Flavor structure from magnetic fluxes and non-Abelian Wilson lines

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Abstract

We study the flavor structure of 4D effective theories, which are derived from extra dimensional theories with magnetic fluxes and non-Abelian Wilson lines. We study zero-mode wavefunctions and compute Yukawa couplings as well as four-point couplings. In our models, we also discuss non-Abelian discrete flavor symmetries such as $D_4$, $\Delta(27)$ and $\Delta(54)$.

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1 Introduction

Recently, extra dimensional field theory, in particular string-derived one, plays an important role in particle physics and cosmology. When we start with extra dimensional field theory, it is one of most important issues how to realize a chiral spectrum in four dimensional (4D) effective field theory. The magnetic flux background is one of interesting ways to realize a 4D chiral theory. Indeed, several field-theoretical models and string models, i.e. magnetized D-brane models, have been studied \cite{1, 2, 3, 4, 5, 6, 7, 8, 9}. Furthermore, magnetized D-brane models are T-duals of intersecting D-brane models. Within the latter type of model building, a number of interesting models have been constructed \cite{4, 5, 6, 10, 11, 12}.

Wavefunction profiles of zero-modes are quasi-localized on the torus with magnetic flux background. The number of zero-modes is determined by the size of magnetic flux. Since we know zero-mode profile explicitly, we can compute concretely 3-point and higher order couplings of 4D effective field theory by overlap integrals of zero-mode profiles \cite{7, 14, 15, 16, 17}. That is an important aspect of magnetized extra dimensional models. Moreover, such a 4D effective field theory can have Abelian and non-Abelian discrete flavor symmetries, which are originated from localization behavior of zero-modes in extra dimensions \cite{18} \cite{2}.

In addition to magnetic fluxes, we can introduce constant gauge backgrounds and non-trivial twisted boundary conditions as well as orbifold boundary conditions \cite{7, 23, 24, 25} \cite{8}. That leads richer structure in model building such as zero-mode spectra and zero-mode profiles.

Non-Abelian Wilson lines, i.e. the so-called toron backgrounds \cite{29}, are also interesting backgrounds \cite{30, 7, 31, 32, 33}. They can break gauge groups with reducing their ranks. For a certain case with magnetic fluxes and non-Abelian Wilson lines, zero-mode profiles have been given \cite{7}. Our purpose of this paper is to study more about models with magnetic fluxes and non-Abelian Wilson lines. We analyze zero-mode profiles in generic case and compute 3-point couplings. Furthermore, we study flavor symmetries. We also study the orbifold compactification.

This paper is organized as follows. In section 2, we give a brief review on the extra dimensional models with magnetic fluxes and non-Abelian Wilson lines. In section 3, we study zero-mode wavefunctions on the torus compactification with magnetic fluxes and non-Abelian Wilson lines. We compute Yukawa couplings in section 4 and study flavor symmetries of our models in section 5. In section 6, we also study the orbifold compactification. Section 7 is devoted to conclusion and discussion. In Appendix A we show useful calculations, which are relevant to Yukawa couplings and in Appendix B we compute four-point couplings as an example of higher order couplings.

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1 See for a review \cite{13} and references therein.
2 Similar non-Abelian discrete flavor symmetries have been derived within the framework of heterotic orbifold models \cite{19, 20, 21}. Analysis on their anomalies are also important \cite{22}.
3 Geometrical backgrounds other than tori and toroidal orbifolds have also been studied \cite{26, 27, 28}. 

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2 Non-Abelian Wilson lines

2.1 Higher dimensional super Yang-Mills theory

Our starting point is $N = 1 \ U(N)$ super Yang-Mills theory in $D = 4 + 2n$ dimensions with $n = 1$ or 3. Its Lagrangian is written by

$$\mathcal{L} = -\frac{1}{4g^2} \text{Tr} (F^{MN} F_{MN}) + \frac{i}{2g^2} \text{Tr} (\bar{\lambda} \Gamma^M D_M \lambda),$$

(1)

where $M, N = 0, \cdots, (D-1)$. Here, $\lambda$ denotes gaugino fields, $\Gamma^M$ is the gamma matrix for $D$ dimensions and the covariant derivative $D_M$ is given as

$$D_M \lambda = \partial_M \lambda - i [A_M, \lambda],$$

(2)

where $A_M$ is the vector field.

Here, we consider the torus $(T^2)^n$ as $2n$-dimensional extra dimensions and denote their coordinates by $y_m (m = 4, \cdots, 2n + 3)$. We use the orthogonal coordinates and choose the torus metric such that $y_m$ is identified by $y_m + 1$ on the torus. The gaugino fields $\lambda$ and the vector fields $A_m$ corresponding to the compact directions are decomposed as

$$\lambda(x, y) = \sum_n \chi_n(x) \otimes \psi_n(y),$$

(3)

$$A_m(x, y) = \sum_n \varphi_{n,m}(x) \otimes \phi_{n,m}(y),$$

(4)

where $x$ denotes the coordinates of four-dimensional uncompact space $R^{3,1}$. Here, we are interested only in zero-modes, $\psi_0(y)$ and $\phi_{0,m}(y)$. Thus, we omit the mode index corresponding to $n = 0$ and write them as $\psi(y)$ and $\phi_m(y)$.

2.2 Non-Abelian Wilson lines

Here, we consider $T^2$ of $(T^2)^n$, whose coordinates are denoted as $(y_4, y_5)$. As a $U(N)$ gauge background, we introduce the following form of (Abelian) magnetic flux,

$$F_{45} = 2\pi \begin{pmatrix} f_a & 1_{N_a} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(5)

where $1_{N_a}$ denotes $(N_a \times N_a)$ identity matrix. For example, we use the following gauge,

$$A_4 = -F_{45} y_5, \quad A_5 = 0,$$

(6)

for the $U(N_a)$ part. Then, their boundary conditions can be written as

$$A_m(y_4 + 1, y_5) = A_m(y_4, y_5) + \begin{pmatrix} \partial_m \chi_4 & 1_{N_a} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \chi_4 = 0,$$

(7)

$$A_m(y_4, y_5 + 1) = A_m(y_4, y_5) + \begin{pmatrix} \partial_m \chi_5 & 1_{N_a} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \chi_5 = -2\pi f_a y_4.$$
This background breaks the gauge group $U(N)$ to $U(N_a) \times U(N - N_a)$. The zero mode $\psi(y)$ corresponding to the gaugino is also decomposed as

$$\psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

depending on their $U(N_a) \times U(N - N_a)$ charges. That is, $A$ and $D$ correspond to the gaugino fields of unbroken symmetries, $U(N_a)$ and $U(N - N_a)$, respectively, while $B$ and $C$ correspond to bi-fundamental representations, $(N_a, N - N_a)$ and $(N_a, N - N_a)$, respectively.

Since only the $U(1)$ part of $U(N_a)$ has the non-trivial background, its charge $q$ is relevant, that is, $A, B, C$ and $D$ have charges $q = 0, 1, -1$ and 0, respectively. For example, the zero-mode of $B$ elements satisfies the following equation,

$$\tilde{\Gamma}^m (\partial_m - i A_m) B(y) = 0,$$

for $m = 4, 5$, where $A_m$ denotes the $U(N_a)$ gauge background [3]. Also, the zero-mode of $C$ elements satisfies $\tilde{\Gamma}^m (\partial_m + i A_m) C(y) = 0$, while the zero-modes of $A$ and $D$ elements satisfy $\tilde{\Gamma}^m \partial_m A(y) = 0$ and $\tilde{\Gamma}^m \partial_m D(y) = 0$. Here, $\tilde{\Gamma}^m$ corresponds to the gamma matrix for the two-dimensional torus $T^2$, e.g.

$$\tilde{\Gamma}^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\Gamma}^5 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and $\psi(y)$ is the two component spinor,

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix},$$

that is, $A, B, C$ and $D$ also have two components, $A_{\pm}, B_{\pm}, C_{\pm}$ and $D_{\pm}$.

In particular, we are interested in matter fields. Thus, let us concentrate on $B$ and $C$ fields. Because of (7), the spinor field, e.g. $B$, satisfies the following boundary conditions,

$$B(y_4 + 1, y_5) = e^{i\chi_4} B(y_4, y_5),$$

$$B(y_4, y_5 + 1) = e^{i\chi_5} B(y_4, y_5).$$

Here, we write these boundary conditions as

$$B(y_4 + 1, y_5) = \Omega_4(y_4, y_5) B(y_4, y_5),$$

$$B(y_4, y_5 + 1) = \Omega_5(y_4, y_5) B(y_4, y_5).$$

The above case corresponds to $\Omega_4(y_4, y_5) = e^{i\chi_4}$ and $\Omega_5(y_4, y_5) = e^{i\chi_5}$. Then, the consistency for the contractible loop, i.e. $(y_4, y_5) \rightarrow (y_4 + 1, y_5) \rightarrow (y_4 + 1, y_5 + 1) \rightarrow (y_4, y_5 + 1) \rightarrow (y_4, y_5)$ requires

$$(\Omega_5^{-1}(y_4, y_5 + 1) \Omega_4^{-1}(y_4 + 1, y_5 + 1) \Omega_5(y_4 + 1, y_5) \Omega_4(y_4, y_5)) B(y_4, y_5) = B(y_4, y_5).$$
The left hand side reduces to $e^{-2\pi i f_a}\psi(y_4, y_5)$ in the above background. This condition for $B$ leads to the quantization condition of the magnetic flux $f_a$. That is, the magnetic flux $f_a$ should be quantized such that $f_a = \text{integer}$. The consistency condition for the $C$ fields also leads to the same quantization condition, i.e. $f_a = \text{integer}$.

When we introduce a non-trivial background for the $SU(N_a)$ part of $U(N_a)$, the situation changes. That modifies the boundary conditions on, for example, $B$,

$$B(y_4 + 1, y_5) = \Omega_4(y_4, y_5)B(y_4, y_5) = e^{i\chi_4}\omega_4 B(y_4, y_5), \quad (17)$$
$$B(y_4, y_5 + 1) = \Omega_5(y_4, y_5)B(y_4, y_5) = e^{i\chi_5}\omega_5 B(y_4, y_5), \quad (18)$$

where $\omega_m$ are constant elements of $SU(N_a)$. Then, the consistency condition (16) reduces to

$$\omega_5^{-1}\omega_4^{-1}\omega_5\omega_4 e^{-2\pi i f_a} = 1_{N_a}. \quad (19)$$

If $\omega_4$ and $\omega_5$ commute each other, that would require again $e^{-2\pi i f_a} = 1$. Thus, it is interesting that $\omega_4$ and $\omega_5$ do not commute each other, that is, non-Abelian Wilson lines. In particular, we consider the case that $\omega_5^{-1}\omega_4^{-1}\omega_5\omega_4$ corresponds to the center of $SU(N_a)$, that is,

$$\omega_5^{-1}\omega_4^{-1}\omega_5\omega_4 = e^{2\pi i M_a/N_a}1_{N_a}, \quad (20)$$

where $M_a$ is an integer. In this case, the consistency condition (19) requires that the magnetic flux should satisfy $f_a = M_a/N_a \pmod 1$.

We denote $P_a = \text{g.c.d.}(M_a, N_a)$, $m_a = M_a/P_a$ and $n_a = N_a/P_a$\footnote{Here, g.c.d. denotes the greatest common divisor.} A solution of Eq. (20) is given as

$$\omega_4 = \hat{P}_a, \quad \omega_5 = \hat{Q}_a^{-m_a}, \quad (21)$$

where

$$\hat{P}_a = \begin{pmatrix} 0 & 1_{P_a} & 0 & 0 \\ 0 & 0 & 1_{P_a} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1_{P_a} & 0 & 0 & 0 \end{pmatrix}, \quad \hat{Q}_a = \rho^{(n_a-1)/2} \begin{pmatrix} 1_{P_a} & 0 & 0 & 0 \\ 0 & \rho^{1_{P_a}} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \rho^{n_a-1}1_{P_a} \end{pmatrix}, \quad (22)$$

with $\rho \equiv e^{2\pi i/n_a}$.

These non-Abelian Wilson lines break the gauge group $U(N_a)$ further. The following condition on the $U(N_a)$ gauge field,

$$A_\mu = w_4A_\mu\omega_4^{-1} = w_5A_\mu\omega_5^{-1}, \quad (23)$$

is required. Then, the gauge group $U(N_a)$ breaks to $U(P_a)$.\footnote{Here, g.c.d. denotes the greatest common divisor.}
3 Matter fields

Here, we consider the following form of $U(N)$ magnetic fluxes,

$$F_{45} = 2\pi \begin{pmatrix} f_1 1_{N_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_n 1_{N_n} \end{pmatrix}. \quad (24)$$

This form of magnetic fluxes breaks $U(N)$ to $\prod_i U(N_i)$ for $f_i =$ integer. Furthermore, the gauge group is broken to $\prod_i U(P_i)$ when we choose $f_i = M_i/N_i$ with $P_i = \text{g.c.d.}(M_i, N_i)$ and non-Abelian Wilson lines such that they satisfy the consistency condition like Eq. (16).

Now, let us focus on the $(N_a + N_b) \times (N_a + N_b)$ block in $U(N)$, which has the magnetic flux,

$$F = 2\pi \begin{pmatrix} m_a 1_{N_a} \\ m_b 1_{N_b} \end{pmatrix}. \quad (25)$$

We use the same gauge as Eq. (6), i.e.

$$A_4 = -2\pi \begin{pmatrix} m_a 1_{N_a} \\ m_b 1_{N_b} \end{pmatrix} y_5, \quad A_5 = 0. \quad (26)$$

Similarly to Eq. (7), we denote their boundary conditions as

$$\begin{align*}
A_m(y_4 + 1, y_5) &= A_m(y_4, y_5) + \begin{pmatrix} \partial_m \chi^a_4 1_{N_a} & 0 \\ 0 & \partial_m \chi^b_4 1_{N_b} \end{pmatrix}, \\
A_m(y_4, y_5 + 1) &= A_m(y_4, y_5) + \begin{pmatrix} \partial_m \chi^a_5 1_{N_a} & 0 \\ 0 & \partial_m \chi^b_5 1_{N_b} \end{pmatrix},
\end{align*} \quad (27)$$

where

$$\chi^a_4 = 0, \quad \chi^b_5 = -2\pi \frac{m_a}{n_a} y_4, \quad \chi^b_4 = 0, \quad \chi^b_5 = -2\pi \frac{m_b}{n_b} y_4. \quad (28)$$

We decompose the gaugino fields of this block in a way similar to Eq. (8). That is, $A$ and $D$ correspond to adjoint matter fields of $U(N_a)$ and $U(N_b)$, respectively, while $B$ and $C$ correspond to bi-fundamental representations, $(N_a, \overline{N}_b)$ and $(\overline{N}_a, N_b)$, respectively. Among them, we concentrate on the field $B$, which satisfies the boundary conditions

$$\begin{align*}
B(y_4 + 1, y_5) &= \Omega^a_4 B(y_4, y_5)(\Omega^b_4)^\dagger = e^{i(\chi^a_4 - \chi^b_4)} \omega^a_4 B(y_4, y_5)(\omega^b_4)^\dagger, \\
B(y_4, y_5 + 1) &= \Omega^a_5 B(y_4, y_5)(\Omega^b_5)^\dagger = e^{i(\chi^a_5 - \chi^b_5)} \omega^a_5 B(y_4, y_5)(\omega^b_5)^\dagger. \quad (29)
\end{align*}$$

Here, $\omega^{a,b}_4$ are non-Abelian Wilson lines, which are given as Eqs. (21) and (22). Then, the gauge symmetries are broken to $U(P_a)$ and $U(P_b)$. We study zero-mode profiles of $B$ fields in what follows.
3.1 Integer magnetic fluxes

Before considering the models with fractional magnetic fluxes and non-Abelian Wilson lines, it would be convenient to review briefly the models with integer magnetic fluxes and no Wilson lines, that is,

\[ n_a = n_b = 1, \quad w_4^{a,b} = w_5^{a,b} = 1. \]  

Then, the boundary condition (29) reduces to

\[
\begin{align*}
B_{pq}(y_4 + 1, y_5) &= B_{pq}(y_4, y_5), \\
B_{pq}(y_4, y_5 + 1) &= e^{-2\pi i m y_4} B_{pq}(y_4, y_5),
\end{align*}
\]

(31)

where \( m = m_a - m_b \). Suppose that \( m > 0 \). Then, the \( B_+ \) component for each \( B_{pq} \) element has \( m \) independent solutions for the zero-mode Dirac equation (9) with above boundary condition (31). These solutions are given by

\[
\Theta^j(y_4, y_5) = \sum_l e^{-m\pi(l + \frac{j}{m})^2 + 2\pi i m(l + \frac{j}{m}) y_4 - \pi m y_5^2 - 2\pi m(l + \frac{j}{m}) y_5} \vartheta \left( \frac{j}{m}, m \tau \right) (mz, m\tau),
\]

(32)

where \( z = y_4 + iy_5 \), \( j = 0, 1, \ldots, m-1 \) and \( \tau = i \). Here, \( \vartheta \left( \frac{j}{m}, m \tau \right) (mz, m\tau) \) denotes the Jacobi theta function. On the other hand, the \( B_- \) component has no normalizable zero-modes. Similarly, the \( C_- \) fields have the same solutions as \( B_+ \), but the \( C_+ \) has no normalizable zero-modes.

When \( m < 0 \), the \( B_- \) and \( C_+ \) fields have the \( |m| \) independent solutions with the same wavefunctions as above except replacing \( m \) by \( |m| \). However, the \( B_+ \) and \( C_- \) fields have no normalizable zero-modes.

3.2 Fractional magnetic fluxes

Here, we study zero-mode profiles in the models with fractional magnetic fluxes and non-Abelian Wilson lines.

3.2.1 \( n_a = n_b \)

First, let us study the magnetic flux (25) for \( n = n_a = n_b \). In this case, the non-Abelian Wilson lines break the gauge group \( U(N_a) \times U(N_b) \) to \( U(P_a) \times U(P_b) \), where \( P_a = N_a/n \) and \( P_b = N_b/n \). Following this breaking pattern, we decompose the fields \( B \) as

\[
B = \begin{pmatrix}
B_{00} & B_{01} & \cdots \\
B_{10} & B_{11} & \cdots \\
\cdots & \cdots & \cdots \\
B_{n-1,0} & B_{n-1,1} & \cdots & B_{n-1,n-1}
\end{pmatrix}.
\]

(33)
Each of $B_{pq}$ components is $(P_a \times P_b)$ matrix-valued fields, which correspond to bi-fundamental $(P_a, P_b)$ fields under $U(P_a) \times U(P_b)$. The boundary condition (29) due to the non-Abelian Wilson lines is written as

$$
B_{pq}(y_4 + 1, y_5) = B_{p+1,q+1}(y_4, y_5),
B_{pq}(y_4, y_5 + 1) = \rho^{-(m_a p - m_b q)} e^{-\frac{2\pi m}{n} y_4} B_{p,q}(y_4, y_5),
$$

(34)

where $m$ is used as $m = m_a - m_b$, and $B_{p+n,q} = B_{p,q+n} = B_{p,q}$. That leads to the boundary condition,

$$
B_{pq}(y_4 + n, y_5) = B_{pq}(y_4, y_5),
B_{pq}(y_4, y_5 + n) = e^{-2\pi i y_4} B_{pq}(y_4, y_5).
$$

(35)

Suppose that $mn > 0$. Then, similarly to section 3.1, the $B_+$ component for $B_{p,q}$ has $nm$ independent solutions for the zero-mode Dirac equation (9) with the above condition (35). These solutions are given by

$$
\Theta^j(y_4, y_5) = \sum_l e^{-nm \pi (l + \frac{j}{nm})^2 + 2\pi i m (l + \frac{j}{nm}) y_4 - \frac{\pi m}{n} y_5 - 2\pi m (l + \frac{j}{nm}) y_5} \theta_n \left[ \frac{j}{nm} \right] (mz, n m \tau),
$$

(36)

where $j = 0, 1, \ldots, nm - 1$ and $\tau = i$. On the other hand, the $B_-$ component has no normalizable zero-modes. One finds that these solutions satisfy the boundary conditions,

$$
\Theta^j(y_4 + 1, y_5) = e^{2\pi i j} \Theta^j(y_4, y_5),
\Theta^j(y_4, y_5 + 1) = e^{-2\pi i j} \Theta^{j+m}(y_4, y_5).
$$

(37)

Thus, the zero-mode solutions with the boundary conditions (34) due to non-Abelian Wilson lines can be written in terms of $\Theta^j$ of Eq. (36) as

$$
B^j_{pq}(y_4, y_5) = c^j_{pq} \sum_{r=0}^{n-1} e^{2\pi i (m_a p - m_b q) \frac{r}{n}} \Theta^{j+mr},
$$

(38)

where $j = 0, 1, \ldots, m - 1$. Here, $c^j_{pq}$ is a constant normalization, which can be determined by the boundary conditions. Note that the boundary condition (34) relates $B_{p,q}$ and $B_{p+1,q+1}$. Thus, there are the series

$$
B_{p,q} \rightarrow B_{p+1,q+1} \rightarrow \cdots \rightarrow B_{p+n,q+n} = B_{p,q}.
$$

(39)

The periodicity of the series is equal to $n$, and there are $n$ independent series. Each of the series has $m$ independent solutions (38) and the total number of independent zero-modes is equal to $mn$.

We have concentrated on the $B_+$ fields. Similarly, when $mn > 0$, the $C_-$ fields have the same solutions as $B_+$. However, the $B_-$ and $C_+$ have no normalizable zero-modes.
for $mn > 0$. On the other hand, when $mn < 0$ the $B_-$ and $C_+$ have normalizable zero-modes with the same wavefunctions as the above, while $B_+$ and $C_-$ have normalizable zero-modes.

We have considered the zero-mode profiles of fermionic fields. If 4D N=1 supersymmetry is preserved, the scalar mode has the same zero-mode profiles as its fermionic superpartner.

### 3.2.2 $n_a \neq n_b$

Next, we study the model with $n_a \neq n_b$. In this case, the non-Abelian Wilson lines break the gauge group $U(N_a) \times U(N_b)$ to $U(P_a) \times U(P_b)$, where $P_a = N_a/n_a$ and $P_b = N_b/n_b$. Similarly to the previous subsection, we decompose the fields $B$ as

$$B = \begin{pmatrix}
  B_{00} & B_{01} & \cdots & B_{0,n_b-1} \\
  B_{10} & B_{11} & \cdots & \\
  \vdots & \vdots & \ddots & \\
  B_{n_a-1,0} & B_{n_a-1,1} & \cdots & B_{n_a-1,n_b-1}
\end{pmatrix}. \tag{40}$$

Each of $B_{pq}$ components is $(P_a \times P_b)$ matrix-valued fields. The boundary condition due to the non-Abelian Wilson lines is written as

$$B_{pq}(y_4 + 1, y_5) = B_{p+1,q+1}(y_4, y_5),$$

$$B_{pq}(y_4, y_5 + 1) = e^{-2\pi i (m_{pa} - m_{qb})} e^{-2\pi i (n_{pa} - n_{qb})} B_{p,q}(y_4, y_5), \tag{41}$$

where $B_{p+n_a,q} = B_{p,q+n_b} = B_{p,q}$. This boundary condition relates $B_{p,q}$ and $B_{p+1,q+1}$. Then, similarly to (39), there are the following series

$$B_{p,q} \to B_{p+1,q+1} \to \cdots \to B_{p+Q_{ab},q+Q_{ab}} = B_{p,q}. \tag{42}$$

Here, the periodicity of the series is obtained by $Q_{ab} \equiv \text{l.c.m.}(n_a, n_b)^5$ and the number of independent series is equal to $k_{ab} \equiv \text{g.c.d.}(n_a, n_b)$. Obviously, there is the relation, $Q_{ab} = n_a n_b / k_{ab}$. The above boundary condition (41) leads to the boundary condition,

$$B_{pq}(y_4 + Q_{ab}, y_5) = B_{pq}(y_4, y_5),$$

$$B_{pq}(y_4, y_5 + Q_{ab}) = e^ {\frac {2\pi i} {k_{ab}} I_{ab} y_4} B_{p,q}(y_4, y_5). \tag{43}$$

Here we have defined 'the intersection number' $I_{ab} \equiv n_b m_a - n_a m_b$ analogous to intersecting brane models. There are $S_{ab} = \frac {n_a n_b} {k_{ab}} I_{ab}$ independent zero-mode solutions, which satisfy the boundary condition (43). Those functions are obtained as

$$\Theta^j(y_4, y_5) = \sum_n e^{-\pi S_{ab}(n + \frac {S_{ab}} {S_{ab}})^2 + \frac {2\pi i S_{ab}} {Q_{ab}} (n + \frac {S_{ab}} {S_{ab}}) y_4 - \frac {S_{ab}} {Q_{ab}} y_5 - 2\pi i \frac {S_{ab}} {Q_{ab}} (n + \frac {S_{ab}} {S_{ab}}) y_5} \cdot \left[ \begin{array}{c}
  j \\
  0
\end{array} \right] ((S_{ab}/Q_{ab}) z, S_{ab} \tau), \tag{44}$$

---

5 Here, l.c.m. denotes the least common multiple.
where \( \tau = i \). These wavefunctions satisfy the following boundary conditions,

\[
\Theta^j(y_4 + 1, y_5) = e^{2\pi i \frac{I_{ab}}{k_{ab}}} \Theta^j(y_4, y_5),
\]

\[
\Theta^j(y_4, y_5 + 1) = e^{2\pi i (\frac{m_a}{n_a} - \frac{m_b}{n_b}) y_4} \Theta^j - \frac{I_{ab}}{k_{ab}}(y_4, y_5).
\]

(45)

Thus, the zero-mode wavefunctions, which satisfy the boundary conditions (41), are obtained as

\[
B^j_{pq}(y_4, y_5) = c^j_{pq} \sum_{r=0}^{Q_{ab}-1} e^{2\pi i (\frac{m_a}{n_a} p - \frac{m_b}{n_b} q) r} \Theta^{j + \frac{I_{ab}}{k_{ab}} r}(y_4, y_5),
\]

(46)

where \( j = 0, 1, \cdots, \frac{I_{ab}}{k_{ab}} - 1 \). Hence, the number of the independent zero-modes is equal to \( M_{ab} = \frac{S_{ab}}{Q_{ab}} = \frac{I_{ab}}{k_{ab}} \) in each of the series (42), and there are the \( k_{ab} \) independent series. Thus, the total number of zero-modes is equal to \( M_{ab} k_{ab} = I_{ab} \).

As an illustrating example, we consider the model with \( n_a = 2, n_b = 4 \) and \( m_a = m_b = 3 \). Then, we have \( k_{ab} = \gcd(n_a, n_b) = 2 \neq 1, Q_{ab} = 4, S_{ab} = 12 \) and \( I_{ab} = 6 \). We decompose the bi-fundamental fields \( B \) with the \( 2 \times 4 \) matrix entries as

\[
B = \begin{pmatrix} B_{00} & B_{01} & B_{02} & B_{03} \\ B_{10} & B_{11} & B_{12} & B_{13} \end{pmatrix}.
\]

(47)

From the wave function formula in Eq. (46), one obtains the three independent solutions labeled by \( j = 0, 1, 2 \) for each component of \( B_{pq} \) and these are represented by linear combinations of \( \Theta^j \) in Eq. (44). For example, the \( B_{00} \) and \( B_{01} \) fields are

\[
B_{00}^j = \Theta^j + \Theta^{j+3} + \Theta^{j+6} + \Theta^{j+9},
\]

\[
B_{01}^j = \Theta^j + e^{-\frac{2\pi i}{3}} \Theta^{j+3} + e^{-3\pi i} \Theta^{j+6} + e^{-\frac{2\pi i}{3}} \Theta^{j+9}.
\]

(48)

Obviously, the \( y_4 \)-direction boundary condition can connect some of \( B_{p,q} \) components follows

\[
B_{00} \rightarrow B_{11} \rightarrow B_{02} \rightarrow B_{13} \rightarrow B_{00},
\]

(49)

\[
B_{01} \rightarrow B_{12} \rightarrow B_{03} \rightarrow B_{10} \rightarrow B_{01}.
\]

(50)

Since there are \( k_{ab} = 2 \) independent series, there are \( I_{ab} = 6 \) zero mode solutions in this background.

### 3.3 Another representation of solutions

In the previous section, we have presented solutions in terms of the \( \Theta^j \) functions. However, by using the properties of the theta function, one can represent the wave functions (38) and (46) as a single theta function as

\[
B_{pq}^j(y_4, y_5) = C_{pq}^j e^{-\pi i a_{pq} y_5^2} \times \vartheta \left[ \begin{array}{c} \frac{j}{M_{ab}} \\ 0 \end{array} \right] \left( I_{ab} z + \left( \frac{m_a}{n_a} p - \frac{m_b}{n_b} q \right), I_{ab} \tau \right),
\]

(51)

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where $\tilde{I}_{ab} = I_{ab}/n_an_b$. The constant $C^j_{pq}$ can be determined by the boundary conditions. The net number of zero-mode multiplicity for each of the series is given by $M_{ab} = I_{ab}/k_{ab}$. Therefore the wave functions $B^j_{pq}(y_4, y_5)$ with $j' = j + M_{ab}$ should be equal to $B^j_{pq}(y_4, y_5)$. Furthermore we impose the $B^j_{p+na,q} = B^j_{p,q+nb} = B^j_{p,q}$ and we have twist boundary condition $B^j_{pq}(y_4 + 1, y_5) = B^j_{p+1,q+1}(y_4, y_5)$. Then these conditions imply the following constraint for the coefficients of $C^j_{pq}$ as

$$e^{2\pi ij\frac{m_a}{M_{ab}} C^j_{p+na,q}} = e^{-2\pi ij\frac{m_b}{M_{ab}} C^j_{p,q+nb}} = C^j_{pq},$$  \hspace{1cm} (52)

$$C^j_{p+1,q+1} = C^j_{p,q}, \hspace{1cm} C^j_{p,q+M_{ab}} = C^j_{p,q},$$  \hspace{1cm} (53)

We start with a certain element, e.g. $C^0_{0,0}$. Then we fix other elements by using the above relations. If there are still unrelated elements, we start with one of those elements and repeat the procedure recursively again.

In general, their solutions for $C^j_{pq}$ can not be determined uniquely. Only in specific cases, we can write $C^j_{pq}$ by a simple form. For example, when $\frac{m_a}{M_{ab}}, \frac{m_b}{M_{ab}} = \text{integer}$, $C^j_{pq}$ is reduced to $C^j_{pq} = \text{const.}$, i.e. independent of $p, q$ and $i$. Most of the following models correspond to this case. As another example for a simple form, if we can find an integer $L$, which satisfies

$$L = \frac{M_{ab}l_a - m_a}{n_a} = - \frac{M_{ab}l_b + m_b}{n_b},$$  \hspace{1cm} (54)

where $l_a$ and $l_b$ are also integers, then we can write $C^j_{pq}$ by the following form

$$C^j_{pq} = e^{2\pi ij\frac{L}{M_{ab}}(p-q)}. \hspace{1cm} (55)$$

Then the forms of wave functions would become simple as

$$B^j_{pq}(y_4, y_5) = e^{2\pi ij\frac{L}{M_{ab}}(p-q)} e^{-\pi \tilde{I}_{ab}y_5^2} \times \vartheta \left[ \begin{array}{c} \frac{j}{M_{ab}} \\ 0 \end{array} \right] \left( \tilde{I}_{ab}z + \left( \frac{m_a}{n_a}p - \frac{m_b}{n_b}q, \tilde{I}_{ab}\tau \right) \right),$$  \hspace{1cm} (56)

up to a normalization factor. However this expression is only valid if there exists such an integer $L$ satisfying the relations. At any rate, in generic case we determine $C^j_{pq}$ recursively as mentioned above.

So far, we have considered the simple $T^2$, where $y_4$ and $y_5$ are identified as $y_4 \sim y_4 + 1$ and $y_5 \sim y_5 + 1$. Similarly, we can study the torus compactification with an arbitrary value of the complex structure modulus $\tau$, although we have fixed $\tau = i$ in the above analysis. Then, we obtain zero-mode wavefunctions similar to Eq. (51) for an arbitrary value of $\tau$ as Eq. (51). We also replace $z = y_4 + iy_5$ in the theta function by $z = y_4 + \tau y_5$. 

11
4 Yukawa couplings

Here, we study Yukawa couplings. Let us consider the following form of the magnetic fluxes,

$$F = \left( \frac{m_a}{n_a} 1_{N_a} \frac{m_b}{n_b} 1_{N_b} \frac{m_c}{n_c} 1_{N_c} \right), \quad (57)$$

and non-Abelian Wilson lines similar to \([21]\). Then, there are three types of matter fields, \((N_a, N_b, N_c)\), and their conjugates under \(U(N_a) \times U(N_b) \times U(N_c)\), although they break to \(U(P_a) \times U(P_b) \times U(P_c)\) by non-Abelian Wilson lines. We consider the case with \(\frac{m_a}{n_a} - \frac{m_b}{n_b} > 0, \frac{m_b}{n_b} - \frac{m_c}{n_c} > 0\) and \(\frac{m_a}{n_a} - \frac{m_c}{n_c} > 0\). Then, the three types of matter fields whose wavefunctions are denoted by \(\Psi^{j,M_1}, \Psi^{k,M_2}\) and \((\Psi^{l,M_3})^*\), appear in the following off-diagonal elements,

$$\begin{pmatrix}
\text{const} & \Psi^{j,M_1} \\
\Psi^{j,M_1} & \text{const} \\
(\Psi^{l,M_3})^\dagger & \text{const}
\end{pmatrix}, \quad (58)$$

where \(M_1 = M_{ab}, M_2 = M_{bc}\) and \(M_3 = M_{ac}\) for simplicity. We use the same indices for \(Q_{ab}\) and others, i.e. \(Q_1 = Q_{ab}, Q_2 = Q_{bc}\) and \(Q_3 = Q_{ac}\). As already explained, in the background with fractional fluxes and non-Abelian Wilson lines, their fields are the matrix valued wave functions. The Yukawa coupling can be calculated by computing the following overlap integral of zero-modes in the \((y_4, y_5)\) compact space,

$$y_{ijkl}^{pq} = \int_0^1 dy_4 \int_0^1 dy_5 [\Psi^{j,M_1}_{pq} \Psi^{k,M_2}_{qr} (\Psi^{l,M_1}_{rp})^*]. \quad (59)$$

That is, the Yukawa coupling \(Y^{ijk}\) in 4D effective theory is obtained as their products on \((T^2)^n\), i.e. \(Y^{ijk} = g_D \prod_{d=1}^n y_d^{ijk}\), where \(y_d^{ijk}\) denotes the overlap integral similar to Eq. \((59)\) for the \(d\)-th torus \((T^2)\) and \(g_D\) is the D-dimensional gauge coupling. From this structure, one can see that the allowed couplings are restricted. In order to see it, we introduce the following parameters as \(k_1 = \text{g.c.d.}(n_a, n_b), k_2 = \text{g.c.d.}(n_b, n_c), k_3 = \text{g.c.d.}(n_a, n_c)\) and \(K = \text{g.c.d.}(k_1, k_2, k_3) = \text{g.c.d.}(n_a, n_b, n_c)\). Then the parameter \(K\) determines the allowed couplings of Yukawa interactions. If \(K = 1\), all of possible combinations \((p, q, r)\) appear in Eq. \((59)\). However, if \(K \neq 1\), only restricted combinations of \((p, q, r)\) appear in Eq. \((59)\), but not all combinations. That is, the couplings are restricted by the \(Z_K\) symmetry. Indeed, allowed combinations of \((p, q, r)\) are controlled by the gauge invariance before the gauge symmetry breaking. This \(Z_K\) symmetry is unbroken symmetry in the original gauge symmetry.

Now, let us consider the following summation of wavefunction products,

$$I_{ijkl}^{pq} = \Psi_{pq}^j \Psi_{qr}^k (\Psi_{rp}^l)^* + \Psi_{p+1,q+1}^j \Psi_{q+1,r+1}^k (\Psi_{r+1,p+1}^l)^* + \cdots + \Psi_{p+Q-1,q+Q-1}^j \Psi_{q+Q-1,r+Q-1}^k (\Psi_{r+Q-1,p+Q-1}^l)^*, \quad (60)$$
where $Q = \text{l.c.m.}(Q_1, Q_2, Q_3)$. One can represent $Q$ as $Q = Q_1q_1 = Q_2q_2 = Q_3q_3$. To compute the integral it is useful to represent the wavefunctions as follows

$$
\tilde{\Psi}^{j',M'_1}(y_4, y_5) = C^{j'}_{pq} e^{-\eta M'_1 y_5^2} \vartheta \left[ \frac{j'}{M'_1} \right] \left( \frac{M'_1}{Q} \right)^2 z + \left( \frac{m_a}{n_a} p - \frac{m_b}{n_b} q \right), \left( \frac{M'_1}{Q} r \right),
$$

$$
\tilde{\Psi}^{k',M'_2}(y_4, y_5) = C^{k'}_{qr} e^{-\eta M'_2 y_5^2} \vartheta \left[ \frac{k'}{M'_2} \right] \left( \frac{M'_2}{Q} \right)^2 z + \left( \frac{m_b}{n_b} q - \frac{m_c}{n_c} r \right), \left( \frac{M'_2}{Q} r \right),
$$

$$
\tilde{\Psi}^{l',M'_3}(y_4, y_5) = C^{l'}_{pr} e^{-\eta M'_3 y_5^2} \vartheta \left[ \frac{l'}{M'_3} \right] \left( \frac{M'_3}{Q} \right)^2 z + \left( \frac{m_a}{n_a} p - \frac{m_c}{n_c} r \right), \left( \frac{M'_3}{Q} r \right),
$$

where $j' = q_1 j$, $k' = q_2 k$, $l' = q_3 l$ and $M'_i = q_i M_i$, $(i = 1, 2, 3)$. Here the relation $M'_1 + M'_2 = M'_3$ holds. By using the product property of the theta function, the product of $\Psi^{j,M_1} \Psi^{k,M_2}$ is presented by the sum of the theta functions as

$$
\tilde{\Psi}_{pq}^{j,M_1} \tilde{\Psi}_{qr}^{k,M_2} = C^{j'}_{pq} C^{k'}_{qr} e^{-\eta M'_{12} y_5^2} \sum_{m \in \mathbb{Z}_{M'_3}} \vartheta \left[ \frac{j' + k' + M_1 m}{M'_3} \right] \left( \frac{M'_3}{Q} \right)^2 z + \left( \frac{m_a}{n_a} p - \frac{m_c}{n_c} r \right), \left( \frac{M'_3}{Q} r \right)
$$

$$
\times \vartheta \left[ \frac{M'_3}{M'_1 M'_2} \right] \left( \frac{m_a M'_2 p - m_b n_b M'_2 q - m_b n_b M'_1 q + m_c n_c M'_1 r}{M'_1 M'_2 M'_3} \left( \frac{M'_3}{Q} r \right) \right). \tag{62}
$$

Here one can use the properties of boundary conditions for non-Abelian Wilson lines. Using the property of $\psi_{pq}^a(y_4 + 1, y_5) = \psi_{p+1,q+1}^a(y_4, y_5)$ we find

$$
\int_0^1 dy_4 \int_0^1 dy_5 I_{pq}^{jk} = \int_0^Q dy_4 \int_0^1 dy_5 \psi_{pq}^j \psi_{pr}^k \psi_{tr}^* \psi_{pq}^j \psi_{tr}^k. \tag{63}
$$

Therefore we can obtain the analytic form of Yukawa couplings and flavor structures similar to the case with Abelian Wilson lines. By using the orthogonal condition for the matrix valued wave functions (see Appendix A) as

$$
\int_0^Q dy_4 \int_0^1 dy_5 \psi_{pq}^j (\psi_{pq}^k)^* = \delta_{j,k}, \tag{64}
$$

one can lead to the following form of Yukawa couplings

$$
\int_0^Q dy_4 \int_0^1 dy_5 \psi_{pq}^j (\psi_{tr}^k)^* = Q \sqrt{\frac{Q}{2M'_3}} \sum_{m \in \mathbb{Z}_{M'_3}} \delta_{j' + k' + M'_1 m, l'(\text{mod}M'_3)}
$$

$$
\times \vartheta \left[ \frac{M'_3}{M'_1 M'_2} \right] \left( \frac{m_a \tilde{I}_{ab} p + m_b \tilde{I}_{ca} q + m_c \tilde{I}_{ab} r}{M'_1 M'_2 M'_3} \left( \frac{M'_3}{Q} r \right) \right). \tag{65}
$$

up to the factor $N_{M_1} N_{M_2} N_{M_3} C^{j'}_{pq} C^{k'}_{qr} C^{l'}_{pr} (C_{pq})^*$. Here, the Kronecker delta $\delta_{j' + k' + M'_1 m, l'(\text{mod}M'_3)}$ leads to the coupling selection rule

$$
j' + k' + M'_1 m = l' \mod M'_3, \tag{66}
$$

13
where \( m = 0, 1, \ldots, M_3' - 1 \). When \( g = \text{g.c.d.}(M_1', M_2', M_3') = \text{g.c.d.}(M_1, M_2, M_3) \), the coupling selection rule is given by

\[
j' + k' = l' \mod g.
\] (67)

That means that we can assign \( Z_g \) charges to all of zero-modes.\(^6\)

Here we study again the \( Z_K \) symmetry, which we showed. The total number of multiplicity of \( \Psi_{ab} \) is nothing but \( |I_{ab}| \), and it is represented by two parameters of \( k_{ab} \) and \( M_{ab} \) as \( I_{ab} = k_{ab}M_{ab} \). If \( K = \text{g.c.d.}(k_{ab}, k_{bc}, k_{ca}) \neq 1 \), they are divided to \( K \) types of zero-modes and distinguished by labeling the component of each matrix. We introduce such a kind of flavor indices as \( \tilde{j} \), \( \tilde{k} \) and \( \tilde{l} \) for \( ab-, bc-, ca-\)sectors, respectively. We define the relation between the flavor labeled by \( \tilde{j} \) and the component of matrix \( p, q \) as \( \tilde{j} = p - q \mod k_1 \). Similarly the other sectors are also defined as \( \tilde{k} = q - r \mod k_2 \) and \( \tilde{l} = p - r \mod k_3 \). Since the allowed couplings must be gauge invariant, there is the coupling selection rule for this kind of flavor indices, which is given by

\[
\tilde{j} + \tilde{k} = \tilde{l} \mod K.
\] (68)

This is because the Yukawa couplings are restricted in the trace of the matrix. Therefore we find two types of coupling selection rules, i.e. the \( Z_g \) and \( Z_K \) symmetries.

We can extend the computation of 3-point couplings to higher order couplings. For example, in appendix B, we show the computation of 4-point couplings.

### 5 Non-Abelian discrete flavor symmetry

Here, we study the non-Abelian flavor symmetries, which can appear in our models.

#### 5.1 The case with \( M_i \neq 1 \) and \( k_i = 1 \)

First, we consider the models with \( k_1 = k_2 = k_3 = 1 \). Then, the number of zero-modes are given by \( |I_{ab}| = M_1, |I_{bc}| = M_2 \) and \( |I_{ca}| = M_3 \). We consider the models with \( g = \text{g.c.d.}(M_1, M_2, M_3) \neq 1 \). The Yukawa couplings do not depend on the matrix components \( (p, q, r) \), and are reduced to the following form

\[
\int_0^Q dy_1 \int_0^1 dy_2 \psi_{pq} \psi_{qr} \psi_{rp} = N_{M_1}N_{M_2}N_{M_3}' \sqrt{Q \frac{Q}{2M_3'}} \sum_{m \in \mathbb{Z} M_3'} \delta_{j' + k' + M_1'm, l'(\mod M_3')} \times \theta \left[ \frac{M_2'j' - M_1'M_2'k' + M_2'M_3'm}{M_1'M_2'M_3'Q} \right] (0, M_1'M_2'M_3'/Q\tau),
\] (69)

where we have taken simply \( p = q = r = 0 \) and the phase factor like \( C_{pq}^{j'q} \) disappears. This form is nothing but the case with integer fluxes and without non-Abelian Wilson lines.

\(^6\)See Refs. [34, 35] for a similar selection rule in intersecting D-brane models.
In this types of Yukawa couplings, 4D effective theory has another flavor symmetry called by the shift symmetry, which corresponds to the transformations of flavor indices as

\[
\begin{align*}
  j' & \rightarrow j' + M'_1/g, \\
  k' & \rightarrow k' + M'_2/g, \\
  l' & \rightarrow l' + M'_3/g,
\end{align*}
\]  

(70)
simultaneously. Under this transformation, Yukawa couplings are invariant. There is also coupling selection rule as shown in the previous section given by the \( Z_g \) symmetry (67). Then, they form the non-Abelian discrete flavor symmetries as the same as the case without non-Abelian Wilson lines.

For simplicity, suppose that \( M'_1 = g \). Then, there are \( g \) zero-modes of \( \Psi_{j',g} \). The selection rule (67) means that 4D effective theory is symmetric under the \( Z_g \) transformation, which acts on \( \Psi_{j',g} \) as

\[
Z \Psi_{j',g} \rightarrow Z \Psi_{j',g},
\]  

(71)
and \( \rho = e^{2\pi i/g} \). Furthermore, the effective theory has another symmetry (70). That can be written as cyclic permutations on \( \Psi_{j',g} \),

\[
\Psi_{j',g} \rightarrow \Psi_{j'+1,g}.
\]  

(72)
That is nothing but a change of ordering and also has a geometrical meaning as a discrete shift of the origin, \( z = 0 \rightarrow z = -\frac{1}{g} \). This symmetry also generates another \( Z_g \) symmetry, which we denote by \( Z_g^{(C)} \) and its generator is represented as

\[
C = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
& & & & & \\
& & & & & \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]  

(73)
on \( \Psi_{j',g} \). These generators, \( Z \) and \( C \), do not commute each other, i.e.,

\[
CZ = \rho ZC.
\]  

(74)
Then, the flavor symmetry corresponds to the closed algebra including \( Z \) and \( C \). Diagonal matrices in this closed algebra are written as \( Z^n (Z')^m \), where \( Z' \) is the generator of another \( Z'_g \) written as

\[
Z' = \begin{pmatrix}
\rho & \cdots & \\
& \ddots & \\
& & \rho
\end{pmatrix},
\]  

(75)
Table 1: $D_4$ representations of zero-modes in the model with $g = 2$.

| $M'$ | Representation of $D_4$ |
|------|-------------------------|
| 2    | 2                       |
| 4    | 1++, 1+, 1-, 1--        |
| 6    | 3 × 2                   |

Table 2: $\Delta(27)$ representations of zero-modes in the model with $g = 3$.

| $M'$ | Representation of $\Delta(27)$ |
|------|--------------------------------|
| 3    | 3                              |
| 6    | 2 × 3                          |
| 9    | 1$_1$, 1$_2$, 1$_3$, 1$_4$, 1$_5$, 1$_6$, 1$_7$, 1$_8$, 1$_9$ |
| 12   | 4 × 3                          |
| 15   | 5 × 3                          |
| 18   | 2 × $\{1$_1$, 1$_2$, 1$_3$, 1$_4$, 1$_5$, 1$_6$, 1$_7$, 1$_8$, 1$_9\}$ |

For example, for $g = 2$ and 3 these flavor symmetries are given as $Z_2 \times Z_2 = D_4$ and $(Z_3 \times Z_3) \times Z_3 = \Delta(27)$, respectively. Then, the fields $\Psi^{j',g}$ correspond to 2 of $D_4$ and 3 of $\Delta(27)$, as shown in Tables 1 and 2 respectively. When $M'/g$ is an integer larger than 1, the $\Psi^{j',M'}$ fields correspond to other representations. For smaller values of $M'/g$, the corresponding representations are shown in Tables 1 and 2.

However we note that their multiplets have several types of representation under this symmetry. Because a $Z_g$ charge of fields labeled by $j$ is not $j$ but $j' = qj$. Therefore even if they have same multiplicities ($M_1 = M_2$), their representations may be different from each other.

### 5.2 The case with $M_i = 1$ and $k \neq 1$

Next, we consider the models with $M_i = 1$ and $k \neq 1$. In this case, we also find flavor structures similar to the case without non-Abelian Wilson lines. Suppose all the components of zero modes are given by $|I_{ab}| = k_1$, $|I_{bc}| = k_2$ and $|I_{ca}| = k_3$. Then it is possible to take phase factors for each of wave functions $C_{pq}^j = 1$. We commonly use $K = \text{g.c.d.}(k_1, k_2, k_3)$. The Yukawa couplings depend on the indices $p, q$ and $r$ only through a combination $\theta_{pqr}$.
\[ \theta_{pqr} = Q \left( \frac{m_a}{n_a} \bar{I}_{bc} p + \frac{m_b}{n_b} \bar{I}_{ca} q + \frac{m_c}{n_c} \bar{I}_{ab} r \right) \]
\[ = Q \left( \frac{m_a}{n_a} \bar{I}_{bc} (\tilde{j} + n_1 k_1) - \frac{m_c}{n_c} \bar{I}_{ab} (\tilde{k} + n_2 k_2) \right), \tag{76} \]

where we have used the relations \( p - q = n_1 k_1 + \tilde{j} \) and \( l - r = n_2 k_2 + \tilde{k} \) with \( n_1, n_2 \in Z \). We find that the Yukawa couplings are invariant under the following transformation as

\[ \tilde{j} \rightarrow \tilde{j} + \frac{m_c I_{ab}}{K}, \]
\[ \tilde{k} \rightarrow \tilde{k} + \frac{m_a I_{bc}}{K}, \]
\[ \tilde{l} \rightarrow \tilde{l} + \frac{m_b I_{ac}}{K}. \tag{77} \]

It is obvious that this transformation is the permutation of flavor index with order \( K \). Therefore we have two symmetries: one is the discrete \( Z_K \) symmetry comes from the coupling selection rule and another is this shift symmetry. By combining these two symmetries, it become the same non-Abelian discrete flavor symmetry as the case without Non-Abelian Wilson-lines. That is, these flavor symmetries are given as \( Z_2 \times Z_2 = D_4 \) for \( K = 2 \), \( (Z_3 \times Z_3) \times Z_3 = \Delta(27) \) for \( K = 3 \) and \( (Z_K \times Z_K) \times Z_K \) for generic \( K \).

We have two aspects of flavor structures which are characterized by the parameters \( M, K \). In the latter case, the origin of flavor symmetry is the gauge symmetry. The background breaks the continuous gauge symmetry, but discrete symmetry remains as the flavor symmetry. In the former case, the flavor would not directly originated from the gauge symmetry. However, T-duals of both cases would correspond to similar intersecting \( D \)-brane models, where \( n_a \) and \( m_a \) have almost the same meaning, that is, winding numbers of \( D \)-branes for different directions. Thus, these two pictures of flavor symmetries are related with each other by T-duality through the intersecting \( D \)-brane picture.

So far, we have considered the models with \( M_i = 1 \) and \( K \neq 1 \) and found the flavor symmetry \( (Z_K \times Z_K) \times Z_K \). Here we comment on generic case with \( M \neq 1 \) and \( K \neq 1 \). Even in such a case, the selection rules due to \( Z_g \) and \( Z_K \) symmetries hold exact. However, the general formula of Yukawa couplings depend on both the indices \( j \) and \( \tilde{j} \). Then, 4D effective Lagrangian is not always invariant under the above (independent) shift transformations \( (70) \) and \( (77) \).

### 5.3 Illustrating examples

We show two illustrating examples. We concentrate on only the \( T^2 \) torus. The first example is the model with \( (I_1, I_2, I_3) = (2, 4, 2) \). The background magnetic flux is taken as

\[ F = 2\pi \left( \frac{1}{2} \bar{1}_{N_a} \quad \frac{3}{8} \bar{1}_{N_b} \quad \frac{1}{4} \bar{1}_{N_c} \right). \tag{78} \]
Then the appearing chiral matters are denoted by

\[
\lambda = \begin{pmatrix}
\text{const} & L_{pq}^{j,M_1=1} \\
H_{rp}^{l,M_3=1} & \text{const} & R_{qr}^{k,M_2=1}
\end{pmatrix},
\]

(79)

where \( p = 0, 1, q = 0, 1, \ldots, 7 \) and \( r = 0, 1, 2, 3 \). The wave functions are represented by following theta functions as

\[
L_{pq}^j(y) = N_{M_1} e^{-\pi/8g^2} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z/8 + (1/2p - 3/8q), \tau/8),
\]

\[
R_{qr}^k(y) = N_{M_2} e^{-\pi/8g^2} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z/8 + (3/8q - 1/4r), \tau/8),
\]

\[
H_{rp}^l(y) = N_{M_3} e^{-\pi/4g^2} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z/4 + (1/2p - 1/4r), 2\tau/8),
\]

(80)

where we take \( j = k = l = 0 \). The several parameters are also given by these fluxes. We have \( k_1 = 2, k_2 = 4, k_3 = 2 \) and \( K = \text{g.c.d.}(k_1, k_2, k_3) = 2 \). The gauge invariant 3-point couplings are divided to four types of Yukawa couplings shown below

\[
\mathcal{L} = \mathcal{L}_{000} + \mathcal{L}_{010} + \mathcal{L}_{001} + \mathcal{L}_{100},
\]

\[
\mathcal{L}_{000} = L_{00} R_{00} H_{00}^\dagger + L_{11} R_{11} H_{11}^\dagger + L_{02} R_{22} H_{02}^\dagger + L_{13} R_{33} H_{13}^\dagger + L_{04} R_{40} H_{00}^\dagger + L_{15} R_{51} H_{11}^\dagger + L_{06} R_{62} H_{02}^\dagger + L_{17} R_{73} H_{13}^\dagger,
\]

\[
\mathcal{L}_{011} = L_{00} R_{01} H_{01}^\dagger + L_{11} R_{12} H_{12}^\dagger + L_{02} R_{23} H_{03}^\dagger + L_{13} R_{30} H_{10}^\dagger + L_{03} R_{41} H_{01}^\dagger + L_{15} R_{52} H_{12}^\dagger + L_{06} R_{63} H_{03}^\dagger + L_{17} R_{70} H_{10}^\dagger,
\]

\[
\mathcal{L}_{101} = L_{10} R_{00} H_{10}^\dagger + L_{01} R_{11} H_{01}^\dagger + L_{12} R_{22} H_{12}^\dagger + L_{03} R_{33} H_{03}^\dagger + L_{14} R_{40} H_{10}^\dagger + L_{05} R_{51} H_{01}^\dagger + L_{16} R_{62} H_{12}^\dagger + L_{07} R_{73} H_{03}^\dagger,
\]

\[
\mathcal{L}_{110} = L_{10} R_{01} H_{11}^\dagger + L_{01} R_{12} H_{02}^\dagger + L_{12} R_{23} H_{13}^\dagger + L_{03} R_{30} H_{00}^\dagger + L_{14} R_{41} H_{11}^\dagger + L_{05} R_{52} H_{02}^\dagger + L_{16} R_{63} H_{13}^\dagger + L_{07} R_{70} H_{00}^\dagger.
\]

As seen in these interaction terms, one finds that all the combinations \((p, q, r)\) are not allowed. This is because it has \( K = \text{g.c.d.}(2, 4, 2) = 2 \). These fields \( L, R, H \) are divided to two classes under the discrete \( Z_2 \) charge. For instance, for \( R \) fields, the flavor index is defined by \( \tilde{k} = q - r \mod 4 \). We assign the \( Z_2 \) charges as

\[
Z_2^+ = \hat{R}^{\tilde{k}=0}, \hat{R}^{\tilde{k}=2},
\]

\[
Z_2^- = \hat{R}^{\tilde{k}=1}, \hat{R}^{\tilde{k}=3},
\]

(81)

and for other fields we also assign the \( Z_2 \) charges as

\[
Z_2^+ = L^{\tilde{k}=0}, H^{\tilde{k}=0},
\]

\[
Z_2^- = L^{\tilde{k}=1}, H^{\tilde{k}=1}.
\]

(82)
That corresponds to the coupling selection rule as \( \tilde{j} + \tilde{k} = \tilde{l} \mod 2 \). The Yukawa couplings \( Y_{pqr}^{jkl} \) are obtained after the overlap integrals as

\[
Y_{pqr}^{jkl} \propto \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (1/2p - 3/4q + 1/4r, \tau/4).
\]

(83)

We also consider about the shift symmetry for this model, i.e.

\[
\begin{align*}
\tilde{j} &\rightarrow \tilde{j} + \frac{m_c I_{ab}}{K} = \tilde{j} + 1 \mod 2, \\
\tilde{k} &\rightarrow \tilde{k} + \frac{m_a I_{ab}}{K} = \tilde{k} + 2 \mod 4, \\
\tilde{l} &\rightarrow \tilde{l} + \frac{m_b I_{ab}}{K} = \tilde{l} + 1 \mod 2.
\end{align*}
\]

(84)

As shown in the previous section, the Yukawa couplings are invariant under this transformation. These two operators make the \( D_4 = \mathbb{Z}_2 \rtimes \mathbb{Z}_2 \) discrete flavor symmetry. One can understand the representation for each field under \( D_4 \) symmetry. As an analysis similar to the previous section, one can find that \( L \) and \( R \) correspond to doublets and \( H \) fields become four non-trivial singlets under \( D_4 \) symmetry.

As another example, we consider the model with \((I_1, I_2, I_3) = (3, 3, 3)\), which is not realized by only integer fluxes. We choose fluxes as

\[
F = 2\pi \begin{pmatrix} 31_{N_a} \\ \frac{3}{2} 1_{N_b} \\ 0 1_{N_c} \end{pmatrix}.
\]

(85)

Then the appearing chiral matter fields are denoted as follows,

\[
\lambda = \begin{pmatrix} \text{const} & L_{0p}^{j,M_1=3} \\ H_{00}^{l,M_3=3} & \text{const} \end{pmatrix} \begin{pmatrix} \text{const} & R_{0q}^{k,M_2=3} \\ 0 1_{N_c} \end{pmatrix},
\]

(86)

where \( p, q = 0, 1 \). This model has \( k_1 = 1 \) and \( Q_1 = 2, Q_2 = 2, Q_3 = 1, Q = 2 \) \((j' = j, k' = k, l' = 2l)\). The gauge invariant 3-point couplings are given as

\[
\mathcal{L} = \text{tr} L_{pq} R_{qr} H_{pr}^\dagger = L_{00} R_{00} H_{00}^\dagger + L_{01} R_{10} H_{00}^\dagger.
\]

(87)

The Yukawa couplings \( Y_{pqr}^{jkl} \) are calculated by overlap integrals as follows

\[
Y_{pqr}^{jkl} = \int_0^1 dy_4 \int_0^2 dy_5 L_{pq}^j(y) R_{qr}^k(y) H_{rp}^l(y)^* \propto \sum_{m \in \mathbb{Z}_6} \delta_{j' + k' + 3m, l'} \vartheta \left[ \frac{3j' - 3k' + 9m}{4} \right] (0, 2\tau),
\]

(88)
where we take \( p = q = r = 0 \). From the structure of Kronecker delta, one can read the selection rule as

\[
\begin{align*}
\quad j' + k' + 3m &= l' \mod 6 \\
\rightarrow \quad j + k - 2l &= 0 \mod 3
\end{align*}
\]  

(89)

Since \( g \) is defined by \( g = \text{g.c.d.}(M_1', M_2', M_3') = 3 \), this model has \( \Delta(27) = (Z_3 \times Z_3) \rtimes Z_3 \) flavor symmetry. Here we mention that the charge assignment is different from the case with Abelian Wilson line. For \( H \) fields, their \( Z_3 \) charges are obtained as \( l' = 2l \), and they correspond to the multiplet of \( 3 \) representations. Other fields, \( L \) and \( R \), correspond to \( 3 \) representations, and they can couple in the language of flavor symmetry. Therefore the extension to the non-Abelian Wilson line case causes to have more various types of representations and flavor structures.

It is possible to introduce the constant gauge potential called by the Abelian Wilson line. We take the previous model with \((I_1, I_2, I_3) = (3, 3, 3)\). We assume \( N_a = 4, N_b = 4, N_c = 2 \). Then the fractional fluxes with non-Abelian Wilson lines can reduce the rank of gauge symmetry, that is, the \( U_b(4) \) gauge group breaks to \( U_b(2) \) and the total gauge symmetry is \( U(4)_a \times U(2)_b \times U(2)_c \). To break the gauge symmetry \( U(4)_a \times U(2)_b \times U(2)_c \) to the standard-model gauge group, Abelian Wilson lines can be introduced. For example, we can introduce the Abelian Wilson lines in \( U(4)_a \) along the following direction,

\[
\begin{pmatrix}
    a_1 \mathbf{1}_3 \\
    a_2 \mathbf{1}_1
\end{pmatrix}
\]

(90)

Then, the gauge group \( U(4)_a \) is broken to \( U(3) \times U(1) \). Similarly, we introduce the Abelian Wilson lines in \( U(2)_c \) along the following direction,

\[
\begin{pmatrix}
    c_1 \mathbf{1}_1 \\
    c_2 \mathbf{1}_1
\end{pmatrix}
\]

(91)

Then, the gauge group \( U(2)_c \) is broken to \( U(1) \times U(1) \). Then the (supersymmetric) standard model with three generations is realized up to \( U(1) \) factors. We can also introduce the Abelian Wilson line along the \( U(1) \) direction of \( U(2)_b \). Since the different Wilson line leads to different Yukawa couplings, that would lead to various flavor structures. For example, the above model leads to the \( \Delta(27) \) flavor symmetry in generic values of Wilson lines as studied in the previous section. However, the flavor symmetry is enhanced to the \( \Delta(54) \) symmetry when Wilson lines vanish. Thus by choosing the particular choice of Abelian Wilson lines, we could realize that the flavor symmetry is large like \( \Delta(54) \) in a subsector, e.g. in the lepton sector, but the other sector, e.g. the quark sector, has the smaller flavor symmetry like \( \Delta(27) \).\[^7\]

\[^7\] Indeed, non-Abelian discrete flavor symmetries such as \( D_4, \Delta(27) \) and \( \Delta(54) \) would lead to phenomenologically interesting models [36, 37, 38].
6 Magnetized orbifold background

We have obtained the explicit wavefunctions on the torus background. Here, we study about the models on the orbifold background. Following Ref. [23], we study the $T^2/Z_2$ orbifold, which is constructed by dividing $T^2$ by the $Z_2$ projection $z \to -z$. Furthermore, we require the field projection of periodic or anti-periodic boundary conditions consistent with the $Z_2$ orbifold,

$$\Psi(-y_4, -y_5) = P\Psi(y_4, y_5),$$

where $P$ is $+1$ or $-1$. One can show that the matter wave functions satisfy the following property

$$\Psi_{pq}^{j}(y_4, y_5) = \Psi_{-p,-q}^{-j}(y_4, y_5).$$

(93)

For the case with $k = 1$, this relation holds, because every sector of $(p, q)$ are related by the boundary conditions, so the labels $(p, q)$ have no meaning. However, in the $k \neq 1$ case, they have $k \times M$ independent zero-modes and we symbolically denote them by $\Psi^{j, \tilde{j}}$ ($j = 0, 1, ..., M - 1$ and $\tilde{j} = 0, 1, ..., k - 1$). For example, in the case with $n_a = n_b = 3$, we may use the following notations

$$\begin{align*}
\Psi_{00}^{j}, \Psi_{11}^{j}, \Psi_{22}^{j} &\to \Psi^{j, 0}, \\
\Psi_{01}^{j}, \Psi_{12}^{j}, \Psi_{20}^{j} &\to \Psi^{j, 1}, \\
\Psi_{02}^{j}, \Psi_{10}^{j}, \Psi_{21}^{j} &\to \Psi^{j, 2},
\end{align*}$$

(94)

where $\tilde{j} = p - q \text{ mod } K$. Then, the above property (93) can be written as

$$\Psi^{j, \tilde{j}}(-y_4, -y_5) = \Psi^{-j,-\tilde{j}}(y_4, y_5).$$

(95)

Then the even and odd wave-functions are easily obtained. For the case with $M = 3$, there are $3 \times 3$ independent fields and they are divided into the following even and odd wave functions

$$\begin{align*}
\text{even} : &\quad \Psi^{0,0}, \Psi^{1,0} + \Psi^{2,0}, \Psi^{0,1} + \Psi^{0,2}, \Psi^{1,1} + \Psi^{2,2}, \Psi^{2,1} + \Psi^{1,2}, \\
\text{odd} : &\quad \Psi^{1,0} - \Psi^{2,0}, \Psi^{1,1} - \Psi^{2,2}, \Psi^{2,1} - \Psi^{1,2}.
\end{align*}$$

(96)

Note that these represent the wave functions e.g. $\Psi_{12}^{1} + \Psi_{21}^{2}$ by $\Psi^{1,1} + \Psi^{2,2}$. As examples, the zero-mode numbers of even and odd wave functions for smaller values of $k$ and $M$ are shown in Table 3.

Yukawa couplings as well as higher order couplings can be computed on the orbifold background by overlap integrals of wavefunctions in a way similar to the torus models.


Table 3: The numbers of even and odd zero-modes

| $k$ | $M$ | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|-----|---|---|---|---|---|---|
|     | even | 1 | 2 | 2 | 3 | 3 | 4 |
|     | odd  | 0 | 0 | 1 | 1 | 2 | 2 |
| $k$ | $M$ | 1 | 2 | 3 | 4 | 5 | 6 |
| 3   | even | 2 | 4 | 5 | 7 | 8 | 10 |
|     | odd  | 1 | 2 | 4 | 5 | 7 | 8 |
| $k$ | $M$ | 1 | 2 | 3 | 4 | 5 | 6 |
|     | even | 3 | 6 | 7 | 10 | 11 | 14 |
|     | odd  | 1 | 2 | 5 | 6 | 9 | 10 |

7 Conclusion

We have studied the flavor structure of 4D effective theories, which are derived from extra dimensional theories with magnetic fluxes and non-Abelian Wilson lines. We have obtained zero-mode wavefunctions for generic case. Their Yukawa couplings as well as four-point couplings have been computed. Furthermore, we have also studied non-Abelian flavor symmetries and some parts of them are originated from gauge symmetries. In addition, the orbifold compactification has been discussed.

We have obtained quite rich flavor structure compared with the magnetized torus/orbifold models with Abelian flavor structures. For example, there are various ways of model building leading to three generation models. Thus, it would be interesting to apply our analysis to phenomenological model building.

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A Integral of wave functions

In this appendix we show some calculations on integral of wave functions, which are used in section 4. The general wave functions are expressed as

$$\Psi_{pq}^{j,M}(y_4, y_5) = C_{pq}^{j} N_M e^{-\frac{M'}{Q} y_5^2} \left[ j' M' \right] \left( \frac{M'}{Q} z + \left( \frac{m_a}{n_a} p - \frac{m_b}{n_b} q \right), \frac{M'}{Q} \right)$$  \hspace{1cm} (97)
where $Q$ is given by integer satisfying the relation $Q = k \times \text{l.c.m.}(n_a, n_b), \; k \in \mathbb{Z}$. The orthogonal condition implies

$$
\int_Q dy_4 \int_0^1 dy_5 \Psi_{pq}^{j,M}(\Psi_{pq}^{k,M})^\dagger = |N_M|^2 C_{pq}^j (C_{pq}^k)^* \int_Q dy_4 \int_0^1 dy_5 e^{2\pi M/Qy_5^2} \times \sum_l e^{-\pi M'/Q(l+j'/M')^2} e^{-2\pi M'/Q}(y_4+iy_5+(\frac{m_a}{n_a}p-\frac{m_b}{n_b}q)),
$$

$$
\times \sum_m e^{-\pi M'/Q(m+j'/M')^2} e^{-2\pi M'/Q}(y_4-iy_5+(\frac{m_a}{n_a}p-\frac{m_b}{n_b}q)).
$$

(98)

Here the integral over $y_4$ is obtained as

$$
\int_Q dy_4 e^{2\pi iy_4 \int_0^1 (l+j'/M')-(m+j'/M')} = Q \delta_{l,m \text{(mod } M)} \delta_{j,k \text{(mod } M)}.
$$

(99)

Then, we obtain

$$
\int_Q dy_4 \int_0^1 dy_5 \Psi_{pq}^{j,M}(\Psi_{pq}^{k,M})^\dagger = Q |N_M|^2 \int_{-\infty}^\infty dy_5 e^{-\pi M'/Qy_5^2} \delta_{j,k \text{(mod } M)} = Q \sqrt{\frac{Q}{2M'}} |N_M|^2 \delta_{j,k \text{(mod } M)}.
$$

(100)

By this we can fix the normalization.

**B N-point coupling**

One can calculate the N-point couplings by using the addition formula of the theta functions as in Ref. [15]. For example, we show here the explicit calculation for general four point couplings. We assume that $\tilde{l}_{ab}, \tilde{l}_{bc}, \tilde{l}_{cd} > 0$ and $\tilde{l}_{da} < 0$. Four zero-mode wavefunctions are written as

$$
\psi_{pq}^{j_1,M_1} = C_{pq} e^{-\pi \frac{M^2}{Q} y^2_5} \begin{bmatrix} j_1'/M'_1 & 0 \end{bmatrix} \left( \frac{M'_1}{Q} z + \left( \frac{m_a}{n_a} p - \frac{m_b}{n_b} q, \frac{M'_1}{Q} \tau \right) \right),
$$

$$
\psi_{qr}^{j_2,M_2} = C_{qr} e^{-\pi \frac{M^2}{Q} y^2_5} \begin{bmatrix} k'/M'_2 & 0 \end{bmatrix} \left( \frac{M'_2}{Q} z + \left( \frac{m_b}{n_b} q - \frac{m_c}{n_c} r, \frac{M'_2}{Q} \tau \right) \right),
$$

$$
\psi_{rs}^{j_3,M_3} = C_{rs} e^{-\pi \frac{M^2}{Q} y^2_5} \begin{bmatrix} l'/M'_3 & 0 \end{bmatrix} \left( \frac{M'_3}{Q} z + \left( \frac{m_c}{n_c} r - \frac{m_d}{n_d} s, \frac{M'_3}{Q} \tau \right) \right),
$$

$$
\psi_{ps}^{j_4,M_4} = C_{ps} e^{-\pi \frac{M^2}{Q} y^2_5} \begin{bmatrix} t'/M'_4 & 0 \end{bmatrix} \left( \frac{M'_4}{Q} z + \left( \frac{m_a}{n_a} p - \frac{m_d}{n_d} s, \frac{M'_4}{Q} \tau \right) \right).
$$
where $Q$ is defined as $Q = \text{l.c.m.}(n_a, n_b, n_c, n_d)$. First, the product of $\psi_{pq}^{j,M_1}$ and $\psi_{qr}^{k,M_2}$ becomes

$$
\psi_{pq}^{j,M_1}\psi_{qr}^{k,M_2} = C_{pq}C_{qr}e^{-\pi \frac{m'}{Q} \hat{\nu}^2} \sum_{m \in Z_{M'}} \vartheta \left[ \frac{j' + k' + M'_1 m}{M'} \right] \left( \frac{M'}{Q} z + \left( \frac{m_a}{n_a} p - \frac{m_c}{n_c} r \right), \frac{M'}{Q} \right) \times
\vartheta \left[ \frac{M'_1 j' - M'_1 k' + M'_1 M'_2 m}{M'_1 M'_2 M'} \right] \left( M'_1 \left( \frac{m_a}{n_a} p_{-} \frac{m_b}{n_b} q_{-} \frac{m_c}{n_c} q_{-} \frac{m_d}{n_d} s_{-} \right), \frac{M'_1 M'_2 M'}{Q} \right) \text{, (101)}
$$

where $M' = M'_1 + M'_2$. Then we repeat this product for $\psi_{rs}^{l,M_3}$ and use the orthogonal condition for the $M'_4$ sector because the relations $M'_1 + M'_2 + M'_3 = M' + M'_3 = M'_4$ hold by definition. Finally we obtain the overlap integral for four wave functions as

$$
Y_{pqr}^{jkl} = C_{pq}^{j} C_{qr}^{k} C_{rs}^{l} (C_{ps}^{c})^{*} Q \sqrt{M'_4 \over Q} \sum_{m \in Z_{M'_4}} \sum_{n \in Z_{M'_4}} \sum_{l \in Z_{M'_4}} \delta_{j' + k' + M'_1 m_1 + M'_1 m_2} \left( M'_1 \left( \frac{m_a}{n_a} p_{-} \frac{m_b}{n_b} q_{-} \frac{m_c}{n_c} q_{-} \frac{m_d}{n_d} s_{-} \right), \frac{M'_1 M'_2 M'}{Q} \right) \text{, (102)}
$$

This result is just the product of two theta functions. By solving the Kronecker delta, we obtain the sum of two theta functions like $\sum_{m} y^{j'k'm} y^{l't' m'}$. Therefore even including the non-Abelian Wilson lines we obtain results which are similar to Ref. [15] for general four point couplings.

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