CERTAIN LINEAR COMBINATIONS OF EXPONENTIAL FUNCTIONS ARE POSITIVE UNDER SEMIDEFINITE LINEAR CONSTRAINTS

ROBERT LIN

ABSTRACT. In this article, I introduce a group-theoretical method to prove positivity of certain linear combinations (with coefficients generally lying in \( \mathbb{C} \)) of exponential functions under a set of semi-definite linear constraints. The basic group-theoretic fact we rely on is the positivity of the fusion coefficients for multiplication of group characters.

1. Introduction

In prior work joint with Jonathan Boretsky\([1]\), it was shown that the positivity of certain linear combinations (with coefficients generally lying in \( \mathbb{C} \)) of exponential functions (parameterized by a single variable \( t \)) implies a set of semi-definite linear constraints. In this work, I will show that the converse is also true, and that in fact, fundamentally, the reason for the positivity of these combinations is due to group theory.

In addition to demonstrating an unexpected connection between semidefinite optimization and group theory, the method I use to prove positivity may be of general interest as well. Possible applications include the study of inequalities arising in the study of semigroups, such as hypercontractive bounds.

2. The Main Theorem (Case where \( G = \mathbb{Z}_N \))

Suppose \( P_t \) is a semigroup that acts on the group algebra \( \mathcal{L}G \) by \( P_t \cdot |g\rangle = e^{-t|g|} |g\rangle \), where \( g \in G \) and \( | \cdot | : G \to \mathbb{R}_{\geq 0} \). Assume that \( |e| = 0 \) and \( |g| = |g^{-1}| \) for all \( g \in G \). In \( \cite{1} \), the sum representation \( P_t = \sum_{i=1}^{N} p_i(t)\sigma_i \) was introduced, where \( \sigma_i \)'s are multipliers in the group element basis induced by the corresponding irreducible characters \( \chi_i \).

In this section, for simplicity, I will present my main theorem in the case \( G = \mathbb{Z}_N \). Thus, \( \sigma_i \) is defined by \( \sigma_i |j\rangle = \chi_{ij} |j\rangle \), where \( \chi_{ij} = q^{ij} \) and \( q = \exp(2\pi i/N) \). The \( \sigma_i \) form a set of mutually orthogonal vectors in the Hilbert space \( B(\mathcal{L}G) \) defined by \( \langle a, b \rangle = \text{Tr}(a^*b) \).

Date: December 2, 2021.
The choice of a length function $|\cdot|$ on the group defines a set of functions $p_i(t)$. We assume that such a decomposition exists for all $t \geq 0$, which is a constraint on the semigroup $P_t$. For $\mathbb{Z}_N$, the existence of a decomposition is guaranteed \[1\] since $|g| = |g^{-1}|$ by assumption.

We can now state the theorem:

**Theorem 2.1.** If there exists $\epsilon > 0$ such that for all $i \in \mathbb{Z}_N$, $p_i(t) \geq 0$ for all $t \leq \epsilon$, then for any $j \in \mathbb{Z}_N$, $p_j(t) \geq 0$ for all $t \geq 0$.

**Proof.** For any $t > 0$, take $n$ large such that $t/n < \epsilon$. Then, $P_t = P_{n\left(\frac{t}{n}\right)} = \left(\sum_{i=1}^{N} p_i(t/n)\sigma_i\right)^n$ since $P_t$ is a semigroup. Since $\sigma_1 \sigma_2 \cdots \sigma_n = \sigma_{a_1 + a_2 + \cdots + a_n}$, we can replace the product by

$$\sum_{a_1, \cdots, a_n=0}^{N-1} p_{a_1}(t/n)p_{a_2}(t/n) \cdots p_{a_n}(t/n)\sigma_{a_1 + \cdots + a_n}. \quad (1)$$

The coefficients in the product are all nonnegative by hypothesis. Since summing over products of nonnegative numbers always yields nonnegative numbers, we get by summing over all indices $a_j$ for fixed $a_1 + \cdots + a_n = i$ that $p_i(t) \geq 0$ for all $t \geq 0$. \qed

**Corollary 2.2.** It suffices to check for all $i$ the values of $p_i(t)$ and $p'_i(t)$ at $t = 0$ in order to check whether $p_j(t) \geq 0$ for all $t \geq 0$. Since $p_j(t)$ is a linear combination of exponentials of the form $e^{-|g|t}$, it follows that we only need to check a set of semidefinite constraints to show that $p_j(t)$ are all positive.

**Example:** For $\mathbb{Z}_6$, $p_{i \neq 0} = 0$ and $p_0(0) = 0$. The inequalities forced by $p'_{i \neq 0}(t = 0) \geq 0$ are clearly necessary. The main theorem implies that these are sufficient. To be explicit, we represent the length function on $\mathbb{Z}_6$ by setting $|0| = 0$, $|1| = |5| = a$, $|2| = |4| = b$, and $|3| = c$. Since the length function completely specifies the semigroup $P_t$, which has the sum representation $P_t = \sum_{i=0}^{5} p_i(t)\chi_i$, we can solve for the $p_i$’s, yielding

$$p_1(t) = \frac{1}{6} \left(1 + e^{-at} - e^{-bt} - e^{-ct}\right) \quad (2)$$

$$p_2(t) = \frac{1}{6} \left(1 - e^{-at} - e^{-bt} + e^{-ct}\right) \quad (3)$$

$$p_3(t) = \frac{1}{6} \left(1 - 2e^{-at} + 2e^{-bt} - e^{-ct}\right). \quad (4)$$

We omit $p_0(t)$ which is a sum of positive quantities \[1\] (since $\chi_0$ corresponds to the trivial representation), hence always positive.
Application of Theorem 2.1 tells us that if for all \( j \in \mathbb{Z}_N, p_j'(0) \geq 0 \), then for all \( i \in \mathbb{Z}_N, p_i(t) \geq 0 \) for all \( t \geq 0 \). For \( a, b, c \geq 0 \), there are only three inequalities resulting from \( p_i'(0) \geq 0 \) (since \( p_i = p_{6-i} \)):

\[
\begin{align*}
a - b - c & \leq 0 \\
-a - b + c & \leq 0 \\
-2a + 2b - c & \leq 0.
\end{align*}
\]

Thus, \( p_i(t) \) are all nonnegative for \( a, b, c \) satisfying the above inequalities.

This set of inequalities for \( p_i(t) \) is quite nontrivial, since generally speaking, one cannot deduce \( p_i(t) \geq 0 \) simply from knowledge that \( p_i'(0) \geq 0 \).

3. Generalization to Arbitrary Finite Groups

The above result can be generalized to any finite group. Let \( \chi_r \) be the character for an irrep labeled by \( r \), and for each irrep \( r \), define \( \sigma_r |g\rangle = \chi_r(g) |g\rangle \). Further assume that \( |\cdot| \) is a class function. Then \( P_t \) admits a decomposition as \( P_t = \sum_r p_r(t) \sigma_r \).

**Theorem 3.1.** If there exists \( \epsilon > 0 \) such that for all irreps \( r \), \( p_r(t) \geq 0 \) for all \( t \leq \epsilon \), then for any irrep \( s \), \( p_s(t) \geq 0 \) for all \( t \geq 0 \).

**Proof.** For any \( t > 0 \), take \( n \) large such that \( t/n < \epsilon \). Then, \( P_t = P_n(t/n) = \left( P_n \right)^n = \left( \sum_{i \text{irrep}} p_i(t/n) \sigma_i \right)^n \) since \( P_t \) is a semigroup. Positivity follows now since the tensor product of irreducible representations of a finite group can be completely reduced, and the multiplicity of the irreducible representation \( \rho \) in \( \rho_a \otimes \rho_b \) is precisely \( n_{ab}^c \), which is always nonnegative. By extension, we can write \( \sigma_{a_1} \sigma_{a_2} \cdots \sigma_{a_n} = n_{a_1a_2\cdots a_n}^b \sigma_b \), where \( n_{a_1a_2\cdots a_n}^b \) is the multiplicity of the irrep \( \rho_b \) in \( \rho_{a_1} \otimes \rho_{a_2} \otimes \cdots \rho_{a_n} \). Thus, the product can be rewritten as

\[
\sum_{a_1,\cdots,a_n} \sum_b p_{a_1}(t/n) p_{a_2}(t/n) \cdots p_{a_n}(t/n) n_{a_1a_2\cdots a_n}^b \sigma_b.
\]

The coefficients in the product are all nonnegative by hypothesis. Since summing over products of nonnegative numbers always yields nonnegative numbers, we get by summing over all indices \( a_j \) for fixed \( b \) that for any irrep \( s \), \( p_s(t) \geq 0 \) for all \( t \geq 0 \).

4. Acknowledgments

I wish to thank Jonathan Boretsky for helpful review of my write-up of my result.
I acknowledge the research support of ARO grant W911NF-20-1-0082 through the MURI project “Toward Mathematical Intelligence and Certifiable Automated Reasoning: From Theoretical Foundations to Experimental Realization.”

References

[1] Manuscript in preparation.

Department of Physics, Harvard University, Cambridge, MA 02138