Nearby Special Lagrangians
Mohammed Abouzaid and Yohsuke Imagi
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Abstract
Let $X$ be a Calabi–Yau manifold and $Q \subset X$ a compact connected embedded special Lagrangian without boundary. We combine methods from symplectic topology and geometric analysis, to study those compact irreducibly immersed special Lagrangians which are $C^0$ close to $Q$ and have unobstructed Floer cohomology. We prove that if the fundamental group $\pi_1 Q$ has no non-abelian free subgroup then every such Lagrangian $L$ is unbranched (that is, the projection $L \to Q$ is a covering map); moreover, if $\pi_1 Q$ is abelian then $L$ is $C^1$ close to $Q \subset X$. We prove a stronger uniqueness theorem when $\pi_1 Q$ is finite. The $\pi_1 Q$ conditions, the Floer cohomology condition and the special Lagrangian condition are all essential as we show by counterexamples.

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1 Introduction
We begin with a definition we use throughout the paper.

Definition 1.1. A Calabi–Yau manifold is a Kähler manifold $X$ equipped with a holomorphic volume form $\Omega$. A special Lagrangian in a Calabi–Yau manifold $X$ is a Lagrangian of phase $0 \in \mathbb{R}/2\pi \mathbb{Z}$ in the sense of Definition 2.32.

Remark 1.2. The definition agrees with that in the literature on special Lagrangians except that Joyce \cite{Joyce} and others call $X$ an almost Calabi–Yau manifold to indicate that this need not be Ricci-flat. We do more symplectic topology
in the present paper and therefore leave out the word almost so as not to be confused with almost complex structures. Special Lagrangians are in general defined as Lagrangians of any constant phase. We mention explicitly what the phases are if they are not $0 \in \mathbb{R}/2\pi\mathbb{Z}$.

Let $X$ be a Calabi–Yau manifold and $Q \subset X$ a closed connected embedded special Lagrangian. Closed means compact without boundary. The $C^1$ nearby problem for $Q \subset X$ is solved by McLean [43]; that is, those special Lagrangians which are $C^1$ close to $Q \subset X$ form a smooth manifold whose tangent space at $Q$ is isomorphic to the (de Rham) cohomology group $H^1(Q, \mathbb{R})$. We study the $C^0$ nearby problem, which is to classify those closed special Lagrangians $L$ contained in a given neighbourhood of $Q \subset X$. Donaldson and He [15, 29, 30] have proved recently an existence theorem about this. We prove on the other hand a uniqueness and non-existence theorem, Theorem 1.3 below.

We make now the definitions we use to state our results. Let $U \subseteq X$ be a Weinstein neighbourhood of $Q$ (which may be embedded in the cotangent bundle $T^*Q$). Let $\iota : \hat{L} \to U$ be a Lagrangian immersion. We suppose that every self-intersection point of $\iota$ is a transverse double point unless otherwise mentioned. In many places we refer only to the image $L := \iota(\hat{L})$ and call this simply the Lagrangian. We say that it is closed if $\hat{L}$ is compact without boundary. We say that it is irreducibly immersed if the domain $\hat{L}$ is connected.

We say that a Lagrangian immersion $\hat{L} \to L \subset U$ is unbranched if the composition with the projection map $U \subset T^*Q \to Q$ is a finite cover $L \to Q$. This is equivalent to saying that there exists a finite cover $P \to Q$ such that the pull-back of $L \subset U$ under the induced map $T^*P \to T^*Q$ is a union of smooth perturbations of $P$ (if $L$ is embedded, this union is disjoint). Here a smooth perturbation means the graph over $P$ of some closed 1-form (so this is a Lagrangian). We say that the Lagrangian is branched if it is not unbranched.

We give also a brief account of Floer cohomology groups and Fukaya categories which we use. We assign to the symplectic manifold $U$ above a filtered strict $A_\infty$ category $\mathcal{F}(U)$. This is a version of the Fukaya categories defined by Akah–Joyce [9] and Fukaya, Oh, Ohta and Ono [21] which we explain in §2. The objects of $\mathcal{F}(U)$ are closed Lagrangians in $U$ equipped with some additional data, the most important of which are bounding cochains in the sense of Fukaya, Oh, Ohta and Ono [21]. It is a property of the Lagrangians that they have these data; that is, some Lagrangians do have the additional data, in which case we say that they have unobstructed Floer cohomology or $HF^*$ unobstructed for short, and others have no such data at all, in which case we say that they have obstructed Floer cohomology or $HF^*$ obstructed for short. Floer cohomology groups are the morphism groups of the cohomology category $HF(\mathcal{F}(U))$ of the strict $A_\infty$ category $\mathcal{F}(U)$. The symbol $HF^*$ is a standard notation for these.

As we are working not in the whole $X$ but in the Weinstein neighbourhood $U$ the zero-section $Q$ (and any Lagrangian that is $C^1$ close to it) has $HF^*$ unobstructed in $\mathcal{F}(U)$ because this does not bound any non-constant holomorphic discs in $U$ (it has possibly $HF^*$ obstructed in the Fukaya category $\mathcal{F}(X)$ of the whole $X$).
Also as $U$ is an exact symplectic manifold we can do the virtual counts in any characteristic $p$. The advantage of having $p > 0$ is that we can cancel out the effect of pseudo-holomorphic discs whose contributions are multiples of $p$; for instance, with $p = 2$ we can do without relative spin structures. Thus by varying $p$ we can include more Lagrangians with $HF^*$ unobstructed. The zero-section $Q \subset U$ has $HF^*$ unobstructed for any $p$ because it has essentially no holomorphic discs. On the other hand, we shall need the nearby special Lagrangians to have $HF^*$ unobstructed only for some $p$. In these circumstances our theorem may be stated as follows:

**Theorem 1.3.** Let $X$ be a Calabi–Yau manifold and $Q \subset X$ a closed connected embedded special Lagrangian; then the following hold.

(i) Let the fundamental group $\pi_1 Q$ be finite and let $U \subseteq X$ be any Weinstein neighbourhood of $Q$. Then every closed special Lagrangian in $U$ with $HF^*$ unobstructed is necessarily of phase $0 \in \mathbb{R}/2\pi\mathbb{Z}$ and equal to $Q$.

(ii) If $\pi_1 Q$ has no non-abelian free subgroup, there exists a Weinstein neighbourhood $U \subseteq X$ of $Q$ such that every closed irreducibly immersed special Lagrangian $L \subset U$ with $HF^*$ unobstructed is unbranched. Moreover, if $\pi_1 Q$ is abelian then $L$ is a McLean $C^1$ perturbation of $Q$.

Here is a sketch of the proof. Suppose first that $\pi_1 Q$ is abelian as in the latter part of (ii). As $L$ has $HF^*$ unobstructed there is a corresponding object $b$ of the Fukaya category $\mathcal{F}(U)$ and we can define the Floer cohomology group $HF^*(\mathbb{T}^\ast Q, b)$, which is a $\mathbb{Z}$-graded vector space. Since the Lagrangian $L$ is special it follows that $HF^*(\mathbb{T}^\ast Q, b)$ is non-zero and supported in a single degree; the non-zero property will be proved in Corollary 2.56 and the single degree property will be proved in Lemma 3.4. We use then the known techniques for dealing with $\mathcal{F}(U)$, which we will explain in §2. The key fact is that $HF^*(\mathbb{T}^\ast Q, b)$ is a certain representation of the fundamental group $\pi_1 Q$ and that this representation determines $b$ up to quasi-isomorphism. As $\pi_1 Q$ is abelian there is a one-dimensional sub-representation of $HF^*(\mathbb{T}^\ast Q, b)$. This may be realized by a McLean $C^1$ perturbation of $Q$ with some local system over it. Then by a Thomas–Yau theorem, Proposition 2.67 (i), we can show that the original special Lagrangian $L$ agrees with the $C^1$ perturbation of $Q$.

The same method applies to $\pi_1 Q$ which has no non-abelian free subgroup as in (ii) above. In this case we shall need to pass to a finite-index subgroup of $\pi_1 Q$ for the algebraic reason. Denote by $P \to Q$ the corresponding finite cover, and by $f : T^* P \to T^* Q$ the induced finite cover between the cotangent bundles. The McLean $C^1$ perturbation $\hat{L}$ will then be defined in $T^* P$ and the original $L$ will be the image of $\hat{L}$ under $f : T^* P \to T^* Q$. So $L$ is unbranched.

Part (i) is proved in §3.1 with simpler algebraic arguments. The conclusion is stronger in two respects: we do not have to shrink the Weinstein neighbourhood $U$, and we can include special Lagrangians of any phase.

Apart from the $HF^*$ hypothesis Theorem 1.3 (i) is stronger than the result of Tsai–Wang [61, 62] which we recall in Proposition 4.1. They prove this by
a simple Riemannian geometry method, supposing that $U$ is Ricci-flat and $Q$ Ricci-positive. Then by Myers’ theorem the fundamental group $\pi_1 Q$ is necessarily finite as in (i) above. On the other hand, Tsai–Wang’s method applies to any singular special Lagrangians in the sense of geometric measure theory\([12]\).

The $HF^*$ hypothesis of our theorem presupposes that the Lagrangians are not very singular, about which we discuss more in Remark\([4,13]\) after we prove the theorem.

The hypothesis on $\pi_1 Q$ is essential to Theorem\([1.3]\) (ii). For instance, if $Q$ is a compact Riemann surface of genus $> 1$ then $\pi_1 Q$ has free non-abelian subgroups and the cotangent bundle of such a surface admits many branched nearby special Lagrangians given by the hyperKähler rotations of Hitchin spectral curves\([31, \S 5.1]\) (where the hyperKähler metric is given by Feix\([17]\) and Kaledin\([35]\)). We give in Example\([5.12]\) a higher-dimensional version of this. He\([29]\) constructs also branched special Lagrangians which are locally modelled upon the product of $S^1$ and a 2-fold spectral curve. His construction applies to many $Q$-homology 3-spheres $Q\([30]\)$ and their fundamental groups have non-abelian free subgroups (more precisely, his examples do not include finite quotients of $S^3$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$; cf. Proposition\([5.16]\)). Thus no contradiction occurs with Theorem\([1.3]\) (ii).

The abelian condition is essential to the latter part of Theorem\([1.3]\) (ii). We prove in Corollary\([4.7]\) that there are multi-valued graphs over a certain flat manifold $Q$. This $Q$ has a non-trivial finite cover by $T^n$ and the fundamental group $\pi_1 Q$ is accordingly not abelian.

We show also that the other hypotheses of Theorem\([1.3]\) are also indispensable. For this we use Lagrangian surgeries obtained from Morse 1-forms on $Q$. We prove in Example\([5.10]\) that there are examples satisfying the conditions of Theorem\([1.3]\) (ii) except that the Lagrangians should be special. We prove in Corollary\([5.3]\) that these Lagrangian are not special even in the weak categorical sense.

We examine now the $HF^*$ hypothesis. This would in general be hard to verify directly (because the Maurer–Cartan equation is defined by using all the holomorphic curve moduli spaces). On the other hand, so far as we know there are no examples in which we cannot apply Theorem\([1.3]\) just because the Lagrangians have $HF^*$ obstructed (which would be hard to verify for the same reason). We change the point of view in Corollary\([5.15]\) we use Theorem\([1.3]\) to find examples with $HF^*$ obstructed. These are branched special Lagrangians and we argue indirectly: if they had $HF^*$ unobstructed, they would satisfy the conclusion of Theorem\([1.3]\) which is impossible because they are branched; they have thus $HF^*$ obstructed. This has a further consequence, Corollary\([5.15]\) which is about Morse 1-forms and perhaps of interest in itself.

We begin in \(|\S 2|\) with the more careful treatment of Fukaya categories. In \(|\S 3|\) we prove Theorem\([1.3]\). In \(|\S 4|\) we recall Tsai–Wang’s result and give the examples related to it. This section is essentially about minimal submanifolds and geometric measure theory, which has nothing directly to do with Floer cohomology groups or Fukaya categories. In \(|\S 5|\) we give the branched examples and prove the relevant results.
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2 Fukaya Categories

In this section we explain the definition and properties of the Fukaya categories. The key result is the split-generation theorem for cotangent bundles, including non-exact Lagrangians; see Theorem 2.84. This is known for exact Lagrangians and we extend it to non-exact Lagrangians. We must include non-exact Lagrangians because of the McLean perturbations, which are non-exact except the trivial perturbations.

We use pseudo-holomorphic curve moduli spaces to define the Fukaya categories. As we include non-exact Lagrangians we cannot follow the direct perturbation method of Seidel [53] and we follow instead the virtual perturbation method of Akaho–Joyce [9] and Fukaya, Oh, Ohta and Ono [21]. But still we cannot use their results as they are. The reason is that we shall be concerned with wrapped Fukaya categories and therefore need to include countably many Lagrangians. This causes a new problem which we explain now. Recall that we can deal only with finitely many kinds of Kuranishi spaces in doing the virtual counts [21, §7.2.3] and the results are accordingly a finite approximation of what we want. These are called gapped $A_N$ categories in our simplified notation, Definition 2.14 (or gapped $A_{NK}$ algebras in their notation [9,21]). We pass the the limit, called gapped $A_\infty$ categories, by the obstruction theory arguments as in the literature [9,21]. But their statements are all about single or finitely many Lagrangians, so we make the version for countably many Lagrangians; see Theorem 2.26.

The gapped $A_\infty$ categories are curved $A_\infty$ categories, including the $m^0$ terms. We make them into strict $A_\infty$ categories by adding bounding cochains, following the references [9,21]. We define then the wrapped Fukaya categories by the localization method, which has been used already in the literature [26] (the idea is due to Paul Seidel and the first author). The advantage of this is that we can make the definition only from the algebraic properties of Hamiltonian continuations, without going back to the study of pseudo-holomorphic curve
moduli spaces.

We prove the split-generation theorem by a method of Paul Seidel and the first author (also from their unpublished work). This is related to the proof of Fukaya, Seidel and Smith \[24\] for exact Lagrangians (they assume that the ground field has characteristic \( \neq 2 \)). They use Lefschetz fibrations, Seidel long exact sequences, and equivariant Fukaya categories, and we replace the last ingredient with the use of Viterbo restriction functors. When working with non-exact Lagrangians, the first two ingredients have already appeared in the literature, so we focus on a discussion of Viterbo restriction functors in the non-exact setting. After providing the argument for this generalisation, the remainder of the proof extends fairly immediately.

There is another method \[2\] of proving the split-generation theorem, using symplectic cohomology (for time-one periodic Hamiltonian orbits). One should then prove that it has the same relation as in the exact case to the algebraic invariants obtained from the Fukaya category. One could do this in closed symplectic manifolds (as will appear in \[6\]) and for monotone Lagrangians (as is done already in \[49\]) but not yet in the circumstances above.

We begin in \S 2.1 with the treatment of Novikov fields. The major difference from the references \[9,21\] is that we extend the definition of Novikov fields used in the references \[9,21\]; see Definition 2.2. We do this because we will use Lie–Kolchin’s theorem, Theorem 3.17 for the proof of Theorem 1.3. We shall therefore need the Novikov fields to be algebraically closed, and our definition will be suitable for this purpose.

In \S 2.2 we define the gapped \(A_\infty\) categories and provide the obstruction theory arguments. Apart from the issue mentioned above, the major difference from the references \[9,21\] is that we extend the definition of gapped \(A_\infty\) algebras; see Remark 2.15. We do this because we include filtered local systems (in the sense of Definition 2.12) in the Fukaya categories.

The gapped structure will be used only for the process of defining the curved \(A_\infty\) categories, which we shall in practice be able to treat as filtered \(A_\infty\) categories (in the sense of Definition 2.14) forgetting the gap conditions.

In \S\S 2.3 and 2.4 we define the curved \(A_\infty\) category \(\mathcal{C}(X)\) of a symplectic manifold \(X\). In \S 2.5 we define the strict \(A_\infty\) categories \(\mathcal{F}_{nc}(X), \mathcal{F}(X)\) including the bounding cochains. In \S 2.6 we recall generalized Thomas–Yau theorems. In \S 2.7 we explain the effect of symplectic and Hamiltonian diffeomorphisms upon the Fukaya categories. In \S 2.8 we introduce the wrapped Fukaya category \(\mathcal{W}(X)\) by the localisation method. In \S 2.9 we define the exact Fukaya category \(\mathcal{F}_{ex}(X)\) and show that its cohomology category \(H\mathcal{F}_{ex}(X)\) is equivalent to that in the literature. In \S 2.10 we turn to the study of cotangent bundles; we prove two lemmas we shall need to deal with McLean perturbations, which we will use to prove Theorem 1.9 (ii). In \S 2.11 we prove the split-generation theorem for the Fukaya categories in cotangent bundles. In \S 2.12 we study the Yoneda functors obtained from the split-generation theorem.
2.1 Novikov Fields

We begin by recalling the definition of non-Archimedean valued fields. We call them valued fields for short.

**Definition 2.1.** By a valued field we mean the pair \((K, v)\) of a field \(K\) and a function \(v : K \rightarrow (-\infty, \infty]\), called a valuation, such that \(v^{-1}(\{0\}) = \{0\}\) and for \(a, b \in K\) we have \(v(ab) = va + vb\) and \(v(a + b) \geq \min\{va, vb\}\). By a valued field extension of \((K, v)\) we mean a valued field \((K', v)\) such that \(K\) is a sub-field of \(K'\) and the two valuations agree on \(K\). We use the same symbol \(v\) for different valuations.

Every valued field \((K, v)\) has a sub-ring \(K^0 \subseteq K\) defined by \(K^0 := \{a \in K : va \geq 0\}\), which is a local ring with maximal ideal \(K^+ := \{a \in K : va > 0\}\); in other words, \(a \in K^0\) is a unit if and only if \(va = 0\).

Let \((K, v)\) be a valued field and define a function \(|\cdot| : K \rightarrow [0, \infty)\) by \(|a| := e^{-va}\) for \(a \in K\). Then for \(a, b \in K\) the strong triangle inequality \(|a + b| \leq \max\{|a|, |b|\}\) holds and there is a distance function \(d : K \times K \rightarrow [0, \infty)\) defined by \((a, b) \mapsto |a - b|\), making \(K\) into a metric space \((K, d)\). We call \((K, v)\) a complete valued field if \((K, d)\) is complete.

We generalize the definition of Novikov fields in the references \([9, 21]\).

**Definition 2.2.** Define the minimum Novikov field \((\Lambda^{\min}, v)\) to be the Novikov field in the literature \([9, 21]\) without the Maslov index factor \(e\) in their notation. We begin by recalling this. Let \(k\) be a field and \(k[\mathbb{R}]\) its group ring. Denote by \(T \in k[\mathbb{R}]\) the formal variable corresponding to \(1 \in \mathbb{R}\) so that every element \(a \in k[\mathbb{R}]\) may be written as \(\sum_{\gamma \in \mathbb{R}} a_{\gamma} T^\gamma\), \(a_{\gamma} \in k\) with \(\supp a := \{\gamma \in \mathbb{R} : a_{\gamma} \neq 0\}\) a finite subset of \(\mathbb{R}\). Define a function \(v : k[\mathbb{R}] \rightarrow (-\infty, \infty]\) by \(a \mapsto \min(\supp a)\). Then the function \(|\cdot| : k[\mathbb{R}] \rightarrow [0, \infty)\) defined by \(|a| := e^{-va}\) for \(a \in \Lambda\) satisfies the strong triangle inequalities, making \(k[\mathbb{R}]\) into a metric space; and \(\Lambda^{\min}\) is its completion. The convolution products extend to the completion, making \(\Lambda^{\min}\) into a ring. This is in fact a field because of the inversion formula \((1-a)^{-1} = 1 + a + a^2 + \ldots\) for \(a \in k[\mathbb{R}]\) with \(va > 0\). The function \(v\) extends to the completion, making it into a complete valued field \((\Lambda^{\min}, v)\). We denote an element of \(\Lambda^{\min}\) by an infinite sum \(\sum_{\gamma \in \mathbb{R}} a_{\gamma} T^\gamma\), \(a_{\gamma} \in k\), with \(\supp a := \{\gamma \in \mathbb{R} : a_{\gamma} \neq 0\}\) a discrete subset of \(\mathbb{R}\).

By a Novikov field we mean a complete valued field extension of \((\Lambda^{\min}, v)\). We call \(k\) the ground field of the Novikov field.

We prove now that algebraically closed Novikov fields exist for arbitrary characteristic. We give two proofs: one is Krasner’s lemma, Lemma 2.5, a general fact about valued fields, and the other is an explicit construction in Proposition 2.6 (ii).

Let \((K, v)\) be a valued field. Define a subring \(K^0 \subseteq K\) defined by \(K^0 := \{a \in K : va \geq 0\}\). This has a unique maximal ideal \(K^+ := \{a \in K : va > 0\}\) and the residue field \(K := K^0/K^+\). The value group \(vg K\) is a subgroup of \(\mathbb{R}\) defined by \(vg K := v(K \setminus \{0\}) \subseteq \mathbb{R}\). We recall the following basic facts:
Proposition 2.3. Let $K$ be a complete valued field, and $L/K$ an algebraic extension; then the valuation of $K$ extends uniquely to $L$ so that:

(a) if $L/K$ has finite degree, the valuation on $L$ is complete; and

(b) if $L/K$ has infinite degree, the valuation of $L$ is strictly not complete.

Also, if $L/K$ is finite then

$$[L : K] \geq [\text{rf } L : \text{rf } K][\text{vg } L : \text{vg } K].$$

(2.1)

More precisely, there exist two intermediate fields of $L/K$, the maximal unramified field $L_{\text{unr}}$ and the maximal tamely-ramified field $L_{\text{tame}}$, such that:

(i) $K \subseteq L_{\text{tame}} \subseteq L_{\text{unr}} \subseteq L$;

(ii) $\text{rf } K \subseteq \text{rf } L_{\text{unr}} = \text{rf } L_{\text{tame}} \subseteq \text{rf } L$;

(iii) $\text{vg } K = \text{vg } L_{\text{unr}} \subseteq \text{vg } L_{\text{tame}} \subseteq \text{vg } L$, $[\text{vg } L_{\text{tame}} : \text{vg } L_{\text{unr}}] = [L_{\text{tame}} : L_{\text{unr}}]$;

(iv) if $\text{rf } K$ has characteristic $p = 0$ then $L = L_{\text{tame}}$, and if $p > 0$ then $[L : L_{\text{tame}}]$ is a power of $p$.

Proof. We consult Neukirch [45]; the referred statements in what follows are all from Chapter II of this book.

The unique extension is by Theorem 4.8. By Proposition 4.9 the extended valuation is equivalent to the $\ell_{\infty}$ norm on a direct sum of $K$, whose elements have only finitely many nonzero components even if they are infinite dimensional. So they are complete if and only if finite-dimensional.

The inequality (2.1) is from Proposition 6.8. The field $L_{\text{unr}}$ is defined in Definition 7.4, and $L_{\text{tame}}$ in Definition 7.10 but under the hypothesis that $\text{rf } K$ has positive characteristic; if $\text{rf } K$ has characteristic 0 we define $L_{\text{tame}} := L$. The parts (i), (ii) and (iv) are then obvious. The part (iii) follows from Proposition 7.7; this is stated again in the positive-characteristic case, but its proof applies also to the characteristic-zero case.

Here the extension $L/L_{\text{tame}}$ is called the wild part of $L/K$, which is in general difficult to study. But it has degree a power of $p$; and for Galois extensions of such a degree, there is a classification result:

Proposition 2.4 (Artin–Schreier). Let $K$ be any field of characteristic $p$. Then for every $a \in K$ either:

(i) the polynomial $f_a = f_a(x) := x^p - x - a \in K[x]$ has a root $b \in K$, in which case its roots $b, b+1, \cdots, b+p-1$ are all in $K$; or

(ii) $f_a \in K[x]$ is irreducible, in which case $K[x]/f_a$ is a degree-$p$ Galois extension of $K$.

Conversely:
(iii) every degree-$p$ Galois extension of $K$ is of the form $K[x]/f_a$ for some $a \in K$; or more generally

(iv) every finite Galois extension of $K$ of degree equal to a power of $p$ is a repeated extension of this kind.

Proof. For (i)–(iii) consult for instance Lang [41, Chapter VI, Theorem 6.4]; and then for (iv) use the following fact [41, Chapter I, Corollary 6.6]: for every finite group $G$ of order a power of $p$ there exists a normal series

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$$

(2.2)

with $[G_i : G_{i-1}] = p$ for every $i$. 

Finally we recall:

Lemma 2.5 (Krasner). Every valued field $K$ has a complete algebraically-closed extension.

Proof. Let $L$ be the algebraic closure of $K$. The valuation extends uniquely to $L$ by the first assertion in Proposition 2.3 because $L$ is the union of finite subextensions of $K$. The completion of this valued field $L$ is algebraically closed, by Krasner’s lemma [45, Chapter II, §6, Exercise 2].

We turn now to the second method. Let $k$ be a field. In Definition 2.2 we have defined the minimum Novikov field $\Lambda_{\text{min}}$ with ground field $k$. We define the maximum Novikov field by

$$\Lambda_{\text{max}} := \left\{ a = \sum_{\gamma \in \mathbb{R}} a_{\gamma} T^\gamma : a_{\gamma} \in k \text{ and } \text{supp} \ a \subseteq \mathbb{R} \text{ well-ordered} \right\}$$

(2.3)

where $\text{supp} \ a := \{ \gamma \in \mathbb{R} : a_{\gamma} \neq 0 \}$. Recall that $\text{supp} \ a$ being well ordered means that every nonempty subset has a minimum. In particular, $\text{supp} \ a$ is half discrete in the sense that for every $\gamma \in \text{supp} \ a$ there exists $\epsilon > 0$ such that

$$\text{supp} \ a \cap (\gamma, \gamma + \epsilon) = \emptyset;$$

(2.4)

because we can take $\gamma + \epsilon$ to be the minimum of $\text{supp} \ a \cap (\gamma, \infty)$. The sums and products in $\Lambda_{\text{max}}$ are defined in the natural way: for $a, b \in \Lambda_{\text{max}}$ we can define

$$a + b := \sum_{\gamma \in \text{supp} \ a \cup \text{supp} \ b} (a_{\gamma} + b_{\gamma}) T^\gamma$$

(2.5)

because $\text{supp} \ a \cup \text{supp} \ b$ is well ordered; and

$$ab := \sum_{\gamma} \left( \sum_{\alpha + \beta = \gamma} a_{\alpha} b_{\beta} \right) T^\gamma$$

(2.6)

because $\sum_{\alpha + \beta = \gamma}$ is finite, which follows since $\text{supp} \ a, \text{supp} \ b$ are half discrete.
The valuation \( v : \Lambda^{\max} \to (-\infty, +\infty] \) is defined by setting \( v(0) := +\infty \) and by taking in (2.3) the least \( \gamma \in \text{supp} a \). Then \((\Lambda^{\max}, v)\) is a complete valued field. We have

\[
rf \Lambda^{\min} \cong rf \Lambda^{\max} \cong k \quad \text{and} \quad vg \Lambda^{\min} \cong vg \Lambda^{\max} = \mathbb{R}.
\]

(2.7)

We prove now the following proposition. Part (ii) with \( k \) algebraically closed is the result we have promised to prove.

**Proposition 2.6.**

(i) \( \Lambda^{\min}, \Lambda^{\max} \) are both perfect.

(ii) \( \Lambda^{\max} \) is maximal in Krull’s sense [40]; that is, \( \Lambda^{\max} \) has no nontrivial valued field extension \( K \) with \( rf K \cong k \) and \( vg K = \mathbb{R} \). In particular, if \( k \) is algebraically closed, so is \( \Lambda^{\max} \).

(iii) If \( k \) is algebraically closed of characteristic 0, so is \( \Lambda^{\min} \).

If \( k \) has positive characteristic \( p \) then \( \Lambda^{\min} \) is not algebraically closed, and its algebraic closure is strictly not complete.

**Remark 2.7.** These results do not assert that \( \Lambda^{\max} \) is the algebraic closure of \( \Lambda^{\min} \). For rational-power series, the algebraic closure is described in Kedlaya [38].

**Proof of Proposition 2.6.** For (i) let \( \Lambda \) be either \( \Lambda^{\min} \) or \( \Lambda^{\max} \). By definition, a field \( \Lambda \) is perfect if either it has characteristic \( p = 0 \) or the ring homomorphism \( \Lambda \to \Lambda \) defined by \( x \mapsto x^p \) is surjective. Suppose \( p > 0 \) and consider the polynomial \( x^p - a \in \Lambda[x], a \in \Lambda \). We show that this has a root in \( \Lambda \). If \( a = 0 \) it has root \( 0 \in \Lambda \). If \( a \neq 0 \) we can write \( a = T^{\omega a}(1 + c) \), \( b \in k^* \) and \( c \in \Lambda^+ \). Let \( b^{1/p} \in k \) be a root of \( x^p - b \in k[x] \), which exists because \( k \) is algebraically closed. Define

\[
(1 + c)^{1/p} := \sum_{k=0}^{\infty} \left( \frac{1}{p} \right)^k c^k \in \Lambda.
\]

(2.8)

Then \( T^{\omega a/p} b^{1/p} (1 + c)^{1/p} \in \Lambda \) is a desired root.

The former part of (ii) is an old result [36, 40]. For the latter, if \( k \) is algebraically closed, then \( rf \Lambda \cong k \) is algebraically maximal and \( vg \Lambda = \mathbb{R} \) is already maximal in \( \mathbb{R} \); so \( \Lambda \) has no nontrivial finite extension, or equivalently \( \Lambda \) is algebraically closed.

The former part of (iii) is proved by Fukaya, Oh, Ohta and Ono [22, Lemma A.1] but we give another proof, using Proposition 2.3 with \( K = \Lambda^{\min} \). Suppose therefore that \( L/K \) is a finite extension, and note that:

- since \( rf K \) is algebraically closed it follows that the inclusions in (ii) are all equalities, so \( K = L^{\text{unr}} \);
- since \( vg K = \mathbb{R} \) it follows that the inclusions in (iii) are all equalities, so \( L^{\text{unr}} = L^{\text{tame}} \); and
- \( L = L^{\text{tame}} \) because \( rf K \) has characteristic zero.
Consequently, $K = L$; that is, $K = \Lambda$ has no nontrivial finite extension, or equivalently, $K$ is algebraically closed.

For the latter part of (iii) let $\gamma < 0$ and consider the Artin–Schreier polynomial

$$x^p - x - T^n \in \Lambda^{\text{min}}[x].$$

If this had a root of the form $x = \sum_{n=1}^{\infty} x_n T^n$ then by induction on $n$ it must be of the form

$$\sum_{n=1}^{\infty} T^{-\gamma/p^n} \in \Lambda^{\text{max}} \setminus \Lambda^{\text{min}};$$

so $\Lambda^{\text{min}}$ is not algebraically closed.

We show next that the algebraic closure of $\Lambda^{\text{min}}$ has infinite degree over $\Lambda^{\text{min}}$. By an Artin–Schreier element over $K := \Lambda^{\text{min}}$ we shall mean a root of an Artin–Schreier polynomial over $K$. Let $L/K$ be the field extension which contains all the Artin–Schreier elements over $K$. We prove the following three facts in this order:

**I** if $a_0, a_1, a_2 \in K$ with $a_0 = a_1 + a_2$ and if $\alpha_1, \alpha_2, \alpha_3 \in L$ are respectively roots of the polynomials $x^p - x - a_i \in K[x]$, $i = 0, 1, 2$, then $K(\alpha_0) \subseteq K(\alpha_1, \alpha_2)$;

**II** if $\alpha$ is an Artin–Schreier element over $K$ then $K(\alpha) \subseteq K(\beta; \delta_1, \cdots, \delta_n)$ where $\beta$ is a root of the polynomial $x^p - x - b \in K[x]$ for some $b \in K$ with $vb \geq 0$, and every $\delta_i$ a root of the polynomial $x^p - x - c_i T^n \in K[x]$ for some $c_i \in k$ and $\gamma_i < 0$; and

**III** if the algebraic closure of $\Lambda^{\text{min}}$ had finite degree over $\Lambda^{\text{min}}$, we should have $L = K(\beta_1, \cdots, \beta_m; \delta_1, \cdots, \delta_n)$ where $\beta_1, \cdots, \beta_m$ are such as $\beta$ in (II) and $\delta_1, \cdots, \delta_n$ are under the same condition as in (II) but that $n$ may of course be different.

For (I) note that $\alpha_1 + \alpha_2$ is a root of the polynomial $x^p - x - a_0 \in K[x]$, so $a_0 = \alpha_1 + \alpha_2 + c$ for some $c \in \{0, 1, \cdots, p-1\}$, which implies $K(\alpha_0) \subseteq K(\alpha_1, \alpha_2)$ as we want. For (II) write $\alpha$ as a root of the polynomial $x^p - x - a \in K[x]$, $a \in K = \Lambda^{\text{min}}$, and recall that $a$ has by definition only finitely many terms of negative powers with respect to $T$; that is, $a = b + \sum c_i T^n$, $vb \geq 0$ and $\gamma_i < 0$. Applying to this the part (I) repeatedly, we see that $K(\alpha) \subseteq K(\beta; \delta_1, \cdots, \delta_n)$ as we want. For (III) it is clear from definition that $L \supseteq K(\beta_1, \cdots, \beta_m; \delta_1, \cdots, \delta_n)$.

On the other hand, by our hypothesis $L/K$ is a finite extension and so is obtained from $K$ by adding finite Artin–Schreier elements. Hence using (II) repeatedly we get the other inclusion we want.

Now in (III) write $\delta_i$ as a root of the polynomial $x^p - x - c_i T^n \in K[x]$ for some $c_i \in k$ and $\gamma_i < 0$. Then $\delta_i = \sum_{n=1}^{\infty} c_i^{1/p^n} T^{\gamma_i/p^n}$ for some $c_i^{1/p^n} \in \bar{k}$, the algebraic closure of $k$, so

$$L \subseteq \left\{ \sum a_\gamma T^\gamma : a_\gamma \in \bar{k} \text{ and if } \gamma < 0 \text{ then } \gamma \in \mathbb{Q}(p; \gamma_1, \cdots, \gamma_n) \right\}. \quad (2.11)$$
But as \( \mathbb{R}/\mathbb{Q} \) is an infinite extension there is some \( \epsilon < 0 \) that is not in \( \mathbb{Q}(p; \gamma_1, \cdots, \gamma_n) \).

Then by (2.11) the Artin–Schreier element \( \sum_{n=1}^{\infty} T^n/p^n \), a root of the polynomial \( x^n - x - T^e \in K[x] \), is not in \( L \), which contradicts the definition of \( L \).

So the algebraic closure of \( \Lambda^{\min} \) has infinite degree over \( \Lambda^{\min} \) and by Proposition 2.3 (b) it is strictly not complete.

**Remark 2.8.** The rôle of Artin–Schreier polynomials is essential to the proof above. In fact, if \( k \) is algebraically closed then every finite normal extension \( L/\Lambda^{\min} \) is a repeated Artin–Schreier extension, as we show now. In the notation of Proposition 2.3 with \( K = \Lambda^{\min} \), we have again \( K = L^{\text{tame}} \). On the other hand, by (i) above, \( L/L^{\text{tame}} \) is automatically Galois. To this we can apply Proposition 2.3 so \( L/\Lambda^{\min} \) is a repeated Artin–Schreier extension as we want.

### 2.2 Gapped \( \Lambda_\infty \) Categories

We introduce the notion of valued vector spaces.

**Definition 2.9.** Let \((\Lambda, v)\) be a Novikov field. By a **valued \( \Lambda \)-vector space** we mean the pair \((E, v)\) of a \( \Lambda \)-vector space \( E \) and a function \( v : E \to (\mathbb{R}, +, \cdot, 0, 1) \), called also the **valuation**, such that \( v^{-1}(\mathbb{R}) = \{0\} \); for \( x, y \in E \) we have \( v(x + y) \geq \min\{v(x), v(y)\} \); and for \( a \in \Lambda \), \( x \in E \) we have \( v(ax) = v(a) + v(x) \) where we use both \( v : \Lambda \to (\mathbb{R}, +, \cdot, 0, 1) \) and \( v : E \to (\mathbb{R}, +, \cdot, 0, 1) \).

Define a function \( | \cdot | : E \to [0, \infty) \) by \( |x| := e^{-v(x)} \) for \( x \in E \). Then the strong triangle inequality \( |x + y| \leq \max\{|x|, |y|\} \) holds for \( x, y \in E \); and in particular, there is a distance function \( d : E \times E \to [0, \infty) \) defined by \( (x, y) \mapsto |x - y| \), making \( E \) into a metric space \((E, d)\). We call \((E, v)\) a **complete** valued vector space if \((E, d)\) is a complete metric space. The valued vector space structure of \((E, v)\) extends to the metric space completion of \((E, d)\), making it into a complete valued vector space. We call this the **completion** of \((E, v)\).

Let \((E_1, v_1), (E_2, v_2), (F, v_3)\) be valued \( \Lambda \)-vector spaces and \( \alpha : E_1 \to F \) a \( \Lambda \)-linear map which is continuous with respect to the metric space structures above. Then there exists \( \|\alpha\| \leq \infty \) by \( \|\alpha\| := \inf_{x \in E_1} (v(\alpha x) - vX) \); that is, \( |\alpha| := e^{-v(\alpha)} \) is the operator norm of the bounded operator \( \alpha : (E_1, | \cdot |) \to (F, | \cdot |) \). We say that \( \alpha \) is filtered if \( \|\alpha\| > 0 \). We call it a **filtered isomorphism** if \( \alpha \) is a linear isomorphism with \( \alpha, \alpha^{-1} \) filtered. The latter is equivalent to \( \|\alpha\| = \|\alpha^{-1}\| = 0 \) or to \(|\alpha| = |\alpha^{-1}| = 1 \) because \( |\alpha\beta| \leq |\alpha||\beta| \) for linear maps \( \alpha, \beta \) between valued vector spaces.

Let \((E_1, v_1), \ldots, (E_d, v_d), (F, v_3)\) be valued \( \Lambda \)-vector spaces and \( \alpha : E_1 \times \cdots \times E_k \to F \) a \( \Lambda \)-multi-linear map. Define then \( \|\alpha\| \leq \infty \) by

\[
\|\alpha\| := \inf_{(x_1, \ldots, x_k) \in E_1 \times \cdots \times E_k} (v(\alpha(x_1, \ldots, x_k)) - v(x_1) - \cdots - v(x_k)). \tag{2.12}
\]

We say that \( \alpha \) is filtered if \( \|\alpha\| > 0 \). We denote by \( \text{hom}(E_1, \ldots, E_k; F) \) the set of filtered \( \Lambda \) multi-linear maps from \( E_1 \times \cdots \times E_k \) to \( F \). The pair \( (\text{hom}(E_1, \ldots, E_k; F), v) \) has the obvious structure of a valued \( \Lambda \)-vector space. If \((F, v)\) is complete then so is \( (\text{hom}(E_1, \ldots, E_k; F), v) \).
Let \((E_i, \nu)_{i \in I}\) be a family of complete valued \(\Lambda\)-vector spaces. Then there exists a valuation \(\bigoplus_{i \in I} E_i \rightarrow (-\infty, \infty]\) defined by \((x_i)_{i \in I} \mapsto \min_{i \in I} \nu x_i\). The complete direct sum \(\bigoplus_{i \in I} E_i, \nu\) is the completion of the direct sum \((\bigoplus_{i \in I} E_i, \nu)\).

We give a basic example of finite-dimensional valued \(\Lambda\)-vector space.

**Example 2.10.** Let \(n > 0\) be an integer and \(E\) an \(n\)-dimensional \(\Lambda\)-vector space. We give \(E\) a certain valuation. Let \(e := \{e_1, \ldots, e_n\} \subset E\) be a basis with \(\nu e_1 = \cdots = \nu e_n = 0\), which exists because \(T^n \subset \Lambda\) for every \(\gamma \in \mathbb{R}\). Then \(E\) has a valuation \(\nu_e : E \rightarrow (-\infty, \infty]\) defined by \(x = x_1 e_1 + \cdots + x_n e_n \mapsto \min\{\nu x_1, \ldots, \nu x_n\}\) for \(x_1, \ldots, x_n \in \Lambda\). By the strong triangle inequality every valuation \(\nu\) on \(E\) satisfies \(\nu x \geq \nu_e x\) for \(x \in E\). In particular, if \(f = \{f_1, \ldots, f_n\} \subset E\) is another basis with \(\nu f_1 f_1 = \cdots = \nu f_n f_n = 0\) then \(\nu f x \geq \nu_e x\) for \(x \in E\). Swapping the roles of \(f, e\) we get the reverse inequality; that is, \(\nu_f = \nu_e\), which we call the \(\ell^\infty\) valuation on \(E\).

We give three other equivalent definitions of the \(\ell^\infty\) valuation on \(E\). Let \(\nu : E \rightarrow (-\infty, \infty]\) be any valuation. We use the subring \(\Lambda^0 \subset \Lambda\) defined in Definition 2.2. Regard \(E\) as an \(\Lambda^0\) module and define a \(\Lambda^0\) submodule \(E^0 \subset E\) by \(E^0 := \{e \in E : \nu e \geq 0\}\). Then \(\Lambda^+ E^0 := \{ae : a \in \Lambda^+, e \in E^0\}\) is a \(\Lambda^0\) submodule of \(E^0\) and the quotient \(\tilde{E} := E^0/\Lambda^+ E^0\) has the vector space structure over \(\Lambda^0/\Lambda^+\) := \(K\). We prove that the following four conditions are equivalent: (i) \(\nu\) is the \(\ell^\infty\) valuation; (ii) \(E^0\) is a rank-\(n\) free \(\Lambda^0\) module; (iii) \(E^0\) is a finitely generated \(\Lambda^0\) module; and (iv) the \(K\)-vector space \(E^0/\Lambda^+ E^0\) has dimension \(\geq n\). Clearly (i) implies (ii) and (ii) implies (iii). We prove that (iii) implies (iv). Take \(e_1, \ldots, e_p \in E^0\) such that their images \(\bar{e}_1, \ldots, \bar{e}_p \in E^0/\Lambda^+ E^0\) are a \(K\)-basis. Then as \(E^0\) is finitely generated we can use Nakayama’s lemma [41, Chapter X, Lemma 4.3] (for non-Noetherian local rings) which implies \(E^0 = \Lambda^0 e_1 + \cdots + \Lambda^0 e_p\). So \(E = \Lambda e_1 + \cdots + \Lambda e_p\) and \(p \geq n\), which is the condition (iv).

We prove now that (iv) implies (i). Take \(e_1, \ldots, e_n \in E^0\) such that their images \(\bar{e}_1, \ldots, \bar{e}_n \in E^0/\Lambda^+ E^0\) are linearly independent over \(K\). Then \(F := \Lambda^0 e_1 + \cdots + \Lambda^0 e_n \leq E^0\) and we show that the equality holds. Otherwise there is some \(x \in E^0 \setminus F\). By re-ordering \(e_1, \ldots, e_n\) we can write \(x = a_1 T^{\gamma_1} e_1 + \cdots + a_m T^{\gamma_m} e_m\) where \(m \leq n\), \(a_1, \ldots, a_m \in E \setminus \{0\}\) with \(\nu a_1 = \cdots = \nu a_m = 0\), \(\gamma_1 = \cdots = \gamma_l < \gamma_{l+1} \leq \cdots \leq \gamma_m\). Then

\[
T^{-\gamma_l} x = a_1 e_1 + \cdots + a_l e_l + a_{l+1} T^{\gamma_{l+1} - \gamma_l} + \cdots + a_m T^{\gamma_m - \gamma_l} e_m
\]

and \(T^{-\gamma_l} x, a_{l+1} T^{\gamma_{l+1} - \gamma_l} + \cdots + a_m T^{\gamma_m - \gamma_l} e_m\) have positive valuation. So \(\bar{a}_1 \bar{e}_1 + \cdots + \bar{a}_l \bar{e}_l = 0\) in \(\tilde{E}\). But none of \(\bar{a}_1, \ldots, \bar{a}_l\) is zero, which contradicts that \(\bar{e}_1, \ldots, \bar{e}_n\) are linearly independent. Thus (i)–(iv) are equivalent.

**Remark 2.11.** Consider the category whose objects are \(n\)-dimensional valued \(\Lambda\)-vector spaces, whose morphisms are filtered homomorphisms. Then there is an equivalence from the category of rank-\(n\) free \(\Lambda^0\) modules to that of \(n\)-dimensional valued vector spaces with \(\ell^\infty\) valuations. The function between the object sets is defined by \(F \mapsto F \otimes_{\Lambda^0} \Lambda\). Each morphism \(\phi \in \text{hom}(F, F')\) is mapped to \(\phi \otimes \text{id}\).
When \( n = 1 \) every valuation is the \( \ell^\infty \) valuation. So the category of rank-1 free \( A^0 \) modules is equivalent to that of 1-dimensional filtered local systems.

We define then the filtered local systems.

**Definition 2.12.** By a filtered local system over a topological space we mean one of non-zero finite-dimensional valued vector spaces whose parallel transports are filtered homomorphisms of valued vector spaces.

We make now the relevant definitions about discrete sub-monoids of \([0, \infty)\).

**Definition 2.13.** Denote by \( \mathbb{N} \) the set of non-negative integers. Let \( \Gamma \subset [0, \infty) \) be a discrete sub-monoid, that is, a discrete subset containing 0 and closed under addition. By a decomposition of \((k, \gamma) \in (\mathbb{N} \times \Gamma) \setminus \{(0,0)\}\) we mean \( l \in \mathbb{N} \setminus \{0\} \), \( a = (a_0, \ldots, a_l) \in \mathbb{N}^{l+1} \) with \( 0 = a_0 \leq \ldots \leq a_l = k \), and \( \alpha = (\alpha_0, \ldots, \alpha_l) \in \Gamma^{l+1} \) with \( a_0 + \cdots + a_l = \gamma \), such that \((l, a_0), (a_1 - a_0, \alpha_1), \ldots, (a_l - a_{l-1}, \alpha_l) \neq (0,0). \) By a reduced decomposition of \((k, \gamma)\) we mean \( p, q \in \mathbb{N} \) and \( \alpha_0, \alpha_1 \in \Gamma \) with \( 0 \leq p \leq q \leq k \), \( \alpha_0 + \alpha_1 = \gamma \) and \((k+p-q+1, \alpha_0), (q-p, \alpha_1) \in (\mathbb{N} \times \Gamma) \setminus \{(0,0)\}\), see Equation (2.13) below for a typical formula where this notion is used. Every decomposition \((l, a, \alpha)\) with \( a_0 = 0, \ldots, a_p = p \) and \( a_{p+1} = q, \ldots, a_l = q + l - p - 1 \) for some \( p, q \) defines indeed the reduced decomposition \((p, q; \alpha_0, \alpha_1)\).

Choose a total order \( \leq \) on \((\mathbb{N} \times \Gamma) \setminus \{(0,0)\}\) such that there is an ordered set isomorphism \((\mathbb{N} \times \Gamma) \setminus \{(0,0)\}, \leq) \cong (\mathbb{N}, \leq)\), which we denote by \((k, \gamma) \mapsto [k, \gamma]\), and if \((l, a, \alpha)\) are as above a decomposition of \((k, \gamma) \in (\mathbb{N} \times \Gamma) \setminus \{(0,0)\}\) then

\[
\max\{[l, a_0], [a_1 - a_0, \alpha_1], \ldots, [a_l - a_{l-1}, \alpha_l]\} \leq [k, \gamma]\]

(2.13)

with equality if and only if at least \( l \) of \((l, a_0), (a_1 - a_0, \alpha_1), \ldots, (a_l - a_{l-1}, \alpha_l)\) are equal to \((1,0)\). Setting \((l, a_0) = (k, \gamma)\) and \((a_1 - a_0, \alpha_1), \ldots, (a_l - a_{l-1}, \alpha_l) = (1,0)\) we get \((1,0) \leq (k, \gamma)\); that is, \((1,0)\) is the least element of \((\mathbb{N} \times \Gamma) \setminus \{(0,0)\}\). In other words, \([k, \gamma] = 0\) if and only if \((k, \gamma) = (1,0)\).

We choose the order \( \leq \) for every discrete sub-monoid \( \Gamma \subset [0, \infty) \) so that if \( \Gamma, \Delta \subset [0, \infty) \) are two discrete sub-monoids with \( \Gamma \subseteq \Delta \) then the induced map \((\mathbb{N} \times \Gamma) \setminus \{(0,0)\} \to (\mathbb{N} \times \Delta) \setminus \{(0,0)\}\) will preserve the orders. Here is an explicit way of doing this. Set \([0] := 0\) and for \( \gamma \in \Gamma \setminus \{0\}\) set

\[
[k, \gamma] := \max\{n \in \mathbb{N} : \gamma = \gamma_1 + \cdots + \gamma_n; \gamma_1, \ldots, \gamma_n \in \Gamma \setminus \{0\}\}.\]

(2.14)

The maximum makes sense because \( \Gamma \subset [0, \infty) \) is discrete and has the least positive element. It is clear that \( [\gamma + \delta] \geq [\gamma] + [\delta] \) for \( \gamma, \delta \in \Gamma \). For \((k, \gamma), (l, \delta) \in (\mathbb{N} \times \Gamma) \setminus \{(0,0)\}\) define \((k, \gamma) \leq (l, \delta)\) by either \( k + [\gamma] + \gamma < l + [\delta] + \delta \) or \( k + [\gamma] + \gamma = l + [\delta] + \delta \) and \( \gamma \leq \delta \). Then we can check all the conditions above.

Let \( (A, \nu) \) be a Novikov field, which we shall fix in this section. We define the notion of gapped \( A_N \) categories, \( N \in \mathbb{N} \cup \{\infty\} \), over \( (A, \nu) \). The morphisms spaces are \( \mathbb{Z} \)-graded. The \( A_\infty \) structures are curved, including the \( m^0 \) terms.

**Definition 2.14.** We begin with a filtered curved \( A_\infty \) category \( (A, m) \) which consists of an object set \( \text{obj}\ A \), a family \( (A_{XY}, \nu)_{X,Y \in \text{obj}\ A} \) of \( \mathbb{Z} \)-graded complete
vector spaces, a discrete sub-monoid $\Gamma \subset \text{obj} \mathcal{A}$ systems. In the references [9, 21] they define $m^k_{x_0, \ldots, x_k} \in \text{hom}^1(\mathcal{A}_{x_0,x_1}[1], \ldots, \mathcal{A}_{x_{k-1},x_k}[1]; \mathcal{A}_{x_0,x_k}[1])$, $k \in \mathbb{N}$ and $x_0, \ldots, x_k \in \text{obj} \mathcal{A}$, such that

$$0 = \sum (-1)^{\# m^{k+p-q+1}(x_1, \ldots, x_p, m^{n-p}(x_{p+1}, \ldots, x_q), x_{q+1}, \ldots, x_k)} \quad (2.15)$$

where $x_1 \in \mathcal{A}_{x_0,x_1}[1], \ldots, x_k \in \mathcal{A}_{x_{k-1},x_k}[1]$; $\# := \deg x_1 + \cdots + \deg x_p$; and $p, q$ are any integers with $0 \leq p \leq q \leq k$. For $k = 0$ we have $m^0_{x} \in \mathcal{A}_{X}^k$ for every $X \in \text{obj} \mathcal{A}$, and (2.15) means $m^k_{X,X} m^0_{x} = 0 \in \mathcal{A}_{X}^k$.

Suppose now given $N \in \mathbb{N} \sqcup \{\infty\}$. By a gapped $A_N$ category $(\mathcal{A}, m)$ we mean an object set $\text{obj} \mathcal{A}$, a family $(\mathcal{A}_{X,Y}, \text{v})_{X,Y \in \text{obj} \mathcal{A}}$ of $\mathbb{Z}$-graded complete valued $\Lambda$-vector spaces, a discrete sub-monoid $\Gamma \subset [0, \infty)$ and a family $(m^k_{x} = m^k_{X,y \ldots, x_k} \in \text{hom}^1(\mathcal{A}_{x_0,x_1}[1], \ldots, \mathcal{A}_{x_{k-1},x_k}[1]; \mathcal{A}_{x_0,x_k}[1]))$, $(k, \gamma) \in (\mathbb{N} \times \Gamma) \setminus \{(0, 0)\}$ with $[k, \gamma] \leq N$ and $x_0, \ldots, x_k \in \text{obj} \mathcal{A}$, such that

$$0 = \sum (-1)^{\# m^{k+p-q+1}_{x_0, \ldots, x_k}(x_1, \ldots, x_p, m^{n-p}_{x_0, x_k}(x_{p+1}, \ldots, x_q), x_{q+1}, \ldots, x_k)} \quad (2.16)$$

where $x_1, \ldots, x_k$, $\#, p, q$ are as in (2.15) and the sum is taken over reduced decompositions $(p, q; \alpha, \beta)$'s of $(k, \gamma)$.

The operators $m^{k+p-q+1}_{x_0, \ldots, x_k}, m^{n-p}_{x_0, x_k}$ are well-defined because $\max\{[q - p, \alpha], [k + p - q + 1, \beta]\} \leq N$ by (2.13).

When $N = \infty$ the gapped $A_{\infty}$ category $(\mathcal{A}, m)$ defines a filtered $A_{\infty}$ category of the same object set, the same morphism spaces, and the $m^k_{x}$ operators defined by $m^k_{X_0, \ldots, x_k} := \sum T^{n} m^k_{X_0, \ldots, x_k}$ where $\gamma \in \Gamma \setminus \{0\}$ for $k = 0$ and $\gamma \in \Gamma$ for $k > 0$. The sum is well-defined because $\text{hom}^1(\mathcal{A}_{X_0,x_1}[1], \ldots, \mathcal{A}_{x_{k-1},x_k}[1]; \mathcal{A}_{X_0,x_k}[1])$ is complete.

**Remark 2.15.** The definition of $\text{hom}^1(\mathcal{A}_{X_0,x_1}[1], \ldots, \mathcal{A}_{x_{k-1},x_k}[1]; \mathcal{A}_{x_0,x_k}[1])$ implies that every $m^k_{X_0, \ldots, x_k}$ is a filtered homomorphism between valued vector spaces. This will be crucial to defining the Fukaya categories with filtered local systems. In the references [9][21] they define $m^k_{X_0, \ldots, x_k}$ to be linear over the residue field $\Lambda^0/\Lambda^+$, determined uniquely from the sum $m^k_{X_0, \ldots, x_k}$: this is not true in our circumstances.

They also work with gapped $A_N$ algebras which are gapped $A_N$ categories with a single object; or they are expressible as the pair $(\mathcal{A}, m)$ of a $\mathbb{Z}$-graded complete valued $\Lambda$-vector space $A$ and a family $m = (m^k_{\gamma} \in \text{hom}^1(\mathcal{A}, A; A))$, $(k, \gamma) \in (\mathbb{N} \times \Gamma) \setminus \{(0, 0)\}$ with $[k, \gamma] \leq N$, satisfying the same equations. If $(\mathcal{A}, m)$ is a gapped $A_N$ category then $\bigoplus_{X,Y \in \text{obj} \mathcal{A}} \mathcal{A}_{X,Y}$, the complete direct sum, has a gapped $A_N$ algebra structure $m$ defined in the obvious way. Conversely, let $(\mathcal{A}, m)$ be a gapped $A_N$ algebra with $A = \bigoplus_{X,Y \in \mathcal{X}} \mathcal{A}_{X,Y}$ such that the induced map $m^k_{x_0} : A_{x_1,Y_1} \times \cdots \times A_{x_k,Y_k} \to A$ is zero unless $Y_1 = X_2, \ldots, Y_{k-1} = X_k$; and if this is the case then $m^k_{x}$ has image in $A_{x_1,Y_1}$. Then $\mathcal{X}', (\mathcal{A}_{XY})_{X,Y \in \mathcal{X}'}$ and $m$ define a gapped $A_N$ category.

We define the truncations of gapped $A_N$ functors, $N \in \mathbb{N} \sqcup \{\infty\}$, and the full subcategories of gapped $A_N$ functors.

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Definition 2.16. Let \( M, N \in \mathbb{N} \cup \{\infty\} \) with \( M \leq N \) and \((\mathcal{A}, m)\) a gapped \( A_N \) category. The \( A_M \) truncation of \((\mathcal{A}, m)\) means the same object set \( \text{obj} \mathcal{A} \), the same morphism spaces \((\mathcal{A}_{X,Y})_{X,Y \in \text{obj} \mathcal{A}}\) and the subfamily \((m_{X_0,\ldots,X_k}^{k,\gamma})_{\gamma \leq M}\): these define a gapped \( A_M \) category. By a full subcategory \((\mathcal{B}, m)\) of \((\mathcal{A}, m)\) we mean a subset \( \text{obj} \mathcal{B} \subseteq \text{obj} \mathcal{A} \), the subfamily \((A_{X,Y})_{X,Y \in \text{obj} \mathcal{B}}\) and the subfamily \((m_{X_0,\ldots,X_k}^{k,\gamma})_{X_0,\ldots,X_k,\gamma \in \text{obj} \mathcal{B}}\): these define a gapped \( A_Y \) category.

We define the gapped \( A_N \) functors, \( N \in \mathbb{N} \cup \{\infty\} \).

Definition 2.17. Let \( \mathcal{A}, \mathcal{B} \) be two filtered \( A_\infty \) categories. By a filtered \( A_\infty \) functor from \( \mathcal{A} \) to \( \mathcal{B} \) we mean a function \( f : \mathcal{A} \rightarrow \mathcal{B} \) between the object sets and a family

\[
(f^k = f^{k}_{X_0,\ldots,X_k} \in \text{hom}^0(\mathcal{A}_{X_0,x_1},\ldots,\mathcal{A}_{X_k-1,x_k}[1]; \mathcal{B}_{f_{X_0}f_{X_k}[1]})),
\]

(2.17)

\( k \in \mathbb{N} \) and \( X_0,\ldots,X_k \in \text{obj} \mathcal{A} \), such that (i) for every \( X \in \text{obj} \mathcal{A} \) the element \( f_X^0 \in \mathcal{A}_{X,X} \) has positive valuation, that is, \( \forall f_X^0 > 0 \); and for \( X_0,\ldots,X_k \in \text{obj} \mathcal{A} \) with \( k \in \mathbb{N} \) we have

\[
\sum (-1)^\# f^{k+p-q+1}(x_1,\ldots,x_p, m^{q-p}(x_{p+1},\ldots,x_q), x_{q+1},\ldots,x_k)
= \sum n^l(f^{a_1}(x_1,\ldots,x_{a_1}),\ldots,f^{a_i-a_i-1}(x_{a_i-1+1},\ldots,x_{a_i}).
\]

(2.18)

where \( x_1,\ldots,x_k,\#, p, q \) are as in (2.15) and \( a_1,\ldots,a_i \) are any integers with \( 0 \leq a_1 < \ldots < a_i = k \). The right-hand side of (2.18) converges by (i) above.

Suppose now that \((\mathcal{A}, m), (\mathcal{B}, n)\) are gapped \( A_N \) categories, \( N \in \mathbb{N} \cup \{\infty\} \). Note that any two discrete sub-monoids of \([0, \infty)\) are contained in some single discrete sub-monoid of \([0, \infty)\). By a gapped \( A_N \) functor from \( \mathcal{A} \) to \( \mathcal{B} \) we mean a function \( f : \text{obj} \mathcal{A} \rightarrow \text{obj} \mathcal{B} \) between the object sets; a discrete sub-monoid \( \Gamma \subseteq [0, \infty) \) with respect to which \( \mathcal{A}, \mathcal{B} \) are gapped; and a family

\[
(f^k = f^{k}_{X_0,\ldots,X_k} \in \text{hom}^0(\mathcal{A}_{X_0,x_1},\ldots,\mathcal{A}_{X_k-1,x_k}[1]; \mathcal{B}_{f_{X_0}f_{X_k}[1]})),
\]

(2.19)

\( (k, \gamma) \in (\mathbb{N} \times \Gamma) \setminus \{(0, 0)\} \) with \( [k, \gamma] \leq N \) and \( X_0,\ldots,X_k \in \text{obj} \mathcal{A} \), such that

\[
\sum (-1)^\# f^{k+p-q+1}(x_1,\ldots,x_p, m^{q-p}(x_{p+1},\ldots,x_q), x_{q+1},\ldots,x_k)
= \sum n^l(f^{a_1}(x_1,\ldots,x_{a_1}),\ldots,f^{a_i-a_i-1}(x_{a_i-1+1},\ldots,x_{a_i}).
\]

(2.20)

where \( x_1,\ldots,x_k,\#, p, q, \alpha, \beta \) are as in (2.15) and \( (l; a_0,\ldots,a_p; a_0,\ldots,a_p) \) any decomposition of \((k, \gamma)\). The operators \( n^l, f^{a_1}, f^{a_i-a_i-1} \) are well-defined because \([l, a_0], [a_1, a_1],\ldots,[a_i - a_i, a_i] \subseteq N \) by (2.12).

When \( N = \infty \) the gapped \( A_\infty \) functor \( f : (\mathcal{A}, m) \rightarrow (\mathcal{B}, n) \) defines a filtered \( A_\infty \) functor \((f^{k}_{X_0,\ldots,X_k})_{k \in \mathbb{N}, X_0,\ldots,X_k \in \text{obj} \mathcal{A}}\) between the corresponding filtered \( A_\infty \) categories, with \( f^{k}_{X_0,\ldots,X_k} := \sum T^f f^{k}_{X_0,\ldots,X_k} \) where \( \gamma \in \Gamma \setminus \{0\} \) for \( k = 0 \) and \( \gamma \in \Gamma \) for \( k > 0 \). The sum is well-defined because \( \text{hom}^0(\mathcal{A}_{X_0,x_1},\ldots,\mathcal{A}_{X_k-1,x_k}[1]; \mathcal{B}_{f_{X_0}f_{X_k}[1]}) \) is complete.
We define the identity functors of gapped $A_N$ categories, $N \in \mathbb{N} \cup \{\infty\}$.

**Definition 2.18.** Let $A$ be a filtered $A_\infty$ category; we leave out $m$ for short.

The *identity* functor $A \to A$ is defined by taking the function $\text{obj} A \to \text{obj} A$ to be the identity; $f^N_X = 0 \in \mathcal{A}^N_{X,X}$ for $X \in \text{obj} A$; $f^N_{XY} : A_{XY} \to A_{XY}$ to be the identity for $X,Y \in \text{obj} A$; and $f^k_{X_0 \ldots X_k} = 0$ for $X_0, \ldots, X_k \in \text{obj} A$ with $k \geq 2$.

Suppose now that $A$ is a gapped $A_N$ category, $N \in \mathbb{N} \cup \{\infty\}$. Then the identity functor $A \to A$ is defined by taking the function $\text{obj} A \to \text{obj} A$ to be the identity; $f^N_{XY} : A_{XY} \to A_{XY}$ to be the identity for $X,Y \in \text{obj} A$; and $f^k_{X_0 \ldots X_k} = 0$ for $X_0, \ldots, X_k \in \text{obj} A$ with $(k,\gamma) \in (\mathbb{N} \times \Gamma) \setminus \{(0,0),(1,0)\}$.

We define the composites of gapped $A_N$ functors, $N \in \mathbb{N} \cup \{\infty\}$.

**Definition 2.19.** Let $A,B,C$ be filtered $A_\infty$ categories and $f : A \to B$, $g : B \to C$ filtered $A_\infty$ functors. Then the *composite* $A_N$ functor $g \circ f : A \to C$ means the ordinary composite $\text{obj} A \to \text{obj} C$ and the family $(g \circ f)^N_{X_0 \ldots X_k}$, $X_0, \ldots, X_k \in \text{obj} A$ with $k \in \mathbb{N}$, defined by

$$(g \circ f)^N(x_1, \ldots, x_k) := \sum (-1)^l g^l(f^{a_1}(x_1, \ldots, x_{a_1}), \ldots, f^{a_{l-1}}(x_{a_1+1}, \ldots, x_{a_l}))$$

where $x_1, \ldots, x_k$, $\#_1, a_1, \ldots, a_l$ are as in (2.18).

Suppose now that $A,B,C$ are filtered $A_\infty$ categories, $N \in \mathbb{N} \cup \{\infty\}$; and $f : A \to B$, $g : B \to C$ filtered $A_\infty$ functors. Then the composite $A_N$ functor $g \circ f : A \to C$ means the ordinary composite $\text{obj} A \to \text{obj} C$; a discrete sub-monoid $\Gamma \subset [0, \infty)$ with respect to which $f,g$ are gapped; and the family $(g \circ f)^N_{X_0 \ldots X_k}$, $(k,\gamma) \in (\mathbb{N} \times \Gamma) \setminus \{(0,0),(1,0)\}$ with $[k,\gamma] \leq N$ and $X_0, \ldots, X_k \in \text{obj} A$, defined by

$$(g \circ f)^N(x_1, \ldots, x_k) := \sum (-1)^l g^l(f^{a_1}(x_1, \ldots, x_{a_1}), \ldots, f^{a_{l-1}}(x_{a_1+1}, \ldots, x_{a_l}))$$

where $x_1, \ldots, x_k$, $\#_1, a_1, \ldots, a_p, \alpha_1, \ldots, \alpha_p$ are as in (2.20). This is independent of the choice of $\Gamma$, in the sense that $(g \circ f)^N_{X_0 \ldots X_k} = 0$ for $\gamma$ not in the smallest choice of $\Gamma$.

We define the *homotopies* of gapped $A_N$ functors, $N \in \mathbb{N} \cup \{\infty\}$. We make a rather strong hypothesis on the functions between object sets.

**Definition 2.20.** Let $(A,m)$, $(B,n)$ be filtered $A_\infty$ categories and $f,g : (A,m) \to (B,n)$ filtered $A_\infty$ functors. Suppose that $f,g$ induce the same function $\text{obj} A \to \text{obj} B$ between the object sets, which we denote by $f$ for short. By a *homotopy* $H : f \Rightarrow g$ we mean a family

$$(H^k = H^k_{X_0 \ldots X_k} \in \text{hom}^{-1}(A_{X_0 X_1[a_1]}, \ldots, A_{X_{k-1} X_k[a_k]}; A_{fX_0 fX_1}[1]));$$

(2.21)

$k \in \mathbb{N}$ and $X_0, \ldots, X_k \in \text{obj} A$, such that (i) for every $X \in \text{obj} A$ the element $H^k_X \in A_{fX X}^{1}$ has positive valuation, that is, $\nu H^k_X > 0$; and (ii) for $X_0, \ldots, X_k \in \text{obj} A$ with $k \in \mathbb{N}$ we have

$$\sum (-1)^b H^{k+p+q+1}(x_1, \ldots, x_p, m_{p-q}(x_{p+1}, \ldots, x_q), x_{q+1}, \ldots, x_k)$$

$$= \sum n_{i,j+1}(f^{a_1}(x_1, \ldots, x_{a_1}), \ldots, f^{a_{j-1}}(x_{a_1+1}, \ldots, x_{a_j}), H^{b_0-a_1}(x_{a_1+1}, \ldots, x_{b_0}), g^{b_0-b_0}(x_{b_0+1}, \ldots, x_{b_1}), \ldots, g^{b_j-b_j}(x_{b_j-1+1}, \ldots, x_{b_j}))$$

(2.21)
where $x_1, \ldots, x_k$, $\#, p, q$ are as in \cite{218}; and $a_1, \ldots, a_i, b_0, \ldots, b_j$ any integers with $0 \leq a_1 \leq \ldots \leq a_i \leq b_0 \leq b_1 \leq \ldots \leq b_j = k$. The right-hand side converges by (i) above.

Suppose now that $(A, m), (B, n)$ are gapped $A_N$ categories, $N \in \mathbb{N} \cup \{\infty\}$; and $f, g : (A, m) \to (B, n)$ gapped $A_N$ functors with the same function $f : \text{obj} A \to \text{obj} B$ between the object sets. By a homotopy $H : f \Rightarrow g$ we mean a discrete sub-monoid $\Gamma \subset [0, \infty)$ with respect to which $f, g$ are both gapped and a family

\begin{equation}
(H^k)^{\gamma} = H^{k^\gamma}_{X_0, \ldots, X_k} \in \text{hom}^{-1}(A_{X_0, X_1[1]}, \ldots, A_{X_{k-1}, X_k[1]; A_{fX_0, fX_1[1]}}),
\end{equation}

$(k, \gamma) \in (\mathbb{N} \times \Gamma) \setminus \{(0, 0)\}$ with $[k, \gamma] \leq N$ and $X_0, \ldots, X_k \in \text{obj} A$, such that

\[
\sum (-1)^{#} H^{\alpha \, p - q + 1}_{\beta} (x_1, \ldots, x_p, m_{\beta}^{q - p} (x_{p+1}, \ldots, x_q), x_{q+1}, \ldots, x_k) = \sum n_{\alpha_0}^{i+j+1} f^{\alpha_1}_{\alpha_0} (x_1, \ldots, x_{a_1}), \ldots, f^{\alpha_{i-1} - a_{i-1}}_{\alpha_i} (x_{a_{i-1}+1}, \ldots, x_{a_i}), g^{b_j - b_{j-1}}_{\beta_j} (x_{b_{j-1}+1}, \ldots, x_{b_j})
\]

where $x_1, \ldots, x_k$, $\#, p, q, \alpha, \beta$ are as in \cite{220} and $i + j + 1$, $(a_0, \ldots, a_i, b_0, \ldots, b_j)$, $(\alpha_0, \ldots, \alpha_i, \beta_0, \ldots, \beta_j)$ any decomposition of $(k, \gamma)$. The operators $n_{\alpha_0}^{i+j+1}, f^{\alpha_1}_{\alpha_0}, \ldots, f^{\alpha_{i-1} - a_{i-1}}_{\alpha_i}, g^{b_j - b_{j-1}}_{\beta_j}$ are well-defined because $[i+j+1, \alpha_0], [a_1, \alpha_1], \ldots, [a_i, \alpha_i], [b_0 - a_i, \beta_0], [b_1 - b_0, \beta_1], \ldots, [b_j - b_{j-1}, \beta_j]$ are all $\leq N$ by \cite{213}.

When $N = \infty$ the gapped $A_N$ homotopy $H : f \Rightarrow g$ defines a filtered $A_{\infty}$ homotopy $(H^k)_{\gamma}^{X_0, \ldots, X_k} \in \text{obj} A$ between the corresponding filtered $A_{\infty}$ functors, with $H^k_{X_0, \ldots, X_k} := \sum T^\gamma H^{k^\gamma}_{X_0, \ldots, X_k}$ where $\gamma \in \Gamma \setminus \{0\}$ for $k = 0$ and $\gamma \in \Gamma$ for $k > 0$. The sum is well-defined because $\text{hom}^{-1}(A_{X_0, X_1[1]}, \ldots, A_{X_{k-1}, X_k[1]; A_{fX_0, fX_1[1]}})$ is complete.

**Remark 2.21.** We can make an equivalent definition by using models of $B \times [0, 1]$ as Fukaya, Oh, Ohta and Ono do \cite{21} §4.2. This is the result of obstruction theory arguments \cite{21} §4.4 which extend readily to our context; the major difference is that mentioned in Remark 2.15.

We define the notion of being homotopic for gapped $A_N$ functors, $N \in \mathbb{N} \cup \{\infty\}$.

**Definition 2.22.** Let $(A, m), (B, n)$ be gapped $A_N$ categories and $f, g : (A, m) \to (B, n)$ gapped $A_N$ functors which induce the same function between the object sets. We say that $f, g$ are homotopic if there is a homotopy from $f$ to $g$.

The following is a straightforward extension of Fukaya, Oh, Ohta and Ono’s result for gapped $A_N$ algebras.

**Lemma 2.23** (cf. \cite{21} Proposition 4.2.37). Let $(A, m), (B, n)$ be gapped $A_N$ categories and suppose given a function $\text{obj} A \to \text{obj} B$. Then being homotopic is an equivalence relation for those gapped $A_N$ functors from $(A, m)$ to $(B, n)$ inducing the given function $\text{obj} A \to \text{obj} B$.

\[\square\]
We define the homotopy equivalences of gapped \(A_N\) categories, \(N \in \mathbb{N} \cup \{\infty\} \).

**Definition 2.24.** Let \((\mathcal{A}, m), (\mathcal{B}, n)\) be gapped \(A_N\) categories of the same object set and \(f : (\mathcal{A}, m) \to (\mathcal{B}, n)\) a gapped \(A_N\) functor which induces the identity on the object set. We call \(f\) a **homotopy equivalence** if there exists a gapped \(A_N\) functor \(g : (\mathcal{B}, n) \to (\mathcal{A}, m)\) which induces the identity on the object set and such that \(g \circ f, f \circ g\) are homotopic to the identity functors of \((\mathcal{A}, m), (\mathcal{B}, n)\) respectively. We say that \((\mathcal{A}, m), (\mathcal{B}, n)\) are **homotopy equivalent** if there exists a homotopy equivalence \((\mathcal{A}, m) \to (\mathcal{B}, n)\).

The following is also a straightforward extension of Fukaya, Oh, Ohta and Ono’s result for gapped \(A_N\) algebras.

**Lemma 2.25** (cf. [21], Corollary 4.2.44 and Remark 7.2.71). Being homotopy equivalent is an equivalence relation for gapped \(A_N\) categories of the same object set.

Let \((\mathcal{A}, m), (\mathcal{B}, n)\) be gapped \(A_N\) categories of the same object set and \(f : (\mathcal{A}, m) \to (\mathcal{B}, n)\) a gapped \(A_N\) functor which induces the identity on the object set. Then \(f\) is a homotopy equivalence if and only if for any \(X, Y \in \text{obj} \mathcal{A}\) the co-chain map \(f_{XY}^*: (A_{XY}, m_{XY}^{10}) \to (B_{XY}, n_{XY}^{10})\) induces an isomorphism of the cohomology groups.

Here is the obstruction theory result for gapped \(A_N\) categories which extends that of Fukaya, Oh, Ohta and Ono [21] Theorem 7.2.72.

**Theorem 2.26.** Let \((\mathcal{B}, n)\) be a gapped \(A_{N+1}\) category, \(N \in \mathbb{N}\), and \((\mathcal{B}', n)\) a full subcategory of the \(A_N\) truncation of \((\mathcal{B}, n)\). Let \((\mathcal{A}', m)\) be a gapped \(A_N\) category with \(\text{obj} \mathcal{A}' = \text{obj} \mathcal{B}'\), and \(g : (\mathcal{A}', m) \to (\mathcal{B}', n)\) a homotopy equivalence which induces the identity on the object set. Then there exist a gapped \(A_{N+1}\) category \((\mathcal{A}, m)\) with \(\text{obj} \mathcal{A} = \text{obj} \mathcal{B}\), which contains \((\mathcal{A}', m)\) as a full subcategory in its \(A_N\) truncation; and a gapped \(A_{N+1}\) functor \(f : (\mathcal{A}, m) \to (\mathcal{B}, n)\) which induces the identity on the object set, whose \(A_N\) truncation agrees with \(g\) on \(\mathcal{A}'\).

**Proof.** Recall from definition that the \(A_0\) truncation of \((\mathcal{A}', m)\) is a family of cochain complexes. We assign to \(X, Y \in \text{obj} \mathcal{A}\) a co-chain complex \((A_{XY}, m_{XY}^{10})\) and a co-chain map \(f_{XY}^*: (A_{XY}, m_{XY}^{10}) \to (B_{XY}, n_{XY}^{10})\). For \(X, Y \in \text{obj} \mathcal{A}\) with \(X \notin \text{obj} \mathcal{A}'\) or \(Y \notin \text{obj} \mathcal{A}'\) set \((A_{XY}, m_{XY}^{10}) : = (A_{XY}, m_{XY}^{10})\) and \(f_{XY}^* : = f_{XY}^*\). For \(X, Y \in \text{obj} \mathcal{A}\) with \(X \notin \text{obj} \mathcal{A}'\) and \(Y \notin \text{obj} \mathcal{A}'\) set \((A_{XY}, m_{XY}^{10}) : = (B_{XY}, n_{XY}^{10})\) and define \(f_{XY}^*: A_{XY} \to B_{XY}\) to be the identity map. In both cases the co-chain map \(f_{XY}^*: (A_{XY}, m_{XY}^{10}) \to (B_{XY}, n_{XY}^{10})\) induces an isomorphism of the cohomology groups.

We extend these co-chain complexes to a gapped \(A_N\) category \((\mathcal{A}, m)\) and these co-chain maps to a gapped \(A_N\) functor \(f^N: (\mathcal{A}, m) \to (\mathcal{B}, n)^N\) where \((\mathcal{B}, n)^N\) is the \(A_M\) truncation of \((\mathcal{B}, n)\). We do this by an induction on \(M \in \{0, \ldots, N - 1\}\). Suppose therefore that we have defined \(m_{X_0 \ldots X_l}^M\) for \((l, \delta) \in (\mathbb{N} \times \Gamma) \setminus \{(0, 0)\}\) with \([l, \delta] \leq M\) and for \(X_0 \ldots X_l \in \text{obj} \mathcal{A}\). We extend these to a gapped \(A_{M+1}\) category \((\mathcal{A}, m)\) and a gapped \(A_{M+1}\) functor \((\mathcal{A}, m) \to (\mathcal{B}, n)^{M+1}\).

Denote by \((k, \gamma) \in (\mathbb{N} \times \Gamma) \setminus \{(0, 0)\}\) the element with \([k, \gamma] = M + 1\). We assign to \(X_0, \ldots, X_k \in \text{obj} \mathcal{A}\) some filtered homomorphisms \(m_{X_0 \ldots X_k}^{k+1}, f_{X_0 \ldots X_k}^k\).
which satisfy the $A_{M+1}$ equations. For $X_0, \ldots, X_k \in \text{obj} \mathcal{A}'$ we use the given $m^{k_1}_{X_0 \ldots X_k}, f^{k_1}_{X_0 \ldots X_k}$; and otherwise we introduce some new $m^{k_1}_{X_0 \ldots X_k}, f^{k_1}_{X_0 \ldots X_k}$. Define a degree-one $\Lambda$-linear map $\partial$ from $\text{hom}^1(\mathcal{A}_{X_0}, x_1, \ldots, \mathcal{A}_{X_{k-1}}, x_k; \mathcal{A}_{X_0}, x_k)$ to itself by

$$\partial \phi := m^1_0 \circ \phi - (-1)^{\text{deg} \phi} \circ (\text{co-derivation of } m^1_0).$$  

(2.23)

This is well-defined because $v \phi \geq 0$ implies $v(\partial \phi) \geq 0$. And $\partial \circ \partial = 0$. Define an obstruction co-cycle $(\text{om})^{k_1}_{X_0 \ldots X_k} \in \text{hom}^2(\mathcal{A}_{X_0}, x_1, \ldots, \mathcal{A}_{X_{k-1}}, x_k; \mathcal{A}_{X_0}, x_k)$ by

$$(x_1, \ldots, x_k) \mapsto \sum (-1)^# \text{m}^{k+p-q+1}(x_1, \ldots, x_p, \text{m}^{q-p}(x_{p+1}, \ldots, x_q), x_{q+1}, \ldots, x_k)$$

where $[k + p - q + 1, \alpha] \leq M$ or $[q - p, \beta] \leq M$. Doing the same computation as Fukaya and others [21] (4.5.2.1) we find $\partial(\text{om})^{k_1}_{X_0 \ldots X_k} = 0$; that is, $(\text{om})^{k_1}_{X_0 \ldots X_k}$ is certainly a co-cycle. Their computation [21] (4.5.2.4) shows also that the $A_M$ homotopy equivalence $f : (\mathcal{A}, m) \to (\mathcal{B}, n)$ induces an isomorphism from the $\partial$-cohomology group $H^2(\text{hom}(\mathcal{A}_{X_0}, x_1, \ldots, \mathcal{A}_{X_{k-1}}, x_k; \mathcal{A}_{X_0}, x_k))$ to the $\partial$-cohomology group $H^2(\text{hom}(\mathcal{B}_{X_0}, x_1, \ldots, \mathcal{B}_{X_{k-1}}, x_k; \mathcal{B}_{X_0}, x_k))$, mapping the cohomology class of $(\text{om})^{k_1}_{X_0 \ldots X_k}$ to that of $(\text{om})^{k_1}_{X_0 \ldots X_k}$. But the latter vanishes for the obvious reason: $(\mathcal{B}, n)$ is already an $A_{M+1}$ category. Thus $(\text{om})^{k_1}_{X_0 \ldots X_k}$ is a co-boundary, written as $\partial \mu^{k_1}_{X_0 \ldots X_k}$ for some

$$\mu^{k_1}_{X_0 \ldots X_k} \in \text{hom}^1(\mathcal{A}_{X_0}, x_1, \ldots, \mathcal{A}_{X_{k-1}}, x_k; \mathcal{A}_{X_0}, x_k).$$  

(2.24)

Define then an obstruction co-cycle in $\text{hom}^1(\mathcal{A}_{X_0}, x_1, \ldots, \mathcal{A}_{X_{k-1}}, x_k; \mathcal{B}_{X_0}, x_k)$ by sending $(x_1, \ldots, x_k)$ to

$$
\left( \sum m^1_{\alpha_0}(f^{a_1}_{\alpha_1}(x_1, \ldots, x_{a_1}), \ldots, f^{a_{i-1}}_{\alpha_i}(x_{a_{i-1}}, \ldots, x_{a_i})) \right) + f^1_0 \circ \mu^{k}_{(x_1, \ldots, x_k)} + \sum (-1)^# f^{k+p-q+1}(x_1, \ldots, x_p, m^{q-p}(x_{p+1}, \ldots, x_q), x_{q+1}, \ldots, x_k)
$$

where $[k + p - q + 1, \alpha] \leq M$ or $[q - p, \beta] \leq M$; and $l, a_1, \ldots, a_i, \alpha_1, \ldots, \alpha_l$ are a decomposition of $(k, \gamma)$ with $(l, \alpha_0) > (1, 0)$ so that $[a_1, \alpha_1], \ldots, [a_{l-1}, \alpha_{l-1}, \alpha_l] \leq M$. As $f^1_0$ is a $k$-chain equivalence we can modify $\mu^{k}_{X_0 \ldots X_k}$ if we need, so that the corresponding obstruction co-cycle will vanish. So there is a gapped $A_{M+1}$ functor $f : (\mathcal{A}, m) \to (\mathcal{B}, n)^{M+1}$ as we want, completing the induction.

We have thus obtained a gapped $A_N$ category $(\mathcal{A}, m)$ and a gapped $A_N$ functor $f : (\mathcal{A}, m) \to (\mathcal{B}, n)^N$; that is, $m^{k_1}_{X_0 \ldots X_k}, f^{k_1}_{X_0 \ldots X_k}$ are given for every $(k, \gamma) \in \mathbb{N} \times \Gamma'$ \{0, 0\} with $[k, \gamma] \leq N$. Extend these to $(k, \gamma)$ with $[k, \gamma] = N + 1$, by repeating the obstruction theory arguments above; then the proof is complete.

\textbf{Remark 2.27.} The first step above has no counterpart in their original proof [21] Theorem 7.2.72. The last step is the closest to it; the difference is only that mentioned in Remark 2.15. The induction steps are similar but we must be more careful: we cannot change the given $m^{k_1}_{X_0 \ldots X_k}, f^{k_1}_{X_0 \ldots X_k}$.
2.3 Fukaya Categories

We begin with the definition of ambient spaces in which we do Floer theory. The example we have in mind is cotangent bundles, but we make a more general definition:

**Definition 2.28.** A compact symplectic manifold \((X,\omega)\) with contact boundary is a symplectic manifold which is given a 1-form \(\lambda\) near \(\partial X \subseteq X\) such that \(d\lambda = \omega\) and the vector field \(v_\lambda\) given by \(v_\lambda \cdot \omega = \lambda\) is outwards pointing on \(\partial X\). We call \(\lambda\) a Liouville 1-form of \((X,\omega)\).

In these circumstances there exists near \(\partial X \subseteq X\) a unique smooth function \(r_\lambda\) with values in \((0,1]\) such that \(r_\lambda|_{\partial X} = 1\) and \(dr_\lambda(v_\lambda) = r_\lambda\). (2.25)

We call \(r_\lambda\) the radius function near \(\partial X\).

We say that an almost complex structures \(J\) is compatible with the contact boundary if it is compatible with \(\omega\) in the usual sense, \(\lambda = Jdr_\lambda\) near \(\partial X\) and \(J\) is invariant near \(\partial X\) under the flow of \(v_\lambda\).

If there exists a global 1-form \(\lambda\) with \(d\lambda = \omega\) and satisfying the conditions above then we call \((X,\lambda)\) a Liouville domain.

We consider the following class of Lagrangians:

**Definition 2.29.** Let \((X,\omega)\) be a compact symplectic manifold with contact boundary, and \(\lambda\) a Liouville 1-form of it. Let \(\tilde{L}\) be a compact manifold possibly with boundary and \(\iota: \tilde{L} \to X\) a Lagrangian immersion. By an abuse of notation we shall usually mention only the image \(L := \iota(\tilde{L}) \subseteq X\) in place of the immersion map, and say that \(L\) is compact if so is \(\tilde{L}\).

We say that \(L\) is generic if its every self-intersection point is a transverse double point outside \(\partial X\). We say that \(L\) has Legendrian collar if \(\iota(\partial \tilde{L}) \subseteq \partial X\), \(\iota^* \lambda = 0\) near \(\partial \tilde{L} \subseteq \tilde{L}\) and \(\iota^* r_\lambda\) has regular value 1 (this is automatically true if \(\tilde{L}\) is closed).

Suppose now that \(L_0\) and \(L_1\) are two Lagrangians in \(X\) given by immersions \(\hat{L}_0 \to X\) and \(\hat{L}_1 \to X\). We say that \(L_0, L_1\) are generic if the immersion \(\hat{L}_0 \cup \hat{L}_1\) is generic.

We recall the notion of \(\mathbb{Z}\)-graded Lagrangians in \(\mathbb{Z}\)-graded symplectic manifolds. There are several ways of doing this \([31]\). We choose that which uses almost complex structures on the ambient symplectic manifold.

**Definition 2.30.** A \(\mathbb{Z}\)-grading of \((X,\omega)\) is the choice up to homotopy of a nowhere-vanishing section \(\Omega^2\) of the squared canonical bundle of \(X\) relative to some \(\omega\)-compatible almost complex structure \(J\).

**Remark 2.31.** A \(\mathbb{Z}\)-grading of \((X,\omega)\) exist if and only if \(2c_1(X) = 0\in H^2(X,\mathbb{Z})\) where \(c_1(X)\) denotes the first Chern class of \(X\).
Given a quadratic volume form $\Omega^2$ representing a $\mathbb{Z}$-grading on $X$, we associate to each Lagrangian immersion $\iota : \hat{L} \to X$ a unique smooth function $\phi : \hat{L} \to \mathbb{R}/2\pi \mathbb{Z}$ with $\iota^* \Omega^2 = \exp(\phi) \vol^2$. Here $\vol^2$ is the squared volume form of $\hat{L}$ with respect to the metric induced from $\omega, J$. To define $\vol^2$ we do not need $\hat{L}$ to be orientable.

**Definition 2.32.** We say that a Lagrangian immersion $\iota : \hat{L} \to X$ is $\mathbb{Z}$-graded if this is equipped with a lift of $\phi$ to the universal cover $\hat{\mathbb{R}} \to \mathbb{R}/2\pi \mathbb{Z} \cong S^1$. We say that $\iota$ is special of phase $\phi$ with respect to $\Omega^\otimes 2$ if $\phi$ is locally constant. When $\hat{L}$ is connected, $\phi$ must be constant and we shall say $\iota$ has phase $\phi \in \mathbb{R}$. We shall also use this constant function for the grading of $\iota$. The shift operator $[1]$ maps the $\mathbb{Z}$-graded Lagrangian $(\iota, \phi)$ to $(\iota, \phi - 2\pi)$.

**Remark 2.33.** The Lagrangian immersion admits a $\mathbb{Z}$-grading if and only if the group homomorphism $\phi_* : H_1(L, \mathbb{Z}) \to H_1(S^1, \mathbb{Z})$ vanishes.

The definition above of special Lagrangians is more general than that in the literature [34]. These agree when $J$ is integrable and $\Omega^2$ is the square of some holomorphic volume form.

**Definition 2.34.** Let $\iota : \hat{L} \to L \subset X$ be a generic Lagrangian immersion. We consider the fibre product

$$L \times_X L := \{(x, y) \in \hat{L} \times \hat{L} : \iota(x) = \iota(y)\} \tag{2.26}$$

which consists of the diagonal $\{(x, x) \in \hat{L} \times \hat{L}\}$, identified with $\hat{L}$, and the self-intersection pairs in $\hat{L}$. Suppose that $L$ is $\mathbb{Z}$-graded by a phase function $\phi : \hat{L} \to \mathbb{R}$ and we define then a Maslov index function $\mu_L : L \times_X L \to \mathbb{Z}$. Set $\mu_L \equiv 0$ on the diagonal $\hat{L}$. We associate to each pair $(x_1, x_2) \in L \times_X L$ an integer called the Maslov index. Put $x := \iota(x_1) = \iota(x_2) \in X$ and take a $\mathbb{C}$-linear isomorphism $(T_x X, J|_x) \cong \mathbb{C}^n$ which maps $\omega$ to $\frac{1}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$, $T_{x_2} L$ to $\mathbb{R}^n \subseteq \mathbb{C}^n$ and $T_x L$ to $\{(e^{i\theta_1} t_1, \cdots, e^{i\theta_n} t_n) \in \mathbb{C}^n : t_1, \cdots, t_n \in \mathbb{R}\}$ for some $\theta_1, \cdots, \theta_n \in (0, \pi)$. These $\theta_1, \cdots, \theta_n$ are unique up to order so we can define

$$\mu_L(x_1, x_2) := \frac{1}{2\pi} [2\theta_1 + \cdots + 2\theta_n + \phi(x_2) - \phi(x_1)]. \quad \tag{2.27}$$

The definition of phase functions implies readily that $\mu_L(x_1, x_2)$ is in fact an integer.

Suppose now that $L_1, L_2 \subseteq X$ are two generic Lagrangians, $\mathbb{Z}$-graded by $\phi_1 : L_1 \to \mathbb{R}$ and $\phi_2 : L_2 \to \mathbb{R}$ respectively. The fibre product $L_1 \times_X L_2$ is then a finite set and we define the Maslov index function $\mu_{L_1, L_2} : L_1 \times_X L_2 \to \mathbb{Z}$ by sending $(x_1, x_2)$ to the right-hand side of (2.27) above with $\phi_1(x_2), \phi_2(x_2)$ in place of $\phi(x_1), \phi(x_2)$ respectively.

**Remark 2.35.** Since $\theta_1, \cdots, \theta_n \in (0, \pi)$ it follows by (2.27) that

$$\mu_L(x_1, x_2) - n < \frac{\phi_2 - \phi_1}{2\pi} < \mu_L(x_1, x_2). \quad \tag{2.28}$$

In particular, if $\phi_2 - \phi_1$ is nearly zero then $\mu_L(x_1, x_2) \in [0, n]$ and if $\phi_1 = \phi_2$ then $\mu_L(x_1, x_2) \in [1, n-1]$. 

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Example 2.36. Take \( X = \mathbb{C}^n \), \( \Omega = dz_1 \wedge \cdots \wedge dz_n \) and \( L_2 = \mathbb{R}^n \subset \mathbb{C}^n \).

Define a \( \mathbb{Z} \)-grading \( \phi_2 : L_2 \to \mathbb{R} \) by \( \phi_2 \equiv 0 \). Take \( \theta_1, \ldots, \theta_m \in \left( \frac{\pi}{2}, \pi \right) \) and \( \theta_{m+1}, \ldots, \theta_n \in (0, \frac{\pi}{2}) \). For \( s \in [0, 1] \) define

\[
L_s := \left\{ (e^{is\theta_1-\pi})t_1, \ldots, (e^{is\theta_m-\pi})t_m, e^{is\theta_{m+1}}t_{m+1}, \ldots, e^{is\theta_n}t_n) : t_1, \ldots, t_n \in \mathbb{R} \right\}
\]

so \( L_0 = L_2 \). Define a \( \mathbb{Z} \)-grading \( \phi_s : L_s \to \mathbb{R} \) by starting with \( \phi_2 \equiv 0 \) on \( L_2 \) and lifting the path \([0, s] \). Then \( \phi_s := 2s((\theta_1 - \pi) + \cdots + (\theta_m - \pi) + \theta_{m+1} + \cdots + \theta_n) \) and in particular \( \phi_1 = 2(\theta_1 + \cdots + \theta_n) - 2m \). The Maslov index of \((L_1, L_2)\) at the intersection point \( 0 \in \mathbb{C}^n \) is now equal to \( \frac{1}{2\pi}[2\theta_1 + \cdots + 2\theta_n + \phi_2 - \phi_1] = m \). This agrees with the Morse index of the quadratic function \( (\tan \theta_1)^2 + \cdots + (\tan \theta_n)^2 \) on \( L_2 \). Our convention is thus the same as Seidel’s [51, §2d, (v)].

We extend the definition of relative spin structures to countably many Lagrangians. These will equip the pseudo-holomorphic curve moduli spaces with natural orientations.

Definition 2.37. Let \( X \) be a symplectic manifold, \( \hat{L} \) a compact manifold and \( \iota : \hat{L} \to X \) a generic Lagrangian immersion; we allow \( \hat{L}, \iota \) to have boundary. By a relative spin structure of \( \iota : \hat{L} \to X \) we mean the data \((\tau, V, o, \sigma, s)\) where \( \tau \) is a triangulation of \( X \), \( V \) an oriented vector bundle over the 3-skeleton \([X, \tau]^3\) of \((X, \tau)\), \( o \) an orientation of \( \hat{L} \), \( \sigma \) a triangulation of \( \hat{L} \) such that \( \iota \) induces a simplicial map \((\hat{L}, \sigma) \to (X, \tau)\), and \( s \) a spin structure of the bundle \( \iota^*V \oplus T\hat{L} \) restricted to the 2-skeleton of \((\hat{L}, \sigma)\).

Suppose now that \((L_N)_{N=0}^\infty\) is a sequence of compact manifolds which may have boundary, and \( \iota : \bigsqcup_{N=0}^\infty L_N \to X \) a generic Lagrangian immersion. By a relative spin structure of \( \iota : \bigsqcup_{N=0}^\infty L_N \to X \) we mean a sequence \((\tau_N, V_N, o_N, \sigma_N, s_N)_{N=0}^\infty\) such that each \((\tau_N, V_N, o_N, \sigma_N, s_N)\) is a relative spin structure of \( \iota|_{L_N} : L_N \to X \), each \((X, \tau_N)\) is a subcomplex of \((X, \tau_{N+1})\), and the restriction of each \( V_{N+1} \) to \([X, \tau_N]^3\) agrees with \( V_N \).

Remark 2.38. Suppose that \( V_0 = V_1 = V_2 = \cdots = V \), that \( V \) is defined over the whole \( X \) and that each \( s_N \) is a spin structure of \( \iota^*V \oplus TL_N \) over the whole \( L_N \). Then each Stiefel–Whitney class \( w_2(\cbar{L}_N) \in H^2(L_N, \mathbb{Z}/2\mathbb{Z}) \) is the pullback of \( w_2(V) \in H^2(X, \mathbb{Z}/2\mathbb{Z}) \). In this case we refer to \( w_2(V) \in H^2(X, \mathbb{Z}/2\mathbb{Z}) \) as the background class of the relative spin structure.

We can in practice take \( X \) to be embedded in the cotangent bundle \( T^*Q \) over a closed oriented manifold \( Q \) and take \( V = \pi^*TQ \), where \( \pi : T^*Q \to Q \) is the projection. So the background class is \( \pi^*w_2(TQ) \) and the zero-section \( Q \) will be given a spin structure of \( V|_Q \oplus TQ \).

We introduce now the notion of branes:

Definition 2.39. Let \((X, \omega)\) be a compact symplectic manifold with contact boundary, and \( \lambda \) a Livouville 1-form of it. By a brane on \( X \) we mean a pair \((L, E)\) where

- \( L \subset X \) is a compact \( \mathbb{Z} \)-graded generically-immersed Lagrangian with Legendrian collar; and
• $E$ is a filtered local system over the domain $\hat{L}$ of the immersion $\hat{L} \to L \subset X$.

We write only $E$ in place of $(L, E)$ when we want to save notation.

We choose countably many branes and a Novikov field.

**Hypothesis 2.40.** Let $(X, \omega)$ be a compact symplectic manifold with contact boundary, and $\lambda$ a Louville 1-form of it. Let $C(X)$ be a countable set of branes such that any two distinct Lagrangian immersions in $C(X)$ are generic in the sense of Definition 2.29; we allow some of the Lagrangian immersions to be the same. Fix a Novikov field $(\Lambda, v)$. When $\Lambda$ has characteristic $\neq 0$ we suppose that $(X, \omega)$ has no non-constant pseudo-holomorphic sphere (cf. [23]). When $\Lambda$ has characteristic $\neq 2$ we suppose that $X$ and the underlying Lagrangians have a relative spin structure with respect to some order of the Lagrangians.

We introduce also the homologically perturbed Floer complexes:

**Definition 2.41.** Let $C(X)$ be as in Hypothesis 2.40 and $(L_0, E_0), (L_1, E_1) \in C(X)$ two elements. Denote by $\pi_- : L_0 \times_X L_1 \to L_0$ and $\pi_+ : L_0 \times_X L_1 \to L_1$ the projections. The *homologically perturbed Floer complex* associated to the pair $(L_0, E_0), (L_1, E_1)$ is the $\mathbb{Z}$-graded finite-dimensional valued $\Lambda$-vector space

$$CF^*(E_0, E_1) := H^*(L_0 \times_X L_1, \text{hom}(\pi_-^* E_0, \pi_+^* E_1))[-\mu_{L_0 L_1}]$$

(2.29)

where $[-\mu_{L_0 L_1}]$ denotes the degree shift by Maslov indices. Since the fibres of $E_0, E_1$ are valued vector spaces it follows that so are $\text{hom}(\pi_-^* E_0, \pi_+^* E_1)$ and $CF^*(E_0, E_1)$.

**Remark 2.42.** The definition (2.29) is actually the result of applying the filtered homological perturbation lemma [21, Theorem A] to the singular de Rham model in the literature. More precisely, in any construction of Lagrangian Floer cohomology groups we have to choose a chain model of the Floer complex $CF^*(E_0, E_1)$ with $L_0 = L_1$. Akaho–Joyce [9] and Fukaya, Oh, Ohta and Ono [21] both take the singular chain model. Fukaya [19] takes the de Rham model assuming that the ground field $k$ contains $\mathbb{R}$.

We come now to the construction of curved Fukaya categories.

**Theorem 2.43.** Let $C(X)$ be as in Hypothesis 2.40 and put $n := \frac{1}{2} \dim X$. Then $C(X)$ has the structure of a gapped $A_\infty$ category whose morphism spaces are given by (2.29) and which has the following properties:

- the $A_\infty$ structure is curved in such a way that if $(L, E) \in C(X)$, if $L$ has no self-intersection point of index 2 or $n - 2$, and if $L$ does not bound any non-constant pseudo-holomorphic discs, then $(L, E)$ has curvature $m^0 = 0$;

- for a pair $E_0, E_1 \in C(X)$, the differential $m^1 : CF^*(E_0, E_1) \to CF^*(E_0, E_1)$ has leading term $m^1_0 = 0$; and

[24]
for a triple $E_0, E_1, E_2 \in \mathcal{C}(X)$ which are supported on the same Lagrangian, the leading term $m_2^0$ of the product map

$$m^2 : CF^*(E_0, E_1) \times CF^*(E_1, E_2) \to CF^*(E_0, E_2)$$

(2.30)

vanishes with the following exceptions: on the diagonal component $\hat{L}$ of $L \times_X L$ this agrees with the usual cup product map

$$H^*(\hat{L}, \hom(E_0, E_1)) \otimes H^*(\hat{L}, \hom(E_1, E_2)) \to H^*(\hat{L}, \hom(E_0, E_2));$$

and on each self-intersection pair $(a, b) \in L \times_X L$ it is given by the map

$$\hom(E_0|_a, E_1|_b)[-\mu_{ab}] \otimes \hom(E_1|_b, E_2|_a)[-\mu_{ba}] \to H^n(\hat{L}, \hom(E_0, E_2)).$$

### 2.4 Proof of Theorem 2.43

We begin by taking the singular chain model of (2.29) as in Remark 2.42:

**Definition 2.44.** Let $L$ be a manifold, and $E$ a filtered local system over $L$. The smooth singular chain complex $C_\ast(L, E)$ is the $\Lambda^0$ module generated by pairs $(\sigma, \epsilon)$ where $\sigma$ is a smooth singular $i$-simplex in $L$, and $\epsilon$ a (flat) section of $\sigma^*E$ of nonnegative valuation, modulo the relation

$$a(\sigma, \epsilon) = (\sigma, a\epsilon)$$

for $a \in \Lambda^0$.

The differential $d$ on $C_\ast(L, E)$ is defined by

$$d(\sigma, \epsilon) = \sum_{j=0}^i (-1)^j(\sigma \circ \delta^j, \epsilon \circ \delta^j)$$

(2.32)

where $\delta^j$ is the $j$th face inclusion map.

If $L$ has boundary, we define $C_\ast(L, \partial L; E) := C_\ast(L, E)/C_\ast(\partial L, E)$. We regard this, by Poincaré duality, as a cochain complex.

In our circumstances, to $(L_i, E_i) \in \mathcal{C}(X)$, $i = 0, 1$, we assign

$$C_\ast(L_0 \times_X L_1, \partial(L_0 \times_X L_1); \hom(\pi^-_E E_0, \pi^+_E E_1))$$

(2.33)

where $\pi_+, \pi_-$ are as in Definition 2.41.

Let $J$ be a compatible almost complex structure on $X$ in the sense of Definition 2.28. We introduce now the moduli spaces of $J$-holomorphic discs. Let $L_0, \cdots, L_k \subseteq X$ be $k+1$ underlying Lagrangians in $\mathcal{C}(X)$.

**Definition 2.45.** We denote by $\mathcal{M}(L_1, \cdots, L_k; L_0)$ the compact moduli space of finite-area stable $J$-holomorphic discs in $X$ with $k+1$ boundary points, whose corresponding segments are ordered counterclockwise and lifted to $L_0, \cdots, L_k$.

More precisely, for each genus-zero prestable bordered Riemann surface $\Sigma$ we choose a continuous map $S^1 \to \partial \Sigma$ as Akaho–Joyce do [9, Definition 4.1]; this is essentially unique. Denote by $\zeta_0, \ldots, \zeta_{k+1} \in S^1$ the points corresponding...
to the marked points on $\partial \Sigma$; and by $\hat{L}_0, \ldots, \hat{L}_k$ the domains of the immersed Lagrangians. For each holomorphic curve $u : \Sigma \to X$ we choose a continuous map $S^1 \setminus \{\zeta_0, \ldots, \zeta_k+1\} \to L_0 \sqcup \cdots \sqcup L_k$ as Akaho-Joyce [9, Definition 4.2]. We require that the $k+1$ connected components of $S^1 \setminus \{\zeta_0, \ldots, \zeta_k+1\}$ should map to $\hat{L}_0, \ldots, \hat{L}_k$ respectively.

For $\gamma \geq 0$ we denote by $\mathcal{M}_\gamma(L_1, \cdots, L_k; L_0)$ the subset made from $J$-holomorphic curves of area $\gamma$.

We recall the maximum principle on Riemann surfaces with Neumann boundary conditions, under the assumption that the almost complex structure of compatible with the contact boundary:

**Lemma 2.46** (Abouzaid–Seidel [8, Lemma 7.2]). There exists a compact set of $X \setminus \partial X$ which contains the image of every non-constant $J$-holomorphic curve included in $\mathcal{M}(L_1, \cdots, L_k; L_0)$.

Hence by the Gromov compactness theorem we get

**Corollary 2.47.** Every $\mathcal{M}_\gamma(L_1, \cdots, L_k; L_0)$ is compact.

Suppose now that $L_0, \ldots, L_k$ are given local systems $E_0, \cdots, E_k$ respectively. Note that for every $i = 1, \cdots, k$ there is an evaluation map

$$\text{ev}_i : \mathcal{M}(L_1, \cdots, L_k; L_0) \to L_{i-1} \times_X L_i$$

where $L_{-1} := L_k$. So $\mathcal{M}(L_1, \cdots, L_k; L_0)$ has local systems of the form $\text{ev}_i \pi^*_i E_j$ for $i, j = 0, \ldots, k$. Using the parallel transport map $\text{ev}_i^* E_i \cong \text{ev}_{i+1}^* E_i$ we get a natural map of local systems

$$\bigotimes_{i=1}^k \hom(\text{ev}_i^* E_{i-1}, \text{ev}_i^* E_i) \to \hom(\text{ev}_0^* E_0, \text{ev}_k^* E_k).$$

Let $\Gamma \subseteq [0, \infty)$ be a discrete sub-monoid which contains the areas of every pseudoholomorphic disc with boundary on the underlying Lagrangians of $C(X)$. The gapped $A_N$ categories in what follows will be gapped with respect to $\Gamma$. We fix an order on the countable set $C(X)$ and use the same order on the underlying Lagrangians of it. Here are the next three steps:

**I** For each $N \in \mathbb{N}$ construct a gapped $A_N$ category $C_N$ whose objects are the first $N$ Lagrangians and whose morphism space between $(L_0, E_0), (L_1, E_1)$ is [23,33]. We denote this morphism space by $\mathcal{C}(E_0, E_1)$. As Fukaya, Oh, Ohta and Ono [23, Remark 13.5] point out we can work with the whole complex [23,33], not with countably generated sub-complexes of it as in the earlier works [9,21].

**II** Construct a homotopy equivalence from $C_N$ to the $A_N$ truncation of $C_{N+1}$.

**III** Lift the gapped $A_N$ structure of $C_N$ to the gapped $A_{N+1}$ structure.
We do (I) as follows. By Corollary 2.47 every moduli space of the form $M_G(L_1, \cdots, L_k; L_0)$ is a finite union of Kuranishi spaces, which are oriented naturally by the relative spin structure. Suppose that for each $i = 1, \ldots, k$ we are given $x_i \in C(E_{i-1}, E_i)$. We choose a multivalued perturbation of the fibre product

$$M_G(L_1, \cdots, L_k; L_0) \times_{ev} (x_1 \times \cdots \times x_k),$$

where $ev := (ev_1, \cdots, ev_k)$, whenever $[k, \gamma] \leq N$. Combining this perturbation with (2.35) we get an element of $C(E_0, E_k)$ which we call the virtual chain for $(x_1, \ldots, x_k)$. It has coefficients in the ground field $\mathbb{A}$ by the result of Fukaya, Oh, Ohta and Ono [23]. Varying $x_1, \cdots, x_k$ we get a map

$$m^k: \prod_{i=1}^{k} C(E_{i-1}, E_i) \to C(E_0, E_k).$$

One can prove that there is a consistent choice of the virtual chains above such that $(m^k)$ defines the gapped $A_N$ structure.

Step (II) is done by a standard cobordism (or bifurcation) argument. Step (III) is a consequence of Theorem 2.26. Once these have been done, we get a gapped $A_\infty$ category whose $A_N$ truncation is homotopy equivalent to $C_N$. Applying the homological perturbation lemma to $C_N$ we get a gapped $A_\infty$ category that we want.

\[\square\]

2.5 Bounding Cochains

We define the units and bounding cochains of filtered $A_\infty$ categories.

**Definition 2.48.** Let $(A, m)$ be a filtered $A_\infty$ category. By units of this we mean a family $(e_X \in A^0_{X,X})_{X \in \text{obj} A}$ of degree-zero morphisms such that $m^1_X e_X = 0$ for $X \in \text{obj} A$: $x = m^0_{X,Y}(e_X, x)$ for $x \in A_{X,Y}$ with $X, Y \in \text{obj} A$: $x = (-1)^{\deg x} m^0_{Y,X}(x, e_X)$ for $x \in A_{Y,X}$ with $X, Y \in \text{obj} A$; and $m^k_{X_0, \ldots, X_k}(x_1, \ldots, x_k) = 0$ for $X_0, \ldots, X_k \in \text{obj} A$ with $k \geq 2$ and for $x_1 \in A_{X_0, X_1}, \ldots, x_k \in A_{X_{k-1}, X_k}$ containing at least one of $e_{X_1}, \ldots, e_{X_k}$.

Let $(A, m)$ be a filtered $A_\infty$ category. We call this a strict $A_\infty$ category if $m^0_X = 0 \in A^0_{X,X}$ for every $X \in \text{obj} A$. If it is strict then define the cohomology category $H A$ to be the associative category consisting of the same objects; the morphism space $(H A)_{XY} := H^*(A_{XY}, m^1_{XY})$ for $X, Y \in \text{obj} A$; and the product map induced from the $m^2$ operators. Define also the category $H^0 A$ to be the subcategory of $HA$ consisting of the same objects and whose every morphism space $H^0 A_{XY}$ is the degree-zero part $H^0(A_{XY}, m^0_{XY})$.

Let $A, B$ be filtered strict $A_\infty$ categories and $f: A \to B$ a filtered $A_\infty$ functor. We call $f$ a strict $A_\infty$ functor if $f^0_X = 0 \in A^0_{X,X}$. This induces then a functor between the cohomology categories, in the obvious way.

Let $X \in \text{obj} A$ be any and $b \in A^1_{X,X}$ a degree-one element with $vb > 0$. We call $b$ a bounding cochain if $\sum_{k=0}^{\infty} m^k_{X, X}(b^{\cdot k}) = 0$. The sum converges because $vb > 0$. We define then a filtered strict $A_\infty$ category $(A', n)$. Define obj $A'$ to be the set of pairs $(X, b)$ with $X \in \text{obj} A$ and $b \in A^1_{X,X}$ a bounding
cochain. For \(Y_0 = (X_0, b_0), Y_1 = (X_1, b_1) \in \text{obj} \mathcal{A}'\) set \(\mathcal{A}'_{Y_0 Y_1} : = \mathcal{A}_{X_0 X_1}\). For \(Y_0 = (X_0, b_0), \ldots, Y_k = (X_k, b_k) \in \text{obj} \mathcal{A}'\) define \(n^k_{Y_0 \ldots Y_k} : \mathcal{A}'_{Y_k Y_1} [1] \times \cdots \times \mathcal{A}'_{Y_0 \ldots Y_k} [1] \to \mathcal{A}'_{Y_0 \ldots Y_k} [1]\) by
\[
(x_1, \ldots, x_k) \mapsto \sum_{p_0, \ldots, p_k=0}^{\infty} m^{p_0+\cdots+p_k} (b_0^{x_1}, b_1^{x_1}, \ldots, b_{k-1}^{x_k}, x_k^{x_k}).
\]
The sum converges because \(vb_0 > 0, \ldots, vb_k > 0\). Since \(vm^{p_0+\cdots+p_k} \geq 0\) it follows that \(vm^k_{Y_0 \ldots Y_k} \geq 0\); that is, \(n^k_{Y_0 \ldots Y_k}\) is a filtered homomorphism. The \(A_\infty\) equations of \((\mathcal{A}, m)\) imply those of \((\mathcal{A}', n)\). Since \(\sum_{k=0}^{\infty} m^k(b^{x_k}) = 0\) it follows that \(n^0 = 0\). Thus \((\mathcal{A}', n)\) is a filtered strict \(A_\infty\) category.

If \(\mathcal{A}\) is a filtered \(A_\infty\) category with units then the strict \(A_\infty\) category \(\mathcal{A}'\) corresponding to it has units too. This is a straightforward extension of the \(A_\infty\) algebra case [21, Lemma 5.2.17].

We introduce now the strict \(A_\infty\) category \(\mathcal{F}_{\text{nc}}(X)\) where ‘nc’ stands for not necessarily closed Lagrangians.

**Definition 2.49.** We define the strict \(A_\infty\) category \(\mathcal{F}_{\text{nc}}(X)\) to be that made from \(\mathcal{C}(X)\) by using bounding cochains as above. We denote an object of \(\mathcal{F}_{\text{nc}}(X)\) by a triple \(\mathbf{b} = (L, E, b)\) where \((L, E)\) is an object of \(\mathcal{C}(X)\) and \(b\) a bounding cochain in \(CF^*(E, E)\). We say that \(\mathbf{b}\) is supported on \(L\) or that \(L\) underlies \(\mathbf{b}\). We define the \(\mathbb{Z}\)-graded \(\Lambda\)-linear associative category \(HF_{\text{nc}}(X)\) to be the cohomology category of \(\mathcal{F}_{\text{nc}}(X)\). We denote by \(HF^*(\mathbf{b}_0, \mathbf{b}_1)\) its morphism space between two objects \(\mathbf{b}_0, \mathbf{b}_1\).

The following is a straightforward extension of the result of Fukaya, Oh, Ohta and Ono [21, §7.3] for embedded Lagrangians.

**Theorem 2.50.** The filtered \(A_\infty\) categories \(\mathcal{C}(X), \mathcal{F}_{\text{nc}}(X)\) have units. \(\square\)

**Remark 2.51.** Strict unitality fails at the level of singular chains, because the perturbations of Kuranishi spaces of the form (2.38) need not be compatible with forgetting marked points. One can however prove that for \((L, E) \in \mathcal{C}(X)\) there exists a homotopy unit of \(CF^*(E, E)\) whose leading term is equal to the fundamental class of \(L\). This may be done by taking fundamental chains of the underlying Lagrangians in \(\mathcal{C}(X)\) and constructing in a consistent way the homotopies between those perturbed Kuranishi spaces for which some of \(x_1, \ldots, x_k\) in (2.39) are the fundamental chains, and those obtained from the forgetful maps. The homotopy units become the ordinary units after passing to the minimal models.

If \(k\) contains \(\mathbb{R}\) we can make the units directly as Fukaya [19] does for embedded Lagrangians, using the de Rham model for (2.29).

We recall the notion of zero objects in \(HF(X)\) and give a corollary to Theorem 2.50.

**Definition 2.52.** We call \(\mathbf{b} \in \text{obj} \mathcal{F}_{\text{nc}}(X)\) a zero object if the unit of \(HF^*(\mathbf{b}, \mathbf{b})\) is zero.
Corollary 2.53. If \( b = (L, E, b) \in \text{obj } \mathcal{F}_\text{nc}(X) \) is a zero object, the Lagrangian \( L \subset X \) has at least one self-intersection point of index \(-1\).

Proof. Since \( HF^*(b, b) \) is \( \mathbb{Z} \)-graded it follows that the unit in \( HF^0(b, b) \) may be cancelled out only by elements of \( CF^{-1}(E, E) \), which come from self-intersection points of \( L \) of index \(-1\). □

Corollary 2.53 gives another proof of the following fact.

Corollary 2.54 (Fukaya, Oh, Ohta and Ono [21, Theorem E]). Every \( b \in \text{obj } \mathcal{F}_\text{nc}(X) \) supported on an embedded Lagrangian is non-zero. □

Take \( \Omega^2, J \) as in Definition 2.30 so that we can speak of special Lagrangians in \( X \). Consider nearly special Lagrangians which have phase sufficiently close (in the \( C^0 \) sense) to a constant function. The estimate (2.28) implies then

Proposition 2.55. If \( b \in \text{obj } \mathcal{F}(X) \) is supported on a generically immersed nearly special Lagrangian then the Floer cochain group \( CF^*(b, b) \) supported in degrees \( 0, \cdots, n \); and accordingly, so is the cohomology group \( HF^*(b, b) \). □

This and Corollary 2.53 imply

Corollary 2.56. Every \( b \in \text{obj } \mathcal{F}_\text{nc}(X) \) supported on a nearly special Lagrangian is non-zero. □

We define the Fukaya category which we use most often.

Definition 2.57. We denote by \( \mathcal{F}(X) \subseteq \mathcal{F}_\text{nc}(X) \) the full subcategory of objects supported on closed Lagrangians; that is, the underlying Lagrangian of an object \( b \) of \( \mathcal{F}(X) \) is a Lagrangian immersion \( L \rightarrow X \) from a compact manifold \( L \) without boundary.

We prove now a Poincaré duality theorem. We are in the right context for doing so; but we shall not need the result for the later treatment, and the reader in a hurry can proceed safely to Proposition 2.66 below.

Theorem 2.58. Let \( b \) be an object of \( \mathcal{F}(X) \) whose local system has one-dimensional fibres. Then there exists a nondegenerate pairing

\[
HF^*(b, b) \otimes HF^{n-*}(b, b) \rightarrow \Lambda.
\] (2.38)

Proof. We begin by recalling the notion of filtered \( A_\infty \) bimodules. Let \( C \) be a \( \mathbb{Z} \)-graded \( \Lambda \)-linear filtered curved \( A_\infty \) algebra. By a bimodule over \( C \) we mean a \( \mathbb{Z} \)-graded \( \Lambda \) module \( \mathcal{P} \) equipped with operations

\[
m^{r|s}_P : (C[1])^\otimes r \otimes \mathcal{P} \otimes (C[1])^\otimes s \rightarrow \mathcal{P},
\] (2.39)

\( r, s = 0, 1, 2, \cdots \), which satisfy the bimodule version of the \( A_\infty \) equation. An \( A_\infty \) bimodule bimodule is filtered if it is equipped with a valuation, so that \( A_\infty \) bimodule operations do not decrease valuations.

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Example 2.59. The $A_\infty$ algebra $C$ itself has the structure of a diagonal bimodule with operations

$$m^{r|s}(x_1, \ldots, x_r, p, y_1, \ldots, y_s) = (-1)^{r+\sum \deg x_i} m^{r+1+s}(x_1, \ldots, x_r, p, y_1, \ldots, y_s),$$

(2.40)

$r, s = 0, 1, 2, \cdots; x_1, \ldots, x_r, p, y_1, \ldots, y_r \in C$.

Example 2.60. If $P$ is an $A_\infty$ bimodule over $C$ then its dual $P^\vee := \text{hom}(P^\vee, \Lambda)$ has the structure of a bimodule given by

$$m_{P^\vee}(y_1, \ldots, y_s, \psi, x_1, \ldots, x_r)(p) = (-1)^{\# \psi} (m_P(x_1, \ldots, x_r, p, y_1, \ldots, y_s)),$$

(2.41)

$r, s = 0, 1, 2, \cdots; x_1, \ldots, x_r, y_1, \ldots, y_r \in C, \psi \in P^\vee, p \in P$ and $\# = 1 + \deg \psi + (\deg y_1 + \cdots + \deg y_s)(\deg x_1 + \cdots + \deg x_r + \deg p + \deg \psi)$.

Next let $b = (L, E, b) \in \text{obj} \mathcal{F}(X)$ and apply to the single brane $(L, E)$ the proof of Theorem 2.43 Then we get a gapped $A_\infty$ algebra $C$. We use the same notation for the diagonal bimodule, and denote by $C^\vee$ its dual bimodule. Recall that the $m^1$ of $C$ has leading term equal to the usual differential, whose cohomology group is

$$CF^*(E, E) := H^*(L \times_X L, \text{hom}(E, E))[-\mu_L].$$

(2.42)

We prove

Proposition 2.61. There exists a $\mathbb{Z}$-graded $\Lambda$-linear natural isomorphism of graded vector spaces

$$CF^*(E, E) \cong CF^*(E, E)^\vee[-n].$$

(2.43)

Proof. If $\Lambda$ has characteristic $\neq 2$ then the condition that $L$ be relatively spin includes the data of an orientation of the domain $\hat{L}$ of the immersed Lagrangian $L \subset X$, so there is a Poincaré duality isomorphism

$$H^*(\hat{L}, \text{hom}(E, E)) \times H^*(\hat{L}, \text{hom}(E, E)) \to \Lambda[-n]$$

(2.44)

where we do not have to change the local-system term because $\text{hom}(E, E)^\vee \cong \text{hom}(E, E)$. For an off-diagonal self-intersection pair $(x, y) \in L \times_X L$ there is also a pairing

$$(x, y) \otimes (y, x) \to \Lambda[-n]$$

(2.45)

which completes the proof.

The following is the heart of the proof of Theorem 2.58

Lemma 2.62. There exists a bimodule homomorphism

$$\Psi : C \to C^\vee[-n]$$

(2.46)

such that the map $\Psi^{0|0} : C \to C^\vee[-n]$ has leading term equal to (2.43).
Proof. Recall that $C$ is by construction an inverse limit of some gapped $A_K$ algebras, which we denote by $(C_K)_{K=0}^\infty$. Regard this as a diagonal bimodule with operations of the form

$$m^{r|1|s} = \sum m^{r|1|s}_{\gamma} T^\gamma,$$

(2.47)

$r, s \geq 0$ and $[r + s + 1, \gamma] \leq K$. For each $K$ we construct a bimodule map $\Psi_K : C_K \to C_K^\vee$ which consists of a collection of

$$\Psi^{r|1|s}_K : C_K^r \times C_K^s \to C_K^\vee[−n]$$

(2.48)

indexed by nonnegative integers $r$ and $s$. We regard the output as an input by duality, so that we have to construct a compatible collection of maps

$$C_K^r \times C_K^s \to \Lambda[−n].$$

(2.49)

Here we use the hypothesis that the local system of $b$ has one-dimensional fibres.

In order to construct this map, we use the stable moduli spaces of pseudoholomorphic discs with $r + s + 2$ boundary points, ordered as in Figure 2.1: the marked point on the right corresponds to the input, and on the left to the output. We choose inductively a family of perturbation data for the fibre products

$$M_{r+s+2}^{q|x_1|\ldots|x_r|p|\tau_1|\ldots|\tau_s} \times_{ev} (q \times x_1 \times \ldots \times x_r \times p \times \tau_1 \times \ldots \times \tau_s)$$

(2.50)

where $ev$ denotes the evaluation maps at the boundary marked points ordered along the boundary and $q, x_1, \ldots, x_r, p, y_1, \ldots, y_s$ are singular simplices in $C$ with $[r + s + 1, \gamma] \leq K$. We make these compatible with the previous perturbations of (2.38) and those for the lower-order maps relative to $r, s$. Then we get a $A_K$ bimodule homomorphism, which we denote by $\Psi_K : C_K \to C_K^\vee[−n]$.

From the $A_K$ bimodule maps of this form we construct an $A_\infty$ bimodule map by the inductions and obstruction theory arguments. The algebraic framework for this is given by Fukaya, Oh, Ohta and Ono [21, §5.2], used originally for the
deformation invariance of Floer cohomology groups; which is in fact a homotopy equivalence of two bimodules [21, Theorem D].

Recall that we have constructed an \(A_K\) homotopy equivalence between \(C_K\) and the \(A_K\) truncation of \(C_{K+1}\). From this we get an \(A_{K+1}\) algebra \(C_{K,K+1}\) which is \(A_{K+1}\) homotopy equivalent to \(C_{K+1}\) and whose \(A_K\) truncation is \(C_K\). The \(A_{K+1}\) bimodule map \(C_{K+1} \to C_{K+1}^\vee[-n]\) which induces an \(A_{K+1}\) bimodule map (over the algebra \(C_{K,K+1}\))

\[
C_{K,K+1} \to C_{K,K+1}^\vee[-n].
\]  

(2.51)

which we denote still by \(\Psi_K\). In the previous stage we have constructed an \(A_K\) bimodule map

\[
\Psi_{K-1} : C_{K-1,K} \to C_{K-1,K}^\vee[-n].
\]  

(2.52)

This is homotopic to the \(A_K\) reduction of (2.51) by a choice of homotopy between the defining perturbation data. Using homological perturbation, and proceeding by induction, we get the desired system of bimodule maps. \(\square\)

Combining Lemma 2.62 with the \(A_\infty\) bimodule version of Whitehead’s theorem [21, Theorem 5.2.35] we get

Corollary 2.63. \(\Psi\) is a homotopy equivalence of bimodules. \(\square\)

The bimodule homotopy equivalence \(\Psi : C \to C^\vee[-n]\) induces for every bounding cochain \(b \in C^1\) another bimodule homotopy equivalence

\[
\Psi_b : C \to C^\vee[-n],
\]  

(2.53)

relative to the \(b\)-deformed \(A_\infty\) structure; and for \(x_1,\ldots,x_r,p,y_1,\ldots,y_s \in CF^*(E,E)\) the term

\[
\Psi_b^{[1]^n}(x_1,\ldots,x_r,p,y_1,\ldots,y_s)
\]  

(2.54)

is given by

\[
\sum \Psi(b,\ldots,b,x_1,b,\ldots,b,x_r,b,\ldots,b,p,b,\ldots,b,y_1,b,\ldots,b,y_s,b,\ldots,b).
\]  

(2.55)

Hence, passing to the cohomology groups, we get an isomorphism

\[
HF^*(b,b) \cong HF^*(b,b)^\vee[-n]
\]  

(2.56)

which completes the proof of Theorem 2.58. \(\square\)

Theorem 2.58 implies

Corollary 2.64. Let \(b\) be a non-zero object of \(F(X)\) whose local system has one-dimensional fibres. Then \(HF^n(b,b) \neq 0\). \(\square\)

Remark 2.65. This is proved already for embedded Lagrangians without local systems by Fukaya, Oh, Ohta and Ono [21, Theorem E]. The second author [33, Lemma 4.4] gives two other proofs, including immersed Lagrangians. One uses the cyclic symmetry [19] assuming that the ground field \(k\) contains \(\mathbb{R}\). The other uses open-closed maps [21, Theorem 6.4.2].

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We end by recalling a result of the second author \cite[Corollary 4.7 (ii)]{33} which we state in a slightly different way because we include local systems.

**Proposition 2.66.** Let $b, b' \in \text{obj} \mathcal{F}(X)$ be isomorphic in $H^0 \mathcal{F}(X)$ and supported on closed special Lagrangians of phase $\phi, \phi' \in \mathbb{R}$ respectively which need not be 0 modulo $2\pi \mathbb{Z}$. Suppose $HF^n(b, b) \neq 0$. Then $\phi - \phi' \in \mathbb{Z}$.

**Proof.** Recall from Corollary \ref{2.56} that $b, b'$ are non-zero objects so that both $HF^0(b, b)$ and $HF^n(b, b)$ are non-zero. Then the same proof works as in \cite[Corollary 4.7 (ii)]{33}. \hfill \Box

Corollary \ref{2.64} gives a sufficient condition for the hypothesis above.

### 2.6 Thomas–Yau Theorems

We recall a few generalizations of Thomas–Yau’s theorem proved by the second author \cite{33}. In this subsection we work with Calabi–Yau manifolds and integrable complex structures.

**Proposition 2.67.** Let $X$ be a Calabi–Yau manifold of complex dimension $n$. Let $b, b' \in \text{obj} \mathcal{F}(X)$ be supported on closed irreducibly immersed special Lagrangians $L, L'$ of phase $\phi, \phi' \in \mathbb{R}$ respectively which need not be 0 modulo $2\pi \mathbb{Z}$. Suppose that either $L$ or $L'$ embedded. Then the following hold.

1. If $\phi = \phi'$ then either $L = L' \subset X$ or $HF^\ast(b, b')$ is supported in degrees $1, \ldots, n-1$.

2. If $b, b'$ are isomorphic in $H^0 \mathcal{F}(X)$ with $HF^n(b, b) \neq 0$ then $L = L' \subset X$.

**Remark 2.68.** The point of (i) is that the degrees are strictly $> 0$ and $< n$. It follows immediately from \ref{2.28} that they belong to $\{0, \ldots, n\}$.

**Proof of Proposition 2.67.** Let $L$ be embedded. Then we can work in a Weinstein neighbourhood of $L$ which is a Stein manifold. This is the condition (ii) of the second author’s result \cite[Theorem 6.2]{33}, which implies (i) above. We prove (ii) now. By Proposition \ref{2.66} the two special Lagrangians have the same phase up to shift. Hence it follows by (i) that either $L = L'$ or $HF^\ast(b, b')$ is supported in $k, \ldots, k+n-2$ for some $k \in \mathbb{Z}$. But the latter is impossible because the graded vector space $HF^\ast(b, b') \cong HF^\ast(b, b)$ has two non-trivial degrees with difference $n$. So $L = L'$.

We recall also the following version. We say that a Calabi–Yau manifold $X$ is real analytic if the Kähler form of $X$ is real analytic with respect to the underlying real analytic structure of the complex manifold $X$.

**Proposition 2.69.** Let $X$ be a real analytic Calabi–Yau manifold and $L, L' \subset X$ two closed irreducibly-immersed Lagrangians which are, near $L \cap L'$, both special and graded by 0 $\in \mathbb{R}$. Then either $L, L'$ agree with each other near $L \cap L'$ or any pair of objects supported on them have Floer cohomology supported in degrees $1, \ldots, n-1$.  

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Suppose \( L \neq L' \) and denote by \( S \subseteq L \cap L' \) the set of intersection points at which \( L, L' \) have at least one common tangent space. The second author [33, Theorem 5.1(i)] proves then that there exists a neighbourhood \( U \subset X \) of \( S \) and arbitrarily-small Hamiltonian perturbations of \( L, L' \) respectively which intersect generically each other with no intersection point in \( U \) of index 0 or \( n \). On the other hand, these have no intersection point of index 0 or \( n \) outside \( U \) [33, Lemma 2.7]. So they have no intersection point at all of index 0 or \( n \). Since \( L, L' \) are both graded by 0 \( \in \mathbb{R} \) it follows by (2.28) that every intersection point of the two perturbed Lagrangians has degree in \( \{1, \ldots, n-1\} \). So their Floer complex is supported in degrees 1, \ldots, \( n-1 \), which completes the proof.

2.7 Hamiltonian Continuations

We return now to the symplectic topology context. We recall the definition of homotopies between strict \( A_\infty \) functors. Let \( A, B \) be strict \( A_\infty \) categories. Then we can define the functor category \( \text{fct}(A, B) \) which is a strict \( A_\infty \) category whose objects are strict \( A_\infty \) functors from \( A \) to \( B \). We say that two objects \( f, g \in \text{obj} \text{fct}(A, B) \) are homotopic if they are isomorphic in \( H^0 \text{fct}(A, B) \). Here we do not impose any conditions on the object sets as we do in Definition 2.20.

We say that a strict \( A_\infty \) functor \( f : A \to B \) is a homotopy equivalence if there exists \( g \in \text{fct}(B, A) \) such that \( g \circ f \) is isomorphic in \( H^0 \text{fct}(A, A) \) to the identity on \( A \) and \( f \circ g \) isomorphic in \( H^0 \text{fct}(B, B) \) to the identity on \( B \). Seidel [53, Theorem 2.9] proves that \( f \) is an homotopy equivalence if and only if the induced functor \( Hf : HA \to HB \) is an equivalence of categories.

We explain now the effect of a symplectomorphism of \( X \) upon the Fukaya categories. Recall from Definition 2.28 that the radius function \( r_\lambda \) is defined near \( \partial X \). Note also that every symplectictomorphism acts naturally upon the objects of \( C(X) \).

Lemma 2.70. Let \( \phi : X \to X \) be a symplectomorphism isotopic to the identity and which agrees near \( \partial X \) with the Hamiltonian diffeomorphism of \( cr_\lambda \) for some \( c > 0 \). Suppose that \( \phi \) induces a bijection of \( C(X) \). Then there exists an equivalence

\[
\phi_* : \mathcal{HF}_{nc}(X) \to \mathcal{HF}_{nc}(X)
\]

such that for every \( b = (L, E, b) \in \text{obj} \mathcal{F}_{nc}(X) \) the image \( \phi b \) is given by a certain bounding cochain \( \phi_* b \) for \( (\phi_* L, \phi_* E) \in C(X) \). The functor \( \phi_* \) is compatible with composition in the sense that, for a pair \( \phi \) and \( \psi \) of boundary-linear symplectormorphisms, the composite \( \psi \circ \phi_* \) is naturally isomorphic to \( (\psi \circ \phi)_* \).

Proof. The strict \( A_\infty \) category \( \mathcal{F}_{nc}(X) \) is made from a gapped \( A_\infty \) category \( C(X) \) whose definition involves the choice of an almost complex structure \( J \) and other data. We can then define another gapped category by using \( \phi_* J \). We want to show that the two gapped \( A_\infty \) categories are homotopy equivalent.

This is done by Fukaya, Oh, Ohta and Ono [21, Theorem A] and by Akaho–Joyce [9, Theorem 11.2(a)] for gapped \( A_\infty \) algebras by interpolating between the almost complex structures; we do so while maintaining compatibility with
the contact boundary, thus ensuring compactness. They use the obstruction theory arguments for gapped $A_N$ functors [21, Lemma 7.2.129] for gapped $A_\infty$ algebras. We extend them to gapped $A_\infty$ categories in the same way as in the proof of Theorem 2.26. Then we get a homotopy equivalence of the two gapped categories. Fukaya [20, Proposition 13.18] proves that this induces a homotopy equivalence of the corresponding strict categories. There is thus an equivalence (2.57) of the cohomology categories.

To find the natural isomorphism between $\psi^* \circ \phi^*$ and $(\psi \circ \phi)^*$ we begin with homotopies of gapped $A_N$ categories. We lift them to gapped $A_\infty$ categories by an obstruction theory for homotopies of homotopies. This is done by Fukaya, Oh, Ohta and Ono [21, Corollary 7.2.219] for gapped $A_N$ algebras. We extend it in the same way as in the proof of Theorem 2.26. Then we get a homotopy between $\psi^* \circ \phi^*$ and $(\psi \circ \phi)^*$ as gapped $A_\infty$ functors. This induces again a homotopy between the strict $A_\infty$ categories, which induces in turn a natural isomorphism that we want.

There is more to say if the symplectomorphism $\phi$ in Lemma 2.70 is a Hamiltonian diffeomorphism:

**Proposition 2.71.** Let $\phi : X \to X$ be the Hamiltonian diffeomorphism of a smooth function $X \to \mathbb{R}$ which agrees near $\partial X$ with $c r_\lambda$ for some $c > 0$. Suppose that $\phi$ induces a bijection of $\mathcal{C}(X)$. Then there exists a natural transformation $\kappa_\phi$ from $\phi^*$ to the identity functor; that is, for any $b_1, b_2 \in \mathcal{F}_{nc}(X)$ there is a commutative diagram

\[
\begin{array}{ccc}
HF^*(b_1, b_2) & \xrightarrow{\phi^*} & HF^*(\phi b_1, \phi b_2) \\
\kappa_{\phi b_1, \phi} & \downarrow & \downarrow \kappa_{\phi b_1, \phi} \\
HF^*(\phi b_1, b_2) & & 
\end{array}
\]

(2.58)

If either $b_1$ or $b_2$ are supported on a compact Lagrangian, then these arrows are isomorphisms. If there is another such Hamiltonian diffeomorphism $\psi : X \to X$ then

\[
\kappa_{\psi \circ \phi} b = \kappa_{\psi}(\phi b) \circ \kappa_{\psi} b, \quad b \in \text{obj} \mathcal{F}(X).
\]

(2.59)

**Proof.** We explain how to deduce the statement from the result of Fukaya, Oh, Ohta and Ono [21, Theorem G (G.4)] who prove that pairs of Hamiltonian diffeomorphisms induce bimodule homomorphisms of Floer groups (Akahori–Joyce explain in [9, Theorem 13.6] how to extend this to immersed Lagrangians, and it is also straightforward to include local systems). As in Lemma 2.70 the construction of the bimodule homomorphisms involves pseudo-holomorphic curve equations with respect to varying almost complex structures on the target, and we require their compatibility with the contact boundary to ensure compactness.

Define $\kappa_\phi b \in HF^*(\phi b, b)$ to be the image of the unit of $HF^*(b, b)$ under the bimodule homomorphism $(\phi, id)_*: HF^*(b, b) \to HF^*(\phi b, b)$ induced by the pair $(\phi, id)$ of Hamiltonian diffeomorphisms. We prove then that the
diagram (2.58) commutes. The pair $(\phi, \text{id})$ of Hamiltonian diffeomorphisms induces the bimodule homomorphism $(\phi, \text{id})_* : HF^*(b_2, b_2) \to HF^*(\phi b_2, b_2)$. In particular, this is a left module homomorphism and there is a commutative diagram

\[
\begin{array}{ccc}
HF^*(b_1, b_2) \times HF^*(b_2, b_2) & \longrightarrow & HF^*(b_1, b_2) \\
\downarrow & & \downarrow \\
HF^*(\phi b_1, \phi b_2) \times HF^*(\phi b_2, b_2) & \longrightarrow & HF^*(\phi b_1, b_2).
\end{array}
\]  

(2.60)

Consider now the direct sum $b_1 \oplus b_2$. If these are supported on distinct Lagrangians $L_1, L_2$ respectively then regard $b_1 \oplus b_2$ as an object of $\mathcal{F}_{nc}(X)$ supported on $L_1 \cup L_2$. If they are supported on the same Lagrangian then regard it as an object of $\mathcal{F}_{nc}(X)$ with local system $E_1 \oplus E_2$ where $E_1, E_2$ are the local systems of $b_1, b_2$ respectively. In either case the pair $(\phi, \text{id})$ induces a bimodule homomorphism $(\phi, \text{id})_* : HF^*(b_1, b_1 \oplus b_2) \to HF^*(\phi b_1, b_1 \oplus b_2)$. In particular, this is a right module homomorphism and there is a commutative diagram

\[
\begin{array}{ccc}
HF^*(b_1, b_1 \oplus b_2) \times HF^*(b_1 \oplus b_2, b_1 \oplus b_2) & \longrightarrow & HF^*(b_1, b_1 \oplus b_2) \\
\downarrow & & \downarrow \\
HF^*(\phi b_1, b_1 \oplus b_2) \times HF^*(b_1 \oplus b_2, b_1 \oplus b_2) & \longrightarrow & HF^*(\phi b_1, b_1 \oplus b_2).
\end{array}
\]

We look at the following component, which commutes too:

\[
\begin{array}{ccc}
HF^*(b_1, b_1) \times HF^*(b_1, b_2) & \longrightarrow & HF^*(b_1, b_2) \\
\downarrow & & \downarrow \\
HF^*(\phi b_1, b_1) \times HF^*(b_1, b_2) & \longrightarrow & HF^*(\phi b_1, b_2).
\end{array}
\]  

(2.61)

This combined with (2.60) implies the commutative diagram (2.58).

We finally prove (2.59). Using (2.61) with $\psi$ in place of $\phi$, $\phi b$ in place of $b_1$, and $b$ in place of $b_2$ we get a commutative diagram

\[
\begin{array}{ccc}
HF^*(\phi b, \phi b) \times HF^*(\phi b, b) & \longrightarrow & HF^*(\phi b, b) \\
\downarrow & & \downarrow \\
HF^*(\psi \phi b, \phi b) \times HF^*(\phi b, b) & \longrightarrow & HF^*(\psi \phi b, b).
\end{array}
\]  

(2.62)

Consider the unit of $HF^0(\phi b, \phi b)$ and $\kappa_\phi b \in HF^*(\phi b, b)$ and look at the image of this pair. Then we see that

\[
\kappa_\psi(\phi b) \circ \kappa_\phi b = (\psi, \text{id})_*(\psi, \text{id})_*(\phi, \text{id})_*(\phi, \text{id})_*, b.
\]  

(2.63)

But Fukaya, Oh, Ohta and Ono [21, Theorem G (G.4)] prove $(\psi, \text{id})_*(\phi, \text{id})_* = (\psi \phi, \text{id})_*$ so the right-hand side of (2.63) is equal to $(\psi, \text{id})_*(\phi, \text{id})_*, b = \kappa_\psi b$. This completes the proof.
2.8 Wrapped Fukaya Categories

In general, if \( \mathcal{A} \) is an \( \mathcal{A}_\infty \) category and \( Z \subseteq H^0 \mathcal{A} \) a set of morphisms, there exists a universal pair consisting of an \( \mathcal{A}_\infty \) category \( \mathcal{A}_{Z^{-1}} \) and an \( \mathcal{A}_\infty \) functor \( \mathcal{A} \to \mathcal{A}_{Z^{-1}} \) such that every element of \( Z \) becomes invertible in \( H^0 \mathcal{A}_{Z^{-1}} \). The key property that we shall use is the fact that this construction is compatible with passage to triangulated categories in the sense that there is a natural equivalence of triangulated categories

\[
H^0(\mathcal{A}_{Z^{-1}}) \cong (H^0\mathcal{A})_{Z^{-1}}
\]

where the right-hand side is the localisation of a triangulated category.

The next result is a key computational tool for computing Fukaya categories from localisations, and also implies the above equivalence of triangulated categories in this specific context:

**Proposition 2.72.** Let \( \mathcal{A} \) be an \( \mathcal{A}_\infty \) category, and \( f : H\mathcal{A} \to H\mathcal{A} \) an equivalence. For objects \( a, b \in \text{obj}\mathcal{A} \) write \( \mathcal{A}(a,b) \) the morphism space of the pair \( (a,b) \). Let \( \kappa \) be a natural transformation from \( f \) to the identity functor; and \( \mathcal{A}_{\kappa^{-1}} \) the localisation of \( \mathcal{A} \) by the set \( \{ \kappa_a \in \mathcal{A}(fa,a) \}_{a \in \text{obj}\mathcal{A}} \). Then for any two objects \( a, b \in \text{obj}\mathcal{A} \) there exist natural isomorphisms

\[
\mathcal{A}_{\kappa^{-1}}(a,b) \cong \lim_{i \to \infty} \mathcal{A}(f^ia,b) \cong \lim_{j \to \infty} \mathcal{A}(a,f^{-j}b) \cong \lim_{i,j \to \infty} \mathcal{A}(f^ia,f^{-j}b)
\]

where \( f^{-1} : H\mathcal{A} \to H\mathcal{A} \) is an inverse to \( f \). Also for \( a, b, c \in \text{obj}\mathcal{A} \) there exists a commutative diagram

\[
\begin{array}{ccc}
\lim_{i \to \infty} \mathcal{A}(f^ia,b) \otimes \lim_{j \to \infty} \mathcal{A}(b,f^{-j}c) & \longrightarrow & \lim_{i,j \to \infty} \mathcal{A}(f^ia,f^{-j}c) \\
\downarrow & & \downarrow \\
\mathcal{A}_{\kappa^{-1}}(a,b) \otimes \mathcal{A}_{\kappa^{-1}}(b,c) & \longrightarrow & \mathcal{A}_{\kappa^{-1}}(a,c)
\end{array}
\]

where the horizontal arrows are products maps and the vertical isomorphisms are obtained from those of \( \text{(2.64)} \).

**Proof.** Assuming that the category \( \mathcal{A} \) is triangulated, the localisation \( \mathcal{A}_{Z^{-1}} \) may be constructed as the categorical quotient of \( \mathcal{A} \) by the subcategory consisting of the cones of all morphisms \( \kappa : a \to f(a) \). Morphisms in the quotient category are then computed by the direct limit of the groups \( \mathcal{A}(a,f^{-j}b) \) as proved in \[55\, \text{Lemma 7.18} \]. The fact that this is isomorphic to the direct limit of the groups \( \mathcal{A}(f^ia,b) \) is then a consequence of iteratively applying the functor, and the expression in terms of the direct limit of the groups \( \mathcal{A}(f^ia,f^{-j}b) \) is implied by the constancy of the colimit with respect to either variable. The expression for the product is an immediate consequence of functoriality.

We shall also need the following observation:
Lemma 2.73. Assume that $K \subseteq Z \subseteq H^0 A$ are two sets of morphisms such that for each element $\beta \in Z$, there exists a morphism $\gamma$ such that $\beta \circ \gamma$ is a composition of morphisms in $K$. Then the natural functor $H A K^{-1} \to H A Z^{-1}$ is an equivalence.

Proof. It suffices to prove that $A K^{-1}$ satisfies the universal property of the localisation away from $Z$, i.e. that every morphism in $Z$ is already invertible in $A K^{-1}$. This is immediate from the assumption because a composition of morphisms in $K$ becomes invertible in $A K^{-1}$, and the only way for a product of morphisms to be invertible is if both factors are so.

We now arrive at the definition of the wrapped category in the contact-type setting:

Definition 2.74. Let $\Phi$ be a set of Hamiltonians $\phi : X \to \mathbb{R}$ which are linear at the boundary in the sense that they agree with $\phi = cr$, for some constant $c > 0$, near $\partial X$. Assume that this set contains Hamiltonians whose slopes are arbitrarily large. Identify these with the time-one maps they generate, and suppose that if $\phi \in \Phi$ and $E \in C(X)$ then $\phi_* E \in C(X)$. Then $W(X)$ is defined as the localisation of $F_{nc}(X)$ by the set $\{ \kappa_\phi \in HF^0(\phi b, b) : \phi \in \Phi, b \in \text{obj} F_{nc}(X) \}$ of continuation morphisms.

As a consequence of Lemma 2.73 this definition does not depend on the set of Hamiltonians which are chosen: if we add or subtract a Hamiltonian diffeomorphism to the set $\Phi$, then the resulting localisation does not change because of the existence of continuation maps to Hamiltonians of larger slope.

The closed Fukaya category $\mathcal{F}(X)$ is embedded naturally as a full subcategory of $W(X)$ because the continuation morphisms are invertible in $H^0 F(X)$.

2.9 Exact Lagrangians

Suppose now that the Liouville 1-form $\lambda$ is given globally on $X$ so that we can speak of exact Lagrangians in $(X, \lambda)$. Denote by $k \subset \Lambda$ the ground field of the Novikov field. Denote by $F_{ex}(X) \subseteq F_{nc}(X)$ the full subcategory of those objects $(L, E)$ for which:

- the Lagrangian $L \subset X$ is embedded and exact; that is, $L$ has no self-intersection point and $\lambda|_L$ is an exact 1-form;

- the local system $E$ is of the form $E \otimes_k \Lambda$ for some $k$ local system $E$ on $L$; and

- $(L, E)$ is given the trivial bounding cochain, which exists because $L$ is embedded and does not bound any nonconstant pseudoholomorphic disc.

For $(L, E) \in F_{ex}(X)$ take any primitive function $h : L \to \mathbb{R}$ with $dh = \lambda|_L$. For $(L_0, E_0), (L_1, E_1) \in F_{ex}(X)$ we define the action function

$$A : L_0 \times_X L_1 \to \mathbb{R} \quad (2.67)$$

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by \( A(x) := h_0(x) - h_1(x) \) where \( h_0, h_1 \) are the primitives of \((L_0, E_0), (L_1, E_1)\) respectively. We recall:

**Proposition 2.75.** Let \((L_0, E_0), \cdots, (L_k, E_k) \in \mathcal{F}_{\text{ex}}(X)\) and \( h_0, \cdots, h_k \) their primitives respectively. As in the proof of Theorem 2.43 take on \( X \) a compatible almost-complex structure \( J \), define the moduli space \( \mathcal{M}(L_1, \cdots, L_k; L_0) \) and define the evaluation map

\[
\mathcal{M}(L_1, \cdots, L_k; L_0) \to \prod_{i=0}^{k-1} (L_i \times_X L_{i+1}) \times (L_0 \times_X L_k).
\]

Then the fibre of this map over a point \((x_1, \cdots, x_k; x_0)\) consists of \( J \)-holomorphic discs of area

\[
\left( \sum_{i=1}^k A(x_i) \right) - A(x_0).
\]

**Proof.** In the cyclic notation, modulo \( k+1 \) for indices, the area is equal to

\[
\sum_{i=1}^{k+1} \int_{x_{i-1}}^{x_i} \lambda = \sum_{i=1}^{k+1} h_{i-1}(x_i) - h_{i-1}(x_{i-1}) = \sum_{i=1}^{k+1} h_{i-1}(x_i) - \sum_{i=0}^{k} h_i(x_i)
\]

\[
= \left( \sum_{i=1}^k h_{i-1}(x_i) - h_i(x_i) \right) + h_k(x_0) - h_0(x_0)
\]

which is equal to (2.69). \( \square \)

This implies that the \( A_\infty \) structure of \( \mathcal{F}_{\text{ex}}(X) \) is reducible to \( k \):

**Corollary 2.76.** \( \mathcal{F}_{\text{ex}}(X) \) has the structure of a \( k \)-linear \( A_\infty \) category whose hom space between two objects \((L_0, E_0), (L_1, E_1)\) is given by

\[
CF^*_{\text{ex}}(E_0, E_1) := H^*(L_0 \times_X L_1, \text{hom}_k(E_0, E_1)[-\mu_{L_0L_1}])
\]

and whose structure maps \( m^k_{\text{ex}} \) are defined by the same disc counts but without the \( T^\gamma \) factor. The \( \Lambda \)-linear extension of this \( A_\infty \) category is isomorphic to that of the subcategory \( \mathcal{F}(X) \) with the same objects, under the maps

\[
T^A : CF^*_{\text{ex}}(E_0, E_1) \otimes_k \Lambda \to CF^*(E_0, E_1), \quad E_i = E_i \otimes_k \Lambda,
\]

defined as the multiplication by \( T^A(x) \) on the component of each \( x \in L_0 \times_X L_1 \). \( \square \)

**Proof.** This follows from the proof of Theorem 2.43; the point is that once we have fixed the data \((x_1, \cdots, x_k; x_0)\) relevant to the \( A_\infty \) operation in Equation (2.37), the \( \gamma \) in the structure map \( m^k_{\text{ex}} \) will be equal to (2.69). \( \square \)
Choose now a countable set of Hamiltonians as in Definition 2.74. Denote by \( \mathcal{W}_{ex}(X) \) the localisation of \( \mathcal{F}_{ex}(X) \) with respect to this. Using (2.72) we regard \( \mathcal{W}_{ex}(X) \) as a full subcategory of \( \mathcal{W}(X) \).

Consider now a Fukaya category \( \mathcal{HF}_{dir}(X) \) with the same object set as \( \mathcal{HF}_{ex}(X) \) but defined by direct perturbations \[53.\]. Denote by \( \mathcal{W}_{dir}(X) \) the localisation of this with respect to the same Hamiltonians. We prove that \( \mathcal{HW}_{ex}(X) \) is equivalent to \( \mathcal{HW}_{dir}(X) \). We consider only the cohomology categories rather than the \( A_{\infty} \) categories, which will do for the treatment in the later sections.

**Proposition 2.77.** \( \mathcal{HW}_{ex}(X) \) is equivalent to \( \mathcal{HW}_{dir}(X) \).

**Proof.** We define a functor \( \mathcal{HF}_{ex}(X) \to \mathcal{HF}_{dir}(X) \). We leave out the local systems to save notation. For two exact embedded Lagrangians \( L_a, L_b \) we define a \( k \)-linear map \( \mathcal{HF}_{ex}^*(L_a, L_b) \to \mathcal{HF}_{dir}^*(L_a, L_b) \). Suppose either \( L_a \) or \( L_b \) has no boundary. Then \( \mathcal{HF}_{dir}^*(L_a, L_b) \) is defined as \( \mathcal{HF}_{ex}^*(\phi_a L_a, \phi_b L_b) \) for some Hamiltonian diffeomorphisms with compact support in \( X \setminus \partial X \). More precisely, the Floer group \( \mathcal{HF}_{ex}^*(\phi_a L_a, \phi_b L_b) \) is defined by the direct perturbations. But these may be regarded as a different choice of virtual chains, so the resulting Floer groups will be canonically isomorphic. This is done by Fukaya, Oh, Ohta and Ono [21] Theorem G (in the context of the present paper the similar arguments have appeared in the proof of Lemma 2.70). There is thus a canonical isomorphism \( \mathcal{HF}_{dir}^*(L_a, L_b) \cong \mathcal{HF}_{ex}^*(\phi_a L_a, \phi_b L_b) \).

Suppose \( L_a, L_b \) have both Legendrian collar. Then the morphisms of \( (L_a, L_b) \) form the wrapped Floer groups \( \lim \mathcal{HF}_{ex}^*(\phi L_a, L_b) \) and \( \lim \mathcal{HF}_{dir}^*(\phi L_a, L_b) \). Since the isomorphism \( \mathcal{HF}_{ex}^*(\phi L_a, L_b) \to \mathcal{HF}_{dir}^*(\phi L_a, L_b) \) exists for every \( \phi \) we get an isomorphism \( \mathcal{HW}_{ex}^*(L_a, L_b) \to \mathcal{HW}_{dir}^*(L_a, L_b) \).

We prove that the \( k \)-linear maps thus defined are compatible with the products. Let \( a, b, c \in \mathbb{N} \) and suppose first that at least two of \( L_a, L_b, L_c \) have no boundary. Write \( \mathcal{HF}_{dir}^*(L_a, L_b) = \mathcal{HF}_{ex}^*(\phi_a L_a, \phi_b L_b) \) and \( \mathcal{HF}_{dir}^*(L_b, L_c) = \mathcal{HF}_{ex}^*(\phi_b L_b, \phi_c L_c) \). Put \( K := \phi_b^{-1} \phi_a L_a \cup \phi_b L_b \). Recall that the isomorphism \( \mathcal{HF}_{ex}^*(K, L_c) \cong \mathcal{HF}_{ex}^*(\phi_b K, \phi_c L_c) \) is a bimodule isomorphism with respect to \((\phi_b)_* : \mathcal{HF}_{ex}^*(K, K) \to \mathcal{HF}_{ex}^*(\phi_b K, \phi_b K) \) and \((\phi_c)_* : \mathcal{HF}_{ex}^*(L_c, L_c) \to \mathcal{HF}_{ex}^*(\phi_c L_c, \phi_c L_c) \).

It is in particular a left module isomorphism with respect to \((\phi_b)_* \) so the diagram

\[
\begin{array}{ccc}
\mathcal{HF}_{ex}^*(K, K) \times \mathcal{HF}_{ex}^*(K, L_c) & \longrightarrow & \mathcal{HF}_{ex}^*(K, L_c) \\
\downarrow & & \downarrow \\
\mathcal{HF}_{ex}^*(\phi_b K, \phi_c L_c) & \longrightarrow & \mathcal{HF}_{ex}^*(\phi_b K, \phi_c L_c)
\end{array}
\]

commutes. The upper arrow contains the product map \( \mathcal{HF}_{ex}^*(L_a, L_b) \times \mathcal{HF}_{ex}^*(L_b, L_c) \to \mathcal{HF}_{ex}^*(L_a, L_c) \) and the lower arrow contains the product map \( \mathcal{HF}_{ex}^*(\phi_a L_a, \phi_b L_b) \times \mathcal{HF}_{ex}^*(\phi_b L_b, \phi_c L_c) \to \mathcal{HF}_{ex}^*(\phi_a L_a, \phi_c L_c) \). These are thus compatible.

Suppose now that at least two of \( L_a, L_b, L_c \) have Legendrian collar. Then \( \mathcal{HW}_{dir}^*(L_a, L_b) = \lim \mathcal{HF}_{ex}^*(\phi^n L_a, L_b) \) and \( \mathcal{HW}_{dir}^*(L_b, L_c) = \lim \mathcal{HF}_{ex}^*(L_b, \phi^{-n} L_c) \).

The isomorphisms \( \mathcal{HF}_{ex}^*(\phi^n L_a, L_b) \to \mathcal{HF}_{dir}^*(\phi^n L_a, L_b) \) and \( \mathcal{HF}_{ex}^*(L_b, \phi^{-n} L_c) \to \mathcal{HF}_{dir}^*(L_b, \phi^{-n} L_c) \) are thus compatible.
\[ HF_{\text{dir}}^*(L_b, \phi^{-n}L_c) \] are compatible with the product maps, completing the proof. \hfill \square

### 2.10 Cotangent Bundles

Let \( Q \) be a compact connected Riemannian manifold without boundary, and \( X \subset T^*Q \) the unit disc sub-bundle. Recall that the wrapped Floer group \( HW^*_{\text{ex}}(T^*_qQ, T^*_qQ) \) is computed by Abbondandolo, Portaluri and Schwarz [1]; they construct a \( \mathbb{Z} \)-graded ring isomorphism

\[ HW^*_{\text{ex}}(T^*_qQ, T^*_qQ) \cong H^{-*}(\Omega_qQ, k) \] (2.73)

where \( \Omega_qQ \) denotes the based-loop space and \( H^{-*}(\Omega_qQ) \) is non-positively graded.

We shall not need an \( A_\infty \) lift of (2.73) as treated by the first author [4].

To define the Fukaya categories \( F(X) \), \( W(X) \) we do the following:

- We take the background class on \( T^*Q \) to be the pull-back by the projection \( T^*Q \to Q \) of the Stiefel–Whitney class \( w_2(Q) \in H^2(Q, \mathbb{Z}/2\mathbb{Z}) \). So the zero-section \( Q \subset T^*Q \) has a relatively spin structure, and we give it a natural one which we shall define shortly below.

- We give \( T^*Q \) the standard Liouville 1-form \( \lambda \) so that \( Q \) is an exact Lagrangian. We fix a point \( q \in Q \) and include in \( W(T^*Q) \) the object supported on \( T^*_qQ \) with the trivial relative spin structure, an arbitrary \( \mathbb{Z} \)-grading, the trivial rank-one local system and the trivial bounding cochain.

Here is the natural relative spin structure on \( Q 

**Lemma 2.78.** The Lagrangian \( Q \subset T^*Q \) has a relative spin structure such that the \( \pi_1Q \) representation on \( HF^*(Q, T^*_qQ) \) is trivial. If \( k \) has characteristic 2 this representation is automatically trivial.

**Proof.** If \( k \) has characteristic 2 then \( Q \) defines an object of the exact Fukaya category over \( \mathbb{Z}/2\mathbb{Z} \) and the \( \pi_1Q \) representation on \( HF^*(Q, T^*_qQ) \) makes sense over \( \mathbb{Z}/2\mathbb{Z} \). But this is automatically trivial because \( \mathbb{Z}/2\mathbb{Z} \) has only one unit.

Suppose now that \( k \) has characteristic \( \neq 2 \) and choose at first any relative spin structure on \( Q \). Then \( Q \) defines an object of the exact Fukaya category over \( \mathbb{Z} \) (it is only here in the present paper that we use Fukaya categories over a ring). The object \( Q \) defines a representation \( \pi_1Q \to \text{GL}_1\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \) and such representations are classified by elements of \( H^1(Q, \mathbb{Z}/2\mathbb{Z}) \). Since the set of relative spin structures is a torsor over the same group \( H^1(Q, \mathbb{Z}/2\mathbb{Z}) \), we can then choose a possibly different relative spin structure which makes this representation trivial. \hfill \square

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We compute the representations corresponding to \( C^1 \) perturbations of the zero-section \( Q \subset T^*Q \).

**Lemma 2.79.** Let \( \alpha \) be a closed 1-form on \( Q \) whose graph \( Q^\alpha \subset T^*Q \) lies in the sub-bundle \( X \subset T^*Q \). Let \( E \) be a filtered local system over \( Q^\alpha \) and regard this as an object of \( \mathcal{F}(X) \), which makes sense because \( H_2(T^*Q, Q^\alpha; \mathbb{Z}) = 0 \) and no non-constant pseudo-holomorphic curve exists for \( (T^*Q, Q^\alpha) \). Denote by \( T^{-[\alpha]} : \pi_1 Q \rightarrow A^* \) the one-dimensional representation \( x \mapsto T^{-[\alpha]} x \). Regard \( E \) as the representation of \( \pi_1 Q \cong \pi_1 Q \). Then \( T^{-[\alpha]} \otimes E \) is isomorphic to the \( \pi_1 Q \) representation on \( HW^*(T^*_q Q, E) \) corresponding to the \( m^2 \) operator \( HW^0(T^*_q Q, T^*_q Q) \times HF^*(T^*_q Q, E) \rightarrow HF^*(T^*_q Q, E) \) under the isomorphism \( HW^0(T^*_q Q, T^*_q Q) \cong \Lambda[\pi_1 Q] \).

**Proof.** Let \( x \in \pi_1(Q, q) \) be any element. Then there exists a corresponding element of \( HW^*_\alpha(T^*_q Q, T^*_q Q) \) under the isomorphism \( HW^*_\alpha(T^*_q Q, T^*_q Q) \cong k[\pi_1 Q] \).

Choose a Hamiltonian diffeomorphism \( \phi : T^*Q \rightarrow T^*Q \) such that \( \phi(T^*_q Q) \) intersects \( T^*_q Q \) transversely at \( q \in Q \) in the zero-section and there exists an element of \( CF^*_\alpha(\phi(T^*_q Q), T^*_q Q) \) corresponding to \( x \).

Denote by \( \tau : T^*Q \rightarrow T^*Q \) the fibrewise translation by \( \alpha \). Put \( \lambda' := \tau \cdot \lambda = \lambda + \alpha \pi \) where \( \pi \) is the projection \( T^*Q \rightarrow Q \). This induces a category isomorphism \( HF^*_\alpha(X, \lambda) \rightarrow HF^*_\alpha(X, \lambda') \) and accordingly a commutative diagram

\[
\begin{array}{ccc}
HF^*_\alpha(\phi(T^*_q Q), T^*_q Q) \times HF^*_\alpha(T^*_q Q, \tau^*E) & \longrightarrow & HF^*_\alpha(\phi(T^*_q Q), \tau^*E) \\
\downarrow & & \downarrow \\
HF^*_\alpha(\phi(T^*_q Q), T^*_q Q) \times HF'_\alpha(T^*_q Q, E) & \longrightarrow & HF'_\alpha(\phi(T^*_q Q), E)
\end{array}
\]

where we have written \( HF' \) for the morphism spaces of \( HF^*_\alpha(X, \lambda') \). Write \( CF^*_\alpha \) for the morphism spaces of \( F^*_\alpha(X, \lambda') \). Since \( CF^*_\alpha(T^*_q Q, Q^\alpha) \) has no differential it follows then that the \( m^2 \) operator

\[
CF^*_\alpha(\phi(T^*_q Q), T^*_q Q) \times CF^*_\alpha(T^*_q Q, E) \rightarrow CF^*_\alpha(\phi(T^*_q Q), E)
\]

agrees with the parallel transport map of \( E|_q \). The areas of the \( J' \)-holomorphic discs are equal to \( \int_0^t \lambda' + \int_0^s \lambda' + \int_0^\ell \lambda' \) where the integrals are as in Figure 2.2.

The first integral is equal to \( h_{\phi(T^*_q Q)}(x) - h_{\phi(T^*_q Q)}(q) = A(x) \). The second integral vanishes. The third integral is equal to \( -[\alpha] \cdot x \). Thus the map

\[
CF^*(\phi(T^*_q Q), T^*_q Q) \times CF^*(T^*_q Q, E) \rightarrow CF^*(\phi(T^*_q Q), E)
\]
maps \((x, q')\) to \(T^{\mathcal{A}(x)-\alpha \cdot \rho(x)} q'\). So \(T^{\mathcal{A}(x)} x\) acts by \(T^{-\alpha \cdot \rho(x)}\) upon \(CF^*(T_q^*Q, E)\). As the functor \(\tau_*: HF(T^*Q, \lambda) \rightarrow HF(T^*Q, \lambda')\) is an isomorphism, \(T^{\mathcal{A}(x)} x\) has the same action upon \(CF^*(T_q^*Q, \tau^*E)\), which completes the proof.

We prove another lemma we shall need to prove Theorem 1.3 (ii). For \(y \in H_1(Q, \mathbb{Z})/(\text{torsion})\) denote by \(|y| \geq 0\) the infimum of the length of a piecewise smooth loop \(\gamma: [0, 1] \rightarrow Q\) with \(\gamma(0) = \gamma(1) = q\) and whose homology class \([\gamma]\) modulo torsion agrees with \([x]\). If \(y \neq 0\) then \(y\) may be represented by some geodesic \(\gamma: [0, 1] \rightarrow Q\) with \(\gamma(0) = \gamma(1) = q\) of length \(|y|\). This may be proved by the standard method in Riemannian geometry, explained by Sakai [50] Chapter V, Lemma 1.5 for instance (although he deals with free homotopy classes in place of homology classes modulo torsion, this does not affect the proof). We call \(|y|\) the least length of \(y\). We say that a geodesic \(\gamma: [0, 1] \rightarrow Q\) is based if \(\gamma(0) = \gamma(1) = q\). For \(x \in \pi_1 Q\) denote by \([x] \in H_1(Q, \mathbb{Z})/(\text{torsion})\) the image of the natural projection \(\pi_1 Q \rightarrow H_1(Q, \mathbb{Z})/(\text{torsion})\), and by \([x]\) the least length of \([x]\) if \([x] \in H_1(Q, \mathbb{Z})/(\text{torsion})\).

**Lemma 2.80.** Let \(b\) be an object of \(\mathcal{F}(X)\) and put \(V := HF^*(T^*_q Q, b)\), a finite-dimensional \(\Lambda\)-vector space. Denote by \(\rho: \pi_1 Q \rightarrow \text{GL}(V)\) the \(\Lambda\)-linear representation defined by the \(m^2\) operator \(HW^0(T^*_q Q, T^*_q Q) \times V \rightarrow V\) and the isomorphism \(HW^0(T^*_q Q, T^*_q Q) \cong \Lambda[\pi_1 Q]\). Let \(\Gamma < \pi_1 Q\) be a subgroup, \(W \subseteq V\) a one-dimensional subspace and \(\sigma: \Gamma \rightarrow \text{GL}(W)\) a sub-representation of \(\rho|_{\Gamma}\). Denote by \(\sigma \circ: \Gamma \rightarrow \mathbb{R}\) the composite \(\sigma \circ: \Gamma \rightarrow \text{GL}(W) \cong \Lambda^* \rightarrow \mathbb{R}\) of group homomorphisms, the last being the original valuation of the Novikov field \(\Lambda\). Then for every \(x \in \Gamma\) with \([x] \neq 0 \in H_1(Q, \mathbb{Z})/(\text{torsion})\) we have \(|\sigma \circ(x)| \leq |x|\).

**Remark 2.81.** We can also define the least length in the based homotopy class \(x\) itself, without passing to the homology class \([x]\) modulo torsion, and the same estimate will still hold. But this is weaker than the estimate above because there are fewer loops contained in \(x\). We shall need the stronger estimate for Proposition 3.1 below.

**Proof of Lemma 2.80.** Define the wrapped Floer group \(HW^*(T^*_q Q, T^*_q Q)\) by choosing a countable set \(\mathcal{H} \subset C^\infty(X, \mathbb{R})\) such that (i) every based geodesic may be lifted to a Hamiltonian chord for some \(H \in \mathcal{H}\); and (ii) there exists a constant \(C > 0\) such that for every Hamiltonian chord \(x: [0, 1] \rightarrow X\) for \(H\) we have \(\sup_{t \in [0, 1]} |H \circ x(t)| \leq C\). Part (i) is possible by the following fact: denote by \(|p|^2: X \rightarrow [0, 1]\) the squared distance function from the zero-section \(Q \subset X\) and let \(F: [0, \infty) \rightarrow \mathbb{R}\) be smooth with derivative \(F' \neq 0\); then Hamiltonian chords of \(F(|p|^2): X \rightarrow \mathbb{R}\) are the lifts of based geodesics in \(Q\). Part (ii) is possible as in the symplectic cohomology case treated by Venkatesh [6] §2.2.

Take any \(x \in \Gamma\) with \([x] \neq 0 \in H_1(Q, \mathbb{Z})/(\text{torsion})\). Choose a geodesic loop \(\gamma: [0, 1] \rightarrow Q\) based at \(q\), of length \(|x|\) and with \(\gamma = [x] \in H_1(Q, \mathbb{Z})/(\text{torsion})\). Abusing notation denote by \([\gamma] \in \pi_1(Q, q)\) the based homotopy class of \(\gamma\). Using the \(\mathbb{R}\)-vector space isomorphism \(\text{hom}(\pi_1 P, \mathbb{R}) \cong \text{hom}(H_1(P, \mathbb{Z})/(\text{torsion}), \mathbb{R})\) we see then that \(\sigma \circ(x) = \sigma \circ([\gamma])\). Choose now some \(H \in \mathcal{H}\) whose Hamiltonian flow contains the lift of \(\gamma\). Recall that \(\sigma([\gamma])|_W := m^2(T^{\mathcal{A}(\gamma)}([\gamma]), \ast)|_W\) and that
$\nu \sigma \gamma = \nu \sigma \gamma + v m^2 \geq A(\gamma)$ and

$$-v \sigma (x) = -A(\gamma) \leq \int_0^1 \gamma^* \lambda - \int_0^1 H \circ \gamma(t) dt \leq |x| + C.$$  \tag{2.74}

Take any $k \in \mathbb{Z} \setminus \{0\}$. Then $[x^k] = k[x] \neq 0 \in H_1(Q, \mathbb{Z})/(\text{torsion})$ so we can apply (2.74) to $x^k$ in place of $x$, which implies

$$-kv \sigma (x) = -v \sigma (x^k) \leq |x^k| + C \leq k|x| + C \tag{2.75}$$

where the first equality follows since $v \sigma : \Gamma \to \mathbb{R}$ is a group homomorphism and the last from the definition of $|x|$. Dividing (2.75) by $k$ and letting $k$ tend to infinity we get $-v \sigma (x) \leq |x|$. Replacing $x$ by $x^{-1}$ we get the same inequality with $v \sigma (x)$ in place of $-v \sigma (x)$. Consequently $|v \sigma (x)| \leq |x|$. \hfill \Box

**Remark 2.82.** The proof above is a piece of action-complete Floer theory [54, 63, 64] and local Fukaya categories [7].

The proof above of Lemma 2.80 implies also

**Corollary 2.83.** In the circumstances of Lemma 2.80 suppose that the underlying Lagrangian of the object $b$ lies within distance $\delta > 0$ from the zero-section $Q \subset T^*Q$. Then $|v \sigma (x)| \leq \delta|x|$ for every $x \in \Gamma$.

**Proof.** Consider the fibrewise rescale of $T^*Q$ by $\delta$. The pull-back by this of the canonical 1-form $\lambda$ on $T^*Q$ is $\delta \lambda$, and the estimate above follows from the proof above. \hfill \Box

### 2.11 The Split-Generation Theorem

Let $\mathcal{A}$ be a strict $A_\infty$ category with units and $\mathcal{F}, \mathcal{G} \subseteq \mathcal{A}$ full subcategories. We say that $\mathcal{F}$ is split-generated by $\mathcal{G}$ if the image of $\mathcal{F}$ under $\mathcal{A} \to \Pi \text{Tw} \mathcal{A}$ lies in the full subcategory split-generated by $\mathcal{G}$. Here $\Pi \text{Tw} \mathcal{A}$ is defined for instance by Seidel [53, Lemma 4.7]. The full subcategory split-generated by $\mathcal{G}$ is defined by Seidel [53, after Lemma 4.8].

Take $\mathcal{A} = \mathcal{W}(X)$, with respect to a fixed background class on $X$ that we omit from the notation, and denote by $\mathcal{G} \subset \mathcal{W}(X)$ the full subcategory consisting of the single object $T^*_Q$. We prove

**Theorem 2.84.** The Fukaya category $\mathcal{F}(X) \subset \mathcal{W}(X)$ is split-generated by $\mathcal{G}$.

We begin by studying the Viterbo restriction functors. We work in a slightly more general context than above. Let $E$ be a Liouville domain and $X \subseteq E$ a Liouville subdomain (i.e. a codimension 0 submanifold of $E$, with smooth contact boundary). We assume that we have fixed a background class in $H^2(E, \mathbb{Z}/2)$ for the construction of Fukaya categories on $E$, and use the restriction of this class to $X$ when discussing Fukaya categories on $X$. We shall assume that all objects of $\mathcal{W}_{ex}(E)$ are supported on Lagrangians which are equipped with a primitive function for the Liouville 1-form, which is constant along the intersection with
∂X. In this setting, Abouzaid and Seidel [8] construct a Viterbo restriction functor
\[ W_{ex}(E) \to W_{ex}(X). \]  
(2.76)

On the other hand, the maximum principle for pseudoholomorphic discs in \( E \) bounded by Lagrangians in \( X \) yields an \( A_\infty \) functor
\[ \mathcal{F}(X) \to \mathcal{F}(E) \]  
(2.77)
on compact Lagrangians. The compatibility between these two functors is given by the following result:

**Proposition 2.85.** The left and right modules over \( W_{ex}(E) \) associated to the image of each object of \( \mathcal{F}(X) \) in \( \mathcal{F}(E) \) are equivalent to the pullbacks of the associated modules over \( \mathcal{F}(X) \) under the Viterbo restriction map.

**Proof.** We make a little modification of Abouzaid–Seidel’s techniques [8]. We begin by constructing a Hamiltonian \( H \) such that for each exact Lagrangian \( L \subset E \) whose primitive vanishes away from \( X \), the generators of the Floer complex \( CF^*(L, L; H) \) which lie away from \( X \) have strictly positive action. In [8] one constructs a family depending on a parameter \( \rho \in (0, 1] \) by rescaling \( H \) along the Liouville flow inside \( X \), yielding a function that does not depend on \( \rho \) away from \( X \); this has the property that the action of all generators in \( X \) are scaled by \( \rho \), while those outside have constant action. We consider instead a family parametrised by \( \rho \in [1, \infty) \) which is constant in \( X \), and obtained by rescaling outside; now the action in \( X \) is constant, but the action outside is scaled.

Consider now a closed (unobstructed) Lagrangian \( K \) in \( X \); we shall omit all notation for the choices of background classes, local systems, and bounding cochains because they are not relevant to our argument, and discuss only the left module structure, since the argument for the right module structure is exactly the same. The goal is to show that the \( A_\infty \) module structure of \( CF^*(K, L) \) over the wrapped Floer cochains of \( L \) in \( E \) is equivalent to the module structure obtained by pullback from the corresponding module structure after restriction to \( X \). This amounts to constructing a homotopy between the given module structure and one for which we can prove the vanishing of any \( A_\infty \) module operation with input including one chord with boundary on \( L \) lying away from \( X \).

The essential fact that we shall use is that, since \( L \) is exact, Stokes’s theorem implies that the topological energy of each holomorphic polygon \( u \) contributing to the \( A_\infty \) module operations is the sum of the negative of the action of the inputs chords with endpoints on \( L \), the difference between the values of the primitive of \( \lambda \) chosen on \( L \) at the two intersection points with \( K \) corresponding to the input and output, and the integral \( \int_{\partial K} u^* \lambda \) along the boundary component mapping to \( K \). Since \( K \) is Lagrangian, the restriction of \( \lambda \) to \( K \) is closed, so this last integral depends only on the homotopy class of the path.

We now consider the homotopy of \( A_\infty \)-modules associated to the above family of Hamiltonians indexed by a finite interval \( \rho \in [1, R] \). We claim that this
defines a homotopy between the initial \( A_\infty \) module structure, and one with the property that the \( A_\infty \) action for a given set of inputs, one of which lies outside of \( X \), has valuation which is bounded by a quantity which goes to \( +\infty \) in the limit \( R \to +\infty \). The argument is exactly the same as the proof of [8, Lemma 7.4]: all of these terms in the expression for the topological energy of a polygon are independent of \( \rho \), except those corresponding to inputs lying outside \( X \); since these are rescaled by \( \rho \) and are negative, we find that the topological energy of a potential polygon is negative for sufficiently large \( \rho \), hence that the moduli space is empty.

Essentially the same argument as in [8] then implies that the homotopy associated to the entire path \( \rho \in [1, \infty) \) is well-defined: even though the indexing family is non-compact, for any given collection of inputs, the contribution with a fixed valuation will be supported in a finite interval, which implies convergence by the completeness of the Novikov field.

We shall use the following consequence:

**Corollary 2.86.** If \( \mathbf{b} \) is an object of \( \mathcal{F}(X) \) whose image in \( \mathcal{W}(E) \) lies in the category split-generated by a collection of exact Lagrangians, then \( \mathbf{b} \) lies in the subcategory of \( \mathcal{W}(X) \) split-generated by their images under the restriction functor.

**Proof.** Let \( \mathcal{L}_b(E) \) and \( \mathcal{R}_b(E) \) denote the left and right modules over the exact category \( \mathcal{W}_{\text{ex}}(E) \) associated to \( \mathbf{b} \). The criterion for split-generation is that the map from the tensor product of these modules to the self-Floer cohomology of \( \mathbf{b} \)

\[
\mathcal{L}_b(E) \otimes_{\mathcal{W}_{\text{ex}}(E)} \mathcal{R}_b(E) \to CF^*(\mathbf{b}, \mathbf{b})
\]  

(2.78)

hits the unit on cohomology. By the above result, these modules are isomorphic to the pullbacks of the modules \( \mathcal{L}_b(X) \) and \( \mathcal{R}_b(X) \) over the image of \( \mathcal{W}_{\text{ex}}(E) \) (which we call \( \mathcal{W}_{\text{ex}}(X) \) associated to \( \mathbf{b} \)). The proof amounts to the homotopy commutativity of the following triangle

\[
\begin{align*}
\mathcal{L}_b(E) \otimes_{\mathcal{W}_{\text{ex}}(E)} \mathcal{R}_b(E) & \to CF^*(\mathbf{b}, \mathbf{b}) \\
\downarrow & \\
\mathcal{L}_b(X) \otimes_{\mathcal{W}_{\text{ex}}(X)} \mathcal{R}_b(X) & \sim CF^*(\mathbf{b}, \mathbf{b})
\end{align*}
\]  

(2.79)

which follows by the same argument as the above proposition: the map in Equation (2.78) is obtained by counting discs with an arbitrary number of inputs that are chords with boundary on \( \mathcal{L} \), and two of which are intersection points of \( K \) and \( L \). The output is a marked point along the interval mapping to \( K \). As before, the integrated maximum principle implies that the image of the discs whose inputs are chords that are contained in \( X \) cannot escape this region, so that the goal is to show that the family associated to rescaling the Hamiltonian outside of \( X \) by a parameter \( \rho \in [1, \infty) \) has the property that there are no discs with prescribed inputs and homotopy class of the path along \( K \) whenever \( \rho \) is sufficiently large. \( \square \)
We return now to the case where $X$ is a sufficiently small disc sub-bundle in $T^*Q$. In [24] Fukaya, Seidel and Smith construct a Lefschetz fibration with total space a Liouville subdomain of $T^*Q$ (which they call $T^*N$), whose critical points are of either disjoint from $Q$ or correspond to the critical points of a Morse function on it (see the discussion preceding [24, Lemma 7]). Taking the inverse image of a small region in the base of the Lefschetz fibration produces a subdomain of $T^*Q$, containing the 0-section, and which has the property that all its Lefschetz thimbles intersect the 0-section transversely at a point, i.e. are isotopic to cotangent fibres. Combining this with Seidel’s doubling construction [53, §18] we get

- a Liouville manifold $E$ (which Seidel denotes by $\tilde{E}$) containing $X$ as a Liouville subdomain and
- a collection $S_1, \ldots, S_k \subset E$ of Lagrangian spheres (which Seidel denotes by $\tilde{\Delta}_i$), whose intersection with $X$ are Lefschetz thimbles associated to a Morse function on $Q$, and with the property that there is a composition $\phi$ of Dehn twists about $S_i$ such that $\phi X$ is disjoint from $X$ (up to Hamiltonian isotopy); see the first paragraph in the proof of [53, Lemma 18.15].

Since we are interested in studying Fukaya categories that are possibly twisted by background classes, we need to know that any background class on $X$ extends to a background class on $E$:

**Lemma 2.87.** The restriction map $H^*(E) \to H^*(X)$ admits a splitting which vanishes on every sphere.

*Proof.* Seidel’s construction exhibits $E$ as a branched double cover of a Liouville domain containing $X$ as a deformation retract. A splitting is then provided by the double covering map. \qed

From these properties we get:

**Lemma 2.88.** If $X \subset E$ is a sufficiently small Weinstein neighbourhood of $Q$, then

- the intersection of each Lagrangian $S_i$ with $X$ is Hamiltonian isotopic to a cotangent fibre, and
- the image of every closed Lagrangian in $X$ under $\phi$ is disjoint from $X$ (up to Hamiltonian isotopy).

Fix a background class on $X$ and the class induced in $E$ by the splitting above. Introduce the Fukaya category $\mathcal{F}(E)$ which contains the Lagrangians $S_i$ with the trivial relative spin structures (which exists because every $S_i$ is the double of a disc in $X$), the trivial rank-one local systems and the trivial bounding cochains.
Remark 2.89. Here we allow the ground field $k$ to have characteristic 2, because we do not take the $\mathbb{Z}/2\mathbb{Z}$-invariant Fukaya category as Seidel does in the discussion of equivariant Fukaya categories in [53], which is used in [24].

The key result about this Fukaya category is:

**Lemma 2.90.** Every object of $\mathcal{F}(E)$ supported in $X$ lies in the category split-generated by the spheres $S_i$.

**Proof.** The proof is the same as that for exact Lagrangians by Seidel [53, Proposition 18.17], using Oh’s result [16 Section 9] for Seidel sequences in the nonexact case. Oh proves indeed that given two objects $b, b'$ and a Lagrangian sphere $S$, with corresponding Dehn twist $\tau_S$, we have a null homotopy for the composition

$$CF^*(\tau_S b, S) \otimes CF^*(S, b') \to CF^*(\tau_S b, b') \to CF^*(b, b')$$  \hspace{1cm} (2.80)

where the first map is given by composition and the second by multiplication with a distinguished closed morphism in $CF^*(b, \tau_S b)$. So the cone of this morphism $b \to \tau_S b$ lies in the category generated by $S$.

Using this sequence repeatedly we see that if $b$ is an object of $\mathcal{F}(E)$ supported in $X$ then there is some closed morphism $\alpha$ from $b$ to $\phi b$ whose cone lies in the category generated by $S_1, \ldots, S_k$. But the Lagrangians underlying $b$ and $\phi b$ may be made disjoint after a Hamiltonian isotopy, so $\alpha$ is exact and the cone of $\alpha$ is isomorphic to a direct sum of $b$ and $\phi b$ (with shifts). This implies that $b$ is a summand of the object lying in the category generated by $S_1, \ldots, S_k$.

We are ready to prove the split-generation result:

**Proof of Theorem 2.84.** Lemma 2.87 implies that any background class on $X$ is the restriction of a background class on $E$. If $\dim Q \neq 2$, the Lagrangian spheres $S_i$ are automatically relatively spin. In the remaining case, we note that the only possible background class is represented by the Poincaré dual of the zero section, which corresponds to a cotangent fibre; and its pullback to $E$ is represented by the inverse image of a cotangent fibre, whose intersection with $S_i$ vanishes with $\mathbb{Z}/2\mathbb{Z}$ coefficients because the submanifolds intersect an even number of times. The result then follows by applying Lemma 2.90 and Corollary 2.86 using as well the fact that all cotangent fibres are equivalent in the wrapped Fukaya category, so that one suffices.

\[ \square \]

2.12 Yoneda Functors

We recall now some standard facts about about $A_\infty$ Yoneda functors. We follow Seidel [53] with minor modifications. For a strict $A_\infty$ category $\mathcal{A}$ with units denote by $\text{mod}\,\mathcal{A}$ the category of left $A_\infty$ modules (Seidel [53 (2f)] works with right modules in place of left modules, and with cohomology units in place of strict units). Seidel [53 (11) and (2g)] defines the *Yoneda functor* $\mathcal{A} \to \text{mod}\,\mathcal{A}$
which is a strict $A_{\infty}$ functor, and proves that this induces a fully faithful functor $H_{\mathcal{A}} \to H(\text{mod} \mathcal{A})$ between the cohomology categories [53, Corollary 2.13].

Seidel [53, (2.13)] proves also that if $\mathcal{A}, \mathcal{B}$ are strict $A_{\infty}$ categories and $F : \mathcal{A} \to \mathcal{B}$ an $A_{\infty}$ functor then the pull-back functor $F^* : \text{mod} \mathcal{B} \to \text{mod} \mathcal{A}$ and the Yoneda functors $\mathcal{A} \to \text{mod} \mathcal{A}, \mathcal{B} \to \text{mod} \mathcal{B}$ fit into the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
\downarrow & & \downarrow \\
\text{mod} \mathcal{A} & \xleftarrow{F^*} & \text{mod} \mathcal{B}
\end{array}
\]

(2.81)

which commute after passing to the cohomology categories.

**Lemma 2.91.** Let $\mathcal{A}$ be a strict $A_{\infty}$ category with units. Let $\mathcal{F}, \mathcal{G} \subseteq \mathcal{A}$ be full subcategories such that $\mathcal{F}$ is split-generated by $\mathcal{G}$. Consider the composite $\mathcal{F} \to \text{mod} \mathcal{G}$ of the inclusion functor $\mathcal{F} \subseteq \mathcal{A}$, the Yoneda functor $\mathcal{A} \to \text{mod} \mathcal{A}$ and the pull-back functor $\text{mod} \mathcal{A} \to \text{mod} \mathcal{G}$. Then the induced functor $H_{\mathcal{F}} \to H(\text{mod} \mathcal{G})$ is fully faithful.

**Proof.** We repeat the argument by Seidel [53, Lemma 4.7]. Denote by $\mathcal{G}' \subseteq \text{mod} \mathcal{A}$ the full subcategory split-generated by $\mathcal{G}$. Then the composite functor $\mathcal{F} \subseteq \mathcal{A} \to \text{mod} \mathcal{A}$ has image in $\mathcal{G}'$. Using (2.81) with $\mathcal{B} = \mathcal{G}$ we get a commutative diagram

\[
\begin{array}{cccc}
H_{\mathcal{F}} & \longrightarrow & H_{\mathcal{G}'} & \longrightarrow & H_{\mathcal{A}} & \longrightarrow & H(\text{mod} \mathcal{A}) \\
& & & & & & \downarrow \\
& & & & & & H_{\mathcal{G}} \longrightarrow H(\text{mod} \mathcal{G}).
\end{array}
\]

(2.82)

Here $H_{\mathcal{G}} \to H(\text{mod} \mathcal{G})$ is induced from the Yoneda functor and accordingly fully faithful. As (2.82) commutes the composite $H_{\mathcal{G}} \to H(\text{mod} \mathcal{G})$ factoring through $H_{\mathcal{G}'}, H_{\mathcal{A}}$ and $H(\text{mod} \mathcal{A})$ is fully faithful. But $\mathcal{G}'$ is split-generated by $\mathcal{G}$ so $H_{\mathcal{G}'} \to H(\text{mod} \mathcal{G})$ is fully faithful too. The arrow $H_{\mathcal{F}} \to H_{\mathcal{G}'}$ is induced from the Yoneda functor $\mathcal{A} \to \text{mod} \mathcal{A}$ and therefore fully faithful too. The composite $H_{\mathcal{F}} \to H_{\mathcal{G}'} \to H(\text{mod} \mathcal{G})$ is thus fully faithful. \qed

We return now to the circumstances of Theorem 2.84 with $\mathcal{A} = \mathcal{W}(X)$, $\mathcal{F} = \mathcal{F}(X)$ and $\mathcal{G} \subseteq \mathcal{W}(X)$ the full subcategory made of the single object $T_{q}^*Q$. Lemma 2.91 implies then

**Corollary 2.92.** The induced functor $H_{\mathcal{F}}(X) \to H(\text{mod} \mathcal{G})$ is fully faithful.

The rest of the section is devoted to showing that this functor is compatible in a certain sense with passing to finite covers of $Q$. Denote by $CW^*(T_{q}^*Q, T_{q}^*Q)$ the endomorphism $A_{\infty}$ algebra of the single object $T_{q}^*Q \in \text{obj} \mathcal{G}$. Then $\text{mod} \mathcal{G}$ is the category of $A_{\infty}$ modules over $CW^*(T_{q}^*Q, T_{q}^*Q)$. Write $X = T_{q}^*Q$ to save notation. The Yoneda functor is then an $A_{\infty}$ functor $\mathcal{F}(T_{q}^*Q) \to \text{mod} CW^*(T_{q}^*Q, T_{q}^*Q)$. Suppose now that $f : P \to Q$ is a finite cover and $p \in f^{-1}(q)$ any point. We define then a corresponding $A_{\infty}$ functor $\mathcal{F}(T_{p}^*P) \to \text{mod} CW^*(T_{p}^*P, T_{p}^*P)$ and compare this with that on $Q$. 49
We denote by the same \( f \) the induced map \( T^*P \to T^*Q \) between the cotangent bundles, which is also a finite cover. We pull back to \( T^*P \) the branes on \( T^*Q \). More precisely, the pull-back of an underlying Lagrangian in \( T^*Q \) is possibly disconnected, in which case we include all the connected components of it; for instance, we include the fibre \( T^*_P P \). We pull back to \( T^*P \) the relative spin structure on \( T^*Q \), with which we introduce the Fukaya category \( \mathcal{F}_{nc}(T^*P) \).

We define then an \( A_\infty \) functor \( f_* : \mathcal{F}_{nc}(T^*P) \to \mathcal{F}_{nc}(T^*Q) \) which acts as the set-theoretic push-forward upon the underlying Lagrangians. Choose a compatible almost complex structure on \( T^*Q \) and pull this back to \( T^*P \). Fix \( N \in \mathbb{N} \) for a moment and recall that \( \mathcal{F}_{nc}(T^*P), \mathcal{F}_{nc}(T^*Q) \) are made from gapped \( A_N \) categories. We define a gapped \( A_N \) functor between these which respects all the \( \mathfrak{m}^\phi_q \) operators, simply by pushing forward the pseudo-holomorphic discs. We define in the same way an inverse system of homotopy equivalences with which we can pass to the limits. Then we get a gapped \( A_\infty \) functor between the curved \( A_\infty \) categories \( \mathcal{C}(T^*P), \mathcal{C}(T^*Q) \). Including the bounding cochains we get an \( A_\infty \) functor \( f_* : \mathcal{F}_{nc}(T^*P) \to \mathcal{F}_{nc}(T^*Q) \) that we want.

We pass now to the wrapped categories. By definition there exist a localization functor \( \mathcal{F}_{nc}(T^*Q) \to \mathcal{W}(T^*Q) \) and accordingly a composite functor \( \mathcal{F}_{nc}(T^*P) \to \mathcal{W}(T^*Q) \). We pull back to \( T^*P \) the set \( \Phi \) of Hamiltonians used to define \( \mathcal{W}(T^*Q) \), with which we define the wrapped category \( \mathcal{W}(T^*P) \). We show that the functor \( \mathcal{F}_{nc}(T^*P) \to \mathcal{W}(T^*Q) \) descends to \( \mathcal{W}_{nc}(T^*P) \). Let \( \psi : T^*P \to T^*P \) be the time-one diffeomorphism of a Hamiltonian we have chosen, which is the pull-back of some Hamiltonian \( \phi \in \Phi \). Denote by the same \( \psi \) the time-one diffeomorphism \( T^*Q \to T^*Q \). Then the push-forward of \( \psi(T^*_P P) \) is equal to \( \phi(T^*_Q Q) \) and the push-forward of the continuation morphism \( \kappa_\psi \in HF^*(\psi(T^*_P P), T^*_P P) \) is the continuation morphism \( \kappa_\phi \in HF^*(\phi(T^*_Q Q), T^*_Q Q) \). The latter is already invertible in \( \mathcal{W}(T^*Q) \) and hence we get by the universal property the functor \( \mathcal{W}(T^*P) \to \mathcal{W}(T^*Q) \) that we want.

We define now a pull-back functor \( f^* : \mathcal{F}(T^*Q) \to \mathcal{F}(T^*P) \) between the Fukaya categories whose underlying Lagrangians are compact without boundary. Take \( (L, E, b) \in \text{obj}\mathcal{F}(T^*Q) \) with \( L \) a Lagrangian, \( E \) a local system and \( b \) a bounding cochain. We define a gapped \( A_\infty \) homomorphism

\[
C_*(L, E) \to C_*(f^*L, f^*E)
\]  

(2.83)

that assigns to each singular chain in \( L \) the sum of all possible inverse images in \( f^*L \). Using the fact that discs are simply connected, we see that the moduli space of discs in \( T^*P \) is a cover of the moduli space of discs in \( T^*Q \), and hence that a choice of virtual chains for \( T^*Q \) determines the corresponding choice for \( T^*P \) by passing to the cover. This implies that (2.83) is compatible with the gapped \( A_N \) algebra structures on the singular chain complexes \( C_*(L, E), C_*(f^*L, f^*E) \). The same argument shows that the homotopies required to construct the inverse system are also compatible with the pull-back by \( f \). So we can take the limit with respect to \( N \) and (2.83) is a gapped \( A_\infty \) homomorphism. This maps the bounding cochain \( b \in C_*(L, E) \) to a bounding cochain in \( C_*(f^*L, f^*E) \), which defines an \( A_\infty \) functor \( f^* : \mathcal{F}(T^*Q) \to \mathcal{F}(T^*P) \) that we want.
Note that for every Lagrangian \( L \subset T^*Q \) we have \( f_* f^* L = L \subset T^*Q \). Hence we get a commutative diagram

\[
\begin{array}{ccc}
\mathcal{F}(T^*Q) & \stackrel{f^*}{\longrightarrow} & \mathcal{F}(T^*P) \\
\downarrow & & \downarrow \\
\mathcal{W}(T^*Q) & \stackrel{f_*}{\longleftarrow} & \mathcal{W}(T^*P).
\end{array}
\] (2.84)

On the other hand, there are a pull-back functor \( \mathcal{W}(T^*Q) \to \text{mod} \mathcal{W}(T^*_qQ, T^*_qQ) \) and the same functor for \( P \) in place of \( Q \). We use also (2.81) with \( B = \mathcal{W}(T^*Q), A = \mathcal{W}(T^*P) \). From these we get a diagram

\[
\begin{array}{ccc}
\mathcal{W}(T^*Q) & \longleftarrow f_* & \mathcal{W}(T^*P) \\
\downarrow & & \downarrow \\
\text{mod} \mathcal{W}(T^*_qQ, T^*_qQ) & \stackrel{f^*}{\longrightarrow} \text{mod} \mathcal{W}(T^*_qP, T^*_qP)
\end{array}
\] (2.85)

which commutes after passing to the cohomology categories, where \( f^* \) denotes the pull-back by \( f_* \). Combining this with (2.84) we get a diagram

\[
\begin{array}{ccc}
\mathcal{F}(T^*Q) & \stackrel{f^*}{\longrightarrow} & \mathcal{F}(T^*P) \\
\downarrow & & \downarrow \\
\text{mod} \mathcal{W}(T^*_qQ, T^*_qQ) & \stackrel{f^*}{\longrightarrow} \text{mod} \mathcal{W}(T^*_qP, T^*_qP)
\end{array}
\] (2.86)

which commutes after passing to the cohomology categories; note that we have got rid of the horizontal arrow in the left direction so that the horizontal arrows of (2.86) are both in the right direction. We have thus proved

**Proposition 2.93.** For every finite cover \( f : (P, p) \to (Q, q) \) the pull-back \( A_\infty \) functor \( f^* : \mathcal{F}(T^*Q) \to \mathcal{F}(T^*P) \), the push-forward \( A_\infty \) algebra homomorphism \( f_* : \mathcal{W}(T^*_qP, T^*_qP) \to \mathcal{W}(T^*_qQ, T^*_qQ) \) and the Yoneda \( A_\infty \) functors fit into the diagram (2.86) which commutes after passing to the cohomology categories.

\[ \square \]

### 3 Proof of Theorem 1.3

Let \( Q \) be a compact connected orientable Riemannian manifold without boundary. We introduce a word we use in what follows.

**Definition 3.1.** A *standard* Calabi–Yau structure near the zero-section \( Q \subset T^*Q \) is a pair consisting of a compatible (integrable) complex structure \( J \) and a \( J \)-holomorphic volume form relative to which \( Q \) is a special Lagrangian.

We begin in §3.1 with the proof of Theorem 1.3 (i). In §3.2 we prove the key lemma for Theorem 1.3 (ii). In §3.3 we prove Theorem 1.3 (ii) when \( \pi_1 Q \) is abelian. In §3.4 we prove a few facts we shall need to deal with finite covers of \( Q \). In §3.5 we prove Theorem 1.3 (ii) in the general case.

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3.1 Proof of Theorem 1.3 (i)

Let \( U \subseteq X \) be a Weinstein neighbourhood of \( Q \) and regard this as embedded in \( T^*Q \) with a standard Calabi–Yau structure. Suppose first that \( \pi_1 Q \) is trivial.

**Proposition 3.2.** If the fundamental group \( \pi_1 Q \) is trivial then the Fukaya category \( \mathcal{F}(U) \) is generated by the single object \( Q \), the zero-section.

**Proof.** Let \( b \) be an object of \( \mathcal{F}(U) \) and consider its image under the Yoneda functor \( H\mathcal{F}(U) \to H \text{mod} CW^*(T^*_q Q, T^*_q Q) \). Applying the homological transfer lemma, we may replace \( CW^*(T^*_q Q, T^*_q Q) \) by an \( A_\infty \) algebra whose underlying graded vector space is given by its cohomology, which is isomorphic to \( H_{-\infty}(\Omega_q Q, \Lambda) \) and hence is supported in non-positive degrees. It follows (by the degree-filtration argument [5, Appendix A]) that the Yoneda image of \( b \) is a repeated extension of ungraded modules over the degree 0 algebra \( HW^0(T^*_q Q, T^*_q Q) \cong H_0(\Omega_q Q, \Lambda) \). But \( H_0(\Omega_q Q, \Lambda) \) is isomorphic to \( \Lambda[\pi_1 Q] \cong \Lambda \) so the image of \( b \) is a repeated extension of ungraded \( \Lambda \)-vector spaces. As \( b \) is supported on a compact Lagrangian without boundary, the number of these extensions is finite and each ungraded piece is finite-dimensional over \( \Lambda \). Recall now that \( Q \) intersects \( T^*_q Q \) transversely and exactly at one point \( q \) and that \( HF^*(T^*_q Q, Q) \) is one-dimensional over \( \Lambda \). Then each ungraded piece of the Yoneda image of \( b \) may be written as \( HF^*(T^*_q Q, Q) \otimes V \) for some finite-dimensional \( \Lambda \)-vector space \( V \). As the functor \( H\mathcal{F}(U) \to H \text{mod} CW^*(T^*_q Q, T^*_q Q) \) is fully faithful, the object \( b \) is isomorphic in the cohomology Fukaya category to a repeated extension of objects of the form \( Q \otimes V \). This completes the proof. \( \Box \)

We use an algebraic lemma by the first author [5, Lemma A.4], which goes back to the first general results on Arnold’s nearby Lagrangian problem [24]. The original statement is over \( \mathbb{Z}/2\mathbb{Z} \) but the proof shows that it applies also to \( \Lambda \):

**Lemma 3.3.** Let \( A \) be a \( \mathbb{Z} \)-graded \( \Lambda \)-linear triangulated \( A_\infty \) category generated by one object \( a \in A \) whose endomorphism algebra \( \text{hom}^1(a, a) \) is supported in non-negative degrees. Then every object \( b \in A \) with \( \text{hom}^1(b, b) \) supported in non-negative degrees is isomorphic in \( H^0 A \) to \( a[k] \otimes V \) for some integer \( k \in \mathbb{Z} \) and for some finite-dimensional \( \Lambda \)-vector space \( V \).

Let \( b \in \text{obj} \mathcal{F}(U) \) be supported on a closed special Lagrangian \( L \subset T^*Q \) relative to a standard Calabi–Yau structure. Then \( b \) is a non-zero object and by Proposition 2.55 the graded group \( HF^*(b, b) \) is supported in non-negative degrees. Proposition 3.2 and Lemma 3.3 imply then that \( b \) is isomorphic in \( H^0 \mathcal{F}(U) \) to some \( Q[k] \otimes V \), \( k \in \mathbb{Z} \), with the trivial bounding cochain. Since \( b \) is non-zero it follows that \( V \) is a non-zero vector space. Note also that there are vector space isomorphisms

\[
HF^n(b, b) \cong HF^n(Q[k] \otimes V, Q[k] \otimes V) \cong HF^n(Q \otimes V, Q \otimes V) \cong HF^n(Q, Q) \otimes \text{End}(V, V),
\]

which is non-zero (by Corollary 2.64 or because \( HF^n(Q, Q) \) is isomorphic to the ordinary cohomology group). Hence it follows by Proposition 2.67 (ii) that \( L = Q \).
Lemma 3.4. For every standard Calabi–Yau structure near \( L \), fix the homology class of \( L \), to area-minimizing currents. This implies that the Maslov index of the pair \((0, L)\) supported on a closed special Lagrangian graded by \( L \) is \( 0 \).

Proof of Lemma 3.4. Suppose first that the Calabi–Yau structure near \( Q \subset T^*Q \) is real analytic. For each \( k = 1, \ldots, n-1 \) we define a compactly-supported Hamiltonian perturbation \( L \) of \( T^*_q Q \), which is special Lagrangian near \( q \in T^*Q \).

To start with, take a \( \mathbb{C} \)-linear isomorphism \( T_q(T^*Q) \cong \mathbb{C}^n \) which maps \( T_q Q \) to \( \mathbb{R}^n \subset \mathbb{C}^n \), \( T_q^* Q \) to \( (i\mathbb{R})^n \), the symplectic form to \( (i/2) \sum_{j=1}^n dz_j \wedge d\bar{z}_j \) and the holomorphic volume form to \( \Lambda_{n-1}^n d\bar{z}_j \) where \( z_1, \ldots, z_n \) are the coordinates of \( \mathbb{C}^n \). Take \( \theta_1, \ldots, \theta_k \in (-\frac{\pi}{2}, 0) \) and \( \theta_{k+1}, \ldots, \theta_n \in (0, \frac{\pi}{2}) \) with \( \theta_1 + \cdots + \theta_n = 0 \) so that the plane \( \Pi := \{ e^{i\theta_1 x_1, \ldots, e^{i\theta_n} x_n} \in \mathbb{C}^n : x_1, \ldots, x_n \in \mathbb{R} \} \) is special Lagrangian with respect to the flat Calabi–Yau structure. Take a small ball in \( \Pi \) about \( 0 \) and embed it into \( T^*Q \) near \( q \), mapping \( 0 \in T_q(T^*Q) \) to \( q \in T^*Q \). Denote the image of the ball by the same \( \Pi \) for short. Take a neighbourhood \( W \) of \( q \) in \( Q \) over which \( T^*Q \) is a product bundle. We can suppose that the ball \( \Pi \) lies in the product bundle \( (T^*Q)|_W \). Denote by \( \text{pr} : (T^*Q)|_W \to T^*_q Q \) the vertical projection of the product bundle. Put \( U_q := \text{pr}(\Pi) \), which is a ball in \( T^*_q Q \) about the origin. Choose a neighbourhood \( U \subset T^*Q \) of \( q \) which is so small that \( (U \cap T^*_q Q) \subset U_q \).

We perturb \( \Pi \) into a special Lagrangian with respect to the Calabi–Yau structure of \( U \). There are two ways of doing this. The first is to use Cartan–Kähler’s theorem as Harvey and Lawson do in [28], Chapter III, Proof of Theorem 5.5, which we can do because we have supposed that our Calabi–Yau structure is real analytic. The second is to use the implicit function theorem again as Harvey and Lawson do in [28], Chapter III, Proof of Corollary 2.14. In either case, we find \( f^* L = P \) so \( L = Q \).
case we get a special Lagrangian near \( q \in T^*Q \). We can suppose that this is a graph over \( U_q \). Using a cut-off function we get then a compactly-supported Hamiltonian perturbation \( L \) of \( T^*_qQ \) such that \( L \cap \text{pr}^*U_q \) is a special Lagrangian in \( U \).

Grade the Lagrangians \( L \) and \( \Pi \) by phase 0 and the Lagrangians \( T^*_qQ \) and \( (i\mathbb{R})^n \) by phase \( \frac{2\pi}{k} \) at \( q \). For \( j = 1, \ldots, k \) define an affine function \([0, 1] \to (-\frac{\pi}{2}, \theta_j)\) by \( t \mapsto -\frac{2\pi\theta_j}{k} + t(\theta_j + \frac{\pi}{2}) \). For \( j = k + 1, \ldots, n \) define an affine function \([0, 1] \to (\frac{\pi}{2}, \theta_j)\) by \( t \mapsto \frac{\pi\theta_j}{k} + t(\theta_j - \frac{\pi}{2}) \). Using these we get a Hamiltonian isotopy from \((i\mathbb{R})^n\) to the plane \( \Pi \) which does not cross \( \mathbb{R}^n \). As \( k(-\frac{\pi}{2}) + (n - k)\frac{\pi}{2} = \frac{2\pi}{k} - k\pi \) there is a graded Hamiltonian isotopy from \((i\mathbb{R})^n[k]\) to \( \Pi \). This induces a \( \mathbb{Z}\)-graded Hamiltonian isotopy from \( T^*_qQ[k] \) to \( L \). Suppose now that \( b \in \text{obj} \mathcal{F}(U) \) is supported on a closed special Lagrangian \( N \subset U \). Since \( (U \cap T^*_qQ) \subset U_q \) it follows then that the special Lagrangian part of \( L \) does not agree with \( N \). We can thus apply Proposition 2.69 to \( N, L \) and consequently \( HF^*(L, b) \cong HF^*(T^*_qQ[k], b) \) is supported in degrees \( 1, \ldots, n - 1 \). So \( HF^*(T^*_qQ, b) \) is supported in degrees \( 1 - k, \ldots, n - 1 - k \). Since this holds for every \( k \in \{1, \ldots, n - 1\} \) it follows that \( \bigcap_{k=1}^{n-1} [1 - k, n - 1 - k] = \{0\} \) as we want.

Finally, if the Kähler form \( \omega \) near \( Q \subset T^*Q \) is not real analytic we can approximate it by a real analytic one (after taking a Kähler potential) and find a compactly-supported Hamiltonian diffeomorphism \( \phi : T^*Q \to T^*Q \) such that \( \phi_*\omega \) is real analytic. Applying to this the result above, we get a corresponding neighbourhood \( V \) of \( Q \subset T^*Q \). Put \( U := \phi^*V \). Let \( b \in \text{obj} \mathcal{F}(U) \) be supported on a closed special Lagrangian (relative to \( \omega \)) and denote by \( \phi b \in \text{obj} \mathcal{F}(V) \) the push-forward. Then \( HF^*(T^*_qQ, \phi b) \) is supported in the single degree \( n \). But as \( \phi \) is compactly supported, we have \( HF^*(T^*_qQ, \phi b) \cong HF^*(T^*_qQ, b) \), which completes the proof.

The effect of the lemma above is that we do not have to deal with the extension problem which we explain now. Let \( b \) be an object of \( \mathcal{F}(U) \). As in the proof of Theorem 1.3 (i) the Yoneda module \( HF^*(T^*_qQ, b) \) is a repeated extension of ungraded modules over \( \Lambda[\pi_1Q] \), that is, representations of \( \pi_1Q \). These extensions may be nontrivial. In Example 5.10, for instance, the direct sum of \( Q \) and the graph of \( \alpha \) are different from their surgery. More precisely, these define non-isomorphic objects of \( H^0\mathcal{F}(U) \). They are made up of the same representations but with different extensions.

### 3.3 The \( \pi_1Q \) abelian Case

We prove Theorem 1.3 (ii) when \( \pi_1Q \) is abelian. We work in a Weinstein neighbourhood of \( Q \subset X \) which we embed in \( T^*Q \). Denote by \( g \) the metric on \( Q \) induced from \( X \), and by \( (J, \Omega) \) the standard Calabi–Yau structure near \( Q \) induced from \( X \). Define a \( C^\infty \) function \( \psi : Q \to (0, \infty) \) by \( \psi^2 \omega^n = (\frac{\pi}{2})^n(-1)^{n(n-1)/2} \Omega \wedge \gamma \). Denote by \( \Psi : C^\infty(T^*Q) \to C^\infty(Q, \mathbb{R}) \) the linear elliptic operator \( \alpha \mapsto d^*(\psi \alpha) \) where \( d^* \) is computed with respect to \( g \). This \( \Psi \) is the linearized operator we use to prove McLean’s theorem 3.4, Proposition
2.13.} By a version of Hodge theory the de Rham cohomology group $H^1(Q, \mathbb{R})$ is isomorphic to the vector subspace $\ker d \cap \ker \Psi \subset C^\infty(T^*P)$. By the de Rham theorem there exists $M > 0$ such that the following holds: suppose given $\delta > 0$ and a group homomorphism $\beta : H_1(Q, \mathbb{Z})/(\text{torsion}) \to \mathbb{R}$ with $|\beta x| \leq \delta|x|$ for $x \in H_1(Q, \mathbb{Z})/(\text{torsion})$; then $\beta$ may be represented by some closed 1-form $\alpha$ on $Q$ with $\Psi \alpha = 0$ and $\sup_P |\alpha|_{C^0} \leq M\delta$ where the pointwise $C^0$ norm is computed with respect to $g$. By McLean’s theorem \[34\] there exists $\epsilon > 0$ such that the following holds: let $\alpha$ be a closed 1-form on $Q$ with $\Psi \alpha = 0$ and $\sup_Q |\alpha|_{C^0} \leq \epsilon$; then the graph $Q^\alpha$ of $\alpha$ is Hamiltonian equivalent to some special Lagrangian near $Q \subset T^*Q$ with respect to $(J, \Omega)$. Denote by $U \subset T^*Q$ the disc sub-bundle of radius $\delta \in (0, M^{-1}\epsilon]$ which is so small too that we can use Lemma \[3.3\].

Let $L \subset U$ be a closed $HF^\ast$-unobstructed special Lagrangian and let $b \in \text{obj} \mathcal{F}(U)$ be supported on $L$. Corollary \[2.68\] implies then that $b$ is a non-zero object. By Lemma \[3.4\] the graded vector space $HF^\ast(T^*_QQ, b)$ is supported in a single degree. Denote by $\rho : \pi_1Q \to \text{GL} HF^\ast(T^*_QQ, b)$ the representation made from the $m^2$ products in $HW(U)$. By hypothesis the group $\pi_1Q$ is abelian so $\rho$ has a one-dimensional sub-representation $\sigma : \pi_1Q \to \Lambda^\ast$. Recall that $\nu \sigma \in \text{hom}(\pi_1Q, \mathbb{R}) \cong H^1(Q, \mathbb{R})$ and denote by $\alpha$ the closed 1-form on $Q$ which represents the de Rham class $\nu \sigma$ and satisfies the equation $\Psi \alpha = 0$. Using the notation of Lemma \[2.79\] write $\sigma \cong T^{-[\alpha]} \otimes E$ where $E$ is some group homomorphism $\pi_1Q \to \text{GL} \Lambda^0$. This defines a rank-one $\Lambda^0$ local system over $Q^\alpha \cong Q$. As in Remark \[2.11\] the category of rank-one free $\Lambda^0$ modules is equivalent to that of one-dimensional valued vector spaces. Hence we get a filtered local system over $Q^\alpha$ which we denote by the same $E$. Lemma \[2.79\] implies then that $\sigma : \pi_1Q \to \Lambda^\ast$ is isomorphic to the representation on $HF^\ast(T^*_QQ, E)$. As $\sigma$ is a sub-representation of $\rho$ there exists a degree-zero non-zero morphism from $\sigma$ to $\rho$. Thus $HF^0(b, E) \neq 0$.

By Corollary \[2.68\] we have $|\nu \sigma(x)| \leq \delta|x|$ for $x \in H_1(Q, \mathbb{Z})/(\text{torsion})$. We have chosen $M > 0$ so that $\sup_Q |\alpha|_{C^0} \leq M\delta \leq \epsilon$. Then the graph $Q^\alpha$ is Hamiltonian equivalent to some special Lagrangian $Q'$. As this Hamiltonian perturbation does not change the isomorphism class of $b$ it follows from Proposition \[2.67\] (i) that $Q' = L$. This completes the proof.

3.4 Finite Covers and McLean’s Theorem

Let $(Q, g)$ be a compact Riemannian manifold without boundary, and $q \in Q$ a point. Recall that for $y \in H_1(Q, \mathbb{Z})/(\text{torsion})$ we have defined $|y| \geq 0$ before stating Lemma \[2.80\]. Let $f : P \to Q$ be a finite cover and denote by $f_* : H_1(P, \mathbb{Z})/(\text{torsion}) \to H_1(Q, \mathbb{Z})/(\text{torsion})$ the homomorphism induced by $f$. We use the $\mathbb{R}$-vector space isomorphism $H^1(P, \mathbb{R}) \cong \text{hom}(H_1(P, \mathbb{Z})/(\text{torsion}), \mathbb{R})$.

**Proposition 3.6.** For every integer $k > 0$ there exists $M > 0$ independent of the finite cover $f : P \to Q$ and so large that every linear map $\tau : H_1(P, \mathbb{Z})/(\text{torsion}) \to \mathbb{R}$ with $|\tau x| \leq |f_* x|$, $x \in H_1(P, \mathbb{Z})/(\text{torsion})$, may be represented by a closed 1-form with $C^k$ norm $\leq M$ with respect to the induced metric $f^*g$.

**Proof.** The proof contains several key steps. We begin by choosing some geodesics.
in $P$ and a triangulation of $P$. We prove in Lemma $3.7$ a uniform estimate for the geodesics; uniform means that the relevant constants are independent of the finite cover $f : P \to Q$. We choose then a subgroup $G < C_1(P,\mathbb{Z})$ of 1-cycles which generate $H_1(P,\mathbb{Z})/(\text{torsion})$. We prove in Lemma $3.8$ a uniform estimate for the elements of $G$. We pass then to the $\mathbb{R}$-coefficients and introduce a 1-cocycle $\sigma \in C^1(P,\mathbb{R})$ of cohomology class $\tau$. We prove in Corollary $3.9$ and Lemma $3.10$ that $\sigma$ satisfies a uniform estimate. We use the de Rham theorem to define a closed 1-form corresponding to $\sigma$, which will satisfy the uniform estimate we want.

We turn now to the formal treatment. Recall that the fundamental group $\pi_1(Q,q)$ of the compact manifold $Q$ is generated by some finitely many elements $x_1, \ldots, x_k$. For $a = 1, \ldots, k$ with $[x_a] \neq 0 \in H_1(Q,\mathbb{Z})/(\text{torsion})$ represent $x_a$ by a geodesic $\gamma_a : [0,1] \to Q$ with $\gamma_a(0) = \gamma_a(1) = q$ and of length $[x_a]$. For $a = 1, \ldots, k$ with $[x_a] = 0 \in H_1(Q,\mathbb{Z})/(\text{torsion})$ take $\gamma_a : [0,1] \to Q$ to be the constant loop with value $q$.

Note that $\gamma_1, \ldots, \gamma_k$ intersect one another only at finitely many points. Choose then a smooth triangulation of $Q$ which contains $\gamma_1 \cup \cdots \cup \gamma_k$ as a sub-complex and is so small that every simplex is contained in some geodesic ball of the Riemannian manifold $(Q,g)$. The latter condition implies that for distinct $x, y \in P$ with $f(x) = f(y)$ the simplices containing $x, y$ are disjoint (for if there were some intersection point $z$ then the geodesic between $x, z$ and that between $y, z$ would be two different lifts of the geodesic between $f(x), f(z)$; but the lift ought to be unique). The finite cover $f : P \to Q$ induces then a triangulation of $P$, which we use in what follows.

Put $l := k \deg f$ and denote by $\delta_1, \ldots, \delta_l : [0,1] \to P$ the lifts under $f : P \to Q$ of $\gamma_1, \ldots, \gamma_k : [0,1] \to Q$. Denote by $|\delta_a| \geq 0$ the length of each $\delta_a$. Regard $\delta_1, \ldots, \delta_l$ as 1-chains in the free abelian group $C_1(P,\mathbb{Z})$. Denote by $|e|_{\ell^p}$, $p \geq 1$, the $\ell^p$ norm on $C_1(P,\mathbb{Z})$; that is, if we denote by $\{e\}$ the set of edges (or 1-simplices) in $P$ then for $y = \sum_y e \in C_1(P,\mathbb{Z})$ we have $|y|_{\ell^p} := (\sum_e |y_e|^p)^{1/p}$.

We prove

Lemma 3.7. There exists $M_0 > 0$ independent of $f : P \to Q$ and so large that $|\delta_a| \leq M_0 |\delta_a|_{\ell^2}$ for every $a = 1, \ldots, l$.

Proof. Put $M_1 := \max\{|\delta_1|, \ldots, |\delta_l|\} = \max\{|\gamma_1|, \ldots, |\gamma_k|\}$, which is independent of $f : P \to Q$. Denote by $\rho_1 > 0$ the greatest length of edges in $P$ and by $\rho_0 > 0$ the least length of edges in $P$, which are also independent of $f : P \to Q$. Fix $a \in \{1, \ldots, l\}$ and denote by $N_a$ the number of edges contained in $\delta_a$; then $N_a \leq \rho_0^{-1} M_1 =: M_2$. On the other hand, the $\ell^1$ norm of $\delta_a$ is the multiple count of the relevant edges so

$$|\delta_a| \leq \rho_1 |\delta_a|_{\ell^1} \leq \rho_1 \sqrt{N_a} |\delta_a|_{\ell^2} \leq \rho_1 \sqrt{M_2} |\delta_a|_{\ell^2}.$$ 

This proves the lemma with $M_0 := \rho_1 \sqrt{M_2} > 0$. \hfill $\square$

Denote by $\ker \partial < C_1(P,\mathbb{Z})$ the subgroup of 1-cycles. We show that the natural projection $\ker \partial \cap (\mathbb{Z} \delta_1 + \cdots + \mathbb{Z} \delta_l) \to H_1(P,\mathbb{Z})$ is surjective. Let $x \in H_1(P,\mathbb{Z})$
be any element and represent this by some loop \( y \) based at \( p \). Then \( f_∗y \) is a loop in \( Q \) based at \( q \), which is based homotopic to some concatenation of \( γ_1, \ldots, γ_k \). Lifting this to \( P \) we see that \( y \) is based homotopic to some concatenation of \( δ_1, \ldots, δ_l \). The corresponding 1-cycle in \( P \) has then homology class \( x \), which proves our claim.

As \( H_1(P, ℤ)/(\text{torsion}) \) is a free abelian group, we can choose a subgroup \( G < \ker ∇(ℤδ_1 + \cdots + ℤδ_l) \) such that the natural projection \( G → H_1(P, ℤ)/(\text{torsion}) \) is surjective. For \( g ∈ G \) denote by \( [g] ∈ H_1(P, ℤ)/(\text{torsion}) \) its homology class modulo torsion.

**Lemma 3.8.** For \( g ∈ G \) we have \( |f_∗[g]| ≤ M_0|g|_{ℓ^2} \).

**Proof.** Write \( g = ∑_{a=1}^l g_aδ_a \) with \( g_a ∈ ℤ \). Recall that each \( f_∗δ_a \) agrees with some \( γ_b \) and that the least length \( |f_∗δ_a| \) of the homology class \( [f_∗g_a] \) is equal to the length \( |f_∗δ_a| = |δ_a| \) of the paths \( f_∗δ_a, δ_a \). So

\[
|f_∗[g]| ≤ ∑_{a=1}^l |g_a| |f_∗δ_a| ≤ ∑_{a=1}^l |μ_a| |f_∗δ_a| = ∑_{a=1}^l |μ_a| |δ_a|.
\]

(3.1)

Lemma 3.7 implies then

\[
|f_∗[g]| ≤ M_0 |∑_{a=1}^l |μ_a| |δ_a|_{ℓ^2} ≤ M_0 |∑_{a=1}^l μ_a δ_a|_{ℓ^2} = M_0 |g|_{ℓ^2}
\]

(3.2)

because \( δ_1, \ldots, δ_l \) have no common edges and are \( ℓ^2 \) orthonormal. This completes the proof.

We pass now to the \( ℜ \)-coefficients. Extend the natural projection \( G → H_1(P, ℤ)/(\text{torsion}) \) to the \( ℜ \)-linear map \( G ⊗ ℜ → H_1(P, ℜ) \). Extend also the given homomorphism \( τ : H_1(P, ℤ)/(\text{torsion}) → ℜ \) to \( H_1(P, ℜ) \). Define an \( ℜ \)-linear map \( τ' : G ⊗ ℜ → ℜ \) to be the composite of the natural projection \( G ⊗ ℜ → H_1(P, ℜ) \) and the extended \( τ : H_1(P, ℜ) → ℜ \). We prove

**Corollary 3.9.** For \( g ∈ G ⊗ ℜ \) we have \( |τ'g| ≤ M_0|g|_{ℓ^2} \).

**Proof.** The hypothesis on \( τ \) and Lemma 3.8 imply that \( |τ'g| ≤ |g|_{ℓ^2} \) for \( g ∈ G \). This extends readily to \( g ∈ G ⊗ ℚ \) because we can cancel out the denominators. Note that \( τ' : G ⊗ ℜ → ℜ \) is an \( ℜ \)-linear map between finite-dimensional vector spaces, which is automatically continuous with respect to the norm topologies. Moreover \( G ⊗ ℚ \) is dense in \( G ⊗ ℜ \) so the estimate holds with \( ℜ \)-coefficients as we want.

We introduce a 1-cocycle \( σ ∈ C^1(P, ℜ) \) of cohomology class \( τ \). Denote by \( \text{im} ∇ \subset C_1(P, ℜ) \) the subspace of 1-boundaries and by \( (\text{im} ∇)⊥ \) its \( ℓ^2 \) orthogonal complement. Put \( G' := (\text{im} ∇)⊥ ∩ (G ⊗ ℜ) \) and denote by \( Π : C_1(P, ℜ) → G' \) the \( ℓ^2 \) orthogonal projection. Define an \( ℜ \)-linear map \( σ : C_1(P, ℜ) → ℜ \) to be the composite of this \( Π \) and that \( τ' \) defined above. Then \( σ \) vanishes on \( \text{im} ∇ \) and is accordingly a 1-cocycle on \( P \). Since \( G' \) agrees with the orthogonal complement

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in \( G \otimes \mathbb{R} \) to \((\im \partial) \cap (G \otimes \mathbb{R}) \) it follows that for \( g \in G \), if we denote by \( g' \) its projection onto \( G' \) then \( g - g' \in (\im \partial) \cap (G \otimes \mathbb{R}) \subseteq \im \partial \) and in particular, \( \tau' g = \tau' g' = \sigma g \), so \( \sigma \) has cohomology class \( \tau \). We prove

**Lemma 3.10.** For every edge \( e \in C_1(P, \mathbb{Z}) \) we have \(|\sigma e| \leq M_0\).

**Proof.** Corollary \[6\] implies that for \( y \in C_1(P, \mathbb{R}) \) we have

\[ |\sigma y| = |\tau' \Pi y| \leq M_0 |\Pi y|_{\ell^2} \leq M_0 |y|_{\ell^2}. \tag{3.3} \]

Taking \( y \) to be an edge \( e \) we get the estimate we want. \( \square \)

We use now the \( \mathbb{R} \)-linear map \( C^1(P, \mathbb{R}) \to \Omega^1_P \) from the \( \mathbb{R} \)-vector space of 1-cochains to that of 1-forms, defined by Singer and Thorpe \[10\] §6.2, Lemma 1], which maps 1-cocycles to closed 1-forms and induces the de Rham isomorphism of the two cohomology groups. Part (4) of their lemma shows that for each edge \( e \in P \) the corresponding 1-form is supported near \( e \) in \( P \). The process of assigning the 1-form is thus local, which we can do essentially in \( Q \) before passing to the finite cover \( P \). Hence it follows by Lemma \[6\] that the 1-form corresponding to \( \sigma \) is bounded with respect to \( f : P \to Q \), completing the proof. \( \square \)

We prove also that McLean’s theorem holds uniformly with respect to finite covers.

**Proposition 3.11.** Let \((Q, g)\) be a compact Riemannian manifold without boundary, \( X \subset T^*Q \) the disc sub-bundle of radius \( \epsilon_X > 0 \), and \((J, \Omega)\) a standard Calabi–Yau structure on \( X \). Define a \( C^\infty \) function \( \psi : Q \to (0, \infty) \) by

\[ \psi^2 - \frac{\Omega}{\mathbb{R}} = \left( \frac{1}{2} \right)^n (-1)^{n(n-1)/2} \Omega \Lambda \Omega. \]

Then there exists \( \epsilon \in (0, \epsilon_X) \) such that the following holds: let \( f : P \to Q \) be a finite cover and \( \alpha \) a closed 1-form on \( P \) with \( \sup_P |\alpha|_{C^0} \leq \epsilon \) and \( d^*(f^*(\psi \alpha)) = 0 \), both computed with respect to the induced metric \( f^* g \); then there exists a \( C^\infty \) function \( v : P \to \mathbb{R} \) such that \( \alpha + dv \) is a special Lagrangian section of \((f^* X, f^* \Omega) \).

**Proof.** For an integer \( k \geq 0 \) denote by \( L^2_k(T^* P) \) the Sobolev space of 1-forms on \( P \) and by \( L^2_k(P, \mathbb{R}) \) that of 0-forms on \( P \). Denote by \( \Xi \subset L^2_k(T^* P) \) the co-dimension one closed linear subspace consisting of \( \xi \in L^2_{k+2}(P, \mathbb{R}) \) with \( \int_P \xi d\mu = 0 \). Denote by \( \Psi : L^2_{k+2}(P, \mathbb{R}) \to \Xi \) the linear elliptic operator \( v \mapsto d^*(\psi dv) \). We regard \( \psi \) as a multiplication operator and write \( \Psi = d^* \psi d \) for instance. By the definition of \( \Xi \) the operator \( \Psi : L^2_{k+2}(P, \mathbb{R}) \to \Xi \) is surjective so there exists a right inverse to it, which we denote by \( G : \Xi \to L^2_{k+2}(P, \mathbb{R}) \). The key fact is that we can choose Sobolev norms on these spaces such that \( G \) is bounded uniformly with respect to the finite cover \( f : P \to Q \).

Those Sobolev norms we get directly from the pull-back metric \( f^* g \) will not do for this purpose. For instance, take \( P = Q = S^1 = \mathbb{R}/\mathbb{Z} \) with \( g = dt \otimes dt \) using the co-ordinate \( t \) on \( \mathbb{R} \). Let \( m > 0 \) be an integer and \( f : S^1 \to S^1 \) the \( m \)-fold cover \( x \mapsto m x \) modulo \( \mathbb{Z} \) for \( x \in S^1 = \mathbb{R}/\mathbb{Z} \). Then \( f^* g = m^2 g \), which contributes \( m^{-2} \) to the Laplacian \( \Psi \) and \( m^2 \) to the Green operator \( G \) by \( m^2 \). The latter is clearly unbounded, which we shall not want to happen.

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We define now the Sobolev norms we use. Let \( \rho > 0 \) be less than half the injectivity radius of the Riemannian manifold \((Q,g)\). For \( q \in Q \) denote by \( B_q \subset Q \) the open geodesic ball of radius \( \rho \) about \( q \). As \( Q \) is compact there exists a finite set \( A \subset Q \) such that \( \bigcup_{a \in A} B_a = Q \). We look now at the finite cover \( f: P \to Q \). For \( x \in P \) with \( f(x) \in A \) denote by \( B_x \subset P \) the lift of \( B_{f(x)} \subset Q \) with respect to \( f: P \to Q \), which exists because \( B_{f(x)} \) is simply connected. Thus \( B_x \) is an open geodesic ball in \((P,f^*g)\). Define then a norm function \( L^k_b(T^*P) \to [0, \infty) \) by

\[
\|u\|_{L^k_b} := \sum_{x \in f^*A} \|u\|_{L^k_b(B_x)}
\]

where the latter is the ordinary Sobolev norm; that is, \( \|u\|_{L^k_b(B_x)} := \int_{B_x} |f_*u|^2 + \cdots + |\nabla^k(f_*u)|^2 d\mu \) where \(|\cdot|, d\mu\) are computed with respect to \( g \). We prove now that \( \Psi \) is bounded below in an \( f \)-uniform way.

**Lemma 3.12.** There exists \( M_0 > 0 \) independent of the finite cover \( f: P \to Q \) and so large that for every \( v \in L^2_b(P, \mathbb{R}) \) we have \( \|v\|_{L^2_b} \leq M_0 \|\Psi v\|_{L^2_b} \).

**Proof.** Recall that if \( Z \) is a compact Riemannian manifold without boundary and \( \psi: Z \to (0, \infty) \) a smooth function then there exists \( M > 0 \) such that for every \( v \in L^2_b(Z, \mathbb{R}) \) with \( \int_Z v d\mu = 0 \) we have \( \|v\|_{L^2_b} \leq M \|d^*\psi dv\|_{L^2_b} \); the constant \( M \) is the inverse of the first non-zero eigenvalue of the elliptic operator \( d^*\psi d \). This extends to a compact Riemannian manifold with boundary, because the estimate follows from that on the double (with a modified metric). Suppose therefore that \( Z \) may now have boundary and note that for every \( v \in L^2_b(Z, \mathbb{R}) \) with \( \int_Z u d\mu = 0 \) we have

\[
\min \psi(dv, dv)_{L^2_b} \leq (\psi dv, dv)_{L^2_b} = (d^*\psi dv, v)_{L^2_b} \leq \|\Psi v\|_{L^2_b} v_{L^2_b} \leq M \|\Psi v\|_{L^2_b}^2.
\]

Thus \( \|dv\|_{L^2_b} \leq (\min \psi)^{-1/2} M^{1/2} \|\Psi v\|_{L^2_b}^2 \). Given an element of \( L^2_b(Y, \mathbb{R}) \) not satisfying the vanishing mean condition, add to this a constant and use the result we have just obtained; then we get the same estimate for it. Consequently, for each \( a \in A \) there exists \( M_a > 0 \) independent of \( f \) and so large that for every \( v_a \in L^2_b(B_a, \mathbb{R}) \) we have \( \|dv_a\|_{L^2_b} \leq M_a \|\Psi v_a\|_{L^2_b} \). Take any \( v \in L^2_b(P, \mathbb{R}) \) and for \( x \in f^*A \) put \( v_x := v|_{B_x} \). Then

\[
\|dv\|_{L^2_b} = \sum_{x \in f^*A} \|d(f_*v_x)\|_{L^2_b(B_{f(x)})} \leq \sum_{x \in f^*A} M_{f(x)} \|\Psi(f_*v_x)\|_{L^2_b(B_{f(x)})} \leq M_0 \|\Psi v\|_{L^2_b}
\]

where \( M_0 := \max_{x \in f^*A} M(x) = \max_{a \in A} M_a \). This completes the proof. \( \Box \)

Fix now \( k > \frac{d}{2} \) so that we have the Sobolev embedding \( L^2_{k+1}(T^*P) \subset C^1(T^*P) \). We prove that the Sobolev embedding and the elliptic regularity are \( f \)-uniform too.

**Lemma 3.13.** There exists \( M_1 > 0 \) independent of the finite cover \( f: P \to Q \) and so large that for every \( u \in L^2_{k+1}(T^*P) \) we have

\[
\sup_P |u|_{C^1} \leq M_1 \|u\|_{L^2_{k+1}} \leq (M_1)^2 (\|u\|_{L^2} + \|(d + d^*f^*\psi)u\|_{L^2_b}). \quad (3.5)
\]
Proof. By the Sobolev embedding on \( P \) there exists \( M > 0 \) independent of \( f \) and so large that for every \( a \in A \) and \( u \in L^2_{k+1}(T^*B_a) \) we have \( \sup_{B_a} |u|_{C^1} \leq M \|u\|_{L^2_{k+1}(B_a)} \). Then for \( u \in L^2_{k+1}(T^*P) \) we have

\[
\sup_{B_x} |u|_{C^1} = \sup_{B_a} |f_x u|_{C^1} \leq M \|f_x u\|_{L^2_{k+1}(B_a)} \leq M \|u\|_{L^2_{k+1}(P)},
\]

which proves the left inequality of (3.5).

We turn now to the other part. We show first that for \( a, b \in A \) with \( B_a \cap B_b \neq \emptyset \) the non-empty intersection is necessarily connected. If there were two components, there would be two distinct geodesics connecting these two. But \( B_a, B_b \) have been chosen to have the common radius \( \rho \) which is less than half the injectivity radius. So \( B_a \cup B_b \) lies in a single geodesic ball, which contradicts that the two geodesics exist. Thus \( B_a \cap B_b \) is connected. The union \( B_a \cup B_b \) is accordingly simply connected.

As \( Q \) is covered by \( B_a \)'s of radius \( \rho > 0 \) there exists \( \sigma \in (0, \rho) \) so close to \( \rho \) that if we denote by \( B'_a \subseteq B_a \) the concentric ball of radius \( \sigma \) then \( \{B'_a\}_{a \in A} \) still covers \( Q \). Then for \( u \in L^2_{k+1}(T^*P) \) we have

\[
\|u\|_{L^2_{k+1}} = \sum_{y \in f^{-1}(A)} \|f_x(u|_{B_y})\|_{L^2_{k+1}(B_{f(y)})} \leq \sum_{y \in f^{-1}(A)} \sum_{B'_y \cap B_{f(y)} \neq \emptyset} \|f_x(u|_{B_y})\|_{L^2_{k+1}(B_{f(y)} \cap B'_y)}.
\]

(3.6)

For \( a \in A \) and \( x \in f^{-1}(a) \) denote by \( Y_{ax} \subseteq f^*A \) the set of \( y \) such that \( B_{f(y)} \cap B'_a \neq \emptyset \) and the simply connected set \( B_{f(y)} \cap B_a \) lifts to \( B_y \cap B_x \). Exchanging the order of the two sums in (3.6) we see then that

\[
\|u\|_{L^2_{k+1}} = \sum_{a \in A} \sum_{x \in f^{-1}(a)} \sum_{y \in Y_{ax}} \|f_x(u|_{B_y})\|_{L^2_{k+1}(B_{f(y)} \cap B'_a)}
\]

\[
\leq \sum_{a \in A} \sum_{x \in f^{-1}(a)} \sum_{y \in Y_{ax}} \|f_x(u|_{B_y})\|_{L^2_{k+1}(B'_a)}.
\]

(3.7)

Note that there is an injection \( Y_{ax} \rightarrow \{z \in A : B'_z \cap B_x \neq \emptyset\} \) defined by \( y \mapsto f(y) \) so that the number \( \#Y_{ax} \) is bounded above by some \( f \)-independent constant \( N > 0 \). It follows then from (3.7) that

\[
\|u\|_{L^2_{k+1}} \leq N \sum_{a \in A} \sum_{x \in f^{-1}(a)} \|f_x(u|_{B_y})\|_{L^2_{k+1}(B'_a)}.
\]

(3.8)

On the other hand, by the ordinary a priori estimates over \( Q \) there exists \( M > 0 \) independent of \( f : P \rightarrow Q \) and so large that for every \( a \in A \) and \( x \in f^{-1}(a) \) we have

\[
\|f_x(u|_{B'_a})\|_{L^2_{k+1}(B'_a)} \leq M(\|f_x(u|_{B_x})\|_{L^2(B_a)} + \|f_x(u|_{B_y})\|_{L^2(B_a)}).
\]
Hence it follows by (3.8) that
\[ \|u\|_{L^2_{k+1}} \leq MN \sum_{a \in A} \sum_{x \in f^{-1}(a)} (\|f_*(u|_{B_x})\|_{L^2(B_a)} + \|(d + d^* \psi)f_*(u|_{B_x})\|_{L^2(B_a)}) \]
\[ \leq MN \sum_{a \in A} \sum_{x \in f^{-1}(a)} (\|u\|_{L^2(B_x)} + \|(d + d^* f^* \psi)u\|_{L^2(B_x)}) \]
\[ = MN(\|u\|_{L^2(P)} + \|(d + d^* f^* \psi)u\|_{L^2(P)}). \]
Thus \( M_1 := MN \) will do for the right part of (3.5).

\[ \square \]

**Corollary 3.14.** There exists \( M_2 \geq 1 \) independent of the finite cover \( f : P \to Q \) and so large that for every \( \xi \in \Xi \) we have \( \|dG\xi\|_{L^2_{k+1}} \leq M_2 \|\xi\|_{L^2_k} \).

**Proof.** Applying Lemma 3.12 to \( G\xi \in L^2_{k+2}(P, \mathbb{R}) \subset L^2_{k}(P, \mathbb{R}) \) in place of \( v \) we see that \( \|dG\xi\|_{L^2} \leq M_0 \|d^* \psi dG\xi\|_{L^2} = M_0 \|\xi\|_{L^2} \). This with the right inequality of (3.5) implies
\[ \|dG\xi\|_{L^2_{k+1}} \leq M_1(\|dG\xi\|_{L^2} + \|(d + d^* \psi)dG\xi\|_{L^2}) \leq M_1(M_0 \|\xi\|_{L^2} + \|\xi\|_{L^2_k}). \]
Thus \( M_2 := M_1(M_0 + 1) \) will do. \( \square \)

We complete now the proof of Proposition 3.11. Let \( \epsilon_0 > 0 \) be so small that if \( \beta \) is a \( C^0 \) section of \( T^*P \) with \( \sup |\beta| \leq \epsilon_0 \) then \( \alpha + \beta \) is a \( C^0 \) section over \( P \) of \( f^*U \subset T^*P \). Note that if \( \beta \) is a \( C^1 \) section of \( T^*P \) with \( \sup_P |\beta| \leq \epsilon_0 \) then we can define the pull-back by \( \alpha + \beta : P \to f^*U \) and in particular the \( n \)-form \( (\alpha + \beta)^* \operatorname{Im} \Omega \). Put \( V := \{v \in L^2_{k+2}(P, \mathbb{R}) : \|dv\|_{L^2_{k+1}} \leq M_0^{-1} \epsilon_0 \} \). The left inequality of (3.5) implies then that for every \( v \in V \) we can define \( dv \) as a \( C^1 \) section over \( P \) of \( f^*U \). By the proof of McLean’s theorem, there exists a non-linear operator \( \Phi : V \to \Xi \) defined by \( \Phi v := -f^* \psi \ast (\alpha + dv)^* \operatorname{Im} f^* \Omega \). We solve the equation \( \Phi v = 0 \). Standard computation shows that \( \Phi \) is differentiable at \( v = 0 \) with respect to the Sobolev norms, whose differential \( \nabla_0 \Phi : L^2_{k+2}(P, \mathbb{R}) \to \Xi \) agrees with \( \Psi \). Corollary 3.14 implies then that for \( \xi \in \Xi \) with \( \|\xi\| \leq M_0^{-1} \epsilon_0 := \epsilon_1 \) we have \( \|dG\xi\|_{L^2_{k+1}} \leq M_0^{-1} \epsilon_0 \) so \( G\xi \in V \). We solve the equation \( \Phi G\xi = 0 \) for \( \xi \in L^2_k(P, \mathbb{R}) \) with \( \|\xi\| \leq \epsilon_1 \). Define \( \Theta : V \to L^2_k(P, \mathbb{R}) \) by \( \Theta := -\Phi + \Psi \). We seek then a fixed point of the composite \( \Theta G \) and use therefore the contraction mapping principle. As \( \Theta \) is the quadratic or higher term made up of \( \psi, \Omega \) there exists \( M > 0 \) independent of \( f \) and so large that the following holds: let \( v, v' \in C^1(P, \mathbb{R}) \) satisfy \( \|dv\|, \|dv'\| < \epsilon_0 \) at every point of \( P \); and put \( \beta = \alpha + dv, \beta' := \alpha + dv' \); then
\[ |\Theta v - \Theta v'|_{C^k} \leq M|\beta - \beta'|_{C^{k+1}}(\|\beta\|_{C^{k+1}} + |\beta'|_{C^{k+1}}) \]
at every point of \( P \). Integrating this and using the Cauchy–Schwarz inequality we get
\[ \int_P |\Theta \beta - \Theta \beta'|_{C^k} d\mu \leq M \sqrt{\int_P \|\beta - \beta'|_{C^{k+1}}^2 d\mu} \sqrt{\int_P (|\beta|_{C^{k+1}} + |\beta'|_{C^{k+1}})^2 d\mu}. \]
(3.9)
Putting $A := |\beta|_{C^{k+1}}$ and $B = |\beta'|_{C^{k+1}}$ we have
\[
\sqrt{\int_P (A + B)^2 \, d\mu} \leq \sqrt{2} \left( \int_P A^2 \, d\mu + \int_P B^2 \, d\mu \right) \leq \sqrt{2} \left( \int_P A^2 \, d\mu + \sqrt{\int_P B^2 \, d\mu} \right).
\]
This with (3.19) implies
\[
\|\Theta v - \Theta v'\|_{L^2_{k+1}} \leq \sqrt{2} M \|\beta - \beta'\|_{L^2_k} \left( \|\beta\|_{L^2_k} + \|\beta'\|_{L^2_k} \right) \leq 2 \sqrt{2} M \left( \|\alpha\|_{L^2_k} + \epsilon \right) \|v - v'\|_{L^2_k}.
\]
Hence choosing $\epsilon \in (0, \min\{(3 \sqrt{2} M (\|\alpha\|_{L^2_k} + \epsilon)^{-1}, \epsilon_1\})$ and using Corollary 3.14 we see that $\Theta G$ is a contraction map from the closed ball $\{\xi \in \Xi : \|\xi\|_{L^2_k} \leq \epsilon\}$ to itself. So there exists $\xi \in \Xi$ with $\Theta G \xi = \xi$. Recall that $k$ has been any integer $> \frac{1}{2}$ and that fixed points are unique; then we see that $\xi \in L^2_k(P, \mathbb{R})$ for every $k > \frac{1}{2}$. Thus $\xi$ is $C^\infty$ and accordingly so is $G \xi$. Now $\xi = \Theta G \xi = -\Phi G \xi + \Psi G \xi = -\Phi G \xi + \xi$ so $\Phi G \xi = 0$. The graph of $\alpha + dG \xi$ is the special Lagrangian we want, which proves Proposition 3.11. \(\square\)

Here is a corollary of Proposition 3.6 and the estimates in the proof above.

**Corollary 3.15.** There exists $M > 0$ independent of the finite cover $f : P \to Q$ and so large that every linear map $\tau : H_1(P, \mathbb{Z})/(\text{torsion}) \to \mathbb{R}$ with $|\tau x| \leq |f, x|$, $x \in H_1(P, \mathbb{Z})/(\text{torsion})$, may be represented by a closed 1-form on $P$ with $C^1$ norm $\leq M$ and $d^*(f^* \psi \alpha) = 0$ where the $C^1$ norm and $d^*$ are computed with respect to the induced metric $f^* g$.

**Proof.** Fix an integer $k > \frac{1}{2}$. Then by Proposition 3.6 there exist a constant $N_0 > 0$ independent of $f$ and a closed smooth 1-form $\beta$ on $P$ with $\|\beta\|_{L^2_{k+1}} \leq N_0$. Define a 1-form $\alpha$ on $P$ by $\alpha := \beta - dG (d^* \psi \beta)$. Then $d\alpha = 0$ and $d^* \psi \alpha = d^* \psi \beta - \Psi G (d^* \psi \beta) = 0$. We estimate the $C^0$ norm of $\alpha$. By Corollary 3.14 there exist constants $N_1, N_2 > 0$ independent of $f, \beta$ and so large that
\[
\|dG (d^* \psi \beta)\|_{L^2_{k+1}} \leq N_1 \|d^* \psi \beta\|_{L^2_k} = N_1 N_2 \|\beta\|_{L^2_{k+1}} \leq N_0 N_1 N_2.
\]
By Lemma 3.13 there exists a constant $N_3$ independent of $f, \beta$ and so large that $\sup_P \|\beta\|_{L^2_{k+1}} \leq N_0 N_3$ and
\[
\sup_P \|dG (d^* \psi \beta)\|_{C^1} \leq N_3 \|dG (d^* \psi \beta)\|_{L^2_{k+1}} \leq N_0 N_1 N_2 N_3.
\]
Thus $|\alpha|_{C^1} \leq |\beta|_{C^1} + |dG (d^* \psi \beta)|_{C^1} \leq N_0 N_3 (1 + N_1 N_2)$, which completes the proof. \(\square\)

### 3.5 Proof of Theorem 1.3 (ii)

We prove Theorem 1.3 in the general case (when $\pi_1 Q$ is not abelian). We begin by choosing a disc sub-bundle $U \subset T^* Q$. We impose two conditions. Firstly, $U$ should be so small that we can use Lemma 3.4. Secondly, let $M > 0$ be as in Corollary 3.15 and $\epsilon > 0$ as in Proposition 3.11 then the radius $\delta$ of $U$ should be $\leq M^{-1} \epsilon$. 

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Let $L \subset U$ be a closed $HF^*$-unobstructed special Lagrangian and $b \in \text{obj} \mathcal{F}(U)$ an object supported on $L$. Then $b$ is a non-zero object. By Lemma 3.4 the graded vector space $HF^*(T_b^*Q, b)$ is supported in a single degree. Denote by $\rho: \pi_1Q \to \text{GL} \cdot HF^*(T^*_bQ, b)$ the representation made from the $m^2$ products in $HW(U)$. By hypothesis the group $\pi_1Q$ has no non-abelian free subgroup, and accordingly its image $\rho(\pi_1Q) \subset \text{GL} \cdot HF^*(T^*_bQ, b)$ does not have such a subgroup either. Since $\pi_1Q$ is finitely generated it follows that so is $\rho(\pi_1Q)$. To this we apply the following two results: one is by Tits [60, Corollary 1] and the other is a Lie–Kolchin theorem stated for instance by Borel [12, Chapter III 10.5]. Both apply to fields of arbitrary characteristic. For the latter statement we need the field to be algebraically closed. We therefore pass to the algebraic closure of $\Lambda$ as in §2.1. This is possible because the bounding cochain equations will still have solutions with the larger field. We denote by the same $\Lambda$ the algebraic closure.

**Theorem 3.16** (Tits). Every finitely-generated linear group has either a non-abelian free subgroup or a finite-index solvable subgroup.

**Theorem 3.17** (Lie–Kolchin). Every solvable linear group over an algebraically closed field has a finite-index triangularisable subgroup.

Consequently there exists a finite-index subgroup of $\pi_1Q$ on which $\rho$ is triangularisable. Let $f: (P, p) \to (Q, q)$ be the corresponding finite cover, and take a one-dimensional sub-representation $\sigma: \pi_1P \to \Lambda^*$ of the triangularisable group.

Recall that $\nu\sigma \in \text{hom}(\pi_1P, \mathbb{R}) \cong H^1(P, \mathbb{R})$ and denote by $\alpha$ the closed 1-form on $P$ that represents the cohomology class $\nu\sigma$ and satisfies the equation $d^*(f^*\psi dx) = 0$. Denote by $P^\alpha \subseteq T^*P$ the graph of $\alpha$. Using the notation of Lemma 2.79 write $\sigma \cong T^{-[\alpha]} \otimes E$ where $E$ is some group homomorphism $\pi_1P \to \text{GL}_1\Lambda^\alpha$. This defines a rank-one $\Lambda^0$ local system over $P^\alpha$. Hence we get a filtered local system over $P^\alpha$ which we denote by the same $E$. Lemma 2.79 implies then that $\sigma: \pi_1P \to \Lambda^*$ is isomorphic to the representation on $HF^*(T^*_bP, E)$. Denote by $f^*\rho$ the restriction to $\pi_1P$ of $\rho: \pi_1Q \to \text{GL} \cdot HF^*(T^*_bQ, b)$. As $\sigma$ is a sub-representation of $f^*\rho$ there exists a degree-zero non-zero morphism from $\sigma$ to $f^*\rho$. Hence it follows by Proposition 2.93 that there exists a degree-zero non-zero morphism from $f^*b$ to $E$. Thus $HF^0(f^*b, E) \neq 0$.

By Corollary 2.83 we have $|\nu\sigma(x)| \leq \delta|x|$ for $x \in H_1(P, \mathbb{Z})/(\text{torsion})$. Corollary 8.15 implies then that $\nu\sigma$ may be represented by a closed 1-form $\alpha$ on $P$ with $d^*(\psi dx) = 0$ and $\sup_P |\alpha_{|C}| \leq M\delta \leq \epsilon$. Proposition 3.11 implies then that the graph of $\alpha$ is Hamiltonian equivalent to some special Lagrangian $P'$. As this Hamiltonian perturbation does not change the isomorphism class of $f^*b$ it follows from Proposition 2.67 (i) that $P'$ is an irreducible component of $f^*L$. Thus $L$ is unbranched, which completes the proof of Theorem 1.3 (ii).

**Remark 3.18.** Theorem 1.3 holds for some badly immersed special Lagrangians. In the known construction of Fukaya categories, the underlying Lagrangians are required to be at least cleanly immersed [20]. On the other hand, in our nearby
problem we want to include as many (special) Lagrangians as possible, so we introduce the following notion \[33, \text{before Corollary 4.6}\].

Let \( \hat{L} \) be a closed manifolds and \( \iota: \hat{L} \to X \) be a Lagrangian immersion. Put \( L := \iota(\hat{L}) \). We say that an object \( b \) in \( HF(X) \) is supported near \( L \) if there exist arbitrarily-small Lagrangian perturbations of \( L \) which underlie some objects in the same isomorphism class \([b]\) in \( H^0F(X) \). More concretely, this means that for every \( \epsilon > 0 \) there exists a perturbation of the immersion \( \hat{L} \to X \) with \( C^k \) norm bounded by \( \epsilon \) for all \( k \), and which supports an object representing the given isomorphism class \([b]\).

Theorem 1.3 holds for a special Lagrangian \( L \) which admits such an isomorphism class \([b]\) as above. The reason is that the results of the paper \[33\] are stated and proved for objects of \( F(X) \) supported near a special Lagrangian. In particular, Proposition 2.67 and Lemma 3.4 are still valid for these objects. So the proof of Theorem 1.3 will work.

But the condition above on \([b]\) is rather strong. Here is one possible way of verifying it. Suppose given a smooth family \( \iota^t: \hat{L} \to X, t \in [0,1] \), such that \( \iota^0 = \iota \) and the other \( \iota^t, t \in (0,1] \), are all generic and mutually isotopic under domain Hamiltonians on \( \hat{L} \) in the sense of Akaho–Joyce \[9, \text{Definition 13.14 (ii)}\] (this is a family version of the classical transversality condition in differential topology). If we could lift such an isotopy to an ambient Hamiltonian isotopy, then it would follow that any isomorphism class in \( H^0F(X) \) supported on the immersion \( \iota^1 \) is also supported on \( \iota^t \) for all \( t \), generalising the Hamiltonian invariance theorem by Fukaya, Oh, Ohta and Ono \[21\]. In general, such a lift does not exist because then those strips which bound arcs between self-intersection points will have constant area under the isotopy. We can perhaps use the methods of \[48\] to prove invariance statements for this perturbation problem.

4 Tsai–Wang’s Method

We begin by recalling the key formula from geometric measure theory. Let \( X \) be a Riemannian manifold. Given a vector field \( v \) on \( X \), a point \( x \in X \), and an \( n \)-dimensional plane \( S \subseteq T_xX \) we denote by \( \mathrm{div}_S v \) the trace of the map \( S \subseteq T_xX \to T_xX \to S \) where the two arrows are the covariant derivative \( \nabla v : T_xX \to T_xX \) and the orthogonal projection \( T_xX \to S \). In particular, if \( v \) is the gradient vector field of a smooth function \( f: X \to \mathbb{R} \) then \( \mathrm{div}_S \nabla f = \mathrm{tr}_S \nabla^2 f \). We use also the notion of rectifiable integral varifolds in \( X \) which we treat, following Simon \[56\], as Radon measures on \( X \). Recall that if \( V \) is a compactly supported stationary integral \( n \)-varifold in \( X \) then for every vector field \( v \) on \( X \) we have \( \int_X \nabla f \cdot v \, dV = 0 \). In particular,

\[
\int_X \mathrm{tr}_{TV} \nabla^2 f \, dV = 0
\]

for every smooth function \( f: X \to \mathbb{R} \).
We recall now the following result from the literature [14, 61, 62]. We prove this to show how to use (4.1).

**Proposition 4.1.** Let $X$ be a Ricci-flat Calabi–Yau manifold of complex dimension $n$, and $Q \subset X$ a closed embedded special Lagrangian which has positive Ricci curvature with respect to the induced metric from $X$. Then there exists a neighbourhood $U \subseteq X$ of $Q$ in which every compactly supported stationary integral $n$-varifold is supported on $Q$.

**Proof.** Tsai and Wang prove that $Q \subset X$ is strongly stable in their sense [62, Proposition A(iii)] and that the following holds [62, Proposition 4.1]: there exists an open neighbourhood $U \subseteq X$ of $Q$ such that for every point $x \in U$ and for every $n$-dimensional plane $S \subseteq T_x U$ we have at $x$

$$\text{tr}_S \nabla^2 d_Q^2 \geq c d_Q^2$$  \hspace{1cm} (4.2)

where $d_Q : U \to [0, \infty)$ is the distance from $Q$ and $c > 0$ depends only upon $Q$. Applying (4.1) with $f = d_Q^2$ and using (4.2) we find $d_Q^2 = 0$ on the support of the varifold $V$, completing the proof. \hfill \Box

**Remark 4.2.** The geometric meaning of the proof is as follows: let $U$ be a tubular neighbourhood and $\phi^t : U \to U$, $t \leq 0$, the flow of the gradient vector field of $d_Q^2$; then $\phi^t$ does not increase the area of $V$.

As is well known, closed special Lagrangian currents are area-minimizing and in particular area-stationary [28]. Combining this with Proposition 4.1 we see immediately that the following holds:

**Corollary 4.3.** In the circumstances of Proposition 4.1 every compactly supported closed special Lagrangian current in $U$ is necessarily non-singular, of phase $0 \in \mathbb{R}/2\pi\mathbb{Z}$ and supported on $Q$. \hfill \Box

**Example 4.4.** Let $X$ be either $T^* S^n$, for $n > 1$, or $T^* \mathbb{C}P^n$ for arbitrary $n$, and let $Q$ denote its zero-section; then $X$ has a Ricci-flat metric constructed respectively by Stenzel [58] and Calabi [13]. In these two cases the hypothesis of Proposition 4.1 holds and moreover its conclusion holds in the whole $X$ rather than in a neighbourhood of $Q$.

We turn now to the example which shows that $\pi_1 Q$ abelian is essential to the latter part of Theorem 1.3 (ii). We begin by proving a lemma in slightly more general circumstances.

**Lemma 4.5.** Let $n, k \geq 0$ be integers. Take $X := T^n \times \mathbb{R}^k$, $Q := T^n \times \{0\}$ and give these the flat metrics. Then every compactly supported stationary integral $n$-varifold in $X$ is parallel to $Q$, that is, supported on $T^n \times A$ for some finite set $A \subseteq \mathbb{R}^k$.

**Proof.** Let $V$ be a compactly supported stationary integral $n$-varifold in $X$. Applying (4.1) again with $v = \text{grad} d_Q^2$ we get $\int_X \text{tr}_{TV} \nabla^2 d_Q^2 \ dV = 0$. But $\nabla^2 d_Q$
is pointwise non-negative and degenerates only in the direction of $T^n \times \{0\}$ so $V$-almost everywhere in $X$ we have $\text{tr}_{TV} \nabla^2 d_Q^2 = 0$ and $TV$ is parallel to $T^n \times \{0\}$.

We recall a lemma of Simon [56, Lemma 19.5]. Let $x$ be a point on the support of $V$ and we work near this so we can regard $x$ as a point of the universal cover $\mathbb{R}^n \times \mathbb{R}^k$ of $T^n \times \mathbb{R}^k$. Let $\epsilon > 0$ be so small that for every point $y$ of distance $< \epsilon$ from $x$ we can define $y - x \in \mathbb{R}^n \times \mathbb{R}^k$. Fix $\delta \in (0,1)$ and let $y$ be a point on the support of $V$ of distance $\delta/4$ from $x$ and such that $y - x$ is not parallel to $\mathbb{R}^n \times \{0\}$. Simon [56, Lemma 19.5] proves then that

$$\Theta_V(x) + \Theta_V(y) \leq (1 - \delta)^{-n} \epsilon^{-n} \omega_n^{-1} V(B_\epsilon)$$  \hspace{1cm} (4.3)

where $\Theta_V$ denotes the density of $V$ and $B_\epsilon$ the ball of radius $\epsilon$ about $x$. The original formula has in fact another term on the right-hand side, which contains the integral over $z \in B_\epsilon$ of $\| pr_{\mathbb{R}^n \times \{0\}} - p_z \|$ where $pr_{\mathbb{R}^n \times \{0\}}, p_z$ are the projections of $\mathbb{R}^n \times \mathbb{R}^k$ onto $\mathbb{R}^n \times \{0\}, T_z V$ respectively. But $T_z V$ is parallel to $\mathbb{R}^n \times \{0\}$ so the additional term vanishes and (4.3) holds.

As $\epsilon, \delta$ tend to 0 the right-hand side of (4.3) tends to $\Theta_V(x)$ and in particular becomes less than $\Theta_V(x) + 1$. But then by (4.3) we have $\Theta_V(y) < 1$, which is impossible. So the hypotheses above fail: that is, if $y$ is a point of the support of $V$ close enough to $x$ then $y - x$ is parallel to $\mathbb{R}^n \times \{0\}$. This completes the proof.

**Remark 4.6.** The hypothesis of Proposition 4.1 is stable under small perturbations of the Ricci-flat Kähler metric (just because positivity is an open condition). On the other hand, Lemma 4.5 will fail under perturbations of the ambient metric. For instance, Kapouleas and Yang [37] construct a sequence of minimal surfaces converging to the flat Clifford torus $T^2 \subset S^3$ with multiplicity two; and a suitable rescaling of the ambient metric converges to the flat metric on the tubular neighbourhood of $T^2 \subset S^3$.

**Corollary 4.7.** Let $X$ be a flat Calabi–Yau manifold, $Q \subset X$ a closed embedded totally geodesic special Lagrangian, and $U \subset X$ a tubular neighbourhood of $Q$. Then every compactly-supported closed special Lagrangian current in $U$ is parallel to $Q$, that is, a locally constant multi-valued graph over $Q$. Every such graph is a disjoint union of single-valued graphs if and only if $\pi_1 Q$ is abelian.

**Proof.** Since $Q$ is a closed flat manifold we get according to Bieberbach a finite Galois cover $f : T^n \rightarrow Q$ such that $\pi_1 T^n$ is a maximal abelian subgroup of $\pi_1 Q$. Denote by the same $f$ the finite cover $T^* T^n \rightarrow T^* Q$ between the cotangent bundles. Identify $U$ with a neighbourhood of the zero-section $Q \subset T^* Q$, and the pull-back $f^* U$ with a neighbourhood of $T^n \subset T^* T^n$. As $T^* T^n$ is a product bundle we can use Lemma 4.5 to show that the pull-back of a compactly supported closed special Lagrangian current $L$ is parallel; and accordingly, so is the original $L$. Now if $\pi_1 Q$ is abelian, this is the maximal abelian subgroup of itself and $Q = T^n$; so every special Lagrangian is a disjoint union of single-valued graphs. If $\pi_1 Q$ is non-abelian, there exist a non-trivial covering transformation.
\( \gamma : T^n \to T^n \) and accordingly an element \( \gamma \in \pi_1 Q \setminus \pi_1 T^n \) which acts non-trivially upon \( H^1(T^n, \mathbb{R}) \). So there exists a constant 1-form \( \alpha \) on \( T^n \) that does change under \( \gamma : T^n \to T^n \). By rescaling \( \alpha \) we can suppose that \( \alpha \) is so small that we can define its graph in \( f^* U \). The push-forward of this defines then a truly multi-valued graph in \( U \).

Example 4.8. Let \( X = U = T^* Q \) with \( Q \) a Klein bottle. Write \( Q \) as the mapping torus of the automorphism \( S^1 \to S^1 \) given by \( x \mapsto -x \) modulo \( \mathbb{Z} \) for \( x \in S^1 = \mathbb{R}/\mathbb{Z} \). Then there are corresponding double covers \( T^2 \to Q \) and \( T^* T^2 \to T^* Q \), under which the graph of \( dx \) is pushed forward to a multi-valued graph. Precisely speaking, \( X \) is not Calabi–Yau and \( Q \) is not even orientable. But \( Q \) is still a special Lagrangian in the extended sense as in Definition 2.32 see also Remark 2.33. The multi-valued graph is also a special Lagrangian in the same sense. It is easy to find multi-valued graphs which are ordinary special Lagrangians. For instance, take \( Q \) to be the mapping torus of the orientation-preserving diffeomorphism \( T^2 \to T^2 \) given by \((x, y) \mapsto (-x, -y)\) where \((x, y) \in T^2 = \mathbb{R}^2/\mathbb{Z}^2\); then the graph of \( dx \) is pushed forward to a multi-valued graph which is an ordinary special Lagrangian.

5 Branched Examples

Here are examples of branched \( C^0 \) nearby closed Lagrangians, which will all be Lagrangian surgeries. We begin by recalling the result of Fukaya, Oh, Ohta and Ono [21, Chapter 10] which describes the effect of applying Lagrangian surgeries to pairs of objects in the Fukaya category.

Theorem 5.1 (cf. [21, Chapter 10]). Let \( L, L' \subseteq X \) be two closed embedded \( \mathbb{Z} \)-graded Lagrangians which intersect each other transversely at least at one point, with the Maslov index function \( \mu_{LL'} : L \times X L' \to \mathbb{Z} \) identically equal to 1. Let \( L \# L' \) be a \( \mathbb{Z} \)-graded Lagrangian surgery [21, Lemma 2.14]. Then for any \( b, b' \in \text{obj} \mathcal{F}(X) \) supported on \( L, L' \) respectively, there exist an object \( b'' \in \text{obj} \mathcal{F}(X) \) supported on \( L \# L' \) and for every other \( a \in \text{obj} \mathcal{F}(X) \) a long exact sequence

\[
\cdots \to HF^*(a, b') \to HF^*(a, b'') \to HF^*(a, b) \to \cdots
\]

where the connecting homomorphism \( HF^*(a, b) \to HF^{*+1}(a, b') \) is the right multiplication by some element of \( HF^1(b, b') \).

Remark 5.2. We allow \( L, L' \) to have two or more intersection points, which is possible as Fukaya, Oh, Ohta and Ono mention [21, after Theorem 55.7] although they deal only with singular intersection points in their formal treatment. So there are many objects \( b'' \) of the form above according to different scales of surgeries at different intersection points. Here are remarks on the proof. Fukaya, Oh, Ohta and Ono use the cobordism method for the moduli spaces of holomorphic discs with boundary on the surgery; the cobordism is associated to stretching the neck (degenerating the
almost complex structure) near the intersection points. The method of Biran–Cornea [18] provides an easier proof which does not rely on a degeneration argument: one can associate to the surgery a Lagrangian cobordism in $X \times \mathbb{C}$, and the count of holomorphic triangles with boundary on this cobordism yields the desired long exact sequence. Although their results are stated for monotone or exact Lagrangians, these assumptions are imposed only to avoid using the theory of virtual chains in case the Lagrangians bound holomorphic discs. We shall in practice take $L, L'$ to be graphical Lagrangians in a cotangent bundle, which do not bound any non-constant holomorphic discs; so their proof can be implemented with no difficulty.

For special Lagrangians we have

**Corollary 5.3.** *In the circumstances of Theorem 5.1 let $L, L'$ be both special Lagrangians. Then $b'' \in \text{obj } F(X)$ is not isomorphic in $H^0 F(X)$ to any other object supported on a closed special Lagrangian of any phase.*

**Proof.** By shifting $L, L'$ we may suppose that $L$ has phase $2i\pi$ for some $i \in \mathbb{Z}$ and that $L'$ has phase 0. Then by (2.28) we have $-i \in (1 - n, 1)$ so $i \geq 0$. Applying (5.1) to $b, b'$ in place of $a$ we get two exact sequences

$$HF^n(b, b'') \to HF^n(b, b) \to HF^{n+1}(b, b') = 0, \quad (5.2)$$

$$0 = HF^{-1}(b', b) \to HF^0(b', b') \to HF^0(b, b''). \quad (5.3)$$

On the other hand $HF^n(b, b)$ and $HF^0(b', b')$ are both nonzero, so $HF^n(b, b'')$ and $HF^0(b', b'')$ are both nonzero. Suppose contrary to our conclusion that $b''$ is isomorphic to an object $a \in F(X)$ supported on a closed special Lagrangian $L''$ of phase $\theta \in \mathbb{R}$. By perturbing $L''$ we may suppose that $a$ is supported on a nearly-special Lagrangian $L^*$ which intersect $L$ and $L'$ generically. Since $HF^n(b, a) \cong HF^n(b, b'') \neq 0$ we get an intersection point $x \in L \cap L^*$ with Maslov index $n$. Hence denoting by $\phi : L^* \to \mathbb{R}$ the phase function and using (2.28) we find $-\phi(x)/2\pi \in (0, n)$ and in particular $\phi(x) < 0$. Also since $HF^0(b', a) \neq 0$ we get an intersection point $x' \in L' \cap L^*$ with $i - \phi(x')/2\pi \in (-n, 0)$ and in particular $\phi(x') > i \geq 0$. But both $\phi(x)$ and $\phi(x')$ are near $\theta$ so $i = \theta = 0$; that is, $L, L'$ and $L''$ have all phase 0. Since $HF^n(b, a)$ and $HF^0(b', a)$ are both non-zero it follows by Proposition 2.47(i) that these three Lagrangians have a common irreducible component. But this is impossible because $L, L'$ intersect each other transversely at least at one point. □

We recall now a key fact about *Morse* 1-forms: a Morse 1-form on a manifold $Q$ is a closed 1-form $\alpha$ on $Q$ such that every point of $Q$ has a neighbourhood on which $\alpha = df$ for some Morse function $f$.

**Theorem 5.4** (Honda [32]). *Let $Q$ be a closed manifold of dimension $n$, and $\alpha$ a Morse 1-form on $Q$ with no critical point of index 0 or $n$. Then there exists on $Q$ a Riemannian metric $g$, and in the de Rham class $[\alpha] \in H^1(Q, \mathbb{R})$, a $g$-harmonic Morse 1-form $\beta$ which has the same distribution of indices as $\alpha$ has; that is, for each $i \in \{0, \cdots, n\}$ the number of index-$i$ critical points of $\beta$ is equal to that of $\alpha$. □*
Remark 5.5. If $Q$ is equipped with any real analytic structure, we can suppose that $g$ and $\beta$ above are real analytic. To see this, perturb the metric from Honda’s theorem to a real analytic Riemannian metric, which we denote still by the same $g$. Choosing the perturbation to be sufficiently small, there exists a unique $g$-harmonic 1-form near $\beta$ in the same de Rham class, which we denote again by the same $\beta$. This $\beta$ is analytic by elliptic regularity.

Here is another fact we will use. We give $T^*Q$ the canonical symplectic and Liouville structures.

Lemma 5.6. Let $(Q, g)$ be a closed oriented real analytic Riemannian manifold. Then there exist a disc sub-bundle $X \subseteq T^*Q$ and a standard Calabi–Yau structure $(J, \Omega)$ (in the sense of Definition 3.1) such that the restriction of the Kähler metric to $Q$ is equal to $g$, and the complex structure is compatible with the Liouville structure of $X$ (in the sense of Definition 2.28).

Remark 5.7. The real analytic structure induced from the complex structure above is possibly different from that induced from the real analytic structures of $Q$ and $T^*Q$.

Proof of Lemma 5.6. Guillemin and Stenzel [27, §5] prove this except the part which has to do with $\Omega$; that is, there exist a sufficiently small disc sub-bundle $X \subseteq T^*Q$ and a complex structure $J$ on $X$ compatible with the Liouville structure of $X$ and such that the restriction to $Q$ of the corresponding Kähler metric is equal to $g$. Making $X$ smaller if we need, extend $g$ to a holomorphic section of $\text{sym}^2 T^*X$ which is locally of the form $g_{ab}(z^1, \ldots, z^n)dz^a dz^b$ where $z^1, \ldots, z^n$ are local holomorphic co-ordinates with which $Q$ is defined by $\text{Im} z^1 = \cdots = \text{Im} z^n = 0$. This is possible because every $g_{ab}$ is real analytic. Since $Q$ is oriented we get, making $X$ yet smaller if we need, a holomorphic volume form on $X$ which is locally of the form $\sqrt{\det g_{ab}} dz^1 \wedge \cdots \wedge dz^n$ and whose restriction to $Q$ agrees with the volume form of $(Q, g)$. This completes the proof.

Remark 5.8. Guillemin and Stenzel [27, §5] do not suppose that $Q$ is oriented. They show also that $J$ is invariant under the involution $\sigma : T^*Q \to T^*Q$ defined by $p \mapsto -p$ and that $J$ is the unique complex structure satisfying this and the other conditions in the lemma.

We give now an example of non-special Lagrangian surgeries. Let $Q$ be a closed connected orientable manifold of dimension $n$, and $\alpha$ a Morse 1-form on $Q$ whose critical points have the same index. The Poincaré duality of Morse–Novikov homology implies then that $n$ must be even and the critical points must have index $\frac{n}{2}$. We prove

Theorem 5.9. There exist a disc sub-bundle $X \subseteq T^*Q$, a closed embedded branched $\mathbb{Z}$-graded Lagrangian $L$, and an object of $HF(X)$ supported on $L$ that is not isomorphic to any other object supported on a closed special Lagrangian (of any phase relative to any standard Calabi–Yau structure).
Proof. Applying Theorem 5.4 to $\alpha$ we get a harmonic 1-form $\beta$ with the same distributions of critical points as $\alpha$ has. Applying McLean’s theorem to $\beta$ we get a $C^1$ nearby special Lagrangian $Q'$ such that the pair $(Q', Q)$ has the same distribution of indices as $\alpha$ has. Consider now $Q'[\frac{n}{2} - 1]\#Q$ as in Theorem 5.1. Corollary 5.3 implies then that this cannot be represented by a special Lagrangian, completing the proof.

Example 5.10. Let $n \geq 4$ be an even number and take $Q$ to be the self-connected sum of $S^{n/2} \times S^{n/2}$; that is, $Q$ is obtained from $S^{n/2} \times S^{n/2}$ by eliminating two points of it and attaching the handle between them. Take a Morse function $S^{n/2} \times S^{n/2} \to \mathbb{R}$ with one critical point of index 0, two of index $\frac{n}{2}$ and one of index $n$. Performing the connected sum construction at the critical point of index 0 and that of index $n$, we obtain a 1-form $\alpha$ on $Q$ satisfying the hypothesis of the theorem above. There is accordingly a branched nearby Lagrangian $Q'[\frac{n}{2} - 1]\#Q$ with $HF^*$ unobstructed. On the other hand, as $n \geq 4$ there is an isomorphism $\pi_1 Q \cong \mathbb{Z}$ so the hypotheses of Theorem 1.3 (ii) holds except the special Lagrangian condition.

Let $b$ be an object of the Fukaya category supported on $Q'[\frac{n}{2} - 1]\#Q$ and let $q \in Q$ be any point. Then the Floer cochain group $CF^*(T^*_q Q, b)$ is non-zero in degrees $\frac{n}{2} - 1$ and 0. As $n \geq 4$ these two degrees differ at least by two and no differentials exist between them. So the cohomology group $HF^*(T^*_q Q, b)$ is non-zero in two different degrees, which is a counterexample to Lemma 3.4.

We give now an example of special Lagrangian surgeries.

Theorem 5.11. Let $n \geq 3$ and let $Q$ have a Morse 1-form $\alpha$ with at least one critical point of index 1, at least one of index $n - 1$ and none of index 0 or $n$. Then near $Q \subset T^*Q$ there exist a standard Calabi–Yau structure and relative to it a closed branched special Lagrangian.

Proof. Following the proof of Theorem 5.9 we get a $C^1$ nearby special Lagrangian $Q'$ such that the Maslov indices of the pair $(Q', Q)$ has the same distributions of the Morse index of $\alpha$. So $(Q', Q)$ has at least one pair intersection points, one of index 1 and the other of index $n - 1$, which thus satisfies the hypothesis of Joyce [34, Theorem 9.7]. We can define then a special Lagrangian surgery $Q'\#Q$ as we want.

The branched special Lagrangian in the preceding theorem is possibly immersed; any pairs of critical points of $\alpha$ of indices 1 and $n - 1$ may be removed by surgery but the other critical points become transverse double points.

Example 5.12. Take $Q$ to be the selfconnected sum of $S^1 \times S^{n-1}$; then $Q$ has a Morse 1-form with exactly two critical points, one of index 1 and the other of index $n - 1$. So with $n \geq 3$ we can apply Theorem 5.11 and if $n \geq 4$ we have $b_2(L) = 0$ so that $L$ is $HF^*$-unobstructed.

But there are also $HF^*$-obstructed special Lagrangian surgeries:
Corollary 5.13. Let \( n, Q, \alpha \) be as in Theorem 5.11 and assume that \( \pi_1 Q \) has no nonabelian free subgroup.

(i) If \( n = 3 \) then near \( Q \subset T^*Q \) there exist a standard Calabi–Yau structure and relative to it a closed embedded \( HF^* \)-obstructed special Lagrangian.

(ii) If \( n \geq 4 \) then near \( Q \subset T^*Q \) there exist a standard Calabi–Yau structure and relative to it a closed generically-immersed \( HF^* \)-obstructed special Lagrangian \( L \).

Proof. For (i) recall from Poincaré–Hopf’s theorem that the numbers of critical points of indices 1 and 2 are the same. So by Theorem 5.11 there is a branched embedded special Lagrangian. By Theorem 1.3 this has \( HF^* \) obstructed as asserted. Part (ii) may be proved in the same way.

Example 5.14. For (i) take \( Q = S^1 \times S^2 \) and regard it as the self-connected sum of \( S^3 \). According to Milnor [11, Lemma 8.2] for every Morse function we can produce, by changing it locally, two additional critical points of any adjacent indices. Applying this to the height function of \( S^3 \) we get a Morse function with exactly one critical point of index 0, exactly one critical point of index 3 and arbitrarily many pairs of critical points of indices 1 and 2. So \( Q \) has such a Morse 1-form as in (i) above.

In the same way, on \( Q = S^1 \times S^{n-1} \) with \( n \geq 4 \) there is such a Morse 1-form as in (ii) above with arbitrarily many pairs of critical points of adjacent indices 1, 2 or \( n - 2, n - 1 \).

Corollary 5.13 implies a statement without Lagrangians in it and perhaps of interest in itself:

Corollary 5.15. Let \( Q \) be a closed connected orientable manifold of dimension \( n \) with \( \pi_1 Q \) having no nonabelian free subgroup. Let \( \alpha \) be a Morse 1-form on \( Q \) with no critical points of index 0 or \( n \). Then

(i) \( n = 2 \) we have \( Q \cong T^2 \) and \( \alpha \) has no critical point at all.

Suppose now that \( n \geq 3 \) and \( \alpha \) has at least one critical point of index 1 and one critical point of index \( n - 1 \). Then

(ii) if \( n = 3 \) there exists a smooth function \( f : Q \to \mathbb{R} \) such that \( \alpha - df \) has no critical point at all; and

(iii) if \( n \geq 4 \) then either \( b_2(Q) > 0 \) or \( \alpha \) has some critical point of index 2 or \( n - 2 \).

Proof. We first prove (i). Since \( \alpha \) has no critical points of index 0 or \( n \) it follows that \( \alpha \) is not exact, so \( b_1(Q) > 0 \). If \( n = 2 \) this condition and the hypothesis on \( \pi_1 Q \) imply \( Q \cong T^2 \), but since the Euler characteristic of \( T^2 \) vanishes, every Morse 1-form on \( T^2 \) with no critical point of index 0 or 2 has no critical point at all, as in (i).
We then prove (iii). Since the immersed Lagrangian \( L \) in Corollary 5.13 (ii) has \( HF^* \) obstructed, it follows that either \( b_2(L) > 0 \) or \( L \) has some self-intersection points of index 2 or \( n - 2 \). So \( b_2(Q) > 0 \) (because \( n \geq 4 \)) or \( \alpha \) has some critical points of index 2 or \( n - 2 \).

We prove (ii) in two different ways. The first proof goes as follows. Let \( x, y \) be two critical points of \( \alpha \), one of index 1 and the other of index 2; such a pair exists by Poincaré duality of Novikov homology. As in Corollary 5.13 (i) there is a double point surgery \( L \) with \( HF^* \) obstructed. In dimension \( n = 3 \) the bounding cochains on \( L \) are critical points of the disc potential function. The relevant homology classes come from those of strips between \( x, y \) with the right orientation. For this we use again Fukaya, Oh, Ohta Ono’s result [21, Chapter 10]; the number of holomorphic discs homologous to this strip with multiplicity one is conserved under the surgery. We prove that the number \( \nu \) of holomorphic discs homologous to the multiplicity-one strip is exactly one. This \( \nu \) is an integer because \( T^*Q \) is an exact symplectic manifold and we can therefore use the result of Fukaya, Oh, Ohta and Ono [23]. We can also work with the Novikov field of any characteristic \( p \) so the Lagrangian \( L \) has \( HF^* \) obstructed in characteristic \( p \); and in particular, \( \nu \) is non-zero modulo \( p \). But \( p \) is any, so \( \nu = 1 \).

Denote by \( Q_\alpha \) the graph of \( \alpha \). As the Floer complex of the pair \((Q_\alpha, Q)\) and the Novikov complex of \( \alpha \) are quasi-isomorphic, the number of gradient trajectories from \( y \) to \( x \) is also one. We can then use the cancellation theorem in Morse theory [44, Theorem 5.4]; that is, we can cancel out \( x, y \) with each other by changing \( \alpha \) in its de Rham cohomology class. Repeating this process we get a Morse 1-form with no critical point at all, as claimed.

For the other proof we use the geometrisation theorem on 3-manifolds, from which we deduce

**Proposition 5.16.** Every closed orientable 3-manifold \( Q \) satisfies one of the following five statements: (i) \( \pi_1 Q \) is finite; (ii) \( Q \cong S^1 \times S^2 \); (iii) \( Q \cong \mathbb{R}P^3 \# \mathbb{R}P^3 \); (iv) there exists a finite cover \( P \to Q \) where \( P \) is the mapping torus of an orientation-preserving diffeomorphism \( T^2 \to T^2 \); or (v) \( \pi_1 Q \) has a nonabelian free subgroup.

**Proof.** We suppose that \( \pi_1 Q \) is infinite and prove that one of (ii)–(v) holds; we shall refer to the book [11] for those results about 3-manifolds which we use, without tracing back their origins in the literature. As \( \pi_1 Q \) is infinite, \( Q \) has no Riemannian metric of positive constant sectional curvature, so \( Q \) is not spherical [11, §1.7]. According to Theorem 1.11.1 of [11], either one of (ii)–(iv) above holds; or \( \pi_1 Q \) has no finite-index solvable subgroup. It thus remains to show that, if \( Q \) is an infinite 3-manifold group with no finite-index solvable subgroup, then it must have a nonabelian free subgroup.

The first step in this direction is to note that [11, p50, (C.6)] asserts that our assumption on the fundamental group implies that one of the following (a)–(d) holds: (a) \( Q \) is reducible [11, p8]; (b) \( Q \) is a hyperbolic manifold and \( \pi_1 Q \) is a linear group over \( \mathbb{Q} \) [11, p51, (C.7)]; (c) \( Q \) is a Seifert manifold and \( \pi_1 Q \) is a...
linear group over $\mathbb{Z}$ \[11\ p52, (C.10) and (C.11)\]; or (d) $Q$ is a Haken manifold with an incompressible torus and (v) holds \[11\ p58, (C.24)\]. Moreover:

- In Case (a) it is well known that either (ii) holds or $Q$ is a connected sum of two closed 3-manifolds $P_1, P_2$ neither of which is a sphere. In the latter case $\pi_1P_1, \pi_1P_2$ are both nontrivial (because Poincaré’s conjecture is true). If $\#\pi_1P_1 = \#\pi_1P_2 = 2$ then $P_1 \cong P_2 \cong \mathbb{R}P^3$ and (iii) holds. Otherwise, $\pi_1Q \cong \pi_1P_1 \ast \pi_1P_2$ has so many elements that (v) holds as follows: one of $\{\pi_1P_1, \pi_1P_2\}$ has at least two nontrivial elements $x, y$ and the other of $\{\pi_1P_1, \pi_1P_2\}$ has at least one nontrivial element $z$ so there is a nonabelian free group generated by $xz, yz$.

- In Cases (b) and (c) we use Tits’ theorem \[60\ Theorem 1\]: in characteristic 0 every linear group has either a finite-index subgroup or a nonabelian free subgroup (we have used already a positive-characteristic version, Theorem 5.16). Since we have supposed that $\pi_1Q$ has no finite-index solvable subgroup, it follows that (v) holds in Cases (b) and (c).

This completes the proof.

Proposition 5.16 implies

**Corollary 5.17.** Let $Q$ be a closed orientable 3-manifold with $\pi_1Q$ having no nonabelian free subgroup. Then every nonzero element of $H^1(Q, \mathbb{R})$ may be represented by a nowhere-vanishing closed 1-form on $Q$.

**Proof.** By Proposition 5.16 one of (i)–(iv) holds. In Cases (i) and (iii) we have $H^1(Q, \mathbb{R}) = 0$ and there is nothing to prove. In Case (ii) we have $Q \cong S^1 \times S^2$ and letting $x$ be the coordinate of $S^1$ we have $H^1(Q, \mathbb{R}) \cong \mathbb{R}[dx]$, which has certainly the desired property. In Case (iv) the mapping torus $P$ is determined by the monodromy matrix $M \in SL_2(\mathbb{Z})$ acting upon $H^1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2$ which we can classify by direct computation into the following three cases: (a) there is an integer $k > 0$ such that $M_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; (b) $M^2$ is conjugate in $GL_2(\mathbb{Z})$ to $\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$ for some $l \in \mathbb{Z} \setminus \{0\}$; (c) $M$ has two distinct real eigenvalues, neither equal to 1. We study these respectively as follows:

- In Case (a) we can rechoose the finite cover $P \to Q$ so that $P = T^3$, from which we get a natural embedding $H^1(P, \mathbb{R}) \subseteq H^1(T^3, \mathbb{R}) = \mathbb{R}[dx] \oplus \mathbb{R}[dy] \oplus \mathbb{R}[dz]$ where $x, y, z$ are the three coordinates of $T^3 = S^1 \times S^1 \times S^1$. This expression implies the desired property of $H^1(P, \mathbb{R})$.

- In Case (b) we can rechoose the finite cover $P \to Q$ so that $M$ will have two eigenvalues both equal to one. Trivialising $P$ over two intervals in $S^1$ and applying to them the Mayer–Vietoris sequence, we get an isomorphism

$$H^1(P, \mathbb{R})/\mathbb{R}[dx] \to H^1(T^2, \mathbb{R})^M$$  \hspace{1cm} (5.4)
where \([dx]\) is obtained from the projection \(x : P \to S^1\) and the right-hand side is the \(M\)-invariant subspace of \(H^1(T^2, \mathbb{R})\). As \(M\) has two eigenvalues both equal to one, there is a coordinate \(y : T^2 \to S^1\) such that \(dy\) is invariant under \(M\) and lifts to \(P\). Then by (5.4) we have \(H^1(P, \mathbb{R}) = \mathbb{R}[dx] \oplus \mathbb{R}[dy]\), in which \(H^1(Q, \mathbb{R})\) has certainly the desired property.

- In Case (c) there is again an isomorphism of the form (5.4) but the right-hand side vanishes because \(M\) has no eigenvalue equal to one. So \(H^1(P, \mathbb{R}) = \mathbb{R}[dx]\), in which \(H^1(Q, \mathbb{R})\) has certainly the desired property.

This completes the proof.

Corollary 5.15 (ii) follows from Corollary 5.17.

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