A Review of $W$ Strings

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ABSTRACT

We review some of the recent developments in the construction of $W$-string theories. These are generalisations of ordinary strings in which the two-dimensional “worldsheet” theory, instead of being a gauging of the Virasoro algebra, is a gauging of a higher-spin extension of the Virasoro algebra—a $W$ algebra. Despite the complexity of the (non-linear) $W$ algebras, it turns out that the spectrum can be computed completely and explicitly for more or less any $W$ string. The result is equivalent to a set of spectra for Virasoro strings with unusual central charge and intercepts.

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1. Introduction

Ordinary string theory has its origins in two-dimensional gravity, which may be thought of as the gauge theory of the Virasoro algebra. The holomorphic and anti-holomorphic general-coordinate symmetries of the two-dimensional action for a set of free scalar fields \(X^\mu\), generated by the energy-momentum tensor components \(T\) and \(\bar{T}\), may be made into local symmetries by introducing gauge fields \(h\) and \(\bar{h}\) with Noether couplings to \(T\) and \(\bar{T}\). The fields \(h\) and \(\bar{h}\), together with an additional field (an overall Weyl scaling factor) that may be introduced because of the Weyl invariance of the theory, constitute the components of the two-dimensional metric tensor. Upon quantisation, ghosts must be introduced for the integrations over the gauge fields \(h\) and \(\bar{h}\), leading to the conclusion that quantum anomalies in the local Virasoro symmetries will be avoided provided that the energy-momentum tensors \(T\) and \(\bar{T}\) have central charges \(c = 26\). The easiest way to achieve this is to take 26 scalars \(X^\mu\), which acquire the interpretation of being coordinates on a 26-dimensional target spacetime. In the BRST description, the anomaly-freedom condition \(c = 26\) for the bosonic string arises from the requirement that the BRST operator \(Q\) be nilpotent.

At the classical level, the equations of motion for the gauge fields \(h\) and \(\bar{h}\) imply that \(T\) and \(\bar{T}\) vanish. At the quantum level, these become operator conditions on physical states, necessary for the vanishing of the expectation values of the operators \(T\) and \(\bar{T}\). In particular, the BRST invariance of states implies that the BRST operator should annihilate the product of the physical and ghost vacua. From the structure of the ghost vacuum, one deduces that the zero modes \(L_0\) and \(\bar{L}_0\) of the energy-momentum tensors \(T\) and \(\bar{T}\) should have eigenvalue 1 on physical states. Thus the physical-state conditions for the bosonic string are

\[
L_0|\text{phys}\rangle = |\text{phys}\rangle, \quad L_n|\text{phys}\rangle = 0, \quad n \geq 1, \quad (1.1)
\]

\[
\bar{L}_0|\text{phys}\rangle = |\text{phys}\rangle, \quad \bar{L}_n|\text{phys}\rangle = 0, \quad n \geq 1
\]

From these conditions, one derives the spectrum of states in the bosonic string.

The construction of \(W\)-string theories proceeds in a very similar manner. Let us, for now, focus on the purely holomorphic sector of such a theory. One introduces gauge fields \(h B, \ldots\) for each of the higher-spin currents \(T\), \(W\), \(\ldots\) in the algebra. Anomaly freedom of the local \(W\) symmetries at the quantum level is again achieved by requiring that the BRST operator be nilpotent. This ensures that one has an anomaly-free theory of \(W\) gravity \([1,2]\). Then BRST invariance of states in the theory determines the intercepts \(\omega_s\) for the zero modes of the matter currents, leading to physical-state conditions

\[
V_0^{(s)}|\text{phys}\rangle = \omega_s|\text{phys}\rangle, \quad V_n^{(s)}|\text{phys}\rangle = 0, \quad n \geq 1, \quad (1.2)
\]

where \(V_n^{(s)}\) denotes the \(n\)'th Laurent mode of the spin-\(s\) current \(V^{(s)}(z)\) in the algebra. There are also analogous conditions for the anti-holomorphic sector of the theory.
In the following sections, we shall review some of the recent results that have been obtained in the study of $W$-string theories, and explain how the steps that have been outlined above are implemented in practice. The most remarkable feature, perhaps, is that despite the great complexity of the $W$ algebras, one can in fact obtain complete results for the physical spectrum in almost all cases. A key reason for this is that it is never necessary to know the explicit form of the algebra itself—it is sufficient to know how it may be realised in terms of free fields. Basic realisations are known in terms of Miura transformations, and from these, more general ones can be obtained that admit string-like interpretations. Further simplifications of the apparently formidable task of constructing $W$-string theories are achieved by observing that one need not struggle to construct primary higher-spin currents, although such exist (the currents that come directly from the Miura transformation are not primary), since the solutions to the physical-state conditions (1.2) are independent of the choice of basis for the currents. Finally, the daunting task of constructing the BRST operator, for the purpose of determining the intercepts $\omega_i$ in (1.2), can be avoided by using a known physical state to read off the intercepts from (1.2). As a consequence of these points, it turns out that the spectrum of $W$-strings can be computed completely. Thus $W$ strings represent a relatively modest and calculable extension of ordinary string theory. In this respect, they contrast rather markedly with another class of possible extensions of string theory—$p$ branes—for which the prospects of exact quantum results seem slender. In fact there is a rather close connection between $W$ algebras and simple Lie algebras, with the Virasoro algebra being associated with $su(2)$, and the extended $W$ algebras being associated with larger simple groups. The study of all $W$ strings rather than merely the Virasoro string is therefore analogous to undertaking a systematic study of all simple Lie groups rather than focussing attention only on $su(2)$.

2. The Physical-state Conditions

The construction of the $W$-gravity theory that forms the basis of the corresponding $W$-string theory is made completely straightforward by making use of the BRST quantisation procedure. We shall not discuss the details here; they may be found, for the example of $W_3$ gravity, in [1,2]. The generalisation to any $W$-gravity theory is immediate. The upshot is that one obtains an anomaly-free $W$-gravity provided that the BRST operator for the corresponding $W$ algebra is nilpotent.

Already for $W_3$, the explicit construction of the BRST operator $Q$ is quite involved [3,4], and in fact this is the only example of a $W$ algebra for which it has been found. Despite the non-linearities of the $W$ algebras, however, one can derive what the central charge that will ensure nilpotence of $Q$ must be in the following way. The potential obstructions to nilpotence come from the central terms in the $W$ algebra; these will give rise to a set of non-vanishing terms with different structures in the expression for $Q^2$. However, there is
just one overall central-charge parameter in a given $W$ algebra, and so it must be the case that when this parameter is chosen so as to make any one anomaly structure in $Q^2$ vanish, all the others will vanish too. The easiest way to determine the required central charge is to look at the anomaly structure in $Q^2$ that corresponds to the spin-2 sector of the algebra. Since this Virasoro subsector is linear, one simply has to add up the contributions from the ghost pairs for each current in the $W$ algebra in order to determine the matter central charge that will cancel all the ghost contributions. The ghosts for a bosonic (fermionic) current of spin $s$ contribute $\mp 2(6s^2 - 6s + 1)$ to the ghostly central charge, and so the matter realisation of the $W$ algebra must have critical central charge $c_{\text{crit}}$ given by

$$c_{\text{crit}} = 2 \sum_{\{s\}_B} (6s^2 - 6s + 1) - 2 \sum_{\{s\}_F} (6s^2 - 6s + 1)$$

in order to achieve nilpotence of $Q$. Here $\{s\}_B$ and $\{s\}_F$ denote the sets of spins of the bosonic and fermionic currents in the $W$ algebra.

The next step is to determine the physical-state conditions for the corresponding $W$-string theory. These are determined by the condition that the product of a physical state with the ghost vacuum be invariant under the action of the BRST operator $Q$. This leads to the conditions (1.2), where the only subtlety that arises is the determination of the intercepts $\omega_s$. These are in principle obtainable once one knows the structure of the ghost vacuum, and the explicit form of the BRST operator. The ghost vacuum is quite easily constructed; the general derivation for a theory with arbitrary numbers of bosonic and fermionic currents is given in [5]. One starts with an $SL(2, C)$-invariant ghost vacuum. The true ghost vacuum is then obtained by acting on this with the appropriate product of ghost creation operators. The ghosts $c^{(s)}$, $b^{(s)}$ for a bosonic current of spin $s$ contribute a factor $c^{(s)}\partial c^{(s)}\partial^2 c^{(s)} \cdots \partial^{s-2} c^{(s)}(0)$. This may be elegantly bosonised, using the rules

$$b^{(s)} \rightarrow e^{-i\phi^{(s)}}, \quad c^{(s)} \rightarrow e^{i\phi^{(s)}}$$

and rewritten as a factor $e^{i(s-1)\phi^{(s)}}(0)$. In a similar way, a fermionic current of spin $s$ implies the inclusion of a factor $e^{-(s-\frac{1}{2})\sigma^{(s)}}(0)$ acting on the $SL(2, C)$ ghost vacuum, where $\sigma^{(s)}$ is the scalar field appearing in the bosonisation of the $\beta, \gamma$ ghost system for a fermionic current of spin $s$ [5]. In general, the non-linearities of the $W$ algebra make the explicit construction of the BRST operator prohibitively complicated. However, as in the $Q^2$ calculation described above, the linearity of the Virasoro subalgebra implies that calculations in the spin-2 sector can be performed quite easily. Here, one finds that the intercept $\omega_2$ for the spin-2 zero-mode $L_0$ is simply given by the negative of the ghostly spin-2 zero mode acting on the ghost vacuum that we have just described. Thus one simply has to count the ghost-number contributions from each factor acting on the $SL(2, C)$ ghost vacuum. The result for the spin-2 intercept is then easily found to be

$$\omega_2 = \frac{1}{2} \sum_{\{s\}_B} s(s - 1) - \frac{1}{2} \sum_{\{s\}_F} (s - \frac{1}{2})^2.$$
This may be recast into the simple form
\[ \omega_2 = \frac{1}{24}(c - 2N_B - N_F), \] (2.4)
where \( c \) is the critical central charge given by (2.1), and \( N_B \) and \( N_F \) are the numbers of bosonic and fermionic currents in the \( W \) algebra [5]. A procedure for determining the higher-spin intercepts, avoiding the problem of first constructing the BRST operator, will be described later.

3. Realisations of \( WA_n \) Algebras

Let us turn now to the question of finding appropriate realisations of the \( W \) algebras, which will enable us to give a string-like interpretation to the \( W \)-gravity theories. The \( unr \)-realisations of \( W \) algebras are provided by the Miura transformation [6]. To be specific, let us consider for now the case of the \( W_N \) algebra. This has one current of each spin \( s \) in the interval \( 2 \leq s \leq N \). Actually, the \( W_N \) algebra is closely associated with the Lie algebra \( su(N) \cong A_{N-1} \), and it will prove to be more convenient to consider \( W_{n+1} \). There is in fact a procedure for associating a \( W \) algebra \( WG \) with every compact semisimple Lie group \( G \), and so we shall denote the \( W_{n+1} \) algebra to be considered here by \( WA_n \).

The Miura transformation provides a realisation of \( WA_n \) in terms of \( n \) free scalar fields \( \varphi^{(n)} = (\varphi_1, \ldots, \varphi_n) \), via the following construction [6]:
\[
\prod_{k=1}^{n+1} \left( \alpha_0 \partial + \vec{h}_k^{(n)} \cdot \partial \varphi^{(n)} \right) = (\alpha_0 \partial)^{n+1} + \sum_{s=2}^{n+1} W_s^{(n)} (\alpha_0 \partial)^{n+1-s},
\] (3.1)

where \( W_s^{(n)} \) denotes the spin-s current of the \( WA_n \) algebra. The \((n + 1)\) vectors \( \vec{h}_k^{(n)} \), which have \( n \) components, satisfy the defining relations
\[
\vec{h}_i^{(n)} \cdot \vec{h}_j^{(n)} = \delta_{ij} - \frac{1}{n+1}, \quad (3.2a)
\]
\[
\sum_{k=1}^{n+1} \vec{h}_k^{(n)} = 0. \quad (3.2b)
\]

Any set of vectors that satisfy these equations will define, \( via \) (3.1), a set of currents that close on the \( WA_n \) algebra. Different choices will yield the algebra in different bases. For \( k = 1, \ldots, n \), the \( \vec{h}_i^{(n)} \) are the weights of the \((n + 1)\)-dimensional representation of \( A_n \) in some basis. It is easy to see from (3.2a, b) that they may be defined recursively as
\[
\vec{h}_i^{(n)} = \left( \vec{h}_i^{(n-1)}, \frac{1}{\sqrt{n(n+1)}} \right), \quad (3.3)
\]
This recursive definition will prove crucial in what follows.

Defining \( \phi_n \equiv \frac{1}{\sqrt{n(n+1)}} \), and making use of (3.3), it follows that the left-hand side of the Miura transformation may be written as [7]

\[
\left( \alpha_0 \partial - n \partial \phi_n \right) e^{-\phi_n/\alpha_0} \prod_{k=1}^{n} \left( \alpha_0 \partial + \vec{h}_{k}^{(n-1)} \cdot \partial \vec{\varphi}^{(n-1)}(n-1) \right) e^{\phi_n/\alpha_0}.
\] (3.4)

But the product of operators sitting between the exponentials in (3.4) is precisely the left-hand side of the Miura transformation (3.1) for the \( W_{A_{n-1}} \) algebra, and so we see that using (3.1) the currents \( W_{n}^{(n)} \) of \( W_{A_n} \) may be written in terms of the currents \( W_{s}^{(n-1)} \) of the \( W_{A_{n-1}} \) algebra together with one free scalar field \( \phi_n \). The explicit expressions are [7]:

\[
W_{k}^{(n)} = \sum_{q=0}^{k} \frac{(n+1+q-k)}{q} \left[ \frac{n+1-k}{n+1+q-k} W_{k-q}(\phi_n) W_{k-q-1}(\phi_n) \right] \]

(3.5)

where \( P_q(\phi_n) \) is defined by

\[
P_q(\phi_n) \equiv e^{-\phi_n/\alpha_0} \left( (\alpha_0 \partial)^q e^{\phi_n/\alpha_0} \right).
\] (3.6)

By applying these results recursively, we may express the currents of \( W_{A_n} \) in terms of the energy-momentum tensor of \( W_{A_1} \cong W_2 \cong \text{Virasoro} \), together with \( (n-1) \) additional scalar fields \( (\phi_2, \ldots, \phi_n) \). Although the Miura transformation gives the energy-momentum tensor explicitly in terms of the single free scalar \( \varphi_1 \), it is clear that since all the scalars commute with each other, we may replace the energy-momentum tensor given in terms of \( \varphi_1 \) by an arbitrary one \( T_{\text{eff}} \), as long as it has the same central charge. This was first discovered for the case of the \( W_3 \) algebra in [8]. In detail, we find from (3.5) that the energy-momentum tensor for the \( W_{A_n} \) algebra is given by

\[
W_{2}^{(n)} = -\frac{1}{2} \partial \vec{\varphi}^{(n)} \cdot \partial \vec{\varphi}^{(n)} + \alpha_0 \tilde{\rho}^{(n)} \cdot \partial^2 \vec{\varphi}^{(n)},
\] (3.7)

where \( \tilde{\rho}^{(n)} \) is the Weyl vector of \( A_n \), given in terms of \( \vec{h}_{k}^{(n)} \) by

\[
\tilde{\rho}^{(n)} = \sum_{j=1}^{n} (n+1-j) \vec{h}_{j}^{(n)}.
\] (3.8)

The central charge for this realisation of the Virasoro algebra is

\[
c_n = n \left( 1 + (n+1)(n+2)\alpha_0^2 \right).
\] (3.9)
The energy-momentum tensor for $\varphi_1$ in (3.7) is

$$-rac{1}{2} (\partial \varphi_1)^2 + \frac{1}{\sqrt{2}} \alpha_0 \partial^2 \varphi_1,$$

which has central charge $(1 + 6\alpha_0^2)$. Thus by replacing (3.10) by $T_{\text{eff}}$, we have a realisation of $WA_n$ in terms of an energy-momentum tensor $T_{\text{eff}}$ with central charge

$$c_{\text{eff}} = 1 + 6\alpha_0^2,$$

(3.11)

together with $(n - 1)$ additional free scalar fields ($\varphi_2, \ldots, \varphi_n$). The central charge of the realisation of $WA_n$ is given by (3.9).

4. $WA_n$ String Theory

The strategy for obtaining a $WA_n$ string theory is now as follows. We first choose a realisation of the energy-momentum tensor $T_{\text{eff}}$ in terms of scalar fields $X^\mu$. The critical central charge condition (2.1), applied to the case of the string based on the $WA_n$ algebra, gives $c_{\text{crit}} = 2n(2n^2 + 6n + 5)$. This implies from (3.9) that the background-charge parameter $\alpha_0$ must take its critical value, given by

$$(\alpha_0)^2 = \frac{(2n + 3)^2}{(n + 1)(n + 2)}.$$  

(4.1)

From (3.11), we see then that the energy-momentum tensor $T_{\text{eff}}$ must have central charge given by

$$c_{\text{eff}} = 26 - \left(1 - \frac{6}{(n + 1)(n + 2)}\right).$$  

(4.2)

Since, for $n \geq 2$, this is not an integer, it follows that to realise $T_{\text{eff}}$ in terms of scalar fields $X^\mu$, we must include a background charge. Without loss of generality, the coordinates $X^\mu$ of the target spacetime may be oriented such that the background-charge vector is aligned along a particular coordinate direction.

The physical states of the theory are defined to be those that satisfy the conditions (1.2). The intercept $\omega_2$ for the $WA_n$ string can be read off from the general results (2.3) or (2.4). The result is

$$\omega_2 = \frac{1}{6} n(n + 1)(n + 2).$$  

(4.3)

As mentioned earlier, a direct computation of the higher-spin intercepts by acting with the BRST operator on the product of a physical state with the ghost vacuum is prohibitively complicated, because of the difficulties of explicitly constructing the BRST operator for such non-linear algebras. Instead we may exploit an observation first made in [9], and subsequently

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developed in [7]. It was noticed in [9] that in the case of the $WA_2 \cong W_3$ algebra, for the 2-scalar $ur$-realisation from the Miura transformation, the known spin-2 and spin-3 intercepts implied that amongst the tachyonic states, described by acting on the vacuum with operators of the form $\exp(\vec{\beta} \cdot \vec{\varphi})$, is a particular state for which the momentum $\vec{\beta}$ is a certain multiple of the Weyl vector $\vec{\rho}$. It was conjectured that for all $W$ strings, such a physical state should exist. The corresponding operator

$$\exp(\lambda \alpha_0 \vec{\rho} \cdot \vec{\varphi}) \quad (4.4)$$

is known as the “cosmological operator.” Since $W_3$ is the only example for which the BRST calculation of the full set of intercepts has been performed [3], no other direct test for the conjecture is available. Assuming the conjecture is correct, it is then easy to determine all the higher-spin intercepts: The unknown constant of proportionality $\lambda$ in $\vec{\beta} = \lambda \alpha_0 \vec{\rho}^{(n)}$ can be calculated from the knowledge of just one intercept. Since we do have an explicit expression for $\omega_2$, this provides the necessary piece of information that determines $\lambda$, and hence all the higher-spin intercepts can now be read off by applying the zero-modes of the currents to the putative physical state. The **lacuna** in this argument is the understanding of why the “cosmological operator” should always give a physical tachyonic state. The best evidence in support of the conjecture comes from the study of $W_N$ strings, where it has been shown explicitly for the cases of the $W_3$, $W_4$, and $W_5$ strings that any values for the intercepts other than those that follow from this conjecture will give rise to a theory that is non-unitary [7]. It seems highly plausible that this feature will persist in all $W$-string theories. It would be very interesting to obtain a proof of this important conjecture.

Proceeding for now under the assumption that the cosmological operator is physical, it follows from (4.3) that the constant $\lambda$ in (4.4) is given by

$$\lambda_\pm = \left(1 \pm \frac{1}{2n + 3}\right). \quad (4.5)$$

Without loss of generality, we may choose the $+$ sign in (4.5), and define this to be the “cosmological solution.” Substituting (4.4) and (4.5) into the physical-state conditions (1.2) now enables one in principle to calculate all the higher-spin intercepts $\omega_s$. The first few examples are given in [7].

The process of calculating the physical spectrum of the $WA_n$ string is now straightforward in principle, since we have realisations in terms of scalar fields $\vec{\varphi}$ and $X^\mu$, and the criticality and intercept conditions for physical states are determined. It turns out that the physical states of the theory can be described in the following way. Let us first consider the situation when we work with the $ur$-realisation of $WA_n$ in terms of the $n$ scalars $\vec{\varphi}$. At the tachyonic level, the physical operators will be of the form

$$e^{\vec{\beta} \cdot \vec{\varphi}}. \quad (4.6)$$
The physical-state conditions for tachyons will clearly simply be the zero-mode intercept conditions in (1.2). The spin-$s$ condition will give a polynomial of degree $s$ in $\vec{\beta}$. Thus in all, we have $n$ equations, coming from the intercept conditions for the currents for spins $2$ up to $n + 1$, on the $n$ components of $\vec{\beta}$. Since the conditions are all independent, it follows that the solutions will comprise a discrete set of values for $\vec{\beta}$. The number of such discrete solutions is given by the product of the degrees of the polynomial equations, and so there will be $(n + 1)!$ solutions for the $WA_n$ string. Amongst them, of course, is the “cosmological solution” (4.4) with $\lambda = \lambda_+$ given by (4.5). Defining a shifted momentum vector $\vec{\gamma}$ by

$$\vec{\beta} = \vec{\gamma} + \alpha_0 \vec{\rho}, \quad (4.7)$$

it is straightforward to show that the system of polynomials in $\vec{\gamma}$ coming from the physical-state conditions (1.2) has a discrete symmetry [6], under the action of the Weyl group of $A_n$ [7]. This group has dimension $(n + 1)!$, and it acts transitively on the Weyl vector $\vec{\rho}$. Thus in fact the discrete set of $(n + 1)!$ solutions of the tachyonic physical-state conditions is generated by taking the cosmological solution itself, and its Weyl-reflected partners. This is the complete set of physical states at the tachyonic level.

Now let us consider the more physically-interesting situation where we replace the energy-momentum tensor (3.10) for $\varphi_1$ by an arbitrary one $T_{\text{eff}}$ built from additional scalars $X^\mu$. It turns out now that the momentum components $(\beta_2, \ldots, \beta_n)$ continue to be frozen to the same sets of values that occurred in the $n$-scalar realisation [7]. Consequently, the tachyonic physical states will have the form

$$|p\rangle = e^{\beta_2 \varphi_2 + \cdots + \beta_n \varphi_n (0)} |p\rangle_{\text{eff}}, \quad (4.8)$$

where $|p\rangle_{\text{eff}}$ describes the effective tachyonic state in the spacetime with coordinates $X^\mu$, and satisfies the effective physical-state conditions

$$L_0^{\text{eff}} |p\rangle_{\text{eff}} = \omega_2^{\text{eff}} |p\rangle_{\text{eff}}, \quad L_n^{\text{eff}} |p\rangle_{\text{eff}} = 0, \ n \geq 1, \quad (4.9)$$

where $L_n^{\text{eff}}$ are the Laurent modes of

$$T^{\text{eff}} = -\frac{1}{2} \eta_{\mu\nu} \partial X^\mu \partial X^\nu + \alpha_\mu \partial^2 X^\mu, \quad (4.10)$$

with $\mu = 0, 1, \ldots, D - 1$, and the background-charge vector $\alpha_\mu$ chosen so that the central charge $D + 12 \alpha_\mu \alpha^\mu$ is equal to the value $c^{\text{eff}}$ given in (4.2), which was required for criticality of the $WA_n$ algebra. Since $T^{\text{eff}}$ replaces the energy-momentum tensor (3.10) for $\varphi_1$, it follows that the effective spin-2 intercept values $\omega_2^{\text{eff}}$ are given in terms of the frozen $\beta_1$ values found above for the $n$-scalar $ur$-realisation. In turn, these may be found by acting with the Weyl group on the shifted momentum vector $\vec{\gamma}$ corresponding to the “cosmological” solution. The result turns out to be [7]

$$\omega_2^{\text{eff}} = 1 - \frac{k^2 - 1}{4(n + 1)(n + 2)}, \quad (4.11)$$
where $k$ is an integer lying in the range $1 \leq k \leq n$.

The above discussion generalises to higher-level states. We must divide these into two categories. The first consists of states for which the excitations take place exclusively in the unfrozen directions $X^\mu$. The second category consists of states where frozen directions are excited too. For reasons that we shall explain later, it turns out that no states in the latter category can be physical. Thus we shall concentrate on the first category of excited states for now. These may be written as

$$|\text{phys}\rangle = e^{\beta_2 \varphi_2 + \cdots + \beta_n \varphi_n(0)} |\text{phys}\rangle_{\text{eff}},$$

(4.12)

where again the values of $\beta_2, \ldots, \beta_n$ are the same set of frozen values found in the $n$-scalar tachyon calculation. The physical-state conditions (1.2) imply that the effective physical states $|\text{phys}\rangle_{\text{eff}}$ must satisfy the effective physical-state conditions

$$L_0^{\text{eff}} |\text{phys}\rangle_{\text{eff}} = \omega_2^{\text{eff}} |\text{phys}\rangle_{\text{eff}}, \quad L_n^{\text{eff}} |\text{phys}\rangle_{\text{eff}} = 0, \ n \geq 1,$$

(4.13)

where again the effective intercept values $\omega_2^{\text{eff}}$ are given by (4.11). The effective physical states $|\text{phys}\rangle_{\text{eff}}$ are built up just as in ordinary string theory, by acting on an effective level-0 state with appropriate operators constructed from products and derivatives of $\partial X^\mu$ contracted into polarisation tensors.

The conclusion of the above discussion is that the $WA_n$ string has a physical spectrum that is the same as that for a set of Virasoro-like string theories, with the non-standard value $c_{\text{eff}}$ for the central charge, given by (4.2), and the set of non-standard values $\omega_2^{\text{eff}}$ for the intercepts, given by (4.11) [7]. Note that the case $k = 1$ in (4.11) gives an effective intercept $\omega_2^{\text{eff}} = 1$, which implies that this sector of the theory gives the same mass spectrum as the Virasoro string. In particular, it describes the usual massless states of string theory. Results for the case of the $WA_2 \cong W_3$ string were found in [10].

The momentum-freezing phenomenon that we met above has a number of important consequences, so we shall now consider this in a bit more detail. A full discussion of the relevant issues may be found in [11]. Although we are calling $\vec{\beta}$ the “momentum” in the $\vec{\varphi}$ directions, it is clear from (4.6) that the true momentum is really $-i\vec{\beta}$. Since the physical-state conditions freeze $\beta$ to real sets of values, it follows that the true momentum is frozen to imaginary values. On the face of it, this sounds rather disturbing, since we are accustomed to dealing only with real momenta in ordinary physics. One way of seeing why momentum normally must be real is that if one calculates correlation functions such as $\int dx e^{-ip'x} e^{ipx}$, the integral will only be well-defined (qua distribution) if the imaginary part of $p' - p$ vanishes; otherwise, the integrand would diverge exponentially. Since it must be well defined for any pair of states one wishes to consider, it follows that $p$ must be real for any state. The situation is different in our case, since we have a background-charge vector $\alpha_0 \vec{\beta}$ in the energy-momentum tensor (3.7), which can be viewed as an injection of momentum $-i\alpha_0 \vec{\beta}$ at
Thus to make correlation functions well defined, it is now necessary that each state have a momentum whose imaginary part is precisely tuned to subtract out the imaginary contribution from the background charge. In terms of $\vec{\beta}$, this means that correlation functions can be made well defined if the frozen components ($\beta_2, \ldots, \beta_n$) have real parts equal to $(\alpha_0 \rho_2, \ldots, \alpha_0 \rho_n)$. But from (4.7), this means that the components ($\gamma_2, \ldots, \gamma_n$) of the shifted momentum would all have to vanish (since, when non-zero, they are always real). However, this can only happen in the special case of the $WA_2$ string; it is a general result from group theory that at most one component of the vector obtained by acting on the Weyl vector of $A_n$ with the Weyl group can be zero.

Fortunately, there is another way in which the exponential divergence of the integrand in correlation functions can be avoided. In a closed $W$ string, there will be both left-moving and right-moving fields, including those corresponding to the frozen directions. It is possible to arrange for the imaginary momentum from the background charge to be cancelled by tuning the total imaginary contribution from the left-moving and right-moving sectors for any state appropriately [11]. This works easily and naturally for any $WA_n$ string, since a particular consequence of the Weyl-group symmetry is that if $\vec{\gamma}$ solves the physical-state conditions, then so does $-\vec{\gamma}$. Thus the left-moving and right-moving momenta for the frozen fields of any state can be paired together, so that the total momentum does exactly cancel the imaginary momentum at infinity. Thus we have the interesting situation that although there are equal numbers of left-moving and right-moving frozen fields ($\varphi_2, \ldots, \varphi_n$) and ($\tilde{\varphi}_2, \ldots, \tilde{\varphi}_n$), they have to have unequal momenta. Since the frozen fields are therefore being treated heterotically, they cannot be assembled into coordinates. Of course the “coordinates” would in any case have been frozen, and unobservable, so it is somewhat academic whether they are left-right symmetric or not. Finally, let us return to the consideration of higher states with excitations in the frozen directions. For these it turns out that the $\vec{\gamma} \to -\vec{\gamma}$ symmetry that played a crucial rôle above no longer occurs. Consequently, one cannot build such states with well-defined correlation functions, and so they will not occur in the physical spectrum [11,7].

5. Generalisations and Discussion

We have seen in the previous section that the physical-state conditions for the $WA_n$ string can be reduced to a set of effective physical-state conditions for Virasoro-like strings. These effective string theories have a non-standard central charge $c_{\text{eff}}$, given by (4.2), and a set of non-standard spin-2 intercept values, given by (4.11). A striking feature of the expression (4.2) for the effective central charge is that it is equal to 26, the critical central charge for the bosonic string, minus the central charge of the $(n + 1, n + 2)$ Virasoro minimal model. Furthermore, the effective intercept values (4.11) can be written as $\omega_2^{\text{eff}} = 1 - \Delta_{k,k}$, where 1 is the intercept for the usual Virasoro string and $\Delta_{r,s}$ are the dimensions of the primary
fields of the \((n + 1, n + 2)\) minimal model. These observations were first made for the \(WA_2\) and \(WA_3\) strings in [9], and developed for \(W_n\) strings in [7]. The underlying significance of this connection between \(WA_n\) strings and minimal models remains somewhat obscure.

The unitarity of the physical spectrum of the \(WA_n\) string can be studied by investigating the unitarity of the spectra of the effective Virasoro string theories. Since these all have an effective central charge which is less than 26, it follows that there will be a range of effective spin-2 intercept values for which unitarity is achieved. In fact this range is precisely spanned by the discrete set of effective intercepts given by (4.11), and so we see that the \(WA_n\) string theories are all unitary [7].

The construction of \(WA_n\) strings that we have described here can be extended to other \(W\) algebras as well. The essential ingredient is that one should be able to find realisations of the algebras in terms of an energy-momentum tensor together with additional fields. This can be done for the \(WD_n\) and \(WB_n\) algebras, based on the Lie algebras \(D_n\) and \(B_n\). As for the case of \(WA_n\), the key point is that the Miura transformations [12,13] can be factorised into a product of transformations for a subalgebra, leading to the above realisations by iteration [14]. In the case of \(WB_n\), there are realisations in terms of an \(N = 1\) super-Virasoro energy-momentum tensor together with additional fields, and so one obtains a description in terms of effective \(N = 1\) superstring theories [14]. Connections with minimal models arise in all cases.

Further generalisations are also possible in which one realises the \(W\) algebra in terms of more than one arbitrary energy-momentum tensor. It was first realised that realisations with two energy-momentum tensors are possible [15], and subsequently this was generalised to multiple energy-momentum tensors [16,17]. The key point here is that for any of the \(A_n\), \(D_n\) or \(B_n\) series one can reduce to a product subalgebra by deleting any vertex in the Dynkin diagram, and factorise the Miura transformation for the original algebra into the product of Miura transformations for the two factors in the subalgebra [16,17]. By applying this iteratively, realisations in terms of many arbitrary energy-momentum tensors can be found [17]. These would give rise to string theories with multiple “spacetimes.”

We have seen that the central charge \(c_{\text{eff}}\) for the effective energy-momentum tensor \(T^{\text{eff}}\) is always non-integer for non-trivial \(W\) algebras; for example it is given by (4.2) for the \(WA_n\) string, with \(25 < c_{\text{eff}} \leq 25 + \frac{1}{2}\). Because of this, the background-charge vector \(\alpha_\mu\) in (4.10) must always be non-zero. This leads to a breaking of the \(D\)-dimensional Poincaré group. Assuming that we take \(\alpha_\mu\) to lie in a spacelike dimension, then it must be imaginary if the number \(D\) of spacetime dimensions exceeds 25. As discussed in [10,7], this implies that the coordinate direction parallel to the background-charge vector appears as a periodic variable in the functional integral, and hence it is automatically compactified on a circle. Since the radius is of Planck size, one is left, à la Kakuza Klein, with a \((D - 1)\)-dimensional observable spacetime with Poincaré invariance in the directions orthogonal to the background-charge vector.
The study of $W$ string theories open up many new lines of investigation. The intriguing relation with minimal models deserves further attention. So far, most of the effort has been concentrated on obtaining realisations of the algebras, and then determining the spectrum of physical states. One of the outstanding problems at present is to try to build in interactions into the theories. Work on this problem is in progress.

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REFERENCES

[1] C.N. Pope, L.J. Romans and K.S. Stelle, Phys. Lett. 268B (1991) 167.
[2] C.N. Pope, L.J. Romans and K.S. Stelle, Phys. Lett. 269B (1991) 287.
[3] J. Thierry-Mieg, Phys. Lett. 197B (1987) 368.
[4] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Comm. Math. Phys. 124 (1989) 87.
[5] H. Lu, C.N. Pope, X.J. Wang and K.W. Xu, “Anomaly Freedom and Realisations for Super-$W_3$ Strings,” preprint CTP TAMU-85/91, to appear in Nucl. Phys. B.
[6] V.A. Fateev and S. Lukyanov, Int. J. Mod. Phys. A3 (1988) 507.
[7] H. Lu, C.N. Pope, S. Schrans and K.W. Xu, “The Complete Spectrum of the $W_N$ String,” preprint CTP TAMU-5/92, KUL-TF-92/1.
[8] L.J. Romans, Nucl. Phys. B352 (1991) 829.
[9] S.R. Das, A. Dhar and S.K. Rama, Mod. Phys. Lett. A6 (1991) 3055;
   “Physical states and scaling properties of $W$ gravities and $W$ strings,” TIFR/TH/91-20.
[10] C.N. Pope, L.J. Romans, E. Sezgin and K.S. Stelle, Phys. Lett. 274B (1992) 298.
[11] H. Lu, C.N. Pope and K.S. Stelle, “Massless States in $W$ Strings,” in preparation.
[12] S.L. Lukyanov and V.A. Fateev, Sov. Scient. Rev. A15 (1990);
    Sov. J. Nucl. Phys. 49 (1989) 925.
[13] A. Bilal and J.-L. Gervais, Nucl. Phys. B314 (1989) 646; B318 (1989) 579.
[14] H. Lu, C.N. Pope, S. Schrans and X.J. Wang, “On Sibling and Exceptional $W$ Strings,”
    preprint CTP TAMU-10/92, to appear in Nucl. Phys. B.
[15] H. Lu, C.N. Pope, S. Schrans and X.J. Wang, “New Realisations of $W$ Algebras and $W$
    Strings,” preprint, CTP TAMU-15/92, to appear in Mod. Phys. Lett. A.
[16] G.M.T. Watts, “A Note on $W$-algebra Realisations,” preprint DUR-CPT 92-15.
[17] H. Lu and C.N. Pope, “On Realisations of $W$ Algebras,” preprint CTP TAMU-22/92.