Blow-up of solutions to a Dirichlet problem for the discrete semi-linear heat equation

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1 Introduction

In this paper, we consider the following partial difference equation with prescribed initial and boundary conditions:

\[ \begin{cases} f_{\vec{n}}^{s+1} = \frac{g_{\vec{n}}^{s}}{(1 - \alpha \delta (g_{\vec{n}}^{s})^{\alpha})^{1/\alpha}} \quad (s \in \mathbb{Z}_{\geq 0}, \, \vec{n} \in \Omega_{D}), \\ f_{\vec{n}}^{0} = a_{\vec{n}} \geq 0, \neq 0 \quad (\vec{n} \in \Omega_{D}), \\ f_{\vec{n}}^{s} = 0 \quad (s \in \mathbb{Z}_{\geq 0}, \, \vec{n} \in \partial \Omega_{D}), \end{cases} \]

(1)

where \( \Omega_{D} \) is a bounded subset of \( \mathbb{Z}^{d} \), \( \partial \Omega_{D} \) is the boundary of \( \Omega_{D} \), \( \Omega_{D}^{\circ} \) is the interior of \( \Omega_{D} \), (namely \( \Omega_{D}^{\circ} := \Omega_{D} \setminus \partial \Omega_{D} \)), \( f_{\vec{n}}^{s} := f(s, \vec{n}) \), \( s \in \mathbb{Z}_{\geq 0}, \, \vec{n} \in \Omega_{D} \). Moreover, we take \( \alpha, \delta > 0 \) and \( g_{\vec{n}}^{s} \) define as:

\[ g_{\vec{n}}^{s} := \sum_{k=1}^{d} \frac{f_{\vec{n}}^{s+1} + f_{\vec{n}}^{s-1}}{2d}, \]

where \( \vec{e}_{k} \) is the unit vector whose \( k \)-th component is 1 and the others are 0.

The difference equation in (1) is investigated [5] as a discretization of the following semi-linear heat equation:

\[ \frac{\partial f}{\partial t} = \Delta f + f^{1+\alpha}, \]

(2)

where \( f := f(t, \vec{x}) \), \( t \geq 0, \, \vec{x} \in \Omega_{C} \subset \mathbb{R}^{d} \) and \( \Delta \) is a \( d \)-dimensional Laplacian.

Solutions of (2) are not necessarily bounded for all \( t \geq 0 \). In general, if there exists a finite time \( T > 0 \) for which the solution of (2) in \( (t, \vec{x}) \in [0, T) \times \Omega_{C} \) satisfies

\[ \limsup_{t \to T-0} \| f(t, \cdot) \|_{L^{\infty}} = \infty, \]
where
\[ \| f(t, \cdot) \|_{L^\infty} := \sup_{\vec{x} \in \Omega_C} |f(t, \vec{x})|, \]
then we say that the solution of Eq. (2) blows up at time \( T \).

The Cauchy problem for Eq. (2) has been studied and a critical exponent which characterises the blow-up of the solutions for Eq. (2) has been discovered and studied by Fujita and et al.\(^1\)\(^2\)\(^3\)\(^4\)

In fact, the difference equation (1) has similar characteristics to the critical exponent known from the continuous case.

Considering Eq. (2) on \([0, T) \times \Omega_C\) with the following initial and boundary conditions
\[
\begin{align*}
    f(0, \vec{x}) &= a(\vec{x}) \geq 0, \not\equiv 0 \quad (\vec{x} \in \Omega_C), \\
    f(t, \vec{x}) &= 0 \quad (t \geq 0, \vec{x} \in \partial\Omega_C),
\end{align*}
\]
where \( \Omega_C \) is a bounded subset of \( \mathbb{R}^d \), the following theorem can be shown to hold.

**Theorem 1** (\(^1\)) The solution of Eq. (2) with initial and boundary conditions (3) does not blow up at any finite time for sufficiently small initial conditions \( a(\vec{x}) \).

In this article, we show that (1) has a property similar to theorem 1. In section 2, we define the blow-up of solutions for (1) and state the main theorem which is a discrete analogue of theorem 1. This theorem is proved in section 3.

### 2 Main theorem

First, we define the blow-up of solutions for (1). Because of the term \( \{1 - \alpha \delta (g_{\vec{n}}^s)\}^{1/\alpha} \), when \( g_{\vec{n}}^s \to (\alpha \delta)^{-1/\alpha} - 0 \), then \( f_{\vec{n}}^{s+1} \to +\infty \). This behaviour may be regarded as an analogue of the blow up of solutions for the semi-linear heat equation. Thus we define a global solution of (1) as follows.

**Definition 2.1** Let \( f_{\vec{n}}^s \) be a solution to (1). When there exists an \( s_0 \in \mathbb{Z}_{\geq 0} \) such that \( g_{\vec{n}}^s \leq (\alpha \delta)^{-1/\alpha} \) for all \( s < s_0 \) and \( \vec{n} \in \Omega_D \), and when there exists \( s_0 \in \Omega_D \) such that \( g_{\vec{n}}^{s_0} \geq (\alpha \delta)^{-1/\alpha} \), then we say that the solution \( f_{\vec{n}}^s \) blows up at time \( s_0 \).

The following theorem is the main theorem of this paper.

**Theorem 2** For \( \Omega_D = \{ \vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d | 0 \leq n_k \leq N_k \ (k = 1, \cdots, d) \} \), the solution of (1) does not blow up at any finite time with sufficiently small initial condition \( a_{\vec{n}} \).

### 3 Proof of the theorem

To prove the theorem, we make use of a comparison theorem.
First, to simplify the equations, we take the scaling \((\alpha \delta)^{1/\alpha} f^s_\vec{n} \to f^s_\vec{n}\) which changes the difference equation in (1) to

\[
f^{s+1}_\vec{n} = \frac{g^s_\vec{n}}{\{1 - (g^s_\vec{n})^\alpha\}^{1/\alpha}}
\]

Now, we construct a majorant solution. Let

\[
\hat{M}(h^s_\vec{n}) := \frac{1}{2d} \sum_{k=1}^d (h^s_{\vec{n} + e_k} + h^s_{\vec{n} - e_k}).
\]

We denote by \(h^s_\vec{n}\) the solution to the initial and boundary condition problem of the linear partial difference equation

\[
\begin{cases}
  h^{s+1}_\vec{n} = \hat{M}(h^s_\vec{n}) & (s \in \mathbb{Z}_{\geq 0}, \vec{n} \in \Omega^\circ_D) \\
  h^0_\vec{n} = a_\vec{n} & (\vec{n} \in \Omega_D) \\
  h^s_\vec{n} = 0 & (s \in \mathbb{Z}_{\geq 0}, \vec{n} \in \partial \Omega_D).
\end{cases}
\]

The majorant solution is \(\bar{f}^s_\vec{n}\) defined as follow:

\[
\bar{f}^s_\vec{n} := \frac{h^s_\vec{n}}{\left(1 - \sum_{k=0}^s |m_k|^\alpha \right)^{1/\alpha}},
\]

where \(m_s\) is defined in terms of (5) as

\[
m_s := \max_{\vec{n} \in \Omega^\circ_D} h^s_\vec{n}.
\]

**Lemma 3.1** When \(\bar{f}^s_\vec{n}\) exists at \(s\), for all \(\vec{n} \in \mathbb{Z}^d\), namely when

\[
1 - \sum_{k=0}^s |m_k|^\alpha > 0
\]

holds, the solution of (1) does not blow up at any time \(s\) and moreover satisfies

\[
\bar{f}^s_\vec{n} \geq f^s_\vec{n}.
\]

**Proof** We precede by induction on \(s\). When \(s = 0\), by the definition of the initial and boundary condition problem, \(f^0_\vec{n}\) exists and (8) holds because

\[
\bar{f}^0_\vec{n} = \frac{h^0_\vec{n}}{\{1 - |m_0|^\alpha\}^{1/\alpha}} \geq h^0_\vec{n} = f^0_\vec{n}.
\]
Suppose that the statement is true up to $s = s_0$ and that $\tilde{f}_{n}^{s_0+1}$ exists. When $\tilde{f}_{n}^{s_0+1} = 0$, we have that

\begin{align*}
\tilde{f}_{n}^{s_0+1} = 0 & \iff h_{n}^{s_0+1} = 0 \\
& \iff M(h_{n}^{s_0}) = 0 \\
& \iff h_{n,k}^{s_0} = 0 \quad (k = 1, 2, \ldots, d) \\
& \iff \tilde{f}_{n,k}^{s_0} = 0 \quad (k = 1, 2, \ldots, d) \\
& \iff f_{n,k}^{s_0} = 0 \quad (k = 1, 2, \ldots, d) \\
& \iff g_{n}^{s_0} = 0 \\
& \iff f_{n}^{s_0+1} = 0.
\end{align*}

Hence (8) holds.

When $\tilde{f}_{n}^{s_0+1} > 0$, if $g_{n}^{s_0} = 0$, then $f_{n}^{s_0+1} = 0$ and the statement is true. Otherwise

\begin{align*}
0 < (\tilde{f}_{n}^{s_0+1})^{-\alpha} = \frac{1 - \sum_{k=0}^{s_0} |m_k|^\alpha}{(h_{n}^{s_0+1})^\alpha} & = \frac{1 - \sum_{k=0}^{s_0} |m_k|^\alpha}{\left(\frac{|m_{s_0+1}|}{h_{n}^{s_0+1}}\right)^\alpha} \\
& \leq \left\{ \frac{M(h_{n}^{s_0})}{\tilde{M}(f_{n}^{s_0})}\right\}^{-\alpha} - 1 = \left\{ \frac{1}{\tilde{M}(f_{n}^{s_0})}\right\}^{-\alpha} - 1 \\
& \leq (g_{n}^{s_0})^{-\alpha} - 1.
\end{align*}

From (4), $(g_{n}^{s_0})^{-\alpha} - 1 = (f_{n}^{s_0+1})^{-\alpha}$ and we find $(\tilde{f}_{n}^{s_0+1})^{-\alpha} \leq (f_{n}^{s_0+1})^{-\alpha}$, i.e. $f_{n}^{s_0+1} \leq f_{n}^{s_0+1}$. Thus, from the induction hypothesis, the statement is true for any non-negative integer $s$.

From this lemma, by proving that $1 - \sum_{k=0}^{s} |m_k|^\alpha > 0$ for all $s \in \mathbb{Z}_{\geq 0}$ with sufficiently small initial condition in (1), one can complete the proof of the main theorem.

The solution of (5) is

\[ h_{\vec{n}}^{s} = \sum_{\vec{n}'' \in \Omega_{0}} \left\{ B_{\vec{n}''}(c_{\vec{n}'})^{s} \prod_{k=1}^{d} \sin \left( \frac{n_{k}' \pi}{N_{k}} \right) \right\}, \]

where $\vec{n} := (n_1, \ldots, n_d)$, $\vec{n}'' := (n_1', \ldots, n_d')$, $c_{\vec{n}'} := \sum_{k=1}^{d} \frac{1}{2} \cos (n_{k}' \pi / N_{k})$ and $B_{\vec{n}''}$ are constants that satisfy $h_{\vec{n}'}^{0} = a_{\vec{n}}$. The following proposition concerning $B_{\vec{n}}$ can be proven.

**Proposition 3.1** If the initial condition of (5) $a_{\vec{n}}$ is fixed, $B_{\vec{n}}$ are determined uniquely.
This property is proved by induction on $d$.

When $d = 1$, put $N := N_1$. Solving $N - 1$ linear equations with $N - 1$ unknowns: $a_{n'} = \sum_{n=1}^{N-1} B_n \sin \left( \frac{n \pi}{N} n' \right)$ ($n' = 1, \ldots, N - 1$), the $B_n$ are determined. If $N - 1$ vectors \( \left( \sin \frac{n \pi}{N}, \ldots, \sin \frac{n(N-1) \pi}{N} \right) \) ($n = 1, \ldots, N - 1$) are linearly independent, then the $B_n$ are determined uniquely. On the other hand, these $N - 1$ vectors are eigenvectors of the following $(N - 1) \times (N - 1)$ matrix:

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 & 1 \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\]

All eigenvector are linearly independent so that the $B_n$ are determined uniquely.

Suppose that the statement is true up to $d = d_0 - 1$. Now we consider the case of $d = d_0$.

If $n_1, \ldots, n_{d_0 - 1}$ are fixed, then each \( \sum B_{n_{d_0 - 1}} \prod_{k=1}^{d_0 - 1} \sin \left( \frac{n_k \pi}{N} n_k' \right) \) is determined uniquely from the case of $d = 1$. Because of the induction hypothesis, the $B_{n_{d_0 - 1}}$ are also determined uniquely. Thus, the statement is true for any $d$.

Now we estimate the infinite series \( \sum_{k=0}^{\infty} |m_k|^\alpha \). Take $B := \max_{n_{d_0}} |B_{n_{d_0}}|$. If one lets $\max_{n_{d_0}} |a_{n_{d_0}}|$ be small, $B$ also becomes small. We consider three cases $\alpha \leq 1$, $\alpha > 1$.

When $\alpha \leq 1$, we obtain

\[
\sum_{k=0}^{\infty} |m_k|^\alpha \leq \sum_{k=0}^{\infty} \left( B \sum_{\bar{n} \in \Omega_D} |c_{\bar{n}}|^k \right)^\alpha
\]

\[
\leq \sum_{k=0}^{\infty} B^\alpha \sum_{\bar{n} \in \Omega_D} |c_{\bar{n}}|^{k\alpha}
\]

\[
= B^\alpha \sum_{\bar{n} \in \Omega_D} \frac{1}{1 - |c_{\bar{n}}|^\alpha} < \infty.
\]

We used the inequality $(x + y)^\alpha \leq x^\alpha + y^\alpha$ ($x, y \geq 0$) in the second line. The inequality above implies that $\sum_{k=0}^{\infty} |m_k|^\alpha$ can take an arbitrarily small value,
if one lets the value of $B$ small. Thus, $\sum_{k=0}^{\infty} |m_k|^\alpha < 1$ with sufficiently small initial condition in $\Omega$ and the statement of theorem 2 holds by lemma 3.1.

When $\alpha > 1$, since $|c_{\vec{n}}| < 1$ ($\vec{n} \in \Omega_0^D$), $|c_{\vec{n}}|^s \to 0$ ($s \to \infty$) for all $\vec{n} \in \Omega_0^D$. Thus, there exists $s_0 \in \mathbb{Z}_{\geq 0}$ such that $\sum_{\vec{n} \in \Omega_0^D} |c_{\vec{n}}|^s < 1$ ($s \geq s_0$). Now we get

$$\sum_{k=0}^{s_0-1} |m_k|^\alpha = \sum_{k=0}^{s_0-1} |m_k|^\alpha + \sum_{k=s_0}^{\infty} |m_k|^\alpha$$

$$\leq \sum_{k=0}^{s_0-1} |m_k|^\alpha + B^\alpha \left( \sum_{\vec{n} \in \Omega_0^D} |c_{\vec{n}}|^k \right)^\alpha$$

$$\leq \sum_{k=0}^{s_0-1} |m_k|^\alpha + B^\alpha \sum_{\vec{n} \in \Omega_0^D} |c_{\vec{n}}|^k$$

$$= \sum_{k=0}^{s_0-1} |m_k|^\alpha + \sum_{\vec{n} \in \Omega_0^D} B^\alpha |c_{\vec{n}}|^s_0 \frac{1}{1-|c_{\vec{n}}|} < \infty.$$ 

$s_0-1 \sum_{k=0}^{s_0-1} |m_k|^\alpha$ can take an arbitrarily small value, if one let the value of $\max_{\vec{n} \in \Omega_0^D} a_{\vec{n}}$ be small so that the inequality above implies that $\sum_{k=0}^{\infty} |m_k|^\alpha$ can take an arbitrarily small value. (if $B$ is sufficiently small.) Thus, $\sum_{k=0}^{\infty} |m_k|^\alpha < 1$ with sufficiently small initial condition in $\Omega$ and the statement of theorem 2 holds by lemma 3.1. This completes the proof of the main theorem.

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