The Second Main Theorem Concerning Small Algebroid Functions.*

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Abstract. In this paper, we firstly give the definition of meromorphic function element and algebroid mapping. We also construct the algebroid function family in which the arithmetic, differential operations is closed. On basis of these works, we firstly proved the Second Main Theorem concerning small algebroid functions for \(v\)-valued algebroid functions.

Keywords. algebroid function, algebroid mapping, corresponding addition, the Second Main Theorem.

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1. INTRODUCTION

In 1925, R. Nevanlinna obtained the Second Main Theorem for meromorphic functions and posed the problem whether the the Second Main Theorem can be extended to small functions (See [1]). Dealing with the problem, Q. T. Chuang proved the Second Main Theorem still holds for small entire functions (See [2], [3]). Until 1986, the problem was solved by N. Steinmetz (See [4]). In 2000, M. Ru proved the Second Main Theorem concerning small meromorphic functions for algebroid functions (See [5]).

It is natural to consider the problem whether the Second Main Theorem for algebroid functions is still true when we replace the small meromorphic functions by small algebroid functions. Before considering the problem, we must define the arithmetic, differential operations over algebroid functions. Hence we give the definition of meromorphic function element, algebroid mapping and construct the algebroid function family \(H_W\). In \(H_W\) the arithmetic, differential operations is closed. On basis of these works, by using the method of Reference [6], we proved the Second Main Theorem concerning small algebroid functions.

Suppose that \(A_v(z), \cdots, A_0(z)\) are analytic functions without common zeros in the complex plane \(C\). Then the binary complex equation

\[
\Psi(z, W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + \cdots + A_0(z) = 0
\]

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defines a \( v \)-valued algebroid function \( W(z) \) in the complex plane \( C \). The above equation can be transformed to the standard equation

\[
\Psi^a(z,W) = W^v + A_{v-1}^*(z)W^{v-1} + \cdots + A_0^*(z) = 0,
\]

where \( A_t^*(z) := \frac{A_t(z)}{A_v(z)} \) \( (t = 0, 1, 2, \cdots, v - 1) \) are meromorphic functions in the complex plane \( C \). Note that for a \( v \)-valued algebroid function \( W(z) \), its standard equation is unique.

If \( \Psi(z, W) \) is irreducible, then the corresponding \( W(z) \) is called a \( v \)-valued irreducible algebroid function. For an irreducible algebroid function \( W(z) \), the points in the complex plane can be divided into two classes. One is a set \( T_W \) of regular points of \( W(z) \), the other is a set \( S_W = C - T_W \) of critical points of \( W(z) \). The set \( S_W \) is an isolated set (See [7], [8]).

In this paper, \( \Psi(z, W) \) needn’t be irreducible in the usual case. A \( v \)-valued algebroid function \( W(z) \) may decompose to \( n \)(\( \geq 1 \)) number of \( v_n \)-valued irreducible algebroid functions (containing the case \( W \) is a complex constant) and \( v = \sum_{j=1}^{v} v_j \).

For a \( v \)-valued reducible algebroid function \( W(z) \), its corresponding binary complex equation \( \Psi(z, W) = 0 \) can be decomposed to the product of \( q(\leq v) \) non-meromorphic coprime factors, namely

\[
\Psi(z, W) = \Psi_1(z, W)\Psi_2(z, W)\cdots\Psi_q(z, W) = 0.
\]

Let \( S_j \) denote the set of critical points of the irreducible complex equation \( \Psi_t(z, W) = 0 \). We define the set of critical points of reducible algebroid function \( W(z) \) by \( S_W := \cup_{j=1}^{q} S_j \). (Since \( \{ S_j \} \) \( (j = 1, \cdots, q) \) are all isolated sets, \( S_W \) is also an isolated set.) The set of regular points of reducible algebroid function \( W(z) \) by \( T_W := C - S_W \).

**Remark 1.1.** If \( q = 1 \), then \( W(z) \) is an irreducible algebroid function.

**Remark 1.2.** If \( (q(z), b) \) is a polar element or a multivalent algebraic function element, then \( b \in S_W \).

**Remark 1.3.** For every \( a \in T_W \), there exist and only exist \( v \) number of regular function elements \( \{(w_t(z), a)\}_{t=1}^{v} \). In this paper, we usually denote \( W(z) = \{w_j(z)\}_{j=1}^{v} \). If there exists \( 1 \leq t < j \leq v \) such that \( w_t(z) \equiv w_j(z) \) in \( S \), then the complex equation \( \psi(z, W) = 0 \) must have non-meromorphic function multiple factor.

In this paper, we use the standard notations of the value distribution for algebroid functions (See [7]).

2. Some Basic Properties of Algebroid Functions

**Definition 2.1.** Let \( W(z) \) and \( M(z) \) be two algebroid functions defined by

\[
\Psi(z, W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + \cdots + A_0(z) = A_v(z) \prod_{j=1}^{v} (W - w_j(z)) = 0, \quad A_v(z) \neq 0
\]

(2.1)
and
\[ \Phi(z, M) = B_s(z)M^s + B_{s-1}(z)M^{s-1} + \cdots + B_0(z) = B_s(z) \prod_{t=1}^{s}(M - m_t(z)) = 0, \quad B_s(z) \neq 0, \]

respectively, \( W(z) \) and \( M(z) \) are called identical, write \( W(z) \equiv M(z) \), provided that \( v = s \) and the corresponding coefficients are proportional, namely
\[ E(z) := \frac{A_v(z)}{B_v(z)} = \frac{A_{v-1}(z)}{B_{v-1}(z)} = \cdots = \frac{A_0(z)}{B_0(z)}. \]

Since the coefficients of the equations (2.1) and (2.2) haven’t common zeros, \( E(z) \) is a nonzero constant or an analytic function without zeros.

**Theorem 2.1.** Suppose that \( W(z) = \{w_j(z)\}_{j=1}^{v} \) and \( M(z) = \{m_t(z)\}_{t=1}^{s} \) are two irreducible algebroid functions defined by (2.1) and (2.2), respectively. The following conditions are equivalent:

1. \( W(z) \equiv M(z) \).
2. There exist some regular function elements \((w_j(z), b)\) of \( W(z) \) and \((m_j(z), b)\) of \( M(z) \) such that \((w_j(z), b) = (m_j(z), b)\).
3. The eliminant \( R(\Psi, \Phi) \equiv 0 \).

**Proof.** \( (1) \Rightarrow (3) \):
\[ R(\Psi, \Phi) = A^s_v(z) \prod_{j=1}^{v} \Phi(z, w_j(z)) = E(z)A^v_z(z) \prod_{j=1}^{v} \Psi(z, w_j(z)) \equiv 0. \]

By the property of the eliminant, the first equal sign holds (See [34]). Then by Definition 2.1, we get the second equal sign. Since \((w_j(z), z))_{j=1}^{v} \) are regular function elements belong to (2.1), \( \Psi(z, w_j(z)) \equiv 0 \) in some neighborhood of \( z \). Combining the identical principle of analytic functions, we get the third equal sign.

\( (3) \Rightarrow (2) \): Since
\[ R(\Psi, \Phi) = A^s_v(z) \prod_{j=1}^{v} \Phi(z, w_j(z)) = A^s_v(z)B^s_s(z) \prod_{j=1}^{s} (w_j(z) - m_t(z)) \equiv 0. \]

there at least exists some term \( w_j(z) - m_t(z) \equiv 0 \). Hence there exist some regular function element \((w_j(z), a)\) of \( W(z) \) and \((m_j(z), a)\) of \( M(z) \) such that \((w_j(z), a) = (m_j(z), a)\).

\( (2) \Rightarrow (1) \): Since the irreducible algebroid function is a connected Riemann surface, the two identical regular function elements can be continued analytically to their Riemann surface respectively, such that the corresponding regular function elements are all identical. Hence \( v = s \). Then combining the Viete theorem, we get
\[ \frac{A_t(z)}{A_v(z)} = \frac{B_t(z)}{B_s(z)} = \sum (-1)^{v-t}w_{n_1}(z)w_{n_2}(z) \cdots w_{n_{v-t}}(z) (t = 0, 1, 2, \cdots, v-1), \]
where \( w_{r_1}(z), w_{r_2}(z), \ldots, w_{r_{v-t}}(z) \) denote any given \( v-t \) distinct elements among \( w_1(z), \ldots, w_v(z) \). From this we can obtain (1). \qedhere

Note that by Theorem 2.1, an irreducible algebroid function \( W(z) \) can not contain two same regular function elements.

**Theorem 2.2.** Suppose that \( W(z) = \{ (w_j(z), B(a, r_a)) \}_{j=1}^{v} \) is a \( v \)-valued algebroid function defined by (2.1). If it contains two same regular function elements, then there exist two same \( m \)-valued \( (2m \leq v) \) algebroid functions decomposed from \( W(z) \). Hence \( W(z) \) is reducible.

**Proof.** Suppose that \( (w_j(z), B(a, r_a)) \equiv (w_l(z), B(a, r_a)) \). Then

\[
R(\Psi, \Psi_W) = (-1)^{\frac{v(v-1)}{2}} A_{w}^{2v-1}(z) \prod_{1 \leq j < t \leq v} (w_j(z) - w_t(z))^2 \equiv 0.
\]

By Theorem 2.4 in reference [7], \( \Psi(z, W) \) must have the non-meromorphic function multiple factor. Hence there exist two same \( m \)-valued \( (2m \leq v) \) algebroid function decomposed from \( W(z) \). So \( W(z) \) is reducible. \qedhere

**Definition 2.2.** Meromorphic function element is defined by \((q(z), B(a, r))\), where \( q(z) \) is analytic in the disc \( B_0(a, r) := \{ 0 < |z - a| < r \} \) and \( a \) is not a essential point. So \( q(z) \) can be expressed by Laurent series \( q(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \) \((a_0 \neq 0)\). We also denote it by \((q(z), a)\). If the above \( t < 0 \), then we call \((q(z), a)\) is a truth meromorphic function element. Especially if \( q(z) \equiv c \) \((c \) denotes a constant.)

Two meromorphic function elements \((q(z), a)\) and \((p(z), b)\) are called identical provided that \( a = b \) and there exists \( r > 0 \) such that \( q(z) \equiv p(z) \) in the disc \( B_0(a, r) \).

If \( \Psi(z, q(z)) = 0 \) holds for any \( z \in B_0(a, r) \), then \((q(z), a)\) is called a meromorphic function element of algebroid function \( W(z) \) or \( \Psi(z, W) = 0 \).

**Remark 2.1.** The regular function element is also the meromorphic function element.

**Definition 2.3.** The regular function element \((p(z), B(b, R_b))\) is called the direct continuation of meromorphic function element \((q(z), B(a, R_a))\) provided that \( b \in B(a, R_a) \) and in the domain \( B(a, R_a) \cap B(b, R_b) \) we have \( p(z) \equiv q(z) \).

For any \( e \in (0, R_a) \), the set of meromorphic function element \((q(z), B(a, R_a))\) and all direct continuation of meromorphic function element \((q(z), B(a, R_a))\) in the disc \( B_0(a, e) \) is called a neighborhood of \((q(z), B(a, R_a))\). We denote it by \( V_e(q(z), a) \).

**Remark 2.2.** For any given point in \( B_0(a, R_a) \), the direct continuation is uniqueness.

**Remark 2.3.** The direct continuation of meromorphic function element must be regular function element. Hence the truth meromorphic function element is isolated.
**Definition 2.4.** Let \( W(z) = \{(w_{a,j}(z), a)\} \) be a \( v \)-valued algebroid function. \( h \) is called an algebroid mapping of \( W(z) \) if \( h \) satisfies the following conditions.

(i) **Uniqueness:** For any regular function element \((w_{a,j}(z), a)\), its image element \( h \circ (w_{a,j}(z), a) = (h \circ w_{a,j}(z), a) \) is meromorphic function element and unique.

(ii) **Continuation:** For any image element \((h \circ w_{a,j}(z), a)\), there exists \( \epsilon = \epsilon(h \circ w_{a,j}(z), a) > 0 \) such that for any regular function element \((w_b(z), b) \in V_{\epsilon}(w_{a,j}, a)\), we have \((h \circ w_b(z), b) \in V_{\epsilon}(h \circ w_{a,j}, a)\).

(iii) **Weak boundary:** If \( a \in S_W \), then \( h \) is weak bounded at the neighborhood of \( a \). Namely there exist integer \( p > 0 \), real numbers \( r > 0 \) and \( M > 0 \), such that for any \( b \in B_0(a, r) := \{ z; 0 < |z - a| < r \} \subset T_W \) and any \( t = 1, 2, \cdots, v \), the corresponding image element \((h \circ w_{b,t}(z), b)\) are all the regular function elements and satisfies \(|(b - a)^p h \circ w_{a,j}(b)| < M\).

**Theorem 2.3.** Let \( h \) be an algebroid mapping of \( v \)-valued algebroid function \( W(z) = \{(w_{a,j}(z), a)\} \). Then

(1) \( h \circ W(z) := \{(h \circ w_{a,j}(z), a)\} \) is a \( v \)-valued algebroid function.

(2) If \( W(z) \) is irreducible, then \( h \circ W(z) \) is irreducible if and only if \( h \) is injective. Namely \( h \circ (w(z), a) \neq h \circ (m(z), b) \) when \( (w(z), a) \neq (m(z), b) \), where \((w(z), a)\) and \((m(z), b)\) are regular function elements.

**Proof.** For any \( z_0 \in T_W \), if there exists some truth meromorphic function element among the corresponding meromorphic image elements \( \{(h \circ w_{z_0,j}(z), z_0)\}_{j=1}^{v} \), then \( z_0 \) is called a pole of \( h \). We denote by \( P_h \) the set of poles of \( h \). By the continuation of \( h \), we know that \( P_h \) is an isolated set.

(1) Firstly we define the analytic functions \( \{H^*_t(z)\}_{t=0}^{v-1} \) in \( T_W - P_h \). For any \( z_0 \in T_W - P_h \), the corresponding image elements \( \{(h \circ w_{z_0,j}(z), z_0)\}_{j=1}^{v} \) are all regular function elements. Set

\[
H^*_t(z_0) = \sum (-1)^{v-t}[h \circ w_{z_0,j_1}(z_0)] \cdot [h \circ w_{z_0,j_2}(z_0)] \cdot \cdots \cdot [h \circ w_{z_0,j_{v-t}}(z_0)], \quad t = 0, 1, 2, \ldots, v-1.
\]

By the continuation of \( h \), there exists \( \epsilon \), such that for any \( y \in B(z_0, \epsilon) \), the corresponding image elements \( \{(h \circ w_{y,j}(z), y)\} \) are the direct continuation of \( \{(h \circ w_{z_0,j}(z), z_0)\} \) respectively. Namely we have \( h \circ w_{y,j}(z) \equiv h \circ w_{z_0,j}(z) \) in the neighborhood of \( y \). So we have

\[
H^*_t(y) = \sum (-1)^{v-t}[h \circ w_{y,j_1}(y)] \cdot [h \circ w_{y,j_2}(y)] \cdot \cdots \cdot [h \circ w_{y,j_{v-t}}(y)]
\]

\[
= \sum (-1)^{v-t}[h \circ w_{z_0,j_1}(y)] \cdot [h \circ w_{z_0,j_2}(y)] \cdot \cdots \cdot [h \circ w_{z_0,j_{v-t}}(y)].
\]

Hence in \( B(z_0, \epsilon) \), for any \( t = 0, 1, \ldots, v-1 \) we have

\[
H^*_t(z) \equiv \sum (-1)^{v-t}[h \circ w_{z_0,j_1}(z)] \cdot [h \circ w_{z_0,j_2}(z)] \cdot \cdots \cdot [h \circ w_{z_0,j_{v-t}}(z)].
\]
So \( \{H_t^*(z)\} \) is analytic in \( B(z_0, \epsilon) \). By Viete theorem, they define the following complex equation

\[
\Phi^*(z, W) = W^v + H_{v-1}^*(z)W^{v-1} + \ldots + H_0^*(z) = \prod_{j=1}^v [W - h \circ w_{20,j}(z)] = 0
\]

and \( \Phi^*(z, h \circ w_{20,j}(z)) = 0 \) in \( B(z_0, \epsilon) \). Since \( z_0 \) is arbitrary, \( \{H_t^*(z)\}_{t=0}^{v-1} \) are analytic in \( T_W - P_h \).

When \( z_0 \in S_W \cup P_h \), since \( h \) is weak bounded, \( z_0 \) is the isolated singular point and is not the essential isolated singular point of \( \{H_t^*(z)\} \). This shows that \( \{H_t^*(z)\}_{t=0}^{v-1} \) are meromorphic in the complex plane and the corresponding complex equation \( \Phi^*(z, W) = 0 \) defines the algebroid function \( h \circ W(z) \).

(2) Suppose that \( h \) is injective. For any two regular image elements \( (h \circ w_{a,j}(z), a) \neq (h \circ w_{b,t}(z), b) \), they define uniquely two different regular primary image elements \( (h \circ w_{a,j}(z), a) \neq (h \circ w_{b,t}(z), b) \). Take a path \( \gamma \subset T_W \cap T_{h0W} \) such that two primary image elements can be continued analytically each other along \( \gamma \). By the continuation of \( h \), we know that \( (h \circ w_{a,j}(z), a) \) and \( (h \circ w_{b,t}(z), b) \) can be connected by \( \gamma \). Hence \( h \circ W(z) \) is irreducible.

Conversely suppose that there exist two different regular function elements \( (w_{a,j}(z), a) \neq (w_{a,t}(z), a)(j \neq t) \) such that the corresponding image elements \( (h \circ w_{j}(z), a) = (h \circ w_{t}(z), a)) \). Then by Theorem 2.2, \( h \circ W(z) \) is reducible.

\[ \square \]

**Definition 2.5.** Suppose that \( W(z) = \{ (w_j(z), a) \} \) is a \( v \)-valued algebroid function defined by the following complex equation

\[
\Psi(z, w) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + \ldots + A_1(z)W + A_0(z) = A_v(z)(W - w_1(z))(W - w_2(z))\ldots(W - w_v(z)) = 0,
\]

and \( f(z) \) is meromorphic in the complex plane \( C \).

1) Define \( h_{-W} \circ (w_j(z), a) := (-w_j(z), a) \). By Viete theorem, the complex equation with respect to \( h_{-W} \circ W(z) \) is

\[
\Psi_{-W}(z, w) := A_v(z)(W - (-w_1(z)))(W - (-w_2(z)))(W - (-w_v(z))) = A_v(z)W^v - A_{v-1}(z)W^{v-1} + \ldots + (-1)^v A_0(z) = 0.
\]

The \( v \)-valued algebroid function \( h_{-W} \circ W(z) \) is called the negative element of \( W(z) \). We denote it by \(-W(z)\), denote the algebroid mapping \( h_{-W} \) by \(-h\).

2) Define \( h_{1/W} \circ (w_j(z), a) := (1/w_j(z), a) \). By Viete theorem, the complex equation with respect to \( h_{1/W} \circ W(z) \) is

\[
\Psi_{1/W}(z, w) := A_v(z)(W - \frac{1}{w_1(z)})(W - \frac{1}{w_2(z)})\ldots(W - \frac{1}{w_v(z)}) = A_0(z)W^v - A_1(z)W^{v-1} + \ldots + A_v(z) = 0.
\]

The \( v \)-valued algebroid function \( h_{1/W} \circ W(z) \) is called the inverse element of \( W(z) \). We denote it by \( \frac{1}{W(z)} \), denote the algebroid mapping \( h_{1/W} \) by \( \frac{1}{h} \).
Remark 2.4. Especially, $W(z) \equiv 0$ is also the algebroid function. Its inverse element is defined as $\frac{1}{W(z)} \equiv \infty$ and $\frac{1}{W(z)}$ is also the algebroid function.

3) Define $h_f \circ (w_j, a) = (f(z), a)$. It is easy to prove that $h_f$ satisfies Definition 2.4. So $h_f$ is an algebroid mapping. By Theorem 2.3, The $v$-valued algebroid function $h_f \circ W(z) = \{f(z)\}$ are $v$ same meromorphic functions $f(z)$. Especially, if $f(z) \equiv c \in \mathbb{C}$, then the algebroid function $h_c \circ W(z) = \{c\}$ degenerates into $v$ same finite or infinite complex constants.

4) Define $h_{W'} \circ (w_j(z), a) = (w'_j(z), a)$. It is easy to prove that $h_{W'}$ satisfies the conditions 1, 2 of Definition 2.4. If $z_0 \in S_W$, then in $B_0(z_0, r) := \{0 < |z - z_0| < r\}$ we have

$$q_t(z) := \sum_{n=0}^{\infty} a_{n,t}(z - a_0)^{n/\lambda_t}, \quad t = 1, 2, ..., m,$$

where $\lambda_t$ is a positive integer, $u_t$ is an integer and $\sum_{t=1}^{m} \lambda_t = v$. It is easy to see that

$$h_{W'} \circ q_t(z) = \sum_{n=0}^{\infty} \frac{n a_{n,t}(z - a_0)^{n/\lambda_t}}{\lambda_t}, \quad t = 1, 2, ..., m$$

is weak bounded. By Theorem 2.3, $h_{W'} \circ W(z)$ defines a $v$-valued algebroid function. We call it the derivative of $W(z)$. We denote it by $h_{W'} \circ W(z) = W'(z)$. The complex equation with respect to $W'(z)$ is

$$\Psi'(z, w) := B_0(z)(W' - w'_1(z))(W' - w'_2(z))...(W' - w'_{v}(z))$$

$$:= B_0(z)(W')^v + B_{v-1}(z)(W')^{v-1} + ... + B_1(z)W' + B_0(z) = 0.$$

Definition 2.6. Let $W(z) = \{(w_j(z), a)\}_{j=1}^{v}$ be a $v$-valued algebroid function. The set of all algebroid mappings of $W(z)$ is denoted by $Y_W$. The set

$$H_W := \{h \circ W(z); h \in Y_W\}$$

is called the algebroid function class of $W(z)$.

Set

$$X_W := \{f \in H_W; T(r, f) = \omega[T(r, W)] \ (r \to \infty, \ r \notin E_f)\},$$

where $E_f$ is a real number set of finite linear measure depending on $f$. $X_W$ is called the small algebroid function set of $W(z)$. The element in $X_W$ is called the small algebroid function of $W(z)$.

Note that the set $X_W$ contains all the finite or infinite complex constants, all the small meromorphic functions and all the small algebroid functions.

Definition 2.7. Let the set of all algebroid mappings of $W(z)$ be $Y_W$ and $H_W := \{h \circ W(z); h \in Y_W\}$. For any $h_1, h_2 \in Y_W$, define

1) Addition: $(h_1 + h_2) \circ W(z) = h_1 \circ W(z) + h_2 \circ W(z)$.
2) Subtraction: $(h_1 - h_2) \circ W(z) = h_1 \circ W(z) - h_2 \circ W(z)$.
3) Multiplication: $(h_1 \cdot h_2) \circ W(z) = (h_1 \circ W(z)) \cdot (h_2 \circ W(z))$. 
Division: \((\frac{b(z)}{a(z)}) \circ W(z) = h_1 \circ W(z) \cdot \frac{1}{h_2} \circ W(z)\).

It is easy to prove that they satisfy Definition 2.4. Hence they are all algebroid mappings. So \(H_W\) is a linear space and is closed with respect to Multiplication and Division.

Suppose that \(\{a_j(z)\},\{b_i(z)\}\) are two group of analytic functions defined in the complex plane \(C\), without no common zeros. The function

\[
q[z, w] := \frac{a_n(z)w^n + a_{n-1}(z)w^{n-1} + \ldots + a_0(z)}{b_m(z)w^m + b_{m-1}(z)w^{m-1} + \ldots + b_0(z)}
\]

is called rational complex function with meromorphic coefficients. The set of all rational complex functions with meromorphic coefficients is denoted by \(Q[z, w]\). By the above definition, Definitions 2.5 and 2.6, for any \(q[z, w] \in Q[z, w]\), \(q \circ \{(w_j(z), a)\} = \{(q[z, w_j(z)], a)\} \in H_W\) is the algebroid function. So \(q[z, w] \in Y_W\).

Especially, when \(Q(z)\) is a single valued rational function defined in the complex plane, \(Q \circ W(z) := \{Q \circ w_j(z), a\}\) is the \(v\)-valued algebroid function. If \(W(z)\) is irreducible and \(Q\) is linear, then \(Q \circ W(z)\) is irreducible. If \(q[z, w] = w \in Q[z, w]\), then \(q \circ \{(w_j(z), a)\} = \{(w_j(z), a)\} = W(z)\) is an identical mapping.

**Theorem 2.4.** Suppose that \(h\) is an algebroid mapping of \(v\)-valued irreducible algebroid function \(W(z) = \{(w_j(z), a)\}\). If \(h \circ W(z)\) is reducible, then it can split to \(n(\geq 1)\) number of \(m\)-valued irreducible algebroid functions and \(v = mn\).

**Proof.** By Theorem 2.3, we know that \(h\) is not injective. Namely there exist two regular function element \((w_1(z), a) \neq (w_2(z), a)\), such that the image elements \((h \circ w_1(z), a) = (h \circ w_2(z), a)\). By Theorem 2.2, \(h \circ W(z) = \{(h \circ w_j(z), a)\}\) can split at least two equal \(m\)-valued \((2m \leq v)\) algebroid functions

\[
h \circ W_1(z) = \{(h \circ w_1(z), a)\} = h \circ W_2(z) = \{(h \circ w_2(z), a)\}.
\]

If \(2m < v\), then there exist the regular function elements

\[
(h \circ w_3(z), a) \in h \circ W(z) - h \circ W_1(z) - h \circ W_2(z)
\]

and \((h \circ w_4(z), a) \in h \circ W_1(z) = \{(h \circ w_1(z), a)\}\) such that \((h \circ w_3(z), a) = (h \circ w_4(z), a)\). Otherwise, since the primary images \((w_3(z), a)\) and \((w_4(z), a)\) are connected, \((h \circ w_3(z), a)\) and \((h \circ w_4(z), a)\) are also connected, which contradicts the fact that \(W_1(z)\) is an algebroid function.). Hence by Theorem 2.1 from \((h \circ w_3(z), a)\) we can continue a \(m\)-valued algebroid function \(h \circ W_3(z)\) such that it equals to \(h \circ W_1(z)\). This work doesn’t stop until we get \(n\) same \(m\)-valued algebroid functions with \(nm = v\). \(\Box\)

**Corollary 2.1.** Suppose that \(h\) is an algebroid mapping of \(v\)-valued irreducible algebroid function \(W(z) = \{(w_j(z), a)\}\). If \(v\) is prime, then \(h \circ W(z)\) is irreducible or \(v\) same meromorphic functions.
Dealing with the addition of two \( v \)-valued algebroid functions, we get the following result.

**Theorem 2.5.** Let \( W(z) = \{ (w_t(z), a) \} \) and \( M(z) = \{ (m_t(z), a) \} \) be two \( v \)-valued algebroid functions. Then

\[
T(r, W + M) \leq T(r, W) + T(r, M) + \log 2.
\]

\[
T(r, W \cdot M) \leq T(r, W) + T(r, M).
\]

**Proof.** Suppose that \( W(z) \) and \( M(z) \) are decomposed to \( v \) simple-valued branch \( \{ W_t(z) \} \) and \( \{ M_t(z) \} \) in the cutting complex plane. Then

\[
m(r, W + M) = \frac{1}{v} \sum_{1 \leq t \leq v} m(r, W_t(z) + M_t(z))
\]

\[
eq \frac{1}{v} \sum_{1 \leq t \leq v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |W_t(re^{i\theta}) + M_t(re^{i\theta})| d\theta
\]

\[
\leq \frac{1}{v}(v \log 2) \sum_{t=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |W_t(re^{i\theta})| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |M_t(re^{i\theta})| d\theta
\]

\[
= m(r, W(z)) + m(r, M(z)) + \log 2.
\]

\[
N(r, W + M) = \frac{1}{v} \int_0^r \frac{n(t, W + M) - n(0, W + M)}{t} dt + \frac{n(0, W + M)}{v} \ln r
\]

\[
\leq \frac{1}{v} \int_0^r \frac{n(t, W)}{t} dt + \frac{n(0, W)}{v} \ln r + \frac{1}{v} \int_0^r \frac{n(t, M) - n(0, M)}{t} dt + \frac{n(0, M)}{v} \ln r
\]

\[
= N(r, W) + N(r, M).
\]

\[
m(r, W \cdot M) = \frac{1}{v} \sum_{1 \leq t \leq v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |W_t(re^{i\theta})M_t(re^{i\theta})| d\theta
\]

\[
\leq \frac{1}{v} \sum_{t=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |W_t(re^{i\theta})| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |M_t(re^{i\theta})| d\theta
\]

\[
= m(r, W(z)) + m(r, M(z)).
\]

\[
N(r, W \cdot M) = \frac{1}{v} \int_0^r \frac{n(t, W \cdot M) - n(0, W \cdot M)}{t} dt + \frac{n(0, W \cdot M)}{v} \ln r
\]

\[
\leq \frac{1}{v} \int_0^r \frac{n(t, W)}{t} dt + \frac{n(0, W)}{v} \ln r + \frac{1}{v} \int_0^r \frac{n(t, M) - n(0, M)}{t} dt + \frac{n(0, M)}{v} \ln r
\]

\[
= N(r, W) + N(r, M).
\]

Hence we get the conclusions of Theorem 2.5. \( \square \)
3. NEVANLINNA’S SECOND MAIN THEOREM CONCERNING SMALL
ALGEBROID FUNCTIONS

Since in $H_W$, elements in $X_W$ can make addition, subtraction, multiplication, division and differential, we have conditions to investigate the theorem concerning small algebroid functions. Referring to the method in [2, 6], we firstly obtain the Second Main Theorem concerning small algebroid functions.

Lemma 3.1. Suppose that $W(z) = \{(w_t(z), a)\}$ is a $v$-valued nonconstant algebroid function in $\{|z| < R\}$, and $\{a_j(z)\}_{j=0}^q \subset X_W$ are $q$ distinct small algebroid function with respect to $W(z)$. Then for any $r \in (0, R)$, we have

$$|m(r, \sum_{j=1}^q \frac{1}{W(z) - a_j(z)})| = S(r, W),$$

where

$$S(r, W) = O(\log(rT(r, f))), \quad (r \to \infty, \ r \notin E),$$

$E$ is a positive real number set of finite linear measure.

Proof. By using the tree $Y$ through all branch points of $W(z)$, we cut $W(z)$ into $v$ singule-valued branch $\{W_t(z)\}_{t=1}^v$. Accordingly, we cut every $a_j(z)$ into $v$ singule-valued branch $\{a_{j,t}(z)\}_{t=1}^v$. For any $t = 1, 2, ..., v$, set

$$F_t(z) := \sum_{j=1}^q \frac{1}{W_t(z) - a_{j,t}(z)}$$

and

$$m(r, F_t) \leq \sum_{j=1}^q m(r, \frac{1}{W_t(z) - a_{j,t}(z)}) + \log q.$$  \hspace{1cm} (3.1)

In order to obtain the lower bound of $m(r, F_t)$, for any $z$, set

$$\delta_t(z) := \min_{1 \leq j < u \leq q} \{|a_{j,t}(z) - a_{u,t}(z)|\} \geq 0.$$  \hspace{1cm} (3.2)

Note that $\delta_t(z)$ is the function of $z$, by the uniqueness theorem, its zeros must be isolated. Take arbitrary $z \in \{z; \delta_t(z) \neq 0\}$.

Case 1. If for any $j \in \{1, 2, ..., q\}$, we have

$$|W_t(z) - a_{j,t}(z)| \geq \frac{\delta_t(z)}{2q},$$

then

$$\sum_{j=1}^q \log^+ \frac{1}{|W_t(z) - a_{j,t}(z)|} \leq q \log^+ \frac{2q}{\delta_t(z)}. \hspace{1cm} (3.3)$$

Case 2. If there exists some $u \in \{1, 2, ..., q\}$ such that

$$|W_t(z) - a_{u,t}(z)| \leq \frac{\delta_t(z)}{2q}. \hspace{1cm} (3.4)$$
Then when \( j \neq u \), we have
\[
|W_t(z) - a_{j,t}(z)| \geq |a_{u,t}(z) - a_{j,t}(z)| - |W_t(z) - a_{u,t}(z)| \geq \delta_t(z) - \frac{\delta_t(z)}{2q} = \frac{2q - 1}{2q} \delta_t(z).
\]
Hence by (3.4) we get
\[
\frac{1}{|W_t(z) - a_{j,t}(z)|} \leq \frac{1}{2q - 1} \frac{2q}{\delta_t(z)} \quad (3.5)
\]
\[
< \frac{1}{2q - 1} \frac{1}{|W_t(z) - a_{u,t}(z)|}.
\quad (3.6)
\]
By (3.1) and (3.6) we get
\[
|F_t(z)| \geq \frac{1}{|W_t(z) - a_{u,t}(z)|} - \sum_{j \neq u} \frac{1}{|W_t(z) - a_{j,t}(z)|}
\geq \frac{1}{|W_t(z) - a_{u,t}(z)|} - \frac{q - 1}{2q - 1} \frac{1}{|W_t(z) - a_{u,t}(z)|} \geq \frac{1}{2|W_t(z) - a_{u,t}(z)|}.
\]
Then by (3.5) we get
\[
\log^+ |F_t(z)| > \log^+ \frac{1}{|W_t(z) - a_{u,t}(z)|} - \log 2
= \sum_{j=1}^{q} \log^+ \frac{1}{|W_t(z) - a_{j,t}(z)|} - \sum_{j \neq u} \log^+ \frac{1}{|W_t(z) - a_{j,t}(z)|} - \log 2
\geq \sum_{j=1}^{q} \log^+ \frac{1}{|W_t(z) - a_{j,t}(z)|} - \sum_{j \neq u} \log^+ \frac{2q}{(2q - 1)\delta_t(z)} - \log 2
> \sum_{j=1}^{q} \log^+ \frac{1}{|W_t(z) - a_{j,t}(z)|} - q \log^+ \frac{2q}{\delta_t(z)} - \log 2.
\]
Combining (3.3), in two cases we have
\[
\log^+ |F_t(z)| > \sum_{j=1}^{q} \log^+ \frac{1}{|W_t(z) - a_{j,t}(z)|} - q \log^+ \frac{2q}{\delta_t(z)} - \log 2.
\quad (3.7)
\]
By definition, for any \( z \in \{z; \delta_t(z) \neq 0\} \), there exists \( j(z) \neq u(z) \) such that
\( \delta_t(z) = a_{j(z),t}(z) - a_{u(z),t}(z) \). Hence we get
\[
\frac{1}{\delta_t(z)} = \frac{1}{a_{j(z),t}(z) - a_{u(z),t}(z)} \leq \sum_{1 \leq j < u \leq q} \frac{1}{a_{j,t}(z) - a_{u,t}(z)}
\]
So
\[
\frac{1}{2\pi} \int_0^{2\pi} \ln^+ \frac{d\theta}{\delta_t(r e^{i\theta})} \leq \sum_{1 \leq j < u \leq q} \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \frac{d\theta}{|a_{j,t}(r e^{i\theta}) - a_{u,t}(r e^{i\theta})|} + O(1)
= \sum m(r, a_{j,t} - a_{u,t}) + O(1) \leq \sum T(r, a_{j,t} - a_{u,t}) + O(1) \leq
= \sum [T(r, a_{j,t}) + T(r, a_{u,t})] + O(1) = S(r, W).
\quad (3.8)
Write \( z = re^{i\theta} \), integrating (3.7) and combining (3.8), we get

\[
m(r, F_t) > \sum_{j=1}^{q} m(r, \frac{1}{|W_t(z) - a_{j,t(z)}|}) + S(r, W).
\]

Then by (3.2), we get

\[
|m(r, F_t) - \sum_{j=1}^{q} m(r, \frac{1}{|W_t(z) - a_{j,t(z)}|})| < S(r, W).
\]

So

\[
|\sum_{j=1}^{q} m(r, \frac{1}{|W_t(z) - a_{j,t(z)}|})| 
= \left| \sum_{t=1}^{v} \frac{1}{v} \sum_{t=1}^{v} m(r, \frac{1}{|W_t(z) - a_{j,t(z)}|}) \right| 
\leq \frac{1}{v} \sum_{t=1}^{v} \sum_{j=1}^{q} m(r, \frac{1}{|W_t(z) - a_{j,t(z)}|}) < S(r, W).
\]

**Lemma 3.2.** Suppose that \( W(z) \) is a \( v \)-valued nonconstant algebroid function in \( \{|z| < R\} \) and \( n \) is a positive integer. Then \( \frac{W^{(n)}}{W} \) is the differential polynomial of \( \frac{W'}{W} \).

**Proof.** When \( n = 1 \), the conclusion holds clearly.

Suppose that for \( n = t \) we have

\[
\frac{W^{(t)}}{W} = P(\frac{W'}{W}),
\]

where \( P(\frac{W'}{W}) \) is the differential polynomial of \( \frac{W'}{W} \). Since

\[
\left( \frac{W^{(t)}}{W} \right)' = \frac{W^{(t+1)}}{W} - \frac{W^{(t)}}{W} \cdot \frac{W'}{W},
\]

\[
\frac{W^{(t+1)}}{W} = \left( \frac{W^{(t)}}{W} \right)' + \frac{W^{(t)}}{W} \cdot \frac{W'}{W}
\]

is the differential polynomial of \( \frac{W'}{W} \).

**Lemma 3.3.** Let \( f_1, f_2, \ldots, f_k, g \in H_W \). Then

\[
W(f_1, f_2, \ldots, f_k) := \begin{vmatrix}
    f_1 & f_2 & \cdots & f_k \\
    f_1' & f_2' & \cdots & f_k' \\
    \cdots & \cdots & \cdots & \cdots \\
    f_1^{(k-1)} & f_2^{(k-1)} & \cdots & f_k^{(k-1)}
\end{vmatrix} = g^k W(\begin{vmatrix}
    \frac{f_1}{g} & \frac{f_2}{g} & \cdots & \frac{f_k}{g}
\end{vmatrix}).
\]
Proof. (1) When $k = 2$, we have
\[
g^2 W \left( \frac{f_1}{g}, \frac{f_2}{g} \right) = g^2 \begin{vmatrix} \frac{f_2}{g} & \frac{f_2}{g} \\ \frac{f_2}{g} & \frac{f_2}{g} \end{vmatrix} = g^2 \begin{vmatrix} \frac{f_1}{g} & \frac{f_2}{g} \\ \frac{f_2}{g} - \frac{f_1}{g} \end{vmatrix} = g^2 \begin{vmatrix} \frac{f_1}{g} & \frac{f_2}{g} \\ \frac{f_2 - f_1}{g} \end{vmatrix}
\]
\[
= g^2 \left( \frac{f_1 f_2}{g^3} - \frac{f_1 f_2'}{g^3} \right) = f_1 f_2' - f_2 f_1' = W(f_1, f_2).
\]

(2) Suppose that for positive integer $k$, we have
\[
g^k W \left( \frac{f_1}{g}, \frac{f_2}{g}, \ldots, \frac{f_k}{g} \right) = W(f_1, f_2, \ldots, f_k).
\]

Then for $k + 1$, we have
\[
g^{k+1} W \left( \frac{f_1}{g}, \frac{f_2}{g}, \ldots, \frac{f_k}{g}, \frac{f_{k+1}}{g} \right) = g^{k+1} \begin{vmatrix} \frac{f_1}{g} & \frac{f_2}{g} & \cdots & \frac{f_k}{g} & \frac{f_{k+1}}{g} \\ \frac{f_1}{g} & \frac{f_2}{g} & \cdots & \frac{f_k}{g} & \frac{f_{k+1}}{g} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{f_1}{g} & \frac{f_2}{g} & \cdots & \frac{f_k}{g} & \frac{f_{k+1}}{g} \\ \frac{f_1}{g} & \frac{f_2}{g} & \cdots & \frac{f_k}{g} & \frac{f_{k+1}}{g} \end{vmatrix}
\]
\[
= g^{k+1} \sum_{n=1}^{k+1} (-1)^{k+1-n} \left( \frac{f_n}{g} \right)^{(k)} W \left( \frac{f_1}{g}, \ldots, \frac{f_{n-1}}{g}, \frac{f_{n+1}}{g}, \ldots, \frac{f_{k+1}}{g} \right)
\]
\[
= g \sum_{n=1}^{k+1} (-1)^{k+1-n} \left( \sum_{j=0}^{k} C_j^k f_n^{(j)} \left( \frac{1}{g} \right)^{(k-j)} \right) W \left( f_1, \ldots, f_{n-1}, f_{n+1}, \ldots, f_{k+1} \right)
\]
\[
= g \sum_{j=0}^{k} C_j^k \left( \frac{1}{g} \right)^{(k-j)} \left( \sum_{n=1}^{k+1} (-1)^{k+1-n} f_n^{(j)} \right) W \left( f_1, \ldots, f_{n-1}, f_{n+1}, \ldots, f_{k+1} \right)
\]
\[
= g \sum_{j=0}^{k} C_j^k \left( \frac{1}{g} \right)^{(k-j)} \begin{vmatrix} f_1 & f_2 & \cdots & f_k & f_{k+1} \\ (f_1)' & (f_2)' & \cdots & (f_k)' & (f_{k+1})' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (f_1)^{(k-j)} & (f_2)^{(k-j)} & \cdots & (f_k)^{(k-j)} & (f_{k+1})^{(k-j)} \\ (f_1) & (f_2)^{(k-j)} & \cdots & (f_k)^{(k-j)} & (f_{k+1})^{(k-j)} \end{vmatrix}
\]
\[
= g C_k^k \left( \frac{1}{g} \right)^{(k-k)} W \left( f_1, \ldots, f_{n-1}, f_n, f_{n+1}, \ldots, f_{k+1} \right).
\]

So the conclusion of Lemma 3.3 holds. □
Lemma 3.4. Suppose that $A_q = \{a_j := a_j(z)\}_{j=1}^q \subset X_W$ are $q \geq 1$ distinct small algebroid functions. Let $L(s, A_q)$ denote the vector space spanned by finitely many $a_1^{p_1} a_2^{p_2} \ldots a_q^{p_q}$, where integer $p_j \geq 0 (j = 1, 2, \ldots, q)$ and $\sum_{j=1}^q p_j = s(\geq 1)$. Let $\dim L(s, A_q)$ denote the dimension of the vector space $L(s, A_q)$. Then for any $\epsilon > 0$, there exists $s \geq 1$ such that

$$\frac{\dim L(s + 1, A_q)}{\dim L(s, A_q)} < 1 + \epsilon.$$

Proof. Let $G(s, A_q)$ denote the set of the form $a_1^{p_1} a_2^{p_2} \ldots a_q^{p_q}$, and let $#(s, A_q)$ denote the number of distinct element of $G(s, A_q)$.

Using mathematical induction, we firstly prove that for any $q > 0, s > 0$, we have

$$#(s + 1, A_q) = C_{q+s}^{s+1}.$$  \hspace{1cm} (3.9)

When $q = 1$, for any integer $s \geq 1$, $#(s + 1, A_1) = 1 = C_{1+s}^{s+1}$. (3.9) holds.

When $q = 2$, for any integer $s \geq 1$, $#(s + 1, A_2) = s + 2 = C_{2+s}^{s+1}$. (3.9) holds.

Suppose that for $q = k$ and any integer $s \geq 1$, we have $#(s + 1, A_k) = C_{k+s}^{s+1}$. Then for $q = k + 1$, we have

$$#(s + 1, A_{k+1}) = #(s + 1, A_k) + #(s, A_k) \cdot #(1, A_1) + #(-s - 1, A_k) \cdot #(-2, A_1) + \ldots + #(1, A_k) \cdot #(s, A_1) + #(s + 1, A_1)$$

$$= C_{k+s+1}^{s+1} + C_{k+s-1}^{s-1} + C_{k+s-2}^{k+1} + \ldots + C_{k+1}^{k+1} + C_k^{k+1} = 1 + \sum_{j=0}^s C_{k+j}^{j+1}.$$

Since $C_{k+j+1}^{j+1} = C_{k+j}^{j+1} + C_{k+j}^{j}$, $C_{k+j}^{j+1} = C_{k+j+1}^{j+1} - C_{k+j}^{j}$. Substituting it into the above equality, we get

$$#(s + 1, A_{k+1}) = 1 + \sum_{j=0}^s (C_{k+j+1}^{j+1} - C_{k+j}^{j}).$$

$$= 1 + (C_{k+s+1}^{s+1} - C_{k+s}^{s}) + (C_{k+s}^{s} - C_{k+s-1}^{s-1}) + (C_{k+s-1}^{s-1} - C_{k+s-2}^{s-2}) + (C_{k+s-2}^{s-2} - C_{k+s-3}^{s-3}) + \ldots$$

$$+ (C_{k+4}^{3} - C_{k+3}^{3}) + (C_{k+3}^{3} - C_{k+2}^{2}) + (C_{k+2}^{2} - C_{k+1}^{1}) + (C_{k+1}^{1} - C_{k}^{0})$$

$$= 1 + C_{k+s+1}^{s+1} - C_{k}^{0} = C_{k+s+1}^{s+1}.$$

Then we prove that for any $q > 0, s > 0$, we have

$$C_{q+s}^{s+1} \leq q(q + 1)s^q.$$  \hspace{1cm} (3.10)

When $q = 1$, for any integer $s \geq 1$, $C_{1+s}^{s+1} = 1 \leq 2s$. (3.10) holds.

When $q = 2$, for any integer $s \geq 1$, $C_{2+s}^{s+1} = s + 2 \leq 6s^2$. (3.10) holds.

Suppose that for $q = k$ and any integer $s \geq 1$, we have $C_{k+s}^{s+1} \leq k(k+1)s^k$.

Then for $q = k + 1$, we get

$$C_{k+s+1}^{s+1} = C_{k+s}^{s+1} k + 1 \leq k(k+1)s^k \frac{k + s + 1}{k}$$
= (k + 1)s^k(k + s + 1) = (k + 1)(k + 2)s^{k+1} \frac{k + s + 1}{ks + 2s} \leq (k + 1)(k + 2)s^{k+1}.

This shows that (3.10) holds. Combining (3.9), for any q > 0, s > 0 we have
\[
\dim L(s + 1, A_q) \leq \#(s + 1, A_q) = C_s^{s+1} \leq q(q + 1)s^q. \tag{3.11}
\]

Finally if Lemma 3.4 doesn’t hold, then for any integer s ≥ 1, we have
\[
\dim L(s + 1, A_q) \geq (1 + \varepsilon) \dim L(s, A_q).
\]

Hence
\[
\dim L(s+1, A_q) \geq (1 + \varepsilon) \dim L(s, A_q) \geq \cdots \geq (1 + \varepsilon)^s \dim L(1, A_q) \geq (1 + \varepsilon)^s.
\]

Combining (3.11), we get
\[
(1 + \varepsilon)^s \leq q(q + 1)s^q. \tag{3.12}
\]

But
\[
\lim_{s \to \infty} \frac{(1 + \varepsilon)^s}{s^q} = \infty.
\]

This contradicts (3.12).
\[\square\]

**Theorem 3.5. (Nevanlinna’s Second Main Theorem)**

Suppose that \( W(z) = \{(w_j(z), a)\} \) is a \( v \)-valued nonconstant algebroid function in the complex plane \( C \). \( \{a_j\}_{j=1}^q \subset X_W \) are \( q \geq 2 \) distinct small algebroid functions of \( W(z) \). Then for any \( \varepsilon \in (0, 1) \) and \( r > 0 \), we have
\[
m(r, W) + \sum_{j=1}^q m(r, \frac{1}{W(z) - a_j}) = (2 + \varepsilon)T(r, W) + 2N_z(r, W) + S(r, W). \tag{3.13}
\]

Its equivalent form is
\[
(q - 1 - \varepsilon)T(r, W) \leq N(r, W) + \sum_{j=1}^q N(r, \frac{1}{W - a_j}) + 2N_z(r, W) + S(r, W) \tag{3.14}
\]
or
\[
(q - 4v + 3 - \varepsilon)T(r, W) \leq N(r, W) + \sum_{j=1}^q N(r, \frac{1}{W - a_j}) + S(r, W). \tag{3.15}
\]

**Proof.** Let \( A_q = \{a_1, a_2, \ldots, a_q\} \) and \( L(s, A_q) \) denote the vector space spanned by finitely many \( a_1^{n_1}a_2^{n_2} \ldots a_q^{n_q} \), where \( n_j \geq 0(j = 1, 2, \ldots, q) \) and \( \sum_{j=1}^q n_j = s \). For given \( s \), set \( \dim L(s, A_q) = n \). Let \( b_1, b_2, \ldots, b_n \) denote a basis of \( L(s, A_q) \). Set \( \dim L(s + 1, A_q) = k \). Let \( B_1, B_2, \ldots, B_k \) denote a basis of \( L(s + 1, A_q) \).

By Lemma 3.4, for any \( \varepsilon > 0 \), there exists some \( s \) such that
\[
1 \leq \frac{k}{n} < 1 + \varepsilon. \tag{3.16}
\]

Let
\[
P(W) := W(B_1, B_2, \ldots, B_k, Wb_1, Wb_2, \ldots, Wb_n).
\]
Since $B_1, B_2, ..., B_k, Wb_1, Wb_2, ..., Wb_n$ are linearly independent, $P(W) \neq 0$. By the definition of the Wronskian determinant, we get

$$P(W) = \sum C_p(z) \prod_{j=0}^{n+k-1} (W^{(j)})^{p_j} = W^n \sum C_p(z) \prod_{j=0}^{n+k-1} \left( \frac{W^{(j)}}{W_t} \right)^{p_j}. \quad (3.17)$$

Since $m(r, W'/W) = S(r, W)$, we get

$$m(r, P(W)) \leq nm(r, W) + S(r, W). \quad (3.18)$$

By Lemma 3.3, we get

$$W(B_1, ..., B_k, Wb_1, ..., Wb_n) = P(W) = W^{n+k} W(\frac{B_1}{W}, ..., \frac{B_k}{W}, Wb_1, ..., Wb_n).$$

(i) Suppose that $(q(z), z_0)$ is a meromorphic function element or multivalent algebraic function element of $W(z)$. If $z_0$ is a $\tau$-fold pole of $q(z)$, by the right of the above equality, it can be see that outside the poles of the small algebroid functions $\{B_i\}, \{b_j\}$, the order of pole of $P(W)$ at $(q(z), z_0)$ is $(n+k)\tau$.

If $z_0$ is a zero of $q(z)$, by the left of the above equality, it can be see that outside the poles of the small algebroid functions $\{B_i\}, \{b_j\}$, $(q(z), z_0)$ isn’t the pole of $P(W)$.

(ii) For any $1 \leq t \leq k$, set

$$W_t(B_1, ..., B_k, Wb_1, ..., Wb_n) := W(B_1, ..., B_{t-1}, B_{t+1}, ..., B_k, Wb_1, ..., Wb_n).$$

When $k < t \leq n + k$, set

$$W_t(B_1, ..., B_k, Wb_1, ..., Wb_n) := W(B_1, ..., B_k, Wb_1, ..., Wb_{t-1}, Wb_{t+1}, ..., Wb_n).$$

Suppose that $(q(z), z_0)$ is any $\lambda$-sheeted algebraic function element of $W(z)$ and $z_0$ isn’t the pole of $q(z)$. Then $z_0$ is at most the pole of $q'(z)$ with the order $\lambda - 1$. By Lemma 3.3 we get

$$P(W) = \sum_{t=1}^{k} \left[ (-1)^{t+1} B_t \cdot W_t(B'_1, ..., B'_{k}, (Wb_1)', ..., (Wb_n)') \right]$$

$$+ \sum_{t=k+1}^{k+n} \left[ (-1)^{t+1} Wb_t \cdot W_t(B'_1, ..., B'_{k}, (Wb_1)', ..., (Wb_n)') \right]$$

$$= \sum_{t=1}^{k} \left[ (-1)^{t+1} B_t \cdot (Wb_t + Wb'_t)^{n+k-1} W_t(\frac{B'_1}{(Wb_t)'}), ..., \frac{B'_k}{(Wb_t)'}), (Wb_1)', ..., (Wb_n)' \right]$$

$$+ \sum_{t=k+1}^{k+n} \left[ (-1)^{t+1} Wb_t \cdot (Wb_t + Wb'_t)^{n+k-1} W_t(\frac{B'_1}{(Wb_t)'}), ..., \frac{B'_k}{(Wb_t)'}), (Wb_1)', ..., (Wb_n)' \right].$$

Hence outside the poles of the small algebroid functions $\{B_i\}, \{b_j\}$, the order of pole of $P(W)$ at $(q(z), z_0)$ is at most $(\lambda - 1)(n + k - 1)$.

Combining (i) and (ii), we get

$$N(r, P(W)) \leq (n + k)N(r, W) + (n + k - 1)N_x(r, W) + S(r, W).$$
By (3.18) we get
\[ T(r, P(W)) \leq nT(r, W) + kN(r, W) + (n + k - 1)N_x(r, W) + S(r, W). \quad (3.19) \]
Suppose that \( a \) is a linear combination of \( \{a_j\} \), then
\[ P(W - a) = W(B_1, B_2, ..., B_k, Wb_1 - ab_1, Wb_2 - ab_2, ..., Wb_n - ab_n) \]
\[ = W(B_1, B_2, ..., B_k, Wb_1, Wb_2, ..., Wb_n) \pm \sum W(B_1, B_2, ..., B_k, ...) \]
where the element "..." behind \( B_k \) in \( \sum W(B_1, B_2, ..., B_k, ...) \) consists of \( ab_j \).

But \( ab_j \) and \( B_1, B_2, ..., B_k \) are linearly dependent, so we get \( \sum W(B_1, B_2, ..., B_k, ...) = 0 \). Hence we get
\[ P(W - a) = P(W). \quad (3.20) \]

By (3.17) and Lemma 3.2, we get
\[ P(W) = W^n \cdot Q\left( \frac{W'}{W} \right), \quad (3.21) \]
where \( Q\left( \frac{W'}{W} \right) \) is the differential polynomial of \( \frac{W'}{W} \). Set
\[ u_j := W - a_j, \quad Q_j := Q\left( \frac{u'_j}{u_j} \right), \quad j = 1, 2, ..., q. \]

By (3.20) and (3.21) we get
\[ P(W) = P(u_j) = u^n_j Q_j, \quad \text{namely} \quad \frac{1}{W - a_j} = \frac{\left| Q_j \right|^{1 \over n}}{|P(W)|^{1 \over n}}. \quad (3.22) \]

Set
\[ F(z) := \sum_{j=1}^{q} \frac{1}{W(z) - a_j}. \]

By Lemma 3.1, we get
\[ m(r, F) = m(r, \sum_{j=1}^{q} \frac{1}{W(z) - a_j}) = \sum_{j=1}^{q} m(r, \frac{1}{W(z) - a_j}) + O(1). \quad (3.23) \]

By (3.22) we get
\[ |F(z)| \leq \sum_{j=1}^{q} \frac{1}{|W(z) - a_j|} \leq \frac{1}{|P(W)|^{1 \over n}} \sum_{j=1}^{q} \left| Q_j \right|^{1 \over n}. \]

Then by (3.19) and (3.16), we get
\[ m(r, F) \leq \frac{1}{n} m(r, \frac{1}{P(W)}) + \frac{1}{n} \sum_{j=1}^{q} m(r, Q_j) + O(1) \]
\[ \leq \frac{1}{n} T(r, P(W)) - \frac{1}{n} N(r, \frac{1}{P(W)}) + S(r, W) \]
\[ \leq T(r, W) + \frac{k}{n} N(r, W) + \frac{n + k - 1}{n} N_x(r, W) - \frac{1}{n} N(r, \frac{1}{P(W)}) + S(r, W) \]
< T(r, W) + \frac{k}{n} N(r, W) + 2N_x(r, W) - \frac{1}{n} N(r, \frac{1}{P(W)}) + S(r, W). \quad (3.24)

By (3.16), (3.23) and (3.24), we get

\[ m(r, W) + \sum_{j=1}^{q} m(r, \frac{1}{W(z) - a_j}) \leq \frac{k}{n} m(r, W) + m(r, F) \]

\[ \leq (1 + \frac{k}{n}) T(r, W) + 2N_x(r, W) + S(r, W) \]

\[ < (2 + \epsilon) T(r, W) + 2N_x(r, W) + S(r, W). \]

Hence we get (3.13).

Note that

\[ m(r, \frac{1}{W(z) - a_j}) \leq T(r, W - a_j) - N(r, \frac{1}{W - a_j}) + O(1) \]

\[ = T(r, W) - N(r, \frac{1}{W - a_j}) + S(r, W). \quad (3.25) \]

Substituting (3.25) into (3.13), we get (3.14).

\[ \square \]

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