ASYMPTOTIC PROPERTIES OF BERGMAN KERNELS FOR POTENTIALS WITH GEVREY REGULARITY

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ABSTRACT. We study the asymptotic properties of the Bergman kernels associated to tensor powers of a positive line bundle on a compact Kähler manifold. We show that if the Kähler potential is in Gevrey class $G^a$ for some $a > 1$, then the Bergman kernel accepts a complete asymptotic expansion in a neighborhood of the diagonal of shrinking size $k^{-\frac{2}{4a+2}+\frac{1}{n}+\varepsilon}$ for every $\varepsilon > 0$. These improve the earlier results in the subject for smooth potentials, where an expansion exists in a $(\frac{\log k}{k})^\frac{1}{2}$ neighborhood of the diagonal. We obtain our results by finding upper bounds of the form $C^m m^{2a+2}$ for the Bergman coefficients $b_m(x, \bar{y})$ in a fixed neighborhood by the method of [BeBeSj08]. We also show that sharpening these upper bounds would improve the rate of shrinking neighborhoods of the diagonal $x = y$ in our results.

1. Introduction

Let $(L, h) \to M$ be a positive Hermitian holomorphic line bundle over a compact complex manifold of dimension $n$. The metric $h$ induces the Kähler form $\omega = -\frac{i}{2} \partial \bar{\partial} \log(h)$ on $M$. For $k \in \mathbb{N}$, let $H^0(M, L^k)$ denote the space of holomorphic sections of $L^k$. The Bergman projection is the orthogonal projection $\Pi_k : L^2(M, L^k) \to H^0(M, L^k)$ with respect to the natural inner product induced by the metric $h^k$ and the volume form $\frac{\omega^n}{n!}$. The Bergman kernel $K_k$, a section of $L^k \otimes L^k$, is the distribution kernel of $\Pi_k$. Given $p \in M$, let $(V, e_L)$ be a local trivialization of $L$ near $p$. We write $|e_L|^2 = e^{-\phi}$ and call $\phi$ a local Kähler potential. In the frame $e_L \otimes e_L$, the Bergman kernel $K_k(x, y)$ is understood as a function on $V \times V$. We note that on the diagonal $x = y$, the function $K_k(x, x) e^{-k\phi(x)}$ is independent of the choice of the local frame, hence it is a globally defined function on $M$ called the Bergman function, which is also equal to $|K_k(x, x)|_{h^k}$.

Zelditch [Ze98] and Catlin [Ca99] proved that on the diagonal $x = y$, the Bergman kernel accepts a complete asymptotic expansion of the form

$$K_k(x, x) e^{-k\phi(x)} \sim \frac{k^n}{\pi^n} \left( b_0(x, \bar{x}) + \frac{b_1(x, \bar{x})}{k} + \frac{b_2(x, \bar{x})}{k^2} + \cdots \right).$$

Near the diagonal, i.e. in a $\sqrt{\frac{\log k}{k}}$-neighborhood of the diagonal, one has a scaling asymptotic expansion for the Bergman kernel (see [ShZe02, MaMa07, MaMa13, LuSh15, HeKeSeXu16]). For $d(x, y) \gg \frac{\sqrt{\log k}}{k}$, where $d$ is the Riemannian distance induced by $\omega$, no useful asymptotics are known for smooth metrics. However, there are off-diagonal upper bounds of Agmon type

$$|K_k(x, y)|_{h^k} \leq C k^n e^{-c \sqrt{d}(x, y)},$$

proved for smooth metrics in [Ch91, De98, Lin01, Be03, MaMa15]. In fact as shown in [Ch13b, HeXu18], one has better decay estimates. More precisely, there exist positive constants $c, C$ and a
function \( f(k) \to \infty \) as \( k \to \infty \) such that

\[
|K_k(x, y)|_{h_k} \leq \begin{cases} 
Ck^n e^{-ckd(x,y)^2}, & d(x, y) \leq f(k)\sqrt{\log k}, \\
Ck^n e^{-c(f(k)\sqrt{k\log k}d(x,y)}, & d(x, y) \geq f(k)\sqrt{\log k}.
\end{cases}
\]

A quantitative version of the above estimate that relates the growth rate of \( f(k) \) to the growth rate of the derivatives of the metric \( h \) is obtained in [HeLuXu18]. In particular, when \( h \) is in the Gevrey a \((a \geq 1)\) class, we get \( f(k) = \frac{k^n}{\sqrt{\log k}} \).

This article generalizes the results in [HeLuXu18] to the setting of Gevrey classes. To be precise, we prove an asymptotic expansion in a \( k^{-\frac{1}{2} + \frac{1}{4a+4\varepsilon}} \) neighborhood of the diagonal for any \( \varepsilon > 0 \) if the metric \( h \) is in the Gevrey a \((a \geq 1)\) class. In particular, we show that in the Gevrey a class, uniformly for all sequences \( x_k \) and \( y_k \) with \( d(x_k, y_k) \leq k^{-\frac{1}{2} + \frac{1}{4a+4\varepsilon}} \), we have

\[
|K_k(x_k, y_k)|_{h_k} \sim \frac{k^n}{\pi^n} e^{-\frac{k\log k}{2}}, \quad \text{as} \quad k \to \infty,
\]

where \( D(x, y) \) is Calabi’s diastasis function (1.7), which is controlled from above and below by \( d^2(x, y) \). Before we state the results we must also mention that in [BeBeSj08], there is an off-diagonal asymptotic expansion for the Bergman kernel of the form

\[
(1.3) \quad K_k(x, y) = e^{k\psi(x,y)} \frac{k^n}{\pi^n} \left( 1 + \sum_{j=1}^{N-1} \frac{b_j(x, \bar{y})}{k^j} \right) + e^{k\left(\frac{\phi(x)}{N} + \frac{\phi(y)}{N}\right)} k^{-N+n} O_N(1),
\]

which holds for all \( d(x, y) \leq \delta \) for some \( \delta > 0 \). Here, \( \psi(x, \bar{y}) \) and \( b_j(x, \bar{y}) \) are almost holomorphic extensions of \( \phi(x) \) and \( b_j(x, \bar{x}) \) from (1.1). However, note that this expansion is only useful when the term \( e^{k\left(\frac{\phi(x)}{N} + \frac{\phi(y)}{N}\right)} k^{-N+n} \) is a true remainder term, i.e. it is less than the principal term \( k^n e^{k\psi(x, \bar{y})} \) in size, which holds only in a neighborhood \( d(x, y) \leq C\sqrt{\log k \over k} \) in general. In the case that \( h \) is real analytic, this is valid in a larger neighborhood \( d(x, y) \leq k^{-1/4} \) [HeLuXu18]. In a recent preprint [RoSjNg18], this is further improved to a fixed neighborhood independent of \( k \).

We now state our main result and its corollaries.

**Theorem 1.1.** Assume that the local Kähler potential \( \phi \) is in the Gevrey class \( G^a(V) \) for some \( a > 1 \), meaning that for some \( C_0 \) and \( C_1 > 0 \), we have

\[
(1.4) \quad \|D^\alpha z D^\beta \overline{z} \phi(z)\|_{L^\infty(V)} \leq C_0 C_1^{\mid \alpha \mid + \mid \beta \mid} \|\alpha \| \|\beta \| a,
\]

Then for every \( \varepsilon > 0 \), there exist positive constants \( \delta \) and \( C \), and an open set \( U \subset V \) containing \( p \), such that for \( N_0(k) = \left[ \left( \frac{k}{\varepsilon} \right)^{2a+2\varepsilon} \right] \) and uniformly for any \( x, y \in U \), we have in the frame \( e^k \otimes e^k \)

\[
K_k(x, y) = e^{k\psi(x, y)} \frac{k^n}{\pi^n} \left( 1 + \sum_{j=1}^{N_0(k)-1} \frac{b_j(x, \bar{y})}{k^j} \right) + e^{k\left(\frac{\phi(x)}{N} + \frac{\phi(y)}{N}\right)} k^{-N+n} O_N(1),
\]

where \( \psi(x, z) \) is a certain almost holomorphic extension of \( \phi(x) \) near the diagonal\(^1\) and \( b_m(x, z) \) are certain almost holomorphic extensions (defined by (5.1)) of the Bergman kernel coefficients \( b_m(x, \bar{x}) \).

As a first corollary of this theorem, we get a complete asymptotic expansion in a \( k^{-\frac{1}{2} + \frac{1}{4a+4\varepsilon}} \) neighborhood of the diagonal.

\(^1\)In the sense of Borel and Hörmander [Ho68]; see our definition (2.4).
Corollary 1.2. Given the same assumptions and notations as in the above theorem, there exist positive constants \( C \) and \( \delta \), and an open set \( U \subset V \) containing \( p \), such that for all \( k \) and \( N \in \mathbb{N} \), we have for all \( x, y \in U \) satisfying \( d(x, y) \leq \delta k^{-\frac{1}{2} + \frac{1}{2n}} \),

\[
K_k(x, y) = e^{-k\psi(x, \bar{y})} \frac{\kappa^k}{\pi^n} \left( 1 + \sum_{j=1}^{N-1} b_j(x, \bar{y}) \frac{1}{k^j} + R_N(x, \bar{y}, k) \right),
\]

where

\[
|R_N(x, \bar{y}, k)| \leq \frac{C_N N^{2\alpha + 2\varepsilon}}{k^N}.
\]

And if we only assume \( b_m(x, z) \) are arbitrary almost holomorphic extensions of Bergman kernel coefficients \( b_m(x, \bar{x}) \), then we still have (1.5). But the remainder term estimate will be weaker: \( |R_N(x, \bar{y}, k)| \leq \frac{C_N}{k^N} \) for some constant \( C_N \).

As another corollary, we obtain the following off-diagonal asymptotic in terms of Calabi’s diastasis \([\text{Cal}53]\) function defined by

\[
D(x, y) = \phi(x) + \phi(y) - \psi(x, \bar{y}) - \psi(x, y).
\]

We point out that near a given point \( p \in M \), we have \( D(x, y) = |x - y|^2 + O(|x - p|^2 + |y - p|^2) \), where \( |z|^p := \sum_{i,j=1}^n \phi_{ij}(p) z_i \bar{z}_j \). If we use Bochner coordinates at \( p \) (introduced in [Bo47]), in which the Kähler potential admits the form \( \phi(x) = |x|^2 + O(|x|^4) \), we have \( D(x, y) = |x - y|^2 + O(|x - p|^4 + |y - p|^4) \).

Corollary 1.3. Under the same assumptions and notations (and the same \( \delta \) and same \( U \)) as in Theorem 1.1, we have uniformly for all \( x, y \in U \) satisfying \( D(x, y) \leq \frac{1}{2} \delta k^{-1 + \frac{1}{2n} + \frac{1}{2n}} \),

\[
\frac{1}{k} \log |K_k(x, y)|_k^h = -\frac{D(x, y)}{2} + \frac{n \log k}{k} - \frac{n \log \pi}{k} + O \left( \frac{1}{k^2} \right).
\]

The following scaling asymptotic is then immediate:

Corollary 1.4. In Bochner coordinates at \( p \), we have uniformly for all \( u, v \in \mathbb{C}^n \) with \( |u|^p \) and \( |v|^p < \frac{\delta}{3} \),

\[
\frac{1}{k^{\frac{1}{2} + \frac{1}{2n}} \log \left| K_k \left( \frac{u}{k^{\frac{1}{2} - \frac{1}{4n} + \varepsilon}}, \frac{v}{k^{\frac{1}{2} - \frac{1}{4n} + \varepsilon}} \right) \right|_k^h = -\frac{|u - v|^2}{2} + \frac{n \log k}{k^{\frac{1}{2n} + \varepsilon}} - \frac{n \log \pi}{k^{\frac{1}{2n} + \varepsilon}} + O \left( \frac{1}{k^{1 + \frac{1}{2n} + \varepsilon}} \right).\]

One of the key ingredients in our proofs is the following estimate on the Bergman kernel coefficients \( b_m(x, z) \). We emphasize again that \( b_m(x, z) \) are particular almost holomorphic extensions of the Bergman kernel coefficients \( b_m(x, \bar{x}) \) appearing in the on-diagonal expansion (1.1) of Zelditch \([\text{Ze}98]\) and Catlin \([\text{Ca}99]\).

Theorem 1.5. Assume the Kähler potential \( \phi \) is in Gevrey class \( C^a(V) \) for some \( a > 1 \). Let \( b_m(x, z) \) be the almost holomorphic extensions (defined by (5.1)) of the Bergman kernel coefficients \( b_m(x, \bar{x}) \). Then, there exists a neighborhood \( U \subset V \) of \( p \), such that for any \( m \in \mathbb{N} \) we have

\[
\|b_m(x, z)\|_{L^\infty(U \times U)} \leq C m^{m^2 + 2\varepsilon},
\]

where \( C \) is a constant independent of \( m \) but dependent on \( \varepsilon \). Moreover, we have the following estimates on the derivatives of \( b_m(x, z) \). Denote \( v = (x, z) \). For any multi-indices \( \alpha \) and \( \beta \) and any \( (x, z) \in U \times U \)

\[
\left| D_v^\alpha D_{\bar{v}}^\beta b_m(x, z) \right| \leq C^{m + |\alpha| + |\beta|} m^{2\alpha + 2\beta + 2\varepsilon} \alpha^{n + \varepsilon} \beta^{n + \varepsilon} \exp \left( -b(1 - \delta_0(|\beta|)) |x - \bar{z}|^{-\frac{1}{a + 1}} \right),
\]
where $C$ is a constant independent of $m, \alpha, \beta$ but dependent on $\varepsilon$, and $\delta_0(\beta) = 1$ only if $\beta = 0$ and is zero otherwise. The constant $b$ is positive and is independent of $\alpha, \beta, m, \varepsilon$. In addition, when we are restricted to the diagonal $z = \bar{x}$, we can choose $\varepsilon = 0$.

**Remark 1.6.** We conjecture that in the Gevrey case, there exist certain almost holomorphic extension $b_m(x, z)$ of the Bergman coefficients $b_m(x, \bar{x})$ such that

$$
\|D^\alpha D^\beta b_m(x, z)\|_{L^\infty(U \times U)} \leq C^{m + |\alpha| + |\beta|} m!^{2a-1} \alpha!^{\rho_m} \beta!^{\rho_m} \exp \left( -b(1 - \delta_0(\beta))|x - \bar{z}|^{-\frac{1}{\alpha - \beta}} \right)
$$

As we show in this paper, if this conjecture holds true, then all of the above results can be improved accordingly. In particular, the quantities $N_0(k) = [(k/C)^{\frac{1}{m+1}}]$ and $e^{-\delta k^{\frac{1}{m+1}}} \exp$ in the remainder estimate of Theorem 1.1 would be replaced by $[(k/C)^{\frac{1}{m+1}}]$ and $e^{-\delta k^{\frac{1}{m+1}}}$, moreover Corollary 1.3 would hold for all $D(x, y) \leq \frac{1}{2} k^{1+\frac{1}{m+1}}$. We expect (1.10) is the best possible result one can seek, because by [LuTi04] the leading term in $b_m(x, \bar{x})$ is $\frac{m}{(m+1)!} \Delta^{m-1} \rho(x)$ where $\rho$ is the scalar curvature, so when the metric is in Gevrey class $G^a$, we have $\frac{m}{(m+1)!} \Delta^{m-1} \rho(x) \approx C^m m!^{2a-1}$. However we are unable to prove this conjecture for general Gevrey Kähler metrics using our method, which is based on a recursive formula of [BeBeSj08]. In Section 6, we discuss the optimality and limitations of this method.

There is a huge literature on Bergman kernels on complex manifolds. Before closing the introduction we only list some related works that were not cited above: [BoSj75, En00, Ch03, Lo04, LuTi04, Lo04, MaMa08, Liu10, LiuLu15, Se15, LuZe16, Ze16, LuSe17]. Applications of the Bergman kernel, and the closely related Szegö kernel, can be found in [Do01], [BlShZe00], [ShZe02], [YuZh16]. The book of Ma and Marinescu [MaMa07] contains an introduction to the asymptotic expansion of the Bergman kernel and its applications. See also the book review [Ze09] for more on the applications of Bergman kernels.

**Organization of the paper.** In Sections 2 and 3, we follow the construction of local Bergman kernel in [BeBeSj08], but we obtain precise estimates for the error term by using the growth rate of Bergman coefficients $b_m(x, z)$ provided by Theorem 1.5. In Section 4, we give the proofs of Theorem 1.1 and Corollaries 1.2 and 1.3. The proof of Theorem 1.5 will be given in Section 5. Section 6 discusses the optimality of our bounds on Bergman coefficients. Section 7 contains the proofs of the properties of almost homomorphic extensions of Gevrey functions.

## 2. Local Bergman kernels

In [BeBeSj08], by using good complex contour integrals, Berman-Berndtsson-Sjöstrand constructed local reproducing kernels (mod $e^{-k^{\delta}}$) for $U = B^n(0, 1) \subset \mathbb{C}^n$, which reproduce holomorphic sections in $U$ up to $e^{-k^{\delta}}$ error terms. These kernels are in general not holomorphic. By allowing more flexibility in choosing the amplitudes in the integral, the authors modified these local reproducing kernels to local Bergman kernels, which means that they are almost holomorphic local reproducing kernels mod $O(k^{-N})$. The global Bergman kernels are then approximated using the standard Hörmander’s $L^2$ estimates.

Throughout this paper, we assume that $\phi$ is in the Gevrey class $G^a(V)$ for some open neighborhood $V \subset M$ of a given point $p$. Let $B^n(0, r)$ be the ball of radius $r$ in $\mathbb{C}^n$. We identify $p$ with $0 \in \mathbb{C}^n$ and $V$ with the ball $B^n(0, 3) \subset \mathbb{C}^n$ and denote $U = B^n(0, 1)$. Let $e_L$ be a local holomorphic frame of $L$ over $V$ as introduced in the introduction. For each positive integer $k$, we denote $H_{k\phi}(U)$ to be
the inner product space of $L^2$-holomorphic functions on $U$ with respect to
\[(u, v)_{k\phi} = \int_U u\bar{v} e^{-k\phi} dVol,\]
where $dVol = \frac{\omega^n}{n!}$ is the natural volume form induced by the Kähler form $\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi$. So the norm of $u \in H_{k\phi}(U)$ is given by
\[\|u\|_{k\phi}^2 = \int_U |u|^2 e^{-k\phi} dVol.\]

Let $\chi \in C_0^\infty(B^n(0, 1))$ be a smooth cut-off function such that $\chi = 1$ in $B^n(0, \frac{1}{2})$ and vanishes outside $B^n(0, \frac{3}{4})$. The following result gives a refinement of the the result of [BeBeSj08] by providing a more precise estimate for the error term when the Kähler potential is in Gevrey class $G^a$. The main ingredient of the proof is Theorem 1.5, whose proof is delayed to Section 5.

**Proposition 2.1.** For each $N \in \mathbb{N}$, there exist $K_{k,x}^{(N)}(y) \in H_{k\phi}(U)$ and a positive constant $C$ independent of $N$ and $k$, such that for all $u \in H_{k\phi}(U)$ we have
\[\forall x \in B^n(0, 1/4) : \quad u(x) = \left(\chi u, K_{k,x}^{(N)}\right)_{k\phi} + k^n e^{-k\phi} \mathcal{R}_{N+1}(\phi, k) \|u\|_{k\phi},\]
where
\[|\mathcal{R}_{N+1}(\phi, k)| \leq \frac{C^{N+1}(N + 1)^{2a + 2\varepsilon}}{k^{N+1}}.\]
The function $K_{k,x}^{(N)}$ is called a local Bergman kernel of order $N$.

**Remark 2.2.** In [BeBeSj08], only the qualitative estimate $\mathcal{R}_{N+1}(\phi, k) = O_N(\frac{1}{k^{N+1}})$ is given.

To prove Proposition 2.1, we first need to recall the techniques of [BeBeSj08].

### 2.1. Review of the method of Berman-Berndtsson-Sjöstrand

The main idea is to construct the local almost holomorphic reproducing kernel (also called local Bergman kernel) by means of the calculus of contour pseudo-differential operators (contour PDO for short) introduced by Sjöstrand [Sj82]. Before we introduce the notion of contour integrals we present some notations and definitions.

Suppose $\phi(x)$ is in Gevrey class $G^a(V)$ and $V = B^n(0, 3)$. By replacing $\phi(x)$ by $\phi(x) - \phi(0)$, we can assume that $\phi(0) = 0$. We then denote $\psi(x, z) = F(\phi)(x, z)$ defined later in Definition 2.4 to be one holomorphic extension of $\phi(x)$. Moreover, since $\phi(x)$ is real-valued, we have $\psi(x, z) = \psi(\bar{z}, \bar{x})$.

We also define
\[\theta(x, y, z) = \int_0^1 (D_x \psi)(tx + (1-t)y, z) dt,\]
where the differential operator $D_x$ is the gradient operator defined by
\[D_x = (D_{x_1}, D_{x_2}, \cdots, D_{x_n}).\]

Note that $\theta(x, y, z) = \psi_x(x, z)$. It is easy to prove that the Jacobian of the map $(x, y, z) \rightarrow (x, y, \theta)$ at $(x, y, z) = (0, 0, 0)$ is non-singular. Thus the map is actually an almost biholomorphic map between two neighborhoods of the origin of $\mathbb{C}^{3n}$. As a result, we can use $(x, y, z)$ or $(x, y, \theta)$ as local coordinates interchangeably. Without loss of generality we can assume that $(x, y, z) \in B^n(0, 3) \times B^n(0, 3) \times B^n(0, 3)$ and $\theta \in W$, where
\[W = \theta(B^n(0, 3) \times B^n(0, 3) \times B^n(0, 3)).\]

Note that $W$ contains the origin because by our assumption $\phi(0) = 0$. 

A fundamental idea of [BeBeSj08] is to use the estimate
\begin{equation}
(2.4) \quad u(x) = c_n \left( \frac{k}{2\pi} \right)^n \int_{\Lambda} e^{k\theta \cdot (x-y)} u(y) \chi(y) d\theta \wedge dy + O(e^{-k\delta})e^{\frac{k\phi(x)}{2}}\|u\|_{k\phi}
\end{equation}
which holds uniformly for \( x \in B^n(0, \frac{1}{4}) \), for any holomorphic function \( u \) defined on \( B^n(0, 1) \). Here, \( c_n = i^{-n^2} \), \( \delta \) is a positive constant, and \( \Lambda = \{(y, \theta) : \theta = \theta(x, y)\} \) is a good contour, which means that there exists \( \delta > 0 \) such that for any \( x, y \) in a neighborhood of the origin,
\begin{equation}
(2.5) \quad 2 \Re \theta \cdot (x-y) \leq -\delta|x-y|^2 - \phi(y) + \phi(x).
\end{equation}
One can easily verify that
\begin{equation}
(2.6) \quad \Lambda = \{(y, \theta) : \theta = \theta(x, y, \bar{y})\},
\end{equation}
with \( \theta(x, y, z) \) defined by (2.3), is a good contour by observing that
\( \theta \cdot (x-y) = \psi(x, \bar{y}) - \psi(y, \bar{y}). \)
To put (2.4) into a useful perspective, one should think of the integral in (2.4) as a contour \( \Psi \text{DO} \) defined as follows. Let \( a = a(x, y, \theta, k) \) be an almost holomorphic symbol in \( B^n(0, 3) \times B^n(0, 3) \times W \), with an asymptotic expansion of the form
\[ a(x, y, \theta, k) \sim a_0(x, y, \theta) + \frac{a_1(x, y, \theta)}{k} + \frac{a_2(x, y, \theta)}{k^2} + \cdots. \]
For simplicity, we will suppress the dependency on \( k \) and write \( a = a(x, y, \theta). \)
A \( \Psi \text{DO} \) associated to a good contour \( \Lambda \) and an amplitude \( a(x, y, \theta) \), is an operator on \( C^\infty_0(U) \) defined by
\[ \text{Op}_\Lambda(a) u = c_n \left( \frac{k}{2\pi} \right)^n \int_{\Lambda} e^{k\theta \cdot (x-y)} a(x, y, \theta) u(y) d\theta \wedge dy. \]
Thus in this language (2.4) means that for \( x \in B^n(0, 1/4) \)
\[ (\chi u)(x) = \text{Op}_\Lambda(1)(\chi u) + O(e^{-k\delta})e^{\frac{k\phi(x)}{2}}\|u\|_{k\phi}. \]
Roughly speaking this says that \( \text{Op}_\Lambda(1) \) is the identity operator mod \( O(e^{-k\delta}) \). We define the integral kernel \( K_{k,x}(y) \) of \( \text{Op}_\Lambda(a) \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{k\phi} \), by
\[ \text{Op}_\Lambda(a) u = (u, K_{k,x})_{k\phi}. \]
The first observation is that the kernel \( K_{k,x}(y) \) of \( \text{Op}_\Lambda(1) \), associated to the contour (2.6), is not almost holomorphic. The idea of [BeBeSj08] is to replace \( \text{Op}_\Lambda(1) \) by \( \text{Op}_\Lambda(1+a) \) where \( a(x, y, \theta) \) is a negligible amplitude and the kernel of \( \text{Op}_\Lambda(1+a) \) is almost holomorphic. An amplitude \( a(x, y, \theta) \) is negligible if
\[ \text{Op}_\Lambda(a)(\chi u) = O(k^{-\infty})e^{\frac{k\phi(x)}{2}}\|u\|_{k\phi}. \]
To find a suitable condition for negligible amplitudes one formally writes
\[ \text{Op}_\Lambda(a) = \text{Op}_\Lambda(Sa|_{x=y}), \]
where \( S \) is a standard operator that is used in microlocal analysis to turn a symbol \( a(x, y, \theta) \) of a \( \Psi \text{DO} \) to a symbol of the form \( \tilde{a}(x, \theta) \). The operator \( S \) is formally defined by
\[ S = e^{-\sum_{m=0}^{\infty} \frac{(D_{\theta} \cdot D_y)^m}{m!k^m}}. \]
Then an amplitude \( a \) is negligible if \( Sa|_{x=y} \sim 0 \) as a formal power series. This implies that there exists an almost holomorphic vector field \( A(x, y, \theta) \) with formal power series
\[ A(x, y, \theta) \sim A_0(x, y, \theta) + \frac{A_1(x, y, \theta)}{k} + \frac{A_2(x, y, \theta)}{k^2} + \cdots. \]
such that
\begin{equation}
Sa \sim k(x - y) \cdot SA \mod \mathcal{I}^\infty,
\end{equation}
where \(\mathcal{I}^\infty\) is the set of functions \(f\) such that for any multi-index \(\alpha\), \(D^\alpha f = 0\) when \(x = y = \bar{z}\). Here \(a_m(x, y, \theta)\) are almost holomorphic functions and \(A_m(x, y, \theta)\) are almost holomorphic vector fields in \(\mathbb{C}^n\), defined on \(B^n(0, 3) \times B^n(0, 3) \times W\).

One particular \(SA\) can be solved as follows. First note that by (2.7) we must have \((SA)_0 = 0\) and \(A_0 = 0\). Then we put
\begin{equation}
SA(x, y, z) = -\frac{1}{k} \int_0^1 (D_y Sa)(x, tx + (1 - t)y, z) dt.
\end{equation}
By taking \(S^{-1}\), \(A\) can be solved uniquely as
\begin{equation}
A(x, y, z) = -\frac{1}{k} S^{-1} \int_0^1 (D_y Sa)(x, tx + (1 - t)y, z) dt.
\end{equation}
Then by the fundamental theorem of calculus we have
\begin{equation}
Sa(x, y, z) = k(x - y) \cdot SA(x, y, z) - \overline{(x - y)} \cdot \int_0^1 (D_y Sa)(x, tx + (1 - t)y, z) dt.
\end{equation}
By using the inverse operator \(S^{-1}\), we have
\begin{equation}
a(x, y, z) = D_\theta \cdot A + k(x - y) \cdot A - \overline{(x - y)} \cdot S^{-1} \int_0^1 (D_y Sa)(x, tx + (1 - t)y, z) dt.
\end{equation}
We use \(a^{(N)}\) and \(A^{(N)}\) to denote the partial sums of \(a\) and \(A\) up to order \(\frac{1}{k^N}\) respectively. And we denote
\(\nabla A := D_\theta \cdot A + k(x - y) \cdot A\).
Since \(A_0 = 0\), we obtain
\begin{equation}
a^{(N)} - \nabla \left( A^{(N+1)} \right) = \frac{D_\theta \cdot A_{N+1}}{k^{N+1}} - \overline{(x - y)} \cdot \left( S^{-1} \int_0^1 (D_y Sa)(x, tx + (1 - t)y, z) dt \right)^{(N)}.
\end{equation}
Next, we observe that the integral kernel of \(\text{Op}_\lambda(1 + a)\) is almost holomorphic if
\begin{equation}
1 + a(x, y, \theta) \sim B(x, z(x, y, \theta)) \Delta_0(x, y, \theta),
\end{equation}
where
\(\Delta_0(x, y, \theta) = \frac{\det \psi_{yz}(y, z)}{\det \theta_z(x, y, z)}\),
and \(B(x, z)\) is almost holomorphic and has an asymptotic expansion of the form
\begin{equation}
B(x, z) \sim b_0(x, z) + \frac{b_1(x, z)}{k} + \frac{b_2(x, z)}{k^2} + \cdots,
\end{equation}
where \(b_m(x, z)\) are almost holomorphic. In fact, as it turns out, \(b_m(x, \bar{x})\) are an almost holomorphic extensions of \(b_m(x, \bar{x})\), the Bergman kernel coefficients of the on-diagonal asymptotic expansion of Zelditch-Catlin (1.1).
If the amplitude \(a\) is negligible, then by applying \(S(\cdot)|_{x=y}\) to both sides of (2.11), we get
\(S(B(x, z(x, y, \theta)) \Delta_0(x, y, \theta))|_{x=y} \sim 1\).
From this, one gets the following recursive equations for Bergman kernel coefficients $b_m(x, z)$, which will play a key role in the proof of Theorem 1.5:

\begin{equation}
 b_m(x, z(x, x, \theta)) = -\sum_{l=1}^{m} \frac{(D_y \cdot D_{\theta})^l}{l!} \left( b_{m-l} (x, z(x, y, \theta)) \Delta_0(x, y, \theta) \right)_{|y=x}.
\end{equation}

Additionally, by comparing the coefficients on both sides of (2.11), we have the following relations between $a_m$ and $b_m$:

\begin{equation}
 a_m(x, y, \theta) = \begin{cases} 
 \Delta_0(x, y, \theta) - 1 & \text{when } m = 0, \\
 b_m(x, z(x, y, \theta)) \Delta_0(x, y, \theta) & \text{when } m \geq 1.
\end{cases}
\end{equation}

These equations will be useful in estimating $a_m$ in terms of the bounds on $b_m$ from Theorem 1.5.

2.2. Almost Holomorphic Extensions of Gevrey functions. In this section, we will review the Gevrey class and consider almost holomorphic extensions of functions in such a class. Indeed, there are many different ways to construct almost holomorphic functions. We will adapt the way in [Ju97] to construct a particular one which is suitable for our analysis. Afterwards, various properties of such an extension are introduced, which will be used for the proof of Proposition 2.1 in Section 3. Although all the properties are natural and elementary, the proofs are however very lengthy. For the convenience of the readers, we shall only state the results we need and postpone the proofs to Section 7.

We recall the definition of Gevrey class $G^a(U)$. For more details, we refer the readers to [Ge18]. Take $\alpha, \beta \in (\mathbb{Z}^{\geq 0})^n$. Here are some standard notations of multi-indices we shall use in the following.

- $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.
- $\alpha \leq \beta$ if $\alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2, \cdots, \alpha_n \leq \beta_n$.
- $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.
- $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$.

**Definition 2.3.** Let $a \in (1, \infty)$ and $U$ be an open subset of $\mathbb{C}^n$. We denote by $G^a(U)$ the set of functions $f(x) \in C^\infty(U, \mathbb{C})$ such that there exists some constant $C_0 = C_0(f) > 0$ and $C_1 = C_1(f) > 0$, satisfying

\begin{equation}
 \|D_x^\alpha D_z^\beta f\|_{L^\infty(U)} \leq C_0 C_1 |\alpha| + |\beta| (a! \beta!)^a,
\end{equation}

for any multi-indices $\alpha, \beta \geq 0$. The space $G^a(U)$ is called the Gevrey class of index $a$. Note that each class $G^a(U)$ forms an algebra which is closed under differentiation and integration.

For any $f \in G^a(U)$, an almost holomorphic extension $F(f)(x, z)$ is a smooth function on $U \times U$ such that $F(f)(x, \bar{x}) = f(x)$ and the anti-holomorphic derivatives have infinite vanishing order along $x = \bar{z}$. We will use the way in [Ju97] to construct a particular almost holomorphic extension. In fact the construction of [Ju97] is adapted from Borel’s method (see also Hörmander [Ho68]). Here, we follow [Ju97] but we use a cut-off function $\chi$ in the Gevrey class $G^{1+\varepsilon}(\mathbb{R})$ where $\varepsilon$ is an arbitrary positive constant, and

\begin{equation}
 \chi(x) = \begin{cases} 
 1 & |x| \leq \frac{1}{2}, \\
 0 & |x| \geq 1.
\end{cases}
\end{equation}

To show the existence of such a cut-off function, one can use the fact that for any $\varepsilon > 0$, the function defined as

\begin{equation}
 f_\varepsilon(x) = \begin{cases} 
 \exp(-x^{-\frac{1}{\varepsilon}}) & x > 0 \\
 0 & x \leq 0,
\end{cases}
\end{equation}
Lemma 2.5. There exist positive constants $m$ and $r$ such that for any $y, z \in D$, there exists a genuine almost holomorphic extension $F(y, z) \in G^a(U)$.

We can take our cut-off function to be $\chi(x) = g(x + 1)g(-x + 1)$.

We now define our almost holomorphic extension of Gevrey functions.

Definition 2.4. Let $a \in (1, \infty)$, $U$ be the unit ball $B(0, 1)$ in $\mathbb{C}^n$, and $f(x) \in G^a(U)$. Let $C_1 = C_1(f)$ be the constant in Definition 2.3. Then for $(y, z) \in U \times U$, we define an almost holomorphic extension

$$F(f)(y, z) = \sum_{\alpha, \beta \geq 0} \frac{D_\alpha f}{\alpha!} \frac{D_\beta f}{\beta!} \left( \frac{y + z}{2} \right)^\alpha \left( \frac{y - z}{2} \right)^\beta \chi \left( |\alpha + \beta|^2 (a-1) 4^{a-1} C_1^2 |y - z|^2 \right).$$

We will justify that $F(f)$ defined as above is genuinely an almost holomorphic extension of $f$ along $y = z$. It is easy to see $F(f)(x, \bar{x}) = f(x)$. And in the next lemma, we will verify that $D_y F(f)(y, z) = O(|y - z|)$, and $D_z F(f)(y, z) = O(|y - z|)$. To be more precise, we show that these quantities vanish at a certain exponential rate along $y = z$.

Lemma 2.5. There exist positive constants $C$ and $b$ such that for any $y, z \in U$, the almost holomorphic extension $F(f)$ satisfies

$$|D_y F(f)(y, z)| \leq C \exp \left( -b |y - z|^{-\frac{1}{a-1}} \right),$$

$$|D_z F(f)(y, z)| \leq C \exp \left( -b |y - z|^{-\frac{1}{a-1}} \right).$$

In particular, $F(f)$ is almost holomorphic along $y = z$.

Indeed, there are various ways to define an almost holomorphic extension besides Definition 2.4. But they are all the same up to an $O(|y - z|)$ error term.

Lemma 2.6. Let $U$ be the unit ball $B(0, 1)$ in $\mathbb{C}^n$ and $f(x) \in C^\infty(U)$. If $F(y, z), \bar{F}(y, z) \in C^\infty(U \times U)$ are both almost holomorphic extensions of $f$, then

$$F(y, z) - \bar{F}(y, z) = O(|y - z|).$$

Next, we show a more general version of Lemma 2.5, which gives estimates on all the derivatives of $F(f)$. It turns out that if $f \in G^a(U)$, then $F(f) \in G^{a+\varepsilon}(U \times U)$ and when the anti-holomorphic derivative appears, it always vanishes to infinite order along $y = z$ at a certain exponential rate.

Lemma 2.7. Take $f \in G^a(U)$. Let $C_0(f)$ and $C_1(f)$ be the constants satisfying (2.3) for $f$. Then for any $\varepsilon > 0$, there exist positive constants $C_1 = C_1(\varepsilon, a, C_1(f))$, $b = b(a, C_1(f))$ and $A = A(a, n)$ such that for any multi-indices $\gamma, \delta, \xi, \eta \geq 0$, we have

$$|D_\gamma D_\delta D_\xi D_\eta F(f)(y, z)| \leq AC_0(f) C_1^{\gamma + \delta + \xi + \eta} |(\gamma! \delta! |\xi|! |\eta|!)^{a+\varepsilon},$$

hence $F(f)(y, z) \in G^{a+\varepsilon}(U \times U)$. Moreover, if $\xi + \eta > 0$, then

$$|D_\gamma D_\delta D_\xi D_\eta F(f)(y, z)| \leq AC_0(f) C_1^{\gamma + \delta + \xi + \eta} |(\gamma! \delta! |\xi|! |\eta|!)^{a+\varepsilon} \exp \left( -b |y - z|^{-\frac{1}{a-1}} \right).$$

In addition, when we are restricted to the diagonal $z = \bar{y}$, we can let $\varepsilon = 0$ in the above estimates.

This motivates us to give the following definition.
Definition 2.8. Let \( U \) be an open neighborhood of the origin in \( \mathbb{C}^{2n} \) and let \( \varepsilon > 0 \) be a constant. A function \( F(y, z) \in C^{\infty}(U) \) is called \( G^{a,\varepsilon} \)-almost holomorphic along the diagonal \( z = \bar{y} \) if there exist positive constants \( C_0 = C_0(F), C_1 = C_1(F) \) and \( b = b(F) \) such that for any multi-indices \( \gamma, \delta, \xi, \eta \geq 0 \), we have

\[
|D^\gamma_y D^\delta_z D^\xi_y D^\eta_z F(y, z)| \leq C_0 C_1^{\gamma + \delta + \xi + \eta} (\gamma! \delta! \xi! \eta!)^{a+\varepsilon}.
\]

And when \( \xi + \eta > 0 \), we have

\[
|D^\gamma_y D^\delta_z D^\xi_y D^\eta_z F(y, z)| \leq C_0 C_1^{\gamma + \delta + \xi + \eta} (\gamma! \delta! \xi! \eta!)^{a+\varepsilon} \exp\left(-b|y - \bar{z}|^{-\frac{1}{a+\varepsilon}}\right).
\]

In addition, when we are restricted to the diagonal \( z = \bar{y} \), we can let \( \varepsilon = 0 \) in the above estimates.

We use \( A^{a,\varepsilon}_\text{diag}(U) \) for the class of such functions. And we also use \( I^{a,\varepsilon}_\text{diag}(U) \) for functions \( F(y, z) \in C^{\infty}(U \times U) \) satisfying (2.21) with no restrictions on \( \xi \) and \( \eta \) (i.e. (2.21) holds even if \( \xi = \eta = 0 \)). And we say a vector belongs to \( A^{a,\varepsilon}_\text{diag}(U) \) or \( I^{a,\varepsilon}_\text{diag}(U) \) if each component function belongs to that class. Obviously, we have \( I^{a,\varepsilon}_\text{diag}(U) \subset A^{a,\varepsilon}_\text{diag}(U) \).

Since the recursive formula for Bergman coefficients requires studying functions of three variables in \( \mathbb{C}^n \), we also present the following definition for functions in \( \mathbb{C}^3 \).

Definition 2.9. Let \( \varepsilon > 0 \) be a constant. Let \( \theta(x, y, z) \) be a function on \( U \) such that \( \Phi : (x, y, z) \to (x, y, \theta(x, y, z)) \) is a diffeomorphism between \( U \) and its image denoted by an open set \( U' \subset \mathbb{C}^3 \). Take \( f(x, y, \theta) \in C^{\infty}(U') \). Denote \( v' = (x, y, \theta) \). We say \( f(x, y, \theta) \) is \( G^{a,\varepsilon} \)-almost holomorphic along \( x = y = \bar{z} \) under \( \theta \), if there exist some positive constants \( C_0 = C_0(f), C_1 = C_1(f) \), and \( b = b(f) \), such that for any multi-indices \( \alpha, \beta \geq 0 \), we have

\[
\left| D^\alpha_x D^\beta_y f(x, y, \theta(x, y, z)) \right| \leq C_0 C_1^{\alpha + \beta} (\alpha! \beta!)^{a+\varepsilon} \exp\left(-b (1 - \delta_0(\beta)) \max\{|x - \bar{z}|, |y - \bar{z}|\}^{-\frac{1}{a+\varepsilon}}\right),
\]

where \( \delta_0(\cdot) \) is the delta function whose value is 1 at 0 and it is zero elsewhere. In addition, when we are restricted to \( x = y = \bar{z} \), we can let \( \varepsilon = 0 \) in the above estimate.

We use \( A^{a,\varepsilon}_\theta(U') \) to denote the set of all \( G^{a,\varepsilon} \)-almost holomorphic functions along \( x = y = \bar{z} \) under \( \theta \) in the above sense. We will also use \( I^{a,\varepsilon}_\theta(U) \) for smooth functions \( f(x, y, \theta) \) such that for any multi-indices \( \alpha, \beta \geq 0 \),

\[
\left| D^\alpha_x D^\beta_y f(x, y, \theta(x, y, z)) \right| \leq C_0 C_1^{\alpha + \beta} (\alpha! \beta!)^{a+\varepsilon} \exp\left(-b \max\{|x - \bar{z}|, |y - \bar{z}|\}^{-\frac{1}{a+\varepsilon}}\right),
\]

And we say a vector belongs to \( A^{a,\varepsilon}_\theta(U) \) or \( I^{a,\varepsilon}_\theta(U) \) if each component function belongs to that class.

In the following, for simplicity we will use the notation

\[
\lambda_{a,\beta}(x, y, z) = \exp\left(-b (1 - \delta_0(\beta)) \max\{|x - \bar{z}|, |y - \bar{z}|\}^{-\frac{1}{a+\varepsilon}}\right).
\]

Remark 2.10. Note that by the above notation, \( A^{a,\varepsilon}_\theta \) (or \( I^{a,\varepsilon}_\theta \)) means \( A^{a,\varepsilon}_\theta \) (or \( I^{a,\varepsilon}_\theta \)) when \( \theta(x, y, z) = z \), which corresponds to the case \( \Phi = I \).

The space \( A^{a,\varepsilon}_\theta(U) \) is closed under algebraic operations and differentiations.

Lemma 2.11. For each \( \theta \) as described in the previous definition, \( A^{a,\varepsilon}_\theta(U) \) is closed under summation, subtraction, multiplication and differentiation. It is also closed under division if the denominator is uniformly away from zero in \( U \).
In particular, suppose \( f, g \in \mathcal{A}_\theta^{a,\varepsilon}(U) \). Then we can choose the constants appearing in (2.22) for the product \( fg \in \mathcal{A}_\theta^{a,\varepsilon}(U) \) as
\[
C_0(fg) = C_0(f)C_0(g), \quad C_1(fg) = 2\max\{C_1(f), C_1(g)\}, \quad \text{and} \quad b(fg) = \min\{b(f), b(g)\}.
\]
And for the differentiation, we can choose the constants as
\[
C_0(D^\alpha_x D^\beta_y f) = C_0(f)(2^a C_1(f))^{1+|\alpha+\beta|}, \quad C_1(D^\alpha_x D^\beta_y f) = 2^a C_1(f), \quad \text{and} \quad b(D^\alpha_x D^\beta_y f) = b(f),
\]
where \( v' = (x, y, \theta) \).

We shall use the following lemma that \( \mathcal{A}_\theta^{a,\varepsilon} \) is closed under certain integrals.

**Lemma 2.12.** If \( f(x, y, z) \in \mathcal{A}_\theta^{a,\varepsilon}(U) \), then \( g(x, y, z) = \int_0^1 f(x, tx + (1-t)y, z) dt \in \mathcal{A}_\theta^{a,\varepsilon} \). And we can choose the constants appearing in Definition 2.9 as
\[
C_0(g) = C_0(f), \quad C_1(g) = 2^{a+\varepsilon+1}C_1(f), \quad b(g) = b(f).
\]
Similarly, If \( f(x, y, z) \in \mathcal{T}_x^{a,\varepsilon} \), then \( g(x, y, z) = \int_0^1 f(x, tx + (1-t)y, z) dt \in \mathcal{T}_x^{a,\varepsilon} \). And we can choose
\[
C_0(g) = C_0(f), \quad C_1(g) = 2^{a+\varepsilon+1}C_1(f), \quad b(g) = b(f).
\]

The space \( \mathcal{A}_\theta^{a,\varepsilon} \) is also closed under composition in the following sense.

**Lemma 2.13.** Let \( f(x, y, z) \in \mathcal{A}_\theta^{a,\varepsilon}(U) \) be a function defined on \( U \subset \mathbb{C}^3n \). Let \( \theta(x, y, z) \) be a map on \( U \) such that \( \Phi : (x, y, z) \to (x, y, \theta(x, y, z)) \) is a diffeomorphism between \( U \) and its image denoted by an open set \( U' \subset \mathbb{C}^3n \). Let \( \Phi^{-1} : (x, y, \theta) \to (x, y, z(x, y, \theta)) \) be the inverse map of \( \Phi \). If \( z = z(x, y, \theta) \in \mathcal{A}_\theta^{a,\varepsilon}(U') \), then the composition function \( f(x, y, \theta) = f(x, y, z(x, y, \theta)) \in \mathcal{A}_\theta^{a,\varepsilon}(U') \).

In particular, if we use \( C_0(f), C_1(f) \) and \( b(f) \) to denote the constants in (2.22) for the function \( f \), then we can choose the constants for \( \tilde{f} \) as
\[
C_0(\tilde{f}) = C_0(f), \quad C_1(\tilde{f}) = 2^{a+\varepsilon+3n}m^{a+\varepsilon-1}C_0(z(x, y, \theta))C_1(f)(C(z(x, y, \theta)),
\]
and
\[
b(\tilde{f}) = \min\{b(f), b(z(x, y, \theta))\},
\]
where \( m = 3n \), \( C_0(z(x, y, \theta)) = \max_{1 \leq i \leq n} C_0(z_i(x, y, \theta)) \), \( C_1(z(x, y, \theta)) = \max_{1 \leq i \leq n} C_1(z_i(x, y, \theta)) \), and \( b(z(x, y, \theta)) = \min_{1 \leq i \leq n} b(z_i(x, y, \theta)) \).

**Remark 2.14.** In Lemma 2.13, if we further assume that \( f(x, y, z) \in \mathcal{T}_x^{a,\varepsilon} \), then the composition \( \tilde{f} \) belongs to \( \mathcal{T}_x^{a,\varepsilon} \) with the same choice of constants.

Now suppose \( U = B(0, 1) \subset \mathbb{C}^n \) and the Kähler potential \( \phi \) belongs to \( G^a(U) \) and let \( \psi = F(\phi) \) be the almost holomorphic extension of \( \phi \) defined by (2.4). Then it is easy to see that \( \psi(y, z) \in \mathcal{A}_\theta^{a,\varepsilon}(U) \).

Further by using Lemma 2.12, if we take \( \theta(x, y, z) = \int_0^1 (D_y \psi)(tx + (1-t)y, z) dt \), then \( \theta \in \mathcal{A}_\theta^{a,\varepsilon} \).

The following lemma says that the implicit functions \( z = z(x, y, \theta) \) belong to \( \mathcal{A}_\theta^{a,\varepsilon} \).

**Lemma 2.15.** Consider the following system of equations:
\[
(2.24) \quad \theta = \int_0^1 (D_y \psi)(tx + (1-t)y, z) dt.
\]
Then the implicit functions \( z = z(x, y, \theta) \) determined by the above equations belong to \( \mathcal{A}_\theta^{a,\varepsilon} \).

As we said at the beginning of this section, the proofs of all the above lemmas will be given in Section 7.

We are now prepared to prove Proposition 2.1.
Let $a_m$, $A_m$, and $b_m$ be given by (2.14), (2.9), and (2.12). Remember that $a^{(N)}$, $A^{(N)}$, and $B^{(N)}$ are the partial sums of $a$, $A$, and $B$ up to order $k^{-N}$. When we apply the method of Berman-Berndtsson-Sjöstrand, the remainder term is closely related to the growth rate of $a_m$, $A_m$ and their derivatives as we will see soon. So we will first make a series of lemmas on estimating $a_m$ and $A_m$ preparing for the proof of Proposition 2.1.

Let’s begin with estimating $a_m$.

**Lemma 3.1.** For each integer $m \geq 0$, we have $a_m(x,y,z) \in A_x^{0,\varepsilon}$. And we can choose the constants appearing in Definition 2.9 as

$$C_0(a_m) = C_0(b_m)C_0(\Delta_0) = C^{m+1}m^{2a+2\varepsilon},$$

$$b(a_m) = b,$$

where $C$ and $b$ are some positive constants independent of $m$.

**Proof.** Recall the relations between $a_m$ and $b_m$ from (2.14):

$$a_m(x,y,z) = \begin{cases} \Delta_0(x,y,z) - 1 & \text{when } m = 0, \\ b_m(x,z)\Delta_0(x,y,z) & \text{when } m \geq 1. \end{cases}$$

Since $\phi \in G^a$, the almost holomorphic extension $\psi(y,z)$ introduced as in Definition 2.4 belongs to $A_x^{0,\varepsilon}$. Recall that by Lemma 2.12, $\theta(x,y,z) = \int_0^1 (D_y \psi)(tx+(1-t)y,z)dt \in A_x^{0,\varepsilon}$. By Lemma 2.11, we know $A_x^{0,\varepsilon}$ is closed under certain algebraic operations and differentiation, $\Delta_0(x,y,z) = \frac{\det \psi_{uz}(y,z)}{\det \psi(x,y,z)}$ is therefore also contained in $A_x^{0,\varepsilon}$. Since $a_m(x,y,z) = b_m(x,z)\Delta_0(x,y,z)$ for $m \geq 1$, by our Lemma 2.11 on the multiplication, we can choose

$$C_0(a_m) = C_0(b_m)C_0(\Delta_0) = C^{m+1}m^{2a+2\varepsilon},$$

$$C(a_m) = 2\max \{C_0(b_m), C(\Delta_0)\},$$

and

$$b(a_m) = \min \{b(b_m), b(\Delta_0)\}.$$ 

Thus, the result follows as $C_0(b_m) = C^{m}m^{2a+2\varepsilon}$ for some positive constant $C$ and $C_1(b_m), b(b_m)$ are both independent of $m$ by Theorem 1.5. In addition, it is easy to see that when we are restricted to $x = y = z$, $\varepsilon$ can be replaced by $0$.

**Lemma 3.2.** Denote $\tilde{a}_m = a_m(x,y,\theta) = a_m(x,y,z(x,y,\theta))$. Then $a_m(x,y,\theta) \in A_0^{a,\varepsilon}$ and we can choose

$$C_0(\tilde{a}_m) = C^{m+1}m^{2a+2\varepsilon},$$

$$C(\tilde{a}_m) = C,$$

$$b(\tilde{a}_m) = b,$$

where $C$ and $b$ are some positive constants independent of $m$.

**Proof.** By Lemma 2.15, we have $z = z(x,y,\theta) \in A_0^{a,\varepsilon}$. Since $\tilde{a}_m$ is obtained from the composition of $a_m(x,y,z)$ and the map $z = z(x,y,\theta)$, by Lemma 2.13

$$C_0(\tilde{a}_m) = C_0(a_m),$$

$$C(\tilde{a}_m) = 2^{a+\varepsilon+9n}(3n)^{a-1+\varepsilon}C_0(z(x,y,\theta))C_1(a_m)C_1(z(x,y,\theta)),$$

$$b(\tilde{a}_m) = \min \{b(a_m), b(z(x,y,\theta))\}.$$

So the result follows directly from Lemma 3.1.

After we obtain the estimates on $a_m(x,y,z)$ and $a_m(x,y,\theta)$, now we proceed to $(Sa)_m(x,y,z)$ and $(Sa)_m(x,y,\theta)$.
Lemma 3.3. For each integer \(m \geq 0\), \((Sa)_m(x, y, \theta) \in A^{a, \varepsilon}_\theta\) and \((Sa)_m(x, y, z) \in A^{a, \varepsilon}_2\). And we can choose

\[
C_0((Sa)_m(x, y, \theta)) = C^{m+1}m^{2a+2\varepsilon}, \quad C_1((Sa)_m(x, y, \theta)) = C, \quad b((Sa)_m(x, y, \theta)) = b,
\]

\[
C_0((Sa)_m(x, y, z)) = C^{m+1}m^{2a+2\varepsilon}, \quad C_1((Sa)_m(x, y, z)) = C, \quad b((Sa)_m(x, y, z)) = b,
\]

where \(C\) and \(b\) are some positive constants independent of \(m\).

Proof. Since

\[
Sa = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(D_\theta \cdot D_y)^i}{i! k^i} \frac{a_j}{k^j} = \sum_{m=0}^{\infty} \sum_{i+j=m} (D_\theta \cdot D_y)^i a_j,
\]

we have

\[
(Sa)_m(x, y, \theta) = \sum_{i+j=m} \frac{(D_\theta \cdot D_y)^i a_j}{i!} (x, y, \theta) = \sum_{i+j=m} \sum_{|\delta|=i} \frac{1}{i!} (\frac{i}{\delta!}) D_\theta^\alpha D_y^\beta a_j(x, y, \theta).
\]

Denote \(\nu' = (x, y, \theta)\). Therefore,

\[
D_\nu^\alpha D_\nu'^\beta (Sa)_m(x, y, \theta) = \sum_{i+j=m} \sum_{|\delta|=i} \frac{1}{i!} (\frac{i}{\delta!}) D_\nu^\alpha (0, \delta, \delta) D_\nu'^\beta a_j(x, y, \theta)
\]

By Lemma 3.2, we have

\[
\left| D_\nu^\alpha D_\nu'^\beta (Sa)_m(x, y, \theta) \right| \leq \sum_{i+j=m} \sum_{|\delta|=i} \frac{1}{i!} (\frac{i}{\delta!}) C^{i+|\beta|+2|\delta|+j+1} j^{2a+2\varepsilon} (\alpha + (0, \delta, \delta))^{a+\varepsilon} \beta^{a+\varepsilon} \lambda_{b,|\beta|}(x, y, z)
\]

\[
\leq \sum_{i+j=m} \frac{n!}{i!} (2a+\varepsilon) (|\alpha|+2i) C^{i+|\beta|+2i+j+1} j^{2a+2\varepsilon} i^{2a+2\varepsilon} \alpha^{a+\varepsilon} \beta^{a+\varepsilon} \lambda_{b,|\beta|}(x, y, z)
\]

\[
\leq C^{m+1+|\alpha|+2a+2\varepsilon} \alpha^{a+\varepsilon} m^{2a+2\varepsilon} \lambda_{b,|\beta|}(x, y, z),
\]

where \(C\) is a constant independent to \(m\), which may vary from line to line, and \(b = b(a_m)\) is also independent to \(m\). So we obtain the result on \((Sa)_m(x, y, \theta)\). Note \(\theta(x, y, z) \in A^{a, \varepsilon}_\theta\). The result on \((Sa)_m(x, y, z)\) follows by Lemma 2.13 and keeping track of these constants.

Next, we will estimate the growth of \((SA)_m(x, y, z)\) and \((SA)_m(x, y, \theta)\).

Lemma 3.4. For each integer \(m \geq 0\), \((SA)_m(x, y, \theta) \in A^{a, \varepsilon}_\theta\) and \((SA)_m(x, y, z) \in A^{a, \varepsilon}_2\). And we can choose

\[
C_0((SA)_m(x, y, \theta)) = C^{m+1}m^{2a+2\varepsilon}, \quad C_1((SA)_m(x, y, \theta)) = C, \quad b((SA)_m(x, y, \theta)) = b,
\]

\[
C_0((SA)_m(x, y, z)) = C^{m+1}m^{2a+2\varepsilon}, \quad C_1((SA)_m(x, y, z)) = C, \quad b((SA)_m(x, y, z)) = b,
\]

where \(C\) and \(b\) are some positive constants independent of \(m\).

Proof. Recall \((SA)_0(x, y, z) = 0\). By (2.8), for \(m \geq 1\), we have

\[
(SA)_m(x, y, z) = - \int_0^1 D_y(SA)_{m-1}(x, tx + (1-t)y, z) dt.
\]

By Lemma 3.3, we have \(D_y(SA)_{m-1}(x, y, z) \in A^{a, \varepsilon}_2\). Then by Lemma 2.12 and Lemma 2.13, we have \((SA)_m(x, y, z) \in A^{a, \varepsilon}_2\) and \((SA)_m(x, y, \theta) \in A^{a, \varepsilon}_\theta\). The remaining part follows straightforward by keeping track of the constants.
We define
\[ d_m(x, y, z) = \int_0^1 D_y(Sa)_m(x, tx + (1 - t)y, z) dt, \]
and
\[ d_m(x, y, \theta) = d_m(x, y, z(x, y, \theta)). \]

Since \((Sa)_m \in A_{x, \varepsilon}^0\), we have \(D_y(Sa)_m \in T_{x, \varepsilon}^a\). By using Lemma 2.12 and Remark 2.14, we can also similarly prove the following estimates on \(d_m(x, y, z)\) and \(d_m(x, y, \theta)\).

**Lemma 3.5.** For each integer \(m \geq 0\), \(d_m(x, y, \theta) \in T_{x, \varepsilon}^a\) and \(d_m(x, y, z) \in T_{x, \varepsilon}^a\). And we can choose
\[
C_0(d_m(x, y, \theta)) = C^{m+1}m!^{2a+2\varepsilon}, \quad C_1(d_m(x, y, \theta)) = C, \quad b(d_m(x, y, \theta)) = b, \\
C_0(d_m(x, y, z)) = C^{m+1}m!^{2a+2\varepsilon}, \quad C_1(d_m(x, y, z)) = C, \quad b(d_m(x, y, z)) = b,
\]
where \(C\) and \(b\) are some positive constants independent of \(m\).

Now that we have the estimates on \((SA)_m\) in Lemma 3.4, by using the operator \(S^{-1}\), we obtain the following estimates on \(A_m\).

**Lemma 3.6.** For each integer \(m \geq 0\), \(A_m(x, y, \theta) \in A_{x, \varepsilon}^a\) and \(A_m(x, y, z) \in A_{x, \varepsilon}^a\). And we can choose
\[
C_0(A_m(x, y, \theta)) = C^{m+1}m!^{2a+2\varepsilon}, \quad C_1(A_m(x, y, \theta)) = C, \quad b(A_m(x, y, \theta)) = b, \\
C_0(A_m(x, y, z)) = C^{m+1}m!^{2a+2\varepsilon}, \quad C_1(A_m(x, y, z)) = C, \quad b(A_m(x, y, z)) = b,
\]
where \(C\) and \(b\) are some positive constants independent of \(m\).

Formally, we write \(d = \sum_{m=0}^\infty \frac{d_m}{m!} \). Similarly, by using Lemma 3.5 and the operator \(S^{-1}\), we obtain the estimates on \((S^{-1}d)_m\).

**Lemma 3.7.** For each integer \(m \geq 0\), \((S^{-1}d)_m(x, y, \theta) \in T_{x, \varepsilon}^a\) and \((S^{-1}d)_m(x, y, z) \in T_{x, \varepsilon}^a\). And we can choose
\[
C_0((S^{-1}d)_m(x, y, \theta)) = C^{m+1}m!^{2a+2\varepsilon}, \quad C_1((S^{-1}d)_m(x, y, \theta)) = C, \quad b((S^{-1}d)_m(x, y, \theta)) = b, \\
C_0((S^{-1}d)_m(x, y, z)) = C^{m+1}m!^{2a+2\varepsilon}, \quad C_1((S^{-1}d)_m(x, y, z)) = C, \quad b((S^{-1}d)_m(x, y, z)) = b,
\]
where \(C\) and \(b\) are some positive constants independent of \(m\).

Since the proof of Lemma 3.6 and 3.7 follow in the same way as that of Lemma 3.3, we omit them here.

We are now ready to estimate \(A_N\), \(A^{(N)}\) and \(D_\theta \cdot A_N\) on the good contour \(\Lambda = \{(x, \theta(x, y, \bar{y})) : x, y \in B^n(0, 1)\}\). For any smooth function \(f(x, y, \theta)\), we denote
\[
\|f(x, y, \theta)\|_{L^\infty(\Lambda)} := \|f(x, y, \theta(x, y, \bar{y}))\|_{L^\infty(B^n(0, 1) \times B^n(0, 1))}.
\]

**Lemma 3.8.** We have
\[
\|A^{(N)}(x, y, \theta(x, y, \bar{y}))\|_{L^\infty(\Lambda)} \leq Ck + \frac{C^N N!^{2a+2\varepsilon}}{k^N},
\]
where \(C\) is some constant independent of \(N\) and \(k\).
Proof. Note that by the definition of $A^{(N)}$ and estimates on each $A_m$ in Lemma 3.6, we have

$$\|A^{(N)}\|_{L^\infty(\Lambda)} \leq C \frac{1^{2\alpha+2\varepsilon}}{k} + \frac{C \cdot 2^{2\alpha+2\varepsilon}}{k^2} + \cdots + \frac{C^N N_1^{2\alpha+2\varepsilon}}{k^N}.$$  

We need to study the function $\frac{C^x x^{(2\alpha+2\varepsilon)x}}{e^{(2\alpha+2\varepsilon)x} k^x}$. To find the minimum of this function we consider

$$f(x) = \log \frac{C^x x^{(2\alpha+2\varepsilon)x}}{e^{(2\alpha+2\varepsilon)x} k^x} = x \log C + (2\alpha + 2\varepsilon) x \log x - (2\alpha + 2\varepsilon) x - x \log k,$$  

for $x \in (0, \infty)$. Since

$$f'(x) = \log C + (2\alpha + 2\varepsilon) \log x - \log k,$$

the only critical point of $f$ is $x_0 = \left(\frac{k}{C}\right)^{\frac{1}{2\alpha+2\varepsilon}}$, and the function $f$ is decreasing on the interval $(0, x_0)$ and increasing on the interval $[x_0, \infty)$. Hence if we take $N_0 = \lceil \left(\frac{k}{C}\right)^{\frac{1}{2\alpha+2\varepsilon}} \rceil$, then by using Stirling’s formula twice

$$\sum_{m=1}^{N} \frac{C^m m!^{2\alpha+2\varepsilon}}{k^m} \leq C' \sum_{m=1}^{N} m^{a+\varepsilon} \frac{C^m m^{2\alpha+2\varepsilon}}{e^{2\alpha+2\varepsilon} k^m} \leq C'' \left( N_{0}^{a+\varepsilon+1} + N^{a+\varepsilon}(N - N_0) \frac{C^N N^{(2\alpha+2\varepsilon)N}}{e^{(2\alpha+2\varepsilon)N} k^N} \right) \leq C'' \left( \left( \frac{k}{C} \right)^{\frac{a+\varepsilon+1}{2\alpha+2\varepsilon}} + N^{a+\varepsilon+1} \frac{C^N N^{(2\alpha+2\varepsilon)N}}{e^{(2\alpha+2\varepsilon)N} k^N} \right) \leq C'' \left( k + \frac{C^N N_1^{2\alpha+2\varepsilon}}{k^N} \right).$$

The result follows by replacing $C'$ and $C''$ by a larger constant $C$.  

We also need the estimates on the anti-holomorphic derivatives of $A_N$ and $A^{(N)}$.

Lemma 3.9. There exists positive constants $C$ and $b$ independent of $N$ and $k$ such that for any $(x, y) \in B^n(0, 1) \times B^n(0, 1)$, we have

$$\left| \left( D_\theta \cdot A^{(N)} \right) (x, y, \theta(x, y, \bar{y})) \right| \leq \left( C k + \frac{C^N N_1^{2\alpha+2\varepsilon}}{k^N} \right) \exp \left( -b|x - y|^{-\frac{1}{a+\varepsilon}} \right),$$

$$\left| \left( D_y \cdot A^{(N)} \right) (x, y, \theta(x, y, \bar{y})) \right| \leq \left( C k + \frac{C^N N_1^{2\alpha+2\varepsilon}}{k^N} \right) \exp \left( -b|x - y|^{-\frac{1}{a+\varepsilon}} \right).$$

We omit the proof of this lemma here since it follows in a similar way as the previous lemma by using Lemma 3.6 and the only difference is the extra exponential factor that comes from the anti-holomorphic derivatives of $A_m$ since $A_m(x, y, \theta) \in A_{\theta}^{a, \varepsilon}$.

Another key lemma is:

Lemma 3.10. There exists positive constants $C$ and $b$ independent of $N$ and $k$, such that for any $(x, y) \in B^n(0, 1) \times B^n(0, 1)$, we have

$$\left| \left( S^{-1} \int_0^1 (D_y S a)(x, t x + (1 - t)y, z) dt \right) \right| \leq C m!^{2\alpha+2\varepsilon} \exp \left( -b|x - y|^{-\frac{1}{a+\varepsilon}} \right),$$

where $S$ is the Bergman kernel.
and
\[
\left| (S^{-1} \int_0^1 (D_\bar{y}S\alpha)(x, tx + (1 - t)y, z) \, dt)^{(N)} (x, y, \bar{y}) \right| \leq \left( Ck + \frac{CN \cdot 2a + 2\varepsilon}{k^N} \right) \exp \left( -b|x - y|^\frac{1}{2-\tau} \right).
\]

Proof. The first inequality directly follows from Lemma 3.7 and the second inequality follows by the same argument as in the proof of Lemma 3.8. \qed

Recalling (2.10) and using Lemma 3.6 and 3.10 together, we obtain the following corollary.

**Corollary 3.11.** There exists positive constants $C$ and $b$ independent of $N$, such that for any $(x, y) \in B^n(0, 1) \times B^n(0, 1)$, we have
\[
\left| a^{(N)} - \nabla \left( A^{(N+1)} \right) (x, y, \bar{y}) \right| \leq \frac{CN^{N+1}(N + 1)^{2a + 2\varepsilon}}{k^{N+1}} + |x - y| \left( Ck + \frac{CN \cdot 2a + 2\varepsilon}{k^N} \right) \exp \left( -b|x - y|^\frac{1}{2-\tau} \right).
\]

Now we are ready to prove Proposition 2.1.

We claim that
\[
(3.4) \quad u(x) = \text{Op}_\Lambda \left( 1 + a^{(N)} \right) (\chi u)(x) + k^n \mathcal{R}_{N+1}(\phi, k) e^{\frac{k\phi(x)}{2}} \|u\|_{k^\phi},
\]
where uniformly for $x \in B(0, \frac{1}{4})$ we have
\[
(3.5) \quad |\mathcal{R}_{N+1}(\phi, k)| \leq \frac{CN^{N+1}(N + 1)^{2a + 2\varepsilon}}{k^{N+1}},
\]
and the integral kernel of $\text{Op}_\Lambda \left( 1 + a^{(N)} \right)$ is almost holomorphic. The complex conjugate of this kernel is given by
\[
(3.6) \quad K^{(N)}_{k, \bar{x}}(y) = \left( \frac{k}{\pi} \right)^n e^{k\phi(x, \bar{y})} \left( 1 + a^{(N)}(x, y, \theta(x, y, \bar{y})) \right) \Delta_0(x, y, \theta(x, y, \bar{y}))^{-1},
\]
which by the relation (2.11) is reduced to
\[
K^{(N)}_{k, \bar{x}}(y) = \left( \frac{k}{\pi} \right)^n e^{k\phi(x, \bar{y})} B^{(N)}(x, \bar{y}).
\]

Hence $K^{(N)}_{k, \bar{x}}(y)$ is almost holomorphic in $y$ because $B(x, z)$ is almost holomorphic.

In the light of (2.4), to prove (3.4) it suffices to show that
\[
\forall x \in B(0, \frac{1}{4}): \quad |\text{Op}_\Lambda \left( a^{(N)} \right) (\chi u)(x)| \leq \frac{CN^{N+1}(N + 1)^{2a + 2\varepsilon}}{k^{N+1-n}} e^{\frac{k\phi(x)}{2}} \|u\|_{k^\phi}.
\]

By definition,
\[
\text{Op}_\Lambda \left( a^{(N)} \right) (\chi u)(x) = c_n \left( \frac{k}{2 \pi} \right)^n \int_A e^{k\theta \cdot (x - y)} u(y) \chi(y) a^{(N)} \, d\theta \wedge dy.
\]
It is easy to see that using integration by parts (see for example the proof of Proposition 2.2 in [BeBeSj08]), we get

\[
\int_\Lambda e^{k\theta \cdot (x-y)} u(y) \chi(y) a^{(N)} \, d\theta \wedge dy = \\
- \int_\Lambda d\chi \wedge u(y) e^{k\theta \cdot (x-y)} A^{(N+1)} \wedge dy + \int_\Lambda e^{k\theta \cdot (x-y)} u(y) \chi(y) \left( a^{(N)} - \nabla \left( A^{(N+1)} \right) \right) \, d\theta \wedge dy \\
- \sum_{i,j} \int_\Lambda e^{k\theta \cdot (x-y)} u(y) \chi(y) \frac{\partial A_i}{\partial \theta_j} d\theta_j \wedge \hat{d}\theta_i \wedge dy - \sum_{i,j} \int_\Lambda e^{k\theta \cdot (x-y)} u(y) \chi(y) \frac{\partial A_i}{\partial \theta_j} d\theta_j \wedge \hat{d}\theta_i \wedge dy
\]

In the first integral, we have identified the \( n \)-vector \( A \) as an \((n - 1, 0)\) form defined by \( A = \sum_{j=1}^n A_j \hat{d}\theta_j \), where \( \hat{d}\theta_j \) is the wedge product of all \( \{d\theta_k\}_{k \neq j} \) such that \( d\theta_j \wedge \hat{d}\theta_j = d\theta \).

We now estimate the integrals on the right hand side of the above equality. For the first integral, as \( d\chi(y) = 0 \) for \( y \in B^n(0, \frac{1}{2}) \), we have \(|x - y| \geq \frac{1}{4} \) for \( x \in B^n(0, 1) \) or otherwise the integrand vanishes. If we take \( \theta^*(x, y, z) = \int_0^1 (D_y \psi)(tx + (1-t)y, z) \, dt \), then by Taylor expansion we have

\[
2 \text{Re} (\theta \cdot (x - y)) + 2 \text{Re} (\theta^* \cdot (x - y)) = 2 \text{Re} (\psi(x, y) - \psi(y, y)) \leq \phi(x) - \phi(y) - \delta|x - y|^2,
\]

where \( \theta^* = \theta^*(x, y, \bar{y}) \) and \( \delta \) is some positive constant. Note that \( \theta^*(x, y, z) \in \mathcal{T}_x^{a, \epsilon} \) by Lemma 2.12. We have \( \theta^*(x, y, \bar{y}) = O(|x - y|^{-\epsilon}) \). Thus by rescaling the unit ball, \( \theta^*(x, y, \bar{y}) \) can be absorbed by \( \delta|x - y|^2 \). Therefore, by changing \( \delta \) to a smaller constant, the integrand of the first integral is bounded by some constant times

\[
\left| u(y) e^{\frac{\delta(x)}{2} - \frac{\delta(y)}{2} - \delta k} \right| A^{(N+1)}.
\]

So by using Cauchy-Schwarz inequality, we obtain the first integral bounded by some constant times

\[
\left\| A^{(N+1)} \right\|_{L^\infty(\Lambda)} e^{\frac{\delta(x)}{2}} \left\| u \right\|_{k\phi} e^{-\delta k}.
\]

By Lemma 3.8 and \( ke^{-\delta k} \leq \frac{(N+2)!}{k^{N+1} N^{N+2}} \), the first integral is bounded by

\[
\frac{C^{N+1} (N + 1) \theta_{a+2\epsilon}}{k N} e^{\frac{\delta(x)}{2}} \left\| u \right\|_{k\phi}.
\]

For the second term, the integrand is bounded by some constant times

\[
\left| u(y) e^{\frac{\delta(x)}{2} - \frac{\delta(y)}{2} - \delta k|x - y|^2} \right| a^{(N)} - \nabla \left( A^{(N+1)} \right) |.
\]

By Corollary 3.11, we have

\[
e^{-k\delta|x - y|^2} \left| a^{(N)} - \nabla \left( A^{(N+1)} \right) \right| \leq C^{N+1} \theta_{a+2\epsilon} + e^{-k\delta|x - y|^2} \left( Ck + \frac{C^N N^{2\theta_{a+2\epsilon}}}{k N} \right) \exp \left( -b|x - y|^{\frac{1}{a-1}} \right).
\]

Note for any positive integer \( M \),

\[
\exp \left( -b|x - y|^{\frac{1}{a-1}} \right) \leq \left( \frac{a - 1}{b} \right)^{M(a-1)} M^{\epsilon-1} |x - y|^M.
\]

Take \( M = 2N + 4 \). Since for any \( x, y \in B^n(0, 1) \),

\[
e^{-k\delta|x - y|^2} |x - y|^{2N+4} \leq \frac{(N + 2)!}{(\delta k)^{N+2}},
\]

\[
\leq \frac{(N + 2)!}{(\delta k)^{N+2}}.
\]

Asymptotic Properties of Bergman Kernels
1. Let \( K_k(x, y) \) be the Bergman kernel of \((M, L^k)\). As we noted before, we also write \( K_k(x, y) \) for the representation of the Bergman kernel in the local frame \( e_L^k \otimes e_L^\alpha \) and we denote \( K_{k, y}(x) := K_k(x, y) \). In the last section, we constructed the local Bergman kernel of order \( N \), which we denoted by \( K_k^{(N)}(x, y) = K_{k, y}^{(N)}(x) \). In this section, we show that \( K_k(x, y) \) is equal to \( K_k^{(N)}(x, y) \) up to order \( k^{-N} \) when \( x, y \) are sufficiently close to each other. Moreover, we will give a precise upper bound for the error term.

**Proposition 4.1.** There exists \( \delta > 0 \) such that whenever \( d(x, y) < \delta \), we have

\[
K_k(x, y) = K_k^{(N)}(x, y) + k^{\frac{3n}{2}} \mathcal{R}_{N+1}(\phi, k) e^{-\frac{k\phi(x)}{2} - \frac{k\phi(y)}{2}},
\]  

(4.1)
where

\[ |\mathcal{R}_{N+1}(\phi, k)| \leq \frac{C^{N+1}(N+1)!2^{2\varepsilon} + 2C}{k^{N+1}}, \]

and the constant \( C \) is independent of \( N, x, y, \) and \( k. \)

**Proof.** We fix \( x \in M \) and assume that \( \phi \) is in Gevrey a class in \( B^n(x, 3). \) Let \( \chi \) be a smooth cut-off function such that

\[ \chi(z) = \begin{cases} 1 & z \in B^n(x, \frac{1}{4}) \\ 0 & z \notin B^n(x, \frac{3}{4}) \end{cases}. \]

We assume \( y \in B^n(x, \frac{1}{4}). \) We first observe that

\[ K_k(y, x) = \left( \chi K_{k,x}, K_{k,y}^{(N)} \right)_{k\phi} + \mathcal{S}_{N+1}(\phi, k)k^{\frac{n}{2}}e^{k\frac{\phi(x)}{2}} + \phi(y), \]

where \( |\mathcal{S}_{N+1}(\phi, k)| \leq \frac{C^{N+1}(N+1)!2^{2\varepsilon}}{k^{N+1}}. \) This is because, by Proposition 2.1, we have

\[ K_{k,x}(y) = \left( \chi K_{k,x}, K_{k,y}^{(N)} \right)_{k\phi} + k^n\mathcal{S}_{N+1}(\phi, k)e^{k\frac{\phi(y)}{2}}\|K_{k,x}\|_{k\phi}, \]

and by the reproducing property of Bergman kernel, we have

\[ \|K_{k,x}\|_{k\phi}e^{-k\frac{\phi(y)}{2}} \leq \left\| K_{k,x} \right\|_{L^2(M, L^k)} \right|_{h^k} = \sqrt{|K_{k,x}(x, x)|_{h^k} \leq Ck^\frac{n}{2}. \]

That why \( |K_k(x, x)|_{h^k} \leq Ck^n \) follows from the extreme property of the Bergman function and also the sub-mean value inequality. For a simple proof see for example Lemma 4.1 of [HeKeSeXu16].

Next, we define

\[ u_{k,y}(z) = \chi(z)K_{k,y}^{(N)}(z) - \left( \chi K_{k,y}^{(N)}, K_{k,z} \right)_{k\phi}. \]

Our goal is to estimate \( |u_{k,y}(x)|. \) Since \( \left( \chi K_{k,y}^{(N)}, K_{k,x} \right)_{k\phi} \) is the Bergman projection of \( \chi K_{k,y}^{(N)}, u_{k,y} \) is the minimal \( L^2 \) solution to the equation

\[ \mathcal{D}u = \mathcal{D}(\chi K_{k,y}^{(N)}). \]

So by using Hörmander’s \( L^2 \) estimates [Ho66] (see [Be10] for an exposition), we have

\[ \|u_{k,y}\|_{L^2}^2 \leq \frac{C}{k} \left\| \mathcal{D} \left( \chi K_{k,y}^{(N)} \right) \right\|_{L^2}^2. \]

We have \( \mathcal{D}(\chi K_{k,y}^{(N)})(z) = \mathcal{D}(z)K_{k,y}^{(N)}(z) + \chi(z)\mathcal{D}K_{k,y}^{(N)}(z). \) Recall that by (3.6)

\[ K_{k,y}^{(N)}(z) = \left( \frac{k}{\pi} \right)^n e^{k\psi(z, y)}B^{(N)}(z, y). \]

For the first term, since \( \mathcal{D}(z) \) is supported in \( d(z, x) \geq \frac{1}{2}, \) if we choose \( d(x, y) \leq \frac{1}{4}, \) then using

\[ \text{Re} \psi(z, y) \leq -\delta|z - y|^2 + \frac{\phi(x)}{2} + \frac{\phi(y)}{2}, \]

we have

\[ \left| \mathcal{D}(z)K_{k,y}^{(N)}(z) \right| \leq Ck^ne^{-k\delta + k\frac{\phi(x)}{2} + k\frac{\phi(y)}{2}}\|B^{(N)}\|_{L^\infty(U \times U)}. \]
We can estimate \( \|B^{(N)}\|_{L^\infty(U \times U)} \) using our Theorem 1.5

\[
\|B^{(N)}\|_{L^\infty(U \times U)} \leq 1 + \frac{1}{k} \|b_1\|_{L^\infty(U \times U)} + \cdots + \frac{1}{k^N} \|b_N\|_{L^\infty(U \times U)}
\leq 1 + \frac{C1^{2a+2\varepsilon}}{k} + \frac{C2^{2a+2\varepsilon}}{k^2} + \cdots + \frac{CN^{12a+2\varepsilon}}{k^N}
\leq C \left(k + \frac{CN^12a+2\varepsilon}{k^N}\right).
\]

Hence,

\[
\left|\overline{\chi}(z)K_{k,y}^{(N)}(z)\right| \leq k^n e^{-kd + k\phi(z) + k \frac{\phi(z)}{2}} \left( Ck + \frac{CN^12a+2\varepsilon}{k^N}\right)
\leq k^n e^{k\phi(z) + \frac{\phi(z)}{2}} \left( Ck + \frac{CN^12a+2\varepsilon}{k^N}\right)
\frac{C^{N+1}(N + 1)!2a+2\varepsilon}{k^{N+1}}.
\]

For the second term, we have

\[
\left|\chi(z)\overline{\partial K}_{k,y}^{(N)}(z)\right| \leq Ck^n e^{-k|z - y|^2 + k\phi(y) + k \frac{\phi(y)}{2}} \left( k |\overline{\partial} \psi(z, y)| |B^{(N)}(z, y)| + |\overline{\partial} B^{(N)}(z, y)| \right).
\]

By using the fact that \( \psi(y, z) \in A_{diag}^{a, \varepsilon} \) and Theorem 1.5,

\[
k |\overline{\partial} \psi(z, y)| |B^{(N)}(z, y)| + |\overline{\partial} B^{(N)}(z, y)| \leq Ck \left(k + \frac{CN^12a+2\varepsilon}{k^N}\right) \exp(-b|z - y|^{-\frac{1}{a-1}}).
\]

Then by the fact that

\[
e^{-k\delta|x - y|^2} \exp(-b|x - y|^{-\frac{1}{a-1}}) \leq \frac{C^{N+1}(N + 1)!2a-1}{k^{N+3}},
\]

we have

\[
\left|\chi(z)\overline{\partial K}_{k,y}^{(N)}(z)\right| \leq k^n e^{k\phi(y) + \frac{\phi(y)}{2}} \frac{C^{N+1}(N + 1)!2a+2\varepsilon}{k^{N+1}}.
\]

So

\[
(4.5) \quad \left|\overline{\partial} \left(\chi K_{k,y}^{(N)}\right)(z)\right| \leq k^n e^{k\phi(y) + \frac{\phi(y)}{2}} \frac{C^{N+1}(N + 1)!2a+2\varepsilon}{k^{N+1}},
\]

and

\[
(4.6) \quad \|u_{k,y}\|^2 \leq \frac{C}{k} \|\overline{\partial} \left(\chi K_{k,y}^{(N)}\right)\|^2_{L^2} \leq Ck^{2n} e^{k\phi(y)} \left(\frac{C^{N+1}(N + 1)!2a+2\varepsilon}{k^{N+1}}\right)^2.
\]

By using Bochner-Martinelli formula in a small Euclidean ball \( B^n(x, r) \), we have

\[
r^{2n-1} |u_{k,y}(x)| \leq C \int_{\partial B^n(x, r)} |u(z)| dS + C \int_{B^n(x, r)} |\overline{\partial} u(z)| \left|\frac{r^{2n-1}}{|z - x|^{2n-1}}\right| dV_0,
\]

where \( dS \) and \( dV_0 \) are respectively the standard volume forms of \( \partial B^n(0, 1) \) and \( B^n(0, 1) \) in Euclidean space. If we use the Bochner coordinates at \( x \), then \( \phi(z) - \phi(x) = O(|z|^2) \), and thus

\[
e^{k\left(\frac{\phi(z)}{2} - \frac{\phi(x)}{2}\right)} \leq C, \quad \text{for any } z \in B^n(x, \frac{1}{\sqrt{k}}).
\]
By integrating the above inequality with respect to $r$ from 0 to $\frac{1}{\sqrt{k}}$, we obtain

$$|u_{k,y}(x)| \leq Ck^n \int_{B^n(x, \frac{1}{\sqrt{k}})} |u(z)|dV_0 + Ck^n \int_0^{\frac{1}{\sqrt{k}}} \int_{B^n(x,r)} \left| \frac{\partial u(z)}{\partial r} \right| \frac{1}{|z-x|^{2n-1}} dV_0 dr$$

$$\leq Ck^n \int_{B^n(x, \frac{1}{\sqrt{k}})} |u(z)|dV_0 + C\|e^{-\frac{k\phi(x)}{2}} \frac{\partial u(z)}{\partial L_\infty} e^{\frac{k\phi(x)}{2}} \int_{B^n(x, \frac{1}{\sqrt{k}})} \frac{1}{|z-x|^{2n-1}} e^{k\left(\frac{\phi(x)}{2} - \frac{\phi(y)}{2}\right)} dV_0$$

$$\leq Ck^n e^{-\frac{k\phi(x)}{2}} \left( \|u\|_{L^2} + \|e^{-\frac{k\phi(x)}{2}} \frac{\partial u(z)}{\partial L_\infty}\right).$$

Therefore, by the estimates (4.5) and (4.6), it follows that

$$|u_{k,y}(x)| \leq Ck^n e^{k\frac{\phi(x)}{2}} e^{k\delta(k)} \left( \frac{C^{N+1}(N+1)!^{2a+2\epsilon}}{k^{N+1}} \right).$$

Combining this estimate with (4.3) and recalling the definition of $u_{k,y}$ in (4.4), we get the result.

We point out that we have renewed the constant $C$ at each step, but the final constant is independent of $k$ and $N$. We also note that the constant $C$ may depend on the point $x$, however by a simple compactness argument one can see that each such $C$ can be bounded by a uniform constant independent of $x$. \hfill $\Box$

Now we are ready to prove Theorem 1.1 and its corollaries.

4.1 Proof of Theorem 1.1. By Proposition 4.1, we just need to show that with $N = N_0 - 1 = [(k/C)^{\frac{1}{2a+2\epsilon}}] - 1$, we have

$$k^{\frac{3n}{2}} e^{k\frac{\phi(x)}{2} + k\frac{\phi(y)}{2}} = e^{k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} e^{-\delta k^{\frac{1}{2a+2\epsilon}} O(1)}.$$

However, by the same proposition we know that

$$|R_{N_0}(\phi, k)| \leq \frac{C^{N_0}N_0!^{2a+2\epsilon}}{k^{N_0}}.$$

Hence it is enough to show that

$$\frac{C^{N_0}N_0!^{2a+2\epsilon}}{k^{N_0}} = e^{-\delta k^{\frac{1}{2a+2\epsilon}} O(1)}.$$

By Stirling’s formula,

$$\frac{C^{N_0}N_0!^{2a+2\epsilon}}{k^{N_0}} \leq C' N_0^{a+\epsilon} e^{(2a+2\epsilon)N_0} k^{-N_0} \leq C' N_0^{a+\epsilon} e^{-(2a+2\epsilon)N_0} \leq C'' k^{\frac{3n}{2}} e^{-\delta k^{\frac{1}{2a+2\epsilon}} O(1)}.$$

Since

$$k^{\frac{3n}{2}} e^{-\delta k^{\frac{1}{2a+2\epsilon}}} \leq C'' e^{-\delta k^{\frac{1}{2a+2\epsilon}}} \leq \delta = \frac{a+\epsilon}{C^{\frac{1}{2a+2\epsilon}}}$$

would do the job.

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2For convenience, we use $N_0$ for $N_0(k) = [(k/C)^{\frac{1}{2a+2\epsilon}}]$. 

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4.2. Proof of Corollary 1.2.

Proof. By Theorem 1.1, uniformly for any \( x, y \in U \), we have

\[
K_k(x, y) = e^{k \psi(x, y)} \frac{k^n}{\pi^n} \left( 1 + \sum_{j=1}^{N_0(k)-1} \frac{b_j(x, \bar{y})}{k^j} \right) + e^{k \frac{(x+y)}{2}} e^{-\delta k \frac{1}{1+2\varepsilon}} O(1).
\]

For any given positive integer \( N \), we rewrite the above formula as follows.

\[
K_k(x, y) = e^{-k \psi(x, \bar{y})} \frac{k^n}{\pi^n} \left( 1 + \sum_{j=1}^{N-1} \frac{b_j(x, \bar{y})}{k^j} + \sum_{j=N}^{N_0(k)-1} \frac{b_j(x, \bar{y})}{k^j} + e^{k \frac{(x+y)}{2}} e^{-\delta k \frac{1}{1+2\varepsilon}} O(1) \right).
\]

Our first observation is that, if \( d(x, y) \leq \sqrt{k} \frac{1}{1+2\varepsilon} \), then

\[
\left| \frac{k}{e^{2k(x+y)}} e^{-\delta k \frac{1}{1+2\varepsilon}} \right| = e^{\frac{k}{2} D(x, y) - \delta k \frac{1}{1+2\varepsilon}} \leq e^{-\frac{k}{2} \delta k \frac{1}{1+2\varepsilon}}.
\]

Now we estimate the term \( \sum_{j=N}^{N_0(k)-1} \frac{b_j(x, \bar{y})}{k^j} \). By Stirling’s formula, we have

\[
|b_j(x, \bar{y})| \leq C_j j^{2a+2\varepsilon} \leq C' j^{2a+2\varepsilon} \frac{C_j j^{2a+2\varepsilon} j}{e^{2a+2\varepsilon} j k^j}.
\]

Since \( \frac{C_j j^{2a+2\varepsilon} j}{e^{2a+2\varepsilon} j k^j} \) is monotonically decreasing for \( 1 \leq j \leq N_0(k) - 1 \) (with the help of Stirling’s formula once more), we get

\[
\left| \sum_{j=N}^{N_0(k)-1} \frac{b_j(x, \bar{y})}{k^j} \right| \leq C' \sum_{j=N+1}^{N_0(k)-1} C_j j^{2a+2\varepsilon} \frac{C_j j^{2a+2\varepsilon} j}{e^{2a+2\varepsilon} j k^j} \leq \frac{C' N_0^{a+\varepsilon} C_j j^{2a+2\varepsilon}}{N_0^{a+\varepsilon} j k^{N_0+1}} \leq \frac{C'' N_0^{2a+2\varepsilon}}{k^{N_0}}.
\]

Therefore,

\[
K_k(x, y) = e^{-k \psi(x, \bar{y})} \frac{k^n}{\pi^n} \left( 1 + \sum_{j=1}^{N-1} \frac{b_j(x, \bar{y})}{k^j} + \frac{C'' N_0^{2a+2\varepsilon}}{k^{N_0}} + e^{-\frac{4k}{\delta} \frac{1}{1+2\varepsilon}} O(1) \right).
\]

By the fact that

\[
e^{-\frac{4k}{\delta} \frac{1}{1+2\varepsilon}} \leq \left( \frac{4}{\delta} \right)^{(2a+2\varepsilon)N} \frac{((2a+2\varepsilon)N)^{(2a+2\varepsilon)N}}{e^{(2a+2\varepsilon)N} k^{N_0}} \leq \left( \frac{8a+8\varepsilon}{\delta} \right)^{(2a+2\varepsilon)N} \frac{N!^{2a+2\varepsilon}}{k^{N_0}},
\]

the first part of our result follows.

Now we prove the second part. Let \( \tilde{b}_m(x, z) \) be another almost holomorphic extension of \( b_m(x, \bar{x}) \). By Lemma 2.6, we have

\[
|b_m(x, \bar{y}) - \tilde{b}_m(x, \bar{y})| = O |x - y|^\infty = O \left( \frac{1}{k^\infty} \right).
\]

The second equality follows from our assumption that \( d(x, y) \leq \delta k^{-\frac{1}{1+2\varepsilon}} \). So the result follows. \( \square \)
Remark 4.2. Let \( \tilde{b}_m \) be the almost holomorphic extension defined in Definition 2.4. If we take \( \tilde{b}_m \) in Corollary 1.2, then (1.5) and (1.6) hold. The reason is as follows.

For any \( M \in \mathbb{N} \), there exists \( C_M \) such that

\[
|b_m(x, \bar{y}) - \tilde{b}_m(x, \bar{y})| \leq C_M |x - y|^{M+1}.
\]

And \( C_M \) depends on the sup norm of the all the \((M + 1)\)th derivatives of \( b_m(x, \bar{y}) \) and \( \tilde{b}_m(x, \bar{y}) \). By Theorem 1.5 and Lemma 2.7, we have

\[
C_M \leq C^{m+M} m!2^{a+2\epsilon} M!(a+\epsilon-1) k^{-N}
\]

where \( C \) is some positive constant independent of \( m \). If we take \( M = [(N - m)^{\frac{4a+4\epsilon}{2a+2\epsilon}}] \), then \( |x - y|^{M+1} \leq 2^{a+m-1} k^{-N} \), whence by Stirling’s formula we obtain

\[
\frac{|b_m(x, \bar{y}) - \tilde{b}_m(x, \bar{y})|}{k^m} \leq C^{m+M} m!2^{a+2\epsilon} M!(a+\epsilon-1) k^{-N}
\]

\[
\leq C'N m!2^{a+2\epsilon} (N - m)(2a+2\epsilon) k^{-N}
\]

\[
\leq C''N m!2^{a+2\epsilon} (N - m)(2a+2\epsilon) k^{-N},
\]

where \( C' \) and \( C'' \) are some positive constants independent of \( m \). We rename \( C'' \) by \( C \) and (1.5), (1.6) follows by

\[
\sum_{j=1}^{N-1} \frac{|b_j(x, \bar{y}) - \tilde{b}_j(x, \bar{y})|}{k^j} \leq \sum_{j=1}^{N-1} \frac{C'_N N!2^{a+2\epsilon}}{k^N} \leq \frac{(2C)_N N!2^{a+2\epsilon}}{k^N}.
\]

4.3. Proof of Corollary 1.3. By Theorem 1.1, we have

\[
K_k(x, y) = e^{\psi(x) k^m} \frac{k^n}{\sigma^n} \left( 1 + \sum_{j=1}^{N_0-1} \frac{b_j(x, \bar{y})}{k^j} \right) + e^{\frac{k\phi(x)}{2}} e^{\frac{k\phi(y)}{2}} e^{-\delta k \frac{1}{2a+2\epsilon}} O(1).
\]

Recall that \( D(x, y) = \phi(x) + \phi(y) - \psi(x, \bar{y}) - \psi(y, \bar{x}) \). Then

\[
\log |K_k(x, y)|^{\alpha_k} = \frac{kD(x, y)}{2} + n \log k - n \log \pi + \log \left| 1 + \sum_{j=1}^{N_0-1} \frac{b_j(x, \bar{y})}{k^j} \right| + e^{\frac{Q(x, y)}{2} k^{-\delta k \frac{1}{2a+2\epsilon}}} O(1),
\]

where \( Q(x, y) = \phi(x) + \phi(y) - 2\psi(x, \bar{y}) \). So it is sufficient to prove

\[
\log \left| 1 + \sum_{j=1}^{N_0-1} \frac{b_j(x, \bar{y})}{k^j} \right| + e^{\frac{Q(x, y)}{2} k^{-\delta k \frac{1}{2a+2\epsilon}}} O(1) = \log \left( 1 + O \left( \frac{1}{k} \right) \right).
\]

To do this we note that by our assumption \( D(x, y) \leq \frac{1}{2}\delta k^{-1+\frac{1}{2a+2\epsilon}} \), hence

\[
\left| e^{\frac{Q(x, y)}{2} k^{-\delta k \frac{1}{2a+2\epsilon}}} \right| = e^{-\frac{D(x, y)}{2} k^{-\delta k \frac{1}{2a+2\epsilon}}} \leq e^{-\frac{\delta k}{4} \frac{1}{2a+2\epsilon}}.
\]

It remains to show that

\[
\sum_{j=1}^{N_0-1} \frac{b_j(x, \bar{y})}{k^j} = O \left( \frac{1}{k} \right).
\]

By the estimates on \( |b_j(x, \bar{y})| \) from Theorem 1.5 and Stirling’s formula, we have

\[
\frac{|b_j(x, \bar{y})|}{k^j} \leq C_j j^{2a+2\epsilon} \leq C' j^{a+\epsilon} C j^{(2a+2\epsilon)j} e^{2j k^j}.
\]
As shown in Lemma 3.8, the function \( f(x) = \log \frac{C^x e^{(2a+2c)x}}{e^{x+k^2}} \) is decreasing on the interval \((0, \left(\frac{x}{k}\right)^{2a+2c}]\), thus for \( j \in [2, N_0 - 1] \),

\[
\frac{|b_j(x, \bar{y})|}{k^j} \leq C''(N_0 - 1)^{a+\varepsilon} C^{2a+4\varepsilon} e^{4\varepsilon k^2} \leq C'' C^2 \frac{N_0^{a+\varepsilon}}{k^2}.
\]

Therefore,

\[
\left| \sum_{j=1}^{N_0-1} \frac{b_j(x, \bar{y})}{k^j} \right| \leq \frac{C}{k} + C'' C^2 \frac{N_0^{a+\varepsilon+1}}{k^2} \leq \frac{C}{k} + C'' C \frac{1}{k} = O \left( \frac{1}{k} \right).
\]

5. Estimates on Bergman Kernel Coefficients

As before, we assume the Kähler metric is in Gevrey class \( G^a(U) \) for some neighborhood \( U \) of \( p \). We will estimate the growth rate of the Bergman kernel coefficients \( b_m(x, z) \) as \( m \to \infty \) for \( x, z \) in \( U \). Our goal is to prove Theorem 1.5.

The key ingredient for the proof is the following recursive formula on \( b_m(x, z) \) established in [BeBeSj08].

\[
b_m(x, z(x, x, \theta)) = - \sum_{l=1}^{m} \left( D_y \cdot D_\theta \right)^l \left( b_{m-l}(x, z(x, y, \theta)) \right) \Delta_0(x, y, \theta) \bigg|_{y=x}.
\]

We will break the proof of Theorem 1.5 into two steps. The first step is to derive from the recursive formula (5.1), a recursive inequality on \( \| D_x^p D_z^q b_m(x, z) \|_{L^\infty(U \times U)} \) for some neighborhood \( U \). The second step is to estimate \( \| D_x^p D_z^q b_m(x, z) \|_{L^\infty(U \times U)} \) by induction.

In the following we shall use the following standard notations for multi-indices.

- \( 1 = (1, 1, \ldots, 1) \).
- \( (\alpha_1, \alpha_2, \ldots, \alpha_n) = \left( \alpha_1, \alpha_2, \ldots, \alpha_n \right) \).
- \( \delta_l = (\delta_1, \delta_2, \ldots, \delta_n) = \frac{p}{\delta_1, \delta_2, \ldots, \delta_n} \) for any non-negative integer \( l \) and multi-index \( \delta \geq 0 \) such that \( |\delta| = l \).

**Lemma 5.1.** Suppose the Kähler potential \( \phi \in G^a(U) \). Let \( W = \{(x, z) \in U \times U : x \neq z\} \) and

\[ b = \min \{ b(\Delta_0(x, y, \theta)), b(\psi_1(x, z)), b(z_i(x, y, \theta)) : 1 \leq i \leq n \} \]

If we denote \( v = (x, z) \) and

\[ b_{m, \mu} = \left\| \frac{1}{\lambda_{b_{|\nu|}}(x, x, z)} D_v^\mu D_\bar{v}^\mu b_m(x, z) \right\|_{L^\infty(W)} , \]

then there exists some positive constant \( C \) independent of \( m, \mu, \nu \), such that

\[
b_{m, \mu} \leq \sum_{l=1}^{m} \sum_{|\delta|=l} \delta^{2a+2c-1} \sum_{\alpha, \beta \leq \delta} \sum_{|\alpha+\beta| \leq \mu} \sum_{\nu_0 \leq \nu} \sum_{\lambda(\xi, \eta) \leq \mu+\nu} C^{\mu - \mu_0 + \nu - \nu_0 + |\delta + \xi + \eta|} \left( \frac{\mu}{\lambda_{\mu_0}} \right)^\nu \left( \frac{\mu}{\lambda_{\mu_0}} \right)^{\nu_0} (\xi, \eta)^{a+\varepsilon},
\]

where \( \xi, \eta \in (\mathbb{Z}^\geq)^n \) and \( \bar{\xi} = (0, \cdots, 0, \xi), \bar{\eta} = (0, \cdots, 0, \eta) \in (\mathbb{Z}^\geq)^{2n} \).

\[3\]We discussed its proof in (2.13).
Proof. We first work on \((D_y \cdot D_\theta)^l (b_{m-l}(x,z(x,y,\theta)) \Delta_0 (x,y,\theta))\). We expand \((D_y \cdot D_\theta)^l\) and obtain

\begin{align}
(5.3) \quad (D_y \cdot D_\theta)^l (b_{m-l}(x,z(x,y,\theta)) \Delta_0 (x,y,\theta)) &= \sum_{|\delta|=l} \binom{l}{\delta} D_y^\delta D_\theta^0 (b_{m-l}(x,z(x,y,\theta)) \Delta_0 (x,y,\theta)) \\
&= \sum_{|\delta|=l} \binom{l}{\delta} \sum_{\alpha,\beta \leq \delta} \binom{\delta}{\alpha} \binom{\delta}{\beta} D_y^\alpha D_\theta^\beta (b_{m-l}(x,z(x,y,\theta))) D_y^{-\alpha} D_\theta^{-\beta} \Delta_0 \\
&= b_{m-l}(x,z(x,y,\theta)) (D_y \cdot D_\theta)^l \Delta_0 (x,y,\theta)
\end{align}

+ \sum_{|\delta|=l} l! \delta! \sum_{\alpha,\beta \leq \delta} \sum_{\alpha+\beta>0} \frac{D^\xi D^\eta b_{m-l}(x,z)}{\xi! \eta!} \frac{D_y^{-\alpha} D_\theta^{-\beta} \Delta_0}{(\delta-\alpha)!(\delta-\beta)!} \sum_{\alpha,\beta \leq \delta} \sum_{1 \leq |\xi|+|\eta| \leq |\alpha+\beta|} \sum_{A_{\alpha,\beta,\xi,\eta}} \prod_{ij} \frac{D^\alpha y D^\beta_\theta z_i}{\alpha_{ij}! \beta_{ij}!} \prod_{ik} \frac{D^\alpha y D^\beta_\theta z_i}{\alpha'_{ik}! \beta'_{ik}!}.

where the index set \(A_{\alpha,\beta,\xi,\eta}\) is defined by

\begin{align}
(5.4) \quad A_{\alpha,\beta,\xi,\eta} &= \left\{ \{\alpha_{ij}, \beta_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq \xi}, \{\alpha'_{ij}, \beta'_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq \eta} : \begin{array}{ll}
\sum_{1 \leq i \leq n, 1 \leq j \leq \xi} \alpha_{ij} + \sum_{1 \leq i \leq n, 1 \leq j \leq \eta} \beta_{ij} &= \alpha, \\
\sum_{1 \leq i \leq n, 1 \leq j \leq \xi} \alpha_{ij} + \sum_{1 \leq i \leq n, 1 \leq j \leq \eta} \beta_{ij} &= \beta, \\
\alpha_{ij} + \beta_{ij} &= 0, \\
\alpha'_{ik} + \beta'_{ik} &= 0
\end{array} \right\}.
\end{align}

We now substitute \((5.3)\) into equation \((5.1)\) and obtain

\begin{align}
b_{m}(x,z(x,y,\theta)) &= -\sum_{l=1}^m \binom{m}{l} b_{m-l}(x,z(x,y,\theta))(D_y \cdot D_\theta)^l \Delta_0 (x,y,\theta) + \sum_{|\delta|=l} l! \delta! \sum_{\alpha,\beta \leq \delta} \sum_{\alpha+\beta>0} \frac{D^\xi D^\eta b_{m-l}(x,z)}{\xi! \eta!} \frac{D_y^{-\alpha} D_\theta^{-\beta} \Delta_0}{(\delta-\alpha)!(\delta-\beta)!} \sum_{\alpha,\beta \leq \delta} \sum_{1 \leq |\xi|+|\eta| \leq |\alpha+\beta|} \sum_{A_{\alpha,\beta,\xi,\eta}} \prod_{ij} \frac{D^\alpha y D^\beta_\theta z_i}{\alpha_{ij}! \beta_{ij}!} \prod_{ik} \frac{D^\alpha y D^\beta_\theta z_i}{\alpha'_{ik}! \beta'_{ik}!} (x,y,\theta).
\end{align}

The correspondence \((x,x,z) \leftrightarrow (x,y,\theta = \psi_x(z,x))\), turns this into

\begin{align}
b_{m}(x,z) &= -\sum_{l=1}^m \binom{m}{l} b_{m-l}(x,z)(D_y \cdot D_\theta)^l \Delta_0 (x,y,\psi_x(z,x)) + \sum_{|\delta|=l} l! \delta! \sum_{\alpha,\beta \leq \delta} \sum_{\alpha+\beta>0} \frac{D^\xi D^\eta b_{m-l}(x,z)}{\xi! \eta!} \frac{D_y^{-\alpha} D_\theta^{-\beta} \Delta_0}{(\delta-\alpha)!(\delta-\beta)!} \sum_{\alpha,\beta \leq \delta} \sum_{1 \leq |\xi|+|\eta| \leq |\alpha+\beta|} \sum_{A_{\alpha,\beta,\xi,\eta}} \prod_{ij} \frac{D^\alpha y D^\beta_\theta z_i}{\alpha_{ij}! \beta_{ij}!} \prod_{ik} \frac{D^\alpha y D^\beta_\theta z_i}{\alpha'_{ik}! \beta'_{ik}!} (x,y,\psi_x(z,x)).
\end{align}

Denote \(v = (x,z)\). Note that in this recursive formula, the coefficients \(b_m\) depend on not only the previous coefficients \(b_{m-l}\) but also derivatives of \(b_{m-l}\). Hence, we need to include derivatives of \(b_m\) in our inductive argument. To do this we apply \(D^\mu y D^\nu_\theta\) on both sides and obtain a recursive formula.
Applying this to our case, we obtain

\[
(D_y \cdot D_\theta)^t \Delta_0(x, x, \psi_x(x, z))
\]

for the derivatives of \(b_m\).

\[
(D_v^\mu D_{\mu}^\nu b_m(x, z)) = - \sum_{l=1}^{m} \sum_{\mu_0 \leq \mu \leq \nu} (\mu_0) (\nu_0) \left( \frac{1}{l!} D_{\mu_0} D_{\mu_1} \cdots D_{\mu_l} b_m - \sum_{|\delta| = l} \frac{D_{\mu_0}^{\mu_1} \cdots D_{\mu_l}^{\mu} b_m}{\xi_{1}!} \right)
\]

\[
\sum_{|\delta| = l} \frac{D_{\mu_0}^{\mu_1} \cdots D_{\mu_l}^{\mu} b_m}{\xi_{1}!} = \sum_{\alpha, \beta \geq 0} \sum_{|\xi| = |\mu|} \frac{D_{\mu_0}^{\mu_1} \cdots D_{\mu_l}^{\mu} b_m}{\xi_{1}!} \]

Denote \( \Phi(x, z) = (\varphi_1, \varphi_2, \cdots, \varphi_{3n}) = (x, x, \psi_x(x, z)) \), and \( w = (x, y, \theta) \). In general, for any smooth function \( f(x, y, \theta) \) and any multi-indices \( \mu, \nu \in (\mathbb{Z}^+)^{2n} \), we have

\[
\frac{D_\mu^\phi D_\nu^\psi}{\mu! \nu!} (f(\Phi(x, z))) = \sum_{0 \leq |\rho + \tau| \leq |\mu + \nu|} \frac{D_\rho^\phi D_\tau^\psi}{\rho! \tau!} \sum_{A_{\mu \nu \rho \tau}} \prod_{ij} \frac{D_{\mu_j}^{\nu_j} D_{\nu_j}^{\psi_j}}{\mu_j! \nu_j!} \prod_{ik} \frac{D_{\mu_k}^{\nu_k} D_{\nu_k}^{\psi_k}}{\mu_k! \nu_k!}
\]

where the index set \( A_{\mu \nu \rho \tau} \) is defined similar as in (5.4) with a minor change that \( 1 \leq i \leq 3n \). Applying this to our case, we obtain

\[
\frac{D_\mu^\phi D_\nu^\psi}{\mu! \nu!} \left( \frac{D_y^\delta D_\theta^\delta \Delta_0}{(\delta - \alpha)!(\delta - \beta)!} (x, x, \psi_x(x, z)) \right) = \sum_{0 \leq |\rho + \tau| \leq |\mu + \nu|} \frac{D_\rho^\phi D_\tau^\psi}{\rho! \tau!} \sum_{A_{\mu \nu \rho \tau}} \prod_{ij} \frac{D_{\mu_j}^{\nu_j} D_{\nu_j}^{\psi_j}}{\mu_j! \nu_j!} \prod_{ik} \frac{D_{\mu_k}^{\nu_k} D_{\nu_k}^{\psi_k}}{\mu_k! \nu_k!}
\]

We will use \( C \) to denote a constant depending on constants \( \varepsilon, \alpha, n \) and \( \text{Kähler potential} \ \phi \) but independent with all the indices \( m, \mu, \nu \), which may vary from line to line. Since \( \Phi(x, z) \in A_{\text{diag}}^{\alpha \varepsilon} \) for each \( 1 \leq i \leq n \), we have

\[
\sum_{A_{\mu \nu \rho \tau}} \prod_{ij} \frac{D_{\mu_j}^{\nu_j} D_{\nu_j}^{\psi_j}}{\mu_j! \nu_j!} \prod_{ik} \frac{D_{\mu_k}^{\nu_k} D_{\nu_k}^{\psi_k}}{\mu_k! \nu_k!} \leq C^{(\mu + \nu)! + |\rho + \tau|} \left( \mu_{ij} + \nu_{ij} \right)^{\alpha + \varepsilon - 1} \left( \mu_{ik} + \nu_{ik} \right)^{\alpha + \varepsilon - 1}.
\]
Apply the combinatorial lemma 7.3 that we will prove later to the two products appearing above, we have
\[
\prod_{ij} (\mu_{ij} + \nu_{ij})! \leq \left( \frac{(2n)!\rho!}{\rho^!} \right) \left( \sum_{ij}(\mu_{ij} + \nu_{ij}) \right)!, \quad \prod_{ik} (\mu'_{ik} + \nu'_{ik})! \leq \left( \frac{(2n)!\tau!}{\tau^!} \right) \left( \sum_{ik}(\mu'_{ik} + \nu'_{ik}) \right)!
\]
Therefore,
\[
\left| \sum_{\mu,\nu} \prod_{ij} D_{\mu_{ij}}^0 D_{\nu_{ij}}^0 \prod_{ik} D_{\mu'_{ik}}^0 D_{\nu'_{ik}}^0 \right| \leq \sum_{\mu,\nu} C^{\mu+\nu+|\rho+\tau|} \left( \frac{(\mu + \nu)!}{\rho!\tau!} \right)^{a+\varepsilon-1}
\]
\[
\leq C^{\mu+\nu+|\rho+\tau|} \left( \frac{\mu!\nu!}{\xi!\eta!} \right)^{a+\varepsilon-1}
\]
The last inequality follows from
\[
\#A_{\mu,\nu,\rho,\tau} \leq \left( \mu + |\rho + \tau| \right) \left( \nu + |\rho + \tau| \right) \leq 2^{\mu+\nu+|\rho+\tau|}.
\]
As \(z(x, y, \theta), \Delta_0(x, y, \theta) \in A_{\theta}^{a, \varepsilon}\) by Lemma 2.13 and Lemma 2.15, after a straightforward calculation, we have
\[
\left| D_{\mu}^0 D_{\nu}^0 \left( \frac{D_{\gamma}^0 - \alpha D_{\theta}^0 - \beta \Delta_0}{(\delta - \alpha)!(\delta - \beta)!} \sum_{A_{\mu,\nu,\rho,\tau}} \prod_{ij} D_{\alpha_{ij}}^0 D_{\beta_{ij}}^0 \prod_{ik} D_{\alpha'_{ik}}^0 D_{\beta'_{ik}}^0 \right) \right| \leq \sum_{\mu,\nu} C^{\mu+\nu+|\beta+\xi+\eta|+1} \left( \frac{\delta^2 \mu!\nu!}{\xi!\eta!} \right)^{a+\varepsilon-1}
\]
\[
\leq C^{\mu+\nu+|\beta+\xi+\eta|+1} \left( \frac{\delta^2 \mu!\nu!}{\xi!\eta!} \right)^{a+\varepsilon-1}
\]
The last inequality follows from the fact that \(\alpha, \beta \leq \delta\),
\[
\left( \frac{\alpha + |\xi + \eta|}{|\xi + \eta|} \right), \quad \left( \frac{\beta + |\xi + \eta|}{|\xi + \eta|} \right) \leq 2^{|\delta|+|\xi+\eta|n}.
\]
Similarly, for any \(\mu, \nu \geq 0\), we also have
\[
\left| \frac{1}{l! \mu! \nu!} \left( (D_y - D_\theta)^l \Delta_0(x, x, \psi_x(x, z)) \right) \right| \leq \sum_{|\delta| = l} \delta^l C^{\mu+\nu+|\delta|+1} \left( \frac{\delta^2 \mu!\nu!}{\xi!\eta!} \right)^{a+\varepsilon-1}
\]
Then (5.5) implies the following inequality
\[
\left| D_{\mu}^0 D_{\nu}^0 b_m(x, z) \right| \leq \sum_{l=1}^{m} \sum_{|\delta| = l} \delta^2 2^{a+2\varepsilon-1} \sum_{\alpha, \beta \leq \delta} \sum_{|\xi+\eta| \leq |\alpha+\beta|} \sum_{\mu_0 \leq \mu} \sum_{\nu_0 \leq \nu} \sum_{\xi!, \eta!} \frac{D_{\xi_{\mu_0}}^0 + \mu_0 D_{\xi_{\nu_0}}^0 + \nu_0 b_{m-l}(x, z)}{(\xi!\eta!)^a+\varepsilon}
\]
\[
\leq C^{\mu, \nu+\nu, \mu_0 + |\beta+\xi+\eta|} \left( \frac{\mu\nu}{\mu_0\nu_0} \right)^{a+\varepsilon} (\nu - \nu_0)^{a+\varepsilon}.
\]
Now we will change all the derivatives $|D_a^b D_b^c m(x, z)|$ to the notation $b_{m, \mu \nu}$ in the above inequality. Note that on the right hand side of (5.5), when $\nu \neq 0$, the anti-holomorphic derivative will hit on either $b_{m-l}(x, z)$ or at least one of the these functions $z_i(x, y, \theta)$, $\Delta_0(x, y, \theta) \in A^a_{\text{diag}}$ and $\varphi_i(x, z) \in A^{a, \bar{a}}_{\text{diag}}$ for $1 \leq i \leq n$. We will consider each case in the following. If the anti-holomorphic derivative hits on $z_i(x, y, \theta)$ or $\Delta_0(x, y, \theta)$, since the derivatives are evaluated at $(x, x, \psi(x, z))$, we will have the extra factor $\exp(-b|x-z|^{-\alpha-1})$ on the right hand side. If the anti-holomorphic derivative hits on $\varphi_i(x, z)$, we also have the extra factor $\exp(-b|x-z|^{-\alpha-1})$ since $\varphi_i(x, z) \in A^{a, \bar{a}}_{\text{diag}}$. The last case is the anti-holomorphic derivative hits on $b_{m-l}(x, z)$, which means $|\eta + \nu| \neq 0$. We again have the extra factor $\exp(-b|x-z|^{-\alpha-1})$ when we change $D^m_\z \bar{D}^{m-l}_\z b_{m-l}(x, z)$ into $b_{m-l} \z$. So no matter in which case, at least one exp($-b|x-z|^{-\alpha-1}$) will appear on the right side when $\nu \neq 0$. And thus the desired result follows. 

Next we use this lemma to prove Theorem 1.5.

5.1. Proof of Theorem 1.5. For convenience, we define

$$a_{m, \mu \nu} = \frac{b_{m, \mu \nu}}{(2m+1)!a+\varepsilon \mu!a+\varepsilon \nu!a+\varepsilon}.$$  

Then by Lemma 5.1

$$a_{m, \mu \nu} \leq \sum_{l=1}^{m} \sum_{|\delta|=l} \sum_{a, \beta \leq \delta} \sum_{|\alpha| \leq |\beta|} \sum_{|\mu| \leq |\nu|} \frac{A^{m-l}_{2l}}{a+\varepsilon} \left( \frac{\bar{\xi} + \mu_0}{\bar{\xi}} \right)^{a+\varepsilon} \left( \frac{\bar{\eta} + \nu_0}{\bar{\eta}} \right)^{a+\varepsilon} C^{l}_{l+1} \sum_{\xi, \eta \in (\mathbb{Z}^{\geq 0})^n} \frac{2^{l+|\xi|+|\eta|}}{|\mu + \nu|} A^{a+\varepsilon \mu |a+\varepsilon \nu |a+\varepsilon},$$

where $\xi, \eta \in (\mathbb{Z}^{\geq 0})^n$ and $\bar{\xi} = (0, \cdots, 0, \xi), \bar{\eta} = (0, \cdots, 0, \eta) \in (\mathbb{Z}^{\geq 0})^n$. Since $b_0(x, z) = 1$, we have

$$a_{0, \mu \nu} = \begin{cases} 1 & \mu = \nu = (0, 0, \cdots, 0), \\ 0 & \text{otherwise}. \end{cases}$$

We will argue by induction on $m$ to prove that for any integer $m \geq 0$ and multi-index $\mu, \nu \geq 0,$

$$a_{m, \mu \nu} \leq \left( \frac{2m + |\mu + \nu|}{|\mu + \nu|} \right)^{a+\varepsilon} A^{a+\varepsilon |\mu + \nu|}.$$ 

where $C$ is the same constant which appears on the right hand side of (5.10) and $A$ is a bigger constant to be selected later. Without losing of generality, we assume $C \geq 1$. Obviously (5.11) implies that (5.12) holds for $m = 0$ and any $\mu, \nu \geq 0$. Assume that (5.12) holds up to $m - 1$ and we proceed to $m$. By (5.10), we have

$$a_{m, \mu \nu} \leq \sum_{l=1}^{m} \sum_{|\delta|=l} \sum_{a, \beta \leq \delta} \sum_{|\alpha| \leq |\beta|} \sum_{|\mu| \leq |\nu|} \frac{A^{m-l}_{2l}}{a+\varepsilon} \left( \frac{\bar{\xi} + \mu_0}{\bar{\xi}} \right)^{a+\varepsilon} \left( \frac{\bar{\eta} + \nu_0}{\bar{\eta}} \right)^{a+\varepsilon} \cdot 2^{l+|\xi|+|\eta|} C^{l+1} \sum_{|\mu + \nu|} \left( \frac{2^{m-l} + |\xi + \mu_0 + \bar{\eta} + \nu_0|}{|\xi + \mu_0 + \bar{\eta} + \nu_0|} \right)^{a+\varepsilon}$$

$$\leq \sum_{l=1}^{m} \sum_{|\delta|=l} \sum_{|\alpha| \leq |\beta|} \sum_{|\mu| \leq |\nu|} \frac{A^{m-l}_{2l}}{a+\varepsilon} \left( \frac{\bar{\xi} + \mu_0}{\bar{\xi}} \right)^{a+\varepsilon} \left( \frac{\bar{\eta} + \nu_0}{\bar{\eta}} \right)^{a+\varepsilon} \cdot 2^{l+|\xi|+|\eta|} C^{l+1} \sum_{|\mu + \nu|} \left( \frac{2^{m-l} + |\xi + \mu_0 + \bar{\eta} + \nu_0|}{|\xi + \mu_0 + \bar{\eta} + \nu_0|} \right)^{a+\varepsilon} \cdot \# \{ \alpha \leq \delta \} \cdot \# \{ \beta \leq \delta \}.$$
Due to the fact
$$\#\{\alpha \leq \delta\} = \#\{\beta \leq \delta\} \leq 2^{\lvert \delta \rvert},$$
it follows that
$$a_{m,\mu \nu} \leq \sum_{l=1}^{m} \sum_{\lvert \delta \rvert = l} \sum_{\lvert \xi + \eta \rvert \leq 2l} \sum_{\mu_0 \leq \mu \leq \nu} \sum_{\nu_0 \leq \nu} \frac{A_{m-l}}{A_{2l}^{a+\varepsilon}} \left( \frac{\bar{\xi} + \mu_0}{\xi} \right)^{a+\varepsilon} \left( \frac{\bar{\eta} + \nu_0}{\eta} \right)^{a+\varepsilon}$$
$$\cdot 2^{\lvert \xi + \eta \rvert \lvert \mu_0 + \nu_0 \rvert + \lvert \delta \rvert \lvert \mu + \nu \rvert + \lvert \delta \rvert + 2\varepsilon \lvert \eta \rvert} \left( 2(m - l) + \lvert \bar{\xi} + \mu_0 + \bar{\eta} + \nu_0 \rvert \right)^{a+\varepsilon}$$
$$\leq A_{m}(2C)^{\lvert \mu + \nu \rvert} \sum_{l=1}^{m} \sum_{\lvert \xi + \eta \rvert \leq 2l} \sum_{\mu_0 \leq \mu \leq \nu} \sum_{\nu_0 \leq \nu} \frac{1}{A_{2l}^{a+\varepsilon}} \left( \frac{\bar{\xi} + \mu_0}{\xi} \right)^{a+\varepsilon} \left( \frac{\bar{\eta} + \nu_0}{\eta} \right)^{a+\varepsilon} A^{-l}$$
$$\cdot 2^{\lvert \mu_0 + \nu_0 \rvert + \lvert \mu + \nu \rvert} \left( \frac{2m - l + \bar{\xi} + \mu_0 + \bar{\eta} + \nu_0}{\bar{\xi} + \mu_0 + \bar{\eta} + \nu_0} \right)^{a+\varepsilon}$$
Moreover, since
$$\#\{\lvert \delta \rvert = l\} = \binom{l + n - 1}{n - 1} \leq 2^{l+n-1} \leq 2^{nl},$$
we have
$$a_{m,\mu \nu} \leq A_{m}(2C)^{\lvert \mu + \nu \rvert} \sum_{l=1}^{m} \sum_{\lvert \xi + \eta \rvert \leq 2l} \sum_{\mu_0 \leq \mu \leq \nu} \sum_{\nu_0 \leq \nu} \frac{1}{A_{2l}^{a+\varepsilon}} \left( \frac{\bar{\xi} + \mu_0}{\xi} \right)^{a+\varepsilon} \left( \frac{\bar{\eta} + \nu_0}{\eta} \right)^{a+\varepsilon}$$
$$\cdot \left( \frac{2m - l + \bar{\xi} + \mu_0 + \bar{\eta} + \nu_0}{\bar{\xi} + \mu_0 + \bar{\eta} + \nu_0} \right)^{a+\varepsilon} 2^{\lvert \mu_0 + \nu_0 \rvert - \lvert \mu + \nu \rvert} \left( \frac{2^{2l}C^{5}}{A} \right)^{l}$$
In the next step we apply the combinatorial inequality
$$\left( \begin{array}{c} \bar{\xi} + \mu_0 \\ \xi + \eta \end{array} \right) \left( \begin{array}{c} \bar{\eta} + \nu_0 \\ \mu_0 + \nu_0 \end{array} \right) \leq \left( \begin{array}{c} \bar{\xi} + \bar{\eta} + \mu_0 + \nu_0 \\ \mu_0 + \nu_0 \end{array} \right) \leq \left( \begin{array}{c} \lvert \xi + \eta \rvert + \lvert \mu_0 + \nu_0 \rvert \\ \lvert \mu_0 + \nu_0 \rvert \end{array} \right)$$
and
$$\left( \begin{array}{c} 2(m - l) + \bar{\xi} + \mu_0 + \bar{\eta} + \nu_0 \\ \lvert \xi + \eta \rvert + \lvert \mu_0 + \nu_0 \rvert \end{array} \right) \left( \begin{array}{c} \lvert \xi + \eta \rvert + \lvert \mu_0 + \nu_0 \rvert \\ \lvert \mu_0 + \nu_0 \rvert \end{array} \right) = \left( \begin{array}{c} 2(m - l) + \bar{\xi} + \mu_0 + \bar{\eta} + \nu_0 \\ \lvert \mu_0 + \nu_0 \rvert \end{array} \right) \left( \begin{array}{c} 2m - 2l + \bar{\xi} + \mu_0 + \bar{\eta} + \nu_0 \\ 2m - 2l \end{array} \right).$$
Observe that, since \( \lvert \xi + \eta \rvert \leq 2l, \mu_0 \leq \mu, \nu_0 \leq \nu, \) we have
$$\left( \begin{array}{c} 2(m - l) + \bar{\xi} + \mu_0 + \bar{\eta} + \nu_0 \\ \lvert \mu_0 + \nu_0 \rvert \end{array} \right) \leq \left( \begin{array}{c} 2m + \lvert \mu + \nu \rvert \\ \lvert \mu + \nu \rvert \end{array} \right).$$
Plugging these into (5.13), we obtain
$$a_{m,\mu \nu} \leq A_{m}(2C)^{\lvert \mu + \nu \rvert} \sum_{l=1}^{m} \frac{1}{A_{2l}^{a+\varepsilon}} \left( \frac{2m + \lvert \mu + \nu \rvert}{\lvert \mu + \nu \rvert} \right)^{a+\varepsilon} \sum_{\lvert \xi + \eta \rvert \leq 2l} \left( \begin{array}{c} 2m - 2l + \bar{\xi} + \mu_0 + \bar{\eta} + \nu_0 \\ \lvert \mu_0 + \nu_0 \rvert \end{array} \right)^{a+\varepsilon}$$
$$\cdot \sum_{\mu_0 \leq \mu \leq \nu} \sum_{\nu_0 \leq \nu} 2^{\lvert \mu_0 + \nu_0 \rvert - \lvert \mu + \nu \rvert} \left( \frac{2^{2l}C^{5}}{A} \right)^{l}$$
Again since
$$\#\{\lvert \xi + \eta \rvert = k\} = \binom{k + 2n - 1}{2n - 1} \leq 2^{k+2n-1},$$
the sum over index $\xi, \eta$ on the right hand side can be estimated as
\[
\sum_{|\xi + \eta| \leq 2l} \left( \frac{2m - 2l + |\xi + \eta|}{2m - 2l} \right)^{a+\varepsilon} = \sum_{k=0}^{2l} \sum_{|\xi + \eta| = k} \left( \frac{2m - 2l + k}{2m - 2l} \right)^{a+\varepsilon} \leq 2^{2l+2n-1} \left( \frac{2m + 1}{2m - 2l + 1} \right)^{a+\varepsilon}.
\]
Therefore,
\[
a_{m,\mu\nu} \leq A^m (2C)^{|\mu + \nu|} \left( \frac{2m + |\mu + \nu|}{|\mu + \nu|} \right)^{a+\varepsilon} \sum_{l=1}^{m} \left( \frac{2^{3n+6}C^5}{A} \right)^l \sum_{\mu_0 \leq \mu} \sum_{\nu_0 \leq \nu} 2^{2|\mu_0| - |\mu|} 2^{2|\nu_0| - |\nu|}
\]
\[
\leq A^m (2C)^{|\mu + \nu|} \left( \frac{2m + |\mu + \nu|}{|\mu + \nu|} \right)^{a+\varepsilon} \sum_{l=1}^{m} \left( \frac{2^{3n+6}C^5}{A} \right)^l 2^{4n}
\]
By taking $A = 2^{7n+7}C^5$, we surely have $\sum_{l=1}^{m} \left( \frac{2^{3n+6}C^5}{A} \right)^l 2^{4n} < 1$, which implies that
\[
a_{m,\mu\nu} \leq A^m (2C)^{|\mu + \nu|} \left( \frac{2m + |\mu + \nu|}{|\mu + \nu|} \right)^{a+\varepsilon}.
\]
So if we write $a_{m,\mu\nu}$ in terms of $D_\mu D_\nu b_m(x, z)$, then by the continuity of each $D_\mu D_\nu b_m(x, z)$ for all $x, z \in U$, we have
\[
|D_\mu D_\nu b_m(x, z)| \leq (2m + 1)!^{a+\varepsilon} (2m + |\mu + \nu|)!^{a+\varepsilon} a_{m,\mu\nu} \lambda_{b,|\nu|}(x, x, z)
\]
\[
\leq (64^{a+\varepsilon} A)^{m + |\mu + \nu|} m!^{2a+2\varepsilon} (2m + |\mu + \nu|)!^{a+\varepsilon} \lambda_{b,|\nu|}(x, x, z).
\]
Thus (1.9) follows by renaming $64^{a+\varepsilon} A$ to $C$.
In particular, when we are restricted to diagonal $z = \bar{x}$,
\[
D_\mu D_\nu b_m(x, \bar{x}) = 0, 
\]
for any multi-indices $\mu \geq 0$ and $|\nu| \neq 0$.
And thus when $z = \bar{x}$, the recursive inequality (5.8) reduces to
\[
|D_\mu b_m(x, \bar{x})| \leq \sum_{l=1}^{m} \sum_{|\xi| = l} \delta^{2\alpha - 2 - 1} \sum_{\alpha, \beta \leq \delta, |\xi| \leq |\alpha + \beta|} \frac{|D_{\xi+\mu_0} b_{m-l}(x, \bar{x})|}{(|\xi|!)^{a+\varepsilon}} C^{l+\xi_0 + |\delta + \xi|} (\mu - \mu_0)^{a+\varepsilon}.
\]
Note that the constant $\varepsilon$ only comes from (5.6) and (5.7) because of the derivatives of $\varphi_i(x, z) \in A_{\text{diag}}^{a,\varepsilon}$ for $1 \leq i \leq n$ and $z(x, y, \theta), \Delta_0(x, y, \theta) \in A_{\theta}^{a,\varepsilon}$. By the definitions of $A_{\text{diag}}^{a,\varepsilon}$ and $A_{\theta}^{a,\varepsilon}$, $\varepsilon$ can be replaced by 0 when we are restricted to $x = y = \bar{x}$. Therefore, (5.8) further reduces to
\[
|D_\mu b_m(x, \bar{x})| \leq \sum_{l=1}^{m} \sum_{|\xi| = l} \delta^{2\alpha - 1} \sum_{\alpha, \beta \leq \delta, |\xi| \leq |\alpha + \beta|} \frac{|D_{\xi+\mu_0} b_{m-l}(x, \bar{x})|}{(|\xi|!)^{a+\varepsilon}} C^{l+\xi_0 + |\delta + \xi|} (\mu - \mu_0)^{a+\varepsilon}.
\]
By using a similar inductive argument as that of estimating $|D_\nu D_\mu b_m(x, z)|$, we obtain for any $x \in U$,
\[
|D_\mu b_m(x, \bar{x})| \leq C^{m+|\mu| m!^{2a} \lambda_0}.
\]
6. Optimality of the upper bounds on Bergman coefficients $b_m$
In this section, we will show that although it would be desirable to improve the estimate (1.9) to
\[
|D_\mu D_\nu b_m(x, z)| \leq C^{m+|\alpha| + |\beta|} m!^{2a+2\varepsilon} \alpha^{a+\varepsilon} \beta^{a+\varepsilon} \exp \left( -b(1 - \delta_0(|\beta|)) |x - \bar{z}|^{-a+\varepsilon} \right) \lambda_{b,|\nu|}(x, x, z),
\]
it is not possible to prove it by the recursive inequality (5.2). Here we provide an example which satisfies (5.2) while fails (6.1). For simplicity, we assume $C = 1$ in (5.2). Let’s consider the worse case when the equality holds, i.e.

(6.2)

$$b_{m,\mu\nu} = \sum_{l=1}^{m} \sum_{|\beta|=l} \alpha^{2a+2\varepsilon-1} \sum_{\alpha,\beta \leq \delta} \sum_{|\xi| \leq |\alpha+\beta|} \sum_{\mu_{0} \leq \mu} \sum_{\nu_{0} \leq \nu} \frac{b_{m-l,\xi+\mu_{0}\eta+\nu_{0}}}{(\xi,\eta)^{a+\varepsilon}} \left( \frac{\mu}{\mu_{0}} \right) \left( \frac{\nu}{\nu_{0}} \right) (\mu - \mu_{0})^{a+\varepsilon} (\nu - \nu_{0})^{a+\varepsilon},$$

One can easily check that this recursive equation uniquely defines $\{b_{m,\mu\nu}\}$ given an initial data $\{b_{0,\nu}\}$. We shall only focus on the terms $b_{m,k\bar{e}_{1}}$ where $e_{1} = (1, 0, \cdots, 0) \in \mathbb{R}^{n}$, $\bar{e}_{1} = (0, \cdots, 0, e_{1}) \in \mathbb{R}^{n}$ and show by induction that

(6.3)

$$b_{m,k\bar{e}_{1}} \geq 2^{-(a+\varepsilon)m} (2m - 2 + k)^{a+\varepsilon}$$

for any $m \geq 1, k \geq 0$.

First let’s check $b_{1,k\bar{e}_{1}}$. Since we know

$$b_{0,\mu\nu} = \begin{cases} 1 & \mu = \nu = (0, 0, \cdots, 0), \\ 0 & \text{otherwise}, \end{cases}$$

by (6.2) we have

(6.4)

$$b_{1,\mu\nu} = \sum_{|\beta|=1} \sum_{\alpha,\beta \leq \delta} \mu_{1}^{a+\varepsilon} \nu_{1}^{a+\varepsilon} \geq \mu_{1}^{a+\varepsilon} \nu_{1}^{a+\varepsilon}.$$

Therefore (6.3) holds for $b_{1,k\bar{e}_{1}}$. Assume that (6.3) holds for $b_{1,k\bar{e}_{1}}, b_{2,k\bar{e}_{1}}, \cdots, b_{m-1,k\bar{e}_{1}}$. Then by only considering the terms with index $l = |\alpha| = |\beta| = 1$, $\mu_{0} = \mu$ and $\xi = 2e_{1}, \eta = 0$ in (6.4), we obtain for $m \geq 2$

$$b_{m,k\bar{e}_{1}} \geq \sum_{|\beta|=1} \sum_{|\alpha|=|\beta|=1} \frac{b_{m-1,2k\bar{e}_{1}}}{2^{a+\varepsilon}} \geq \frac{b_{m-1,2(k+2)\bar{e}_{1}}}{2^{a+\varepsilon}} \geq 2^{-(a+\varepsilon)m} (2m - 2 + k)^{a+\varepsilon}.$$

Note that if in particular we put $k = 0$ into (6.3), then we get

$$b_{m,0} \geq \left( \frac{1}{8} \right)^{(a+\varepsilon)m} m!^{2a+2\varepsilon},$$

which show that up to an exponential factor $C^{m}, m!^{2a+2\varepsilon}$ is the best upper bound one can hope from the recursive inequality (5.2).

7. Proofs of main lemmas on almost holomorphic extensions of Gevrey functions

In this section, we will complete all the proofs skipped in Section 2.2.

Proof of Lemma 2.5. We will prove the estimate on $D_{\bar{z}}F(f)$. The other one follows in the same way. For simplicity, we denote

$$\bar{\chi}(|\alpha + \beta|) = \chi(|\alpha + \beta|^{2(a-1)}4^{a-1}C_{1}^{2}|y - \bar{z}|^{2}).$$
Thus the result follows. □

For any $1 \leq i \leq n$, we have

$$D_{z_i} F(f)$$

$$= \frac{1}{2} \sum_{a, \beta \geq 0} \frac{D_x^{a+\epsilon_i} D^\beta_x f}{a! \beta!} \left( \frac{y - \bar{z}}{2} \right)^{a} \left( \frac{z - \bar{y}}{2} \right)^{\beta} \left( \bar{\chi}(\alpha + \beta) - \bar{\chi}(\alpha + \beta + \epsilon_i) \right)$$

$$+ 2 \sum_{a, \beta \geq 0} \frac{D_x^a D^\beta_x \bar{f}}{a! \beta!} \left( \frac{y - \bar{z}}{2} \right)^{a} \left( \frac{z - \bar{y}}{2} \right)^{\beta+\epsilon_i} |\alpha + \beta|^{2(a-1)} 4^{a-1} C_1^2 \chi' \left( |\alpha + \beta|^{2(a-1)} 4^{a-1} C_1^2 |y - \bar{z}|^2 \right) .$$

$$:= I + II.$$

We will use $C$ to denote a constant depend on $a, \varepsilon, C_1(f)$ and the cut-off function $\chi$, which may change from line to line. Since $f \in G^a(U)$ and by Stirling’s formula, each term in $I$ is bounded by

$$C_0 C_1^{[\alpha + \beta] + 1} (\alpha_i + 1)^a (\alpha! \beta!) a^{-1} \left| \frac{y - \bar{z}}{2} \right|^{[\alpha + \beta]}$$

$$\leq CC_0 C_1^{[\alpha + \beta] + 1} (|\alpha + \beta| + 1)^a \frac{|\alpha + \beta|}{e} \left( \frac{y - \bar{z}}{2} \right)^{[\alpha + \beta]} .$$

Note that the difference of cut-off functions in $I$ is zero unless

$$|\alpha + \beta| \in \left[ \frac{1}{2} \left( \sqrt{2} C_1 |y - \bar{z}| \right)^{-\frac{1}{a-1}} - 1, \frac{1}{2} (C_1 |y - \bar{z}|)^{-\frac{1}{a-1}} \right] .$$

It implies that each term in $I$ is bounded by

$$CC_0 (|\alpha + \beta| + 1)^a \frac{|\alpha + \beta|}{2} e^{-|\alpha + \beta| a^{-1}} \leq CC_0 e^{-\frac{a-1}{2} (\sqrt{2} C_1 |y - \bar{z}|)^{-\frac{1}{a-1}}} .$$

Since there are less than $\left( \frac{1}{2} (C_1 |y - \bar{z}|)^{-\frac{1}{a-1}} + 1 \right)^{2n}$ many terms in $I$, we have

$$|I| \leq CC_0 e^{-b|y - \bar{z}|^{-\frac{1}{a-1}}} ,$$

where $b$ is a positive constant depending on $a, C_1 = C_1(f)$.

For the second term $II$, similarly we have (7.1) or $\chi'$ vanishes otherwise. And thus each term is bounded by

$$CC_0 C_1^{[\alpha + \beta] + 2} \left( \frac{|\alpha + \beta|}{e} \right)^{2(a-1)|\alpha + \beta|} \left| \frac{y - \bar{z}}{2} \right|^{[\alpha + \beta] + 2} \frac{\alpha + \beta}{2} (a-1)$$

$$\leq CC_0 e^{-\left( \frac{\alpha - 1}{2} |\alpha + \beta| \right) a^{-1} \alpha + \beta} \frac{1}{2} (a-1)$$

$$\leq CC_0 e^{-\frac{a-1}{2} (\sqrt{2} C_1 |y - \bar{z}|)^{-\frac{1}{a-1}}}$$

$$\leq CC_0 e^{-b|y - \bar{z}|^{-\frac{1}{a-1}}} .$$

So we have

$$|D_{z_i} F(f)(y, \bar{z})| \leq CC_0 \exp \left( -b|y - \bar{z}|^{-\frac{1}{a-1}} \right) \quad \text{for } 1 \leq i \leq n .$$

Thus the result follows.
**Proof of Lemma 2.6.** Since $F$ is an almost holomorphic extension of $f$, by taking the Taylor expansion, for any $N \in \mathbb{N}$ we have

$$F(x + y, \bar{x} + z) = \sum_{|\alpha + \beta| \leq N-1} \frac{D^\alpha D^\beta F}{\alpha! \beta!} (x, \bar{x}) y^\alpha z^\beta + O\left(||(y, z)||^N\right)$$

$$= \sum_{|\alpha + \beta| \leq N-1} \frac{D^\alpha D^\beta f}{\alpha! \beta!} (x) y^\alpha z^\beta + O\left(||(y, z)||^N\right).$$

If we take $x = \frac{y + \bar{z}}{2}$ and replace $(y, z)$ by $(\frac{y - \bar{z}}{2}, \frac{z - \bar{y}}{2})$, then

$$F(y, z) = \sum_{|\alpha + \beta| \leq N-1} \frac{D^\alpha D^\beta f}{\alpha! \beta!} \left(\frac{y + \bar{z}}{2}\right) \left(\frac{y - \bar{z}}{2}\right)^\alpha \left(\frac{z - \bar{y}}{2}\right)^\beta + O\left(||y - \bar{z}||^N\right).$$

Similarly, the same identity holds for $\bar{F}(y, z)$. Therefore, for any $N \in \mathbb{N}$, we have

$$F(y, z) - \bar{F}(y, z) = O\left(||y - \bar{z}||^N\right).$$

□

**Proof of Lemma 2.7.** To prove this lemma we first need to obtain some estimates on the derivatives of our cut-off function $\chi$.

**Lemma 7.1.** Let $\varepsilon > 0$ be a constant and $\chi \in C^{1+\varepsilon}(\mathbb{R})$ be the cut-off function constructed in (2.16). Then there exists some positive constant $C = C(\chi)$ such that for any multi-indices $\gamma, \delta, \xi, \eta \geq 0$, we have

$$\|D^\gamma_y D^\delta_z D^\xi_y D^\eta_z \chi\|_{L^\infty(\mathbb{C}^2)} \leq \left(2^{2-1} C_1 |\alpha + \beta| a^{-1}\right)^{\gamma + \eta + \xi + \delta} (\gamma + \eta + \xi + \delta)!^{1+\varepsilon}.$$ 

**Proof.** By a straightforward calculation, we have

$$D^\gamma_y \left(\chi \left(|\alpha + \beta| 2^{(a-1)} 4^{a-1} C_1^2 |y - \bar{z}|^2\right)\right) = \left(|\alpha + \beta| 2^{(a-1)} 4^{a-1} C_1^2 \right)^\gamma \chi^{(|\gamma|)}$$

$$D^\gamma_y D^\delta_z \left(\chi \left(|\alpha + \beta| 2^{(a-1)} 4^{a-1} C_1^2 |y - \bar{z}|^2\right)\right) = (-1)^{\eta\gamma} \left(|\alpha + \beta| 2^{(a-1)} 4^{a-1} C_1^2 \right)^{\gamma + \eta} \chi^{(|\gamma| + \eta)}.$$ 

Therefore,

$$D^\gamma_y D^\delta_z D^\xi_y D^\eta_z \left(\chi \left((\alpha + \beta) 2^{(a-1)} 4^{a-1} C_1^2 |y - \bar{z}|^2\right)\right)$$

$$= \sum_{\xi_0 \leq \xi, \delta_0 \leq \delta} \binom{\xi}{\xi_0} \binom{\delta}{\delta_0} (-1)^{\eta_0 + \xi_0 + \delta_0 - \delta_0} \left(|\alpha + \beta| 2^{(a-1)} 4^{a-1} C_1^2 \right)^{|\gamma + \eta + \xi_0 + \delta_0|}$$

$$\frac{(\gamma + \eta)!}{(\gamma + \eta + \xi_0 + \delta_0 - \xi - \delta)!} (\bar{y} - z)^{\gamma + \eta - (\xi - \xi_0) - (\delta - \delta_0)} (\bar{z} - y)^{\xi_0 + \delta_0} \chi^{(|\gamma + \eta + \xi_0 + \delta_0|)}.$$
Since the cut-off function $\chi$ is in $G^{1+\varepsilon}(R)$, there exists a positive constant $C = C(\chi)$, which may vary from line to line, such that

$$
\left| D_y^\gamma D_z^\delta D_y^\xi D_z^\eta F(f)(y, z) \right|
\leq \sum_{\xi_0 \leq \varepsilon, \delta_0 \leq \delta} \left( \frac{\xi}{\xi_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta} \left( \frac{\xi}{\delta_0} \right)^{\delta} \left( \frac{\xi}{\delta_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta}
\times \left| \sum_{\xi_0 \leq \varepsilon, \delta_0 \leq \delta} \left( \frac{\xi}{\xi_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta} \left( \frac{\xi}{\delta_0} \right)^{\delta} \left( \frac{\xi}{\delta_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta}
\right|_{\gamma + \eta + \xi_0 + \delta_0 + \xi + \delta} \cdot \left| \sum_{\xi_0 \leq \varepsilon, \delta_0 \leq \delta} \left( \frac{\xi}{\xi_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta} \left( \frac{\xi}{\delta_0} \right)^{\delta} \left( \frac{\xi}{\delta_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta_0} \left( \frac{\xi}{\delta_0} \right)^{\delta}
\right|_{\gamma + \eta + \xi_0 + \delta_0 + \xi + \delta}.
\right)

Our result follows by using that for any $y, z \in C^n$,

$$
|\alpha + \beta|^{2(a-1)4^{a-1}C_1^2} |y - z|^2 \leq 1.
$$

\[\square\]

We now estimate the derivatives on $F(f)$. By a straightforward calculation, we have

$$
D_y^\gamma D_z^\delta D_y^\xi D_z^\eta F(f)(y, z)
= \sum_{\gamma_0 + \gamma_1 + \gamma_2 = \gamma} \sum_{\xi_0 + \xi_1 + \xi_2 = \xi} \sum_{\delta_0 + \delta_1 + \delta_2 = \delta} \left( \begin{array}{c}
\gamma_0, \gamma_1, \gamma_2 \\
\xi_0, \xi_1, \xi_2
\end{array} \right) \left( \begin{array}{c}
\delta_0, \delta_1, \delta_2
\end{array} \right) \left( \begin{array}{c}
\eta_0, \eta_1, \eta_2
\end{array} \right)
\times \left( \frac{1}{2} \right)^{\gamma + \gamma_1 + \gamma_2} \left( \frac{1}{2} \right)^{\xi + \xi_1 + \xi_2} \left( \frac{1}{2} \right)^{\delta + \delta_1 + \delta_2} \left( \frac{1}{2} \right)^{\eta + \eta_1 + \eta_2}
\times (-1)^{\gamma_1 + \xi_1} \frac{D_x^{\alpha + \gamma_0 + \eta_0} D_x^{\beta + \delta_0 + \xi_0} f}{(\alpha - \gamma_1 - \eta_1)(\beta - \delta_1 - \xi_1)}
\times \left( \begin{array}{c}
m - \gamma_1 \\
\xi_0 - \gamma_1
\end{array} \right) \left( \begin{array}{c}
\beta - \delta_1
\end{array} \right) \left( \chi \left( \frac{|\alpha + \beta|^{2(a-1)4^{a-1}C_1^2} |y - z|^2}{2} \right) \right).
\right)
\right)

Let $C_0 = C_0(f)$ introduced in Definition 2.3. We use $C$ to denote a constant which depends on $a$, $C_1(f)$ introduced in Definition 2.3 and the cut-off function $\chi$, which may vary from line to line. By the fact that $f \in G^a(U)$ and the previous lemma, it follows that

$$
\left| D_y^\gamma D_z^\delta D_y^\xi D_z^\eta F(f)(y, z) \right|
\leq C_0 |\gamma + \delta + \xi + \eta|
\times \sum_{\gamma_0 + \gamma_1 + \gamma_2 = \gamma} \sum_{\xi_0 + \xi_1 + \xi_2 = \xi} \sum_{\delta_0 + \delta_1 + \delta_2 = \delta} \left( \begin{array}{c}
\gamma_0, \gamma_1, \gamma_2 \\
\xi_0, \xi_1, \xi_2
\end{array} \right) \left( \begin{array}{c}
\delta_0, \delta_1, \delta_2
\end{array} \right) \left( \begin{array}{c}
\eta_0, \eta_1, \eta_2
\end{array} \right)
\times \left( \begin{array}{c}
\gamma_0, \gamma_1, \gamma_2 \\
\xi_0, \xi_1, \xi_2
\end{array} \right) \left( \begin{array}{c}
\delta_0, \delta_1, \delta_2
\end{array} \right) \left( \begin{array}{c}
\eta_0, \eta_1, \eta_2
\end{array} \right)
\times \left( \begin{array}{c}
m - \gamma_1 \\
\xi_0 - \gamma_1
\end{array} \right) \left( \begin{array}{c}
\beta - \delta_1
\end{array} \right) \left( \chi \left( \frac{|\alpha + \beta|^{2(a-1)4^{a-1}C_1^2} |y - z|^2}{2} \right) \right).
\right)
\right)
Using the fact \(|\alpha + \beta|^2(\alpha - 1)4^{a - 1}C_1^2 |y - z|^2 \leq 1\) and Stirling’s formula, it is bounded by

\[
C_0C^{\gamma + \delta + \varepsilon + \eta} \sum_{\gamma_0 + \gamma_1 + \gamma_2 = \gamma} \sum_{\delta_0 + \delta_1 + \delta_2 = \delta} \sum_{\eta_0 + \eta_1 + \eta_2 = \eta} \left( \gamma_0, \gamma_1, \gamma_2 \right) \left( \delta_0, \delta_1, \delta_2 \right) \left( \xi, \xi, \xi \right) \left( \eta, \eta, \eta \right) \]

\[
\frac{(\alpha + \gamma_0 + \eta_0)!^a (\beta + \delta_0 + \xi_0)!^a}{(\alpha - \gamma_1 - \eta_1)!^a (\beta - \delta_1 - \xi_1)!^a} \left( \frac{C_1}{2} \right)^{\alpha + \beta} (\gamma_2 + \eta_2 + \xi_2 + \delta_2)^{1+\varepsilon}
\]

\[
\cdot \left(2^{a-1}\alpha + \beta\right)^{a\alpha + \varepsilon} \left(\gamma_2 + \eta_2 + \xi_2 + \delta_2\right)^{1+\varepsilon}
\]

\[
|\alpha + \beta|^{(a-1)(\gamma_1 + \delta_1 + \varepsilon_1 + \eta_1 + \gamma_2 + \delta_2 + \varepsilon_2 + \eta_2 + 1)} e^{-\frac{1}{2}(a-1)|\alpha + \beta|}
\]

\[
\leq 2^{a-1}(|\gamma_1 + \delta_1 + \varepsilon_1 + \eta_1 + \gamma_2 + \delta_2 + \varepsilon_2 + \eta_2 + 1|)^{a+\varepsilon}
\]

Therefore,

\[
|D_y^\varepsilon D_z^\delta F(f)(y, z)| \leq 2^{a-1}C_0C^{\gamma + \delta + \varepsilon + \eta} (\gamma! \delta! \varepsilon! \eta!)^{a+\varepsilon} \sum_{\alpha, \beta \geq 0} e^{-\frac{1}{2}(a-1)|\alpha + \beta|}
\]

Note that when \(\xi + \eta > 0\), we have (7.1). So

\[
|D_y^\varepsilon D_z^\delta D_y^\eta D_z^\zeta F(f)(y, z)| \leq 2^{a-1}C_0C^{\gamma + \delta + \varepsilon + \eta} (\gamma! \delta! \varepsilon! \eta!)^{a+\varepsilon} e^{-\frac{a-1}{8} (\sqrt{2}C_1|y-z|)^{1-\varepsilon} \sum_{\alpha, \beta \geq 0} e^{-\frac{1}{2}(a-1)|\alpha + \beta|}
\]

The result follows as \(\sum_{\alpha, \beta \geq 0} e^{-\frac{1}{2}(a-1)|\alpha + \beta|} \leq (\frac{4}{a-1})^{2n} e^{-\frac{(a-1)n}{2}}\).

In addition, when \(z = \bar{y}\), note all the derivatives of \(\chi\) vanish and \(|\chi| \leq 1\), whence we can replace \(\varepsilon\) by zero.

\[\square\]

**Proof of Lemma 2.11.** It is easy to see that \(A^{a,\xi}_0\) is closed under summation, subtraction and differentiation. Now we consider multiplication. Take \(f, g \in A^{a,\xi}_0\). We will use \(C_0(f), C_1(f), b(f)\) and \(C_0(g), C_1(g), b(g)\) to denote the constants in (2.22) corresponding to \(f, g\) respectively. Take
$C_1 = \max\{C_1(f), C_1(g)\}$ and $b = \min\{b(f), b(g)\}$. Let $v' = (x, y, \theta)$. Then

$$
\left| D_v^\alpha D_{v'}^\beta (fg)(x, y, \theta(x, y, z)) \right|
\leq \sum_{\alpha_0 \leq \alpha, \beta_0 \leq \beta} \binom{\alpha}{\alpha_0} \binom{\beta}{\beta_0} \left| D_v^\alpha D_{v'}^\beta f(x, y, \theta(x, y, z)) \right| \left| D_v^{\alpha-\alpha_0} D_{v'}^{\beta-\beta_0} g(x, y, \theta(x, y, z)) \right|
\leq \sum_{\alpha_0 \leq \alpha, \beta_0 \leq \beta} C_0(f) C_0(g) \binom{\alpha}{\alpha_0} \binom{\beta}{\beta_0} C_1^{\alpha+\beta} |(\alpha!\beta!)^{a+\varepsilon} \lambda_{b,|\beta|}(x, y, z)|
= C_0(f) C_0(g) (2C_1)^{\alpha+\beta} |(\alpha!\beta!)^{a+\varepsilon} \lambda_{b,|\beta|}(x, y, z)|.
$$

In addition, when we are restricted to $x = y = \bar{z}$, it is easy to see that we can replace $\varepsilon$ by 0. Therefore, $fg \in A_0^{a,\varepsilon}$. And we can choose $C(f, g) = C_0(f) C_0(g)$, $C(f, g) = 2 \max\{C_1(f), C_1(g)\}$ and $b(f, g) = \min\{b(f), b(g)\}$.

To prove $A_0^{a,\varepsilon}$ is closed under division, we will verify that if $f \in A_0^{a,\varepsilon}(U')$ and $\inf_{U'} |f| \geq C_2 > 0$, then the reciprocal $\frac{1}{f} \in A_0^{a,\varepsilon}(U')$. Define $h(w) = \frac{1}{w}$ for $w \in \mathbb{C} \setminus \{0\}$. Then for any $v_0 \in U'$, we have

$$
D_v^\alpha D_{v'}^\beta (h \circ f)(v_0) = \alpha! \beta! \sum_{k=0}^{\min\{\alpha+\beta\}} \frac{(-1)^k}{f(v_0)^{k+1}} \sum_{\begin{subarray}{c}
\alpha_1 + \alpha_2 + \ldots + \alpha_k = \alpha \\
\beta_1 + \beta_2 + \ldots + \beta_k = \beta \\
\alpha_1 + \beta_1 > 0, \ldots, \alpha_k + \beta_k > 0
\end{subarray}} \frac{D_{v'}^\alpha D_v^\beta f(v_0)}{\alpha_1! \beta_1!} \ldots \frac{D_{v'}^{\alpha_k} D_v^{\beta_k} f(v_0)}{\alpha_k! \beta_k!}.
$$

We use $C_0 = C_0(f), C_1 = C_1(f)$ and $b = b(f)$ to denote the constants in (2.22) for $f$. Without losing of generality, we can assume $C_0 > 1$ and $C_2 < 1$. Then

$$
\left| D_v^\alpha D_{v'}^\beta (h \circ f)(x, y, \theta(x, y, z)) \right|
\leq \frac{1}{C_{k+1}^2} \sum_{\begin{subarray}{c}
\alpha_1 + \alpha_2 + \ldots + \alpha_k = \alpha \\
\beta_1 + \beta_2 + \ldots + \beta_k = \beta \\
\alpha_1 + \beta_1 > 0, \ldots, \alpha_k + \beta_k > 0
\end{subarray}} (\alpha!\beta!)^{a+\varepsilon} C_0^{\alpha+\beta} C_1^{\alpha+\beta} \lambda_{b,|\beta|}(x, y, z)
\leq \sum_{k=0}^{\min\{\alpha+\beta\}} \frac{\alpha + \beta}{k!} C_0 C_1 \left( \frac{C_0}{C_2} \right)^{\alpha+\beta} |(\alpha!\beta!)^{a+\varepsilon} \lambda_{b,|\beta|}(x, y, z)|
\leq \frac{1}{C_2} \left( \frac{2^{6n+2} C_0 C_1}{C_2} \right)^{\alpha+\beta} |(\alpha!\beta!)^{a+\varepsilon} \lambda_{b,|\beta|}(x, y, z)|.
$$

In addition, when we are restricted to $x = y = \bar{z}$, it is easy to see that we can replace $\varepsilon$ by 0. Therefore, $\frac{1}{f} = h \circ f \in A_0^{a,\varepsilon}(U')$.

**Proof of Lemma 2.12.** Denote $v = (x, y, z)$ and $C_0 = C_0(f), C_1 = C_1(f), b = b(f)$. As $t \in [0, 1]$, we have

$$
\left| D_v^\alpha D_{v'}^\beta g(x, y, z) \right| \leq \max_{t \in [0, 1]} \left| D_v^\alpha D_{v'}^\beta \left( f(x, tx + (1-t)y, z) \right) \right|
$$
We write $D^0_\alpha = D^0_\alpha D^0_y D^0_z$ and $D^\beta_\alpha = D^\beta_\alpha D^\beta_y D^\beta_z$. Then
\[
\left| D^\alpha_\beta D^\beta_\alpha g(x, y, z) \right|
\leq \max_{t \in [0,1]} \left| \sum_{\alpha_1' \leq \alpha_1} \sum_{\alpha_1' \leq \alpha_1} \left( \frac{\alpha_1}{\alpha_1'} \right) \left( \frac{\beta_1}{\beta_1'} \right) \sum_{\alpha_2' \leq \alpha_2} \sum_{\alpha_2' \leq \alpha_2} \left( \frac{\alpha_2}{\alpha_2'} \right) \left( \frac{\beta_2}{\beta_2'} \right) f(x, tx + (1 - t)y, z) \right|
\leq \sum_{\alpha_1' \leq \alpha_1} \sum_{\alpha_1' \leq \alpha_1} \left( \frac{\alpha_1}{\alpha_1'} \right) \left( \frac{\beta_1}{\beta_1'} \right) C_0 (2^{a+\varepsilon} C_1)^{1/\alpha_1+\varepsilon} \left( \frac{\alpha_1}{\beta_1} \right) \lambda_{b_1|\beta_1}(x, tx + (1 - t)y, z)
\leq C_0 (2^{a+\varepsilon} C_1)^{1/\alpha_1+\varepsilon} \lambda_{a_1+\varepsilon|\beta_1+\varepsilon}(x, y, z).
\]
The last inequality follows from $|tx + (1 - t)y - z| \leq \max\{|x - z|, |y - z|\}$ for any $t \in [0,1]$. In addition, as $f \in A_\beta^{a,\varepsilon}$, when restricted to $x = y = z$, we can replace $\varepsilon$ by 0. So we get the first part of the lemma. The second part follows by the same argument.

**Proof of Lemma 2.13.** Let $m = 3n$. We denote $v = (x, y, z)$, $v' = (x, y, \theta)$. By a straightforward calculation, we have
\[
\frac{D^0_\alpha D^\beta_\alpha}{\alpha!\beta!} f
= \left( \sum_{0 \leq |\xi + \eta| \leq |\alpha + \beta|} \frac{D^\xi D^\eta f}{\xi!\eta!} \right) \sum_{A_{\alpha \beta \eta}} \frac{D^{\alpha_1} D^{\beta_1} v_1 \cdots D^{\alpha_m} D^{\beta_m} v_m}{\alpha_1!\beta_1! \cdots \alpha_m!\beta_m!} \left( \frac{v}{v'} \right)^{a_1+\varepsilon} \left( \frac{\alpha_1}{\alpha_1'} \right) \left( \frac{\beta_1}{\beta_1'} \right) \lambda_{b_1|\beta_1}(x, y, z),
\]
where $A_{\alpha \beta \eta}$ is defined in (5.4). Then since $f \in A_\beta^{a,\varepsilon}$ and $v = (x, y, z(x, y, \theta)) \in A_\beta^{a,\varepsilon}$, by taking $b = \min\{b(f), b(v)\}$, $C_0(v) = \max_i C_i(v_i)$ and $C_1(v) = \max_i C_1(v_i)$, we have
\[
\left| \frac{D^0_\alpha D^\beta_\alpha}{\alpha!\beta!} f(x, y, \theta(x, y, z)) \right|
\leq \left( \sum_{0 \leq |\xi + \eta| \leq |\alpha + \beta|} C_0(f) C_1(f)^{1/2} \sum_{A_{\alpha \beta \eta}} C_0(v) C_1(v)^{1/2} \left( \frac{v}{v'} \right)^{a_1+\varepsilon} \left( \frac{\alpha_1}{\alpha_1'} \right) \left( \frac{\beta_1}{\beta_1'} \right) \lambda_{b_1|\beta_1}(x, y, z).\right.
\]
Now we prove two combinatorial lemmas to estimate of $\xi!\eta! \prod_{i,j} \alpha_{ij}! \beta_{ij}! \prod_{i,k} \alpha'_{ik}! \beta'_{ik}!$ appearing in the above inequality.

**Lemma 7.2.** For any integers $k, i_1, i_2, \ldots, i_k \in \mathbb{Z}^+$, we have
\[
(7.3) \quad k! i_1! i_2! \cdots i_k! \leq (i_1 + i_2 + \cdots + i_k)!. \]

**Proof.** We will do induction on $k$. When $k = 1$, the result follows trivially. Assume it is true for $k - 1$ and we proceed to the case $k$. For simplicity, we denote $i = i_1 + i_2 + \cdots + i_k$. Then by using the result by induction, we have
\[
(7.4) \quad k! i_1! i_2! \cdots i_k! \leq k(i_1 + i_2 + \cdots + i_{k-1})! i_k! = \frac{k!}{i_k!} i_k!.
\]
Note that $\binom{i_k}{i_k} \geq i \geq k$ and thus the result follows.

**Lemma 7.3.** For any multi-indices, $\alpha_1, \alpha_2, \ldots, \alpha_k \in (\mathbb{Z}_\geq 0)^n$, if $|\alpha_i| > 0$ for each $1 \leq i \leq k$, then we have
\[
(7.5) \quad \alpha_1! \alpha_2! \cdots \alpha_k! \leq \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_k)!}{k!} n^k.
\]
Proof. We denote \( \alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) \) and define \( l_j = \#\{1 \leq i \leq k, \alpha_{ij} \neq 0\} \) for \( 1 \leq j \leq n \). Then by applying Lemma 7.2 to the \( j \)-th component of each \( \alpha_i \) for \( 1 \leq i \leq k \), we have
\[
\alpha_{1j}! \alpha_{2j}! \cdots \alpha_{kj}! \leq \frac{(\alpha_{1j} + \alpha_{2j} + \cdots + \alpha_{kj})!}{l_j!}.
\]
Therefore,
\[
(7.6) \quad \alpha_1! \alpha_2! \cdots \alpha_k! \leq \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_k)!}{l_1! l_2! \cdots l_n!}.
\]
Since \( |\alpha_i| \neq 0 \) for each \( i \), we have \( l := l_1 + l_2 + \cdots l_n \geq k \). Then we can find nonnegative integers \( k_1, k_2, \ldots, k_n \) such that \( k_1 + k_2 + \cdots + k_n = k \) and \( k_j \leq l_j \) for \( 1 \leq j \leq n \). Therefore
\[
(7.7) \quad l_1! l_2! \cdots l_n! \geq k_1! k_2! \cdots k_n! = \frac{k!}{(k_1, k_2, \ldots, k_n)} \geq \frac{k!}{n^k}.
\]
Plug this back into (7.6), we have the result. \( \square \)

By using Lemma 7.3, we have the upper bound for the factorials on the right hand side as
\[
\xi! \eta! \prod_{i,j} \alpha_{ij}! \beta_{ij}! \prod_{i,k} \alpha_{ik}'! \beta_{ik}'! \leq m^{[\xi + \eta]} \left( \sum_{ij} \alpha_{ij} + \beta_{ij} \right)! \left( \sum_{ik} \alpha_{ik}' + \beta_{ik}' \right)! \leq m^{[\xi + \eta]} (\alpha + \beta)!.
\]
Therefore,
\[
\frac{D_\alpha^\beta D_\eta^\gamma \bar{f}(x, y, \theta(x, y, z))}{\alpha! \beta! \gamma!}
\leq \sum_{0 \leq [\xi + \eta] \leq [\alpha + \beta]} C_0(f) C_1(f) \sum_{A_{\alpha \beta \xi \eta}} C_0(v)^{[\xi + \eta]} C_1(v)^{[\alpha + \beta]} \left( m^{[\xi + \eta]} (\alpha + \beta)! \right)^{a-1+\varepsilon} \lambda_{b,|\beta|} (x, y, z)
\leq \sum_{0 \leq [\xi + \eta] \leq [\alpha + \beta]} C_0(f) \left( m^{a-1+\varepsilon} C_1(f) C_0(v) C_1(v) \right)^{[\alpha + \beta]} (\alpha + \beta)!^{a-1+\varepsilon} \lambda_{b,|\beta|} (x, y, z) \cdot \#A_{\alpha \beta \xi \eta}.
\]
Note the cardinality of \( A_{\alpha \beta \xi \eta} \) has the following upper bounded
\[
\#A_{\alpha \beta \xi \eta} \leq \begin{pmatrix} \alpha + ([\xi + \eta]) \mathbb{I} \\ [\xi + \eta] \mathbb{I} \end{pmatrix} \begin{pmatrix} \beta + ([\xi + \eta]) \mathbb{I} \\ [\xi + \eta] \mathbb{I} \end{pmatrix}.
\]
So we get
\[
\frac{D_\alpha^\beta D_\eta^\gamma \bar{f}(x, y, \theta(x, y, z))}{\alpha! \beta! \gamma!}
\leq \sum_{0 \leq [\xi + \eta] \leq [\alpha + \beta]} C_0(f) \left( m^{a+\varepsilon + 2m^{a-1+\varepsilon} C_1(f) C_0(v) C_1(v)} \right)^{[\alpha + \beta]} (\alpha! \beta!)^{a-1+\varepsilon} \lambda_{b,|\beta|} (x, y, z)
\leq C_0(f) \left( m^{a+3m+\varepsilon} C_1(f) C_0(v) C_1(v)} \right)^{[\alpha + \beta]} (\alpha! \beta!)^{a-1+\varepsilon} \lambda_{b,|\beta|} (x, y, z).
\]
And if we keep track of the constant \( \varepsilon \), it is easy to see that \( \varepsilon \) comes from derivatives of since \( f \) and \( v = (x, y, z(x, y, \theta)) \). Since \( f \in A_{\alpha \beta \xi}^\varepsilon \) and \( v = (x, y, z(x, y, \theta)) \in A_{\beta \xi}^\varepsilon \), we can replace \( \varepsilon \) by 0 when restricted to \( x = y = \tilde{z} \). Therefore, we can take
\[
C_0(\tilde{f}) = C_0(f), \quad C_1(\tilde{f}) = 2^{a+3m+\varepsilon} m^{a-1+\varepsilon} C_0(v) C_1(f) C_1(v).
\]
2.15. We are going to prove the following more general lemma. Note that we can assume \(\psi(y, z) = yz + O((|y, z|)^4)\) by using the Bochner coordinates at 0. Then the Lemma 2.15 follows directly by taking \(F(x, y, z, \theta) = \int_0^1 (D_y \psi)(tx + (1 - t)y, z) dt - \theta\) and \(\theta(x, y, z) = \int_0^1 (D_y \psi)(tx + (1 - t)y, z) dt\).

**Lemma 7.4.** Consider smooth maps \(\theta(x, y, z) = (\theta_1(x, y, z), \theta_2(x, y, z), \ldots, \theta_n(x, y, z))\) and \(F(x, y, z, \theta) = (F_1(x, y, z, \theta), F_2(x, y, z, \theta), \ldots, F_n(x, y, z, \theta))\) satisfying the system of equations \(F(x, y, z, \theta(x, y, z)) = 0\). Assume that for any \(x, y, z \in U = B^n(0, 1)\) and multi-indices \(\alpha, \beta \geq 0\), we have

\[
\left| \left( D^\alpha_{(x,y,z,\theta)} D^\beta_{(\bar{x},\bar{y},\bar{z},\bar{\theta})} F \right) (x, y, z, \theta(x, y, z)) \right| \leq C_0 C_1^{(\alpha+\beta)\gamma} F^{\alpha+\epsilon} \lambda_{\alpha,\beta}(x, y, z),
\]

where \(C_0 = C_0(F), C_1 = C_1(F)\) and \(b = b(F)\) are some positive constants. And \(\epsilon\) can be replaced by 0, when we are restricted to \(x = y = z\).

If the \(2n \times 2n\) matrix \(\frac{\partial F}{\partial x} \frac{\partial F}{\partial y} \frac{\partial F}{\partial z} \frac{\partial F}{\partial \theta}\) is the identity matrix at \((x_0, y_0, z_0, \theta_0)\), then the implicit functions \(z = z(x, y, \theta)\) near \((x_0, y_0, \theta_0)\) determined by the equation \(F(x, y, z, \theta) = 0\) belong to \(A^\alpha_\theta\).

We first consider a special case when \(F\) is a function in the following lemma.

**Lemma 7.5.** Consider smooth maps \(\theta(x, y, z) = (\theta_1(x, y, z), \theta_2(x, y, z), \ldots, \theta_n(x, y, z))\) and function \(f(x, y, z, \theta)\) such that \(f(x, y, z, \theta(x, y, z)) = 0\). And for any \(x, y, z \in U\) and multi-indices \(\alpha, \beta \geq 0\), we have

\[
\left| \left( D^\alpha_{(x,y,z,\theta)} D^\beta_{(\bar{x},\bar{y},\bar{z},\bar{\theta})} f \right) (x, y, z, \theta(x, y, z)) \right| \leq C_0 C_1^{(\alpha+\beta)\gamma} F^{\alpha+\epsilon} \lambda_{\alpha,\beta}(x, y, z),
\]

where \(C_0, C_1\) and \(b\) are some positive constants. And \(\epsilon\) can be replaced by 0, when we are restricted to \(x = y = z\).

Assume at \((x_0, y_0, z_0, \theta_0 = \theta(x_0, y_0, z_0))\), the matrix \(\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial z} \frac{\partial f}{\partial \theta}\) is non-singular. Then the implicit function \(z_n = z_n(x, y, z_1, z_2, \ldots, z_{n-1}, \theta)\) determined by the equation \(f(x, y, z, \theta) = 0\) satisfies that for any multi-indices \(\alpha, \beta \geq 0\),

\[
\left| \left( D^\alpha_{(x,y,z,\theta)} D^\beta_{(\bar{x},\bar{y},\bar{z},\bar{\theta})} z_n \right) (x, y, z', \theta(x, y, z)) \right| \leq C_0' C_1'^{(\alpha+\beta)\gamma} F^{\alpha+\epsilon} \lambda_{\alpha,\beta}(x, y, z),
\]

where \(C_0', C_1'\) and \(b'\) are some positive constants and \(z' = (z_1, z_2, \ldots, z_{n-1})\). In addition, when we are restricted to \(x = y = \bar{z}\), \(\epsilon\) can be replace by 0.

**Proof.** For simplicity, we denote \(v = (x, y, z_1, z_2, \ldots, z_{n-1}, \theta)\). Near some point \((v, z_n)\), we have the Taylor series of \(f\) as

\[
f(v', z_n') = \sum_{\alpha, \beta \geq 0, i, j \geq 1} a_{\alpha \beta ij}(v' - v)^\alpha (z_n' - z_n)^j (z_n - z_n')^j,
\]

where \(a_{\alpha \beta ij} = D^\alpha D^\beta f_{\alpha \beta ij} (v, z_n)\). The equation \(f = 0\) implies

\[
a_{0000} + a_{0001}(z_n' - z_n) + a_{0001}(z_n - z_n')
\]

\[
= \sum_{|\alpha|+|\beta| > 0} (a_{\alpha \beta 00} + a_{\alpha \beta 10}(z_n' - z_n) + a_{\alpha \beta 01}(z_n' - z_n)) (v' - v)^\alpha (z_n' - z_n)^j.
\]

\[(7.11)\]
Assume near \( v \), the Taylor series of \( z_n = z_n(v) \) is as follows.

\[
z_n' - z_n = \sum_{|\gamma + \delta| > 0} b_{\gamma \delta} (v' - v)^\gamma (v - v)^\delta,
\]

where \( b_{\gamma \delta} = \frac{D^\gamma_{z}(v)}{\gamma!} \).

We define the following index sets for simplicity.

\[
A_{\alpha \beta \gamma \delta} = \left\{ i, j, \{\xi_k, \eta_k\}_{1 \leq k \leq i}, \{\xi'_l, \eta'_l\}_{1 \leq l \leq j} : \begin{array}{ll}
\alpha + \sum_{1 \leq k \leq i} \xi_k + \sum_{1 \leq l \leq j} \xi'_l = \gamma, \\
\beta + \sum_{1 \leq k \leq i} \eta_k + \sum_{1 \leq l \leq j} \eta'_l = \delta, \\
\xi_k + \xi'_l > 0, \eta_k + \eta'_l > 0, i + j \geq 2 \end{array} \right\}.
\]

\[
B_{\gamma \delta} = \{\alpha, \beta, \xi, \eta : \alpha + \xi = \gamma, \beta + \eta = \delta, |\alpha + \beta| > 0 \}.
\]

When restrict (7.11) to points \((x, y, z, \theta(x, y, z))\), \(a_{0000} = 0\). By comparing the coefficients of (7.11), for any multi-indices \(|\gamma + \delta| > 0\), we have

\[
a_{0010} b_{\gamma \delta} + \overline{a_{0010} b_{\gamma \delta}} = -a_{\gamma \delta 00} - \sum_{B_{\gamma \delta}} a_{\alpha \beta 10} b_{\beta \xi 0} - \sum_{B_{\gamma \delta}} a_{\alpha \beta \gamma 0} \overline{b_{\beta \eta 0}} - \sum_{A_{\alpha \beta \gamma \delta}} a_{\alpha \beta \gamma l} b_{\beta \xi_l \eta_l} \cdots b_{\beta \xi_l \eta_l} \overline{b_{\beta \eta_l}} \overline{b_{\beta \eta_l}} \overline{b_{\beta \eta_l}}.
\]

Taking the conjugate and switch the multi-indices \(\gamma\) and \(\delta\), we have

\[
\overline{a_{0010} b_{\gamma \delta}} + \overline{a_{0010} b_{\gamma \delta}} = -a_{\gamma \delta 00} - \sum_{B_{\gamma \delta}} a_{\alpha \beta 10} b_{\beta \xi 0} - \sum_{B_{\gamma \delta}} a_{\alpha \beta \gamma 0} \overline{b_{\beta \eta 0}} - \sum_{A_{\alpha \beta \gamma \delta}} a_{\alpha \beta \gamma l} b_{\beta \xi_l \eta_l} \cdots b_{\beta \xi_l \eta_l} \overline{b_{\beta \eta_l}} \overline{b_{\beta \eta_l}} \overline{b_{\beta \eta_l}}.
\]

Then for any \(|\gamma + \delta| > 0\), by solving \(b_{\gamma \delta}\), we obtain the following recursive formula on the coefficients \(b_{\gamma \delta}\).

\[
(7.12) \quad b_{\gamma \delta} = \frac{a_{0010}}{|a_{0010}|^2 - |a_{0011}|^2} \left( a_{\gamma \delta 00} + \sum_{B_{\gamma \delta}} a_{\alpha \beta 10} b_{\beta \xi 0} + \sum_{B_{\gamma \delta}} a_{\alpha \beta \gamma 0} \overline{b_{\beta \eta 0}} + \sum_{A_{\alpha \beta \gamma \delta}} a_{\alpha \beta \gamma l} b_{\beta \xi_l \eta_l} \cdots b_{\beta \xi_l \eta_l} \overline{b_{\beta \eta_l}} \overline{b_{\beta \eta_l}} \overline{b_{\beta \eta_l}} \right)
\]

\[
+ \frac{a_{0011}}{|a_{0010}|^2 - |a_{0011}|^2} \left( a_{\gamma \delta 00} + \sum_{B_{\gamma \delta}} a_{\alpha \beta 10} b_{\beta \xi 0} + \sum_{B_{\gamma \delta}} a_{\alpha \beta \gamma 0} \overline{b_{\beta \eta 0}} + \sum_{A_{\alpha \beta \gamma \delta}} a_{\alpha \beta \gamma l} b_{\beta \xi_l \eta_l} \cdots b_{\beta \xi_l \eta_l} \overline{b_{\beta \eta_l}} \overline{b_{\beta \eta_l}} \overline{b_{\beta \eta_l}} \right).
\]

By (7.9), when \(\theta = \theta(x, y, z)\), the Taylor coefficients \(a_{\alpha \beta kl}\) satisfies that

\[
|a_{\alpha \beta kl}| \leq C_0 C_1^{[\alpha + \beta] + k + l} (\alpha ! \beta ! k! l!)^{\alpha - 1 + \varepsilon} \lambda_{\alpha |\beta| + l}(x, y, z),
\]

where \(\lambda\) is as defined in (2.23). We normalized \(a_{\alpha \beta kl}\) to \(\tilde{a}_{\alpha \beta kl}\) as

\[
(7.13) \quad \tilde{a}_{\alpha \beta kl} = \frac{|a_{\alpha \beta kl}|}{(\alpha ! \beta ! k! l!)^{\alpha - 1 + \varepsilon} \lambda_{\alpha |\beta| + l}(x, y, z)},
\]

which is dominated by \(C_0 C_1^{[\alpha + \beta] + k + l}\). Similarly, we define

\[
(7.14) \quad \tilde{b}_{\gamma \delta} = \frac{|b_{\gamma \delta}|}{(\gamma ! \delta!)^{\alpha - 1 + \varepsilon} \lambda_{|\gamma| + \delta}(x, y, z)}.
\]
Since the matrix \( \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \theta} \end{pmatrix} \) is non-singular at \( (x_0, y_0, z_0, \theta_0) \), by choosing a sufficiently small neighborhood \( U \) of \( (x_0, y_0, z_0, \theta_0) \), we have \( |a_{0016}|^2 - |a_{0001}|^2 \geq A > 0 \). By \((7.13),(7.14)\) and using triangle inequalities and Lemma 7.3, we write \((7.12)\) the following recursive inequality on \( b_{\gamma\delta} \).

\[
\tilde{b}_{\gamma\delta} \leq \frac{C_0 C_1}{A} \left( \sum_{B_{\gamma\delta}} a_{\alpha_1\beta_1} \tilde{b}_{\xi\eta} + \sum_{B_{\gamma\delta}} a_{\alpha_1\beta_1} \tilde{b}_{\eta\xi} \right) + C_0 C_1 \left( \sum_{B_{\gamma\delta}} a_{\alpha_1\beta_1} \tilde{b}_{\xi\eta} + \sum_{B_{\gamma\delta}} a_{\alpha_1\beta_1} \tilde{b}_{\eta\xi} \right).
\]

(7.15)

Recall the definition of majorant for power series as follows.

**Definition 7.6** (Majorant). Consider two power series in variables \( x \in \mathbb{R}^n \).

\[
f(x) \sim \sum_{\alpha \geq 0} a_{\alpha} x^{\alpha}, \quad g(x) \sim \sum_{\alpha \geq 0} b_{\alpha} x^{\alpha}.
\]

We say that \( g \) is a majorant of \( f \), or \( b_{\alpha} \) is a majorant of \( a_{\alpha} \), if \( |a_{\alpha}| \leq b_{\alpha} \) for any \( \alpha \geq 0 \). And we denote this by \( f << g \).

For multi-indices \( |\alpha + \delta| > 0 \), we define \( d_{\alpha\delta} \) recursively as

\[
d_{\gamma\delta} = \frac{C_0^2 C_1}{A} \left( C_1^{\gamma+\delta} + \sum_{B_{\gamma\delta}} C_2^{\alpha+\beta+1} d_{\xi\eta} + \sum_{B_{\gamma\delta}} C_2^{\alpha+\beta+1} d_{\eta\xi} + \sum_{A_{\gamma\delta}} C_2^{\alpha+\beta+1+i+j} d_{\xi\eta \cdots \eta\xi} \right) + \frac{C_0^2 C_1}{A} \left( C_1^{\gamma+\delta} + \sum_{B_{\gamma\delta}} C_2^{\alpha+\beta+1} d_{\xi\eta} + \sum_{B_{\gamma\delta}} C_2^{\alpha+\beta+1} d_{\eta\xi} + \sum_{A_{\gamma\delta}} C_2^{\alpha+\beta+1+i+j} d_{\xi\eta \cdots \eta\xi} \right),
\]

(7.16)

where \( C_2 = 6nC_1 \). Since \( C_0 C_2^{\alpha+\beta+1+i+j} \) is a majorant of \( \bar{a}_{\alpha_1\beta_1} \), \( d_{\gamma\delta} \) defined as above is a majorant of \( b_{\gamma\delta} \) in \((7.15)\) for any \( |\gamma + \delta| > 0 \). Now we will solve \( d_{\gamma\delta} \) by the recursive equation \((7.16)\). Formally, we define \( d(u, v) = \sum_{|\gamma + \delta| > 0} d_{\gamma\delta} u^{\gamma} v^{\delta} \). Then \((7.16)\) is equivalent to

\[
d(u, v)
\]

\[
= \frac{2C_0^2 C_1}{A} \left( \frac{1}{1 - C_2 d(u, v)} - \prod_{i=1}^{m} \frac{1}{(1 - C_2 u_i)(1 - C_2 v_i)} - 1 - C_2 d(u, v) \right),
\]

where \( m = 3n \). It is easy to see that \( d(u, v) = d(v, u) \) and thus

\[
d(u, v) = \frac{2C_0^2 C_1}{A} \left( \frac{1}{(1 - C_2 d(u, v))^2} - \prod_{i=1}^{m} \frac{1}{(1 - C_2 u_i)(1 - C_2 v_i)} - 1 - 2C_2 d(u, v) \right).
\]

(7.17)

Observe that \((u, v, d) = 0\) satisfies the equation and there is no linear term of \( d \) on the right hand side. By the Implicit Theorem for real analytic functions (See [KP02] for more details), it follows that \( d(u, v) \) is real analytic near the origin. Therefore, there exists some constant \( C_3 \) such that \( \tilde{b}_{\gamma\delta} \leq d_{\gamma\delta} \leq C_3^{\gamma+\delta} \) for any \( |\gamma + \delta| > 0 \). By using \((7.14)\), we obtain the desired bounds for \( b_{\alpha\delta} \). In addition, note that the constant \( \varepsilon \) only comes from the estimate of \( f \) in \((7.9)\). Therefore, when we are restricted to \( x = y = \tilde{z} \), the constant \( \varepsilon \) can be replaced by \( 0 \). \( \square \)
Now we will do induction on the dimension $n$. When $n = 1$, the result directly follows from the previous lemma. We assume the result holds for $n - 1$ and proceed to $n$. First, we consider the equation $F_n(x, y, z, \theta) = 0$. Since the matrix $\begin{pmatrix} \frac{\partial F_i}{\partial x} & \frac{\partial F_i}{\partial y} & \frac{\partial F_i}{\partial z} \\ \frac{\partial F_i}{\partial \bar{z}} & \frac{\partial F_i}{\partial \bar{y}} & \frac{\partial F_i}{\partial \bar{z}} \end{pmatrix}$ is identity at $(x_0, y_0, z_0, \theta_0)$, by using the previous lemma again, we have the implicit function $z_n = h_n(x, y, z', \theta)$, which satisfies 

$$F_n(x, y, z', h_n, \theta) = 0.$$ 

Take the derivative with respect to $z_j$ and $\bar{z}_j$ for $1 \leq j \leq n - 1$. Then at $(x_0, y_0, z_0, \theta_0)$,

$$\frac{\partial h_n}{\partial z_j} = \frac{\partial h_n}{\partial \bar{z}_j} = 0.$$ 

Define $G_i(x, y, z', \theta) = F_i(x, y, z', h_n(x, y, z', \theta))$ for $1 \leq i \leq n - 1$. Since functions $F$ and $h_n$ satisfy (7.8) and (7.10) respectively, the composition function $G_i(x, y, z', \theta)$ for $1 \leq i \leq n - 1$ also satisfy the estimates on the derivatives (7.8) by a similar argument as in the proof of Lemma 2.13. On the other hand, we have for any $1 \leq i, j \leq n - 1$, at $(x_0, y_0, z_0, \theta_0)$

$$\frac{\partial G_i}{\partial z_j} = \frac{\partial F_i}{\partial z_j} = \delta_{ij}, \quad \frac{\partial G_i}{\partial \bar{z}_j} = \frac{\partial F_i}{\partial \bar{z}_j} = 0.$$ 

Therefore, the matrix $\begin{pmatrix} \frac{\partial G_i}{\partial x} & \frac{\partial G_i}{\partial y} & \frac{\partial G_i}{\partial z} \\ \frac{\partial G_i}{\partial \bar{z}} & \frac{\partial G_i}{\partial \bar{y}} & \frac{\partial G_i}{\partial \bar{z}} \end{pmatrix}$ is identity at $(x_0, y_0, z_0, \theta_0)$. Using the conclusion from the induction, we have the implicit functions $z_i = h_i(x, y, \theta)$ of the equations $G_i(x, y, z', \theta) = 0$ for $1 \leq i \leq n - 1$. It is easy to verify that $z_i = h_i(x, y, \theta)$ for $1 \leq i \leq n - 1$ and $z_n = h_n(x, y, h_1, h_2, \cdots, h_{n-1}, \theta_n)$ satisfy all the requirements and our result follows.

\[ \square \]

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