COEFFICIENT BOUNDS FOR A CERTAIN SUBCLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, we introduce and investigate a new subclass of the analytic and bi-univalent functions in the open unit disk in the complex plane. For the functions belonging to this class, we obtain estimates on the first three coefficients in their Taylor-Maclaurin series expansion. Some interesting corollaries and applications of the results obtained here are also discussed.

1. Introduction and Preliminaries

Let $A$ denote the class of all complex-valued analytic functions in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ in the complex plane of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in U. \quad (1.1)$$

Furthermore, by $S$ we shall denote the class of all functions in $A$ which are univalent in $U$. Some of the important and well-investigated subclasses of $S$ include the class $S^*(\alpha)$ of starlike functions of order $\alpha$ and the class $C(\alpha)$ of convex functions of order $\alpha$ ($\alpha \in [0, 1]$).

By definition

$$S^*(\alpha) = \left\{ f \in S : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ z \in U \right\}, \ \alpha \in [0, 1)$$

and

$$C(\alpha) = \left\{ f \in S : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ z \in U \right\}, \ \alpha \in [0, 1).$$

The above mentioned function classes have been recently investigated rather extensively in [10, 20, 26, 29] and the references therein.

It is well-known that every function $f \in S$ has an inverse $f^{-1}$, defined by $f^{-1}(f(z)) = z$, $z \in U$ and $f(f^{-1}(w)) = w$, $w \in D = \{ w \in \mathbb{C} : |w| < r_0(f) \}$, $r_0(f) \geq 1/4$ where $f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_3^2 - 5a_2 a_3 + a_4) w^4 + \cdots$.

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An analytic function \( f \) is subordinate to an analytic function \( \phi \), written \( f(z) < \phi(z) \), provided there is an analytic function \( u : U \to U \) with \( u(0) = 0 \) and \( |u(z)| < 1 \) satisfying \( f(z) = \phi(u(z)) \) (see, for example, [14]).

Ma and Minda [12] unified various subclasses of starlike and convex functions for which either of the quantity \( zf_0'(z)/f(z) \) or \( 1 + zf''(z)/f(z) \) is subordinate to a more superordinate function. For this purpose, they considered an analytic function with positive real part in \( U \), with \( \phi(0) = 1 \), \( \phi'(0) > 0 \) and \( \phi \) maps \( U \) onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike and Ma-Minda convex functions consists of functions \( f \in A \) satisfying the subordination \( zf_0'(z)/f(z) < \phi(z) \) and \( 1 + zf''(z)/f(z) < \phi(z) \), respectively. These classes denoted, respectively, by \( S^* (\phi) \) and \( C(\phi) \).

An analytic function \( f \in S \) is said to be bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both \( f \) and \( f^{-1} \) are, respectively, Ma-Minda starlike or Ma-Minda convex functions. These classes are denoted, respectively, by \( S^*_\Sigma (\phi) \) and \( C^*_\Sigma (\phi) \). In the sequel, it is assumed that \( \phi \) is an analytic function with positive real part in \( U \), satisfying \( \phi(0) = 1 \), \( \phi'(0) > 0 \) and \( \phi(U) \) is starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the following form:

\[
\phi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots, \quad b_1 > 0.
\]  

(1.2)

A function \( f \in A \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent. Let \( \Sigma \) denote the class of bi-univalent functions in \( U \) given by (1.1).

Examples of functions in the class \( \Sigma \) are

\[
\frac{z}{1-z}, \quad \ln \frac{1}{1-z}, \quad \ln \sqrt{\frac{1+z}{1-z}}.
\]

However, the familiar Koebe function is not a member of \( \Sigma \). Other common examples of functions in \( A \) such as

\[
\frac{2z - z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}
\]

are also not members of \( \Sigma \).

Earlier, Brannan and Taha [3] introduced certain subclasses of bi-univalent function class \( \Sigma \), namely bi-starlike function of order \( \alpha \) denoted \( S^*_\Sigma (\alpha) \) and bi-convex function of order \( \alpha \) denoted \( C^*_\Sigma (\alpha) \) corresponding to the function classes \( S^*(\alpha) \) and \( C(\alpha) \), respectively. Thus, following Brannan and Taha [3], a function \( f \in \Sigma \) is in the classes \( S^*_\Sigma (\alpha) \) and \( C^*_\Sigma (\alpha) \), respectively, if each of the following conditions is satisfied:

\[
\text{Re} \left( \frac{zf''(z)}{f(z)} \right) > \alpha, \ z \in U, \ \text{Re} \left( \frac{zg'(w)}{g(w)} \right) > \alpha, \ w \in D
\]
and \[ \text{Re} \left( 1 + \frac{zf'(z)}{f(z)} \right) > \alpha, z \in U, \quad \text{Re} \left( 1 + \frac{zg'(w)}{g(w)} \right) > \alpha, w \in D. \]

For each of the function classes \( S^*_2(\alpha) \) and \( C^*_2(\alpha) \), they found non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \).

Lewin [11] investigated bi-univalent function class and showed that \( |a_2| < 1.51 \). Subsequently, Brannan and Clunie [2] conjectured that \( |a_2| < \sqrt{2} \).

For a brief history and interesting examples of functions which are in the class, together with various other properties of this bi-univalent function class, one can refer the work of Srivastava et al. [22] and references therein. In [22], Srivastava et al. reviewed the study of coefficient problems for bi-univalent functions. Also, various subclasses of bi-univalent function class were introduced and non-sharp estimates on the first two coefficients in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example, [1, 4, 5, 6, 7, 8, 9, 13, 15, 19, 21, 23, 24, 25, 27, 28]. Recently, Orhan et al. [17] reviewed the study of coefficient problems for the subclass \( NP^{\beta\lambda}_2(\beta, h) \) of bi-univalent functions.

However, the problem to find the coefficient bounds on \( |a_n|, n = 3, 4, \ldots \) for functions \( f \in \Sigma \) is presumably still an open problem (see, for example [2, 11, 16]).

Inspired by the aforementioned works, we define a subclass of \( \Sigma \) as follows.

**Definition 1.1.** A function \( f \in \Sigma \) given by (1.1) is said to be in the class \( M^*_2(\beta, \phi) \), \( \beta \geq 0 \), where \( \phi \) is an analytic function given by (1.2), if the following conditions are satisfied:

\[
\left( \frac{zf'(z)}{f(z)} \right)^\beta \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\beta} \prec \phi(z), \quad z \in U,
\]
\[
\left( \frac{zg'(w)}{g(w)} \right)^\beta \left( 1 + \frac{zg''(w)}{g'(w)} \right)^{1-\beta} \prec \phi(w), \quad w \in D,
\]

where \( g = f^{-1} \).

**Remark 1.2.** Taking \( \beta = 1 \), we have \( M^*_2(\phi, 1) = S^*_2(\phi) \); that is, \( \frac{zf'(z)}{f(z)} \prec \phi(z), z \in U \) and \( \frac{zg'(w)}{g(w)} \prec \phi(w), w \in D \) if and only if \( f \in S^*_2(\phi) \), where \( g = f^{-1} \).

**Remark 1.3.** Taking \( \beta = 0 \), we have \( M^*_2(\phi, 0) = C^*_2(\phi) \); that is, \( 1 + \frac{zf'(z)}{f(z)} \prec \phi(z), z \in U \) and \( 1 + \frac{zg'(w)}{g(w)} \prec \phi(w), w \in D \) if and only if \( f \in C^*_2(\phi) \), where \( g = f^{-1} \).

**Remark 1.4.** These classes \( S^*_2(\phi) \) and \( C^*_2(\phi) \) were investigated by Ma and Minda [12].

The object of this paper is to introduce a new subclass \( M^*_2(\phi, \beta) \) of the function class \( \Sigma \) that is wider (respect to \( \beta \)) to the subclasses examined so far and to find
estimates on the first three Taylor-Maclaurin coefficients $|a_2|, |a_3|$ and $|a_4|$ for the functions in this class.

To prove our main results, we have to recall the following well-known Lemma [18].

**Lemma 1.5.** Let $P$ be the class of all analytic functions $p(z)$ of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

satisfying $\Re(p(z)) > 0$, $z \in U$ and $p(0) = 1$. Then,

$$2p_2 = p_1^2 + (4 - p_1^2) x,$$

$$4p_4 = p_1^4 + 2 \left( 4 - p_1^2 \right) p_1 x - (4 - p_1^2) p_1 x^2 + 2 \left( 4 - p_1^2 \right) \left( 1 - |x|^2 \right) z,$$

for some $x$, $z$ with $|x| \leq 1$, $|z| \leq 1$ and $p_1 \in [0, 2]$.

**2. Coefficient bounds for the function class $M_\Sigma(\phi, \beta)$**

In this section, we will try to find the estimates on the coefficients $|a_2|, |a_3|$ and $|a_4|$ for the functions in the class $M_\Sigma(\phi, \beta)$.

**Theorem 2.1.** Let the function $f(z)$ given by (1.1) be in the class $M_\Sigma(\phi, \beta)$, $\beta \in [0, 1]$, where $\phi$ is an analytic function given by (1.2). Then,

$$|a_2| \leq \frac{b_1}{2 - \beta}, |a_3| \leq \begin{cases} \frac{b_1^3}{(2 - \beta)^3}, & \text{if } b_1 \leq \frac{(2 - \beta)^2}{2(3 - 2\beta)}, \\ \frac{b_1}{2(3 - 2\beta)}, & \text{if } b_1 > \frac{(2 - \beta)^2}{2(3 - 2\beta)} \end{cases},$$

and

$$|a_4| \leq \min \left\{ \frac{b_1^3 \varphi(\beta) - 6(2 - \beta)^3 \Lambda + 6(2 - \beta)^3 |2b_2 - b_1|}{18(2 - \beta)^3(4 - 3\beta)} b_1, \frac{b_1}{3(4 - 3\beta)} \right\},$$

where $\varphi(\beta) = \beta^3 - 3\beta^2 - 46\beta + 60 > 0$ and $\Lambda = \Lambda(b_1, b_2, b_3) = b_1 - 2b_2 + b_3$.

**Proof.** Let $f \in M_\Sigma(\phi, \beta)$, $\beta \in [0, 1]$, where $\phi$ is an analytic function given by (1.2) and $g = f^{-1}$. Then, there are analytic functions $u : U \to U$, $v : D \to D$ with $u(0) = 0 = v(0)$, $|u(z)| < 1$, $|v(w)| < 1$ and satisfying

$$\left( \frac{z f'(z)}{f(z)} \right)^{\beta} \left( 1 + \frac{z f''(z)}{f'(z)} \right)^{1-\beta} = \phi(u(z))$$

and

$$\left( \frac{w g'(w)}{g(w)} \right)^{\beta} \left( 1 + \frac{w g''(w)}{g'(w)} \right)^{1-\beta} = \phi(v(w)). \quad (2.1)$$

Let us define the functions $p(z)$ and $q(w)$ by

$$p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in U$$

and

$$q(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + \sum_{n=1}^{\infty} q_n w^n, \quad w \in D.$$
Using (2.2) and (2.3) in (1.2), we can easily write

\[ u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left\{ p_1 z + \left[p_2 - \frac{p_1^2}{2}\right] z^2 + \left[p_3 - p_1 p_2 + \frac{p_1^3}{4}\right] z^3 + \cdots \right\} \] (2.2)

and

\[ v(w) = \frac{q(w) - 1}{q(w) + 1} = \frac{1}{2} \left\{ q_1 w + \left[q_2 - \frac{q_1^2}{2}\right] w^2 + \left[q_3 - q_1 q_2 + \frac{q_1^3}{4}\right] w^3 + \cdots \right\} \] (2.3)

Using (2.2) and (2.3) in (1.2), we can easily write

\[ \phi(u(z)) = 1 + \frac{b_1 p_1}{2} z + \left[\frac{b_2}{2} \left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4} b_2 p_1^2\right] z^2 + \left[\frac{b_3}{2} \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) + \frac{b_3 p_1}{2} \left(p_2 - \frac{p_1^2}{2}\right) + \frac{b_3 p_1^3}{8}\right] z^3 + \cdots \] (2.4)

and

\[ \phi(v(w)) = 1 + \frac{b_1 q_1}{2} w + \left[\frac{b_2}{2} \left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4} b_2 q_1^2\right] w^2 + \left[\frac{b_3}{2} \left(q_3 - q_1 q_2 + \frac{q_1^3}{4}\right) + \frac{b_3 q_1}{2} \left(q_2 - \frac{q_1^2}{2}\right) + \frac{b_3 q_1^3}{8}\right] w^3 + \cdots \] (2.5)

Also, using (2.4) and (2.5) in (2.1) and equating the coefficients, we get

\[ (2 - \beta) a_2 = \frac{b_1 p_1}{2}, \] (2.6)

\[ 2 (3 - 2\beta) a_3 + \frac{1}{2} (\beta^2 + 5\beta - 8) a_2 = \frac{b_1}{2} \left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4} b_2 p_1^2, \] (2.7)

\[ 3 (4 - 3\beta) a_4 + (4\beta^2 + 11\beta - 18) a_2 a_3 - \frac{1}{6} (\beta^3 + 21\beta^2 + 20\beta - 48) a_3^2 = \frac{b_1}{2} \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) + \frac{b_3 p_1}{2} \left(p_2 - \frac{p_1^2}{2}\right) + \frac{b_3 p_1^3}{8} \] (2.8)

and

\[ -(2 - \beta) a_2 = \frac{b_1 q_1}{2}, \] (2.9)

\[ -2 (3 - 2\beta) a_3 + \frac{1}{2} (\beta^2 - 11\beta + 16) a_2^2 = \frac{b_1}{2} \left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4} b_2 q_1^2, \] (2.10)

\[ -3 (4 - 3\beta) a_4 + (4\beta^2 - 34\beta + 42) a_2 a_3 + \frac{1}{6} (\beta^3 - 27\beta^2 + 158\beta - 192) a_3^2 = \frac{b_1}{2} \left(q_3 - q_1 q_2 + \frac{q_1^3}{4}\right) + \frac{b_3 q_1}{2} \left(q_2 - \frac{q_1^2}{2}\right) + \frac{b_3 q_1^3}{8} \] (2.11)

From (2.6) and (2.9), we have

\[ a_2 = \frac{b_1 p_1}{2 (2 - \beta)} = \frac{-b_1 q_1}{2 (2 - \beta)}, \] (2.12)

which is equivalent to

\[ p_1 = -q_1. \] (2.13)

By subtracting from (2.7) to (2.10) and considering (2.12) and (2.13), we can easily obtain
On the other hand, subtracting (2.11) from (2.8) and considering (2.12) and (2.14), we get

\[ a_4 = \frac{b_1 p_4}{4 (2 - \beta)^2} + \frac{b_1 (p_2 - q_2)}{8 (3 - 2\beta)}. \]  

(2.14)

where \( \beta = \theta_3 - 3 \beta^2 - 46 \beta + 60 > 0 \) and \( \Lambda = \Lambda (b_1, b_2, b_3) = b_1 - 2b_2 + b_3. \)

Since \( p_1 = -q_1, \) according to Lemma 1.5 we write

\[ p_2 - q_2 = \frac{4 - p_1^2}{2} (x - y), \quad p_2 + q_2 = p_1^2 + \frac{4 - p_1^2}{2} (x + y) \]  

(2.16)

and

\[ p_3 - q_3 = \frac{p_3^3}{2} + \frac{p_1 (4 - p_1^2)}{4} (x + y) - \frac{p_1 (4 - p_1^2)}{16} \left[ (1 - |x|^2) z - (1 - |y|^2) w \right]. \]  

(2.17)

for some \( x, y, z, w \) with \( |x| \leq 1, |y| \leq 1, |z| \leq 1, |w| \leq 1. \) In this case, since \( p_1 \in [0, 2], \) we may assume without any restriction that \( t \in [0, 2], \) where \( t = |p_1|. \)

Hence, we find from (2.12) that

\[ |a_3| \geq \frac{b_1}{2 - \beta}. \]

Applying triangle inequality on the last equation and taking \( \xi = |x|, \eta = |y|, \) we have

\[ |a_3| \leq c_1(t) + c_2(t) (\xi + \eta), \]  

(2.18)

where

\[ c_1(t) = \frac{b_1 t^2}{4 (2 - \beta)^2} \geq 0, \quad c_2(t) = \frac{b_1 (4 - t^2)}{16 (3 - 2\beta)} \geq 0, \quad t \in [0, 2]. \]

Let us define the function \( F : \mathbb{R}^3 \rightarrow \mathbb{R} \) as follows:

\[ F(\xi, \eta, t) = c_1(t) + c_2(t) (\xi + \eta), \quad (\xi, \eta) \in \Omega, \quad t \in [0, 2], \]  

(2.19)

where \( \Omega = \{(\xi, \eta) : \xi, \eta \in [0, 1]\}. \)

From (2.18) and (2.19), we can write
\[ |a_3| \leq \min \\{ \max \{ F(\xi, \eta, t) : (\xi, \eta) \in \Omega \} : t \in [0, 2] \} . \] (2.20)

We can easily show that
\[ \max \{ F(\xi, \eta, t) : (\xi, \eta) \in \Omega \} = F(1, 1, t) = c_1(t) + 2c_2(t), \quad t \in [0, 2] . \] (2.21)

Now, let us define the function \( H : \mathbb{R} \to \mathbb{R} \) as follows:
\[ H(t) = c_1(t) + 2c_2(t), \quad t \in [0, 2] . \]

Substituting the value of \( c_1(t) \) and \( c_2(t) \) in the above function, we have
\[ H(t) = \frac{b_1}{2(3-2\beta)} + \frac{\Delta(\beta, b_1)}{8(3-2\beta)(2-\beta)^2} t^2, \] (2.22)
where \( \Delta(\beta, b_1) = 2(3-2\beta)b_1^2 - (2-\beta)^2b_1 \).

Differentiating both sides of (2.22), we get
\[ H'(t) = \frac{\Delta(\beta, b_1)}{4(3-2\beta)(2-\beta)^2} t. \]

It is clear that \( H'(t) \leq 0 \) if \( 0 < b_1 \leq \frac{(2-\beta)^2}{2(3-2\beta)} \); that is, \( H(t) \) is a decreasing function. Therefore,
\[ \min \{ H(t) : t \in [0, 2] \} = H(2) = \frac{b_1^2}{(2-\beta)^2}. \] (2.23)

Let \( b_1 > \frac{(2-\beta)^2}{2(3-2\beta)} \), then \( H'(t) > 0 \), so \( H(t) \) is a strictly increasing function. Therefore,
\[ \min \{ H(t) : t \in [0, 2] \} = H(0) = \frac{b_1}{2(3-2\beta)}. \] (2.24)

Consequently, from (2.21), (2.24) and (2.20), we have
\[ |a_3| \leq \begin{cases} \frac{b_1^2}{(2-\beta)^2}, & \text{if } b_1 \leq \frac{(2-\beta)^2}{2(3-2\beta)}, \\ \frac{b_1}{2(3-2\beta)}, & \text{if } b_1 > \frac{(2-\beta)^2}{2(3-2\beta)}. \end{cases} \] (2.25)

Substituting the expressions (2.16) and (2.17) in (2.15), we obtain
\[ a_4 = \frac{b_1(4-\rho_1^2)}{24(4-3\beta)} \left[ (1 - |x|^2) z - (1 - |y|^2) w - \frac{b_1(4-\rho_1^2)p_1}{48(4-3\beta)} (x^2 + y^2) \right] + \frac{b_2(4-\rho_1^2)p_1}{24(4-3\beta)} (x + y) + \frac{5b_2(4-\rho_1^2)p_1}{64(2-\beta)(3-2\beta)} (x - y) + \frac{b_2(4-\rho_1^2)p_1}{144(2-\beta)^2(4-3\beta)} (x^2 + y^2). \]

Applying triangle inequality on the last equation, we have
where

\[ d_1(t) = \frac{b_1 (4 - t^2) (t - 2)}{48 (4 - 3\beta)} \leq 0, \]

\[ d_2(t) = \frac{(4 - t^2) t [8 |b_2| (2 - \beta) (3 - 2\beta) + 15b_1^2 (4 - 3\beta)]}{192 (2 - \beta) (3 - 2\beta) (4 - 3\beta)} \geq 0, \]

\[ d_3(t) = \frac{b_3 \varphi(\beta) - 6 (2 - \beta)^3 A}{144 (2 - \beta)^3 (4 - 3\beta)} t^3 + \frac{b_1 (4 - t^2)}{12 (4 - 3\beta)} \geq 0. \]

Let us define the function \( G : \mathbb{R}^3 \to \mathbb{R} \) as follows:

\[ G(\xi, \eta, t) = d_1(t) (\xi^2 + \eta^2) + d_2(t) (\xi + \eta) + d_3(t), \quad (\xi, \eta) \in \Omega, \quad t \in [0, 2]. \]  \hspace{1cm} (2.27)

From (2.26) and (2.27), we can write

\[ |a_4| \leq \min \{ \max \{ G(\xi, \eta, t) : (\xi, \eta) \in \Omega \} : t \in [0, 2] \}. \]  \hspace{1cm} (2.28)

Firstly, we need investigate maximum of the function \( G(\xi, \eta, t) \) on the closed square \( \Omega \) for each \( t \in [0, 2] \). Since the coefficients of the function \( G(\xi, \eta, t) \) is dependent to variable \( t \), we must investigate this maximum respect to \( t \) taking into account these cases: \( t = 0, \ t \in (0, 2) \) and \( t = 2 \).

For \( t = 0 \) we have

\[ G_0(\xi, \eta) = G(\xi, \eta, 0) = \frac{-b_1}{6 (4 - 3\beta)} (\xi^2 + \eta^2) + \frac{b_1}{3 (4 - 3\beta)}, \quad (\xi, \eta) \in \Omega. \]

We can easily show that the maximum of the function \( G_0(\xi, \eta) \) occurs at \( (\xi, \eta) = (0, 0) \), and

\[ \max \{ G_0(\xi, \eta) : (\xi, \eta) \in \Omega \} = G_0(0, 0) = \frac{b_1}{3 (4 - 3\beta)}. \]  \hspace{1cm} (2.29)

In the case \( t \in (0, 2) \), by simple differentiation, we get

\[ G_x(\xi, \eta, t) = 2d_1(t) \xi + d_2(t), \quad G_\eta(\xi, \eta, t) = 2d_1(t) \eta + d_2(t), \]

\[ G_{xx}(\xi, \eta, t) = G_{xx}(\xi, \eta, t) = 2d_1(t), \quad G_{x\eta}(\xi, \eta, t) = G_{x\eta}(\xi, \eta, t) = 0. \]

From the first and second equations above, we see that \( (\xi_0, \eta_0) \), where \( \xi_0 = \eta_0 = \frac{-d_2(t)}{2d_1(t)} \), is critical and likely a extremal point for of the function \( G(\xi, \eta, t) \).

Since

\[ \Delta(\xi_0, \eta_0) = G_{xx}(\xi_0, \eta_0) G_{xx}(\xi_0, \eta_0) - \left[ G_{x\xi}(\xi_0, \eta_0) \right]^2 = 4d_1^2(t) > 0 \]
and \( G''_{\xi} (\xi, \eta, t) = G''_{\eta} (\xi, \eta, t) = 2d_1 (t) < 0 \), \((\xi_0, \eta_0)\) is a likely maximum point for the function \( G(\xi, \eta, t) \). But, it is clear that \((\xi_0, \eta_0)\) is not a local maximum point if \(-\frac{d_2(t)}{2d_1(t)} > 0\); that is if \((\xi_0, \eta_0) \notin \Omega\). We assume that \((\xi_0, \eta_0) \in \Omega\). In this case \((\xi_0, \eta_0)\) is a local maximum point for the function \( G(\xi, \eta, t) \).

Therefore,

\[
\max \{ G(\xi, \eta, t) : (\xi, \eta) \in \Omega \} = G(\xi_0, \eta_0, t) = d_3 (t) - \frac{d_2^2 (t)}{2d_1 (t)}.
\]

Let us define the function \( h : \mathbb{R} \to \mathbb{R} \) by

\[
h(t) = d_3 (t) - \frac{d_2^2 (t)}{2d_1 (t)}, \quad t \in (0, 2).
\]

Substituting the value \( d_1(t), d_2(t) \) and \( d_3(t) \) in the above function, we have

\[
h(t) = h_1 t^3 + h_2 t^2 + h_3, \quad t \in (0, 2),
\]

where

\[
\begin{align*}
h_1 &= \frac{\vert b_1 \vert \beta (3 - 2 \beta)^3 A + 6 (2 - \beta)^3 |b_2|}{144 (2 - \beta)^3 (3 - 3 \beta)} \nonumber \\
&\quad + \frac{1536 (2 - \beta)^3 (3 - 2 \beta)^2 (4 - 3 \beta) b_1}{4 (3 - 3 \beta) b_1}, \quad h_1 > 0, \\
h_2 &= \frac{8 \vert b_2 \vert (2 - \beta) (3 - 2 \beta) + 15 b_1^2 (4 - 3 \beta)^2}{768 (2 - \beta)^3 (3 - 2 \beta)^2 (4 - 3 \beta) b_1} - \frac{b_1}{12 (4 - 3 \beta)}, \\
h_3 &= \frac{b_1}{3 (4 - 3 \beta)} > 0.
\end{align*}
\]

Also, we consider the function \( \bar{h} : \mathbb{R} \to \mathbb{R} \) as follows:

\[
\bar{h}(t) = h_1 t^3 + h_2 t^2 + h_3, \quad t \in (0, 2),
\]

where

\[
\bar{h}_2 = h_2 + \frac{b_1}{12 (4 - 3 \beta)} = \frac{8 \vert b_2 \vert (2 - \beta) (3 - 2 \beta) + 15 b_1^2 (4 - 3 \beta)^2}{768 (2 - \beta)^3 (3 - 2 \beta)^2 (4 - 3 \beta) b_1} > 0.
\]

Since \( h(t) < \bar{h}(t) \) for all \( t \in (0, 2) \), we can write

\[
\min \{ h(t) : t \in (0, 2) \} \leq \min \{ \bar{h}(t) : t \in (0, 2) \}.
\]

Now, we will investigate minimum of the function \( \bar{h}(t) \) on the open interval \((0, 2)\).

Differentiating both sides of \((2.31)\), we have

\[
\bar{h}'(t) = (3h_1 t + 2h_2) t, \quad t \in (0, 2).
\]

Since \( h_1 > 0, \bar{h}_2 > 0 \), the function \( \bar{h}(t) \) is a strictly increasing function on \((0, 2)\).

Therefore,

\[
\min \{ \bar{h}(t) : t \in (0, 2) \} = \bar{h}(0) = \lim_{t \to 0+} \bar{h}(t) = \frac{b_1}{3 (4 - 3 \beta)}.
\]
Finally, let $t = 2$. In this case the function $G(\xi, \eta, 2)$ is a constant as follows:

$$G_2(\xi, \eta) = G(\xi, \eta, 2) = d_3(2) = \frac{b_1^3 \varphi(\beta) - 6(2 - \beta)^3 \Lambda + 6(2 - \beta)^3 |2b_1 - b_2|}{18(2 - \beta)^3(4 - 3\beta)} \quad (2.34).$$

Thus, from (2.29)-(2.34) and (2.28), we obtain

$$|a_4| \leq \min \left\{ \left| \frac{b_1^3 \varphi(\beta) - 6(2 - \beta)^3 \Lambda + 6(2 - \beta)^3 |2b_1 - b_2|}{18(2 - \beta)^3(4 - 3\beta)} \right| \frac{b_1}{3(4 - 3\beta)} \right\}.$$

With this, the proof of Theorem 2.1 is completed.

The following theorems are direct results of Theorem 2.1.

**Theorem 2.2.** Let the function $f(z)$ given by (1.1) be in the class $S^\omega_\Sigma(\phi)$, where $\phi$ is an analytic function given by (1.2). Then,

$$|a_2| \leq b_1, |a_3| \leq \begin{cases} b_1^2, & \text{if } b_1 \leq \frac{1}{2}, \\ b_1, & \text{if } b_1 > \frac{1}{2} \end{cases}$$

and

$$|a_4| \leq \min \left\{ \left| \frac{2b_1^3 - \Lambda}{3} + |2b_1 - b_2| \right| \frac{b_1}{3} \right\},$$

where $\Lambda = \Lambda(b_1, b_2, b_3) = b_1 - 2b_2 + b_3$.

**Theorem 2.3.** Let the function $f(z)$ given by (1.1) be in the class $C^\omega_\Sigma(\phi)$, where $\phi$ is an analytic function given by (1.2). Then,

$$|a_2| \leq \frac{b_1}{2}, |a_3| \leq \begin{cases} b_1^2, & \text{if } b_1 \leq \frac{2}{3}, \\ b_1^2, & \text{if } b_1 > \frac{2}{3} \end{cases}$$

and

$$|a_4| \leq \min \left\{ \left| \frac{5b_1^3 - 4\Lambda}{48} + 4|2b_2 - b_1| \right| \frac{b_1}{12} \right\},$$

where $\Lambda = \Lambda(b_1, b_2, b_3) = b_1 - 2b_2 + b_3$.

3. **Concluding remarks**

If the function $\phi(z)$, aforementioned in study, is given by

$$\phi(z) = \frac{1 + az}{1 + bz} = 1 + (a - b)z - b(a - b)z^2 + b^2(a - b)z^3 + \cdots \quad (-1 \leq b < a \leq 1), \quad (3.1)$$

then $b_1 = (a - b)$, $b_2 = -b(a - b)$ and $b_3 = b^2(a - b)$. 


Taking $a = 1 - 2\alpha$, $b = -1$ in (3.1), we have
\[
\phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha)z + 2(1 - \alpha)z^2 + 2(1 - \alpha)z^3 + \cdots \quad (0 \leq \alpha < 1).
\]
(3.2)

Hence, $b_1 = b_2 = b_3 = 2(1 - \alpha)$.

Choosing $\phi(z)$ of the form (3.1) and (3.2) in Theorem 2.1, we can readily deduce the following results, respectively.

**Corollary 3.1.** Let the function $f(z)$ given by (1.1) be in the class $M_\Sigma\left(\frac{1+z}{1-2z}, \beta\right)$ \((-1 \leq b < a \leq 1, 0 \leq \beta \leq 1\). Then,
\[
|a_2| \leq \frac{a-b}{2-\beta}, \quad |a_3| \leq \left\{\begin{array}{ll}
\frac{(a-b)^2}{2(\beta-2)}, & \text{if } a - b \leq \frac{2(\beta-2)}{2(\beta-2)}, \\
\frac{a-b}{2(\beta-2)}, & \text{if } a - b > \frac{2(\beta-2)}{2(\beta-2)},
\end{array}\right.
\]
and $|a_4| \leq \frac{a-b}{3(4-3\beta)}$.

**Corollary 3.2.** Let the function $f(z)$ given by (1.1) be in the class $M_\Sigma\left(\frac{1+2(1-\alpha)z}{1-2z}, \beta\right)$ \(= M_\Sigma(\alpha, \beta), \alpha \in [0,1), \beta \in [0,1]\). Then,
\[
|a_2| \leq \frac{2(1-\alpha)}{2-\beta}, \quad |a_3| \leq \left\{\begin{array}{ll}
\frac{1-\alpha}{2}, & \text{if } 0 \leq \alpha < 1 - \alpha_0, \\
\frac{4(1-\alpha)^2}{(2-\beta)^2}, & \text{if } 1 - \alpha_0 \leq \alpha < 1,
\end{array}\right.
\]
and $|a_4| \leq \frac{2(1-\alpha)}{3(4-3\beta)}$.

Also, taking $\alpha = 0$ in (3.2), we get
\[
\phi(z) = \frac{1+z}{1-2z} = 1 + 2z + 2z^2 + 2z^3 + \cdots .
\]
(3.3)

Hence, $b_1 = b_2 = b_3 = 2$.

Choosing $\phi(z)$ of the form (3.3) in Theorem 2.1, we arrive at the following corollary.

**Corollary 3.3.** Let the function $f(z)$ given by (1.1) be in the class $M_\Sigma\left(\frac{1+z}{1-2z}, \beta\right)$, $\beta \in [0,1]$. Then,
\[
|a_2| \leq \frac{2}{2-\beta}, \quad |a_3| \leq \frac{1}{3-2\beta} \quad \text{and} \quad |a_4| \leq \frac{2}{3(4-3\beta)}.
\]

Choosing $\phi(z)$ of the form (3.1) and (3.2) in Theorem 2.2, we can readily deduce the following results, respectively.

**Corollary 3.4.** Let the function $f(z)$ given by (1.1) be in the class $S_\Sigma^a\left(\frac{1+z}{1+2z}\right)$ \((-1 \leq b < a \leq 1\). Then,
\[
|a_2| \leq a-b, \quad |a_3| \leq \left\{\begin{array}{ll}
\frac{(a-b)^2}{a+b}, & \text{if } a - b \leq \frac{a+b}{2}, \\
\frac{a-b}{2}, & \text{if } a - b > \frac{a+b}{2},
\end{array}\right.
\]
and $|a_4| \leq \frac{a-b}{3}$.

**Corollary 3.5.** Let the function $f(z)$ given by (1.1) be in the class $S_\Sigma^a\left(\frac{1+2(1-\alpha)z}{1-2z}\right)$ \(= S_\Sigma^a(\alpha), \alpha \in [0,1]\). Then,
\[
|a_2| \leq 2(1-\alpha), \quad |a_3| \leq \left\{\begin{array}{ll}
1-\alpha, & \text{if } 0 \leq \alpha < \frac{3}{4}, \\
4(1-\alpha)^2, & \text{if } \frac{3}{4} \leq \alpha < 1 \quad \text{and} \quad |a_4| \leq \frac{2(1-\alpha)}{3}.
\end{array}\right.
\]
Remark 3.6. In the special case, we can also obtain Corollary 3.4 from Corollary 3.1 and Corollary 3.5 from Corollary 3.2 for $\beta = 1$.

Moreover, taking, for example, $\alpha = \frac{3}{4}$ in (3.2), we have

$$\phi(z) = \frac{2 - z}{2(1 - z)} = 1 + \frac{1}{2} z + \frac{1}{2} z^2 + \frac{1}{2} z^3 + \cdots.$$ (3.4)

Hence, $b_1 = b_2 = b_3 = \frac{1}{2}$.

Choosing $\phi(z)$ of the form (3.4) in Theorem 2.2, we arrive at the following corollary.

Corollary 3.7. Let the function $f(z)$ given by (1.1) be in the class $S_{\Sigma} \left( \frac{z}{2(1 - z)} \right)$. Then,

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{4} \quad \text{and} \quad |a_4| \leq \frac{1}{8}.$$ 

Remark 3.8. In the special case, we can also obtain Corollary 3.7 from Corollary 3.5 for $\alpha = \frac{3}{4}$.

Choosing $\phi(z)$ of the form (3.5) in Theorem 2.3, we can readily deduce the following results, respectively.

Corollary 3.9. Let the function $f(z)$ given by (1.1) be in the class $C_{\Sigma} \left( \frac{1 + \alpha z}{1 + \beta z} \right)$ ($-1 \leq b < a \leq 1$). Then,

$$|a_2| \leq \frac{a - b}{2}, \quad |a_3| \leq \begin{cases} \frac{(a - b)^2}{a - b}, & \text{if } a - b \leq \frac{2}{3}, \\ \frac{a - b}{6}, & \text{if } a - b > \frac{2}{3}, \end{cases} \quad \text{and} \quad |a_4| \leq \frac{a - b}{12}.$$ 

Corollary 3.10. Let the function $f(z)$ given by (1.1) be in the class $C_{\Sigma} \left( \frac{1 + (1 - 2\alpha)z}{1 - z} \right)$ = $C_{\Sigma}(\alpha), \; \alpha \in [0, 1)$. Then,

$$|a_2| \leq 1 - \alpha, \quad |a_3| \leq \begin{cases} \frac{1 - \alpha}{3}, & \text{if } 0 \leq \alpha < \frac{2}{5}, \\ (1 - \alpha)^2, & \text{if } \frac{2}{5} \leq \alpha < 1 \quad \text{and} \quad |a_4| \leq \frac{1 - \alpha}{6}. \end{cases}$$

Moreover, taking, for example, $\alpha = \frac{2}{3}$ in (3.2), we get

$$\phi(z) = \frac{3 - z}{3(1 - z)} = 1 + \frac{2}{3} z + \frac{2}{3} z^2 + \frac{2}{3} z^3 + \cdots.$$ (3.5)

Hence, $b_1 = b_2 = b_3 = \frac{2}{3}$.

Choosing $\phi(z)$ of the form (3.5) in Theorem 2.3, we arrive at the following corollary.

Corollary 3.11. Let the function $f(z)$ given by (1.1) be in the class $C_{\Sigma} \left( \frac{3 - z}{3(1 - z)} \right)$. Then,

$$|a_2| \leq \frac{1}{5}, \quad |a_3| \leq \frac{1}{5} \quad \text{and} \quad |a_4| \leq \frac{1}{15}.$$ 

Remark 3.12. In the special case, we can also obtain Corollary 3.11 from Corollary 3.10 for $\alpha = \frac{2}{3}$.
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