Normal form for maps with nilpotent linear part

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The normal form for an n-dimensional map with irreducible nilpotent linear part is determined using sl₂-representation theory. We sketch by example how the reducible case can also be treated in an algorithmic manner. The construction (and proof) of the sl₂-triple from the nilpotent linear part is more complicated than one would hope for, but once the abstract sl₂ theory is in place, both the description of the normal form and the computational splitting to compute the generator of the coordinate transformation can be handled explicitly in terms of the nilpotent linear part without the explicit knowledge of the triple. If one wishes one can compute the normal form such that it is guaranteed to lie in the kernel of an operator and one can be sure that this is really a normal form with respect to the nilpotent linear part; one can state that the normal form is in sl₂-style. Although at first sight the normal form theory for maps is more complicated than for vector fields in the nilpotent case, it turns out that the final result is much better. Where in the vector field case one runs into invariant theoretical problems when the dimension gets larger if one wants to describe the general form of the normal form, for maps we obtain results without any restrictions on the dimension. In the literature only the two-dimensional nilpotent case has been described so far, as far as we know.

1. Introduction

Normal form theory (cf. ([1], appendix A) for historical remarks) aims to transform a given system to some canonical form, called the normal form, in order to analyse the possible bifurcations when the parameters of the system pass a critical value or to compute...
approximate orbits. If one is lucky, the normal form is simpler or has more symmetry than the original system, but this is not part of the definition.

For vector fields, this theory is well developed, see for instance [1,2], and many different viewpoints have been developed over the years, involving many different branches of mathematics, such as Lie theory, cohomology theory and representation theory (cf. [1]).

However, for discrete dynamical systems the development of normal form theory has received little attention in the literature.

Remark 1.1. In the literature, for instance [3], maps are often seen as Poincaré maps for some vector field and if the linearization is of the type identity plus nilpotent matrix, it is called nilpotent, since it is related to the Poincaré map of a vector field with nilpotent linear part. This is not our definition of nilpotent.

There are few papers regarding the study of normal forms for maps [4–7], and there seems to be no general and systematic approach to the construction of normal forms of maps. However, the recent appearance of [8] confirms that there is indeed a need for the theoretical development of normal form theory for maps.

One of the main differences between maps and vector fields lies in fact that the so-called homological operator is not linear with respect to the linear part of the map. This plays a role when we want to lift decompositions, like the additive semisimple–nilpotent decomposition, or when we want to compute Lie brackets. For instance, for vector fields the additive semisimple–nilpotent splitting for the linear part of the vector field induces an additive semisimple–nilpotent splitting for the homological operator. In the case of maps, the additive semisimple–nilpotent splitting of the linear part does not induce an additive semisimple–nilpotent splitting of the homological operator, due to the lack of linearity. The multiplicative semisimple–nilpotent splitting does not have this problem and might be the more natural splitting to consider. But here nilpotent means identity plus nilpotent, so it will not be a trivial application of the theory in this paper.

Similarly, for vector fields with nilpotent linear part, embedding the linear part into an \( \mathfrak{sl}_2 \)-triple induces an \( \mathfrak{sl}_2 \)-triple in the homological operator.

Nilpotent normal forms for vector fields have been studied in the literature and over the years it has become clear that there is intricate interplay between nilpotent normal forms and the finite dimensional representations of \( \mathfrak{sl}_2 \), see [9–11]. The representation theory of \( \mathfrak{sl}_2 \) has also played a role in the normalization of nilpotent vector fields depending on parameters. For instance, in [12] the authors have the nilpotent normal form of versal deformations of nilpotent and non-semisimple vector fields.

The construction of the normal form for maps with nilpotent linear part has received little attention in the literature. The only case that we are aware of is the two-dimensional case, see [5].

In this paper, we construct from the nilpotent linear part an \( \mathfrak{sl}_2 \)-action that can be used to compute the normal form and we give an explicit expression for the normal form in the irreducible case, followed by several reducible cases, which tend to be slightly more complicated to analyse.

The Jacobson–Morozov theorem ([1,12], Section 12.5) guarantees that any nilpotent element of a reductive Lie algebra can be embedded in an \( \mathfrak{sl}_2 \)-triple, which consists of three elements \( \langle N, H, M \rangle \) that satisfy the commutator relations

\[
[M,N] = H, \quad [H,N] = -2N \quad \text{and} \quad [H,M] = 2M.
\]

This is isomorphic to the Lie algebra \( \mathfrak{sl}_2 \), which is spanned by the matrices

\[
n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad m = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

For further details on the study of Lie algebras and the finite-dimensional representations of \( \mathfrak{sl}_2 \), we refer the reader to [13]. The idea of normal form theory is to transform a map \( f \) through a
conjugation by a near identity transformation to a new map \( \tilde{f} \), in such a way that if we were to repeat the process, \( \tilde{f} \) would prove to be stable under it. This gives rise to the so-called homological operator (see definition 2.1), which plays a central role in normal form theory. In analogy to normal form theory for vector fields the aim is to remove the terms in the image of the homological operator. Equivalently, the transformed map \( \tilde{f} \) is in normal form when its nonlinear part lies in a complement to the image of the homological operator. This is where the finite-dimensional representations of \( \mathfrak{sl}_2 \) come into play.

For an \( \mathfrak{sl}_2 \)-triple \( (N,H,M) \) acting on a finite-dimensional vector space, the space decomposes as \( \text{im} N \oplus \ker M \). Therefore, embedding the homological operator into an \( \mathfrak{sl}_2 \)-triple provides an explicit description of the normal form style (as the kernel of the operator \( M \)) of a map with nilpotent linear part. Moreover, one can in general compute \( \ker M \) and the \( H \)-eigenvalues of the elements in the kernel. The dimension of the irreducible representation generated as an \( N \)-orbit of the kernel element is its eigenvalue plus one. This allows one to check that the computational results are correct, using the Cushman–Sanders test [1,9]. We remark that the correctness argument involves two steps: first one shows that the candidate description of \( \ker M \) does not contain any linear relations, then one computes the generating function to show that the dimensions at all degrees are correct, comparing the result to the known generating functions of general polynomials and polynomial vector fields.

The explicit construction of an \( \mathfrak{sl}_2 \)-triple for general maps with nilpotent linear part is the main result of this paper. In the vector field case, see ([1], Section 9.1), the operator is also linear with respect to the linear part of the vector field, actually a Lie algebra homomorphism. An embedding of the linear part therefore induces a \( \mathfrak{sl}_2 \)-triple in the homological operator.

The challenge for the case of maps is the nonlinearity with respect to the linear part. Therefore, we indeed have to construct \( \mathfrak{sl}_2 \)-triples for each homogeneous degree separately. We are able to generalize this to all degrees such that the construction only depends on the nilpotency index of the linear part. The nilpotency index for a nilpotent matrix \( N \in \mathfrak{gl}(\mathbb{R}) \) is defined as the least number \( p \) such that \( N^p = 0 \), where \( \mathfrak{gl}_n(\mathbb{R}) \) is the space of real valued \( n \times n \) matrices.

The construction of \( \mathfrak{sl}_2 \)-triples is first shown for nilpotent matrices in Jordan form.

Our approach is to first split the homological operator into two commuting operators by considering the map as an element in the tensor product of a space of polynomials and a vector space. This allows us to embed them in two different \( \mathfrak{sl}_2 \)-triples and from these two we construct an \( \mathfrak{sl}_2 \)-triple for the homological operator. The first operator is a multiplication operator which is a Lie algebra homomorphism and the second is a substitution operator. Below we give a more detailed treatment of the matter. Problems arise with the second operator, which only is an antihomomorphism. Therefore, our construction is specifically tailored to work with homomorphisms and antihomomorphisms. This is first done for matrices and then it is shown how the interplay with homomorphisms allows for the general construction with relative ease.

The paper is organized as follows. In §2, we describe the normal form procedure for maps. In §3, we find relations between the nilpotent and its transpose, followed by the construction of an \( \mathfrak{sl}_2 \)-triple in §4. In §5, this triple is used to compute the normal form in the general irreducible case using transvectants, with the two- and three-dimensional case as explicit example. In the electronic supplementary material, we treat several reducible cases with two blocks, namely the (2, 3)- and (2, 2)-case and the \((k_1, k_2)\)-case. Here we show the effectiveness of the \( \mathfrak{sl}_2 \) representation theory to prove the correctness of the computed normal form. The results in the §§2–4 are based on the MSc-thesis [14].

2. Preliminaries

Consider a smooth map \( f : \mathbb{R}^n \to \mathbb{R}^n \), such that \( f(0) = 0 \).

Define the vector space \( P_k \) as the span of the set of homogeneous monomials of degree \( k \) and let \( P_k^H = P_{k+1} \otimes \mathbb{R}^n \).

Then set \( P = \prod_{k \in \mathbb{N}} P_k^H \), the direct product of the homogeneous spaces (product, since we allow infinite summations). With this notation, any smooth map \( f \in P \) with \( f(0) = 0 \) can be
written as
\[ f = Ax + \sum_{i=1}^{\infty} f_i(x), \quad f_i \in \mathcal{P}_i^n, \] (2.1)
where \( f_i \) are the terms of degree \( i + 1 \) in the multivariate Taylor expansion of \( f \). Without loss of generality, we assume that the linear part \( A := Df(0) \) is in some sort of desired form, usually real block diagonal form or Jordan normal form.

A near identity transformation is a map \( \varphi \in \mathcal{P}^n \) such that \( \varphi(0) = 0, D\varphi(0) = I \) and any near identity transformation can therefore be written as
\[ \varphi(x) = x + \varphi_1(x) + \varphi_2(x) + \cdots, \quad \varphi_i \in \mathcal{P}_i^n. \]

Let \( A \in \mathfrak{gl}_n(\mathbb{R}) \) and define the map \( A_* : \mathcal{P}^n \to \mathcal{P}^n \) as
\[ (A_* \varphi)(x) = \varphi(Ax), \] (2.2)
where \( \mathcal{P}^n \) is the space of vector polynomials. The following definition plays a key role in the theory of normal forms and has a direct analogue in the vector field case.

**Definition 2.1.** The linear operator,
\[ \nabla_A : P_{k+1} \otimes \mathbb{R}^n \to P_{k+1} \otimes \mathbb{R}^n, \quad \nabla_A = 1 \otimes A - A_* \otimes 1, \] (2.3)
is called the homological operator. This notation is inspired by the fact that if we consider the tensor product \( \mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}^n \), then this is similar to the notation \( \Delta(X) = X \otimes 1 + 1 \otimes X \) that is commonly used in the context of actions on tensor products.

Consider a near identity transformation \( \varphi \in \mathcal{P}^n \). Then any map (2.1) can be transformed to a new map \( \tilde{f} \) via \( \tilde{f} = \varphi \circ f \circ \varphi^{-1} \). By expanding the right-hand side of \( \tilde{f} \) and collecting terms of the same degree, we can write the transformed map as
\[ \tilde{f}(x) = Ax + f_1(x) - (\nabla_A \varphi_1)(x) + f_2(x) - (\nabla_A \varphi_2)(x) + r_2(x) + \cdots, \] (2.4)
where \( f_i \in \mathcal{P}_i^n \) denote the terms in the original map, \( \nabla_A \) is defined as in (2.3) and the terms \( r_i \in \mathcal{P}_i^n \) are additional terms that depend on \( \varphi_i \), \( 1 \leq k < i \) and \( f_{k'}, 1 \leq k' < i \).

Each homogeneous space \( \mathcal{P}_k^n \) of vector monomials can be decomposed as \( \mathcal{P}_k^n = \text{im} \nabla_A \oplus C_k \) where \( C_k \) is a complement to \( \text{im} \nabla_A \), determining the style \( C \) of the normal form. Any term in \( f_i \in \mathcal{P}_i^n \) that lies in the image of \( \nabla_A \) can be transformed away by choosing \( \varphi_i \in \mathcal{P}_i^n \) such that \( \nabla_A \varphi_i = f_i \). Therefore the normal form of a map \( f : \mathbb{R}^n \to \mathbb{R}^n \) with respect to the linear part \( A \) can be defined as follows.

**Definition 2.2.** The map (2.1) is said to be in normal form (with style \( C \)) with respect to its linear part \( A \) if \( f_k \in C_k, \forall k \in \mathbb{N} \) where \( \mathcal{P}_k^n = \text{im} \nabla_A \oplus C_k \) and \( \nabla_A \) is defined as in (2.3).

**Remark 2.3.** The terminology style has been introduced by James Murdock and tries to convey that it is a matter of choice, at times depending on taste and fashion.

From (2.4), the normal form of \( f \) can be found by recursively solving the equations
\[ \nabla_A \varphi_1 = f_1 - \tilde{f}_1, \]
\[ \nabla_A \varphi_2 = f_2 + r_2 - \tilde{f}_2, \]
\[ \nabla_A \varphi_3 = f_3 + r_3 - \tilde{f}_3, \]
\[ \vdots \]
for some general \( \tilde{f}_k \in C_k \) and \( \varphi_k \in \mathcal{P}_k^n \). The terms \( \tilde{f}_k \) that remain constitute the normal form of \( f \). Then (2.3) can be written as
\[ \nabla_A = A - A_* \] (2.5)
The following lemma provides some properties of \( A_* \) as defined by equation (2.2).
Lemma 2.4. The following holds true for all $A, B \in \mathfrak{gl}(\mathbb{R})$.

1. The maps $A$ and $B_*$ commute, or equivalently $[A, B_*] = 0$.
2. For any $k \in \mathbb{N}$, $(A_*)^k = (A^k)_*$, and we can write this as $A_*^k$.
3. If $A$ is nilpotent, then $A_*$ is nilpotent.
4. The map $A \mapsto A_*$ is an algebra antihomomorphism $A_*B_* = (BA)_*$.

This does not imply it is a Lie algebra antihomomorphism.

Proof. For the first item we have then

$$(A, B_*) \varphi(x) = AB_\ast \varphi(x) - B_\ast A \varphi(x) = A \varphi(Bx) - A \varphi(Bx) = 0.$$  

For the second item

$$(A_*)^k \varphi(x) = (A^{k-1}_\ast \varphi)(Ax) = \cdots = \varphi(A^k x) = (A^k)_\ast \varphi(x).$$

For the third one, since $A^p = 0$, then for an arbitrary $\varphi \in \mathcal{P}_n$ and by using the previous item we obtain

$$(A_\ast^p \varphi)(x) = ((A^p)_\ast \varphi)(x) = \varphi(A^p x) = 0.$$

As $\varphi$ was arbitrary, we find that $A^p_\ast$ maps every element to zero, and hence is the zero operator. For the last one by direct computation we have then

$$A_*B_* \varphi(x) = B_* \varphi(Ax) = \varphi(BAx) = (BA)_* \varphi(x).$$

$\blacksquare$

Corollary 2.5. If $A^p = 0$ then $\nabla_{A^p} = 0$.

We state some important properties of the homological operator in the following lemma.

Lemma 2.6 (6), Lemma 2.1. Let $A, B \in \mathfrak{gl}_n(\mathbb{R})$ and let $\nabla_A$ be as in (2.3). Then

(a) If $A$ is nilpotent, then $\nabla_A$ is nilpotent (see corollary 2.5).

(b) If $AB = BA$, then $\nabla_A \nabla_B = \nabla_B \nabla_A$.

In principle, one can rely on the Jacobson–Morozov theorem to extend a nilpotent matrix to an $s_{12}$-triple. The main difficulty lies in the embedding of $n_\ast$ into a triple, because the map $A \mapsto A_\ast$ for $A \in \mathfrak{gl}_n(\mathbb{R})$ is not a linear map (implicitly assuming it is has simplified some ‘proofs’ in the literature). One could for fixed dimension and degree compute the matrix of $n_\ast$, and then apply Jacobson–Morozov, but we prefer to give a construction for which we can vary the dimension and get explicit formulas for the $s_{12}$-triple. This has the advantage of having a uniform definition of the normal form style.

We hasten to point out that this is the main difficulty that needs to be overcome. To illustrate that the map $A \mapsto \nabla_A$ is not a Lie algebra homomorphism, one computes for $A, B \in \mathfrak{gl}_n(\mathbb{R})$,

$$\nabla_{[A, B]} = [A, B] - [A, B]_\ast.$$  

On the other hand, the bracket of $\nabla_A$ and $\nabla_B$ expands as

$$[\nabla_A, \nabla_B] = (A - A_\ast)(B - B_\ast) - (B - B_\ast)(A - A_\ast).$$  

$$= AB - AB_\ast - A_\ast B + A_\ast B_\ast - BA + BA_\ast + B_\ast A - B_\ast A_\ast$$  

$$= AB + A_\ast B_\ast - BA - B_\ast A_\ast = (AB) - (BA) + [A_\ast, B_\ast]$$  

$$= [A, B] + [A_\ast, B_\ast].$$

The two quantities are not equal because $\varphi$ is not a linear map and hence we cannot conclude that $(\nabla_{[A, B]} \varphi)(x)$ is in general equal to $([\nabla_A, \nabla_B] \varphi)(x)$, since $\varphi((AB - BA)x)$ is not in general equal to $(AB)_\ast \varphi(x) - (BA)_\ast \varphi(x)$. Therefore, the map $A \mapsto \nabla_A$ is not a Lie algebra homomorphism.

The following definition is used to construct a triple for the nilpotent matrix $n \in \mathfrak{gl}_n(\mathbb{R})$. 
Define 2.7. Let $n \in \mathfrak{gl}_n(\mathbb{R})$ be a nilpotent matrix and let $N$ be the Jordan normal form of $n$, the block diagonal matrix with nilpotent Jordan blocks on the block diagonal. Let $P \in \text{GL}_n(\mathbb{R})$ be an invertible matrix such that $n = P^{-1}NP$. Then we define $m = P^{-1}MP$, where $M$ is the transpose of $N$. Note that $n^t = P^t Pm(P^t P)^{-1}$, so unless $P$ is orthogonal, $m$ is not the transpose of $n$. We call $m$ the conjugate transpose of $n$.

Remark 2.8. The nilpotent linear part of the map $f \in \mathcal{P}^n$ to be normalized will be denoted by $n$. The main goal is to embed the homological operator $\nabla_n$, see definition 2.1, into a triple. We mention here that the construction of such an $s_2$-triple is not trivial and the formulas may look cumbersome. On the bright side, once the triple is in place, almost all our computations can be done with $\nabla_n$. The triple then defines the normal form style, but is not needed in the actual computations, since we then rely on the abstract $s_2$-representation theory and the Clebsch–Gordan formalism.

First we will provide an explicit construction of $s_2$-triples for nilpotent matrices. Then we use the fact that $n_*$ is an antihomomorphism to construct an $s_2$-triple for $n_*$.

3. Relations

By convention, the Jordan normal form of a matrix $A \in \mathfrak{gl}_n(\mathbb{R})$ is the upper triangular matrix with the eigenvalues on the main diagonal, either ones or zeros on the super diagonal and zeros elsewhere.

Lemma 3.1. Let $N \in \mathfrak{gl}_n(\mathbb{R})$ be a nilpotent matrix in Jordan form and $M$ be its transpose. Then $N M N = N$ and $M N M = M$.

Proof. Let $N \in \mathfrak{gl}_n(\mathbb{R})$ consist of a single nilpotent Jordan block. By block diagonality, it is sufficient to prove the result for one nilpotent Jordan block.

A straightforward computation shows that

$$N M N = \begin{pmatrix} 0_{n-1,1} & I_{n-1} \\ 0 & 0_{1,n-1} \end{pmatrix} \begin{pmatrix} 0_{1,n-1} & 0 \\ I_{n-1} & 0_{n-1,1} \end{pmatrix} \begin{pmatrix} 0_{n-1,1} & I_{n-1} \\ 0 & 0_{1,n-1} \end{pmatrix} = N,$$

(3.1)

where $I_k$ is the $k \times k$ identity and $0_{l,k}$ is the $l \times k$-matrix filled with zeros. This proves the first identity. The second follows by transposition. □

Theorem 3.2. Let $n$ be nilpotent and $m$ its conjugate transpose. Then $n m n = n$ and $m n m = m$.

Proof. Let $P \in \text{GL}_n(\mathbb{R})$ be the invertible matrix such that $n = P^{-1}NP$. By definition 2.7, $m = P^{-1}MP$. Then $n m n = P^{-1}NPP^{-1}MPP^{-1}NP = P^{-1}NNPP = P^{-1}N = n$ and $m n m = P^{-1}MPP^{-1}NPP^{-1}MP = P^{-1}MMP = P^{-1}MP = m$. □

In the proof of theorem 4.2, we will make extensive use of the following relations.

Theorem 3.3. Let $N \in \mathfrak{gl}_n(\mathbb{R})$ consist of a single nilpotent Jordan block (that is $N$ is irreducible) and let $M$ be its transpose. Then for any $0 \leq l \leq k, 1 \leq k \in \mathbb{N}$ the following relations hold.

\begin{enumerate}
    \item $N^l M^k N^l \equiv M^{k-l} N^k$,
    \item $M^k N^l M^l \equiv M^k N^{k-l}$.
\end{enumerate}
Proof. For \( k \geq n \), the left-hand side and right-hand side are zero in (a) and it is only necessary to prove the result for \( k < n \). Assume that \( N \) is an \( n \times n \) matrix. A computation shows that for \( k < n \)

\[
N^k = \begin{pmatrix} 0_{n-k,k} & I_{n-k} \\ O_k & O_{n-k} \end{pmatrix},
\] (3.2)

where \( O_k \) is the zeromatrix. Denote by \( J_k \) a single \( k \times k \) nilpotent Jordan block and \( N_{n,k} \) the \( n \times k \) zero matrix with a one in the lower left corner. In the special case that \( n = k = 1 \), we define \( N_{1,1} = (1) \) and \( J_1 = (0) \).

Using this partition and (3.4), we have

\[
M^k N^k = \begin{pmatrix} 0_{n-k,k} & O_k \\ I_{n-k} & 0_{n-k,k} \end{pmatrix} \begin{pmatrix} 0_{n-k,k} & I_{n-k} \\ O_k & 0_{k,n-k} \end{pmatrix} = \begin{pmatrix} O_k & 0_{k,n-k} \\ 0_{n-k,k} & I_{n-k} \end{pmatrix}.
\] (3.3)

For \( l = 0 \) the statement is trivial. Now, we prove the statement for \( l = 1 \) by computing the left-hand side and right-hand side of the equality to show the result. Then \( N \) can be partitioned as

\[
N = \begin{pmatrix} J_k & N_{k,n-k} \\ 0_{n-k,k} & J_{n-k} \end{pmatrix}.
\] (3.4)

Using this partition and (3.4), we have

\[
NM^k N^k = \begin{pmatrix} J_k & N_{k,n-k} \\ 0_{n-k,k} & J_{n-k} \end{pmatrix} \begin{pmatrix} O_k & 0_{k,n-k} \\ 0_{n-k,k} & I_{n-k} \end{pmatrix} = \begin{pmatrix} O_k & N_{k,n-k} \\ 0_{n-k,k} & J_{n-k} \end{pmatrix}
= \begin{pmatrix} O_{k-1} & 0_{k-1,n-k+1} \\ 0_{n-k+1,k-1} & J_{n-k+1} \end{pmatrix} = M^{k-1} N^k.
\]

Now the result follows by induction on \( l \). The second item of the theorem follows by transposing the first item.

The above theorem readily generalizes to the case of a general nilpotent matrix.

**Corollary 3.4.** Let \( n \) be any nilpotent matrix and let \( m \) be its conjugate transpose (as defined in definition 2.7). Then

(a) \( n^l m^k n^k = m^{(k-l)} n^k \),
(b) \( m^k n^k m^l = n^k m^{k-l} \).

Since we can switch the role of \( n \) and \( m \) we also have

(c) \( m^l n^k m^k = n^{(k-l)} m^k \),
(d) \( n^k m^k n^l = n^k m^{k-l} \).

Furthermore, taking \( l = k \) we see that \( n^k m^k n^k = n^k \) and \( m^k n^k m^k = m^k \).

**Lemma 3.5.** For \( l \geq k \) we have \( n^l m^k n^k = n^l \) and \( m^k n^k m^l = m^l \).

**Proof.** Assume \( l \geq k \). Then \( n^l m^k n^k = n^{(l-k)+k} m^k n^k = n^{(l-k)} n^k = n^l \) and \( m^k n^k m^l = m^k n^k m^{(k-l)+l} = m^l \).

**Corollary 3.6.** Let \( n \) be any nilpotent matrix and let \( m \) be its conjugate transpose and assume \( k, l \in \mathbb{N}, k \geq 1 \). Let \( k \oplus l = \max(k, l) \). Then

- \( n^l m^k n^k = m^{\max(0,k-l)} n^{\max(k,l)} = m^{(k-l)} n^k \),
- \( m^k n^k m^l = m^{\max(k,l)} n^{\max(0,k-l)} = m^{(k-l)} n^k \),
- \( m^l n^k m^k = n^{\max(0,k-l)} m^{\max(k,l)} = n^{(k-l)} m^k \),
- \( n^k m^k n^l = n^{\max(k,l)} m^{\max(0,k-l)} = n^{(k-l)} m^k \).

**Corollary 3.7.** Let \( n \) be any nilpotent matrix and let \( m \) be its conjugate transpose and assume \( k, l \in \mathbb{N}, k \geq 1 \). Let \( k \oplus l = \max(k, l) \). Then
\[ n^i_n^j = (n^k n^m)^i_j = (n^k n^m)^i_j = (n^k n^m)^i_j = \cdots = (n^k n^m)^i_j = (n^k n^m)^i_j = (n^k n^m)^i_j = (n^k n^m)^i_j. \]

**Lemma 3.8.** Let \( n \) be any nilpotent matrix and let \( m \) be its conjugate transpose and assume \( i, l \in \mathbb{N} \). Let \( \pi_l = n^i_n^j n^m n^l \). Then \( \pi_l \) is a projection operator.

**Proof.**
\[ \pi_l \cdot \pi_l = n^i_n^j n^m n^l n^m n^l = n^i_n^j n^m n^l n^m n^l = n^i_n^j = \pi_l. \]

We need the following lemma in the proof of lemma 4.6.

**Lemma 3.9.** Let \( N \) be in Jordan normal form and let \( P_i^l = N^i_n^j N^m n^l \). Then \( P_i^l \) is a diagonal projection operator and it projects on the \((p-i)\)-dimensional space spanned by \( (e_{1-i+1}, \ldots, e_{p-i}) \). In particular, \( \sigma_l = P_{i-p}^l \) projects on \( e_l \) and the \( \sigma_l, l = 1, \ldots, p \) are complete:
\[ \sum_{i=1}^{p} \sigma_i = id_p. \]

Furthermore, \( P_i^l = \sum_{j=1}^{p-i} \sigma_j \) and \( \epsilon_p := 1 + \sum_{j=1}^{p-i} \sigma_j = \sigma_1 + \sum_{j=1}^{p-i} (p-j+1)(j-1)\sigma_j. \)

**Proof.** The projection part follows from lemma 3.8. Diagonality follows from the Jordan normal form of \( N \) and \( M \). Then, assuming \( N \) is irreducible,
\[ \epsilon_p = 1 + \sum_{j=1}^{p-i} \sigma_j = \sigma_1 + \sum_{j=1}^{p-i} \sigma_j = \sigma_1 + \sum_{j=1}^{p-i} \sigma_j = \]
\[ = \sigma_1 + \sum_{j=1}^{p-i} \sigma_j = \sigma_1 + \sum_{j=1}^{p-i} \sigma_j = \sigma_1 + \sum_{j=1}^{p-i} \sigma_j = \]
\[ = \sigma_1 + \sum_{j=1}^{p-i} (j-1)\sigma_j = \sigma_1 + \sum_{j=1}^{p-i} (j-1)\sigma_j, \]
with trace \( 1 + \frac{1}{2}p(p+1)(p-1). \)

**Lemma 3.10.** \( \text{Tr} P_i^l = p - i. \)

**Proof.** Since \( M^i_n^j \) projects on the last \( p - i \) coordinates, one finds
\[ \text{Tr}(P_i^l) = \text{Tr}(N^i_n^j N^m n^l) = \text{Tr}(M^i_n^j) = p - i. \]

**4. An \( sl_2 \)-triple for a nilpotent matrix**

In the following theorem 4.2, an explicit construction of an \( sl_2 \)-triple for a nilpotent matrix \( n \in \mathfrak{gl}_n(\mathbb{R}) \) is given, using its conjugate transpose \( m \) as defined in definition 2.7. It follows from lemmas 3.9 and 4.6 that this construction coincides with the triples constructed in ([15], Section 2.5) for nilpotent matrices in Jordan form.

Before we can prove the theorem, we need the technical lemma 4.1. In the proof of theorem 4.2, the computation of the relation \( [\mathfrak{h}, \mathfrak{m}] = 2\mathfrak{m} \) is done by showing that this relation holds for the
The generators of $\mathfrak{m}$. The generators are given by the expression $n' m' n^{l-1}$ and the Lie bracket to be computed is given by

$$[[m^k, n^k], n' m' n^{l-1}].$$

Here $n \in \mathfrak{gl}_n(\mathbb{R})$ is a nilpotent matrix with nilpotency index $p \in \mathbb{N}$ and $\mathfrak{m}$ is as in definition 2.7. The powers of $n$ and $\mathfrak{m}$ satisfy the relations $1 \leq i \leq p$, $0 \leq l \leq i$ and $1 \leq k \leq p$. The following lemma computes the Lie bracket (4.1).

**Lemma 4.1.** Let $n \in \mathfrak{gl}_n(\mathbb{R})$ be a nilpotent matrix with nilpotency index $p$ and $\mathfrak{m}$ be as in definition 2.7. Fix $i, l \in \mathbb{N}$ such that $0 \leq i, l \leq p$. Then for any natural number $1 \leq k \leq p$ the Lie bracket (4.1) equals either one of the following.

(a) If $k \leq \min(i, i - l - 1)$ then

$$[[m^k, n^k], n' m' n^{i-1}] = 0.$$

(b) If $l < k \leq i - l - 1$, then

$$[[m^k, n^k], n' m' n^{i-1}] = n^{k-1} m^{k+i-1} n^{i-1} - n^k m^{k+i-l} n^{l-1}.$$

(c) If $i - l - 1 < k \leq l$, then

$$[[m^k, n^k], n' m' n^{i-1}] = n' m^k + n^{k+1} - n' m^k + n^{k+1} n^k.$$

(d) If $k > \max(i, i - l - 1)$, then

$$[[m^k, n^k], n' m' n^{i-1}] = n^{k-1} m^{k+i-1} n^{i-1} - n^k m^{k+i-l} n^{l-1} + n' m^{k+i} n^{k+1} - n' m^{k+i+1} n^k.$$

**Proof.** The proof proceeds by expanding (4.1) for each of the four cases and then we use corollary 3.4 to rewrite each of the terms to obtain the identities. We first expand (4.1) as

$$m^k n^{k+i} m^{i-1} - n^k m^{k+i} m^{i-1} - n' m^{i-1} m^k + n' m^{i-1+k} m^k.$$ (4.2)

For the first term in (4.2), we find

$$m^k n^{k+i} m^{i-1} = n' m^{k+i} n^{i-1} n^{l-1}.$$

where we underline terms that we combine to apply corollary 3.6. If we assume that $k \leq i - l - 1$, this reduces to $n' m^{i-1}$. If we assume that $k > i - l - 1$, this reduces to $n' m^{k+i} n^{i-1}$.

For the second term in (4.2), we find

$$n^k m^{k+i} m^{i-1} = n^{k+i} n^{k+i} n^{i-1} = n^{k+i} n^{k+i} n^{i-1}.$$

If we assume that $k \leq l$, this reduces to $n^{i-1}$. If we assume that $k > l$, this reduces to $n^{k+i} n^{i-1}$.

For the third term in (4.2), we find

$$n' m^{i-1} m^k n^{k} = n' m^{k+1} m^{i-1} n^{k+1} = n^{k+i} m^{k+1} m^{i-1}.$$

If we assume that $k \leq i - l - 1$, this reduces to $n^{i-1}$. If we assume that $k > i - l - 1$, this reduces to $n^{k+i} n^{i-1}$.

For the fourth term in (4.2), we calculate

$$n' m^{i-1+k} m^k = n' m^{i-1+k} m^{i-1} = n' m^{i-1+k} m^{i-1}.$$

If we assume that $k \leq l$, this reduces to $n' m^{i-1}$.

But if $l < k$, it reduces to $n^{k-1} m^{k+1} n^{i-1}$. 

If we assume that $k \leq l$, this reduces to $n' m^{i-1}$. But if $l < k$, it reduces to $n^{k-1} m^{k+1} n^{i-1}$. 


By adding the four terms with the appropriate signs and keeping track of the inequalities, we see that we have proved the lemma.

**Theorem 4.2.** Let \( n \in gl_n(\mathbb{R}) \) be a nilpotent matrix with nilpotency index \( p \geq 2 \) and let \( m \) its conjugate transpose, as defined in definition 2.7. Then the triple \((\bar{n}, h, \bar{m})\), where

\[
\bar{n} := n, \quad \bar{m} := \sum_{i=1}^{p-1} \sum_{l=0}^{i-1} n^i m^i n^{i-1-l} - \sum_{i=1}^{p-1} \sum_{l=0}^{i-1} n n^i m^i n^{i-1-l}, \quad \bar{h} := \sum_{i=1}^{p-1} [m^i, n^i],
\]

(4.3)
is an sl₂-triple.

**Remark 4.3.** Notice that one might change the \( p - 1 \) upper bound to \( \infty \); this does not change the definition; it does make it look more universal, but obviously it adds a small worry on first reading.

**Proof.** We verify that \( \bar{n}, \bar{m} \) and \( \bar{h} \) satisfy the following relations:

\[
[\bar{m}, \bar{n}] = \bar{h}, \quad [\bar{h}, \bar{n}] = -2\bar{n}, \quad [\bar{h}, \bar{m}] = 2\bar{m}.
\]

By a straightforward calculation, one has

\[
[\bar{m}, \bar{n}] = \left[ \sum_{i=1}^{p-1} \sum_{l=0}^{i-1} n^i m^i n^{i-1-l} - \sum_{i=1}^{p-1} \sum_{l=0}^{i-1} n n^i m^i n^{i-1-l} \right] = \sum_{i=2}^{p-1} \sum_{l=0}^{i-1} \left( m^i n^l + \sum_{k=1}^{i-2} \sum_{l=0}^{k-1} n^k m^k n^{k-1-l} - \sum_{l=0}^{i-2} n^{i-1-l} n^{i-1} + n^i m^i n^{i-1-l} - n^i m^i \right)
\]

\[
= [m, n] + \sum_{i=2}^{p-1} \left( m^i n^l + \sum_{k=1}^{i-2} \sum_{l=0}^{k-1} n^k m^k n^{k-1-l} - \sum_{l=0}^{i-2} n^{i-1-l} n^{i-1} + n^i m^i n^{i-1-l} - n^i m^i \right)
\]

\[
= [m, n] + \sum_{i=2}^{p-1} [m^i, n^i] = \bar{h}.
\]

For the second commutator relation in (4), we find

\[
[\bar{h}, \bar{n}] = \left[ \sum_{k=1}^{p-1} [m^k, n^k], n \right] = \sum_{k=1}^{p-1} (m^k n^k - n^k m^k) n - \sum_{k=1}^{p-1} n(m^k n^k - n^k m^k)
\]

\[
= \sum_{k=1}^{p-1} (m^k n^{k+1} - n^k m^k n - n m^k n^k + n^{k+1} m^k)
\]

\[
= \sum_{k=1}^{p-1} (m^k n^{k+1} - n^k m^{k-1} - m^{k-1} n^k + n^{k+1} m^k).
\]

The equality between the second-last and last step follows by applying corollary 3.4 for \( l = 1 \). Notice that in the last expression the first and the third form a telescoping series as well as the
The last step uses the nilpotency of \( \bar{\bar{n}} \) of lemma 4.1. The second and fourth term. The terms are rearranged and summation over \( k \) yields

\[
[\bar{\bar{n}}, \bar{n}] = \sum_{k=1}^{p-1} (\bar{m}^k n^{k+1} - n^k \bar{m}^{k-1} - \bar{m}^{k-1} n^k + n^{k+1} \bar{m}) \\
= \sum_{k=1}^{p-1} (\bar{m}^k n^{k+1} - \bar{m}^{k-1} n^k - n^k \bar{m}^{k-1} + n^{k+1} \bar{m}) \\
= m^{p-1} n^p - n + n^p m^{p-1} - n = -2n = -2\bar{n},
\]

where the nilpotency of \( n \) implies \( n^p = 0 \).

To prove the last equation in (4), it is sufficient by linearity, to prove that

\[
[\bar{\bar{n}}, n'm'n^{i-l-1}] = 2n'm'n^{i-l-1},
\]

holds for all \( 0 \leq l \leq i - 1 \) and \( 0 \leq i < p \). To prove that (4.4) holds, we consider three distinct cases.

Case I First we assume that \( 2l < i - 1 \) and expand the bracket as,

\[
[\bar{\bar{n}}, n'm'n^{i-l-1}] = \sum_{k=1}^{p-1} ([m^k, n^k], n'm'n^{i-l-1}) \\
+ \sum_{k=1}^{i-l-1} ([m^k, n^k], n'm'n^{i-l-1}) + \sum_{k=l+1}^{p-1} ([m^k, n^k], n'm'n^{i-l-1}) \\
+ \sum_{k=i-l}^{p-1} ([m^k, n^k], n'm'n^{i-l-1}).
\]

(4.5)

We notice that the first summation in (4.5) satisfies the condition of lemma 4.1(a). The second summation satisfies the condition of lemma 4.1(b) and the third summation satisfies the condition of lemma 4.1(d). Using lemma 4.1, we compute the Lie product of (4.5) as

\[
[\bar{\bar{n}}, n'm'n^{i-l-1}] = \sum_{k=1}^{i-l-1} (n^{k-1} m^{k+i-l-1} n^{i-l-1} - n^k m^{k+i-l} n^{i-l-1}) \\
+ \sum_{k=1}^{p-1} (n'm^{k+l} n^{k-1} - n'm^{k+i-l} n^{i-l-1}) + \sum_{k=1}^{p-1} (n'^{k-1} m^{k+l-1} n^{i-l-1} - n'^k m^{k+l-1} n^k) \\
= \sum_{k=1}^{p-1} (n^{k-1} m^{k+i-l-1} n^{i-l-1} - n^k m^{k+i-l} n^{i-l-1}) + \sum_{k=1}^{p-1} (n'm^{k+l} n^{k-1} - n'm^{k+i-l} n^{i-l-1}) \\
= n'm'n^{i-l-1} - n^p m^{i-l} n^{i-l-1} + n'm'n^{i-l-1} - n'm^{i-l} n^{i-l-1} = 2n'm'n^{i-l-1}.
\]

The last step uses the nilpotency of \( n \) and the fact that \( l \geq 0 \) and \( i-l-1 \geq 0 \). Case II Second, we assume that \( i - 1 < 2l \) and expand the Lie product as

\[
[\bar{\bar{n}}, n'm'n^{i-l-1}] = \sum_{k=1}^{i-l-1} ([m^k, n^k], n'm'n^{i-l-1}) \\
+ \sum_{k=1}^{i-l-1} ([m^k, n^k], n'm'n^{i-l-1}) + \sum_{k=1}^{i-l-1} ([m^k, n^k], n'm'n^{i-l-1}) \\
+ \sum_{k=1}^{i-l-1} ([m^k, n^k], n'm'n^{i-l-1}).
\]

(4.6)

The first summation in (4.6) satisfies the condition of lemma 4.1(a), the second satisfies the condition of lemma 4.1(c), and lastly the third summation satisfies the condition of lemma 4.1(d).
Therefore, (4.6) can be computed as

\[
\langle \hat{h}, n'm'n'i-l-1 \rangle = \sum_{k=-l}^{l} (n'm^{k+l}n^{k-1} - n'm^{k+l+1}n^{k})
\]

\[+ \sum_{k=-l+1}^{p-1} (n'm^{k+l}n^{k-1} - n'm^{k+i-l}n^{i-l-1}) + \sum_{k=l+1}^{p-1} (n^{k-1}m^{i+k-l}n^{i-l} - n'm^{k+l+1}n^{k})
\]

\[= \sum_{k=-l}^{p-1} (n'm^{k+l}n^{k-1} - n'm^{k+l+1}n^{k}) + \sum_{k=l+1}^{p-1} (n^{k-1}m^{i+k-l}n^{i-l} - n'm^{k+i-l}n^{i-l-1})
\]

\[= n'm'n'i-l-1 - n'm^{p+l}n^{p-1} + n'm'n'i-l-1 - n^{p-1}m^{p+i-l}n^{i-l-1} = 2n'm'n'i-l-1.
\]

Here the last step follows because of nilpotency of n and m. Case III The last case we treat is when \(2l = i - 1\). The Lie product is expanded as

\[
\langle \hat{h}, \hat{m} \rangle = \sum_{k=1}^{p-1} [n'm^k - m^kn^k, n'm'n'i-l-1]
\]

\[= \sum_{k=1}^{l} [n'm^k - m^kn^k, n'm'n'i-l-1] + \sum_{k=l+1}^{p-1} [n'm^k - m^kn^k, n'm'n'i-l-1].
\] (4.7)

The first summation in (4.7) satisfies the conditions of lemma 4.1(a) and the second summation satisfies the conditions of lemma 4.1(d). Therefore, (4.7) can be computed as

\[
\langle \hat{h}, \hat{m} \rangle = \sum_{k=1}^{p-1} (n'm^{k+l}n^{k-1} - n'm^{k+i-l}n^{i-l-1} - n'm^{k+l+1}n^{k} + n^{k-1}m^{i+k-l}n^{i-l})
\]

\[= n'm'n'i-l-1 - n'm^{p+l}n^{p-1} + n^{p-1}m^{p+i-l}n^{i-l-1} + n'm'n'i-l-1 = 2n'm'n'i-l-1.
\]

Here the last step follows because of nilpotency of n and m.

This concludes the proof.

**Corollary 4.4.** Let \(n \in \mathfrak{gl}_p(\mathbb{R})\) be a nilpotent matrix with nilpotency index \(p \geq 2\) and let \(m\) be defined as in definition 2.7. Then the triple \((\bar{n}_*, \bar{m}_*, \bar{h}_*)\), where

\[\bar{n}_* := n_* = \sum_{i=1}^{p-1} \sum_{l=0}^{i-1} n'_i \cdot n'_i n'_i n'_i \] \[\bar{m}_* := \sum_{i=1}^{p-1} \sum_{l=0}^{p-1} m'_i \cdot m'_i n'_i n'_i n'_i \]

\[\bar{h}_* := \sum_{i=1}^{p-1} [m'_i, n'_i],\]

is an \(sl_2\)-triple. This follows immediately from corollary 3.7.

**Remark 4.5.** Notice that \(\bar{m}_*\) denotes the bar of \(m_*\), not the star of \(\bar{m}\).

**Lemma 4.6.** \(\text{ker } \bar{M} = \text{ker } \bar{M}_* = \text{ker } \bar{M}_*\).

**Proof.** First we notice that \(N_i' M_i' N_i' - 1 = N_i' M_i' N_i' - 1\) using the relations. Then

\[\bar{M} = M + \sum_{i=2}^{p-1} \sum_{l=0}^{i-1} N_i' M_i' N_i' - 1 = M + \sum_{i=2}^{p-1} \sum_{l=0}^{i-1} N_i' M_i' N_i' - 1 = (1 + \sum_{i=2}^{p-1} \sum_{l=0}^{i-1} P_i) M = E_p M.
\]

where the \(P_i\) are as in lemma 3.9. The matrix \(E_p\) is diagonal, with strictly positive integer entries on the diagonal (since the \(P_i\) being projection operators, only contribute zeros or ones on the diagonal), so it is invertible. It follows that \(\text{ker } \bar{M} = \text{ker } \bar{M}\). To lift this proof to the starred case, we construct a monomial basis \(e_i\) of the space of polynomials in such a way that \(N_i e_i\) is either \(e_i+1\) or zero, and \(N_i\) is in Jordan normal form and \(M_i\) is its transpose.

**Corollary 4.7.** \(\text{ker } M = \text{ker } \bar{M}\) and \(\text{ker } M_* = \text{ker } \bar{M}_*\).
Proof. The $P$-conjugate of $\mathcal{E}_p$ is also invertible.

Remark 4.8. The diagonal matrix $\mathcal{E}_p$ in the irreducible case, is given by

$$(1, 1 \cdot (p - 1), 2 \cdot (p - 2), \ldots, (p - 1) \cdot 1),$$

as one would expect from $\mathfrak{sl}_2$-representation theory and as has been shown in lemma 3.9. In the reducible case, it consists of blocks of this type.

Remark 4.9. While working on lemma 3.9, we found the following interesting identity:

$$M = \tilde{M} + \sum_{i=1}^{p-1} (-1)^{i+1} \sum_{l=1}^{i} \frac{(i-1)!(i-l)}{l!(i-l)!} \tilde{N}^{i-l} M \tilde{N}^{l-1} = \tilde{M} + \sum_{i=1}^{p-1} (-1)^{i+1} \sum_{l=1}^{i} \frac{1}{l!(i-l)!} \tilde{N}^{i-1} M \tilde{N}^{l-1},$$

a formula that we did not prove, but checked in the irreducible case (and some reducible cases) for $p < 10$. Since $i > i - l - 1$, $\tilde{M}$ can be moved to the right using the $\mathfrak{sl}_2$ bracket relations, showing that $\ker \tilde{M} \subset \ker M$.

Theorem 4.10. Let $\Delta_\tilde{h} = \tilde{h} + \tilde{h}_\ast$. Then $\langle \nabla_\tilde{m}, \Delta_\tilde{h}, \nabla_\tilde{n} \rangle$ is an $\mathfrak{sl}_2$-triple.

Remark 4.11. This has as a consequence that $\text{im } \nabla_\tilde{n}$ has $\ker \nabla_\tilde{m}$ as a natural complement.

Proof. We compute

$$[\nabla_\tilde{m}, \nabla_\tilde{n}] = \nabla_\tilde{m} \nabla_\tilde{n} - \nabla_\tilde{n} \nabla_\tilde{m} = (\tilde{m} - \tilde{m}_\ast)(\tilde{n} - \tilde{n}_\ast) - (\tilde{n} - \tilde{n}_\ast)(\tilde{m} - \tilde{m}_\ast)$$

$$= \tilde{m} \tilde{n} - \tilde{m} \tilde{n}_\ast + \tilde{m}_\ast \tilde{n} - \tilde{m}_\ast \tilde{n}_\ast - \tilde{n} \tilde{m} + \tilde{n} \tilde{m}_\ast - \tilde{n}_\ast \tilde{m} + \tilde{n}_\ast \tilde{m}_\ast$$

$$= [\tilde{m}, \tilde{n}] + [\tilde{m}_\ast, \tilde{n}_\ast] = \tilde{h} + \tilde{h}_\ast = \Delta_\tilde{h},$$

$$[\Delta_\tilde{h}, \nabla_\tilde{m}] = \Delta_\tilde{h} \nabla_\tilde{m} - \nabla_\tilde{m} \Delta_\tilde{h} = (\tilde{h} + \tilde{h}_\ast)(\tilde{m} - \tilde{m}_\ast) - (\tilde{m} - \tilde{m}_\ast)(\tilde{h} + \tilde{h}_\ast)$$

$$= \tilde{h} \tilde{m} - \tilde{h} \tilde{m}_\ast + \tilde{h}_\ast \tilde{m} - \tilde{h}_\ast \tilde{m}_\ast - \tilde{m} \tilde{h} + \tilde{m}_\ast \tilde{h} + \tilde{m}_\ast \tilde{h}_\ast$$

$$= [\tilde{h}, \tilde{m}] - [\tilde{h}_\ast, \tilde{m}_\ast] = 2\tilde{m} - 2\tilde{m}_\ast = 2\nabla_\tilde{m}$$

and

$$[\Delta_\tilde{h}, \nabla_\tilde{n}] = \Delta_\tilde{h} \nabla_\tilde{n} - \nabla_\tilde{n} \Delta_\tilde{h} = (\tilde{h} + \tilde{h}_\ast)(\tilde{n} - \tilde{n}_\ast) - (\tilde{n} - \tilde{n}_\ast)(\tilde{h} + \tilde{h}_\ast)$$

$$= \tilde{h} \tilde{n} - \tilde{h} \tilde{n}_\ast + \tilde{h}_\ast \tilde{n} - \tilde{h}_\ast \tilde{n}_\ast - \tilde{n} \tilde{h} + \tilde{n}_\ast \tilde{h} + \tilde{n}_\ast \tilde{h}_\ast$$

$$= [\tilde{h}, \tilde{n}] - [\tilde{h}_\ast, \tilde{n}_\ast] = -2\tilde{n} + 2\tilde{n}_\ast = -2\nabla_\tilde{n}.$$
Then we compute as follows: $N_k |k, l| = |N(k, l)| = |k + 1, l + 1|$ if $l < n$ and 0 if $l = n$. The kernel of $M_\ast$ (and of $\tilde{M}_\ast$) is spanned by $|1, k|, k = 1, \ldots, n$.

$$\ker M_\ast = \bigoplus_{l=0}^{n-2} u^l x_{n-l} \mathbb{R}[u^n u^{-1}, \ldots, u^{n_{n-l}}] \oplus u^{n-1} x_1 \mathbb{R}[u^{n-1} x_1].$$

The $\tilde{M}_\ast$-eigenvalue of $|1, k|$ is $n - k$. Thus we have a decomposition

$$\mathcal{P}^\ast = (V_{n-1} \oplus V_{n-2} \oplus \cdots \oplus V_0) \otimes V_{n-1},$$

generated by $|1, 1| |n|, |1, 2| |n|, \ldots, |1, n| |n|$. Since $V_{k-1} \otimes V_{n-1} = \bigoplus_{i=0}^{n+k} V_i$ by Clebsch–Gordan, we have

$$\mathcal{P}^\ast = \bigoplus_{k=1}^{n} V_{k-1} \otimes V_{n-1} = \bigoplus_{k=1}^{n} \bigoplus_{l=n-k}^{n+k-2} V_l,$$

where the elements in $V_l$ have $\Delta_l$-eigenvalue $l$.

**Definition 5.2.** We define the $\mathit{p}$th transvectant of $w_{k-1} \otimes v_{n-1}$, where $w_{k-1}$ is a top weight vector with eigenvalue $k - 1$ and $v_{n-1}$ a top weight vector with eigenvalue $n - 1$ (think of $|1, n - k + 1|$ and $|n|$, respectively),

$$\kappa_{k+n-2p-2}: W_{k-1} \otimes V_{n-1} \rightarrow U_{k+n-2p-2} \cap \ker \nabla_M,$$

as

$$\kappa_{k+n-2p-2} w_{k-1} \otimes v_{n-1} = \sum_{i+j=p} \binom{p}{i} \binom{k-1}{k-j} \otimes \binom{n-1}{n-j}, \quad k = 1, \ldots, n, p = 0, \ldots, \min(k - 1, n - 1),$$

where $w_{k-1} = \frac{1}{n!} W_k w_{k-1}$ and $v_{n-1} = \frac{1}{n!} W_n v_{n-1}$, $0 \leq i \leq n - 1$ and $0 \leq j \leq k - 1$.

**Remark 5.3.** The transvectant was one of the main tools of Classical Invariant Theory to compute covariants, that is, irreducible $\mathfrak{sl}_2$-representations. In the modern theory, it is replaced by the much more general Young–Weyl tableaux to cover more general Lie algebras. But in normal form theory, there is no need for such generality (yet), and we can enjoy working with explicit formulas to implement the Clebsch–Gordan decomposition.

**Remark 5.4.** In the usual transvectant definition, there is always a factor $(−1)^i$; here we do not have this sign appearing because it is present in the definition of $\nabla_M$.

In our example

$$w_{k-1}^{(i)} = \frac{1}{j!} W_k |1, n - k + 1| = \frac{1}{j!} |j + 1, n - k + j + 1|$$

and

$$v_{n-1}^{(i)} = \frac{1}{n!} W_n |n| = \frac{1}{n!} |n - i|.$$

The transvectant now reads

$$\kappa_{k+n-2p-2} |1, n - k + 1| |n| = \sum_{i+j=p} \frac{1}{i! j! n!} \frac{\binom{p}{i}}{\binom{n-1}{n-j} \binom{k-1}{k-j}} |j + 1, n - k + j + 1| |n - i|.$$ (5.1)

Here we can replace $|i + 1, n - k + i + 1|$ by $x_{i+1} x_{n-k+i+1} F_{n-k+1}^p (x_{i+1}, \ldots, x_{n-k+i+1})$ where $F_{n-k+1}^p$ is an arbitrary polynomial and $|n - i|$ by $e_{n-i}$ to obtain the corresponding term in the general normal form formula.
**Definition 5.5.** Define the Clebsch–Gordan coefficient, also known as 3j-symbol, by (cf. [16])

\[
\binom{m}{i} \binom{n}{j} \binom{m+n-2p}{k} = \sum_{r+q=k} (-1)^{i-k+r} \frac{(p)}{(i)} \frac{(q)}{(j)} \frac{(r)}{(k)}, \quad i + j = k + p,
\]

with \(j_1 = m, j_2 = n\) and \(j_3 = n + m - 2p\) as the three \(j\)'s. Here all the \(n\) and \(k\) are arbitrary and not determined by their previous meaning.

**Remark 5.6.** We consider the special case \(k = 0\):

\[
\binom{m}{i} \binom{n}{j} \binom{m+n-2p}{0} = \frac{(p)}{(i)} \frac{(q)}{(j)}, \quad i + j = p.
\]

Now recall the inversion formula for the Clebsch–Gordan coefficients [16]:

\[
\psi_{i}^{(j)} \otimes v_{m}^{(n)} = \sum_{p=k+i+j} \binom{m}{i} \binom{n}{j} \binom{m+n-2p}{k} \frac{(p)}{(i)} \frac{(q)}{(j)} \frac{(r)}{(k)} \times_{m+n-2p} w_{n} \otimes v_{m},
\]

with the proviso that \(p \leq \min(m, n)\). This allows us to project an arbitrary expression onto \(\ker \bar{\nabla}\) by taking \(k = 0\):

\[
\pi_{\ker \bar{\nabla}} \psi_{i}^{(j)} \otimes v_{m}^{(n)} = \sum_{p=i+j} \binom{m}{i} \binom{n}{j} \binom{m+n-2p}{k} \frac{(p)}{(i)} \frac{(q)}{(j)} \frac{(r)}{(k)} \times_{m+n-2p} w_{n} \otimes v_{m}
\]

\[
= \sum_{p=i+j} \frac{(p)}{(m+n-p+1)} \times_{m+n-2p} w_{n} \otimes v_{m} = \frac{(i+j)}{(m+n-i+j+1)} \times_{m+n-2(i+j)} w_{n} \otimes v_{m}.
\]

**Remark 5.7.** Notice that the projection on \(\ker \bar{\nabla}\) is done without using \(\bar{M}\); only \(\bar{N}\) is used. The Clebsch–Gordan inversion formula not only allows us to project on \(\ker \bar{\nabla}\), but also to find the preimage of \(\ker \bar{\nabla}\) by lowering \(k\) to \(k - 1\) and dividing by \(k\). Again, only \(\bar{N}\) is used.

In our example

\[
\psi_{i}^{(j)} = \frac{1}{i!} \times_{n} w_{n},
\]

and

\[
\psi_{i}^{(j)} = \frac{1}{j!} \times_{n} w_{n} = \frac{1}{j!} \times_{n} w_{n} = \frac{1}{j!} \times_{n} w_{n}.
\]

It follows that

\[
\pi_{\ker \bar{\nabla}} |j + 1, n - k + j + 1|n - i\rangle = \frac{(i+j)(n-1)}{(n+k-(i+j)-1)} \times_{n+k-2(i+j)} |1, n - k + 1|n\rangle.
\]

or

\[
\pi_{\ker \bar{\nabla}} |I, J|K\rangle = \frac{(n-1)!}{(K-I)!} \frac{(n-I-L-1)}{(n-K+I-1)} \times_{K-J-1} |1, J - I + 1|n\rangle.
\]

(5.2)

The transvectant now reads

\[
\times_{k+n-2p-2} |1, n - k + 1|n\rangle = \sum_{i+j=p} \frac{1}{i!} \frac{(p)}{(i)} \frac{(q)}{(j)} \times_{n-k+j+1}|1, n - k + j + 1|n - i\rangle.
\]

This allows us to compute the general form of the normal form (the description problem) and to do the actual computations for any dimension, only using \(\bar{N}\) and the knowledge of \(\ker M\) and \(\ker M_{*}\) (thanks to lemma 3.9).
The normal form of the two-dimensional nilpotent map

The transvectant in (5.1) now reads, with $n = 2$,

$$\varpi_{k-2p} |1, 3-k| 2 = \sum_{i+j=p} \frac{1}{i!j!} \binom{l}{j} (j+1, 3-k+j|2-i), \quad k = 1, 2, p = 0, k-1.$$

Here we can replace $|j+1, 3-k+j| 2-i$ by $x_{j+1} x_{3-k+j} F_{3-k}^p(x_{j+1}, \ldots, x_{3-k+j})$ where $F_{3-k}^p$ is an arbitrary polynomial and $|2-i|$ by $e_{2-i}$ to obtain the corresponding term in the general normal form formula.

It follows from the Clebsch–Gordan formula that $0 \leq p \leq k-1$.

For $k = 1$ and $p = 0$ we have, $\varpi_1 |1, 2| 2 = |1, 2| 2$, corresponding to

$$\begin{pmatrix} 0 \\ x_1 x_2 F_0^0(x_1, x_2) \end{pmatrix}.$$

For $k = 2$ and $p = 0$, we have, $\varpi_2 |1, 1| 2 = |1, 1| 2$, corresponding to

$$\begin{pmatrix} 0 \\ x_1 F_0^0(x_1) \end{pmatrix}.$$ 

For $k = 2$ and $p = 1$, we have, $\varpi_0 |1, 1| 2 = |j+1, 1+j| 2-i = |2, 2| 2 + |1, 1| 1$, corresponding to

$$\begin{pmatrix} x_1 F_1^1(x_1) \\ x_2 F_1^1(x_2) \end{pmatrix}.$$

The general normal form map in ker $\nabla_M$-style is now

$$F(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ x_1 x_2 F_0^0(x_1, x_2) \end{pmatrix} + \begin{pmatrix} 0 \\ x_1 F_0^0(x_1) \end{pmatrix} + \begin{pmatrix} x_1 F_1^1(x_1) \\ x_2 F_1^1(x_2) \end{pmatrix}.$$

Here we can combine the $F_2^0$ and $F_1^0$ to

$$\begin{pmatrix} 0 \\ x_1 p(x_1, x_2) \end{pmatrix}$$

to get a shorter expression (but which is less natural with respect to $\mathfrak{sl}_2$). Notice that the versal deformation of the linear part is given by

$$\begin{pmatrix} \alpha_1^1 & 1 \\ \alpha_0^1 & \alpha_1^1 \end{pmatrix},$$

where the indices of $\alpha$ correspond to those of $F$. This is the same as in the vector field case since the map is linear.

Remark 5.8. In ([5], Example 2.4), the authors compute the normal form of this map, without the use of $\mathfrak{sl}_2$ theory:

$$\tilde{f}(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sum_{n=2}^{\infty} \sum_{k=0}^{n} \alpha_{n,k} \begin{pmatrix} 0 \\ x_1^{k} x_2^{n-k} \end{pmatrix}.$$

One could call this the normal form in ker $\mathfrak{M}$-style, since the normal form terms are in $\mathbb{R}[x_1, x_2] \otimes \ker \mathfrak{M}$. Is this style in general equivalent to the ker $\nabla_M$-style? If so, it would certainly simplify the description problem of the general nilpotent normal form. We will try to shed some light on the answer to this question in the following.
If we apply the projection formula (5.2) on \( \ker \nabla \overline{M} \) to the term \( \left( x_2 F_1(x_2) \right) \mid 2, 2 \rangle = \left( x_1 F_1(x_1) \right) \rangle \), we obtain
\[
\pi_{\ker \nabla \overline{M}} \left| 2, 2 \rangle \right> = \frac{1}{2} \left| 1, 1 \rangle \right> + \left| 2, 2 \rangle \right>.
\]

This leads us to the conclusion that for \( n = 2 \) we can identify \( \ker \nabla \overline{M} \) with \( \mathbb{R}[x_1, x_2] \otimes \ker M \). This last formulation of the normal form style has an interesting computational aspect: in order to find a normal form transformation it is enough to solve \( \psi \) from the equation
\[
M(\nabla_N \psi(x) - F(x)) = 0,
\]
at some fixed degree, which looks a lot better than solving from
\[
\nabla \overline{M}(\nabla_N \psi(x) - F(x)) = 0,
\]
which is the usual computational method.

We may conjecture that this holds in general, i.e. \( \ker \nabla \overline{M} \equiv \mathbb{R}[x_1, \ldots, x_n] \otimes \ker M \). Indeed, if we project it on \( \ker M \) we find
\[
\pi_{\ker M} \left| 2n - 2l + 1 - j \mid 1, j \rangle \right> = \frac{(n - j - l + 1)!}{(n - j)!} \mid I, I + j - 1 \mid \mid n\rangle, \quad j = 1, \ldots, n.
\]

This shows that \( \ker M \equiv \ker \nabla \overline{M} \) for irreducible \( N \). We show in the electronic supplementary material that this no longer holds in general for the reducible case.

(i) Example of a normal form calculation

We give a very simple example, just to give the reader an impression of how this might work in general. One should keep in mind that this is more suitable for an automated calculation and not very convincing as a hand calculation. Consider the map
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_2 + a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2 \\ b_{11} x_1^2 + 2b_{12} x_1 x_2 + b_{22} x_2^2 \end{pmatrix}.
\]

We see that \( N \) acts as \( N \mid 1 \rangle = 0 \) and \( N \mid 2 \rangle = \mid 1 \rangle \). And \( M \) acts as \( M \mid 1 \rangle = \mid 2 \rangle \) and \( M \mid 2 \rangle = 0 \). Thus \( \ker M \) is spanned by \( \mid 2 \rangle \). Furthermore, \( N \mid 1, 1 \rangle = \mid 2, 2 \rangle \), \( N \mid 1, 2 \rangle = 0 \) and \( N \mid 2, 2 \rangle = 0 \). And \( M \mid 1, 1 \rangle = 0 \), \( M \mid 1, 2 \rangle = 0 \) and \( M \mid 2, 2 \rangle = \mid 1, 1 \rangle \). Thus \( \ker M \) is spanned by \( \mid 1, 1 \rangle, \mid 1, 2 \rangle \). In our example
\[
\psi^{(0)}_1 = \mid 2 \rangle, \quad \psi^{(1)}_1 = N \mid 2 \rangle = \mid 1 \rangle.
\]

and
\[
\psi^{(0)}_0 = \mid 1, 2 \rangle, \quad \psi^{(0)}_1 = \mid 1, 1 \rangle, \quad \psi^{(1)}_1 = N \mid 1, 1 \rangle = \mid 2, 2 \rangle.
\]
Now recall the inversion formula for the Clebsch–Gordan coefficients [16]:

\[
|1, 2||2\rangle = w_0^{(0)} \otimes v_1^{(0)} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \triangledown^{(0)}_1 w_0 \otimes v_1 = w_0 \otimes v_1,
\]

\[
|1, 2||1\rangle = w_0^{(0)} \otimes v_1^{(1)} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \triangledown^{(1)}_1 w_0 \otimes v_1 = \triangledown_N w_0 \otimes v_1,
\]

\[
|1, 1||2\rangle = w_0^{(1)} \otimes v_1^{(0)} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \triangledown^{(0)}_2 w_1 \otimes v_1 = w_1 \otimes v_1,
\]

\[
|1, 1||1\rangle = w_0^{(1)} \otimes v_1^{(1)} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \quad \triangledown^{(1)}_2 w_1 \otimes v_1 + \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \triangledown^{(0)}_0 w_1 \otimes v_1
\]

\[
= \frac{1}{2} \triangledown_N w_1 \otimes v_1 - \frac{1}{2} \triangledown^{(0)}_0 w_1 \otimes v_1,
\]

\[
|2, 2||2\rangle = w_1^{(1)} \otimes v_1^{(0)} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \triangledown^{(0)}_1 w_1 \otimes v_1 + \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \triangledown^{(0)}_0 w_1 \otimes v_1
\]

\[
= \frac{1}{2} \triangledown_N w_1 \otimes v_1 + \triangledown^{(0)}_0 w_1 \otimes v_1.
\]

and

\[
|2, 2||1\rangle = w_1^{(1)} \otimes v_1^{(1)} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \quad \triangledown^{(1)}_0 w_1 \otimes v_1 = \frac{1}{2} \triangledown_N \triangledown^{(1)}_2 w_1 \otimes v_1.
\]

It follows that

\[
a_{11}|1, 1||1\rangle + \frac{1}{2} a_{12}|1, 2||1\rangle + a_{22}|2, 2||1\rangle + b_{11}|1, 2||2\rangle + \frac{1}{2} b_{12}|1, 2||2\rangle + b_{22}|2, 2||2\rangle
\]

\[
= -a_{11} \triangledown_0 w_1 \otimes v_1 + b_{11} \triangledown_1 w_1 \otimes v_1 + \frac{1}{2} b_{12} \triangledown_0 w_1 \otimes v_1 + b_{22} \triangledown_0 w_1 \otimes v_1
\]

\[
+ \frac{1}{2} \triangledown_N(a_{11} \triangledown_1 w_1 \otimes v_1 + a_{12} \triangledown_0 w_1 \otimes v_1 + a_{22} \triangledown^{(1)}_1 w_1 \otimes v_1 + b_{22} w_1 \otimes v_1).
\]

The first-order normal form is now

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} b_{22} - \frac{1}{2} a_{11} \\ b_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} b_{12} \end{pmatrix} \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix}.
\]

We see that the number of terms is half that of the original map. In the \(n\)-dimensional case, one would expect \(1/n\) terms in the normal form, at least in the irreducible case.

(b) The three-dimensional nilpotent normal form

The transvectant in (5.1) now reads, with \(n = 3\),

\[
\triangledown_{k-2p+1}|1, 4 - k||3\rangle = \sum_{i+j=p} \frac{1}{i! j! (i+j)!} \chi(i) \chi(j) |j + 1, 4 - k + j||3 - i\rangle, k = 1, 2, 3, p = 0, \ldots, k - 1.
\]

We now do \(k = 1, \ldots, 3\) and \(p = 0, \ldots, k - 1\) (playing computer):

For \(k = 1\) and \(p = 0\) we have, \(\triangledown_2|1, 3||3\rangle = |1, 3||3\rangle\), corresponding to

\[
\begin{pmatrix} 0 \\ 0 \\ x_1 x_2 x_3 \end{pmatrix}.
\]

For \(k = 2\) and \(p = 0\), we have \(\triangledown_3|1, 2||3\rangle = |1, 2||3\rangle\), corresponding to

\[
\begin{pmatrix} 0 \\ 0 \\ x_1 x_2 \end{pmatrix}.
\]
For $k = 3$ and $p = 0$, we have $\kappa_4 |1, 1||3| = |1, 1||3|$, corresponding to
\[
\begin{pmatrix}
0 \\
0 \\
x_1 F_1^0(x_1)
\end{pmatrix}.
\]

For $k = 2$ and $p = 1$, we have
\[
\kappa_1 |1, 2||3| = \sum_{i+j=1} \frac{1}{(i!)^2} |j + 1, 4 - k + j||3 - i| = |2, 3||3| + \frac{1}{2} |1, 2||2|,
\]
corresponding to
\[
\frac{1}{2} \begin{pmatrix}
0 \\
x_1 x_2 F_2^1(x_1, x_2) \\
x_2 x_3 F_2^1(x_2, x_3)
\end{pmatrix}.
\]

For $k = 3$ and $p = 1$, we have
\[
\kappa_2 |1, 1||3| = \sum_{i+j=1} \frac{1}{(i!)^2} |j + 1, 4 - k + j||3 - i| = \frac{1}{2} |2, 2||3| + \frac{1}{2} |1, 1||2|,
\]
corresponding to
\[
\frac{1}{2} \begin{pmatrix}
0 \\
x_1 F_1^1(x_1) \\
x_2 F_1^1(x_2)
\end{pmatrix}.
\]

For $k = 3$ and $p = 2$, we have
\[
\kappa_0 |1, 1||3| = \sum_{i+j=2} \frac{1}{(i!)^2} \frac{1}{j!} |j + 1, 4 - k + j||3 - i| = \frac{1}{2} |3, 3||3| + \frac{1}{2} |2, 2||2| + \frac{1}{2} |1, 1||1|,
\]
corresponding to
\[
\frac{1}{2} \begin{pmatrix}
x_1 F_2^2(x_1) \\
x_2 F_2^2(x_2) \\
x_3 F_2^2(x_3)
\end{pmatrix}.
\]

The normal form is now (where we ignore the superfluous $\frac{1}{2}$s):
\[
F(x) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
x_1 x_2 x_3 F_3^0(x_1, x_2, x_3)
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
x_1 x_2 F_2^0(x_1, x_2)
\end{pmatrix}
\]
\[
+ \begin{pmatrix}
0 \\
0 \\
x_1 F_1^0(x_1)
\end{pmatrix} + \begin{pmatrix}
\frac{1}{2} x_1 x_2 F_2^1(x_1, x_2) \\
x_2 x_3 F_2^1(x_2, x_3)
\end{pmatrix} + \begin{pmatrix}
0 \\
x_1 F_1^1(x_1) \\
x_2 F_1^2(x_2)
\end{pmatrix} + \begin{pmatrix}
x_3 F_1^2(x_3)
\end{pmatrix}.
\]

Notice that the versal deformation of the linear part is given by
\[
\begin{pmatrix}
\alpha_1^2 & 1 & 0 \\
\alpha_1^1 & \alpha_1^2 & 1 \\
\alpha_0^0 & \alpha_1^1 & \alpha_1^2
\end{pmatrix},
\]
where the indices of $\alpha$ correspond to those of $F$. This is the same as in the vector field case since the map is linear.
6. Concluding remarks

One of the ideas one might get from this paper is that the extension of a nilpotent action into an $\mathfrak{sl}_2$-triple leads to constructions that hardly need the triple in the actual computations. It is as if the existence of the triple is enough, and this would readily follow from the finite dimensionality of the representations and the Jacobson–Morozov theorem. This impression is created by the fact that the $\mathfrak{sl}_2$-equivariant decomposition is given automatically if the nilpotent is in Jordan normal form and therefore acts as a simple shift operator on the monomials. It is this simplicity that allows us to give such a general result on the normal form of maps with nilpotent linear part, much more general than is possible in the corresponding vector field case.

Data accessibility. The data are provided in the electronic supplementary material [17].

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All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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