Conditional Variational Inference with Adaptive Truncation for Bayesian Nonparametric Models

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Abstract

The scalable inference for Bayesian nonparametric models with big data is still challenging. Current variational inference methods fail to characterise the correlation structure among latent variables due to the mean-field setting and cannot infer the true posterior dimension because of the universal truncation. To overcome these limitations, we build a general framework to infer Bayesian nonparametric models by maximising the proposed nonparametric evidence lower bound, and then develop a novel approach by combining Monte Carlo sampling and stochastic variational inference framework. Our method has several advantages over the traditional online variational inference method. First, it achieves a smaller divergence between variational distributions and the true posterior by factorising variational distributions under the conditional setting instead of the mean-field setting to capture the correlation pattern. Second, it reduces the risk of underfitting or overfitting by truncating the dimension adaptively rather than using a prespecified truncated dimension for all latent variables. Third, it reduces the computational complexity by approximating the posterior functionally instead of updating the stick-breaking parameters individually. We apply the proposed method on hierarchical Dirichlet process and gamma–Dirichlet process models, two essential Bayesian nonparametric models in topic analysis. The empirical study on three large datasets including arXiv, New York Times and Wikipedia reveals that our proposed
method substantially outperforms its competitor in terms of lower perplexity and much clearer topic-words clustering.

Some key words: Gibbs sampling; Hierarchical Dirichlet process; Nonparametric evidence lower bound; Stochastic variational inference; Topic modelling.

1 Introduction

Bayesian nonparametric models, differing from parametric models by relaxing the fixed dimension assumption, are widely used in bioinformatics, language processing, computer vision and network analysis (Dunson & Park, 2008; Sudderth & Jordan, 2009; Caron & Fox, 2017; Ranganath & Blei, 2018). For example, in natural language processing, Teh et al. (2006) develop a hierarchical Dirichlet process, which extends the latent Dirichlet allocation model (Blei et al., 2003) from a nonparametric perspective. Hierarchical Dirichlet process is defined on a countable dimensional simplex to replace the finite-dimensional Dirichlet distribution in latent Dirichlet allocation. Within such model, the number of topics is regarded as a random variable instead of a fixed value and hence can be inferred from data.

The inference of Bayesian nonparametric models is more complicated than its parametric counterpart. Due to the infinite-dimensional nature of Bayesian nonparametric models, a finite-dimensional truncation is needed to approximate the posterior. However, the selection of the optimal truncation level poses extra challenges. The traditional Markov chain Monte Carlo methods (Teh et al., 2006; Papaspiliopoulos & Roberts, 2008) can produce an adaptive selection of the truncated dimension but are not computationally scalable especially for big data. On the other hand, standard variational inference methods (Teh et al., 2008; Wang et al., 2011; Hoffman et al., 2013; Roychowdhury & Kulis, 2015) can accelerate the computation but suffer from an universal selection of the truncation level, that is, truncating the dimension of all latent variables to a prespecified value. However, a subjective selection of the fixed truncation level would lead to a low predictive accuracy due to the possible
overfitting or underfitting. In this sense, such universal truncation method contradicts the motivation and advantages of using Bayesian nonparametric models.

In this paper, we propose a general framework with novel and efficient algorithms to infer a large class of Bayesian nonparametric models in the following steps. First, we derive the nonparametric evidence lower bound based on finite and measurable partitions. Second, we propose the conditional setting when factorising variational distributions by letting variables in the middle layers conditional on two adjacent layers. Third, to handle big data, we develop the corresponding stochastic variational inference framework (Hoffman et al., 2013; Blei et al., 2017) under our conditional setting. Finally, within our framework, we adopt Monte Carlo sampling to generate samples for local latent variables, and further update the variational parameters for global latent variables based on the empirical distribution generated from these samples. Meanwhile, we truncate the dimension of variational distributions to that of the empirical distribution.

Our proposed method, named conditional variational inference with adaptive truncation, benefits from both the accuracy of Monte Carlo sampling and the efficiency of variational inference as follows. First, our method rebuilds the correlation structure and hence attains a smaller divergence between the variational distribution and the true posterior. Such procedure removes the unrealistic mean-field assumption, and searches an optimal variational distribution over a wider family. Second, our method assigns a probability of increasing the dimension of variation distributions adaptive to the goodness-of-fit. As the inference proceeds, it reaches a stable level balancing the goodness-of-fit and model complexity. Therefore, it provides an adaptive selection of the truncated dimension and reduces the risk of overfitting or underfitting. Finally, our method achieves better prediction without sacrificing computational efficiency. With the optimal variational distributions for global variables, the local Markov chain converges fast, which is demonstrated in our empirical study.

To assess the empirical performance of the proposed method, we develop detailed algorithms for hierarchical Dirichlet process model and gamma–Dirichlet process model (Jordan,
2010), and apply them on the topic analysis of three large datasets including arXiv, New York Times and Wikipedia. The results show that the algorithms for our proposed method consistently outperform traditional online variational inference (Wang et al., 2011) in three examples by substantially reducing the hold-out perplexity. Furthermore, our method gives much clearer topic-words clustering by removing replicated topics and providing room to further add new topics. We provide the code at https://github.com/yiruiliu110/ConditionalVI.

2 Hierarchical Bayesian nonparametric models

Suppose that $(\Omega, \mathcal{F})$ is a Polish sample space, $\Theta$ is the set of all bounded measures on $(\Omega, \mathcal{F})$ and $\mathcal{M}$ is a $\sigma$-algebra on $\Theta$. A random measure $G$ on $(\Omega, \mathcal{F})$ is a transition kernel from $(\Theta, \mathcal{M})$ into $(\Omega, \mathcal{F})$ such that (i) $G \mapsto G(A)$ is $\mathcal{M}$-measurable for any $A \in \mathcal{F}$ and (ii) $A \mapsto G(A)$ is a measure for any realisation of $G$ (Ghosal & Van der Vaart, 2017). For example, a Dirichlet process $\mathcal{P}$ (Ferguson, 1973) with base measure $P_0$ satisfies

\[
(P(A_1), P(A_2), \ldots, P(A_n)) \sim \text{Dirichlet}(P_0(A_1), P_0(A_2), \ldots, P_0(A_n))
\]

for any partition $\Omega = (A_1, \ldots, A_n)$ of $\Omega$, that is, a finite number of measurable, nonempty and disjoint sets such that $\bigcup_{i=1}^n A_i = \Omega$. The Dirichlet process is denoted by $P \sim \text{DP}(P_0)$ or $P \sim \text{DP}(\alpha H)$ with prior precision $\alpha = P_0(\Omega)$ and center measure $H = \alpha^{-1}P_0$. Moreover, a random measure is called a completely random measure (Kingman, 1993) if it also satisfies (iii) $P(A_i)$ is independent of $P(A_j)$ for any disjoint subsets $A_i$ and $A_j$ in $\Omega$. See Appendix A.1 for a short review. Completely random measures and their normalisations (Regazzini et al., 2003), for example, gamma process and Dirichlet process, respectively, are commonly used as priors for infinite-dimensional latent variables in Bayesian nonparametric models, because their realisations are atomic measures with a countable-dimensional support.

As an important subclass of Bayesian nonparametric models, hierarchical Bayesian nonparametric models use random measures for priors in multiple layers. Consider the following
model,
\[
G_0 \mid H \sim P(H), \quad \beta \mid \lambda \sim p(\beta \mid \lambda),
\]
\[
G_j \mid G_0 \sim R(G_0) \quad (j = 1, \ldots, J),
\]
\[
z_{ji} \mid G_j \sim G_j, \quad x_{ji} \sim f(x_{ji} \mid \beta, z_{ji}) \quad (i = 1, \ldots, N_j; j = 1, \ldots, J),
\]
whose two-layer hierarchical structure is summarised in Figure 1. Specifically, in the top layer, \( G_1, \ldots, G_J \) are generated from a random measure \( R \) with common base measure \( G_0 \), while in the bottom layer, \( G_0 \) itself is a realisation of random measure \( P \) with base measure \( H \). To ensure exchangeability, \( G_1, \ldots, G_J \) are assumed to be identical and independent given \( G_0 \). The global parameter \( \beta \) is assigned a prior \( p(\beta \mid \lambda) \). In addition, each local latent variable \( z_{ji} \) is sampled from \( G_j \) independently and the observation \( x_{ji} \) is generated from a likelihood function \( f \), which is parameterised by both global latent variable \( \beta \) and local latent variable \( z_{ji} \).

We next illustrate the necessity of hierarchical structure in Bayesian nonparametric models, using the example of hierarchical Dirichlet process model in topic modelling (Teh et al., 2006), where \( P \) and \( R \) in (1) are both Dirichlet processes,
\[
G_0 \mid H \sim DP(\alpha H), \quad G_j \mid G_0 \sim DP(\gamma G_0) \quad (j = 1, \ldots, J).
\]
Suppose a corpus has \( J \) documents, each document \( j \) has \( N_j \) words and each word is chosen from a vocabulary with \( W \) terms. We describe the generative model as follows. First, \( G_0 = \sum_{k=0}^{\infty} G_{0k} \delta_{\phi_k} \) is generated from \( DP(\alpha H) \), and for each document \( j \) a topic proportion \( G_j = \sum_{k=0}^{\infty} G_{jk} \delta_{\phi_k} \) is independently sampled from \( DP(\gamma G_0) \). Second, for any topic \( k \).
the distribution of words over vocabulary is independently sampled from a $W$-dimensional Dirichlet distribution parameterised by $\eta, \beta_k \sim \text{Dir}(\eta)$. Third, for each word $i$ in document $j$, a topic assignment $z_{ji}$ is allocated by $z_{ji} \sim \text{Multinomial}(G_j)$, where $z_{ji}$ represents the topic $k$ if $z_{ji} = \phi_k$. Finally, the observation $x_{ji}$ is independently generated from the assigned topic and the corresponding within-topic word distribution, $x_{ji} \mid \{z_{ji} = \phi_k\} \sim \text{Multinomial}(\beta_k)$. Within such hierarchical Dirichlet process model, in the top layer, if $G_1, \ldots, G_J$ are sampled from a Dirichlet process with a diffuse base measure instead of an atomic $G_0$, the supports of $G_1, \ldots, G_J$ do not overlap almost surely, which results in no share of topics among different documents. To solve this issue, we let the base measure $G_0$ have an atomic and infinite-dimensional support, for example, assigning a Dirichlet process prior for $G_0$.

Generally speaking, it is not necessary to restrict the prior for $G_0$ to be a Dirichlet process or other probability random measures. The essential point here is to equip $G_0$ with an infinite-dimensional and atomic support. Therefore, other completely random measures and their normalisation can also be used as priors for $G_0$. For example, the gamma–Dirichlet process model (Jordan, 2010),

$$G_0 \mid H \sim \Gamma P(\alpha H), \quad G_j \mid G_0 \sim \text{DP}(G_0) \quad (j = 1, \ldots, J).$$  \hfill (3)

The gamma–Dirichlet process allows a more flexible model by removing the constraint on the prior precision in the top layer. Other choices of prior for $G_0$ include beta process, $\sigma$-stable process and inverse Gaussian process (Ghosal & Van der Vaart, 2017).

### 3 Conditional variational inference

#### 3.1 Kullback–Leibler divergence between random measures

The object of variational inference is to minimise the divergence between the variational distribution and the true posterior. For infinite-dimensional random measures, their Kullback–Leibler divergence is well defined although the corresponding density function does not exist.
with respect to Lebesgue measure. Suppose two random measures $P$ and $Q$ from $(\Theta, \mathcal{M})$ into $(\Omega, \mathcal{F})$, the Radon–Nikodym derivative $dQ/dP$ exists if $Q$ is absolutely continuous with respect to $P$. Their Kullback–Leibler divergence is defined as,

$$\text{KL}(Q \parallel P) = \int_\Theta \log \frac{dQ}{dP} dQ,$$

which is computationally intractable due to the infinite-dimensional integral. By contrast, we calculate this divergence by the limit superior of the divergence between corresponding finite-dimensional induced measures, that is,

$$\text{KL}(Q \parallel P) = \limsup_{\Omega} \text{KL}(p^\Omega \parallel q^\Omega), \quad (4)$$

where $p^\Omega$ and $q^\Omega$ are respectively induced measures from $P$ and $Q$ on a finite-dimensional partition $\Omega = (A_1, \ldots, A_n)$, such that $p^\Omega(A_i) = P(A_i)$ and $q^\Omega(A_i) = Q(A_i)$ for each $A_i \in \Omega$.

With an induced random variable $Z^\Omega: \Theta \rightarrow \mathbb{R}^n$, we can also denote the induced measures by $p(Z^\Omega)$ and $q(Z^\Omega)$. The result in (4) is justified in Appendix A.2.

### 3.2 Nonparametric evidence lower bound

The parametric variational inference algorithm uses a finite-dimensional variational distribution to approximate the true posterior by maximising the evidence lower bound (Blei et al., 2017), while, for nonparametric models, we need to use a random measure as variational distribution due to the infinite dimensionality of latent variables. Based on the Kullback–Leibler divergence between random measures in (4), we propose a general variational inference framework for Bayesian nonparametric models by defining the corresponding nonparametric evidence lower bound as,

$$\text{NPELBO} = \liminf_{\Omega} \left[ E_{q(Z^\Omega)} \left\{ \log p(X, Z^\Omega) \right\} - E_{q(Z^\Omega)} \left\{ \log q(Z^\Omega) \right\} \right], \quad (5)$$

where $p(X, Z^\Omega)$ and $q(Z^\Omega)$ correspond to the induced measures from the joint distribution and the variational distribution on $\Omega$, with $Z$ and $X$ denoting the observations and latent
variables, respectively. Provided the result that

\[ \text{KL}(Q(Z) \mid\mid P(X \mid Z)) + \text{NPELBO} = \log p(X), \]  

our proposed framework considers maximising the nonparametric evidence lower bound in (5), which is equivalent to minimising the Kullback–Leibler divergence between variational distribution \( Q(z) \) and true posterior \( P(z \mid x) \). See Appendix A.3 for the proof of equation (6). To simplify the notation, we will use \( p(\cdot) \) and \( q(\cdot) \) to denote the true and variational distributions, respectively, where the context is clear.

### 3.3 Conditional variational distribution

The hierarchical Bayesian nonparametric model in (1) has multiple layers, and hence \( Z \) in (5) includes several latent variables, which are global latent variable \( \beta \), local latent variables \( \{z_{ji}\}_{1 \leq j \leq J, 1 \leq i \leq N_j} \), global prior \( G_0 \), and local priors \( \{G_j\}_{1 \leq j \leq J} \). To factorise the variational distribution \( q(\beta, \{z_{ji}\}, G_0, \{G_j\}) \), the traditional variational inference algorithms typically consider the mean-field setting, \( q(\beta, \{z_{ji}\}, G_0, \{G_j\}) = q(\beta)q(G_0)\prod_{j=1}^J q(G_j)\prod_{j=1}^J \prod_{i=1}^{N_j} q(z_{ji}) \), where variables in different layers are assumed to be independent. However, this assumption is not valid in nonparametric variational inference because the independence between \( \{G_j\}_{1 \leq j \leq J} \) and \( G_0 \) contradicts the fact that the support of each \( G_j \) is fully determined by \( G_0 \). As the updatings of \( q(G_j) \) and \( q(G_0) \) are independent in the procedure of iterations, they are likely to have different supports, which contradicts their definitions. Moreover, the mean-field assumption fails to account for the possibly high correlation among \( G_0, \{G_j\} \) and \( \{z_{ji}\} \).

In contrast to the traditional variational inference under the mean-field setting, we factorise the variational distribution as,

\[ q(\beta, \{z_{ji}\}, G_0, \{G_j\}) = q(\beta)q(G_0)\prod_{j=1}^J q(G_j \mid G_0, z_j)\prod_{j=1}^J \prod_{i=1}^{N_j} q(z_{ji}), \]  

in the sense of the probability law. On one hand, our conditional setting eliminates the contradiction in the mean-field setting, because we consider the variational distribution of
Conditional on $G_0$, which ensures that $G_j$ shares the same support of $G_0$. On the other hand, such conditional design facilitates the recovery of the dependence structure among $G_0$, \{G_j\} and \{z_{ji}\}.

Combining (5) and (7), our proposed conditional variational inference seeks to maximise the following nonparametric evidence lower bound,

$$\text{NP}ELBO = \liminf_{\Omega} \left[ E_{q(\beta, z_j, G_0, G_j \mid G_0)} \left\{ \log p(x_j, \beta, z_j, G_0, G_j) \right\} \right]$$

$$- \sum_{j=1}^J E_{q(G_0)} \left\{ \log q(G_0) \right\} - \sum_{j=1}^J \sum_{i=1}^{N_j} E_{q(z_{ji})} \left\{ \log q(z_{ji}) \right\}$$

$$- \sum_{j=1}^J E_{q(G_j \mid G_0, z_j)} E_{q(z_j)} E_{q(G_j \mid G_0, z_j)} \left\{ \log q(G_j \mid G_0, z_j) \right\} \right],$$

where $x_j = \{x_{ji}\}_{1 \leq i \leq N_j}$, $z_j = \{z_{ji}\}_{1 \leq i \leq N_j}$, and $\Omega$ is a partition of the sample space $\Omega$ for $G_0$ and \{G_j\}_{1 \leq j \leq J}.

### 3.4 Conditional coordinate ascent

To maximise the nonparametric evidence lower bound in (8), we first seek the optimal variational distribution of $G_j$ given $G_0$ and $z_j$ for each $j$. As $p(x_j, \beta, z_j, G_0, G_j) = p(G_0, \{z_j\}) \prod_{j=1}^J p(G_j \mid G_0, z_j) p(x_j \mid \beta, z_j)$, the non-constant term in (8) with respect to $q(G_j \mid G_0, z_j)$ is

$$\liminf_{\Omega} \left[ \sum_{j=1}^J E_{q(G_0)} E_{q(z_j)} E_{q(G_j \mid G_0, z_j)} \left\{ \log p(G_j \mid G_0, z_j) - \log q(G_j \mid G_0, z_j) \right\} \right].$$

It is worth noting that the above expression can be viewed as the negative of the Kullback–Leibler divergence whose maximum is zero. Therefore, the optimal conditional variational distribution for $G_j$ is $q(G_j \mid G_0, z_j) = p(G_j \mid G_0, z_j)$ as the divergence equals zero if and only if $q(G_j \mid G_0, z_j) = p(G_j \mid G_0, z_j)$ for any partition $\Omega$. This result is also intuitive because the best variational distribution to approximate the posterior given other variables is the conditional posterior itself. Benefiting from the conjugacy in Bayesian nonparametric models, the analytical form of such conditional posterior is easy to derive.
We then implement a coordinate ascent approach by iterating the following three steps until convergence. The first step considers obtaining the optimal $q(G_0)$ conditional on other parameters. To achieve this, in Appendix A.4, we rely on (8) to derive the evidence lower bound under $\Omega$ with respect to $q(G_0)$,

$$\text{ELBO}^\Omega = E_{q(G_0^\Omega)} \{ \log \frac{p(G_0^\Omega)}{q(G_0^\Omega)} \} + \sum_{j=1}^{J} E_{q(G_j^\Omega)} E_{q(z_j)} \left[ \log E_{p(G_j^\Omega | G_0^\Omega)} \{ p(z_j | G_j^\Omega) \} \right] + \text{constant}, \quad (9)$$

where $E_{p(G_j^\Omega | G_0^\Omega)}$ is with respect to the prior distribution $p(G_j^\Omega | G_0^\Omega)$ instead of the variational distribution. Consequently, this expectation can be easily calculated due to its analytical representation. As the nonparametric evidence lower bound $\text{NPELBO} = \lim \inf_{\Omega} \text{ELBO}^\Omega$, if we can find a random measure $q(G_0)$ with its induced measure $q^\Omega(G_0)$ satisfying

$$q(G_0^\Omega) \propto p(G_0^\Omega) \exp \left( \sum_{j=1}^{J} E_{q(z_j)} \left[ \log E_{p(G_j^\Omega | G_0^\Omega)} \{ p(z_j | G_j^\Omega) \} \right] \right), \quad (10)$$

for any partition $\Omega$, then this $q(G_0)$ is the optimal variational random measure. In cases where it is difficult to find a simple random measure satisfying (10), we need to restrict the variational distribution in a special family and optimise the parameters. Provided with the updated $q(G_0)$ and other parameters, the second step considers optimising the variational distribution for $z_j$ in the form of

$$q(z_j) \propto \exp \left( E_{q(G_0)} \left[ \log E_{p(G_j | G_0)} \{ p(z_j | G_j) \} \right] + E_{q(\beta)} \left\{ \log p(x_j | z_j, \beta) \right\} \right). \quad (11)$$

Finally, the optimal variational distribution for the global latent variable $\beta$ given other updated parameters is

$$q(\beta) \propto p(\beta) \exp \left[ \sum_{j=1}^{J} E_{q(z_j)} \left\{ \log p(x_j | z_j, \beta) \right\} \right]. \quad (12)$$

4 Adaptive truncation

4.1 Stochastic variational inference

Whereas the coordinate ascent formulas in (10)–(12) provide a general framework, they are difficult to be directly implemented especially for big data, because updating all local
latent variables in each iteration is not computationally efficient. By contrast, stochastic variational inference (Hoffman et al., 2013) is widely used in practice, where the computation is accelerated by randomly selecting a small batch of data and iteratively updating the parameters with a random but unbiased gradient of evidence lower bound. Specifically, for an evidence lower bound ELBO(ξ) as a function of parameter ξ, if there exists a random function h(ξ) satisfying E[h′(ξ)] = ELBO′(ξ), ξ can be updated in the τ-th iteration by ξ(τ) = ξ(τ−1) + ρt h′(ξ(τ−1)), where the step size ρt satisfies the Robbins–Monro condition (Robbins & Monro, 1951).

For hierarchical Bayesian nonparametric models, the traditional stochastic variational inference methods suffer from the mean-field assumption and the universal truncation (Hoffman et al., 2013; Wang et al., 2011). To overcome these disadvantages, we propose a new approach by integrating Monte Carlo sampling scheme into the stochastic variational inference framework under the conditional variational setting, namely conditional variational inference with adaptive truncation. The proposed method not only benefits from the fast speed of stochastic variational inference but also overcomes the challenges of nonparametric variational inference discussed in Section 3.3. Moreover, it can automatically truncate the dimension of variational distributions in an adaptive fashion. We will show the detailed procedures in Sections 4.2 and 4.3.

4.2 The hybrid of optimisation and sampling

Under the conditional setting, we rely on the conditional variational inference framework in Section 3.4 to infer global variables, while approximate the optimal distributions of local variables via Monte Carlo sampling instead of analytical optimisation.

For the variational inference part, we approximate the posterior distribution for global prior G0 and global latent variable β. From the entire data x = {x1, . . . , xJ}, we randomly sample a subset {xs : xs ∈ x}Ss=1, where S is the batch size with S ≪ J. Assuming that a partition Ω is given to obtain the limit inferior of nonparametric evidence lower bound,
we aim to update the parameters for \( q(G_0^\Omega) \) conditional on the updated \( q(\beta) \) and \( \{q(z_s)\}_{s=1}^S \).

While standard stochastic variational inference uses the analytical way to update parameters, we draw \( T_s \) samples from \( q(z_s) \) for each \( z_s \) in the batch, \( \hat{z}_s = \{\hat{z}_{s,t} : \hat{z}_{s,t} \sim q(z_s)\}_{t=1}^{T_s} \), so as to get a random nonparametric evidence lower bound with respect to \( q(G_0^\Omega) \),

\[
\text{NPPELBO} = E_{q(G_0^\Omega)} \left[ \log \frac{p(G_0^\Omega)}{q(G_0^\Omega)} + \frac{J}{S} \sum_{s=1}^S \sum_{t=1}^{T_s} \log E_{p(G_0^\Omega | G_0^\Omega)} \{p(\hat{z}_{s,t} | G_0^\Omega)\} \right] + \text{constant.} \tag{13}
\]

It is obvious that \( E(\text{NPPELBO}) = \text{NPPELBO} \) and hence the random gradient is unbiased, which satisfies the key condition for stochastic variational inference. Therefore, according to (13), we can use the random gradient generated from \( \hat{z}_s \) to update the parameters of \( q(G_0^\Omega) \).

Analogously, the random nonparametric evidence lower bound with respect to \( q(\beta) \) is,

\[
\text{NPPELBO} = E_{q(\beta)} \left[ \log \frac{p(\beta)}{q(\beta)} + \frac{J}{S} \sum_{s=1}^S \sum_{t=1}^{T_s} \log p(x_s | \hat{z}_{s,t}, \beta) \right] + \text{constant,} \tag{14}
\]

and then we can update its parameter with the corresponding random gradient in a similar way.

For the Monte Carlo sampling part, given the updated \( q(G_0^\Omega) \) and \( q(\beta) \) from (13) and (14), we draw samples \( \hat{z}_s \) for each \( z_s \) in the batch using Markov chain Monte Carlo. It is difficult to get an closed-form formula for optimal \( q(z_s) \) due to the lack of conjugacy between \( G_0 \) and \( z_s \). Moreover, since \( G_s \) is integrated out, the local latent variables \( \{z_{si}\}_{i=1}^{N_s} \) are conditionally dependent and cannot be i.i.d. sampled. Therefore, we propose the following Gibbs sampling approach to get the samples under optimal variational distributions. Conditional on \( q(G_0^\Omega) \) and samples \( \hat{z}_{s,i^-} = \{\hat{z}_{sl} : l = 1, \ldots, N_s, l \neq i\} \), it follows from (11) that the optimal variational distribution of \( q(z_{si}) \) is

\[
q(z_{si}) \propto \exp \left\{ E_{q(G_0^\Omega)} E_{p(\Omega | G_0^\Omega)} [p(z_{si}, \hat{z}_{s,i^-} | G_0^\Omega)] + E_{q(\beta)} [\log p(x_{si} | z_{si}, \beta)] \right\}. \tag{15}
\]

Then we sample \( \hat{z}_{si} \sim q(z_{si}) \) for each \( i \) iteratively, which constructs a Markov chain. As \( \hat{z}_{si} \) is sampled from the optimised variational distribution conditional on \( \hat{z}_{s,i^-} \) in (15), the joint distribution generated from the Markov chain will converge to the optimal variational
distribution, which achieves the maximum nonparametric evidence lower bound. After the convergence, we can sample \( \hat{z}_{s,1}, \ldots, \hat{z}_{s,T_s} \) from the stable Markov chain.

To maximise the nonparametric evidence lower bound, we iterate the following three steps, (i) randomly selecting a small batch from the entire data, (ii) sampling \( \{\hat{z}_s\}_{s=1}^S \) by Monte Carlo method, and (iii) updating \( q(G_0^0) \) and \( q(\beta) \) in the stochastic variational inference framework. Moreover, the partition \( \Omega \) in our method is data-adaptive as demonstrated in Section 4.3.

### 4.3 Partition refinement and adaptive truncation

In this section, we illustrate the approach to determine the finite and measurable partition \( \Omega \), which could reach the limit inferior of nonparametric evidence lower bound. Rather than fixed on a universal truncation level, in our framework, the dimension of \( \Omega \) gradually increases to a stable level. This partition or truncation is dependent on data fitting and embedded within the optimisation process, which provides another key advantage of integrating the Monte Carlo sampling scheme into the stochastic variational inference framework.

We first define the partition \( \Omega \). Note that samples \( \{\hat{z}_s\}_{s=1}^S \) used to simulate the optimal variational distribution have finite-dimensional atomic support, denoted by \( \phi_1, \ldots, \phi_K \), where \( K \) is a finite integer. We therefore partition the sample space \( \Omega \) into \( K+1 \) disjoint subsets including \( K \) probability mass atoms \( \phi_1, \ldots, \phi_K \) and one complement set \( \phi_0 = \Omega/\{\phi_1, \ldots, \phi_K\} \).

We then update the partition \( \Omega \) in the inference procedure. If all points in \( \{\hat{z}_s\}_{s=1}^S \) are sampled before, we keep the current partition \( \Omega \). Otherwise, if a sample \( \hat{z}_{si} \in \phi_0 \), which means it is distinct from \( \phi_1, \ldots, \phi_K \), we draw a new \( \phi_{K+1} \) and refine the partition as \( \{\phi_0, \phi_1, \ldots, \phi_K, \phi_{K+1}\} \), where we update \( \phi_0 \) as \( \Omega/\{\phi_1, \ldots, \phi_K, \phi_{K+1}\} \). With the dynamic partition \( \Omega \) defined above, the prior is proportional to the posterior on \( \phi_0 \), \( p(G_0(\phi_0)) \propto q(G_0(\phi_0)) \), due to the lack of data information. Therefore, the nonparametric evidence lower bound will remain constant under any further partition, which means the partition \( \Omega \) enables the nonparametric evidence lower bound to attain its limit inferior.
Algorithm 1: Conditional variational inference with adaptive truncation.

1. Initialise the partition $\Omega$, the parameters for $q(G_0), q(\beta)$ and set up the step-size $\{\rho_r\}_{r \geq 1}$;

2. repeat
   3. Randomly select $x_1, \ldots, x_S$ from the entire dataset;
   4. for $s \in \{1, \ldots, S\}$ do
      5. Initialise the values for $\{\hat{z}_{si}\}_{i=1}^{Ns}$;
      6. repeat
         7. for $i \in \{1, \ldots, N_s\}$ do
            8. Sample $\hat{z}_{si}$ conditional on $q(G_0), q(\beta)$ and $\hat{z}_{s,i-}$ according to (15);
            9. if Sampling a new $\hat{z}_{si}$ then
               10. Refine the partition $\Omega$;
            until Convergence;
      11. Sample $\{\hat{z}_{s,t}\}_{t=1}^{Ts}$ from the stable Markov chain;
      12. Update the parameters for $q(G_0)$ and $q(\beta)$ given the samples $\{\hat{z}_s\}_{s=1}^{S}$ with step-size $\rho_r$ according to (13) and (14);
   until Convergence;

In our framework, we start from a low-dimensional partition when the variational distributions are far from the optimal, then update the partition and gradually increase its dimension according to data fitting. When the inference is close to convergence with a large enough dimensional partition, it is less likely to refine the partition and hence the dimension of variational distributions attains a stable level. This data-adaptive truncation reflects a balance between the goodness-of-fit and model complexity. We summarise the above inference procedure in Algorithm 1.
5 Applications in topic modelling

5.1 Inference of hierarchical Dirichlet process model

We apply the proposed conditional variational inference with adaptive truncation method to the hierarchical Dirichlet process model. Specifically, we factorise the variational distributions in the conditional setting according to (7) and specify the variational family as follows. First, the variational distribution of $G_s$ for each $s$ is given by $q(G_s \mid G_0, z_s) = \text{DP}(\sum_{k=1}^{\infty} n_{sk} \delta_{\phi_k} + G_0)$, where $n_{sk} = \sum_{i=1}^{N_s} I(z_{si} = \phi_k)$ with $I(\cdot)$ being the indicator function. Second, $q(\beta_k)$ for each topic $k$ is set as a $W$-dimensional Dirichlet distribution, $q(\beta_k) = \text{Dirichlet}(\lambda_k)$, where $\lambda_k = (\lambda_{k1}, \ldots, \lambda_{kW})^T$ is the parameter of vocabulary distribution for topic $k$. To make prediction, $\lambda_k$ serves as the core task of inference. Specially, the variational distributions for the topics without any observation remain the same as the prior. Therefore, we regard them as the zeroth topic without loss of generality and denote the corresponding variational distribution on vocabulary by $q(\beta_0) = \text{Dirichlet}(\eta)$. Third, we propose the variational family for $G_0$ as,

$$q(G_0) = \sum_{k=1}^{K} m_k \delta_{\phi_k} + m_0 \text{DP}(\alpha H),$$

such that $\sum_{k=1}^{K} m_k = 1$ and $\phi_k \sim H$ due to the lack of posterior information for $\phi_k$. Taking into account the tradeoff between inferential accuracy and computational efficiency, in (16) we assume that $q(G_0)$ have deterministic probability mass on $\phi_k$s, as the main purpose of $G_0$ is to provide a discrete and infinite-dimensional support to ensure that $G_j$s share the same topics $\phi_k$s. This kind of spike and slab methodology is widely used in Bayesian analysis (Andersen et al., 2017). Under such scenario, the optimised $\{m_k\}_{k=1}^{K}$ coincide with maximum-a-posteriori estimation. Finally, following (15) we use Monte Carlo sampling to get samples $\{\hat{z}_s\}_{s=1}^{S}$ and hence do not need to parametrise their variational distributions. Based on these settings, we can infer the hierarchical Dirichlet process model by applying Algorithm 1 in the following steps.
The partition $\Omega$. As different samples in $\{\hat{z}_s\}_{s=1}^S$ are used to represent different topic clusters in topic modelling, their exact values in sample space do not contain any statistical information. We then index the topics with observations from 1 to $K$ and denote the different clusters by distinct points $\phi_1, \ldots, \phi_K$ in $\Omega$. With the samples $\{\hat{z}_s\}_{s=1}^S$, we define $\hat{n}_{sk,t} = \sum_{s=1}^{N_s} I(\hat{z}_{si,t} = \phi_k)$. Then the number of topics with observations is $K = \sum_{k=0}^{S} \sum_{s=1}^{K} T_s^{-1} \sum_{t=1}^{T_s} \hat{n}_{sk,t} > 0$. We partition $\Omega$ to a $(K+1)$-dimensional $\Omega$ including $K$ single points $\{\phi_k\}_{k=1}^K$ and one complement set $\phi_0 = \Omega/\{\phi_k\}_{k=1}^K$.

Inference for $G_0$. With the partition $\Omega$ defined above, $G_0^\Omega$ conditional on $G_0^\Omega$ is a $(K+1)$-dimensional Dirichlet distribution. By (13), we derive the random nonparametric evidence lower bound with respect to $q(G_0)$ in Appendix A.5,

$$\text{NPPELBO} = - \sum_{k=0}^{K} \log m_k + \alpha \log m_0 + \frac{J}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} T_s^{-1} \sum_{t=1}^{T_s} \log \frac{\Gamma(\gamma m_k + \hat{n}_{sk,t})}{\Gamma(\gamma m_k)} + \text{constant.} \quad (17)$$

However, there is no closed-form expression for the probability proportion parameters $\{m_k\}_{k=1}^K$ to attain the maximum in (17). Moreover, the standard gradient descent algorithm fails in this case, because $\{m_k\}_{k=1}^K$ may easily exceed the simplex during the updating procedure. Instead, given the parameters $\{m_k^{(\tau)}\}_{s=1}^S$ in the $\tau$-th iteration, we define

$$m_k^* \begin{cases} J S^{-1} \gamma \sum_{s=1}^{S} \{ T_s^{-1} \sum_{t=1}^{T_s} \Phi(\gamma m_k^{(\tau)} + \hat{n}_{sk,t}) - \Phi(\gamma m_k^{(\tau)}) \} m_k^{(\tau)} - 1 \quad (k = 1, \ldots, K), \\ \alpha - 1 \quad (k = 0), \end{cases} \quad (18)$$

such that $\sum_{k=0}^{K} m_k^* = 1$, and update the parameters by $m^{(\tau+1)} = (1 - \rho_t)m^{(\tau)} + \rho m_k^*$. In Appendix A.6, we also show that this updating algorithm is consistent to the gradient descent after the inverse logit transformation. In the process of updating, the condition $\sum_{k=0}^{K} m_k^* = 1$ always holds and hence we eliminate the risk of exceeding the simplex.

Inference for $\beta$. By (14), we update the parameters for $q(\beta)$ using samples $\{\hat{z}_s\}_{s=1}^S$. We define $\lambda_{kw}^*$ for topic $k$ and word $w$ as,

$$\lambda_{kw}^* = \eta + JS^{-1} \sum_{s=1}^{S} \sum_{t=1}^{T_s} \sum_{i=1}^{T_i} I(\hat{z}_{si,t} = \phi_k, x_{si} = w), \quad (19)$$
and update the parameter $\lambda_k$ by $\lambda_k^{(r+1)} = (1 - \rho_t)\lambda_k^{(r)} + \rho_t\lambda^*_{k}$ for each $k$, where $\lambda^*_{k} = (\lambda^*_{k1}, \ldots, \lambda^*_{kW})^T$.

**Sampling for $z$.** According to (15) we sample $\hat{z}_{si}$ conditional on $q(G_0)$ and $\hat{z}_{si-}$ by

$$
q(\hat{z}_{si} = \phi_k) \propto \begin{cases} 
(\gamma m_k + \hat{n}^k_{si}) \exp \left( \Phi(\lambda_{kx_{si}}) - \Phi(\sum_{w=1}^{W} \lambda_{kw}) \right) & (k = 1, \ldots, K), \\
\gamma m_0 \exp \left( \Phi(\eta) - \Phi(W\eta) \right) & (k = 0),
\end{cases}
$$

(20)
to construct the Markov chain, where $\hat{n}^k_{si} = \sum_{l\in I_i} I(\hat{z}_{sl} = \phi_k)$. Whenever the sampled $\hat{z}_{si}$ is $\phi_0$, which means $\hat{z}_{si}$ forms a new point not belonging to $\{\phi_1, \ldots, \phi_K\}$, we need to update the partition and add a new topic indicated by $\phi_{K+1}$. Otherwise we stick to the same partition dimension. Iterating the sampling scheme till convergence, we obtain the samples $\{\hat{z}_{si,t}\}_{1\leq s\leq S,1\leq i\leq N_s,1\leq t\leq T_s}$ and corresponding $\{\hat{n}_{sk,t}\}_{1\leq s\leq S,1\leq k\leq K,1\leq t\leq T_s}$ for the selected chunk.

According to Algorithm 1, we repeatedly select documents in a batch with randomness, sample $z$ and update parameters for $G_0$, $\beta$ by iterating (18)–(20) until the nonparametric evidence lower bound converges to its maximum.

Our method is different from other nonparametric inference methods. **Wang & Blei (2012)** replace analytical updating for local parameters by the locally collapsed Gibbs sampling. But their work cannot maximise the evidence lower bound, especially when $q(\beta)$ has large variance. **Bryant & Sudderth (2012)** use split-merge algorithms to generate new dimensions and remove redundant dimensions. However, to check the split-merge criterion, their method needs to calculate the training likelihood, which is computationally inefficient. Moreover, both methods are based on the mean-field assumption and hence ignore the correlation structure among latent variables.

### 5.2 Extension under a general completely random measure

The algorithm of conditional variational inference with adaptive truncation can also be applied to a general class of hierarchical Bayesian nonparametric models, where the global
prior $G_0$ is generated from a completely random measure. For example, gamma–Dirichlet process model uses gamma process to generate $G_0$ and Dirichlet process to generate $\{G_j\}_{j=1}^J$. In these models, concentration parameter for any $G_j$ is not fixed and $G_0$ is not restricted to be a probability measure. The corresponding inference algorithm is similar to that of hierarchical Dirichlet process model, but requires a new parameter $\mu$ to approximate $G_0(\Omega)$. We choose the variational family for global prior $G_0$ as,

$$Q(G_0) = \mu \left( \sum_{k=1}^{K} m_k \delta_{\phi_k} + m_0 \tilde{N}(\alpha H) \right),$$

(21)

where $\tilde{N}$ is the normalisation of the corresponding completely random measure and $\sum_{k=0}^{K} m_k = 1$. Similarly, we derive the random nonparametric evidence lower bound in Appendix A.7,

$$\text{NPELBO} = K \log \mu + \sum_{k=1}^{K} \log v(\mu m_k) + \log u(\mu m_0)$$

$$+ \frac{J}{S} \sum_{s=1}^{S} \left\{ \log \frac{\Gamma(\mu)}{\Gamma(\mu + N_s)} + T_s^{-1} \sum_{t=1}^{T_s} \sum_{k=1}^{K} \log \frac{\Gamma(\mu m_k + \hat{n}_{sk,t})}{\Gamma(\mu m_k)} \right\} + \text{constant},$$

(22)

where $v(\cdot)$ is the weight intensity measure (Appendix A.1) for the completely random measure and $u$ is the density function for $G_0(\Omega)$ that can be derived given the Laplace transform of the completely random measure. Therefore, we can update $\{m_k\}_{k=0}^{K}$ in the same way as the hierarchical Dirichlet model. Following Algorithm 1 and its application in Section 5.1, we can also update $\mu$ by the stochastic gradient descent. To illustrate with an example, we consider the gamma–Dirichlet model, whose inference algorithm is provided in Appendix A.7.

### 5.3 Real data analysis

We apply the algorithm of conditional variational inference with adaptive truncation to three large datasets.

1. **arXiv**: The corpus includes the descriptive metadata of all articles on arXiv, a free distribution service and an open archive for scholarly articles, up to September 1, 2019, which includes 1.03M documents and 44M words from a vocabulary of 7,500 terms after preprocessing.
2. **New York Times**: The corpus combines all articles published by *New York Times* from January 1, 1987 to June 19, 2007 ([Sandhaus, 2008](#)), which has 1.56M documents and 176M words from a vocabulary of 7,600 terms after preprocessing.

3. **Wikipedia**: The corpus collects the entire entries on English *Wikipedia* websites on January 1, 2019, which contains 4.03M documents and 423M words from a vocabulary of 8,000 terms after preprocessing.

In the preprocessing, stemming and lemmatisation are used to clean the raw text data. Moreover, words with too high or too low frequency and common stop words are both removed before experiments.

To evaluate the performance of our proposed method, we set aside a test set for 10,000 documents for each dataset and calculate the hold-out perplexity ([Ranganath & Blei, 2018](#)),

$$
\text{perplexity}_{\text{hold-out}} = \exp \left\{ - \frac{\sum_{j \in D_{\text{test}}} \log p(x_j^{\text{test}} | x_j^{\text{train}}, D_{\text{train}})}{\sum_{j \in D_{\text{test}}} |x_j^{\text{test}}|} \right\},
$$

where $D_{\text{train}}$ and $D_{\text{test}}$ represent the training and test data, respectively, and $x_j^{\text{train}}$ and $x_j^{\text{test}}$ are the training and test words in test document $j$, respectively, and $|x_j^{\text{test}}|$ is the number of words in $x_j^{\text{test}}$. The perplexity measures the uncertainty of the fitted model, and hence a better language model with more accurate inference will have a higher predictive likelihood and thus a lower perplexity. Since the exact computation for perplexity is not tractable, the standard routine uses $D_{\text{train}}$ to get the variational distribution for $\beta$ and $G_0$, obtains the variational distribution for $G_j$ based on $G_0$ and $x_j^{\text{test}}$, and then approximates the likelihood by $p(x_j^{\text{test}} | x_j^{\text{train}}) = \prod_{w \in x_j^{\text{test}}} \sum_{k=0}^{K} \overline{G}_{jk} \overline{\beta}_k$, where $\overline{G}_j = (\overline{G}_{j0}, \ldots, \overline{G}_{jK})^T$ and $\overline{\beta}_k = (\overline{\beta}_{k1}, \ldots, \overline{\beta}_{kW})^T$ are the variational expectations of $G_j$ and $\beta_k$, respectively ([Blei et al., 2003](#)). We model three datasets under both hierarchical Dirichlet process and gamma–Dirichlet process models. For hierarchical Dirichlet process model we set the hyperparameters as $\alpha = \gamma = \eta = 5$, where $\alpha$ and $\gamma$ are the concentration parameters for $G_0$ and $\{G_j\}$ respectively, and $\eta$ is the hyperparameter of prior on the distribution of words. We choose a batch size of 256 and adopt the Robbins Monro learning rate $(64 + t)^{-0.6}$ in updating ([Hoffman et al., 2010](#)). The
Figure 2: Top row: plots for the perplexity vs the running time up to 5 hours. Bottom row: plots for the number of topics vs the running time up to 5 hours. Left, middle and right columns correspond to datasets *arXiv*, *New York Times* and *Wikipedia*, respectively. The black dotted line corresponds to traditional online variational inference method for hierarchical Dirichlet process model. The red solid and blue dashed lines correspond to conditional variational inference with adaptive truncation method for hierarchical Dirichlet process model and for gamma–Dirichlet process model, respectively.
initial number of topics is chosen as 100. For gamma–Dirichlet process model, we use the same hyperparameters but discard $\gamma$. To make comparison, we keep the default settings in traditional online variational inference (Wang et al., 2011).

The top row of Figure 2 plots the hold-out perplexity as a function of running time for three comparison methods on three datasets. Table 1 reports numerical summaries. Several conclusions can be drawn here. First, on all three datasets, our conditional variational inference with adaptive truncation method consistently outperforms the traditional online variational inference method. The improvement is highly significant especially for arXiv and Wikipedia. For New York Times, such improvement is moderate probably due to the long length of documents in this corpus. Second, for each dataset, the gamma–Dirichlet process model attains a lower perplexity than hierarchical Dirichlet process model, which makes sense due to the fact that the gamma–Dirichlet process model removes a restriction of the hierarchical Dirichlet process model and hence is more flexible. Third, the proposed method is computationally efficient. Although it involves Monte Carlo sampling, the perplexity converges at a fast speed. This is because the convergence of local Markov chain to assign words to topics is accelerated by a clear topic-words clustering as the global variational distributions approach to the optimal.

The bottom row of Figure 2 plots the number of topics in the process of inference. For traditional online variational inference, the number of topics remains constant, while for our method, it first has a steep increase and then converges to a stable level. For example, the number of topics in Wikipedia drastically increase from 100 to around 190 for hierarchical Dirichlet process model and around 200 for gamma–Dirichlet process model. The sharp increase is driven by the data complexity, while the stable level is due to the dimension penalty of hierarchical Dirichlet process model. Although the estimation of the number of topics is not consistent, the proposed truly nonparametric inference method can provide some useful information about topics in data. For instance, arXiv, containing the abstract descriptions of scientific articles, has the smallest number of topics because its topics are
Table 1: A summary of hold-out perplexity results on three datasets. Relative improvements in percentage over TOVI for HDP model are shown in parentheses.

| Inference method | arXiv   | New York Times | Wikipedia  |
|------------------|---------|----------------|------------|
| HDP model        | TOVI    | 1005           | 1681       | 1422       |
| HDP model        | CVIAT   | 832 (17.21%)   | 1569 (6.66%) | 1207 (15.12%) |
| ΓDP model        | CVIAT   | 808 (19.60%)   | 1536 (8.62%) | 1157 (18.64%) |

TOVI, traditional online variational inference; CVIAT, conditional variational inference with adaptive truncation. HDP, hierarchical Dirichlet process; ΓDP, gamma–Dirichlet process.

restricted to the quantitative subjects including computer science, mathematics, statistics and physics. By contrast, *New York Times* is a compilation of all new articles covering a wider range of areas, and hence consists of more topics. *Wikipedia* has the largest number of topics as it contains almost every aspect of an encyclopedia. One key point here is that we do not need to set a fixed number of topics before the inference. Instead, our method starts from an initial value, for example 100 in our experiments, then automatically reaches the optimal number of topics after iterations, and finally keeps it at a stable level.

Moreover, our method reveals better linguistic results. To compare our proposed method with traditional online inference for hierarchical Dirichlet process model, we report the top 12 words in the top 10 topics with biggest weights for both methods on datasets *arXiv* and *Wikipedia* in Tables 2a and 2b, respectively. We observe a few apparent patterns. First, our method does not contain replicated topics. The traditional online variational inference method results in very similar word components, for example, columns 1-6 in the bottom part of Table 2a. An ideal topic-word clustering should allocate these words into just one topic. But since the prespecified number of topics is fixed at 150, which seems larger than the truth, the inference generates multiple replicated topics. By contrast, the topic-word
Table 2: Top 12 words in top 10 topics for datasets *arXiv* and *Wikipedia*

(a) *arXiv*

|   |  1   |  2   |  3   |  4   |  5   |  6   |  7   |  8   |  9   | 10  |
|---|------|------|------|------|------|------|------|------|------|-----|
| **CVIAT** | galaxy | cluster | halo | star | galaxy | CVIAT | tell | effect | want | friend |
|   | group | algebra | source | xray | neutrino | survey | tell | process | case | process |
|   | network | learn | abundance | emiss | neutrino | cluster | emotion | vocal | song | feel |
|   | neutrino | higg | region | star | line | emiss | track | word | tradition | rate |
|   | star | dwarf | gas | line | neutrino | xray | idea | review | charles | decide |
|   | gaug | string | star | neutrino | xray | mass | richard | frank | polit |  
|   | prove | bound | boson | neutrino | xray | view | michael | frank | battle | 
|   | algorithm | optim | dwarf | neutrino | xray | view | agreement | proposal |  
|   | collision | product | dwarf | neutrino | xray | view | military | soldier | 
|   | test | error | dwarf | neutrino | xray | view | test | experience | 
|   | | | | | | | | | 
| **TOVI** | galaxy | cluster | halo | star | galaxy | TOVI | album | episode | song | studio |
|   | group | algebra | source | xray | neutrino | album | actor | album | song | love |
|   | network | learn | abundance | emiss | neutrino | episode | character | character | character | character |
|   | neutrino | higg | region | xray | neutrino | character | character | character | character | character |
|   | star | dwarf | gas | neutrino | xray | review | character | character | character | character |
|   | gaug | string | star | neutrino | xray | review | character | character | character | character |
|   | prove | bound | boson | neutrino | xray | review | character | character | character | character |
|   | algorithm | optim | dwarf | neutrino | xray | review | character | character | character | character |
|   | collision | product | dwarf | neutrino | xray | review | character | character | character | character |
|   | test | error | dwarf | neutrino | xray | review | character | character | character | character |
|   | | | | | | | | | 

(b) *Wikipedia*

|   |  1   |  2   |  3   |  4   |  5   |  6   |  7   |  8   |  9   | 10  |
|---|------|------|------|------|------|------|------|------|------|-----|
| **CVIAT** | tell | increase | band | human | james | claim | armies | process | polit | album |
|   | effect | album | natur | robert | issue | battle | model | parti | chart |  
|   | want | case | guitar | tradition | charles | announce | attack | inform | union | song |
|   | friend | process | vocal | term | david | critic | troop | effect | communists | track |
|   | ask | measure | track | idea | thomas | controversi | command | problem | movement | video |
|   | leave | require | drum | word | michael | agreement | military | test | social | label |
|   | feel | require | drum | word | michael | agreement | military | test | social | label |
|   | decide | rate | bass | theorie | frank | polit | fight | example | republic | peak |
|   | turn | example | song | philosophies | peter | allocation | tanks | research | leader | week |
|   | good | reduce | tour | culture | andrew | statement | brigade | specific | worker | digitated |
|   | away | possibilities | studio | believe | brown | agree | german | individual | socialist | hot |
|   | believe | occur | label | conception | henry | minister | capture | object | liberal | remix |

| **TOVI** | album | episode | actor | album | album | episode | character | novel | animal | ship |
|   | band | tell | movi | song | song | televis | kill | character | episode | navies |
|   | song | character | character | band | chart | drama | human | love | character | class |
|   | track | kill | critic | tour | video | actor | earth | poem | voice | boat |
|   | rock | tried | televis | artist | love | movi | episode | tell | air | vessel |
|   | vocal | leave | review | blue | billboard | actress | reveal | king | video | command |
|   | review | rock | label | theatre | com | fiction | dvd | submarine |  
|   | chart | need | scene | track | version | voice | fight | narrated | televis | gun |
|   | studio | love | theatre | label | week | uncredit | doctor | friend | ray | fleet |
|   | bass | reveal | picture | chart | peak | nominal | battle | mother | blu | sail |
|   | drum | mother | love | singer | remix | cast | voice | critic | song | destroy |
clustering by our proposed method does not have such redundancy. It is apparent that our top 10 topics are distinct. Second, our method leads to much clearer topic-word clustering. For both datasets, our results indicate that all of our detected words within any column are highly relevant and should intuitively be grouped into one cluster with clear linguistic meaning. For example, our method in column 7 of Table 2b presents several words all related to military, but words in the same column for traditional online variational inference seems to be a mixture of several loosely connected topics including ‘human, character, reveal, episode, comic, voice’, ‘human, earth’ and ‘human, kill, attack, fight, battle, doctor’. This mixture of topics makes the topic-word clustering within this column ambiguous. Furthermore, our method in column 5 of Table 2b identifies a topic about popular English given names. Although these given names are not shown in one document, our method can successfully discover that they belong to one topic, while the traditional online variational inference fails. This is because our method does not force the topics to merge together if the prespecified number of topics is not large enough, and hence can largely reduce the noise.

6 Discussion

Within the proposed general framework, Algorithm 1 can also be applied to other hierarchical Bayesian nonparametric models including hierarchical Pitman–Yor process model (Teh & Jordan, 2010) and hierarchical beta process model (Thibaux & Jordan, 2007), which are used to present the power law and the sparsity in latent features, respectively. In such cases, other Monte Carlo methods, for example, slice sampling (Neal, 2003), retrospective Markov chain Monte Carlo (Papaspiliopoulos & Roberts, 2008) or unbiased Markov chain Monte Carlo methods with couplings (Jacob et al., 2019), could possibly be used. We expect that our proposed method provides more advantages in these applications, because hierarchical Pitman–Yor process with heavy tail behaviour and hierarchical beta process with sparse structure may suffer more from the universal truncation.
A Appendix

A.1 A short review of completely random measures

A completely random measure (Kingman, 1993) is characterised by its Laplace transform,

\[ E\{e^{-tP(A)}\} = \exp \left\{ - \int_A \int_{[0, \infty]} (1 - e^{-tw})v^c(dx, ds) \right\}, \]

where \( A \) is any measurable subset of \( \Omega \) and \( v^c(dx, ds) \) is called the Lévy measure. If \( v^c(dx, ds) = \kappa(dx)v(ds) \), where \( \kappa(\cdot) \) and \( v(\cdot) \) are measures on \( \Omega \) and \((0, \infty] \), respectively, the completely random measure is homogeneous (Ghosal & Van der Vaart, 2017). In such case, we call \( v(\cdot) \) the weight intensity measure. We can view completely random measure as a Poisson process on the product space \( \Omega \times (0, \infty] \) using its Lévy measure as the mean measure.

A.2 Derivation for (4)

By definition of induced measure, \( q^\Omega(d\Theta) = Q(d\Theta) \) for any \( \mathcal{M} \)-measurable \( d\Theta \), we have

\[ \int_{\Theta} \log \frac{dq^\Omega}{dp^\Omega}dq^\Omega = \int_{\Theta} \log \frac{dq^\Omega}{dp^\Omega}dQ. \]

It follows from \( \limsup_{\Omega} dq^\Omega / dp^\Omega = dQ / dP \) and the monotone convergence theorem that

\[ \limsup_{\Omega} \int_{\Theta} \log \frac{dq^\Omega}{dp^\Omega}dQ = \int_{\Theta} \log \frac{dQ}{dP}dQ. \]

Combining the above equations yields (4). Furthermore, suppose there exists a sequence of partition \( \{\Omega_i\}_{i \geq 1} \) such that \( \limsup \Omega_i = \Omega \), we have

\[ \limsup_{\Omega_i} \int_{\Theta} \log \frac{dq_i^\Omega}{dp^\Omega_i}dq_i^\Omega = \limsup_{\Omega_i} \int_{\Theta} \log \frac{dq_i^\Omega}{dp^\Omega_i}dQ = \int_{\Theta} \log \frac{dq^\Omega}{dp^\Omega}dQ = \int_{\Theta} \log \frac{dq^\Omega}{dp^\Omega}dq^\Omega. \]

Hence \( \limsup KL(q^{\Omega_i} \parallel p^{\Omega_i}) = KL(q^{\Omega} \parallel p^{\Omega}) \), which will be used in Appendix A.5.

A.3 Derivation for (6)

By \( p(X, Z) = p(Z \mid X)p(X) \), we have

\[ \int \log \frac{p(X, Z^\Omega)}{q(Z^\Omega)}q(dZ^\Omega) = \log p(X) + \int \log \frac{p(Z^\Omega)}{q(Z^\Omega)}q(dZ^\Omega). \]
Taking the limit inferior on both sides, we have
\[
\lim_{\alpha} \inf \left\{ \int \log \frac{p(X, Z^{\alpha})}{q(Z^{\alpha})} q(dZ^{\alpha}) \right\} = \log p(X) - \limsup_{\alpha} \left\{ - \int \log \frac{p(Z^{\alpha})}{q(Z^{\alpha})} q(dZ^{\alpha}) \right\}.
\]
Combing the above equation with the definition of nonparametric evidence lower bound in (5) and Kullback–Leibler divergence in (4) yields (6).

### A.4 Derivation for (9)

By \( p(G^{\alpha}_{0}, \{z_j\}_{j=1}^{J}) = \int \cdots \int p(G^{\alpha}_{0}, \{G_j\}_{j=1}^{J}, \{z_j\}_{j=1}^{J}) dG_1 dG_2 \cdots dG_J \) and the hierarchical generative structure, the evidence lower bound under partition \( \Omega \) with respect to \( q(G^{\alpha}_{0}) \) equals,
\[
\text{ELBO}^{\Omega} = E_{q(G^{\alpha}_{0})} \left[ \log p(G^{\alpha}_{0}, \{z_j\}_{j=1}^{J}) \right] - E_{q(G^{\alpha}_{0})} \left[ \log q(G^{\alpha}_{0}) \right] + \text{constant}
\]
\[
= E_{q(G^{\alpha}_{0})} \left[ \log q(G^{\alpha}_{0}) \prod_{j=1}^{J} \int p(G_j^{\alpha} | G_0^{\alpha}) p(z_j | G_j^{\alpha}) dG_j \right] - E_{q(G^{\alpha}_{0})} \left[ \log q(G^{\alpha}_{0}) \right] + \text{constant}
\]
\[
= \sum_{j=1}^{J} E_{q(G^{\alpha}_{0})} \left[ \log E_{p(G_j^{\alpha} | G_0^{\alpha})} \left[ p(z_j | G_j^{\alpha}) \right] \right] + E_{q(G^{\alpha}_{0})} \left[ \log p(G^{\alpha}_{0}) - \log q(G^{\alpha}_{0}) \right] + \text{constant}.
\]
Furthermore, based on the equation above, (8) can be expressed as NPELBO = \( \liminf_{\alpha} \text{ELBO}^{\Omega} \).

### A.5 Derivation for (17)

By the formula of moments for Dirichlet-distributed random variables, we obtain
\[
E_{p(G^{\alpha}_{0} | G_0^{\alpha})} \left[ p(\hat{z}_{s,t} | G^{\alpha}_{0}) \right] = \frac{\Gamma(\gamma)}{\Gamma(\gamma + N_s)} \prod_{k=1}^{K} \frac{\Gamma(\gamma G_{0k} + \hat{n}_{sk,t})}{\Gamma(\gamma G_{0k})}.
\]
Based on the points \( \{\phi_k\}_{k=1}^{K} \) defined in Section 5.1, we propose a sequence of partition \( \{\Omega_c : \Omega_c = \bigcup_{k=0}^{K} \Omega_{ck} \}_{c \geq 1} \) to approach \( \Omega \), where \( \Omega_{ck} = (\phi_k - c^{-1}, \phi_k + c^{-1}] \) for \( k = 1, \ldots, K \) and \( \Omega_{c0} \) is the corresponding complement. Under \( \Omega_c \), \( q(G^{\alpha}_{0c}) = d_{K+1}(m_0^{-1}(G^{\alpha}_{0} - M^{\alpha_{c}})) \) and \( p(G^{\alpha}_{0c}) = d_{K+1}(G^{\alpha_{c}}) \), where \( d_{K+1}(\cdot) \) denotes the density function for \((K + 1)\)-dimensional Dirichlet distribution, \( M = \sum_{k=1}^{K} m_k \delta_{\phi_k} \) and \( M^{\alpha_{c}} \) is the corresponding induced random variable. By (13), the random nonparametric evidence lower bound under \( \Omega_c \) is
\[
\text{NPELBO}^{\Omega_c} = E_{q(G^{\alpha}_{0c})} \left\{ \sum_{k=1}^{K} (\alpha H_k^{\Omega_c} - 1) \log \frac{m_0 G_{0k}}{G_{0k} - m_k} + (\alpha H_k^{\Omega_c} - 1) \log m_0 \right. \\
+ \left. \frac{J}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} T_s^{-1} \sum_{t=1}^{T_s} \log \frac{\Gamma(\gamma G_{0k} + \hat{n}_{sk,t})}{\Gamma(\gamma G_{0k})} \right\} + \text{constant},
\]
where $H_k^{C} = H(\Omega_k)$. Since, $(G_{0k} - m_k)/m_0 \sim \text{Beta}(H_k^{C})$ under $q(G_0^{C})$, the term $E_{q(G_0^{C})}\{(\alpha H_k^{C} - 1) \log m_0(G_{0k} - m_k)^{-1}\}$ is constant with respect to parameters $\{m_k\}_{k=0}^K$. Taking limsup on both sides of the above equation with $\limsup_{\Omega_C} E_{q(G_0^{C})}(\log G_{0k}) = \log m_k$, $\limsup_{\Omega_C} H_k^{C} = 0$ for $k = 1, \ldots, K$ and $\limsup_{\Omega_C} H_0^{C} = 1$, we obtain equation (17).

### A.6 Derivation for (18)

Consider the Lagrange multiplier of constrained optimisation,

$$L' = -\sum_{k=1}^{K} \log m_k + (\alpha - 1) \log m_0 + \frac{J}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} T_{s}^{-1} \sum_{t=1}^{T_s} \log \frac{\Gamma(\gamma m_k + \hat{n}_{sk,t})}{\Gamma(\gamma m_k)} - \lambda(\sum_{k=1}^{K} m_k - 1),$$

its first order conditions satisfy,

$$\begin{align*}
&JS^{-1}\gamma \sum_{s=1}^{S} \{T_{s}^{-1} \sum_{t=1}^{T_s} \Phi(\gamma m_k + \hat{n}_{sk,t}) - \Phi(\gamma m_k)\} m_k - 1 = m_k \lambda, \quad (k = 1, \ldots, K), \\
&\alpha - 1 = m_0 \lambda, \\
&(k = 0).
\end{align*}$$

Dividing $\lambda$ on both sides of the above equations, the definition of $\{m^*_k\}_{k=0}^{K}$ in (18) follows.

We next show that this updating is consistent to the gradient descent after the inverse logit transformation, that is, transforming $\{m_k\}_{k=0}^{K}$ by $m_k = e^{\theta_k} / \sum_{t=0}^{K} e^{\theta_t}$ to remove the constraint of $\sum_{k=0}^{K} m_k = 1$. By $\partial m_k / \partial \theta_k = m_k - m_k^2$, $\partial m_l / \partial \theta_k = -m_k m_l$ for $l \neq k$, and the chain rule, we have

$$\begin{align*}
\frac{\partial L}{\partial \theta_k} &= J S^{-1} \gamma \sum_{s=1}^{S} \{T_{s}^{-1} \sum_{t=1}^{T_s} \Phi(\gamma m_k + \hat{n}_{sk,t}) - \Phi(\gamma m_k)\} m_k - 1 - \Lambda m_k \quad (k = 1, \ldots, K), \\
&\alpha - 1 - \Lambda m_k \quad (k = 0),
\end{align*}$$

where $L$ denotes NPELBO in (17) and

$$\Lambda = \alpha - 1 + \sum_{k=1}^{K} \left[JS^{-1} \gamma \sum_{s=1}^{S} \{T_{s}^{-1} \sum_{t=1}^{T_s} \Phi(\gamma m_k + \hat{n}_{sk,t}) - \Phi(\gamma m_k)\} m_k - 1\right].$$

As $\partial L / \partial \theta_k = \Lambda (m_k^* - m_k)$, $(m_k^* - m_k)$ represents the gradient with respect to $\{\theta_k\}_{k=0}^{K}$ after the inverse logit transformation.
A.7 Derivation for the extension in Section 5.2

Without restriction on probability random measure,

\[
\log E_{p(G_s^2 | G_0^2)}[p(\hat{z}_{s,t} \mid G_s^2)] = \log \frac{\Gamma(\sum_{k=0}^{K} G_{0k})}{\Gamma(\sum_{k=0}^{K} G_{0k} + N_s)} \prod_{k=1}^{K} \frac{\Gamma(G_{0k} + \hat{n}_{sk,t})}{\Gamma(G_{0k})},
\]

In analogy to Appendix A.5, under a partition \( \Omega_c \), the random nonparametric evidence lower bound equals,

\[
\text{NPELBO}^{\Omega_c} = K \log \mu + E_{q(G_0^c)} \left[ \sum_{k=1}^{K} \log p(G_{0k}^c) + \log p(G_{00}^c) \right.
\]
\[\left. + \frac{J}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} T_s^{-1} \sum_{t=1}^{T_s} \log \frac{\Gamma(\gamma G_{0k}^c + \hat{n}_{sk,t})}{\Gamma(\gamma G_{00k}^c)} \right] + \text{constant},
\]

where \( K \log \mu \) comes from the Jacob matrix from \( G_0, G_1, \ldots, G_K \) to \( \mu, m_1, \ldots, m_K \). As the partition converges to single points and the corresponding complement, \( \limsup_{\Omega_c} p(G_{0k}^c) = v(G_{0k}^c) \), \( \limsup_{\Omega_c} p(G_{00}^c) = u(G_{00}^c) \), we can have (22) by \( \limsup_{\Omega_c} G_{0k} = \mu m_k \) for \( k \neq 0 \) and \( \limsup_{\Omega_c} G_{00} = \mu m_0 \). Specially, for the gamma–Dirichlet model,

\[
\text{NPELBO} = - \mu - \sum_{k=1}^{K} \log m_k + (\alpha - 1) \log \mu m_0
\]
\[+ \frac{J}{S} \sum_{s=1}^{S} \left\{ \log \frac{\Gamma(\mu)}{\Gamma(\mu + N_s)} + \sum_{k=1}^{K} T_s^{-1} \sum_{t=1}^{T_s} \log \frac{\Gamma(\mu m_k + \hat{n}_{sk,t})}{\Gamma(\mu m_k)} \right\} + \text{constant}.
\]

Therefore, its gradient with respect to \( \mu \) is,

\[\frac{-1 + (\alpha - 1) \mu}{\mu} + \frac{J}{S} \sum_{s=1}^{S} \left\{ \Phi(\mu) - \Phi(\mu + N_s) + \sum_{k=1}^{K} T_s^{-1} \sum_{t=1}^{T_s} m_k \left( \Phi(\mu m_k + \hat{n}_{sk,t}) - \Phi(\mu m_k) \right) \right\}\]

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