Gravitational collapse of massless scalar field
and radiation fluid

Anzhong Wang∗
Departamento de Física Teórica, Universidade do Estado do Rio de Janeiro,
Rua São Francisco Xavier 524, Maracanã, 20550-013 Rio de Janeiro – RJ, Brazil
and
Observatório Nacional – CNPq,
Rua General José Cristino 77, São Cristóvão, 20921-400 Rio de Janeiro – RJ, Brazil

J.F. Villas da Rocha†, N.O. Santos‡
Departmento de Astrofísica, Observatório Nacional – CNPq,
Rua General José Cristino 77, São Cristóvão, 20921-400 Rio de Janeiro – RJ, Brazil

∗Internet: wang@symbcomp.uerj.br; wang@on.br
†Internet: roch@on.br
‡Internet: nos@on.br
Abstract

Several classes of conformally-flat and spherically symmetric exact solutions to the Einstein field equations coupled with either a massless scalar field or a radiation fluid are given, and their main properties are studied. It is found that some represent the formation of black holes due to the gravitational collapse of the matter fields. When the spacetimes have continuous self-similarity (CSS), the masses of black holes take a scaling form $M_{BH} \propto (P - P^*)^\gamma$, where $\gamma = 0.5$ for massless scalar field and $\gamma = 1$ for radiation fluid. The reasons for the difference between the values of $\gamma$ obtained here and those obtained previously are discussed. When the spacetimes have neither CSS nor DSS (Discrete self-similarity), the masses of black holes always turn on with finite non-zero values.

PACS numbers: 96.60.Lf, 04.20Jb, 04.40.+c.
I. INTRODUCTION

Gravitational collapse is one of the fundamental problems in General Relativity (GR). The collapse generally has three kinds of possible final states. The first is simply the halt of the processes in a self-sustained object or the dispersion of a matter or gravitational field. The second is the formation of black holes with outgoing gravitational radiation and matter, while the third is the formation of naked singularities. For the last case, however, the cosmic censorship hypothesis [1] declares that these naked singularities do not occur in Nature. The study of gravitational collapse has been mainly guided by these three possibilities.

However, due to the mathematical complexity of the Einstein field equations, we are frequently forced to impose some symmetries on the concerned systems in order to make the problem tractable. Spacetimes with spherical symmetry are one of the cases. In particular, the gravitational collapse of a minimally coupled massless scalar field in such spacetimes has been studied both analytically [2] and numerically [3], and some fundamental theorems have been established. Quite recently this problem has further attracted attention, due to Choptuik’s discovery of critical phenomena that were hitherto unknown [4]. By using a very sophisticated method, Choptuik showed numerically the following intriguing features: Let the initial distribution of
the massless scalar field be parameterized smoothly by a parameter $P$ that characterizes the strength of the initial conditions, such that the collapse of the scalar field with the initial data $P > P^*$ forms a black hole, while the one with $P < P^*$ does not. Then, it was found that: a) the critical solution with $P = P^*$ is universal in the sense that in all the one-parameter families of the solutions considered it approaches an identical spacetime; b) the critical solution is periodic in the logarithm of spacetime scale, with a period of $\Delta \approx 3.44$. This is usually referred to as “echoing” or discrete self-similarity (DSS); c) near the critical solution (but with $P > P^*$), the black hole mass is given by

$$M_{BH} = K(P - P^*)^\gamma,$$

where $K$ is a family-dependent constant, but $\gamma$ is an apparently universal scaling exponent, which has been numerically determined as $\gamma \approx 0.37$. These phenomena were soon found also in the collapse of axisymmetric gravitational waves \[5\] and radiation fluid \[6\]. Therefore, it seems that the phenomena are not due to the particular choice of the matter fields, but rather are generic features of GR. Further numerical evidence to support this conclusion can be found in \[7\].

Parallel to the above numerical investigations, there have been analytical efforts in understanding the physics behind these phenomena \[8, 9, 10\]. While the universality of the critical solution and its self-similarity (echoing) have
been found in most cases, the universality of the exponent $\gamma$ does not. In particular, Maison [11] showed that $\gamma$ is matter-dependent. For the collapse of the perfect fluid with the equation of state $p = k\rho$, it strongly depends on $k$, where $p$ and $\rho$ are respectively the pressure and energy density of the fluid, and $k$ is a constant. The same conclusion was also reached both analytically [12] and numerically [13]. Thus, one might expect that $\gamma$ is universal only within a particular family of matter fields.

However, even in this sense $\gamma$ is still not universal. Lately, Oliveira and one of the present authors [14] constructed an analytic model that represents the collapse of massless scalar wave packets by using the so-called “cut-paste” method to the model studied in [8], and found that $\gamma = 0.5$ for spacetimes with continuous self-similarity (CSS). This is different from the value $\gamma \approx 0.374$ for spacetimes with DSS [4]. Thus, the exponent $\gamma$ depends not only on matter fields but also on self-similarities, continuous or discrete. A natural question now is: What will happen when the collapse has neither CSS nor DSS?

In this paper we shall first present several classes of exact solutions to

\footnote{Note that the original Roberts solutions [15] are not regular at the center $R = 0$. However, as shown in [8], for the subcritical solutions the hypersurface $R = 0$ is always time-like and with negative mass. Since the past and future self-similarity horizons carry flat-space null data, one can replace the negative mass part of the spacetime with flat one in both the past and future light cones of the singularity, so that the pieced spacetime has a regular center [16]. Of course, this will give up the analytic condition, by which, together with the condition of a regular center, it was shown that the critical solution found in [3] is unique. In sequel, the exponent $\gamma$ is uniquely determined [9, 10].}
Einstein’s field equations coupled either with a massless scalar field or with a radiation fluid, and then study their physical properties. To derive these solutions, we assume that the spacetimes are spherically symmetric and conformally flat. One might argue that spacetimes with conformal flatness are not very realistic, and the total mass of the spacetime is usually infinite. To overcome this shortage, one may match the spacetimes to an asymptotically flat exterior by the so-called “cut-paste” method [14]. In the present paper, we shall briefly discuss this possibility, and leave the details to be reported somewhere else. The rest of the paper is organized as follows: In section II several classes of exact solutions to the Einstein field equations coupled with either a massless scalar field or a radiation fluid are derived, while in section III their physical interpretations are studied. The paper is closed with section IV where our main conclusions are presented.

II. ANALYTIC SOLUTIONS OF MASSLESS SCALAR FIELD AND RADIATION FLUID

The general spherically symmetric spacetime is described by the metric [16]

\[ ds^2 = G(t, r)dt^2 + 2H(t, r)dt dr - J(t, r)dr^2 - K(t, r)d\Omega^2, \]

where \( d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2 \), and \( \{ x^\mu \} \equiv \{ t, r, \theta, \varphi \} \ (\mu = 0, 1, 2, 3) \) are the usual spherical coordinates. Due to the arbitrariness in the choice of coordinates, it is subject to the following coordinate transformation, \( t = \)
\( T(t', r'), \ r = R(t', r') \). Making use of this freedom, we can set \( H(t, r) = 0 \).

If we further consider the shear-free case [17], we have \( K(t, r) = r^2 J(t, r) \).

Then, the metric takes the form

\[
 ds^2 = G(t, r) dt^2 - K(t, r) \left( dr^2 + r^2 d\Omega^2 \right).
\]

Note that the above metric is still subject to the transformation \( \bar{t} = f(t) \), where \( f(t) \) is an arbitrary function. Later on, we shall use this freedom to further simplify the metric. With the above form of metric, one can show that the conformal-flatness condition \( C_{\mu\nu\lambda\sigma} = 0 \), where \( C_{\mu\nu\lambda\sigma} \) denotes the Weyl tensor, now reads

\[
 C_{,rr} - \left( \frac{C}{r} \right) ,r - \frac{C}{r^2} = 0,
\]

where \( C \equiv \sqrt{G(t, r)/K(t, r)} \), and \( (,)_x \equiv \partial(\)/\( \partial x \). The above equation has the general solution

\[
 C(t, r) = f_1(t) + f_2(t) r^2,
\]

where \( f_1 \) and \( f_2 \) are two arbitrary functions of \( t \). Thus, there are three possibilities:

\[
i) \ f_1(t) \neq 0, f_2(t) = 0, \quad ii) \ f_1(t) = 0, f_2(t) \neq 0, \quad iii) \ f_1(t) \neq 0, f_2(t) \neq 0.
\]

In case i), by introducing a new coordinate \( \bar{t} \equiv \int f_1(t) dt \) we can bring the metric to a form that is conformally flat to the Minkowski metric. Thus,
without loss of generality, in this case we can set $f_1(t) = 1$. By a similar argument, we can set $f_2(t) = 1$ in cases ii) and iii). Once this is done, cases i) and ii) are not independent. In fact, by a coordinate transform $r = 1/\bar{r}$, the metric of case ii) will reduce to that of case i). Therefore, the general metric for spherically symmetric spacetimes with conformal flatness takes the form

$$ds^2 = G(t, r) \left[ dt^2 - h(t, r) \left( dr^2 + r^2 d\Omega^2 \right) \right], \quad (2.1)$$

where

$$h(t, r) = \frac{1}{C(t, r)} = \begin{cases} 1, \\ \frac{1}{f_1(t) + r^2}, \end{cases} \quad (2.2)$$

with $f_1(t) \neq 0$. In the following, we shall refer solutions with $C(t, r) = 1$ to as Type A solutions, and solutions with $C(t, r) = f_1(t) + r^2$ to as Type B solutions.

The concept of CSS (or homotheticity) is defined in a relativistic context as the existence of a conformal Killing vector field $\xi^\mu$, satisfying

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 2g_{\mu\nu}. \quad [18]$$

Because of the spherical symmetry, we can write $\xi^\mu$ as $\xi^\mu = \xi^0 \delta^t_t + \xi^1 \delta^t_r$, where $\xi^0$ and $\xi^1$ are functions of $t$ and $r$. Substituting this expression into the above equations, we find

$$\xi^1 R_{tr} + \xi^0 R_{rt} = R,$$
\[\xi^1\nu_r + \xi^0\nu_t + \xi^1_r = 1,\]
\[\xi^1\lambda_r + \xi^0\lambda_t + \xi^0_t = 1,\]
\[e^{2\nu}\xi^1_t - e^{2\lambda}\xi^0_{tr} = 0,\]  
(2.3)

where \(\lambda \equiv \frac{1}{2} \ln G,\, \nu \equiv \frac{1}{2} \ln(h^2 G),\) and \(R \equiv rhG^{1/2}.\)

The concept of DSS is defined as follows [10]: If there exists a diffeomorphism \(\phi\) and a real constant \(\Delta\) such that, for any integer \(n,\)
\[(\phi_*)^n g_{ab} = e^{2n\Delta} g_{ab},\]
then the corresponding spacetime is said to have DSS, where \(\phi_*\) denotes the pullback of \(\phi.\) For the metric (2.1) it can be shown that the diffeomorphism implies that
\[G(t, r) = G(e^{n\Delta} t, e^{n\Delta} r),\, \, h(t, r) = h(e^{n\Delta} t, e^{n\Delta} r).\]  
(2.4)

To see the connection between CSS and DSS, one may define a vector field \(\xi \equiv \partial/\partial \sigma,\) where \(\sigma\) is one of the four coordinates of the spacetime such that if a point \(p\) has the coordinate \((\sigma, x^a),\) its image \(\phi(p)\) has the coordinate \((\sigma+\Delta, x^a).\) The discrete diffeomorphism \(\phi\) is then realized as the Lie dragging along \(\xi\) by a distance \(\Delta.\) Clearly, CSS corresponds to DSS for infinitesimally small \(\Delta.\) In this sense, CSS can be considered as a degenerate case of DSS. For the details, we would like to refer readers to [10].

The functions \(G(t, r)\) and \(f_1(t)\) are determined by the Einstein field equations \(R_{\mu\nu} - g_{\mu\nu}R/2 = -8\pi T_{\mu\nu}.\) Note that in this paper we choose units such
that $G = 1 = c$, where $G$ is the gravitational constant, and $c$ the speed of light. For the metric (2.1), the non-vanishing components of the Ricci tensor are given by

$$R_{00} = \frac{3}{2h} \left( \frac{hG_{,t}}{G} \right)_{,t} + \frac{3h_{,tt}}{h} - \left( \frac{hr^2G_{,r}}{2r^2h^3G} \right)_{,r},$$

$$R_{01} = \frac{h}{G} \left( \frac{G_{,r}}{h} \right)_{,t} - \frac{3G_{,t}G_{,r}}{2G^2} + \left( \frac{2h_{,r}}{h} \right)_{,t},$$

$$R_{11} = \left( \frac{3G_{,r}}{2G} \right)_{,r} + \frac{hG_{,r}}{2r^2G} \left( \frac{r^2}{h} \right)_{,r} + \frac{2}{r} \left( \frac{rh_{,r}}{h} \right)_{,r} - \frac{h(hG_{,tt}+5G_{,r}h_{,r})}{2G} - \frac{(h^2h_{,r})_{,t}}{h},$$

$$R_{22} = \sin^2 \theta R_{33} = \frac{r^2}{2} \left\{ \frac{(h^3G_{,r})_{,r}}{h^3G} + \frac{(h^6G^4)_{,r}}{rh^6G^4} + \frac{2h_{,rr}}{h} \right\} - \frac{(h^5G_{,t})_{,t}}{h^3G} - \frac{2(h^2h_{,t})_{,t}}{h}. $$

To solve the Einstein field equations, we need to specify the matter fields. In the following we shall consider two particular cases, one is a massless scalar field, and the other is a radiation fluid.

**A. Exact solutions of massless scalar field**

The Einstein field equations for the massless scalar field can be written as

$$R_{\mu\nu} = -8\pi\phi_{,\mu} \phi_{,\nu}. $$

Because of the spherical symmetry, without loss of generality, we assume that $\phi = \phi(t, r)$. Then, the above equation immediately yield $R_{22} = 0$. In view of
Eq. (2.8), this can be written as

\[
\frac{(h^3 G_{,r})_{,r}}{h^3 G} + \frac{(h^0 G^4)_{,r}}{r h^6 G^4} + \frac{2h_{,rr}}{h} = \frac{(h^5 G_{,t})_{,t}}{h^3 G} + \frac{2(h^2 h_{,t})_{,t}}{h}. \tag{2.10}
\]

To solve this equation, let us consider Types $A$ and $B$ solutions separately.

**Type A Solutions.** It can be shown that for Type $A$ solutions, Eq. (2.10) has the general solution,

\[
G(t, r) = c_1 - \frac{f_{,u}(u) - g_{,v}(v)}{r^2} + \frac{f(u) + g(v)}{r^3}, \tag{2.11}
\]

where $c_1$ is an arbitrary constant, $f(u)$ and $g(v)$ are arbitrary functions of their arguments, with $u \equiv t + r$ and $v \equiv t - r$. It should be noted that when other components of the Einstein field equations (2.5) – (2.7) are considered, $f(u)$ and $g(v)$ are not really arbitrary. They have to satisfy the integrability condition for the massless scalar field, which can be written as

\[
R_{00}R_{11} - R_{01}^2 = 0. \tag{2.12}
\]

Once this condition is satisfied, the massless scalar field can be obtained by integrating the other two independent field equations (2.5) and (2.7), which can be written in the form

\[
\phi_{,tt}^2 = -\frac{1}{8\pi} R_{00}, \quad \phi_{,rr}^2 = -\frac{1}{8\pi} R_{11}. \tag{2.13}
\]

To solve Eqs. (2.12) and (2.13) for the general solution of $G$ given by Eq. (2.11) turns out to be complicate. Therefore, in the following we consider some particular solutions, which are sufficient for our present purpose.
**Case α** If we choose the two functions $f$ and $g$ as

$$f(u) = \frac{c_2 u^4}{16}, \quad g(v) = -\frac{c_2 v^4}{16},$$

where $c_2$ is an arbitrary constant, we have

$$G(t, r) = c_1 - c_2 t. \quad (2.14)$$

For such a choice, one can show that Eq.(2.12) is automatically satisfied, while the integration of Eq.(2.13) yields

$$\phi(t, r) = \pm \left(\frac{3}{16\pi}\right)^{1/2} \ln |c_1 - c_2 t| + \phi_0, \quad (2.15)$$

where $\phi_0$ is an integration constant. Note that this solution was also obtained recently in [19].

**Type B Solutions.** For Type B solutions, it can be shown that Eq.(2.10) has solutions only when $f_1(t)$ is a constant, and they are given by

$$G(t, r) = \sum_{n=0}^{\infty} r^{-2} \cosh \left[ (k_n - 16 f_1)^{1/2} (t + t_0) \right] (f_1 + r^2)^{3/2} \exp \left\{ \frac{1}{2} N(r) \right\} \left( a_n + b_n \int e^{N(r)} dr \right), \quad (2.16)$$

where

$$N(r) \equiv 2 \ln r - \ln (f_1 + r^2) - 2 \ln \left[ (f_1 - r^2) - (k_n - 16 f_1)^{1/2} r \right] - 2 \left( \frac{k_n - 16 f_1}{f_1} \right)^{1/2} \arctan \left( \frac{r}{\sqrt{f_1}} \right), \quad (2.17)$$

and $t_0, k_n, a_n$ and $b_n$ are arbitrary constants, subject to Eq.(2.12).
Case $\beta$) One particular solution of Eq. (2.16) is given by

$$G(t, r) = A \cosh(4\alpha t) - B \sinh(4\alpha t),$$

(2.18)

where $\alpha \equiv \sqrt{-f_1}$ ($f_1 < 0$), $A$ and $B$ are two arbitrary constants. For such a choice it can be shown that the integrability condition (2.12) is satisfied, and Eq. (2.13) has the solution

$$\phi(t, r) = \pm \left( \frac{3}{16\pi} \right)^{1/2} \ln \left| \frac{(A - B)e^{2\alpha t} - (B^2 - A^2)^{1/2}e^{-2\alpha t}}{(A - B)e^{2\alpha t} + (B^2 - A^2)^{1/2}e^{-2\alpha t}} \right| + \phi_0,$$

(2.19)

where $\phi_0$ is another integration constant. Since $\phi$ is real, we require $B^2 - A^2 \geq 0$.

B. Exact solutions of radiation fluid

For a perfect fluid, the energy-momentum tensor takes the form $T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}$, where $u_\mu$ denotes the four velocity of the fluid. In the present case, we can assume that it has only two non-vanishing components, $u_\mu = \{u_0, u_1, 0, 0\}$. Then, one can show that only four of the ten Einstein field equations are independent, and can be written in the form [20]

$$R_0^0 - R_2^2)(R_1^1 - R_2^2) - R_1^0 R_0^1 = 0,$$

(2.20)

$$\rho = -\frac{1}{16\pi}(R_0^0 + R_1^1 - 4R_2^2),$$

(2.21)

$$p = -\frac{1}{16\pi}(R_0^0 + R_1^1),$$

(2.22)

$$u_0^2 = \frac{g_{00}(R_0^0 - R_2^2)}{R_0^0 + R_1^1 - 2R_2^2}.$$  

(2.23)
For the radiation fluid, we have \( p = \rho/3 \). Then, Eqs.(2.21) and (2.22) give

\[
R_0^0 + R_1^1 + 2R_2^2 = 0. 
\] (2.24)

In the following, we shall first solve Eq.(2.24) to get the general form of \( G \), and then consider the constraint equation (2.20). Once this is done, we shall use Eqs.(2.21) and (2.23) to get the energy density of the fluid and its four velocity. It can be seen that such obtained \( \rho \) is not always positive. Therefore, to have physical meaningful solutions, we shall further impose the condition \( \rho \geq 0 \).

Before proceeding further, we would like to mention that all the conformally flat perfect fluid solutions are known [21]. However, here we shall re-derive them in a different system of coordinates (2.1) for the convenience of the study of their gravitational collapse.

**Type A Solutions.** When \( h(t, r) = 1 \), Eq.(2.24) takes the form

\[
\left( \frac{r^2G_{,r}}{G^{1/2}} \right)_r = \left( \frac{r^2G_{,t}}{G^{1/2}} \right)_t, 
\] (2.25)

which has the general solution,

\[
G(t, r) = \left[ \frac{f(u) + g(v)}{r} \right]^2, 
\] (2.26)

where \( f(u) \) and \( g(v) \) are two arbitrary functions, subjected to Eq.(2.20) and the condition \( \rho \geq 0 \).
Case $\gamma$) If we choose $f(u)$ and $g(v)$ as

$$f(u) = \frac{c_2}{4} \left( u - \frac{c_1}{c_2} \right)^2, \quad g(v) = -\frac{c_2}{4} \left( v - \frac{c_1}{c_2} \right)^2,$$

where $c_1$ and $c_2$ are arbitrary constants, we find

$$G(t, r) = \left( c_1 - c_2 t \right)^2. \quad (2.27)$$

For such a choice, one can easily show that Eq. (2.20) is satisfied, while Eqs. (2.21) and (2.23) give

$$\rho = 3p = \frac{c_2^4}{8\pi (c_1 - c_2 t)^4}, \quad u_0 = \frac{1}{c_1 - c_2 t}. \quad (2.28)$$

The above solution belongs to the general Friedmann-Robertson-Walker solutions.

Type B Solutions. When $h(t, r) = [f_1(t) + r^2]^{-1}$, Eq. (2.24) takes the form

$$G_{rt} + 6G_{rt} \frac{G_{rt}}{f_1 + r^2} - \frac{12G^2 f_{1, t}^2}{(f_1 + r^2)^2} + \frac{4G^2 f_{1, t}}{f_1 + r^2} - 16 f_1 G^2 =$$

$$- (f_1 + r^2)^2 \left[ 2G_{rr} - G_{rr}^2 + \frac{4f_1 GG_{rr}}{r(f_1 + r^2)} \right]. \quad (2.29)$$

As in the case of the massless scalar field, the above equation has solutions only when $f_1(t)$ is a constant, and the corresponding solutions are given by

$$G(t, r) = \sum_{n=0}^{\infty} \cosh^2[k_n(t + t_0)] \frac{(f_1 + r^2)^n}{r^2} \exp[N(r)]$$

$$\left( a_n + b_n \int \exp[-N(r)]dr \right)^2, \quad (2.30)$$

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where
\[ N(r) \equiv \ln[2(f_1 + r^2)] + \left(\frac{4k_n}{f_1}\right)^{1/2} \arctan\left(\frac{r}{\sqrt{f_1}}\right), \]  
\hspace{1cm} (2.31)
and \( k_n, a_n \) and \( b_n \) are arbitrary constants.

**Case \( \delta \)** A particular case of Eq.(2.30) that satisfies Eq.(2.20) is
\[ G(t, r) = [A \cosh(2\alpha t) - B \sinh(2\alpha t)]^2, \]  
\hspace{1cm} (2.32)
where \( A \) and \( B \) are two constants, and \( \alpha \) is defined as that in Case \( \gamma \). The corresponding physical quantities are
\[ \rho = 3p = \frac{3\alpha^2(B^2 - A^2)}{2\pi[A \cosh(2\alpha t) - B \sinh(2\alpha t)]^4}, \]  
\[ u_0 = \frac{1}{A \cosh(2\alpha t) - B \sinh(2\alpha t)}. \]  
\hspace{1cm} (2.33)

**III. PHYSICAL INTERPRETATION OF THE EXACT SOLUTIONS**

To study the solutions given in the last section, let us first introduce the mass function \( m(t, r) \) via the relation
\[ 1 - \frac{2m(t, r)}{R} = -R_{\alpha\beta} g^{\alpha\beta}, \]  
\hspace{1cm} (3.1)
where \( R \) is the physical radius of the two-sphere \( t, r = \text{Const.} \), and is defined as that in Eq.(2.3). On the apparent horizon
\[ R_{\alpha\beta} g^{\alpha\beta} = 0, \]  
\hspace{1cm} (3.2)
the mass function reads
\[ M_{AH}(t, r) = \frac{R_{AH}}{2}, \quad (3.3) \]
where \( R_{AH} \) is a solution of Eq.\((3.2)\).

A. Massless scalar field

Case \( \alpha \): In this case, the solutions are given by Eqs.\((2.14)\) and \((2.15)\) with \( h = 1 \). According to the values of the constant \( c_1 \), the solutions can be further divided into two sub-cases, \( c_1 > 0 \) and \( c_1 < 0 \). However, it is easy to show that with the replacement \( t \) by \(-t\), we can get one from the other. Thus, without loss of generality, we need only consider the case \( c_1 > 0 \). Introducing a quantity \( P \equiv c_2/c_1 \), the metric can be written as \( ds^2 = c_1(1 - Pt)ds^2_M \), where \( ds^2_M \) denotes the Minkowski metric. From this expression we can see that the amplitude of \( c_1 \) does not play any significant role, hence in the following we shall set \( c_1 = 1 \). Then, the metric takes the form
\[ ds^2 = (1 - Pt)[dt^2 - dr^2 - r^2 d^2 \Omega]. \quad (3.4) \]

From Eqs.\((3.1)\) and \((3.2)\) we find that the corresponding mass function is given by
\[ m(t, r) = \frac{P^2 r^3}{8(1 - Pt)^{3/2}}, \quad (3.5) \]
and that the apparent horizon is located on the hypersurface
\[ r_{AH}^2 = \frac{4(1 - Pt)^2}{P^2}. \quad (3.6) \]
On the other hand, the corresponding Kretschmann scalar is given by

\[ R \equiv R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{45P^4}{8(1 - Pt)^6}, \]

which shows that the spacetime is singular on the hypersurface \( t = P^{-1} \).

The nature of the singularity depends on the signature of the parameter \( P \).

In fact, when \( P > 0 \) the singularity is hidden behind the apparent horizon given by Eq. (3.6), and the solution represents the formation of black holes, due to the collapse of the massless scalar field. The corresponding Penrose diagram is given by Fig. 1(a). From Eqs. (3.5) and (3.6) we can see that the total mass of the black hole is infinitely large. To get a black hole with finite mass, we can cut the spacetime along a hypersurface and then join it to an asymptotically flat exterior [14]. Equation (2.13) shows that the coordinates are comoving with the massless scalar field. Thus, we can cut the spacetime along the hypersurface \( r = r_0 \), where \( r_0 \) is a constant, and then join the part \( r \leq r_0 \) to an asymptotically-flat exterior. Once this is done, the total mass of the scalar field that falls inside the black hole is given by Eq. (3.3) at the point where the hypersurface \( r = r_0 \) intersects with the apparent horizon (3.6), which is

\[ M_{BH} = \left( \frac{r_0^3}{8} \right)^{1/2} P^{1/2}. \]  

When \( P = 0 \), the massless scalar field \( \phi \) becomes a constant, and the corresponding metric (3.4) becomes that of Minkowski.
When $P < 0$, the solutions are actually the time inverse of the ones with $P > 0$. Thus, they represent white holes, and the corresponding Penrose diagram is given by Fig.1(b).

The above analysis shows that, although this class of solutions does not exhibit critical phenomenon in the sense of Choptuik [4], the mass of black holes does take a scaling form (3.7) with the exponent $\gamma$ being 0.5, which is exactly the same as that given in the collapse of a massless scalar wave packet with CSS [8, 14]. This result is a little bit surprising. However, a closer exam of the solutions shows that they also have CSS. In fact, Eq.(2.3) has the solution

$$\xi^0 = -\frac{2(1 - Pt)}{3P}, \quad \xi^1 = \frac{2r}{3}.$$ 

Moreover, introducing two new coordinates

$$\bar{t} = \frac{2(1 - Pt)^{3/2}}{3P}, \quad \bar{r} = r^{3/2},$$

the metric (3.4) can be written as

$$ds^2 = d\bar{t}^2 - \left(\frac{4P \bar{r}}{9}\right)^{2/3} d\bar{r}^2 - \left(\frac{2P \bar{r}}{3}\right)^{2/3} \bar{r}^2 d^2 \Omega,$$

where $x = \bar{t}/\bar{r}$ is the self-similar variable. The above expression takes the exact form for the solutions with CSS [18].

Solutions with CSS were extensively studied by Brady [23], and all of them were divided into two classes. One can show that the above solutions
belong to Brady’s second class. To show this, let us write Eq.(2.15) as

$$\phi = \pm \kappa (\ln |x| - \ln \bar{r}) + \bar{\phi}_0,$$

where $\bar{\phi}_0$ is another constant, and $\kappa \equiv (12\pi)^{-1/2}$. Thus, we have $4\pi \kappa^2 = 1/3 < 1$, which falls into Class II solutions of Brady [23]. It should be noted that Class II solutions were further divided into three sub-classes: i) $4\pi \kappa_c^2 < 4\pi \kappa^2 < 1$; ii) $0 < 4\pi \kappa^2 \leq 4\pi \kappa_c^2$; and iii) $\kappa = 0$, where $\kappa_c$ is an undetermined constant. In case i), it was conjectured that the collapse always forms black holes, while in cases ii) and iii) critical phenomena exist. The only difference between the last two cases is that in case ii) the critical solution separates black holes from naked singularities, while in case iii) it separates black holes from those solutions that represent the dispersion of the collapse [8]. Clearly, our above solutions belong to case i).

**Case $\beta$** In this case the solutions are given by Eqs.(2.18) and (2.19) with $h = (r^2 - \alpha^2)^{-1}$. Since $B^2 - A^2 \geq 0$, we can introduce a constant $t_0$ such that $\sinh(4\alpha t_0) = A/\sqrt{B^2 - A^2}$. Then, the metric coefficient $G(t,r)$ can be written as $G(t,r) = (B^2 - A^2)^{1/2} \sinh[4\alpha(t_0 - \epsilon t)],$ where $\epsilon = \text{sign}(B)$. Clearly, the factor $(B^2 - A^2)^{1/2}$ does not play any significant role to the properties of the solutions. Without loss of generality, we shall set it equal to one. On the other hand, introducing a new radial coordinate $\bar{r}$ by

$$\bar{r} = -\int h(t,r)dr = \frac{1}{2\alpha} \ln \frac{\alpha + r}{\alpha - r},$$

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the corresponding metric becomes
\[ ds^2 = \sinh[4\alpha(t_0 - \epsilon t)] \left\{ dt^2 - d\bar{r}^2 - \frac{\sinh^2(2\alpha\bar{r})}{4\alpha^2} d^2 \Omega \right\}, \]
(3.8)
and Eq.(2.19) reads
\[ \phi(t, \bar{r}) = \pm \left( \frac{3}{16\pi} \right)^{1/2} \ln|\tanh[2\alpha(t_0 - \epsilon t)]| + \tilde{\phi}_0, \]
(3.9)
where \( \tilde{\phi}_0 \) is another constant. To have a correct signature for the metric, we require \( t_0 - \epsilon t \geq 0 \).

The physical relevant quantities now are given by
\[ m(t, \bar{r}) = \frac{\sinh^3(2\alpha\bar{r})}{4\alpha \sinh^{3/2}[4\alpha(t_0 - \epsilon t)]}, \]
\[ \mathcal{R} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{1440\alpha^4}{\sinh^4[4\alpha(t_0 - \epsilon t)]}, \]
(3.10)
while the apparent horizon is given by
\[ \bar{r}_{AH} = 2(t_0 - \epsilon t), \quad (\epsilon = \pm 1). \]
(3.11)
Equation (3.10) shows that the solutions are singular on the hypersurface \( t_0 - \epsilon t = 0 \). When \( \epsilon = +1 \), the singularity is hidden behind the apparent horizon, and the solutions represent the formation of black holes. The corresponding Penrose diagram is similar to that of Fig.1(a). When \( \epsilon = -1 \), the apparent horizon is behind the singularity, and the solutions represent white holes. The corresponding Penrose diagram is that of Fig.1(b). Thus, only the solutions with \( \epsilon = +1 \) represent the gravitational collapse of the massless scalar field.
Eqs. (3.10) and (3.11) show that in this case the total mass of the black hole is also infinitely large. To obtain a black hole with finite mass, we can cut the spacetime and then join it to an asymptotically-flat exterior. Equation (3.9) shows that the scalar field depends only on $t$. That is, the coordinate system $\{t, \bar{r}, \theta, \varphi\}$ is comoving. Therefore, we can cut the spacetime along the hypersurface $\bar{r} = \bar{r}_0$, where $\bar{r}_0$ is a constant. Once this is done, the total mass that the scalar wave packet falls inside the black hole should be given by Eqs. (3.10) and (3.11) with $t_0 - t = \bar{r}_0/2$, namely,

$$M_{BH} = \frac{\sinh^{3/2}(2\alpha \bar{r}_0)}{4\alpha},$$

(3.12)

which, unlike the last case, is finite and non-zero for any given scalar wave packet. Therefore, this model represents the formation of black holes, which turns on always at finite masses.

It can be shown that this class of solutions does not have either CSS or DSS.

B. Radiation fluid

Case $\gamma$) In this case the solutions are given by Eqs. (2.27) and (2.28) with $h = 1$. Similar to Case $\alpha$, now only the parameter $c_2$ is essential. Thus, without loss of generality, we set $c_1 = 1$ and $P = c_2$. Then, the physically
relevant quantities are

\[ m(t, r) = \frac{P^2 r^3}{2(1 - Pt)}, \]

\[ \mathcal{R} \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{36P^4}{(1 - Pt)^8}, \]  

(3.13)

while the apparent horizons now are located on

\[ r_{AH} = \frac{|1 - Pt|}{|P|}. \]

(3.14)

Equation (3.13) shows that the spacetime is singular on the hypersurface \( 1 - Pt = 0 \). The nature of the singularity depends on the signature of \( P \). In fact, when \( P > 0 \), it is space-like and hidden behind the apparent horizon. The corresponding solutions represent the formation of black holes.

The Penrose diagram is given by Fig.2(a), which is quite similar to Fig.1(a), except that now the apparent horizon is null. When \( P = 0 \) the metric reduces to Minkowski. When \( P < 0 \), the singularity precedes the apparent horizon, and the corresponding solutions represent white holes [cf. Fig.2(b)].

The mass function (3.13) on the apparent horizon (3.14) takes the form

\[ M_{AH} = \frac{P r_{AH}^2}{2}, \]

(3.15)

which diverges as \( r_{AH} \to +\infty \). That is, the total masses of black holes are infinite. To obtain black holes with finite masses, following the previous cases, we can cut the spacetime along the hypersurface \( r = \text{Const.} \), say, \( r_0 \),

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since now the radiation fluid is comoving, too [cf. Eq. (2.28)]. Then, we can see that the total mass that the fluid falls inside the black hole is given by Eq. (3.15) with \( r_{AH} = r_0 \),

\[
M_{BH} = \frac{r_0^2}{2} P.
\]  

(3.16)

That is, in this case \( M_{BH} \) also takes a scaling form but with the exponent \( \gamma \) being equal to 1. This is different from the value \( \gamma \approx 0.36 \) found in \([5, 9]\).

Note that the solutions studied here and the ones studied in \([5, 9]\) all have CSS. As a matter of fact, Eq. (2.3) has the solution

\[
\xi^0 = -\frac{1 - Pt}{2P}, \quad \xi^1 = \frac{r}{2}.
\]

Therefore, the difference between the values of the exponent \( \gamma \) are not due to the different self-similarities, as that in Case \( \alpha \). We believe that this is due to the regular condition at the center. In \([9]\) it was shown that if the solutions are analytic and have a regular center, the mass of black holes with CSS must take a scaling form with \( \gamma \approx 0.36 \). Thus, we conjecture that if we give up the analytic condition, replacing, for example, by the condition that the metric is \( c^1 \), as did in \([8]\), one should find solutions that represent critical collapse with a scaling form of mass and \( \gamma = 1 \). Since in the present case the solutions do not represent the critical collapse, we can not verify this point here.

**Case \( \delta \)** In this case the solutions are given by Eqs. (2.32) and (2.33) with
\( h(t, r) = (r^2 - \alpha^2)^{-1} \). Similar to Case \( \beta \), we can introduce a constant \( t_0 \) by 
\[
\sinh(2\alpha t_0) = A/(B^2 - A^2)^{1/2},
\]
and write the metric in the form
\[
ds^2 = \sinh^2[2\alpha(t_0 - t)] \left\{ dt^2 - d\bar{r}^2 - \frac{\sinh^2(2\alpha \bar{r})}{4\alpha^2} d^2\Omega \right\}, \tag{3.17}
\]
where \( \bar{r} \) is defined as that in Case \( \beta \). Then, we find that
\[
m(t, \bar{r}) = \frac{\sinh^3(2\alpha \bar{r})}{4\alpha \sinh[2\alpha(t_0 - ct)]},
\]
\[
\mathcal{R} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{576\alpha^4}{\sinh^8[2\alpha(t_0 - ct)]}, \tag{3.18}
\]
and
\[
\bar{r}_{AH} = t_0 - ct, \tag{3.19}
\]
where \( \epsilon \equiv \text{sign}(B) \). The above expressions show that when \( \epsilon = +1 \), the solutions represent the formation of black holes, and the corresponding Penrose diagram is that of Fig.2(a). When \( \epsilon = -1 \), they represent white holes. The corresponding Penrose diagram is that of Fig.2(b).

The masses of black holes are infinite, as we can see from the above expressions. But, since the coordinates are comoving with the fluid [cf. Eq.(2.33)], we can cut the spacetime along the hypersurface \( \bar{r} = \bar{r}_0 \), where \( \bar{r}_0 \) is a constant, and then join the part \( \bar{r} \leq \bar{r}_0 \) with an asymptotically-flat exterior. Once this is done, the mass that the fluid falls inside the black hole is
\[
M_{BH} = \frac{\sinh^2(2\alpha \bar{r}_0)}{4\alpha}, \tag{3.20}
\]
which is finite and non-zero for any given collapsing shell of radiation fluid. Therefore, this case also represents the formation of black holes, which turns on with finite masses.

Using Eqs. (2.3) and (2.4), one can show that the solutions in this case have neither CSS nor DSS.

IV. CONCLUDING REMARKS

In this paper, we have presented several classes of conformally flat and spherically symmetric exact solutions to the Einstein field equations, coupled with either a massless scalar field or a radiation fluid. Some of these solutions represent the formation of black holes, due to the gravitational collapse of the matter fields. However, since the masses of black holes are all infinite, we have discussed the possibility of cutting the spacetime along a hypersurface, and then joining the internal part with an asymptotically-flat exterior, so that the resulting masses of black holes are finite. Once this is done, we have shown that the masses of such formed black holes always take a scaling form for spacetimes with CSS for both massless scalar field and radiation fluid. The corresponding exponent $\gamma$ is 0.5 for the massless scalar field, and 1 for the radiation fluid. In [9] it was shown that the masses of black holes formed from the critical collapse of radiation fluid with CSS always take a scaling form but with $\gamma \approx 0.36$. This seems to contradict with the results
obtained here. However, it should be pointed out that the results obtained in [9, 10] are based on the requirement that the solutions be analytic and have a regular center. When we give up one of the two conditions, we would expect that the results would be different, in particular, we should be able to construct solutions that represent critical collapse of a radiation fluid with masses of black holes taking a scaling form and the exponent being 1. Of course, this is just a speculation, since our solutions constructed here do not really represent critical collapse. To clarify this point, it would be very useful to consider solutions with less requirements than those in [9, 10].

On the other hand, the masses of black holes formed from the collapse that has neither CSS nor DSS always turn on at finite values, which supports our conjecture made in [14]. Thus, if astrophysically interesting black holes are all with finite non-zero mass, Nature seems to forbid solutions with CSS or DSS.

ACKNOWLEDGMENT

One of the authors (AW) would like to thank Henrique P. de Oliveira for useful discussions. The financial assistance from CNPq is gratefully acknowledged.

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**Figure Captions**

Fig. 1 The Penrose diagram for Case $\alpha$) defined in text: (a) for $P > 0$ and (b) for $P < 0$. The spacetime singularities are represented by dash lines. The apparent horizons (AH) are space-like.

Fig. 2 The Penrose diagram for Case $\gamma$): (a) for $P > 0$ and (b) for $P < 0$. The spacetime singularities, represented by dash lines, are space-like, while the apparent horizons (AH) are null.