3D symplectic map

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Abstract

Quantum 3D R – matrix in the classical (i.e. functional) limit gives a symplectic map of dynamical variables. The corresponding 3D evolution model is considered. An auxiliary problem for it is a system of linear equations playing the role of the monodromy matrix in 2D models. A generating function for the integrals of motion is constructed as a determinant of the auxiliary system.

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1 Introduction

In this paper we formulate a classical discrete space–time model (DSTM)\(^1\), which is in some sense the classical counterpart of the Zamolodchikov–Bazhanov–Baxter 3D lattice spin model \([4, 5, 6]\). One way to obtain a classical model associated with ZBB one is to consider the path integration motivated quasi-classical limit of ZBB, this was done in Ref. \([7]\).

But here we choose another way: we consider a symplectic map given by the functional limit of the operator 3D R–matrix \([8, 9]\). In this approach we elude the usage of the Hirota–Miwa equation, because of this map is the local symplectic map for three pairs of dynamical variables. This map we use to construct 3D DSTM with dynamical variables belonging to the vertices of the kagome lattice.

An auxiliary linear problem for such maps in the operator as well as in the functional approaches were found recently in \([10]\). This problem is an over-defined set of linear equations with compatibility conditions giving the map. Analogous system of linear equations for whole two dimensional lattice plays the role of the monodromy matrix in 2D models, and remarkably, the determinant of this system corresponds to the trace of the monodromy matrix and gives a generating function for integrals of motion.

In the main part of this paper (section 2: “The model”) we formulate the evolution model, introduce linear variables and show how the integrals of motion can be derived. An interpretation of the results and some aspects of further investigations are given in section 3: “Discussion”.

Section 2 contains all the essential results of the paper, but in such short form they look artificially. All cumbersome details concerning the derivation of the model, its geometrical interpretation and correct application of the original linear problem from Ref. \([10]\) are given in Appendix.

2 The model

2.1 Algebraic definition of the evolution

Consider a \(Z^2\) span of a two dimensional plane generated by two independent shifts \(a\) and \(b\) such that point \((n, m)\) of the span we present as \(P = a^n \cdot b^m\). Assign to each point \(P\) three pairs of dynamical variables:

\[
\begin{bmatrix}
  u_1(P), w_1(P), u_2(P), w_2(P), u_3(P), w_3(P)
\end{bmatrix}.
\]

For this discrete space system introduce a discrete time evolution

\[
[u_k(n, P), w_k(n, P)] \rightarrow [u_k(n + 1, P), w_k(n + 1, P)], \quad k = 1, 2, 3,
\]

defined by the following relations:

\[
\begin{align*}
  w_1(n + 1, P) &= \frac{w_1(n, P) w_2(n, P) + w_2(n, P) u_2(n, P) + \kappa^2 u_3(n, P) w_3(n, P)}{w_3(n, P)}, \\
  u_1(n + 1, P) &= \frac{u_1(n, P) u_2(n, P) w_2(n, P) + u_2(n, P) w_3(n, P) + \kappa^2 u_1(n, P) w_3(n, P)}{u_1(n, P) w_2(n, P) + u_2(n, P) w_3(n, P) + \kappa^2 u_1(n, P) w_3(n, P)}.
\end{align*}
\]

\(^1\)Ref. \([1, 2, 3]\) are the rather uncompleted list of the publications which are to be mentioned.
Thus \( \psi \) and \( \kappa \) conserved the Poisson brackets

\[
\{ u_k(n, P), w_m(n, Q) \} = u_k(n, P) w_m(n, Q) \delta_{k,m} \delta_{P,Q},
\]

i. e. map (2.2) is the symplectic map.

### 2.2 Auxiliary linear problem

Introduce a set of auxiliary variables \( \psi^*_k(n, P) \) assigned to the points of the 2D lattice. Let the evolution of \( \psi^*_k(n, P) \) be

\[
\psi^*_1(n + 1, P) = \psi^*_5(n, P), \quad \psi^*_3(n + 1, P) = \psi^*_2(n, a^{-1}b^{-1} \cdot P),
\]

and

\[
\psi^*_2(n + 1, P) = \psi^*_1(n, P) \kappa \left( \frac{u_3(n, P)}{u_1(n, P)w_2(n, P)} - \frac{w_1(n, P)}{u_2(n, P)w_3(n, P)} \right) \\
- \psi^*_5(n, a \cdot P) \frac{\kappa^2}{u_1(n + 1, P)w_3(n, P)} \\
+ \psi^*_5(n, P) \frac{u_3(n, P)}{u_2(n + 1, a \cdot P)} \\
+ \psi^*_5(n, b \cdot P) \frac{w_1(n + 1, P)}{w_2(n, P)}.
\]

Thus \( \psi^*_k(n, P) \) are the linear variables. Define functions \( J^*_k(n, P) \) as follows:

\[
J^*_1(n, P) = \psi^*_1(n, P)w_1(n, P) - \psi^*_2(n, P)u_1(n, P) + \psi^*_5(n, aP)k + \psi^*_5(n, bP)ku_1(n, P)w_1(n, P),
\]

\[
J^*_2(n, P) = \psi^*_1(n, P)k + \psi^*_5(n, a^{-1}P)ku_2(n, P)w_2(n, P) + \psi^*_5(n, P)w_2(n, P) - \psi^*_5(n, bP)u_2(n, P),
\]

\[
J^*_3(n, P) = -\psi^*_1(n, P)u_3(n, P) + \psi^*_5(n, b^{-1}P)w_3(n, P) + \psi^*_5(n, aP)k + \psi^*_5(n, P)ku_3(n, P)w_3(n, P).
\]
One can verify easily

\[ J_1^*(n + 1, P) = J_2^*(n, P) \frac{u_1(n + 1, P)w_1(n + 1, P)}{u_2(n, P)w_2(n, P)} + J_3^*(n, P) \frac{\kappa}{w_3(n, P)}, \]

\[ J_2^*(n + 1, a \cdot P) = J_1^*(n, P) \frac{u_2(n + 1, a \cdot P)}{u_1(n, P)} + J_3^*(n, P) \frac{w_2(n + 1, a \cdot P)}{w_3(n, P)}, \]

\[ J_3^*(n + 1, b \cdot P) = -J_1^*(n, P) \frac{\kappa}{u_1(n, P)} + J_2^*(n, P) \frac{w_3(n + 1, b \cdot P)}{u_2(n, P)w_2(n, P)}, \]

(2.10)

Regarding \( \psi^* k(n, P) \) and \( J^* k(n, P) \) as components of row vectors \( \Psi^*(n) \) and \( J^*(n) \), we obtain three formal matrix equations: first, Eq. (2.9) we rewrite as

\[ J^*(n + 1) = \Psi^*(n) \cdot D(n), \]

(2.11)

second, Eqs. (2.7,2.8) we rewrite as

\[ \Psi^*(n + 1) = \Psi^*(n) \cdot K(n), \]

(2.12)

and finally, Eq. (2.10) – as

\[ J^*(n + 1) = J^*(n) \cdot M(n). \]

(2.13)

Hence

\[ J^*(n + 1) = \Psi^*(n) \cdot K(n) \cdot D(n + 1) = \Psi^*(n) \cdot D(n) \cdot M(n), \]

(2.14)

so the main relation of this subsection arises:

\[ K(n) \cdot D(n + 1) = D(n) \cdot M(n). \]

(2.15)

Note, there is some ambiguity in the definition of \( K \) and \( M \). Namely, for (2.7) given \( \psi^*_2(n + 1, P) \) can be obtained in general from

\[ J_1^*(n + 1, P) = J_1^*(n, P) \alpha(n, P) + J_2^*(n, P) \beta(n, P) + J_3^*(n, P) \gamma(n, P) \]

(2.16)

with arbitrary \( \alpha, \beta, \gamma \). For \( K \) given – \( M \) can be restored directly.

### 2.3 Finite lattice and integrals of motion

Reduce now the infinite \( \mathbb{Z}^2 \) span of the two dimensional plane to the finite \( \mathbb{Z}^2_N \) span of a two dimensional torus. Namely, let

\[ u_k(n, a^N \cdot P) = u_k(n, b^N \cdot P) = u_k(n, P), \]

\[ w_k(n, a^N \cdot P) = w_k(n, b^N \cdot P) = w_k(n, P), \]

(2.17)

\( k = 1, 2, 3 \), for all \( P \). Obviously, these conditions are conserved by evolution (2.3-2.5). Pure periodic boundary conditions are natural for the dynamical variables, while for auxiliary \( \psi^*_k(n, P) \) one can impose quasiperiodical boundary conditions:

\[ \psi^*_k(n, a^N \cdot P) = \psi^*_k(n, P) x, \quad \psi^*_k(n, b^N \cdot P) = \psi^*_k(n, P) y. \]

(2.18)
Variables $x, y$ play the rôle of the spectral parameters. Now all matrices $K, M, D$ are finite $(3N^2 \times 3N^2)$ dimensional matrix functions on $x, y$.

Take now the determinants of the left and right hand sides of Eq. (2.15). It is easy to see that
\[
\det K(n, x, y) = x^{-N}y^{-N} \det K(n, 1, 1) = x^{-N}y^{-N} K(n),
\]
and
\[
\det M(n, x, y) = x^{-N}y^{-N} \det M(n, 1, 1) = x^{-N}y^{-N} M(n).
\]
Denote further
\[
D(n, x, y) = \det D(n, x, y).
\]
Hence Eq. (2.13) gives
\[
D(n + 1, x, y) = \frac{M(n)}{K(n)} D(n, x, y).
\]
Taking the series decomposition of $D$,
\[
D(n, x, y) = \sum_{l, m} x^l y^m D_{l,m}(n),
\]
we obtain the integrals of motion as the ratios of $D_{k,m}$:
\[
I_{l,m}[u_k(n, P), w_k(n, P)] = \frac{D_{l,m}(n)}{D_{N,-N}(n)},
\]
where $I_{l,m}$ are functionals of $u_k(n, P), w_k(n, P)$ and
\[
I_{l,m}[u_k(n, P), w_k(n, P)] = I_{l,m}[u_k(n + 1, P), w_k(n + 1, P)].
\]

3 Discussion

So, we have described the completely integrable classical discrete space–time model. This model can be quantized (see Appendix for the details), and general approach giving relation (2.15) has a quantum counterpart. This gives a way to construct a 3D analogue of the Bethe ansatz. Several papers concerning eigenvectors and eigenfunctions for 3D transfer matrices and evolution operators are to be mentioned here: [14] and [15].

Mention also the evolution operator for the affine Toda field theory (ATFT), given in Ref. [16]. First, the classical limit of ATFT is closely connected with our evolution: the limit case when $\kappa^2 = 0$ and $u_3(P) w_3(P) u_2(P) \rightarrow 0$ for all $P$, turning the frame of reference $n, a, b$ appropriately and hiding one space direction into the rank of $A_N$, one can obtain a map corresponding to ATFT. Second, the approach to 3D models as to 2D models with high rank symmetry group implies the tedious technique of the nested Bethe ansatz while we try to avoid this using “relativistic” approach.

\footnote{This operation in terms of classical variables is defined badly, and in operator language gives some central elements.}
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Appendix

A Definition of map $R$

Consider the following map:

\[
[u_1, u_1; u_2, u_2; u_3, u_3] \xrightarrow{R} [\dot{u}_1, \dot{u}_1; \dot{u}_2, \dot{u}_2; \dot{u}_3, \dot{u}_3],
\]

(A.1)

where $\dot{u}_k, \dot{w}_k$ are the following functions of $u_k, w_k$:

\[
\begin{align*}
\dot{w}_1 &= \frac{w_1 w_2 + w_2 u_3 + \kappa^2 u_3 u_3}{w_3}, & u_1 &= \frac{u_1 u_2 w_2}{u_1 w_2 + u_2 w_3 + \kappa^2 u_3 u_3}, \\
\dot{w}_2 &= \frac{w_1 w_2 + w_2 u_3 + \kappa^2 u_3 u_3}{w_3}, & u_2 &= \frac{w_1 u_2 + u_2 u_3 + \kappa^2 u_1 w_1}{u_1 w_2 + u_2 w_3 + \kappa^2 u_3 u_3}, \\
\dot{w}_3 &= \frac{u_2 w_2 w_3}{u_1 w_2 + u_2 w_3 + \kappa^2 u_3 u_3}, & u_3 &= \frac{u_1 u_2}{u_1 w_2 + u_2 w_3 + \kappa^2 u_3 u_3}.
\end{align*}
\]

(A.2)

Inverse action, $R^{-1}$, is given by

\[
\begin{align*}
w_1 &= \frac{\dot{w}_1 \dot{u}_2 \dot{u}_3}{\dot{w}_1 \dot{w}_3 + \dot{u}_3 \dot{w}_3 + \kappa^2 \dot{u}_3 \dot{u}_3}, & u_1 &= \frac{\dot{u}_1 \dot{w}_2 + \dot{u}_2 \dot{w}_3 + \kappa^2 \dot{u}_2 \dot{u}_2}{\dot{w}_3}, \\
w_2 &= \frac{\dot{u}_1 \dot{w}_3 + \dot{u}_3 \dot{u}_3 + \kappa^2 \dot{u}_1 \dot{u}_2}{\dot{u}_3}, & u_2 &= \frac{\dot{u}_1 \dot{u}_1 + \dot{u}_1 \dot{u}_3 + \kappa^2 \dot{u}_2 \dot{u}_3}{\dot{u}_3}, \\
w_3 &= \frac{\dot{u}_1 \dot{u}_2 \dot{u}_3 \dot{u}_3 + \kappa^2 \dot{u}_2 \dot{u}_3}{\dot{u}_1 \dot{u}_1 + \dot{u}_1 \dot{u}_3 + \kappa^2 \dot{u}_2 \dot{u}_3}. & u_3 &= \frac{\dot{u}_1 \dot{u}_2 \dot{u}_3 \dot{u}_3 + \kappa^2 \dot{u}_2 \dot{u}_3}{\dot{u}_1 \dot{u}_1 + \dot{u}_1 \dot{u}_3 + \kappa^2 \dot{u}_2 \dot{u}_3}.
\end{align*}
\]

(A.3)

The reader can easily recognize $R$ in our evolution formulae (2.3) up to some re-enumeration of final states’ coordinates. The map $R$ can be extracted from the operator function $\mathbf{R}$ (see (9)). In few words, in the operator formulation $\mathbf{R}$ is a function of three independent commutative pairs $\mathbf{u}_k, \mathbf{w}_k, k = 1, 2, 3$, such that

\[
\mathbf{u}_k \cdot \mathbf{w}_k = q \mathbf{w}_k \cdot \mathbf{u}_k,
\]

(A.4)

and

\[
\dot{\mathbf{u}}_k = \mathbf{R} \cdot \mathbf{u}_k \cdot \mathbf{R}^{-1}, \quad \dot{\mathbf{w}}_k = \mathbf{R} \cdot \mathbf{w}_k \cdot \mathbf{R}^{-1}.
\]

(A.5)

Map (A.2) appears in the limit $q \to 1, q^{1/2} \to -1, u \to u, w \to w$. As the consequence of (A.4) and (A.5), $\mathbf{R}$ conserves the Poisson brackets (2.4).

Note that the map $\mathbf{R}$ obey the Functional Tetrahedron Equation [11, 10, 12, 13], this fact provides the integrability.

Few words concerning the structure of $\mathbf{R}$. Introduce useful variables $\overline{u}, \overline{w}, u, w, s$ as follows:

\[
\overline{w} = w_1 w_2, \quad \overline{u} = u_2 u_3, \quad s \overline{u} \overline{w} = \frac{u_1}{w_3};
\]

(A.6)

\[
u = \frac{w_3}{w_2}, \quad w = \frac{w_1}{u_3}, \quad s \frac{\overline{u}}{\overline{w}} = \frac{u_1}{u_2}.
\]

(A.7)

Then $\mathbf{R}$ conserves $\overline{u}, \overline{w}$ and $s$:

\[
\overline{w} = \overline{u}, \quad \overline{w} = \overline{w}, \quad \dot{s} = s.
\]

(A.8)
and $R$ acts on $u, w$ as follows:

\[
\begin{align*}
\dot{u} &= \frac{1}{u} \left( \frac{1 + w + \kappa^2 u}{s + w + s \kappa^2 u} \right), \\
\dot{w} &= \frac{s}{w} \left( \frac{1 + w + \kappa^2 u}{1 + w + s \kappa^2 u} \right).
\end{align*}
\]  

(A.9)

Define the function $H$:

\[
H(u, w) = w + \frac{s}{w} + \kappa^2 \left( 1 + w \right) \left( \frac{1}{u} + u \frac{s}{w} \right).
\]

(A.10)

It can be easily verified,

\[
H(u, w) = H(\dot{u}, \dot{w}).
\]

(A.11)

Note, $H(u, w) = h$ is a genus one elliptic curve. Thus $H(u, w)$ can be considered as the Hamiltonian, corresponding to $R$, and the evolution

\[
H(u, w) \tau \frac{d u(\tau)}{d \tau} = \{ H(u, w), u \}, \quad H(u, w) \tau \frac{d w(\tau)}{d \tau} = \{ H(u, w), w \}
\]

(A.12)

gives map (A.9) as the first rational point.

**B Evolution**

Define now the geometrical evolution model with the help of the map (A.1,A.2).

Consider the kagome lattice formed by $3N^2$ oriented lines as it is shown in Fig. A.

The lattice consists of two types of triangles and one type of hexagons. Choose the triangles marked in Fig. A for the enumeration of the lattice elements. A shift of the triangle $P$ to any other triangle of
the same type is a superposition of two independent shifts, and because of the existence of three similar directions of the kagome lattice it is convenient to use the homogeneous notation for a multiplicative shift operator

\[ \tau_1^\alpha \cdot \tau_2^\beta \cdot \tau_3^\gamma : \alpha + \beta + \gamma = 0, \]  

with the action of \( \tau_1^{-1} \tau_2 \), \( \tau_1^{-1} \tau_3 \) and \( \tau_2^{-1} \tau_3 \) defined in Fig. A.

Consider now triangle \( P \) in details. Enumerate the vertices of \( P \) by the numbers 1, 2 and 3. Define the evolution \( U \) of the lattice as the mutual change of all the \( P \) - type triangles in the spirit of the Yang – Baxter equivalence as it is shown in Fig. B.

![Fig. B. The action of \( U \).](image)

The lattice obtained is again the kagome lattice with the similar set of \( P \) - type triangles marked as \( \tau_k P \) in Fig. B. We regard this change as the evolution from time \( n \) to time \( n + 1 \), and obviously the time can be extracted from \( P \) as a power of \( \tau_k \) (recall, translations (B.1) for a time fixed are of zero power). Thus three shifts \( \tau_1, \tau_2, \tau_3 \) are the elementary shifts of a three dimensional rectangular lattice, and the evolution \( U \) is the saw transfer matrix of the 3D lattice (see Refs. [14, 15] for geometrical details).

Assign now the pair of dynamical variables \( u_k(P), w_k(P) \) to vertex \( k \) of triangle \( P \). Define the evolution for these variables via the map \( R(A.1,A.2) \) as follows:

\[
U : [u_1(P), w_1(P); u_2(P), w_2(P); u_3(P), w_3(P)] \xrightarrow{R} [u_1(\tau_1 P), w_1(\tau_1 P); u_2(\tau_2 P), w_2(\tau_2 P); u_3(\tau_3 P), w_3(\tau_3 P)].
\]  

(B.2)

Denote now for the shortness the translations for the time fixed as

\[
a = \tau_1^{-1} \cdot \tau_2, \quad b = \tau_1^{-1} \cdot \tau_3,
\]  

(B.3)

and exhibit the “time” as a power of \( \tau_1 \) in the decomposition of three dimensional \( P \) with respect to \( \tau_1, a, b \):

\[
P = \tau_1^n \cdot a^\alpha \cdot b^\beta \mapsto (n, P = a^\alpha \cdot b^\beta).
\]  

(B.4)

with this definition of time and space coordinates, map (B.2) gives exactly the evolution [2.3, 2.5].
C  Basic auxiliary linear problem

A version of an auxiliary linear problem for 3D models was suggested in Ref. [10]. Here we recall it briefly for the case of commutative dynamical variables and show how integrals of the motion can be derived with a help of it.

We start from the basic definition.
Consider some vertex $V$ from our (kagome) lattice. Complete pair $u, w$ assigned to $V$ by an “internal current” $\phi$, while observable (or “external”) currents we are intending to measure in the faces. Internal current as produces additive contributions into four external currents of four faces surrounding the vertex. The values of the contributions are shown in Fig. C.

The rules of the game are quite simple:

- any observable face current is the sum of the surrounding vertices contributions, and
- the face current for any closed face is zero.

This indeed is the decent linear problem due to the following remarkable feature: for a given geometry of a graph the relations between the outer face currents (with the conservation laws being taken into account) define the dynamical variables of the elements of the graph unambiguously.

Consider now the problem of the equivalence of two Yang – Baxter – type graph as in Fig. B. In Fig. D these graphs are redrawn with the face currents’ contributions marked out.
First, the conservation laws for the closed faces $\phi_a = \phi_h = 0$. Second, the equivalence means that the outer currents $\phi_b, ..., \phi_g$ for the left and right hand side graphs coincide. Thus restoring the values of the vertex contributions into the face currents, one obtains the following system:

\[
\begin{align*}
\phi_h &:= w_1 \phi_1 + k \phi_2 - u_3 \phi_3 = 0 , \\
\phi_a &:= - \dot{u}_1 \phi_1 + k \dot{u}_2 \dot{w}_2 \phi_2 + \dot{w}_3 \phi_3 = 0 , \\
\phi_b &:= \dot{w}_1 \phi_1 = w_2 \phi_2 + k w_3 \phi_3 , \\
\phi_c &:= k \dot{\phi}_2 = k \phi_1 + k \phi_3 , \\
\phi_d &:= - \dot{u}_3 \phi_3 = k u_1 \phi_1 - u_2 \phi_2 , \\
\phi_e &:= - u_1 \phi_1 = - \dot{u}_2 \phi_2 + k \phi_3 , \\
\phi_f &:= k u_2 w_2 \phi_2 = k \dot{u}_1 \dot{w}_1 \phi_1 + k \dot{u}_3 \dot{w}_3 \phi_3 , \\
\phi_g &:= w_3 \phi_3 = k \phi_1 + \dot{w}_2 \phi_2 .
\end{align*}
\]

This system is eight homogeneous linear equations for six internal currents $\phi_k, \dot{\phi}_k$. The initial graph (let hand side of Fig. D) is described by two independent currents, so the rank of $8 \times 6$ matrix of the coefficients must be $6 - 3 = 4$. This is the zero curvature condition, and the solution of it is given by

\[
\begin{align*}
\dot{w}_1 &= w_2 \Lambda_3 , & \dot{u}_1 &= \Lambda_2^{-1} w_3^{-1} , \\
\dot{w}_2 &= \Lambda_3^{-1} w_1 , & \dot{u}_2 &= \Lambda_1^{-1} u_3 , \\
\dot{w}_3 &= \Lambda_2^{-1} \dot{u}_1^{-1} , & \dot{u}_3 &= u_2 \Lambda_1 ,
\end{align*}
\]

(Fig. D. The equivalence.)
This is exactly the map $R$.

### D Linear problem for the whole lattice

Consider now the current system for the whole kagome lattice. All faces are closed, so all the face currents are zeros.

Describe the system of “vacuum currents” in details. As it was mentioned, the lattice consists of two types of the triangles and one type hexagons. The elementary “puzzle” is shown in Fig. E.

Thus for each “puzzle” three “vacuum” equations are:

\[
J_1(P) \overset{\text{def}}{=} w_1(P) \phi_1(P) + \kappa \phi_2(P) - u_3(P) \phi_3(P) = 0 \quad (D.1)
\]

\[
J_2(P) \overset{\text{def}}{=} -u_1(P) \phi_1(P) + \kappa u_2(aP) w_2(aP) \phi_2(aP) + w_3(bP) \phi_3(bP) = 0 \quad (D.2)
\]

\[
J_3(P) \overset{\text{def}}{=} w_2(P) \phi_2(P) + \kappa u_3(P) w_3(P) \phi_3(P) + \kappa \phi_1(a^{-1}P) + \kappa \phi_3(a^{-1}P)
\]
\[ +\kappa u_1(b^{-1}P)w_1(b^{-1}P)\phi_1(b^{-1}P) - u_2(b^{-1}P)\phi_2(b^{-1}P) = 0. \]  
(D.3)

We are looking for a nontrivial solution of this system, i.e. corresponding determinant must be zero. This means that the following quadratic form is degenerative:

\[ M(\psi_k^*, \phi_k) = \sum_{P,k} \psi_k^*(P) \cdot J_k(P) \equiv \sum_{P,k} J_k^*(P) \cdot \phi_k(P). \]  
(D.4)

In this way we obtain the alternative equivalent system of \( J_k^*(P) \) (2.9), used in sections 2.2 and 2.3, with three types co-vectors \( \psi_1^*(P), \psi_2^*(P), \psi_3^*(P) \). The transition from the original linear problem \( \phi \leftrightarrow J \) to the co-linear problem \( \psi^* \leftrightarrow J^* \) is important, because one can’t define an evolution for \( \phi \) (but hence there is the ambiguity in the definition of the evolution for \( \psi^* \)).

Introduce the translations for \( \psi_k^* \):

\[ T_a \cdot \psi_k^*(P) = \psi_k^*(aP), \quad T_b \cdot \psi_k^*(P) = \psi_k^*(bP). \]  
(D.5)

Then system (2.9) can be written as

\[ J_1^* = w_1 \cdot \psi_1^* - u_1 \cdot \psi_2^* + \kappa (T_a + u_1w_1 T_b) \cdot \psi_3^* = 0, \]
\[ J_2^* = \kappa \cdot \psi_1^* + \kappa u_2w_2 T_a^{-1} \cdot \psi_2^* + (w_2 - u_2 T_b) \cdot \psi_3^* = 0, \]
\[ J_3^* = -u_3 \cdot \psi_1^* + w_3 T_b^{-1} \cdot \psi_2^* + \kappa (T_a + u_3w_3) \cdot \psi_3^* = 0, \]  
(D.6)

thus the following \( 3 \times 3 \) block form of the “monodromy matrix” \( D \) arises:

\[ \mathbf{D}^*(n) = \begin{pmatrix} W_1(n) & -U_1(n) & \kappa T_a + \kappa U_1(n)W_1(n)T_b \\ \kappa & \kappa U_2(n) W_2(n) T_a^{-1} & W_2(n) - U_2(n) T_b \\ -U_3(n) & W_3(n) T_b^{-1} & \kappa T_a + \kappa U_3(n) W_3(n) \end{pmatrix}, \]  
(D.7)

where block notations \( U_k(n) = \text{diag}_P u_k(n, P), W_k(n) = \text{diag}_P w_k(n, P). \)

As an example one can consider the case \( N = 1 \). The determinant of \( \mathbf{D} \) gives immediately all the invariants (A.6,A.10) of single R.