Another look at Bootstrapping the Student t-statistic

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Dedicated to the memory of Sándor Csörgő

Abstract

Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with mean $\mu = \mathbb{E}X$. Let $\{v^{(n)}_1, \ldots, v^{(n)}_n\}_{n=1}^\infty$ be vectors of non-negative random variables (weights), independent of the data sequence $\{X_1, \ldots, X_n\}_{n=1}^\infty$, and put $m_n = \sum_{i=1}^n v^{(n)}_i$. Consider $X^*_1, \ldots, X^*_{m_n}$, $m_n \geq 1$, a bootstrap sample, resulting from re-sampling or stochastically re-weighing a random sample $X_1, \ldots, X_n$, $n \geq 1$. Put $\bar{X}_n = \sum_{i=1}^n X_i/n$, the original sample mean, and define $X^*_m = \sum_{i=1}^n v^{(n)}_i X_i/m_n$, the bootstrap sample mean. Thus, $\bar{X}^*_m - \bar{X}_n = \sum_{i=1}^n (v^{(n)}_i/m_n - 1/n)X_i$. Put $V^2_n = \sum_{i=1}^n (v^{(n)}_i/m_n - 1/n)^2$ and let $S^2_n, S^2_{m_n}$ respectively be the the original sample variance and the bootstrap sample variance. The main aim of this exposition is to study the asymptotic behavior of the bootstrapped t-statistics $T^*_m := (\bar{X}^*_m - \bar{X}_n)/(S_n V_n)$ and $T^*_m := \sqrt{m_n}(\bar{X}^*_m - \bar{X}_n)/S^*_{m_n}$ in terms of conditioning on the weights via assuming that, as $n, m_n \to \infty$, $\max_{1 \leq i \leq n}(v^{(n)}_i/m_n - 1/n)^2/V^2_n = o(1)$ almost surely or in probability on the probability space of the weights. In consequence of these maximum negligibility conditions of the weights, a characterization of the validity of this approach to the bootstrap is obtained as a direct consequence of the Lindeberg-Feller central limit theorem. This view of justifying the validity of the bootstrap of i.i.d. observables is believed to be new. The need for it arises naturally in practice when exploring the nature

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of information contained in a random sample via re-sampling, for example. Unlike in the theory of weighted bootstrap with exchangeable weights, in this exposition it is not assumed that the components of the vectors of non-negative weights are exchangeable random variables. Conditioning on the data is also revisited for Efron’s bootstrap weights under conditions on $n, m_n$ as $n \to \infty$ that differ from requiring $m_n/n$ to be in the interval $(\lambda_1, \lambda_2)$ with $0 < \lambda_1 < \lambda_2 < \infty$ as in Mason and Shao \cite{19}. Also, the validity of the bootstrapped $t$-intervals is established for both approaches to conditioning. Moreover, when conditioning on the sample, our results in this regard are new in that they are shown to hold true when $X$ is in the domain of attraction of the normal law (DAN), possibly with infinite variance, while the ones for $E_X X^2 < \infty$ when conditioning on the weights are first time results per se.

Keywords: Conditional Central Limit Theorems, Stochastically Weighted Partial Sums, Weighted Bootstrap.

1 Introduction to the approach taken

The main objective of the present paper is to address the possibility of investigating and concluding the validity of bootstrapped partial sums via conditioning on the random weights. The term bootstrapping here will refer to both re-sampling, like Efron’s, and stochastically re-weighing the data. We show that a direct consequence of the Lindeberg-Feller central limit theorem (CLT) as stated in Lemma 5.1, which is also known as the Hájek-Sidák theorem (cf., e.g., Theorem 5.3 in DasGupta \cite{10}), is the only required tool to establish the validity of bootstrapped partial sums of independent and identically distributed (i.i.d.) random variables. As a consequence of Lemma 5.1, Theorem 2.1 characterizes valid schemes of bootstrap in general, when conditioning on the weights. Accordingly, the bootstrap weights do not have to be exchangeable in order for the bootstrap scheme to be valid. This is unlike the method of studying the consistency of the generalized bootstrapped mean that was initiated by Mason and Newton \cite{18} in terms of conditioning on the sample (cf. their Theorem 2.1 on the thus conditioned asymptotic normality of linear combination of exchangeable arrays). The latter approach relies on Theorem 4.1 of Hájek \cite{15} concerning the asymptotic normality of linear rank statistics.

We also investigate the validity of Efron’s scheme of bootstrap and also that of the scheme of stochastically re-weighing the observations by verifying
how the respective bootstrap weights satisfy the required maximal negligibility conditions (cf. Corollaries 2.1 and 2.2 respectively).

To illustrate the different nature of the two approaches to conditioning, we also study Efron’s scheme of bootstrap applied to i.i.d. observations via conditioning on the data. When doing this, we view a bootstrap partial sum as a randomly weighted sum of centered multinomial random variables. This enables us to derive conditional central limit theorems for these randomly weighted centered multinomial random variables via results of Morris [20]. The proofs of our Theorems 3.1 and 3.2 in this regard will be seen to be significantly shorter and simpler in comparison to similar results on bootstrapped partial sums when conditioning on the data.

For throughout use, let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. real valued random variables with mean $\mu := E(X)$. For a random sample $X_1, \ldots, X_n$, $n \geq 1$, Efron’s scheme of bootstrap, cf. [11], is a procedure of re-sampling $m_n \geq 1$ times with replacement from the original data in such a way that each $X_i$, $1 \leq i \leq n$, is selected with probability $1/n$ at a time. The resulting sub-sample will be denoted by $X^*_1, \ldots, X^*_{m_n}$, $m_n \geq 1$, and is called the bootstrap sample. The bootstrap partial sum is a stochastically re-weighted version of the original partial sum of $X_1, \ldots, X_n$, i.e.,

$$\sum_{i=1}^{m_n} X^*_i = \sum_{i=1}^{n} w_i^{(n)} X_i,$$

where, $w_i^{(n)} := \#$ of times the index $i$ is chosen in $m_n$ draws with replacement from $1, \ldots, i, \ldots, n$ of the indices of $X_1, \ldots, X_i, \ldots, X_n$.

**Remark 1.1.** In view of the preceding definition of $w_i^{(n)}$, $1 \leq i \leq n$, they form a row-wise independent triangular array of random variables such that $\sum_{1 \leq i \leq n} w_i^{(n)} = m_n$, and for each $n \geq 1$,

$$(w_1^{(n)}, \ldots, w_n^{(n)}) \overset{d}{=} \text{multinomial}(m_n; \frac{1}{n}, \ldots, \frac{1}{n}),$$

i.e., a multinomial distribution of size $m_n$ with respective probabilities $1/n$. Clearly, for each $n$, $w_i^{(n)}$ are independent from the random sample $X_i$, $1 \leq i \leq n$. Weights denoted by $w_i^{(n)}$ will stand for triangular multinomial random variables in this context throughout.
The randomly weighted representation of $\sum_{1 \leq i \leq m_n} X_i^*$ as in (1), in turn, enables one to think of bootstrap in a more general way in which the scheme of bootstrap is restricted to neither Efron’s nor to re-sampling in general. In this exposition the term bootstrap will refer both to re-sampling, such as Efron’s, as well as to stochastically re-weighing the sample. Both of these schemes of bootstrap can be viewed and treated as weighted bootstraps. As such, throughout this paper, the notation $v_i^{(n)}$, $1 \leq i \leq n$, will stand for bootstrap weights that are to be determined by the scheme of bootstrap in hand. Thus, to begin with, we consider a sequence $\{v_1^{(n)}, \ldots, v_n^{(n)}\}_{n \geq 1}$ of vectors of non-negative random weights, independent of the data sequence $\{X_1, \ldots, X_n\}_{n \geq 1}$, and put $m_n = \sum_{i=1}^n v_i^{(n)}$, $m_n \geq 1$. We do not assume that the components of the vectors of the non-negative weights in hand are exchangeable random variables.

Consider now a bootstrap sample $X_1^*, \ldots, X_{m_n}^*$, $m_n \geq 1$, which is a result of some weighted bootstrap via re-sampling or stochastically re-weighing the original random sample $X_1, \ldots, X_n$, $n \geq 1$. Define the bootstrap sample mean $X_{m_n}^* := \sum_{i=1}^n v_i^{(n)} X_i / m_n$ and the original sample mean $\bar{X}_n := \sum_{i=1}^n X_i / n$. In view of the above setup of bootstrap weights one can readily see that

$$
\bar{X}_{m_n}^* - \bar{X}_n = \sum_{i=1}^n \left( \frac{v_i^{(n)}}{m_n} - \frac{1}{n} \right) X_i
$$

$$
= \sum_{i=1}^n \left( \frac{v_i^{(n)}}{m_n} - \frac{1}{n} \right) (X_i - \mu).
$$

Hence, when studying bootstrapped $t$-statistics via $\{\bar{X}_{m_n}^* - \bar{X}_n\}_{n \geq 1}$ in the sequel, it is important to remember that, to begin with, the latter sequence of statistics has no direct information about the parameter of interest $\mu := E(X)$.

In particular, in this paper, the following two general forms of bootstrapped $t$-statistics will be considered.

$$
T_{m_n}^* = \frac{\sum_{i=1}^n \left( \frac{v_i^{(n)}}{m_n} - \frac{1}{n} \right) X_i}{S_n \sqrt{\sum_{i=1}^n \left( \frac{v_i^{(n)}}{m_n} - \frac{1}{n} \right)^2}}.
$$

(2)
\[ T_{mn}^{**} = \frac{\sum_{i=1}^{n} \left( \frac{v_i^{(n)}}{m_n} - \frac{1}{n} \right) X_i}{S_{mn}^*/\sqrt{m_n}}, \]

where \( S_n^2 \) and \( S_m^2 \) are respectively the original sample variance and the bootstrapped sample variance, i.e.,

\[ S_n^2 = \sum_{1 \leq i \leq n} (X_i - \bar{X}_n)^2 / n \]

and

\[ S_{mn}^2 = \sum_{1 \leq i \leq m_n} (X_i^* - \bar{X}_{mn}^*)^2 / m_n. \]

Remark 1.2. In this exposition, both \( T_m^* \) and \( T_{mn}^{**} \) will be called bootstrapped versions of the well-known Student \( t \)-statistic

\[ T_n := \frac{X_n}{S_n/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i}{S_n\sqrt{n}}. \]

Remark 1.3. In Efron’s scheme of bootstrap \( v_i^{(n)} = w_i^{(n)} \), 1 \( \leq \) i \( \leq \) n, and \( (3) \) is seen to be the well-known Efron bootstrapped \( t \)-statistic. When the parameter of interest is \( \mu = E(X) \), Weng [22] suggests the use of \( \sum_{i=1}^{n} \zeta_i X_i / m_n \) as an estimator of \( \mu \), where \( \zeta_i \) are i.i.d. Gamma(4, 1) random variables which are assumed to be independent from the random sample \( X_i \), 1 \( \leq \) i \( \leq \) n, and \( m_n = \sum_{i=1}^{n} \zeta_i \). This approach is used in the so-called Bayesian bootstrap (cf., e.g., Rubin [21]). This scheme of bootstrap, in a more general form, shall be viewed in Corollary 2.2 below in the context of conditioning on the i.i.d. positive random variables \( \{\zeta_1, \ldots, \zeta_n\} \) as specified there.

The main objective of this exposition is to show that in the presence of the introduction of the extra randomness, \( v_i^{(n)} \), 1 \( \leq \) i \( \leq \) n, as a result of re-sampling or re-weighing, conditional distributions of the bootstrapped \( t \)-statistics \( T_m^* \) and \( T_{mn}^{**} \) will asymptotically coincide with that of the original \( t \)-statistic \( T_n \). In this paper this problem will be studied by both of the two approaches to conditioning in hand. In Section 2, based on the Lindeberg-Feller CLT we conclude a characterization of the asymptotic behavior of the bootstrapped mean via conditioning on the bootstrap weights, \( v_i^{(n)} \), 1 \( \leq \) i \( \leq \) n, in terms of a manifold conditional Lindeberg-Feller type CLT for \( T_m^* \) and
Then we show that the validity of Efron’s scheme of bootstrap results directly from Theorem 2.1 for both of the latter bootstrapped $t$-statistics when conditioning on $w_i^{(n)}$ as in Remark 1.1 (cf. Corollary 2.1). As another example, in Corollary 2.2 the weights $\zeta_i/m_n$, where $\zeta_i$ are positive i.i.d. random variables independent of $\{X_i, 1 \leq i \leq n\}_{n \geq 1}$, are considered for re-weighing the original sequence. It is shown that under appropriate moment conditions for $\zeta_i$, the validity of bootstrapping the $t$-statistic $T_n$ via conditioning on $\zeta_i$, $1 \leq i \leq n$, also follows from Theorem 2.1 for both $T_n^*$ and $T_n^{**}$ in these terms as well. In Section 3, we continue the investigation of the limiting conditional distribution of $T_n^{**}$, but this time via conditioning on the sample $X_i, 1 \leq i \leq n$, $n \geq 1$, and only for Efron’s bootstrap scheme, on assuming that $X \in DAN$ (cf. Theorem 3.1).

The aim of weighted bootstrap via conditioning on the bootstrap weights as in Theorem 2.1 is to provide a scheme of bootstrapping that suites the observations in hand. In other words, it specifies a method of re-weighing or re-sampling that leads to the same limit as that of the original $t$-statistic. This view of justifying the validity of the bootstrap is believed to be new for the two general forms of the bootstrapped Student $t$-statistics $T_n^*$ and $T_n^{**}$. The need for this approach to the bootstrap in general arises naturally in practice when exploring the nature of information contained in a random sample that is treated as a population, via re-sampling it, like as in Efron [11], for example, or by re-weighing methods in general.

In Section 4, we demonstrate the validity of the bootstrapped $t$-intervals for both approaches to conditioning. In particular, when conditioning on the sample, our results in this regard are new in that they are shown to hold true when $X \in DAN$, possibly with infinite variance, while the ones with $E_{X}X^2 < \infty$ when conditioning on the weights are first time results per se.

All the proofs are given in Section 5.

**Notations.** Conditioning on the bootstrap weights $v_i^{(n)}$ and conditioning on the data $X_i$, call for proper notations that distinguish the two approaches. Hence, the notation $(\Omega_X, \mathcal{F}_X, P_X)$ will stand for the probability space on which $X, X_1, X_2, \ldots$ are defined, while $(\Omega_v, \mathcal{F}_v, P_v)$ will stand for the probability space on which the triangular arrays of the bootstrap weights $v_1^{(1)}, (v_1^{(2)}, v_2^{(2)}), \ldots, (v_1^{(n)}, \ldots, v_n^{(n)}), \ldots$ are defined. In view of the independence of these sets of random variables, jointly they live on the direct product probability space $(\Omega_X \times \Omega_v, \mathcal{F}_X \otimes \mathcal{F}_v, P_{X,v} = P_X \times P_v)$. Moreover, for use throughout, for each $n \geq 1$, we let $P_{X^{(n)}}$ be a short hand notation.
for the conditional probability $P(.)|\mathcal{F}_v^{(n)}$ and, similarly, $P(.)|X(.)$ will stand for the conditional probability $P(.)|\mathcal{F}_X^{(n)}$, where $\mathcal{F}_v^{(n)} := \sigma(v_1^{(n)}, \ldots, v_n^{(n)})$ and $\mathcal{F}_X^{(n)} := \sigma(X_1, \ldots, X_n)$, respectively, with corresponding conditional expected values $E(.)|v$ and $E(.)|X$. In case of Efron’s scheme of bootstrap, we will use $w$ instead of $v$ in all these notations whenever convenient.

2 CLT via conditioning on the bootstrap weights

In this section we explore the asymptotic behavior of the weighted bootstrap via conditioning on the bootstrap weights. The major motivation for conditioning on the weights is that, when bootstrapping the i.i.d. observables $X, X_1, X_2, \ldots$, these random variables should continue to be the prime source of stochastic variation and, hence, the random samples should be the main contributors to establishing conditional CLT’s for the bootstrapped $t$-statistics as defined in (2) and (3). The following Theorem 2.1 formulates the main approach of this paper to the area of weighted bootstrap for the Student $t$-statistic. Based on a direct consequence of the Lindeberg-Feller CLT (cf. Lemma 5.1), it amounts to concluding appropriate equivalent Lindeberg-Feller type CLT’s respectively, corresponding to both versions of the following statement: as $n, m_n \rightarrow \infty$,

$$
M_n := \max_{1 \leq i \leq n} \left( \frac{v_i^{(n)}}{m_n} - \frac{1}{n} \right)^2 \Bigg/ \sum_{i=1}^{n} \left( \frac{v_i^{(n)}}{m_n} - \frac{1}{n} \right)^2 \begin{cases} 
\frac{o(1)}{n} \text{ a.s.} - P_v, \\
\frac{o_P(1)}{n}.
\end{cases}
$$

**Theorem 2.1.** Let $X, X_1, X_2, \ldots$ be real valued i.i.d. random variables with mean 0 and variance $\sigma^2$, and assume that $0 < \sigma^2 < \infty$. Put $V_{i,n} := \left| \left( \frac{v_i^{(n)}}{m_n} - \frac{1}{n} \right)X_i \right|$, $1 \leq i \leq n$, $V_n^2 := \sum_{i=1}^{n} \left( \frac{v_i^{(n)}}{m_n} - \frac{1}{n} \right)^2$, $M_n := \max_{1 \leq i \leq n} \left( \frac{v_i^{(n)}}{m_n} - \frac{1}{n} \right)^2 \Bigg/ \sum_{i=1}^{n} \left( \frac{v_i^{(n)}}{m_n} - \frac{1}{n} \right)^2$, and let $Z$ be a standard normal random variable throughout. Then, as $n, m_n \rightarrow \infty$, having
\[ M_n = o(1) \text{ a.s.} - P_v \]  \hspace{1cm} (5)

is equivalent to concluding the respective statements of (6) and (7) simultaneously as follows

\[ P_{X|v}(T_{m_n}^* \leq t) \longrightarrow P(Z \leq t) \text{ a.s.} - P_v \text{ for all } t \in \mathbb{R} \]  \hspace{1cm} (6)

and

\[ \max_{1 \leq i \leq n} P_{X|v}(V_{i,n}/(S_n V_n) > \varepsilon) = o(1) \text{ a.s.} - P_v, \text{ for all } \varepsilon > 0, \]  \hspace{1cm} (7)

and, in a similar vein, having

\[ M_n = o_P(1) \]  \hspace{1cm} (8)

is equivalent to concluding the respective statements of (9) and (10) simultaneously

\[ P_{X|v}(T_{m_n}^* \leq t) \longrightarrow P(Z \leq t) \text{ in probability} - P_v \text{ for all } t \in \mathbb{R} \]  \hspace{1cm} (9)

and

\[ \max_{1 \leq i \leq n} P_{X|v}(V_{i,n}/S_n > \varepsilon) = o_P(1), \text{ for all } \varepsilon > 0. \]  \hspace{1cm} (10)

Moreover, assume that, as \( n, m_n \to \infty \), we have for any \( \varepsilon > 0 \),

\[ P_{X|v}\left(\left| \frac{S_{m_n}^2/m_n}{\sigma^2 \sum_{i=1}^{n} \left( \frac{v_i(n)}{m_n} - \frac{1}{n} \right)^2} - 1 \right| > \varepsilon \right) = \begin{cases} o(1) \text{ a.s.} - P_v & \text{if } \varepsilon \to 0 \\ o_P(1) & \text{else} \end{cases} \]  \hspace{1cm} (11)

Then, as \( n, m_n \to \infty \), via (11), the statement of (5) is also equivalent to having (13) and (14) simultaneously as below

\[ P_{X|v}(T_{m_n}^{**} \leq t) \longrightarrow P(Z \leq t) \text{ a.s.} - P_v \text{ for all } t \in \mathbb{R} \]  \hspace{1cm} (13)

and

\[ \max_{1 \leq i \leq n} P_{X|v}(V_{i,n}/(S_{m_n}/\sqrt{m_n}) > \varepsilon) = o(1) \text{ a.s.} - P_v, \text{ for all } \varepsilon > 0, \]  \hspace{1cm} (14)

and, in a similar vein, via (12), the statement (8) is also equivalent to having (15) and (16) simultaneously as below

\[ P_{X|v}(T_{m_n}^{**} \leq t) \longrightarrow P(Z \leq t) \text{ in probability} - P_v \text{ for all } t \in \mathbb{R} \]  \hspace{1cm} (15)

and

\[ \max_{1 \leq i \leq n} P_{X|v}(V_{i,n}/(S_{m_n}/\sqrt{m_n}) > \varepsilon) = o_P(1), \text{ for all } \varepsilon > 0. \]  \hspace{1cm} (16)
For verifying the technical conditions (11) and (12) as above, one does not need to know the actual finite value of $\sigma^2$.

The essence of Theorem 2.1 is that for i.i.d. data with a finite second moment, a scheme of bootstrap for the Student $t$-statistic is valid if and only if the random weights in hand satisfy either one of the maximal negligibility conditions as in (5) or (8) for $M_n$. Thus, when conditioning on the weights, Theorem 2.1 provides an overall approach for obtaining CLT’s for bootstrap means in this context, a role that is similar to that of Theorem 2.1 of Mason and Newton [18] that provides CLT’s for generalized bootstrap means of exchangeable weights when conditioning on the sample. Incidentally, conclusion (9) of our Theorem 2.1 under the maximal negligibility conclusion (8) is a non-parametric version of the scaler scaled (not self-normalized) Theorem 3.1 of Arena-Gutiérrez and Martán [2] under their more restrictive conditions E1-E5 for exchangeable weights, where condition E4 and E5 combined yield our condition (8) in terms of exchangeable weights. In this regard we also note in passing that, at the end of Section 1.2 of his lectures on some aspects of the bootstrap [13], Giné notes that checking conditions E4-E5 of [2] sometimes require ingenuity.

When the scheme of bootstrap is specified to be Efron’s, then Corollary 2.1 hereupon to Theorem 2.1 implies the validity of this scheme for both $T_{m_n}$ and $T_{m_n}^{**}$ as follows.

**Corollary 2.1.** Consider $v_i^{(n)} = w_i^{(n)}, 1 \leq i \leq n, n \geq 1, \text{ and } M_n$ of Theorem 2.1 in terms of these re-sampling weights as in Remark 1.1, i.e., Efron’s scheme of bootstrap. Assume that $0 < \sigma^2 = \text{var}(X) < \infty$.

(a) If $m_n, n \to \infty$, in such a way that $m_n = o(n^2)$, then, mutatis mutandis, (8) is equivalent to having (9) and (10) simultaneously, and spelling out only (9), in this context it reads

$$P_{X|w}(T_{m_n}^* \leq t) \to P(Z \leq t) \text{ in probability} - P_w \text{ for all } t \in \mathbb{R}, \quad (17)$$

(b) If $m_n, n \to \infty$ in such a way that $m_n = o(n^2)$ and $n = o(m_n)$, then, mutatis mutandis again, (8) is also equivalent to having (15) and (16) simultaneously, and spelling out only (15), in this context it reads as follows

$$P_{X|w}(T_{m_n}^{**} \leq t) \to P(Z \leq t) \text{ in probability} - P_w \text{ for all } t \in \mathbb{R}. \quad (18)$$

**Remark 2.1.** It is noteworthy to note that, along the lines of the proof of the preceding corollary (cf. the second part of the proof of our Lemma 5.3), it
will be seen that for a finite number of observations $X_1, \ldots, X_n$ in hand, $S^{*2}_{m_n}$, i.e., the bootstrap version of the sample variance $S^2_n$, is an in probability-$P_{X,w}$ consistent estimator of $S^2_n$, as only $m_n =: m \to \infty$. In other words, when $EX^2_1 < \infty$, on taking $n$ to be fixed as $m_n = m \to \infty$, we have that

$$S^{*2}_{m_n} \longrightarrow S^2_n \text{ in probability} - P_{X,w}.$$  \hspace{1cm} (19)

Consequently, the bootstrap sample variance of only one large enough bootstrap sub-sample yields a consistent estimator for the sample variance $S^2_n$ of the original sample. Moreover, a similar result can be shown to also hold true for estimating the mean of the original sample $\bar{X}_n$ via taking only one large enough bootstrap sub-sample and computing its mean $\bar{X}^{*}_{m_n}$ when $n$ is fixed. In fact, in a more general setup, the consistency result (19) for characteristics of the original sample which are of the form of $U$-statistics can be found in Csörgő and Nasarî [6] (cf. Part (a) of Theorem 3.2). These results provide an alternative to the classical method, as suggested, for example, by Efron and Tibshirani [12], where the average of the bootstrapped estimators, $\bar{X}^*(b)$ of $B$ bootstrap sub-samples drawn repeatedly and independently from the original sample, is considered as an estimator for a characteristic of the sample in hand, such as $\bar{X}_n$ and $S^2_n$, for example. The validity of the average of these $B$ bootstrap estimators is then investigated as $B \to \infty$.

**Remark 2.2.** In probability-$P_w$, part (b) of Corollary 2.1 parallels (1.11) of Theorem 1.1 of Mason and Shao [19] in which they conclude that, when $EX^2 < \infty$, then for almost all realizations of the sample (i.e., for almost all samples), the conditional (on the data) distribution of $T^{**}_{m_n}$ will coincide with the standard normal distribution whenever $\lambda_1 \leq m_n/n \leq \lambda_2$ for all $n$ large enough and some constants $0 < \lambda_1 < \lambda_2 < \infty$. It would be desirable to have an a.s.-$P_w$ version of our Corollary 2.1 and to extend the in probability-$P_w$ validity of its present form to having $X \in \text{DAN}$ with $EX^2 = \infty$.

Now suppose that $v_i^{(n)} = \zeta_i$, $1 \leq i \leq n$, where $\zeta_i$ are positive i.i.d. random variables. In this case the bootstrapped $t$-statistic $T^*_{m_n}$ defined by (2) is of the form:

$$T^*_{m_n} = \frac{\sum_{i=1}^n \left( \frac{\zeta_i}{m_n} - \frac{1}{n} \right) X_i}{S_n \sqrt{\sum_{i=1}^n \left( \frac{\zeta_i}{m_n} - \frac{1}{n} \right)^2}}, \hspace{1cm} (20)$$
where \( m_n = \sum_{i=1}^{n} \zeta_i \).

The following Corollary 2.2 to Theorem 2.1 establishes the validity of this scheme of bootstrap for \( T_{m_n}^* \), as defined by (20), via conditioning on the bootstrap weights of the latter.

**Corollary 2.2.** Assume that \( 0 < \sigma^2 = \text{var}(X) < \infty \), and let \( \zeta_1, \zeta_2, \ldots \) be a sequence of positive i.i.d. random variables which are independent of \( X_1, X_2, \ldots \). Then, as \( n \to \infty \),

(a) if \( E_\zeta(\zeta_1^4) < \infty \), then, mutatis mutandis, condition (5) is equivalent to having (6) and (7) simultaneously, and spelling out only (6), in this context it reads

\[
P_{X|\zeta}(T_{m_n}^* \leq t) \to P(Z \leq t) \text { a.s.} - P_\zeta, \text{ for all } t \in \mathbb{R},
\]

(b) if \( E_\zeta(\zeta_1^2) < \infty \), then, mutatis mutandis, (8) is equivalent (9) and (10) simultaneously, and spelling out only (9), in this context it reads

\[
P_{X|\zeta}(T_{m_n}^* \leq t) \to P(Z \leq t) \text { in probability} - P_\zeta, \text{ for all } t \in \mathbb{R},
\]

where \( Z \) is a standard normal random variable.

### 3 CLT via conditioning on the sample

Efron’s bootstrapped partial sums via conditioning on the data have been the subject of intensive study and many remarkable papers can be found in the literature in this regard.

Conditioning on the data which are assumed to be in DAN, Hall [17] proved that if \( m_n, n \to \infty \), and \( \lambda_1 \leq m_n/n \leq \lambda_2 \), where \( 0 < \lambda_1 < \lambda_2 < \infty \), then there exists a sequence of positive numbers \( \{\gamma_n\}_{n=1}^\infty \) such that

\[
\frac{\sqrt{m_n}(X_{m_n}^* - \bar{X}_n)}{\gamma_n} \xrightarrow{d} N(0,1) \text{ in probability} - P_X.
\]

In the same year S. Csörgő and Mason [9] showed that under the same conditions as those assumed by Hall, i.e., \( X \in \text{DAN} \) and \( m_n/n \in [\lambda_1, \lambda_2] \) with \( 0 < \lambda_1 < \lambda_2 < \infty \) as before, the numerical constants \( \gamma_n \) in (23) can be replaced by the sample standard deviation \( S_n \), and the conclusion of (23)
remains true. Furthermore, Mason and Shao \cite{19} replaced $S_n$ by the bootstrapped sample standard deviation $S_n^*$ and, under the conditions assumed by Hall \cite{17} and S. Csörgő and Mason \cite{9}, i.e., when $m_n/n \in [\lambda_1, \lambda_2]$, they concluded that

$$T_{m_n}^* \xrightarrow{d} N(0,1) \text{ in probability}$$

if and only if $X \in DAN$, possibly with $EX^2 = \infty$. As mentioned already (cf. Remark 2.2), when $m_n/n \in [\lambda_1, \lambda_2]$, Mason and Shao \cite{19} also characterized the almost sure-$P_X$ validity (asymptotic normality) of $T_{m_n}^*$ via conditioning on the data when their variance is positive and finite.

Thus, whenever $m_n/n \in [\lambda_1, \lambda_2]$, via conditioning on the data which are in $DAN$, Mason and Shao \cite{19} established the validity in probability-$P_X$ of the Efron bootstrapped version of the $t$-statistics as in (3), as well as its almost sure-$P_X$ validity when $EX^2$ is positive and finite (cf. (1.10) and (1.11), respectively, of their Theorem 1.1). Under its condition (25) the respective conclusions of our (26) and (27) of our forthcoming Theorem 3.1 parallel those of (1.10) and (1.11) of Theorem 1.1 of Mason and Shao \cite{19}, who also noted the desirability of having (24) holding true when the data are in $DAN$ and $m_n = n$. Theorem 3.2 below relates to this question in terms of $(S_{m_n}^*/S_n)T_{m_n}^*$ (cf. (30) and Remark 3.2).

**Remark 3.1.** For a rich source of information on the topic of bootstrap we refer to the insightful survey by S. Csörgő and Rosalsky \cite{8}, in which various types of limit laws are studied for bootstrapped sums.

Among those who explored weighted bootstrapped partial sums, we mention S. Csörgő \cite{7} and Arenal-Gutiérrez et al. \cite{2}, who studied the unconditional strong law of large numbers for the bootstrap mean.

Mason and Newton \cite{18} introduced the idea of the generalized bootstrap for the sample mean that is to replace the multinomial Efron bootstrap as in our Remark 1.1 by another vector of exchangeable non-negative random variables that are also independent of the $X_i$. Their basic tool for establishing the almost sure-$P_X$ CLT consistency of their generalized bootstrap mean, as in their Theorem 2.1, is Theorem 4.1 of Hájek \cite{15} concerning the asymptotic normality of linear rank statistics. Accordingly, their Theorem 2.1 deals with the a.s.-$P_X$ asymptotic normality of exchangeable arrays of self-normalized partial sums when conditioning on the sample.

Taking a different approach form that of Mason and Newton \cite{18}, Arenal-Gutiérrez and Matrán \cite{3} developed a technique by which they derived a
Conditioning on the sample, in this section we study the validity of Efron’s scheme of bootstrap when applied to sums of i.i.d. random variables. As will be seen, in establishing a conditional CLT, given the data, the weights, $w_i^{(n)}$, as random variables, weighted by conditioning on the data, will play the dominant role. This is in contrast to the previous section, in which a weighted i.i.d. version of the Lindeberg-Feller CLT for the data, $X$, played the dominant role in deducing our Theorem 2.1.

Clearly (cf., e.g., Lemma 1.2 in S. Csörgő and Rosalsky [8]), unconditional central limit theorems result from the conditional ones in $P_v$ or $P_X$ under their respective conditions, and, in turn, this is the way bootstrap works when taking repeated bootstrap samples (cf. our Section 4). S. Csörgő and Rosalsky [8] indicate that the laws of unconditional bootstrap are “less frequently spelled out in the literature”. Hall [16], however, addresses both conditional and unconditional laws for bootstrap. S. Csörgő and Rosalsky [8] also note that, according to Hall, conditional laws are of interest to statisticians who are interested in the probabilistic aspects of the sample in hand, while the unconditional laws of bootstrap have the “classical frequency interpretation”. Accordingly, and as noted already, our approach in Section 2 is that of a statistician interested in studying the probabilistic aspects of a sample that is treated as a population, by means of conditioning on re-sampling, and/or, re-weighing the data in hand.

We wish to emphasize that in this section only Efron’s scheme of bootstrap will be considered. This is so, since the validity and establishment of the results here, to a large extent, rely on the multinomial structure of the random weights, $w_i^{(n)}$, in this scheme. On the other hand, the data are assumed to be in $DAN$, possibly with infinite variance, and studied under conditions on $n, m_n$, as $n \to \infty$, that differ from requiring $m_n/n$ to be in the interval $[\lambda_1, \lambda_2]$ with $0 < \lambda_1 < \lambda_2 < \infty$ as in Mason and Shao [19].

It is well-known that the $t$-statistic converges in distribution to a standard normal random variable if and only if the data are in $DAN$ (cf. Giné et al. [14]). The following Theorem 3.1 establishes the validity (asymptotic normality) of the Efron bootstrapped version of the $t$-statistics as in [3], based on random samples on $X \in DAN$ via conditioning on the data. It is to be compared to the similarly conditioned Theorem 1.1 of Mason and Shao [19].
Theorem 3.1. Let $X, X_1, \ldots$ be i.i.d. random variables with $X \in \text{DAN}$. Consider $T_{m_n}^{**}$ as in (3) with $X \in \text{DAN}$ and Efron’s bootstrap $\{w_i^{(n)}, 1 \leq i \leq n\}, n \geq 1$, scheme of re-sampling from random samples $\{X_i, 1 \leq i \leq n\}_{n \geq 1}$ as in (7) and Remark 1.1. If, as $n, m_n \to \infty$ so that $$\frac{m_n}{2n \log n} \to \infty,$$ then, for all $t \in \mathbb{R}$, $$P_{w_X} (T_{m_n}^{**} \leq t) \to P(Z \leq t) \text{ in probability }- P_X,$$ and, when $E_{X}X^2 < \infty$, then $$P_{w_X} (T_n^{**} \leq t) \to P(Z \leq t) \text{ a.s. }- P_X,$$ where, $Z$ is a standard normal random variable. Further to (27), if $n, m_n \to \infty$ so that, instead of (25), we have $$\frac{m_n}{n} \to \infty$$ then, when $E_{X}X^2 < \infty$, (27) continues to hold true in probability-$P_X$.

The next result relates to a question raised by Mason and Shao [19] asking if the conditional CLT in (26) held true when $m_n = n$. According to the following Theorem 3.2, the answer is positive if one replaces $T_{m_n}^{**}$ by

$$T_{m_n, S_n}^{**} := \frac{\sum_{i=1}^{n} (\frac{w_i^{(n)}}{m_n} - \frac{1}{n})X_i}{S_n/\sqrt{m_n}} = \frac{S^*_n}{S_n} T_{m_n}^{**}. \quad (29)$$

Theorem 3.2. Let $X, X_1, \ldots$ be i.i.d. random variables with $X \in \text{DAN}$. Consider Efron’s bootstrap scheme as in Theorem 3.1. If, as $n, m_n \to \infty$, so that for an arbitrary $\varepsilon > 0$ we have $\frac{m_n}{n} \geq \varepsilon > 0$, then, for all $t \in \mathbb{R}$,$$P_{w_X} (T_{m_n, S_n}^{**} \leq t) \to P(Z \leq t) \text{ in probability }- P_X,$$ where $Z$ is a standard normal random variable.

Remark 3.2. On taking $m_n = n$, Theorem 3.2 continues to hold true as before, but now in terms of $$T_{n, S_n}^{**} = \frac{S^*_n}{S_n} T_n^{**}.$$
The conclusion of (30) coincides with that of (5.2) of S. Csörgő and Mason [9], who, as mentioned right after (23) above, concluded it for $X \in DAN$ whenever, $m_n/n \in [\lambda_1, \lambda_2]$ with $0 < \lambda_1 < \lambda_2 < \infty$. Thus, for our conclusion in (30), we may take $m_n/n \in [\lambda_1, \lambda_2]$ with $0 < \lambda_1 < \lambda_2 < \infty$. Thus, for our conclusion in (30), we may take $m_n/n \in (\lambda_1, \lambda_2)$ and conclude also Remark 3.2 with $m_n = n$ that was first established by Athreya [4]. For further comments along these lines we refer to Section 5 of S. Csörgő and Mason [9].

4 Validity of Bootstrapped $t$-intervals

In order to establish an asymptotic confidence bound for $\mu = E(X)$ with an asymptotic probability coverage of size $\alpha$, $0 < \alpha \leq 1$, using the classical CLT, one can use the classical Student pivot $T_n$ via setting $T_n \leq z_\alpha$, where $P(Z \leq z_\alpha) = \alpha$. One can also establish an asymptotic size $\alpha$ bootstrap confidence bound for $\mu$ by taking $B \geq 1$ bootstrap sub-samples of size $m_n$ via re-sampling or by generating $B$ sets of stochastically reweighed bootstrap sub-samples of \{\$X_i, 1 \leq i \leq n\$\} independently (i.e., each set of the $B$ bootstrap weights are independent). The latter can be done by simulating $B$ sets of independent i.i.d. weights $(\zeta_1^{(b)}, \ldots, \zeta_n^{(b)})$, $1 \leq b \leq B$. Obviously, the independence of the bootstrap weights with respect to the probability $P_v$ does not imply the independence of the thus generated sub-samples with respect to the joint distribution of the data and the bootstrap weights. One will have $B$ values of $T_{m_n}^*(b)$ and/or $T_{m_n}^{**}(b)$ or $T_n^*(b)$, $1 \leq b \leq B$, and respective asymptotic 100 $\alpha\%$ bootstrap confidence bounds will result, as in the upcoming Theorems 4.1 and 4.2 from the inequalities

$$T_n \leq C_{s, \alpha}^{(B)}(b), \quad s = 1, 2, 3, 4$$

where

$$C_{1, \alpha}^{(B)} := \inf \{t : \frac{1}{B} \sum_{b=1}^{B} I(T_{m_n}^*(b) \leq t) \geq \alpha\},$$

$$C_{2, \alpha}^{(B)} := \inf \{t : \frac{1}{B} \sum_{b=1}^{B} I(T_{m_n}^{**}(b) \leq t) \geq \alpha\},$$

$$C_{3, \alpha}^{(B)} := \inf \{t : \frac{1}{B} \sum_{b=1}^{B} I(T_{m_n,S_n}^{**}(b) \leq t) \geq \alpha\},$$

$$C_{4, \alpha}^{(B)} := \inf \{t : \frac{1}{B} \sum_{b=1}^{B} I(T_n^*(b) \leq t) \geq \alpha\},$$

$$15$$
and $T_n$ is the Student $t$-statistic as in (4).

Observe that $C_{s,\alpha}^{(B)}$, $s = 1, 2, 3, 4$, are bootstrap estimations of the respective 100$\alpha$ percentile of the distributions $P_{X,v}(T_{m_n} \leq t)$, $P_{X,v}(T_{m_n}^{**} \leq t)$, $P_{X,v}(T_{m_n,S_n} \leq t)$ and $P_{X,v}(T_n^{**} \leq t)$. Moreover, since $C_{s,\alpha}^{(B)}$ are the 100$\alpha$ percentiles of their respective empirical distributions, therefore they coincide with their respective order statistics $T_{m_n}^{(l)}$, $T_{m_n}^{**(l)}$, $T_{m_n,S_n}^{(l)}$ and $T_n^{**(l)}$, where $l = \lceil \alpha(B + 1) \rceil$.

We note that $C_{s,\alpha}^{(B)}$, $s = 1, 2, 3, 4$, are natural extensions of S. Csörgő and Mason’s [9] approach to establishing the validity of bootstrapped empirical processes. Some ideas that are used in the proofs of the results in this section were borrowed from [9] and adapted accordingly.

The objective of this section is to show that in the light of Theorems 2.1, 3.1 and 3.2, the confidence bounds obtained from (31) will achieve the nominal coverage probability $\alpha$ as $n, m_n$ and $B \to \infty$. More precisely, in Theorem 4.1 below we consider the confidence bound as in (31) and Efron’s scheme of bootstrap, and show that the asymptotic nominal coverage probability $\alpha$ will be achieved. Moreover, the latter will be shown to be true via conditioning on the bootstrap weights and also via conditioning on the data. In Theorem 4.2 we consider the confidence bound in (31) with $C_{4,\alpha}^{(B)}$ when the scheme of bootstrap is stochastically re-weighing and via conditioning on the bootstrap weights, we show that the asymptotic nominal coverage probability $\alpha$ will again be achieved.

Thus, both approaches to the bootstrap will be shown to work, namely, as in (a) of Theorem 4.1 and as in Theorem 4.2 when conditioning on the weights, and as in (b) and (c) of Theorem 4.1 when conditioning on the data.

In order to state the just mentioned conclusions, one needs to define an appropriate probability space for accommodating the presence of $B$ bootstrap sub-samples, as $B \to \infty$. This means that one has to incorporate $B$ i.i.d. sets of weights

$$\left( (v_1^{(1)}(b), (v_1^{(2)}(b), v_2^{(2)}(b)), \ldots, (v_1^{(n)}(b), \ldots, v_n^{(n)}(b)), \ldots), \right),$$

which live on their respective probability spaces $(\Omega_v(b), \mathcal{F}_v(b), P_v(b))$, $b \geq 1$. In view of this, and due to the fact that $n, m_n$ and $B$ will approach $\infty$, we let $(\otimes_{b=1}^{\infty} \Omega_v(b), \otimes_{b=1}^{\infty} \mathcal{F}_v(b), \otimes_{b=1}^{\infty} P_v(b))$ be the probability space on which the
following row-wise i.i.d. array of bootstrap weights are defined:

\[
\begin{align*}
&v^{(1)}_1, (v^{(2)}_1, v^{(2)}_2), (v^{(3)}_1, v^{(3)}_2, v^{(3)}_3), \ldots, \\
v^{(1)}_2, (v^{(2)}_1, v^{(2)}_2), (v^{(3)}_1, v^{(3)}_2, v^{(3)}_3), \ldots, \\
&: : : : : \\
v^{(1)}_i, (v^{(2)}_1, v^{(2)}_2), (v^{(3)}_1, v^{(3)}_2, v^{(3)}_3), \ldots.
\end{align*}
\]

In what follows, we let \((\otimes_{b=1}^{\infty} \Omega_{X,v(b)}; \otimes_{b=1}^{\infty} \mathfrak{F}_{X,v(b)}; \otimes_{b=1}^{\infty} P_{X,v(b)})\) be the joint probability space of the \(X\)'s and the preceding array of the weights \(v(b)\), \(b \geq 1\).

**Theorem 4.1.** Consider Efron’s scheme of bootstrap, i.e., \(v_i^{(n)} = w_i^{(n)}\), \(1 \leq i \leq n, n \geq 1\).

(a) Assume the conditions of Corollary 2.1. Then, as \(n,m,n,B \to \infty\),

\[
C_{1,\alpha}^{(B)} \to z_{\alpha} \text{ in probability } \quad \otimes_{b=1}^{\infty} P_{X,w(b)}.
\]

(b) Assume the conditions of Theorems 3.1. Then, as \(n,m,n,B \to \infty\),

\[
C_{2,\alpha}^{(B)} \to z_{\alpha} \text{ in probability } \quad \otimes_{b=1}^{\infty} P_{X,w(b)}.
\]

(c) Assume the conditions of Theorem 3.2. Then, as \(n,m,n,B \to \infty\),

\[
C_{3,\alpha}^{(B)} \to z_{\alpha} \text{ in probability } \quad \otimes_{b=1}^{\infty} P_{X,w(b)}.
\]

**Theorem 4.2.** Suppose that \(v_i^{(n)} = \zeta_i, 1 \leq i \leq n\), and put \(m_n = \sum_{i=1}^{n} \zeta_i\). Assume the conditions of Corollary 2.2. Then, as \(n,B \to \infty\),

\[
C_{4,\alpha}^{(B)} \to z_{\alpha} \text{ in probability } \quad \otimes_{b=1}^{\infty} P_{X,\zeta(b)}.
\]

When conditioning on the sample, the validity of the bootstrap confidence intervals was also studied by Hall [16] when, with some \(\delta > 0\), \(E_X X^{4+\delta} < \infty\) and \(m_n = n\). Our conclusions in (b) and (c) hold true when \(X \in DAN\), possibly with infinite variance. Conclusion (a) of Theorem 4.1 and that of Theorem 4.2 are first time results for establishing the validity of bootstrap confidence intervals via conditioning on the weights when \(E_X X^2 < \infty\).
5 Proofs

The proof of Theorem 2.1 is based on the following Lemma 5.1 that amounts to a realization of the Lindeberg-Feller CLT.

**Lemma 5.1.** Let $X, X_1, \ldots$ be real valued i.i.d. random variables with mean 0 and variance $0 < \sigma^2 < \infty$ on $(\Omega_X, \mathcal{F}_X, P_X)$, as before, and let $\{a_{i,n}\}_{i=1}^n$, $n \geq 1$ be a triangular array of real valued constants. Then, as $n \to \infty$,

$$M_n = \frac{\max_{1 \leq i \leq n} a_{i,n}^2}{\sum_{i=1}^n a_{i,n}^2} \to 0,$$  

(32)

if and only if

$$\frac{\sum_{i=1}^n a_{i,n}X_i}{\sigma \sqrt{\sum_{i=1}^n a_{i,n}^2}} \to_d N(0,1),$$

and, for all $\varepsilon > 0$, $\max_{1 \leq i \leq n} P_X\left(\frac{|a_{i,n}X_i|}{\sigma \sqrt{\sum_{i=1}^n a_{i,n}^2}} > \varepsilon\right) \to 0$  

(33)

or, equivalently, if and only if

$$\frac{\sum_{i=1}^n a_{i,n}X_i}{S_n \sqrt{\sum_{i=1}^n a_{i,n}^2}} \to_d N(0,1),$$

and, for all $\varepsilon > 0$, $\max_{1 \leq i \leq n} P_X\left(\frac{|a_{i,n}X_i|}{S_n \sqrt{\sum_{i=1}^n a_{i,n}^2}} > \varepsilon\right) \to 0$,  

(34)

where $N(0,1)$ stands for a standard normal random variable, and $S_n$ is the sample variance of the first $n \geq 1$ of the mean 0 and variance $\sigma^2$ i.i.d. sequence $X, X_1, X_2, \ldots$ of random variables.

**Proof of Lemma 5.1**

The equivalence of the respective two statements of (33) and (34) is an immediate consequence of Slutsky’s theorem via having $S_n^2 \to \sigma^2$ in probability as $n \to \infty$. Hence, it suffices to establish the equivalence of the statement (32) to the two simultaneous statements of (33).

First assume that we have (32) and show that it implies Lindeberg’s conditions that in our context reads as follows: with $F(x) = P_X(X \leq x)$,

$$L_n(\varepsilon) := \frac{1}{\sigma^2 \sum_{i=1}^n a_{i,n}^2} \sum_{i=1}^n a_{i,n}^2 \int_{|a_{i,n}x| > \varepsilon \sigma \sqrt{\sum_{i=1}^n a_{i,n}^2}} x^2 dF(x) \to 0$$  

(35)
for each $\varepsilon > 0$, as $n \to \infty$. Now observe that $L_n(\varepsilon)$ can be bounded above by

$$
\frac{1}{\sigma^2} \int_{|x| > \sigma} \sqrt{\frac{\sum_{i=1}^{n} a_{i,n}^2}{\max_{1 \leq i \leq n} a_{i,n}^2}} x^2 dF(x) \to 0, \text{ as } n \to \infty,
$$

(36)
on assuming (32) and $EX^2 = \int x^2 dF(x)$, i.e., (32) implies (35). The latter, in turn, implies the Lindeberg CLT statement of (33). Moreover, by Chebyshev’s inequality, via (32) we conclude also the second, the so-called uniform asymptotic uniform negligibility condition statement of (33). Thus, we now have that (32) implies (33).

Conversely, on assuming now (33), its Lindeberg-Feller type simultaneous conclusions imply the Lindeberg condition of (35), as per the Lindeberg-Feller CLT, and (35) yields (32). □

**Proof of Theorem 2.1**

In view of Lemma 5.1, the a.s.-$P_v$ equivalence of (5) to (6)-(7) and, via (11), that of (5) to (13)-(14) hold true along a set $N \in \mathcal{F}_v$ with $P_v(N) = 1$.

As for the in probability-$P_v$ equivalence of (8) to (9)-(10) and, via (12), also to (15)-(16), they hold true via the characterization of convergence in probability in terms of a.s. convergence of subsequences. Accordingly, for each subsequence $\{n_k\}_k$ of $n$, $n \geq 1$, there exists a further subsequence $\{n_{k\ell}\}_\ell$ along which, as $\ell \to \infty$, by virtue of Lemma 5.1, the latter two in probability-$P_v$ equivalencies reduce to appropriate a.s.-$P_v$ equivalences. This also completes the proof of Theorem 2.1. □

**Proof of Corollary 2.1**

Here the bootstrap weights $v_i^{(n)} = w_i^{(n)}$, $1 \leq i \leq n$, $n \geq 1$, are as in Remark 1.1, i.e., for each $n \geq 1$,

$$
\left( w_1^{(n)}, \ldots, w_n^{(n)} \right) \overset{d}{=} \text{multinomial}\left( m_n, \frac{1}{n}, \ldots, \frac{1}{n} \right),
$$

with $m_n = \sum_{i=1}^{n} w_i^{(n)}$. In view of Theorem 2.1 part (a) of Corollary 2.1 will follow from the following Lemma 5.2 and Lemmas 5.2 and 5.3 together will conclude part (b). □

We now state and prove Lemmas 5.2 and Lemma 5.3.
Lemma 5.2. Consider Efron’s scheme of bootstrap and assume that $\sigma^2 = \text{var}(X) < \infty$. If $m_n, n \to \infty$ in such a way that $m_n = o(n^2)$, then,

$$\begin{align*}
M_n &= \frac{\max_{1 \leq i \leq n} \left( \frac{w(n)}{m_n} - \frac{1}{n} \right)^2}{\sum_{i=1}^{n} \left( \frac{w(n)}{m_n} - \frac{1}{n} \right)^2} 
&\to 0 \quad \text{in probability} - P_w.
\end{align*}$$

Lemma 5.3. Consider Efron’s scheme of bootstrap and assume that $0 < \sigma^2 = \text{var}(X) < \infty$. As $m_n, n \to \infty$ in such a way that $m_n = o(n^2)$ and $n = o(m_n)$, then,

$$\begin{align*}
P_{X|w} \left( \left| \frac{S_{m_n}^{\ast 2}}{\sigma^2 \sum_{i=1}^{n} (\frac{w(n)}{m_n} - \frac{1}{n})^2} - 1 \right| > \varepsilon \right) &\to 0 \quad \text{in probability} - P_w.
\end{align*}$$

Proof of Lemma 5.2

In order to prove this lemma, for $\varepsilon, \varepsilon' > 0$, we write:

$$\begin{align*}
P_w \left( \frac{\max_{1 \leq i \leq n} \left( \frac{w(n)}{m_n} - \frac{1}{n} \right)^2}{\sum_{i=1}^{n} \left( \frac{w(n)}{m_n} - \frac{1}{n} \right)^2} > \varepsilon \right)
&\leq P_w \left( \frac{\max_{1 \leq i \leq n} \left( \frac{w(n)}{m_n} - \frac{1}{n} \right)^2}{\sum_{i=1}^{n} \left( \frac{w(n)}{m_n} - \frac{1}{n} \right)^2} > \varepsilon, \left| \frac{m_n}{(1 - \frac{1}{n})} \sum_{i=1}^{n} \left( \frac{w(n)}{m_n} - \frac{1}{n} \right)^2 - 1 \right| \leq \varepsilon' \right)

&+ P_w \left( \left| \frac{m_n}{(1 - \frac{1}{n})} \sum_{i=1}^{n} \left( \frac{w(n)}{m_n} - \frac{1}{n} \right)^2 - 1 \right| > \varepsilon' \right)

&= P_w \left( \max_{1 \leq i \leq n} \left( \frac{w(n)}{m_n} - \frac{1}{n} \right)^2 > \varepsilon \frac{(1 - \varepsilon')(1 - \frac{1}{n})}{m_n} \right)

&+ P_w \left( \left| \sum_{i=1}^{n} \left( \frac{w(n)}{m_n} - \frac{1}{n} \right)^2 - \frac{1}{m_n} \right| > \varepsilon' \frac{(1 - \frac{1}{n})}{m_n} \right)

=: L_1(n) + L_2(n).
\end{align*}$$
An upper bound for $L_1(n)$ is:

$$L_1(n) \leq nP_w\left(\left|\frac{w_i^{(n)}}{m_n} - \frac{1}{n}\right| > \sqrt{\frac{\varepsilon(1 - \varepsilon')(1 - \frac{1}{n})}{m_n}}\right)$$

$$\leq n \exp\left\{\frac{\varepsilon^2(1 - \varepsilon')(1 - \frac{1}{n})}{2\left(\frac{\sum m_i}{n} + \sqrt{\varepsilon(1 - \varepsilon')(1 - \frac{1}{n})}\right)}\right\}.$$  

The preceding relation, which is due to Bernstein’s inequality, is a general term of a finite series when $m_n = O(n^2)$. As for $L_2(n)$, we first note that for each $i, 1 \leq i \leq n$,

$$E_w\left(\frac{w_i^{(n)}}{m_n} - \frac{1}{n}\right)^2 = E_w\left(\frac{w_i^{(n)}}{m_n} - \frac{1}{n}\right)^2 = \frac{(1 - \frac{1}{n})^2}{n m_n}.$$  

We now employ Chebyshev’s inequality to bound $L_2(n)$ above as follows.

$$L_2(n) \leq \frac{m_n^2}{\varepsilon^2 (1 - \frac{1}{n})^2}E_w\left(\sum_{i=1}^{n}\left(\frac{w_i^{(n)}}{m_n} - \frac{1}{n}\right)^2 - \frac{(1 - \frac{1}{n})^2}{m_n}\right)^2$$

$$= \frac{m_n^2}{\varepsilon^2 (1 - \frac{1}{n})^2}\left\{E_w\left(\sum_{i=1}^{n}\left(\frac{w_i^{(n)}}{m_n} - \frac{1}{n}\right)^2 - \frac{(1 - \frac{1}{n})^2}{m_n}\right)^2\right\}$$

$$= \frac{m_n^2}{\varepsilon^2 (1 - \frac{1}{n})^2}\left\{nE_w\left(\frac{w_1^{(n)}}{m_n} - \frac{1}{n}\right)^4 + n(n-1)E_w\left(\left(\frac{w_1^{(n)}}{m_n} - \frac{1}{n}\right)^2\left(\frac{w_2^{(n)}}{m_n} - \frac{1}{n}\right)^2\right) - \frac{(1 - \frac{1}{n})^2}{m_n}\right\}.$$  

In view of the fact that $w_i^{(n)}, 1 \leq i \leq n$ have multinomial distribution, after computing $E_w\left[(w_1^{(n)})^a(w_2^{(n)})^b\right]$, where $a, b$ are two integers such that $0 \leq a, b \leq 2$, followed by some algebra, we can bound the preceding term by

$$\frac{m_n^2}{\varepsilon^2 (1 - \frac{1}{n})^2}\left\{(1 - \frac{1}{n})^4 + \frac{(1 - \frac{1}{n})^4}{m_n^3} + \frac{(m_n - 1)(1 - \frac{1}{n})^2}{nm_n^3} + \frac{4(n-1)}{n^3m_n^3} + \frac{1}{nm_n^2} - \frac{1}{n^2m_n^2} + \frac{n-1}{n^3m_n^2} + \frac{4(n-1)}{n^2m_n^3} - \frac{(1 - \frac{1}{n})^2}{m_n^2}\right\}$$

$$\sim \frac{1}{\varepsilon^2}\left\{\frac{4m_n}{n^2} + \frac{1}{n^3m_n} + \frac{1}{m_n} + \frac{1}{n^2} + \frac{4}{nm_n}\right\},$$

where $a_n \sim b_n$ stands for the asymptotic equivalence of numerical sequences $a_n$ and $b_n$.

Clearly, as $n, m_n \to \infty$, the preceding relation approaches zero when $m_n = o(n^2)$. Now the proof of Lemma 5.2 is complete. □
Proof of Lemma 5.3

For $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ we have,

$$P_w(P_{X|w} \left( \left| \frac{S^2_{m_n}}{m_n} - \sigma^2 \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right)^2 \right| > \varepsilon_1 \right) > \varepsilon_2)$$

$$= P_w \left( \left\{ P_{X|w} \left( \left| \frac{S^2_{m_n}}{m_n} - \sigma^2 \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right) \right| > \varepsilon_1 \right) > \varepsilon_2 \right) \right)$$

$$\leq P_w \left( P_{X|w} \left( \left| \frac{S^2_{m_n}}{m_n} - \sigma^2 \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right)^2 \right| > \varepsilon_1 \right) > \varepsilon_2, \left| \frac{m_n}{(1 - \frac{1}{n})} \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right)^2 - 1 \right| \leq \varepsilon_3 \right)$$

$$+ P_w \left( \sum_{i=1}^n \left| \frac{m_n}{(1 - \frac{1}{n})} \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right) \right| > \varepsilon_3 \right)$$

$$\leq P_w \left( P_{X|w} \left( \left| \frac{S^2_{m_n}}{m_n} - \sigma^2 \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right)^2 \right| > \frac{\sigma^2 \varepsilon_1 (1 - \varepsilon_3) (1 - \frac{1}{n})}{m_n} \right) > \varepsilon_2 \right)$$

$$+ P_w \left( \sum_{i=1}^n \left| \frac{m_n}{(1 - \frac{1}{n})} \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right) \right| > \varepsilon_3 \right)$$

$$=: t_1(n) + t_2(n).$$

We note that along the lines of the proof of Lemma 5.2, it was already shown that, when $m_n = o(n^2)$, as $n \to \infty$, then $t_2(n) \to 0$.

To show that $t_1(n) \to 0$, as $n \to \infty$, we proceed as follows.

$$t_1(n) \leq P_w(P_{X|w} \left( \frac{S^2_{m_n}}{m_n} > \frac{\sigma^2 \varepsilon_1 (1 - \varepsilon_3) (1 - \frac{1}{n})}{m_n} \right) > \varepsilon_2)$$

$$+ P_w \left( \frac{S^2_{m_n}}{m_n} > \frac{\sigma^2 (1 - \frac{1}{n})}{m_n} \right) > \varepsilon_2$$

$$+ P_w \left( \frac{\sum_{i=1}^n \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right)^2 - (1 - \frac{1}{n})}{m_n} > \frac{\varepsilon_1 (1 - \varepsilon_3) (1 - \frac{1}{n})}{m_n} \right) > \varepsilon_2$$

$$=: t_1^{(1)}(n) + t_1^{(2)}(n) + t_1^{(3)}(n).$$

Now from the $U$-statistic representation of the sample variance we have that
\[ S^2_{m_n} - S^2_n = \sum_{1 \leq i \neq j \leq n} \left( \frac{w_i^{(n)} w_j^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right) (X_i - X_j)^2. \]

Therefore, \( t_1^{(1)}(n) \) can be bounded above by

\[
P_w \left( \left| \sum_{1 \leq i \neq j \leq n} \left( \frac{w_i^{(n)} w_j^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right) |X_i - X_j|^2 \right| > \sigma^2 \varepsilon_1 (1 - \varepsilon_3)(1 - \frac{1}{n}) > \frac{\varepsilon_2}{3} \right)
\]

\[
\leq P_w \left( E_{X|w} \left( \left| \sum_{1 \leq i \neq j \leq n} \left( \frac{w_i^{(n)} w_j^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right) \right| > \sigma^2 \varepsilon_1 (1 - \varepsilon_3)(1 - \frac{1}{n}) \frac{\varepsilon_2}{3} \right) \right)
\]

\[
\leq P_w \left( \sum_{1 \leq i \neq j \leq n} \left| \frac{w_i^{(n)} w_j^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right| E_{X|w} (X_i - X_j)^2 > \sigma^2 \varepsilon_1 (1 - \varepsilon_3)(1 - \frac{1}{n}) \frac{\varepsilon_2}{3} \right)
\]

\[
\leq P_w \left( \sum_{1 \leq i \neq j \leq n} \left| \frac{w_i^{(n)} w_j^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right| > \varepsilon_1 (1 - \varepsilon_3)(1 - \frac{1}{n}) \frac{\varepsilon_2}{6} \right).
\]

For ease of notation we set \( \varepsilon_n := \varepsilon_1 (1 - \varepsilon_3)(1 - \frac{1}{n}) \frac{\varepsilon_2}{6} \). Using this, the preceding term can be bounded above by

\[
\varepsilon_n^{-2} \left\{ n(n - 1) E_w \left( \left( \frac{w_1^{(n)} w_2^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right)^2 \right) \right. \\
+ n(n - 1)(n - 2) E_w \left( \left| \frac{w_1^{(n)} w_2^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right| \left| \frac{w_3^{(n)} w_4^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right| \right) \\
+ n(n - 1)(n - 2)(n - 3) E_w \left( \left| \frac{w_1^{(n)} w_2^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right| \left| \frac{w_3^{(n)} w_4^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right| \right) \left\} \right.
\]

\[
\leq \varepsilon_n^{-2} \left\{ n(n - 1) E_w \left( \left( \frac{w_1^{(n)} w_2^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right)^2 \right) \right. \\
+ n(n - 1)(n - 2) E_w \left( \left( \frac{w_1^{(n)} w_2^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right)^2 \right) \\
+ n(n - 1)(n - 2)(n - 3) E_w \left( \left( \frac{w_1^{(n)} w_2^{(n)}}{m_n(m_n - 1)} - \frac{1}{n(n - 1)} \right)^2 \right) \right. \\
\sim \varepsilon_n^{-2} \left\{ \frac{n^2}{n^2 m_n^2} + \frac{n^3}{n^2 m_n^2} + \frac{n^4}{n^2 m_n^2} \right\} \rightarrow 0.
\]
The preceding conclusion, which implies that \( t_1^{(1)}(n) \to 0 \), is true since, as \( n \to \infty \), \( \varepsilon_n \to \varepsilon_1(1 - \varepsilon_3)\frac{\varepsilon_2}{6} \) and \( n = o(m_n) \) by assumption, as \( n, m_n \to \infty \). Moreover, we note that the preceding convergence to 0 also takes place when \( n \), the number of the original observations, is fixed and \( m_n := m \to \infty \) (cf. Remark 2.1).

To show \( t_1^{(2)}(n) \to 0 \), as \( n \to \infty \), we note that

\[
t_1^{(2)} \leq P \left( |S_n^2 - \sigma^2| > \sigma^2 \varepsilon_1(1 - \varepsilon_3)(1 - \frac{1}{n}) \right) > \frac{\varepsilon_2}{3}
\]

\[
\leq 3\varepsilon_2^{-1} P \left( |S_n^2 - \sigma^2| > \sigma^2 \varepsilon_1(1 - \varepsilon_3)(1 - \frac{1}{n}) \right) \to 0.
\]

To deal with \( t_1^{(3)}(n) \), we observe that it can be bounded above by

\[
3\varepsilon_2^{-1} P \left( \left| \frac{1 - \frac{1}{n}}{m_n} \sum_{i=1}^{n} \frac{w_i^{(n)} - \frac{1}{n}}{m_n} \right| > \varepsilon_1(1 - \frac{1}{n}) \right).
\]

Once again we note that during the proof of Lemma 5.2 it was shown that when \( m_n = o(n^2) \), as \( n \to \infty \), the preceding term approaches zero, i.e., \( t_1^{(3)}(n) \to 0 \). We now conclude that, as \( n \to \infty \), \( t_1(n) \to 0 \), and the latter also completes the proof of Lemma 5.3. \( \square \)

**Proof of Corollary 2.2**

Recall that \( m_n := \sum_{i=1}^{n} \zeta_i = n \frac{m_n}{\bar{\zeta}_n} = n \bar{\zeta}_n \). In view of Theorem 2.1, the proof of parts (a) and (b) of Corollary 2.2 will result from showing that, as \( n \to \infty \),

\[
M_n = \frac{\max_{1 \leq i \leq n} \left( \frac{\zeta_i}{m_n} - \frac{1}{n} \right)^2}{\sum_{i=1}^{n} \left( \frac{\zeta_i}{m_n} - \frac{1}{n} \right)^2} = \begin{cases} o(1) \text{ a.s. - } P \zeta \text{ when } E\zeta(\zeta_1^4) < \infty \quad (37) \\ o_P(1) \text{ when } E\zeta(\zeta_1^2) < \infty. \quad (38) \end{cases}
\]

Since \( \zeta_i \)'s are positive random variables, we have

\[
M_n = \frac{\max_{1 \leq i \leq n} \left( \frac{\zeta_i}{m_n} - \frac{1}{n} \right)^2}{\sum_{i=1}^{n} \left( \frac{\zeta_i}{m_n} - \frac{1}{n} \right)^2} = \frac{\max_{1 \leq i \leq n} (\zeta_i - \bar{\zeta}_n)^2}{\sum_{1 \leq i \leq n} (\zeta_i - \bar{\zeta}_n)^2}.
\]

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In view of Kolmogorov’s strong law of large numbers, when \( E_\zeta(\zeta_1^2) < \infty \), we have that, as \( n \to \infty \),

\[
\sum_{i=1}^{n} (\zeta_i - \bar{\zeta}_n)^2 / n \to \text{var}(\zeta_1) \text{ a.s.} - P_\zeta.
\]

Also,

\[
\max_{1 \leq i \leq n} \left| \zeta_i - \bar{\zeta}_n \right| \leq \frac{2 \max_{1 \leq i \leq n} \zeta_i}{\sqrt{n}}.
\]

Therefore, to prove parts (a) and (b) of Corollary (2.2), it suffices to, respectively, show that, as \( n \to \infty \),

\[
\max_{1 \leq i \leq n} \frac{\zeta_i}{\sqrt{n}} = \begin{cases} 
  o(1) \text{ a.s.} - P_\zeta \text{ when } E_\zeta(\zeta_1^4) < \infty \\
  o_p(1) \text{ when } E_\zeta(\zeta_1^2) < \infty.
\end{cases}
\]  

(39)  

(40)

To establish (39), for \( \varepsilon > 0 \), we write

\[
\sum_{n=1}^{\infty} n P_\zeta(\zeta_1 > \varepsilon \sqrt{n}) \leq \sum_{n=1}^{\infty} E_\zeta\{\zeta_1^2 I(|\zeta_1| > \varepsilon \sqrt{n})\}
\]

\[
= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} E_\zeta\{\zeta_1^2 I(\varepsilon \sqrt{k} < \zeta_1 \leq \varepsilon \sqrt{k+1})\}
\]

\[
\leq \sum_{k=1}^{\infty} E_\zeta\{\zeta_1^2 I(\varepsilon \sqrt{k} < \zeta_1 \leq \varepsilon \sqrt{k+1})\}
\]

\[
\leq \sum_{k=1}^{n} E_\zeta\{\zeta_1^4 I(\varepsilon \sqrt{k} < \zeta_1 \leq \varepsilon \sqrt{k+1})\}
\]

\[
= \varepsilon E_\zeta(\zeta_1^4) < \infty.
\]

In order to prove (40) for \( \varepsilon > 0 \), we continue as follows.

\[
nP_\zeta(\zeta_1 > \varepsilon \sqrt{n}) \leq \varepsilon^{-2} E_\zeta(\zeta_1^2 I(\zeta_1 > \varepsilon \sqrt{n})) \to 0, \text{ as } n \to \infty.
\]

This also completes the proof of (40) and that of Corollary 2.2. \( \square \)

**Proof of Theorem 3.1 and Theorem 3.2**

We first prove Theorem 3.2.
Proof of Theorem 3.2

In order to prove this theorem, first define

\[ T_{m_n}^*(\mu) := \frac{\sum_{i=1}^{n} \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right) (X_i - \mu)}{\sqrt{\frac{(1 - \frac{1}{n})}{n m_n} \sum_{i=1}^{n} (X_i - \mu)^2}}. \]  

(41)

Recall that \((w_1^{(n)}, \ldots, w_n^{(n)}) \sim \text{multinomial}(m_n, \frac{1}{n}, \ldots, \frac{1}{n})\) for each \(n \geq 1\). Hence, by virtue of Corollary 4.1 of [20], conditioning on the data, \(T_{m_n}^*(\mu)\) is a properly normalized linear function of \(w_1^{(n)}, \ldots, w_n^{(n)}\). The term properly normalized is used since

\[ \sum_{i=1}^{n} \text{var}_{w|X} \left( \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right) (X_i - \mu) \right) = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{nm_n} (1 - \frac{1}{n}). \]

The latter normalizing sequence is that of the CLT of Corollary 4.1 of [20]. It will be first shown that the conditions under which Corollary 4.1 of Morris [20] holds true are satisfied in probability \(P_X\). Then, by making use of the characterization of convergence in probability by almost sure convergence of subsequences, it will be concluded that for each subsequence \(\{n_{\ell}\}_{\ell=1}^{\infty}\) of \(\{n\}_{n=1}^{\infty}\), there is a further subsequence \(\{n_{\ell_s}\}_{s=1}^{\infty}\), along which, from Corollary 4.1 of Morris [20], \(T_{m_n}^*(\mu)\), conditionally on the sample, converges in distribution to standard normal a.s. \(- P_X\). The latter means that, \(\forall t \in \mathbb{R}\)

\[ P_{w|X}(T_{m_n}^*(\mu) \leq t) \to P(Z \leq t) \text{ in probability} - P_X, \]  

(42)

where \(Z \overset{d}{=} \mathcal{N}(0, 1)\).

The conditions of Corollary 4.1 of Morris [20] are satisfied, for one has

(a) \(\frac{m_n}{n} \geq \varepsilon > 0\), assumed,

(b) \(\max_{1 \leq i \leq n} \left( \frac{1}{n} \right) \to 0\), as \(n \to \infty\),

(c) \(\frac{\max_{1 \leq i \leq n} \text{var}_{w|X} \left( \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right) (X_i - \mu) \right)}{\sum_{i=1}^{n} \text{var}_{w|X} \left( \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right) (X_i - \mu) \right)} = \frac{\max_{1 \leq i \leq n} (X_i - \mu)^2}{\sum_{i=1}^{n} (X_i - \mu)^2} \to 0.\)
The latter holds true in probability-$P_X$.

Conclusion (c) is a characterization of $X \in DAN$ (cf., e.g., [14]). In view of (a), (b) and (c), one can conclude that (42) holds true.

Now observe that for $T_{m_n,s_n}^*$, as defined in (29), we have

$$T_{m_n,s_n}^* = \frac{S_n}{\sqrt{(1-\frac{1}{n}) \sum_{i=1}^{n} (X_i - \mu)^2}} T_{m_n}^*(\mu).$$

Via Slutsky’s theorem in probability—$P_X$, one will have, $\forall t \in \mathbb{R}$, as $n, m_n \to \infty$ as in (25)

$$P_{w|X}(T_{m_n,s_n}^* \leq t) \to P(Z \leq t) \text{ in probability} - P_X,$$

if it is shown that, for $\varepsilon_1, \varepsilon_2 > 0$, as $n \to \infty$,

$$P_X(\frac{1}{S_n^2} | S_n^2 \left| \sum_{i=1}^{n} (X_i - \mu)^2 \right| > \varepsilon_1 > \varepsilon_2) \to 0. \quad (44)$$

In order to prove the preceding result, for $X \in DAN$, without loss of generality we first assume that $\mu = 0$ and write

$$P_X(\frac{\bar{X}_n^2}{S_n^2} > \varepsilon_1) > \varepsilon_2 \to 0, \text{ as } n \to \infty.$$

The preceding relation is due to the laws of large numbers when $E_X X^2 < \infty$. When $E_X X^2 = \infty$, then we also make use of Raikov’s theorem (cf., e.g., [14]). Hence (43) is valid, and the proof of Theorem 3.2 is complete. □

**Proof of Theorem 3.1**

Due to Slutsky’s theorem in probability—$P_X$, Theorem 3.1 will follow if one shows that, for $\varepsilon > 0$, as $n, m_n \to \infty$ as in (25), we have

$$P_{w|X}(\frac{1}{S_n^2} | S_{m_n}^2 - S_n^2 \left| > \varepsilon_1 \right| \to 0 \text{ in probability} - P_X. \quad (45)$$

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Using the $U$-statistic representation of the sample variance, for $S_{m_n}^2$ and $S_n^2$ we write
\begin{align*}
S_n^2 &= \frac{1}{n^2} \sum_{1 \leq i \neq j \leq n} (X_i - X_j)^2 = \frac{n-1}{n} \cdot \frac{1}{2n(n-1)} \sum_{1 \leq i \neq j \leq n} (X_i - X_j)^2 \\
S_{m_n}^2 &= \frac{1}{2m_n(m_n-1)} \sum_{1 \leq i \neq j \leq n} w_i^{(n)} w_j^{(n)} (X_i - X_j)^2.
\end{align*}

To establish (45), we first note that when $E X^2 = +\infty$, as $n \to +\infty$, $S_n^2/\ell^2(n) \to 1$ in $P_X$ and for $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3 > 0$, on using the above $U$-statistic representations, we have
\begin{align*}
P_X\{P_{w|X}(\frac{1}{\ell^2(n)} |S_{m_n}^2 - S_n^2| > \varepsilon_1 > \varepsilon_2) \}
&\leq P_X\{P_{w|X}(\frac{|S_{m_n}^2 - S_n^2|}{\ell^2(n)} > \varepsilon_1, \bigcap_{1 \leq i \neq j \leq n} |\frac{w_i^{(n)} w_j^{(n)}}{m_n(m_n-1)} - \frac{1}{n^2}| \leq \frac{\varepsilon_3}{n^2 \sqrt{\log n}} > \frac{\varepsilon_2}{2}\}
&\leq P_X\{I(\frac{\sum_{1 \leq i \neq j \leq n} (X_i - X_j)^2}{2n^2 \ell^2(n) \sqrt{\log n}} > \frac{\varepsilon_1}{\varepsilon_3}) > \frac{\varepsilon_2}{2}\}
&\leq P_X\{I(\frac{\sum_{1 \leq i \neq j \leq n} (X_i - X_j)^2}{2n^2 \ell^2(n) \sqrt{\log n}} > \frac{\varepsilon_1 \varepsilon_2}{2\varepsilon_3})\}
&\leq P_X\{I(\frac{\sum_{1 \leq i \neq j \leq n} (X_i - X_j)^2}{2n^2 \ell^2(n) \sqrt{\log n}} > \frac{\varepsilon_1 \varepsilon_2}{2\varepsilon_3})\}
&\leq \frac{2}{\varepsilon_2} P_{w}(\bigcup_{1 \leq i \neq j \leq n} |\frac{w_i^{(n)} w_j^{(n)}}{m_n(m_n-1)} - \frac{1}{n^2}| > \frac{\varepsilon_3}{n^2 \sqrt{\log n}})
&\leq o(1) + \frac{2}{\varepsilon_2} n^2 \exp \{ - \frac{m_n(m_n-1)}{n^2 \log n} \frac{\varepsilon_3^2}{2(1 + \frac{\varepsilon_1}{\sqrt{\log n}})} \}
&= o(1).
\end{align*}

The relation (46) is due to the fact that $X \in DAN$, and an application of Bernstein’s inequality for $w_i^{(n)} w_j^{(n)}$, viewed as $\sum_{1 \leq s \leq m_n(m_n-1)} I(Y_s = 1)$,
where, \( Y_s, 1 \leq s \leq m_n(m_n - 1) \), are i.i.d. random variables which are uniformly distributed on the set \( \{1, \ldots, n^2\} \). And this completes the proof of (26).

In order to prove (27) we first note that, as \( n \to \infty \),
\[
\frac{\max_{1 \leq j \leq n}(X_j - \mu)^2}{\sum_{i=1}^n (X_i - \mu)^2} \to 0 \text{ a.s. - } P_X
\]
once again from Corollary 4.1 of Morris [20], on taking \( m_n = n \) and as \( n \to \infty \), for \( T_n^*(\mu) \) as defined in (11) we conclude that, for all \( t \in \mathbb{R} \),
\[
P_{w|X}(T_n^*(\mu) \leq t) \to P(Z \leq t) \text{ a.s. - } P_X.
\]
Now, in view of (44) and Slutsky’s theorem, the proof of (27) follows if we show that for \( \varepsilon_1, \varepsilon_2 > 0 \), as \( n, m \to \infty \) such that \( m_n/n \to \infty \),
\[
P_X\{ \limsup_{n \to \infty} P_{w|X}(|S_{m_n}^* - S_n^2| > \varepsilon_1) > \varepsilon_2 \} = 0
\]
Observing now that \( P_{w|X}(|S_{m_n}^* - S_n^2| > \varepsilon_1) \) asymptotically is bounded above by
\[
\sum_{1 \leq i < j \leq n} \varepsilon^{-1} E_w\left|\frac{w_i^{(n)} w_j^{(n)}}{m_n(m_n - 1)} - \frac{1}{n^2}\left|\frac{(X_i - X_j)^2}{2} - \sigma^2\right|\right|
\]
where \( \sigma^2 = \text{var}_X(X) \). We note that
\[
E_w\left|\frac{w_i^{(n)} w_j^{(n)}}{m_n(m_n - 1)} - \frac{1}{n^2}\right| \leq \sqrt{\text{var}_w\left(\frac{w_i^{(n)} w_j^{(n)}}{m_n(m_n - 1)}\right)} \sim \frac{1}{m_n n}
\]
Observing now that, as \( n \to \infty \), \( n^{-2} \sum_{1 \leq i < j \leq n} \left|\frac{(X_i - X_j)^2}{2} - \sigma^2\right| \) is convergent a.s.-\( P_X \) and that \( n/m_n \to 0 \), completes the proof of (27) and also that of Theorem 3.1. \( \square \)

**Proof of Theorem 4.1**

Observe that, as \( n, m_n \) approach infinity, the asymptotic equivalence of \( S_{m_n}^2(b)/m_n \), \( S_n^2 \sum_{i=1}^n \left(\frac{w_i^{(n)}(b)}{m_n} - \frac{1}{n}\right)^2 \) and \( \sigma^2 \sum_{i=1}^n \left(\frac{w_i^{(n)}(b)}{m_n} - \frac{1}{n}\right)^2 \) with respect to the conditional probability \( P_{X|w} \), for each \( 1 \leq b \leq B \), yields asymptotic equivalence
for $T^*_m(b)$, $T^*_m(b)$ and $T^*_{m,n}(b)$, where the latter is defined by

$$T^*_{m,n}(b) := \frac{\sum_{i=1}^{n} \left( \frac{w^{(n)}(b_i)}{m_n} - \frac{1}{n} \right) X_i}{\sigma \sqrt{\sum_{i=1}^{n} \left( \frac{w^{(n)}(b_i)}{m_n} - \frac{1}{n} \right)^2}}, \quad 1 \leq b \leq B.$$ 

Therefore, we only give the proof of this theorem for $T^*_{m,n}$ and its associated bootstrapped quantile which is defined by

$$C_{\sigma,\alpha}^{(B)} := \inf\{t : \frac{1}{B} \sum_{b=1}^{B} I(T^*_{m,n}(b) \leq t) \geq \alpha\}.$$ 

In other words, we shall show that, as $n, m, B \to \infty$, we have

$$C_{\sigma,\alpha}^{(B)} \to z_\alpha \text{ in probability} - \bigotimes_{b=1}^{\infty} P_{X,w(b)}.$$ 

To do so, we first note that in view of the asymptotic normality of $T^*_{m,n}(b)$, for each $1 \leq b \leq B$, one can conclude the asymptotic conditional independence of $T^*_{m,n}(b)$ and $T^*_{m,n}(b')$ for each $1 \leq b \neq b' \leq B$, from the fact that conditionally they are asymptotically uncorrelated. The latter is established in the following Lemma 5.4.

**Lemma 5.4.** Assume the conditions of Theorem 4.1. As $n, m, B \to \infty$, for each $1 \leq b \neq b' \leq B$, we have

$$E\left( T^*_{m,n}(b) T^*_{m,n}(b') \mid (w_1^{(n)}(b), \ldots, w_n^{(n)}(b)), (w_1^{(n)}(b'), \ldots, w_n^{(n)}(b')) \right) \to 0 \text{ a.s. } P_w.$$ 

**Proof of Lemma 5.4.**

For ease of notation, we let $E_{|b}(\cdot)$ and $E_{|b,b'}(\cdot)$ be the respective short hand notations for the conditional expectations $E\left( \cdot \mid (w_1^{(n)}(b), \ldots, w_n^{(n)}(b)) \right)$ and $E\left( \cdot \mid (w_1^{(n)}(b), \ldots, w_n^{(n)}(b)) \right)$, $(w_1^{(n)}(b'), \ldots, w_n^{(n)}(b'))$. Similarly, we let $P_{|b}(\cdot)$ and $P_{|b,b'}(\cdot)$ stand for the conditional probabilities $P\left( \cdot \mid (w_1^{(n)}(b), \ldots, w_n^{(n)}(b)) \right)$ and $P\left( \cdot \mid (w_1^{(n)}(b), \ldots, w_n^{(n)}(b)) \right)$, $(w_1^{(n)}(b'), \ldots, w_n^{(n)}(b'))$, respectively.
Now observe that from the independence of the $X_i$’s, we conclude that

$$E_{X|b,b'}(T_{m_n,\sigma}^*(b) T_{m_n,\sigma}^*(b')) = \frac{\sum_{i=1}^n (w_i^{(n)}(b) - \frac{1}{n})(w_i^{(n)}(b') - \frac{1}{n})}{\sqrt{\sum_{k=1}^n (w_k^{(n)}(b) - \frac{1}{n})^2 \sqrt{\sum_{l=1}^n (w_l^{(n)}(b) - \frac{1}{n})^2}}}
$$

By this, with $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3 > 0$, we can write

$$P\left(\left|E_{X|b,b'}(T_{m_n,\sigma}^*(b) T_{m_n,\sigma}^*(b'))\right| > \varepsilon_1\right) \leq P\left(\left|\frac{m_n}{(1-\frac{1}{n})} \sum_{i=1}^n (w_i^{(n)}(b) - \frac{1}{n})(w_i^{(n)}(b') - \frac{1}{n})\right| > \varepsilon_1(1-\varepsilon_2)(1-\varepsilon_3)\right)$$

$$+ P\left(\left|\frac{m_n}{(1-\frac{1}{n})} \sum_{i=1}^n (w_i^{(n)}(b) - \frac{1}{n})^2 - 1\right| > \varepsilon_2\right)$$

$$+ P\left(\left|\frac{m_n}{(1-\frac{1}{n})} \sum_{i=1}^n (w_i^{(n)}(b') - \frac{1}{n})^2 - 1\right| > \varepsilon_3\right).$$

The last two terms in the preceding relation have already been shown to approach zero as $\frac{m_n}{n^2} \to 0$. We now show that the first term approaches zero as well in view of the following argument which relies on the facts that $w_i^{(n)}$, $1 \leq i \leq n$ are multinomially distributed and that for each $1 \leq i, j \leq n$, $w_i^{(n)}(b)$ and $w_j^{(n)}(b')$ are i.i.d.’s (in terms of $P_x$) when $b \neq b'$.

In what will follow, for the ease of notation we put $\varepsilon_4 := \varepsilon_1(1-\varepsilon_2)(1-\varepsilon_3)$.

$$P\left(\left|\frac{m_n}{(1-\frac{1}{n})} \sum_{i=1}^n (w_i^{(n)}(b) - \frac{1}{n})(w_i^{(n)}(b') - \frac{1}{n})\right| > \varepsilon_4\right) \leq \varepsilon_4^{-2} \frac{m_n^2}{(1-\frac{1}{n})^2} \left\{ nE_b^2\left(\frac{w_i^{(n)}(b)}{m_n} - \frac{1}{n}\right)^2 + n(n-1)E_b^2\left[\left(\frac{w_i^{(n)}(b)}{m_n} - \frac{1}{n}\right)\left(\frac{w_i^{(n)}(b')}{m_n} - \frac{1}{n}\right)\right] \right\}$$

$$= \varepsilon_4^{-2} \frac{m_n^2}{(1-\frac{1}{n})^2} \left\{ n\left(\frac{1}{m_n} - \frac{1}{n}\right)^2 + n(n-1)(\frac{1}{m_n n^2})^2 \right\}$$

$$\leq \varepsilon_4^{-2} \left(\frac{1}{n} + \frac{1}{n^2(1-\frac{1}{n})^2}\right) \to 0.$$

Now the proof of Lemma 5.4 is complete. □
We now continue the proof of Theorem 4.1 by showing that for any \( \varepsilon > 0 \), as \( n, m_n, B \to \infty \),

\[
\lim_{b=1}^{\infty} P_{X,w(b)}(C_{\sigma,\alpha}^{(B)} \leq z_{\alpha} - \varepsilon) \to 0, \quad \text{(47)}
\]

\[
\lim_{b=1}^{\infty} P_{X,w(b)}(C_{\sigma,\alpha}^{(B)} > z_{\alpha} + \varepsilon) \to 0. \quad \text{(48)}
\]

Observe that we have

\[
\lim_{b=1}^{\infty} P_{X,w(b)}(C_{\sigma,\alpha}^{(B)} \leq z_{\alpha} - \varepsilon) \leq \lim_{b=1}^{\infty} P_{X,w(b)}\left(\frac{1}{B} \sum_{b=1}^{B} I(T_{m_n,\sigma}^*(b) \leq z_{\alpha} - \varepsilon) \geq \alpha\right) \quad \text{(49)}
\]

and

\[
\lim_{b=1}^{\infty} P_{X,w(b)}(C_{\sigma,\alpha}^{(B)} > z_{\alpha} + \varepsilon) \leq \lim_{b=1}^{\infty} P_{X,w(b)}\left(\frac{1}{B} \sum_{b=1}^{B} I(T_{m_n,\sigma}^*(b) \leq z_{\alpha} + \varepsilon) < \alpha\right) \quad \text{(50)}
\]

In view of (49) and (50), the relations (47) and (48) will follow if for each \( a \in \mathbb{R} \), one shows that, as \( n, m_n, B \to \infty \),

\[
\lim_{b=1}^{\infty} P_{X,w(b)}\left(\frac{1}{B} \sum_{b=1}^{B} \left|I(T_{m_n,\sigma}^*(b) \leq a) - \Phi(a)\right| > \varepsilon\right) \to 0, \quad \text{(51)}
\]

where \( \Phi(.) \) is the standard normal distribution function.
To establish (51), we write
\[
\bigotimes_{b=1}^{\infty} P_{X,w(b)} \left( \frac{1}{B} \sum_{b=1}^{B} |I(T_{m_n,\sigma}^{*}(b) \leq a) - \Phi(a)| > \varepsilon \right)
\]
\[
= E \left\{ P \left( \frac{1}{B} \sum_{b=1}^{B} |I(T_{m_n,\sigma}^{*}(b) \leq a) - \Phi(a)| > \varepsilon \right) \bigotimes_{b=1}^{\infty} \mathcal{F}_{w(b)} \right\}
\]
\[
\leq E \left\{ \frac{1}{B^2} \sum_{b=1}^{B} E_{X|b} \left[ (I(T_{m_n,\sigma}^{*}(b) \leq a) - \Phi(a)) (I(T_{m_n,\sigma}^{*}(b') \leq a) - \Phi(a)) \right] \right\}
\]
\[
+ E \left\{ \frac{1}{B^2} \sum_{1 \leq b \neq b' \leq B} E_{X|b,b'} \left[ (I(T_{m_n,\sigma}^{*}(b) \leq a) - \Phi(a)) (I(T_{m_n,\sigma}^{*}(b') \leq a) - \Phi(a)) \right] \right\}
\]
\[
\leq \frac{1}{B} + E \left\{ E_{X|1,2} \left[ (I(T_{m_n,\sigma}^{*}(1) \leq a) - \Phi(a)) (I(T_{m_n,\sigma}^{*}(2) \leq a) - \Phi(a)) \right] \right\}
\]
\[
\rightarrow 0, \text{ as } n, m_n, B \to \infty.
\]

The preceding relation is true since, in view of Lemma 5.4, for large enough \( n, m_n \) we have that

\[
E \left\{ E_{X|1,2} \left[ (I(T_{m_n,\sigma}^{*}(1) \leq a) - \Phi(a)) (I(T_{m_n,\sigma}^{*}(2) \leq a) - \Phi(a)) \right] \right\}
\]
\[
\approx E \left\{ E_{X|1} \left[ (I(T_{m_n,\sigma}^{*}(1) \leq a) - \Phi(a)) \right] E \left\{ E_{X|2} \left[ (I(T_{m_n,\sigma}^{*}(2) \leq a) - \Phi(a)) \right] \right\} \right\}
\]
\[
= E \left\{ P_{X|1} \left[ (T_{m_n,\sigma}^{*}(1) \leq a) - \Phi(a) \right] \right\} E \left\{ P_{X|2} \left[ (T_{m_n,\sigma}^{*}(2) \leq a) - \Phi(a) \right] \right\}
\]
\[
\rightarrow 0, \text{ as } n, m_n \to \infty.
\]

The preceding relation is due to part (a) of Corollary 2.1 with \( \sigma^2 \) replacing \( S_n^2 \) therein, and Lemma 1.2 in [8]. Now the proof of part (a) of Theorem 4.1 is complete.

To prove parts (b) and (c), we first conclude the asymptotic in probability equivalence of \( T_{m_n}^{**} \) and \( T_{m_n,\mu}^{*} \), as \( n, m_n \to \infty \), in terms of the conditional probability \( P_{w|X} \) (cf. the proof of Theorem 3.1). The same equivalence holds true between \( T_{m_n,S_n}^{*} \) and \( T_{m_n,\mu}^{*} \) by virtue of Theorem 3.2 (cf. the proof of Theorem 3.2). Therefore, parts (b) and (c) will follow if we show that, as \( n, m_n, B \to \infty \),

\[
C_{\mu,\alpha}^{(B)} \longrightarrow z_{\alpha} \text{ in probability} - \bigotimes_{b=1}^{\infty} P_{X,w(b)},
\]

33
where $C_{\mu,\alpha}^{(B)} := \inf\{ t : \frac{1}{B} \sum_{b=1}^{B} I(T^*_{m_n,\mu}(b) \leq t) \geq \alpha \}$. To do so, similarly to what we did in the proof of part (a), we shall show that for any $\varepsilon > 0$, as $n, m_n, B \to \infty$, we have

$$\bigotimes_{b=1}^{\infty} \mathbb{P}_{X,w(b)}(C_{\mu,\alpha}^{(B)} \leq z_\alpha - \varepsilon) \to 0 \quad (52)$$

and

$$\bigotimes_{b=1}^{\infty} \mathbb{P}_{X,w(b)}(C_{\mu,\alpha}^{(B)} > z_\alpha + \varepsilon) \to 0. \quad (53)$$

Now observe that

$$\bigotimes_{b=1}^{\infty} \mathbb{P}_{X,w(b)}(C_{\mu,\alpha}^{(B)} \leq z_\alpha - \varepsilon) \leq \bigotimes_{b=1}^{\infty} \mathbb{P}_{X,w(b)}\left( \frac{1}{B} \sum_{b=1}^{B} I(T^*_{m_n,\sigma}(b) \leq z_\alpha - \varepsilon) \geq \alpha \right) \quad (54)$$

and

$$\bigotimes_{b=1}^{\infty} \mathbb{P}_{X,w(b)}(C_{\mu,\alpha}^{(B)} > z_\alpha + \varepsilon) \leq \bigotimes_{b=1}^{\infty} \mathbb{P}_{X,w(b)}\left( \frac{1}{B} \sum_{b=1}^{B} I(T^*_{m_n,\sigma}(b) \leq z_\alpha + \varepsilon) < \alpha \right). \quad (55)$$

In view of (54) and (55), the relations (52) and (53) will follow if for each $a \in \mathbb{R}$, one shows that, as $n, m_n, B \to \infty$,

$$\bigotimes_{b=1}^{\infty} \mathbb{P}_{X,w(b)}\left( \frac{1}{B} \sum_{b=1}^{B} |I(T^*_{m_n,\mu}(b) \leq a) - \Phi(a)| \geq \varepsilon \right) \to 0. \quad (56)$$

We establish the preceding relation in a similar way we established (51) of part (a), on noting that the proof here will be done via conditioning on the sample. Before sorting out the details, it is important to note that, via conditioning on the sample, $T^*_{m_n,\mu}(b)$ and $T^*_{m_n,\mu}(b')$ are independent for each
1 \leq b \neq b' \leq B. We have

\[ \bigotimes_{b=1}^{\infty} P_{X,u(b)} \left( \frac{1}{B} \sum_{b=1}^{B} |I(T_{m_n,\mu}^*(b) \leq a) - \Phi(a)| > \varepsilon \right) \]

\[ = E \left\{ P \left( \frac{1}{B} \sum_{b=1}^{B} |I(T_{m_n,\mu}^*(b) \leq a) - \Phi(a)| > \varepsilon \right| X \right\} \]

\[ \leq E \left\{ \frac{1}{B^2} \sum_{b=1}^{B} E \left[ \left( |I(T_{m_n,\mu}^*(b) \leq a) - \Phi(a)| \right)^2 \right| X \right\} \]

\[ + E \left\{ \frac{1}{B^2} \sum_{1 \leq b \neq b' \leq B} E \left[ \left( |I(T_{m_n,\mu}^*(b) \leq a) - \Phi(a)| \right) \left( |I(T_{m_n,\mu}^*(b') \leq a) - \Phi(a)| \right) \right| X \right\} \]

\[ \leq \frac{1}{B} + E \left\{ \left( P(T_{m_n,\mu}^*(1) \leq a|X) - \Phi(a) \right) \left( P(T_{m_n,\mu}^*(2) \leq a|X) - \Phi(a) \right) \right\} \]

\[ \rightarrow 0, \text{ as } n, m_n, B \rightarrow \infty. \]

The preceding relation is true due to the fact that, as \( n, m_n \rightarrow \infty, \)

\[ P(T_{m_n,\mu}^* \leq a|X) \rightarrow \Phi(a) \text{ in probability} - P_X \]

and Lemma 1.2 in S. Csörgő and Rosalsky [8]. Now the proof of (56) and, consequently that of parts (b) and (c) are complete. Hence the proof of Theorem 4.1 is also complete. □

**Proof of Theorem 4.2**

Once again, in view of the fact that, as \( n \rightarrow \infty, S_n^2 \rightarrow \sigma^2 \text{a.s.} - P_X \) we replace \( T_n^* \) with \( T_{n,\sigma}^* \), which is defined by

\[ T_{n,\sigma}^* := \frac{\sum_{i=1}^{n} (\zeta_i - \bar{\zeta}_n)X_i}{\sigma \sqrt{\sum_{i=1}^{n} (\zeta_i - \bar{\zeta}_n)^2}} \text{ a.s.} - P_\zeta \]

The proof of this theorem essentially consists of the same steps as those of part (a) of Theorem 4.1. Hence, once again, the asymptotic normality of \( T_{n,\sigma}^*(b) \), for each \( 1 \leq b \leq B \), conclude the asymptotic conditional independence of \( T_{n,\sigma}^*(b) \) and \( T_{n,\sigma}^*(b') \) for each \( 1 \leq b \neq b' \leq B \), from the fact that
conditionally they are asymptotically uncorrelated. The latter is established in the following Lemma 5.5.

**Lemma 5.5.** Assume the conditions of Theorem 4.2. As $n, m_n \to \infty$, for each $1 \leq b \neq b' \leq B$, we have that

$$E\left(T_{n,\sigma}^*(b) T_{n,\sigma}^*(b') \mid (\zeta_1(b), \ldots, \zeta_n(b)), (\zeta_1(b'), \ldots, \zeta_n(b'))\right) \to 0 \text{ in probability} - P_{\zeta}.$$  

To prove this lemma, without loss of generality we assume that $E_{\zeta}(\zeta_1) = 0$, and let $E_{|b,b'|}$ be a short hand notation for $E\left(\cdot \mid (\zeta_1(b), \ldots, \zeta_n(b)), (\zeta_1(b'), \ldots, \zeta_n(b'))\right)$. Now, similarly to the proof of Lemma 5.5, we note that

$$E_{X|b,b'}(T_{n,\sigma}^*(b) T_{n,\sigma}^*(b')) = \frac{\sum_{i=1}^{n} (\zeta_i(b) - \bar{\zeta}(b)) (\zeta_i(b') - \bar{\zeta}(b'))}{\sqrt{\sum_{k=1}^{n} (\zeta_k(b) - \bar{\zeta}(b))^2} \sqrt{\sum_{l=1}^{n} (\zeta_l(b) - \bar{\zeta}(b'))^2}}.$$

In view of the preceding statement, to complete the proof, with $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, we proceed as follows:

$$P\left(\frac{\left|\sum_{i=1}^{n} (\zeta_i(b) - \bar{\zeta}(b)) (\zeta_i(b') - \bar{\zeta}(b'))\right|}{\sqrt{\sum_{k=1}^{n} (\zeta_k(b) - \bar{\zeta}(b))^2} \sqrt{\sum_{l=1}^{n} (\zeta_l(b) - \bar{\zeta}(b'))^2}} > \varepsilon_1\right)$$

$$\leq P\left(\left|\sum_{i=1}^{n} (\zeta_i(b) - \bar{\zeta}(b)) (\zeta_i(b') - \bar{\zeta}(b'))\right| > \varepsilon_1(1 - \varepsilon_2)(1 - \varepsilon_3)\right)$$

$$+ P\left(\left|\frac{\sum_{k=1}^{n} (\zeta_k(b) - \bar{\zeta}(b))^2}{n} - 1\right| > \varepsilon_2\right)$$

$$+ P\left(\left|\frac{\sum_{l=1}^{n} (\zeta_l(b') - \bar{\zeta}(b'))^2}{n} - 1\right| > \varepsilon_3\right).$$

Clearly, the last two relations approach zero as $n \to \infty$. Hence, it only remains to show the asymptotic negligibility of the first term of the preceding three. To do so, we let $\varepsilon_4 := \varepsilon_1(1 - \varepsilon_2)(1 - \varepsilon_3)$ and apply Chebyshev’s inequality to arrive at

$$P\left(\left|\sum_{i=1}^{n} (\zeta_i(b) - \bar{\zeta}(b)) (\zeta_i(b') - \bar{\zeta}(b'))\right| > \varepsilon_4\right)$$

$$\leq \varepsilon_4^{-2} n^{-2} \{nE^2(\zeta_1(b) - \bar{\zeta}(b)) + n(n - 1)E^2[(\zeta_1(b) - \bar{\zeta}(b)) (\zeta_2(b) - \bar{\zeta}(b))]\}$$

$$\leq \varepsilon_4^{-2} n^{-2} \{nE^2(\zeta_1^2) + \frac{n(n - 1)}{n^2} E^2(\zeta_1^2)\} \to 0, \text{ as } n \to \infty.$$
This completes the proof of Lemma 5.5 □

Due to similarity of the rest of the proof of this theorem and that of (51) of part (a) in the proof of Theorem 4.1, the details are omitted. Now the proof of Theorem 4.2 is complete. □

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