The Kagan characterization theorem on Banach spaces

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Abstract

A. Kagan introduced classes of distributions $D_{m,k}$ in $m$-dimensional space $\mathbb{R}^m$. He proved that if the joint distribution of $m$ linear forms of $n$ independent random variables belong to the class $D_{m,m-1}$ then the random variables are Gaussian. If $m = 2$ then the Kagan theorem implies the well-known Darmois-Skitovich theorem, where the Gaussian distribution is characterized by the independence of two linear forms of $n$ independent random variables. In the paper we describe Banach spaces where the analogue of the Kagan theorem is valid.

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1 Introduction

In 1953 G. Darmois and V.P. Skitovich independently proved the following theorem which characterizes the Gaussian distribution on the real line.

The Darmois-Skitovich theorem ([2], [28], see also [14] Sec. 3) Let $\xi_1, \ldots, \xi_n$, $n \geq 2$, be independent random variables. Let $a_j, b_j$ be nonzero numbers. If linear forms $L_1 = a_1\xi_1 + \cdots + a_n\xi_n$ and $L_2 = b_1\xi_1 + \cdots + b_n\xi_n$ are independent then $\xi_j$ are Gaussian random variables.

The generalizations of this characterization theorem to various algebraic structures were considered in numerous works (see, e.g., [5] – [8], [19]– [21], [23], for more references one can see [4]), in which the coefficients of forms are topological automorphisms of the corresponding algebraic structures. In 1962, S.G. Ghurye and I. Olkin generalized the Darmois-Skitovich theorem to the case of random variables taking values in $\mathbb{R}^n$, where the coefficients of forms are non-degenerate matrices ([8]). Later, A. Zinger simplified their proof (see [14], Sec. 3.2). The Darmois-Skitovich theorem was also generalized to the case of random variables taking values
in a Hilbert space (see [16], [18]). In 1985, W. Krakowiak generalized the Darmois-Skitovich theorem to the case of random variables taking values in an arbitrary separable Banach space $X$, where the coefficients of forms are invertible linear bounded operators $A_j$ and $B_j$ ([17]). In 2008, M. Myronyuk gave another proof of the Darmois-Skitovich theorem on Banach spaces ([20]). In this paper the Heyde theorem on Banach spaces was proved as well. Later in [23] the analogues of these theorems were proved for Q-independent random variables.

Consider a random vector $L = (L_1, ..., L_m)$. The distribution $\mu_L$ of the random vector $L$ belongs to the class $D_{m,k}$, $1 \leq k \leq m$, if the characteristic function $\hat{\mu}_L(t_1, ..., t_m)$ admits the following factorization

$$\hat{\mu}_L(t_1, ..., t_m) = \mathbb{E}[e^{i\langle L, (t_1, ..., t_m) \rangle}] = \prod_{i_1, ..., i_k} R_{i_1, ..., i_k}(t_{i_1}, ..., t_{i_k}), \quad t_i \in \mathbb{R},$$

where $R_{i_1, ..., i_k}(t_{i_1}, ..., t_{i_k})$ are continuous functions such that $R_{i_1, ..., i_k}(0, ..., 0) = 1$, and all indexes $(i_1, ..., i_k)$ in this product satisfy the conditions $1 \leq i_1 < \cdots < i_k \leq m$. A. Kagan proved the following generalization of the Darmois-Skitovich theorem.

The Kagan theorem ([13]) Let $\xi_1, ..., \xi_n$, $n \geq 2$, be independent random variables. Let $A_{j_1}$ be nonzero numbers. Put $L_j = A_{j_1}\xi_1 + \cdots + A_{jn}\xi_n$, $j = 1, 2, ..., m$. If $L = (L_1, ..., L_m) \in D_{m,m-1}$ then all $\xi_j$ are Gaussian random variables.

The class $D_{2,1}$ contains all distributions of $\mathbb{R}^2$ with independent components. Therefore the Darmois-Skitovich is a partial case of the Kagan theorem. Note that the Kagan theorem was generalized for locally compact Abelian groups ([6]). In this paper we prove analogues of the Kagan theorem for Banach spaces.

2 Notation and definitions

We use standard facts of the theory of probability distributions in a Banach space (see, e.g., [29]). Let $X$ be a separable real Banach space and let $X^*$ be its dual space. Denote by $L_{inv}(X)$ the set all linear bounded invertible operators in $X$. Denote by $A^*$ the adjoint operator to a linear bounded operator $A$. Denote by $I$ the identity operator. Denote by $\langle x, f \rangle$ the value of a functional $f \in X^*$ at an element $x \in X$. Denote by $\mathcal{M}^1(X)$ the convolution semigroup of probability distributions on $X$. For $\mu \in \mathcal{M}^1(X)$ we denote by $\overline{\mu}$ the distribution defined by the formula $\overline{\mu}(E) = \mu(-E)$ for any Borel set $E \subset X$. Let $\xi$ be a random variable with values in $X$ and with a distribution $\mu$. The characteristic functional of $\mu$ is defined by the formula

$$\hat{\mu}(f) = \mathbb{E}[e^{i\langle \xi, f \rangle}] = \int_X e^{i\langle x, f \rangle} d\mu(x), \quad f \in X^*.$$

Note that $\overline{\hat{\mu}}(f) = \hat{\overline{\mu}}(f)$. It is known that $\hat{\mu}(f)$ is positive definite and continuous in the norm topology ([29, Ch.4, §2]).

In the paper, unless otherwise specified, we work in a norm topology.
Let $\psi(f)$ be a complex-valued function on $X^*$, and let $h$ be an arbitrary element of $X^*$. We denote by $\Delta_h$ an operator of the finite difference
\[ \Delta_h \psi(f) = \psi(f + h) - \psi(f). \]

A function $\psi(f)$ on $X^*$ is called a polynomial if
\[ \Delta_h^{n+1} \psi(f) = 0 \]
for some $n$ and for all $f, h \in X^*$.

**Definition 2.1** A random variable $\xi$ is called Gaussian if for any $f \in X^*$ the real-valued random variable $\langle \xi, f \rangle$ is Gaussian. In other words $\mu \in \mathcal{M}^1(X)$ is called Gaussian if its one-dimensional image $\mu_f = \mu \circ f^{-1}$ is a Gaussian distribution for all $f \in X^*$ and hence there exist real numbers $m_f$ and $\sigma_f \geq 0$ such that $\hat{\mu}_f(t) = e^{im_f t - \frac{1}{2} \sigma_f t^2}$, $t \in \mathbb{R}$.

The following proposition follows immediately from Definition 2.1.

**Proposition 2.1** A distribution $\mu$ on $X$ is Gaussian if and only if for any $f \in X^*$ the characteristic function $\hat{\mu}_f(t)$ of a random variable $\langle \xi, f \rangle$ is a characteristic function of a Gaussian distribution on $\mathbb{R}$.

Proposition 2.1 allows easily to obtain analogues of the well-known classical theorems for a real separable Banach space such as the Kac-Bernstein theorem (II, [10]) on characterization of a Gaussian distribution by the independence of the sum of independent random variables and their difference, the Cramer theorem on the decomposition of a Gaussian distribution ([27, §4.1]), and the Marcinkiewicz theorem ([27, §3.13]). The proofs of these theorems reduces to the description of solutions of the corresponding functional equations that admit restrictions to each one-dimensional subspace.

For the convenience of references we formulate the analogues of the Cramer theorem and the Marcinkiewicz theorem for Banach spaces as propositions.

**Proposition 2.2** Let $\xi$ be a Gaussian random variable with values in a real separable Banach space $X$. Let $\xi = \xi_1 + \xi_2$, where $\xi_1, \xi_2$ are independent random variables with values in $X$. Then $\xi_1, \xi_2$ are Gaussian random variables.

**Proposition 2.3** Let $X$ be a real separable Banach space, $\mu \in \mathcal{M}^1(X)$. Let $\psi(f)$ be a polynomial on $X^*$. If
\[ \hat{\mu}(f) = e^{\psi(f)} \] (1)
in a certain neighborhood of zero, then $\mu$ is a Gaussian distribution on $X$.

Note that the anologue of the Darmois-Skitovich theorem for a Banach space can not be proved in a such simple way.
2.1 The main results

Let $X$ be a separable real Banach space. Consider a random vector $\zeta = (\zeta_1, \ldots, \zeta_m)$ in values in $X^m$. Following A.M. Kagan ([13]) we say that the distribution of the random vector $\zeta$ belongs to the class $\mathcal{D}_{m,k}$, $1 \leq k \leq m$, if the characteristic functional $\hat{\mu}_{\zeta}(f_1, \ldots, f_m)$ admits the following factorization

$$\hat{\mu}_{\zeta}(f_1, \ldots, f_m) = \mathbb{E}[e^{i\langle \zeta, (f_1, \ldots, f_m) \rangle}] = \prod_{i_1, \ldots, i_k} R_{i_1, \ldots, i_k}(f_{i_1}, \ldots, f_{i_k}), \quad f_i \in X^*, \quad 1 \leq i_1 < \cdots < i_k \leq m,$$

where $R_{i_1, \ldots, i_k}(f_1, \ldots, f_k)$ are continuous functions such that $R_{i_1, \ldots, i_k}(0, \ldots, 0) = 1$, and in the product all indexes $(i_1, \ldots, i_k)$ satisfy the conditions $1 \leq i_1 < \cdots < i_k \leq m$.

We formulate an analogue of this characterization for independent random variables with values in a Banach space.

**Theorem 2.1** Let $X$ be a real reflexive separable Banach space. Let $A_{pj}$, $p = 1, m$, $i = 1, n$, be linear continuous operators of $X$ such that $\text{Ker} A_{pj} = \{0\}$. Put

$$G_i = \{(A_{i_1}x, \ldots, A_{i_n}x) \in X^m : x \in X\}, \quad i = 1, n.$$  \hspace{1cm} (2)

Suppose that the condition

$$G_i \cap G_j = \{0\}, \quad i, j = 1, n, i \neq j.$$  \hspace{1cm} (3)

holds. Let $\xi_i$, $i = 1, 2, \ldots, n$, be independent random variables with values in $X$. Consider the linear forms $L_j = A_{j_1}\xi_1 + \cdots + A_{j_n}\xi_n$, $j = 1, 2, \ldots, m$. If the random vector $L = (L_1, \ldots, L_m) \in \mathcal{D}_{m,m-1}$ then all $\xi_i$ are Gaussian random variables.

**Theorem 2.1** is a spesial case of **Theorem 2.2** which is formulated and is proved further for $Q$-independent random variables.

Let $\xi_1, \ldots, \xi_n$ be random variables with values in separable Banach space $X$. Following A. Kagan and G. Székely ([15]), we say that random variables $\xi_1, \ldots, \xi_n$ are $Q$-independent if the characteristic functional of the distribution of the vector $(\xi_1, \ldots, \xi_n)$ can be represented in the form

$$\hat{\mu}_{(\xi_1, \ldots, \xi_n)}(f_1, \ldots, f_n) = \mathbb{E}[e^{i\langle (\xi_1, \ldots, \xi_n), (f_1, \ldots, f_n) \rangle}] =$$

$$= \left(\prod_{j=1}^n \hat{\mu}_{\xi_j}(f_j)\right) \exp\{q(f_1, \ldots, f_n)\}, \quad f_j \in X^*,$$  \hspace{1cm} (4)

where $q(f_1, \ldots, f_n)$ is a continuous polynomial on $(X^*)^n$ such that $q(0, \ldots, 0) = 0$.

The paper [15] has led to new researches of characterization problems on locally compact Abelian groups ([13, 21, 22]) and Banach spaces ([23]).

It is obvious that if random variables $\xi_1, \ldots, \xi_n$ are independent then they are $Q$-independent (in this case the function $q \equiv 0$). Therefore if a statement is valid for $Q$-independent random variables then it is valid for independent random variables (for example **Theorem 2.2** implies **Theorem 2.1**).
The following $Q$-analogue of the Proposition 2.2 (the Cramer theorem in a Banach space) follows directly from Definition 2.1 and the $Q$-analogue of the Cramer theorem on a real line (see [15]).

**Proposition 2.4** Let $\xi$ be a Gaussian random variable with values in a real separable Banach space $X$. Let $\xi = \xi_1 + \xi_2$, where $\xi_1, \xi_2$ are $Q$-independent random variables with values in $X$. Then $\xi_1, \xi_2$ are Gaussian random variables.

The following theorem is the $Q$-analogue of Theorem 2.1.

**Theorem 2.2** Let $X$ be a real reflexive separable Banach space. Let $A_{pj}, p = 1, m, i = 1, n,$ be linear continuous operators of $X$ such that $\text{Ker} A_{pj} = \{0\}$. Let $G_i$ be the sets defined by (2). Suppose that the condition (3) holds. Let $\xi_i, i = 1, 2, \ldots, n$, be $Q$-independent random variables with values in $X$. Consider the linear forms $L_j = A_{j1}\xi_1 + \cdots + A_{jn}\xi_n, j = 1, 2, \ldots, m$. If the random vector $L=(L_1, \ldots, L_m) \in D_{m,m-1}$ then all $\xi_i$ are Gaussian random variables.

**Proof.** Consider the continuous linear functions $T_i : X \to X^m$ of the form

$$T_i(x) = (A_{i1}x, \ldots, A_{mi}x) \in G_i$$

and the factor mappings

$$P_i : X^m \to X^m/G_i.$$

We will construct the compositions

$$F_{ij} = P_iT_j$$

and will check that $\text{Ker} F_{ij} = \{0\}$ for $i \neq j$. Let $x \in \text{Ker} F_{ij}$. Then $F_{ij}(x) = P_iT_j(x) = P_i(A_{ij}x, \ldots, A_{mj}x) \in \text{Ker} T_j$ that is $(A_{ij}x, \ldots, A_{mj}x) \in G_i$. It follows from condition (3) that $x = 0$ for $i \neq j$. Thus $\text{Ker} F_{ij} = \{0\}$ for $i \neq j$. Since $X$ is a reflexive space, it follows from this that

$$F_{ij}^*(X^*) = X^*$$

(5)

in the norm topology.

Note that $F_{ij}^* = T_j^*P_i^*$. Put $C_{pi} = A_{pi}^*, p = 1, m, i = 1, n$. It is easy to verify that the mapping $T_i^* : (X^*)^m \to X^*$ has the form

$$T_i^*(f_1, \ldots, f_m) = C_{i1}f_1 + \cdots + C_{mi}f_m, \quad i = 1, n.$$

Put

$$\overline{G_i} = \{ (f_1, \ldots, f_m) \in (X^*)^m : \langle x, f_1 \rangle + \cdots + \langle x, f_m \rangle = 0 \quad \forall (x_1, \ldots, x_m) \in G_i \}.$$ 

It is clear that

$$\overline{G_i} = \{ (f_1, \ldots, f_m) \in (X^*)^m : \langle A_{i1}x, f_1 \rangle + \cdots + \langle A_{mi}x, f_m \rangle = 0 \quad \forall x \in X \} =$$

$$= \{ (f_1, \ldots, f_m) \in (X^*)^m : \langle x, C_{i1}f_1 + \cdots + C_{mi}f_m \rangle = 0 \quad \forall x \in X \} = \text{Ker} T_i^*.$$
Thus $P_i^*$ is a natural embedding of $\text{Ker} T_i^*$ into $(X^*)^n$.

Let $\mu_i$ be a distribution of the random variable $\xi_i$. We consider the characteristic functional $\hat{\mu}_{\mathbf{L}}(f_1, \ldots, f_m)$ of the random vector $\mathbf{L}$. Since the random variables $\xi_i$ are $\mathcal{Q}$-independent, the functional $\hat{\mu}_{\mathbf{L}}(f_1, \ldots, f_m)$ has the form

$$
\hat{\mu}_{\mathbf{L}}(f_1, \ldots, f_m) = \mathbb{E}[e^{i\mathbf{L} \cdot (f_1, \ldots, f_m)}] = \mathbb{E}[e^{i\sum_{i=1}^{m} A_i \xi_i + A_{m+1} \xi_{m+1} + \cdots + A_n \xi_n f_j}] = \mathbb{E}[e^{i\sum_{i=1}^{m} (A_i \xi_i + A_{m+1} \xi_{m+1} + \cdots + A_n \xi_n f_j)\sum_{i=1}^{n} (\xi_i C_{1i} f_1 + \cdots + C_{mi} f_m)\exp\{g(f_1, \ldots, f_m)\}, \quad f_j \in X^*.
$$

(6)

where $g(f_1, \ldots, f_m)$ is a continuous polynomial on $(X^*)^n$ such that $g(0, \ldots, 0) = 0$.

Since $\mathbf{L} \in \mathcal{D}_{m,m-1}$, we have

$$
\hat{\mu}_{\mathbf{L}}(f_1, \ldots, f_m) = \prod_{j=1}^{m} R_j(f_1, \ldots, f_j, f_{j+1}, \ldots, f_m), \quad f_j \in X^*,
$$

(7)

where $R_{i_1,\ldots,i_k}(f_1, \ldots, f_k)$ are continuous functions such that $R_{i_1,\ldots,i_k}(0, \ldots, 0) = 1$, and all indexes $(i_1, \ldots, i_k)$ satisfy the conditions $1 \leq i_1 < \cdots < i_k \leq m$.

It follows from (6) and (7) that

$$
\prod_{i=1}^{n} \hat{\mu}_i(C_{1i} f_1 + \cdots + C_{mi} f_m)\exp\{g(f_1, \ldots, f_m)\} = \prod_{j=1}^{m} R_j(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_m), \quad f_j \in X^*.
$$

(8)

Put $\nu_i = \mu_i \ast \hat{\mu}_i$. Then $\hat{\nu}_i(f) = |\mu_i(f)|^2 \geq 0$ and the characteristic functionals $\hat{\nu}_i(f)$ also satisfy (8), in which all factors in the left and right parts of the equation are non-negative. If we prove that all $\nu_i$ are Gaussian distributions then Proposition 2.4 implies that all $\mu_i$ are Gaussian distributions. Thus, we may assume from the beginning that all factors in the left and right parts of the equation (8) are non-negative.

Since $\hat{\nu}_i(0) = 1$, it follows from the continuity of the characteristic functionals $\hat{\nu}_i(f)$ in the norm topology that there exists a ball $U_R$ around zero such that all $\hat{\nu}_i(f) > 0$ for $f \in U_R$. We choose in $U_R$ a ball $V_r$, where the radius $r = \frac{B}{m \cdot \max \|C_{mi}\|}$ ($p = 1, m, i = 1, \ldots, n$). Then

$$
C_{1i}(V_r) + \cdots + C_{mi}(V_r) \subset U_R, \quad i = 1, \ldots, n.
$$

(9)

Put $\psi_i = \log \hat{\nu}_i$ in $V_r$, $i = 1, n$, $s_j = \log R_j, \quad j = 1, 2, \ldots, m$. It follows from (8) that the functions $\psi_i$ and $s_j$ satisfy equation

$$
\sum_{i=1}^{n} \psi_i(C_{1i} f_1 + \cdots + C_{mi} f_m) + q(f_1, \ldots, f_m) =
$$

6
where \( s_j \) are arbitrary functions.

We use the finite difference method to solve equation (10). Let \( g_{nj}, j = 1, m \), be arbitrary elements of \( V_r \) such that

\[
C_{1n}g_{n1} + \cdots + C_{mn}g_{nm} = 0.
\]

Note that the existence of such elements follows from (5) and the form of \( F_{ij}^{*} \). Substitute \( f_j + g_{nj} \) for \( f_j \) for all \( j \) in equation (10). We obtain

\[
\sum_{i=1}^{n-1} \Delta h_{in} \psi_i (C_{1i}f_1 + \cdots + C_{mi}f_m) + \Delta \varphi_g q(f_1, \ldots, f_m) =
\]

\[
= \sum_{j=1}^{m} \Delta \varphi_g s_j (f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_m), \quad f_j \in V_r,
\]

where \( h_{in} = C_{1i}g_{n1} + \cdots + C_{mi}g_{nm}, i = 1, n - 1, \varphi_{jn} = (g_{n1}, \ldots, g_{n,j-1}, g_{n,j+1}, \ldots, g_{nm}), j = 1, m \), \( g_n = (g_{n1}, \ldots, g_{nm}) \). The left-hand side of equation (11) no longer contains the function \( \psi_n \).

Let \( g_{n-1,j}, j = 1, m \), be arbitrary elements of \( V_r \) such that

\[
C_{1n-1}g_{n-1,1} + \cdots + C_{m,n-1}g_{n-1,m} = 0.
\]

Reasoning similarly we exclude the function \( \psi_{n-1} \) from the left-hand side of equation (11). We obtain

\[
\sum_{i=1}^{n-2} \Delta h_{i,n-1} \psi_{i} (C_{1i}f_1 + \cdots + C_{mi}f_m) + \Delta \varphi_g \Delta \varphi_{g_{n-1}} q(f_1, \ldots, f_m) =
\]

\[
= \sum_{j=1}^{m} \Delta \varphi_{g_{j,n-1}} \Delta \varphi_g s_j (f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_m), \quad f_j \in V_r,
\]

where \( h_{i,n-1} = C_{1i}g_{n-1,1} + \cdots + C_{mi}g_{n-1,m}, i = 1, n - 2, \varphi_{j,n-1} = (g_{n-1,1}, \ldots, g_{n-1,j-1}, g_{n-1,j+1}, \ldots, g_{n-1,m}), j = 1, m, \varphi_{g_{n-1}} = (g_{n-1,1}, \ldots, g_{n-1,m}) \).

By excluding the functions \( \psi_j \) from the left-hand side of equation (10), after \( n - 1 \) steps we come to the equation of the form

\[
\Delta h_{12} \cdots \Delta h_{1n} \psi_1 (C_{11}f_1 + \cdots + C_{m1}f_m) + \Delta \varphi_g \Delta \varphi_g q(f_1, \ldots, f_m) =
\]

\[
= \sum_{j=1}^{m} \Delta \varphi_g r_j (f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_m), \quad f_j \in V_r,
\]

where

\[
h_{1j} = C_{11}g_{j1} + \cdots + C_{m1}g_{jm}, \quad j = 2, n,
\]
\( \mathcal{g}_j = (g_{j1}, \ldots, g_{jm}) \) and \( r_j \) are arbitrary functions. Note that we chose elements \( g_{ij} \) at every step in such a manner that the equality

\[ C_{11}g_{i1} + \cdots + C_{m1}g_{im} = 0, \quad i = \overline{2,n}, \]  

is fulfilled.

Let \( k_m \) be an arbitrary element of \( X^* \). Substitute \( f_m + k_m \) for \( f_m \) in (13) and subtract equation (13) from the resulting equation. We get

\[ \Delta_{\mathcal{C}m} \Delta_{\mathcal{h}_{12}} \cdots \Delta_{\mathcal{h}_{1n}} \psi_1(C_{11}f_1 + \cdots + C_{m1}f_m) + \Delta_{\mathcal{r}_m} \Delta_{\mathcal{g}_{n}} \cdots \Delta_{\mathcal{g}_{2}} q(f_1, \ldots, f_m) = = \sum_{j=1}^{m-1} r_j(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_m), \quad f_j \in V_r, \]  

where \( \Delta_{\mathcal{r}_m} = (0, \ldots, 0, k_m) \). Note that equation (16) does not contain the function \( r_m \).

By repeating this operation, we consistently exclude all functions \( r_j \) from the right-hand side of the resulting equations. After \( m \) steps we get

\[ \Delta_{C_{11}k_1} \cdots \Delta_{\mathcal{C}m} \Delta_{h_{12}} \cdots \Delta_{h_{1n}} \psi_1(C_{11}f_1 + \cdots + C_{m1}f_m) + + \Delta_{r_1} \cdots \Delta_{\mathcal{r}_m} \Delta_{g_{n}} \cdots \Delta_{g_{2}} q(f_1, \ldots, f_m) = 0, \quad f_j \in V_r. \]

(17)

Since \( q(f_1, \ldots, f_m) \) is a polynomial, we have

\[ \Delta_{(s_1, \ldots, s_m)}^{l+1} q(f_1, \ldots, f_m) = 0, \quad f_j \in X^*, \]  

for some \( l \) and arbitrary elements \( s_1, \ldots, s_m \in X^* \).

Taking into account (18), it follows from (17) that

\[ \Delta_{C_{11}s_1 + \cdots + C_{m1}s_m}^{l+1} \Delta_{C_{11}k_1} \cdots \Delta_{\mathcal{C}m} \Delta_{h_{12}} \cdots \Delta_{h_{1n}} \psi_1(C_{11}f_1 + \cdots + C_{m1}f_m) = 0, \quad f_j \in V_r. \]  

(19)

Since \( X \) is a reflexive space and \( \text{Ker} A_p = \{0\} \), the subgroups \( C_{j1}(X^*) \) are dense in \( X^* \) in the norm topology. Therefore we can put \( C_{11}k_1 = \cdots = C_{m1}k_m = g \), where \( g \in V_r \). Since elements \( h_{ij} \) are defined by (14) and belong to \( F_{ij}^*(X^*) \), it follows from (5) that we can put \( h_{12} = \cdots = h_{1n} = g \). Since \( C_{11}s_1 + \cdots + C_{m1}s_m \) belong to \( F_{ij}^*(X^*) \), it follows from (5) that we can put \( C_{11}s_1 + \cdots + C_{m1}s_m = g \). Thus, it follows from (19) that the function \( \psi_1(f) \) satisfies the equation

\[ \Delta_{g}^{l+m+n} \psi_1(f) = 0, \quad f, g \in V_r, \]  

(20)

i.e. \( \psi_1(f) \) is a polynomial in a neighborhood of zero of \( X^* \). In the same way we can prove that all functions \( \psi_j(f), j = \overline{2,n} \), are polynomials in a neighborhood of zero of \( X^* \). It follows from Proposition 2.3 that all \( \mu_i \) are Gaussian distributions. □
It is known that a characteristic functional is continuous in the norm topology and is sequentially continuous in the topology of pointwise convergence ([29, P.4, §2]). We use only the continuity in the norm topology to prove Theorem 2.2. An essential fact, which was used in the proof, is the following assertion: the equality $\text{Ker}A = \{0\}$ for some bounded operator $A$ implies that $\overline{\text{Im}A^*} = X^*$. This assertion is not valid for arbitrary Banach spaces, but it is true for reflexive ones.

Suppose that $X$ is not a reflexive space. For which maximally wide class of Banach spaces does Theorem 2.2 remain true?

Let us also assume at first that the random variables $\xi_i, i = 1, 2, \ldots, n$, have the distributions $\mu_i$ such that their characteristic functions do not vanish. Then equation (19) is fulfilled on $X^*$.

Putting $f_1 = f, f_2 = \cdots = f_m = 0, s_1 = s, s_2 = \cdots = s_m = 0$ in (19), we get

$$\Delta_{C_{11}s}^{l+1} \Delta_{C_{11}g_1} \cdots \Delta_{C_{m19}g_m} \Delta_{h_{12}} \cdots \Delta_{h_{1n}} \psi_1(C_{11}f_1) = 0, \quad f \in X^*. \quad (21)$$

Elements $C_{11}f, C_{11}s, C_{11}g_1, \ldots, C_{m19}g_m, h_{12}, \ldots, h_{1n}$ belong to images of the operators conjugate to operators with zero kernel. Since $X$ is not a reflexive space, we can not obtain from (21) the equation of type (20) on $X^*$.

Now we will use the fact that a characteristic functional is sequentially continuous in the topology of pointwise convergence. It means that if a sequence of elements $\{f_n\}$ of $X^*$ converges pointwise to $f \in X^* (\{f_n(x) \to f(x)\})$ for any $x \in X$ then $\hat{\mu}(f_n) \to \hat{\mu}(f)$. We need the following definition:

**Definition 2.2** A Banach space $X$ is said to be quasireflexive if $X$ has finite codimension in its second dual $X^{**}$.

We need the following proposition.

**Proposition 2.5** Let $X$ be a quasireflexive separable Banach space. Let $A$ be a linear bounded operator of $X$ such that $\text{Ker}A = \{0\}$. Let $f$ be an arbitrary element of $X^* \setminus \text{Im}(A^*)(X^*)$. Then there exists a sequence of elements $\{f_n\}$ of the image $\text{Im}(A^*)$ such that a sequence of elements $\{f_n\}$ converges pointwise to $f$.

As V. Kadec informed the author, Proposition 2.5 holds (comments of V. Kadec see in Appendix). Note that readers can also find useful information about quasireflexive spaces and additional references in Appendix.

Using Proposition 2.5 we obtain from (21) the following equation:

$$\Delta_{g}^{l+m+n} \psi_1(f) = 0, \quad f, g \in X^*. \quad (22)$$

Now we can formulate the following theorem.

**Theorem 2.3** Let $X$ be a real quasireflexive separable Banach space. Let $A_{pj}, p = 1, m, i = 1, n$, be linear continuous operators of $X$ such that $\text{Ker}A_{pj} = \{0\}$. Let $G_i$ be the sets defined by (2). Suppose that the condition (3) holds. Let $\xi_i, i = 1, 2, \ldots, n$, be $Q$-independent
random variables with values in $X$ and with non vanishing characteristic functionals. Consider the linear forms $L_j = A_{j1}\xi_1 + \cdots + A_{jn}\xi_n$, $j = 1, 2, \ldots, m$. If the random vector $L = (L_1, \ldots, L_m)$ belongs to the class $D_{m,m-1}$ then all $\xi_i$ are Gaussian random variables.

Note that we cannot omit the condition that the characteristic functionals of the distributions $\mu_i$ do not vanish because otherwise we cannot claim that the equation (22) is fulfilled in some ball.

Theorem 2.3 implies the following theorem.

**Theorem 2.4** Let $X$ be a real quasireflexive separable Banach space. Let $A_{pj}$, $p = 1, \ldots, m$, $i = 1, \ldots, n$, be linear continuous operators of $X$ such that $\ker A_{pj} = \{0\}$. Let $G_i$ be the sets defined by (2). Suppose that the condition (3) holds. Let $\xi_i$, $i = 1, 2, \ldots, n$, be independent random variables with values in $X$ and with non vanishing characteristic functionals. Consider the linear forms $L_j = A_{j1}\xi_1 + \cdots + A_{jn}\xi_n$, $j = 1, 2, \ldots, m$. If the random vector $L = (L_1, \ldots, L_m)$ belongs to the class $D_{m,m-1}$ then all $\xi_i$ are Gaussian random variables.

### 3 Appendix

The content of this section belongs to V. Kadec and is published with his consent.

#### 3.1 Density and sequential density in weak-star topology

Let $X$ and $Y$ be normed spaces over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). We denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from $X$ into $Y$. By $B_X$ and $S_X$ we denote the closed unit ball and the unit sphere of $X$ respectively.

**Definition 3.1** Let $X$ be a Banach space. A set $F \subset X^*$ is called total over $X$, if for any $y \in X \setminus \{0\}$ there exists an $f \in F$ such that $f(y) \neq 0$.

Evidently, $F$ is total over $X$ if and only if its linear span $\text{Lin}\, F$ is total, and if and only if the norm-closure $\overline{\text{Lin}\, F}$ of the linear span is total. The bipolar theorem applied to the dual pair $(X^*, X)$ implies that $F \subset X^*$ is total over $X$ if and only if $\text{Lin}\, F$ is $w^*$-dense in $X^*$.

**Definition 3.2** Let $X$ be a Banach space and $\theta \in (0, 1]$. A set $F \subset X^*$ is said to be $\theta$-norming over $X$ if

$$\sup_{f \in F \setminus \{0\}} \frac{|f(x)|}{\|f\|} \geq \theta \|x\|$$

for all $x \in X$. The set $F \subset X^*$ is said to be norming if there exists a $\theta \in (0, 1]$ such that $F$ is $\theta$-norming over $X$.

The following well-known result can be deduced from the bipolar theorem, applied to the dual pair $(X^*, X)$.
Proposition 3.1 ([12, Section 17.2.4, Exercise 2]) Let $X$ be a Banach space, and $\theta \in (0, 1]$. A set $F \subset S_{X^*}$ is $\theta$-norming for $X$ if and only if the $w^*$-closure of the absolute convex hull of $F$ contains $\theta B_{X^*}$.

Proposition 3.2 ([12, Section 17.2.4, Corollary 4]) Let $X$ be a separable Banach space. Then the $w^*$-topology $\sigma(X^*, X)$ is metrizable on the bounded subsets of the space $X^*$.

The very first example of quasireflexive nonreflexive space was constructed by R.C. James in [9]. Since then, quasireflexive spaces form an important part of Banach space theory.

The following result by W.J. Davis and J. Lindenstrauss ([3]) is highly non-trivial. Two years later it was rediscovered by A. Plichko ([26]). The proof in both papers develops ideas by Yuri Petunin ([24]). One can find a detailed exposition with all necessary preliminaries in the book [25].

Proposition 3.3 A Banach space $X$ has a total nonnorming subspace in its dual if and only if $X$ is not quasireflexive.

Definition 3.3 Let $X$ be a Banach space. A set $F \subset X^*$ is called $w^*$-sequentially dense in $X^*$, if for every $g \in X^*$ there exists a sequence $f_n \in F$ such that $\lim_{n \to \infty} f_n(x) = g(x)$ for all $x \in X$.

Lemma 3.1 Let $X$ be a separable Banach space, $F \subset X^*$ be a linear subspace. Then the following assertions are equivalent: (i) $F$ is $w^*$-sequentially dense in $X^*$, and (ii) $F$ is norming.

Proof. (i) $\Rightarrow$ (ii). Assume $F$ is $w^*$-sequentially dense in $X^*$. Denote $U$ the $w^*$-closure of the unit ball $B_F$ of $F$. Thanks to Banach-Steinhaus theorem [12, Section 10.4.2, Theorem 1], the sequence $f_n$ from Definition 3.3 is bounded, hence it belongs to a set of the form $nB_F$. Consequently, every $g \in X^*$ belongs to some set of the form $nU$, that is

$$X^* = \bigcup_{n \in \mathbb{N}} nU. \quad (23)$$

The application of Baire theorem gives that $U$ contains some ball $\theta B_{X^*}$. Since $U$ is the $w^*$-closure of the absolute convex hull of $S_F$, Proposition 3.1 says that $S_F$ is norming, so $F$ is norming as well.

(ii) $\Rightarrow$ (i). Evidently, a linear subspace $F$ is norming if and only if its unit sphere is norming, so in notations of the previous implication Proposition 3.1 says that $U$ contains some ball $\theta B_{X^*}$. So, (23) holds true. Consequently, every $g \in X^*$ belongs to the $w^*$-closure of some ball $nB_F$. But, thanks to metrizability (Proposition 3.2), this means that $g$ is a $w^*$-limit of some sequence from $nB_F \subset F$. □

After this, Proposition 3.3 reformulates as follows:

**Theorem 3.1** For a separable Banach space $X$ the following assertions are equivalent: (i) every $w^*$-dense subspace of $X^*$ is $w^*$-sequentially dense, and (ii) $X$ is quasireflexive.
3.2 Applications to the range of adjoint operator

Recall that for arbitrary Banach spaces $X, Y$ and for arbitrary $T \in \mathcal{L}(X, Y)$ the $w^*$-density of $T^*(Y^*)$ in $X^*$ is equivalent to the injectivity of $T$ ([12 Section 17.1.3, Theorem 5]). This and Theorem [3.1] give the following corollary:

**Theorem 3.2** Let $X$ be a separable quasireflexive Banach space, $Y$ be a Banach space and $T \in \mathcal{L}(X, Y)$ be injective. Then $T^*(Y^*)$ is $w^*$-sequentially dense in $X^*$.

It is not clear for us whether for every norm-closed total subspace $F \subset X^*$ there exist a Banach space $Y$ and an injective operator $T \in \mathcal{L}(X, Y)$ such that $T^*(Y^*) \subset F$. In the case of positive answer one would have the following inverse to Theorem 3.2 result:

**Hypothesis.** Let $X$ be a separable Banach space that is not quasireflexive. Then there exist a Banach space $Y$ and an injective operator $T \in \mathcal{L}(X, Y)$ such that $T^*(Y^*)$ is not $w^*$-sequentially dense in $X^*$.

We can demonstrate a partial result. At first, a technical tool.

**Lemma 3.2** Let $X, Y$ be infinite-dimensional Banach spaces, and let $F \subset X^*$ be a $w^*$-separable closed linear subspace. Then there exists a $T \in \mathcal{L}(X, Y)$ such that $T^*(Y^*) \subset F$ and $T^*(Y^*)$ is $w^*$-dense in $F$.

**Proof.** Fix a $w^*$-dense in $S_F$ countable set $\{f_n\}_{n \in \mathbb{N}} \subset S_F$ and a normalized basic sequence $\{e_n\}_{n \in \mathbb{N}} \subset S_Y$ (the existence of the latter was demonstrated by S. Masur, see a detailed exposition in [11 Theorem 6.3.3]). Denote $\{e^*_n\}_{n \in \mathbb{N}} \subset Y^*$ there corresponding coordinate functionals extended to the whole $Y$ by the Hahn-Banach theorem. Define $T \in \mathcal{L}(X, Y)$ by formula

$$Tx = \sum_{n=1}^{\infty} 2^{-n} f_n(x) e_n.$$

Then, the adjoint operators $T^* : Y^* \to X^*$ acts on every $g \in Y^*$ as

$$T^* g = \sum_{n=1}^{\infty} 2^{-n} g(e_n) f_n.$$

Evidently, $T^*(Y^*) \subset \overline{\text{Lin}} \{f_n\}_{n \in \mathbb{N}} \subset F$. On the other hand, for every $k \in \mathbb{N}$ we have $T^* e^*_k = 2^{-k} f_k$, consequently $T^*(Y^*)$ contains the $w^*$-dense in $F$ set $\text{Lin} \{f_n\}_{n \in \mathbb{N}}$. □

**Corollary 3.1** $X, Y$ be infinite-dimensional Banach spaces, $X$ and $X^*$ are both separable, and $X$ is not quasireflexive. Then there exist an injective operator $T \in \mathcal{L}(X, Y)$ such that $T^*(Y^*)$ is not $w^*$-sequentially dense in $X^*$.

**Proof.** Indeed, Theorem 3.1 gives the existence of a $w^*$-dense norm-closed linear subspace $F \subset X^*$ which is not $w^*$-sequentially dense. By separability of $X^*$ the subspace $F$ is separable, and consequently $F$ is $w^*$-separable. Applying Lemma 3.2 we obtain a $T \in \mathcal{L}(X, Y)$ such that
$T^*(Y^*) \subset F$ and $T^*(Y^*)$ is $w^*$-dense in $F$. Then $T^*(Y^*)$ is $w^*$-dense in $X^*$, so $T$ is injective. But the inclusion $T^*(Y^*) \subset F$ implies that $T^*(Y^*)$ is not $w^*$-sequentially dense in $X^*$. □  

Remark, that for separable $X$ the dual space $X^*$ is $w^*$-separable ([12] Section 17.2.4, Corollary 2]). Unfortunately, this does not guaranty the $w^*$-separability of subspaces of $X^*$. This is the reason, why in the Corollary 3.1 we make the assumption of separability of $X^*$ in norm topology. We don’t know if in the case of separable non-quasireflexive $X$ the corresponding $w^*$-dense but not $w^*$-sequentially dense $F \subset X^*$ can be always selected to be $w^*$-separable. We cannot extract this additional property neither from the demonstration by Davis-Lindenstrauss ([3]), nor from Plichko’s demonstration, but such a possibility does not contradict our intuition.

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