MECHANICS OF CONTINUOUS MEDIA IN 
$$(\mathcal{L}_n, g)$$-SPACES.

IV. Stress (Tension) Tensor

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Abstract

Basic notions of continuous media mechanics are introduced for spaces with affine connections and metrics. Stress (tension) tensors are considered, obtained by the use of the method of Lagrangians with covariant derivatives (MLCD). On the basis of the covariant Noether’s identities for the energy-momentum tensors, Navier-Stokes’ identities are found and generalized Navier-Cauchy as well as Navier-Stokes’ equations are considered over $$(\mathcal{L}_n, g)$$-spaces.

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#### 1 Introduction

The important relations between stresses (tensions) and deformations and friction could also be established in continuous media mechanics in such comprehensive spaces as spaces with affine connections and metrics $([T_n, g])$-spaces. The main task of this paper is the consideration of the structures connected to the relations between stresses and deformations as a basis for the description of the dynamics of physical system (continuous media and fluids) in $([T_n, g])$-spaces.

In Section 2 energy-momentum tensors are considered, obtained by the use of the method of Lagrangians with covariant derivatives (MLCD). In Section 3 the structure of the stress (tension) tensor is discussed. In Section 4 relations between the kinematic characteristics of the relative velocity, of the friction velocity and the stress (tension) tensors are considered. In Section 5, on the basis of the covariant Noether’s identities for the energy-momentum tensors, Navier-Stokes’ identities are found and generalized Navier-Cauchy and Navier-Stokes’ equations are considered over $([T_n, g])$-spaces. The last Section 6 comprises some concluding remarks. The whole picture of a continuum media mechanics in $([T_n, g])$-spaces could be related to a classical (non-quantized) field theory.

All considerations are given in details (even in full details) for those readers who are not familiar with the considered problems.

Remark. The present paper is the fourth part of a larger research report on the subject with the title “Contribution to continuous media mechanics in $([T_n, g])$-spaces” with the following contents:

I. Introduction and mathematical tools.
II. Relative velocity and deformations.
III. Relative accelerations.
IV. Stress (tension) tensor.

The parts are logically self-dependent considerations of the main topics considered in the report.

#### 1.1 Invariant projections of a mixed tensor field of second rank

In the relativistic continuum media mechanics notions are introduced as generalizations of the same notions of the classical continuum media mechanics. This has been done by means of the projections of the (canonical, symmetric of Belinfante or symmetric of Hilbert) energy-momentum tensors along or orthogonal to a non-isotropic (non-null) contravariant vector field.

There are possibilities for using the projections for finding out the physical interpretations of the determined energy-momentum tensors. In an analogous way as in (pseudo) Riemannian spaces without torsion ($V_n$-spaces), the different relations
where \( \alpha, \beta \) and contravariant basic vector fields respectively, \( (e^i = \nabla u \in \psi \text{type } 1 \text{tensor fields} ) \), juxtaposed and vice versa by the use of the contravariant and covariant metric spaces.

The energy-momentum tensors are obtained as mixed tensor fields of second rank \( (\pi \text{and } \pi^* \text{between the quantities with well known physical interpretations can be considered as 1 by the use of the covariant and contravariant tensor fields} ) \).

To every covariant basic vector field a contravariant basic vector field can be juxtaposed and vice versa by the use of the contravariant and covariant metric tensor fields

\[
G = G^\alpha_\beta \cdot e_\beta \otimes e^\alpha = G_{ij} \cdot \partial_j \otimes dx^i \tag{1}
\]

in contrast to the mixed tensor fields of second rank of type 2

\[
\overline{G} = \overline{G}^\beta_\alpha \cdot e^\alpha \otimes e_\beta = \overline{G}^j_i \cdot dx^i \otimes \partial_j . \tag{2}
\]

The sets of vectors \( \{e^\alpha\} \) and \( \{e_\alpha\} \) are non-co-ordinate (non-holonomic) covariant and contravariant basic vector fields respectively, \( (\alpha, \beta = 1, ..., n) \).

The set of vectors \( \{dx^i\} \) and \( \{\partial_i\} \) are co-ordinate (holonomic) covariant and contravariant basic vector fields respectively, \( (i, j = 1, ..., n), \dim(L_n, g) = n \).

To every covariant basic vector field a contravariant basic vector field can be juxtaposed and vice versa by the use of the contravariant and covariant metric tensor fields

\[
g(e_\gamma) = g_\gamma e^{\alpha}, \quad g(\partial_j) = g_j \cdot dx^i, \quad \overline{g}(e^\gamma) = g^\alpha_\gamma e^{\alpha}, \quad \overline{g}(dx^i) = g_\gamma \cdot \partial_i . \tag{3}
\]

On this basis a tensor field of type 2 can be related to a tensor field of type 1 by the use of the covariant and contravariant tensor fields \( g = g_{ij} \cdot dx^i \cdot dx^j \) and \( \overline{g} = g^{ij} \cdot \partial_i \partial_j \text{[}dx^i \cdot dx^j = (1/2)(dx^i \otimes dx^j + dx^j \otimes dx^i), \partial_i \partial_j = (1/2)(\partial_i \partial_j + \partial_j \partial_i)\text{]} \)

\[
\overline{G} = g(G)\overline{g} = G^\beta_\alpha \cdot e^\alpha \otimes e_\beta = g_{\alpha \gamma} \cdot G^\gamma_\delta \cdot g_\delta \cdot e^\alpha \otimes e_\beta , \tag{4}
\]

\[
G^\beta_\alpha = g_{\alpha \gamma} \cdot G^\gamma_\delta \cdot g_\delta , \tag{5}
\]

\[
G = \overline{g}(G)g = G^\alpha_\beta \cdot e_\beta \otimes e^\alpha = g^{\delta \gamma} \cdot \overline{G}^\gamma_\delta \cdot g_{\alpha \gamma} \cdot e_\beta \otimes e^\alpha , \tag{6}
\]

\[
G^\alpha_\beta = g^{\delta \gamma} \cdot \overline{G}^\gamma_\delta \cdot g_{\gamma \alpha} . \tag{7}
\]

The Kronecker tensor field appears as a mixed tensor field of second rank of type 1

\[
K_r = g^\alpha_\beta \cdot e_\beta \otimes e^\alpha = g_j \cdot \partial_i \otimes dx^j
\]

and can be projected by means of the non-isotropic (non-null) contravariant vector field \( u \) and its projection metrics \( h_u \) and \( h^u \) \( [h_u = g - \frac{1}{e} \cdot g(u) \otimes g(u), h^u = \overline{g} - \frac{1}{e} \cdot u \otimes u, e = g(u, u) \neq 0] \)

\[
K_r = \varepsilon_{K_r} \cdot u \otimes g(u) + u \otimes g(K_r u) + K_r s \otimes g(u) + (K_r S)g ,
\]

where

\[
\varepsilon_{K_r} = \frac{1}{e^2} \cdot [g(u)](K_r)u = \frac{1}{e^2} \cdot u_{\overline{\pi}} \cdot u^\pi = \frac{1}{e^2} \cdot g_{\overline{\pi} \overline{\pi}} \cdot u^\pi \cdot u^\beta = \frac{1}{e^2} \cdot f^\rho_{\beta} \cdot f^\sigma_{\delta} \cdot g_{\rho \sigma} \cdot u^\gamma \cdot u^\delta = \frac{1}{e^2} \cdot u^i \cdot u^j = \frac{1}{e^2} \cdot g_{ik} \cdot u^i \cdot u^k = \frac{1}{e} \cdot k , \tag{8}
\]

\[
k = \frac{1}{e} \cdot [g(u)](K_r)u = \frac{1}{e} \cdot u^i \cdot u^\pi , \quad u^\pi = g_{\overline{\pi} \overline{\pi}} \cdot u^3 , \tag{9}
\]
does not appear in general as a Kronecker tensor field.

isotropic (non-null) contravariant vector field \( u \) can be written in the form

\[ G = \varepsilon \cdot e^i \cdot e_j, \]

\( \varepsilon \) interpretation can also be accepted for the projections of the energy-momentum tensors found by means of the method of Lagrangians with covariant derivatives (MLCD).

The representation of the tensor fields of the type 1 by the use of the non-isotropic (non-null) contravariant vector field \( u \) and its projective metrics \( h^u \) and \( h_u \) corresponds in its form to the representation of the viscosity tensor and the energy-momentum tensors in the continuum media mechanics in \( V_3 \) - or \( V_4 \)-spaces, where \( \varepsilon_G \) is the inner energy density, \( \varepsilon^u \) is the conductive energy flux density and \( \varepsilon^s \) is the stress tensor density. An analogous interpretation can also be accepted for the projections of the energy-momentum tensors found by means of the method of Lagrangians with covariant derivatives (MLCD).

\section{Energy-momentum tensors and the rest mass density}

The covariant Noether identities (generalized covariant Bianchi identities) can be considered as identities for the components of mixed tensor fields of second rank of the first type. The second covariant Noether identity

\[ \bar{\sigma}^\alpha{}^\beta - \varepsilon^s T^{\alpha}{}^\beta = -F^\alpha{}^\beta \]

can be written in the form

\[ \theta - s T = Q, \]
where
\[ \theta = \overline{\theta}_{\alpha}^{\beta} \cdot e_{\beta} \otimes e^{\alpha} = \overline{\theta}_{i}^{j} \cdot \partial_{j} \otimes dx^{i}, \]
\[ sT = \overline{sT}_{\alpha}^{\beta} \cdot e_{\beta} \otimes e^{\alpha} = \overline{sT}_{i}^{j} \cdot \partial_{j} \otimes dx^{i}, \]
\[ Q = \overline{Q}_{\alpha}^{\beta} \cdot e_{\beta} \otimes e^{\alpha} = \overline{Q}_{i}^{j} \cdot \partial_{j} \otimes dx^{i}, \] (20)

The tensor of second rank \( \theta \) is the generalized canonical energy-momentum tensor (GC-EMT) of the type 1; the tensor \( sT \) is the symmetric energy-momentum tensor of Belinfante (S-EMT-B) of the type 1; the tensor \( Q \) is the variational energy-momentum tensor of Euler-Lagrange (V-EMT-EL) of the type 1.

The second covariant Noether identity for the energy-momentum tensors of the type 1 is called \textit{second covariant Noether identity of type 1}.

By means of the non-isotropic contravariant vector field \( u \) and its corresponding projective metric the energy-momentum tensors can be represented in an analogous way as the mixed tensor fields of the type 1.

The structure of the generalized canonical energy-momentum tensor and the symmetric energy-momentum tensor of Belinfante for the metric and non-metric tensor fields has similar elements and they can be written in the form
\[ G = kG - L \cdot Kr, \]
\[ \theta = k\theta - L \cdot Kr, \]
\[ sT = T - L \cdot Kr, \] (21)

where
\[ k\theta = k\overline{\theta}_{\alpha}^{\beta} \cdot e_{\beta} \otimes e^{\alpha} = k\overline{\theta}_{i}^{j} \cdot \partial_{j} \otimes dx^{i}. \] (22)

On the analogy of the notions of the continuum media mechanics \( kG \) is called \textit{viscosity tensor field}.

\( G \) and \( kG \) can be written by means of \( u, h^{u} \) and \( h_{u} \) in the form
\[ G = \varepsilon_{G} \cdot u \otimes g(u) + u \otimes g^{(G)\pi} + g_{s} \otimes g(u) + (G S) g, \]
\[ kG = \varepsilon_{k} \cdot u \otimes g(u) + u \otimes g^{(k)\pi} + k_{s} \otimes g(u) + (k S) g. \] (23)

From the relation [21], the relations between the different projections of \( G \) and \( kG \) follow. If we introduce the abbreviation \( \varepsilon_{G} = \rho_{G} \), then
\[ \varepsilon_{k} = \rho_{k} + \frac{1}{c^{2}} \cdot L \cdot k, \]
\[ k_{S} = G_{S} + L \cdot Kr S, \]
\[ G_{S} = k G, \] (24)

and \( G \) can be written by means of [21] in the form
\[ G = \left( \rho_{G} + \frac{1}{c^{2}} \cdot L \cdot k \right) \cdot u \otimes g(u) - L \cdot Kr + u \otimes g^{(k)\pi} + k_{s} \otimes g(u) + (k S) g, \] (25)

where
\[ \rho_{G} = \frac{1}{c^{2}} \cdot \lbrack g(u) \rbrack (G)(u) \]

is the \textit{rest mass density} of the energy-momentum tensor \( G \) of the type 1. This type of representation of a given energy-momentum tensor \( G \) by means of the projective metrics of \( u \) and \( \rho_{G} \) is called \textit{representation of \( G \) by means of the projective metrics of the contravariant non-isotropic (non-null) vector field \( u \) and the rest mass density \( \rho_{G} \)}.

If the viscosity tensor field \( kG = 0 \) then the energy-momentum tensor \( G \sim (\theta, sT) \) will have the simplest form \( G = -L \cdot Kr \). At the same time, the relation \( \varepsilon_{k} = \rho_{k} + \frac{1}{c^{2}} \cdot L \cdot k = 0 \) is valid. It leads to the relation \( \rho_{G} = - \frac{1}{c^{2}} \cdot L \cdot k \). For \( L = p \), it follows that \( p = - \frac{k}{c^{2}} \cdot \rho_{G} \cdot e \), i.e. the pressure \( p \) is proportional to the mass density \( \rho_{G} \) and to the rest mass energy \( \rho_{G} \cdot e \) respectively. Therefore, if a flow has no viscosity
then the rest mass density (and rest mass energy respectively) are acting in their motion as a pressure of the flow. Since 
$$k = \frac{1}{c} \cdot [g(u)](Kr)u = \frac{1}{e} \cdot u^\alpha \cdot u_\alpha,$$
we have
$$p = -\frac{1}{u^\alpha \cdot u_\alpha} \cdot \rho_G \cdot e^2.$$

Special case: V\text{a}-spaces: 
$$S = C, \ k = 1, \ u^\alpha \cdot u_\alpha = u^i \cdot u_i = e, \ p = -\rho_G \cdot e.$$
Therefore, the pressure in a viscous-free flow is caused by the existence of a rest mass density of the flow. If we can measure a pressure in a viscous-free flow we can conclude that its rest mass density is different from zero.

There are other possibilities for representation of \(G\) by means of \(u\) and its corresponding projective metrics.

If we introduce the abbreviation
$$p_G = \rho_G \cdot u + {G\pi}, \quad (26)$$
where \(p_G\) is the momentum density of the energy-momentum tensor \(G\) of the type 1, then \(G\) can be written in the form
$$G = u \otimes g(\rho_G \cdot u + {G\pi}) + {G\alpha} \otimes g(u) + (G)g,$$
$$G = u \otimes g(p_G) + {G\alpha} \otimes g(u) + (G)g. \quad (27)$$

The representation of \(G\) by means of the last relation is called representation of \(G\) by means of the projective metrics of the contravariant non-isotropic contravariant vector field \(u\) and the momentum density \(p_G\).

By the use of the relations
$$g({G\pi}, u) = 0, \quad (G)g(u) = 0, \quad (28)$$
valid (because of their constructions) for every energy-momentum tensor \(G\) and the definitions
$$e_G = G(u) = (G)(u) = e \cdot (\rho_G \cdot u + {G\pi}), \quad g(u, u) = e \neq 0, \quad (29)$$
where \(e_G\) is the energy flux density of the energy-momentum tensor \(G\) of the type 1, the tensor field \(G\) can be written in the form
$$G = (\rho_G \cdot u + {G\alpha}) \otimes g(u) + u \otimes g({G\pi}) + (G)g,$$
$$G = \frac{1}{e} \cdot e_G \otimes g(u) + u \otimes g({G\pi}) + (G)g. \quad (30)$$

The representation of \(G\) by means of the last expression is called representation of \(G\) by means of the projective metrics of the contravariant non-isotropic vector field \(u\) and the energy flux density \(e_G\).

The generalized canonical energy-momentum tensor \(\theta\) can be represented, in accordance with the above described procedure, by the use of the projective metrics of \(u\) and the rest mass density \(\rho_\theta\)
$$\theta = k \theta - L \cdot Kr, \quad k \theta = \theta + L \cdot Kr, \quad (31)$$
$$\theta = (\rho_\theta + \frac{1}{e} \cdot L \cdot K) \cdot u \otimes g(u) - L \cdot Kr + u \otimes g(\theta) + \theta_{\alpha} \otimes g(u) + (\theta)g, \quad (32)$$
where
$$k \theta = k \bar{\theta}_{\alpha}^\beta \cdot \bar{e}_\beta \otimes \bar{e}_\alpha, \quad k \bar{\theta}_{\alpha}^\beta = \bar{T}_{\alpha}^\beta - \bar{K}_{\alpha}^\beta - \bar{W}_{\alpha}^{\beta \gamma} + L \cdot g_{\alpha}^\beta, \quad (33)$$
$$\rho_\theta = \frac{1}{e^2} \cdot [g(u)](\theta)(u), \quad k = \frac{1}{e} \cdot [g(u)](Kr)(u), \quad (34)$$
\[\rho_\theta = \frac{1}{e^2} \cdot g_{\sigma \tau} \cdot u^\beta \cdot \overline{\theta} \gamma \cdot u^\sigma = \frac{1}{e^2} \cdot \overline{\theta} \gamma \cdot u^\sigma \cdot u^\tau,\]
\[\varepsilon_{\theta \theta} = \rho_\theta + \frac{1}{e} \cdot L \cdot k,\]
\[\delta_\pi = \frac{1}{e} \cdot \left[ g(u) \right] (k\theta) h^u = \theta_\pi^\gamma \cdot e_\alpha = \theta_\pi^\gamma \cdot \partial_i,\]
\[\delta_\pi^\alpha = \frac{1}{e} \cdot k \theta \gamma \cdot u^\tau \cdot h^\alpha \tau, \quad \theta_\pi = \frac{1}{e} \cdot k \theta k \cdot u^\tau \cdot h^\tau,\]
\[\delta_\pi^\alpha = \frac{1}{e} \cdot h^\alpha \gamma \cdot \overline{\theta} \delta \gamma \cdot u^\delta, \quad \delta_\pi^\alpha = \frac{1}{e} \cdot h^\gamma j \cdot g_\delta \cdot k \theta k \cdot u^\tau,\]
\[\delta_\pi^\alpha = \frac{1}{e} \cdot \overline{\theta} \gamma \cdot g_\delta \cdot \theta_\pi \cdot k \theta k \cdot u^\tau, \quad \delta_\pi^\beta = \frac{1}{e} \cdot h^\gamma j \cdot k \theta k \cdot u^\tau,\]
\[\delta_\pi^\alpha \delta_\pi^\beta = \theta_\pi^\alpha \cdot e_\alpha \otimes e_\beta = \theta_\pi^\beta \cdot \partial_i \otimes \partial_j,\]
\[\theta_\gamma = (\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot u \otimes u - L \cdot K r (\overline{\theta} \gamma) + u \otimes \theta_\pi + \theta_\pi \otimes u + \delta_\pi = \theta_\alpha \beta \cdot e_\alpha \otimes e_\beta = \theta_\alpha \gamma \cdot \partial_i \otimes \partial_j,\]
\[\theta_\gamma \theta_\alpha \beta = \theta_\gamma \cdot g_\theta \delta \cdot \theta_\pi \gamma, \quad \theta_\gamma \delta = \theta_\delta k \cdot g_\gamma \gamma,\]
\[g(\theta) = (\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot g(u) \otimes g(u) - L \cdot K r (\overline{\theta} \gamma) + g(u) \otimes g(u) + g(\theta_\pi) + g(u) + g(\theta_\pi),\]
\[g(K r) = g_\alpha \gamma \cdot e_\alpha \otimes e_\beta = g_\gamma \cdot d x^i \otimes d x^j, \quad g_{\alpha \gamma} = g_{\alpha \gamma} \cdot f^\gamma \beta = g_{\gamma \alpha} \cdot f^\gamma \beta = g_{\gamma \alpha},\]
\[g_{\gamma \alpha} = g_{\gamma \alpha}, \quad (K r)_{\gamma} = K r (\overline{\theta} \gamma) = g_{\gamma \alpha} \cdot e_\alpha \otimes e_\beta = g_{\gamma \beta} \cdot \partial_i \otimes \partial_j.\]

In a co-ordinate basis \( \theta_i \), \( \theta_\gamma \) and \( g(\theta) \) can be represented in the forms
\[\theta_i = (\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot u_i \cdot u^j - L \cdot g^j_i \cdot u^j + \theta_\gamma \cdot u^i - g_\gamma \cdot e_\alpha \otimes e_\beta = \theta_\gamma \cdot \partial_i + \theta_\gamma \cdot \partial_j,\]
\[\theta^i = (\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot u^j \cdot u^j - L \cdot g^i_j \cdot u^j + \theta_\gamma \cdot u^i - g_\gamma \cdot e_\alpha \otimes e_\beta = \theta_\gamma \cdot \partial_i + \theta_\gamma \cdot \partial_j,\]
\[\theta_ij = g_{\gamma \alpha} \cdot \gamma = g_{\gamma \alpha} \cdot e_\alpha \otimes e_\beta = g_{\gamma \beta} \cdot \partial_i \otimes \partial_j,\]
\[u_i = g_{\gamma \alpha} \cdot \partial_i + \theta_\alpha \cdot e_\alpha \otimes e_\beta = g_{\gamma \alpha} \cdot e_\alpha \otimes e_\beta = \theta_\alpha \cdot \partial_i \otimes \partial_j.\]

The symmetric energy-momentum tensor of Belinfante \( sT \) can be represented in an analogous way by the use of the projective metrics \( h^u \) and \( h_u \) and the rest mass density \( \rho_T \) in the form
\[sT = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u \otimes g(u) - L \cdot K r + u \otimes g(\theta_\pi) + T_\gamma \otimes g(u) + (\overline{\theta} \gamma) g,\]
\[sT = sT \alpha, \quad e_\alpha \otimes e_\alpha = \theta_\gamma \cdot \theta_\gamma \cdot \partial_i \otimes d x^i,\]
\[sT = sT + L \cdot K r, \quad sT = sT + L \cdot K r = T, \quad \varepsilon_T = \rho_T + \frac{1}{e} \cdot L \cdot k.\]
\begin{align}
\rho_T &= \frac{1}{e^2} \cdot |g(u)|(sT)(u) = \frac{1}{e^2} \cdot g_{\pi \sigma} \cdot u^\beta \cdot s T_\gamma ^\alpha \cdot u^\gamma = \frac{1}{e^2} \cdot s T_\gamma ^\alpha \cdot u^\gamma \cdot u^\sigma \\
&= \frac{1}{e^2} \cdot g_{\pi \sigma} \cdot u^\gamma \cdot s T_\gamma ^i \cdot u^i = \frac{1}{e^2} \cdot s T_\gamma ^i \cdot u^i \cdot u^\sigma,
\end{align}
(55)

\begin{align}
T_\pi &= \frac{1}{e} \cdot |g(u)|(T)h^u = \frac{1}{e} \cdot |g(u)|(s_k T)h^u = T_\pi ^i \cdot e_\alpha = T_\pi ^i \cdot \partial_i,
\end{align}
(56)

\begin{align}
T_\pi ^i &= \frac{1}{e} \cdot T_\gamma ^\beta \cdot u^\pi \cdot h_\pi ^\alpha, \\
T_\pi ^i &= \frac{1}{e} \cdot T_1 ^i \cdot u^\pi \cdot h_\pi ^i,
\end{align}
(57)

\begin{align}
T_\pi ^i &= \frac{1}{e} \cdot h^\alpha (g)(T)(u) = T_\pi ^i \cdot e_\alpha = T_\pi ^i \cdot \partial_i,
\end{align}
(58)

\begin{align}
T_\pi ^i &= \frac{1}{e} \cdot h^\alpha \cdot g_{\pi \sigma} \cdot T_\beta ^\gamma \cdot u^\pi, \\
T_\pi ^i &= \frac{1}{e} \cdot h^{ij} \cdot g_{jk} \cdot T_1 ^k \cdot u^i,
\end{align}
(59)

\begin{align}
T_\pi ^i &= h^\alpha \cdot g_{\pi \sigma} \cdot T_\beta ^\gamma \cdot h_\pi ^\alpha, \\
T_\pi ^i &= h^{ij} \cdot g_{jk} \cdot T_1 ^k \cdot h_\pi ^i,
\end{align}
(60)

\begin{align}
\rho_T &= \rho_T + \frac{1}{e} \cdot L \cdot k \cdot u \otimes u - L \cdot K \cdot \pi, \\
\rho_T &= \rho_T + \frac{1}{e} \cdot T \pi + T \pi \otimes u + T_\pi = \rho_T + \frac{1}{e} \cdot T \pi \cdot \partial_i \otimes \partial_j,
\end{align}
(62)

\begin{align}
sT_\alpha ^\beta &= g_{\pi \sigma} \cdot T_\gamma ^\alpha \cdot g_\pi ^\sigma, \\
sT_\alpha ^\beta &= sT_\alpha ^\beta \cdot g_\pi ^\sigma \cdot T_\gamma ^\alpha \cdot g_\pi ^\sigma,
\end{align}
(63)

\begin{align}
g(sT) &= \rho_T + \frac{1}{e} \cdot L \cdot k \cdot g(u) \otimes u - L \cdot g(K0) + g(u) \otimes g(T \pi) + g(T_\pi), \\
g(sT) &= \rho_T + \frac{1}{e} \cdot L \cdot k \cdot g(u) \otimes u - L \cdot g(K0) + g(u) \otimes g(T \pi) + g(T_\pi),
\end{align}
(64)

\begin{align}
sT_\alpha ^\beta &= sT_\alpha ^\beta \cdot e_\alpha \otimes e_\beta, \\
sT_\alpha ^\beta &= sT_\alpha ^\beta \cdot \delta_\alpha \otimes \delta_\beta,
\end{align}
(65)

In a coordinate basis \( sT \), (sT)\( g \) and \( g(sT) \) can be represented in the forms

\begin{align}
sT_1 ^i &= \rho_T + \frac{1}{e} \cdot L \cdot k \cdot u_i \cdot u_j - L \cdot g_1 ^i \cdot u^j + T_\pi ^i \cdot u^j + u_i \cdot T_\pi ^j + g_{ik} \cdot T_\pi ^j, \\
sT_1 ^i &= \rho_T + \frac{1}{e} \cdot L \cdot k \cdot u_i \cdot u_j - L \cdot g_1 ^i \cdot u^j + T_\pi ^i \cdot u^j + T_\pi ^j
\end{align}
(66)

\begin{align}
sT_1 ^i &= sT_1 ^i \cdot g_\pi ^i, \\
sT_1 ^i &= sT_1 ^i \cdot T_\pi ^i, \\
sT_1 ^i &= sT_1 ^i \cdot T_\pi ^i, \\
sT_1 ^i &= sT_1 ^i \cdot T_\pi ^i
\end{align}
(67)

\begin{align}
sT_1 ^i &= g_{ik} \cdot sT_1 ^k = \rho_T + \frac{1}{e} \cdot L \cdot k \cdot u_i \cdot u_j - L \cdot g_1 ^i \cdot u^j + u_i \cdot T_\pi ^j + T_\pi ^j \cdot u_j + g_{ik} \cdot T_\pi ^j \cdot g_\pi ^j,
\end{align}
(68)

where

\begin{align}
T_\pi ^i &= g_\pi ^i \cdot T_\pi ^i, \\
T_\pi ^i &= g_\pi ^i \cdot T_\pi ^i, \\
T_\pi ^i &= g_\pi ^i \cdot T_\pi ^i,
\end{align}

The variational energy-momentum tensor of Euler-Lagrange \( Q \) can be represented in the standard manner by the use of the projective metrics \( h^\pi \), \( h_\pi \) and the rest mass density \( \rho_Q \) in the form

\begin{align}
Q &= -\rho_Q \cdot u \otimes g(u) - u \otimes g(Q_\pi) - Q_\pi \otimes g(u) - (Q_\pi) g,
\end{align}
(69)

where

\begin{align}
\rho_Q &= -\frac{1}{e^2} \cdot |g(u)|(Q)(u),
\end{align}
(70)

\begin{align}
\rho_Q &= -\frac{1}{e^2} \cdot g_{\pi \sigma} \cdot u^\beta \cdot \overline{Q}_\gamma ^\alpha \cdot u^\gamma = -\frac{1}{e^2} \cdot \overline{Q}_\gamma ^\alpha \cdot u^\gamma \cdot u^\sigma, \\
\rho_Q &= -\frac{1}{e^2} \cdot g_{ij} \cdot u^j \cdot \overline{Q}_k ^i \cdot u^k = -\frac{1}{e^2} \cdot \overline{Q}_k ^i \cdot u^k \cdot u^\sigma,
\end{align}
(71)

\begin{align}
Q_\pi &= -\frac{1}{e} \cdot [g(u)](Q)h^u = Q_\pi ^\alpha \cdot e_\alpha = Q_\pi ^i \cdot \partial_i,
\end{align}
(72)

\begin{align}
Q_\pi ^\alpha &= -\frac{1}{e} \cdot \overline{Q}_\gamma ^\beta \cdot u^\pi \cdot h_\gamma ^\alpha, \\
Q_\pi ^i &= -\frac{1}{e} \cdot \overline{Q}_k ^i \cdot u^k \cdot h_\pi ^i,
\end{align}
(73)

\begin{align}
Q_\pi &= -\frac{1}{e} \cdot h^\alpha (g)(Q)(u) = Q_\pi ^\alpha \cdot e_\alpha = Q_\pi ^i \cdot \partial_i,
\end{align}
(74)
\[ Q_{s\alpha} = -\frac{1}{c} \cdot h^{\alpha\beta} \cdot g_{\gamma\delta} \cdot u^\gamma \cdot u^\delta, \quad Q_{s1} = -\frac{1}{c} \cdot h^{ij} \cdot g_{\alpha\beta} \cdot Q_{ij} \cdot u^i \cdot u^j, \]

\[ Q_S = -h^u (g(Q)) h^u = Q S^{\alpha\beta} \cdot e_\alpha \otimes e_\beta = Q S^{ij} \cdot \partial_i \otimes \partial_j, \]

\[ Q S^{\alpha\beta} = -h^{\alpha\gamma} \cdot g_{\gamma\delta} \cdot \bar{Q} \delta_{\bar{\gamma}} \cdot h^\delta_\beta, \quad Q S^{ij} = -h^{ik} \cdot g_{\kappa\mu} \cdot \bar{Q}_{m} \cdot i_{h^\mu} \cdot h^\gamma_j, \]

\[ (Q)g = -\rho Q \cdot u \otimes u - u \otimes Q \pi - Q S \otimes u - Q S = \]

\[ = Q S^{\alpha\beta} \cdot e_\alpha \otimes e_\beta = Q S^{ij} \cdot \partial_i \otimes \partial_j, \]

\[ Q^{\alpha\beta} = g^{\alpha\gamma} \cdot \bar{Q} \gamma_{\bar{\alpha}} \cdot g^\beta_\gamma, \quad Q^{ij} = \bar{Q}_{i k} \cdot g_{j}^k, \]

\[ g(Q) = -\rho Q \cdot g(u) \otimes g(u) - g(u) \otimes g(Q \pi) - g(Q S) \otimes g(u) - g(Q S) g, \]

\[ g(Q) = Q_{\alpha\beta} \cdot e^\alpha \otimes e^\beta = Q_{ij} \cdot dx^i \otimes dx^j, \quad Q_{\alpha\beta} = g_{\alpha\gamma} \cdot \bar{Q} \gamma_{\bar{\beta}} \cdot Q_{ij} = g_{i\kappa} \cdot \bar{Q}^k_{j}, \]

In a co-ordinate basis \( Q \), \( (Q)g \) and \( g(Q) \) can be represented in the forms

\[ \bar{Q}^{ij} = -\rho Q \cdot u^i \cdot u^j - Q \pi_{ij} \cdot u^j - Q S_{ij} - g_{\kappa\mu} \cdot Q S^{ij}, \]

\[ Q_{ij} = g_{\kappa\mu} \cdot \bar{Q}^k_{j} = -\rho Q \cdot u^i \cdot u^j - Q \pi_{ij} \cdot u^j - Q S_{ij} - g_{\kappa\mu} \cdot Q S^{kl} \cdot g_{ij}, \]

where

\[ Q_{\pi_{ij}} \equiv g_{\kappa\mu} \cdot Q_{\pi}^k, \quad Q_{s_{ij}} \equiv g_{\kappa\mu} \cdot Q_s^l, \quad Q_{s_{ij}} \equiv g_{\kappa\mu} \cdot Q S^{kl} \cdot g_{ij}. \]

The introduced abbreviations for the different projections of the energy-momentum tensors have their analogous forms in \( V_3 \) and \( V_4 \)-spaces, where their physical interpretations have been proposed [4], [5], [6] (S.383-385). The stress tensor in \( V_3 \)-spaces has been generalized to the energy-momentum tensor \( sT \) in \( V_4 \)-spaces. The viscosity stress tensor \( s_k T \) appears as the tensor \( T \) in the structure of the symmetric energy-momentum tensor of Belinfante \( sT \).

On the analogy of the physical interpretation of the different projections, the following definitions can be proposed for the quantities in the representations of the different energy-momentum tensors:

A. Generalized canonical energy-momentum tensor of the type 1 ....... \( \theta \)
(a) Generalized viscous energy-momentum tensor of the type 1 ......... \( k\theta \)
(b) Rest mass density of the generalized canonical energy-momentum tensor \( \theta \) ................................................................. \( \rho \theta \)
(c) Conductive momentum density of the generalized canonical energy-momentum tensor \( \theta \) ................................................................. \( \rho \pi \)
(d) Conductive energy flux density of the generalized canonical energy-momentum tensor \( \theta \) ................................................................. \( e \cdot \theta \pi \)
(e) Stress tensor of the generalized canonical energy-momentum tensor \( \theta \) ................................................................. \( \theta \pi \)

B. Symmetric energy-momentum tensor of Belinfante of the type 1 ....... \( sT \)
(a) Symmetric viscous energy-momentum tensor of the type 1 ......... \( T \)
(b) Rest mass density of the symmetric energy-momentum tensor of Belinfante \( sT \) ................................................................. \( \rho_T \)
(c) Conductive momentum density of the symmetric energy-momentum tensor of Belinfante \( sT \) ................................................................. \( \pi_T \)
(d) Conductive energy flux density of the symmetric energy-momentum tensor of Belinfante \( sT \) ................................................................. \( e \cdot T \pi \)
(e) Stress tensor of the symmetric energy-momentum tensor of Belinfante \( sT \) ................................................................. \( T \pi \)
C. Variational (active) energy-momentum tensor of Euler-Lagrange. 

(a) Rest mass density of the variational energy-momentum tensor of Euler-Lagrange \( Q \). .................................................. \( \rho_Q \)

(b) Conductive momentum density of the variational energy-momentum tensor of Euler-Lagrange \( Q \). .................................................. \( \rho_{\pi} \)

(c) Conductive energy flux density of the variational energy-momentum tensor of Euler-Lagrange \( Q \). .................................................. \( \dot{\varepsilon} \cdot Q_s \)

(d) Stress tensor of the variational energy-momentum tensor of Euler-Lagrange \( Q \). .................................................. \( Q_S \)

The projections of the energy-momentum tensors have properties which are due to their construction, the orthogonality of the projective metrics \( h_\pi \) and \( h^u \) correspondingly, and to the vector fields \( u \) and \( g(u) \) \( h_\pi(u) = 0, \ h^u[g(u)] = 0 \)

\[
g(u, \pi) = g(\pi, u) = 0, \ g(u, T) = 0, \ g(u, Q) = 0, \quad (84)
\]

\[
g(u, \pi) = g(\pi, u) = 0, \ g(u, T) = 0, \ g(u, Q_s) = 0, \quad (85)
\]

\[
g(u)^T(\pi) = 0, \ (\pi)g(u) = 0, \ g(u)^T(S) = 0, \ (T)g(u) = 0, \ g(u)(Q_S) = 0, \quad (86)
\]

From the properties of the different projections, it follows that the conductive momentum density \( \pi \) (or \( \pi \)) is a contravariant vector field orthogonal to the vector field \( u \). The conductive energy flux density \( \dot{\varepsilon} \cdot s \) (or \( \pi \)) is also a contravariant vector field orthogonal to \( u \). The stress tensor \( S \) (or \( \pi \)) is orthogonal to \( u \) independently of the side of the projection by means of the vector field \( u \).

The second covariant Noether identity \( \theta - \pi T \equiv Q \) can be written by the use of the projections of the energy-momentum tensors in the form

\[
(\rho_\pi - \rho_T + \rho_Q) \cdot u \otimes g(u) + u \otimes g(\pi - T) + Q \pi) +
+(\pi - T + Q_s) \otimes g(u) + (\pi - T + Q)g(u) = 0.
\]

(87)

After contraction of the last expression consistently with \( u \) and \( \pi \) and taking into account the properties \( [\pi] : [\pi] \) the second covariant Noether identity disintegrates in identities for the different projections of the energy-momentum tensors

\[
\rho_\pi \equiv \rho_T - \rho_Q, \; \pi \pi \equiv T - Q, \; \pi \pi \equiv T - Q_s, \; \pi \pi \equiv T - Q.
\]

(88)

If the covariant Euler-Lagrange equations of the type \( \delta L / \delta V^A_B = 0 \) are fulfilled for the non-metric tensor fields of a Lagrangian system and \( \rho_Q = 0 \), then the variational energy-momentum of Euler-Lagrange \( Q = \rho_Q + \rho_Q \) is equal to zero.

This fact leads to vanishing the invariant projections of \( Q (\rho_Q = 0, \pi = 0, Q_s = 0, Q = 0) \). The equality which follows between \( \theta \) and \( \pi T \) has as corollaries the identities

\[
\rho_\pi \equiv \rho_T, \quad \pi \pi \equiv T, \quad \pi \pi \equiv T, \quad \pi \pi \equiv T.
\]

(89)

From the first identity \( \rho_\pi \equiv \rho_T \) and the identity \( [\pi] \) for \( \rho \), it follows that the covariant Euler-Lagrange equations of the type \( \delta L / \delta V^A_B = 0 \) for non-metric fields \( V \) and \( \rho Q = 0 \) appear as sufficient conditions for the unique determination of the notion of rest mass density \( \rho \) for a given Lagrangian system.

**Proposition 1** The necessary and sufficient condition for the equality

\[
\rho_\pi = \rho_T
\]

(90)
is the condition $\rho Q = 0$.

Proof: It follows immediately from the first identity in (88).

The condition $\rho Q \neq 0$ leads to the violation of the unique determination of the notion of rest mass density and to the appearance of three different notions of rest mass density corresponding to the three different energy-momentum tensors for a Lagrangian system. Therefore, the violation of the covariant Euler-Lagrange equations $\delta v L/\delta V^A B = 0$ for the non-metric tensor fields or the existence of metric tensor fields in a Lagrangian density with $\rho Q \neq 0$ induce a new rest mass density (a new rest mass respectively) for which the identity (88) is fulfilled.

The identity $\rho_0 \equiv \rho_T - \rho_Q$ can be related to the physical hypotheses about the inertial, passive and active gravitational rest mass densities in models for describing the gravitational interaction. To every energy-momentum tensor a non-null rest mass density corresponds. The existence of the variational energy-momentum tensor of Euler-Lagrange is connected with the existence of the gravitational interaction in a Lagrangian system in Einstein’s theory of gravitation [7] and therefore, with the existence of a non-null active gravitational rest mass density. When a Lagrangian system does not interact gravitationally, the active gravitational rest mass density is equal to zero and the principle of equivalence between the inertial and the passive rest mass density is fulfilled [8, 9].

From the second covariant Noether identity of the type 1 by means of the relations

$$\overline{G} = g(G)\overline{g} , \quad G = \overline{g}(G)g ,$$

one can find the second covariant Noether identity of the type 2 for the energy-momentum tensors of the type 2 in the form

$$\overline{\theta} - s\overline{T} \equiv \overline{Q} , \quad (91)$$

where

$$\overline{\theta} = g(\theta)\overline{g} , \quad s\overline{T} = g(sT)\overline{g} , \quad \overline{Q} = g(Q)\overline{g} . \quad (92)$$

The invariant representation of the energy-momentum tensors by means of the projective metrics $h^\alpha, h_\alpha$ and the rest mass density allows a comparison of these tensors with the well known energy-momentum tensors from the continuum mechanics (for instance, with the energy-momentum tensor of an ideal liquid in $V_4$-spaces:

$$sT_{i}^j = (\rho + \frac{1}{e} \cdot p) \cdot u_i \cdot u^j - p \cdot g_i^j , \quad e = \text{const.} \neq 0, \quad k = 1 . \quad (93)$$

It follows from the comparison that the Lagrangian invariant $L$ can be interpreted as the pressure $p = L$ characterizing the Lagrangian system. This possibility for an other physical interpretation than the usual one (in the mechanics $L$ is interpreted as the difference between the kinetic and the potential energy) allows a description of Lagrangian systems on the basis of phenomenological investigations determining the dependence of the pressure on other dynamical characteristics of the system. If these relations are given, then by the use of the method of Lagrangians with covariant derivatives (MLCD) the corresponding covariant Euler-Lagrange equations can be found as well as the energy-momentum tensors.

3 Stress (tension) tensor

Let us now consider closer the introduced stress tensors $G S (G \sim \theta, sT, Q)$ obtained from the different energy-momentum tensors $\theta, sT, Q$. 

3.1 Representation of the stress (tension) viscosity tensor field

The tensor $G^S$ has the structure of a contravariant tensor of second rank of the type

$$G^S = h^u(g(G)h^u = h^{ik} \cdot g_{\pi \pi} \cdot G_{\pi \pi}^{-1} \cdot h^{mj} \cdot \partial_i \otimes \partial_j =$$

$$= \overline{g}(h_u)\overline{g}(g)(G)\overline{g}(h_u)\overline{g}(h_u)\overline{g} = G^S_{ij} \cdot \partial_i \otimes \partial_j \text{ , }$$

where

$$h^u = \overline{g}(h_u)\overline{g} \text{ , } G = G^S_{ij} \cdot \partial_i \otimes \partial_j \text{ . }$$

The corresponding covariant tensor of second rank $g(G)h_u$ will have the forms

$$g(G)h_u = g(\overline{g})(h_u)\overline{g}(h_u)\overline{g}(h_u)\overline{g}(g) = (h_u)(G)\overline{g}(h_u) =$$

$$= h_{ik} \cdot G_{\pi \pi} \cdot g_{\pi \pi} \cdot h_{ij} \cdot dx^i \otimes dx^j =$$

$$= h_{ik} \cdot G_{kl} \cdot h_{ij} \cdot dx^i \otimes dx^j =$$

$$= h_u(\overline{B})h_u := \overline{A} \text{ , }$$

where

$$\overline{B} = (G)\overline{g} \text{ , } h_u = h_u(\overline{g})h_u = g(h^u)g \text{ . }$$

An energy-momentum tensor $G (G \sim \theta, sT, Q)$ can now be represented in the form

$$g(G) = \varepsilon_{G} \cdot g(u) \otimes g(u) + g(u) \otimes g(G^{\pi}) + g(G^{S}) \otimes g(u) + g(G^{S})g =$$

$$= \varepsilon_{G} \cdot g(u) \otimes g(u) + g(u) \otimes g(G^{\pi}) + g(G^{S}) \otimes g(u) + h_u(\overline{B})h_u \text{ . }$$

On the other side, the stress tensor $G^S$ could be expressed by its viscosity part $kS$ and its pressure part $L \cdot KrS$

$$G^S = kS - L \cdot KrS \text{ , } L = p \text{ . }$$

Since

$$G = kG - L \cdot Kr \text{ , }$$

we will concentrate our attention to the representation of the viscosity tensor field $kG$. It can be represented in analogous way as the energy-momentum tensor $G$

$$kG = \varepsilon_k \cdot u \otimes g(u) + u \otimes g(k^{\pi}) + k^S \otimes g(u) + (k^S)g \text{ , }$$

$$g(kG) = \varepsilon_k \cdot g(u) \otimes g(u) + g(u) \otimes g(k^{\pi}) + g(k^S) \otimes g(u) + g(k^S)g \text{ . }$$

The covariant tensor of second rank $g(k^S)g$, called stress (tension) viscosity tensor, could now be written in the forms

$$g(k^S)g = g(\overline{g})(h_u)(kG)\overline{g}(h_u)\overline{g}(g) = (h_u)(kG)\overline{g}(h_u) =$$

$$= h_{ik} \cdot k_{G_{\pi \pi}} \cdot g_{\pi \pi} \cdot h_{ij} \cdot dx^i \otimes dx^j =$$

$$= h_{ik} \cdot kS^{kl} \cdot h_{ij} \cdot dx^i \otimes dx^j =$$

$$= h_u(\overline{B})h_u := \overline{X} \text{ , }$$

where

$$\overline{B} = (kG)\overline{g} = kG_{m \cdot i} \cdot g_{\pi \pi} \cdot \partial_i \otimes \partial_j = \overline{B}^{ij} \cdot \partial_i \otimes \partial_j \text{ . }$$

The tensor $\overline{B}$ can further be expressed by its symmetric and antisymmetric parts

$$s\overline{B} = \frac{1}{2} \cdot (\overline{B}^{ij} + \overline{B}^{ji}) \cdot \partial_i \otimes \partial_j \text{ , } a\overline{B} = \frac{1}{2} \cdot (\overline{B}^{ij} - \overline{B}^{ji}) \cdot \partial_i \otimes \partial_j \text{ . }$$
The stress (tension) viscosity tensor $\overline{A}$ can be represented as a sum, containing three terms: a trace-free symmetric term, an antisymmetric term and a trace term

$$\overline{A} = k_s \overline{D} + k \overline{W} + \frac{1}{n-1} \cdot k \overline{U} \cdot h_u ,$$  \hspace{1cm} (106)

where

$$k_s \overline{D} = k \overline{D} - \frac{1}{n-1} \cdot \mathbf{g}[k \overline{D}] \cdot h_u = k \overline{D} - \frac{1}{n-1} \cdot k \overline{U} \cdot h_u ,$$ \hspace{1cm} (107)

$$\mathbf{g}[k \overline{D}] = g_{ij} \cdot k \overline{D}_{ij} = 0 ,$$ \hspace{1cm} (108)

$$k \overline{D} = h_u (\, s \overline{B} \,) h_u , \quad k \overline{U} = \mathbf{g}[k \overline{D}] = \mathbf{g}[s \overline{A}] ,$$ \hspace{1cm} (109)

$$s \overline{A} = \frac{1}{2} \cdot (\overline{A}_{ij} + \overline{A}_{ji}) \cdot dx^i dx^j ,$$ \hspace{1cm} (110)

$$k \overline{U} = \frac{1}{2} \cdot (s \overline{A}_{ij}) = g_{ij} \cdot s \overline{A}_{ij} .$$ \hspace{1cm} (111)

The trace-free symmetric tensor $k_s \overline{D}$ is called shear stress (tension) viscosity tensor (stress deviator), the antisymmetric tensor $k \overline{W}$ is called rotation (vortex) stress viscosity tensor, the invariant $k \overline{U}$ is called expansion stress viscosity invariant. The viscosity tensor field $kG$ could now be written in the forms

$$g(kG) = \varepsilon_k \cdot g(u) \otimes g(u) + g(u) \otimes g^k \pi + g^k s \otimes g(u) + g^k (kS)g =$$

$$= \varepsilon_k \cdot g(u) \otimes g(u) + g(u) \otimes g^k \pi + g^k s \otimes g(u) + \overline{A} =$$

$$= \varepsilon_k \cdot g(u) \otimes g(u) + g(u) \otimes g^k \pi + g^k s \otimes g(u) +$$

$$+ k_s \overline{D} + k \overline{W} + \frac{1}{n-1} \cdot k \overline{U} \cdot h_u .$$ \hspace{1cm} (112)

The energy-momentum tensor $G$ will have the form

$$G = (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot u \otimes g(u) - L \cdot K r + u \otimes g^k \pi + k s \otimes g(u) +$$

$$+ \mathbf{g}[k \overline{D}] + \mathbf{g}[k \overline{W}] + \frac{1}{n-1} \cdot k \overline{U} \cdot \mathbf{g}(h_u) ,$$ \hspace{1cm} (113)

or the form

$$g(G) = (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot g(u) \otimes g(u) - L \cdot g(K r) + g(u) \otimes g^k \pi + g^k s \otimes g(u) +$$

$$+ k_s \overline{D} + k \overline{W} + \frac{1}{n-1} \cdot k \overline{U} \cdot h_u .$$ \hspace{1cm} (114)

The vector field

$$kG(\xi(a) \perp) = g(k^\pi, \xi(a) \perp) \cdot u + \mathbf{g}[k \overline{D}](\xi(a) \perp) + (k \overline{W})(\xi(a) \perp) + \frac{1}{n-1} \cdot k \overline{U} \cdot \xi(a) \perp$$  \hspace{1cm} (115)

is called stress vector field. It describes the stress along the vector field $\xi(a) \perp$ but is not collinear to $\xi(a) \perp$. The following relation is also fulfilled

$$K r(\xi(a) \perp) = g^j_i \cdot \xi(a) \perp = g^j_i \cdot g^k \pi \cdot h_{k l} \cdot \xi(\pi) \cdot \partial_i = g^k \pi \cdot h_{k l} \cdot \xi(\pi) \cdot \partial_i =$$

$$= f^i_m \cdot g^m \pi \cdot h_{k l} \cdot \xi(\pi) \cdot \partial_i .$$ \hspace{1cm} (116)

If we write the vector field $\xi(a) \perp$ in the form

$$\xi(a) \perp = r \cdot n(a) \perp , \quad g(n(a) \perp, n(a) \perp) = \varepsilon \ , \ \varepsilon = \pm 1 ,$$ \hspace{1cm} (117)
then
\[ \mathcal{A}(\xi_{(a)}, \xi_{(a)}) = r^2 \cdot \mathcal{A}(n_{(a)}\perp, n_{(a)}\perp) = r^2 \cdot \sigma_N = r^2 \cdot k s \mathcal{D}(n_{(a)}\perp, n_{(a)}\perp) \pm \frac{1}{n - 1} \cdot k \mathcal{U} \cdot r^2, \] (118)

where
\[ h_u(n_{(a)}\perp, n_{(a)}\perp) = \pm r^2, \quad \sigma_N = k s \mathcal{D}(n_{(a)}\perp, n_{(a)}\perp) \pm \frac{1}{n - 1} \cdot k \mathcal{U} . \] (119)

The hypersurface
\[ \mathcal{A}(\xi_{(a)}, \xi_{(a)}) = \sigma_N \cdot r^2 = \pm k^2, \quad k = \text{const}. \] (120)
is called Cauchy stress hypersurface. It follows, that for a Cauchy stress hypersurface the relation
\[ \sigma_N = k s \mathcal{D}(n_{(a)}\perp, n_{(a)}\perp) \pm \frac{1}{n - 1} \cdot k \mathcal{U} = \pm \frac{k^2}{r^2} \] (121)
is fulfilled.

Special case: \( k s \mathcal{D} := 0: \)
\[ \sigma_N = \pm \frac{1}{n - 1} \cdot k \mathcal{U} = \pm \frac{k^2}{r^2}, \quad k \mathcal{U} = (n - 1) \cdot \frac{k^2}{r^2}. \] (122)

The trace term
\[ \frac{1}{n - 1} \cdot k \mathcal{U} \cdot h_u \] (123)
could be written by the use of the explicit form of the covariant projective metric
\[ h_u \]
\[ \frac{1}{n - 1} \cdot k \mathcal{U} \cdot h_u = \frac{1}{n - 1} \cdot k \mathcal{U} \cdot [g - \frac{1}{e} \cdot g(u) \otimes g(u)] = \frac{1}{n - 1} \cdot [k \mathcal{U} \cdot g - \frac{1}{e} \cdot k \mathcal{U} \cdot g(u) \otimes g(u)] = \frac{1}{n - 1} \cdot k \mathcal{U} \cdot g - \frac{1}{n - 1} \cdot \frac{1}{e} \cdot k \mathcal{U} \cdot g(u) \otimes g(u). \] (124)

By means of the last expression we can represent \( g(G) \) in the form
\[ g(G) = (\rho_G - \frac{1}{n - 1} \cdot \frac{1}{e} \cdot k \mathcal{U} + \frac{1}{e} \cdot L \cdot k) \cdot g(u) \otimes g(u) - L \cdot g(K \mathcal{R}) + \frac{1}{n - 1} \cdot k \mathcal{U} \cdot g + +g(u) \otimes g(k \pi) + g(k s) \otimes g(u) + k s \mathcal{D} + k \mathcal{W} \] (125)
\[ g(G) = (\rho_G - \frac{1}{e} \cdot k E + \frac{1}{e} \cdot L \cdot k) \cdot g(u) \otimes g(u) - L \cdot g(K \mathcal{R}) + k E \cdot g + +g(u) \otimes g(k \pi) + g(k s) \otimes g(u) + k s \mathcal{D} + k \mathcal{W} , \] (126)
\[ g(G) = [\rho_G + \frac{1}{e} \cdot (L \cdot k - k E)] \cdot g(u) \otimes g(u) - L \cdot g(K \mathcal{R}) + k E \cdot g + +g(u) \otimes g(k \pi) + g(k s) \otimes g(u) + k s \mathcal{D} + k \mathcal{W} , \] (127)

where
\[ k E = \frac{1}{n - 1} \cdot k \mathcal{U} \] (128)
is the inner energy density. The pressure $p$ is considered as a hydrostatic stress invariant. The tensor $p \cdot Kr + kE \cdot \bar{g}(g)$ is called hydrostatic stress tensor. It contains a part with the pressure $p$ and a part induced by the inner energy density $kE$.

**Special case:** $(L_n, g)$-spaces: $S = C$. In this special case $f^i_j = g^i_j$, $k = 1$,

$$g(Kr) = g(g^i_j \cdot \partial_i \otimes dx^j) = g_{ij} \cdot dx^k \otimes dx^j = g_{kj} \cdot dx^k \otimes dx^j = g \ ,$$

(129)

$$g(G) = (\rho_G - \frac{1}{e} \cdot kE + \frac{1}{e} \cdot L) \cdot g(u) \otimes g(u) - (L - kE) \cdot g +$$

$$+ g(u) \otimes g(k\pi) + g(ks) \otimes g(u) + k_s \bar{D} + k \bar{W} \ ,$$

(130)

$$g(G) = (\rho_G + \frac{1}{e} \cdot \overline{p}) \cdot g(u) \otimes g(u) - \overline{p} \cdot g +$$

$$+ g(u) \otimes g(k\pi) + g(ks) \otimes g(u) + k_s \bar{D} + k \bar{W} \ ,$$

(131)

where

$$\overline{p} = L - kE = p - kE \ .$$

(132)

It is obvious from the last expressions, that the structure of the invariant $\overline{p}$ includes not only the pressure $p$ but also the inner energy density $kE$, which acts in the flow on the one side as a rest mass density $(1/e) \cdot kE$ and, on the other side, as an additional pressure $kE$. Usually $\overline{p}$, the invariant $\overline{p}$ is interpreted as the pressure of a system in a more general sense than as the hydrostatic pressure. The invariant $\overline{p}$ could vanish under two different types of conditions:

(a) $\overline{p} = 0$ if $p = 0$ and $kE = 0$.

(b) $\overline{p} = 0$ if $p = kE$.

In both cases we have for $g(G)$

$$g(G) = \rho_G \cdot g(u) \otimes g(u) + g(u) \otimes g(k\pi) + g(ks) \otimes g(u) + k_s \bar{D} + k \bar{W} \ .$$

(133)

For $k\pi = 0$, $ks = 0$, $k_s \bar{D} = 0$, and $k \bar{W} = 0$ the energy-momentum tensors for dust matter follow

$$g(G) = \rho_G \cdot g(u) \otimes g(u) \ .$$

(134)

The last result shows that dust matter could exist under different inner conditions for the hydrostatic pressure and the inner energy.

4 Relations between stresses (tensions) and deformations

The considered notions of deformation velocity and deformation acceleration are kinematic quantities of a continuous media (flow). On the other side, the energy-momentum tensors and their corresponding viscosity tensors are related to the dynamic characteristics of a continuous media. The assumption that deformations generate stresses and motions in a media (flow) does not lead to a unique (logically consistent) way for finding relations between stresses and deformations. For any specific branch of the continuous media mechanics (elasticity theory, plasticity theory, viscosity theory, hydrodynamics etc.) different relations between stresses and deformations are assumed. In general, the most of these assumptions are included in the precondition.
The vector fields $\xi_{(a)\perp}$ and their corresponding infinitesimal vectors $\xi'_{(a)\perp} = d\lambda^a \cdot \xi_{(a)\perp}$ (over $a$ is not summarized), introduced in the preconditions, are interpreted as deformation vector fields, orthogonal to the velocity of the continuous media in the space-time. This interpretation of $\xi_{(a)\perp}$ and $\xi'_{(a)\perp}$ coincides with the interpretation of $\xi_{(a)\perp}$ as the distance between two material points $\mathcal{P}$ and $\mathcal{P}$ in a flow with the corresponding co-ordinates $\mathcal{P} = x^i(\tau_0, \lambda^a_0) + d\lambda^a$ after a deformation. The point $\mathcal{P}$ is the point $P$ after a deformation, and the point $P$ is the point $P'$ taking the place of the point $P$ after the same deformation, i.e. after a deformation we have the changes of the points $P'$ and $P$

$$P' \rightarrow P = P_{(a)\perp} \mathcal{P},$$

$$\mathcal{P}^i = x^i(\tau_0, \lambda^a_0) + d\lambda^a = x^i(\tau_0, \lambda^a_0) + \frac{dx^i}{d\lambda^a} \bigg|_{(\tau_0, \lambda^a_0)} = x^i + \xi'_{(a)\perp}. \quad \text{(136)}$$

The vector $\xi_{(a)\perp}$ is then called deformation vector. The act of deformation itself is not considered. Only the result of the deformation (the deformation vector $\xi_{(a)\perp}$) is taken into account in all investigations. This means that at a given moment (time-point $\tau_0$) we have a picture of a continuous media after a (possible) deformation and all future deformations, stresses and motions of this media are further considered by the use of the tools of continuous media mechanics.

Remark. The interpretation of $\xi_{(a)\perp}$ as a deformation vector is not necessary for the considerations of the relative velocity and the relative acceleration with their kinematic characteristics. It is needed if we wish to consider the different relations from elasticity theory (Hook’s law) and hydrodynamics (different types of flows) from a general point of view.

Conjecture. The stress (tension) viscosity tensor $g^{(kS)}g = \overline{A}$ could be considered in general as a function of the friction deformation velocity tensor $R$ and the deformation velocity tensor $d$

$$\overline{A} = \overline{A}(R, d),$$

$$\overline{A} = \overline{A}_{ij} \cdot dx^i \otimes dx^j,$$

$$d = d_{ij} \cdot dx^i \otimes dx^j,$$

$$R = R_{ij} \cdot dx^i \otimes dx^j \quad \text{(137)}$$

On the basis of different specializations of the form of dependence of $\overline{A}$ on $d$ and $R$, we can find generalizations of well known schemes from the classical continuous media mechanics in Euclidean spaces $E_n$ ($n = 3$).

4.1 Linear elasticity theory

In a linear elasticity theory the viscosity tensor $g^{(kS)}g = \overline{A}$ depends linearly only on the friction deformation velocity tensor $R$

$$\overline{A}_{ij} = C_{ij}^{\ km} \cdot R_{km}, \quad \overline{A} = C[R],$$

$$C = C_{ij}^{\ km} \cdot dx^i \otimes dx^j \otimes \partial_k \otimes \partial_m \in \otimes^2(M),$$

$$R = R_{ij} \cdot dx^i \otimes dx^j \in \otimes^2(M),$$

$$R = \sigma R + \omega R + \frac{1}{n-1} \cdot \sigma R \cdot h_u,$$

$$\overline{A} = k_s \overline{D} + k \overline{W} + \frac{1}{n-1} \cdot k \overline{U} \cdot h_u \quad \text{,}$$

16
\[ \overline{\mathbf{A}}[C] = C[\sigma R + \omega R + \frac{1}{n-1} \cdot \partial R \cdot h_u] = \\
= C[\sigma R] + C[\omega R] + \frac{1}{n-1} \cdot \partial R \cdot C[h_u] \] (138)

where \( C_{ij}^{km} \) are components of the tensor of elasticity coefficients. They should obey the conditions

\[ C_{ij}^{km} \cdot \mathbf{u} = 0 \quad , \quad C_{ij}^{km} \cdot \mathbf{u} = 0 \] (139)

On the other side, we have to decompose the tensor of elasticity coefficients \( C = C_{ij}^{km} \cdot dx^i \otimes dx^j \partial_k \otimes \partial_m \) with respect to the basis \( dx^1 \otimes dx^2 \) in a trace-free symmetric part, antisymmetric part, and trace part [in correspondence with the structure of \( \overline{\mathbf{A}} \)]

\[ C = sC + aC + \frac{1}{n-1} \cdot \overline{[C]} : h_u \]

where

\[ sC = sC - \frac{1}{n-1} \cdot \overline{[C]} : h_u \quad , \quad \overline{[C]} = 0 \] (140)

\[ sC = \frac{1}{2} \cdot (C_{ij}^{km} + C_{ji}^{km}) \cdot dx^i \otimes dx^j \partial_k \otimes \partial_m = \\
= C_{(ij)}^{km} \cdot dx^i dx^j \partial_k \otimes \partial_m \]

\[ \overline{[C]} = \overline{[sC]} = g^{ij} \cdot C_{ij}^{km} \cdot \partial_k \otimes \partial_m \]

\[ aC = \frac{1}{2} \cdot (C_{ij}^{km} - C_{ji}^{km}) \cdot dx^i \otimes dx^j \partial_k \otimes \partial_m = \\
= C_{[ij]}^{km} \cdot dx^i \wedge dx^j \partial_k \otimes \partial_m \]

For the viscosity tensor \( \overline{\mathbf{A}} \), the following expressions are valid:

\[ \overline{\mathbf{A}} = C[R] = (sC + aC + \frac{1}{n-1} \cdot \overline{[C]} : h_u)[R] = \\
= sC[R] + aC[R] + \frac{1}{n-1} \cdot \overline{[C]}[R] \cdot h_u = \\
= ksD + kW + \frac{1}{n-1} \cdot kU \cdot h_u \]

\[ \overline{[C]}[R] = (\overline{[sC]}[R]) = g^{ij} \cdot C_{ij}^{km} \cdot R_{km} \]

The above relation for \( \overline{\mathbf{A}} : \overline{\mathbf{A}} = C[R] \) is called generalized Hook’s law. Since \( \overline{\mathbf{A}} = ksD + kW + \frac{1}{n-1} \cdot kU \cdot h_u \), the generalized Hook’s law could be decomposed in its three parts

\[ ksD = sC[R] = sC[\sigma R + \omega R + \frac{1}{n-1} \cdot \partial R \cdot h_u] = \\
= sC[\sigma R] + sC[\omega R] + \frac{1}{n-1} \cdot \partial R \cdot sC[h_u] \] (142)

\[ kW = aC[R] = aC[\sigma R + \omega R + \frac{1}{n-1} \cdot \partial R \cdot aC[h_u] = \\
= aC[\sigma R] + aC[\omega R] + \frac{1}{n-1} \cdot \partial R \cdot aC[h_u] \] (143)

\[ kU = (\overline{[C]}[R]) = (\overline{[sC]}[R] + \overline{[sC]}[\omega R + \frac{1}{n-1} \cdot \partial R \cdot h_u] = \\
= (\overline{[sC]}[\sigma R] + (\overline{[sC]})[\omega R] + \frac{1}{n-1} \cdot \partial R \cdot (\overline{[sC]})[h_u] \] (144)
Remark. Usually, only the relation \( k_s \mathcal{D} = \mathcal{C}[\sigma R] + \frac{1}{n-1} \cdot \varrho R \cdot \mathcal{C}[\varrho h] \) is considered as generalized Hook’s law in \( E_3 \)-spaces. The other two relations (for \( k_s \mathcal{D} = \mathcal{C}[\omega R] \) and for \( k_s \mathcal{W} = \mathcal{C}[\varrho R] \)) are not taken into account because of the assumption \( \omega R = 0 \) and consideration of the symmetric energy-momentum \( \omega T \) (for which \( k_s \mathcal{W} = 0 \)) instead of the (generalized) canonical energy-momentum tensor \( \varrho T \).

4.1.1 Isotropic elastic media

Definition 2 A continuous media (flow) for which the components \( C_{ij}^{km} \) of the tensor of elasticity coefficients \( C \) has the forms

\[
C_{ij}^{km} = \lambda_1 \cdot h_{ij} \cdot h_{jn} \cdot g^{ik} \cdot g^{jm} + \lambda_2 \cdot h_{ij} \cdot g^{km} ,
\]

\[
C = \lambda_1 \cdot h_u(\overline{\eta}) \otimes h_u(\overline{\eta}) + \lambda_2 \cdot h_u \otimes \overline{\eta}
\]

is called isotropic elastic media \([?]\).

For the isotropic elastic media the following relations are fulfilled:

\[
\overline{A}_{ij} = \lambda_1 \cdot \sigma R_{ij} + \lambda_2 \cdot \varrho R \cdot h_{ij} = \lambda_1 \cdot (\sigma R_{ij} + \omega R_{ij}) + \left( \frac{\lambda_1}{n-1} + \lambda_2 \right) \cdot \varrho R \cdot h_{ij} ,
\]

or

\[
\overline{A}_{ij} = \lambda_1 \cdot (\sigma R_{ij} + \omega R_{ij}) + \overline{\lambda} \cdot \varrho R \cdot h_{ij} ,
\]

\[
\overline{\lambda} = \frac{\lambda_1}{n-1} + \lambda_2 .
\]

The constant \( \lambda_1 \) and \( \overline{\lambda} \) are called Lame constant.

Since the viscosity tensor \( g^{(k)S}g = \overline{A} \) could be written in the forms

\[
g^{(k)S}g = \overline{A} = k_s \mathcal{D} + k_s \mathcal{W} + \frac{1}{n-1} \cdot k_s \mathcal{U} \cdot h_u = \lambda_1 \cdot (\sigma R + \omega R) + \overline{\lambda} \cdot g \cdot h_u \]

we can decompose the generalized Hook’s law for isotropic elastic media in its three parts

\[
k_s \mathcal{D} = \lambda_1 \cdot \sigma R ,
\]

\[
k_s \mathcal{W} = \lambda_1 \cdot \omega R ,
\]

\[
k_s \mathcal{U} = \overline{\lambda} \cdot g \cdot h_u .
\]

The energy-momentum tensors \( G \sim (\varrho s, T) \) will have for isotropic elastic media the form

\[
g(G) = (\varrho \mathcal{C} + \frac{1}{c} \cdot p \cdot k) \cdot g(u) \otimes g(u) - p \cdot g(Kr) + g(u) \otimes g(k \pi) + g^{(k)S} \otimes g(u) +
+\lambda_1 \cdot (\sigma R + \omega R) + \overline{\lambda} \cdot g \cdot h_u .
\]

Let us find now the explicit form of the part \( h_u^{(k)G}(\xi^a)_{\perp} \) of the stress vector field \( kG(\xi^a)_{\perp} \) orthogonal to the vector field \( u \) for isotropic elastic media

\[
(h_u^{(k)G})(\xi^a)_{\perp} = [(k_s \mathcal{D})(\xi^a)_{\perp} + (k_s \mathcal{W})(\xi^a)_{\perp}] + \frac{1}{n-1} \cdot k_s \mathcal{U} \cdot h_u(\xi^a)_{\perp} = \overline{A}(\xi^a)_{\perp} + \lambda_1 \cdot (\sigma R + \omega R) + \overline{\lambda} \cdot g \cdot h_u(\xi^a)_{\perp} .
\]

\[
h_u^{(k)G}(\xi^a)_{\perp} = \lambda_1 \cdot (\sigma R + \omega R) + \overline{\lambda} \cdot g \cdot h_u(\xi^a)_{\perp} .
\]

The considered relations could be implemented for description of isotropic elastic media in \( (T_n, g) \)-spaces.
4.2 Hydrodynamics

In the theory of fluids two major types of fluids are considered [?]:
(a) Stokes’ fluids
(b) Newton’s fluids.

Definition 3 Stokes’ fluid. A flow with \( \overline{\mathbf{A}} = \overline{\mathbf{A}}(d) \) [\( \overline{\mathbf{A}}_{ij} = \overline{\mathbf{A}}_{ij}(d_{kl}) \)] is called Stokes’ fluid.

Definition 4 Newton’s fluid. A flow with \( \overline{\mathbf{A}} = Nk[d] \) [\( \overline{\mathbf{A}}_{ij} = Nk_{ij} km \cdot d_{km} \)] and \( C = Nk \) is called Newton’s fluid.

4.2.1 Newton’s fluids

In the Newton fluid the viscosity tensor \( g(k) g = \overline{\mathbf{A}} \) depends linearly only on the deformation velocity tensor \( d \)

\[
\begin{align*}
\overline{\mathbf{A}}_{ij} &= Nk_{ij} km \cdot d_{km}, \\
Nk &= Nk_{ij} km \cdot dx^i \otimes dx^j \otimes \partial_k \otimes \partial_m \in \otimes 2(M), \\
d &= d_{ij} \cdot dx^i \otimes dx^j \in \otimes 2(M), \\
d &= \sigma + \omega + \frac{1}{n-1} \cdot \theta \cdot h_u, \\
\overline{\mathbf{A}} &= k_s \overline{D} + k_W + \frac{1}{n-1} \cdot k_U \cdot h_u, \\
\overline{\mathbf{A}}[d] &= Nk[\sigma + \omega + \frac{1}{n-1} \cdot \theta \cdot h_u] = \\
&= Nk[\sigma] + Nk[\omega] + \frac{1}{n-1} \cdot \theta \cdot Nk[h_u]. \quad (155)
\end{align*}
\]

The components \( Nk_{ij} km \) of the tensor \( Nk \in \otimes 2(M) \) are called viscosity coefficients. They should obey the conditions

\[
Nk_{ij} km \cdot u^j = 0 \hspace{1cm} Nk_{ij} km \cdot u^i = 0. \quad (156)
\]

On the other side, we have to decompose the tensor of viscosity coefficients \( Nk = Nk_{ij} km \cdot dx^i \otimes dx^j \otimes \partial_k \otimes \partial_m \) with respect to the basis \( dx^i \otimes dx^j \) in a trace-free symmetric part, antisymmetric part, and trace part [in correspondence with the structure of \( \overline{\mathbf{A}} \)]

\[
Nk = sNk + aNk + \frac{1}{n-1} \cdot \overline{\mathbf{g}}[Nk] \cdot h_u, \quad (157)
\]

where

\[
\begin{align*}
sNk &= sNk - \frac{1}{n-1} \cdot \overline{\mathbf{g}}[Nk] \cdot h_u, \\
sNk &= \frac{1}{2} \cdot (Nk_{ij} km + Nk_{ji} km) \cdot dx^i \otimes dx^j \otimes \partial_k \otimes \partial_m = \\
&= Nk_{(ij)} km \cdot dx^i dx^j \otimes \partial_k \otimes \partial_m, \\
\overline{\mathbf{g}}[Nk] &= \overline{\mathbf{g}}[sNk] = g^{ij} \cdot Nk_{ij} km \cdot \partial_k \otimes \partial_m, \\
aNk &= \frac{1}{2} \cdot (Nk_{ij} km - Nk_{ji} km) \cdot dx^i \otimes dx^j \otimes \partial_k \otimes \partial_m = \\
&= Nk_{[ij]} km \cdot dx^i \wedge dx^j \otimes \partial_k \otimes \partial_m.
\end{align*}
\]
For the viscosity tensor $\mathbf{\underline{A}}$, the following expressions are valid:

$$
\mathbf{\underline{A}} = N k[d] = (sNk + a_N k + \frac{1}{n - 1} \cdot \mathbf{g}[Nk] \cdot h_u)[d] = 
$$

$$
= sNk[d] + a_N k[d] + \frac{1}{n - 1} \cdot (\mathbf{g}[Nk])[d] \cdot h_u = 
$$

$$
= k_s \mathbf{\underline{D}} + k \mathbf{\underline{W}} + \frac{1}{n - 1} \cdot \mathbf{\underline{U}} \cdot h_u ,
$$

$$
(\mathbf{g}[Nk])[d] = (\mathbf{g}[Nk])[\sigma] = \text{g}^{ij} \cdot N k_{ij} km \cdot d_{\underline{\mathbf{m}}} .
$$

The above relation for $\mathbf{\underline{A}} : A = N k[d]$ is called generalized viscosity tensor for Newton’s fluids. Since $\mathbf{\underline{A}} = k_s \mathbf{\underline{D}} + k \mathbf{\underline{W}} + \frac{1}{n - 1} \cdot k \mathbf{\underline{U}} \cdot h_u$, the generalized Hook’s law could be decomposed in its three parts

$$
k_s \mathbf{\underline{D}} = sNk[d] = sNk[\sigma + \omega + \frac{1}{n - 1} \cdot \theta \cdot h_u] = 
$$

$$
= sNk[\sigma] + sNk[\omega] + \frac{1}{n - 1} \cdot \theta \cdot sNk[h_u], \quad (160)
$$

$$
k \mathbf{\underline{W}} = a_N k[d] = a_N k[\sigma + \omega + \frac{1}{n - 1} \cdot \theta \cdot h_u] = 
$$

$$
= a_N k[\sigma] + a_N k[\omega] + \frac{1}{n - 1} \cdot \theta \cdot a_N k[h_u], \quad (161)
$$

$$
k \mathbf{\underline{U}} = (\mathbf{g}[Nk])[d] = (\mathbf{g}[Nk])[\sigma + \omega + \frac{1}{n - 1} \cdot \theta \cdot h_u] = 
$$

$$
= (\mathbf{g}[Nk])[\sigma] + (\mathbf{g}[Nk])[\omega] + \frac{1}{n - 1} \cdot \theta \cdot (\mathbf{g}[Nk])[h_u]. \quad (162)
$$

Remark. Usually, only the relation $k_s \mathbf{\underline{D}} = sNk[\sigma] + \frac{1}{n - 1} \cdot \theta \cdot sNk[h_u]$ is considered as generalized viscosity tensor for Newton’s fluids in $E_3$-spaces. The other two relations (for $k_s \mathbf{\underline{D}} = sNk[\omega]$ and for $k \mathbf{\underline{W}} = a_N k[d]$) are not taken into account because of the assumption $\omega = 0$ and consideration of the symmetric energy-momentum $\mathbf{T}$ (for which $k \mathbf{\underline{W}} = 0$) instead of the (generalized) canonical energy-momentum tensor $\theta$.

4.2.2 Isotropic homogeneous Newton’s fluids

For an isotropic homogeneous Newton’s fluid the components $N k_{ij} km$ will have a form analogous to this for isotropic elastic media

$$
N k_{ij} km = k_1 \cdot h_{ii} \cdot h_{jj} \cdot g^{ij} \cdot g^{km} + k_2 \cdot h_{ij} \cdot g^{km} , \quad (163)
$$

$$
N k = k_1 \cdot h_u(\mathbf{\underline{g}}) \otimes h_u(\mathbf{\underline{g}}) + k_2 \cdot h_u \otimes \mathbf{\underline{g}} .
$$

The constant $k_1$ is called first viscosity coefficient. For the components $\mathbf{\underline{A}}_{ij}$ of the viscosity tensor $g(kS)g = \mathbf{\underline{A}}$ and for the tensor $\mathbf{\underline{A}}$ we obtain

$$
\mathbf{\underline{A}}_{ij} = (k_1 \cdot h_{ii} \cdot h_{jj} \cdot g^{ij} \cdot g^{km} + k_2 \cdot h_{ij} \cdot g^{km}) \cdot d_{\underline{\mathbf{m}}} = 
$$

$$
= k_1 \cdot h_{ii} \cdot h_{jj} \cdot g^{ij} \cdot d_{\underline{\mathbf{m}}} + k_2 \cdot h_{ij} \cdot g^{km} \cdot d_{\underline{\mathbf{m}}} = 
$$

$$
= k_1 \cdot d_{ij} + k_2 \cdot h_{ij} \cdot g^{km} \cdot d_{\underline{\mathbf{m}}} = k_1 \cdot d_{ij} + k_2 \cdot \theta \cdot h_{ij} , \quad (164)
$$

$$
\mathbf{\underline{A}} = k_1 \cdot d + k_2 \cdot \theta \cdot h_u = k_1 \cdot (\sigma + \omega + \frac{1}{n - 1} \cdot \theta \cdot h_u) + k_2 \cdot \theta \cdot h_u = 
$$

$$
= k_1 \cdot (\sigma + \omega) + \left(\frac{k_1}{n - 1} + k_2\right) \cdot \theta \cdot h_u = 
$$

$$
= k_1 \cdot (\sigma + \omega) + k_2 \cdot \theta \cdot h_u , \quad k_2 = \frac{k_1}{n - 1} + k_2 . \quad (165)
$$
Since the viscosity tensor \( g^{(k)S})g = \overline{A} \) could be written in the forms
\[
g^{(k)S})g = \overline{A} = k_s \overline{D} + k_W + \frac{1}{n - 1} \cdot k_\overline{U} \cdot h_u = \\
= k_1 \cdot (\sigma + \omega) + \overline{\epsilon}_2 \cdot \theta \cdot h_u ,
\]
we can decompose the generalized Hook’s law for isotropic elastic media in its three parts
\[
k_s \overline{D} = k_1 \cdot \sigma , \quad k_W = k_1 \cdot \omega , \quad k_\overline{U} = (n - 1) \cdot \overline{\epsilon}_2 \cdot \theta = \overline{\epsilon} \cdot \theta .
\]

The energy-momentum tensors \( G \sim (\theta, sT) \) will have for isotropic elastic media the form
\[
g(G) = (\rho C + \frac{1}{c} \cdot p \cdot k) \cdot g(u) \otimes g(u) - p \cdot g(Kr) + g(u) \otimes (g^{(k)\pi}) + g^{(k)s} \otimes g(u) + \\
+ k_1 \cdot (\sigma + \omega) + \overline{\epsilon}_2 \cdot \theta \cdot h_u .
\]

For \( \overline{g[\overline{A}]} = g^{ij} \cdot \overline{\overline{A}}_{ij} = \overline{g^{ij}} \cdot \overline{\overline{A}}_{ij} \), it follows the expression
\[
\overline{g[\overline{A}]} = g^{ij} \cdot \overline{\overline{A}}_{ij} = \left( \frac{k_1}{n - 1} + k_2 \right) \cdot \theta \cdot \theta = \left[ k_1 + (n - 1) \cdot k_2 \right] \cdot \theta = \overline{\epsilon} \cdot \theta = (n - 1) \cdot \overline{\epsilon}_2 \cdot \theta .
\]

The constant \( \overline{\epsilon} \) is called volume viscosity coefficient. The condition \( \overline{\epsilon} = 0 \ (n \neq 1) \) is called Stokes’ condition. It leads to \( \overline{g[\overline{A}]} = 0 \) and to vanishing of the inner energy (enthalpy) density
\[
k_\overline{E} = \frac{1}{n - 1} \cdot k_\overline{U} = 0 ,
\]
because of \( k_\overline{U} = \overline{g[\overline{A}]} = \overline{g[\overline{A}]} = 0 \).

For a fluid with \( k_\overline{E} = 0 \) we have the canonical form of the energy-momentum tensors \( G \sim (\theta, sT) \)
\[
G = (\rho C + \frac{1}{c} \cdot p \cdot k) \cdot u \otimes g(u) - p \cdot Kr + u \otimes g^{(k)\pi} + k_s \otimes g(u) + \\
+ \overline{g[k_s \overline{D}]} + \overline{g[k_W]} .
\]

**Special case:** Isotropic homogeneous Newton’s fluid with \( k_s = 0 \). For \( k_s = 0 \), it follows that
\[
N_{k_{ij}} = k_1 \cdot h_{il} \cdot h_{jn} \cdot g^{jk} \cdot g_{\overline{mm}} , \quad N_k = k_1 \cdot h_u(\overline{g}) \otimes h_u(\overline{g}) ,
\]
\[
\overline{\overline{A}}_{ij} = k_1 \cdot h_{il} \cdot h_{jn} \cdot g^{jk} \cdot g_{\overline{mm}} \cdot d_{\overline{mm}} = k_1 \cdot d_{ij} , \quad \overline{\overline{A}} = k_1 \cdot d , \quad (176)
\]
\[
k_\overline{U} = \overline{g[\overline{A}]} = k_1 \cdot \overline{g[d]} = k_1 \cdot \theta , \quad k_\overline{E} = \frac{k_1}{n - 1} \cdot \theta .
\]

Let us now consider the trace \( \overline{g[\overline{g(G)}]} = g^{ij} \cdot G_{ij} \) of the energy-momentum tensors \( G \sim (\theta, sT) \). By the use of the relations
\[
\overline{g[\overline{g(u)} \otimes g(\xi)]} = g^{ij} \cdot g_{ik} \cdot u^i \cdot g_{\overline{j} \cdot \xi} = g(u, \xi) ,
\]
\[
\overline{g[\overline{g(G)}]} = \rho \overline{G} \cdot c + \{ k - \overline{g[\overline{g(Kr)}]} \} \cdot p ,
\]
\[
g(Kr) = g^{ij} \cdot dx^i \otimes dx^j ,
\]
\[
\overline{g[\overline{g(Kr)}]} = g^{ij} \cdot g_{ij} = f^k_j \cdot g_{ik} \cdot g^{ij} = f^k_j \cdot g_{ik} = f^k_k = f .
\]
the explicit form for \( \mathcal{F}[g(G)] \) follows as

\[
\mathcal{F}[g(G)] = \rho_G \cdot e + (k - f) \cdot p .
\]  

(178)

Now we can express the pressure \( p \) by means the rest mass density \( \rho_G \) and the trace \( \mathcal{F}[g(G)] \) of the energy-momentum tensor \( G \)

\[
p = \frac{1}{f - k} \cdot (\rho_G \cdot e - \mathcal{F}[g(G)]) = \\
= \frac{1}{f - k} \cdot (\rho_G \cdot g_{ij} \cdot u^i \cdot u^j - g_{ij} \cdot G_{ij}) .
\]  

(179)

If we knew the explicit form of \( \rho_G \) and \( G_{ij} \) (on the basis of experimental data) we can find the corresponding Euler-Lagrange’s equations for the unknown field variables on which the pressure \( p = L \) depends.

**Special case:** \((L_n,g)-spaces: S = C : f^i \cdot j = g^i_j , \ f = g^k_k = n, \ k = 1\)

\[
\mathcal{F}[g(G)] = \rho_G \cdot e + (1 - n) \cdot p ,
\]

\[
p = \frac{1}{n - 1} \cdot (\rho_G \cdot e - \mathcal{F}[g(G)]) .
\]  

(180)

### 4.2.3 Barotropic systems

**Definition 5** Dynamical system for which the pressure \( p \) is depending only on \( \rho_G \cdot e \) is called barotropic system [?].

All isotropic homogeneous Newton’s fluids with vanishing volume viscosity coefficient \( k = 0 \) and vanishing trace \( \mathcal{F}[g(G)] \cdot \mathcal{F}[g(G)] = 0 \) of their energy-momentum tensors \( G \) are barotropic systems fulfilling the condition

\[
L = p = \frac{1}{f - k} \cdot \rho_G \cdot e .
\]  

(181)

**Special case:** \((L_n,g)-spaces: S = C : f^i \cdot j = g^i_j , \ f = g^k_k = n, \ k = 1, \ \mathcal{F}[g(G)] = 0\)

\[
p = \frac{1}{n - 1} \cdot \rho_G \cdot e .
\]  

(182)

For \( n = 4 \) :

\[
p = \frac{1}{3} \cdot \rho_G \cdot e .
\]  

(183)

### 5 Navier-Stokes’s identities. Generalized Navier-Stokes’ equation

By the use of the method of Lagrangians with covariant derivatives (MLCD) [?]
the different energy-momentum tensors and the covariant Noether’s identities for a field theory as well as for a theory of continuous media can be found. On the basis of the \((n - 1) + 1\) projective formalism and by the use of the notion of covariant divergency of a tensor of second rank the corresponding covariant divergencies of the energy-momentum tensors could be found. They lead to Navier-Stokes’ identity and to the corresponding generalized Navier-Stokes’ equations.
5.1 Covariant divergency of the energy-momentum tensors and the rest mass density

The covariant divergency $\delta G$ of the energy-momentum tensor $\delta G$ ($G \sim \theta^, T$, $Q$) can be represented by the use of the projective metrics $h^u$, $h_u$ of the contravariant vector field $u$ and the rest mass density for the corresponding energy-momentum tensor. In this case the representation of the energy-momentum tensor is in the form

$$G = (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot u \otimes g(u) - L \cdot Kr + u \otimes g^k u + k u \otimes g(u) + (k S) g, \quad (184)$$

where

$$k^\pi = \overline{G^\pi}, \quad k^s = \overline{G^s}, \quad k S = \overline{G S}.$$

By the use of the relations

$$u(\rho_G + \frac{1}{c} \cdot L \cdot k) = u \rho_G + L \cdot u(\frac{1}{c} \cdot k) + \frac{1}{c} \cdot k \cdot (u L) =$$

$$= [\rho_G / \alpha + L \cdot (\frac{1}{c} \cdot k) / \alpha + \frac{1}{c} \cdot k \cdot L / \alpha] \cdot u^\alpha =$$

$$= [\rho_G / \alpha + L \cdot (\frac{1}{c} \cdot k) / \alpha + \frac{1}{c} \cdot k \cdot L / \alpha] \cdot u^\alpha,$$

$$Kr L = L / \alpha \cdot e^\alpha = L / \alpha \cdot dx^i = \nabla_{Kr} L, \quad (187)$$

$$\delta (L \cdot Kr) = Kr L + L \cdot \delta Kr, \quad (188)$$

$$\delta (L \cdot Kr) = \frac{1}{c} \cdot [\nabla_{[\alpha} L \cdot \nabla_{\beta]} Kr] g = (L \cdot g^l_{[\alpha} / \beta) e^\beta = (L \cdot g^l_{[\alpha} / \beta) dx^i =$$

$$= (L / \beta, \cdot g^l_{[\alpha} / \beta) e^\beta = (L, i + L \cdot g^l_{[\alpha} / \beta) dx^i,$$

$$\delta (u \otimes g(\overline{G^\pi})) = \delta u \cdot g(\overline{G^\pi}) + g(\nabla_u \overline{G^\pi}) + (\nabla_u g)(\overline{G^\pi}), \quad (189)$$

$$\delta (\overline{G^\alpha} \otimes g(u)) = \delta \overline{G^\pi} \cdot g(u) + g(\nabla_u g(u) + (\nabla_u g)(u),$$

$$\delta (\overline{G S} g) = (u \pi, \cdot \overline{G S}^\beta) / \beta \cdot e^\alpha [\text{see } (??,)]$$

$\delta G$ and $\overline{\mathcal{g}}(\delta G)$ can be found in the forms

$$\delta G = (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot g(\alpha)+$$

$$+ [u(\rho_G + \frac{1}{c} \cdot L \cdot k) + (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta \overline{G^\alpha}] \cdot g(u) -$$

$$- L \cdot Kr - L \cdot \delta Kr + u \cdot g(\overline{G^\pi}) + g(\nabla_u \overline{G^\pi}) + g(\nabla_u g(u) +$$

$$+ (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla_u g(u) + (\nabla_u g)(\overline{G^\pi}) + (\nabla_u g)(u) +$$

$$+ \delta (\overline{G S} g) g, \quad (189)$$

$$\overline{\mathcal{g}}(\delta G) = (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot a+$$

$$+ [u(\rho_G + \frac{1}{c} \cdot L \cdot k) + (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta \overline{G^\alpha}] \cdot u-$$

$$- L \cdot \overline{\mathcal{g}}(Kr L) \cdot L \cdot \overline{\mathcal{g}}(\delta Kr) + \delta u \cdot \overline{G^\pi} + \nabla_u \overline{G^\pi} + \nabla_u g(u) +$$

$$+ (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot \overline{\mathcal{g}}(\nabla_u g(u) \overline{G^\pi}(\overline{G^\pi}) + (\overline{G^\pi} g(u) +$$

$$+ \overline{\mathcal{g}}(\delta (\overline{G S} g)). \quad (192)$$

In a co-ordinate basis $\delta G$ and $\overline{\mathcal{g}}(\delta G)$ will have the forms

$$G_i, j = (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot a_i+$$

$$+ [((\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot u^j + (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot u^j + \overline{G^\alpha} / \beta)] \cdot u_i-$$

$$- L, i - L \cdot g^j_i / \jmath + u / j \cdot \overline{G^\alpha} + \overline{G^\alpha} / \beta \cdot u^j / \jmath + \overline{G^\alpha} / \beta +$$

$$+ g_{i, j} (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot u^j + \overline{G^\alpha} / \beta \cdot u^j + \overline{G^\alpha} / \beta +$$

$$+ \overline{G^\alpha} (g_{i, j}, k) \cdot (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot a_i+$$

$$+ [((\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot u^j + (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot u^j + \overline{G^\alpha} / \beta)] \cdot u_i-$$

$$- L, j - L \cdot g^j_i / \jmath + u / j \cdot \overline{G^\alpha} + \overline{G^\alpha} / \beta \cdot u^j / \jmath + \overline{G^\alpha} / \beta +$$

$$+ g_{i, j} (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot u^j + \overline{G^\alpha} / \beta \cdot u^j + \overline{G^\alpha} / \beta +$$

$$+ \overline{G^\alpha} (g_{i, j}, k) \cdot (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot a_i+ \quad (194)$$

$$\overline{\mathcal{g}}(G_i^k) / j = (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot a_i+$$

$$+ [((\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot u^j + (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot u^j + \overline{G^\alpha} / \beta)] \cdot u_i-$$

$$- L, j - L \cdot g^j_i / \jmath + u / j \cdot \overline{G^\alpha} + \overline{G^\alpha} / \beta \cdot u^j / \jmath + \overline{G^\alpha} / \beta +$$

$$+ g_{i, j} (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot u^j + \overline{G^\alpha} / \beta \cdot u^j + \overline{G^\alpha} / \beta +$$

$$+ \overline{G^\alpha} (g_{i, j}, k) \cdot (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot a_i+ \quad (195)$$
5.2 Covariant divergency of the energy-momentum tensors for linear elastic theory

For linear elastic theory we have to specialize the structure of the term \( \delta((C^s)g) \) in the expression for the divergency of the energy-momentum tensor \( G \sim (\theta_s T) \). Since \( (C^s)g = (k S)g = \overline{g}(g(k S)g) = \overline{g}(A) \), we have

\[
\delta((C^s)g) = \delta(\overline{g}(A)) \ .
\] (196)

5.2.1 Representation of \( \delta(\overline{g}(A)) \)

For \( \overline{g}(A) = \overline{g}(k_s D) + \overline{g}(k W) + \frac{1}{n-1} \cdot k U \cdot \overline{g}(h_u) \) we obtain by the use of the general formulae for covariant divergency of a tensor field

\[
\delta(\overline{g}(A)) = \delta(\overline{g}(k_s D)) + \delta(\overline{g}(k W)) + \frac{1}{n-1} \cdot \delta(k U \cdot \overline{g}(h_u)) \ ,
\] (197)

where

\[
\begin{align*}
\delta(\overline{g}(k_s D)) &= \delta(\overline{g}(s C[s R])) + \delta(\overline{g}(s C[s \theta])) + \frac{1}{n-1} \cdot \delta(s R \cdot \overline{g}(s C[h_u])) , \\
\delta(\overline{g}(k W)) &= \delta(\overline{g}(a C[a R])) + \delta(\overline{g}(a C[a \theta])) + \frac{1}{n-1} \cdot \delta(a R \cdot \overline{g}(a C[h_u])) , \\
\delta(k U \cdot \overline{g}(h_u)) &= \delta((C)(R) \cdot \overline{g}(h_u)) , \\

\overline{g}(k_s D) &= g^{ij} \cdot k_s D_{kj} \cdot \partial_i \otimes dx^j , \\
\delta(\overline{g}(k W)) &= (g^{ij} \cdot k_s D_{kj})_i \cdot dx^j = (g^{ij} \cdot k_s D_{jk} + g^{ij} \cdot k_s D_{jk:i}) \cdot dx^k , \\
\delta(k U \cdot \overline{g}(h_u)) &= \delta(k U \cdot g^{ij} \cdot h_jk \cdot \partial_i \otimes dx^k) = \\
&= [k U, i \cdot g^{ij} \cdot h_jk + k U \cdot (g^{ij} \cdot h_jk + g^{ij} \cdot h_jk)] \cdot dx^k , \\
g^{ij} &= f^j I \cdot g^{ij} + f^j I \cdot g^{ij} = g^{ij} , \\
\overline{g}(s C[s R]) &= g^{ij} \cdot s C_{jk}^{mn} \cdot R_{mn} \cdot \partial_i \otimes dx^k , \\
\delta(\overline{g}(s C[s \theta])) &= (g^{ij} \cdot s C_{jk}^{mn} \cdot \sigma R_{mn})_i \cdot dx^k , \\
(g^{ij} \cdot s C_{jk}^{mn} \cdot \sigma R_{mn};i) &= g^{ij} \cdot s C_{jk}^{mn} \cdot \sigma R_{mn} + \\
&+ g^{ij} \cdot (s C_{jk}^{mn};i \cdot \sigma R_{mn} + s C_{jk}^{mn} \cdot \sigma R_{mn})(200) \\
\delta((C[R]) \cdot \overline{g}(h_u)) &= \delta(g^{ij} \cdot C_{rl}^{mn} \cdot R_{mn} \cdot g^{ij} \cdot h_{jk}) \cdot \partial_i \otimes dx^k , \\
\delta(g^{ij} \cdot C_{rl}^{mn} \cdot R_{mn} \cdot g^{ij} \cdot h_{jk}) &= g^{rl} \cdot C_{rl}^{mn} \cdot R_{mn} \cdot g^{ij} \cdot h_{jk} + \\
&+ g^{ij} \cdot (C_{rl}^{mn};i \cdot R_{mn} + C_{rl}^{mn} \cdot R_{mn};i) \cdot g^{ij} \cdot h_{jk} + \\
&+ g^{ij} \cdot C_{rl}^{mn} \cdot R_{mn} \cdot (g^{ij} \cdot h_{jk} + g^{ij} \cdot h_{jk:i}) .
\] (202)
5.2.2 Isotropic elastic media

For isotropic elastic media $\mathbf{A} = \lambda_1 \cdot (\sigma R + \omega R) + \chi \cdot \partial R \cdot h_u$. Then, it follows for $(^kS)g = \mathbf{g}(\mathbf{A})$

$$\mathbf{g}(\mathbf{A}) = \lambda_1 \cdot \mathbf{g}(\sigma R + \omega R) + \chi \cdot \partial R \cdot \mathbf{g}(h_u) =$$

$$= \lambda_1 \cdot \mathbf{g}(\sigma R) + \chi \cdot \partial R \cdot \mathbf{g}(h_u), \quad \lambda_1, \chi = \text{const.},$$

$$\delta(\mathbf{g}(\mathbf{A})) = \lambda_1 \cdot \delta(\mathbf{g}(\sigma R)) + \chi \cdot \delta(\partial R \cdot \mathbf{g}(h_u)),$$

where

$$\delta(\mathbf{g}(\sigma R)) = (\delta \mathbf{g}^k_{i,j}) i \cdot dx^k = (g^j_{ij} \cdot \sigma R^k_{ji} + g^j_{ij} \cdot \omega R^k_{ij}) \cdot dx^k,$$

$$\delta(\mathbf{g}(\omega R)) = (\delta \mathbf{g}^k_{i,j}) i \cdot dx^k = (g^j_{ij} \cdot \omega R^k_{ji} + g^j_{ij} \cdot \omega R^k_{ij}) \cdot dx^k,$$

$$\delta(\partial R \cdot \mathbf{g}(h_u)) = \delta(\partial R \cdot g^j_{ij}) i \cdot dx^k = [\partial R^i_{,j} \cdot g^k_{ij} \cdot h_{ki} + \partial R \cdot (g^{jk}_{ij} \cdot h_{ki} + g^{jk}_{ij} \cdot h_{ki})] \cdot dx^i.$$ (204)

For the covariant divergence $\delta(\mathbf{g}(\mathbf{A}))$ of the viscosity tensor $(^kS)g = \mathbf{g}(\mathbf{A})$ we obtain in a co-ordinate (or in a non-co-ordinate basis if $dx^k \to e^k$)

$$\delta((^kS)g) = \delta(\mathbf{g}(\mathbf{A})) =$$

$$= \{\lambda_1 \cdot (g^j_{ij} \cdot \sigma R^k_{ji} + g^j_{ij} \cdot \omega R^k_{ij}) +$$

$$+ \chi \cdot \partial R^i_{,j} \cdot g^k_{ij} \cdot h_{ki} + \partial R \cdot (g^{jk}_{ij} \cdot h_{ki} + g^{jk}_{ij} \cdot h_{ki})]\} \cdot dx^i.$$ (205)

The energy-momentum tensors $G \sim (\theta, sT)$ could be found in the forms

$$\delta G = \left(\rho g + \frac{1}{c} \cdot L \cdot k \cdot g(u) +
+ [u(\rho g + \frac{1}{c} \cdot L \cdot k) \cdot g(u) + (\rho g + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta \mathbf{g}^k] \cdot g(u) -
- K \tau L \cdot L + K \tau r + \delta u \cdot G^\pi \cdot g(\nabla u \cdot \mathbf{g}^\pi) + g(\mathbf{g}^\pi \cdot \mathbf{g}^\pi) + g(\nabla \varphi \cdot \mathbf{g}^\pi) +$$

$$+ (\rho g + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla u g)(u) + (\nabla u g)(\mathbf{g}^\pi) + (\nabla \varphi g)(u) +$$

$$+ \chi \cdot \delta(\mathbf{g}(\sigma R)) + \lambda_1 \cdot \delta(\mathbf{g}(\omega R)) + \chi \cdot \delta(\partial R \cdot \mathbf{g}(h_u)),$$

where

$$G^i_{ij} = \left(\rho g + \frac{1}{c} \cdot L \cdot k \right) \cdot a_i +$$

$$+ [(\rho g + \frac{1}{c} \cdot L \cdot k) \cdot u_i + (\rho g + \frac{1}{c} \cdot L \cdot k) \cdot w_j + G^v_{ij} \cdot u_i] -$$

$$- L, i - L \cdot g^i_{jk} \cdot w_j + G^i_{jk} \cdot u_i + G^\pi_{jk} \cdot G^\pi + G^\pi \cdot \mathbf{g}^\pi + (G^\pi \cdot \mathbf{g}^\pi) + G^\pi \cdot \mathbf{g}^\pi +$$

$$+ \chi \cdot \partial R^i_{,j} \cdot g^k_{ij} \cdot h_{ki} + \partial R \cdot (g^{jk}_{ij} \cdot h_{ki} + g^{jk}_{ij} \cdot h_{ki}) \} \cdot dx^i.$$ (205)

5.3 Covariant divergence of the energy-momentum tensors for Newton’s fluids

Analogous considerations as in the case of linear elasticity theory could be made for Newton’s fluids. A comparison of $\mathbf{A} = \chi k [d], \mathbf{A}_{ij} = \chi k_{ij,km} \cdot \mathbf{g}^{km}$ with $\mathbf{A} = C[R], \mathbf{A}_{ij} = C_{ij,km} \cdot R^{km}$ shows that the results for the Newton fluids could be formally found from the results for linear elasticity theory by the use of the following substitutions

$$C \to \chi k, \quad R \to \frac{d}{c} \sigma R \to \sigma, \quad \omega R \to \omega, \quad \partial R \to \theta,$$

$$\lambda_1 \to k_1, \quad \chi \to \frac{k_2}{k_1}.$$ (206)
For \( \overline{\mathcal{A}} = \overline{\mathcal{A}}_{(kS)} + \overline{\mathcal{A}}_{(kW)} + \frac{1}{n-1} \cdot k \overline{\mathcal{U}} \cdot \overline{\mathcal{G}}(h_u) \) we obtain by the use of the general formulae for covariant divergence of a tensor field

\[
\delta(\overline{\mathcal{A}}) = \delta(\overline{\mathcal{A}}_{(kS)}) + \delta(\overline{\mathcal{A}}_{(kW)}) + \frac{1}{n-1} \cdot \delta(k \overline{\mathcal{U}} \cdot \overline{\mathcal{G}}(h_u)) \quad ,
\]

where

\[
\delta(\overline{\mathcal{A}}_{(kS)}) = (g^{ij} \cdot \overline{\mathcal{A}}_{jk})_{;i} \cdot dx^k = (g^{ij} \cdot \overline{\mathcal{A}}_{jk} + g^{ij} \cdot \overline{\mathcal{A}}_{jk;i}) \cdot dx^k =
\]

\[
= g^{ij} \cdot (k \overline{\mathcal{D}}_{jk} + k \overline{W}_{jk} + \frac{1}{n-1} \cdot k \overline{\mathcal{U}} \cdot h_{jk}) \cdot dx^k +
\]

\[
+ g^{ij} \cdot [k \overline{D}_{jk;i} + k \overline{W}_{jk;i} + \frac{1}{n-1} \cdot (k \overline{U}, i \cdot h_{jk} + k \overline{U}, h_{jk;i})] \cdot dx^k \quad ,
\]

\[
\delta(\overline{\mathcal{A}}_{(kW)}) = \delta(\overline{\mathcal{G}}_{(\sigma + \omega)} + \delta(\overline{\mathcal{G}}_{(\alpha \beta \gamma \delta)} + \frac{1}{n-1} \cdot \delta(\theta \cdot \overline{\mathcal{G}}(h_u))) \quad ,
\]

\[
\delta(k \overline{\mathcal{U}} \cdot \overline{\mathcal{G}}(h_u)) = \delta(k \overline{U} \cdot g^{ij} \cdot h_{jk} \cdot \partial_i \otimes dx^k) =
\]

\[
= [k \overline{U};i \cdot g^{ij} \cdot h_{jk} + k \overline{U} \cdot (g^{ij} \cdot h_{jk} + g^{ij} \cdot h_{jk;i})] \cdot dx^k ,
\]

\[
g^{ij} = f^j_{\cdot l} \cdot g^{il} = f^j_{\cdot l} \cdot g^{il} = g^{ji} \quad ,
\]

\[
\delta(\overline{\mathcal{G}}_{(\sigma + \omega)} = g^{ij} \cdot \overline{\mathcal{K}}_{jk}^{mn} \cdot \sigma_{mn} \cdot \partial_i \otimes dx^k ,
\]

\[
\delta(\overline{\mathcal{G}}_{(\alpha \beta \gamma \delta)}) = (g^{ij} \cdot \overline{\mathcal{K}}_{jk}^{mn} \cdot \sigma_{mn})_{;i} \cdot dx^k ,
\]

\[
(g^{ij} \cdot \overline{\mathcal{K}}_{jk}^{mn} \cdot \sigma_{mn})_{;i} = g^{ij} \cdot \overline{\mathcal{K}}_{jk}^{mn} \cdot \sigma_{mn} +
\]

\[
+ g^{ij} \cdot (\overline{\mathcal{K}}_{jk}^{mn} \cdot \sigma_{mn} + \overline{\mathcal{K}}_{jk}^{mn} \cdot \sigma_{mn}) \quad ,
\]

\[
\delta((\overline{\mathcal{G}}_{(\alpha \beta \gamma \delta)} = \delta(g^{ij} \cdot \overline{\mathcal{K}}_{jk}^{mn} \cdot d_{mn} \cdot g^{ij} \cdot h_{jk}) \cdot \partial_i \otimes dx^k ,
\]

\[
\delta(g^{ij} \cdot \overline{\mathcal{K}}_{jk}^{mn} \cdot d_{mn} \cdot g^{ij} \cdot h_{jk}) = g^{il} \cdot \overline{\mathcal{K}}_{kl}^{mn} \cdot d_{mn} \cdot g^{ij} \cdot h_{jk} +
\]

\[
+ g^{ij} \cdot \overline{\mathcal{K}}_{jk}^{mn} \cdot d_{mn} \cdot (g^{ij} \cdot h_{jk} + g^{ij} \cdot h_{jk;i}) \quad ,
\]

5.3.1 Isotropic homogeneous fluid

For isotropic homogeneous fluid \( \overline{\mathcal{A}} = k_1 \cdot (\sigma + \omega) + k_2 \cdot \theta \cdot h_u \). Then, it follows for

\[
(kS)g = \overline{\mathcal{G}}_{(\mathcal{A})}
\]

\[
\overline{\mathcal{G}}(\mathcal{A}) = k_1 \cdot \overline{\mathcal{G}}(\sigma + \omega) + k_2 \cdot \theta \cdot \overline{\mathcal{G}}(h_u) =
\]

\[
= k_1 \cdot \overline{\mathcal{G}}(\sigma) + \overline{\mathcal{G}}(\omega) + k_2 \cdot \theta \cdot \overline{\mathcal{G}}(h_u) ,
\]

\[
k_1 \cdot k_2 = \text{const} \quad ,
\]

\[
\delta(\overline{\mathcal{G}}_{(\mathcal{A})}) = k_1 \cdot \delta(\overline{\mathcal{G}}(\sigma)) + k_1 \cdot \delta(\overline{\mathcal{G}}(\omega)) + k_2 \cdot \delta(\theta \cdot \overline{\mathcal{G}}(h_u)) \quad ,
\]

26
where

\[
\delta(\bar{\mathcal{F}}(\sigma)) = \delta(\bar{\mathcal{F}}(\sigma)) = (g^{ij} \cdot \sigma_{jk})_i \cdot dx^k = (g^{ij} \cdot \sigma_{jk} + g^{ij} \cdot \sigma_{jk;i}) \cdot dx^k \tag{213}
\]

\[
\delta(\bar{\mathcal{F}}(\omega)) = \delta(\bar{\mathcal{F}}(\omega)) = (g^{ij} \cdot \omega_{jk})_i \cdot dx^k = (g^{ij} \cdot \omega_{jk} + g^{ij} \cdot \omega_{jk;i}) \cdot dx^k \tag{214}
\]

\[
\delta(\theta \cdot \bar{\mathcal{F}}(\omega)) = \delta(\theta \cdot g^{ij} \cdot h_{jk} \cdot \partial_i \otimes dx^k) = [\theta \cdot g^{ij} \cdot h_{jk} + \theta \cdot (g^{ij} \cdot h_{jk} + g^{ik} \cdot h_{jk;i})] \cdot dx^k. \tag{215}
\]

For the covariant divergence \(\delta(\bar{\mathcal{F}}(A))\) of the viscosity tensor \((kS)g = \bar{\mathcal{F}}(A)\) we obtain in a co-ordinate (or in a non-co-ordinate basis if \(dx^k \to e^k\))

\[
\delta((kS)g) = \delta(\bar{\mathcal{F}}(A)) = [k_1 \cdot (g^{ik} \cdot \sigma_{jk} + g^{ik} \cdot \sigma_{jk} + g^{ik} \cdot \omega_{jk} + g^{ik} \cdot \omega_{jk;i}) + \kappa_2 \cdot (\theta \cdot h_{jk} + \delta(\theta \cdot \bar{\mathcal{F}}(\omega)))]. \tag{216}
\]

The energy-momentum tensors \(G \sim (\theta, sT)\) could be found in the forms

\[
\delta G = \left(\rho_G + \frac{1}{c} \cdot L \cdot k\right) \cdot g(a) + \\
+ [u(\rho_G + \frac{1}{c} \cdot L \cdot k) + (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta G] \cdot g(u) - \\
- K\tau L - L \cdot \delta K\tau \cdot g(\nabla u) \cdot g(\nabla u) + \\
+ (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla_u g)(u) + (\nabla_u g)(\nabla u) \cdot g(u) + \\
+ k_1 \cdot (\bar{\mathcal{F}}(\sigma)) + \kappa_2 \cdot (\theta \cdot \bar{\mathcal{F}}(\omega)), \tag{217}
\]

\[
G_{ij,k} = \left(\rho_G + \frac{1}{c} \cdot L \cdot k\right) \cdot a_i + \\
+ \left[(\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla u) + (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla u) + \delta G\right] \cdot \delta u - \\
- L \cdot (\nabla u) \cdot (\nabla u) + \delta G \cdot \delta u + \nabla u \cdot \nabla u + \\
+ g_{ij,k} \cdot (\rho_G + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta G \cdot \delta u + \nabla u \cdot \nabla u + \delta G \cdot \delta u, \tag{218}
\]

### 5.4 Explicit form of the energy-momentum tensors \(\theta, sT\) and \(Q\)

On the grounds of the representations of \(\delta G\) and \(\bar{\mathcal{F}}(\delta G)\) the representation of the different energy-momentum tensors \(\theta, sT\) and \(Q\) can be found.

The covariant divergence \(\delta \theta\) of the generalized canonical energy-momentum tensor (GC-EMT) \(\theta\) can be written in the form

\[
\delta \theta = \left(\rho_\theta + \frac{1}{c} \cdot L \cdot k\right) \cdot g(a) + \\
+ [u(\rho_\theta + \frac{1}{c} \cdot L \cdot k) + (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta G] \cdot g(u) - \\
- K\tau L - L \cdot \delta K\tau \cdot g(\nabla u) \cdot g(\nabla u) + \\
+ (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla_u g)(u) + (\nabla_u g)(\nabla u) \cdot g(u) + \\
+ \theta \cdot \bar{\mathcal{F}}(\omega)), \tag{219}
\]

or in the form

\[
\bar{\mathcal{F}}(\delta \theta) = \left(\rho_\theta + \frac{1}{c} \cdot L \cdot k\right) \cdot a + \\
+ [u(\rho_\theta + \frac{1}{c} \cdot L \cdot k) + (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta G] \cdot u - \\
- \bar{\mathcal{F}}(K\tau L) - L \cdot \bar{\mathcal{F}}(K\tau) \cdot u + \nabla_u \cdot \delta \theta + \nabla_u \cdot \nabla \theta + \nabla \cdot u + \\
+ (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla_u g)(u) + (\nabla_u g)(\nabla u) \cdot g(u) + \\
+ \bar{\mathcal{F}}(\delta(\theta S)g) \tag{220}
\]
In a co-ordinate basis $\delta \theta$ and $\overline{\mathbf{g}}(\delta \theta)$ will have the forms

\[
\overline{\mathbf{g}}_{ij;j} = (\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot a_i + \\
+ [(\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot a_i + (\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot u_i + (\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot u^j \cdot \Psi^j + \theta^i_{\Psi^j}] \cdot u_i - \\
L_{ij} - L_{ij} \cdot g_{ij} + u_i \cdot u^j \cdot \Psi^{ij} + g_{ij} \cdot (\theta^i_{\Psi^j}) = u_i + \\
+ g_{ij;k} \cdot [(\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot u_i \cdot u^k + \theta^i_{\Psi^j} \cdot u^k + u^j \cdot \theta^{ik}] + \\
+ (g_{ik} \cdot \theta^{ijk}) \cdot u_i,
\]

(221)

The covariant divergency $\delta T$ of the symmetric energy-momentum tensor of Belinfante (S-EMT-B) $\mathbf{S} T$, represented in the form

\[
s T = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u \otimes g(u) - L \cdot K r + u \otimes g(T \Psi) + T \Psi \otimes g(u) + (T S) g,
\]

can be found in the form

\[
\delta T = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot g(u) + \\
+ [u(\rho_T + \frac{1}{e} \cdot L \cdot k) + (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot \delta u + \delta T \Psi] \cdot g(u) = \\
- K r L - L \cdot \delta K r + \delta u \cdot g(T \Psi) + g(\nabla_u T \Psi) + g(\nabla_r T \Psi) + \\
+ (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot (\nabla_u g)(u) + (\nabla_u g)(T \Psi) + (\nabla_r T \Psi)(u) + \\
+ \delta((T S) g),
\]

(223)

or in the form

\[
\overline{\mathbf{g}}(\delta s T) = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u + \\
+ [u(\rho_T + \frac{1}{e} \cdot L \cdot k) + (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot \delta u + \delta T \Psi] \cdot u = \\
- \mathbf{g}(K r L) - L \cdot \mathbf{g}(\delta K r) + \delta u \cdot T \Psi + \nabla_u T \Psi + \nabla_r T \Psi + \\
+ (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot \mathbf{g}(\nabla_u g)(u) + \mathbf{g}(\nabla_u g)(T \Psi) + \mathbf{g}(\nabla_r T \Psi)(u) + \\
+ \mathbf{g}(\delta((T S) g)).
\]

(224)

In a co-ordinate basis $\delta s T$ and $\overline{\mathbf{g}}(\delta s T)$ will have the forms

\[
\mathbf{s} T_{ij;j} = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot a_i + \\
+ [(\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot a_i + (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u_i + (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u^j \cdot \Psi^j + \theta^i_{\Psi^j}] \cdot u_i - \\
L_{ij} - L_{ij} \cdot g_{ij} + u_i \cdot u^j \cdot \Psi^{ij} + g_{ij} \cdot (\theta^i_{\Psi^j}) = u_i + \\
+ g_{ij;k} \cdot [(\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u_i \cdot u^k + \theta^i_{\Psi^j} \cdot u^k + u^j \cdot \theta^{ik}] + \\
+ (g_{ik} \cdot \theta^{ijk}) \cdot u_i,
\]

(225)

The covariant divergency $\delta Q$ of the variational energy-momentum tensor of Euler-Lagrange (V-EMT-EL) $Q$, represented in the form

\[
Q = - \rho_Q \cdot u \otimes g(u) - u \otimes g(Q \Psi) - Q_s \otimes g(u) - (Q S) g,
\]
follows in the form
\[
\delta Q = - \rho Q \cdot g(a) - (u \rho Q + \rho Q \cdot \delta u + \delta Q s) \cdot g(u) - \\
- \delta u \cdot g(Q \pi) - g(\nabla u \cdot Q \pi) - g(\nabla q \cdot u) - \\
- \rho Q \cdot (\nabla g)(u) - (\nabla g Q \pi) - (\nabla q \cdot g)(u) - \delta(\cdot(Q)g),
\]
(227)
or in the form
\[
\mathcal{F}(\delta Q) = - \rho Q \cdot a - (u \rho Q + \rho Q \cdot \delta u + \delta Q s) \cdot u - \\
- \delta u \cdot Q \pi - \nabla u \cdot Q \pi - \nabla q \cdot u - \\
- \rho Q \cdot \mathcal{F}(\nabla g)(u) - \mathcal{F}(\nabla g Q \pi) - \mathcal{F}(\nabla q \cdot g)(u) - \mathcal{F}(\delta(\cdot(Q)g)).
\]
(228)

In a co-ordinate basis \( \delta Q \) and \( \mathcal{F}(\delta Q) \) will have the forms
\[
\overline{\Theta}^{i \cdot j} = - \rho Q \cdot a_i - (\rho Q \cdot w^j + \rho Q \cdot u^j \cdot j + Q s^{j \cdot j}) \cdot u_i - \\
- u^j \cdot Q \pi_i - g_{ij} \cdot (Q \pi \cdot u^k + u^j \cdot k) - \\
g_{ij} \cdot (\rho Q \cdot u^j \cdot u^k + Q \pi \cdot u^j \cdot u^k) - \\
- g^{\cdot j} \cdot g_{ij} \cdot (\rho Q \cdot u^i \cdot u^k + Q \pi \cdot u^i \cdot u^k) - g^{\cdot j}(g_{ik} \cdot Q s^{j \cdot k}),
\]
(229)
\[
\overline{\mathcal{T}}^{i \cdot j} = - \rho Q \cdot a_i - (\rho Q \cdot w^j + \rho Q \cdot u^j \cdot j + Q s^{j \cdot j}) \cdot u_i - \\
- u^j \cdot Q \pi_i - Q \pi \cdot u^j \cdot j - Q s^{j \cdot j} - \\
- g^{\cdot j} \cdot g_{ij} \cdot (\rho Q \cdot u^i \cdot u^k + Q \pi \cdot u^i \cdot u^k) - g^{\cdot j}(g_{ik} \cdot Q s^{j \cdot k}).
\]
(230)

### 5.5 Covariant Noether’s identities and relations between their structures

The covariant Noether identities
\[
\overline{\mathcal{F}}_\alpha + \theta_\alpha \cdot \beta / \beta \equiv 0, \quad \overline{\Theta}_i + \theta_i \cdot j \equiv 0,
\]
\[
\overline{\Theta}_i - \mathcal{F}_i \equiv \overline{Q}_\alpha \cdot \beta , \quad \overline{\theta}_j - \mathcal{T}_j \equiv \overline{Q}_\cdot j,
\]
for the mixed tensor fields of second rank of the type 1 could be written in index-free form. \( \theta, sT \) and \( Q \) can be written in index-free form by the use of the covariant divergence as
\[
F + \delta \theta \equiv 0, \quad \theta - sT \equiv Q,
\]
\[
\mathcal{F}(F) + \mathcal{F}(\delta \theta) \equiv 0, \quad (\theta) \mathcal{F} - (sT) \mathcal{F} \equiv (Q) \mathcal{F},
\]
(231)
where
\[
F = \nu F + g F, \quad (232)
\]
\[
\nu F = \nu a F + \nu W, \quad g F = g a F + g W, \quad \nu F = \nu F + g W, \quad \nu W = \nu W a \cdot e^\alpha, \quad \nu F = \nu F_\alpha \cdot e^\alpha, \quad (233)
\]
\[
g a F = \frac{\delta g L}{\delta g \beta j} \cdot g_{\alpha j / \alpha} \cdot e_\alpha, \quad g W = g W a \cdot e^\alpha.
\]
(234)

From the second Noether identity \( \theta - sT \equiv Q \) the relation between the covariant divergencies of the energy-momentum tensors \( \theta, sT \) and \( Q \) follows \( \delta \theta \equiv \delta s T + \delta Q, \delta s T \equiv \delta \theta - \delta Q \).

**Definition 6** Local covariant conserved quantity \( G \) of the type of an energy-momentum tensor of the type 1. Mixed tensor field \( G \) of the type 1 with vanishing covariant divergence, i. e. \( \delta G = 0, \, G_\alpha / \beta = 0, \, G_i / j = 0 \).

If a given energy-momentum tensor has to fulfil conditions for a local covariant conserved quantity, then relations follow from the covariant Noether identities (CNIs) between the covariant divergencies of the other energy-momentum tensors and the covariant vector field \( F \).
| No. | Condition for $\delta G$ | Condition for $F$ | Corollaries from CNIs |
|-----|--------------------------|-------------------|-----------------------|
| 1.  | $\delta \theta = 0$      | $F = 0$           | $\delta s T = - \delta Q$ |
| 2.  | $\delta s T = 0$         | $F \neq 0$        | $\delta \theta = \delta Q = - F$ |
| 3.  | $\delta Q = 0$           | $F \neq 0$        | $\delta \theta = \delta s T = - F$ |

Special case:

$$\frac{\delta_s L}{\delta V^A_B} = 0, \quad \frac{\delta g L}{\delta g_{\alpha \beta}} = 0.$$ (235)

$$v_a F = 0, \quad g_a F = 0, \quad Q = 0, \quad v F = v W, \quad g F = g W,$$ (236)

$$a F = v_a F + g_a F = 0, \quad F = W : W + \delta \theta = 0, \quad \theta = s T, \quad \delta \theta = \delta s T = - W.$$ (237)

For $W = 0 : \delta \theta = 0, \delta s T = 0$.

The finding out the covariant Noether identities for a given Lagrangian density $L = \sqrt{-g} \cdot L$ along with the energy-momentum tensors $\theta$, $s T$ and $Q$ allow the construction of a rough scheme of the structures of a Lagrangian theory over a differentiable manifold with contravariant and covariant affine connections and a metric:

![Diagram](image)

**Fig. 1. Scheme of the main structure of a Lagrangian theory**

The symmetric energy-momentum tensor of Hilbert $g_{sh} T$ appears as a construction related to the functional variation of the metric field variables $g_{\alpha \beta}$ [as a part of the variables $K^A_B \sim (V^A_B, g_C)$] and interpreted as a symmetric energy-momentum tensor of a Lagrangian system. This tensor does not exist as a relevant element of the scheme for obtaining Lagrangian structures by the method of Lagrangians with covariant derivatives (MLCD). It takes in the scheme a separate place and has different relations than the usual for the other elements.
Classical field theories involve relations between the different structures of Lagrangian systems. For the most part of Lagrangian systems equations of the type of the Euler-Lagrange equations have been imposed and the symmetric energy-momentum tensor of Hilbert has been used.

5.6 Navier-Stokes’ identities

If we consider the projections of the first Noether’s identity along a non-null (non-isotropic) vector field \( u \) and its corresponding contravariant and covariant projective metrics \( h^u \) and \( h_u \) we will find the first and second Navier-Stokes’ identities.

From the Noether’s identities in the form

\[
\mathcal{g}(F) + \mathcal{g}(\delta \theta) \equiv 0, \quad \text{(first Noether’s identity)},
\]

\[
(\theta \mathcal{g} - (\ast T)\mathcal{g} \equiv (Q)\mathcal{g}, \quad \text{(second Noether’s identity)},
\]

we can find the projections of the first Noether’s identity along a contravariant non-null vector field \( u = u^i \cdot \partial_i \) and orthogonal to \( u \).

Since

\[
g(\mathcal{g}(F), u) = g_{ik} \cdot g^{ij} \cdot F_i \cdot u^j = g_{ik} \cdot F_i \cdot u^j = F_i \cdot u^j \equiv F(u), \quad F = F_k \cdot d\mathbf{x}^k
\]

\[
g(\mathcal{g}(\delta \theta), u) = (\delta \theta)(u)
\]

we obtain the first Navier-Stokes’ identity in the form

\[
F(u) + (\delta \theta)(u) \equiv 0.
\]

By the use of the relation

\[
\mathcal{g}[h_u(\mathcal{g}(F)) = \mathcal{g}(h_u(\mathcal{g}(F)) = h^u(F), \quad \mathcal{g}(h_u)\mathcal{g} = h^u,
\]

\[
\mathcal{g}[h_u(\mathcal{g}(\delta \theta))] = \mathcal{g}(h_u(\mathcal{g}(\delta \theta))) = h^u(\delta \theta),
\]

the first Noether’s identity could be written in the forms

\[
h_u[\mathcal{g}(F)] + h_u[\mathcal{g}(\delta \theta)] \equiv 0, \quad (243)
\]

\[
h^u(F) + h^u(\delta \theta) \equiv 0. \quad (244)
\]

The last two forms of the first Noether’s identity represent the second Navier-Stokes’ identity.
If the projection $h^n(F)$, orthogonal to $u$, of the volume force $F$ is equal to zero, we obtain the *generalized Navier-Stokes’ equation* in the form

$$ h^n(\delta \theta) = 0 \ , \quad (245) $$

or in the form

$$ h_u[\mathcal{F}(\delta \theta)] = 0 \ . \quad (246) $$

Let us now find the explicit form of the first and second Navier-Stokes’ identities and the explicit form of the generalized Navier-Stokes’ equation. For this purpose we can use the explicit form of the covariant divergency $\delta \theta$ of the generalized canonical energy-momentum tensor $\theta$.

(a) The first Navier-Stokes’ identity follows in the form

$$ F(u) + (\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot g(a, u) + $$

$$ + \epsilon \cdot [u(\rho_0 + \frac{1}{e} \cdot L \cdot k) + (\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot \delta u + \delta \theta] - $$

$$ - (K r L)(u) - L \cdot (\delta Kr)(u) + g(\nabla_u \theta, u) + g(\nabla_u u, u) + $$

$$ + (\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot (\nabla_u g)(u, u) + (\nabla_u g)(\theta, u) + (\nabla_u g)(\delta, u) + $$

$$ + [\delta((F g)](u) = 0 \ . \quad (247) $$

(b) The second Navier-Stokes’ identity can be found in the form

$$ h_u[\mathcal{F}(F)] + h_u[\mathcal{F}(\delta \theta)] = $$

$$ = (\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot h_u(a) - $$

$$ - h_u[\mathcal{F}(K r L)] - L \cdot h_u[\mathcal{F}(\delta Kr)] + \delta u \cdot h_u(\theta, u) + $$

$$ + h_u(\nabla_u \theta) + h_u(\nabla_u u) + $$

$$ + (\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot h_u[\mathcal{F}(\nabla_u g)] + h_u[\mathcal{F}(\nabla_u g)(\theta, u)] + $$

$$ + h_u[\mathcal{F}(\nabla_u g)(\delta, u)] + h_u[\mathcal{F}(\delta((F g)])(u) = 0 \ . \quad (248) $$

(c) The generalized Navier-Stokes’ equation $h_u[\mathcal{F}(\delta \theta)] = 0$ follows from the second Navier-Stokes’ identity under the condition $h_u[\mathcal{F}(F)] = 0$ or under the condition $F = 0$

$$ (\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot h_u(a) - $$

$$ - h_u[\mathcal{F}(K r L)] - L \cdot h_u[\mathcal{F}(\delta Kr)] + \delta u \cdot h_u(\theta, u) + $$

$$ + h_u(\nabla_u \theta) + h_u(\nabla_u u) + $$

$$ + (\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot h_u[\mathcal{F}(\nabla_u g)] + h_u[\mathcal{F}(\nabla_u g)(\theta, u)] + $$

$$ + h_u[\mathcal{F}(\nabla_u g)(\delta, u)] + h_u[\mathcal{F}(\delta((F g)])(u) = 0 \ , \quad (249) $$

$$ h_u(a) = g(a) - \frac{1}{e} \cdot g(u, a) \cdot g(u) \ . \quad (250) $$

*Special case: $(L_n, g)$-spaces: $S = C$, $f^i_j = g^j_i$, $g(u, u) = e = \text{const.} \neq 0$, $k = 1$. $\delta K r = 0$, \quad (251)
(a) First Navier-Stokes’ identity

\[ F(u) + \left( \rho_0 + \frac{1}{e} \cdot L \right) \cdot g(a, u) + \]
\[ + e \cdot [u(\rho_0 + \frac{1}{e} \cdot L) + (\rho_0 + \frac{1}{e} \cdot L) \cdot \delta u + \delta^a \mathcal{S}] - \]
\[ -(KrL)(u) + g(\nabla_a e, u) + g(\nabla\varpi u, u) + \]
\[ +(\rho_0 + \frac{1}{e} \cdot L) \cdot (\nabla_u g)(a, u) + (\nabla_u g)(e, u) + (\nabla\varpi g)(u, u) + \]
\[ + [\delta((^0\mathcal{S}) g)](u) \equiv 0 \ . \] (252)

(b) Second Navier-Stokes’ identity

\[ (\rho_0 + \frac{1}{e} \cdot L) \cdot h_u(a) - \]
\[ -h_u[\mathcal{G}(KrL)] + \delta u \cdot h_u(e, \pi) + \]
\[ +h_u(\nabla_u e, \pi) + h_u(\nabla\varpi u) + \]
\[ +(\rho_0 + \frac{1}{e} \cdot L) \cdot h_u[\mathcal{G}(u)] + h_u[\mathcal{G}(\nabla_u g)(u)] + \]
\[ +h_u[\mathcal{G}(\nabla\varpi g)(u)] + h_u[\mathcal{G}(\delta((^0\mathcal{S}) g))] \]
\[ + h_u[\mathcal{G}(F)] \equiv 0 \ . \] (253)

(c) Generalized Navier-Stokes’ equation \( h_u[\mathcal{G}(\delta\theta)] = 0 \)

\[ (\rho_0 + \frac{1}{e} \cdot L) \cdot h_u(a) - \]
\[ -h_u[\mathcal{G}(KrL)] + \delta u \cdot h_u(e, \pi) + \]
\[ +h_u(\nabla_u e, \pi) + h_u(\nabla\varpi u) + \]
\[ +(\rho_0 + \frac{1}{e} \cdot L) \cdot h_u[\mathcal{G}(u)] + h_u[\mathcal{G}(\nabla_u g)(u)] + \]
\[ +h_u[\mathcal{G}(\nabla\varpi g)(u)] + h_u[\mathcal{G}(\delta((^0\mathcal{S}) g))] + h_u[\mathcal{G}(F)] \]
\[ = 0 \ . \] (254)

Special case: \( V_n\)-spaces: \( S = C, f^i_j = g^i_j, \nabla\xi g = 0 \) for \( \forall \xi \in T(M), g(u, u) = e = \) const. \( \neq 0, k = 1, g(a, u) = 0. \)

(a) First Navier-Stokes’ identity

\[ F(u) + \]
\[ + e \cdot [u(\rho_0 + \frac{1}{e} \cdot L) + (\rho_0 + \frac{1}{e} \cdot L) \cdot \delta u + \delta^a \mathcal{S}] - (KrL)(u) + \]
\[ + [\delta((^0\mathcal{S}) g)](u) \equiv 0 \ . \] (255)

(b) Second Navier-Stokes’ identity

\[ (\rho_0 + \frac{1}{e} \cdot L) \cdot h_u(a) - h_u[\mathcal{G}(KrL)] + \delta u \cdot h_u(e, \pi) + h_u[\mathcal{G}(\delta((^0\mathcal{S}) g))] + h_u[\mathcal{G}(F)] \equiv 0 \ . \] (256)

Generalized Navier-Stokes’ equation \( h_u[\mathcal{G}(\delta\theta)] = 0 \)
If we express the stress (tension) tensor \((\vartheta \overline{S})g\) by the use of the shear stress tensor \(k_s \overline{D}\), rotation (vortex) stress tensor \(k \overline{W}\), and the expansion stress invariant \(k \overline{U}\) then the covariant divergency of the corresponding tensors could be found and at the end we will have the explicit form of the Navier-Stokes’ identities and the generalized Navier-Stokes’ equation including all necessary tensors for further applications.

5.7 Navier-Stokes’ identities in linear elasticity theory

If we express the stress (tension) tensor \((\vartheta \overline{S})g\) by its explicit form for linear elasticity theory we can find the Navier-Stokes’ identities in linear elasticity theory

(a) First Navier-Stokes’ identity

\[
F(u) + (\rho_0 + \frac{1}{c} \cdot L) \cdot h_u(a) - h_u[\overline{g}(K r L)] + \delta u \cdot h_u(\vartheta \overline{\pi}) + h_u[\overline{g}(\delta(\vartheta \overline{S}))] = 0 . \tag{257}
\]

(b) The second Navier-Stokes’ identity can be found in the form

\[
h_u[\overline{g}(F)] + h_u[\overline{g}(\delta \theta)] =
\]

\[
= (\rho_0 + \frac{1}{c} \cdot L \cdot k) \cdot h_u(a) - h_u[\overline{g}(K r L)] - L \cdot h_u[\overline{g}(\delta K r)] + \delta u \cdot h_u(\vartheta \overline{\pi}) + h_u[\overline{g}(\vartheta \overline{\pi})] + h_u[\overline{g}(\delta \vartheta \overline{u})] + h_u[\overline{g}(\delta(\vartheta \overline{S}))] + \overline{g}(\delta(\vartheta \overline{F}))(u) + \frac{1}{n-1} \cdot \overline{g}(\delta(\vartheta \overline{U} \cdot \overline{g}(h_u))) + h_u[\overline{g}(F)] \equiv 0 . \tag{259}
\]

(c) The generalized Navier-Stokes’ equation \(h_u[\overline{g}(\delta \theta)] = 0\) follows from the second Navier-Stokes’ identity under the condition \(h_u[\overline{g}(F)] = 0\) or under the condition \(F = 0\)
\[ + h_u \left[ \mathcal{g} \left( \delta \left( \mathcal{g} \left( k \mathbf{D} \right) \right) \right) \right] + \mathcal{g} \left( \delta \left( \mathcal{g} \left( k \mathbf{W} \right) \right) \right) + \frac{1}{n-1} \cdot \mathcal{g} \left( \delta (k \mathbf{U} \cdot \mathcal{g} (h_u)) \right) = 0 \] , \quad (260)

\[ h_u (a) = g(a) - \frac{1}{c} \cdot g(u, a) \cdot g(u) \] . \quad (261)

From the first Noether identity \( \mathcal{g}(F) + \mathcal{g}(\delta \theta) \equiv 0 \) for the case of isotropic elastic media with \( \mathcal{g}[F] = 0 \), the equation \( \mathcal{g}(\delta \theta) = 0 \) follows in the form

\[ \mathcal{g}^{\mathcal{F}} \cdot \partial_j g^i_{\ j} = \left( \rho_0 + \frac{1}{c} \cdot L \cdot k \right) \cdot a^i + \\
+ \left[ (\rho_0 + \frac{1}{c} \cdot L \cdot k) \cdot u^i \right] - L \cdot g^{\theta \mathbf{\pi}}_{\ j} \cdot g^{i \ j} + u^j \cdot \delta^j = 0 \] , \quad (262)

(b) The second Navier-Stokes identity can be found in the form

\[ + \left( \rho_0 + \frac{1}{c} \cdot L \cdot k \right) \cdot h_u (a) - \\
- h_u \left[ \mathcal{g} \left( K R L \right) \right] - L \cdot h_u \left[ \mathcal{g} \left( \delta K R \right) \right] - \delta u \cdot h_u (\theta \mathbf{\pi}) + \\
+ h_u \left[ \mathcal{g} \left( \delta \left( \mathcal{g} \left( k \mathbf{U} \right) \right) \right) \right] + \frac{1}{n-1} \cdot \mathcal{g} \left( \delta (k \mathbf{U} \cdot \mathcal{g} (h_u)) \right) \] . \quad (263)

(b) The second Navier-Stokes’ identity can be found in the form

\[ h_u \left[ \mathcal{g}(F) \right] + h_u \left[ \mathcal{g}(\delta \theta) \right] \equiv \]

\[ \equiv \left( \rho_0 + \frac{1}{c} \cdot L \cdot k \right) \cdot h_u (a) - \\
- h_u \left[ \mathcal{g} \left( K R L \right) \right] - L \cdot h_u \left[ \mathcal{g} \left( \delta K R \right) \right] - \delta u \cdot h_u (\theta \mathbf{\pi}) + \\
+ h_u \left[ \mathcal{g} \left( \delta \left( \mathcal{g} \left( k \mathbf{U} \right) \right) \right) \right] + \frac{1}{n-1} \cdot \mathcal{g} \left( \delta (k \mathbf{U} \cdot \mathcal{g} (h_u)) \right) \] . \quad (264)
(c) The generalized Navier-Stokes’ equation $h_u[\mathcal{G}(\delta \theta)] = 0$ follows from the second Navier-Stokes’ identity under the condition $h_u[\mathcal{G}(F)] = 0$ or under the condition $F = 0$

$$\left(\rho_0 + \frac{1}{c} \cdot L \cdot k\right) \cdot h_u(a) -$$

$$-h_u[\mathcal{G}(KrL)] - L \cdot h_u[\mathcal{G}(\delta Kr)] + \delta u \cdot h_u(\theta \bar{\theta}) +$$

$$+ h_u(\nabla_u \theta^\sigma) + h_u(\nabla_\theta \bar{\theta}) +$$

$$(\rho_0 + \frac{1}{c} \cdot L \cdot k) \cdot h_u[\mathcal{G}(\nabla_u g)(u)] + h_u[\mathcal{G}(\nabla_u g)(\theta \bar{\theta})] +$$

$$+ h_u[\mathcal{G}(\delta \mathcal{G}(kU))] + \mathcal{G}(\delta(\mathcal{G}(kW))) + \frac{1}{n-1} \cdot \mathcal{G}(\delta(kU \cdot \mathcal{G}(h_u))] = 0 , \quad (265)$$

$$h_u(a) = g(a) - \frac{1}{c} \cdot g(u, a) \cdot g(u) .$$

From the first Noether identity $\mathcal{G}(F) + \mathcal{G}(\delta \theta) \equiv 0$ for the case of Newton’s fluids with $\mathcal{G}[F] = 0$, the equation $\mathcal{G}(\delta \theta) = 0$ follows in the form

$$\mathcal{G}(\mathcal{G}(\delta \mathcal{G}(kU))) + \mathcal{G}(\delta(\mathcal{G}(kW))) + \frac{1}{n-1} \cdot \mathcal{G}(\delta(kU \cdot \mathcal{G}(h_u))] = 0 , \quad (266)$$

$$h_u(a) = g(a) - \frac{1}{c} \cdot g(u, a) \cdot g(u) .$$

If the vector field $\mathbf{u}$ is a given vector field $|e = g(u, u) \neq 0|$ and $\sigma, \omega, \text{and } \theta$ are given in their explicitly form as function of the components $u^i$ and their covariant derivatives $u^i_{;\lambda}$ of the velocity vector $\mathbf{u}$ then the last equation with respect to $\mathbf{u}$ is called generalized Navier-Stokes equation.

On the basis of the considered results many problems of the continuous media mechanics could be further specialized and solved.

6 Conclusion

In the present paper continuous media mechanics is consider over $(\mathcal{L}_n, g)$-spaces with respect to its basic notions from the point of view of the differential-geometric structures related to

(a) deformations,

(b) stresses (tensions),

(c) relation between stressed and deformations on the basis of the method of Lagrangians with covariant derivatives.

The presented results were specialized for elasticity theory and hydrodynamics in $(\mathcal{L}_n, g)$-spaces. These results could also be used in all spaces considered as special cases of the $(\mathcal{L}_n, g)$-spaces [Euclidean $E_n$-spaces, (pseudo) Euclidean $M_n$-spaces, (pseudo) Riemannian $\mathcal{V}_n$- and $\mathcal{U}_n$-spaces, Weyl’s spaces $\mathcal{W}_n$ and $\mathcal{Y}_n$ etc.] as well as in general relativity as an extension in some aspects of the relativistic continuous media mechanics. Many new features of the continuous media mechanics are found at a local level [e.g. the existence of local rotations velocity tensors for isotropic elastic media and Newton’s fluids, existence of stresses generated by rotation velocity tensor etc.].

The existence of classical field theories for describing dynamical systems in different models of space or space-time allows a closer comparison with the structure
of a theory of continuous media. All basic notions of continuous mechanics could be used in a classical field theory and vice versa - basic notions of a classical field theory could be applied in a theory of continuous media.

The covariant and contravariant metrics introduced over differentiable manifolds with contravariant and covariant affine connections allow applications for mathematical models of dynamic systems described over \((\mathcal{L}_n, g)\)-spaces. On the other side, different type of geometries can be considered by imposing certain additional conditions of the type of metric transports on the metric. Additional conditions determined by different "draggings along" of the metric can have physical interpretation connected with changes of the length of a vector field and with changes of the angle between two vector fields.

The introduction of contravariant and covariant projective metrics corresponding to a non-isotropic (non-null) contravariant vector field allows the evolution of tensor analysis over sub manifolds of a manifold with contravariant and covariant connections and metrics and its applications for descriptions of the evolution of physical systems over \((\mathcal{L}_n, g)\)-spaces.

The kinematic characteristics, connected with the introduced notions of relative velocity and relative acceleration can be used for description of different dynamic systems by means of mathematical models, using differentiable manifold \(M\) with contravariant and covariant affine connections and metrics as a model of space-time \((\dim M = 4)\) \[\text{ETG in } V_n\text{-spaces, Einstein-Cartan theory in } U_n\text{-spaces, metric-affine theories in } (L_n, g)\text{-spaces}], or as a model for the consideration of dynamic characteristics of some physical systems \[\text{theories of the type of Kaluza-Klein in } V_n\text{-spaces } (n > 4)\text{, relativistic hydrodynamics etc.}\]. All basic notions related to the relative velocity and relative acceleration could be related to deformations and stresses (tensions) in the continuous media mechanics in \((\mathcal{L}_n, g)\)-spaces. At the same time the kinematic characteristics can be used for a more correct formulation of problems, connected with the experimental check-up of modern gravitational theories.

In the case of general relativity theory one of the propositions can be used for describing the characteristics of gravitational detectors: If test particles are considered to move in an external gravitational field \((R_{ij} = 0)\), then their relative acceleration will be caused only by the curvature shear acceleration. Therefore, gravitational wave detectors have to be able to detect accelerations of the type of shear acceleration (and not of the type of expansion acceleration), if the energy-momentum tensor of the detector is neglected as a source of a gravitational field.

The conjecture connecting the stresses and deformations in continuous media mechanics on the basis of the kinematic characteristics and the energy-momentum tensors could lead to a more exact and comprehensive description of dynamical systems in continuous media mechanics and in the classical (non-quantized) field theories in \(V_n\text{-, } U_n\text{-, } (L_n, g)\text{-, and } (\mathcal{L}_n, g)\)-spaces as well as in their special cases.

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