Ray Transforms and Vector Fields

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Abstract

We review and extend a technique for recovering a smooth function from its averages over a wide class of curves in a general region of Euclidean space. The method is based on complexification of the underlying vector fields defining the transport and recasting the problem in terms of complex-analytic function theory. Conditions on the validity of prior formulae appearing in [HB10] as well as stability estimates are then discussed first for the case of vector fields with polynomial coefficients and later for more general cases.

Keywords: X-ray transforms, explicit inversion, complex analysis, transport equation, harmonic calculus

1 Background and Motivation

The following filtered backprojection formula appeared in [Bal05]:

\[ f(z) = \frac{(1 - |z|^2)^2}{4\pi} \int_0^{2\pi} \frac{1}{|1 - ze^{-i\theta}|^4} H_s \frac{\partial}{\partial s} I_f(s(ze^{-i\theta}), e^{i\theta}) d\theta \]  

(1)

where \( f(z) \) is a smooth enough function and \( I_f \) its geodesic ray transform in the Poincaré disc. Formula (1), once obtained, subsequently gave a holomorphic integrating factor to derive a similar, though more complicated, formula for the attenuated radon transform (AtRT) \( I_{\text{AtRT}}f \) on the same space. The method used in that paper, which we call the method of complexification, was an extension of one first used in [Nov02] and rests on the introduction of a complex parameter \( \lambda \) into the governing transport equation and a subsequent analysis of the behavior of solutions in terms of this new parameter.

Recently, in [HB10], we obtained a strikingly similar result to (1), namely

\[ f(z) = \frac{1}{4\pi} \int_0^{2\pi} P(\lambda, \theta) X_\theta^+ H(I_{\theta}f)(s(ze^{-i\theta}), e^{i\theta}) d\theta \]  

(2)

The aim of this paper is to outline, briefly, the method which resulted in the above formula as well as to further classify the breadth of its validity. Although the original impetus for the above was the inversion of the AtRT, formula (2) is interesting in its own right. Much of the material in this article may be found in more detail in [Hoe11].

The structure of this paper is as follows. Section 2 contains the entire cast of characters needed throughout the article. Since our goal is partly expository, section 3 is a concise review of the method of complexification as well as a discussion of what we term “H-ness” and its...
limitations. In section 5 we examine H-ness in more detail first for the case of polynomials and eventually for more general real-analytic vector fields. We generalize the situation in the penultimate section 7 where we present results relating H-ness to reasonable frequency constraints.

2 Generic Preliminaries

Our setup will be as follows. Let $\gamma : \mathbb{R}^2 \ni (t, s) \mapsto \gamma(t, s) \in \Sigma \subset \mathbb{R}^2$ be a real-analytic diffeomorphism on a simply-connected domain $\Sigma$ generating the linear, stationary transport operator

$$X_\mathbf{x} = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}, \quad \mathbf{x} = (x, y) \in \Sigma$$

We regard $\mathbb{R}^2 \cong \mathbb{C}$ via the standard isomorphism so that $\gamma$ is identified with $\gamma^1(t, s) + i\gamma^2(t, s)$. Defining complex $w = \gamma(t, s)$, we see that $(w, \overline{w})$ are now (independent) complex coordinates on $\Sigma$. The regularity of the curves $\gamma(t, s)$ show us that $\gamma_{\gamma}^\ast \partial / \partial z$ is a non-degenerate field on $\Sigma$, $X|_w = \mu(w)\frac{\partial}{\partial w} + \overline{\mu}(w)\frac{\partial}{\partial \overline{w}}$ where $(\phi_* X)(f) = X(\phi_* f)$ defines the pushforward $\phi_*$. The equation of interest is the stationary transport boundary value problem $X|_w u(w) = f(w)$, for $w \in \Sigma$, $f(w) \in C^\infty_0(\Sigma)$ with $\lim_{\gamma^{-\infty}} u(w(t, s)) = 0$, i.e. the BVP

$$\mu(w)\frac{\partial u}{\partial w} + \overline{\mu}(w)\frac{\partial u}{\partial \overline{w}} = f(w), \quad w \in \Sigma$$

$$u|_{\partial \Sigma} = 0$$

One key difference between this formulation of the problem and that considered in [Ba05] is that there is, a priori, no immediately obvious object to “complexify” since we no longer have a parameterization of the points of impact on $\partial \Sigma$ as was the case then. To circumvent the aforementioned difficulty, we appeal to the Riemann mapping theorem ([Neh52, GK06]) which guarantees a unique biholomorphism $z : \Sigma \to D^+$ satisfying $z(\zeta) = 0$, $z'(\zeta) > 0$ for $\zeta \in D^+$, where $D^+$ is the unit disc $\{z \in \mathbb{C}; |z| < 1\}$. Since the Riemann mapping is conformal it is necessarily (infinitesimally) factorable (as in e.g. [Pal04]) with respect to the subvarieties of integral curves of $X_z$. Because of this equivalence between our initial domain $\Sigma$ and the unit disc all further results will henceforth be presented in the disc.

Since $\gamma^\ast z$ maps $\mathbb{R}^2$ into $D^+$ we use $(z, \overline{z})$ as our coordinates on $D^+$ and have a new vector field $X|z = z_* X|z(w)$ where $\mu \mapsto \{z_* \mu\} \frac{\partial}{\partial z} \circ z^{-1}$ and likewise for $\overline{\mu}$. By a forgivable abuse of notation we denote $\{z_* \mu\} \frac{\partial}{\partial z} \circ z^{-1}$ by $\mu(z)$ and $\{z_* \overline{\mu}\} \frac{\partial}{\partial \overline{z}} \circ z^{-1}$ by $\overline{\mu}(z)$ so that

$$X|z = \mu(z)\frac{\partial}{\partial z} + \overline{\mu}(z)\frac{\partial}{\partial \overline{z}}, \quad z \in D^+, \quad |\mu| > 0$$

is our governing differential operator. Defining $t(z) = z_* w_* t$ and $s(z) = z_* w_* s$, smooth functions on $D^+$, the method of characteristics gives the solution to the BVP $X|_z u(z) = f(z)$, $u(z(-\infty, s)) = 0$ as

$$u(z) = (D_1 f)(z) = \frac{1}{2} \int_{\mathbb{R}} f(z(t_0, s)) \text{sign}(t(z) - t_0) dt_0$$

and since $\gamma^\ast z^* = (z \circ \gamma)^\ast$. we define the X-ray transform of a function $f(z)$ over the integral curves of $X|z$, indexed by the transverse parameter $s$, to be

$$(I f)(s) = \int_{\mathbb{R}} f(z(t, s)) dt$$
The main players we need at our disposal are as follows;

**Symmetric Beam Transform**

\[
(D_0 \psi)(z) = \frac{1}{2} \int_{\mathbb{R}} \psi(e^{i\theta} z(t_0, s(z e^{-i\theta}))) \text{sign}(t(z e^{-i\theta}) - t_0) dt_0, \quad \psi \in L^1(D^+)
\]

**Ray Transform**

\[
(I \psi)(s, e^{i\theta}) = (I_0 \psi)(s) = \int_{\mathbb{R}} \psi(e^{i\theta} z(t, s)) dt, \quad \psi \in L^1(D^+)
\]

**Hilbert Transform**

\[
(H \psi)(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{\psi(y)}{x - y} dy, \quad \psi \in L^p(\mathbb{R}), \quad p > 1
\]

as well as the classical Poisson kernel \(P(z, \theta) = \frac{1 - |e^{i\theta} z|^2}{|1 - e^{-i\theta} z|^2}\) which for \(z \in D^+, \theta \in T\) generates the harmonic function on the unit disc. Occasionally we will use the nonstandard notation \(I_0^f\) to indicate the line integral of \(f\) over the integral curves of the vector field \(X\).

### 3 Complexification in a Nutshell

The main result of [HB10] is the following.

**Theorem 3.1** Under suitable conditions on \(X|_z\) and \(s(z)\) there exists a function \(\lambda(z)\) on \(D^+\) such that

\[
f(z) = \frac{1}{4\pi} \int_0^{2\pi} P(\lambda, \theta) X_\theta^+ H(I_\theta f)(s(z e^{-i\theta}), e^{i\theta}) d\theta
\]

provides a reconstruction for the function \(f\) based on the data \(I_\theta f\) of ray transforms of \(f\) over the integral curves of \(X_\theta = \theta_s(X|_z)\).

We now review the method outlined in that paper which was used to obtain this result.

#### 3.1 Symmetrizing and Symmetry-Breaking

Let \(\lambda = \theta \in T = \partial D^+\) and define the conformal map \(\lambda : (z, \bar{z}) \to (\lambda z, \frac{1}{\lambda} \bar{z})\). If, for each \(s\), \(\Phi(\cdot, s)\) is a set of integral curves of \(D^+\), then \(z^{-1}(\lambda^* \Phi(\cdot, s))\) are conformally related curves in \(\Sigma\). For \(\lambda \in D^+/\{0\}\) we then consider \(X^\lambda_\theta = \lambda_s X|_z\) to be the so-called "complexification of \(X|_z\)." Explicitly, \(\lambda_s X|_z\) takes the form \(\mu(\xi, \lambda z) \frac{\partial}{\partial z} + \mu(\xi, \lambda \bar{z}) \frac{\partial}{\partial \bar{z}}\) or \(X^\lambda_\theta = \xi(z, \lambda) \frac{\partial}{\partial z} + \rho(z, \lambda) \frac{\partial}{\partial \bar{z}}\) with \(\frac{1}{\lambda} \xi(z, \lambda) = \mu(z, \lambda) \equiv \lambda_s \mu(z)\) and \(\lambda \rho(z, \lambda) = \lambda \bar{\mu}(z, \lambda) = \lambda_s \bar{\mu}(z)\).

Define \(X^\lambda_\theta = \pm i(\xi(z, \lambda) \frac{\partial}{\partial z} + \rho(z, \lambda) \frac{\partial}{\partial \bar{z}})\) as a vector field orthogonal to \(X^\lambda_\theta\) when \(\lambda = e^{i\theta}\).

N.B. \(e^{i\theta}\) and \(\theta\) will be used interchangeably, their meaning clear from context.

For functions \(k(z, \lambda), \frac{\partial}{\partial z}\) and \(k_{\bar{z}}\) are equivalent, as are \(\frac{\partial}{\partial z}\) and \(k_{\bar{z}}\), and we will use them interchangeably.

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3
3.2 Analysis, Asymptotics, and H-ness

Our complexified transport equation now reads as follows:

\[ X_\lambda u(z, \lambda) = f(z), \quad \lambda \in D^+ \]  

(9)

where it should be stressed that the parameter \( \lambda \) has no obvious relation to the original particle transport that started this rigmarole. The method used to obtain Theorem 3.1 involves solving equation (9) for \( u(z, \lambda) \) and showing analytic dependence of the solution on this parameter, i.e. \( \partial \lambda u(z, \lambda) = 0 \).

A restricted class of vector fields known as type H was identified which ensure the ensuing steps work out as we need. The following is a revised version of that definition better suited to the purposes of this paper:\[ 3 \]

**Definition** A real vector field \( X|_z \), complexified in the manner above

\[ \lambda_s X|_z = a(z, \lambda) \frac{\partial}{\partial z} + b(z, \lambda) \frac{\partial}{\partial \bar{z}}, \quad \lambda \in D^+/\{0\} \]

is said to be of type H if the following holds:

1. \( a(z, \lambda) \) is a holomorphic function of \( \lambda \) for \( \lambda \in D^+ \) and has at least one zero \( \lambda = \lambda_i(z) \in D^+ \)

2. \( b(z, \lambda) \) is a meromorphic function of \( \lambda \) for \( \lambda \in D^+ \) and has no zeroes in \( D^+/\{0\} \)

3. \( \frac{a(z, \lambda)}{b(z, \lambda)} \) is a holomorphic function of \( \lambda \in D^+ \)

4. \( s(z, \lambda), \frac{\partial s(z, \lambda)}{\partial z}, \frac{\partial s(z, \lambda)}{\partial \bar{z}} \) are meromorphic functions of \( \lambda \) for \( \lambda \in D^+ \)

where, as in the above, \( s(z, \lambda) = \lambda_s s(z) \) is the complexified parameter specifying transverse foliation of the integral curves of \( X_\lambda \).

This condition H is the “suitable condition” mentioned in 3.1 and we will assume our vector field is of this type (i.e. \( a = \mu \) and \( b = \rho \)). The \( \lambda_i(z) \) mentioned previously are the zeroes of the complexified \( \frac{\partial}{\partial z} \) coefficient of our initial field. Note that condition H is strong insofar as holomorphy itself is a rather stringent condition. The above criteria will heretofore be called “H-ness”.

3.3 A Proof Sketch

We give a scandalously brief sketch of the proof leading to (8), highlighting where H-ness comes into play. First of all, by the third condition in 3.2 of nondegeneracy we see that the Jacobian

\[ \partial s(z) = |s_z(z, \lambda)|^2 - |s_{\bar{z}}(z, \lambda)|^2 \neq 0 \]

holds on \( \lambda \in D^+/\{0\} \) since the inequality

\[ 0 \neq \left| \frac{\partial s(z, \lambda)}{\partial z} \frac{\partial t(z, \lambda)}{\partial \bar{z}} - \frac{\partial s(z, \lambda)}{\partial \bar{z}} \frac{\partial t(z, \lambda)}{\partial z} \right| \leq |s_z(z, \lambda)|(|t_z(z, \lambda)| + |\frac{\xi(z, \lambda)}{\rho(z, \lambda)}| |t_z(z, \lambda)|) \]

guarantees that \( |s_z(z, \lambda)|^2 \neq 0 \) on that same region.

\[ ^3 \text{Notice that the maximum principle causes the third condition to follow automatically if, in the second condition, } D^+/\{0\} \text{ is replaced by } D^+ \text{ which was the situation examined in [HB10]. Since the first and last of the above are in agreement with those already considered, we need only check that the middle conditions square with what we need.} \]
We may therefore make a change of variables in \( s \) to get 
\[ s^* X_\lambda = s^* X_\lambda \delta(z - z_0) \]
whereby our fundamental equation 
\[ X_\lambda G_\lambda(z; z_0) = \delta(z - z_0) \]
is solved explicitly by 
\[ G_\lambda(z, z_0) = -\gamma \frac{\partial (t, s)}{\partial (z, \bar{z})} \bigg|_{z_0} X_\perp s(z_0) \]
(10)

Checking against a bump function extends this to hold weakly at \( \lambda \to 0 \) and density shows 
that \( u(z, \lambda) \) is holomorphic in \( \lambda \) as needed. A similar argument works on 
\( u_\pm(z, \lambda) \) and \( u_{\bar{z}}(z, \lambda) \) by invoking the final condition of H-ness. From here, Hilbert’s relations on the boundary values 
of complex-analytic functions become viable as the following result shows.

**Proposition 3.2**

\[ u_\pm(z, e^{i\theta}) \overset{\text{lim}}{\to} D_{\pm} \lambda \to e^{i\theta} u(z, \lambda) = \mp \frac{1}{2i} (HI_\theta f)(s(e^{-i\theta} z), \theta) + (D_\theta f)(z) \]

where the Hilbert transform \( H \) is taken with respect to the first variable.

The proof of this comes from an explicit analysis of (10) with \( \lambda = 1 - \epsilon \) and deriving the relation 
\[ X_1 i s'(z, 1) = \frac{1}{2} \left( \frac{\partial}{\partial \rho} \right)_{\lambda=1} X_\perp s(z, 1) \]
(11)

It can be shown that by invoking the third condition of [3.2] we have 
\[ \text{sign}(is'(z, 1) - is'(z_0, 1)) = \text{sign}(t(z, 1) - t(z_0, 1)) \]

from which the Sokhotskyi-Plemelj formula allows us to obtain the advertised proposition.

### 3.4 Reconstruction

We use Proposition 3.2 together with the classical representation of complex-analytic functions 
on the unit disc. By definition of \( \lambda_i \) and \( X_\perp \), one has 
\( iX_\lambda u(z, \lambda_i) = X_\perp \rho(z, 1) \) so that on equating real and imaginary parts we have 
\[ \frac{1}{2\pi} \int_{0}^{2\pi} P(\lambda_i, \theta) X_\perp (D_\theta f)(z) d\theta = -\frac{1}{4\pi} \int_{0}^{2\pi} P(\lambda_i, \theta) X_\rho H(I_\theta f)(s(ze^{-i\theta}), e^{i\theta}) d\theta \]
and 
\[ f(z) = \frac{1}{4\pi} \int_{0}^{2\pi} P(\lambda_i, \theta) X_\perp H(I_\theta f)(s(ze^{-i\theta}), e^{i\theta}) d\theta \]
(12)

which is the result we sought. Notice that \( H \) always denotes the Hilbert transform with respect 
to the \( s \) variable.

### 4 Statement of Results

The goal of this paper is to establish, over the next two sections, the following result (viz. 
Theorem 7.4 and Corollary 6.4 respectively).
Theorem 4.1 Let $\epsilon > 0$ be given. Suppose that $\mu(z, \bar{z}) = \sum_{p+q \geq 0} a_{pq} z^p \bar{z}^q$ is a real-analytic function on $D^+$ and that $f \in C^\infty_c(D^+)$. Define $c_j(z, \bar{z}) = \sum_{q-p=r} a_{pq} z^p \bar{z}^q$. Let $l(z)$ and $k(z)$ be the max and min respectively of the $j$ such that $c_j(z) \neq 0$. Suppose that $\lambda s, \lambda s_\bar{z}$ and $\lambda s_\bar{z}$ are meromorphic for $\lambda \in D^+$. 

- If there are only finitely many nonzero $c_j(z)$, and both $l(z) + k(z) + 2 \geq 0$ and $0 < |c_k(z)| < |c_l(z)|$ holds for all nonzero $z \in D^+$, then there exists a vector field $Y_{\bar{\theta}}$ such that we have a perfect reconstruction

$$f(z) = \frac{1}{4\pi} \int_0^{2\pi} P(\lambda_i, \theta) Y_{\bar{\theta}} H(I_{\theta} f)(s(ze^{-i\theta}), e^{i\theta})d\theta$$

- If, for all $z \in D^+/\{0\}$, there exist infinitely many $j \in \mathbb{Z}$ such that $c_j(z)$ and $c_{j+1}(z)$ are both nonzero, and if $\limsup_{j \to \infty} |c_{j+1}(z)/c_j(z)| < 1$ then there exists a vector field $Y_{\bar{\theta}}$ such that

$$||I_Y f - I_X f||_{L^q(D^+)} < C\epsilon \quad 1 \leq q \leq \infty$$

If, in addition, we have the Fréchet bound $||H_\delta I_{\theta} f - H I_{\theta} f||_S < \delta(\epsilon)$ then there exists a function $\lambda_i : z \to \lambda_i(z)$ satisfying

$$\sup_{z \in D^+} \left| f(z) - \frac{1}{4\pi} \int_0^{2\pi} P(\lambda_i, \theta) Y_{\bar{\theta}} H(I_{\theta} f)(s(ze^{-i\theta}), e^{i\theta})d\theta \right| \leq C(\delta(\epsilon)) \quad (13)$$

where $H I_{\theta} f$ is the Hilbert transform in the $s$ variable of the trace of $f$ over the integral curves of $\theta(s)(\mu \partial + \bar{\mu} \bar{\partial})$

Clearly (20) generalizes (8) in the sense that it allows an approximate reconstruction, to arbitrary accuracy, for a large class of vector fields. We will, in the sequel, prove Theorem 4.1 and help explain just how broad its applicability is. The case of polynomial fields is as good as one could hope for. Our methodology is to establish results first for the case of polynomial vector field coefficients and later to reinterpret the terms $c_j(z)$ as the frequencies of the complexified vector field’s coefficients in the case of non-polynomial fields.

5 Polynomial Vector Fields

5.1 The polynomial space $\Gamma(\Omega)$

Consider the case in which $\mu(z)$ is a nonvanishing polynomial, i.e. $\mu(z, \bar{z}) = \sum_{p+q \geq 0} a_{pq} z^p \bar{z}^q$ for $z \in \Omega \supset \{0\}$. The complexified coefficients of the field $X_\lambda$ around $\lambda = 0$ are then

$$\xi(z, \lambda) = \sum_{p+q \geq 0} b_{pq}(z) \lambda^{q-p+1}, \quad \rho(z, \lambda) = \sum_{p+q \geq 0} d_{pq}(z) \lambda^{q-p-1}$$

with $b_{pq} = a_{pq} z^p \bar{z}^q$ and $d_{pq} = \bar{b}_{pq}$. In order for H-ness to hold\(^4\) we will need that $\partial_\lambda \xi(z, \lambda) = 0$, which a priori we do not have since $q-p+1$ may very well be negative. If $q-p+1 \geq 0$ for all $(p, q)$-pairs then we are (provided we have roots and the rest of condition H) in the position

\(^4\)For the moment we will be ignoring any possible problems with $s(z, \lambda)$
of the previous section. If not, i.e. if $p > q + 1$ holds for some $(p, q)$-pair, then we proceed as follows.

First of all, we will mostly be using the local irreducible exponents $k, l$ given by

$$k(z) = \min_{c_j(z) \neq 0} (j) \quad \text{and} \quad l(z) = \max_{c_j(z) \neq 0} (j) \quad (14)$$

where

$$c_r(z, \bar{z}) = \sum_{p,q \text{ such that } q-p=r} a_{pq} z^p \bar{z}^q$$

so that the Laurent expansion of $\mu(z, \lambda)$ around $\lambda = 0$ is given by $\sum_{r=k}^l c_r(z, \bar{z}) \lambda^r$. Occasionally we will need the global exponents defined as

$$k_{\mu} = \min_{z \in \Omega} k(z) \quad \text{and} \quad l_{\mu} = \max_{z \in \Omega} l(z) \quad (15)$$

Obviously $-N \leq k(\mu)$ and similarly $0 \leq l(\mu) \leq N$, and our previous assumption is equivalent to the condition $k(\mu) + 1 < 0$. Notice $k \geq k_{\mu}$ and that $l \leq l_{\mu}$ depending on $z \in \Omega$. To be clear, if there is no $\mu$ we are referring to the local irreducible exponents. Since $|\mu| > 0$ we can be certain that $k, l$ always exist (even if they may be equal). Also, $k(0) = l(0) = 0$ with $c_0(0) = a_{00}$.

We define the following polynomial space;

$$\Gamma(\Omega) \doteq \{ \mu = \sum_{p+q} a_{pq} z^p \bar{z}^q; \ |\mu| > 0 \text{ and } |k(z)| \leq l + 2 \}$$

for reasons which will be made clearer in the sequel.

5.2 The rescaling scalar

Consider the function $w(z) \doteq 2 - z^{|k_{\mu}|^{-1}} - \bar{z}^{|k_{\mu}|^{-1}}$, which has two important properties:

1. $w(z) \in \mathbb{R}$ for $z \in \mathbb{C}$
2. $0 < |w| < 2 \ll \infty$ for $z \in D^+$

The first of the above guarantees that the field $Y|_z \doteq \frac{1}{w(z)} X|_z = a(z) \frac{\partial}{\partial z} + b(z) \frac{\partial}{\partial \bar{z}}$ has the same integral curves as $X|_z$. The second fact ensures that this rescaling introduces no artificial degeneracies into the field, in the sense that $|a| = |\frac{\partial \mu}{\partial w}| > 0$. This amounts to a change in variables generated via

$$\tilde{t}(t, s) = \int_0^t \frac{dp}{2 - z^{|k_{\mu}|^{-1}}(p, s) - \bar{z}^{|k_{\mu}|^{-1}}(p, s)}$$

If $l(\mu) < 0$ then we should use the complementary complexification $\lambda : (z, \bar{z}) \rightarrow (\frac{1}{\lambda}z, \lambda \bar{z})$, for $\lambda \in D^+$ and get a holomorphic $\frac{1}{\lambda} \lambda_{*}\mu$. Since this is a situation which was dealt with in the previous section we may assume that $l(\mu) \geq 0$. 

7
5.3 The first three conditions of H-ness

Our first result towards establishing H-ness in the case of vector fields with polynomial coefficients is the following simple lemma.

**Lemma 5.1** \( a(z, \lambda) = \lambda a(z) \) is holomorphic for \( \lambda \in D^+ \)

**Proof** We have \( a(z, \lambda) = \frac{\sum_{k=0}^{l} c_k(z) \lambda^{k-\mu} \lambda^{\mu} - z^{(k+1) - \mu}}{2 \lambda^{k+1} - (k+1) \lambda^{1+\mu}} \) where the numerator contains only positive powers of \( \lambda \). Notice \( a(0, \lambda) = \frac{a_0 \lambda^{k(0) + 1}}{2} \sim \lambda \). The quadratic formula shows that away from the origin \( z = 0 \) the solution to \( w(z, \lambda) = 0 \) is given by \( |\Lambda(z)| = \frac{1 \pm \sqrt{1 - |z|^{2(k+1) - 1}}}{z^{k+1}} \). By the triangle inequality with \( z \neq 0 \) we see that

\[
1 \leq \left| \frac{1}{z^2} - \frac{1}{z \sqrt{1 - |z|^2}} \right|^2 \leq \left| \frac{1 \pm \sqrt{1 - |z|^2}}{z} \right|^2
\]

with equality holding only when \( |z| = 1 \), and therefore \( \Lambda \notin D^+ \) for \( |z| < 1 \), and ipso facto \( w(z, \lambda) \neq 0 \) for \( \Lambda \in D^+ \).

We can now obtain a positive answer on the first criterion of H-ness.

**Proposition 5.2** If \( \mu(z, \bar{z}) \in \Gamma(D^+ \) and if

\[
\log |c_k(z)| < \frac{1}{2\pi} \int_0^{2\pi} \log \left| \sum_{j=k}^{l} c_r(z) e^{i\theta(j-k)} \right| d\theta
\]

for \( z \in D^+ / \{0\} \) then the coefficient \( a(z, \lambda) = \frac{\sum_{k=0}^{l} c_k(z) \lambda^{k-\mu} \lambda^{\mu} - z^{(k+1) - \mu}}{2 \lambda^{k+1} - (k+1) \lambda^{1+\mu}} \) has a root \( \lambda_i(z) \in D^+ \) and the first condition of H-ness is met.

**Proof** Recall Jensen’s formula for a meromorphic function \( h(z) \) with roots \( \alpha_{\nu} \) and \( \beta_{\tau} \) in a region \( R = \{ z, |z| < R \} \),

\[
\log |h(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |h(Re^{i\theta})| d\theta + \sum_{\nu} \log \frac{\alpha_{\nu}}{R} - \sum_{\tau} \log \frac{\beta_{\tau}}{R}
\]

provided \( |h(0)| \neq 0, \infty \) (see e.g. [W. 64, Nev70]). Since the polynomial \( P_{l-k}(\lambda) = c_k(z) \lambda^{k-\mu} + c_{k+1}(z) \lambda^{k-\mu+1} + \cdots + c_l(z) \lambda^{l-\mu} \) has no poles and since \( \lambda = 0 \) is not a root when \( k = k_\mu \), we may apply Jensen’s formula in that case to \( P_{l-k}(\lambda) \) and \( R = 1 \) to yield

\[
\log |c_k(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \sum_{j=k}^{l} c_r(z) e^{i\theta(j-k)} \right| d\theta + \sum_{i} \log |\lambda_i(z)|
\]

where \( P_{l-k}(\lambda_i(z)) = 0 \) and the result is immediate. If \( k > k_\mu, \lambda = 0 \) is a root of local order \( k_\mu(z) - k_\mu \) and there’s nothing to prove. At \( z = 0 \) there is likewise nothing to prove.

The next theorem uses similar arguments to address the second condition of H-ness.

**Proposition 5.3** If \( \mu(z, \bar{z}) \in \Gamma(D^+ \) and if

\[
\frac{1}{2\pi} \int_0^{2\pi} \log \left| \sum_{j=l}^{k} \tilde{c}_j(z) e^{i\theta(k_\mu - j - 2)} \right| d\theta \leq \log |\tilde{c}_j(z)|
\]

then \( b(z, \lambda) = \frac{\tilde{c}_j(z) \lambda^{k_\mu - l - 2} \cdots \tilde{c}_l(z) \lambda^{k_\mu - k - 2}}{2 \lambda^{k_\mu - l - 2} \cdots \lambda^{k_\mu - k - 2}} \) is nonvanishing for \( (z, \lambda) \in D^+ \times D^+ / \{0\} \).
Proof Since $\bar{\mu}$ was given as a polynomial we are guaranteed meromorphy of the term $b(z, \lambda)$.
Looking at $b(z, \lambda) = \bar{c}_l(z)\lambda^{|k_\mu|-l-2} + \cdots + \bar{c}_k(z)\lambda^{|k_\mu|-k-2}$ we see that $b(z, \lambda = 0)$ is nonzero for $z \neq 0$ and $|k_\mu| - l = 2$ since $\bar{c}_l \neq 0$. If $z \neq 0$ and $|k_\mu| - l < 2$ then of course $\lim_{|\lambda| \to 0} b(z, \lambda) = \infty$ with local order $|k_\mu| - \mu(z) = 2$. If $z = 0$ then $b(z, \lambda) \sim \frac{1}{\lambda}$ near $|\lambda| = 0$. The lack of vanishing of the denominator together with the way $\bar{\mu}$ was complexified ensure that $b(z, \lambda)$ has no other singularities within $D^+ \times D^+$.

In the Jensen formula \((16)\), if $h(z)$ had a zero of order $m$ at $z = 0$ then $h = h_0 z^m + \cdots$ in a vicinity of the origin. In that case the function $\Upsilon(z) \equiv \frac{R^m h(z)}{z^m}$ has the same modulus on $\partial R$ but is nonvanishing at the origin, its value there being $R^m h_0$. The Jensen formula applied to $\Upsilon(z)$ would yield
\[
\log |h_0| = \frac{1}{2\pi} \int_0^{2\pi} \log |h(Re^{i\theta})| \, d\theta + \sum_{\nu} \log \frac{|\alpha_{\nu}|}{R} - \sum_{\tau} \log \frac{|\beta_{\tau}|}{R} - m \log R
\]
Assuming that there exists at least one $\Lambda_j(z) \in D^+/\{0\}$ such that $b(z, \Lambda_j) \equiv 0$, we use \((17)\) with $h(\lambda) = c_l(z)\lambda^{|k_\mu|-l-2} + \cdots + c_k(z)\lambda^{|k_\mu|-k-2}$ to get
\[
\log |c_l(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log |c_l(z) e^{i\theta(|k_\mu|-l-2)} + \cdots + c_k(z) e^{i\theta(|k_\mu|-k-2)}| \, d\theta + \sum_j \log |\Lambda_j(z)|
\]
Whence
\[
\log |c_l(z)| < \frac{1}{2\pi} \int_0^{2\pi} \log \left| \sum_{j=l}^k c_j(z) e^{i\theta(|k_\mu|-j-2)} \right| \, d\theta
\]  \((17)\)
The result follows from the above inequality by contradiction.

Remark Since $a(\lambda, \lambda) = \frac{\sum c_{\mu} \lambda^\mu}{\sum c_{\mu} \lambda^{-|\mu|-2}}$, the origin $\lambda = 0$ is the only spot where analyticity may fail. But since $\mu \in \Gamma(\Omega)$ we see that $l + k + 2 \geq 0$ and $|c_{\mu} c_{\bar{\mu}}| < 1$ keep $\frac{\lambda}{\bar{\lambda}}$ bounded as $\lambda \to 0$ so that by Riemann’s theorem $\frac{\bar{\lambda}}{\lambda}$ is analytic on all $D^+$ as required in condition H.

The preceding results combine in the following important corollary.

Corollary 5.4 If $\mu(z, \bar{z}) \in \Gamma(D^+)$ and if for $z \neq 0$ we have
\[
\log |c_k(z)| < \log |c_l(z)|
\]
then $\frac{1}{w(z)} X|_z = a(z, \lambda) \frac{\partial}{\partial z} + b(z, \lambda) \frac{\partial}{\partial \bar{z}}$ meets the first three conditions of H-ness.

Since polynomials are the building blocks of real-analytic functions, we extend these results in the subsequent sections.

6 H-ness in the space $\mathcal{H}_{k,l}(\Omega)$
We begin with a definition extending our previous notion of $\Gamma(\Omega)$.

Definition A real-analytic function $\mu(z) = \sum_{j \geq 0} a_{pq} z^p \bar{z}^q$ on $\Omega \supset \{0\}$ with $a_{00} \neq 0$ belongs to the space $\mathcal{G}_{k_\mu, l_\mu}(\Omega)$ when
1. \( \sum_{p,q} |a_{pq}| < \infty \)

2. Both \( k(\mu) \) and \( l(\mu) \), defined as in (15) are finite with \( k(\mu) < -1 \) and \( l(\mu) \geq 0 \)

3. \(-k(z) \leq l(z) + 2 \) for all \( z \in \Omega \)

4. \( 0 < |c_k(z)| < |c_l(z)| \) holds for all \( z \in \Omega/\{0\} \) where \( c_r(z, \bar{z}) = \sum_{q-p=r} a_{pq}z^p\bar{z}^q \) with \( k, l \) the local irreducible exponents of (14)

We drop subscripts on \( G_{k,\mu}(\Omega) \) since the notation \( G_{k,\mu}(\Omega) \) is more concise and the global meaning is obvious. Clearly \( |c_k(0)| = |c_l(0)| = |a_{00}| > 0 \). The condition on absolute summability ensures that \( c_r(z) \) is well-defined. The conditions guarantee we are left with a complexified \( \lambda \mu(z_{1/\lambda}, \bar{z} \lambda) \) which has a finite Laurent series in \( \lambda \). We have thereby established our main result with regard to polynomials.

**Theorem 6.1** Let \( \mu \in G_{k,\mu}(D^+ \lambda) \) and let \( w(z) = 2 - z^{k_\mu} - z^{l_\mu} \). Then the vector field

\[
X_\lambda = \lambda \left( \frac{\mu(z)}{w(z)} \frac{\partial}{\partial z} + \frac{\bar{\mu}(z)}{w(z)} \frac{\partial}{\partial \bar{z}} \right)
\]

satisfies the first three conditions of H-ness.

With that in mind, we make the following

**Definition** Denoting the meromorphic functions in \( \lambda \in \Omega \) as \( \mathcal{M}(\Omega) \) we define

\[
\mathcal{H}_{k,\mu}(\Omega) = \{ \mu \in G_{k,\mu}(\Omega); s(z, \lambda), \partial s(z, \lambda), \bar{\partial} s(z, \lambda) \in \mathcal{M}(\Omega) \}
\]

### 6.1 The Fourth Condition

We now address the fourth and final condition of H-ness, namely meromorphy of \( s(z, \lambda) \) and its \( z \) and \( \bar{z} \) derivatives. To start with, this condition is already more relaxed than the initial three since meromorphy itself is less restrictive than holomorphy and there is no constraint on existence (or lack thereof) of roots. Secondly, for the space \( HL_p(G, \Omega) \) defined as all \( f \) satisfying both

1. \( \frac{f(z)-f(z_0)}{z-z_0} \in L_p(\Omega), \forall z_0 \in G \)

2. \( ||f||_{HL_p(G, \Omega)} = ||f||_{L_p(\Omega)} + \sup \{ ||f(z)-f(z_0)||_{L_p(\Omega)}; z_0 \in G \} < \infty \)

we have the result ([Ren89], Thm. 3.2)

**Theorem 6.2** For domain \( \Omega, z_0 \in \Omega \) and \( \mu \in HL_p(G, \Omega) \) for \( p > 2, |\mu| < 1 \) and \( \forall z \in \Omega \) we have that if \( u(z) \) solves \( \partial u = -\mu(z) \bar{\partial} u \) on \( \Omega \) and \( u(z) \) has a zero/pole of order \( m \) at the point \( z_0 \) then

\[
u(z) = c\{(z-z_0) + b(z-z_0)\}^m + \bar{c}n\{(z-z_0) + \bar{b}(z-z_0)\}^m + O(|z-z_0|^{m+\alpha})
\]

for some \( \alpha > 0 \) and the \( \pm \) picked according to whether \( z_0 \) is a zero or pole respectively.
Theorem 6.3 \ If \( \mu = \sum a_{pq}z^p\bar{z}^q \in \mathcal{H}_{k,l}(D^+) \), then the field

\[
X_\lambda \doteq \lambda_\ast \left( \frac{\mu(z)}{w(z)} \frac{\partial}{\partial z} + \frac{\bar{\mu}(z)}{w(z)} \frac{\partial}{\partial \bar{z}} \right)
\]

with \( w(z) \doteq 2 - z^{k_\mu} - \bar{z}^{k_\mu} \) satisfies condition H.

The above result allows us to reconstruct functions over what are initially non-type H fields as in the following easy corollary.

Corollary 6.4 \ If \( \mu \in \mathcal{H}_{k,l}(D^+) \), \( \lambda_\ast |_{\lambda_\ast(z)} = 0 \) and \( X^\perp_\lambda = i\theta_\ast(-\frac{\mu(z)}{w(z)} \frac{\partial}{\partial z} + \frac{\bar{\mu}(z)}{w(z)} \frac{\partial}{\partial \bar{z}}) \) and \( f \in C^\infty_0(D^+) \) then

\[
f(z) = \frac{w(z)}{4\pi} \int_0^{2\pi} P(\lambda_\ast, \theta) X^\perp_\lambda H(\bar{I}_\theta f)(s(ze^{-i\theta}), e^{i\theta})d\theta
\]

where \( \bar{I}_\theta f \) is the ray transform of \( f \) over the integral curves of \( Y_\theta = \theta_\ast(\mu\partial + \bar{\mu}\bar{\partial}) \).

Proof \ Consider the equation \( X|_z u(z) = g(z) \) for \( g \doteq \frac{f(z)}{w(z)} \in C^\infty_0(D^+) \). Then by \( \{ 0, 3\} \) \( \lambda_\ast X|_z \) is type H and has zeros \( \lambda_\ast(z) \in D^+ \). Thus, by \( \{ 3\} \)

\[
g(z) = \frac{1}{4\pi} \int_0^{2\pi} P(\lambda_\ast, \theta) X^\perp_\lambda H(\bar{I}_\theta g)(s(ze^{-i\theta}), e^{i\theta})d\theta
\]

where \( \bar{I}_\theta g \) is the trace of \( g \) over the integral curves of \( X_\theta \). However \( f \) was arbitrary in \( C^\infty_0(D^+) \) and since \( e^{i\theta}w \in \mathbb{R} \) is both finite and nonvanishing on \( D^+ \) the integral curves of \( e^{i\theta}(\mu\partial + \bar{\mu}\bar{\partial}) \) and of \( X_\theta \) are the same. In particular, under a change of variables, \( I_\theta g = \bar{I}_\theta f \). The result follows since \( s \) was unchanged.

7 Some Harmonic Analysis: Onward and Upward

7.1 The Projection Operator

The Fourier expansion of a smooth function \( a(z, e^{i\theta}) \) on the unit disc given is by

\[
a(z, e^{i\theta}) = \sum_{n \in \mathbb{Z}} \hat{a}_n(z)e^{in\theta} \quad \text{with} \quad \hat{a}_n(z) \doteq \frac{1}{2\pi} \int a(z, \theta)e^{-in\theta}d\theta \quad (18)
\]

Let \( f \mapsto \bar{f} \) be the conjugation operator, determining the harmonic conjugate of a smooth function. Defining the Bergman space \( H^2 \) as all complex-analytic and Lebesgue square-integrable
functions on the unit disc, then the orthogonal projection from $L^2(D^+)$ to $H^2$ is defined (e.g. [Gar09]) by the operator $P$ via

$$P : f \mapsto \frac{1}{2}(f + i\hat{f}) + \frac{1}{2}\hat{a}_0$$

or explicitly

$$P(\sum_{n\in\mathbb{Z}} \hat{a}_n(z)e^{in\theta}) = \sum_{n\in\mathbb{Z}} \hat{a}_n(z)e^{in\theta}$$

The operator $P : L^2 \to H^2$ then can easily been seen as removing negative frequencies from the initial signal.

7.2 Scaling Redux

As usual we let $\mu(z, \bar{z})$ be real-analytic, absolute-summable and nonvanishing. Then $\theta_* \sum_{p+q=n} a_{pq}z^p\bar{z}^q$ takes the form

$$e^{i\theta} \mu = \sum_{n\in\mathbb{Z}} c_n(z)e^{in\theta}, \quad \text{with} \quad c_r(z, \bar{z}) = \sum_{p,q} a_{pq}z^p\bar{z}^q$$

(19)

We conveniently now view the $c_n$’s as Fourier coefficients of the function $\mu(z, \theta)$ i.e. $c_j(z) = \hat{\mu}(z, j)$. Define the operator $P_{k,l}$ on smooth functions via

$$P_{k,l} \mu = e^{ik\theta} P(e^{-ik\theta} \mu) - e^{i(l+1)\theta} P(e^{-i(l+1)\theta} \mu)$$

$$= \sum_{n=k} l c_n(z)e^{in\theta}$$

Let $\mathcal{R}(\Omega)$ be the space of real-analytic functions of two variables on a region $\Omega$. Then define the following space:

$$\hat{\mathcal{C}}(\Omega) = \left\{ g(z) \in \mathcal{R}(\Omega); \forall z \in \Omega / \{0\} \text{ there are infinitely many } n \text{ such that } \hat{g}(z, n), \hat{g}(z, n+1) \neq 0 \text{ and } \limsup_{n \to \infty} \left| \frac{\hat{g}(z, n+1)}{\hat{g}(z, n)} \right| < 1 \right\}$$

Clearly $\hat{\mathcal{C}}(\Omega)$ is “most” of $\mathcal{R}(\Omega)$ since it accounts for, in some sense, those real-analytic functions with “non-sparse” spectrums. The classical ratio test for infinite series ensures that $\mu \in \hat{\mathcal{C}}$ are also absolute-summable. We have the following result about convergence on compact subsets.

**Proposition 7.1** Let $K \subset \Omega$ be compact and let $\mathcal{G} = \bigcup_{(k,l)\in\mathbb{Z}^2} \mathcal{G}_{k,l}$. Then $\mathcal{G}(K)$ is dense in $\hat{\mathcal{C}}(K)$ with respect to the uniform norm.

**Proof** Let $\omega(z) \in \hat{\mathcal{C}}(K)$ and let $0 \neq z_j \in \text{supp } \omega \subset K$. Then, $e^{i\theta} \omega(z) = \sum_{n} \omega_n(z)e^{in\theta}$ with $\omega_n(z_j) \neq 0$ for infinitely many $n \in \mathbb{Z}$. We may pick an $l(z_j)$ such that $\hat{\omega}(z_j, l(z_j)) \neq 0$. By the assumptions of $\hat{\mathcal{C}}$, there exists a finite $k(z_j) < -1$ such that $k(z_j) + l(z_j) + 2 \geq 0$ and $0 < |\hat{\omega}(z_j, k(z_j))| < |\hat{\omega}(z_j, l(z_j))|$, namely $k(z_j) = -(l(z_j) + 1)$.

\*\*In the reasonable, informal way rather than a measure-theoretic sense
The varieties \( \{ z; \tilde{\omega}_r(z, \tilde{z}) = 0 \} \) define (possibly degenerate) circles. Therefore, there is an \( \epsilon \)-neighborhood \( N_{\epsilon_j}(z_j) = N_j \) around \( z_j \) on which there are two simple functions, \( -\infty < k_j(z) \leq k(z_j) \) and \( l(z) \leq l_j(z) < \infty \) on \( z \in N_j \), for which

\[
0 < |\tilde{\omega}_j(z, k_j(z))| < |\tilde{\omega}_j(z, l_j(z))| \quad \forall z \in N_j / \{ \}
\]

and

\[
l_j(z) + k_j(z) + 2 \geq 0 \quad \forall z \in N_j
\]

Define

\[
\tilde{\omega}_j(z) \doteq \chi(N_j)P_{k(j),l(j)}e^{i\theta} \omega
\]

where \( k(j) = \min_{z \in N_j} k_j(z) \) and \( l(j) = \max_{z \in N_j} l_j(z) \). Then \( K \subset \bigcup N_j \) provides an open cover reducible to

\[
supp \omega \subset \bigcup_{j=1}^p N_j
\]

Consider the following function

\[
\tilde{\Omega}_p(z) \doteq \sum_{j=1}^p \tilde{\omega}_j(z)
\]

where \( k_i(z) = \min_{z \in \Omega} k_j(z) \) and \( l_j(z) = \max_{z \in \Omega} l_j(z) \) on \( z \in \cap N_j \neq \emptyset \) in the case of overlapping neighborhoods. By design we have that

\[
\tilde{\Omega}_p(z) \in \mathcal{G}_{k(\omega),l(\omega)}(K)
\]

with \( k(\omega) \doteq \min_{z \in \Omega} k_j(z) \) and \( l(\omega) = \max_{z \in \Omega} l_j(z) \). Further, if \( S_R(z) \) is the \( R \)’th partial Fourier sum of \( \omega(z) \) notice that

\[
|\tilde{\Omega}_p(z) - S_R(z)| = O\left(\frac{1}{L^\delta}\right), \quad \delta > 0
\]

with \( L = R - \max\{|k(\omega)|, l(\omega)\} \). The classical bound

\[
|\omega(z) - S_R(z)| < \sum_{|n|>R} \omega_n(z)
\]

guarantees that on letting \( \min\{|k|, l\} \not\to \infty \) and \( \sum \epsilon_j \not\to 0 \), that \( \lim_{p \to \infty} \tilde{\Omega}_p(z) = \omega(z) \) uniformly since the Fourier series can be brought as close as wanted in the mesh limit.

The following corollary is then immediate.

**Corollary 7.2** Let \( K \subset \Omega \) be compact and let \( \mathcal{H} = \bigcup_{(k,l) \in \mathbb{Z}^2} \mathcal{H}_{k,l} \). Then \( \mathcal{H}(K) \) is dense in \( \mathcal{C}(K) \) with respect to the uniform norm.

### 7.3 Putting it all together

Define

\[
\mathcal{O}(\Omega) \doteq \{ \mu \in \mathcal{R}(\Omega) \text{ satisfying condition 4 of H-ness} \}
\]

and let \( \mathcal{D} = \mathcal{C} \cap \mathcal{O} \). Then, if \( K \) is compact, for \( \mu \in \mathcal{D}(K) \) we see that \( \tilde{X} = \tilde{\Omega}_p(z) \partial + \overline{\tilde{\Omega}_p(z)} \overline{\partial} \) can be chosen to approximate \( X = \mu \partial + \mu \overline{\partial} \) so that their integral curves are arbitrarily close in \( L^p(K) \).
for $1 \leq p \leq \infty$ via Poincaré’s inequality. Then $||x - \tilde{x}||_{L^p} \leq C||\mu - \tilde{\Omega}(z)||_{L^p} + ||(x - \tilde{x})||_{L^p} \leq C'(\epsilon + \text{diam}\{K/\text{suppf}\})$. We then make the obvious choice setting $K = \text{suppf}$.

Let $\rho_{\alpha,\beta}$ be the seminorm on the Fréchet space $S$ of Schwartz-class functions on $C$, namely $\rho_{\alpha,\beta}(\phi) = \sup_{x \in C} |x^\alpha \partial^\beta \phi|$, which generates the usual topology on $S$. If $f \in C_c^\infty(D^+)$ then clearly $I_\theta f(s) \in S$. We let $\tilde{s}$ be the transverse flow induced by $X^\perp$ from which the corresponding Hilbert transform $H_\tilde{s}$ is defined. By continuity of Hilbert transforms on Schwartz functions, we assume that $H_\tilde{s}I_\theta^\perp f$ is small in the induced norm $\rho$ on $S$. That being the case, then since $\tilde{X}^\perp = \theta_*(\partial + i\partial)$ for some function $\alpha(z)$ and because differentiation is continuous on $S$, we see that $\rho\{\tilde{X}^\perp(H_\tilde{s}I_\theta^\perp f - H\lambda f)\}$ may therefore be made as small as desired.

We have therefore established the following corollary.

**Corollary 7.3** Let $\epsilon > 0$, $f \in C_c^\infty(D^+)$, $\mu \in \mathcal{D}(D^+)$ and let $I_\theta f$ be the ray transform of $f$ over the integral curves of $Y_\theta = \theta_*(\mu \partial + \bar{\mu} \bar{\partial})$. Then there exists functions $w_\epsilon(z)$, and $\tilde{\Omega}_p(z)$ such that $||f^\perp f - I^\perp f||_{L^q(D^+)} < C\epsilon$ for $C = C(\text{supf})$ constant and where $\tilde{X}_\theta = \theta_*(\tilde{\Omega}_p \partial + \bar{\Omega}_p \bar{\partial})$. Suppose further that $||H_\tilde{s}I_\theta^\perp f - H\lambda f||_S < \delta(\epsilon)$. Then there exists $\lambda_\epsilon(z)$ such that

$$
\sup_{z \in D^+} \left| f(z) - \frac{w_\epsilon(z)}{4\pi} \int_0^{2\pi} P(\lambda_\epsilon, \theta)\tilde{X}_\theta^\perp e^{i\theta} d\theta \right| \leq \epsilon
$$

We may now summarize our stability and approximation results in the following theorem.

**Theorem 7.4** Let $\epsilon > 0$ and $H(I_\theta f)(s(ze^{i\theta}), e^{i\theta})$ be given. Suppose that $\mu(z, \bar{z}) = \sum_{p+q \geq 0} a_{pq} z^p \bar{z}^q$ is a real-analytic function on $D^+$ and that $f \in C_c^\infty(D^+)$. Define $c_\lambda(z, \bar{z}) \equiv \sum_{p-q = \lambda} a_{pq} z^p \bar{z}^q$. Furthermore, suppose $\lambda_s, \lambda_z, \lambda_{sz}$, and $\lambda_{sz}$ are meromorphic for $\lambda \in D^+$. If, for all $z \in D^+ \setminus \{0\}$, there exist infinitely many $j \in \mathbb{Z}$ such that $c_j(z)$ and $c_{j+1}(z)$ are both nonzero, and if $\limsup_{j \to \infty} |\frac{c_{j+1}(z)}{c_j(z)}| < 1$, then there exist functions $w_\epsilon(z)$, and $\tilde{\Omega}_p(z)$ such that

$$
||f^\perp f - I^\perp f||_{L^q(D^+)} < C\epsilon
$$

for $C = C(\text{supf})$ constant and where $\tilde{X}_\theta = \theta_*(\tilde{\Omega}_p \partial + \bar{\Omega}_p \bar{\partial})$. Define $\tilde{\lambda}_\lambda(z, \bar{z}) \equiv \sum_{p-q = \lambda} a_{pq} z^p \bar{z}^q$. Furthermore, suppose $\lambda_s, \lambda_z, \lambda_{sz}$, and $\lambda_{sz}$ are meromorphic for $\lambda \in D^+$. If, for all $z \in D^+ \setminus \{0\}$, there exist infinitely many $j \in \mathbb{Z}$ such that $c_j(z)$ and $c_{j+1}(z)$ are both nonzero, and if $\limsup_{j \to \infty} |\frac{c_{j+1}(z)}{c_j(z)}| < 1$, then there exist functions $w_\epsilon(z)$, and $\tilde{\Omega}_p(z)$ such that

$$
||f^\perp f - I^\perp f||_{L^q(D^+)} < C\epsilon
$$

for $C = C(\text{supf})$ constant and where $\tilde{X}_\theta = \theta_*(\tilde{\Omega}_p \partial + \bar{\Omega}_p \bar{\partial})$. Define $\tilde{\lambda}_\lambda(z, \bar{z}) \equiv \sum_{p-q = \lambda} a_{pq} z^p \bar{z}^q$. Furthermore, suppose $\lambda_s, \lambda_z, \lambda_{sz}$, and $\lambda_{sz}$ are meromorphic for $\lambda \in D^+$. If, for all $z \in D^+ \setminus \{0\}$, there exist infinitely many $j \in \mathbb{Z}$ such that $c_j(z)$ and $c_{j+1}(z)$ are both nonzero, and if $\limsup_{j \to \infty} |\frac{c_{j+1}(z)}{c_j(z)}| < 1$, then there exist functions $w_\epsilon(z)$, and $\tilde{\Omega}_p(z)$ such that

$$
||f^\perp f - I^\perp f||_{L^q(D^+)} < C\epsilon
$$

If, in addition, we have $||H_\tilde{s}I_\theta^\perp f - H\lambda f||_S < \delta(\epsilon)$ then there exists a function $\lambda_\epsilon : z \to \lambda_\epsilon(z)$ satisfying the following inequality

$$
0 \leq \sup_{z \in D^+} \left| f(z) - \frac{w_\epsilon(z)}{4\pi} \int_0^{2\pi} P(\lambda_\epsilon, \theta)\tilde{X}_\theta^\perp e^{i\theta} d\theta \right| \leq \epsilon
$$

where $I_\theta f$ is the trace of $f$ over the integral curves of $X_\theta = \theta_*(\mu \partial + \bar{\mu} \bar{\partial})$ and where $\tilde{X}_\theta^\perp = i\theta_\cdot \frac{1}{w_\epsilon}(-\tilde{\Omega}_p \partial + \bar{\Omega}_p \bar{\partial})$.

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