An elementary proof of the Voros connection formula for WKB solutions to the Airy equation with a large parameter

Dedicated to Professor Yoshitsugu Takei on his 60th birthday

By

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Abstract

The Voros connection formula for WKB solutions to the Airy equation with a large parameter is proved by using cubic equations. Some parts of the results are generalized to the Pearcey system, which is a two-variable version of the Airy equation, is given.

§ 1. Introduction

In [3], the Voros connection formula (cf. [12]) for the WKB solutions to Schrödinger-type ordinary differential equations is proved from the viewpoint of microlocal analysis [6]. The proof consists of two parts. In the first part, the Schrödinger-type ordinary differential equation with an analytic potential is formally transformed to the Airy equation with a large parameter near a simple turning point. The formal series appearing in the transformation can be justified by using microdifferential operators. In the second part, the Voros connection formula for the Airy equation is proved by computing the Borel transform of the WKB solutions directly in terms of the Gauss hypergeometric...
functions. The classical connection formulas for the hypergeometric functions yield the Voros formula for the Airy equation. Combining these two parts, we have the Voros formula for general equations. (See [7] also.)

In this article, we focus on the second part. We observe that the parameters in the hypergeometric functions used in the proof in [3, 7] are contained in the Schwarz list [11]. In other words, they are algebraic functions. Our aim is to prove the connection formula for the WKB solutions to the Airy equation by using analytic continuation of algebraic functions, not of the hypergeometric functions. This point of view gives some generalization. We find a similar structure in the Pearcey system (cf. [9]) with a large parameter. This system is a natural generalization of the Airy equation to the two-variable case. We see that the Borel transforms of the suitably normalized WKB solutions to the Pearcey system are also algebraic functions. Hence such WKB solutions are resurgent. We hope that this example provides a part of basics of the prospective theory for the exact WKB analysis of holonomic systems.

§ 2. The Airy equation with a large parameter and its WKB solutions

The differential equation

\[ \left( -\frac{d^2}{dz^2} + z \right) w = 0 \]

is called the Airy equation (cf. [10]). Airy used a solution to this equation expressed by an integral, known as the “Airy function” nowadays (see § 3), effectively in his theory of the rainbow (caustics) [11]. We introduce a positive large parameter \( \eta \) by setting \( z = \eta^{2/3} x, \psi(x, \eta) = w(\eta^{2/3} x) \). Then we have a differential equation of the form

\[ \left( -\frac{d^2}{dx^2} + \eta^2 x \right) \psi = 0. \]

We call this the Airy equation with the large parameter, or simply, the Airy equation. This equation plays a fundamental role in the exact WKB analysis. Let \( S \) denote the logarithmic derivative of the unknown function \( \psi \), namely \( S = \frac{d}{dx} \log \psi \). Then \( S \) should satisfy the following Riccati-type equation:

\[ \frac{dS}{dx} + S^2 = \eta^2 x. \]

This equation has a formal solution \( S = \sum_{j=-1}^{\infty} \eta^{-j} S_j \) defined by the recurrence relation

\[ \begin{cases} S_{-1}^2 = x, \\ S_{j+1} = -\frac{1}{2S_{-1}} \left( \frac{dS_j}{dx} + \sum_{k=0}^{j} S_k S_{j-k} \right) \end{cases} \quad (j = -1, 0, 1, 2, 3, \ldots). \]
We consider the exponential of the integral of $S$:

$$\psi = \exp \left( \int S \, dx \right),$$

which formally satisfies (2.2). We call this a WKB solution to (2.2). We easily see that

$$S_{-1} = x^{1/2}, \quad S_0 = -\frac{1}{4x} \ldots$$

and if we fix the branch of the square root, say, as $x^{1/2} > 0$ for $x > 0$, we have a formal solution $S^{(+)}$. Another choice of the branch also gives a formal solution $S^{(-)}$. If we set

$$S_{\text{odd}} = \sum_{j=-1}^{\infty} \eta^{-2j-1} S_{2j+1} = \eta x^{1/2} - \eta^{-1} \frac{5}{32} x^{-5/2} - \eta^{-3} \frac{1105}{2048} x^{-11/2} + \ldots,$$

$$S_{\text{even}} = \sum_{j=0}^{\infty} \eta^{-2j} S_{2j} = -\frac{1}{4x} - \eta^{-2} \frac{15}{64} x^{-4} - \frac{1695}{1024} x^{-7} + \ldots,$$

then we have

$$S^{(\pm)} = \pm S_{\text{odd}} + S_{\text{even}}$$

satisfies (2.3) and

$$S_{\text{even}} = -\frac{1}{2} \frac{d}{dx} \log S_{\text{odd}}$$

holds. Hence we may take $-1/2 \log S_{\text{odd}}$ as a primitive of $S_{\text{even}}$ and have the special WKB solutions of the form

(2.5) $$\psi_{\pm} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp \left( \pm \int_0^x S_{\text{odd}} \, dx \right).$$

Here the integral is defined by one half of the term-by-term contour integral of $S_{\text{odd}}$ starting from $x$ on the second sheet of the Riemann surface of $\sqrt{x}$, going around the origin counterclockwise and back to the $x$ on the first sheet:

$$\int_0^x S_{\text{odd}} \, dx = \frac{2}{3} x^{3/2} + \eta^{-1} \frac{5}{48} x^{-3/2} + \eta^{-3} \frac{1105}{9216} x^{-9/2} + \ldots.$$ 

We observe that $S_{\text{odd}}$ has the weighted homogeneity

$$S_{\text{odd}}(\lambda^2 x, \lambda^{-3} \eta) = \lambda^{-2} S_{\text{odd}}(x, \eta).$$

Hence $\psi_{\pm}$ satisfies

$$\psi_{\pm}(\lambda^2 x, \lambda^{-3} \eta) = \lambda \psi_{\pm}(x, \eta).$$
This relation and (2.2) imply that $\psi_{\pm}$ are formal solutions to the following system of partial differential equations in the variable $(x, \eta)$:

\[
\left\{\begin{array}{l}
\left( -\frac{\partial^2}{\partial x^2} + \eta^2 x \right) \psi = 0, \\
\left( 2x \frac{\partial}{\partial x} - 3\eta \frac{\partial}{\partial \eta} - 1 \right) \psi = 0.
\end{array}\right.
\] (2.6)

§ 3. Integral representation

There are two standard solutions to (2.1), which are called the Airy functions [10]:

\[
\text{Ai}(z) = \frac{1}{2\pi i} \int_{\gamma_1} \exp \left( -z\xi + \frac{\xi^3}{3} \right) d\xi,
\]

(3.1)

\[
\text{Bi}(z) = \frac{1}{2\pi} \int_{-\gamma_2 + \gamma_3} \exp \left( -z\xi + \frac{\xi^3}{3} \right) d\xi.
\]

(3.2)

Here the paths $\gamma_j$ ($j = 1, 2, 3$) are taken as shown in Fig. 3.1. More precisely, the arguments of three half-line asymptotes of the paths are $\pm \pi/3, \pi$.

Hence the integrals

\[
\varphi_j = \int_{\gamma_j} \exp \left( \eta \left( \frac{t^3}{3} - xt \right) \right) dt \quad (j = 1, 2, 3)
\]

(3.3)

are solutions to (2.2) satisfying $\varphi_1 + \varphi_2 + \varphi_3 = 0$. Here $\gamma_j$’s are suitably deformed. We can see that $\varphi_j$ have the same homogeneity as that of $\psi_{\pm}$, namely,

$$\varphi_j(\lambda^2 x, \lambda^{-3} \eta) = \lambda \varphi_j(x, \eta).$$

This implies $\psi = \varphi_j$ ($j = 1, 2, 3$) satisfy (2.6).
Remark. The Airy functions are expressed in terms of $\varphi_j$ as

\begin{equation}
\text{Ai}(\eta^{2/3} x) = \frac{\eta^{1/3}}{2\pi i} \varphi_1(x, \eta), \quad \text{Bi}(\eta^{2/3} x) = \frac{\eta^{1/3}}{2\pi} (\varphi_3(x, \eta) - \varphi_2(x, \eta)).
\end{equation}

We rewrite (3.3) by setting $t^{3/3} - xt = -y$:

\begin{equation}
\varphi_j = \int_{c_j} g(x, y) \exp(-y\eta) dy.
\end{equation}

Here $c_j$ is the image of $\gamma_j$ by the mapping $t \mapsto y$ and

$$g(x, y) = \left. \frac{1}{x - t^2} \right|_{t=t(x, y)}.$$ 

The branch of the root $t = t(x, y)$ of the cubic equation

\begin{equation}
t^{3/3} - xt = -y
\end{equation}

is taken suitably (see §6). We note that (3.5) looks like the Laplace integral defining the Borel sum of WKB solutions (see (5.1)) and hence $g$ is expected to have some relation with the Borel transform of WKB solutions.

\textbf{Lemma 3.1.} The function $g = g(x, y)$ defined as above satisfies the cubic equation

\begin{equation}
(9y^2 - 4x^3)g^3 + 3xg + 1 = 0.
\end{equation}

\textbf{Proof.} Eliminating $t$ from the relations

$$(x - t^2)g = 1, \quad \frac{t^3}{3} - xt = -y,$$

we have (3.7). \hfill \Box

\textbf{Proposition 3.2.} The algebraic function $g$ defined by the cubic equation (3.7) satisfies the system of partial differential equations

\begin{equation}
\begin{cases}
- \frac{\partial^2}{\partial x^2} + x \frac{\partial^2}{\partial y^2} + 2x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} + 2 \right) g = 0.
\end{cases}
\end{equation}

\textbf{Proof.} We can express $g_x$, $g_{xx}$, $g_y$ and $g_{yy}$ in terms of $g$ by differentiating (3.7) twice in $x$ and in $y$. Putting them into the left-hand sides of (3.8) and using (3.7), we see they vanish. \hfill \Box
We observe that the characteristic variety of the system (3.8) is the conormal bundle of the curve $4x^3 - 9y^2 = 0$ and hence the system is holonomic. The second equation of (3.8) implies that the unknown function can be written in the form

$$g(x, y) = \frac{1}{x} h \left( \frac{y}{x^{3/2}} \right)$$

by using a function $h$ of one variable. It is easy to find a second-order ordinary differential equation for $h$ by using the first equation. This implies the rank of the holonomic system equals 2. Hence we have

**Theorem 3.3.** Let $g_j$ ($j = 1, 2, 3$) be three branches of the algebraic function $g$ defined by the cubic equation (3.7). Then any two of them form a basis of the analytic solution space of the holonomic system

$$\begin{cases} 
- \frac{\partial^2}{\partial x^2} + x \frac{\partial^2}{\partial y^2} u = 0, \\
2x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} + 2 u = 0.
\end{cases}$$

(3.9)

§ 4. The Borel transform of WKB solutions

Let $\psi_{\pm, B}$ denote the Borel transform of $\psi_{\pm}$. Explicitly, $\psi_{\pm}$ have the forms

$$\psi_{\pm} = \eta^{-\frac{1}{4}} x^{-\frac{1}{4}} \left( 1 - \frac{5}{32} \eta^{-2} x^{-3} - \frac{1105}{2048} \eta^{-4} x^{-6} - \cdots \right)^{-\frac{1}{2}} \times \exp \left( \pm \left( \frac{2}{3} \eta^{-\frac{3}{2}} + \eta^{-2} \frac{5}{48} x^{-\frac{3}{2}} + \eta^{-3} \frac{1105}{9216} x^{-\frac{3}{2}} + \cdots \right) \right).$$

Setting

$$A = -\frac{5}{32} x^{-3} - \frac{1105}{2048} \eta^{-2} x^{-6} - \cdots$$

and

$$B = \frac{5}{48} x^{-\frac{3}{2}} + \eta^{-2} \frac{1105}{9216} x^{-\frac{3}{2}} + \cdots,$$

we may rewrite them as

$$\eta^{-\frac{1}{2}} x^{-\frac{1}{4}} (1 + \eta^{-2} A(x, \eta))^{-\frac{1}{2}} \exp \left( \pm \frac{2}{3} \eta^{-\frac{3}{2}} \right) \exp(\pm \eta^{-1} B(x, \eta))$$

$$= \eta^{-\frac{1}{4}} x^{-\frac{1}{4}} \exp \left( \pm \frac{2}{3} \eta^{-\frac{3}{2}} \right) \left( 1 - \frac{1}{2} \eta^{-2} A - \frac{3}{8} \eta^{-4} A^2 + \cdots \right) \times \left( 1 + \eta^{-1} B + \eta^{-2} \frac{1}{2} B^2 + \cdots \right)$$

$$= \eta^{-\frac{1}{4}} x^{-\frac{1}{4}} \exp \left( \pm \frac{2}{3} \eta^{-\frac{3}{2}} \right) (1 + b_{1}^\pm \eta^{-1} x^{-\frac{3}{2}} + b_{2}^\pm (\eta^{-1} x^{-\frac{3}{2}})^2 + \cdots)$$
with some constants $b_j^\pm$. Then $\psi_{\pm, B}$ can be written in the form

$$(4.1) \quad \psi_{\pm, B}(x, y) = \frac{1}{x^{\frac{1}{\pi}}} \left( \frac{y}{x^{\frac{1}{\pi}}} \pm \frac{2}{3} \right)^{-\frac{1}{2}} \left\{ \frac{1}{\Gamma(\frac{3}{2})} \left( \frac{y}{x^{\frac{1}{\pi}}} \pm \frac{2}{3} \right) + \cdots \right\} \times \frac{\sqrt{\pi b_j^\pm}}{\Gamma(j + \frac{3}{2})} \left( \frac{y}{x^{\frac{1}{\pi}}} \pm \frac{2}{3} \right)^j + \cdots \right\}. $$

We introduce a new variable $s = \frac{3y}{4x^{\frac{1}{\pi}}} + \frac{1}{2}$. Then we may rewrite $\psi_{\pm, B}$ as follows:

$$(4.2) \quad \psi_+, B = \frac{\sqrt{3}}{2\sqrt{\pi} x} s^{-\frac{1}{4}} \left( 1 + \frac{b_1^+}{3} s + \cdots \right),$$

$$(4.3) \quad \psi_-, B = \frac{\sqrt{3}}{2\sqrt{\pi} x} (s - 1)^{-\frac{1}{4}} \left( 1 + \frac{b_1^-}{3} (s - 1) + \cdots \right).$$

Here we take the branches as $s^{1/2} > 0$ if $s > 0$ and $(s - 1)^{1/2} > 0$ if $s > 1$.

On the other hand, it follows from the definition of the Borel transform that $\psi_{\pm, B}$ satisfies the formal Borel transform of $(2.0)$:

$$(4.4) \quad \left\{ \left( -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} x \right) \psi_{\pm, B} = 0 \right\}$$

or, equivalently,

$$(4.5) \quad \left\{ \left( -\frac{\partial^2}{\partial x^2} + x \frac{\partial^2}{\partial y^2} \right) \psi_{\pm, B} = 0, \right\}$$

This system coincides with $(3.9)$. Hence Theorem 3.3 yields

**Proposition 4.1.** The Borel transforms $\psi_{\pm, B}$ of the WKB solutions $\psi_\pm$ can be written as linear combinations of any two of $g_j$’s. Here $g_j$ ($j = 1, 2, 3$) are defined in Theorem 3.3. Especially, $\psi_{\pm, B}$ are algebraic functions.

**Remark.** The explicit forms of $\psi_{\pm, B}$ are given by using hypergeometric functions in $[3, 7]$:

$$(4.6) \quad \left\{ \begin{array}{l}
\psi_+, B = \frac{\sqrt{3}}{2\sqrt{\pi} x} s^{-\frac{1}{4}} \binom{\frac{5}{6}}{\frac{1}{2}} F_1 \left( \frac{5}{6}, \frac{1}{2}; s \right), \\
\psi_-, B = \frac{\sqrt{3} i}{2\sqrt{\pi} x} (1 - s)^{-\frac{1}{4}} \binom{\frac{5}{6}}{\frac{1}{2}} F_1 \left( \frac{5}{6}, \frac{1}{2}; 1 - s \right).
\end{array} \right.$$
The classical connection formulas for the hypergeometric functions were used in [3, 7] for the derivation of the Voros connection formula for the Airy equation. In the present article, we do not utilize (4.6).

We set

\[ g = \frac{X}{x s^{1/2}(1-s)^{1/2}} \]

in (3.7). Then we have the following cubic equation for \( X \):

\[ 16X^3 - 3X - s^{1/2}(1-s)^{1/2} = 0. \]  

If \( s = 0 \), we have \( X = 0, \pm \sqrt{3}/4 \) and we find three roots \( X_1, X_2, X_3 \) of (4.7) near \( s = 0 \) with expansions

\[
\begin{align*}
X_1 &= \frac{\sqrt{3}}{4} + \frac{1}{6} s^{1/2} - \frac{1}{6\sqrt{3}} s + \cdots, \\
X_2 &= -\frac{\sqrt{3}}{4} + \frac{1}{6} s^{1/2} + \frac{1}{6\sqrt{3}} s + \cdots, \\
X_3 &= -\frac{1}{3} s^{1/2} - \frac{5}{162} s^{3/2} - \cdots.
\end{align*}
\]

We take the analytic continuation of \( X_j \) to \( s = 1 \). If \( s = 1 \), we also have \( X = 0, \pm \sqrt{3}/4 \). For \( s = 1/2 \), we have a double root \( X = -1/4 \) and a simple root \( X = 1/2 \). The expansion of the branches of \( X \) which merge at \( s = 1/2 \) are

\[ X = -\frac{1}{4} \pm \frac{1}{2\sqrt{3}} \left( s - \frac{1}{2} \right) + \frac{1}{18} \left( s - \frac{1}{2} \right)^2 \pm \cdots. \]

The graphs of \( Y = 16X^3 - 3X - s^{1/2}(1-s)^{1/2} \) for \( s = 0, 1/2, 1 \) are shown in Figures 4.1–4.3. Since the coefficients of \( (s - 1/2) \) of (4.9) do not vanish, two roots \( X_2 \) and \( X_3 \) pass each other at \( s = 1/2 \) and the coefficients of the leading terms of the expansions of \( (X_1, X_2, X_3) \) at \( s = 1 \) are \( (\sqrt{3}/4, 0, -\sqrt{3}/4) \).
Replacing $s$ by $1-s$ in the expansions of $X_j$’s at $s=0$ and comparing the leading coefficients, we have

$$
\begin{align*}
X_1 &= \frac{\sqrt{3}}{4} + \frac{1}{6} (1-s)^{1/2} - \frac{1}{6\sqrt{3}} (1-s) + \cdots, \\
X_2 &= -\frac{1}{3} (1-s)^{1/2} - \frac{5}{162} (1-s)^{3/2} + \cdots, \\
X_3 &= -\frac{\sqrt{3}}{4} + \frac{1}{6} (1-s)^{1/2} + \frac{1}{6\sqrt{3}} (1-s) + \cdots.
\end{align*}
$$

(4.10)

We set

$$
g_j = \frac{X_j}{xs^{1/2}(1-s)^{1/2}} \quad (j=1,2,3).
$$

Then (4.8) yields the expansion of $g_j$ at $s=0$:

$$
g_1 = \frac{1}{xs^{1/2}} \left( \frac{\sqrt{3}}{4} + \frac{1}{6} s^{1/2} + \frac{5}{24\sqrt{3}} s + \cdots \right), \\
g_2 = \frac{1}{xs^{1/2}} \left( -\frac{\sqrt{3}}{4} + \frac{1}{6} s^{1/2} - \frac{5}{24\sqrt{3}} s + \cdots \right), \quad g_3 = -g_1 - g_2.
$$

Using Proposition 4.1 and comparing the leading terms of these expansions with (4.2), we have

$$
\psi_{+,B} = \frac{1}{\sqrt{\pi}} (g_1 - g_2).
$$

(4.11)

Similarly, (4.10) yields the expansion of $g_j$ at $s=1$:

$$
g_1 = \frac{1}{x(1-s)^{1/2}} \left( \frac{\sqrt{3}}{4} + \frac{1}{6} (1-s)^{1/2} + \frac{5\sqrt{3}}{24} (1-s) + \cdots \right), \\
g_2 = \frac{1}{x} \left( -\frac{1}{3} + \frac{16}{81} (1-s) + \cdots \right), \\
g_3 = \frac{1}{x(1-s)^{1/2}} \left( -\frac{\sqrt{3}}{4} + \frac{1}{6} (1-s)^{1/2} - \frac{5\sqrt{3}}{24} (1-s) + \cdots \right).
$$

We compare the leading terms with (4.13). Then we obtain

$$
\psi_{-,B} = \frac{i}{\sqrt{\pi}} (g_1 - g_3).
$$

Using Proposition 4.1 and comparing the leading terms of these expansions with (4.2), we have

$$
\psi_{+,B} = \frac{1}{\sqrt{\pi}} (g_1 - g_2), \quad \psi_{-,B} = \frac{i}{\sqrt{\pi}} (g_1 - g_3).
$$

These relations can be also written in the form

$$
\psi_{+,B} = \frac{1}{\sqrt{\pi}} \Delta_{-\frac{3}{4},3/2} g_2(x,y), \quad \psi_{-,B} = \frac{i}{\sqrt{\pi}} \Delta_{\frac{3}{4},3/2} g_3(x,y).
$$

Here $\Delta_{x} g(x,y)$ designates the discontinuity of $g(x,y)$ at $y=\alpha$. 

**Theorem 4.2.** The Borel transforms $\psi_{\pm,B}$ of the WKB solutions $\psi_{\pm}$ to the Airy equation are expressed in terms of three branches $g_j$’s (specified as above) of the algebraic function $g$ defined by (3.7) as follows:

$$
\psi_{+,B} = \frac{1}{\sqrt{\pi}} (g_1 - g_2), \quad \psi_{-,B} = \frac{i}{\sqrt{\pi}} (g_1 - g_3).
$$

These relations can be also written in the form

$$
\psi_{+,B} = \frac{1}{\sqrt{\pi}} \Delta_{-\frac{3}{4},3/2} g_2(x,y), \quad \psi_{-,B} = \frac{i}{\sqrt{\pi}} \Delta_{\frac{3}{4},3/2} g_3(x,y).
$$

Here $\Delta_{x} g(x,y)$ designates the discontinuity of $g(x,y)$ at $y=\alpha$. 

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Note that if \( x \neq 0 \), then \( \pm 2/3x^{3/2} \) are singularities of square root-type of \( \psi_{+,B} \) and of \( \psi_{-,B} \) in \( y \)-variable.

§ 5. The Voros connection formula for the Airy equation

The Stokes curve (cf. [7]) of (2.2) is defined by

\[
\text{Im} \int_0^x \sqrt{x} \, dx = 0.
\]

It consists of three half-lines

\[
\arg x = 0, \pm 2/3 \pi.
\]

Let \( \mathcal{R}_I \) and \( \mathcal{R}_{II} \) denote the sectors (two of the Stokes regions)

\[
\{ x \in \mathbb{C} | -2/3 \pi < \arg x < 0 \} \quad \text{and} \quad \{ x \in \mathbb{C} | 0 < \arg x < 2/3 \pi \},
\]

respectively. Let \( \ell_{\pm}(x) \) be half-lines

\[
\left\{ \pm \frac{2}{3} x^{3/2} + t \bigg| t \geq 0 \right\}
\]

with the positive orientation in the \( y \)-plane and we set

\[
(5.1) \quad \Psi^J_{\pm} = \int_{\ell_{\pm}(x)} \psi_{\pm,B}(x,y) \exp(-y\eta) \, dy \quad \text{for} \quad x \in \mathcal{R}_J \quad (J = I, II).
\]

Figures 5.1 and 5.2 show \( \mathcal{R}_J \) and \( \ell_{\pm}(x) \) for \( x \in \mathcal{R}_J \) \((J = I, II)\), respectively. The wavy lines designate the branch cuts for \( \psi_{+,B} \) in \( y \)-plane. Theorem 4.2 implies that these functions are well-defined and each \( \Psi^J_{\pm} \) gives the Borel sum of \( \psi_{\pm} \) in \( \mathcal{R}_J \) \((J = I, II)\), respectively.

We take the analytic continuation of \( \Psi^I_{\pm} \) to \( \mathcal{R}_{II} \). If a point \( x \in \mathcal{R}_I \) with \( \text{Re} \, x > 0 \) moves up to \( \mathcal{R}_{II} \) across the positive real axis, the singular point \( y = 2/3x^{3/2} \) of \( \psi_{+,B} \) crosses \( \ell_+(x) \) (see Figures 5.2.3 and 5.2.1). Hence we have to take a path \( \tilde{\ell}_+(x) \) of integration shown in Figure 5.3 instead of \( \ell_+(x) \) for \( x \in \mathcal{R}_{II} \) for the analytic continuation.
Thus the analytic continuation of $\Psi_I^+$ has an expression

$$\Psi_I^+ = \int_{\ell_+(x)} \psi_{+,B}(x,y) \exp(-y\eta) dy$$

for $x \in \mathcal{R}_I$. If we take a path $\Gamma$ surrounding the branch cut (or $\ell_-(x)$) clockwise (as shown in Figure 5.4), we have

$$\tilde{\ell}_+(x) \sim \ell_+(x) + \Gamma$$

as the path of integration for $x \in \mathcal{R}_I$. Hence we have

(5.2) $\Psi_I^+ = \int_{\ell_+(x)} \psi_{+,B}(x,y) \exp(-y\eta) dy + \int_{\Gamma} \psi_{+,B}(x,y) \exp(-y\eta) dy.$

The first term equals $\Psi_{II}^I$. It follows from Theorem 4.2 that the second term can be written in terms of $g_j$'s:

$$\int_{\Gamma} \frac{1}{\sqrt{\pi}} (g_1 - g_2) \exp(-y\eta) dy.$$

Since $g_2$ is holomorphic on a neighborhood of $\ell_-(x)$ in the $y$-variable, it does not contribute to the value of the integration. Therefore we have

(5.3) $\Psi_I^+ = \Psi_{II}^I + \frac{1}{\sqrt{\pi}} \int_{\Gamma} g_1 \exp(-y\eta) dy.$

The second term of the right-hand side equals

$$-\frac{1}{\sqrt{\pi}} \int_{\ell_-(x)} (\Delta \frac{2}{3} x^{3/2} g_3) \exp(-y\eta) dy.$$

Hence Theorem 4.2 yields

$$\Psi_I^+ = \Psi_{II}^I - \frac{1}{i} \int_{\ell_-(x)} \psi_{-,B}(x,y) \exp(-y\eta) dy,$$

namely,

$$\Psi_I^+ = \Psi_{II}^I + i \Psi_{II}^I.$$

On the other hand, the singularity $y = -2/3x^{3/2}$ does not meet $\ell_-(x)$ when $x$ moves from $\mathcal{R}_I$ to $\mathcal{R}_II$ across the positive real axis. This implies $\Psi_\perp^- = \Psi_{II}^I$. Hence we have obtained the Voros connection formula for the WKB solutions to the Airy equation without using any knowledge concerning the hypergeometric functions:

$$\begin{cases} 
\Psi_I^+ = \Psi_{II}^I + i \Psi_{II}^I, \\
\Psi_\perp^- = \Psi_{II}^I.
\end{cases}$$
§ 6. Relation between WKB solutions and Airy functions

In this section, we relate the Airy functions and the WKB solutions. We employ the same notation as in the previous sections.

**Theorem 6.1.** The Airy functions $Ai$ and $Bi$ are expressed in terms of the Borel sums $\Psi_1^\pm$ of WKB solutions $\psi_\pm$ to the Airy equation in $\mathcal{R}_1$ as follows:

\[
\begin{aligned}
\text{Ai}(\eta^2 x) &= \eta^2 \frac{1}{2\sqrt{\pi}} \Psi_1^-, \\
\text{Bi}(\eta^2 x) &= \eta^2 \frac{1}{\sqrt{\pi}} \Psi_1^+ - \eta^2 \frac{i}{2\sqrt{\pi}} \Psi_-.
\end{aligned}
\]

**Proof.** We choose the path $\gamma_j$ of integration (3.3) more carefully. If $x \in \mathcal{R}_1$, we take $\gamma_1$ (resp. $\gamma_2$) as the steepest descent path of the phase function of the integral (3.3) passing through the saddle point $\sqrt{x}$ (resp. $-\sqrt{x}$). Here we take the branch as \(|\arg \sqrt{x}| < \pi/3\) for $x \in \mathcal{R}_1 \cup \mathcal{R}_{11}$. As a set, $\gamma_1$ (resp. $\gamma_2$) is included in

\[
\left\{ t \in \mathbb{C} \mid \text{Im} \left( \frac{t^3}{3} + xt + \frac{2}{3} x^{3/2} \right) = 0, \text{Re} \left( \frac{t^3}{3} - xt + \frac{2}{3} x^{3/2} \right) \leq 0 \right\}
\]

(resp. \(\left\{ t \in \mathbb{C} \mid \text{Im} \left( \frac{t^3}{3} - xt - \frac{2}{3} x^{3/2} \right) = 0, \text{Re} \left( \frac{t^3}{3} - xt - \frac{2}{3} x^{3/2} \right) \leq 0 \right\}\)).

We go back to (3.5) and specify the branch of $g$ more precisely. Let us divide $\gamma_1$ into two parts by cutting it at the saddle point $\sqrt{x}$. We denote by $\gamma_1^-$ (resp. $\gamma_1^+$) the lower (resp. upper) part. Let $c_1^\pm$ be the image of $\gamma_1^\pm$ by the mapping $t \mapsto y$. Then $c_1^-$ (resp. $c_1^+$) coincides with $-\ell_-(x)$ (resp. $\ell_-(x)$). If $y \in c_1^-$ (resp. $y \in c_1^+$) is close to $2/3x^{3/2}$, the branch of the root $t$ of (3.6) should be taken as

\[
t = x^{1/2} \left( 1 + \frac{2\sqrt{3}}{3} (1 - s)^{1/2} + \cdots \right) \quad \text{resp.} \quad t = x^{1/2} \left( 1 - \frac{2\sqrt{3}}{3} (1 - s)^{1/2} + \cdots \right).
\]

Here we choose the branch as $(1 - s)^{1/2} = (s - 1)^{1/2} e^{-\pi i/2}$ for $s > 1$. Hence the branch of $g = \frac{1}{x-t^2}$ should be taken as $g_1$ on $c_1^-$ (resp. $g_3$ on $c_1^+$). By the definition of $g_j$, (3.5) for $j = 1$ should be understood as

\[
\varphi_1 = \int_{c_1^-} g_1(x, y) \exp(-y\eta)dy + \int_{c_1^+} g_3(x, y) \exp(-y\eta)dy = \int_{-\ell_-(x)} g_1(x, y) \exp(-y\eta)dy + \int_{\ell_+(x)} g_3(x, y) \exp(-y\eta)dy = -\int_{\ell_-(x)} (g_1(x, y) - g_3(x, y)) \exp(-y\eta)dy.
\]

Using Theorem 4.2 we have

\[
\varphi_1 = -\frac{\sqrt{\pi}}{i} \int_{\ell_-(x)} \psi_{-,B}(x, y) \exp(-y\eta)dy.
\]
Thus the first equation of (3.4) shows
\[
\text{Ai}(\eta^{2/3} x) = \frac{\eta^{1/3}}{2\pi i} \left( -\frac{\sqrt{\pi}}{i} \int_{\ell_-(x)} \psi_{-B}(x, \eta) \exp(-y\eta) \, dy \right) = \frac{\eta^{1/3}}{2\pi i} \Psi_1^1.
\]

Next we divide \( \gamma_2 \) into two parts at the saddle point \(-\sqrt{x}\). We denote by \( \gamma_2^- \) (resp. \( \gamma_2^+ \)) the right (resp. left) part. Let \( c_2^- \) (resp. \( c_2^+ \)) be the image of \( \gamma_2^- \) (resp. \( \gamma_2^+ \)) by the mapping \( t \mapsto y \). Then \( c_2^- \) (resp. \( c_2^+ \)) coincides with \(-\ell_+(x)\) (resp. \( \ell_+(x) \)). If \( y \in c_2^- \) (resp. \( y \in c_2^+ \)) is close to \(-2/3x^{3/2}\), the branch of the root \( t \) of (3.6) should be taken as
\[
t = x^{1/2} \left( -1 + \frac{2\sqrt{3}}{3} s^{1/2} + \cdots \right) \quad \text{(resp. } t = x^{1/2} \left( -1 - \frac{2\sqrt{3}}{3} s^{1/2} + \cdots \right) \).
\]

Hence the branch of \( g = \frac{1}{x-t^2} \) must be taken as \( g_1 \) on \( c_1^- \) (resp. \( g_2 \) on \( c_1^+ \)). Then we have
\[
\varphi_2 = \int_{c_2^-} g_1(x, y) \exp(-y\eta) \, dy + \int_{c_2^+} g_2(x, y) \exp(-y\eta) \, dy
= -\int_{\ell_+(x)} (g_1(x, y) - g_2(x, y)) \exp(-y\eta) \, dy
= -\sqrt{\pi} \int_{\ell_+(x)} \psi_{-B}(x, y) \exp(-y\eta) \, dy.
\]

Hence the second equation in (3.4) yields
\[
\text{Bi}(\eta^{2/3} x) = \frac{\eta^{1/3}}{2\pi i} \left( -\varphi_1(x, \eta) - 2\varphi_2(x, \eta) \right)
= \frac{\eta^{1/3}}{2\pi i} \sqrt{\pi} \int_{\ell_-(x)} \psi_{-B}(x, y) \exp(-y\eta) \, dy
+ \frac{\eta^{1/3}}{2\pi} \int_{\ell_+(x)} \sqrt{\pi} \psi_{+B}(x, y) \exp(-y\eta) \, dy
= \frac{\eta^{1/3}}{\sqrt{\pi}} \Psi_1^- - \frac{\eta^{1/3}}{2\sqrt{\pi}} i \Psi_1^1.
\]

This completes the proof. \( \square \)

Remark. Using the Voros connection formula, we obtain
\[
\begin{cases}
\text{Ai}(\eta^{2/3} x) = \eta^{1/3} \frac{1}{2\sqrt{\pi}} \Psi_1^-, \\
\text{Bi}(\eta^{2/3} x) = \eta^{1/3} \frac{1}{\sqrt{\pi}} \Psi_1^+ + \eta^{1/3} \frac{i}{2\sqrt{\pi}} \Psi_1^1.
\end{cases}
\]

Conversely, \( \Psi_\pm^1 \) are written in terms of \( \text{Ai} \) and \( \text{Bi} \) as follows:
\[
\begin{cases}
\Psi_+^1 = \eta^{-\frac{1}{3}} \sqrt{\pi} i \text{Ai}(\eta^{\frac{2}{3}} x) + \eta^{-\frac{1}{3}} \sqrt{\pi} \text{Bi}(\eta^{\frac{2}{3}} x), \\
\Psi_-^1 = \eta^{-\frac{1}{3}} 2\sqrt{\pi} \text{Ai}(\eta^{\frac{2}{3}} x), \\
\Psi_+^{11} = -\eta^{-\frac{1}{3}} \sqrt{\pi} i \text{Ai}(\eta^{\frac{2}{3}} x) + \eta^{-\frac{1}{3}} \sqrt{\pi} \text{Bi}(\eta^{\frac{2}{3}} x), \\
\Psi_-^{11} = \eta^{-\frac{1}{3}} \sqrt{\pi} 2\sqrt{\pi} \text{Ai}(\eta^{\frac{2}{3}} x).
\end{cases}
\]
Watson’s lemma and the expressions given above reproduce the classical asymptotic formulas for the Airy functions (cf. [10]):

\[
\begin{align*}
\text{Ai}(\eta^2 x) & \sim \eta^{\frac{1}{3}} \frac{1}{2\sqrt{\pi}} \psi_+ - \eta^{\frac{1}{3}} \frac{i}{2\sqrt{\pi}} \psi_- \quad (x \in \mathcal{R}_1, \eta \to \infty), \\
\text{Bi}(\eta^2 x) & \sim \eta^{\frac{1}{3}} \frac{1}{2\sqrt{\pi}} \psi_+ + \eta^{\frac{1}{3}} \frac{i}{2\sqrt{\pi}} \psi_- \quad (x \in \mathcal{R}_2, \eta \to \infty), \\
\end{align*}
\]

Explicit forms of \( \psi_\pm \) are written as follows:

\[
\psi_\pm = e^{\pm \frac{2}{3} x^{3/2}} \eta \sum_{n=0}^{\infty} \eta^{-n-\frac{1}{2}} \left( \pm \frac{3}{4} \right)^n \frac{\Gamma(n + \frac{1}{6})\Gamma(n + \frac{5}{6})}{n!} x^{-\frac{3}{2}n - \frac{1}{2}}.
\]

**§ 7. Some generalization**

We consider the following integral:

\[
v = \int \exp \left( \eta \left( t^4 + x_2 t^2 + x_1 t \right) \right) dt.
\]

Here \( x_1, x_2 \) are complex variables and the path of integration is taken suitably. This is a natural extension of (3.3) to two-variable case. It is called the Pearcey integral ([10]) with the large parameter \( \eta \). Most parts of the discussions developed in §§2–6 can be generalized to the system of partial differential equations that characterizes this integral. We review some results concerning this system without proof. Details will be given in [4].

We can easily see that \( \psi = v \) satisfies the system of partial differential equations

\[
\begin{align*}
(4\partial_1^3 + 2x_2 \eta^2 \partial_1 + x_1 \eta^3) \psi &= 0, \\
(\eta \partial_2 - \partial_1^2) \psi &= 0,
\end{align*}
\]

where we set \( \partial_1 = \partial/\partial x_1, \partial_2 = \partial/\partial x_2 \). WKB solutions to this system is constructed in [2, 5] and the connection problem of the solutions was discussed in [5]. We employ another system of partial differential equations. We set

\[
\begin{align*}
P_1 &= 4\partial_1 \partial_2 + 2\eta x_2 \partial_1 + \eta^2 x_1, \\
P_2 &= 4\partial_2^2 + \eta x_1 \partial_1 + 2\eta x_2 \partial_2 + \eta, \\
P_3 &= \eta \partial_2 - \partial_1^2.
\end{align*}
\]

It is easy to see that \( \psi = v \) is a solution to the system (cf. [9])

\[
\begin{align*}
P_1 \psi &= 0, \\
P_2 \psi &= 0, \\
P_3 \psi &= 0.
\end{align*}
\]
In fact, we have
\[ P_1 v = \eta \int \partial_t \exp(\eta (t^4 + x_2 t^2 + x_1 t)) \, dt = 0, \]
\[ P_2 v = \eta \int \partial_t (t \exp(\eta (t^4 + x_2 t^2 + x_1 t))) \, dt = 0, \]
\[ P_3 v = \eta^2 \int (t^2 - t^2) \exp(\eta (t^4 + x_2 t^2 + x_1 t)) \, dt = 0. \]

Setting
\[ Q_1 = 4 \partial_1^3 + 2 x_2 \eta^2 \partial_1 + x_1 \eta^2 \quad \text{and} \quad Q_2 = \eta \partial_2 - \partial_1^2 (= P_3), \]
we can confirm the following relations:
\[ (7.7) \quad P_1 = \eta^{-1} (Q_1 + 4 \partial_1 Q_2), \]
\[ (7.8) \quad P_2 = \eta^{-2} \partial_1 Q_1 + (4 \eta^{-2} Q_2 + 8 \eta^{-2} \partial_1^2 + 2 x_2) Q_2, \]
\[ (7.9) \quad Q_1 = \eta P_1 - 4 \partial_1 P_3. \]

Hence if \( \eta \neq 0 \) is fixed, \((7.6)\) is equivalent to \((7.2)\). We also note that
\[ P_2 = \eta^{-1} \partial_1 P_1 + 2(2 \eta^{-1} \partial_2 + x_2) P_3 \]
holds. Next we consider \( \eta \) as an independent complex variable. Then the systems \((7.2)\) and \((7.6)\) are subholonomic. To find another independent differential equation for \( v \), we look at the weighted homogeneity of \((7.1)\) in \((x_1, x_2, \eta)\). We can see that
\[ v(\lambda^3 x_1, \lambda^2 x_2, \lambda^{-4} \eta) = \lambda v(x_1, x_2, \eta) \]
holds for \( \lambda \neq 0 \) and hence \( \psi = v \) is a solution to
\[ (7.10) \quad (3 x_1 \partial_1 + 2 x_2 \partial_2 - 4 \eta \partial_\eta - 1) \psi = 0, \]
where \( \partial_\eta = \partial/\partial \eta \). We set
\[ P_4 = 3 x_1 \partial_1 + 2 x_2 \partial_2 - 4 \eta \partial_\eta - 1 \]
and consider the system of partial differential equations
\[ P_j \psi = 0 \quad (j = 1, 2, 3, 4). \]

To be more specific, let \( \mathcal{D} \) denote the Weyl algebra of the variable \((x_1, x_2, \eta)\) and \( \mathcal{I} \) the left ideal in \( \mathcal{D} \) generated by \( P_j \) \((j = 1, 2, 3, 4)\). We can prove the following theorem (cf. [8]):

**Theorem 7.1.** Let \( \mathcal{M} \) denote the left \( \mathcal{D} \)-module defined by \( \mathcal{I} : \)
\[ \mathcal{M} = \mathcal{D}/\mathcal{I}. \]

Then \( \mathcal{M} \) is a holonomic system of rank 3.

Thus \( \mathcal{M} \) characterizes the 3-dimensional linear subspace spanned by \((7.1)\) in the space of analytic functions. Note that there are four valleys of the integral \((7.1)\) and hence six infinite paths of integration connecting distinct two valleys. Any three of them are independent, which give a basis of the solution space.
Next we construct WKB solutions to $\mathcal{M}$. We set $S = \partial_1 \psi/\psi$ and $T = \partial_2 \psi/\psi$. Since we have (7.7) and (7.8), we can use (7.2) to find $S, T$. That is, we see that $S$ and $T$ should satisfy

$$4S^3 + 2\eta^2 x_2 S + \eta^3 x_1 + 12 S \partial_1 S + 4 \partial_1^2 S = 0$$

and

$$\eta T - \partial_1 S - S^2 = 0.$$

We seek formal solutions to these equations of the forms

$$S = \sum_{k=-1}^{\infty} \eta^{-k} S_k, \quad T = \sum_{k=-1}^{\infty} \eta^{-k} T_k.$$

Putting these expressions into the above equations, we see that $S_k$ and $T_k$ are obtained by the following recursion relations and initial conditions:

(7.11) \hspace{1cm} 4S_{-1}^3 + 2x_2 S_{-1} + x_1 = 0,

(7.12) \hspace{1cm} S_0 = -\frac{1}{2} \partial_1 \log(6S_{-1}^2 + x_2),

(7.13) \hspace{1cm} S_k = -\frac{2}{6S_{-1}^2 + x_2} \left( \sum_{k_1+k_2+k_3=k-2} S_{k_1} S_{k_2} S_{k_3} \right.

+ 3 \sum_{k_1+k_2=k-2} S_{k_1} \partial_1 S_{k_2} + \partial_1^2 S_{k-2} \left. \right) \quad (k \geq 1),

(7.14) \hspace{1cm} T_{-1} = S_{-1}^2,

(7.15) \hspace{1cm} T_k = \partial_1 S_{k-1} + \sum_{j=-1}^{k} S_j S_{k-j-1} \quad (k \geq 0).

This construction is the same as that given in [2, 5] and hence the 1-form of formal series

$$\omega = S dx_1 + T dx_2$$

is closed. In these references, a formal solution of the form

$$\psi = \eta^{-1/2} \exp \left( \int \omega \right)$$

is called a WKB solution to (7.2). Here $(a_1, a_2)$ is a suitably fixed point.

Now we consider the WKB solutions to $\mathcal{M}$ of the form

$$\psi = \eta^{-1/2} \exp \left( \int \omega \right).$$

In addition to (7.6) (or (7.2)), $\psi$ should satisfy (7.10) and hence the choice of the primitive of

$$\omega = \sum_{k=-1}^{\infty} \eta^{-k} \omega_k = \sum_{k=-1}^{\infty} \eta^{-k} (S_k dx_1 + T_k dx_2)$$

is constrained by this equation (up to genuine
An elementary proof of the Voros connection formula for the Airy equation

(7.16)

\[
\begin{align*}
\int \omega_0 &= -\frac{1}{2} \log(6S_{-1}^2 + x_2), \\
\int \omega_k &= -\frac{1}{4k} (3x_1 S_k + 2x_2 T_k) \quad (k \neq 0).
\end{align*}
\]

This choice is consistent with the construction of \( S_k \) and \( T_k \). In fact, we can confirm the first equation of (7.16) by direct computation and the second by using the homogeneity of \( S_k \) and \( T_k \). From now on, we take special WKB solutions of the form

(7.17)

\[
\psi = \frac{1}{\eta(6S_{-1}^2 + x_2)^{1/2}} \exp \left( \eta \int \omega - 1 + \sum_{k=1}^{\infty} \eta^{-k} \int \omega_k \right)
\]

with the primitives given by (7.16). Let \( S_{-1}^{(j)} \) \((j = 1, 2, 3)\) denote the three roots of (7.11) and set \( T_{-1}^{(j)} = (S_{-1}^{(j)})^2 \). According to this choice, we have three WKB solutions \( \psi_j \) \((j = 1, 2, 3)\) of the form (7.17).

Let \( \psi_{j,B} \) be the Borel transform of \( \psi_j \) \((j = 1, 2, 3)\) and \( P_{k,B} \) the formal Borel transform of \( P_k \) \((k = 1, 2, 3, 4)\). The explicit forms of \( P_{k,B} \)'s are given as follows:

\[
\begin{align*}
P_{1,B} &= 4\partial_1 \partial_2 + 2x_2 \partial_y \partial_1 + x_1 \partial_y^2, \\
P_{2,B} &= 4\partial_2^2 + x_1 \partial_y \partial_1 + 2x_2 \partial_y \partial_2 + \partial_y, \\
P_{3,B} &= \partial_y \partial_2 - \partial_1, \\
P_{4,B} &= 3x_1 \partial_1 + 2x_2 \partial_2 - 4\partial_y (-y) - 1 \\
&= 3x_1 \partial_1 + 2x_2 \partial_2 + 4y \partial_y + 3.
\end{align*}
\]

Then \( P_{k,B} \psi_{j,B} = 0 \) holds for \( j = 1, 2, 3; k = 1, 2, 3, 4 \). We denote by \( \mathcal{I}_B \) the left ideal in \( \mathcal{D}_B \) generated by \( P_{k,B} \) \((k = 1, 2, 3, 4)\). Here \( \mathcal{D}_B \) denotes the Weyl algebra of the variable \((x_1, x_2, y)\). Then the following theorem can be proved in a similar manner to Theorem 7.1.

**Theorem 7.2.** Let \( \mathcal{M}_B \) denote the left \( \mathcal{D}_B \)-module defined by \( \mathcal{I}_B \):

\[
\mathcal{M}_B = \mathcal{D}_B / \mathcal{I}_B.
\]

Then \( \mathcal{M}_B \) is a holonomic system of rank 3.

Thus \( \mathcal{M}_B \) characterizes the subspace of analytic functions spanned by \( \psi_{j,B} \) \((j = 1, 2, 3)\).

We go back to (7.1). We set \( t^4 + x_2 t^2 + x_1 t = -y \) and rewrite (7.1) as

\[
v = \int \exp(-\eta y) g(x_1, x_2, y) dy.
\]

Here \( g \) is defined by

\[
g(x_1, x_2, y) = \left. \frac{1}{4t^3 + 2x_2 t + x_1} \right|_{t = t(x_1, x_2)}
\]

and the path of integration is suitably modified. The following lemma can be proved in a similar way to the proof of Lemma 3.1.
Lemma 7.3. The function $g$ defined as above satisfies the quadratic equation

$$(4x_1^2x_2(36y - x_2^2) + 16y(x_2^2 - 4y)^2 - 27x_1^4)g^4 + 2(-8x_2y + 2x_2^3 + 9x_1^2)g^2 - 8x_1g + 1 = 0$$

and it is a solution to the holonomic system $M_B$.

We note that the singular locus of $M_B$ coincides with the zero-point set of the leading coefficient of the above quadratic equation. For general $(x_1,x_2,y)$, there are four roots $g_k$ ($k = 1, 2, 3, 4$) of the quadratic equation, which satisfy $g_1 + g_2 + g_3 + g_4 = 0$. Looking at the singularity of $g_k$, we find that any three of $g_k$’s are linearly independent. Thus we have the following theorem.

Theorem 7.4. The Borel transform $\psi_{j,B}$ of the WKB solution $\psi_j$ ($j = 1, 2, 3$) can be written as a linear combination of any three of $g_k$’s. In particular, $\psi_{j,B}$’s are algebraic and hence they are resurgent.

We can write down $\psi_{j,B}$ in terms of $g_k$’s. Explicit forms will be given in [4].

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