SUPERSYMMETRIES AND CONSTANTS OF
MOTION IN TAUB-NUT SPINNING SPACE

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Abstract

We review the geodesic motion of pseudo-classical spinning particles
in curved spaces. Investigating the generalized Killing equations for
spinning spaces, we express the constants of motion in terms of Killing-
Yano tensors. The general results are applied to the case of the four-
dimensional Euclidean Taub-NUT spinning space. A simple exact
solution, corresponding to trajectories lying on a cone, is given.

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1 Introduction

The models of relativistic particles with spin have been proposed for a long time. The first published work concerning the lagrangian description of the relativistic particle with spin was the paper by Frenkel which appeared in 1926 [1]. After that the literature on the particle with spin grew vast [2].

The models involving only conventional coordinates are called the classical models while the models involving anticommuting coordinates are generally called pseudo-classical.

In this paper we shall confine ourselves to discuss the relativistic spin one half particle models involving anticommuting vectorial degrees of freedom which are usually called the spinning particles. Spinning particles are in some sense the classical limit of the Dirac particles. After the first quantization these new anticommuting variables are mapped into the Dirac matrices and they disappear from the theory [3,4].

The action of spin one half relativistic particle with spinning degrees of freedom described by Grassmannian (odd) variables was first proposed by Berezin and Marinov [5] and soon after that was discussed and investigated in the papers [6-10].

In spite of the fact that the anticommuting Grassmann variables do not admit a direct classical interpretation, the lagrangians of these models turn out to be suitable for the path integral description of the quantum dynamics. The pseudo-classical equations acquire physical meaning when averaged over inside the functional integrals [5,11]. In the semi-classical regime, neglecting higher order quantum correlations, it should be allowed to replace some combinations of Grassmann spin variables by real numbers. Using these ideas the motion of spinning particles in external fields have been studied in Refs. [5, 12-14].

On the other hand, generalizations of Riemannian geometry based on anticommuting variables have been proved to be of mathematical interest. Therefore the study of the motion of the spinning particles in curved spacetime is well motivated.

In the present paper we investigate the motion of pseudo-classical spinning point particles in curved spaces. The generalized Killing equations for the configuration space of spinning particles (spinning space) are analysed and the solutions of the homogeneous part of these equations are expressed in terms of Killing-Yano tensors. We mention that the existence of a Killing-
Yano tensor is both a necessary and a sufficient condition for the existence of a new supersymmetry for the spinning space [15-17].

The general results are applied to the case of the four-dimensional Euclidean Taub-NUT spinning space. The motivation to carry out this example is twofold. First of all, in the Taub-NUT geometry there are known to exist four Killing-Yano tensors [18]. From this point of view the spinning Taub-NUT space is an exceedingly interesting space to exemplify the effective construction of all conserved quantities in terms of geometric ones, namely Killing-Yano tensors. On the other hand, the Taub-NUT geometry is involved in many modern studies in physics. For example the Kaluza-Klein monopole of Gross and Perry [19] and of Sorkin [20] was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein theory. Remarkably the same object has re-emerged in the study of monopole scattering. In the long distance limit, neglecting radiation, the relative motion of slow Bogomolny-Prasad-Sommerfield monopoles is described by the geodesics of this space [21,22]. The dynamics of well-separated monopoles is completely soluble and has a Kepler type symmetry [18,23-25]. The problem of geodesic motion in this metric has therefore its own interest, independently of monopole scattering.

The plan of this paper is as follows. In Sec. 2 we summarize the relevant equations for the motions of spinning points in curved spaces. The generalized Killing equations for spinning spaces are investigated and the constants of motion are derived in terms of the solutions of these equations. In Sec. 3 we analyse the motion of pseudo-classical spinning particles in the Euclidean Taub-NUT space. We examine the generalized Killing equations for this spinning space and describe the derivation of the constants of motion in terms of the Killing-Yano tensors. In Sect. 4 we solve the equations given in the previous Section for the special case of motion on a cone. This case represents an extension of the scalar particle motions in the usual Taub-NUT space in which the orbits are conic sections [18,23-25]. An explicit exact solution is given and, in spite of its simplicity, this solution is far from trivial. Our comments and concluding remarks are presented in Sec. 5.
2 Spinning spaces and Killing equations

Spinning particles, such as Dirac fermions, can be described by pseudo-classical mechanics models involving anticommuting c-numbers for the spin degrees of freedom. The configuration space of spinning particles (spinning space) is an extension of an ordinary Riemannian manifold, parametrized by local coordinates \( \{x^\mu\} \), to a graded manifold parametrized by local coordinates \( \{x^\mu, \psi^\mu\} \), with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anticommuting) [3-17].

The dynamics of spinning point-particles in a curved space-time is described by the one-dimensional \( \sigma \)-model with the action:

\[
S = \int_a^b d\tau \left( \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x) \psi^\mu \frac{D\psi^\nu}{D\tau} \right). \tag{1}
\]

Here and in the following, the overdot denotes an ordinary proper-time derivative \( d/d\tau \), whilst the covariant derivative of \( \psi^\mu \) is defined by

\[
\frac{D\psi^\mu}{D\tau} = \dot{\psi}^\mu + \dot{x}^\lambda \Gamma^\mu_{\lambda\nu} \psi^\nu. \tag{2}
\]

The trajectories, which make the action stationary under arbitrary variations \( \delta x^\mu \) and \( \delta \psi^\mu \) vanishing at the end points, are given by:

\[
\frac{D^2 x^\mu}{D\tau^2} = \ddot{x}^\mu + \Gamma^\mu_{\lambda\nu} \dot{x}^\lambda \dot{x}^\nu = \frac{1}{2i} \psi^\kappa \psi^\lambda R^\mu_{\kappa\lambda \nu} \dot{x}^\nu, \tag{3}
\]

\[
\frac{D\psi^\mu}{D\tau} = 0. \tag{4}
\]

The antisymmetric tensor

\[
S^{\mu\nu} = -i \psi^\mu \psi^\nu \tag{5}
\]

can formally be regarded as the spin polarization tensor of the particle. The first equation of motion (3) implies the existence of a spin dependent gravitational force [14]

\[
\frac{D^2 x^\mu}{D\tau^2} = \frac{1}{2} S^{\kappa\lambda} R^\mu_{\kappa\lambda \nu} \dot{x}^\nu \tag{6}
\]

which is analogous to the electromagnetic force, with spin replacing the electric charge as the coupling constant. The second equation of motion (4)
can be expressed in terms of this tensor (5) and it asserts that the spin is covariantly constant
\[
\frac{DS^{\mu\nu}}{D\tau} = 0. \tag{7}
\]

The interpretation of \( S^{\mu\nu} \) as spin tensor is corroborated by studying electromagnetic interaction of the particle \([5,9,13,14]\). From such an analysis it results that the space-like components are proportional to the magnetic dipole moment of the particle, whilst the time-like components \( S_{0i} \) represent the electric dipole moment. The requirement that for free Dirac particles the electric dipole moment vanishes in the rest frame can be written as a covariant constraint \([3]\)
\[
g_{\nu\lambda}S_{\mu\nu}\dot{x}^\lambda = 0 \tag{8}
\]
which, in terms of the Grassmann coordinates, it is equivalent to
\[
g_{\mu\nu}\dot{x}^\mu\psi^\nu = 0. \tag{9}
\]

The concept of Killing vector can be generalized to the case of spinning manifolds. For this purpose we consider the world-line hamiltonian given by
\[
H = \frac{1}{2}g^{\mu\nu}\Pi_\mu\Pi_\nu \tag{10}
\]
where
\[
\Pi_\mu = g_{\mu\nu}\dot{x}^\nu \tag{11}
\]
is the covariant momentum.

For any constant of motion \( \mathcal{J}(x, \Pi, \psi) \), the bracket with \( H \) vanishes
\[
\{ H, \mathcal{J} \} = 0 \tag{12}
\]
where the Poisson-Dirac brackets for functions of the covariant phase space variables \((x, \Pi, \psi)\) is defined by
\[
\{ F, G \} = \mathcal{D}_\mu F \frac{\partial G}{\partial \Pi_\mu} - \frac{\partial F}{\partial \Pi_\mu} \mathcal{D}_\mu G - \mathcal{R}_{\mu\nu}\frac{\partial F}{\partial \Pi_\mu} \frac{\partial G}{\partial \Pi_\nu} + \imath (-1)^{\alpha_F} \frac{\partial F}{\partial \psi^\mu} \frac{\partial G}{\partial \psi^\nu} \tag{13}
\]
The notations used are
\[
\mathcal{D}_\mu F = \partial_\mu F + \Gamma^\lambda_{\mu\nu} \Pi_\lambda \frac{\partial F}{\partial \Pi_\nu} - \Gamma^\lambda_{\mu\nu} \psi^\rho \frac{\partial F}{\partial \psi^\lambda} ; \quad \mathcal{R}_{\mu\nu} = \frac{\imath}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\mu\nu} \tag{14}
\]
and $a_F$ is the Grassmann parity of $F$: $a_F = (0,1)$ for $F=(\text{even,odd})$.

If we expand $\mathcal{J}(x,\Pi,\psi)$ in a power series in the covariant momentum

$$
\mathcal{J} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{J}^{(n)}_{\mu_1...\mu_n}(x,\psi) \Pi_{\mu_1} \ldots \Pi_{\mu_n}
$$

(15)

then the bracket $\{H,\mathcal{J}\}$ vanishes for arbitrary $\Pi_\mu$ if and only if the components of $\mathcal{J}$ satisfy the generalized Killing equations [3,15,26]:

$$
\mathcal{J}^{(n)}_{(\mu_1...\mu_n;\mu_{n+1})} + \frac{\partial \mathcal{J}^{(n)}_{(\mu_1...\mu_n)}}{\partial \psi^\lambda} \Gamma^{\lambda}_{\mu_{n+1}} \psi^\lambda = \frac{i}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\nu(\mu_{n+1})} \mathcal{J}^{(n+1)\nu}_{\mu_1...\mu_n}
$$

(16)

where the parentheses denote symmetrization with norm one over the indices enclosed.

In the scalar case, neglecting the Grassmann variables $\{\psi^\mu\}$, all the generalized Killing equations (16) are homogeneous and decoupled. The first equation ($n=0$) shows that $\mathcal{J}^{(0)}$ is a trivial constant, the next one ($n=1$) is the equation for the Killing vectors and so on. In general, for a given $n$, neglecting all spin degrees of freedom, eq.(16) defines a Killing tensor of valence $n$

$$
\mathcal{J}^{(n)}_{(\mu_1...\mu_n;\mu_{n+1})}(x) = 0
$$

(17)

and from eq.(15)

$$
\mathcal{J} = \mathcal{J}^{(n)}_{\mu_1...\mu_n}(x) \Pi^{\mu_1} \ldots \Pi^{\mu_n}
$$

(18)

is a first integral of the geodesic equation [27].

In the spinning case the symmetries can be divided into two classes. First, there are four independent generic symmetries which exist in any theory:

1. Proper-time translations generated by the hamiltonian $H$ (10);

2. Supersymmetry generated by the supercharge

$$
Q_0 = \Pi_\mu \psi^\mu;
$$

(19)

3. Chiral symmetry generated by the chiral charge

$$
\Gamma_* = \frac{i}{d!} \sqrt{g} \epsilon_{\mu_1...\mu_d} \psi^{\mu_1} \ldots \psi^{\mu_d};
$$

(20)
4. Dual supersymmetry generated by the dual supercharge

\[ Q^* = i\{\Gamma^*_s, Q_0\} = \frac{i[\not\gamma]}{(d-1)!} \sqrt{g} \epsilon_{\mu_1...\mu_d} \Pi^{\mu_1} \psi^{\mu_2}...\psi^{\mu_d} \]  \hspace{1cm} (21)

where \( d \) is the dimension of space-time.

As a rule we have the freedom to choose the value of the supercharge \( Q_0 \) and any choice gives a consistent model. The condition for the absence of an intrinsic electric dipole moment of physical fermions (leptons and quarks) as formulated in eq.(9) implies

\[ Q_0 = 0. \]  \hspace{1cm} (22)

However, for the time being, we shall not fix the value of the supercharge, keeping the presentation as general as possible.

The second kind of conserved quantities, called *non-generic*, depend on the explicit form of the metric \( g_{\mu \nu}(x) \). In the recent literature there are exhibited the constants of motion in the Schwarzschild [28], Taub-NUT [17,29-33], Kerr-Newman [15,16] spinning spaces.

In what follows we shall deal with the *non-generic* constants of motion in connection with the Killing eqs.(16) looking for the general features of the solutions. The spinning particle constants of motion can be seen either as extensions of the constants from the scalar case or new ones depending on the Grassmann-valued spin variables \( \{\psi^\mu\} \).

Let us assume that the number of terms in the series (15) is finite. That means that, for a given \( n \), \( J^{(n+1)}_{\mu_1...\mu_n+1} \) vanishes and the last non-trivial generalized Killing equation from the system (16) is in fact homogeneous:

\[ J^{(n)}_{\mu_1...\mu_n;\mu_{n+1}} + \frac{\partial J^{(n)}_{\mu_1...\mu_n}}{\partial \psi'^{\sigma}} \Gamma^{\sigma}_{\mu_{n+1}} \lambda \psi'^{\lambda} = 0. \]  \hspace{1cm} (23)

The line of action to solve the system of coupled differential equations (16) is standard. One starts with a \( J^{(n)}_{\mu_1...\mu_n} \) solution of the homogeneous eq.(23) which has to be introduced in the right-hand side (RHS) of the generalized Killing equations (16) for \( J^{(n-1)}_{\mu_1...\mu_{n-1}} \) and the iteration must be carried on to \( n = 0 \).

For the beginning let us note that eq.(23) has solutions which do not depend on the Grassmann coordinates. These are the Killing tensors of valence \( n \), as it can be seen comparing eq.(17) with eq.(23) in which all spin
degrees of freedom are neglected. However, for the spinning particles, the generalized Killing equations (16) are not decoupled. Even if one starts with a Killing tensor of valence \( n \) we get from the remaining Killing equations the components \( J_{\mu_1 \ldots \mu_m}^{(m)}(m < n) \) with non-trivial spin contributions.

Therefore the quantity (18) is no more conserved and the actual constant of motion is

\[
\mathcal{J} = \sum_{m=0}^{n} \frac{1}{m!} J_{\mu_1 \ldots \mu_m}^{(m)}(x, \psi) \Pi^{\mu_1} \ldots \Pi^{\mu_m}
\]  

(24)
in which \( J_{\mu_1 \ldots \mu_m}^{(m)}(x, \psi) \) with \( m < n \) has a non-trivial spin dependent expression.

The construction of the conserved quantity (24) in which the last term \( J_{\mu_1 \ldots \mu_m}^{(n)} \) is a Killing tensor can be done effectively. We shall illustrate this construction with a few examples. For \( n = 0 \) eq.(17) is satisfied by a simple, irrelevant constant. The first non-trivial case is \( n = 1 \). In this case eq.(17) is satisfied by a Killing vector \( R_{\mu} : \)

\[
R_{\mu;\nu} = 0.
\]  

(25)

Introducing this Killing vector in the RHS of the generalized Killing eq.(16) for \( n = 0 \) one obtains for the \( \mathcal{J}^{(0)} \) the expression [17]:

\[
\mathcal{J}^{(0)} = \frac{i}{2} R_{[\mu;\nu]} \psi^{\mu} \psi^{\nu}
\]  

(26)

where the square bracket denotes antisymmetrization with norm one. Consequently, starting with a Killing vector \( R_{\mu} \), we get in the spinning case the conserved quantity (24) in the form

\[
\mathcal{J} = \frac{i}{2} R_{[\mu;\nu]} \psi^{\mu} \psi^{\nu} + R_{\mu} \Pi^{\mu}.
\]  

(27)

A more involved example is given by a Killing tensor \( J_{\mu\nu}^{(2)} = K_{\mu\nu} \) satisfying eq.(17) for \( n = 2 \):

\[
K_{(\mu;\nu;\lambda)} = 0.
\]  

(28)

This solution must be introduced in the RHS of the generalized Killing equation (16) for \( \mathcal{J}^{(1)} \) and then we have to evaluate the new \( \mathcal{J}^{(0)} \). Unfortunately it is not possible to find closed, analytic expressions for \( \mathcal{J}^{(1)} \) and \( \mathcal{J}^{(0)} \) involving the spin variables using directly the components of the Killing
tensor $K_{\mu\nu}$. But assuming that the Killing tensor $K_{\mu\nu}$ can be written as a symmetrized product of two Killing-Yano tensors, the construction of the conserved quantity (24) is feasible.

We remind that a tensor $f_{\mu_1...\mu_r}$ is called a Killing-Yano tensor of valence $r$ [27,34] if it is totally antisymmetric and it satisfies the equation

$$f_{\mu_1...\mu_{r-1}(\mu_r;\lambda)} = 0. \tag{29}$$

It is known that the Killing-Yano tensors play a key role in the Dirac theory on a curved space-time [35]. The study of the generalized Killing equations strengthens the connection of the Killing-Yano tensors with the supersymmetric classical and quantum mechanics on curved manifolds.

For the generality, let us assume that the Killing tensor $K_{\mu\nu}$ can be written as a symmetrized product of two different Killing-Yano tensors

$$K_{\mu\nu}^{ij} = \frac{1}{2} (f_i^{\mu} f_j^{\nu} + f_j^{\mu} f_i^{\nu}) \tag{30}$$

where $f_i^{\mu}$ is a Killing Yano tensor of valence 2 and type $i$. We use for the Killing tensor $K_{\mu\nu}^{ij}$ two additional indices $i, j$ to emphasize the fact that it is formed from two different Killing-Yano tensors.

Introducing the Killing tensor $K_{\mu\nu}^{ij}$ in the form (30) in the RHS of eq.(16) for $n = 1$ we can express the solution $\mathcal{J}^{(1)}_{\mu}^{ij}$ in terms of the Killing-Yano tensors and their derivatives [15,32]:

$$\mathcal{J}^{(1)}_{ij} = \frac{i}{2} \psi^{\lambda} \psi^\sigma (f_i^{\lambda} D_{\nu} f_j^{\mu} + f_j^{\mu} D_{\nu} f_i^{\lambda}) + \frac{1}{2} f_i^{\mu\rho} c_{ij\lambda\rho} + f_j^{\mu\rho} c_{ij\lambda\rho} \tag{31}$$

where the tensor $c_{ij\mu\nu}$ is

$$c_{ij\mu\nu} = -2 f_{i[j(\mu,\nu].} \tag{32}$$

Finally, using $\mathcal{J}^{(1)}_{ij}$ in the RHS of eq.(16) for $n = 0$ we get for $\mathcal{J}^{(0)}_{ij}$

$$\mathcal{J}^{(0)}_{ij} = -\frac{1}{4} \psi^{\lambda} \psi^\sigma \psi^\rho \psi^\tau (R_{\mu\nu\lambda\sigma} f_i^{\mu} f_j^{\nu} + \frac{1}{2} c_{ij\rho\pi} c_{ij\rho\pi}) \tag{33}$$

Collecting the quantities (30),(31) and (33) in eq.(24) we get the corresponding conserved quantity:

$$\mathcal{J}_{ij} = \frac{1}{2!} K_{ij}^{\mu\nu} \Pi_{\mu} \Pi_{\nu} + \mathcal{J}_{ij}^{(1)} \Pi_{\mu} + \mathcal{J}_{ij}^{(0)}. \tag{34}$$
Higher orders of the generalized Killing eq.(16) can be treated similarly, but the corresponding expressions are quite involved. On the other hand, for practical purposes (see Section 3), it turns out to be sufficient to consider in detail the Killing tensors up to the valence 2.

In what follows we shall return to the eq.(23) looking for solutions depending on the Grassmann variables \( \{ \psi^\mu \} \). The existence of such kind of solutions of the Killing equation is one of the specific features of the spinning particle models.

Even the lowest order eq.(23) with \( n = 0 \) has a non-trivial solution \[ J^{(0)} = \frac{i}{4} f_{\mu \nu} \psi^\mu \psi^\nu \] (35)

where \( f_{\mu \nu} \) is a Killing-Yano tensor covariantly constant. Moreover, from eq.(24), we infer that \( J^{(0)} \) is a separately conserved quantity.

The next eq.(23) with \( n = 1 \) can have different kinds of solutions. The most remarkable class of solutions is represented by \[ J^{(1)}_{\mu_1} = f_{\mu_1 \mu_2 \ldots \mu_r} \psi^{\mu_2} \ldots \psi^{\mu_r} \] (36)
generated from a Killing-Yano tensor of valence \( r \). Again, introducing this quantity in the RHS of eq.(16) for \( n = 0 \) we get for \( J^{(0)} \):

\[ J^{(0)} = \frac{i}{r+1} (-1)^{r+1} f_{[\mu_1 \ldots \mu_r ; \mu_{r+1}]} \cdot \psi^{\mu_1} \ldots \psi^{\mu_{r+1}} \] (37)

and the constant of motion corresponding to these solutions of the generalized Killing equations is [31]:

\[ Q_f = f_{\mu_1 \ldots \mu_r} \Pi^{\mu_1} \psi^{\mu_2} \ldots \psi^{\mu_r} + \frac{i}{r+1} (-1)^{r+1} f_{[\mu_1 \ldots \mu_r ; \mu_{r+1}]} \cdot \psi^{\mu_1} \ldots \psi^{\mu_{r+1}}. \] (38)

This quantity is a superinvariant

\[ \{ Q_f, Q_0 \} = 0 \] (39)

for the bracket defined by eq.(13). A similar result was obtained in ref.[36] in which it is discussed the role of the generalized Killing-Yano tensors, with the framework extended to include electromagnetic interactions. This result
extends the analysis from refs.[15-17] where it is established that the existence of a Killing-Yano tensor of the usual type \((r = 2)\) is both a necessary and a sufficient condition for the existence of a new supersymmetry of the type (38) obeying the superinvariance condition (39).

To conclude, we mention that eq.(23) with \(n = 1\) has also many other solutions by forming combinations of different Killing-Yano tensors. As a rule, the corresponding constants of motion are not completely new and they can be expressed in terms of the quantities described above. We shall illustrate this fact choosing for the solution of eq.(23) with \(n = 1\) the quantity

\[
\mathcal{J}_\mu^{(1)} = R_\mu^\lambda f_{\lambda\sigma} \bar{\psi}_\lambda \psi_\sigma
\]  

(40)

where \(R_\mu\) is a Killing vector (Killing-Yano tensor with \(n = 1\)) and \(f_{\lambda\sigma}\) is a Killing-Yano tensor covariantly constant. Introducing this solution in the RHS of the eq.(16) with \(n = 0\), after some calculations, we get for \(\mathcal{J}^{(0)}\) [31,32]:

\[
\mathcal{J}^{(0)} = \frac{i}{2} R_{[\mu;\nu]} f_{\lambda\sigma} \bar{\psi}^\mu \psi^\nu \bar{\psi}_\lambda \psi_\sigma.
\]  

(41)

Combining eqs.(40) and (41) with the aid of eq.(24) we get the constant of motion :

\[
\mathcal{J} = f_{\mu\nu} \bar{\psi}^\mu \psi^\nu \left( R_{\lambda} \Pi^\lambda + \frac{i}{2} R_{[\lambda;\sigma]} \bar{\psi}^\lambda \psi_\sigma \right).
\]  

(42)

As expected, we recognize in this expression the conserved quantities (27) and (35).

In the next Section we shall apply these general results concerning the solutions of the Killing equations for spinning spaces to the case of the four-dimensional Euclidean Taub-NUT spinning space.
3 EUCLIDEAN TAUB-NUT SPINNING SPACE

The Kaluza-Klein monopole [19,20] was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional theory, adding the time coordinate in a trivial way. Its line element is expressed as:

\[
 ds_5^2 = -dt^2 + ds_4^2 \\
 = -dt^2 + V^{-1}(r)[dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2] \\
 + V(r)[dx^5 + \vec{A}(\vec{r}) \ dr_\vec{r}]^2
\]  

(43)

where \( \vec{r} \) denotes a three-vector \( \vec{r} = (r, \theta, \phi) \) and the gauge field \( \vec{A} \) is that of a monopole

\[
 A_r = A_\theta = 0, \quad A_\phi = 4m(1 - \cos \theta) \\
 \vec{B} = \text{rot} \vec{A} = \frac{4m \vec{r}}{r^3}.
\]  

(44)

The function \( V(r) \) is

\[
 V(r) = \left(1 + \frac{4m}{r}\right)^{-1}
\]  

(45)

and the so called NUT singularity is absent if \( x^5 \) is periodic with period \( 16\pi m \) [37].

It is convenient to make the coordinate transformation

\[
 4m(\chi + \phi) = -x^5
\]  

(46)

with \( 0 \leq \chi < 4\pi \), which converts the four-dimensional line element \( ds_4 \) into

\[
 ds_4^2 = V^{-1}(r)[dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2] + 16m^2V(r)[d\chi + \cos \theta d\phi]^2.
\]  

(47)

Spaces with a metric of the form given above have an isometry group \( SU(2) \times U(1) \). The four Killing vectors are

\[
 D_A = R_A^\mu \partial_\mu, \quad A = 0, 1, 2, 3,
\]  

(48)
where
\[
D_0 = \frac{\partial}{\partial \chi},
\]
\[
D_1 = -\sin \varphi \frac{\partial}{\partial \theta} - \cos \varphi \cot \theta \frac{\partial}{\partial \varphi} + \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \chi},
\]
\[
D_2 = \cos \varphi \frac{\partial}{\partial \theta} - \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} + \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \chi},
\]
\[
D_3 = \frac{\partial}{\partial \varphi}.
\]
(49)

\(D_0\) which generates the \(U(1)\) of \(\chi\) translations, commutes with the other Killing vectors. In turn the remaining three vectors, corresponding to the invariance of the metric (47) under spatial rotations \((A = 1, 2, 3)\), obey an \(SU(2)\) algebra with
\[
[D_1, D_2] = -D_3, \text{ etc.}
\]
(50)

In the purely bosonic case these invariances would correspond to conservation of the so called “relative electric charge” and the angular momentum [18,23-25]:
\[
q = 16m^2 V(r) (\dot{\chi} + \cos \theta \dot{\varphi}),
\]
(51)
\[
\vec{j} = \vec{r} \times \vec{p} + q \frac{\vec{r}}{r},
\]
(52)

where \(\vec{p} = V^{-1}(r)\hat{r}\) is the “mechanical momentum” which is only part of the momentum canonically conjugate to \(\vec{r}\).

As observed in [18], the Taub-NUT geometry also possesses four Killing-Yano tensors of valence 2. The first three are rather special: they are covariantly constant (with vanishing field strength)
\[
f_i = 8m (d\chi + \cos \theta d\varphi) \wedge dx_i - \epsilon_{ijk}(1 + \frac{4m}{r}) dx_j \wedge dx_k,
\]
\[
D_{\mu} f_{i\lambda} = 0, \quad i = 1, 2, 3.
\]
(53)

Moreover, they are mutually anticommuting and square the minus unity:
\[
f_i f_j + f_j f_i = -2\delta_{ij}.
\]
(54)

Thus they are complex structures realizing the quaternion algebra. Indeed, the Taub-NUT manifold defined by (47) is hyper-Kähler and, as a
consequence, the corresponding supersymmetric σ-model has an $N = 4$ supersymmetry.

The fourth Killing-Yano tensor is

$$f_Y = 8m(d\chi + \cos \theta d\varphi) \wedge dr + 4r(r + 2m)(1 + \frac{r}{4m}) \sin \theta d\theta \wedge d\varphi$$  \hspace{1cm} (55)

and has only one non-vanishing component of the field strength

$$f_{Y\theta\varphi} = 2(1 + \frac{r}{4m})r \sin \theta.$$  \hspace{1cm} (56)

In the Taub-NUT case there is a conserved vector analogous to the Runge-Lenz vector of the Kepler-type problem whose existence is rather surprising in view of the complexity of the equations of motion. This conserved vector is:

$$\vec{K} = \frac{1}{2} \vec{K}_{\mu
u} \Pi^\mu \Pi^\nu = \vec{p} \times \vec{j} + \left( \frac{q^2}{4m} - 4mE \right) \frac{\vec{r}}{r} \hspace{1cm} (57)$$

where the conserved energy $E$, from eq. (10), is

$$E = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu = \frac{1}{2} V^{-1}(r) \left[ \ddot{r}^2 + \left( \frac{q}{4m} \right)^2 \right] + \frac{4m + r}{2} \dot{r}^2 + \frac{1}{2} (4m + r) r \dot{\theta}^2 + \frac{1}{2} (4m + r) r \sin^2 \theta \dot{\varphi}^2 + 8m^2 \frac{r}{4m + r} (\cos \theta \dot{\varphi} + \dot{\chi})^2.$$  \hspace{1cm} (58)

The components $K_{i\mu\nu}$ involved with the Runge-Lenz type vector (57) are Killing tensors and they can be expressed as symmetrized products of the Killing-Yano tensors $f_i$ (53) and $f_Y$ (55) as in eq. (30) [17,18,33]:

$$K_{i\mu\nu} = m \left( f_{Y\mu\lambda} f_i^{\lambda \nu} + f_{Y\nu\lambda} f_i^{\lambda \mu} \right) + \frac{1}{8m} (R_{0\mu} R_{i\nu} + R_{0\nu} R_{i\mu}).$$  \hspace{1cm} (59)

This equation corrects some old formulas from the literature [18].

Using these conservation laws one can determine the orbits. Eq.(52) implies that

$$\vec{j} \cdot \frac{\vec{r}}{r} = |\vec{j}| \cos \theta = q.$$  \hspace{1cm} (60)
which fixes the relative motion to lie on a cone whose vertex is at the origin and whose axis is $\vec{j}$. Moreover, taking into account the existence of the Runge-Lenz vector (57), one finds that the trajectories lie simultaneously on the cone (60) and also in the plane perpendicular to

$$\vec{n} = q\vec{K} + \left(4mE - \frac{q^2}{4m}\right)\vec{j}.$$  

Thus they are conic sections.

Starting with these results from the bosonic sector of the Taub-NUT space one can proceed with the spin contribution to the conserved quantities (51),(52) and (57).

First of all, corresponding to the generic symmetries described in the previous Section, there are four universal conserved charges. For the Taub-NUT spinning space these are:

1. The energy (58);

2. The supercharge (19): 

$$Q_0 = \frac{4m + r}{r} \dot{r} \psi^r + (4m + r) \dot{\theta} \psi^\theta + \left[(4m + r) r \sin^2 \theta \dot{\varphi} + q \cos \theta\right] \psi^\varphi + q \psi^x; \quad (62)$$

3. The chiral charge 

$$\Gamma_1 = 4m(4m + r) \sin \theta \psi^r \psi^\theta \psi^\varphi \psi^x; \quad (63)$$

4. The dual supercharge 

$$Q^* = 4m(4m + r) \sin \theta \dot{r} \psi^\theta \dot{\psi}^r \dot{\psi}^x - \dot{\theta} \psi^r \psi^\varphi \dot{\psi}^x + \dot{\varphi} \psi^r \dot{\psi}^\varphi - \dot{\chi} \psi^r \psi^\theta \dot{\psi}^x.$$  

From eq.(4) which shows that $\psi^\mu$ is covariantly constant, we find that the rate of change of the spins is:

$$\dot{\psi}^r = \frac{2m}{r(4m + r)} \dot{r} \psi^r + \frac{r^2 + 2mr}{4m + r} \dot{\theta} \psi^\theta.$$  

15
\begin{equation}
\dot{\psi}^\theta = -\frac{r + 2m}{r(4m + r)}(\dot{r} \psi^\theta + \dot{\psi}^r) + \frac{8m^2 + 8mr + r^2}{(4m + r)^2} \frac{\cos \theta}{\sin \theta} (\dot{\theta} \psi^\theta \\
+ \dot{\psi}^\theta) + \frac{8m^2}{(4m + r)^2} \frac{1}{\sin \theta} (\dot{\theta} \psi^\theta + \dot{\chi} \psi^\theta),
\end{equation}

\begin{equation}
\dot{\psi}^\phi = \frac{\cos \theta}{(4m + r)}(\dot{r} \psi^\phi + \dot{\phi} \psi^r) - \frac{2m}{r(4m + r)}(\dot{r} \psi^\phi + \dot{\chi} \psi^\theta) \\
+ \left(\frac{8m^2 + 8mr + r^2 \cos^2 \theta}{(4m + r)^2} \frac{\cos \theta}{\sin \theta} + \frac{1}{2} \sin \theta\right) (\dot{\theta} \psi^\phi + \dot{\psi}^\phi) \\
- \frac{8m^2}{(4m + r)^2} \frac{\cos \theta}{\sin \theta} (\dot{\theta} \psi^\phi + \dot{\chi} \psi^\phi).
\end{equation}

As a rule, the complicated eqs. (3) and (4) should be integrated to obtain the full solution of the equations of motion for the usual coordinates \(\{x^\mu\}\) and Grassmann coordinates \(\{\psi^{\mu}\}\). In addition to the brute force method of trying to solve these equations there are the general prescriptions described in the previous Section which are considerable simpler. In what follows we shall use the Killing-Yano tensors to generate the constants of motion for spinning particles.

We start with the observation that the angular momentum (52) and the “relative electric charge” (51) are constructed with the aid of the Killing vectors (49). The corresponding conserved quantities in the spinning case are the followings:

\begin{equation}
\vec{J} = \vec{B} + \vec{j},
\end{equation}

\begin{equation}
J_0 = B_0 + q
\end{equation}

where we used eq. (27), and we introduced the notation: \(\vec{J} = (J_1, J_2, J_3), \vec{B} = \)
(B_1, B_2, B_3). From eq.(26), the scalars \( B_A \)

\[
B_A = \frac{i}{2} R_{A[\mu\nu]} \psi^\mu \psi^\nu
\]  

(68)

have the following detailed expressions:

\[
B_0 = \frac{32 m^3 \cos \theta}{(4m + r)^2} S^\varphi + \frac{32 m^3}{(4m + r)^2} S^\chi - \frac{8 m^2 r \sin \theta}{4m + r} S^\varphi, \\
B_1 = -\sin \varphi \left( (2m + r) S^\varphi + \frac{8 m^2 r \sin \theta}{4m + r} S^\chi \right) \\
+ \cos \varphi \left[ \left( \frac{32 m^3}{(4m + r)^2} - (2m + r) \right) \sin \theta \cos \theta S^\varphi \right. \\
+ \frac{32 m^3 \sin \theta}{(4m + r)^2} S^\chi + \left. \frac{8 m^2 r + (8 m r^2 + r^3) \sin^2 \theta}{4m + r} \right] \\
+ \frac{8 m^2 r \cos \theta}{4m + r} S^\varphi \\
B_2 = \cos \varphi \left( (2m + r) S^\varphi + \frac{8 m^2 r \sin \theta}{4m + r} S^\chi \right) \\
+ \sin \varphi \left[ \left( \frac{32 m^3}{(4m + r)^2} - (2m + r) \right) \sin \theta \cos \theta S^\varphi \right. \\
+ \frac{32 m^3 \sin \theta}{(4m + r)^2} S^\chi + \left. \frac{8 m^2 r + (8 m r^2 + r^3) \sin^2 \theta}{4m + r} \right] \\
+ \frac{8 m^2 r \cos \theta}{4m + r} S^\varphi \\
B_3 = \left[ (2m + r) \sin^2 \theta + \frac{32 m^3 \cos \theta^2}{(4m + r)^2} \right] S^\varphi + \frac{32 m^3 \cos \theta}{(4m + r)^2} S^\chi \\
+ \frac{(8 m r^2 + r^3) \sin \theta \cos \theta S^\varphi}{4m + r} - \frac{8 m^2 r}{4m + r} \sin \theta S^\chi. 
\]  

(69)

We mention that the above constants of motion are superinvariant:

\[
\{ J_A, Q_0 \} = 0 , \quad A = 0, \ldots, 3. 
\]  

(70)
Also, the components (66) of the angular momentum satisfy, as expected, the $SO(3)$ algebra:

$$\{J_i, J_j\} = \epsilon_{ijk} J_k, \quad i, j, k = 1, 2, 3.$$  \hfill (71)

We consider now the Killing-Yano tensors of valence 2 and we search for those constants of motion built of them.

Using eq.(38) we can construct from the Killing-Yano tensors (53) and (55) the supercharges $Q_i$ and $Q_Y$. The supercharges $Q_i$ together $Q_0$ from eq.(19) realize the $N = 4$ supersymmetry algebra [17]:

$$\{Q_A, Q_B\} = -2i\delta_{AB} H, \quad A, B = 0, \ldots, 3$$  \hfill (72)

making manifest the link between the existence of the Killing-Yano tensors (53) and the hyper-Kähler geometry of the Taub-NUT manifold. Moreover, the supercharges $Q_i$ transform as vectors at spatial rotations

$$\{Q_i, J_j\} = \epsilon_{ijk} Q_k, \quad i, j, k = 1, 2, 3$$  \hfill (73)

while $Q_Y$ and $Q_0$ behave as scalars.

We note also that the bracket of $Q_Y$ with itself can be expressed in terms of the hamiltonian, angular momentum and “relative electric charge”:

$$\{Q_Y, Q_Y\} = -2i \left( H + \frac{\vec{J}^2 - J_0^2}{4m^2} \right).$$  \hfill (74)

On the other hand, the existence of the Killing-Yano covariantly constant tensors $f_i$ (53) is connected with three new constants of motion as shown in eq.(35):

$$S_i = \frac{i}{4} f_{i\mu} \psi^\mu \psi^\nu, \quad i = 1, 2, 3$$  \hfill (75)

which realize an $SO(3)$ Lie-algebra similar to that of the angular momentum (71):

$$\{S_i, S_j\} = \epsilon_{ijk} S_k, \quad i, j, k = 1, 2, 3.$$  \hfill (76)

These components of the spin are separately conserved and can be combined with the angular momentum $\vec{J}$ to define a new improved form of the angular momentum $I_i = J_i - S_i$ with the property that it preserves the algebra [17]:

$$\{I_i, I_j\} = \epsilon_{ijk} I_k, \quad i, j, k = 1, 2, 3$$  \hfill (77)
and that it commutes with the $SO(3)$ algebra generated by the spin $S_i$

$$\{I, S_j\} = 0. \quad (78)$$

Let us note also the following Dirac brackets of $S_i$ with supercharges

$$\{S_i, Q_0\} = -\frac{Q_i}{2}, \quad \{S_i, Q_j\} = \frac{1}{2}(\delta_{ij}Q_0 + \epsilon_{ijk}Q_k). \quad (79)$$

To get the spin correction to the Runge-Lenz vector (57) it is necessary to investigate the generalized Killing eqs. (16) for $n = 1$ with the Killing tensor $\vec{K}_{\mu\nu}$ in the RHS. For an analytic expression of the solution of this equation we shall use the decomposition (59) of the Killing tensor $\vec{K}_{\mu\nu}$ in terms of Killing-Yano tensors. Starting with this decomposition of the Runge-Lenz vector $\vec{K}$ from the scalar case, it is possible to express the corresponding conserved quantity $\vec{K}$ in the spinning case [33]:

$$\vec{K}_i = 2m \left( -i\{Q_Y, Q_1\} + \frac{1}{8m^2}J_iJ_0 \right) \quad (80)$$

This expression differs from previous results presented in the literature [17,31] and the difference has the origin in the corrected form of relation (59). A detailed expression of the components $\mathcal{K}_{i\mu\nu}$ is:

$$\mathcal{K}_i = 2m \left[ \left( f_Y f_i(\mu\nu) + \frac{1}{8m^2}R_{i(\mu}R_{0\nu)} \right) \Pi^\mu\Pi^\nu \right. \\
+ \left( f_i^\lambda\beta f_Y_{\mu\alpha;\lambda} + f_i^\lambda\mu f_Y_{\alpha\beta;\lambda} \\
- \frac{1}{16m^2}(R_{i\alpha;\beta}R_{0\mu} + R_{0\alpha;\beta}R_{i\mu}) \right) S^{\alpha\beta}\Pi^\mu \\
+ \frac{1}{32m^2}S^{\alpha\beta}S^{\gamma\delta}R_{i\alpha;\beta}R_{0\gamma;\delta} \right]. \quad (81)$$

More explicitly, we can write the Runge-Lenz vector $\vec{K}$ for the spinning case as in eq.(34):

$$\vec{K} = \frac{1}{2}\vec{K}_{\mu\nu}\Pi^\mu\Pi^\nu + \vec{S}_\mu\Pi^\mu + \vec{S} \quad (82)$$
where the first term is the Runge-Lenz vector (57) from the scalar case and the last two terms represent the specific spin contribution. A detailed expression of this contribution is [30,33]:

\[ S_{1\mu} \cdot \Pi^\mu = \left[ -(4m + r) \cos \theta \cos \varphi \cdot S^{r\theta} \\
+ 4mr \cos \theta \sin \varphi \cdot S^{\theta \varphi} + 4mr \sin \varphi \cdot S^{\theta \chi} \\
+ (4m + r) \sin \theta \sin \varphi \cdot S^{r \varphi} + 4mr \sin \theta \cos \varphi \cdot S^{r \chi} \right] \dot{r} \\
+ \left[ r(4m + r) \sin \theta \cos \varphi \cdot S^{r \theta} - \frac{4mr(6m + r)}{4m + r} \cos \theta \sin \varphi \cdot S^{r \varphi} \\
- \frac{4mr(6m + r)}{4m + r} \sin \varphi \cdot S^{r \chi} \right] \dot{\theta} \\
+ r^2(6m + r) \sin \theta \sin \varphi S^{\theta \varphi} - 4mr^2 \sin^2 \theta \cos \varphi S^{r \chi} \right] \dot{\varphi} \\
+ \left[ \frac{4mr(6m + r)}{4m + r} \cos \theta \sin \varphi S^{r \theta} \\
+ \left( r(4m + r) \sin^3 \theta \cos \varphi \cos \chi + \frac{256m^4r}{(4m + r)^3} \sin \theta \cos^2 \theta \cos \varphi \right) S^{r \varphi} \\
- \left( 4mr + \frac{8m^2r}{4m + r} - \frac{256m^4r}{(4m + r)^3} \right) \sin \theta \cos \theta \cos \varphi S^{r \chi} \\
+ \left( \frac{r^2(32m^3 + 64m^2r + 14mr^2 + r^3)}{(4m + r)^2} \right) \sin^2 \theta \cos \theta \cos \varphi \\
+ \frac{32m^3r^2}{(4m + r)^2} \cos^3 \theta \cos \varphi \right) S^{\theta \varphi} \\
+ \left( 4mr^2 \sin^2 \theta \cos \varphi + \frac{32m^3r^2}{(4m + r)^2} \cos^2 \theta \cos \varphi \right) S^{\theta \chi} \\
- \frac{32m^3r^2}{(4m + r)^2} \sin \theta \cos \theta \sin \varphi S^{r \chi} \right] \dot{\varphi} \\
+ \left[ \frac{4mr(6m + r)}{4m + r} \sin \varphi S^{r \theta} + \frac{256m^4r}{(4m + r)^3} \sin \theta \cos \varphi S^{r \chi} \\
+ \left( 4mr + \frac{8m^2r}{4m + r} + \frac{256m^4r}{(4m + r)^3} \right) \sin \theta \cos \theta \cos \varphi S^{r \varphi} \right] \]
\[
S_1 = \frac{8m^2r^2(8m + r)}{(4m + r)^3} \sin^2 \theta S^{\varphi \varphi} \cos \varphi S^{\vartheta \varphi} S^{\varphi \chi} \dot{\chi},
\]

\[
S_3 = \frac{8m^2r^2(8m + r)}{(4m + r)^3} \sin \theta S^{\vartheta \vartheta} S^{\varphi \chi} \dot{\chi}.
\]

The components \(S_2 \Pi^\mu\) and \(S_2\) can be obtained from \(S_1 \Pi^\mu\) respectively \(S_1\) with the substitutions:

\[
\sin \varphi \rightarrow - \cos \varphi,
\]

\[
\cos \varphi \rightarrow \sin \varphi.
\]
Therefore, in the spinning case, the Runge-Lenz vector contains additional terms linear and quadratic in the spin. The presence of a contribution quadratic in the spin, non-existent in Refs.[17,30], is again related to the term $J_iJ_0$ from eq.(80). We would like to emphasize that the contribution of the $J_iJ_0$ term in eq.(80) is essential in reproducing the known vectorial expression of the Runge-Lenz vector in the scalar case, and gives the correct Poisson-Dirac bracket between two components of the Runge-Lenz vector.

The Dirac brackets involving the Runge-Lenz vector (80) are (after some algebra):

\[
\{K_i, Q_0\} = 0,
\{K_i, J_j\} = \epsilon_{ijk}K_k,
\{K_i, K_j\} = \epsilon_{ijk}J_k \left[ \frac{J_0^2}{16m^2} - 2H \right]
\]

and they are similar to those known from the scalar case. The Runge-Lenz vector $\vec{K}$ together with the total angular momentum $\vec{J}$ generates an $SO(4)$ or $SO(3,1)$ algebra depending upon the sign of the quantity $\left( \frac{J_0^2}{16m^2} - 2E \right) |_{\psi=0}$ is positive or negative.

In conclusion, all conserved quantities for motions in spinning Taub-NUT space have been expressed in terms of the geometric objects (49), (53) and (55). Other expressions involving the Killing-Yano tensors will produce conserved quantities which are not new, but rather combinations of the above primary conserved quantities. For example, let us consider a solution of the homogeneous eq.(23) for $n = 1$ of the type (40):

\[
J_{A(1)}^{(1)} = R_{A\mu}f_{j\lambda\sigma}^{\lambda}\psi^\sigma, \\
A = 0, \ldots, 3, \quad j = 1, 2, 3.
\]

(87)

After some algebra we get the constants of motion of the form (42):

\[
J_{A(j)} = f_{j\lambda\sigma}^{\lambda}\psi^\sigma \left( R_{A\mu}\Pi^\mu + \frac{i}{2}R_{A[\alpha;\beta]}\psi^\alpha\psi^\beta \right) \\
= -4iS_jJ_A, \quad A = 0, \ldots, 3, \quad j = 1, 2, 3.
\]

(88)

As expected, the constants $J_{A(j)}$ are not new, being expressed in terms of the constants $J_A$ (62), (63) and $S_j$ (75). However, the combinations (88) arise
in a natural way as solutions of the generalized Killing equations and appear only in the spinning case. Moreover, we can form a sort of Runge-Lenz vector involving only Grassmann components:

\[ L_i = \frac{1}{m} \epsilon_{ijk} S_j J_k, \quad i, j, k = 1, 2, 3 \]  

(89)

with the commutation relations like in eqs. (86):

\[
\{ L_i, J_j \} = \epsilon_{ijk} L_k, \\
\{ L_i, L_j \} = \left( \vec{S} \vec{J} - \vec{S}^2 \right) \frac{1}{m^2} \epsilon_{ijk} J_k.
\]  

(90)

Note also the following Dirac brackets of \( L_i \) with supercharges:

\[
\{ L_i, Q_0 \} = -\frac{1}{2m} \epsilon_{ijk} Q_j J_k, \\
\{ L_i, Q_j \} = -\frac{1}{2m} (\epsilon_{ijk} Q_0 J_k - \delta_{ij} Q_k M^-_k + Q_i M^-_j).
\]  

(91, 92)
4 SPECIAL SOLUTION

In spite of the fact that all conserved quantities have been expressed in terms of geometric ones in a close form, their detailed expressions (69), (83), (84) are quite intricate.

We wish to consider a special class of solution of the equations of motion which is very simple, but not at all trivial. In this Section we confine ourselves to the motion on a cone on the analogy of the scalar case where the trajectories are conic sections.

For this purpose let us choose the $z$ axis along $\vec{J}$ so that the motion of the particle may be conveniently described in terms of polar coordinates

$$\vec{r} = r\vec{e} (\theta, \varphi)$$

with

$$\vec{e} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta).$$

For this choice of the axis we have:

$$(4m + r)\dot{r} = -(2m + r)S^{r\theta} - \frac{8m^2 r}{4m + r} \sin \theta S^{\varphi x},$$

$$\dot{\varphi} = \frac{q}{r(4m + r) \cos \theta} - \left[ \left( \frac{32m^3}{r(4m + r)^3} - \frac{2m + r}{r(4m + r)} \right) S^{r\varphi} + \frac{32m^3}{r(4m + r)^3 \cos \theta} S^{r\varphi} + \frac{8m^2 + (8mr + r^2) \sin \theta^2}{(4m + r)^2 \sin \theta \cos \theta} S^{\theta \varphi} + \frac{8m^2}{(4m + r)^2 \sin \theta} S^{\theta \varphi} \right]$$

and from eqs.(66), (67) and (69)

$$J_0 - \frac{\vec{J} \vec{r}}{r} = -r(4m + r)S^{\theta \varphi} \sin \theta.$$
polarization tensor $S^\mu_\nu$ (5) as follows:

\begin{align*}
\dot{S}^{r_\theta} &= - \frac{\dot{r}}{4m + r} S^{r_\theta} + \frac{8m + r}{2(4m + r)^3} q \sin \theta S^{r_\phi} \\
&\quad - \frac{8m^2}{r(4m + r)^3} \frac{q \sin \theta}{\cos \theta} S^{r_\chi} \frac{r \sin^2 \theta + 2m}{(4m + r)^2} \frac{q}{\cos \theta} S^{\theta_\phi} \frac{2mq}{(4m + r)^2} S^{\theta_\chi}, \\
\dot{S}^{r_\phi} &= - \frac{q}{2r(4m + r) \sin \theta} S^{r_\theta} - \frac{\dot{r}}{4m + r} S^{r_\phi} - \frac{2mq}{(4m + r)^2} S^{\phi_\chi}, \\
\dot{S}^{r_\chi} &= \frac{q}{2(4m + r) \sin \theta \cos \theta} S^{r_\theta} + \frac{\dot{r}}{4m + r} S^{r_\phi} \frac{(r \sin^2 \theta + 2m) q}{(4m + r)^2 \cos \theta} S^{r_\chi}, \\
\dot{S}^{\theta_\phi} &= \frac{(2m + r) q}{r^2(4m + r)^2 \cos \theta} S^{r_\theta} - \frac{2m + r}{r(4m + r)} \frac{\dot{r}}{4m + r} \frac{S^{\theta_\phi}}{S^{\phi_\chi}}, \\
\dot{S}^{\theta_\chi} &= \frac{q}{8m(4m + r)^2} S^{r_\theta} + \frac{\dot{r}}{4m + r} \cos \theta S^{\theta_\phi} - \frac{\dot{r}}{r} S^{\theta_\chi} + \frac{8m + r}{2(4m + r)^3} q \sin \theta S^{\phi_\chi}, \\
\dot{S}^{\phi_\chi} &= \frac{q}{8m(4m + r)^2} S^{r_\phi} - \frac{(2m + r) q}{r^2(4m + r)^2 \cos \theta} S^{r_\chi} \\
&\quad - \frac{2r(4m + r) \sin \theta \cos \theta}{2r(4m + r) \sin \theta} S^{\theta_\phi} - \frac{q}{2r(4m + r) \sin \theta} S^{\theta_\chi} - \frac{\dot{r}}{r} S^{\phi_\chi}.
\end{align*}

Since we are looking for solutions with $\dot{\theta} = 0$ we have from eq.(95)

\begin{align*}
S^{r_\theta} + \frac{8m^2 r \sin \theta}{(2m + r)(4m + r)} S^{\phi_\chi} &= 0.
\end{align*}
This relation implies that the special solution investigated in this Section is situated in the sector with
\[ \Gamma_* = 0. \]  
\[ (100) \]

We mention that in this sector the system of eqs.(98) is satisfied even if the angle \( \theta \) is not constant.

Using eq.(99) we can express \( S^{r\theta} \) through \( S^{\varphi\chi} \) and the following equations are equivalent to the system (98)

\[ \frac{d}{dt} [(4m + r)S^{r\varphi}] = \frac{r}{4m + r} qS^{\varphi\chi}, \]
\[ \frac{d}{dt} [\cos \theta S^{r\varphi} + S^{r\chi}] = 0, \]
\[ \frac{d}{dt} [r(4m + r)S^{\theta\varphi}] = -2 \frac{\sin \theta}{\cos \theta} \frac{r}{4m + r} qS^{\varphi\chi}, \]
\[ \frac{d}{dt} [r \cos \theta S^{\theta\varphi} + r S^{\theta\chi}] = -\frac{\sin \theta}{4m} \frac{r}{4m + r} qS^{\varphi\chi}. \]  
\[ (101) \]

Thus the equations of motion for \( S^{\mu\nu} \) are written in a more tractable form and the solution follows without difficulties. If we take into consideration the constraint coming from eq.(97), namely:

\[ \frac{d}{dt} [(4m + r)S^{\theta\varphi}] = 0 \]
\[ (102) \]

then we have to impose

\[ q \cdot S^{\varphi\chi} = 0. \]  
\[ (103) \]
on the system (101). We shall analyze both solutions \( S^{\varphi\chi} = 0 \) and \( q = 0 \) successively.

For \( S^{\varphi\chi} = 0 \), from eq.(99) we have also

\[ S^{\theta\varphi} = 0. \]  
\[ (104) \]

In spite of this drastic simplification, eqs.(101) have a non-trivial solution:

\[ S^{r\varphi} = \frac{(\sin \theta \mp 1)}{\cos \theta(4m + r)} \Sigma. \]
\[ S^{rx} = \frac{\sin \theta}{4m} \Sigma - \frac{(\sin \theta \pm 1)}{4m + r} \Sigma, \]
\[ S^{\theta \varphi} = \frac{1}{r(4m + r)} \Sigma, \]
\[ S^{\theta \chi} = -\frac{(\sin \theta \pm 1) \tan \theta}{4m r} \Sigma - \frac{\cos \theta}{r(4m + r)} \Sigma \]

(105)

where \( \Sigma \) is a Grassmann constant, commuting with \( \psi^\mu \), and anticommuting with itself.

In the case of this particular solution, from eqs.(67)-(69) we get that the spin contribution to the “relative electric charge” vanishes \( (B_0 = 0) \) and

\[ J_0 = q. \]

(106)

Therefore the “relative electric charge” has the same expression as in the scalar case. However the total angular momentum is modified by the spin contribution:

\[ J_0 - \frac{\vec{J} \cdot \vec{r}}{r} = q - J \cos \theta = -\Sigma \sin \theta. \]

(107)

Here \( J \) is the magnitude of the total angular momentum and eq.(107) fixes the angle \( \theta \) in terms of the constants \( q, J \) and \( \Sigma \). Also the equations for \( \varphi \) and \( \chi \) are modified:

\[ \dot{\varphi} = \frac{q}{r(4m + r) \cos \theta} \pm \frac{\Sigma}{(4m + r)^2 \cos \theta}, \]
\[ \dot{\chi} = \frac{8m + r}{16m^2 (4m + r)} q \mp \frac{\Sigma}{(4m + r)^2}. \]

(108)

At last, \( \dot{r} \) can be derived from the energy, eq.(58).

Concerning the second possibility, namely \( q = 0 \) in eq.(103), we have from eqs.(101):

\[ S^{r\varphi} = \frac{C^{r\varphi}}{4m + r}, \]
\[ S^{rx} = C^{rx} - \frac{\cos \theta}{4m + r} C^{r\varphi}, \]
\[ S^{\theta \varphi} = \frac{C^{\theta \varphi}}{r(4m + r)}. \]
\[ S^{\theta x} = \frac{C^{\theta x}}{r} - \cos \theta \frac{C^{\theta \varphi}}{r(4m+r)}, \]
\[ S^{r \theta} = \frac{C^{r \theta}}{4m+r}, \]
\[ S^{\varphi \chi} = \frac{C^{\varphi \chi}}{r}, \] (109)

where \( C^{\mu \nu} \) are Grassmann constants of the same kind as \( \Sigma \).

Taking into account the constraints:
\[ J_0 = B_0 = \frac{8m^2(4mC^{r \chi} - \sin \theta C^{\theta \varphi})}{(4m+r)^2}, \] (110)

and, from (99), where we substitute (109):
\[ C^{r \theta} + \frac{8m^2 \sin \theta C^{\varphi \chi}}{2m+r} = 0 \] (111)

we get
\[ C^{r \theta} = 0 \quad , \quad C^{\varphi \chi} = 0 \] (112)

and
\[ C^{\varphi \chi} = 0 \quad , \quad C^{\theta \varphi} = 0 \] (113)

or
\[ C^{r \chi} = \sin \frac{\theta}{4m} C^{\theta \varphi}. \] (114)

In conclusion, the case \( q = 0 \) is included into the previous case \( (S^{\varphi \chi} = 0) \). Practically we must impose in addition to the previous solution the condition that \( q = 0 \) and we get either the solution (112), (114) or a trivial one without any spin contribution (112), (113).

Concerning the Runge-Lenz vector for \( \theta = \text{constant} \) we have from eqs. (83) and (84):
\[ S_{1\mu} \Pi^\mu = \mp 2 \tan \theta \sin \varphi \Sigma \mp 2r \tan \theta \sin^2 \theta \cos \varphi \dot{\Sigma} \]
\[ \mp \frac{(8m+r) \sin \theta \cos \varphi}{4m(4m+r)} q \Sigma, \]
\[ S_{3\mu} \Pi^\mu = \mp 2r \sin^2 \theta \dot{\varphi} \Sigma \]
\[ \mp \tan \theta ((8m+r) \sin \theta \mp (4m+r)) \frac{1}{4m(4m+r)} q \Sigma. \] (115)
Again the component $S_{2\mu} \Pi^\mu$ can be obtained from $S_{1\mu} \Pi^\mu$ with the substitution (85).

Using then:

\[
\vec{S}\vec{p} = -\frac{(4m + r)\dot{r} \sin \theta q\Sigma}{4mr},
\]

\[
\vec{p}\vec{r} = \frac{(4m + r)\dot{r} r}{r},
\]

\[
\vec{J}\vec{p} = (q + \Sigma \sin \theta) \frac{\vec{p}\vec{r}}{r},
\]

we get to the conclusion that, for the case of motion lying on a cone, there is a conserved vector $\vec{n}$ orthogonal on $\vec{p}$:

\[
\vec{n} = q\vec{K} + \frac{(q - \Sigma \sin \theta)\vec{J}}{q} \left(4mE - \frac{q(q - \Sigma \sin \theta)}{4m}\right)
\]

which is similar to the scalar case, eq.(61). Physical observables are obtained by averaging with some suitable density over the anticommuting parameters. After the integration was performed we may treat $\Sigma$ as a classical variable. Therefore the trajectories of a spinning particle constrained to the motion on a cone are conic sections determined by the condition $\vec{n}\vec{p} = 0$. 

29
5 CONCLUDING REMARKS

The spinning particle model is a world line supersymmetric extension of the theory of a scalar particle. It describes a relativistic particle with spin one half.

It is a theory which describes in a pseudo-classical way a Dirac particle moving in an arbitrary $d$-dimensional space-time. In addition to the usual space-time coordinates, the model involves anticommuting vectorial coordinates which take into account the spin degrees of freedom. It is worth emphasizing that along the world line of the particle there is a supersymmetry between the fermionic spin variables and the bosonic position coordinates. The model is a one-dimensional supersymmetric field theory on the world line.

The term pseudo-classical refers to the fact that there is no classical interpretation for the anticommuting variables. To get the ”observable” trajectories one has to average over the spin variables. On the other hand it is possible to quantize the model giving rise to supersymmetric quantum mechanics. After quantization the conservation law for the supercharge becomes the Dirac equation.

The constants of motion of a scalar particle in a curved space-time are determined by the symmetries of the manifold, i.e. if a space-time admits a Killing tensor $K_{\mu_1 \ldots \mu_r}$ of valence $r$, then the quantity $K_{\mu_1 \ldots \mu_r} \Pi^\mu_1 \cdots \Pi^\mu_r$ is conserved along the geodesic.

In the spinning case the generalized Killing equations (16) are more involved and new procedures should be conceived. The aim of this paper was to point out the important role of the Killing-Yano tensors to generate solutions of the generalized Killing equations. We presented a detailed discussion on how to construct conserved quantities out of Killing-Yano tensors for the Taub-NUT spinning space. Finally we solved the equation of motion for the case when the angle $\theta$ is held fixed. This solution is most simple but far from trivial and the trajectories are the analogous of the ones of a scalar particle, being conic sections.

The extension of these results for the motion of spinning particles in spaces with torsion [38] and/or interacting with background fields [3,36] will be discussed elsewhere [39].
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References

[1] J.Frenkel, Z.für Physik, 37, (1926) 243.

[2] A.Frydryszak, Lagrangian models of particles with spin: The first seventy years in From field theory to quantum groups, Birthday volume dedicated to Jerzy Lukierski, World Scientific, Singapore (1996).

[3] R.H.Rietdijk, Applications of supersymmetric quantum mechanics, Ph. D. Thesis, Univ. Amsterdam (1992).

[4] D.M.Gitman, Amer.Math.Soc.Transl.(Series 2) 177 (1996) 83.

[5] F.A.Berezin and M.S.Marinov, Pis’ma Zh.Eksper.Teoret.Fiz. 21 (1975) 678; English transl. JETP Lett. 21 (1975) 320; Ann.Phys.N.Y. 104 (1977) 336.

[6] R.Casalbuoni, Nuovo Cimento A33 (1976) 115; id, A33 (1976) 389.

[7] A.Barducci, R.Casalbuoni and L.Lusanna, Nuovo Cimento A35 (1976) 377.

[8] L.Brink, S.Deser, P.di Vecchia and P.Lowe, Phys.Lett. B64 (1976) 435.

[9] L.Brink, P.di Vecchia and P.Howe, Nucl.Phys. B118 (1977) 76.

[10] A.P.Balachandran, P.Salomonson, B.Skagerstan and J.Winnberg, Phys. Rev. D15 (1977) 2308.

[11] A.Barducci, R.Casalbuoni and L.Lusanna, Nucl.Phys. B180 [FS2] (1981) 141.

[12] A.Barducci, R.Casalbuoni and L.Lusanna, Nucl.Phys. B124 (1977) 93; id, 521.
[13] A.Barducci, F.Bondi and R.Casalbuoni, Nuovo Cimento 64B (1981) 287.
[14] J.W.van Holten, Nucl.Phys. B356 (1991) 3.
[15] G.W.Gibbons, R.H.Rietdijk and J.W.van Holten, Nucl.Phys. B404 (1993) 42.
[16] J.W.van Holten, in Proceedings of ”Quarks-94”, Vladimir, Russia (1994).
[17] J.W.van Holten, Phys.Lett. B342 (1995) 47.
[18] G.W.Gibbons and P.J.Ruback, Phys.Lett. B188 (1987) 226; Commun.Math.Phys. 115 (1988) 267.
[19] D.J.Gross and M.J.Perry, Nucl.Phys. B226 (1983) 29.
[20] R.D.Sorkin, Phys.Rev.Lett. 51 (1983) 87.
[21] N.S.Manton, Phys.Lett. B110 (1985) 54; id, B154 (1985) 397; id, (E) B157 (1985) 475.
[22] M.F.Atiyah and N.Hitchin, Phys.Lett. A107 (1985) 21.
[23] G.W.Gibbons and N.S.Manton, Nucl.Phys. B274 (1986) 183.
[24] L.Gy.Feher and P.A.Horvathy, Phys.Lett. B182 (1987) 183; id, (E) B188 (1987) 512.
[25] B.Cordani, L.Gy.Feher and P.A.Horvathy, Phys.Lett. B201 (1988) 481.
[26] R.H. Rietdijk and J.W.van Holten, Class.Quant.Grav. 7 (1990) 247.
[27] W. Dietz and R.Rüdinger, Proc.R.Soc.London A375 (1981) 361.
[28] R.H. Rietdijk and J.W.van Holten, Class.Quant.Grav. 10 (1993) 375.
[29] M.Visinescu, Class.Quant.Grav. 11 (1994) 1867.
[30] M.Visinescu, Phys.Lett. B339 (1994) 28.
[31] D.Vaman and M.Visinescu, Phys.Rev. D54 (1996) 1398.
[32] M.Visinescu, Nucl.Phys.B (Proc.Suppl.) 56B (1997) 142.
[33] D.Vaman and M.Visinescu, Phys.Rev. D57 (1998) 3790.
[34] K.Yano, Ann.Math. 55 (1952) 328.
[35] B.Carter and R.G.McLenaghan, Phys.Rev. D19 (1970) 1093.
[36] M.Tanimoto, Nucl.Phys. B442 (1995) 549.
[37] C.W.Misner, Journ.Math.Phys. 4 (1980) 924.
[38] R.H.Rietdijk and J.W.van Holten, Nucl.Phys. B472 (1996) 427.
[39] D.Vaman and M.Visinescu, in preparation.