We review some recent results concerning gauge theories in various dimensions. In particular, we discuss RG fixed points and “mirror” symmetry duality in 3d \( N = 4 \) supersymmetric gauge theories and a classification of non-trivial RG fixed points in 5d \( N = 1 \) supersymmetric theories.

1. Introduction

By using exact results which can be obtained for supersymmetric gauge theories, it has recently been seen that these theories exhibit a variety of interesting phenomena and phases. For example, non-trivial RG fixed points arise in many susy gauge theories. Also some exhibit duality, i.e. universality: different UV theories can flow to the same RG fixed point in the IR. There is also, in some examples, the phenomenon of composite gauge invariance: strong coupling dynamics can lead to new gauge fields via collective excitations. The classic examples of these phenomena, due to Seiberg, are \( N = 1 \) supersymmetric QCD for various numbers of flavors and colors. These theories have non-trivial RG fixed points for \( \frac{2}{3} N_c < N_f < 3 N_c \). There are dual \( SU(N_f - N_c) \) gauge theories which flow to the same RG fixed points. There is composite, IR free, \( SU(N_f - N_c) \) gauge invariance for \( N_c + 2 \leq N_f \leq \frac{3}{2} N_c \). Other examples can be found, for example, in [2] and references therein.

As an aside, we point out that the composite gauge invariance does not violate the theorems of Weinberg and Witten [3], which state: (1) Given a Lorentz-covariant, conserved \( J_\mu \), there can be no composite or elementary massless particles carrying its charge with spin \( j > \frac{1}{2} \); (2) If there is a Lorentz-covariant, conserved \( T_{\mu\nu} \), there can be no composite or elementary massless particles with \( j > 1 \). The point is that the new massless composite \( j = 1 \) gauge fields do not carry charge under any Lorentz-covariant conserved \( J_\mu \). The new gauge symmetry appears “out of thin air” [4].

There are still many open questions, for example: Properties of RG fixed points are not well understood in general. Is there a better way to describe fixed points than in terms of the original UV Lagrangians? What is the general statement of duality? What is the duality operator map? Is there a framework where duality is manifest? Is there a framework where duality is exact, even away from the fixed point?
There has also been recent interest in exploring gauge theories in other dimensions, both with the hope of gaining insight into these issues and because of powerful new connections between gauge theories and string theory.

Note that, in dimensions other than four, the gauge coupling is dimensionful:

$$d^d x \frac{1}{g^2} F_{\mu\nu}^2 \sim \frac{1}{g^2} L^{d-4},$$

so $g \sim L^{(d-4)/2}$.

Therefore, for $d < 4$, all gauge theories are asymptotically free in the UV and strongly coupled in the IR. For $d > 4$, all gauge theories (with finite $g$) are infrared free and have a “Landau-pole” singularity in the ultraviolet. They can only be regarded as the low energy limit of some other theory.

While these statements are very general, different theories do exhibit a wide variety of phenomena. For example many theories, both in $d < 4$ and in $d > 4$ have non-trivial RG fixed points. Non-trivial RG fixed points for $d > 4$ mean that these gauge theories really “exist.” The IR free theory with finite $g$ can be regarded as a perturbation away from the RG fixed point by $\frac{1}{g^2} F_{\mu\nu}^2$.

2. Composite Gauge Invariance in String Theory

In string theory, singular geometry or gauge fields can lead to composite gauge invariance or RG fixed points. There are two (related) basic ways in which non-perturbative enhanced gauge symmetry arises in string theory. One is via singular geometry or gauge fields associated with compactification. The other is via D-branes, which are collective excitations of various dimensions with supersymmetric gauge theories living in their world-volume.

For type II strings, the supersymmetric gauge theories living in the world-volume of D-branes have 16 super-charges. For example, $N$ coincident type IIB 5-branes have a 6d $\mathcal{N} = (1,1)$ supersymmetric theory living in their world-volume with gauge group $U(N)$. The 6d $\mathcal{N} = (1,1)$ gauge multiplet consists of a $\mathcal{N} = (1,0)$ gauge multiplet, which contains no scalars, and a $\mathcal{N} = (1,0)$ adjoint matter multiplet. The expectation values of the 4$N$ scalars in the Cartan of the adjoint matter multiplet, which break $U(N) \to U(1)^N$, correspond to the locations of the $N$ 5-branes in the four transverse directions. The separation of branes gives a geometric picture of the Higgs mechanism: the $W$ bosons arise from strings stretching between the branes; their mass is proportional to their length, which is the separation of the branes.

For type I or heterotic strings, the supersymmetric gauge theories living in the world-volume of D-branes have 8 super-charges. In particular, $N$ coincident type I 5-branes, which corresponds to $N$ point-like $SO(32)$ instantons, have a 6d $\mathcal{N} = (1,0)$ supersymmetric gauge theory living in their world-volume with gauge group $Sp(N)$, 16 matter hypermultiplets in the $2N$ fundamental, and a hypermultiplet in the antisymmetric $N(2N - 1)$. There are $N$ hypermultiplet flat directions associated with the matter in the antisymmetric representation, which break $Sp(N) \to Sp(1)^N$,
corresponding to the locations of the $N$ 5-branes in the four transverse directions. Again, the separation of the branes gives a geometric picture of the Higgs mechanism.

The other basic mechanism for composite gauge invariance in string theory is via branes which wrap surfaces in the internal compactification space. The wrapped branes can yield gauge fields in the remaining uncompactified dimensions, with masses proportional to the area of the surfaces which they wrap. When the surfaces degenerate, the gauge fields become massless, giving another geometric picture of un-Higgsing a gauge symmetry. Much as in $\mathbb{C}^2/\Gamma_G$, the new massless gauge fields give the quantum resolution of the classical singularity in the moduli space where the internal space degenerates.

The original example of this was in type IIA string theory compactified to 6d on a K3 surface $\mathbb{C}^2/\Gamma_G$. The local behavior of K3 near a singularity looks like an ALE (asymptotically locally Euclidean) space, which is a possibly blown-up version of the orbifold $\mathbb{C}^2/\Gamma_G$, where $\Gamma_G$ is a discrete subgroup of $SU(2)$ which acts on the $S^3$ surrounding a point in $\mathbb{C}^2$. The subscript $G$ on $\Gamma_G \subset SU(2)$ reflects the famous correspondence between the discrete subgroups of $SU(2)$, cyclic, dihedral, tetrahedral, octahedral, and icosahedral, and the ADE groups $G = A_r, D_r, E_6, 7, 8$. The ALE space has $r = \text{rank}(G)$ two-cycles and thus $3r$ real blowing up parameters.

The IIA string on the ALE space has an additional $r$ real parameters, corresponding to the integral of the NSNS two-form $B$ field over the $r$ two-cycles. All together, the $4r$ real scalars combine with $U(1)^r$ gauge fields, coming from reducing the IIA three-form potential on the $r$ two-cycles, and fermionic partners, to form a $\mathcal{N} = (1,1)$ supersymmetric $U(1)^r$ gauge theory in the remaining six dimensions. Classically, the $U(1)^r$ gauge theory has no charged matter. However, it was argued in $\mathbb{C}^2/\Gamma_G$ that two-branes wrapped around the $r$ two-cycles yield the $W$ bosons needed to extend $U(1)^r$ to $G$. For generic values of the $r$ hypermultiplet blowing up modes and $B$ fields $\mathbb{C}^2/\Gamma_G$, $G$ is broken to $U(1)^r$ by the Higgs mechanism – the two-cycles are non-degenerate and thus the $W$ bosons are massive – but, when the blowing up modes and $B$ fields vanish, the non-perturbative, composite $G$ gauge symmetry is restored. The composite gauge symmetry is necessary for the duality between IIA on K3 and the heterotic string on $T^4$, where the enhanced gauge symmetry $G$ is visible classically.

The above analysis can obviously be reduced to fewer dimensions by additional compactification. It can also be pushed up to 7d and 8d by compactification of M theory or F theory, respectively, on the K3. In all cases, it yields supersymmetric theories with 16 super-charges. See $\mathbb{C}^2/\Gamma_G$ for discussion of such theories in various $d$.

It is also possible to obtain via compactification a variety of gauge theories with 8 super-charges in 6d, 5d, or 4d (and fewer by further reduction) via compactification of F theory, M theory, or type IIA, respectively, on Calabi-Yau threefolds. In these cases, composite gauge invariance occurs when a complex surface shrinks to zero size along a complex curve in the Calabi-Yau. It is possible to get essentially any gauge group: $A_r, B_r, C_r, D_r, E_{6,7,8}, F_4$ and $G_2$ from various types of singularities $\mathbb{C}^2/\Gamma_G$. 
Finally, by combining compactifications with having branes in the transverse directions, it is possible to obtain a variety of interesting gauge theories living in the world-volume of the branes. The dynamics of the world-volume gauge theories “probes” that of space-time. For example, it was shown in 13 that a type II brane living at a point on an ALE space in the transverse directions has a supersymmetric gauge theory living in its world-volume whose Higgs branch is isomorphic to the ALE space itself, giving a physical realization of the “hyper-Kahler quotient” construction 14 of the ALE spaces. Moving the brane around in the transverse ALE space corresponds to changing hypermultiplet expectation values on the Higgs branch. The general connections between the dynamics of the world-volume gauge theory on the “probe” brane and that of space-time are:

1. Global symmetries on the brane ↔ gauge symmetries in space-time.

2. Moduli space of gauge theory on brane ↔ moduli space of the brane in spacetime, which is the space-time itself. Further, the metric of the corresponding moduli on the brane is the same (locally) as the space-time metric.

In this way, known results concerning either the world-volume gauge theories or the space-time string theories which they probe, can lead to new results about the other. This has lead to a better understanding of the field theories which can be obtained on branes and also a better understanding of string theory and string duality.

3. “Tensionless strings” and non-trivial RG fixed points

It was pointed out in 15 that the IIB theory at a $\mathbb{C}^2/\Gamma_G$ singularity of K3 is quite different from the IIA story reviewed in the previous section. Indeed, the theory in the remaining six dimensions has $\mathcal{N} = (2,0)$ supersymmetry, whose multiplets contain five real scalars and a self-dual tensor field, along with fermion partners, with no gauge fields at all. There are $r = \text{rank}(G)$ such matter multiplets on $\mathbb{C}^2/\Gamma_G$. There are BPS strings, which are obtained by wrapping the three-brane of the IIB theory on the $r$ two-cycles of $\mathbb{C}^2/\Gamma_G$, whose BPS tension goes to zero at the point where the $5r$ scalar expectation values go to zero. Rather than having massless gauge fields, we get “tensionless strings.”

The above “tensionless string” theories can also be obtained in the world-volume of IIA or M theory 5-branes. The theory in the world-volume of $N$ parallel 5-branes is the same as that of the IIB theory at a $\mathbb{C}^2/\mathbb{Z}_N$ singularity. The 5-branes are connected by 2-branes, which intersect the 5-branes along strings. These strings become tensionless when the positions of the branes in the transverse directions coincide.

There are also 6d “tensionless string” theories with $\mathcal{N} = (1,0)$ rather than $\mathcal{N} = (2,0)$ supersymmetry. The canonical example is the small $E_8$ instanton, which has a moduli space with two branches which intersect at a point at the origin, where there is a “tensionless string” theory. Omitting the free decoupled hypermultiplet
corresponding to the position of the 5-brane in the 4 transverse directions, one branch has 29 hypermultiplets and is isomorphic to the moduli space of an $E_8$ instanton. The other branch has a $\mathcal{N} = (1, 0)$ tensor multiplet rather than the 29 hypermultiplets; it is labeled by the expectation value $\langle a \rangle \in \mathbb{R}^+$ of the real scalar component of a 6d $\mathcal{N} = (1, 0)$ tensor super-multiplet. In the context of the realization of $\mathcal{E}$ of the $E_8 \times E_8$ heterotic string as $\text{M}$ theory on $S^1/\mathbb{Z}_2$, the theory arises in the world-volume of the $\text{M}$ theory 5-brane in the presence of the parallel 9-brane $\mathcal{E}$. The branch with the tensor multiplet corresponds to separating the 5-brane and 9-brane a distance $\langle a \rangle$ in the 11-th dimension of $\text{M}$ theory. The 5-brane and 9-brane are connected by the 2-brane of $\text{M}$ theory, which intersects the 5-brane along a string of tension proportional to $\langle a \rangle$. At the “tensionless string” point at the origin, where the two branches touch, there is a six-dimensional, scale-invariant theory with a global $E_8$ symmetry.

The above “tensionless string” theories can be considered in the limit $M_P \to \infty$ where, as usual, gravity effectively decouples and we are usually left with a low-energy effective quantum field theory. It is not a-priori clear what to make of these “tensionless string” theories in this limit and there are many questions one might ask: “Do they have to be formulated in terms of the original string theory even though gravity and such can be decoupled in the $M_P \to \infty$ limit? Do they satisfy the axioms of quantum field theory? Are they local? Can they be described in terms of interacting quantum field theories which have string-like excitations (like QCD) or is it necessary to formulate them as some sort of non-critical theory of fundamental strings? If it is possible to describe them as an interacting field theory, is there a manifestly field theoretic UV free Lagrangian which flows to the same theory in the IR?”

It was argued in $\mathcal{E}$ that, because these theories occur in flat six-dimensional Minkowski space, they do likely satisfy the axioms of a local quantum field theory, with local energy momentum tensors and other local fields. To accommodate the tensionless string excitations, these quantum field theories would have to be at interacting, scale-invariant fixed points. At present, there are no known 6d UV free Lagrangian field theories which flow to the same RG fixed points in the IR but, as will be discussed below, it is known that there are such theories upon reduction to fewer dimensions. This definitely shows that, at least upon dimensional reduction, the “tensionless string” theories really are interacting local quantum field theories$^a$.

4. 3d $\mathcal{N} = 4$ supersymmetric theories

In $\mathcal{E}$ certain aspects of 3d $\mathcal{N} = 4$ supersymmetric (i.e. 8 super-charges, the

$^a$More recently, it has been argued in $\mathcal{E}$ that the theories obtained in the world-volume of type II or heterotic 5-branes, in the limit $M_P \to \infty$ with $M_s$ held fixed, are not local quantum field theories. They have string-like excitations and exhibit T-duality upon compactification. The local quantum field theory discussed above (upon compactification to 3d) is only the low-energy limit of the new theory associated with $E_8$ instantons. At energy $\sim M_s$, string-like excitations become important.
same number as $\mathcal{N} = 2$ supersymmetry in 4d) $U(1)$ and $SU(2)$ gauge theories were solved by using connections with string theory and string duality. The results were subsequently rederived purely via a field theory analysis\(^{24}\), giving a non-trivial check of string duality. The argument of\(^{23}\) is as follows: 3d $\mathcal{N} = 4$ $U(1)$ or $SU(2)$ theories arise in the world-volume of a type I (or heterotic) 5-brane (or small instanton) which is compactified on $T^3$, with the 5-brane wrapping $T^3$ to yield a 2-brane. The heterotic or type I theory on $T^3$ is dual to $M$ theory compactified on $K3$, where the 2-brane in the uncompactified 7d space-time must arise as the fundamental $M$ theory 2-brane at a point in $K3$. The moduli space of the 3d $U(1)$ or $SU(2)$ supersymmetric gauge theories must probe, and thus reproduce, the local geometry of $K3$.

The theory in the uncompactified 7d space-time can have $G = SU(N)$, $SO(N)$, or $E_6, E_7, E_8$ gauge symmetry. In the heterotic description, the gauge symmetry $G$ arises perturbatively for particular $T^3$ moduli while, in the $M$ theory description, it is composite gauge invariance associated with compactification on a $K3$ with a $C^2/\Gamma_G$ singularity, where $\Gamma_G$ is the discrete subgroup of $SU(2)$ corresponding to group $G$. The 3d theory living in the world-volume of the 2-brane probe has a moduli space with two branches which touch at a point at the origin, where there is an unbroken, global $G$ symmetry. One branch is isomorphic to the moduli space of a $G$ instanton. The other branch is isomorphic to the $\mathbb{C}^2/\Gamma_G$ orbifold space which the theory probes.

The 3d world-volume theory for $G = SU(N)$ is $U(1)$ with $N$ electrons. The fact that this theory has a Higgs branch, where the electrons get expectation values, which is isomorphic to the moduli space of an $SU(N)$ instanton is a classical result which is not modified by quantum effects (and independent of 3d, depending only on there being 8 super-charges). This theory also has a Coulomb branch where the scalars in the photon vector-multiplet get expectation values. Classically, the Coulomb branch is $\mathbb{R}^3 \times S^1$, with the $S^1$ the scalar $\gamma$ which is dual to the photon in 3d via $*dA = d\gamma$. The above argument of\(^{23}\) shows that, via quantum effects, the Coulomb branch must become $\mathbb{C}^2/\mathbb{Z}_N$. Similarly, the 3d world-volume theory for $G = SO(N)$ is $SU(2)$ with $N$ fundamental quarks. The fact that this theory has a Higgs branch which is isomorphic to the moduli space of an $SO(N)$ instanton is a classical result (which is crucial in\(^{23}\)) which is not modified by quantum effects. The Coulomb branch, on the other hand, is modified by quantum effects and must become $\mathbb{C}^2/\Gamma_{SO(N)}$. The singularities at the origin of these 3d $U(1)$ and $SU(2)$ theories for $N > 1$ implies that they lead to non-trivial RG fixed points with global $SU(N)$ or $SO(N)$ symmetry respectively\(^{23}\).

Finally, the 3d theories associated with $G = E_6, E_7, E_8$ correspond to reduction of the 6d small $E_8$ instanton “tensionless string” theories discussed earlier.

It was pointed out in\(^{23}\) that there can be a “mirror symmetry” duality for 3d, $N = 4$ supersymmetric theories; this involves dual descriptions of RG fixed points with the exchanges: Higgs branch ↔ Coulomb branch, classical ↔ quantum, mass terms ↔ Fayet-Iliopoulos terms, and manifest global symmetries ↔ hidden global
symmetries not present in the ultra-violet Lagrangian. Examples include duals to $U(1)$ with $N$ electrons, $SU(2)$ with $N$ quarks, and the $E_{6,7,8}$ “tensionless string” theories mentioned above. The dual descriptions of the 3d RG fixed points with global $G = SU(N)$, $SO(N)$, and $E_{6,7,8}$ symmetry are given, for all $G$, by $\prod_i U(n_i)$ gauge theories, with each $U(n_i)$ factor associated with a node of the corresponding $G$ Dynkin diagram and matter given by bi-fundamentals corresponding to the links of the extended Dynkin diagram. For the $G = E_{6,7,8}$ cases, this gives local, UV free, Lagrangian quantum field theories which flow in the IR to the same RG fixed points as the small $E_8$ instanton “tensionless string” theories! This shows, at least in 3d, that these are local quantum field theories at non-trivial RG fixed points.

The 3d mirror symmetry of $\overline{\mathbb{C}_3}$ was recently related to string theory dualities. In $\overline{\mathbb{C}_3}$, it was connected with $M$ theory on $K3 \times K3$, with the mirror symmetry related to an exchange of the (non-compact) $K3$s. In $\overline{\mathbb{C}_3}$, it was connected with intersecting branes of type IIB theory, with the mirror symmetry related to the $SL(2,\mathbb{Z})$ S-duality of the type IIB theory. More recently, it was shown in $\overline{\mathbb{C}_3}$ that mirror symmetry, in gauge theories realized geometrically via type II string theory on a Calabi-Yau threefold and an additional circle, is tantamount to T-duality.

5. 5d $N = 1$ supersymmetric theories

We now turn to 5d $N = 1$ susy theories (which have 8 super-charges, the same number as the 3d theories discussed in the previous section). For finite coupling, as discussed at the end of sect. 1, these theories are not asymptotically free in the UV and are infrared free. However, it was argued in $\overline{\mathbb{C}_3}$ that for gauge group $SU(2)$ and $N_f \leq 7$ flavors there are non-trivial RG fixed points at $g_{cl} = \infty$; in other words, the theories with finite coupling can be regarded as perturbations of the fixed point by the relevant operator $\Delta L = g^{-2}_{cl} F^2$, which drives the theory to flow to a free theory in the IR. Thus the $SU(2)$ theories with $N_f \leq 7$ really “exist,” as they can be defined in the UV at the scale-invariant fixed point with $m_0 \equiv g^{-2}_{cl} \to 0$. (Note that $g^{-2}_{cl}$ has dimensions of mass in 5d.) It was further argued in $\overline{\mathbb{C}_3}$ that the $SO(2N_f) \times U(1)_I$ global symmetry of $SU(2)$ with $N_f$ flavors for $m_0 \neq 0$, where $U(1)_I$ has current $j_I = * (F \wedge F)$, is extended to $E_{N_f+1}$ at $m_0 = 0$. In particular, the RG fixed point with $N_f = 7$ and $m_0 = 0$ is that of the small $E_8$ instanton theory reduced to 5d.

The argument of $\overline{\mathbb{C}_3}$ is via probing type I ↔ heterotic duality when compactified on $S^1$. Wrapping the type I 5-brane around the $S^1$ (with $SO(32)$ Wilson lines included) leads to a 4-brane with a 5d $SU(2)$ world-volume gauge theory with $N_f \leq 16$ flavors. The full theory is T-dual to type I’ on an interval with sixteen D8 branes and one D4 brane probe in the bulk and orientifold planes at each of the two ends. When the probe and $N_f$ of the D8 branes are near an orientifold plane, the theory on the probe is $SU(2)$ with $N_f$ flavors. The distance between the probe and the orientifold plane is $b$See, however, the earlier footnote.
$\langle a \rangle \in \mathbb{R}^+$, the real scalar in the $SU(2)$ gauge multiplet whose expectation value breaks $SU(2)$ to $U(1)$, giving the Coulomb branch. The distance between the D8 branes and the orientifold planes gives the mass of the $SU(2)$ flavors. When, for example, the $N_f$ flavors are massless, the effective $U(1)$ gauge coupling on the Coulomb branch is $t(a)F_{\mu\nu}^2$, where $t(a)$ is exactly given by supersymmetry, 5d gauge invariance, and a one-loop calculation to be $t(a) = m_0 + (8 - N_f)a$; the 8 is the contribution of $W$ bosons and the $-N_f$ is the contribution of quarks running in the loop. $t(a)$ gives the exact space-time metric (and thus dilaton), showing that a result of $\mathbb{H}$ is exact.

Consider the effective gauge coupling $t(a) = m_0 + (8 - N_f)a$ as a function of $\langle a \rangle \in \mathbb{R}^+$. For $N_f > 8$, there is a “Landau pole” on the Coulomb branch for $a \geq m_0/(N_f - 8); m_0 \neq 0$ is needed as a UV cutoff. For $N_f = 8$, $t(a) = m_0$ is a constant and it is impossible to take the limit $m_0 \to 0$ without having the effective gauge coupling degenerate. For $N_f < 8$ it is possible to take $m_0 \to 0$ while preserving $t(a) = \alpha_{\text{eff}}^2 \geq 0$ on the entire Coulomb branch $\langle a \rangle \in \mathbb{R}^+$. The theory at $a = 0$ is scale invariant and interacting – thus it is a non-trivial RG fixed point.

While this argument of $\mathbb{H}$ shows that RG fixed points could exist for $N_f \leq 7$, one might wonder if they really do exist. Probing type I–heterotic duality $\mathbb{H}$ shows that they do indeed exist, with $E_{N_f+1}$ global symmetry. In the context of type 1/heterotic on $S^1$, $m_0$ corresponds to the radius of the $S^1$, with $m_0 = 0$ corresponding to $R = R_{\text{critical}}$, the value for the radius where spacetime gauge symmetry is enhanced from $SO(2N_f) \times U(1)$ to $E_{N_f+1}$. For the $SO(32)$ heterotic string on $S^1$, this is visible at the classical level via winding modes which become massless at $R = R_{\text{critical}}$. For $N_f \leq 7$, at $m_0 = a = 0$, there is an $E_{N_f+1}$ gauge symmetry in spacetime and thus an $E_{N_f+1}$ global symmetry for the theory on the brane. For $m_0 \neq 0$ the Higgs branch, which intersects the Coulomb branch at the origin, is isomorphic to the moduli space of an $SO(2N_f)$ instanton; for $m_0 \to 0$ it becomes isomorphic to the moduli space of an $E_{N_f+1}$ instanton. The interacting, scale invariant theories at the origin of the moduli space correspond to the $E_8$ “tensionless string” theory, reduced to 5d.

Unlike the situation in 3d $\mathbb{E}$, reviewed in the previous section, in 5d there are no known UV free Lagrangian field theories which flow into the small $E_8$ instanton “tensionless string” RG fixed points in the IR. Nevertheless, these are local quantum field theories at RG fixed points $\mathbb{E}$. The Lagrangian $SU(2)$ field theories with $N_f \leq 7$ flavors flow out of these fixed points upon perturbation by a relevant operator $\mathbb{E}$.

It is also possible to get 5d $N = 1$ supersymmetric $SU(2)$ with $N_f$ flavors as “composite” gauge invariance and matter from $M$ theory on a singular Calabi-Yau threefold. The physics/geometry dictionary is that a ruled complex surface, with $N_f$ singularities along the ruling, collapses to a complex curve of size $\sim m_0$. $M$ theory on the CY leads to 5d Chern-Simons interactions $c_{ijk}A^i \wedge F^j \wedge F^k$, where $c_{ijk}$ is the intersection form for $H_4$ classes. These terms are related by 5d susy to the effective gauge coupling. Since we know $t(a) = m_0 + (8 - N_f)$, this means that the intersection

$^c$See, however, the earlier footnote.
form $c$ must satisfy $c = 8 - N_f$ for the singularity which gives $SU(2)$ with $N_f$ flavors; this is indeed the case, giving a check of $M$ theory and its composite gauge invariance. Now one can consider the limit $m_0 \to 0$, which is where the complex surface $S$ is a “del Pezzo” surface which can collapse to a point. The gauge theory results for $N_f \leq 7$ perfectly matches and extend the del Pezzo classification.

In the work, we consider general aspects of 5d susy gauge theories, including an anomaly which renders some theories inconsistent and others consistent only by including a Wess-Zumino type Chern-Simons term. We discuss the general necessary conditions for non-trivial RG fixed points and find all possible gauge groups and matter content which satisfy them. The gauge theory results are connected, via $M$ theory, to results about Calabi-Yau geometry. In particular, the classification of non-trivial RG fixed points related to gauge theory yields a new classification of the higher codimension analog of “del Pezzo” singularities of CY threefolds. It is thereby verified using the geometry that the new 5d RG fixed points do indeed exist.

5d $N = 1$ susy gauge theories have a real adjoint scalar $\Phi$ and thus there is always a Coulomb branch moduli space with $\langle \Phi \rangle$ in the Cartan, breaking $G \to U(1)^r$. The Coulomb branch is a wedge parameterized by Cartan scalar moduli $\phi^i$ in $\mathbb{R}^r/\mathbb{W}$, where $\mathbb{W}$ is the Weyl group of $G$. The unbroken $U(1)^r$ is enhanced to larger $G$ subgroups at the walls of the Coulomb branch Weyl chamber. 5d $N = 1$ susy implies that the low energy effective theory on the Coulomb branch is given by a prepotential $\mathcal{F}(\phi^i)$, which leads to $U(1)^r_f$ effective gauge couplings $t(\phi)_{ij} F^i F^j$ and moduli space metric $ds^2 = t(\phi)_{ij} d\phi^i d\phi^j$, where $t(\phi)_{ij} = \partial_i \partial_j \mathcal{F}(\phi)$ must be continuous. In addition, there is a Chern-Simons term $c_{ijk} A^i \wedge F^j \wedge F^k$, where $c_{ijk} = \partial_i \partial_j \partial_k \mathcal{F}$. The Chern-Simons term is not gauge invariant and thus it is necessary to have $c_{ijk} \in \mathbb{Z}$, for all $i, j, k$, in order for the action to be well-defined mod $2\pi l$, with $l$ integer. This requires $c_{ijk}$ to be a constant independent of $\phi^i$; thus, gauge invariance implies that the prepotential can be at most locally cubic: $\mathcal{F}(\phi^i) = \frac{1}{2} t(\phi)_{ij} \phi^i \phi^j + \frac{1}{6} c_{ijk} \phi^i \phi^j \phi^k$.

The quantization condition $c_{ijk} \in \mathbb{Z}$ implies that some theories are only soluble when a classical Chern-Simons term is included. For example, $U(1)$ gauge theory with $N_f$ electrons has Chern-Simons coefficient $c = c_{\text{classical}} + c_{\text{quantum}}$ where, via a one-loop calculation, $c_{\text{quantum}} = -\frac{1}{2} N_f \text{sign}(\phi)$; this result is exact. Gauge invariance then implies that $c_{\text{classical}} + \frac{1}{2} N_f \in \mathbb{Z}$; thus, for odd $N_f$, $c_{\text{classical}} \neq 0$. There is a similar situation for non-Abelian theories. At the classical level, $\mathcal{F} = \frac{1}{2} m_0 \text{Tr} \Phi^2 + \frac{1}{6} c_{\text{cl}} \text{Tr} \Phi^3$, where the first term gives the classical gauge coupling and the second term, which is only possible for groups with a non-trivial cubic Casimir, i.e. only for $G = SU(N)$ with $N \geq 3$, gives a 5d non-Abelian Chern Simons term. For groups $G$ with $\pi_3(G) \neq 0$ there can be a global anomaly analogous to the 4d $\pi_3$ anomaly of $\mathbb{R}^4$. The non-trivial cases are $\pi_3(SU(N)) = \mathbb{Z}$ for $N \geq 3$ and $\pi_3(Sp(N)) = \mathbb{Z}_2$. For $SU(N)$, the $\pi_3$ anomaly implies that the theory is only consistent if $c = c_{\text{cl}} + c_{\text{quantum}} \in \mathbb{Z}$, with $N_f$ fundamentals, this requires $c_{\text{cl}} + \frac{1}{2} N_f \in \mathbb{Z}$ and thus $c_{\text{cl}} \neq 0$ for $N_f$ odd. For $Sp(N)$, the $\pi_3$ anomaly implies that the theory with an odd number of half-hypermultiplets
is inconsistent.

Because the prepotential is at most cubic, it is exactly given by a one-loop calculation, which yields the cubic term. The result, which is valid for any gauge group $G$ and matter multiplets in representations $r_f$ with masses $m_f$ is

$$F = \frac{1}{6}m_0 h_{ij} \phi^i \phi^j + \frac{1}{6}c_{kl}d_{ijkl} \phi^i \phi^j \phi^k + \frac{1}{12} \left( \sum_R |R \cdot \phi|^3 - \sum_f \sum_{w \in W_f} |w \cdot \phi + m_f|^3 \right), \quad (1)$$

where the last two terms are the contributions, respectively, of the vector and matter multiplets in the loop. In this expression, $m_0 = g_s^{-2}$, $h_{ij} \equiv \text{Tr}(T_i T_j)$, $d_{ijkl} \equiv \frac{1}{2} \text{Tr}(T_i \{T_j, T_k\})$, and $R$ are the roots of $G$ and $W_f$ are the weights of $G$ in representation $r_f$.

It is necessary for the theory to have $ds^2 = t(\phi)_{ij} d\phi^i d\phi^j \geq 0$ and thus $\partial_i \partial_j F$ must have non-negative eigenvalues on the Coulomb branch. When the $c_{ijkl} \phi^k$ part has negative eigenvalues, the theory is at best sensible in a subspace of the Coulomb wedge near the origin by taking $m_0 > 0$. On the other hand, when this part is positive, there can be a scale invariant RG fixed point at the origin with $m_0 = 0$. A necessary condition for a RG fixed point is thus that, with $m_0 = 0$, $\partial_i \partial_j F d\phi^i d\phi^j \geq 0$ throughout the entire Coulomb wedge. This is equivalent to the condition that $F$ be a convex function over the entire Coulomb wedge – i.e. for any points $x$ any $y$ in the Coulomb wedge, $F(\lambda x + (1 - \lambda)y) \leq \lambda F(x) + (1 - \lambda) F(y)$ for all $0 \leq \lambda \leq 1$. Using the above general expression for $F$ it is possible to classify all possible gauge groups and matter content satisfying this condition.

Note that the gauge contribution to $F$, $\frac{1}{12} \sum_R |R \cdot \phi|^3$, is purely convex whereas the matter contribution, $-\frac{1}{12} \sum_f \sum_{w \in W_f} |w \cdot \phi + m_f|^3$, is purely concave. Thus there can be a RG fixed point provided there isn’t too much matter. This is analogous to the condition of asymptotic freedom in four dimensions. Some general comments are: $G$ must be non-Abelian to have a fixed point. Any non-Abelian theory with no matter can have a RG fixed point. There can be no fixed point for theories with matter in representations with weights $|W_f| \geq |R|$. If $F$ is convex, giving mass to some matter and integrating it out yields a low-energy $\mathcal{F}_{\text{low}}$ which is even more convex; this is consistent with flowing between RG fixed points. There can be no new fixed points associated with product groups – matter which couples the groups necessarily leads to non-convex $F$.

Consider, for example, $G = Sp(N)$ with $n_A$ matter fields in the two index antisymmetric representation and $N_f$ in the fundamental representation. The Coulomb branch wedge is given by $\Phi = \text{diag}(a_1, \ldots, a_N, -a_1, \ldots, -a_N)/\text{permutations}$ and can thus be chosen to be $a_1 \geq a_2 \geq \ldots \geq a_N \geq 0$. The prepotential for $m_0 = g_s^{-2}$ is $\mathcal{F} = \frac{1}{8} \sum_{i<j} [(a_i - a_j)^3 + (a_i + a_j)^3] (1 - n_A) + \sum_i a_i^3 (8 - N_f))$. This function is convex on the entire Coulomb wedge provided either $n_A = 1$ and $N_f \leq 7$ or $n_A = 0$ and $N_f \leq 2N + 4$. The RG fixed points with $n_A = 1$ and $N_f \leq 7$ can be shown to exist by probing type I ↔ heterotic duality on $S^1$ with $N$ four brane probes. This argument shows that the RG fixed points have a $SP(1) \times E_{N_f+1}$ global symmetry and there is
a Higgs branch which is the moduli space of $N$ $E_{N_f+1}$ instantons. The existence of the RG fixed points for $n_A = 0$ can be shown via $M$ theory on a singular CY, which reproduces exactly the condition $N_f \leq 2N + 4$.

For $G = SU(N)$, the Coulomb wedge can be taken to be $a_1 \geq a_2 \geq \ldots \geq a_N$ with $\sum_i a_i = 0$. The prepotential (1) with $m_0 = 0$ is convex on the entire Coulomb wedge when there are $N_f$ matter fields in the fundamental representation and $c_d$, with $c_d + \frac{1}{2}N_f \in \mathbb{Z}$ for gauge invariance, which satisfy $N_f + 2|c_d| \leq 2N$. All these can be obtained from $N_f = 2N$ and $c_d = 0$ by adding masses of various signs and RG flow. For $N_f < 8$ the prepotential is also convex with $n_A = 1$ two index antisymmetric representation matter field provided $N_f + 2|c_d| \leq 8 - N$. For $SU(4)$ there are also solutions with $n_A = 2$.

For $G = SO(N)$, we find that $\mathcal{F}$ is convex and thus there can be RG fixed points for $n_V \leq N - 4$. For $N \leq 12$ there can also be spinors, with $n_S \leq 2^{(12-N)/2}$ for $N$ even and $n_S \leq 2^{(11-N)/2}$ for $N$ odd. For $G_2$, the necessary condition for RG fixed points is $N_f \leq 4$; for $F_4$, it is $n_{26} \leq 3$; for $E_6$, it is $n_{27} \leq 4$; for $E_7$, $n_{56} \leq 3$; for $E_8$, there can be no added matter fields. This gives a complete classification of possible $5d \mathcal{N} = 1$ RG fixed points related to gauge theories. The next question is if they all really exist. This will be discussed below.

We now turn to the Calabi-Yau interpretation, with the 5d gauge theories obtained via $M$ theory on singular CY geometries. Comparing with the 5d gauge theory results gives a highly non-trivial check of the relation in $M$ theory between singularities in CY geometry and the composite gauge invariance in space-time. In particular, the geometry of intersecting surfaces on the CY must yield an intersection form which reproduces the group theory of our general result eq. (1) for $\mathcal{F}$. The fact that this indeed works in every case is quite astonishing.

For example, to get $SU(3)$, we need two intersecting, ruled, complex surfaces, $S_1$ and $S_2$, to collapse to a complex curve of size $m_0 \sim g_{cd}^{-2}$. The surfaces $S_i$ correspond to Cartan $U(1)_i$ generated by $(T_i)_{jk} = \delta_{jk}(\delta_{ij} - \delta_{i+1,j})$, $i = 1, 2$. Their intersections are $S_1^3 = 8$, $S_2^3 = 8 - N_f$, $S_1^2S_2 + S_1S_2^2 = -2$, where, by an arbitrary choice, the singularities in the ruling corresponding to the $N_f$ flavors were put in the $S_2$ component. These relations yield $S_1^2S_2 = -1 + c'$ and $S_1S_2^2 = -1 - c'$, where $c'$ is an undetermined integer. These relations give

$$6\mathcal{F} = \sum_{ijk} c_{ijk} \phi^i \phi^j \phi^k = \sum_{i<j} (a_i - a_j)^3 + (c' + \frac{1}{2}N_f) \sum_i a_i^3 - \frac{1}{2}N_f \sum_i |a_i|^3,$$

which exactly agrees with the field theory result (1)! The term $c' + \frac{1}{2}N_f$ is the Chern-Simons term $c_{cd}$ and indeed satisfies the quantum quantization condition $c + \frac{1}{2}N_f \in \mathbb{Z}$.

Now, if it is mathematically possible to take the size $m_0$ of the complex curve to zero in the Calabi-Yau construction of enhanced gauge symmetry $G$ with a given matter content, we’ll have a string theory construction of the corresponding RG fixed point theory. We have thus verified in $\square$ that, in (essentially) every case, the above
classification for RG fixed points perfectly matches and extends the mathematics of higher codimension del-Pezzo collapses. (We expect it to also work in the few, more technically difficult, cases which were not checked in \[33\].) Thus, all of the 5d RG fixed points in our classification do indeed exist.

To conclude, gauge theories have interesting dynamics in various dimensions. There are several types of interplay between gauge and string dynamics. These interconnections have led to new insights into field theory, string theory, and mathematics.

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