A symmetrization approach to hypermatrix SVD

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Abstract

We propose a new hypermatrix singular value decomposition based upon the spectral decomposition of the symmetric products of transposes.

1 Introduction

One of the most fruitful ideas in matrix theory is that of matrix decomposition or canonical form. Of the many matrix canonical forms discussed in the literature, the Singular Value Decomposition (or SVD for short), is by far the most widely used. Recall that for an arbitrary $A \in \mathbb{C}^{n \times n}$, the SVD of $A$ is expressed by

$$A = (U \sqrt{\text{diag}(A)}) (\sqrt{\text{diag}(A)} V)$$

such that

$$UU^* = I = V^* V$$

Calculating the SVD consists of finding the eigenvalues and eigenvectors of the Hermitian products of $A$ and $A^*$. Important information about the matrix $A$ is obtained through decomposition such as the matrix rank, the orthonormal basis vectors and the diagonal matrix of the scaling values, all of which are useful to be extended to higher dimensions. Over the past decades, considerable progress has been made in generalizing the matrix SVD to higher order hypermatrices. Two predominant approaches to hypermatrix canonical forms are now well established as the CANDECOMP-PARAFAC (CP) model [CC70, Har70] and the Tucker model [Tuc66], where the former is a special case to the later. Based on Tucker model, De Lathauwer, De Moor, and Vandewalle pioneered a multilinear generalization of the matrix SVD to hypermatrices in [DLDMV00], namely the Higher-Order Singular Value Decomposition (HOSVD). The classical models of CP and Tucker or HOSVD generally express the decomposition of a hypermatrix as a sum of outer products of vectors, also referred to as the $n$-mode product in the form of “hypermatrix times matrices” [KB09]. In particular, the $n$-mode product enables the hypermatrix SVD through performing matrix SVDs following the mode-$n$ flattening (unfolding) of the original hypermatrix into matrices, and then assemble results into

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a hypermatrix of the same order. One of the advantages of the classical models and the method of HOSVD is that
the obtained results guarantee orthogonality to some extent: the singular vectors are entries of orthogonal matrices,
and the core hypermatrix coordinating singular values meets a property of all-orthogonality that is a relaxation to the
diagonality property in the matrix SVD. Thorough discussions on the classical methods and applications have been
reviewed in [KB09]. Other more recent studies also explored alternative representations of a hypermatrix SVD as a
sum of outer products of matrices, which is a generalization based on a different hypermatrix multiplication scheme
in the form of "hypermatrix times hypermatrix" [KMP08 KM11].

While the aforementioned generalizations to higher-order SVD have been widely used in applications, they often
reduce the problems to matrix SVDs through the folding and unfolding schemes. By contrast to the matrix case, such
higher-order SVD methods do not stem from a hypermatrix formulation of the spectral theorem. Recent works in
[GF17 GER11] motivated by the generalization of the spectral theorem to hypermatrices suggest new ways to extend
matrix SVD to hypermatrix SVD while retaining the link to the spectra. In the present note, we discuss in analogy
to matrix SVD the new approach to obtain orthogonal hypermatrices and diagonal scaling hypermatrix via spectral
decomposition of symmetric products of transposes. Our work is based on the the Bhattacharya-Mesner algebra (BM
algebra) introduced in [MB94 GF17 GF20], which has enabled the generalization of many important matrix concepts
including the rank, inverse, and spectral decompositions to hypermatrices. In addition to the hypermatrix SVD, we also
expand the list of concepts to the BM algebra to include definitions of tensorial orbits and invariants of hypermarices,
and hypermatrix orthorgonality and unitarity.

2 Overview of the Bhattacharya-Mesner (BM) algebra.

Hypermatrices are multidimensional matrices. More precisely, a hypermatrix is a finite multiset whose elements (called
entries) are indexed by members of some fixed Cartesian product of the form
\[ \{0, \cdots , n_0 - 1\} \times \{0, \cdots , n_1 - 1\} \times \cdots \times \{0, \cdots , n_{m-1} - 1\} . \]
Such a hypermatrix is of order \( m \) and of size \( n_0 \times n_1 \times \cdots \times n_{m-1} \). A hypermatrix is cubic of side length \( n \) if
\( n_0 = n_1 = \cdots = n_{m-1} = n \).

Hypermatrix algebras arise from natural generalizations of classical matrix notions and algorithms [MB94 GKZ94
Ker08 GER11 GF17 MB90]. The important distinction between hypermatrices and tensors closely mirrors the dis-
tinction between matrices and abstract linear transformations. Recall that an abstract linear transformation specified
over finite dimensional \( \mathbb{K} \)-vector spaces is identified with a matrix orbit. For instance, let \( \mathbf{M} \in \mathbb{K}^{m \times n} \) be associated
with some abstract linear transformation specified relative to the standard basis for \( \mathbb{K}^{n \times 1} \) and \( \mathbb{K}^{1 \times m} \). The tensorial
orbit of the linear transformation (accounting for all possible coordinate changes) is the matrix set
\[ \{ \mathbf{A} \cdot \mathbf{M} \cdot \mathbf{B} : \mathbf{A} \in \text{GL}_m(\mathbb{K}) \text{ and } \mathbf{B} \in \text{GL}_n(\mathbb{K}) \} . \]
A matrix property common to every member of a tensorial orbit is a tensorial invariant.

Classically, third order hypermatrices in $K^{m \times n \times p}$ arise from tensorial orbits induced by the action of various appropriate subgroups of the general linear group on canonical embeddings of $K$-vector spaces: $K^{m \times 1 \times 1}$, $K^{1 \times n \times 1}$ and $K^{1 \times 1 \times p}$ respectively. Incidentally, classical tensorial invariants such as the rank and singular values are defined by analogy to their matrix counterparts.

Hypermatrix multiplication, named the Bhattacharya-Mesner product (BM-product), is a generalization to the matrix multiplication [MB90, MB94]. Occasionally, the product of a conformable matrix pair $A \in K^{m \times \ell}$, $B \in K^{\ell \times n}$, can be written using the BM-product notation as $\text{Prod} (A, B)$ for consistency and such a product is specified entry-wise by

$$\text{Prod} (A, B) [i, j] = \sum_{0 \leq t < \ell} A [i, t] B [t, j], \quad \forall \begin{cases} 0 \leq i < m \\ 0 \leq j < n \end{cases}.$$  

Similarly, the BM-product of a conformable triple of third order hypermatrices $A \in K^{m \times \ell \times p}, B \in K^{m \times n \times \ell}$ and $C \in K^{\ell \times n \times p}$, is noted $\text{Prod} (A, B, C)$ and specified entry-wise by

$$\text{Prod} (A, B, C) [i, j, k] = \sum_{0 \leq t < \ell} A [i, t, k] B [i, j, t] C [t, j, k], \quad \forall \begin{cases} 0 \leq i < m \\ 0 \leq j < n \\ 0 \leq k < p \end{cases}.$$  

Furthermore, we recall that the general Bhattacharya-Mesner product of a conformable triple $A \in K^{m \times \ell \times p}, B \in K^{m \times n \times \ell}$ and $C \in K^{\ell \times n \times p}$, taken with an additional cubic background hypermatrix $M \in K^{\ell \times \ell \times \ell}$ (similar to metric tensors first introduced in differential geometry [RLC00, Gau28]) is denoted $\text{Prod}_M (A, B, C) \in K^{m \times n \times p}$ and specified entry-wise by

$$\text{Prod}_M (A, B, C) [i, j, k] = \sum_{0 \leq t_0, t_1, t_2 < \ell} A [i, t_0, k] B [i, j, t_1] C [t_2, j, k] M [t_0, t_1, t_2].$$  

The original BM-product is thus recovered from the general BM-product by setting the cubic background hypermatrix $M$ to be equal to the Kronecker delta hypermatrix denoted $\Delta$, whose entries are specified by

$$\Delta [i_0, i_1, i_2] = \begin{cases} 1 & \text{if } 0 \leq i_0 = i_1 = i_2 < n \\ 0 & \text{otherwise} \end{cases}.$$  

The general Bhattacharya-Mesner product of conformable matrices $A \in K^{m \times \ell}, B \in K^{\ell \times n}$,
taken with the background matrix $M \in \mathbb{K}^{t \times t}$ is given by

$$\text{Prod}_M(A, B)[i, j] = \sum_{0 \leq t_0, t_1 < t} A[i, t_0] B[t_1, j] M[t_0, t_1], \quad \forall \left\{ \begin{array}{l} 0 \leq i < m \\ 0 \leq j < n \end{array} \right.$$

We further recall that the transpose of an arbitrary hypermatrix $A \in \mathbb{K}^{m \times n \times p}$, denoted as $A^\top \in \mathbb{K}^{p \times n \times m}$, results from a cyclic permutation on the indices and is specified entry-wise as follows

$$A^\top[i, j, k] = A[k, i, j].$$

We adopt the convention

$$A^{\top 2} := (A^\top)^\top, \quad A^{\top 3} := (A^{\top 2})^\top = A.$$

$$\Rightarrow A^{\top u} = A^{\top v} \text{ if } u \equiv v \mod 3.$$

Note that when $\mathbb{K}$ is commutative

$$\text{Prod}(A, B, C)^\top = \text{Prod}(B^\top, C^\top, A^\top).$$

### 3 Tensorial matrix orbits.

Let $\mathbb{K}$ denote an arbitrary field (not necessarily commutative) and let $\text{GL}_n(\mathbb{K})$ denote the general linear group of invertible $n \times n$ matrices whose entries belong to $\mathbb{K}$. When investigating matrices, it is of interest to determine matrix attributes which are independent of the chosen coordinate system. For this purpose we associate with an arbitrary matrix $M \in \mathbb{K}^{m \times n}$ a tensorial orbit induced by the action on $M$ of the group $\text{GL}_m(\mathbb{K}) \times \text{GL}_n(\mathbb{K})$ as follows

$$\mathcal{T}(M) := \left\{ A \cdot M \cdot B : \begin{array}{l} A \in \text{GL}_m(\mathbb{K}) \\ \text{and} \\ B \in \text{GL}_n(\mathbb{K}) \end{array} \right\}. \quad (3)$$

For instance, the tensorial orbit of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ whose entries are taken from the finite field with two elements denoted $\mathbb{F}_2$ is

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

In particular the tensorial orbit of a zero matrix is a singleton

$$\mathcal{T}(0_{m \times n}) = \{0_{m \times n}\}.$$

Recall that

$$\forall M \in \text{GL}_n(\mathbb{K}), \quad \mathcal{T}(M) = \text{GL}_n(\mathbb{K}).$$
When $\mathbb{K} = \mathbb{F}_p$, for a prime $p$ we have

$$|\mathcal{T}(M)| = \prod_{0 \leq i < n} \left( p^{k_n} - p^{k_i} \right).$$

The cardinality $|\mathcal{T}(M)|$ is by definition a tensorial invariant, whereas the property of being symmetric (i.e. $M = M^\top$) is not in general a tensorial invariant. Classical matrix attributes well known to be tensorial invariants include:

- The rank of $M \in \mathbb{C}^{m \times n}$ defined as
  $$\min_{A \in GL_m(\mathbb{C})} \min_{B \in GL_n(\mathbb{C})} \|AB\|_{\ell_0}.$$

- The nullity of $M \in \mathbb{C}^{m \times n}$ defined as
  $$\text{Dimension of } \{ x \in \mathbb{C}^{1 \times m} : xA = 0, \forall y \in \mathbb{C}^{n \times 1} \} = \min (m, n) - \min_{A \in GL_m(\mathbb{C})} \min_{B \in GL_n(\mathbb{C})} \|AB\|_{\ell_0}.$$

- Singular values of $M \in \mathbb{C}^{m \times n}$, defined as multiset of moduli of diagonal entries of any diagonal matrix element in the sub-orbit of $\mathcal{T}(M)$
  $$\left\{ A \in U_m(\mathbb{C}) \text{ and } B \in U_n(\mathbb{C}) : A \right\},$$
  where $U_m(\mathbb{C})$ and $U_n(\mathbb{C})$ respectively denote the unitary subgroup of $GL_m(\mathbb{C})$ and $GL_n(\mathbb{C})$.

- The eigenvalues of $M \in \mathbb{C}^{n \times n}$ defined as
  $$\left\{ \lambda \in \mathbb{C} : 0 = \det (\lambda I_n - A \right\} \left\{ A, B \in GL_n(\mathbb{C}) \text{ and } I_n = AB \right\}.$$  

4 Classical hypermatrix tensorial orbits and their invariants

Classical hypermatrix tensorial orbits are similar to matrix tensorial orbits in that they are both resulted from the action of the general linear group. Hypermatrix tensorial orbits are often simply called tensors in the literature, which are defined as elements of the tensor product of vector spaces [dSL08, Lim13]. The classical tensorial orbit of the hypermatrix $H \in \mathbb{K}^{m \times n \times p}$, resulted from the action of the group $GL_m(\mathbb{K}) \times GL_n(\mathbb{K}) \times GL_p(\mathbb{K})$ on $H$ is given by

$$\mathcal{T}(H) := \left\{ H \times_0 A \times_1 B \times_2 C : A \in GL_m(\mathbb{K}) \text{ and } B \in GL_n(\mathbb{K}) \text{ and } C \in GL_p(\mathbb{K}) \right\}.$$
The notation above refers to the $n$-mode product introduced by De Lathauwer, De Moor, and Vandewalle in [DLDMV00]. We also note that this notation is equivalent to the multilinear multiplication in some earlier works denoted as $(A, B, C) \cdot H$ in lieu of $H \times_0 A \times_1 B \times_2 C$ [dSL08].

By analogy to the matrix case, classical third order tensorial invariants include:

- The tensor rank of $H \in \mathbb{K}^{m \times n \times p}$ defined as
  $$\min_{A \in \text{GL}_m(\mathbb{K}), B \in \text{GL}_n(\mathbb{K}), C \in \text{GL}_p(\mathbb{K})} \| H \times_0 A \times_1 B \times_2 C \|_{\ell_0}$$

- The nullity of $H \in \mathbb{K}^{m \times n \times p}$ defined as
  Dimension of $\{ (x, y, z) \in \mathbb{C}^m \times \mathbb{C}^n : 0 = H \times_0 x \times_1 y \times_2 z, \forall z \in \mathbb{C}^p \}$

- Singular values of $H \in \mathbb{K}^{m \times n \times p}$, defined as the multiset of moduli super-diagonal entries of diagonal elements of the tensorial sub-orbit
  $$\left\{ H \times_0 A \times_1 B \times_2 C : A \in U_m(\mathbb{K}), B \in U_n(\mathbb{K}), C \in U_p(\mathbb{K}) \right\}$$

5 Non-classical tensorial orbits and their invariants

Historically, the study of classical tensorial orbits has been the predominant approach to investigating hypermatrices [Gor69, Hil90, RLC00, Gau28]. Unfortunately, two main drawbacks plague the classical tensorial orbits. The first drawback is conceptual in nature. It results from the fact that classical tensorial invariants do not suggest a distinct hypermatrix analog of the general linear group, nor do they suggest any generalization to hypermatrices of such notions as inverse, nullity, determinant, spectral decomposition, Rayleigh quotient inequality, resolution of identity, Parseval identity, unitarity and Fourier transforms. The second drawback is somewhat related to the first one but is of a computational nature. Classical tensorial invariant do not suggest any generalization of classical matrix algorithms such as the rank revealing LU decomposition and the The Gram-Schmidt orthogonalization process among others. These drawback have been recently addressed by the proposing new non-classical tensorial orbits and invariants [GF20, GF17]. For instance, new hypermatrix invariants which extend matrix notions and algorithms to hypermatrices arise from the BM algebra [GF20]. To be more specific, the BM algebra suggests a generalization to higher order hypermatrices of notions such as inverse and rank so as to enable the generalization to hypermatrices of the classical Rank Nullity theorem [GF20]. On the computational side, the BM approach also suggest a generalization to hypermatrices of the rank revealing LU factorization as well as the orthogonalization procedure, and higher order generalization of the Fourier transforms [GF20, GF17]. The BM algebra also enables a hypermatrix formulation of the spectral decomposition which we can extend to the symmetrization formulation of the third order hypermatrix SVD. This latter topic is the main subject of the present note and will be discussed at length.
We briefly recall here for the readers’ benefit an example of a non-classical tensorial orbit. Recall that the matrix
general linear group over an arbitrary field \( K \) (possibly non-commutative) is the matrix set
\[
\text{GL}_m(K) := \{ A \in K^{m \times m} : \exists B \in K^{m \times m} \text{ s.t. } B \cdot A \cdot X = X, \forall X \in K^{m \times n} \}. \tag{4}
\]

In contrast to the matrix general linear groups, their third order hypermatrix analog does not form a group. On the other hand, third order hypermatrix analog to general linear groups are defined similarly to Eq. (4) as follows :
\[
\text{GL}_{m \times n \times p}(m \times p \times p, p \times n \times p) := \{ (A, B) \in K^{m \times p \times p} \times K^{p \times n \times p} : \exists (C, D) \in K^{m \times p \times p} \times K^{p \times n \times p} \text{ s.t. } \text{Prod}(C, \text{Prod}(A, X, B), D) = X, \forall X \in K^{m \times n \times p} \}. \tag{5}
\]

Just as in the matrix case, third order hypermatrix analog of general linear groups are defined in terms of hypermatrix
inverse pairs.

Note that over any field (not necessarily commutative) there are subsets of invertible hypermatrix pairs which do form a group with respect to the BM product. The simplest example is the third order hypermatrix analog of the subgroup diagonal matrices. We call such hypermatrices scaling hypermatrices.

A pair \((A, B) \in K^{m \times p \times p} \times K^{p \times n \times p}\) is an invertible scaling hypermatrix pair if
\[
A[i, t, k] = \begin{cases} 
\alpha_{it} \in K \setminus \{0\} & \text{if } 0 \leq t = k < p \\
0 & \text{otherwise}
\end{cases} \forall \begin{cases} 
0 \leq i < m \\
0 \leq t < p \\
0 \leq k < p
\end{cases},
\]
\[
B[t, j, k] = \begin{cases} 
\beta_{tj} \in K \setminus \{0\} & \text{if } 0 \leq t = k < p \\
0 & \text{otherwise}
\end{cases} \forall \begin{cases} 
0 \leq t < p \\
0 \leq j < n \\
0 \leq k < p
\end{cases},
\]

\[\implies \text{Prod}(A, X, B)[i, j, k] = \alpha_{ik} X[i, j, k] \beta_{kj}, \quad \forall \begin{cases} 
0 \leq i < m \\
0 \leq j < n \\
0 \leq k < p
\end{cases}.\]

The corresponding inverse pair is \((C, D) \in K^{m \times p \times p} \times K^{p \times n \times p}\) such that
\[
C[i, t, k] = \begin{cases} 
\alpha_{it}^{-1} & \text{if } 0 \leq t = k \leq p \\
0 & \text{otherwise}
\end{cases} \forall \begin{cases} 
0 \leq i < m \\
0 \leq t < p \\
0 \leq k < p
\end{cases},
\]
\[
D[t, j, k] = \begin{cases} 
\beta_{tj}^{-1} & \text{if } 0 \leq t = k \leq p \\
0 & \text{otherwise}
\end{cases} \forall \begin{cases} 
0 \leq t < p \\
0 \leq j < n \\
0 \leq k < p
\end{cases}.
\]
Examples of non-classical tensorial orbits associated with $H \in \mathbb{K}^{m \times n \times p}$ are

\[
\begin{align*}
\{ \text{Prod} \left( P^T, Q^T, \text{Prod} \left( \text{Prod} \left( U, H, V \right), E^T, F^T \right) \right) \} : (U, V), (E, F) (P, Q) \in \text{GL}_{m \times n \times p} (m \times p \times p, p \times n \times p, \mathbb{K}) \}, \\
\{ \text{Prod} \left( P^T, Q^T, \text{Prod} \left( U, \text{Prod} \left( H, E^T, F^T \right), V \right) \right) : (U, V), (E, F) (P, Q) \in \text{GL}_{m \times n \times p} (m \times p \times p, p \times n \times p, \mathbb{K}) \}, \\
\{ \text{Prod} \left( \text{Prod} \left( P^T, Q^T, \text{Prod} \left( U, H, V \right) \right), E^T, F^T \right) : (U, V), (E, F) (P, Q) \in \text{GL}_{m \times n \times p} (m \times p \times p, p \times n \times p, \mathbb{K}) \}, \\
\{ \text{Prod} \left( U, \text{Prod} \left( P^T, Q^T, H \right), E^T, F^T \right), V \right) : (U, V), (E, F) (P, Q) \in \text{GL}_{m \times n \times p} (m \times p \times p, p \times n \times p, \mathbb{K}) \}, \\
\{ \text{Prod} \left( U, \text{Prod} \left( P^T, Q^T, H \right), E^T, F^T \right), V \right) : (U, V), (E, F) (P, Q) \in \text{GL}_{m \times n \times p} (m \times p \times p, p \times n \times p, \mathbb{K}) \}.
\end{align*}
\]

For convenience we adopt the notational convention such that

\[
\text{Prod} \left( C, \text{Prod} \left( A, X, B \right), D \right) = X, \ \forall \ X \in \mathbb{K}^{m \times n \times p} \Leftrightarrow \begin{cases}
C = A^{-1} \\
D = B^{-1}
\end{cases}
\]

\[
\text{Prod} \left( \text{Prod} \left( X^T, B^T, A^T \right), D^T, C^T \right) = X, \ \forall \ X \in \mathbb{K}^{m \times n \times p} \Leftrightarrow \begin{cases}
C^T = (A^T)^{-1} \\
D^T = (B^T)^{-1}
\end{cases}
\]

and

\[
\text{Prod} \left( D^T, C^T, \text{Prod} \left( B^T, A^T, X^T \right) \right) = X, \ \forall \ X \in \mathbb{K}^{m \times n \times p} \Leftrightarrow \begin{cases}
C^{T^2} = (A^{T^2})^{-1} \\
D^{T^2} = (B^{T^2})^{-1}
\end{cases}
\]

### 6 SVD via Symmetrization.

Recall the canonical $\mathbb{R}^{2 \times 2}$ representation of the field $\mathbb{C}$ is prescribed by the correspondence

\[
(a + b\sqrt{-1}) \leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.
\]

(6)

We therefore express an arbitrary $M \in \mathbb{C}^{n \times n}$ as a new matrix $M' \in \mathbb{R}^{2n \times 2n}$ obtained by replacing each entry of $M$ by the corresponding $2 \times 2$ real matrix representation. It follows that no loss of generality incurs from restricting the discussion to real matrices.
It is well known that the Singular Value Decomposition (or SVD for short) of \( A \in \mathbb{R}^{n \times n} \) is obtained by solving for matrices \( U, V, \text{diag}(\mu) \) and \( \text{diag}(\nu) \) in the constraints

\[
\begin{align*}
\begin{cases}
(\mathbf{A} \mathbf{A}^\top)^k = (\mathbf{U} \text{diag}(\mu)^k) (\mathbf{U} \text{diag}(\mu)^k)^\top, \quad \forall k \in \{0, 1\}. \\
(\mathbf{A}^\top \mathbf{A})^k = (\text{diag}(\nu)^k \mathbf{V}) (\text{diag}(\nu)^k \mathbf{V})^\top.
\end{cases}
\end{align*}
\]

A distinctive feature of SVD constraints is that it can be equivalently formulated as a pair of fixed point constraints of the form

\[
\begin{align*}
\begin{cases}
(\mathbf{A} \mathbf{A}^\top) (\text{diag}(\mu)^k) = \mathbf{U} \text{diag}(\mu) \\
(\text{diag}(\nu)^k \mathbf{V}) = \mathbf{V} \text{diag}(\nu).
\end{cases}
\end{align*}
\]

(7)

The fixed point formulation in Eq. (7) lies at the heart of iterative procedures for SVD numerical approximation schemes which fortunately extend to hypermatrices. Characteristic polynomials which eliminate the entries of \( U \) and \( V \) from the SVD constraints in Eq. (1) are

\[
\begin{align*}
\begin{cases}
\text{Rank}(\mathbf{A} \mathbf{A}^\top - (\mathbf{U} \mu_i \mathbf{I}_n) (\mathbf{U} \mu_i \mathbf{I}_n)^\top) < \text{Rank}(\mathbf{A} \mathbf{A}^\top) \quad \Rightarrow \quad \det(\mathbf{A} \mathbf{A}^\top - \mu_i^2 \mathbf{I}_n) = 0 \\
\text{and} \quad \det(\mathbf{A} \mathbf{A}^\top - \nu_i^2 \mathbf{I}_n) = 0
\end{cases}, \quad \forall 0 \leq i < n \quad (8)
\end{align*}
\]

It is well known that \( \{\mu_i^2 : 0 \leq i < n\} = \{\nu_i^2 : 0 \leq i < n\} \), and as a result we can take

\[
\text{diag}(\mu) = \text{diag}(\sigma) = \text{diag}(\nu).
\]

Once the singular values are known, we simultaneously solve for entries of \( U \) via constraints given by

\[
(\mathbf{I}_n \otimes \text{Vandermonde} \{\sigma \circ \sigma\}) \text{vec}(\mathbf{U}[i,k] \mathbf{U}[j,k] : 0 \leq i < j < n, 0 \leq k < n) = \text{vec}\left(\left(\mathbf{A} \mathbf{A}^\top\right)^k[i,j] : 0 \leq i < j < n, 0 \leq k < n\right)
\]

and also simultaneously solve for all entries of \( V \) via constraints given by

\[
(\mathbf{I}_n \otimes \text{Vandermonde} \{\sigma \circ \sigma\}) \text{vec}(\mathbf{V}[k,i] \mathbf{V}[k,j] : 0 \leq i < j < n, 0 \leq k < n) = \text{vec}\left(\left(\mathbf{A}^\top \mathbf{A}\right)^k[i,j] : 0 \leq i < j < n, 0 \leq k < n\right).
\]

Note that the constraints above express a composition of constraints of type one and two as described in [GG18]. We now extend to third order hypermatrices the matrix symmetrization formulation of the SVD. For an arbitrary
$A \in \mathbb{C}^{n \times n \times n}$, the three products of transposes which necessarily result in a symmetric hypermatrix are

\[
\begin{align*}
\text{Prod} \left( A, A^T, A^T \right)^T &= \text{Prod} \left( A, A^T, A^T \right) \\
\text{Prod} \left( A^T, A, A^T \right)^T &= \text{Prod} \left( A^T, A, A^T \right) \\
\text{Prod} \left( A^T, A^T, A \right)^T &= \text{Prod} \left( A^T, A^T, A \right)
\end{align*}
\]

Just as was done for matrices, we devise the SVD from the spectral decomposition of these symmetric products of transposes. Recall that the scaling hypermatrices described in section 5 are hypermatrix analog of diagonal matrices and characterized by the constraints

$$D^{\circ 3} \in \left\{ \text{Prod} \left( D^T, D^T, D \right), \text{Prod} \left( D, D^T, D^T \right), \text{Prod} \left( D^T, D, D^T \right) \right\},$$

where $D^{\circ 3}$ represents the Hadamard exponent of the scaling hypermatrix $D$. Here we recall that the Hadamard exponent $H^{\circ z}$ is defined for an arbitrary $H \in \mathbb{C}^{m \times n \times p}$ and $z \in \mathbb{C}$ as follows

$$H^{\circ z}[i,j,k] = \begin{cases} 
(H[i,j,k])^z & \text{if } H[i,j,k] \neq 0 \\
0 & \text{otherwise}
\end{cases}.$$

The above constraints are thus the hypermatrix diagonality constraints generalized from the following matrix constraints

$$\text{Prod} \left( D^T, D \right) = D^{\circ 2} = \text{Prod} \left( D, D^T \right).$$

Note that in contrast to the matrix case, scaling hypermatrices are not necessarily symmetric. For simplicity we describe the detailed derivation of the SVD for an arbitrary side length two cubic hypermatrix $A \in \mathbb{C}^{2 \times 2 \times 2}$ whose entries are given by

$$A[:, :, 0] = \begin{pmatrix} a_{000} & a_{010} \\
a_{100} & a_{110} \end{pmatrix}, \quad A[:, :, 1] = \begin{pmatrix} a_{001} & a_{011} \\
a_{101} & a_{111} \end{pmatrix},$$
associated with the spectral decomposition constraints

\[
\begin{align*}
\text{Prod} \left( A, A^\top, A^\top \right) &= \text{Prod} \left( \text{Prod} \left( U, D_\mu, D_\mu^\top \right), \text{Prod} \left( U, D_\mu, D_\mu^\top \right)^2, \text{Prod} \left( U, D_\mu, D_\mu^\top \right)^3 \right) \\
D_\mu^{3} &= \text{Prod} \left( D_\mu^\top, D_\mu^{2^2}, D_\mu \right) \\
D_\mu \left[ ; ; ; 0 \right] &= \begin{pmatrix} \mu_{00} & 0 \\ \mu_{01} & 0 \end{pmatrix} \\
D_\mu \left[ ; ; ; 1 \right] &= \begin{pmatrix} 0 & \mu_{01} \\ 0 & \mu_{11} \end{pmatrix} \\
\text{Prod} \left( U, D_\mu, D_\mu^\top \right) \left[ i, j, k \right] &= \mu_{\min \{i,j\} \max \{i,j\}} \mu_{\min \{j,k\} \max \{j,k\}} u_{ijk}
\end{align*}
\]

\[
\begin{align*}
\text{Prod} \left( A^\top, A, A^\top \right) &= \text{Prod} \left( \text{Prod} \left( D_\nu^\top, V, D_\nu \right)^\top, \text{Prod} \left( D_\nu^\top, V, D_\nu \right), \text{Prod} \left( D_\nu^\top, V, D_\nu \right)^2 \right) \\
D_\nu^{3} &= \text{Prod} \left( D_\nu^\top, D_\nu^{2^2}, D_\nu^\top \right) \\
D_\nu \left[ ; ; ; 0 \right] &= \begin{pmatrix} \nu_{00} & \nu_{01} \\ 0 & 0 \end{pmatrix} \\
D_\nu \left[ ; ; ; 1 \right] &= \begin{pmatrix} 0 & 0 \\ \nu_{01} & \nu_{11} \end{pmatrix} \\
\text{Prod} \left( D_\nu^\top, V, D_\nu \right) \left[ i, j, k \right] &= \nu_{\min \{i,k\} \max \{i,k\}} \nu_{\min \{j,k\} \max \{j,k\}} v_{ijk}
\end{align*}
\]
and

\[
\begin{align*}
\text{Prod} \left( A^{T2}, A^{T}, A \right) &= \text{Prod} \left( \text{Prod} \left( D_{\omega}, D_{\omega}^{T}, W \right)^{T2}, \text{Prod} \left( D_{\omega}, D_{\omega}^{T}, W \right)^{T}, \text{Prod} \left( D_{\omega}, D_{\omega}^{T}, W \right) \right) \\
D_{\omega}^{3} &= \text{Prod} \left( D_{\omega}, D_{\omega}^{T}, D_{\omega}^{T2} \right) \\
D_{\omega}[:,:,:0] &= \begin{pmatrix} \omega_{00} & 0 \\ 0 & \omega_{01} \end{pmatrix} \\
D_{\omega}[:,:,:1] &= \begin{pmatrix} \omega_{01} & 0 \\ 0 & \omega_{11} \end{pmatrix} \\
\text{Prod} \left( D_{\omega}, D_{\omega}^{T}, W \right)_{[i,j,k]} &= \omega_{\min\{i,k\}} \omega_{\min\{i,j\}} \omega_{\max\{i,k\}} \omega_{\max\{i,j\}} w_{ijk}
\end{align*}
\]

The hypermatrices \( U, V \) and \( W \) whose individual slices correspond to eigenmatrices are subject to the following third order orthogonality constraints

\[
\text{Prod} \left( U, U^{T2}, U^{T} \right)_{[i,j,k]} = \text{Prod} \left( V^{T}, V, V^{T2} \right)_{[i,j,k]} = \text{Prod} \left( W^{T2}, W^{T}, W \right)_{[i,j,k]} = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}
\]

A distinctive feature of SVD constraints quite analogous to the matrix setting is the equivalent formulation as fixed point constraints of the form

\[
\begin{align*}
\text{Prod} \left( U, D_{\mu}, D_{\mu}^{T} \right) &= \text{Prod} \left( \text{Prod} \left( A^{T}, A, A^{T2} \right), \left( \text{Prod} \left( U, D_{\mu}, D_{\mu}^{T} \right)^{T2} \right)^{-11}, \left( \text{Prod} \left( U, D_{\mu}, D_{\mu}^{T} \right)^{T} \right)^{-12} \right), \\
\text{Prod} \left( D_{\nu}^{T}, V, D_{\nu} \right) &= \text{Prod} \left( \left( \text{Prod} \left( D_{\nu}^{T}, V, D_{\nu} \right)^{T} \right)^{-10}, \text{Prod} \left( A^{T}, A, A^{T2} \right), \left( \text{Prod} \left( D_{\nu}^{T}, V, D_{\nu} \right)^{T2} \right)^{-12} \right), \\
\text{Prod} \left( D_{\omega}, D_{\omega}^{T}, W \right) &= \text{Prod} \left( \left( \text{Prod} \left( D_{\omega}, D_{\omega}^{T}, W \right)^{T2} \right)^{-10}, \left( \text{Prod} \left( D_{\omega}, D_{\omega}^{T}, W \right)^{T} \right)^{-11}, \text{Prod} \left( A^{T}, A, A^{T2} \right) \right).
\end{align*}
\]

Just as in the matrix case, the characteristic polynomials which determine the entries of the scaling hypermatrices (hypermatrix analog of the singular values) are given by constraints of the form

\[
\forall 0 \leq i < 2, \quad \text{Rank} \left( A, A^{T2}, A^{T} \right) > \text{Rank} \left\{ \text{Prod} \left( A, A^{T2}, A^{T} \right) - \text{Prod} \left( \bar{U}_{i}, \bar{U}_{i}^{T2}, \bar{U}_{i}^{T} \right) \right\}
\]

where

\[
\bar{U}_{i} = \text{Prod} \left( U, D_{\mu}^{[i]}, D_{\mu}^{[i]} \right)^{T}.
\]
∀ 0 ≤ i < 2, \quad \text{Rank} \left( A^T, A, A^{-2} \right) > \text{Rank} \left\{ \text{Prod} \left( A^T, A, A^{-2} \right) - \text{Prod} \left( \tilde{V}_i^T, \tilde{V}_i, \tilde{V}_i^{-2} \right) \right\}

where

\tilde{V}_i = \text{Prod} \left( \left( D^i_{\nu} \right)^T, V, D^i_{\nu} \right).

and

∀ 0 ≤ i < 2, \quad \text{Rank} \left( A^{-2}, A^T, A \right) > \text{Rank} \left\{ \text{Prod} \left( A^{-2}, A^T, A \right) - \text{Prod} \left( \tilde{W}_i^{-2}, \tilde{W}_i^T, \tilde{W}_i \right) \right\}

where

\tilde{W}_i = \text{Prod} \left( D^i_{\nu}, \left( D^i_{\nu} \right)^T, W \right).

The entries of the scaling hypermatrices above are given by

\begin{align*}
D^i_\mu [\ :, : , 0] &= \begin{pmatrix}
\mu_{00} & 0 \\
\mu_{01} & 0
\end{pmatrix} \\
D^i_\nu [\ :, : , 0] &= \begin{pmatrix}
\nu_{00} & \nu_{01} \\
0 & 0
\end{pmatrix} \\
D^i_\omega [\ :, : , 0] &= \begin{pmatrix}
\omega_{00} & 0 \\
0 & \omega_{01}
\end{pmatrix} \\
D^i_\mu [\ :, : , 1] &= \begin{pmatrix}
0 & \mu_{00} \\
0 & \mu_{01}
\end{pmatrix} \\
D^i_\nu [\ :, : , 1] &= \begin{pmatrix}
0 & 0 \\
\nu_{00} & \nu_{01}
\end{pmatrix} \\
D^i_\omega [\ :, : , 1] &= \begin{pmatrix}
0 & \omega_{00} \\
0 & \omega_{01}
\end{pmatrix} \\
D^i_\mu [\ :, : , 0] &= \begin{pmatrix}
\mu_{01} & 0 \\
\mu_{11} & 0
\end{pmatrix} \\
D^i_\nu [\ :, : , 0] &= \begin{pmatrix}
\nu_{01} & \nu_{11} \\
0 & 0
\end{pmatrix} \\
D^i_\omega [\ :, : , 0] &= \begin{pmatrix}
\omega_{01} & 0 \\
0 & \omega_{11}
\end{pmatrix} \\
D^i_\mu [\ :, : , 1] &= \begin{pmatrix}
0 & \mu_{01} \\
0 & \mu_{11}
\end{pmatrix} \\
D^i_\nu [\ :, : , 1] &= \begin{pmatrix}
0 & 0 \\
\nu_{01} & \nu_{11}
\end{pmatrix} \\
D^i_\omega [\ :, : , 1] &= \begin{pmatrix}
\omega_{01} & 0 \\
0 & \omega_{11}
\end{pmatrix}
\end{align*}

Consequently characteristic polynomial constraints are expressed by

\begin{align}
\forall 0 \leq i < 2, \quad &\begin{cases}
0 = \text{det} \left\{ \text{Prod} \left( A, A^{-2}, A^T \right) - \text{Prod} \left( \tilde{U}_i, \tilde{U}_i^{-2}, \tilde{U}_i^T \right) \right\}, \\
0 = \text{det} \left\{ \text{Prod} \left( A^T, A, A^{-2} \right) - \text{Prod} \left( \tilde{V}_i^T, \tilde{V}_i, \tilde{V}_i^{-2} \right) \right\}, \\
0 = \text{det} \left\{ \text{Prod} \left( A^{-2}, A^T, A \right) - \text{Prod} \left( \tilde{W}_i^{-2}, \tilde{W}_i^T, \tilde{W}_i \right) \right\}.
\end{cases} \\
\tag{14}
\end{align}

Using the hypermatrix determinant formula introduced by Gnang and Yuval in [GF17], the corresponding constraints
are expressed as

\[
\begin{align*}
0 &= (\mu^6_{01} - a^3_{101} - a^3_{111}) (a_{000}a_{001}a_{100} + a_{010}a_{011}a_{110})^3 - (\mu^6_{00} - a^3_{000} - a^3_{010}) (a_{001}a_{100}a_{101} + a_{011}a_{110}a_{111})^3 \\
0 &= (\mu^6_{11} - a^3_{111}) (a_{000}a_{010}a_{100} + a_{010}a_{011}a_{110})^3 - (\mu^6_{01} - a^3_{000} - a^3_{010}) (a_{001}a_{100}a_{101} + a_{011}a_{110}a_{111})^3
\end{align*}
\]

Once we have determined the entries of the scaling values, we simultaneously solve for all entries of \(U\) via constraints given by

\[
\begin{pmatrix}
\mu^6_{00} & \mu^6_{01} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu^4_{00} & \mu^4_{01} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu^2_{00} & \mu^2_{01} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu^6_{01} & \mu^6_{11}
\end{pmatrix}
\begin{pmatrix}
u^3_{000} \\
u^3_{u10} \\
\nu^3_{u00}u_01u_{100} \\
u^3_{u10}u_{011}u_{110} \\
u^3_{u00}u_{100}u_{101} \\
u^3_{u011}u_{101}u_{111} \\
u^3_{v10} \\
u^3_{v111}
\end{pmatrix}
= \begin{pmatrix}
a^3_{000} + a^3_{100} \\
a^3_{001} \\
a^3_{000}u_{010}a_{100} + a_{010}a_{011}a_{110} \\
0 \\
a_{001}a_{100}a_{101} + a_{011}a_{110}a_{111} \\
1 \\
a^3_{101} + a^3_{111}
\end{pmatrix},
\]

solve for all entries of \(V\) via constraints given by

\[
\begin{pmatrix}
\nu^6_{00} & \nu^6_{01} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \nu^4_{00} & \nu^4_{01} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \nu^2_{00} & \nu^2_{01} & \nu^2_{01} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \nu^6_{01} & \nu^6_{11}
\end{pmatrix}
\begin{pmatrix}
u^3_{v00} \\
u^3_{v01} \\
\nu^3_{v00}v_{01}v_{100} \\
\nu^3_{v01}v_{011}v_{101} \\
\nu^3_{v01}v_{100}v_{110} \\
\nu^3_{v01}v_{101}v_{111} \\
\nu^3_{v10} \\
\nu^3_{v111}
\end{pmatrix}
= \begin{pmatrix}
a^3_{000} + a^3_{101} \\
a^3_{000} \\
a_{000}a_{010}a_{100} + a_{010}a_{011}a_{110} \\
0 \\
a_{001}a_{100}a_{101} + a_{011}a_{110}a_{111} \\
1 \\
a^3_{101} + a^3_{111}
\end{pmatrix},
\]

and also solve for all entries of \(W\) via constraints given by

\[
\begin{pmatrix}
\omega^6_{00} & \omega^6_{01} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \omega^4_{00} & \omega^4_{01} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega^2_{00} & \omega^2_{01} & \omega^2_{01} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \omega^6_{01} & \omega^6_{11}
\end{pmatrix}
\begin{pmatrix}
u^3_{w00} \\
u^3_{w10} \\
u^3_{w00}w_{001}w_{010} \\
u^3_{w10}w_{011}w_{110} \\
u^3_{w00}w_{010}w_{011} \\
u^3_{w01}w_{101}w_{111} \\
u^3_{w11} \\
u^3_{w111}
\end{pmatrix}
= \begin{pmatrix}
a^3_{000} + a^3_{100} \\
a^3_{000} + a^3_{100} \\
a_{000}a_{001}a_{010} + a_{100}a_{011}a_{110} \\
0 \\
a_{001}a_{010}a_{011} + a_{101}a_{110}a_{111} \\
1 \\
a^3_{011} + a^3_{111}
\end{pmatrix},
\]
Note that constraints above correspond to a composition of constraints of type one and two discussed in [GG18]. The hypermatrix SVD is thus expressed by the following sum of outer products

\[ A = \sum_{0 \leq i,j,k < 2} \sigma_{i,j,k} \text{Prod} \left( \tilde{U} [:, i, :], \tilde{V} [:, :, j], \tilde{W} [k, :, :] \right) \]

(15)

where

\[ \tilde{U} = \text{Prod} \left( U, D_{\mu}, D_{\mu}^\top \right), \quad \tilde{V} = \text{Prod} \left( D_{\nu}^\top, V, D_{\nu} \right), \quad \text{and} \quad \tilde{W} = \text{Prod} \left( D_{\omega}, D_{\omega}^\top, W \right). \]

The coefficients \( \{ \sigma_{i,j,k} : 0 \leq i, j, k < 2 \} \subset \mathbb{C} \) of the linear combination in Eq. (15) are obtained through solving a system of linear equations. The expansion in Eq. (15) is equivalently expressed as

\[ A = \text{Prod} \left( U', V', W' \right), \]

where \( U' \in \mathbb{C}^{2 \times \| \sigma \|_{\ell_0} \times 2}, \ V' \in \mathbb{C}^{2 \times \| \sigma \|_{\ell_0} \times 2}, \) and \( W' \in \mathbb{C}^{\| \sigma \|_{\ell_0} \times 2 \times 2}, \) and \( \sigma \) is the vector whose entries are made up of the coefficients \( \{ \sigma_{i,j,k} : 0 \leq i, j, k < 2 \} \) in the linear combination. As an illustration, consider the task of expressing the SVD of hypermatrices of arbitrary side lengths generated from \( 2 \times 2 \times 2 \) hypermatrices by taking combinations of direct sums and Kronecker products. As shown in [GM18], and similarly to the matrix case, when given the SVD of hypermatrices \( A_0 \in \mathbb{C}^{m \times m} \) and \( A_1 \in \mathbb{C}^{n \times n} \)

\[ A_0 = \text{Prod} \left( U'_0, V'_0, W'_0 \right), \quad A_1 = \text{Prod} \left( U'_1, V'_1, W'_1 \right), \]

then the SVD of \( A_0 \oplus A_1 \) and \( A_0 \otimes A_1 \) are expressed by

\[ A_0 \otimes A_1 = \text{Prod} \left( U'_0 \otimes U'_1, V'_0 \otimes V'_1, W'_0 \otimes W'_1 \right) \in \mathbb{C}^{mn \times mn \times mn}, \]

\[ A_0 \oplus A_1 = \text{Prod} \left( U'_0 \oplus U'_1, V'_0 \oplus V'_1, W'_0 \oplus W'_1 \right) \in \mathbb{C}^{(m+n) \times (m+n) \times (m+n)}, \]

7 Action on vector spaces and orthogonality.

7.1 The matrix case.

The action of a matrix in \( \mathbb{C}^{n \times n} \) on the vector space \( \mathbb{C}^{n \times 1} \) can be seen as a special instance of a more general (not necessarily linear) map introduced in [GF17] specified in terms of a matrix pair \( (A, B) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \) as follows

\[ \mathcal{T}_{A, B} : \mathbb{C}^{n \times 1} \to \mathbb{C}^{n \times 1}, \quad y = \mathcal{T}_{A, B} (x), \]

such that

\[ \forall 0 \leq k < n, \quad y [k] = \sqrt{\text{Prod}_{P_k} (x^\top, x)} \quad \text{where} \quad P_k = \text{Prod}_{I_{n, k}, I_{n, k}, I_{n, k}} (A, B). \]

(16)

Note that the map \( \mathcal{T}_{A, B} \) is determined up to the sign of the entries of its output. Invertibility in this context means that neither of the \( n \) univariate polynomials in

\[ \text{Resultant}_x \left\{ \text{Prod}_{P_k} (x^\top, x) : 0 \leq k < n \right\}, \]

15
is an identically non-zero constant. For instance when \( n = 2 \) and

\[
A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \quad B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix},
\]

the map \( \mathcal{T}_{A,B} \) is invertible if neither of the polynomials in

\[
\{Q_0(x_0), Q_1(x_1)\} = \text{Resultant}_x \left\{ \prod_{k=1}^n \left( x^\top \right)^k : 0 \leq k < 2 \right\}
\]

explicitly given by

\[
Q_0(x_0) = (a_{01} b_{10} x_0^2 - y_1^2) a_{10}^2 b_{01}^2 - 2 (a_{00} b_{00} x_0^2 - y_0^2) (a_{01} b_{10} x_0^2 - y_1^2) a_{10} a_{11} b_{01} b_{11} + (a_{00} b_{00} x_0^2 - y_0^2)^2 a_{11}^2 b_{11}^2 + (-1) (a_{01} b_{10} x_0^2 - y_2^2) (a_{10} b_{00} x_0 + a_{00} b_{01} x_0) (a_{11} b_{10} x_0 + a_{01} b_{11} x_0) a_{10} b_{01} + (a_{01} b_{10} x_0^2 - y_1^2) (a_{10} b_{00} x_0 + a_{00} b_{01} x_0)^2 a_{11} b_{11} - (a_{00} b_{00} x_0^2 - y_0^2) (a_{10} b_{00} x_0 + a_{00} b_{01} x_0) (a_{11} b_{10} x_0 + a_{01} b_{11} x_0) a_{11} b_{11}.
\]

and

\[
Q_1(x_1) = (a_{11} b_{11} x_1^2 - y_1^2) a_{00}^2 b_{00}^2 - 2 (a_{10} b_{01} x_1^2 - y_0^2) (a_{11} b_{11} x_1^2 - y_1^2) a_{00} a_{01} b_{00} b_{10} + (a_{10} b_{01} x_1^2 - y_0^2)^2 a_{01}^2 b_{10}^2 + (-1) (a_{11} b_{11} x_1^2 - y_2^2) (a_{10} b_{00} x_1 + a_{00} b_{01} x_1) (a_{11} b_{10} x_1 + a_{01} b_{11} x_1) a_{00} b_{00} + (a_{11} b_{11} x_1^2 - y_1^2) (a_{10} b_{00} x_1 + a_{00} b_{01} x_1)^2 a_{01} b_{10} - (a_{10} b_{01} x_1^2 - y_0^2) (a_{10} b_{00} x_1 + a_{00} b_{01} x_1) (a_{11} b_{10} x_1 + a_{01} b_{11} x_1) a_{01} b_{10}.
\]

is an identically non-zero constant. Furthermore when \( A = B^{-1} \) the map \( \mathcal{T}_{A,B} \) is subject to the resolution of identity

\[
y = \mathcal{T}_{A,B}(x) \quad \implies \quad \text{Prod}(y^\top, y) = \text{Prod}(x^\top, x).
\]

In other words the map preserves the sum of squares of the entries. Also note that when \( A = B^\top \), the map \( \mathcal{T}_{A,B} \) expresses up to the sign of the entries a linear transformation. In particular, when \( A B = I_n \) and \( B = A^\top \in \mathbb{R}^{n \times n} \) the map \( \mathcal{T}_{A,A^\top} \) expresses up to the entry signs a linear isometry of \( \mathbb{R}^{n \times 1} \), thereby emphasizing the importance of matrix orthogonality. Recall for illustration purposes that

\[
X = \begin{pmatrix} x_0 & x_2 \\ x_1 & x_3 \end{pmatrix},
\]

is orthogonal if \( X \cdot X^\top = I_2 \). Hence

\[
X \cdot X^\top = I_2 \quad \implies \quad \begin{cases} 
\frac{x_0^2}{2} + \frac{x_2^2}{2} = 1 \\
x_0 x_1 + x_2 x_3 = 0 \\
x_1^2 + \frac{x_3^2}{2} = 1
\end{cases}.
\]
On the one hand,

\[
0 = \left( \prod_{0 \leq i < 4} x_i \right) \implies X \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.
\]

On the other hand, when \( 0 \neq \left( \prod_{0 \leq i < 4} x_i \right) \) implies that

\[
0 = x_0 x_1 + x_2 x_3 \iff x_0 x_1 x_2^{-1} x_3^{-1} = -1, \ \forall k \in \mathbb{Z}.
\]

\[
\implies \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s/t \r \\ r \s \\ s \t \end{pmatrix} \implies X = \begin{pmatrix} -s/t & s \\ r & t \end{pmatrix}
\tag{17}
\]

By normalizing the row of \( X \) we obtain the following parametrization of the orthogonal matrices

\[
X = \begin{pmatrix} \frac{-s}{\sqrt{(-s/t)^2 + s^2}} & \frac{s}{\sqrt{(-s/t)^2 + s^2}} \\ \frac{r}{\sqrt{r^2 + t^2}} & \frac{t}{\sqrt{r^2 + t^2}} \end{pmatrix}, \ s \in \{-1, 1\}, \ \text{and} \ r \neq t\sqrt{-1}.
\tag{18}
\]

To express some important invariants of orthogonal matrices, consider the index rotation operation introduced in [GM13], noted \( A^{R_\theta} \) for \( \theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\} \), which generalizes the matrix transpose operation and is defined for an arbitrary \( A \in \mathbb{C}^{n \times n} \) as

\[
A^{R_\theta} = A, \ A^{R_{\frac{\pi}{2}}} = A^\top Q, \ A^{R_{\pi}} = QAQ, \ A^{R_{\frac{3\pi}{2}}} = QA^\top,
\]

where

\[
Q = \sum_{0 \leq i < n} I_n [i, n-i-1] I_n [i, :].
\]

Alternatively, we can also express the index rotation operation entry-wise as

\[
(A^{R_\theta})_{[i,j]} = A \left[ \left( i - \frac{n-1}{2} \right) \cos \theta + \left( \frac{n-1}{2} - j \right) \sin \theta + \frac{n-1}{2}, \left( i - \frac{n-1}{2} \right) \sin \theta - \left( \frac{n-1}{2} - j \right) \cos \theta + \frac{n-1}{2} \right].
\]

For instance for a given 3 \( \times \) 3 matrix \( A \), we have

\[
A^{R_\theta} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}, \ A^{R_{\frac{\pi}{2}}} = \begin{pmatrix} a_{20} & a_{10} & a_{00} \\ a_{01} & a_{11} & a_{01} \\ a_{22} & a_{12} & a_{02} \end{pmatrix}, \ A^{R_{\pi}} = \begin{pmatrix} a_{22} & a_{12} & a_{02} \\ a_{21} & a_{11} & a_{01} \\ a_{00} & a_{10} & a_{20} \end{pmatrix}, \ A^{R_{\frac{3\pi}{2}}} = \begin{pmatrix} a_{02} & a_{12} & a_{22} \\ a_{01} & a_{11} & a_{21} \\ a_{00} & a_{10} & a_{20} \end{pmatrix}.
\]

Following immediately from the orthogonal matrix parametrization in Eq. (18), we can obtain the properties of orthogonal matrices,

\[
XX^\top = I_2 \implies X^{R_{\frac{2\pi k}}}(X^{R_{\frac{2\pi k}}})^\top = I_2, \ \forall k \in \{0, 4\} \cap \mathbb{Z}
\]
In particular, this distinction will enable us to generalize the matrix conjugate transpose operation. Note that if

These operations distinguish actions defined on individual block matrices from actions defined on the whole matrix.

Furthermore, given \( XX^\top = I_2 = YY^\top \) we have

\[
(XY)(XY)^\top = I_2, \quad (X \otimes Y)(X \otimes Y)^\top = I_2 \oplus I_2, \quad \text{and} \quad (X \otimes Y)(X \otimes Y)^\top = I_2 \otimes I_2.
\]

The canonical matrix representation of complex number described in Eq. (6) motivates a variant of the transpose and index rotation operation which operates block partitioned matrices. More precisely, consider the variant of the transpose and index rotation operations defined on block matrices where each block is a square matrix of the same size

\[
\begin{pmatrix}
A_{00} & A_{01} \\
A_{10} & A_{11}
\end{pmatrix}^R_{b,2\pi k} = \sum_{0 \leq i,j < 2} (I_2 [:, i] \cdot I_2 [j, :]) R_{b,2\pi k} \otimes A_{ij},
\]

where \( \{A_{00}, A_{01}, A_{10}, A_{11}\} \) correspond square matrix blocks all of the same size. Similarly

\[
\begin{pmatrix}
A_{00} & A_{01} \\
A_{10} & A_{11}
\end{pmatrix}^R_{b,2\pi k} = \sum_{0 \leq i,j < 2} (I_2 [:, i] \cdot I_2 [j, :]) R_{e,2\pi k} \otimes A_{ij}.
\]

These operations distinguish actions defined on individual block matrices from action defined on the whole matrix. This distinction will enable us to generalize the matrix conjugate transpose operation. Note that if \( A \in (O_m(\mathbb{C}))^{n \times n} \) then

\[
\forall 0 \leq i < n, \quad \left( \frac{A}{\sqrt{n}} \left( \frac{A}{\sqrt{n}} \right)^{T_b} \right)^{T_b} [i, i] = I_m = \left( \frac{A}{\sqrt{n}} \right)^{T_b} \frac{A}{\sqrt{n}} [i, i]
\]

It therefore follows that

\[
\frac{A}{\sqrt{n}} \left( \frac{A}{\sqrt{n}} \right)^{T_b} = 0_{n \times n} \otimes 0_{m \times m} \implies A = 0_{n \times n} \otimes 0_{m \times m}.
\]

The non-negativity property still holds if each block entry of \( A \) is positive scaling on an orthogonal matrix i. e.

\[
\forall 0 \leq i, j < n, \quad A[i, j] = r_{ij} A_{ij}, \quad \text{where} \quad r_{ij} > 0 \quad \text{and} \quad A_{ij} \in O_m(\mathbb{C}).
\]

In particular \( A \in (\mathbb{C}^{m \times m})^{n \times n} \) such that

\[
\forall 0 \leq i, j < n, \quad A[i, j] = r_{ij} A_{ij}, \quad \text{where} \quad r_{ij} > 0 \quad \text{and} \quad A_{ij} \in O_m(\mathbb{C})
\]
is called block unitary if

\[
\frac{A}{\sqrt{n}} \left( \frac{A}{\sqrt{n}} \right)^{T_b} = I_n \otimes I_m = \left( \frac{A}{\sqrt{n}} \right)^{T_b} \frac{A}{\sqrt{n}}
\]

In the case of \( 2 \times 2 \) block matrices, block unitary constrains for matrix blocks

\[
\{A_{00}, A_{01}, A_{10}, A_{11}\} \subset O_m(\mathbb{C})
\]
are expressed by
\[
\begin{pmatrix}
\frac{A_{00}}{\sqrt{2}} & \frac{A_{01}}{\sqrt{2}} \\
\frac{A_{10}}{\sqrt{2}} & \frac{A_{11}}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
\frac{A_{00}^T}{\sqrt{2}} & \frac{A_{01}^T}{\sqrt{2}} \\
\frac{A_{10}^T}{\sqrt{2}} & \frac{A_{11}^T}{\sqrt{2}}
\end{pmatrix}
= \begin{pmatrix}
I_m & \frac{A_{00}A_{10}^T + A_{01}A_{11}^T}{2} \\
\frac{A_{10}A_{00}^T + A_{11}A_{01}^T}{2} & I_m
\end{pmatrix}
\]

which yields the constraints
\[
\begin{align*}
A_{00}A_{10}^T + A_{01}A_{11}^T &= 0_{m \times m} \\
A_{10}A_{00}^T + A_{11}A_{01}^T &= 0_{m \times m}
\end{align*}
\]

\[
\Rightarrow \begin{pmatrix}
\frac{A_{00}}{\sqrt{2}} & \frac{A_{01}}{\sqrt{2}} \\
\frac{A_{10}}{\sqrt{2}} & \frac{A_{11}}{\sqrt{2}}
\end{pmatrix}
= \begin{pmatrix}
\frac{(-1)A_{01}A_{11}^T}{\sqrt{2}} & \frac{A_{01}}{\sqrt{2}} \\
\frac{A_{10}}{\sqrt{2}} & \frac{A_{11}}{\sqrt{2}}
\end{pmatrix}
\]

Recall the canonical representation of the complex numbers by $2 \times 2$ matrices described in Eq. (6), an unitary matrix $U \in \mathbb{C}^{n \times n}$ can therefore be seen as an $n \times n$ matrix of $2 \times 2$ block denoted $A \in (\mathbb{R}^{2 \times 2})^{n \times n}$ such that
\[
A[i,j] = \begin{pmatrix}
\Re (U[i,j]) & -\Im (U[i,j]) \\
\Im (U[i,j]) & \Re (U[i,j])
\end{pmatrix}
\]

It follows that
\[
UU^* = I_n \iff A (A^T_c)^T_c = I_n \otimes I_2.
\]

It is therefore apparent that the algebra of complex numbers closely relate to the algebra of $2 \times 2$ matrices and of real orthogonal matrices in particular.

### 7.2 The hypermatrix case.

We now extend the discussion in section 7.1 to the hypermatrix case to emphasize the compelling similarities. By analogy to the matrix case, the action on the vector space $\mathbb{C}^{n \times 1 \times 1}$ is specified in terms of a triple $A, B, C \in \mathbb{C}^{n \times n \times n}$ as
\[
\mathcal{T}_{A,B,C} : \mathbb{C}^{n \times 1 \times 1} \to \mathbb{C}^{n \times 1 \times 1}, \quad y = \mathcal{T}_{A,B,C} (x),
\]

such that
\[
\forall 0 \leq k < n, \quad \begin{cases}
y[k] = \sqrt{\text{Prod}_k (x^T, x, x)} \\
P_k = \text{Prod}_{\Delta^{(k)}} (A, B, C)
\end{cases}
\]

where $\Delta^{(t)}[i,j,k] = \begin{cases} 1 & \text{if } 0 \leq t = i = j < n \\ 0 & \text{otherwise} \end{cases}$.
Invertibility in this context means that neither of the polynomials in
\[
0 \neq \text{Resultant}_x \left\{ \text{Prod} \left( x^{T^2}, x^T, x \right) : 0 \leq k < n \right\}.
\]
is an identically non-zero constant. Recall that a triple \( \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{C}^{n \times n \times n} \) form an \textit{uncorrelated triple} if
\[
\text{Prod} \left( \mathbf{A}, \mathbf{B}, \mathbf{C} \right) [i, j, k] = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}.
\]
In the case where \( \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{C}^{n \times n \times n} \) form an uncorrelated triple, the map \( \mathcal{T}_{\mathbf{A}, \mathbf{B}, \mathbf{C}} \) is subject to the resolution of identity
\[
y = \mathcal{T}_{\mathbf{A}, \mathbf{B}, \mathbf{C}} (\mathbf{x}) \implies \text{Prod} \left( y^{T^2}, y^T, y \right) = \text{Prod} \left( x^{T^2}, x^T, x \right).
\]
In other words, the map preserves the sum of cubes of the entries.
In the case where \( \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{C}^{n \times n \times n} \) form an uncorrelated triple, \( \mathbf{B} = \mathbf{A}^{T^2} \) and \( \mathbf{C} = \mathbf{A}^T \), the map \( \mathcal{T}_{\mathbf{A}, \mathbf{A}^{T^2}, \mathbf{A}^T} \) is the third order hypermatrix analog of the vector isometry. This latter observation therefore emphasizes the importance of orthogonal hypermatrices. Recall that
\[
\mathbf{X}[:, :, 0] = \begin{pmatrix} x_0 & x_2 \\ x_1 & x_3 \end{pmatrix}, \quad \mathbf{X}[:, :, 1] = \begin{pmatrix} x_4 & x_6 \\ x_5 & x_7 \end{pmatrix},
\]
is orthogonal if
\[
\text{Prod} \left( \mathbf{X}, \mathbf{X}^{T^2}, \mathbf{X}^T \right) [:, :, 0] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{Prod} \left( \mathbf{X}, \mathbf{X}^{T^2}, \mathbf{X}^T \right) [:, :, 1] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]
The corresponding constraints are therefore given by the polynomial constraints
\[
\begin{cases}
x_1 x_4 x_5 + x_3 x_6 x_7 = 0 \\
x_0 x_1 x_4 + x_2 x_3 x_6 = 0 \\
x_0^3 + x_2^3 = 1 \\
x_3^3 + x_7^3 = 1
\end{cases}.
\]
When \( 0 \neq \prod_{0 \leq i < 8} x_i \), the above system of equations yields the equivalence of
\[
\begin{cases}
0 = x_1 x_4 x_5 + x_3 x_6 x_7 \\ 0 = x_0 x_1 x_4 + x_2 x_3 x_6
\end{cases} \iff \begin{cases}
x_1 x_3^{-1} x_4 x_5 x_6^{-1} x_7^{-1} = 1 \\
x_0 x_1 x_2^{-1} x_3^{-2} x_4 x_6^{-1} = 1
\end{cases}.
\]
\[
\begin{pmatrix}
  x_0 \\
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
  x_7
\end{pmatrix}
\quad \Rightarrow \quad
\begin{pmatrix}
  x_0 \\
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
  x_7
\end{pmatrix}
= \begin{pmatrix}
  v_0 \\
  v_1 \\
  v_2 \\
  v_3 \\
  v_4 \\
  v_5 \\
\end{pmatrix}
\]

\[
\Rightarrow \quad X[:,0] = \begin{pmatrix}
  v_0 \\
  v_1 \\
\end{pmatrix} \quad X[:,1] = \begin{pmatrix}
  v_2 \\
  v_3 \\
\end{pmatrix}
\]

We account for the sum of cube constraints by normalizing appropriate rows as follows

\[
X[:,0] = \begin{pmatrix}
  \frac{v_0 v_1 v_3}{\sqrt{v_0^2 + v_1^2}} \\
  \frac{v_0 v_5}{v_2 v_3} \\
\end{pmatrix} \quad X[:,1] = \begin{pmatrix}
  \frac{v_1}{\sqrt{v_0^2 + v_1^2}} \\
  \frac{v_0}{\sqrt{v_0^2 + v_1^2}} \\
\end{pmatrix}
\]

where

\[
\exp\left( \frac{2\pi k \sqrt{-1}}{3} \right) : 0 \leq k < 3
\]

When \(0 = \prod_{0 \leq i < 8} x_i\), The variables to be assigned zero entries are indicated in the table below:

| \(x_0 = 0, x_3 = 0, x_1 = 0\) | \(x_0 = 0, x_6 = 0, x_1 = 0\) | \(x_0 = 0, x_3 = 0, x_7 = 0, x_1 = 0\) |
| \(x_0 = 0, x_7 = 0, x_6 = 0, x_1 = 0\) | \(x_0 = 0, x_4 = 0, x_3 = 0\) | \(x_0 = 0, x_3 = 0, x_5 = 0\) |
| \(x_0 = 0, x_4 = 0, x_6 = 0\) | \(x_0 = 0, x_4 = 0, x_3 = 0, x_7 = 0\) | \(x_0 = 0, x_4 = 0, x_7 = 0, x_6 = 0\) |
| \(x_0 = 0, x_6 = 0, x_5 = 0\) | \(x_3 = 0, x_2 = 0, x_1 = 0\) | \(x_2 = 0, x_6 = 0, x_1 = 0\) |
| \(x_0 = 0, x_2 = 0, x_1 = 0\) | \(x_3 = 0, x_2 = 0, x_1 = 0\) | \(x_6 = 0, x_1 = 0\) |
| \(x_3 = 0, x_2 = 0, x_1 = 0, x_5 = 0\) | \(x_4 = 0, x_3 = 0, x_2 = 0, x_5 = 0\) | \(x_4 = 0, x_2 = 0, x_6 = 0\) |
| \(x_4 = 0, x_7 = 0, x_2 = 0\) | \(x_2 = 0, x_6 = 0, x_1 = 0, x_5 = 0\) | \(x_4 = 0, x_2 = 0, x_6 = 0, x_5 = 0\) |
| \(x_4 = 0, x_3 = 0\) | \(x_3 = 0, x_1 = 0, x_5 = 0\) | \(x_4 = 0, x_3 = 0, x_5 = 0\) |
| \(x_4 = 0, x_6 = 0\) | \(x_4 = 0, x_3 = 0, x_7 = 0\) | \(x_4 = 0, x_7 = 0, x_6 = 0\) |
| \(x_6 = 0, x_1 = 0, x_5 = 0\) | \(x_4 = 0, x_6 = 0, x_5 = 0\) |

To express some important invariants of orthogonal hypermatrices, we extend the index rotation operation to third order hypermatrices and is denoted by \(A^{R[x, \theta_y, \theta_z]}\) for \(\theta_x, \theta_y, \theta_z \in \{0, \frac{2\pi}{4}, 1, \frac{2\pi}{4}, 2, \frac{2\pi}{4}, 3, \frac{2\pi}{4}\}\) such that \(A^{R[x, y, z]}\) denotes the hypermatrix which result from performing the index rotation by angle \(\theta_x\) to each row slices of \(A\). Similarly, \(A^{R[y, \theta_y, \theta_z]}\) denotes the hypermatrix which result from performing the index rotation by angle \(\theta_y\) to each column slice of \(A\) and finally \(A^{R[x, \theta_y, \theta_z]}\) denotes the hypermatrix which result from performing the index rotation by angle \(\theta_z\) to each depth.
slice of \( A \). The index rotation \( A^{R[\theta_x,\theta_y,\theta_z]} \) is performed relative to the axis \( x, y \) and \( z \) in that order. For instance we have

\[
(A^{R[\theta_0,\theta_z]})_{i,j,k} = A \left[ (i - \frac{n-1}{2}) \cos \theta + \left( \frac{n-1}{2} - j \right) \sin \theta + \frac{n-1}{2}, (i - \frac{n-1}{2}) \cos \theta + \frac{n-1}{2}, k \right].
\]

\[
\text{Prod}(X^T \otimes X^T) = \Delta_2 \implies \text{Prod}(X^{R[2\pi k_0, 2\pi k_1, 2\pi k_2]} \otimes X^{R[2\pi k_0, 2\pi k_1, 2\pi k_2]})^{T^2} = \Delta
\]

where \( [k_0 2\pi, k_1 2\pi, k_2 2\pi] \) belong to values indicated in the table below

| \( i, j, k \) | \( \pi, 0, 0 \) | \( \pi, \pi, 0 \) | \( \pi, \pi, \pi \) |
|----------------|----------------|----------------|----------------|
| \( \pi, 0, 0 \) | \( \pi, \pi, 0 \) | \( \pi, \pi, \pi \) |
| \( 0, 0, 0 \) | \( \pi, 0, 0 \) | \( \pi, \pi, 0 \) |
| \( 0, 0, \pi \) | \( \pi, 0, \pi \) | \( \pi, \pi, \pi \) |

As shown in [GF17] if \( \text{Prod}(X^T \otimes X^T) = \Delta = \text{Prod}(Y^T \otimes Y^T) \) then we have

\[
\text{Prod}((X \oplus Y), (X \oplus Y)^T, (X \oplus Y)^T) = \Delta \oplus \Delta
\]

and

\[
\text{Prod}((X \otimes Y), (X \otimes Y)^T, (X \otimes Y)^T) = \Delta \otimes \Delta
\]

Consider block operation of hyratrices

\[
A[:,0] = \begin{pmatrix} A_{000} & A_{010} \\ A_{100} & A_{110} \end{pmatrix}, \quad A[:,1] = \begin{pmatrix} A_{001} & A_{011} \\ A_{101} & A_{111} \end{pmatrix}
\]

\[
A^T = \sum_{0 \leq i,j,k < 2} \text{Prod}(K_0[:,i,j], K_1[:,i,j], K_2[k,i,j])^T \otimes A_{ijk}
\]

s.t.

\[
K_0[:,0] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_0[:,1] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

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\[
K_1[;:, 0] = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; \quad K_1[;:, 1] = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}
\]
\[
K_2[;:, 0] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}; \quad K_2[;:, 1] = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}
\]
\[
A^{T_e} = \sum_{0 \leq i, j, k < 2} \text{Prod}(K_0 [:, i:], K_1 [:, :, j], K_2[k, :, :]) \otimes A^{T_e}_{ijk}
\]
\[
A^{R_{b, \frac{2\pi k}{4}}} = \sum_{0 \leq i, j, k < 2} \text{Prod}(K_0 [:, i:], K_1 [:, :, j], K_2[k, :, :]) \otimes A^{R_{\frac{2\pi k}{4}}}_{ijk}
\]
\[
A^{R_{e, \frac{2\pi k}{4}}} = \sum_{0 \leq i, j, k < 2} \text{Prod}(K_0 [:, i:], K_1 [:, :, j], K_2[k, :, :]) \otimes A^{R_{\frac{2\pi k}{4}}}_{ijk}
\]

Similarly to the matrix case, if \( A \) is block hypermatrix whose invididual blocks are orthogonal hypermatrices all of the same size and all subject to
\[
\text{Prod}(X, X^{T_2}, X^{T}) = \Delta
\]
then it follows that
\[
\forall 0 \leq i < n, \quad \text{Prod} \left( \frac{A}{\sqrt{n}}, \left( \frac{A^{T_e}}{\sqrt{n}} \right)^{T_b}, \left( \frac{A^{T_e}}{\sqrt{n}} \right)^{T_b} \right)[i, i] = \Delta
\]
In which case
\[
\text{Prod} \left( \frac{A}{\sqrt{n}}, \left( \frac{A^{T_e}}{\sqrt{n}} \right)^{T_b}, \left( \frac{A^{T_e}}{\sqrt{n}} \right)^{T_b} \right) = 0 \implies A = 0.
\]
In the case of \( 2 \times 2 \times 2 \) block hypermatrix
\[
A[:, :, 0] = \frac{1}{\sqrt{2}} \begin{pmatrix} A_{000} & A_{010} \\ A_{100} & A_{110} \end{pmatrix}, \quad A[:, :, 1] = \frac{1}{\sqrt{2}} \begin{pmatrix} A_{001} & A_{011} \\ A_{101} & A_{111} \end{pmatrix}
\]
where
\[
\forall X \in \{A_{000}, A_{100}, A_{010}, A_{110}, A_{001}, A_{101}, A_{011}, A_{111}\},
\]
we have
\[
\text{Prod} \left( X, X^{T_2}, X^{T} \right) = \Delta
\]
is expressed by
\[
\text{Prod} \left( A, \left( A^{T_e} \right)^{T_b}, \left( A^{T_e} \right)^{T_b} \right)[:, :, 0] =
\]

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\[
\begin{pmatrix}
\Delta \\
\frac{\text{Prod}(A_{100}, A_{001}^T, A_{000}^T) + \text{Prod}(A_{110}, A_{011}^T, A_{010}^T)}{2} \\
\frac{\text{Prod}(A_{100}, A_{101}^T, A_{100}^T) + \text{Prod}(A_{110}, A_{111}^T, A_{110}^T)}{2}
\end{pmatrix}
\]

A necessary condition for the resulting block hypermatrix to be orthogonal is specified by the constraints

\[
\begin{align*}
\text{Prod}(A_{000}, A_{100}^T, A_{001}^T) + \text{Prod}(A_{010}, A_{110}^T, A_{011}^T) &= 0 \\
\text{Prod}(A_{100}, A_{001}^T, A_{000}^T) + \text{Prod}(A_{110}, A_{011}^T, A_{010}^T) &= 0 \\
\text{Prod}(A_{100}, A_{101}^T, A_{001}^T) + \text{Prod}(A_{110}, A_{111}^T, A_{011}^T) &= 0 \\
\text{Prod}(A_{001}, A_{000}^T, A_{100}^T) + \text{Prod}(A_{011}, A_{010}^T, A_{110}^T) &= 0 \\
\text{Prod}(A_{001}, A_{100}^T, A_{101}^T) + \text{Prod}(A_{011}, A_{110}^T, A_{111}^T) &= 0 \\
\text{Prod}(A_{101}, A_{001}^T, A_{100}^T) + \text{Prod}(A_{111}, A_{011}^T, A_{110}^T) &= 0
\end{align*}
\]

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