ON THE NUMBER OF COMMUTATION CLASSES OF THE LONGEST ELEMENT IN THE SYMMETRIC GROUP

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Abstract. Using the standard Coxeter presentation for the symmetric group $S_n$, two reduced expressions for the same group element are said to be commutation equivalent if we can obtain one expression from the other by applying a finite sequence of commutations. The resulting equivalence classes of reduced expressions are called commutation classes. How many commutation classes are there for the longest element in $S_n$?

Original proposer of the open problem: Donald E. Knuth
The year when the open problem was proposed: 1992 [11, §9]

A Coxeter system is a pair $(W, S)$ consisting of a distinguished (finite) set $S$ of generating involutions and a group

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

called a Coxeter group, where $e$ is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. It turns out that the elements of $S$ are distinct as group elements and that $m(s,t)$ is the order of $st$. Since the elements of $S$ have order two, the relation $(st)^{m(s,t)} = e$ can be written to allow the replacement

$$\underbrace{sts \cdots} \mapsto \underbrace{tst \cdots}_{m(s,t)}$$

which is called a commutation if $m(s,t) = 2$ and a braid move if $m(s,t) \geq 3$.

Given a Coxeter system $(W, S)$, a word $w = s_{x_1}s_{x_2} \cdots s_{x_m}$ in the free monoid $S^*$ is called an expression for $w \in W$ if it is equal to $w$ when considered as a group element. If $m$ is minimal among all expressions for $w$, the corresponding word is called a reduced expression for $w$. In this case, we define the length of $w$ to be $\ell(w) = m$. According to [8], every finite Coxeter group contains a unique element of maximal length, which we refer to as the longest element and denote by $w_0$.

Let $(W, S)$ be a Coxeter system and let $w \in W$. Then $w$ may have several different reduced expressions that represent it. However, Matsumoto’s Theorem [7, Theorem 1.2.2]
states that every reduced expression for \( w \) can be obtained from any other by applying a finite sequence of commutations and braid moves.

Following [13], we define a relation \( \sim \) on the set of reduced expressions for \( w \). Let \( w \) and \( w' \) be two reduced expressions for \( w \) and define \( w \sim w' \) if we can obtain \( w' \) from \( w \) by applying a single commutation. Now, define the equivalence relation \( \approx \) by taking the reflexive transitive closure of \( \sim \). Each equivalence class under \( \approx \) is called a commutation class.

The Coxeter system of type \( A_{n-1} \) is generated by \( S(A_{n-1}) = \{s_1, s_2, \ldots, s_{n-1}\} \) and has defining relations (i) \( s_is_i = e \) for all \( i \); (ii) \( s_is_j = s_js_i \) when \( |i - j| > 1 \); and (iii) \( s_is_js_i = s_js_is_j \) when \( |i - j| = 1 \). The corresponding Coxeter group \( W(A_{n-1}) \) is isomorphic to the symmetric group \( S_n \) under the correspondence \( s_i \mapsto (i, i+1) \). It is well known that the longest element in \( S_n \) is given in 1-line notation by

\[
  w_0 = [n, n-1, \ldots, 2, 1]
\]

and that \( \ell(w_0) = \binom{n}{2} \).

Let \( c_n \) denote the number of commutation classes of the longest element in \( S_n \). The longest element \( w_0 \) in \( S_4 \) has length 6 and is given by the permutation \((1, 4)(2, 3)\). There are 16 distinct reduced expressions for \( w_0 \) while \( c_4 = 8 \). The 8 commutation classes for \( w_0 \) are given in Figure 1 where we have listed the reduced expressions that each class contains. Note that for brevity, we have written \( i \) in place of \( s_i \).

In [12], Stanley provides a formula for the number of reduced expressions of the longest element \( w_0 \) in \( S_n \). However, the following question is currently unanswered.

**Open Problem.** What is the number of commutation classes of the longest element in \( S_n \)?

To our knowledge, this problem was first introduced in 1992 by Knuth in Section 9 of [11], but not using our current terminology. A more general version of the problem appears in Section 5.2 of [9]. In the paragraph following the proof of Proposition 4.4 of [14], Tenner explicitly states the open problem in terms of commutation classes.

According to sequence A006245 of The On-Line Encyclopedia of Integer Sequences [1], the first 10 values for \( c_n \) are 1, 1, 2, 8, 62, 908, 24698, 1232944, 112018190, 18410581880. To
date, only the first 15 terms are known. The current best upper-bound for $c_n$ was obtained by Felsner and Valtr. They prove that for sufficiently large $n$, $c_n \leq 2^{0.6571n^2}$ [5, Theorem 2], although their result is stated in terms of arrangements of pseudolines.

The commutation classes of the longest element of the symmetric group are in bijection with a number of interesting objects. It turns out that $c_n$ is equal to the number of

- heaps for the longest element in $S_n$ [13, Proposition 2.2];
- primitive sorting networks on $n$ elements [2, 10, 11, 15, 16];
- rhombic tilings of a regular $2n$-gon (where all side lengths of the rhombi and the $2n$-gon are the same) [3, 14];
- oriented matroids of rank 3 on $n$ elements [6, 9];
- arrangements of $n$ pseudolines [4, 5, 11].

In Figure 2, we have drawn lattice point representations of the 8 heaps that correspond to the commutation classes for the longest element in $S_4$. Note that our heaps are sideways versions of the heaps that usually appear in the literature. The minimum ladder lotteries (or ghost legs) corresponding to the 8 primitive sorting networks on 4 elements are provided in Figure 3. The 8 distinct rhombic tilings of a regular octagon are depicted in Figure 4.
Figure 4. Rhombic tilings of a regular octagon.

Very little is known about the number of commutation classes of the longest element in other finite Coxeter groups.

REFERENCES

[1] The On-Line Encyclopedia of Integer Sequences (OEIS). http://oeis.org, 2016.
[2] D. Armstrong. The sorting order on a Coxeter group. J. Combin. Theory, Ser. A, 2009.
[3] S. Elnitsky. Rhombic Tilings of Polygons and Classes of Reduced Words in Coxeter Groups. J. Combin. Theory, Ser. A, 77(2), 1997.
[4] S. Felsner. On the Number of Arrangements of Pseudolines. Discrete Comput. Geom., 18(3), 1997.
[5] S. Felsner and P. Valtr. Coding and Counting Arrangements of Pseudolines. Discrete Comput. Geom., 46(3), 2011.
[6] J. Folkman and J. Lawrence. Oriented matroids. J. Combin. Theory, Ser. B, 25(2), 1978.
[7] M. Geck and G. Pfeiffer. Characters of finite Coxeter groups and Iwahori–Hecke algebras. 2000.
[8] J.E. Humphreys. Reflection Groups and Coxeter Groups. Cambridge University Press, Cambridge, 1990.
[9] M.M. Kapranov and V.A. Voevodsky. Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher Bruhat orders (list of results). Cahiers de topologie et géométrie différentielle catégoriques, 32(1), 1991.
[10] J. Kawahara, T. Saitoh, R. Yoshinaka, and S. Minato. Counting Primitive Sorting Networks by πDDs. TCS Technical Report, 2011.
[11] D. Knuth. Axioms and Hulls. Springer-Verlag, Berlin, 1992.
[12] R.P. Stanley. On the number of reduced decompositions of elements of Coxeter groups. European J. Combin, 5(4), 1984.
[13] J.R. Stembridge. On the fully commutative elements of Coxeter groups. J. Algebraic Combin., 5, 1996.
[14] B.E. Tenner. Reduced decompositions and permutation patterns. J. Algebraic Combin., 24(3), 2006.
[15] K. Yamanaka and S. Nakano. Efficient Enumeration of All Ladder Lotteries with k Bars. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences, E97-A(6), 2009.
[16] K. Yamanaka, S. Nakano, Y. Matsui, R. Uehara, and K. Nakada. Efficient Enumeration of All Ladder Lotteries and Its Application. Theoretical Computer Science, 411, 2010.

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