Proof of a Conjecture of Farkas and Kra

Nian Hong Zhou

Abstract. In this paper we prove a conjecture of Farkas and Kra, which is a modular equation involving a half sum of certain modular form of weight 1 for congruence subgroup \( \Gamma_1(k) \) with any prime \( k \). We prove that their conjecture holds for all odd integers \( k \geq 3 \). A new modular equation of Farkas and Kra type is also established.

1. Introduction and statement of results

In this paper, we let \( z \in \mathbb{C}, \tau \in \mathbb{C} \) with \( \Im(\tau) > 0 \) and \( q = e^{2\pi i \tau} \). The theta function with characteristic \( [\epsilon, \epsilon'] \in \mathbb{R}^2 \) is defined by

\[
\theta[\epsilon, \epsilon'](z, \tau) = \sum_{n \in \mathbb{Z}} \exp\left\{ 2\pi i \left( \frac{n}{2} + \frac{\epsilon}{2} \right)^2 \tau + \left( n + \frac{\epsilon}{2} \right) \left( z + \frac{\epsilon'}{2} \right) \right\},
\]

which is a generalization of the Jacobi theta functions. The theory of above theta function was systematically studied by Farkas and Kra [2], which plays an important role in combinatorial number theory, algebraic geometry and physics.

In [2, Chapter 4], Farkas and Kra treated the theta function (1.1) with \( \epsilon, \epsilon' \in \mathbb{Q} \) and \( z = 0 \), that is, the theta constants with rational characteristics. Their derived many interesting results, one of them is the following (see [2, Theorem 9.8, p. 318] and [3]):

**Theorem 1.1.** For each odd prime \( k \) and all \( \tau \in \mathbb{C} \) with \( \Im(\tau) > 0 \),

\[
\frac{d}{d\tau} \log \left( \frac{\eta(k\tau)}{\eta(\tau)} \right) + \frac{1}{2\pi i(k-2)} \sum_{0 \leq \ell \leq (k-3)/2} \left( \frac{\theta'[1/((1+2\ell)/k)](0, \tau)}{\theta[1/((1+2\ell)/k)](0, \tau)} \right)^2
\]

is a cusp 1-form (cusp form of weight 1) for the Hecke congruence subgroup \( \Gamma_0(k) \). This form is identically zero provided \( k \leq 19 \). Here \( \eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n) \) is the Dedekind eta function and

\[
\theta'[1/((1+2\ell)/k)](0, \tau) = \frac{\partial}{\partial z} \theta[1/((1+2\ell)/k)](z, \tau) \bigg|_{z=0}.
\]
They then in [2, Conjecture 9.10, p. 320] (see also [3]) conjectured that \((1.2)\) is identically zero for each odd prime \(k\) and all \(\tau \in \mathbb{C}\) with \(\Im(\tau) > 0\).

**Remark 1.2.** We remark that for odd integers \(k, \ell\) with \(k \geq 3,\)
\[
\left[ \frac{\partial}{\partial z} \log \left( \theta \left[ 1/\ell/k \right] (0,\tau) \right) \right]^2
\]
is a modular 1-form (modular form of wight 1) for the group:
\[
G(k) = \Gamma_1(k) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (\mod k) \right\}.
\]
This fact and more related results can be found in [2,3].

The aim of this paper is to give a proof of the conjecture of Farkas and Kra of above. For the simplicity of the proof, we shall introduce the Jacobi theta function \(\theta_2(z,q)\), which is defined by (see for example [4])
\begin{equation}
\theta_2(z,q) = \sum_{n \in \mathbb{Z}} q^{(2n+1)^2/8} e^{i(2n+1)z}.
\end{equation}
Hence it is clear that
\[
\theta \left[ \frac{1}{e'} \right] (z,\tau) = \theta_2 \left( \pi z + \frac{e'\pi}{2},q \right)
\]
and the conjecture of our concern is equivalent to the following.

**Conjecture 1.3.** For each odd prime \(k\) and all \(\tau \in \mathbb{C}\) with \(\Im(\tau) > 0,\)
\[
4(k-2)q \frac{d}{dq} \log \frac{\eta(k\tau)}{\eta(\tau)} - \sum_{0 \leq \ell < k \atop \ell \equiv 1 (\mod 2)} \left[ \frac{\partial}{\partial z} \log \theta_2 \left( \frac{\ell}{2k},q \right) \right]^2 = 0.
\]

We shall prove a more general result than Conjecture [1,3]. To state our main result, we define the following half sum
\begin{equation}
S_\delta(k) := \sum_{0 \leq \ell < k \atop \ell-k \equiv \delta (\mod 2)} \left[ \frac{\partial}{\partial z} \log \theta_2 \left( \frac{\ell}{2k},q \right) \right]^2
\end{equation}
for each integer \(k \geq 2\) and each \(\delta \in \{0,1\}\). Our main result is the following modular equations.

**Theorem 1.4.** For all \(\tau \in \mathbb{C}\) with \(\Im(\tau) > 0,\) we have if \(\delta = 0\) then
\[
S_\delta(k) = 4(k-2)q \frac{d}{dq} \log \left( \frac{\eta(k\tau)}{\eta(\tau)} \right),
\]
and if \(\delta = 1\) then
\[
S_\delta(k) = 4q \frac{d}{dq} \log \left( \frac{\eta(2k\tau)^{2k-2}}{\eta(\tau)^k \eta(k\tau)^{k-2}} \right).
\]
We immediately obtain the proof of Conjecture 1.3 by setting $k \in 2\mathbb{Z}_+ + 1$ and $\delta = 0$ in Theorem 1.4.

**Corollary 1.5.** Conjecture 1.3 holds for all odd integers $k \geq 3$. In particular, Conjecture 1.3 is true.

We shall give some consequences of Theorem 1.4. For this purpose we first use Lemma 2.2 below to deduce a proposition as follows.

**Proposition 1.6.** We have

$$S_\delta(k) = \sum_{0 \leq \ell < k, \ell \equiv k \equiv \delta \pmod{2}} \left[ \tan \left( \frac{\ell \pi}{2k} \right) - 4 \sum_{h=1}^{2k} (-1)^h \sin \left( \frac{\ell h \pi}{k} \right) \sum_{n \geq 1} \frac{q^{hn}}{1 - q^{2kn}} \right]^2.$$ 

By setting $q = 0$ in Theorem 1.4, applying Proposition 1.6 and (2.3) below we obtain the following trigonometric identity, which has been proved in [2, 3] by using the theory of modular form.

**Corollary 1.7.** For each integer $k \geq 2$,

$$\sum_{0 \leq \ell < k, \ell \equiv k \equiv \delta \pmod{2}} \left[ \tan \left( \frac{\ell \pi}{2k} \right) \right]^2 = \begin{cases} (k - 1)(k - 2)/6 & \text{if } \delta = 0, \\ k(k - 1)/6 & \text{if } \delta = 1. \end{cases}$$

From Theorem 1.4, Proposition 1.6 and (2.3), by choosing different pair $(k, \delta)$ one can obtain many Lambert series identities. For example, if we set $(k, \delta) = (3, 1)$, then it is easy to see

**Corollary 1.8.** We have

$$\left( 1 + 2 \sum_{n \geq 1} \frac{q^n + q^{2n} - q^{4n} - q^{5n}}{1 - q^{6n}} \right)^2 = 1 + 4 \sum_{n \geq 1} \left( \frac{nq^n}{1 - q^n} + \frac{nq^{3n}}{1 - q^{3n}} - \frac{8nq^{6n}}{1 - q^{6n}} \right).$$

Our proof of the main theorem is based on the series expansion (1.3) of $\theta_2(z, q)$ and the Jacobi triple product identity. The rest of this paper is organized as follows. In the next section, we first establish some primary results for $\theta_2(z, q)$. In Section 3 we prove Theorem 1.4.

2. Primaries

We shall need the following primary results, which will be used to prove main results of this paper.
Proposition 2.1. We have
\[
\left( \frac{\partial}{\partial z} \log \theta_2(z, q) \right)^2 = T_{z,q}(\log \theta_2(z, q)),
\]
where and throughout, $T_{z,q}$ is a linear operator defined as
\[
T_{z,q} = -8q \frac{\partial}{\partial q} - \frac{\partial^2}{\partial z^2}.
\]

Proof. By (1.3) it is clear that
\[
\left( 8q \frac{\partial}{\partial q} + \frac{\partial^2}{\partial z^2} \right) \theta_2(z, q) = 0,
\]
which means that
\[
\frac{1}{\theta_2(z, q)} \frac{\partial^2}{\partial z^2} \theta_2(z, q) = -8q \frac{\partial}{\partial q} \log \theta_2(z, q).
\]
Then from the basic fact that
\[
\frac{\partial^2}{\partial z^2} \log \theta_2(z, q) = \frac{1}{\theta_2(z, q)} \frac{\partial^2}{\partial z^2} \theta_2(z, q) - \left( \frac{\partial}{\partial z} \log \theta_2(z, q) \right)^2
\]
we complete the proof of the proposition. \(\square\)

We need the Jacobi triple product identity for $\theta_2(z, q)$ (see for example [1,4]),

(2.1) \[ \theta_2(z, q) = q^{1/8} e^{-iz} \prod_{n \geq 1} (1 - q^n)(1 + e^{-2iz} q^n)(1 + e^{2iz} q^{n-1}). \]

Lemma 2.2. For each $\ell, k \in \mathbb{Z}$ with $\ell \neq k$ and $k > 0$,
\[
-\frac{\partial}{\partial z} \log \theta_2 \left( \frac{\ell \pi}{2k}, q \right) = \tan \left( \frac{\ell \pi}{2k} \right) - 4 \sum_{h=1}^{2k} (-1)^h \sin \left( \frac{\ell h \pi}{k} \right) \sum_{n \geq 0} q^{hn} \frac{1 - q^{2kn}}{1 - q^{2kn}}.
\]

Proof. Taking the logarithmic derivative of $\theta_2(z, q)$ respect to $z$ by (2.1), we have the well known Fourier expansion
\[
\frac{\partial}{\partial z} \log \theta_2(z, q) = -\tan(z) + 4 \sum_{n \geq 1} \frac{(-1)^n q^n}{1 - q^n} \sin(2nz).
\]

Noticing that
\[
\sum_{n \geq 1} \frac{(-1)^n q^n}{1 - q^n} \sin \left( \frac{2n \ell \pi}{2k} \right) = \sum_{h=1}^{2k} \sum_{n \geq 0} (-1)^h q^{2nk+h} \frac{1}{1 - q^{2nk+h}} \sin \left( \frac{\ell h \pi}{k} \right)
\]
\[
= \sum_{h=1}^{2k} (-1)^h \sin \left( \frac{\ell h \pi}{k} \right) \sum_{n \geq 0} \sum_{\ell \geq 1} q^{(2nk+h)\ell}
\]
Proof of a Conjecture of Farkas and Kra

and (2.2) we immediately obtain that
\[ \frac{\partial}{\partial z} \log \theta_2 \left( \frac{ell \pi}{2k}, q \right) = -\tan \left( \frac{ell \pi}{2k} \right) + 4 \sum_{h=1}^{2k} (-1)^h \sin \left( \frac{ell h \pi}{k} \right) \sum_{n \geq 1} \frac{q^{hn}}{1 - q^{2kn}}. \]

This completes the proof of the lemma.

The following lemma will be used to prove Theorem 1.4 in the next section.

**Lemma 2.3.** For each \( k \in \mathbb{Z}_+ \),
\[ T_{z,q}(\log \theta_2(kz,q^k)) \bigg|_{z=0} = 8(k-1)q \frac{d}{dq} \log \left( \frac{\eta(2k\tau)^2}{\eta(k\tau)} \right) \]
and
\[ T_{z,q} \left( \log \left( \frac{\theta_2(kz - \pi/2, q^k)}{\theta_2(z - \pi/2, q)} \right) \right) \bigg|_{z=0} = 8q \frac{d}{dq} \log(\eta(k\tau)^{k-3}\eta(\tau)^2). \]

**Proof.** By (2.2) we have
\[ \frac{\partial^2}{\partial z^2} \log \theta_2(z,q) = -\tan^2(z) - 1 + 8 \sum_{n \geq 1} \frac{(-1)^n q^n}{1 - q^n} \cos(2nz) \]
and
\[ \frac{\partial^2}{\partial z^2} \log \theta_2(z - \pi/2, q) = -\cot^2(z) - 1 + 8 \sum_{n \geq 1} \frac{nq^n}{1 - q^n} \cos(2nz). \]

Hence we obtain that
\[ \frac{\partial^2}{\partial z^2} \log \theta_2(z,q) \bigg|_{z=0} = -1 + 8 \sum_{n \geq 1} \frac{(-1)^n q^n}{1 - q^n} \]
\[ = -1 + 16 \sum_{n \geq 1} \frac{nq^{2n}}{1 - q^{2n}} - 8 \sum_{n \geq 1} \frac{nq^n}{1 - q^n} \]
and
\[ \frac{\partial^2}{\partial z^2} \log \left( \frac{\theta_2(kz - \pi/2, q^k)}{\theta_2(z - \pi/2, q)} \right) \bigg|_{z=0} \]
\[ = \lim_{z \to 0} \left( \cot^2(z) + 1 - k^2(\cot^2(kz) + 1) \right) + 8 \sum_{n \geq 1} \left( \frac{k^2 nq^{kn}}{1 - q^{kn}} - \frac{nq^n}{1 - q^n} \right) \]
\[ = \frac{1 - k^2}{3} + 8k^2 \sum_{n \geq 1} \frac{nq^{kn}}{1 - q^{kn}} - 8 \sum_{n \geq 1} \frac{nq^n}{1 - q^n}. \]

Using the fact that
\[ q \frac{d}{dq} \log \eta(\alpha \tau) = \frac{\alpha}{24} - \sum_{n \geq 1} \frac{\alpha nq^{an}}{1 - q^{an}}, \quad \alpha \in \mathbb{R}_+, \]

(2.3)
and the above we obtain
\begin{equation}
\frac{\partial^2}{\partial z^2} \log \theta_2(z, q) \bigg|_{z=0} = 8q \frac{d}{dq} \log \left( \frac{\eta(\tau)}{\eta(2\tau)^2} \right)
\end{equation}
and
\begin{equation}
\frac{\partial^2}{\partial z^2} \log \left( \frac{\theta_2(kz - \pi/2, q^k)}{\theta_2(z - \pi/2, q)} \right) \bigg|_{z=0} = 8q \frac{d}{dq} \log \left( \frac{\eta(\tau)}{\eta(k\tau)^k} \right).
\end{equation}

Moreover, by \([2.1]\) and the definition of \(\eta(\tau)\), it is easy to see that
\begin{equation}
\theta_2(0, q) = 2 \frac{\eta(2\tau)^2}{\eta(\tau)}
\end{equation}
and
\begin{equation}
\lim_{z \to 0} \frac{\theta_2(z - \pi/2, q)}{z} = 2\eta(\tau)^3.
\end{equation}

Thus for integer \(k \geq 1\), application of \((2.4)\) and \((2.6)\) imply that
\[
T_{z,q}(\log \theta_2(kz, q^k)) \bigg|_{z=0} = -8q \frac{d}{dq} \log \theta_2(0, q^k) - \frac{\partial^2}{\partial z^2} \log \theta_2(kz, q^k) \bigg|_{z=0}
\]
\[
= -8q \frac{d}{dq} \log \left( \frac{\eta(2k\tau)^2}{\eta(k\tau)^2} \right) + k^2 \left( -8q \frac{d}{dq} \log \left( \frac{\eta(k\tau)}{\eta(2k\tau)^2} \right) \right)
\]
\[
= 8(k-1)q \frac{d}{dq} \log \left( \frac{\eta(2k\tau)^2}{\eta(k\tau)^2} \right),
\]
and application of \((2.5)\) and \((2.7)\) imply that
\[
T_{z,q} \left( \log \left( \frac{\theta_2(kz - \pi/2, q^k)}{\theta_2(z - \pi/2, q)} \right) \right) \bigg|_{z=0}
\]
\[
= -8q \frac{d}{dq} \log \left( \frac{\eta(k\tau)}{\eta(\tau)^3} \right) - \frac{\partial^2}{\partial z^2} \log \left( \frac{\theta_2(kz - \pi/2, q^k)}{\theta_2(z - \pi/2, q)} \right) \bigg|_{z=0}
\]
\[
= -24q \frac{d}{dq} \log \left( \frac{\eta(k\tau)}{\eta(\tau)^3} \right) - 8q \frac{d}{dq} \log \left( \frac{\eta(\tau)}{\eta(k\tau)^k} \right)
\]
\[
= 8q \frac{d}{dq} \log(\eta(k\tau)^{k-3}\eta(\tau)^2),
\]
which completes the proof of the lemma.

We need the following half product formula for Jacobi theta function \(\theta_2\), which will be used to prove Theorem \([1.4]\) in the next section.

**Lemma 2.4.** For integer \(k \geq 1\) and \(\delta \in \{0, 1\}\),
\[
\prod_{0 \leq \ell < 2k \atop \ell - k \equiv \delta \pmod{2}} \theta_2 \left( z + \frac{\ell}{2k} \pi, q \right) = C_{k,\delta} \frac{\eta(\tau)^k}{\eta(k\tau)^k} \theta_2 \left( kz + \frac{(\delta - 1)\pi}{2}, q^k \right),
\]
where \(C_{k,\delta} = e^{\frac{i\pi}{2}(\delta - 1)(k+1)\eta(\tau)^2 \pmod{2}}\). Here and throughout, \(1_{\text{condition}} = 1\) if the ‘condition’ is true, and equals to \(0\) if the ‘condition’ is false.
Proof. From (2.1) we have

\[
\prod_{0 \leq \ell < 2k, \ell - k \equiv \delta \pmod{2}} \theta_2(z + \frac{\ell \pi}{2k}, q) = \prod_{0 \leq \ell < 2k, \ell - k \equiv \delta \pmod{2}} \left( q^{1/8} e^{-i(z + \frac{\ell \pi}{2k})} \prod_{n \geq 1} (1 - q^n) \right)
\]

\times \prod_{n \geq 1} \prod_{0 \leq \ell < 2k, \ell - k \equiv \delta \pmod{2}} \left( 1 + q^n e^{-2iz - \frac{\ell \pi}{k}} \right) \left( 1 + q^n e^{2iz + \frac{\ell \pi}{k}} \right).

It is easy to check that

\[
\prod_{0 \leq \ell < 2k, \ell - k \equiv \delta \pmod{2}} (1 + xe^{\pm \ell \pi i/2k}) = 1 - e^{\delta \pi i} x^k \quad \text{and} \quad \sum_{0 \leq \ell < 2k, \ell - k \equiv \delta \pmod{2}} \ell = k(k - 1) + k \mathbf{1}_{k \not\equiv \delta \pmod{2}}.
\]

Thus we obtain that

\[
\prod_{0 \leq \ell < 2k, \ell - k \equiv \delta \pmod{2}} \theta_2(z + \frac{\ell \pi}{2k}, q) = q^{k/12} \eta(\tau)^k e^{-i(kz)\pi} e^{-\frac{i\pi}{2}(k-1) + k \mathbf{1}_{k \not\equiv \delta \pmod{2}}}
\]

\times \prod_{n \geq 1} \left( 1 - e^{-2iz - \delta \pi i} q^{kn} \right) \left( 1 - e^{2iz + \delta \pi i} q^{k(n-1)} \right)

= C_{k, \delta} \theta_2(kz + (\delta - 1)\pi/2, q_0) \eta(\tau)^k \eta(k\tau)

with

\[
C_{k, \delta} = e^{\frac{i\pi(\delta - 1)}{2}} - \frac{i\pi(k-1) + k \mathbf{1}_{k \not\equiv \delta \pmod{2}}}{2} = e^{\frac{i\pi}{2}(\delta - k + 1 \mathbf{1}_{k \not\equiv \delta \pmod{2}})},
\]

which completes the proof of the lemma.

\[
3. \text{Proof of Theorem 1.4}
\]

First of all, we define

\[
G_{\delta,k}(z, q) := \sum_{0 \leq \ell < k, \ell - k \equiv \delta \pmod{2}} \left[ \frac{\partial}{\partial z} \log \theta_2 \left( z + \frac{\ell \pi}{2k}, q \right) \right]^2,
\]

then from (1.4) we have \(S_{\delta}(k) = G_{\delta,k}(0, q)\). By Proposition 2.1 we get

\[
G_{\delta,k}(z, q) = \sum_{0 \leq \ell < k, \ell - k \equiv \delta \pmod{2}} T_{z,q} \left( \log \theta_2 \left( z + \frac{\ell \pi}{2k}, q \right) \right)
\]

\[
= T_{z,q} \left( \sum_{0 \leq \ell < k, \ell - k \equiv \delta \pmod{2}} \log \theta_2 \left( z + \frac{\ell \pi}{2k}, q \right) \right).
\]
We shall define the auxiliary function as follows:

\[ A_{\delta,k}(z, q) = T_{z,q} \left( \sum_{0 \leq \ell < 2k \atop \ell \equiv -k \equiv \delta (\text{mod } 2)} \log \theta_2 \left( z + \frac{\ell \pi}{2k}, q \right) - \mathbf{1}_{\delta=0} \log \theta_2 \left( z + \frac{\pi}{2}, q \right) \right). \]

It is clear that \( A_{\delta,k}(0, q) := \lim_{z \to 0} A_{\delta,k}(z, q) \) exists. We claim that

\[
G_{\delta,k}(0, q) = \frac{1}{2} A_{\delta,k}(0, q).
\]

In fact, by \( \theta_2(z + \pi, q) = -\theta_2(z, q) \) and \( \theta_2(z, q) = \theta_2(-z, q) \) we have

\[
A_{\delta,k}(z, q) = \sum_{0 \leq \ell < 2k \atop \ell \equiv -k \equiv \delta (\text{mod } 2)} T_{z,q} \left( \log \theta_2 \left( z + \frac{\ell \pi}{2k}, q \right) + \log \theta_2 \left( z + \frac{(2k - \ell) \pi}{2k}, q \right) \right)
\]

Further, by the definition of \( T_{z,q} \), we see that

\[
T_{z,q} \left( \log \theta_2 \left( -\theta_2 \left( -z + \frac{\ell \pi}{2k}, q \right) \right) \right) \bigg|_{z=0} = -8q \frac{\partial}{\partial q} \left( \log \theta_2 \left( \frac{\ell \pi}{2k}, q \right) \right) - \frac{\partial^2}{\partial z^2} \left( \log \theta_2 \left( -z + \frac{\ell \pi}{2k}, q \right) \right) \bigg|_{z=0}
\]

which immediately obtain the proof of (3.1).

On the other hand, from Lemma 2.4 we find that

\[
\sum_{0 \leq \ell < 2k \atop \ell \equiv -k \equiv \delta (\text{mod } 2)} \log \theta_2 \left( z + \frac{\ell \pi}{2k}, q \right) = \prod_{0 \leq \ell < 2k \atop \ell \equiv -k \equiv \delta (\text{mod } 2)} \theta_2 \left( z + \frac{\ell \pi}{2k}, q \right)
\]

which implies that

\[
A_{\delta,k}(z, q) = T_{z,q} \left( \log \theta_2 \left( kz + \frac{(\delta - 1) \pi}{2}, q^k \right) + \log \left( C_{k,\delta} \frac{\eta(\tau)^k}{\eta(k\tau)} \theta_2 \left( kz + \frac{(\delta - 1) \pi}{2}, q^k \right) \right) \right)
\]

which implies that

\[
A_{\delta,k}(z, q) = T_{z,q} \left( \log \theta_2 \left( kz + \frac{(\delta - 1) \pi}{2}, q^k \right) + \log \left( C_{k,\delta} \frac{\eta(\tau)^k}{\eta(k\tau)} \theta_2 \left( kz + \frac{(\delta - 1) \pi}{2}, q^k \right) \right) \right)
\]

\[
+ T_{z,q} \left( \mathbf{1}_{k \equiv \delta (\text{mod } 2)} \log \theta_2(z, q) - \mathbf{1}_{\delta=0} \log \theta_2 \left( z + \frac{\pi}{2}, q \right) \right).
\]
Further by Lemma 2.3 we obtain that
\[ A_{0,k}(0, q) = T_{z,q} \left( \log \left( \frac{\theta_2(kz - \pi/2, q^k)}{\theta_2(z - \pi/2, q)} \right) \right) \bigg|_{z=0} \]
\[ - 8q \frac{\partial}{\partial q} \log \left( \frac{\eta(\tau)^k}{\eta(k\tau)} \right) + 1_{k \equiv 0 \pmod{2}} T_{z,q} (\log \theta_2(z, q)) \bigg|_{z=0} \]
\[ = 8q \frac{d}{dq} \log(\eta(k\tau)^{-3}\eta(\tau)^2) - 8q \frac{d}{dq} \log \left( \frac{\eta(\tau)^k}{\eta(k\tau)} \right) \]
\[ = 8(k - 2)q \frac{d}{dq} \log \left( \frac{\eta(k\tau)}{\eta(\tau)} \right) \]
and
\[ A_{1,k}(0, q) = T_{z,q} (\log \theta_2(kz, q^k)) \bigg|_{z=0} \]
\[ - 8q \frac{\partial}{\partial q} \log \left( \frac{\eta(\tau)^k}{\eta(k\tau)} \right) + 1_{k \equiv 1 \pmod{2}} T_{z,q} (\log \theta_2(z, q)) \bigg|_{z=0} \]
\[ = 8(k - 1)q \frac{d}{dq} \log \left( \frac{\eta(2k\tau)^2}{\eta(k\tau)} \right) - 8q \frac{d}{dq} \log \left( \frac{\eta(\tau)^k}{\eta(k\tau)} \right) \]
\[ = 8q \frac{d}{dq} \log \left( \frac{\eta(2k\tau)^{2k-2}}{\eta(\tau)^k\eta(k\tau)^{k-2}} \right), \]
which completes the proof of Theorem 1.4 by noting that \( S_\delta(k) = G_{\delta,k}(0, q) = \frac{1}{2} A_{\delta,k}(0, q). \)

Acknowledgments

The author would like to thank the anonymous referees for their very helpful comments and suggestions. The author also thank his advisor Zhi-Guo Liu for consistent encouragement and useful suggestions.

References

[1] G. E. Andrews, A simple proof of Jacobi’s triple product identity, Proc. Amer. Math. Soc. 16 (1965), 333–334.

[2] H. M. Farkas and I. Kra, Theta Constants, Riemann Surfaces and the Modular Group: An introduction with applications to uniformization theorems, partition identities and combinatorial number theory, Graduate Studies in Mathematics 37, American Mathematical Society, Providence, RI, 2001.

[3] ______, On theta constant identities and the evaluation of trigonometric sums, in: Complex Manifolds and Hyperbolic Geometry (Guanajuato, 2001), 115–131, Contemp. Math. 311, Amer. Math. Soc., Providence, RI, 2002.
[4] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis: An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions*, Fourth edition, Cambridge University Press, New York, 1962.

Nian Hong Zhou
School of Mathematical Sciences, East China Normal University, 500 Dongchuan Road, Shanghai 200241, P. R. China

*E-mail address*: nianhongzhou@outlook.com