COMPACT ACTIONS AND UNIQUENESS OF THE GROUP MEASURE SPACE DECOMPOSITION OF II_1 FACTORS

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Abstract. We prove that any II_1 factor \( L^\infty(X) \rtimes \Gamma \) arising from a compact, free, ergodic, probability measure preserving action of a countable group \( \Gamma \) with positive first \( \ell^2 \)-Betti number, has a unique group measure space Cartan subalgebra, up to unitary conjugacy.

1. Introduction

The group measure space construction of Murray and von Neumann associates to every probability measure preserving (p.m.p.) action \( \Gamma \acts (X, \mu) \) of a countable group \( \Gamma \), a finite von Neumann algebra \( M = L^\infty(X) \rtimes \Gamma [MvN36] \). If the action is (essentially) free and ergodic, then \( M \) is a II_1 factor and \( A = L^\infty(X) \) is a Cartan subalgebra, i.e. a maximal abelian von Neumann subalgebra whose normalizer, \( N_M(A) = \{ u \in U(M) | uAu^* = A \} \), generates \( M \), as a von Neumann algebra.

During the last decade, S. Popa’s deformation/rigidity theory has led to spectacular progress in the classification of group measure space II_1 factors (see the surveys [Po07], [Va10a]). In particular, several large families of group measure space II_1 factors \( L^\infty(X) \rtimes \Gamma \) have been shown to have a unique Cartan subalgebra [OP07], [OP08], [CS11], [CSU11], [PV11a], [PV11b] or a unique group measure space Cartan subalgebra [Pe09], [PV09], [Io10], [FV10], [PV10], [CP10], [HPV10], [Va10b], [Io11].

Such “unique Cartan subalgebra” results are extremely useful because they enable one to reduce the classification of the factors \( L^\infty(X) \rtimes \Gamma \), up to isomorphism, to the classification of the corresponding actions, up to orbit equivalence. Indeed, by [Si55], [FM77], two free, ergodic actions \( \Gamma \acts X \) and \( \Lambda \acts Y \) are orbit equivalent if and only if there exists an isomorphism of group measure space factors \( L^\infty(X) \rtimes X \cong L^\infty(Y) \rtimes Y \) which identifies the Cartan subalgebras \( L^\infty(X) \) and \( L^\infty(Y) \).

Recall that the actions \( \Gamma \acts X \) and \( \Lambda \acts Y \) are orbit equivalent if there exists an isomorphism of probability spaces \( \theta : X \to Y \) such that \( \theta(\Gamma \cdot x) = \Lambda \cdot \theta(x) \), almost everywhere; for recent progress on orbit equivalence, see the surveys [Fu09], [Ga10].

In their breakthrough work [OP07], S. Popa and N. Ozawa proved that II_1 factors \( L^\infty(X) \rtimes \Gamma \) associated with free, ergodic, compact actions of free groups \( \Gamma = F_n \) and their direct products \( \Gamma = F_{n_1} \times \ldots \times F_{n_k} \) have a unique Cartan subalgebra, up to unitary conjugacy. Their proof makes crucial use of the fact that free groups satisfy a strong form of weak amenability, called the complete metric approximation property (c.m.a.p.). Subsequently, the results of [OP07] have been extended to other classes of groups \( \Gamma \): lattices in products of \( SO(n, 1) \) and \( SU(n, 1) \) [OP08], groups having positive first \( \ell^2 \)-Betti number and the c.m.a.p. [Da10], [Si10], hyperbolic groups [CST1] and direct products of hyperbolic groups [CSU11] (the last two results notably using [Oz07] and [Oz10]). Note, however, that in all of these cases one requires that both the action \( \Gamma \acts X \) is compact and, implicitly, that the group \( \Gamma \) is weakly amenable.

In a very recent breakthrough, S. Popa and S. Vaes obtained wide-ranging unique Cartan subalgebra results by removing the compactness assumption on the action. More precisely, they showed that if \( \Gamma \) is either a weakly amenable group with \( \beta_1^{(2)}(\Gamma) > 0 \) [PV11a] or a hyperbolic group [PV11b] (or...
a direct product of groups in one of these two classes), then \( \Pi_1 \) factors \( L^\infty(X) \rtimes \Gamma \) associated with arbitrary free, ergodic actions of \( \Gamma \) have a unique Cartan subalgebra, up to unitary conjugacy.

The main result of \([OP07]\) shows in particular that any \( \Pi_1 \) factor arising from a free, ergodic, compact action of a free group \( \mathbb{F}_n \) has a unique group measure space Cartan subalgebra. The original motivation for our paper was to give a proof of this result which does not rely on the weak amenability of \( \mathbb{F}_n \). By combining our recent work \([Io11]\) with techniques from \([OP07]\) and \([CP10]\), we were able to not only prove this, but to show the following:

**Theorem 1.1.** Let \( \Gamma \subset (X, \mu) \) be a free, ergodic, compact, p.m.p. action of a countable group \( \Gamma \) with \( \beta_{\Gamma}^{(2)}(\Gamma) > 0 \).

Then the \( \Pi_1 \) factor \( M = L^\infty(X) \rtimes \Gamma \) has a unique group measure space Cartan subalgebra, up to unitary conjugacy. More precisely, if \( M = L^\infty(Y) \rtimes \Lambda \), for any free, ergodic, p.m.p. action \( \Lambda \subset (Y, \nu) \), then we can find a unitary \( u \in M \) such that \( uL^\infty(X)u^* = L^\infty(Y) \).

Recall that a p.m.p. action \( \Gamma \subset (X, \mu) \) is called compact if we can decompose \( L^2(X) = \bigoplus_n \mathcal{H}_n \), where \( \{\mathcal{H}_n\}_n \) is a sequence of \( \Gamma \)-invariant, finite dimensional subspaces of \( L^2(X) \). For instance, if \( G \) is a compact group containing a dense copy of \( \Gamma \) and \( m_G \) denotes the Haar measure of \( G \), then the left multiplication action \( \Gamma \rtimes (G, m_G) \) is a free, ergodic, p.m.p. compact action.

The condition \( \beta_{\Gamma}^{(2)}(\Gamma) > 0 \) is equivalent to \( \Gamma \) being non-amenable and having an unbounded cocycle into its regular representation \( \lambda : \Gamma \to \mathcal{U}(L^2(\Gamma)) \) \([PT07]\), and is satisfied by any free product group \( \Gamma = \Gamma_1 \ast \Gamma_2 \) with \( |\Gamma_1| \geq 2 \) and \( |\Gamma_2| \geq 3 \).

Note that in the case \( \Gamma \) does not have Haagerup’s property, Theorem 1.1 is a consequence of the main result of \([Pe09]\).

A far-reaching conjecture predicts that \( \Pi_1 \) factors \( M = L^\infty(X) \rtimes \Gamma \) arising from free, ergodic, p.m.p. actions of groups \( \Gamma \) with \( \beta_{\Gamma}^{(2)}(\Gamma) > 0 \) have a unique Cartan subalgebra. Recently, there has been some progress on the “group measure space” version of the conjecture. Firstly, it has been shown in \([PV09]\) that if \( \Gamma = \Gamma_1 \ast \Gamma_2 \), where \( \Gamma_1 \) is an infinite property (T) group and \( \Gamma_2 \) is a non-trivial group, then \( M \) has a unique group measure space Cartan subalgebra. More generally, the same has been proven in \([CP10]\) (see also \([Va10]\)) under the assumption that \( \Gamma \) admits a non-amenable subgroup with the relative property (T). Lastly, we proved in \([Io11]\) that \( M \) also has a unique group measure space Cartan subalgebra if the action \( \Gamma \rtimes (X, \mu) \) is rigid.

As mentioned above, this conjecture has been very recently established in full generality for arbitrary actions of weakly amenable groups \( \Gamma \) with \( \beta_{\Gamma}^{(2)}(\Gamma) > 0 \) \([PV11a]\).

Theorem 1.1 provides further positive evidence towards this conjecture. It implies that any residually finite group \( \Gamma \) satisfying \( \beta_{\Gamma}^{(2)}(\Gamma) > 0 \) admits at least one action whose \( \Pi_1 \) factor has a unique group measure space decomposition. Indeed, if \( \{\Gamma_n\}_n \) is a descending chain of normal, finite index subgroups of \( \Gamma \) with trivial intersection, then the left multiplication action \( \Gamma \rtimes (G, m_G) \) on the profinite completion \( G = \varprojlim \Gamma_n \) is a free, ergodic, compact p.m.p. action.

To outline the proof of Theorem 1.1 denote \( A = L^\infty(X) \) and \( M = A \rtimes \Gamma \). Consider another group measure space decomposition \( M = B \rtimes \Lambda \) and let \( \Delta : M \to M \boxtimes M \) be the associated comultiplication \([PV09]\). In the first part of the proof, we show that a corner of \( \Delta(A) \) embeds into \( M \boxtimes A \), in the sense of \([Po03]\). To achieve this, we combine the fact that the action \( \Delta(\Gamma) \subset \Delta(A) \) is compact (hence, a fortiori, weakly compact) with a result from \([OP07]\) relating weak compactness of actions to uniform convergence of deformations, and results of \([CP10]\) on malleable deformations coming from cocycles into the regular representation (see Section 4). In the second part of the proof, we use techniques from \([Io11]\) to conclude that if \( \Gamma \) has positive first \( L^2 \)-Betti number, then the condition \( \Delta(A) \prec M \boxtimes A \) automatically implies that \( A \) and \( B \) are conjugate (see Section 3).
2. Preliminaries

Throughout the paper we work with tracial von Neumann algebras \((M, \tau)\), i.e. von Neumann algebras \(M\) endowed with a faithful normal tracial state \(\tau\), and assume that all von Neumann algebras have separable predual.

2.1. Intertwining-by-bimodules. We first recall from [Po03, Theorem 2.1 and Corollary 2.3] S. Popa’s powerful intertwining-by-bimodules technique.

**Theorem 2.1.**[Po03] Let \((M, \tau)\) be a tracial von Neumann algebra and \(P, Q \subset M\) be von Neumann subalgebras. Then the following are equivalent:

- There exist non-zero projections \(p \in P, q \in Q\), a \(*\)-homomorphism \(\phi : pPp \to qQq\) and a non-zero partial isometry \(v \in qMq\) such that \(\phi(x)v = vx\), for all \(x \in pPp\).
- There is no sequence \(u_n \in U(P)\) satisfying \(\|E_Q(xu_ny)\|_2 \to 0\), for all \(x, y \in M\).

If one of these conditions holds true, then we say that a corner of \(P\) embeds into \(Q\) inside \(M\) and write \(P \lesssim_M Q\).

2.2. Relative amenability. Let \((M, \tau)\) be a tracial von Neumann algebra. Recall that \(M\) is said to be amenable if we can find a sequence \(\xi_n \in L^2(M)\overline{\otimes}L^2(M)\) such that \(\langle x\xi_n, \xi_n \rangle \to \tau(x)\) and \(\|x\xi_n\|_2 \to 0\), for every \(x \in M\). By A. Connes’ celebrated theorem [Co76] this is equivalent to \(M\) being approximately finite dimensional.

Now, let \(Q \subset M\) be a von Neumann subalgebra. Jones’ basic construction \(\langle M, e_Q \rangle\) is defined as the von Neumann subalgebra of \(\B(L^2(M))\) generated by \(M\) and the orthogonal projection \(e_Q\) from \(L^2(M)\) onto \(L^2(Q)\). Recall that \(\langle M, e_Q \rangle\) is equipped with a faithful semi-finite trace given by \(Tr(xe_QyL) = \tau(xy)\) for all \(x, y \in M\). We denote by \(L^2(\langle M, e_Q \rangle)\) the associated Hilbert space and endow it with the natural \(M\)-bimodule structure.

Following [OP07, Definition 2.2] we say that a (not necessarily unital) von Neumann subalgebra \(P \subset M\) is amenable relative to \(Q\) inside \(M\) if there exists a sequence \(\xi_n \in L^2(\langle M, e_Q \rangle)\) such that \(\langle x\xi_n, \xi_n \rangle \to \tau(x)\), for every \(x \in M\), and \(\|y\xi_n\|_2 \to 0\), for every \(y \in P\).

**Lemma 2.2.**[IPV10] Let \((B, \tau)\) be a tracial von Neumann algebra and \(\Lambda \subset (B, \tau)\) be a trace preserving action. Denote \(M = B \rtimes \Lambda\). Define the \(*\)-homomorphism \(\Delta : M \to M \overline{\otimes} M\) by letting \(\Delta(b) = b \otimes 1\), for all \(b \in B\), and \(\Delta(v_s) = v_s \otimes v_s\), for every \(s \in \Lambda\). Let \(P, Q \subset M\) be von Neumann subalgebras such that \(P\) has no amenable direct summand and \(Q\) is amenable.

Then there is no non-zero projection \(p \in \Delta(P) \cap M \overline{\otimes} M\) such that the von Neumann algebra \(\Delta(P)p\) is amenable relative to \(M \overline{\otimes} Q\).

**Proof.** Since \(B\) and \(Q\) are amenable, the \(M-M \overline{\otimes} M\)-bimodule \(\Delta(M)L^2(\langle M \overline{\otimes} M, e_{M \overline{\otimes} Q} \rangle)M \overline{\otimes} M\) is weakly contained in the coarse \(M-M \overline{\otimes} M\)-bimodule, \(L^2(M) \overline{\otimes} L^2(M \overline{\otimes} M)\). Repeating the proof of [IPV10, Proposition 7.2.4] now gives the conclusion. 

2.3. Weakly compact actions. We next recall N. Ozawa and S. Popa’s notion of weakly compact actions. Let \((P, \tau)\) be a tracial von Neumann algebra and denote by \(P = \{\bar{x} | x \in P\}\) the complex conjugate von Neumann algebra.

**Definition 2.3.**[OP07, Definition 3.1] A trace preserving action \(\Gamma \subset \tau(P, \tau)\) is weakly compact if we can find a sequence of unit vectors \(\eta_n \in L^2(\bar{P} \overline{\otimes} P)\) such that

- \(\|\eta_n - (v \otimes \bar{v})\eta_n\|_2 \to 0\), for every \(v \in U(P)\).
- \(\|\eta_n - (\sigma_g \otimes \bar{\sigma}_g)\eta_n\|_2 \to 0\), for every \(g \in \Gamma\).
• \( \langle (x \otimes 1)\eta_n, \eta_n \rangle = \tau(x) = \langle \eta_n, (1 \otimes \bar{x})\eta_n \rangle \), for every \( x \in P \) and every \( n \).

Note that these conditions force \( P \) to be amenable. On the other hand, by [OP07, Proposition 3.2], if \( P \) is amenable and \( \Gamma \curvearrowleft^\sigma P \) is a compact action (i.e. if \( L^2(P) \) is the direct sum of finite dimensional \( \sigma(\Gamma) \)-invariant subspaces) then \( \sigma \) is weakly compact.

N. Ozawa and S. Popa showed in [OP07, Theorem 4.9] that weakly compact actions can be used to deduce relative amenability. An obvious modification of their proof (in the case \( p = 1 \)) gives the following:

**Theorem 2.4.** [OP07] Theorem 4.9 Let \((M, \tau)\) be a tracial von Neumann algebra and \(Q_1, \ldots, Q_k \subset M\) be von Neumann subalgebras. Assume that there are a tracial \( v \) on von Neumann algebra \( \tilde{N} \) and automorphisms \( \{\alpha_t\}_{t \in \mathbb{R}} \) of \( M \) such that

1. \( \lim_{t \to 0} ||\alpha_t(x) - x||_2 = 0 \), for all \( x \in M \), and
2. \( L^2(\tilde{M}) \otimes L^2(M) \) is isomorphic as an \( M \)-bimodule to a submodule of a multiple of \( \bigoplus_{j=1}^k L^2((M, e_{Q_j})) \).

Let \( P \subset M \) be a von Neumann subalgebra and define \( c_t := \inf_{v \in \mathcal{L}(P)} ||E_M(\alpha_t(v))||_2 \), for \( t \in \mathbb{R} \). Let \( \mathcal{G} \subset N_M(P) \) be a subgroup which acts weakly compactly on \( P \) by conjugation and denote \( N = \mathcal{G}'' \).

Then either \( c_t \to 1 \), as \( t \to 0 \), or there exist \( j \in \{1, \ldots, k\} \) and a non-zero projection \( p_j \in N' \cap M \) such that \( Np_j \) is amenable relative to \( Q_j \).

4.2. **Deformations coming from group cocycles.** Let \( \Gamma \curvearrowleft (A, \tau) \) be a trace preserving action and set \( M = A \rtimes \Gamma \). Let \( \pi : \Gamma \to \mathcal{O}(H_A) \) be an orthogonal representation, where \( H_A \) is a separable real Hilbert space. Also, let \( b : \Gamma \to H_A \) be a cocycle, i.e. a map satisfying the identity \( b(gh) = b(g) + \pi(g)b(h) \), for all \( g, h \in \Gamma \).

From this data, T. Sinclair constructed a *malleable deformation* in the sense of S. Popa, i.e. a tracial von Neumann algebra \( \tilde{M} \supset M \) and a 1-parameter group of automorphisms \( \{\alpha_t\}_{t \in \mathbb{R}} \) of \( \tilde{M} \) such that \( ||\alpha_t(x) - x||_2 \to 0 \), for all \( x \in \tilde{M} \) (see [Si10, Section 3] and [Ya10b, Section 3.1]).

To recall this construction, let \( (D, \tau_0) \) be the unique tracial von Neumann generated by unitary elements \( \omega(\xi), \xi \in H_A \), subject to the relations \( \omega(\xi + \eta) = \omega(\xi)\omega(\eta), \omega(\xi)^* = \omega(-\xi) \) and \( \tau_0(\omega(\xi)) = \exp(-||\xi||^2) \), for all \( \xi, \eta \in H_A \). Consider the Gaussian action \( \Gamma \curvearrowleft D \otimes A \) which on the generating functions \( \omega(\xi) \) is given by \( \sigma_g(\omega(\xi)) = \omega(\pi_g(\xi)) \). Finally, let \( \Gamma \curvearrowleft D \otimes A \) be the diagonal action and define \( M = (D \otimes A) \rtimes \Gamma \).

Then the formula

\[
\alpha_t(g) = (\omega(tb(g)) \otimes 1)g, \quad \text{for all } g \in \Gamma, \quad \text{and} \quad \alpha_t(x) = x, \quad \text{for all } x \in D \otimes A
\]

gives the desired 1-parameter group of automorphisms \( \{\alpha_t\}_{t \in \mathbb{R}} \) of \( \tilde{M} \).

Moreover, the formula \( \beta(\omega(\xi)) = \omega(-\xi) \), for all \( \xi \in H_A \), and \( \beta(x) = x \), for all \( x \in M \), defines an automorphism of \( \tilde{M} \) which satisfies \( \beta^2 = id \) and \( \beta \circ \alpha_t = \alpha_{-t} \circ \beta \), for all \( t \in \mathbb{R} \). As shown in [Po06a, Lemma 2.1] the existence of such a \( \beta \) implies a “transversality” property for \( \alpha_t \):

**Lemma 2.5.** [Po06a] For every \( x \in M \) and \( t \in \mathbb{R} \) we have that

\[
||\alpha_t(x) - x||_2 \leq 2||\alpha_t(x) - EM(\alpha_t(x))||_2.
\]

Next we record a direct consequence of S. Popa’s *spectral gap principle* [Po06b, Lemma 1.5].

**Lemma 2.6.** [Po06b] Assume that \( \pi \) is weakly contained in the regular representation of \( \Gamma \). Let \( B \subset M \) be a von Neumann subalgebra with no amenable direct summand.

Then \( \alpha_t \to id \) in \( ||.||_2 \) uniformly on \( (B' \cap M)_1 \) (the unit ball of the commutant of \( B \)).
For a proof, see e.g. [Io11] Lemma 2.2.

Now, let $B \subset M$ be a von Neumann subalgebra. J. Peterson [Pe06, Theorem 4.5] and I. Chifan and J. Peterson [CP10, Theorem 2.5] proved that if $\alpha_t \to \text{id}$ uniformly on $(B)_1$ and $B \not\approx_M A$ then $\alpha_t \to \text{id}$ uniformly on $N_M(B)$.

**Theorem 2.7.** [Pe06, CP10] Assume that $\pi$ is mixing. Let $B \subset M$ be a von Neumann subalgebra. Suppose that $\alpha_t \to \text{id}$ in $\|\cdot\|_2$ uniformly on $(B)_1$ and that $B \not\approx_M A$. Denote $P = N_M(B)^\prime$. Then there exists $g \in \Lambda$ such that $\Lambda \ni \alpha_t \to \text{id}$ uniformly on $N_M(B)$.

Conversely, I. Chifan and J. Peterson proved in [CP10, Theorem 3.2] that if $B$ is abelian and $\alpha_t \to \text{id}$ uniformly on a “large” sequence $u_n \in N_M(B)$, then $\alpha_t \to \text{id}$ uniformly on $(B)_1$.

**Theorem 2.8.** [CP10] Assume that $\pi$ is mixing. Let $B \subset M$ be an abelian von Neumann subalgebra. Assume that we can find a sequence $u_n \in N_M(B)$ such that

1. $\alpha_t \to \text{id}$ in $\|\cdot\|_2$ uniformly on $\{u_n\}_{n \geq 1}$ and
2. $\|E_A(xu_ny)\|_2 \to 0$, for all $x, y \in M$.

Then $\alpha_t \to \text{id}$ in $\|\cdot\|_2$ uniformly on $(B)_1$.

Theorems 2.7 and 2.8 have been proven in [Pe06] and [CP10] using J. Peterson’s technique of unbounded derivations [Pe06]. For proofs using the automorphisms $\{\alpha_t\}_{t \in \mathbb{R}}$, see S. Vaes’s paper [Va10b, Theorems 3.9 and 4.1].

### 3. Conjugacy of group measure space Cartan subalgebras

In this section we assemble together the main technical results of [Io11] to prove a new conjugacy criterion for group measure space Cartan subalgebras of $L^\infty(X) \rtimes \Gamma$ arising from actions of groups with positive first $l^2$-Betti number.

**Theorem 3.1.** Let $\Gamma$ be a countable group satisfying $\beta_1(\mathbb{Z}^2)(\Gamma) > 0$. Let $\Gamma \acts (X, \mu)$ be a free ergodic p.m.p. action and set $M = L^\infty(X) \rtimes \Gamma$. Assume that $M = L^\infty(Y) \rtimes \Lambda$ for a free ergodic p.m.p. action $\Lambda \acts (Y, \nu)$. Let $\Delta : M \to M \otimes M$ be the $*$-homomorphism given by $\Delta(b) = b \otimes 1$ for all $b \in L^\infty(Y)$ and $\Delta(v_g) = v_g \otimes v_g$ for all $g \in \Lambda$, where $\{v_g\}_{g \in \Lambda} \subset M$ denote the canonical unitaries. If $\Delta(L^\infty(X)) \leq_M \otimes \Delta \otimes L^\infty(X)$, then $L^\infty(X)$ and $L^\infty(Y)$ are unitarily conjugate.

**Proof.** Denote $A = L^\infty(X)$ and $B = L^\infty(Y)$. Since $\beta_1(\mathbb{Z}^2)(\Gamma) > 0$, by [Io11, Theorem 4.2], in order to get the conclusion, it suffices to find an amenable subgroup $\Lambda_0 < \Lambda$ such that $A \preceq_N B \times \Lambda_0$. Let us assume by contradiction that this is false.

Since $\Delta(A) \preceq_M \otimes A$, the second condition in Theorem 2.1 holds true. Thus, we can find a finite set $F \subset (M \otimes M)_1$ and $\delta > 0$ such that $\sum_{x \in F} \|E_{M \otimes A}(\Delta(a) x)\|_2^2 \geq 2\delta$, for every $a \in \mathcal{U}(A)$. By Kaplansky’s theorem we may assume that $F \subset 1 \otimes (M)_1$, i.e. we have

$$\sum_{x \in F} \|E_{M \otimes A}(\Delta(a)(1 \otimes x))\|_2^2 \geq 2\delta, \forall a \in \mathcal{U}(A) \tag{3.1}$$

For $g \in \Lambda$, we define $f(g) = \sum_{x \in F} \|E_A(v_g x)\|_2^2$. Then we have the following:

**Claim.** Let $g_1, \ldots, g_m, h_1, \ldots, h_m \in \Lambda$ and $\Lambda_1, \ldots, \Lambda_m < \Lambda$ be amenable subgroups. Put $S = \bigcup_{i=1}^m g_i \Lambda_i h_i$. Then there exists $g \in \Lambda \setminus S$ such that $f(g) \geq \delta$. 
Proof of the claim. Since \( \Lambda_i \) is amenable by our assumption we have that \( A \not\prec_M B \times \Lambda_i \), for all \( i \in \{1, \ldots, m\} \). It follows from S. Popa’s intertwining-by-bimodules technique (see Theorem 2.1 and e.g. [Io11] Remark 1.2)) that we can find \( a \in \mathcal{U}(A) \) such that

\[
(3.2) \quad \sum_{i=1}^{m} \| E_{B \rtimes \Lambda_i}(v_g^a, av_n^a) \|_2^2 \leq \frac{\delta}{|F|}
\]

If we write \( a = \sum_{g \in \Lambda} b_g v_g \), where \( b_g \in B \), then (3.2) implies that \( \sum_{g \in S} \| b_g \|_2^2 \leq \frac{\delta}{|F|} \). On the other hand, since \( E_{M^\infty A}(\Delta(a)(1 \otimes x)) = \sum_{g \in A} b_g v_g \otimes E_A(v_g x) \), for every \( x \in M \), (3.1) can be rewritten as \( \sum_{g \in A} f(g) \| b_g \|_2^2 \geq 2\delta \). Since \( f(g) \leq |F| \), for all \( g \in \Lambda \), combining the last two inequalities yields that \( \sum_{g \in A \setminus S} f(g) \| b_g \|_2^2 \geq \delta \). Finally, since \( \sum_{g \in \Lambda \setminus S} \| b_g \|_2^2 \leq \| a \|_2^2 = 1 \), the claim follows. \( \square \)

The claim shows that the conclusion of [Io11] Lemma 3.2] holds true. The second part of the proof of [Io11] Theorem 3.1], which only uses the conclusion of [Io11] Lemma 3.2], provides a decreasing sequence \( \{\Lambda_k\}_{k \geq 1} \) of non-amenable subgroups of \( \Lambda \) such that \( A \not\prec_M B \times (\cup_{k \geq 1} C(\Lambda_k)) \), where \( C(\Lambda_k) \) denotes the centralizer of \( \Lambda_k \) in \( \Lambda \).

Since \( \beta_{(2)}^2(\Gamma) > 0 \), there is an unbounded cocycle \( b : \Gamma \to \ell_2^2 \Gamma \) for the regular representation [PT07].

Let \( \tilde{M} \supset M \) and \( \{\alpha_t\}_{t \in \mathbb{R}} \subset \text{Aut}(\tilde{M}) \) be the malleable deformation constructed in Section 2.4.

Now, the group \( \cup_{k \geq 1} C(\Lambda_k) \) must be non-amenable by our assumption. Thus, \( C(\Lambda_k) \) is non-amenable for some \( k \geq 1 \) and therefore its von Neumann algebra, \( L(C(\Lambda_k)) \), has no amenable direct summand. Lemma 2.6 implies that \( \alpha_t \to \text{id} \) in \( \| \cdot \|_2 \) uniformly on \( (L(\Lambda_k))_1 \).

Since \( \Lambda_k \) is non-amenable and \( A \) is abelian, we have that \( L(\Lambda_k) \not\prec_M A \). By [Po03] Corollary 2.3] can find a sequence \( g_n \in \Lambda_k \) such that \( \| E_A(x v_{g_n} y) \|_2 \to 0 \), for all \( x, y \in M \). Because \( v_{g_n} \in N_M(B) \) and \( \alpha_t \to \text{id} \) in \( \| \cdot \|_2 \) uniformly on \( \{v_{g_n}\}_{n \geq 1} \), applying Theorem 2.8 gives that \( \alpha_t \to \text{id} \) in \( \| \cdot \|_2 \) uniformly on \( (B)_1 \).

Finally, if \( B \not\prec_M A \), then by [Po03] Theorem A.1] we get that \( A \) and \( B \) are conjugate. Otherwise, if \( B \not\prec_M A \), then [Po03] Theorem 2.7] implies that \( \alpha_t \to \text{id} \) in \( \| \cdot \|_2 \) uniformly on \( (M)_1 \). In particular, we can find \( t > 0 \) such that \( \mathbb{R} \tau(\alpha_t(u_g)u_g^* y) \geq \exp(-\frac{1}{t}) \), for all \( g \in \Gamma \). Since \( \tau(\alpha_t(u_g)u_g) = \exp(-t^2 \| b(g) \|_2^2) \) we derive that \( \| b(g) \|_2 \leq \frac{1}{2t} \) for all \( g \in \Gamma \). This shows that \( b \) is bounded, a contradiction. \( \square \)

4. PROOF OF THEOREM 1.1

Let \( \Gamma \rhd (X, \mu) \) be a free ergodic p.m.p. compact action. Denote \( A = L^\infty(X) \) and \( M = A \rtimes \Gamma \). Assume that \( \Gamma \) has positive first \( \ell^2 \)-Betti number and let \( b : \Gamma \to \ell_2^2 \Gamma \) be an unbounded cocycle into the regular representation of \( \Gamma \) [PT07].

Let \( \Lambda \rhd (Y, \nu) \) be a free ergodic p.m.p. action such that \( M = B \rtimes \Lambda \), where \( B = L^\infty(Y) \). Let \( \{v_s\}_{s \in \Lambda} \subset M \) be the canonical unitaries. Let \( \Delta : M \to M^\infty \Lambda \) be the \( * \)-homomorphism given by \( \Delta(b) = b \otimes 1 \) for \( b \in B \) and \( \Delta(v_s) = v_s \otimes v_s \) for \( s \in \Lambda \).

Our goal is to show that \( A \) and \( B \) are unitarily conjugate. If we denote \( A = M^\infty \Lambda \) and \( M = M^\infty \Lambda \) then by Theorem 3.1] it suffices to prove that \( \Delta(A) \not\prec_M A \). Assume this to be false.

Towards a contradiction, let \( \tilde{M} \supset M \) and \( \{\alpha_t\}_{t \in \mathbb{R}} \subset \text{Aut}(\tilde{M}) \) be the malleable deformation defined in Section 2.4. We let \( \tilde{M} = M^\infty \tilde{M} \) and denote by \( \tilde{\alpha}_t \) the automorphism \( \text{id} \otimes \alpha_t \) of \( \tilde{M} \) for every \( t \in \mathbb{R} \). Note that \( \tilde{M} \supset M \) and \( \{\tilde{\alpha}_t\}_{t \in \mathbb{R}} \subset \text{Aut}(\tilde{M}) \) is precisely the malleable deformation associated to the cocycle \( b \) and the obvious crossed product decomposition \( M = A \rtimes \Gamma \).
Since $b$ is a cocycle for the regular representation, it is easy to see that the $\mathcal{M}$-bimodule $L^2(\hat{M}) \otimes L^2(M)$ is isomorphic to $\bigoplus_{n \in \mathbb{Z}} L^2((\mathcal{M}, \varepsilon_A))$ (see [Va10b, Lemma 3.5]). Since the action $\Gamma \actson A$ is compact, the action $\Delta(\Gamma) \actson \Delta(A)$ is compact and thus weakly compact. By applying Theorem 2.3 to $P = \Delta(A)$ and $N = \Delta(M)$ we deduce that either (1) $\Delta(M)p$ is amenable relative to $\mathcal{A}$ for some non-zero projection $p \in \Delta(M)' \cap M$ or (2) $\inf_{v \in \ell^2(\Delta(A))} ||E_M(\tilde{\alpha}_t(v))||_2 \to 1$ as $t \to 0$.

Since $M$ has no amenable direct summand, the first case is ruled out by Lemma 2.5.

In the second case, we get that $\sup_{v \in \ell^2(\Delta(A))} ||\tilde{\alpha}_t(v) - E_M(\tilde{\alpha}_t(v))||_2 \to 0$ as $t \to 0$. By Lemma 2.5 we deduce that $\tilde{\alpha}_t \to \text{id}$ in $||.||_2$ uniformly on $(\Delta(A))'$. Since $\Delta(A) \not\cong \mathcal{A}$ by assumption and the von Neumann algebra generated by the normalizer of $\Delta(A)$ in $\mathcal{M}$ contains $\Delta(M)$, Theorem 2.7 gives that $\tilde{\alpha}_t \to \text{id}$ in $||.||_2$ uniformly on $(\Delta(M))'$. Since $E_M(v_s) = v_s \otimes \alpha_t(v_s)$ for $s \in \Lambda$ we derive that $\alpha_t \to \text{id}$ in $||.||_2$ uniformly on $\{v_s\}_{s \in \Lambda}$. Since $\Lambda$ is non-amenable and $A$ is abelian we can find a sequence $\{s_n\}_{n \geq 1}$ satisfying $||E_A(xv_{s_n}y)||_2 \to 0$ for all $x, y \in M$. Continuing exactly as in the end of the proof of Theorem 3.1 provides the desired contradiction.

**Remark 4.1.** It is easy to adapt the proof of the main result to show that if $t > 0$ and $M^t = L^\infty(Y) \rtimes \Lambda$, for some free ergodic p.m.p. action $\Lambda \actson (Y, \nu)$, then $L^\infty(Y)$ is unitarily conjugate to $L^\infty(X)^t$. It follows from D. Gaboriau’s work on $\ell^2$-Betti numbers for equivalence relations [Ga01] that $M$ has trivial fundamental group, $\mathcal{F}(M) = \{1\}$.

**Remark 4.2.** Let $\mathbb{Z}_p$ be the ring of $p$-adic integers, for some prime $p$. Consider the profinite action $\Gamma := \mathbb{Z}_p^2 \rtimes \text{SL}_2(\mathbb{Z}) \actson \lim \mathbb{Z}_p^2$ and denote by $M$ the associated II$_1$ factor. By [OP08, Corollary D] and [PV09, Section 5.5], $M$ has two non-conjugate group measure space Cartan subalgebras, $L^\infty(\lim \mathbb{Z}_p^2)$ and $L(\mathbb{Z})$. This shows that the assumption that $\beta_1^{(2)}(\Gamma) > 0$ from the hypothesis of our main result cannot be replaced by the assumption that $\Gamma$ is merely non-amenable.

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