Covariant - tensor method for quantum groups and applications I : $SU(2)_q$

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Abstract

A covariant - tensor method for $SU(2)_q$ is described. This tensor method is used to calculate $q$-deformed Clebsch-Gordan coefficients. The connection with covariant oscillators and irreducible tensor operators is established. This approach can be extended to other quantum groups.
1. Introduction

In recent years there has been considerable interest in q-deformations of Lie algebras (quantum groups) [1] and their applications in physics [2]. The main goal of these applications is a generalization of the concept of symmetry. The properties of quantum groups are similar to those of classical Lie groups with q not being a root of unity. However, it is still not clear to what extent the familiar tensor methods, used in the representation theory of Lie algebras, are applicable to the case of q - deformations.

Different types of the tensor calculus for $SU(2)_q$ were proposed and applied in references [3,4,6,9,10]. However, no simple covariant - tensor calculus for $SU(n)_q$ was presented. In this paper we propose a simple covariant - tensor method for $SU(2)_q$ which can be extended to the general $SU(n)_q$. Details for $SU(n)_q$ and especially for $SU(3)_q$ will be published separately.

The plan of the paper is the following. In Section 2 we recall the basics of the $SU(2)_q$ algebra, its fundamental representation and invariants. In Section 3 we construct the general $SU(2)_q$ - covariant tensors and invariants. In Section 4 we apply this tensor method to calculate q - deformed Clebsch-Gordan coefficients and in Section 5 we demonstrate their symmetries. We point out that this method is simpler than that used in previous calculations [5,6] and can be generalized to other quantum groups. Finally, in Section 6 we connect covariant tensors with covariant q - oscillators and construct unit irreducible tensor operators.

2. $SU(2)_q$ - algebra, its fundamental representation and invariants

Let us recall that three generators of $SU(2)_q$ obey the following commutation relations [1] (we take q real)

$$[J^0, J^\pm] = \pm J^\pm$$

$$[J^+, J^-] = [2J^0]_q = \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}}$$

The coproduct $\Delta : SU(2)_q \rightarrow SU(2)_q \otimes SU(2)_q$ is defined as

$$\Delta(J^\pm) = J^\pm \otimes J^0 + q^{-J^0} \otimes J^\pm$$

$$\Delta(J^0) = J^0 \otimes 1 + 1 \otimes J^0$$

Let $V_2$ be a two - dimensional space spanned by the basis $|e_a>, a = 1, 2$, and $|v> = \sum_a |e_a > v_a \in V_2$. The $SU(2)_q$ generators $J^k(k = \pm, 0)$ act as
\[ J^k|e_a\rangle = \sum_b (J^k)_{ba}|e_b\rangle \]

\[ J^k|v\rangle = \sum_{a,b} (J^k)_{ba}|e_b\rangle v_a = \sum_b |e_b\rangle (J^k|v\rangle)_b = \sum_b |e_b\rangle v_b. \quad (2.3) \]

In the fundamental representation of \( SU(2)_q \) the generators \( J^k \)s are ordinary 2x2 Pauli matrices. Let \((V_2)^*\) be a dual space with the basis \(<e_a| = (|e_a\rangle)^+\) and \(|v| = (|v\rangle)^+ = \sum_a v^*_a < e_a|\). The dual basis is orthonormal, i.e. \(<e_a|e_b> = \delta_{ab}\). We note that the components of the vector \(|v\rangle, v_a,\ (\text{or} \ v^*_a \text{of} <v|)\) are not defined as real or complex numbers. Their algebraic properties will follow from \( SU(2)_q \)-invariance requirements. Here we identify (for the spin \( j = 1/2 \))

\[ |e_a\rangle = |\frac{1}{2}, m_a\rangle \]

\[ <e_a| = <\frac{1}{2} m_a| \]

\[ m_a = \pm \frac{1}{2} \quad (2.4) \]

and the matrix elements of the generators \( J_k \) are

\[ <e_a|J^0|e_a\rangle = m_a \]

\[ <e_1|J^+|e_2\rangle = <e_2|J^-|e_1\rangle = 1 \quad (2.5) \]

We define a scalar product as \(<u|v> = \sum_a u^*_a v_a\) and the norm as \(<v|v> = \sum_a v^*_a v_a\). This scalar product (and the norm) are not \( SU(2)_q \)-invariant. Instead, the quantity

\[ <v|q^{-J^0}|v> \quad (2.6) \]

is invariant under the action of the coproduct (2.2) in the following sense:

\[ \Delta( J^\pm )\langle v|q^{-J^0}|v\rangle = (J^\pm \langle v|)\langle v\rangle + (q^{-J^0}\langle v|)J^\pm q^{-J^0}\langle v\rangle = -\langle v|J^\pm |v\rangle + \langle v|J^\pm |v\rangle = 0, \quad (2.7) \]

\[ \Delta( J^0 )\langle v|q^{-J^0}|v\rangle = (J^0 \langle v|)q^{-J^0}\langle v\rangle + \langle v|J^0 q^{-J^0}\langle v\rangle = -\langle v|J^0 q^{-J^0}\langle v\rangle + \langle v|J^0 q^{-J^0}\langle v\rangle = 0. \quad (2.8) \]

The quadratic forms

\[ \sum_a u^*_a q^{-J^0} v_a = \sum_a u^*_a q^{-m_a} v_a \quad (2.9) \]

and

\[ \sum_a v_a q^{m_a} u^*_a \quad (2.10) \]
are $SU(2)_q$ -invariant. Note that the first quadratic form (2.8) can be written as $< u| q^{-J_0} | v >$. If we demand $\sum_a v_a^* q^{-m_a} v_a = \sum_a v_a q^{m_a} v_a^*$, it follows that $v_1^* v_1 = q v_1 v_1^*$ and $v_2^* v_2 = q^{-1} v_2 v_2^*$.

In addition to the $< u| q^{-J_0} | v >$ - invariant form we consider another form,

$$\epsilon_{ab}| e_a \rangle| e_b \rangle$$

with

$$\epsilon_{ab} = \begin{pmatrix} 0 & q^\frac{1}{2} \\ -q^{-\frac{1}{2}} & 0 \end{pmatrix}$$

$$\epsilon_{ab} \epsilon_{bc} = -\delta_{ac}$$

$$(\epsilon_{ab})_q = - (\epsilon_{ab})_q^{-1}$$

where $\bar{1} = 2$ and $\bar{2} = 1$. Note that the $q$ -antisymmetric combination $v_a v_b \epsilon_{ab}$ is $SU(2)_q$ -invariant, showing that $v_a$ and $v_b$ do not commute. Instead, they $q$ -commute, i.e. $v_2 v_1 = q v_1 v_2$.

### 3. General $SU(2)_q$ - tensors and invariants

Let us consider the tensor - product space $(V_2)^{\otimes k} = V_2 \otimes ... \otimes V_2$ with the basis $| e_{a_1} \rangle \otimes ... \otimes | e_{a_k} \rangle$, $a_1, ... a_k = 1, 2$. Then we write an element of the tensor space $(V_2)^{\otimes k}$ as tensor $| T \rangle$ of the form

$$| T \rangle = | e_{a_1} \rangle ... | e_{a_k} \rangle T^{a_1} ... T^{a_k} = | e_{a_1} ... e_{a_k} \rangle T^{a_1 ... a_k}$$

(3.1)

We have the following proposition:

The tensor $| T \rangle$ transforms under the $SU(2)_q$ algebra as an irreducible representation of spin $j = k/2$ if and only if $T^2 T^1 = q T^1 T^2$.

Let us assume $T^2 T^1 = q T^1 T^2$. Then

$$| T_{j=\frac{k}{2}} \rangle = | e_{a_1} ... e_{a_k} \rangle T^{a_1 ... a_k} = \sum_{m=-j}^{+j} | j m \rangle T^{j m}$$

(3.2)

The vectors $| j m \rangle$ span the space $V_{2j+1}$ of the irreducible representation with spin $j$. From $T^2 T^1 = q T^1 T^2$ it follows that

$$T^{a_1 ... a_k} = q^{\chi(a_1 ... a_k)} : T^{a_1 ... a_k} :$$

(3.3)

where $: T :$ means the normal order of indices (1’s on the left of 2’s), i.e. $T^{11...22}$ and index 1 (2) appears $n_1(n_2)$ times, respectively. $\chi$ is the number of inversions with respect to the normal order. Hence,
\[
| jm \rangle = | e \{ a_1 \ldots a_k \} \rangle = \frac{1}{\sqrt{f}} q^{-\frac{M}{2}} \sum_{\text{perm}(a_1 \ldots a_k)} q^{\chi(a_1 \ldots a_k)} | e_{a_1 \ldots a_k} \rangle
\]

(3.4)

where the curly bracket \( \{ a_1 \ldots a_k \} \) denotes the q-symmetrization. The summation runs over all the allowed permutations of the fixed set of indices \((n_1 1's\) and \(n_2 2's\)) and

\[
M = n_1 n_2 = (j + m)(j - m)
\]

\[
j = \frac{1}{2}(n_1 + n_2) \quad m = \frac{1}{2}(n_1 - n_2)
\]

(3.5)

\[
f = \left( \frac{2j}{j + m} \right)_q = \frac{[2j]_q!}{[j + m]_q! [j - m]_q!}
\]

The important relation is

\[
f = q^{-M} \sum_{\text{perm}(a_1 \ldots a_k)} q^{2\chi(a_1 \ldots a_k)}
\]

(3.6)

From equation (3.4) and the definition of the coproduct \( \Delta(J^\pm) \) (2.2) we can reproduce

\[
\Delta(J^\pm) | jm \rangle = \sqrt{[j \pm m]_q[j \pm m + 1]_q} | jm \pm 1 \rangle \\
\Delta(J^0) | jm \rangle = m | jm \rangle
\]

(3.7)

From (3.2) and (3.4) we immediately obtain the relation between \( T^{jm} \) and the components of \( T^{a_1 \ldots a_k} \):

\[
T^{jm} = q^{\frac{M}{2}} \sqrt{f} : T^{a_1 \ldots a_k} : \\
T^{j-m} = q^{\frac{M}{2}} \sqrt{f} : T^{\overline{a}_1 \ldots \overline{a}_k} :
\]

(3.8)

where \( \overline{1} = 2, \overline{2} = 1 \) and \( T^{j-m} = (T^{jm})_{a_1 \rightarrow a_2} \). In the dual space \((V_2 \otimes k)^*\) we define

\[
\langle e_{a_k \ldots a_1} | = (| e_{a_1 \ldots a_k} \rangle)^+ \\
\langle e_{a_k \ldots a_1} | e_{b_1 \ldots b_k} \rangle = \delta_{a_1 b_1} \ldots \delta_{a_k b_k}
\]

and in the dual space \((V_{2j+1})^*\) we define

\[
\langle jm | = (| jm \rangle)^+ = \langle e_{a_k \ldots a_1} | =
\]

\[
= \frac{1}{\sqrt{f}} q^{-\frac{M}{2}} \sum_{\text{perm}(a_1 \ldots a_k)} q^{\chi(a_1 \ldots a_k)} (| e_{a_1 \ldots a_k} \rangle)^+ =
\]

(3.10)

\[
= \frac{1}{\sqrt{f}} q^{-\frac{M}{2}} \sum_{\text{perm}(a_1 \ldots a_k)} q^{\chi(a_1 \ldots a_k)} \langle e_{a_k \ldots a_1} | .
\]
As a consequence of equations (3.4),(3.6) and ((3.9) we obtain

$$\langle jm_1 | jm_2 \rangle = \frac{1}{T} q^{-M} \sum_{\text{perm}(a_1...a_k)} q^{2\chi(a_1...a_k)} \delta_{m_1m_2} = \delta_{m_1m_2}. \quad (3.11)$$

The SU(2)$_q$ - invariant quantity built up of the tensors $< T |$ and $| U >$ of spin $j = k/2$ is

$$I = \langle T | q^{-J_0} | U \rangle = (T^{a_k...a_1})^* q^{-J_0} U^{a_1...a_k} = \sum_{m=-j}^{+j} (T^{jm})^* U^{jm} q^m. \quad (3.12)$$

The second type of the SU(2)$_q$ - invariant quantity built up of the tensors $| T >$ and $| U >$ of spin $j = k/2$ is

$$I' = T^{a_k...a_1} U^{b_1...b_k} \epsilon_{a_1 b_1} \epsilon_{a_2 b_2} ... \epsilon_{a_k b_k} \quad (3.13)$$

with $\epsilon_{ab}$ given in (2.11). Of course, $T^a T^b \epsilon_{ab} = 0$ if $T^a$ and $T^b$ q-commute. Furthermore, using equation (3.3) we can also write

$$I = q^{\chi(s)} (T^{a_k...a_1})^* q^{-J_0} U^{(a_1...a_k)}$$

$$I' = q^{\chi(s)} T^{a_k...a_1} U^{(b_1...b_k)} \epsilon_{a_1 b_1} ... \epsilon_{a_k b_k} \quad (3.14)$$

where $s \in S_k$ is a fixed permutation of the indices $a_1...a_k$ and $\chi(s) = \chi(a_1...a_k) - \chi(s(a_1...a_k))$ is the number of inversions with respect to the $(a_1...a_k)$ order.

4. q - Clebsch-Gordan coefficients

Here we present a new simple method for calculating the q-deformed Clebsch-Gordan coefficients. It can be immediately extended and applied to SU(n)$_q$ and other quantum groups. This method is a consequence of the previously described tensor method and construction of invariants. Our notation is

$$|JM\rangle = \sum_{m_1, m_2} \langle j_1 m_1 j_2 m_2 | JM \rangle_q | j_1 m_1 \rangle | j_2 m_2 \rangle. \quad (4.1)$$

For $q \in R$, C - G coefficients are real

$$\langle j_1 m_1 j_2 m_2 | JM \rangle_q^* = \langle j_1 m_1 j_2 m_2 | JM \rangle_q \quad (4.2)$$

and

$$\langle j_1 m_1 j_2 m_2 | JM \rangle_q = \langle JM | j_1 m_1 j_2 m_2 \rangle_q \quad (4.3)$$

Using the tensor notation $| jm \rangle = | e_{(a_1...a_k)} \rangle ((3.4), (3.9) and (3.10) )$, we first calculate C - G coefficient for $j_1 \otimes j_2 \rightarrow j_1 + j_2$:
\[ \langle j_1 + j_2 \, m_1 + m_2 \mid j_1 m_1 \, j_2 m_2 \rangle_q = \]

\[ = \langle e_{\{b_1 \ldots b_1, a_k \ldots a_k\}} \mid e_{\{a_1 \ldots a_k\}} e_{\{b_1 \ldots b_l\}} \rangle = \]

\[ = \langle e_{\{b,a\}} \mid e_{\{a\}} e_{\{b\}} \rangle = \]

\[ = \frac{1}{\sqrt{f_1 f_2 f_3}} q^{-\frac{1}{4}(M_1 + M_2 + M_3)} \sum_{\text{perm}(a),(b)} q^{\chi(a) + \chi(b) + \chi(a,b)} = (4.4) \]

\[ = \sqrt{\frac{f_1 f_2}{f_3}} q^{\frac{1}{4}(M_1 + M_2 - M_3)} q^{(j_1 - m_1)(j_2 + m_2)} = \]

\[ = \sqrt{\frac{f_1 f_2}{f_3}} q^{j_1 m_2 - j_2 m_1}. \]

where we have used

\[ \chi(a, b) = \chi(a) + \chi(b) + (j_1 - m_1)(j_2 + m_2) \quad (4.5) \]

and equation (3.6) together with the abbreviations

\[ k = 2j_1 \quad l = 2j_2 \quad j_3 = j_1 + j_2 \quad m_3 = m_1 + m_2 \]

\[ M_i = (j_i + m_i)(j_i - m_i) \]

\[ f_i = \binom{2j_i}{j_i + m_i} \quad (4.6) \]

\[ \frac{f_1 \cdot f_2}{f_3} = \frac{[2j_1]_q ![2j_2]_q ![j_3 + m_3]_q ![j_3 - m_3]_q}{[j_1 + m_1]_q ![j_1 - m_1]_q ![j_2 + m_2]_q ![j_2 - m_2]_q ![2j_3]_q} \]

The main observation is that any $C - G$ coefficient $\langle j_1 m_1 j_2 m_2 \mid JM \rangle$ can be written in the form (4.4). Namely, the $C - G$ coefficient $\langle j_1 m_1 j_2 m_2 \mid JM \rangle$ is projection of the state $\langle j_1 m_1 \mid \otimes \langle j_2 m_2 \rangle = \langle e_{\{a_1 \ldots a_2j_1\}} e_{\{b_1 \ldots b_2j_2\}} \mid$ from the tensor product space $V_{2j_1+1}^* \otimes V_{2j_2+1}^*$ to the state $|JM\rangle = |e_{\{a_1 \ldots [a_2j_1 - n+1 \ldots a_2j_1 \ldots b_{n+1}, b_{2j_2}\}] \rangle$ (with the appropriate symmetry of $2j_1 + 2j_2$ indices) in the space $V_{2J+1} \subset V_{2j_1+1} \otimes V_{2j_2+1}$. Here, the square brackets $[.]$ denote $q$-antisymmetrization and $n = 2j = j_1 + j_2 - J$. 

Furthermore, the state $|e_{[a_1 a_2 \ldots [a_n b_n] \ldots b_1]} \rangle > \propto \epsilon_{a_n b_n} \ldots \epsilon_{a_1 b_1}$ transforms as a singlet, i.e., it is invariant under the coproduct action in the tensor product space $V_n \otimes V_n$. Hence, using the equation (3.4), we can write

$$\langle j_1 m_1 j_2 m_2 | J M \rangle_q = \mathcal{N} \sum_{\text{perm}(a,b),(c,d)} \langle e_{\{a,b\}} e_{\{c,d\}} | e_{\{a,d\}} \rangle \cdot (\epsilon_{(b,c)})_n \tag{4.7}$$

where the length of $b(c)$ is $n = j_1 + j_2 - J$, $(\epsilon_{(b,c)})_n = \epsilon_{b_1 c_1} \ldots \epsilon_{b_n c_n}$ and

$$\mathcal{N} = \left( \frac{[2j_1]_q! [2j_2]_q! [2J + 1]_q}{[j_1 + j_2 - J]_q! [j_1 - j_2 + J]_q! [-j_1 + j_2 + J]_q! [j_1 + j_2 + J + 1]_q} \right)^{1/2}. \tag{4.8}$$

Expression (4.7) is efficient for practical calculation of $C - G$ coefficients (see Appendix). We also present a simple derivation of the standard expression for $q - C - G$ coefficients [6]. Using the decomposition

$$\langle j_1 m_1 \rangle = \sum_{m=-j}^{+j} < j_1 | m_1 - m > < j_1 - j | m > < jm | < j_m | < j m_1 - m | < jm |$$

$$\langle j_2 m_2 \rangle = \sum_{m=-j}^{+j} < j_2 m_2 | j - m > < j_2 - j | m_2 + m > < j - m | < j_2 - j | m_2 + m | \tag{4.9}$$

$$| JM > = \sum_{m=-j}^{+j} < j_1 - j | m_1 - m | j_2 - j | m_2 + m | JM > < j_1 - j | m_1 - m > < j_2 - j | m_2 + m >$$

we immediately write

$$\langle j_1 m_1 j_2 m_2 | J M >_q = \mathcal{N} \sum_{m=-j}^{+j} < j_1 m_1 | j_1 - j | m_1 - m > | jm > \tag{4.10}$$

$$\times < j_2 m_2 | j - m | j_2 - j | m_2 + m > < jm | j - m | 00 >$$

$$\times < j_1 - j | m_1 - m | j_2 - j | m_2 + m | J M >_q$$

where $\mathcal{N}$ is the norm depending on $j_1, j_2$ and $J$. Three of the four $C - G$ coefficients appearing on the right-hand side have the simple form (4.4). The fourth coefficient $< jm | j - m | 00 >$ also has a simple form. Namely, for $n = 2j$ we have

$$< jm | j - m | 00 >_q = \frac{1}{\sqrt{|n + 1|}_q} \epsilon_{a_1 b_1} \ldots \epsilon_{a_n b_n} =$$
\[
\frac{1}{\sqrt{[2j + 1]_q}} q^{j_{m_1}} (-q^{-\frac{1}{2}})^{n_2} = (-j^{-m}) \frac{1}{\sqrt{[2j + 1]_q}} q^m. \tag{4.11}
\]

The denominator \(\sqrt{[2j + 1]}\) comes from the orthonormality condition. Finally, inserting equations (4.4) and (4.11) into equation (4.10) we find

\[
\langle j_1 m_1 j_2 m_2 | JM \rangle_q = N \sum_{m=-j}^{+j} (-j^{-m}) q^{j_{m_2} - j_{m_1}} \times
\]

\[
q^{m(2J + 2j + 1)} \left( \begin{array}{c}
2j \\
j + m
\end{array} \right)_q \left( \begin{array}{c}
2j_1 - 2j \\
(j_1 - j + m_1 - m)
\end{array} \right)_q \left( \begin{array}{c}
2j_2 - 2j \\
j_2 - j + m_2 + m
\end{array} \right)_q \sqrt{\left( \begin{array}{c}
2J \\
J + M
\end{array} \right)_q \left( \begin{array}{c}
2j_1 \\
j_1 + m_1
\end{array} \right)_q \left( \begin{array}{c}
2j_2 \\
j_2 + m_2
\end{array} \right)_q}
\tag{4.12}
\]

with \(j_1 + j_2 - j = J + j\). This result agrees with the result found by Ruegg [6] if the normalization factor \(N\) is taken as

\[
N = \left\{ \frac{[2j_1]_q ![2j_2]_q ![2J + 1]_q ![j_1 + j_2 - J + 1]_q!}{[j_1 + j_2 - J]_q ![j_1 - j_2 + J]_q ![\pm j_1 + j_2 + J]_q ![j_1 + j_2 + J + 1]_q!} \right\}^{\frac{1}{2}} \tag{4.13}
\]

We point out that our tensor method is simple and can be easily applied to \(SU(n)_q\) for \(n \geq 3\). We also mention that it can be applied to multiparameter quantum groups. For example, it can be shown [7] that \(C - G\) coefficients for the two-parameter \(SU(2)_{p,q}[8]\) depend effectively on one parameter only.

5. Symmetry relations

For completeness we rederive the known symmetry relations for \(q - C - G\) coefficients and \(q - 3 - j\) symbols. From equation (4.4) immediately follow symmetry relations

\[
\langle j_1 - m_1 j_2 - m_2 | j_1 + j_2 - m_1 - m_2 >_q = \\
\langle j_2 m_2 j_1 m_1 | j_1 + j_2 m_1 + m_2 >_q = \\
\langle j_1 m_1 j_2 m_2 | j_1 + j_2 m_1 + m_2 >_q^{-1}
\tag{5.1}
\]

and

\[
\langle j_1 - m_1 j_1 + j_2 m_1 + m_2 | j_2 m_2 >_q = (-j^{i+m_1}) q^{-m_1}
\]

\[
\times \sqrt{\frac{[2j_2 + 1]_q}{[2j_1 + 2j_2 + 1]_q}} < j_1 m_1 j_2 m_2 | j_1 + j_2 m_1 + m_2 >_q. \tag{5.2}
\]
Furthermore, from equation (4.11) we have

\[
<j - m \ j m | 00 >_q = (-)^{2j} < j m \ j m - 00 >_{q-1} \quad < j m \ 00 >_q = 1.
\]

(5.3)

The symmetry relations (5.1)-(5.3) are sufficient to derive the symmetries of the general \( C - G \) coefficients. From equation (4.10) we obtain

\[
<j_1 - m_1 \ j_2 - m_2 | J - M >_q = < j_2 m_2 \ j_1 m_1 | JM >_q = (-)^{j_1 + j_2 - J} < j_1 m_1 \ j_2 m_2 | JM >_{q-1},
\]

and

\[
<j_1 - m_1 \ JM | j_2 m_2 >_q = (-)^{J - j_2 + m_1} q^{-m_1}
\]

\[
\times \sqrt{\frac{[2j_2 + 1]_q}{[2j + 1]_q}} < j_1 m_1 \ j_2 m_2 | JM >_q.
\]

(5.5)

(One can deduce this directly from (4.7))

We can define the \( q \)-deformed \( 3 - j \) symbol as

\[
\left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right)_q = q^{\frac{1}{3} (m_2 - m_1)} (-)^{j_1 - j_2 - m_3} \sqrt{\frac{[2j_3 + 1]_q}{[2j + 1]_q}} < j_1 m_1 \ j_2 m_2 | j_3 - m_3 >_q
\]

(5.6)

where the additional factor \( q^{1/3(m_2 - m_1)} \) comes from the requirement that symmetry relations for the \( (3 - j)_q \) coefficients should not contain explicit \( q \)-factors:

\[
\left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{array} \right)_q = \left( \begin{array}{ccc} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{array} \right)_q = (-)^{j_1 + j_2 + j_3} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right)_q^{-1}
\]

(5.7)

and that the \( (3 - j)_q \) coefficients are invariant under cyclic permutations. Note that the \( SU(2)_q \) invariant, built up of the three states \( |j_1 m_1 >, |j_2 m_2 > \) and \( |j_3 m_3 > \), is

\[
\sum_{m_1, m_2, m_3} < j_3 - m_3 \ j_3 m_3 | 00 >_q < j_1 m_1 \ j_2 m_2 | j_3 - m_3 >_q |j_1 m_1 > |j_2 m_2 > |j_3 m_3 > =
\]

\[
= \sum_{m_1, m_2, m_3} q^{\frac{1}{3} (m_1 - m_3)} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right)_q |j_1 m_1 > |j_2 m_2 > |j_3 m_3 > =
\]

(5.8)

\[
= \sum_{m_1, m_2, m_3} N_{123} (\epsilon_{(b,c)}) k_1 (\epsilon_{(d,e)}) k_2 (\epsilon_{(a,f)}) k_3 |e_{(a,b)} > |e_{(c,d)} > |e_{(e,f)} > .
\]

Now we identify
\[<j_1m_1,j_2m_2,j_3-m_3>_q<jj_3-m_3j_3m_3|00>_q = \]
\[= q^{\frac{1}{2}(m_1-m_3)} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \]
\[= N_{123}(\epsilon_{(b,c)})_{k_1}(\epsilon_{(d,e)})_{k_2}(\epsilon_{(a,f)})_{k_3} \]
where, for example, \((\epsilon_{(b,c)})_{k} = \epsilon_{b_1c_1}...\epsilon_{b_kc_k}\) with
\[k_1 = j_1 + j_2 - j_3 \quad k_2 = -j_1 + j_2 + j_3 \quad k_3 = j_1 - j_2 + j_3 \quad (5.10)\]
and \(N_{123}\) is the normalization factor fully symmetric in indices \((123)\). Equation \((5.9)\) represents the connection with the tensor notation used.

### 6. Covariant q-oscillators and irreducible tensor operators

Let us define the q-bosonic operators \(a_i\) and \(a^+_i\) \((i = 1, 2)\) such that \(|e_i> = a^+_i|0,0>_F\) and \(<e_i| = _F<0,0|a_i\), where \(|0,0>_F\) denotes the (Fock) vacuum state invariant under \(SU(2)_q\). Hence, \(a^+_1\) and \(a^+_2\) are covariant operators transforming as an \(SU(2)_q\) doublet. Therefore, analogously as in equation \((3.2)\), they q-commute
\[a^+_2a^+_1 = qa^+_1a^+_2. \quad (6.1)\]

Furthermore, we define the projector \(P_{(j=k/2)}\) from the tensor space \((V_2)^{\otimes k}\) to the totally q-symmetric space carrying an irreducible representation of spin \(j = k/2\)
\[P_{(j=k/2)}|e_{i_1...i_k}> = \frac{1}{\sqrt{|k|}_q!}a^+_{i_1}...a^+_{i_k}|0,0>_F = \]
\[= \frac{1}{\sqrt{|k|}_q!}q^{\chi_{(i_1...i_k)}}(a^+_1)^{n_1}(a^+_2)^{n_2}|0,0>_F. \quad (6.2)\]

We find from equation \((3.4)\) that
\[|jm> = q^M(a^+_1)^{n_1}(a^+_2)^{n_2}\sqrt{|n_1|}_q!|n_2|_q!|0,0>_F \]
\[j = \frac{1}{2}(n_1 + n_2) \quad m = \frac{1}{2}(n_1 - n_2) \quad (6.3)\]

We define the number operators \(N_i\) and \(N\) as
\[N_i |jm> = N_i |n_1, n_2> = n_i |n_1, n_2> \]
\[N = N_1 + N_2 \quad [N, N_i] = 0 \quad [N_1, N_2] = 0 \quad \]
\[\left[ N_i, a^+_j \right] = \delta_{ij}a^+_i \quad [N_i, a_j] = -\delta_{ij}a_i \quad \]
\[\left[ N, a^+_i \right] = a^+_i \quad [N, a_i] = -a_i. \quad (6.4)\]
The action of $a_i^+$ and $a_i$ on the basis vectors $|jm\> \text{ is}$

$$
\begin{align*}
a_i^+ |jm\> &= q^{-\frac{j+1}{2}} \sqrt{[n_1+1]_q} \ |j + \frac{1}{2}, m + \frac{1}{2}\> \\
a_i^2 |jm\> &= q^{-\frac{j+1}{2}} \sqrt{[n_2+1]_q} \ |j + \frac{1}{2}, m - \frac{1}{2}\> \\
a_i |jm\> &= q^{-\frac{j+1}{2}} \sqrt{[n_1]_q} \ |j - \frac{1}{2}, m - \frac{1}{2}\> \\
a_2 |jm\> &= q^{\frac{j+1}{2}} \sqrt{[n_2]_q} \ |j - \frac{1}{2}, m + \frac{1}{2}\>. 
\end{align*}
$$

(6.5)

The commutation relations between $a_i$ and $a_j^+$ follow immediately:

$$
\begin{align*}
a_2^+ a_1^+ &= q a_1^+ a_2^+ \quad a_2 a_1 = q^{-1} a_1 a_2 \\
a_2 a_1^+ &= a_1^+ a_2 \quad a_1 a_2^+ = a_2^+ a_1 
\end{align*}
$$

(6.6)

and

$$
\begin{align*}
a_1 a_1^+ &= q^{-N_2} [N_1+1]_q \quad a_1^+ a_1 = q^{-N_2} [N_1]_q \\
a_2 a_2^+ &= q^{+N_1} [N_2+1]_q \quad a_2^+ a_2 = q^{+N_1} [N_2]_q \\
H &= a_1^+ a_1 + a_2^+ a_2 = [N]_q
\end{align*}
$$

(6.7)

Then

$$
\begin{align*}
a_1 a_1^+ - q a_1^+ a_1 &= q^{-N} \\
a_2 a_2^+ - q^{-1} a_2^+ a_2 &= q^{+N}
\end{align*}
$$

(6.8)

and

$$
\begin{align*}
a_1 a_1^+ - q^{-1} a_1^+ a_1 &= q^{2J^0} \\
a_2 a_2^+ - q a_2^+ a_2 &= q^{2J^0}
\end{align*}
$$

(6.9)

The generators $J^\pm$ and $J^0$ can be represented as

$$
\begin{align*}
J^+ &= q^{-J^0+\frac{1}{2}} a_1^+ a_2 \\
J^- &= q^{-J^0-\frac{1}{2}} a_2^+ a_1 \\
2J^0 &= N_1 - N_2 \\
[J^+, J^-] &= [2J^0]_q = [N_1 - N_2]_q \\
[N, J^\pm] &= [N, J^0] = 0.
\end{align*}
$$

(6.10)

We point out that the oscillator operators $a_i$ and $a_i^+$ are covariant since the corresponding tensors $|e_{i_1, \ldots, i_k}\>$, equation (3.4), are covariant and irreducible by construction.

We note that the covariant q-Bose operators $a, a^+$ (6.1) are the same as in [9], where they were constructed using the Wigner $D^{(j)}$-functions. A different set of covariant operators was constructed in ref.[10]. Other constructions [11] are non-covariant in the sense that operators do not transform as $SU(2)_q$ doublet. In the non-covariant approach one has to solve an additional problem of constructing covariant, irreducible tensor operators [12].
The definition of the irreducible tensor operators of $SU(2)_q$ is

$$(J^{\pm} T_{km} - q^{-m} T_{km} J^{\pm}) q^{-J^0} =$$

$$= \sqrt{[k \mp m]_q [k \pm m + 1]_q} T_{km}$$

$$[J^0, T_{km}] = m T_{km} \quad \text{(6.11)}$$

$$|jm> = T_{jm}|0, 0>_F$$

According to equations (6.1-6.3) we define a unit tensor operator as

$$T_{jm} = q^{\frac{1}{2} n_1 n_2} (a_1^+)^{n_1} (a_2^+)^{n_2} \frac{1}{\sqrt{n_1! n_2!}} \quad \text{(6.12)}$$

which is covariant and irreducible by construction and satisfies the requirements (6.11) automatically. Note that $(T_{km})^+$ transforms as contravariant tensor. One can define the tensor

$$V_{k\mu} = (-)^{k-\mu} q^{\mu} T_{k-\mu}^+ \quad \text{(6.13)}$$

which transforms as covariant, irreducible tensor. In the tensor notation we have

$$V_{\{i_1...i_k\}} = \epsilon_{i_1i_2...}\epsilon_{i ki_k} T_{\{j_1...j_k\}} = (-)^{n_2} q^{\frac{1}{2}(n_1-n_2)} T_{k-\mu} \quad \text{(6.14)}$$

For completeness, we present relations between the Biedenharn operators $b_i, b_i^+$ of ref.[11], $t_i, t_i^+$ of ref.[10] and $a_i, a_i^+$ of the present paper:

$$b_1 = q^{-N_2-\frac{1}{2}N_1} \quad t_1 = q^{\frac{1}{2}N_2} a_1$$
$$b_2 = q^{-\frac{1}{2}N_2} t_2 = q^{-\frac{1}{2}N_1} a_2$$
$$b_1^+ = t_1^+ q^{-N_2-\frac{1}{2}N_1} = a_1^+ q^{\frac{1}{2}N_2}$$
$$b_2^+ = t_2^+ q^{-\frac{1}{2}N_2} = a_2^+ q^{-\frac{1}{2}N_1} \quad \text{(6.15)}$$

We point out that the general covariant oscillators (e.g. $t_i, t_i^+$ and $a_i, a_i^+$) are characterized by the anyonic type $q$-commutation relation (6.1). Actually, equation (6.1) is a consequence of underlying braid group symmetry and can be also obtained from the $SU(2)_q$ R-matrix [10].

Finally, we give the Borel-Weil realization

$$a_i^+ \equiv X_i \quad a_1 \equiv D_i \quad i = 1, 2 \quad \text{(6.16)}$$

which is covariant automatically. The commutation relations are
\[ X_2 X_1 = q \ X_1 X_2 \quad D_2 D_1 = q^{-1} \ D_1 D_2 \]
\[ D_1 X_1 = q \ X_1 D_1 + q^{-N} \quad D_2 X_2 = q^{-1} X_2 D_2 + q^{N} \]
\[ [D_i, X_j] = 0 \quad i \neq j \quad (6.17) \]

or

\[ D_1 X_1 = q^{-1} X_1 D_1 + q^{2J_0} \]
\[ D_2 X_2 = q \ X_2 D_2 + q^{2J_0} \quad (6.18) \]

where

\[ N_i = X_i \partial_i \]
\[ \partial_i = \frac{\partial}{\partial X_i} \quad (6.19) \]

It follows that

\[ D_i X_i^n = [n]_q X_i^{n-1} \]
\[ D_1 = \frac{1}{X_1} [X_1 \partial_1]_q q^{-X_2 \partial_2} \quad (6.20) \]
\[ D_2 = \frac{1}{X_2} [X_2 \partial_2]_q q^{X_1 \partial_1} \]
Appendix

We demonstrate usefulness of the equation (4.7) for the practical calculations. Using equations (4.5) and (4.11) we write:

\[ \chi(a, b) = \chi(a) + \chi(b) + n_2(a)n_1(b) \]
\[ \chi(c, d) = \chi(c) + \chi(d) + n_2(c)n_1(d) \]
\[ \chi(a, d) = \chi(a) + \chi(d) + n_2(a)n_1(d) \]
\[ \chi(b) = \chi(c) \]
\[ (\epsilon_{b,c})_n = (-)^{n_2(b)}q^\frac{1}{2}(n_1(b)-n_2(b)) \]  

(A.1)

where

\[ n = 2j = j_1 + j_2 - J \]
\[ n_1(b) = n_2(c) = j + m \]
\[ n_2(b) = n_1(c) = j - m \]
\[ n_1(a) = j_1 - j + m_1 - m \]
\[ n_2(a) = j_1 - j - m_1 + m \]
\[ n_1(d) = j_2 - j + m_2 + m \]
\[ n_2(d) = j_2 - j - m_2 - m \]  

(A.2)

After inserting equation (3.6) into equation (4.7), we immediately obtain the final result, equation (4.12):

\[ N \frac{q^{-\frac{1}{2}(M_1+M_2+M_J)}}{\sqrt{f_1 f_2 f_J}} \sum_{n_1(b)=0}^{2j} \sum_{\text{perm}(a)} \sum_{\text{perm}(b)} \sum_{\text{perm}(d)} \]
\[ \times q^{n_2(a)n_1(b)+n_1(b)n_1(d)+n_2(a)n_1(d)} \]
\[ \times q^{2\chi(a)+2\chi(b)+2\chi(d)} (\epsilon_{b,c})_{2j} = \]
\[ = N \frac{q^{-\frac{1}{2}(M_1+M_2+M_J)}}{\sqrt{f_1 f_2 f_J}} \sum_{m=-j}^{+j} q^{n_2(a)n_1(b)+n_1(b)n_1(d)+n_2(a)n_1(d)} \]
\[ \times f_a f_b f_d q^{n_1(a)n_2(a)+n_1(b)n_2(b)+n_1(d)n_2(d)} (\epsilon_{b,c})_{2j} \]  

(A.3)

We extend this simple calculation of the $SU(2)_q$ C.-G. coefficients to the $SU(N)_q$ quantum groups in the forthcoming paper.
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