COSMETIC SURGERY AND THE $SL(2, \mathbb{C})$ CASSON INVARIANT FOR TWO-BRIDGE KNOTS

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Dedicated to Professor Makoto Sakuma on the occasion of his 60th birthday

Abstract. We consider the cosmetic surgery problem for two-bridge knots in the 3-sphere. It is seen that all the two-bridge knots at most 9 crossings other than $9_{27} = S(49, 19) = C[2, 2, -2, 2, 2, -2]$ admits no purely cosmetic surgery pairs. Then we show that any two-bridge knot of the Conway form $[2x, 2, -2x, 2, 2, -2x]$ with $x \geq 1$ admits no cosmetic surgery pairs yielding homology 3-spheres, where $9_{27}$ appears for $x = 1$. Our advantage to prove this is using the $SL(2, \mathbb{C})$ Casson invariant.

1. Introduction

Dehn surgery can be regarded as an operation to make a ‘new’ 3-manifold from a given one. Of course the trivial Dehn surgery leaves the manifold unchanged, but ‘most’ non-trivial ones would change the topological type. In fact, Gordon and Luecke showed as the famous result in [10] that any non-trivial Dehn surgery on a non-trivial knot in the 3-sphere $S^3$ never yields $S^3$.

As a natural generalization, the following conjecture was raised.

Cosmetic Surgery Conjecture ([14, Problem 1.81(A)]): Two surgeries on inequivalent slopes are never purely cosmetic.

Here we say that two slopes are equivalent if there exists a homeomorphism of the exterior of a knot $K$ taking one slope to the other, and two surgeries on $K$ along slopes $r_1$ and $r_2$ are purely cosmetic if there exists an orientation preserving homeomorphism between the pair of the surgered manifolds.

Remark 1.1. The Cosmetic Surgery Conjecture for “chirally cosmetic” case is not true: there exist counter-examples given by Mathieu [16, 17]. In fact, for example, $(18k + 9)/(3k + 1)$- and $(18k + 9)/(3k + 2)$-surgeries on the right-hand trefoil knot in $S^3$ yield orientation-reversingly homeomorphic pairs of 3-manifolds for any non-negative integer $k$, and it can be shown that such pairs of slopes are inequivalent. That is to say, the trefoil knot admits chirally cosmetic surgery pairs along inequivalent slopes.

In this paper, we consider cosmetic surgeries on a well-known class of knots in $S^3$, the two-bridge knots. First, by using known results, we have the following in Section 2.

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Proposition 1.1. All the two-bridge knots of at most 9 crossings other than $9_{27} = S(49, 19) = C[2, 2, -2, 2, 2, -2]$ admit no purely cosmetic surgery pairs.

Here the knot $9_{27}$ in the Rolfsen’s knot table is the two-bridge knot of the Schubert form $S(49, 19)$ and the Conway form $C[2, 2, -2, 2, 2, -2]$.

In view of this, let us focus on the knot $9_{27}$. Previously, for the same reason, the first author considered this knot in [13], and it was shown that some pairs of surgeries give distinct manifolds.

In this paper, for a family of knots including the knot $9_{27}$, we have the following.

Theorem 1.2. Let $K_x$ be a two-bridge knot of the Conway form $[2x, 2, -2x, 2x, 2, -2x]$ with $x \geq 1$. Then $K_x$ admits no cosmetic surgery pairs yielding homology 3-spheres, i.e., any $\frac{1}{n}$- and $\frac{1}{m}$-surgeries on $K_x$ are not purely cosmetic for $m \neq n$. In other words, all the homology 3-spheres obtained by Dehn surgeries on $K_x$ are mutually distinct.

Remark 1.2. This cannot be achieved by using known invariants; the (original) Casson invariant and the $\tau$-invariant defined by Ozsváth-Szabó in [20], and the correction term in Heegaard Floer homology. See Section 5 for details.

Our advantage in this paper is to use the $SL(2, \mathbb{C})$ version of the Casson invariant. Very roughly speaking, for a closed orientable 3-manifold $\Sigma$, the $SL(2, \mathbb{C})$ Casson invariant $\lambda_{SL(2, \mathbb{C})}(\Sigma)$ is defined by counting the (signed) equivalence classes of representatives of the fundamental group $\pi_1(\Sigma)$ in $SL(2, \mathbb{C})$. Based on the method to enumerate the boundary slopes for two-bridge knots developed in [18], we give calculations of the $SL(2, \mathbb{C})$ Casson invariant for the knots $K_x$’s. The calculations will be given in Section 4. Before that the formulae and the method used in the calculations will be explained in Section 3.

Practically our method can be applied further. However it seems not enough to prove that all the $K_x$’s have no purely cosmetic surgery pairs.

Here we recall basic definitions and terminology about Dehn surgery.

A Dehn surgery is the following operation for a given knot $K$ (i.e., an embedded circle) in a 3-manifold $M$. Take the exterior $E(K)$ of $K$ (i.e., the complement of an open tubular neighborhood of $K$ in $M$), and then, glue a solid torus to $E(K)$. Let $\gamma$ be the slope (i.e., an isotopy class of a non-trivial unoriented simple loop) on the peripheral torus of $K$ in $M$ which is represented by the curve identified with the meridian of the attached solid torus via the surgery. Then, by $K(\gamma)$, we denote the manifold which is obtained by the Dehn surgery on $K$, and call it the 3-manifold obtained by Dehn surgery on $K$ along $\gamma$. In particular, the Dehn surgery on $K$ along the meridional slope is called the trivial Dehn surgery.

When $K$ is a knot in $S^3$, by using the standard meridian-longitude system, slopes on the peripheral torus are parametrized by rational numbers with 1/0. Thus, when a slope $\gamma$ corresponds to a rational number $r$, we call Dehn surgery along $\gamma$ $r$-surgery, and use $K(r)$ in stead of $K(\gamma)$.

2. Two-bridge knots

For two-bridge knots, we use standard definitions based on [5]. See also [3] [18].

To show Proposition 1.1 we use the following two known results.
One ingredient is the Casson invariant of 3-manifolds introduced by Casson. By using the Casson invariant, Boyer and Lines in [1] proved that a knot $K$ in $S^3$ satisfying $\Delta''_K(1) \neq 0$ has no cosmetic surgeries. Here $\Delta_K(t)$ denotes the (symmetrized) Alexander polynomial for $K$. That is, $\Delta_K(t)$ satisfies that $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$.

The other one is the following excellent result recently obtained by Ni and Wu in [19]. Suppose that $K$ is a non-trivial knot in $S^3$ and $r_1, r_2 \in \mathbb{Q} \cup \{0/1\}$ are two distinct slopes such that the surgered manifolds $K(r_1), K(r_2)$ are orientation-preservingly homeomorphic. Then $r_1, r_2$ satisfy that (a) $r_1 = -r_2$, (b) $q^2 \equiv -1 \mod p$ for $r_1 = p/q$, (c) $\tau(K) = 0$, where $\tau$ is the invariant defined by Ozsváth-Szabó in [20]. This result is obtained by using Heegaard Floer homology. We remark that, for alternating knots, $\tau(K) = -\sigma(K)$ holds [20] Theorem 1.4], where $\sigma(K)$ denotes the signature of $K$.

Now Proposition 1.1 follows from Table 1. To fill the table, we use the values given in Knotinfo [6]. Also we can use the facts that the half of $\Delta''_K(1)$ is equal to the second coefficient of the Conway polynomial. This well-known fact is due to Casson, and, for details, see [1] and [12 Section 1] for example.

Table 1. Two-bridge knots of at most 9 crossings with trivial $\tau$-invariant

| Name | Schubert Form | Alexander Polynomial | $\Delta''_K(1)$ |
|------|--------------|---------------------|----------------|
| 4    | $S(5,2)$     | $-t^{-1} + 3 - t$   | -2             |
| 6    | $S(9,7)$     | $-2t^{-1} + 5 - 2t$ | -4             |
| 63   | $S(13,5)$    | $t^{-2} - 3t^{-1} + 5 - 3t + t^2$ | 2             |
| 77   | $S(21,8)$    | $t^{-2} - 5t^{-1} + 9 - 5t + t^2$ | -2           |
| 81   | $S(13,11)$   | $-3t^{-1} + 7 - 3t$ | -6             |
| 83   | $S(17,4)$    | $-4t^{-1} + 9 - 4t$ | -8             |
| 88   | $S(25,9)$    | $2t^{-2} - 6t^{-1} + 9 - 6t + 2t^2$ | 4             |
| 89   | $S(25,7)$    | $-t^{-3} + 3t^{-2} - 5t^{-1} + 7 - 5t + 3t^2 - t^3$ | -4           |
| 812  | $S(29,12)$   | $t^{-2} - 7t^{-1} + 13 - 7t + t^2$ | -6           |
| 813  | $S(29,11)$   | $2t^{-2} - 7t^{-1} + 11 - 7t + 2t^2$ | 2            |
| 914  | $S(37,14)$   | $2t^{-2} - 9t^{-1} + 15 - 9t + 2t^2$ | -2           |
| 919  | $S(41,16)$   | $2t^{-2} - 10t^{-1} + 17 - 10t + 2t^2$ | -4           |
| 927  | $S(49,19)$   | $-t^{-3} + 5t^{-2} - 11t^{-1} + 15 - 11t + 5t^2 - t^3$ | 0            |

3. $SL(2, \mathbb{C})$ Casson invariant

We here recall briefly the definition of the $SL(2, \mathbb{C})$ Casson invariant, denoted by $\lambda_{SL(2,\mathbb{C})}$, based on [23]. Let $M$ be a closed, orientable 3-manifold with a Heegaard splitting $H_1 \cup F H_2$ with handlebodies $H_1, H_2$ and a Heegaard surface $F$, that is, $H_1 \cup H_2 = M$ and $\partial H_1 = \partial H_2 = H_1 \cap H_2 = F$. Then the inclusion maps $F \to H_i$ and $H_i \to M$ for $i = 1, 2$ induce surjections on the fundamental groups. It then follows that $X(M) = X(H_1) \cap X(H_2) \subset X(F)$, where $X(M), X(H_1), X(H_2)$ and $X(F)$ denote the $SL(2, \mathbb{C})$-character varieties for $M, H_1, H_2$ and $F$ respectively. There are natural orientations on all the character varieties determined by their complex structures. The invariant $\lambda_{SL(2,\mathbb{C})}$ is (roughly) defined as an oriented intersection number of the subspaces of characters of irreducible representations in
$X(H_1)$ and $X(H_2)$, which counts only compact, zero-dimensional components of the intersection. See [7] and [8], also [3] for detailed definition.

For the 3-manifolds obtained by Dehn surgeries on two-bridge knots, Boden and Curtis studied the $SL(2, \mathbb{C})$ Casson invariant $\lambda_{SL(2, \mathbb{C})}$ in detail in [3], and showed that $\lambda_{SL(2, \mathbb{C})}$ can be calculated as follows ([3, Theorem 2.5]): Let $K$ be a two-bridge knot with Schubert form $S(\alpha, \beta)$ and $K_{p/q}$ the 3-manifold obtained by $p/q$-surgery on $K$. Suppose that $p/q$ is not a strict boundary slope and no $p'$-th root of unity is a root of $\Delta_K(t)$, where $p'$ is $p$ if $p$ is odd and $p'/2$ if $p$ is even. Then

$$\lambda_{SL(2, \mathbb{C})}(K_{p/q}) = \begin{cases} \frac{\|p/q\|_T}{2} & \text{if } p \text{ is even}, \\ \frac{\|p/q\|_T}{2} - \frac{\alpha - 1}{4} & \text{if } p \text{ is odd}. \end{cases}$$

Here $\|p/q\|_T$ denotes the total Culler-Shalen seminorm of $p/q$.

Recall that a slope on the boundary of a knot exterior $M$ is called a boundary slope if there exists an essential surface $F$ embedded in $M$ with nonempty boundary representing the slope, and a boundary slope is called strict if it is the boundary slope of an essential surface that is not the fiber of any fibration over the circle.

In this paper, we omit the detailed definition of the total Culler-Shalen norm (see [3] for example), while the calculation of the total Culler-Shalen seminorm of a slope for a two-bridge knot was essentially given in [22]. In fact, the following explicit formula is presented as [3, Proposition 2.3].

$$\|p/q\|_T = \frac{1}{2} \left( -|p| + \sum W_i \Delta(p/q, N_i) \right)$$

Here $N_1, \ldots, N_n$ denote the boundary slopes for a two-bridge knot $K$. By the result given in [11], a boundary slope for a two-bridge knot $S(\alpha, \beta)$ is associated to a continued fraction expansion of $\alpha/\beta$. Then $W_i$ is set to be $\prod_j (|n_j| - 1)$ for the continued fraction expansion $[n_1, \ldots, n_m]$ associated to $N_i$.

Combining these formulae, we see the following.

$$\lambda_{SL(2, \mathbb{C})}(M_{p/q}) - \lambda_{SL(2, \mathbb{C})}(M_{-p/q}) = \frac{1}{2} \left( \|p/q\|_T - \|\frac{-p}{q}\|_T \right) = \frac{1}{4} \sum W_i \left( \Delta \left( \frac{p}{q}, N_i \right) - \Delta \left( \frac{-p}{q}, N_i \right) \right) = \frac{1}{4} \sum W_i (|p - qN_i| - |p - qN_i|).$$
In particular, we have the following when $p = 1$.

$$\lambda_{SL(2,\mathbb{C})}(M_{1/q}) - \lambda_{SL(2,\mathbb{C})}(M_{-1/q})$$

$$= \frac{1}{2} \sum_{q,T} \left| \frac{1}{q} \right|_T - \left| -\frac{1}{q} \right|_T$$

$$= \frac{1}{4} \sum_i W_i \left( \Delta \left( \frac{1}{q}N_i \right) - \Delta \left( -\frac{1}{q}N_i \right) \right)$$

$$= \frac{1}{4} \sum_i W_i (|1-qN_i| - |1-qN_i|)$$

$$= \frac{1}{4} \left( \sum_{N_i > 0} W_i ((qN_i - 1) - (1+qN_i)) + \sum_{N_i < 0} W_i ((1-qN_i) - (-1-qN_i)) \right)$$

$$= \frac{1}{2} \left( -\sum_{N_i > 0} W_i + \sum_{N_i < 0} W_i \right)$$

Consequently, together with the result of Ni and Wu given in [19], a two-bridge knot has no purely cosmetic surgery pairs yielding homology 3-spheres if $-\sum_{N_i > 0} W_i + \sum_{N_i < 0} W_i \neq 0$ holds.

On the other hand, in [18, Theorem 2], the following method to enumerate all the continued fractions associated to boundary slopes for a two-bridge knot was given. The boundary slopes of a two-bridge knot with Schubert form $S(\alpha, \beta)$ are associated to the continued fractions obtained by applying the following substitutions at non-adjacent positions in the simple continued fraction (i.e., the unique one with all terms positive and the last term greater than 1) of $\beta/\alpha$. The following exhibit the substitutions at position 2.

Substitution 1:

$[b_0, 2b_1, b_2, b_3, \ldots, b_n] \mapsto [b_0 + 1, (-2, 2)^{b_1-1}, -2, b_2 + 1, b_3, \ldots, b_n]$

Substitution 2:

$[b_0, b_1 + 1, b_2, b_3, \ldots, b_n] \mapsto [b_0 + 1, (-2, 2)^{b_1}, -b_2 - 1, -b_3, \ldots, -b_n]$

Let us recall how to calculate the boundary slopes from a continued fraction.

By the result given in [11], a continued fraction expansion is associated to a boundary slope if it has partial quotients which are all at least two in absolute value. We call such a continued fraction a boundary slope continued fraction.

Given a two-bridge knot with Schubert form $S(\alpha, \beta)$, consider a boundary slope continued fraction expansion $[c, b_0, b_1, \cdots, b_n]$ of $\beta/\alpha$ with integer part $c$ and $|b_i| \geq 2$ for $0 \leq i \leq n$. Compare the signs of the terms $b_1, \cdots, b_n$ to the pattern $[+---\cdots]$, and let $n^+$ (resp. $n^-$) be the number of terms matching (resp. not matching) the pattern. Note that, among the boundary slope continued fractions, there is a unique one having all terms even; that is associated to the longitude (i.e., the boundary slope of a Seifert surface). Let $n^+_0$ and $n^-_0$ be the corresponding values for the continued fraction associated to the longitude. Then the boundary slope associated to the continued fraction is presented as $2((n^+-n^-) - (n^+_0 - n^-_0))$.

4. Calculation

In this section, we give a proof of Theorem [1.2]
As explained in the previous section, to prove the theorem, it suffices to enumerate all the boundary slopes by using the substitution method, and calculate $\sum_{N > 0} W_i$ and $\sum_{N < 0} W_i$ for the obtained boundary slopes.

First we consider the case $x = 1$, that is, the case of $9_{27}$. We start with the simple continued fraction of $18/49$, which is represented as the continued fraction $[0, 2, 1, 2, 1, 1, 2]$. We use 6-tuples of the form $(b_1, b_2, b_3, b_4, b_5, b_6)$ with $b_j = 0, 1$ to show where substitutions are applied. As an example, $(0, 0, 1, 0, 0, 1)$ means the substitution rule is applied at positions 3 and 6. Then we have a boundary slope continued fraction $[0, 2, 2, -2, 2, -2]$ which is the longitude continued fraction. Hence we see that $n_0^+ = 3$ and $n_0^- = 3$.

Here recall that each term of boundary slope continued fractions must be at least two in absolute value. Hence $(0, 0, 0, b_4, b_5, b_6)$ does not fit in our case since the term of 1 at position 2 remains after substitutions. Similarly, we can eliminate the possibility of $(b_1, b_2, 0, 0, 0, b_6)$ and $(b_1, b_2, b_3, 0, 0, 0)$. We also note that no two terms of 1 are adjacent in a 6-tuple. It is therefore enough to consider the following 10 cases to obtain all the boundary slope continued fractions.

Case 1. $(0, 0, 1, 0, 0, 1)$.
Then we have $[0, 2, 2, -2, 2, 0, 0, 0]$. Hence $n_1^+ = 3$, $n_1^- = 3$ and $N_1 = 0$.

Case 2. $(0, 0, 1, 1, 0, 1)$.
Then we have $[0, 2, 2, -2, 3, 0, 0, 0]$. Hence $n_2^+ = 1$, $n_2^- = 4$, $N_2 = -6$ and $W_2 = 4$.

Case 3. $(0, 1, 0, 0, 1, 0)$.
Then we have $[0, 3, -3, -2, 3, 0, 0, 0]$. Hence $n_3^+ = 2$, $n_3^- = 2$ and $N_3 = 0$.

Case 4. $(0, 1, 0, 1, 0, 0)$.
Then we have $[0, 3, -4, 2, 2, 0, 0, 0]$. Hence $n_4^+ = 3$, $n_4^- = 1$, $N_4 = 4$ and $W_4 = 6$.

Case 5. $(0, 1, 0, 1, 0, 1)$.
Then we have $[0, 3, -4, 3, -2, 0, 0, 0]$. Hence $n_5^+ = 4$, $n_5^- = 0$, $N_5 = 8$ and $W_5 = 12$.

Case 6. $(1, 0, 0, 0, 1, 0)$.
Then we have $[1, -2, 2, 2, 0, 0, 0]$.

Hence $n_6^+ = 1$, $n_6^- = 4$, $N_6 = -6$ and $W_6 = 2$.

Case 7. $(1, 0, 0, 1, 0, 0)$.
Then we have $[1, -2, 2, 3, -2, 0, 0, 0]$. Hence $n_7^+ = 2$, $n_7^- = 3$, $N_7 = -2$ and $W_7 = 2$.

Case 8. $(1, 0, 0, 1, 0, 1)$.
Then we have $[1, -2, 2, 3, -3, 2, 0, 0, 0]$. Hence $n_8^+ = 2$, $n_8^- = 3$, $N_8 = -2$ and $W_8 = 4$.

Case 9. $(1, 0, 1, 0, 0, 1)$.
Then we have $[1, -2, 3, -2, 2, 2, -2, -2, -2, -2]$. 


Hence $n_9^+ = 2, n_9^- = 4, N_9 = -4$ and $W_9 = 2$.

Case 10. $(1,0,1,0,1,0)$. Then we have $[1,-2,3,-2,3,-3]$. Hence $n_{10}^+ = 0, n_{10}^- = 5, N_{10} = -10$ and $W_{10} = 8$.

We therefore see that

$$\lambda_{SL(2,\mathbb{C})}(M_{1/q}) - \lambda_{SL(2,\mathbb{C})}(M_{-1/q}) = \frac{1}{2} \left(- \sum_{N_i > 0} W_i + \sum_{N_i < 0} W_i \right) = 4.$$ 

Next we consider the general case, where $x \geq 2$.

We remark that the Schubert form of the knot $K_x$ is described as $S((8x^2 - 1)^2, 32x^3 - 8x^2 - 8x + 2)$. Thus its simple continued fraction is given as $[0, 2x, 1, 1, 2x - 2, 1, 2x - 1, 1, 1, 2x - 1]$. We in turn use 9-tuples of the form $(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9)$ with $b_j = 0, 1$ to show where substitutions are applied. The longitude continued fraction is obtained from $(0, 0, 1, 0, 1, 0, 0, 1, 0)$ and is $[0, 2x, 2, -2x, 2x, 2, -2x]$.

There is no possibility of $(0, 0, 0, b_4, b_5, b_6, b_7, b_8, b_9), (b_1, 0, 0, b_5, b_6, b_7, b_8, b_9), (b_1, b_2, b_3, 0, 0, b_7, b_8, b_9), (b_1, b_2, b_3, b_4, b_5, 0, 0, b_9)$ and $(b_1, b_2, b_3, b_5, b_6, 0, 0, 0)$ since each term of boundary slope continued fractions is at least two in absolute value. We again note that no two terms of 1 are adjacent in a 9-tuple. It is therefore enough to consider the following 25 cases to obtain all the boundary slope continued fractions.

Case 1. $(0, 0, 1, 0, 1, 0, 0, 1)$. Then we have $[0, 2x, 2, -2x + 1, -2, (2, -2)x^{-1}, 2, 2, (-2, 2)x^{-1}]$. Hence $n_1^+ = 2x + 1, n_1^- = 2x + 1$ and $N_1 = 0$.

Case 2. $(0, 0, 1, 0, 0, 1, 0, 1, 0)$. Then we have $[0, 2x, 2, -2x + 1, -2, (2, -2)x^{-1}, 3, -2x]$. Hence $n_2^+ = 2x + 2, n_2^- = 2, N_2 = 4x$ and $W_2 = 4(x - 1)(2x - 1)^2$.

Case 3. $(0, 0, 1, 0, 1, 0, 0, 1, 0)$. Then we have $[0, 2x, 2, -2x, 2x, 2, -2x]$. Hence $n_3^+ = 3, n_3^- = 3$ and $N_3 = 0$.

Case 4. $(0, 0, 1, 0, 1, 0, 1, 0, 0)$. Then we have $[0, 2x, 2, -2x, 2x + 1, -2, -2x + 1]$. Hence $n_4^+ = 2, n_4^- = 4, N_4 = -4$ and $W_4 = 4x(x - 1)(2x - 1)^2$.

Case 5. $(0, 0, 1, 0, 1, 0, 1, 0, 1)$. Then we have $[0, 2x, 2, -2x, 2x + 1, -3, (2, -2)x^{-1}]$. Hence $n_5^+ = 1, n_5^- = 2x + 2, N_5 = -4x - 2$ and $W_5 = 4x(2x - 1)^2$.

Case 6. $(0, 1, 0, 0, 0, 1, 0, 0, 1)$. Then we have $[0, 2x + 1, -2, -2x + 2, -2, (2, -2)x^{-1}, 2, 2, (-2, 2)x^{-1}]$. Hence $n_6^+ = 2x + 2, n_6^- = 2x, N_6 = 4$ and $W_6 = 2x(2x - 3)$.

Case 7. $(0, 1, 0, 0, 0, 1, 0, 1, 0)$. Then we have $[0, 2x + 1, -2, -2x + 2, -2, (2, -2)x^{-1}, 3, -2x]$. Hence $n_7^+ = 2x + 3, n_7^- = 1, N_7 = 4x + 4$ and $W_7 = 4x(2x - 1)(2x - 3)$.
Case 8. \((0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0)\).
Then we have \([0, 2x + 1, -2, -2x + 1, 2x, 2, -2x]\).
Hence \(n^+ = 4\), \(n^- = 2\), \(N = 4\) and \(W = 4x(x - 1)(2x - 1)^2\).

Case 9. \((0, 1, 0, 0, 1, 0, 1, 0, 0, 0)\).
Then we have \([0, 2x + 1, -2, -2x + 1, 2x + 1, -2, -2x + 1]\).
Hence \(n^+_9 = 4\), \(n^- = 3\) and \(N = 0\).

Case 10. \((0, 1, 0, 0, 1, 0, 1, 0, 1)\).
Then we have \([0, 2x + 1, -2, -2x + 1, 2x + 1, -3, (2, -2)^{-1}]\).
Hence \(n^+_1 = 2\), \(n^- = 2x + 1\), \(N = -4x + 2\) and \(W = 16x^2(x - 1)\).

Case 11. \((0, 1, 0, 1, 0, 0, 0, 1, 0, 0)\).
Then we have \([0, 2x + 1, -3, (2, -2)^{-1}, -2x + 1, -2, 2x]\).
Hence \(n^+_1 = 2x + 2\), \(n^- = 1\), \(N = 4x + 2\) and \(W = 8x(x - 1)(2x - 1)\).

Case 12. \((0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0)\).
Then we have \([0, 2x + 1, -3, (2, -2)^{-1}, -2x, 2x, 1]\).
Hence \(n^+_1 = 2x + 1\), \(n^- = 2\), \(N = 4x - 2\) and \(W = 8x(x - 1)(2x - 1)\).

Case 13. \((0, 1, 0, 1, 0, 0, 1, 0, 1)\).
Then we have \([0, 2x + 1, -3, (2, -2)^{-1}, -2x, 3, (-2, 2)^{-1}]\).
Hence \(n^+_1 = 2x\), \(n^- = 2x\) and \(N = 0\).

Case 14. \((0, 1, 0, 1, 0, 1, 0, 0, 1, 0)\).
Then we have \([0, 2x + 1, -3, (2, -2)^{-1}, -2x, 2x, 1]\).
Hence \(n^+_1 = 4x - 1\), \(n^- = 2x - 1\), \(N = 4x + 4\) and \(W = 8x\).

Case 15. \((0, 1, 0, 1, 0, 0, 1, 0, 1)\).
Then we have \([0, 2x + 1, -3, (2, -2)^{-2}, 2, -3, (2, -2)^{-1}, 2, 2, (-2, 2)^{-1}]\).
Hence \(n^+_1 = 4\), \(n^- = 0\), \(N = 8x\) and \(W = 16x(2x - 1)\).

Case 16. \((1, 0, 0, 1, 0, 0, 0, 1, 0, 0)\).
Then we have \([1, (-2, 2)^{x}, 2, (-2, 2)^{-1}, 2x - 1, 2, -2x]\).
Hence \(n^+_1 = 2x + 1\), \(n^- = 2x + 1\) and \(N = 0\).

Case 17. \((1, 0, 1, 0, 0, 1, 0, 0, 1, 0)\).
Then we have \([1, (-2, 2)^{x}, 2, (-2, 2)^{-1}, 2x, -2, -2x + 1]\).
Hence \(n^+_1 = 2x\), \(n^- = 2x + 2\), \(N = -4\) and \(W = 2(x - 1)(2x - 1)\).

Case 18. \((1, 0, 0, 1, 0, 0, 1, 0, 1)\).
Then we have \([1, (-2, 2)^{x}, 2, (-2, 2)^{-1}, 2x, -3, (2, -2)^{-1}]\).
Hence \(n^+_1 = 2x - 1\), \(n^- = 4x\), \(N = -4x - 2\) and \(W = 2(2x - 1)\).

Case 19. \((1, 0, 0, 1, 0, 1, 0, 0, 1, 0)\).
Then we have \([1, (-2, 2)^{x}, 2, (-2, 2)^{-2}, -2, 3, (-2, 2)^{-1}, -2, -2, (2, -2)^{-1}]\).
Hence \(n^+_1 = 4x - 2\), \(n^- = 4x - 1\), \(N = -2\) and \(W = 2\).

Case 20. \((1, 0, 0, 1, 0, 1, 0, 1)\).
Then we have \([1, (-2, 2)^{x}, 2, (-2, 2)^{-2}, -2, 3, (-2, 2)^{-1}, -3, 2x]\).
Hence \(n^+_1 = 4x - 1\), \(n^- = 2x\), \(N = 4x - 2\) and \(W = 4(2x - 1)\).
Case 21. \((1,0,1,0,0,1,0,0,1)\).
Then we have \([1,(-2,2)^{x-1},-2,3,-2x+1,-2,(2,-2)^{x-1},2,2,(-2,2)^{x-1}]\).
Hence \(n^+_{21} = 2x, n^-_{21} = 4x, N_{21} = -4x\) and \(W_{21} = 4(x-1)\).

Case 22. \((1,0,1,0,0,1,0,0,1)\).
Then we have \([1,(-2,2)^{x-1},-2,3,-2x+1,-2,(2,-2)^{x-1},3,-2x]\).
Hence \(n^+_{22} = 2x + 1, n^-_{22} = 2x + 1\) and \(N_{22} = 0\).

Case 23. \((1,0,1,0,0,1,0,0,1)\).
Then we have \([1,(-2,2)^{x-1},-2,3,-2x,2x,2,-2x]\).
Hence \(n^+_{23} = 2, n^-_{23} = 2x + 2, N_{23} = -4x\) and \(W_{23} = 2(2x-1)^3\).

Case 24. \((1,0,1,0,0,1,0,0,0)\).
Then we have \([1,(-2,2)^{x-1},-2,3,-2x,2x,2,-2x+1]\).
Hence \(n^+_{24} = 1, n^-_{24} = 2x + 3, N_{24} = -4x - 4\) and \(W_{24} = 8x(x-1)(2x-1)\).

Case 25. \((1,0,1,0,1,0,0,1,0,1)\).
Then we have \([1,(-2,2)^{x-1},-2,3,-2x,2x+1,-3,(2,-2)^{x-1}]\).
Hence \(n^+_{25} = 0, n^-_{25} = 4x + 1, N_{25} = -8x - 2\) and \(W_{25} = 8x(2x-1)\).

Since we are assuming \(x \geq 2\),

\[
\lambda_{SL(2,\mathbb{C})}(M_{1/q}) - \lambda_{SL(2,\mathbb{C})}(M_{-1/q}) = \frac{1}{2} N \left( \sum_{N > 0} W_i + \sum_{N < 0} W_i \right) = 8x^2 - 12x + 2
\]

\[
= 8 \left( x - \frac{3}{4} \right) - \frac{5}{2}
\]

\[
> 0.
\]

5. **Alexander Polynomial**

In this section, we justify Remark \([12]\) in Section 1 as follows.

**Proposition 5.1.** Let \(K_x\) be a two-bridge knot with Conway form \(C[2x,2,-2x,2x,2,-2x]\) for \(x \geq 1\). Then \(\Delta'_{K_x}(1) = 0\) and \(\tau(K_x) = 0\) hold. Here \(\Delta_{K_x}(t)\) denotes the Alexander polynomial of \(K_x\) normalized to be symmetric and to satisfy \(\Delta_{K_x}(1) = 1\).

**Proof.** Let \(K_x\) be a two-bridge knot \(K\) with Conway form \(C[2x,2,-2x,2x,2,-2x]\) for \(x \geq 1\). Then \(K_x\) is a slice knot, originally observed by Casson and Gordon, and see \([13]\) Lemma 8.2 for a proof. On the other hand, the invariant \(\tau\) must vanish for slice knots as shown in \([20]\) Corollary 1.3. Thus we have \(\tau(K_x) = 0\) for our knot \(K_x\) with \(x \geq 1\).

Now let us calculate the Alexander polynomial for \(K_x\). This is just a straightforward calculation, but we include it for readers’ convenience.

In general, a two-bridge knot with Conway form \([2A,-2B,2C,-2D,2E,-2F]\) is depicted as in Figure 11. Note that such a knot is of genus three, and any two-bridge knot of genus three has such a Conway form. In the figure, \(A\) to \(F\) denote the numbers of horizontal full-twists with signs of the twists.
Figure 1. A to F denote the numbers of full-twists.

Such a Seifert surface of genus three can be deformed into the one as shown in Figure 2. To calculate the Seifert matrix, we set a basis $a_1, \cdots, a_6$ of the first homology group of the surface, as illustrated in Figure 2.

Then we have the Seifert matrix as follows.

$$M = \begin{pmatrix}
A & 0 & 0 & 0 & 0 & 0 \\
1 & B & 1 & 0 & 0 & 0 \\
0 & 0 & C & 0 & 0 & 0 \\
0 & 0 & 1 & D & 1 & 0 \\
0 & 0 & 0 & 0 & E & 0 \\
0 & 0 & 0 & 0 & 1 & F \\
\end{pmatrix}$$

Then $\Delta_K_x(t) = \det(M - t^tM)$ is obtained as

$$\det \begin{pmatrix}
(1-t)A & -t & 0 & 0 & 0 & 0 \\
1 & (1-t)B & 1 & 0 & 0 & 0 \\
0 & -t & (1-t)C & -t & 0 & 0 \\
0 & 0 & 1 & (1-t)D & 1 & 0 \\
0 & 0 & 0 & -t & (1-t)E & -t \\
0 & 0 & 0 & 0 & 1 & (1-t)F \\
\end{pmatrix}$$

We then have the following polynomial of degree 6;

$$ABCDEF(1-t)^6 + ((A+C)DEF - ABC(D+F) + ABEF)t(1-t)^4 + (AB+EF)t^2(1-t)^2 + t^3$$

Now we consider the Conway form $[2x, 2, -2x, 2x, 2, -2x]$, that is,

$$A = x, B = -1, C = -x, D = -x, E = 1, F = x.$$  

This implies that $\Delta_K_x(t) = -x^4(1-t)^6 - x^2t(1-t)^4 + t^3$.

After normalization, we have the following.

$$\Delta_K_x(t) = -x^4(t^{-3} + t^3) + (6x^4 - x^2)(t^{-2} + t^2) - (15x^4 - 4x^2)(t^{-1} + t) + 20x^4 - 6x^2 + 1$$
It follows that;
\[
\Delta'_{K_x}(t) = -x^4(-3t^{-4} + 3t^2) + (6x^4 - x^2)(-2t^{-3} + 2t) + (-15x^4 + 4x^2)(-t^{-2} + 1)
\]
\[
\Delta''_{K_x}(t) = -x^4(12t^{-5} + 6t) + (6x^4 - x^2)(6t^{-4} + 2) + (-15x^4 + 4x^2)(2t^{-3})
\]
\[
\Delta''_{K_x}(1) = -18x^4 + 8(6x^4 - x^2) + 2(-15x^4 + 4x^2) = 0.
\]

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References

[1] S. Akbulut and J. D. McCarthy, Casson’s invariant for oriented homology 3-spheres, Mathematical Notes, 36, Princeton Univ. Press, Princeton, NJ, 1990.
[2] S. A. Bleiler, C. D. Hodgson and J. R. Weeks, Cosmetic surgery on knots, in Proceedings of the Kirbyfest (Berkeley, CA, 1998), 23–34 (electronic), Geom. Topol. Monogr., 2, Geom. Topol. Publ., Coventry.
[3] H. U. Boden and C. L. Curtis, The $SL(2, \mathbb{C})$ Casson invariant for Dehn surgeries on two-bridge knots, Algebr. Geom. Topol. 12 (2012), no. 4, 2095–2126.
[4] S. Boyer and D. Lines, Surgery formulae for Casson’s invariant and extensions to homology lens spaces, J. Reine Angew. Math. 405 (1990), 181–220.
[5] G. Burde and H. Zieschang, Knots, de Gruyter Studies in Mathematics, 5, de Gruyter, Berlin, 1985.
[6] J. C. Cha and C. Livingston, KnotInfo: Table of Knot Invariants, http://www.indiana.edu/~knotinfo, May 20, 2016.
[7] C. L. Curtis, An intersection theory count of the $SL_2(\mathbb{C})$-representations of the fundamental group of a 3-manifold, Topology 40 (2001), no. 4, 773–787.
[8] C. L. Curtis, Erratum to: “An intersection theory count of the $SL_2(\mathbb{C})$-representations of the fundamental group of a 3-manifold” [Topology 40 (2001), no. 4, 773–787], Topology 42 (2003), no. 4, 929.
[9] N. Dunfield, Program to compute the boundary slopes of a 2-bridge or Montesinos knot, Available online at http://www.computop.org
[10] C. McA. Gordon and J. Luecke, Knots are determined by their complements, J. Amer. Math. Soc. 2 (1989), no. 2, 371–415.
[11] A. Hatcher and W. Thurston, Incompressible surfaces in 2-bride knot complements, Invent. Math. 79 (1985), no. 2, 225–246. MR0778125 (86g:57063)
[12] J. Hoste, A formula for Casson’s invariant, Trans. Amer. Math. Soc. 297 (1986), no. 2, 547–562.
[13] K. Ichihara, Cosmetic surgeries and non-orientable surfaces. Proceedings of the Institute of Natural Sciences, Nihon University, 48 (2013), 169–174.
[14] Problems in low-dimensional topology. Edited by Rob Kirby. AMS/IP Stud. Adv. Math., 2, Geometric topology (Athens, GA, 1993), 35–473, Amer. Math. Soc., Providence, RI, 1997.
[15] P. Lisca, Lens spaces, rational balls and the ribbon conjecture, Geom. Topol. 11 (2007), 429–472.
[16] Y. Mathieu, Sur les nœuds qui ne sont pas déterminés par leur complément et problèmes de chirurgie dans les variétés de dimension 3, Thèse, L’Université de Provence, 1990.
[17] Y. Mathieu, Closed 3-manifolds unchanged by Dehn surgery, J. Knot Theory Ramifications 1 (1992), No.3, 279–296.
[18] T. W. Mattman, G. Maybrun and K. Robinson, 2-bride knot boundary slopes: diameter and genus, Osaka J. Math. 45 (2008), no. 2, 471–489.
[19] Y. Ni and Z. Wu, Cosmetic surgeries on knots in $S^3$, J. reine angew. Math., Ahead of Print, DOI 10.1515/crelle-2013-0067
[20] P. Ozsváth and Z. Szabó, Knot Floer homology and the four ball genus, Geom. Topol. 7 (2003), 615–643.

[21] P. Ozsváth and Z. Szabó, Knot Floer homology and rational surgeries, Algebr. Geom. Topol. 11 (2011), 1–68.

[22] T. Ohtsuki, Ideal points and incompressible surfaces in two-bridge knot complements, J. Math. Soc. Japan 46 (1994), no. 1, 51–87. MR1248091 (94k:57016)

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