markov property at times \( T, 2T, \ldots \) that the number of event occurring before a time horizon \( nT \) is bounded by \( kN_{\text{max}} \bar{a} \), and hence that it is finite almost surely.

This concludes the proof of Proposition 3.9.

### 3.3. Numerical properties of the constrained branching process

The aim of this section is to discuss some of the numerical properties of the \( N_{\text{min}} - N_{\text{max}} \) process defined in Remark 2.4 of Section 2.1, which we restate here for convenience. For two integers \( N_{\text{min}} \neq 1 \) and \( N_{\text{max}} \), with \( 0 \leq N_{\text{min}} \leq N_{\text{max}} \), we define the \( N_{\text{min}} - N_{\text{max}} \) process as the particle system defined in Section 2.1 with the parameter choices

\[
p^{0,i}(x_i, i \in s) = 1_{|s| = N_{\text{min}}} \quad \text{and} \quad q^{0,i}(x_i, i \in s) = 1_{|s| = N_{\text{max}}}, \quad i_0 \in s.
\]

In Section 3.3.1, we first make comparisons between this algorithm and the fixed size Moran type algorithm studied in \([22, 23]\). We will often use the abbreviation FV IPS to refer to the latter, in reference to the name Fleming Viot interacting particle system, which has been used in several instances in the literature. In Section 3.3.2, we introduce a filtering method to deal with numerical discrepancies occurring in situations where \( \Pi^A_T \Pi^B_T \) has high variance.

#### 3.3.1. Comparison with the FV IPS

We recall that the FV IPS simulates a set of \( N \) sub-Markov processes, until the first time one of the particles dies, at which point a resampling event occurs by uniformly selecting one of the remaining \( N - 1 \) particles and duplicating it. Note that the FV IPS is a particular instance of the \( N_{\text{min}} - N_{\text{max}} \) IPS, with a null branching rate and \( N_{\text{min}} = N_{\text{max}} = N \). In particular, Theorem 2.6 applies to the FV IPS (see \([66]\) for an analogous result in the FV case).

In some situations, it is possible to use the FV IPS to approximate quantities associated with branching processes. Indeed, it is well known that for a branching process \( X = (X_t)_{t \geq 0} \), under fairly weak assumptions, the following many-to-one formula holds

\[
\psi_t[f](x) := \mathbb{E}_{x_0}[\{f, X_t\}] = \mathbb{E}_x \left[ e^{\int_0^t \beta(Y_s)ds} f(Y_t) \right],
\]

where \( \beta \) is determined by the branching rate and mean offspring of \( X \), and the law of \( Y \) is uniform amongst all paths of \( X \). In the case where \( \beta \) is uniformly bounded above, one may compensate the right-hand side above by its maximum, \( \tilde{\beta} \), to introduce a penalised process: \( \psi_t[f](x) := e^{-\tilde{\beta}t} \psi_t[f](x) \). The penalisation has the effect of killing \( Y \) at an additional rate \( \tilde{\beta} - \beta \), thus yielding a sub-Markov process with semigroup \( \psi_t \). This then allows one to use the FV IPS to obtain estimates of this semigroup and, in some situations, of its associated quasi-stationary distribution, for example. In general, the use of the FV IPS for the approximation of quantities related to some non-conservative semigroups is widespread, see for instance \([8, 10−12, 22, 38, 40, 60, 65]\).
In this section, we show on a simple example that the \(N_{\min}-N_{\max}\) algorithm allows one to overcome some of the limitations of the FV IPS. For simplicity, we only consider the situation where \(N_{\min} = N_{\max}\). In this situation, the \(N_{\min}-N_{\max}\) algorithm is the classical Moran interacting particle system, and identical strategies have already been discussed in the literature (see e.g. [15, 22, 60]). Our aim is to illustrate some numerical properties of these strategies for a toy model.

**A simple branching birth and death process.** We consider a birth and death process on \(E = \{1, \ldots, M\}\) for some parameter \(M \in \mathbb{N}\). For a particle occupying state \(x\), one of the following things may occur:

- the particle jumps with rate \(x^2\), to state \(1 \vee (x-1)\) with probability \(\frac{x}{x+1}\) and to state \((x+1) \wedge M\) with probability \(\frac{1}{x+1}\).
- at rate \(b(x) = x\), a new particle is produced at the same site, which will continue independently the same behaviour as the original particle.

In this situation where the state space is finite and irreducible, it is well known that the semigroup \((\psi_t)_{t \geq 0}\) admits a limiting distribution up to an exponential rescaling (see for instance [20]): there exist a constant \(\lambda_M \in \mathbb{R}\), a positive function \(g_M : \{1, \ldots, M\} \to (0, +\infty)\) and a probability measure \(\nu_M\) on \(\{1, \ldots, M\}\) such that

\[
\sup_{f, x} \left| e^{-\lambda_M t} \psi_t[f](x) - g_M(x) \nu_M(f) \right| \leq C_M e^{-\gamma_M t}, \quad t \geq 0,
\]

for some positive constants \(C_M, \gamma_M\), where the supremum is taken over all functions \(f : \{1, \ldots, M\} \to (0, +\infty)\), \(\|f\|_\infty \leq 1\) and all \(x \in \{1, \ldots, M\}\).

The associated birth and death process with killing evolves on \(\{1, \ldots, M\}\) with the same jump rates, without branching, and with the additional killing rate \(\kappa(x) = M - x\). Denoting as above by \(\psi^k\) the associated sub-Markov semigroup, we have \(\psi^k_t = e^{-\lambda_M t} \psi_t\) and hence

\[
\sup_{f, x} \left| e^{-(\lambda_M - \gamma_M) t} \psi^k_t[f](x) - g_M(x) \nu_M(f) \right| \leq C_M e^{-\gamma_M t},
\]

where the supremum is taken over all functions \(f : \{1, \ldots, M\} \to (0, +\infty)\), \(\|f\|_\infty \leq 1\) and all \(x \in \{1, \ldots, M\}\).

**Theoretical convergence of the \(N_{\min}-N_{\max}\) and of the FV IPSs.** In what follows, \(m^M_N\) refers to the empirical measure of the \(N_{\min}-N_{\max}\) particle system at time \(T\), associated to the above branching birth and death process, and \(\mu^M_N\) refers to the empirical measure of the FV IPS with \(N := N_{\min} = N_{\max}\) particles at time \(T\) associated to the above killed birth and death process.

In particular, it is well known (see e.g. Theorem 3.27 in [26]) that, in the situation where \(N_{\min} = N_{\max} = N\),

\[
\left\| \frac{m^M_N(f)}{N} - \nu_M(f) \right\|_2 \leq \frac{C_{FV} \|f\|_\infty}{N^{\alpha/2}} + C_N e^{-\gamma_N T} \|f\|_\infty.
\]

for some \(\alpha > 0\) and \(C_{FV} > 0\). Using the fact that the \(N_{\min}-N_{\max}\) process is ergodic, we deduce that its normalised empirical stationary distribution \(\mathcal{X}_N^M\) (the normalised empirical distribution of a random vector of particles distributed according to the stationary distribution of the \(N_{\min}-N_{\max}\) process) satisfies

\[
\mathcal{X}_N^M(f) \xrightarrow{\mathcal{L}_2} \nu_M(f),
\]

for all function \(f : \{1, \ldots, M\} \to \nu_M\). Similarly, the normalised empirical stationary distribution \(\mathcal{Y}_N^M\) of the FV particle system satisfies

\[
\mathcal{Y}_N^M(f) \xrightarrow{\mathcal{L}_2} \nu_M(f).
\]

These theoretical convergence results ensure that both algorithms can be used to approximate the limiting distribution \(\nu_M\). It remains to compare their performances via numerical simulations.

**Numerical comparison of the \(N_{\min}-N_{\max}\) and the FV IPSs.** We now consider the problem of numerically approximating the mean of \(\nu_M\), that is \(\nu_M(f_M)\) with \(f_M(x) = x\) for all \(x \in \{1, \ldots, M\}\). The estimator is \(\hat{\theta}_N = \mathcal{X}_N^M(f_M)\) (resp. \(\hat{\theta}_N = \mathcal{Y}_N^M(f_M)\)) for the \(N_{\min}-N_{\max}\) (resp. FV) IPS, and we compare three important metrics determining the performance of the algorithms:
\[ N = 10 \quad |E(\hat{\theta}_N) - \nu_M(f_M)| \quad \text{Std}(\hat{\theta}_N) \quad (A_T + B_T)/T \]

| \( M = 10 \) | \( N_{min} - N_{max} \) | 0.08 | 0.30 | 14.0 |
| \( M = 100 \) | \( N_{min} - N_{max} \) | 0.10 | 0.41 | 87.2 |
| \( M = 1000 \) | \( N_{min} - N_{max} \) | 0.20 | 0.51 | 988 |
| \( M = +\infty \) | \( N_{min} - N_{max} \) | 0.22 | 0.53 | 9989 |

Table 1

For each value of \( M \in \{10, 100, 1000, +\infty\} \), we display the bias of the estimator \( \hat{\theta}_N \), its standard deviation, and the number of events, for the \( N_{min} - N_{max} \) algorithm for FV IPS (for which \( \hat{\theta}_N = \chi_N^M(f_M) \)) and for the FV IPS (for which \( \hat{\theta}_N = \chi_N^M(f_M) \)) with \( N = N_{min} = N_{max} = 10 \) particles.

Note that the FV IPS is not defined when \( M = +\infty \).

- the bias of the estimator, \( |E(\hat{\theta}_N) - \nu_M(f_M)| \),
- the standard deviation of the estimator, \( \text{Std}(\hat{\theta}_N) \),
- the number of interaction events, \( (A_T + B_T)/T \), which corresponds to the number \( B_T \) of selection events in the \( N_{min} - N_{max} \) algorithm and the number \( A_T \) of resampling events in FV IPS, per unit of time.

The last metric \( (A_T + B_T)/T \) is particularly important since it entails that the total number of interaction events grows (at least) linearly in time.

For \( N = 10 \) and then \( N = 100 \), we let the upper bound of the state space \( M \) vary in \( \{10, 100, 1000, +\infty\} \), and present the results in Table 1 for \( N = 10 \), and in Table 2 for \( N = 100 \). For each metric, a lower number indicates that the algorithm performs better.

\| \( N = 100 \) | \( |E(\hat{\theta}_N) - \nu_M(f_M)| \) | \( \text{Std}(\hat{\theta}_N) \) | \( (A_T + B_T)/T \) |
| \( M = 10 \) | \( N_{min} - N_{max} \) | 0.01 | 0.12 | 144 |
| \( M = 100 \) | \( N_{min} - N_{max} \) | 0.02 | 0.18 | 857 |
| \( M = 1000 \) | \( N_{min} - N_{max} \) | 0.10 | 0.39 | 9866 |
| \( M = +\infty \) | \( N_{min} - N_{max} \) | 0.20 | 0.50 | 99873 |

Table 2

For each value of \( M \in \{10, 100, 1000, +\infty\} \), we display the bias of the estimator \( \hat{\theta}_N \), its standard deviation, and the number of events, for the \( N_{min} - N_{max} \) algorithm for FV IPS (for which \( \hat{\theta}_N = \chi_N^M(f_M) \)) and for the FV IPS (for which \( \hat{\theta}_N = \chi_N^M(f_M) \)) with \( N = N_{min} = N_{max} = 100 \) particles.

Note that the FV IPS is not defined when \( M = +\infty \).

We can see that, for this particular example, the \( N_{min} - N_{max} \) algorithm performs better in all situations, with respect to all metrics. Moreover, the bias of the FV approximation increases when \( M \) increases, while the variance of the estimator also increases and the computation cost increases. Without surprise, increasing \( M \) does not change the dynamic of the \( N_{min} - N_{max} \) algorithm, since very few particles, if any, reach the boundary \( M \) even when \( M = 10 \). Further, this suggests that it should be possible to extend the \( N_{min} - N_{max} \) algorithm to the case of unbounded branching rates, and this is supported by the numerical simulations when \( M = +\infty \) (for which the FV IPS is of course not defined), although it is not covered by our main result. We intend to address these questions in future work.

Remark 3.10. We note that the birth-death process studied above is a typical example of the type of setting where the \( N_{min} - N_{max} \) process may outperform the FV IPS. Here, we briefly outline another example where this is also the case.

We consider a branching random walk on \( \{0, \ldots, n\} \) for some \( n \geq 1 \) with a time inhomogeneous branching rate. More precisely, let \( (X_t)_{t \geq 0} \) denote a continuous time random walk on \( \mathbb{Z} \) that moves one step to the right with probability
Now let $b, k : \{0, \ldots, n\} \to [0, \infty)$ be bounded measurable functions and $\mathcal{E}_b, \mathcal{E}_k$ be exponential random variables, that are independent of each other and $X$, and define

$$\tau^b_1 = \inf\{t > 0 : \int_0^t b(X_s)ds > \mathcal{E}_b\},$$

and

$$\tau^k_1 = \inf\{t > 0 : \int_0^t k(X_s)ds > \mathcal{E}_k\}.$$

Further, set $\tau_1 = \tau^b_1 \land \tau^k_1$. If $\tau_1 = \tau^b_1$, a new particle is added to the system at position $X_{\tau^b_1}$, which continues the same behaviour as the original particle. If $\tau_1 = \tau^k_1$, the particle is removed from the system. We can then define $\tau_2$ to be the first time after $\tau_1$ that one of the particles alive at time $\tau_1$ branches or is killed, and this process continues iteratively.

Letting $N_t$ denote the number of particles alive at time $t \geq 0$, and $\{X^i_t : i = 1, \ldots, N_t\}$ their positions, the branching random walk is defined as

$$Z_t = \sum_{i=1}^{N_t} \delta_{X^i_t}, \quad t \geq 0.$$

We now consider a time-inhomogeneous version of the above process. We consider a sequence of switching times $0 = T_0^{\text{off}} < T_1^{\text{on}} < T_1^{\text{off}} < T_2^{\text{on}} < T_2^{\text{off}} < \ldots$, defined via

$$T_i^{\text{on}} = T_{i-1}^{\text{off}} + e_{i-1}(s_{\text{on}}), \quad i \geq 1,$$

and

$$T_i^{\text{off}} = T_i^{\text{on}} + e_i(s_{\text{off}}), \quad i \geq 1,$$

where $e_i(s_{\text{off}})$ and $e_i(s_{\text{on}})$ are independent (of each other and everything else) exponential random variables with rate $s_{\text{off}}$ and $s_{\text{on}}$, respectively. For $T_i^{\text{off}} \leq t < T_i^{\text{on}}$, $Z_t$ evolves as above but with $b = 0$, i.e. each particle moves as a random walk until it is (possibly) killed, until time $T_i^{\text{on}}$. At time $T_i^{\text{on}}$, we sample from an exponential random variable with rate $B \in (0, \infty)$ and set $b$ to be this value. For $T_i^{\text{on}} \leq t < T_i^{\text{off}}$, $Z_t$ then evolves as the branching random walk with branching rate $b$ for each particle. At time $T_i^{\text{off}}$, we again set the branching rate to be 0, and continue this process until some terminal time $T$.

We can see that in this case, while the branching rate is bounded almost surely on any compact time interval, we may simulate the $N_{\text{min}} - N_{\text{max}}$ process. However, we cannot use the fixed size Moran type algorithm since we do not know the upper bound for the branching rate.

### 3.3.2. Two-level Monte Carlo methods for estimating Lyapunov coefficients

In many applications, it can be of interest to estimate numerically the growth rate of the semigroup $Q$ for large times. A natural conjecture is that for large times $T$, $m_0Q_Tf \approx \langle m_0, \psi \rangle \langle \varphi, f \rangle e^{\lambda T}$, where $\varphi$ and $\psi$ are the left eigenmeasure and right-eigenfunction respectively of the operator $Q$, and $\lambda$ is the Lyapunov exponent for $Q$. One can try to estimate $\lambda$ by computing the quantity $m_0Q_Tf$ for large times $T$.

By Theorem 2.6, one method for estimating $\lambda$ would be through equation (6), so that we can simulate the $(m_T, \Pi_T^A, \Pi_T^B)$-system and estimate $\lambda$ by:

$$\hat{\lambda} = \frac{1}{T} \log \left( \Pi_T^A\Pi_T^B m_T(1) \right) = \frac{1}{T} \log \left( \Pi_T^A\Pi_T^B \right),$$

for $T$ large. In practice, however, this is numerically a poor choice of estimator, due to large-deviation effects. The presence of the ‘exponential’ functionals, $\Pi_T^A, \Pi_T^B$ mean that significant contributions to the expectation will be made by paths of the $N_{\text{min}} - N_{\text{max}}$ process which have probability $e^{-\delta T}$, and so infeasibly many simulations will be needed to compute an unbiased estimate for $\hat{\lambda}$. A simple solution is to replace the estimator by

$$\bar{\lambda} = \frac{1}{\delta t} \log \left[ \frac{1}{n} \sum_{i<n} \left( \frac{\Pi_{t_i}^A\Pi_{t_i}^B}{\Pi_{t_{i-1}}^A\Pi_{t_{i-1}}^B} \right) \right].$$
where \( t_i = T \frac{i}{N} \) and \( \delta t = \frac{T}{N} \). In many examples, under conditions which ensure that the process is ergodic and thus converges to a stationary distribution, as \( T \to \infty \), for fixed \( \delta t \), the estimator \( \hat{\lambda} \) can be shown to converge to \( \lambda \) (e.g. \[30\]).

In some cases the expression in (41) may not be valid and we can only use estimates of the form in (40). Examples may be when the process is not ergodic, or the system is not time-homogenous, such as the example given in Remark 3.10. In such circumstances, we propose a novel Monte Carlo method to compute estimates of \( \lambda \). Specifically, we note that the \( N_{\text{min}}-N_{\text{max}} \) process \( X_t := \{X_t^1, \ldots, X_t^{N_t}\} \) defines a continuous time Markov process, while the processes \( A_t \) and \( B_t \) corresponding to the number of selection and resampling events, are themselves additive functionals of the Markov process. Estimation of \( \lambda \) corresponds to computing a Feynman-Kac type exponential functional for a Markov process, and it is well known that estimating such functionals can be performed efficiently using interacting particle simulations, \[22, 23, 25, 26\].

To try to manage this error, we propose the following algorithm. Informally, we simulate multiple independent copies of the \( N_{\text{min}}-N_{\text{max}} \) process, in continuous time, using the algorithm described in Section 2.1. At fixed time intervals, we perform a resampling of the independent \( N_{\text{min}}-N_{\text{max}} \) processes, based on the individual weights accrued since the last resampling.

This suggests an algorithm as follows, assuming that \( b^i, \kappa^i, p^i \) and \( q^i \) have been chosen:

**Two-level Monte Carlo algorithm for computing Lyapunov Exponents**

1. Fix \( T, N^*, \delta t, N_0, x_0 \). Initiate with \( t = 0 \), and \( \mathcal{X}^1, \ldots, \mathcal{X}^{N^*} \) each containing \( N_0 \) particles of initial value \( x_0 \). Set \( W = 1 \).
2. Run each \( \mathcal{X}^i \) from time \( t \) to time \( t' = \min\{t + \delta t, T\} \) according to the \( N_{\text{min}}-N_{\text{max}} \) algorithm described in Section 2.1, setting \( w_i := \Pi A_{t'} \Pi B_{t'} \).
3. Set \( W := W \times \left( \frac{1}{N^*} \sum_{i=1}^{N^*} w_i \right) \).
4. If \( t' < T \), set \( t = t' \), resample each \( \mathcal{X}^i \) from the existing particles \( \{\mathcal{X}^1, \ldots, \mathcal{X}^{N^*}\} \) with probabilities proportional to \( w_1, \ldots, w_{N^*} \), independently (multinomial resampling). Return to step 2.
5. If \( t' = T \), stop, and return

\[
\hat{\lambda} := \frac{1}{T} \log (W) .
\]

Here, the choice of multinomial resampling could be replaced by other resampling schemes. Similarly, the resampling could be conditional on properties of the particles (e.g. the effective sample size). We compare the effectiveness of the two methods in Figure 3, for a simple Markov process where the true value of the Lyapunov exponent can be calculated accurately numerically. We note that both methods have similar computational effort, however the iterated method achieves a much better numerical estimate of \( \lambda \).
Fig 3: We use two methods described above to estimate the eigenvalue $\lambda$. The first simulation (left) estimates $\lambda$ using (40), based on a single trajectory run up to time $T = 4000$. The true value of $\lambda$ is highlighted with the orange line, the green line represents an analytical estimate of the convergence value for the simulation (in this case, an estimate of the average growth of the exponential weights under the stationary measure for the system). In the second figure (right), $\lambda$ is estimated using an interacting particle system, as described above. In this case, the particle filter has 100 particles, and is run for a total time of $T = 40$, giving comparable computational effort. It should be noted however that the second case also has a computational cost associated with the resampling scheme applied, and so the overall computational cost will be higher depending on the resampling scheme applied, and the frequency of resampling events.

4. Proof of Theorem 2.6

This section is dedicated to the proof of Theorem 2.6, which is rather long and so we break it up into five steps. In the first three steps, the main idea is to define two martingales $M$ and $\mathcal{M}$ such that

$$
\nu_f^t - \nu_0 = \Pi_t^A \Pi_t^B (M_t - M_{\tau_{c_1}}) + \sum_{n=1}^{A_t} \Pi_{\rho_n}^A \Pi_{\rho_n}^B (M_{\rho_n} - M_{\rho_n-}) + \sum_{n=1}^{B_t} \Pi_{\sigma_n}^A \Pi_{\sigma_n}^B (M_{\sigma_n} - M_{\sigma_n-})
$$

+ \sum_{n=1}^{C_t} \Pi_{\tau_{n-1}}^A \Pi_{\tau_{n-1}}^B (M_{\tau_n} - M_{\tau_{n-1}}),
$$
(42)

where $\sigma_n$ and $\rho_n$ denote the $n^{th}$ selection and resampling events, respectively, and

$$
\nu_f^t = \Pi_t^A \Pi_t^B \sum_{i \in S_t} Q_{T-t} f(X_i^t).
$$

The martingale $M$ compensates the evolution of the branching process between resampling/selection times. The martingale $\mathcal{M}$ compensates the evolution of the particle system during resampling/selection events.

The many-to-one formula (6) then follows. The final two steps make use of the following technical lemma to control the $L^2$ norms of the martingale increments in order to obtain (7).

**Lemma 4.1.** For all $\ell \geq 1$, for all $T \geq 0$,

$$
\mathbb{E} \left( (\Pi_T^B)^\ell \right) \leq \exp(c_\ell ||b||_\infty T),
$$

where

$$
c_\ell := \sup_{x \geq 2} x \left( (1 + 1/x)^\ell - 1 \right).
$$

In addition, denoting by $\beta_T$ the total number of branching events (with or without selection) up to time $T$, we have

$$
\mathbb{E} \left( \beta_T (\Pi_T^B)^2 \right) \leq N_0 \exp((3 + c_4) ||b||_\infty T/2).
$$
The above estimates are the main reason why we require \( b \) to be bounded. These types of estimates are not difficult to derive in some particular situations where the branching rate is unbounded (typically for time inhomogeneous branching rates, or in situations where the branching rates is high in rarely visited regions), and in these situations, our proofs adapt easily. We defer the proof of the above lemma to the end of the section and now proceed to the proof of Theorem 2.6.

**Proof of Theorem 2.6.**

**Step 1.** The martingale \( \mathcal{M} \). For all \( i \in \mathbb{N} \), we define respectively the birth time \( B_i \) and the death time \( D_i \) of the particle with index \( i \) as

\[
B_i = \inf \{ t \geq 0, \ i \in \bar{S}_t \} \quad \text{and} \quad D_i = \inf \{ t \geq B_i, \ i \notin \bar{S}_t \}.
\]

Fix \( T > 0 \) and \( t \in [0, T) \). For all particles \( i \in \mathbb{N} \), we set

\[
M_i^{t,n} = \begin{cases} 
0, & \text{if } t \leq B_i \lor \tau_n, \\
Q_{T-t}f(X^i_t) - Q_{T-B_i \lor \tau_n} f(X^i_{B_i \lor \tau_n}), & \text{if } t \in [B_i \lor \tau_n, D_i \lor \tau_{n+1}), \\
(1 - 1_{D^n(i)} + 1_{B^n(i)}) Q_{T-D_i \lor \tau_{n+1}} f(X^i_{D_i \lor \tau_{n+1}}) - Q_{T-B_i \lor \tau_n} f(X^i_{B_i \lor \tau_n}), & \text{if } t \geq D_i \lor \tau_{n+1},
\end{cases}
\]

where \( B^n(i) \) denotes the event “\( \tau_{n+1} \) is a branching time for the \( i \)-th particle” and \( D^n(i) \) denotes the event “\( \tau_{n+1} = D_i \) is a soft killing time for the \( i \)-th particle”.

By definition of the process between the time intervals \([\tau_n, \tau_{n+1})\), to show that \( \mathcal{M} \) is indeed a martingale, it is sufficient to prove that

\[
\mathcal{M}_t^{1,0} := \begin{cases} 
Q_{T-t}f(X^1_t) - Q_T f(X^1_0), & \text{if } t \in [0, \tau_1), \\
(1 - 1_{D(1)} + 1_{B(1)}) Q_{T-\tau_1} f(X^1_{\tau_1}) - Q_T f(X_0), & \text{if } t \geq \tau_1,
\end{cases}
\]

is a martingale. We observe that, by construction, \( \tau_1 = \tau^b \wedge \tau^k \wedge \tau^h \) where

\[
\begin{align*}
\tau^b &= \inf_{j \in S_0} \inf \{ t \geq 0, \ \int_0^t b^j((X^i_u)_{u \in \bar{S}_u}) \, du \geq e^b \}, \\
\tau^k &= \inf_{j \in S_0} \inf \{ t \geq 0, \ \int_0^t \kappa^j((X^i_u)_{u \in \bar{S}_u}) \, ds \geq e^k \}, \\
\tau^h &= \inf_{j \in S_0} \inf \{ t \geq 0 X^i_t \in \partial \}.
\end{align*}
\]

Then, for all \( s, t \geq 0 \) such that \( s + t \leq T \), setting \( \mathbb{E}^t = \mathbb{E}[ \cdot | \sigma(\bar{S}_u, X_u, 0 \leq u \leq t) ] \) and using the fact that \( Q_u f \equiv 0 \) on \( \partial \) for all \( u \geq 0 \), we have

\[
\mathbb{E}^t(M_1^{t,0}) = \mathbb{E}^t \left[ (Q_{T-(t+s)} f(X^1_{t+s}) - Q_T f(X_0)) 1_{t+s < \tau_1} \right] \\
+ \mathbb{E}^t \left[ ((1 - 1_{D(1)} + 1_{B(1)}) Q_{T-\tau_1} f(X^1_{\tau_1}) - Q_T f(X_0)) 1_{\tau_1 \leq t} \right]
\]

(44)

Using the fact that the particles are independent copies of \( X \) before time \( \tau_1 \), and the Markov property at time \( t \), we obtain

\[
\mathbb{E}^t \left[ Q_{T-(t+s)} f(X^1_{t+s}) 1_{t+s < \tau_1} \right] = \mathbb{E}^t \left[ Q_{T-(t+s)} f(Y^1_{t+s}) e^{-\int_{t+s}^{t+u} h_u \, du} 1_{t+s < \tau_0} \right] 1_{t < \tau_1},
\]

where \( \mathbb{E}^t \) is the expectation on a probability space \( \Omega' \) such that \( (Y^i_u)_{u \geq t}, i \in \bar{S}_0 \), are independent copies of \( X \), starting from \( X^i_0 \) at time \( t \), and where

\[
h_u := \sum_{j \in S_0} b^j((Y^i_u)_{u \in \bar{S}_u}) + \kappa^j((Y^i_u)_{u \in \bar{S}_u}).
\]
Then, using the Markov property at time $t+s$, we obtain
\[
\mathbb{E}_x^t [Q_{T-(t+s)} f(X_{t+s}^1) 1_{t+s < \tau}] = E^t \left[ e^{T_s} b(Y^1_s) - \kappa(Y^1_s) \right] 1_{t+s < \tau} \\
= E^t \left[ e^{T_s} b(Y^1_s) - \kappa(Y^1_s) \right] 1_{t < \tau},
\]
where $h_u^1 := 2b^1((Y^1_u)_{i \in S_0}) + \sum_{j \in S_0, j \neq 1} b^1((Y^1_u)_{i \in S_0}) + \kappa^1((Y^1_u)_{i \in S_0})$. Similarly, since $\kappa^1$ is the intensity of the soft killing time for the first particle, and $b^1$ is the intensity of its branching time, we obtain
\[
\mathbb{E}_x^t \left[ (1 - 1D(1) + 1B(1)) Q_{T-\tau} f(X_{t}^1) 1_{t < \tau < t+s} \right] = E^t \left[ \int_t^{(t+s) \wedge \tau} h_u - \kappa^1((Y^1_u)_{i \in S_0}) + b^1((Y^1_u)_{i \in S_0}) \right] Q_{T-u} f(Y^1_u) e^{-\int_0^u h_s du} du 1_{t < \tau} + E^t \left[ 1_{Y_{t < \tau} \in \partial} e^{-\int_0^\tau h_s du} 1_{\tau < t+s} Q_{T-\tau} f(Y^1_\tau) \right] 1_{t < \tau}.
\]
Hence, using the fact that $Y$ satisfies the strong Markov property at time $\tau_0$, and the fact that $Q_{T-\tau} f(Y^1_{\tau_0}) = 0$ on the event $\{Y^1_{\tau_0} \in \partial\}$, we deduce that
\[
\mathbb{E}_x^t \left[ (1 - 1D(1) + 1B(1)) Q_{T-\tau_1} f(X_{t}^1) 1_{t < \tau_1 < t+s} \right] = E^t \left[ \int_t^{(t+s) \wedge \tau_1} h_u e^{T_s} b(Y^1_s) - \kappa(Y^1_s) \right] f(Y^1_s) e^{-\int_0^u h_s du} du 1_{t < \tau_1} + E^t \left[ 1_{Y_{t < \tau_1} \in \partial} e^{-\int_0^\tau h_s du} 1_{\tau_1 < t+s} Q_{T-\tau_1} f(Y^1_{\tau_1}) \right] 1_{t < \tau_1},
\]
where we used the assumption $b(Y^1_t) - \kappa(Y^1_t) = b^1((Y^1_t)_{i \in S_0}) - \kappa^1((Y^1_t)_{i \in S_0})$. But
\[
e^{-\int_0^u h_s du} 1_{t+s < \tau} + \int_t^{(t+s) \wedge \tau} h_u e^{-\int_0^u h_s du} du 1_{t < \tau} = 1.
\]
Hence we deduce from (44) that
\[
\mathbb{E}_x^t (M_{t+s}^{1,0}) = (1 - 1D(1) + 1B(1)) (Q_{T-\tau_1} f(X_{t}^1) - Q_T f(X_0)) 1_{\tau_1 < t} + E^t \left[ e^{T_s} b(Y^1_s) - \kappa(Y^1_s) \right] f(Y^1_s) 1_{t < \tau_1} = (1 - 1D(1) + 1B(1)) Q_{T-\tau_1} f(X_{t}^1) 1_{t < \tau_1} + Q_{T-t} f(X^1_t) 1_{t < \tau_1} - Q_T f(x) = M_{t}^{1,0}.
\]
This concludes Step 1.

**Step 2. A first martingale decomposition.**

Fix $t \geq 0$ and recall that $C_t = \inf \{n \geq 0 : t < \tau_{n+1}\}$. In what follows, we denote by $\rho_1, \rho_2, \ldots$ the sequence of times at which a resampling event occurs, and by $\sigma_1, \sigma_2, \ldots$ the sequence of times at which a selection event occurs. Note that, if $\tau_n \in \{\rho_1, \rho_2, \ldots\}$ or $\tau_n \in \{\sigma_1, \sigma_2, \ldots\}$, then $N_n = N_{n-1} \in \{N_{\min}, \ldots, N_{\max}\}$.

We will show that, for all $t \geq 0$, we have
\[
M_t := \sum_{n=0}^{C_t} \sum_{i \in S_t} Q_{T-t} f(X^1_i) - \sum_{i \in S_0} Q_T f(X^1_0) = M_t^{1,0}.
\]
where we recall that $A_t$ is the number of resampling events up to time $t$, $B_t$ is the number of selection events up to time $t$, and where we have set $i_t$ to denote the index of the particle added at time $\rho_t$, and $j_{n-1}$ is the particle killed (due to a selection event) at time $\sigma_n$ (as above, $X_{\rho_{n}}^{i_{t}}$ denotes the position of the particle before the selection event).

We will show that (45) holds for $n = 0, 1$, since the general formula then follows using similar arguments.

Noting that for all $t \in [\tau_0, \tau_1)$, $C_t = A_t = B_t = 0$, in this case we have

$$
\sum_{n=0}^{C_t} \sum_{i \in S_t} M^{i,n}_t = \sum_{i \in S_0} M^{i,0}_t + \sum_{i \in S_1} M^{i,1}_t
$$

as required.

Let us now describe the evolution of the martingale at time $\tau_1$, depending on the type of event that occurs.

**Case $\tau_1 = \rho_1$.** In this case, for all $t \in [\tau_1, \tau_2)$, we have $A_t = 1$ and $B_t = 0$, and $\bar{S}_t = \{1 + \max S_0 \cup \bar{S}_0 \setminus \{i_0\}$, where $i_0$ is the particle removed at time $\tau_1$. Thus

$$
\sum_{n=0}^{C_t} \sum_{i \in S_t} M^{i,n}_t = \sum_{i \in S_0} M^{i,0}_t + \sum_{i \in S_1} M^{i,1}_t
$$

$$
\sum_{i \in S_0} [(1 - 1_{i = j_0}) Q_{T - \tau_1} f(X_{\tau_1}^i) - Q_T f(X_{\tau_1}^i)] + \sum_{i \in S_1} [Q_{T - \tau_1} f(X_{\tau_1}^i) - Q_{T - \tau_1} f(X_{\tau_1}^i)]
$$

$$
\sum_{i \in S_1} Q_{T - \tau_1} f(X_{\tau_1}^i) - \sum_{i \in S_0} Q_T f(X_0^i) - Q_{T - \tau_1} f(X_{\tau_1}^i)
$$

as claimed.

**Case $\tau_1 = \sigma_1$.** Similarly, if $\tau_1 = \sigma_1$, then, for all $t \in [\tau_1, \tau_2)$, we have $A_t = 0$ and $B_t = 1$, and $\bar{S}_t = \{1 + \max S_0 \cup \bar{S}_0 \setminus \{j_0\}$. Denoting by $j_0$ the particle duplicated at time $\tau_1$, we have

$$
\sum_{n=0}^{C_t} \sum_{i \in S_t} M^{i,n}_t = \sum_{i \in S_0} M^{i,0}_t + \sum_{i \in S_1} M^{i,1}_t
$$

$$
\sum_{i \in S_1} [(1 + 1_{i = j_0}) Q_{T - \tau_1} f(X_{\tau_1}^i) - Q_T f(X_{\tau_1}^i)] + \sum_{i \in S_1} [Q_{T - \tau_1} f(X_{\tau_1}^i) - Q_{T - \tau_1} f(X_{\tau_1}^i)]
$$

$$
\sum_{i \in S_1} Q_{T - \tau_1} f(X_{\tau_1}^i) - \sum_{i \in S_0} Q_T f(X_0^i) + Q_{T - \tau_1} f(X_{\tau_1}^j_{0})
$$

as claimed.

**Case $\tau_1 \notin \{\rho_1, \sigma_1\}$.** If $\tau_1 \notin \{\rho_1, \sigma_1\}$, we have, for all $t \in [\tau_1, \tau_2)$, $A_t = B_t = 0$. If $\tau_1$ is a branching event of the particle with index $j_0 \in S_0$, then $\bar{S}_t = \{1 + \max S_0 \cup S_0$.

$$
\sum_{n=0}^{C_t} \sum_{i \in S_t} M^{i,n}_t = \sum_{i \in S_0} [(1 + 1_{i = j_0}) Q_{T - \tau_1} f(X_{\tau_1}^i) - Q_T f(X_{\tau_1}^i)] + \sum_{i \in S_1} [Q_{T - \tau_1} f(X_{\tau_1}^i) - Q_{T - \tau_1} f(X_{\tau_1}^i)]
$$

$$
\sum_{i \in S_1} Q_{T - \tau_1} f(X_{\tau_1}^i) - \sum_{i \in S_0} Q_T f(X_0^i),
$$
Step 3. A second martingale decomposition.

For all time \( t \geq 0 \) and all bounded measurable function \( f : E \to \mathbb{R} \), we denote the integral of \( f \) with respect to the occupation measure of the particle system at time \( t \) by

\[
\mu_t^f := \sum_{i \in S_t} Q_{T-t} f(X_t^i).
\]

Further, recall the definitions of \( \Pi_t^A, \Pi_t^B \) and \( \nu_t^f = \Pi_t^A \Pi_t^B \mu_t^f \).

In this step, we prove that equation (42) holds with \( \mathcal{M} \) defined as in the previous steps, and \( \mathcal{M} \) to be defined shortly. To this end, we first note that from Step 2 we obtain

\[
\mu_t^f - \mu_0^f = \sum_{n=0}^{C_t} \sum_{i \in S_t} M_t^{i,n} + \sum_{n=1}^{A_t} Q_{T-\rho_n} f(X_{\rho_n}^{i,n-1}) - \sum_{n=1}^{B_t} Q_{T-\sigma_n} f(X_{\sigma_n}^{j,n-1}).
\]

For \( n \in \mathbb{N} \), consider the following martingale increments. We define

\[
\mathcal{M}_{\rho_n} - \mathcal{M}_{\rho_n-} := Q_{T-\rho_n} f(X_{\rho_n}^{i,n-1}) - \frac{1}{N_{\rho_n}-1} \sum_{i \in S_{\rho_n-} \setminus \{i_{n-1}'\}} Q_{T-\rho_n} f(X_{\rho_n}^i),
\]

where \( i_{n-1}' \) is the index of the particle duplicated at time \( \rho_n \) and \( i_{n-1}' \) is the index of the particle removed at time \( \rho_n \) (note also that \( \bar{N}_{\rho_n} = \bar{N}_{\rho_n-} = |S_{\rho_n-}| \)). We also define

\[
\mathcal{M}_{\sigma_n} - \mathcal{M}_{\sigma_n-} := \frac{1}{N_{\sigma_n}+1} \left( \sum_{i \in S_{\sigma_n-}} Q_{T-\sigma_n} f(X_{\sigma_n}^i) + Q_{T-\sigma_n} f(X_{\sigma_n}^{j,n-1}) \right) - Q_{T-\sigma_n} f(X_{\sigma_n}^{j,n-1}),
\]

where we recall that \( j_{n-1} \) is the particle removed at time \( \sigma_n \) and \( j_{n-1}' \) is the index of the particle duplicated during the branching event at this time (note also that \( \bar{N}_{\sigma_n} = \bar{N}_{\sigma_n-} = |S_{\sigma_n-}| \)). We assume further that \( \mathcal{M} \) is constant except at these jump times.

Then,

\[
\mu_t^f - \mu_0^f = \mathcal{M}_t + \mathcal{M}_t + \frac{1}{N_{\rho_n}-1} \sum_{i \in S_{\rho_n-} \setminus \{i_{n-1}'\}} Q_{T-\rho_n} f(X_{\rho_n}^i) - \sum_{n=1}^{B_t} \frac{1}{N_{\sigma_n}+1} \left( \sum_{i \in S_{\sigma_n-}} Q_{T-\sigma_n} f(X_{\sigma_n}^i) + Q_{T-\sigma_n} f(X_{\sigma_n}^{j,n-1}) \right).
\]

For each \( n \geq 1 \), we define \( \mu_{\tau_n-}^f \) and \( \nu_{\tau_n-}^f \) by

\[
\mu_{\tau_n-}^f = \sum_{i \in S_{\tau_n-}} f(X_{\tau_n}^i) - 1_{\tau_n \in \mathcal{K}} f(X_{\tau_n-}^{j,n-1}) + 1_{\tau_n \in \mathcal{B}} f(X_{\tau_n-}^{j,n-1}) \quad \text{and} \quad \nu_{\tau_n-} = \Pi_{\tau_n-}^A \Pi_{\tau_n-}^B \mu_{\tau_n-},
\]

where \( \mathcal{K} \) is the set of killing times and \( \mathcal{B} \) is the set of branching times. Informally, \( \mu_{\tau_n-} \) and \( \nu_{\tau_n-} \) represent the state of the particle system at time \( \tau_n \) before the resampling or selection eventually occurring at time \( \tau_n \). Note that, at any time \( \tau_n \) that is not a resampling or a selection time, we have \( \mu_{\tau_n-} = \mu_{\tau_n} \) and \( \nu_{\tau_n-} = \nu_{\tau_n} \). Hence, denoting by \( (\theta_n)_n \) the sequence of events from the families \( \rho \) or \( \sigma \), we obtain for all \( t \geq 0 \),

\[
\nu_t^f - \nu_0^f = \nu_t - \nu_{\tau_{\theta_t}} + \sum_{n=1}^{C_t} (\nu_{\tau_n} - \nu_{\tau_n-}) + \sum_{n=1}^{C_t} (\nu_{\tau_n-} - \nu_{\tau_{\theta_t}}).
\]
which entails (42) and thus concludes Step 3.

At resampling times $\rho_n$, we have $\Pi^B_{\rho_n} - \Pi^B_{\rho_n-} = 0$ and hence,

$$
\nu^f_{\rho_n} - \nu^f_{\rho_n-} = \Pi^A_{\rho_n} \Pi^B_{\rho_n} \nu^f_{\rho_n} - \Pi^A_{\rho_n-} \Pi^B_{\rho_n-} \nu^f_{\rho_n-}
$$

$$
= \Pi^A_{\rho_n} \Pi^B_{\rho_n} (\nu^f_{\rho_n} - \nu^f_{\rho_n-}) + \Pi^B_{\rho_n} \nu^f_{\rho_n-} (\Pi^A_{\rho_n} - \Pi^A_{\rho_n-})
$$

$$
= \Pi^A_{\rho_n} \Pi^B_{\rho_n} Q_{T^{-\rho_n}} f(X^{i_{\rho_n}-1}) + \Pi^B_{\rho_n} \nu^f_{\rho_n-} (\Pi^A_{\rho_n} - \Pi^A_{\rho_n-})
$$

(51)

For the increment of $\Pi^A$, we note that

$$
\Pi^A_{\rho_n} - \Pi^A_{\rho_n-} = \Pi^A_{\rho_n} \left( \frac{\bar{N}_n}{N_{\rho_n} - 1} \right) \Pi^A_{\rho_n} = -\frac{1}{N_{\rho_n} - 1} \Pi^A_{\rho_n}.
$$

Substituting this into (51) and using (48), we obtain

$$
\nu^f_{\rho_n} - \nu^f_{\rho_n-} = \Pi^A_{\rho_n} \Pi^B_{\rho_n} \left( Q_{T^{-\rho_n}} f(X^{i_{\rho_n}-1}) - \frac{1}{N_{\rho_n} - 1} \nu^f_{\rho_n-} \right)
$$

$$
= \Pi^A_{\rho_n} \Pi^B_{\rho_n} (\mathcal{M}_{\rho_n} - \mathcal{M}_{\rho_n-})
$$

(52)

Similarly, for $n \in \{1, \ldots, B_t\}$,

$$
\nu^f_{\sigma_n} - \nu^f_{\sigma_n-} = \Pi^A_{\sigma_n} \Pi^B_{\sigma_n} (\nu^f_{\sigma_n} - \nu^f_{\sigma_n-}) + \Pi^A_{\sigma_n} \nu^f_{\sigma_n-} (\Pi^B_{\sigma_n} - \Pi^B_{\sigma_n-})
$$

(53)

In this case, we have

$$
\Pi^B_{\sigma_n} - \Pi^B_{\sigma_n-} = \Pi^B_{\sigma_n} \left( \frac{\bar{N}_{\sigma_n}}{N_{\sigma_n} + 1} \right) \Pi^B_{\sigma_n} = \frac{1}{N_{\sigma_n} + 1} \Pi^B_{\sigma_n}
$$

(54)

and

$$
\nu^f_{\sigma_n} - \nu^f_{\sigma_n-} = \Pi^A_{\sigma_n} \Pi^B_{\sigma_n} \left( -Q_{T^{-\sigma_n}} f(X^{j_{\sigma_n}-1}) + \frac{1}{N_{\sigma_n} + 1} \nu^f_{\sigma_n-} \right)
$$

(55)

$$
= \Pi^A_{\sigma_n} \Pi^B_{\sigma_n} (\mathcal{M}_{\sigma_n} - \mathcal{M}_{\sigma_n-})
$$

where $j_{\sigma_n}$ is the index of the particle removed from the system at time $\sigma_n$. Again, substituting these equalities back into (53) and using (49) gives,

$$
\nu^f_{\sigma_n} - \nu^f_{\sigma_n-} = \Pi^A_{\sigma_n} \Pi^B_{\sigma_n} \left( -Q_{T^{-\sigma_n}} f(X^{j_{\sigma_n}-1}) + \frac{1}{N_{\sigma_n} + 1} \nu^f_{\sigma_n-} \right)
$$

(56)

Finally, we have $\nu_{\tau_n-} - \nu_{\tau_n-} = \Pi^A_{\tau_{n-1}} \Pi^B_{\tau_{n-1}} (\mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n-})$ for all $n \geq 1$, hence

$$
\nu^f_{\tau_n-} - \nu^0_{\tau_n-} = \Pi^A_{\tau_n} \Pi^B_{\tau_n} (\mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n-}) + \sum_{n=1}^{B_t} \Pi^A_{\tau_n} \Pi^B_{\tau_n} (\mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n-})
$$

$$
+ \sum_{n=1}^{C_t} \Pi^A_{\tau_{n-1}} \Pi^B_{\tau_{n-1}} (\mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n-})
$$

which entails (42) and thus concludes Step 3.

Step 4. Control of the $L^2$ norm of the martingale increments. In this section, we assume without loss of generality (taking $e^{||0||_{\infty} T} f/||f||_{\infty}$ instead of $f$) that

$$
\sup_{t \in [0,T]} ||Q_t f||_{\infty} \leq 1.
$$
We now show that the $L^2$ norm of each of these quantities on the righthand side of (42) is finite. To this end, let us first consider $M_{\rho_n} - M_{\rho_n^-}$. We have, according to (48),

\begin{equation}
|M_{\rho_n} - M_{\rho_n^-}| = \left| Q_{T-\rho_n} f(X_{\rho_n}^{i_n-1}) - \frac{1}{N_{\rho_n} - 1} \sum_{i \in S_{\rho_n^-} \setminus \{i_{n-1}'\}} Q_{T-\rho_n} f(X_{\rho_n}^i) \right| \leq 2. \tag{57}
\end{equation}

Then, setting $R_T = \max_{t \in [0, T]} N_t$, we obtain

\[
E \left[ \sum_{n=1}^{A_T} \Pi_{\rho_n}^A \Pi_{\rho_n}^B (M_{\rho_n} - M_{\rho_n^-}) \right]^2
= E \left[ \left( \frac{N_{\rho_n} - 1}{N_{\rho_n}} \sum_{n=1}^{A_T} \Pi_{\rho_n}^A \Pi_{\rho_n}^B (M_{\rho_n} - M_{\rho_n^-}) \right)^2 \right]
\leq 4 E \left[ \sum_{n=1}^{A_T} \left( \frac{R_T - 1}{R_T} \right)^{2A_n} (\Pi_{\rho_n}^B)^2 \right]
\leq 4 E \left[ (\Pi_B^B)^2 \sum_{n=1}^{A_T} \left( \frac{R_T - 1}{R_T} \right)^{2n} \right].
\tag{58}
\]

To control the sum term, since $1/R_T \leq \frac{1}{2}$, we have

\[
\sum_{n=1}^{A_T} \left( 1 - \frac{1}{R_T} \right)^{2n} \leq \sum_{n=1}^{+\infty} \left( 1 - \frac{1}{R_T} \right)^{2n} = \frac{(1 - \frac{1}{R_T})^2}{1 - (1 - \frac{1}{R_T})^2} = R_T \frac{1 - 2/R_T + R_T^{-2}}{2 - 1/R_T} \leq R_T/2 \leq (N_0 + \beta_T)/2,
\]

where $\beta_T$ denotes the total number of branching events before time $T$. We now make use of Lemma 4.1, stated and proved at the end of this section, which entails that

\begin{equation}
E \left[ (\Pi_B^B)^2 (N_0 + \beta_T)/2 \right] \leq N_0 \exp(c\|b\|_{\infty} T), \tag{59}
\end{equation}

for some (explicit) constant $c > 0$. Substituting these estimates into (58), we obtain

\begin{equation}
E \left[ \sum_{n=1}^{A_T} \Pi_{\rho_n}^A \Pi_{\rho_n}^B (M_{\rho_n} - M_{\rho_n^-}) \right]^2 \leq 4 N_0 \exp(c\|b\|_{\infty} T). \tag{60}
\end{equation}

Let us now consider $\Delta_{\sigma_n} M$. Using (49), we have

\[
|M_{\sigma_n} - M_{\sigma_n^-}| \leq 2.
\]

Then

\begin{equation}
E \left[ \sum_{n=1}^{B_T} \Pi_{\sigma_n}^A \Pi_{\sigma_n}^B (M_{\sigma_n} - M_{\sigma_n^-}) \right]^2
\leq E \left[ \left( \sum_{n=1}^{B_T} \Pi_{\sigma_n^-}^A \Pi_{\sigma_n^-}^B \frac{N_{\sigma_n^-} + 1}{N_{\sigma_n^-}} (M_{\sigma_n} - M_{\sigma_n^-}) \right)^2 \right]
\leq 4 E \left[ \sum_{n=1}^{B_T} (\Pi_{\sigma_n}^B)^2 \right] \leq 4 E \left[ \beta_T (\Pi_B^B)^2 \right]. \tag{62}
\end{equation}
Using Lemma 4.1 stated and proved at the end of this section, we deduce that

\[ E \left[ \left( \sum_{n=1}^{B_T} \Pi^A_{\sigma_n} \Pi^B_{\sigma_n} (M_{\sigma_n} - M_{\sigma_n-}) \right)^2 \right] \leq 4N_0 \exp(c\|b\|_\infty T), \]

where \( c \) can be chosen to be the same as in (60).

It remains to control the increments of \( \bar{M} \). For all \( n \geq 1 \), let \( E_n \subset \mathbb{N} \) be the set of indices of particles defined by

\[ E_n = \{ i \in \mathbb{N} \text{ s.t. the } i^{th} \text{ particle is branching, killed, born, removed or duplicated at time } \tau_n \}. \]

Observe that \( |E_n| \leq 4 \) for all \( n \geq 1 \). We also define, for all \( i \in \mathbb{N} \), the sequence of times \((\tau^i_n)_{n \geq 0}\) by \( \tau^i_0 = 0 \) and \( \tau^i_{n+1} = \min\{ \tau_k > \tau^i_n, \text{ with } k \geq 0 \text{ and } i \in E_k \} \).

We write

\[ \bar{M}_T = \sum_{i \in \mathbb{N}} \sum_{n \geq 1} \bar{M}^i_{\tau^i_n \wedge T} - \bar{M}^i_{\tau^i_{n-1} \wedge T}. \]

Since all the martingales increments are orthogonal, we have, for all \( i \in \mathbb{N} \),

\[
\Delta := E \left[ \left( \Pi^A_{\tau^i_T}(\bar{M}_T - \bar{M}_{\tau^i_T}) + \sum_{n=1}^{C_T} \Pi^A_{\tau^i_{n-1}} \Pi^B_{\tau^i_{n-1}} (M_{\tau^i_n} - M_{\tau^i_{n-1}}) \right)^2 \right]
\]

\[
= E \left[ \left( \sum_{i \in \mathbb{N}} \sum_{n \geq 1} \Pi^A_{\tau^i_{n-1}} \Pi^B_{\tau^i_{n-1}} (M^i_{\tau^i_n \wedge T} - M^i_{\tau^i_{n-1} \wedge T}) \right)^2 \right]
\]

\[
= E \left[ \sum_{i \in \mathbb{N}} \sum_{n \geq 1} \left( \Pi^A_{\tau^i_{n-1}} \Pi^B_{\tau^i_{n-1}} \right)^2 (M^i_{\tau^i_n \wedge T} - M^i_{\tau^i_{n-1} \wedge T})^2 \right].
\]

But the increments \( \bar{M}^i_{\tau^i_{n+1} \wedge T} - \bar{M}^i_{\tau^i_{n-1} \wedge T} \) are almost surely bounded by 3, so that

\[
\Delta \leq 9 E \left[ \sum_{i \in \mathbb{N}} \sum_{n \geq 1} \left( \Pi^A_{\tau^i_{n-1}} \Pi^B_{\tau^i_{n-1}} \right)^2 1_{\tau^i_{n-1} < T} \right]
\]

\[
\leq 9 E \left[ \sum_{n \geq 1} \sum_{i \in E_n} \left( \Pi^A_{\tau^i_{n-1}} \Pi^B_{\tau^i_{n-1}} \right)^2 1_{\tau^i_{n-1} < T} \right]
\]

\[
\leq 36 E \left[ \sum_{n \geq 1} \left( \Pi^A_{\tau^i_{n-1}} \Pi^B_{\tau^i_{n-1}} \right)^2 1_{\tau^i_{n-1} < T} \right]
\]

Hence, denoting by \( \beta_T \) (resp. \( \kappa_T \)) the number of branching (resp. killing without resampling) events occurring before time \( T \geq 0 \) (so that \( C_T = \beta_T + \kappa_T + A_T \)), we have

\[ \Delta \leq 36 E \left[ \left( \Pi^B_{\tau^i_T} \right)^2 (1 + \beta_T + \kappa_T + \sum_{n=1}^{A_T} \left( \Pi^A_{\tau^i_{n-1}} \right)^2) \right] \]

We already computed that, almost surely,

\[
\sum_{n=1}^{A_T} \left( \Pi^A_{\tau^i_{n-1}} \right)^2 \leq \sum_{k=0}^{+\infty} \left( \frac{R_T - 1}{R_T} \right)^{2k} \leq R_T/2 \leq (N_0 + \beta_T)/2.
\]
where we used, as above, Lemma 4.1. We thus obtained, using the last inequality and (64), that
\[
0 \leq \text{E} \left[ (\Pi_T^B)^2 (1 + \beta_T + \kappa_T + \sum_{n=1}^{A_T} (\Pi_{n-1}^A)^2) \right] \leq \text{E} \left[ (\Pi_T^B)^2 (\beta_T + 3N_0) \right] \leq 4N_0 \exp(c\|b\|_{\infty}T),
\]
where we used, as above, Lemma 4.1. We thus obtained, using the last inequality and (64), that
\[
\Delta \leq 144N_0 \exp(c\|b\|_{\infty}T).
\]
Using (42), (60), (63) and the last inequality, we deduce that
\[
\text{E} \left[ (\nu_T^f - \nu_0^f)^2 \right] \leq 152N_0 \exp(c\|b\|_{\infty}T).
\]
Without assuming that \(\sup_{t \in [0,T]} \|Q_t f\|_{\infty} \leq 1\), we thus obtain
\[
\text{E} \left[ (\nu_T^f - \nu_0^f)^2 \right] \leq 152N_0 \exp(c\|b\|_{\infty}T) \left( \sup_{t \in [0,T]} \|Q_t f\|_{\infty} \right)^2.
\]

Step 5. Conclusion.
Recall the occupation measure \(m_t\) defined in (5) for all \(t \geq 0\). With this notation, the conclusion of Step 4 reads
\[
\text{E} \left[ (m_0Q_T f - \Pi_T^A \Pi_T^B m_T f)^2 \right] \leq 152N_0 \exp(c\|b\|_{\infty}T) \left( \sup_{t \in [0,T]} \|Q_t f\|_{\infty} \right)^2.
\]
We deduce that
\[
\left\| m_0Q_T f_{1_E} - m_T f_{1_E} 1_{T \neq 0} - m_0Q_T f_{1_E} \right\|_2 \leq \left\| (m_0Q_T f_{1_E} - \Pi_T^A \Pi_T^B m_T f_{1_E}) \frac{m_T f_{1_E}}{m_T f_{1_E} 1_{T \neq 0}} \right\|_2 + \left\| \Pi_T^A \Pi_T^B m_T f_{1_E} - m_0Q_T f_{1_E} \right\|_2 \leq 2 \sqrt{152}N_0 \exp(c\|b\|_{\infty}T/2) \|f\|_{\infty} \sup_{t \in [0,T]} \|Q_t f_{1_E}\|_{\infty}.
\]

We conclude that
\[
\left\| \frac{m_0Q_T f_{1_E} - m_T f_{1_E} 1_{T \neq 0}}{m_0Q_T f_{1_E} / N_0} \right\|_2 \leq \frac{2 \sqrt{152} \exp(c\|b\|_{\infty}T/2) \|f\|_{\infty} \sup_{t \in [0,T]} \|Q_t f_{1_E}\|_{\infty}}{m_0Q_T f_{1_E} / N_0} \frac{1}{\sqrt{N_0}}.
\]
Observing that \(\sup_{t \in [0,T]} \|Q_t f_{1_E}\|_{\infty} \leq e\|b\|_{\infty}T\), this concludes the proof of Theorem 2.6.

Proof of Lemma 4.1. We use the same notations as in the proof of Theorem 2.6.
Fix \(C > c_{\ell}\|b\|_{\infty}\) and \(\delta > 0\) such that \(C - \delta C^2 2^{\ell} \geq c_{\ell}\|b\|_{\infty}\). We have
\[
\text{E} \left[ (\Pi_{T_1}^B)^{\ell} e^{-C(T_1 \wedge \delta)} \right] = \text{E} \left[ \left( \frac{N_0 + 1}{N_0} \right)^{\ell} 1_{T_1 = T_1 \wedge \delta} e^{-C(T_1 \wedge \delta)} \right] + \text{E} \left( 1_{T_1 > T_1 \wedge \delta} e^{-C(T_1 \wedge \delta)} \right)
\]
\[
= \left( \frac{N_0 + 1}{N_0} \right)^{\ell} \text{E} \left( e^{-C(T_1 \wedge \delta)} \right) + \left( 1 - \left( \frac{N_0 + 1}{N_0} \right)^{\ell} \right) \text{E} \left( 1_{T_1 > T_1 \wedge \delta} e^{-C(T_1 \wedge \delta)} \right).
\]
On the one hand, we have
\[ \mathbb{E}\left(e^{-C(\tau_1 \wedge \delta)}\right) \leq 1 - e^{-\delta C} \mathbb{E}(\tau_1 \wedge \delta) \leq 1 - (C - \delta C^2) \mathbb{E}(\tau_1 \wedge \delta), \]
and, on the other hand,
\[ \mathbb{E}\left(1_{\tau_1 > \tau_1 \wedge \delta} e^{-C(\tau_1 \wedge \delta)}\right) \geq \mathbb{E}\left(e^{-\delta C} (N_0 ||b||_\infty + C)\right) \leq 1 - (N_0 ||b||_\infty + C) \mathbb{E}(\tau_1 \wedge \delta). \]

Setting, for the sake of readability, \( \alpha = \mathbb{E}(\tau_1 \wedge \delta) \) and \( \beta = \left(\frac{N_0 + 1}{N_0}\right) \epsilon \), we deduce that
\[ \mathbb{E}\left(\left(\Pi^B_{\tau_1 \wedge \delta}\right)^\ell e^{-C(\tau_1 \wedge \delta)}\right) \leq 1 - \alpha (C - \delta C^2 \beta + N_0 ||b||_\infty (1 - \beta)) \]
\[ \leq 1 - \alpha (C - \delta C^2 2^\ell - ||b||_\infty \epsilon) \leq 1, \]
where we used the fact that \( \beta \leq 2^\ell \) and \( \beta - 1 \leq c_\ell/N_0. \)

We define the sequence of stopping times \( (\theta_n)_{n \geq 0} \) by \( \theta_0 = 0 \) and
\[ \theta_{n+1} := \inf\{((n+1)\delta) \wedge \tau_k, k \geq 1, \tau_k > \theta_n\}. \]

Then
\[ \mathbb{E}\left(\left(\Pi^B_{T \wedge \theta_{n+1}}\right)^\ell e^{-C(T \wedge \theta_{n+1})} | \mathcal{F}_{T \wedge \theta_n}\right) = \left(\Pi^B_{T \wedge \theta_n}\right)^\ell e^{-C(T \wedge \theta_n)} \times \mathbb{E}_{T \wedge \theta_n}\left(\left(\Pi^B_{(T-u) \wedge \theta_1}\right)^\ell e^{-C((T-u) \wedge \theta_1)}\right)_{|u=T \wedge \theta_n}. \]

Since \( T - u \leq \delta \) almost surely in the right hand term, we deduce from (66) that
\[ \mathbb{E}\left(\left(\Pi^B_{T \wedge \theta_{n+1}}\right)^\ell e^{-C(T \wedge \theta_{n+1})} | \mathcal{F}_{T \wedge \theta_n}\right) \leq \left(\Pi^B_{T \wedge \theta_n}\right)^\ell e^{-C(T \wedge \theta_n)}. \]

In particular, using Fatou’s Lemma, we deduce, letting \( n \to +\infty \), that
\[ \mathbb{E}\left(\left(\Pi^B_T\right)^\ell e^{-C T}\right) \leq 1. \]

Since this is true for any \( C > c_\ell ||b||_\infty \), this concludes the proof of the first part of the lemma.

For the second part, we observe that \( \beta_T \) is stochastically dominated by a continuous time counting process \( (Z_t)_{t \geq 0} \) on \( \mathbb{N} \) starting from \( N_0 \) and with increment rate \( ||b||_\infty n \) from state \( n \) to state \( n+1 \). In particular, denoting by \( L \) the infinitesimal generator of this process and \( V(n) = n^2 \) for all \( n \geq 1 \), we obtain
\[ LV(n) = ||b||_\infty n((n+1)^2 - n^2) = ||b||_\infty n(2n + 1) \leq 3||b||_\infty V(n), \]
and deduce that
\[ \mathbb{E}(\beta_T^2) \leq \mathbb{E}(Z_T^2) \leq e^{3||b||_\infty T} Z_0^2 = e^{3||b||_\infty T} N_0^2. \]

Hence, we obtain
\[ \mathbb{E}\left(\beta_T \left(\Pi^B_T\right)^2\right) \leq \sqrt{\mathbb{E}(\beta_T^2)} \sqrt{\mathbb{E}\left(\left(\Pi^B_T\right)^4\right)} \leq N_0 \exp((3 + c_4) ||b||_\infty T/2). \]

This concludes the proof of the second part of the Lemma.

\[ \square \]

Remark 4.2. Our proof adapts to the case where \( \tau_\infty < +\infty \) with positive probability (that is when Assumption 2 does not holds true), using the following strategy. In this situation, one shows using the same approach that, for all \( n \geq 0 \),
\[ m_0 Q_T f = \mathbb{E}\left(\Pi^A_{\tau_n \wedge T} \Pi^B_{\tau_n \wedge T} m_{\tau_n \wedge T} Q_{T-\tau_n \wedge T} f\right). \]
Since the branching rate is uniformly bounded and since $\Pi^A_{\tau_n \wedge T}$ is bounded by 1, one checks that $\Pi^A_{\tau_n \wedge T} \Pi^B_{\tau_n \wedge T} m_{\tau_n \wedge T} Q_{T-\tau_n \wedge T} f$ can be uniformly bounded (in $n$) by an integrable random variable. Since in addition $\Pi^A_{\tau_n \wedge T} \Pi^B_{\tau_n \wedge T} m_{\tau_n \wedge T} Q_{T-\tau_n \wedge T} f \to 0$ on the event $\tau_\infty \leq T$ (since this events necessarily corresponds to the accumulation of resampling events) and $\Pi^A_{\tau_n \wedge T} \Pi^B_{\tau_n \wedge T} m_{\tau_n \wedge T} Q_{T-\tau_n \wedge T} f \to \Pi^A_T \Pi^B_T m_T f$ on the complementary event, one deduces from the dominated convergence theorem that

$$m_0 Q_T f = \mathbb{E} \left( \Pi^A_T \Pi^B_T m_T f \mathbf{1}_{T < \tau_\infty} \right).$$

The rest of the proof (see Steps 4 and 5) then proceeds almost identically.

Remark 4.3. We comment on some direct and natural continuations of the proof, which are easily derived from the fact that the proof is based on the control of martingale increments.

First, if the martingale $\nu'_{T} - \nu'_{0}$ is càdlàg, then Doob’s maximal inequality and (65) entail that

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \left( \nu'_{T} - \nu'_{0} \right)^2 \right] \leq c_T \left\| f \right\|_2^2 \sup_{t \in [0,T]} \| Q_t \mathbf{1}_E \|_\infty^2 N_0$$

for some (explicit) constant $c_T$. Then the conclusion reads

$$\left\| \sup_{t \in [0,T]} \left| \frac{m_0 Q_t f}{m_0 Q_t \mathbf{1}_E} \right| \mathbf{1}_{N_t \neq 0} - \frac{m_T f}{m_0 Q_T \mathbf{1}_E / N_0} \right\|_2 \leq \frac{c_T \left\| f \right\|_\infty}{m_0 Q_T \mathbf{1}_E / N_0} \frac{1}{\sqrt{N_0}},$$

Second, if the killing rate is bounded, then the statement of Lemma 4.1 holds true with $\beta_T$ replaced by the total number of killing/branching events $C_T$, and with different constants. Then, for any $p \geq 2$, using the same approach as in Step 4 of the proof, but using the Burkholder-Davis-Gundy inequality instead, one deduces similarly that

$$\mathbb{E} \left[ \left| \nu'_{T} - \nu'_{0} \right|^p \right] \leq c_T N_0^{p/2},$$

and hence

$$\left\| \frac{m_0 Q_t f}{m_0 Q_t \mathbf{1}_E} - \frac{m_T f}{m_0 Q_T \mathbf{1}_E} \right\|_p \leq \frac{c_T \left\| f \right\|_\infty}{m_0 Q_T \mathbf{1}_E / N_0} \frac{1}{\sqrt{N_0}},$$

for some (explicit) constants $c_T$ and $c_T'$. Note that the assumption that the killing rate is uniformly bounded is technical: after using Burkholder-Davis-Gundy inequality, one needs to control the term $\mathbb{E} \left( C_T^{p/2} \left( \frac{N+1}{N} \right)^p c_T \right)$, however, depending on the particular process at hand, there may be other ways to obtain upper bounds on this quantity which do not require the killing rate to be bounded.

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