Abstract

Nonlinear effects on the early stage of phase ordering are studied using Adomian’s decomposition method for the Ginzburg-Landau equation for a nonconserved order parameter. While the long-time regime and the linear behavior at short times of the theory are well understood, the onset of nonlinearities at short times and the breaking of the linear theory at different length scales are less understood. In the Adomians decomposition method, the solution is systematically calculated in the form of a polynomial expansion for the order parameter, with a time dependence given as a series expansion. The method is very accurate for short times, which allows to incorporate the short-time dynamics of the nonlinear terms in a analytical and controllable way.

Key words: Phase ordering, Ginzburg-Landau equation, Short-time dynamics, Adomians decomposition method
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1 Introduction

The study of phase ordering kinetics is of fundamental importance to the understanding of nonequilibrium dynamics of phase transitions and is of interest
in diverse branches of physics, ranging from cosmology and elementary particle physics to the different areas of condensed matter physics. The typical situation for phase ordering occurs when a system consisting of a two-phase mixture in a homogeneous phase is rapidly driven across the critical coexistence temperature $T_c$ into a nonequilibrium state. Fluctuations around the initial homogeneous state will develop and the system will ultimately break into domains of different phases in space, forming a new equilibrium state.

The dynamics of the time evolution towards equilibrium is a much-studied problem is nonequilibrium statistical mechanics - for reviews see Refs. [1] and [2]. If the order parameter $\phi$ that characterizes the phases of the system is not conserved, the time evolution of $\phi$ is usually described by the phenomenological Ginzburg-Landau (GL) equation [2]

$$\frac{\partial \phi(\vec{x}, t)}{\partial t} = -M \frac{\delta F[\phi(\vec{x}, t)]}{\delta \phi} + \zeta(\vec{x}, t),$$

where $F[\phi]$ is the coarse-grained free energy functional, and $M$ is a constant mobility, and $\zeta(\vec{x}, t)$ is a noise term that mimics the thermal fluctuations. For the purposes of the present paper, we take for $F[\phi]$ the usual Landau-Ginzburg double-well form

$$F[\phi] = \int d^3 x \left[ R^2 \frac{1}{2} \left( \nabla \phi \right)^2 + \frac{1}{2} \epsilon \phi^2 + \frac{1}{4} u \phi^4 \right].$$

Here, $R^2$ is a measure of the interaction range, $\epsilon = (T - T_c)/T_c$ is the reduced temperature, and $u$ is a positive constant. In a situation of a temperature quench, $\epsilon = -|\epsilon|$, the equilibrium (long time) values of the order parameter correspond to $\phi^2 = |\epsilon|/u$, the binodal points. The unstable (spinodal) region corresponds to $\phi^2 \leq |\epsilon|/(3u)$. There are several time regimes associated with the time evolution towards equilibrium. When $\phi \approx 0$ at $t = 0$, one has the linear regime. The nonlinear terms in Eq. (1) are small for small $t$ and can be neglected; the equation becomes linear [3]. In this case, the equation can be integrated very easily and the (noiseless) solution can be written as

$$\phi(\vec{x}, t) = \int d^3 k \exp \left[ i \vec{k} \cdot \vec{x} - M(R^2 k^2 + \epsilon) t \right] \tilde{\phi}(\vec{k}), \quad \text{for} \quad t \approx 0,$$

where $\tilde{\phi}(\vec{k})$ is the Fourier transform of $\phi(\vec{x}, t = 0)$. In a situation of a temperature quench, there is an exponential growth of the order parameter for wave-lengths corresponding to $k^2 \lesssim |\epsilon|/R^2$. Because of the fast exponential growth for long wave-lengths the nonlinearities rapidly set in, the growth process is slowed down and for longer times interfaces start to be produced, and the linear theory cannot be employed anymore. The time scale that characterizes the onset of the nonlinear regime depends on $M$ and $R^2$. For much longer
time scales, one will reach the final stage of the coarsening process and the reach of equilibrium.

In the present paper we are mostly interested in the study of the breakdown of the linear approximation at early stage of the phase ordering. In particular, we are interested in studying systems with initial configurations that consist of domains where the order parameter is not necessarily small. This situation occurs for example if the initial state of the system consists of domains composed of metastable matter, i.e. $|\epsilon|/u \geq \phi^2 \geq |\epsilon|/(3u)$. Our aim here is to provide a semi-analytical method to study the short-time evolution of nonlinearities in the Ginzburg-Landau equation for such situations. The motivation is of course to obtain an understanding of the time and length scales of the breakdown of the linear approximation in a systematic and controllable way, without resorting to some sort of discretization methods which can easily result in massive numerical computation.

Specifically, we propose to employ Adomian’s decomposition method [4] to represent the solution of Eq. (1) in terms of a functional series expansion. The method has been used in several instances [5] and more recently it has been used for the KdV equation [6] and the coupled system of the Schrödinger-KdV equations [7]. For the GL equation the use of the method has been outlined in Ref. [8], but no explicit solutions were obtained nor comparisons with complete solutions were done. In the present letter we show that the decomposition method can be a useful analytical tool to investigate the dynamics of the early stage of phase ordering kinetics. We focus primarily on situations where the linear theory certainly fails, as in the case when the initial state consists of large domains with metastable matter. Since the early works of Ref. [9], the study of the early stage of phase ordering has gained renewed interest in recent years [10]-[15]. Another direction that the decomposition method can be useful is in the study of the scaling behavior at early stages of phase ordering [16], a subject of current interest for thin films roughness studies [17].

In this first application, to illustrate the applicability and reliability of the method for the GL equation, we use a simple double-well free energy functional and present numerical results for the one-dimensional case only. The initial condition is chosen to be an oscillatory function to mimic a configuration of domains with large order parameter. A more detailed investigation at higher dimensions and different free-energies and initial configurations is reserved for a future publication. In the next Section we explain the decomposition method for the GL equation and obtain the lowest-order terms of the series expansion. In Section 3 we show that in the linear approximation the series can be explicitly summed and gives the exact result. For the nonlinear regime we use lowest-order terms of decomposition to compare results with the numerical solution of the GL equation. Our conclusions and future perspectives are presented in Section 4.
2 The decomposition method for the Ginzburg-Landau equation

For the free energy of the form given in Eq. (2), the Ginzburg-Landau equation (we neglect the noise term since in a quench situation $\epsilon < 0$ the thermal fluctuations being proportional to $\sqrt{T}$ become smaller after the quench) can be written as

$$\frac{\partial \phi(\vec{x}, t)}{\partial t} = \left( \gamma \nabla^2 + \beta \right) \phi(\vec{x}, t) - \alpha A[\phi(\vec{x}, t)],$$

(4)

where $A[\phi]$ is the nonlinear term

$$A[\phi(\vec{x}, t)] = \phi^3(\vec{x}, t),$$

(5)

and $\gamma = MR^2$, $\beta = M|\epsilon|$ and $\alpha = Mu$. Adomian’s decomposition method consists in expressing $\phi$ in the form of a functional series expansion, where the terms of the expansion are determined recursively. Specifically, the iterative procedure consists in writing $\phi(\vec{x}, t)$ and the nonlinear term $A[\phi]$ as the expansions

$$\phi(x, t) = \sum_{n=0}^{\infty} \phi_n(x, t), \quad A[\phi] = \sum_{n=0}^{\infty} A_n[\phi_0, \phi_1, \ldots, \phi_n],$$

(6)

where the $A_n$’s are the Adomian polynomials, which can be determined from

$$A_n[\phi_0, \phi_1, \ldots, \phi_n] = \frac{1}{n!} \left\{ \frac{d^n}{d\lambda^n} A[\phi_0 + \lambda \phi_1 + \cdots + \lambda^n \phi_n] \right\}_{\lambda=0}.$$  

(7)

Rewriting Eq. (4) as

$$\phi(x, t) = \phi(x, 0) + \left( \gamma \nabla^2 + \beta \right) \int_0^t dt' \phi(x, t') - \alpha \int_0^t dt' A[\phi(x, t')],$$

(8)

and substituting the expansions of Eq. (6), one obtains

$$\sum_{n=0}^{\infty} \phi_n(x, t) = \phi(\vec{x}, 0) + \left( \gamma \nabla^2 + \beta \right) \int_0^t dt' \sum_{n=0}^{\infty} \phi_n(x, t')$$

$$- \alpha \int_0^t dt' \sum_{n=0}^{\infty} A_n[\phi_0(x, t'), \phi_1(x, t'), \ldots, \phi_n(x, t')].$$

(9)
If one identifies $\phi_0(x,t) = \phi(x,0) \equiv \phi_0(x)$, Adomian’s iteration procedure consists in matching the $(n+1)$-th order term on the l.h.s. of this equation to the $n$-th order term on the r.h.s. as

$$\phi_{n+1}(x,t) = \left( \gamma \nabla^2 + \beta \right) \int_0^t dt' \phi_n(x,t') - \alpha \int_0^t dt' A_n[\phi_0(x,t'), \phi_1(x,t'), \cdots, \phi_n(x,t')] .$$

(10)

For the case when $A[\phi]$ is a cubic form, the $A_n$’s can be written in a compact way as

$$A_n[\phi_0, \phi_1, \cdots \phi_n] = \frac{1}{n!} \left\{ \frac{d^n}{d\lambda^n} \sum_{i,j,k=0}^n \lambda^{i+j+k} \phi_i \phi_j \phi_k \right\}_{\lambda=0}
= \sum_{i,j,k=0}^n \theta(i+j+k-n) \phi_i \phi_j \phi_k ,$$

(11)

where $\theta$ is the Heaviside step function. The general expression for the $(n+1)$-th term of the expansion of $\phi(\vec{x}, t)$ is given as

$$\phi_{n+1}(\vec{x}, t) = \int_0^t dt' \left[ (\gamma \nabla^2 + \beta) \phi_n(\vec{x}, t') - \alpha \sum_{i,j,k=0}^n \theta(i+j+k-n) \phi_i(\vec{x}, t') \phi_j(\vec{x}, t') \phi_k(\vec{x}, t') \right] .$$

(12)

In general, it is not possible to express the solution in closed form. As will be shown in the next section, when the nonlinear term is neglected, the series can be summed and the solution is precisely Eq. (3). The solution will be given in terms of a power series expansion in $t$ as

$$\phi_n(\vec{x}, t) = \bar{\phi}_n(\vec{x}) \frac{t^n}{n!} ,$$

(13)

where the $\bar{\phi}_n$’s are easily obtained from Eq. (12). The first few terms are given by

$$\bar{\phi}_0(\vec{x}) = \phi_0(\vec{x}) ,$$

(14)
\[ \tilde{\phi}_1(\vec{x}) = (\gamma \nabla^2 + \beta) \phi_0(x) - \alpha \phi_0^3(\vec{x}), \]  
\[ \tilde{\phi}_2(\vec{x}) = (\gamma \nabla^2 + \beta) \tilde{\phi}_1(x) - 3 \alpha \left[ \phi_0^2(\vec{x}) \tilde{\phi}_1(\vec{x}) \right], \]  
\[ \tilde{\phi}_3(\vec{x}) = (\gamma \nabla^2 + \beta) \phi_0^3(\vec{x}) - 3 \alpha \left[ \phi_0(\vec{x}) \phi_0^2(\vec{x}) + \phi_0(\vec{x}) \tilde{\phi}_1(\vec{x}) \phi_0^2(\vec{x}) \right], \]  
\[ \tilde{\phi}_4(\vec{x}) = (\gamma \nabla^2 + \beta) \phi_0^3(\vec{x}) - 3 \alpha \left[ 3 \phi_0^2(\vec{x}) \tilde{\phi}_1(\vec{x}) + 3 \phi_0^2(\vec{x}) \tilde{\phi}_2(\vec{x}) \right], \]  
\[ \tilde{\phi}_5(\vec{x}) = (\gamma \nabla^2 + \beta) \phi_0^3(\vec{x}) - 3 \alpha \left[ 3 \phi_0^2(\vec{x}) \tilde{\phi}_1(\vec{x}) + 3 \phi_0^2(\vec{x}) \tilde{\phi}_2(\vec{x}) + 3 \phi_0^2(\vec{x}) \tilde{\phi}_3(\vec{x}) \right] + 6 \phi_0(\vec{x}) \tilde{\phi}_2(\vec{x}) \phi_0^2(\vec{x}). \]

In the next section we present explicit solutions for the one-dimensional case. We obtain the exact result in the linear approximation and for the full nonlinear equation we compare results obtained with the decomposition method with the full numerical solution of the Ginzburg-Landau.

3 Explicit solutions

We start considering the linear equation ($\alpha = 0$). The $n$-th term is given by
\[ \phi_n(\vec{x}, t) = \frac{1}{n!} \left[ (\gamma \nabla^2 + \beta) t \right]^n \phi_0(\vec{x}). \]  

The series can be explicitly summed, giving
\[ \phi(\vec{x}, t) = \sum_{n=0}^{\infty} \phi_n(\vec{x}, t) = \exp \left[ (\gamma \nabla^2 + \beta) t \right] \phi_0(\vec{x}). \]  

Using the Fourier transform of the initial condition and applying the exponential operator to it, one obtains precisely Eq. (3) with $\epsilon = -|\epsilon|$.

Next, we consider the nonlinear terms. We use an initial configuration that mimics a system with domains of phases characterized by an order parameter that is not small. We also consider a quench to zero temperature ($\epsilon = -1$). For $u^2 = 1$, we have that the unstable (spinodal) region corresponds to $-\sqrt{1/3} \leq \phi \leq +\sqrt{1/3}$, and the equilibrium values for the order parameter are $\phi_{\text{equil}} = \pm 1$. Specifically, we use as an initial configuration a periodic function of the
form

\[ \phi(x, 0) = \phi_0 \cos(ax) . \]  

(23)

In the regions of space in the neighborhood of \( x = \pm n\pi/a, \ n = 0, 1, \cdots \) the order parameter is close to \( \phi_0 \); as one departs from these regions the order parameter starts diminishing until it reaches zero, independently of the value of \( \phi_0 \). Note that the average value of \( \phi(x, t) \) is zero over a large region of space.

It is instructive to take a look at the first few terms of the \( \bar{\phi} \)'s of Eqs. (14)-(20). We present the explicit forms of the \( \bar{\phi}_n \) up to \( n = 4 \),

\[ \bar{\phi}_0(x) = \phi_0 \cos(ax) , \]  

(24)

\[ \bar{\phi}_1(x) = \phi_0 \cos(ax) \left[ (\beta - a^2 \gamma) - \alpha \phi_0^2 \cos^2(ax) \right] , \]  

(25)

\[ \bar{\phi}_2(x) = \phi_0 \cos(ax) \left[ (\beta - a^2 \gamma)^2 - \alpha \left( 3 \beta - 5 a^2 \gamma \right) \phi_0^2 \cos^2(ax) \right. \]
\[ \left. + 2 \alpha^2 \phi_0^4 \cos^2(ax) - 6 \alpha a^2 \gamma \phi_0^2 \sin^2(ax) \right] , \]  

(26)

\[ \bar{\phi}_3(x) = \phi_0 \cos(ax) \left[ (\beta - a^2 \gamma)^3 + \alpha^2 \left( 17 \beta - 31 a^2 \gamma \right) \phi_0^4 \cos^4(ax) \right. \]
\[ \left. - 9 \alpha^3 \phi_0^6 \cos^6(ax) + 24 \alpha a^2 \gamma \left( -\beta + 3 a^2 \gamma \right) \phi_0^2 \sin^2(ax) \right. \]
\[ \left. + \alpha \left( -9 \beta^2 + 26 a^2 \beta \gamma - 33 a^4 \gamma^2 + 58 a^2 \alpha \gamma \phi_0^2 \sin^2(ax) \right) \right] \times \phi_0^2 \cos^2(ax) , \]  

(27)

\[ \bar{\phi}_4(x) = \phi_0 \cos(ax) \left\{ (\beta - a^2 \gamma)^4 - 3 \alpha^3 \left( 31 \beta - 67 a^2 \gamma \right) \phi_0^6 \cos(ax) \right. \]
\[ \left. + 10 \alpha^4 \phi_0^8 \cos^2(ax)^2 + \alpha^2 \phi_0^4 \cos(ax)^4 \left[ 71 \beta^2 - 260 a^2 \beta \gamma \right. \right. \]
\[ \left. + 409 a^4 \gamma^2 - 598 a^2 \alpha \gamma \phi_0^2 \sin^2(ax) \right] \left. + 348 \alpha a^4 \gamma^2 \phi_0^4 \sin^2(ax) \right. \]
\[ \left. - 6 \alpha a^2 \gamma \left( 13 \beta^2 - 66 a^2 \beta \gamma + 117 a^4 \gamma^2 \right) \phi_0^2 \sin^2(ax) \right. \]
\[ \left. + \alpha \phi_0^2 \cos^2(ax) \left[ -19 \beta^3 + 83 a^2 \beta^2 \gamma - 189 a^4 \beta \gamma^2 + 253 a^6 \gamma^3 \right. \right. \]
\[ \left. - 2 \alpha a^2 \gamma \left( -253 \beta + 929 a^2 \gamma \right) \phi_0^2 \sin^2(ax) \right] \right\} . \]  

(28)

The effects of the nonlinearity are encoded by the terms proportional to \( \alpha \) and involve powers of \( \phi_0 \) higher than two. It seems clear that when \( \phi_0 \) is small, the higher powers of \( \phi_0 \) will give small contributions. We verified explicitly that at small times one is still in the linear regime at short times when \( \phi_0 \) is small and the decomposition method gives the correct answer for all values of \( x \). When \( \phi_0 \) is large, the nonlinearities will modify the linear term - in the regions close to the coordinate points \( x = \pm n\pi/a, \ n = 0, 1, \cdots \) - at relatively low order \( n \), even at very short times, as we shall see shortly. Therefore, despite the average value of \( \phi \) is zero over a large region of space, for local regions with large values of \( \phi \) the local approximation breaks down. This is
agreement with the discussions in Refs. [11] and [12] on the breakdown of the linear approximation at different length scales.

For longer times, it seems that one has to go to large orders in the expansion to obtain a good approximation to the solution. The situation can be improved if one is able to perform a (partial) summation of the series. In the literature [18] Padé approximation has been used with success to improve the convergence of the series. We are not going to follow this path here, since we are interested in the short-time behavior of the solutions.

In order to obtain a quantitative estimate of the effects of the nonlinear terms, we compare results with a numerical solution of the GL equation. The numerical solution is obtained using a semi-implicit finite-difference scheme for the time evolution and a Fast Fourier Transform for the spatial dependence [19]. Specifically, we use a domain of length $L$, with node points $x_j, j = 0, \cdots, N$, with spacing given by $h = L/N$. The value of $\phi(x, t)$ at point $x_j$ at time $t_n$ is denoted by $\phi^n_j$. The spatial and time derivatives are discretized as

$$
\frac{\partial^2 \phi(x, t)}{\partial x^2} \rightarrow \frac{\phi^n_{j+1} - 2\phi^n_j + \phi^n_{j-1}}{h^2}, \quad \frac{\partial \phi(x, t)}{\partial t} \rightarrow \frac{\phi^n_j - \phi^{n-1}_j}{\Delta t}. \quad (29)
$$

Next, we write the Fourier series for $\phi^n_j$ and $U(\phi^n_j) = \beta \phi^n_j - \alpha (\phi^n_j)^3$ as

$$
\phi^n_j = \sum_{k=0}^{N-1} \exp \left( i \frac{2\pi}{N} j k \right) \tilde{\phi}^n(k), \quad U(\phi^n_j) = \sum_{k=0}^{N-1} \exp \left( i \frac{2\pi}{N} j k \right) \tilde{U}^n(k). \quad (30)
$$

Substituting these into the GL equation, one can write the following semi-implicit equation for $\tilde{\phi}$ at momentum $k$ and time $n$

$$
\tilde{\phi}^n(k) = \frac{1}{1 + \gamma \lambda_k \Delta t} \left[ \tilde{\phi}^{n-1}(k) + \Delta t \tilde{U}^{n-1}(k) \right], \quad (31)
$$

where $\lambda_k = [2 - 2 \cos(2\pi k/N)]/h^2$. This scheme has been shown to be very efficient previously [19], and seems to be adequate for our purposes here.

In Figure 1 we plot the solutions at short times ($t \leq 1/M$) obtained with the decomposition method, for the linear (dashed lines) and nonlinear (long-dashed lines) approximations, and with the numerical method just described (solid lines). We present the results for $x = 0$. We use $a = \pi/4$, $L = 8$ and $\phi_0 = 0.5$ and $\phi_0 = 0.75$ for the initial condition in Eq. (23); the first value corresponds to unstable matter and the the second to metastable matter in the region close to $x = 0$ and $x = L$. Note that the average value of $\phi$ over our spatial length $L$ is zero (we use periodic boundary conditions).
Fig. 1. The solutions at short times at $x = 0$ of the noiseless GL equation. The solid line represents the numerical solution using Eq. (31) and the dashed and long-dashed lines represent the solutions obtained with the decomposition method up to 6-th order, in the linear and nonlinear approximations respectively.

From the plots in the Figure, one clearly sees that the linear approximation fails dramatically at all time scales. It is also clear that the decomposition method describes very well the exact solution at short times, up to the characteristic “diffusion” time scale given by $1/M$.

In order to analyze the breakdown of the linear approximation and the rate of convergence of the expansion we consider the partial sum

$$\phi_N(x, t) = \sum_{n=0}^{N} \phi_n(x, t).$$

(32)

The exact solution is obtained for $N \to \infty$. In Table 1 we present the ratios of $\phi_N$ to the exact solution $\phi_{\text{exact}}(x = 0, t)$ for the linear ($\phi^l_N$) and nonlinear ($\phi^n_N$) approximations for $t = 1/(2M)$ and $t = 1/M$. One clearly sees again that the decomposition method provides an excellent approximation to the solution at short times at relatively low order. For $N = 4$, in the nonlinear case, it approximates the solution to 2.5% accuracy at $t = 1/(2M)$ and to 10% at $t = 1/M$. The Table also indicates that convergence becomes slower after such an accuracy has been achieved and a resummation seems necessary to further improve convergence.
Table 1
Ratios of the linear $\phi^l_N(x=0,t)$ and nonlinear $\phi^{nl}_N(x=0,t)$ approximations to the exact solution $\phi_{exact}(x=0,t)$, for $t = 1/(2M)$ and $t = 1/M$.

| N | $\phi^l_N/\phi_{exact}$ | $\phi^{nl}_N/\phi_{exact}$ | $\phi^l_N/\phi_{exact}$ | $\phi^{nl}_N/\phi_{exact}$ |
|---|----------------|----------------|----------------|----------------|
| 1 | 1.088 | 0.997 | 1.185 | 1.036 |
| 2 | 1.178 | 1.031 | 1.478 | 1.146 |
| 3 | 1.193 | 1.026 | 1.576 | 1.115 |
| 4 | 1.194 | 1.025 | 1.600 | 1.100 |
| 5 | 1.195 | 1.025 | 1.605 | 1.100 |
| 6 | 1.195 | 1.025 | 1.606 | 1.100 |

4 Conclusions and Future Perspectives

We have proposed to use Adomian’s decomposition method to study in an analytical way the early stage of phase ordering in the context of the Ginzburg-Landau equation for a nonconserved order parameter. The method seems to be particularly useful for studying the time evolution of the order parameter for systems with regions in space containing metastable matter, where the linear approximation breaks down. Although we have used a particular free energy functional and a simple initial configuration, they were sufficiently representative for the purposes of the present paper.

The results presented here apply to a model A type of equation, with a double-well free-energy functional. We expect the method to be useful for other types of phase ordering models, such as model B, governed by a Cahn-Hilliard type of equation. Work in this direction is in progress. The coupled problem of a conserved and a nonconserved order parameter (model C) seems to be particularly interesting, since very little is known about the short time dynamics for nontrivial couplings.

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References

[1] J.D. Gunton, M. San Miguel, and P.S. Sahni, in Phase Transitions and Critical Phenomena, edited by C. Domb and J.L. Lebowitz, Academic, New York, 1983, Vol. 8, p. 267.

[2] A.J. Bray, Adv. Phys. 43 (1994) 357.

[3] J.W. Cahn and H.E. Hilliard, J. Chem. Phys. 28 (1958) 258.

[4] G. Adomian, Solving Frontier Problems in Physics: The Decomposition Method, Kluwer, Boston, 1994.

[5] A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema, Rotterdam, 2002.

[6] D. Kaya and S.M. El-Sayed, Phys. Lett. A 310 (2003) 44.

[7] D. Kaya and S.M. El-Sayed, Phys. Lett. A 313 (2003) 28.

[8] G. Adomian and R.E. Meyers, Comp. Phys. Appl. 29 (1995) 3.

[9] K. Binder, Phys. Rev. A 29 (1984) 341, K.R. Elder, T.M. Rogers, and R.C. Desai, Phys. Rev. B 38 (1988).

[10] C. Yeung, N. Gross, and Michael Costolo, Phys. Rev. E 52 (1996) 6025.

[11] N.A. Gross, W. Klein, and K. Ludwig, Phys. Rev. Lett. 73 (1994) 2639.

[12] N.A. Gross, W. Klein, and K. Ludwig, Phys. Rev. E 56 (1997) 5160.

[13] S. Villain-Guillot and C. Josserand, Phys. Rev. E 66 (2002) 036308.

[14] S. Villain-Guillot and C. Josserand, Eur. Phys. J. B 29 (2002) 305.

[15] B.A. Berg, U.M. Heller, H. Meyer-Ortmanss, and A. Velytsky, Phys. Rev. D 69 (2004) 034501.

[16] A. Coniglio, P. Ruggiero, and M. Zannetti, Phys. Rev. E 50 (1994) 1994; F. Corberi, A. Coniglio, and M. Zanetti, Phys. Rev. E 51 (1995) 5469.

[17] C. Castellano and J. Krug, Phys. Rev. B 62 (2000) 2879.

[18] Y.C. Jiao, Y. Yamamoto, C. Dang and Y. Hao, Computers Math. Applic. 43 (2002) 783.

[19] M.I.M. Copetti and C.M. Elliot, Mat. Sci. Tecn. 6 (1990) 273.