GENERATING HYPERBOLIC SINGULARITIES IN COMPLETELY INTEGRABLE SYSTEMS

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Abstract. Let \((M, \Omega)\) be a connected symplectic 4-manifold and let \(F = (J, H): M \to \mathbb{R}^2\) be a completely integrable system on \(M\) with only non-degenerate singularities and for which \(J: M \to \mathbb{R}\) is a proper map. Assume that \(F\) does not have singularities with hyperbolic blocks and that \(p_1, \ldots, p_n\) are the focus-focus singularities of \(F\). For each subset \(S = \{i_1, \ldots, i_j\}\) we will show how to modify \(F\) locally around any \(p_i, i \in S\), in order to create a new integrable system \(\tilde{F} = (J, \tilde{H}): M \to \mathbb{R}^2\) such that its classical spectrum \(\tilde{F}(M)\) contains \(j\) smooth curves of singular values corresponding to non-degenerate transversally hyperbolic singularities of \(\tilde{F}\). Moreover the focus-focus singularities of \(\tilde{F}\) are precisely \(p_i, i \in \{1, \ldots, n\} \setminus S\), and each of these \(p_i\) is non-degenerate. The proof is based on Eliasson’s linearization theorem for non-degenerate singularities, and properties of the Hamiltonian Hopf bifurcation.

1. Introduction

This paper intends to shed some light on the following question in the theory of finite dimensional completely integrable Hamiltonian systems:

Question 1.1 Suppose that \(f_1, \ldots, f_n: M \to \mathbb{R}\) form an integrable system on a \(2n\)-dimensional symplectic manifold \((M, \omega)\). What is the relation between the following conditions?

1. the flow of at least one of the functions \(f_i, i \in \{1, \ldots, n\}\), is periodic;
2. existence of a set \(B \subset M\) of non-degenerate focus-focus singularities of the joint map \((f_1, \ldots, f_n): M \to \mathbb{R}^n\);
3. existence of a set \(C \subset M\) of non-degenerate hyperbolic singularities of \((f_1, \ldots, f_n)\);
4. existence of set \(D \subset M\) of degenerate singularities of \((f_1, \ldots, f_n)\).

Several results are known which give partial answers to this question. For instance, under general assumptions on \(f_1, \ldots, f_n\) and \(M\), if condition (1) holds for all \(i = 1, \ldots, n\), then

\[ B = C = D = \emptyset. \]

In fact, there is a complete theory for these systems - called toric integrable systems - when \(M\) is compact, by Atiyah, Guillemin-Sternberg, and Delzant [At82, GuSt82, De88].
The results of this paper imply that there are integrable systems for which (1), (2), (3), and (4) hold simultaneously with $B \neq \emptyset$, $C \neq \emptyset$, and $D \neq \emptyset$ (this is the content of Theorems 1.2, 3.3, and 5.1). Moreover, it seems quite plausible that under some general condition, (1), (2) and (3) with $B \neq \emptyset$ and $C \neq \emptyset$ will imply that $D \neq \emptyset$ in (4) (see Question 5.2).

Because several of the technical tools we use are exclusive to dimension four, from now on we assume that $2n = 4$. Let $(M, \Omega)$ be a connected symplectic 4-manifold, that is, $M$ is a smooth connected 4-manifold, and $\Omega$ is a non-degenerate closed 2-form on $M$, i.e. a symplectic form. A smooth function $f: M \to \mathbb{R}$ induces a vector field $\mathcal{X}_f$ on $M$ by means of Hamilton’s equation:

$$\Omega(\mathcal{X}_f, \cdot) = -df.$$  

The vector field $\mathcal{X}_f$ is called the *Hamiltonian vector field* induced by $f$. Given any two smooth functions $f, g: M \to \mathbb{R}$ we may define their Poisson bracket

$$\{f, g\} := \Omega(\mathcal{X}_f, \mathcal{X}_g).$$

An *integrable system*\(^1\) is a triple

$$(M, \Omega, (J, H))$$

where $(M, \Omega)$ is a connected symplectic 4-manifold and $J, H: M \to \mathbb{R}$ are smooth functions for which $\{J, H\} \equiv 0$ on $M$, and such that the differentials $dJ, dH$ are linearly independent almost everywhere on $M$. Near each point in $M$ there are coordinates $(x, y, \xi, \eta)$ in which the symplectic form $\Omega$ is given by $d\xi \wedge dx + d\eta \wedge dy$, and the equation $\{J, H\} \equiv 0$ may be written as a partial differential equation

$$\frac{\partial J}{\partial \xi} \frac{\partial H}{\partial x} - \frac{\partial J}{\partial x} \frac{\partial H}{\partial \xi} + \frac{\partial J}{\partial \eta} \frac{\partial H}{\partial y} - \frac{\partial J}{\partial y} \frac{\partial H}{\partial \eta} = 0,$$

which is equivalent to $J$ (respectively $H$) being constant along the flow lines of $\mathcal{X}_H$ (respectively $\mathcal{X}_J$). The following result gives a method to attach hyperbolic singularities to an integrable system, by modifying locally the system near its focus-focus singularities. Focus-focus singularities come endowed with a Hamiltonian circle action near the focus-focus singular fiber, it is the Hamiltonian action generated by the $J$-component of the system in local normal form (as explained in Section 4). The image of the joint map $F := (J, M): M \to \mathbb{R}^2$ is called by physicists the *classical spectrum* of $F$ (in analogy with the semiclassical spectrum of quantum mechanics).

Throughout the paper we assume that each focus-focus singularity is in a singular fiber in which it is the only singularity (a system satisfying this property is often called “simple”). Topologically, this means that the singular fiber containing a focus-focus singularity is a torus pinched precisely once (as opposed to a multipinched torus).

\(^1\)More precisely, a “finite dimensional completely integrable Hamiltonian system”.


Theorem 1.2. Let \((M, \Omega, F := (J, H): M \to \mathbb{R}^2)\) be an integrable system on a connected symplectic 4-manifold \((M, \Omega)\) with only non-degenerate singularities and for which \(J: M \to \mathbb{R}\) is a proper map. Assume that \(F\) does not have singularities with hyperbolic blocks and that \(p_1, \ldots, p_n\) are all the focus-focus singularities of \(F\), each of which is simple. For each subset \(S = \{i_1, \ldots, i_j\}, 1 \leq j \leq n\), there is an integrable system

\[(M, \Omega, \tilde{F}_S := (J, \tilde{H}): M \to \mathbb{R})\]

such that its classical spectrum \(\tilde{F}_S(M)\) contains \(j\) loops consisting of three piecewise smooth curves of singular values. One smooth piece corresponds to non-degenerate transversally hyperbolic singularities of \(\tilde{F}_S\) and two pieces correspond to non-degenerate transversally elliptic singularities. Moreover the focus-focus singularities of \(\tilde{F}_S\) are precisely \(p_i, i \in \{1, \ldots, n\} \setminus S\), and each \(p_i, i \in \{1, \ldots, n\} \setminus S\), is non-degenerate.

The proof of Theorem 1.2 uses Eliasson’s linearization theorem for non-degenerate singularities, and properties of the Hamiltonian Hopf bifurcation.

Remark 1.3. The properness of \(J\) in Theorem 1.2 means that the preimage under \(J\) of a compact set is compact in \(M\).

Structure of the paper. In Section 2 we review the different types of singularities an integrable system can have, state Eliasson’s linearization theorem, and recall the notion of degenerate singularity. In Section 3 we introduce hyperbolic semitoric systems, and prove a version of Theorem 1.2 for semitoric systems that takes into account the additional symmetry given by the semitoric property. In Section 4 we explain the existence of a global Hamiltonian circle action semiglobally near a singularity of focus-focus type, using Eliasson’s linearization theorem. In Section 5 we state the general theorem of the paper, Theorem 5.1, of which Theorem 1.2 is a consequence. In Section 6 we briefly recall the basics about Hopf bifurcations which we need for the proof of Theorem 5.1. The remaining of the paper is devoted to the proof of Theorem 5.1.

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2. Eliasson’s Linearization of Singularities

Let \((M, \Omega, F := (J, H): M \to \mathbb{R}^2)\) be an integrable system and let \(p \in M\). We say that \(p\) is a regular point if \(d_p F\) has rank 2. A point \(c \in \mathbb{R}^2\) is called a regular value if every point in \(F^{-1}(c)\) is regular; in this case, \(F^{-1}(c)\) is called a regular fiber. We say that \(p\) is a critical point (or a singularity), if the rank of \(d_p F\) is 0 or 1. The fiber \(F^{-1}(c)\) is a singular fiber if it contains one or more critical points.
Let $X$ be a connected component of a regular fiber $F^{-1}(c)$ and assume that the Hamiltonian vector fields $\mathcal{X}_{f_1}, \mathcal{X}_{f_2}$ are complete on $F^{-1}(c)$. Then it follows from the definition of integrable system that $X$ is diffeomorphic to $T^2$, $S^1 \times \mathbb{R}$, or $\mathbb{R}^2$. In the particular case that $F^{-1}(c)$ is compact, then $\mathcal{X}_{f_1}, \mathcal{X}_{f_2}$ are complete, and therefore $X$ is diffeomorphic to $T^2$; this is always true for instance if at least one of $f_1, f_2, F$ is a proper map.

2.1. Non degeneracy. Throughout this paper we use the notion of non-degeneracy of a singularity. Suppose that $(M, \Omega)$ is a connected symplectic four-manifold. Let $F := (f_1, f_2)$ be an integrable system on $(M, \Omega)$ and $p \in M$ a critical point of $F$.

If $d_p F = 0$, then $p$ is called non-degenerate if the Hessians $\text{Hess} f_j(p)$, $j = 1, 2$, span a Cartan subalgebra of the symplectic Lie algebra of quadratic forms on the tangent space $(T_p M, \Omega_p)$ to $M$ at $p$. It follows from the work of Williamson [Wi36] that such a Cartan subalgebra has a basic building block of three types: two uni-dimensional ones (the elliptic block $x^2 + \xi^2$ and the real hyperbolic block: $x\xi$) and a two-dimensional block called focus-focus:

$$
J_1 = x\eta - y\xi \\
J_2 = x\xi + y\eta.
$$

If rank $(d_p F) = 1$ one may assume that $d_p f_1 \neq 0$. Let $\iota : S \to M$ be an embedded local 2-dimensional symplectic submanifold through the point $p$, such that $T_p S \subset \ker(d_p f_1)$ and $T_p S$ is transversal to the Hamiltonian vector field $\mathcal{X}_{f_1}$ defined by $f_1$. The critical point $p$ of $F$ is transversally non-degenerate if $\text{Hess}(\iota^* f_2)(p)$ is a non-degenerate symmetric bilinear form on $T_p S$.

Remark 2.1. The notion of non-degeneracy is independent of the choice we make of $S$. The existence of $S$ follows from the Flow Box Theorem (see, eg. [AbMa78, Theorem 5.2.19], also called the Darboux-Carathéodory theorem [PeVN11, Theorem 4.1]).

Remark 2.2. For the notion of non-degeneracy of a critical point in arbitrary dimension see [Ve78], [VN06, Section 3].

2.2. Linearization theorem. Non-degenerate critical points may be characterized ([El90, El84, VNWa14]) using the Williamson normal form [Wi36]. Eliasson’s theorem stated below describes the local normal form around non-degenerate singularities of an integrable system; the result holds in any dimension, but we state it here in dimension four since it is the case we are concerned with in the present paper. The analytic version of the theorem is due to Vey [Ve78].

**Theorem 2.3** (Eliasson [El90, El84, VNWa14]). The non-degenerate critical points of an integrable system $F : M \to \mathbb{R}^2$ are linearizable. That is, if $p \in M$ is a nondegenerate critical point of the integrable system $F = (f_1, f_2) : M \to \mathbb{R}^2$ then there exist symplectic coordinates $(x, y, \xi, \eta)$
Figure 1. Local description of the possible images of $F$ near a regular or singular value. The transversally-hyperbolic, elliptic-hyperbolic, and hyperbolic-hyperbolic cases are not possible if $F$ is semitoric (Definition 3.1), but they are possible if $F$ is hyperbolic semitoric (Definition 3.2).

near $p$, in which $p$ is represented as $(0, 0, 0, 0)$, such that $\{f_i, J_j\} = 0$, for all $i, j \in \{1, 2\}$, where we have the following possibilities for each $J_i$, $i \in \{1, 2\}$, each of which is defined on a neighborhood of $(0, 0, 0, 0) \in \mathbb{R}^4$:

(i) Elliptic component: $J_i = (x^2 + \xi^2)/2$ or $J_i = (y^2 + \eta^2)/2$.
(ii) Hyperbolic component: $J_i = x\xi$ or $J_i = y\eta$.
(iii) Focus-focus component: $J_1 = x\eta - y\xi$ and $J_2 = x\xi + y\eta$ (note that this component appear in “pairs”).
(iv) Non-singular component: $J_i = \xi$ or $J_i = \eta$.

Moreover if $p$ does not have any hyperbolic block, then the system of commuting equations $\{f_i, J_j\} = 0$, for all indices $i, j \in \{1, 2\}$, may be replaced by the single equation

$$(F - F(p)) \circ \varphi = g \circ (J_1, J_2),$$

where $\varphi = (x, y, \eta, \xi)^{-1}$ and $g$ is a diffeomorphism from a small neighborhood of the origin in $\mathbb{R}^4$ into another such neighborhood, such that

$$g(0, 0, 0, 0) = (0, 0, 0, 0).$$
See Figure 1 for a local description of the singularities appearing in Theorem 2.3.

**Remark 2.4.** The analytic case of Theorem 2.3 was proved by Rüßmann in [Ru64] when $2n = 4$, and then in any dimension by Vey [Ve78].

A simple way to check Eliasson non-degeneracy is as follows:

**Lemma 2.5.** Let $F := (f_1, f_2): M \to \mathbb{R}^2$ be an integrable system. A critical point $p$ of $F$ of rank 0 is non-degenerate if the Hessians $\text{Hess}f_1(p)$ and $\text{Hess}f_2(p)$ are linearly independent, and there is a linear combination

$$\alpha B \text{Hess}f_1(p) + \beta B \text{Hess}f_2(p)$$

for which there are no multiple eigenvalues. Here $B$ is the symplectic matrix corresponding to the symplectic form $\Omega$. In particular if $B \text{Hess}f_1(p)$ has no multiple eigenvalues then $p$ is non-degenerate.

This criterion is based on the fact (see for instance [BoFo04]) that a commutative subalgebra of the symplectic algebra is a Cartan subalgebra if and only if it is two-dimensional and if it contains an elements whose eigenvalues are different.

### 3. Hyperbolic semitoric systems

Theorem 1.2 may be applied in particular to enlarge the category of semitoric systems, these are systems which have an additional circular Hamiltonian symmetry coming from a global Hamiltonian action of the circle $S^1$. Many integrable systems from classical mechanics (see eg. the book by Holm [Ho11], or the article [PeVN12b]), exhibit symmetries of this nature including the Lagrange Top, the two-body problem, and the spherical pendulum. Recall that an action of the circle $S^1$ on a symplectic manifold $(M, \Omega)$ by symplectomorphisms is Hamiltonian if there exists a smooth map $J: M \to \mathbb{R}$, called the momentum map such that

$$\Omega(X_M, \cdot) = -dJ,$$

where $X_M$ is the vector field (or infinitesimal generator) of the $S^1$-action.

The category of semitoric systems includes many examples from the physics literature such as integrable systems of Jaynes-Cummings type, and the Jaynes-Cummings system. In this section we state a theorem which allows us to construct a semitoric system with hyperbolic singularities, from a semitoric system without hyperbolic singularities; below we give the precise definitions of these notions.

#### 3.1. The Jaynes-Cummings system

The famous *Jaynes-Cummings system* [JaCu63, Cu65] is given on phase space $S^2 \times \mathbb{R}^2$ by

$$J := \frac{u^2 + v^2}{2} + z \quad \text{and} \quad H := \frac{1}{2}(ux + vy),$$

(1)
where $S^2$ is the unit sphere in $\mathbb{R}^3$ with coordinates $(x, y, z)$, and $\mathbb{R}^2$ is equipped with coordinates $(u, v)$. We endow $S^2 \times \mathbb{R}^2$ with the product symplectic structure $\omega_{S^2} \oplus \omega_0$ where $\omega_{S^2}$ is the standard symplectic form on $S^2$ and $\omega_0$ is the standard symplectic form on $\mathbb{R}^2$. The $J$ component is the momentum map of the Hamiltonian $S^1$-action that simultaneously rotates about the vertical axes of $S^2$ and the origin in $\mathbb{R}^2$. Under the flow of $J$, the points $(x, y, z)$ and $(u, v)$ move along the flows of $z$ and $(u^2 + v^2)/2$, respectively, with same angular velocity, so $\langle (x, y), (u, v) \rangle = ux + vy = 2H$ is constant and commutes with $J$. The completely integrable system $F: S^2 \times \mathbb{R}^2$ given by (1) has been extensively studied, and recently attracted a lot of interest in both the physics and mathematics communities, see for instance [BaDo13a, BaDo13c, PeVN12]. In [PeVN12, Corollary 2.2] it was proved that $(S^2 \times \mathbb{R}^2, \omega_{S^2} \oplus \omega_0, (J,H))$ is an example of a so called semitoric system, which we define in general next.

### 3.2. Semitoric systems

We start with the notion of a semitoric system ([PeVN09, Definition 2.1]).

**Definition 3.1.** A semitoric system (or semitoric integrable system) is an integrable system $(M, \Omega, F := (J,H) : M \to \mathbb{R}^2)$ such that:

1. $J$ is the momentum map for a Hamiltonian $S^1$-action;
2. $J$ is proper;
3. $F$ has only non-degenerate singularities, without hyperbolic blocks.

If $F$ is a semitoric system in the sense of Definition 3.1, we have the following possibilities for the map $(J_1, J_2)$ in Theorem 2.3, depending on the rank of the singularity:

1. if $p$ is a singularity of $F$ of rank zero, then the building blocks are
   - (i) $J_1 = (x^2 + \xi^2)/2$ and $J_2 = (y^2 + \eta^2)/2$.
   - (ii) $J_1 = x\eta - y\xi$ and $J_2 = x\xi + y\eta$.
2. if $p$ is a singularity of $F$ of rank one, then
   - (iii) $J_1 = (x^2 + \xi^2)/2$ and $J_2 = \eta$.

Semitoric systems as in Definition 3.1 are classified in [PeVN09, PeVN11] in terms of five symplectic invariants. This classification excluded systems with hyperbolic singularities, which are however prominent in the physics literature. By eliminating conditions (ii) and (iii) in Definition 3.1 one would expand significantly the class of integrable systems which are “semitoric” in spirit, that is, those for which the component $J$ still generates a Hamiltonian $S^1$-action. For the purpose of this paper, we keep item (ii) due technical reasons, and show that there are many interesting examples of systems

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2it can probably be weakened to the condition that $F$ is proper as a map into $\mathbb{R}^2$, as in [PeRaVN15, PeRaVN14], but this still requires substantial work.
which satisfy the assumptions (i) and (ii) but not (iii); precisely, we will be concerned with hyperbolic semitoric systems.

**Definition 3.2.** A hyperbolic semitoric system is an integrable system $(M, \Omega, F := (J, H) : M \to \mathbb{R}^2)$ for which:

(a) $J$ is the momentum map for a Hamiltonian $S^1$-action;
(b) $J$ is proper;
(c.1) the set of hyperbolic singularities of $F$ is non-empty;
(c.2) the set of degenerate singularities $F$ is isolated.

Our next goal is to construct hyperbolic semitoric systems from semitoric systems.

### 3.3. From semitoric to hyperbolic semitoric systems.

The main theorem of this paper (Theorem 5.1) implies that any semitoric system in the more strict sense of Definition 3.1 may be suitably modified to create a new “semitoric” system in the more relaxed sense of Definition 3.2. That is, we have the following result.

**Theorem 3.3.** Given a semitoric system

$$(M, \Omega, F := (J, H))$$

with focus-focus singularities $p_1, \ldots, p_{mf}$ and an integer $1 \leq k < m_f$, there exists a smooth Hamiltonian function $\tilde{H}^k : M \to \mathbb{R}$ such that

$$(M, \Omega, \tilde{F} := (J, \tilde{H}^k))$$

is a hyperbolic semitoric system with non-degenerate focus-focus singularities at $p_1, \ldots, p_k$.

The new aspect of this theorem is that it says that the global Hamiltonian $S^1$-action of the original semitoric system may be preserved, in addition to the other information about the old and newly created singularities given in Theorem 1.2.

### 4. Focus-focus singularities and Hamiltonian $S^1$-actions

The proof in Section 7 is based on the knowledge of Eliasson’s normal form near a focus-focus singular point. Here we quickly review the ingredients we use.

**Definition 4.1.** A smooth map $F = (H_1, H_2) : M \to \mathbb{R}^2$ is a momentum map on $U$ if $dF$ is surjective almost everywhere in $U$ and $\{H_1, H_2\} = 0$.

**Definition 4.2.** A singular Liouville foliation $\mathcal{F}$ is a union of connected subsets of $M$, called the leaves of the foliation, which are pairwise disjoint and such that there is a momentum map $F : W \to \mathbb{R}^2$ in the sense of Definition 4.1 such that the leaves of $\mathcal{F}$ coincide with the connected components of the fibers $F^{-1}(c)$ for $c$ varying in an open subset of $\mathbb{R}^2$. 
Figure 2. The figure displays elliptic (red dot) and hyperbolic singularities (red star) on a surface.

Let \((M, \Omega, F := (f_1, f_2))\) be an integrable system on a connected symplectic 4-manifold \((M, \Omega)\) (semitoric or not). Let \(F\) be the singular Liouville foliation of \(M\) associated to \(F\), as in Definition 4.2. The leaves of this foliation are the connected components of the fibers \(F^{-1}(c)\). Let \(p\) be a critical point of focus-focus type. For simplicity suppose that \(F(p) = 0\) and that the fiber \(\Lambda_0 := F^{-1}(0)\) contains no critical point other than \(p\). It is well known that \(\Lambda_0\) is a “pinched torus” (that is, an immersion of the sphere \(S^2\) with a transversal double point). All the fibers of \(F\) in a neighborhood of the fiber \(\Lambda_0\) are 2-tori. By Theorem 2.3 there exist symplectic coordinates \((x, y, \xi, \eta)\) in a neighborhood \(V\) about the focus-focus point \(p\) in which if

\[
J_1 = x\eta - y\xi
\]

\[
J_2 = x\xi + y\eta
\]

then \((J_1, J_2)\) is a momentum map for \(\mathcal{F}\); here the critical point \(p\) corresponds to \((0, 0, 0, 0)\). Near \(p\), the Hamiltonian flow of \(J_1\) is periodic, and assuming \(V\) to be invariant with respect to this flow the associated \(S^1\)-action is free in \(V \setminus \{p\}\). That is, we always have existence of a Hamiltonian action of the circle \(S^1\) that commutes with the flow of the system semiglobally near a focus-focus singularity (i.e. in a neighborhood of the singular fiber that contains the focus-focus singular point \(p\)). This is a property of focus-focus singularities which is essential for Theorem 3.3 and our upcoming Theorem 5.1.

Fix a point \(A \in \Lambda_0 \cap (V \setminus \{p\})\). Let \(\Sigma\) denote a small 2-dimensional manifold transversal to \(\mathcal{F}\) at \(A\). Since \(\mathcal{F}\) in a neighborhood of \(\Sigma\) is regular for \(F\) and \((J_1, J_2)\) simultaneously, there exists a diffeomorphism \(\varphi\) from a neighborhood \(U\) of \(F(A)\) into a neighborhood of \((0,0)\) in \(\mathbb{R}^2\) such that \((J_1, J_2) = \varphi \circ F\). Thus there exists a smooth momentum map \(\Phi = \varphi \circ F\) (in the sense of Definition 4.1) for \(\mathcal{F}\), defined on a neighborhood \(F^{-1}(U)\) of \(\Lambda_0\), which agrees with \((J_1, J_2)\) on \(V\). Write \(\Phi := (H_1, H_2)\) and \(\Lambda_c := \Phi^{-1}(c)\) (notice that \(\Lambda_0 = \mathcal{F}_p\)). It follows from equations (2) and (3) that near \(p\)
the orbits corresponding to the Hamiltonian $H_1$ are periodic. On the other hand, the vector field $\mathcal{X}_{H_2}$ is hyperbolic and it has a local stable manifold in the $(\xi, \eta)$-plane transversal to its local unstable manifold in the $(x, y)$-plane. Moreover, the vector field $\mathcal{X}_{H_2}$ is radial in the sense that the flows approaching the origin do not spiral on the local (un)stable manifolds (for further details see [VN03, Section 3] and [PeVN11, Section 5.2.3]).

5. Main theorem

Note that, by definition, a semitoric system can never be hyperbolic semitoric, or conversely. The main theorem of this paper is the following result about attaching hyperbolic singularities to an integrable system with non-degenerate focus-focus singularities $p_1, \ldots, p_n$, by modifying the system locally near a certain subset

$$\{p_i \mid i \in S\}$$

of the set of focus-focus singularities. By relabeling if needed we may assume that $S = \{1, \ldots, k\}$ for some $k \leq n$.

**Theorem 5.1.** Let

$$(M, \Omega, F := (J, H) : M \to \mathbb{R}^2)$$

be an integrable Hamiltonian system where $J : M \to \mathbb{R}$ is a proper map. Assume that among the singularities of $F$ are $m_f$ simple non-degenerate focus-focus singularities $p_1, \ldots, p_{m_f}$, where $0 \leq m_f < \infty$. Then for any $1 \leq k < m_f$ there exists an integrable system

$$(\tilde{M}, \tilde{\Omega}, \tilde{F} := (\tilde{J}, \tilde{H}) : \tilde{M} \to \mathbb{R}^2)$$

with the following properties:
(1) $M = \tilde{M}$, $\Omega = \tilde{\Omega}$, and $J = \tilde{J}$;
(2) $\tilde{F}$ has non-degenerate singularities at $p_1, \ldots, p_k$ of focus-focus type;
(3) $\tilde{F}$ has non-degenerate singularities at $p_{k+1}, \ldots, p_{m_f}$ of elliptic-elliptic type;
(4) There exist $m_f - k$ closed piecewise smooth curves with corners in the classical spectrum $\tilde{F}(M)$, each of which consists of three smooth curves

$$\gamma^j_i: [0,1] \to \tilde{F}(M), \quad j = 1, 2, 3, \quad k + 1 \leq i \leq m_f$$

such that for each fixed $i$ the images of the interior $(0,1)$ under each of $\gamma^1_i$ and $\gamma^2_i$ consists of non-degenerate transversally elliptic singular values, and under $\gamma^3_i$ it consists of non-degenerate transversally hyperbolic singular values. The two endpoints where the elliptic and hyperbolic families meet non-transversally are degenerate singularities. The third endpoint is where the two elliptic families meet transversally at the non-degenerate elliptic-elliptic singularity $p_i$ (see Figure 5).

(5) If $(M, \Omega, F)$ is semitoric (see Definition 3.1) then $(M, \Omega, \tilde{F})$ is hyperbolic semitoric (see Definition 3.2).

We will prove Theorem 5.1 in Section 7.

**Question 5.2** Are there any hyperbolic semitoric systems

$$(M, \Omega, F := (J, H): M \to \mathbb{R}^2)$$

for which the set of degenerate singularities is empty?

The proof of Theorem 5.1 in Section 7 gives evidence that the answer to the question is probably going to be no, at least under some fairly general conditions. The answer is no in all the physical examples we are aware of. That is, it is quite possible that the existence of hyperbolic singularities satisfying (a),(b) and (c.1) in Definition 3.2 forces the existence of at least some degenerate singularities in item (c.2), at the points where the various non-degenerate families of singularities under consideration meet with each other. If this were not the case, there would be loops of singularities, which we cannot rule out at the time being (it would be crucial to understand whether the existence of the global $S^1$-action may not be compatible with the existence of such a loop).

6. The Hopf bifurcation

Another way of formulating the theorem is to say that for a non-degenerate focus-focus equilibrium point in an integrable system it is possible to locally modify the Hamiltonian so that a Hamiltonian Hopf bifurcation is induced. The main point is that this modification can be achieved without destroying the integrability of the system. Here we are going to present a brief review
of some background on the Hopf bifurcation. The original references are [MeSch71, So77, vdM85].

A Hamiltonian Hopf bifurcation in the linear approximation occurs when pure imaginary eigenvalues of an equilibrium collide and move into the complex plane. At the collision point the linearization is not diagonalizable. At the bifurcation point the Hamiltonian may be put into Sokolskii’s normal form

\[ \hat{H}_0 = \omega \Gamma_1 + \sigma \Gamma_2 + CT_1^2 + 2BT_1 \Gamma_3 + 2D\Gamma_3^2 + h(\Gamma_1, \Gamma_3) \]

up to flat terms where \( h \) contains cubic and higher order terms. Here the abbreviations

\[
\begin{align*}
\Gamma_1 &= \hat{x}\hat{\eta} - \hat{y}\hat{\zeta} \\
\Gamma_2 &= \frac{\hat{\xi}^2 + \hat{\eta}^2}{2} \\
\Gamma_3 &= \frac{\hat{x}^2 + \hat{y}^2}{2}
\end{align*}
\]

are used. Adding the bifurcation parameter \( \nu \) gives the unfolding

\[ \hat{H}_\nu = \sigma \nu (a \Gamma_1 + b \Gamma_3) + \hat{H}_0. \]

This is the Hamiltonian Hopf bifurcation normal form, where \( \omega \neq 0, \sigma = \pm 1, D \neq 0, b \neq 0 \). There are two different cases, depending on the sign of \( \sigma D \). The cases \( \sigma D < 0 \) (respectively \( \sigma D > 0 \)) are called the subcritical (respectively supercritical) Hamiltonian Hopf bifurcation. When the elliptic-elliptic equilibrium point looses stability in the subcritical case there is a family of stable periodic orbits near the unstable focus-focus equilibrium point. In the supercritical case there is no such family.

Up to flat terms the Hamiltonian Hopf bifurcation normal form \( \hat{H}_\nu \) is a family of integrable system

\[(\mathbb{R}^4, \Omega, \hat{F} = (\Gamma_1, \hat{H}_\nu)), \quad \Omega = d\hat{\xi} \wedge d\hat{x} + d\hat{\eta} \wedge d\hat{y}.\]

To find the image of \( \hat{F} \) and its critical values introduce symplectic polar coordinates

\[
\begin{align*}
\hat{x} &= \sqrt{2z} \cos \theta, \\
\hat{y} &= \sqrt{2z} \sin \theta, \\
2zp_z &= \hat{x}\hat{\zeta} + \hat{y}\hat{\eta}, \\
p_\theta &= \hat{x}\hat{\eta} - \hat{y}\hat{\zeta}
\end{align*}
\]

so that

\[ \Omega = p_z \wedge z + p_\theta \wedge \theta. \]

In these coordinates

\[
\begin{align*}
\Gamma_1 &= p_\theta, \\
\Gamma_2 &= zp_z^2 + p_\theta^2/(4z),
\end{align*}
\]
and
\[ \Gamma_3 = z. \]
Hence define a reduced Hamiltonian with \( p_\theta = J \) as a parameter and canonical variables \((z, p_z)\) as
\[ \hat{H}_\nu = \omega J + \sigma \left( z p_z^2 + \frac{J^2}{4z} + \nu(aJ + b z) \right) + CJ^2 + 2BJz + 2Dz^2. \]
Solving this equation for \( p_z(z; J, \hat{H}) \) gives the reduced action
\[ \oint p_z(z) dz \]
and the discriminant of the polynomial
\[ Q(z) = z^2 p_z(z)^2 \]
contains the set of critical values of \( \hat{F} = (J, \hat{H}) \). For more details on this derivation using singular reduction instead of polar coordinates see [DuIv05]. Since \( z = \Gamma_3 \geq 0 \) only when the double root is non-negative does the corresponding part of the discriminant surface of \( Q(z) \) belong to the set of critical values of \( \hat{F} \).

7. Proof of Theorem 5.1

We are going to deform the Hamiltonian \( H \) to a new Hamiltonian \( \tilde{H} \) in a neighborhood of each focus-focus singularity. The deformation vanishes outside a sufficiently small neighborhood. We will show that we can choose a deformation which turns a focus-focus point into an elliptic-elliptic point with the properties described in (4). This can be done independently for each focus-focus point, and hence we can create \( 1 \leq m_f - k \leq m_f \) elliptic-elliptic points.

In the construction we use Eliasson’s coordinates near each focus-focus point, and we preserve the \( S^1 \) action which always exist near a focus-focus point, see Section 4. If we are in the semitoric setting, then one of Eliasson’s integrals is the global \( S^1 \) action \( J \), and hence the global Hamiltonian \( S^1 \) action is preserved, establishing (5).

To establish (3) and (4) for a single equilibrium point there are three steps.

*Step 1.* (A new integrable system) Let \((x, y, \xi, \eta)\) be the Eliasson coordinates given by Theorem 2.3 near \( p \) of the original semitoric system and let the Hamiltonian of the original system in these coordinates be \( H \). Then there exists a smooth function \( G: M \to \mathbb{R} \) such that
\[ \tilde{H} = H + G, \]
with
\[ G = G(J_1, J_2, K_1, K_2), \]
Figure 4. Types of roots of the polynomial $P(\lambda) = a + b\lambda^2 + \lambda^4$. The discriminant of $P(\lambda)$ is marked in bold (red) lines, delineating the regions hyperbolic-hyperbolic (HH) with 4 real roots, elliptic-elliptic (EE) with 4 pure imaginary roots, elliptic-hyperbolic (EH) with 2 real and 2 pure imaginary roots, and focus-focus (FF) with a complex quadruplet. For eigenvalues coming from $H_2$ the grey region of hyperbolic-hyperbolic and elliptic-hyperbolic is not accessible.

where

\[ J_1 = x\eta - y\xi \]
\[ J_2 = x\xi + y\eta \]

and

\[ K_1 = \frac{1}{2}(x^2 + y^2) \]
\[ K_2 = \frac{1}{2}(\xi^2 + \eta^2) \]

Now $J_1, J_2, K_1, K_2$ all have vanishing Poisson bracket with $J = J_1$. Thus

\[ \{\tilde{H}, J\} = 0 \]

and $(M, \Omega, J, \tilde{H})$ is an integrable system.

Step 2. (Type of equilibrium) The quadratic part of the original semitoric integrable system in Eliasson coordinates near the focus-focus singularity is

\[ H_2 = \omega J_1 + \alpha J_2, \]

with given real parameters $\omega$ and $\alpha$. The quadratic part of the modified integrable system is

\[ \tilde{H}_2 = \tilde{\omega} J_1 + \tilde{\alpha} J_2 + \gamma K_1 + \delta K_2, \]

with real parameters $\tilde{\omega}, \tilde{\alpha}, \gamma, \delta$. We can choose these parameters by choosing the function $G$ from Step 1. We are now going to show that by choosing these parameters we can make $\tilde{H}_2$ have a non-degenerate elliptic-elliptic equilibrium point at the origin. This does not seem to help in creating
hyperbolic singularities, but we will see in Step 3 that by adding appropriate higher order terms in $G$ we can create hyperbolic-transverse singularities.

Except for cases with multiple roots the type of equilibrium of $\tilde{H}_2$ is determined by the characteristic polynomial

$$P(\lambda) = \text{det}(B \text{Hess}\tilde{H} - \lambda) = a + b\lambda^2 + \lambda^4$$

where

$$a = (\tilde{\alpha}^2 + \tilde{\omega}^2 - \gamma\delta)^2 \geq 0$$

and

$$b = 2(\gamma\delta - \tilde{\alpha}^2 + \tilde{\omega}^2).$$

The discriminant of $P(\lambda)$ is

$$16a(4a - b^2)^2.$$  

The line $a = 0$ and the parabola $a = b^2/4$ divide the plane into four regions, see Figure 4. When $0 < a < b^2/4$ and $b > 0$ there are four distinct pure imaginary eigenvalues

$$\pm \sqrt{-b/2 \pm \sqrt{b^2/4 - a}}.$$  

Now we verify that by a choice of $\tilde{\alpha}, \tilde{\omega}, \gamma, \delta$ we can satisfy these conditions. Note that

$$b^2/4 - a = 4\tilde{\omega}^2(\gamma\delta - \tilde{\alpha}^2)$$

and by choosing $\gamma\delta > \tilde{\alpha}^2$ this can be made positive. Now $\gamma\delta > \tilde{\alpha}^2$ also implies $b > 0$. Finally we can achieve $a > 0$ by increasing $\gamma\delta$ if necessary. Since the eigenvalues of the linearization of the Hamiltonian vectorfield of $\tilde{H}_2$ are distinct the singularity at the origin is non-degenerate, according to Lemma 2.5. The construction may be done independently at each focus-focus point, and thus we established part (3) of the theorem.

**Remark 7.1.** Because of the way $a$ and $b$ depend on $\tilde{\alpha}, \tilde{\omega}, \gamma, \delta$ it is not possible to have $a < 0$ nor is it possible to have $0 < a < b^2/4$ and $b < 0$. Thus the idea to directly choose the parameters $\tilde{\alpha}, \tilde{\omega}, \gamma, \delta$ so that $\tilde{H}_2$ has a singularity of hyperbolic-hyperbolic type or elliptic-hyperbolic type does not work.

**Remark 7.2.** It is easy to see that when $a = b^2/4$ and $b > 0$ and in addition $\tilde{\omega} \neq 0$ the eigenvalues are pure imaginary, the matrix $B \text{Hess}\tilde{H}_2$ is not semi-simple, and the singularity is degenerate.

**Remark 7.3.** It is possible to choose a smooth family $\tilde{H}_2(s)$ with parameter $s \in [0, 1]$ such that $\tilde{H}_2(0) = H_2$, $\tilde{H}_2(s)$ has a non-degenerate focus-focus singularity for $s \in [0, s^*]$, $\tilde{H}_2(s)$ has a non-degenerate elliptic-elliptic singularity for $s \in [s^*, 1]$, and only $\tilde{H}_2(s^*)$ is degenerate.
Figure 5. Schematic structure of the image of $\tilde{F}$ after the Hopf bifurcation. Elliptic-elliptic is the non-degenerate elliptic equilibrium point, $E$ denotes the two transversally elliptic non-degenerate families of singularities (stable isolated periodic orbits) attached to the equilibrium, $H$ denotes the transversally hyperbolic non-degenerate families of singularities (unstable isolated periodic orbits). $nT$ denotes that there are $n$ tori in the preimage of the corresponding regular values. The cusps where the $E$ and $H$ families meet are degenerate singularities (saddle-centre bifurcation of periodic orbits).

**Step 3.** (Higher order terms) We are now going to determine higher order terms in $\tilde{H}$ that will generate a family of hyperbolic-transverse singularities. This will be done by transforming a special case of the general Hopf normal form $\tilde{H}_r$ into $\tilde{H}$. We are going to show that the image of $\tilde{F}$ has the form schematically shown in Figure 5 as described precisely in the statement part (4) of our main theorem.

To obtain the Hopf normal form we need to introduce a new set of local coordinates related to Eliasson’s coordinates by a linear symplectic transformation. We now choose a parameter $\hat{\gamma}$ such that

$$\bar{\alpha}^2 = \hat{\gamma}\delta, \; \bar{\alpha} \neq 0, \; \bar{\omega} \neq 0, \; \delta \neq 0,$$

and set $\sigma = \text{sign}(\delta)$.

The transformation is

$$p = T\hat{p}, \; p = (x, y, \xi, \eta),$$

where

$$T = \begin{pmatrix} \sqrt{|\delta|} & 0 & 0 & 0 \\ 0 & \sqrt{|\delta|} & 0 & 0 \\ -\sqrt{|\hat{\gamma}|}\text{sign}(\delta\bar{\alpha}) & 0 & 1/\sqrt{|\delta|} & 0 \\ 0 & -\sqrt{|\hat{\gamma}|}\text{sign}(\delta\bar{\alpha}) & 0 & 1/\sqrt{|\delta|} \end{pmatrix}. $$
The transformation $T$ has the property that it maps the quadratic part of $\hat{H}_\nu$ into $\hat{H}_2$. Specifically it transforms the function $\Gamma_i$ as defined in Section 6 as follows:

\[
\begin{align*}
\Gamma_1 &= J_1 \circ T, \\
\Gamma_2 &= \sigma(\tilde{\alpha}J_2 + \hat{\gamma}K_1 + \delta K_2) \circ T, \\
\Gamma_3 &= \sigma(K_1/\delta) \circ T.
\end{align*}
\]

E.g. the first identity for $J_1 = x\eta - y\xi$
follows from

\[
\sqrt{|\delta|} \hat{x} (\sqrt{|\hat{\gamma}|} \text{sign}(\hat{\gamma}\tilde{\alpha}) \hat{y} + \hat{\eta}/\sqrt{|\delta|}) - \sqrt{|\delta|} \hat{y} (\sqrt{|\hat{\gamma}|} \text{sign}(\hat{\gamma}\tilde{\alpha}) \hat{x} + \hat{\xi}/\sqrt{|\delta|}) \\
= \hat{x}\hat{\eta} - \hat{y}\hat{\xi}
\]

We now specialize the Hopf normal form $\hat{H}_\nu$ discussed in Section 6 to $a = B = C = 0, b = 1$, because that is sufficient for our purpose. Transforming the Hopf normal form (given in variables with hat) and with sufficiently small $\nu$

\[
\hat{H}_\nu = \tilde{\omega} \Gamma_1 + \sigma(\Gamma_2 + \nu \Gamma_3) + 2D\Gamma_3^2
\]

into variables without hat gives

\[
\tilde{H} = \tilde{\omega}J_1 + \tilde{\alpha}J_2 + \left(\hat{\gamma} + \frac{\nu}{\delta}\right) K_1 + \delta K_2 + 2D(\alpha J_2 + \hat{\gamma}K_1 + \delta K_2)^2.
\]

Thus the quadratic part of $\tilde{H}$ has parameters in the notation of Step 2 given by $\tilde{\omega}, \tilde{\alpha}, \delta$ and

$\gamma = \hat{\gamma} + \nu/\delta$.

Now we choose the function $G$ by setting

$G = \tilde{H} - H$,

so that

$H + G = \tilde{H}$.

Note that $G$ is not only producing the desired $\tilde{H}$, but it also annihilates any unwanted higher order terms which may be present in $H$.

Recall that

$\tilde{\alpha}^2 = \hat{\gamma}\delta, \tilde{\alpha} \neq 0,$

and $\tilde{\omega} \neq 0$, and therefore the parameters in Step 2 are

$\tilde{\omega}, \tilde{\alpha}, \gamma = \hat{\gamma} + \nu/\delta, \delta,$

and they put $\tilde{H}$ in the elliptic-elliptic region as long as $\nu$ is positive. This may be seen by computing the eigenvalues of the linearized vector field of $\tilde{H}_\nu$ at the origin:

- when $\nu < 0$ the equilibrium is of type focus-focus with eigenvalues

$\pm \sqrt{-\nu} \pm i\tilde{\omega}$
when \( \nu > 0 \) it is of type elliptic-elliptic with eigenvalues

\[ \pm i (\sqrt{\nu} \pm \tilde{\omega}). \]

Hence for sufficiently small and positive \( \nu \) the equilibrium is of elliptic-elliptic type and it is non-degenerate because the eigenvalues are distinct.

We are now going to establish the properties of the image of \( \tilde{F} \) claimed in (4) of the main theorem. See Figure 6 and 7 for an illustration of the various cases.

First of all notice that \( \bar{F} \) and \( \tilde{F} \) are related by a linear symplectic transformation, so we may as well study the image of \( \tilde{F} \). As described in Section 6 the critical values of \((\Gamma_1, H_\nu)\) are contained in the discriminant surface of the cubic polynomial

\[ Q(z) = 4z\sigma (\bar{H} - \omega J - \sigma \nu z - 2Dz^2) - J^2. \]
Figure 7. Image of the map $\hat{F}$ for the supercritical Hopf bifurcation with $\sigma D > 0$. Critical values of $\hat{F}$ are shown in red. In addition, the vanishing of the discriminant of $Q(z)$ is shown in blue. The image is to the left for $D > 0$ and to the right for $D < 0$. The parameter values are $\omega = 1$, $\nu = \pm 1/2$, and $D = 1$ or $D = -2$.

The discriminant of this polynomial has a rational parametrization that can be found by the Ansatz

$$Q(z) = -4\sigma D(z - d)^2(2z + \sigma s^2/D)$$

and solving for the double root $d$, $J$ and $\hat{H}$. The result is

$$(J_c(s), \hat{H}_c(s)) = \left(\frac{1}{2D}s(s^2 - \nu), \frac{1}{4s}J_c(s)(\nu + 4s\omega + 3s^2)\right),$$

where the double roots of $Q(z)$ occur at

$$z = \Gamma_3 = d(s) = \sigma J_c(s)/(2s).$$

Assume that $\nu > 0$. Three piecewise smooth curves of critical values are found for $s \in [-\sqrt{\nu}, -\sqrt{\nu/3}]$, for $s \in [-\sqrt{\nu/3}, \sqrt{\nu/3}]$, and $s \in [\sqrt{\nu/3}, \sqrt{\nu}]$. Both derivatives

$$\frac{d}{ds} (J_c(s), \hat{H}_c(s)) = \frac{3s^2 - \nu}{2D}(1, s + \omega).$$
vanishes at $s = \pm \sqrt{\nu/3}$ only. We call these corresponding critical values the cusp. The other segment endpoints $\pm \sqrt{\nu}$ map to the same critical value $(0,0)$.

We now establish the type and the (non-)degeneracy of the critical values on the three curves. We need to compute the determinant of the Hessian of the reduced Hamiltonian $\hat{H}(z, p_z)$ from section 6 at the critical points. The determinant of the Hessian at an arbitrary point is

$$-4p_z^2 + 8\sigma Dz + J^2/z^2,$$

and evaluating this at $J = J_c(s)$, $z = d(s)$, and $p_z = 0$ gives

$$2(3s^2 - \nu),$$

which is non-zero unless $s = \pm \sqrt{\nu/3}$, corresponding to the cusps. Moreover for $s \in (-\sqrt{\nu/3}, \sqrt{\nu/3})$ the determinant is negative, so this segment is transversally-hyperbolic. Similarly, the other two segments $(-\sqrt{\nu}, -\sqrt{\nu/3})$ and $(\sqrt{\nu/3}, \sqrt{\nu})$ lead to a positive determinant and are hence transversally-elliptic. Finally at the cusps the determinant of the Hessian vanishes and hence these are degenerate critical values.

The three curves are graphs over $J$ because

$$\frac{dJ_c(s)}{ds} \neq 0$$

on the interior of the three segments, and when

$$\frac{dJ_c(s)}{ds} = 0$$

then also

$$\frac{d\hat{H}_c(s)}{ds} = 0,$$

creating the two cusps. When the two curves of transversally elliptic values intersect at the origin they have slopes

$$\omega \pm \sigma \sqrt{\nu},$$

and hence the intersection is transversal for $\nu > 0$.

Not all points in the discriminant surface of $Q$ are actually critical values of $\hat{F} = (J, \hat{H})$. To be in the image of $\hat{F}$ we need $\Gamma_3 \geq 0$. The double root

$$d(s) = \sigma (s^2 - \nu)/(4D)$$

is non-negative for $s \in [-\sqrt{\nu}, \sqrt{\nu}]$ only when $\sigma D < 0$. Hence we are now choosing $D$ such that $\sigma D < 0$. Accordingly we have one of the two right situations shown in Figure 6.

A particular point on the transversally-hyperbolic curve is obtained from $s = 0$ where

$$(J, \hat{H}) = (0, -\nu^2/(8D)).$$
Hence the sign of $D$ determines whether this critical value is above or below the equilibrium point at the origin.

**Remark 7.4.** The cusps corresponding to $s = \pm \sqrt{\nu/3}$ are created by a saddle-centre bifurcation of an elliptic and a hyperbolic periodic orbit. This is one of the typical degenerate singularities described in [BoFo04]. For a detailed analysis of the universal features of the dynamics near this bifurcation using an integrable model see [DuIv05b].

**Remark 7.5.** As in Step 2, we can think of the function $G$ as a one-parameter family that smoothly deforms the original Hamiltonian into the new Hamiltonian $\tilde{H}$. From this point of view it makes sense to consider $\nu < 0$ as well, and the corresponding sets of critical values of $\hat{F}$ are shown on the left column of Figure 6 and 7.

### 8. Extensions

The main result of this paper is stated in the form of several theorems, but in fact it is more a method than a result. The following Theorem has a similar proof.

**Theorem 8.1.** Each focus-focus singular point in a semitoric integrable system $(M, \Omega, F)$ may be continuously deformed via a path of semitoric systems into an elliptic-elliptic point on the boundary of the image of the new integrable system $(M, \Omega, \tilde{F})$ constructed in Section 7, either at the top boundary (maximal $\tilde{H}$ for fixed $J$) or at the bottom boundary (minimal $\tilde{H}$ for fixed $J$). This may be done simultaneously for any subset of the set of focus-focus singular points of the system.

The proof is as above, the only difference is that now we choose $G$ with $\sigma D > 0$. The corresponding critical values are shown in the two right cases in Figure 7. The sign of $D$ determines which one of the two cases occurs.

Under what conditions the resulting semitoric system is in fact toric is an open question.

### 9. Examples

A deformation of the spherical pendulum that models floppy triatomic molecules (eg. HCN) has been studied in [Ef05]. Strictly speaking this is not an example for our theorem since the momentum map $J$ of the spherical pendulum is not proper. But if $J$ is not proper the same idea still works as long as $\hat{F}$ is proper, which can be easily checked in this particular example. In fact, our theorem can be generalised to the situation where $J$ is not proper by adding higher order terms to $G$ that make $\tilde{H}$ proper.

Here we give a brief description of a deformation of the semitoric coupled spin-oscillator [PeVN12] (Jaynes-Cummings system) that makes it pass through a subcritical Hopf bifurcation with $\sigma D > 0$. A family for which the
focus-focus point is driven into the boundary in a supercritical Hopf bifurcation with $\sigma D < 0$ has been presented in [VN07]. The image of the momentum map of the Jaynes-Cummings is shown in Figure 8.

In particular examples it may be simpler to avoid the Eliasson normal form, since it may not be easy to find. Instead we start with $F = (J, H)$ in the original variables $(u, v, x, y, z)$ where

$$H = \frac{(xu + yv)}{2}$$

and

$$J = \frac{(u^2 + v^2)}{2} + z.$$

Then we ask for a function $G$ that commutes with $J$, and define

$$\tilde{H} = H + G.$$ 

In an example it will also typically be nicer to work with a global function $G$, instead of one that is only defined in a neighborhood of the equilibrium under consideration.

For the spin-oscillator any function of the form

$$G(z, vx - uy, u^2 + v^2, xu + vy)$$

works. We chose $G = G(z)$, and the characteristic polynomial of the linearization at the equilibrium point $(0, 0, 0, 0, 1)$ is given by

$$\frac{1}{16} - \frac{1}{2} \lambda^2(1 - 2G'(1)^2) + \lambda^4$$

and undergoes a Hopf bifurcation when $G'(1)^2 = 1$. Now set

$$G(z) = \gamma z^2.$$ 

In the $(b, a)$ plane we obtain the horizontal line

$$(b, a) = (1/2 - 8\gamma^2, 1/16).$$

For $\gamma > 1/2$ the singularity is in the elliptic-elliptic region. The non-linear analysis of this example is somewhat involved and will be presented in detail.
in a forthcoming paper. Here we simply present the image of \( \hat{F} \) for \( \gamma = 4/5 \), see Figure 3.

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