PRINCIPAL REALIZATION FOR THE EXTENDED AFFINE
LIE ALGEBRA OF TYPE $sl_2$ WITH COORDINATES IN A
SIMPLE QUANTUM TORUS WITH TWO GENERATORS

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Dedicated to the memory of Professor Magdi Assem

Abstract. We construct an irreducible representation for the extended affine algebra of
type $sl_2$ with coordinates in a quantum torus. We explicitly give formulas using vertex
operators similar to those found in the theory of the infinite rank affine algebra $A_\infty$.

Introduction. The purpose of this paper is to construct a module for the Extended
Affine Lie algebra (EALA for short), we call it $\hat{L}$, of type $sl_2$ with coordinates in a
quantum torus $\mathbb{C}_q$. EALAs were first introduced in the paper [HKT] as a natural
generalization of affine Kac-Moody algebras and we note that the reported motivation for
this work was from quantum gauge theories. Roughly one can think of EALA’s as higher
dimensional generalizations of loop algebras, and in [HKT] all examples came from certain
central extensions of the Lie algebra of polynomial maps of an $n$-dimensional torus $T^n$
into a finite dimensional simple Lie algebra $\mathfrak{g}$ over the field of complex numbers. That is, they
used Laurent polynomials in many variables as their coordinates. Subsequently, in [BGK],
it became clear that the set of axioms for EALA’s allowed other coordinate algebras as
well. Indeed, the quantum torus $\mathbb{C}_q$ studied in [M], as well as certain alternative and
Jordan algebra coordinates can appear as the coordinate algebra for an EALA. In this
paper we focus on what is, in many respects, the easiest example of a coordinate algebra.
Thus, our coordinates will be a quantum torus, $\mathbb{C}_q$, parametrized by one nonzero scalar
$q$ which, in addition, we assume to be generic, in the sense that it is not a root of unity,
and where $\mathbb{C}_q$ is generated by two elements $s$ and $t$ together with their inverses. These
generators then satisfy

$$ts = qst.$$
Such a quantum torus is simple as an algebra, and we note it is an algebraic version of the noncommutative torus studied in [C]. The module we construct is obtained through a process which follows the usual construction, by vertex operators, of the principal module for the affine Kac Moody Lie algebra $A_1^{(1)}$ as found in [W], [LM], [K]. Thus, we identify a Heisenberg subalgebra $\hat{H}$ and use the standard irreducible module for this where the center acts as the identity to construct our module for $\hat{L}$. Letting $\mathcal{S} = \mathbb{C}[x_1, x_2, x_3, \ldots]$ denote this Heisenberg module we will let

$$\mathcal{F} = \mathbb{C}[v, v^{-1}] \otimes \mathcal{S},$$

be the space on which we will define an action of $\hat{L}$ to obtain our module. One cannot fail to notice that $\mathcal{F}$ is isomorphic to the fermionic Fock space by the boson-fermion correspondence [K]. This type of fermionic Fock module arose in the study of the infinite rank affine algebra of type $A_\infty$ and the Kadomtsev-Petviashvili hierarchy [DJKM]. Our formulas are recorded in the bosonic picture, that is using the symmetric algebra $\mathcal{S}$ rather than the exterior algebra used for the fermionic description. In the bosonic picture the variable $v$ corresponds to the grading with respect to the variable $t$ of our quantum torus while the central element of our Heisenberg algebra, $c_s$, is so chosen that it is tied up with the variable $s$ of our algebra $\hat{L}$. One may consult [AABGP] for more on the basics theory of EALA’s as well as [BGK] and [BGKN] for how our algebra $\hat{L}$ fits into this theory. In [ABGP] one sees how the affine Lie algebras fit into this theory. However, not very much of the material from these works is needed to understand the present work, and what is needed will be recalled and the appropriate references will be cited. Also, it is clear to us that the module we construct is very closely related to the basic module of $A_\infty$ [DJKM] and [K].

Previous to this there has been some work on the representation theory of toroidal Lie algebras, where we recall these are the universal central extensions of Lie algebras of the form

(1) \[ \hat{\mathfrak{g}} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}] \]

where $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ is just the algebra of Laurent polynomials in the commuting variables $t_1, \ldots, t_n$ and $\hat{\mathfrak{g}}$ is a finite dimensional simple Lie algebra over $\mathbb{C}$. Indeed in [EY] and [EMY1] one finds vertex operator representations given in the so-called homogeneous (or untwisted) picture for these toroidal algebras with $\hat{\mathfrak{g}}$ simply laced while in [Bi1] one finds the principal (or twisted) picture dealt with. In these works it is important that one deals with the universal central extension of the algebra in (1) and also that the coordinates $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ are commutative. In fact, in these works the authors consider and add certain derivations, via a generalization of a semi-direct product construction involving a cocycle, to get a module for a Lie algebra containing
the given toroidal algebra. However, this bigger algebra is not necessarily an EALA. Thus, in the present work one finds, for the first time, a theory dealing with algebras having noncommutative coordinates as well as one which gives closed form formulas for the actions of the derivations involved in our algebras. Other works closely related to these are [BC] where one finds the beginnings of a theory of Verma type modules of some related algebras and the work [Bi2] where the author constructs an extension of the KdV hierarchy related to his principal representation given in [Bi1]. The very nice work [Bi2] is the first to point to definite applications of EALA’s to the theory of integrable systems. All the works referred to so far are algebraic in character. On the analytic side there are several papers discussing central extensions of current groups, that is Maps(X; G) where X is a compact manifold and G is a finite dimensional Lie group. Here we mention [EF] and [LMNS]. There is also an interesting work [IKUX] in which a toroidal algebra is identified as the current algebra of the four-dimensional Kähler WZW model.

In §1 we recall in detail the definition of our Lie algebra \( \hat{\mathcal{L}} \) and the basic properties which we will make use of. In particular, we define a particular maximal Heisenberg subalgebra, \( \hat{H} \), which plays a major role in the rest of the paper. The very existence of this Heisenberg subalgebra depends on the noncommutative nature of our coordinates as it includes certain elements (in the s direction) from the space \([C_q, C_q]I\) which is in our Lie algebra \( \hat{\mathcal{L}} \). Of course in the commutative case, when \( q = 1 \), there are no such elements available as then the space \([C_q, C_q]\) is zero. Thus, it is the very noncommutativity of the coordinates which allows us to define this Heisenberg subalgebra and hence, in the final analysis, a space which will eventually become a module for \( \hat{\mathcal{L}} \). The rest of §1 is devoted to showing that the algebra \( \hat{\mathcal{L}} \) is, in fact, the universal central extension of \( sl_2(C_q) \). It is becoming more and more clear (see [Bi1], [Bi2], [EM], [EMY1], [EMY2], [Ya]) that in order to have a robust representation theory of \( sl_2(C_q) \) one needs to study its universal central extension and vertex representations thereof. The main support for that comes undoubtedly from the affine theory and the underlying physical principle that the central extension is a result of a c-number anomaly brought about by quantization of currents [GSW, vol.1, page 302]. It is the universal central extension that in principle classifies all such anomalies. We also note here that in proving that \( \hat{\mathcal{L}} \) is in fact the universal central extension of \( sl_2(C_q) \) we need to make use of the structure of our quantum torus \( C_q \) as a Jordan algebra. This is interesting in its own right and we present complete proofs of the necessary details.

In §2 we display the algebra via formal variables. Here we follow the method used in [FLM] for the affine Lie algebras and so work with formal power series with coefficients in one of our Lie algebras. As is usual the formulas involve certain delta functions, and although we list all of the delta function identities we use, we do not offer proofs for them as most are quite easy and well-known or follow from well-know ones. We note here that in doing this one is essentially giving a very nice basis of the Lie algebra and the basic motivation for making the right definitions for the power series we consider comes from
both the affine theory as well as our Lemma 2.9. We summarize our results in Theorem 2.29, which should be consulted after reading the basic definitions, by any reader wishing to skip some of the more computational details of §2.

In the final section, §3, we define our space $\mathcal{F}$ and action of our algebra $\hat{L}$ on this space and then proceed to show this action makes $\mathcal{F}$ into an $\hat{L}$ module. Here the motivation for making the correct definitions comes from the fact that we have a maximal Heisenberg subalgebra in our centrally closed algebra as well as the usual result, namely Lemma 14.5 in [K], on solutions to certain eigenvalue equations. This seems to go back to at least [KKLW]. We note that in the basic Definition 3.5 there is a scalar factor, $b(m, \epsilon)$, which appears and that the motivation for what this should be is provided in the computations of Lemma 3.10. We have written the proof of this Lemma in such a way as to display this. Our main results are stated at the end of this section and we close by making some remarks about the structure of $\mathcal{F}$. Thanks go to Professor Yun Gao for pointing out some inaccuracies in an earlier version of this paper.

This work began as discussions between the two authors and our dear friend and colleague Prof. Magdi Assem who came to a sudden death during 1996 while at the Institute for Advanced Studies in Princeton. There is no doubt that he would have been a co-author to this paper had he not been taken from us, and it is with the deepest respect that we dedicate this paper to him.

Section 1. Basics on the Lie Algebra. We will work over the complex field $\mathbb{C}$ and begin by recalling the definition of the quantum torus in two variables over $\mathbb{C}$. Let $q$ be any non-zero complex number and let $I_q$ be the ideal of the group algebra of the free group on two generators $s$ and $t$ over $\mathbb{C}$ generated by the relation $ts = qst$. We let $\mathbb{C}_q$ denote the factor algebra and as usual we identify $s$ and $t$ with their images in $\mathbb{C}_q$. $\mathbb{C}_q$ is called the quantum torus on the two generators $s$ and $t$. Thus $\mathbb{C}_q$ has a basis consisting of monomials $s^{a_1}t^{a_2}$ where $a := (a_1, a_2)$ is in $\mathbb{Z}^2$. We will sometimes write $m(a)$ for the monomial $s^{a_1}t^{a_2}$ so we have that

\begin{equation}
(1.1) \quad m(a)m(b) = q^{a_2b_1} m(a + b).
\end{equation}

Basic facts about $\mathbb{C}_q$ can be found in [BGK] Proposition 2.44 and Remark 2.45. See [M] for how it is related to other studies. For our purposes we need to recall that $\mathbb{C}_q$ is simple if and only if $q$ is not a root of unity, which is the same as saying that the center of $\mathbb{C}_q$ is just the scalars $\mathbb{C}$ in $\mathbb{C}_q$. One says that $q$ is generic in this case and from now on we take this as a basic assumption. Thus we will be working with a quantum torus satisfying the following condition.

Assumption 1.2. $\mathbb{C}_q$ is a simple associative algebra over $\mathbb{C}$.

Considering $\mathbb{C}_q$ as a Lie algebra under the commutator product we have, since $q$ is generic, that the derived algebra, $[\mathbb{C}_q, \mathbb{C}_q]$, satisfies

\begin{equation}
(1.3) \quad [\mathbb{C}_q, \mathbb{C}_q] = \oplus_{a \in \mathbb{Z}^2 \setminus \{0\}} \mathbb{C} m(a).
\end{equation}
We also have that

\[(1.4) \quad \mathbb{C}_q = [\mathbb{C}_q, \mathbb{C}_q] \oplus \mathbb{C}.\]

Next, we recall the definition of the map $\varepsilon : \mathbb{C}_q \to \mathbb{C}$.

**Definition 1.5.** Define a $\mathbb{C}$-linear map $\varepsilon : \mathbb{C}_q \to \mathbb{C}$ by saying that

\[
\varepsilon(m(a)) = \begin{cases} 
1 & \text{if } a = 0, \\
0 & \text{if } a \in \mathbb{Z}^2, a \neq 0.
\end{cases}
\]

This mapping $\varepsilon$ will play a role throughout the paper. For now, we note that $\mathbb{C}_q$ is a $\mathbb{Z}^2$-graded algebra in the obvious way, and hence, we have degree derivations $d_s, d_t$ defined by saying

\[(1.6) \quad d_s(m(a)) = a_1 m(a), \quad d_t(m(a)) = a_2 m(a),\]

where $a = (a_1, a_2)$. We have $\varepsilon(d_s(m(a))) = 0$, for all monomials $m(a), a \in \mathbb{Z}^2$, and similarly for $d_t$ so that,

\[(1.7) \quad \varepsilon(d_s(x)) = 0 = \varepsilon(d_t(x)), \text{ for all } x \in \mathbb{C}_q.\]

We also remark that as $q$ is generic then $d_s$ and $d_t$ span the space of outer derivations of $\mathbb{C}_q$ when considered as an associative algebra over $\mathbb{C}$. This follows from [BGK] Remark 2.52.

We next want to define the Lie algebras which will play a major role in this paper. Thus, we let $gl_2(\mathbb{C}_q)$ denote the Lie algebra of all $2 \times 2$ matrices with entries in $\mathbb{C}_q$. Of course $gl_2(\mathbb{C}_q)$ is also an associative algebra over $\mathbb{C}$ and it is the usual commutator product which we use when considering this as a Lie algebra. If $A = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$ is in $gl_2(\mathbb{C}_q)$ we define $tr(A)$ by,

\[(1.8) \quad tr(A) = (x_{1,1} + x_{2,2}) + [\mathbb{C}_q, \mathbb{C}_q] \in \frac{\mathbb{C}_q}{[\mathbb{C}_q, \mathbb{C}_q]}.\]

It is clear that $tr(AB) = tr(BA)$ for any $A, B \in gl_2(\mathbb{C}_q)$. Also note that identifying $\mathbb{C}$ and $\frac{\mathbb{C}_q}{[\mathbb{C}_q, \mathbb{C}_q]}$ we have that $tr(A) = \varepsilon(x_{1,1} + x_{2,2})$. Moreover we let

\[(1.9) \quad sl_2(\mathbb{C}_q) = \{ A \in gl_2(\mathbb{C}_q) | tr(A) = 0 \},\]
so that $sl_2(\mathbb{C}_q)$ is a Lie subalgebra of $gl_2(\mathbb{C}_q)$. As is usual we let $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and let $I$ denote the $2 \times 2$ identity matrix. First, we define a new product on $\mathbb{C}_q$ by:

\[(1.10) \quad x \circ y = \frac{xy + yx}{2} \text{ for all } x,y \in \mathbb{C}_q.\]

We note that this product makes $\mathbb{C}_q$ into a Jordan algebra with identity over $\mathbb{C}$. This will play a role later on when we compute central extensions. We also have that if $A = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$, then

\[(1.11) \quad A = x_{1,2}E + x_{2,1}F + \frac{1}{2}(x_{1,1} - x_{2,2})H + \frac{1}{2}(x_{1,1} + x_{2,2})I.\]

It follows from this that $sl_2(\mathbb{C}_q)$ is of codimension one in $gl_2(\mathbb{C}_q)$. In fact $sl_2(\mathbb{C}_q)$ is a simple Lie algebra and is the derived algebra of $gl_2(\mathbb{C}_q)$. Notice also that the algebras $gl_2(\mathbb{C}_q)$ and $sl_2(\mathbb{C}_q)$ are both graded by $\mathbb{Z}^3$ in the following way. Elements of the one-dimensional space

\[(1.12) \quad \mathbb{C}m(a)E \text{ have degree } (1, a_1, a_2) \in \mathbb{Z}^3,\]

\[(1.13) \quad \mathbb{C}m(a)F \text{ have degree } (-1, a_1, a_2) \in \mathbb{Z}^3,\]

\[(1.14) \quad \mathbb{C}m(a)H \text{ have degree } (0, a_1, a_2) \in \mathbb{Z}^3,\]

for all $a = (a_1, a_2) \in \mathbb{Z}^2$, while elements in

\[(1.15) \quad \mathbb{C}m(b)I \text{ have degree } (0, b_1, b_2) \in \mathbb{Z}^3,\]

for all elements $b = (b_1, b_2) \in \mathbb{Z}^2 \setminus \{0\}$. Thus, letting

\[(1.16) \quad \Delta = \{(n, a_1, a_2) \in \mathbb{Z}^3 | n \in \{-1, 0, 1\}, a = (a_1, a_2) \in \mathbb{Z}^2\},\]

we have that

\[(1.17) \quad n = (n_1, n_2, n_3) \in \mathbb{Z}^3 \text{ is a degree of a non-zero space in } sl_2(\mathbb{C}_q) \text{ if and only if } n \in \Delta.\]
Similar remarks apply to the algebra $gl_2(\mathbb{C}_q)$ and in particular, this is graded by $\mathbb{Z}^3$ both as a Lie algebra as well as an associative algebra. It follows that the derivations $d_s, d_t$ of $\mathbb{C}_q$ have natural lifts to degree derivations of $gl_2(\mathbb{C}_q)$, and we again denote these by $d_s, d_t$.

Note that we have a bilinear form defined on $gl_2(\mathbb{C}_q)$ as follows. For $X,Y \in gl_2(\mathbb{C}_q)$ we let

$$ (X,Y) = tr(XY) \in \frac{\mathbb{C}_q}{[\mathbb{C}_q, \mathbb{C}_q]} \cong \mathbb{C}. $$

Here, as before, we have identified $\frac{\mathbb{C}_q}{[\mathbb{C}_q, \mathbb{C}_q]}$ with $\mathbb{C}$. It is clear that this defines a symmetric bilinear form which is invariant in the sense that for $X,Y,Z \in gl_2(\mathbb{C}_q)$ we have

$$ ([X,Y], Z) = (X, [Y,Z]). $$

Moreover, the restriction of this form to $sl_2(\mathbb{C}_q)$ is non-degenerate.

Restricting the derivations $d_s, d_t$ to $sl_2(\mathbb{C}_q)$ lets us define a central extension, which we denote as $\mathcal{L}$. As a vector space over $\mathbb{C}$, $\mathcal{L}$ is given by

$$ \mathcal{L} = sl_2(\mathbb{C}_q) \oplus \mathbb{C}c_s \oplus \mathbb{C}c_t, $$

where $c_s, c_t$ are two new symbols. Multiplication in $\mathcal{L}$ is given as follows. Let $X,Y \in sl_2(\mathbb{C}_q)$, then as elements in $\mathcal{L}$ we define

$$ [X,Y] = XY - YX + (d_s X, Y)c_s + (d_t X, Y)c_t, $$

$$ [c_s, \mathcal{L}] = \{0\} = [c_t, \mathcal{L}]. $$

That this defines a Lie algebra is clear from the fact that both $d_s$ and $d_t$ are skew-symmetric relative to the above bilinear form on $sl_2(\mathbb{C}_q)$. Thus, the subspace $sl_2(\mathbb{C}_q)$ of $\mathcal{L}$ has co-dimension 2. We will soon see that, in fact, $\mathcal{L}$ is the universal central extension of $sl_2(\mathbb{C}_q)$.

We can now extend the Lie algebra $\mathcal{L}$ to a Lie algebra $\hat{\mathcal{L}}$ by adding the derivations $d_s, d_t$ to $\mathcal{L}$. Formally, we let

$$ \hat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C}d_s \oplus \mathbb{C}d_t, $$

with multiplication extending (1.21) and (1.22) and satisfying

$$ [d, X] = d(X), [d, c] = 0, \text{ for } X \in sl_2(\mathbb{C}_q), d \in \{d_s, d_t\}, c \in \{c_s, c_t\}. $$
Letting the elements $c_s, c_t, d_s, d_t$ have degree $(0, 0, 0) \in \mathbb{Z}^3$ we see that both of our algebras have $\mathbb{Z}^3$ gradings with the degrees of the non-zero spaces being in $\Delta$ as in 1.16. Thus, we can write

\[(1.25) \quad \mathcal{L} = \oplus_{n \in \Delta} \mathcal{L}_n \quad \text{and} \quad \hat{\mathcal{L}} = \oplus_{n \in \Delta} \hat{\mathcal{L}}_n.\]

Moreover, we have that

\[(1.26) \quad \dim \mathcal{L}_n = \dim \hat{\mathcal{L}}_n = 1 \quad \text{if} \quad n = (\pm 1, a_1, a_2),\]

\[(1.27) \quad \dim \mathcal{L}_n = \dim \hat{\mathcal{L}}_n = 2 \quad \text{if} \quad n = (0, a_1, a_2), \quad \text{and} \quad a = (a_1, a_2) \neq 0,\]

\[(1.28) \quad \dim \mathcal{L}_0 = 3 \quad \text{and} \quad \dim \hat{\mathcal{L}}_0 = 5.\]

Notice also that $\mathcal{L}$ is the derived algebra of $\hat{\mathcal{L}}$. We extend the form on $sl_2(\mathbb{C}_q)$ to a symmetric form on all of $\hat{\mathcal{L}}$ by setting

\[(1.29) \quad (c_s, c_t) = (d_s, d_t) = 0 = (c_s, d_t) = (c_t, d_s) \quad \text{and} \quad (c_s, d_s) = (c_t, d_t) = 1,\]

and then declaring that the subspace spanned by $c_s, c_t, d_s, d_t$ is perpendicular to $sl_2(\mathbb{C}_q)$. That this form is a non-degenerate symmetric bilinear form on $\hat{\mathcal{L}}$ which is invariant is easy to check. In fact, $\hat{\mathcal{L}}$ is an extended affine Lie algebra of type $A_1$, in the sense of [AABGP], whose core is just $\mathcal{L}$.

**Remark 1.30.** In the above we have used the Lie algebra $sl_2(\mathbb{C}_q)$ to build the algebras $\mathcal{L}$ and $\hat{\mathcal{L}}$ but we could have very well used $gl_2(\mathbb{C}_q)$ in it’s place. At several points in what follows it will be convenient to work in these bigger algebras obtained by ”adding the identity as a central element”.

Our next goal is to introduce a Heisenberg subalgebra of $\hat{\mathcal{L}}$ which will play a major role in the rest of the paper. It is here that the non-commutative nature of the coordinates $\mathbb{C}_q$ first makes itself felt as we need to use some elements from the space $[\mathbb{C}_q, \mathbb{C}_q] = \oplus_{a \in \mathbb{Z}^2 \setminus \{0\}} \mathbb{C}m(a)$, which is non-trivial in our case. The basis elements which we need are given in the following definition.

**Definition 1.31.** Define elements $E_{2k+1}, A_m$ for $k \in \mathbb{Z}, m \in \mathbb{Z} \setminus \{0\}$ by saying

\[E_{2k+1} = s^k E + s^{k+1} F \quad \text{and} \quad A_m = s^m I\]

Furthermore we let $\hat{\mathcal{H}}$ be the span of all of these elements together with the element $c_s$. 

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Lemma 1.32. The subspace $\hat{H}$ is an infinite dimensional Heisenberg Lie algebra which is maximal in the sense that it is not properly contained in any other Heisenberg subalgebra of $\hat{L}$.

Proof. It is easy to check that the following formulas hold.

(1.33) $[E_{2k+1}, E_{2j+1}] = (2k + 1)\delta_{k+j+1,0}c_s$,

(1.34) $[A_m, A_n] = 2m\delta_{m+n,0}c_s$,

(1.35) $[E_{2k+1}, A_m] = 0$, for all $k, j \in \mathbb{Z}$ and $m, n \in \mathbb{Z} \setminus \{0\}$.

From these it is clear that $\hat{H}$ is a Heisenberg subalgebra of $\hat{L}$. To see this is maximal, first note that in the Lie algebra $gl_2(\mathbb{C}_q)$ the maximal commuting Lie subalgebra containing the element $E_1$ above consists of elements of the form $f_1(s)I + f_2(s)E_1$ where both $f_1(s)$ and $f_2(s)$ are just Laurent polynomials of the form $a_js^j + \cdots + a_k s^k$ for $j \leq k$ integers. It now easily follows from this that $\hat{H}$ is maximal. □

Our next goal is to show that $L$ is the universal central extension (uce for short) of the Lie algebra $sl_2(\mathbb{C}_q)$. From knowing the situation described in [BGK] one would certainly guess that this is the case. Indeed, in that work the uce of $sl_n(\mathbb{C}_q)$ is described for $n \geq 3$ and any quantum torus in any number of variables using the results from [KL]. See also [BGKN] for related material. In these cases it turns out that the Connes first cyclic homology group, $HC_1(\mathbb{C}_q)$ is two dimensional when the quantum torus, $\mathbb{C}_q$, is simple and generated by two elements and their inverses, as in the case we are considering in this paper. This homology group, $HC_1(\mathbb{C}_q)$, is defined using the associative algebra structure of $\mathbb{C}_q$ while in the case we are interested in here, namely with $sl_2(\mathbb{C}_q)$, it is the Jordan algebra structure, under the Jordan product $\circ$ of (1.10), which comes into play. Roughly speaking, $sl_2(\mathbb{C}_q)$ is too small so that it only “sees” the Jordan algebra structure of the coordinates and not the associative algebra structure. The reader should see [AG],[ABC] and [G] for related material. Note also that the paper [Y] leads one to expect that $HC_1(\mathbb{C}_q)$ (so considering $\mathbb{C}_q$ as an associative algebra) will not give this uce when $q$ is chosen to be an even root of unity.

There are various methods we could use here and we have chosen the one that closely resembles that used in [BGK] and goes back to Tits [T]. Thus we closely follow the presentation in [Se], pages 61-65 and we only indicate the argument until we come to the end where we must compute using the Jordan product on $\mathbb{C}_q$.

To begin, we let $\mathfrak{g}$ be the uce of $sl_2(\mathbb{C}_q)$ which exists as this latter algebra is perfect. Then we have a surjective homomorphism

(1.36) $\phi : \mathfrak{g} \to sl_2(\mathbb{C}_q)$. 
Pulling back the subalgebra of $sl_2(\mathbb{C}_q)$ generated by our three elements $E, F, H$ we can get (after normalizing by a central element) an $sl_2$-triple of elements in $\mathfrak{g}$. Then as in [Se] one decomposes $\mathfrak{g}$ into a direct sum of $sl_2(\mathbb{C})$ modules each of which is either isomorphic to the trivial one dimensional module or to the adjoint module. We thus obtain,

\[(1.37) \quad \mathfrak{g} = \mathcal{D} \oplus (sl_2(\mathbb{C}) \otimes \mathbb{C}_q),\]

where there is a product $\{\cdot, \cdot\} : \mathbb{C}_q \times \mathbb{C}_q \to \mathcal{D}$ as well as an action of $\mathcal{D}$ on $\mathbb{C}_q$ by derivations when $\mathbb{C}_q$ is considered as a Jordan algebra under $\circ$. One has $\mathcal{D} = \{\mathbb{C}_q, \mathbb{C}_q\}$ and with proper normalizations the following formulas hold,

\[(1.38) \quad [x \otimes a, y \otimes b] = B(x,y)\{a,b\} + [x,y] \otimes a \circ b,\]

\[(1.39) \quad \{\{a,b\}, x \otimes c\} = x \otimes (\{a,b\}(c)),\]

\[(1.40) \quad \{\{a,b\}, \{c,d\}\} = \{\{a,b\}(c), d\} + \{c, \{a,b\}(d)\},\]

where $x, y \in sl_2(\mathbb{C}), a, b, c, d \in \mathbb{C}_q$ and where $B(\cdot, \cdot)$ is the Killing form on $sl_2(\mathbb{C})$. In the above we have

\[(1.41) \quad \{a, b\}(c) = \frac{1}{2}(a \circ (b \circ c) - b \circ (a \circ c)) = \frac{1}{8}[[a,b],c],\]

\[(1.42) \quad \{a, b\} = -\{b, a\}, \quad \text{and}\]

\[(1.43) \quad \{a \circ b, c\} + \{b \circ c, a\} + \{c \circ a, b\} = 0.\]

Thus, the homomorphism $\phi$ maps $x \otimes a$ to $ax \in sl_2(\mathbb{C}_q)$ for $x \in \{E, F, H\}$. We also have that $\phi(\{a, b\}) = \frac{1}{8}[a, b]I \in \mathbb{C}_q$.

The next step is to define a universal object satisfying the conditions satisfied by $\mathcal{D}$ above, so conditions 1.42 and 1.43. Thus, we let

\[(1.44) \quad \langle \mathbb{C}_q, \mathbb{C}_q \rangle = \frac{\mathbb{C}_q \otimes \mathbb{C}_q}{T},\]

where $T$ is the subspace of $\mathbb{C}_q \otimes \mathbb{C}_q$ spanned by the following elements for $a, b, c \in \mathbb{C}_q$

\[(1.45) \quad a \otimes b + b \otimes a, a \circ b \otimes c + b \circ c \otimes a + c \circ a + b \text{ for } a, b, c \in \mathbb{C}_q.\]
Notice that here we have to use the Jordan product and this is what distinguishes this case from that considered in [BGK]. Using the object \( \langle C_q, C_q \rangle \) we form the space \( \hat{g} \) defined by

\[
(1.46) \quad \hat{g} = \langle C_q, C_q \rangle \oplus (sl_2(\mathbb{C}) \otimes C_q)
\]

and make this into an algebra by using formulas 1.38, 1.39, 1.40, and 1.41 but with \( \langle \cdot, \cdot \rangle \) replacing \( \{ \cdot, \cdot \} \). It is straightforward to verify this is in fact a well-defined Lie algebra which is a perfect central extension of \( sl_2(\mathbb{C}_q) \). Thus we get a Lie algebra homomorphism \( \mu \) from \( g \) onto \( \hat{g} \). But using the universal property inherent in our definition of \( \langle C_q, C_q \rangle \) together with the definition of multiplication in \( \hat{g} \) we get a Lie algebra homomorphism \( \nu \) from \( \hat{g} \) onto \( g \). As the compositions \( \nu \circ \mu \) and \( \mu \circ \nu \) are both identity maps we get that \( g \) and \( \hat{g} \) are isomorphic. Thus, \( \hat{g} \) is the uce of \( sl_2(\mathbb{C}_q) \).

From what we have so far we know, as \( L \) is a perfect central extension of \( sl_2(\mathbb{C}_q) \), that there is a surjective Lie algebra homomorphism \( \psi : \hat{g} \to L \). Noting that in \( L \) we have

\[
(1.47) \quad [m(a)E, m(b)F] = (m(a) \circ m(b))H + \frac{1}{2}[m(a), m(b)]I + \delta_{a+b,0}q^{-a_1a_2}(a_1c_s + a_2c_t), \quad \text{and}
\]

\[
(1.48) \quad [m(a)H, m(b)H] = [m(a), m(b)]I + 2\delta_{a+b,0}q^{-a_1a_2}(a_1c_s + a_2c_t) \quad \text{for} \quad a, b \in \mathbb{Z}^2,
\]

we have that the homomorphism \( \psi \) can be taken to satisfy the following formulas,

\[
(1.49) \quad \psi(X \otimes m(a)) = m(a)X, \quad \text{for} \quad X \in \{E, F, H\},
\]

and

\[
(1.50) \quad \psi(\langle m(a), m(b) \rangle) = \frac{1}{8}[m(a), m(b)]I + \frac{1}{4}\delta_{a+b,0}q^{-a_1a_2}(a_1c_s + a_2c_t).
\]

Also, it is clear that \( \langle C_q, C_q \rangle \) is graded by \( \mathbb{Z}^2 \) where the degree of \( \langle m(a), m(b) \rangle \) is just \( a + b \). Thus, we have that \( \psi \) induces a graded homomorphism between the graded vector spaces \( \langle C_q, C_q \rangle \) and \( [C_q, C_q] \oplus C_{c_s} \oplus C_{c_t} \). Here we have given \( c_s, c_t \) degree 0. Writing \( \langle C_q, C_q \rangle_a \) for the subspace of \( \langle C_q, C_q \rangle \) of degree \( a \) we find, because of 1.3, that

\[
(1.51) \quad \dim \langle C_q, C_q \rangle_a \geq 1 \quad \text{if} \quad a \neq 0
\]

\[
(1.52) \quad \dim \langle C_q, C_q \rangle_0 \geq 2.
\]

Thus, in order to show that \( L \) is the uce of \( sl_2(\mathbb{C}_q) \) we need only show the two inequalities (1.51) and (1.52) are in fact equalities. We do this in our next results but need to develop a little more notation first.
For \(a, b \in \mathbb{Z}^2\) we let

\[
\sigma(a, b) = q^{-n_2b_1}, \quad f(a, b) = \sigma(a, b)\sigma(b, a)^{-1}, \quad \text{and} \quad 2\rho(a, b) = \sigma(a, b)(1 + f(b, a)).
\]

Then we have the following formulas for \(a, b \in \mathbb{Z}^2\).

\[
m(a)m(b) = \sigma(a, b)m(a + b),
\]

\[
m(a)m(b) = f(a, b)m(b)m(a),
\]

\[
m(a) \circ m(b) = \rho(a, b)m(a + b).
\]

Moreover, we have \(\rho(a, b) = \rho(b, a)\), and as \(q\) is not an even root of unity, \(\rho(a, b) \neq 0\). Notice that we have for any \(x, y, z \in \mathbb{C}_q\) that

\[
\langle x \circ y, z \rangle + \langle y \circ z, x \rangle + \langle z \circ x, y \rangle = 0,
\]

and that \(\langle \cdot, \cdot \rangle\) is skew-symmetric. This gives us for any \(b, c, d \in \mathbb{Z}^2\), that

\[
\rho(b, c)\langle m(b + c), m(d) \rangle = \rho(c, d)\langle m(b), m(c + d) \rangle + \rho(b, d)\langle m(c), m(b + d) \rangle.
\]

Clearly, by (1.57) we have \(\langle 1, x \rangle = 0\) and \(\langle x, x \rangle = 0\) for any \(x \in \mathbb{C}_q\).

**Lemma 1.59.** For \(n \in \mathbb{Z}, n \neq -1\), we have that, \(\langle s^n, s \rangle = \langle t^n, t \rangle = 0\).

**Proof.** For \(a, b \in \mathbb{Z}\) we have that \(s^a\) and \(s^b\) commute so that \(s^a \circ s^b = s^{a+b}\). Thus from (1.57) we have that \(\langle s^a, s^b \rangle = \langle s, s^{a+b-1} \rangle + \langle s^{a-1}, s^{b+1} \rangle\). Assuming \(n \geq 1\) we obtain \(\langle s^{n+1}, s \rangle = \langle s^n \circ s, s \rangle = -\langle s^{n+1}, s \rangle + \langle s^{n-1}, s^2 \rangle\). Doing this again to the second term we get that \(\langle s^{n+1}, s \rangle = -2\langle s^{n+1}, s \rangle + \langle s^{n-1}, s^3 \rangle\). Proceeding in this way after \(n\) steps we get that \(\langle s^{n+1}, s \rangle = -n\langle s^{n+1}, s \rangle + \langle s^{n-(n-1)}, s^{n+1} \rangle\). But this equals \(-(n+1)\langle s^{n+1}, s \rangle\) and so we get that \(\langle s^{n+1}, s \rangle = 0\).

For the other case note that \(\langle 1, s \rangle = 0\), while we know that \(\langle s^{-1}, s \rangle \neq 0\) since it’s image under \(\psi\) is \(-\frac{1}{2}c_s\). For \(n \geq 2\) we have that \(\langle s^{-n}, s \rangle = \langle s^{-(n-1)} \circ s^{-1}, s \rangle = -\langle s^{-n+2}, s^{-1} \rangle\). If \(n = 2\) this is zero. Otherwise, this is \(-\langle s^{-n+3} \circ s^{-1}, s^{-1} \rangle = \langle s^{-n+2}, s^{-1} \rangle - \langle s^{-n+3}, s^{-2} \rangle\). Repeating this process \(n-2\) times yields that \(\langle s^{-n}, s \rangle = -\langle s^{-n+2}, s^{-1} \rangle = (n-2)\langle s^{-n+2}, s^{-1} \rangle\). It follows that \(\langle s^{-n}, s \rangle = 0\). The assertion with \(t\) replacing \(s\) is proved in the same way. \(\square\)

We now let \(\epsilon_1 = (1, 0)\) and \(\epsilon_2 = (0, 1)\) in \(\mathbb{Z}^2\).
Lemma 1.60. For all \( n \in \mathbb{Z}, e \in \mathbb{Z}^2 \) we have

\begin{align*}
(1) \quad & \langle s^n, m(e) \rangle \in \mathbb{C} \langle s, m(e + (n-1)\epsilon_1) \rangle, \\
(2) \quad & \langle t^n, m(e) \rangle \in \mathbb{C} \langle t, m(e + (n-1)\epsilon_2) \rangle.
\end{align*}

In particular, \( \dim \langle \mathbb{C}_q, \mathbb{C}_q \rangle_e \leq 2 \), for all \( e \in \mathbb{Z}^2 \).

Proof. We only need to prove (1) as (2) is similar. For \( n = 0 \) the element above is zero while when \( n = 1 \) the result is clear. For \( n \geq 2 \) we use induction on \( n \) and have that \( \langle s^n, m(e) \rangle = \langle s^{n-1} \circ s, m(e) \rangle = \langle s^{n-1}, s \circ m(e) \rangle + \langle s, m(e) \circ s^{n-1} \rangle \). Clearly we have that \( \langle s, m(e) \circ s^{n-1} \rangle \in \mathbb{C} \langle s, m(e + (n-1)\epsilon_1) \rangle \), while by induction we get that \( \langle s^{n-1}, s \circ m(e) \rangle \in \mathbb{C} \langle s, m(e + (n-1)\epsilon_1) \rangle \). In a similar way we have, for \( n \geq 1 \), that \( \langle s^{-n}, m(e) \rangle \in \mathbb{C} \langle s^{-1}, m(e - (n-1)\epsilon_1) \rangle \). Finally, note that for any \( d \in \mathbb{Z}^2 \) we have, by (1.58), that \( \rho(d - \epsilon_1, \epsilon_1)\langle m(d), s^{-1} \rangle = \rho(d - \epsilon_1, \epsilon_1)\langle m(d - \epsilon_1 + \epsilon_1), s^{-1} \rangle = \rho(\epsilon_1, -\epsilon_1)\langle m(d - \epsilon_1), m(0) \rangle + \rho(d - \epsilon_1, -\epsilon_1)\langle s, m(d - \epsilon_1) \rangle \in \mathbb{C} \langle s, m(d - \epsilon_1) \rangle \). This is what we want. The last statement of the Lemma follows from the first two together with the fact that if \( a = (a_1, a_2) \in \mathbb{Z}^2 \) then \( \rho(a_1\epsilon_1, a_2\epsilon_2)\langle m(a_1\epsilon_1 + a_2\epsilon_2), m(b) \rangle = \rho(b, a_2\epsilon_2)\langle m(a_1\epsilon_1), m(b + a_2\epsilon_2) \rangle + \rho(b, a_1\epsilon_1)\langle m(a_2\epsilon_2), m(b + a_1\epsilon_1) \rangle \). \( \square \)

We can now complete the proof of our result on universal central extensions.

Theorem 1.61. \( \mathcal{L} \) is the universal central extension of the Lie algebra \( \mathfrak{sl}_2(\mathbb{C}_q) \).

Proof. We know that we only need to show equality holds in (1.51) and (1.52). However we already know from the previous two Lemmas that \( \dim \langle \mathbb{C}_q, \mathbb{C}_q \rangle_0 = 2 \), and that if \( a = (a_1, a_2) \) with exactly one of \( a_1, a_2 \) equal to zero then \( \dim \langle \mathbb{C}_q, \mathbb{C}_q \rangle_a \) equals 1. Finally we suppose that \( a = (a_1, a_2) \) where both \( a_1, a_2 \) are non-zero. Then we have, by (1.58) that \( \langle m(a - \epsilon_2), t \rangle = \langle m(a_1\epsilon_1 + (a_2 - 1)\epsilon_2), t \rangle = \alpha \langle m(a_1\epsilon_1), m(a_2\epsilon_2) \rangle + \beta \langle m((a_2 - 1)\epsilon_2), m(a_1\epsilon_1 + \epsilon_2) \rangle \), for some non-zero scalars \( \alpha, \beta \). Since \( q \) is generic and \( a_1, a_2 \) are non-zero we have that \( m(a_1\epsilon_1), m(a_2\epsilon_2) \) \( \neq 0 \), so that \( \langle m(a_1\epsilon_1), m(a_2\epsilon_2) \rangle \neq 0 \). Thus we obtain that the element \( \alpha \langle m(a_1\epsilon_1), m(a_2\epsilon_2) \rangle = \langle m(a - \epsilon_2), t \rangle - \beta \langle m((a_2 - 1)\epsilon_2), m(a_1\epsilon_1 + \epsilon_2) \rangle \) is non-zero and so, by the previous Lemma in the space \( \mathbb{C} \langle t, m(\mathbf{a} - \epsilon_2) \rangle \cap \mathbb{C} \langle s, \mathbf{a} - \epsilon_1 \rangle \). It follows that \( \langle \mathbb{C}_q, \mathbb{C}_q, \mathbf{a} \rangle \) is one dimensional. \( \square \)

Section 2. Displaying the algebra via formal variables. In this section we will give what amounts to a, particularly nice, multiplication table for our Lie algebra \( \mathcal{L} \) using the approach via formal variables as in [FLM]. As the basic set up is very well known in the affine case we will be brief and assume the basics on delta functions and the identities they satisfy. Thus, as has become customary we will only state the ones we use and leave their verification to the reader who should consult [FLM].

To begin with we note that we will be working with spaces of the form \( V[[z, z^{-1}]] \), where \( V \) is a vector space over \( \mathbb{C} \) and where

\begin{equation}
V[[z, z^{-1}]] = \{ \sum_{n \in \mathbb{Z}} v_n z^{-n} | v_n \in V \}.
\end{equation}
Often for us in this section the space \( V \) will be \( \mathcal{L} \) or \( \hat{\mathcal{L}} \). However in a few places we use the versions coming from \( gl_2(C_q) \) as mentioned in Remark1.30. Later \( V \) will be a space of the form \( \text{End}(\mathcal{F}) \) for an appropriately chosen space \( \mathcal{F} \). Recall also that the delta function, \( \delta(z) \) is defined by

\[
\delta(z) = \Sigma_{n \in \mathbb{Z}} z^n.
\]

We will also have need to use \( z \frac{d}{dz} \delta(z) \) which we denote by \( \delta^{(1)}(z) \). Thus,

\[
\delta^{(1)}(z) = \Sigma_{n \in \mathbb{Z}} nz^n.
\]

We define the following elements,

\[
e(k, m) := s^k t^m E, \quad f(k, m) := s^k t^m F, \quad \text{and} \quad h(k, m) := s^k t^m H \quad \text{for} \quad k, m \in \mathbb{Z}.
\]

One should notice that these elements, together with the elements \( s^k t^m I, c_s, c_t \) for \( k, m \in \mathbb{Z} \), and not both zero, form a basis of \( \mathcal{L} \). The series that shall concern us most in this section are given in the following definition.

**Definition 2.5.** We let, for \( \epsilon = \pm 1 \) and \( m \in \mathbb{Z} \setminus \{0\} \),

\[
W^\epsilon_m(z) := \Sigma_{k \in \mathbb{Z}} ((e(k, m) + \epsilon q^m f(k+1, m)) z^{-(2k+1)} + (\frac{\epsilon q^m/2 - 1}{2}) h(k, m) + (\frac{\epsilon q^m/2 + 1}{2}) s^k t^m I) z^{-2k}).
\]

We also let

\[
W_0^{-1}(z) := \Sigma_{k \in \mathbb{Z}} ((e(k, 0) - f(k + 1, 0)) z^{-(2k+1)} - h(k, 0) z^{-2k}) + (1/2) c_s, \quad \text{and} \quad \alpha(z) := \Sigma_{k \in \mathbb{Z}} ((e(k, 0) + f(k + 1, 0)) z^{-(2k+1)} + \Sigma_{k \in \mathbb{Z} \setminus \{0\}} s^k I z^{-2k}).
\]

Here we have fixed, once and for all, a square root of \( q \) so the two square roots of \( q^m \) are \( \epsilon q^m/2, \epsilon = \pm 1 \). Notice that as \( q \) is generic, so not a root of unity, the entries \( \epsilon q^m/2, \epsilon q^m/2 - 1, \epsilon q^m/2 + 1 \), in the above series are non-zero. It easily follows from this that the linear span of the coefficients of \( z^n \) for \( n \in \mathbb{Z} \), together with \( c_s \) and \( c_t \), is all of \( \mathcal{L} \) and moreover that these coefficients are linearly independent. Sometimes these coefficients are called the moments of the series in question, so we have that the moments of \( W^\epsilon_m(z) \), \( W_0^{-1}(z) \), and \( \alpha(z) \), together with \( c_s, c_t \) form a basis of \( \mathcal{L} \). Another way to write these series, using the notation developed in Definition 1.31 together with delta functions, which will be very useful for our purposes, is as follows. The verification that these formulas hold is immediate.

\[
W^\epsilon_m(z) = \delta(z^2/s) \left( \frac{\epsilon q^m/2}{\epsilon q^m/2 z} \right) t^m, \quad \text{for} \quad \epsilon = \pm 1, m \in \mathbb{Z} \setminus \{0\}.
\]

\[
W_0^{-1}(z) = \delta(z^2/s) \left( \frac{1}{\epsilon q^m/2 z} \right) + (1/2) c_s.
\]

\[
\alpha(z) = \Sigma_{k \in \mathbb{Z} \setminus \{0\}} A_k z^{-2k} + \Sigma_{k \in \mathbb{Z}} E_{2k+1} z^{-(2k+1)}.
\]

Our first Lemma shows how the basis elements of our Heisenberg algebra \( \hat{H} \) interact with the elements \( W^\epsilon_m(z) \) for all possible values of \( \epsilon \) and \( m \).
Lemma 2.9.

1. \([E_{2k+1}, W_m^\epsilon(z)] = z^{2k+1}(1 - (\epsilon q^{m/2})^{2k+1})W_m^\epsilon(z),\) and
2. \([A_n, W_m^\epsilon(z)] = z^{2n}(1 - q^{mn})W_m^\epsilon(z),\) for \(m \in \mathbb{Z} \setminus \{0\}, \epsilon = \pm 1\) and also for \(\epsilon = -1, m = 0.\)

Proof. Writing \(W_m^\epsilon(z)\) as above we obtain that

\[
[E_{2k+1}, W_m^\epsilon(z)] = E_{2k+1}\delta(z^2/s) \left( \frac{\epsilon q^{m/2}}{\epsilon q^{m/2}z} z^{-1} \right) t^m - \delta(z^2/s) \left( \frac{\epsilon q^{m/2}}{\epsilon q^{m/2}z} z^{-1} \right) t^m E_{2k+1} +
\]

\[
tr((d_s E_{2k+1})\delta(z^2/s) \left( \frac{\epsilon q^{m/2}}{\epsilon q^{m/2}z} z^{-1} \right) t^m)c_s + tr((d_t E_{2k+1})\delta(z^2/s) \left( \frac{\epsilon q^{m/2}}{\epsilon q^{m/2}z} z^{-1} \right) t^m)c_t,
\]

a sum of four terms. Putting in that \(E_{2k+1} = \begin{pmatrix} 0 & s^k \\ s^{k+1} & 0 \end{pmatrix}\) (see 1.3.1) gives us the first term is

\[
\delta(z^2/s) \left( \frac{\epsilon q^{m/2}}{\epsilon q^{m/2}z} z^{-1} \right) t^m = \delta(z^2/s) \left( \frac{\epsilon q^{m/2}}{\epsilon q^{m/2}z} z^{-1} \right) t^m =
\]

\[
z^{2k+1}\delta(z^2/s) \left( \frac{\epsilon q^{m/2}}{\epsilon q^{m/2}z} z^{-1} \right) t^m.
\]

Here we have used the identity

\[(2.10) \quad f(s, z)\delta(z^2/s) = f(z^2, z)\delta(z^2/s),\]

which holds for Laurent polynomials \(f(s, z)\). As for the second term, we must bring \(E_{2k+1}\) across \(t^m\) so we must use that \(ts = qst\). Thus we have

\[
t^m E_{2k+1} = \begin{pmatrix} 0 & s^k q^{km} \\ s^{k+1} q^{(k+1)m} & 0 \end{pmatrix} t^m.
\]

Therefore we have that the second term is

\[
\delta(z^2/s) \left( \frac{\epsilon q^{m/2}}{\epsilon q^{m/2}z} z^{-1} \right) \left( \begin{array}{cc} 0 & s^k q^{km} \\ s^{k+1} q^{(k+1)m} & 0 \end{array} \right) t^m =
\]

\[
\delta(z^2/s) \left( \frac{\epsilon q^{m/2}}{\epsilon q^{m/2}z} z^{-1} \right) \left( \frac{\epsilon q^{m/2} q^{km}}{\epsilon q^{m/2} z^{2k+1} q^{km}} \right) t^m =
\]

\[
\epsilon(q^{m/2}z)^{2k+1}\delta(z^2/s) \left( \frac{\epsilon q^{m/2}}{\epsilon q^{m/2}z} z^{-1} \right) t^m;
\]

as \(\epsilon = \pm 1\) and \(2k + 1\) is odd.
Here we have used identity (2.10) again.

To deal with the third term we recall that \( \varepsilon(s^k t^m) = 0 \) whenever \( m \neq 0 \) so that this third term is

\[
\delta_{m,0} tr\left( \begin{pmatrix} 0 & k s^k \\ (k+1)s^{k+1} & 0 \end{pmatrix} \delta(z^2/s) \begin{pmatrix} 1 & z^{-1} \\ -z & 1 \end{pmatrix} \right) c_s = \\
\delta_{m,0} \varepsilon(\delta(z^2/s)(-kz^{2k+1} + (k+1)z^{2k+1})) c_s = \delta_{m,0} \varepsilon(\delta(z^2/s)z^{2k+1}) c_s = \delta_{m,0} z^{2k+1}c_s.
\]

Finally we note the fourth term is zero because we have \( d_t(E_{2k+1}) = 0 \).

Thus we now have that

\[
[E_{2k+1}, W^\varepsilon_m(z)] = (z^{2k+1} - (\epsilon q^{m/2})^{2k+1}) \delta(z^2/s) \left( \begin{pmatrix} \epsilon q^{m/2} & z^{-1} \\ \epsilon q^{m/2}z & 1 \end{pmatrix} \right) t^m + \delta_{m,0} z^{2k+1} c_s = \\
z^{2k+1}(1 - (\epsilon q^{m/2})^{2k+1}) W^\varepsilon_m(z),
\]

if \( m \neq 0, \epsilon = \pm 1 \). In the case when \( \epsilon = -1 \) and \( m = 0 \) we get

\[
2z^{2k+1}W_0^{-1}(z) = z^{2k+1}(1 - (\epsilon q^{m/2})^{2k+1}) W_0^{-1}(z).
\]

This proves (1).

To prove (2) we have

\[
[A_n, W^\varepsilon_m(z)] = [(s^n 0) \begin{pmatrix} 0 & s^n \\ s^n & 0 \end{pmatrix} \delta(z^2/s) \left( \begin{pmatrix} \epsilon q^{m/2} & z^{-1} \\ \epsilon q^{m/2}z & 1 \end{pmatrix} \right) t^m] = \\
(s^n I) \delta(z^2/s) \left( \begin{pmatrix} \epsilon q^{m/2} & z^{-1} \\ \epsilon q^{m/2}z & 1 \end{pmatrix} \right) t^m - \delta(z^2/s) \left( \begin{pmatrix} \epsilon q^{m/2} & z^{-1} \\ \epsilon q^{m/2}z & 1 \end{pmatrix} \right) t^m (s^n I) + \\
tr((d_s(s^n I)) \delta(z^2/s) \begin{pmatrix} \epsilon q^{m/2} & z^{-1} \\ \epsilon q^{m/2}z & 1 \end{pmatrix} t^m) c_s + \\
tr((d_t(s^n I)) \delta(z^2/s) \begin{pmatrix} \epsilon q^{m/2} & z^{-1} \\ \epsilon q^{m/2}z & 1 \end{pmatrix} t^m) c_t.
\]

Notice that the last term here is zero, as \( d_t(s^n) = 0 \). Thus the above becomes

\[
(z^{2n} - z^{2n} q^{mn}) \delta(z^2/s) \left( \begin{pmatrix} \epsilon q^{m/2} & z^{-1} \\ \epsilon q^{m/2}z & 1 \end{pmatrix} \right) t^m + \delta_{m,0} tr(n s^n \delta(z^2/s) \begin{pmatrix} 1 & z^{-1} \\ -z & 1 \end{pmatrix}) c_s = \\
z^{2n}(1 - q^{mn}) W^\varepsilon_m(z),
\]

if \( m \neq 0 \). This is also true if \( m = 0 \) and \( \epsilon = -1 \) as then the right hand side, as well as the left hand side is zero. This completes the proof of (2) and hence our Lemma. □
We next want to obtain formulas which express the commutators of the various series $W^n(z)$. Of course, for this we need to use different formal variables, so we will consider the product $[W^{e_1}_{m_1}(z_1), W^{e_2}_{m_2}(z_2)]$ where $z_1, z_2$ are formal variables. Here it is convenient to simplify the notation and define

$$Q^e_{m}(z) := \begin{pmatrix} \epsilon q^{m/2} & z^{-1} \\ \epsilon q^{m/2} z & 1 \end{pmatrix}. \tag{2.11}$$

Thus we have taking the product that

$$Q^e_{m_1}(z_1)Q^e_{m_2}(z_2) = \begin{pmatrix} \epsilon_1 \epsilon_2 q^{(m_1+m_2)/2} + \epsilon_2 q^{m_2/2}z_1^{-1} z_2 & \epsilon_1 q^{m_1/2} z_2^{-1} + z_1^{-1} \\ \epsilon_1 \epsilon_2 q^{(m_1+m_2)/2} z_1 + \epsilon_2 q^{m_2/2} z_2 & (\epsilon_1 q^{m_1/2} z_1/z_2) + 1 \end{pmatrix}. \tag{2.12}$$

**Lemma 2.13.** If $m_1 + m_2 \neq 0$ then we have

$$[W^{e_1}_{m_1}(z_1), W^{e_2}_{m_2}(z_2)] = W^{e_1 e_2}_{m_1 + m_2}(z_1)\delta(z_2/\epsilon_1 z_1 q^{m_1/2}) - W^{e_1 e_2}_{m_1 + m_2}(z_2)\delta(z_1/\epsilon_2 z_2 q^{m_2/2}).$$

**Proof.** We can express the commutator in question as a sum of four terms as follows.

$$[W^{e_1}_{m_1}(z_1), W^{e_2}_{m_2}(z_2)] = \delta(z_2^2/s)Q^e_{m_1}(z_1)t^{m_1} \delta(z_2^2/s)Q^e_{m_2}(z_2)t^{m_2} - \delta(z_2^2/s)Q^e_{m_2}(z_2)t^{m_2} \delta(z_2^2/s)Q^e_{m_1}(z_1)t^{m_1} + tr((d_s(W^{e_1}_{m_1}(z_1))W^{e_2}_{m_2}(z_2))c_t + tr((d_t(W^{e_1}_{m_1}(z_1))W^{e_2}_{m_2}(z_2))c_s.$$

Using the delta function identities

$$t^m \delta(z_2^2/s) = \delta(z_2^2/q^{m}s)t^m, \tag{2.14}$$

$$\delta(z_2^2/s)\delta(z_2^2/q^{m_1}s) = \delta(z_1^2/s)\delta(z_1^2/q^{m_2}z_1^2), \tag{2.15}$$

$$\delta(z_2^2/q^{m_1}z_1^2) = (1/2)(\delta(z_2/\epsilon q^{m_1/2}z_1) + \delta(z_2/-\epsilon q^{m_1/2}z_1)), \tag{2.16}$$

for $\epsilon = \pm 1$, on the first term above we get

$$(1/2)\delta(z_2^2/s)(\delta(z_2/\epsilon_1 q^{m_1/2}z_1)Q^e_{m_1}(z_1)Q^e_{m_2}(z_2) + \delta(z_2/-\epsilon_1 q^{m_1/2}z_1)Q^e_{m_1}(z_1)Q^e_{m_2}(z_2))t^{m_1+m_2}.$$

Using (2.12) and (2.10) we find that

$$\delta(z_2/\epsilon_1 q^{m_1/2}z_1)Q^e_{m_1}(z_1)Q^e_{m_2}(z_2) = 2\delta(z_2/\epsilon_1 q^{m_1/2}z_1)Q^e_{m_1+m_2}(z_2),$$

and
Thus, the first two terms in our expansion of $[W_{m_1}^\epsilon(z_1), W_{m_2}^\epsilon(z_2)]$ become
\[
\delta(z_2/\epsilon q^{m/2}z_1)Q_{m_1}^{\epsilon} Q_{m_2}^{\epsilon}(z_2) = 0.
\]

Finally we note that $tr((d_s(W_{m_1}^\epsilon(z_1))W_{m_2}^\epsilon(z_2))c_s = tr((d_t(W_{m_1}^\epsilon(z_1))W_{m_2}^\epsilon(z_2))c_t = 0$ because of the hypothesis $m_1 + m_2 \neq 0$. This is what we want. \(\square\)

In our next Lemma we treat the case of products of the form $[W_m^\epsilon(z_1), W_{-m}^{-\epsilon}(z_2)]$ for $m \neq 0$. The proof begins the same as in the last Lemma but here the analysis of the coefficients of $c_s$ and $c_t$ is a lot more delicate.

**Lemma 2.17.** For $m \neq 0$, $\epsilon = \pm 1$ we have
\[
[W_m^\epsilon(z_1), W_{-m}^{-\epsilon}(z_2)] = W_0^{-1}(z_1)\delta(z_2/\epsilon z_1 q^{m/2}) - W_0^{-1}(z_2)\delta(z_1/ - \epsilon z_2 q^{-m/2}).
\]

**Proof.** Exactly as in Lemma 2.13 we obtain that
\[
[W_m^\epsilon(z_1), W_{-m}^{-\epsilon}(z_2)] = \delta(z_2^2/s)\delta(z_2/\epsilon q^{m/2}z_1)Q_0^{-1}(z_1) - \delta(z_2^2/s)\delta(z_1/ - \epsilon q^{-m/2}z_2)Q_0^{-1}(z_2) +
\]
\[
tr(d_s(\delta(z_2^2/s)Q_m^\epsilon(z_1))t^m)\delta(z_2^2/s)Q_{-m}^{-\epsilon}(z_2)t^{-m})c_s +
\]
\[
tr(d_t(\delta(z_2^2/s)Q_m^\epsilon(z_1))t^m)\delta(z_2^2/s)Q_{-m}^{-\epsilon}(z_2)t^{-m})c_t.
\]

Using that $W_0^{-1}(z) - (1/2)c_s = \delta(z^2/s)Q_0^{-1}(z)$ we obtain that the first two terms in the expansion of $[W_m^\epsilon(z_1), W_{-m}^{-\epsilon}(z_2)]$ become
\[
\delta(z_2/\epsilon q^{m/2}z_1)W_0^{-1}(z_1) - \delta(z_1/ - \epsilon q^{-m/2}z_2)W_0^{-1}(z_2)
\]
\[
(2.18) - (1/2)\delta(z_2/\epsilon q^{m/2}z_1)c_s + (1/2)\delta(z_1/ - \epsilon q^{-m/2}z_2)c_s.
\]

We will see that the third term in our expansion of $[W_m^\epsilon(z_1), W_{-m}^{-\epsilon}(z_2)]$ will cancel with $-(1/2)\delta(z_2/\epsilon q^{m/2}z_1)c_s + (1/2)\delta(z_1/ - \epsilon q^{-m/2}z_2)c_s$ in (2.18). Towards this end we have that this third term is
\[
tr(d_s(\delta(z_2^2/s)Q_m^\epsilon(z_1))t^m)\delta(z_2^2/s)Q_{-m}^{-\epsilon}(z_2)t^{-m})c_s =
\]
\[
tr(Q_m^\epsilon(z_1)Q_{-m}^{-\epsilon}(z_2))\epsilon((d_s(\delta(z_2^2/s))\delta(z_2^2/q^m s))c_s =
\]
\[
tr(Q_m^\epsilon(z_1)Q_{-m}^{-\epsilon}(z_2))\delta^{(1)}(z_2^2/q^m z_1^2)c_s.
\]

Here we have used the following delta function identity.
\[
(2.19) \epsilon((d_s(\delta(z_2^2/s))\delta(z_2^2/q^m s)) = \delta^{(1)}(z_2^2/q^m z_1^2).
\]
For simplicity we let

\[ \mu(z_1, z_2) = \text{tr}(Q_m'(z_1)Q_m^{-\epsilon}(z_2)). \]

Also, we will next use the following two delta function identities, where as usual \( f(z_1, z_2) \) denotes a Laurent polynomial and \( a \) is any non-zero scalar.

\[ \delta^{(1)}(z^2) = (1/4)(\delta^{(1)}(\epsilon z) + \delta^{(1)}(-\epsilon z)), \text{ for } \epsilon = \pm 1. \]

\[ f(z_1, z_2)\delta^{(1)}(az_1/z_2) = f(z_1, a z_1)\delta^{(1)}(az_1/z_2) + (D_{z_2} f(z_1, z_2))\delta(az_1/z_2) = \]

\[ f(a^{-1}z_2, z_2)\delta^{(1)}(az_1/z_2) - (D_{z_1} f(z_1, z_2))\delta(az_1/z_2). \]

Here we are using the notation that \( D_{z_1} \) and \( D_{z_2} \) denotes the degree derivation with respect to the variable \( z_1 \) and \( z_2 \) respectively. That is

\[ D_{z_i} = z_i \frac{d}{dz_i} \text{ for } i = 1, 2. \]

Our third term now becomes

\[
(1/4)\mu(z_1, z_2)(\delta^{(1)}(z_2/\epsilon q^{m/2}z_1) + \delta^{(1)}(z_2/ -\epsilon q^{m/2}z_1))c_s + \\
((1/4)\mu(\epsilon q^{-m/2}z_2, z_2)\delta^{(1)}(z_2/\epsilon q^{m/2}z_1) + (1/4)(D_{z_2} \mu(z_1, z_2))\delta(z_2/\epsilon q^{m/2}z_1))c_s + \\
((1/4)\mu(-\epsilon q^{-m/2}z_2, z_2)\delta^{(1)}(-z_2/\epsilon q^{m/2}z_1) + (1/4)(D_{z_1} \mu(z_1, z_2))\delta(z_2/ -\epsilon q^{m/2}z_1))c_s
\]

Using (2.12) we find that

\[ \mu(z_1, z_2) = \epsilon(q^{m/2}z_1/z_2 - q^{-m/2}z_2/z_1). \]

Thus, doing the necessary evaluations we get that,

\[
\begin{align*}
\mu(q^{-m/2}z_2, z_2) &= \epsilon(\epsilon - \epsilon) = 0, \\
\mu(-\epsilon q^{-m/2}z_2, z_2) &= \epsilon(-\epsilon + (-\epsilon)) = 0, \\
D_{z_1}(\mu(z_1, z_2)) &= \epsilon(q^{m/2}z_1/z_2 + q^{-m/2}z_2/z_1), \\
D_{z_1}(\mu(z_1, z_2)) |_{z_2=q^{m/2}z_1} &= \epsilon(q^{m/2}q^{-m/2} + q^{-m/2}q^{m/2}) = 2\epsilon^2 = 2, \\
D_{z_1}(\mu(z_1, z_2)) |_{z_2=-q^{m/2}z_1} &= -2.
\end{align*}
\]
This now gives us that the third term in our expansion of \([W^\epsilon_m(z_1), W_-^\epsilon_m(z_2)]\) is

\[
(1/2)\delta(z_2/\epsilon q^{m/2} z_1)c_s - (1/2)\delta(z_1/ - \epsilon q^{-m/2} z_2)c_s
\]
as we claimed above.

Finally we need to deal with the coefficient of \(c_t\) which comes from the fourth term above. This is

\[
tr(d_t(\delta(z_1^2/s) Q^\epsilon_m(z_1) t^m)\delta(z_2^2/s) Q_-^\epsilon_m(z_2) t^{-m}) c_t =
\]
\[
m(\epsilon(\delta(z_1^2/s) Q^-_m(z_2)) t^m) c_t =
\]
\[
m(\epsilon((q^{m/2} z_1/z_2) - (q^{-m/2} z_2/z_1))\delta(z_2/\epsilon q^{m/2} z_1)) c_t.
\]

Here we have used (2.24). This now becomes

\[
m(\epsilon((q^{m/2} z_1/z_2) - (q^{-m/2} z_2/z_1))\delta(z_2/\epsilon q^{m/2} z_1)) c_t =
\]
\[
m(\epsilon((q^{m/2} z_1/z_2) - (q^{-m/2} z_2/z_1))(1/2)\delta(z_2/\epsilon q^{m/2} z_1) + (z_2/ - \epsilon q^{-m/2} z_1)) c_t.
\]

Here we have used formula (2.16) as well as the following delta function identity.

\[
(2.25) \quad \epsilon(\delta(z_1^2/s) Q^-_m(z_2)) = \delta(z_2^2/\epsilon q^{m/2} z_1 z_1).
\]

Finally we see that this is

\[
(1/2)m\epsilon(q^{m/2}(\epsilon q^{-m/2}) - q^{-m/2}(\epsilon q^{m/2})) c_t,
\]

which is easily seen to be zero. Thus we now have that

\[
[W^\epsilon_m(z_1), W_-^\epsilon_m(z_2)] = W^{-1}_0(z_1)\delta(z_2/\epsilon z_1 q^{m/2}) - W^{-1}_0(z_2)\delta(z_1/ - \epsilon z_2 q^{-m/2})
\]

which is what we want. \(\square\)

The next Lemma deals with products of the form \([W^\epsilon_m(z_1), W^\epsilon_m(z_2)]\). Here we will allow the case of \(m = 0\) and \(\epsilon = -1\). As many of the arguments are similar to the previous two Lemmas we will be brief.

**Lemma 2.26.** For \(m \in \mathbb{Z} \setminus \{0\}\) and \(\epsilon = \pm 1\) we have that

\[
[W^\epsilon_m(z_1), W^\epsilon_m(z_2)] = (\alpha(z_1) - \alpha(z_2))\delta(z_2/\epsilon q^{m/2} z_1) + (1)((z_2/\epsilon q^{m/2} z_1)c_s + 2m\delta(z_2/\epsilon q^{m/2} z_1)) c_t.
\]

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Proof. Expanding as in Lemmas 2.13 and 2.17 we get that

\[
[W_m^\epsilon(z_1), W_m^{-\epsilon}(z_2)] = \\
\delta(z_2/\epsilon q^{m/2} z_1)\delta(z_2^2/s)Q_0^1(z_1) - \delta(z_2/\epsilon q^{-m/2} z_2)\delta(z_2^2/s)Q_0^1(z_2) + \\
(tr(Q_m^\epsilon(z_1)Q_m^{-\epsilon}(z_2)))\delta^{(1)}(z_2^2/q^{m} z_1^2)c_s + \\
m_\delta(z_2^2/s)\delta(z_2^2/q^{m}s)(tr(Q_m^\epsilon(z_1)Q_m^{-\epsilon}(z_2))^c_t.
\]

Now

\[
Q_0^1(z) = \begin{pmatrix} 1 & z^{-1} \\ z & 1 \end{pmatrix} = I + z^{-1} \begin{pmatrix} 0 & 1 \\ z^2 & 0 \end{pmatrix},
\]

so that recalling \( E_1 = \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix} \) we obtain that

\[
\delta(z_2^2/s)Q_0^1(z_1) = \delta(z_1^2/s)(I + z_1^{-1} \begin{pmatrix} 0 & 1 \\ z_1^2 & 0 \end{pmatrix}) = \\
\delta(z_2^2/s)I + z_1^{-1}\delta(z_1^2/s)E_1 = \\
I + \sum_{k \in \mathbb{Z} \setminus \{0\}} A_k z^{-2k} + \sum_{k \in \mathbb{Z}} E_{2k+1} z^{-(2k+1)} = \alpha(z_1) + I.
\]

Note that here we are in the \( gl_2(\mathbb{C}_q) \) version of our spaces as in Remark 1.30. Obviously we have that

\[
\delta(z_2/\epsilon q^{m/2} z_1) = \delta(z_1/\epsilon q^{-m/2} z_2),
\]

so that in the first two terms in our expansion of \([W_m^\epsilon(z_1), W_m^{-\epsilon}(z_2)]\) the identity term above cancels and we get

\[
(\alpha(z_1) - \alpha(z_2))\delta(z_2/\epsilon q^{m/2} z_1).
\]

As for the third term it is

\[
(tr(Q_m^\epsilon(z_1)Q_m^{-\epsilon}(z_2)))\delta^{(1)}(z_2^2/q^{m} z_1^2)c_s = \\
(1/4)\nu(z_1, z_2)(\delta^{(1)}(z_2/\epsilon q^{m/2} z_1) + \delta^{(1)}(-z_2/\epsilon q^{m/2} z_1))c_s,
\]

where we have used (2.21) and have let

\[
(2.27) \quad \nu(z_1, z_2) = tr(Q_m^\epsilon(z_1)Q_m^{-\epsilon}(z_2)).
\]

Next we use (2.22) as well as the following delta function identity

\[
(2.28) \quad \delta^{(1)}(z) = -\delta^{(1)}(1/z),
\]
to get the third term which is

\[
(-1/4)\nu(z_1, z_2)\delta^{(1)}(\epsilon q^{m/2}z_1/z_2) + (1/4)\nu(z_1, z_2)\delta^{(1)}(-\epsilon q^{m/2}z_1/z_2) =
\]
\[
(-1/4)(\nu(z_1, \epsilon q^{m/2}z_1)\delta^{(1)}(\epsilon q^{m/2}z_1/z_2) + (D_{z_2}\nu(z_1, z_2))\delta^{(1)}(\epsilon q^{m/2}z_1/z_2) -
\]
\[
(1/4)\nu(z_1, -\epsilon q^{m/2}z_1)\delta^{(1)}(-\epsilon q^{m/2}z_1/z_2) + (D_{z_2}\nu(z_1, z_2))\delta^{(1)}(-\epsilon q^{m/2}z_1/z_2).
\]

It is easy to see that

\[
\nu(z_1, z_2) = 1 + \epsilon q^{-m/2}z_1^{-1}z_2 + \epsilon q^{m/2}z_1z_2^{-1} + 1,
\]
\[
\nu(z_1, \epsilon q^{m/2}z_1) = 4,
\]
\[
\nu(z_1, -\epsilon q^{m/2}z_1) = 0,
\]
\[
D_{z_2}\nu(z_1, z_2) = \epsilon q^{-m/2}z_2z_1^{-1} - \epsilon q^{m/2}z_1z_2^{-1},
\]
\[
(D_{z_2}\nu(z_1, z_2))\delta(\epsilon q^{m/2}z_1/z_2) = 0,
\]
\[
(D_{z_2}\nu(z_1, z_2))\delta(-\epsilon q^{m/2}z_1/z_2) = 0.
\]

Thus, the third term is

\[
-\delta^{(1)}(\epsilon q^{m/2}z_1/z_2)c_s = \delta^{(1)}(z_2/\epsilon q^{m/2}z_1)c_s.
\]

Finally, using (2.25) and (2.16) we get that the fourth term is

\[
m\nu(z_1, z_2)\delta(z_2^2/q^{m/2}z_1^2)c_t =
\]
\[
m\nu(z_1, z_2)(1/2)(\delta(z_2/\epsilon q^{m/2}z_1) + \delta(-z_2/\epsilon q^{m/2}z_1))c_t =
\]
\[
(m/2)(\nu(z_1, \epsilon q^{m/2}z_1)\delta(z_2/\epsilon q^{m/2}z_1) + \nu(z_1, -\epsilon q^{m/2}z_1)\delta(-z_2/\epsilon q^{m/2}z_1)c_t =
\]
\[
2m\delta(z_2/\epsilon q^{m/2}z_1)c_t,
\]

which is what we want. □

For the convenience of the reader we now summarize our previous work so that we have an easy reference in one place for these results.
Theorem 2.29. The following formulas hold.

1. \[ E_{2k+1}, E_{2j+1} = (2k + 1)\delta_{k+j+1,0}c_s, \]
2. \[ [A_m, A_n] = 2m\delta_{m+n,0}c_s, \]
3. \[ [E_{2k+1}, A_m] = 0, \text{ for all } k, j \in \mathbb{Z} \text{ and } m, n \in \mathbb{Z} \setminus \{0\}, \]
4. \[ \alpha(z) = \sum_{k \in \mathbb{Z} \setminus \{0\}} A_k z^{-2k} + \sum_{k \in \mathbb{Z}} E_{2k+1} z^{-(2k+1)}, \]
5. \[ [E_{2k+1}, W_m^\epsilon(z)] = z^{2k+1} (1 - (eq^{m/2})^{2k+1}) W_m^\epsilon(z), \text{ and} \]
6. \[ [A_n, W_m^\epsilon(z)] = z^{2n} (1 - q^{m\epsilon}) W_m^\epsilon(z), \]
for \( k \in \mathbb{Z}, m, n \in \mathbb{Z} \setminus \{0\}, \epsilon = \pm 1, \) and also for \( \epsilon = -1, m = 0, \)
7. \[ [W_{m_1}^{\epsilon_1}(z_1), W_{m_2}^{\epsilon_2}(z_2)] = W_{m_1+m_2}^{\epsilon_1\epsilon_2}(z) \delta(z_2/\epsilon_1 z_1 q^{m_1/2}) - W_{m_1+m_2}^{\epsilon_1\epsilon_2}(z_2) \delta(z_1/\epsilon_2 z_2 q^{m_2/2}) \]
for \( m_1 + m_2 \neq 0, \)
8. \[ [W_0^\epsilon(z_1), W_m^{-\epsilon}(z_2)] = W_0^{-1}(z_1) \delta(z_2/\epsilon_1 z_1 q^{m/2}) - W_0^{-1}(z_2) \delta(z_1/ - \epsilon_z q^{-m/2}), \]
for \( m \neq 0, \epsilon = \pm 1, \)
9. \[ [W_0^\epsilon(z_1), W_m^\epsilon(z_2)] = (\alpha(z_1) - \alpha(z_2)) \delta(z_2/\epsilon q^{m/2} z_1) + \delta^{(1)}(z_2/\epsilon q^{m/2} z_1) c_s + 2m \delta(z_2/\epsilon q^{m/2} z_1) c_t, \]
for \( m \in \mathbb{Z} \setminus \{0\} \) and \( \epsilon = \pm 1. \)

Remark 2.30. Since the moments of both generating functions \( W_m^\epsilon(z) \) and \( \alpha(z) \), together with the central elements \( c_s, c_t \) form a basis of \( \mathcal{L} \), we have that in order to define a representation of \( \mathcal{L} \) it is enough to find a countably infinite dimensional Heisenberg algebra and a module for it, together with formal series in \( z \) having coefficients in the endomorphisms of this module, which satisfy (1) thru (9) of Theorem 2.29 as well as having that \( c_s, c_t \) are central. This will be the method we use in the next section.

Section 3. The Principal Representation for our Algebra. In this section we will construct a representation for the algebra \( \mathcal{L} \). The method we use will be to first construct a module, \( \mathcal{F} \), for a countably infinite dimensional Heisenberg Lie algebra isomorphic to \( \hat{H} \). This module \( \mathcal{F} \) will be the tensor product of a standard irreducible Heisenberg module with central charge 1 and Laurent polynomials, \( \mathbb{C}[v, v^{-1}] \), where the variable \( v \) is introduced to keep track of the grading relative to the variable \( t \) in our algebra \( \mathcal{L} \). The central element \( c_s \) will act as the identity on \( \mathcal{F} \) while the central element \( c_t \) will act as zero. We will then construct elements of the space \( \text{End}(\mathcal{F})[[z, z^{-1}]] \) which we show satisfy the formulas of Theorem 2.29.

To begin we let \( \mathcal{S} \) be the polynomial algebra in the variables \( x_1, x_2, x_3, \ldots \), over \( \mathbb{C} \) so that

\[
\mathcal{S} = \mathbb{C}[x_1, x_2, x_3, \ldots].
\]
As is usual we let \( x_k \) denote multiplication on \( S \) by the variable \( x_k \) while we let \( \frac{\partial}{\partial x_k} \) denote the partial derivative on \( S \) corresponding to the variable \( x_k \). Denoting the identity operator on \( S \) by \( c_S \) (this will eventually be the image of the element \( c_S \) of \( L \)) we have

\[
[\frac{\partial}{\partial x_n}, m x_m] = \delta_{n,m} mc_S.
\]

It follows from 1.33, 1.34, and 1.35, that the assignment

- \( E_{2k+1} \mapsto -(2k+1)x_{-(2k+1)} \), for \( k < 0 \),
- \( E_{2k+1} \mapsto \frac{\partial}{\partial x_{2k+1}} \), if \( k \geq 0 \),

defines a faithful representation of our Heisenberg algebra \( \hat{H} \) on \( S \). Next, we let

\[
\begin{align*}
A_n &\mapsto -2nx_n - (2^n + 1), \\
A_n &\mapsto \frac{\partial}{\partial x_{2n}} \text{ for } n \geq 1,
\end{align*}
\]

\( c_S \mapsto c_S = \text{identity on } S \),

defines a faithful representation of our Heisenberg algebra \( \hat{H} \) on \( S \). Next, we let

\[
F = \mathbb{C}[v, v^{-1}] \otimes S,
\]

where \( v \) is a variable and the tensor product is over \( \mathbb{C} \). As is usual we still let \( x_n \) denote multiplication on \( F \) by the variable \( x_n \) so that actually we now have \( x_n = Id_{\mathbb{C}[v, v^{-1}] \otimes x_n} \). Similarly we will use the notation \( \frac{\partial}{\partial x_n} = Id_{\mathbb{C}[v, v^{-1}] \otimes \frac{\partial}{\partial x_n}} \) as well as \( c_S = Id_F \).

We are now ready to define the analogues of our operators \( \alpha(z) \), and \( W^\epsilon_m(z) \) which here we will denote by \( a(z) \) and \( X^\epsilon_m(z) \), respectively.

**Definition 3.5.** Let \( \epsilon = \pm 1 \) and \( m \in \mathbb{Z} \setminus \{0\} \) or let \( \epsilon = -1 \) and \( m = 0 \). We define

\[
\begin{align*}
(1) \quad b(m, \epsilon) &:= 1/(\epsilon q^{-m/2} - 1), \\
(2) \quad a(z) &:= \Sigma_{m \geq 1} (\frac{\partial}{\partial x_m} z^{-m} + mx_m z^m), \text{ and} \\
(3) \quad X^\epsilon_m(z) &:= v^m b(m, \epsilon) \exp(\Sigma_{k \geq 1} z^k (1 - (\epsilon q^{m/2})^k) x_k) \exp(-\Sigma_{k \geq 1} \frac{z^k (1 - (\epsilon q^{m/2})^{-k})}{k} \frac{\partial}{\partial x_k}).
\end{align*}
\]

Several remarks are in order. First, we note that the term \( v^m \) appearing in (3) above is to be understood as multiplication by \( v^m \otimes Id_S \) on \( F \). Also, that these terms give well-defined elements of the space \( \text{End}(F)[[z, z^{-1}]] \) is by now standard and follows as in [FLM]. We will see later why we need to pick the scalars \( b(m, \epsilon) \) as we do in (1) above. For the time being we begin to establish the various formulas satisfied by these elements.
Lemma 3.6. With $m$ and $\epsilon$ as in Definition 3.5 and any $n \geq 1$ we have

(1) \[ \frac{\partial}{\partial x_n}, X_m^\epsilon(z) = z^n(1 - (\epsilon q^{m/2})^n)X_m^\epsilon(z), \text{ and} \]

(2) \[ [nx_n, X_m^\epsilon(z)] = z^{-n}(1 - (\epsilon q^{m/2})^{-n})X_m^\epsilon(z). \]

Proof. Clearly we have that

\[ \left[ \frac{\partial}{\partial x_n}, \exp(z^n(1 - (\epsilon q^{m/2})^n)x_n) \right] = z^n(1 - (\epsilon q^{m/2})^n)\exp(z^n(1 - (\epsilon q^{m/2})^n)x_n). \]

It follows from this that

\[ \left[ \frac{\partial}{\partial x_n}, X_m^\epsilon(z) \right] = z^n(1 - (\epsilon q^{m/2})^n)X_m^\epsilon(z). \]

Similarly we note that

\[ [nx_n, \exp((-z^{-n}(1 - (\epsilon q^{m/2})^{-n}) \frac{\partial}{\partial x_n})) = z^{-n}(1 - (\epsilon q^{m/2})^{-n})\exp((-z^{-n}(1 - (\epsilon q^{m/2})^{-n}) \frac{\partial}{\partial x_n}), \]

and (2) follows easily from this. □

Remark 3.7. The representation in 3.3 clearly lifts to a representation, $\pi$ where

\[ \pi : \hat{H} \to \text{End}(\mathcal{F}) \subset \text{End}(\mathcal{F})[[z, z^{-1}]]. \]

Then we find that the previous result implies that (5) and (6) of Theorem 2.29 holds in the present setting. That is, we have

\[ \pi(E_{2k+1}), X_m^\epsilon(z) = z^{2k+1}(1 - (\epsilon q^{m/2})^{2k+1})X_m^\epsilon(z), \text{ and} \]

\[ \pi(A_n), X_m^\epsilon(z) = z^{2n}(1 - q^{mn})X_m^\epsilon(z), \]

for $k \in \mathbb{Z}$ and $m, n \in \mathbb{Z} \setminus \{0\}, \epsilon = \pm 1$, and also for $\epsilon = -1, m = 0$.

We now go towards verifying that (7),(8), and (9) of Theorem 2.29 hold for the operators $X_m^\epsilon(z)$. To begin with we introduce some notation. We let, as above, $z_1, z_2$ be variables and then let

\[ \tau(z_1, z_2, \omega_1, \omega_2) = \frac{(1 - (z_2/z_1))(1 - (\omega_2/\omega_1))}{(1 - (z_2/\omega_1))(1 - (\omega_2/z_1))}. \]

Here $\omega_1, \omega_2$ are taken to be rational functions of the variables $z_1, z_2$ such that the above expression makes sense.
Lemma 3.9. For \( m_i \in \mathbb{Z} \setminus \{0\}, \epsilon_i = \pm 1 \) and for \( m_i = 0, \epsilon_i = -1 \) where \( i = 1, 2 \) we have that

\[
\exp(-\sum_{k \geq 1} \frac{-1}{k} \left( \frac{1}{1 - \epsilon_1 q^{m_1/2}} \right)^{-k} \frac{\partial}{\partial x_k}) \exp(\sum_{k \geq 1} \frac{-1}{k} \left( \frac{1}{1 - \epsilon_2 q^{m_2/2}} \right)^{k} x_k) =
\]

\[
\tau(z_1, z_2, \omega_1, \omega_2) \exp(\sum_{k \geq 1} \frac{1}{k} \left( -\frac{1}{1 - \epsilon_1 q^{m_1/2}} \right)^{-k} \frac{\partial}{\partial x_k}) \exp(-\sum_{k \geq 1} \frac{-1}{k} \left( \frac{1}{1 - \epsilon_2 q^{m_2/2}} \right)^{k} x_k),
\]

where \( \omega_i = \epsilon_i q^{m_i/2} z_i \), for \( i = 1, 2 \).

Proof. We have that the left hand side of the above is

\[
\exp(-\sum_{k \geq 1} \frac{z_1^{-k} - \omega_1^{-k}}{k} \frac{\partial}{\partial x_k}) \exp(\sum_{k \geq 1} (z_2^k - \omega_2^k) x_k).
\]

Using the Campbell-Baker-Hausdorff formula together with the fact that in a Heisenberg Lie algebra the derived algebra equals the center of the algebra we have that this term is

\[
P \exp(\sum_{k \geq 1} (z_2^k - \omega_2^k) x_k) \exp(-\sum_{k \geq 1} \frac{z_1^{-k} - \omega_1^{-k}}{k} \frac{\partial}{\partial x_k}) \quad \text{where}
\]

\[
P = \exp(-\sum_{k \geq 1} \frac{\omega_2 - \omega_1}{k} (z_2^k - \omega_2^k)).
\]

Thus we have that since \( \ln(1 - z) = -\sum_{k \geq 1} \frac{z^k}{k} \) then

\[
P = \exp(-\sum_{k \geq 1} \frac{\omega_2 - \omega_1}{k} (z_2^k - \omega_2^k))
\]

\[
= \exp(\ln(1 - (z_2/z_1)))\exp(\ln(1 - (\omega_2/\omega_1))^{-1})\exp(\ln(1 - (z_2/\omega_1))^{-1})\exp(\ln(1 - (\omega_2/\omega_1)))
\]

\[
= \frac{(1 - (z_2/z_1))(1 - (\omega_2/\omega_1))}{(1 - (z_2/\omega_1))(1 - (\omega_2/z_1))} = \tau(z_1, z_2, \omega_1, \omega_2)
\]

which is what we want. \( \square \)

We now use this result to get our analogue of (7) and (8) of Theorem 2.29.

Lemma 3.10. Let \( \epsilon_1, \epsilon_2 \) and \( m_1, m_2 \) be as usual but assume that either \( m_1 + m_2 \neq 0 \) or that \( m_1 + m_2 = 0 \) but \( \epsilon_1 \epsilon_2 = -1 \). Then we have that

\[
[X_{m_1}^{\epsilon_1}(z_1), X_{m_2}^{\epsilon_2}(z_2)] =
\]

\[
X_{m_1+m_2}^{\epsilon_1\epsilon_2}(z_1) \delta(z_2/\epsilon_1 q^{m_1/2} z_1) - X_{m_1+m_2}^{\epsilon_1\epsilon_2}(z_2) \delta(z_1/\epsilon_2 q^{m_2/2} z_2).
\]

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Proof. As in Lemma 3.9 we let \( \omega_i = \epsilon_i q^{m_{i/2}} z_i \), for \( i = 1, 2 \). Clearly we have that

\[
[X^{\epsilon_1}_{m_1}(z_1), X^{\epsilon_2}_{m_2}(z_2)] = 
\nu^{m_1 + m_2} b(m_1, \epsilon_1) b(m_2, \epsilon_2) (\tau(z_1, z_2, \omega_1, \omega_2) - \tau(z_2, z_1, \omega_2, \omega_1)) E
\]

where \( E = \exp(\sum_{k \geq 1} (z_1^k - \omega_1^k + z_2^k - \omega_2^k) x_k) \exp(-\sum_{k \geq 1} z_k^{-k} - \omega_k^{-k} \frac{\partial}{\partial x_k}) \).

We will use the following easily established delta function identity,

\[
(3.11) \quad \delta(z) = \frac{1}{1-z} + \frac{1}{1-z^{-1}} - 1.
\]

We let \( x = z_2/z_1 \) and \( \alpha_i = \epsilon_i q^{m_{i/2}} \) for \( i = 1, 2 \). Then

\[
\tau(z_1, z_2, \omega_1, \omega_2) = \frac{(1-x)(1-(\alpha_2/\alpha_1)x)}{(1-\alpha_1^{-1}x)(1-\alpha_2 x)} \quad \text{and}
\]

\[
\tau(z_2, z_1, \omega_2, \omega_1) = \frac{(1-x^{-1})(1-(\alpha_1/\alpha_2)x^{-1})}{(1-\alpha_2^{-1}x^{-1})(1-\alpha_1 x^{-1})}.
\]

It follows easily, using partial fractions, that we have

\[
\tau(z_1, z_2, \omega_1, \omega_2) = 1 + C[\frac{1}{1-\alpha_1^{-1}x} - \frac{1}{1-\alpha_2 x}],
\]

\[
\tau(z_2, z_1, \omega_2, \omega_1) = 1 + C[\frac{1}{1-\alpha_2^{-1}x^{-1}} - \frac{1}{1-\alpha_1 x^{-1}}],
\]

where

\[
C = \frac{(1-\alpha_1)(1-\alpha_2)}{(1-\alpha_1 \alpha_2)}.
\]

Thus, using (3.11) we have that

\[
\tau(z_1, z_2, \omega_1, \omega_2) - \tau(z_2, z_1, \omega_2, \omega_1) = 
C[\frac{1}{1-\alpha_1^{-1}x} - \frac{1}{1-\alpha_2 x} - \frac{1}{1-\alpha_2^{-1}x^{-1}} + \frac{1}{1-\alpha_1 x^{-1}}] = 
C[\delta(\alpha_1^{-1}x) - \delta(\alpha_2^{-1}x^{-1})] = 
C[\delta(z_2/w_1) - \delta(z_1/w_2)].
\]
Again we use that $\omega_i = \epsilon_i q^{m_i/2} z_i$, for $i = 1, 2$ to obtain that

$$E\delta(\omega_2/z_1) = \exp(\Sigma_{k \geq 1} (z_2^k - (\epsilon_1 q^{m_1/2} z_1)^k) x_k) \exp(-\Sigma_{k \geq 1} \frac{z_2^{-k} - (\epsilon_1 q^{m_1/2} z_1)^{-k}}{k} \partial_x) \delta(\epsilon_2 q^{m_2/2} z_2/z_1) =$$

$$\exp(\Sigma_{k \geq 1} z_2^k (1 - (\epsilon_1 \epsilon_2 q^{m_1+m_2/2} z_1^k) x_k) \times \exp(-\Sigma_{k \geq 1} \frac{z_2^{-k} (1 - (\epsilon_1 \epsilon_2 q^{m_1+m_2/2} z_1^{-k})}{k} \partial_x) \delta(\epsilon_2 q^{m_2/2} z_2/z_1).$$

Similarly one also gets that

$$E\delta(\omega_1/z_2) = \exp(\Sigma_{k \geq 1} z_1^k (1 - (\epsilon_1 \epsilon_2 q^{m_1+m_2/2} z_1^k) x_k) \times \exp(-\Sigma_{k \geq 1} \frac{z_1^{-k} (1 - (\epsilon_1 \epsilon_2 q^{m_1+m_2/2} z_1^{-k})}{k} \partial_x) \delta(\epsilon_1 q^{m_1/2} z_1/z_2).$$

Finally we note that (recall Definition 3.5 (1))

$$C = \frac{(1 - \alpha_1)(1 - \alpha_2)}{(1 - \alpha_1 \alpha_2)} = \frac{b(m_1 + m_2, \epsilon_1 \epsilon_2)}{b(m_1, \epsilon_1) b(m_2, \epsilon_2)}.$$

Thus we obtain that

$$[X_{m_1}^{\epsilon_1}(z_1), X_{m_2}^{\epsilon_2}(z_2)] = X_{m_1+m_2}^{\epsilon_1 \epsilon_2}(z_1) \delta(z_2/\epsilon_1 q^{m_1/2} z_1) - X_{m_1+m_2}^{\epsilon_1 \epsilon_2}(z_2) \delta(z_1/\epsilon_2 q^{m_2/2} z_2).$$

This is what we want. □

**Remark 3.12.** It is in proving the previous Lemma that one is led to the correct definition of the constants $b(m, \epsilon)$ which appear in our operators $X_\mu(z)$. Indeed, knowing Theorem 2.29 (7) and (8), as well as the computation in the last Lemma, shows that we must take $b(m, \epsilon) := 1/(\epsilon q^{-m/2} - 1)$ as we did in Definition 3.5.

We deal with the one remaining case in our next Lemma.

**Lemma 3.13.** For $m \in \mathbb{Z} \setminus \{0\}, \epsilon = \pm 1$ we have that

$$[X_m^{\epsilon}(z_1), X_{-m}^{\epsilon}(z_2)] = (a(z_1) - a(z_2)) \delta(z_2/\epsilon q^{m/2} z_1) + \delta^{(1)}(z_2/\epsilon q^{m/2} z_1) c_s.$$
Proof. We let \( \omega_i = \epsilon q^{m/2} z_i \) for \( i = 1, 2 \) so we have as above that

\[
[X_m^e(z_1), X_{-m}^e(z_2)] = b(m, \epsilon)b(-m, \epsilon)(\tau(z_1, z_2, \omega_1, \omega_2) - \tau(z_2, z_1, \omega_2, \omega_1))E,
\]

where \( E \) is as in the previous Lemma and \( \tau \) is as in (3.8). Letting

\[
R(z_1, z_2) = \tau(z_1, z_2, \omega_1, \omega_2) - \tau(z_2, z_1, \omega_2, \omega_1),
\]

we have that

\[
R(z_1, z_2) = \frac{(1 - (z_2/z_1))(1 - (q^{-m}z_2/z_1))}{(1 - (z_2/\epsilon q^{m/2} z_1))^2} - \frac{(1 - (z_1/z_2))(1 - (q^m z_1/z_2))}{(1 - (\epsilon q^{m/2} z_1/z_2))^2}.
\]

Next, we use the delta function identity

\[
\frac{1}{(1 - z)^2} = z^{-1} \delta^{(1)}(z) + \frac{z^{-2}}{(1 - z^{-1})^2},
\]

with \( z = z_2/\epsilon q^{m/2} z_1 \) to get

\[
R(z_1, z_2) = (1 - (z_2/z_1))(1 - (q^{-m}z_2/z_1)) \frac{\epsilon q^{m/2} z_1}{z_2} \delta^{(1)}(z_2/\epsilon q^{m/2} z_1) + \frac{q^m z_1^2}{z_2^2} \frac{1}{(1 - (\epsilon q^{m/2} z_1/z_2))^2} - \frac{(1 - (z_1/z_2))(1 - (q^m z_1/z_2))}{(1 - (\epsilon q^{m/2} z_1/z_2))^2} = (1 - (z_2/z_1))(1 - (q^{-m}z_2/z_1)) \frac{\epsilon q^{m/2} z_1}{z_2} \delta^{(1)}(z_2/\epsilon q^{m/2} z_1) + \frac{((z_1/z_2) - 1)((q^m z_1/z_2) - 1)}{(1 - (\epsilon q^{m/2} z_1/z_2))^2} - \frac{(1 - (z_1/z_2))(1 - (q^m z_1/z_2))}{(1 - (\epsilon q^{m/2} z_1/z_2))^2} = (1 - (z_2/z_1))(1 - (q^{-m}z_2/z_1)) \frac{\epsilon q^{m/2} z_1}{z_2} \delta^{(1)}(z_2/\epsilon q^{m/2} z_1).
\]

We now let

\[
g_1(z_1, z_2) = b(m, \epsilon)b(-m, \epsilon)(1 - (z_2/z_1))(1 - (q^{-m}z_2/z_1)) \frac{\epsilon q^{m/2} z_1}{z_2},
\]

\[
g_2(z_1, z_2) = \Sigma_{k \geq 1} (z_1^k - \omega_1^k + z_2^k - \omega_2^k) x_k,
\]

\[
g_3(z_1, z_2) = -\Sigma_{k \geq 1} \frac{(z_1^{-k} - \omega_1^{-k} + z_2^{-k} - \omega_2^{-k})}{k} \partial x_k.
\]
Thus we have that
\[ [X^e_m(z_1), X^e_m(z_2)] = g_1(z_1, z_2) \exp(g_2(z_1, z_2)) \exp(g_3(z_1, z_2)) \delta^{(1)}(z_2/\epsilon q^{m/2} z_1). \]

From (2.22) and (2.28) we get the following identity for any non-zero scalar \( b \).
\[ f(z_1, z_2) \delta^{(1)}(z_2/bz_1) = f(z_1, bz_1) \delta^{(1)}(z_2/bz_1) - (D_{z_2} f(z_1, z_2)) \delta(z_2/bz_1) \]
Using this identity on the above (with \( b = \epsilon q^{m/2} \)) we get that
\[ [X^e_m(z_1), X^e_m(z_2)] = g_1(z_1, \epsilon q^{m/2} z_1) \exp(g_2(z_1, \epsilon q^{m/2} z_1)) \exp(g_3(z_1, \epsilon q^{m/2} z_1)) \delta^{(1)}(z_2/\epsilon q^{m/2} z_1) \]
\[ - (D_{z_2} g_1(z_1, z_2)) \exp(g_2(z_1, z_2)) \exp(g_3(z_1, z_2)) \delta(z_2/\epsilon q^{m/2} z_1) \]
\[ - g_1(z_1, z_2) (D_{z_2} g_2(z_1, z_2)) \exp(g_2(z_1, z_2)) \exp(g_3(z_1, z_2)) \delta(z_2/\epsilon q^{m/2} z_1) \]
\[ - g_1(z_1, z_2) \exp(g_2(z_1, z_2)) (D_{z_2} g_3(z_1, z_2)) \exp(g_3(z_1, z_2)) \delta(z_2/\epsilon q^{m/2} z_1). \]

It is easy to see that
\[ g_1(z_1, \epsilon q^{m/2} z_1) = 1, \]
\[ g_i(z_1, \epsilon q^{m/2} z_1) = 0 \text{ for } i = 2, 3, \]
\[ D_{z_2} g_1(z_1, z_2) \big|_{z_2 = \epsilon q^{m/2} z_1} = 0, \]
\[ (D_{z_2} g_2(z_1, z_2)) \delta(z_2/\epsilon q^{m/2} z_1) = (\Sigma_{k \geq 1} k(z_2^k - z_1^k) x_k) \delta(z_2/\epsilon q^{m/2} z_1), \]
\[ (D_{z_2} g_3(z_1, z_2)) \delta(z_2/\epsilon q^{m/2} z_1) = (\Sigma_{k \geq 1} (z_2^k - z_1^k) \frac{\partial}{\partial x_k}) \delta(z_2/\epsilon q^{m/2} z_1) \]

Our result follows easily from this. \( \square \)

Recalling Remark 2.30 as well as Definition 2.5 we have that the Lemmas proved so far in this section imply that we have now established the following result.

**Theorem 3.15.** The space \( \mathcal{F} = \mathbb{C}[v, v^{-1}] \otimes S \) is a module for the Lie Algebra \( \mathcal{L} \) where the central element \( c_\epsilon \) acts as the identity of \( \mathcal{F} \) and the central element \( c_t \) acts as zero. Let \( \pi \) be the corresponding representation. Then the action of \( \mathcal{L} \) on \( \mathcal{F} \) extends the action of \( \hat{H} \) on \( \mathcal{F} \) given in (3.3) and satisfies that for \( m \neq 0, \epsilon = \pm 1, k \in \mathbb{Z} \)
\[ \pi(c(k, m) + \epsilon q^{m/2} f(k, 1, m)) \text{ equals the coefficient of } z^{-(2k+1)} \text{ in } X^\epsilon_m(z), \]
\[ \pi\left(\frac{\epsilon q^{m/2} - 1}{2} h(k, m) + \frac{(\epsilon q^{m/2} + 1)}{2} s^{kt m} I\right) \text{ equals the coefficient of } z^{-2k} \text{ in } X^\epsilon_m(z), \]
\[ 30 \]
while for $m = 0$ and $\epsilon = -1, k \in \mathbb{Z}$ we have

\[
\pi(e(k, 0) - f(k + 1, 0)) \text{ equals the coefficient of } z^{-(2k+1)} \text{ in } X_0^{-1}(z) \text{ and }
\]

\[
\pi(-h(k, 0) + \delta_{k,0}(1/2)Id_F) \text{ equals the coefficient of } z^{-2k} \text{ in } X_0^{-1}(z).
\]

We next go towards showing that the representation $\pi$ above can be lifted to a representation of the Lie algebra $\hat{\mathcal{L}}$. Thus, recalling 1.23 we see we only need to define operators $\pi(d_s), \pi(d_t)$ and then check that they satisfy the appropriate formulas. However, it turns out that it is more convenient to work with the element $\hat{d}_s$ which is defined by

\[
(3.16) \quad \hat{d}_s := 2d_s + (1/2)ad\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = 2d_s + (1/2)ad(H).
\]

Notice that we have

\[
[d_t, W_m^\epsilon(z)] = mW_m^\epsilon(z),
\]
\[
[d_t, \alpha(z)] = 0,
\]
\[
[d_t, c_s] = [d_t, c_t] = 0, \text{ and}
\]
\[
[d_t, \hat{d}_s] = 0.
\]

The element $d_t$ will be represented by the degree derivation with respect to our variable $v$ on the module $F$ so that we will let

\[
(3.17) \quad \pi(d_t) := D_v := v\frac{\partial}{\partial v}.
\]

Then, as follows easily from Definition 3.5, we have that

\[
[D_v, X_m^\epsilon(z)] = mX_m^\epsilon(z),
\]
\[
[D_v, \alpha(z)] = 0,
\]
\[
[D_v, \pi(c_t)] = [D_v, 0] = 0,
\]
\[
[D_v, \pi(c_s)] = [D_v, Id_F] = 0, \text{ and so}
\]
\[
[D_v, \pi(\hat{d}_s)] = 0.
\]

This shows that it is easy to represent the element $d_t$. To deal with $\hat{d}_s$ is a little more complicated but follows along the usual lines. This is dealt with in the proof of our next result.
Corollary 3.18. The representation $\pi$ of $\mathcal{L}$ extends to a representation, which we also denote by $\pi$, of the Lie algebra $\hat{\mathcal{L}}$ satisfying

$$\pi(d_t) = D_v$$

and

$$\pi(\hat{d}_s) = -\Sigma_{k \geq 1} k x_k \frac{\partial}{\partial x_k}.$$ 

Proof. For the case when $m \neq 0$ and $\epsilon = \pm 1$, and also for the case when $m = 0, \epsilon = -1$ we have that (since $c_s$ is central)

$$[\hat{d}_s, W_m^\epsilon(z)] = [2d_s + (1/2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \delta(z^2/s) \begin{pmatrix} \frac{\epsilon q^{m/2}}{\epsilon q^{m/2} z} & z^{-1} \\ \epsilon q^{m/2} z & 1 \end{pmatrix} t^m] =$$

$$2d_s(\delta(z^2/s)) \begin{pmatrix} \frac{\epsilon q^{m/2}}{\epsilon q^{m/2} z} & z^{-1} \\ \epsilon q^{m/2} z & 1 \end{pmatrix} t^m +$$

$$(1/2)\delta(z^2/s)(\begin{pmatrix} \frac{\epsilon q^{m/2}}{\epsilon q^{m/2} z} & z^{-1} \\ -\epsilon q^{m/2} z & -1 \end{pmatrix} - \begin{pmatrix} \frac{\epsilon q^{m/2}}{\epsilon q^{m/2} z} & -z^{-1} \\ \epsilon q^{m/2} z & -1 \end{pmatrix}) t^m =$$

$$2d_s(\delta(z^2/s)) \begin{pmatrix} \frac{\epsilon q^{m/2}}{\epsilon q^{m/2} z} & z^{-1} \\ \epsilon q^{m/2} z & 1 \end{pmatrix} t^m + \delta(z^2/s) \begin{pmatrix} 0 & z^{-1} \\ -\epsilon q^{m/2} z & 0 \end{pmatrix} t^m.$$

It is straightforward to see that

$$2d_s(\delta(z^2/s)) = -D_z(\delta(z^2/s)),$$

so using this in the above gives us that

$$[\hat{d}_s, W_m^\epsilon(z)] = (-D_z(\delta(z^2/s))) \begin{pmatrix} \frac{\epsilon q^{m/2}}{\epsilon q^{m/2} z} & z^{-1} \\ \epsilon q^{m/2} z & 1 \end{pmatrix} t^m +$$

$$\delta(z^2/s)(-D_z(\begin{pmatrix} \frac{\epsilon q^{m/2}}{\epsilon q^{m/2} z} & z^{-1} \\ \epsilon q^{m/2} z & 1 \end{pmatrix}) t^m) =$$

$$- D_z(W_m^\epsilon(z)),$$

so that we have

$$[\hat{d}_s, W_m^\epsilon(z)] = -D_z(W_m^\epsilon(z)).$$

In a similar way we also obtain that

$$[\hat{d}_s, \alpha(z)] = -D_z(\alpha(z)).$$
We must show similar formulas hold for $\pi(\hat{d}_s)$. In doing this we will make use of the following two formulas.

\[(3.22) \quad [z \frac{\partial}{\partial z}, z^m] = mz^m. \]
\[(3.23) \quad [z \frac{\partial}{\partial z}, (\frac{\partial}{\partial z})^m] = -m(\frac{\partial}{\partial z})^m. \]

Using these one easily obtains that

\[\left[ kx_k \frac{\partial}{\partial x_k}, \exp(z^k(1 - (\epsilon q^{m/2})^k)x_k)\exp\left(\frac{-z^{-k}(1 - (\epsilon q^{m/2})^{-k})}{k} \frac{\partial}{\partial x_k}\right) \right] = \]
\[kx_k z^k(1 - (\epsilon q^{m/2})^k)x_k\exp(z^k(1 - (\epsilon q^{m/2})^k)x_k)\exp\left(\frac{-z^{-k}(1 - (\epsilon q^{m/2})^{-k})}{k} \frac{\partial}{\partial x_k}\right) + \]
\[\exp(z^k(1 - (\epsilon q^{m/2})^k)x_k)\exp\left(\frac{-z^{-k}(1 - (\epsilon q^{m/2})^{-k})}{k} \frac{\partial}{\partial x_k}\right) k\left(\frac{-z^{-k}(1 - (\epsilon q^{m/2})^{-k})}{k} \frac{\partial}{\partial x_k}\right) = \]
\[D_z(\exp(z^k(1 - (\epsilon q^{m/2})^k)x_k)\exp\left(\frac{-z^{-k}(1 - (\epsilon q^{m/2})^{-k})}{k} \frac{\partial}{\partial x_k}\right)). \]

It follows from this that we have

\[(3.24) \quad [\pi(\hat{d}_s), X_m(\epsilon)] = -D_z(X_m(\epsilon)). \]

Similarly, one easily gets that

\[(3.25) \quad [\pi(\hat{d}_s), a(\epsilon)] = -D_z(a(\epsilon)). \]

Looking at (3.20) and (3.21) we see this is exactly what we need. □

We close with several results concerning the structure of the module $\mathcal{F}$. First, we let $\mathcal{F}_m := \mathbb{C}v^m \otimes \mathcal{S}$ for all $m \in \mathbb{Z}$. Clearly then, we have that $\mathcal{F}_m$ is an irreducible $\hat{H}$-module for each $m \in \mathbb{Z}$ and we have that

\[(3.26) \quad \mathcal{F} = \oplus_{m \in \mathbb{Z}} \mathcal{F}_m. \]

Viewing the vertex operator $X_{\pm 1}(\epsilon)$ as a map from $\mathcal{F}_m$ to $\mathcal{F}_{m \pm 1}[\{z, z^{-1}\}]$ we find that we have

\[X_{\pm 1}(\epsilon)(v^m \otimes 1) = v^{m \pm 1}b(\pm 1, \epsilon)e^{\Sigma_{k \geq 1} z^k(1 - (\epsilon q^{\pm 1/2}^k)x_k)} = v^{m \pm 1}b(\pm 1, \epsilon)[1 + z(1 - \epsilon q^{\pm 1/2})x_1 + \ldots] \]

which shows that for any $m \in \mathbb{Z}$ the $\hat{L}$-module generated by any $\mathcal{F}_m$ is all of $\mathcal{F}$. These observations make the following result almost clear.
Proposition 3.27. \( \mathcal{F} \) is an irreducible module for the Lie algebra \( \hat{\mathcal{L}} \).

Proof. If \( f \) is a non-zero element we can write \( f = \sum_{m \in \mathbb{Z}} f_m \), where each element \( f_m \) is in \( \mathcal{F}_m \). We call the number of non-zero terms in this sum the length of \( f \). The remarks proceeding the statement of the Proposition show that any element of length one generates all of \( \mathcal{F} \) as an \( \hat{\mathcal{L}} \) module. On the other hand if we have a non-zero submodule \( M \) of \( \mathcal{F} \) and if \( f \) is a non-zero element of \( M \) whose component \( f_m \neq 0 \) then we can write

\[
f = f_m + \sum_{n \in \mathbb{Z}, n \neq m} f_n
\]

and so we obtain that if the length of \( f \) is greater than one then the element \( mf - \pi(d_v)f \) is again in \( M \) and is non-zero and has length shorter that \( f \). Our result follows from this. \( \square \)

Remark 3.28. Let \( \mathcal{H} \) be the span of the elements \( H, c_s, c_t, d_s, d_t \) so that \( \mathcal{H} \) is the five dimensional Cartan subalgebra of the EALA \( \hat{\mathcal{L}} \). It is straightforward to see that the module \( \mathcal{F} \) is a direct sum of finite dimensional \( \mathcal{H} \) weight spaces.

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