MUL TIBUBBLE BLOW-UP ANAL YSIS FOR THE BREZIS-NIRENBERG PROBLEM IN THREE DIMENSIONS

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ABSTRACT. For a smooth bounded domain \( \Omega \subset \mathbb{R}^3 \) and smooth functions \( a \) and \( V \), we consider the asymptotic behavior of a sequence of positive solutions \( u_\varepsilon \) to
\[-\Delta u_\varepsilon + (a + \varepsilon V)u_\varepsilon = u_\varepsilon^5 \text{ on } \Omega \text{ with zero Dirichlet boundary conditions, which blow up as } \varepsilon \to 0.\]
We derive the sharp blow-up rate and characterize the location of concentration points in the general case of multiple blow-up, thereby obtaining a complete picture of blow-up phenomena in the framework of the Brezis-Peletier conjecture in dimension \( N = 3 \).

1. INTRODUCTION

For an open bounded set \( \Omega \subset \mathbb{R}^3 \), let us consider a sequence of solutions \((u_\varepsilon)\) to the problem
\[-\Delta u_\varepsilon + (a + \varepsilon V)u_\varepsilon = u_\varepsilon^5 \quad \text{on } \Omega,\]
\[u_\varepsilon > 0 \quad \text{on } \Omega,\]
\[u_\varepsilon = 0 \quad \text{on } \partial \Omega.\] (1.1)

We will assume throughout the paper that \( a \in C(\overline{\Omega}) \cap C^{1,\sigma}_{\text{loc}}(\Omega) \) for some \( \sigma \in (0,1) \) and \( V \in C(\overline{\Omega}) \cap C^1(\Omega) \), but it is perfectly meaningful to think of \( a \) and \( V \) being constants. Moreover, we always assume that \(-\Delta + a\) is coercive, which is in fact a necessary condition in this context, see Appendix C of [19], and that the boundary of \( \Omega \) is \( C^2 \).

The study of this equation has been initiated in the seminal work [2] by Brezis and Nirenberg. The understanding of the behavior of solutions of this equation is pivotal in the Yamabe problem, see for instance [8] and reference therein. Subsequently, Brezis and Peletier [3] initiated the study of (1.1) in the case where there is at least one blow-up point \( x_0 \in \overline{\Omega} \), i.e. there is a sequence \( x_\varepsilon \to x_0 \) such that \( u_\varepsilon(x_\varepsilon) \to \infty \) as \( \varepsilon \to 0 \). In [3] the authors conjecture an asymptotic expression for \( \|u_\varepsilon\|_\infty \) in the case where \((u_\varepsilon)\) has precisely one blow-up point.

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Date: January 9, 2023

Partial support through ANR BLADE-JC ANR-18-CE40-002 is acknowledged.
To discuss this in more depth, let us introduce the object that largely governs the asymptotic behavior of \( (u_\varepsilon) \), namely the Green’s function \( G_a : \Omega \times \Omega \to \mathbb{R} \). This is the unique function satisfying, for each fixed \( y \in \Omega \),

\[
\begin{cases}
-\Delta_x G_a(x, y) + a(x) G_a(x, y) = \delta_y & \text{in } \Omega, \\
G_a(\cdot, y) = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(1.2)

Note that \( G_a(x, y) > 0 \) for every \( x, y \in \Omega \) as a consequence of coercivity. The regular part \( H_a \) of \( G_a \) is defined by

\[
H_a(x, y) := \frac{1}{4\pi|x-y|} - G_a(x, y).
\]  

(1.3)

It is well-known that for each \( y \in \Omega \) the function \( H_a(\cdot, y) \), which is originally defined in \( \Omega \setminus \{y\} \), extends to a continuous function in \( \Omega \). Thus we may define the Robin function

\[
\phi_a(y) := H_a(y, y).
\]

It is proved in [3, 29, 15] that single-blow-up sequences of solutions to (1.1) must concentrate at critical points \( x_0 \) of \( \phi_a \).

For space dimension \( N \geq 4 \) and \( a \equiv 0, \ V \equiv -1 \), sequences of solutions with a single blow-up point \( x_0 \) exist as a consequence of the Brezis–Nirenberg existence result [2]. As observed conjecturally in [3] and confirmed rigorously in [29, 15], the blow-up behavior of such \( u_\varepsilon \) is governed by the value \( \phi_0(x_0) \) in the sense that

\[
\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|_\infty^{\frac{2(N-4)}{N-2}} = d_N \phi_0(x_0),
\]  

(1.4)

where \( d_N \) is a constant only depending on \( N \) only. (Note that \( \phi_0(x_0) > 0 \) by the maximum principle for \( H_0(x_0, \cdot) \).)

However, the conjectures in [3] leave open what happens in the Brezis–Nirenberg-critical dimension \( N = 3 \), even in the case of one blow-up point \( x_0 \). Indeed, by [7] single blow-up sequences must satisfy \( \phi_a(x_0) = 0 \) in that case, so that the leading order of \( \|u_\varepsilon\|_\infty \) can no longer be captured by the right side of (1.4). In particular, since \( \phi_0 > 0 \) we necessarily must have \( a \neq 0 \) if blow-up occurs.

Dealing with the case where \( \phi_a(x_0) = 0 \) in the context of blow-up asymptotics is a formidable problem. The reason for this is that the value \( \phi_a(x_0) \) appears as the leading coefficient of a certain energy expansion related to \( u_\varepsilon \), see e.g. [29, eq. (23)], [13, eq. (3.28)]. As long as \( \phi_a(x_0) > 0 \), this term determines the asymptotic behavior of \( \|u_\varepsilon\|_\infty \) as in (1.4). Now if \( \phi_a(x_0) = 0 \), it is the next term in this expansion that becomes relevant for the asymptotics of \( \|u_\varepsilon\|_\infty \). To extract this term, the expansion needs in turn to be computed at a higher precision, which is a considerable analytic challenge.
Indeed, even for a single blow-up point $x_0$, the asymptotics replacing (1.4) in case $N = 3$ (and hence $\phi_a(x_0) = 0$) have been derived only recently in [13] under a non-degeneracy assumption on $\phi_a$, see also [14, 12] for the special case of least-energy solutions. They read

$$\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|_\infty^2 = \frac{\sqrt{3}}{4} \frac{|a(x_0)|}{\int_{\Omega} V(y)G_a(x_0, y)^2 \, dy}.$$ 

(In this statement it is assumed that $a(x_0) < 0$. If at the same time $\int_{\Omega} V(y)G_a(x, y)^2 \, dy = 0$, then (1.5) continues to hold with right side equal to $+\infty$.)

2. Main results

In this paper we achieve a complete analysis of blowing-up solutions $u_\varepsilon$ to (1.1) in the spirit of Brezis and Peletier, in the general case where the sequence of solutions $u_\varepsilon$ to (1.1) may present multiple (a priori even infinitely many) blow-up points.

In particular, we are able to describe precisely the $L^\infty$ asymptotics near each concentration point, generalizing (1.5). The appropriate expression, see (2.5) below, involves an interaction between the blow-up points through new quantities and cannot be guessed easily from (1.5).

To state our result precisely, we introduce some more notation.

For any number $n \in \mathbb{N}$ of concentration points, let 

$$\Omega^n_* := \{ \mathbf{x} = (x_1, ..., x_n) \in \Omega^n : x_i \neq x_j \text{ for all } i \neq j \}.$$ 

For $\mathbf{x} \in \Omega^n_*$ we denote $M_a(\mathbf{x}) \in \mathbb{R}^{n \times n} = (m_{ij})_{i,j=1}^n$ the matrix with entries 

$$m_{ij}(\mathbf{x}) := \begin{cases} \phi_a(x_i) & \text{for } i = j, \\ -G_a(x_i, x_j) & \text{for } i \neq j. \end{cases}$$ 

(2.1)

Its lowest eigenvalue $\rho_a(\mathbf{x})$ is simple and the corresponding eigenvector can be chosen to have strictly positive components, see Lemma 4.5. We denote by $\Lambda(\mathbf{x}) \in \mathbb{R}^n$ the unique vector such that 

$$M_a(\mathbf{x}) \cdot \Lambda(\mathbf{x}) = \rho_a(\mathbf{x})\Lambda(\mathbf{x}), \quad (\Lambda(\mathbf{x}))_1 = 1.$$ 

Moreover, we define the Aubin–Talenti type bubble function 

$$B(x) := \left(1 + \frac{|x|^2}{3}\right)^{-1/2}$$ 

and, for every $\mu > 0$ and $x_0 \in \mathbb{R}^3$ its rescaled and translated versions 

$$B_{\mu,x_0}(x) = \mu^{-1/2}B \left(\frac{x - x_0}{\mu}\right) = \frac{\mu^{1/2}}{(\mu^2 + \frac{|x-x_0|^2}{3})^{1/2}}.$$ 

Notice that the normalizations are chosen here so that $-\Delta B_{\mu,x_0} = B_{\mu,x_0}^5$ on $\mathbb{R}^3$, for every $\mu > 0$ and $x_0 \in \mathbb{R}^3$. 

The $B_{\mu, x_0}(x)$ are easily found to represent the leading order profile of $u_\varepsilon$ around each of its concentration points, see Proposition 3.1 below. However, as explained above, a higher precision is needed for our purposes. Thus we shall need to introduce the following explicit correction function. Let $W$ be the unique radial solution to

$$\begin{align*}
-\Delta W - 5W B^4 &= -B, \\
W(0) &= \nabla W(0) = 0,
\end{align*}
$$

see Lemma B.5 below. In analogy to the notation for $B_{\mu, x_0}$ we set

$$W_{\mu, x_0}(x) := \mu^{-1/2} W \left( \frac{x - x_0}{\mu} \right).$$

Here is our main result.

**Theorem 2.1.** Let $(u_\varepsilon)$ be a sequence of solutions to \((1.1)\) with \(\|u_\varepsilon\|_\infty \to \infty\). Then there exists \(n \in \mathbb{N}\) and \(n\) sequences of points \(x_{1,\varepsilon}, \ldots, x_{n,\varepsilon} \in \Omega\) such that \(\mu_{i,\varepsilon} := u_\varepsilon(x_{i,\varepsilon})^{-2} \to 0\) as \(\varepsilon \to 0\) and \(\nabla u_\varepsilon(x_{i,\varepsilon}) = 0\) for every \(\varepsilon > 0\).

Moreover, the following holds.

(i) **Properties of concentration points:** There is \(x_0 := (x_{1,0}, \ldots, x_{n,0}) \in \Omega^n\) such that up to a subsequence, \((x_{1,\varepsilon}, \ldots, x_{n,\varepsilon}) \to x_0\). Moreover, \(\rho_\varepsilon(x_0) = \nabla_x \rho_\varepsilon(x_0) = 0\).

The matrix \(M_\varepsilon(x_0)\) is semi-positive definite with simple lowest eigenvalue \(\rho_\varepsilon(x_0)\).

The associated eigenvector is \(\Lambda(x_0) = (\Lambda_{1,0}, \ldots, \Lambda_{n,0})\) with \(\Lambda_{i,0} = \lim_{\varepsilon \to 0} \mu_{i,\varepsilon}^{1/2} \in (0, \infty)\) for every \(i\).

(ii) **Global asymptotics:** \(u_{1,\varepsilon}^{-1/2} u_\varepsilon(x) \to 4 \pi \sqrt{3} \sum_i \Lambda_{i,0} G_a(x_i, x) := G(x)\) uniformly away from \(\{x_{1,0}, \ldots, x_{n,0}\}\).

(iii) **Refined local asymptotics:** Let \(B_{i,\varepsilon} := B_{\mu_{i,\varepsilon}, x_{i,\varepsilon}}\) and \(W_{i,\varepsilon} := (a + \varepsilon V)(x_{i,\varepsilon}) W_{\mu_{i,\varepsilon}, x_{i,\varepsilon}}\).

Then, for \(\delta > 0\) small enough and every \(0 < \nu < 1\),

$$|u_\varepsilon - B_{i,\varepsilon} - \mu_{i,\varepsilon} W_{i,\varepsilon}| \lesssim \mu_{i,\varepsilon}^{1-\nu} |x - x_{i,\varepsilon}|^{2+\nu} \quad \text{on } B(x_{i,\varepsilon}, \delta).$$

(iv) **Blow-up rate:** Assume either (a) that \(\rho_0\) is $C^2$ in $x_0$ with $D_\rho^2 \rho_0(x_0) \geq c$ for some $c > 0$, in the sense of quadratic forms, or (b) that \(\rho_\varepsilon\) is real-analytic in $x_0$.

Let $G$ be as in (ii). Then

$$\lim_{\varepsilon \to 0} \varepsilon u_\varepsilon(x_{i,\varepsilon})^2 = 12 \pi^2 \sqrt{3} \Lambda_{i,0}^{-2} \sum_{j=1}^n a(x_{j,0}) \Lambda_{j,0}^4 \int_\Omega V G^2 dx,$$

provided that both of the quantities $\sum_j a(x_{j,0}) \Lambda_{j,0}^4$ and $\int_\Omega V G^2 dx$ are non-zero. If one of them equals zero, but not the other one, \((2.5)\) remains true, with the right side being equal to $0$, respectively $+\infty$.

We emphasize that contrary to previous result on the Brezis-Peletier conjecture no bound on the number of blow-up points of $u_\varepsilon$ is assumed. In fact it was already
known from the work of Li-Zhu [23] that in dimension 3 the blow-up points must be isolated, see also [21, 22]. In fact, points (i) and (ii) of the theorem follow directly from [10] and have been reproved in [19], see also reference therein, but we chose to include them in the theorem for a complete statement.

This theorem is an exhaustive description of blow-up phenomena of equation (1.1) in dimension \( N = 3 \). Its main points are items (iii) and (iv), namely the strong (superquadratic) pointwise bound (2.4) of \( u_\varepsilon \) near each blow-up point, and the explicit asymptotic expression for the blow-up rates \( u_\varepsilon(x_{i,\varepsilon}) \) in (2.5) derived from it.

Let us give several more remarks to put this result into context.

Remarks 2.2. (a) The appearance of the matrix \( M_a \) and its lowest eigenvalue \( \rho_a \) in the asymptotic expansions relative to multiple blow-up is well-known, see e.g. [1, 27, 28, 5, 24]. E.g. in [28] solutions to (1.1) blowing up in points \((x_1, ..., x_n) = x\) are constructed under the assumption \( \rho_a(x_0) = \nabla \rho_a(x_0) = 0 \), which is optimal as Theorem 2.1 shows.

Our new contribution is, in this context, to deal with the vanishing \( \rho_a(x_0) = 0 \) and to extract the next-order term determining the asymptotics (2.5), compare the discussion leading to (1.5). This is the main difficulty overcome in our paper. To underline the novelty of (2.5), one may remark that even for the solutions constructed in [28], only the information \( \mu_i,\varepsilon = O(\varepsilon) \) is obtained through the existence argument, which is less precise than (2.5).

(b) The conditions (a) and (b) in item (iv) can be thought of as non-degeneracy assumptions. This is clear for (a), which is the natural generalization of [13, Assumption 1.1(d)] to the case of multiple blow-up points. The \( C^2 \)-differentiability of \( \rho_a \) is guaranteed under the slightly stronger assumption \( a \in C^{0,1}(\Omega) \cap C_{\text{loc}}^{2,\sigma}(\Omega) \) for some \( \sigma \in (0, 1) \), see Lemma 4.7.

Remarkably, in assumption (b), no positivity condition at all is needed, only higher regularity of \( \rho_a \), more precisely real-analyticity. This observation is new even for the case of one blow-up point. We prove in Lemma 4.8 that \( \rho_a \) is real-analytic if \( a \equiv \text{const.} \). But in view of similar results, e.g. [17, 18, 11], it is reasonable to expect that \( \phi_a \), and hence \( \rho_a \), is real-analytic whenever \( a \) is; see also the remarks after the proof of Lemma 4.8. This is an open question to the best of our knowledge, and it would be very interesting to obtain an answer to it.

(c) Since Theorem 2.1 only makes a statement about space dimension \( N = 3 \), a natural question is to determine the asymptotic behavior of a sequence of solutions
to $-\Delta u_\varepsilon + \varepsilon V u_\varepsilon = u_\varepsilon^{\frac{N+2}{N-2}}$ when $N \geq 4$. We are not aware of a result like this in the literature, even though in view of the above discussion it should be easier to obtain than Theorem 2.1. Analogous computations as those used in the proof of Theorem 2.1 lead us to believe that in this case (say $N \geq 5$ for simplicity)

$$
(M_0(x_0) \cdot \lambda_0)_i = -c_N \left( \lim_{\varepsilon \to 0} \varepsilon \mu_{i,\varepsilon}^N V(x_{i,0}) \lambda_{i,0} \right)
$$

for some dimensional constant $c_N > 0$. Here,

$$
\mu_{i,\varepsilon} = u(x_{i,\varepsilon}) \quad \text{and} \quad \lambda_{i,0} = \lim_{\varepsilon \to 0} \left( \frac{\mu_{i,\varepsilon}}{\mu_{1,\varepsilon}} \right)^\frac{N-2}{2} \in (0, \infty),
$$

and we have employed the above notations otherwise. We will come back to a thorough analysis of this problem in future work.

(d) Another problem closely related to (1.1) is (for $N \geq 3$)

$$
-\Delta u_\varepsilon + au_\varepsilon = u_\varepsilon^{\frac{N+2}{N-2}}, \quad u_\varepsilon > 0, \quad u_\varepsilon|_{\partial \Omega} = 0,
$$

whose single-blow-up asymptotics as $\varepsilon \to 0+$ in the case $a \equiv 0$ have as well been determined by [3, 29, 15], see also [16]. The case of multiple concentration points in (2.7), still for $a \equiv 0$, has subsequently been studied in [31, 1, 32]. There, the authors derive an asymptotic formula for (essentially) $u_\varepsilon(x_{i,\varepsilon})$ similar to (iv) under the condition that $\rho_0(x_0) > 0$. In the spirit of our above discussion, this should be viewed as the analogue of the simpler case (1.4). For single-blow-up in $N = 3$, the analogue of the harder formula (1.5) has been proved in [13, Theorem 1.3].

On the other hand, for multiple blow-up in (2.7), we are not aware of a formula analogous to (2.5) for $\rho_a(x_0) = 0$, not even when $a \equiv 0$. We believe that our methods can yield such a formula, but we leave this question to future work. Such belief may be justified by the results in [13], where $L^\infty$ single-blow-up asymptotics are obtained for both (1.1) and (2.7) when $\phi_a(x_0) = 0$ by arguments very similar to each other, which are however different from the ones employed here.

A crucial tool which we use repeatedly in the asymptotic analysis leading to Theorem 2.1 is the non-degeneracy of the bubble $B$ as a solution to the equation $-\Delta u = u^5$. The non-degeneracy property roughly says that the solutions of the linearized equation around $B$, i.e. $-\Delta v = 5B^4 v$, with polynomial growth are either the one you may

\[1\] We take $a \equiv 0$ here so that $\phi_a$ remains well-defined when $N \geq 4$. But treating more general $a \neq 0$ appears equally possible by using the appropriate asymptotic expansion of $G_a(x, \cdot)$, which contains additional singular terms.

\[2\] A subtle, yet interesting difference between (1.1) and (2.7) is that when single blow-up happens (say in $x_0$), one automatically has $\phi_a(x_0) = 0$ in the former problem [13, Theorem 1.5], but not in the latter [6, Theorem 2.(b)]. Hence $\rho_0(x_0) > 0$ may well be satisfied, even when $N = 3$.\]
expect, that is to say the one you can construct from the family $B_{\mu,x_0}$, or some function equivalent to the same homogeneous polynomial both at 0 and $+\infty$. In fact this non-degeneracy property does not depend on the fact that $N = 3$, we state it, see Proposition A.4 in the appendix, for general dimension $N \geq 3$. Proposition A.4 substantially improves previous statements of the same kind with decreasing behavior at infinity, see for example in [30, Appendix D] and [4, Lemma 2.4], the result is derived assuming that $\nabla v \in L^2(\mathbb{R}^N)$, respectively that $|v(x)| = o(1)$ as $|x| \to \infty$. However, as pointed out to us by an anonymous reviewer, to whom we are grateful, a kind of analysis suitable for our purposes was already performed by Korevaar, Mazzeo, Pacard and Schoen, see Section 2.2 of [20] and references therein. In their setting they deal with singular decreasing solutions of the linearization of $-\Delta u = u^{N+2}N^{-2}$ about a singular solution, nevertheless the proof contains all ingredients to be applied in our setting. The full needed non-degeneracy statement and a sketch of its proof is postponed to Appendix A.

2.1. Structure of the paper. Section 3 is devoted to the proof of part (iii) of Theorem 2.1. Our starting point is a qualitative result on blow-up sequences from [10], Proposition 3.1, which we refine in two iteration steps.

In Section 4, the precise expansion of $u_\varepsilon$ from Section 3 is used in turn to derive two asymptotic Pohozaev-type identities involving $\rho_a(x_\varepsilon)$ and $\nabla \rho_a(x_\varepsilon)$, respectively. Together with some linear-algebraical arguments on the matrix $M_a(x_\varepsilon)$ and using either the non-degeneracy assumption $D^2 \rho_a(x_0) \geq c$ or the analyticity of $\rho_a$, the combination of these identities yields the asymptotic expression of $u_\varepsilon(x_i,\varepsilon)$ claimed in part (iv) of Theorem 2.1.

In Appendix A, as already mentioned above, we give some details concerning the non-degeneracy property of the limit equation $-\Delta u = u^{N+2}N^{-2}$ under polynomial growth conditions (for general $N = 3$).

Finally, a second appendix contains some explicit computations involving the functions $G_a$ and $W$.

2.2. Notation. Let $f, g : X \to \mathbb{R}_+$ be nonnegative functions defined on some set $X$. We write $f(x) \lesssim g(x)$ if there is a constant $C > 0$ independent of $x$ such that $f(x) \leq C g(x)$ for all $m \in M$, and accordingly for $\gtrsim$. If $f \lesssim g$ and $g \lesssim f$, we write $f \sim g$.

Let $f : X^n \to \mathbb{R}$ be a function of $n$ variables for $X \subset \mathbb{R}^3$. We write $\nabla x_i$ and $\partial^{ri}_k f$ to denote the gradient, respectively the $k$-th partial derivative, of $f$ with the $i$-th variable. When $n = 2$, we also write $\nabla x_1 = \nabla x$, $\partial^{ri}_k f = \partial^x_k$ and $\nabla x_2 = \nabla y$, $\partial^{ri}_k f = \partial^y_k$. 
3. Asymptotic analysis of $u_\varepsilon$

The following proposition follow almost directly from \cite{10} and it has been reproved in this exact frame in \cite{19}, see proposition B.1. It is the starting point of our analysis.

**Proposition 3.1.** Let $(u_\varepsilon)$ be a sequence of solutions to (1.1). Then, up to extracting a subsequence, there exists $n \in \mathbb{N}$ and points $x_{1,\varepsilon}, \ldots, x_{n,\varepsilon}$ such that the following holds.

(i) $x_{i,\varepsilon} \to x_i \in \Omega$ for some $x_i \in \Omega$ with $x_i \neq x_j$ for $i \neq j$.

(ii) $\mu_{i,\varepsilon} := u_\varepsilon(x_{i,\varepsilon})^{-2} \to 0$ as $\varepsilon \to 0$ and $\nabla u_\varepsilon(x_{i,\varepsilon}) = 0$ for every $i$.

(iii) $\lambda_{i,0} := \lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} := \lim_{\varepsilon \to 0} \frac{\mu_{i,\varepsilon}^{1/2}}{\mu_{i,\varepsilon}}$ exists and lies in $(0, \infty)$ for every $i$.

(iv) $\mu_{i,\varepsilon}^{1/2} u_\varepsilon(x_{i,\varepsilon} + \mu_{i,\varepsilon} x) \to B$ in $C^1_{\text{loc}}(\mathbb{R}^n)$.

(v) There is $C > 0$ such that $u_\varepsilon \leq C \sum_i B_{i,\varepsilon}$ on $\Omega$.

Here and in the following, all sums are over $1, \ldots, n$ unless specified otherwise.

Since the $\mu_{i,\varepsilon}$ are all of comparable size by Proposition 3.1, it will be convenient in the following to state error estimates in terms of $\mu_{\varepsilon} := \max_i \mu_{i,\varepsilon} \lesssim \min_i \mu_{i,\varepsilon}$.

Proposition 3.1 says near $x_{i,\varepsilon}$, the function $u_\varepsilon$ is well approximated by $B_{i,\varepsilon}$. Our goal in this section is to extract the next term in the asymptotic development of $u_\varepsilon$ near $x_{i,\varepsilon}$. This term will turn out to involve the function

$$W_{i,\varepsilon}(x) := (a(x_{i,\varepsilon}) + \varepsilon V(x_{i,\varepsilon}))W_{\mu_{i,\varepsilon},x_{i,\varepsilon}}, \quad (3.1)$$

with $W_{\mu,0}$ as in (2.3).

We also define the small ball

$$b_{i,\varepsilon} := B(x_{i,\varepsilon}, \delta_0)$$

around $x_{i,\varepsilon}$, with some number $\delta_0 > 0$ independent of $\varepsilon$ and chosen so small that $\delta_0 < \frac{1}{2} \min_{i \neq j} |x_{i,\varepsilon} - x_{j,\varepsilon}|$ for all $\varepsilon > 0$ small enough.

Here is the main result of this section.

**Theorem 3.2.** Let $u_\varepsilon$ be a sequence of solutions to (1.1) and adopt the notations from Proposition 3.1. For every $i = 1, \ldots, n$, denote

$$r_{i,\varepsilon} := u_{i,\varepsilon} - B_{i,\varepsilon}, \quad q_{i,\varepsilon} := r_{i,\varepsilon} - \mu_{i,\varepsilon}^2 W_{i,\varepsilon} = u_{\varepsilon} - B_{i,\varepsilon} - \mu_{i,\varepsilon}^2 W_{i,\varepsilon}.$$

Then, for every $x \in b_{i,\varepsilon}$, we have the bounds

$$|r_{i,\varepsilon}(x)| \lesssim \mu_{\varepsilon}^{\frac{1}{2} - \vartheta} |x - x_{i,\varepsilon}|^{1 + \vartheta}, \quad \text{for every } 0 < \vartheta < 1, \quad (3.2)$$

$$|q_{i,\varepsilon}(x)| \lesssim \mu_{\varepsilon}^{\frac{1}{2} - \nu} |x - x_{i,\varepsilon}|^{2 + \nu}, \quad \text{for every } 0 < \nu < 1. \quad (3.3)$$
Our proof of Theorem 3.2 is in the spirit of [9] and related works. It consists of two iterative steps carried out in Subsections 3.1 and 3.2 below. The structure of each step is similar: through a well-chosen asymptotic analysis ansatz, the desired bound is ultimately deduced from the non-degeneracy of solutions to some limit equation. This is precisely where Corollary A.2 enters. We emphasize again that to obtain the precision required in (3.3), Corollary A.2 needs to be applied with \( \tau \in (2, 3) \), in which case solutions to the linearized equation (A.1) may in general take a non-standard form like (A.4).

### 3.1. A first quantitative bound

In a first step, we now prove the cruder one of the two bounds stated in Theorem 3.2. Let us for convenience restate the result of this subsection as follows.

**Proposition 3.3.** Let \( i = 1, \ldots, n \). As \( \varepsilon \to 0 \), for every \( 0 < \vartheta < 1 \),

\[
| (u_\varepsilon - B_{i,\varepsilon})(x) | \lesssim \mu_\varepsilon^{\frac{1}{2} - \vartheta} |x - x_{i,\varepsilon}|^{1+\vartheta}, \quad \text{for all } x \in b_{i,\varepsilon}.
\]

**Proof.** We denote \( r_{i,\varepsilon} = u_\varepsilon - B_{i,\varepsilon} \). We fix some \( 0 < \vartheta < 1 \) and denote

\[
R_{i,\varepsilon}(x) := \frac{r_{i,\varepsilon}(x)}{|x - x_{i,\varepsilon}|^{1+\vartheta}}. \tag{3.4}
\]

Fix some \( z_{i,\varepsilon} \in b_{i,\varepsilon} \) such that

\[
R_{i,\varepsilon}(z_{i,\varepsilon}) \geq \frac{1}{2} \| R_{i,\varepsilon} \|_{L^\infty(b_{i,\varepsilon})}. \tag{3.5}
\]

Moreover, we denote \( d_{i,\varepsilon} := |x_{i,\varepsilon} - z_{i,\varepsilon}| \). Let us define the rescaled and normalized version

\[
\tilde{r}_{i,\varepsilon}(x) := \frac{r_{i,\varepsilon}(x + d_{i,\varepsilon}x)}{r_{i,\varepsilon}(z_{i,\varepsilon})}, \quad x \in B(0, d_{i,\varepsilon}^{-1}d_0). \tag{3.6}
\]

Then (3.5) implies

\[
\tilde{r}_{i,\varepsilon}(x) \lesssim |x|^{1+\vartheta}, \quad x \in B(0, d_{i,\varepsilon}^{-1}d_0), \tag{3.7}
\]

in particular \( \tilde{r}_{\varepsilon} \) is uniformly bounded on compacts of \( \mathbb{R}^3 \).

Abbreviating \( a_\varepsilon := a + \varepsilon V, \) we have, on \( B(0, d_{i,\varepsilon}^{-1}d_0) \),

\[
-\Delta \tilde{u}_{i,\varepsilon} + d_{i,\varepsilon}^2 \tilde{u}_{i,\varepsilon} \frac{\tilde{u}_{i,\varepsilon}}{r_{i,\varepsilon}(z_{i,\varepsilon})} = \tilde{r}_{i,\varepsilon} d_{i,\varepsilon}^2 \left( \tilde{u}_{i,\varepsilon}^4 + \tilde{u}_{i,\varepsilon}^3 \tilde{B}_{i,\varepsilon} + \tilde{u}_{i,\varepsilon}^2 \tilde{B}_{i,\varepsilon}^2 + \tilde{u}_{i,\varepsilon} \tilde{B}_{i,\varepsilon}^3 + \tilde{B}_{i,\varepsilon}^4 \right). \tag{3.8}
\]

Here we wrote \( \tilde{u}_{i,\varepsilon}(x) := u_\varepsilon(x_{i,\varepsilon} + d_{i,\varepsilon}x) \) and likewise \( \tilde{a}_{i,\varepsilon}(x) := a_\varepsilon(x_{i,\varepsilon} + d_{i,\varepsilon}x) \) and \( \tilde{B}_{i,\varepsilon}(x) := B_{i,\varepsilon}(x_{i,\varepsilon} + d_{i,\varepsilon}x) = \mu_{i,\varepsilon}^{-1/2} B(\mu_{i,\varepsilon}^{-1}d_{i,\varepsilon}x) \).

We treat three cases separately, depending on the ratio between \( \mu_\varepsilon \) and \( d_{i,\varepsilon} \).

**Case 1.** \( \mu_\varepsilon >> d_{i,\varepsilon} \) as \( \varepsilon \to 0 \). In that case, we have \( \tilde{B}_{i,\varepsilon} \lesssim \mu_{i,\varepsilon}^{-1/2} \) uniformly on \( \mathbb{R}^3 \). Since \( \tilde{u}_{i,\varepsilon} \lesssim \tilde{B}_{i,\varepsilon} \) on \( b_{i,\varepsilon} \), the right side of (3.8) therefore tends to zero uniformly on compacts in that case because \( d_{i,\varepsilon}^2 \mu_{\varepsilon}^{-2} \to 0 \).
Using \( \bar{u}_{i,\varepsilon} \lesssim \bar{B}_{i,\varepsilon} \lesssim \mu_{i,\varepsilon}^{-1/2} \) and \( \frac{1}{d_{i,\varepsilon}(z_{i,\varepsilon})} \lesssim d_{i,\varepsilon}^{-1-\theta/2} \|R_{i,\varepsilon}\|_\infty \) by (3.5), the second summand on the left side of (3.8) is bounded by

\[
\left| d_{i,\varepsilon}^2 \bar{a}_{i,\varepsilon} \frac{\bar{u}_{i,\varepsilon}}{R_{i,\varepsilon}(z_{i,\varepsilon})} \right| \lesssim \frac{d_{i,\varepsilon}^{-\theta} \mu_{i,\varepsilon}^{1/2}}{\|R_{i,\varepsilon}\| L_\infty(b_{i,\varepsilon})} \lesssim \frac{\mu_{i,\varepsilon}^{1/2-\theta}}{\|R_{i,\varepsilon}\| L_\infty(b_{i,\varepsilon})}.
\]

Now suppose that for contradiction that \( \|R_{i,\varepsilon}\| L_\infty(b_{i,\varepsilon}) \gg \mu_{i,\varepsilon}^{1/2-\theta} \) as \( \varepsilon \to 0 \). Then this term goes to zero uniformly. Thus the limit \( \bar{r}_{i,0} := \lim_{\varepsilon \to 0} \bar{r}_{i,\varepsilon} \) satisfies

\[
-\Delta \bar{r}_{i,0} = 0 \quad \text{on } \mathbb{R}^3.
\]

By Liouville’s theorem, the growth bound (3.7) implies that \( \bar{r}_{i,0}(x) = b \cdot x + c \) for some \( b \in \mathbb{R}^3 \), \( c \in \mathbb{R} \). On the other hand, still by (3.7), we find \( b = \nabla \bar{r}_{i,0}(0) = 0 \) and \( c = \bar{r}_{i,0}(0) = 0 \), and hence \( \bar{r}_{i,0} \equiv 0 \). But by the choice of \( d_{i,\varepsilon} \), there is \( \xi_{i,\varepsilon} := \frac{\bar{z}_{i,\varepsilon} - \bar{x}_{i,\varepsilon}}{d_{i,\varepsilon}} \in S^2 \) such that \( \bar{r}_{i,\varepsilon}(\xi_{i,\varepsilon}) = 0 \). Up to a subsequence, \( \xi_{i,0} := \lim_{\varepsilon \to 0} \xi_{i,\varepsilon} \in S^2 \) exists and satisfies \( \bar{r}_{i,0}(\xi_{i,0}) = 1 \). This contradicts \( \bar{r}_{i,0} \equiv 0 \).

Thus we must have \( \|R_{i,\varepsilon}\| L_\infty(b_{i,\varepsilon}) \lesssim \mu_{i,\varepsilon}^{1/2-\theta} \), i.e. \( r_{i,\varepsilon}(x) \lesssim \mu_{i,\varepsilon}^{1/2-\theta} |x - x_{i,\varepsilon}|^{1+\theta} \).

**Case 2.a)** \( \mu_{\varepsilon} << d_{i,\varepsilon} \ll 1 \) as \( \varepsilon \to 0 \). This case works similarly, but we need to argue a little more carefully. This is because the relevant bound \( B_{i,\varepsilon} \lesssim \mu_{\varepsilon}^{1/2} d_{i,\varepsilon}^{-1} |x|^{-1} \lesssim \mu_{\varepsilon}^{1/2} d_{i,\varepsilon}^{-1} \) now only holds on compacts of \( \mathbb{R}^3 \setminus \{0\} \) and no convergence holds at the origin. Nevertheless, we have

\[
\bar{r}_{i,\varepsilon} d_{i,\varepsilon}^2 (\bar{u}_{i,\varepsilon} + ... + \bar{B}_{i,\varepsilon}^4) \lesssim d_{i,\varepsilon}^2 \bar{B}_{i,\varepsilon}^4 \lesssim \mu_{\varepsilon} d_{i,\varepsilon}^{-2} \to 0
\]

and

\[
\left| d_{i,\varepsilon}^2 \bar{a}_{i,\varepsilon} \frac{\bar{u}_{i,\varepsilon}}{R_{i,\varepsilon}(z_{i,\varepsilon})} \right| \lesssim \frac{d_{i,\varepsilon}^{-\theta} \mu_{\varepsilon}^{1/2}}{\|R_{i,\varepsilon}\| L_\infty(b_{i,\varepsilon})} \lesssim \frac{\mu_{\varepsilon}^{1/2-\theta}}{\|R_{i,\varepsilon}\| L_\infty(b_{i,\varepsilon})}
\]

uniformly on compacts of \( \mathbb{R}^3 \setminus \{0\} \). If \( \|R_{i,\varepsilon}\| \gg \mu_{\varepsilon}^{1/2-\theta} \), then, using that still \( d_{i,\varepsilon} \to 0 \), \( \bar{r}_{i,0} := \lim_{\varepsilon \to 0} \bar{r}_{i,\varepsilon} \) satisfies

\[
-\Delta \bar{r}_{i,0} = 0 \quad \text{on } \mathbb{R}^3 \setminus \{0\}.
\]

But by (3.7), \( \bar{r}_{i,0} \) is bounded near \( 0 \) and thus can be extended to a harmonic function on all of \( \mathbb{R}^3 \). A Taylor expansion together with (3.7) now shows that \( \bar{r}_{i,0}(0) = \nabla \bar{r}_{i,0}(0) = 0 \). As in Case 1, we can now derive a contradiction.

Thus we must have \( \|R_{i,\varepsilon}\| L_\infty(b_{i,\varepsilon}) \lesssim \mu_{i,\varepsilon}^{1/2-\theta} \), i.e. \( r_{i,\varepsilon}(x) \lesssim \mu_{i,\varepsilon}^{1/2-\theta} |x - x_{i,\varepsilon}|^{1+\theta} \), also in this case.

**Case 2.b)** \( d_{i,\varepsilon} \sim 1 \) as \( \varepsilon \to 0 \). In this case there is no need for a blow-up argument. Instead, we can simply bound, by the definition of \( z_{i,\varepsilon} \),

\[
\frac{|r_{i,\varepsilon}(x)|}{|x - x_{i,\varepsilon}|^{1+\theta}} \lesssim \frac{|r_{i,\varepsilon}(z_{i,\varepsilon})|}{d_{i,\varepsilon}^{1+\theta}} \lesssim \frac{|r_{i,\varepsilon}(z_{i,\varepsilon})|}{\mu_{i,\varepsilon}^{1/2}}.
\]
where the last inequality simply comes from the bound $|u_\varepsilon| \lesssim B_{i,\varepsilon}$ on $b_{i,\varepsilon}$ and the observation that $d_{i,\varepsilon} \sim 1$ implies $B_{i,\varepsilon}(z_{i,\varepsilon}) \lesssim \mu_{i,\varepsilon}^{1/2}$. Thus

$$|r_{i,\varepsilon}(x)| \lesssim \mu_{i,\varepsilon}^{1/2} |x - x_{i,\varepsilon}|^{1+\theta} \leq \mu_{i,\varepsilon}^{1/2-\theta} |x - x_{i,\varepsilon}|^{1+\theta},$$

which completes the discussion of this case.

**Case 3.** $\mu_{i,\varepsilon} \sim d_{i,\varepsilon}$ as $\varepsilon \to 0$. This is the most delicate case because the right side of (3.8) now tends to a non-trivial limit. Indeed, $\beta_{i,0} := \lim_{\varepsilon \to 0} \beta_{i,\varepsilon} := \lim_{\varepsilon \to 0} \frac{\mu_{i,\varepsilon}}{d_{i,\varepsilon}}$ exists and $\beta_{i,0} \in (0, \infty)$. Then

$$d_{i,\varepsilon}^{1/2} \bar{B}_{i,\varepsilon} = \frac{\beta_{i,\varepsilon}^{1/2}}{(\beta_{i,\varepsilon}^2 + \frac{|x|^2}{3})^{1/2}} \to \frac{\beta_{i,0}^{1/2}}{(\beta_{i,0}^2 + \frac{|x|^2}{3})^{1/2}} =: B_{0,\beta_{i,0}}.$$

By the convergence of $u_{\varepsilon}$ from Proposition 3.1, we also have $d_{i,\varepsilon}^{1/2} \bar{u}_{i,\varepsilon} \to B_{0,\beta_{i,0}}$ uniformly on compacts of $\mathbb{R}^3$. Moreover

$$\left| \frac{d_{i,\varepsilon} \bar{a}_{i,\varepsilon}}{r_{i,\varepsilon}(z_{i,\varepsilon})} \right| \lesssim \frac{d_{i,\varepsilon}^{1/2-\theta}}{\|R_{i,\varepsilon}\|_{L^\infty(b_{i,\varepsilon})}} \lesssim \frac{\mu_{i,\varepsilon}^{1/2-\theta}}{\|R_{i,\varepsilon}\|_{L^\infty(b_{i,\varepsilon})}}.$$ 

If $\|R_{i,\varepsilon}\|_{L^\infty(b_{i,\varepsilon})} \gg \mu_{i,\varepsilon}^{1/2-\theta}$, we therefore recover the limit equation

$$-\Delta \bar{r}_{i,0} = 5 \bar{r}_{i,0} B_{0,\beta_{i,0}}^4 \quad \text{on } \mathbb{R}^3,$$

which is precisely the linearized equation (A.1), up to a harmless rescaling. By (3.7), we have $|r_{i,0}(x)| \lesssim |x|^{1+\theta}$ for all $x \in \mathbb{R}^3$. Thus by Corollary A.2 we conclude $\bar{r}_{i,0} \equiv 0$. This contradicts $\bar{r}_{i,0}(z_{i,\varepsilon}) = 1$, as desired.

Thus we have shown $\|R_{i,\varepsilon}\|_{L^\infty(b_{i,\varepsilon})} \lesssim \mu_{i,\varepsilon}^{1/2-\theta}$, i.e. $r_{i,\varepsilon}(x) \lesssim \mu_{i,\varepsilon}^{1/2-\theta} |x - x_{i,\varepsilon}|^{1+\theta}$, also in the third and final case. \hfill $\square$

### 3.2. A refined expansion.

With Proposition 3.3 at hand, we now complete the proof of Theorem 3.2 by proving the bound (3.3) on

$$q_{i,\varepsilon} = u_{i,\varepsilon} - B_{i,\varepsilon} - \mu_{i,\varepsilon}^2 W_{i,\varepsilon},$$

where $W_{i,\varepsilon}$ is defined in (3.1). Again, we restate the bound here for convenience.

**Proposition 3.4.** As $\varepsilon \to 0$, for all $0 < \nu < 1$,

$$|q_{i,\varepsilon}(x)| \lesssim \mu_{i,\varepsilon}^{\frac{1}{2} - \nu} |x - x_{i,\varepsilon}|^{2+\nu}, \quad \text{for all } x \in b_{i,\varepsilon}.$$

**Proof.** Abbreviating $a_{i,\varepsilon} = a + \varepsilon V$, it is easily checked that $q_{i,\varepsilon}$ satisfies the equation

$$-\Delta q_{i,\varepsilon} + a_{i,\varepsilon} r_{i,\varepsilon} + (a_{i,\varepsilon} - a_{i,\varepsilon}(x_{i,\varepsilon})) B_{i,\varepsilon} = 5 B_{i,\varepsilon}^4 q_{i,\varepsilon} + O(\mu_{i,\varepsilon}^2 B_{i,\varepsilon}^3).$$

(3.9)
We now insert the bounds $|a_\varepsilon| \lesssim 1$, and $|a_\varepsilon(x) - a_\varepsilon(x_{i,\varepsilon})| \lesssim |x - x_{i,\varepsilon}|$ because $a, V \in C^1(\Omega)$. Together with the previously proved bounds $u_\varepsilon \lesssim B_{i,\varepsilon}$ on $b_{i,\varepsilon}$ from Proposition 3.1 and $r_{i,\varepsilon} \leq \mu_\varepsilon^{1/2-\nu} |x - x_{i,\varepsilon}|^{1+\nu}$ on $b_{i,\varepsilon}$ from Proposition 3.3, we find that on $b_{i,\varepsilon}$,

$$\left| -\Delta q_{i,\varepsilon} - 5q_{i,\varepsilon}B_{4,i,\varepsilon}^4 \right| \lesssim |x - x_{i,\varepsilon}|B_{i,\varepsilon} + \mu_\varepsilon^{1/2-\nu} |x - x_{i,\varepsilon}|^{1+\nu} + \mu_\varepsilon^{1-2\nu} |x - x_{i,\varepsilon}|^{1+\nu} + \mu_\varepsilon^{1-2\nu} |x - x_{i,\varepsilon}|^{2+2\nu}B_{4,i,\varepsilon}^3 \quad (3.10)$$

Now for some $0 < \nu < 1$, let

$$Q_{i,\varepsilon}(x) := \frac{q_{i,\varepsilon}(x)}{|x - x_{i,\varepsilon}|^{2+\nu}}$$

and denote by $z_{i,\varepsilon} \in b_{i,\varepsilon}$ a point where $Q_{i,\varepsilon}(z_{i,\varepsilon}) \geq \frac{1}{2}\|Q_{i,\varepsilon}\|_{L^\infty((b_{i,\varepsilon})}$ and set $d_{i,\varepsilon} := |z_{i,\varepsilon} - x_{i,\varepsilon}|$. We introduce the function

$$\tilde{q}_{i,\varepsilon}(x) := \frac{q_{i,\varepsilon}(x_i,\varepsilon + d_{i,\varepsilon}x)}{q_{i,\varepsilon}(z_{i,\varepsilon})}, \quad x \in B(0, d_{i,\varepsilon}^{-1} \delta_0),$$

which satisfies

$$\tilde{q}_{i,\varepsilon}(x) \lesssim |x|^{2+\nu}. \quad (3.11)$$

Moreover, multiplying (3.10) by $\frac{d_{i,\varepsilon}^2}{q_{i,\varepsilon}(z_{i,\varepsilon})}$ and observing that $\frac{1}{d_{i,\varepsilon}(z_{i,\varepsilon})} \leq \frac{\mu_\varepsilon^{d_{i,\varepsilon}^2}}{\|Q_{i,\varepsilon}\|_{L^\infty((b_{i,\varepsilon})}}$, we get

$$\left| -\Delta \tilde{q}_{i,\varepsilon} - 5d_{i,\varepsilon}^2 \tilde{q}_{i,\varepsilon}B_{4,i,\varepsilon}^4 \right| \lesssim \frac{1}{\|Q_{i,\varepsilon}\|_{L^\infty((b_{i,\varepsilon})}} (d_{i,\varepsilon}^{1-\nu} \tilde{B}_{i,\varepsilon} + \mu_\varepsilon^{1/2-\nu} d_{i,\varepsilon} + \mu_\varepsilon^{1-2\nu} d_{i,\varepsilon}^{2+\nu} \tilde{B}_{3,i,\varepsilon}) \quad (3.12)$$

locally on $\mathbb{R}^3$. Now we again distinguish three cases. Since the argument is analogous to that of Proposition 3.3, we shall be a bit briefer here.

**Cases 1 and 2.a)** $\mu_\varepsilon >> d_{i,\varepsilon}$ or $\mu_\varepsilon << d_{i,\varepsilon} << 1$ as $\varepsilon \to 0$. In these cases $B_{i,\varepsilon} \lesssim \mu_\varepsilon^{-1/2}$ and $\tilde{B}_{i,\varepsilon} \lesssim d_{i,\varepsilon}^{-1} \mu_\varepsilon^{1/2}$ respectively. In both cases one finds again that $d_{i,\varepsilon}^2 \tilde{B}_{4,i,\varepsilon} \to 0$. Moreover, using $0 < \nu < 1$ the right side of (3.12) can be bounded by

$$\frac{\mu_\varepsilon^{\frac{1}{2} - \nu}}{\|Q_{i,\varepsilon}\|_{L^\infty((b_{i,\varepsilon})}}.$$

If $\|Q_{i,\varepsilon}\|_{L^\infty((b_{i,\varepsilon})} >> \mu_\varepsilon^{\frac{1}{2} - \nu}$, then the limit $\tilde{q}_{i,0}$ satisfies

$$-\Delta \tilde{q}_{i,0} = 0 \quad \text{on } \mathbb{R}^3, \quad \tilde{q}_{i,0}(x) \lesssim |x|^{2+\nu} \quad \text{on } \mathbb{R}^3.$$ 

This bound implies on the one hand

$$\tilde{q}_{i,0}(0) = 0, \quad \nabla \tilde{q}_{i,0}(0) = 0, \quad D^2\tilde{q}_{i,0}(0) = 0,$$

and on the other hand

$$\tilde{q}_{i,0}(x) = a + b \cdot x + \langle x, Cx \rangle$$

for some $a \in \mathbb{R}, b \in \mathbb{R}^3, C \in \mathbb{R}^{3\times 3}$, where we can assume $C$ symmetric. It is easy to see that in fact $a = \tilde{q}_{i,0}(0) = 0$, $b = \nabla \tilde{q}_{i,0}(0) = 0$ and $C = \frac{1}{2} D^2 \tilde{q}_{i,0}(0) = 0$. Hence $\tilde{q}_{i,0} \equiv 0$. As before, this contradicts the fact that $\tilde{q}_{i,0}(\xi_{i,0}) = 1$ for some $\xi_{i,0} \in S^2$. Thus

$$\|Q_{i,\varepsilon}\|_{L^\infty((b_{i,\varepsilon})} \lesssim \mu_\varepsilon^{\frac{1}{2} - \nu}, \quad \text{which implies the proposition in Cases 1 and 2.a)}.$$

**Case 2.b)** $d_{i,\varepsilon} \sim 1$ as $\varepsilon \to 0$. 

As above, the definition of $z_{i,\varepsilon}$ gives, on $b_{i,\varepsilon}$,
\[
\frac{|q_{i,\varepsilon}(x)|}{|x - x_{i,\varepsilon}|^{2+\theta}} \lesssim \frac{|q_{i,\varepsilon}(z_{i,\varepsilon})|}{\delta_{i,\varepsilon}^{2+\theta}} \lesssim |q_{i,\varepsilon}(z_{i,\varepsilon})| \lesssim \mu_{i,\varepsilon}^{1/2}.
\]
The last inequality comes from the bound $|u_{\varepsilon}| \lesssim B_{i,\varepsilon}$ on $b_{i,\varepsilon}$ and the observation that $d_{i,\varepsilon} \sim 1$ implies $B_{i,\varepsilon}(z_{i,\varepsilon}) \lesssim \mu_{i,\varepsilon}^{1/2}$. This gives the claimed bound in this case.

**Case 3.** $\mu_{i,\varepsilon} \sim d_{i,\varepsilon}$ as $\varepsilon \to 0$. Let $\beta_{i,0} = \lim_{\varepsilon \to 0} \beta_{i,\varepsilon} = \lim_{\varepsilon \to 0} \frac{\mu_{i,\varepsilon}}{d_{i,\varepsilon}}$, then $d_{i,\varepsilon}B_{i,\varepsilon}^4 \to \frac{\beta_{i,0}^2}{(\beta_{i,0}^2 + |x|^2/3)^2} = B_{0,\beta_{i,0}}^4$. If $\|Q_{i,\varepsilon}\|_{L^\infty(b_{i,\varepsilon})} \gg \mu_{i,\varepsilon}^{3/2-\nu}$, then by the same estimates on (3.12) as in Case 1, the right side of (3.12) tends to zero and the limit function $\bar{q}_{i,0}$ satisfies
\[
-\Delta \bar{q}_{i,0} = 5\bar{q}_{i,0}B_{0,\beta_{i,0}}^4 \quad \text{on } \mathbb{R}^3, \quad |\bar{q}_{i,0}(x)| \lesssim |x|^{2+\nu} \quad \text{on } \mathbb{R}^3.
\]
Now we can invoke Corollary A.2 to obtain $\bar{q}_{i,0}(0) = 0$, which contradicts $\bar{q}_{i,0}(\xi_{i,0}) = 1$. Thus $\|Q_{i,\varepsilon}\|_{L^\infty(b_{i,\varepsilon})} \lesssim \mu_{i,\varepsilon}^{3/2-\nu}$ also in this case, and the proof of the proposition is complete. □

4. **Proof of Theorem 2.1**

4.1. **The main expansions.** By applying the expansion of $u_{\varepsilon}$ near the concentration points derived in Theorem 3.2, we can prove the following expansions.

We will also need the matrix $\tilde{M}_{i}(x) \in \mathbb{R}^{n \times n} = (\tilde{m}_{ij}(x))_{i,j=1}$ with entries
\[
\tilde{m}_{ij}(x) := \begin{cases}
\partial_i \phi_a(x_i) & \text{for } i = j, \\
-2\partial_i^x G_a(x_i, x_j) & \text{for } i \neq j.
\end{cases} \quad (4.1)
\]

Finally, recall that we have defined in Proposition 3.1 $\lambda_{i,0} = \lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = \lim_{\varepsilon \to 0} \frac{\mu_{i,\varepsilon}^{1/2}}{\mu_{i,\varepsilon}}$.

Now define $\tilde{G} := 4\pi \sqrt{3} \sum_i \lambda_{i,0} G_a(x, x_i)$ and denote
\[
Q_V(y) := \int_{\Omega} V(x)\tilde{G}(x)G_a(x, y) \, dx.
\]

Then the following expansions hold.

**Proposition 4.1.** Let $x_{\varepsilon} = (x_{1,\varepsilon}, \ldots, x_{n,\varepsilon}) \in \mathbb{R}^{3n}$ and $\lambda_{\varepsilon} = (\lambda_{1,\varepsilon}, \ldots, \lambda_{n,\varepsilon})$ be as in Proposition 3.1. As $\varepsilon \to 0$, we have
\[
\varepsilon(Q_V(x_{i,\varepsilon}) + o(1)) = -4\pi \sqrt{3}(M_a(x_{\varepsilon}) \cdot \lambda_{\varepsilon})_i + 3\pi(a(x_{i,\varepsilon}) + o(1))\lambda_{i,\varepsilon}\mu_{i,\varepsilon} \quad (4.2)
\]
and, for every $0 < \nu < 1$,
\[
(\tilde{M}_{i}(x_{\varepsilon}) \cdot \lambda_{\varepsilon})_i = \mathcal{O}(\varepsilon + \mu_{\varepsilon}^\nu), \quad (4.3)
\]

Before giving the proof of Proposition 4.1, we observe the following property of the function $\tilde{G}$ defined at the beginning of this section.
Lemma 4.2. As $\varepsilon \to 0$, we have $\mu_{1,\varepsilon}^{-1/2} u_{\varepsilon} \to \bar{G}$ uniformly away from $\{x_{1,0}, \ldots, x_{n,0}\}$.

Remark 4.3. Note that $\bar{G}$ is defined in terms of the $\lambda_{i,0}$ from Proposition 3.1, while the function $G$ appearing in Theorem 2.1 is defined in terms of the eigenvector $\Lambda_0$. We shall however prove in Lemma 4.5 below that $\lambda_0 = \Lambda_0$ and hence in fact $\bar{G} = G$.

Proof of Lemma 4.2. By applying $(-\Delta + a)^{-1}$ to the equation satisfied by $u_{\varepsilon}$, we obtain

$$u_{\varepsilon}(x) = \int_{\Omega} G_a(x, y)(u_{\varepsilon}^5(y) + \varepsilon V(y)u(y)) \, dy$$

for every $x \in \Omega$. By developing $u_{\varepsilon} = B_{i,\varepsilon} + r_{i,\varepsilon}$ near $x_{i,\varepsilon}$ and using $|r_{i,\varepsilon}(x)| \lesssim \mu_{1/2} |x - x_{i,\varepsilon}|^{1+\theta}$ for every $\theta \in (0,1)$ by Theorem 3.2, we get

$$\int_{\Omega} G_a(x, y) u_{\varepsilon}^5(y) \, dy = \sum_i \left( \int_{B_{i,\varepsilon}} G_a(x, y) u_{\varepsilon}^5(y) \, dy \right) + \int_{\Omega \setminus B_{i,\varepsilon}} G_a(x, y) u_{\varepsilon}^5(y) \, dy$$

$$= \sum_i \mu_{i,\varepsilon}^{1/2} \int_{\mathbb{R}^3} B^5 \, dy G_a(x, x_{i,\varepsilon}) + o(\mu_{\varepsilon}^{1/2})$$

uniformly for $x$ in compacts of $\Omega \setminus \{x_{1,0}, \ldots, x_{n,0}\}$. Moreover, the bound $u_{\varepsilon} \lesssim B_{i,\varepsilon}$ near $x_{i,\varepsilon}$ from Proposition 4.1 easily gives

$$\varepsilon \int_{\Omega} G_a(x, y)V(y)u(y) \, dy = O(\varepsilon \mu_{\varepsilon}^{1/2}) = o(\mu_{\varepsilon}^{1/2})$$

uniformly for $x \in \Omega$.

Since $\int_{\mathbb{R}^3} B^5 \, dy = 4\pi \sqrt{3}$, combining all of the above, dividing by $\mu_{1,\varepsilon}^{1/2}$ and recalling the definition of $\lambda_{i,0}$ gives the conclusion. \qed

Proof of Proposition 4.1. Proof of (4.2). Integrate equation (1.1) for $u_{\varepsilon}$ against $G_a(x_{i,\varepsilon}, \cdot)$ to get

$$\int_{\Omega} (-\Delta + a) u_{\varepsilon} G_a(x, x_{i,\varepsilon}) \, dx + \varepsilon \int_{\Omega} V u_{\varepsilon} G_a(x, x_{i,\varepsilon}, \cdot) \, dx = \int_{\Omega} u_{\varepsilon}^5 G_a(x_{i,\varepsilon}, \cdot) \, dx \quad (4.4)$$

Then by the definition of $G_a(x_{i,\varepsilon}, \cdot)$ and by the convergence $\mu_{1,\varepsilon}^{-1/2} u_{\varepsilon} \to \bar{G}$ from Lemma 4.2, the left side of (4.4) equals

$$u_{\varepsilon}(x_{i,\varepsilon}) + \varepsilon \mu_{1/2}^{-1} \lambda_{i,\varepsilon}^{-1} (Q_V(x_{i,\varepsilon}) + o(1)) = \mu_{i,\varepsilon}^{-1/2} + \varepsilon \mu_{1/2}^{-1} \lambda_{i,\varepsilon}^{-1} (Q_V(x_{i,\varepsilon}) + o(1)).$$

Evaluating the right side of (4.4) requires some more care. We start by writing

$$\int_{\Omega} u_{\varepsilon}^5 G_a(x, x_{i,\varepsilon}, \cdot) \, dx = \sum_j \int_{B_{i,\varepsilon}} u_{\varepsilon}^5 G_a(x, x_{i,\varepsilon}, \cdot) \, dx + \sum_{\Omega \setminus B_{i,\varepsilon}} u_{\varepsilon}^5 G_a(x, x_{i,\varepsilon}, \cdot) \, dx.$$

Clearly, on $\Omega \setminus \bigcup B_{i,\varepsilon}$ we have $u_{\varepsilon} \lesssim \sum B_{i,\varepsilon} \lesssim \mu_{\varepsilon}^{1/2}$, and so

$$\int_{\mathbb{R}^3} u_{\varepsilon}^5 G_a(x, x_{i,\varepsilon}, \cdot) \, dx \lesssim \mu_{\varepsilon}^{5/2} = o(\mu_{\varepsilon}^{3/2}).$$
It therefore remains to evaluate the integral over the balls $b_{j,\varepsilon}$, up to $o(\mu_{\varepsilon}^{3/2} + \varepsilon \mu_{\varepsilon}^{1/2})$ precision. We shall consider the cases $j = i$ and $j \neq i$ separately.

Case $j = i$.

Careful, but straightforward computations and estimates give
\[
\int_{b_{i,\varepsilon}} B_{i,\varepsilon}^5 G_a(x_{i,\varepsilon}, \cdot) \, dx = \mu_{i,\varepsilon}^{-1/2} - 4\pi \sqrt{3} \phi_a(x_{i,\varepsilon}) \mu_{i,\varepsilon}^{1/2} + 3a(x_{i,\varepsilon}) \mu_{i,\varepsilon}^{3/2} + o(\mu_{\varepsilon}^3),
\]
see Lemma B.1 below.

To evaluate the error term on the right side, we need the full precision of the asymptotic expansion of $u_{\varepsilon}$ derived in Theorem 3.2. By that theorem, on $b_{i,\varepsilon}$ we may write
\[
u_{i,\varepsilon} = 5\mu_{i,\varepsilon}^2 W_{i,\varepsilon} B_{i,\varepsilon}^4 + o(|q_{i,\varepsilon}| B_{i,\varepsilon}^4 + r_{i,\varepsilon} B_{i,\varepsilon}^3),
\]
with the remainders $r_{i,\varepsilon}$ and $q_{i,\varepsilon}$ satisfying the bounds $|r_{i,\varepsilon}| \lesssim \mu_{\varepsilon}^{-\vartheta} |x - x_{i,\varepsilon}|^{1+\vartheta}$ and $|q_{i,\varepsilon}| \lesssim \mu_{\varepsilon}^{-4-\nu} |x - x_{i,\varepsilon}|^{2+2\nu}$ on $b_{i,\varepsilon}$ respectively (with $0 < \nu, \vartheta < 1$). Thus we get
\[
\int_{b_{i,\varepsilon}} |r_{i,\varepsilon}| H_a(x_{i,\varepsilon}, \cdot) + \frac{q_{i,\varepsilon}}{|x - x_{i,\varepsilon}|} \, B_{i,\varepsilon}^4 \, dx + \int_{b_{i,\varepsilon}} |u_{\varepsilon} - B_{i,\varepsilon}|^2 B_{i,\varepsilon}^3 G_a(x_{i,\varepsilon}, \cdot) \, dx 
\lesssim \mu_{\varepsilon}^{-\vartheta/2} + \mu_{\varepsilon}^{-4-\nu} + \mu_{\varepsilon}^{-2\vartheta} = o(\mu_{\varepsilon}^{3/2}).
\]

Thus
\[
\int_{b_{i,\varepsilon}} (\nu_{i,\varepsilon} - B_{i,\varepsilon}^5) G_a(x_{i,\varepsilon}, \cdot) \, dx = \frac{5}{4\pi} \mu_{i,\varepsilon}^2 \int_{b_{i,\varepsilon}} W_{i,\varepsilon} |x - x_{i,\varepsilon}|^{-1} B_{i,\varepsilon}^4 \, dx + o(\mu_{\varepsilon}^{3/2})
\]
\[
= \mu_{i,\varepsilon}^{3/2} \frac{5}{4\pi} (a(x_{i,\varepsilon}) + \varepsilon V(x_{i,\varepsilon})) \int_{\mathbb{R}^3} \frac{W(x) B(x)^4}{|x|} \, dx + o(\mu_{\varepsilon}^{3/2})
\]
\[
= 3a(x_{i,\varepsilon})(\pi - 1) \mu_{i,\varepsilon}^{3/2} + o(\mu_{\varepsilon}^{3/2}).
\]

where the final identity is computed in Lemma B.4. Collecting all expansions, we conclude the proof of (4.2).

Case $j \neq i$. The expansion of the cross terms with $j \neq i$ is simpler because $G_a(x_{i,\varepsilon}, \cdot)$ is bounded on $b_{j,\varepsilon}$ in that case, and we give a bit fewer details. Using the bound (3.2) on $r_{i,\varepsilon}$ from Theorem 3.2 we write
\[
\int_{b_{j,\varepsilon}} u_{\varepsilon}^5 G_a(x_{i,\varepsilon}, \cdot) \, dx = \int_{b_{j,\varepsilon}} B_{j,\varepsilon}^5 G_a(x_{i,\varepsilon}, \cdot) \, dx + O\left(\mu_{\varepsilon}^{1/2-\vartheta} \int_{b_{j,\varepsilon}} B_{j,\varepsilon}^4 |x - x_{j,\varepsilon}|^{1+\vartheta} \, dx\right)
\]
\[
= 4\pi \sqrt{3} \mu_{j,\varepsilon}^{1/2} G_a(x_{i,\varepsilon}, x_{j,\varepsilon}) + o(\mu_{\varepsilon}^{3/2}).
\]

In summary, inserting everything into (4.4), we have proved
\[
\mu_{i,\varepsilon}^{-1/2} + \varepsilon \mu_{i,\varepsilon}^{1/2} \lambda_{i,\varepsilon}^{-1}(Q_V(x_{i,\varepsilon}) + o(1)) = 4\pi \sqrt{3} (-\phi_a(x_{i,\varepsilon}) + G_a(x_{i,\varepsilon}, x_{j,\varepsilon}) + 3a(x_{i,\varepsilon}) + o(1)) \mu_{i,\varepsilon}^{3/2}.
\]

Dividing by $\mu_{i,\varepsilon}^{1/2}$ and recalling the definitions of $\lambda_{i,\varepsilon}$ and $M_a(x_{\varepsilon})$, we obtain (4.2).

Proof of (4.3).
We proceed similarly to the proof of expansion (4.2) and multiply equation (1.1) for \( u_\varepsilon \) against \( \nabla x G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \).

Then
\[
\int_\Omega (-\Delta + a + \varepsilon V) u_\varepsilon \nabla x G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \, dy = \nabla x \int_\Omega (-\Delta + a + \varepsilon V) u_\varepsilon G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \, dy = \nabla u_\varepsilon(x_{i,\varepsilon}) = 0.
\]

The right side of equation (1.1) integrated against \( \nabla x G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \) can be written as
\[
\int_\Omega u_\varepsilon^5 \nabla x G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \, dx = \sum_j \left( \int_{b_{j,\varepsilon}} u_\varepsilon^5 \nabla x G_{a}(x_{i,\varepsilon}, \cdot) \, dx \right) + \int_{\Omega \setminus \bigcup_j b_{j,\varepsilon}} u_\varepsilon^5 \nabla x G_{a}(x_{i,\varepsilon}, \cdot) \, dx.
\]

The last term on the right side, since \( u_\varepsilon \lesssim \sum_k B_{k,\varepsilon} \lesssim \mu_\varepsilon^{1/2} \) on \( \Omega \setminus \bigcup_j b_{j,\varepsilon} \), is bounded by
\[
\int_{\Omega \setminus \bigcup_j b_{j,\varepsilon}} u_\varepsilon^5 \nabla x G_{a}(x_{i,\varepsilon}, \cdot) \, dx \lesssim \mu_\varepsilon^{5/2}.
\]

It remains to evaluate the integral over the balls \( b_{j,\varepsilon} \). We again consider the cases \( j = i \) and \( j \neq i \) separately.

**Case** \( j = i \). We write
\[
\int_{b_{i,\varepsilon}} u_\varepsilon^5 \nabla x G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \, dy = \int_{b_{i,\varepsilon}} B_{i,\varepsilon}^5 \nabla x G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \, dx + 5 \int_{b_{i,\varepsilon}} r_{i,\varepsilon} B_{i,\varepsilon}^4 \nabla x G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \, dx \\
+ O \left( \int_{b_{i,\varepsilon}} r_{i,\varepsilon}^2 \frac{B_{i,\varepsilon}^3}{|x - x_{i,\varepsilon}|^2} \, dx \right).
\]

Let us treat the terms on the right side one by one. The first term, by explicit computations carried out in Lemma B.2, is
\[
\int_{b_{i,\varepsilon}} B_{i,\varepsilon}^5 \nabla x G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \, dy = -2\pi \sqrt{3} \nabla \phi_{a+\varepsilon V}(x_{i,\varepsilon}) \mu_{i,\varepsilon}^{1/2} + o(\varepsilon^{1/2} + \nu) \\
= -2\pi \sqrt{3} \nabla \phi_{a}(x_{i,\varepsilon}) \mu_{i,\varepsilon}^{1/2} + O(\varepsilon^{1/2} + \nu)
\]
for every \( 0 < \nu < 1 \). The last equality comes from the fact that by the resolvent formula,
\[
\nabla \phi_{a+\varepsilon V}(x_{i,\varepsilon}) = \nabla \phi_{a}(x_{i,\varepsilon}) + \varepsilon \nabla_x \int_\Omega G_{a}(x, y)V(y)G_{a+\varepsilon V}(x, y) \, dy \bigg|_{x=x_{i,\varepsilon}} = \nabla \phi_{a}(x_{i,\varepsilon}) + o(\varepsilon),
\]

3 The choice of multiplying against \( G_{a+\varepsilon V} \) rather than \( G_{a} \) is made on technical grounds. Indeed, by doing so, one does not need to bound \( \int \varepsilon V \nabla G_{a} u_\varepsilon \lesssim \varepsilon \mu_\varepsilon^{1/2} \) on the left side, but instead needs to bound \( \nabla \phi_{a+\varepsilon V}(x_{i,\varepsilon}) - \nabla \phi_{a}(x_{i,\varepsilon}) \) on the right side. The latter can be better handled with the bounds we have so far. In fact, plugging in the (slightly non-optimal) bounds on \( r_{i,\varepsilon} \) (or \( q_{i,\varepsilon} \)) into the first term leads to an estimate of the type \( \int \varepsilon V \nabla G_{a} u_\varepsilon \lesssim \varepsilon \mu_\varepsilon^{1/2} \lesssim (\varepsilon^{1/2} + \nu) \mu_\varepsilon^{1/2} \). To get the desired bound, we would need \( \varepsilon \lesssim \mu_\varepsilon^{\nu} \), which is unclear at this stage. For the bound \( \nabla \phi_{a+\varepsilon V}(x_{i,\varepsilon}) - \nabla \phi_{a}(x_{i,\varepsilon}) \), proved in Lemma B.3, a similar issue does not arise.
see Lemma B.3 for details.

To estimate the last term on the right side of (4.6), the bound (3.2) on $r_{i,\varepsilon}$ from Theorem 3.2 gives

$$\int_{b_{i,\varepsilon}} r_{i,\varepsilon}^2 \frac{B_i^3}{|x - x_{i,\varepsilon}|^2} \, dx \lesssim \mu_{\varepsilon}^{1 - 2\vartheta} \int_{\Omega} B_i^3 |x - x_{i,\varepsilon}|^{2\vartheta} \, dx \lesssim \mu_{\varepsilon}^{\frac{7}{2} - 2\vartheta}.$$  

Finally, let us show that the second term on the right side of (4.6) is negligible. Here is where we use the full strength of Theorem 3.2, i.e. the bound (3.3) on $\vartheta = 4\pi - \varepsilon$ is where we use the full strength of Theorem 3.2, i.e. the bound (3.3) on $\vartheta = 4\pi - \varepsilon$, once more after using it to get (4.5) in the proof of (4.2). We write

$$\text{Case } j \neq i.$$  

Again, this case is less involved and we will be briefer. We have, using the bound on $|u_{\varepsilon} - B_{j,\varepsilon}|$ from Proposition 3.3,

$$\int_{b_{j,\varepsilon}} u_{\varepsilon}^5 \nabla_x G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \, dx$$  

\begin{align*}
&= \int_{b_{j,\varepsilon}} B_{j,\varepsilon}^5 \nabla_x G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \, dx + O \left( \mu_{\varepsilon}^{\frac{1}{4} - \vartheta} \int_{b_{j,\varepsilon}} B_{j,\varepsilon}^4 |x - x_{j,\varepsilon}|^{1 + \vartheta} \nabla_x G_{a+\varepsilon V}(x_{i,\varepsilon}, \cdot) \, dx \right) \\
&= 4\pi \sqrt{\mu_{\varepsilon}^{1/2}} \nabla_x G_{a}(x_{i,\varepsilon}, x_{j,\varepsilon}) + o(\mu_{\varepsilon}^{3/2}).
\end{align*}  

Now collecting all the estimates, choosing $\vartheta > 0$ small enough and dividing by $\mu_{1,\varepsilon}$ gives (4.3) \[\square\]

Remark 4.4. Arguing as in Section 3, one can deduce, for all $0 < \nu < 1$, the bound $|q_{i,\varepsilon}(x)| \lesssim \mu_{\varepsilon}^{1/2} |x - x_{i,\varepsilon}|^{1 + \nu}$ on $b_{i,\varepsilon}$, which is weaker than (3.3) near $x_{i,\varepsilon}$. This bound
can however be checked to be just enough to appropriately bound the terms in \( q_{i,\varepsilon} \) in the previous proof of Proposition 4.1.

4.2. Properties of the matrix \( M_a(x) \) and the eigenvector \( \rho_a(x) \). For every \( x \in \Omega_n^* \), recall that we denote by \( \rho_a(x) \) the lowest eigenvalue of the matrix \( M_a(x) \).

Moreover, denote by \( x_0 = (x_{1,0}, \ldots, x_{n,0}) \) and \( \lambda_0 \) the limit points of \( x_\varepsilon \) and \( \lambda_\varepsilon \) respectively. (The vector \( \lambda_\varepsilon \) is defined in Proposition 3.1.)

**Lemma 4.5.** (i) For every \( x \in \Omega_n^* \), \( \rho_a(x) \) is a simple eigenvalue. The associated eigenvector can be chosen so that all of its entries are strictly positive. All other eigenvectors of \( M_a(x) \) have both strictly negative and strictly positive entries.

(ii) For \( x = x_0 \), we have \( \rho_a(x_0) = 0 \) with eigenvector \( \Lambda(x_0) = \lambda_0 \). Moreover, \( \nabla \rho_a(x_0) = 0 \).

In the following, for \( \varepsilon \geq 0 \) let us abbreviate \( \Lambda_\varepsilon := \Lambda(x_\varepsilon) \).

**Proof.** Assertion (i) follows by the Perron–Frobenius argument detailed for the case \( a = 0 \) in [1, Appendix A]. This argument still applies because it only relies on the strict negativity of all off-diagonal entries \( -G_a(x_i, x_j) \), which is fulfilled in our case.

For assertion (ii), expansion (4.2) plainly gives \( M_a(x_0) \cdot \lambda_0 = 0 \) by passing to the limit, hence \( \lambda_0 \) is an eigenvector with eigenvalue 0. Since \( \lambda_0 \) has strictly positive entries, it must be the lowest one by part (i) of the lemma.

It remains to prove that \( \nabla \rho_a(x_0) = 0 \). To see this, note on the one hand that a direct calculation gives

\[
\partial_l^ε(\Lambda_ε, M_a(x) \cdot \lambda_ε)|_{x=x_ε} = \lambda_{i,ε}(M_a^{ε}(x_ε) \cdot \lambda_ε)|_i
\]

for every \( l, 1, 2, 3, i = 1, \ldots, n \) and \( ε > 0 \). Hence by expansion (4.3)

\[
\partial_l^ε(\Lambda_ε, M_a(x) \cdot \lambda_ε)|_{x=x_ε} = O(ε + \mu_l^ε)
\]

for every \( 0 < \nu < 1 \). On the other hand, we can evaluate the same quantity as follows: Decompose, for \( x \) close to \( x_ε \), the vector \( \lambda_ε = \sigma_ε(x) + δ_ε(x) \), where \( \sigma_ε(x)||\Lambda(x) \) and \( δ_ε(x) \perp \Lambda(x) \). Then we can express

\[
\langle \lambda_ε, M_a(x) \cdot \lambda_ε \rangle = \rho_a(x)\|\sigma_ε(x)\|^2 + \langle δ_ε(x), M_a(x) \cdot δ_ε(x) \rangle.
\]

Using the fact that the dependence of all quantities on \( x \) is \( C^1 \), we obtain

\[
\partial_l^ε(\Lambda_ε, M_a(x) \cdot \lambda_ε)|_{x=x_ε} = \partial_l^ε \rho_a(x) + O(|\rho_a(x)| + |\delta_ε(x)|).
\]

Combining these estimates, we get

\[
|\nabla \rho_a(x_ε)| = O(ε + \mu_ε^ν + |\rho_a(x_ε)| + |\delta_ε(x_ε)|).
\]

Now the facts that \( \rho_a(x_0) = 0 \) and \( \lambda_ε \to \lambda_0 \) as \( ε \to 0 \) clearly imply \( |\rho_a(x_ε)| + |\delta_ε(x_ε)| = o(1) \), which gives the conclusion.
In the following, we will decompose the vector $\lambda_\varepsilon$ as

$$\lambda_\varepsilon = \sigma_\varepsilon + \delta_\varepsilon,$$

(4.8)

where $\sigma_\varepsilon \parallel \Lambda_\varepsilon$ and $\delta_\varepsilon \perp \Lambda_\varepsilon$.

**Lemma 4.6.** As $\varepsilon \to 0$, we have

$$|\delta_\varepsilon| \lesssim \varepsilon + \mu_\varepsilon + |\rho_a(x_\varepsilon)|,$$

(4.9)

and, for all $0 < \nu < 1$,

$$|\nabla \rho_a(x_\varepsilon)| \lesssim \varepsilon + \mu_\varepsilon + |\rho_a(x_\varepsilon)|.$$

(4.10)

**Proof.** Writing $M_a(x_\varepsilon) \cdot \lambda_\varepsilon = \rho_a(x_\varepsilon) + M_a(x_\varepsilon) \cdot \delta_\varepsilon$, from (4.2) we plainly get

$$|M_a(x_\varepsilon) \cdot \delta_\varepsilon| \lesssim \varepsilon + \mu_\varepsilon + |\rho_a(x_\varepsilon)|.$$

Since $M_a(x_\varepsilon)$ has bounded inverse (independently of $\varepsilon$) on the spectral subspace containing $\delta_\varepsilon$, this gives (4.9).

The bound (4.10) simply follows by inserting (4.9) into the a priori bound (4.7) which was already obtained in the proof of Lemma 4.5. $\square$

### 4.3. Proof of Theorem 2.1.

Part (i) of Theorem 2.1 is contained in Proposition 3.1 and Lemma 4.5, and part (ii) in Lemmas 4.2 and 4.5. Part (iii) is precisely the statement of Theorem 3.2. So it only remains to prove part (iv) of Theorem 2.1.

To prove (iv), the crucial step is to realize that both assumptions (a) and (b) on $\rho_a$ imply

$$\rho_a(x_\varepsilon) = o(\varepsilon + \mu_\varepsilon).$$

(4.11)

Indeed, let us assume first that assumption (a) holds, that is, $D^2\rho_a(x_0) \geq c$ for some $c > 0$, in the sense of quadratic forms. Then [13, Lemma 4.2] applies to give that

$$\rho_a(x_\varepsilon) \lesssim |\nabla \rho_a(x_\varepsilon)|^2.$$

Using (4.10) with $\nu > 1/2$, and absorbing $\rho_a(x_\varepsilon)^2 = o(\rho_a(x_\varepsilon))$ on the left side, (4.11) follows.

On the other hand, assume that (b) holds, that is, $\rho_a$ is real-analytic.

Up to extracting a subsequence, we may assume that $\frac{x_\varepsilon - x_0}{|x_\varepsilon - x_0|} \to \eta \in S^{3n-1}$. (If $x_\varepsilon = x_0$, then (4.11) is trivially true.) Let

$$f(t) := \rho_a(x_0 + t\eta).$$

Then $f$ is well-defined on some open interval containing 0 and $f$ is analytic because $\rho_a$ is. Moreover, since $|\rho_a(x)| \to \infty$ as $x \to \partial \Omega_\varepsilon^*$, we must have $f \neq 0$. As a consequence,
there is a smallest \( k \geq 2 \) such that \( f^{(k)}(0) \neq 0 \). Defining

\[
f_\varepsilon(t) := \rho_a \left( x_0 + t \frac{x_\varepsilon - x_0}{|x_\varepsilon - x_0|} \right),
\]

we clearly have \( f^{(k)}_\varepsilon(0) \rightarrow f^{(k)}(0) \) as \( \varepsilon \rightarrow 0 \) and thus \( f^{(k)}_\varepsilon(0) \neq 0 \) for all \( \varepsilon \) small enough.

Let us assume for definiteness that \( f^{(k)}_\varepsilon(0) > 0 \). By analyticity and the choice of \( k \), we may write

\[
f_\varepsilon(t) = \frac{f^{(k)}_\varepsilon(0) + o(1)}{k!} t^k
\]

and

\[
f'_\varepsilon(t) = \frac{f^{(k)}_\varepsilon(0) + o(1)}{(k-1)!} t^{k-1}.
\]

Here \( o(1) \) denotes a quantity that tends to zero as \( t \rightarrow 0 \), uniformly in \( \varepsilon \). Solving the second equation for \( t \) and inserting it into the first, we obtain

\[
f_\varepsilon(t) = (c_k + o(1)) f'_\varepsilon(t) \frac{t^k}{k!}, \quad \text{with} \quad c_k = \frac{1}{k} \left( \frac{(k-1)!}{f^{(k)}_\varepsilon(0)} \right)^{x_{t-1}} > 0.
\]

Taking \( t = t_\varepsilon := |x_\varepsilon - x_0| \), we get

\[
|\rho_a(x_\varepsilon)| = |f_\varepsilon(t_\varepsilon)| \leq (c_k + o(1)) f'_\varepsilon(t_\varepsilon) \frac{t_\varepsilon^k}{k!} \lesssim |\nabla \rho_a(x_\varepsilon) \cdot \frac{x_\varepsilon - x_0}{|x_\varepsilon - x_0|}| \lesssim |\nabla \rho_a(x_\varepsilon)|.
\]

Now by using (4.10) with \( \nu < 1 \) so large that \( \nu \frac{k}{k-1} > 1 \) and again absorbing \( \rho_a(x_\varepsilon)^{\nu \frac{k}{k-1}} = o(\rho_a(x_\varepsilon)) \), (4.11) follows also under assumption (b).

Armed with (4.11), it is now straightforward to conclude the proof of Theorem 2.1.(iv).

Since \( \lim_{\varepsilon \rightarrow 0} \varepsilon \mu_{j,\varepsilon} = \lambda_j^2 \lim_{\varepsilon \rightarrow 0} \varepsilon \mu_{1,\varepsilon} \), we only need to evaluate the limit \( \lim_{\varepsilon \rightarrow 0} \varepsilon \mu_{1,\varepsilon} \). Using (4.8) together with (4.11) and (4.9), identity (4.2) becomes

\[
\varepsilon \lambda_{1,\varepsilon} (Q_V(x_{i,\varepsilon}) + o(1)) = -4\pi \sqrt{3} \sigma_{i,\varepsilon} (M_a(x_\varepsilon) \cdot \delta_{\varepsilon})_i + 3\pi (a(x_{i,\varepsilon}) + o(1)) \lambda_{i,\varepsilon}^2 \mu_{1,\varepsilon}. \tag{4.12}
\]

Now write \( \mu_{i,\varepsilon} = \lambda_{i,\varepsilon}^2 \mu_{1,\varepsilon} \) and sum over \( i \). Since \( \sigma_{\varepsilon} \perp \delta_{\varepsilon} \), the first term on the right side of (4.12) vanishes in the sum. Moreover, recalling that \( G = \tilde{G} \) from Remark 4.3, we can write

\[
4\pi \sqrt{3} \sum_i Q_V(x_{i,\varepsilon}) = \int_\Omega V(x) G(x)^2 \, dx.
\]

Thus we obtain

\[
3\pi \mu_{1,\varepsilon} \sum_i (a(x_{i,\varepsilon}) + o(1)) \lambda_{i,\varepsilon}^4 = \frac{\varepsilon}{4\pi \sqrt{3}} \left( \int_\Omega V G^2 \, dx + o(1) \right). \tag{4.13}
\]

If \( \int_\Omega V G^2 \, dx \neq 0 \), by passing to the limit \( \varepsilon \rightarrow 0 \) in (4.13) and recalling that \( \lambda_0 = \Lambda_0 \) by Lemma 4.5, we clearly obtain (2.5). If \( \int_\Omega V G^2 \, dx = 0 \), suppose that \( \sum_i a(x_{i,0}) \lambda_{i,0}^4 < 0 \), say. Then the right side of (4.13) must be strictly negative. Hence the quotient \( (\sum_i (a(x_{i,\varepsilon}) + o(1)) \lambda_{i,\varepsilon}^4) \int_\Omega V G^2 \, dx + o(1))^{-1} \) is positive and tends to \( +\infty \) as \( \varepsilon \rightarrow 0 \). When \( \sum_i a(x_{i,0}) \lambda_{i,0}^4 > 0 \), the argument is analogous.

The proof of Theorem 2.1 is thus complete.
4.4. Regularity of $\phi_a$ and $\rho_a$. We end this section by discussing sufficient conditions for $C^2$-differentiability and real-analyticity\textsuperscript{4} of the Robin function $\phi_a$, and, in turn, $\rho_a$.

We first observe some sufficient conditions for regularity of $\rho_a$.

**Lemma 4.7.** (i) If $\phi_a$ and $a$ are real-analytic on $\Omega$ then $\rho_a$ is real-analytic on $\Omega^*_a$.

(ii) If $a \in C^{0,1}(\Omega) \cap C^{2,\sigma}_{loc}(\Omega)$ for some $\sigma \in (0, 1)$, then $\phi_a \in C^2(\Omega)$ and $\rho_a \in C^2(\Omega^*_a)$.

In the statement of Lemma 4.7, we chose to assume global analyticity of $a$ and $\phi_a$ for simplicity. Since analyticity is a local property, it would of course be equally possible to conclude analyticity of $\rho_a$ in a neighborhood of some $x_0 = (x_1, \ldots, x_n)$ by assuming $a$ and $\phi_a$ to be analytic on neighborhoods of $x_1, \ldots, x_n$.

**Proof.** It is a general fact that if a matrix $M(\xi)$ depends analytically on some parameter $\xi \in \mathbb{R}^n$ in a neighborhood of $\xi_0 \in \mathbb{R}^n$, then its simple eigenvalues also depend analytically on these parameters. This is a direct consequence of the analytic implicit function theorem applied to $p(\xi, \lambda) := \det(M(\xi) - \lambda \text{Id})$ and the fact that $p(\xi_0, \lambda_0) = 0 \neq \partial_\lambda p(\xi_0, \lambda_0))$ if and only if $\lambda_0$ is a simple eigenvalue of $M(\xi_0)$.

We apply this fact to the matrix $M(x)$ with parameter $x \in \Omega^*_a$. The off-diagonal entries $G_a(x_i, x_j)$ are always analytic in $x_i, x_j$ because $x_i \neq x_j$. This follows from elliptic regularity and the fact that $G_a(x, y)$ solves the PDE $-\Delta_y G_a(x, y) = -a(x)G_a(x, y)$ on $\Omega \setminus \{x\}$ with analytic coefficient $a(x)$. Now by assumption, the diagonal entries of $M(x)$ also depend analytically on $x$. This completes the proof of (i).

For (ii), we know from [13, Lemma 4.1] that $\phi_a \in C^2(\Omega)$. Then the $C^2$-differentiability follows as above using the implicit function theorem formulated for $C^2$ functions. □

In the simplest case where $a$ is a constant, we can prove that the hypothesis of Lemma 4.7(i) is indeed fulfilled.

**Lemma 4.8.** Suppose that $a \equiv \text{const.}$. Then $\phi_a$ is real-analytic on $\Omega$ and $\rho_a$ is real-analytic on $\Omega^*_a$.

**Proof.** We write $H_a(x, y)$ as

$$H_a(x, y) = \eta(x, y) + \sum_{k=0}^{\infty} h_k(x, y),$$

for some sequence of functions $h_k(x, y)$ satisfying

$$-\Delta_x h_0(x, y) = \frac{a}{4\pi|x - y|} \quad \text{for } x, y \in \Omega,$$

\textsuperscript{4}We will use the terms **analytic** and **real-analytic** interchangeably in the following.
and recursively, for $k \geq 1$,
\[-\Delta_x h_k(x, y) = -ah_{k-1}(x, y) \quad \text{for } x, y \in \Omega. \tag{4.16}\]
It is easy to verify that $h_k(x, y)$ given by
\[h_k(x, y) = \frac{a^{k+1}}{4\pi(2k+2)!} |x - y|^{2k+1}, \tag{4.17}\]
satisfies (4.15)–(4.16). In particular, with this choice the sum in (4.14) converges indeed on all of $\mathbb{R}^3$. By construction, the remainder function $\eta(x, y)$ satisfies
\[-\Delta_x \eta(x, y) + a(x)\eta(x, y) = 0.\]
Since $a$ is analytic, elliptic regularity theory implies that $\eta(x, y)$ is analytic as a function of $x$. On the other hand, $\eta(x, y)$ is symmetric in $x$ and $y$ because $H_a(x, y)$ is and so are all the $h_k(x, y)$ by their explicit expressions given in (4.17). Hence as a function of $y$, $\eta(x, y)$ satisfies
\[-\Delta_y \eta(x, y) + a(y)\eta(x, y) = 0,\]
and as above we conclude that $\eta(x, y)$ is analytic in $y$.

Since
\[h_k(x, x) = 0 \tag{4.18}\]
by (4.17), from (4.14) we obtain $\phi_a(x) = \eta(x, x)$ and hence $\phi_a$ is analytic. \qed

The above proof, in particular the ansatz given by (4.14)–(4.16), still appears to be a promising approach to prove analyticity of $\phi_a$ in the more general case where $a$ is analytic, but non-constant. However, there are several obstructions to a straightforward adaptation. Firstly, the $h_k(x, y)$ will not have a simple expression as in (4.17) because of additional terms coming from derivatives of $a$. In particular, it is much harder to find a way to simultaneously justify (4.18) and the convergence of the sum in (4.14). Secondly, it is not clear how to prove analyticity of $\eta(x, y)$ in the second variable $y$. In particular, the symmetry of $h_k$ in $x$ and $y$ seems problematic to ensure.

**Appendix A. A Liouville Type Result for the Solutions of the Linearized Equation**

In this section, we give a general statement for the classification of the solution of the linearized equation (A.1). This kind of result is not new, see for instance [20, Proposition 2] or [25, Proposition 3]. The cited results concern solutions to (A.1) which are singular in the origin, and the linearization takes place, somewhat more generally, about a singular solution to $-\Delta u = u^{\frac{N+2}{N-2}}$ instead of the bubble. Nevertheless the proof of those former results can also be applied in our framework. For the sake of completeness and accessibility, we now make a precise statement in our context and give a quick sketch of its proof.
We consider the equation
\[ -\Delta v = N(N + 2)\tilde{B}^{p-1}v \quad \text{on } \mathbb{R}^N, \] (A.1)
with \( p = \frac{N+2}{N-2} \) and \( \tilde{B}(x) = (1 + |x|^2)^{-\frac{N+2}{2}} \). Plainly, (A.1) is the linearization of \( -\Delta u = N(N-2)u^p \) at the solution \( \tilde{B} \). (Note that the normalization of the bubble we choose here is different from the one employed in Theorem 2.1. This turns out to be more natural and convenient for some of the related functions appearing below.)

The canonical solutions to (A.1) are linear combinations of the functions
\[ w_0(x) := \frac{1 - |x|^2}{(1 + |x|^2)^{N/2}} = \frac{2}{2 - N} \partial_\mu \left( \mu^{-\frac{N-2}{2}} \tilde{B}(\mu^{-1}x) \right) \bigg|_{\mu=1} \] (A.2)
and
\[ w_i(x) := \frac{x_i}{(1 + |x|^2)^{N/2}} = \frac{1}{N - 2} \partial_{y_i} \left( \tilde{B}(x - y) \right) \bigg|_{y=0}, \quad i = 1, \ldots, N, \] (A.3)
which arise as derivatives of \( \tilde{B} \) with respect to its symmetry parameters.

The main non-degeneracy result reads as follows.

**Proposition A.1.** Let \( v \) be a solution to (A.1) and suppose that \( |v(x)| \lesssim |x|^{\tau} \) for all \( |x| \geq 1 \), for some \( \tau \geq -N + 2 \). Then
\[ v(x) = \sum_{i=0}^{3} c_i w_i(x) + \sum_{k=2}^{\lfloor \tau \rfloor} v_k^-(r) Y_k(\omega) \] (A.4)
for some smooth functions \( v_k^- \) defined on \( (0, \infty) \). Here \( r = |x|, \omega = x/|x| \) and \( Y_k \) is a spherical harmonic on \( S^{N-1} \) of degree \( k \).

Moreover, for every \( k \geq 2 \), we have either \( v_k^- \equiv 0 \) or \( v_k^-(r) \sim r^k \) both as \( r \to 0 \) and as \( r \to \infty \).

**Proof.** Since \( -\Delta \) is diagonal with respect to spherical harmonics and \( \tilde{B} \) is radial, we can write a solution to (A.1) as
\[ v = \sum_{k=0}^{\infty} v_k(r) Y_k(\theta), \]
where \( Y_k \) is a suitable spherical harmonic of degree \( k \) and \( v_k \) solves the equation
\[ v_k'' + \frac{N-1}{r} v_k' + \left( N(N+2)\tilde{B}^{p-1} - \frac{k(k+N-2)}{r^2} \right) v_k = 0. \] (A.5)

It follows from the discussion on [26, p. 310-311], that the only solutions for degrees \( k = 0, 1 \) which satisfy \( |v_k(r)| \lesssim r^\tau \) are constant multiples of \( v_0(r) = \frac{1-r^2}{(1+r^2)^{N/2}} \) and \( v_1(r) = \frac{r}{(1+r^2)^{N/2}} \).
For \( k \geq 2 \), passing to logarithmic coordinates via
\[
v_k(r) = r^{-\frac{N-2}{2}} \psi_k(\ln r),
\]
the new unknown function \( \psi_k \) satisfies
\[
\mathcal{L}_k \psi_k := \psi_k''(t) - \mu_k^2 \psi_k(t) + g(t) \psi_k(t) = 0,
\]
where we have set \( \mu_k := \frac{N-2}{2} + k \) and \( g(t) := \frac{N(N+2)}{4} \cosh(t)^{-2} \).

It follows from arguments developed in [26], see also [20, Section 2.2], that for each \( k \geq 2 \) there are precisely two linearly independent solutions \( \psi_k^\pm \) to (A.7), satisfying
\[
|\psi_k^\pm(t)| \sim e^{\mp \mu_k t} \text{ for } t \in \mathbb{R}.
\]
(This could alternatively also be proved via elementary ODE analysis.)

The solutions \( v_k^\pm \) associated to \( \psi_k^\pm \) via (A.6) then satisfy, if not identically equal to 0,
\[
|v_k^-(r)| \sim r^k, \quad |v_k^+(r)| \sim r^{-\alpha-2-k} \quad \text{for } r > 0.
\]
In particular, \( v_k^+ \) is singular near zero. Since \( v \) is bounded near the origin by assumptions, \( v_k \) must be proportional to \( v_k^- \) for every \( k \geq 2 \). Moreover, because of the growth bound \( |v(x)| \lesssim |x|^\tau \) as \( |x| \to \infty \), (A.8) forces \( v_k^- \equiv 0 \) for every \( k > \lceil \tau \rceil \). This yields the claimed expression for \( v \).

In our proof of Theorem 2.1, we use the non-degeneracy statement of Proposition A.1 in the form of the following corollary.

**Corollary A.2.** Let \( v \) be a solution to (A.1) and suppose that \( |v(x)| \lesssim |x|^\tau \) on \( \mathbb{R}^N \) for some \( \tau \in (1, \infty) \setminus \mathbb{N} \). Then \( v \equiv 0 \).

**Proof.** By Proposition A.1, \( v \) is of the form (A.4). But now it is easy to see that the assumption \( v(x) \lesssim |x|^\tau \) as \( |x| \to 0 \) with \( \tau > 1 \), together with (A.8), forces \( c_i = 0 \) for \( i = 0, ..., N \). If \( \tau < 2 \), we are done. If \( \tau > 2 \), we have
\[
\infty > C \geq \lim_{|x| \to 0} \frac{|v(x)|}{|x|^\tau} = \sum_{k=2}^{\infty} \frac{|v_k^-(r)|}{r^\tau} Y_k(\theta).
\]
Now, if some \( v_k^- \) is not identically equal to 0, then we consider \( k_0 \) smallest \( k \) such that \( v_k^- \neq 0 \). Since \( \tau \) is non-integer, \( k_0 < \tau \), and we get, by (A.8),
\[
\frac{|v(x)|}{|x|^\tau} \sim |x|^{k_0-\tau} \text{ at } 0.
\]
This yields a contradiction with (A.9), hence \( v \equiv 0 \) as claimed. \( \Box \)
The preceding proof also shows that the restriction \( \tau \notin \mathbb{N} \) is necessary for Corollary A.2 to hold. Indeed, for any \( k \in \mathbb{N} \) and any spherical harmonic \( Y_k \) of degree \( k \), the function \( v(x) = v_k(r)Y_k(\omega) \) satisfies (A.1) with \( |v(x)| \lesssim |x|^k \) for all \( x \in \mathbb{R}^N \), while certainly \( v \not\equiv 0 \).

**Appendix B. Some computations**

**Lemma B.1.** Let \( a \in C(\overline{\Omega}) \cap C^{1,\sigma}_{\text{loc}}(\Omega) \) for some \( \sigma > 0 \). As \( \varepsilon \to 0 \),

\[
\int_{b_i,\varepsilon} B^5_{i,\varepsilon} G_a(x_{i,\varepsilon}, \cdot) \, dx = \mu_{i,\varepsilon}^{-1/2} - 4\pi \sqrt{3} \phi_a(x_{i,\varepsilon}) \mu_{i,\varepsilon}^{1/2} + 3a(x_{i,\varepsilon}) \mu_{i,\varepsilon}^{3/2} + o(\mu_{i,\varepsilon}^{3/2}).
\]

**Proof.** Under our assumptions on \( a \), [13, Lemma B.2] asserts that

\[
G_a(y,z) = \frac{1}{4\pi z-y} - \phi_a(y) - \frac{1}{2} \nabla \phi_a(y) \cdot (z-y) + \frac{a(y)}{8\pi} |z-y| + O(|z-y|^{1+\nu}), \quad (B.1)
\]

for every \( 0 < \nu < 1 \). Recall \( b_i,\varepsilon = B(x_{i,\varepsilon}, \delta_0) \) with \( \delta_0 > 0 \) independent of \( \varepsilon \), and pick \( y = x_{i,\varepsilon} \) in (B.1). We compute

\[
\int_{b_i,\varepsilon} \frac{1}{4\pi z-x_{i,\varepsilon}} B^5_{i,\varepsilon} \, dz = \frac{1}{4\pi} \mu_{i,\varepsilon}^{-1/2} \int_{B(0,\mu_{i,\varepsilon}^{-1} \delta_0)} B^5 \frac{1}{|z|} \, dz = \mu_{i,\varepsilon}^{-1/2} + o(\mu_{i,\varepsilon}^{3/2}).
\]

Here we used that

\[
\int_{\mathbb{R}^3} \frac{1}{|x|(1 + |x|^2)^{3/2}} \, dx = 6\pi \int_0^\infty \frac{1}{(1 + s)^{5/2}} \, ds = 6\pi B(1,3/2) = 4\pi,
\]

where \( B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \) is the Beta function.

Next,

\[
\phi_a(x_{i,\varepsilon}) \int_{b_i,\varepsilon} B^5_{i,\varepsilon} \, dz = \mu_{i,\varepsilon}^{1/2} \phi_a(x_{i,\varepsilon}) \mu_{i,\varepsilon}^{1/2} \int_{B(0,\mu_{i,\varepsilon}^{-1} \delta_0)} B^5 \, dz = 4\pi \sqrt{3} \phi_a(x_{i,\varepsilon}) \mu_{i,\varepsilon}^{1/2} + o(\mu_{i,\varepsilon}^{3/2}).
\]

Here we used that

\[
\int_{\mathbb{R}^3} \frac{1}{(1 + |x|^2/3)^{5/2}} \, dx = 6\pi \sqrt{3} \int_0^\infty \frac{s^{1/2}}{(1 + s)^{5/2}} \, ds = 6\pi \sqrt{3} B(3/2,1) = 4\pi \sqrt{3}.
\]

By antisymmetry of the integrand,

\[
\int_{b_i,\varepsilon} \nabla \phi_a(x_{i,\varepsilon}) \cdot (z-x_{i,\varepsilon}) B^5_{i,\varepsilon} \, dz = 0.
\]

Next,

\[
\frac{a(x_{i,\varepsilon})}{8\pi} \int_{b_i,\varepsilon} |z-x_{i,\varepsilon}| B^5_{i,\varepsilon} \, dx = \frac{a(x_{i,\varepsilon})}{8\pi} \mu_{i,\varepsilon}^{3/2} \int_{B(\mu_{i,\varepsilon}^{-1} \delta_0)} B^5 |x| \, dx = 3a(x_{i,\varepsilon}) \mu_{i,\varepsilon}^{3/2} + o(\mu_{i,\varepsilon}^{3/2}).
\]

Here we used that

\[
\int_{\mathbb{R}^3} \frac{1}{(1 + |x|^2/3)^{5/2}} \, dx = 18\pi \int_0^\infty \frac{s}{(1 + s)^{5/2}} \, ds = 18\pi B(2,1/2) = 24\pi.
\]
Finally,
\[ \int_{\Omega} B^5_{i,\varepsilon} |z - x_{i,\varepsilon}|^{1+\nu} \, dx \lesssim \mu_{\varepsilon}^{3+\nu} = o(\mu_{\varepsilon}^{3/2}). \]

Combining all of the above, the lemma follows. \qed

**Lemma B.2.** Let \( a \in C(\overline{\Omega}) \cap C^{1,\sigma}_{\text{loc}}(\Omega) \) for some \( \sigma > 0 \). As \( \varepsilon \to 0 \), we have
\[ \int_{b_{i,\varepsilon}} B^5_{i,\varepsilon} \nabla x G_a(x_{i,\varepsilon}, z) \, dz = -2\pi \sqrt{3} \nabla \phi_a(x_{i,\varepsilon}) \mu_{i,\varepsilon}^{1/2} + O(\mu_{\varepsilon}^{1+\nu}), \]
for every \( 0 < \nu < 1 \).

**Proof.** The argument in [13, Lemma B.2] in fact also shows
\[ \nabla x G_a(y, z) = \frac{y - z}{4\pi |y - z|^3} - \frac{1}{2}\nabla \phi_a(z) + \frac{a(z)}{8\pi |y - z|} + O(|y - z|^\nu), \]
for every \( 0 < \nu < 1 \). Picking \( y = x_{i,\varepsilon} \), and observing the cancellations by antisymmetry, this identity gives
\[ \int_{b_{i,\varepsilon}} B^5_{i,\varepsilon}(z) \nabla x G_a(x_{i,\varepsilon}, z) \, dz = -2\pi \sqrt{3} \nabla \phi_a(x_{i,\varepsilon}) + O(\mu_{\varepsilon}^{1+\nu}). \]
This is the assertion. \qed

**Lemma B.3.** Let \( \varepsilon > 0 \), \( a \in C(\overline{\Omega}) \) and \( V \in C^1(\overline{\Omega}) \cap C^{0,\sigma}_{\text{loc}}(\Omega) \) for some \( \sigma \in (0, 1) \) be such that the Green’s functions \( G_a \) and \( G_{a+\varepsilon V} \) exist. Then
\[ \phi_{a+\varepsilon V}(x) - \phi_a(x) = \varepsilon \int_{\Omega} G_a(x, y)^2 V(y) \, dy + O(\varepsilon) \quad (B.2) \]
and
\[ \nabla \phi_{a+\varepsilon V}(x) - \nabla \phi_a(x) = O(\varepsilon). \quad (B.3) \]

The bounds are uniform for \( x \) in compact subsets of \( \Omega \).

**Proof.** By the resolvent formula, we have
\[ H_{a+\varepsilon V}(x, y) - H_a(x, y) = G_a(x, y) - G_{a+\varepsilon V}(x, y) = \varepsilon \int_{\Omega} G_a(x, z) V(z) G_{a+\varepsilon V}(z, y) \, dz. \quad (B.4) \]
In particular, \( G_{a+\varepsilon V}(x, y) = G_a(x, y) + O(\varepsilon) \). Plugging this back into the right side of (B.4) and evaluating at \( x = y \) gives (B.2).

To prove (B.3), some more care needs to be taken because the derivative of the integrand in (B.2) behaves like \( |x - y|^{-3} \), which is not integrable. To overcome this issue,
Lemma B.4. Let $W$ be the unique radial solution to
\[-\Delta W - 5WB^4 = -B, \quad W(0) = \nabla W(0) = 0.\]
Then we have
\[5 \int_{\mathbb{R}^3} \frac{W(x)B(x)^4}{|x|} \, dx = 12\pi(\pi - 1).\]

Proof. By the equation, we have, for every $R > 0$,
\[5 \int_{B_R} \frac{W(x)B(x)^4}{|x|} \, dx = -\int_{B_R} \frac{\Delta W}{|x|} \, dx + \int_{B_R} \frac{B(x)}{|x|} \, dx. \quad (B.5)\]
We need to compute the asymptotics as $R \to \infty$ of the two integrals on the right side. The second one is straightforward to evaluate. We have
\[\int_{B_R} \frac{B(x)}{|x|} \, dx = 4\pi \int_0^R \frac{r \, dr}{(1 + \frac{r^2}{3})^{1/2}} = 4\pi \left[3(1 + \frac{r^2}{3})^{1/2}\right]_0^R = 12\pi(1 + \frac{R^2}{3})^{1/2} - 12\pi = 4\pi\sqrt{3}R - 12\pi + O(R^{-1}) \quad (B.6)\]
as $R \to \infty$. To evaluate the first integral on the right side of (B.5), we integrate by parts. By Green’s formula and since $W(0) = 0$, we have
\[-\int_{B_R} \frac{\Delta W}{|x|} \, dx = \int_{\partial B_R} \left( W \frac{\partial}{\partial n} \frac{1}{|x|} - \frac{1}{|x|} \frac{\partial W}{\partial n} \right) \, d\sigma(x)\]
\[= -4\pi W(R) - 4\pi W'(R)R. \quad (B.7)\]
By Lemma B.5 below, we have
\[W(r) = v(r)\varphi(r),\]
where $v$ is the solution to the homogeneous equation $-\Delta v = 5vB^4$ given by
\[v(r) = \frac{3 - r^2}{(3 + r^2)^{3/2}} = -\frac{2}{\sqrt{3}} \frac{d}{d\mu}_{\mu=1} B_{\mu,0}(r) \quad (B.8)\]
and \( \varphi(r) = \int_0^r \psi(s) \, ds \), with
\[
\psi(r) = \sqrt{3} \left( -r + 2\sqrt{3} \arctan \left( \frac{r}{\sqrt{3}} \right) - \frac{3r}{r^2 + 3} \right) \frac{(3 + r^2)^3}{r^2(3 - r^2)^2}. \tag{B.9}
\]
From these expressions, we can easily read off the asymptotic behavior of \( W(R) \) and \( W'(R) \) to the precision necessary to evaluate (B.7) as \( R \to \infty \). Indeed, we have
\[
\psi(R) = -\sqrt{3} R + 3\pi + O(R^{-1}),
\]
hence
\[
\varphi(R) = -\frac{\sqrt{3}}{2} R^2 + 3\pi R + O(\ln R).
\]
on the other hand,
\[
v(R) = -R^{-1} + O(R^{-3}),
\]
which yields
\[
W(R) = \frac{\sqrt{3}}{2} R - 3\pi + o(1).
\]
Moreover,
\[
v'(R) = 3R \frac{R^2 - 3}{(3 + R^2)^{5/2}} = R^{-2} + O(R^{-4})
\]
and thus
\[
W'(R) = \psi(R) v(R) + \varphi(R) v'(R)
\]
\[
= (\sqrt{3} - 3\pi R^{-1}) + \left( \frac{-\sqrt{3}}{2} + 3\pi R^{-1} \right) + o(R^{-1})
\]
\[
= \frac{\sqrt{3}}{2} + o(R^{-1}).
\]
Inserting these expansions into (B.7) above, we obtain
\[
- \int_{B_R} \frac{\Delta W}{|x|} \, dx = -4\pi \sqrt{3} R + 12\pi^2 + o(1).
\]
Coming back to (B.5) and inserting this expansion as well as (B.6), the divergent terms in \( R \) cancel and we get
\[
5 \int_{\mathbb{R}^3} \frac{W(x) B(x)^4}{|x|} \, dx = 5 \lim_{R \to \infty} \int_{B_R} \frac{W(x) B(x)^4}{|x|} \, dx = 12\pi (\pi - 1),
\]
as claimed. \(\square\)

**Lemma B.5.** Let \( W \) be the unique radial solution to
\[
-\Delta W - 5WB^4 = -B, \quad W(0) = \nabla W(0) = 0. \tag{B.10}
\]
Then \( W \) is given by
\[
W(r) = v(r) \int_0^r \psi(s) \, ds, \tag{B.11}
\]
with \( v \) as in (B.8) and \( \psi \) as in (B.9).
Notice that indeed $W'(0) = 3^{-1/2} \psi(0) = 0$. This is not directly obvious from the definition of $\psi$, but follows by noting that for

$$h(r) := -r + 2\sqrt{3} \arctan\left( \frac{r}{\sqrt{3}} \right) - \frac{3r}{r^2 + 3}$$

one has $h(0) = h'(0) = h''(0) = 0$. This implies $h(r) = O(r^3)$ and thus $\psi(r) = O(r)$ as $r \to 0$.

It can of course be verified by straightforward computation that $W$ given by (B.11) solves (B.10). In the following proof we actually sketch how to find (B.11) using the method of the variation of constants.

**Proof.** Setting $W = v \varphi$ with the new unknown $\varphi$, solving (B.10) becomes equivalent to solving

$$\varphi'' + \left( 2 \left( \frac{1}{r} + \frac{v'}{v} \right) \right) \varphi' = \frac{B}{v}, \quad \varphi(0) = \varphi'(0) = 0,$$

or, with $\varphi(r) = \int_0^r \psi(s) \, ds$ and $H := 2 \left( \frac{1}{r} + \frac{v'}{v} \right)$,

$$\psi' + H \psi = \frac{B}{v}, \quad \psi(0) = 0,$$

To solve this first-order equation, we make a second time the variation of constants ansatz $\psi = \psi_0 \eta$, where $\psi'_0 + H \psi_0 = 0$ and $\eta$ needs to solve

$$\eta' = \frac{B}{v \psi_0}, \eta(0) = 0.$$

The solution $\psi_0$ can be determined directly as

$$\psi_0(r) = \exp \left( - \int_1^r H(s) \, ds \right) = \frac{1}{r^2 v^2},$$

and hence

$$\eta(r) = \int_0^r B(s) s^2 \psi(s) \, ds = \sqrt{3} \int_0^r \frac{s^2(3 - s^2)}{(3 + s^2)^2} \, ds = 3 \int_0^{r/\sqrt{3}} \frac{s^2(1 - s^2)}{(1 + s^2)^2} \, ds.$$

To evaluate this integral, we write

$$\frac{s^2(1 - s^2)}{(1 + s^2)^2} = -1 + \frac{3s^2 + 1}{(1 + s^2)^2} = -1 + \frac{3}{1 + s^2} - \frac{2}{(1 + s^2)^2}.$$

It can be verified by direct computation that $(\arctan s + \frac{s}{s^2 + 1})' = \frac{2}{(1 + s^2)^2}$. From here, we can thus explicitly compute $\eta$, and via $\psi = \psi_0 \eta$ we easily obtain the claimed expression for $\psi$. \qed
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