A KEY EQUATION AND THE COMPUTATION OF ERROR VALUES FOR CODES FROM ORDER DOMAINS

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Abstract. We study the computation of error values in the decoding of codes constructed from order domains. Our approach is based on a sort of analog of the key equation for decoding Reed-Solomon and BCH codes. We identify a key equation for all codes from order domains which have finitely-generated value semigroups; the field of fractions of the order domain may have arbitrary transcendence degree, however. We provide a natural interpretation of the construction using the theory of Macaulay’s inverse systems and duality. O’Sullivan’s generalized Berlekamp-Massey-Sakata (BMS) decoding algorithm applies to the duals of suitable evaluation codes from these order domains. When the BMS algorithm does apply, we will show how it can be understood as a process for constructing a collection of solutions of our key equation.

§1. Introduction

The theory of error control codes constructed using ideas from algebraic geometry (including the geometric Goppa and related codes) has recently undergone a remarkable extension and simplification with the introduction of codes constructed from order domains. Interestingly, this development has been largely motivated by the structures utilized in the Berlekamp-Massey-Sakata decoding algorithm with Feng-Rao-Duursma majority voting for unknown syndromes.

We will review the definition of an order domain in §2; for now we will simply say that the order domains form a class of rings having many of the same properties as the rings $R = \cup_{m=0}^{\infty} L(mQ)$ underlying the one-point geometric Goppa codes constructed from curves. The general theory gives a common framework for these codes, $n$-dimensional cyclic codes, as well as many other Goppa-type codes constructed from varieties of dimension $> 1$. Høholdt, Pellikaan, and van Lint have given an exposition of order domains in [HPL], synthesizing work of many others in the coding theory community, and this is probably the best general reference for this topic.

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More recently, Geil and Pellikaan ([GP], [Gei]) and O'Sullivan ([OS1]) have studied the structure of order domains whose fields of fractions have arbitrary transcendence degree. Moreover, O'Sullivan ([OS2]) has shown that the Berlekamp-Massey-Sakata decoding algorithm (abbreviated as the BMS algorithm in the following) and the Feng-Rao procedure extend in a natural way to a suitable class of codes in this much more general setting.

The decoding problem here can be divided into two parts: determination of the error locations, then determination of the corresponding error values. When it applies, the BMS algorithm produces a Gröbner basis for what is known in the usual terminology as the error-locator ideal corresponding to the error vector, hence sufficient information to determine the error locations. Here, we will consider the problem of determining the error values in conjunction with the BMS algorithm or some other algorithm that determines the error locator ideal.

For the Reed-Solomon codes (the simplest examples of codes from order domains, or geometric Goppa codes), the Berlekamp-Massey decoding algorithm (the precursor of BMS) can be phrased as a method for solving a key equation. For a Reed-Solomon code with minimum distance \(d = 2t + 1\), the key equation has the form

\[
(fS \equiv g \mod \langle X^{2t} \rangle).
\]

Here \(S\) is a known univariate polynomial in \(X\) constructed from the error syndromes, and \(f, g\) are unknown polynomials in \(X\). If the error vector \(e\) satisfies \(wt(e) \leq t\), there is a unique solution \((f, g)\) with \(\deg(f) \leq t\), and \(\deg(g) < \deg(f)\) (up to a constant multiple). The polynomial \(f\) is known as the error locator because its roots give the inverses of the error locations; the polynomial \(g\) is known as the error evaluator because the error values can be determined from values of \(g\) at the roots of \(f\), via the Forney formula.

O'Sullivan has introduced a generalization of this key equation for one-point geometric Goppa codes from curves in [OS3] and shown that the BMS algorithm can be modified to compute the analogs of the error-evaluator polynomial together with error locators. His definitions make heavy use of the particular features of the curve case, however. For instance the objects corresponding to \(S\) and \(g\) in (1.1) are differentials on the underlying curve.

Our main goals in this article are the following. First, we wish to identify an analog of the key equation (1.1) for codes from order domains. We will only consider order domains whose value semigroups are finitely generated. In these cases, the ring \(R\) can be presented as an affine algebra \(R \cong \mathbb{F}[X_1, \ldots, X_s]/I\), where the ideal \(I\) has a Gröbner basis of a very particular form (see [GP] and §2 below). Although O'Sullivan has shown how more general order domains arise naturally from valuations on function fields, it is not clear to us how our approach applies to those examples. On the positive side, by basing all constructions on algebra in polynomial rings, all codes from these order domains can be treated in a uniform way. Second, we also propose to study the relation between the BMS algorithm and the process of solving this key equation in the cases where BMS is applicable.
Finally, we wish to show how solutions of our key equation can be used to determine error values and complete the decoding process.

Our key equation generalizes the key equation for \( n \)-dimensional cyclic codes studied by Chabanne and Norton in [CN]. Results on the algebraic background for their construction appear in [Norton1]. See also [Norton2] for connections with the more general problem of finding shortest linear recurrences, and [NS] for a generalization giving a key equation for codes over commutative rings. In the present article, we will point out another natural interpretation of these ideas in the context of Macaulay’s inverse systems for ideals in a polynomial ring (see [Mo], [EI]) and the theory of duality.

In spirit, our approach is also quite close to the treatment of one-point geometric Goppa codes from curves by Heegard and Saints in [HS], in that we essentially treat all of our codes as (subcodes of) punctured \( n \)-dimensional cyclic codes.

The present article is organized as follows. In \( \S 2 \) we will briefly review the definition of an order domain, evaluation codes and dual evaluation codes. We will also introduce some standard examples. \( \S 3 \) contains a quick summary of the basics of Macaulay inverse systems and duality for quotients of a polynomial ring by zero-dimensional ideals. In \( \S 4 \) we introduce the key equation. We will also relate the BMS algorithm to the process of solving this equation. \( \S 5 \) is devoted to a discussion of how the key equation can be used to determine error values. The major idea appears already for the case of \( n \)-dimensional cyclic codes in [CN]. However, our results apply more generally and include a few improvements. Finally, in \( \S 6 \) we present two detailed decoding examples using these methods.

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\( \S 2. \) Codes from Order Domains

In this section we will briefly recall the definition of order domains and explain how they can be used to construct error control codes. We will use the following formulation.

\( \text{(2.1) Definition.} \) Let \( R \) be a \( \mathbb{F}_q \)-algebra and let \( (\Gamma, +, \succ) \) be a well-ordered semigroup. An order function on \( R \) is a surjective mapping \( \rho : R \to \{-\infty\} \cup \Gamma \) satisfying:

\begin{enumerate}
  \item \( \rho(f) = -\infty \iff f = 0, \)
  \item \( \rho(cf) = \rho(f) \) for all \( f \in R \), all \( c \neq 0 \) in \( \mathbb{F}_q \),
  \item \( \rho(f + g) \preceq \max \prec \{\rho(f), \rho(g)\} \),
  \item if \( \rho(f) = \rho(g) \neq -\infty \), then there exists \( c \neq 0 \) in \( \mathbb{F}_q \) such that \( \rho(f) \prec \rho(f - cg) \),
  \item \( \rho(fg) = \rho(f) + \rho(g) \).
\end{enumerate}

We call \( \Gamma \) the value semigroup of \( \rho \).

The terminology “order function” is supposed to suggest the existence of \( \mathbb{F}_q \)-bases of \( R \) whose elements have distinct \( \rho \)-values, and are hence ordered by \( \rho \). This
is a consequence of Axiom 4. It is also possible to reindex the corresponding bases by the natural numbers and define order functions in a different but equivalent way. This is done, for instance, in [OS1] and [OS2].

Axioms 1 and 5 in this definition imply that \( R \) must be an integral domain. In the cases where the transcendence degree of \( R \) over \( \mathbb{F}_q \) is at least 2, a ring \( R \) with one order function will have many others too. For this reason an order domain is formally defined as a pair \( (R, \rho) \) where \( R \) is an \( \mathbb{F}_q \)-algebra and \( \rho \) is an order function on \( R \). However, from now on, we will only use one particular order function on \( R \) at any one time. Hence we will often omit it in referring to the order domain, and we will refer to \( \Gamma \) as the value semigroup of \( R \).

From one point of view, order functions come from valuations on \( K = QF(R) \). As noted by O’Sullivan in [OS1], in fact \( S = \{ f/g : \rho(g) \geq \rho(f) \} \) is a valuation ring of \( K \). From now on, we will restrict our attention to the case that \( \Gamma \) is a sub-semigroup of \( \mathbb{Z}^r_{\geq 0} \), for some \( r \geq 1 \), hence is finitely generated. Without loss of generality, then, we may assume \( r = \text{tr.deg.}_{\mathbb{F}_q}(K) \). To obtain a well-ordering on \( \mathbb{Z}^r_{\geq 0} \) we can fix a monomial order, \( \succ \).

As noted in the introduction, order domains give a common generalization of several types of rings that have been used in the construction of codes. For instance, the order domains used in the construction of one-point geometric Goppa codes are the following. If \( Y \) is a smooth projective curve defined over \( \mathbb{F}_q \), and \( Q \) is an \( \mathbb{F}_q \)-rational point on \( Y \), then \( R = \bigcup_{m=0}^{\infty} L(mQ) \) is an order domain. \( \Gamma \) is equal to the Weierstrass semigroup of \( Y \) at \( Q \) (the sub-semigroup of \( \mathbb{Z}_{\geq 0} \) consisting of all pole orders of rational functions on \( X \) with poles only at \( Q \)), and \( \rho(f) = -v_Q(f) \), where \( v_Q \) is the discrete valuation at \( Q \) on the function field of \( Y \). The polynomial ring \( R = \mathbb{F}_q[X_1, \ldots, X_r] \) is an order domain, where \( \Gamma = \mathbb{Z}^r_{\geq 0}, \succ \) is a monomial order, and \( \rho(f) \) for \( f \neq 0 \) is defined by \( \rho(f) = \alpha \) if \( LT_\succ(f) = X^\alpha \). These examples of order domains feature in the construction of Reed-Muller and other multidimensional cyclic codes. Many other classes of examples are considered in [Gei] and [GP].

Geil and Pellikaan (see [GP]) have proved a characterization of order domains with finitely generated \( \Gamma \), which we will now review. In the following statement, \( M \) is an \( r \times s \) matrix with entries in \( \mathbb{Z}_{\geq 0} \) with linearly independent rows. For \( \alpha \in \mathbb{Z}^s_{\geq 0} \) (written as a column vector), the matrix product \( M\alpha \) is a vector in \( \mathbb{Z}^r_{\geq 0} \). We will call this the \( M \)-weight of the monomial. We write \( \langle M \rangle \) for the subsemigroup of \( \mathbb{Z}^r_{\geq 0} \) generated by the columns of \( M \), ordered by any convenient monomial order \( \succ \) on \( \mathbb{Z}^r_{\geq 0} \) (for instance the \( \text{lex} \) order as in Robbiano’s characterization of monomial orders by weight matrices). We will make use of the monomial orders \( \succ_{M, \tau} \) on \( \mathbb{F}_q[X_1, \ldots, X_s] \) defined as follows: \( X^\alpha \succ_{M, \tau} X^\beta \) if \( M\alpha \succ M\beta \), or if \( M\alpha = M\beta \) and \( X^\alpha \succ_{\tau} X^\beta \), where \( \tau \) is another monomial order used to break ties.

**(2.2) Theorem.** (Geil-Pellikaan)

1. Let \( \Gamma = \langle M \rangle \subset \mathbb{Z}^r_{\geq 0} \) be a semigroup. Let \( I \subset \mathbb{F}_q[X_1, \ldots, X_s] \) be an ideal, and let \( G \) be the reduced Gröbner basis for \( I \) with respect to a weight order \( \succ = \succ_{M, \tau} \) as above. Suppose that every element of \( G \) has exactly two monomials of highest \( M \)-weight in its support, and that the monomials in the com-
plement of $LT_>(I)$ (the “standard monomials” or monomials in the “footprint of the ideal”) have distinct $M$-weights. Then $R = \mathbb{F}_q[X_1, \ldots, X_s]/I$ is an order domain with value semigroup $\Gamma$ and order function $\rho$ defined as follows: Writing $f$ in $R$ as a linear combination of the monomials in the complement of $LT_>(I)$, $\rho(f) = \max_\succ \{ M \beta : X^\beta \in \text{supp}(f) \}$.

(2) Every order domain with semigroup $\Gamma = \langle M \rangle$ has a presentation $R \cong \mathbb{F}_q[X_1, \ldots, X_s]/I$ such that the reduced Gröbner basis of $I$ with respect to $\succ_{M, \tau}$ and the standard monomials have the form described in part (1).

In principle, this result gives a method to construct the order domains with a given value semigroup $\Gamma$, as in following example.

(2.3) Example. Take $r = 2$, $\Gamma = \langle M \rangle \subset \mathbb{Z}_{\geq 0}^2$, ordered by $\succ$ the lexicographic order, where

$$M = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 1 & 0 \end{pmatrix}.$$ 

By the definition, the order function $\rho$ must be surjective, so there exist $x, y, z \in R$ with $\rho(x) = (0, 2)$, $\rho(y) = (1, 1)$, $\rho(z) = (3, 0)$, $R$ is generated by $x, y, z$, and $\rho(x^a y^b z^c)$ is equal to the $M$-weight $M(a, b, c)^t$ for all monomials $x^a y^b z^c$. It follows that there is a surjective ring homomorphism

$$\phi : \mathbb{F}_q[X, Y, Z] \rightarrow R,$$

where $\phi(X) = x$, $\phi(Y) = y$, and $\phi(Z) = z$. We consider the monomial order $\succ_{M, \text{lex}}$ on $\mathbb{F}_q[X, Y, Z]$. It is easy to see that all $\mathbb{Z}$-relations between $\rho(x)$, $\rho(y)$, $\rho(z)$ are generated by $3 \rho(x) + 2 \rho(y) = 6 \rho(y)$. For Definition (2.1) to hold, we must have $\rho(x^2 z^3 - cy^6) < \rho(x^2 z^3)$ for some $c \neq 0$. Hence $R \cong \mathbb{F}_q[X, Y, Z]/I$, where $I = \langle F \rangle$ for some $F = X^2 Z^3 - cY^6 + H(X, Y, Z)$, where every term in $H$ is less than $Y^6$ in the $\succ_{M, \text{lex}}$ order. The monomials in the complement of $\langle X^2 Z^3 \rangle$ have distinct $M$-weights and $G = \{ F \}$ is a Gröbner basis for $I$ of the required form, so all such $R$ are order domains by Theorem (2.2). Note that all are deformations of the monomial algebra $\mathbb{F}_q[u^2, uv, uv^3]$. Indeed, Theorem (2.2) can be reinterpreted as saying that the order domains with semigroup $\Gamma$ are all flat deformations of the monomial algebra $\mathbb{F}_q[\Gamma]$. This point of view is exploited in [L] to construct order domains in the function fields of varieties such as Grassmannians and flag varieties.

The most direct way to construct codes from an order domain given by a particular presentation $R \cong \mathbb{F}_q[X_1, \ldots, X_s]/I$ is to generalize Goppa’s construction in the case of curves:

(2.4) Construction of Codes.

(1) Let $X_R$ be the variety $V(I) \subset \mathbb{A}^s$ and let

$$X_R(\mathbb{F}_q) = \{ P_1, \ldots, P_n \}$$

be the set of $\mathbb{F}_q$-rational points on $X_R$. 


(2) Define an evaluation mapping
\[ ev : R \to \mathbb{F}_q^n \]
\[ f \mapsto (f(P_1), \ldots, f(P_n)) \]
(3) Let \( V \subset R \) be any finite-dimensional vector subspace. Then the image \( ev(V) \subseteq \mathbb{F}_q^n \) will be a linear code in \( \mathbb{F}_q^n \). One can also consider the dual code \( ev(V)^\perp \).
(4) Of particular interest here are the codes constructed as follows. Let \( \Delta \) be the ordered basis of \( R \) given by the monomials in the complement of \( LT_>(I) \). Note that this basis comes equipped with an ordering by \( \rho \)-value, or equivalently by the \( M \)-weights ordered by \( \succ \) in \( \mathbb{Z}_{\geq 0} \). Let \( \ell \in \mathbb{N} \) and let \( V_\ell \) be the span of the first \( \ell \) elements of the ordered basis \( \Delta \). In this way, we obtain evaluation codes \( Ev_\ell = ev(V_\ell) \) and dual codes \( C_\ell = Ev_\ell^\perp \) for all \( \ell \).

The BMS algorithm is specifically tailored for this last class of codes. If the \( C_\ell \) codes are used to encode messages, then the \( Ev_\ell \) codes describe the parity checks and the syndromes used in the decoding algorithm.

§3. PRELIMINARIES ON INVERSE SYSTEMS

A natural setting for our formulation of a key equation for codes from order domains is the theory of inverse systems of polynomial ideals originally introduced by Macaulay ([Ma]). There are several different versions of this theory. For modern versions using the language of differentiation operators, see [Mo] or [El]. Here, we will summarize a number of more or less well-known results, using an alternate formulation of the definitions that works in any characteristic. A reference for this approach is [North].

Let \( k \) be a field, let \( S = k[X_1, \ldots, X_s] \) and let \( T \) be the formal power series ring \( k[[X_1^{-1}, \ldots, X_s^{-1}]] \) in the inverse variables. \( T \) is an \( S \)-module under a mapping
\[ c : S \times T \to T \]
\[ (f, g) \mapsto f \cdot g, \]
sometimes called contraction, defined as follows. First, given monomials \( X^\alpha \) in \( S \) and \( X^{-\beta} \) in \( T \), \( X^\alpha \cdot X^{-\beta} \) is defined to be \( X^{\alpha - \beta} \) if this is in \( T \), and 0 otherwise. We then extend by linearity to define \( c : S \times T \to T \).

Let \( Hom_k(S, k) \) be the usual linear dual vector space. It is a standard fact that the mapping
\[ \phi : Hom_k(S, k) \to T \]
\[ \Lambda \mapsto \sum_{\beta \in \mathbb{Z}_{\geq 0}^s} \Lambda(X^\beta)X^{-\beta} \]
is an isomorphism of $S$-modules, if we make $\text{Hom}_k(S, k)$ into an $S$-module in the usual way by defining $(q\Lambda)(p) = \Lambda(qp)$ for all polynomials $p, q$ in $S$. In explicit terms, the $k$-linear form on $S$ obtained from an element of $g \in T$ is mapping $\Lambda_g$ defined as follows. For all $f \in S$,

$$\Lambda_g(f) = (f \cdot g)_0,$$

where $(t)_0$ denotes the constant term in $t \in T$. In the following we will identify elements of $T$ with their corresponding linear forms on $S$.

For each ideal $I \subseteq R$, we can define the annihilator, or inverse system, of $I$ in $T$ as

$$I^\perp = \{\Lambda \in T : \Lambda(p) = 0, \forall p \in I\}.$$  

It is easy to check that $I^\perp$ is an $S$-submodule of $T$ under the module structure defined above. Similarly, given an $S$-submodule $H \subseteq T$, we can define

$$H^\perp = \{p \in S : \Lambda(p) = 0, \forall \Lambda \in H\},$$

and $H^\perp$ is an ideal in $R$.

The key point in this theory is the following duality statement.

(3.1) Theorem. The ideals of $R$ and the $S$-submodules of $T$ are in inclusion-reversing bijective correspondence via the constructions above, and for all $I, H$ we have:

$$(I^\perp)^\perp = I, \quad (H^\perp)^\perp = H.$$  

See [North] for a proof.

We will be interested in applying Theorem (3.1) when $I$ is the ideal of some finite set of points in the $n$-dimensional affine space over $k$ (e.g. when $k = \mathbb{F}_q$ and $I$ is an error-locator ideal arising in decoding – see §4 below).

(3.2) Lemma. Let

$$I = m_{P_1} \cap \cdots \cap m_{P_t},$$

where $m_{P_i}$ is the maximal ideal of $S$ corresponding to the point $P_i$, and $t \geq 1$. The submodule of $T$ corresponding to $I$ has the form

$$H = I^\perp = (m_{P_1})^\perp \oplus \cdots \oplus (m_{P_t})^\perp.$$  

Proof. In Proposition 2.6 of [Ger], Geramita shows that $(I \cap J)^\perp = I^\perp + J^\perp$ for any pair of ideals. The idea is that $I^\perp$ and $J^\perp$ can be constructed degree by degree, so the corresponding statement from the linear algebra of finite-dimensional vector spaces applies. The equality $(I + J)^\perp = I^\perp \cap J^\perp$ also holds from linear algebra (and no finite-dimensionality is needed). The sum in the statement of the Lemma is a direct sum since $m_{P_i} + \cap_{j \neq i} m_{P_j} = S$, hence $(m_{P_i})^\perp \cap \Sigma_{j \neq i} (m_{P_j})^\perp = \{0\}$. □

We can also give a concrete description of the elements of $(m_P)^\perp$.  

(3.3) Proposition. Let $P = (a_1, \ldots, a_s) \in \mathbb{A}^s$ over $k$, and let $L_i$ be the coordinate hyperplane $X_i = a_i$ containing $P$.

(1) $(m_P)^\perp$ is the cyclic $S$-submodule of $T$ generated by

$$h_P = \sum_{u \in \mathbb{Z}_{\geq 0}^s} P_u X^{-u},$$

where if $u = (u_1, \ldots, u_s)$, $P_u$ denotes the product $a_1^{u_1} \cdots a_s^{u_s}$ ($X^u$ evaluated at $P$).

(2) $f \cdot h_P = f(P) h_P$ for all $f \in S$, and the submodule $(m_P)^\perp$ is a one-dimensional vector space over $k$.

(3) Let $I_{L_i}$ be the ideal $\langle X_i - a_i \rangle$ in $S$ (the ideal of $L_i$). Then $(I_{L_i})^\perp$ is the submodule of $T$ generated by

$$h_{L_i} = \sum_{j=0}^{\infty} a_i^j X_i^{-j}.$$

(4) In $T$, we have

$$h_P = \prod_{i=1}^s h_{L_i}.$$

Proof. (1) First, if $f \in m_P$, and $g \in S$ is arbitrary then

$$\Lambda_{g \cdot h_P}(f) = (f \cdot (g \cdot h_P))_0 = ((fg) \cdot h_P)_0 = f(P) g(P) = 0.$$

Hence the $S$-submodule $\langle h_P \rangle$ is contained in $(m_P)^\perp$. Conversely, if $h \in (m_P)^\perp$, then for all $f \in m_P$,

$$0 = \Lambda_{h}(f) = (f \cdot h)_0.$$

An easy calculation using all $f$ of the form $f = x^\beta - a^\beta \in m_P$ shows that $h = c h_P$ for some constant $c$. Hence $(m_P)^\perp = \langle h_P \rangle$.

(2) The second claim follows by a direct computation of the contraction product $f \cdot h_P$.

(3) Let $f \in I_{L_i}$ (so $f$ vanishes at all points of the hyperplane $L_i$), and let $g \in S$ be arbitrary. Then

$$\Lambda_{g \cdot h_{L_i}}(f) = (f \cdot (g \cdot h_{L_i}))_0$$

$$= ((fg) \cdot h_{L_i})_0$$

$$= f(0, \ldots, 0, a_i, 0, \ldots, 0)g(0, \ldots, 0, a_i, 0, \ldots, 0)$$

$$= 0,$$
since the only nonzero terms in the product \(((fg) \cdot h_{L_i})\) come from monomials in \(fg\) containing only the variable \(X_i\). Hence \(\langle h_{L_i} \rangle \subset T\) is contained in \(I_{L_i}^\perp\). Then we show the other inclusion as in the proof of (1).

(4) We have \(m_P = I_{L_1} + \cdots + I_{L_s}\). Hence \((m_P)^\perp = (I_{L_1})^\perp \cap \cdots \cap (I_{L_s})^\perp\), and the claim follows. We note that a more explicit form of this equation can be derived by the formal geometric series summation formula:

\[
h_P = \sum_{u \in \mathbb{Z}_{\geq 0}} P^u X^{-u} = \prod_{i=1}^{s} \frac{1}{1 - a_i/X_i} = \prod_{i=1}^{s} h_{L_i}, \quad \square
\]

Finally, we note that both the polynomial ring \(S\) and the formal power series ring \(T\) can be viewed as subrings of the field of formal Laurent series in the inverse variables,

\[
K = k((X_1^{-1}, \ldots, X_s^{-1})),
\]

which is the field of fractions of \(T\). Hence there is a natural interpretation of the (full) product \(fg\) for \(f \in S\) and \(g \in T\) as an element of \(K\). The contraction product \(f \cdot g\) can be understood as a projection of \(fg\) into \(T \subset K\) (image under the linear projection with kernel spanned by all monomials not in \(T\)). In the sequel, we will also need to make use of the projection of \(fg\) into \(S_+ = \langle X_1, \ldots, X_s \rangle \subset S \subset K\) under the linear projection with kernel spanned by all monomials not in \(S_+\). We will denote this by \((fg)_+\). Hence \((fg)_+\) gives the sum of all terms in \(fg\) with all exponents nonnegative and some exponent strictly positive, while \(f \cdot g\) gives the sum of all terms in \(fg\) with nonpositive exponents. Any “mixed terms” in \(fg\) (i.e. those terms with some positive and some negative exponents) will be irrelevant in our applications. We will use the following fact.

**Proposition 3.4.** Let \(f \in k[X_i]\) be a univariate polynomial satisfying \(f(P) = 0\). Then

\[
(fh_P)_+ = X_i g,
\]

where \(g(P) = f'(P)\) (formal derivative).

**Proof.** This follows by a direct computation using (3.3). \(\square\)

§4. The Key Equation and its Relation to the BMS Algorithm

In this section, we will introduce our key equation for codes from order domains and relate it to the Berlekamp-Massey-Sakata decoding algorithm. Let \(C\) be one of the codes \(C = ev(V)\) or \(ev(V)^\perp\) constructed from an order domain \(R \cong \mathbb{F}_q[X_1, \ldots, X_s]/I\) as in §2 above. Consider an error vector \(e \in \mathbb{F}_q^s\) (where entries are indexed by the elements of the set \(X_R(\mathbb{F}_q)\)). In the usual terminology, the error-locator ideal corresponding to \(e\) is the ideal \(I_e \subset \mathbb{F}_q[X_1, \ldots, X_s]\) defining the set of error locations:

\[
I_e = \{ f \in \mathbb{F}_q[X_1, \ldots, X_s] : f(P) = 0, \quad \forall P \text{ s.t. } e_P \neq 0 \}.
\]
(Since $I_e \supset I$, one could also consider the ideal corresponding to $I_e$ in $R$. However, following the general philosophy of Heegard and Saints in [HS], we will find it more convenient to work with $I_e$ as an ideal in the polynomial ring.)

We will also use a slightly different notation and terminology in the following because we want to make a systematic use of the observation that this ideal depends only on the support of $e$, not on the error values. Indeed, many different error vectors yield the same ideal defining the error locations. For this reason we will introduce $\mathcal{E} = \{ P : e_P \neq 0 \}$, and refer to the error-locator ideal for any $e$ with $\text{supp}(e) = \mathcal{E}$ as $I_{\mathcal{E}}$.

For each monomial $X^u \in \mathbb{F}[X_1, \ldots, X_s]$, we let

$$E_u = \langle e, ev(X^u) \rangle = \sum_{P \in X_R(\mathbb{F}_q)} e_P P^u$$

be the corresponding syndrome of the error vector. (As in (3.3), $P^u$ is shorthand notation for the evaluation of the monomial $X^u$ at $P$.)

In the practical decoding situation, of course, for a code $C = ev(V)^\perp$ where $V$ is a subspace of $R$ spanned by some set of monomials, only the $E_u$ for the $X^u$ in a basis of $V$ are initially known from the received word.

In addition, the elements of the ideal $I + \langle X_1^q - X_1, \ldots, X_s^q - X_s \rangle$ defining the set $X_R(\mathbb{F}_q)$ give relations between the $E_u$. Indeed, the $E_u$ for $u$ in the ordered basis $\Delta$ for $R$ with all components $\leq q - 1$ determine all the others, and these syndromes still satisfy additional relations. Thus the $E_u$ are, in a sense, highly redundant.

To package the syndromes into a single algebraic object, we define the syndrome series

$$S_e = \sum_{u \in \mathbb{Z}_{\geq 0}} E_u X^{-u}$$

in the formal power series ring $T = \mathbb{F}_q[[X_1^{-1}, \ldots, X_s^{-1}]]$. (This depends both on the set of error locations $\mathcal{E}$ and on the error values.) Chabanne and Norton considered the same type of expression in [CN] for $n$-dimensional cyclic codes. As in §3, we have a natural interpretation for $S_e$ as an element of the dual space of the ring $S = \mathbb{F}_q[X_1, \ldots, X_s]$.

A fundamental tool in our considerations will be the following expression for the syndrome series $S_e$. We substitute from (4.1) for the syndrome $E_u$ and change the order of summation to obtain:

$$S_e = \sum_{u \in \mathbb{Z}_{\geq 0}} E_u X^{-u}$$

$$= \sum_{u \in \mathbb{Z}_{\geq 0}} \sum_{P \in X_R(\mathbb{F}_q)} e_P P^u X^{-u}$$

$$= \sum_{P \in X_R(\mathbb{F}_q)} e_P \sum_{u \in \mathbb{Z}_{\geq 0}} P^u X^{-u}$$

(4.2)

$$= \sum_{P \in X_R(\mathbb{F}_q)} e_P h_P,$$
where $h_P$ is the generator of $(m_P)^\perp$ from (3.3). The sum in (4.2), taking the terms with $e_P \neq 0$, gives the decomposition of $S_e$ in the direct sum expression for $I_E^\perp$ as in (3.2).

The following result is well-known in a sense; it is a translation of the standard fact that error-locators give linear recurrences on the syndromes. But to our knowledge, this connection has not been considered from exactly our point of view in this generality (see [AD] for a special case).

(4.3) Theorem. With all notation as above,

1. $f \in I_E$ if and only if $f \cdot S_e = 0$ for all error vectors $e$ with $\text{supp}(e) = \mathcal{E}$.
2. For each $e$ with $\text{supp}(e) = \mathcal{E}$, $I_E = \langle S_e \rangle^\perp$ in the duality from Theorem (3.1).
3. If $e, e'$ are two error vectors with the same support, then $\langle S_e \rangle = \langle S_{e'} \rangle$ as submodules of $T$.

Proof. For (1), we start from the expression for $S_e$ from (4.2). Then by (3.3), we have

$$ f \cdot S_e = \sum_{P \in \mathcal{E}} e_P (f \cdot h_P) = \sum_{P \in \mathcal{E}} e_P f(P) h_P. $$

If $f \in I_E$, then clearly $f \cdot S_e = 0$ for all choices of error values $e_P$. Conversely, if $f \cdot S_e = 0$ for all $e$ with $\text{supp}(e) = \mathcal{E}$, then $f(P) = 0$ for all $P \in \mathcal{E}$, so $f \in I_E$.

Claim (2) follows from (1).

The perhaps surprising claim (3) is a consequence of (2). Another way to prove (3) is to note that there exist $g \in R$ such that $g(P)e_P = e'_P$ for all $P \in \mathcal{E}$. We have

$$ g \cdot S_e = \sum_{P \in \mathcal{E}} e_P (g \cdot h_P) = \sum_{P \in \mathcal{E}} e_P g(P) h_P = \sum_{P \in \mathcal{E}} e'_P h_P = S_{e'}. $$

Hence $\langle S_{e'} \rangle \subseteq \langle S_e \rangle$. Reversing the roles of $e$ and $e'$, we get the other inclusion as well, and (3) follows. □

The following explicit expression for the terms in $f \cdot S_e$ is also useful. Let $f = \sum_m f_m X^m \in S$. Then

$$ f \cdot S_e = (\sum_m f_m X^m) \cdot (\sum_{u \in \mathbb{Z}_{\geq 0}} E_u X^{-u}) $$

$$ = \sum_{r \in \mathbb{Z}_{\geq 0}} (\sum_m f_m E_{m+r}) X^{-r}. \tag{4.4} $$

Hence $f \cdot S_e = 0 \iff \sum_m f_m E_{m+r} = 0$ for all $r \geq 0$.

The equation $f \cdot S = 0$ from (1) in (4.3) is the prototype, so to speak, for our generalizations of the key equation to all codes from order domains, and we will refer to it as the key equation in the following. It also naturally generalizes all the various key equations that have been developed in special cases, as we will
demonstrate shortly. Before proceeding with that, however, we wish to make several
comments about the form of this equation.

Comparing the equation $f \cdot S_e = 0$ with the familiar form (1.1), several differences
may be apparent. First, note that the syndrome series $S_e$ will not be entirely known
from the received word in the decoding situation. The same is true in the Reed-
Solomon case, of course. The polynomial $S$ in the congruence in (1.1) involves only
the known syndromes, and (1.1) is derived by accounting for the other terms in the
full syndrome series. With a truncation of $S_e$ in our situation we would obtain a
similar type of congruence (see the discussion following (4.14) below, for instance).

It is apparently rare, however, that the portion of $S_e$ known from the received
word suffices for decoding up to half the minimum distance of the code. As first
noted for the one-point geometric Goppa codes from curves, it is often the case
that additional syndromes (or other extra information about the error) must be
determined in order to exploit the code’s full error correcting capacity. For this
reason, even though we have not made any hypotheses so far on how our code was
constructed (i.e. on how the vector subspace $V \subset R$ was chosen), the key equation
will be most useful in the case that $C$ is one of the codes $C_\ell = Ev_\ell$ defined in §2,
for which the Feng-Rao majority voting process for unknown syndromes and the
generalized BMS algorithm are applicable.

Another difference is that there is no apparent analog of the error-evaluator
polynomial $g$ from (1.1) in the equation in $f \cdot S_e = 0$. In §5, we will see that the
way to obtain error evaluators in this situation is to consider the “purely positive
parts” $(fS_e)_+$ for certain solutions of our key equation.

We now turn to several examples that show how our key equation relates to
several special cases that have appeared in the literature.

(4.5) Example. We begin by providing more detail on the precise relation between
(4.3), part (1) in the case of a Reed-Solomon code and the usual key equation from
(1.1). These codes are constructed from the order domain $R = \mathbb{F}_q[X]$ (where
$\Gamma = \mathbb{Z}_{\geq 0}$ and $\rho$ is the degree mapping), according to (4.4). The key equation (1.1)
applies to the code $Ev_\ell = ev(V_\ell)$, where $V_\ell = \text{span}\{1, X, X^2, \ldots, X^{\ell-1}\}$, and the
evaluation takes place at all $\mathbb{F}_q$-rational points on the affine line, omitting 0.

For the $Ev_\ell$ Reed-Solomon codes, the known syndromes are $E_1, \ldots, E_{d-1}$, and
$S$ is the syndrome polynomial:

$$S = E_1 + E_2X + \cdots + E_{d-1}X^{d-2}.$$

In the special solution $(f, g)$ of (1.1) used for decoding,

$$f = \prod_{i=1}^{\left| \mathcal{E} \right|} (1 - \alpha^{e_i} x),$$

where $\alpha^{-e_i}$ are the error locations. Moreover,

$$g = \sum_{i=1}^{\left| \mathcal{E} \right|} e_i \alpha^i \prod_{j \neq i} (1 - \alpha^{e_j} x).$$
If (1.1) is written as an equation

$$fS = g + x^{2t}h,$$

then $h$ is another polynomial of degree $|\mathcal{E}| - 1$ sometimes called the error coevaluator:

$$h = \sum_{i=1}^{|\mathcal{E}|} e_i \alpha^{(2t+1)i} \prod_{j \neq i} (1 - \alpha^{e_j}x).$$

Either $g$ or $h$ can be used to solve for the error values $e_i$ once the roots of $f$ are determined.

Our key equation in this case is closely related, but not precisely the same. The natural way to apply (4.3) here is to the dual code $C_\ell = Ev^\perp_\ell$. Our prototype key equation $f \cdot S_e = 0$ uses the full syndrome series, but of course, we could also consider the truncation of $S_e$ using only the known syndromes $E_0, \ldots, E_{\ell-1}$ and obtain a congruence close in form to (1.1).

Starting from (4.4) and using the formal geometric series summation formula as in (3.3) part (4), we can write:

$$S_e = \sum_{P \in \mathcal{E}} e_P h_P$$

$$= \sum_{P \in \mathcal{E}} e_P \sum_{u \geq 0} P^u X^{-u}$$

$$= \sum_{P \in \mathcal{E}} e_P \frac{1}{1 - P/X}$$

$$= X \sum_{P \in \mathcal{E}} e_P \prod_{Q \in \mathcal{E}, Q \neq P} (X - Q) \prod_{P \in \mathcal{E}} (X - P)$$

Hence, in this formulation, $S_e = Xq/p$, where $p$ is the generator of the (actual) error locator ideal. By considering the truncated form of $f \cdot S_e = 0$, it can be seen that our $q$ is actually the analog of the error coevaluator as above. Moreover if $f = p$, then $(pS_e)_+ = Xq$ gives the error (co)evaluator. There are no “mixed terms” in the products $fS_e$ in this one-variable situation.

(4.6) Example. The key equation for $s$-dimensional cyclic codes introduced by Chabanne and Norton in [CN] has the form:

$$\sigma S_e = \left( \prod_{i=1}^s X_i \right) g,$$

where

$$\sigma = \prod_{i=1}^s \sigma_i(X_i),$$
and $\sigma_i$ is the univariate generator of the elimination ideal $I_\ell \cap \mathbb{F}_q[X_i]$. Our version of the Reed-Solomon key equation from (4.5) is a special case of (4.7). Moreover, (4.7) is clearly the special case of (4.3), part (1) for these codes where $f = \sigma$ is the particular error locator polynomial $\prod_{i=1}^s \sigma_i(X_i) \in I_\ell$. For this special choice of error locator, $\sigma \cdot S_e = 0$, and $(\sigma S_e)_+ = (\prod_{i=1}^s X_i) g$ for some polynomial $g$. This last claim can be established using (4.4). We see that $S_e$ can be written as

$$S_e = \sum_P e_P h_P = \left( \prod_{i=1}^s X_i \right) \sum_P e_P \frac{1}{\prod_{i=1}^s (X_i - X_i(P))}$$

and the product $\sigma S_e = (\sigma S_e)_+$ reduces to a polynomial (again, there are no “mixed terms”).

In order to use (4.7) for decoding, Chabanne and Norton propose iterated applications of the one-variable Berlekamp-Massey algorithm to find the factors of the product $\sigma$ one at a time. In §5 and §6 we will see that the more general BMS algorithm gives additional flexibility for decoding these codes, although the equation (4.7) will still lead most directly to determination of the error values.

(4.8) Example. We now turn to the key equation for one-point geometric Goppa codes introduced by O’Sullivan in [OS3]. Let $\mathcal{X}$ be a smooth curve over $\mathbb{F}_q$ of genus $g$, and consider one-point codes constructed from $R = \cup_{m=0}^{\infty} L(mQ)$ for some point $Q \in \mathcal{X}(\mathbb{F}_q)$, O’Sullivan’s key equation has the form:

$$f \omega_e = \phi.$$  

Here $\omega_e$ is the syndrome differential, which can be expressed as

$$\omega_e = \sum_{P \in \mathcal{X}(\mathbb{F}_q)} e_P \omega_{P,Q},$$

where $\omega_{P,Q}$ is the differential of the third kind on $Y$ with simple poles at $P$ and $Q$, no other poles, and residues

$$\text{res}_P(\omega_{P,Q}) = 1, \quad \text{res}_Q(\omega_{P,Q}) = -1.$$  

For any $f \in R$, we have

$$\text{res}_Q(f \omega_e) = \sum_P e_P f(P),$$

the syndrome of $e$ corresponding to $f$. (We only defined syndromes for monomials above; taking a presentation $R = \mathbb{F}_q[X_1, \ldots, X_s]/I$, however, any $f \in R$ can be expressed as a linear combination of monomials and the syndrome of $f$ is defined accordingly.) The right-hand side of (4.9) is also a differential. In this situation, (4.9) furnishes a key equation in the following sense: $f$ is an error locator (i.e. $f$ is in the ideal of $R$ corresponding to $I_\ell$) if and only if $\phi$ has poles only at $Q$. 

In the special case that \((2g - 2)Q\) is a canonical divisor (the divisor of zeroes of some differential of the first kind \(\omega_0\) on \(\mathcal{X}\)), (4.9) can be replaced by the equivalent equation

\[
(4.10) \quad f o_e = g,
\]

where \(o_e = \omega_e/\omega_0\) and \(g = \phi/\omega_0\) are rational functions on \(\mathcal{X}\). Since \(\omega_0\) is zero only at \(Q\), the key equation is now that \(f\) is an error locator if and only if (4.9) is satisfied for some \(g \in R\).

For instance, when \(\mathcal{X}\) is a smooth plane curve \(V(F)\) over \(\mathbb{F}_q\) defined by \(F \in \mathbb{F}_q[X, Y]\), with a single point \(Q\) at infinity, then it is true that \((2g - 2)Q\) is canonical.

O’Sullivan shows in Example 4.2 of [OS3] (using a slightly different notation) that

\[
(4.11) \quad o_e = \sum_{P \in \mathcal{X}(\mathbb{F}_q)} e_P H_P,
\]

where if \(P = (a, b)\), then \(H_P = \frac{F(a,Y)}{(X-a)(Y-b)}\). This is a function with a pole of order 1 at \(P\), a pole of order \(2g - 1\) at \(Q\), and no other poles.

To relate this to our approach, note that we may assume from the start that \(Q = (0 : 1 : 0)\) and that \(F\) is taken in the form from Theorem (2.2), that is

\[
F(X, Y) = X^\beta - cY^\alpha + G(X, Y)
\]

for some relatively prime \(\alpha < \beta\) generating the value semigroup at \(Q\). Every term in \(G\) has \((\alpha, \beta)\)-weight less than \(\alpha \beta\).

Then we can proceed as in Example (4.3) of [OS3] to relate \(H_P\) to an element of \(T = \mathbb{F}_q[[X^{-1}, Y^{-1}]]\). First we rearrange to obtain

\[
H_P = \frac{F(a,Y)}{(X-a)(Y-b)} = \frac{a^\beta - cY^\alpha + G(a,Y)}{(X-a)(Y-b)} = \frac{(a^\beta - X^\beta) + F(X, Y) + (G(a,Y) - G(X, Y))}{(X-a)(Y-b)}
\]

The \(F(X, Y)\) term in the numerator does not depend on \(P\). We can collect those terms in the sum (4.11) and factor out the \(F(X, Y)\). We will see shortly that those terms can in fact be ignored. The \(G(a,Y) - G(X, Y)\) in the numerator furnish terms that go into the error evaluator \(g\) here. The remaining portion is

\[
\frac{-(X^\beta - a^\beta)}{(X-a)(X-b)} = -\frac{X^{\beta-1}}{Y} \sum_{i=0}^{\beta} \sum_{j=0}^{\infty} \frac{a^i b^j}{X^i Y^j}.
\]
The sum here looks very much like that defining our $h_P$ from (3.3), except that it only extends over the monomials in complement of $\langle LT(F) \rangle$. Call this last sum $h'_P$. As noted before the full series $h_P$ (and consequently $S$) are redundant. For example, every ideal contained in $m_P$ (for instance the ideal $I = \langle F \rangle$ defining the curve), produces relations between the coefficients. From the duality theorem (3.1), we have that $I \subset m_P$ implies $(m_P)^\perp \subset I^\perp$, so $F \cdot h_P = 0$.

The relation $F \cdot h_P = 0$ says in particular that the terms in $h'_P$ are sufficient to determine the whole series $h_P$. Indeed, we have

$$h_P = h'_P + \frac{(cY^\alpha - G)}{X^\beta} \cdot h'_P + \left(\frac{(cY^\alpha - G)}{X^\beta}\right)^2 \cdot h'_P + \cdots$$

$$= \left(\frac{1}{1 - \frac{(cY^\alpha - G)}{X^\beta}}\right) \cdot h'_P$$

$$= \left(\frac{X^\beta}{F}\right) \cdot h'_P$$

It follows that O’Sullivan’s key equation and ours are equivalent.

We now turn to the precise relation between solutions of our key equation and the polynomials generated by steps of the BMS decoding algorithm applied to the $C_\ell = Ev^\perp_\ell$ codes from order domains $R$. We will see that the steps of the BMS algorithm systematically produce successively better approximations to solutions of $f \cdot Se = 0$, so that in effect, the BMS algorithm is a method for solving the key equation for these codes. In addition to [OS3] cited previously, a similar interpretation of the Berlekamp-Massey algorithm in the Reed-Solomon case (and related cases) was developed by Fitzpatrick in [F] (see also [CLO], Chapter 9, §4).

We recall the key features of O’Sullivan’s presentation of BMS. For our purposes, it will suffice to consider the “Basic Algorithm” from §3 of [OS2], in which all needed syndromes are assumed known and no sharp stopping criteria are identified. The syndrome mapping corresponding to the error vector $e$ is

$$Syn_e : R \rightarrow \mathbb{F}_q$$

$$f \mapsto \sum_{P \in E} e_P f(P),$$

where as above $E$ is the set of error locations. The same reasoning used in the proof of our Theorem (4.2) shows

$$(4.12) \quad f \in I_E \iff Syn_e(fg) = 0, \forall g \in R.$$

From Definition (2.1) and Geil and Pellikaan’s presentation theorem (2.2), we have an ordered monomial basis of $R$:

$$\Delta = \{X^{\alpha(j)} : j \in \mathbb{N}\},$$
whose elements have distinct $\rho$-values. As in the construction of the $E\nu_\ell$ codes, we write $V_\ell = \text{Span}\{1 = X^{\alpha(1)}, \ldots, X^{\alpha(\ell)}\}$. The $V_\ell$ exhaust $R$, so for $f \neq 0 \in R$, we may define

$$o(f) = \min\{\ell : f \in V_\ell\},$$

and (for instance) $o(0) = -1$. Indeed, all properties of order domains can be restated in terms of $o$, and O’Sullivan uses this function rather than $\rho$ in [OS1] and [OS2]. In particular the semigroup $\Gamma$ in our presentation carries over to a (nonstandard) semigroup structure on $\mathbb{N}$ defined by the addition operation

$$i \oplus j = k \iff o(X^{\alpha(i)}X^{\alpha(j)}) = k.$$  

Given $f \in R$, one defines

$$\text{span}(f) = \min\{\ell : \exists g \in V_\ell \text{ s.t. } \text{Syn}_e(fg) \neq 0\}$$

$$\text{fail}(f) = o(f) \oplus \text{span}(f).$$

When $f \in I_\mathcal{E}$, $\text{span}(f) = \text{fail}(f) = \infty$.

The BMS algorithm, then, is an iterative process which produces a Gröbner basis for $I_\mathcal{E}$ with respect to the monomial order $> = >_{M,\tau}$ in (2.2). The strategy is to maintain data structures for all $m \geq 1$ as follows. The $\Delta_m$ are an increasing sequence of sets of monomials, converging to the monomial basis for $I_\mathcal{E}$ as $m \to \infty$. $\delta_m$ is the set of maximal elements of $\Delta_m$ with respect to $>$ (the “interior corners of the footprint”). Similarly, we consider $\Sigma_m = \mathbb{Z}_{\geq 0}^* \setminus \Delta_m$, and $\sigma_m$, the set of minimal elements of $\Sigma_m$ (the “exterior corners”). For sufficiently large $m$, the elements of $\sigma_m$ will be the leading terms of the elements of the Gröbner basis of $I_\mathcal{E}$, and $\Sigma_m$ will the be set of monomials in $LT_>(I_\mathcal{E})$.

For each $m$, the algorithm also produces collections of polynomials $F_m = \{f_m(s) : s \in \sigma_m\}$ and $G_m = \{g_m(c) : c \in \delta_m\}$ satisfying:

$$o(f_m(s)) = s, \quad \text{fail}(f_m(s)) > m$$

and

$$\text{span}(g_m(c)) = c, \quad \text{fail}(g_m(c)) \leq m.$$  

In the limit as $m \to \infty$, by (4.12), the $F_m$ yield the Gröbner basis for $I_\mathcal{E}$.

We record the following simple observation.

(4.13) Proposition. With all notation as above, suppose $f \in R$ satisfies $o(f) = s$, $\text{fail}(f) > m$. Then

$$f \cdot S_e \equiv 0 \text{ mod } W_{s,m},$$

where $W_{s,m}$ is the $\mathbb{F}_q$-vector subspace of the formal power series ring $T$ spanned by the $X^{-\alpha(j)}$ such that $s \oplus j > m$.

Proof. By the definition, $\text{fail}(f) > m$ means that $\text{Syn}_e(fX^{\alpha(k)}) = 0$ for all $k$ with $o(f) \oplus k \leq m$. By the definitions of $S_e$ and the contraction product, $\text{Syn}_e(fX^{\alpha(k)})$ is exactly the coefficient of $X^{-\alpha(k)}$ in $f \cdot S_e$. □
The subspace $W_{s,m}$ in (4.13) depends on $s = o(f)$. In our situation, though, note that if $s' = \max\{o(f) : f \in F_m\}$, then (4.13) implies

\begin{equation}
(4.14) \quad f \cdot S_e \equiv 0 \mod W_{s',m} \quad \text{for all } f = f_m(s) \text{ in } F_m.
\end{equation}

Moreover, only finitely many terms from $S_e$ enter into any one of these congruences, so (4.14) is, in effect, a sort of general analog of (1.1).

The $f_m(s)$ from $F_m$ can be understood as approximate solutions of key equation (where the goodness of the approximation is determined by the subspaces $W_{s',m}$, a decreasing chain, tending to $\{0\}$ in $T$, as $m \to \infty$). The BMS algorithm thus systematically constructs better and better approximations to solutions of the key equation. O’Sullivan’s stopping criteria ([OS2]) show when further steps of the algorithm make no changes. Also note that the Feng-Rao theorem shows that any additional syndromes needed for this can be determined by the majority-voting process when $wt(e) \leq \left\lfloor \frac{d_{FR}(C_{\ell}) - 1}{2} \right\rfloor$.

We conclude this section by noting that O’Sullivan has also shown in [OS3] that, for codes from curves, the BMS algorithm can be slightly modified to compute error locators and error evaluators simultaneously in the situation studied in Example (4.7). The same is almost certainly true in our general setting, although we have not worked out all the details. One reason we have not done so is that it is not clear that all of the purely positive parts $(fS_e)_+$ for $f \in I_\mathcal{E}$ are directly useful for determining error values. That seems to be true only for special $f \in I_\mathcal{E}$ (in particular, for the univariate polynomials in the elimination ideals $I_\mathcal{E} \cap \mathbb{F}_q[X_i]$).

In the practical decoding situation, once the BMS algorithm is executed, the next step would be to solve a system of polynomial equations to determine the error locations, i.e. to find the variety $V(I_\mathcal{E})$ using the computed Gröbner basis for $I_\mathcal{E}$. Many of the same techniques useful for that process can efficiently produce the needed univariate polynomials as a byproduct. Hence we will not consider the sort of modification of BMS proposed in [OS3].

§5. Determination of Error Values

In this section, we will see how solutions $f$ of the key equation (4.3), part (1) can be used to determine error values. The method is the same as that presented in [CN]; our proofs are significantly simplified by the use of the formalism from §3.

We will begin with some general results concerning the polynomials $(fS)_+$ for univariate $f \in I_\mathcal{E}$. First we consider a simple special case. Let $\mathcal{E} = \{P_1, \ldots, P_t\} = \text{supp}(e)$ for the error vector $e$. We will say that $\mathcal{E}$ is in general position with respect to $X_i$ if the $X_i$-coordinates of the $P_j$ are distinct.

(5.1) Proposition. Let $e$ be an error vector such that $\mathcal{E}$ is in general position with respect to $X_i$. Let $f$ be the monic generator of the elimination ideal $I_\mathcal{E} \cap \mathbb{F}_q[X_i]$, then $(fS_e)_+ = X_ig$ for some $g \in \mathbb{F}_q[X_i]$. Moreover, if $P$ is any one of the points in $\mathcal{E}$, the error value $e_P$ may be recovered by computing

\[ e_P = \frac{g(P)}{f'(P)}, \]
where $f'$ is the formal derivative.

Note that this is exactly the way error values are usually determined in Reed-Solomon decoding. The formal derivative does not vanish at $P$ because the roots of $f$ are distinct.

**Proof.** We use the formula (4.4) for $S_e$ and (3.3), retaining only terms with $e_P \neq 0$:

$$S_e = \sum_{P \in \mathcal{E}} e_P \left( \sum_{u \in \mathbb{Z}_{\geq 0}} P^u X^{-u} \right).$$

Since $f$ is a univariate polynomial in $X_i$, nonzero terms in the purely positive part $(fS_e)_+$ can only come from terms in $S_e$ where the monomial $X^u$ contains no variable other than $X_i$. (Any other terms in the product are “mixed” and project to zero.) As a result

$$(fS_e)_+ = \left( f \sum_{P \in \mathcal{E}} e_P \left( \sum_{j \geq 0} X_i(P)^j X_i^{-j} \right) \right)_+ = \left( f \sum_{P \in \mathcal{E}} e_P \frac{X_i}{X_i - X_i(P)} \right)_+ = X_i \sum_{P \in \mathcal{E}} e_P \prod_{Q \in \mathcal{E}, Q \neq P} (X_i - X_i(Q)).$$

The polynomial $g$ appears on the right of the final line here, and the other claims now follow from the usual analysis in the univariate case or (3.4). $\square$

The same reasoning shows that in case $\mathcal{E}$ is not in general position with respect to $X_i$ and $f$ is the generator for $I_\mathcal{E} \cap \mathbb{F}_q[X_i]$, then we still have $(fS)_+ = X_i g$ for $g \in \mathbb{F}_q[X_i]$, but now for each root $a$ of $f(X_i) = 0$,

$$\frac{g(a)}{f'(a)} = \sum_{P \in \mathcal{E}, X_i(P) = a} e_P.$$

Even when $\mathcal{E}$ is not in general position with respect to any of the variables, the error values $e_P$ can be recovered from $S_e$ and the univariate polynomials $f_i(X_i)$, $i = 1, \ldots, s$ generating the collection of elimination ideals $I_\mathcal{E} \cap \mathbb{F}_q[X_i]$. We illustrate the idea in a simple example with $s = 2$ before giving the general statement.

(5.2) Example. Let

$$\mathcal{E} = \{P_1, \ldots, P_4\} = \{(0,0), (0,1), (1,1), (\alpha,1)\}$$
in \( \mathbb{A}^2 \) over \( \mathbb{F}_q \), where \( \alpha \neq 0,1 \). Note that \( \mathcal{E} \) is not in general position with respect to either \( X \) or \( Y \). We have univariates \( f_1(X) = X(X-1)(X-\alpha) \) and \( f_2(Y) = Y(Y-1) \). Using (3.3), (4.4), and computations as in Example (4.6), we have

\[
S_e = e_{P_1} + e_{P_2} \frac{Y}{Y-1} + e_{P_3} \frac{XY}{(X-1)(Y-1)} + e_{P_4} \frac{XY}{(X-\alpha)(Y-1)}.
\]

(Note the special form of \( h_P \) when one coordinate is zero.) Hence

\[
(f_1 f_2 S)_+ = XY(e_{P_1}(X-1)(X-\alpha)(Y-1) + e_{P_2}(X-1)(X-\alpha)Y)
\]

\[
e_{P_3}X(X-\alpha)Y + e_{P_4}X(X-1)Y).
\]

Write \( g(X,Y) \) for the factor in the parentheses on the right. Note that if we substitute the points of \( \mathcal{E} \) in to \( g \), only one term is nonzero each time, and this allows us to determine the \( e_{P_i} \):

\[
g(P_1) = g(0,0) = -\alpha e_{P_1},
\]

\[
g(P_2) = g(0,1) = \alpha e_{P_2},
\]

\[
g(P_3) = g(1,1) = (1-\alpha)e_{P_3},
\]

\[
g(P_4) = g(\alpha,1) = \alpha(\alpha-1)e_{P_4},
\]

because the factor multiplying \( e_{P_i} \) is the product

\[
\prod_{\gamma : f_1(\gamma) = 0, \gamma \neq X(P_i)} (X(P_i) - \gamma) \prod_{\delta : f_2(\delta) = 0, \delta \neq Y(P_i)} (Y(P_i) - \delta) \neq 0.
\]

There is another useful expression for (5.4). This product is the same as

\[
f'_1(X(P_i))f'_2(Y(P_i)).
\]

Note also that if we divide the term multiplying \( e_{P_i} \) in \( g(X,Y) \) by (5.4) we get one of the polynomials in a multivariable *Lagrange interpolation basis* for \( \mathbb{F}_q[X,Y]/I_{\mathcal{E}} \), that is a collection of polynomials satisfying \( g_i(P_k) = 0 \) if \( k \neq i \), and \( g_i(P_k) = 1 \) if \( k = i \). The same is true in general as we will now show.

**Proposition 5.5**. Let \( \mathcal{E} = \{P_1, \ldots, P_t\} \) be a finite set in \( \mathbb{A}^s \) over \( \mathbb{F}_q \). Let \( f_i \) be the monic generator of \( I_{\mathcal{E}} \cap \mathbb{F}_q[X_i] \), \( i = 1, \ldots, s \). Then

\[
(f_1 f_2 \cdots f_s h_{P_i})_+ = (\prod_{i=1}^s X_i)g_i,
\]

where the polynomials \( g_i \) satisfy

\[
g_i(P_i) = \prod_{\ell=1}^s f'_\ell(P_i),
\]
and \( g_i(P_k) = 0 \) if \( k \neq i \). As a result, the \( g_i(X)/g_i(P_i) \) form a Lagrange interpolation basis for \( \mathbb{F}_q[X_1, \ldots, X_s]/I_E \).

**Proof.** This follows immediately from part (4) of (3.3) and (3.4). \( \square \)

From (5.5) we have

\[
(f_1 f_2 \cdots f_s S_e)_+ = \left( \prod_{i=1}^t X_i \right) g
\]

where \( g = \sum_{i=1}^t e_{P_i} g_i \). Hence \( g(P_i) = e_{P_i} g_i(P_i) \), so by (5.5),

\[
e_{P_i} = \frac{g(P_i)}{\prod_{i=1}^t f'_i(P_i)}, (5.6)
\]

and this allows us to determine the error values.

We close this section with a comment about the problem of determining the univariate error locator polynomials \( f_i \). This can be done easily given any Gröbner basis \( G \) of \( I_E \) (for instance the output of the BMS algorithm), using the linear algebra techniques in \( \mathbb{F}_q[X_1, \ldots, X_s]/I_E \) described, for instance, in [CLO], Chapter 2, Section 2. Using normal form calculations with respect to \( G \), to determine \( f_i \), we would simply determine the smallest \( k \) for which the normal forms of 1, \( X_i \), \( X_i^2 \), \ldots, \( X_i^k \) give a linearly dependent set in \( \mathbb{F}_q[X_1, \ldots, X_s]/I_E \). The corresponding dependence equation gives the univariate polynomial \( f_i \). Computations of this type would also be used, for instance, to convert the Gröbner basis \( G \) to a lexicographic Gröbner basis via the FGLM algorithm to solve for the error locations by elimination.

§6. Two Examples

In this section we will present two examples illustrating the results of the previous sections.

(6.1) Example. For our first example, we consider Hermitian codes, in particular codes constructed from the order domain \( R = \mathbb{F}_{16}[X, Y]/(X^5 + Y^4 + Y) \), the affine coordinate ring of the Hermitian curve over \( \mathbb{F}_{16} \). In the set-up from §2, we have \( r = 1 \), \( \rho(X) = 4 \), \( \rho(Y) = 5 \), and \( \Gamma = \langle 4, 5 \rangle \subset \mathbb{Z}_{\geq 0} \). Taking the \( >_{(4,5),\text{lex}} \) monomial order the monomials in \( \Delta = \{X^i Y^j : 0 \leq i \leq 4, j \geq 0 \} \) are an \( \mathbb{F}_{16} \)-basis for \( R \). As is well-known, there are 64 affine \( \mathbb{F}_{16} \)-rational points on the Hermitian curve \( X_R \).

By the Feng-Rao bound, the minimum distance of the \( C_{21} = ev(V_{21})^+ \) code is at least 15, so we expect to be able to correct any 7 errors in a received word. In the order defined previously,

\[
V_{21} = \text{Span}\{1, X, Y, X^2, \ldots, Y^5\}.
\]
Hence all syndromes $E_{(i,j)}$ with $i + j \leq 4$ and

$$E_{(4,1)}, E_{(3,2)}, E_{(2,3)}, E_{(1,4)}, E_{(0,5)}$$

are known initially from the received word. In addition, using the equation of the curve, we determine $E_{(5,0)} = E_{(0,4)} + E_{(0,1)}$, and $E_{(6,0)} = E_{(1,4)} + E_{(1,1)}$.

To normalize the field, we take $\mathbb{F}_16 = \mathbb{F}_2[\beta] / (\beta^4 + \beta + 1)$, so $\beta$ is a primitive element. We consider the error of weight 7 for which

$$\mathcal{E} = \{(\beta, \beta^6), (\beta^2, \beta^{14}), (\beta^4, \beta^6), (\beta^5, \beta^{14}), (\beta^8, \beta^3), (\beta^{11}, \beta^{12}), (0, 0)\}$$

and the corresponding error values are

$$\beta, \beta^4, \beta^{12}, 1, 1, \beta, \beta.$$

As is usual for these codes, the known syndromes do not suffice to determine the error locations and values. Running the BMS algorithm with Feng-Rao majority voting, additional syndromes $E_{(4,2)}, E_{(3,3)}, E_{(2,4)}$ are computed, and the curve equation furnishes the values of $E_{(6,1)}$ and $E_{(7,0)}$. The output of the BMS algorithm is the following Gröbner basis for $I_\mathcal{E}$:

$$p_1 = X^2Y + (\beta^3 + \beta^2 + \beta)X + (\beta^2 + \beta)Y + (\beta + 1)XY + (\beta^3 + \beta^2)Y^2 + (\beta^3 + \beta^2 + 1)X^3,$$

$$p_2 = XY^2 + (\beta^2 + 1 + \beta)Y + (\beta^3 + \beta^2 + \beta)Y + (\beta^2 + 1)X^2 + (\beta^3 + \beta + 1)YX + (\beta^2 + \beta)Y^2 + (\beta^3 + \beta^2 + \beta)X^3,$$

$$p_3 = Y^3 + (\beta^3 + \beta + 1)X + (\beta^2 + \beta)Y + \beta^2X^2 + \beta^3XY + (\beta^3 + \beta^2 + 1)X^3,$$

$$p_4 = X^4 + (\beta + 1)XY + (\beta^3 + \beta^2)X^3 + \beta^2X^2 + \beta^3Y^2 + \beta^2Y.$$

The leading terms are written first in each case, so the “footprint” of the ideal $I_\mathcal{E}$ (the set of monomials in the complement of $LT_{>}(I_\mathcal{E})$) is

$$\Delta_\mathcal{E} = \{1, X, Y, X^2, XY, Y^2, X^3\},$$

and consists of the first 7 monomials in $\mathbb{F}_16[X, Y]$ in the $>_{(4,5),\text{lex}}$ order. This is the “generic” case for errors of weight exactly 7 with this ordering.

At this point if we write the polynomials in (6.2) as $f = \sum_{m,n} X^m Y^n$, then all solve a system of equations of the form in (4.4):

$$\sum_{m,n} f_{m,n} E_{(m+r,n+s)} = 0$$

for all

$$(r, s) \in \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), (3, 0)\}.$$
Hence they are solutions of the truncated key equation

\[(6.4) \quad f \cdot S_e \equiv 0 \mod W\]

where \(W = \text{Span}\{X^{-a}Y^{-b} : (a, b) >_{(4,5), \text{lex}} (3, 0)\}\). The polynomials in (6.2) could also be found of course by directly solving the linear equations (6.3). If \(W\) in (6.4) is replaced by any \(W' \subset W\), the set of solutions will be the same.

To determine the error values in a systematic way, we could now exhaustively search for solutions of the system \(p_1 = \cdots = p_4 = 0\), or proceed as follows:

1. Convert the Gröbner basis \(\{p_1, p_2, p_3, p_4\}\) to a lex Gröbner basis using the FGLM algorithm. (If we made \(X\) the smallest variable we would note that \(\mathcal{E}\) is in general position with respect to \(X\) as in §5; with \(Y\) as the smallest variable, we would see that \(\mathcal{E}\) is not in general position with respect to \(Y\).)
2. Solve the corresponding system to find the points in \(\mathcal{E}\).
3. Form any other needed univariate polynomials in \(I_{\mathcal{E}}\) from the solutions.

Then (5.1) with the univariate polynomial in \(X\), or (5.6) will recover the error values.

One of the important things to realize about the results in this article is that even though this first example was constructed using a code from an order domain with \(r = 1\) (a well-studied example of a geometric Goppa code from a curve), the actual process of applying the BMS algorithm and determining the error values would be \textit{exactly the same for any other example of a C\(\ell\) code}. This is the real lesson of [HS] (although the real power of that approach was probably not noticed at the time because order domains of arbitrary transcendence degree had not been used to construct codes as of yet). At the fundamental level, we are always working with the ideal \(I_{\mathcal{E}}\) of a finite set of points in \(\mathbb{A}^s\), and the determination of error locations and values can be performed in a totally uniform fashion.

For example, here is the same sort of computation for a two-dimensional extended cyclic code. (This is the dual of the extended code corresponding to one of Hansen’s toric codes, see [H].)

\textbf{(6.4) Example.} Let \(\mathbb{F}_8 = \mathbb{F}_2[\alpha]/\langle \alpha^3 + \alpha + 1 \rangle\), and consider order domain structure on \(R = \mathbb{F}_8[X,Y]\) induced by the graded lexicographic order with \(X > Y\). We have

\[
V_{10} = \text{Span}\{1, Y, X, Y^2, XY, X^2, Y^3, XY^2, X^2Y, X^3\}
\]

and these give the known syndromes for \(C_{10} = Ev_{10}^1\) (where the evaluation code is formed using all 64 \(\mathbb{F}_8\) rational points in \(\mathbb{A}^2\)). By the Feng-Rao theorem, this code has \(d \geq 5\), so we consider an error vector with \(\mathcal{E} = \{P, Q\} = \{(1,1), (\alpha, \alpha^2)\}\) and \(e_P = 1, e_Q = \alpha^2 + 1\).

In this case the known syndromes are sufficient to determine a Gröbner basis for \(I_{\mathcal{E}}\) by BMS; we are in effect solving the truncated key equation

\[f : \overline{S_e} \equiv 0 \mod W,\]
where $S$ is the known part of the syndrome series, and $W = \text{Span}\{X^{-m}Y^{-n} : m + n \geq 2\}$. The output is
\[
\{x + (\alpha^2 + \alpha)y + \alpha^2 + \alpha + 1, y^2 + (\alpha^2 + \alpha)y + \alpha^2\}
\]
which is the graded lex Gröbner basis for $I_E$. The error values are determined using (5.1) or (5.6).

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