On Conformal Deformations II

Barak Kol

Racah Institute of Physics
The Hebrew University
Jerusalem 91904
Israel

barak.kol@phys.huji.ac.il

ABSTRACT: The conformal index counts the number of exactly marginal deformations. In 4d the index is given by the number of chiral primary operators of dimension 3 moded out by the complexified global group, where the quotient is defined as usual by imposing a D-term. Here we show its consistency with the Leigh-Strassler method for weakly coupled theories, and we test it against known examples. In several examples this method discovers extra exactly marginal deformations beyond those of Leigh-Strassler. [This is an unpublished paper dated 3.3.03.]

KEYWORDS: .
Preface

This is an unpublished paper dated 3.3.2003. Its purpose was to promote both by refining the formalism and by analyzing useful examples. introduced the role of the D-term of the global group. Here it was refined in terms of the holomorphic quotient into the central result, eq. (1.3). The paper ended up not being published because the logical step involved in the above-mentioned refinement was well known and was judged not to merit publication. presented the arguments which led me to this result, and here it was further tested by many examples and found to be precisely correct. Now I was extremely pleased that this central result was rigorously derived by Green et. al. [arXiv:1005.3546]. Note that the superconformal index introduced in (and further discussed here) can be considered to be realized by their derivation.

\footnote{Files dated by the computer operating system. This version is practically the same as the original. Changes include: a new title page but no change in the abstract, added preface and new acknowledgements, and a couple of minor non-scientific proof-reading corrections.}
1. Introduction

Given a field theory, the first priority usually is to determine the vacuum structure, and in particular the moduli space of vacua, $\mathcal{M}$ when one exists. Similarly, given a conformal field theory we would like to know the conformal moduli space, $\mathcal{M}_c$, namely the space generated by exactly marginal deformations. However, whereas we know much about the moduli space of vacua following the progress made in supersymmetric theories during the mid 90’s, we still know little about the conformal moduli space.

There are several properties of $\mathcal{M}_c$ which one would like to know on the way to a complete solution: the dimension, the local geometric structure (complex manifold, any non-trivial holonomy or special geometry), and finally determination of the metric, singularities and global issues. However, the dimension which is simply the number of exactly marginal deformations is the only topic which was studied so far.

Leigh and Strassler (1995) \cite{1} discovered that exactly marginal deformations are generic in 4d $\mathcal{N} = 1$ supersymmetric field theories and used exact $\mathcal{N} = 1$ relations, including the NSVZ formula \cite{7} to compute $\dim(\mathcal{M}_c)$. In section \ref{section2} we will describe their result in full, but very roughly it is

$$\dim(\mathcal{M}_c) = #(\beta) - #(\gamma) \quad (1.1)$$

Here $#(\beta)$ is the number of supersymmetric marginal operators, (or their associated couplings and $\beta$-functions), and $#(\gamma)$ is the number of anomalous dimensions ($\gamma$-functions) of the fundamental fields (including possibly mixing). However, this groundbreaking formulation had some disadvantages: for a general CFT, not given by a Lagrangian the concept of fundamental fields may not be well defined and so is $#(\gamma)$, moreover, the $\gamma$’s suffer from non-gauge-invariance, and both $\beta$’s and $\gamma$’s are scheme dependent.

In \cite{3} we studied the translation of this mechanism under the AdS/CFT correspondence \cite{6} for the case of 4d $\mathcal{N} = 4$. The translation is not straightforward since both the $\beta$ and $\gamma$ functions do not have a well understood translation. Following the supergravity analysis we claimed \cite{2} that in supergravity $\dim(\mathcal{M}_c)$ is given by the index of the supersymmetry variation operator, and hence purely in field theory

$$\dim(\mathcal{M}_c) = \text{Index}[\delta_{\text{superconf}}] \quad (1.2)$$

where $\delta_{\text{superconf}}$ is the superconformal variation operator considered to operate on the space of operators. We defer a direct discussion of the index for later work and concentrate here on the more explicit claim\footnote{\cite{2} included the main idea, but without the full details. The relation between the $\gamma$ functions and the complexified global transformation was discussed in \cite{3}.} that locally at the origin of $\mathcal{M}_c$

$$\mathcal{M}_c \simeq \text{supermarginals}/G_C \quad (1.3)$$

where $G_C$ stands for the complexified global group, and the holomorphic quotient is defined as usual by imposing the D-term (for the global group) and then dividing by $G$. 

Definition: “supermarginals ” $\equiv$ chiral primary operators of dimension 3,
Consequently, the generic dimension is
\begin{equation}
\dim(M_c) = \dim(R) - (\dim(G) - \dim(G_0)) \tag{1.4}
\end{equation}
where $R$ denotes the vector space of supermarginals, $G_0 \subseteq G$ are global symmetries unbroken by any of the supermarginal couplings. Interesting sub-generic cases do exist in which the dimension of the holomorphic quotient is strictly smaller than the one above (1.4). In this case our analysis shows that (1.3) is correct to lowest order in the couplings, and although in principle higher order contributions could change that, this did not happen in several examples.

The merits of this “index” formulation are that it is valid for any 4d $\mathcal{N} = 1$ CFT, all quantities are physical, and it requires only knowledge of the chiral primary spectrum. The global group is seen to play a central role on $M_c$ analogous to the gauge group on an $M$.

We should mention here two other motivations for the D-term. The first is that when an AdS dual exists, the global symmetry group becomes gauged and then the D-term is a necessary condition for a susy vacuum (see [9] for the state-of-the-art on 5d supergravity). The second is that $M_c$ is a complex space, and the only way to perform a quotient by the global group while keeping holomorphy is to impose the D-term.

The relation (1.3) is at the center of the current paper. Here it is precisely formulated\(^2\), and it is confirmed and confronted against the LS formulation. In section 2 we show that when the LS formulation is valid it coincides with (1.3), and that essentially they computed the index for zero couplings. Then in section 3 we re-analyze many of the examples of [1] and some others. For each example we find the full set of exactly marginal operators, which is often strictly larger than the ones found by [1]. Table 1 summarizes the local description of $M_c$ for all the examples.

2. An Equivalence with Leigh-Strassler

2.1 Set-up

Let us start by setting up the notation for an arbitrary $\mathcal{N} = 1$ gauge theory. The local symmetry group is a product $L = \prod_{i=1}^{n_L} L_i$. The fundamental matter fields (generators of the chiral ring) are the chiral fields $\phi_{s,t_s}$ where $s = 1, \ldots, n_R$ runs over the distinct representations $R_s$ of $L$ and $t_s$ runs over $t_s = 1, \ldots, T_s$, where $T_s$ is the multiplicity. Finally one should specify a superpotential $W = W(\phi)$.

The classical global group (namely, no anomalies) for $W = 0$ is $\prod_{s=1}^{n_R} U(T_s).$\(^3\) “Holomorphic” couplings in 4d $\mathcal{N} = 1$, namely those which appear in that part of the Lagrangian which is integrated over “half of superspace” ($\int d^2\theta$), are complex and may be divided into gauge couplings $g_i$, $i = 1, \ldots, n_G$, \(^4\) and superpotential parameters $h_j$. In addition there

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\(^3\)A comment is due on the $U(1)$ factors in the global group. One may choose to gauge an anomaly-free $U(1)$, thereby removing it from the global group, and at the same time constraining the possible operators that can be added to the superpotential.

\(^4\)It is conventional to use the complex coupling $\tau_I = \theta_I/(2\pi) + 4\pi i/g_I^2$, where $\theta_I$ is the associated theta angle, and $g_{\theta,I} = \exp(2\pi i \tau_I)$
are the “non-holomorphic” couplings in the Kahler potential, which we will not need, and henceforth “couplings” will denote the holomorphic ones unless stated otherwise.

The holomorphic marginal operators are generated from dimension 3 operators integrated over $d^2 \theta$. The classical supermarginals are the gauge coupling $g_i, i = 1, \ldots, n_L$ with the associated operators $\text{Tr}(W_\alpha W^\alpha)$, and the superpotential supermarginal, the coefficients of all dimension 3 gauge-invariant operators which may be added to $W, h_j, j = 1, \ldots, n_C$. Sometimes we shall refer to both $g_i$ and $h_j$ collectively as $\hat{g}_k$.

### 2.2 The Leigh-Strassler formulation

Let us now describe the original formulation of Leigh-Strassler. A set of couplings $\hat{g}_k$ is exactly marginal if and only if all their beta functions $\beta_k$ vanish (for all $\hat{g}_k$). $\mathcal{N} = 1$ imposes exact formulas for the $\beta$-functions in terms of the $\gamma$-functions of the charged fields. Note that the $\gamma$-functions of a specific representation $R_s$ are in the adjoint of $U(T_s)$, since these fields can mix in the two point function.

For a gauge coupling, $g_i$, the exact formula is the NSVZ formula

$$\beta_{g_i} \sim f(g) [\beta_0 i - \sum_{s=1}^{n_R} T(R_s, L_i) \gamma_s] \quad (2.1)$$

where $\beta_0$ is the 1-loop beta function, $T_{s,i} \equiv T(R_s, L_i)$ is the quadratic index of the representation $\text{Tr}_{L_s,R_s}(T^A T^B) = T_{s,i} \delta^{A,B}$ and $\gamma_s$ is the $U(1)$ component of the gamma functions. For the beta function of a superpotential parameter $h$ ($\Delta W = h \mathcal{O}$) the exact $\beta$ function is

$$\beta_{h_j} \sim [\beta_0 + \sum_{s=1}^{n_R} \gamma_s] h_j \quad (2.2)$$

where $\beta_0 \propto -\Delta_W + \Delta_\mathcal{O} = -3 + \Delta_\mathcal{O}$, $\gamma_s$ is in the adjoint of $U(T_s)$, and hence the $U(1)$ charge of $h$ is $d(\phi_s, \mathcal{O}_j) = \frac{\partial \log(\mathcal{O}_j)}{\partial \log(\phi_s)}$, namely the degree of $\phi_s$ in $\mathcal{O}_j$.

We concentrate on marginal couplings ($\beta_0 = 0$) and combine the relations (2.1,2.2) into

$$0 = \beta_k \propto \sum_{s=1}^{n_R} q_{g_i}^{s,A_s} \gamma_{s,A_s} \quad (2.3)$$

where $A_s$ is an index in the adjoint of $U(T_s)$ and the matrix $q$ is given by

$$
\begin{array}{c|c|c|}
\hline
 & U(1)_1, \ldots, U(1)_n_R & SU(T_1), \ldots, SU(T_s), \ldots, SU(T_{n_R}) \\
\hline
g_i & -T(R_s, L_i) & - \\
\hline
& & \\
\hline
h_j & d(\phi_s, \mathcal{O}_j) & T^{A_s} \\
& & \\
\hline
\end{array} \quad (2.4)
$$

From this linear dependence it is deduced that in order to find exactly marginals it is enough to set to zero all the $\gamma$-functions

$$\gamma_{s,A_s} = \gamma_{s,A_s}(g_i, h_j) = 0, \quad (2.5)$$
or more precisely, it is enough to require the vanishing of \( \text{rank}(q) \) \( \gamma \)-functions which have non-zero \( q \) coefficients. Generically there will be solutions when there are fewer (independent) \( \gamma \)-functions than couplings, and the generic dimension of the solution space is

\[
\dim M_c = \#(\beta) - \text{rank}(q) \tag{2.6}
\]

### 2.3 Comparing with Leigh-Strassler

We now compare our formulae (1.3, 1.4) with (2.6) at zero couplings. First, let us make the D-term constraint, implicit in (1.3) explicit. The marginal operators are of course gauge-invariant, but they may be charged under the global group. The \( h \)'s have a standard charge given by \( d(\phi_s, \mathcal{O}_j) \), as in the \( q \)-matrix. We define the charges of the \( g \)'s by using the \( q \)-matrix to be \( T(R_s, L_i) \). Altogether the D-term has the form

\[
D^{A_s} = \sum_k \hat{g}_k^\dagger T^{A_s} \hat{g}_k = \sum_k \hat{g}_k^\dagger q^{A_s} \hat{g}_k = \\
= \sum_i T(R_s, L_i) \delta^{A_s,1} g_i^2 - \sum_j h_j^\dagger T^{A_s} h_j \tag{2.7}
\]

where \( \delta^{A_s,1} \) selects only the \( U(1) \) generators and the generator \( T^{A_s} \) acts according to the representation of \( h \).

The couplings of LS are precisely the supermarginals for \( W = g_i = 0 \) (when one of these conditions is violated some of the LS couplings cease to be supermarginal as we will see soon). In order for the two methods to agree it must be that the \( \gamma \)-function equations coincide with the D-term equations to lowest order in the coupling. First we should check that in both cases there is the same number of equations. Indeed, since mixing in the two-point-function is allowed only between fields in the same representation, the \( \gamma \)-functions are valued in the global group. So (1.1) can be rewritten as

\[
\dim(M_c) = \#(\beta) - \#(\gamma) = \#(\text{dim 3, chiral primaries for } W = 0) - \dim(\text{classical global group}). \tag{2.8}
\]

Moreover, picking only the independent \( \gamma \)-functions as in (2.4) is equivalent to correcting for the dimension of \( G_0 \) the subgroup under which no coupling is charged, as in (1.4), since when we turn on a coupling \( \hat{g}_k \) then the violation of the global group generator \( T^{A_s} \) is given by the \( q \)-matrix \( q_i^{A_s} \), and so dependent columns in \( q \) generate \( G_0 \).

Next we would like to show that to lowest order \( O(g^2, h^2) \) the \( \gamma \)-functions coincide with the D-term. Looking at the 1-loop contribution of the gauge bosons to the two-point-function, figure [a], we see that it is indeed proportional to \( \text{Tr}_{R_i, R_s}(T^A T^B) g^2 \). Similarly the \( O(h^2) \) contribution is given by the diagram in figure [b]. The vertices come from \( \partial_{s_1, s_2} W \psi_{s_1} \psi_{s_2} \) and its complex conjugate, and the diagram is proportional to

\[
- \sum_j d(\phi_j, \mathcal{O}_i) h_j^\dagger T^{A_s} h_j, \text{ where the minus sign comes from the fermionic loop.}
\]

Note that our “consistency” check at zero couplings is very close to a proof that the two methods are identical, the only part missing being a confirmation of all constants in the D-term constraint (2.7).
Going beyond zero coupling, we need to study the effect of turning on a coupling. The Konishi anomaly [5] tells us that

\[
\{ \bar{Q}, \bar{\psi}_{s,t} \phi_{s,u} \} = \sum_i T(R_s, L_i) \text{tr}(\lambda_i \lambda_i) + \frac{\partial W}{\phi_{s,t}} \phi_{s,u} \tag{2.9}
\]

where \( i = 1, \ldots, n_L \) runs over the gauge groups, \( s = 1, \ldots, n_R \) runs over the various matter representations and \( t_s, u_s = 1, \ldots, T_s \) where \( T_s \) is the representation multiplicity.

Therefore whether the global group is broken classically by \( W \) or quantum mechanically by an instanton (for \( g > 0 \)), for each global group generator which is broken on the LHS of (2.9) the RHS ceases to be chiral primary since it is expressed as \( \bar{Q} \text{(something)} \). Thus the Konishi anomaly guarantees the index nature of (1.3) since the difference in (1.4) stays constant as couplings are turned on.

The argument above may continue to hold in some cases which are not weakly coupled, but are still defined by a Lagrangian through an RG flow. If one knows the fundamental fields (with non-trivial anomalous dimensions this time) and the supermarginal operators, then the exact formulae for the \( \beta \)-functions in terms of the D-terms holds, only they are not homogeneous anymore since \( \beta_0 = 0 \) does not necessarily hold. However, since we assume that a fixed point does exist, then the dimension of the solution space for the non-homogeneous problem reduces to that of the homogeneous problem, which is the one we analyzed above.

Since the general claim (1.3) was shown to coincide with Leigh-Strassler both at zero coupling and at small coupling, and since it is phrased in terms of physically well-defined quantities, we consider it to be plausible.

2.4 Summary of method

We summarize now our method (based on the principles in [2]):

- Identify the global group \( G \), and the supermarginals. The supermarginals consist of all gauge couplings and the dimension 3 gauge-invariant chiral primaries.

- Define the q-matrix of coupling charges as in (2.4).

- Find the subgroup \( G_0 \subseteq G \) under which no coupling is charged. This is equivalent to finding the number of dependent columns in the \( U(1) \) part of the q-matrix.
• Determine the generic dimension to be $1.4$.

• Continue to perform the holomorphic quotient $(1.3)$ according to the D-term constraints $(2.7)$, followed by a division by $G$. Often a $U(1)$ factor in $G$ can be cancelled against a supermarginal coupling, leading to simplification.

In the process of the imposing the D-terms it may happen that a certain global $U(1)$ has only positive (gauge) or only negative (superpotential) couplings. In that case all those couplings will be forced to zero and the total dimension will be sub-generic, at least to lowest order in the coupling. In principle it could happen that higher order corrections lift this degeneracy, but this did not happen is some examples: in an $\mathcal{N} = 2$ orbifold of $\mathcal{N} = 4$ studied in [8] this phenomena was seen first and it was shown to persist at least up to third order; in a scalar theory (subsection 3.5) it is known that there are no exactly marginals, to all orders.

3. Examples

In this section we apply our method to a series of examples, most of them from [1]. For each example we identify the global group, $G$, and the “supermarginal” (chiral primaries of dimension 3) operators, $R$ (at the origin of $\mathcal{M}_r$). The set of exactly marginal operators is then given by the holomorphic quotient $R/G$. The exactly marginal operators of [1] were found by applying intuition and special discrete symmetries to solve their equations, but were not claimed to exhaust the whole set of exactly marginals, and indeed we often find a strictly larger set, the first case being subsection 3.3. Table I summarizes the local description of $\mathcal{M}_c$ for all the examples.

3.1 $\mathcal{N} = 4$

We worked out this case in detail in [3] and we include it here for completeness. In an $\mathcal{N} = 1$ language the matter content is three adjoints, and the superpotential is $W_0 = g \text{tr} ([\phi_1, \phi_2] \phi_3)$. The global group is $SO(6)_R$ out of which $SU(3) \times U(1)_R$ is manifest in $\mathcal{N} = 1$ language.

For any simple gauge group the theory has one exactly conformal parameter which preserves $\mathcal{N} = 4$, namely the complex gauge coupling $5$. For some gauge groups cubic invariants exist, in particular for $SU(N)$ one has the symmetric invariant $d^{ABC} = \text{Tr} ([T^A, T^B]T^C)$ where $T^A$ are group generators. Therefore $S^3(3) = 10$ of $SU(3)$ are supermarginals. In addition, for zero coupling $W_0$ is supermarginal as well. The $q$-matrix is

$$
\begin{array}{c|cc|c}
& U(1) & SU(3) \\
\hline
U(1) & 1 & 1 \\
SU(3) & 3 & 3 \\
\end{array}
$$

\begin{equation}
(3.1)
\end{equation}

\begin{equation}
\begin{array}{c|cc|c}
& U(1) & SU(3) \\
\hline
U(1) & 1 & 1 \\
SU(3) & 3 & 3 \\
\end{array}
\end{equation}

\begin{equation}
(3.1)
\end{equation}

\begin{equation}
\begin{array}{c|cc|c}
& U(1) & SU(3) \\
\hline
U(1) & 1 & 1 \\
SU(3) & 3 & 3 \\
\end{array}
\end{equation}

\begin{equation}
(3.1)
\end{equation}

\begin{equation}
In particular the one loop beta function vanishes $b_0 \propto 3 - 3 = 0$.\]
Theory | Global group, $G$ | supermarginals, $R$
---|---|---
$\mathcal{N} = 4$ | $SU(3)$ | $10 + 1$
$\mathcal{N} = 2$ w. $A, S$ | $SU(N_f)_{\text{diag}} \times U(1)_B$ | $12_0 + 1_{-2,0} + 1_{0,0}$
$\mathcal{N} = 2$ w. $N_f = 2 N_c$ | $SU(N_f)_{\text{diag}} \times U(1)_B$ | $(84, 1)_3 + (1, 84)_{-3}$
$\text{SQCD w. } N_c = 3, N_f = 9$ | $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ | $(70, 1)_4 + (1, 70)_{-4}$
$\text{SQCD w. } N_c/N_f = 1/2$ | $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ | (baryons)
$\text{same w. } N_c = 4$ | $SU(N_f)_{\text{diag}} \times U(1)_B$ | no exactly marginals due to sub-generic D-term
$\text{scalar theory}$ | no exactly marginals due to sub-generic D-term
$SU(N) \times SU(N)$ | $SU(3) \times SU(3) \times U(1)_V$ | no exactly marginals
w. $3[(\mathbf{N}, \bar{\mathbf{N}}) + (\bar{\mathbf{N}}, \mathbf{N})]$ | $1 + (10, 1)_3 + (1, 10)_{-3}$
$\text{same w. } N = 3$ | $SU(2) \times SU(2)$ w. $(3, 3)$ | $1 + 1$
$E(6) \text{ w. } 12 \cdot 27$ | $SU(12)$ | $S^3(12) = 182$
$SU(4) \text{ w. } 8(4 + 4) + 4 \cdot 6$ | $SU(8)_Q \times SU(8)_{\bar{Q}} \times SU(4)_A \times U(1)^2$ | $(28, 1, 4)_{2,0} + (1, 28, 4)_{-2,0}$

Table 1: Summary of examples and results. For each theory we state the global group, and the supermarginal operators – the dimension 3 chiral primaries (in a complex representation). $g > 0$ is assumed when relevant. Locally around the origin $\mathcal{M}_c$ is given by $\mathcal{M}_c \simeq \mathbb{R}/G$ – note that $G$ “plays the role” of a local group on $\mathcal{M}_c$ and the appearance of a D-term in the quotient. *The specified global group does not include a $U(1)_R$ factor. In cases with extended supersymmetry and higher R-symmetry, only the commutant with $U(1)_R$ is given.

A combination of $g$ and $W_0$ produces the fixed line of $\mathcal{N} = 4$ and the $q$-matrix reduces to

$$
\begin{array}{c|c}
g & W_0 \\
\hline
\phi^3 & 10 \\
\end{array}
$$

Hence

$$
\mathcal{M}_c \simeq 10/SL(3, \mathbb{C}) + 1_{\mathbb{C}}.
$$

In the zero coupling limit the global group enhances $SU(3)_{\phi} \to U(3)_{\phi}$ and this effect in cancelled by a new supermarginal, namely $W_0$, which is allowed since $W = 0$.

In [1] IV.D two exactly marginal deformations are demonstrated $\phi_1 \phi_2 \phi_3$ and $\phi_1^3 + \phi_2^3 + \phi_3^3$ based on some special discrete subgroup of $SU(3)$ under which these operators are invariant. Of course these operators could be rotated by $SU(3)$ so we should think of them as being representatives. Since [1] finds here $2_{\mathbb{C}} = 10 - 8$ operators they exhaust all exactly marginals, except for the subtlety that the $SU(3)$ action actually induces some discrete identification by some finite subgroup of $SU(3)$ (see [2], appendix).

3.2 $\mathcal{N} = 2$

Take an $\mathcal{N} = 2$ theory with matter in the symmetric and anti-symmetric. In $\mathcal{N} = 1$ language an $\mathcal{N} = 2$ matter hypermultiplet in representation $R$ doubles into chiral multiplets...
in $R$ and in its complex conjugate $\tilde{R}$, and in this case we have matter in $S, \tilde{S}, A, \tilde{A}$. In addition the $\mathcal{N} = 2$ vector multiplet contributes an adjoint $\phi$. The superpotential is of the form $W_0 = g (\tilde{A} \phi A + \tilde{S} \phi S)$.

At zero couplings the global symmetry is $U(1)^5 = U(1)_S \times U(1)_{\tilde{S}} \times U(1)_A \times U(1)_{\tilde{A}} \times U(1)_{\phi}$. The supermarginals are the gauge coupling ($b_0 = 3N - N_\phi - (N + 2)S - (N - 2)A = 0$), and the five dimension 3 operators $\phi^3, \tilde{S} \phi S, \tilde{A} \phi A, \tilde{S} \phi A, \tilde{A} \phi A$, and the q-matrix is

$$
\begin{array}{cccccc}
U(1)_S & U(1)_{\tilde{S}} & U(1)_A & U(1)_{\tilde{A}} & U(1)_\phi \\
g & -(N + 2)/2 & -(N + 2)/2 & -(N - 2)/2 & -N \\
S \phi S & 1 & 1 & 0 & 0 & 1 \\
A \phi A & 0 & 0 & 1 & 1 & 1 \\
S \phi A & 0 & 1 & 1 & 0 & 1 \\
A \phi S & 0 & 1 & 0 & 1 & 1 \\
\phi^3 & 1 & 0 & 0 & 0 & 3 \\
\end{array}
$$

(3.4)

One notices that there is one combination which is preserved by all supermarginals, namely, $G_0 = U(1)_{(S - \tilde{S}) + (A - \tilde{A})}$. Therefore the (generic) dimension is $6 - 4 = 2C$.

Let us study this space in more detail. We can eliminate one coupling, and one global $U(1)$ by noticing that the D-term equation for $U(1)_\phi - (S + \tilde{S} + A + \tilde{A})$ is simply $h_{\phi^3} = 0$. Next, there is one exactly marginal (the $\mathcal{N} = 2$ coupling) which is a mixture of $g, \tilde{S} \phi S$ and $\tilde{A} \phi A$. This can be seen by looking at the appropriate rows in the q-matrix, and noticing that for these rows the reduced q-matrix degenerates as $U(1)_S = U(1)_{\tilde{S}}, U(1)_A = U(1)_{\tilde{A}}$ and hence there are 2 equations for 3 couplings. Finally, on the $\mathcal{N} = 2$ fixed line the global symmetry is $U(1)_S \times U(1)_A$ and there are two additional supermarginals $\tilde{S} \phi A, \tilde{A} \phi S$. Since there is the abovementioned combination of the global $U(1)$’s which preserves both, we get a second exact marginal.

All in all we find the same exact marginals as in section IV.E.

Another conformal $\mathcal{N} = 2$ example is given by a $\mathcal{N} = 2$ SQCD with $N_f = 2N_c$. This example was treated by Seiberg-Witten (and not in [1]), and is known to have $\dim(M_c) = 1C$ namely the complex gauge coupling.

Let us confirm this result using our formulation. In terms of $\mathcal{N} = 1$ at zero coupling we have the q-matrix

$$
\begin{array}{cccccc}
U(1)_L & U(1)_R & U(1)_\phi & SU(N_f)_L & SU(N_f)_R \\
g & -N_c & -N_c & -N_c & - & - \\
Q \phi Q & 1 & 1 & 1 & N_f & N_f \\
\phi^3 & 0 & 0 & 3 & 1 & 1 \\
\end{array}
$$

(3.5)

As in the previous example $U(1)_\phi$ and $\phi^3$ can be both eliminated after considering the D-term equation for $U(1)_{\phi - (R + L)/2}$. Next we notice that $U(1)_B \equiv U(1)_{R - L}$ is preserved

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6Actually $U(1)_\phi$ should be replaced by the $\mathcal{N} = 2 SU(2)_R$ which is not manifest in $\mathcal{N} = 1$. 

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by all supermarginals. Hence the q-matrix reduces to

\[
\begin{array}{c|cc|cc}
& U(1)_{L-R} & SU(N_f)_L & SU(N_f)_R \\
\hline
\text{g} & -2N_c & - & - \\
\text{Q} & 2 & N_f & N_f \\
\end{array}
\]

(3.6)

Performing the quotient \((N_f, N_f)/(SL(N_f, \mathbb{C}) \times SL(N_f, \mathbb{C}))\) a singlet remains, and then the D-term for the \(U(1)\) determines \(g\).

Alternatively, the analysis can be repeated on the \(\mathcal{N} = 2\) fixed line. Now the global group is broken down to \(SU(N_f)_\text{diag} \times U(1)_B\) (the axial \(U(1)\) is broken by instantons). The supermarginals are in \((\text{adjoint} + 1)_0\), and so after the division by \(G\) only the singlet remains.

### 3.3 SQCD with \(N_c = 3, N_f = 9\)

This is our first example without extended supersymmetry, and the first where we display more exact marginals than in [1]. The number of colors is especially tuned to allow for cubic invariants.

The matter content of 9 flavors guarantees \(b_0 = 3N_c - N_f = 0\) and generates a classical global group \(U(1)_Q \times U(1)_{\tilde{Q}} \times SU(9)_Q \times SU(9)_{\tilde{Q}}\). The supermarginals are \(g\) and the gauge invariant cubics, namely \(Q^3\) and \(\tilde{Q}^3\). The q-matrix is

\[
\begin{array}{c|cc|c|c}
& U(1)_Q & U(1)_{\tilde{Q}} & SU(9)_Q & SU(9)_{\tilde{Q}} \\
\hline
\text{g} & -9/2 & -9/2 & - & - \\
\text{Q} & 3 & 0 & S^4(9) = 84 & 1 \\
\text{Q} & 0 & 3 & 1 & 84 \\
\end{array}
\]

(3.7)

Instantons break the \(U(1)^2\) part into \(U(1)_B\) whose charges are defined by \(q_B = q_Q - q_{\tilde{Q}}\). After eliminating \(g\) against the broken axial \(U(1)\) the q-matrix reduces to

\[
\begin{array}{c|cc|c}
& U(1)_B & SU(9)_Q & SU(9)_{\tilde{Q}} \\
\hline
\text{Q} & 3 & S^3(9) = 84 & 1 \\
\text{Q} & -3 & 1 & 84 \\
\end{array}
\]

(3.8)

### 3.4 SQCD with \(N_c/N_f = 1/2\)

In this example we find a large \(\mathcal{M}_c\) for one of the simplest (and relatively physical) theories.

SQCD within the “conformal window” \(1/3 < N_c/N_f < 2/3\) flows in the IR to a non-trivial CFT. The dimension of the meson fields \(M = Q \tilde{Q}\) is given by \(D(M) = 3(1 - N_c/N_f)\). If one wants to construct supermarginal operators, it is necessary to choose \(N_c/N_f = 1/2\) and hence

\[
D(M) = 3/2
\]

(3.9)

The global symmetry is known to be \(U(1)_B \times SU(N_f)_Q \times SU(N_f)_{\tilde{Q}}\) (the axial \(U(1)\) is broken by the instantons).
The supermarginals are of the form $M^2 = Q^2 \tilde{Q}^2$ and are in the representation $S^2(N_f, N_f)_0 = (S^2(N_f), S^2(N_f))_0 + (A^2(N_f), A^2(N_f))_0$.

The q-matrix is simply

$$U(1)_B \begin{array}{c|cc}
& SU(N_f) & SU(N_f) \\
\hline
M^2 & S^2(N_f) & S^2(N_f) \\
0 & A^2(N_f) & A^2(N_f)
\end{array}$$  \hspace{1cm} (3.10)

and so

$$\mathcal{M}_c \simeq S^2(N_f, N_f)/SL(N_f, \mathbb{C}) \times SL(N_f, \mathbb{C})$$  \hspace{1cm} (3.11)

which has dimension $N_c^2 (N_f^2 - 3)/2 + 2$.

For the special case $N_c = 4$, $N_f = 8$ there are additional baryonic supermarginals, $Q^4$ and $\tilde{Q}^4$. They lie in representation $A^4(8)_4 + c.c. = 70_4 + 70_{-4}$. Now the D-term constraint for $U(1)_B$ is not trivial anymore and should be added.

3.5 Scalar theory

It is well known that theories with no gauge fields do not have exactly marginals \(\text{[11]}\), and actually any marginal perturbation leads to a Landau pole in the UV. From our point of view this is a non-generic case where the dimension formula (3.4) and the quotient formula (1.3) disagree, as a single D-term equation can impose the vanishing of several couplings. In this example we see that the quotient formula is correct.

The simplest example is a theory with 2 scalars $A_i$, $i = 1, 2$. The q-matrix is given by

$$\begin{array}{c|cc}
& U(1)_A & SU(2)_A \\
\hline
A^i & 3 & 4
\end{array}$$  \hspace{1cm} (3.12)

Although there are 4 supermarginals (parametrized by $h_1 A^3 + h_2 A^2 B + h_3 A B^2 + h_4 B^3$) and only 2 global $U(1)$’s, there are no exact marginals as can be seen from the 1-loop $\gamma$-functions

$$\gamma_A = 3 |h_1|^2 + 2 |h_2|^2 + |h_3|^2 = 0$$  \hspace{1cm} (3.13)
$$\gamma_B = |h_2|^2 + 2 |h_3|^2 + 3 |h_4|^2 = 0$$  \hspace{1cm} (3.14)

These two equations force all $h$’s to vanish. One could be concerned whether higher order contributions to the $\gamma$’s could change the picture, such as adding $-|h_4|^4$ to $\gamma_A$, but then the general theorem \(\text{[11]}\) forbids such a correction.

3.6 $SU(N_c) \times SU(N_c)$ with 3 bifundamentals

This example shows that you cannot get exactly marginals starting with only gauge couplings.

The theory has gauge group $SU(N_c) \times SU(N_c)$ and matter in $3 [Q + \tilde{Q}] \equiv 3 [(N_c, \bar{N}_c) + (\bar{N}_c, N_c)]$. Each factor of the gauge group has effectively $N_f = 3 N_c$ and so $b_0 = 0$. The
classical global group is $U(3)_Q \times U(3)_{\tilde{Q}}$. For generic $N_c$ the only supermarginals are the gauge couplings and the q-matrix is

\[
\begin{array}{c|cc|cc}
 & U(1)_Q & U(1)_{\tilde{Q}} & SU(3)_Q & SU(3)_{\tilde{Q}} \\
g_1 & -3N_c/2 & -3N_c/2 & - & - \\
g_2 & -3N_c/2 & -3N_c/2 & - & - \\
\end{array}
\] (3.16)

We see that the q-matrix is degenerate, the $U(1)_B$ combination is unbroken, and we change basis to $U(1)_B, U(1)_Q + U(1)_{\tilde{Q}}$. The q-matrix becomes

\[
\begin{array}{c|cc|cc}
 & U(1)_{Q+\tilde{Q}} & U(1)_B & SU(3)_Q & SU(3)_{\tilde{Q}} \\
g_1 & -3N_c & 0 & - & - \\
g_2 & -3N_c & 0 & - & - \\
\end{array}
\] (3.17)

At first it looks like due to the degeneracy of the q-matrix we are left with one constraint on two couplings. However, this D-term constraint has only negative charges $0 = -3N_c g_1^2 - 3N_c g_2^2$ and hence there is no solution (this result is robust against corrections since the quadratic form $g_1^2 + g_2^2$ is non-degenerate and thus the local behavior will not be changed by higher order corrections). This is not surprising as we expect the balance of $g$’s and $h$’s to be necessary.

For $N_c = 3$ $h$’s can be formed, and dim($\mathcal{M}_c$) returns to be generic. The additional supermarginals are $Q^3$, $\tilde{Q}^3$ in the $(10,1)_3$, $(1,10)_{-3}$. The q-matrix is

\[
\begin{array}{c|cc|cc}
 & U(1)_{Q+\tilde{Q}} & U(1)_B & SU(3)_Q & SU(3)_{\tilde{Q}} \\
g_1 & -3N_c & 0 & - & - \\
g_2 & -3N_c & 0 & - & - \\
Q^3 & 3 & 3 & 10 & 1 \\
\tilde{Q}^3 & 3 & -3 & 1 & 10 \\
\end{array}
\] (3.18)

The non-Abelian quotient gives two $10/SL(3,\mathbb{C})$ factors, the $U(1)_B$ D-term equates the absolute values of these two factors, and finally the $U(1)_{Q+\tilde{Q}}$ D-term gives $N_c (g_1^2 + g_2^2) = |h_{Q^3}|^2 + |h_{\tilde{Q}^3}|^2$ so that only the sum $g_1^2 + g_2^2$ is fixed and thus an additional exactly marginal is reclaimed. Thus altogether we get

\[
[10_3/SL(3,\mathbb{C}) + 10_{-3}/SL(3,\mathbb{C})]/\mathbb{C}^* + 1
\] (3.19)

where the last 1 arises from the gauge couplings.

3.7 Others

Here we discuss and compare several other cases that appeared in [4].

$SU(2) \times SU(2)$ with $(3,3)$

The matter is denoted by $Q \equiv (3,3)$. For each gauge group factor we have effectively 3 adjoints and so $b_0 = 0$. The classical global group is $U(1)_Q$, and the supermarginals are
$g$ and $Q^3$. The resulting q-matrix is

$$
\begin{array}{c|cc}
\hline
 & U(1)_Q & \\
\hline
g_1 & -3 \cdot 2 & \\
g_2 & -3 \cdot 2 & \\
\hline
Q^3 & 3 & \\
\end{array}
$$

(3.20)

The instantons break the $U(1)_Q$ and we are left with two exactly marginals (one of them gauge like in the previous example?) Here our results agree with [1] section IV.B.

**E$_6$ with 12 27**

This example was brought in [1] as an example with chiral matter. From our perspective the gauge group and its chiral representations do not play a role, and the only difference is that the global group will not include factors for both $Q$ and $\tilde{Q}$.

The matter content was chosen so that $b_0 = 3T(\mathbf{78}) - 12T(\mathbf{27}) = 3 \cdot 24 - 12 \cdot 6 = 0$. The classical global group is $U(12)_Q$ broken by instantons to $SU(12)_Q$. In order to form cubic in $Q$ it is important that $E_6$ has an invariant cubic symmetric tensor (for the 27), and so $Q^3$ sits in $S^3(\mathbf{12})_3 = 182$. Altogether we find

$$\mathcal{M}_c \simeq 182/SL(12, \mathbb{C})$$

(3.21)

This 39$_C$ dimensional space is much larger than the special cases found in [1] section IV.F.

**SU(4) with 8 flavors and 4 antisymmetric**

This theory is brought up in [1] as an example with a global $U(1)$ which is unbroken by instantons (“undetermined R charge”). For us this is not a new feature, but it is an interesting application for our method.

The matter content satisfies $b_0 = 3N_c - N_f - 4T(A) = 12 - 8 - 4 = 0$, where $T(A) = (N - 2)/2 = 1$. The classical global group is $U(8)_Q \times U(8)_{\tilde{Q}} \times U(4)_A$ and the supermarginals are the gauge coupling and $Q^2 A, \tilde{Q}^2 A$. The q-matrix is given by

$$
\begin{array}{c|ccccc}
\hline
& U(1)_Q & U(1)_{\tilde{Q}} & U(1)_A & SU(8)_Q & SU(8)_{\tilde{Q}} & SU(4) \\
\hline
\hline
g & -4 & -4 & -4 & - & - & - \\
Q^2 A & 2 & 0 & 1 & A^2(8) = 28 & 1 & 4 \\
\hline
\end{array}
$$

(3.22)

Out of the $U(1)^3$ two are unbroken by instantons, and can be chosen to be $U(1)_{2A-Q-\tilde{Q}}$ (which is preserved by all supermarginals) and $U(1)_{B} = U(1)_{Q-\tilde{Q}}$. The reduced q-matrix becomes

$$
\begin{array}{c|cccc}
\hline
& U(1)_B & U(1)_{2A-Q-\tilde{Q}} & SU(8)_Q & SU(8)_{\tilde{Q}} & SU(4) \\
\hline
\hline
Q^2 A & 2 & 0 & A^2(8) = 28 & 1 & 4 \\
\hline
\end{array}
$$

(3.23)

from which we deduce

$$\mathcal{M}_c \simeq [(28, 4)_2/SL(8, \mathbb{C}) + (28, 4)_{-2}/SL(8, \mathbb{C})] / (SL(4, \mathbb{C}) \times \mathbb{C}^*) .$$

(3.24)
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