NONEQUILIBRIUM FLUCTUATIONS OF ONE-DIMENSIONAL BOUNDARY DRIVEN WEAKLY ASYMMETRIC EXCLUSION PROCESSES

PATRÍCIA GONÇALVES, CLAUDIO LANDIM, AND ANIURA MILANÉS

ABSTRACT. We prove the nonequilibrium fluctuations of one-dimensional, boundary driven, weakly asymmetric exclusion processes through a microscopic Cole-Hopf transformation.

1. INTRODUCTION

Nonequilibrium fluctuations of interacting particle systems around the hydrodynamic limit is one of the main open problems in the field. It has only been derived for few one-dimensional dynamics and no progress has been made in the last twenty years. We refer to the last section of [14, Chapter 11] for references and an historical account.

We examine in this article the dynamical nonequilibrium fluctuations of one-dimensional weakly asymmetric exclusion processes in contact with reservoirs. In a future work, following the strategy presented in [17] for the symmetric simple exclusion process, we use the results presented here to prove the stationary fluctuations of the density field.

The motivations are twofold. On the one hand, the investigation of the steady states of boundary driven interacting particle systems has attracted a lot of attention in these last fifteen years, mainly after [7, 11]. The density fluctuations at the steady state is an important part of the theory and it can only be seized through the dynamical nonequilibrium fluctuations [17]. On the other hand, several published results [6] still wait for rigorous proofs.

Denote by $\mu_{ss}^N$ a stationary state of a one-dimensional weakly asymmetric exclusion processes in contact with reservoirs. The stationary density fluctuation field, denoted by $\mathcal{Y}_N$, acts on smooth functions $H : [0, 1] \to \mathbb{R}$ as

$$\mathcal{Y}_N(H) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N-1} H(k/N) \left[ \eta_k - \theta_N(k) \right],$$

where $\eta$ represents a configuration and $\theta_N(k) = E_{\mu_{ss}^N}[\eta_k]$. Not much information is available on $\theta_N(k)$, besides discrete difference equations which involve second-order covariance terms. It follows from Theorem 2.1 below and some straightforward arguments, presented in [17] in the case of boundary driven
symmetric simple exclusion processes, that we may replace $\theta_N(k)$ by $\bar{\rho}(k/N)$ in the definition of the density fluctuation field, where $\bar{\rho}$ is the solution of the stationary hydrodynamic equation, provided

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N-1} H(k/N) \left\{ \bar{\rho}(k/N) - \theta_N(k) \right\}$$

is uniformly bounded. Note that we do not need to prove that this expression vanishes in the limit, as one would expect from the definition of the density fluctuation field, but just that it is uniformly bounded.

The proof of the nonequilibrium density fluctuations we present here relies on a microscopic Cole-Hopf transformation introduced by Gärtner [12] to investigate the hydrodynamic behavior of weakly asymmetric exclusion processes on $\mathbb{Z}$, and used by Dittrich and Gärtner [9] to prove the nonequilibrium fluctuations of the same models.

As in PDE, the microscopic Cole-Hopf transformation turns a nonlinear problem involving local functions into a linear one. For this reason, it permits to avoid proving a nonequilibrium Boltzmann-Gibbs principle [14, Section 11.1], introduced by H. Rost [3], which is the main technical difficulty in the proof of density fluctuations.

The proof of the nonequilibrium fluctuations relies on sharp estimates of the moments of the microscopic Cole-Hopf variables, and on sharp estimates of the fundamental solution of initial-boundary value semi-discrete linear partial differential equations. These results are presented in the last two sections of this article. The bounds on the fundamental solutions are derived in a similar way as hypercontractivity is proven for ergodic Markov chains.

2. Notation and Results

2.1. The model. Fix $E > 0$, $\alpha$, $\beta$ in $(0, 1)$ and $N \geq 1$. Denote by $\{\eta^N_t : t \geq 0\}$, the speeded-up, one-dimensional, boundary driven, weakly asymmetric simple exclusion process with state space $\Sigma_N = \{0, 1\}^{\{1, \ldots, N-1\}}$. The configurations of the state space are denoted by the symbol $\eta$, so that $\eta(j) = 1$ if site $j$ is occupied for the configuration $\eta$ and $\eta(j) = 0$ if site $j$ is empty. The infinitesimal generator of the Markov process is denoted by $\mathcal{L}_N$ and acts on functions $f : \Sigma_N \to \mathbb{R}$ as

$$(\mathcal{L}_N f)(\eta) = N^2 \sum_{j=0}^{N-1} c_{j,j+1}(\eta) \left\{ f(\sigma^{j+1} \eta) - f(\eta) \right\},$$
where, for $1 \leq j \leq N - 2$,

$$
c_{j,j+1}(\eta) = \left(1 + \frac{E}{N}\right) \eta(j) [1 - \eta(j+1)] + \eta(j+1) [1 - \eta(j)],
$$

$$
c_{0,1}(\eta) = \left(1 + \frac{E}{N}\right) \eta(0) [1 - \eta(1)] + \eta(1) [1 - \eta(0)],
$$

$$
c_{N-1,N}(\eta) = \left(1 + \frac{E}{N}\right) \eta(N-1) [1 - \eta(N)] + \eta(N) [1 - \eta(N-1)],
$$

with the convention, adopted throughout the article, that

$$
\eta(0) = \alpha, \quad \eta(N) = \beta.
$$

In these formulas, $\sigma^{j,j+1}\eta$, $1 \leq j \leq N - 2$, is the configuration obtained from $\eta$ by exchanging the occupation variables $\eta(j), \eta(j+1)$,

$$
(\sigma^{j,j+1}\eta)(k) = \begin{cases} 
\eta(j+1), & k = j, \\
\eta(j), & k = j+1, \\
\eta(k), & k \neq j, j+1,
\end{cases}
$$

while $\sigma^0\eta = \sigma^1\eta, \sigma^{N-1,N}\eta = \sigma^{N-1}\eta$ are the configurations obtained by flipping the occupation variables $\eta(1), \eta(N-1)$, respectively,

$$
(\sigma^{j}\eta)(k) = \begin{cases} 
\eta(k), & k \neq j, \\
1 - \eta(k), & k = j.
\end{cases}
$$

2.2. Hydrodynamic limit. Let $D(\mathbb{R}_+, \Sigma_N)$ be the space of $\Sigma_N$-valued functions which are right continuous with left limits, endowed with the Skorohod topology. For a probability measure $\mu_N$ on $\Sigma_N$, denote by $P_{\mu_N}$ the measure on $D(\mathbb{R}_+, \Sigma_N)$ induced by the Markov process $\eta^N_t$ with initial distribution $\mu_N$. We represent by $E_{\mu_N}$ the expectation with respect to $P_{\mu_N}$ and by $E_{\mu_N}$ the expectation with respect to $\mu_N$.

Let $\pi^N_t(du), t \geq 0$, be the positive random measure on $[0,1]$ obtained by rescaling space by $N^{-1}$ and by assigning mass $N^{-1}$ to each particle:

$$
\pi^N_t(dx) = \frac{1}{N} \sum_{j=1}^{N-1} \eta^N_t(j) \delta_{j/N}(dx),
$$

where $\delta_{j/N}$ is the Dirac mass at $j/N$.

Fix a measurable density profile $\rho_0 : [0,1] \to [0,1]$ and let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures on $\Sigma_N$ associated to $\rho_0$ in the sense that for every continuous function $G : [0,1] \to \mathbb{R}$ and every $\delta > 0$,

$$
\lim_{N \to +\infty} \mu_N \left( \frac{1}{N} \sum_{k=1}^{N-1} G(k/N) \eta(k) - \int_0^1 G(x) \rho_0(x) \, dx \right) > \delta = 0.
$$
Then, for each $t \geq 0$, $\pi^N_t$ converges in $\mathbb{P}_{\mu_N}$-probability to a measure which is absolutely continuous with respect to the Lebesgue measure and whose density $\rho(t,x)$ is the unique weak solution of the viscous Burgers equation with Dirichlet’s boundary conditions:

\[
\begin{cases}
\partial_t \rho = \partial_x^2 \rho - E \partial_x b(\rho), \\
\rho(t,0) = \alpha, \quad \rho(t,1) = \beta, \quad t \geq 0 \\
\rho(0,x) = \rho_0(x), \quad 0 \leq x \leq 1,
\end{cases}
\]

(2.2)

where $b(\rho) = \rho(1 - \rho)$. We refer to [12, 5, 14, 2, 11] and references therein.

2.3. Nonequilibrium fluctuations. To define the space in which the fluctuations take place, denote by $C^2_0([0,1])$ the space of twice continuously differentiable functions on $(0,1)$ which are continuous on $[0,1]$ and which vanish at the boundary. Let $-\Delta$ be the positive operator, essentially self-adjoint on $L^2[0,1]$, defined by

\[
-\Delta = -\frac{d^2}{dx^2}, \quad \mathcal{D}(-\Delta) = C^2_0([0,1]).
\]

Its eigenvalues and corresponding (normalized) eigenfunctions have the form $\lambda_n = (n\pi)^2$ and $e_n(x) = \sqrt{2}\sin(n\pi x)$ respectively, for any $n \geq 1$. By the Sturm-Liouville theory, $\{e_n, \, n \geq 1\}$ forms an orthonormal basis of $L^2[0,1]$.

We denote with the same symbol the closure of $-\Delta$ in $L^2[0,1]$. For any non-negative integer $k$, we define the Hilbert spaces $\mathcal{H}_k = D\left(\{-\Delta\}^{k/2}\right)$, with inner product $\langle f, g \rangle_k = \sum_{n=1}^{+\infty} (n\pi)^{2k} \langle f, e_n \rangle \langle g, e_n \rangle$.

Moreover, if $\mathcal{H}_{-k}$ denotes the topological dual space of $\mathcal{H}_k$,

\[
\mathcal{H}_{-k} = \{ f \in \mathcal{D}'(0,1) : \sum_{n=1}^{+\infty} n^{-2k} \langle f, e_n \rangle^2 < \infty \},
\]

\[
\langle f, g \rangle_{-k} = \sum_{n=1}^{+\infty} (n\pi)^{-2k} \langle f, e_n \rangle \langle g, e_n \rangle,
\]

where $\mathcal{D}'(0,1)$ represents the space of distributions on $(0,1)$ and $\langle f, \cdot \rangle$ the action of the distribution $f$ on test functions.

Fix a continuous density profile $\rho_0 : [0,1] \to [0,1]$, and denote by $\rho(t,x)$ the unique weak solution of the viscous Burgers equation (2.2). Let $Y^N_t$ represent
the density fluctuation field which acts on functions $H$ in $C^{1}([0,1])$ as

$$Y_{t}^{N}(H) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N-1} H(k/N)\{\eta_{t}(k) - \rho(t,k/N)\}.$$ 

Fix $t > 0$ and a function $G$ in $C_{0}^{2}([0,1])$. Recall that we denote by $\rho(s, x) = \rho_{s}(x)$ the solution of the viscous Burgers equation \(\text{(2.2)}\). Let $(T_{t,s}G)(x) = G(s, x)$, $0 \leq s \leq t$, be the solution of the backward linear equation with final condition

$$
\begin{cases}
-\partial_{s}G = \partial_{x}^{2}G + E(1 - 2\rho_{s})\partial_{x}G, \\
G(t, x) = G(x), \quad 0 \leq x \leq 1, \\
G(s, 0) = G(s, 1) = 0, \quad 0 \leq s \leq t.
\end{cases}
$$

(2.3)

Denote by $D([0,T], \mathcal{H}_{-k})$ the set of trajectories $Y : [0, T] \to \mathcal{H}_{-k}$ which are right continuous and have left limits, endowed with the Skorohod topology.

**Theorem 2.1.** Fix $T > 0$, a positive integer $k > 7/2$, and a density profile $\rho_{0} : [0,1] \to [0,1]$ in $C^{4}([0,1])$ such that $\rho_{0}(0) = \alpha$, $\rho_{0}(1) = \beta$. Let $\{\mu_{N} : N \geq 1\}$ be a sequence of probability measures on $\Sigma_{N}$ for which there exists a finite constant $A_{2}$ such that

$$\sup_{N \geq 1} \max_{1 \leq k \leq N-1} E_{\mu_{N}}\left[\left(\frac{1}{\sqrt{N}} \sum_{j=1}^{k} \{\eta_{0}(j) - \rho_{0}(j/N)\}\right)^{4}\right] \leq A_{2}. \quad \text{(2.4)}$$

Let $Q_{N}$ be the probability measure on $D([0,T], \mathcal{H}_{-k})$ induced by the density fluctuation field $Y_{N}$ and the probability measure $\mu_{N}$. Then, all limit points $Q^{*}$ of the sequence $Q_{N}$ are concentrated on paths $Y$ such that for all $t \geq 0$ and $G$ in $C_{0}^{5}([0,1])$,

$$W(t, G) := Y_{t}(G) - Y_{0}(T_{t,0}G)$$

are mean-zero Gaussian random variables with covariances given by

$$E_{Q^{*}}[W(t,G)W(s,H)] = 2 \int_{0}^{t} \int_{0}^{s} \sigma(\rho(r,x)) (\partial_{x}T_{t,r}G)(x)(\partial_{x}T_{s,r}H)(x) \, dx \, dr,$$

(2.5)

for all $0 \leq s, t \leq T$. In this formula, $\sigma(\rho)$ represents the mobility which is given by $\sigma(\rho) = \rho(1-\rho)$. Moreover, for all $G$ and $H$ in $C_{0}^{5}([0,1])$, and $t > 0$,

$$E_{Q^{*}}[W(t,G)Y_{0}(H)] = 0.$$

**Corollary 2.2.** In addition to the hypotheses of Theorem 2.1 assume that $Y_{0}^{N}$ converges to a zero-mean Gaussian field $Y$ with covariance denoted by $\ll \cdot, \cdot \gg$, so that for all $G, H$ in $C^{2}([0,1])$,

$$\lim_{N \to \infty} E_{\mu_{N}}[Y_{0}^{N}(H)Y_{0}^{N}(G)] = \ll H, G \gg.$$
Then, the sequence \( Q^N \) converges to a mean-zero Gaussian measure \( Q \) whose covariances are given by

\[
E_Q[Y_t(G)Y_s(H)] = \ll T_{t,0}G, T_{s,0}H \gg \\
+ 2 \int_0^{t \wedge s} \int_0^1 \sigma(\rho(r,x)) \left( \partial_x T_{t,r}G(x) \right) \left( \partial_x T_{s,r}H(x) \right) dx \, dr .
\]

for all \( 0 \leq s, t \leq T, \; G, \; H \in C_0^5([0,1]) \).

This result is an immediate consequence of Theorem 2.1. Under any limit point \( Q^* \) of the sequence \( Q^N \), for any function \( G \) in \( C_0^5([0,1]) \), \( Y_t(G) \) can be written as the sum of two uncorrelated mean-zero Gaussian variables \( W(t,G) \) and \( Y_0(T_{t,0}G) \).

Since under the measure \( Q \), \( W(t,G) \) is a Brownian motion changed in time, the process \( Y_t \) may be understood as a generalized Ornstein-Uhlenbeck process described by the formal stochastic partial differential equation

\[
dY_t = L_t Y_t \, dt + \sqrt{2} \sigma(\rho_t) \nabla dW_t ,
\]

where \( L_t \) is the linear differential operator

\[
\partial^2_x + (1 - 2\rho_t)E \partial_x .
\]

The article is organized as follows. In Section 3 we introduce the microscopic Cole-Hopf transformation and we write the density fluctuation field as the sum of a current field and a remainder. In Section 4 we prove Theorem 2.1 and Corollary 2.2 assuming that the density field \( Y_t^N \) is tight and that three estimates are in force. In Sections 5–7 we prove these three estimates, and in Section 8 we prove tightness of \( Y_t^N \). All proofs rely on estimates on the moments of the microscopic Cole-Hopf variables, presented in Section 9, and on estimates of the solutions of certain semi-discrete equations, presented in Section 10.

3. A MICROSCOPIC COLE-HOPF TRANSFORMATION

To keep notation simple, from now on we drop the superscript \( N \) on the process \( \eta_t^N \). Following [9, 12] we define in this section a microscopic Cole-Hopf transformation of the process \( \eta_t \). For \( N \geq 1 \), let

\[
\Lambda^-_N = \{1, \ldots, N-1\}, \quad \Lambda_N = \{0, \ldots, N-1\}, \quad \Lambda^+_N = \{0, \ldots, N\} .
\]

Denote by \( \Omega = \Omega^N \) the linear operator defined on functions \( f : \Lambda_N \to \mathbb{R} \) by

\[
\begin{cases}
(\Omega f)(0) = -\alpha EN f(0) + N(\nabla^+_N f)(0) , \\
(\Omega f)(j) = (\Delta_N f)(j) - E(\nabla^-_N f)(j) , & 1 \leq j \leq N-2 , \\
(\Omega f)(N-1) = \beta EN f(N-1) - N \left( 1 + \frac{\rho_0}{N} \right) (\nabla^-_N f)(N-1) .
\end{cases}
\tag{3.1}
\]

In this formula,

\[
(\nabla^+_N f)(j) = N[f(j+1) - f(j)] , \quad (\nabla^-_N f)(j) = -N[f(j-1) - f(j)] .
\]
\[ (\Delta N f)(j) = N^2[f(j + 1) + f(j - 1) - 2f(j)]. \]

Let \( \lambda_t = \lambda_t^N \) be the solution of the linear equation
\[
\begin{cases}
(\partial_t \lambda_t)(j) = (\Omega \lambda_t)(j), & 0 \leq j \leq N - 1, \\
\lambda_0(j) = \exp\left\{ - (\gamma/N) \sum_{k=1}^j \rho_k(k/N) \right\},
\end{cases}
\]
where \( \gamma = \gamma_N \leq 0 \) is chosen so that \( e^{-\gamma/N} = 1 + E/N \), and \( \rho_0 : [0, 1] \to [0, 1] \) is a density profile satisfying the assumptions of Theorem 2.1. For \( j \in \Lambda_N^- \), let
\[
r_t(j) = -\frac{1}{\gamma} [\nabla^- N \ln(\lambda_t)](j).
\]

Denote by \( \tilde{Y}_t^N \), \( t \geq 0 \), the modified density fluctuation field defined on functions \( G \) in \( C^1([0, 1]) \) by
\[
\tilde{Y}_t^N(G) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N-1} G(j/N) \left\{ \eta_t(j) - r_t(j) \right\}.
\]

Next result asserts that the original density fluctuation field \( Y^N_t \) is close to the modified density field \( \tilde{Y}_t^N \).

**Proposition 3.1.** For each \( T > 0 \),
\[
\sup_{N \geq 1} \sup_{0 \leq t \leq T} \max_{1 \leq j \leq N-1} N \left| r_t(j) - \rho(t, j/N) \right| < \infty.
\]

For \( 0 \leq j, k \leq N \) with \( |j - k| = 1 \), denote by \( J_t^{j,k} \), the total number of jumps from \( j \) to \( k \) in the time interval \( [0, t] \), and let \( W_t^{j,j+1} \) be the total current over the bond \( \{j, j+1\} \), that is
\[
W_t^{j,j+1} = J_t^{j,j+1} - J_t^{j+1,j}.
\]

In this formula, \( J_t^{0,1} \) (resp. \( J_t^{1,0} \)) stands for the total number of particles created (resp. removed) at the left boundary, with a similar convention at the right boundary.

For \( j \in \Lambda_N \), let
\[
\xi_t(j) = \exp\left\{ (\gamma/N) [W_t^{j,j+1} - \sum_{k=1}^j \eta_t(k)] \right\}.
\]

Since
\[
\xi_t(j) - \xi_0(j) = \int_0^t \xi_{j-}(s) \left[ e^{\gamma/N} - 1 \right] dJ_s^{j,j+1} + \int_0^t \xi_{j-}(s) \left[ e^{-\gamma/N} - 1 \right] dJ_s^{j+1,j},
\]
\( \xi_t(j) \) can be written as
\[
\xi_t(j) = \xi_0(j) + \int_0^t \xi_{j-}(s) \eta_s(j) ds + \mathcal{M}_t^N(j),
\]
where
\[
\mathcal{M}_t^N(j) = \sum_{k=1}^j \int_0^t \xi_{j-}(s) \left( \eta_s(k) - \rho(s, k/N) \right) ds.
\]
where, in view of the definition of $\gamma$ and of the convention (2.1),

$$g_{j,j+1}(\eta) = EN[\eta(j+1) - \eta(j)] ,$$

and $\mathcal{M}^N_t(j)$ is a martingale with quadratic variation given by

$$\langle \mathcal{M}^N_t(j), \mathcal{M}^N_t(k) \rangle_t = \delta_{j,k} E^2 \int_0^t \xi_s(j)^2 h_j(\eta_s) \, ds.$$  \hfill (3.6)

In this formula, $\delta_{j,k}$ is the delta of Kroenecker and

$$h_j(\eta) := e^{\gamma/N} \eta(j) [1 - \eta(j+1)] + \eta(j+1) [1 - \eta(j)].$$  \hfill (3.7)

By the continuity equation, for $1 \leq j \leq N - 1$,

$$W_{t^{-1},j} - W_{t^{j+1}} = \eta(j) - \eta(0) .$$

As a consequence, for $0 \leq j \leq N - 2$, $1 \leq k \leq N - 1$,

$$\begin{align*}
\xi_t(j + 1) - \xi_t(j) &= \xi_t(j)\eta_t(j + 1) [\exp(-\gamma/N) - 1] , \\
\xi_t(k - 1) - \xi_t(k) &= \xi_t(k)\eta_t(k) [\exp(\gamma/N) - 1] .
\end{align*}$$

These equations explain the term $\sum_{1 \leq k \leq j} \eta_t(k)$ in the definition of $\xi_t(j)$. In view of the previous identities, by definition of $g_{j,j+1}$, and by the choice of $\gamma$,

$$\xi_t(j) = \xi_0(j) + \int_0^t (\Omega_s\eta)(j) \, ds \pm \mathcal{M}^N_t(j) .$$

The advantage of the process $\xi_t$ compared to the original process $\eta_t$ is that it evolves according to the linear equation (3.9). Of course, the original process $\eta_t$ can be recovered from $\xi_t$, since from (3.4) and by the continuity equation appearing right below (3.7), for $1 \leq j \leq N - 1$,

$$\eta_t(j) = -\frac{1}{\gamma} [\nabla_N \ln(\xi_t)](j) .$$

Denote by $J^N_t$, $t \geq 0$, the current fluctuation field defined on functions $G \in C^1([0,1])$ by

$$J^N_t(G) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \frac{(\nabla_N^+ G)(j/N)}{\gamma \lambda_t(j)} (\xi_t(j) - \lambda_t(j)) .$$

By the formula for $\eta_t(j)$ in terms of $\xi_t(j)$, and by (3.3), a summation by parts yields that for functions $G \in C^1_0([0,1])$

$$\tilde{J}^N_t(G) = J^N_t(G) + R^N_t(G) ,$$  \hfill (3.10)

where the remainder $R^N_t(G)$ is given by

$$R^N_t(G) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \frac{1}{\gamma} (\nabla_N^+ G)(j/N) \left[ \ln \left( \frac{\xi_t(j)}{\lambda_t(j)} \right) + 1 - \frac{\xi_t(j)}{\lambda_t(j)} \right] .$$

Notice that both the current field $J^N_t$ and the remainder $R^N_t$ depend only on the process $\xi_t$. Sometimes, by abuse of notation, we consider that $R^N_t$ acts
on discrete functions \( g : \{0, \ldots, N\} \to \mathbb{R} \) instead of continuous functions \( G : [0,1] \to \mathbb{R} \). This is the case in the next proposition.

The second result of this section asserts that the modified density fluctuation field \( \hat{Y}_t^N \) is close to the current fluctuation field \( J_t^N \).

**Proposition 3.2.** Fix \( T > 0 \) and a function \( \phi : [0, T] \times \Lambda_N^+ \to \mathbb{R} \) such that
\[
\sup_{0 \leq t \leq T} \max_{j \in \Lambda_N} \frac{|(\nabla^+_N \phi_t)(j)|}{\lambda_t(j)} < \infty.
\]
Then, for any \( \delta > 0 \),
\[
\lim_{N \to +\infty} \mathbb{P}_{\mu_N} \left[ \sup_{0 \leq t \leq T} |R^N_t(\phi_t)| > \delta \right] = 0.
\]

4. PROOF OF THEOREM 2.1

Fix a density profile \( \rho_0 \) satisfying the assumptions of the theorem and denote by \( \rho(t,x) \) the solution of the viscous Burgers equation (2.2) with initial condition \( \rho_0 \). Let \( \{\mu_N : N \geq 1\} \) be a sequence of probability measures on \( \Sigma_N \) for which (2.4) holds.

Let \( \phi : \Lambda_N \to \mathbb{R} \) be a strictly positive function. Denote by \( A_\phi = A_\phi^N \) the difference operator which acts on functions \( g : \Lambda_N^+ \to \mathbb{R} \) by
\[
\begin{cases}
(A_\phi g)(0) = (A_\phi g)(N) = 0, \\
(A_\phi g)(j) = (\Delta_N g)(j) + E \frac{[1 - \theta_\phi(j)]}{1 + (E/N) \theta_\phi(j)} (\nabla^+_N g)(j) - E \theta_\phi(j) (\nabla^- N g)(j)
\end{cases}
\]
for \( 1 \leq j \leq N - 1 \), where
\[
\theta_\phi(j) = \frac{(\nabla^- N \phi)(j)}{E \phi(j - 1)}.
\]
Denote by \( \lambda_s \) the solution of (3.2). For \( s \geq 0 \), let \( A_s = A_{\lambda_s} \), and let
\[
\tilde{r}_s(j) := \theta_{\lambda_s}(j) = \frac{(\nabla^-_{\lambda_s} \lambda_s)(j)}{E \lambda_s(j - 1)}, \quad 1 \leq j \leq N - 1.
\]
By Lemma 5.2 below, \( |\tilde{r}_s(j) - \rho(t,j/N)| \leq C_0/N \) uniformly in \( 0 \leq t \leq T \) and \( 1 \leq j \leq N - 1 \). Moreover, as \( (A_s g)(0) = (A_s g)(N) = 0 \), the solution of the semi-discrete equation
\[
\begin{cases}
-(\partial_s g)(s,j) = (A_s g)(s,j), \quad 0 \leq j \leq N, \\
g(t,j) = G(j/N), \quad 0 \leq j \leq N,
\end{cases}
\]
for some \( t > 0 \) and some \( G \) in \( C^2_0([0,1]) \), is such that \( g_s(0) = g_s(N) = 0 \) for all \( 0 \leq s \leq t \). Hence, the semi-discrete equation (4.2) has to be understood as a discrete approximation of the differential equation (2.3).

Fix a function \( G \) in \( C^2_0([0,1]) \) and \( t > 0 \). Let \( g_s(j) = g^{N,t}_s(j) \) be the solution of (4.2). A long computation yields that for \( 0 \leq s \leq t \),
\[
M^N_s(t,G) := J^N_s(g_s) - J^N_0(g_0) = \frac{1}{\sqrt{N}} \sum_{j \in \Lambda_N} \int_0^t \frac{(\nabla^+_N g)(j)}{\gamma \lambda_r(j)} dM^N_r(j),
\]
where $M^N_s(j)$ is the martingale introduced in (3.5). We present some details of this computation below equation (7.2).

**Proposition 4.1.** Fix a density profile $\rho_0 : [0,1] \to [0,1]$ and a sequence $\{\mu_N : N \geq 1\}$ of probability measures on $\Sigma_N$ satisfying the assumptions of Theorem 2.1. Then, for each function $G$ in $C^\infty_0([0,1])$ and $t > 0$, there exists a finite constant $C_0$, depending only on $G$ and $t$, such that for all $N \geq 1$,

\[
E_{\mu_N} \left[ \sup_{0 \leq s \leq t} M^N_s(t,G)^4 \right] \leq C_0 , \quad E_{\mu_N} \left[ (M^N(t,G))^2 \right] \leq C_0 .
\]

If $G$ belongs to $C^\infty_0([0,1])$, then the sequence of martingales $M^N_s(t,G)$, $0 \leq s \leq t$, converges in $D([0,t],\mathbb{R})$ to a mean-zero, continuous martingale, denoted by $M_s(t,G)$. For $G_1$, $G_2$ in $C^\infty_0([0,1])$, $t_1, t_2 > 0$, and $0 \leq s_j \leq t_j$, the covariances of $M_{s_1}(t_1,G_1)$ and $M_{s_2}(t_2,G_2)$ are given by

\[
E[M_{s_1}(t_1,G_1)M_{s_2}(t_2,G_2)] = 2 \int_0^{s_1 \land s_2} \int_0^1 \sigma(\rho(r,x)) (\partial_x T_{t_1,r,G_1})(x) (\partial_x T_{t_2,r,G_2})(x) \, dx \, dr .
\]

Since $M_s(t,G)$ is a continuous martingale whose quadratic variation is deterministic, $M_s(t,G)$ is a Brownian motion changed in time. In particular, $M_t(t,G)$ is a mean-zero Gaussian random variable.

**Proof of Theorem 2.1.** Let $Q^*$ be a limit point of the sequence $Q_N$. Fix a function $G \in C^\infty_0([0,1])$ and $t > 0$. Let $g_s(j) = g_s^{N,t}(j)$ be the solution of (4.2) with final condition equal to $G$. By (3.10), Proposition 3.1 and (4.3),

\[
Y^N_t(G) - Y^N_0(g_0) = M^N_t(t,G) + R^N_t(G) - R^N_0(g_0) + \frac{C_N}{\sqrt{N}} ,
\]

where $C_N$ is a sequence of numbers uniformly bounded. By Proposition 4.1 and in view of the remark made just after that result, $M^N_t(t,G)$ converges in distribution to a mean-zero Gaussian random variable, denoted by $W(t,G)$, whose variance is given by the right hand side of (2.5), with $H = G$, $s = t$.

Let $\psi(s,j) = (\nabla_N g_{t-s})(j)/\lambda_{t-s}(j)$, $j \in \Lambda_N$, $0 \leq s \leq t$. By Remark 7.2 and by Proposition 3.2, $R^N_t(G)$ and $R^N_0(g_0)$ converges to 0 in probability. Recall that we denote by $T_{t,s,G}$ the solution of equation (2.3). By Lemma 7.4, $Y^N_t(g_0) - Y^N_0(T_{t,0,G})$ is absolutely bounded by $C_0/\sqrt{N}$. In conclusion, $Y^N_t(G) - Y^N_0(T_{t,0,G})$ converges in distribution to $W(t,G)$.

The covariance between $Y_0(H)$ and $W_t(t,G)$ vanishes because $W_s(t,G)$, $0 \leq s \leq t$ is a martingale which vanishes at $s = 0$.

To complete the proof, it remains to compute the covariance between $W(t,G)$ and $W(s,H)$. Assume that $s \leq t$. Since $W_s(t,G)$, $0 \leq r \leq t$, is a martingale,

\[
E_{Q^*}[W(t,G)W(s,H)] = E_{Q^*}[W_s(t,G)W(s,H)] .
\]

By the polarization identity, we may express the covariance of a pair of random variables $(X,Y)$ in terms of the variances of the variables $X+Y$ and $X-Y$. □
5. Proof of Proposition 3.1

The main result of this section asserts that the solution \( \lambda_t \) of the linear equation (3.2) (satisfied by the expectation of the Cole-Hopf variables \( \xi_t \)), is close to the Cole-Hopf transformation of the solution of the viscous Burgers equation (2.2).

Fix a profile \( \rho_0 : [0, 1] \to [0, 1] \) in \( C^4([0, 1]) \), and denote by \( \rho(t, x) \) the solution of the hydrodynamic equation (2.2). Let \( K(t, x) \) be the Cole-Hopf transformation of \( \rho(t, x) \):

\[
K(t, x) = \exp \left\{ E \left[ \int_0^t \partial_x \rho(s, x) - E b(\rho(s, x)) \right] \, ds + \int_0^x \rho_0(y) \, dy \right\}.
\]

Since \( \partial_t K = KE[\partial_x \rho - Eb(\rho)] \) and \( \partial_x K = EK \rho \), \( K \) satisfies the linear parabolic conditions

\[
\begin{cases}
\partial_t K = \partial_x^2 K - E \partial_x K, \\
(\partial_x K)(t, 0) = E \alpha K(t, 0), \quad (\partial_x K)(t, 1) = E \beta K(t, 1), \quad 0 < t \leq T, \\
K(0, x) = \exp \left\{ E \int_0^x \rho_0(y) \, dy \right\}, \quad 0 \leq x \leq 1.
\end{cases}
\]

As \( \rho_0 \) belongs to \( C^4([0, 1]) \), \( K_0 \) belongs to \( C^5([0, 1]) \), and, by Lemma 10.1, \( K \) belongs to \( C^2_R(\mathbb{R} \times [0, 1]) \).

Denote by \( \|f\|_M \) the sup norm of a function \( f : \Lambda_N, \Lambda_N^+ \to \mathbb{R} \):

\[
\|f\|_M = \max_j |f(j)|,
\]

where the maximum is carried over the domain of definition of \( f \). By abuse of notation, if \( G \) belongs to \( C([0, 1]) \), \( \|G\|_M \) represents \( \max_{0 \leq j \leq N} |G(j/N)| \).

**Lemma 5.1.** Let \( \lambda_t \) and \( K_t \) be the solutions of (3.2) and (5.1), respectively. Then, for every \( T > 0 \),

\[
\sup_{N \geq 1} \sup_{0 \leq t \leq T} \max_{0 \leq j \leq N - 1} N |\lambda_t(j) - K_t(j/N)| < +\infty,
\]

\[
\sup_{N \geq 1} \sup_{0 \leq t \leq T} \max_{1 \leq j \leq N - 1} N |(\nabla_N^{-1} \lambda_t)(j) - (\partial_x K_t)(j/N)| < +\infty.
\]

**Proof:** Fix \( T > 0 \). In this proof, \( C_0 \) represents a finite constant which may depend on the parameters \( E, \beta, \alpha \), on the initial condition \( \rho_0 \), and on \( T \). Let \( w_t(j) := \lambda_t(j) - K_t(j/N) \). A simple computation shows that

\[
(\partial_t w_t)(j) = (\Omega w_t)(j) + \varphi(t, j),
\]

where \( \Omega \) has been introduced in (3.1) and where \( \varphi(t, j) \) is given by

\[
\begin{cases}
N \{ (\nabla_N^{-1} K_t)(j/N) - \alpha E K_t(j/N) \} - (\partial_t K_t)(j/N), \quad j = 0, \\
|\nabla_N^{-1} - \partial_x^2| K_t(j/N) - E |(\nabla_N^{-1} - \partial_x) K_t|(j/N), \quad 1 \leq j \leq N - 2, \\
E \beta NK_t(j/N) - (N + E) (\nabla_N K_t)(j/N) - (\partial_x K_t)(j/N), \quad j = N - 1.
\end{cases}
\]

In view of the boundary conditions satisfied by \( K_t \), we may replace in the previous equation \( \alpha E K_t(0) \) by \( (\partial_x K_t)(0) \) and \( E \beta K_t([N - 1]/N) \) by \( E \beta \{ K_t([N -
where derivatives, namely $\frac{\partial}{\partial_s K_t}(1)$. After these replacements, recalling that $K_t$ and $\rho_0$ belong to $C^4([0,1])$, we obtain that $\varphi(t,j)$ is absolutely bounded by $C_0N^{-1}$ for $j$ in $\{1,\ldots,N-2\}$ and by $C_0$ for $j=0$ and for $j=N-1$.

Let $G_t(j) = \varphi_t(j)1\{1 \leq j \leq N-2\}$, $U_t(j) = \varphi_t(j) - G_t(j)$ so that $|G_t(j)| \leq C_0N^{-1}$. We may represent the solution $w_t$ of (5.2) as

$$w_t = e^{\Omega t}w_0 + \int_0^t e^{\Omega(t-s)}(G_s + U_s) \, ds.$$  

By Lemma 10.4, $\|e^{\Omega t}w_0\|_M$ is bounded by $C_0e^{C_0t}\|w_0\|_M \leq C_0N^{-1}$ and $\|e^{\Omega(t-s)}G_s\|_M$ is absolutely bounded by $C_0e^{C_0(t-s)}N^{-1} \leq C_0N^{-1}$. Furthermore, since $U_s$ vanishes everywhere except at two points, by Corollary 10.7, $\|e^{\Omega(t-s)}U_s\|_M \leq C_0(t-s)^{-1/2}N^{-1}$ for all $N$ large enough. Putting together all the previous estimates, we conclude that $\|w_t\|_M$ is bounded by $C_0N^{-1}$, proving the first assertion of the lemma.

We turn to the second assertion. Let

$$\gamma_t(j) = \begin{cases} 
[N/(N+E)]\alpha E \lambda_t(0), & j = 0 \\
(\nabla_N^{-1}\lambda_t)(j), & 1 \leq j \leq N-1, \\
\beta E \lambda_t(N-1), & j = N. 
\end{cases}$$

It is not difficult to show that for $1 \leq j \leq N-1$, $\gamma_t$ solves the equation

$$\partial_t \gamma_t(j) = (\Delta_N \gamma_t(j)) - E(\nabla_N^{-1}\gamma_t)(j).$$

Clearly, $(\partial_t K)$ satisfies a similar equation where the discrete differential operators are replaced by continuous ones. Therefore, in view of (5.1), $w_t(j) = \{\gamma_t(j) - (\partial_t K)(t,j/N)\}$, $0 \leq j \leq N-1$, satisfies

$$\begin{align*}
\partial_t w_t(j) &= (\Delta_N w_t(j)) - E(\nabla_N^{-1}w_t)(j) + \varphi(t,j), & 1 \leq j \leq N-1, \\
w_t(0) &= (\partial_t K)(0,j/N), \\
w_t(N) &= (\partial_t K)(N-1,j/N) - K(t,j/N),
\end{align*}$$

where $\varphi(t,j)$ accounts for the difference between the discrete and continuous derivatives, namely

$$\varphi(t,j) = (\Delta_N v_t)(j/N) - (\partial_t^2 v)(t,j/N) - E\{\nabla_N^{-1}v_t)(j/N) - (\partial_x v)(t,j/N)\},$$

where $v(t,j) = (\partial_x K)(t,j/N)$.

Since $K_t$ belongs to $C^4([0,1])$, $\varphi$ is absolutely bounded by $C_0N^{-1}$ uniformly in $t$ and $j$. By the first part of the proof and by Lemma 10.4, $w_t(0)$ and $w_t(N)$ are also absolutely bounded by $C_0N^{-1}$.

Let $w^*_t(j)$ be the solution of (5.3) with the same initial condition satisfied by $w_t(j)$, but with boundary conditions $w^*_t(0) = C/N$, $w^*_t(N) = C/N$, where $C$ is a finite constant such that $w_t(0) \lor w_t(N) \leq C/N$ for all $0 \leq t \leq T$. By the maximum principle, $w_t(j) \leq w^*_t(j)$ for $0 \leq t \leq T$, $0 \leq j \leq N$. Denote by $\Omega_t$
the generator of a weakly asymmetric random walk on \{0, \ldots, N\} absorbed at 0 and N. We may represent \(w_t^*\) as
\[
w_t^* = e^{\Omega^* t} w_0 + \int_0^t e^{\Omega^* (t-s)} \varphi_s \, ds,
\]
and repeat the arguments presented in the first part of the proof to conclude that \(\|w_t^*\|_M \leq C_0/N\). This provides an upper bound for \(w_t\). A lower bound can be derived along the same lines.

Recall the definition of \(\tilde{r}_t\), given in (4.1).

**Lemma 5.2.** For every \(T > 0\),
\[
\sup_{N \geq 10} \sup_{0 \leq t \leq T} \max_{1 \leq j \leq N-1} N |\tilde{r}_t(j) - \rho(t, j/N)| < \infty.
\]

**Proof:** By definition of \(\tilde{r}_t\) and by the uniform lower bound for \(\lambda_t\), proved in Lemma 10.5,
\[
|\tilde{r}_t(j) - \rho(t, j/N)| \leq C_0 \big| (\nabla N \lambda_t)(j) - E\lambda_t(j-1)\rho(t, j/N) \big|
\]
for some finite constant \(C_0\), whose value may change from line to line. Since \((\partial_x K_t)(j/N) = E\rho(t, j/N) K_t(j/N)\) and since \(\rho\) is bounded, the right hand side of the previous expression is less than or equal to
\[
C_0 \left\{ \big| (\nabla N \lambda_t)(j) - (\partial_x K_t)(j/N) \big| + \big| K_t(j/N) - \lambda_t(j-1) \big| \right\}.
\]
The result follows from Lemma 5.1 and the smoothness of \(K\).

**Lemma 5.3.** For every \(T > 0\),
\[
\sup_{N \geq 10} \sup_{0 \leq t \leq T} \max_{1 \leq j \leq N-2} |\nabla N \tilde{r}_t(j)| < \infty.
\]

**Proof:** Write
\[
|\nabla N \tilde{r}_t(j)| \leq N |\tilde{r}_t(j+1) - \rho(t, [j+1]/N)| + N |\rho(t, [j+1]/N) - \rho(t, j/N)| + N |\rho(t, j/N) - \tilde{r}_t(j)|.
\]
The first and third terms on the right hand side of the last expression are bounded by the previous lemma. To complete the proof it remains to recall that \(\rho\) is of class \(C^{1,2}\).

**Proof of Proposition 3.1** By Lemma 5.2 it is enough to show that
\[
\sup_{0 \leq t \leq T} \max_{1 \leq j \leq N-1} N |r_t(j) - \tilde{r}_t(j)| \leq C_0.
\]  \hspace{1cm} (5.4)

By definition of \(r_t\) and \(\gamma\), for \(1 \leq j \leq N-1\)
\[
r_t(j) = \frac{\log \left( 1 + [E/N] \tilde{r}_t(j) \right)}{\log(1 + [E/N])}.
\]
Since, by Lemma 10.3
\[
0 \leq \tilde{r}_t(j) \leq 1,
\]
for $1 \leq j \leq N - 1$, $0 \leq t \leq T$, (5.4) holds, which completes the proof of the proposition.

6. PROOF OF PROPOSITION 3.2

Fix $T > 0$ and a sequence of probability measures $\{\mu_N : N \geq 1\}$ fulfilling (2.4).

Lemma 6.1. For every $T > 0$ and $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \sup_{0 \leq t \leq T} \frac{1}{\sqrt{N}} \sum_{j \in \Lambda_N} [\xi_t(j) - \lambda_t(j)]^2 > \delta \right] = 0.$$  (6.1)

Proof: Fix $T > 0$ and $\tau > 0$. It is enough to show that for an appropriate choice of $\tau$, for each $\delta > 0$,

$$\lim_{N \to \infty} \sup_{0 \leq t \leq T} \frac{1}{\tau} \mathbb{P}_{\mu_N} \left[ \sup_{t \leq s \leq t + \tau} \frac{1}{N} \sum_{j \in \Lambda_N} [\xi_s(j) - \lambda_s(j)]^2 > \delta \right] = 0.$$  (6.2)

A long and simple computation shows that for $t \leq s$,

$$\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} [\xi_t(j) - \lambda_t(j)]^2 = \int_t^s \frac{2}{\sqrt{N}} \sum_{j=0}^{N-1} (\xi_r(j) - \lambda_r(j)) [\Omega(\xi_r(j) - \lambda_r(j))] dr$$

$$+ \int_t^s \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \Omega(\xi_r(j))^2 dr$$

$$- \int_t^s a_N \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \xi_r(j) \eta_r(j) \eta_r(j+1) dr + \{M_s - M_t\},$$

where $a_N = N^2 \{e^{\gamma/N} - e^{-\gamma/N} + e^{-2\gamma/N} - 1\}$ is a positive constant and $M_t$ a martingale.

Consider a sequence $\tau = \tau_N$ such that $N^{-1} \ll \tau_N \ll N^{-2/3}$. We show below that with this choice (6.1) holds for each term of the previous decomposition. For instance, by Lemma 9.2 and Tchebycheff inequality,

$$\lim_{N \to \infty} \sup_{0 \leq t \leq T} \frac{1}{\tau} \mathbb{P}_{\mu_N} \left[ \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} [\xi_t(j) - \lambda_t(j)]^2 > \delta \right] = 0.$$  (6.2)

Hence, (6.1) holds for the second term on the left hand side of (6.2) provided $N^{-1} \ll \tau_N$.

Repeating the arguments presented in the proof of Lemma 10.2 we can show that the expression inside the first integral on the right hand side of (6.2) is bounded by

$$\frac{C_0}{\sqrt{N}} \sum_{j=0}^{N-1} [\xi_r(j) - \lambda_r(j)]^2$$
for some finite constant $C_0$. To show that (6.1) holds for this term it is therefore enough to apply Markov inequality and to recall the statement of Lemma 9.2. No condition on $\tau_N$ is needed in this argument due to the time integral.

The expression inside the integral in the second term on the right hand side of (6.2) is bounded by

$$E \left[ \frac{C_0}{\sqrt{N}} \sum_{j=0}^{N-1} N^2 |\xi_r(j+1) - \xi_r(j)|^2 + N \xi_r(0)^2 + N \xi_r(N-1)^2 \right]$$

for some finite constant $C_0$. By (9.1), $\xi_r(0)^2$ and $\xi_r(N-1)^2$ are bounded above by $C_0 N^{-1} \sum_{j \in \Lambda_N} \xi_r(j)^2$, and $|\xi_r(j+1) - \xi_r(j)|$ is less than or equal to $(e^{-\gamma/N} - 1) \xi_r(j)$. The previous expression is thus less than or equal to $C_0 N^{-1/2} \sum_{j \in \Lambda_N} \xi_r(j)^2$.

By Tchebycheff and Hölder inequalities,

$$\mathbb{P}_{\mu_N} \left[ \sup_{t \leq s \leq t+\tau} \int_t^s \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \xi_r(j)^2 \, dr > \delta \right] \leq \frac{N^2 \tau^3}{\delta^2} \mathbb{E}_{\mu_N} \left[ \int_t^{t+\tau} \frac{1}{N} \sum_{j=0}^{N-1} \xi_r(j)^8 \, dr \right].$$

By Lemma 9.1, this expression is bounded above by $C_0 N^2 \tau^4 \delta^{-4}$. The contribution of the second term on the right hand side of (6.2) to (6.1) is thus bounded by $C_0 N^{2/3} \tau^4 \delta^{-4}$, which vanishes, as $N \to \infty$, provided $\tau_N \ll N^{-2/3}$.

Since the third term in (6.2) is negative, it remains to consider the martingale $M_t$. Its quadratic variation $\langle M \rangle_t$, is such that

$$\langle M \rangle_s - \langle M \rangle_t \leq \int_t^s \frac{C_0}{N} \sum_{j=0}^{N-1} \xi_r(j)^2 \left( \frac{1}{N^2} \xi_r(j)^2 + [\xi_r(j) - \lambda_r(j)]^2 \right) \, dr$$

for some finite constant $C_0$ and all $t \leq s$. Therefore, by Doob’s inequality,

$$\mathbb{P}_{\mu_N} \left[ \sup_{t \leq s \leq t+\tau} |M_s - M_t| > \delta \right] \leq \frac{C_0}{\delta^2} \mathbb{E}_{\mu_N} \left[ \int_t^{t+\tau} \frac{1}{N} \sum_{j=0}^{N-1} \xi_r(j)^2 \left( \frac{1}{N^2} \xi_r(j)^2 + [\xi_r(j) - \lambda_r(j)]^2 \right) \, dr \right].$$

By Lemmas 9.1 and 9.2, this expectation is bounded above by $C_0 \tau N^{-1}$. Hence, (6.1) holds for the martingale part in (6.2), which proves the lemma.

**Corollary 6.2.** For every $T > 0$, $\delta > 0$ and $a < 1$,

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \sup_{0 \leq t \leq T} |\xi_t(0) - \lambda_t(0)| > \delta \right] = 0, \quad \lim_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \inf_{0 \leq t \leq T} \xi_t(0) - \lambda_t(0) < a \right] = 0.$$

**Proof:** By the triangular inequality, by Lemma 10.3 and by (9.1), $|\xi_t(0) - \lambda_t(0)|^2$ is bounded by

$$C_1 \left( \frac{1}{N} \right)^2 \xi_t(0)^2 + [\xi_t(j) - \lambda_t(j)]^2 + \left( \frac{1}{N} \right)^2 \xi_t(0)^2,$$

for some finite constant $C_1$ and all $j \in \Lambda_N$. In view of Lemma 9.1 and Lemma 10.4, averaging over $0 \leq j \leq \epsilon N$, the first assertion of the corollary follows from Lemma 6.1.
By Lemma [10.5] there exists a positive constant \( c_0 \), depending only on \( \rho_0 \), \( E \), \( \alpha \), \( \beta \) and \( T \), such that \( \lambda_t(j) \geq c_0 \) for all \( 0 \leq t \leq T \), \( 0 \leq j \leq N - 1 \). Let \( \delta = c_0(1 - a) > 0 \) so that

\[
P_{\mu_N} \left[ \inf_{0 \leq t \leq T} \frac{\xi_t(0)}{\lambda_t(0)} < a \right] \leq P_{\mu_N} \left[ \sup_{0 \leq t \leq T} |\xi_t(0) - \lambda_t(0)| > \delta \right].
\]

Hence, the second assertion of the corollary follows from the first one. \( \square \)

**Proof of Proposition 3.2** By Lemma [10.3] and by (9.1), \( \xi_t(j)/\lambda_t(j) \geq e^\gamma \xi_t(0)/\lambda_t(0) \) for all \( j \in \Lambda_N \). Therefore, by the second assertion of Corollary 6.2, for every \( a < e^\gamma \),

\[
\lim_{N \to \infty} P_{\mu_N} \left[ \inf_{0 \leq t \leq T} \min_{0 \leq j \leq N-1} \frac{\xi_t(j)}{\lambda_t(j)} < a \right] = 0.
\]

Fix \( a < e^\gamma \) and denote by \( \Lambda_\gamma \) the previous set of trajectories.

For each \( 0 < \delta < 1 \) there exists a finite constant \( C(\delta) \) such that

\[
|\log(z) + 1 - z| \leq C(\delta)|1 - z|^2, \quad z \geq \delta.
\]

Therefore, on the set \( \Lambda_\alpha \), by Lemma [10.5] applied to the function \( \lambda_t \), for every function \( \phi : [0, T] \times \Lambda_N \to \mathbb{R} \) satisfying the assumptions of the proposition,

\[
|R^N_t(\phi_t)| \leq \frac{C_1}{\sqrt{N}} \sum_{j=0}^{N-1} (\nabla_N^+ \phi_t)(j) \left( \frac{\xi_t(j) - \lambda_t(j)}{\lambda_t^2(j)} \right)^2 \leq \frac{C_1}{\sqrt{N}} \sum_{j=0}^{N-1} (\xi_t(j) - \lambda_t(j))^2,
\]

for some finite constant \( C_1 \). Hence, the assertion of the proposition follows from Lemma 6.1. \( \square \)

7. PROOF OF PROPOSITION 4.1

Fix a density profile \( \rho_0 \) satisfying the assumptions of Theorem 2.1 and denote by \( \rho(t, x) \) the solution of the viscous Burgers equation (2.2) with initial condition \( \rho_0 \). Let \( \{\mu_N : N \geq 1\} \) be a sequence of probability measures on \( \Sigma_N \) for which (2.4) holds.

Denote by \( \Omega^* \) the adjoint operator of \( \Omega \) with respect to the counting measure. An elementary computation gives that

\[
\begin{align*}
(\Omega^* f)(0) &= (1 - \alpha)EN f(0) \quad + \quad (E + N)(\nabla_N^+ f)(0), \\
(\Omega^* f)(j) &= (\Delta_N f)(j) \quad + \quad E(\nabla_N^+ f)(j), \quad 1 \leq j \leq N - 2, \\
(\Omega^* f)(N - 1) &= -(1 - \beta)EN f(N - 1) \quad - \quad N(\nabla_N^+ f)(N - 1).
\end{align*}
\]

Note that \( \Omega^* \) has exactly the same structure as \( \Omega \). Fix a function \( \psi : \Lambda_N \to \mathbb{R} \), and denote by \( \psi(s, j) \), \( j \in \Lambda_N \), \( s \geq 0 \) the solution of

\[
\begin{align*}
\partial_s \psi_s &= \Omega^* \psi_s, \\
\psi_0(j) &= \psi(j).
\end{align*}
\] (7.1)
Lemma 7.1. Assume that $F$ belongs to $C^4([0, 1])$ and let $F(t, x)$ be the solution of the linear equation

$$
\begin{cases}
\partial_t F = \partial_x^2 F + E\partial_x F , \\
F(0, x) = F(x) , \quad x \in [0, 1],
\end{cases}
$$

with boundary conditions

$$(\partial_x F)(s, 0) = -(1-\alpha)EF(s, 0) , \quad (\partial_x F)(s, 1) = -(1-\beta)EF(s, 1) , \quad s \geq 0 .$$

Suppose that there exists a finite constant $C_0$ such that

$$\max_{j \in \Lambda_N} |\psi(j) - F(j/N)| \leq C_0/N .$$

Then, for every $T > 0$, there exists a finite constant $C_0$ such that

$$\sup_{0 \leq t \leq T} \max_{j \in \Lambda_N} |\psi_t(j) - F_t(j/N)| \leq C_0/N .$$

Proof: By the note following Lemma 10.1 $F$ belongs to $C^{1,3}(\mathbb{R}_+ \times [0, 1])$. As in the proof of Lemma 5.1 let $w_t(j) := \psi_t(j) - F_t(j/N)$. As $F$ belongs to $C^{1,3}(\mathbb{R}_+ \times [0, 1])$, equation (5.2) holds with $\Omega$ replaced by $\Omega^*$ for some function $\varphi(t, j)$ which is absolutely bounded by $C_0N^{-1}$ for $j$ in $\{1, \ldots, N-2\}$ and by $C_0$ for $j = 0$ and for $j = N - 1$. Since, by assumption, the initial condition $w_0$ is also uniformly bounded by $C_0/N$, the arguments presented in the proof of the first assertion of Lemma 5.1 yield that $\psi_t(j) - F_t(j/N)$ is uniformly bounded by $C_0/N$. \qed

Recall the definition of the operator $A_\phi$ introduced at the beginning of Section 4. The proof of Proposition 4.1 relies on the following remarkable identity, derived from a long, but elementary, computation. For every pair of functions $g : \Lambda_N^+ \to \mathbb{R}$, $\phi : \Lambda_N \to \mathbb{R}$,

$$\Omega^* \left( \frac{\nabla^+ g}{\phi} \right)(j) - \left( \nabla^+ g \right)(j) \frac{(\Omega \phi)(j)}{\phi(j)^2} = \frac{[\nabla^+ (A_\phi g)](j)}{\phi(j)} , \quad j \in \Lambda_N . \quad (7.2)$$

Identity (7.2) explains the second identity in (4.3). Indeed, for a time-independent function $g : \Lambda_N^+ \to \mathbb{R}$, since $\partial_s \lambda_s^{-1} = -\lambda_s^{-2} \Omega \lambda_s$, due to (3.2), (3.9) and an integration by parts,

$$J_s^N(g) - J_0^N(g) = \frac{1}{\sqrt{N}} \sum_{j \in \Lambda_N} \int_0^s \frac{(\nabla^+ g)(j)}{\gamma \lambda_r(j)} d\mathcal{M}_r^N(j) \quad (7.3)$$

$$+ \frac{1}{\gamma \sqrt{N}} \sum_{j \in \Lambda_N} \int_0^s \left\{ \Omega^* \left( \frac{\nabla^+ g}{\lambda_s} \right)(j) - \left( \nabla^+ g \right)(j) \frac{(\Omega \lambda_s)(j)}{\lambda_s(j)^2} \right\} \xi_r(j) dr .$$

By (7.2), the expression inside braces in the previous equation is equal to $[\nabla^+ (A_\lambda g)(j)]/\lambda_r(j)$, where $A_\lambda = A_{\lambda_s}$. Hence, if we consider a time-dependent function $g_s$ which solves (4.2), the additive part in the previous decomposition of $J_s^N(g_s) - J_0^N(g_0)$ vanishes, yielding (4.3).
Remark 7.2. Fix a function $G$ in $C^2_t([0,1])$ and $t > 0$. Let $g_s$ be the solution of (4.2) with final condition equal to $G$, $g(t, j) = G(j/N)$, and let $\psi(s, j) = (\nabla_N g_{t-s})(j)/\lambda_{t-s}(j)$, $j \in \Lambda_N$, $0 \leq s \leq t$. By (4.2) and (7.2), in the time interval $[0, t]$, $\psi(s, j)$ solves the equation (7.1) with initial condition

$$\psi_0(j) = (\nabla_N G)(j/N)/\lambda_t(j).$$

In particular, by Lemmas 10.5 and 10.4 there exists a finite constant $C_0$ such that for all $N \geq 1$,

$$\sup_{0 \leq s \leq t} \|\psi_s\|_M \leq C_0. \quad (7.4)$$

Remark 7.3. Similarly, let $G(s, x)$ be the solution of (2.3) with final condition $G(t, x) = G(x)$. A computation, based on a continuous version of equation (7.2), shows that in the time interval $[0, t]$, the function $F_s = \partial_s G_{t-s}/K_{t-s}$ solves the equation appearing in the statement of Lemma 7.1 with initial condition $F(0, x) = (\partial_x G)(x)/K(t, x)$.

Therefore, if $G$ belongs to $C^5_0([0,1])$, since $K$ belongs to $C^2,4(\mathbb{R}_+ \times [0,1])$, $F(0, x) = (\partial_x G)(x)/K_t(x)$ belongs to $C^4([0,1])$. Moreover, by Lemmas 10.5 and 5.1, $\psi(0, j) - F(0, j/N)$ is uniformly bounded by $C_0/N$. Therefore, by Lemma 7.1 there exists a finite constant $C_0$ for which for all $N \geq 1$,

$$\sup_{0 \leq s \leq t} \max_{j \in \Lambda_N} |\psi_s(j) - F_s(j/N)| \leq C_0/N. \quad (7.5)$$

Lemma 7.4. Fix $G$ in $C^5_0([0,1])$ and $t > 0$. Denote by $G(s, x)$ the solution of (2.3) with final condition equal to $G$, and by $g$ the solution of (4.2) with the same final condition. Then, there exists a finite constant $C_0$ such that for all $N \geq 1$,

$$\|G(0, \cdot) - g(0, \cdot)\|_M \leq C_0/N.$$

Proof: Since $G(s, 0) = g_s(0) = 0$ for $0 \leq s \leq t$, for every $j \in \Lambda_N$, by Remarks 7.2 and 7.3

$$|G(0, j/N) - g_{0}(j)| = \frac{1}{N} \sum_{k=0}^{j-1} |(\nabla_N G)(0, k/N) - (\nabla_N g_0)(k/N)|$$

$$= \frac{1}{N} \sum_{k=0}^{j-1} \left| N \int_{k/N}^{(k+1)/N} F(t, y)K(0, y) dy - \psi_0(k/N)\lambda_0(k) \right|.$$

We have seen just before the statement of the lemma, that under the assumptions that $G$ belongs to $C^5_0([0,1])$, $F(0, \cdot)$ belongs to $C^4([0,1])$. Therefore, by the proof of Lemma 7.1 $F$ belongs to $C^{1,3}([0, t] \times [0, 1])$. The assertion of the lemma follows from this remark, from the fact that $\rho_0$ belongs to $C^4([0,1])$ and from (7.5).
Lemma 7.5. For each function $G$ in $C^0_0([0, 1])$ and $t > 0$, the quadratic variation $\langle M^N(t, G) \rangle_s$ of the martingale $M^N_s(t, G)$ converges in $L^1(\mathbb{P}_\mu)$ to

$$2 \int_0^s \int_0^1 \sigma(r, x) \left[ (\partial_x T_{t,r}(G)(x))^2 \right] dx \, dr,$$

where $T_{t,r}(G)$ is the solution of (2.3).

Proof: With the notation introduced just before the statement of the lemma, the quadratic variation of the martingale $M^N_s(t, G)$ can be written as

$$\langle M^N(t, G) \rangle_s = \int_0^s \frac{E^2}{\gamma^2 N} \sum_{j \in \Lambda_N} \xi_r(j)^2 h_j(\eta_r) \psi_{t-r}(j)^2 \, dr. \tag{7.6}$$

By (7.4), $\psi$ is uniformly bounded in the time interval $[0, t]$. Since the cylinder functions $h_j$ are also bounded, by Lemma 9.2, we may replace $\xi_r(j)^2$ by $\lambda_r(j)^2$ in the previous formula paying the price of an error which converges to 0 in $L^1(\mathbb{P}_\mu)$.

For two functions $f, g : \Lambda_N \to \mathbb{R}$, and $1 \leq \ell \leq N/2$, since $b^2 - a^2 = (b-a)(b+a)$,

$$\frac{1}{N} \sum_{j=\ell}^{N-1-\ell} \frac{1}{2\ell + 1} \sum_{k=-\ell}^{\ell} |f(j+k)^2 - f(j)^2| g(j) \leq \frac{4\|f\| \|g\| M}{N} \sum_{j=0}^{N-2} \sum_{j=0}^{N-2} |f(j+1) - f(j)|.$$ 

Applying this identity to $\ell = \epsilon N$, $f = \lambda_r \psi_{t-r}$ and $g(j) = h_j$, by Lemma 10.2 we may replace the quadratic variation of $M^N_s(t, G)$ the term $\lambda_r(j)^2 \psi_{t-r}(j)^2$ by an average of these quantities over a macroscopic interval of length $\epsilon N$, paying the price of an error which vanishes in $L^1(\mathbb{P}_\mu)$, as $N \uparrow \infty$ and then $\epsilon \downarrow 0$. A summation by parts yields that

$$\langle M^N(t, G) \rangle_s = \int_0^s \frac{E^2}{\gamma^2 N} \sum_{j=\epsilon N}^{(1-\epsilon)N} \lambda_r(j)^2 \psi_{t-r}(j)^2 V_{j,\epsilon N}(\eta_{t-r}) \, dr + O(\epsilon),$$

where $V_{j,\epsilon N}(\eta_r) = (2\epsilon N + 1)^{-1} \sum_{|k| \leq \epsilon N} h_{j+k}(\eta_r)$. By Lemma 7.8 below, we may replace $V_{j,\epsilon N}(\eta_r)$ by $2\rho_r(j/N)[1 - \rho_r(j/N)] = 2\sigma(\rho_r(j/N))$ with an error of the same type.

Up to this point we proved that

$$\langle M^N(t, G) \rangle_s = 2 \int_0^s \frac{E^2}{\gamma^2 N} \sum_{j=\epsilon N}^{(1-\epsilon)N} \lambda_r(j)^2 \psi_{t-r}(j)^2 \sigma(\rho_r(j/N)) \, dr + O(\epsilon) + R_{N,\epsilon},$$

where $R_{N,\epsilon}$ is an error which vanishes in $L^1(\mathbb{P}_\mu)$, as $N \uparrow \infty$ and then $\epsilon \downarrow 0$. Note that the first term on the right hand side is deterministic.

By Lemma 5.1 $\lambda_s$ converges to $K_s$, and, by (7.5), $\psi_s$ converges to $F_s = \partial_x G_{t-s}/K_{t-s}$ uniformly in time and space. Since $K_r^2 p^2 = (\partial_x G_r)^2$ and since $\gamma$ converges to $E$, the lemma is proved. \qed
Lemma 7.6. For each function $G$ in $C^0_0((0,1))$ and $t > 0$, there exists a finite constant $C_0$, depending only on $G$ and $t$, such that for all $N \geq 1$, 
\[
\mathbb{E}_{\mu_N} \left[ (M^N(t,G))^2 \right] \leq C_0 , \quad \mathbb{E}_{\mu_N} \left[ \sup_{0 \leq s \leq t} |M^N_s(t,G)|^4 \right] \leq C_0 .
\]

Proof: We first estimate the quadratic variation $\langle M^N(t,G) \rangle_s$, given by (7.6). By (7.4), the solution $\psi_s$ of equation (7.1) is uniformly bounded. As the cylinder function $h_j$ is also bounded, $\langle M^N(t,G) \rangle_s$ is less than or equal to 
\[
C_0 \int_0^s \frac{1}{N} \sum_{j \in \Lambda_N} \xi_r(j)^2 \, dr .
\]

The first assertion of the lemma follows therefore from Lemma 9.1 with $n = 2$. We turn to the second assertion of the lemma. By the Burkholder-Davis-Gundy inequality and by [9, Lemma 3], the second expectation appearing in the statement of the lemma is bounded above by 
\[
C_0 \left\{ \mathbb{E}_{\mu_N} \left[ (M^N(t,G))^2 \right] + \mathbb{E}_{\mu_N} \left[ \sup_{0 \leq s \leq t} |M^N_s(t,G) - M^N_{s-}(t,G)|^4 \right] \right\}
\]
for some finite constant $C_0$. In view of the first part of the proof, it remains to estimate the fourth moment of the jumps. Clearly, $|M^N(t,G) - M^N_{s-}(t,G)| = |J^N_s(g_s) - J^N_{s-}(g_s)|$. By the definition of $J^N_s$ and of $\psi_s$, since $|\xi_{s-}(j)/\xi_s(j)| \leq \varepsilon^{-N}$, and since $\psi_s$ is uniformly bounded, this latter quantity is less than or equal to 
\[
\frac{1}{\gamma N} \sum_{j=0}^{N-1} |(\psi_{t-s})(j)||\xi_s(j) - \xi_{s-}(j)| \leq \frac{C_0}{\gamma^{N/2}} \sum_{j=0}^{N-1} \xi_s(j) .
\]

The second assertion of the lemma follows from Schwarz inequality and from Lemma 9.1. \hfill \Box

Lemma 7.7. Fix $G$ in $C^0_0([0,1])$ and $t > 0$. The sequence of martingales $M^N_s(t,G)$ introduced in (4.3) converges in $D([0,t],\mathbb{R})$ to a mean-zero, continuous martingale, denoted by $M_s(t,G)$. For $G_1$, $G_2$ in $C^0_0([0,1])$, $t_1$, $t_2 > 0$, and $0 \leq s_j \leq t_j$, the covariances of $M_{s_1}(t_1,G_1)$ and $M_{s_2}(t_2,G_2)$ are given by 
\[
\mathbb{E}[M_{s_1}(t_1,G_1) M_{s_2}(t_2,G_2)] = 2 \int_0^{s_1 \wedge s_2} \int_0^1 \sigma(p(r,x)) \left( \partial_x T_{t_1,r} G_1(x) \right) \left( \partial_x T_{t_2,r} G_2(x) \right) \, dx \, dr .
\]

Proof: The proof of the convergence in $D([0,t],\mathbb{R})$ of the martingales $M^N_s(t,G)$ to a mean-zero, continuous martingale, whose quadratic variation is given by the right hand side of the displayed equation appearing in the statement of the lemma with $G_j = G$ and $t_j = t$, relies on [13] Theorem VIII.3.12. We claim that conditions (3.14) and b-(iv) are fulfilled. Condition $[\gamma_3,D]$ (defined in 3.3 page 470 of [13]) follows from Lemma 7.5. By Assertion VIII.3.5 in [13], condition $[\delta_5,D]$ and condition (3.14) are a consequence of 
\[
\lim_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \sup_{s \leq t} |M^N_s(t,G) - M^N_{s-}(t,G)| \right] = 0 .
\]
an assertion which has been proved in the previous lemma.

It remains to prove the formula for the covariances. Fix $G_1, G_2$ in $C_0^\infty([0,1])$, $t_1, t_2 > 0$, $0 \leq s \leq t_1$, and let $s = s_1 \wedge s_2$. Since $M^N_s(t_j, G_j)$, $0 \leq s \leq t_j$, are martingales in $L^2(\mathbb{P}_{\mu_N})$, $\mathbb{E}_{\mu_N} [M^N_s(t_1, G_1) M^N_s(t_2, G_2)] = \mathbb{E}_{\mu_N} [M^N_s(t_1, G_1) M^N_s(t_2, G_2)]$.

By the polarization identity, the computation of the covariance is reduced to the computation of the variance of the martingales $M^N_s(t_1, G_1) \pm M^N_s(t_2, G_2)$. In view of (4.3), the martingale $M^N_s(t_1, G_1) \pm M^N_s(t_2, G_2)$ can be represented as a martingale $M^N_s(t_1, t_2, G_1, G_2)$. The proof of Lemma 7.5 shows that the quadratic variation of this martingale converges in $L^1(\mathbb{P}_{\mu_N})$ to

$$2 \int_0^s \int_0^1 \sigma(r(x)) \left[ \partial_x T_{t_1}(G_1 + T_{t_2}(G_2)(x)) \right]^2 dx dr . \quad (7.7)$$

By the first part of the proof, the martingale $M^N_s(t_1, G_1) \pm M^N_s(t_2, G_2)$ converges in distribution to the martingale $M_s(t_1, G_1) \pm M_s(t_2, G_2)$. As the limit is continuous, the convergence in the Skorohod topology entails convergence in distribution at fixed times. Since, by Lemma 7.6, $M^N_s(t_1, G_1) \pm M^N_s(t_2, G_2)$ is bounded in $L^4(\mathbb{P}_{\mu_N})$,

$$\mathbb{E} \left[ \left\{ M_s(t_1, G_1) \pm M_s(t_2, G_2) \right\}^2 \right] = \lim_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \left\{ M^N_s(t_1, G_1) \pm M^N_s(t_2, G_2) \right\}^2 \right]$$

which completes the proof of the lemma since the right hand side converges to (7.7).

We conclude this section stating a result which permits to replace cylinder functions by functions of the empirical measure. Denote by $\nu_\rho$, $0 \leq \rho \leq 1$, the Bernoulli product measure on $\{0,1\}^\mathbb{Z}$ with density $\rho$. For a function $\hat{h} : \{0,1\}^\mathbb{Z} \to \mathbb{R}$ which depends only on a finite number of sites, let $\hat{h}(\rho) = E_{\nu_\rho} [h(\eta)]$. Denote by $\tau_\eta j, j \in \mathbb{Z}, \eta \in \{0,1\}^\mathbb{Z}$, the configuration $\eta$ translated by $j$: $(\tau_\eta j) (k) = \eta(j + k), k \in \mathbb{Z}$. For a cylinder function $h$, whose support is represented by $\Lambda \subset \mathbb{Z}$, and for a configuration $\eta \in \Sigma_N$ the meaning of $h(\tau_\eta j)$ is clear provided $j + \Lambda \subset \{1, \ldots, N\}$.

**Lemma 7.8.** Let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures in $\Sigma_N$. For every continuous function $G : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ and every cylinder function $h$,

$$\lim_{N \to \infty} \sup \mathbb{E}_{\mu_N} \left[ \int_0^t \left| \frac{1}{N} \sum_j G(s, j/N) h(\tau_j \eta_s) - \int_0^1 G(s, x) \hat{h}(\rho(s, x)) dx \right| ds \right] = 0 ,$$

where $\rho(s, x)$ is the solution of the hydrodynamic equation (2.2) and where the sum over $j$ is carried over all $j$'s for which the support of $h$ is contained in $\Sigma_N - j$.

The proof of this result is similar to the one presented in [14], given the estimate presented in [2] Lemma 3.1.
8. Tightness of the Density Field

We prove in this section that the sequence \( \{ Y^N_t : N \geq 1 \} \) is tight in \( D(\mathbb{R}_+, \mathcal{H}_{-k}) \) for \( k > 7/2 \). Recall from Section 2.3 the definition of the eigenfunctions \( \{ e_n : n \geq 1 \} \) and of the eigenvalues \( \{ \lambda_n : n \geq 1 \} \) of the operator \( -\Delta \) defined on \( C^2_0([0,1]) \). Denote by \( \| \cdot \|_{-k} \) the norm of \( \mathcal{H}_{-k} \), defined as
\[
\| f \|_{-k}^2 = \sum_{n \geq 1} \lambda_n^{-2k} (f, e_n)^2.
\]

By Propositions 3.1, 3.2, and by (3.10), to prove that the sequence \( \{ Y^N_t : N \geq 1 \} \) is tight it is enough to show that the sequence \( \{ J^N_t : N \geq 1 \} \) is tight:

We claim that for every \( k > 7/2, T > 0, \epsilon > 0 \),
\[
\lim_{A \to \infty} \limsup_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \sup_{0 \leq t \leq T} \| J^N_t \|_{-k} > A \right] = 0,
\]
\[
\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \omega_{\delta}(J^N_t) \geq \epsilon \right] = 0,
\]
where, for \( \delta > 0 \),
\[
\omega_{\delta}(J^N_t) = \sup_{|s-t| < \delta} \| J^N_t - J^N_s \|_{-k}.
\]

The first condition in the penultimate displayed equation is a consequence of part (a) of Corollary 8.2. The second condition follows from part (b) of that corollary and from Lemma 8.3.

**Lemma 8.1.** There exists a finite constant \( C_0 \), such that for every \( n \geq 1 \),
\[
\mathbb{E}_{\mu_N} \left[ \sup_{0 \leq t \leq T} \left( J^N_t \frac{1}{\gamma \lambda_t} \nabla_N e_n \right)^2 \right] \leq C_0 n^6.
\]

**Proof:** By (7.2) and (7.3),
\[
J^N_t(e_n) = J^N_0(e_n) + \int_0^t J^N_s(A_s e_n) ds + \mathcal{M}^N_t(e_n),
\]
where \( \mathcal{M}^N_t(e_n) \) is the martingale appearing on the right hand side of (7.3) with \( g = e_n \). We estimate separately each term of the previous expression. By Schwarz inequality,
\[
\mathbb{E}_{\mu_N} \left[ J^N_0(e_n)^2 \right] \leq \frac{1}{\gamma^2} \sum_{j=0}^{N-1} \frac{\left( \nabla_N e_n \right)^2(j/N)^2}{\lambda_0(j)^2} \mathbb{E}_{\mu_N} \left[ (\xi_0(j) - \lambda_0(j))^2 \right].
\]

By assumption (2.4), the expectation is bounded by \( C_0/\gamma \). Hence, since \( \lambda_0 \) is bounded below by a strictly positive constant, the previous sum is less than or equal to \( C_0 n^2 \).
We turn to the time integral term in the decomposition of $J_t^N(e_n)$. By Schwarz inequality, and by the definition of $J_t^N$,

$$
\mathbb{E}_{\mu_N} \left[ \sup_{0 \leq t \leq T} \left( \int_0^t J_s^N(A_s^N e_n) \, ds \right)^2 \right] 
\leq T \int_0^T \frac{1}{\gamma^2 N} \sum_{j=0}^{N-1} \frac{\| \nabla_N^+(A_s e_n)(j) \|^2}{\lambda_s(j)^2} \varphi_s(j, j) \, ds 
+ T \int_0^T \frac{1}{\gamma^2 N} \sum_{j \neq k} \frac{\| \nabla_N^+(A_s e_n)(j) \| \| \nabla_N^+(A_s e_n)(k) \|}{\lambda_s(j) \lambda_s(k)} \varphi_s(j, k) \, ds,
$$

where $\varphi_s(j, j) = \mathbb{E}_{\mu_N} \{|\xi_s(j) - \lambda_s(j)|\}^2$, $\varphi_s(j, k) = \mathbb{E}_{\mu_N} \{|\xi_s(j) - \lambda_s(j)| \{\xi_s(k) - \lambda_s(k)\}\}$. Recall that $\lambda_s(j)$ is bounded below by a strictly positive constant. By Lemma 9.2, $\sup_{0 \leq t \leq T} \max_{j,k} |\varphi_s(j, k)| \leq C_0/N$. On the other hand, in view of Lemma 5.3 by a Taylor expansion and since $(A_s e_n)(0) = (A_s e_n)(N) = 0$,

$$
\sup_{0 \leq s \leq T} \max_{1 \leq j \leq N-2} \left| |\nabla_N^+(A_s e_n)(j)| \right| \leq C_0 n^3,
\sup_{0 \leq s \leq T} \max_{k=0, N-1} \left| |\nabla_N^+(A_s e_n)(k)| \right| \leq C_0 n^2 N.
$$

It follows from these bounds that the penultimate displayed equation is bounded by $C_0 n^6$.

It remains to examine the martingale term in the decomposition of $J_t^N(e_n)$. By the definition (7.3) of the martingale $M_t^N(e_n)$, by Doob’s inequality, and by (3.6),

$$
\mathbb{E}_{\mu_N} \left[ \sup_{0 \leq t \leq T} M_t^N(e_n)^2 \right] \leq \mathbb{E}_{\mu_N} \left[ \int_0^T \frac{4 E^2}{\gamma^2 N} \sum_{j=0}^{N-1} \frac{(\nabla_N^+ e_n)(j)^2}{\lambda_s(j)^2} \xi_s(j) \, ds \right].
$$

Since the cylinder functions $h_j$ are bounded and since, by Lemma 10.5, $\lambda_s$ is uniformly bounded below, by Lemma 9.1 this expression is less than or equal to $C_0 n^2$. This completes the proof of the lemma.

**Corollary 8.2.** For each $k > 7/2$

(a) $\limsup_{N \to +\infty} \mathbb{E}_{\mu_N} \left[ \sup_{0 \leq t \leq T} \| J_t^N \|_{-k}^2 \right] < \infty$

(b) $\lim_{m \to +\infty} \limsup_{N \to +\infty} \mathbb{E}_{\mu_N} \left[ \sup_{0 \leq t \leq T} \sum_{n \geq m} \langle J_t^N, e_n \rangle^2 \lambda_n^{-2k} \right] = 0.$

**Proof:** This result is a consequence of the previous lemma and of the observation that

$$
\sup_{0 \leq t \leq T} \| J_t^N \|_{-k}^2 \leq \sum_{n \geq 1} \lambda_n^{-2k} \sup_{0 \leq t \leq T} | J_t^N(e_n) |^2.
$$
Lemma 8.3. For every \( n \geq 1 \) and every \( \epsilon > 0 \),
\[
\lim_{\delta \to 0} \lim_{N \to +\infty} \sup_{t} \mathbb{P}_{\mu_N} \left[ \sup_{|s-t| < \delta} \left| J^N_t(e_n) - J^N_s(e_n) \right| > \epsilon \right] = 0.
\]

Proof. Recall the decomposition of \( J^N_t(e_n) \) presented at the beginning of the proof of Lemma 8.1. We first claim that for every \( \epsilon > 0 \),
\[
\lim_{\delta \to 0} \lim_{N \to +\infty} \sup_{t} \mathbb{P}_{\mu_N} \left[ \sup_{|s-t| < \delta} \left| \mathcal{M}_t^N(e_n) - \mathcal{M}_s^N(e_n) \right| > \epsilon \right] = 0. \tag{8.1}
\]

Denote by \( \omega_\delta(x) \) the modified modulus of continuity of a path \( x \) in \( D([0, T], \mathbb{R}) \). Since \( \omega_\delta(x) \leq 2\omega_\delta(x) + \sup_{t \leq T} |x_t - x_{t-}| \), to prove (8.1) it is enough to show that for every \( \epsilon > 0 \)
\[
\lim_{\delta \to 0} \lim_{N \to +\infty} \mathbb{P}_{\mu_N} \left[ \omega_\delta(\mathcal{M}_t^N(e_n)) > \epsilon \right] = 0, \tag{8.2}
\]
\[
\lim_{N \to +\infty} \mathbb{P}_{\mu_N} \left[ \sup_{t \leq T} |\mathcal{M}_t^N(e_n) - \mathcal{M}_t(e_n)| > \epsilon \right] = 0.
\]

Clearly, \( |\mathcal{M}_t^N(e_n) - \mathcal{M}_t(e_n)| = |J^N_t(e_n) - J^N_t(e_n)| \). By definition of \( J^N_t \) and since \( |\xi_{t-}(j)/\xi_t(j)| \leq e^{-\gamma/N} \) this latter quantity is less than or equal to
\[
\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \frac{1}{\lambda_t(j)} |\xi_t(j) - \xi_{t-}(j)| \leq \frac{C_0}{N^{3/2}} \sum_{j=0}^{N-1} \xi_t(j).
\]

The second condition of (8.2) follows from the previous estimate, from Markov inequality and from the fact that the expectation of \( \xi_t(j) \) (which is equal to \( \lambda_t(j) \)) is uniformly bounded.

We turn to the first condition of (8.2). By Aldous criterium, it is enough to show that for every \( \epsilon > 0 \)
\[
\lim_{\delta \to 0} \lim_{N \to +\infty} \mathbb{P}_{\mu_N} \left[ \sup_{0 \leq \theta \leq \delta} \left| \mathcal{M}_{\tau+\theta}(e_n) - \mathcal{M}_{\tau}(e_n) \right| > \epsilon \right] = 0,
\]
where \( \Xi_{\tau} \) represents the set of stopping times bounded by \( T \). By Tchebychev inequality and by the explicit expression for the quadratic variation of \( \mathcal{M}_t^N(e_n) \), the previous probability is bounded by
\[
E_{\mu_N} \left[ \int_{\tau}^{\tau+\theta} E_2^2 \sum_{j=0}^{\tau-1} \xi_t(j)^2 \eta_\lambda(j) \sum_{j=0}^{\tau-1} \xi_t(j)^2 \eta_\lambda(j) \right] ds.
\]

By Lemma 9.1 and Lemma 10.5 the previous expectation is bounded above by \( C_0 n^2 \delta / \epsilon^2 \), proving the first assertion of (8.2). This proves (8.1).

We claim that for every \( \epsilon > 0 \)
\[
\lim_{\delta \to 0} \lim_{N \to +\infty} \mathbb{P}_{\mu_N} \left[ \sup_{0 \leq s, t \leq T} \left| \int_{s}^{t} J^N_r(A_e(e_n)) dr \right| > \epsilon \right] = 0 \tag{8.3}
\]
By Tchebychev inequality, the previous probability is bounded by
\[ \frac{\delta}{\epsilon^2} \mathbb{E}_{\mu_N} \left[ \int_0^T \left( \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \nabla^+_N(A_n \epsilon_n)(j/N) \frac{[\xi_r(j) - \lambda_r(j)]}{\lambda_r(j)} \right)^2 dr \right]. \]

The computations performed in the proof of Lemma 8.1 yield that the previous expression is bounded by \( C_0 n^6 \delta/\epsilon^2 \). This proves (8.3).

The assertion of the lemma is a consequence of (8.1), (8.3). \( \square \)

9. EXPONENTIAL ESTIMATES

We present in this section some bounds on the process \( \xi_t \). By (3.8) and by the definition of the variables \( \xi_t(j) \), for \( 0 \leq j \leq N - 2 \),
\[ \xi_t(j) \leq \xi_t(j + 1) \leq e^{-\gamma/N} \xi_t(j). \] (9.1)

**Lemma 9.1.** Fix \( n \geq 1 \), \( T > 0 \) and a sequence of probability measures \( \{\mu_N : N \geq 1\} \) on \( \Sigma_N \). There exists a finite constant \( C_1 \) and \( N_0 \geq 1 \), depending only on \( n, \beta, E \) and \( T \), such that for all \( 0 \leq j \leq N - 1 \) and all \( N \geq N_0 \),
\[ \mathbb{E}_{\mu_N} \left[ \sup_{0 \leq t \leq T} \xi_t(j)^n \right] \leq C_1. \] (9.2)

**Proof:** Fix \( n \geq 1 \) and \( T > 0 \). In the proof \( C_1 \) represents a finite constant which depends only on \( n, \beta, T \) and \( E \) and which may change from line to line. We first claim that
\[ \sup_{0 \leq t \leq T} \max_{0 \leq j \leq N - 1} \mathbb{E}_{\mu_N} [\xi_t(j)^n] \leq C_1. \] (9.3)

A similar computation to the one performed just after (3.4) shows that for each \( 0 \leq j \leq N - 1 \)
\[ \xi_t(j)^n = \xi_0(j)^n + \int_0^t \{[\Omega_n \xi_t^n](j) + A_n(s, j)\} ds + \mathcal{M}_n^N(t, j). \] (9.4)

In this formula, \( \mathcal{M}_n^N(\cdot, j) \) is a zero-mean martingale; \( \Omega_n \) is the linear operator equal to \( \Omega \) in the interior of \( \Lambda_N \) and given at the boundary by
\[
\begin{align*}
(\Omega_n f)(0) &= -\alpha N R_n f(0) + N(\nabla^+_N f)(0), \\
(\Omega_n f)(N-1) &= \beta N S_n f(N-1) - N \left(1 + \frac{E}{N}\right)(\nabla^-_N f)(N-1),
\end{align*}
\] (9.5)

where
\[ R_n = N \left(1 + \frac{E}{N}\right) \left(1 - e^{\nu \gamma/N}\right), \quad S_n = N \left(e^{-\gamma \nu/N} - 1\right); \]

and
\[ A_n(t, j) = -N^2 \left\{ \left(1 + \frac{E}{N}\right) \left(e^{\gamma \nu/N} - 1\right) + (e^{-\gamma \nu/N} - 1) \right\} \xi_t(j)^n \eta_t(j) \eta_t(j + 1). \]

Notice that \( A_1(t, j) = 0 \) and that \( R_1 = S_1 = E \) so that \( \Omega_1 = \Omega \).
It follows from the previous computations that \( f_n(t, j) = E_{\mu_N} \left[ \xi_t(j)^n \right] \) satisfies the differential inequality

\[
\partial_t f(t, j) \leq (\Omega_n) f(t, j).
\]

Let \( F_n(t, \cdot) \) be the solution of equation (3.2), with \( \Omega_n \) instead of \( \Omega \) and initial condition \( F_n(0, j) = f_n(0, j) \). By the maximum principle, \( f_n(t, \cdot) \leq F_n(t, \cdot) \) for all \( t \geq 0 \). Claim (9.2) follows from Lemma 10.4 and the bound \( F_n(0, j) \leq \exp\{-\gamma n\} \).

It remains to bring the supremum inside the expectation. Since, by (9.1), \( \xi_t(j) \) is increasing in \( j \), it is enough to prove the lemma for \( j = N - 1 \). However, by (9.1), \( \xi_t(N - 1) \leq e^{-\gamma} \xi_t(j) \) so that

\[
E_{\mu_N} \left[ \sup_{0 \leq t \leq T} \xi_t(N - 1)^n \right] \leq e^{-\gamma n} E_{\mu_N} \left[ \sup_{0 \leq t \leq T} \frac{1}{N} \sum_{j=0}^{N-1} \xi_t(j)^n \right].
\]

By (9.3),

\[
\xi_t(j)^n \leq \xi_0(j)^n + \int_0^t [\Omega_n \xi^n_s(j)] \, ds + M^N_s(t, j).
\]

We need therefore to estimate three terms. The first one is given by

\[
\frac{1}{N} \sum_{j=0}^{N-1} \xi_0(j)^n \leq e^{-\gamma n}.
\]

The second one is also simple to handle. Since

\[
\frac{1}{N} \sum_{j=0}^{N-1} [\Omega_n \xi^n_j](j) \leq E \xi(0)^n + \beta N (e^{-\gamma n/N} - 1) \xi(N - 1)^n,
\]

we have that

\[
E_{\mu_N} \left[ \sup_{0 \leq t \leq T} \int_0^t \frac{1}{N} \sum_{j=0}^{N-1} [\Omega_n \xi^n_s(j)] \, ds \right] \leq C_1 E_{\mu_N} \left[ \int_0^T \{\xi_s(0)^n + \xi_s(N - 1)^n\} \, ds \right].
\]

By (9.2), this expression is bounded by a constant independent of \( N \). To estimate the martingale term, apply Doob’s inequality and use the fact that the martingales \( M^N_0(t, \cdot) \) are orthogonal to get that

\[
E_{\mu_N} \left[ \left( \sup_{0 \leq t \leq T} \frac{1}{N} \sum_{j=0}^{N-1} M^N_s(t, j) \right)^2 \right] \leq E_{\mu_N} \left[ \int_0^T \frac{C_1}{N^2} \sum_{j=0}^{N-1} \xi_s(j)^{2n} \, dt \right].
\]

By (9.2), this expression is bounded by \( C_1 N^{-1} \), which concludes the proof of the lemma.

Lemma 9.2. Let \( \{\mu_N : N \geq 1\} \) be a sequence of measures on \( \Sigma_N \) satisfying (2.4). Then, for each fixed \( T > 0 \), there exist finite constants \( C_1 \) and \( N_0 \geq 1 \), depending only on \( E, \beta, T \) and \( A_2 \) such that

\[
\sup_{0 \leq t \leq T} \max_{e \in E_N} E_{\mu_N} \left[ (\xi_t(j) - \lambda_t(j))^4 \right] \leq \frac{C_1}{N^2}.
\]
Proof: For $0 \leq k \leq N - 1$ and $t \geq 0$, let $q_t(k, \cdot)$ be the solution of equation (3.2) with initial condition $q_0(k, j) = \delta_{k,j}$. By (3.9),

$$\xi_t(j) = \sum_{k=0}^{N-1} \xi_0(k) q_t(k, j) + \sum_{k=0}^{N-1} \int_0^t q_{t-s}(k, j) dM^N_s(k),$$

so that

$$\xi_t(j) - \lambda_t(j) = \sum_{k=0}^{N-1} \left( \xi_0(k) - \lambda_0(k) \right) q_t(k, j) + \sum_{k=0}^{N-1} \int_0^t q_{t-s}(k, j) dM^N_s(k). \quad (9.5)$$

To prove the lemma we need to estimate the fourth moments of the terms on the right hand side of (9.5).

By Hölder’s inequality,

$$E_{\mu_N} \left[ \left( \sum_{k=0}^{N-1} \left( \xi_0(k) - \lambda_0(k) \right) q_t(k, j) \right)^4 \right] \leq E_{\mu_N} \left[ \sum_{k=0}^{N-1} \left( \xi_0(k) - \lambda_0(k) \right)^4 q_t(k, j) \right] \left( \sum_{k=0}^{N-1} q_t(k, j) \right)^3.$$

Notice that

$$|\xi_0(k) - \lambda_0(k)| \leq C_1 \frac{k}{N} \sum_{j=1}^k \left\{ \eta_0(j) - \rho_0 \left( \frac{j}{N} \right) \right\}$$

for some finite constant $C_1$ which depends only on $E, \beta, T, A_2$, and whose value may change from line to line. Therefore, by assumption (2.4) and since, by (10.10), $\sum_{k=0}^{N-1} q_s(k, j)$ is uniformly bounded in $j$ and $0 \leq s \leq T$, the fourth moment of the first term on the right hand side of (9.5) is bounded by $C_1/N^2$.

We turn to the martingale term in (9.5). For $0 \leq r \leq t$, let $\mathcal{M}^N_{j,t}(r)$ be the martingale defined by

$$\mathcal{M}^N_{j,t}(r) = \sum_{k=0}^{N-1} \int_0^r q_{t-s}(k, j) dM^N_s(k).$$

By the Burkholder-Davis-Gundy inequality and Lemma 3, there exists a finite constant $C_0$ such that

$$E_{\mu_N} [M^N_{j,t}(t)^4] \leq C_0 \left\{ E_{\mu_N} \left[ (\mathcal{M}^N_{j,t}(t))^2 \right] + E_{\mu_N} \left[ \sup_{0 \leq s \leq t} |\mathcal{M}^N_{j,t}(s) - \mathcal{M}^N_{j,t}(s-)|^4 \right] \right\},$$

where $<\mathcal{M}^N_{j,t}>_r$ stands for the quadratic variation of the martingale $\mathcal{M}^N_{j,t}$.

We first estimate the jump term. By (9.5) and by definition of $\xi_s$, $|\mathcal{M}^N_{j,t}(s) - \mathcal{M}^N_{j,t}(s-)| = |\xi_s(j) - \xi_{s-}(j)| \leq (C_0/N) \xi_s(j)$. Hence, by Lemma 9.1, the second expectation on the right hand side of the previous formula is bounded above by $C_0/N^4$. 


It remains to examine the quadratic variation. By (3.6) the quadratic variation of the martingale $M_{j,t}(r)$ is bounded above by

$$C_1 \int_0^r \sum_{k=0}^{N-1} q_{t-s}(k,j)^2 \xi_s(k)^2 \, ds$$

$$\leq C_1 \int_0^r \max_{0 \leq k \leq N-1} q_{t-s}(k,j) \sum_{k=0}^{N-1} q_{t-s}(k,j) \xi_s(k)^2 \, ds.$$

By remark (10.10), $\sum_{k=0}^{N-1} q_{s}(k,j)$ is uniformly bounded in $j$ and $0 \leq s \leq T$, and by Corollary 10.7, max$_{0 \leq k \leq N-1} q_{t-s}(k,j)$ is bounded above by $C_1 \{N^2(t-s)\}^{-1/2}$ for all $N$ large enough and all $j$. Since, by (9.1), $\xi_s(k) \leq \xi_s(N-1)$, $0 \leq k \leq N-1$, the previous expression is less than or equal to

$$C_1 \int_0^r \frac{1}{N \sqrt{t-s}} \xi_s(N-1)^2 \, ds.$$

Hence, by the Cauchy-Schwarz's inequality,

$$\mathbb{E}_{\mu_N} \left[ (M_{j,t}(r))^2 \right] \leq \frac{C_1}{N^2} \mathbb{E}_{\mu_N} \left[ \int_0^t \frac{1}{\sqrt{t-s}} \xi_s(N-1)^4 \, ds \right],$$

which concludes the proof of the lemma in view of Lemma 9.1

10. THE OPERATORS $\Omega_n$

We prove in this section some properties of the solutions of the differential equation $\partial_t f_t = \Omega_n f_t$, where $\Omega_n$ is the linear operator defined by (3.1) and (9.4).

We start with a result on classical solutions of the viscous Burgers equation (2.2).

**Lemma 10.1.** Let $\rho_0$ be density profile in $C^4([0, 1])$. Then, the solution of the viscous Burgers equation (2.2) belongs to $C^{2,3}([0, \infty) \times [0, 1])$ and the solution of the linear equation (5.1) belongs to $C^{2,4}([0, \infty) \times [0, 1])$.

**Proof:** Since $\rho_0$ belongs to $C^4([0, 1])$, $K_0$ defined by (6.1) belongs to $C^{2m+1}([0, 1])$ with $m = 2$. Therefore, the (generalized) Fourier series expansion of the solution $K$ of (5.1) with initial condition $K_0$, provided by the method of separation of variables, yields that $K \in C^{m,2m}([0, \infty) \times [0, 1])$. Moreover, since the semigroup corresponding to (5.1) is positivity improving and since $K_0$ is bounded below by a positive constant, so is $K_t$. Thus, $\rho(t,x) = \partial_x K / EK$, which solves the viscous Burgers equation, is well defined and belongs to $C^{2,3}([0, \infty) \times [0, 1])$. Uniqueness of classical solutions of (2.2) completes the proof.

**Note:** With the same notation as in the previous lemma, assume that $K_0$ belongs to $C^{2m+2}([0, 1])$, $m \geq 0$, so that $\partial_x K_0 \in C^{2m+1}([0, 1])$. Since $\partial_x K$ satisfies the same equation as $K$, one obtains by the previous argument that $\partial_x K \in C^{m,2m}([0, \infty) \times [0, 1])$, so that $K \in C^{m,2m+1}([0, \infty) \times [0, 1])$. 
We turn to the operator $\Omega_n$, which should be understood as a small perturbation of $\Omega_0$, obtained from $\Omega_n$ by setting $\alpha = \beta = 0$, and which represents the generator of a weakly asymmetric random walk on $\Lambda_N$ with reflection at the boundary.

Let $m_N$ be the measure given by

$$m_N(k) = \left(1 + \frac{E}{N}\right)^{-k}, \quad 0 \leq k \leq N - 1.$$ 

Denote by $\langle \cdot, \cdot \rangle_{m_N}$ the scalar product in $L^2(m_N)$. A calculation shows that for each $n \geq 0$, $\Omega_n$ is self-adjoint in $L^2(m_N)$, that is

$$\langle g, \Omega_n f \rangle_{m_N} = \langle \Omega_n g, f \rangle_{m_N}, \quad f, g \in L^2(m_N).$$

For $p \geq 0$, denote by $\| \cdot \|_p$, the $L^p$ norm with respect to $m_N$ and by $D_N$ the Dirichlet form associated to $\Omega_0$ and $m_N$:

$$D_N(f) = \langle f, -\Omega_0 f \rangle_{m_N} = N^2 \sum_{k=0}^{N-2} \left[f(k+1) - f(k)\right]^2 m_N(k).$$

The logarithmic Sobolev inequality for the weakly asymmetric random walk on $\Lambda_N$ with reflection at the boundary \cite[Example 3.6]{example} states that there exists a finite constant $A_0$, depending only on $E$, such that

$$\sum_{k=0}^{N-1} f(k)^2 \log f(k)^2 m_N(k) \leq A_0 D_N(f) \quad (10.1)$$

for all functions $f$ such that $\|f\|_2 = 1$ and all $N \geq 2$.

Fix $n \geq 1$, an initial condition $f : \Lambda_N \to \mathbb{R}$ and denote by $f^{(n)}$ the solution of the linear differential equation

$$\partial_t f^{(n)}_t = \Omega_n f^{(n)}_t, \quad f^{(n)}_0 = f. \quad (10.2)$$

It is not difficult to prove a maximum principle for the solution of this linear equation,

$$f^{(n)}_t(t) \geq 0 \text{ for all } t \geq 0 \text{ if } f \geq 0,$$

and to deduce the existence of a unique solution.

**Lemma 10.2.** Fix $n \geq 1$ and let $f_t = f^{(n)}_t$ be the solution of (10.2). There exists a finite constant $C_0$, depending only on $E$, $\beta$ and $n$, such that for any $t \geq 0$

$$\|f_t\|_2^2 + \int_0^t D_N(f_s) \, ds \leq e^{C_0 t} \|f_0\|_2^2.$$

**Proof:** Fix $n \geq 1$. Differentiating $\|f_t\|_2^2$ yields

$$\frac{1}{2} \frac{d}{ds} \|f_t\|_2^2 = -\alpha N R_n f_s(0)^2 m_N(0) + \beta N S_n f_s(N - 1)^2 m_N(N - 1) - D_N(f_s). \quad (10.3)$$
For every $1 \leq m \leq N$ and every $s \geq 0$,
\[
f_s(N-1)^2 \leq 2e^{-\gamma \left(\frac{m}{N^2}D_N(f_s) + \frac{1}{m}\langle f_s, f_s \rangle_{mN}\right)}.
\] (10.4)

Indeed, fix $1 \leq m \leq N$. By Young’s inequality,
\[
f_s(N-1)^2 \leq 2\left(f_s(N-1) - \frac{1}{m}\sum_{k=N-m}^{N-1} f_s(k)\right)^2 + 2\left(\frac{1}{m}\sum_{k=N-m}^{N-1} f_s(k)\right)^2.
\]

By Schwarz inequality and since $mN(k) \geq e^\gamma$ for $0 \leq k \leq N - 1$, the second term on the right hand side is less than or equal to
\[
\frac{2}{m}\sum_{k=N-m}^{N-1} f_s(k)^2 \leq \frac{2e^{-\gamma}}{m}\sum_{k=0}^{N-1} f_s(k)^2 m_N(k) = \frac{2e^{-\gamma}}{m}\langle f_s, f_s \rangle_{mN}.
\]

The first term on the right hand side can be rewritten as
\[
2\left(\frac{1}{m}\sum_{k=N-m}^{N-1} \sum_{j=k}^{N-2} [f_s(j+1) - f_s(j)]\right)^2 \leq 2\sum_{k=N-m}^{N-1} \sum_{j=k}^{N-2} [f_s(j+1) - f_s(j)]^2.
\]

Since $m_N(k) \geq e^\gamma$ this sum is bounded above by
\[
2me^{-\gamma}\sum_{j=0}^{N-2} [f_s(j+1) - f_s(j)]^2 m_N(j) = 2e^{-\gamma}\frac{m}{N^2}D_N(f_s),
\]

which proves (10.4). Set $m = \lceil Ne^\gamma/4\beta S_n \rceil \wedge N$, where $[a]$ represents the integer part of $a$. Putting together (10.3) and (10.4) yields
\[
\frac{d}{ds}\langle f_s, f_s \rangle_{mN} \leq -D_N(f_s) + C_0(f_s, f_s)_{mN}.
\]

To conclude the proof it remains to apply Gronwall’s inequality. \(\square\)

Next result shows that the solutions of (10.2) are monotone.

**Lemma 10.3.** Fix $n \geq 1$ and a non-negative initial condition $f_0 : \Lambda_N \to \mathbb{R}$ such that $f_0(j) \leq f_0(j+1)$, $0 \leq j < N - 1$. Then, the solution $f_t = f_t^{(n)}$ of (10.2) conserves the monotonicity:
\[
f_t(j) \leq f_t(j+1)
\]
for all $t \geq 0$ and $0 \leq j < N - 1$. Conversely, if the non-negative initial condition is such that $f_0(j+1) \leq e^{-\gamma n/N} f_0(j)$, $0 \leq j < N - 1$, the same property holds at later times:
\[
f_t(j+1) \leq e^{-\gamma n/N} f_t(j)
\]
for all $t \geq 0$ and $0 \leq j < N - 1$. 


Proof: For \( t > 0, j \in \{1, \ldots, N-1\} \), let \( g_t(j) = f_t(j) - f_t(j-1) \). It is easy to show that \( g_t \) satisfies an equation of the form

\[
\frac{d}{dt} g_t = \tilde{\Omega}_n g_t + \psi_t ,
\]

where all the entries in \( \psi_t \) are null except for the first and the last which are equal to \( \alpha N R_n f_t(0) \) and \( \beta N S_n f_t(N-1) \), respectively.

Moreover, \( \tilde{\Omega}_n \) is a tridiagonal matrix whose diagonal elements are equal to \(-N^2(2+E/N)\), upper off-diagonal elements equal \( N^2 \) and lower off-diagonal elements are equal to \( N^2(1+E/N) \).

We may now apply the maximum principle to conclude the proof of the first assertion of the lemma because, as already seen, the solution \( f_t \) is non-negative. Alternatively, we can recall the observation (see [18, Exercise 97, pag. 375]) that for any \( t > 0 \) the exponential \( e^{At} \) of a matrix \( A \) has all its entries positive if and only if all the off-diagonal elements of \( A \) are non-negative. Since that holds for \( \tilde{\Omega}_n \) and \( \Omega_n \), then \( g_t \), which can be written as

\[
g_t = e^{\tilde{\Omega}_n t} g_0 + \int_0^t e^{\tilde{\Omega}_n (t-s)} \psi_s \, ds ,
\]

is non-negative.

The same argument applies to the second assertion. For \( t > 0, j \in \{1, \ldots, N-1\} \), let \( g_t(j) = e^{-\gamma_j/N} f_t(j) - f_t(j-1) \). Then, \( g_t \) satisfies the equation

\[
f_t(j) = e^{\tilde{\Omega}_n t} g_0 + \int_0^t e^{\tilde{\Omega}_n (t-s)} \psi_s \, ds ,
\]

where all the entries in \( \psi_t \) are null except for the first and the last which are equal to \( N(N+E)(1-\alpha)(e^{-\gamma_N/N} - 1)f_t(0) \) and \( N^2(1-\beta)(e^{-\gamma_N/N} - 1)f_t(N-1) \), respectively.

\[\square\]

Lemma 10.4. Fix \( n \geq 1 \) and let \( f_t = f_t^{(n)} \) be the solution of (10.2). There exists a finite constant \( C_0 \), depending only on \( E, \beta \) and \( n \), such that for any \( t \geq 0 \)

\[
\|f_t\|_M \leq C_0 e^{C0 t}\|f_0\|_M .
\]

for all \( t \geq 0 \).

Proof: Let \( g_0 \) be the function which is constant and equal to \( \|f_0\|_M \) and denote by \( g_t \) the solution of (10.2) with initial condition \( g_0 \). By the maximum principle, \( f_t(j)^2 \leq g_t(j)^2 \), for all \( 1 \leq j \leq N, t \geq 0 \).

By Lemma 10.3 \( e^{\gamma_j} g_t(k) \leq g_t(j) \leq e^{-\gamma_j} g_t(k) \) for all \( 0 \leq j, k \leq N-1 \), \( t \geq 0 \), which together with \( m_N(j) \geq c^\gamma, 0 \leq j \leq N-1 \), gives that \( \|g_t\|_M^2 \leq e^{(2n+1)\gamma N^{-1}} \|g_t\|_2^2 \). By Lemma 10.2 \( \|g_t\|_2^2 \leq C_0 e^{C0 t}\|g_0\|_2^2 \). In conclusion,

\[
f_t(j)^2 \leq C_0 e^{C0 t} N^{-1}\|g_0\|_2^2 \leq C_0 e^{C0 t}\|g_0\|_M^2 = C_0 e^{C0 t}\|f_0\|_M^2 ,
\]

which proves the lemma. \[\square\]

Fix \( n \geq 1 \) and denote by \( q_t(j, \cdot) = q_t^{(n)}(j, \cdot) \) the solution of the linear equation (10.2) with initial condition \( q_0(j,k) = \delta_{j,k} \). Fix a function \( f : \Lambda_N \to \mathbb{R} \).

We may represent the solution \( f_t \) of (10.2) with initial condition \( f \) as \( f_t(k) =
In the particular case where \( f(k) = 1 \) for all \( k \in \Lambda_N \), by Lemma 10.4
\[
\max_{k \in \Lambda_N} \sum_{j \in \Lambda_N} q_t(j, k) = \max_{k \in \Lambda_N} f_t(k) \leq C_0 e^{C_0 t}.
\]

**Lemma 10.5.** Fix \( n \geq 1 \), a strictly positive initial condition \( f_0 : \Lambda_N \to \mathbb{R} \) and let \( f_t \) be the solution of (10.2). For every \( T > 0 \), there exists a positive constant \( c_0 \), depending only on \( f_0 \), \( E \), \( \alpha \), \( \beta \) and \( T \), such that
\[
c_0 \leq f_t(j)
\]
for all \( 0 \leq t \leq T \), \( j \in \Lambda_N \).

**Proof.** By the maximum principle, it is enough to prove the lemma for a constant initial profile. Assume, therefore, that \( f_0(j) = a \) for all \( j \in \Lambda_N \) and for some \( a > 0 \). A simple computation shows that
\[
\frac{d}{dt} \frac{1}{N} \sum_{j=0}^{N-1} f_t(j) m_N(j) = \frac{1}{N} \sum_{j=0}^{N-1} (\Omega_n f_t)(j) m_N(j) \geq -\alpha R_n f_t(0) m_N(0).
\]

By Lemma 10.3 \( f_t(0) \leq N^{-1} \sum_{0 \leq j \leq N-1} f_t(j) \). On the other hand, \( m_N(0) = 1 \leq m_N(j) e^{-\gamma} \) for all \( j \in \Lambda_N \). Hence,
\[
\frac{d}{dt} \frac{1}{N} \sum_{j=0}^{N-1} f_t(j) m_N(j) \geq -\alpha R_n e^{-\gamma} \frac{1}{N} \sum_{j=0}^{N-1} f_t(j) m_N(j).
\]

Therefore, by Gronwall’s inequality and since \( R_n \) is bounded above by a finite constant independent of \( N \),
\[
\frac{1}{N} \sum_{j=0}^{N-1} f_t(j) m_N(j) \geq e^{-\alpha t} \frac{1}{N} \sum_{j=0}^{N-1} f_0(j) m_N(j) \geq ae^{-\gamma} e^{-\alpha t}.
\]

A constant profile satisfies both conditions of Lemma 10.3. We may therefore apply this lemma to bound above \( N^{-1} \sum_{j \in \Lambda_N} f_t(j) \) by \( C_0 \min_{k \in \Lambda_N} f_t(k) \), which completes the proof since \( m_N(j) \leq 1 \).

Next result provides a bound for the fundamental solution of (10.2). The proof is based on the classical arguments of hypercontractivity [4][8]. We need, however, to estimate additional terms which appear because \( \Omega_n \) is not a generator.

For \( \epsilon > 0 \), let \( \delta = \epsilon/(1 + \epsilon) \), and let \( \varphi_{\epsilon} : [0, 1] \to [\delta, 1 - 2\epsilon] \) be given by
\[
\varphi_{\epsilon}(t) = \begin{cases} \sqrt{\delta^2 + t}, & \text{for } 0 \leq t \leq 1/8, \\ 1 - \sqrt{4\epsilon^2 + 1 - t}, & \text{for } 7/8 \leq t \leq 1. \end{cases}
\]

We complete the definition of \( \varphi_{\epsilon} \) in the interval \([1/8, 7/8]\) in a way to obtain an increasing \( C^1 \) function whose derivative in the interval \([1/8, 7/8]\) is bounded by
2. Note that this bound is compatible with \( \varphi'_{\epsilon}(1/8) \) and \( \varphi'_{\epsilon}(7/8) \), which are both bounded by \( \sqrt{2} \).

Actually, the exact form of \( \varphi_{\epsilon} \) is irrelevant for the proof of Lemma 10.6. The only important properties needed are that

\[
\int_0^1 \frac{1}{\varphi_{\epsilon}(t)[1 - \varphi_{\epsilon}(t)]} dt < \infty, \quad \text{and} \quad \int_0^1 \varphi_{\epsilon}(t) \log \frac{\dot{\varphi}_{\epsilon}(t)}{\varphi_{\epsilon}(t)[1 - \varphi_{\epsilon}(t)]} dt < \infty,
\]

where \( \dot{\varphi}_{\epsilon}(t) \) represents the derivative of \( \varphi_{\epsilon} \).

**Lemma 10.6.** Fix \( n \geq 1 \) and recall that we denote by \( q_t(j, \cdot) \) the solution of the linear equation (10.2) with initial condition \( q_0(j, k) = \delta_{j,k} \). Assume that \( N \geq n + 1 \) and let \( A_1 = -\gamma n \beta \). There exists a finite constants \( C_0 \), depending only on \( E, \beta \) and \( n \), such that

\[
\max_{0 \leq j,k \leq N-1} q_T(j, k) \leq C_0 e^{C_0 T} \sqrt{N^2 T}
\]

for all \( T \) such that

\[
\log(T N^2) \geq 16, \quad \log(T N^2) \leq \sqrt{\frac{T N^2}{8A_0}}, \quad \log(T N^2) \leq N \left( 1 + \frac{1}{8e^E A_1} \right) .
\]

(10.6)

where \( A_0 \) is given in (10.1).

**Proof:** Here we follow [15] [16]. In this proof \( C_0 \) represents a finite constant depending only on \( \beta, E \) and \( n \), which may change from line to line.

Fix \( 0 \leq k \leq N - 1 \) and \( T \) in the range (10.6). Let \( \epsilon^{-1} = \log(T N^2) \), \( p : [0, T) \to [1 + \epsilon, 2 \epsilon^{-1}] \) be given by \( p(t) = [1 - \varphi_{\epsilon}(t/T)][1 + \epsilon]^{-1} \). Set \( f_t(x) = q_t(x, \cdot) \), \( u_t^2 = f_t^2 \), \( v_t^2 = u_t^2/\|u_t\|_2^2 \). An elementary computation, identical to the one presented at the beginning of the proof of Theorem 2.1 in [15], gives that

\[
\frac{d}{dt} \log \|f_t\|_{p(t)} \leq \frac{\dot{p}(t)}{p(t)^2} \int v_t^2 \log v_t^2 \, dm_N - \frac{2[p(t) - 1]}{p(t)^2} D_N(v_t) + A_1 N v_t(N - 1)^2 .
\]

(10.7)

Set

\[
\ell(t)^2 = N^2 \left\{ \frac{p(t) - 1}{4A_0 p(t)} \wedge 1 \right\} = \frac{T N^2}{A_0} \left\{ \frac{\varphi_{\epsilon}(t/T)[1 - \varphi_{\epsilon}(t/T)]}{4\dot{\varphi}_{\epsilon}(t/T)} \wedge \frac{A_0}{T} \right\} .
\]

By the second condition in (10.6), \( \ell(t) \geq 1 \). Divide the interval \( \Lambda_N \) in subintervals of length \( \ell(t) \). The last interval has length between \( \ell(t) \) and \( 2\ell(t) - 1 \). By the logarithmic Sobolev inequality (10.1) and by the the proof of Lemma 4.3 of [15], since \( m_N(k) \geq e^\gamma \), the first term on the right hand side of (10.7) is less than or equal to

\[
\frac{\dot{p}(t)}{p(t)^2} \left\{ A_0 \frac{4\ell(t)^2}{N^2} D_N(v_t) - \log[e^\gamma \ell(t)] \right\} .
\]
By definition of \( \ell(t) \), the right hand side of (10.7) is bounded by
\[
- \frac{\dot{p}(t)}{2p(t)^2} \log[e^{2\gamma \ell(t)^2}] - \frac{[p(t) - 1]}{p(t)^2} D_N(v_t) + A_1 N v_t(N - 1)^2 .
\] (10.8)

Let
\[
m(t) = N \frac{p(t) - 1}{p(t)^2} \left\{ \frac{1}{2eE A_1} \wedge 4 \right\} = N \varphi_c(t/T)[1 - \varphi_c(t/T)] \left\{ \frac{1}{2eE A_1} \wedge 4 \right\}.
\]

Notice that \( m(t) \leq N \), because \( 0 \leq p(t)^{-1} \leq 1 \). On the other hand, as \( p(t)^{-1}[1 - p(t)^{-1}] \geq \{4 \log(TN^2)\}^{-1} \) and \( N \geq \log(TN^2)\{8eE A_1 \vee 1\} \), we have that \( m(t) \geq 1 \).

Adding and subtracting the average of \( v_t(j) \) over the interval \( \{N - m(t), \ldots, N - 1\} \), and repeating the same argument as in the proof of Lemma 10.2, since \(-\gamma \leq E\), we obtain that
\[
v_t(N - 1)^2 \leq 2m(t) \sum_{j=N-m(t)}^{N-2} \{v_t(j + 1) - v_t(j)\}^2 + \frac{2}{m(t)} \sum_{j=N-m(t)}^{N-1} v_t(j)^2
\]
\[
\leq 2eE m(t) \frac{N^2}{N^2} D_N(v_t) + \frac{2eE}{m(t)}
\]
because \( \|v(t)\|_2 = 1 \). By definition of \( m(t) \), the first term of this expression multiplied by \( A_1 N \) may be absorbed by the Dirichlet form in (10.8). Hence, (10.8) is less than or equal to
\[
- \frac{\dot{p}(t)}{2p(t)^2} \log[e^{2\gamma \ell(t)^2}] + C_0 \frac{p(t)^2}{p(t) - 1} .
\]

Up to this point, we proved that
\[
\log \left( \frac{\|f_T\|_{L^1}}{\|f_0\|_{L^1}} \right) \leq - \int_0^T \frac{\dot{p}(t)}{2p(t)^2} \log[e^{2\gamma \ell(t)^2}] \, dt + C_0 \int_0^T \frac{p(t)^2}{p(t) - 1} \, dt .
\] (10.9)

Since \( \dot{p}(t)p(t)^2 = T^{-1} \varphi_c(t/T) \), in view of (10.6), the first term on the right hand side is less than or equal to
\[
- \frac{1}{2} \log(TN^2) + C_0 + \frac{1}{2} \int_0^1 \varphi_c(t) \log \left\{ \varphi_c(t) \sqrt{T/4A_0} \frac{1}{\varphi_c(t)[1 - \varphi_c(t)]} \right\} \, dt .
\]

Since \( \log(a \vee b) \leq \log_+ a + \log_+ b \), where \( \log_+ a = \log a \vee 0 \), the previous integral can be estimated by the sum of two terms. The first one is \( \log_+ (T/4A_0) \leq C_0 T \), while the second one is
\[
\frac{1}{2} \int_0^1 \varphi_c(t) \log \left\{ \frac{\varphi_c(t)}{\varphi_c(t)[1 - \varphi_c(t)]} \right\} \, dt .
\]

On the interval \([1/8, 7/8]\), \( \varphi_c(t) \) is bounded by 2 and \( \varphi_c(t)[1 - \varphi_c(t)] \) is bounded below by a positive constant independent of the parameters. On the other hand, on the interval \([0, 1/8]\), in view of (10.6), \( \varphi_c(t)/\{\varphi_c(t)[1 - \varphi_c(t)]\} \geq [\delta^2 + t]^{-1} \geq 1 \). Hence, in this interval, the previous integral is bounded by
\[
\frac{1}{4} \int_0^{1/8} \frac{1}{\sqrt{\delta^2 + t}} \log \frac{1}{\delta^2 + t} \, dt \leq C_0 .
\]
A similar analysis can be carried out in the interval $[7/8, 1]$. The second term on the right hand side of (10.9) is equal to

$$C_0 T \int_0^1 \frac{1}{\varphi_r(t)[1 - \varphi_r(t)]} dt \leq C'_0 T.$$ 

Therefore,

$$\log \left( \frac{\|f_T\|_{p(T)}}{\|f_0\|_{p(0)}} \right) \leq - (1/2) \log \{N^2 T\} + C_0 + C_0 T.$$ 

To conclude the proof of the lemma, it remains to observe that $\|f_T\|_M \leq e^{E/2} \|f_T\|_{p(T)}$, $\|f_0\|_{p(0)} \leq 1$. □

**Corollary 10.7.** Fix $n \geq 1$, $T_0 > 0$, and denote by $q_t(j, \cdot)$ the solution of the linear equation (10.2) with initial condition $q_0(j, k) = \delta_{j,k}$. There exist a finite constant $C_0$ and $N_0 \geq 1$, depending only on $E$, $\beta$ and $n$, such that

$$q_t(j, k) \leq C_0 e^{C_0 t} \leq \sqrt{N^2 t}$$

for all $0 \leq t \leq T_0$, $N \geq N_0$, and $0 \leq j, k \leq N - 1$.

**Proof:** Fix $n \geq 1$, $T_0 > 0$, and $0 \leq j \leq N - 1$. There exists $N_0 \geq n + 1$ for which the last condition in (10.6) is satisfied for all $0 \leq t \leq T_0$, $N \geq N_0$.

There exists $a > 0$ such that $\sup_{x \geq a} \log x/\sqrt{x} \leq (8A_0)^{-1/2}$. Let $b = \max\{a, e^{16}\}$.

Fix $0 \leq t \leq T_0$. If $tN^2 \leq b$, by Lemma 10.4

$$\max_{0 \leq k \leq N-1} q_t(j, k) \leq C_0 e^{C_0 t} \leq \sqrt{bC_0 e^{C_0 t}} \leq \sqrt{N^2 t}.$$ 

On the other hand, if $tN^2 \geq b$, $t$ fulfills all the assumptions of the previous lemma. This ends the proof. □

We conclude this section with a remark used several times in the previous sections. Let $f_t(k) = \sum_{j \in \Lambda_N} q_t(j, k)$. Thus, $f$ is the solution of (10.2) with initial condition $f(k) = 1$ for all $k \in \Lambda_N$. By Lemma 10.4, for all $T > 0$, there exists a finite constant $C_0$, depending only on $E$, $\beta$ and $T$ such that

$$\sup_{0 \leq t \leq T} \max_{k \in \Lambda_N} \sum_{j \in \Lambda_N} q_t(j, k) = \sup_{0 \leq t \leq T} \max_{k \in \Lambda_N} f_t(k) \leq C_0.$$  \hspace{1cm} (10.10)

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Departamento de Matemática, PUC-RIO, Rua Marquês de São Vicente, no. 225, 22453-900, Rio de Janeiro, RJ-Brazil and CMAT, Centro de Matemática da Universidade do Minho, Campus de Gualtar, 4710-057, Braga, Portugal

E-mail address: patricia@mat.puc-rio.br

IMPA, Estrada Dona Castorina 110, CEP 22460 Rio de Janeiro, Brasil and CNRS UMR 6085, Université de Rouen, Avenue de l’Université, BP.12, Technopôle du Madrillet, F76801 Saint-Étienne-du-Rouvray, France.

E-mail address: landim@impa.br

Departamento de Matemática, ICEx, UFMG, Campus Pampulha, CEP 31270-901, Belo Horizonte, Brasil.

E-mail address: aniura@mat.ufmg.br