Unifying Approaches in Integrable Systems: Quantum and Statistical, Ultralocal and Nonultralocal

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Abstract

The aim of this review is to present the list of by now a significant collection of quantum integrable models, ultralocal as well as nonultralocal, in a systematic way stressing on their underlying unifying algebraic structures. We restrict to quantum and statistical models belonging to trigonometric and rational classes with $2 \times 2$ Lax operators. The ultralocal models can be classified successfully through their associated quantum algebra and are governed by the Yang-Baxter equation, while the nonultralocal models, the theory of which is still in the developmental stage, allow systematization through the braided Yang-Baxter equation. Along with the known integrable models some possible directions for investigation in this field and generation of such new models are suggested.

Key words: Integrable Quantum and Statistical Vertex Models; Quantum Algebras, Yang-Baxter Equation, Braided Extension, Algebraic Bethe Ansatz, Ultralocal and Nonultralocal Models

1 Introduction

By quantum integrable systems we will mean the systems with sufficient number of higher conserved quantities including the Hamiltonian of the model. Such a notion of integrability in the Liouville sense allows description through action-angle variables with the conserved quantities, which are now operators, playing the role of action variables. For integrable systems the conserved quantities, being functionally independent should form a commuting set of operators $[c_n, c_m] = 0$, $n, m = 1, 2, \ldots, N$, such that their total number would match with the degree of freedom of the system. For example, an one dimensional lattice model of $l$-sites describing $d$-mode pseudo-particle should have number of

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conserved quantities $N = d l$. Note that for spin-$\frac{1}{2}$ chains we have $d = 1$, while spin-1 and electron models account for $d = 2$. In this review we will stick to single-mode: $d = 1$ systems for simplicity and consider mainly periodic lattice models with $N < \infty$, where the algebraic structures can be seen in their exact form. At the lattice constant $\Delta \to 0$ the field models will be generated from their exact lattice versions, whenever possible. Integrable field models with $N \to \infty$ consequently needs to have infinite number of conservation laws.

Integrable systems therefore are restrictive systems with a very rich symmetry. The beauty of such models is that they allow exact solutions for the eigenvalue problem simultaneously for all conserved operators including the Hamiltonian. Moreover such 1-dimensional quantum systems are related also to the corresponding 2-dimensional classical statistical models with a fluctuating variable. Therefore parallel to a quantum mechanical model one can in principle exactly solve also a related vertex-type model on a 2-dimensional lattice using almost the same techniques and similar results [1]. Celebrated examples of such interrelated integrable quantum and statistical systems are $XYZ$ quantum spin-$\frac{1}{2}$ chain and the 8-vertex statistical model, $XXZ$ spin chain and the 6-vertex model, spin-1 chain and the 19-vertex model etc.

For describing an integrable system with such an involved structure, one naturally can no longer start from the Hamiltonian of the model as customary in physics, since now the Hamiltonian is merely one among many commuting conserved charges. It therefore needs to adopt certain abstractions which are formalized by the quantum inverse scattering method and the algebraic Bethe ansatz (see [2, 3]). Though we would use the same language, we take here a slightly different viewpoint since we intend to describe integrable systems belonging to both ultralocal and nonultralocal classes. For effective description of integrable systems it is convenient to define a generating function called transfer matrix $\tau(\lambda)$, depending on some extra parameter $\lambda$ known as the spectral parameter, such that one can recover the infinite number of conserved quantities as the expansion coefficients of $\tau(\lambda)$ or any function of it like $\ln \tau(\lambda) = \sum_j c_j \lambda^j$. The crucial integrability condition may then be defined in a compact form as

$$[\tau(\lambda), \tau(\mu)] = 0,$$

(1.1)

from which the commutativity of $c_j$'s follows immediately by comparing the coefficients of different powers of $\lambda, \mu$.

However for solving the eigenvalue problem as well as for identifying the structure of the model we require a more general matrix formulation, from where the integrability condition may be derived. At the same time we need to transit from the global to the local description defined at each lattice point, where some individual properties of a model are well expressed. At this local level, as we see now, the difference between the ultralocal and the nonultralocal models become prominent. An integrable system allowing the needed abstraction may be represented by an unusual type of matrix called the Lax operator $L_{aj}(\lambda)$ defined at each site $j$ in a 1-dimensional discretized lattice. The index $a$ defines the matrix or the auxiliary space, while $j$ designates the quantum space. The matrix elements of the Lax operator, unlike in usual matrices are operators acting on some Hilbert space. The models with the Lax operators commuting at different lattice sites:

$$L_{aj}(\lambda)L_{bk}(\mu) = L_{bk}(\mu)L_{aj}(\lambda), \quad a \neq b, \quad j \neq k,$$

(1.2)
are known as the *ultralocal* models, while the integrable models for which the above ultralocality condition does not hold are classified as the *nonultralocal* models. Note that in expressions like (1.2) different auxiliary spaces mean different tensor products like $L_{1j}(\lambda) = L_j(\lambda) \otimes I$ and $L_{2j}(\mu) = I \otimes L_j(\mu)$. The ultralocal property (1.2) generally reflects the involvement of canonical operators with commutation relations like $[u(x), p(y)] = i\delta(x - y)$ or $[\psi(x), \psi(y)] = \delta(x - y)$ in the Lax operator giving trivial commutator at points $x \neq y$. In nonultralocal models on the other hand the basic fields may be of noncanonical type, e.g. $[j_1(x), j_1(y)] = \delta'(x - y)$ or derivatives of the canonical fields may appear in their Lax operators violating the ultralocal condition and bringing additional complications, which might not always be resolved. Due to this reason the theory and application for the nonultralocal models are still in the process of development and are far from completion. In spite of many important models belonging to this class, it is rather disappointing to note that, this category of models has not received the required attention in the literature.

## 2 Integrable structures in ultralocal models

We focus first on the ultralocal systems due to their relative simplicity and formulate a unifying scheme for generating such quantum and statistical integrable models. For ensuring the integrability of an ultralocal model it is sufficient to impose certain matrix commutation relation known as the *quantum Yang-Baxter equation* (QYBE) on its representative Lax operator in the form

$$R_{ab}(\lambda - \mu) L_{aj}(\lambda) L_{bj}(\mu) = L_{bj}(\mu) L_{aj}(\lambda) R_{ab}(\lambda - \mu),$$

defined at each lattice site $j = 1, 2, \ldots, N$. The above QYBE expresses actually the commutation relations among different matrix elements of the $L$-operator, given in a compact matrix form, where the structure constants are determined by the spectral parameter dependent $c$-number elements of the $R(\lambda - \mu)$-matrix. The $R$-matrix in turn should satisfy a similar but simpler YBE

$$R_{ab}(\lambda - \mu) R_{ac}(\lambda - \gamma) R_{bc}(\mu - \gamma) = R_{bc}(\mu - \gamma) R_{ac}(\lambda - \gamma) R_{ab}(\lambda - \mu).$$

Since our intention is to establish the integrability which is a global property, we have to switch from this local picture at each site $j$ to a global one by defining a matrix, known as the monodromy matrix

$$T_a(\lambda) = \prod_{j=1}^N L_{aj}(\lambda), \quad T(\lambda) \equiv \begin{pmatrix} A(\lambda), & B(\lambda) \\ C(\lambda), & D(\lambda) \end{pmatrix}. \quad (2.3)$$

Multiplying therefore the QYBE (2.1) for $j = 1, 2, \ldots, N$ and using the ultralocality condition (1.2), thanks to which one can treat the objects at different lattice points as commuting objects as in the classical case and drag $L_{aj}(\lambda)$ through all $L_{bk}(\mu)$’s for $k \neq j, b \neq a$ to arrive at the global QYBE

$$R_{ab}(\lambda - \mu) T_a(\lambda) T_b(\mu) = T_b(\mu) T_a(\lambda) R_{ab}(\lambda - \mu). \quad (2.4)$$

Note that the local and the global QYBE have exactly the same structural form. Invariance of the algebraic form also for the tensor product of the algebras, as revealed here, indicates the occurrence of the coproduct related to a deep Hopf algebra structure underlying all integrable systems [4]. We will
see below that for nonultralocal models such a structure is modified a bit to include additional braiding relations. For the periodic ultralocal models defining further the transfer matrix as \( \tau(\lambda) = tr_a T_\alpha(\lambda) \), taking trace from both sides of the global YBE (2.4) and canceling the \( R \)-matrices due to the cyclic rotation of matrices under the trace we reach finally for \( \tau(\lambda) \) at the trace identity (1.1) defining the quantum integrability of the system. Therefore we may conclude that the local QYBE (2.1) in association with the ultralocality condition (1.2) is the sufficient condition for quantum integrability of an ultralocal system. Consequently we may define such an integrable system by its representative Lax operator together with the associated \( R \)-matrix satisfying these criteria. Note that we are concerned here only with the systems with periodic boundary condition. For models with open boundaries, the QYBE should however be modified with the inclusion of a reflection matrix, which was introduced in detail in [5].

2.1 List of well known ultralocal models

To have a concrete picture before us, we furnish a list of the well known ultralocal models together with their \( L \)-operators and \( R \)-matrices. We will however restrict here for simplicity only to the quantum models with \( 2 \times 2 \)-matrix Lax operators associated with \( 4 \times 4 \) \( R \)-matrices. We show in the next section how these Lax operators can be generated in a systematic way confirming their integrability. The \( R^{\alpha\beta}_{\gamma\delta} \)-matrix that satisfies the YBE relation (2.2), with the indices taking values 1, 2 only, can be given in a simple form by defining its nontrivial elements as [6]

\[
R^{11}_{11} = R^{22}_{22} = a(\lambda), \quad R^{12}_{12} = R^{21}_{21} = b(\lambda), \quad R^{12}_{21} = R^{21}_{12} = c. \tag{2.5}
\]

These elements may be expressed explicitly through trigonometric functions in spectral parameters as

\[
a(\lambda) = \sin(\lambda + \alpha), \quad b(\lambda) = \sin \lambda, \quad c = \sin \alpha \tag{2.6}
\]

or as its \( \alpha \to 0, \lambda \to 0 \) limit, through rational functions as

\[
a(\lambda) = \lambda + \alpha, \quad b(\lambda) = \lambda, \quad c = \alpha. \tag{2.7}
\]

Moreover under a twisting transformation

\[
R(\lambda) \to \tilde{R}(\lambda, \theta) = F(\theta)R(\lambda)F(\theta), \quad \text{with } F_{ab}(\theta) = e^{i\theta(\sigma_3^a - \sigma_3^b)} \tag{2.8}
\]

one gets twisted trigonometric and rational \( R \)-matrix solution of (2.2), which may be given by (2.5) with the difference \( R^{12}_{12} = b(\lambda)e^{i\theta}, R^{21}_{21} = b(\lambda)e^{-i\theta} \). Apart from these \( R \)-matrices there can be elliptic \( R \)-matrix solution, for example that related to the XYZ spin chain and the 8-vertex model [1]. All models we consider here however are associated with the trigonometric or the rational \( R \)-matrices and in the list presented below we group them accordingly, denoting \( H \) for the Hamiltonian and \( \mathcal{L} (L_n) \) for the Lax operator related to field (lattice) models.

I. Models associated with trigonometric \( R \)-matrix \( (q = e^{i\alpha}, \xi = e^{i\lambda}) \)

i) Field models

1. Sine-Gordon model [7]

\[
u_{tt} - u_{xx} = \frac{m^2}{\alpha} \sin(\alpha u), \quad \mathcal{L} = \begin{pmatrix} ip, & m \sin(\lambda - \alpha u) \\ m \sin(\lambda + \alpha u), & -ip \end{pmatrix}, \quad p = \dot{u}. \tag{2.9}
\]
2. Liouville model [8]
\[ u_{tt} - u_{xx} = e^{i\alpha u}, \quad L = i\left( \frac{p_x}{2e^{i\alpha u}} - p \right), \quad [u(x), p(y)] = i\delta(x-y). \] (2.10)

3. A derivative NLS (DNLS) model [9]
\[ i\psi_t - \psi_{xx} + 4i\psi^\dagger \psi \psi_x = 0, \quad L = i\left( -\frac{1}{4}\xi^2 + k_- N, \xi \psi^\dagger \right) , \quad N = \psi^\dagger \psi, \quad [\psi(x), \psi^\dagger(y)] = \delta(x-y) \] (2.11)

4. Massive Thirring (bosonic) model (MTM) [6]
\[ H = \int dx \left[ -i\psi^\dagger (\alpha^3 \partial_x + \alpha^2) \psi + 2\psi^\dagger (1) \psi (2) \psi (1) \right], \quad \psi^\dagger = (\psi^\dagger (1)^\dagger, \psi (2)^\dagger), \quad [\psi^\dagger(a)(x), \psi (b)^\dagger(y)] = \delta_{ab}\delta(x-y) \]
\[ L = i\left( f^+(\xi, N^a), \xi \psi (1)^\dagger + \frac{1}{2} \psi (2)^\dagger \right), \quad f^\pm(\xi, N^a) = \pm\left( \frac{1}{4} \left( \frac{1}{\xi^2} - \xi^2 \right) + k_{\pm} N^a \right) \] (2.12)

ii) Lattice Models
1. Anisotropic XXZ spin chain [10]
\[ H = \sum_n \sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \cos \alpha \sigma_n^3 \sigma_{n+1}^3, \quad L_n(\xi) = \sin(\lambda + \alpha \sigma_n^3 \sigma_{n+1}^3) + \sin \alpha (\sigma_+^n \sigma_-^n + \sigma_-^n \sigma_+^n) \] (2.13)

2. Lattice SG model [11]
\[ L_n(\lambda) = \left( \frac{g(u_n) e^{ip_n \Delta}}{m \Delta \sin(\lambda + \alpha u_n)}, \frac{m \Delta \sin(\lambda - \alpha u_n)}{e^{-ip_n \Delta} g(u_n)} \right), \quad g^2(u_n) = 1 + m^2 \Delta^2 \cos \alpha (2u_n + 1) \] (2.14)

3. Lattice Liouville model [8]
\[ L_n(\xi) = \left( \frac{e^{ip_n \Delta}}{\Delta \xi e^{i\alpha u_n}}, \frac{\Delta \xi e^{i\alpha u_n}}{f(u_n) e^{-ip_n \Delta}} \right), \quad f^2(u_n) = 1 + \Delta^2 e^{i\alpha (2u_n + 1)}. \] (2.15)

4. Lattice DNLS model [12]
\[ L_n(\xi) = \left( \frac{1}{2} q^{-N_n} - i\xi \Delta q^{N_n+1}, \frac{\kappa A_n^\dagger}{\kappa A_n}, \frac{1}{2} q^{N_n} + i\xi \Delta q^{-(N_n+1)} \right), \quad [A_n, A_m^\dagger] = \delta_{nm} \frac{\cos \alpha (2N_n + 1)}{\cos \alpha} \] (2.16)

5. Lattice MTM [13]

   Exact lattice version of MTM (2.12).
   Lax operator: \[ L_n = \tilde{L}_n^{(1)} \tilde{L}_n^{(2)} \] (each factor is a realization of (2.16) for a bosonic mode).
6. Discrete-time or relativistic quantum Toda chain [14]
\[ H = \sum_i \left( \cosh 2\alpha p_i + \alpha^2 \cosh \alpha (p_i + p_{i+1}) e^{(u_i - u_{i+1})} \right), \quad L_n(\xi) = \left( \frac{1}{2} e^{\alpha p_n} - \xi e^{-\alpha p_n}, \alpha e^{u_n}, -\alpha e^{-u_n} \right) \] (2.17)
Ia. Models associated with twisted trigonometric $R$-matrix

6.a) Quantum Suris discrete-time Toda chain [15, 14]

$$L_k(\xi) = \begin{pmatrix} \frac{1}{2}e^{2\alpha p_k} - \xi, & \alpha e^{u_k} \\ -\alpha e^{2\alpha p_k - u_k}, & 0 \end{pmatrix},$$ (2.18)

7. Ablowitz-Ladik model [16, 6]

$$ib_{j,t} + (1 + \alpha b_j^* b_j)(b_{j+1} + b_{j-1}) = 0, \quad L_k(\xi) = \begin{pmatrix} \frac{1}{2}, & b_k^* \\ b_k, & \xi \end{pmatrix}, \quad [b_k, b_l^*] = \delta_{kl}(1 - b_k^* b_k)$$ (2.19)

II. Models associated with rational $R$-matrix

i) Field models:

1. Nonlinear Schrödinger equation (NLS)

$$i\psi_t + \psi_{xx} + (\psi^*\psi)\psi = 0, \quad L(\lambda) = \begin{pmatrix} \lambda, & \psi \\ \psi^*, & -\lambda \end{pmatrix},$$ (2.20)

ii) Lattice Models:

1. Isotropic XXX spin chain [10]

$$H = \sum_{n} \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}, \quad L_{an}(\lambda) = \lambda I + \alpha P_n, \quad P_n = \frac{1}{2}(I + \vec{\sigma}_a \cdot \vec{\sigma}_n)$$ (2.21)

2. Gaudin model [18]

In the simplest case the Hamiltonians

$$H_k = \sum_{n \neq k} \frac{1}{\epsilon_k - \epsilon_l} (\vec{\sigma}_k \cdot \vec{\sigma}_l), \quad k = 1, 2, \ldots, N, \quad L_{ak}(\lambda) = (\lambda - \epsilon_k)I + \alpha P_k.$$ (2.22)

3. Lattice NLS model [11]

$$L_n(\lambda) = \begin{pmatrix} \lambda + s - \Delta N_n & \Delta^2 (2s - \Delta N_n)^2 \psi_n^* \\ \Delta^2 \psi^* (2s - \Delta N_n)^2 & \lambda + s + \Delta N_n \end{pmatrix}, \quad N_n = \psi_n^* \psi_n, \quad [\psi_k, \psi_l^*] = \delta_{kl}.$$ (2.23)

4. Simple lattice NLS [19]

$$L_n(\lambda) = \begin{pmatrix} \lambda + s - N_n & \psi_n^* \\ \psi_n & -1 \end{pmatrix}.$$ (2.24)

5. Discrete self trapping dimer model [20]

$$H = -\left[ \frac{1}{2} \sum_a (s_a - N^{(a)})^2 + (\psi^{(1)}(1)\psi^{(2)}(2) + \psi^{(1)}(2)\psi^{(1)}(1)) \right], \quad [\psi^{(a)}, \psi^{(b)}] = \delta_{ab}, a, b = 1, 2$$ (2.25)

Lax operator $L(\lambda) = L^{(1)}(\lambda)L^{(2)}(\lambda)$, (each factor as (2.24) for each of two bosonic modes).

6. Toda chain (nonrelativistic) [6]

$$H = \sum_i \left( \frac{1}{2}p_i^2 + e^{(u_i - u_{i+1})} \right), \quad L_n(\lambda) = \begin{pmatrix} p_n - \lambda & e^{u_n} \\ -e^{-u_n} & 0 \end{pmatrix}. \quad (2.26)$$
3 Unifying algebraic approach in ultralocal models

Though the QYBE itself represents an unifying approach for all ultralocal models, we intend to specify here a common algebraic structure independent of the spectral parameter that will not only systematize the models including those listed above, but also identify their common integrable origin, establishing naturally the quantum integrability for all of them, simultaneously. From the above list of models, one may observe that, different integrable models have their representative Lax operators in diverse forms with varied dependence on the spectral parameter as well as on the basic operators like spin, bosonic or the canonical operators. However the R-matrices associated with all of them are given by the same form (2.5) with known trigonometric (2.6) or its limiting rational (2.7) solutions.

To explain this intriguing observation we may look for a common origin for the Lax operators linked with these properties hold also for (3.2) defining it as a Hopf algebra. Referring the interested readers to the original works [21] for more mathematical treatment of the noncocommutative Hopf algebra, we may look for a common origin for the Lax operators linked with a general underlying algebra free from spectral parameters, though derivable from the QYBE. We propose to take the Lax operator of such ancestor model in the form [17]

$$L^{(\text{anc})}_{\text{trig}}(\xi) = \left( \frac{\xi c_1^+ e^{i\alpha S^3} + \xi^{-1} c_1^- e^{-i\alpha S^3}}{\epsilon_- S^+}, \frac{\epsilon_+ S^-}{\xi c_2^+ e^{-i\alpha S^3} + \xi^{-1} c_2^- e^{i\alpha S^3}} \right), \quad \xi = e^{i\alpha \lambda}, \epsilon_\pm = 2 \sin \alpha \xi^\pm_1, \quad (3.1)$$

where $S$ and $c_a^\pm$, $a = 1, 2$ are some operators, the algebraic properties of which are specified below. The structure of (3.1) becomes clearer if we notice the decomposition $L^{(\text{anc})}_{\text{trig}}(\xi) = \xi L_+ + \xi^{-1} L_-$, where $L_\pm$ are spectral parameter $\xi$-independent upper and lower triangular matrices similar to the construction of [4]. Inserting (3.1) in QYBE together with its associated R-matrix (2.5) with trigonometric solution (2.6) and matching different powers of the $\xi$ we obtain the underlying general algebra as

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = \left( M^+ \sin(2\alpha S^3) + M^- \cos(2\alpha S^3) \right) \frac{1}{\sin \alpha}, \quad [M^\pm, \_] = 0, \quad (3.2)$$

with $M^\pm = \frac{1}{2} \sqrt{-1}(c_1^+ c_2^- \pm c_1^- c_2^+)$ behaving as central elements with arbitrary values of $c$’s. As we have mentioned above, the integrable systems are associated with an important Hopf algebra $A$, exhibiting the properties like 1) coproduct $\Delta(x) : A \rightarrow A \otimes A$, 2) antipode or 'inverse' $S : A \rightarrow A$, 3) counit $\epsilon : A \rightarrow k$, 4) multiplication $M : A \otimes A \rightarrow A$ and 5) unit $\alpha : k \rightarrow A$. It can be shown that all these properties hold also for (3.2) defining it as a Hopf algebra. Referring the interested readers to the original works [21] for more mathematical treatment of the noncocommutative Hopf algebra, we give here only some simple and intuitive arguments in its constructions. For example the coproduct $\Delta(x)$, the most important of these characteristics, can be derived for algebra (3.2) by exploiting a QYBE property that the product of two Lax operators $L_{aj} L_{aj+1}$ is again a solution of the QYBE and may be given in the explicit form as

$$\Delta(S^+) = c_1^+ c_\alpha S^3 \otimes S^+ + S^+ \otimes c_2^+ e^{-i\alpha S^3}, \quad \Delta(S^-) = c_2^- c_\alpha S^3 \otimes S^- + S^- \otimes c_1^- e^{-i\alpha S^3}$$

$$\Delta(S^3) = I \otimes S^3 + S^3 \otimes I, \quad \Delta(c_a^\pm) = c_a^\pm \otimes c_a^\pm. \quad (3.3)$$

The multiplication property mentioned above is also in agreement with the ultralocality condition, which is used for transition from local to global QYBE following the multiplication like

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

with $A = L_i(\lambda), B = L_i(\mu), C = L_{i+1}(\lambda), D = L_{i+1}(\mu). \quad (3.4)$$
Note that (3.2) is a $q$-deformed algebra and a generalization of the well known quantum algebra $[21] U_q(su(2))$.

In fact different choices of the central elements $c_a^\pm$ reduce this algebra to the $q$-spin, $q$-boson as well as various other $q$-deformed algebras along with their undeformed limits. Therefore we can obtain easily the coproduct for these algebras, whenever admissible, from their general form (3.3) in a systematic way by taking the corresponding values of $c$'s.

### 3.1 Generation of models

We know that the well known integrable models listed above were discovered at different points of time, mostly in an isolated way and generally by quantization of the existing classical models. However, as we will see, they can actually be generated in a systematic way through various realizations of the same Lax operator (3.1) giving a unifying picture of integrable ultralocal models. For this we find first a representation of (3.2) like

$$ S^\uparrow = u, \quad S^+ = e^{-ip}g(u), \quad S^- = g(u)e^{ip}, \quad (3.5) $$

in physical variables with $[u, p] = i$, where the operator function

$$ g(u) = (\kappa + \sin \alpha(s - u)(M^+ \sin \alpha(u + s + 1) + M^- \cos \alpha(u + s + 1)))\frac{1}{\sin \alpha}, \quad (3.6) $$

containing free parameters $\kappa$ and $s$. We demonstrate now that the Lax operator (3.1), which represents a generalized lattice SG like model for (3.5) may serve as an ancestor model (with possible realizations also in other physical variables like bosonic $\psi, \psi^\dagger$ or spin $s^\pm, s^3$ operators) for generating all integrable ultralocal quantum as well as statistical systems. As an added advantage, the Lax operators of these models are derived automatically from (3.1), while the $R$-matrix is simply inherited. The underlying algebras of the models are also given by the corresponding representations of the ancestor algebra (3.2), which being a direct consequence of the QYBE ensures the quantum integrability of all its descendant models, that we construct here. It should be stressed that due to the symmetry of the solution (2.5): $[R(\lambda - \mu), \sigma^a \otimes \sigma^a] = 0, \ a = 1, 2, 3$ the Lax operator (3.1) as a solution of QYBE may be right or left multiplied by any $\sigma^a$. We shall use this freedom in our following constructions, whenever needed.

Note that we may generate also the quantum field models by taking properly the continuum limit of their lattice variants with the lattice spacing $\Delta \rightarrow 0$. Though in general such transitions to the field limit might be tricky and problematic we suppose their validity assuming the lattice operators to go smoothly to the field operators $p_j \rightarrow p(x), \psi_j \rightarrow \psi(x)$, with the corresponding commutators: $[\psi_j, \psi_k^\dagger] = \frac{1}{\Delta}\delta_{jk} \rightarrow [\psi(x), \psi(y)] = \delta(x - y)$ etc. The lattice Lax operator therefore should reduce to its field counterpart $L(x, \lambda)$ as $L_j(\lambda) \rightarrow I + i\Delta L(x, \lambda) + O(\Delta^2)$. The associated $R$-matrix however remains the same, since it does not contain lattice constant $\Delta$. Thus integrable field models like sine-Gordon, Liouville, NLS or the derivative NLS models can be recovered from their exact lattice versions and having the same quantum $R$-matrix, though all discrete models may not always have such a direct field limit.
3.1.1 Models belonging to trigonometric class

1.) Choosing trivially all central elements as $c_0^\pm = 1$, $a = 1$, which gives $M^- = 0$, $M^+ = 1$, (3.2) reduces clearly to the well known quantum algebra $U_q(su(2))$ [21] given by

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = [2S^3]_q,$$  \hspace{2cm} (3.7)

with the known form of its coproduct recovered easily from (3.3). The simplest representation $S = \frac{1}{2} \sigma$ for this case derives from (3.1) the integrable XXZ spin chain (2.13). On the other hand, representation (3.5) with the corresponding reduction of (3.6) as $g(u) = \frac{1}{2 \sin \alpha} \left[ 1 + \cos \alpha (2u + 1) \right]^{\frac{1}{2}}$ with suitable choice of parameters $s, \kappa$ recovers the Lax operator of lattice sine-Gordon model (2.14) directly from (3.1) and at its field limit the field Lax operator (2.9). Note that the spectral dependence in $\epsilon_\pm$ appearing in (3.1) can be easily removed through a simple gauge transformation [4] and therefore we ignore them in our construction and use the freedom of translational symmetry of the spectral parameters $\lambda \rightarrow \lambda + \text{const.}$, whenever needed.

2.) An unusual exponentially deformed algebra can be generated from (3.2) by fixing the elements as $c_1^+ = c_2^+ = 1$, $c_1^- = c_2^- = 0$, which gives $M^\pm = \pm \frac{1}{2} \sqrt{\pm 1}$ and

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = \frac{e^{2i\alpha S^3}}{2i \sin \alpha}$$  \hspace{2cm} (3.8)

and reduces (3.6) to $g(u) = \frac{(1 + e^{i\alpha (2u+1)})^{\frac{1}{2}}}{\sqrt{2 \sin \alpha}}$. This algebra and the corresponding realization yields clearly from (3.1) the Lax operator of the lattice Liouville model (2.15) and at its field limit that of the Liouville field model (2.10).

It is interesting to observe here that though the underlying algebraic structure and hence its realization are fixed by the choice of $M^\pm$, the Lax operator (3.1) depends explicitly on the set of $c^\prime$s and therefore may take different forms for the same model. For example in the present case $c_1^- \neq 0$ would give again the same value for $M^\pm$ but a different Liouville Lax operator [22] more convenient for the Bethe ansatz solution.

This opens up therefore interesting possibilities for obtaining systematically different useful Lax operators for the same integrable model, as well as for constructing new nonultralocal models [23].

3.) Recall that the well known $q$-bosonic algebra may be given by [24] $[A, N] = A$, $[A^\dagger, N] = -A^\dagger$, $AA^\dagger - q^{-2}A^\dagger A = q^{2N}$ or in its conjugate form with $q \rightarrow q^{-1}$. Combining these two forms we can easily write the commutator of such $q$-bosons as

$$[A, N] = A, \quad [A^\dagger, N] = -A^\dagger, \quad [A, A^\dagger] = \frac{\cos(\alpha(2N + 1))}{\cos \alpha}.$$  \hspace{2cm} (3.9)

It is interesting to find that for the choice of the central elements $c_1^+ = c_2^+ = 1$, $c_1^- = -iq$, $c_2^- = \frac{i}{q}$ compatible with $M^+ = 2 \sin \alpha$, $M^- = 2 \cos \alpha$ we may get a realization

$$S^+ = -\kappa A, \quad S^- = \kappa A^\dagger, \quad S^3 = -N, \quad \kappa = -i(\cot \alpha)^{\frac{1}{2}},$$  \hspace{2cm} (3.10)

with (3.2) reducing directly to the relation (3.9), which gives thus a new integrable $q$-boson model. It is important to note now that either using (3.5) which simplifies (3.6) to $g^2(u) = [-2u]_q$ or directly
taking the mapping of the $q$-bosons to standard bosons: $A = \psi(2N)_{\theta,\alpha}^{(2N)}$, $N = \psi^\dagger \psi$, we may convert (3.1) with (3.10) to an exact lattice version of the quantum derivative nonlinear Schrödinger (QDNLS) equation (2.16) and consequently to the QDNLS field model (2.11). The QDNLS is related also to the interacting bose gas with derivative $\delta$-function potential [25].

4.) Since the matrix product of Lax operators with each factors representing different Lax operator realization for the same model should give again a QYBE solution, we can construct multi-mode integrable extensions by taking the product of single-mode Lax operators. Using this trick, i.e. by combining two QDNLS models constructed above as $L(c_1^+, c_2^\pm, \psi^{(1)}) L(c_2^+, c_1^\pm, \psi^{(2)}) = L(\lambda)$, we can create further an integrable exact lattice version of the massive Thirring model [13]. At the continuum limit it goes to the bosonic massive Thirring model introduced in [6], the field Lax operator (2.12) of which can be given simply by the superposition $L = L^{(1)}(\xi, k_\pm, \psi^{(1)}) + \sigma^3 L^{(2)}(\xi, k_\mp, \psi^{(2)})\sigma^3$, where $1 \pm ik_\pm \sin \alpha = e^{\pm i\theta}$ and the constituting operators $L^{(a)}$ is given clearly by the DNLS Lax operator (2.11) for each of its two bosonic modes.

5.) Since the general algebra permits trivial eigenvalues for central elements, one may choose both $M^\pm = 0$, which might correspond to different sets of choices like i) $c_a^+ = 1$, $a = 1, 2$, or ii) $c_a^- = 1$, $a = 1, 2$, or iii) $c_1^\pm = \pm 1$, or iv) $c_1^\mp = 1$, with the rest of $c$’s being zero. It is easy to see that all of these sets lead to the same underlying algebra

$$[S^+, S^-] = 0, \quad [S^3, S^\pm] = \pm S^\pm,$$  \hspace{1cm} (3.11)

though generating different Lax operators from (3.1).

As here (3.6) gives simply $g(u) = \text{const.}$, interchanging canonically $u \rightarrow -ip, p \rightarrow -iu$, from (3.5) one gets

$$S^3 = -ip, \quad S^\pm = \alpha e^{\mp u},$$  \hspace{1cm} (3.12)

which evidently generates from the same general Lax operator (3.1) the discrete-time or relativistic quantum Toda chain (2.17). Note that, iii) and iv) give two different Lax operators found in [14] and [26] for the relativistic Toda chain. Case i) and ii) on the other hand could be used for constructing nonultralocal quantum models, namely light-cone SG and the mKdV model [23].

Models in twisted trigonometric class: Under twisting when the $R$-matrix changes as (2.8) the associated Lax operator is also transformed similarly as $L_n(\lambda) \rightarrow \tilde{L}_n(\lambda, \theta) = F_n(\theta)L_n(\lambda)F_n(\theta)$, with $F_n(\theta) = e^{i\theta(\sigma_3^a - S_3^0)}$. As a result the ancestor model (3.1) associated with the trigonometric twisted $R$-matrix (2.8) gets deformed with its operator elements changing as

$$c_a^+ \rightarrow c_a^+ e^{-i\theta S^3_k}, \quad S_k^\pm \rightarrow \tilde{S}_k^\pm = e^{-i\frac{\theta}{2} S^3_k} S_k^\mp e^{-i\frac{\theta}{2} S^3_k},$$  \hspace{1cm} (3.13)

and as a consequence the diagonal elements of the twisted Lax operator take the form $e^{i(\theta \pm \alpha)} S_k^3$, with obvious preference for the choice $\theta = \pm \alpha$.

6.) We may generate the quantum analog of Suris discrete-time Toda chain belonging to the twisted class by starting from the ancestor model with the change (3.13), but by fixing the parameter $\theta = -\alpha$ (an equivalent model is obtained by the choice $\theta = \alpha$). Using the same realization (3.12) we arrive now at the explicit form (2.18).

7.) However if we start from the same twisted ancestor model with the same value $\theta = -\alpha$ of the twisting parameter, but take the central elements as $c_1^+ = c_2^- = 0$ with $c_1^- = c_2^+ = 1$ giving
$M^\pm = \frac{1}{2}\sqrt{\mp 1}$ (compare with the Liouville case!), all noncommuting operators clearly vanish from the diagonal elements of the resulting Lax operator. Moreover renaming the deformed operators as $b_k = 2\sin \alpha \tilde{S}_k^i$, we get their modified algebra as a type of $q$-boson: $[b_k, b_k^\dagger] = \delta_{kl}(1 - b_k^\dagger b_k)$ and thus generate finally the exact form of the Ablowitz-Ladik model (2.19).

The domain of the models considered can therefore be considerably extended if we use twisting and some other allowed transformations [27] that preserves integrability.

### 3.1.2 Models belonging to the rational class

One of the crucial parameters inbuilt in both $R$-matrix and above Lax operators is the deformation parameter $q = e^{i\alpha}$, the physical meaning of which is as anisotropic or relativistic parameter. We consider now the undeformed limit $q \to 1$ or $\alpha \to 0$ related to isotropic or nonrelativistic models belonging to the rational class, which reduces various $\alpha$-dependent objects as $S^\pm \to is^\pm, \{c_\alpha^\pm\} \to \{c_\alpha^i\}$, $M^+ \to -m^+, M^- \to -\alpha m^-$, $\xi \to 1 + i\alpha \lambda$. This transforms (3.2) to a $q$-independent algebra with

$$[s^+, s^-] = 2m^+ s^3 + m^-, \quad [s^3, s^\pm] = \pm s^\pm,$$

(3.14)

were $m^+ = c_0^1 c_0^2$, $m^- = c_1^0 c_2^0 + c_0^1 c_1^2$ and $c_\alpha^i, i = 0, 1$ are central to (3.14). Note that (3.14) is a generalization of spin as well as the bosonic algebra and its coproduct can be obtained as a limit of (3.3). Consequently, the general Lax operator (3.1) is converted into

$$L_{\text{rat}}^{(\text{unc})}(\lambda) = \begin{pmatrix} c_0^1 \lambda + s^3 + c_1^1 & s^- \\ s^+ & c_0^2 \lambda - s^3 - c_1^2 \end{pmatrix},$$

(3.15)

and the quantum $R$-matrix (2.5) is reduced to its rational form (2.7).

We would see that the ultralocal integrable systems belonging to the rational class can be generated in a similar way now from the Lax operator (3.15) with algebra (3.14), all sharing the same rational $R$-matrix (2.7). It is not difficult to check by a variable change $(u,p) \to (\psi, \psi^\dagger)$ that at the limit $\alpha \to 0$ (3.5) reduces to a generalized Holstein-Primakov transformation (HPT)

$$s^3 = s - N, \quad s^+ = g_0(N)\psi, \quad s^- = \psi^\dagger g_0(N), \quad g_0^2(N) = m^- + m^+(2s - N), \quad N = \psi^\dagger \psi,$$

(3.16)

which is also an exact realization of (3.14). Therefore Lax operator (3.15) with such a realization may be considered as a generalized lattice NLS, which would serve as a generating model for all quantum integrable models belonging to the rational class.

1.) The choice $m^+ = 1, m^- = 0$, clearly reduces (3.14) to $su(2)$ algebra $[s^+, s^-] = 2s^3, \quad [s^3, s^\pm] = \pm s^\pm$. A compatible choice $c_0^1 = 1, c_0^2 = 0$ yields from (3.15) for the spin-$\frac{1}{2}$ representation the Lax operator of the $XXX$ spin chain (2.21).

Taking spin-$\frac{1}{2}$ and spin-1 realizations alternatively along the lattice we can construct now the integrable alternate spin model discovered in [29].

Note that a slightly different choice $c_1^1 = -c_0^1 = 1, c_0^2 = 0$ giving $m^+ = -1, m^- = 0$, generates on the other hand the corresponding model with $su(1,1)$ algebra.

The bosonic realization (3.16) in present cases with $m^+ = \pm 1, m^- = 0$, is simplified to the standard HPT with $g_0^2(N) = \pm(2s - \psi^\dagger \psi)$, which reproduces from (3.15) the exact lattice NLS
model (2.23) and at the continuum limit the more familiar *NLS field model* (2.20), with $+(-)$ sign in the HPT corresponding to the attractive (repulsive) interaction.

2.) A complementary choice $m^+ = 0, m^- = 1$, on the other hand converts (3.16) to $s^+ = \psi, s^- = \psi^\dagger, s^3 = s - N$ due to $g_0(N) = 1$ and reduce (3.14) directly to the standard bosonic relations $[\psi, N] = \psi, \; [\psi^\dagger, N] = -\psi^\dagger, \; [\psi, \psi^\dagger] = 1$. Remarkably, (3.15) with this realization generates yet another *simple lattice NLS* model with Lax operator (2.24).

3.) Combining two such bosonic Lax operators (2.24), constructed above: $L^{(1)}(\lambda)L^{(2)}(\lambda) = L(\lambda)$ and considering them to be inserted at a single site we can construct the Lax operator of an integrable model involving two-bosonic modes, which yields the quantum discrete self trapping model (2.25).

4.) Note that the trivial choice $m^\pm = 0$ gives again algebra (3.11) and hence the realization (3.12). This however yields from (3.15) the Lax operator of the *nonrelativistic Toda chain* (2.26) associated with the rational $R$-matrix. It is interesting to note that in [30] the Lax operators like (3.1) and (3.15) appeared in their bosonic realization and were shown to be the most general possible form within their respective class.

Therefore these quantum Lax operators are in the core of the ultralocal integrable models, both discrete and continuum, which can be constructed from them in a unified way. Models belonging to trigonometric and rational class are generated from (3.1) and its limiting form (3.15) respectively, and therefore inherit the same corresponding $R$-matrices (2.6) and (2.7).

### 3.2 Fundamental and regular models

The Lax operator $L_{aj}(\lambda)$ in general acts on the product space $V_a \otimes h_j$, of the common auxiliary space $V_a$ and the quantum space $h_j$ at site $j$. The models with $V_a$ isomorphic to all $h_j, j = 1, \ldots N$ and given by the fundamental representation are called *fundamental* models. For such models the finite-dimensional matrix representations of auxiliary and quantum spaces become equivalent and may lead to $L_{al}(\lambda) \equiv R_{al}(\lambda)$. However for clarifying a misconception prevalent in the literature, we should stress that a model is represented by its Lax operator only, the associated $R$-matrix accounts for the commutation property of the elements of this Lax operator through QYBE. Therefore even for a fundamental model the Lax operator may differ from its $R$-matrix. We may however demand for the fundamental models an useful additional property: $L_{al}(0) = P_{al}$, known as the *regularity* condition given through the permutation operator, which may be expressed in the general case as a $n^2 \times n^2$ matrix $P_{al}^{(n)} = \sum_{\beta=1}^{n} E^a_{\beta \alpha} E^l_{\beta \alpha}$, $E_{\alpha \beta}$ being a matrix with its $(\alpha, \beta)$ element as 1 and the rest 0.

Recall that the global operator $\tau(\lambda)$ is constructed from the local Lax operators $L_{aj}(\lambda)$, $j = 1, \ldots N$ as $\tau(\lambda) = tr_a (L_{a1}(\lambda) \ldots L_{aN}(\lambda))$, where the transfer matrix $\tau(\lambda)$ acts on the total quantum space $\mathcal{H} = \otimes_{j=1}^{N} h_j$. Therefore, from the knowledge of Lax operators it is possible to derive all conserved quantities including the Hamiltonian, which in general would be nonlocal objects. The above regularity condition on Lax operators however allows to overcome this difficulty and obtain Hamiltonians with nearest-neibour (NN) interactions. Let us pay some special attention to this specific group of models since, as we will see below, the most important integrable models applicable to condensed matter physics related problems are given by the regular models with $n = 2, 3, 4$ etc. For this reason, though our main concern in this paper is $2 \times 2$ auxiliary matrix space, we describe
here more general $n$ cases and demonstrate that Hamiltonians of such different physical models interestingly have similar forms, when expressed through the permutation operator $P_{jj+1}$. Such a permutation operator exhibits space interchanging property $P_{aj}L_{ak} = L_{jk}P_{aj}$, along with $P^2 = 1$ and $tr_a(P_{aj}) = 1$. For all regular and periodic models, using the freedom of cyclic rotation of matrices under the trace, we can express the transfer matrix as

$$\tau(0) = tr_a(P_{aj}P_{aj+1}...P_{aN}P_{a1}...P_{a(j-1)}) = (P_{jj+1}...P_{jN}P_{j1}...P_{jj-1})tr_a(P_{aj})$$

(3.17)

for any $j$ and as its derivative with respect to $\lambda$ we similarly get

$$\tau'(0) = tr_a \sum_{j=1}^{N} \left( P_{aj}L'_{aj+1}(0)...P_{aN}P_{a1}...P_{a(j-1)} \right) = \sum_{j=1}^{N}(L'_{jj+1}(0)...P_{jN}P_{j1}...P_{jj-1})tr_a(P_{aj}),$$

(3.18)

where the periodic boundary condition: $L_{aN+j} = L_{aj}$ is assumed. Defining now $H = c_1 = \frac{d}{d \lambda} \ln \tau(\lambda)|_{\lambda=0} = \tau'(0)\tau^{-1}(0)$ and using (3.17), (3.18) we may construct the related Hamiltonian as

$$H = \sum_{j=1}^{N} L'_{jj+1}(0) P_{jj+1}$$

(3.19)

with only NN interactions, where all nonlocal factors are canceled out due to relevant properties of the permutation operator. Similarly taking higher derivatives of $\ln \tau(\lambda)$, higher conserved quantities $c_j, j = 2, 3, \ldots N$ can be constructed for these regular models. Note that the conserved operator $c_j$ involves interactions of $j + 1$ neighbors.

For the simplest case of $n = 2$, we may take the Lax operator as the $R$-matrix given by (2.5), which satisfies clearly the regularity condition $L(0) \equiv R(0) = P$ for both trigonometric and rational cases. Moreover for (2.6) the part $L'_{jj+1}(0)$ in (3.19) introduces anisotropy reproducing the Hamiltonian of the $XXZ$ spin chain (2.13). However since for the rational case (2.7) $L'_{jj+1}(0) = 1$, using the expression $P_{jj+1}^{(2)} = \sum_{\alpha=1}^{2} E_{\alpha}^j E_{\alpha}^{j+1} \equiv \frac{1}{2}(H_{jj+1}^\sigma + 1)$, where $H_{jj+1}^\sigma = \tilde{\sigma}_j \tilde{\sigma}_{j+1}$, (3.19) is reduced clearly to the isotropic spin-$\frac{1}{2}$ Hamiltonian $H^\sigma = \sum_j H_{jj+1}^\sigma$ (2.21).

It is intriguing to note that for the rational class the same form of Hamiltonian $H = \sum_{j=1}^{N} P_{jj+1}^{(n)}$ with several higher values of $n$ describes most of the important integrable models, though their physical forms are given mainly through various representations of the permutation operator. For example, for $n = 3$ corresponding to $SU(3)$ group we can express the permutation operator $P_{jj+1}^{(3)} = \sum_{\alpha=1}^{3} E_{\alpha}^j E_{\alpha}^{j+1}$ through spin-1 operators $\tilde{S}$ giving a variant of the integrable spin-1 model

$$H = \sum_j \tilde{S}_j \tilde{S}_{j+1} + \epsilon(\tilde{S}_j \tilde{S}_{j+1})^2,$$

(3.20)

with $\epsilon = \pm 1$.

Considering a supersymmetric invariant $gl(1, 2)$ case, i.e. realizing the corresponding graded permutation operator $P_{jj+1}^{(1,2)}$ using fermionic $(c_{aj}, c^\dagger_{aj})$, $a = \uparrow, \downarrow$ and spin $\tilde{S}$ operators we may construct again from the Hamiltonian density $H_{jj+1}^{1j} = (2P_{jj+1}^{(1,2)} - 1)$ the well known integrable $t - J$ model [31]

$$H_{jj+1}^{1j} = \sum_j H_{jj+1}^{1j} = \sum_j -tP \left( \sum_{\sigma=\uparrow, \downarrow} c_{aj}^\dagger c_{a(j+1)} + h.c. \right) P + J \left( S_j S_{j+1} - \frac{1}{4} n_j n_{j+1} \right) + n_j + n_{j+1}$$

(3.21)
with $J = 2t = 2$, where $P$ projects out the double occupancy states.

A different 4-dimensional realization of the fermion operators on the other hand converts the same Hamiltonian to an integrable correlated electron model proposed in [32].

Similarly for $n = 4$, i.e. for $SU(4)$, realizing $P_{jj+1}^{(4)} = P_{jj+1}^{(2)} \otimes P_{jj+1}^{(2)}$ in the factorized form we can get

$$ H = \sum_j P_{jj+1}^{(4)} = \frac{1}{4} \sum_j (H_{jj+1}^{(\sigma)} + 1)(H_{jj+1}^{(\tau)} + 1) = \frac{1}{4} [H^{\sigma} + H^{\tau} + \sum_j (H_{jj+1}^{(\sigma)} H_{jj+1}^{(\tau)} + 1)] ,$$

(3.22)

with $H^{\sigma,\tau}$ representing isotropic spin-$\frac{1}{2}$ Hamiltonians (2.21). Adding now interaction along the rung:

$$ H_{rung} = J \sum_j \vec{\sigma}_j \vec{\tau}_j, \quad [H, H_{rung}] = 0 \text{ to (3.22), where } \sigma, \tau \text{ represent the spins along two legs of the ladder, we may construct a model which is nothing but the integrable spin-$\frac{1}{2}$ ladder discovered recently [33].}

On the other hand, from the same form of Hamiltonian but by considering a supersymmetric extension $SU(2,2)$ we may realize $P_{jj+1}^{(2,2)}$ again through fermion operators $(c_{aj}, c_{aj}^\dagger)$, $a = \uparrow, \downarrow$, to construct an integrable extension of the Hubbard model proposed in [34].

One can repeat the above construction of the spin-ladder model for generating also an integrable $t-J$ ladder model introduced in [35], which would therefore corresponds to a similar construction in $n = 6$ with Hamiltonian

$$ H = \sum_j P_{jj+1}^{(6)} = \frac{1}{4} \sum_j (H_{jj+1}^{(1)} + 1)(H_{jj+1}^{(2)} + 1) = \frac{1}{4} [H^{(1)} + H^{(2)} + \sum_j (H_{jj+1}^{(1)} H_{jj+1}^{(2)} + 1)] ,$$

(3.23)

where $H^{(a)\mu\nu}, a = 1, 2$ are $t-J$ Hamiltonians (3.21) along two legs. Adding a suitable $H_{rung}$ to (3.23) with $[H, H_{rung}] = 0$, defining the interaction along the rung we finally obtain the integrable $t-J$ ladder model.

Apart from the above applications of integrable systems having similar structure from the algebraic point of view, we should mention some other important models like Hubbard model and the Kondo problem, which also falls in the class of exactly solvable problems in one-dimension [36]. Employing further twisting and gauge transformations on multi-fermion or multi-spin integrable models one can generate another type of integrable models of current interest [37]. Importance of solvable models in physical systems, their relevance to experiments and related issues are discussed in [38]. For detailed and involved application of Bethe ansatz technique including that for the theory of correlation functions to various integrable systems like $\delta$-bose gas, NLS, sine-Gordon etc. the readers are referred to [3].

### 3.3 Fusion method

We have constructed spin, boson as well as the q-spin and q-boson models through realization of particular Lax operators with inequivalent auxiliary and quantum spaces. However in case of finite-dimensional higher rank spin representations there exists an intriguing method, known as the fusion method, for obtaining higher spin models by fusing the elementary $R$-matrices like (2.5). Thus by
fusion of only the quantum spaces one can construct spin-s Lax operators with a spin-$\frac{1}{2}$ auxiliary space, as obtainable also directly from (3.1) as a particular realization. Fusing further the auxiliary spaces the higher-spin Lax operator with spin-s auxiliary space may be constructed as

$$L_{ab} = (P^+_a \otimes P^+_b) \prod_{j=1}^{s} \prod_{k=1}^{s} R_{aj,bk}(\lambda + i\alpha(2s - k - j))(P^+_a \otimes P^+_b)g_{2s}(\lambda)$$

(3.24)

with $P^+_a(b)$ as the symmetrizer in the fused spin-s space $a(b)$ and $g_{2s}$ some normalizing factor [39]. For the rational $R$-matrix corresponding to (2.7) one obtains from (3.24) the integrable spin-s Babujian-Takhtajan model [39], which for $s = 1$ may be given in the same form as Hamiltonian (3.20), but with $\epsilon = -1$. Similarly for the trigonometric case (2.6) the fused model would correspond to integrable anisotropic higher-spin chain.

It may however be stressed that such fusion technique, as far as we know, has not been formulated yet for bosonic and q-bosonic models. Such extension, at least for the restricted values of $q$, needs therefore more attention.

3.4 Construction of classical models

The systematic procedure for constructing quantum integrable models as various reductions of the same ancestor model, as described here, is applicable naturally also to the corresponding classical models by taking the classical limit $\hbar \to 0$. At this limit all field operators would be transformed to ordinary functions with their commutators reducing to the Poisson brackets. Note also that parameter $\alpha$ appearing in the $R$-matrix is scaled actually as $\hbar \alpha$, which yields the classical $r$-matrix: $R(\lambda) = I + \hbar r(\lambda) + O(\hbar^2)$ and reduces QYBE (2.1) to its classical limit $\{L_{ai}(\lambda), L_{bj}(\mu)\} = \delta_{ij}[r_{ab}(\lambda - \mu), L_{ai}(\lambda)L_{bj}(\mu)]$. The classical Lax operator reduced from (3.1) would remain however almost in the same form, though the corresponding quantum algebras would change into their corresponding Poisson algebras. This aspect of classical integrable systems is given in great detail in the excellent monograph [40]. Using these classical analogs of quantum systems one can therefore apply the algebraic scheme formulated above for generating quantum models also in classical context and construct systematically the corresponding classical integrable models [41].

4 Integrable statistical systems: vertex models

$D$-dimensional quantum systems are known to be related to $(1 + D)$-dimensional classical statistical models, which is true naturally also for $D = 1$, where the integrability of models might get manifested. Interestingly, integrable quantum spin chain and the corresponding vertex model share the same quantum $R$-matrix and have the same representation for the transfer matrix, commutativity of which: $[\tau(\lambda), \tau(\mu)] = 0$ guarantees their integrability. However, while the spin chain Hamiltonian $H_s$ is expressed through the transfer matrix as $ln\tau(\lambda) = I + \lambda H_s + O(\lambda^2)$, the partition function $Z$ of the vertex model is constructed from $\tau(\lambda)$ as $Z = tr(\tau(\lambda)^M)$. The known integrable vertex models are usually related to the quantum fundamental models described above.

In conventional vertex models each bond connecting $N \times M$ arrays in a 2-dimensional lattice can take $n$ different possible random values with certain probabilities, which for a configuration $i,j;k,l$
of bonds meeting at each vertex point is given by the Boltzmann weights $w_{ij,kl}$. These Boltzmann weights may be assigned as matrix elements $w_{ij,kl}(\lambda) = R_{kl}^{ij}(\lambda)$ of a $R$-matrix (though it might be of a more general $L$-operator, as we will see below), which for integrable models must satisfy the Yang-Baxter equation (2.1) and correspond to a quantum integrable model. The partition function of these vertex models may be expressed as $Z = \sum_{\text{config}} \prod_{a,b,j,k} \omega_{a,j,b,k}(\lambda)$.

The simplest among the vertex models for $n = 2$ is the 6-vertex model [1], which corresponds to the XXZ spin chain and may be defined on a square lattice with a random direction on each bond (left or right on the horizontal, up or down on the vertical), constrained by the ice rule, that the number of incoming and outgoing arrows at each vertex are the same. This leaves only 6 possible configurations and the corresponding Boltzmann weights may be given by 6 nontrivial matrix elements of the $R$-matrix (2.5) with (2.6). It is fascinating that this model may describe the possible configurations of Hydrogen (H) ions around Oxygen (O) atoms in an ice crystal having two different (close-removed) positions of the H-ions relative to the O-atom in the H-bonding, while the ice rule corresponds to the charge neutrality of the water molecule.

A more general 6-vertex model may be obtained if instead we assign its Boltzmann weights directly to the spin-1/2 matrix representation of the general ancestor Lax operator (3.1). The parameters $c_1^+ = -c_1^- = \rho_+$, $c_2^+ = -c_2^- = \rho_-$ present in the Lax operator may be combined to serve as the horizontal $h \sim \ln \rho_+ \rho_-$ and vertical $v \sim \ln \frac{\rho}{\rho}$ fields acting on the model, which recovers amazingly the most general 6-vertex model proposed many years ago [42] through a different construction. This also confirms the fact that the Lax operator (3.1) is indeed in the core of integrable quantum as well as statistical models. Using twisting transformation one can recover also the 6V(1) vertex model introduced in [27].

We may consider higher vertex models with $n > 2$, which may be obtained from the $R$-matrix (or the Lax operators) of the corresponding quantum integrable fundamental models with higher-dimensional auxiliary spaces. The well known examples are the 19-vertex model [43] related to the Babujian-Takhtajan integrable spin-1 model [39], the Boltzmann weights of which may be given by the matrix elements of Lax operator (3.24) with $s = 1$. Similarly one may construct the vertex models equivalent to the Hubbard model, supersymmetric t-J model, Bariev chain etc. [44].

In a following section new type of vertex models will be constructed from our ancestor Lax operator using nonfundamental representations.

5 Directions for constructing new classes of ultralocal models

The same unified scheme described in sect. 3 for constructing integrable models may be used also to indicate various directions for generating new integrable classes of quantum and statistical models.

5.1 Inhomogeneous models

In all above constructions the central elements in the ancestor models (3.1) or (3.15) are chosen as constant parameters. However if they are chosen as site dependent (or may even time dependent) functions we can get an inhomogeneous class of models. In these cases the $c$’s would be attached with
site indices as $c'_j$'s in the Lax operators and similarly in (3.6) $M_j^\pm$ would appear as functions, leading to the corresponding inhomogeneous extensions of the known integrable models, namely inhomogeneous lattice sine-Gordon, Liouville, Toda chain, NLS model etc. However since the local algebra remains same as the original model, they have the same quantum $R$-matrices. Though similar inhomogeneous Toda chain, NLS models etc. were proposed earlier as classical systems, they seem to be new and yet unstudied as quantum models. Recall that the impurity models proposed earlier [45] fall into this class and are obtained by a particular choice of inhomogeneous $c_j$'s which amounts to a shifting of the spectral parameter. Implementing the same idea to the XXX spin chain we notice that, if in its constructing along with $c_0^0 = 1$ we choose $c_2^1 = -c_1^1 = \epsilon_j$ resulting again $m^+ = 1, m^- = 0$, we get the same form of the Lax operator, but with a shift $L_j(\lambda - \epsilon_j)$, resulting that of the Gaudin model (2.22).

Similarly higher spin representations as well as $su(1,1)$ variant would yield other generalizations of the same model. The commuting set of Hamiltonians for the Gaudin model may be generated from its transfer matrix at the limit $\alpha \to 0$ [18] as $H_j = \alpha^2 (\prod_{k,j} \frac{1}{(\lambda - \epsilon_k)} )^\tau (\lambda \to \epsilon_j)$, $j = 1, 2, \ldots, N$. Remarkably, the Gaudin model may be mapped into the integrable BSC model, which is of immense contemporary interest [28].

Physically such inhomogeneities may be interpreted as impurities, varying external fields, incommensuration etc.

### 5.2 Hybrid models

Another way of constructing new models is to use different realizations of algebras (3.2) or (3.14) at different lattice sites, depending on the type of the $R$-matrix. For example one may consider spin-$\frac{1}{2}$ and spin-1 representations of $su(2)$ at alternate lattice sites, which was realized actually in [29]. However we can build more general inhomogeneous integrable models by considering different underlying algebras and different Lax operators at differing sites. The basic idea is that the Lax operators representing different models that are descended from the same ancestor model and share the same $R$-matrix can be combined together to build various hybrid models preserving quantum integrability. For example, we may consider fermion-boson or spin-boson interacting models by inserting alternatively spin-$\frac{1}{2}$ and bosonic (or q-bosonic) Lax operators at alternate sites. One of such physical constructions would be the celebrated Jaynes-Cummings model. It is possible also to construct some exotic hybrid integrable models, an example of which could be a hybrid sine-Gordon-Liouville model, where for $x \geq 0$ it would follow the sine-Gordon dynamics, while for $x < 0$ the Liouville dynamics!

### 5.3 Nonfundamental statistical models

Vertex models, as mentioned, are described generally by the $R$-matrix of a regular quantum integrable model. However one can construct a new class of integrable vertex models by exploiting a richer variety of nonfundamental systems, where we define the Boltzmann weights as matrix elements of the generalized Lax operator (3.1): $L_{ab}^{jk}(u) = \omega_{a,jib,k}(u)$, with the use of the explicit matrix representation for the basic operators $S^\pm, S^3$ as

$$<s, \bar{m}|S^3|m, s> = m\delta_{m,\bar{m}}, \quad <s, \bar{m}|S^\pm|m, s> = f^\pm_s(m)\delta_{m,\pm\bar{m}}. \quad (5.1)$$
Here \( f^+_s(m) = f^-_s(m + 1) \equiv g(m) \) is defined as in (3.6). Such general Boltzmann weights would now represent an ancestor vertex model analogous to the quantum case and would generate through various reductions new series of vertex models, linked to q-spin and q-boson with generic \( q \), \( q \) roots of unity and \( q \to 1 \) [46]. In all these models, generalizing the usual approach the horizontal (h) and vertical (v) links may become inequivalent and independent at every vertex point. The h links, which are related to the auxiliary space admit 2 values, while the v links, which correspond to the quantum space may have richer possibilities with \( j, k \in [1, D] \), \( D \) being the dimension of the nonfundamental matrix-representation of the q-algebras. The familiar ice-rule is generalized here as the ‘color’ conservation \( a + j = b + k \) for determining nonzero Boltzmann weights. Note that, alternatively finite-dimensional higher spin and q-spin vertex models can also be constructed using the fusion technique [39].

An interesting possibility of regulating dimension for the matrix representation opens up at \( q^p = \pm 1 \), when a variety of new q-spin and q-boson vertex models with finite-dimensional representation can be generated [46].

As in quantum models we can also construct here a rich collection of hybrid models by combining different vertex models of the same class and inserting their defining Boltzmann weights along the vertex points \( l = 1, 2, \ldots, N \) in each row, in any but in the same manner. Due to the association with the same \( R \)-matrix the integrability of such statistical models is naturally preserved.

6 Unified Bethe ansatz solution

In physical models our aim usually is to solve the eigenvalue problem for the Hamiltonian only. Solvable models allow such exact solutions \( H \mid m \rangle = E_m \mid m \rangle \) through coordinate formulation of the Bethe ansatz (CBA) [47], which was used successfully in many condensed matter physics related problems like spin-chain, attractive and repulsive \( \delta \)-Bose gas, Hubbard model etc [48]. Nevertheless CBA depends heavily on the structure of the Hamiltonian of individual models and lacks consequently the unified approach of its algebraic formulation. We would focus here briefly only on the algebraic Bethe ansatz (ABA) [2, 3], which under certain conditions can solve the eigenvalue problem for the spectral parameter-dependent transfer matrix \( \tau(\lambda) \mid m \rangle = \Lambda_m(\lambda) \mid m \rangle \) and hence through its expansion the eigenvalue problem for the whole set of conserved operators, simultaneously. Moreover, the ABA due to its predominantly model-independent features, which we will demonstrate below, appears to be a fairly universal method.

Since the eigenvectors are common for all commuting conserved operators, by expanding \( \text{ln} \Lambda_m(\lambda) \) simply as

\[
c_1 \mid m \rangle = \Lambda'_m(0)\Lambda^{-1}_m(0) \mid m \rangle, \quad c_2 \mid m \rangle = (\Lambda'_m(0)\Lambda^{-1}_m(0))' \mid m \rangle
\]

etc. we obtain their respective values, where one may take \( H = c_1 \) or other combinations of \( c \)’s as the Hamiltonian, depending on the concrete model. This powerful method applicable to both integrable quantum and statistical systems requires however explicit knowledge of the associated Lax operator and the \( R \)-matrix.

It may be noticed that the off-diagonal element \( B(\lambda) \) (\( C(\lambda) \)) of the monodromy matrix (2.3) acts generally like creation (annihilation) operator for the pseudoparticles, induced by the local creation
(annihilation) operator as the matrix elements in $L_j(\lambda)$ acting on the quantum space at $j$. Therefore the $m$-particle state $|m \rangle$ may be created by acting $m$ times with $B(\lambda_a)$ on the pseudovacuum $|0 \rangle = \prod_j^N |0 \rangle_j$, giving $|m \rangle = B(\lambda_1)B(\lambda_2) \cdots B(\lambda_m) |0 \rangle$, where we suppose the crucial annihilation condition $C(\lambda_a) |0 \rangle = 0$.

Now for solving the eigenvalue problem of $\tau(\lambda) = A(\lambda) + D(\lambda)$ exactly, we have to drag this operator through the string of $B(\lambda_a)$'s without spoiling their structure and finally hit the pseudovacuum giving $A(\lambda) |0 \rangle = \alpha(\lambda) |0 \rangle$ and $D(\lambda) |0 \rangle = \beta(\lambda) |0 \rangle$. For this purpose therefore one requires commutation relations between the elements of (2.3), which for ultralocal models may be derived from the QYBE (2.4). This, apart from ensuring the integrability of the system, is another important role played by (2.4), yielding the relations

\[ A(\lambda)B(\lambda_a) = f(\lambda_a - \lambda)B(\lambda_a)A(\lambda) - f(\lambda_a - \lambda)B(\lambda)A(\lambda_a), \]
\[ D(\lambda)B(\lambda_a) = f(\lambda - \lambda_a)B(\lambda_a)D(\lambda) - f(\lambda - \lambda_a)B(\lambda)D(\lambda_a), \]
\[ \text{(6.2)} \]

together with the trivial commutators $[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [D(\lambda), D(\mu)] = [A(\lambda), D(\mu)] = 0$ etc., where $f(\lambda) = \frac{\alpha(\lambda)}{\beta(\lambda)}$, $f_1(\lambda) = \frac{\alpha(\lambda)}{\lambda - \lambda_a}$ are combinations of the elements from the $R(\lambda)$-matrix (2.5).

We notice that (6.2) are almost the right kind of relations but for the second terms in both the RHS, where the argument of $B$ has changed spoiling the structure of the eigenvector. However, if we put the sum of all such unwanted terms $= 0$, we should be able to achieve our goal. In field models such unwanted terms vanish automatically, while in lattice models their removal amounts to the Bethe equations, which may be induced independently by the periodic boundary condition, giving

\[ \left( \frac{\alpha(\lambda_a)}{\beta(\lambda_a)} \right)^N \prod_{b \neq a} \frac{f(\lambda_a - \lambda_b)}{f(\lambda_b - \lambda_a)} = \prod_{a=1}^m f(\lambda_a - \lambda), \]
\[ \text{(6.3)} \]

Therefore the ABA finally solves the eigenvalue problem for $\tau(\lambda)$ yielding

\[ \Lambda_m(\lambda) = \left( \prod_{a=1}^m f(\lambda_a - \lambda) \right) \alpha(\lambda) + \left( \prod_{a=1}^m f(\lambda - \lambda_a) \right) \beta(\lambda), \]
\[ \text{(6.4)} \]

where the Bethe equation (6.3), which is equivalent also to the singularity-free condition of the eigenvalue (6.4) serves in turn as the set of equations for determining the parameters $\lambda_a$.

Note that in both the above equations $\alpha(\lambda) = (\langle 0 | L_j^{11}(\lambda) | 0 \rangle)^N$ and $\beta(\lambda) = (\langle 0 | L_j^{22}(\lambda) | 0 \rangle)^N$ are the only model dependent parts given by the action of the upper and lower diagonal operator elements $\hat{L}_j^{ii}(\lambda), i = 1, 2$ of the Lax operator of the model on the pseudovacuum. For vertex models, for which the ABA formulation goes parallely, the Lax operator elements in the above equations should be replaced by their matrix representations expressed through the Boltzmann weights as $\langle 0 | L_j^{11}(\lambda) | 0 \rangle = \omega_{+1,1} + 1(\lambda)$, $\langle 0 | L_j^{22}(\lambda) | 0 \rangle = \omega_{-1,-1}(\lambda)$. It is remarkable that the rest of the terms in (6.4) and (6.3) are given solely through the $R$-matrix elements $f(\lambda)$ and therefore depend only on the related class (2.6) or (2.7). Recall that in integrable models, as described in sect. 3, the $R$-matrix remains same for all models belonging to a particular class, while the $L$-operators differ and may be obtained through various reductions from the same ancestor Lax operator.

Therefore taking the Lax operator elements in (6.4) and (6.3) as those from the general Lax operator (3.1), one may consider the above eigenvalue and the Bethe equation to be the unifying
equations for exact solution of all integrable ultralocal quantum and statistical models constructed here. Consequently models like the DNLS, SG, Liouville and the XXZ chain together with the 6-vertex model, belonging to the trigonometric class (2.6) should share similar eigenvalue relations with individual differences appearing only in the form of $\alpha(\lambda)$ and $\beta(\lambda)$ coefficients. Thus this deep rooted universality in integrable systems helps to solve the eigenvalue problem for the whole class of models and for the full hierarchy of their conserved currents in a systematic way. Let us present the explicit example of XXZ chain with Lax operator (2.13), defining $|0\rangle$ as all spin up state which gives $\alpha(\lambda) = \sin^N(\lambda + \alpha)$, $\beta(\lambda) = \sin^N \lambda$ in Bethe equation (6.3) (with a shift $\lambda \rightarrow \lambda + \frac{\alpha}{2}$) resulting

$$
\left( \frac{\sin(\lambda_a + \frac{\alpha}{2})}{\sin(\lambda_a - \frac{\alpha}{2})} \right)^N = \prod_{b \neq a}^{m} \frac{\sin(\lambda_a - \lambda_b + \alpha)}{\sin(\lambda_a - \lambda_b - \alpha)}
$$

for $a = 1, 2, \ldots, m$. Similarly (6.4) gives the eigenvalue

$$
\Lambda_{m}^{XXZ}(\lambda) = \sin^N(\lambda + \alpha) \prod_{a=1}^{m} \frac{\sin(\lambda_a - \lambda + \frac{\alpha}{2})}{\sin(\lambda_a - \lambda - \frac{\alpha}{2})} + \sin^N \lambda \prod_{a=1}^{m} \frac{\sin(\lambda - \lambda_a + 3\frac{\alpha}{2})}{\sin(\lambda - \lambda_a + \frac{\alpha}{2})}
$$

yielding for $H_{xxz} = c_1$, the energy spectrum

$$
E_{xxz}^{(m)} = \Lambda_{m}(\lambda)'\Lambda_{m}^{-1}(\lambda) |_{\lambda=0} = \sin \alpha \sum_{a=1}^{m} \frac{1}{\sin(\lambda_a - \frac{\alpha}{2}) \sin(\lambda_a + \frac{\alpha}{2})} + N \cot \alpha.
$$

At the limit $\alpha \rightarrow 0$, $\sin \lambda \rightarrow \lambda$, when the $R$-matrix along with its associated models reduce to the rational class, one can derive the corresponding Bethe ansatz results by taking the rational limit of the above equations. For example the relevant equations for the isotropic XXX chain can be obtained directly from those for the XXZ chain presented above. Intriguingly the corresponding result for the NLS lattice model, which belongs to the same rational class, should also show close similarity with that of the XXX chain.

7 Quantum integrable nonultralocal models

Though many celebrated classical integrable models like KdV, mKdV, nonlinear $\sigma$-model, derivative NLS etc. belong to the class of nonultralocal models, successful quantum generalization could be made only for handful of them. The reason, as mentioned already, is the violation of the ultralocality condition. Recall that this condition helps to transit from local QYBE to its global form and consequently establish the integrability for ultralocal systems. Therefore the key equations and the related formulation for the integrability theory of the nonultralocal models must be suitably modified.

7.1 Braided extensions of QYBE

For understanding first the algebraic structures underlying the nonultralocal systems we have to note that the trivial multiplication property (3.4) valid for ultralocal models needs to be generalized here as $(A \otimes B)(C \otimes D) = \psi_{BC}(A(C \otimes B)D)$ where the braiding $\psi_{BC}$ takes into account the noncommutativity of $B_2, C_1$. In spite of such braided extension of the multiplication rule, the associated
coproduct structure of the underlying Hopf algebra, crucial for transition to the global QYBE, must be preserved. Such a braided extension of the Hopf algebra [49, 50] was implemented in formulating the integrability theory of nonultralocal models through an unified approach [51]. The basic idea is to complement the commutation rule for the Lax operators at the same site with their braiding property at different lattice sites. Note however that in general the braiding may differ widely and with arbitrarily varying ranges the picture might become too complicated for explicit description. Therefore let us limit first to the nearest-neighbor (NN) type braiding

\[ L_{2j+1}(\mu)Z_{21}^{-1}L_{1j}(\lambda) = L_{1j}(\lambda)L_{2j+1}(\mu) \]  

assuming that the ultralocality holds starting from the next neighbors. A pictorial description of this condition is given in Fig. 1a).

The local QYBE at the same time must also be generalized to incorporate the braiding relations, such that the transition to its global form becomes possible again. Such braided extension of the QYBE (BQYBE) compatible with (7.1) takes the form (see Fig. 1b) )

\[ R_{12}(\lambda - \mu)Z_{21}^{-1}L_{1j}(\lambda)L_{2j}(\mu) = Z_{12}^{-1}L_{2j}(\mu)L_{1j}(\lambda)R_{12}(\lambda - \mu). \]  

We list below the known nonultralocal integrable models that can be described by the above braided equations. Note that the quantum R-matrix appearing here is the same (2.5) as for the ultralocal systems. However the additional braiding matrix Z, unlike the R-matrix seems to be model-dependent and generally independent of the spectral parameter, though similar to the R-matrix it satisfies the YBE like equations and might also become spectral parameter dependent for specific models [51].

The next step is the global extension of the BQYBE for the monodromy matrix (2.3) and it is not difficult to check that due to the braiding relation (7.1), the form of BQYBE is preserved for global matrices like \( T_{a}^{[k,j]}(\lambda) = \prod_{j=1}^{N} L_{aj}(\lambda) \) (see Fig. 1c)). However since for the periodic boundary condition one imposes \( L_{aN+1}(\lambda) = L_{a1}(\lambda) \), the Lax operators \( L_{aj}(\lambda) \) for \( j = 1 \) and \( j = N \) again become NN entries and hence modify the equation due to the appearance of an extra Z matrix from the braiding relation (7.1), leading finally to the global BQYBE

\[ R_{12}(\lambda - \mu)Z_{21}^{-1}T_{1}(\lambda)Z_{12}^{-1}T_{2}(\mu) = Z_{12}^{-1}T_{2}(\mu)Z_{21}^{-1}T_{1}(\lambda)R_{12}(\lambda - \mu). \]  

Though this equation is similar to (2.4), the commutation of the transfer matrices ensuring the integrability of the systems through factorization of the trace identity becomes problematic due to the presence of Z-matrix. Detail discussion of this problem and the classification of the Z-matrices allowing factorization is given in [51]. Investigations of some nonultralocal systems from a different angle were done in [52]. It is easy to see that from the corresponding equations for the nonultralocal models presented above one can recover the known relations for the ultralocal models by supposing the braiding matrix \( Z = 1 \) (see also the caption in Fig 1)
Fig. 1: Pictorial description of a) braiding relation (7.1), b) local braided QYBE (7.2) for the Lax operators $L_{aj}(\lambda_a)$ and c) global braided QYBE for $T_{a[1,k]}(\lambda_a) = \prod_{j=1}^{k} L_{aj}(\lambda_a)$, $k < N$. Note that putting $Z = 1$, i.e. removing braiding by undoing the crossing of dashed lines in above figures 1a,b,c) one can recover the corresponding pictures for the ultralocal models [1], namely ultralocality condition (1.2), local (2.1) and global QYBE (2.4), respectively.
7.2 List of quantum integrable nonultralocal models

Nonultralocal models are mostly nonfundamental systems with infinite dimensional representations defined in some Hilbert space. They may correspond to integrable models with spectral parameter dependent Lax operator and \( R(\lambda) \)-matrix or may describe only nonultralocal algebras having spectral parameterless \( L \)-operator and \( R(\lambda)_{\lambda \to +\infty} \rightarrow R_\ell^+ \)-matrix. Nevertheless the nonultralocal quantum models listed below should be described through the same braided relations (7.1,7.2,7.3) or their corresponding spectral-less form in a systematic way. Therefore we present only the explicit form of their braiding matrix \( Z \) and the \( L \) operator, indicating the class of \( R \)-matrix they belong to. These inputs should be enough to obtain all individual equations and derive the related results.

I. Systems with spectral parameterless \( R \)-matrix

1. Current algebra in WZWN model [53]

   The model involves the nonultralocal current algebra

   \[
   \{L_1(x), L_2(y)\} = \frac{\gamma}{2} [C, L_1(x) - L_2(y)] \delta(x - y) + \gamma C \delta'(x - y) \tag{7.4}
   \]

   with \( C_{12} = 2P_{12} - 1 \), where \( P_{12} \) is the permutation operator, \( L = \frac{1}{2}(J_0 + J_1) \) with \( J_\mu = \partial_\mu g g^{-1} \), is the current and \( g \in SU(N) \) the chiral field. Discretized and quantized version of this algebra may be cast as the spectral-free limit of the above braided YBE relations with \( R_\ell^+ \) as the R-matrix, current \( L \) as the Lax operator and \( Z_{12} = R_{q_{21}}^+ \) as the braiding matrix, which takes the form

   \[
   R_{q_{21}}^+ L_{1j} L_{2j} = L_{2j} L_{1j} R_{q_{12}}^+, \quad L_{1j} L_{2j+1} = L_{2j+1} (R_{q_{12}}^+)^{-1} L_{1j} \tag{7.5}
   \]

   For the details and an interesting quantum group relation of this model the readers are referred to the original works [53].

2. Coulomb gas picture of CFT [54]

   The Drinfeld-Sokolov linear problem : \( Q_x = \mathcal{L}(x)Q \) describing this system may be given in the simplest case by the linear operator \( \mathcal{L}(x) = v(x) \sigma^3 - \sigma^+ \) with a nonultralocal property due to current-like relation \( \{v(x), v(y)\} = \delta'(x - y) \). Discretized and quantized forms of the current-like operator defined through the commutation relations

   \[
   [v^\pm_k, v^\pm_l] = \pm i \frac{\alpha}{2} (\delta_{k,l+1} - \delta_{k+1,l}), \quad [v^+_k, v^-_l] = \pm i \frac{\alpha}{2} (\delta_{k+1,l} - \delta_{k,l+1}) \tag{7.6}
   \]

   construct the corresponding discretized linear operator as \( L_k = e^{-iv^+_k \sigma^3} + \Delta e^{iv^+_k \sigma^+} \), which similar to the above case satisfies the spectral-free braided YBE and other relations with \( R_\ell^+ \) as \( R \) and \( Z = q^{-\sigma^3 \otimes \sigma^3} \) as the braiding matrix. Generalization of this model for \( SU(N) \) has also been constructed similarly in [54].

II. Models with rational \( R(\lambda) \)-matrix

3. Nonabelian Toda chain [55]

   The Lax operator of the model given by

   \[
   L_k(\lambda) = \begin{pmatrix} \lambda - A_k & -B_{k-1} \\ 0 & I \end{pmatrix}, \quad A_k = \hat{g}_k g_k^{-1}, \quad B_k = g_{k+1} g_k^{-1}, \quad g_k \in SU(N), \tag{7.7}
   \]
represents nonultralocal integrable model and solves all braided relations including the BQYBE with spectral dependent rational $R(\lambda) = P - i\hbar\lambda I$ and the braiding matrix $Z_{12} = 1 + i\hbar(e_{22} \otimes e_{12})\pi$, where $P$ and $\pi$ are permutation operators. For further details on this model including its gauge relation with an ultralocal model we refer to the original work [55].

4. **Nonultralocal quantum mapping** [56]

The system is described by the Lax operator $L_n = V_{2n}V_{2n-1}$, with $V_n = \lambda_n\sigma^- + \sigma^+ + \frac{1}{2}v_n(1 + \sigma^3)$, where the discretized operator $v_k \equiv v_k^\pm$ involves nonultralocal algebra (7.6) and yields at the continuum limit $\Delta \to 0$ the current-like field: $v_k \to i\Delta v(x)$. This nonultralocal quantum integrable model satisfies again integrable braided relations with spectral-dependent rational $R(\lambda_1 - \lambda_2)$-matrix similar to the above case but now with a spectral dependent braiding matrix $Z_{12}(\lambda_2) = I - \frac{\hbar}{\lambda_2}\sigma^- \otimes \sigma^+$ and $Z_{21}(\lambda_1)$. For generalization of this model to higher rank groups and other details we refer again to the original work [56].

III. **Models with trigonometric $R(\lambda)$-matrix**

5. **Quantum mKdV model** [57]

This well known nonultralocal model may be raised to the quantum level with discrete Lax operator

\[
L_k(\xi) = \begin{pmatrix}
(W_k^-)^{-1} & i\Delta \xi W_k^+ \\
-i\Delta \xi (W_k^+)^{-1} & W_k^-
\end{pmatrix},
\]

where $W_j^\pm = e^{iv_j^\pm}$ with $v_k^\pm$ obeying the nonultralocal relations like (7.6). $R$-matrix (2.6) and the braiding matrix $Z_{12} = Z_{21} = q^{-\frac{1}{2}\sigma^3 \otimes \sigma^3}$ are associated with this nonultralocal integrable system [57]. Bethe ansatz solution of quantum mKdV and its generalizations can be found in detail in [58]. It is seen easily that one can recover the well known Lax operator of the mKdV field model: $U(x, \xi) = \frac{i}{2}(iv(x)\sigma^3 + \xi\sigma^2)$ from (7.8) at the field limit when $v_k^\pm \to \sqrt{\pi}\Delta v(x)$, as $L_k = I + \Delta U(x, \xi) + O(\Delta^2)$.

6. **Quantum light-cone sine-Gordon model**

It is known that this well known equation: $\partial^2_u u = 2\sin 2u$ may be represented by the zero curvature condition: $\partial_- U_+ - \partial_+ U_- + [U_+, U_-] = 0$ of the Lax pair $U_{\pm}$ with $U_-(x) = e^\frac{i}{2}\partial_- u(x)\sigma^3 + \xi(e^{-iu(x)}\sigma^+ + e^{iu(x)}\sigma^-)$ and similarly for $U_+(x)$. Recently quantum as well as exact lattice versions of the nonultralocal Lax operator have been constructed [23], which in particular for $U_-(x)$ may be given in the form

\[
L_j^{(\text{-lcsg})}(\lambda) = e^{i(p_j - \alpha \nabla u_j)\sigma^3 + \Delta \xi \left(e^{-i(p_j + \alpha u_{j+1})}\sigma^+ + e^{i(p_j + \alpha u_{j+1})}\sigma^-\right)}, \quad \nabla u_j \equiv u_{j+1} - u_j.
\]

It may be shown also that (7.9) obeys exactly the above BQYBE and the braiding relation with the trigonometric $R$-matrix (2.6) and the braiding matrix $Z_{12}^{(\text{-lcsg})} = e^{i\sigma^3 \otimes \sigma^3}$, and consequently represent a genuine quantum integrable nonultralocal model.

Some other nonultralocal models known in the literature need introduction of braiding beyond NN, basic formulation of which can be found in [50, 51]. Examples of such models having same braiding between any two different sites are i.) **Integrable model on moduli space** [59], ii.) **Supersymmetric models** [60, 51], iii.) **Braided algebra** [49], iv) **NUL extension of YBE** [61] etc. Their unified description can be found in [51, 62].
7.3 Algebraic Bethe ansatz

The solution of the eigenvalue problem for integrable nonultralocal models by diagonalizing the transfer matrix may be formulated through algebraic Bethe ansatz exactly in analogy with the ultralocal models, whenever the factorization of the trace problem, as mentioned above, could be resolved. The key equation that is to be used for nonultralocal models for finding the commutation relations analogous to (6.2) in the ABA scheme should naturally be given by BQYBE (7.3). We however skip all details of this ABA formulation for nonultralocal models, which can be found in explicit form on the example of the nonultralocal quantum mKdV model in [57, 58].

7.4 Open directions in nonultralocal models

Since some of the nonultralocal models like nonabelian Toda chain, WZWN current algebra, mKdV etc. described above can be connected to ultralocal models through operator dependent local gauge transformation, it would be challenging to discover similar relation, if any, for the rest of the quantum integrable nonultralocal models [23].

Other challenging problems undoubtedly are the possible quantum integrable formulation of the famous nonultralocal models like nonlinear $\sigma$-model, complex sine-Gordon model, derivative NLS equation etc. through braided YBE.

As we know, there is a remarkable interconnection between the integrable quantum and statistical models. However this connection is discovered until now only for the ultralocal models as we have also seen here. Therefore it should be a new direction of study to investigate whether there could be any meaningful statistical model corresponding to the integrable nonultralocal models described here.

Another problem worth looking into would be to formulate fundamental nonultralocal models, if any, which then could be used possibly for generalizing spin and electron models with nonultralocality.

Anyway since this vast branch of integrable systems has received significantly insufficient attention, we may hope to have many hidden excitements in this area.

8 Concluding remarks

Quantum integrable systems can be divided into two broad classes, ultralocal (UL) and nonultralocal (NUL). We have presented here a brief description of such models with references for further details and demonstrated that the models belonging to both these classes can be described systematically through a set of algebraic relations signifying integrability of these systems. For UL models these relations are the ultralocality condition and the QYBE involving Lax operator $L$ and the $R$ -matrix, while for NUL models they are extended to braiding relation and braided QYBE with an additional entry of braiding matrix $Z$. The $L$ operator representing an individual model is naturally model dependent and the same seems to be true also for the $Z$ matrix. The $R$-matrix on the other hand is mainly of two types (elliptic case is not considered here), trigonometric and rational depending on the class of models that are associated with q-deformed and undeformed algebras, respectively. This induces a significantly model-independent approach also in the ABA method for solving the
eigenvalue problem. For UL systems, the theory of which is more developed, one can go beyond and prescribe an unifying algebraic scheme for generating individual Lax operators realized from a single ancestor model in a systematic way. It would be a challenge to extend the formulation of this scheme also for the NUL models. The integrable statistical vertex models can be related to the corresponding quantum models, which as a rule belong to UL systems. Possible systematic extension of such relation to NUL systems would be another challenging problem.

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