THE RATIONAL HULL OF RUDIN’S KLEIN BOTTLE

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Abstract. In this paper, we compute the rational hull of an explicit Klein bottle in $\mathbb{C}^2$ which was first constructed by Rudin as an example of a totally real nonorientable surface in $\mathbb{C}^2$. In contrast to its polynomial hull — which was shown to contain an open set by the first author in 2011 — we show that its rational hull is two-dimensional. Using the same technique, we also describe the rational hulls of certain other surfaces in $\mathbb{C}^2$.

1. Introduction

In [8], Rudin gave an explicit smooth embedding of the Klein bottle into $\mathbb{C}^2$ so that the image is totally real, i.e., no tangent space to the image in $\mathbb{C}^2$ contains a nontrivial complex subspace. Later, Givental (see [6]) constructed totally real embeddings into $\mathbb{C}^2$ of all nonorientable surfaces that are connected sums of $n$ Klein bottles, where $n$ is odd and at least three. In fact, Givental’s embeddings are Lagrangian, i.e., the pull-backs of the standard Kähler form via the embeddings vanish. The question of whether or not the Klein bottle admits any Lagrangian embedding into $\mathbb{C}^2$ was settled much later (in the negative) by Shevchishin (see [10]). This places certain constraints on the convexity properties of Klein bottles in $\mathbb{C}^2$, as discussed below.

Given a compact set $X \subset \mathbb{C}^n$, its rational hull is defined as

$$\hat{X}_{\text{rat}} = \{ z \in \mathbb{C}^n : P(z) \in P(X), \text{ for all polynomials } P : \mathbb{C}^n \to \mathbb{C} \}.$$ 

The set $X$ is said to be rationally convex if its rational hull is trivial, i.e., $\hat{X}_{\text{rat}} = X$. In [5], Duval and Sibony show that for any $n$-dimensional totally real submanifold of $\mathbb{C}^n$, being rationally convex is equivalent to being Lagrangian with respect to some Kähler form on $\mathbb{C}^n$. So, although there are rationally convex topological Klein bottles in $\mathbb{C}^2$ (see [9]), by Shevchishin’s result, every totally real Klein bottle in $\mathbb{C}^2$ must have a nontrivial rational hull. It is natural to ask whether there is a constraint on the dimension of this hull. In this paper, we show that the rational hull of Rudin’s Klein bottle is 2-dimensional. More precisely, the hull consists of the Klein bottle and an attached analytic annulus (see Section 2).

We note that a similar question regarding the polynomial hull of Rudin’s Klein bottle was addressed in [3]. The polynomial hull of a compact set $X \subset \mathbb{C}^n$ is

$$\{ z \in \mathbb{C}^n : |P(z)| \leq \sup_X |P|, \text{ for all polynomials } P : \mathbb{C}^n \to \mathbb{C} \}.$$ 

It is known that the polynomial hull of any compact $n$-dimensional manifold in $\mathbb{C}^n$ must be of dimension at least $n + 1$ (see [1]). In [3], the first author shows that the polynomial hull of Rudin’s Klein bottle contains an open set in $\mathbb{C}^2$. On the other hand, he produces a totally real Klein bottle in $\mathbb{C}^2$ whose polynomial hull is

We would like to thank Alexander Izzo for playing a key role in facilitating this collaboration.
of dimension 3. In spite of admitting a smaller polynomial hull, we show that the rational hull of his Klein bottle has the same size and structure as that of Rudin’s Klein bottle (see Section 3).

The proofs in this paper all follow the same blueprint. First, the given compact set is projected into $\mathbb{C}$ via a suitable rational function. Then, peak functions are used to concentrate representing measures of points in the rational hull on a single fiber of this projection (this device is used in [2]). Here, we use the identification of $\hat{X}_{\text{rat}}$ with the spectrum of the uniform algebra $\mathcal{R}(X)$, the space of functions on $X$ that are uniformly approximable by rational functions without poles on $X$ (see [11, Section 1.2]). The choice of the projecting function forces the nontrivial portion of the rational hull to be in an annulus contained in a single fiber of this projection. The annulus is then shown to be in the rational hull by an application of the argument principle. This technique can also be used to compute the rational hulls of certain tori considered by Duval and Gayet in [4]. We do this in Section 4.

Lastly, we note that each surface $S$ considered in this paper has the property that $\hat{S}_{\text{rat}} \setminus S$ is a smooth 2-manifold. This answers the question raised by Alexander Izzo (private communication) of whether such surfaces exist in $\mathbb{C}^2$.

2. Rudin’s Klein Bottle

In this section, we will compute the rational hull of

$$K = \{(e^{2i\theta}g^2(\phi), e^{i\theta}g(\phi)h(\phi)) : -\pi \leq \phi < \pi\},$$

where

$$g(\phi) = a + b \cos \phi,$$

$$h(\phi) = \sin \phi + i \sin 2\phi$$

for some fixed real numbers $0 < b < a$.

**Theorem 2.1.** The rational hull of $K$ in $\mathbb{C}^2$ is $K \cup A$, where

$$A = \{(z,0) \in \mathbb{C}^2 : (a-b)^2 \leq |z| \leq (a+b)^2\}.$$

**Proof.** We first show that

$$\hat{K}_{\text{rat}} \subseteq K \cup A.$$

For this, set $q(z, w) = \frac{w^2}{z}$. Since $p(z, w) = z$ does not vanish on $K$, $q$ is holomorphic near $K$. Let

$$\Gamma = q(K) = \{(\sin \phi + i \sin 2\phi)^2 : -\pi \leq \phi < \pi\}.$$

Note that $\Gamma$ is a simple closed curve in $\mathbb{C}$ that is smooth everywhere except at the origin. Let $\Omega$ denote the simply connected domain bounded by $\Gamma$.

First, we describe the fibers of $q$ restricted to $K$. Given $\lambda \in \Gamma$, let

$$K_\lambda = \{(z, w) \in K : q(z, w) = \lambda\}.$$

We claim that

$$K_\lambda = \begin{cases} C_{-\pi} \cup C_0, & \text{when } \lambda = 0, \\ C_\phi, & \text{for some } \phi \in (0, \pi), \text{ when } \lambda \neq 0. \end{cases}$$

where $C_\phi = \{(e^{2i\theta}g^2(\phi), e^{i\theta}g(\phi)h(\phi)) : \theta \in \mathbb{R}\}$ is a circle on $K$. Indeed, if $\lambda = 0$, then $e^{2i\theta}g^2(\phi)$ and $e^{i\theta}g(\phi)h(\phi) \in K_\lambda$ only if $h(\phi) = 0$. For $\phi \in [-\pi, \pi)$, this is only possible when $\phi$ is either $0$ or $-\pi$. On the other hand, if $\lambda \neq 0$, then $h^2(\phi) = \lambda$. 

yields exactly two solutions in the interval \((-\pi, \pi)\), since \(h(-\phi) = -h(\phi)\). Let \(\phi\) be the solution that lies in \((0, \pi)\). Then,
\[
K_\lambda = C_\phi \cup C_{-\phi}.
\]

However,
\[
C_\phi = \{ (e^{2i\theta}g^2(\phi), e^{i\theta}g(\phi)h(\phi)) : \theta \in \mathbb{R} \}
\]
\[
= \{ (e^{2i\eta}g^2(-\phi), e^{i\eta}g(-\phi)h(-\phi)) : \eta \in \mathbb{R} \} = C_{-\phi}.
\]

Thus, we have the second part of (2.1). We also note that the circles
\[
C_{-\pi} = \{ (z, 0) \in \mathbb{C}^2 : |z| = (a-b)^2 \},
\]
\[
C_0 = \{ (z, 0) \in \mathbb{C}^2 : |z| = (a+b)^2 \}
\]
are the inner and outer boundaries, respectively, of the annulus \(A\), i.e., \(A\) is attached to the fiber \(K_0\).

Now, suppose \((z_0, w_0) \in \tilde{K}_{\text{rat}}\). Then, \(z_0 \neq 0\), since \(p(z, w) = z\) does not vanish on \(K\). Thus, \(q\) is holomorphic near \((z_0, w_0)\). If \(\lambda_0 = q(z_0, w_0) \notin \Gamma\), then the polynomial \(w^2 - \lambda_0 z\) vanishes on \((z_0, w_0)\), but not on \(K\). This is a contradiction. Thus, \(\lambda_0\) must be in \(\Gamma\).

We first assume that \(\lambda_0 \neq 0\). Since \(\Omega\) is a domain bounded by a simple closed curve, \(\lambda_0 \in \partial \Omega\) is a peak point for \(\mathcal{P}(\overline{\Omega})\). Thus, there is an \(f \in \mathcal{P}(\overline{\Omega})\) such that \(f(\lambda_0) = 1\) and \(|f| < 1\) on \(\overline{\Omega \setminus \{\lambda_0\}}\). For all positive integers \(n\), the function \(z(f^n \circ q)\) is in \(\mathcal{R}(K)\). If \(\mu\) is a representing measure for \((z_0, w_0)\), then
\[
|z_0| = \left| \int_K z f^n \left( \frac{w^2}{z} \right) d\mu(z, w) \right|.
\]

But, as \(n \to \infty\), the right-hand side of (2.2) converges to
\[
\left| \int_{K_{\lambda_0}} z f^n (\lambda_0) d\mu(z, w) \right| = \left| \int_{C_{\phi_0}} zd\mu(z, w) \right| \leq g^2(\phi_0),
\]
where \(\phi_0 \in (0, \pi)\) is such that \(K_{\lambda_0} = C_{\phi_0}\) (see (2.1)). Thus, \(|z_0| \leq g^2(\phi_0)|\). Repeating the above argument for \(1/z\), which is in \(\mathcal{R}(K)\), since \(z\) does not vanish on \(K\), we also have that \(|z_0| \geq g^2(\phi_0)|\). Thus,
\[
|z_0| = g^2(\phi_0).
\]

So, we may write \(z_0 = e^{2i\theta}g^2(\phi_0)\) for some \(\theta \in [0, \pi]\). It follows that
\[
w_0^2 = z_0 q(z_0, w_0) = e^{2i\theta}g^2(\phi_0)h^2(\phi_0).
\]

In other words,
\[
(2.3) \quad (z_0, w_0) \in C_{\phi_0} \subset K.
\]

Now, suppose \(\lambda_0 = q(z_0, w_0) = 0\). We repeat the above argument, now choosing an \(f \in \mathcal{P}(\overline{\Omega})\) such that \(f(0) = 1\) and \(|f| < 1\) on \(\overline{\Omega \setminus \{0\}}\). We obtain that
\[
|z_0| = \left| \int_{C_{-\pi} \cup C_0} zd\mu(z, w) \right| \leq g^2(0) = (a+b)^2,
\]
and
\[
\frac{1}{|z_0|} \leq \int_{C_{-\pi} \cup C_0} \frac{1}{z} d\mu(z, w) \leq \frac{1}{g^2(-\pi)} = \frac{1}{(a-b)^2}.
\]
Thus, \((a - b)^2 \leq |z_0| \leq (a + b)^2\) and \(w_0 = 0\). So, we have that

\[
(2.4) \quad (z_0, w_0) \in A.
\]

Combining (2.3) and (2.4), we conclude that

\[
\hat{K}_{\text{rat}} \subseteq K \cup A.
\]

Next, we show that \(A \subseteq \hat{K}_{\text{rat}}\). Let \(\mathbb{D} \) denote the open unit disc in \(\mathbb{C}\). Consider the continuous family of maps \(F_\phi : \mathbb{D} \to \mathbb{C}^2\) given by

\[
F_\phi : \zeta \mapsto (\zeta^2g(\phi), \zeta g(\phi)h(\phi))
\]

where \(\phi \in [-\pi, 0]\). Then, \(\{F_\phi(\cdot)\}_{(-\pi, 0]}\) is a continuous family of holomorphic discs in \(\mathbb{C}^2\) whose boundaries are attached to \(K\). Note that \(\zeta \mapsto F_\phi(\zeta)\) is a two-to-one map on \(\mathbb{D} \setminus 0\), when \(\phi\) is either \(-\pi\) or \(0\). Moreover,

\[
F_{-\pi}(\mathbb{D}) = \{(z, 0) \in \mathbb{C}^2 : |z| \leq (a - b)^2\}
\]

\[
F_0(\mathbb{D}) = \{(z, 0) \in \mathbb{C}^2 : |z| \leq (a + b)^2\}.
\]

Thus, for any holomorphic function \(f\) defined on \(\mathbb{C}^2\),

\[
(2.5) \quad \#\mathcal{Z}(f \circ F_0) - \#\mathcal{Z}(f \circ F_{-\pi}) = 2 \#\mathcal{Z}(f|_A),
\]

for \(f\) nonvanishing on \(C_{-\pi}\), where \(\#\mathcal{Z}(g)\) denotes the number of zeros of \(g\) (counting multiplicities).

Now, suppose \(P\) is a polynomial that vanishes at some point of \(\mathbb{A} \setminus K\) but does not vanish anywhere on \(K\). Set \(f_\phi(\zeta) = P(F_\phi(\zeta))\). Then, each \(f_\phi\) is holomorphic on \(\mathbb{D}\) and continuous up to the boundary. Since \(P\) does not vanish on \(K\), each \(f_\phi\) is nonvanishing on \(\partial \mathbb{D}\). So,

\[
G : \phi \mapsto \#\mathcal{Z}(f_\phi) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f_\phi'(\zeta)}{f_\phi(\zeta)} d\zeta
\]

is a continuous integer-valued function on \([-\pi, 0]\) — therefore, it must be a constant. But, from (2.5) and the assumption on \(P\), we have have that

\[
G(0) - G(-\pi) = \#\mathcal{Z}(f_0) - \#\mathcal{Z}(f_{-\pi}) = 2 \#\mathcal{Z}(P|_A) \geq 2
\]

This is a contradiction. Hence, \(P\) must vanish somewhere on \(K\) forcing every point of \(A\) to be in \(\hat{K}_{\text{rat}}\). This completes the proof of Theorem 2.1. \(\square\)

Since \(K\) is not rationally convex, we know that \(\mathcal{R}(K)\) is a proper subalgebra of \(\mathcal{C}(K)\). The complete description of \(\hat{K}_{\text{rat}}\) allows us to give a characterization of the functions in \(\mathcal{R}(K)\).

**Proposition 2.2.** Let \(A^o\) denote the interior of the annulus \(A\) relative to the plane \(w = 0\). Then,

\[
\mathcal{R}(K) = \{f \in \mathcal{C}(K) : f\text{ extends holomorphically to } A^o\}.
\]

**Proof.** First, observe that since \(\mathcal{R}(K)\) can be identified with \(\mathcal{R}(\hat{K}_{\text{rat}})\), the inclusion \(\mathcal{R}(K) \subseteq \{f \in \mathcal{C}(K) : f\text{ extends holomorphically to } A^o\}\) is clear.

Next, we work with \(q(z, w) = w^2/z\), as above. Note that \(\mathcal{R}(K)\) contains all the functions of the form \(f \circ q\) with \(f \in \mathcal{C}(\Gamma)\). Now, the polynomial \(p\) defined by \(p(z, w) = z\) is zero-free on \(K\), so \(p^j|_K \in \mathcal{R}(K)\) for all positive and negative integers \(j\). Also, if \(r\) is given by \(r(z, w) = w\), then \(r^j|_K \in \mathcal{R}(K)\) for all positive integers \(j\). It follows that \(\mathcal{R}(K)|_{C_\phi} = \text{the algebra of all restrictions of } f|_{C_\phi}\text{ for } f \in \mathcal{R}(K)\) — contains (the restriction to \(C_\phi\) of) \(e^{ik\theta}\) for all integers \(k\).
Consider now the closed ideal \( \mathcal{I} = \{ f \in \mathcal{R}(K) : f|_{\partial A} = 0 \} \) of \( \mathcal{R}(K) \). This ideal is a closed subalgebra of the space \( C_0(K \setminus A) \), the algebra of continuous functions on \( K \) that vanish on the boundary of the annulus \( A \). It contains the space \( \mathcal{B} \) of all (restrictions to \( K \) of) Laurent polynomials

\[
P(z, w) = \sum_{j=-N}^{N} f_j(q(z, w)) z^j
\]

with functions \( f_j \in C(\Gamma) \) that vanish at the origin. Note that \( \mathcal{B} \) separates points on \( K \setminus A \), and is closed under multiplication. Indeed, if \( P \) is as in (2.6), the complex conjugate of \( P \) on \( K \) agrees with \( \overline{P(z, w)} = \sum_{j=-N}^{N} \overline{f_j(q(z, w))} \overline{z}^{-j} \), which is a Laurent polynomial of the kind under consideration. The Stone-Weierstrass theorem implies that the closed subalgebra of \( C_0(K \setminus \partial A) \) generated by \( \mathcal{B} \) is all of \( C_0(K \setminus \partial A) \). But, \( \mathcal{B} \subseteq \mathcal{I} \subseteq C_0(K \setminus \partial A) \). So, \( \mathcal{I} = C_0(K \setminus \partial A) \).

We may now complete our proof via the following argument. Since 0 is a peak point of \( \mathcal{P}(\Gamma) \), the set \( \partial A \) is a peak set for \( \mathcal{R}(K) \). Thus, the algebra \( \mathcal{R}(K)|_{\partial A} \) is closed in \( C(\partial A) \). It is, moreover, dense in the space of restrictions to \( \partial A \) of functions in \( \mathcal{H}(A) \) — the space of functions continuous on \( A \) and holomorphic on \( A^\circ \). Consequently, every function \( f \) in \( \mathcal{H}(A) \) extends to a function, which we will denote by \( f^\ast \), in \( \mathcal{R}(K) \). Now, suppose \( g \) is continuous on \( K \cup A \) and holomorphic on \( A^\circ \). Let \( g^\ast \) denote an extension in \( \mathcal{R}(K) \) (as discussed above) of the function \( g|_A \). Then, \( g - g^\ast \) lies in \( C_0(K \setminus A) \), and so in \( \mathcal{R}(K) \). It follows that \( g = (g - g^\ast) + g^\ast \) lies in \( \mathcal{R}(K) \). Hence, the claim. \( \square \)

We now invoke [7] (and the references cited therein) to obtain the following criterion as a corollary. It is immediate from the corollary that the (finite, regular, Borel) measures orthogonal to \( \mathcal{R}(K) \) are those measures on \( K \) that are concentrated on \( \partial A \) and that are orthogonal to the algebra \( \mathcal{R}(A) \).

Corollary 2.3. The function \( f \in \mathcal{C}(K) \) lies in \( \mathcal{R}(K) \) if and only if

\[
\int_{\partial A} f(z, 0)g(z, 0) dz = 0
\]

for all functions \( g \) holomorphic on a neighborhood in the \( z \)-plane of the annulus \( A \).

3. Another family of Klein Bottles

In [3], the first author modifies Rudin’s Klein bottle to obtain a totally real Klein bottle in \( \mathbb{C}^2 \) whose polynomial hull is shown to be of dimension three. We repeat the technique used in the previous section to show that the rational hull of this set looks exactly like that of Rudin’s Klein bottle.

Let

\[
K^\ast = \left\{ \left( e^{2i\theta} g^2(\phi) e^{-i\phi h(\phi)/g(\phi)} \right) : -\pi \leq \theta, \phi < \pi \right\},
\]

where \( g \) and \( h \) are as in Section 2.

Theorem 3.1. The rational hull of \( K^\ast \) is \( K^\ast \cup A \), where

\[
A = \{(z, 0) \in \mathbb{C}^2 : (a - b)^2 \leq |z| \leq (a + b)^2 \}.
\]
Proof. The first part of the proof is almost identical to that of Theorem 2.1. Here, we make a slightly different choice of $q$:

$$q(z, w) = zw^2.$$ 

The resulting curve, $\Gamma = q(K^*)$, is the same as in the previous section. If $\lambda_0 = q(z_0, w_0) \notin \Gamma$, then $q(z, w) - \lambda_0$ is a polynomial whose zero set passes through $(z_0, w_0)$ but not through $K^*$. Thus, $q(K^*) \subseteq \Gamma$. The rest of the argument can be repeated verbatim to conclude that $\hat{K}^* \subseteq \hat{K}^* \cup \hat{A}$.

To obtain the other inclusion, we consider a continuous family of ‘rational discs’ as follows. Let

$$F_\phi : \zeta \mapsto \left(\zeta^2 g^2(\phi), \frac{h(\phi)}{g(\phi)\zeta}\right), \quad \phi \in [-\pi, 0],$$

be a family of holomorphic maps on the punctured disc $\mathbb{D} \setminus \{0\}$. Observe that $F_\phi(\partial \mathbb{D}) \subset K^*$, and for any polynomial $P$ on $\mathbb{C}^2$, $P \circ F_\phi$ is a rational function on $\mathbb{D}$ that is continuous up to the boundary and only has poles (if any) at the origin. Further, note that when $\phi$ is either $-\pi$ or 0, $\zeta \mapsto F_\phi(\zeta)$ is holomorphic on $\mathbb{D}$, and is two-to-one on $\mathbb{D} \setminus 0$. As before, $F_{-\pi}(\partial \mathbb{D})$ and $F_0(\partial \mathbb{D})$ are the inner and outer boundaries of $\hat{A}$.

Suppose $P$ is a polynomial that vanishes at some point of $A \setminus K^*$ but does not vanish anywhere on $K^*$. Set $f_\phi(\zeta) = P(F_\phi(\zeta))$. As discussed above, each $f_\phi$ is rational on $\mathbb{D}$ (with possible poles at 0). Since $P$ does not vanish on $K^*$, each $f_\phi$ is nonvanishing on $\partial \mathbb{D}$. So,

$$\phi \mapsto \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f_\phi'(\zeta)}{f_\phi(\zeta)} d\zeta$$

is a continuous integer-valued function on $[-\pi, 0]$ that counts the discrepancy between the number of zeroes and poles of $f_\phi$ in $\mathbb{D}$ — thus, it must be constant. Since $P$ vanishes on $A$, $f_0$ has at least one more zero than $f_{-\pi}$ (both are holomorphic on $\mathbb{D}$). This is a contradiction. Hence, $A \subset \hat{K}^*$. Combining the two inclusions, we have our proof. \qed

Remark 3.2. The main results of Section 2 and Section 3 are formally equivalent in the sense that either can be deduced from the other by invoking an appropriate automorphism. Indeed, consider the automorphism

$$J : (z, w) \mapsto \left(\frac{z}{z^2}, \frac{w}{z^2}\right)$$

on the domain $T = \{(z, w) \in \mathbb{C}^2 : z \neq 0\}$. The inverse of $J$ is given by $J^{-1}(z, w) = (z, z^2w)$. It is easy to see that the automorphism $J$ carries the Klein bottle $K$ onto the Klein bottle $K^*$, and, of course, $J^{-1}(K^*) = K$. Furthermore, $J$ effects an isomorphism between $\mathcal{R}(K^*)$ and $\mathcal{R}(K)$: given a rational function $f$ on $\mathbb{C}^2$, $f$ is holomorphic on a neighborhood of $K^*$ if and only if $f \circ J$ is holomorphic on a neighborhood of $K$. It follows that $J$ restricts to a bijection between $\hat{K}^*$ and $\hat{K}^*$. The same reasoning also gives the analogues of Proposition 2.2 and Corollary 2.3 for $\mathcal{R}(K^*)$.

4. A FAMILY OF TOTALLY REAL TORI

In [4], Duval and Gayet provide a dichotomy for any generic totally real unknotted torus embedded in $S^3$ — either it is rationally convex and fillable by holomorphic discs, or its rational hull contains a holomorphic annulus. As an example
of the latter, they consider a family of tori that come naturally from a fibration $S^1 \to S^3 \to S^2$. The techniques used in the previous sections can also be utilised to compute the precise rational hulls of these tori.

Let $\Theta : S^3 \to S^2 \subset \mathbb{C} \times \mathbb{R}$ be the conjugate Hopf fibration given by

$$(z, w) \mapsto (2zw, |z|^2 - |w|^2).$$

Let $\pi$ and $\pi_{\mathbb{R}}$ denote the projections of $\mathbb{C} \times \mathbb{R}$ onto $\mathbb{C}$ and $\mathbb{R}$, respectively. For any embedded closed curve $\gamma$ in $S^2$, the set

$$T^\gamma = \Theta^{-1}(\gamma)$$

is a torus in $S^3$. Furthermore, if $\pi(\gamma) \subset \mathbb{D}$ is immersed, then $T^\gamma$ is totally real. Fix a $\gamma$ so that $\pi(\gamma) \subset \mathbb{D}$ is a figure eight avoiding the origin. After suitable rotations, we can assume that the double point of $\gamma$ is in the interval $(0,1)$. For the sake of computational convenience, we further assume that the double point of $\pi(\gamma)$ is at $\frac{3}{5} \in \mathbb{D}$, and refer to $T^\gamma$ simply as $T$. A similar computation can be carried out for any other choice of double point in $(0,1)$.

**Theorem 4.1.** Let $\gamma$ and $T$ be as chosen above. Then,

$$\hat{T}_{\text{rat}} = T \cup A,$$

where $A = \{(z, w) \in \mathbb{C}^2 : 2zw = \frac{3}{5}; -\frac{1}{5} \leq |z|^2 - |w|^2 \leq \frac{1}{5}\}$.

**Proof.** We will first show that $T_{\text{rat}} \subseteq T \cup A$ using the same technique as in the proofs of Theorems 2.1 and 3.1. Here, let

$$q(z, w) = 2zw.$$

Set $\Gamma = q(T) = \pi(\gamma)$. So, $\Gamma$ is a smooth immersed curve in $\mathbb{D}$ with exactly one double point. For $\lambda \in \Gamma \setminus \{\frac{3}{5}\}$, let $\tilde{\lambda}$ denote the unique lift of $\lambda$ to $\gamma \subset S^2$.

As before, we describe the fibers $T_{\lambda} = \{(z, w) \in T : q(z, w) = \lambda\}$ for all $\lambda \in \Gamma$. Note that since $q(e^{i\theta}z, e^{-i\theta}w) = q(z, w)$, each $T_{\lambda}$ contains circles. We make this more precise. Any $(z, w) \in T_{\lambda}$ satisfies the following equations.

\begin{align*}
|2zw| &= |\lambda|, \\
|2zw|^2 + (|z|^2 - |w|^2)^2 &= 1. 
\end{align*}

(4.1) \hspace{1cm} (4.2)

The second equation follows from the fact that $(z, w) \in S^3$. Substituting (4.1) into (4.2), we obtain that

$$|z|^2 - |w|^2 = \pm \sqrt{1 - |\lambda|^2}.$$

As long as $\lambda \neq \frac{3}{5}$, any $\lambda \in \gamma$ lifts to a unique point $\tilde{\lambda} \in S^2$. So, if $(z, w) \in T_{\lambda}$, $(2zw, |z|^2 - |w|^2) = \tilde{\lambda}$, and the sign of $|z|^2 - |w|^2$ coincides with the sign of $\pi_{\mathbb{R}}(\lambda)$. If $\pi_{\mathbb{R}}(\lambda) > 0$ (i.e., $\tilde{\lambda}$ is in the northern hemisphere of $S^3$), we have that

$$|z|^2 - |w|^2 = \sqrt{1 - |\lambda|^2}.$$

(4.3)

We simultaneously solve (4.1) and (4.3) for $|w|^2$ to obtain that

$$|w|^2 = \frac{1 - \sqrt{1 - |\lambda|^2}}{2}.$$
We can similarly solve for \( r(\lambda) = |w| \) when \( \pi_\mathbb{R}(\lambda) \leq 0 \) to obtain that

\[
(4.4) \quad r^2(\lambda) = \begin{cases} \frac{1-\sqrt{1-|\lambda|^2}}{2}, & \text{if } \pi_\mathbb{R}(\lambda) > 0; \\ \frac{1+\sqrt{1-|\lambda|^2}}{2}, & \text{if } \pi_\mathbb{R}(\lambda) < 0; \\ \frac{|\lambda|}{2}, & \text{if } \pi_\mathbb{R}(\lambda) = 0. \end{cases}
\]

So, when \( \lambda \neq \frac{3}{5} \), we get that

\[
T_\lambda = \left\{ \left( \frac{\lambda}{2r(\lambda)} e^{-i\theta}, r(\lambda)e^{i\theta} \right) : \theta \in \mathbb{R} \right\},
\]

where \( r(\lambda) > 0 \) is as above. Thus, \( T_\lambda \) is a circle in \( T \). It is polynomially convex, and \( P(T_\lambda) = C(T_\lambda) \). As \( T_\lambda \) is a peak set for \( \tilde{T}_{\text{rat}} \), we further have that \( R(T)|T_\lambda = C(T_\lambda) \). Also, observe that \( |z| \) is the constant \( |\lambda|/2r(\lambda) \) on \( T_\lambda \). On the other hand,

\[
T_{\tilde{\gamma}} = \left\{ \left( \frac{3e^{-i\theta}}{\sqrt{10}}, \frac{e^{i\theta}}{\sqrt{10}} \right) : \theta \in \mathbb{R} \right\} \cup \left\{ \left( \frac{e^{-i\theta}}{\sqrt{10}}, \frac{3e^{i\theta}}{\sqrt{10}} \right) : \theta \in \mathbb{R} \right\}
\]

is the boundary of \( A \) within the variety \( \{2zw = 3/5\} \) in \( \mathbb{C}^2 \). We also note that

\[
\lim_{\Gamma \ni \lambda \to \frac{3}{5}} r(\lambda) = \frac{1}{\sqrt{10}} \text{ or } \frac{3}{\sqrt{10}}
\]

depending on the branch of approach.

Now, let \( (z_0, w_0) \in \tilde{T}_{\text{rat}} \). As before, \( \lambda_0 = q(z_0, w_0) \in \Gamma \). Let \( \Omega \) denote the open set bounded by \( \gamma \) (\( \Omega \) has two components). Again, \( \lambda_0 \) is a peak point of \( P(\Omega) \); let \( f \in P(\Omega) \) peak there. For all positive integers \( n \), the function \( z(f^n \circ q) \) is in \( R(T) \). If \( \mu \) is a representing measure for \( (z_0, w_0) \), then

\[
(4.5) \quad |z_0| = \left| \int_T z f^n(2zw) d\mu(z, w) \right|.
\]

But, as \( n \to \infty \), the right-hand side of \((4.5)\) converges to

\[
\left| \int_{T_{\lambda_0}} zd\mu(z, w) \right| \leq \begin{cases} \frac{|\lambda_0|}{2r(\lambda_0)}, & \text{if } \lambda_0 \neq \frac{3}{5}; \\ \frac{1}{\sqrt{10}}, & \text{if } \lambda_0 = \frac{3}{5}. \end{cases}
\]

Repeating the above argument for \( 1/z \), which is in \( R(T) \), since \( z \) does not vanish on \( T \), we also have that

\[
|z_0| \geq \begin{cases} \frac{|\lambda_0|}{2r(\lambda_0)}, & \text{if } \lambda_0 \neq \frac{3}{5}; \\ \frac{1}{\sqrt{10}}, & \text{if } \lambda_0 = \frac{3}{5}. \end{cases}
\]

Hence, we obtain that

\[
|z_0| = \frac{|\lambda_0|}{2r(\lambda_0)}, \quad \text{if } \lambda_0 \neq \frac{3}{5},
\]

and

\[
\frac{1}{10} \leq |z_0|^2 \leq \frac{9}{10}, \quad \text{if } \lambda_0 = \frac{3}{5}.
\]

In the former case, since \( 2z_0w_0 = \lambda_0 \), we obtain that \( (z_0, w_0) \in T_{\lambda_0} \). In the latter case, we have that

\[
|w_0|^2 = \left| \frac{3}{10z_0} \right|^2 = \frac{9}{100|z_0|^2}.
\]
Thus,
\[ -\frac{4}{5} \leq |z_0|^2 - |w_0|^2 \leq \frac{4}{5}. \]
So, \((z_0, w_0) \in A\). Therefore, \(\tilde{T}_{\text{rat}} \subseteq T \cup A\).

Next, let \(\Gamma'\) denote one of the two closed loops of \(\Gamma\). Observe that the boundary of the annulus \(A\) bounds the cylinder
\[ \bigcup_{\lambda \in \Gamma'} T_\lambda \]
in \(T\). Since \(\partial A\) bounds a surface in \(T\), a classical argument using Stokes’ theorem shows that \(T \cup A \subseteq \tilde{T}_{\text{rat}}\). For the sake of completeness and uniformity, we provide a slightly different argument for this that follows the idea used in Section 2. Suppose \(P\) is a polynomial in \(\mathbb{C}^2\) that vanishes on \(A \setminus T\) but does not vanish anywhere on \(T\). Fix a parametrization \(s : [0, 1] \to \Gamma'\) so that \(s(0) = s(1) = \frac{3}{5}\), \(\lim_{t \to 0} r(s(t)) = \frac{1}{\sqrt{10}}\) and \(\lim_{t \to 1} r(s(t)) = \frac{3}{\sqrt{10}}\). Then,

\[ F_t : \zeta \mapsto \begin{cases} \left( \frac{3}{\sqrt{10}}, \frac{\zeta}{\sqrt{10}} \right), & \text{if } t = 0, \\ \left( \frac{3}{2\pi t}, \frac{r(s(t)) \zeta}{1 - \sqrt{10}}, \frac{3}{\sqrt{10}} \right), & \text{if } t \in (0, 1), \\ \left( \frac{3}{\sqrt{10}}, \frac{\zeta}{\sqrt{10}} \right), & \text{if } t = 1 \end{cases} \]

is a continuous family of holomorphic maps on the punctured unit disc \(\mathbb{D} \setminus \{0\}\) which are continuous and non-vanishing on \(\partial \mathbb{D}\) and have poles at the origin. Thus,

\[ G : t \mapsto \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{(P \circ F_t)'(\zeta)}{P \circ F_t(\zeta)} d\zeta \]
is a continuous integer-valued function on \([0, 1]\). But, since \(G(1) - G(0)\) counts the zeros of \(P\) on \(A\), we have that \(|G(1) - G(0)| \geq 1\). This is a contradiction. Hence, \(P\) must vanish on \(T\). We have therefore established the second inclusion, and indeed the theorem itself. \(\square\)

As in the case of the Klein bottles discussed earlier, we can provide a more accurate description of \(\mathcal{R}(T)\).

**Proposition 4.2.** Let \(A^o\) denote the interior of the annulus \(A\) relative to the variety \(\{2zw = 3/5\}\). Then,

\[ \mathcal{R}(T) = \{ f \in \mathcal{C}(T) : f \text{ extends holomorphically to } A^o \}. \]

**Proof.** We first observe that \(\mathcal{R}(T)\) contains all the functions of the form \(f \circ \pi\) for all \(f \in \mathcal{C}(\Gamma)\). Thus, if \(\mu\) is an extreme point of the unit ball of \(\mathcal{R}(T)^\perp\) — the space of all finite regular Borel measures on \(T\) that are orthogonal to \(\mathcal{R}(T)\) — then \(\mu\) is supported on the fiber \(T_\lambda\) for some \(\lambda \in \Gamma\). But, as observed earlier, if \(\lambda \neq \frac{3}{5}\), each \(T_\lambda\) satisfies \(\mathcal{P}(T_\lambda) = \mathcal{C}(K)\). Thus, \(\mu|_{T_\lambda} \equiv 0\) for such \(\lambda\), and \(\mu\) is concentrated on \(T_{\frac{3}{5}}\). In fact, this yields that any measure orthogonal to \(\mathcal{R}(T)\) is supported on \(T_{\frac{3}{5}}\). So, \(\mathcal{R}(T)\) contains the ideal

\[ \mathcal{I} = \{ f \in \mathcal{C}(T) : f \text{ vanishes on } \partial A \}. \]

Since \(3/5\) is a peak point of \(\mathcal{P}(\Gamma)\), the set \(\partial A\) is a peak set for \(\mathcal{R}(T)\). Thus, the algebra \(\mathcal{R}(T)|_{\partial A}\) is closed in \(\mathcal{C}(\partial A)\). It is, moreover, dense in the space of restrictions to \(\partial A\) of functions in \(\mathcal{H}(A)\) — the space of function continuous on \(A\) and holomorphic on \(A^o\). Consequently, every function \(f\) in \(\mathcal{H}(A)\) extends to a
function, which we will denote by $f^*$, in $R(T)$. Now, suppose $g$ is continuous on $T \cup A$ and holomorphic on $A^\circ$. Let $g^*$ denote an extension in $R(T)$ (as discussed above) of the function $g|_A$. Then, $g - g^*$ lies in $I$, and so in $R(T)$. It follows that $g = (g - g^*) + g^*$ lies in $R(T)$. Hence, the claim. □

**Remark 4.3.** The results of this section can be extended to the case where $\pi(\gamma)$ has $n$ self-intersections in $\mathbb{D}$ (and avoids 0). In this case, $T^\gamma$ is a torus in $\mathbb{C}^2$ whose rational hull consists precisely of the torus and $n$ attached annuli.

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