The Asymptotic Behaviour of Tilted Bianchi type VI\(_0\) Universes

Sigbjørn Hervik*
DAMTP,
Centre for Mathematical Sciences,
Cambridge University
Wilberforce Rd.
Cambridge CB3 0WA, UK

July 13, 2018

Abstract

We study the asymptotic behaviour of the Bianchi type VI\(_0\) universes with a tilted \(\gamma\)-law perfect fluid. The late-time attractors are found for the full 7-dimensional state space and for several interesting invariant subspaces. In particular, it is found that for the particular value of the equation of state parameter, \(\gamma = 6/5\), there exists a bifurcation line which signals a transition of stability between a non-tilted equilibrium point to an extremely tilted equilibrium point. The initial singular regime is also discussed and we argue that the initial behaviour is chaotic for \(\gamma < 2\).

1 Introduction

In a recent paper we analysed how tilted fluids affect the general late-time behaviour of cosmological models of Bianchi type [1]. As the asymptotic behaviour of all Bianchi models with a non-tilted \(\gamma\)-law perfect fluid are determined [2–4], the sensitivity of the non-tilted equilibrium points was checked with regards to inclusion of tilted fluids. In this paper, we will consider one of the models in full generality; namely the Bianchi type VI\(_0\) model containing a tilted \(\gamma\)-law perfect fluid.

The last decades, there have been some investigations of Bianchi models with a tilted fluid [5–7]; in particular, type II [8] and type V [9–11]. Apart from these special Bianchi types, the general behaviour of tilted Bianchi universes seems to be poorly understood. The work [1] gave us some hints where interesting behaviour may occur, but a more elaborate analysis is needed in

*S.Hervik@damtp.cam.ac.uk
order to understand the more general behaviour of universes with tilted fluids. More specifically, in [1] it was shown that for the Bianchi type $V_{I_0}$ model there are no stable non-tilted equilibrium points for equation of state parameter obeying $\gamma > 6/5$. The value $\gamma = 6/5$ signals the onset of an instability with respect to tilt implying that at late times the peculiar velocities of the fluid is a major contributor to the cosmological shear. The Bianchi type $V_{I_0}$ is not the most general model but is sufficiently general to account for many interesting phenomena. Thus we will assume that the universe contains a $\gamma$-law perfect fluid ($0 < \gamma < 2$) which in general can be tilted. The state space is thus 7-dimensional [2] (compared to the 8-dimensional state space for the most general models).

The dynamical systems approach – which is the method adopted here – requires that we find all equilibrium points for the system of equations. These equilibrium points play a special role in the evolution of the system. Not only do they correspond to exact self-similar solutions to Einstein’s equations, but some of them may even serve as attractors for more general solutions. However, not many exact solutions with tilted fluids are known for the Bianchi models. Some type II solutions have been found [12], and Rosquist and Jantzen found some type $V_{I_0}$ solutions [13,14]. Here, we will show that the solutions (for $6/5 < \gamma < 3/2$) found by Rosquist and Jantzen are attractors in a four-dimensional invariant subspace of the full state space. However, in the full state space, there always exists an unstable mode. In the full state space, the future attractor for $6/5 < \gamma$ is an extremely tilted model which is connected via a line bifurcation at $\gamma = 6/5$ to a non-tilted equilibrium point. This $\gamma = 6/5$ bifurcation, which correspond to a one-parameter family of exact tilted solutions, seems to have been first discussed in a recent paper by Apostolopoulos [15]. However, not much discussion is devoted to this solution as the solution itself is implicitly given in terms of a cubic. Here, we have introduced a different parameterisation which makes it possible to explicitly write down the solution in terms of the expansion-normalised variables.

We also discuss the initial singular regime, which is a lot more complicated. In fact, there are indications that the initial singular regime possesses an oscillatory, very likely even a chaotic behaviour similar to that found in other Bianchi models [16–24].

The paper is organised as follows. Next, in section 2 we write down the equations of motion using the orthonormal frame formalism. In section 3 we discuss the equilibrium points which are important for the late-time asymptotic behaviour. Then, in section 4 we discuss their stability and present some general results regarding the asymptotic behaviour of the tilted type $V_{I_0}$ model. The initial singular regime is discussed in section 5 before we finally summarise our results in section 6.
2 Equations of motion

The energy-momentum tensor for a tilted perfect fluid is

\[ T_{\mu\nu} = (\dot{\rho} + \dot{p})\dot{u}_\mu \dot{u}_\nu + \dot{p}g_{\mu\nu}, \]  

(1)

where \( \dot{u}^\mu = (\cosh \beta, \sinh \beta c^a) \) is the fluid velocity. The spatial vector \( c^a \) is chosen to be a unit vector in the tangent space of the surfaces of homogeneity; i.e. \( c^a c_a = 1 \). We will further assume that the fluid obey the barotropic equation of state,

\[ \dot{\rho} = (\gamma - 1)\dot{\rho}, \quad 0 < \gamma < 2. \]  

(2)

In terms of the unit normal vector \( u = e_0 \) to the group orbits the energy-momentum tensor takes the imperfect fluid form

\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu + \dot{p}g_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \pi_{\mu\nu}, \]  

(3)

where

\[ \rho = (1 + \gamma \sinh^2 \beta)\dot{\rho}, \]  

(4)

\[ p = \left( \gamma - 1 + \frac{1}{3}\gamma \sinh^2 \beta \right)\dot{\rho}, \]  

(5)

\[ q_a = \gamma \dot{\rho} \cosh \beta \sinh \beta c_a, \]  

(6)

\[ \pi_{ab} = \gamma \dot{\rho} \sinh^2 \beta \left( c_ac_b - \frac{1}{3}h_{ab} \right). \]  

(7)

For the Bianchi cosmologies – which admit a simply transitive symmetry group acting on the spatial hypersurfaces – we can always write the line-element as

\[ ds^2 = -dt^2 + \delta_{ab}\omega^a \omega^b, \]

where \( \omega^a \) is a triad of one-forms obeying

\[ d\omega^a = -\frac{1}{2}C^a_{bc} \omega^b \wedge \omega^c, \]

and \( C^a_{bc} \) depend only on time and are the structure constants of the Bianchi group type under consideration. The structure constants \( C^a_{bc} \) can be split into a vector part \( a_b \), and a trace-free part \( n^{ab} \) by \([25] \)

\[ C^a_{bc} = \varepsilon_{bcd}n^d_a - \delta^a_b a_c + \delta^a_c a_b. \]

The matrix \( n^{ab} \) is symmetric, and, using the Jacobi identity, \( a_b = (1/2)C^a_{ba} \) is in the kernel of \( n^{ab} \)

\[ n^{ab}a_b = 0. \]

For the type VI\(_0\) model, \( a_c = 0 \) and \( n_{ab} \) has two non-zero eigenvalues with opposite sign. This implies that we can choose a frame such that the structure constants can be written

\[ n_{ab} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \bar{n} & n \\ 0 & n & \bar{n} \end{bmatrix}, \quad a_b = 0. \]  

(8)
Furthermore, the type VI$_0$ has $\bar{n}^2 < n^2$.\footnote{Note that the Bianchi type II is the limit where $\bar{n}^2 = n^2$, and that Bianchi type VII$_0$ has $\bar{n}^2 > n^2$.} The equations of motion can now be written down.

Following [2] we introduce expansion-normalised variables to write the system as an autonomous system of differential equations. In the notation of ref. [26] we define

$$
\Sigma_{ab} = \begin{bmatrix}
-2\Sigma_+ & \sqrt{3}\Sigma_{12} & \sqrt{3}\Sigma_{13} \\
\sqrt{3}\Sigma_{12} & \Sigma_+ + \sqrt{3}\Sigma_- & \sqrt{3}\Sigma_{23} \\
\sqrt{3}\Sigma_{13} & \sqrt{3}\Sigma_{23} & \Sigma_+ - \sqrt{3}\Sigma_-
\end{bmatrix},
$$

$$N_{22} = N_{33} = \sqrt{3}\bar{N}, \quad N_{23} = N_{32} = \sqrt{3}N.
$$

We also introduce the three-velocity $V$ by

$$
sinh \beta = \frac{V}{\sqrt{1 - V^2}}, \quad 0 \leq V < 1.
$$

For the expansion-normalised variables the equations of motion are:

$$
\Sigma_+’ = (q - 2)\Sigma_+ + 3(\Sigma_{12}^2 + \Sigma_{13}^2) - 2N^2 + \frac{1}{2}[4N\Sigma_+ v_1 + (N\Sigma_{12} + \bar{N}\Sigma_{13})v_2 - (N\Sigma_{13} + \bar{N}\Sigma_{12})v_3]
$$

$$
\Sigma_-’ = (q - 2)\Sigma_- + \sqrt{3}(\Sigma_{12}^2 - \Sigma_{13}^2) - 2R_1\Sigma_{23} + \frac{\sqrt{3}}{2}[((N\Sigma_{12} + \bar{N}\Sigma_{13})v_2 + (N\Sigma_{13} + \bar{N}\Sigma_{12})v_3]
$$

$$
\Sigma_{12}’ = (q - 2 - 3\Sigma_+ - \sqrt{3}\Sigma_- + \sqrt{3}Nv_1)\Sigma_{12} - (R_1 + \sqrt{3}\Sigma_{23} - \sqrt{3}\bar{N}v_1)\Sigma_{13}
$$

$$
\Sigma_{13}’ = (q - 2 - 3\Sigma_+ + \sqrt{3}\Sigma_- - \sqrt{3}Nv_1)\Sigma_{13} - (-R_1 + \sqrt{3}\Sigma_{23} + \sqrt{3}\bar{N}v_1)\Sigma_{12}
$$

$$
\Sigma_{23}’ = (q - 2)\Sigma_{23} - 2\sqrt{3}\bar{N}N + 2R_1\Sigma_- + 2\sqrt{3}\Sigma_{12}\Sigma_{13} - \sqrt{3}v_2(N\Sigma_{13} + \bar{N}\Sigma_{12})
$$

$$
N’ = (q + 2\Sigma_+)N + 2\sqrt{3}\Sigma_{23}\bar{N}
$$

$$
\bar{N}’ = (q + 2\Sigma_+)\bar{N} + 2\sqrt{3}\Sigma_{23}N
$$

$$
\Omega’ = \frac{\Omega}{1 + (\gamma - 1)V^2}\left\{2q - (3\gamma - 2) + [2q(\gamma - 1) - (2 - \gamma) - \gamma\mathcal{S}]V^2\right\}
$$

$$
V’ = \frac{V(1 - V^2)}{1 - (\gamma - 1)V^2}\left\{(3\gamma - 4) - \mathcal{S}\right\}
$$
where

\[ q = 2\Sigma^2 + \frac{1}{2}(3\gamma - 2) + (2 - \gamma)V^2 \Omega \]
\[ \Sigma^2 = \Sigma_+^2 + \Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 + \Sigma_{23}^2 \]
\[ S = \Sigma_{ab}c^ae^b, \quad c^ae_a = 1, \quad v^a = Vc^a \]
\[ V^2 = v_1^2 + v_2^2 + v_3^2. \]  

(20)

These variables are subject to the constraints

\[ 1 = \Sigma^2 + N^2 + \Omega \]  
\[ 0 = 2\Sigma_- N \left[ 1 + (\gamma - 1)V^2 \right] + \gamma\Omega v_1 \]  
\[ 0 = - (\Sigma_{12} N + \Sigma_{13} \bar{N}) \left[ 1 + (\gamma - 1)V^2 \right] + \gamma\Omega v_2 \]  
\[ 0 = (\Sigma_{13} N + \Sigma_{12} \bar{N}) \left[ 1 + (\gamma - 1)V^2 \right] + \gamma\Omega v_3 \]  
\[ 0 = NR_1 - \sqrt{3\Sigma_- N}. \]  

(21) (22) (23) (24) (25)

The parameter \( \gamma \) will be assumed to be in the interval \( \gamma \in (0, 2) \).

Note that \( R_1 \), which is the component of the rotation tensor describing rotations with respect to the axis \( e_1 \), is implicitly defined via eq. (25). For the type VI\(_0\) model we will in practice solve this equation by introducing the parameter \( \lambda \) instead of \( \bar{N} \), by \( \bar{N} = \lambda N \). For the type VI\(_0\) model, this parameter is bounded by \( \lambda^2 \leq 1 \).

### 2.1 The state space

The constraint (21) implies that \( \Sigma_\pm, \Sigma_{12}, \Sigma_{13}, \Sigma_{23}, N \) and \( \Omega \) are all bounded. Combining the equations for \( N \) and \( \bar{N} \) we get the equations

\[ (N \pm \bar{N})' = \left( q + 2\Sigma_+ \pm 2\sqrt{3\Sigma_{23}} \right) (N \pm \bar{N}). \]  

(26)

Thus if the initial data have \( \bar{N}^2 < N^2 \), then this will hold for all times. Hence, for type VI\(_0\), \( \bar{N} \) will be bounded as well. The invariant subspaces \( N \pm \bar{N} = 0 \) correspond to Bianchi type II universes. Note that, given the set \((\Sigma_\pm, \Sigma_{12}, \Sigma_{13}, \Sigma_{23}, N, \bar{N})\) we can determine \( \Omega \) from eq. (21), and \( v^1, v^2 \) and \( v^3 \) from eqs. (22), (23) and (24), respectively. We also require \( 0 \leq V \leq 1 \) for physical reasons (we are not allowing superluminal velocities). This implies the bounds

\[ \Sigma_+^2 + \Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 + \Sigma_{23}^2 + N^2 \leq 1 \]
\[ \bar{N}^2 \leq N^2 \]
\[ 4\Sigma_2^2 N^2 + (\Sigma_{12} N + \Sigma_{13} \bar{N})^2 + (\Sigma_{13} N + \Sigma_{12} \bar{N})^2 \leq 1. \]  

(27)

However, as can be shown, the last of these inequalities is redundant due to the fact that the first two imply the third (see Appendix A). Hence, the state space can be considered a subspace of a compact region in \( \mathbb{R}^7 \).
There are also some discrete symmetries under which this system is invariant. These are (of course, also compositions of these maps are symmetries)

\[ \phi_1: (\Sigma_-, \Sigma_{12}, \Sigma_{13}) \mapsto (-\Sigma_-, -\Sigma_{13}, -\Sigma_{12}) \]  
(28)

\[ \phi_2: (\Sigma_{12}, \Sigma_{23}, \bar{N}) \mapsto (-\Sigma_{12}, -\Sigma_{23}, -\bar{N}) \]  
(29)

\[ \phi_3: (\Sigma_{13}, \Sigma_{23}, \bar{N}) \mapsto (-\Sigma_{13}, -\Sigma_{23}, -\bar{N}). \]  
(30)

To understand the physical importance of these maps we can apply them to the vector \( v^a \): 

\[ \phi_1: (v_1, v_2, v_3) \mapsto (-v_1, v_3, v_2) \]

\[ \phi_2: (v_1, v_2, v_3) \mapsto (v_1, -v_2, v_3) \]

\[ \phi_3: (v_1, v_2, v_3) \mapsto (v_1, v_2, -v_3). \]

Hence, these discrete symmetries correspond to different permutation and inversions of the axes. These symmetries are intrinsic to the type VI\(_0\) geometry.

### 2.2 Invariant subspaces

The invariant subspaces play an important role in the evolution of Bianchi type VI\(_0\) universes. They are important to study since they are invariant under the evolution of the system; i.e. if a point \( p \) lie in one invariant subspace, then also the maximal extended evolution of \( p \) will lie entirely inside it.

Some of the physically interesting invariant subspaces are as follows (we will also assume that the boundaries are included):

1. \( T(VI_0) \): The full state space of tilted type VI\(_0\).
2. \( F(VI_0) \): The set of fixed points of the map \( \phi_1 \). Given by \( \Sigma_- = 0, \Sigma_{12} = -\Sigma_{13} \).
3. \( T_2(VI_0) \): A Bianchi type VI\(_0\) with a two-component tilted fluid. It is the set of fixed points of \( \phi_2 \) (or \( \phi_3 \)). Given by \( \Sigma_{12} = \Sigma_{23} = \bar{N} = 0 \) (or \( \Sigma_{13} = \Sigma_{23} = \bar{N} = 0 \)).
4. \( T_1(VI_0) \): Bianchi type VI\(_0\) with a one-component tilted fluid. Given by \( \Sigma_{12} = \Sigma_{13} = 0 \). This is the fixed-point-set of \( \phi_2 \circ \phi_3 \).
5. \( B(VI_0) \): Non-tilted Bianchi type VI\(_0\). Given by \( \Sigma_- = \Sigma_{12} = \Sigma_{13} = V = 0 \).
6. \( T^\pm(II) \): The tilted type II boundary. Given by \( N = \pm \bar{N} \). Note that these two subspaces can be mapped onto each other using \( \phi_2 \) or \( \phi_3 \). Because of this, we will in most cases not differentiate between these two invariant subspaces.
7. \( B(II) \): Non-tilted type II. Given by \( N^2 = \bar{N}^2 \) and \( V = 0 \).
8. \( B(I) \): Bianchi type I universes. Given by \( N = \bar{N} = V = 0 \).
Table 1: The dimension of the state space for different invariant subspaces. The asterisk indicates that the true number of physical degrees of freedom should be reduced due to unphysical gauge freedoms.

9. \( \partial_{\nu} B(I) \): The Bianchi type I vacuum boundary. Given by \( N = \bar{N} = V = \Omega = 0 \).

3 Equilibrium points of importance for the late-time behaviour

The system of equations has numerous equilibrium points which correspond to fixed points in the evolution of the system. That is, if the state variables are written as a vector \( X \), and we write the system of equations as

\[
X' = F(X),
\]

then a point \( X_0 \) is an equilibrium point if

\[
F(X_0) = 0.
\]

These equilibrium points usually play a particular role in the evolution of the system. Not only are they exact solutions, but some of these equilibrium points may act as past or future attractors for the set of solutions. Hence, for example, at late times these equilibrium points may be a good approximation of more general solutions.

Some of the physically interesting equilibrium points for the future evolution are given below. If some equilibrium points are related via one of the discrete symmetries, permutations, or other gauge symmetries, we only give one of them because the solutions they represent are equivalent. We also indicate the smallest invariant subspace mentioned in section 2.2 to which they belong.

3.1 Non-tilted

1. \( \mathcal{I}(I) \): FRW

\[
\Sigma^2 = N = \bar{N} = V = 0, \quad \Omega = 1, \quad q = \frac{1}{2}(3\gamma - 2).
\]

Invariant subspace: \( B(I) \).
2. \( CS(II) \): Collins-Stewart type II \((2/3 < \gamma < 2)\)
\[
\Sigma_- = \Sigma_{12} = \Sigma_{13} = V = 0, \Sigma_+ = -\frac{1}{16}(3\gamma - 2), \bar{N} = N
\]
\[
\Sigma_{23} = -\frac{\sqrt{3}}{16}(3\gamma - 2), N^2 = \frac{3}{64}(3\gamma - 2)(2 - \gamma), \Omega = \frac{3}{16}(6 - \gamma), q = \frac{1}{2}(3\gamma - 2).
\]
Invariant subspace: \( B(II) \)

3. \( C(VI_0) \): Collins VI\(_0\) \((2/3 < \gamma < 2)\)
\[
\Sigma_- = \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = \bar{N} = V = 0, \Sigma_+ = -\frac{1}{16}(3\gamma - 2), N^2 = \frac{3}{16}(3\gamma - 2)(2 - \gamma), \Omega = \frac{3}{8}(2 - \gamma), q = \frac{1}{2}(3\gamma - 2)
\]
Invariant subspace: \( B(VI_0) \)

### 3.2 Intermediately tilted

1. \( \mathcal{H}(II) \): Hewitt’s tilted type II \((10/7 < \gamma < 2)\) [12]
\[
\Sigma_- = 0, \Sigma_+ = \frac{1}{8}(9\gamma - 14), \Sigma_{23} = -\frac{2\sqrt{3}}{8}(5\gamma - 6), \Sigma_{12} = \Sigma_{13} = \frac{\sqrt{6}}{8}\sqrt{2 - \gamma(17\gamma - 10)(7\gamma - 10)}, \bar{N} = N, \]
\[
N^2 = \frac{3(2 - \gamma)(5\gamma - 4)(3\gamma - 4)}{4(17\gamma - 18)}, V^2 = \frac{(3\gamma - 4)(7\gamma - 10)}{(11\gamma - 10)(5\gamma - 4)}, \]
\[
\Omega = \frac{3(2 - \gamma)}{4(17\gamma - 18)}(21\gamma^2 - 24\gamma + 4), q = \frac{1}{2}(3\gamma - 2).
\]
Invariant subspace: \( T^+(II) \)

2. \( \mathcal{L}(II) \): Type II line bifurcation \((\gamma = 14/9)\) [8]
\[
\Sigma_\pm = 0, \Sigma_{23} = -\frac{2\sqrt{3}}{3}, \bar{N} = N, N^2 = \frac{1}{17}(2b + 1)(17 - 8b), \Sigma_{12} = \Sigma_{13} = -\frac{2\sqrt{3}b}{3}, \]
\[
V^2 = \frac{3(4b + 1)(2b + 1)}{(17 - 8b)(8 - 3b)}, \Omega = \frac{2}{17}(16b^2 - 45b + 59), \]
\[0 < b < 1, q = \frac{4}{3}.
\]
Invariant subspace: \( T^+(II) \)

3. \( \mathcal{R}(VI_0) \): Rosquist-Jantzen \((6/5 < \gamma < 3/2)\) [14]
\[
\Sigma_{13} = \Sigma_{23} = \bar{N} = 0, \Sigma_+ = -\frac{1}{2}(3\gamma - 2), q = \frac{1}{2}(3\gamma - 2),
\]
\[
c_1 = \frac{1}{2} \left\{ \frac{(5\gamma - 6)(9\gamma^2 - 13\gamma + 6 + 3(2 - \gamma)s)}{\gamma(9\gamma - 10)} \right\}^{1/2}
\]
\[
c_2 = \frac{1}{2} \left\{ \frac{(2 - \gamma)(45\gamma^2 - 65\gamma + 18 - 3(5\gamma - 6)s)}{\gamma(9\gamma - 10)} \right\}^{1/2}
\]
\[
c_3 = 0
\]
where \( s = \sqrt{(\gamma - 1)(9\gamma - 1)} \).
\[
\Sigma_- = -\frac{1}{2}\delta c_1, \quad \Sigma_{12} = \delta c_2
\]
where $\delta = \sqrt{3}(3\gamma - 2) - 2(2 - \gamma)c_2^2)/(6c_1c_2^2)$.

\[
N^2 = \frac{(5\gamma - 6)^2 - c_1^2(3\gamma - 4)}{12c_1^2c_2^2} \left[11\gamma - 6 - (3\gamma + 2)c_1^2\right]
\]

\[
V = \frac{\lambda}{N}, \quad \Omega = 1 - \Sigma - N^2,
\]

where $N < 0$, and

\[
\lambda = \frac{1}{2\sqrt{3}}(2 - \gamma)c_1 + \frac{1}{\sqrt{3}}(5\gamma - 6)c_2.
\]

Invariant subspace: $T_2(VI_0)$

4. $\mathcal{L}(VI_0)$: Type VI$_0$ line bifurcation ($\gamma = 6/5)^2$

\[
\Sigma_\pm = \Sigma_{23} = 0, \, \Sigma_+ = -\frac{2}{5}, \, \Sigma_- = 0, \, q = \frac{4}{5},
\]

\[
\bar{N} = \lambda N, \, N = \frac{\sqrt{2}}{\lambda} \sqrt{\frac{5V^2 + 3\lambda + 1}{2V^2 + 3\lambda + 1}},
\]

\[
\Sigma_{12} = -\Sigma_{13} = \frac{16V^2 - 6\lambda(3V^2 + 4)}{2V^2 + 3\lambda + 1}, \quad \Omega = \frac{3}{25} \left(\frac{\lambda^2 + 5(1 + 5\lambda)}{2V^2 + 3\lambda + 1}\right),
\]

\[
V^2 = \frac{15 + 211\lambda + 397\lambda^2 + 25\lambda^3 + 3(5\lambda + 1)\sqrt{s}}{10(3 - 5\lambda)(1 - \lambda)^2}
\]

where

\[
s = \left[(\lambda + 24 - 9\sqrt{5})^2 - 16(61 - 27\sqrt{5})\right] \times \left[(\lambda + 24 + 9\sqrt{5})^2 - 16(61 + 27\sqrt{5})\right]
\]

\[-24 - 9\sqrt{5} + 4\sqrt{61 + 27\sqrt{5}} \leq \lambda < 0.
\]

Invariant subspace: $T_2(VI_0)$

### 3.3 Extremely tilted

1. $\mathcal{E}(II)$: Extremely tilted type II ($0 < \gamma < 2$)

\[
\Sigma_\pm = 0, \, \Sigma_{23} = -\frac{2\sqrt{5}}{5}, \, \Sigma_+ + \Sigma_{12} = -\frac{10}{3} \sqrt{\frac{2}{5}},
\]

\[
\Sigma_{13} - \Sigma_{12} = -\frac{2}{3} \sqrt{\frac{4}{5}}, \, \bar{N} = N = \sqrt{\frac{3}{10}}, \, V = 1, \, \Omega = \frac{20}{5}, \, q = \frac{4}{5}.
\]

Invariant subspace: $T^+(II)$

2. $\mathcal{E}_1(VI_0)$: Extremely tilted type VI$_0$ ($0 < \gamma < 2$)

\[
V = 1, \, \Sigma_{13} = \Sigma_{23} = \bar{N} = 0, \, \Sigma_+ = -\frac{2}{5}, \, \Sigma_- = \frac{2\sqrt{2}}{5}, \, \Sigma_{12} = -\frac{1}{5}, \, N = -\frac{1}{5}, \, \Omega = \frac{1}{5}, \, q = \frac{5}{3}.
\]

Invariant subspace: $T_2(VI_0)$

---

These solutions seem to be found first by Apostolopoulos [15]. However, he writes the solutions implicitly as solutions to a cubic. As can be seen, we managed to write them explicitly down in terms of one free variable which makes it easier to interpret the solutions.
3. $\mathcal{E}_2(VI_0)$: Extremely tilted type VI$_0$ ($0 < \gamma < 2$)

$V = 1$, $\Sigma_+ = \Sigma_{23} = N = 0$, $\Sigma_+ = -\frac{2}{3}$,

$\Sigma_{12} = -\Sigma_{13} = \frac{\sqrt{6}}{10}$, $N = \frac{2\sqrt{3}}{5}$, $\Omega = \frac{6}{2\pi}$, $q = \frac{4}{3}$.

Invariant subspace: $F(VI_0)$

4 The late-time asymptotic behaviour

An important key to understanding the late-time behaviour of the dynamical system is to consider the future stable equilibrium points. Assume that $X_0$ is an equilibrium point; i.e. $F(X_0) = 0$. Then we can write the equations of motion as

$$\delta X' = A\delta X + O(\delta X^2).$$

The local stability of the equilibrium point, $X_0$, depends on the eigenvalues of $A$; if all eigenvalues have negative real parts, then $X_0$ is a future attractor. Every eigenvalue with a positive real part signals an unstable mode.

The (local) future stable equilibrium points are for the different invariant subspaces given in Table 2 and the various connections between them are illustrated in Fig. 1. The eigenvalues of the linearised system are discussed in Appendix B.

For the non-tilted subspaces $B(I)$, $B(II)$, and $B(VI_0)$, it is proven that generic solutions approach the respective local future attractors [2]. In the tilted case we can only partially prove that the local attractors are also global.
Table 2: The future stable equilibrium points for various invariant (sub)spaces.

| Space     | Matter        | Stable point |
|-----------|---------------|--------------|
| $B(I)$    | $0 < \gamma < 2$ | $\mathcal{I}(I)$ |
| $B(II)$   | $0 < \gamma < 2/3$ | $\mathcal{I}(I)$ |
|           | $2/3 < \gamma < 2$ | $\mathcal{C}(II)$ |
| $T^\pm(II)$ | $0 < \gamma < 2/3$ | $\mathcal{I}(I)$ |
|           | $2/3 < \gamma < 10/7$ | $\mathcal{C}(II)$ |
|           | $10/7 < \gamma < 14/9$ | $\mathcal{H}(II)$ |
|           | $\gamma = 14/9$ | $\mathcal{L}(II)$ |
|           | $14/9 < \gamma < 2$ | $\mathcal{E}(II)$ |
| $B(VI_0)$ | $0 < \gamma < 2/3$ | $\mathcal{I}(I)$ |
|           | $2/3 < \gamma < 2$ | $\mathcal{C}(VI_0)$ |
| $T_1(VI_0)$ | $0 < \gamma < 2/3$ | $\mathcal{I}(I)$ |
|           | $2/3 < \gamma < 2$ | $\mathcal{C}(VI_0)$ |
| $T_2(VI_0)$ | $0 < \gamma < 2/3$ | $\mathcal{I}(I)$ |
|           | $2/3 < \gamma < 6/5$ | $\mathcal{C}(VI_0)$ |
|           | $6/5 < \gamma < 3/2$ | $\mathcal{R}(VI_0)$ |
|           | $3/2 < \gamma < 2$ | $\mathcal{E}_1(VI_0)$ |
| $F(VI_0)$ | $0 < \gamma < 2/3$ | $\mathcal{I}(I)$ |
|           | $2/3 < \gamma < 6/5$ | $\mathcal{C}(VI_0)$ |
|           | $\gamma = 6/5$ | $\mathcal{L}(VI_0)$ |
|           | $6/5 < \gamma < 2$ | $\mathcal{E}_2(VI_0)$ |
| $T(VI_0)$ | $0 < \gamma < 2/3$ | $\mathcal{I}(I)$ |
|           | $2/3 < \gamma < 6/5$ | $\mathcal{C}(VI_0)$ |
|           | $\gamma = 6/5$ | $\mathcal{L}(VI_0)$ |
|           | $6/5 < \gamma < 2$ | $\mathcal{E}_2(VI_0)$ |
attractors. The subspace $T^\pm(II)$ was discussed in [8]. We will use these results in the following way. Assume we have a subspace, $S$ (given by $y = 0$, say), for which a function $Z$ is strictly monotonically decreasing; i.e.

$$Z'|_{y=0} = \alpha Z|_{y=0}, \quad \alpha \leq \epsilon < 0.$$\ 

Away from $S$, we can write $Z' = \alpha(y)Z$ where $\alpha(0) \leq \epsilon < 0$. The function $\alpha(y)$ is continuous in $y$; hence, there will exist a $\delta > 0$ such that

$$\alpha(y) < 0, \quad \text{for } |y| < \delta.$$  \hspace{1cm} (33)

So suppose the solution curves have the property $y \to 0$, then after sufficiently long time the function $Z$ will be monotonic along the solution curves. The late time analysis can therefore to some extent be extracted from the monotonic functions in $S$.

In the following, $\omega(p)$ will mean the future asymptotic set; i.e. for a solution curve $c(\tau)$ having $c(0) = p$, then $\omega(p) = \lim_{\tau \to \infty} c(\tau)$.

**Theorem 1 (No-hair).** For $\gamma$ obeying $0 < \gamma < 2/3$, any $p \in T(V I_0)$ for which $V < 1$, and $\Omega > 0$ has $\omega(p) = I(I)$.

**Proof.** Part of the proof relies on the observation that the function $S = \Sigma_{abc}e^ae^b$ obeys the bound $|S| \leq 2\Sigma \leq 2$ (see Appendix A). From the equation for $V$ we thus have

$$V' \leq (3\gamma - 2)\frac{V(1 - V^2)}{1 - (\gamma - 1)V^2}.$$\ 

Hence, if $0 < \gamma < 2/3$, $V$ will be monotonically decreasing.

There is also another monotonically increasing function, namely

$$(\beta \Omega)' = \left[2q - (3\gamma - 2)\right](\beta \Omega),$$  \hspace{1cm} (34)

$$\beta \equiv \frac{(1 - V^2)^{2-\gamma}}{1 + (\gamma - 1)V^2}.$$  \hspace{1cm} (35)

Note that

$$2q - (3\gamma - 2) = 3(2 - \gamma)\Sigma^2 + (2 - 3\gamma)N^2 + \frac{\gamma(4 - 3\gamma)V^2}{1 + (\gamma - 1)V^2}.$$

Hence, it follows that for $0 < \gamma < 2/3$, $\lim_{\tau \to \infty} \Omega = 1$, and $\lim_{\tau \to \infty} V = 0$. \hspace{1cm} $\square$

**Theorem 2 (Global attractors for $T_1(V I_0)$).** For $2/3 < \gamma < 4/3$, any $p \in T_1(V I_0)$ with $\Omega > 0$, $N^2 > \bar{N}^2$ and $V < 1$, has $\omega(p) = C$.

**Proof.** We note that in the subspace $T_1(V I_0)$ we have

$$\Sigma_+ = (q - 2)\Sigma_+ - 2N^2 - \frac{\gamma \Omega v^2}{1 + (\gamma - 1)V^2}.$$  \hspace{1cm} (36)
Since $q \leq 2$ (which can easily be checked) there exist a $\tau_1$ such that $\Sigma_+ \leq 0$ for all $\tau > \tau_1$. Furthermore, in $T_1(V I_0)$ we also have

$$V' = \frac{V(1-V^2)}{1+(\gamma-1)V^2} [3\gamma - 4 + 2\Sigma_+]. \quad (37)$$

Hence, for $\gamma < 4/3$, $V$ will be monotonically decreasing for $\tau > \tau_1$, and $\lim_{\tau \to \infty} V = 0$. This means that the solutions will approach the non-tilted subspace $B(V I_0)$ in the limit $\tau \to \infty$. Thus after sufficiently long time, the monotone functions in $B(V I_0)$ (given in [2]) will also be monotone along orbits in $T(V I_0)$. The point $C$ is an isolated equilibrium point, thus we can use the monotone functions which implies $\omega(p) = C$.

There are strong reasons to believe that $C$ is a global attractor in $T_1(V I_0)$ for all $0 < \gamma < 2$. As evidence for this is the existence of a monotonic function in $T_1(V I_0)$. Define $\sigma = \Sigma_+^2 + \Sigma_2^3$, which in $T_1(V I_0)$ has the evolution equation

$$\sigma' = 2(q-2)\sigma - 4\sqrt{3}N\bar{N}\Sigma_{23}. \quad (38)$$

In $T_1(V I_0)$ we have the monotonically increasing function:

$$Z_1 = \frac{N^2 + \sigma}{N^2 + \bar{\sigma}}, \quad Z'_1 = \frac{4\sigma(N^2 - \bar{N}^2)(\Sigma_+ + 1)}{(N^2 + \sigma)(N^2 + \bar{\sigma})} - Z_1. \quad (39)$$

As $\tau \to \infty$ there are thus three possibilities:

$$\sigma, \quad \Sigma_+ + 1, \quad \text{or} \quad N^2 - \bar{N}^2 \to 0.$$ 

If $\sigma \to 0$, the non-tilted analysis can again be applied. The case $\Sigma_+ = -1$ is unstable in the future so this cannot happen for general solutions. If $N^2 - \bar{N}^2 \to 0$, the tilted analysis of type II can be applied. All the late-time asymptotes of the type II case have an unstable direction into the interior of $T_1(V I_0)$ (which exactly corresponds to $N^2 - \bar{N}^2$). Unfortunately, for the tilted type II model the local attractors have not been rigorously proven to be global attractors [8].

Apart from these two theorems we have not been able to show any global late time attractors for the various (sub)spaces. However, numerical analysis seems to imply that the local attractors also are global attractors.

5 The initial singular regime

Let us consider the initial singular regime; i.e. where $\tau \to -\infty$. This case is a lot more subtle than at late times due to the oscillatory behaviour of the system of equations as one approach $\tau \to -\infty$. In fact, the tilted type II seems to have an initial oscillatory regime [8], and hence, one would expect a similar – if not more complex – behaviour for the tilted type VI_0 model.

In the following we should emphasise that we consider a non-stiff fluid. For a stiff fluid ($\gamma = 2$) it has been pointed out that the Bianchi models allow
for a stable past attractor which would remove this chaotic behaviour into the past \[28, 29\].

In the study of the initial singular regime it is useful to introduce the variable \( \lambda \), instead of \( \bar{N} \), as

\[
\bar{N} = \lambda N.
\]

(40)

The equations for \( N \) and \( \lambda \) are then

\[
N' = (q + 2\Sigma_+ + 2\sqrt{3}\lambda\Sigma_{23})N
\]

(41)

\[
\lambda' = 2\sqrt{3}\Sigma_{23}(1 - \lambda^2).
\]

(42)

In this case, the invariant subspaces \( \lambda = \pm 1 \) correspond to \( T(II)^\pm \). In this case, there are two Kasner circles, which differ by an orientation of frame. They are as follows.\(^3\)

1. \( K^{\pm}(II) \): Kasner “type II” (0 < \( \gamma \) < 2)
   \[
   \Sigma_+ + \Sigma_{23} = 1, \lambda = \pm 1, q = 2
   \]
   \[
   \Sigma_\mp = \Sigma_{13} = \Sigma_{12} = N = \Omega = 0.
   \]

2. \( K(VI_0) \): Kasner “type VI_0” (0 < \( \gamma \) < 2)
   \[
   \Sigma_+ + \Sigma_- = 1, \lambda = 0, q = 2
   \]
   \[
   \Sigma_{23} = \Sigma_{13} = \Sigma_{12} = N = \Omega = 0.
   \]

Each of them also have extremely tilted Kasner sets and corresponding bifurcations. These are discussed in detail in [30]. We will not discuss these here as they do not change our conclusion radically. However, bear in mind that these equilibrium points exist and that there are transitions between them.

The behaviour near the initial depends on three types of heteroclinic orbits. These are

1. \( T_R \): Frame rotations.
2. \( T_N \): Taub type II vacuum orbits.
3. \( T_\lambda \): Frame rotations between \( K^{\pm}(II) \) and \( K(VI_0) \).

Let us consider, for illustration, the Kasner circle \( K^{+}(II) \). The analysis for the Kasner circle \( K^{-}(II) \) can be obtained from \( K^{+}(II) \) by using the map \( \phi_2 \) (or \( \phi_3 \)).

\( K^{+}(II) \)

Here, the frame rotations, and Taub type II orbits are given by (when projected onto the \((\Sigma_+, \Sigma_{23})\)-plane):

1. Three frame rotations:
   \[
   \Sigma_+ = C_1
   \]
   \[
   \Sigma_+ \pm \sqrt{3}\Sigma_{23} = C_\pm
   \]

\(^3\)With a slightly abuse of notation because they are really type I solutions.
Figure 2: The $K^+(II)$ frame rotations projected onto the $(\Sigma_+, \Sigma_{23})$-plane. Arrows are future-directed.

Figure 3: Taub orbits, $T_{N+}$, projected onto the $(\Sigma_+, \Sigma_{23})$-plane. Arrows are future-directed.
Figure 4: Flow diagram for the transitions between the Kasner circles $K^\pm(II)$ and $K(VI_0)$ projected onto the $(\Sigma_{23}, \lambda)$-plane. Here, the constant $r$ is given by $r = \sqrt{1 - \Sigma_{23}^2}$. Arrows are future-directed.

2. Taub type II vacuum orbits:

$\Sigma_+ + 1 = C(\Sigma_{23} + \sqrt{3})$.

All of these orbits maps the Kasner circle onto itself: $K^+(II) \mapsto K^+(II)$. The frame rotations are illustrated in Fig.2 and the Taub orbits are illustrated in Fig.3.

For the Kasner circle $K(VI_0)$ there are two frame rotations of a similar kind as for $K^\pm(II)$; namely the ones given by $\Sigma_+ \pm \sqrt{3}\Sigma_- = C_\pm$. These orbits also maps the Kasner circle onto itself: $K(VI_0) \mapsto K(VI_0)$.

In addition to these homoclinic orbits there are homoclinic orbits between the three types of Kasner circles. They are given by the set of differential equations

\[
\begin{align*}
\lambda' &= 2\sqrt{3}\Sigma_{23}(1 - \lambda^2) \\
\Sigma_-' &= -2\sqrt{3}\lambda\Sigma_-\Sigma_{23} \\
\Sigma_{23}' &= 2\sqrt{3}\lambda\Sigma_-^2 \\
1 &= \Sigma_+^2 + \Sigma_-^2 + \Sigma_{23}^2, \quad \Sigma_+ = \text{constant.}
\end{align*}
\]

A flow diagram for this system projected onto the $(\lambda, \Sigma_{23})$-plane is shown in Fig.4. We can see that there are homoclinic orbits between $K^\pm(II)$ and $K(VI_0)$. However, these only form a set of measure zero. Most orbits connect $K^\pm(II)$. Nonetheless, orbits can come arbitrary close to $K(VI_0)$ and hence, there might be frame rotations within the Kasner circle $K(VI_0)$ before the orbit go back to $K^\pm(II)$. 

16
From this fairly simple, but far from complete analysis, we believe that the initial behaviour of the tilted type VI$_0$ models is fairly complicated. There are infinite sequences of homoclinic orbits which consists of frame rotations and Taub vacuum orbits. These orbits are believed to be chaotic in general and thus we conjecture that the tilted type VI$_0$ has a chaotic behaviour to the past.

6 Summary

Here, for the first time, we have analysed the late-time behaviour of general tilted Bianchi type VI$_0$ universes. Our results are summarised in Table 2. We performed a local analysis and found all the future stable equilibrium points for various subclasses of tilted type VI$_0$ models as well as for the general tilted type VI$_0$. In particular, we confirmed the observation in [1] on the existence of new self-similar solutions for $\gamma \geq 6/5$. These solutions also proved to be important for the late-time behaviour; the Rosquist-Jantzen solutions are late-time attractors for a certain class of models with a two-component tilted fluid; and the line bifurcation at $\gamma = 6/5$ is, in fact, the late-time attractor for general tilted type VI$_0$ solutions. For $\gamma > 6/5$, the late-time attractor is an extremely tilted model.$^4$

It is interesting to note that the general late-time attractors lie in the fixed-point-set of $\phi_1$. This means that in general the solutions are asymptotically $\phi_1$-symmetric. More specifically, $\phi_1$-symmetric implies $v_1 = 0$, and $v_2^2 = v_3^2$.

At early times the analysis suggests chaotic behaviour as $\tau \to -\infty$. This behaviour seems to be a generic property of tilted Bianchi models [30]. So this behaviour was not very surprising, in particular considering the fact that the tilted type II model – which is part of the boundary of type VI$_0$ – was known to be chaotic [8].

Let us finish off with some comments about future research and how this work may relate to other Bianchi types. Firstly, the work [1] indicates that there may be some similarities between the more general type VI$_h$ models and the type VI$_0$. Hence, some of the features of the late-time behaviour found in this work may also appear in the type VI$_h$ models. As goes for the class VII$_0$ model, which is given by $\bar{N}^2 > N^2$, there is one obvious difference. The general non-tilted VII$_0$ model is not asymptotically self-similar [31, 32]. This happens because the type VII$_0$ state space is not compact; there is no upper bound on $\bar{N}$. In the terminology of [33], the type VII$_0$ model is extremely Weyl dominant while the type VI$_0$ model is Weyl-Ricci balanced at late times. Hence, we would expect a fairly different behaviour for the tilted type VII$_0$ model at late times than that found here. Moreover, in this work the extremely tilted invariant sets have not been emphasised. For example, a two-fluid model where one fluid is extremely tilted is if particular interest. Further investigations in this case is

$^4$It should be noted that the $\gamma = 6/5$ line bifurcation looks remarkably similar to the Wainwright $\gamma = 10/9$ line bifurcation of the exceptional model [27]. This makes us wonder if there are unknown line bifurcations along the entire line given by $\gamma = 2(3 + \sqrt{-h})/(5 + 3\sqrt{-h})$ for the more general type VI$_h$ models [1].
required.

Nonetheless, there are quite a few unanswered questions regarding Bianchi models with a perfect fluid. This work has answered some of them. Hopefully, future work will answer more of them.

Acknowledgments

The author would like to thank J.D. Barrow, A.A. Coley and S.T.C. Siklos for discussions related to this work. This work was funded by the Research Council of Norway and an Isaac Newton Studentship.

A Some simple proofs

On bound (27)

We start with assuming that the first two bounds in eq. (27) hold. Then, using the Schwarz inequality, we have

\[
4\Sigma_{2}N^2 + (\Sigma_{12}N + \Sigma_{13}\bar{N})^2 + (\Sigma_{13}N + \Sigma_{12}\bar{N})^2 \\
\leq 4\Sigma_{2}N^2 + 2\left(\Sigma_{12}^2 + \Sigma_{13}^2\right)(N^2 + \bar{N}^2) \\
\leq 4\left(\Sigma_{2}^2 + \Sigma_{12}^2 + \Sigma_{13}^2\right)N^2. \quad (44)
\]

Consider now the maximal value of the function \(F(X, Y) = 4X^2Y^2\) inside the unit disc

\[X^2 + Y^2 \leq 1.\]

One easily finds that the maximal value is at \(X^2 = Y^2 = 1/2\). Thus

\[F(X, Y) \leq 1.\]

Hence, by identifying \(X^2 = N^2\) and \(Y^2 = \Sigma_{2} + \Sigma_{12}^2 + \Sigma_{13}^2\), we obtain from eq. (44)

\[4\Sigma_{2}N^2 + (\Sigma_{12}N + \Sigma_{13}\bar{N})^2 + (\Sigma_{13}N + \Sigma_{12}\bar{N})^2 \leq 1. \quad (45)\]

Thus the last inequality in eq. (27) is redundant.

Showing \(|\mathcal{S}| \leq 2\Sigma \leq 2\)

We will consider the function

\[|\mathcal{S}| = |\Sigma_{ab}c^ac^b|,\]

where the matrix \(\Sigma_{ab}\) is symmetric and trace-free, and \(c^a c_a = 1\). This implies that the maximal value of \(|\mathcal{S}|\) occurs when \(c^a\) is parallel to one of the eigenvectors of \(\Sigma_{ab}\). Thus, if \(\lambda_i\) are the eigenvalues, we have

\[|\Sigma_{ab}c^ac^b| \leq \max(|\lambda_1|, |\lambda_2|, |\lambda_3|). \quad (46)\]
The eigenvalues obey the relations

\[ \Sigma_a \lambda_1 = \lambda_1 + \lambda_2 + \lambda_3 = 0 \]
\[ \Sigma_{ab} \lambda_2 \lambda_3 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 6 \Sigma^2. \]

These two equations imply that the maximal value of an eigenvalue occur when \( \lambda_1 = \pm 2 \Sigma \), and \( \lambda_2 = \lambda_3 = \mp \Sigma \) (or a permutation thereof). Hence, according to eqs. (46) and (21) we have

\[ |S| \leq 2 \Sigma \leq 2. \]

\section*{B Eigenvalues of Equilibrium points}

In this appendix we discuss some of the eigenvalues for the various equilibrium points.

\subsection*{B.1 Non-tilted}

1. \( \mathcal{I}(I) \): FRW

\[ \lambda_{1,2,3,4,5} = -\frac{3}{2}(2 - \gamma), \quad \lambda_{6,7} = \frac{1}{2}(3 \gamma - 2). \]

2. \( \mathcal{C}S(II) \): Collins-Stewart type II (2/3 < \( \gamma < 2 \))

The essential unstable eigenvalue is for all equilibrium points in \( T(II) \):

\[ \lambda_7 = -4 \sqrt{3} \Sigma_{23}. \]

Hence, since \( \Sigma_{23} < 0 \) this point is unstable.

3. \( \mathcal{C}(VI_0) \): Collins VI\(_0\) (2/3 < \( \gamma < 2 \))

\[ \lambda_{1,2} = -\frac{4}{3}(2 - \gamma) \left(1 \pm \sqrt{5\gamma - 6}\right), \quad \lambda_{3,4} = -\frac{4}{3}(2 - \gamma) \left(1 \pm \sqrt{\frac{10 - 15\gamma}{2 - \gamma}}\right), \]
\[ \lambda_{5,6,7} = -\frac{4}{3}(2 - \gamma), \quad \lambda_{6,7} = -\frac{4}{3}(6 - 5\gamma). \]

Here, \( \lambda_{1,2,3,4} \) correspond to the non-tilted case, \( \lambda_{5,6,7} \) are the eigenvalues in the \( v_1, v_2 \) and \( v_3 \) directions, respectively.

\subsection*{B.2 Intermediately tilted}

1. \( \mathcal{H}(II) \): Hewitt’s tilted type II (10/7 < \( \gamma < 2 \))

Unstable due to the eigenvalue

\[ \lambda_7 = -4 \sqrt{3} \Sigma_{23}. \]
2. $\mathcal{L}(II)$: Type II line bifurcation ($\gamma = 14/9$)
   Unstable due to the eigenvalue
   \[ \lambda_7 = -4\sqrt{3}\Sigma_{23}. \]

3. $\mathcal{R}(VI_0)$: Rosquist-Jantzen ($6/5 < \gamma < 3/2$)
   Due to the complex character of these solutions, one has to part of the stability analysis numerically. Some eigenvalues are possible to find analytically. The following seems to hold:
   \[ \text{Re}(\lambda_{1,2,3,4}) = -\frac{3}{4}(2 - \gamma), \quad \lambda_5 + \lambda_6 = -\frac{3}{2}(2 - \gamma), \quad \text{Re}(\lambda_5), \text{Re}(\lambda_6) < 0, \]
   \[ \lambda_7 = \frac{3}{2}(5\gamma - 6). \]
   Here, $\lambda_{1,2,5,6}$ correspond to directions along the invariant subspace $T_2(VI_0)$.

4. $\mathcal{L}(VI_0)$: Type VI$_0$ line bifurcation ($\gamma = 6/5$)
   Again we have to rely on some numerical analysis. However, analytic combined with numerics seem to indicate that
   \[ \lambda_1 = 0, \quad \text{Re}(\lambda_{2,3,4,5}) = -\frac{3}{4}, \]
   \[ \lambda_6 + \lambda_7 = -\frac{6}{5}, \quad \text{Re}(\lambda_6), \text{Re}(\lambda_7) < 0. \]
   Here, $\lambda_{1,2,3,4,5}$ correspond to directions along the invariant subspace $F(VI_0)$.

B.3 Extremely tilted

1. $\mathcal{E}(II)$: Extremely tilted type II ($0 < \gamma < 2$)
   Unstable due to the eigenvalue
   \[ \lambda_7 = -4\sqrt{3}\Sigma_{23}. \]

2. $\mathcal{E}_1(VI_0)$: Extremely tilted type VI$_0$ ($0 < \gamma < 2$)
   \[ \lambda_{1,2} = -\frac{1}{6} (3 \pm i\sqrt{183}), \quad \lambda_{3,4} = -\frac{1}{9} \left(3 \pm i\sqrt{53 + 8\sqrt{3}}\right), \]
   \[ \lambda_5 = -\frac{3}{4}, \quad \lambda_6 = -\frac{3(2\gamma - 3)}{2\gamma}, \quad \lambda_7 = \frac{3}{4} (5\gamma - 6). \] (48)
   Here, $\lambda_{1,2,5,6}$ correspond to directions along the invariant subspace $T_2(VI_0)$.

3. $\mathcal{E}_2(VI_0)$: Extremely tilted type VI$_0$ ($0 < \gamma < 2$)
   \[ \lambda_1 = \frac{6(6-5\gamma)}{5(2-\gamma)}, \quad \lambda_{2,3} = -\frac{3}{5} (1 \pm i\sqrt{19}), \]
   \[ \lambda_{4,5} = -\frac{3}{5} (1 \pm i\sqrt{11}), \quad \lambda_{6,7} = -\frac{3}{5} \left(1 \pm i\sqrt{14 + 5\sqrt{2}}\right). \] (49)
   Here, $\lambda_{1,2,3,4,5}$ correspond to directions along the invariant subspace $F(VI_0)$. 

20
References

[1] J.D. Barrow and S. Hervik, *Class. Quantum Grav.* **20** (2003) 2841

[2] C.G. Hewitt and J. Wainwright in *Dynamical Systems in Cosmology*, eds: J. Wainwright and G.F.R. Ellis, Cambridge University Press (1997)

[3] J.T. Horwood, M.J. Hancock, D. The, J. Wainwright, *Class. Quantum Grav.* **20** (2003) 1757

[4] C.G. Hewitt, J.T. Horwood, J. Wainwright, *Class. Quantum Grav.* **20** (2003) 1743

[5] A.R. King and G.F.R. Ellis, *Commun. Math. Phys.* **31** (1973) 209

[6] O.I. Bogoyavlenskii and S.P. Novikov, *Sel. Math. Sov.* **2** (1982) 159; originally published as *Trudy Sem. Petrovsk.* **1** (1975) 7

[7] J.D. Barrow and D.H. Sonoda, *Phys. Reports* **139** (1986) 1

[8] C.G. Hewitt, R. Bridson, J. Wainwright, *Gen.Rel.Grav.* **33** (2001) 65

[9] I.S. Shikin, *Sov. Phys. JETP* **41** (1976) 794

[10] C.B. Collins, *Comm. Math. Phys.* **39** (1974) 131

[11] C.G. Hewitt and J. Wainwright, *Phys. Rev.* **D46** (1992) 4242

[12] C.G. Hewitt, *Class. Quantum Grav.* **8** (1991) L109

[13] K. Rosquist, *Phys. Lett.* **97A** (1983) 145

[14] K. Rosquist and R.T. Jantzen, *Phys. Lett.* **107A** (1985) 29

[15] P.S. Apostolopoulos, *gr-qc/0310033*

[16] B.L. Spokoiny, *Phys. Lett.* **A 81** (1981) 493

[17] V.A. Belinsky, I.M. Khalatnikov and E. M. Lifshitz, *Adv. Phys.* **19** (1970) 525

[18] J.D. Barrow, *Phys. Rev. Lett.* **46** (1981) 963

[19] J.D. Barrow, *Phys. Reports* **85** (1982) 1

[20] D. Chernoff and J.D. Barrow, *Phys. Rev. Lett.* **50** (1983) 134

[21] J. Demaret, M. Henneaux, and P. Spindel, *Phys. Lett.* **B164** (1985) 27

[22] T.Damour, M.Henneaux, B.Julia and H.Nicolai, *Phys. Lett.* **B509** (2001) 323

[23] T. Damour and M. Henneaux, *Phys. Rev. Lett.* **86** (2001) 4749-4752
[24] D. Hobill, A.B. Burd and A.A. Coley, eds, *Deterministic chaos in general relativity*, NATO ASI Series B 332 (Plenum Press, New York).

[25] G.F.R. Ellis and M.A.H. MacCallum, *Comm. Math. Phys.* 12 (1969) 108

[26] H. van Elst, *Hubble normalised orthonormal frame equations for perfect fluid cosmologies in component form*, available at www.maths.qmul.ac.uk/~hve/research.html.

[27] J. Wainwright, *Gen. Rel. Grav.* 16 (1984) 657

[28] J.D. Barrow, *Nature* 272 (1978) 211

[29] A.A. Coley, *Dynamical Systems and Cosmology*, Kluwer Academic Publishers, (2003)

[30] C. Uggla, H. van Elst, J. Wainwright, and G.F.R. Ellis, gr-qc/0304002

[31] J. Wainwright, M.J. Hancock and C. Uggla, *Class. Quantum Grav.* 16 (1999) 2577

[32] U.S. Nilsson, M.J. Hancock and J. Wainwright, gr-qc/9912019

[33] J.D. Barrow and S. Hervik, *Class. Quant. Grav.* 19 (2002) 5173