POINTED FINITE TENSOR CATEGORIES OVER
ABELIAN GROUPS

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ABSTRACT. We give a characterization of finite pointed tensor categories obtained as de-equivariantizations of finite-dimensional pointed Hopf algebras over abelian groups only in terms of the (cohomology class of the) associator of the pointed part. As an application we prove that every coradically graded pointed finite braided tensor category is a de-equivariantization of a finite dimensional pointed Hopf algebras over an abelian group.

1. Introduction

Let $k$ be an algebraically closed field of characteristic zero. Via Tannakian reconstruction, tensor categories with a quasi-fiber functor with values in the category of finite dimensional $k$-vector spaces correspond to categories $H \mathcal{M}$ of finite dimensional corepresentations of a coquasi-Hopf algebra $H$ over $k$.

A tensor category is pointed if every simple object is invertible; this condition ensures the existence of a quasi-fiber functor as above. Hence any finite pointed tensor category is equivalent to the category of comodules over a finite dimensional pointed coquasi-Hopf algebra.

In a previous paper [4] written jointly with M. Pereira, we studied de-equivariantizations of Hopf algebras by applying Tannakian techniques. We explicitly constructed a coquasi-bialgebra such that its tensor category of comodules realizes the de-equivariantization of a Hopf algebra, [4, Theorem 2.8]. As an application, we defined a big family of pointed coquasi-Hopf algebras $A(H, G, \Phi)$ attached to a coradically graded pointed Hopf algebra $H$ over $k$ and some extra group-theoretical data [4, Proposition 3.3, Definition 3.5].

In this paper we pursue the study of de-equivariantizations of Hopf algebras initiated in [4]. We characterize pointed finite tensor categories over abelian groups constructed as de-equivariantizations of tensor categories of comodules over finite dimensional pointed Hopf algebras. For Hopf algebras, the de-equivariantization process generalizes the theory of central extensions.
of Hopf algebras. However, the central quotient is not necessarily a Hopf algebra but a coquasi-Hopf algebra.

Given a tensor category $\mathcal{C}$, we denote by $G(\mathcal{C})$ the group of isomorphism classes of invertible objects and by $\omega(\mathcal{C}) \in H^3(G(\mathcal{C}), k^\times)$ the cohomology class defining the associator of the tensor subcategory of invertible objects.

Breen [9, Proposition 4.1] defined for every abelian group $\Lambda$ a group homomorphism $\psi : H^3(\Lambda, k^\times) \to \text{Hom}(\wedge^3 \Lambda, k^\times)$, that measures if the category of Yetter-Drinfeld modules of $k \omega \Gamma$ (the coquasi-Hopf algebra defined over the group algebra $k \Gamma$ with associator $\omega$) is pointed, see Theorem 3.2.

A tensor category $\mathcal{C}$ is coradically graded if $\mathcal{C}$ is equivalent to the category of comodules over a coradically graded coalgebra, see [13, Section 1.13] for a more categorical definition. By [5] every finite-dimensional pointed Hopf algebra $H$ with abelian group of group-like elements $\Gamma$ is a cocycle deformation of $B(V)^\# k \Gamma$, where $V \in k \Delta$ $\Delta$ denotes the infinitesimal braiding of $H$ and $B(V)$ is the Nichols algebra of $V$. In particular $H \mathcal{M}$ and $B(V)^\# k \Gamma \mathcal{M}$ are tensor equivalent, so the pointed tensor categories obtained from $H$ or $B(V)^\# k \Gamma$ are the same. Therefore we may restrict just to coradically graded coquasi-Hopf algebras. Our main result can be summarized as:

**Theorem 1.1.** A finite tensor category $\mathcal{C}$ is tensor equivalent to a de-equivariantization of a pointed Hopf algebra over an abelian group if and only if $\mathcal{C}$ is coradically graded, $G(\mathcal{C})$ is abelian and $\psi_{G(\mathcal{C})}(\omega_{\mathcal{C}}) \equiv 1$. Moreover, $\mathcal{C}$ is realized as the corepresentations of a finite dimensional coquasi-Hopf algebra of the form $B(V)^\# k \omega \Gamma$.

Theorem 1.1 is proved in Section 4 where a pointed Hopf algebra of the form $B(V)^\# k \Gamma$ is explicitly constructed. As a consequence of this result we obtain that every coradically graded pointed finite braided tensor category is tensor equivalent to a de-equivariantization of a coradically graded pointed Hopf algebra over an abelian group.

While the paper was at final stages of preparation, Huang, Liu, Yang and Ye posted the paper [17] containing results close to some of ours. They studied finite dimensional coquasi-Hopf algebras of the form $B(V)^\# k \omega \Lambda$, with $\Lambda$ abelian and $\omega$ trivializable. Their ideas for the construction of coquasi-Hopf algebras are different to our techniques, mainly they do not use the concept of de-equivariantization. Instead, they find an specific representative for the cohomology class of a trivializable 3-cocycle and an specific trivialization, suitable for the construction of a pointed Hopf algebra of the form $H := B(V)^\# k \Gamma$ such that the coquasi-Hopf algebra $B(V)^\# k \omega \Lambda$ is a quotient of $H$. These examples correspond to the general construction in [4].

The organization of the paper is as follows. Section 2 is devoted to preliminaries. In Section 3 we define the map $\psi : H^3(\Lambda, k^\times) \to \text{Hom}(\wedge^3 \Lambda, k^\times)$ and characterizations of the condition $\psi_{\Lambda}(\omega) = 1$, which are used in the sequel.
In Section 2, we prove generation in degree one for coradically graded coquasi-Hopf algebras $A$ with associator $\omega \in H^3(G(A), \mathbb{k}^\times)$ such that $\psi_{G(A)}(\omega) = 0$; hence we prove [13, Conjecture 1.32.1] under the previous condition on the cohomology class determined by the associator. We prove Theorem 1.1. We finish the section with an example of a coradically graded coquasi-Hopf algebra over $\Lambda = (\mathbb{Z}/2\mathbb{Z})^\oplus 3$ with associator $\omega \in Z^3(\Lambda, \mathbb{k}^\times)$, such that $\psi_{\Lambda}(\omega) \neq 1$.

**Notation.** Throughout the paper algebras and coalgebras are always defined over $k$. We use Sweedler’s notation for coalgebras omitting the sum symbol: $\Delta(c) = c_1 \otimes c_2$ for all $c \in C$, $(C, \Delta, \varepsilon)$ a coalgebra. Given a group $\Gamma$, $\hat{\Gamma}$ denotes the group of characters of $\Gamma$ over $k$, and $(\cdot, \cdot) : \hat{\Gamma} \times \Gamma \to k^\times$ is the evaluation map. For each $\theta \in \mathbb{N}_0$, we call $I_\theta = \{ n \in \mathbb{N} : n \leq \theta \}$, or simply $I$ if $\theta$ is clear from the context. Also, $\delta_{x,y}$ is the Kronecker delta.

By a tensor category we mean a $k$-linear abelian category $C$ with finite dimensional Hom spaces and objects of finite length, endowed with a rigid $k$-bilinear monoidal structure and such the unit object $1$ is simple [13]. A tensor category is finite if it is $k$-linearly equivalent to the category of finite dimensional comodules over a finite dimensional $k$-coalgebra.

## 2. Preliminaries

In this section we recall some definitions and results about coquasi-Hopf algebras and tensor categories.

### 2.1. Coquasi-bialgebras.

A coquasi-bialgebra $(H, m, u, \omega, \Delta, \varepsilon)$ is a coalgebra $(H, \Delta, \varepsilon)$ together with coalgebra morphisms:

- the multiplication $m : H \otimes H \to H$ (denoted $m(g \otimes h) = gh$),
- the unit $u : k \to H$ (where we call $u(1) = 1_H$),

and a convolution invertible element $\Omega \in (H \otimes H \otimes H)^*$ such that

1. $h_1(g_1k_1)\Omega(h_2,g_2,k_2) = \Omega(h_1,g_1,k_1)(h_2g_2)k_2$,
2. $1_Hh = h1_H = h$,
3. $\Omega(h_1g_1,k_1,l_1)\Omega(h_2,g_2,k_2l_2) = \Omega(h_1,g_1,k_1)\times\Omega(h_2,g_2,k_2l_2)\Omega(g_3,k_3,l_2)$,
4. $\Omega(h,1_H,g) = \varepsilon(h)\varepsilon(g)$,

for all $h,g,k,l \in H$. Note that

\[ \Omega(1_H, h, g) = \Omega(h, g, 1_H) = \varepsilon(h)\varepsilon(g) \quad \text{for all } g, h \in H. \]

A coquasi-bialgebra $H$ is a coquasi-Hopf algebra if there is a coalgebra map $S : H \to H^{op}$ (the antipode) and elements $\alpha, \beta \in H^*$ such that

1. $\alpha(h)1_H = S(h_1)\alpha(h_2)h_3$,
2. $\beta(h)1_H = h_1\beta(h_2)S(h_3)$,
3. $\varepsilon(h) = \omega(h_1\beta(h_2), S(h_3), \alpha(h_4)h_5) = \omega^{-1}(S(h_1), \alpha(h_2)h_3\beta(h_4), S(h_5))$, for all $h \in H$. 

Example 2.1. Let $G$ be a discrete group. Recall that a (normalized) 3-cocycle $\omega \in Z^3(G, k^\times)$ is a map $\omega : G \times G \to G \to k^\times$ such that

$$\omega(gh, c, l)\omega(g, h, kl) = \omega(g, h, k)\omega(g, hk, l)\omega(h, k, l), \quad \omega(g, 1, h) = 1,$$

for all $g, h, k, l \in G$.

Given $\omega \in Z^3(G, k^\times)$, we define the coquasi-Hopf algebra $k\omega G$, with structure $(kG, \Omega_{\omega}, S, \alpha, \beta)$, where $kG$ is the group algebra with the usual comultiplication $\Delta(g) = g \otimes g$ for all $g \in G$, and $\Omega_{\omega}(g, h, k) = \omega(g, h, k)$ for all $g, h, k \in G$. The antipode structure is given by

$$S(g) = g^{-1}, \quad \alpha(g) = 1, \quad \beta(g) = \omega(g, g^{-1}, g)^{-1},$$

for all $g \in G$.

Let $H$ be a coquasi-Hopf algebra. The category of left $H$-comodules $H\mathcal{M}$ is rigid and monoidal, where the tensor product is $\otimes = \otimes_k$, the comodule structure of the tensor product is the codiagonal one and the associator is

$$\phi_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

$$\phi_{U,V,W}((u \otimes v) \otimes w) = \Omega(u_{-1}, v_{-1}, w_{-1})u_0 \otimes (v_0 \otimes w_0)$$

for $u \in U, v \in V, w \in W$ and $U, V, W \in H\mathcal{M}$. The dual coactions are given by $S$ and $S^{-1}$, as in the case of Hopf algebras.

Example 2.2. Let $G$ be a discrete group and $\omega \in Z^3(G, k^\times)$. The tensor category $k\omega G\mathcal{M}$ is $\text{Vec}_G^\omega$, the category of $G$-graded vector spaces with associator induced by $\omega$.

2.2. Braided tensor categories. A tensor category $\mathcal{C}$ is called braided if it is endowed with a natural isomorphism

$$c_{X,Y} : X \otimes Y \to Y \otimes X, \quad X, Y \in \mathcal{C},$$

satisfying the hexagon axioms, see [18].

Example 2.3 (Pointed braided fusion categories). Let $\mathcal{B}$ be a pointed braided fusion category. The set of isomorphism classes of simple objects $\Gamma := G(\mathcal{B})$ is an abelian group with product induced by the tensor product.

The associativity constraint defines a 3-cocycle $\omega \in Z^3(\Gamma, k^\times)$. The braiding defines a function $c : \Gamma \times \Gamma \to k^\times$ satisfying the following equations:

$$c(g, hk)/c(g,h)c(g,k) = \omega(g, h, k)\omega(h, k, g)/\omega(h, g, k),\quad c(gh, k)/c(g, k)c(h, k) = \omega(g, h, k)\omega(k, g, h)/\omega(g, k, h),$$

for all $g, h, k \in \Gamma$.

These equations come from the hexagon axioms. A pair $(\omega, c)$ satisfying (8) is called an abelian 3-cocycle. Following [11], we denote by $Z^3_{\text{ab}}(\Gamma, k^\times)$ the abelian group of all abelian 3-cocycles $(\omega, c)$. 
An abelian 3-cocycle \((\omega, c) \in Z_3^{\text{ab}}(\Lambda, k^\times)\) is called an \emph{abelian 3-coboundary} if there is \(\alpha : \Lambda^{\times 2} \to k^\times\), such that

\[
\omega(g, h, k) = \frac{\alpha(g, h)\alpha(gh, k)}{\alpha(g, hk)\alpha(h, k)}
\]

for all \(g, h, k \in \Lambda\).

\(B_3^{\text{ab}}(\Lambda, k^\times)\) denotes the subgroup of \(Z_3^{\text{ab}}(\Lambda, k^\times)\) of abelian 3-coboundaries. The quotient group \(H_3^{\text{ab}}(\Lambda, k^\times) := Z_3^{\text{ab}}(\Lambda, k^\times)/B_3^{\text{ab}}(\Lambda, k^\times)\) is called the \emph{third group of abelian cohomology} of \(\Lambda\).

**Example 2.4** (Corepresentations of coquasitriangular coquasi-Hopf algebras). A \emph{coquasitriangular} coquasi-Hopf algebra is a pair \((H, r)\), where \(H\) is a coquasi-Hopf algebra and \(r : H \otimes H \to k\) is a convolution invertible map such that

\[
(10) \quad r(x_1, y_1)x_2y_2 = y_1x_1r(y_2, x_2),
\]

\[
(11) \quad r(x, yz) = \Omega(y_1, z_2, x_2)r(x_2, z_3)r(x_1, y_3)\Omega(x_3, y_4, z_4)
\]

\[
(12) \quad r(xy, z) = \Omega(z_1, x_1, y_1)^{-1}r(x_2, z_2)\Omega(x_3, z_3, y_2)r(y_3, z_4)\Omega^{-1}(x_4, y_4, z_5),
\]

for all \(x, y, z \in H\). If \((H, r)\) is a coquasitriangular coquasi-Hopf algebra the \(r\)-form defines a braiding by

\[
\kappa_{V, W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto r(v_{-1}, w_{-1})w_0 \otimes v_0.
\]

Hence \((H\mathcal{M}, c)\) is a braided tensor category.

**Example 2.5** (Center construction). An important example of a braided tensor category is the center \(Z(C)\) of a tensor category \((C, a, 1)\). The center construction produces a braided tensor category \(Z(C)\) from any tensor category \(C\). Objects of \(Z(C)\) are pairs \((Z, c_{-Z})\), where \(Z \in C\) and \(c_{-Z} : - \otimes Z \to Z \otimes -\) is a natural isomorphism such that the diagram

\[
\begin{align*}
Z \otimes (X \otimes Y) & \xrightarrow{a_{Z,X,Y}} (Z \otimes X) \otimes Y \quad \xrightarrow{c_{X,Z} \otimes \text{id}_Y} (Z \otimes X) \otimes Y \\
(X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}^{-1}} X \otimes (Y \otimes Z) \xrightarrow{\text{id}_X \otimes c_{Y,Z}} X \otimes (Y \otimes Z) \\
X \otimes (Y \otimes Z) & \xrightarrow{a_{X,Y,Z}} X \otimes (Z \otimes Y)
\end{align*}
\]

commutes for all \(X, Y, Z \in C\). The braided tensor structure is the following:
the tensor product is $(Y, e_{-Y}) \otimes (Z, e_{-Z}) = (Y \otimes Z, e_{-Y \otimes Z})$, where
\[ c_{X,Y \otimes Z} : X \otimes Y \otimes Z \rightarrow Y \otimes Z \otimes X, \quad X \in \mathcal{C}, \]
\[ c_{X,Y \otimes Z} = a_{Y,Z,X}(id_Y \otimes c_{X,Z})a_{Y,X,Z}^{-1}(c_{X,Y} \otimes id_Z)a_{X,Y,Z}. \]

- the braiding is the morphism $c_{X,Y}$,

**Example 2.6** (The Drinfeld center of $\text{Vec}_\Lambda^\omega$). Let $\Lambda$ be a discrete group and $\omega \in Z^3(\Lambda, k^*)$. The Drinfeld center of $\text{Vec}_\Lambda^\omega$ is equivalent to $k^\omega \mathcal{YD}$, the category of Yetter-Drinfeld modules over $k^\omega \Lambda$. The objects of $k^\omega \mathcal{YD}$ are $\Lambda$-graded vector spaces $V = \bigoplus_{g \in \Lambda} V_g$ with a linear map $\triangleright : k^\omega \Lambda \otimes V \rightarrow V$ such that $1 \triangleright v = v$ for all $v \in V$,
\[ (gh) \triangleright v = \frac{\omega(g, hkh^{-1}, h)}{\omega(g, h, k)\omega(ghk^{-1}, g^{-1}, g)}(g \triangleright (h \triangleright v)), \quad g, h, k \in \Lambda, \quad v \in V_k, \]
satisfying the following compatibility condition:
\[ g \triangleright V_h \subseteq V_{ghg^{-1}} \quad \text{for all } g, h \in \Lambda. \]

Morphisms in $k^\omega \mathcal{YD}$ are $\Lambda$-linear $\Lambda$-homogeneous maps. The tensor product of $V = \bigoplus_{g \in \Lambda} V_g$ and $W = \bigoplus_{g \in \Lambda} w$ is $V \otimes W$ as vector space, with
\[ (V \otimes W)_h = \bigoplus_{h \in G} V_h \otimes W_{h^{-1}g}, \]
and for all $v \in V_g, w \in W_l$,
\[ h \triangleright (v \otimes w) = \frac{\omega(hgh^{-1}, hhl^{-1}, l)\omega(h, g, l)}{\omega(hgh^{-1}, h, l)}(h \triangleright v) \otimes (h \triangleright w). \]

The associativity constraints are the same as $\text{Vec}_\Lambda^\omega$. The category is tensor braided, with braiding $c_{V,W} : V \otimes W \rightarrow W \otimes V, V, W \in k^\omega \mathcal{YD}$,
\[ c_{V,W}(v \otimes w) = (g \triangleright w) \otimes v, \quad g \in \Lambda, \quad v \in V_g, \quad w \in W. \]

2.3. **Bosonization for coquasi-Hopf algebras.** Now we recall the notation and results from [3] but restricted to pointed coquasi-Hopf algebras.

Given a Hopf algebra $R$ in $k^\omega \mathcal{YD}$ with multiplication $\cdot : R \otimes R \rightarrow R$ and comultiplication $\Delta : R \rightarrow R \otimes R$, $\Delta(r) = r^{(1)} \otimes r^{(2)}$, the **bosonization** of $R$ by $k^\omega \Lambda$ [3] Definition 5.4] is the coquasi-Hopf algebra $R \# k^\omega \Lambda$ with underlying vector space $R \otimes k\Lambda$ and the following structure maps:
\[ (r \# g)(s \# h) = \frac{\omega(g, l, h)\omega(k, l, gh)}{\omega(k, g, lh)\omega(l, g, h)} r \cdot (g \triangleright s) \# gh, \]
\[ \Delta(r \# g) = \frac{1}{\omega(k_{j^{-1}}, j, g)} r^{(1)} \# l g \otimes r^{(2)} \# g, \]
\[ \Omega(r \# g, s \# h, t \# k) = \varepsilon(r)\varepsilon(s)\varepsilon(t)\omega(g, h, k), \]
for all $g, h, k, l \in \Lambda, r \in R_k, s \in R_l, t \in R, where r^{(1)} \otimes r^{(2)} \in \bigoplus_j R_{kj^{-1}} \otimes R_j$.

We have two canonical coquasi-Hopf algebra maps
\[ \pi : R \# k^\omega \Lambda \rightarrow k^\omega \Lambda, \quad \pi(r \# g) = \varepsilon(r)g, \quad \iota : k^\omega \Lambda \rightarrow R \# k^\omega \Lambda, \quad \iota(g) = 1 \# g, \]
such that $\pi \circ \iota = \text{id}_{k^\omega \Lambda}$.

Reciprocally, let $H$ be a coquasi-Hopf algebra and assume that there exist coquasi-Hopf algebra maps $\pi : H \to k^\omega \Lambda$, $\iota : k^\omega \Lambda \to H$ such that $\pi \circ \iota = \text{id}_{k^\omega \Lambda}$. Then $H \simeq R\# k^\omega \Lambda$, where $R = H^{co \pi}$ admits a structure of Hopf algebra in $k^\omega \Lambda \mathcal{YD}$ \cite[Theorem 5.8]{6}.

In particular this applies for $H = \oplus_{n \geq 0} H_n$ coradically graded such that $H_0 = k^\omega \Lambda$ \cite[6.1]{6}. Here, $R$ is a graded Hopf algebra in $k^\omega \Lambda \mathcal{YD}$:

$$R = \oplus_{n \geq 0} R_n, \quad \text{with } R_n = R \cap H_n, \quad n \geq 0, \quad \text{so } R_0 = k1.$$

2.4. **Nichols algebras.** Nichols algebras can be defined over any abelian braided tensor category see \cite{22}. In particular we may consider Nichols algebra over $C = Z^H(\mathcal{M})$ or $C = Z_H^H \mathcal{YD}$, where $H$ is a coquasi-bialgebra, see \cite{2} for the definition when $H$ is a Hopf algebra and \cite{17} for $H = k^\omega \Lambda$.

Given an object $V \in C$ and $n \geq 3$, $V^\otimes n$ denotes $\cdots ((V \otimes V) \otimes \cdots) \otimes V$, $n$ copies of $V$. We consider the following (graded) Hopf algebras in $C$:

- the tensor algebra $T(V) = \oplus_{n \geq 0} V^\otimes n$, with product given by the canonical isomorphism $V^\otimes m \otimes V^\otimes n \simeq V^\otimes (m+n)$; the coproduct $\Delta : T(V) \to T(V) \otimes T(V)$ is the unique graded algebra map such that $\Delta_{0,1} : V \to k \otimes V$ and $\Delta_{1,0} : V \to V \otimes k$ are the canonical isomorphisms.

- the tensor coalgebra $C(V) = \oplus_{n \geq 0} V^\otimes n$, with coproduct

$$\Delta = \oplus_{m,n \geq 0} : C(V) \to C(V) \otimes C(V), \quad \Delta_{n,m} : V^\otimes (m+n) \rightrightarrows V^\otimes m \otimes V^\otimes n;$$

the product $\Delta : T(V) \to T(V) \otimes T(V)$ is the unique graded coalgebra map induced by the canonical isomorphisms $k \otimes V \simeq V \otimes k$.

There exists a unique graded Hopf algebra map $T(V) \to C(V)$ in $C$, which is the identity on $V$. The **Nichols algebra** $\mathcal{B}(V)$ of $V$ is the image of this map: it is a graded Hopf algebra in $C$.

We may identify $\mathcal{B}(V)$ as a quotient $\mathcal{B}(V) = T(V)/\mathcal{J}(V)$ with the following universal property: $\mathcal{J}(V)$ is the largest coideal of $T(V)$ spanned by elements of $\mathbb{N}$-degree $\geq 2$. There are other characterizations of $\mathcal{B}(V)$ \cite{22}.

A **post-Nichols algebra** of $V$ is a graded Hopf subalgebra $\mathcal{E} = \oplus_{n \in \mathbb{N}_0} \mathcal{E}_n$ of $C(V)$ in $C$ such that $\mathcal{E}_1 = V$, see \cite{1}; hence $\mathcal{B}(V) \subseteq \mathcal{E}$ and the set of primitive elements of $\mathcal{E}$ is exactly $\mathcal{E}_1$.

3. TRIVIALIZATIONS OF ELEMENTS IN $H^3(\Lambda, k^\times)$

In this section we study a family of 3-cocycles of finite abelian groups called **trivializable.** These are the cocycles considered as associators for pointed tensor categories in Section \cite{1}.

Let $\Lambda$ be a finite abelian group. We denote by $\Lambda^n$ the $n$-th exterior power of $\Lambda$, viewed as a $\mathbb{Z}$-module. For each $\omega \in Z^3(\Lambda, k^\times)$, Breen \cite[Proposition 4.1]{9} defined an alternating trilinear map

$$\psi_\Lambda(\omega)(l_1, l_2, l_3) = \prod_{\sigma \in \mathfrak{S}_3} \omega(l_{\sigma(1)}, l_{\sigma(2)}, l_{\sigma(3)})^{\text{sgn}(\sigma)}, \quad l_1, l_2, l_3 \in \Lambda.$$
The group homomorphism $\psi_\Lambda : Z^3(\Lambda, k^\times) \to \text{Hom}(\wedge^3 \Lambda, k^\times)$ induces a group homomorphism

$$\psi_\Lambda : H^3(\Lambda, k^\times) \to \text{Hom}(\wedge^3 \Lambda, k^\times).$$

Note that $\text{Hom}(\Lambda^{\otimes 3}, k^\times) \subset Z^3(\Lambda, k^\times)$. Hence, if $\Lambda$ is finite the restriction of $\psi_\Lambda$ to $\text{Hom}(\Lambda^{\otimes 3}, k^\times)$ is surjective. Thus $\psi_\Lambda$ is surjective.

Given $\omega \in Z^3(\Lambda, k^\times)$, we denote by $p^*\omega \in Z^3(\Gamma, k^\times)$ the pull-back of $\omega$ by $p$; that is, the 3-cocycle defined by

$$p^*\omega(g, h, k) = \omega(p(g), p(h), p(k)), \quad g, h, k \in \Gamma.$$

**Proposition 3.1.** Let $\Lambda = \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_m\mathbb{Z}$, and

$$p : \Gamma := \mathbb{Z}/2n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2n_m\mathbb{Z} \to \Lambda := \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_m\mathbb{Z},$$

the canonical epimorphism. For any $(\omega, c) \in Z^3_{ab}(\Lambda, k^\times)$ the pull back $p^*\omega \in H^3(\Gamma, k^\times)$ is trivial.

**Proof.** If $(\omega, c) \in Z^3_{ab}(\Lambda, k^\times)$ the map

$$q : \Lambda \to k^\times, \quad q(l) = c(l, l), \quad l \in \Lambda,$

is a quadratic form on $\Lambda$; that is, $q(l^{-1}) = q(l)$ for all $l \in \Lambda$ and the map

$$b_q(k, l) = q(kl)q(k)^{-1}q(l)^{-1}, \quad k, l \in \Lambda,$

is a bicharacter.

The quadratic form $q$ determines completely the abelian cohomology class of the pair $(w, c)$, see [12 Theorem 26.1]. Using the map $q$, Quinn [21] defined an explicit abelian 3-cocycle $(h, c)$ with $c(l, l) = q(l)$ for all $l \in \Lambda$. Assume that $\Lambda = \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_m\mathbb{Z}$. For each $i \in \{1, \ldots, m\}$ let $g_i := q(e_i)$ and $h_i \in Z^3(\mathbb{Z}/n_i\mathbb{Z}, k^\times)$ defined by

$$h_i(a, b, c) = \begin{cases} 1, & \text{if } b + c < n_i, \\ q_i^{n_i}a, & \text{if } b + c \geq n_i, \end{cases} \quad 0 \leq a, b, c < n_i.$$

Then by [21] and [12 Theorem 26.1], $h \in Z^3(\Lambda, k^\times)$ given by

$$h(x, y, z) = h_1(x_1, y_1, z_1)h_2(x_2, y_2, z_2) \cdots h_m(x_m, y_m, z_m),$$

is a 3-cocycle cohomologous to $\omega$.

For any $n \in \mathbb{N}$ and $a \in \Gamma$, we have $q(a^n) = q(a)^{2m^2}$. Hence $h_i$ has order at most two. By Example 3.3 via the epimorphism

$$p : \mathbb{Z}/2n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2n_m\mathbb{Z} \to \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_m\mathbb{Z},$$

the pull-back $p^*h$ is trivial, and also $p^*\omega$. \hfill $\square$

**Theorem 3.2.** Let $\omega \in H^3(\Lambda, k^\times)$. The following statements are equivalent:

(a) $\psi_\Lambda(\omega) \equiv 1$.

(b) The braided fusion category $\mathcal{YD}^{\Lambda}$ is pointed.

(c) There exists a finite abelian group $\Gamma$ and a group epimorphism $p : \Gamma \to \Lambda$ such that the pullback $p^*\omega \in H^n(\Gamma, k^\times)$ is trivial.
Proof. \([a] \iff [b]\). For each \(l \in \Lambda\), the map
\[
\beta_l : \Lambda \times \Lambda \to k^\times, \quad \beta_l(g, h) = \frac{\omega(g, l, h)}{\omega(g, h, l) \omega(l, g, h)}
\]
is a 2-cocycle; that is, it satisfies the equation
\[
\beta_l(g, h) \beta_l(gh, k) = \beta_l(gh, k) \beta_l(h, k),
\]
for all \(g, h, k \in \Lambda\).

By \([16, \text{Example 6.3}]\) we have an exact sequence of groups
\[
0 \to \hat{\Lambda} \to \text{Inv}(k^\omega \Lambda \YD) \to \Lambda \omega \to 0,
\]
where \(\Lambda \omega = \{ l \in \Lambda : 0 = [\beta_l] \in H^2(\Lambda, k^\times) \}\). Then \(k^\omega \Lambda \YD\) is pointed if and only if \(0 = [\beta_l]\) for all \(l \in \Lambda\).

Since \(k^\times\) is divisible, \(\beta_l\) has trivial cohomology class if and only if \(\beta_l\) is symmetric. In conclusion, \(k^\omega \Lambda \YD\) is pointed if and only if \(\beta_l(g, h) = \beta_l(h, g)\) for all \(l, g, h \in \Lambda\). Since \(\beta_l\) has trivial cohomology class if and only if \(\beta_l\) is symmetric. In conclusion, \(k^\omega \Lambda \YD\) is pointed if and only if \(\psi(\omega) = 1\).

(b) \(\iff\) (c). Assume that (b) holds. Then there is a finite abelian group \(\Gamma\) and an abelian 3-cocycle \((\alpha, c) \in Z^3_{ab}(\Gamma, k^\times)\) such that \(k^\omega \Lambda \YD \cong \text{Vec}_{\Gamma}(\alpha, c)\) as braided fusion categories. The forgetful functor \(k^\omega \Lambda \YD \to \text{Vec}_{\Lambda}\) defines a group epimorphism \(\pi_1 : \Gamma \to \Lambda\) such that \(\pi_1^*([\omega]) = [\alpha]\). By Proposition \(3.1\), there exists an abelian group \(\Gamma_2\) and an epimorphism \(\pi_2 : \Gamma_2 \to \Gamma_1\) such that \(\pi_2^*([\alpha]) = 0\), hence \(\pi_2 \circ \pi_1 : \Gamma_2 \to \Lambda\) trivializes \(\omega\). \(\square\)
**Definition 3.3.** Let $\omega \in H^3(\Lambda, k^\times)$. We say that $\omega$ is *trivializable* if it satisfies one of the equivalent conditions of Theorem 3.2. If $p : \Gamma \to \Lambda$ is an epimorphism of abelian groups such that the pullback $p^*\omega \in H^3(\Gamma, k^\times)$ is trivial, we say that $\omega$ is *$p$-trivial*.

**Example 3.4.** Let $C_n$ be the cyclic group of order $n$ generated by $\sigma$. Then

$$\cdots \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{\sigma-1} \mathbb{Z}C_n \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{\sigma-1} \mathbb{Z}C_n \xrightarrow{\cdot} \mathbb{Z},$$

where $N = 1 + \sigma + \sigma^2 + \cdots + \sigma^{n-1}$ is a free resolution of $\mathbb{Z}$. Thus,

$$H^3(C_n, k^\times) = \mathbb{G}_m(n) := \{a \in k^\times : a^n = 1\}.$$ 

Let $m, n \in \mathbb{N}$ such that $n|m$ and $\pi : C_m \to C_n$ be the canonical group epimorphism. The induced map is

$$\pi^* : H^3(C_n, k^\times) \to H^3(C_m, k^\times),$$

$q \mapsto q^m$.

Hence, if $q \in H^3(C_n, k^\times)$ has order $s$, the canonical epimorphism $\pi : C_{sn} \to C_n$ trivializes $q$. Thus $\pi : C_{n^2} \to C_n$ trivializes all elements in $H^3(C_n, k^\times)$.

**Example 3.5.** Let $\zeta$ be a $n$-th root of unity, $\Lambda = (\mathbb{Z}/n\mathbb{Z})^{\oplus 3}$. We define

$$\omega \in \mathbb{Z}^3(\Lambda, k^\times), \quad \omega(\vec{x}, \vec{y}, \vec{z}) = \zeta^{x_1y_2z_3},$$

where $\vec{x}, \vec{y}, \vec{z} \in \Lambda$.

Then, $\psi_{/\Lambda}(\vec{x}, \vec{y}, \vec{z}) = \zeta^{\det([\vec{x}, \vec{y}, \vec{z}])}$, so $\psi_{/\Lambda}(\omega) \neq 0$ and $\langle \psi(\omega) \rangle = \text{Hom}(\Lambda^3\Lambda, k^\times)$. It follows by Theorem 3.2 that $\omega$ is not trivializable.

**Remark 3.6.** In the proof of Proposition 3.1, we prove that every cohomology class of a 3-cocycle $\omega$ that admits an abelian structure $(\omega, c)$ has order two.

If $\Lambda$ is a cyclic group of odd order and $\omega \in H^3(\Lambda, k^\times)$ is a non-zero element, there is not $c \in C^2(\Lambda, k^\times)$ such that $(\omega, c) \in Z^3_{ab}(\Lambda, k^\times)$, however $\psi_{/\Lambda}(\omega) = 0$.

4. Pointed coradically graded coquasi-Hopf algebras

Let $\Gamma$ and $\Lambda$ be abelian groups and $p : \Gamma \to \Lambda$ a group epimorphism. We fix a section $\iota : \Lambda \to \Gamma$ of $p$, and a 3-cocycle $\omega \in H^3(\Lambda, k^\times)$.

We assume that there is $\alpha : \Gamma \times \Gamma \to k^\times$, such that $\delta(\alpha) = p^*\omega$; that is,

$$p^*\omega(g, h, k) = \frac{\alpha(h, g)\alpha(gh, k)}{\alpha(gh, k)}, \quad g, h, k \in \Gamma,$$

4.1. **Trivializing the non-associativity of Nichols algebras.** We consider the functor $k^\omega_{/\Lambda} \mathcal{YD} \rightarrow k_{/p^*\omega_{/\Gamma}} \mathcal{YD}$ given on the objects by

$$V \mapsto \hat{V},$$

with $\Gamma$-grading

$$\hat{V}_g = \begin{cases} V_k & g = \iota(k), \\ 0 & g \notin \iota(\Lambda), \end{cases}$$

and $\Gamma$-action via $p$; on the morphisms, it is just the identity. As $\delta(\alpha) = p^*\omega$, there is a braided tensor equivalence

$$(F_\alpha, \overline{\alpha})_{k_{/p^*\omega_{/\Gamma}}} \mathcal{YD} \rightarrow k_{/\Gamma} \mathcal{YD},$$
where $F_\alpha(V) = V$ as $\Gamma$-graded vector spaces, with $\Gamma$-action

$$g \cdot v = \frac{\alpha(h, g)}{\alpha(g, h)} g \triangleright v,$$

the functor is the identity for morphisms; the isomorphism constraints are

$$\overline{\alpha}_{V,V'} : F_\alpha(V \otimes V') \to F_\alpha(V) \otimes F_\alpha(V')$$

$$v \otimes v' \mapsto \alpha(g, h) v \otimes v', \quad g, h \in \Gamma, \quad v, v' \in V_g.$$  

Fix $V \in \mathcal{YD}^{k\Lambda}_{\Lambda}$. Hence $W = F_\alpha(V) \in \mathcal{YD}^{1\Lambda}_{\Lambda}$ is a braided vector space of diagonal type: there exists a basis $(x_i)_{i \in I}$, elements $g_i \in \Gamma$, $\chi_i \in \hat{\Gamma}$ such that $x_i \in W_{g_i}$, so the braiding is

$$c(x_i \otimes x_j) = g_i \cdot x_j \otimes x_i = q_{ij} x_j \otimes x_i, \quad q_{ij} := \chi_j(g_i), \quad i, j \in I.$$  

Coming back to $V$, let $\ell_i = p(g_i) \in \Lambda$, $i \in I$. As $V = W$ as vector spaces and the $\Gamma$-grading on $W$ is induced by $\iota$, we have that $g_i = \iota(\ell_i)$ and $x_i \in V_{\ell_i}$ for all $i \in I$. The quasi-braiding in $\mathcal{YD}^{k\Lambda}_{\Lambda}$ is given by

$$c_V(x_i \otimes x_j) = g_i \triangleright x_j \otimes x_i = q_{ij} \frac{\alpha(\ell_i, \ell_j)}{\alpha(\ell_j, \ell_i)} x_j \otimes x_i, \quad i, j \in I.$$  

4.2. Generation in degree one. We recall some results about the FRT construction. Let $H(W)$ be the bialgebra corresponding to the diagonal braided vectors space $(W, c) \in \mathcal{YD}^{k\Lambda}_{\Lambda}$. As an algebra, $H(W)$ is presented by generators $T_i^j$, $i, j \in I$ and relations

$$q_{ij} T_i^m T_j^m - q_{nm} T_i^m T_j^m, \quad i, j, m, n \in I.$$  

Hence $H(W)$ is a quantum linear space, so in particular it is $\mathbb{Z}^I$-graded, with $\deg T_i^j = \alpha_i$, $i, j \in I$. The coproduct satisfies

$$\Delta(T_i^j) = \sum_{k \in I} T_i^k \otimes T_k^j, \quad i, j \in I.$$  

Hence $W$ is an $H(W)$-comodule with coaction

$$\rho : W \to H(W) \otimes W, \quad \rho(x_i) = \sum_{j \in I} T_i^j \otimes x_j, \quad i \in I.$$  

The $R$-matrix $r : H(W) \otimes H(W) \to \mathbb{C}$ is determined by

$$r(T_i^m \otimes T_j^n) = q_{ji} \delta_{i,m} \delta_{j,n}, \quad i, j, m, n \in I.$$  

and $c$ is also the braiding in the category of $H(W)$-comodules.

**Theorem 4.1.** Let $R = \oplus_{n \geq 0} R_n \in \mathcal{YD}^{k\Lambda}_{\Lambda}$ be a post-Nichols algebra of $V = R_1$ such that $\dim R < \infty$. Then $R = B(V)$.

**Proof.** By abuse of notation, let $\alpha : H(W) \otimes H(W) \to \mathbb{C}$,

$$\alpha(T_{i_1}^{m_1} \ldots T_{i_s}^{m_s}, T_{j_1}^{n_1} \ldots T_{j_t}^{n_t}) = \delta_{i_1, m_1} \ldots \delta_{i_s, m_s} \delta_{j_1, n_1} \ldots \delta_{j_t, n_t} \alpha(g_{i_1} \ldots g_{i_s}, g_{j_1} \ldots g_{j_t}), \quad s, t \in \mathbb{N}, \; i_k, m_k, j_t, n_t \in I.$$

Afterwards, let $\alpha : H(W) \otimes H(W) \to \mathbb{C}$ be a post-Nichols algebra of $V = R_1$ such that $\dim R < \infty$. Then $R = B(V)$. 

**Proof.** By abuse of notation, let $\alpha : H(W) \otimes H(W) \to \mathbb{C}$,

$$\alpha(T_{i_1}^{m_1} \ldots T_{i_s}^{m_s}, T_{j_1}^{n_1} \ldots T_{j_t}^{n_t}) = \delta_{i_1, m_1} \ldots \delta_{i_s, m_s} \delta_{j_1, n_1} \ldots \delta_{j_t, n_t} \alpha(g_{i_1} \ldots g_{i_s}, g_{j_1} \ldots g_{j_t}), \quad s, t \in \mathbb{N}, \; i_k, m_k, j_t, n_t \in I.$$

Finally, let $\alpha : H(W) \otimes H(W) \to \mathbb{C}$ be a post-Nichols algebra of $V = R_1$ such that $\dim R < \infty$. Then $R = B(V)$.
As \( H(W) \) is \( \mathbb{Z}^1 \)-graded, the map is well-defined, and \( \alpha(1, x) = \alpha(x, 1) = \epsilon(x) \) for all \( x \in H(W) \). Hence we may consider the coquasi-bialgebra \( H(W)^\alpha \) obtained by a 2-cocycle deformation by \( \alpha \).

Notice that \( (V, c_V) \) is the image of \( (W, c) \) under the braided equivalence \( H(W) \mathcal{M} \rightarrow H(W)^\alpha \mathcal{M} \) induced by the 2-cocycle \( \alpha \), and this equivalence takes post-Nichols algebras of \( (W, c) \) to post-Nichols algebras of \( (V, c_V) \). Hence \( R \) is the image of a post-Nichols algebra \( R' \) of \( (W, c) \), which is of diagonal type. By [3], \( R' = \mathcal{B}(W) \), so \( R = \mathcal{B}(V) \). \( \square \)

### 4.3. Pointed coquasi-Hopf algebras and de-equivariantization.

Let \( H \) be a coquasi-Hopf algebra and \( G \) be an affine group scheme over \( \mathbb{k} \).

A central inclusion of \( G \) in \( H \) is a full braided embedding \( \iota : \text{Rep}(G) \rightarrow \mathcal{Z}(H \mathcal{M}) \) such that the composition \( \iota \circ U : \text{Rep}(G) \rightarrow H \mathcal{M} \) is full, where \( U : \mathcal{Z}(H \mathcal{M}) \rightarrow H \mathcal{M} \) is the forgetful functor.

Let \( \mathcal{O}(G) \) be the algebra of regular function over \( G \). The algebra \( \mathcal{O}(G) \) is a commutative algebra in the symmetric category \( \text{Rep}(G) \), and thus a commutative algebra in the braided tensor category \( \mathcal{Z}(H \mathcal{M}) \). Following [14], we define the de-equivariantization \( H \mathcal{M}(G) \) of \( H \mathcal{M} \) by \( G \), as the monoidal category of left \( \mathcal{O}(G) \)-modules in \( H \mathcal{M} \), with the tensor product \( M \otimes_{\mathcal{O}(G)} N \).

Now we prove that each coradically graded pointed coquasi-Hopf algebra with trivializable 3-cocycle comes from the construction above.

**Theorem 4.2.** Let \( A \) be a finite-dimensional coradically graded coquasi-Hopf algebra such that \( A_0 \simeq \mathbb{k}^\omega \Lambda \), where \( \omega \) is trivializable. Then \( A \mathcal{M} \) is a de-equivariantization of a coradically graded pointed Hopf algebra over an abelian group.

**Proof.** By [3] there exists a post-Nichols algebra \( R = \bigoplus_{n \geq 0} R_n \in \mathbb{k}^\omega \Lambda \mathcal{YD} \) of \( V = R_1 \) such that \( A \simeq R \# \mathbb{k}^\omega \Lambda \); hence \( \dim R < \infty \), and by Theorem [11] \( R = \mathcal{B}(V) \). We consider \( \hat{V} \in \mathbb{k}^{p^\omega \Gamma} \mathcal{YD} \): as the braiding is the same, \( \mathcal{B}(V) \simeq \mathcal{B}(\hat{V}) \) as braided Hopf algebras, and

\[
\pi := (\text{id} \otimes p) : B := \mathcal{B}(\hat{V}) \# \mathbb{k}^{p^\omega \Gamma} \rightarrow A = \mathcal{B}(V) \# \mathbb{k}^\omega \Lambda
\]

is a projection of coquasi-Hopf algebras.

Given an epimorphism \( f : H \rightarrow Q \) of finite dimensional coquasi-Hopf algebras, it follows by [10] Proposition 5.1] that

\[
H^{cof} := \{ b \in B : (\text{id} \otimes f) \Delta(b) = b \otimes 1 \}
\]

admits a structure of commutative algebra in \( \mathcal{Z}(H \mathcal{M}) \) such that the tensor category of left \( H^{cof} \)-modules in \( H \mathcal{M} \) is tensor equivalent to \( Q \mathcal{M} \).

Set \( K = \ker p \). We claim that there is a central inclusion \( \iota : \text{Rep}(\hat{K}) \rightarrow \mathcal{Z}(\hat{B} \mathcal{M}) \), such that the central algebra \( \mathcal{O}(\hat{K}) = \mathbb{k}K \) is the central algebra associated to the epimorphism [15].

The inclusion \( \mathbb{k}K \rightarrow B, a \mapsto 1 \# a \), is an injective coquasi-Hopf algebra morphism, that induces a full tensor embedding

\[
\text{Rep}(\hat{K}) = \text{Vec}_K \rightarrow B \mathcal{M}.
\]
Let $V_a = k^v \in kK\mathcal{M}$ be a one-dimensional comodule with $\rho(v) = a \otimes v$, $a \in K$. As $1\# a$ is a central group-like of $B$, for any $M \in B\mathcal{M}$, the flip map

$$c_{M,V_a} : M \otimes V_a \to V_a \otimes M \quad m \otimes v \mapsto v \otimes m,$$

is an isomorphism of $B$-comodules. Now (13) follows from the fact that

$$p^* \omega(a, g, h) = p^* \omega(g, a, h) = p^* \omega(g, h, a) = 1 \quad \text{for all } g, h \in \Gamma, a \in K.$$

Since $kK = B^{\text{co } \pi} := \{b \in B : \text{id} \otimes \pi \Delta(b) = b \otimes 1\}$, the central algebra associated to the surjective tensor functor $\pi : B\mathcal{M} \to A\mathcal{M}$ is exactly $kK$. By [10] Proposition 5.1, $A\mathcal{M}$ is a de-equivariantization of $B\mathcal{M}$ by $\tilde{K}$.

Recall that there is a map $\alpha : \Gamma \times \Gamma \to k^\times$ such that $\delta(\alpha) = p^* \omega$; we can extend $\alpha$ linearly to a map $\alpha : B \otimes B \to k$ such that

$$\alpha(B_n \otimes B) = \alpha(B \otimes B_n) = 0 \quad \text{for all } n > 0,$$

and $\alpha$ is a twist. Hence $H := B^\alpha$ is a Hopf algebra; as a coalgebra, $H \cong B$ is coradically graded, with $H_0 = k\Gamma$. Hence $H \cong R^\prime \# k\Gamma$ for some graded Hopf algebra $R^\prime \in k^\times \mathcal{Y}\mathcal{D}$, where $R^\prime_0 = F_{\alpha}(V)$, so $H \cong B(V) \# k\Gamma$. Since $H\mathcal{M}$ is tensor equivalent to $B\mathcal{M}$, we have that $A\mathcal{M}$ is a de-equivariantization of the Hopf algebra $H$ by the group $\tilde{K}$. \hfill $\Box$

**Proof of Theorem 1.1.** Let $\mathcal{C}$ be a de-equivariantization of $H\mathcal{M}$, where $H$ is a finite-dimensional pointed Hopf algebra with abelian coradical. By [5] we may assume that $H$ is coradically graded. By [4] Proposition 3.3, $\mathcal{C}$ is the category of comodules over a finite-dimensional coradically graded coquasi-Hopf algebra with trivializable 3-cocycle.

On the other hand, assume that $\mathcal{C}$ is a coradically graded tensor category such that $G(\mathcal{C})$ is abelian and $\psi_{G(\mathcal{C})}(\omega_C) \equiv 1$; by hypothesis, $\mathcal{C}$ is the category of comodules over a finite-dimensional coradically graded coalgebra. As $\mathcal{C}$ is pointed, this coalgebra has a structure of pointed coquasi-Hopf algebra, see [15] Proposition 2.6, and Theorem 4.2 applies. \hfill $\Box$

**Corollary 4.3.** Let $\mathcal{C}$ be a coradically graded finite tensor category admitting a braiding. Then $\mathcal{C}$ can be realized as the de-equivariantization of the category of comodules of a finite dimensional coradically graded pointed Hopf algebra over an abelian group.

**Proof.** Since $\mathcal{C}$ is braided, the full subcategory generated by the invertible objects is also braided. Thus, the associator of $\mathcal{C}$ is an abelian 3-cocycle. By Proposition 4.1 and Theorem 1.1, $\mathcal{C}$ is tensor equivalent to the de-equivariantization of a Hopf algebra of the form $B(V) \# k\Lambda$. Hence $(\mathcal{C}, c)$ is realized as the comodules on a coquasitriangular coquasi-Hopf algebra of the form $(B(V) \# k\omega, r)$. \hfill $\Box$

Let $(\mathcal{B}, c)$ be a braided tensor category. The Mueger’s center or symmetric center $Z_2(\mathcal{B})$ is the full tensor subcategory of all objects $Y \in \mathcal{B}$ such that $c_{X,Y} \circ c_{Y,X} = \text{id}_{Y \otimes X}$ for all $X \in \mathcal{B}$. A braided tensor category is called symmetric if $\mathcal{B} = Z_2(\mathcal{B})$. 


Example 4.4. Let \( \mathcal{B} = \mathrm{Vec}_\Lambda^{(\omega, c)} \) be a pointed fusion category. The Mueger’s center of \( \mathcal{B} \) is the fusion subcategory with simple objects given by

\[
\mathcal{Z}_2(\Lambda) := \{ g \in \Lambda : c(g, h)c(h, g) = 1, \text{ for all } h \in \Lambda \}.
\]

Recall that the map \( q : \Lambda \to \k^\times, g \mapsto c(g, g) \) is a quadratic form. Since \( q(gh)q(g)^{-1}q(h)^{-1} = c(g, h)c(h, g) \), we have that

\[
\mathcal{Z}_2(\Lambda) := \{ g \in \Lambda : q(gh) = q(g)q(h), \text{ for all } h \in \Lambda \}.
\]

Since \( q \) only depends on the abelian cohomology class of \((\omega, c)\), the same is true of \( \mathcal{Z}_2(\Lambda) \). The restriction \( q|_{\mathcal{Z}_2(\Lambda)} \) is a group homomorphism in \( \text{Hom}(\mathcal{Z}_2(\Lambda), \mathbb{Z}/2\mathbb{Z}) \). Hence \( \mathcal{Z}_2(\mathcal{B}) \) has a (possibly trivial) \( \mathbb{Z}/2\mathbb{Z} \)-grading.

Let \( (\mathcal{B}(V)\# k^\omega[\Lambda], r) \) be a coquasitriangular coquasihopf algebra. The \( r \)-form defines an abelian structure \((\omega, c) \in Z^3(\Lambda, k^\times)\), where \( c := r_{G(H) \times G(H)} \). Assume without loss of generality that \( \omega|_{\mathcal{Z}_2(\Lambda)\times_3} \equiv 1 \).

As in [7, Section 4], the condition (11) implies that \( V \in \mathcal{Z}_2(\mathrm{Vec}_\Lambda^{(\omega, c)}) \). Hence \( \mathcal{B}(V)\# k\mathcal{Z}_2(\Lambda) \) is a coquasitriangular Hopf subalgebra of \( \mathcal{B}(V)\# k^\omega\Lambda \). Then [7, Theorem 1.1], can be applied to \( \mathcal{B}(V)\# k\mathcal{Z}_2(\Lambda) \). Hence, \( r_1 := r|_{V \otimes V} \) is a morphism in \( \mathcal{Z}_2(\mathrm{Vec}_\Lambda^{(\omega, c)}) \).

In conclusion, as in [7], every coquasitriangular structure on a coradially graded coquasi-Hopf algebra is determined by the following data: an abelian 3-cocycle \((\omega, c) \in Z^3_{ab}(\Lambda, k^\times)\), an object \( V \in \mathcal{Z}_2(\mathrm{Vec}_\Lambda^{(\omega, c)}) \), and a morphism \( r_1 := r|_{V \otimes V} \) in \( \mathcal{Z}_2(\mathrm{Vec}_\Lambda^{(\omega, c)}) \).

4.4. A pointed coquasi-Hopf algebra over an abelian group with non-trivializable associator. Let \( A \) and \( B \) be finite abelian groups and \( \alpha \in Z^2(A, \hat{B}) \) a 2-cocycle, that is, a map \( \alpha : A \times A \to \hat{B} \) such that

\[
\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z), \quad x, y, z \in A.
\]

We denote by \( \hat{B} \times_\alpha A \) the central extension of \( A \) by \( \hat{B} \) associated to \( \alpha \). Explicitly, \( \hat{B} \times_\alpha A = \hat{B} \times A \) as a set, and the product is given by

\[
(x, a)(x', a') = (xx'\alpha(a, a'), aa'), \quad a, a' \in A, \quad x, x' \in B.
\]

The function

\[
\omega_\alpha((x_1, a_1), (x_2, a_2), (x_3, a_3)) = \alpha(a_1, a_2)(x_3),
\]

is a 3-cocycle \( \omega_\alpha \in Z^3(A \oplus B, k^\times) \). It is easy to see that \( \psi_{A \oplus B}(\omega_\alpha) = 0 \) if and only if \( \alpha(a_1, a_2) = \alpha(a_2, a_1) \) for all \( a_1, a_2 \in A \). By [23, Theorem 3.6], the braided categories \( k^\omega\Lambda_A \text{Vec}_{\hat{B}\times_\alpha A}^B \mathcal{D} \) and \( k^\omega\Lambda_A \text{Vec}_{\hat{B}\times_\alpha A}^B \mathcal{D} \) are equivalent.

Example 4.5. Let \( A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) and \( B = \mathbb{Z}/2\mathbb{Z} \). We define

\[
\alpha : A \times A \to \hat{B}, \quad \alpha((m_1, m_2), (n_1, n_2)) = \chi^{m_1n_2},
\]

where \( \chi : \mathbb{Z}/2\mathbb{Z} \to k^\times \) is the non-trivial character.
The associated 3-cocycle is

$$\omega_\alpha(\vec{x}, \vec{y}, \vec{z}) = (-1)^{x_1 y_2 z_3},$$

where $\vec{x}, \vec{y}, \vec{z} \in (\mathbb{Z}/2\mathbb{Z})^3$. Then,

$$\psi_\Lambda(\omega_\alpha)(\vec{x}, \vec{y}, \vec{z}) = (-1)^{\det([\vec{x}, \vec{y}, \vec{z}])}.$$

Thus, $\psi(\omega) \neq 0$. It follows by Theorem 3.2 that $\omega$ is not trivializable.

The group $\hat{B} \rtimes_\alpha A$ is isomorphic to $D_4$, the dihedral group of order 8: it is a non-abelian group of order eight with two elements of order four.

In [20 Example 6.5], Milinski and Schneider constructed a Nichols algebra $\mathcal{B}(V)$ of dimension 64 over $D_4$. Since the braided categories $k_{\omega}^A \oplus B \mathcal{YD}$ and $k_{\hat{B} \rtimes_\alpha A} \mathcal{YD}$ are equivalent, there is a Nichols algebra $\mathcal{B}(V')$ of dimension 64 in $k_{\omega}^A \oplus B \mathcal{YD}$. Hence, the bosonization $A := \mathcal{B}(V')#k_{\omega}(\mathbb{Z}/2\mathbb{Z})^3$ is a coradically graded coquasi-Hopf algebra with non-trivializable associator.

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