EFFECTIVE FIELD THEORY OF QED VACUUM FLUCTUATIONS

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Abstract

Only photons are dynamical degrees of freedom in the QED vacuum at energies well below the electron mass \( m \). Their interactions via couplings to virtual electron-positron pairs are described to lowest order by an effective theory incorporating the Uehling and Euler-Heisenberg interactions as dominant terms. By a redefinition of the electromagnetic field, the Uehling term is shown not to contribute in the absence of matter. The Stefan-Boltzmann energy for blackbody radiation at temperature \( T \) is then modified by a term proportional to \( T^8/m^4 \). Correspondingly, the Casimir force between two parallel plates with separation \( L \) gets an additional contribution proportional to \( 1/L^8 m^4 \). Higher order corrections to these results are discussed.

Before the Standard Model was established, effective field theories were said to be given by phenomenological Lagrangians. The reason was that they were in general non-renormalizable in the sense of involving interactions with dimensions larger than four. As quantum theories they could therefore only give meaningful results at tree level, i.e. from Feynman diagrams without loops. Besides the Fermi Lagrangian for weak interactions, the prototype was the non-linear \( \sigma \)-model for low-energy pion interactions. In the exponential representation it can be written in the compact form

\[
\mathcal{L} = \frac{1}{4} f_\pi^2 \text{Tr} \left[ \partial_\mu U \partial^\mu U^\dagger + m_\pi^2 (U + U^\dagger) \right]
\]

with the \( SU(2) \) matrix \( U = \exp(i \tau \cdot \pi / f_\pi) \) where the isospin vector \( \pi = (\pi^+, \pi^0, \pi^-) \) contains the pion fields and \( f_\pi \) is the pion decay constant. When expanded in powers of the field, one obtains to lowest non-trivial order the interacting Lagrangian

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \pi \cdot \partial^\mu \pi - m_\pi^2 \pi \cdot \pi \right)
\]

\[
+ \frac{m_\pi^2}{24 f_\pi^2} (\pi \cdot \pi)^2 + \frac{1}{6 f_\pi^2} \left[ (\pi \cdot \partial_\mu \pi)^2 - (\pi \cdot \pi)(\partial_\mu \pi)^2 \right] + \ldots
\]

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It was shown by Weinberg that in lowest order perturbation theory these interaction terms reproduced the scattering lengths obtained from current algebra. This was equivalent to using the Lagrangian just as a classical theory. It took twelve years before it was considered a full-fledged quantum theory and one-loop corrections were calculated, again by Weinberg. A much more detailed investigation of the theory and its physical content were then later initiated by Gasser and Leutwyler. Divergences from loop integrations are absorbed by the coupling constants of higher order interactions.

One can only speculate why this development took so many years. One reason can be that the interactions in Eq.(2) involve time-derivatives which were notoriously difficult to handle before when path integrals were not generally used in field theory. Also, divergences must be regularized by a method which preserves chiral invariance. Today one uses dimensional regularization which was first fully developed in the mid-seventies.

A similar and even slower development has taken place in the understanding of quantum effects in Einstein’s theory of gravity described by the Hilbert action

$$S[g] = \frac{2}{\kappa^2} \int d^4x \sqrt{-g} R$$

with \(\kappa^2 = 32\pi G_N\). Here \(g\) is the determinant of the metric \(g_{\mu\nu}(x)\) and \(R\) is the scalar curvature given by derivatives of the metric. Expanding around flat spacetime by writing \(g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\) one gets in addition to the free Lagrangian for the graviton field \(h_{\mu\nu}\), an infinite series of interactions of dimensions more than four. The theory is thus not renormalizable in the textbook sense. But a few years ago, Donoghue showed that treating the above action as describing an effective theory for energies below the Planck energy \(1/\kappa\), one can systematically calculate quantum corrections to classical results. For example, Newton’s constant \(G_N\) becomes effectively smaller at shorter distances which is characteristic for a non-abelian gauge theory. These quantum effects are completely negligible in practically all realistic physical situations.

Interactions between photons and electrons are described by QED. This is a renormalizable quantum theory and can thus in principle be used to arbitrarily high energies in the absence of other particles or interactions. At energies below the electron mass \(m\) only photon degrees of freedom can be excited from the vacuum and one can then construct an effective but non-renormalizable theory for these excitations alone. Formally this can be done by integrating out the electron field in the full QED partition function. One can then arrange the result as an expansion in Lorentz and gauge invariant operators of increasing dimensions. Including operators up to dimension \(D = 8\) we then have for the
The effective Lagrangian 
\[ \mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu \nu}^2 + \mathcal{L}_U + \mathcal{L}_{\text{EH}} + \ldots \]

The first term
\[ \mathcal{L}_U = \frac{\alpha}{60 \pi m^2} F_{\mu \nu} \Box F^{\mu \nu} \]  
(4)
is the Uehling interaction due to the lowest-order vacuum polarization loop where \( \alpha = e^2 / 4 \pi \) is the fine structure constant and \( \Box \equiv \partial_{\mu} \partial^{\mu} \). The next term
\[ \mathcal{L}_{\text{EH}} = \frac{\alpha^2}{90 m^4} \left[ (F_{\mu \nu} F^{\mu \nu})^2 + \frac{7}{4} (F_{\mu \nu} \tilde{F}^{\mu \nu})^2 \right] \]  
(5)
is the lowest order Euler-Heisenberg interaction where \( \tilde{F}_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma} \) is the dual field strength.

The equation of motion for the free field is \( F_{\mu \nu} = 0 \) and thus we see that the Uehling term can be effectively set equal to zero as long as there is no matter present. Since the equation of motion is only satisfied by on-shell photons, one may question the validity of this simplification where the interactions are used to generate loop diagrams with virtual photons. But since one is in general allowed to shift integration variables in the corresponding functional integrals, we see that under the transformation
\[ A_{\mu} \rightarrow A_{\mu} + \frac{\alpha}{30 \pi m^2} \Box A_{\mu} \]  
(6)
the Uehling interaction is again removed. We are thus left with the result
\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu}^2 + \frac{\alpha^2}{90 m^4} \left[ (F_{\mu \nu} F^{\mu \nu})^2 + \frac{7}{4} (F_{\mu \nu} \tilde{F}^{\mu \nu})^2 \right] \]  
(7)
for the effective theory describing interacting photons at low energies in the absence of matter.

The energy density of free photons at non-zero temperature \( T \) is given by the Stefan-Boltzmann law \( E_{\text{S-B}} = \frac{\pi^2 T^4}{15} \). From the above effective theory we can now calculate the first quantum correction to this classical result. For dimensional reasons we see that it must thus vary with the temperature like \( T^8 / m^4 \). Its magnitude is most directly calculated using the Matsubara formalism where the photon field is periodic in imaginary time. In lowest order perturbation theory the correction to the free energy density is then obtained directly from the partition function as \( \Delta F = \langle \Delta \mathcal{L}_{\text{E}} \rangle \) where \( \Delta \mathcal{L}_{\text{E}} \) is the Euclidean version of the Euler-Heisenberg interaction \( \mathcal{L}_{\text{EH}} \). We thus have
\[ \Delta F = \frac{\alpha^2}{90 m^4} \langle 7 F_{\mu \nu} F_{\nu \beta} F_{\beta \alpha} F_{\alpha \mu} - \frac{5}{2} F_{\mu \nu} F_{\nu \mu} F_{\beta \alpha} F_{\alpha \beta} \rangle \]  
(8)
Expanding the expectation values using Wick’s theorem, we see that the result will follow from the two-loop diagram in Fig.1. This is just a product of two one-loop diagrams and is therefore much easier to evaluate than the corresponding three-loop diagram which would be needed in QED. For our purpose

![Figure 1: Euler-Heisenberg correction to the free energy.](image1)

we need the gauge invariant correlator of the electromagnetic field tensor

\[
\langle F_{\mu\nu}(k)F_{\alpha\beta}(-k) \rangle = \frac{1}{k^2} \left[ k_\mu k_\beta \delta_{\nu\alpha} - k_\mu k_\alpha \delta_{\nu\beta} + k_\nu k_\alpha \delta_{\mu\beta} - k_\nu k_\beta \delta_{\mu\alpha} \right]
\]

When the photon field is in thermal equilibrium at temperature \( T \), the fourth component of the four-momentum vector \( k_\mu \) is quantized,

\[
k_4 = \omega_n = 2\pi T n
\]

where the Matsubara index \( n \) takes all positive and negative integer values for bosons. Loop integrations are then done by the sum-integral

\[
\sum_{k} \int \frac{d^3k}{(2\pi)^3} = T \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3}
\]

where the 3-dimensional momentum integration is dimensionally regularized and the Matsubara summation is regularized using zeta-functions. In our case Eq.(8) reduces to

\[
\Delta F = -\frac{22\alpha^2}{45 m^4} \sum_{p} \sum_{q} \frac{(p \cdot q)^2}{p^2 q^2}
\]

where the basic sum-integral is

\[
\sum_{k} \frac{k_\mu k_\nu}{k^2} = \frac{\pi^2 T^4}{90} (\delta_{\mu\nu} - 4\delta_{\mu 4} \delta_{\nu 4})
\]

Many other more complex sum-integrals have been calculated by Arnold and Zhai\( ^8 \) and Braaten and Nieto\( ^9 \). We thus obtain\( ^{10} \)

\[
\Delta F = -\frac{22\pi^4 \alpha^2}{3^5 \cdot 5^3} \frac{T^8}{m^4}
\]

This result has previously been derived by Barton\( ^{11} \) using a semi-classical method and treating the interacting photon gas as a material medium. It
gives directly the pressure in the gas as \( P = -F \). The entropy is given by the derivative with respect to temperature and thus the energy density follows from \( \mathcal{E} = (1 - T \partial / \partial T) F \) as

\[
\mathcal{E} = \frac{\pi^2}{15} T^4 + \frac{7 \cdot 22 \pi^4 \alpha^2}{3^5 \cdot 5^3} \frac{T^8}{m^4}
\]

(12)

Obviously the corrections due to the Euler-Heisenberg interaction are negligible for ordinary temperatures.

Instead of this global derivation of the energy density one can retrieve the same results from a local calculation based upon the energy-momentum tensor \( T_{\mu\nu}(x) \) of the interacting photon field. In a curved spacetime it is in general defined by

\[
T_{\mu\nu}(x) = -\frac{2}{\sqrt{g}} \frac{\delta S[A]}{\delta g^{\mu\nu}(x)}
\]

(13)

where \( S[A] \) is the corresponding action functional for the system and \( g \) is the determinant of the Euclidean metric \( g_{\mu\nu} \). The result in flat spacetime now follows from the effective Lagrangian (7). It is given by the standard tensor

\[
T_{\mu\nu}^M = F_{\mu\lambda} F_{\nu\lambda} - \frac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta}
\]

(14)

in free Maxwell theory plus a correction

\[
T_{\mu\nu}^{EH} \quad = \quad \frac{\alpha^2}{45 m^4} \left( -4 F_{\mu\lambda} F_{\nu\lambda} (F_{\alpha\beta} F_{\alpha\beta}) + 7 F_{\mu\lambda} F_{\nu\lambda} (F_{\alpha\beta} \tilde{F}_{\alpha\beta}) \right.
\]

\[
+ \quad \frac{1}{2} \delta_{\mu\nu} \left( (F_{\alpha\beta} F_{\alpha\beta})^2 - \frac{7}{4} (F_{\alpha\beta} \tilde{F}_{\alpha\beta})^2 \right)
\]

(15)

due to the Euler-Heisenberg interaction. The energy density (12) can now be obtained from the expectation value \( \mathcal{E} = -\langle T_{44} \rangle \). This approach represents a much more technically cumbersome approach than the previous method based upon thermodynamics (10).

Instead of having the photon field at finite temperature corresponding to periodic boundary conditions in the imaginary time dimension, we can consider the field confined between two parallel plates with separation \( L \gg 1/m \). The vacuum energy of free photons then give rise to the Casimir force between the plates. It can be obtained from the fluctuations of the transverse \( (E_x, E_y) \) and longitudinal \( E_z \) components of the electric field \( E \) where the \( z \)-axis is normal to the plates. In terms of the function

\[
F(\theta) = -\frac{1}{2} \frac{d^3}{d\theta^3} \cot \theta = \frac{3}{\sin^4 \theta} - \frac{2}{\sin^2 \theta}
\]

(16)
where $\theta = z\pi/L$ gives the distance from one plate, one has

\[
\langle E_z^2(x) \rangle = \frac{\pi^2}{48L^4} \left( \frac{1}{15} - F(\theta) \right)
\]

after regularization. Similar expressions obtain for the fluctuations of the magnetic field, i.e. $\langle B_x^2 \rangle = -\langle E_y^2 \rangle$ and $\langle B_y^2 \rangle = -\langle E_z^2 \rangle$. The vacuum energy density is then simply

\[
\mathcal{E} = \frac{\pi^2}{2} \cdot \frac{1}{5} \cdot \frac{1}{7} \cdot \frac{3}{5} \cdot m^4 L^7
\]

since the position dependence via the function $F(\theta)$ drops out in the sum.

Since we know the free photon propagator between the plates, it is now possible to calculate the first quantum correction to the Casimir energy due to the Euler-Heisenberg interaction in the effective Lagrangian (7). In lowest order perturbation theory it is given by

\[
\Delta E = - \int d^3x \langle \mathcal{L}_{EH} \rangle
\]

The field correlators diverge near the plates, but these divergences disappear after integration over the volume between the plates. One then finds

\[
\Delta E = -\frac{11\alpha^2\pi^4}{2^7 \cdot 3^6 \cdot 5^3 m^4 L^7}
\]

Dividing by the plate separation $L$ to get an energy density, we see that it can also be obtained from the free energy correction (11) by the substitution $T \to 1/2L$ just as for the leading term of the Casimir energy. It represents a tiny, additional contribution to the attractive force between the plates.

Obviously, the same result for the vacuum energy in the interacting theory should be obtainable from full QED. Such a calculation has been attempted by Robaschik and collaborators. They obtain a correction to the Casimir force varying with the separation as $1/L^5$ while the above approach gives a correction going like $1/L^8$. They assume that the electrons don’t feel the presence of the plates and their result is due to properties of the photon propagator at the high energy scale $m$. At such short scales also properties of the confining plates should in principle enter the result. On the other hand, in the effective theory only photon degrees of freedom with much smaller momenta are included.

From the interacting energy-momentum tensor (15) one can find the energy density and pressure between the plates. This calculation has not yet been completed. It is much more difficult than in the finite-temperature case since there is no translation invariance normal to the plates.
The Euler-Heisenberg interaction (5) was originally derived for constant fields only. Here we have used it as an effective Lagrangian for general fields. One might worry that it then no longer incorporates all the physics. But here the crucial concept of matching comes in. It is essential in the construction of effective theories. From gauge and Lorentz invariance we know that the structure of the dimension $D = 8$ interaction must be of the form (5) plus the dimension $D = 8$ Uehling interaction. This latter is of the same form as (4) but involving two operators. It can also be removed by a transformation similar to (6). The only uncertainty in the $D = 8$ operators is then their coefficients. In order to find these to lowest order in $\alpha$, we go to the special case of constant fields and use the Euler-Heisenberg result. Higher order corrections to the same coefficients have been calculated by Ritus and confirmed by Reuter, Schmidt and Schubert. Having fixed the coupling constants in this way, we then have the effective Lagrangian (7) which is valid for arbitrary fields both classically and as a quantum theory.

Historically, the Euler-Heisenberg interaction was first used by Euler to calculate the cross-section for elastic scattering of light by light at energies below the electron mass. An attempt to calculate the lowest order quantum correction to this result was apparently first made by Halter who considered the contribution from a one-loop diagram with two Euler-Heisenberg four-photon vertices. It was later pointed out that this contribution is much smaller than $O(\alpha)$ corrections to the coupling constants at tree level. At somewhat higher energies a dimension $D = 10$ interaction comes in and gives the dominant contribution as recently shown by Dicus, Kao and Repko. They find that it can be written as

$$L_{DKR} = \frac{\alpha^2}{945m^8} \left[ (\partial^\alpha F_{\mu\nu} F_{\nu\lambda}) F_{\lambda\rho} F^{\lambda\rho} + 3(\partial^\alpha F_{\mu\nu}) (\partial^\alpha F^{\mu\nu}) F_{\lambda\rho} F^{\lambda\rho} + 11 (\partial^\alpha F_{\mu\nu}) (\partial^\alpha F_{\rho\lambda}) F^{\rho\mu} F^{\lambda\nu} \right]$$

In addition, they construct a $D = 12$ four-photon vertex operator with four derivatives which contributes at even higher energies to the scattering amplitude. At this order also the $D = 12$ Euler-Heisenberg interaction which couples six photons, can contribute by contracting a pair of photon fields. However, using dimensional regularization this contribution vanishes. All these new operators will contribute to the QED vacuum energy when calculating to higher orders. We will then also experience that the tree-level coupling constants in the corresponding effective theory must be renormalized as in ordinary, renormalizable quantum field theories.

References
1. S. Weinberg, *Phys. Rev. Lett.* **17**, 616 (1966).
2. S. Weinberg, *Physica* **A96**, 327 (1979).
3. J. Gasser and H. Leutwyler, *Ann. Phys. (NY)* **158**, 142 (1984); *Nucl. Phys. B250*, 465 (1985).
4. For a recent review, see G. Ecker, *Prog. Part. Nucl. Phys*, **36**, 71 (1996).
5. J.F. Donoghue, *Phys. Rev.* **D50**, 3874 (1994).
6. E.A. Uehling, *Phys. Rev.* **48**, 55 (1935).
7. E. Euler and W. Heisenberg, *Zeit. Phys.* **98**, 714 (1936).
8. P. Arnold and C. Zhai, *Phys. Rev. D51*, 1906 (1995).
9. E. Braaten and A. Nieto, *Phys. Rev. D51*, 6990 (1995).
10. X. Kong and F. Ravndal, [hep-ph/9803216](http://arxiv.org/abs/hep-ph/9803216) (to be published in *Nucl. Phys. B*).
11. G. Barton, *Ann. Phys. (NY)* **205**, 49 (1991).
12. C.A. Lütken and F. Ravndal, *Phys. Rev. A31*, 2082 (1985).
13. X. Kong and F. Ravndal, *Phys. Rev. Lett.* **79**, 545 (1997).
14. M. Bordag, D. Robaschik and E. Wieczorek, *Ann. Phys. (NY)* **165**, 192 (1985); D. Robaschik, K. Scharnhorst and E. Wieczorek, *Ann. Phys. (NY)* **174**, 401 (1987).
15. E. Euler, *Ann. Phys. B26*, 398 (1936).
16. J. Halter, *Phys. Lett. B316*, 155 (1993).
17. V.I. Ritus, *Proc. Lebed. Phys. Inst. 168*, Nova Science Pub., New York, 1986.
18. M. Reuter, M.G. Schmidt and C. Schubert, *Ann. Phys. (NY)* **259**, 313 (1997).
19. F. Ravndal in *Beyond the Standard Model V*, eds. G. Eigen, P. Osland and B. Stugu (AIP, New York, 1997).
20. D.A. Dicus, C. Kao and W.W. Repko, *Phys. Rev. D57*, 2443 (1998).