LS-sequences of points in the unit square

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Abstract

We define a countable family of sequences of points in the unit square: the LS-sequences of points à la Halton. They reveal a very strange and interesting behaviour, as well as resonance phenomena, for which we have not found an explanation, so far. We conclude with three open problems.

Keywords Uniform distribution, Discrepancy, Quasi-Monte Carlo methods.

Mathematics Subject Classification 11K06, 11K31, 11K38, 11K45.

1 LS-sequences in $[0, 1[$

In [1] the first author introduced a countable family of LS-sequences of points in $[0, 1[$, whose construction strongly depends on the LS-sequences of partitions of $[0, 1[$ introduced in the same article. Actually, the former are obtained by reordering the left endpoints of the intervals of the latter.

In this paper we make the first step in the direction of their generalization to higher dimension, presenting two possible extensions to dimension 2: the LS-point sets à la van der Corput, and the LS-sequences à la Halton.

The interest on low discrepancy sequences of points in dimension two or higher is motivated by their application in Quasi-Monte Carlo methods (see [3]).

In this section we recall the definition of the LS-sequences of partitions and points and their main properties.

The LS-sequences of partitions are obtained as a particular case of the $\rho$-refinements introduced by the third author in [10].
Definition 1. Consider any non trivial finite partition $\rho$ of $[0,1]$. The $\rho$-refinement of a partition $\pi$ of $[0,1]$ (which will be denoted by $\rho\pi$) is obtained by subdividing only the interval(s) of $\pi$ having maximal length homotetically to $\rho$. Denote by $\rho^n\pi$ the $\rho$-refinement of $\rho^{n-1}\pi$, and by $\{\rho^n\pi\}_{n \geq 1}$ the sequence of successive $\rho$-refinements.

If $\rho = \{[0,\alpha],[\alpha,1]\}$ and $\omega = \{[0,1]\}$ is the trivial partition of $[0,1]$, the sequence of successive $\rho$-refinements is actually the splitting procedure introduced by Kakutani [6].

In [10] it has been proved that the sequence $\{\rho^n\omega\}_{n \geq 1}$ is uniformly distributed, which means that if $\rho^n\omega = \{[y_i^{(n)},y_{i+1}^{(n)}]: 1 \leq i \leq t_n\}$, the sequence satisfies
\[
\lim_{n \to \infty} \frac{1}{t_n} \sum_{i=1}^{t_n} \chi_{[a,b]}(y_i^{(n)}) = b - a ,
\]
for every pair of real numbers $a,b$, with $0 \leq a < b \leq 1$ (see [10] for the first formal definition of this concept).

An LS-sequence of partitions, denoted by $\{\rho_{L,S}^n\}_{n \geq 1}$, is obtained as successive $\rho_{L,S}$-refinements of $\omega$, where $\rho_{L,S}$ is the partition made by $L$ long intervals having length $\gamma$ followed by $S$ short ones having length $\gamma^2$, and $\gamma$ is the positive root of the equation $L\gamma + S\gamma^2 = 1$.

It is clear that each partition $\rho_{L,S}^n$ has only long intervals having length $\gamma^n$ and short ones having length $\gamma^{n+1}$. The sequence $\{t_n\}_{n \geq 1}$ of the number of intervals of $\{\rho_{L,S}^n\}_{n \geq 1}$ satisfies the difference equation $t_n = Lt_{n-1} + St_{n-2}$ with $t_0 = 1$ and $t_1 = L + S$.

In [1] it has been proved that when $S \leq L$ the LS-sequence of partitions $\{\rho_{L,S}^n\}$ has low discrepancy, namely there exists a constant $C > 0$ depending on $L$ and $S$ such that $t_n D(\rho_{L,S}^n) \leq C$ for any $n$. Here $D(\rho_{L,S}^n)$ denotes the discrepancy of $\rho_{L,S}^n$ defined by
\[
D(\rho_{L,S}^n) = \sup_{0 \leq a < b \leq 1} \left| \frac{1}{t_n} \sum_{j=1}^{t_n} \chi_{[a,b]}(y_j^{(n)}) - (b - a) \right| .
\]

In fact, we have more generally

Theorem 2. If $S \leq L$ there exist $c_1, c_1' > 0$ such that $c_1' \leq t_n D(\rho_{L,S}^n) \leq c_1$ for any $n \in \mathbb{N}$.

If $S = L + 1$ there exist $c_2, c_2' > 0$ such that $c_2' \log t_n \leq t_n D(\rho_{L,S}^n) \leq c_2 \log t_n$ for any $n \in \mathbb{N}$.
If \( S \geq L + 2 \) there exist \( c_3, c'_3 > 0 \) such that \( c'_3 t_n^{1-\tau} \leq t_n D(\rho_{L,S}^n) \leq c_3 t_n^{1-\tau} \) for any \( n \in \mathbb{N} \), where \( 1 - \tau = -\frac{\log(S\gamma)}{\log \gamma} > 0 \).

In [2] the first author (improving [1]) introduced an algorithm which associates to each \( LS \)-sequence of partitions a sequence (of points), called \( LS \)-sequence of points and denoted by \( \{\xi^n_{L,S}\}_{n \geq 1} \). The sequences \( \{\xi^n_{L,S}\}_{n \geq 1} \) can be seen in terms of the representation in base \( L+S \) of a suitable subsequence of natural numbers. We describe briefly this construction (see [2] for further details).

Any positive integer can be written as

\[
n = \sum_{k=0}^{M} a_k(n)(L+S)^k,
\]

where \( a_k(n) \in \{0, 1, 2, \ldots, L+S-1\} \) for all \( 0 \leq k \leq M \) and \( M = \lfloor \log_{L+S} n \rfloor \) (here \( \lfloor \cdot \rfloor \) denotes the integer part, as usual).

Let us denote by \( \mathbb{N}_{L,S} \) the infinite set of all positive integers \( n \), ordered by magnitude, such that, for each \( k \in \mathbb{N} \), \((a_k(n), a_{k+1}(n)) \notin \{L, L+1, \ldots, L+S-1\} \times \{1, \ldots, L+S-1\} \). Define on \( \mathbb{N}_{L,S} \) the function

\[
\phi_{L,S}(n) = \sum_{k=0}^{M} \tilde{a}_k(n) \gamma^{k+1},
\]

where \( \tilde{a}_k(n) = a_k(n) \) if \( 0 \leq a_k(n) \leq L-1 \), while \( \tilde{a}_k(n) = L + \gamma(a_k(n) - L) \) if \( L \leq a_k(n) \leq L+S-1 \).

The sequence \( \{\phi_{L,S}(n)\}_{n \geq 0} \) defined on \( \mathbb{N}_{L,S} \) is the \( LS \)-sequence of points.

The most important property these sequences show is that whenever the \( LS \)-sequence of points has low discrepancy, the corresponding \( LS \)-sequence of points obtained by the algorithm described above has low discrepancy too. More precisely, in [1] upper bounds for their discrepancy have been given (see [7] for the general theory on uniform distribution and discrepancy).

**Theorem 3.** If \( S \leq L \) there exists \( k_1 > 0 \) such that for any \( N \in \mathbb{N} \) we have

\[
N D(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}) \leq k_1 \log N.
\]

If \( S = L + 1 \) there exists \( k_2, c'_2 > 0 \) such that for any \( N \in \mathbb{N} \) we have

\[
c'_2 \log N \leq N D(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}) \leq k_2 \log^2 N.
\]
If \( S \geq L + 2 \) there exists \( k_3, c'_3 > 0 \) such that for any \( N \in \mathbb{N} \) we have \( c'_3 N^{1-\tau} \leq N D(\xi^{1}_{L,S}, \xi^{2}_{L,S}, \ldots, \xi^{N}_{L,S}) \leq k_3 N^{1-\tau} \log N \), where \( 1 - \tau = \frac{-\log(S\gamma)}{\log \gamma} > 0 \).

## 2 LS-sequences of points in the unit square

In this section we see how the LS-sequences of points can be used to produce sequences in the unit square.

One possibility is to imitate the van der Corput idea [9], combining an LS-sequence of points with the points (ordered by magnitude) associated to the Knopowski partition \( \left\{ \left\lfloor \frac{i-1}{N}, \frac{i}{N} \right\rfloor, 1 \leq i \leq N \right\} \) of order \( N \). Hammersley [5] extended this definition to higher dimension.

The other possibility is to put on the two coordinates two different LS-sequences of points, imitating what Halton did in [4] when he paired two van der Corput sequences having different bases. He proved that whenever these bases are coprime, the sequence has low discrepancy in the unit square.

The first idea produces the following

**Definition 4.** For each LS-sequence of points \( \{\xi^n_{L,S}\}_{n \geq 1} \), the finite set of points

\[
P_{L,S}(N) = \left\{ \left( \frac{n-1}{N}, \xi^n_{L,S} \right), \quad n = 1, \ldots, N \right\}
\]

is called LS-point set à la van der Corput of order \( N \) in the unit square.

The main result concerning these LS-points sets à la van der Corput is given by the following

**Proposition 5.** If \( S \leq L \) there exists \( C_1 > 0 \) such that for any \( N \in \mathbb{N} \) we have

\[
N D\left( P_{L,S}(N) \right) \leq C_1 \log N.
\]

If \( S = L + 1 \) there exists \( C_2 > 0 \) such that for any \( N \in \mathbb{N} \) we have

\[
N D\left( P_{L,S}(N) \right) \leq C_2 \log^2 N.
\]

If \( S \geq L + 2 \) there exists \( k_3, c'_3 > 0 \) such that for any \( N \in \mathbb{N} \) we have

\[
N D\left( P_{L,S}(N) \right) \leq C_3 N^{1-\tau} \log N, \text{ where } 1 - \tau = \frac{-\log(S\gamma)}{\log \gamma} > 0.
\]
Proof. Let us fix a rectangle $R = [0, a \times ]0, b]$ and an LS-sequence of points $\{\xi_{L,S}^n\}_{n \geq 1}$. If we denote by $f_{L,S}$ one of the three upper bounds appearing in Theorem 3, a simple calculation gives.

$$\left| \frac{1}{N} \sum_{j=1}^{N} \chi_{R} \left( \frac{j-1}{N}, \xi_{L,S}^j \right) - ab \right| = \left| \frac{1}{N} \sum_{j=1}^{N} \chi_{[0,a]} \left( \frac{j-1}{N} \right) \chi_{[0,b]}(\xi_{L,S}^j) - ab \right|$$

$$\leq \left| \frac{1}{N} \sum_{j=1}^{[Na]+1} \chi_{[0,b]}(\xi_{L,S}^j) - \frac{[Na] + 1}{N} b \right| + \left| \frac{[Na] + 1}{N} b - ab \right|$$

$$= \frac{[Na] + 1}{N} \left| \frac{1}{[Na] + 1} \sum_{j=1}^{[Na]+1} \chi_{[0,b]}(\xi_{L,S}^j) - b \right| + b \left| \frac{[Na] + 1}{N} - a \right|$$

$$\leq \frac{1}{N} \left| \frac{1}{[Na] + 1} \sum_{j=1}^{[Na]+1} \chi_{[0,b]}(\xi_{L,S}^j) - b \right| + \frac{b}{N}$$

$$\leq \frac{1}{N} f_{L,S}(\lceil Na \rceil) \leq c f_{L,S}(N) .$$

Taking the supremum over all the rectangles $R$ in the unit square, the theorem is completely proved. \hfill \square

**Remark** Comparing the above proposition to Theorem 3, we conclude that any LS-point set à la van der Corput has low discrepancy when $S \leq L$.

Let us now give the second and more interesting generalization.

**Definition 6.** Given two LS-sequences of points $\{\xi_{L_1,S_1}^n\}_{n \geq 1}$ and $\{\xi_{L_2,S_2}^n\}_{n \geq 1}$, the sequence

$$\{\xi_{L_1,S_1}^n, \xi_{L_2,S_2}^n\}_{n \geq 1}$$

is called LS-sequence of points à la Halton in the unit square.

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Halton proved that two van der Corput sequences generate a uniformly distributed sequence in the square if and only if they have coprime bases.

The situation is not clear at all if we use pairs of \textit{LS-sequences of points}, as it can be seen in Figures 1 and 2, where numerical experiments suggest that there is no uniform distribution, while Figures 3 and 4 suggest that for those pairs of \textit{LS-sequences of points} we do have uniform distribution.

The following proposition may give an explanation for the “resonance” which appears in Figure 1.
Proposition 7. If for each $n \in \mathbb{N}$ we denote by $t_n$ the total number of intervals of $\rho^n_{1,1}$ and by $t'_n$ the total number of intervals of the partition $\rho^n_{4,1}$, then we have that

$$t'_n = t_{3n} \text{ for all } n \in \mathbb{N}.$$

Proof. We use induction on $n \geq 1$. If $n = 1$ we have $t'_1 = 5 = t_3$.

Assume $t'_{n-1} = t_{3(n-1)}$ holds. The relations $t_n = t_{n-1} + t_{n-2}$ and $t'_n = 4t'_{n-1} + t'_{n-2}$ implies $t_{3n} = 4t_{3n-3} + t_{3n-6}$. From the inductive assumption we have $4t_{3(n-1)} + t_{3(n-2)} = 4t'_{n-1} + t'_{n-2}$, and therefore the proposition is proved. \qed

Open problems

1. When is $\{\xi^n_{L_1,S_1}, \xi^n_{L_2,S_2}\}_{n \geq 1}$ uniformly distributed?

2. What is its discrepancy?

3. What happens in higher dimension?

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* Research supported by the Doctoral Fellowship in Mathematics and Informatics of University of Calabria in cotutelle with Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria.