Local fractional Moisil–Teodorescu operator in quaternionic setting involving Cantor-type coordinate systems

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1 INTRODUCTION

Vector analysis,1,2 a branch of mathematics that deals with scalar and vector quantities, has proven to be a powerful tool for developing a mathematical setting able to comprise several differential field equations in different areas of engineering. One of the problems that arise in vector analysis in Euclidean spaces, when dealing with operators such as divergence, gradient, and curl for analyzing the behavior of scalar- and vector-valued functions, is understanding that each of these operators must conform to all common differential operators and vector identities regardless of the considered coordinate system.

It is well known that in vector analysis in \( \mathbb{R}^3 \), the vector product does not allow the formation of an algebra. In this sense, various attempts were made to construct such structure. Quaternion vector algebra, which marked the beginning of modern vector analysis, was discovered by William Rowan Hamilton on October 16, 1843. The fact that Hamilton started with triplets and ended up with quaternions implies immediately his concern about the special role of what he called vector:

\[
\mathbf{q} = \sum_{j=1}^{3} q_j \mathbf{e}_j, \quad \text{where } \{\mathbf{e}_j\}, j = 1, 2, 3 \text{ denotes the standard orthonormal basis of } \mathbb{R}^3.
\]

The new multiplication rules with the basis vectors, as he used since then,

\[
e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1,
\]

contain the solution of the problem.3,4
With the introduction of the Nabla-operator in Cartesian coordinates given by

\[ \nabla = \sum_{j=1}^{3} e_j \partial_j, \]  

(2)

where \( \partial_j \) denotes a partial derivative, Hamilton also invented the other essential technical ingredient for the vector calculus: the vector differential operator, which is used to describe the gradient of a scalar function as well as the divergence and the curl of a vector valued function.

There is a debate over whether it is correct that the action of the vector Laplacian is the action of the scalar Laplacian component by component; nevertheless, this happens only for the case of the Cartesian coordinates, and the result totally differ in any other orthogonal curvilinear system (see other studies5-8).

In the conventional approach, the Laplacian \( \nabla^2 \) is an operator that can act on both scalar \( q \) and vector \( \vec{q} \) fields. The operator on a scalar can be written as

\[ \nabla^2 q = \nabla \cdot \nabla q, \]  

(3)

which will produce another scalar field. Meanwhile, on a vector field it can be expressed as

\[ \nabla^2 \vec{q} = \nabla(\nabla \cdot \vec{q}) - \nabla \times (\nabla \times \vec{q}), \]  

(4)

which will produce another vector field.

However, in Cartesian coordinates, both operators coincide with

\[ \nabla^2 = \sum_{j=1}^{3} \partial_j^2, \]  

(5)

where it is evident that operation on a scalar (vector) field transforms into a scalar (vector) field.

An overwhelming majority of physically meaningful problems can be simplified by using of non-Cartesian coordinates, especially orthogonal curvilinear coordinates.

Regarding the three-dimensional nature of many problems in mathematical physics, handling them with quaternion analysis techniques seems to be adequate. The ideas and techniques of quaternion analysis were originally introduced by Rudolf Fueter.9-11 So far, motivated by these works, mathematicians have been interested in developing various approaches along classical lines in the study of this theory, see for example, other studies.12-14

The Moisil–Teodorescu operator (a determined first-order elliptic operator that can be expressed in terms of the usual divergence, gradient and curl operators, see Moisil15) is nowadays considered to be a good analogue of the usual Cauchy–Riemann operator of complex analysis in the framework of quaternionic analysis, and it is a square root of the scalar Laplace operator in \( \mathbb{R}^3 \). For a close relationship of the Moisil–Teodorescu operator with many spatial models of mathematical physics we refer the reader to other studies.16-19

In their study, Bory and Pérez-De La Rosa20 describe how the meaning of the Moisil–Teodorescu operator on open subsets of \( \mathbb{R}^3 \) involving orthogonal curvilinear coordinates can benefit from an approach within the framework of quaternionic analysis.

The enormous success and large application of local and nonlocal fractional calculus (see other studies21-44) is a strong motivation to show its connection with quaternionic analysis. In particular, the main goal of the present work is to develop a quaternionic structure for the local fractional Moisil–Teodorescu operator in Cantor-type cylindrical and spherical coordinates.

The Helmholtz equation represents the time-independent form of the wave equation obtained while applying the technique of separation of variables. It is applied to deal with problems in such fields as electromagnetic radiation, seismology, transmission, and acoustics. In the framework of fractional calculus, in Samuel and Thomas,45 an analytic solution for the fractional Helmholtz equation in terms of the Mittag-Leffler function is derived while in Hao et al.,26 the Cantor-type cylindrical-coordinate method is applied to handle the corresponding local fractional Helmholtz equation.

The paper is organized as follows: in Section 2 we introduce the basic facts on local fractional calculus developed in Yang et al.41 as well as the rudiments of quaternionic analysis. Next, we introduce the local fractional Moisil–Teodorescu operator in Cantor-type cylindrical and spherical coordinates in Section 3 in addition to the local fractional Laplacian and the local fractional Bitsadze operators also in Cantor-type cylindrical and spherical coordinates. Additionally, in Section 4
the quaternionic Helmholtz equation associated with local fractional derivative operators involving the Cantor-type cylindrical and spherical coordinates is considered.

2 | PRELIMINARIES

2.1 | Few aspects of local fractional calculus

We begin our study by recalling some facts on local fractional calculus that can be found in other studies. 36,41

Definition 1. Let \( f(x) \) be a function defined on a fractal set of fractal dimension \( a \) \((0 < a < 1)\); the function \( f(x) \) is said to be local fractional continuous at \( x = x_0 \) if for each \( \varepsilon > 0 \), there exists a corresponding \( \delta > 0 \) such that

\[
| f(x) - f(x_0) | < \varepsilon^a,
\]

whenever \( 0 < |x - x_0| < \delta \).

The set of all local fractional continuous functions on the interval \((a, b)\) will be denoted by \( C_\alpha(a, b) \).

Definition 2. The local fractional derivative of \( f(x) \in C_\alpha(a, b) \) of order \( a \) \((0 < a < 1)\) at \( x = x_0 \) is defined as

\[
D^{(a)} f(x_0) = f^{(a)}(x_0) = \lim_{x \to x_0} \frac{\Delta^a [f(x) - f(x_0)]}{(x - x_0)^a},
\]

provided the limit exists, where \( \Delta^a [f(x) - f(x_0)] \equiv \Gamma(1 + a) [f(x) - f(x_0)] \) with Euler's gamma function \( \Gamma(1 + a) := \int_0^\infty \mu^{a-1} \exp(-\mu) \, d\mu \).

The set of all functions such that its local derivative exists for all \( x \) on the interval \((a, b)\) will be denoted by \( D_\alpha(a, b) \). The set \( D_\alpha(a, b) \) is called a \( \alpha \)-local fractional derivative set.

Let \( f, g \in D_\alpha(a, b) \), then

\[
D^{(a)} \left[ f(x) \pm g(x) \right] = D^{(a)} f(x) \pm D^{(a)} g(x),
\]

\[
D^{(a)} \left[ f(x) g(x) \right] = g(x) D^{(a)} f(x) + f(x) D^{(a)} g(x),
\]

\[
D^{(a)} \left[ \frac{f(x)}{g(x)} \right] = \frac{[D^{(a)} f(x)] g(x) - f(x) [D^{(a)} g(x)]}{g^2(x)},
\]

provided \( g(x) \neq 0 \).

Suppose that \( f(x) = (\phi \circ \varphi)(x), \, x \in (a, b) \). Then, we have

\[
f^{(a)}(x) = \phi^{(a)}(\varphi(x)) \left( \varphi^{(1)}(x) \right)^a,
\]

provided \( \phi^{(a)}(\varphi(x)) \) and \( \varphi^{(1)}(x) \) exist.

The generalized functions defined on Cantor sets are given by

\[
E_\alpha (x^\alpha) = \sum_{k=0}^\infty \frac{x^{k\alpha}}{\Gamma(1 + k\alpha)},
\]

The sine and cosine functions on a fractal set are given, respectively, by

\[
\sin_\alpha(x^\alpha) = \frac{E_\alpha (i^a x^\alpha) - E_\alpha (-i^a x^\alpha)}{2i^a} = \sum_{k=0}^\infty \frac{(-1)^k x^{(2k+1)a}}{\Gamma(1 + (2k + 1)a)},
\]

\[
\cos_\alpha(x^\alpha) = \frac{E_\alpha (i^a x^\alpha) + E_\alpha (-i^a x^\alpha)}{2} = \sum_{k=0}^\infty \frac{(-1)^k x^{2ka}}{\Gamma(1 + 2ka)},
\]

where \( x \in \mathbb{R}, \, 0 < \alpha < 1 \) and \( i^a \) is a imaginary unit of a fractal set.
The previously defined functions satisfy the following identities:

\[
D^{(a)} \left[ \frac{x^{na}}{\Gamma(1 + na)} \right] = \frac{x^{(n-1)a}}{\Gamma(1 + (n-1)a)},
\]

\[ D^{(a)} \left[ E_a \left( x^a \right) \right] = E_a \left( x^a \right), \tag{13} \]

\[ D^{(a)} \left[ \sin_a \left( x^a \right) \right] = \cos_a \left( x^a \right), \tag{14} \]

\[ D^{(a)} \left[ \cos_a \left( x^a \right) \right] = -\sin_a \left( x^a \right). \tag{15} \]

### 2.2 Rudiments of quaternionic analysis

In this subsection, we follow Kravchenko and Shapiro\textsuperscript{18} in reviewing some standard facts on quaternionic analysis to be used in this paper.

We work with the skew field (a complex noncommutative, associative algebra with zero divisors) \( \mathbb{H}(\mathbb{C}) \) of complex quaternions, that is, each \( a \in \mathbb{H}(\mathbb{C}) \) is of the form \( q = \sum_{k=0}^{3} q_k i_k \), with \( \{q_k\} \subset \mathbb{C} \); \( i_0 = 1 \) and \( i_1, i_2, i_3 \) stand for the quaternionic imaginary units. By definition, the complex imaginary unit in \( \mathbb{C} \), denoted by \( i \), commutes with all the quaternionic imaginary units.

For \( q = \sum_{k=0}^{3} q_k i_k \in \mathbb{H}(\mathbb{C}) \), we will write \( q_0 = : \text{Sc}(q) \), \( \vec{q} := \sum_{k=1}^{3} q_k i_k = : \text{Vec}(q) \), so \( q = q_0 + \vec{q} \). We call \( q_0 \) and \( \vec{q} \) the scalar and vector parts of \( q \), respectively. Then \( \{\text{Vec}(q) : q \in \mathbb{H}(\mathbb{C})\} \) is identified with \( \mathbb{C}^3 \).

For any \( p, q \in \mathbb{H}(\mathbb{C}) \):

\[
p \cdot q = p_0 q_0 - \langle \vec{p}, \vec{q} \rangle + p_0 \vec{q} + q_0 \vec{p} + [\vec{p}, \vec{q}].
\]

where

\[
\langle \vec{p}, \vec{q} \rangle = \sum_{k=1}^{3} p_k q_k, \quad [\vec{p}, \vec{q}] := \left| \begin{array}{ccc} i_1 & i_2 & i_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{array} \right|.
\]

In particular, if \( p_0 = q_0 = 0 \) then \( p \cdot q = -\langle \vec{p}, \vec{q} \rangle + [\vec{p}, \vec{q}] \).

The Moisil–Teodorescu operator, denoted by \( D_{MT} \), is defined to be

\[
D_{MT}[f] := i_1 \frac{\partial f}{\partial x} + i_2 \frac{\partial f}{\partial y} + i_3 \frac{\partial f}{\partial z}. \tag{16} \]

It is worth noting that \( D_{MT} \) is the resemblance to \( V \) and factorizes the scalar Laplacian. In fact, it holds

\[
-D_{MT}^2 = \Delta_{\mathbb{R}^3}, \tag{17} \]

which implies several advantages in the applications to physical problems.

The operator \( \Delta_{\mathbb{R}^3} \) is a scalar operator; it acts separately on every coordinate function \( f_k \) of \( f \) as

\[
\Delta_{\mathbb{R}^3}[f] := \Delta[f_0] + i_1 \Delta[f_1] + i_2 \Delta[f_2] + i_3 \Delta[f_3].
\]

This property guarantees that any hyperholomorphic function is also harmonic.

For an \( \mathbb{H}(\mathbb{C}) \)-valued function \( f := f_0 + \vec{f} \), the action of the operator \( D_{MT} \) can be represented as follows:

\[
D_{MT}[f] = -\text{div}[\vec{f}] + \text{grad}[f_0] + \text{curl}[\vec{f}]. \tag{18} \]

This is an immediate consequence of the quaternionic product. Moisil–Teodorescu operator can act on the right in which case the notation \( D'_{MT}[f] \) for the same \( f = f_0 + \vec{f} \) means that

\[
D'_{MT}[f] := \frac{\partial f}{\partial x} i_1 + \frac{\partial f}{\partial y} i_2 + \frac{\partial f}{\partial z} i_3. \tag{19} \]
3 | LOCAL FRACTIONAL MOISIL–TEODORESCU OPERATOR IN CANTOR-TYPE COORDINATES

3.1 | The Cantor-type cylindrical coordinate system

Consider the coordinate system of the Cantor-type cylindrical coordinates developed in other studies.35,36,40,41

Consider the Cantor-type cylindrical coordinates

\[
\begin{align*}
  x^a &= r^a \cos(\theta^a), \\
  y^a &= r^a \sin(\theta^a), \\
  z^a &= z^a.
\end{align*}
\]  

If we denote by

\[
\begin{align*}
  e^{\theta}_r &= \cos(\theta) i_1 + \sin(\theta) i_2, \\
  e^{\phi}_\theta &= -\sin(\theta) i_1 + \cos(\theta) i_2, \\
  e^{\phi}_\phi &= i_3,
\end{align*}
\]

then the vector operations gradient, divergence, and curl in orthogonal curvilinear coordinates can be expressed as follows:

If \( f = f_0 \) a scalar function, then one has

\[
\text{grad}^{\alpha}[f_0] = \frac{\partial^\alpha f_0}{\partial r^a} e^a_r + \frac{1}{r^a} \frac{\partial^\alpha f_0}{\partial \theta^a} e^a_\theta + \frac{\partial^\alpha f_0}{\partial z^a} e^a_\phi.
\]  

If \( f = f = f_1 e^a_r + f_2 e^a_\theta + f_3 e^a_\phi \), then we have

\[
\text{div}^{\alpha}[f] = \frac{\partial^\alpha f_1}{\partial r^a} + \frac{1}{r^a} \frac{\partial^\alpha f_2}{\partial \theta^a} + \frac{\partial^\alpha f_3}{\partial z^a}.
\]  

and

\[
\text{curl}^{\alpha}[f] = \left[ \frac{1}{r^a} \frac{\partial^\alpha f_3}{\partial \theta^a} - \frac{\partial^\alpha f_2}{\partial z^a} \right] e^a_r + \left[ \frac{\partial^\alpha f_1}{\partial z^a} - \frac{\partial^\alpha f_3}{\partial r^a} \right] e^a_\theta
\]

\[+ \left[ \frac{\partial^\alpha f_2}{\partial r^a} - \frac{1}{r^a} \frac{\partial^\alpha f_1}{\partial \theta^a} + \frac{f_3}{r^a} \right] e^a_\phi.
\]

3.2 | The Cantor-type spherical coordinate system

Consider the coordinate system of the Cantor-type spherical coordinates developed in other studies.35,36,40,41

Consider the Cantor-type spherical coordinates

\[
\begin{align*}
  x^a &= r^a \sin(\theta^a) \cos(\phi^a), \\
  y^a &= r^a \sin(\theta^a) \sin(\phi^a), \\
  z^a &= r^a \cos(\theta^a).
\end{align*}
\]

If we denote by

\[
\begin{align*}
  e^{\phi}_r &= \sin(\phi) i_1 + \sin(\phi \cos(\phi) i_2 + \cos(\phi \cos(\phi) i_3, \\
  e^{\phi}_\theta &= \cos(\phi) i_1 + \cos(\phi \sin(\phi) i_2 + \sin(\phi \sin(\phi) i_3, \\
  e^{\phi}_\phi &= -\sin(\phi) i_1 + \cos(\phi) i_2,
\end{align*}
\]

then the vector operations gradient, divergence, and curl in orthogonal curvilinear coordinates can be expressed as follows:

If \( f = f_0 \) a scalar function, then one has

\[
\text{grad}^{\alpha}[f_0] = \frac{\partial^\alpha f_0}{\partial r^a} e^a_r + \frac{1}{r^a} \frac{\partial^\alpha f_0}{\partial \theta^a} e^a_\theta + \frac{1}{r^a \sin(\theta^a)} \frac{\partial^\alpha f_0}{\partial \phi^a} e^a_\phi.
\]
If \( f = \tilde{f} = f_1 \mathbf{e}_r + f_2 \mathbf{e}_\theta + f_3 \mathbf{e}_\phi \), then we have

\[
\text{div}^{(\alpha)}[\tilde{f}] = \frac{\partial^{\alpha} f_1}{\partial r^\alpha} + \frac{2 f_1}{r^\alpha} + \frac{1}{r} \frac{\partial^{\alpha} f_2}{\partial \theta^\alpha} + \frac{1}{r^2 \sin \alpha} \left( \frac{\partial^{\alpha} f_3}{\partial \psi^\alpha} + f_2 \cos \alpha \right),
\]

(28)

and

\[
\text{curl}^{(\alpha)}[\tilde{f}] = \left[ \frac{1}{r^\alpha} \frac{\partial^{\alpha} f_1}{\partial \theta^\alpha} + \frac{1}{r^2 \sin \alpha} \frac{\partial^{\alpha} f_2}{\partial \psi^\alpha} + \frac{f_3 \cos \alpha}{r^2 \sin \alpha} \right] \mathbf{e}_r
\]

\[
+ \left[ \frac{1}{r^2 \sin \alpha} \frac{\partial^{\alpha} f_1}{\partial \psi^\alpha} + \frac{1}{r} \frac{\partial^{\alpha} f_2}{\partial \theta^\alpha} - \frac{\partial^{\alpha} f_3}{\partial r^\alpha} \right] \mathbf{e}_\theta
\]

\[
+ \left[ \frac{\partial^{\alpha} f_2}{\partial r^\alpha} - \frac{1}{r^2} \frac{\partial^{\alpha} f_1}{\partial \theta^\alpha} + \frac{f_2}{r^2} \right] \mathbf{e}_\phi.
\]

(29)

### 3.3 Local fractional Moisil–Teodorescu operator in Cantor-type cylindrical and spherical coordinates

Taking advantage the results from the previous two subsections, we have that the local fractional Moisil–Teodorescu operator for a quaternion-valued function in Cantor-type cylindrical coordinates is given by

\[
D^{(\alpha)}_{MT}[f] = - \text{div}^{(\alpha)}[\tilde{f}] + \text{grad}^{(\alpha)}[f_0] + \text{curl}^{(\alpha)}[\tilde{f}]
\]

\[
= - \left[ \frac{\partial^{\alpha} f_1}{\partial r^\alpha} + \frac{2 f_1}{r^\alpha} + \frac{1}{r} \frac{\partial^{\alpha} f_2}{\partial \theta^\alpha} + \frac{1}{r^2 \sin \alpha} \left( \frac{\partial^{\alpha} f_3}{\partial \psi^\alpha} + f_2 \cos \alpha \right) \right] \mathbf{e}_r
\]

\[
+ \left[ \frac{1}{r^2 \sin \alpha} \frac{\partial^{\alpha} f_1}{\partial \psi^\alpha} + \frac{1}{r} \frac{\partial^{\alpha} f_2}{\partial \theta^\alpha} - \frac{\partial^{\alpha} f_3}{\partial r^\alpha} \right] \mathbf{e}_\theta
\]

\[
+ \left[ \frac{\partial^{\alpha} f_2}{\partial r^\alpha} - \frac{1}{r^2} \frac{\partial^{\alpha} f_1}{\partial \theta^\alpha} + \frac{f_2}{r^2} \right] \mathbf{e}_\phi.
\]

(30)

while in Cantor-type spherical coordinates is given by

\[
D^{(\alpha)}_{MT}[f] = - \text{div}^{(\alpha)}[\tilde{f}] + \text{grad}^{(\alpha)}[f_0] + \text{curl}^{(\alpha)}[\tilde{f}]
\]

\[
= - \left[ \frac{\partial^{\alpha} f_1}{\partial r^\alpha} + \frac{2 f_1}{r^\alpha} + \frac{1}{r} \frac{\partial^{\alpha} f_2}{\partial \theta^\alpha} + \frac{1}{r^2 \sin \alpha} \left( \frac{\partial^{\alpha} f_3}{\partial \psi^\alpha} + f_2 \cos \alpha \right) \right] \mathbf{e}_r
\]

\[
+ \left[ \frac{1}{r^2 \sin \alpha} \frac{\partial^{\alpha} f_1}{\partial \psi^\alpha} + \frac{1}{r} \frac{\partial^{\alpha} f_2}{\partial \theta^\alpha} - \frac{\partial^{\alpha} f_3}{\partial r^\alpha} \right] \mathbf{e}_\theta
\]

\[
+ \left[ \frac{\partial^{\alpha} f_2}{\partial r^\alpha} - \frac{1}{r^2} \frac{\partial^{\alpha} f_1}{\partial \theta^\alpha} + \frac{f_2}{r^2} \right] \mathbf{e}_\phi.
\]

(31)

### 3.4 Local fractional Laplacian in Cantor-type cylindrical and spherical coordinates

The local fractional Moisil–Teodorescu operator \( D^{(\alpha)}_{MT} \) factorizes the Laplacian \( \Delta^{(\alpha)}_\mathbb{H} \) as follows:

\[
\left( D^{(\alpha)}_{MT} \right)^2 [f] = - \Delta^{(\alpha)}_\mathbb{H} [f].
\]

(32)

where

\[
\Delta^{(\alpha)}_\mathbb{H} [f] := \Delta_0^{(\alpha)} [f_0] + \tilde{\Delta}^{(\alpha)} [\tilde{f}].
\]

(33)
with $\Delta^{(a)}_{0}[f_0] := \text{div}^{(a)}[\text{grad}^{(a)}[f_0]]$ and $\tilde{\Delta}^{(a)}[\tilde{f}] := \text{grad}^{(a)}[\text{div}^{(a)}[\tilde{f}]] - \text{curl}^{(a)}[\text{curl}^{(a)}[\tilde{f}]]$.

Then, in Cantor-type cylindrical coordinates the previous operators take the form:

$$
\Delta^{(a)}_{0}[f_0] = \frac{\partial^2 f_0}{\partial r^{2a}} + \frac{1}{r^{2a}} \frac{\partial^2 f_0}{\partial \theta^{2a}} + \frac{1}{r^a} \frac{\partial^a f_0}{\partial r^a} + \frac{\partial^2 f_0}{\partial z^{2a}},
$$

and

$$
\tilde{\Delta}^{(a)}[\tilde{f}] = \left[ \frac{\partial^2 f_1}{\partial r^{2a}} + \frac{1}{r^{2a}} \frac{\partial^2 f_1}{\partial \theta^{2a}} - \frac{1}{r^{2a}} \frac{\partial^2 f_1}{\partial \phi^{2a}} - \frac{2}{r^a} \frac{\partial^a f_1}{\partial r^a} + \frac{\partial^2 f_1}{\partial z^{2a}} \right] \mathbf{e}_r^a
$$

Moreover, in Cantor-type spherical coordinates these operators take the form:

$$
\Delta^{(a)}_{0}[f_0] = \frac{\partial^2 f_0}{\partial r^{2a}} + \frac{2}{r^a} \frac{\partial^a f_0}{\partial r^a} + \frac{1}{r^{2a}} \frac{\partial^2 f_0}{\partial \theta^{2a}} + \frac{\cos_a(\theta^a)}{r^{2a} \sin_a(\theta^a)} \frac{\partial^a f_0}{\partial \theta^a}
$$

and

$$
\tilde{\Delta}^{(a)}[\tilde{f}] = \left[ \frac{\partial^2 f_1}{\partial r^{2a}} + \frac{2}{r^a} \frac{\partial^a f_1}{\partial r^a} - \frac{1}{r^{2a}} \frac{\partial^2 f_1}{\partial \theta^{2a}} - \frac{2 \cos_a(\theta^a)}{r^{2a} \sin_a(\theta^a)} \frac{\partial^a f_1}{\partial \theta^a} \right] \mathbf{e}_r^a
$$

3.5 Local fractional Bitsadze operator

The local fractional quaternionic Bitsadze operator reads as

$$
\Delta^{(a)}_{\frac{\partial}{\partial t}}[f] := \Delta^{(a)}_{0}[f_0] + \tilde{\Delta}^{(a)}[\tilde{f}],
$$

with $\tilde{\Delta}^{(a)}[\tilde{f}] := \text{grad}^{(a)}[\text{div}^{(a)}[\tilde{f}]] + \text{curl}^{(a)}[\text{curl}^{(a)}[\tilde{f}]]$.

Notice that $D_{MT}^{(a)}$ and $D_{MT}^{(a)}$ factorize the local fractional quaternionic Bitsadze operator $\Delta^{(a)}_{\frac{\partial}{\partial t}}$ as follows:

$$
D_{MT}^{(a)} D_{MT}^{(a)}[f] = -\Delta^{(a)}_{\frac{\partial}{\partial t}}[f].
$$

Our purpose is to derive the local fractional quaternionic Bitsadze operator on the Cantor sets by using the Cantor-type cylindrical and spherical coordinates, which extend the quaternionic Bitsadze operator of Bory and Pérez-De La Rosa based upon the standard derivative operators.
We are reduced to handle the corresponding local fractional expressions of $\tilde{\Delta}^{(\alpha)}$ in Cantor-type cylindrical and spherical coordinates.

### 3.5.1 Cantor-type cylindrical coordinates

\[
\tilde{\Delta}^{(\alpha)}[f] = \left[ \frac{\partial^2 f_1}{\partial r^{2\alpha}} + \frac{2}{r^2} \frac{\partial^2 f_2}{\partial r^2 \partial \theta^\alpha} + \frac{1}{r^2} \frac{\partial^\alpha f_1}{\partial \theta^\alpha} - \frac{f_1}{r^{2\alpha}} + 2 \frac{\partial^2 f_3}{\partial r^2 \partial z^\alpha} - 1 \frac{\partial^2 f_1}{r^{2\alpha}} \right] \mathbf{e}_r \\
+ \left[ 2 \frac{\partial^2 f_1}{r^2 \partial r^2 \partial \theta^\alpha} + \frac{1}{r^2} \frac{\partial^2 f_2}{\partial \theta^2 \partial z^\alpha} \right] \mathbf{e}_\theta \\
+ \left[ 2 \frac{\partial^2 f_1}{r^2 \partial r^2 \partial z^\alpha} + \frac{2}{r^2} \frac{\partial^2 f_2}{\partial \theta^2 \partial z^\alpha} - \frac{2 \partial^\alpha f_1}{r^2 \partial \theta^2 \partial z^\alpha} - \frac{\partial^2 f_2}{r^{2\alpha}} \right] \mathbf{e}_z.
\]

### 3.5.2 Cantor-type spherical coordinates

\[
\tilde{\Delta}^{(\alpha)}[f] = \left[ \frac{\partial^2 f_1}{\partial r^{2\alpha}} + \frac{2}{r^2} \frac{\partial^2 f_2}{\partial r^2 \partial \theta^\alpha} - \frac{2}{r^{2\alpha}} f_1 - \frac{1}{r^2} \frac{\partial^2 f_1}{\partial \theta^2 \partial \phi^\alpha} - \frac{\cos \theta^\alpha}{r^2 \sin \alpha^\alpha} \frac{\partial^\alpha f_1}{\partial \theta^\alpha} \right] \mathbf{e}_r \\
+ \left[ \frac{2}{r^2 \sin \alpha^\alpha} \frac{\partial^2 f_3}{\partial \theta^2 \partial \phi^\alpha} \mathbf{e}_\theta \\
+ \left[ - \frac{\partial^2 f_2}{r^2 \partial r^2 \partial \phi^\alpha} - \frac{2}{r^2} \frac{\partial^2 f_2}{\partial \phi^2 \partial \theta^\alpha} - \frac{1}{r^2 \sin \alpha^\alpha} \frac{\partial^2 f_3}{\partial \phi^2 \partial \theta^\alpha} - \frac{1}{r^2} \frac{\partial^2 f_1}{\partial \theta^2 \partial \phi^\alpha} \right] \mathbf{e}_\phi \\
+ \left[ \frac{1}{r^2 \sin \alpha^\alpha} \frac{\partial^2 f_3}{\partial \theta^2 \partial \phi^\alpha} \mathbf{e}_\lambda \right] \right] \mathbf{e}_\lambda.
\]

### 4 THE CANTOR-TYPE CYLINDRICAL AND SPHERICAL-COORDINATE METHODS TO THE LOCAL FRACTIONAL QUATERNIONIC HELMHOLTZ EQUATION

In this section, we derive interesting formulas for the component equations of the local fractional quaternionic Helmholtz equation

\[
\Delta^{(\alpha)}_{\mathbb{H}}[f] + \lambda^2 f = 0, \quad \lambda \in \mathbb{C},
\]

on the Cantor sets by using the Cantor-type cylindrical and spherical coordinate methods.
We have
\[
- \left( D^{(a)}_{MT} - \lambda I \right) \left( D^{(a)}_{MT} + \lambda I \right) = \Delta^{(a)}_{\mathbb{H}} + \lambda^2 I, \tag{43}
\]
where \( I \) denotes the identity operator.

Seen this factorization, the null solutions of the operator \( D^{(a)}_{MT} + \lambda I \), called the perturbed local fractional Moisil–Teodorescu operator, are special solutions of (42).

On substituting Equations (34) and (35) into Equation (40), we get the local fractional quaternionic Helmholtz equation in the Cantor-type cylindrical coordinates.

\[
0 = \left[ \frac{\partial^2 f_0}{\partial r^{2a}} + \frac{2}{r^a} \frac{\partial f_0}{\partial r^a} + \frac{1}{r^{2a}} \frac{\partial^2 f_0}{\partial \theta^{2a}} + \frac{\cos \alpha (\theta^a)}{r^{2a} \sin \alpha (\theta^a)} \frac{\partial^2 f_0}{\partial \psi^{2a}} + \lambda^2 f_0 \right]
\]
\[
+ \left[ \frac{\partial^2 f_1}{\partial r^{2a}} + \frac{2}{r^a} \frac{\partial f_1}{\partial r^a} - \frac{2}{r^{2a}} f_1 + \frac{\cos \alpha (\theta^a)}{r^{2a} \sin \alpha (\theta^a)} \frac{\partial^2 f_1}{\partial \theta^{2a}} + \lambda^2 f_1 \right] e^a
\]
\[
+ \left[ \frac{\partial^2 f_2}{\partial r^{2a}} + \frac{2}{r^a} \frac{\partial f_2}{\partial r^a} + \frac{2}{r^{2a}} \frac{\partial^2 f_2}{\partial \theta^{2a}} + \frac{2}{r^{2a}} \frac{\partial^2 f_2}{\partial \psi^{2a}} + \lambda^2 f_2 \right] e^a
\]
\[
+ \left[ \frac{\partial^2 f_3}{\partial r^{2a}} + \frac{1}{r^a} \frac{\partial f_3}{\partial r^a} + \frac{1}{r^{2a}} \frac{\partial^2 f_3}{\partial \theta^{2a}} + \lambda^2 f_3 \right] e^a. \tag{44}
\]

Equating each component in (44) to zero, we get

\[
\begin{align*}
\Delta^{(a)}_0 [f_0] + \lambda^2 f_0 &= 0, \\
\Delta^{(a)}_0 [f_1] - \frac{2}{r^a} \frac{\partial f_1}{\partial \psi^a} + \left( \lambda^2 - \frac{1}{r^a} \right) f_1 &= 0, \\
\Delta^{(a)}_0 [f_2] + \frac{2}{r^a} \frac{\partial f_2}{\partial \psi^a} + \left( \lambda^2 - \frac{1}{r^a} \right) f_2 &= 0, \\
\Delta^{(a)}_0 [f_3] + \lambda^2 f_3 &= 0. \tag{45}
\end{align*}
\]

One can notice that the components of the local fractional quaternionic Helmholtz operator in Cantor-type cylindrical coordinates are different from the corresponding local fractional scalar Helmholtz operator (there are several extra terms).

In a similar way, substituting Equations (36) and (37) into Equation (42), we obtain the local fractional quaternionic Helmholtz equation in the Cantor-type spherical coordinates.

\[
0 = \left[ \frac{\partial^2 f_0}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial f_0}{\partial \rho^a} + \frac{1}{\rho^2} \frac{\partial^2 f_0}{\partial \theta^2 a} + \frac{\cos \theta (\theta^a)}{\rho^2 \sin \theta (\theta^a)} \frac{\partial^2 f_0}{\partial \phi^2 a} + \lambda^2 f_0 \right]
\]
\[
+ \left[ \frac{\partial^2 f_1}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial f_1}{\partial \rho^a} + \frac{2}{\rho^2} \frac{\partial^2 f_1}{\partial \theta^2 a} + \frac{2}{\rho^2} \frac{\partial^2 f_1}{\partial \phi^2 a} + \lambda^2 f_1 \right] e^a
\]
\[
+ \left[ \frac{\partial^2 f_2}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial f_2}{\partial \rho^a} + \frac{2}{\rho^2} \frac{\partial^2 f_2}{\partial \theta^2 a} + \frac{2}{\rho^2} \frac{\partial^2 f_2}{\partial \phi^2 a} + \lambda^2 f_2 \right] e^a
\]
\[
+ \left[ \frac{\partial^2 f_3}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f_3}{\partial \rho^a} + \frac{1}{\rho^2} \frac{\partial^2 f_3}{\partial \theta^2 a} + \lambda^2 f_3 \right] e^a. \tag{46}
\]
Now equate each component in (46) to zero; we get

\[
\begin{align*}
\Delta_0^{(\alpha)} [f_0] + \lambda^2 f_0 &= 0, \\
\Delta_0^{(\alpha)} [f_1] - \frac{2}{r^2} \frac{\partial^2 f_2}{\partial \theta^2} - \frac{2 \cos(\theta)}{r^2 \sin(\theta)} \frac{\partial f_2}{\partial \phi} - \frac{2}{r^2 \sin(\theta)} \frac{\partial f_3}{\partial \phi} \\
&+ \left( \lambda^2 - \frac{2}{r^2} \right) f_1 = 0, \\
\Delta_0^{(\alpha)} [f_2] + \frac{2}{r^2} \frac{\partial^2 f_1}{\partial \theta^2} - \frac{2 \cos(\theta)}{r^2 \sin^2(\theta)} \frac{\partial^2 f_1}{\partial \phi^2} \\
&+ \left( \lambda^2 - \frac{1}{r^2 \sin^2(\theta)} \right) f_2 = 0, \\
\Delta_0^{(\alpha)} [f_3] + \frac{2}{r^2 \sin(\theta)} \frac{\partial^2 f_1}{\partial \phi^2} + \frac{2 \cos(\theta)}{r^2 \sin^2(\theta)} \frac{\partial f_3}{\partial \phi} \\
&+ \left( \lambda^2 - \frac{1}{r^2 \sin^2(\theta)} \right) f_3 = 0.
\end{align*}
\]

(47)

Again, notice that the components of the local fractional quaternionic Helmholtz operator in Cantor-type spherical coordinates are different from the corresponding local fractional scalar Helmholtz operator.

Observe that the first component equation in (44) shows that the Helmholtz equation in Cantor-type cylindrical coordinates obtained in Hao et al.\textsuperscript{[26]} has exactly the form of the scalar part of the local fractional quaternionic Helmholtz equation in the Cantor-type cylindrical coordinates.

Similarly, the first component in equation (46) reflects that the homogeneous version of the Helmholtz equation in Cantor-type spherical coordinates obtained in Rahmat et al.\textsuperscript{[35]} is identical with the scalar part of the local fractional quaternionic Helmholtz equation in the Cantor-type spherical coordinates.

5 | CONCLUDING REMARKS

In the present work, we developed a general quaternionic structure for the local fractional Moisil–Teodorescu operator in Cantor-type cylindrical and spherical coordinate systems. The mathematical model introduced in this work emerges as a good generalization of the standard local fractional vector calculus in Cantor-type cylindrical and spherical coordinates. Furthermore, we presented two examples for the Helmholtz equation with local fractional derivatives on the Cantor sets by making use of the local fractional Moisil–Teodorescu operator that reveal the capacity and adaptability of the methods.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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