Order $\alpha_s^3 \ln^2 (1/\alpha_s)$ Corrections to Heavy-Quarkonium Creation and Annihilation

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Abstract

In the framework of nonrelativistic QCD, we compute the leading double-logarithmic corrections of order $\alpha_s^3 \ln^2 (1/\alpha_s)$ to the heavy-quark-antiquark bound-state wave function at the origin, which determines the production and annihilation rates of heavy quarkonia. The phenomenological implications for the top-antitop and $\Upsilon$ systems are discussed.

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1 Introduction

Nonrelativistic quantum chromodynamics (NRQCD) [1,2,3] is a powerful tool for the investigation of heavy-quark threshold dynamics. Recent developments of the NRQCD effective-theory approach [4,5,6,7,8,9,10,11,12,13,14] led to the complete next-to-next-to-leading-order (NNLO) description of the production of heavy quark-antiquark pairs at threshold. Important applications include Υ sum rules and toponium phenomenology [15,16,17,18,19,20,21,22,23,24,25,26,27]. A review of the recent progress in the perturbative study of heavy quark-antiquark systems may be found, for example, in Ref. [28]. In view of the surprising significance of the NNLO corrections, it appears indispensable to also gain control over the next-to-next-to-next-to-leading order (N3LO) in order to improve the reliability of the theoretical predictions and our understanding of the structure and the peculiarities of the threshold expansion.

Some specific classes of N3LO corrections were analyzed in literature. The one-loop renormalization of the \(1/m_q^2\) operators was obtained in Refs. [5,29]. The retardation effects arising from the emission and absorption of virtual ultrasoft gluons by the heavy quarks were studied in Ref. [30]. The leading logarithmic \(O(\alpha_s^2 \ln(1/\alpha_s))\) corrections to the heavy-quark bound-state energies \(E_n\), where \(n\) is the principal quantum number, were obtained in Ref. [31].

In this paper, we take the next step in this direction and investigate a particular class of N3LO corrections, namely the leading logarithmic \(O(\alpha_s^3 \ln^2(1/\alpha_s))\) corrections to the wave functions at the origin \(\psi_n(0)\) of the heavy quark-antiquark bound states which are not generated by the renormalization group (RG). As is well known, \(\psi_n(0)\) are key parameters in the analysis of the creation and annihilation of heavy quarkonia. The origin of these logarithmic corrections is the presence of several scales in the threshold problem. In fact, \(\ln(1/\alpha_s)\) appears as a logarithm of a ratio of scales. These corrections are related to the anomalous dimensions of the operators in the effective Hamiltonian. They can be found by analyzing the divergences of the effective theory or by direct inspection of the regions of the logarithmic integration. We shall verify that both methods lead to the same result. As a by-product of our analysis, we shall also reproduce the \(O(\alpha_s^3 \ln(1/\alpha_s))\) corrections to \(E_n\) recently obtained in Ref. [31].

In N3LO, there are in general also logarithmic corrections of the form \(\alpha_s^3 \ln^m(\mu/\alpha_s m_q)\) \((m = 1, 2, 3)\), where \(\alpha_s m_q\) represents the soft or potential scales (see Section 2). These corrections may be directly extracted from the NNLO result via the RG equation, and they may be resummed by an appropriate choice of the normalization point, \(\mu \approx \alpha_s m_q\).

We note in passing that, in the \(\overline{\text{MS}}\) scheme, the optimal choice in NNLO is \(\mu \approx \text{few units} \times \alpha_s m_q\) [21,20].

This paper is organized as follows. In Section 2, we recall the potential-NRQCD (pNRQCD) formalism, from which \(E_n\) and \(\psi_n(0)\) can be extracted. In Section 3, we evaluate the non-RG \(O(\alpha_s^3 \ln^2(1/\alpha_s))\) and \(O(\alpha_s^3 \ln(1/\alpha_s))\) corrections to \(\psi_n(0)\) and \(E_n\),

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1In the NRQCD effective theory, there are two expansion parameters, the strong coupling constant \(\alpha_s\) and the heavy-quark velocity \(\beta\), and the perturbative order of some correction is determined by their total power. E.g., terms of \(O(\alpha_s^2)\), \(O(\alpha_s \beta)\), and \(O(\beta^2)\) contribute at NNLO.
respectively, using the effective-theory approach. In Section 4, we repeat this calculation in the conventional approach, by inspecting the logarithmically divergent phase-space integrals. In Section 5, we present a numerical analysis and discuss phenomenological implications of our results. Section 6 contains our conclusions.

2 Vacuum-polarization function near threshold in NRQCD

We investigate the near-threshold behavior of the vacuum-polarization function \( \Pi(q^2) \) of a heavy-quark vector current \( j_\mu = \bar{q}\gamma_\mu q \),

\[
(q_\mu q_\nu - g_\mu_\nu q^2) \Pi(q^2) = i \int d^4x e^{iq\cdot x} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle.
\] (1)

Its imaginary part is related to the normalized cross section of \( q\bar{q} \) production in \( e^+e^- \) annihilation at energy \( s = q^2 \),

\[
R(s) = \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)},
\] (2)

by

\[
R(s) = 12\pi Q_q^2 \text{Im}\Pi(s + i\epsilon),
\] (3)

where \( Q_q \) is the fractional charge of quark \( q \).

Near threshold, the heavy quarks are nonrelativistic, so that one may consider the quark velocity \( \beta = \sqrt{1 - 4m_q^2/s} \) as a small parameter. An expansion in \( \beta \) may be performed directly in the Lagrangian of QCD by using the framework of effective field theory. In the threshold problem, there are four different scales \([12]\): (i) the hard scale (energy and momentum scale like \( m_q \)); (ii) the soft scale (energy and momentum scale like \( \beta m_q \)); (iii) the potential scale (energy scales like \( \beta^2 m_q \), while momentum scales like \( \beta m_q \)); and (iv) the ultrasoft scale (energy and momentum scale like \( \beta^2 m_q \)). The ultrasoft scale is only relevant for gluons. By integrating out the hard scale of QCD, one arrives at the effective theory of NRQCD \([13,4]\). If one alsointegrates out the soft scale and the potential gluons, one obtains the effective theory of pNRQCD, which contains potential quarks and ultrasoft gluons as active particles \([10]\). The dynamics of the quarks is governed by the effective, nonrelativistic Schrödinger equation and by their interaction with the ultrasoft gluons. To get a regular perturbative expansion within pNRQCD, this interaction should be expanded in multipoles. The corrections from harder scales are contained in the Wilson coefficients, leading to an expansion in \( \alpha_s \), as well as in the higher-dimensional operators of the nonrelativistic Hamiltonian, corresponding to an expansion in \( 1/m_q \) or \( \beta \).

The nonrelativistic expansion in \( \alpha_s \) and \( \beta \) provides us with the following representation of the heavy-quark vacuum-polarization function near threshold:

\[
\Pi(E) = \frac{N_c}{2m_q^2} C_h(\alpha_s) G(0, 0, E) + \ldots,
\] (4)
where \( E = \sqrt{s} - 2m_q \) is the \( q\bar{q} \) energy counted from the threshold, \( C_h(\alpha_s) \) is the square of the hard matching coefficient of the nonrelativistic vector current, and the ellipsis stands for the higher-order terms in \( \beta \). \( G(x, y, E) \) is the nonrelativistic Green function, which sums up the \( (\alpha_s/\beta)^n \) terms singular near the threshold. It is determined by the Schrödinger equation which describes the the propagation of the nonrelativistic quark-antiquark pair in pNRQCD,

\[
(H - E) G(x, y, E) = \delta^{(3)}(x - y),
\]

where \( H \) is the nonrelativistic Hamiltonian defined by

\[
H = -\frac{\partial^2}{\partial x^2} + V(x) + \ldots, \quad V(x) = V_C(x) + \ldots.
\]

Here, \( V_C(x) = -C_F \alpha_s / x \) is the Coulomb potential, \( C_F = 4/3 \) is the eigenvalue of the quadratic Casimir operator of the fundamental representation of the colour group, \( x = |x| \), and the ellipses stand for the higher-order terms in \( \alpha_s \) and \( 1/m_q \). The Green function has the spectral representation

\[
G(x, y, E) = \sum_{n=1}^{\infty} \frac{\psi^*_n(x)\psi_n(y)}{E_n - E} + \int_0^{\infty} \frac{d^3k}{(2\pi)^3} \frac{\psi^*_k(x)\psi_k(y)}{k^2/m_q - E},
\]

where \( \psi_m \) and \( \psi_k \) are the wave functions of the \( q\bar{q} \) bound and continuum states, respectively.

Below the threshold, the (perturbative) vacuum-polarization function of a stable heavy quark is essentially determined by the bound-state parameters. For the leading-order Coulomb (C) Green function, the energy levels and wave functions at the origin read

\[
E^C_n = -\frac{\lambda^2_s}{m_q n^2}, \quad \left| \psi^C_n(0) \right|^2 = \frac{\lambda^3_s}{\pi n^2}.
\]

where \( \lambda_s = \alpha_s C_F m_q / 2 \). We are interested in the corrections to \( \left| \psi^C_n(0) \right|^2 \). Note that, for the study of bound-state parameters, we have \( \beta \approx \alpha_s \), so that we are only dealing with one expansion parameter.

### 3 Non-RG leading logarithmic corrections from the effective-theory approach

The NNLO non-RG leading logarithmic corrections to the wave functions at the origin are generated by the following operators in the effective nonrelativistic Hamiltonian:

\[
\Delta' H = -\frac{C_F C_A \alpha_s^2}{2m_q x^2} + \frac{C_F \alpha_s}{2m_q^2} \left\{ \frac{1}{x}, \frac{\partial^2}{\partial x^2} \right\} + \left( 1 + \frac{4}{3} S^2 \right) \frac{\pi C_F \alpha_s}{m_q^2} \delta(x) - \frac{\partial^4}{4m_q^2},
\]

\( \Delta' H \)
where $C_A = 3$ is the eigenvalue of the quadratic Casimir operator of the adjoint representation of the colour group, $S$ represents the spin of the quark-antiquark system, and $\{\ldots\}$ denotes the anticommutator. The first term of Eq. (9) is the non-Abelian potential \[32,33,34\], and the rest is the standard Breit potential and the kinetic-energy correction. Note that the representation \[9\] of the nonrelativistic Hamiltonian is not unique and can be related to other representations found in literature by the use of the equations of motion. However, the corresponding corrections to the Green function \[18,19,22\] are independent of the specific representation. In the vicinity of the $n$th bound-state pole, the resulting correction to the Green function reads \[22\]

$$
\Delta G(0,0,E)|_{E \to E_n} = \left| \frac{\psi_n^C(0)}{E_n - E} \right|^2 \left( 4 - \frac{4}{3} S(S + 1) \right) C_F + 2C_A \right\} G_C(0,0,E) + \ldots,
$$

(10)

where $G_C(x,y,E)$ is the Coulomb Green function. The spin-independent term proportional to $C_F$ in Eq. (11) comes about as $(4 - 1 + 1)C_F$, where the contributions arise from the anticommutator, $\delta$-function, and $1/m^3_q$ terms of Eq. (9). Writing

$$
|\psi_n(0)|^2 = \left| \psi_n^C(0) \right|^2 \left( 1 + \Delta \psi_n^2 \right),
$$

(11)

we have

$$
\Delta \psi_n^2(0) = \frac{2\pi \alpha_s}{m_q^2} \left\{ \left[ 4 - \frac{4}{3} S(S + 1) \right] C_F + 2C_A \right\} G_C(0,0,E_n) + \ldots.
$$

(12)

These corrections are singular, since the Coulomb Green function at the origin is ultraviolet (UV) divergent. In dimensional regularization, with $d = 4 - 2\epsilon$ space-time dimensions, it takes the form \[26\]

$$
G_C(0,0,k) = \frac{1}{2\epsilon} \left( \frac{\mu}{k} \right)^{2\epsilon} \frac{m_q \lambda_s}{2\pi} + \ldots = \frac{m_q \lambda_s}{2\pi} \left( \frac{1}{2\epsilon} + \ln \frac{\mu}{k} \right) + \ldots,
$$

(13)

where $k^2 = -m_q E$ and the ellipsis stands for the nonlogarithmic contribution. The pole term in Eq. (13) is canceled by the $O(\alpha_s^2)$ infrared (IR) pole of the hard matching coefficient $C_h(\alpha_s)$ \[13,17\]. Thus, the scale $\mu$ in the logarithm is to be identified with $m_q$, and the sought correction reads \[19,21\]

$$
\Delta \psi_n^2(0) = C_F \alpha_s^2 \left\{ \left[ 2 - \frac{2}{3} S(S + 1) \right] C_F + C_A \right\} \ln \frac{1}{\alpha_s}.
$$

(14)

The residual operators in the NNLO effective nonrelativistic Hamiltonian which are not contained in Eq. (9) correspond to the purely perturbative corrections to the static Coulomb potential. The corresponding corrections to the wave functions at the origin \[19,21,26\] contain RG logarithms of the form $\alpha_s^2 \ln^m(\mu/\alpha_s m_q)$ ($m = 1, 2$), which vanish for $\mu = \alpha_s m_q$. This also holds in N$^3$LO for the RG logarithms of the form $\alpha_s^3 \ln^m(\mu/\alpha_s m_q)$ ($m = 1, 2, 3$) because the ultrasoft effects enter the stage only in N$^3$LO, so that the corresponding running of the strong coupling constant at the ultrasoft scale only becomes
relevant in N^4LO. An important point here is that, starting from NNLO, the hard matching coefficient \( C_h(\alpha_s) \) receives a nonvanishing anomalous dimension. Therefore, starting from N^3LO, not only the running of \( \alpha_s \) should be taken into account, but also the effective-theory RG should be used for the evolution of the hard matching coefficient from \( \mu = \alpha_s m_q \) down to \( \mu = \alpha_s m_q \). 

At N^3LO, the non-RG leading logarithmic corrections to the wave functions at the origin are produced by the one-loop renormalization of the operators in Eq. (9). In dimensional regularization, the pole part of the correction is

\[
\Delta''H = \frac{1}{2\epsilon} \frac{C_F \alpha_s}{\pi} \left\{ - \left( \frac{4}{3} C_F + \frac{2}{3} C_A \right) \frac{C_A \alpha_s^2}{m_q x^2} + \frac{2}{3} \frac{C_A \alpha_s}{m_q^2} \left\{ \partial_x^2 \frac{1}{x} \right\} - \left( \frac{16}{3} C_F - \frac{8}{3} C_A \right) \frac{\pi \alpha_s}{m_q^2} \delta(x) \right\} + \left[ \frac{2}{3} C_F + \left( \frac{17}{3} - \frac{7}{3} S^2 \right) C_A \right] \frac{\pi \alpha_s}{m_q^2} \delta(x),
\]

where the first three terms contained within the parentheses represent the IR divergence, while the fourth one embodies the UV divergence of the potential. The IR poles are canceled by the ultrasoft contribution, with the characteristic scale \( \alpha_s^2 m_q \), and may be read off from Refs. [30,31], while the UV poles are canceled by the IR poles of the hard coefficients and may be extracted from Refs. [5,29]. Evaluating the corrections to the wave functions at the origin due to Eq. (15) in the same way as Eq. (12) was obtained, we find the N^3LO non-RG leading logarithmic corrections to the wave functions at the origin to be

\[
\Delta''\psi_n^2(0) = -\frac{C_F \alpha_s^3}{\pi} \left\{ \frac{3}{2} C_F + \left[ \frac{41}{12} - \frac{7}{12} S(S+1) \right] C_F C_A + \frac{2}{3} C_A^2 \right\} \ln^2 \frac{1}{\alpha_s}.
\]  

In the derivation of this result, the factor \((\mu/k)^{2\epsilon}\) in Eq. (13) needed to be expanded up to \(O(\epsilon^2)\). The contributions to Eq. (16) related to the IR and UV poles of Eq. (15) are of opposite signs. This may be understood by observing that the IR poles introduce a factor \( \ln(E_0^C/\lambda_s) \approx \ln \alpha_s \), while the UV poles contribute a factor \( \ln(m_q/\lambda_s) \approx \ln(1/\alpha_s) \). The Abelian \( C_F^3 \) term in Eq. (16) agrees with the QED result of Ref. [35]. In contrast to the QED case, the leading logarithmic QCD corrections are spin dependent.

In the remainder of this section, we revisit the leading logarithmic corrections to the bound-state energy levels, which may be obtained in a way similar to the case of the wave functions at the origin. These corrections start from N^3LO, since there are no relevant singularities in NNLO. In addition to Eq. (15), we now have to take into account the IR-singular contribution to the Coulomb potential \([36,37]\),

\[-\frac{1}{2\epsilon} \frac{C_F C_A^3 \alpha_s^4}{12 x}, \]

which is canceled by the ultrasoft contribution \([30]\). Obviously, this term gives no contributions to the wave functions. Writing

\[ E_n = E_n^C + \Delta E_n, \]

\[ E_n^C \]
we find

\[ \Delta E_n = -E_n \frac{\alpha_s^3}{\pi} \left\{ \frac{3}{n} C_F^3 + \left[ \frac{41}{6n} - \frac{7}{6n} S(S + 1) - \frac{2}{3n^2} \right] C_F^2 C_A + \frac{4}{3n} C_F C_A^2 + \frac{1}{6} C_A^3 \right\} \times \ln \frac{1}{\alpha_s}, \]  

(19)
in agreement with the result for \( l = 0 \) of Ref. [31].

4 Non-RG leading logarithmic corrections from the traditional approach

An alternative method of finding the leading logarithmic corrections is to directly inspect the regions of logarithmic integration [35,38,39,40]. Let us first consider the NNLO corrections to the wave functions. Substituting the continuum part of the spectral representation (7) into Eq. (12), we obtain

\[ \Delta' \psi_n^2(0) = \frac{2\pi \alpha_s}{m_q^2} \left\{ \left[ 4 - \frac{4}{3} S(S + 1) \right] C_F + 2C_A \right\} \int_0^\infty \frac{d^3 k}{(2\pi)^3} \frac{|\psi_k^C(0)|^2}{k^2/m_q - E_n}. \]  

(20)
The integral over \( k \) logarithmically diverges at large momentum and should be cut at scale \( m_q \), where the nonrelativistic approximation becomes inapplicable. At low momentum, the logarithmic integration over \( k \) is effectively cut at the Coulomb scale \( \alpha_s m_q \). Inserting in Eq. (20) the well-known expression for the Coulomb wave function at the origin,

\[ |\psi_k^C(0)|^2 = \frac{2\pi \lambda_s}{k} \frac{1}{1 - \exp\left(-2\pi \lambda_s / k\right)} = 1 + \frac{\pi \lambda_s}{k} + \ldots, \]  

(21)
where only the second term is relevant for our purposes, we find, with logarithmic accuracy,

\[ \Delta' \psi_n^2(0) = C_F \alpha_s^2 \left\{ \left[ 2 - \frac{2}{3} S(S + 1) \right] C_F + C_A \right\} \int_{\alpha_s m_q}^{m_q} \frac{dk}{k} + \ldots. \]  

(22)
Performing in Eq. (22) the integration over \( k \), we recover Eq. (14).

The N^3LO leading (double) logarithmic corrections to the wave function at the origin are of the form

\[ \Delta'' \psi_n^2(0) = -C_F \alpha_s^2 \left\{ \left( \frac{8}{3} C_F^2 + 4 C_A C_F + \frac{4}{3} C_A^2 \right) \int_{\alpha_s m_q}^{m_q} \frac{dk}{k} \int_{-E}^{E} \frac{dk'}{k'} \right. \]
\[ \left. + \left[ \frac{1}{3} C_F^2 + \left( \frac{17}{6} - \frac{7}{6} S(S + 1) \right) C_F C_A \right] \int_{\alpha_s m_q}^{m_q} \frac{dk}{k} \int_{\alpha_s m_q}^{m_q} \frac{dk'}{k'} \right\}. \]  

(23)
Here \( k \) is the momentum of the potential heavy quark, which can be as small as \( \alpha_s m_q \), while \( k' \) is the momentum of the virtual gluon, which can even be ultrasoft, of order
The integrals over $k$ have the same origin as those in Eq. (22). The integrals over $k'$ represent the logarithmic corrections to the potential due to the virtual-gluon exchanges \[5,29,30]. In the first integral on the right-hand side of Eq. (23), the quark momentum $k$ plays the role of an UV cutoff, and the quark energy $-E = k^2/m_q$ acts as an IR cutoff, so that this contribution corresponds to the IR poles of Eq. (15). In the second integral, the momentum $k$ acts as an IR cutoff, and this contribution corresponds to the UV poles of Eq. (15). Integrating Eq. (23) over $k'$ and $k$, we arrive at Eq. (16). A similar analysis for QED bound states may be found in Ref. [35].

The $N^3$LO leading (single) logarithmic corrections to the energy levels may be obtained by the method introduced in Refs. [38,39,40], where the regions of the virtual-photon momentum which lead to logarithmic contributions have been studied in the QED case. In this way, we obtain

\[
\Delta E_n = -E_n \frac{\alpha_s^3}{\pi} \left\{ \left[ \frac{8}{3n} C_F^3 + \left( \frac{31}{6n} - \frac{2}{3n^2} \right) C_F^2 C_A + \frac{4}{3n} C_F C_A^2 + \frac{1}{6} C_A^3 \right] \int_{\alpha_s m_q}^{E_n} \frac{dk'}{k'} + \left[ \frac{1}{3n} C_F^3 + \left( \frac{17}{6n} - \frac{7}{6n} S(S + 1) - \frac{2}{3n^2} \right) C_F^2 C_A \right] \int_{\alpha_s m_q}^{m_q} \frac{dk'}{k'} \right\}, \tag{24}
\]

where again the first (second) integral corresponds to the IR (UV) poles of Eq. (15). After integration, we recover Eq. (19).

5 Phenomenological applications

Our results affect two processes of primary interest, namely the threshold production of top and bottom quark-antiquark pairs. The relatively large width $\Gamma_t$ of the top quark serves as an efficient IR cutoff for long-distance effects. Because the relevant scale $\sqrt{m_t \Gamma_t}$ is much larger than the asymptotic scale parameter $\Lambda_{QCD}$, but comparable to the Coulomb scale, the cross section in the threshold region may be described by the NRQCD perturbation expansion if singular Coulomb effects are properly taken into account [41]. In order to analyze the significance of the $N^3$LO leading logarithmic corrections to the cross section, we start from the NNLO calculation of Ref. [26] and add the contributions from Eqs. (16) and (19). In Fig. 1, the normalized cross section $R$ thus obtained is compared with the pure NNLO result. The input parameters are taken to be $\alpha_s(M_Z) = 0.118$, $m_t = 175$ GeV, and $\Gamma_t = 1.43$ GeV. The soft renormalization scale is determined from the condition $\mu_s = 2\alpha_s(\mu_s) m_t$, and hard renormalization scale is chosen to be $\mu_h = m_t$. The effect of the $N^3$LO leading logarithms is twofold. The normalization of the cross section is reduced by about 7% around the $1S$ peak and below, and the energy gap between the $1S$ peak and the nominal threshold is decreased by roughly 10%.

In the case of bottom quark-antiquark production, the nonperturbative effects are much more significant, and one is led to use the sum rule approach [13,14] to get them under control. Specifically, appealing quark-hadron duality, one matches the theoretical
results for the moments of the spectral density,
\[ M_n = \frac{12\pi^2}{n!} \left(4m_q^2\right)^n \frac{d^n \Pi(s)}{ds^n} \bigg|_{s=0} = \left(4m_q^2\right)^n \int_0^\infty ds \frac{R(s)}{s^{n+1}} \]
with their experimental counterparts evaluated from the expressions in the second line of Eq. (25). For large \( n \), the moments are saturated by the near-threshold region. Then, the main contribution to the experimental moments comes from the \( \Upsilon \) resonances, which are measured with high precision. On the other hand, for \( n \) of \( O \left(\frac{1}{\alpha_s^2}\right) \), the Coulomb effects should be properly taken into account on the theoretical side. In order to analyze the N\( ^3 \)LO leading logarithmic corrections to the \( \Upsilon \) sum rules, we upgrade the NNLO result of Ref. [21] by including Eqs. (16) and (19). We fix the strong coupling constant by \( \alpha_s(M_Z) = 0.118 \) and focus on the determination of the bottom-quark mass. At present, this appears to be the most interesting application of Eq. (25). We find that the inclusion of the N\( ^3 \)LO leading logarithms in the sum rules leads to a reduction of the extracted mass value by approximately 30 MeV for moderate values of \( n, 5 < n < 15 \), and if the soft renormalization point \( \mu_s \) is chosen to be \( \mu_s = 4\alpha_s(\mu_s) m_b \). This result does not depend on whether the energy denominators of the Green function are expanded around the Coulomb values or not (see Ref. [21] for details) and on which mass parameter, pole or \( \overline{\text{MS}} \) mass, is considered. On the other hand, the result essentially depends on \( \mu_s \) because the \( \mu_s \) dependence of \( \alpha_s \) is compensated only by higher-order terms. For example, for \( \mu_s = 2\alpha_s(\mu_s) m_b \), the correction to the mass parameter reaches \(-70\) MeV. However, the perturbative result can hardly be trusted at such a low renormalization point [21,26].

For completeness, we also present the numerical corrections to the parameters of the 1S \( \Upsilon \) resonance. For \( m_b = 4.8 \) GeV and \( \mu_s = 4\alpha_s(\mu_s) m_b \), the wave-function correction is \(-19\)%, and the correction to the binding energy is 25 MeV.

6 Discussion and conclusions

We studied a special class of NNLO and N\( ^3 \)LO corrections to the key parameters of heavy quark-antiquark bound states, namely those which are enhanced by a maximum power of \( \ln(1/\alpha_s) \) and are not generated by the RG. By contrast, the RG logarithms are well known and may be resummed by an appropriate scale choice. Such non-RG leading logarithmic corrections first arise for the wave functions at the origin in NNLO [19,21] and for the energy levels in N\( ^3 \)LO [31]. Specifically, they are of the forms \( \alpha_s^2 \ln(1/\alpha_s) \) and \( \alpha_s^3 \ln(1/\alpha_s) \), respectively. We confirmed these results and completed the knowledge of non-RG leading logarithmic corrections in N\( ^3 \)LO by computing the \( O \left(\alpha_s^3 \ln^2(1/\alpha_s)\right) \) corrections to the wave functions at the origin.

We applied our result to top and bottom quark-antiquark production at threshold. In the case of top, the resulting correction to the production cross section near the 1S peak reaches 10%. In the case of bottom, the mass extracted from the \( \Upsilon \) sum rules is shifted by
approximately −30 MeV. The latter value is comparable to the uncertainty of ±60 MeV, which is usually assigned to this kind of bottom-quark mass determination on the basis of the renormalization scale dependence of the strong coupling constant in the NNLO result. In fact, the non-RG leading logarithmic contributions are about two times smaller than the contribution from the N$^3$LO RG logarithms, which may be estimated from the renormalization scale dependence of the NNLO result \[21,22\]. Thus, the resummation of the RG logarithms \[25\] seems to be very justified. At the same time, the scale of the corrections is close to the estimate given in Ref. \[30\] on the basis of the analysis of the ultrasoft contributions.

Although, in contrast to QED, ln(1/\(\alpha_s\)) is not a big number, especially for the case of bottom, the leading logarithmic terms can be considered as typical representatives of the N$^3$LO corrections. For comparison, at NNLO, the leading logarithmic term accounts for approximately one half (third) of the total correction to the \(n = 0\) wave function at the origin in the case of top (bottom). Obviously, the N$^3$LO corrections are comparable to the NNLO ones and reach 10% in magnitude, even in the case of top, where \(\alpha_s \approx 1/10\). This tells us that the NRQCD threshold expansion is not a fast convergent series for the physical value of the strong coupling constant.

A final comment refers to the resummation of the non-RG leading logarithmic corrections to the wave functions at the origin. A part of the non-RG leading logarithmic corrections to the heavy-quark threshold cross section was resummed in Ref. \[23\] by using the RG equation of the effective theory for the evolution of the hard matching coefficient \(C_h(\alpha_s)\) from scale \(m_q\) down to scale \(\beta m_q\). This evolution equation is obtained by studying the dependence of \(C_h(\alpha_s)\) on the scale \(\mu\), which cancels the \(\mu\) dependence of Eqs. (12) and (13). This effectively sums up the higher-order NRQCD corrections due to the tree-level operators of Eq. (9). The terms resummed in this way are of the form \(\alpha_s(\alpha_s \ln(1/\beta))^{2n-1}\), with \(n = 1, 2, \ldots\), \(i.e.\) they include only even powers of \(\alpha_s\). For the bound-state parameters, we have \(\beta \approx \alpha_s\), and Eq. (14) gives the first term of this series. In order to resum all correction of the form \(\alpha_s(\alpha_s \ln(1/\alpha_s))^{3n}\), it is necessary to add the terms with odd powers of \(\alpha_s\), of the form \(\alpha_s(\alpha_s \ln(1/\alpha_s))^{2n}\), which are generated by Eq. (15). For this end, one has to take into account not only the evolution of \(C_h(\alpha_s)\), but also the evolution of the potential \(\Phi\) from the hard scale down to the ultrasoft scale. Numerically, however, the effect of the resummation may not be essential for phenomenological applications.

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Fig. 1. Normalized cross section $R(E)$ of $e^+e^- \rightarrow t\bar{t}$ in NNLO (dotted line) and with the leading logarithmic N$^3$LO corrections included (solid line), as a function of the centre-of-mass energy $E$ counted from the nominal threshold at $2m_t$. 