FOURIER TRANSFORM OVER FINITE FIELD AND
IDENTITIES BETWEEN GAUSS SUMS

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INTRODUCTION

This paper is a continuation of [7]. In [7] we considered distributions of the form

\[ C\psi(Q(x)) \prod_{j=1}^{k} \chi_j(P_j(x)) \]

over local fields, where \( Q(x) \) is a rational function of several variables, \( P_j \) are polynomials, \( C \in \mathbb{C}^* \), \( \psi \) is a non-trivial additive character, \( \chi_j \) are multiplicative characters. The main problem we posed in [7] is to determine when the Fourier transform of such a distribution is again a distribution of the same type. It turned out that there is a necessary condition: the map \( x \mapsto dQ(x) \) (which we assume to be dominant) should be birational. The simplest example when this condition is satisfied is the monomial case:

\[ Q(x_1, \ldots, x_k) = a \prod_{i=1}^{k} x_i^{n_i}, \]

where all \( n_i \) are non-zero integers, the degree of \( Q \) is either 0 or 2. In this case it is natural to consider \( P_j = x_j \), so we are looking at the distributions of the form

\[ C\psi(a \prod_{i=1}^{k} x_i^{n_i}) \prod_{i=1}^{k} \chi_j(x_i). \quad (0.1) \]

We have shown in [7] that the Fourier transform of this distribution is again of the same form if and only if certain monomial identity between the values of gamma-functions is satisfied. The structure of these identities is well-known. In the archimedean case all such identities follow from the multiplication law and the functional equation. In the non-archimedean case to get interesting identities we have to consider characters of various extensions of a given local field. The monomial identities between gamma-functions of a collection characters are governed by the linear relations between the corresponding induced representations of the Galois group (see [5]).

In this work we will consider an analogue of this picture over finite fields. The intuition coming from representation theory tells us that in this case instead of distributions we have to consider \( l \)-adic perverse sheaves. Namely, the Goersky-MacPherson extension of perverse sheaves allows us to associate irreducible perverse sheaves on \( \mathbb{A}^n \) (equipped with an action of the Frobenius) to an expression of the form (0.1). We show that the analogue of the main result of [7] holds in this context. For this we develop the theory of monomial identities between gamma-functions of characters over a finite field \( \mathbb{F}_q \) (i.e. Gauss sums). More precisely, we are interested in identities which hold universally over arbitrary finite extension of our finite field \( \mathbb{F}_q \). In section 3 we show that such identities are governed by linear relations between divisors (formal linear combinations of points) on the set...
of multiplicative characters of \( \overline{\mathbb{F}_q} \) of finite order. This reminds of the situation over \( \mathbb{C} \) where gamma-function is meromorphic and one has to look at its divisor of poles and zeroes. Over finite field instead of poles and zeroes we have the jump in the absolute value of gamma-function occurring at the trivial character. In order to use this jump effectively we have to invoke the theory of Kloosterman sheaves due to N. Katz (see [10]).

In section 2 we compute the traces and eigenvalues of Frobenius acting on the stalks of perverse sheaves on \( \mathbb{A}^n \) associated with expressions of the form (1.1). This allows us to extract elementary identities for finite Fourier transform from the identities involving geometric Fourier transforms of these perverse sheaves. Here we observe another similarity with the situation over \( \mathbb{C} \): the conditions under which our perverse sheaves have zero stalks over the union of coordinate hyperplanes are very similar to the conditions guaranteeing that the corresponding distributions over \( \mathbb{C} \) have no poles (see proposition 4.8 of [7]). This phenomenon might suggest that there exists a generalized Riemann-Hilbert correspondence which includes sheaves and \( D \)-modules generated by exponents of rational functions. One can consider as further evidence for existence of such a correspondence the work [11] where monodromy groups of “exponential” perverse sheaves are compared with differential Galois groups, as well as the works [4] and [5] where sheaf-theoretic methods of computing determinants of cohomology are transferred into the realm of irregular connections.

Finally, in section 5 we study “forms” of the identities for the Fourier transform of perverse sheaves considered above. More precisely, we can replace powers of variables in the expression (1.1) by norms in finite extensions of our field. We show that all our results can be generalized to this case.

Notation. All our schemes are of finite type over a finite field (or over its algebraic closure). By a sheaf on such a scheme \( S \) we mean an object of the derived category of constructible \( l \)-adic sheaves on \( S \) defined in [6]. If \( i: x \to S \) is a closed point, \( F \) is a sheaf on \( S \) then we denote the stalk \( i^*F \) by \( F_x \) or by \( F|_x \).

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1. Fourier transform

1.1. Definition. Let \( \mathbb{F}_q \) be the finite field with \( q \) elements, \( \psi: \mathbb{F}_q \to \mathbb{C}^* \) be a non-trivial additive character. For every finite-dimensional vector space \( V \) over \( \mathbb{F}_q \) we define the Fourier transform of a function \( f: V \to \mathbb{C} \) by the formula

\[
\hat{f}(x^*) = \sum_{x \in V} f(x) \psi(\langle x^*, x \rangle)
\]

where \( x^* \in V^* \), \( \langle \cdot, \cdot \rangle \) denotes the natural pairing between \( V^* \) and \( V \). We have the following formulas:

\[
\hat{f}(x) = q^n f(\overline{-x}),
\]

\[
(\hat{f}, \hat{g}) = q^n (f, g)
\]

where \( n = \dim V \), the scalar product \( (f, g) \) is defined by the formula

\[
(f, g) = \sum_{x} f(x) \overline{g(x)}.
\]
Let us choose a prime \( l \) such that \( (l, q) = 1 \) and an identification \( \overline{\mathbb{Q}}_l \simeq \mathbb{C} \). We denote by \( L_\psi \) the Artin-Schreier sheaf on \( \mathbb{A}^1 \) associated with \( \psi \). For every scheme \( S \) and a morphism \( f : S \to \mathbb{A}^1 \) we denote \( L_\psi(f) = f^*L_\psi \). Let \( \mathcal{D}(S) \) denotes the derived category of constructible complexes of \( l \)-adic sheaves on \( S \) (see [3]).

We consider vector spaces \( V \) and \( V^* \) as schemes over \( \mathbb{F}_q \). Then the Fourier-Deligne transform \( \mathcal{F} = \mathcal{F}_\psi : \mathcal{D}(V) \to \mathcal{D}(V^*) \) is defined by the formula
\[
\mathcal{F}(A) = R p_{V*} p_V^*(A \otimes L_\psi([x^*], x))[n]
\]
where \( p_V \) and \( p_{V*} \) are the projections of \( V \times V^* \) onto its factors.

One can associate with every object \( K \in \mathcal{D}(V) \) its trace function \( t_K : V(\mathbb{F}_q) \to \mathbb{C} \) by considering traces of the Frobenius acting on the fibers of \( K \). Then we have
\[
t_{\mathcal{F}(K)} = (-1)^n t_K.
\]

We refer to [1] for the definition of perverse \( l \)-adic sheaves. The important property of the Fourier-Deligne transform is that it sends perverse sheaves on \( V \) to perverse sheaves on \( V^* \) (see [3]).

1.2. Gauss sums. Let \( \lambda : \mathbb{F}_q^* \to \mathbb{C}^* \) be a non-trivial multiplicative character of \( \mathbb{F}_q^* \). Then we can consider \( \lambda \) as a function on \( \mathbb{F}_q \) by setting \( \lambda(0) = 0 \). Then we have
\[
\hat{\lambda} = g(\lambda)\lambda^{-1}
\]
where
\[
g(\lambda) = g(\lambda, \psi) = \sum_{x \in \mathbb{F}_q^*} \lambda(x)\psi(x)
\]
is the Gauss sum associated with \( \lambda \). Applying the Fourier transform twice we derive that
\[
g(\lambda)g(\lambda^{-1}) = \lambda(-1)q.
\]

Also we have
\[
\overline{g(\lambda)} = \lambda(-1)g(\lambda^{-1}).
\]

It is convenient to extend this notation to the case of the trivial character by setting
\[
g(1) = \sum_{x \in \mathbb{F}_q^*} \psi(x) = -1.
\]

Then for any multiplicative character \( \lambda \) we have the following formulas:
\[
\sum_{x \in \mathbb{F}_q^*} \psi(xy)\lambda(x) = g(\lambda)\lambda^{-1}(y)
\]
for any \( y \in \mathbb{F}_q^* \),
\[
g(\lambda^{-1}) \equiv (\lambda(-1)g(\lambda)^{-1}) \mod q\mathbb{Z}
\]
where we denote by \( q^\mathbb{Z} \) the subgroup in \( \mathbb{C}^* \) consisting integer powers of \( q \).

Let \( \lambda \) be a non-trivial character of \( \mathbb{F}_q^* \). There is a smooth sheaf \( L_\lambda \) of rank 1 on \( \mathbb{G}_m \) such that \( \lambda = t_{L_\lambda} \). If \( j : \mathbb{G}_m \to \mathbb{A}^1 \) is the natural inclusion then we have \( Rj_*L_\lambda = jL_\lambda \). One has
\[
\mathcal{F}(jL_\lambda[1]) \simeq G(\lambda) \otimes j_!L_{\lambda^{-1}}[1]
\]
where
\[
G(\lambda) = G(\lambda, \psi) = H^1_c(\mathbb{G}_m, L_\lambda \otimes L_\psi)
\]
(1.1)
is a one-dimensional \( \overline{\mathbb{Q}}_l \)-space equipped with an action of \( \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \) such that \( \text{Frob}_q \) acts as \( -g(\lambda, \psi) \). It is convenient to extend the notation to the case of the trivial character by setting

\[
G(1) = G(1, \psi) = H^1_\psi(\mathbb{G}_m, L_\psi) = \overline{\mathbb{Q}}_l
\]

(1.2)

(so the Galois action on this space is trivial).

Consider the extension of fields \( \mathbb{F}_q \subset \mathbb{F}_{q^t} \) of degree \( d \). Then we have

\[
L_\psi \otimes_{\mathbb{F}_q} \mathbb{F}_{q^t} \simeq L_{\psi \circ \text{Tr}_d}
\]

\[
L_\lambda \otimes_{\mathbb{F}_q} \mathbb{F}_{q^t} \simeq L_{\lambda \circ \text{Nm}_d}
\]

where \( \text{Tr}_d : \mathbb{F}_{q^t} \rightarrow \mathbb{F}_q \) is the trace, \( \text{Nm}_d : \mathbb{F}_{q^t}^* \rightarrow \mathbb{F}_q^* \) is the norm homomorphism.

It follows that \( G(\lambda \circ \text{Nm}_d, \psi \circ \text{Tr}_d) \) as \( \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^t}) \)-representation is obtained from \( \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \)-representation \( G(\lambda, \psi) \) by restriction. Thus, we arrive to the identity

\[
-g(\lambda \circ \text{Nm}_d, \psi \circ \text{Tr}_d) = (-g(\lambda, \psi))^d
\]

(1.3)

which is due to Hasse and Davenport (see [9]).

Another important identity for Gauss sums is the Hasse-Davenport product formula (see [10]):

\[
g(\lambda^n) = \lambda(n^n) \prod_{i=0}^{n-1} \frac{g(\lambda^n_i)}{g(\epsilon^n_i)}
\]

(1.4)

where \( \epsilon_n \) is a character of order \( n \) on \( \mathbb{F}_q^* \) (so \( n|(q - 1) \)), in particular, \( n \neq 0 \) in \( \mathbb{F}_q \).

The geometric interpretation of this identity is more involved (see [10], prop. 5.6.2) and it seems unlikely that it admits an elegant geometric proof.

1.3. Main lemma. Let us consider the standard embedding \( j : \mathbb{G}_m^n \rightarrow \mathbb{A}^n \). Assume that we have simple perverse sheaves \( K \) and \( K' \) on \( \mathbb{G}_m^n \). We are looking for a criterion checking that

\[
\mathcal{F}(j_*, K) \simeq j_*, K'.
\]

For every \( q_1 = q^d \) let us denote by \( f_{q_1} \) (resp. \( f_{q_1}' \)) the trace function of \( K \otimes \mathbb{F}_{q_1} \) (resp. \( K' \otimes \mathbb{F}_{q_1} \)). These are \( \mathbb{C} \)-valued functions on \( (\mathbb{F}_{q_1}^*)^n \). For every collection of characters \( \lambda_1, \ldots, \lambda_n \) of \( \mathbb{F}_{q_1}^* \), let us denote by \( \lambda_1 \otimes \ldots \otimes \lambda_n \) the corresponding function \( \lambda_1(x_1) \ldots \lambda_n(x_n) \) on \( (\mathbb{F}_{q_1}^*)^n \).

**Lemma 1.3.1.** Assume that for every \( d > 0 \) and for every collection of non-trivial characters \( (\lambda_1, \ldots, \lambda_n) \) of \( \mathbb{F}_{q_1}^* \), where \( q_1 = q^d \) one has

\[
(-q_1)^n (f_{q_1}, \lambda_1 \otimes \ldots \otimes \lambda_n) = \prod_{i=1}^{n} g(\lambda_i, \psi \circ \text{Tr}_d) \cdot (f_{q_1}', \lambda_{i}^{-1} \otimes \ldots \otimes \lambda_n^{-1})
\]

and that this number is not zero for at least one collection of non-trivial characters \( (\lambda_1, \ldots, \lambda_n) \). Then

\[
\mathcal{F}(j_*, K) \simeq j_*, K'.
\]

**Proof.** Since \( K \) and \( K' \) are irreducible it suffices to prove that \( j^* \mathcal{F}(j_*, K) \simeq K' \). For every scheme \( S \) let us denote by \( K_0(S) \) the Grothendieck group of the category of perverse sheaves on \( S \). The Fourier-Deligne transform induces a homomorphism \( \mathcal{F} : K_0(\mathbb{A}^n) \rightarrow K_0(\mathbb{G}_m^n) \). Let us denote by \( \overline{K}_0(\mathbb{G}_m^n) \) the quotient of \( K_0(\mathbb{G}_m^n) \) by
the sum of the subgroups \( p_i^*(K_0(G_{m_i}^{n-1})) \) where \( p_i : \mathbb{G}_{m_i}^n \rightarrow \mathbb{G}_{m_i}^{n-1} \) is the projection omitting the \( i \)-th factor. Then \( \mathcal{F} \) induces a well-defined homomorphism

\[
\mathcal{F} : K_0(\mathbb{G}_{m_i}^n) \rightarrow K_0(\mathbb{G}_{m_i}^{n-1}).
\]

Indeed, for a sheaf \( A \) on \( \mathbb{G}_{m_i}^n \) we can choose a sheaf \( \widetilde{A} \) on \( \mathbb{A}^n \) such that \( A \simeq j^*\widetilde{A} \) and set \( \mathcal{F}([A]) = [j^*\mathcal{F}(\widetilde{A})] \). Since \( K_0(\mathbb{G}_{m_i}^n) \) is the quotient of \( K_0(\mathbb{A}^n) \) by the subgroup generated by sheaves supported on coordinate hyperplanes and the Fourier transform interchanges such sheaves with sheaves of the form \( p_i^*A \), the map \( \mathcal{F} \) is well-defined. The involutivity of the Fourier transform implies that \( \mathcal{F} \) is an isomorphism. We have a natural basis in \( K_0(\mathbb{G}_{m_i}^n) \) corresponding to simple perverse sheaves on \( \mathbb{G}_{m_i}^n \) which are not constant on any factor \( \mathbb{G}_{m_i} \) (i.e. do not belong to \( p_i^*\mathcal{D}(\mathbb{G}_{m_i}^{n-1}) \) for any \( i \)), and the map \( \mathcal{F} \) induces some permutation on this basis.

Our assumption that the scalar product of \( f_{q_i} \) (resp. \( f'_{q_i} \)) with some non-trivial multiplicative characters is non-zero implies that \( K \) (resp. \( K' \)) is not constant on any factor \( \mathbb{G}_{m_i} \). Hence, \( \mathcal{F}([K]) \) and \( [K'] \) are both elements of the basis in \( K_0(\mathbb{G}_{m_i}^n) \).

Note that a function \( f \) on \( (\mathbb{F}_q)^n \) is completely determined by the scalar products with functions of the form \( \lambda_1 \otimes \ldots \otimes \lambda_n \), where \( \lambda_i \) are characters of \( \mathbb{F}_q^* \). Moreover, \( f \) can be represented in the form

\[
f = \sum_{i=1}^n p_i^*f_i
\]

if and only if

\[
(f, \lambda_1 \otimes \ldots \otimes \lambda_n) = 0
\]

for all collections of non-trivial characters \( \lambda_1, \ldots, \lambda_n \). Now we claim that an element \( x \in K_0(\mathbb{G}_{m_i}^n) \) lies in \( \sum_i p_i^*K_0(\mathbb{G}_{m_i}^{n-1}) \) if and only if for all extensions \( \mathbb{F}_q \subset \mathbb{F}_{q_i} \), the trace function \( t_x \) of \( x \) over \( \mathbb{F}_{q_i} \) has form \( \sum_i p_i^*f_i \). Indeed, a function \( f \) can be represented in the form \( \sum_i p_i^*f_i \) if and only if the following equality holds:

\[
\sum_{I \subset [1, \ldots, n]} (-1)^{|I|} p_I^* \sigma_I f = 0
\]

where \( p_I : \mathbb{G}_{m_i}^n \rightarrow \mathbb{G}_{m_i}^I \) is the natural projection, \( \sigma_I : \mathbb{G}_{m_i}^I \rightarrow \mathbb{G}_{m_i}^n \) is the embedding which sends \( (a_i)_{i \in I} \) to \( (b_i)_{1 \leq i \leq n} \) where \( b_i = a_i \) for \( i \in I \), \( b_i = 1 \) otherwise. Thus, our condition implies that all the trace functions of the element

\[
\sum_{I \subset [1, \ldots, n]} (-1)^{|I|} p_I^* \sigma_I x
\]

are zero. Hence, this element is zero by Theorem 1.1.2 of [13].

Thus, an element of \( K_0(\mathbb{G}_{m_i}^n) \) is completely determined by the scalar products of its trace functions with \( \lambda_1 \otimes \ldots \otimes \lambda_n \), where \( \lambda_i \) are non-trivial characters.

Now if \( x \in K_0(\mathbb{G}_{m_i}^n) \) and \( (\lambda_1, \ldots, \lambda_n) \) is a collection of non-trivial characters of \( \mathbb{F}_q^* \) then

\[
(-q)^n(t_x, \lambda_1 \otimes \ldots \otimes \lambda_n) = \prod_i g(\lambda_i)(t_x, \lambda_i^{-1} \otimes \ldots \otimes \lambda_i^{-1}).
\]

Thus, our assumptions imply that \( \mathcal{F}([K]) = [K'] \) in \( K_0(\mathbb{G}_{m_i}^n) \), hence \( j^*(\mathcal{F}(j_!K)) \simeq K' \).
2. Computation of Goresky-MacPherson extensions

2.1. Vanishing of stalks. Let \( \psi \) be a non-trivial additive character of \( F_q, \chi_1, \ldots, \chi_k \) are characters of \( F_q^*, n_1, \ldots, n_k \) are integers such that \( (n_i, q) = 1 \). Then we define \( F^{n_1, \ldots, n_k}_{\chi_1, \ldots, \chi_k} \) as the Goresky-MacPherson extension to \( \mathbb{A}^k \) of the smooth perverse sheaf

\[
L(\psi(\prod_i x_i^{n_i}) \prod_i \chi_i(x_i))[k] := L(\psi(\prod_i x_i^{n_i}) \otimes L_{\chi_1} \boxtimes \ldots \boxtimes L_{\chi_k}[k]
\]

on \( \mathbb{G}_m^k \). We are interested in computing the stalk \( (F^{n_1, \ldots, n_k}_{\chi_1, \ldots, \chi_k})_0 \) where \( 0 = (0, \ldots, 0) \in \mathbb{A}^k \). Of course, the interesting case is when some of \( n_i \) are negative.

Let us also denote

\[
\tilde{F}^{n_1, \ldots, n_k}_{\chi_1, \ldots, \chi_k} = R^j_*(L(\psi(\prod_i x_i^{n_i}) \prod_i \chi_i(x_i)))
\]

where \( j \) is the embedding into \( \mathbb{A}^k \) of the complement to the subspace \( x_i = 0 \) for all \( i \) such that \( n_i < 0 \) or \( \chi_i \neq 1 \), the sheaf \( L(\psi(\prod_i x_i^{n_i}) \prod_i \chi_i(x_i)) \) is smooth on this complement.

Consider the restriction of \( \tilde{F}^{n_1, \ldots, n_k}_{\chi_1, \ldots, \chi_k} \) to the open subset \( x_1 \neq 0 \). Passing to the étale covering \( z_1 = x_1^{1/n_2} \) we can make a change of variables \( x_2 = x_2^i x_i, x'_i = x_i \) for \( i > 2 \), so that

\[
\psi(\prod_i x_i^{n_i}) \prod_i \chi_i(x_i) = (\chi_1^n \chi_2^{-n_1})(z_1) \cdot \psi(\prod_{i \geq 2} (x'_i)^{n_i}) \prod_{i \geq 2} \chi_i(x'_i).
\]

Thus, the pull-back of our sheaf under this étale covering is the external tensor product of \( L_{\chi_1}^{n_2} \otimes \mathbb{A}^k_{\chi_2} \) and \( \tilde{F}^{n_2, \ldots, n_k}_{\chi_2, \ldots, \chi_k} \). Therefore, the calculation of all the stalks of \( \tilde{F}^{n_1, \ldots, n_k}_{\chi_1, \ldots, \chi_k} \) reduces to the calculation of the stalks \( (\tilde{F}^{n_2, \ldots, n_k}_{\chi_2, \ldots, \chi_k})_0 \) for all subsets \( \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\} \). Similar reasoning works for the sheaves \( F^{n_1, \ldots, n_k}_{\chi_1, \ldots, \chi_k} \).

In the case \( k = 1 \) we have

\[
(F^{-n}_x)_0 = (\tilde{F}^{-n}_x)_0 = 0
\]

for any \( n > 0 \) and any character \( \chi \). This follows from the fact that the Swan conductor of \( L(\psi(1/x^n)) \) at \( x = 0 \) is equal to \( n \). Now let us consider the case \( k = 2 \).

Part of the following theorem is contained in Theorem 3.1.1 of [12] (which is due to Deligne). Our statement is slightly more general, however, the proof is based on the same idea.

**Theorem 2.1.1.** Let \( m, n > 0 \). Set \( m = m'd, n = n'd, \) where \( d = \text{gcd}(m, n) \).

Then

(i) \( (\tilde{F}^{-m, -n}_{\chi_1, \chi_2})_0 = (F^{-m, -n}_{\chi_1, \chi_2})_0 = 0 \) for any \( \chi_1, \chi_2 \),

(ii) \( (\tilde{F}^{m, -n}_{\chi_1, \chi_2})_0 = (F^{m, -n}_{\chi_1, \chi_2})_0 = 0 \) if \( \chi_1^n \chi_2^{-n} \neq 1 \),

(iii) \( (\tilde{F}^{m, -n}_{\chi_1, \chi_2})_0 = H^1(\mathbb{A}^1, \mathbb{F}_p(t^d) \otimes L_{\chi})[1], \) where \( L_{\chi} \) is extended to \( \mathbb{A}^1 \) by zero for \( \chi \neq 1 \) while \( L_1 = \mathbb{G}_m(\mathbb{A}^1) \). If \( \chi = 1 \) the dimension of this cohomology space is \( d - 1 \), while for \( \chi \neq 1 \) it is equal to \( d \).

**Proof.** Consider the Galois covering

\[
r : \mathbb{A}^2 \rightarrow \mathbb{A}^2 : (x, y) \mapsto (x^m, y = y')
\]

with Galois group \( G = \mu_{m'} \times \mu_{m''} = \mu_{m'm''} \). Then we have

\[
(F^{m, -n}_{\chi_1, \chi_2})_0 = H^0(\mathbb{A}^1, (\tilde{F}^{m, m'd, m''d}_{\chi_1, \chi_2})_0).
\]
This shows that for the proof of (i) and (ii) it suffices to consider the case \( m = n \).

(i) Let \( \pi : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2 \) be the blow-up of \( \mathbb{A}^2 \) at 0. Then there is a natural embedding \( \tilde{j} : \mathbb{A}^1 \times \mathbb{G}_m \to \tilde{\mathbb{A}}^2 \) and we have

\[
(\tilde{E}^{0,-n}_{\chi_1,\chi_2})_0 = R\Gamma(E, \mathcal{G})
\]

where \( E = \pi^{-1}(0) \) is the exceptional divisor,

\[
\mathcal{G} = \tilde{j}_*(L(\psi(y^{-n}) \chi_1(x) \chi_2(y)))|_E.
\]

Consider the open chart in \( \tilde{\mathbb{A}}^2 \) with coordinates \( x, u \) such that \( y = xu \). In this chart \( E \) is given by the equation \( x = 0 \) while \( u \) is the local coordinate on \( E \). We have

\[
\psi(y^{-n}) \chi_1(x) \chi_2(y) = \psi(xu)^{-n} \chi_1(x) \chi_2(u).
\]

Thus, for \( u \neq 0 \) we have \( \mathcal{G}_u = 0 \) while \( \mathcal{G}_0 = (\tilde{E}^{0,-n}_{\chi_1,\chi_2})_0 \). Now (2.1) implies that \( \mathcal{G}_0 \) is a direct summand in \( (\tilde{E}^{0,-n}_{\chi_1,\chi_2})_0 = 0 \). Therefore, \( \mathcal{G}_0 = (\tilde{E}^{0,-n}_{\chi_1,\chi_2})_0 = 0 \).

(ii) Consider the standard action of \( \mathbb{G}_m \) on \( \mathbb{A}^2 \). Then with respect to an action of \( t \in \mathbb{G}_m \) the sheaf \( \tilde{\mathbb{A}}^{n,n}_{\chi_1,\chi_2} \) gets tensored with \( (L_{\chi_1,\chi_2}) \). Since \( \chi_1\chi_2^{-1} \) is non-trivial this immediately implies the result.

(iii) Consider first the case \( m = n = d \). Then using the above notation we can write

\[
(\tilde{E}^{d,d}_{\chi,\chi^{-1}})_0 = R\Gamma(E, \tilde{j}_*(L(\psi(\frac{x^d}{y^d}) \chi(x))^)}(\frac{x}{y}))|_E).
\]

On the open chart \( U_1 \subset \tilde{\mathbb{A}}^2 \) with coordinates \( x, u \) such that \( y = xu \) we have

\[
L(\psi(\frac{x^d}{y^d}) \chi(\frac{x}{y})) = L(\psi(\frac{x}{u^d}) \chi^{-1}(u)).
\]

On the second open chart \( U_2 \subset \tilde{\mathbb{A}}^2 \) with coordinates \( v, y \) such that \( x = vy \) we have

\[
L(\psi(\frac{x^d}{y^d}) \chi(\frac{v}{y})) = L(\psi(v^d) \chi(v)).
\]

Hence,

\[
(\tilde{E}^{d,d}_{\chi,\chi^{-1}})_0 = R\Gamma(E, L(\psi(v^d) \chi(v)))
\]

where the sheaf \( L(\psi(v^d) \chi(v)) \) is extended by zero from \( \mathbb{A}^1 \) to \( \mathbb{P}^1 = E \). This implies immediately that

\[
(\tilde{E}^{d,d}_{\chi,\chi^{-1}})_0 = H^1_c(\mathbb{A}^1, L(\psi(v^d) \chi(v)))[1].
\]

Now in the general case we have

\[
(\tilde{E}^{m,-n}_{\chi^{m'},\chi^{-n'}})_0 = H^0(G, (\tilde{E}^{m',d}_{\chi^{m'},\chi^{-n'}})_0) = H^0(G, H^1_c(\mathbb{A}^1, L(\psi(m'^{-d}) \chi(v^{m'}\psi)))[1].
\]

The latter space up to a shift of degree has form \( H^0(G, R\Gamma_c(\mathbb{P}^1 \cup L(\psi(v^d) \chi(v)))) \) where \( f : \mathbb{A}^1 \to \mathbb{A}^1 : v \mapsto v^{m'} \) is a covering with Galois group \( G \). Therefore, this space is isomorphic to \( R\Gamma_c(L(\psi(v^d) \chi(v))) \). \( \square \)

The theorem 2.1.1 admits the following partial generalization to higher dimensions.
Theorem 2.1.2. Let $n_i > 0$, $i = 1, \ldots, k$, $m_j$, $j = 1, \ldots, l$ and $n > 0$ be integers which are prime to $q$. Set $m_j = m'_j$ where $d = \gcd(m_1, \ldots, m_l)$. Then

(i) $(F_{\chi_1, \ldots, \chi_k})_{0} = (F_{\chi_1, \ldots, \chi_k})_{0} = 0$ for any $\chi_1, \ldots, \chi_k$.

(ii) $(F_{m_1, \ldots, m_l})_{0} = (F_{m_1, \ldots, m_l})_{0} = 0$ unless there exists a character $\eta$ such that $\eta_i = \eta_{m_i}^i$ for all $i$.

(iii) $(\widetilde{F}_{1, \ldots, 1})_{0} = 0$ provided that $\gcd(n, n_1, \ldots, n_k) = 1$.

Proof. (i) Considering the covering of $\mathbb{A}^{k+1}$ induced by $(x_1, x_2) \mapsto (x_1^{n_1}, x_2^{n_2})$ we reduce to the case $n_1 = n_2$. Then the argument with blow-up along $x_1 = x_2 = 0$ similar to the case (i) of Theorem 2.1.1 allows the induction in $k$.

(ii) Consider the natural action of the torus $\mathbb{G}_m$ on $\mathbb{A}^k$. Then under the action of $(t_1, \ldots, t_l) \in T$ both the sheaves $\widetilde{F}_{m_1, \ldots, m_l}$ and $F_{m_1, \ldots, m_l}$ get tensored with the space $(L_{\eta_1})_{t_1} \otimes \cdots \otimes (L_{\eta_l})_{t_l}$. Now unless there exists $\eta$ such that $\eta_i = \eta_{m_i}^i$ for all $j$ we can find a one-parameter subgroup $i : \mathbb{G}_m \to T$ such that $i^*(L_{\eta_1} \otimes \cdots \otimes L_{\eta_l}) \simeq L_{\chi}$ where $\chi$ is a non-trivial character. Then under the action of $i(t)$ our sheaves get tensored with $(L_{\chi})_t$ which immediately implies the vanishing of their stalks at zero.

(iii) We proceed by induction in $k$. The case $k = 1$ follows from Theorem 2.1.1 (iii). Now assume that $k > 1$. Let us denote the variables by $(x, x_1, \ldots, x_k)$. Let $n = n'd$, $n_1 = n'_1d$ where $d = \gcd(n, n_1)$. Consider first the Galois covering $(x, x_1) \mapsto (x = \mathbb{F}^{u_1}, x_1 = \mathbb{F}^{\xi_1})$ with Galois group $G = \mu_{n'\eta_1}$. Then we have

$$\widetilde{F}_{1, \ldots, 1} = H^0(G, (R_j, L(\psi(x^{\frac{n_1'}{n_1}}_{x_1} \cdots )))_0).$$

Let $\pi : \mathbb{A}^{k+1} \to \mathbb{A}^{k+1}$ be the blow-up of $\mathbb{A}^{k+1}$ along the subspace $x = x_1 = 0$. Let $j : \mathbb{A}^1 \times \mathbb{G}_m^k \to \mathbb{A}^{k+1}$ be the natural embedding where the first factor $\mathbb{A}^1$ corresponds to the variable $x$. We have

$$\widetilde{F}_{1, \ldots, 1} = R\Gamma(E_0, R_j^*L(\psi(x^{\frac{n_1'}{n_1}}_{x_1} \cdots )))_0.$$

where $E_0 = \pi^{-1}(0) \simeq \mathbb{P}^1$.

On the open chart $U_1 \subset \mathbb{A}^{k+1}$ with coordinates $x, u, x_2, \ldots$ such that $x_1 = xu$ we have

$$L(\psi(x^{\frac{n_1'}{n_1}}_{x_1} \cdots )) = L(\frac{1}{u^{n_1'}x_2 \cdots x_k}).$$

The extension of this sheaf has zero stalks over $E_0 \cap U_1$ by (i). On the second open chart $U_2 \subset \mathbb{A}^{k+1}$ with coordinates $v, x_1, x_2, \ldots$ such that $x = vx_1$ we have

$$L(\psi(x^{\frac{n_1'}{n_1}}_{x_1} \cdots )) = L(\frac{1}{x_2 \cdots x_k}).$$

Thus, the sheaf

$$R_j^*L(\psi(x^{\frac{n_1'}{n_1}}_{x_1} \cdots ))_0$$
is supported at one point \( v = 0 \) and we get

\[
(R_j \psi \left( \frac{x_1^{dn'_1} \cdots x_k^{dn'_k}}{x_1^{n_2} \cdots x_k^{n_k}} \right))_0 = (R_j \psi \left( \frac{x_1^{dn'_1} \cdots x_k^{dn'_k}}{x_1^{n_2} \cdots x_k^{n_k}} \right))_0.
\]

Hence,

\[
(F_{1,1, \ldots, 1}^{n,-n_1, \ldots, -n_k})_0 = H^0(G, (R_j \psi \left( \frac{x_1^{dn'_1} \cdots x_k^{dn'_k}}{x_1^{n_2} \cdots x_k^{n_k}} \right))_0) = (R_j \psi \left( \frac{x_1^{dn'_1} \cdots x_k^{dn'_k}}{x_1^{n_2} \cdots x_k^{n_k}} \right))_0.
\]

Since \( \gcd(d, n_2, \ldots, n_k) = \gcd(n, n_1, \ldots, n_k) = 1 \) this space is zero by induction assumption.

\[2.2. \quad \text{Traces of Frobenius on the stalks of Goreski-MacPherson extensions.}\]

In this section we show how to compute the traces of \( \text{Frob}_q \) on the stalks of the sheaves \( F_{\chi_1, \ldots, \chi_k}^{n_1, \ldots, n_k} \) on \( \mathbb{A}^k \) introduced above. It suffices to consider the stalk at zero, and according to Theorem 2.1.2(ii) the non-zero stalk at zero can appear only for the sheaf of the form \( F_{\chi_1, \ldots, \chi_k}^{dn_1, \ldots, dn_k} \). First let us consider the particular case when \( n_i = \pm 1 \) for all \( i \). In order to formulate the answer we have to introduce two families of polynomials in \( q \):

\[
a(n, m) = \begin{cases} 
\sum_{i=0}^{m-1} \binom{m-1}{i} \binom{n-1}{i+1} q^{i+1}, & m \geq 1 \\
1, & m = 0
\end{cases} \quad (2.2)
\]

\[
b(n, m) = \sum_{i=0}^{m-1} \binom{m-1}{i} \binom{n-1}{i} q^i. \quad (2.3)
\]

These polynomials satisfy the following recursive relations

\[
b(n, m) - b(n, m - 1) = a(n, m - 1),
\]

\[
a(n, m) - a(n - 1, m) = qb(n - 1, m).
\]

For a local system \( L \) on \( U \xrightarrow{j} \mathbb{A}^N \) let us denote

\[
j_{!*}L := j_{!*}(L[N])[-N].
\]

**Theorem 2.2.1.** Let \( \chi \) be a non-trivial character, \( d \geq 1 \). Then

\[
\text{Tr}(\text{Frob}_q, j_{!*}L(\psi(\frac{x_1^d \cdots x_n^d}{y_1^d \cdots y_m^d})))_0 = a(n, m) + b(n, m) \cdot \sum_{t \in \mathbb{F}_q} \psi(t^d), \quad (2.4)
\]

\[
\text{Tr}(\text{Frob}_q, j_{!*}L(\psi(\frac{x_1^d \cdots x_n^d}{y_1^d \cdots y_m^d} \chi(\frac{x_1 \cdots x_n}{y_1 \cdots y_m})))_0 = (b(n, m) \cdot \sum_{t \in \mathbb{F}_q} \psi(t^d) \chi(t). \quad (2.5)
\]

**Proof.** Consider the blow-up \( \pi: \mathbb{A}^{n+m} \to \mathbb{A}^{n+m} \) along the subspace \( x_1 = y_1 = 0 \). Let \( \tilde{j} \) be the natural embedding of \( \mathbb{G}^m_{n+m} \) into \( \mathbb{A}^{n+m} \). We claim that

\[
R\pi_* (j_{!*}L(\psi(\frac{x_1^d \cdots x_n^d}{y_1^d \cdots y_m^d}))) \quad (2.6)
\]

is the Goreski-MacPherson extension from \( \mathbb{G}^m_{n+m} \), and the similar statement holds for \( L(\psi(\frac{x_1^d \cdots x_n^d}{y_1^d \cdots y_m^d}) \chi(\frac{x_1 \cdots x_n}{y_1 \cdots y_m})) \). Let us denote by \( S(k) \) the subset of \( \mathbb{A}^{n+m} \) where exactly \( k \) coordinates vanish (so \( k \geq 2 \)). It suffices to check that the stalks of the sheaf
over points of $S(k)$ with $x_1 = y_1 = 0$ are concentrated in degrees $< k$. Now for $p = (0, a_2, \ldots, a_n, 0, b_2, \ldots, b_m) \in S(k)$ we have

$$R\pi_*(\tau_s \psi(x_1^{d_1} \cdots x_n^{d_n})|_{\pi^{-1}(p)}) = R\Gamma(\pi^{-1}(p), \tau_s \psi(x_1^{d_1} \cdots x_n^{d_n}))|_{\pi^{-1}(p)}.$$ 

We can cover $\mathbb{A}^{n+m}$ by two open affine charts: $U_1$ with coordinates $(u, x_2, \ldots, x_n, y_1, \ldots, y_m)$ such that $x_1 = uy_1$ and $U_2$ with coordinates $(x_1, \ldots, x_n, v, y_2, \ldots, y_m)$ such that $y_1 = vx_1$. Both $U_1$ and $U_2$ are isomorphic to $\mathbb{A}^{n+m}$ and we will denote by $S(k)$ the strata given by vanishing of coordinates in $U_1$ and $U_2$. Let us represent $\pi^{-1}(p)$ as the disjoint union of $\pi^{-1}(a) \cap U_1 \cap U_2 \simeq \mathbb{C}_m$ and two points $p_1, p_2$, where $p_1 \in U_1$ has coordinates $(u = 0, a_2, \ldots, a_n, 0, b_2, \ldots, b_m)$ and $p_2 \in U_2$ has coordinates $(0, a_2, \ldots, a_n, v = 0, b_2, \ldots, b_m)$. We have $\pi^{-1}(p) \cap U_1 \cap U_2 \subset S(k-2)$ while $p_1$ and $p_2$ are in $S(k-1)$. Consider the exact triangle

$$R\Gamma(\pi^{-1}(p) \cap U_1 \cap U_2, \tau_s \psi(u^{d_1} \cdots x_n^{d_n} y_1^{d_1} \cdots y_m^{d_m})) \to R\Gamma(\pi^{-1}(p), \tau_s \psi(u^{d_1} \cdots x_n^{d_n} y_1^{d_1} \cdots y_m^{d_m})) \to \cdots$$

Assume first that $k > 2$. Then making the change of variables $x_2' = ux_2$ (and leaving all the other variables the same) we can rewrite the first term of the above triangle as

$$R\Gamma_c(\mathbb{C}_m, \tau_s \psi(x_2'^{d_1} \cdots x_n^{d_n} y_1^{d_1} \cdots y_m^{d_m}))$$

where $p' = (a_2, \ldots, a_n, b_2, \ldots, b_m)$. Since $p' \in S(k-2)$ and $R\Gamma_c(\mathbb{C}_m, \psi)$ lives in degrees 1 and 2, this term is concentrated in degrees $< k$. Now assume that $k = 2$, i.e. $a_2 \ldots a_n b_2 \ldots b_m \neq 0$. Then we can make the change of variables $u' = ux_2 \ldots x_n y_1^{-1} \ldots y_m^{-1}$ so that the first term of the above triangle takes form

$$R\Gamma(\mathbb{C}_m, \psi(u^{d_1} \cdots x_n^{d_n} y_1^{d_1} \cdots y_m^{d_m}))$$

which is concentrated in degree $1 < k = 2$. On the other hand, since $p_1 \in S(k-1)$ the last term of the above triangle is concentrated in degrees $< k - 1$. Therefore, our claim follows, so the sheaf (2.6) is the Goreski-MacPherson extension. The similar argument works for the sheaf involving the non-trivial character $\chi$.

Let us denote

$$A(n, m) := \text{Tr}(\text{Frob}_q, j_\ast \psi(x_1^{d_1} \cdots x_n^{d_n} y_1^{d_1} \cdots y_m^{d_m}))_0).$$

Then for $n + m > 2$ the above exact triangle shows that

$$A(n, m) = (q-1)A(n-1, m-1) + A(n-1, m) + A(n, m-1). \quad (2.7)$$

Notice that $a(n, m)$ and $b(n, m)$ are solutions of (2.4) with initial values $a(0, m) = b(0, m) = 0$, $a(n, 0) = 0$, $a(1, 1) = 0$, $b(n, 0) = 0$, $b(1, 1) = 1$. Since $A(1, 1) = \sum_{t \in \mathbb{Z}_q} \psi(t^d)$ by Theorem 2.1.1 (iii), $A(0, m) = 0$ by Theorem 2.1.2 (i) and $A(n, 0) = 1$ this proves (2.4). The proof of (2.4) is similar.

In the general case we can proceed as follows. The stalk at zero of the sheaf

$$j_\ast \psi(x_1^{n_1} \cdots x_k^{n_k} y_1^{m_1} \cdots y_m^{m_k}) \chi(x_1^{n_1} \cdots x_k^{n_k} y_1^{m_1} \cdots y_m^{m_k}) \quad (2.8)$$
where \( n_i > 0, m_j > 0, \gcd(n_i, q) = \gcd(m_j, q) = 1 \), can be computed as \( G \)-invariants of the stalk at zero of the sheaf

\[
j_* L\left( \frac{x_1^{N_1} \cdots x_d^{N_d}}{y_1^{N_1} \cdots y_d^{N_d}} \right) \end{equation}

where \( N = \operatorname{lcm}(n_1, \ldots, n_k, m_1, \ldots, m_l) \) (by \( \operatorname{lcm} \) we mean the least common multiple), \( G \) is the product of the groups \( \mu(N/n_1), \ldots, \mu(N/m_l) \) acting on the variables in the natural way. Now we can use the blow-up along \( x_1 = y_1 = 0 \) and the \( G \)-equivariant exact triangle similar to the one considered in the above proof. As a result we get the following recursion relation for the trace of \( \text{Frob}_q \) at zero:

\[
a_{n_1, m_2, \ldots, m_l}^n = (q-1)a_{n_1, m_2, \ldots, m_l}^{n_1} + a_{n_2, m_1, \ldots, m_l}^{m_2} + a_{n_3, m_1, \ldots, m_l}^{m_3} \]

for \( k + l > 2 \) where \( n_i' = \operatorname{lcm}(n_1, m_1, n_2) \), \( n = \operatorname{lcm}(n_1, m_1) \). Using this recursion relation and Theorem 2.1.3 we can in principle compute all these traces.

2.3. **Pointwise purity.** Recall (see [3]) that an object \( K \in \mathcal{D}^b_{\text{r}}(\mathbb{A}^n) \) is called pointwise pure of weight \( w \) if for every closed point \( x \in \mathbb{A}^n \) such that \( k(x) = \mathbb{F}_q \), the endomorphism \( \text{Frob}_q \) acts on \( H^i K|_x \) with eigenvalues which are algebraic numbers with all conjugates of absolute value \( q^{\frac{w}{n}} \).

**Lemma 2.3.1.** Let \( K \) be a \( \mathbb{G}_m \)-equivariant sheaf on \( \mathbb{A}^n \) where \( \mathbb{G}_m \) acts on \( \mathbb{A}^n \) by

\[
t(x_1, \ldots, x_n) = (t^{d_1} x_1, \ldots, t^{d_n} x_n).
\]

Assume that all weights \( d_i \) are positive integers. Then the natural map

\[
\text{R} \Gamma(\mathbb{A}^n, K) \rightarrow K|_0
\]

is an isomorphism.

**Proof.** Consider the coordinate stratification of \( \mathbb{A}^n \). It suffices to prove that for any stratum \( S \subset \mathbb{A}^n \setminus \{0\} \) and a \( \mathbb{G}_m \)-equivariant sheaf \( K \) on \( S \) one has \( H^0(\mathbb{A}^n, j_* K) = 0 \) for any \( q \) where \( j : S \rightarrow \mathbb{A}^n \) is the embedding. Without loss of generality we can assume that \( S \) is the open stratum: \( S = \mathbb{G}_m^n \). Let \( d \) be the greatest common divisor of \( d_1, \ldots, d_n \). Consider the covering

\[
\pi : \mathbb{A}^n \rightarrow \mathbb{A}^n : (x_1, \ldots, x_n) \mapsto (x_1^d, \ldots, x_n^d).
\]

Set \( K' = (\pi|_{\mathbb{G}_m^n})^* K \). Then \( K' \) is a \( \mathbb{G}_m^n \)-equivariant sheaf on \( \mathbb{G}_m^n \) with respect to the action with the weights \( (d_1/d, \ldots, d_n/d) \). Since \( j_* K \) is the direct summand in \( \pi_* (j_* K') \) it suffices to prove that cohomologies of \( j_* K \) vanish. Thus, we can assume from the beginning that \( d = 1 \). Then the action of \( \mathbb{G}_m \) on \( \mathbb{G}_m^n \) is free. Let \( p : \mathbb{G}_m^n \rightarrow T \) be the quotient under this action (so that \( T \) is a torus). We have \( K = p^* L \) for some sheaf \( L \) on \( T \). Let \( f : \tilde{\mathbb{A}}^n \rightarrow \mathbb{A}^n \) be the weighted blow-up of \( \mathbb{A}^n \) along the origin, i.e. \( \tilde{\mathbb{A}}^n = \text{Proj} \ k[x_1, \ldots, x_n, x_1 t^{d_1}, \ldots, x_n t^{d_n}] \) where \( x_i \) are coordinates on \( \mathbb{A}^n \) (\( \deg x_i = 0 \)), \( t \) is an independent variable of degree 1. The morphism \( f \) is proper and the inclusion \( j : \mathbb{G}_m^n \rightarrow \tilde{\mathbb{A}}^n \) factors through the inclusion \( j : \mathbb{G}_m^n \rightarrow \tilde{\mathbb{A}}^n \). Hence, it suffices to prove that \( H^q (\tilde{\mathbb{A}}^n, j_* K) = 0 \). Note that there is a natural projection \( \tilde{p} : \tilde{\mathbb{A}}^n \rightarrow \mathbb{P}(d_1, \ldots, d_n) \) where \( \mathbb{P}(d_1, \ldots, d_n) = \text{Proj} \ k[y_1, \ldots, y_n] \) (\( \deg y_i = d_i \)) is the corresponding weighted projective space. We can identify \( T \)
with an open subset of \( \mathbb{P}(d_1, \ldots, d_n) \) defined by \( y_1 \ldots y_n \neq 0 \) so that the following diagram is commutative.

\[
\begin{array}{ccc}
\mathbb{G}_m^n & \xrightarrow{p} & \tilde{\mathbb{P}}^{-1}(T) \\
& \searrow & \downarrow \tilde{\mathbb{P}} \\
T & \xrightarrow{\tilde{\pi}} & \mathbb{P}(d_1, \ldots, d_n)
\end{array}
\] (2.9)

Choosing a section of the homomorphism of tori \( p : \mathbb{G}_m^n \rightarrow T \) we can lift the natural action of \( T \) on \( \mathbb{P}(d_1, \ldots, d_n) \) to an action of \( T \) on \( \mathbb{A}^n \). Hence, the projection \( \tilde{\pi}^{-1}(T) \rightarrow T \) is a locally trivial fibration with fiber \( F \) which is equal to a generic \( \mathbb{G}_m \)-orbit on \( \mathbb{A}^n \): \( F = \{ (t^{d_1}, \ldots, t^{d_n}) : t \in \mathbb{A}^1 \} \). Furthermore there is a canonical zero section \( \sigma : T \rightarrow \tilde{\pi}^{-1}(T) \) and \( \mathbb{G}_m^n \subset \tilde{\pi}^{-1}(T) \) is the complement to \( \sigma(T) \). Now the fact that \( H^q(F, k_{(l,F-0)}) = 0 \) implies easily that \( R^q\pi_* \mathcal{F}(\tilde{\pi}, K) = 0 \).

**Proposition 2.3.2.** The perverse sheaves \( \mathcal{F}_{\chi_1, \ldots, \chi_k} \) are pointwise pure.

**Proof.** It suffices to prove that the stalk at zero of the sheaf \( \mathcal{F}_{\chi_1, \ldots, \chi_k} \) is pure of weight 0. By Galois covering argument it suffices to prove the purity of the stalk at zero of the sheaf

\[
K = \mathfrak{g}_* L(\psi(x_1^{d_1} \ldots x_n^{d_n}) \chi(x_1^{n_1} \cdots x_k^{n_k}))
\]

where \( m, n > 0 \). Note that \( K \) is pure of weight 0 (see [1], 5.3.2). In particular, \( K|_0 \) is of weight \( \leq 0 \). On the other hand, by the principal theorem of [1] (3.3.1, 6.2.3) the complex \( R\Gamma(\mathbb{A}^n, K) \) is of weight \( \geq 0 \). Applying the above lemma to the \( \mathbb{G}_m \)-action

\[
t(x_1, \ldots, x_n, y_1, \ldots, y_m) = (t^m x_1, \ldots, t^m x_n, t^n y_1, \ldots, t^n y_m)
\]

we deduce that \( K|_0 \) is pure of weight 0.

**Corollary 2.3.3.** Let \( l \) be the number of negative integers among \( (n_1, \ldots, n_k), N \) be the least common multiple of \( (|n_1|, \ldots, |n_k|) \). Then the eigenvalues of \( \text{Frob}_\eta \) on the stalk of the sheaf

\[
\mathfrak{g}_* L(\psi(x_1^{d_1} \ldots x_k^{d_k}) \chi(x_1^{n_1} \cdots x_k^{n_k}))
\]

at 0 for \( \chi \neq 1 \) have form \( q^i \lambda \) where \( i \in \mathbb{Z}, 0 \leq i < N \), and \( \lambda' = -g(\eta) \) for some character \( \eta \) of \( \mathbb{F}_q^* \) satisfying \( \eta^{dN} = \chi^N \circ N\text{tr} \). If \( \chi = 1 \) in addition the eigenvalues of the form \( q^{i+1} \) for \( 0 \leq i < N \) can appear.

3. **Identities between Gauss sums**

3.1. **Relations between cyclotomic divisors.** For every \( N \in \mathbb{Z}_{>0} \) let us denote by \( A_N \) the abelian group generated by symbols \([s,n]_N\) where \( s \in \mathbb{Z}/N\mathbb{Z}, n \in \mathbb{Z}_{>0} \), subject to the following relations: for every \( d|N \) we have

\[
[s,dn]_N = \sum_{i=0}^{d-1} [s + i\frac{N}{d}, n]_N.
\]

(3.1)
Also for every prime $p$ we can consider the group $A_N^{(p)}$ defined in the same way as $A_N$ except that we allow only symbols $[s, n]_N$ with $(n, p) = 1$, and the relation (3.1) is imposed for every $d/N$ such that $(d, p) = 1$.

**Lemma 3.1.1.** The elements $[s, n]_N$ such that $\gcd(s, n, N) = 1$ form a basis of $A_N$ (resp. $A_N^{(p)}$). In particular, $A_N^{(p)}$ is a subgroup in $A_N$.

The proof is straightforward and is left to the reader.

For every set $S$ let us denote by $\text{Div}(S)$ the group $\oplus_S \mathbb{Z}$, i.e. the group of formal linear combinations of elements of $S$ with integer coefficients. We’ll call elements of $\text{Div}(S)$ divisors on $S$. In particular, we want to consider $\text{Div}(\mathbb{Q}/\mathbb{Z})$. For every pair $(r, n)$ where $r \in \mathbb{Q}$, $n \in \mathbb{Z}_{>0}$, we consider the divisor

$$D_{r, n} = (r) + (r + 1/n) + \ldots + (r + (n - 1)/n)$$

(3.2) on $\mathbb{Q}/\mathbb{Z}$. The divisor $D_{r, n}$ depends only on $n$ and on the residue class of $r$ modulo $1/n \mathbb{Z}$. We have the following relations between these divisors:

$$D_{r, dn} = \sum_{i=0}^{d-1} D_{r+i/dn, n}.$$

This means that we can define the homomorphism

$$\alpha_N : A_N \to \text{Div}(\mathbb{Q}/\mathbb{Z})$$

by the formula $\alpha_N([s, n]_N) = D_{s/n, n}$.

One immediately checks that for every $M \in \mathbb{Z}_{>0}$ there is a homomorphism $\phi_{M, N} : A_N \to A_{MN}$ sending $[s, n]_N$ to $[Ms, n]_{MN}$. Moreover, one has $\phi_{K, MN} \circ \phi_{M, N} = \phi_{KM, N}$ and $\alpha_N = \alpha_{MN} \circ \phi_{M, N}$.

**Lemma 3.1.2.** The homomorphism $\phi_{M, N}$ identifies $A_N$ (resp. $A_N^{(p)}$) with a direct summand of $A_{MN}$ (resp. $A_{MN}^{(p)}$).

**Proof.** It suffices to consider the case when $M$ is prime. Let us look at the images of basis elements $[s, n]_N$, $\gcd(s, n, N) = 1$. If $\gcd(n, M) = 1$ then $\phi_{M, N}([s, n]_N) = [Ms, n]_{MN}$ is a basis element in $A_{MN}$. Otherwise, $M|n$ and we have

$$\phi_{M, N}([s, n]_N) = \sum_{i=0}^{M-1} [s + iN, \frac{n}{M}]_{MN}.$$ 

Since $\gcd(s, n, N) = 1$ it follows that $\gcd(s, M, N) = 1$, therefore, replacing $s$ by $s + N$ if necessary we can assume that $\gcd(s, M) = 1$. Then $[s, \frac{n}{M}]_{MN}$ is a basis element in $A_{MN}$. Clearly, the basis elements obtained in this way are all different which implies our statement.

**Theorem 3.1.3.** One has $\ker(\alpha_N) = 0$.

Assume that $\alpha_N(x) = 0$ where $x = \sum_i m_i[s_i, n_i]_N$ with $\gcd(s_i, n_i, N) = 1$. Set $M = \prod_i n_i$. We claim that $\phi_{M, N}(x) = 0$. Indeed, we have

$$\phi_{M, N}([s_i, n_i]_N) = [Ms_i, n_i]_{MN} = \sum_{j=0}^{n_i-1} [M\bar{s}_i + j \frac{MN}{n_i}, 1]_{MN}$$

(3.3)

where $\bar{s}_i \in \mathbb{Z}$ is a representative of $s_i$. By Lemma 3.1.1 the map $\alpha_{MN}$ is injective on the subgroup generated by elements $[\bar{s}_i, 1]_{MN}$. Therefore, $\phi_{M, N}(x) = 0$. Hence, $x = 0$ by Lemma 3.1.2.
3.2. Application to Gauss sums. One can generalize the content of 3.1 to the case of cyclic groups without fixed generators. Namely, for every finite cyclic group $G$ we can define the abelian group $A(G)$ generated by symbols $[g, n]$ where $g \in G$, $n \in \mathbb{Z}_{>0}$ subject to relations

$$[g^d, dn] = \sum_{h \in H} [gh, n]$$

(3.4)

for every subgroup $H \subset G$, where $d = |H|$. Given a prime number $p$ we can define similarly the group $A^p(G)$ using only the symbols $[g, n]$ with $gcd(n, p) = 1$. Thus, we have $A_N = A(\mathbb{Z}/N\mathbb{Z})$, $A_N^p = A^p(\mathbb{Z}/N\mathbb{Z})$.

Note that Lemma 3.1.2 can be reformulated as follows: for every inclusion of finite cyclic groups $H \subset G$ the induced homomorphism $A(H) \to A(G)$ identifies $A(H)$ with the direct summand of $A(G)$. Similar property holds for the homomorphism $A^p(H) \to A^p(G)$.

Now let us fix a finite field $\mathbb{F}_q$, where $q = p^s$. For every $d > 0$ we denote by $X_d = X(\mathbb{F}^*_{q^d})$ the group of characters of $\mathbb{F}^*_{q^d}$. For every $d_1 | d_2$ we have the inclusion $X_{d_1} \to X_{d_2}$ induced by the norm homomorphism $\mathbb{F}^*_{q^{d_2}} \to \mathbb{F}^*_{q^{d_1}}$. Let us denote by $X$ the direct limit of the system $(X_d)$ with respect to these inclusions. The group $X$ is isomorphic non-canonically to the $q$-prime part of $\mathbb{Q}/\mathbb{Z}$.

Let us define the homomorphism $\alpha^d : A^p(X_d) \to \text{Div}(X)$ by sending $[\chi, n]$, $\chi \in X_d$, $n \in \mathbb{Z}_{>0}$, to the divisor

$$D_{\chi, n} = \sum_{\xi \in X : \xi^n = \chi} (\xi).$$

(3.5)

Then Theorem 3.1.3 implies that all the homomorphisms $\alpha^d$ are injective. Moreover, the induced injective homomorphism

$$\lim_d A^p(X_d) \to \text{Div}(X)$$

(3.6)

is clearly surjective, therefore, it is an isomorphism.

Let us fix a non-trivial additive character $\psi : \mathbb{F}_q \to \mathbb{C}^*$. Then to every generator $[\chi, n]$ of $A^p(X(\mathbb{F}^*_q))$, we can associate the following function $f_{\chi, n}$ on $X(\mathbb{F}^*_q)$:

$$f_{\chi, n}(\lambda) = \frac{g(\lambda^n \chi)}{\lambda(n^\chi) g(\chi)}$$

(where by our convention $g(1) = -1$). It is easy to see that Hasse-Davenport formula (3.4) implies that the map $[\chi, n] \mapsto f_{\chi, n}$ extends to a homomorphism $A^p(X(\mathbb{F}^*_q)) \to \mathcal{C}(X(\mathbb{F}^*_q), \mathbb{C}^*)$ where $\mathcal{C}(S, \mathbb{C}^*) = (\mathbb{C}^*)^S$ is the group of $\mathbb{C}^*$-valued functions on $S$.

It is convenient to set

$$D_{\chi^{-1}, -n} = -D_{\chi, n}.$$  

(3.7)

We have the following corollary of Theorem 3.1.3.

**Corollary 3.2.1.** Assume that $\sum_{i=1}^k D_{\chi_i, n_i} = 0$ in $\text{Div}(X)$, where $\chi_i \in X(\mathbb{F}^*_q)$, $gcd(n_i, q) = 1$. Then for every character $\lambda$ of $\mathbb{F}^*_q$ one has

$$\prod_i \frac{g(\lambda^{n_i} \chi_i)}{\lambda(n_i) \chi_i} = q^{m(\lambda)}$$

for some $m(\lambda) \in \mathbb{Z}$. In particular, if $\lambda^{n_i} \chi_i \neq 1$ for all $i$ then $2m(\lambda)$ is the number of $i$ such that $\chi_i = 1$. 

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Proof. This follows from the fact that \( g(\lambda^{-1}) \equiv \lambda(-1)g(\lambda)^{-1} \mod q^2 \) for any \( \lambda \).

In fact, the above corollary in some sense describes all multiplicative identities between Gauss sums which hold universally over all extensions of a given finite field. Here is a more precise statement.

**Theorem 3.2.2.** Let \((\chi_1, \ldots, \chi_k)\) be a collection of characters of \( \mathbb{F}_q^* \), \((n_1, \ldots, n_k)\) be integers such that \( \gcd(n_i, q) = 1 \). Assume that for any \( d \geq 1 \) and for any character \( \lambda \in X(\mathbb{F}_q^*) \) such that \( \lambda^{-1} \neq \chi_i \circ \text{Nm}_d \) for any \( i \), one has

\[
\prod_{i=1}^{k} (-g(\lambda^{n_i}(\chi_i \circ \text{Nm}_d), \psi \circ \text{Tr}_d)) = c^d \cdot \lambda(a)
\]

for some constants \( c \in \mathbb{C}^* \), \( a \in \mathbb{F}_q^* \). Then \( \sum_{i=1}^{k} D_{\chi_i, n_i} = 0 \).

**Proof.** Since \( D_{\chi_i, n_i} = D_{\chi, n} \) we can pass to any extension \( \mathbb{F}_{q^d} \) of \( \mathbb{F}_q \). Thus, we can assume that \( \chi_i \) is \( n_i \)-th power of some character and applying Hasse-Davenport identity \([4]\) we reduce ourselves to the case \( n_i = \pm 1 \). In other words, it suffices to prove that if for two collections of characters of \( \mathbb{F}_q^* \): \((\chi_1, \ldots, \chi_k)\) and \((\eta_1, \ldots, \eta_l)\) and for some constants \( c \in \mathbb{C}^* \) and \( a \in \mathbb{F}_q^* \) one has

\[
\prod_{i=1}^{k} (-g(\lambda^{n_i}(\chi_i \circ \text{Nm}_d), \psi \circ \text{Tr}_d)) = c^d \cdot \lambda(a) \cdot \prod_{j=1}^{l} (-g(\lambda^{n_j}(\eta_j \circ \text{Nm}_d), \psi \circ \text{Tr}_d))
\]

for all \( \lambda \in X(\mathbb{F}_q^*) \) such that \( \lambda^{-1} \) is different from all \( \chi_i \circ \text{Nm}_d \) and \( \eta_j \circ \text{Nm}_d \), then \( \chi_i = \eta_j \) for some \( (i, j) \). Let

\[
\{\chi_1, \ldots, \chi_k, \eta_1, \ldots, \eta_l\} = \{\mu_1, \ldots, \mu_r\}
\]

where the characters \( \mu_i \) are all different. Then we can rewrite the above identity as follows

\[
\prod_{i=1}^{k} (-g(\lambda^{n_i}(\chi_i \circ \text{Nm}_d), \psi \circ \text{Tr}_d)) - \sum_{i=1}^{r} c_i^d \delta(\lambda^{n_i} \circ \text{Nm}_d) =
\]

\[
c^d \cdot \lambda(a) \cdot \left( \prod_{j=1}^{l} (-g(\lambda^{n_j}(\eta_j \circ \text{Nm}_d), \psi \circ \text{Tr}_d)) - \sum_{i=1}^{r} b_i^d \delta(\lambda^{n_i} \circ \text{Nm}_d) \right)
\]

for some constants \( c_i, b_i \), where \( \delta \) is the delta-function at the trivial character. Let us denote

\[
K_{\psi; \chi_1, \ldots, \chi_k}(t) = \sum_{x_1, \ldots, x_k = t} \psi(x_1 + \ldots + x_k)\chi_1(x_1)\ldots \chi_k(x_k)
\]

for \( t \in \mathbb{F}_{q^*}^\times \). Then the multiplicative Fourier transform of \( K_{\psi; \chi_1, \ldots, \chi_k} \) is the function

\[
\lambda \mapsto \prod_{i=1}^{k} g(\lambda \chi_i).
\]
Thus, applying the inverse Fourier transform to the above equality we obtain the following equality of functions on $F^*_q$:

$$(-1)^k K_{\psi \circ Tr_d; \chi_1 \circ Nm_d, \ldots, \chi_k \circ Nm_d} - \frac{1}{q-1} \sum_{i=1}^r c_i^d \mu_i \circ Nm_d = c_{t^*_a} \left( (-1)^l K_{\psi \circ Tr_d; \eta_1 \circ Nm_d, \ldots, \eta_l \circ Nm_d} - \frac{1}{q-1} \sum_{i=1}^r b_i^d \mu_i \circ Nm_d \right)$$

where $t^*_a \cdot f(x) := f(a^{-1}x)$. According to Theorem 7.8 of [4] there exists an irreducible local system on $G_m$ whose trace functions over extensions of $F_q$ are given by $K_{\psi \circ Tr_d; \chi_1 \circ Nm_d, \ldots, \chi_n \circ Nm_d}$. Therefore, by Theorem 1.1.2 of [13] the above equality implies the similar equality in the Grothendieck group of local systems of $G_m$. Hence, we necessarily should have

$$(-1)^k K_{\psi; \chi_1, \ldots, \chi_k} = c_{t^*_a} K_{\psi; \eta_1, \ldots, \eta_l}.$$ 

Making the multiplicative Fourier transform we conclude that the equality

$$\prod_{i=1}^k (-g(\lambda \chi_i)) = c \cdot \lambda(a) \cdot \prod_{j=1}^l (-g(\lambda \eta_j))$$

holds for all $\lambda \in X(F^*_q)$. Now considering the jumps of the absolute value of both sides we immediately derive that the sets $\{\chi_1, \ldots, \chi_k\}$ and $\{\eta_1, \ldots, \eta_l\}$ are the same. 

4. IDENTITIES WITH THE FOURIER TRANSFORM

4.1. Main theorem. Let $\lambda_1, \ldots, \lambda_k$ be characters of $F^*_q$, $(n_1, \ldots, n_k)$ be a collection of integers such that $(n_i, q) = 1$ for every $i$. Let us denote

$$I_{n_1, \ldots, n_k}(a) = \sum_{(x_1, \ldots, x_k) \in (F^*_q)^k} \psi(a \prod_{i=1}^k x_i^{n_i}) \lambda_1(x_1) \ldots \lambda_k(x_k)$$

where $a \in F^*_q$.

Lemma 4.1.1. Let $\lambda$ be a character of $F^*_q$. Then for any $d > 0$ such that $(d, q) = 1$ and any $a \in F^*_q$, one has $I_d^\lambda(a) = 0$ unless there exists a character $\mu$ such that $\lambda = \mu^d$. On the other hand,

$$I_d^\mu(a) = \sum_{\chi: \chi^d = 1} g(\mu \chi)(\mu \chi)(a^{-1}).$$

Proof. If $\lambda$ is not of the form $\mu^d$ then the restriction of $\lambda$ to the subgroup of roots of unity of $d$-th order is non-trivial. Thus, summing over cosets of this subgroup we get $I_d^\lambda(a) = 0$. Let $[d] : F_q \to F_q^*$ be the homomorphism of raising to the $d$-th power. Then we have

$$\sum_{\chi^d = 1} \chi(x) = \begin{cases} 0 & x \notin [d](F_q^*), \\ d_1 & x \in [d](F_q^*). \end{cases}$$
where \( d_1 = \gcd(d, q - 1) \). Hence,
\[
\sum_{t \in \mathbb{F}_q^*} \psi(at^d)\mu(t^d) = d_1 \cdot \sum_{x \in [d](\mathbb{F}_q^*)} \psi(ax)\mu(x) = \sum_{x \in \mathbb{F}_q^*, \chi(x) = 1} \psi(ax)(\mu\chi)(x).
\]

\[\square\]

**Lemma 4.1.2.** One has \( I_{\lambda_1, \ldots, \lambda_k}^{n_1, \ldots, n_k}(a) = 0 \) unless there exists \( \lambda \) such that \( \lambda_i = \lambda^{n_i} \) for all \( i \). One has
\[
I_{\lambda_1, \ldots, \lambda_k}^{n_1, \ldots, n_k}(a) = (q - 1)^{k-1} \sum_{t \in \mathbb{F}_q^*} \psi(at^d)\lambda(t) = (q - 1)^{k-1} \sum_{\chi: \chi^n = 1} g(\lambda\chi)(\lambda\chi)(a^{-1}).
\]
where \( d = \gcd(n_1, \ldots, n_k) \).

**Proof.** Let \( n_i = n_i' d \). Since \( \gcd(n_1', \ldots, n_k') = 1 \) we can choose new coordinates \( y_i = \prod_j x_{ij}^{a_{ij}} \) on \( \mathbb{G}^k \) such that \( y_1 = \prod x_j^{n_j} \). Then we have
\[
I_{\lambda_1, \ldots, \lambda_k}^{n_1, \ldots, n_k}(a) = \sum_{(y_1, \ldots, y_k) \in (\mathbb{F}_q^*)^k} \psi(ay_1^{d}) \prod_i \lambda_i(y_i^{b_{ij}})
\]
where \( (b_{ij}) \) is the inverse matrix to \( (a_{ij}) \). Therefore, we get zero unless \( \prod_i \lambda_i^{b_{ij}} = 1 \) for every \( j > 1 \), i.e.
\[
\prod_{i,j} \lambda_i^{b_{ij}}(y_j) = \lambda'(y_1)
\]
where \( \lambda' = \prod_i \lambda_i^{b_{ij}} \). But the LHS is just \( \prod_i \lambda_i(x_i) \). Thus, the condition is that
\[
\prod_i \lambda_i(x_i) = \lambda'(\prod_i x_i^{n_i'})
\]
i.e. \( \lambda_i = (\lambda')^{n_i'} \). It remains to apply Lemma 4.1.1. \[\square\]

Let us denote by \( F^{n_1, \ldots, n_k}(a) = F_{\lambda_1, \ldots, \lambda_k}^{n_1, \ldots, n_k}(a, \psi) \) the simple perverse sheaf on \( \mathbb{A}^k \) obtained as the Goreski-MacPherson extension of the smooth perverse sheaf
\[
L_{\psi}(a \prod_i x_i^{n_i}) \otimes L_{\chi_1} \boxtimes \cdots \boxtimes L_{\chi_k}[k]
\]
on \( \mathbb{G}^k \). For \( a = 1 \) we get the sheaf which we earlier denoted \( F_{\chi_1, \ldots, \chi_k}^{n_1, \ldots, n_k} \).

Recall that for every character \( \chi \in X(\mathbb{F}_q) \) and \( n \neq 0 \) we have defined the divisor \( D_{\chi,n} \in \text{Div}(X) \) by (3.3) and (3.7).

**Theorem 4.1.3.** Assume that \( (n_i, q) = 1 \) for all \( i \). One has an isomorphism
\[
\mathcal{F}(F^{n_1, \ldots, n_k}_{\chi_1, \ldots, \chi_k}(a)) \cong V \otimes F_{\eta_1, \ldots, \eta_k}^{m_1, \ldots, m_k}(b)
\]
where \( V \) is a one-dimensional \( \mathbb{Q}_l \)-vector space with \( \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) \)-action in the following situations:
(i) \( \sum n_i = 2 \), \( m_i = n_i \) for all \( i \); \( \eta_i = \chi^{n_i} \chi_i^{-1} \) where the characters \( \chi, \chi_i \in X(\mathbb{F}_q) \) satisfy
\[
D_{1,1} + D_{\chi^{-1},1} = \sum_i D_{\chi_i^{-1}, m_i};
\]
(4.2)
if \( \gcd(n_1, \ldots, n_k) = 2 \) then we require that \( \chi \) is the non-trivial character of order 2 (so \( q \) should be odd):

\[
ab = -\prod_i n_i^{-n_i};
\]  

(4.3)

\[
V = G(\chi^{-1}) \otimes (\bigotimes_i G(\chi_i)) \otimes (L_{\chi})_{-b}(-m)
\]  

(4.4)

where \( G(\lambda) \) are defined by (4.2) and (4.4). \( F \mapsto F(1) \) is the Tate twist (the action of \( \text{Frob}_q \) is multiplied by \( q^{-1} \)), \( 2m+1 \) is the number of trivial characters among \( \chi, \chi_1, \ldots, \chi_k \).

(ii) \( \sum n_i = 0 \), \( m_i = -n_i \) for all \( i \); \( \eta_i = \chi^{n_i} \chi_i^{-1} \) where the characters \( \chi, \chi_i \in X(F_q^*) \) satisfy

\[
D_{1,1} - D_{\chi^{-1},1} = \sum_i D_{\chi_i^{-1},m_i};
\]  

(4.5)

if \( \gcd(n_1, \ldots, n_k) > 1 \) then we require \( \chi = 1 \):

\[
\frac{a}{b} = \prod_i n_i^{-n_i};
\]  

(4.6)

\[
V = G(\chi) \otimes (\bigotimes_i G(\chi_i)) \otimes (L_{\chi^{-1}})_{-b}(-m)
\]  

(4.7)

where \( 2m+1 \) is the number of trivial characters among \( \chi, \chi_1, \ldots, \chi_k \).

Proof. Note that for any field extension \( F_q \subseteq F_{q_i} \), the extension of scalars of \( F_{q_i}^{n_1, \ldots, n_k}(a, \psi) \) to \( F_{q_i} \) is isomorphic to \( F_{q_i}^{x_1, \ldots, x_k}(a, \psi \circ \text{Tr}) \). Notice also that our assumptions do not change after arbitrary extension of scalars. Thus, by Lemma 1.3.1 it suffices to prove the identity

\[
(-q)^k (\psi(a \prod_i x_i^{n_i} \chi_i(x_i)), \lambda_1 \circ \ldots \circ \lambda_k) = c \prod_{i=1}^k g(\lambda_i) (\psi(b \prod_i x_i^{m_i} \eta_i(x_i)), \lambda_i^{-1} \circ \ldots \circ \lambda_k^{-1})
\]

for every collection of non-trivial characters \( \lambda_i \), where \( c = \text{Tr}(\text{Frob}_{q_i}, V) \). Using the previous notation we can write this identity as follows:

\[
(-q)^k I_{n_1, \ldots, n_k}^{m_1, \ldots, m_k} (a) = c \prod_{i=1}^k g(\lambda_i) \cdot I_{\eta_1, \ldots, \eta_k, \lambda_1, \ldots, \lambda_k} (b).
\]

(4.8)

Now let us specialize to different cases.

(i) In this situation both sides are zero unless there exists \( \lambda \) such that

\[
\frac{\chi_i}{\lambda_i} = \lambda^{n_i}.
\]

Then we have \( \lambda_i = \chi_i \lambda^{-n_i} \), so that \( \eta_i \lambda_i = (\chi \lambda^{-1})^{n_i} \). Assume first that \( \gcd(n_1, \ldots, n_k) = 1 \). Then according to Lemma 1.1.2 our identity (4.8) takes form

\[
(-q)^k g(\lambda) (\lambda a^{-1}) = c \prod_{i=1}^k g(\chi_i, \lambda^{-n_i}) \cdot g(\chi \lambda^{-1}) \cdot (\chi \lambda^{-1})(b^{-1})
\]

Notice that the relation (4.2) implies that there exists \( i \) such that \( \chi_i = 1 \) and \( j \) such that \( \chi_j = \chi^{n_j} \). Therefore, the non-triviality of all the characters \( \lambda_i = \chi_i \lambda^{-n_i} \) implies the non-triviality of \( \lambda \) and of \( \chi \lambda^{-1} \). On the other hand, we have \( |c| = q^{\frac{k}{2}} \).
Therefore, both sides of our identity have the same absolute value so it suffices to prove the identity modulo $q^Z$. Then we can rewrite it as follows:

\[ (-1)^k g(\lambda)g(\lambda^{-1}) \equiv c \cdot \left( \prod_i \chi_i(-1) \cdot (\chi)(-b^{-1})\lambda(-ab) \cdot \prod_{i=1}^k g(\chi_i^{-1}\lambda^{n_i}) \right) \mod q^Z. \]

On the other hand, by Corollary 3.2.1 and (4.2) we have

\[ g(\lambda)g(\lambda^{-1}) \equiv -g(\chi^{-1}) \prod_i \frac{g(\chi_i^{-1}\lambda^{n_i})}{\lambda(n_i^{\alpha_i})g(\chi_i^{-1})} \mod q^Z. \]

Substituting this in the previous identity and using (4.3) we get

\[ c \equiv (-1)^{k+1} (\chi)(-b)g(\chi^{-1}) \prod_{i=1}^k g(\chi_i) \mod q^Z \]

which follows from (4.3).

Now consider the case $\gcd(n_1, \ldots, n_k) = 2$. Then using Lemma 4.1.2 and the equality $s = N/2$ the identity (4.8) can be rewritten as

\[ (-1)^k (g(\lambda)g(\lambda^{-1}) + g(\lambda\epsilon_2)(\lambda\epsilon_2)(a^{-1})) = c \cdot \prod_{i=1}^k g(\chi_i\lambda^{-n_i}) \cdot (g(\epsilon_2\lambda^{-1})(\epsilon_2\lambda^{-1})(b^{-1}) + g(\lambda^{-1}\lambda^{-1})(b^{-1})) \]

where $\epsilon_2$ is the non-trivial character of order 2. Since the characters $\lambda$ and $\lambda\epsilon_2$ are non-trivial we have $g(\lambda^{-1}) = g(\lambda^{-1})g(\lambda^{-1})$ and $g(\lambda^{-1}\epsilon_2) = g(\lambda^{-1})\epsilon_2(-1)g(\lambda\epsilon_2)^{-1}$. Notice also that since $n_i$ are even we have $\epsilon_2(-ab) = 1$. Therefore, our identity follows from

\[ (-1)^k g(\lambda)g(\lambda\epsilon_2) = g(\epsilon_2(-ab))g(\epsilon_2(-b^{-1}))c \cdot \prod_{i=1}^k g(\chi_i\lambda^{-n_i}), \]

which can be proven as in the previous case.

(ii) Again both sides of (4.8) are zero unless there exists $\lambda$ such that $\lambda_i = \chi_i\lambda^{-n_i}$ so that $\eta_i\lambda_i = (\chi^{-1}\lambda^{-1})\chi^{-1} = \chi^{-1}\lambda^{n_i}$. Assume first that $\gcd(n_1, \ldots, n_k) = 1$. Then according to Lemma 4.1.2 the identity (4.8) takes form

\[ (-1)^k (g(\lambda)\lambda(a^{-1}) = c \cdot \prod_{i=1}^k g(\chi_i\lambda^{-n_i}) \cdot g(\chi^{-1}\lambda)(\chi^{-1}\lambda)(b^{-1}). \] (4.9)

In the case $\gcd(n_1, \ldots, n_k) = d > 1$ the identity (4.8) is equivalent to

\[ (-1)^k \sum_{\eta^d=1} g(\lambda\eta)(\lambda\eta)(a^{-1}) = c \cdot \prod_{i=1}^k g(\chi_i\lambda^{-n_i}) \cdot \sum_{\eta^d=1} g(\lambda\eta)(\lambda\eta)(b^{-1}). \]

which reduces to (4.9) with $\chi = 1$ since $\eta(a) = \eta(b)$ for any character $\eta$ of order $d$ (note that by (4.6) the ratio $a/b$ is the $d$-th power). Note that $|c| = q^k$. On the other hand, if $\chi \neq 1$ then the non-triviality of characters $\chi_i^{-1}\lambda^{n_i}$ and the equality (4.5) imply the non-triviality of $\chi$ and $\chi^{-1}\lambda$. Thus, both sides of (4.9) have the same absolute value so we can work modulo $q^Z$. The remaining part of the proof is similar to the case (i).
4.2. **Hypergeometric sheaves.** If we work over the algebraic closure of a finite field then the isomorphisms of Theorem 4.1.3 follow easily from the theory of hypergeometric sheaves developed in [10], [11] and [8]. Indeed, let \( f : \mathbb{G}_m^n \to \mathbb{G}_m \) be a non-constant homomorphism of tori. Then for any character \( \chi \) of \( \mathbb{G}_m^n(\mathbb{F}_q) \) we can consider the sheaf \( j_1^*L(\psi(f(x))\chi(x))[n] \) on \( \mathbb{A}^n \) where \( j : \mathbb{G}_m^n \to \mathbb{A}^n \) is the standard open embedding. By a simple coordinate change one can see that
\[
j^*F j_1^*L(\psi(f(x))\chi(x))[n] \simeq (f^{-1})^*j_1^*F j_1^*H
\]
where \( j_1 : \mathbb{G}_m \to \mathbb{A}^1 \) is the standard embedding, \( H \) is the hypergeometric sheaf on \( \mathbb{G}_m \) defined as follows:
\[
H = \text{im}(f_L(\psi(\sum x_i)\chi(x))[n] \to f_*L(\psi(\sum x_i)\chi(x))[n]).
\]
To see that \( H \) is a hypergeometric and to compute it explicitly we notice that according to Proposition 5.6.2 of [10] for any multiplicative character \( \eta \) one has
\[
[N]_*L(\psi(t)\eta(t))[1] \simeq \text{Hyp}(!, \psi; (\eta); \emptyset)
\]
where \([N] : \mathbb{G}_m \to \mathbb{G}_m\) is the morphism of raising to the \( N \)-th power \((N \text{ is assumed to be relatively prime to } q)\), \((\eta)\) is set of \( N \)-th roots of \( \eta \) considered as a character of \( \overline{\mathbb{F}}_q^* \), \( \text{Hyp}(!, \psi; (\eta); \emptyset) \) is the hypergeometric sheaf defined as the \(!\)-convolution of the \( N \) sheaves \( L(\psi(t)\eta(t))[1] \) on \( \mathbb{G}_m \). On the other hand, the Cancellation Theorem 8.4.7 of [11] implies that for arbitrary collections of characters \((\eta_i)\) and \((\rho_j)\) the unique simple quotient of the \(!\)-convolution of the sheaves \( L(\psi(t)\eta_i(t))[1] \) and \( L(\psi(-t^{-1})\rho_j)[1] \) on \( \mathbb{G}_m \) depends only on the divisor \( \sum (\eta_i) - \sum (\rho_j) \). According to Proposition 8.1.4 of [8] this unique simple quotient coincides with the image of the natural morphism from the \(!\)-convolution to the \(*\)-convolution of the same collection of sheaves. Thus, \( H \) is the (irreducible) hypergeometric sheaf corresponding to the divisor \( \sum D_{\chi_i, \eta_i} \) where \( f(x) = \prod x_i^{n_i}, \chi(x) = \prod \chi_i(x_i) \). It remains to notice that \( j_1^*F (j_1^*H) \) is the image of the natural morphism \( L_{\psi,* \text{inv}} H \to L_{\psi,* \text{inv}} H \), where \( \text{inv} : \mathbb{G}_m \to \mathbb{G}_m \) is the inversion. Thus, it is also a hypergeometric sheaf which can be computed by Cancellation Theorem. This gives isomorphisms of Theorem 4.1.3 over an algebraically closed field. On the other hand, this argument allows to compute the rank of the Fourier transform of the sheaf
\[
j^*L(\psi(\prod x_i^{m_i} \prod \chi_i(x_i) \prod \eta_j(y_j)))
\]
for generic characters \( \chi_i, \eta_j \) (i.e. when there is no cancellation), where \( m_i \) and \( n_j \) are positive and prime to \( q \). Namely, using Theorem 8.4.2 of [11] we find that this rank is equal to \( \max(\sum m_i, \sum n_j + 1) \).

4.3. **Examples.** In the case \( k = 1 \) the conditions of Theorem 4.1.3 are satisfied only in the case \( n_1 = N = 2, \chi_1 = 1, \chi = e_2 \), which corresponds to the isomorphism
\[
F(\mathcal{L}_\psi(ax^2)) \simeq \mathcal{G}(e_2) \otimes (L_{e_2}a) \otimes \mathcal{L}_\psi(-\frac{x^2}{4a}).
\]
For \( k = 2 \) we have the following examples:
1. \( n_1 = n_2 = 1, \chi_1 = \eta_2 = 1, \chi_2 = \eta_1 = \chi \), where \( \chi \) is an arbitrary character, the corresponding isomorphism is
\[
F(\mathcal{L}_\psi(axy)\chi(x)) \simeq (L_{\psi})_{-a^{-1}}(-1) \otimes (L_{\psi}(-a^{-1} xy)\chi(y)).
\]
2. Let $f_{3,-1}: \mathbb{F}_q^2 \to \mathbb{C}$ be the trace function of $F_{3,-1}^{3,-1}$. Then according to Theorem 2.1.1, we have

$$f_{3,-1}(x, y) = \begin{cases} \psi(x^3) \epsilon_3(y), & y \neq 0, \\ 0, & y = 0, x \neq 0, \\ g(\epsilon_3^{-1}), & x = y = 0. \end{cases}$$

Now Theorem 4.1.3 implies that

$$\hat{f}_{3,-1}(x, y) = qf_{3,-1}(x, 27y).$$

3. Let $f_{4,-2}^a: \mathbb{F}_q^2 \to \mathbb{C}$ be the trace function of $F_{4,-2}^{4,-2}(a)$. Then using Theorem 2.1.1 and Lemma 4.1.1, we find

$$f_{4,-2}^a(x, y) = \begin{cases} \psi(a^2 x) \epsilon_2(y), & y \neq 0, \\ 0, & y = 0, x \neq 0, \\ g(\epsilon_4(a^{-1}) + g(\epsilon_4^{-1}) \epsilon_4(a), & x = y = 0 \end{cases}$$

if $q \equiv 1 \mod(4)$. In case $q \equiv 3 \mod(4)$ the function $f_{4,-2}^a(x, y)$ is just an extension by zero of $\psi(a^2 x) \epsilon_2(y)$. Now Theorem 4.1.3 implies that

$$\hat{f}_{4,-2}^a(x, y) = qe_2(a)f_{4,-2}^a(x, 32y).$$

4. Let $f_{n,-2}^a: \mathbb{F}_q^2 \to \mathbb{C}$ be the trace function of $F_{n,-2}^{n,-2}(a)$. Then using Theorem 2.1.1 and Lemma 4.1.1, we find

$$f_{n,-2}^a(x, y) = \begin{cases} \psi(a^n x) \epsilon_n(y), & y \neq 0, \\ 0, & y = 0, x \neq 0, \\ g(\epsilon_n(a^{-1}) + g(\epsilon_n^{-1}) \epsilon_n(a), & x = y = 0 \end{cases}$$

if $q \equiv 1 \mod(4)$. In case $q \equiv 3 \mod(4)$ the function $f_{n,-2}^a(x, y)$ is just an extension by zero of $\psi(a^n x) \epsilon_n(y)$. Now Theorem 4.1.3 implies that

$$\hat{f}_{n,-2}^a(x, y) = qe_n(a)f_{n,-2}^a(x, 32y).$$

Now let us consider some higher-dimensional examples.
5. If the conditions of Theorem 4.1.3 are satisfied for the collection \((n_1, \ldots, n_k), (\chi_1, \ldots, \chi_k)\), then they are also satisfied for the collection \((n_1, \ldots, n_k, 1, -1), (\chi_1, \ldots, \chi_k, \chi, \chi^{-1})\) where \(\chi\) is any character. For example, for any characters \(\chi_1, \ldots, \chi_{k+1}\) we have

\[
\mathcal{F}(j_1 L(\psi(\frac{x_1 \cdots x_{k+2}}{y_1 \cdots y_k}) \chi_1(\frac{x_1}{y_1}) \chi(\frac{x_k}{y_k}) \chi_{k+1}(x_{k+1}))[2k+2]) \simeq \overline{Q}_l(-k-1) \otimes \\
j_1 L(\psi((-1)^{k+1} \frac{x_1 \cdots x_{k+2}}{y_1 \cdots y_k}) \chi_{k+1}(\frac{x_1}{y_1}) \chi(\frac{x_k}{y_k}) \chi_{k+1}(-x_{k+2}))[2k+2],
\]

\[
\mathcal{F}(j_1 L(\psi(\frac{x_1 \cdots x_k}{y_1 \cdots y_k}) \chi_1(\frac{x_1}{y_1}) \chi(\frac{x_k}{y_k}) \chi_{k+1}(y_{k+1}))[2k+2]) \simeq \overline{Q}_l(-k-1) \otimes \\
j_1 L(\psi((-1)^k \frac{y_1 \cdots y_k}{x_1 \cdots x_{k+1}}) \chi_1(\frac{y_1}{x_1}) \chi(\frac{y_k}{x_k}) \chi_{k+1}(1)(-1)^{k+1}x_{k+1}))[2k+2],
\]

6. More generally, if the conditions of Theorem 4.1.3 are satisfied for the expression \(\psi(\prod_i x_i^{n_i}) \prod_i \chi_i(x_i),\) they are also satisfied for the expressions

\[
\psi(\prod_i x_i^{n_i}, \frac{u^k}{v_1 \cdots v_k}) \prod_i \chi_i(x_i) \cdot \frac{\prod_{j=1}^k (\chi\epsilon_j^i(v_j))}{\chi^k(u)}
\]

and

\[
\psi(\prod_i x_i^{n_i}, \frac{v_1 \cdots v_k}{u^k}) \prod_i \chi_i(x_i) \cdot \frac{\prod_{j=1}^k (\chi\epsilon_j^i(v_j))}{\chi^k(u)}
\]

where \(\chi\) is any character.

For example, for any \(k > 0\) and any characters \(\chi, \eta\) we have

\[
\mathcal{F}(j_1 L(\psi(\frac{x_1 \cdots x_{k+2}}{y^k}) \prod_{i=1}^k (\chi\epsilon_i^j(x_i)) \cdot \eta(x_{k+1}))[k+3]) \simeq \overline{Q}_l(-2) \otimes \\
(\bigotimes_{i=1}^k G(\epsilon_i^j)) j_1 L(\psi((-1)^{k+1} \frac{x_1 \cdots x_{k+2}}{y^k}) \prod_{i=1}^k (\chi^{-1} \eta \epsilon_i^{-1}(x_i)) \cdot \eta(-x_{k+2}))[k+3],
\]

\[
\mathcal{F}(j_1 L(\psi(\frac{x_1 \cdots x_k}{y^k}) \prod_{i=1}^k (\chi\epsilon_i^j(x_i))[k+1]) \simeq \\
\overline{Q}_l(-1) \otimes (\bigotimes_{i=1}^k G(\epsilon_i^j)) \otimes j_1 L(\psi((-1)^k \frac{y^k}{k^k x_1 \cdots x_k}) \prod_{i=1}^k (\chi^{-1} \epsilon_i^{-1}(x_i)) \chi^{-k}(-y/k))[k+1].
\]

7. Example 2 has the following generalization: for any \(k > 0\) and any \(1 \leq i \leq k+1\) one has

\[
\mathcal{F}(j_1 L(\psi(\frac{x^{k+2}}{y_1 \cdots y_k}) \prod_{1 \leq j < i} \epsilon_{k+2}^j(y_j) \prod_{i \leq j \leq k} \epsilon_{k+2}^{i+1}(y_j))[k+1]) \simeq \\
(\bigotimes_{j=1}^{k+1} G(\epsilon_{k+2}^j)) \otimes j_1 L(\psi((-1)^{k+1} \frac{x^{k+2}}{(k+2)^{k+2} y_1 \cdots y_k}) \prod_{1 \leq j < i} \epsilon_{k+2}^{i-j}(y_j) \prod_{i \leq j \leq k} \epsilon_{k+2}^{i-j-1}(y_j))[k+1]
\]
8. Here is an example involving the monom $\frac{x_1 \cdots x_k}{y^k}$ which is different from the one considered in example 6. Let $\chi$ be any character, then for any $i$, $1 \leq i \leq k$, we have
\[
\mathcal{F}(j_\ast L(\psi(\frac{x_1 \cdots x_k}{y^k}) \prod_{1 \leq j < 1} (\chi\epsilon_k^j(x_j)) \prod_{1 \leq j \leq k-1} (\chi\epsilon_k^{j+1}(x_j)))[k + 1]) \simeq \\
\bigotimes_{j=1}^{k-1} (-1) \otimes (\bigotimes_{i=1}^{j} G(\epsilon_i^k)) \otimes \\
\mathcal{F}(j_\ast L(\psi((-1)^k \frac{y^k}{k! x_1 \cdots x_k}) \prod_{1 \leq j < 1} \epsilon_k^{j-j}(x_j) \prod_{1 \leq j \leq k-1} \epsilon_k^{j-1}(x_j)) (\chi^{-1} \epsilon_k^j)((-x_k))[k + 1])
\]

4.4. Identities with binomial coefficients. Combining the example 5 of the previous section with Theorem 2.2.1 we obtain some identities with polynomials (2.2) and (2.3). We need the following simple lemma.

Lemma 4.4.1. For any character $\chi$ of $\mathbb{F}_q^*$ and any $a, \hat{x}_1, \ldots, \hat{x}_n, \hat{y}_1, \ldots, \hat{y}_n \in \mathbb{F}_q^*$ one has
\[
\sum_{(x,y)\in(\mathbb{F}_q^*)^{2n}} \psi(\frac{x_1 \cdots x_n}{y_1 \cdots y_n}) + \sum_{m=1}^{n} (x_m \hat{x}_m + y_m \hat{y}_m)) \chi(\frac{x_1 \cdots x_n}{y_1 \cdots y_n}) = \\
q^n \psi((-1)^n \frac{\hat{y}_1 \cdots \hat{y}_n}{\hat{x}_1 \cdots \hat{x}_n}) \chi((-1)^n \hat{y}_1 \cdots \hat{y}_n) - q^n - 1 \cdot q - 1 \cdot g(\chi).
\]

Proof. Use the equality (4.10) and induction in $n$. 

Consider the isomorphism
\[
\mathcal{F}(j_\ast L(\psi(\frac{x_1 \cdots x_n}{y_1 \cdots y_n}))) \simeq \bigotimes_{j=1}^{k-1} (-1) \otimes j_\ast L(\psi((-1)^k \frac{\hat{y}_1 \cdots \hat{y}_n}{\hat{x}_1 \cdots \hat{x}_n})).
\]

Let us restrict it to the point with $\hat{y}_1 = \ldots = \hat{y}_r = \hat{x}_1 = \ldots = \hat{x}_s = 0$ and $\hat{y}_{r+1} \ldots \hat{y}_n, \hat{x}_{s+1} \ldots \hat{x}_n \neq 0$. Taking the traces of Frobenius we get the identity
\[
\sum_{(x,y)\in(\mathbb{F}_q^*)^{2n}} \psi(\frac{x_1 \cdots x_n}{y_1 \cdots y_n}) + \sum_{m=1}^{n} (x_m \hat{x}_m + y_m \hat{y}_m) + \\
\sum_{(i,j,k,l)\neq(0,0,0,0)} N(i,j,k,l) a(i+j,k+l) = \\
\left\{ \begin{array}{ll} 
q^n a(r,s), & (r,s) \neq (0,0) \\
q^n \psi((-1)^n \frac{\hat{y}_1 \cdots \hat{y}_n}{\hat{x}_1 \cdots \hat{x}_n}), & r = s = 0 
\end{array} \right.
\]

where
\[
N(i,j,k,l) = \sum_{(x,y)\in S(i,j,k,l)} \psi(\sum_{m=1}^{n} (x_m \hat{x}_m + y_m \hat{y}_m))
\]

and $S(i,j,k,l)$ is the set of $(x,y)\in \mathbb{F}_q^* \times \mathbb{F}_q^*$ with exactly $i$ coordinates $(x_1, \ldots, x_r)$ vanish, $j$ coordinates $(x_{r+1}, \ldots, x_n)$ vanish, $k$ coordinates $(y_1, \ldots, y_s)$ vanish, and $l$ coordinates $(y_{s+1}, \ldots, y_n)$ vanish. Thus,
\[
N(i,j,k,l) = \binom{r}{i} \binom{n-r}{j} \binom{s}{k} \binom{n-s}{l} (q-1)^{r+s-i-k} (-1)^{r+s+k+l}.
\]

On the other hand, it is easy to see that for $(r,s) \neq (0,0)$ we have
\[
\sum_{(x,y)\in(\mathbb{F}_q^*)^{2n}} \psi(\frac{x_1 \cdots x_n}{y_1 \cdots y_n}) + \sum_{m=1}^{n} (x_m \hat{x}_m + y_m \hat{y}_m) = (q-1)^{r+s} (-1)^{r+s+1}
\]
Thus, we arrive to the following identity for any \((r, s) \neq (0, 0)\):

\[
\sum_{(i, j, k, l) \neq (0,0,0,0)} \binom{n-r}{i} \binom{n-s}{j} \binom{n}{k} \binom{1-i-k}{l} (q-1)^{r+s-i-k}(1)^{r+s+j+l} a(i + j, k + l) = q^n a(r, s) + (q-1)^{r+s-1}(1)^{r+s}.
\]  

(4.11)

In the case \(r = s = 0\) using Lemma [4.4.1] we get the identity

\[
\sum_{(j, l) \neq (0, 0)} \binom{n}{j} \binom{n}{l} (-1)^{j+l} a(j, l) = -\frac{q^n - 1}{q - 1}.
\]  

(4.12)

Similarly the isomorphism

\[
\mathcal{F}(j_L, L)(\psi(\frac{x_1 \ldots x_n}{y_1 \ldots y_n})\chi(\frac{x_1 \ldots x_n}{y_1 \ldots y_n})) \approx \mathcal{O}(\psi((-1)^{n} \frac{y_1 \ldots y_n}{x_1 \ldots x_n})\chi((-1)^{n} \frac{y_1 \ldots y_n}{x_1 \ldots x_n}))
\]

for a non-trivial character \(\chi\) leads to the identities

\[
\sum_{(i, j, k, l) \neq (0,0,0,0)} \binom{n-r}{i} \binom{n-s}{j} \binom{n}{k} \binom{1-i-k}{l} (q-1)^{r+s-i-k}(1)^{r+s+j+l} b(i + j, k + l) = q^n b(r, s) + (q-1)^{r+s-1}(1)^{r+s+1}
\]  

(4.13)

for \((r, s) \neq (0, 0)\) and

\[
\sum_{(j, l) \neq (0, 0)} \binom{n}{j} \binom{n}{l} (-1)^{j+l} b(j, l) = \frac{q^n - 1}{q - 1}.
\]  

(4.14)

5. IDENTITIES CONTAINING NORMS

5.1. More identities with Gauss sums. We want to generalize Corollary [3.2.1] to include the identities for Gauss sums containing norms. In other words, we want to employ systematically both Hasse-Davenport identities [1.3] and [1.4]. Let us fix a finite field \(\mathbb{F}_q\). For every \(d > 0\), a character \(\chi \in X(\mathbb{F}_q^*)\), and an integer \(n > 0\), \(\text{gcd}(n, q) = 1\), consider the function on \(X(\mathbb{F}_q^*)\) defined by

\[
f_{\chi, n}(\lambda) = \frac{g((\lambda^n \circ \text{Nm})\chi)}{\lambda(n^ad)g(\chi)}
\]

For \(d = 1\) this definition coincides with the one we considered before. As before the map \([\chi, n] \rightarrow f_{\chi, n}\) extends to a homomorphism from \(A^p(X(\mathbb{F}_q^*))\) to \(\mathcal{C}(X(\mathbb{F}_q^*), \mathbb{C}^*)\) due to the identity [1.4].

Let us define the abelian group \(A_q\) as the quotient of \(\oplus_d A^p(X(\mathbb{F}_q^*))\) by the subgroup generated by the elements of the form \(k[\chi, n] - [\chi \circ \text{Nm}_k, n]\) for all \(\chi \in X(\mathbb{F}_q^*), n, k \in \mathbb{Z}_{>0}\). The identity [1.3] implies that for every \(k > 0\) we have

\[
f_{\chi \circ \text{Nm}_k, n} = f_{\chi, n}^k.
\]

Hence, the map \([\chi, n] \rightarrow f_{\chi, n}\) extends to a homomorphism from \(A_q\) to \(\mathcal{C}(X(\mathbb{F}_q^*), \mathbb{C}^*)\)

On the other hand, we can define a homomorphism

\[
\beta : A_q \rightarrow \text{Div}(X)
\]

by sending \([\chi, n]\) to the divisor \(d \cdot D_{\chi, n}\), where \(\chi \in X(\mathbb{F}_q^*), D_{\chi, n}\) is defined by [1.5].

**Theorem 5.1.1.** \(\ker \beta = 0\).
Proof. By definition the homomorphism $\beta$ can be factorized as follows:

$$\beta : A_q \xrightarrow{\beta'} \lim_{d \to} A^{(p)}(X(\mathbb{F}_{q^d})) \to \text{Div}(X)$$

where the last arrow is the isomorphism $\beta'(x)$, $\beta'$ is defined by the formula $\beta'([\chi, n]) = d[l, n]$ for $\chi \in X(\mathbb{F}_{q^d})$. Clearly, $\beta'$ is an isomorphism modulo torsion so it suffices to prove that the group $A_q$ is torsion-free. Fix a prime number $l$. Assume that we have $lx = 0$ for some $x \in A_q$. We can write $x$ in the form

$$x = \sum_i \pm[i, n_i]$$

with some $\chi_i \in X(\mathbb{F}_{q^{n_i}})$. Let us write the degrees $d_i$ in the form $d_i = d_i' l^{s_i}$ where $gcd(d_i', l) = 1$. Set $d = \prod_i d_i'$. Using the relations in $A_q$ we obtain that $x' = dx$ has form

$$x' = \sum_j \pm[j, n_j]$$

with $\chi_j \in X(\mathbb{F}_{q^{n_j}})$. Since $d$ is relatively prime to $l$ it suffices to prove that $x' = 0$.

Let $A_q(d, l)$ be the quotient of $\oplus_i A^{(p)}(\mathbb{F}_{q^{d_i}})$ by the subgroup generated by the elements $l[\chi, n] - [\chi \circ \text{Nm}_i, n]$. Then we have a sequence of homomorphisms

$$A_q(d, l) \xrightarrow{\beta_{d,l}} \lim_{i \to} A^{(p)}(X(\mathbb{F}_{q^{d_i}})) \to \text{Div}(X)$$

where $\beta_{d,l}([\chi, n]) = d[l, n]$ for $\chi \in X(\mathbb{F}_{q^{d_i}})$, the last arrow is an embedding induced by $\text{Div}$. Notice, that by Lemma 3.1.2 the group $A_q(d, l)$ has no torsion. Therefore, the homomorphism $\beta_{d,l}$ being an isomorphism modulo torsion should be injective. Hence, the composed map

$$\beta_{d,l} : A_q(d, l) \to \text{Div}(X)$$

is injective. Since $\beta_{d,l}$ is a composition of $\beta$ and of the natural homomorphism $A_q(d, l) \to A_q$ we deduce that $A_q(d, l)$ is a subgroup in $A_q$. Now we have $x' = A_q(d, l)$ and $lx = 0$. Since $A_q(d, l)$ has no torsion this implies that $x' = 0$. \hfill $\Box$

This theorem allows to write the identities between Gauss sums containing norms in the following form.

**Corollary 5.1.2.** Let $\chi_1, \ldots, \chi_k$ be the collection of characters, $\chi_i \in X(\mathbb{F}_{q^{n_i}})$; $n_1, \ldots, n_k$ be the collection of integers such that $gcd(n_i, q) = 1$. Assume that $\sum_{i=1}^k d_i D_{\chi_i, n_i} = 0$ in $\text{Div}(X)$. Then for every character $\lambda$ of $\mathbb{F}_q^*$ one has

$$\prod_i g((\lambda^{n_i} \circ \text{Nm}_{d_i})\chi_i) / (\lambda(n_{d_i})g(\chi_i)) = q^{m(\lambda)}$$

for some $m(\lambda) \in \mathbb{Z}$. In particular, if $(\lambda^{n_i} \circ \text{Nm}_{d_i})\chi_i \neq 1$ for all $i$ then $2m(\lambda)$ is the number of $i$ such that $\chi_i = 1$.

One can rewrite this corollary in yet another form. Namely, let $k$ be a finite étale $\mathbb{F}_q$-algebra, so that $k = \prod_i \mathbb{F}_{q^{e_i}}$. Let $X(k^*)$ be the group of characters of $k^*$. For every $\chi \in X(k^*)$ let us denote

$$g(\chi) = \sum_{x \in k^*} \chi(x)\psi(\text{Tr}_{k/\mathbb{F}_q}(x)).$$

(5.1)
In fact, if \( \chi = \prod_i \chi_i \), where \( \chi_i \in X(F_{_q^{*i}}) \) then

\[
g(\chi) = \prod_i g(\chi_i, \psi \circ \text{Tr}_{d_i}).
\]

Let \( V \) be a virtual finite module over \( k \), then \( V \) is determined by the collection of integers \( (n_i = \text{rk}_i V) \) such that \( V = \sum_i n_i[F_{_q^{*i}}] \). For every character \( \chi = \prod_i \chi_i \) of \( k^* \) let us denote

\[
D_{\chi, V} = \sum_i d_i D_{\chi_i, r_k, V}.
\]

Also set

\[
p(V) = \prod_i n_i^{n_i d_i}.
\]

On the other hand, we can associate to \( V \) the homomorphism

\[
det_V = \text{det}_V/F_q : k^* \to F_{_q^*} : (x_i) \mapsto \prod_i \text{Nm}_{d_i}(x_i)^{n_i}.
\]

Then we have

\[
g((\lambda \circ \text{det}_V)\chi) = \prod_i g((\lambda^{n_i} \circ \text{Nm}_{d_i})\chi_i, \psi \circ \text{Tr}_{d_i}).
\]

Thus, Corollary 5.1.2 leads to the following statement.

**Theorem 5.1.3.** Let \( V \) be a virtual finite \( k \)-module such that \( \text{gcd}(p(V), q) = 1 \), \( \chi \) be a character of \( k^* \). Assume that \( D_{\chi, V} = 0 \). Then for every \( \lambda \in X(F_{_q^*}) \) one has

\[
g((\lambda \circ \text{det}_V)\chi) = q^{m(\lambda)} \cdot p(V) \cdot g(\chi)
\]

where \( m(\lambda) \) is an integer depending on \( \lambda \).

**Remark.** In fact, our method allows to prove a stronger result. Namely, consider the natural homomorphism \( \text{Div}(X) \to \text{Div}(X/\text{Frob}_q) \). Then one can replace the assumption \( D_{\chi, V} = 0 \) by the weaker assumption that the image of \( D_{\chi, V} \) in \( \text{Div}(X/\text{Frob}_q) \) is zero. In this way one gets much more identities between Gauss sums (cf. [5],5.13). However, the importance of the identities of theorem 5.1.3 is that they hold universally over any finite extension of a given finite field \( F_q \).

### 5.2. Identities with the Fourier transform containing norms.

Recall (see e.g. [4]) that for every finite étale \( F_q \)-algebra \( k \) one can define the ring scheme \( \mathbb{A}^1_k \) over \( F_q \) in a natural way, so that for every \( F_q \)-algebra \( A \) on has \( \mathbb{A}^1_k(A) = k \otimes_{F_q} A \). Note that as an \( F_q \)-scheme \( \mathbb{A}^1_k \) is non-canonically isomorphic to \( \mathbb{A}^d \), where \( d = [k : F_q] \). There is a natural morphism of \( F_q \)-schemes

\[
\text{Tr} : \mathbb{A}^1_k \to \mathbb{A}^1
\]

inducing the usual trace map on points. We also have an open subscheme \( \mathbb{G}_m k \subset \mathbb{A}^1_k \) of invertible elements such that \( \mathbb{G}_m k(F_q) \simeq k^* \). For every character \( \chi \in X(k^*) \) we can define a rank 1 smooth \( l \)-adic sheaf \( L_\chi \) on \( \mathbb{G}_m k \). Using the multiplication morphism

\[
m : \mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^1_k
\]

and the sheaf \( \text{Tr}^* L_\psi \) we can define the Fourier transform for sheaves on \( \mathbb{A}^1_k \).
Let us call a character \( \chi \in X(k^*) \) non-degenerate if \( k = \prod q_i \) and \( \chi = \prod \chi_i \) where \( \chi_i \) are non-trivial characters of \( \mathbb{F}_{q_i}^* \). For a non-degenerate character \( \chi \in X(k^*) \) we have

\[
\mathcal{F}(j_iL_\chi) \simeq G(\chi) \otimes j_iL_{\chi^{-1}}
\]

where

\[
G(\chi) = H^d_c(\mathbb{G}_m,k,L_\chi),
\]

is the one-dimensional \( \overline{\mathbb{Q}}_l \)-space on which Frob acts as \((-1)^d g(\chi)\), where \( d = [k : \mathbb{F}_q] \). As before we will use the definition (5.2) also in the case of degenerate characters.

For any extension \( \mathbb{F}_q \subset \mathbb{F}_q' \) we have

\[
\mathbb{A}k \otimes_{\mathbb{F}_q} \mathbb{F}_q \simeq \mathbb{A}k'
\]

where \( k' = k \otimes_{\mathbb{F}_q} \mathbb{F}_q' \), and for any \( \chi \in X(k^*) \)

\[
L_\chi \otimes_{\mathbb{F}_q} \mathbb{F}_q \simeq L_{\chi \circ \text{Nm}_{k'/k}}
\]

Let \( V \) be a virtual finite \( k \)-module such that \( \gcd(p(V),q) = 1 \). We can define the scheme-theoretic version of the homomorphism \( \det_V \) considered above:

\[
\det_V : \mathbb{G}_m,k \to \mathbb{G}_m.
\]

Let \( \chi \) be a character of \( k^* \), \( a \) be an element of \( \mathbb{F}_q^* \). We denote by \( F_{V,\chi}(a) = F_{V,\chi}(a,\psi) \) the simple perverse sheaf on \( \mathbb{A}^1k \) obtained as the Goreski-MacPherson extension of the smooth perverse sheaf

\[
L_\psi(a \det_V(x)) \otimes L_\chi[d]
\]

on \( \mathbb{G}_m,k \), where \( d = [k : \mathbb{F}_q] \).

We need a slight generalization of Lemma 4.1.2. Let us denote

\[
I_{V,\chi}(a) = \sum_{x \in k^*} \psi(a \det_V(x)) \lambda(x)
\]

where \( V \) is a virtual finite \( k \)-module, \( \lambda \in X(k^*) \), \( a \in \mathbb{F}_q^* \). For a virtual \( k \)-module \( V \) with \( \text{rk}_i V = n_i \) let us denote

\[
d(V) = [F_q^* : \text{det}_V(k^*)] = \gcd(n_1, \ldots, n_r).
\]

**Lemma 5.2.1.** Assume that \( \gcd(p(V),q) = 1 \). One has \( I_{V,\chi}(a) = 0 \) unless there exists \( \mu \) such that \( \lambda = \mu \circ \det_V \). One has

\[
I_{V,\mu \circ \det_V}(a) = \frac{|k^*|}{q-1} \sum_{\nu \in X(F_q^*) : \nu \circ \det_V = 1} g(\mu \nu)(\mu \nu(a)^{-1})
\]

For a virtual \( k \)-module \( V = \sum_i n_i[\mathbb{F}_{q_i}] \) let us denote \( \text{rk}_{\mathbb{F}_q} V = \sum_i n_i d_i \).

**Theorem 5.2.2.** One has an isomorphism

\[
\mathcal{F}(F_{V,\chi}(a)) \simeq H \otimes F_{W,\eta}(b)
\]

where \( H \) is a one-dimensional \( \overline{\mathbb{Q}}_l \)-vector space with \( \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \)-action in the following situations:

1. \( \text{rk}_{\mathbb{F}_q} V = 2, W = V, \eta = (\nu \circ \det_V)^{-1} \), where the characters \( \nu \in X(\mathbb{F}_q^*), \chi \in X(k^*) \) satisfy

\[
D_{1,1} + D_{\nu^{-1},1} = D_{\chi^{-1},V};
\]
if $d(V) = 2$ then we require that $\nu$ is the non-trivial character of order $2$ (so $q$ should be odd):

$$ab = -p(V)^{-1};$$

$$H = G(\nu^{-1}) \otimes G(\chi) \otimes (L_\nu)_-b(-m)$$

where $m \in \mathbb{Z}$.

(ii) $\text{rk}_{F_q} V = 0$, $W = -V$; $\eta = (\nu \circ \det_V) \chi^{-1}$, where the characters $\nu \in X(F_q^*)$, $\chi \in X(k^*)$ satisfy

$$D_{1, 1} - D_{\nu^{-1}, 1} = D_{\chi^{-1}, V};$$

if $d(V) > 1$ then we require $\nu = 1$;

$$\frac{a}{b} = p(V);$$

$$H = G(\nu) \otimes G(\chi) \otimes (L_{\nu^{-1}})_-b(-m)$$

where $m \in \mathbb{Z}$.

Proof. For every extension $F_q \subset F_{q_1}$ let us denote $k_{q_1} = k \otimes_{F_q} F_{q_1}$. Then the trace function of the sheaf $V_{\chi; q_1}$ over $F_{q_1}$ is given by

$$f_{V, \chi; q_1}(x) = (-1)^d \psi(\text{Tr}(a \det_{V_{q_1}}(x))) \chi(\text{Nm}_{k_{q_1}/k}(x))$$

for $x \in k_{q_1}$, where $V_{q_1} = V \otimes_k k_{q_1}$.

An argument similar to that of Lemma 1.3.1 shows that it suffices to check that for every extension $F_q \subset F_{q_1}$ and every non-degenerate character $\lambda$ of $k_{q_1}^*$, one has

$$(-q_1)^d (f_{V, \chi; q_1}, \lambda) = g(\lambda) \cdot (f_{W, \eta; q_1}, \lambda^{-1})$$

and that this number is not zero for at least one non-degenerate character $\lambda$. More precisely, we replace the embedding $\mathbb{G}_m \to \mathbb{A}^1 k$ by the embedding $j : \mathbb{G}_m k \to \mathbb{A}^1 k$. Then we replace the group $K_0(\mathbb{G}_m^n)$ from the proof of Lemma 1.3.3 by the quotient of $K_0(\mathbb{G}_m k)$ by the subgroup $j^* \mathcal{F}(K_0(Z))$, where $Z$ is the complement to the image of $j$. Then almost the same proof goes through. The only point where one needs a different argument is in showing that an element $x \in K_0(\mathbb{G}_m k)$ belongs to $j^* \mathcal{F}(K_0(Z))$ if and only if a similar condition holds for all its trace functions. For $k$ split over $F_q$ this statement is proven in Lemma 1.3.1. Thus, it suffices to check that if $x \otimes F_{q_1} \in j^* (\mathcal{F}(K_0(Z \otimes F_{q_1})))$ then $x \in j^* \mathcal{F}(K_0(Z))$. Equivalently, we have to check that if an element $y \in K_0(\mathbb{A}^1 k)$ satisfies

$$y \otimes F_{q_1} \in K_0(Z \otimes F_{q_1}) + \mathcal{F}(K_0(Z \otimes F_{q_1}))$$

then $y$ itself satisfies the similar condition. Let us write $y = \sum a_i [F_i] + \sum b_j [G_j]$ where $F_i$ and $G_j$ are simple perverse sheaves, each sheaf $F_i$ satisfies either supp $F_i \subset Z$ of supp $\mathcal{F}(F_i) \subset Z$, while each $G_j$ satisfies neither of these conditions. Then we should have

$$\sum b_j [G_j \otimes F_{q_1}] \in K_0(Z \otimes F_{q_1}) + \mathcal{F}(K_0(Z \otimes F_{q_1}))$$

which implies $b_j = 0$ since the $[G_j \otimes F_{q_1}]$ (and $\mathcal{F}([G_j \otimes F_{q_1}])$) are linear combinations of simple perverse sheaves not supported on $Z \otimes F_{q_1}$.

The rest of the proof goes similar to that of Theorem 4.1.3 using Theorem 5.1.3 and Lemma 7.2.1.
REFERENCES

[1] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Asterisque 100 (1982), 7–172.
[2] S. Bloch, H. Esnault, Gauss-Manin determinants for rank one irregular connections on curves, preprint math.AG/9904098.
[3] S. Bloch, H. Esnault, Gauss-Manin determinant connections and periods for irregular connections, preprint math.AG/9912095.
[4] P. Deligne, Applications de la formule des traces aux sommes trigonométriques, in Cohomologie Etale (SGA 4 1/2), pp. 168–232. Lecture Notes in Math. 569, Springer, 1977.
[5] P. Deligne, Les constantes des équations fonctionnelles des fonctions L, in Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 501–597. Lecture Notes in Math. 349, Springer, 1973.
[6] P. Deligne, La conjecture de Weil, II, Publ. Math. IHES 52 (1980), 313–428.
[7] P. Etingof, D. Kazhdan, A. Polishchuk, When is the Fourier transform of an elementary function elementary?, preprint.
[8] O. Gabber, F. Loeser, Faisceaux pervers l-adiques sur un tore, Duke Math J. 83 (1996), 501–606.
[9] H. Hasse, H. Davenport, Die Nullstellen der Kongruenzzetafunktionen in gewissen zyklischen Fallen, J. Reine Angew. Math. 172 (1934), 151–182.
[10] N. Katz, Gauss sums, Kloosterman sums, and monodromy groups, Princeton University Press, 1988.
[11] N. Katz, Exponential sums and differential equations, Princeton University Press, 1990.
[12] G. Laumon, Majorations de sommes trigonométriques (d’après P. Deligne et N. Katz), in Caractéristique d’Euler-Poincaré, Astérisque, 83–83 (1981), 221–258.
[13] G. Laumon, Transformation de Fourier, constantes d’équations fonctionnelles et conjecture de Weil, IHES Publ. Math. No. 65 (1987), 131–210.