Duality family of scalar field

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Abstract: We show that there exists a duality family of self-interacting massive scalar fields. The scalar field in a duality family are related by a duality transformation. Such a duality of scalar fields is a field version of the Newton-Hooke duality in classical mechanics. The duality transformation preserves the type of the field equation: transforming a Klein-Gordon type equation to another Klein-Gordon type equation with a different self-interacting potential. Once a field in a duality family is solved, all other family members are solved by the transformation. That is, a series of exactly solvable models can be constructed from one exactly solvable model. The dual field of the power-interaction field, the sine-Gordon field, etc., are considered. Moreover, as a comparison, we show an analogue of the duality in classical and quantum mechanics.

Keywords: Duality family; Exact solution of field equation.

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1. Introduction

The scalar field with the self-interaction potential $V(\phi)$ determined by the Klein-Gordon type equation

$$\Box \phi + m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} = 0 \quad (1)$$

is an important field-theory model, such as the $\phi^n$-field and the sine-Gordon field. The field equation (1) is usually a nonlinear equation. The solution of such field equations, e.g., the soliton solution, is important in studying the nonperturbation aspect of fields.

In this paper, we show that there exists a duality family of self-interaction scalar fields. The family members are connected by a duality transform. The duality transformation transforms a Klein-Gordon type equation to another Klein-Gordon type equation, while, for comparison, a Klein-Gordon type equation after an arbitrary transformation is usually no longer a Klein-Gordon type equation. Two dual fields are related by a duality transformation. Once a field equation is solved, the solution of its dual field can be obtained by the dual transformation immediately. A field has not only one dual fields. All fields who are dual to each other form a duality family. Every field belongs to a certain duality family. A duality family consists of an infinite number of family members. Different family members may have different masses and different self-interaction potentials. So long as one field in the duality family is solved, all other fields in the duality family are solved by the duality transformation. For example, starting with a solution of the sine-Gordon equation, we can solve all the fields in the sine-Gordon-field duality family. The duality of the power-interaction field, the sine-Gordon field, the sinh-Gordon field, etc., are considered as examples.

In order to help understand the duality of scalar fields, we can refer to its analogue in classical mechanics. In classical mechanics, Newton discovered a duality in his *Principia* (Corollary III of Proposition VII) [1]. Newton considered such a problem: for a power law of centripetal attraction, does there exist a dual law for which a body with the same constant of area describes the same orbit [1]. Newton’s result is today known as the Newton-Hooke duality, for it is a duality between the Newtonian gravitational potential ($1/r$-potential) and the Hookian harmonic-oscillator potential ($r^2$-potential). Chandrasekhar reexpressed the Newton-Hooke duality in a more modern language [1]. E. Kasner and V. I. Arnol’d independently generalized the Newton-Hooke duality to general power potentials: two power potentials $U(r) = k r^a$ and $V(r) = m r^A$ are dual, if $\frac{a+2}{a} = \frac{A+2}{A}$ [2–4]. T. Needham intuitively explains the Newton-Hooke duality and its generalization, the Kasner-Arnol’d theorem, in Refs. [4,5]. R. W. Hall and K. Josic reviewed the power-potential duality in Ref. [6]. In appendix B, we generalize the classical mechanical general power-potential duality to arbitrary potentials in classical mechanics and in quantum mechanics in arbitrary dimensions.

Various dualities reveal underlying connections among different physical problems. The gauge/gravity duality is a profound relation [7–9] and has been applied in many problems [10–17]. The fluid/gravity duality is a duality between spacetime manifolds and fluids [18–30]. The gravoelectric duality is useful to seek the solution of the Einstein equation [31–35]. The strong–weak duality bridges a strongly coupled theory to an equivalent weak coupling theory: the duality between fermions and strongly-interacting bosonic Chern-Simons-matter theories [36], the electric-magnetic duality [37–39], the duality in the couple of gauge fields to gravities [40], the duality in higher spin gauge fields [41], the duality in quantum many-body systems [42], the duality in string theory [43–46]. In condensed matter physics, there are also dualities, such as the duality between the Ising and the Heisenberg models and the gauge theory [47]. Moreover, the theory of the quantum sine-Gordon equation is equivalent to the theory of massive Thirring model when the parameter satisfying certain conditions [48,49]; such a duality exists in the lattice sine-Gordon model [50]. By this duality, the
soliton solution of the non-linear Schrödinger model can be studied by the Thirring model fermion in the non-relativistic limit [51].

In section 2, we present the duality relation. In section 3, we show that the duality relation can serve as a method of solving field equations. In sections 4, 5, and 6, as examples, we discuss the duality of the power-interaction field, the sine-Gordon field, etc. In appendix A, we sketch a similar duality in classical mechanics and in quantum mechanics as a comparison. In appendix B, we calculate a traveling wave solution of scalar field equation.

2. The duality transformation and the duality family

In the section, we show that there exists a duality transformation between self-interacting massive scalar fields. All fields connected by the duality transformation form a family.

\[ \Box \phi + m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} = 0, \]  
\[ \Box \varphi + M^2 \varphi + \frac{\partial U(\varphi)}{\partial \varphi} = 0 \]

with \( m \) and \( M \) the masses, if the potentials \( V(\phi) \) and \( U(\varphi) \) are related by

\[ \frac{1}{m^2} \phi^{-2} [G - V(\phi)] = \frac{1}{M^2} \varphi^{-2} [G - U(\varphi)], \]
\[ \phi \leftrightarrow \varphi^\sigma, \]

where

\[ G = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + V(\phi), \]
\[ \mathcal{G} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} M^2 \varphi^2 + U(\varphi), \]

the fields \( \phi(x) \) and \( \varphi(y) \) are related by the duality transformation:

\[ \phi \leftrightarrow \varphi^\sigma, \]
\[ x^\mu \leftrightarrow \frac{M}{m} \sigma y^\mu, \quad \mu = 0, 1, \ldots \]

The constant \( \sigma \) can be chosen arbitrarily.

The above duality relation can be verified directly.

The duality relation (4) is an explicit expression of the dual potential. Given a potential \( V(\phi) \), the dual potential by Eq. (4) reads

\[ U(\varphi) = \frac{M^2}{m^2} \phi^{-2} [V(\phi) - G(x)] \bigg|_{\phi = \varphi^\sigma} \varphi^2 + \mathcal{G}(y). \]

Note that \( G \) defined by Eq. (6) does not explicitly depend on \( \phi \), i.e.,

\[ \frac{\partial G}{\partial \phi} = 0. \]
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This can be verified as follows. Without loss of generality, we consider the 1 + 1-dimensional case. In 1 + 1 dimensions, Eq. (6) becomes

$$G = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{2} m^2 \phi^2 + V(\phi)$$  \hspace{1cm} (12)

and the field equation (2) is

$$\frac{\partial^2 \phi}{\partial t^2} \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} = 0.$$  \hspace{1cm} (13)

By Eq. (12) we have

$$\frac{\partial G}{\partial \phi} = \frac{\partial \phi}{\partial t} \frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial \phi} \right) - \frac{\partial \phi}{\partial x} \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial \phi} \right) + \frac{\partial}{\partial \phi} \left( m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} \right)$$

$$= \frac{\partial \phi}{\partial t} \left( \frac{\partial t}{\partial \phi} \frac{\partial \phi}{\partial t} \right) + \frac{\partial \phi}{\partial x} \left( \frac{\partial x}{\partial \phi} \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial \phi} \left( m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} \right)$$

$$= \frac{\partial^2 \phi}{\partial t^2} + m^2 \phi + \frac{\partial^2 V(\phi)}{\partial \phi},$$  \hspace{1cm} (14)

where $\frac{\partial \phi}{\partial t} = \frac{\partial (\phi, t)}{\partial (t, x)}$, $\frac{\partial \phi}{\partial x} = \frac{\partial (\phi, t)}{\partial (x, t)}$, $\frac{\partial t}{\partial \phi} = \frac{\partial (t, x)}{\partial (\phi, t)}$, and $\frac{\partial t}{\partial x} = \frac{\partial (t, x)}{\partial (\phi, x)}$ with $\frac{\partial (u, v)}{\partial (x, y)} = \left( \begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array} \right)$ the Jacobian determinant are used. Comparing Eq. (14) with the field equation (13) gives Eq. (11).

Generally speaking, $G$ is a function of $t$ and $x$, i.e., $G = G(t, x)$. We show in appendix B that for traveling wave solutions $G = c$, a constant, while for nontraveling wave solutions $G = G(t, x)$.

Comparing the field equations (2) and (3) shows that the duality transformation preserves the type of the field equation: a Klein-Gordon type equation is still a Klein-Gordon type equation after the duality transformation.

Duality family. In the duality relation, the constant $\sigma$ can be chosen arbitrarily and different choices give different dual fields, so a field has an infinite number of dual fields. All the fields who are dual to each other form a duality family. In a duality family, as long as one equation is solved, the solutions of other family members can be obtained directly by the duality transformation, just like that in classical mechanics the solution of the Newtonian gravitational potential can be obtained from the solution of the harmonic-oscillator potential by the Newton-Hooke duality transformation.

The duality transformation with $\sigma = 1$ is a special duality transformation which only changes the coupling constant. To see this, we extract the coupling constant $\lambda$ out of the potential and rewrite the potential $V(\phi)$ as $\lambda V(\phi)$. The dual field of $\lambda V(\phi)$ is $\frac{m^2}{\lambda^2} V(\phi)$ (choosing $\frac{m^2}{\lambda^2} G = \phi$). That is, the coupling constant $\lambda$ is transformed to $\frac{m^2}{\lambda^2}$. This allows us to transform a large-mass strongly interacting field to a small-mass weakly interacting field which can be dealt with perturbatively.

3. Solving field equations by the duality

In the above, we show that if two fields satisfy the duality relation (4), the solutions of the field equations (2) and (3) are related by the duality transformations (8) and (9). This provides an approach for solving field equations: starting from a solved field, we can obtain the solution of its dual field by the duality transformation.
First we show that the duality relation transforms the solution of a field equation to the solution of the dual field equation. The field equation (2) has an implicit solution:

$$\beta_\mu x^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2\left(\frac{1}{2}m^2 \phi^2 + V(\phi) - G\right)}} d\phi = 0 \quad (15)$$

with $\beta^2 = \beta_\mu \beta^\mu$ a constant. This is a traveling wave solution of the scalar field equation, see Appendix B. Note that for traveling wave solutions $G$ is a constant.

Substituting the duality relations (8) and (9) into the solution (15),

$$\beta_\mu y^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2\left(\frac{1}{2}M^2 \phi^2 + \frac{M^2}{m^2} \phi^{2(1-\sigma)} |V(\phi') - G|\right)}} d\phi = 0, \quad (16)$$

we arrive at an implicit solution of a field equation with the potential $U(\phi) = \frac{M^2}{m^2} \phi^{2(1-\sigma)} |V(\phi') - G| + \gamma$:

$$\beta_\mu y^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2\left(\frac{1}{2}M^2 \phi^2 + U(\phi) - \gamma\right)}} d\phi = 0. \quad (17)$$

Next we show that the solutions of a field equation and its dual field equation are related by the duality relation. Eq. (4) gives

$$G = V(\phi) - \frac{m^2 \phi^2}{M^2 \phi^2} \left[U(\phi) - \gamma\right]. \quad (18)$$

Rewriting the solution of the potential $V(\phi)$, Eq. (15), as

$$\beta_\mu \frac{d x^\mu}{d\phi} + \frac{\sqrt{-\beta^2}}{\sqrt{2\left(\frac{1}{2}m^2 \phi^2 + V(\phi) - G\right)}} = 0 \quad (19)$$

and substituting Eq. (18) into Eq. (19) give

$$\beta_\mu \frac{m \phi}{M \phi} \frac{d x^\mu}{d\phi} + \frac{\sqrt{-\beta^2}}{\sqrt{2\left(\frac{1}{2}M^2 \phi^2 + U(\phi) - \gamma\right)}} = 0 \quad (20)$$

Eq. (20) should be a solution of the field equation with the potential $U(\phi)$, i.e., it must be of the form

$$\beta_\mu \frac{d y^\mu}{d\phi} + \frac{\sqrt{-\beta^2}}{\sqrt{2\left(\frac{1}{2}M^2 \phi^2 + U(\phi) - \gamma\right)}} = 0. \quad (21)$$

Comparing Eqs. (20) and (21) gives

$$\frac{m \phi}{M \phi} \frac{d x^\mu}{d\phi} = \frac{d y^\mu}{d\phi}. \quad (22)$$

We have

$$\frac{m}{M} \frac{d x^\mu}{d\phi} = \frac{d \ln \phi}{d \ln \varphi} = \sigma, \quad (23)$$

where $\sigma$ is an arbitrary constant. Solving Eq. (23) gives the duality transformations (8) and (9).
4. Power interactions

The duality of a power interaction $V(\phi) = \lambda \phi^a$, generally, is no longer a power interaction. If requiring that duality of a power interaction is still a power interaction, we arrive at the following conclusion.

The dual potential of the power potential

$$V(\phi) = \lambda \phi^a,$$

by Eq. (4), is

$$U(\varphi) = \frac{M^2}{m^2} \lambda \phi^{2 + (a-2)\sigma} - \frac{M^2}{m^2} G \phi^{2(1-\sigma)} + \mathcal{G}.$$  

If requiring the dual potential is still a power potential, i.e.,

$$U(\varphi) = \eta \varphi^A,$$

there are two choices. One choice is $2 + (a - 2)\sigma = 0$ and $A = 2(1 - \sigma)$, i.e.,

$$\sigma = \frac{2}{2-a}, \quad \frac{2}{2-a} = \frac{2}{2} - \frac{A}{2}.$$  

This gives

$$U(\varphi) = \frac{M^2}{m^2} G \phi^{\frac{2a}{2-a}}.$$  

Another choice is $A = 2 + (a-2)\sigma$ and $2(1 - \sigma) = 0$, i.e.,

$$\sigma = 1, \quad A = a.$$  

This gives

$$U(\varphi) = \frac{M^2}{m^2} \lambda \phi^a.$$  

In this case, the power of the potential does not change after the duality transformation, but the coupling constant becomes $\frac{M^2}{m^2} \lambda$. That is, the duality transformation of $\sigma = 1$ transforms a field of mass $m$ and coupling constant $\lambda$ into a field of mass $M$ and coupling constant $\frac{M^2}{m^2} \lambda$. When $M \ll m$, the coupling constant of the dual field $\frac{M^2}{m^2} \lambda \ll \lambda$. In this case, the duality connects a large-mass and strong-coupling field to a small-mass and weak-coupling field.

Especially, when $M = m$ the duality transformation with $\sigma = 1$ is an identity transformation.

4.1. The $\phi^4$-field: self-duality

The $\phi^4$-field is self-dual. By self-dual we mean that the duality of a $\phi^4$-field is also a $\phi^4$-field. That is, the fields

$$V(\phi) = \lambda \phi^4 \quad \text{and} \quad U(\varphi) = \eta \varphi^4$$

are dual. The duality transformations are

$$\phi \to \varphi^{-1}, \quad x^\mu \to -\frac{M}{m} y^\mu, \quad \frac{m^2}{M^2} \eta \to -G.$$
It can be checked that the field equation with \( V(\phi) = \lambda \phi^4 \),
\[
\Box \phi + m^2 \phi + 4 \lambda \phi^3 = 0,
\]
has a soliton solution
\[
\phi = \frac{m}{2\sqrt{\lambda}} \tan \left( \alpha t + \beta x_1 + \gamma x_2 - x_3 \sqrt{\alpha^2 - \beta^2 - \gamma^2 + \frac{m^2}{2} + \delta} \right). \tag{34}
\]
The solution (34) gives \( G = \frac{1}{2} \partial \mu \phi \partial ^\mu \phi + \frac{m^2}{2} \phi^2 + \lambda \phi^4 = - \frac{m^4}{16\lambda} \). The Lorentz invariance of the solution can be verified by directly performing the Lorentz transformation.

By the duality relation, the solution of the field equation with the potential \( U(\phi) = \eta \phi^4 \),
\[
\Box \phi + \frac{M^2}{2} \phi + 4 \lambda \phi^3 = 0, \tag{35}
\]
is
\[
\phi = \frac{M}{2\sqrt{\eta}} \tan \left( \alpha' \tau + \beta' y_1 + \gamma' y_2 - y_3 \sqrt{\alpha'^2 - \beta'^2 - \gamma'^2 + \frac{M^2}{2} + \delta'} \right), \tag{36}
\]
where \( \alpha' = \alpha \frac{M}{m}, \beta' = \beta \frac{M}{m}, \gamma' = \gamma \frac{M}{m} \), and \( \delta' = \delta + \frac{\pi}{2} \). After the duality transformation, the mass \( m \) is transformed to \( M \).

The \( \phi^4 \)-field is self-dual, i.e., the duality of a \( \phi^4 \)-field is still a \( \phi^4 \)-field. It should be emphasized that even in the case that the masses of the field and its dual field are the same, the duality transformation (\( \sigma = -1 \)) is not an identity transformation (\( \sigma = 1 \)).

Now we consider the duality between a weak coupling field and a strong coupling field, taking the \( \phi^4 \)-field as an example.

The dual field of a \( \phi^4 \)-field with mass \( m \) and coupling constant \( \lambda \), by the duality transformation (32), is a \( \phi^4 \)-field with mass \( M \) and coupling constant \( \eta = -\frac{M^2}{m^2} G = \frac{m^4 M^2}{16\lambda} \). If the mass of the dual field \( \phi \) is chosen as \( M = \frac{1}{m} \), then the coupling constant becomes
\[
\eta = \frac{1}{16\lambda}. \tag{37}
\]
The field equation of the dual field is
\[
\Box \phi + \frac{1}{m^2} \phi^3 = 0. \tag{38}
\]

This means that the dual field of a \( \phi^4 \)-field with mass \( m \) and coupling constant \( \lambda \) is a \( \phi^4 \)-field with mass \( \frac{1}{m} \) and coupling constant \( \frac{1}{m^2} \). When the coupling constant \( \lambda \) of the field \( \phi \) is large (strongly interacting), the coupling constant \( \frac{1}{m^2} \) of its dual field \( \phi \) will be small (weakly interacting). That is, the duality in this case is a strong-weak coupling duality. In principle, this strong-weak coupling duality also exists in more general cases, not limited to \( \phi^4 \)-field. Especially, if \( m \), the mass of the field \( \phi \), is small, then \( \frac{1}{m} \), the mass of the dual field \( \phi \), is large. This implies that one can construct an effective theory for the field \( \phi \) with a heavy mass propagator, like that in the four-fermion effective theory.

### 4.2. The \( \phi^1 \)-field and the \( \phi^{-2} \)-field

The dual field of the \( \phi^1 \)-field is the \( \phi^{-2} \)-field, i.e., the fields
\[
V(\phi) = \lambda \phi \quad \text{and} \quad U(\phi) = \eta \phi^{-2} \tag{39}
\]
are dual to each other. The duality transformations are

\[ \phi \rightarrow \phi^2, \; x^\mu \rightarrow -2 \frac{M}{m} y^\mu, \; \frac{m^2}{M^2} \eta \rightarrow -G. \]  

(40)

It can be checked that the field equation with \( V(\phi) = \lambda \phi \) has a 1 + 3-dimensional traveling wave solution

\[ \phi = \exp \left( \alpha t + \beta x_1 + \gamma x_2 + \sqrt{m^2 + \alpha^2 - \beta^2 - \gamma^2} x_3 \right) - \frac{\lambda}{m^2}. \]  

(41)

The solution (41) gives \( G = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \lambda \phi = -\frac{\lambda^2}{2m^2} \). For the traveling wave solution, \( G \) is a constant.

By the duality transformation, the solution of the \( \phi^2 \)-field is

\[ \phi = \left[ \exp \left( 2 \left( \alpha' \tau + \beta' y_1 + \gamma' y_2 + \sqrt{M^2 + \alpha'^2 - \beta'^2 - \gamma'^2} y_3 \right) \right) - \frac{\sqrt{2\eta M}}{M} \right]^{1/2}, \]  

(42)

where \( \alpha' = \frac{\alpha M}{m}, \beta' = \frac{\beta M}{m}, \) and \( \gamma' = \frac{\gamma M}{m} \).

Moreover, \( \phi^1 \)-field has a 1 + 1-dimensional nontraveling wave solution

\[ \phi = e^{\alpha t} \sinh \left( \sqrt{\alpha^2 + m^2} x \right) - \frac{\lambda}{m^2}. \]  

(43)

For this nontraveling wave solution, \( G = -\frac{\lambda^2}{2m^2} - \frac{a^2 + m^2}{2} e^{2\alpha t} \) is not a constant but depends on \( t \).

This gives \( \eta = -\frac{M^2}{m^2} G = \frac{3^2 M^2}{2m^2} + \frac{a^2 + m^2}{2m^2} M^2 e^{2\alpha t} \). By the duality transformation, the corresponding solution of the \( \phi^{-2} \)-field is

\[ \phi = \left[ e^{2\frac{M}{m} \alpha t} \sinh \left( \frac{2M}{m} \sqrt{\alpha^2 + M^2} y \right) - \frac{\lambda}{m^2} \right]^{1/2}. \]  

(44)

This is just the solution of the field equation (3) with the potential

\[ U(\varphi) = \left( \frac{\lambda^2 M^2}{2m^3} + \frac{a^2 + m^2}{2m^3} M^2 e^{4\alpha \tau} \right) \varphi^{-2}. \]  

(45)

This potential is a \( \phi^{-2} \)-potential with a time-dependent coefficient.

In a word, the function \( G \) for traveling wave solutions is a constant, and for nontraveling wave solution is not a constant (see Appendix B).

**4.3. The \( \phi^3 \)-field and the \( \phi^6 \)-field**

The dual field of a \( \phi^3 \)-field is a \( \phi^6 \)-field, i.e., the fields

\[ V(\phi) = \lambda \phi^3 \; \text{and} \; U(\varphi) = \eta \varphi^6 \]  

(46)

are dual to each other. The duality transformations are

\[ \phi \rightarrow \varphi^{-2}, \; x^\mu \rightarrow -\frac{M}{m} y^\mu, \; \frac{m^2}{M^2} \eta \rightarrow -G. \]  

(47)

It can be checked that the field equation with \( V(\phi) = \lambda \phi^3 \) has a solution

\[ \phi = -\frac{m^2}{6\lambda} \left[ 3 \tanh^2 \left( \alpha x_1 + \beta x_2 + \gamma x_3 + \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \left( \frac{m}{\sqrt{2}} \right)^2} t \right) - 1 \right]. \]  

(48)
The solution (48) gives \( G = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \lambda \phi^3 = \frac{m^6}{5443}. \)

By the duality transformation, the solution of the \( \phi^6 \)-field is

\[
\varphi = \left\{ \frac{\sqrt{-6\eta}}{2M} \left[ 3 \tanh^2 \left( 2 \left( \alpha' y_1 + \beta' y_2 + \gamma' y_3 + \sqrt{\alpha'^2 + \beta'^2 + \gamma'^2} \left( \frac{M}{2} \right)^2 \right) - 1 \right) \right] \right\}^{-1/2},
\]

where \( \alpha' = \frac{a M}{m}, \beta' = \frac{b M}{m}, \) and \( \gamma' = \gamma \frac{M}{m}. \)

5. The sine-Gordon equation, the sinh-Gordon equation, and all that

The sine-Gordon equation, sinh-Gordon equation, and all field equations of this type can be compactly written as

\[
\Box \phi - a e^{\beta \phi} + b e^{-\beta \phi} = 0
\]

which recovers the sine-Gordon equation when \( a = b = -\frac{1}{2} \frac{m^3}{\sqrt{\lambda}} \) and \( \beta = i \frac{\sqrt{\lambda}}{m}, \) recovers the sinh-Gordon equation when \( a = b = -\frac{1}{2} \) and \( \beta = 1, \) and so on.

The potential corresponding to the field equation (50) is

\[
V(\phi) = -a \beta e^{\beta \phi} + b \beta e^{-\beta \phi}.
\]

The field equation (50) has the following solution (52):

\[
\phi(t, x) = \frac{4}{\beta} \arctanh \left( \exp \left( \frac{2\beta \sqrt{ab}}{\mu^2 - v^2} (\mu t + \nu x + \theta) \right) \right) + \frac{1}{2} \ln \frac{b}{a}.
\]

For the potential (51), by Eqs. (6) and (52), we have \( G = -2\sqrt{ab}/\beta. \)

The dual potential of the potential (51) can be obtained by the duality relations (4). For massless fields, \( m^2/M^2 = 1. \) The function \( \mathcal{G}(y) \) can be chosen arbitrarily, because the choice of \( \mathcal{G}(y) \) does not influence the field equation. Choosing \( \mathcal{G}(y) = 0 \) gives

\[
U(\varphi) = \frac{1}{\beta} \varphi^{2(1-\sigma)} \left[ 2\sqrt{ab} - \left( a e^{\beta \varphi} + b e^{-\beta \varphi} \right) \right].
\]

Different choices of the constant \( \sigma \) in Eq. (53) give different dual potentials.

The dual field equation by the duality transformations (8) and (9) is

\[
\Box \varphi - \frac{2(1-\sigma)}{\beta} \varphi^{1-2\sigma} \left( a e^{\beta \varphi} + b e^{-\beta \varphi} - 2\sqrt{ab} \right) - \sigma \varphi^{1-\sigma} \left( a e^{\beta \varphi} - b e^{-\beta \varphi} \right) = 0,
\]

and the solution of the dual field,

\[
\varphi(\tau, y) = \left[ \frac{4}{\beta} \arctanh \left( \exp \left( \frac{2\beta \sqrt{ab}}{\mu^2 - v^2} (\mu \sigma \tau + v \sigma y + \theta) \right) \right) + \frac{1}{2} \ln \frac{b}{a} \right]^{1/\sigma}.
\]

Note that different solutions lead to different \( G \) and then leads to different coefficients in the dual potential.
5.1. The sine-Gordon equation

The field equation (50) recovers the sine-Gordon equation when \( a = b = -\frac{1}{2} \frac{m^3}{\sqrt{\lambda}} \) and \( \beta = i \frac{\sqrt{\lambda}}{m} \).

\[
\Box \phi + \frac{m^3}{\sqrt{\lambda}} \sin \frac{\sqrt{\lambda}}{m} \phi = 0.
\] (56)

The potential is

\[
V(\phi) = 2 \frac{m^4}{\lambda} \sin^2 \left( \frac{\sqrt{\lambda} \phi}{2m} \right).
\] (57)

The solution of the sine-Gordon equation by Eq. (52) is

\[
\phi(t,x) = \frac{4m}{\sqrt{\lambda}} \arctan \left( \exp \left( \frac{m}{\sqrt{\nu^2 - \mu^2}} (\mu t + \nu x + \theta) \right) \right).
\] (58)

For the sine-Gordon field \( G = -2\sqrt{ab}/\beta = -m^4/\lambda \). The dual potential of the sine-Gordon potential then is

\[
U(\varphi) = 2 \frac{m^4}{\lambda} \varphi^{2(1-\sigma)} \sin^2 \left( \frac{\sqrt{\lambda} \varphi^{\sigma}}{2m} \right).
\] (59)

The dual equation is

\[
\Box \varphi + 4(1-\sigma) \frac{m^4}{\lambda} \varphi^{1-2\sigma} \sin^2 \left( \frac{\sqrt{\lambda} \varphi^{\sigma}}{2m} \right) + \sigma \frac{m^3}{\sqrt{\lambda}} \varphi^{1-\sigma} \sin \left( \frac{\sqrt{\lambda} \varphi^{\sigma}}{m} \right) = 0.
\] (60)

The solution of the dual field equation (60) is

\[
\varphi(\tau,y) = \left[ \frac{4m}{\sqrt{\lambda}} \arctan \left( \exp \left( \frac{m}{\sqrt{\nu^2 - \mu^2}} (\sigma \mu \tau + \sigma \nu y + \theta) \right) \right) \right]^{1/\sigma}.
\] (61)

Moreover, the parameter \( m \) in the sine-Gordon equation (56) can be explained as the mass of the field, which can be seen by expanding Eq. (56). If regarding the parameter \( m \) as the mass, the potential then is

\[
V(\phi) = 2 \frac{m^4}{\lambda} \sin^2 \left( \frac{\sqrt{\lambda} \phi}{2m} \right) - \frac{1}{2} m^2 \phi^2.
\] (62)

The dual potential of the sine-Gordon potential then is

\[
U(\varphi) = 2 \frac{M^4}{\lambda} \varphi^{2(1-\sigma)} \sin^2 \left( \frac{\sqrt{\lambda} \varphi^{\sigma}}{2M} \right) - \frac{1}{2} M^2 \varphi^2.
\] (63)

The dual equation is

\[
\Box \varphi + 4(1-\sigma) \frac{M^4}{\lambda} \varphi^{1-2\sigma} \sin^2 \left( \frac{\sqrt{\lambda} \varphi^{\sigma}}{2M} \right) + \sigma \frac{M^3}{\sqrt{\lambda}} \varphi^{1-\sigma} \sin \left( \frac{\sqrt{\lambda} \varphi^{\sigma}}{M} \right) = 0.
\] (64)

The solution of the dual field equation (64) is

\[
\varphi(\tau,y) = \left[ \frac{4M}{\sqrt{\lambda}} \arctan \left( \exp \left( \frac{M}{\sqrt{\nu^2 - \mu^2}} (\sigma \mu \tau + \sigma \nu y + \theta) \right) \right) \right]^{1/\sigma}.
\] (65)
5.2. The sinh-Gordon equation

The field equation (50) recovers the sinh-Gordon equation when \( a = b = -\frac{1}{2} \) and \( \beta = 1 \):

\[
\Box \phi + \sinh \phi = 0. \tag{66}
\]

The potential is

\[
V(\phi) = 2 \sinh^2 \left( \frac{\phi}{2} \right). \tag{67}
\]

The solution of the sinh-Gordon equation by Eq. (52) is

\[
\phi(t, x) = 2 \ln \left( \coth \frac{\mu t + \nu x + \theta}{2 \sqrt{\nu^2 - \mu^2}} \right). \tag{68}
\]

For the sinh-Gordon field, by Eq. (53), we have \( G = 1 \). The dual potential of the sinh-Gordon potential then is

\[
U(\phi) = 2 \phi^{2(1-\sigma)} \sinh^2 \left( \frac{\phi^\sigma}{2} \right). \tag{69}
\]

The dual equation is

\[
\Box \phi + 4(1 - \sigma) \phi^{1-2\sigma} \sinh^2 \left( \frac{\phi^\sigma}{2} \right) + \sigma \phi^{1-\sigma} \sinh(\phi^\sigma) = 0. \tag{70}
\]

The solution of the dual field equation (70) is

\[
\phi(\tau, y) = \left[ 2 \ln \left( \coth \frac{\sigma \mu \tau + \sigma \nu y + \theta}{2 \sqrt{\nu^2 - \mu^2}} \right) \right]^{1/\sigma}. \tag{71}
\]

6. \( \Box \phi - a e^{\beta \phi} - b e^{2\beta \phi} = 0 \)

Consider the scalar field equation

\[
\Box \phi - a e^{\beta \phi} - b e^{2\beta \phi} = 0. \tag{72}
\]

The potential is

\[
V(\phi) = -\frac{a}{\beta} e^{\beta \phi} - \frac{b}{2\beta} e^{2\beta \phi}. \tag{73}
\]

It can be checked that the field equation (72) has the solution:

\[
\phi(t, x) = -\frac{1}{\beta} \ln \left( \frac{\alpha}{\mu^2 - \nu^2} + \frac{1 + \sqrt{1 + \frac{b}{a^2 \beta} (\mu^2 - \nu^2) \sin(\mu t + \nu x + \theta)}}{1 + \sqrt{1 + \frac{b}{a^2 \beta} (\mu^2 - \nu^2) \sin(\mu t + \nu x + \theta)}} \right). \tag{74}
\]

For the field \( \phi \) with the potential (73), we have \( G = (\nu^2 - \mu^2)/(2\beta^2) \). The dual potential then by Eq. (4) is

\[
U(\phi) = \phi^{2(1-\sigma)} \left( -\frac{a}{\beta} e^{\beta \phi^\sigma} - \frac{b}{2\beta} e^{2\beta \phi^\sigma} + \frac{\mu^2 - \nu^2}{2\beta^2} \right). \tag{75}
\]

Here \( M^2/m^2 = 1 \) for the field is massless and we choose \( \mathcal{G} = 0 \). By the duality transformations (8) and (9), the dual field equation is

\[
\Box \phi - 2(1 - \sigma) \phi^{1-2\sigma} \left( \frac{a}{\beta} e^{\beta \phi^\sigma} + \frac{b}{2\beta} e^{2\beta \phi^\sigma} - \frac{\mu^2 - \nu^2}{2\beta^2} \right) - \sigma \phi^{1-\sigma} e^{\beta \phi^\sigma} (a + b e^{\beta \phi^\sigma}) = 0 \tag{76}
\]
and the solution of the dual field equation is

$$\varphi(\tau, y) = \left\{ -\frac{1}{\beta} \ln \left( \frac{a\beta}{\mu^2 - \nu^2} \left[ 1 + \sqrt{1 + \frac{b}{a^2\beta} (\mu^2 - \nu^2) \sin(\sigma\mu\tau + \sigma\nu y + \theta)} \right] \right) \right\}^{1/\sigma}. \quad (77)$$

For more examples see Ref. [53].

7. Conclusion and outlooks

We show that there exits a duality of scalar fields. The duality transformation preserves the type of the field equation.

A field has an infinite number of dual fields. All dual fields form a duality family. In a duality family, as long as one field is solved, all other fields can be solved by the duality relation. This provides a high-efficiency approach to solve field equations.

The existence of the duality family inspires us to classify fields based on the duality. A duality family is a duality class. In future works, we will discuss the property of the duality family.

The duality relation also relates various qualities of fields, such as heat kernels, effective actions, vacuum energies, spectral counting functions, etc. In further works, we will consider the quantum theory of the duality, such as the duality relation of the Feynman rule. Especially, in quantum field theory we will consider the duality in the heat kernel method [54] and in the scattering spectrum method [55–57]. In these methods we can calculate the one-loop effective action and the vacuum energy [58,59]. We may observe the relation of the one-loop effective action and the vacuum energy of dual fields. A similar duality also appears in the Gross–Pitaevskii equation [60]. Moreover, we will consider the duality of spinor fields and vector fields.

Appendix A  The duality in classical and quantum mechanics

In the above, we discuss the duality of scalar fields. In this appendix, we show that a similar duality also exists in classical and quantum mechanics.

Appendix A.1  The duality in classical mechanics

The equation of motion in classical mechanics is the Newton equation.

*Two equations of motion with the one-dimensional potentials $U(x)$ and $V(\zeta)$*

$$\frac{dt}{dx} = \frac{1}{\sqrt{2[E - U(x)]}}, \quad (A.1)$$

$$\frac{d\tau}{d\xi} = \frac{1}{\sqrt{2[E - V(\zeta)]}}, \quad (A.2)$$

where $E$ and $E$ are energies, if the potentials $U(x)$ and $V(\zeta)$ are related by

$$x^{-2}[E - U(x)] = \xi^{-2}[E - V(\zeta)] \quad (A.3)$$

with

$$x \leftrightarrow \xi^\sigma, \quad (A.4)$$

the solutions of the equations of motion (A.1) and (A.2) are related by the transformation

$$t \leftrightarrow \sigma \tau. \quad (A.5)$$
Here $\sigma$ is a constant chosen arbitrarily.

**Proof.** By Eqs. (A.1) and (A.2) we have

\[
E - U(x) = \frac{1}{2} \left( \frac{dx}{dt} \right)^2, \quad \text{(A.6)}
\]

\[
\varepsilon - V(\zeta) = \frac{1}{2} \left( \frac{d\zeta}{d\tau} \right)^2. \quad \text{(A.7)}
\]

Substituting into Eq. (A.3) gives

\[
x^{-2} \left[ \frac{1}{2} \left( \frac{dx}{dt} \right)^2 \right] = \zeta^{-2} \left[ \frac{1}{2} \left( \frac{d\zeta}{d\tau} \right)^2 \right]. \quad \text{(A.8)}
\]

This gives

\[
\frac{dt}{d\tau} = \frac{d\ln x}{d\ln \zeta}. \quad \text{(A.9)}
\]

Because $t$ and $\tau$ are independent of $x$ and $\zeta$, we have

\[
\frac{dt}{d\tau} = \frac{d\ln x}{d\ln \zeta} = \sigma, \quad \text{(A.10)}
\]

where $\sigma$ is an arbitrary constant. Solving Eq. (A.10) gives the duality transformations (A.4) and (A.5).

**Two orbit equations with the three-dimensional central potentials $U(r)$ and $V(\rho)$,**

\[
\frac{d\theta}{dr} = \frac{l/r^2}{\sqrt{2[E - l^2/(2r^2)] - U(r)}}, \quad \text{(A.11)}
\]

\[
\frac{d\phi}{d\rho} = \frac{\ell/\rho^2}{\sqrt{2[\varepsilon - \ell^2/(2\rho^2)] - V(\rho)}}, \quad \text{(A.12)}
\]

where $E$ and $\varepsilon$ are energies and $l$ and $\ell$ are angular momenta, if the potentials $U(r)$ and $V(\rho)$ are related by

\[
\frac{r^2}{l^2} [E - U(r)] = \frac{\rho^2}{\ell^2} [\varepsilon - V(\rho)], \quad \text{(A.13)}
\]

with

\[
r \rightarrow \rho^{1/2}, \quad \text{(A.14)}
\]

the solutions of the equations of motion (A.11) and (A.12) are related by the transformation

\[
\theta \rightarrow \frac{l}{\ell} \sigma \phi. \quad \text{(A.15)}
\]

Here $\sigma$ is a constant chosen arbitrarily.

**Proof.** By Eqs. (A.11) and (A.12) we have

\[
E - U(r) = \frac{1}{2} \frac{l^2}{r^4} \left( \frac{dr}{d\theta} \right)^2 + \frac{l^2}{2r^2}, \quad \text{(A.16)}
\]

\[
\varepsilon - V(\rho) = \frac{1}{2} \frac{\ell^2}{\rho^4} \left( \frac{d\rho}{d\phi} \right)^2 + \frac{\ell^2}{2\rho^2}. \quad \text{(A.17)}
\]
Substituting into Eq. (A.13) gives
\[ \frac{r^2}{l^2} \left[ \frac{1}{r^2} \left( \frac{dr}{d\theta} \right)^2 + \frac{i^2}{2r^2} \right] = \frac{\rho^2}{\ell^2} \left[ \frac{1}{\rho^2} \left( \frac{d\rho}{d\phi} \right)^2 + \frac{i^2}{2\rho^2} \right]. \] (A.18)

This gives
\[ \left( \frac{d \ln r}{d \theta} \right)^2 = \left( \frac{d \ln \rho}{d \phi} \right)^2, \] (A.19)

or,
\[ \frac{d \theta}{d \phi} = \frac{d \ln r}{d \ln \rho}. \] (A.20)

Because \( \theta \) and \( \phi \) are independent of \( r \) and \( \rho \), we have
\[ \frac{d \theta}{d \phi} = \frac{d \ln r}{d \ln \rho} = \frac{A}{\ell}, \] (A.21)

where \( \sigma \) is an arbitrary constant. Solving Eq. (A.21) gives the duality transformations (A.14) and (A.15).

Three-dimensional power potentials. Consider three-dimensional power potentials as an example. The duality of a power potential, generally speaking, is no longer a power potential. However, if requiring that the dual potential is still a power potential, we have the following result. The power potentials \( U(r) = \xi r^a \) and \( V(\rho) = \eta \rho^A \) are dual to each other, if \( \frac{a+2}{2} = \frac{A+2}{2} \).

The orbit of the potential \( U(r) \) with the energy \( E \) and the orbit of its dual potential \( V(\rho) \) with the energy \( \mathcal{E} \) can be obtained from each other by the replacement, \( r \rightarrow \rho^{\frac{a+2}{2}} \) and \( \theta \rightarrow \frac{A}{a+2} \phi \).

The Newton-Hooke duality. An important special case of the duality is the Newton-Hooke duality. The Newton-Hooke duality is a duality between the Newtonian gravitational potential and the harmonic-oscillator potential, which is revealed by Newton in his Principia [1]. The Newtonian gravitational potential, in fact, has an infinite number of dual potentials corresponding to various choices of the parameter \( \sigma \). The Newton-Hooke duality, the duality between \( U(r) = \xi r^a \) and \( V(\rho) = \eta \rho^2 \), corresponds to \( \sigma = 2 \). The energy of the Newtonian gravitational potential system becomes the coupling constant of its dual potential.

Appendix A.2 The duality in quantum mechanics

The equation of motion in quantum mechanics is the Schrödinger equation.

Two one-dimensional stationary Schrödinger equations with potentials \( U(x) \) and \( V(\xi) \),
\[ \frac{d^2 u(x)}{dx^2} + [E - U(x)] u(x) = 0, \] (A.22)
\[ \frac{d^2 v(\xi)}{d\xi^2} + [\mathcal{E} - V(\xi)] v(\xi) = 0, \] (A.23)

where \( E \) and \( \mathcal{E} \) are eigenvalues, if the potentials \( U(x) \) and \( V(\xi) \) are related by
\[ \sigma \left[ x^2 [U(x) - E] + \frac{1}{4} \right] = \frac{1}{\sigma} \left[ \xi^2 [V(\xi) - \mathcal{E}] + \frac{1}{4} \right], \] (A.24)

with
\[ x \rightarrow \xi^\sigma, \] (A.25)
The solutions are related by the duality transformation
\[ u(x) \rightarrow \zeta^{(\alpha-1)/2} v(\xi). \]  
(A.26)

Here \( \alpha \) is a constant chosen arbitrarily.

The duality of the Poschl-Teller potential. For the Poschl-Teller potential \( U(x) = a \operatorname{sech}^2 x \), the stationary Schrödinger equation has the solution, \( u(x) = P_l^{\sqrt{E}/(\sqrt{1-4\alpha} - 1/2)} \) (tanh \( x \)) with \( P_l^m(z) \) the associated Legendre polynomial. The dual potential of the Poschl-Teller potential is \( V(\xi) = \frac{1}{4}(\sigma^2 - 1) \xi^2 + \sigma^2 \xi^{2(\alpha-1)}(a \operatorname{sech}^2 \xi^\alpha - \mathcal{E}) + \mathcal{E} \) and its solution \( v(\xi) = \xi^{(\alpha-1)/2} P_l^{\sqrt{E}/(\sqrt{1-4\alpha} - 1/2)} \) (tanh \( \xi^\alpha \)). Different choices of \( \alpha \) give different dual potentials. The constant \( \mathcal{E} \) in the dual potential \( V(\xi) \) can also be chosen arbitrarily, since it is a constant added in the potential.

For the radial equation of \( n \)-dimensional central potential \( U(r) \) and the radial equation of \( m \)-dimensional central potential \( V(\rho) \),
\[
\frac{d^2 u_l(r)}{dr^2} + \left[ E - \frac{\frac{\ell - \frac{3}{2} + \frac{m}{2}}{2} \frac{\frac{\ell - \frac{3}{2} + \frac{m}{2}}{2}}{r^2} - U(r) \right] u_l(r) = 0, \quad (A.27)
\]
\[
\frac{d^2 v_\ell(\rho)}{d\rho^2} + \left[ \mathcal{E} - \frac{\frac{\ell - \frac{3}{2} + \frac{m}{2}}{2} \frac{\frac{\ell - \frac{3}{2} + \frac{m}{2}}{2}}{\rho^2} - V(\rho) \right] v_\ell(\rho) = 0, \quad (A.28)
\]

where \( E \) and \( \mathcal{E} \) are eigenvalues, if the potentials \( U(r) \) and \( V(\rho) \) are related by
\[
\frac{r^2}{(l + \frac{n}{2} - 1)^2} [U(r) - E] = \frac{\rho^2}{(\ell + \frac{m}{2} - 1)^2} [V(\rho) - \mathcal{E}] \quad (A.29)
\]
with
\[
r \rightarrow \rho^\alpha, \quad (A.30)
\]
the eigenfunctions \( u_l(r) \) and \( v_\ell(\rho) \) are related by the duality transformation
\[
u_l(r) \rightarrow \rho^{(\alpha-1)/2} v_\ell(\rho). \quad (A.31)
\]

The relation between the angular momenta of the dual systems, then, is
\[
l + \frac{n}{2} - 1 \rightarrow \frac{1}{\alpha}(\ell + \frac{m}{2} - 1). \quad (A.32)
\]

Here \( \alpha \) is a constant chosen arbitrarily.

It should be emphasized that the duality relation of the angular momentum, Eq. (A.32), is a result of the dual transformations (A.30) and (A.31).

Three-dimensional power potentials. A three-dimensional power potential has infinite number of dual potentials. If requiring the dual potential of a power potential is also a power potential, we have the following result.

The power potentials \( U(r) = \xi^\alpha \) and \( V(\rho) = \eta \rho^A \) are dual to each other, if \( \frac{A + 2}{2} = \frac{2}{\alpha + 2} \). The solution are related by the transformation \( r \rightarrow \rho^{2(\alpha+2)} \) and \( u_l(r) \rightarrow \rho^{-a/2(\alpha+2)} v_\ell(\rho) \).

The Newton-Hooke duality. In quantum mechanics there still exists the Newton-Hooke duality, i.e., the duality between the Newtonian gravitational potential and the harmonic-
oscillator potential. The solution of radial equation of the Newtonian gravitational potential \( U(r) = \frac{\xi}{r} \),

\[
u_{\ell}(r) = A e^{-\sqrt{2E}/r}(2\sqrt{-E})^{\ell+\frac{3}{2}} \frac{\xi}{2\sqrt{-E}} r(\ell+\frac{3}{2})F_1\left(\ell+\frac{3}{4}, \frac{\xi}{2\sqrt{-E}}, \ell+\frac{3}{2}, 2\sqrt{-E}r^2\right),
\]

(A.33)

and the solution of radial equation of the harmonic-oscillator potential \( V(\rho) = -4E\rho^2 \),

\[
u_{\ell}(\rho) = A e^{-\sqrt{-E}\rho^2}(2\sqrt{-E})^{\ell+\frac{3}{2}} \frac{\xi}{2\sqrt{-E}} \rho^{\ell+\frac{3}{2}} F_1\left(\ell+\frac{3}{4}, \frac{\xi}{2\sqrt{-E}}, \ell+\frac{3}{2}, 2\sqrt{-E}\rho^2\right),
\]

(A.34)

are related by the transformation \( r \leftrightarrow \rho^2, u_{\ell}(r) \leftrightarrow \rho^{1/2} v_{\ell}(\rho), \) and \( l+\frac{1}{2} \leftrightarrow \frac{1}{2}(\ell+\frac{1}{2}) \).

For more examples see Ref. [61].

Appendix B The traveling wave solution of scalar field equation

In this appendix we derive the solution of the field equation, Eq. (15), in section 3.

**Approach 1.** The traveling wave solution is a solution satisfying

\[
\phi(x^\mu) = \phi(\beta_\mu x^\mu) = \phi(z),
\]

where \( z = \beta_\mu x^\mu \). A familiar special case is the \( 1+1 \)-dimensional case with \( \beta_\mu = (1,-1) \) and in this case \( z = t-x \).

Substituting

\[
\Box \phi = \partial_\mu \partial_\mu \phi = \frac{\partial z}{\partial x_\mu} \frac{d}{dz} (\frac{\partial z}{\partial x^\mu} \frac{d\phi}{dz}) = \beta^2 \frac{d^2\phi}{dz^2}
\]

(B.2)

into the field equation, \( \Box \phi + m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} = 0 \), gives

\[
\beta^2 \frac{d^2\phi}{dz^2} + m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} = 0.
\]

(B.3)

Multiplying both sides by \( \frac{d\phi}{dz} \) and integrating over \( z \) give

\[
\frac{1}{2} \beta^2 \left( \frac{d\phi}{dz} \right)^2 + \frac{1}{2} m^2 \phi^2 + V(\phi) - c = 0,
\]

(B.4)

where \( c \) is the integration constant. By Eq. (B.4) we have

\[
dz = d\phi \sqrt{-\beta^2} \frac{1}{\sqrt{2[\frac{1}{2} m^2 \phi^2 + V(\phi) - c]}}.
\]

(B.5)

Integrating both sides gives

\[
\beta_\mu x^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2[\frac{1}{2} m^2 \phi^2 + V(\phi) - c]}} d\phi = 0;
\]

(B.6)

note that \( \beta_\mu x^\mu = z \).

Rewriting Eq. (B.6) as

\[
\beta_\mu x^\mu = - \int_{\phi_0}^{\phi} \frac{\sqrt{-\beta^2}}{\sqrt{2[\frac{1}{2} m^2 \phi^2 + V(\phi) - c]}} d\phi
\]

(B.7)
and taking the derivative with respect to $x^\mu$ give

\[ \partial_\mu \phi = -\beta_\mu \frac{\sqrt{2 \left[ \frac{1}{2} m^2 \phi^2 + V(\phi) - c \right]}}{\sqrt{-\beta^2}}. \quad (B.8) \]

Substituting Eq. (B.8) into the expression of $G$, Eq. (6), gives

\[ G = c. \quad (B.9) \]

Substituting $c = G$ into the solution (B.6) gives the solution (15), i.e.,

\[ \beta_\mu x^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2 \left( \frac{1}{2} m^2 \phi^2 + V(\phi) - G \right)}} d\phi = 0. \quad (B.10) \]

**Approach 2.** Alternatively, we can also directly verify that Eq. (B.10) is a solution of the field equation.

Rewrite Eq. (B.10) as

\[ F_1(x^\mu, \phi) = 0, \quad (B.11) \]

where

\[ F_1(x^\mu, \phi) \equiv \beta_\mu x^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2 \left( \frac{1}{2} m^2 \phi^2 + V(\phi) - G \right)}} d\phi. \quad (B.12) \]

By the formula for derivative of implicit function, we have

\[ \partial_\mu \phi = -\frac{\partial F_1(x^\mu, \phi)}{\partial \phi} = \frac{\beta_\mu \sqrt{2 \left[ \frac{1}{2} m^2 \phi^2 + V(\phi) - G \right]}}{\sqrt{-\beta^2}}. \quad (B.13) \]

Rewrite Eq. (B.13) as

\[ F_2(\phi, \partial_\mu \phi) = 0, \quad (B.14) \]

where

\[ F_2(\phi, \partial_\mu \phi) = \partial_\mu \phi + \frac{\beta_\mu \sqrt{2 \left[ \frac{1}{2} m^2 \phi^2 + V(\phi) - G \right]}}{\sqrt{-\beta^2}}. \quad (B.15) \]

By the formula for derivative of implicit function, we have

\[ \square \phi = \partial^\mu (\partial_\mu \phi) = \left[ \frac{\partial}{\partial \phi} (\partial_\mu \phi) \right] \partial_\mu \phi = \frac{\partial F_2(\phi, \partial_\mu \phi)}{\partial \phi} \partial_\mu \phi = \left( m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} \right), \quad (B.16) \]

where \( \frac{\partial F_2(\phi, \partial_\mu \phi)}{\partial \phi} = -\left( m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} \right) \) and note that \( \partial_\mu \phi = \frac{\partial \phi}{\partial x_\mu} \).

This proves that Eq. (B.10) is a solution of the field equation.

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