SOME $L_p$-ESTIMATES FOR ELLIPTIC AND PARABOLIC OPERATORS WITH MEASURABLE COEFFICIENTS

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Abstract. We consider linear elliptic and parabolic equations with measurable coefficients and prove two types of $L_p$-estimates for their solutions, which were recently used in the theory of fully nonlinear elliptic and parabolic second order equations in [1]. The first type is an estimate of the $\gamma$th norm of the second-order derivatives, where $\gamma \in (0, 1)$, and the second type deals with estimates of the resolvent operators in $L_p$ when the first-order coefficients are summable to an appropriate power.

Let $d \geq 1$ be an integer and let $\mathbb{R}^d$ be a Euclidean space of points $x = (x^1, \ldots, x^d)$. Consider an operator $L$ of the form

$$L = \partial_t + a^{ij}(t, x)D_{ij} + b^i(t, x)D_i - c(t, x),$$

where and below in the article the summation convention is enforced,

$$D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_iD_j, \quad \partial_t = \frac{\partial}{\partial t},$$

$a(t, x) = (a^{ij}(t, x))$ is a uniformly nondegenerate and bounded matrix-valued, $b(t, x) = (b^i(t, x))$ is an $\mathbb{R}^d$-valued, and $c(t, x)$ is a real-valued measurable functions defined on $\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$.

In this article we are going to discuss two types of estimates for operators like $L$, which were recently used in the theory of fully nonlinear elliptic and parabolic second order equations in [1].

The first type (see Theorems 1.8 and 1.9) is about the possibility to estimate the integrals of $|D^2u|^\gamma$ with some $\gamma \in (0, 1)$ through the $L_p$-norm of $Lu$ and the sup norm of $u$, where $D^2u$ is the Hessian of $u$. This seemingly very weak estimate, discovered for elliptic equations by F.H. Lin, recently played a crucial role in the theory of fully nonlinear elliptic and parabolic equations with VMO “coefficients” (see [1]). In [1] we use a result stated in [8] without proof. Even though the proof

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is not difficult it is still worth presenting it with all details especially because on our way we obtain some new nontrivial information such as Lemma 1.6 or its probabilistic counterpart Theorem 3.1. One more point worth mentioning is that unlike F.H. Lin, who used a rather delicate reversed Hölder’s inequality which was proved an the basis of Gehring’s lemma, we are using a basic result of Krylov-Safonov, which provided the foundation of the theory of fully nonlinear elliptic and parabolic second-order equations. In fact, we need its version obtained in [8] just by analyzing the corresponding arguments in [6]. To obtain the above mentioned estimate we assume that $b$ and $c$ are bounded. Similar estimates we give for $|Du|$, where $Du$ is the gradient of $u$.

The second type of results deals with estimates of the $L_p$-norm of $\mu u$ through the $L_p$-norm of $\mu u - Lu$ if $\mu > 0$ with a constant independent of $\mu$ if $\mu$ is large (see Theorems 4.4 and 5.3). As we have noted these theorems are also used in [1] in particular cases when the drift coefficients are bounded. However, even in this case we could not find a direct reference to the result we needed and, therefore, our explanation in [1] contains the words such as “by analyzing the proof...”. Here we prove the corresponding result with all details and also give its generalization for the case in which $b$ is in $L_q$ with an appropriate $q \leq p$.

1. Estimates of $|D^2u|$

Fix a $\delta \in (0, 1)$ and introduce $S_\delta$ as the set of symmetric $d \times d$-matrices $a = (a^{ij})$ such that for any $\xi \in \mathbb{R}^d$ we have

$$\delta |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq \delta^{-1} |\xi|^2.$$

For constants $K \geq 0$ denote by $\mathcal{L}_{\delta,K}$ the set of operators $L$ of type (0.1), when $a(t, x) = (a^{ij}(t, x))$ is $S_\delta$-valued and $b$ and $c$ are such that

$$|b| + c \leq K, \quad c \geq 0.$$

Let $\mathcal{L}_{\delta,K}^0$ be a subset of $\mathcal{L}_{\delta,K}$ consisting of operators with infinitely differentiable coefficients.

For $\rho, r > 0$ introduce

$$B_r = \{ x \in \mathbb{R}^d : |x| < r \},$$

$$C_{\rho, r} = (0, \rho) \times B_r, \quad \partial' C_{\rho, r} = ([0, \rho] \times \partial B_r) \cup \{ \rho \} \times B_r,$$

$$C_{\rho, r}(t, x) = (t, x) + C_{\rho, r}, \quad \partial' C_{\rho, r}(t, x) = (t, x) + \partial' C_{\rho, r}.$$

Our first goal in this section is to prove the following parabolic version of the main result of [9] by F.H. Lin.
Theorem 1.1. There are constants \( \gamma \in (0, 1] \) and \( N \), depending only on \( \delta, K, \) and \( d \), such that for any \( L \in \mathfrak{L}_{\delta,K} \) and \( u \in W^{1,2}_{d+1,\text{loc}}(C_{2,1}) \cap C(\bar{C}_{2,1}) \) we have
\[
\int_{C_{1,1}(1,0)} |D^2u|^\gamma \, dx \, dt \leq N \left( \int_{C_{2,1}} |L u|^{d+1} \, dx \, dt \right)^{\gamma/(d+1)} + N \sup_{\partial C_{2,1}} |u|^\gamma.
\] (1.1)

This theorem is stated as Corollary 4.2 in [8] but no proof is given there. We fill this gap in this article.

The following theorem is proved in [8].

Theorem 1.2. Let \( u \in W^{1,2}_{d+1}(C_{2,1}) \) and assume that \( u \geq 0 \) on \( \partial C_{2,1} \) and there exists an operator \( \tilde{L} \in \mathfrak{L}_{\delta,K} \) such that \( \tilde{L} u \leq 0 \) in \( C_{2,1} \). Then there exist constants \( \gamma = \gamma(\delta,d,K) \in (0,1] \) and \( N = N(\delta,d,K) \) such that for any \( \lambda > 0 \)
\[
|C_{1,1}(1,0) \cap \{ (t,x) : -\tilde{L} u(t,x) \geq \lambda \}| \leq N \lambda^{-\gamma} u^\gamma(0,0).
\] (1.2)

Corollary 1.3. Under the conditions of Theorem 1.2 for any \( \gamma' \in (0, \gamma) \) we have
\[
\int_{C_{1,1}(1,0)} |L u|^{\gamma'} \, dx \, dt \leq N u^{\gamma'}(0,0),
\] (1.3)
where \( N = N(\delta,d,K,\gamma') \).

Indeed, the left-hand side of (1.3) equals
\[
\int_0^\infty |C_{1,1}(1,0) \cap \{ -\tilde{L} u \geq \lambda^{1/\gamma'} \}| \, d\lambda \\
\leq \int_0^\mu |C_{1,1}(1,0)| \, d\lambda + N u^{\gamma'}(0,0) \int_\mu^\infty \lambda^{-\gamma/\gamma'} \, d\lambda,
\]
where \( \mu = u^{\gamma'}(0,0) \). Upon computing the last integral we arrive at (1.3).

For elliptic operators we have the following version of Theorem 1.2.

Theorem 1.4. Let \( u \in W^2_d(B_1) \) and assume that \( u \geq 0 \) on \( \partial B_1 \) and there exists an operator \( \tilde{L} \in \mathfrak{L}_{\delta,K} \) with coefficients independent of \( t \) such that \( \tilde{L} u \leq 0 \) in \( B_1 \). Then there exist constants \( \gamma = \gamma(\delta,d,K) \in (0,1] \) and \( N = N(\delta,d,K) \) such that for any \( \lambda > 0 \)
\[
|B_1 \cap \{ x : -\tilde{L} u(x) \geq \lambda \}| \leq N \lambda^{-\gamma} u^\gamma(0).
\] (1.4)

Proof. First assume that \( u \in C^2(\bar{B}_1) \) and define a function \( v = v(t,x) \) by \( v(t,x) = u(x) \). By the maximum principle \( u \geq 0 \) in \( B_1 \). Therefore \( v \) satisfies the assumptions of Theorem 1.2 and (1.4) in this particular case follows from (1.2).
In the general case, introduce \( f = -Lu \) and find a sequence of operators \( L_n \in \mathfrak{L}_{\delta, K} \), \( n = 1, 2, \ldots \), with smooth coefficients converging (a.e.) to the corresponding coefficients of \( L \). Also let \( f_n \in C^1(B_1) \), \( n = 1, 2, \ldots \), be a sequence of nonpositive functions such that \( f_n \to f \) in \( L_d(B_1) \). Define \( u_n \in C^2(\overline{B_1}) \) as unique solutions of equations \( L_n u_n = -f_n \) with zero boundary condition. Since, \( u_n \leq u \) on \( \partial B_1 \) and \( L_n(u_n - u) = -f_n + f + (L - L_n)u \to 0 \) in \( L_d(D_1) \), we have that
\[
\lim_{n \to \infty} u_n \leq u
\]
in \( \overline{B_1} \) owing to the Alexandrov estimate. Now we recall that the convergence almost everywhere implies the convergence in distribution and conclude that
\[
F(\lambda) := |B_1 \cap \{ x : f(x) \geq \lambda \}| = \lim_{n \to \infty} |B_1 \cap \{ x : f_n(x) \geq \lambda \}|
\leq N \lambda^{-\gamma} \lim_{n \to \infty} u_n^{\gamma}(0) \leq N \lambda^{-\gamma} u^{\gamma}(0)
\]
at all \( \lambda > 0 \) at which \( F(\lambda) \) is continuous. Since the right-hand side of \( (1.4) \) is continuous in \( \lambda \) and the left-hand side is right continuous, we have \( (1.4) \) for all \( \lambda > 0 \) and the theorem is proved.

As in the case of Corollary 1.3 we have the following.

**Corollary 1.5.** Under the conditions of Theorem 1.4 for any \( \gamma' \in (0, \gamma) \) we have
\[
\int_{B_1} |Lu|^{\gamma'} \, dx \leq Nu^{\gamma'}(0),
\]
where \( N = N(\delta, d, K, \gamma') \).

Here is a useful generalization of Corollary 1.3.

**Lemma 1.6.** There are constants \( \gamma \in (0, 1] \) and \( N \), depending only on \( \delta, K, \) and \( d \), such that, if \( L \in \mathfrak{L}_{\delta, K} \) and \( u \in W^{1,2}_{d+1, \text{loc}}(C_{2,1}) \cap C(\overline{C_{2,1}}) \) and \( Lu = g - f \) in \( C_{2,1} \) with \( f \geq 0 \), then we have
\[
\int_{C_{1,1}(1,0)} |f|^{\gamma} \, dx \, dt \leq N \left( \int_{C_{2,1}} |g_+|^{d+1} \, dx \, dt \right)^{\gamma/(d+1)} + N|u_+(0,0)|^{\gamma} + N \sup_{\partial C_{2,1}} |u_-|^{\gamma}.
\]

**Proof.** First we reduce the general case to the one in which \( u \in W^{1,2}_{d+1}(C_{2,1}) \). To do that we introduce “shifted and dilated” \( u \), that is for \( \varepsilon \in (0, 1) \) we define
\[
u_\varepsilon(t, x) = u(\varepsilon^2 t + 1 - \varepsilon^2, \varepsilon x)).
\]
Obviously, \( u_\varepsilon \in C(\overline{C_1}) \cap W^{1,2}_{d+1}(C_{2,1}) \). We also modify the coefficients of \( L \) in such a way that the new \( g \) and \( f \) are just shifted and dilated
original $g$ and $f$, respectively, times $\varepsilon^2$. If (1.5) holds for $u_{\varepsilon}$, then we obtain it as is by letting $\varepsilon \uparrow 1$ by the monotone convergence theorem owing to the continuity of $u$ in $C_{2,1}$. By the way, we do not assume that the integral in the right-hand side of (1.5) is finite. Thus indeed we may concentrate on $u \in W_{d+1}^{1,2}(C_{2,1})$.

Then observe that if the integral in the right-hand side of (1.5) is infinite, we have nothing to prove. Therefore, we may assume that it is finite. Then $g \in L_{d+1}(C_{2,1})$ since $g+g \geq Lu$. It follows that $f \in L_{d+1}(C_{2,1})$ as well.

Now take an operator $L' \in L^{0}_{0}$ and introduce

$$f' = f + [(L' - L)u]_{-}, \quad g' = g + [(L' - L)u]_{+},$$

so that $L'u = g' - f'$ and $f', g' \in L_{d+1}(C_{2,1})$ and $f' \geq 0$. Obviously, if (1.5) were true with $f', g'$ in place of $f, g$ for any $L'$, then by approximating $L$ by operators $L'$ we would obtain (1.5) in its original form.

Therefore, in the rest of the proof without losing generality we assume that $L \in L^{0}_{0}$ and introduce functions $v$ and $w$ as $W_{d+1}^{1,2}(C_{2,1})$ solutions of

$$Lv = -f, \quad Lw = -g$$

with zero condition on $\partial' C_{2,1}$ for $v$ and with condition $w = -u$ on $\partial' C_{2,1}$. The existence and uniqueness of $v$ and $w$ is a classical result (see, for instance, [11]).

Clearly, $v = u + w$ and by the maximum principle $v \geq 0$. By Corollary 1.3, for an appropriate $\gamma$, the left-hand side of (1.5) is less than a constant times

$$v^\gamma(0, 0) \leq u^\gamma_{+}(0, 0) + w^\gamma_{+}(0, 0).$$

After that it only remains to use the parabolic Alexandrov estimate. The lemma is proved.

For elliptic operators Lemma 1.6 becomes the following.

**Lemma 1.7.** There are constants $\gamma \in (0, 1]$ and $N$, depending only on $\delta$, $K$, and $d$, such that, if $L \in L^{0}_{0}$ has coefficients independent of $t$ and $u \in W_{d,\text{loc}}^{2}(B_{1}) \cap C(B_{1})$ and $Lu = g - f$ in $B_{1}$ with $f \geq 0$, then we have

$$\int_{B_{1}} |f|^\gamma dx \leq N\left( \int_{B_{1}} |g+|^d dx \right)^{\gamma/d} + N|u_{+}(0)|^\gamma + N \sup_{\partial B_{1}} |u_{-}|^\gamma.$$

The proof is based on Corollary 1.5 and consists of repeating the proof of Lemma 1.6 with obvious changes. Of course, at the last step one applies the original Alexandrov estimate rather than its parabolic version.
Proof of Theorem 1.1. Introduce $h = Lu$, take an operator $L' \in \mathcal{L}_{\delta/2,K}$, and observe that
\[ L'u = g - f, \quad g := h + 2[(L'-L)u]_+ , \quad f := |(L'-L)u|. \]
According to (1.5) and the parabolic Alexandrov estimate
\[ \int_{C_{1,1}(1,0)} |(L'-L)u|^{\gamma} \, dxdt \leq N\|[(L'-L)u]_+\|^\gamma_{L_{d+1}(C_{2,1})} \]
\[ + N\|h\|^\gamma_{L_{d+1}(C_{2,1})} + N \sup_{\partial C_{2,1}} |u|^{\gamma}. \quad (1.6) \]

We now use the arbitrariness of $L'$. Obviously, there exists an $\varepsilon = \varepsilon(\delta,d) > 0$ and an operator $L' \in \mathcal{L}_{\delta/2}$ with lower order coefficients coinciding with the ones of $L$ and such that
\[ L'u = Lu - \varepsilon|D^2u|. \]
With such an operator (1.6) becomes (1.1). The theorem is proved.

The reader understands that the following result is obtained by mimicking the proof of Theorem 1.1 and using Lemma 1.7 instead of Lemma 1.6.

Theorem 1.8. There are constants $\gamma \in (0,1]$ and $N$, depending only on $\delta$, $K$, and $d$, such that for any $L \in \mathcal{L}_{\delta,K}$ with the coefficients independent of $t$ and $u \in W^2_{d,loc}(B_1) \cap C(B_1)$ we have
\[ \int_{B_1} |D^2u|^{\gamma} \, dx \leq N\left( \int_{B_1} |Lu|^d \, dx \right)^{\gamma/d} + N \sup_{\partial B_1} |u|^\gamma. \quad (1.7) \]

Next result is stronger than Theorem 1.1 and looks like the right parabolic counterpart of Theorem 1.8. It is proved in [1] on the basis of Theorem 1.1. We give it with a proof just for completeness.

Theorem 1.9. Let $u \in C(\bar{C}_1) \cap W^{1,2}_{d+1,loc}(C_1)$. Then there are constants $\gamma \in (0,1]$ and $N$, depending only on $\delta,d$, and $K$, such that for any $L \in \mathcal{L}_{\delta,K}$ we have
\[ \int_{C_1} |D^2u|^{\gamma} \, dx \leq N \sup_{\partial C_1} |u|^{\gamma} + N \left( \int_{C_1} |Lu|^{d+1} \, dx \right)^{\gamma/(d+1)}. \quad (1.8) \]

Proof. First as in the proof of Lemma 1.6 one reduces the general situation to the one in which $u \in W^{1,2}_{d+1}(C_1)$.

Then we may also assume that the coefficients of $L$ are infinitely differentiable in $\mathbb{R}^{d+1}$. Now set $f = Lu$ in $C_1$ and extend $f(t,x)$ for $t \leq 0$ as zero. Also set $u(t,x) = u(-t,x)$ for $t \leq 0$. Observe that the new $u$ belongs to $W^{1,2}_{d+1}((-1,1) \times B_1)$. After that define $v(t,x)$ as a unique $W^{1,2}_{d+1,loc}((-1,1) \times B_1) \cap C([-1,1] \times B_1)$ solution of $Lv = f$ with
terminal and lateral conditions being \( u \). The existence and uniqueness of such a solution is a classical result (see, for instance, Theorem 7.17 of [11]). By uniqueness \( v = u \) in \( C_1 \), so that owing to Theorem 1.1,

\[
\int_{C_1} |D^2 u|^{\gamma} \ dx \ dt = \int_{C_1} |D^2 v|^{\gamma} \ dx \ dt \leq N \left( \int_{(−1,1) \times B_1} |f|^{d+1} \ dx \ dt \right)^{\gamma/(d+1)}
\]

\[+N \sup_{\partial(−1,1) \times B_1} |v|^{\gamma} = N \left( \int_{C_1} |f|^{d+1} \ dx \ dt \right)^{\gamma/(d+1)} + N \sup_{\partial C_1} |u|^{\gamma}.
\]

The theorem is proved.

2. Estimating \( |Du| \)

**Lemma 2.1.** There are constants \( \gamma \in (0, 1] \) and \( N \), depending only on \( \delta, K \), and \( d \), such that, if \( L \in \mathfrak{L}_{\delta,K} \) and \( u \in W^{1,2}_{d+1,loc}(C_{2,1}) \cap C(\bar{C}_{2,1}) \), then we have

\[
\int_{C_{1,1}(1,0)} |Du|^{\gamma} \ dx \ dt \leq N \left( \int_{C_{2,1}} |Lu|^{d+1} \ dx \ dt \right)^{\gamma/(d+1)} + N \sup_{\partial C_{2,1}} |u|^{\gamma}.
\]  

(2.1)

Proof. It certainly suffices to concentrate on smooth \( u \). In that case observe that

\[
L(−u^2) = g − f, \quad g := −2uLu − cu^2, \quad f = 2a^{ij}(D_i u)D_j u.
\]

By Lemma 1.6 with an appropriate \( \gamma \)

\[
\int_{C_{2,1}} |Du|^{2\gamma} \ dx \ dt \leq N \sup_{C_{2,1}} |u|^{\gamma} \left( \int_{C_{2,1}} |Lu|^{d+1} \ dx \ dt \right)^{(d+1)/\gamma} + N \sup_{C_{2,1}} |u|^{2\gamma}.
\]

After that it only remains to use Jensen’s inequality and again the parabolic Alexandrov estimate. The lemma is proved.

We also have (2.2) for elliptic operators. Therefore, as above, Lemma 1.7 yields

**Theorem 2.2.** There are constants \( \gamma \in (0, 1] \) and \( N \), depending only on \( \delta, K \), and \( d \), such that, if \( L \in \mathfrak{L}_{\delta,K} \) has the coefficients independent of \( t \) and \( u \in W^2_{d,loc}(B_1) \cap C(\bar{B}_1) \), then we have

\[
\int_{B_1} |Du|^{\gamma} \ dx \leq N \sup_{\partial B_1} |u|^{\gamma} + N \left( \int_{B_1} |Lu|^{d} \ dx \right)^{\gamma/d}.
\]

Here is our estimate of \( Du \) in the parabolic case.
Theorem 2.3. Let \( u \in C(C_1) \cap W^{1,2}_{d+1,\text{loc}}(C_1) \). Then there are constants \( \gamma \in (0,1] \) and \( N \), depending only on \( \delta \), \( K \), and \( d \), such that for any \( L \in L_{\delta,K} \) we have

\[
\int_{C_1} |Du|^{\gamma} \, dx \, dt \leq N \left( \int_{C_1} |Lu|^{d+1} \, dx \, dt \right)^{\gamma/(d+1)} + N \sup_{\partial C_1} |u|^{\gamma}.
\]

This theorem is derived from Lemma 2.1 in the same way as Theorem 1.9 is derived from Theorem 1.1.

3. Probabilistic versions

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space endowed with an increasing filtration of \( \sigma \)-fields \( \mathcal{F}_t \subset \mathcal{F} \), \( t \geq 0 \), each of which is complete with respect to \( P, \mathcal{F} \). By \( P \) we denote the predictable \( \sigma \)-field on \( \Omega \times (0, \infty) \) generated by \( \{\mathcal{F}_t, t \geq 0\} \). Let \( w_t, t \geq 0 \), be a \( d_1 \)-dimensional \( \mathcal{F}_t \)-Wiener process on \( \Omega \), where \( d_1 \geq d \) is an integer. Assume that on \( \Omega \times (0, \infty) \) we are given \( P \)-measurable functions \( \sigma_t = \sigma_t(\omega) \) and \( b_t = b_t(\omega) \) with values in the set of \( d \times d_1 \)-matrices and \( \mathbb{R}^d \), respectively. Suppose that \( a_t := (1/2)\sigma_t \sigma_t^* \in S_\delta \) and \( |b_t| \leq K \) for all \((\omega,t)\), where \( K \) and \( \delta \) are fixed constants.

Theorem 3.1. Introduce

\[
x_t = \int_0^t \sigma_s \, dw_s + \int_0^t b_s \, ds,
\]

\[
\tau = \inf \{t \geq 0 : (t, x_t) \notin C_{2,1}\}.
\]

Let \( f(t,x), (t,x) \in (0, \infty) \times \mathbb{R}^d \), be a nonnegative Borel function such that \( f(t,x) = 0 \) for \( t \leq 1 \). Then

\[
\left( \int_{C_{2,1}} f^\gamma \, dx \, dt \right)^{1/\gamma} \leq NE \int_0^\tau f(t,x_t) \, dt, \tag{3.1}
\]

where \( \gamma = \gamma(\delta,d,K) > 0 \) and \( N = N(\delta,d,K) < \infty \).

Proof. First assume that \( f \) is infinitely differentiable in \((t,x)\). Consider the following Bellman’s equation:

\[
\partial_t u + \inf_{a \in S_\delta, |b| \leq K} \left[ a^{ij} \partial_{ij} u + b^i \partial_i u + f \right] = 0 \tag{3.2}
\]

in \( C_{2,1} \) with zero boundary data on \( \partial C_{2,1} \). By Theorem 6.4.1 of [6] this problem has a unique solution bounded and continuous in \( C_{2,1} \) and having bounded and continuous in \( C_{2,1} \) derivatives \( \partial_t u, Du, \) and \( D^2 u \). Actually, to apply Theorem 6.4.1 of [6] directly we need to have the term \( -u \) in the left-hand side of (3.2). However, this is easily achieved
by introducing a new function \( v \) such that \( u = e^{-t} v \). By the maximum principle \( u \geq 0 \).

Obviously,

\[
\partial_t u(t, x_t) + a^{ij}_i D_{ij} u(t, x_t) + b^i_i D_i u(t, x_t) + f(t, x_t) \geq 0, \tag{3.3}
\]

for \( t < \tau \). Furthermore, it is easy to see that there exists an operator \( L \in \mathbb{L}_{\delta, K} \) such that \( Lu = -f \), so that by Lemma 1.6

\[
\int_{C_{1,1}(1,0)} |f|^\gamma \, dx \, dt \leq Nu^\gamma(0,0). \tag{3.4}
\]

Due to (3.3), by Itô’s formula

\[
u(t \wedge \tau, x_{t \wedge \tau}) = u(0, 0) + m_t + \int_0^{t \wedge \tau} \left[ \partial_t u(t, x_t) + a^{ij}_i D_{ij} u(t, x_t) + b^i_i D_i u(t, x_t) \right] dt \geq u(0, 0) - \int_0^{t \wedge \tau} f(s, x_s) \, ds + m_t,
\]

where \( m_t \) is a martingale. Upon plugging in \( t = 4 \), observing that \( 4 \wedge \tau = \tau \) and \( u(\tau, x_\tau) = 0 \), and taking the expectations of the extreme terms we obtain

\[
E \int_0^\tau f(t, x_t) \, dt \geq u(0, 0).
\]

After that to prove (3.1) for infinitely differentiable \( f \), it only remains to use (3.4).

The parabolic Alexandrov estimate in probabilistic terms (see, for instance, Theorem 2 of [4] or Theorem 2.2.4 of [5]) implies that for any Borel \( g \geq 0 \)

\[
E \int_0^\tau g(t, x_t) \, dt \leq N \|g\|_{L_{d+1}(C_{2,1})},
\]

where \( N = N(d, \delta, K) \). This easily allows us to extend (3.1) to the set of bounded Borel \( f \) (vanishing for \( t \leq 1 \)). Finally, applying the monotone convergence theorem we get the desired result. The theorem is proved.

We have derived Theorem 3.1 from Lemma 1.6, but actually Theorem 3.1 is equivalent to Lemma 1.6

In probabilistic terms Lemma 1.7 means the following.

**Theorem 3.2.** There exist constants \( \gamma \in (0, 1) \) and \( N \in (0, \infty) \) depending only on \( \delta, K \) and \( d \), such that if \( f(x) \) is a nonnegative function on \( B_1 \) and

\[
\tau = \inf \{ t \geq 0 : |x_t| = 1 \},
\]


then
\[
\left( \int_{B_1} f^i \, dx \right)^{1/\gamma} \leq NE \int_0^\tau e^{-Kt} f(x_t) \, dt \leq NE \int_0^\tau f(x_t) \, dt. \tag{3.5}
\]

We leave it to the reader to follow closely the proof of Theorem 3.1 by using the corresponding results for elliptic equations from [6].

The probabilistic versions of Lemmas 1.6 and 1.7 allow one to give different proofs of Theorems 1.8 and 1.9. We only show this on the example of Theorem 1.8.

**Proof of Theorem 1.8.** First as in the proof of Lemma 1.6 we may assume that \( u \in W^2_{d}(B_1) \). Then we find an \( \varepsilon = \varepsilon(\delta, d) > 0 \) and an operator \( L' \in \mathcal{L}_{\delta/2,K} \) with coefficients independent of \( t \) such that
\[
L'u = Lu - \varepsilon|D^2u|.
\]

Let \( a = a(x) = (a^{ij}(x)) \), \( b = b(x) = (b^i(x)) \), and \( c = c(x) \) be the coefficients of \( L' \). Define \( \sigma = \sqrt{2a} \). One knows (see, for instance, [2] or [5]) that there always exist \( (\Omega, \mathcal{F}, P) \), \( \mathcal{F}_t \), and \( w_t \) as in the beginning of the section, and there exists an \( \mathcal{F}_t \)-adapted continuous \( \mathbb{R}^d \)-valued process \( x_t, t \geq 0 \), on \( \Omega \) such that with probability one for all \( t \geq 0 \)
\[
x_t = \int_0^t \sigma(x_s) \, dw_s + \int_0^t b(x_s) \, ds.
\]

From [4] (see the comments after Theorem 3 there and see Theorem 4 of [2]) or [5] we know that Itô’s formula is applicable to
\[
u(x_{t\wedge \tau}) \exp \left[ - \int_0^{t\wedge \tau} c(x_s) \, ds \right].
\]

Therefore,
\[
u(0) = Eu(x_{\tau}) \exp \left[ - \int_0^\tau c(x_s) \, ds \right]
\]
\[
- E \int_0^\tau L'u(x_t) \exp \left[ - \int_0^t c(x_s) \, ds \right] \, dt.
\]

By using the probabilistic version of the Alexandrov estimate and the fact that \( 0 \leq c \leq K \) we conclude
\[
\varepsilon E \int_0^\tau |D^2u|(x_t)e^{-Kt} \, dt \leq \varepsilon E \int_0^\tau |D^2u|(x_t) \exp \left[ - \int_0^t c(x_s) \, ds \right] \, dt
\]
\[
= u(0) + E \int_0^\tau Lu(x_t) \exp \left[ - \int_0^t c(x_s) \, ds \right] \, dt
\]
\[
- Eu(x_{\tau}) \exp \left[ - \int_0^{t\wedge \tau} c(x_s) \, ds \right] \leq u(0) + N\|Lu\|_{L^1(B_1)} + \sup_{\partial B_1} |u|,
\]
so that by Theorem 3.2
\[
\left( \int_{B_1} |D^2u|^\gamma \, dx \right)^{1/\gamma} \leq N|u(0)| + N\|Lu\|_{L^d(B_1)} + N \sup_{\partial B_1} |u|. \tag{3.6}
\]

Now observe that the above argument is applicable for \( \varepsilon = 0 \) and \( L' = L \) in which case we get
\[
0 = u(0) + E \int_0^\tau Lu(x_t) \exp \left[ - \int_0^t c(x_s) \, ds \right] dt
- Eu(x_\tau) \exp \left[ - \int_0^\tau c(x_s) \, ds \right],
\]
\[
|u(0)| \leq |E \int_0^\tau Lu(x_t) \exp \left[ - \int_0^t c(x_s) \, ds \right] dt| + \sup_{\partial B_1} |u|.
\]

and to obtain (1.7) from (3.6) it only remains to use the probabilistic version of the Alexandrov estimate once more. The theorem is proved.

**Remark 3.3.** One of consequences of Theorem 3.2 is obtained when one takes \( f \) to be the indicator function of a Borel \( G \subset B_1 \). Then estimate (3.5) says that \( |G|^{1/\gamma} \) is less than a constant \( N \) times the average time spent by \( x_t \) in \( G \) before exiting from \( B_1 \), where \( |G| \) is the Lebesgue measure of \( G \).

It turns out that even in such estimates of the average time spent by \( x_t \) in \( G \) before exiting from \( B_1 \) the constant \( \gamma \) can be very small when \( \delta \) is small, so that there is no hope to get (3.5) with large \( \gamma \) for arbitrary \( f \).

For instance, take \( d = 2, b = c = 0, \)
\[
\bar{a}^{ij}(x) = \delta^{ij} - \varepsilon \frac{x^ix^j}{|x|^2}
\]
for \( x \neq 0 \) and \( \bar{a}^{ij}(x) = \delta^{ij} \), where \( \varepsilon = 1 - \delta \). Then let \( e_1 \) be the first basis vector and set
\[
a^{ij}(x) = \bar{a}^{ij}(x - e_1/2)
\]
Also let \( G \) be the indicator of \( B_r + e_1/2 \), where \( r \in (0, 1/2) \).

Next solve the equation
\[
a^{ij}D_{ij}u(x) = -1 \tag{3.7}
\]
in \( B_{3/2} + e_1/2 \) with zero boundary condition. Then the value at zero of this solution will be the average time spent by \( x_t \) in \( G \) before exiting from \( B_{3/2} + e_1/2 \) and since the latter contains \( B_1 \supset G \), \( u(0) \) is greater
than the average time spent by $x_t$ in $G$ before leaving $B_1$. It turns out that
\[
u(0) = \frac{1 - \varepsilon}{\varepsilon(2 - \varepsilon)} 2^{\varepsilon/(1 - \varepsilon)} (1 - 3 - \varepsilon/(1 - \varepsilon)) r^{(2 - \varepsilon)/(1 - \varepsilon)}
\]
which equals a constant times $|G|^{1/\gamma}$ with
\[
\gamma = \frac{2(1 - \varepsilon)}{2 - \varepsilon}.
\]
Thus, $\gamma$ can be made as small as we wish on the account of taking $\delta$ small enough or $\varepsilon$ close to 1.

One solves (3.7) in polar coordinates with pole at $e_1/2$. Then, if $\rho$ is the polar radius, our function $\nu(x)$ is written as $v(\rho)$ and $v$ satisfies
\[
(1 - \varepsilon)v'' + \frac{1}{\rho} v' = -I_{[0,r]}(\rho)
\]
with boundary conditions $v'(0) = 0$ and $v(3/2) = 0$. The latter equation is easily solvable by using an appropriate integrating factor, yields a function $v$ with bounded second-order derivative, and after noting that $u(0) = v(1/2)$ leads to (3.8).

4. Estimates in $L_p$ of resolvent operators. Parabolic case

For a domain $Q \in \mathbb{R}^{d+1}$ denote by $\partial Q$ the parabolic boundary of $Q$, that is the set of all points $(t_0, x_0) \in \partial Q$, for each of which there exists a $\kappa > 0$ and a continuous $\mathbb{R}^d$-valued function $x(t)$ defined on $[t_0 - \kappa, t_0]$ such that $x(t_0) = x_0$ and $(t, x(t)) \in Q$ for $t \in [t_0 - \kappa, t_0)$. In case $Q = \mathbb{R}^{d+1}$ we have $\partial Q = \partial' Q = \emptyset$.

Take $p \in [d + 1, \infty)$ and introduce
\[
\tilde{W}^{1,2}_p(Q) = \bigcap_{G \subset Q} W^{1,2}_p(G),
\]
where the intersection is taken over all bounded open subsets $G$ of $Q$.

Set
\[
W^{1,2}_p = W^{1,2}_p(\mathbb{R}^{d+1}), \quad L_p = L_p(\mathbb{R}^{d+1})
\]
and denote by $C(\bar{Q})$ the set of bounded continuous functions on $\bar{Q}$.

Next, let
\[
L_0 = \partial_t + a^{ij}(t, x) D_{ij},
\]
where $a(t, x) = (a^{ij}(t, x))$ is a $d \times d$ symmetric matrix-valued function. Let $\mathbb{R}^d$-valued function $b(t, x) = (b^1(t, x), \ldots, b^d(t, x))$ and real-valued function $c(t, x)$ be defined on $\mathbb{R}^{d+1}$. Set
\[
L = L_0 + b^i D_i - c,
\]
fix a $\delta > 0$ and $K \in [0, \infty)$ and impose the following.
**Assumption 4.1.** (i) For any \( \xi \in \mathbb{R}^d \) and all values of the arguments

\[ a^{ij}(t,x)\xi^i \xi^j \geq \delta |\xi|^2, \quad \text{tr} a(t,x) + 1 \leq K, \]

(ii) We have \( b = b_1 + b_2 \), where \( b_1 \) is bounded and \( b_2 \in L_{d+1} \). The function \( c \) is nonnegative and bounded.

Our main goal in this section is to establish estimates like

\[ \lambda(\mu) \| u \|_{L_p(Q)} \leq N \| \mu u - Lu \|_{L_p(Q)}, \tag{4.1} \]

for \( u \in W^{1,2}_p(Q) \) vanishing on \( \partial' Q \), where \( N \) is a constant and the function \( \lambda(\mu) > 0 \) for \( \mu > 0 \) behaves like \( \mu \) as \( \mu \to \infty \). The linear behavior of \( \lambda(\mu) \) for large \( \mu \) is, of course, the best one could expect.

In a particular case of bounded \( b \neq 0 \) as we will see from Corollary 4.7 one can take \( \lambda(\mu) = \mu^2 \) for \( \mu \) close to 0.

**Remark 4.2.** For \( \theta \in (0, \infty) \) we introduce \( \mu_\theta(\lambda) \) as a continuous non-negative increasing function of \( \lambda > 0 \) such that

\[ \| (|b| - \mu_\theta(\lambda))_+ \|_{L_{d+1}} \leq \theta \lambda^{-1/(2d+2)}. \tag{4.2} \]

By our Assumption 4.1 (ii), for any \( \theta, \lambda \in (0, \infty) \), there exists a such a \( \mu_\theta(\lambda) \) satisfying (4.2).

On the other hand, if, for some \( \theta, \lambda \in (0, \infty) \), we can find an appropriate constant \( \mu \), then Assumption 4.1 (ii) is satisfied and one can find \( \mu_\theta(\lambda) \) satisfying (4.2) for all \( \theta, \lambda \in (0, \infty) \). Indeed, then \( (|b| - \mu)_+ \in L_{d+1} \) for some \( \mu \in [0, \infty) \). The latter means that \( b = b_1 + b_2 \), where

\[ b_1 = b I_{|b| \leq \mu} + \frac{b}{|b|} \mu I_{|b| > \mu} \]

is bounded and

\[ b_2 = b I_{|b| > \mu} - \frac{b}{|b|} \mu I_{|b| > \mu} \in L_{d+1}. \]

To satisfy our requirement for \( \mu_\theta(\lambda) \) to be increasing and continuous, as is easy to see, one can just take

\[ \mu_\theta(\lambda) = \inf \{ \mu \geq 0 : \| (|b| - \mu)_+ \|_{L_p} \leq \theta \lambda^{-1/(2d+2)} \} \]

**Remark 4.3.** If \( b_2 \in L_{d+2} \), then for any \( \nu > 0 \)

\[ \int_{\mathbb{R}^{d+1}} \nu(|b_2| - \nu)^{d+1}_+ dxdt \leq \int_{\mathbb{R}^{d+1}} |b_2|^{d+2} dxdt. \]

In particular, for any \( \lambda, \nu \in (0, \infty) \) and \( M := \sup |b_1| \)

\[ \int_{\mathbb{R}^{d+1}} (|b| - \nu \lambda^{1/2} - M)^{d+1}_+ dxdt \leq (\| b_2 \|^{d+2}_{L_{d+2}} / \nu) \lambda^{-1/2} \]
and one can take \( \nu_\theta \lambda^{1/2} + M \) as \( \mu_\theta(\lambda) \) in Remark 4.2 if one chooses

\[
\nu_\theta \geq \|b_2\|_{L^{d+2}} \theta^{-d-1}.
\]

In the following main result of the section the case \( Q = \mathbb{R}^{d+1} \) is allowed. Of course, in that case no conditions on the values of \( u \) on \( \partial'Q \) are necessary.

**Theorem 4.4.** There is a constant \( \theta = \theta(\delta, d) > 0 \) such that, if \( \lambda > 0 \), \( u \in \hat{W}^{1,2}_{d+1}(Q) \cap C(\hat{Q}) \), \( u \leq 0 \) on \( \partial'Q \), and in case \( Q \) is unbounded

\[
\lim_{t+|x| \to \infty, (t,x) \in Q} u(t,x) \leq 0,
\]

then we have:

(i) For \( p = d + 1 \) and \( \mu \geq K \lambda + \mu_\theta(\lambda) \lambda^{1/2} \), it holds that

\[
\lambda \|u_+\|_{L^p(Q)} \leq N \|\mu u - Lu\|_{L^p(Q)};
\]

(ii) For \( p \geq d + 1 \) and \( \mu \geq K \lambda + \mu_\theta(\lambda) \lambda^{1/2} \), it holds that

\[
\lambda^{(d+1)/p} \mu^{1-(d+1)/p} \|u_+\|_{L^p(Q)} \leq N \|\mu u - Lu\|_{L^p(Q)}.
\]

In all cases \( N = N(d, p, \delta) \).

Before we prove this theorem we extract a few corollaries.

**Corollary 4.5.** If \( b_2 \in L_{d+2} \), then for any \( p \geq d + 1 \), \( \mu > 0 \), and \( u \in \hat{W}^{1,2}_{d+1}(Q) \cap C(\hat{Q}) \), such that \( u \leq 0 \) on \( \partial'Q \) and condition (4.3) is satisfied, we have

\[
\mu \|u_+\|_{L^p(Q)} \leq N(K + \nu_\theta + 1)^{(d+1)/p} \|\mu u - Lu\|_{L^p(Q)}
\]

for \( \mu \geq (K + \nu_\theta + 1)M^2 \), where \( \nu_\theta \) and \( M \) are taken from Remark 4.3, and

\[
\mu^{1+(d+1)/p} \|u_+\|_{L^p(Q)} \leq N[(K + \nu_\theta + 1)M^{2(d+1)/p} \|\mu u - Lu\|_{L^p(Q)}
\]

for \( \mu \leq (K + \nu_\theta + 1)M^2 \). In all cases \( N = N(d, p, \delta) \).

Indeed, take \( \mu_\theta(\lambda) \) from Remark 4.3. Then for any \( \mu > 0 \) one can find \( \lambda(\mu) > 0 \) such that

\[
\mu = K \lambda(\mu) + \mu_\theta(\lambda) \sqrt{\lambda(\mu)} = (K + \nu_\theta) \lambda(\mu) + M \sqrt{\lambda(\mu)},
\]

which implies that (4.5) holds with \( \lambda = \lambda(\mu) \). After that it only remains to prove that

\[
\lambda(\mu) \geq \mu/(K + \nu_\theta + 1) \quad \text{if} \quad \mu \geq (K + \nu_\theta + 1)M^2,
\]

\[
\lambda(\mu) \geq \mu^2(K + \nu_\theta + 1)^{-2}M^{-2} \quad \text{if} \quad 0 < \mu \leq (K + \nu_\theta + 1)M^2
\]

This is easily done after observing that \( x := \mu/M^2 \) and \( y := \sqrt{\lambda(\mu)}/M \) satisfy \( x = (K + \nu_\theta)y^2 + y \) and the rest is left to the reader.
Here is a particular case of Corollary 4.5 when $p = d + 1$.

**Corollary 4.6.** If $b_2 \in L_{d+2}$, then for any $\mu > 0$ and $u \in \hat{W}^{1,2}_{d+1}(Q) \cap C(\bar{Q})$, such that $u \leq 0$ on $\partial'Q$ and condition (4.3) is satisfied, we have

$$\mu \|u_+\|_{L_{d+1}(Q)} \leq N(K + \nu_0 + 1)\|\mu u - Lu\|_{L_{d+1}(Q)}$$

if $\mu \geq (K + \nu_0 + 1)M^2$, and

$$\mu^2 \|u_+\|_{L_{d+1}(Q)} \leq N(K + \nu_0 + 1)^2M^2\|\mu u - Lu\|_{L_{d+1}(Q)}$$

if $\mu \leq (K + \nu_0 + 1)M^2$, where $N = N(d, \delta)$.

In the case of bounded $b$ we have a version of Theorem 4.4, which is easier to memorize. The first part of the following result was used, for instance, in [1].

**Corollary 4.7.** Assume that $b$ is bounded and set $M = \sup |b|$. Then for any $\mu > 0$ and $u \in \hat{W}^{1,2}_{d+1}(Q) \cap C(\bar{Q})$, such that $u \leq 0$ on $\partial'Q$ and condition (4.3) is satisfied, we have

$$\mu \|u_+\|_{L_{d+1}(Q)} \leq N(K + 1)\|\mu u - Lu\|_{L_{d+1}(Q)} \quad \text{if} \quad \mu \geq (K + 1)M^2,$$

$$\mu^2 \|u_+\|_{L_{d+1}(Q)} \leq N(K + 1)^2M^2\|\mu u - Lu\|_{L_{d+1}(Q)} \quad \text{if} \quad \mu \leq (K + 1)M^2,$$

where $N = N(\delta, d, K)$.

Indeed, we have $\nu_0 = 0$ and $(K + 1)^{(d+1)/p} \leq K + 1$ whereas

$$\mu^{1+(d+1)/p} \geq \mu^2[(K + 1)M]^{-2+2(d+1)/p}$$

if $\mu \leq (K + 1)M^2$.

**Remark 4.8.** The fact that in Corollary 4.7 we have the factors $\mu$ and $\mu^2$ in different ranges of $\mu$ may look suspicious. In Example 5.6 we give an argument partially explaining this effect.

In the general case, if $b_2 \in L_{d+1}$ one can still obtain estimates like in Corollaries 4.6 and 4.7 for $\mu$ small.

**Corollary 4.9.** Take a $\lambda' > 0$ and set $\mu' := K\lambda' + \mu_0(\lambda')\sqrt{\lambda'}$. Then for any $0 < \mu \leq \mu'$ and $u \in \hat{W}^{1,2}_{d+1}(Q) \cap C(\bar{Q})$, such that $u \leq 0$ on $\partial'Q$ and condition (4.3) is satisfied, we have

$$\mu^{1+(d+1)/p}\|u_+\|_{L_{d+1}(Q)} \leq N(\mu')^{2(d+1)/p}(\lambda')^{-(d+1)/p}\|\mu u - Lu\|_{L_{d+1}(Q)},$$

where $N$ is the constant in (4.5).

Indeed, define $\lambda > 0$ from

$$\mu = K\lambda + \mu_0(\lambda)\sqrt{\lambda}.$$  

(4.7)
This is possible since \( \mu_\theta(\lambda) \) is an increasing and continuous function of \( \lambda \). Then (4.5) holds. Since \( \mu \leq \mu' \) we have that \( \lambda \leq \lambda' \) and
\[
\mu = K\lambda + \mu_\theta(\lambda)\sqrt{\lambda} \leq \sqrt{\lambda}(K\sqrt{\lambda'} + \mu_\theta(\lambda')) = \sqrt{\lambda'}(\lambda')^{-1/2},
\]
\[
\lambda \geq \mu^2(\mu')^{-2}\lambda'.
\]
Finally,
\[
\lambda^{(d+1)/p}\mu^{-1-(d+1)/p} \geq \mu^{1+(d+1)/p}(\mu')^{-2(1-d+1)/p}(\lambda')^{(d+1)/p}.
\]

Remark 4.10. Unfortunately, in general, there is no control on how fast \( \mu_\theta(\lambda) \) may grow to infinity as \( \lambda \to \infty \). Accordingly, we do not know how slow the solution of (4.7) may go to infinity as \( \mu \to \infty \), so that we were able to prove the natural rate \( \mu^{-1} \) of decay of the resolvent operator \( R_\mu \) of \( L \) only if \( b_2 \in L_{d+2} \). Actually, the author conjectures that for some \( b \), if \( p \in [p+1, p+2) \), the \( L_p \to L_p \)-norm of \( R_\mu \) may have as slow power decay as we wish as \( \mu \to \infty \). Still from our results we have that \( \lambda \to \infty \) as \( \mu \to \infty \) so that the norm of \( R_\mu \) as an operator in \( L_p \) does go to zero as \( \mu \to \infty \). We will see later that in the elliptic case we will not have this issue.

**Proof of Theorem 4.4.** Take a \( \varepsilon > 0 \) and define
\[
Q_\varepsilon = \{(t, x) \in Q : u(t, x) > \varepsilon \}.
\]
Obviously \( Q_\varepsilon \) is a bounded domain, \( u - \varepsilon = 0 \) on \( \partial Q_\varepsilon \), and \( u - \varepsilon \in W_{d+1}^{1,2}(Q_\varepsilon) \). If the assertions of the theorem are true when \( Q \) is bounded, \( u \in W_{d+1}^{1,2}(Q) \cap C(\bar{Q}) \), and \( u \leq 0 \) on \( \partial Q \), then, applying them to \( Q_\varepsilon \) and \( u - \varepsilon \) and passing to the limit as \( \varepsilon \downarrow 0 \) on the basis of the monotone convergence theorem, we obtain the assertions in full generality. Therefore, we may assume that \( Q \) is bounded, \( u \in W_{d+1}^{1,2}(Q) \cap C(\bar{Q}) \), and \( u \leq 0 \) on \( \partial Q \).

Next let \( b_n = bI_{|b| \leq n} \) and observe that the original \( \mu_\theta(\lambda) \) is satisfying (4.2) with \( b_n \) in place of \( b \). Hence, if the theorem is true under one more additional assumption that \( b \) is bounded, then under the conditions in (i)
\[
\lambda \|u_+\|_{L_{d+1}(Q)} \leq N(d, \delta)(\|\mu u - L^n u\|_{L_{d+1}(Q)}),
\]
where \( L^n = L + (b_n^i - b_i^i)D_i \). Observe that if \( (b_i^i D_i u)_- \notin L_{d+1}(Q) \), then the right-hand side of (4.4) is infinite and we have nothing to prove. In case \( (b_i^i D_i u)_- \in L_{d+1}(Q) \), we can pass to the limit in (4.8) and obtain (4.4). Similar situation occurs in the case of assertion (ii) of the theorem. Therefore, in the rest of the proof of the theorem we may assume that \( b \) is bounded.

Using approximations we convince ourselves that we may also assume that \( a^{ij}, c \in C_b^\infty \). In that case introduce \( I \) as the set of \( \mu > 0 \) for
each of which the operator $\mu - L$ as an operator from $W^{1,2}_{d+1}$ to $L_{d+1}$ is onto, invertible, and the inverse $R_\mu := (\mu - L)^{-1}$ is bounded as an operator from $L_{d+1}$ onto $W^{1,2}_{d+1}$. Obviously $I$ is an open subset of $(0, \infty)$. It is well known (see, for instance, [7]) that all large $\mu$ are in $I$.

Therefore, it makes sense to introduce $\mu'$ as the smallest number such that $(\mu', \infty) \subset I$.

Also notice that, if $u \in W^{1,2}_{d+1}(Q) \cap C(\bar{Q})$, $u \leq 0$ on $\partial' Q$, and $\mu > \mu'$, then by the maximum principle (see, for instance, Theorem 3.4.2 of [6]) in $Q$ we have

$$u \leq R_\mu[(\mu u - Lu) + I_Q], \quad u_+ \leq R_\mu[(\mu u - Lu) + I_Q].$$

It follows that for any $p$

$$\|u_+\|_{L_p(Q)} \leq \|R_\mu[(\mu u - Lu) + I_Q]\|_{L_p}$$

and reduces the proof of the theorem to proving that $\mu' < K\lambda + \mu_\theta(\lambda)\lambda^{1/2}$ for any $\lambda > 0$ and that

$$\lambda^{(d+1)/p}\mu^{1-(d+1)/p}\|R_\mu f\|_{L_p} \leq N\|f\|_{L_p} \quad (4.9)$$

as long as $f \geq 0$ and $\mu \geq K\lambda + \mu_\theta(\lambda)\lambda^{1/2}$.

First we deal with $p = d + 1$ and then we use the Marcinkiewicz interpolation theorem. For $\mu > \mu'$ denote by $N_\mu$ the norm of $R_\mu$ as an operator acting from $L_{d+1}$ into $L_{d+1}$, that is the least constant $N$ such that

$$\|R_\mu g\|_{L_{d+1}} \leq N\|g\|_{L_{d+1}} \quad (4.10)$$

for all $g \in L_{d+1}$.

Our first goal is to show that

$$N_\mu \leq N\lambda^{-1} \quad (4.11)$$

as long as $\mu > \mu'$ and $\mu \geq K\lambda + \mu_\theta(\lambda)\lambda^{1/2}$, where $N = N(d, \delta)$, $\lambda > 0$ is fixed, and $\theta(\delta, d) > 0$ is to be chosen appropriately.

By the maximum principle $|R_\mu g| \leq R_\mu |g|$, so that $N_\mu$ is also the least constant $N$ for which (4.10) holds for all nonnegative $g \in L_{d+1}$.

Observe that owing to [4] for any nonnegative $f \in C^\infty_0$ there exists a nonnegative function $\psi_\lambda$, which is $\lambda$-convex in $x$, decreasing in $t$ and such that for any $\varepsilon > 0$ (see equation (29) in [4])

$$L_0\psi_\lambda^\varepsilon - \lambda(\text{tr} a + 1)\psi_\lambda^\varepsilon \leq -f^\varepsilon, \quad (4.12)$$

where the notation $v^\varepsilon$ stands for a standard mollification of $v$ with kernel of support diameter $\varepsilon$. Furthermore,

$$\sup \psi_\lambda \leq N\lambda^{-d/(2d+2)}\|f\|_{L_{d+1}}, \quad (4.13)$$

$$\|\psi_\lambda\|_{L_{d+1}} \leq N\lambda^{-1}\|f\|_{L_{d+1}}, \quad (4.14)$$
where the first estimate is a combination of estimates (12) and (13) of [4] and the second one is obtained before Theorem 3 of [4]. The above cited results of [4] are obtained there by using the theory of controlled diffusion processes. The more PDE oriented reader may like the above cited results of [4] and the second one is obtained before Theorem 3 of [4]. The above cited results of [4] and the second one is obtained before Theorem 3 of [4]. The above cited results of [4] and the second one is obtained before Theorem 3 of [4]. The above cited results of [4] and the second one is obtained before Theorem 3 of [4].

We have proved (4.17) for nonnegative $\mu$ depend only on $d$ and $\delta$. We also know that $|D\psi^\varepsilon_\lambda| \leq \psi^\varepsilon_\lambda \sqrt{\lambda}$. Therefore, (4.12) implies that

$$L\psi^\varepsilon_\lambda - \mu \psi^\varepsilon_\lambda \leq -f^\varepsilon + (|b|\sqrt{\lambda} + \lambda(tr \ a + 1) - \mu)\psi^\varepsilon_\lambda.$$  

(4.15)

By the maximum principle

$$\psi_\lambda \geq R_\mu f^\varepsilon - R_\mu (|b|\sqrt{\lambda} + \lambda(tr \ a + 1) - \mu)\psi^\varepsilon_\lambda$$

and (4.13) and (4.14) yield

$$\|R_\mu f^\varepsilon\|_{L_{d+1}} \leq N\lambda^{-1}\|f\|_{L_{d+1}} + N\lambda^{-d/(2d+2)}\|f\|_{L_{d+1}} \|R_\mu (|b|\sqrt{\lambda} + \lambda(tr \ a + 1) - \mu)_+\|_{L_{d+1}}$$

$$\leq N\|f\|_{L_{d+1}} (\lambda^{-1} + \lambda^{-d/(2d+2)}N\|(|b|\sqrt{\lambda} + \lambda(tr \ a + 1) - \mu)_+\|_{L_{d+1}}),$$

(4.16)

where $N' = N'(d, \delta)$.

By using the Alexandrov estimate and Fatou’s lemma we can pass to the limit in (4.16) as $\varepsilon \downarrow 0$ and then we obtain

$$\|R_\mu f\|_{L_{d+1}} \leq N'\|f\|_{L_{d+1}} (\lambda^{-1} + \lambda^{-d/(2d+2)}N\|(|b|\sqrt{\lambda} + \lambda(tr \ a + 1) - \mu)_+\|_{L_{d+1}}),$$

(4.17)

We have proved (4.17) for nonnegative $f \in C_0^\infty$. The Alexandrov estimate and Fatou’s lemma allow us to carry over this estimate to arbitrary nonnegative $f \in L_{d+1}$. After that we recall what was said about $N_\mu$ in connection with (4.10) and by the definition of $N_\mu$ we conclude that

$$N_\mu \leq N'\lambda^{-1} + N'\lambda^{-d/(2d+2)}N\|(|b|\sqrt{\lambda} + K\lambda - \mu)_+\|_{L_{d+1}}.$$  

(4.18)

For $\mu \geq K\lambda + \mu_\theta(\lambda)\lambda^{1/2}$ the factor of $N_\mu$ in (4.18) is dominated by

$$N'\lambda^{-d/(2d+2)}\|(|b|\sqrt{\lambda} - \mu_\theta(\lambda)\sqrt{\lambda})_+\|_{L_{d+1}}$$

$$= N'\lambda^{1/(2d+2)}\|(|b| - \mu_\theta(\lambda))_+\|_{L_{d+1}},$$

which by the definition of $\mu_\theta(\lambda)$ is less than or equal to $N'\theta$. Thus, if $\theta = \theta(d, \delta) > 0$ is chosen in such a way that $N'\theta \leq 1/2$, then $N_\mu \leq 2N'\lambda^{-1}$, which is (4.11). Thus, (4.11) holds if $\mu > \mu'$ and $\mu \geq K\lambda + \mu_\theta(\lambda)\lambda^{1/2}$.

To finish considering the case that $p = d + 1$ it suffices to show that

$$\mu' < K\lambda + \mu_\theta(\lambda)\lambda^{1/2}.$$
To this end suppose that $\mu' \geq K\lambda + \mu_\theta(\lambda)\lambda^{1/2}$. Then by the above for any $u \in W^{1,2}_{d+1}$ the inequality

$$\lambda\|u\|_{L_{d+1}} \leq N\|\mu u - Lu\|_{L_{d+1}}$$

(4.19) holds if $\mu > \mu'$ and by continuity if $\mu = \mu'$ as well. Furthermore, as is well known, under our additional assumptions on the coefficients of $L$, there are constants $M_i < \infty$ such that for any $u \in W^{1,2}_{d+1}$

$$\|u\|_{W^{1,2}_{d+1}} \leq M_1(\|Lu\|_{L_{d+1}} + \|u\|_{L_{d+1}}) \leq M_2(\|\mu u - Lu\|_{L_{d+1}} + (1 + \mu)\|u\|_{L_{d+1}}).$$

Owing to (4.19) (recall that $\lambda > 0$ is fixed)

$$\|u\|_{W^{1,2}_{d+1}} \leq M_1(\|Lu\|_{L_{d+1}} + \|u\|_{L_{d+1}}) \leq M_3(1 + \mu)\|\mu u - Lu\|_{L_{d+1}},$$

where $M_3$ is independent of $u$ and $\mu \geq \mu'$. By the method of continuity applied with respect to $\mu$ this estimate implies that $\mu' \in I$, which yields the desired contradiction with the definition of $\mu'$. This proves that (4.9) holds for $p = d + 1$ and any $f \in L_p$ as long as $\mu \geq K\lambda + \mu_\theta(\lambda)\lambda^{1/2}$.

Next observe that the maximum principle implies that $R_\mu$ is well defined as an operator in $L_\infty$ and its norm is less than or equal to $\mu^{-1}$ for any $\mu > 0$. Then we obtain that (4.9) holds for $p \geq d + 1$ and all $f \in L_p$ by the Marcinkiewicz interpolation theorem. The positivity of the operator $R_\mu$ and the monotone convergence theorem allows us to conclude that (4.9) holds for $p \geq d + 1$ and all $f \geq 0$ as long as $\mu \geq K\lambda + \mu_\theta(\lambda)\lambda^{1/2}$. The theorem is proved.

5. Estimates in $L_p$ of the resolvent operators. Elliptic case

Take $p \in [d, \infty)$, a domain $Q \subset \mathbb{R}^d$, introduce $W_p^2(Q)$ as the usual Sobolev space and $W^2_p(Q)$ as the collection of all $u$ which belong to $W^2_p(G)$ for any bounded subdomain of $G$. Also denote

$$W_p^2 = W_p^2(\mathbb{R}^d), \quad L_p = L_p(\mathbb{R}^d).$$

Introduce

$$L_0 = a^{ij}(x)D_{ij},$$

where $a(x) = (a^{ij}(x))$ is a $d \times d$ symmetric matrix-valued function. Let $\mathbb{R}^d$-valued function $b(x) = (b^1(x), ..., b^d(x))$ and real-valued function $c(x)$ be defined on $\mathbb{R}^d$. Set

$$L = L_0 + b^i D_i - c,$$

fix a $\delta > 0$ and $K \in [0, \infty)$ and impose the following.
Assumption 5.1. (i) For any $x, \xi \in \mathbb{R}^d$
\[ a^{ij}(x)\xi^i\xi^j \geq \delta |\xi|^2, \quad \text{tr} \, a(x) \leq K, \]
(ii) We have $b = b_1 + b_2$, where $b_1$ is bounded and $b_2 \in L_d$. The function $c$ is nonnegative and bounded.

Remark 5.2. For $\theta \in (0, \infty)$ introduce $\mu_\theta \in (0, \infty)$ in such a way that
\[ \|(|b| - \mu_\theta)_+\|_{L_d} \leq \theta. \]
As in Remark 4.2 Assumption 5.1(ii) is satisfied if and only if there exists a $\theta \in (0, \infty)$ such that one can find an appropriate $\mu_\theta$ and in this case one can find an appropriate $\mu_\theta$ for any $\theta \in (0, \infty)$.

In the following theorem the case $Q = \mathbb{R}^d$ is allowed. Of course, in that case no conditions on the values of $u$ on $\partial Q$ are necessary.

Theorem 5.3. There exists a constant $\theta = \theta(\delta, d) > 0$ such that, if $u \in \hat{W}^2_d(Q) \cap C(\bar{Q}), u \leq 0$ on $\partial Q$, and in case $Q$ is unbounded
\[ \lim_{{|x| \to \infty, \ x \in Q}} u(t, x) \leq 0, \quad (5.1) \]
then for any $\lambda > 0$ and $\mu \geq K\lambda + \mu_\theta \lambda^{1/2}$ we have
\[ \lambda^{d/p} \mu^{1-d/p} \|u_+\|_{L_p(Q)} \leq N \|((\mu u) - Lu)_+\|_{L_p(Q)}, \quad (5.2) \]
where $N = N(d, \delta, p)$.

From the proof of Corollary 4.5 we know that if $\mu = K\lambda + \mu_\theta \lambda^{1/2}$, then
\[ \lambda \geq \mu/(K + 1) \quad \text{for} \quad \mu \geq (K + 1)\mu_\theta^2, \]
\[ \lambda \geq \mu^2(K + 1)^{-2} \mu_\theta^{-2} \quad \text{for} \quad \mu \leq (K + 1)\mu_\theta^2. \]
This leads to the following.

Corollary 5.4. For any $\mu > 0$ and $u \in \hat{W}^2_d(Q) \cap C(Q)$, such that $u \leq 0$ on $\partial Q$ and (5.1) holds, we have
\[ \mu \|u_+\|_{L_p(Q)} \leq N(K + 1)^{d/p} \|((\mu u) - Lu)_+\|_{L_p(Q)} \]
if $\mu \geq (K + 1)\mu_\theta^2$ and
\[ \mu^{1+d/p} \|u_+\|_{L_p(Q)} \leq N[(K + 1)\mu_\theta]^{2d/p} \|((\mu u) - Lu)_+\|_{L_p(Q)} \]
if $\mu \leq (K + 1)\mu_\theta^2$, where $N$ is the constant in (5.2).

In the case of bounded $b$ our results lead to a simpler statement.
Corollary 5.5. Assume that $b$ is bounded and set $M = \sup |b|$. Then for any $\mu > 0$ and $u \in \hat{W}^2_d(Q) \cap C(\bar{Q})$, such that $u \leq 0$ on $\partial Q$ and (5.1) holds, we have
\[
\mu \| u_+ \|_{L^p(Q)} \leq N(K + 1) \| (\mu u - Lu)_+ \|_{L^p(Q)} \quad \text{if} \quad \mu \geq (K + 1)M^2,
\]
\[
\mu^2 \| u_+ \|_{L^p(Q)} \leq N(K + 1)^2 M^2 \| (\mu u - Lu)_+ \|_{L^p(Q)} \quad \text{if} \quad \mu \leq (K + 1)M^2,
\]
where $N$ is the constant in (5.2).

The proof is almost identical to the proof of Corollary 4.7. It turns out that for $p = d = 1$ the result of Corollary 5.5 (especially concerning the case of small $\mu$) is rather sharp.

Example 5.6. Take $d = p = 1$ a constant $M > 0$ and for $\mu > 0$ consider the equation $u'' + bu' - \mu u = -f$, where $b(x) = -M\text{sign } x$. If $f$ approaches the $\delta$-function concentrated at the origin, its $L_1$-norm tends to one. The limit of $L_1$-norms of the corresponding solution will be the $L_1$-norm of the fundamental solution with “pole” at the origin. This solution is $e^{-\nu|x|}/(2\nu)$, where
\[
\nu = \frac{\sqrt{M^2 + 4\mu} - M}{2} = \frac{2\mu}{\sqrt{M^2 + 4\mu} + M}.
\]
Hence the limit $L_1$-norm of the solutions is
\[
\frac{1}{\nu^2} = \left[\frac{\sqrt{M^2 + 4\mu} + M}{4\mu^2}\right]^2.
\]
The last expression should be multiplied by $\mu$ in order to become bounded for large $\mu$ and by $\mu^2$ in order to become bounded for small $\mu$. This shows that the dichotomy is unavoidable and has exactly the form as in Corollary 5.5.

Proof of Theorem 5.3. We first concentrate on the case that $p = d$. As in the proof of Theorem 4.4 we may assume that $Q$ is bounded $u \in W^2_p(Q) \cap C(\bar{Q})$, $b$ is bounded, and $a^{ij}$ and $c$ are infinitely differentiable with bounded derivatives. In that case it is well known (see, for instance, Theorem 11.6.2 in [7]) that for any $\mu > 0$ the operator $\mu - L$ as an operator from $W^2_p$ to $L_d$ is onto, invertible, and the inverse $R_\mu := (\mu - L)^{-1}$ is bounded as an operator from $L_d$ onto $W^2_d$. Denote by $N_\mu$ the norm of $R_\mu$ as an operator acting from $L_d$ into $L_d$, that is the least constant $N$ such that
\[
\| R_\mu g \|_{L_d} \leq N \| g \|_{L_d}
\]
for all $g \in L_d$. By the same reasons as in the proof of Theorem 4.4 to prove the theorem for $p = d$ we need only show that $N_\mu \leq N\lambda^{-1}$. 

Observe that, if \( \lambda = 1 \), then, owing to [3], for any nonnegative \( f \in C_0^\infty \) there exists a nonnegative function \( \psi_\lambda \), which is \( \lambda \)-convex in \( x \) and
\[
L_0 \psi_\lambda^\varepsilon - \lambda \text{tr} a \psi_\lambda^\varepsilon \leq -f^\varepsilon,  
\] (5.4)
where the notation \( v^\varepsilon \) stands for a standard mollification of \( v \) with kernel of support diameter \( \varepsilon \) (see the proof of Lemma 1 of [3]). Furthermore (see equation (22) in [3] and the end of the proof of Theorem 2 of [3]),
\[
\sup \psi_\lambda \leq N \lambda^{-1/2} \| f \|_{L_d},  
\] (5.5)
\[
\| \psi_\lambda \|_{L_d} \leq N \lambda^{-1} \| f \|_{L_d},  
\] (5.6)
where \( N = N(d, \delta) \). Actually, dilations show that one can take any \( \lambda > 0 \). These results are obtained in [3] by probabilistic methods. In terms of PDEs the existence of \( \psi_\lambda \) with the properties described above can be found in Theorem 3.2.3 of [6].

We also know that \( |D\psi_\lambda^\varepsilon| \leq \psi_\lambda^\varepsilon \sqrt{\lambda} \). Therefore (5.4) implies that
\[
L\psi_\lambda^\varepsilon - \mu \psi_\lambda^\varepsilon \leq -f^\varepsilon + (|b| \sqrt{\lambda} + \lambda \text{tr} a - \mu) \psi_\lambda^\varepsilon.  
\] (5.7)

By the maximum principle
\[
\psi_\lambda \geq R_\mu f^\varepsilon - R_\mu (|b| \sqrt{\lambda} + \lambda \text{tr} a - \mu) \psi_\lambda^\varepsilon  
\]
and (5.5) and (5.6) yield
\[
\| R_\mu f^\varepsilon \|_{L_d} \leq N \lambda^{-1} \| f \|_{L_d}
\]
\[
+ N \lambda^{-1/2} \| f \|_{L_d} \| R_\mu (|b| \sqrt{\lambda} + \lambda \text{tr} a - \mu)_+ \|_{L_d}
\]
\[
\leq N' \| f \|_{L_d} \left( \lambda^{-1} + \lambda^{-1/2} \mu \text{tr} a - \mu \right)_+ \|_{L_d},  
\] (5.8)
where \( N' = N'(d, \delta) \).

By using the Alexandrov estimate and Fatou’s lemma we can pass to the limit in (5.8) as \( \varepsilon \downarrow 0 \) and then we obtain
\[
\| R_\mu f \|_{L_d} \leq N \lambda^{-1} \| f \|_{L_d}
\]
\[
\leq N' \| f \|_{L_d} \left( \lambda^{-1} + \lambda^{-1/2} \mu \text{tr} a - \mu \right)_+ \|_{L_d}.  
\] (5.9)

We have proved (5.9) for nonnegative \( f \in C_0^\infty \). The Alexandrov estimate and Fatou’s lemma allow us to carry over this estimate to arbitrary nonnegative \( f \in L_d \). As in the proof of Theorem 4.4 constant \( N_\mu \) is also the smallest constant for which (5.3) holds for all nonnegative \( g \). Therefore, now (5.9) implies that
\[
N_\mu \leq N' \lambda^{-1} + N' \lambda^{-1/2} N_\mu \left( |b| \sqrt{\lambda} + K \lambda - \mu \right)_+ \|_{L_d}.  
\] (5.10)

For \( \mu \geq K \lambda + \mu_0 \lambda^{1/2} \) the factor of \( N_\mu \) in (4.18) is dominated by
\[
N' \lambda^{-1/2} \| (|b| \sqrt{\lambda} - \mu_0 \sqrt{\lambda})_+ \|_{L_d} = N' \| (|b| - \mu_0)_+ \|_{L_d} \leq N' \theta.  
\]
Thus, if $\theta$ is chosen in such a way that $N'\theta \leq 1/2$, then $N'_{\mu} \leq 2N'\lambda^{-1}$, which is equivalent to (5.2) for $p = d$ as it was explained above.

This proves the theorem for $p = d$. For general $p \geq d$ it suffices to use the Marcinkiewicz interpolation theorem as in the proof of Theorem 4.4. The theorem is proved.

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