Machine learning and the Continuum Hypothesis

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In January 2019 the journal Nature reported on an exciting development in Machine Learning: the very first issue of journal Nature Machine Intelligence contains a paper that describes a learning problem whose solvability is neither provable nor refutable on the basis of the standard ZFC axioms of Set Theory.

In this note I describe what the fuss is all about and indicate that maybe the problem is not so undecidable after all.

Introduction

In the paper [1], in Nature Machine Intelligence, its authors exhibit an abstract machine-learning situation where the learnability is actually neither provable nor refutable on the basis of the axioms of ZFC. This was deemed so exciting that the mother journal Nature actually devoted two commentaries to this: see [9] and [2].

The first of these, [9], is rather matter-of-fact in its description of the problem but the second manages, in just a few lines, to mix up Gödel’s Incompleteness Theorems and the undecidability of the Continuum Hypothesis. It misstates the former — “Gödel discovered logical paradoxes” — and misinterprets the latter: “a paradox known as the Continuum Hypothesis”.

In popular parlance one can say that Gödel employed the Liar’s Paradox in his proof of his incompleteness theorems but those are not paradoxes, they are, as the name indicates, theorems. And the Continuum Hypothesis is not a paradox; it is ‘simply’ a statement that cannot be proved nor refuted on the basis of the usual axioms of Set Theory.

In the first part of this note I will explain what the set theory behind the paper is and where the undecidability comes from. In the second part I will show why I think that the problem is not undecidable at all: there is no algorithm that solves this particular learning problem.

The amount of Set Theory needed to appreciate the arguments in this paper is not too large. We shall meet the cardinal numbers $\aleph_k$ for $k \leq \omega$ as well as the cardinality of the continuum: $2^{\aleph_0}$. We shall also use the ordinals $\omega_k$ as ‘typical’ well-ordered sets of cardinality $\aleph_k$. The first chapter of Kunen’s book [5] more than suffices for our purposes.

1. The Learning Problem

The following is a summary of the parts of [1] that lead to the undecidability result.

The authors start with the following real-life situation as an instance of their general learning problem. A website has a collection of advertisements that it can show to its visitors; each advertisement, $A$, comes with a set, $F_A$, of visitors for whom it is of interest: say if $A$ advertises running shoes then $F_A$ contains avid runners (or people who just like snazzy shoes). Choosing the optimal advertisement to display amounts to choosing a finite set from a population while maximizing the probability that the visitor is actually in that set. The problem is that the probability distribution is unknown.

Rather than dwell on this particular example the authors make an abstraction: Given a set $X$ and a family $\mathcal{F}$ of subsets of $X$ find a member of $\mathcal{F}$ whose measure with respect to an unknown probability distribution is close to maximal. This should be done based on a finite sample generated i.i.d. from the unknown distribution.

The undecidability manifests itself when we let $X$ be the unit interval $I$ and $\mathcal{F}$ the family $\operatorname{fin}_I$ of finite subsets of $I$. 

1.1. Learning functions. In the general situation the abstract problem described above is made more explicit and quantitative as follows.

For the unknown probability distribution \( P \) on \( X \) find \( F \in \mathcal{F} \) such that \( E_P(F) \) is quite close to \( \text{Opt}(P) \), which is defined to be \( \sup_{Y \in \mathcal{F}} E_P(Y) \).

To quantify this further a learning function for a \( \mathcal{F} \) is defined to be a function
\[
G : \bigcup_{k \in \mathbb{N}} X^k \to \mathcal{F}
\]
with certain desirable properties.

In this case the desirable properties are captured in the following definition of an \( (\epsilon, \delta) \)-EMX learner for \( \mathcal{F} \). This is a function \( G \) as above such that for some \( d \in \mathbb{N} \), depending on \( \epsilon \) and \( \delta \), the following inequality holds
\[
\Pr_{S \sim \mathcal{P}_d} \left[ E_P(G(S)) \leq \text{Opt}(P) - \varepsilon \right] \leq \delta
\]
for all distributions \( P \) with finite support.

The letters EMX abbreviate ‘estimating the maximum’.

2. A combinatorial translation

The first step in \( [1] \) is to translate the existence of a suitable function \( G \) into a statement that is a bit more amenable to set-theoretic treatment.

This translation involves what the author call monotone compression schemes. Here and later we use \( [X]^m \) to denote the family of \( n \)-element subsets of \( X \).

Definition 2.1. Let \( m \) and \( d \) be two natural numbers with \( m > d \). An \( m \to d \) monotone compression scheme for a family \( \mathcal{F} \) of finite subsets of a set \( X \) is a function \( \eta : [X]^d \to \mathcal{F} \) such that whenever \( A \) is an \( m \)-element subset of \( X \) it has a \( d \)-element subset \( B \) such that \( A \subseteq \eta(B) \), where we identify \( B \) with a point in \( X \) that enumerates it.

This definition is slightly different from the formulation of Definition 2 in \( [1] \), which leaves open the possibility that \( |A| < m \) and that \( |B| < d \), as it uses indexed sets. It is clear from the results and their proofs that our definition captures the essence of the notion.

There is a second unnamed function implicit in Definition 2.1 the choice of the subset \( B \) of \( A \), we call this function \( \sigma \). So our schemes consist of a pair of functions: \( \sigma : [X]^m \to [X]^d \) and \( \eta : [X]^d \to \mathcal{F} \); they should satisfy \( A \subseteq (\eta \circ \sigma)(A) \) for all \( A \).

Also, in the cases that we are interested in the set \( X \) comes with a linear order \( \prec \); in that case we can identify \( [X]^m, \) the family of \( m \)-element subsets with a subset of the product \( X^m \). Every set corresponds to its monotone enumeration: \( [X]^m = \{ x \in X^m : (i < j < m) \to (x_i < x_j) \} \).

The translation is now as follows.

Lemma 2.2 ([1] Lemma 1.1). For an upward-directed family \( \mathcal{F} \) of finite sets the existence of a \( (\frac{1}{3}, \frac{2}{3}) \)-EMX learning function is equivalent to the existence of a natural number \( m \) and an \( (m+1) \to m \) monotone compression scheme for \( \mathcal{F} \).

The proof of necessity takes the natural number \( d \) in the learning function and produces a monotone \( (m + 1) \to m \) compression scheme with \( m = \lceil \frac{d}{2} \rceil \).

At this point the authors turn to the special case of the unit interval \( [1] \) and its family \( [1] \) of finite subsets and prove the following.

Theorem 2.3. There is a monotone \( (m + 1) \to m \) compression scheme for \( [1] \) for some \( m \in \mathbb{N} \) if and only if \( 2^{\aleph_0} < \aleph_n \).

As the inequality \( 2^{\aleph_0} < \aleph_n \) is both consistent with and independent of the axioms of ZFC the same holds for the existence of a compression scheme and for the existence of a \( (\frac{1}{3}, \frac{2}{3}) \)-EMX learning function.

Theorem 2.3 is an immediate consequence of the set of equivalences in the following theorem.

Theorem 2.4 ([1] Theorem 1]). Let \( k \in \mathbb{N} \) and let \( X \) be a set. Then there is a \( (k + 2) \to (k + 1) \) monotone compression scheme for the finite subsets of \( X \) if and only if \( |X| = \aleph_k \).

Indeed, \( 2^{\aleph_0} < \aleph_n \) if and only if \( |[1]| = \aleph_k \) for some \( k \in \mathbb{N} \).

In the next section we take a closer look at monotone compression schemes and point out a connection with an old result of Kuratowski's.

3. On compression schemes and decompositions

We begin by giving an equivalent description of monotone compression schemes that does not mention the function \( \eta \).

This shows that it is \( \sigma \) that is doing the compressing.

Proposition 3.1. Let \( m \) and \( d \) be natural numbers and let \( X \) be a set. There is an \( m \to d \) monotone compression scheme for the finite subsets of \( X \) if and only if there is a finite-to-one function \( \sigma : [X]^m \to [X]^d \) such that \( \sigma(x) \subseteq x \) for all \( x \).

Proof. If the pair \( (\eta, \sigma) \) determines an \( m \to d \) monotone compression scheme then \( \sigma \) is finite-to-one. For let \( y \in [X]^d \) then \( \sigma(x) = y \) implies \( x \subseteq \eta(y) \), hence there are at most \( \binom{m}{d} \) such \( x \), where \( M = |\eta(y)| \).

Conversely, if \( \sigma \) is as in the statement of the proposition then we can let \( \eta(y) = \bigcup\{x : \sigma(x) = y\} \).

3.1. Kuratowski’s decompositions. The following theorem, proved by Kuratowski in \([6]\), provides one direction in his characterization of when a set has cardinality at most \( \aleph_k \).

Theorem 3.2. The power \( \omega_k^{k+2} \) can be written as the union of \( k + 2 \) sets, \( \{ A_i : i < k + 2 \} \), such that for every \( i < k + 2 \) and every point \( x_j : j < k + 2 \) in \( \omega_k^{k+2} \) the set of points \( y \in A_i \) that satisfy \( y_j = x_j \) for \( j \neq i \) is finite.

In Kuratowski’s words “\( A_i \) is finite in the direction of the \( i \)th axis”.

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Theorem 3.3. For a set $X$ and a natural number $k$ the following are equivalent:

1. $|X| \leq \aleph_k$.
2. $X^{k+2}$ admits a Kuratowski-type decomposition into $k + 2$ sets.
3. there is a $(k+2) \to (k+1)$ monotone compression scheme for the finite subsets of $X$.

We sketch the proof of that last implication for completeness sake. Both it and Kuratowski’s necessity proof use a form of the following lemma.

Lemma 3.4. Let $k$, $l$, and $m$ be natural numbers with $m > l$. Assume $\sigma: [\omega_k]^{m+1} \to [\omega_l]^{m+1}$ determines an $(m+1) \to (l+1)$ monotone compression scheme. Then there is an $m \to l$ monotone compression scheme for $\omega_k$.

Proof. We start by determining an ordinal $\delta$ as follows. Let $\delta_0 = \omega_k$. Given $\delta_n$, use the fact that $\sigma$ is finite-to-one to find an ordinal $\delta_{n+1} > \delta_n$ such that every $x \in [\omega_k]^{m+1}$ that satisfies $\sigma(x) \in [\delta_{n+1}]^{m+1}$ is in $[\delta_n]^{m+1}$.

In the end let $\delta = \sup_n \delta_n$. Then $\delta$ satisfies: every $x \in [\omega_k]^{m+1}$ that satisfies $\sigma(x) \in [\delta]^{m+1}$ is in $[\delta]^{m+1}$.

We define an $m \to l$ monotone compression scheme for $\delta$. If $x \in [\delta]^m$ then $y = x \setminus \{\delta\}$ is in $[\omega_{k+1}]^{m+1}$ and so $\sigma(y) \subseteq y$. It is not possible that $\sigma(y) \subseteq x$ by the choice of $\delta$, hence $\delta \in \sigma(y)$ and so setting $\varsigma(x) = \sigma(y) \setminus \{\delta\}$ defines a map \( \varsigma : [\delta]^m \to [\delta] \). This map is finite-to-one and satisfies $\varsigma(x) \subseteq x$ for all $x$.

To finish the proof of necessity we argue by induction and contradiction. If $|X| = \aleph_k$ and there is a finite-to-one $\sigma : [X]^{k+2} \to [X]^{k+1}$ with $\sigma(x) \subseteq x$ for all $x$ then there is a subset $Y$ of $X$ with $|Y| = \aleph_k$ and a finite-to-one $\varsigma : [Y]^{k+1} \to [Y]^{k}$ with $\varsigma(x) \subseteq x$ for all $x$. This would contradict the obvious inductive assumption. We leave it as an exercise to the reader to ponder what absurdity would arise in the case $k = 0$.

4. Algorithmic considerations

In this section we address a point already raised by the authors in [1]: the functions that are used in the previous sections are quite arbitrary and not related to any recognizable algorithm. Indeed, the constructions of the compression schemes for uncountable sets blatantly applied the Axiom of Choice: once by assuming that the underlying sets were well-ordered and again when in every step of the induction a choice of well-orders of type $\omega_k$ needed to be made.

One may therefore wonder what happens if we impose some structure on the maps in question. One possible way of separating out ‘algorithmic’ functions is by requiring them to have nice descriptive properties. If ‘nice’ is taken to mean ‘Borel measurable’ then the desired functions do not exist.

Sketch of the proof. The case $k = 0$ is easy: let $A_0 = \{(m,n) : m \leq n\}$ and $A_1 = \{(m,n) : m > n\}$.

The rest of the proof proceeds by induction on $k$. We give the step from $k = 0$ to $k = 1$ in some detail and leave the other steps to the reader.

To decompose $\omega^3_1$ into three sets $A_0$, $A_1$ and $A_2$ we apply the Axiom of Choice to choose (simultaneously) for each infinite ordinal $\omega$ a decomposition $\{X(\alpha,0), X(\alpha,1)\}$ of $(\alpha + 1)^2$, say by choosing well-orders of type $\omega$ and then using the decomposition obtained for $k = 0$.

- One puts $\langle \alpha, \beta, \gamma \rangle$ into $A_0$ if $\beta$ is the largest coordinate and $\langle \alpha, \gamma \rangle \in X(\beta,0)$ or if $\gamma$ is the largest coordinate and $\langle \alpha, \beta \rangle \in X(\gamma,0)$.
- One puts $\langle \alpha, \beta, \gamma \rangle$ into $A_1$ if $\alpha$ is the largest coordinate and $\langle \beta, \gamma \rangle \in X(\alpha,0)$ or if $\gamma$ is the largest coordinate and $\langle \alpha, \beta \rangle \in X(\gamma,1)$.
- One puts $\langle \alpha, \beta, \gamma \rangle$ into $A_2$ if $\alpha$ is the largest coordinate and $\langle \beta, \gamma \rangle \in X(\alpha,1)$ or if $\beta$ is the largest coordinate and $\langle \alpha, \gamma \rangle \in X(\gamma,0)$.

To see that $A_0$ is finite in the direction of the 0th coordinate take $\langle \beta, \gamma \rangle \in \omega^2_1$, then $\langle \alpha, \beta, \gamma \rangle \in A_0$ implies $\beta$ is largest and $\langle \alpha, \gamma \rangle \in X(\beta,0)$, or $\gamma$ is largest and $\langle \alpha, \beta \rangle \in X(\gamma,0)$; in either case $\alpha$ belongs to a finite set.

A similar argument works for $A_1$ and $A_2$ of course.

The inductive steps for larger $k$ are modelled on this step. \[\square\]

We now show how Theorem 3.2 can be used to prove sufficiency in Theorem 2.4.

Constructing a compression scheme from a decomposition. From a decomposition as in Theorem 3.2 we construct a finite-to-one function $\sigma: [\omega_k]^{k+2} \to [\omega_k]^{k+1}$ such that $\sigma(x) \subseteq x$ for all $x$. We assume, without loss of generality, that the sets $A_i$ are disjoint.

Let $x \in [\omega_k]^{k+2}$ (so $i < j < k + 2$ implies $x_i < x_j$). Take (the unique) $i$ such that $x \in A_i$ and let $\sigma(x)$ be the point in $\omega_k^{k+1}$ that is $x$ but without its coordinate $x_i$. In terms of sets we would have set $\sigma(x) = x \setminus \{x_i\}$.

This function is finite-to-one: if $y \in [\omega_k]^{k+1}$ then for each $i < k + 2$ there are only finitely many $x$ in $A_i$ with $y = \sigma(x)$.

As mentioned above Kuratowski’s result works both ways: if $X^{k+2}$ admits a decomposition as above for $\omega^{k+2}$ then $|X| \leq \aleph_k$. This suggests that the necessity in Theorem 2.4 is related to the converse of Theorem 3.2. This is indeed the case: one can construct a Kuratowski-type decomposition from a compression scheme, but because of our definition of the schemes we only get a decomposition of the subset $[\omega_k]^{k+2}$ of the whole power. This can be turned into one for the whole power but the process is a bit messy so we leave it be.

The proof of necessity from [1] closes the circle of implications that proves the following.
4.1. Continuity and Borel measurability. Here we show, for arbitrary \( m \in \mathbb{N} \), that there does not exist an 
\((m+1) \to m\) monotone compression scheme for the finite subsets of \( I \) where the function \( \sigma \) is Borel measurable. To this end let \( m \) be a natural number and let \( \sigma : [I]^{m+1} \to [I]^m \) be a function such that \( \sigma(x) \subseteq x \) for all \( x \).

If \( \sigma \) is continuous then \( \sigma \) is not finite-to-one. One can apply \([3, \text{ Theorem VI.7}]\) and deduce that there is a point \( y \) such that the fiber \( \sigma^{-1}(y) \) is one-dimensional, but in this case there is an elementary and more informative argument.

To this end let \( x \in [I]^{m+1} \) and assume for notational convenience that \( \sigma(x) = (x_i : i \leq m) \), i.e., that the coordinate \( x_{m_i} \) is left out of \( x \) when forming \( \sigma(x) \).

Let \( \varepsilon = \frac{1}{m} \min\{x_{i+1} - x_i : i < m\} \) and let \( \delta > 0 \) be such that \( \delta < \varepsilon \) and for all \( y \in [I]^{m+1} \) with \( \|y - x\| < \delta \) we have \( \|\sigma(y) - \sigma(x)\| < \varepsilon \).

Now if \( y \in [I]^{m+1} \) and \( \|y - x\| < \delta \) then \( y_i - x_i < \varepsilon \) for all \( i < m \). Also, when \( i < j \) we have \( x_j - x_i > 3\varepsilon \). It follows that \( y_m - x_m > \varepsilon \) for all \( i < m \). This implies that \( \sigma(y) = (y_i : i < m) \) for all \( y \) with \( \|y - x\| < \delta \).

This shows that for every \( i \) the set \( O_i = \{x \in [I]^{m+1} : \sigma(x) = x \setminus \{x_i\}\} \) is open. Because \([I]^{m+1}\) is connected there is one \( i \) such that \( O_i \subseteq [I]^{m+1} \). This shows that \( \sigma \) cannot be finite-to-one.

The above proof can be used/adapted to show that if \( \sigma \) is Borel measurable it is not finite-to-one either.

If \( \sigma \) is Borel measurable then \( \sigma \) is not finite-to-one.

There is a dense \( G_\delta \)-set \( G \) in \([I]^{m+1}\) such that the restriction of \( \sigma \) to \( G \) is continuous, see \([2, \S 31 I] \).

Let \( x \in G \). As in the previous proof we assume \( \sigma(x) = (x_i : i < m) \) and we obtain a \( \delta > 0 \) such that \( \sigma(y) = (y_i : i < m) \) for all \( y \in G \) that satisfy \( \|y - x\| < \delta \).

By the Kuratowski-Ulam theorem, \([3]\), we can find a point \( y \) in \( G \) with \( \|y - x\| < \delta \) such that the set of points \( t \) in the interval \( (x_m - \delta, x_m + \delta) \) for which \( y_t = \sigma(y) * (t) \) belongs to \( G \) is co-meager. But for every such point we have \( \sigma(y) = \sigma(y) \) and this shows that \( \sigma \) is not finite-to-one.

4.2. EMX learning is impossible. As we saw above a learning function is a function \( G \) from the union \( \bigcup_{k \in \mathbb{N}} I^k \) to the family of finite subsets of \( I \). We can call such a function continuous or Borel measurable if its restriction to each individual power is.

In the construction of an \((m+1) \to m\) compression scheme from a learning function the authors use its restriction to just one of these powers \( I^d \), where \( d \leq m \).

The definition of \( \eta(S) \) involves taking the union of \( G(T) \) for all \( d \)-element subsets \( T \) of \( S \), hence a union of \([m]\) many sets.

The definition of \( \sigma \) involves choosing one \( m \)-element subset with a certain property from of a given \( m+1 \)-element set.

The latter choice can be made explicit using a Borel linear order on the family of all finite subsets of \( I \), or just \([I]^m\).

An analysis of this procedure shows that if \( G \) is Borel measurable then so are \( \sigma \) and \( \eta \).

The results of this section then imply that a Borel measurable learning function does not exist. In this author’s opinion that means that the title of \([1]\) should be emended to “EMX-learning is impossible”.

4.3. On the other hand... One may argue that the choice of the unit interval in \( I \) is a bit of a red herring. None of the arguments in the paper use the structure of \( I \) in any significant way.

In the step from the problem of the advertisements to the more abstract problem there is no real need to go to the unit interval. One may equally well use the set of rational numbers to code or rank the elements of the learning set.

In that case there is, as we have seen, a \( 2 \to 1 \) monotone compression scheme for the finite subsets of \( \mathbb{N} \): simply let \( \sigma(x) = \max x \); the corresponding function \( \eta \) is defined by \( \eta(n) = \{i : i \leq n\} \).

It is an easy matter to transfer this scheme to the family of finite subsets of the rational numbers. Whether this scheme gives rise to a useful EMX learning function remains to be seen.

References
[1] Shai Ben-David, Pavel Hrubes, Shay Moran, Amir Shpilka, and Amir Yehudayoff, Learnability can be undecidable, Nature Machine Intelligence 1 (2019), 44–48, DOI: 10.1038/s42256-018-0002-3.
[2] Davide Castelvecchi, Machine learning leads mathematicians to unsolvable problem, Nature 565 (2019), 277, DOI: 10.1038/d41586-019-00083-3.
[3] Ryszard Engelking, General topology, 2nd ed., Sigma Series in Pure Mathematics 6, Heldermann Verlag, Berlin (1989), MR1039321.
[4] Witold Hurewicz and Henry Wallman, Dimension Theory, Princeton Mathematical Series, v. 4, Princeton University Press, Princeton, N. J. (1941), MR0006493.
[5] Kenneth Kunen, Set theory. An introduction to independence proofs, Studies in Logic and the Foundations of Mathematics 102, North-Holland Publishing Co., Amsterdam-New York (1980), MR597342.
[6] Casimir Kuratowski, Sur une caractérisation des alephs, Fundamenta Mathematicae 38 (1951), 14–17, MR0048518, DOI: 10.4064/fm-38-1-14-17.
[7] K. Kuratowski., Topology. Vol. I, Academic Press, New York-London; Pafnutievoj Wydawnictwo Naukowe, Warsaw (1966), MR0217751.
[8] C. Kuratowski and St. Ulam, Quelques propriétés topologiques du produit combinatoire, Fundamenta Mathematicae 19 (1932), 247–251, ZBL 0005.18301.
[9] Lev Reyzin, Unprovability comes to machine learning, Nature 565 (2019), 166–167, DOI: 10.1038/d41586-019-00012-4.