The four-propagator three-loop vacuum integral by the hypergeometry

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Abstract

Hypergeometric function method is proposed to calculate the scalar integrals of Feynman diagrams. For the scalar integral of three-loop vacuum diagram with four-propagator, we verify the equivalency of Feynman parametrization and the hypergeometric technique. The result can be described as generalized hypergeometric functions of triple variables. Based on the triple hypergeometric functions, we also establish the systems of homogeneous linear partial differential equations(PDEs) satisfied by the mentioned scalar integral. The continuation of the scalar integral from its convergent regions to whole kinematic domains can be made numerically through the system of homogeneous linear PDEs with the help of the element method.

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I. INTRODUCTION

Applying the method of integration by part (IBP)\cite{1}, the general scalar integrals can be reduced to a linear combination of scalar integrals. Calculating scalar integral is an obstacle to predict the electroweak observables precisely in the standard model. Those one-loop scalar integrals are computed\cite{2, 3}, however the calculations of the multi-loop scalar integrals are not advanced enough. A number of useful methods are introduced to evaluate those scalar integrals in literature\cite{4}. An analytic expression for the planar massless two-loop vertex diagrams is given in Ref.\cite{5} by Feynman parametrization method. The Mellin-Barnes (MB) method is sometimes used to compute some massless scalar integrals\cite{6, 7}. In the paper\cite{8}, the Mellin-Barnes representation is used to obtain expressions for some classes of single-loop massive Feynman integrals and vertex type. The results are presented in the form of hypergeometric functions. Furthermore, multivariable hypergeometric functions are presented giving explicit series for small and large momenta for two-loop self-energy diagrams\cite{9}. However, the technique of multiple MB representations will be very cumbersome for multi-loop diagram. For scalar integral, the author of Refs.\cite{10–24} derives a set of differential equations based on the IBP relationship. Another method is recommended to analyze the scalar integrals which is called “dimensional recurrence and analyticity”\cite{25–31}. The asymptotic expansions in momenta and masses can be employed to approach the scalar integral relies on kinematic invariants and masses\cite{32}. A novel method\cite{33, 34} to compute Feynman integrals by constructing and solving a system of ordinary differential equations (ODEs).

The class of two-loop massive scalar self-energy diagrams with three propagators is studied in an arbitrary number of dimensions\cite{35}, and they can be described by generalized hypergeometric functions of several variables, namely Laricella functions. The results can be generalized to N loop massive scalar self-energy diagrams with N + 1 propagators. But they only get the analytical results in the converges. The continuation from its convergent regions to whole kinematic domain has not been finished.

For scalar integrals, we can get the analytic expressions through the hypergeometric theory. According to the series representations of modified Bessel functions and some integrals in hypergeometric theory, our previous work\cite{36} have obtained the generalized hypergeometric functions of the one-loop $B_0$ function, two-loop vacuum integral, the scalar integrals from two-loop sunset and one-loop triangle diagrams. And we establish the sys-
FIG. 1: three-loop vacuum diagram, \( m_i \) denotes the mass of the \( i \)-th particle and \( p_j \) denotes the corresponding momentum.

Systems of linear homogeneous PDEs satisfied by the scalar integrals in the kinematic region. Furthermore, the \( C_0 \) function have been calculated under the guidance of MB representations. The continuation to the whole kinematic domain can be finished with help of finite element. The point specified here is that the system of homogeneous linear PDEs differs from that presented in literatures. The detailed description is in Ref.\([36]\). Note that the three-loop vacuum integrals with arbitrary masses are considered numerically in Refs.\([38, 39]\), which are the different methods compared with ours. The results of this paper are consistent with the corresponding results of Ref.\([38]\).

This paper aims at computing the scalar integral of three-loop vacuum diagram FIG.1. Our presentation is organized as follows. In section II, the equivalence of traditional Feynman parametrization and the hypergeometric theory for this scalar integral is proved. In section III, we obtain the generalized hypergeometric functions in independent kinematic variables for the scalar integral, which are convergent in the connected region. Meanwhile, we write down the systems of homogeneous linear partial differential equations (PDEs) satisfied by the corresponding the generalized hypergeometric functions. According to the PDEs, the continuation can be finished from the convergent domain to whole kinematic regions with the help of the finite element method. As the special case, the analytical results of the three-loop vacuum integral in the convergence region are presented in section IV. Finally our conclusions are summarized in section V.
II. THE EQUIVALENCY BETWEEN FEYNMAN PARAMETRIZATION AND THE HYPERGEOMETRIC METHOD

The modified Bessel functions can be written in the following form \[40–43\]

\[
\frac{2(m^2)^{D/2-\alpha}}{(4\pi)^{D/2} \Gamma(\alpha)} k_{D/2-\alpha}(mx) = \int \frac{d^D q}{(2\pi)^D} \frac{\exp[-i\mathbf{q} \cdot \mathbf{x}]}{(q^2 + m^2)^\alpha},
\]

\[
\frac{\Gamma(D/2 - \alpha)}{(4\pi)^{D/2} \Gamma(\alpha)} \left(\frac{x}{2}\right)^{\alpha-D} = \int \frac{d^D q}{(2\pi)^D} \frac{\exp[-i\mathbf{q} \cdot \mathbf{x}]}{(q^2)^\alpha}, \tag{1}
\]

where \(\mathbf{q}, \mathbf{x}\) are vectors in the \(D\)-dimension Euclid space, \(m\) is the mass of the corresponding particle. After trivial cancellations of numerator and denominator terms, the general analytic expression for the scalar integral of three-loop vacuum diagram FIG.1 can be written in the form

\[
U_3(m^2_1, m^2_2, m^2_3, m^2_4) = (\mu^2)^{6-3D/2} \prod_{i=1}^{3} \frac{d^D p_i}{(2\pi)^D} \frac{1}{(p_i^2 - m^2_i)(p_2^2 - m^2_2)(p_3^2 - m^2_3)((p_1 + p_2 + p_3)^2 - m^2_4)}, \tag{2}
\]

where \(D = 4 - 2\varepsilon\) is the number of dimensions in dimensional regularization and \(\mu\) denotes the renormalization energy scale. With the Wick rotation and Eq.(1), the three-loop \(U_3\) function is formulated as

\[
U_3(m^2_1, m^2_2, m^2_3, m^2_4) = \frac{(-i)^{2^4}(\mu^2)^{6-3D/2}}{\Gamma(D/2)(4\pi)^{3D/2}} \prod_{i=1}^{4} (m_i^2)^{D/2-1} \int_0^\infty dx \left(\frac{x}{2}\right)^{D-1} k_{D/2-1}(m, x). \tag{3}
\]

The integral representation of the Bessel function can be applied to Eq.(3)

\[
k_{\mu}(x) = \frac{1}{2} \int_0^\infty t^{-\mu-1} \exp\{-t - \frac{x}{4t}\} dt, \quad \Re(x^2) > 0. \tag{4}
\]

Thus the \(U_3\) function is written as

\[
U_3 = \frac{(-i)^{\sum_{i=1}^{4} m_i^2}^{D/2-1} 2^{1-D} (\mu^2)^{3D/2-6}}{(4\pi)^{3D/2} \Gamma(D/2)} \int_0^\infty dt_1 t_1^{-D/2} \int_0^\infty dt_2 t_2^{-D/2} \int_0^\infty dt_3 t_3^{-D/2} \int_0^\infty dt_4 t_4^{-D/2} \\
\times \exp\{-t_1 - t_2 - t_3 - t_4\} \int_0^\infty dx x^{D-1} \exp\{-\frac{m_1^2}{4t_1} - \frac{m_2^2}{4t_2} - \frac{m_3^2}{4t_3} - \frac{m_4^2}{4t_4}\} x^2 \\
= \frac{(-i)(\mu^2)^{6-3D/2}}{(4\pi)^{2D}} \int_0^\infty dt_1 t_1^{-D/2} \int_0^\infty dt_2 t_2^{-D/2} \int_0^\infty dt_3 t_3^{-D/2} \int_0^\infty dt_4 t_4^{-D/2} \\
\times \exp\{-m_1^2 t_1 - m_2^2 t_2 - m_3^2 t_3 - m_4^2 t_4\} \int dx x \exp\{-\frac{t_1 t_2 t_3 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4}{4t_1 t_2 t_3 t_4}\} \\
= \frac{(-i)(\mu^2)^{6-3D/2}}{(4\pi)^{3D/2}} \int_0^\infty dt_1 t_1^{-D/2} \int_0^\infty dt_2 t_2^{-D/2} \int_0^\infty dt_3 t_3^{-D/2} \int_0^\infty dt_4 t_4^{-D/2} \\
\times \exp\{-m_1^2 t_1 - m_2^2 t_2 - m_3^2 t_3 - m_4^2 t_4\} \frac{t_1 t_2 t_3 t_4}{t_1 t_2 t_3 t_4 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4} \tag{5}
\]
Performing the variable transformation

\[ t_1 = \varrho y_1, \ t_2 = \varrho y_2, \ t_3 = \varrho y_3, \ t_4 = \varrho (1 - y_1 - y_2 - y_3), \]  

(6)

the Jacobian of the transformation is

\[ \frac{\partial(t_1, t_2, t_3, t_4)}{\partial(y_1, y_2, y_3, \varrho)} = \varrho^3, \]  

(7)

finally we have

\[ U_3 = \frac{(-i)(\mu^2)^{6-3D/2}}{(4\pi)^{3D/2}} \int_0^1 dy_1 \int_0^1 dy_2 \int_0^1 dy_3 \]
\[ \times \int_0^\infty d\varrho^{3-3D/2} \exp\{(-m_1^2 y_1 - m_2^2 y_2 - m_3^2 y_3 - m_4^2 (1 - y_1 - y_2 - y_3))\varrho\} \]
\[ \times (y_1 y_2 y_3 + y_1 y_3 (1 - y_1 - y_2 - y_3) + y_2 y_3 (1 - y_1 - y_2 - y_3) + y_3 (1 - y_1 - y_2 - y_3))^{D/2} \]
\[ = \frac{(-i)\Gamma(4 - 3D/2)}{(4\pi)^{3D/2}(\mu^2)^{3D/2-6}} \int_0^1 dy_1 \int_0^1 dy_2 \int_0^1 dy_3 \int_1^\infty \varrho \delta(1 - y_1 - y_2 - y_3 - y_4) \]
\[ \times (m_1^2 y_1 + m_2^2 y_2 + m_3^2 y_3 + m_4^2 y_4)^{3D/2-4} \]
\[ \times (y_1 y_2 y_3 + y_1 y_2 y_4 + y_1 y_3 y_4 + y_2 y_3 y_4)^{D/2}. \]  

(8)

One can get the same result as Eq. (6) while using the Feynman parametrization for \( U_3 \) function.

\section*{III. THE SYSTEM OF HOMOGENEOUS LINEAR PDES FOR THREE-LOOP VACUUM DIAGRAM FIG.1: GENERAL CASE}

In order to obtain the triple hypergeometric series for the scalar integral from three-loop vacuum diagram, the modified Bessel functions in power series an be written as

\[ k_\mu(x) = \frac{1}{2} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left[ \Gamma(-\mu - n) \left( \frac{x}{2} \right)^{2n} + \Gamma(\mu - n) \left( \frac{x}{2} \right)^{2(n-\mu)} \right] \]
\[ = \frac{\Gamma(\mu)\Gamma(1 - \mu)}{2} \sum_{n=0}^\infty \frac{1}{n!} \left[ - \frac{1}{\Gamma(1 + \mu + n)} \left( \frac{x}{2} \right)^{2n} + \frac{1}{\Gamma(1 - \mu + n)} \left( \frac{x}{2} \right)^{2(n-\mu)} \right]. \]  

(9)

And one can present radial integral\[40\] as,

\[ \int_0^\infty dt \left( \frac{t}{2} \right)^{2\varrho-1} k_\mu(t) = \frac{1}{2} \Gamma(\varrho)\Gamma(\varrho - \mu). \]  

(10)

According to the topology of FIG.1, we can get the similar analytic expression whichever \( m_i (i = 1, 2, 3, 4) \) is the maximum mass. So, we take the \( m_4 \) maximum mass as an example to illustrate the calculation process. As \( m_4 > max(m_1, m_2, m_3) \), inserting the expressions
of \( k_{D/2-1}(m_1 x), k_{D/2-1}(m_2 x), k_{D/2-1}(m_3 x) \) into the Eq. (3) and then applying Eq. (10), the \( U_3 \) function is written as

\[
U_3(m_1^2, m_2^2, m_3^2, m_4^2) = \frac{-i (m_4^2)^{3D/2-4} (\mu^2)^{6-3D/2}}{\Gamma(D/2)(4\pi)^{3D/2}} \Gamma^3 \left( \frac{D}{2} - 1 \right) \Gamma^3 \left( \frac{D}{2} - D \right) \phi(x_1, x_2, x_3), \tag{11}
\]

with \( x_1 = \frac{m_1^2}{m_2^2}, x_2 = \frac{m_2^2}{m_3^2}, x_3 = \frac{m_3^2}{m_4^2} \), and the function \( \phi(x_1, x_2, x_3) \) is defined as

\[
\phi(x_1, x_2, x_3) = -\frac{1}{\Gamma^2(D/2)}(x_1 x_2 x_3)^{D/2-1} F_{3}^{(3)} \left( \begin{array}{c}
1, \frac{D}{2}; \frac{D}{2}, \frac{D}{2}, \frac{D}{2}; x_1, x_2, x_3
\end{array} \right)
+ \frac{1}{\Gamma^2(D/2)}(x_1 x_2)^{D/2-1} F_{3}^{(3)} \left( \begin{array}{c}
1, \frac{D}{2}; \frac{D}{2}, \frac{D}{2}, \frac{D}{2}; x_1, x_2, x_3
\end{array} \right)
+ \frac{1}{\Gamma^2(D/2)}(x_1 x_3)^{D/2-1} F_{3}^{(3)} \left( \begin{array}{c}
1, \frac{D}{2}; \frac{D}{2}, \frac{D}{2}, \frac{D}{2}; x_1, x_2, x_3
\end{array} \right)
+ \frac{1}{\Gamma^2(D/2)}(x_2 x_3)^{D/2-1} F_{3}^{(3)} \left( \begin{array}{c}
1, \frac{D}{2}; \frac{D}{2}, \frac{D}{2}, \frac{D}{2}; x_1, x_2, x_3
\end{array} \right)
- \frac{\Gamma(3-D)}{\Gamma(D/2)\Gamma(2-D/2)} (x_1)^{D/2-1} F_{3}^{(3)} \left( \begin{array}{c}
2-D/2, 3-D; \frac{D}{2}, \frac{D}{2}, \frac{D}{2}; x_1, x_2, x_3
\end{array} \right)
- \frac{\Gamma(3-D)}{\Gamma(D/2)\Gamma(2-D/2)} (x_2)^{D/2-1} F_{3}^{(3)} \left( \begin{array}{c}
2-D/2, 3-D; \frac{D}{2}, \frac{D}{2}, \frac{D}{2}; x_1, x_2, x_3
\end{array} \right)
- \frac{\Gamma(3-D)}{\Gamma(D/2)\Gamma(2-D/2)} (x_3)^{D/2-1} F_{3}^{(3)} \left( \begin{array}{c}
2-D/2, 3-D; \frac{D}{2}, \frac{D}{2}, \frac{D}{2}; x_1, x_2, x_3
\end{array} \right)
+ \frac{\Gamma(3-D)\Gamma(4-3D/2)}{\Gamma^3(2-D/2)} F_{3}^{(3)} \left( \begin{array}{c}
3-D, 4-3D/2; \frac{D}{2}, \frac{D}{2}, \frac{D}{2}; x_1, x_2, x_3
\end{array} \right). \tag{12}
\]

Here \( F_{3}^{(3)} \) is the Lauricella function of three independent variables

\[
F_{3}^{(3)} \left( \begin{array}{c}
a, b; \frac{D}{2}, \frac{D}{2}, \frac{D}{2}; x_1, x_2, x_3
\end{array} \right) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \frac{(a)_{n_1+n_2+n_3} (b)_{n_1+n_2+n_3}}{n_1! n_2! n_3! (c_1)_{n_1} (c_2)_{n_2} (c_3)_{n_3}} x_1^{n_1} x_2^{n_2} x_3^{n_3}, \tag{13}
\]

with the connected convergent region \( \sqrt{|x_1|} + \sqrt{|x_2|} + \sqrt{|x_3|} \leq 1 \).

For the case \( m_3 > \max(m_1, m_2, m_3, m_4) \), one similarly derives

\[
U_3(m_1^2, m_2^2, m_3^2, m_4^2) = \frac{-i (m_4^2)^{3D/2-4} (\mu^2)^{6-3D/2}}{\Gamma(D/2)(4\pi)^{3D/2}} \Gamma^3 \left( \frac{D}{2} - 1 \right) \Gamma^3 \left( \frac{D}{2} - D \right) \phi(y_1, y_2, y_3), \tag{14}
\]
with \( y_1 = \frac{m_2}{m_3} = \frac{x_1}{x_3}, \ y_2 = \frac{m_2}{m_3} = \frac{x_2}{x_3}, \ y_3 = \frac{m_2}{m_3} = \frac{1}{x_3} \). We specify here that \( \phi(y_1, y_2, y_3) = (x_3)^{4-3D/2} \phi(x_1, x_2, x_3) \). For the case \( m_2 > \max(m_1, m_3, m_4) \) and \( m_1 > \max(m_2, m_3, m_4) \), we can get the similar results.

In other words, when \( D = 4 - 2\varepsilon \), the analytic expression of the \( U_3 \) function can be formulated as

\[
U_3(m_1^2, m_2^2, m_3^2, m_4^2) = -\frac{i\Gamma^3(1-\varepsilon)\Gamma^3(\varepsilon)}{\Gamma(2-\varepsilon)(4\pi)^4} \left( \frac{m_2^2}{4\pi} \right)^{2-3\varepsilon} (\mu^2)^{3\varepsilon} \Phi_U(x_1, x_2, x_3),
\]

where

\[
\Phi_U(x_1, x_2, x_3) = \begin{cases}
\phi(x_1, x_2, x_3), & \sqrt{|x_1|} + \sqrt{|x_2|} + \sqrt{|x_3|} \\ (x_3)^{D/2-4}\phi(\frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{1}{x_3}), & 1 + \sqrt{|x_1|} + \sqrt{|x_2|} \leq \sqrt{|x_3|}, \\
(x_2)^{D/2-4}\phi(\frac{x_1}{x_2}, \frac{x_3}{x_2}, \frac{1}{x_2}), & 1 + \sqrt{|x_1|} + \sqrt{|x_3|} \leq \sqrt{|x_2|}, \\
(x_1)^{D/2-4}\phi(\frac{x_1}{x_1}, \frac{x_2}{x_1}, \frac{1}{x_1}), & 1 + \sqrt{|x_2|} + \sqrt{|x_3|} \leq \sqrt{|x_1|}.
\end{cases}
\]

Here, \( \Phi_U(x_1, x_2, x_3) \) satisfies the system of homogeneous linear PDEs \cite{44}

\[
\left[ \left( \sum_{i=1}^{3} \hat{\partial}_{x_i} + 3 - D \right) \left( \sum_{i=1}^{3} \hat{\partial}_{x_i} + 4 - \frac{3D}{2} \right) - \frac{1}{x_i} \hat{\partial}_{x_i} (\hat{\partial}_{x_i} + 1 - \frac{D}{2}) \right] \Phi_U(x_1, x_2, x_3) = 0,
\]

with \( \hat{\partial}_{x_i} = x_i \partial / \partial x_i, \ i = 1, 2, 3 \).

The \( \Phi_U \) function under the restriction \( x_2 = x_3 = 0 \) is given as

\[
\Phi_U(x_1, 0, 0) = F(x_1) = \begin{cases}
\phi(x_1, 0, 0), & |x_1| \leq 1, \\
(x_1)^{D/2-4}\phi(\frac{1}{x_1}, 0, 0), & |x_1| \geq 1.
\end{cases}
\]

And one derives \( \phi(x_1, 0, 0) = (x_1)^{D/2-4}\phi(\frac{1}{x_1}, 0, 0) \). In the whole \( x_1 \)-coordinate axis, \( F(x_1) \) is a continuous differentiable function. And \( F(x_1) \) satisfies the first PDE with the condition \( x_2 = x_3 = 0 \) in Eq.\( \text{(17)} \). In the similarly way, one can get the analytic expressions for \( F(x_2), F(x_3) \) in the whole \( x_2, x_3 \)-coordinate axis. Taking the \( \Phi_U(x_1, 0, 0) = F(x_1) \) and \( \Phi_U(0, x_2, 0) = F(x_2) \) as boundary conditions, we can get the numerical solutions of \( \Phi_U \) on the entire \( x_1-x_2 \) plane by the first two PDEs when \( x_3 = 0 \). Using the similar method, the continuation of \( \Phi_U \) to whole three dimension space can be finished through the system of PDEs in Eq.\( \text{(17)} \).

We give the Laurent series of \( \Phi_U \) function around space-time dimensions \( D = 4 \) in order to make the numerical continuation of \( \Phi_U \) to whole kinematic regions,

\[
\Phi_U(x_1, x_2, x_3) = \frac{\phi_U^{(-3)}(x_1, x_2, x_3)}{\varepsilon^3} + \frac{\phi_U^{(-2)}(x_1, x_2, x_3)}{\varepsilon^2} + \frac{\phi_U^{(-1)}(x_1, x_2, x_3)}{\varepsilon} + \sum_{i=0}^{\infty} \varepsilon^i \phi_U^{(i)}(x_1, x_2, x_3).
\]
The systems of linear PDEs in appendix A are derived, which satisfy the functions \( \phi_{U}^{-3}, \phi_{U}^{-2} \) and \( \phi_{U}^{(i)}(i = -1, 0, 1, 2, \cdots) \).

Through the systems of PDEs in appendix A, the continuation of the numerical solution of the triple hypergeometric series can be made to whole kinematic domain. One derives \( \phi_{U}^{-3} = (x_1 + x_2 + x_3)/2 \) satisfies Eq. (A1) explicitly. After obtaining the solutions \( \phi_{U}^{-2} \) and \( \phi_{U}^{-1} \), one writes \( F = x_1^{-1/2}x_2^{-1/2}x_3^{-1/2}\phi_{U}^{(n)} \) satisfies the system of linear PDEs

\[
2x_1 \frac{\partial^2 F}{\partial x_1^2} - x_2 \frac{\partial^2 F}{\partial x_2^2} - x_3 \frac{\partial^2 F}{\partial x_3^2} + 2 \frac{\partial F}{\partial x_1} - \frac{\partial F}{\partial x_2} - \frac{\partial F}{\partial x_3} + \left( - \frac{1}{2x_1} + \frac{1}{4x_2} + \frac{1}{4x_3} \right) F = \phi_{U}^{(n-1)} - 1 \frac{\partial F}{\partial x_1} + \frac{1}{4x_3} \, \phi_{U}^{(n-2)} - x_1^{-1/2}x_2^{-1/2}x_3^{-1/2}(g_1 - g_2 - g_3) = 0 ,
\]

\[
\frac{\partial^2 F}{\partial x_2^2} + \frac{\partial^2 F}{\partial x_3^2} + \frac{\partial F}{\partial x_2} + \frac{\partial F}{\partial x_3} + \left( - \frac{1}{2x_1} + \frac{1}{4x_2} + \frac{1}{4x_3} \right) F - x_1^{-1/2}x_2^{-1/2}x_3^{-1/2}(g_1 - g_2 - g_3) = 0 ,
\]

\[
\frac{\partial^2 F}{\partial x_1^2} + x_1^2 \frac{\partial^2 F}{\partial x_2^2} + x_1 \frac{\partial}{\partial x_1} \frac{\partial^2 F}{\partial x_2^2} + x_1 \frac{\partial}{\partial x_1} \frac{\partial^2 F}{\partial x_3^2} + x_2 \frac{\partial}{\partial x_2} \frac{\partial^2 F}{\partial x_3^2} + x_1 \frac{\partial}{\partial x_1} \frac{\partial^2 F}{\partial x_3^2}
\]

\[
+ x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + (x_1 - 1) \frac{\partial F}{\partial x_3} + \left( \frac{1}{4} - \frac{1}{4x_3} \right) F - x_1^{-1/2}x_2^{-1/2}x_3^{-1/2}g_3 = 0 ,
\]

and

\[
g_1(x_1, x_2, x_3) = -(1 - 5x_1) \frac{\partial \phi_{U}^{(n-1)}}{\partial x_1} + 5x_2 \frac{\partial \phi_{U}^{(n-1)}}{\partial x_2} + 5x_3 \frac{\partial \phi_{U}^{(n-1)}}{\partial x_3} - 7\phi_{U}^{(n-1)} + 6\phi_{U}^{(n-2)} ,
\]

\[
g_2(x_1, x_2, x_3) = 5x_1 \frac{\partial \phi_{U}^{(n-1)}}{\partial x_1} - (1 - 5x_2) \frac{\partial \phi_{U}^{(n-1)}}{\partial x_2} + 5x_3 \frac{\partial \phi_{U}^{(n-1)}}{\partial x_3} - 7\phi_{U}^{(n-1)} + 6\phi_{U}^{(n-2)} ,
\]

\[
g_3(x_1, x_2, x_3) = 5x_1 \frac{\partial \phi_{U}^{(n-1)}}{\partial x_1} + 5x_2 \frac{\partial \phi_{U}^{(n-1)}}{\partial x_2} - (1 - 5x_3) \frac{\partial \phi_{U}^{(n-1)}}{\partial x_3} - 7\phi_{U}^{(n-1)} + 6\phi_{U}^{(n-2)} .
\]

The second and third PDEs in Eq. (20) are recognized as the constraints of the function \( F(x_1, x_2, x_3) \). The system of PDEs in Eq. (20) is recognized as the modified functional based on the constraint variational principle.

\[
\Pi_{U}^* (F) = \Pi_{U} (F)
\]

\[
+ \int_{\Omega} \left\{ x_1 \frac{\partial^2 F}{\partial x_1^2} - x_2 \frac{\partial^2 F}{\partial x_2^2} - x_3 \frac{\partial^2 F}{\partial x_3^2} + 2 \frac{\partial F}{\partial x_1} - \frac{\partial F}{\partial x_2} - \frac{\partial F}{\partial x_3} + \left( - \frac{1}{2x_1} + \frac{1}{4x_2} + \frac{1}{4x_3} \right) F - x_1^{-1/2}x_2^{-1/2}x_3^{-1/2}(g_1 - g_2 - g_3) \right\} dx_1 dx_2 dx_3
\]

\[
+ \int_{\Omega} \left\{ x_1 \frac{\partial^2 F}{\partial x_1^2} + x_2 \frac{\partial^2 F}{\partial x_2^2} + x_3 \frac{\partial}{\partial x_3} \frac{\partial^2 F}{\partial x_2^2} + x_1 \frac{\partial}{\partial x_1} \frac{\partial^2 F}{\partial x_3^2} + x_2 \frac{\partial}{\partial x_2} \frac{\partial^2 F}{\partial x_3^2}
\]

\[
+ 2x_1 \frac{\partial F}{\partial x_1} + 2x_2 \frac{\partial F}{\partial x_2} + x_3 \frac{\partial F}{\partial x_3} + (x_1 - 1) \frac{\partial F}{\partial x_3} + \left( \frac{1}{4} - \frac{1}{4x_3} \right) F - x_1^{-1/2}x_2^{-1/2}x_3^{-1/2}g_3 \right\} dx_1 dx_2 dx_3 ,
\]
where $\chi_{23}(x_2, x_3)$, $\chi_{123}(x_1, x_2, x_3)$ are Lagrange multipliers. $\Omega$ represents the kinematic domain of numerical solution can be made, and $\Pi_u(F)$ is the functional of the first PDE in Eq. (20):

$$
\Pi_u(F) = \int_\Omega \left\{ x_1 \left( \frac{\partial F}{\partial x_1} \right)^2 - \frac{x_2}{2} \left( \frac{\partial F}{\partial x_2} \right)^2 - \frac{x_3}{2} \left( \frac{\partial F}{\partial x_3} \right)^2 - [ - \frac{1}{4x_1} + \frac{1}{8x_2} + \frac{1}{8x_3} ] F^2 + x_1^{-1/2} x_2^{-1/2} x_3^{-1/2} \left( 2g_1 - g_2 - g_3 \right) F \right\} dx_1 dx_2 dx_3.
$$

Firstly, the corresponding function expression of one coordinate axis is taken as the boundary condition, the numerical solution of the planes can be obtained. Then the whole kinematic region can get the numerical solution by the finite element method. In the following, we will discuss the special case of the three-loop vacuum integral FIG.1.

IV. SPECIAL CASE

Now, we will deal with the situations different from above.

A. special case one: $m_1 = m_2 = m, m_3 \neq 0, m_4 \neq 0$

When a variable is equal to one in Eq. (16), the analytic expression of the $U_j$ function can be simplified. We adopt the following formulae:

$$
_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \right| 1 \right) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c-a)\Gamma(c-b)},
$$

if $Rl(c-a-b) > 0$. With the help of Eq. (24), the $F_c^{(3)}$ function can turn to be

$$
F_c^{(3)} \left( \begin{array}{c} a, b; \\ c_1, c_2, c_3 \end{array} \right| 1, x_2, x_3 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F^{4:0}_{2:1} \left( \begin{array}{c} a, b, 1+a-c_1, 1+b-c_1; \\ \frac{1+a+b-c_1}{2}, \frac{2+a+b-c_1}{2}, c_2, c_3; \frac{x_2}{4}, \frac{x_3}{4} \end{array} \right)
$$

and $F^{4:0}_{2:1}$ is the Kampé de Fériet function.
By the Eq.\((25)\) and after some simplification, we get the results

\[
U_3(m^2, m^2, m^2, m^2) = \frac{(-i)(m^2)^{3D/2-4}(\mu^2)^{6-3D/2}}{(4\pi)^{3D/2}\Gamma(D/2)} \omega\left(\frac{x}{4}, \frac{y}{4}\right),
\tag{27}
\]

and the \(\omega(x, y)\) is given by

\[
\omega\left(\frac{x}{4}, \frac{y}{4}\right) = \frac{\Gamma(4 - 3D/2)\Gamma^2(3 - D)\Gamma(2 - D/2)\Gamma^2(D/2 - 1)}{\Gamma(6 - 2D)}
\times F^{3:0}_{1:1}\left(\begin{array}{ccc}
3 - D, & 2 - D/2, & 1; \\
4 - 3D/2, & 3 - D, & 2 - D/2;
\end{array}; \frac{x}{4}, \frac{y}{4}\right) + \frac{\Gamma(3 - D)\Gamma^2(2 - D/2)\Gamma(D/2 - 1)\Gamma(1 - D/2)}{\Gamma(4 - D)}
\times F^{3:0}_{1:1}\left(\begin{array}{ccc}
2 - D/2, & 1, & D/2; \\
3/2, & D/2, & D/2;
\end{array}; \frac{x}{4}, \frac{y}{4}\right),
\tag{28}
\]

with \(x = m^2_3/m^2, \ y = m^2_4/m^2\). And the \(F^{3:0}_{1:1}\) convergent region is \(|\frac{x}{4}| + |\frac{y}{4}| \leq 1\). If we use the method of Mellin-Barnes representation on Eq.\((2)\), we can get the same result as Eq.\((28)\). And the function \(\omega(x, y) = \omega(\frac{1}{4}, \frac{1}{4})\) when \(m_1 = m_2 = m_3 = m_4 = m\).

**B. special case two:** \(m_1 = 0, m_2 \neq 0, m_3 \neq 0, m_4 \neq 0\)

We discuss the another special case as \(m_1 = 0\). Assuming \(m_4 > max(m_2, m_3)\), the \(U_3\) function can be formulated as

\[
U_3(0, m^2_2, m^2_3, m^2_4) = \frac{(-i)2^3(m^2_2m^2_3m^2_4)^{D/2-1}(\mu^2)^{6-3D/2}}{(D/2 - 1)(4\pi)^{3D/2}} \int_0^\infty dx\left(\frac{x}{2}\right) k_{D/2-1}(m_2x)k_{D/2-1}(m_3x)k_{D/2-1}(m_4x)
\]

\[
= \frac{(-i)(m^2_2)^{3D/2-4}(\mu^2)^{6-3D/2}}{(D/2 - 1)(4\pi)^{3D/2}}\Gamma^2(D/2 - 1)\Gamma^2(2 - D/2) \psi(x_1, x_2),
\tag{29}
\]
with \( x_1 = m_2^2/m_4^2, \ x_2 = m_3^2/m_4^2 \). Meanwhile the double hypergeometric series \( \psi(x_1, x_2) \) is

\[
\psi(x_1, x_2) = \frac{\Gamma(2 - D/2)}{\Gamma(D/2)^2} F_4 \left( \begin{array}{c} 1, 2 - D/2 \\ D/2, D/2 \end{array} \middle| x_1, x_2 \right)
\]

\[
- \frac{\Gamma(3 - D)}{\Gamma(D/2)x_1^{D/2 - 1}} F_4 \left( \begin{array}{c} 2 - D/2, 3 - D \\ D/2, 2 - D/2 \end{array} \middle| x_1, x_2 \right)
\]

\[
- \frac{\Gamma(3 - D)}{\Gamma(D/2)x_2^{D/2 - 1}} F_4 \left( \begin{array}{c} 2 - D/2, 3 - D \\ 2 - D/2, D/2 \end{array} \middle| x_1, x_2 \right)
\]

\[
+ \frac{\Gamma(3 - D)\Gamma(4 - 3D/2)}{\Gamma(2 - D/2)\Gamma(2 - D/2)} F_4 \left( \begin{array}{c} 3 - D, 4 - 3D/2 \\ 2 - D/2, 2 - D/2 \end{array} \middle| x_1, x_2 \right),
\]

one can get the same result by the method of Mellin-Barnes representation. And \( F_4 \) is the Apell function

\[
F_4 \left( \begin{array}{c} a, b \\ c_1, c_2 \end{array} \middle| x_1, x_2 \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{m!n!(c_1)_{m}(c_2)_{n}} x_1^{m} x_2^{n}, \]

whose convergent region is \( \sqrt{|x_1|} + \sqrt{|x_2|} < 1 \). For the case \( m_3 > \max(m_2, m_4) \), one similarly derives

\[
U_3(0, m_2^2, m_3^2, m_4^2) = \frac{(-1)(m_3^2)^{3D/2-4}(\mu^2)^{6-3D/2}}{(D/2 - 1)(4\pi)^{3D/2}} \Gamma^2(D/2 - 1) \Gamma^2(2 - D/2) \psi(y_1, y_2),
\]

with \( y_1 = m_2^2/m_3^2 = x_1/x_2, \ y_2 = m_4^2/m_3^2 = 1/x_2, \) and \( \psi(y_1, y_2) = (x_2)^{4-3D/2}\psi(x_1, x_2). \) In other words, the analytic expression of the \( U_3 \) function can be formulated as

\[
U_3(0, m_2^2, m_3^2, m_4^2) = \frac{(-1)(m_3^2)^{3D/2-4}(\mu^2)^{6-3D/2}}{(D/2 - 1)(4\pi)^{3D/2}} \Gamma^2(D/2 - 1) \Gamma^2(2 - D/2) \psi_U(x_1, x_2),
\]

where

\[
\psi_U(x_1, x_2) = \begin{cases} \psi(x_1, x_2), & \sqrt{|x_1|} + \sqrt{|x_2|} < 1, \\ (x_2)^{3D/2-4}\psi(x_2, x_1), & 1 + \sqrt{|x_1|} \leq \sqrt{|x_2|}, \\ (x_1)^{3D/2-4}\psi(x_1, x_2), & 1 + \sqrt{|x_2|} \leq \sqrt{|x_1|}. \end{cases}
\]

Correspondingly the double hypergeometric series \( \Psi_\nu(x_1, x_2) \) satisfies the system of homogeneous linear PDEs

\[
\left\{ (\partial_{x_1} + \partial_{x_2} + 3 - D)(\partial_{x_1} + \partial_{x_2} + 4 - 3D/2) - \frac{1}{x_1} \partial_{x_1}(\partial_{x_1} + 1 - D/2) \right\} \Psi_U = 0,
\]
with \( \hat{\theta}_i = x_i \partial / \partial x_i \), \( i = 1, 2 \).

The \( \Psi_U \) function under the restriction \( x_2 = 0 \) is

\[
\Psi_{U}(x_1, 0) = G(x_1) = \begin{cases} 
\psi(x_1, 0), & |x_1| \leq 1 \\
(x_1)^{3D/2-4}\psi(1/x_1', 0), & |x_1| \geq 1 
\end{cases}.
\]

(36)

One can get the analytic expressions \( G(x_1) \) in the whole \( x_1 \)-coordinate axis. And the \( G(x_1) \) function satisfies the first PDE with the restriction \( x_2 = 0 \) in Eq.(35). Similarly, \( \psi(0, x_2) = G(x_2) \) satisfies the second PDE with the restriction \( x_1 = 0 \) in Eq.(35).

Based on the constraint variational principle [46], the system of PDEs in Eq.(B1) can be finished numerically by its analytic expression on the whole \( x_i (i = 1, 2) \)-axis and the system of PDEs in Eq.(B3).

We give the Laurent series of \( \Psi_U \) function around space-time dimensions \( D = 4 \) in order to make the numerical continuation of \( \Psi_U \) to whole kinematic regions,

\[
\Psi_{U}(x_1, x_2) = \frac{\psi^{(-3)}(x_1, x_2)}{\varepsilon^2} + \frac{\psi^{(-2)}(x_1, x_2)}{\varepsilon} + \sum_{i=-1}^{\infty} \varepsilon^i \psi^{(i)}(x_1, x_2).
\]

(37)

The systems of linear PDEs in appendix B are derived, which satisfied by the functions \( \psi^{(-3)}_U, \psi^{(-2)}_U, \psi^{(-1)}_U \) and \( \psi^{(n)}_U \) (\( n = 0, 2, \cdot \cdot \cdot \)). The numerical continuation of the Apell function can be made by the systems of PDEs in appendix B. One derives \( \psi^{(-3)}_U = (x_1 + x_2)/2 \) which satisfies the system of PDEs in Eq.(B1) explicitly. After obtaining the solutions \( \psi^{(n-2)}_U, \psi^{(n-1)}_U \) in the whole \( x_1 - x_2 \) plane, one writes the system of linear PDEs satisfied by

\[
H = x_1^{-1/2}x_2^{-1/2}\psi^{(n)}_U
\]
as

\[
-x_1 \frac{\partial^2 H}{\partial x_1^2} + x_2 \frac{\partial^2 H}{\partial x_2^2} - x_2 \frac{\partial H}{\partial x_1} + x_1 \frac{\partial H}{\partial x_2} + (\frac{1}{4x_1} - \frac{1}{4x_2})H + x_1^{-1/2}x_2^{-1/2}(g_1 - g_2) = \frac{1}{4} \sum_{i=1}^{\infty} \frac{1}{\varepsilon^i} \psi^{(i)}(x_1, x_2).
\]

(38)

and

\[
g_1(x_1, x_2) = -(1 - 5x_1) \frac{\partial \psi^{(n-1)}_U}{\partial x_1} + 5x_2 \frac{\partial \psi^{(n-1)}_U}{\partial x_2} - 7\psi^{(n-1)}_U + 6\psi^{(n-2)}_U,
\]

\[
g_2(x_1, x_2) = 5x_1 \frac{\partial \psi^{(n-1)}_U}{\partial x_1} - (1 - 5x_2) \frac{\partial \psi^{(n-1)}_U}{\partial x_2} - 7\psi^{(n-1)}_U + 6\psi^{(n-2)}_U.
\]

(39)

Based on the constraint variational principle [46], the system of PDEs in Eq.(38) can be treated as the modified functional with stationary conditions

\[
\Pi^*_U(H) = \Pi_U(H)
\]
\[ + \int_{\Omega} \chi_{12} \left\{ x_1 (2x_1 - 1) \frac{\partial^2 H}{\partial x_1^2} + x_2 (2x_2 - 1) \frac{\partial^2 H}{\partial x_2^2} + 4x_1 x_2 \frac{\partial^2 H}{\partial x_1 \partial x_2} + (2x_1 - 1) \frac{\partial H}{\partial x_1} \right\} \, dx_1 dx_2 , \]

\[ +(2x_2 - 1) \frac{\partial H}{\partial x_2} + \left( \frac{1}{4x_1} + \frac{1}{4x_2} \right) H + x_1^{-1/2} x_2^{-1/2} (g_1 + g_2) \right\} \, dx_1 dx_2 , \]

(40)

where \(\chi_{12}(x_1, x_2)\) are Lagrange multipliers, \(\Omega\) represents the kinematic domain where the numerical continuation is made, and \(\Pi_\nu(F)\) is the functional of the first PDE in Eq. (48):

\[ \Pi_\nu(H) = \int_{\Omega} \left\{ \left( \frac{x_1}{2} \frac{\partial H}{\partial x_1} \right)^2 - \frac{x_2}{2} \left( \frac{\partial H}{\partial x_2} \right)^2 + \frac{1}{8x_1} - \frac{1}{8x_2} \right\} H^2 + x_1^{-1/2} x_2^{-1/2} (g_1 - g_2) H \, dx_1 dx_2 . \]

(41)

Because of the boundary conditions \(\Psi_\nu(x_1, 0) = G(x_1)\), one can perform the continuation of the solution to whole kinematic region numerically through finite element method [46] from Eq. (40).

The function \(\psi(y_1, y_2)\) will turn to be \(\psi(1, y_2)\) as \(m_2 = m_3 = m\). In order to get the simper result, we need the reduction formulae [49]

\[ F_4 \left( \begin{array}{c|c|c|c} a, b & 1, y \\ c, d \end{array} \right) = \frac{\Gamma(c-a-b) \Gamma(c-a) \Gamma(c-b) \Gamma(a+b-c) \Gamma(1+a-c) \Gamma(1+b-c)}{\Gamma(a)} F_3 \left( \begin{array}{c|c|c|c|c} a, b, 1+a-c, 1+b-c & \frac{y}{4} \\ d, (a+b-c+2)/2, (a+b-c+1)/2 \end{array} \right) . \]

(42)

Using the above reduction formula Eq. (42), we obtain

\[ U_\alpha(0, m^2, m^2, m_4^2) \]

\[ = \frac{(-1)(\mu^2)^{3D/2-4}(\mu^2)^{6-3D/2}}{(4\pi)^{3D/2} \Gamma(D/2-1)} f\left( \frac{y_2}{4} \right) , \]

(43)

with \(y_2 = m_i^2/m^2\), and \(f\left( \frac{y_2}{4} \right)\) is as follows:

\[ f\left( \frac{y_2}{4} \right) = \frac{\Gamma(4-3D/2) \Gamma(3-D) \Gamma(2-D/2) \Gamma(2-D/2-1)}{\Gamma(6-2D)} \times F_2 \left( \begin{array}{c|c|c|c} 4 - 3D/2, 3 - D, 2 - D/2 & \frac{y_2}{4} \\ 7/2 - D, 2 - D/2 \end{array} \right) \]

\[ + \frac{\Gamma(3-D) \Gamma(2-D/2) \Gamma(D/2-1) \Gamma(1-D/2)}{\Gamma(4-D)} \times F_2 \left( \begin{array}{c|c|c|c} 1, 3 - D, 2 - D/2 & \frac{y_2}{4} \\ 5/2 - D/2, D/2 \end{array} \right) . \]

(44)

If \(x = 0\) in Eq. (28), then the result is consistent with Eq. (44). And the Eq. (44) is in agreement with Eq. (1.28) of Ref. [50].
C. special case three: \( m_1 = m_2 = 0, m_3 \neq 0, m_4 \neq 0 \)

When \( m_1 = m_2 = 0 \), the \( U_3 \) function can be formulated as

\[
U_3(0, 0, m_1^2, m_4^2) = \frac{(-i)2^2(m_2^2m_4^2)^{D/2-1}(\mu^2)^{6-3D/2}\Gamma^2(D/2-1)\int_0^\infty dx x^{3-D}k_{D/2-1}(m_3)xk_{D/2-1}(m_4)x}{\Gamma(D/2)(4\pi)^{3D/2}}
\]

\[
= \frac{(-i)(m_2^2m_4^2)^{3D/2-4}(\mu^2)^{6-3D/2}\Gamma^3(D/2-1)\Gamma(2-D/2)}{\Gamma(D/2)(4\pi)^{3D/2}}
\]

\[
\times \left\{ -\frac{\Gamma(2-D/2)\Gamma(3-D)}{\Gamma(D/2)} x^{D/2-1} 2\_\text{F}_1 \left( \begin{array}{c} 2-D/2, 3-D \neg x^{D/2} \end{array} \right) 
+ \frac{\Gamma(3-D)\Gamma(4-3D/2)}{\Gamma(2-D/2)} 2\_\text{F}_1 \left( \begin{array}{c} 3-D, 4-3D/2 \neg x^{2-D/2} \end{array} \right) \right\},
\]

where \( x = m_1^2/m_4^2 \). At that case, the result from the method of Mellin-Barnes representation coincided with the hypergeometric function method. And \( 2\_\text{F}_1 \) is the hypergeometry function

\[
2\_\text{F}_1 \left( \begin{array}{c} a, b \neg c \neg x \end{array} \right) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{n!(c)_n} x^n,
\]

whose convergent region is \( |x| \leq 1 \), \( (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \). Using the properties of hypergeometric functions, the analytical results can be extended from \( |x| \leq 1 \) to the domain of \( |x| > 1 \).

When \( m_1 = m_2 = 0 \) and \( m_3 = m_4 = m \), with the help of the formulae (24), Eq.(45)

\[
U_3(0, 0, m^2, m^2) = (-i)\frac{(m^2)^2}{(4\pi)^6} 1\_\varepsilon\varphi_3^{(-3)} + 1\_\varepsilon^2\varphi_3^{(-2)} + 1\_\varepsilon^3\varphi_3^{(-1)} + \varphi_3^{(0)} + \mathcal{O}(\varepsilon) \]

where \( \varphi_3^{(-3)}, \varphi_3^{(-2)}, \varphi_3^{(-1)}, \varphi_3^{(0)} \) are written as

\[
\varphi_3^{(-3)} = \frac{1}{3}, \\
\varphi_3^{(-2)} = \frac{1}{6}(7 - 6\gamma_E + 6\ln\frac{4\pi\mu^2}{m^2}) \\
\varphi_3^{(-1)} = \frac{1}{12}(25 - 42\gamma_E + 18\gamma_E^2 + \pi^2 + 42\ln\frac{4\pi\mu^2}{m^2} - 36\gamma_E\ln\frac{4\pi\mu^2}{m^2} + 18\ln^2\frac{4\pi\mu^2}{m^2}) \\
\varphi_3^{(0)} = \frac{1}{24}(-5 - 150\gamma_E + 126\gamma_E^2 - 36\gamma_E^3 + 7\pi^2 - 6\gamma_E\pi^2 + 6\ln\frac{4\pi\mu^2}{m^2}(25 - 42\gamma_E + 18\gamma_E^2 + \pi^2) \\
- 18\ln^2\frac{4\pi\mu^2}{m^2}(-7 + 6\gamma_E) + 36\ln^3\frac{4\pi\mu^2}{m^2} + 56\zeta(3)).
\]

When \( m_1 = m_2 = m_3 = 0; m_4 = m \), the \( U_3 \) function can be formulated as

\[
U_3(0, 0, 0, m^2) = (-i)\frac{(m^2)^2}{(4\pi)^6} 1\_\varepsilon\varphi_4^{(-2)} + 1\_\varepsilon^2\varphi_4^{(-1)} + \varphi_4^{(0)} + \mathcal{O}(\varepsilon) \]

(49)
where $\varphi_4^{(-2)}$, $\varphi_4^{(-1)}$, $\varphi_4^{(0)}$ are written as

$$\varphi_4^{(-2)} = -\frac{1}{12},$$
$$\varphi_4^{(-1)} = \frac{1}{8}(-5 + 2\gamma_E - 2\ln \frac{4\pi \mu^2}{m^2}),$$
$$\varphi_4^{(0)} = \frac{1}{48}(-145 + 90\gamma_E - 18\gamma_E^2 - 5\pi^2 - 90\ln \frac{4\pi \mu^2}{m^2} + 36\gamma_E \ln \frac{4\pi \mu^2}{m^2} - 18\ln^2 \frac{4\pi \mu^2}{m^2}).$$

(50)

the result of the Eq.(48) and Eq.(50) are consistent with the result of Ref. [38].

V. SUMMARY

Using the integral representations for modified Bessel functions, we verify the equivalency of Feynman parametrization and the hypergeometric technique to calculate the scalar integrals of Feynman diagrams in this work. For the three-loop vacuum integrals, we have presented only diagram FIG.1 to elucidate the technique in detail. For scalar integrals of diagram FIG.1, the multiple series of representations which are convergent in certain connected regions of kinematic invariants can be derived. The systems of linear homogeneous PDEs can be established by the scalar integrals in whole kinematic domain. The continuation of the analytic representations of scalar integrals from the convergent regions to whole kinematic domain through numerical methods can be performed when recognizing the system of linear PDEs as stationary conditions. For this purpose, the finite element method can be applied. For the special case of the three-loop vacuum diagram FIG.1, we derives the analytic result in the convergence domain. We will apply this technique to numerically evaluate the scalar integrals from multi-loop diagrams elsewhere in the near future.

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Appendix A: The system of linear PDEs for Laurent expansion around $D = 4$

Correspondingly we present the systems of linear PDEs satisfied by $\phi_\nu^{(-3)}$, $\phi_\nu^{(-2)}$ and $\phi_\nu^{(n)}$ ($n = -1, 0, 1, 2, \ldots$):

\[
\begin{align*}
\{( \sum_{i=1}^{3} \hat{\theta}_{x_i} - 1)(\sum_{i=1}^{3} \hat{\theta}_{x_i} - 2) - \frac{1}{x_1} \hat{\theta}_{x_1} (\hat{\theta}_{x_1} - 1)\} \phi_\nu^{(-3)} &= 0 , \\
\{( \sum_{i=1}^{3} \hat{\theta}_{x_i} - 1)(\sum_{i=1}^{3} \hat{\theta}_{x_i} - 2) - \frac{1}{x_2} \hat{\theta}_{x_2} (\hat{\theta}_{x_2} - 1)\} \phi_\nu^{(-3)} &= 0 , \\
\{( \sum_{i=1}^{3} \hat{\theta}_{x_i} - 1)(\sum_{i=1}^{3} \hat{\theta}_{x_i} - 2) - \frac{1}{x_3} \hat{\theta}_{x_3} (\hat{\theta}_{x_3} - 1)\} \phi_\nu^{(-3)} &= 0 , \\
\end{align*}
\]
\[
\begin{align*}
\{( \sum_{i=1}^{3} \hat{\theta}_{x_i} - 1)(\sum_{i=1}^{3} \hat{\theta}_{x_i} - 2) - \frac{1}{x_1} \hat{\theta}_{x_1} (\hat{\theta}_{x_1} - 1)\} \phi_\nu^{(-2)} \\
-\left( \frac{1}{x_1} \hat{\theta}_{x_1} - 5 \sum_{i=1}^{3} \hat{\theta}_{x_i} + 7 \right) \phi_\nu^{(-3)} &= 0 , \\
\{( \sum_{i=1}^{3} \hat{\theta}_{x_i} - 1)(\sum_{i=1}^{3} \hat{\theta}_{x_i} - 2) - \frac{1}{x_2} \hat{\theta}_{x_2} (\hat{\theta}_{x_2} - 1)\} \phi_\nu^{(-2)} \\
-\left( \frac{1}{x_2} \hat{\theta}_{x_2} - 5 \sum_{i=1}^{3} \hat{\theta}_{x_i} + 7 \right) \phi_\nu^{(-3)} &= 0 , \\
\{( \sum_{i=1}^{3} \hat{\theta}_{x_i} - 1)(\sum_{i=1}^{3} \hat{\theta}_{x_i} - 2) - \frac{1}{x_3} \hat{\theta}_{x_3} (\hat{\theta}_{x_3} - 1)\} \phi_\nu^{(-2)} \\
-\left( \frac{1}{x_3} \hat{\theta}_{x_3} - 5 \sum_{i=1}^{3} \hat{\theta}_{x_i} + 7 \right) \phi_\nu^{(-3)} &= 0 ,
\end{align*}
\]
\[
\begin{align*}
\{( \sum_{i=1}^{3} \hat{\theta}_{x_i} - 1)(\sum_{i=1}^{3} \hat{\theta}_{x_i} - 2) - \frac{1}{x_1} \hat{\theta}_{x_1} (\hat{\theta}_{x_1} - 1)\} \phi_\nu^{(n)} \\
-\left( \frac{1}{x_1} \hat{\theta}_{x_1} - 5 \sum_{i=1}^{3} \hat{\theta}_{x_i} + 7 \right) \phi_\nu^{(n-1)} + 6 \phi_\nu^{(n-2)} &= 0 , \\
\{( \sum_{i=1}^{3} \hat{\theta}_{x_i} - 1)(\sum_{i=1}^{3} \hat{\theta}_{x_i} - 2) - \frac{1}{x_2} \hat{\theta}_{x_2} (\hat{\theta}_{x_2} - 1)\} \phi_\nu^{(n)} \\
-\left( \frac{1}{x_2} \hat{\theta}_{x_2} - 5 \sum_{i=1}^{3} \hat{\theta}_{x_i} + 7 \right) \phi_\nu^{(n-1)} + 6 \phi_\nu^{(n-2)} &= 0 , \\
\{( \sum_{i=1}^{3} \hat{\theta}_{x_i} - 1)(\sum_{i=1}^{3} \hat{\theta}_{x_i} - 2) - \frac{1}{x_3} \hat{\theta}_{x_3} (\hat{\theta}_{x_3} - 1)\} \phi_\nu^{(n)} \\
-\left( \frac{1}{x_3} \hat{\theta}_{x_3} - 5 \sum_{i=1}^{3} \hat{\theta}_{x_i} + 7 \right) \phi_\nu^{(n-1)} + 6 \phi_\nu^{(n-2)} &= 0 ,
\end{align*}
\]
\[
\begin{align*}
\cdots \cdots \cdots \cdots ,
\end{align*}
\]
\[
\begin{align*}
\{( \sum_{i=1}^{3} \hat{\theta}_{x_i} - 1)(\sum_{i=1}^{3} \hat{\theta}_{x_i} - 2) - \frac{1}{x_1} \hat{\theta}_{x_1} (\hat{\theta}_{x_1} - 1)\} \phi_\nu^{(n)} \\
-\left( \frac{1}{x_1} \hat{\theta}_{x_1} - 5 \sum_{i=1}^{3} \hat{\theta}_{x_i} + 7 \right) \phi_\nu^{(n-1)} + 6 \phi_\nu^{(n-2)} &= 0 ,
\end{align*}
\]
\[
\begin{align*}
\cdots \cdots \cdots \cdots ,
\end{align*}
\]
Appendix B: The system of linear PDEs for Laurent expansion around $D = 4$

Correspondingly we present the systems of linear PDEs satisfied by $\psi^{(-3)}_U$, $\psi^{(-2)}_U$ and $\psi^{(n)}_U$ ($n = -1, 0, 1, 2, \cdots$):

\[
\begin{align*}
\{ &\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 1)(\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 2) - \frac{1}{x_1} \hat{\varphi}_{x_{1}} (\hat{\varphi}_{x_{1}} - 1) \} \psi^{(-3)}_U = 0 , \\
\{ &\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 1)(\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 2) - \frac{1}{x_2} \hat{\varphi}_{x_{2}} (\hat{\varphi}_{x_{2}} - 1) \} \psi^{(-3)}_U = 0 , \\
\end{align*}
\]

(B1)

\[
\begin{align*}
\{ &\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 1)(\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 2) - \frac{1}{x_1} \hat{\varphi}_{x_{1}} (\hat{\varphi}_{x_{1}} - 1) \} \psi^{(-2)}_U = 0 , \\
\{ &\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 1)(\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 2) - \frac{1}{x_2} \hat{\varphi}_{x_{2}} (\hat{\varphi}_{x_{2}} - 1) \} \psi^{(-2)}_U = 0 , \\
\end{align*}
\]

(B2)

\[
\begin{align*}
\{ &\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 1)(\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 2) - \frac{1}{x_1} \hat{\varphi}_{x_{1}} (\hat{\varphi}_{x_{1}} - 1) \} \psi^{(n)}_U = 0 , \\
\{ &\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 1)(\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 2) - \frac{1}{x_2} \hat{\varphi}_{x_{2}} (\hat{\varphi}_{x_{2}} - 1) \} \psi^{(n)}_U = 0 , \\
\end{align*}
\]

(B3)

\[
\begin{align*}
\{ &\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 1)(\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 2) - \frac{1}{x_1} \hat{\varphi}_{x_{1}} (\hat{\varphi}_{x_{1}} - 1) \} \psi^{(n-1)}_U + 6 \psi^{(n-2)}_U = 0 , \\
\{ &\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 1)(\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 2) - \frac{1}{x_2} \hat{\varphi}_{x_{2}} (\hat{\varphi}_{x_{2}} - 1) \} \psi^{(n)}_U = 0 , \\
\end{align*}
\]

(B3)

\[
\begin{align*}
\{ &\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 1)(\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 2) - \frac{1}{x_1} \hat{\varphi}_{x_{1}} (\hat{\varphi}_{x_{1}} - 1) \} \psi^{(n-1)}_U + 6 \psi^{(n-2)}_U = 0 , \\
\{ &\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 1)(\sum_{i=1}^{2} \hat{\varphi}_{x_{i}} - 2) - \frac{1}{x_2} \hat{\varphi}_{x_{2}} (\hat{\varphi}_{x_{2}} - 1) \} \psi^{(n)}_U = 0 , \\
\end{align*}
\]

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