NOTES ON $C_0$-REPRESENTATIONS AND THE HAAGERUP PROPERTY

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Abstract. For any locally compact group $G$, we show the existence and uniqueness up to quasi-equivalence of a unitary $C_0$-representation $\pi_0$ of $G$ such that the coefficient functions of $C_0$-representations of $G$ are exactly the coefficient functions of $\pi_0$. The present work, strongly influenced by [4] (which dealt exclusively with discrete groups), leads to new characterizations of the Haagerup property: $G$ has that property if and only if the representation $\pi_0$ induces a $*$-isomorphism of $C^*(G)$ onto $C^*_{\pi_0}(G)$. When $G$ is discrete and countable, we also relate the Haagerup property to relative strong mixing properties in the sense of [9] of the group von Neumann algebra $L(G)$ into finite von Neumann algebras.

1. Introduction

Throughout this article, $G$ denotes a locally compact group. We associate to $G$ a unitary representation $(\pi_0, H_0)$ which has the following properties:

- it is a $C_0$-representation: every coefficient function $s \mapsto \langle \pi_0(s) \xi | \eta \rangle$ associated with $\pi_0$ tends to 0 as $s \to \infty$;
- the coefficient functions of $\pi_0$ are exactly the coefficient functions of $C_0$-representations of $G$;
- the representation $\pi_0$ is the unique $C_0$-representation, up to quasi-equivalence, which satisfies the above properties.

The key idea is to use G. Arsac’s notion of $A_\pi$-spaces from [1]. Using the same arguments as in Theorem 3.2 and Corollary 3.4 of [4], we deduce that:

Proposition A. Let $G$ be a group as above. Then it has the Haagerup property if and only if the maximal $C^*$-algebra $C^*(G)$ is $*$-isomorphic to the $C^*$-algebra $C^*_{\pi_0}(G)$.

The preceding proposition deserves a comment which we owe to A. Valette: the Haagerup property of a group $G$ is exactly property $C_0$ in the sense of V. Bergelson and J. Rosenblatt in Definition 2.4 of [3]. Moreover, Theorem 2.5 of the same article states the density of $C_0$-representations in the set of all (classes of) unitary representations on a fixed Hilbert space, and this suffices to prove that there is a $C_0$-representation whose extention to the maximal $C^*$-algebra $C^*(G)$ is faithful.

Date: May 11, 2014.

2010 Mathematics Subject Classification. Primary 22D10, 22D25; Secondary 46L10.

Key words and phrases. Locally compact groups, unitary representations, $C^*$-algebras, von Neumann algebras, strong mixing, Følner sequences.
In the last part of the present notes, we assume that $G$ is discrete and countable. We relate the Haagerup property of $G$ to the embedding of its von Neumann algebra $L(G)$ as a strongly mixing subalgebra of some finite von Neumann algebra $M$ in the sense of [9]: this means that, for all $x, y \in M$ such that $E_{L(G)}(x) = E_{L(G)}(y) = 0$ and for any sequence of unitary operators $(u_n) \subset L(G)$ which converges weakly to 0, one has
\[
\lim_{n \to \infty} \|E_{L(G)}(xu_ny)\|_2 = 0.
\]

In Section 3, we prove the following result which uses some results from Chapter 2 of [5]:

**Theorem B.** Let $G$ be an infinite, countable group. Then it has the Haagerup property if and only if $L(G)$ can be embedded into some finite von Neumann algebra $M$ in such a way that $L(G)$ is strongly mixing in $M$ and that there is a sequence of elements $(x_k)_{k \geq 1} \subset M \ominus L(G)$ such that $\|x_k\|_2 = 1$ for every $k$, and
\[
\lim_{k \to \infty} \|\lambda(g)x_k - x_k\lambda(g)\|_2 = 0
\]
for every $g \in G$.

**Acknowledgements.** We warmly thank A. Valette for his comment about Bergelson and Rosenblatt result mentioned above, and the referee for having detected separability problems in a previous version of the present article and for many valuable comments.

## 2. An enveloping $C_0$-representation

In order to give precise statements of our results, we need to recall some notations and facts on spaces of coefficient functions of unitary representations ($A_\pi$-spaces of G. Arsac) from [1] and from P. Eymard’s article [7].

The Banach algebra of all continuous functions on $G$ which tend to 0 at infinity is denoted by $C_0(G)$, and its dense subalgebra formed by all continuous functions with compact support is denoted by $K(G)$.

Let $(\pi, H)$ be a unitary representation of $G$. If $\xi, \eta \in H$, we denote by
\[
\xi *_{\pi} \bar{\eta}(s) = \langle \pi(s)\xi | \eta \rangle \quad (s \in G)
\]
the coefficient function associated to $\xi$ and $\eta$. These functions are denoted by $\xi *_{\pi} \eta$ in [1] for instance, but our notation reminds the fact that $\xi *_{\pi} \bar{\eta}$ is linear in $\xi$ and antilinear in $\eta$.

A representation $(\pi, H)$ of $G$ is a $C_0$-representation if, for all $\xi, \eta \in H$, the associated coefficient function $\xi *_{\pi} \bar{\eta}$ belongs to $C_0(G)$.

The Fourier-Stieltjes algebra is the set of all coefficient functions as above. It is denoted by $B(G)$ ([7]).

Recall that $B(G)$ is a Banach algebra with respect to the norm
\[
\|\varphi\|_B = \inf\{\|\xi\|\|\eta\| : \varphi = \xi *_{\pi} \bar{\eta}\}.
\]
It is the dual space of the enveloping $C^*$-algebra $C^*(G)$ under the duality bracket defined on the dense $*$-subalgebra $K(G)$ by
\[
\langle \varphi, f \rangle = \int_G \varphi(s)f(s)ds \quad \forall \varphi \in B(G), \ f \in K(G).
\]
Every unitary representation \((\pi, H)\) of \(G\) gives rise to a natural \(*\)-homomorphism, still denoted by \(\pi\), from \(C^*(G)\) onto \(C^*_\pi(G)\), which extends the map \(f \mapsto \pi(f)\) defined on \(K(G)\). (Recall that \(C^*_\pi(G)\) is the \(C^*\)-algebra generated by \(\{\pi(f) : f \in K(G)\}\).)

If \(E(G)\) is any subset of \(B(G)\), we set
\[
E_1(G) = \{\varphi \in E(G) : \|\varphi\|_B = 1\}
\]
the intersection with the unit sphere of \(B(G)\).

A continuous function \(\varphi : G \to \mathbb{C}\) is \emph{positive definite} if, for all \(s_1, \ldots, s_n \in G\) and all \(t_1, \ldots, t_n \in \mathbb{C}\), one has
\[
\sum_{i,j=1}^{n} \bar{t}_i t_j \varphi(s_i^{-1}s_j) \geq 0.
\]
We denote by \(P(G)\) the set of all positive definite functions on \(G\). For instance, every coefficient function \(\xi \ast \pi \tilde{\xi}\) is positive definite, and, conversely, for every \(\varphi \in P(G)\), there exists a unique (up to unitary equivalence) triple \((\pi_\varphi, H_\varphi, \xi_\varphi)\) where \((\pi_\varphi, H_\varphi)\) is a unitary representation of \(G\) and \(\xi_\varphi\) is a cyclic vector for \(\pi_\varphi\) that satisfies
\[
\varphi = \xi_\varphi \ast \pi_\varphi \tilde{\xi}_\varphi.
\]
We recall that \(\|\varphi\|_B = \varphi(1)\) for every positive definite function \(\varphi\).

If \(\varphi \in B(G)\), the \emph{adjoint} \(\varphi^*\) of \(\varphi\) is defined by \(\varphi^*(s) = \overline{\varphi(s^{-1})}\) for every \(s \in G\). We say that \(\varphi\) is \emph{selfadjoint} if \(\varphi^* = \varphi\) and we denote by \(B_{sa}(G)\) the real Banach algebra of all selfadjoint elements of \(B(G)\). Every element \(\varphi \in B_{sa}(G)\) admits a unique decomposition, called \emph{Jordan decomposition}, as
\[
\varphi = \varphi^+ - \varphi^-
\]
where \(\varphi^\pm \in P(G)\) and \(\|\varphi\|_B = \|\varphi^+\|_B + \|\varphi^-\|_B\). Thus \(B_{sa}(G) = P(G) - P(G)\).

The obvious decomposition of any \(\psi \in B(G)\)
\[
\psi = \frac{1}{2}(\psi + \psi^*) + i \cdot \frac{1}{2i}(\psi - \psi^*)
\]
and the Jordan decomposition imply that
\[
B(G) = P(G) - P(G) + iP(G) - iP(G).
\]

We also need to recall the definition and a few facts on \(A_\pi\)-spaces in the sense of G. Arsac [1] since they play an important role in the present notes. If \((\pi, H)\) is a unitary representation of \(G\), \(A_\pi(G)\) is the norm closed subspace of \(B(G)\) generated by the coefficient functions \(\xi \ast \pi \tilde{\eta}\) of \(\pi\). Every element \(\varphi \in A_\pi(G)\) can be written as
\[
\varphi = \sum_n \xi_n \ast \pi \tilde{\eta}_n
\]
where \(\xi_n, \eta_n \in H\) for every \(n\), \(\sum_n \|\xi_n\| \|\eta_n\| < \infty\), and where
\[
\|\varphi\|_B = \inf\{\sum_n \|\xi_n\| \|\eta_n\| : \varphi = \sum_n \xi_n \ast \pi \tilde{\eta}_n\}.
\]
The Banach space \(A_\pi(G)\) identifies with the predual of the von Neumann algebra \(L_\pi(G) := \pi(G)^\prime \subset B(H)\) under the duality bracket
\[
\langle \varphi, \pi(f) \rangle = \int_G \varphi(g)f(g)dg
\]
for every \( \varphi \in A_\pi(G) \) and every \( f \in K(G) \).

As is usually the case, \( \lambda \) denotes the left regular representation of \( G \), and \( L(G) = L_\lambda(G) \) is its associated von Neumann algebra. In this case, \( A(G) = A_\lambda(G) \) is the Fourier algebra of \( G \) \cite{7}, Chapter 3.

If \( M \) is a von Neumann algebra, its predual is denoted by \( M_* \), and if \( \varphi \in M_* \) and \( a \in M \), we define \( a\varphi \) and \( \varphi a \in M_* \) by
\[
\langle a\varphi, x \rangle = \langle \varphi, xa \rangle \quad \text{and} \quad \langle \varphi a, x \rangle = \langle \varphi, ax \rangle \quad \forall x \in M.
\]
Hence, one has \( \varphi \), \( \lambda \), \( L \), \( \supseteq \), \( \Pi \), \( \Sigma \), \( \emptyset \), \( \subset \), \( \subsetneq \), \( \pi \), \( \lambda \)

\[
\sum_{n} \xi_n \star_x \tilde{n} \in A_\pi(G),
\]
then
\[
\langle \varphi, x \rangle = \sum_{n} \langle x\xi_n|\eta_n \rangle \quad \forall x \in L_n(G).
\]
If \( a \in L_n(G) \), it is easily checked that
\[
a\varphi = \sum_{n} (a\xi_n) \star_x \tilde{n} \quad \text{and} \quad \varphi a = \sum_{n} \xi_n \star_x a^{-1}\tilde{n}.
\]
Finally, if \( (\pi_1, H_1) \) and \( (\pi_2, H_2) \) are two unitary representations of \( G \), then:

1. we say that they are quasi-equivalent if the map \( \pi_1(f) \mapsto \pi_2(f) \), from \( \pi_1(K(G)) \) to \( \pi_2(K(G)) \), extends to an isomorphism of \( L_{\pi_1}(G) \) onto \( L_{\pi_2}(G) \);
2. we say that they are disjoint if no non-zero subrepresentation of \( \pi_1 \) is equivalent to some subrepresentation of \( \pi_2 \).

It follows from Propositions 3.1 and 3.12 of \cite{1} that:

(a) the representations \( \pi_1 \) and \( \pi_2 \) are quasi-equivalent if and only if \( A_{\pi_1}(G) = A_{\pi_2}(G) \);

(b) the representations \( \pi_1 \) and \( \pi_2 \) are disjoint if and only if \( A_{\pi_1}(G) \cap A_{\pi_2}(G) = \{0\} \).

Let us now introduce one of the main objects of the present article: let \( A_0(G) = B(G) \cap C_0(G) \) be the space of all elements of \( B(G) \) that tend to 0 at infinity. We also put \( P_0(G) = P(G) \cap C_0(G) \), and let \( A_{0,sa}(G) \) be the real subspace of selfadjoint elements of \( A_0(G) \).

The following result is inspired by \cite{4}.

**Proposition 2.1.** The set \( A_0(G) \) is a closed two-sided ideal of \( B(G) \), it is equal to the set of all coefficient functions of all \( C_0 \)-representations and every \( \varphi \in A_0(G) \) can be expressed as
\[
\varphi = \varphi_1 - \varphi_2 + i\varphi_3 - i\varphi_4
\]
with \( \varphi_j \in P_0(G) \) for all \( j = 1, \ldots, 4 \).

**Proof.** The space \( A_0(G) \) is obviously a two-sided ideal of \( B(G) \). It is closed because of the following inequality, which holds for every element \( \varphi \in B(G) \):
\[
\|\varphi\|_{\infty} \leq \|\varphi\|_B.
\]
Finally, the decomposition of \( \varphi \) as
\[
\varphi = \frac{1}{2}(\varphi + \varphi^*) + i \cdot \frac{1}{2i}(\varphi - \varphi^*)
\]
shows that it suffices to prove that for every selfadjoint element \( \varphi \in A_0(G) \), the positive definite functions \( \varphi^\pm \) of the Jordan decomposition \( \varphi = \varphi^+ - \varphi^- \) both belong to \( C_0(G) \). But it is proved in Lemme 2.12 of [7] that \( \varphi^+ \) and \( \varphi^- \) are uniform limits on \( G \) of linear combinations of right translates \( s \mapsto \varphi(sg) \) of \( \varphi \). As every such translate belongs to \( C_0(G) \), this proves the claim. \( \square \)

The reason why we denote the intersection \( B(G) \cap C_0(G) \) by \( A_0(G) \) instead of \( B_0(G) \) for instance is that we will see that it is an \( A_\pi \)-space for some suitable representation that we introduce now.

We choose some dense directed set \( (\varphi_i)_{i \in I} \) in \( P_{0,1}(G) \) and, for every \( i \in I \), let \( (\pi_i, H_i, \xi_i) \) be the associated cyclic representation. Put first \( K_0 = \bigoplus_{i \in I} H_i \) and \( \sigma_0 = \bigoplus_{i \in I} \pi_i. \) For instance, if \( G \) is assumed to be discrete, one can set \( \varphi_1 = \delta_1 \), so that \( \pi_1 = \lambda \) is the left regular representation of \( G \). Next, set

\[
H_0 = K_0 \otimes \ell^2(\mathbb{N}) \quad \text{and} \quad \pi_0 = \sigma_0 \otimes 1_{\ell^2(\mathbb{N})}.
\]

Notice that both \( \sigma_0 \) and \( \pi_0 \) are \( C_0 \)-representations.

**Proposition 2.2.** Let \( G \) be a locally compact, second countable group, and let \( (\pi_0, H_0) \) be the above representation. Then:

1. For every \( C_0 \)-representation \( \pi \) of \( G \), one has \( A_{\pi}(G) \subset A_0(G) \).
2. One has \( A_0(G) = A_{\sigma_0}(G) \), and every coefficient function of any \( C_0 \)-representation is a coefficient function associated to \( \pi_0 \).
3. The unitary representation \( \pi_0 \) is the unique \( C_0 \)-representation such that \( A_0(G) = A_{\pi_0}(G) \), up to quasi-equivalence.

**Proof.** (1) Observe that every coefficient function \( \varphi \) of the \( C_0 \)-representation \( \pi \) is a linear combination of four elements in \( P_{0,1}(G) \), by the same argument as in the proof of Proposition 2.1. As \( A_0(G) \) is closed, this proves the first assertion. In particular, \( A_{\sigma_0}(G) \) and \( A_{\pi_0}(G) \) are contained in \( A_0(G) \).

(2) First, if \( \varphi \in P_{0,1}(G) \), then it is a norm limit of a subsequence \( (\psi_k)_{k \geq 1} \) of \( (\varphi_i)_{i \in I} \). This shows that \( \varphi \in A_{\sigma_0}(G) \), and Proposition 2.1 proves that \( A_0(G) \subset A_{\sigma_0}(G) \subset A_{\pi_0}(G) \). Next, let \( \varphi \in A_0(G) \). Let us prove that it is a coefficient function of \( \pi_0 \). As \( A_{\pi_0}(G) = A_0(G) \), there exist sequences of vectors \( (\xi_n)_{n \geq 1}, (\eta_n)_{n \geq 1} \subset K_0 \) such that

\[
\sum_{n} \|\xi_n\| \cdot \|\eta_n\| < \infty
\]

and

\[
\varphi = \sum_{n} \xi_n *_{\sigma_0} \eta_n.
\]

Replacing \( \xi_n \) by \( \sqrt{\frac{\|\xi_n\|}{\|\eta_n\|}} \xi_n \) and \( \eta_n \) by \( \sqrt{\frac{\|\xi_n\|}{\|\eta_n\|}} \eta_n \), we assume that

\[
\sum_{n} \|\xi_n\|^2 = \sum_{n} \|\eta_n\|^2 = \sum_{n} \|\xi_n\| \cdot \|\eta_n\| < \infty.
\]

Put \( \xi = \bigoplus_{n} \xi_n, \eta = \bigoplus_{n} \eta_n \in H_0 \). Then \( \varphi = \xi *_{\pi_0} \eta \).

(3) follows immediately from (1) and (2). \( \square \)

**Definition 2.3.** The representation \( (\pi_0, H_0) \) is called the enveloping \( C_0 \)-representation of \( G \).
Remark 2.4. (1) As is well known, the left regular representation of $G$ is a $C_0$-representation. Hence the Fourier algebra $A(G)$ is contained in $A_0(G)$. In fact, one can have equality $A(G) = A_0(G)$ as well as strict inclusion $A(G) \subseteq A_0(G)$. Indeed, on the one hand, I. Khalil proved in [10] that if $G$ is the $ax + b$-group over $\mathbb{R}$, then $A(G) = A_0(G)$, and, on the other hand, A. Figà-Talamanca [8] proved that if $G$ is unimodular and if its von Neumann algebra $L(G)$ is not atomic (e.g. it is the case whenever $G$ is infinite and discrete), then $A(G) \subseteq A_0(G)$.

(2) We are grateful to the referee for the following observation: the proofs of Propositions 2.1 and 2.2 show that they hold with $A_0(G)$ replaced by any norm-closed, $G$-invariant subspace of $B(G)$.

The next proposition is strongly inspired by, and is a slight generalization of Theorem 3.2 of [4]. It will be used to give characterizations of the Haagerup property in terms of the enveloping $C_0$-representation.

Proposition 2.5. Let $G$ be locally compact group and let $(\pi, H)$ be a unitary representation of $G$, and let us assume that the space $A_\pi(G)$ is an ideal of $B(G)$. Then $\pi : C^*_\pi(G) \to C^*_\pi(G)$ is a $*$-isomorphism if and only if there is a sequence $(\varphi_n)_{n \geq 1} \subset A_\pi(G) \cap P_1(G)$ such that $\varphi_n \to 1$ uniformly on compact subsets of $G$.

Proof. Assume first that $\pi$ is a $*$-isomorphism. We can suppose that $C^*_\pi(G)$ contains no non-zero compact operator. Let $\chi$ be the state on $C^*_\pi(G)$ which comes from the trivial character $f \mapsto \int f(s)ds$ on $K(G) \subset C^*_\pi(G)$. By Glimm's Lemma, there exists an orthonormal sequence $(\xi_n)_{n \geq 1} \subset H$ such that

$$\chi(x) = \lim_{n \to \infty} \langle x\xi_n | \xi_n \rangle$$

for every $x \in C^*_\pi(G)$. Put $\varphi_n = \xi_n * \pi \xi_n \in A_\pi(G) \cap P_1(G)$ for every $n$. Then one has for every $f \in K(G)$:

$$\lim_{n \to \infty} \int_G \varphi_n(t)f(t)dt = \lim_{n \to \infty} \langle \pi(f)\xi_n | \xi_n \rangle = \int_G f(t)dt.$$

Theorem 13.5.2 of [6] implies that $\varphi_n \to 1$ uniformly on compact subsets of $G$.

Conversely, if there exists a sequence $(\varphi_n)_{n \geq 1} \subset A_\pi(G) \cap P_1(G)$ such that $\varphi_n \to 1$ uniformly on compact subsets of $G$, let $x \in \ker(\pi)$. We have to prove that $\langle \varphi, x^*x \rangle_{B,C^*} = 0$ for every state $\varphi$ on $C^*(G)$. Observe first that, for every $\psi \in A_\pi(G)$ and every $y \in C^*(G)$, one has

$$\langle \psi, y \rangle_{B,C^*} = \langle \psi, \pi(y) \rangle_{A_\pi, C^*}$$

Indeed, if we write $\psi = \sum_k \xi_k * \pi \eta_k$, and if $f \in K(G)$, we have

$$\langle \psi, f \rangle_{B,C^*} = \int_G \psi(s)f(s)ds = \sum_k \int_G \langle \pi(s)\xi_k | \eta_k \rangle f(s)ds = \langle \psi, \pi(f) \rangle_{A_\pi, C^*}$$

and the formula holds by density of $K(G)$ in $C^*(G)$.

Let us fix such a state $\varphi \in P_1(G)$ and set $\psi_n = \varphi \varphi_n \in A_\pi(G) \cap P_1(G)$ for every $n$. As $\psi_n$ is a state on $L_\pi(G)$, its restriction to $C^*_\pi(G)$ is still a state, and $\langle \psi_n, x^*x \rangle = \langle \psi_n, \pi(x^*x) \rangle = 0$ for every $n$. As $\psi_n \to \varphi$ in the weak* topology of $B(G) = C^*(G)^*$, one has $\langle \varphi, x^*x \rangle = 0$.

$\square$
3. The Haagerup property

As in the first section, $G$ denotes a locally compact group and $(\pi_0, H_0)$ denotes its enveloping $C_0$-representation.

Following M. Bekka [2], we say that $(\pi, H)$ is an amenable representation if $\pi \otimes \bar{\pi}$ weakly contains the trivial representation. Equivalently, this means that there exists a net of unit vectors $(\xi_i) \subset H \otimes \bar{H}$ such that

$$\langle \pi \otimes \bar{\pi}(s)\xi_i | \xi_i \rangle \to 1$$

uniformly on compact subsets of $G$; notice that $\pi \otimes \bar{\pi}$ is unitarily equivalent to the representation $(T, g) \mapsto \pi(g)T\pi(g^{-1})$ acting on the space $HS(H)$ of all Hilbert-Schmidt operators.

If $G$ is moreover second countable, we say that it has the Haagerup property if there exists a sequence $(\varphi_n)_{n \geq 1} \subset P_{0,1}(G)$ which tends to $1$ uniformly on compact sets. Note that it is equivalent to say that $G$ admits an amenable, $C_0$-representation. See [3] for more information on the Haagerup property.

The next result generalizes partly, and is inspired by Corollary 3.4 of [4].

**Proposition 3.1.** Let $G$ and $(\pi_0, H_0)$ be as above. Then the following conditions are equivalent:

1. $G$ has the Haagerup property;
2. $C^*(G) = C^*_{\pi_0}(G)$, i.e. the $*$-homomorphism $\pi_0 : C^*(G) \to C^*_{\pi_0}(G)$ is an isomorphism;
3. the representation $\pi_0$ weakly contains the trivial representation;
4. the representation $\pi_0$ is amenable.

**Proof.** (1) $\Rightarrow$ (2). There exists a sequence $(\varphi_n)_{n \geq 1} \subset P_{0,1}(G)$ which converges to $1$ uniformly on compact sets. The assertion follows readily from Proposition 2.5.

(2) $\Rightarrow$ (3). It follows also from Proposition 2.5.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are obvious. $\square$

**Remark 3.2.** As $A(G) \subset A_{\pi_0}(G)$, there exists a $*$-homomorphism $\Phi$ from $L_{\pi_0}(G)$ onto $L(G)$ such that $\Phi(\pi_0(f)) = \lambda(f)$ for every $f \in K(G)$. Thus, let $z_A \in L_{\pi_0}(G)$ be the central projection such that $L_{\pi_0}(G)z_A$ is $*$-isomorphic to $L(G)$. This allows us to consider the following two subrepresentations of $\pi_0$: set $\pi_{00}(s) = \pi_0(s)(1 - z_A)$ and $\lambda_0(s) = \pi_0(s)z_A$ for all $s \in G$. Then $\lambda_0$ is quasi-equivalent to $\lambda$, and since $\pi_{00}$ is disjoint from $\lambda$, we have $A_{\pi_{00}}(G) \cap A(G) = \{0\}$. It would be interesting to get more information on $\pi_{00}$, in particular when $G$ has the Haagerup property.

From now on, we assume that $G$ is an infinite, discrete, countable group. Following [4], for any (not necessarily closed) ideal $D \subset \ell^\infty(G)$, we say that a unitary representation $(\pi, H)$ of $G$ is a $D$-representation if $H$ contains a dense subspace $K$ such that the coefficient function $\xi \mapsto \eta \in D$ for all $\xi, \eta \in K$. We associate to $D$ the following $C^*$-algebra $C^*_D(G)$: it is the completion of $K(G)$ with respect to the $C^*$-norm

$$\|f\|_D := \sup(\|\pi(f)\| : \pi \text{ is a } D \text{-representation}).$$

When $D = C_0(G)$, one gets $C^*_D(G) = C^*_{\pi_0}(G)$. This makes the link between Proposition 3.1 above and the main results of N. Brown and E. Guentner in [4].

We end the present notes with a relationship between the Haagerup property for discrete groups and strongly mixing von Neumann subalgebras in the sense of [9],
Definition 1.1. We need to recall some definitions and facts from [9] first and from Chapter 2 of [5] next.

Let \( 1 \in B \subset M \) be finite von Neumann algebras (with separable preduals) endowed with a normal, finite, faithful, normalized trace \( \tau \). We denote by \( E_B \) the \( \tau \)-preserving conditional expectation from \( M \) onto \( B \), and by \( M \ominus B = \{ x \in M : E_B(x) = 0 \} \). We assume that \( B \) is diffuse.

**Definition 3.3.** Let \( B \subset M \) be a pair as above. We say that \( B \) is **strongly mixing in** \( M \) if
\[
\lim_{n \to \infty} \| E_B(xu_n y) \|_2 = 0
\]
for all \( x, y \in M \ominus B \) and all sequences \( (u_n) \subset U(B) \) which converge to 0 in the weak operator topology.

This definition is motivated by the following situation: if a countable group \( G \) acts in a trace-preserving way on some finite von Neumann algebra \((Q, \tau)\) and if we put \( B := L(G) \subset M := Q \rtimes G \), then \( B \) is strongly mixing in \( M \) if and only if the action of \( G \) on \( Q \) is strongly mixing in the usual sense: for all \( a, b \in Q \), one has
\[
\lim_{g \to \infty} \tau(a \sigma_g(b)) = \tau(a) \tau(b).
\]

Let now \( G \) be a countable group with the Haagerup property. By Theorems 2.1.5, 2.2.2 and 2.3.4 of [5], there exists a trace preserving and strongly mixing action of \( G \) on some finite von Neumann algebra \((Q, \tau)\) which contains non trivial asymptotically invariant sequences and Følner sequences in the sense below. For instance, if \( G \) has the Haagerup property, there exists an action \( \alpha \) of \( G \) on the hyperfinite type II\(_1\)-factor \( R \) such that:

- \( \alpha \) is strongly mixing;
- the fixed point algebra \((R_\omega)_\alpha\), that is, the set of all (classes of) central sequences \( x = [(x_n)] \in R_\omega \) such that \( \alpha_\omega^g(x) = x \) for all \( g \in G \), is of type II\(_1\).

**Definition 3.4.** Let \( 1 \in B \subset M \) be a pair of finite von Neumann algebras as above, and let \( (e_k)_{k \geq 1} \subset M \) be a sequence of projections in \( M \).

1. We say that \( (e_k)_{k \geq 1} \) is a **non trivial asymptotically invariant sequence** for \( B \) if \( E_B(e_k) = \tau(e_k) \) for every \( k \), if
\[
\lim_{k \to \infty} \| be_k - e_k b \|_2 = 0
\]
for every \( b \in B \) and if
\[
\inf_k \tau(e_k)(1 - \tau(e_k)) > 0.
\]

2. We say that \( (e_k)_{k \geq 1} \) is a **Følner sequence** for \( B \) if \( E_B(e_k) = \tau(e_k) \) for every \( k \), if \( \lim_k \| e_k \|_2 = 0 \) and if
\[
\lim_{k \to \infty} \frac{\| be_k - e_k b \|_2}{\| e_k \|_2} = 0
\]
for every \( b \in B \).

In general, the existence of a non trivial asymptotically invariant sequence for \( B \) implies the existence of a Følner sequence for \( B \), but the converse does not hold. See [3], p. 19, for more details.

Combining these types of properties, we get:
Theorem 3.5. Let $G$ be an infinite, countable group. Then it has the Haagerup property if and only if it satisfies one of the following equivalent conditions:

(1) (resp. (1')) There exists a finite von Neumann algebra $M$ containing $L(G)$ such that $L(G)$ is strongly mixing in $M$ and $M$ contains a Følner sequence for $L(G)$ (resp. a non trivial asymptotically invariant sequence for $L(G)$).

(2) There exists a finite von Neumann algebra $M$ containing $L(G)$ such that $L(G)$ is strongly mixing in $M$ and there is a sequence of elements $(x_k)_{k \geq 1} \subset M \otimes B$ such that $\|x_k\|_2 = 1$ for every $k$, and

$$\lim_{k \to \infty} \|\lambda(g)x_k - x_k\lambda(g)\|_2 = 0$$

for every $g \in G$.

Proof. If $G$ has the Haagerup property, then each condition (1), (1') and (2) holds, by Theorem 2.3.4 of [3], and there are plenty of non trivial asymptotically invariant or Følner sequences in the hyperfinite type $\mathrm{II}_1$-factor $R$. Thus, assume that condition (1) holds and that $B := L(G)$ embeds into some finite von Neumann algebra $M$ such that $B := L(G)$ is strongly mixing in $M$ and that $M$ contains a Følner sequence for $B$. We have to show the existence of a sequence $(\varphi_k)_{k \geq 1} \subset P_{01}(G)$ which tends to 1 pointwise.

Recall first that to any completely positive map $\Phi : M \to M$, one associates a function $\varphi$ on $G$ by

$$\varphi(g) = \tau(\Phi(\lambda(g))\lambda(g^{-1})) \quad (g \in G),$$

and that $\varphi$ is positive definite. In particular, for every $x \in M \otimes B$, the function $\varphi_x : G \to \mathbb{C}$ defined by

$$\varphi_x(g) = \tau(\mathbb{E}_B(x^*\lambda(g)x)\lambda(g^{-1})) = \tau(x^*\lambda(g)x\lambda(g^{-1})) \quad (g \in G)$$

is positive definite. Moreover, since $B$ is strongly mixing in $M$ and since $\lambda(G)$ is an orthonormal set, one has

$$|\varphi_x(g)| \leq \|\mathbb{E}_B(x^*\lambda(g)x)\|_2 \to 0$$

as $g \to \infty$, which shows that $\varphi_x \in P_{0}(G)$ for every $x \in M \otimes B$.

Next, let $(e_k)_{k \geq 1} \subset M$ be a Følner sequence for $B$ and choose $c > 0$ and an integer $k_0 > 0$ such that

$$1 - \tau(e_k) \geq c$$

holds for every $k \geq k_0$. Define then

$$x_k = \frac{e_k - \tau(e_k)}{\sqrt{\tau(e_k)(1 - \tau(e_k))}} = x_k^* \quad (k \geq 1)$$

and put $\varphi_k = \varphi_{x_k}$ for every $k$. One has, for every integer $k \geq k_0$ and every $g \in G$:

$$\varphi_k(g) = \tau(x_k\lambda(g)x_k\lambda(g^{-1}))$$

$$= \frac{1}{\tau(e_k)(1 - \tau(e_k))} \cdot \tau((e_k - \tau(e_k))\lambda(g)(e_k - \tau(e_k))\lambda(g^{-1}))$$

$$= \frac{1}{\tau(e_k)(1 - \tau(e_k))} \cdot \tau(e_k\lambda(g)e_k\lambda(g^{-1}) - \tau(e_k)^2)$$

$$= \frac{\tau(e_k\lambda(g)e_k\lambda(g^{-1}) - e_k)}{\tau(e_k)(1 - \tau(e_k))} + 1.$$
Hence, by Cauchy-Schwarz Inequality,
\[
|\varphi_k(g) - 1| \leq \frac{1}{c} \frac{\|e_k\|_2 \|\lambda(g)e_k\lambda(g^{-1}) - e_k\|_2}{\|e_k\|_2^2} = \frac{1}{c} \frac{\|\lambda(g)e_k - e_k\lambda(g)\|_2}{\|e_k\|_2} \to 0
\]
as \(k \to \infty\) for every \(g \in G\). A similar argument works if \((e_k)\) is a non trivial asymptotically invariant sequence.

Finally, assume that \(G\) satisfies condition (2), and let \((x_k) \subset M \ominus B\) be as above. Define \(\varphi_k(g) = \tau(x_k^* \lambda(g) x_k \lambda(g^{-1}))\) exactly as above. Then by the same arguments, \(\varphi_k \in P_{0,1}(G)\) for every \(k\), and, for fixed \(g \in G\), one has:
\[
|\varphi_k(g) - 1| = |\tau(x_k^* \lambda(g) x_k \lambda(g^{-1})) - \tau(x_k^* x_k)|
\]
\[
= |\langle \lambda(g)x_k \lambda(g) - x_k | x_k \rangle|
\]
\[
\leq \|\lambda(g)x_k \lambda(g^{-1}) - x_k\|_2 \|x_k\|_2
\]
\[
= \|\lambda(g)x_k \lambda(g^{-1}) - x_k\|_2 \to 0
\]
as \(k \to \infty\). \(\square\)

Remark 3.6. Assume that \(G\) has the Haagerup property. One can ask whether there exists a group \(\Gamma\) containing \(G\) and such that the pair of finite von Neumann algebras \(L(G) \subset L(\Gamma)\) satisfies condition (2) in Theorem 3.5. Unfortunately, it is only the case when \(G\) is amenable, and this has no real interest. Indeed, assume for simplicity that \(G\) is torsion free, that it embeds into some group \(\Gamma\) and that the pair \(L(G) \subset L(\Gamma)\) satisfies condition (2) above. Then, on the one hand, by Lemma 2.2 and Proposition 2.3 of \([9]\), the pair of groups \(G \subset \Gamma\) satisfies condition (ST), which means that, for every \(\gamma \in \Gamma \setminus G\), the subgroup \(\gamma G \gamma^{-1} \cap G\) is finite, hence trivial. In other words, \(G\) is malnormal in \(\Gamma\). On the other hand, by classical arguments, the existence of a sequence \((x_k) \subset L(\Gamma) \ominus L(G)\) as above implies that the action \(G \curvearrowleft X \colonequals \Gamma \setminus G\) defined by \((g, x) \mapsto gxg^{-1}\) has an invariant mean. This means that the associated representation \(\lambda_X\) weakly contains the trivial representation. But the first condition implies that this action is free, hence that \(\lambda_X\) is equivalent to a multiple of the regular representation. This forces \(G\) to be amenable.

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