ON SOME PERMANENCE PROPERTIES OF (DERIVED) SPLINTERS

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ABSTRACT. We show that Noetherian splinters ascend under essentially étale homomorphisms. Along the way, we also prove that the henselization of a Noetherian local splinter is always a splinter and that the completion of a local splinter with geometrically regular formal fibers is a splinter. Finally, we give an example of a (non-excellent) Gorenstein local splinter with mild singularities whose completion is not a splinter. Our results provide evidence for a strengthening of the direct summand theorem, namely that regular maps preserve the splinter property.

1. INTRODUCTION

Recall that a Noetherian ring $R$ is a splinter if any finite ring map $R \to S$ that induces a surjection on $\text{Spec}$ has a left-inverse in the category of $R$-modules [Ma88]. Perhaps owing to their simple definition, basic questions about splinters are often devilishly difficult to answer. For example, Hochster’s direct summand conjecture (now a theorem) is the modest assertion that a regular ring of any characteristic is a splinter. However, it took the advent of perfectoid geometry for this conjecture to be settled by André in mixed characteristic [And18] (see also [Bha18, Hei02]), decades after Hochster’s verification of the equal characteristic case using Frobenius techniques [Hoc73(a)].

The direct summand theorem justifies thinking of a splinter as a characteristic independent notion of singularity. The goal of this paper is to show that splinters satisfy some basic permanence properties enjoyed by other classes of singularities. Our first main result follows below, which to the best of our knowledge has not appeared previously.

**Theorem A.** Let $\varphi: R \to S$ be an essentially étale homomorphism of Noetherian rings of arbitrary characteristic. If $R$ is a splinter, then $S$ is a splinter.

Most notions of singularities such as reduced, normal, Gorenstein, complete intersection, Cohen–Macaulay in fact ascend under regular maps, that is, flat maps of Noetherian rings with geometrically regular fibers. Theorem A provides characteristic independent evidence suggesting the same is true for splinters. However, a proof of this stronger result is likely difficult as ascent of splinters under regular maps would imply the direct summand theorem (see Remark 4.0.1(2)). On the other hand, Theorem A and an application of Néron–Popescu desingularization show that the principal remaining difficulty lies in verifying that a polynomial ring over a splinter remains a splinter (Remark 4.0.1(1)).

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The second author was supported in part by NSF grant DMS #1602070 and #1707661, and by a fellowship from the Sloan Foundation.
Splinters behave very differently depending on the characteristic of the ring. In equal characteristic 0, for instance, splinters starkly contrast with a more restrictive derived variant. Following [Bha12], we say a Noetherian ring $R$ of arbitrary characteristic is a derived splinter if for any proper surjective morphism $f : X \to \text{Spec}(R)$, the induced map $R \to R\Gamma(X, \mathcal{O}_X)$ splits in the derived category $D(\text{Mod}_R)$. In equal characteristic 0, splinters have long been known to coincide with normal rings via an argument involving the trace map [Bha12, Example 1.1]. On the other hand, a ring essentially of finite type over a field of characteristic 0 is a derived splinter precisely when it has rational singularities [Kov00] (c.f. [Bha12, Theorem 2.12]). Hence such rings are not only normal but even Cohen–Macaulay.

The situation is remarkably different in prime characteristic, where Bhatt showed that splinters coincide with derived splinters [Bha12, Theorem 1.4]. Furthermore, in this setting splinters are conjecturally equivalent to $F$-regular singularities. $F$-regularity was introduced by Hochster and Huneke in the celebrated theory of tight closure [HH90], and correspond via standard reduction techniques to the Kawamata log terminal singularities fundamental in complex birational geometry [Smi97, MS97, Har98]. While an $F$-regular singularity is always a splinter (Remark 2.4.1), the converse is known only in the $\mathbb{Q}$-Gorenstein setting [Sin99]. The conjectural equivalence of splinters and $F$-regularity would also resolve some important and long-standing localization questions in tight closure theory.

It is perhaps not surprising that (derived) splinters remain quite mysterious in mixed characteristic, where sophisticated techniques were required to prove the direct summand theorem. Bhatt improved upon André’s result in [Bha18], showing that regular rings in mixed characteristic are even derived splinters. Astonishingly, Bhatt’s forthcoming work further indicates that splinters and derived splinters also coincide in mixed characteristic [Bha].

Given the close relationship between splinters and their derived variant, it is natural to wonder about the derived analogue of Theorem A. In equal characteristic zero, this follows from [Kov00] because rational singularities are preserved by essentially étale maps (see Corollary 3.3.5). In prime and mixed characteristic, Theorem A also implies the derived analogue by Bhatt’s work.

1.1. **Structure of the proof of Theorem A**. We prove Theorem A by reducing to the case where $\varphi : R \to S$ is finite étale, which first necessitates an understanding of how splinters behave under henselization. To that end, we establish the following result:

**Theorem B.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then $R$ is a splinter if and only if its henselization $R^{\text{h}}$ is a splinter.

Strict henselizations of splinters are also splinters (see Corollary 3.3.3), which in turn implies that Theorems A and B are equivalent (Remark 3.3.4).

Preservation of the splinter property under henselization raises the natural question of whether the completion of a local splinter is a splinter. In equal characteristic zero this fails by Nagata’s example of a normal local ring whose completion is not reduced [Nag55] because splinters are always reduced (Lemma 2.1.1). The same example does not work in prime characteristic where local splinters are always analytically unramified (Remark 2.4.1(b)).
Nevertheless, we show that there exist positive characteristic local splinters with even Gorenstein $F$-regular singularities whose completions are not splinters (Example 3.2.1). At the same time, any such example cannot be excellent because we also prove that splinters behave well under completions for local rings that usually arise in arithmetic and geometry.

**Theorem C.** Let $(R, m)$ be a Noetherian local ring such that $R \to \hat{R}$ is regular (for example, if $R$ is excellent). Then $R$ is a splinter if and only if $\hat{R}$ is a splinter.

Theorem C further corroborates our belief that regular maps should preserve the splinter property.

The proof of Theorem C uses an ideal-theoretic result of Smith for excellent normal local rings [Smi94, Proposition 5.10]. In our setting, we are able to establish an analogue of Smith’s result for henselizations of arbitrary normal local rings, not just excellent ones. This key result, highlighted below, allows us to prove Theorem B without any restrictions on formal fibers.

**Proposition 3.1.4.** Let $(R, m)$ be a normal local domain (not necessarily Noetherian), and let $R^h$ and $R^{sh}$ denote its henselization and strict henselization respectively. Then $R^h$ and $R^{sh}$ are normal domains, and if $I$ is an ideal of $R$, we have

$$I(R^h)^+ \cap R = IR^+ \cap R = I(R^{sh})^+ \cap R.$$

Here $R^+$ denotes the *absolute integral closure* of a domain $R$, that is, $R^+$ is the integral closure of $R$ in an algebraic closure of its fraction field. Armed with Proposition 3.1.4 and [Smi94, Proposition 5.10], Theorems B and C then have almost similar proofs.

### 1.2. Outline of the paper.

We begin Section 2 by discussing some elementary properties of splinters (Subsection 2.1). We next investigate descent of splinters under pure and, the closely related notion of, cyclically pure maps. Crucial to our comparison of purity versus cyclic purity for splinters is Hochster’s notion of approximately Gorenstein rings (Definition 2.2.4). These are Noetherian rings for which purity and cyclic purity coincide. Splinters are also related to absolute integral closures, a connection hinted at by the aforementioned Proposition 3.1.4. Indeed, it is well-known that in order for a Noetherian domain $R$ to be a splinter, it is necessary and sufficient for $R \to R^+$ to be a pure map (a.k.a. universally injective map). Thus, we briefly discuss how the absolute integral closure interacts with the notion of a splinter. We end Section 2 by highlighting connections between the splinter condition and notions of singularities defined in prime characteristic via the Frobenius map. In Section 3 we prove our main theorems. We end our paper by collecting some basic questions, which, as far as we know, are still open for splinters (Section 4).

### 1.3. Conventions.

All rings in this paper are commutative with identity. Although our results are primarily about Noetherian rings, we often use the absolute integral closure of a domain which is highly non-Noetherian. Diverging from usual practice in commutative algebra, by a ‘local ring’ we mean a ring with a unique maximal ideal which is not necessarily Noetherian. When we want the local ring to be Noetherian, we will explicitly say so. We will
sometimes talk about normal rings in a non-Noetherian setting. Recall that an arbitrary commutative ring $R$ is normal if for all prime ideals $p$ of $R$, $R_p$ is a domain which is integrally closed in its fraction field \cite{Sta19, Tag 00GV}. A reduced ring $R$ (not necessarily Noetherian) with finitely many minimal primes is normal precisely when it is integrally closed in its total quotient ring, and in this case $R$ decomposes as a finite product of normal domains \cite{Sta19, Tag 030C}.

2. Properties of Splinters

Throughout this section, $R$ will denote a Noetherian ring of arbitrary characteristic unless otherwise specified. We begin by highlighting some well-known properties of splinters.

2.1. Basic properties. If $R$ is a splinter, then $R$ is reduced because the canonical map $R \rightarrow R_{\text{red}}$ splits. Note that a reduced ring $R$ is a splinter if and only if every finite extension $R \hookrightarrow S$ splits in the category of $R$-modules. This is because if $R$ is reduced, a finite map $R \rightarrow S$ induces a surjection on Spec if and only if it is injective.

A local splinter is not just reduced, but even a domain. In the literature this is usually deduced as a consequence of splinters being normal, but we provide a direct elementary proof here.

**Lemma 2.1.1.** A Noetherian local splinter $(R, m)$ is a domain. In fact, $R$ is normal.

**Proof.** Assume $R$ is not a domain. Since $R$ is reduced, it has more than one minimal prime, say $p_1, \ldots, p_n$, where $n \geq 2$. Consider the projection

$$\varphi: R \rightarrow R/p_1 \times \cdots \times R/p_n.$$ 

This is finite and surjective on Spec, and so, it splits as a map of $R$-modules. Let $\phi$ be a left inverse of $\varphi$, and $e_i$ be the standard idempotent of $R/p_1 \times \cdots \times R/p_n$ with a 1 in the $i$-th spot. Since $e_i$ is annihilated by $p_i$ when $R/p_1 \times \cdots \times R/p_n$ is viewed as an $R$-module, it follows that $\phi(e_i) \in m$, for all $i$. But then,

$$1 = \phi(1) = \sum_{i=1}^{n} \phi(e_i) \in m,$$

which is a contradiction. So $n = 1$, and $R$ is a domain because it is reduced.

We now show $R$ is normal. Let $K = \text{Frac}(R)$, and $a/b \in K$ be integral over $R$. As the module-finite extension

$$R \hookrightarrow R[a/b]$$

splits, it must be an isomorphism since $R$ and $R[a/b]$ are torsion-free $R$-modules of the same rank. Thus, $a/b \in R$, proving normality. \hfill \Box

**Remark 2.1.2.** Lemma [2.1.1] implies that a (non-local) splinter decomposes into a finite product of normal domains \cite{Sta19, Tag 030C}, each of which is easily checked to also be a splinter. Conversely, a finite direct product of splinters is a splinter. Hence most questions about splinters reduce to the domain case.
The following result shows that the property of being a splinter is local:

**Lemma 2.1.3.** Let $R$ be a Noetherian ring. Then the following are equivalent:

1. $R$ is a splinter.
2. For all prime ideals $p \in \text{Spec}(R)$, $R_p$ is a splinter.
3. For all maximal ideals $m \in \text{Spec}(R)$, $R_m$ is a splinter.

Hence, if $R$ is a splinter, then for any multiplicative set $S \subset R$, $S^{-1}R$ is a splinter.

**Sketch of proof.** For (1) $\implies$ (2) it suffices to assume $R$ is a domain, and then the result follows by a simple spreading out argument, while (2) $\implies$ (3) is obvious. Assuming (3), note that it suffices to show that any finite extension $R \hookrightarrow S$ splits. But such a splitting can be checked locally at the maximal ideals where it always holds by the hypothesis of (3).

The final assertion follows from the equivalence of (1)-(3) because localizations of $S^{-1}R$ at its prime ideals coincide with localizations of $R$ at primes that do not intersect $S$. □

### 2.2. Purity and descent of splinters.

This subsection is the technical heart of this paper, as it provides a more tractable criterion for verifying the splinter condition (Lemma 2.2.6). Among applications, we show that splinters descend under a notion that is substantially weaker than faithful flatness (Proposition 2.2.8), which we now introduce.

For any commutative ring $A$, we say that a map of $A$-modules $M \to N$ is *pure* if for all $A$-modules $P$, the induced map $M \otimes_A P \to N \otimes_A P$ is injective. Pure maps are sometimes also called *universally injective maps* [Sta19, Tag 058I]. Closely related to the notion of purity is that of cyclic purity, which may be less familiar to the reader.

**Definition 2.2.1.** Given a ring $A$, we say that a map of $A$-modules $M \to N$ is *cyclically pure* if for all cyclic $A$-modules $A/I$, the induced map $M/IM \to N/IN$ is injective.

**Remark 2.2.2.** Taking the cyclic module to be $A$ itself, it follows that cyclically pure maps are injective. Pure maps are obviously cyclically pure, and it is easy to check that faithfully flat ring maps are pure, hence cyclically pure [Bou89, Chapter I, §3.5, Proposition 9]. Purity and cyclic purity are significantly weaker than faithful flatness. For example, Kunz showed that the Frobenius map of a Noetherian ring of prime characteristic is faithfully flat precisely when the ring is regular [Kun69]. However, non-regular rings for which the Frobenius map is pure (equivalently, cyclically pure) are abundant and are at the heart of Frobenius splitting techniques in positive characteristic algebra and geometry.

A split map of modules is pure, and the converse is true under a mild restriction.

**Lemma 2.2.3.** [HR76, Corollary 5.2] If $\varphi : M \to N$ is a pure map of $A$-modules whose cokernel is finitely presented, then $\varphi$ splits, that is, it admits a left inverse in $\text{Mod}_A$.

It is natural to ask when the notions of cyclic purity and purity coincide for ring maps. Hochster discovered a surprising algebraic property that characterizes those Noetherian rings $A$ for which any cyclically pure ring map $A \to B$ is pure. We now introduce this property.
Definition 2.2.4. ([Hoc77, Definitions (1.1) and (1.3)]) A Noetherian local ring \((R, m)\) is approximately Gorenstein if it satisfies the following equivalent conditions:

(i) For every integer \(N > 0\), there is an ideal \(I \subseteq m^N\) such that \(R/I\) is Gorenstein.

(ii) For every integer \(N > 0\), there is an \(m\)-primary irreducible ideal \(I \subseteq m^N\).

We say a Noetherian ring \(R\) (not necessarily local) is approximately Gorenstein if \(R_m\) is an approximately Gorenstein local ring for all maximal ideals \(m\).

Remark 2.2.5.

(1) The key point is that if \(R\) is approximately Gorenstein, then a ring homomorphism \(R \to S\) is pure if and only if it is cyclically pure [Hoc77, Theorem 2.6].

(2) If \((R, m)\) is Noetherian local, then for every ideal \(I \subseteq m^N\) such that \(R/I\) is Gorenstein, there exists an \(m\)-primary ideal \(J\) such that \(I \subseteq J \subseteq m^N\) and \(R/J\) is Gorenstein. Namely, if \(x_1, \ldots, x_k \in R\) such that their images in \(R/I\) form a system of parameters of \(R/I\), then one can take \(J = I + (x_1^N, \ldots, x_k^N)\). Note that \(J\) is irreducible because the zero ideal of a zero-dimensional Gorenstein ring is irreducible.

The next lemma is a crucial in our proofs of Theorems A, B and C, and establishes a connection between approximately Gorenstein rings and splinters. Although the result is essentially contained in [Hoc77], we include a proof for the reader’s convenience.

Lemma 2.2.6. ([Hoc77]) Let \((R, m)\) be a Noetherian normal local ring. Then for any ring map \(R \to S\), the following are equivalent:

(1) \(R \to S\) is pure.

(2) \(R \to S\) is cyclically pure.

(3) For all \(m\)-primary ideals \(I\) of \(R\), the induced map \(R/I \to S/IS\) is injective.

(4) There exists a decreasing sequence \(\{I_t\}_{t \in \mathbb{N}}\) of \(m\)-primary ideals of \(R\) cofinal with powers of \(m\) such that for all \(t\), \(R/I_t\) is a Gorenstein ring, and the induced map \(R/I_t \to S/I_t S\) is injective.

In particular, if \(R\) is a splinter, then it satisfies the equivalent conditions (1) – (4).

Proof. The implications (1) ⇒ (2) ⇒ (3) follow by the definitions of purity and cyclic purity. For (3) ⇒ (4) we need to construct a decreasing sequence \(\{I_t\}\) of \(m\)-primary ideals cofinal with powers of \(m\) such that each \(R/I_t\) is Gorenstein. For this, it suffices to show by Remark 2.2.5 that a Noetherian normal local ring is approximately Gorenstein. We may assume that \(\dim(R) \geq 2\). Otherwise \(R\) is a regular local ring, and regular local rings are clearly approximately Gorenstein. If \(\dim(R) \geq 2\) and \(R\) is normal, then \(R\) has depth \(\geq 2\). Hence the depth of the completion \(\widehat{R}\) is also \(\geq 2\), and so, \(R\) is approximately Gorenstein by [Hoc77, Theorem (5.2)] because \(\widehat{R}\) has no associated primes of coheight \(\leq 1\).

It remains to show (4) ⇒ (1). Let \(E = E_R(R/m)\) be the injective hull of the residue field of \(R\). Recall that by Matlis duality, \(R \to S\) is pure if and only if the induced map \(E \to E \otimes_R S\) is injective [HH95, Lemma 2.1(e)]. Since every element of \(E\) is annihilated
by a power of $m$, it follows by the hypotheses on the collection $\{I_t\}$ that

$$E = \bigcup_t (0 :_E I_t).$$

Now $(0 :_E I_t) \cong \text{Hom}_R(R/I_t, E)$ is an injective $R/I_t$-module by $\text{Hom-}\otimes$ adjunction. One then checks that $(0 :_E I_t)$ is in fact the injective hull of the residue field of $R/I_t$, and hence coincides with the latter zero-dimensional Gorenstein local ring [LL+07, Theorem A.29]. Thus $E$ is the union of $R$-modules isomorphic to $R/I_t$, and injectivity of $E \to E \otimes_R S$ then follows because for every $t$, $R/I_t \to S/I_t S$ is injective by the hypothesis of (4). □

**Remark 2.2.7.** Lemma 2.2.6 is only stated for a Noetherian normal ring, although its proof holds for any approximately Gorenstein ring. Apart from Noetherian normal rings, the class of approximately Gorenstein rings includes Noetherian local rings that are analytically unramified, excellent reduced rings, and Noetherian rings of depth at least 2.

Lemma 2.2.6 can be used to show that splinters descend under cyclically pure maps.

**Proposition 2.2.8.** Let $\varphi: R \to S$ be a map of Noetherian rings such that $S$ is splinter.

1. If $\varphi$ is pure, then $R$ is a splinter. In particular, splinters descend under faithfully flat maps.

2. If $\varphi$ is cyclically pure and maps nonzerodivisors of $R$ to nonzerodivisors of $S$ (for example, if $S$ is a domain or $\varphi$ is flat), then $R$ is a splinter.

**Proof.** Note that in both cases $R$ is reduced since $S$ is reduced and $\varphi$ is injective. Assertion (2) follows from (1). Indeed, we claim that the hypotheses of (2) imply that $\varphi$ is pure. This follows by Lemma 2.2.6 if we can show that $R$ is normal. Since $R$ is Noetherian, it suffices for us to show that it is integrally closed in its total quotient ring [Sta19, Tag 030C]. Note that $S$ is normal since $S$ is a splinter (Lemma 2.1.1). Let $a/b$ be an element in the total quotient ring of $R$ (hence $b$ is a nonzerodivisor) that is integral over $R$. Then $\varphi(a)/\varphi(b)$ is an element in the total quotient ring of $S$ that is integral over $S$ ($\varphi(b)$ remains a nonzerodivisor in $S$). Therefore $aS \subseteq bS$ since $S$ is normal, and so, $aR = aS \cap R \subseteq bS \cap R = bR$ by cyclic purity of $\varphi$. This shows $a/b \in R$, and so, $R$ is normal.

For the proof of (1), let $R \to T$ be a finite map such that the induced map $\text{Spec}(T) \to \text{Spec}(R)$ is surjective. By base change, $S \to T \otimes_R S$ has the same properties (a surjective morphism of schemes is ‘universally surjective’ by [EGAII, Chapter 1, Proposition (3.5.2)(ii)]). Therefore, the map

$$S \to T \otimes_R S$$

splits in the category of $S$-modules. In particular, $S \to T \otimes_R S$ is a pure map of $S$-modules (hence also $R$-modules). Consider the commutative diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\varphi} & S \\
\downarrow & & \downarrow \\
T & \longrightarrow & T \otimes_R S
\end{array}
$$
Since the composition $R \xrightarrow{\varphi} S \rightarrow S \otimes_R T$ is pure as a map of $R$-modules, it follows that $R \rightarrow T$ is also pure. But $\coker(R \rightarrow T)$ is a finitely presented $R$-module since $R$ is Noetherian and $R \rightarrow T$ is finite, and so, $R \rightarrow T$ splits by Lemma [2.2.3] \hfill \square

2.3. Absolute integral closure and splinters. Recall that if $R$ is a domain, then its absolute integral closure, denoted $R^+$, is the integral closure of $R$ in a fixed algebraic closure of its fraction field. Absolute integral closures allow one to verify the splinter condition by checking cyclic purity of a single map, as highlighted in the following result.

**Lemma 2.3.1.** Let $R$ be a Noetherian domain. Then $R$ is a splinter if and only if the map $R \rightarrow R^+$ is cyclically pure.

**Proof.** Suppose $R$ is a splinter. The map $R \rightarrow R^+$ is a filtered colimit of extensions of form $R \hookrightarrow T$, where $T$ is a finitely generated $R$-subalgebra of $R^+$. But $T$ is then a finite $R$-module (since it is integral over $R$), and so, $R \hookrightarrow T$ splits. Since a filtered colimit of split maps is pure [HH95, Lemma 2.1(i)], we see that $R \rightarrow R^+$ is cyclically pure.

Conversely, suppose $R \rightarrow R^+$ is cyclically pure. Because $R^+$ is integrally closed in its fraction field, the same reasoning as in the proof of Proposition [2.2.8(2)] implies that $R$ is normal. Thus, $R \rightarrow R^+$ is pure by Lemma [2.2.6]. Let $R \rightarrow S$ be a module finite ring map that is surjective on $\text{Spec}$, and $\mathfrak{p} \in \text{Spec}(S)$ be a prime ideal that contracts to $(0) \in \text{Spec}(R)$. Then $R \hookrightarrow S/\mathfrak{p}$ is module finite extension of domains, and hence, $S/\mathfrak{p}$ embeds in $R^+$. As the composition $R \hookrightarrow S/\mathfrak{p} \hookrightarrow R^+$ is pure, so is $R \hookrightarrow S/\mathfrak{p}$. But then $R \hookrightarrow S/\mathfrak{p}$ splits by Lemma [2.2.3], which shows that $R \rightarrow S$ also splits. \hfill \square

2.4. Connection with $F$-singularities. Splinters have long been known to be related to singularities in prime characteristic defined via the Frobenius map. This connection will be important in our analysis of how splinters behave under completions (see Example [3.2.1]). Hence we briefly recall some relevant definitions.

If $I$ is an ideal of a Noetherian domain $R$ of prime characteristic $p > 0$, then the **tight closure of $I$**, denoted $I^*$, is the collection of elements $r \in R$ for which there exists a nonzero $c \in R$ such that $cr^p \in I^{[p^e]}$, for all $e \geqslant 0$. Here $I^{[p^e]}$ denotes the ideal generated by $p^e$-th powers of elements of $I$. We say that $R$ is **weakly $F$-regular** if all ideals of $R$ are tightly closed, that is, $I^* = I$, for any ideal $I$ of $R$. We say an ideal $I$ of $R$ is a **parameter ideal** if $I$ is generated by elements $r_1, \ldots, r_n \in I$ such that for any prime ideal $\mathfrak{p}$ of $R$ that contains $I$, the images of $r_i$ in $R_\mathfrak{p}$ form part of a system of parameters of $R_\mathfrak{p}$. We say $R$ is **$F$-rational** if every parameter ideal of $R$ is tightly closed. Thus, weakly $F$-regular rings are $F$-rational, and the converse holds when $R$ is Gorenstein [HH94(a), Corollary 4.7].

The connections between splinters and $F$-singularities are summarized below.

**Remark 2.4.1.**

1. If $R$ is a splinter of prime characteristic, then the Frobenius map is pure, that is, $R$ is $F$-pure. More generally, any integral map $R \rightarrow S$ which is surjective on $\text{Spec}$ is pure. If $R$ is in addition local, then purity of Frobenius implies that the completion
is reduced, that is, \( R \) is formally reduced. Note that local splinters in equal characteristic 0 need not be formally reduced because there exist Noetherian normal local rings which are not formally reduced \([\text{Nag55}]\).

(2) If \( R \) is a weakly \( F \)-regular domain, then \( \widehat{R} \) is a splinter. Indeed, for any ideal \( I \) of \( R \),

\[
IR^+ \cap R = I \tag{2.4.1.1}
\]

where the containment in (2.4.1.1) follows by \([\text{HH94(b), Corollary 5.23}]\). This shows that \( R \rightarrow R^+ \) is cyclically pure, and so, \( R \) is a splinter by Lemma 2.3.1.

(3) Lemma 2.3.1 combined with Hochster and Huneke’s famous characterization of \( R^+ \) being a big Cohen-Macaulay algebra \([\text{HH92, Theorem 5.15}]\) can be used to deduce that a locally excellent Noetherian splinter of prime characteristic is always Cohen-Macaulay. Since splinters in equal characteristic 0 are equivalent to normal rings, it follows that splinters are not always Cohen-Macaulay in general.

(4) Smith showed that if \( R \) is a locally excellent Noetherian domain of prime characteristic, then for any parameter ideal \( I \) of \( R \),

\[
I = IR^+ \cap R \tag{2.4.1.1.1}
\]

where the containment in (2.4.1.1.1) follows by \([\text{HH94(b), Corollary 5.23}]\). This shows that \( R \rightarrow R^+ \) is cyclically pure, and so, \( R \) is a splinter by Lemma 2.3.1.

3. Proofs of the main theorems

The goal of this section is to prove Theorems A, B and C. Since the proof of Theorem A uses Theorem B we prove the latter result first.

3.1. Henselization of a splinter. The henselization of a local ring is constructed as a filtered colimit of certain essentially étale \( R \)-algebras, so we briefly review the notion of essentially étale maps first.

A local homomorphism of local rings (not necessarily Noetherian)

\[
\varphi: (R, m) \rightarrow (S, n)
\]

is an étale homomorphism of local rings if \( \varphi \) is flat, \( S \) is the localization of a finitely presented \( R \)-algebra, \( mS = n \) and the induced map \( \kappa(m) \rightarrow \kappa(n) \) is finite separable. A homorphism \( \varphi: R \rightarrow S \) of rings (not necessarily local) is étale at \( q \in \text{Spec}(S) \) if the induced map \( R_{\varphi^{-1}(q)} \rightarrow S_q \) is an étale homomorphism of local rings. We say \( \varphi \) is essentially étale (resp. étale) if \( S \) is the localization of a finitely presented (resp. is a finitely presented) \( R \)-algebra and \( \varphi \) is étale at all \( q \in \text{Spec}(S) \). In particular, essentially étale maps are flat.

**Remark 3.1.1.** If \( \varphi: (R, m) \rightarrow (S, n) \) is an étale homomorphism of local rings, choose a finitely presented \( R \)-algebra \( S' \) such that \( S \) is the localization of \( S' \) at a prime ideal \( q \) of \( S' \). Since the locus of primes of \( \text{Spec}(S') \) at which \( \text{Spec}(S') \rightarrow \text{Spec}(R) \) is étale is open,

\(^{1}\)A Noetherian ring \( R \) is locally excellent if for all maximal ideals \( m \) of \( R \), \( R_m \) is excellent.
Let \((R, \mathfrak{m})\) be a local ring (not necessarily Noetherian), and let \(R^h\) denote its henselization. Recall that \(R^h\) is realized as a filtered colimit of étale homomorphisms of local rings \((R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})\) such that the residue field \(\kappa(m)\) of \(R\), one also has the strict henselization \(R^{sh}\) of \(R\) which is a filtered colimit of étale homomorphisms of local rings \((R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})\) such that \(\kappa(m) \subseteq \kappa(n) \subseteq \kappa(m)^{sep}\) \cite[Chapter VIII, Théorème 2]{Ray70}. It follows that if \(R\) is reduced, regular, normal, Cohen-Macaulay or Gorenstein, then so are \(R^h\) and \(R^{sh}\) (see \cite[Tag 07QL]{Sta19}). Additionally, the associated local maps

\[ R \to R^h \to R^{sh} \]

are always faithfully flat \cite[Tag 07QM]{Sta19}.

When \((R, \mathfrak{m})\) is Noetherian local, the completion of \(R^h\) at its maximal ideal \(\mathfrak{m}R^h\) coincides with \(\hat{R}\), that is, the induced map \(\hat{R} \to \hat{R}^h\) on completions is an isomorphism \cite[Tag 06LJ]{Sta19}. This implies the following relation between \(\mathfrak{m}\)-primary ideals of \(R\) and \(\mathfrak{m}R^h\)-primary (resp. \(\mathfrak{m}\hat{R}\)-primary) ideals of \(R^h\) (resp. \(\hat{R}\)).

**Lemma 3.1.2.** Let \(\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})\) be a local homomorphism of Noetherian local rings such that the induced map on completions \(\hat{R} \to \hat{S}\) is an isomorphism (for example, if \(S = R^h, \hat{R}\)). Then expansion and contraction of ideals via \(\varphi\) induces a bijection between \(\mathfrak{m}\)-primary ideals of \(R\) and \(\mathfrak{n}\)-primary ideals of \(S\).

**Proof.** Our hypothesis implies that the composition \(R \xrightarrow{\varphi} S \to \hat{S}\) coincides, up to isomorphism, with the completion \(R \to \hat{R}\) which is faithfully flat. In particular, \(\varphi\) is a pure map, which implies that expansion of ideals gives an injective map from the collection of ideals of \(R\) to the collection of ideals of \(S\). Thus, to prove the lemma, it remains to show that any \(\mathfrak{n}\)-primary ideal of \(S\) is the expansion of an \(\mathfrak{m}\)-primary ideal of \(R\).

Now because \((\mathfrak{m}S)\hat{S} = (\mathfrak{m}\hat{R})\hat{S} = n\hat{S}\), faithful flatness of \(S \to \hat{S}\) shows that \(\mathfrak{m}S = n\). Tensoring the isomorphism \(\hat{R} \sim \hat{S}\) by \(R/\mathfrak{m}^n\) then implies that for all \(n \geq 1\), we have

\[ R/\mathfrak{m}^n \sim R/\mathfrak{m}^n \hat{R} \sim \hat{S}/\mathfrak{n}^n \hat{S} \sim S/\mathfrak{n}^n. \]

Let \(I\) be an \(\mathfrak{n}\)-primary ideal of \(S\), and choose \(n \geq 1\) such that \(\mathfrak{n}^n \subseteq I\). If \(\overline{I}\) is the image of \(I\) in \(S/\mathfrak{n}^n\), then by the isomorphism \(R/\mathfrak{m}^n \sim S/\mathfrak{n}^n\), there exist \(r_1, \ldots, r_m \in R\) whose images in \(S/\mathfrak{n}^n\) generate the ideal \(\overline{I}\). Then \(I\) is the expansion of the \(\mathfrak{m}\)-primary ideal \(\mathfrak{m}^n + (r_1, \ldots, r_m)\).

**Remarks 3.1.3.**

1. Lemma 3.1.2 does not hold if \(S = R^{sh}\) is the strict henselization of \(R\), since in this case the maps \(R/\mathfrak{m}^n \to R^{sh}/\mathfrak{m}^n R^{sh}\) are no longer isomorphisms.
(2) In the statement of Lemma 3.1.2 if the induced map $\widehat{R} \to \widehat{S}$ is surjective, then every $n$-primary ideal of $S$ is the expansion of an $m$-primary ideal of $\widehat{R}$. However, uniqueness is lost.

The goal in the rest of this subsection is to show that the splinter condition is preserved under henselization (Theorem [3]). But first, we establish an ideal theoretic result inspired by [Smi94, Proposition 5.10]. Smith’s result is proved under excellence hypothesis and deals with the completion of Noetherian local rings. Surprisingly, it turns out that an analogue of her result holds for henselizations of arbitrary non-Noetherian normal local domains. Thus we carefully prove the result we need, which should be of independent interest.

**Proposition 3.1.4** (c.f. [Smi94, Proposition 5.10]). Let $(R, m)$ be a normal local domain (not necessarily Noetherian), and let $R^h$ and $R^{sh}$ denote its henselization and strict henselization respectively. Then we have the following:

1. $R^h$ (resp. $R^{sh}$) is a normal domain.
2. If $I$ is an ideal of $R$, then $I(R^h)^+ \cap R = IR^+ \cap R = I(R^{sh})^+ \cap R$.

**Proof.** (1) The fact that $R^h$ and $R^{sh}$ are normal local domains is a consequence of the ascent of normality under étale maps [Sta19, Tag 06DI].

(2) Since $(R^h)^+$ and $(R^{sh})^+$ are $R^+$-algebras, $IR^+ \cap R$ is contained in $I(R^h)^+ \cap R$ and $IR^{sh} \cap R$. Let $z \in I(R^h)^+ \cap R$ (resp. $z \in I(R^{sh})^+ \cap R$). Then there exists a module finite extension domain $T$ of $R^h$ (resp. of $R^{sh}$) contained in $(R^h)^+$ (resp. $(R^{sh})^+$) such that $z \in IT$. Let $i_1, \ldots, i_n \in I$ and $t_1, \ldots, t_n \in T$ such that

$$z = i_1 t_1 + \cdots + i_n t_n.$$ 

Note that the $t_i$ are integral over $R^h$ (resp. $R^{sh}$). Since $R^h$ (resp. $R^{sh}$) is a filtered colimit of étale local $R$-algebras, there exists an étale homomorphism of local rings $(R, m) \to (S, n)$ such that $S \subseteq R^h$ (resp. $S \subseteq R^{sh}$) and $t_1, \ldots, t_n$ are integral over $S$.

By Remark 3.1.1 choose an étale $R$-algebra $S'$ such that $S$ is the localization of $S'$ at a prime ideal $q$ of $S'$. Since $R$ is a normal domain and $R \to S'$ is étale, $S'$ is a normal ring by [Sta19, Tag 033C]. Moreover, $S'$ has only finitely many minimal primes because all minimal primes of $S'$ must contract to $(0)$ in $R$ by going down [Sta19, Tag 00HS], and the generic fiber of $R \to S'$ is a finite product of finite separable field extensions of $\text{Frac}(R)$. In particular, $S'$ decomposes as a finite product of normal domains by [Sta19, Tag 030C], and since $S$ is the localization of one these factors, we may assume $S'$ is a domain. Thus, $S' \subseteq S \subseteq T$. Furthermore, after localizing $S'$ at a suitable element not in $q$, we may also assume $t_1, \ldots, t_n$ are integral over $S'$. Let $T'$ be the $S'$-subalgebra $S'[t_1, \ldots, t_n]$ of $T$. Then $T'$ is a module-finite extension domain of $S'$ by construction, and

$$z \in IT' \cap R.$$ 

Note that $T'$ is not necessarily an integral extension of $R$ (otherwise we would be done). However, the composition $R \to S' \to T'$ is quasi-finite since étale and finite maps are quasi-finite [Sta19, Tag 00US] and [Tag 00PM] and the composition of quasi-finite maps is quasi-finite [Sta19, Tag 00PO]. Moreover, the induced map $\text{Spec}(T') \to \text{Spec}(R)$ is surjective.
because $R \to S'$ is faithfully flat (it is étale, $(R, \mathfrak{m})$ is local and $\mathfrak{q}$ lies over $\mathfrak{m}$) and $S' \to T'$ is a finite extension. Then $z \in IR^+ \cap R$ using the following Lemma which makes no further use of the henselian property.

**Lemma 3.1.5.** Let $R \hookrightarrow S$ be a quasi-finite extension of domains (not necessarily Noetherian) such that the induced map on $\text{Spec}$ is surjective. Suppose also that $R$ is integrally closed in its fraction field, that is, $R$ is normal. Then we have the following:

1. $S$ can be identified as a subring of $\text{Frac}(R^+)$.
2. With the identification from (1), if $I$ is an ideal of $R$ and $z \in S$ such that $z \in IS \cap R$, then $z \in IR^+ \cap R$.
3. With the identification from (1), if $I$ is an ideal of $R$, then $IS^+ \cap R = IR^+ \cap R$.

Recall that a finite type map $\varphi : R \to S$ is quasi-finite at $q \in \text{Spec}(S)$, if $q$ is isolated in its fibre. We say $\varphi$ is quasi-finite if it is quasi-finite at all $q \in \text{Spec}(S)$ [Sta19, Tag 00PL], or equivalently, if $\varphi$ has finite fibers [Sta19, Tag 00PM]. Finite, unramified and étale maps are quasi-finite. Quasi-finite maps are ‘close’ to being finite by Zariski’s Main Theorem – if $R \to S$ is quasi-finite at $q \in \text{Spec}(S)$ and $\overline{S}$ is the integral closure of $R$ in $S$, then there exists $g \in \overline{S}$ such that $g \notin q$ and $\overline{S}_g \sim S_q$ [Sta19, Tag 00QQ9]. Since $S_q$ is of finite type over $R$, it is then easy to see that $S_q = T_q$, for an $R$-subalgebra $T \subseteq \overline{S}$ such that $R \to T$ is finite.

**Proof of Lemma 3.1.5.** We first prove (1) and (2). Note that once $S$ is identified as a subring of $\text{Frac}(R^+)$, (2) will follow if we can show that for each prime ideal $\mathfrak{p}$ of $R^+$,

$$z \in (IR^+)^\mathfrak{p} = IR^\mathfrak{p}_q.$$

Choose any $\mathfrak{p} \in \text{Spec}(R^+)$, and let $\mathfrak{p} := \mathfrak{p} \cap R$. Let $\overline{S}$ denote the integral closure of $R$ in $S$, and $q$ be a prime ideal of $S$ such that

$$q \cap R = \mathfrak{p}.$$

Such a prime exists because $R \hookrightarrow S$ is surjective on $\text{Spec}$. Then by Zariski’s Main Theorem [Sta19, Tag 00QQ9], there exists $g \in \overline{S}$, $g \notin q$ and an isomorphism

$$\overline{S}_g \sim S_g.$$

This implies $\text{Frac}(S) = \text{Frac}(\overline{S})$, and so, $\text{Frac}(S)$ is an algebraic extension of $\text{Frac}(R)$. In particular, we may identify $\text{Frac}(S)$, hence also $S$ and $\overline{S}$, as subrings of $\text{Frac}(R^+)$, which proves (1).

With the identification from (1), we have $R \subseteq \overline{S} \subseteq R^+$. Thus, $\overline{S} \subseteq R^+$ is also an integral extension, and consequently,

$$\overline{S}_g = S_g \subseteq R_g^+$$

is an integral extension as well. Let $\mathfrak{q}$ be a prime ideal of $R^+$ not containing $g$ such that $\mathfrak{q}R_g^+$ lies over $qS_g$. Since $q$ contracts to $\mathfrak{p}$ in $R$ by our choice of $q$, it follows that $\mathfrak{q}$ also lies over $\mathfrak{p}$ in $R$. Moreover, $z \in IS \cap R$ implies

$$z \in IS_g \subseteq IR_g^+ \subseteq IR^\mathfrak{p}_g.$$

As $R$ is integrally closed in $\text{Frac}(R)$ and $\text{Frac}(R^+)/\text{Frac}(R)$ is a normal algebraic field extension, the elements of $\text{Aut}(\text{Frac}(R^+)/\text{Frac}(R))$ (which induce automorphisms of $R^+$
over $R$) act transitively on the prime ideals of $R^+$ that lie over $p$ \cite[Chapter V, §2.3, Proposition 6(i)]{Bou89}. Since $\Sigma$ and $\mathfrak{P}$ are prime ideals of $R^+$ that both lie over $p$ and $z \in R$ by the hypothesis of (2), we then get

$$z \in IR^+_\Sigma \Rightarrow z \in IR^+_\mathfrak{P}.$$  

However, $\mathfrak{P}$ is an arbitrary prime ideal of $R^+$, and so, $z \in IR^+$, hence also $z \in IR^+ \cap R$, proving (2).

(3) The non-trivial inclusion is $IS^+ \cap R \subseteq IR^+ \cap R$. Suppose $z \in IS^+ \cap R$. Since $S^+$ is a union of module-finite $S$-subalgebras, there exists a module-finite extension $T$ of $S$ contained in $S^+$, such that $z \in IT \cap R$. However, the composition $R \hookrightarrow S \hookrightarrow T$ is quasi-finite since $R \hookrightarrow S$ and $S \hookrightarrow T$ are quasi-finite. Moreover, the induced map $\text{Spec}(T) \to \text{Spec}(R)$ remains surjective. Then $z \in IT \cap R$ implies $z \in IR^+ \cap R$ by (2). \hfill $\square$

Armed with Proposition \[3.1.4\] the proof of Theorem B is now a formal exercise.

**Proof of Theorem B.** If $R^h$ is a splinter, then by descent of splinters along the faithfully flat map $R \to R^h$ (Lemma \[2.2.8\]), it follows that $R$ is a splinter. We now show the converse, that is, we show that if a Noetherian local ring $(R, m)$ is a splinter, then $R^h$ is a splinter. Since $R$ is normal (see Lemma \[2.1.1\]), $R^h$ is normal. To show that $R^h$ is a splinter, it then suffices to show by Lemma \[2.3.1\] that the map

$$R^h \to (R^h)^+$$

is cyclically pure. By normality of $R^h$, purity of the map $R^h \to (R^h)^+$ will follow by Lemma \[2.2.6\] if we can show that for every $mR^h$-primary ideal $I$ of $R^h$, the induced map

$$R^h/I \to (R^h)^+/I(R^h)^+$$

is injective.

Therefore, let $I$ be an $mR^h$-primary ideal. By Lemma \[3.1.2\] there exists a unique $m$-primary ideal $J$ of $R$ such that

$$I = JR^h.$$ We then have

$$(I(R^h)^+ \cap R^h) \cap R = J(R^h)^+ \cap R = JR^+ \cap R = J. \quad \text{(3.1.5.1)}$$

where the first equality is obvious, the second equality follows from Proposition \[3.1.4\] and the third equality follows from cyclic purity of $R \to R^+$ because $R$ is a splinter (Lemma \[2.3.1\]). Observe that the ideal

$$I(R^h)^+ \cap R^h$$

is $mR^h$-primary. Indeed, $I(R^h)^+ \cap R^h$ contains the $mR^h$-primary ideal $I$, and it is also contained in $mR^h$ because $J \neq R$. The upshot of this observation is that $I(R^h)^+ \cap R^h$ must also be expanded from a unique $m$-primary ideal $J'$ of $R$. Then uniqueness and \[3.1.5.1\] forces $J' = J$. Thus, $I(R^h)^+ \cap R^h = JR^h = I$, which is equivalent to the injectivity of $R^h/I \to (R^h)^+/I(R^h)^+$. This completes the proof. \hfill $\square$

\footnote{Bourbaki uses the term quasi-Galois extension in \cite[Chapter V, §2.3, Proposition 6]{Bou89} as a synonym for a normal algebraic field extension. See the footnote on \cite[Pg. 331]{Bou89} as well as \cite[Chapter V, §9, no. 3, Definition 2]{Bou03}.}
Remark 3.1.6. Theorem B immediately implies a special case of Theorem A, namely that if \((R, m) \hookrightarrow (S, n)\) is an étale homomorphism of Noetherian local rings such that the induced map on residue fields is an isomorphism, then \(R\) is a splinter if and only if \(S\) is a splinter. In this case the henselization of \((S, n)\) is also \(R^h\) and hence there is a faithfully flat local map \(S \to R^h = S^h\). Since \(R^h\) is a splinter if \(R\) is by Theorem B, \(S\) is a splinter by descent of splinters along faithfully flat maps (Proposition 2.2.8).

As far as we are aware, it is not known whether the henselization of an arbitrary weakly \(F\)-regular Noetherian local ring of prime characteristic \(p > 0\) is weakly \(F\)-regular. The best known result, due to Hochster and Huneke, is that weak \(F\)-regularity is preserved under henselization for local \(G\)-rings [HH94(a), Theorem (7.24)]. Recall that we say a Noetherian ring \(R\), not necessarily local, is a \(G\)-ring if for all maximal ideals \(m\) of \(R\), the formal fibers of \(R_m\) are geometrically regular (c.f. Remark 3.2.3). At first glance, [HH94(a), Theorem (7.24)] implies the seemingly more general statement that weak \(F\)-regularity of a Noetherian local ring \((R, m)\) is preserved under henselization if \(R^h\) is a \(G\)-ring and the singular locus of \(R^h\) is closed. However, \(R^g\) is a \(G\)-ring if and only if \(R\) is a \(G\)-ring [Gre76, Theorem 5.3], and the singular locus of a local \(G\)-ring is always closed [ILO14, Exposé I, Proposition 5.5.1]. Thus, the requirement for the singular locus of the target to be closed is unnecessary in the statement of [HH94(a), Theorem (7.24)], and the best known result is indeed the one stated above.

Nevertheless, Theorem B has the following consequence for arbitrary weakly \(F\)-regular local rings.

Corollary 3.1.7. Let \((R, m)\) be a Noetherian local ring of prime characteristic \(p > 0\). If \(R\) is weakly \(F\)-regular (i.e. all ideals are tightly closed), then \(R^h\) is a splinter.

Proof. Weakly \(F\)-regular local rings are splinters (Remark 2.4.1(2)). Now apply Theorem B. □

3.2. Completion of a splinter. We want to show that if \((R, m)\) is a \(G\)-ring, then \(\hat{R}\) is also a splinter. Before we prove this result, we give an example that illustrates that the completion of a local splinter may not be always be a splinter, even for rings with very mild singularities. Our example comes from the theory of \(F\)-singularities summarized in Subsection 2.4.

Example 3.2.1. Let \((R, m)\) be a Noetherian local ring of prime characteristic \(p > 0\). It is well-known that if \(R\) is Gorenstein, then \(R\) is weakly \(F\)-regular if and only if \(R\) is \(F\)-rational [HH94(a), Corollary 4.7]. Loepp and Rotthaus construct an example of a non-excellent Gorenstein local domain \((R, m)\) which is weakly \(F\)-regular (equivalently \(F\)-rational), but whose completion is not weakly \(F\)-regular (equivalently \(F\)-rational) [LR01, Section 5]. However, any weakly \(F\)-regular domain of prime characteristic is a splinter (Remark 2.4.1(2)). Thus, Loepp and Rotthaus’ construction gives a Gorenstein local splinter domain, whose completion is not \(F\)-rational. Since \(\hat{R}\) is excellent, if it is a splinter, then it will also be \(F\)-rational by Remark 2.4.1(4), which shows that \(\hat{R}\) cannot be a splinter.
The main technical result used in the proof of Theorem C is [Smi94, Proposition 5.10], which is the analogue for completion of Proposition 3.1.4. We need the following minor generalization of Smith’s result.

**Proposition 3.2.2** (c.f. [Smi94, Proposition 5.10]). Suppose \((R, m)\) is a Noetherian normal local ring, and let \(\hat{R}\) denote the completion of \(R\). If \(R\) is a \(G\)-ring, then for any ideal \(I\) of \(R\),

\[
I(\hat{R})^+ \cap R = IR^+ \cap R = I(R^h)^+ \cap R.
\]

**Proof.** Recall that we have a chain of faithfully flat maps \(R \to R^h \to \hat{R}\), and \(\hat{R}\) is the \(mR^h\)-adic completion of \(R^h\). Since \(R\) is a \(G\)-ring, so is its henselization \(R^h\) [Gre76, Theorem 5.3]. Moreover, a Henselian \(G\)-ring is excellent [ILO14, Exposé 1, Corollaire 6.3(ii)], and so, \(R^h\) is an excellent, normal local domain. Therefore, \(\hat{R} = \hat{R^h}\) is also normal.

Let \(I\) be an ideal of \(R\). The equality \(IR^+ \cap R = I(R^h)^+ \cap R\) follows from Proposition 3.1.4. By the discussion in the previous paragraph, applying [Smi94, Proposition 5.10] to the map of normal excellent local rings \(R^h \to \hat{R^h} = \hat{R}\), we get

\[
I(R^h)^+ \cap R^h = I(\hat{R})^+ \cap R^h. \tag{3.2.2.1}
\]

Intersecting (3.2.2.1) with \(R\) then gives us

\[
IR^+ \cap R = I(R^h)^+ \cap R = (I(R^h)^+ \cap R^h) \cap R \stackrel{\text{3.2.2.1}}{=} (I(\hat{R})^+ \cap R^h) \cap R = I(\hat{R})^+ \cap R,
\]

which completes the proof of the proposition. \(\square\)

**Remark 3.2.3.** The proof of Proposition 3.2.2 uses that if \((R, m)\) is a Noetherian local \(G\) ring, then \(R^h\) is excellent. The converse is also true. Namely, if \(R^h\) is excellent, then \(R \to \hat{R}\) is regular. To see this, note that \(R \to \hat{R}\) factors as \(R \to R^h \to \hat{R}\), and \(\hat{R}\) is also the completion of \(R^h\). Thus \(R \to \hat{R}\) is regular because \(R \to R^h\) is regular [Sta19, Tag 07QQ], \(R^h \to \hat{R}\) is regular by excellence of \(R^h\), and the composition of regular maps of Noetherian rings is regular [Sta19, Tag 07QI].

We can now prove Theorem C.

**Proof of Theorem C** Again, by faithfully flat descent of the splinter property, the non-trivial implication is to show that if \((R, m)\) is a \(G\)-ring which is a splinter, then \(\hat{R}\) is also a splinter.

So suppose \(R\) is a splinter. By Lemma 2.1.1 \(R\) is a normal domain. The proof that \(\hat{R}\) is a splinter is now a formal consequence of Proposition 3.2.2 and Lemma 3.1.2 with the argument proceeding in the same manner as in the proof of Theorem B upon replacing \(R^h\) by \(\hat{R}\) and \((R^h)^+\) by \((\hat{R})^+\). Thus, the details are omitted. \(\square\)

Theorem C allows us to generalize results on excellent local splinters in prime characteristic to splinters that are \(G\)-rings.
Corollary 3.2.4. Let \((R, m)\) be a Noetherian local \(G\)-ring of prime characteristic. If \(R\) is a splinter, then we have the following:

1. \(R\) is Cohen–Macaulay.
2. If \(R\) is also Gorenstein, then \(R\) is weakly \(F\)-regular, and hence \(F\)-rational.

Proof. (1) Since \(\hat{R}\) is an excellent local splinter, it is Cohen–Macaulay by Remark 2.4.1(3). Thus, \(R\) is also Cohen–Macaulay.

(2) Since \(\hat{R}\) is an excellent splinter, it is \(F\)-rational by Remark 2.4.1(4). However, a Gorenstein \(F\)-rational local ring is weakly \(F\)-regular [HH94(a), Corollary 4.7]. Thus \(\hat{R}\) is weakly \(F\)-regular, and since weak \(F\)-regularity descends under faithfully flat maps [HH90, Proposition 4.12], it follows that \(R\) is weakly \(F\)-regular. Consequently, \(R\) is also \(F\)-rational.

\(\square\)

Remark 3.2.5. Since the completion of a regular local ring is always regular, the direct summand theorem shows that the completion of a regular local ring is always a splinter, even though it is not difficult to construct regular local rings whose formal fibers are not geometrically regular. Thus, while \(G\)-rings are sufficient for the splinter property to ascend under completions, they are by no means necessary.

3.3. Étale ascent. To prove the ascent of the splinter property under essentially étale maps (Theorem A), we will reduce to the finite étale case. For the reduction, we need the following result which is well-known to experts. However, a proof is included for the sake of completeness.

Lemma 3.3.1. Let \(\varphi: (R, m) \to (S, n)\) be a local homomorphism of local rings (not necessarily Noetherian). Suppose \(S\) is the localization of a finite type \(R\)-algebra \(S'\) at a prime ideal \(q\) such that \(R \to S'\) is quasi-finite at \(q\). Then the induced map \(\varphi^h: R^h \to S^h\) is finite.

In particular, if \(\varphi: (R, m) \to (S, n)\) is an étale homomorphism of local rings, then \(\varphi^h: R^h \to S^h\) is a finite étale map.

Proof. Note that \(q\) lies over \(m\) because \(n\) lies over \(m\). Since the locus of primes of \(\text{Spec}(S')\) at which \(\text{Spec}(S') \to \text{Spec}(R)\) is quasi-finite is open [Sta19, Tag 00QA], we may assume \(R \to S'\) is quasi-finite. Then by base change, \(R^h \to R^h \otimes_R S'\) is also quasi-finite [Sta19, Tag 00PP, part (3)]. Note that

\[
(R^h \otimes_R S') \otimes_{S'} \kappa(q) = (R^h \otimes_R \kappa(m)) \otimes_{\kappa(m)} \kappa(q) = \kappa(q).
\]

Hence the expansion of \(q\) in \(R^h \otimes_R S = (R^h \otimes_R S') \otimes_{S'} S'_q\) is a prime ideal of \(R^h \otimes_R S\), which implies by [Sta19, Tag 05WP] that

\[
S^h = (R^h \otimes_R S')_q(R^h \otimes_R S).
\]

Let \(\Omega\) be the contraction of the maximal ideal of \(S^h\) to \(R^h \otimes_R S'\). It is clear that \(\Omega\) lies over the maximal ideal \(mR^h\) of \(R^h\). Since \(R^h\) is Henselian and \(R^h \to R^h \otimes_R S'\) is quasi-finite,
§2.3, Proposition 4(e) [see also [Sta19] Tag 04GG, part (13)] shows that we can choose \( f \notin \Omega \) such that

\[
(R^h \otimes_R S')_f
\]

is a finite \( R^h \)-algebra, which consequently decomposes as a finite product of \( R^h \)-module finite Henselian local rings [Sta19] Tag 04GG, part (10) and Tag 04GH. Since \( \Omega \) lies over the maximal ideal \( mR^h \) of \( R^h \), the prime xideal \( \Omega(R^h \otimes_R S')_f \) is maximal in \((R^h \otimes_R S')_f \) by module finiteness of \( R^h \to (R^h \otimes_R S')_f \). Thus, \( S^h = (R^h \otimes_R S')_{\varphi(R^h \otimes_R S')} = (R^h \otimes_R S')_{\Omega} \) coincides with one of the \( R^h \)-module-finite factors of \((R^h \otimes_R S')_f \), proving the first assertion.

If \( \varphi \) is an étale homomorphism of local rings, we may assume that \( S' \) is an étale \( R \)-algebra. Then \( S^h \) is a localization of the étale \( R^h \)-algebra \( R^h \otimes_R S' \), and so \( \varphi^h \) is an étale homomorphism of local rings which is finite by what we proved above because étale maps are quasi-finite [Sta19] Tag 00US.\( \square \)

**Remark 3.3.2.** The analogue of Lemma 3.3.1 also holds when we replace ‘henselizations’ by ‘completions’ in the Noetherian setting, i.e., if \((R, m) \to (S, n)\) is an étale (even unramified) homomorphism of Noetherian local rings, then the induced map on completions \( \hat{R}^m \to \hat{S}^n \) is module-finite. This fact is easier to prove [Sta19] Tag 039H, part (1).

We finally have all the tools in our arsenal to prove Theorem A.

**Proof of Theorem A** Let \( \varphi: R \to S \) be an essentially étale homomorphism of Noetherian rings such that \( R \) is a splinter. We want to show that \( S \) is a splinter. Let \( q \) be a prime ideal of \( S \). By Lemma 2.1.3 it suffices to show \( S_q \) is a splinter. Replacing \( \varphi: R \to S \) by the induced map \( R_{\varphi(q)} \to S_q \), we may assume \( R = (R, m) \) and \( S = (S, n) \) are local, and \( \varphi \) is an étale local homomorphism of Noetherian local rings. Consider the commutative diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\varphi} & S \\
\downarrow & & \downarrow \\
R^h & \xrightarrow{\varphi^h} & S^h
\end{array}
\]

where the vertical maps are faithfully flat (hence pure) and \( \varphi^h \) is finite étale by Lemma 3.3.1. Note that \( R^h \) is a splinter by Theorem B and so, to show that \( S \) is a splinter, it is enough to show \( S^h \) is a splinter by faithfully flat descent (see Lemma 2.3.1).

Thus, we reduce the proof of Theorem A to the case where \( \varphi: (R, m) \to (S, n) \) is a finite étale local homomorphism of Noetherian local rings. In particular, \( \varphi \) induces a surjective map on \( \text{Spec} \) since it is faithfully flat. Let \( \phi: S \to T \) be a finite map such that the induced map on \( \text{Spec} \) is surjective. In order to show that \( \phi \) splits, it suffices to exhibit an \( S \)-linear map \( T \to S \) that maps \( 1 \in T \) to a unit in \( S \).

The composition \( R \xrightarrow{\varphi} S \xrightarrow{\phi} T \) is a finite map and \( \text{Spec}(\phi \circ \varphi) \) is surjective. Thus, since \( R \) is a splinter, there exists an \( R \)-linear map

\[
g: T \to R
\]
such that \( g(1) = 1 \). The retraction \( g \) induces an \( S \)-linear map

\[
\Psi_g : T \to \text{Hom}_R(S, R)
\]
defined as follows: for all \( t \in T \) and \( s \in S \),

\[
\Psi_g(t)(s) := g(\phi(s)t).
\]

We then have a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\Psi_g} & \text{Hom}_R(S, R) \\
\downarrow{g} & & \downarrow{\text{ev}_1} \\
R & & R
\end{array}
\]

where the vertical map is evaluation at 1. Now by Grothendieck duality for a proper smooth map [Har66, Chapter VII, Theorem 4.1] applied to the finite étale map \( \varphi \), we have isomorphisms of \( S \)-modules

\[
\text{Hom}_R(S, R) = \text{Hom}_R(\varphi_* S, R) \cong \varphi_*(\text{Hom}_S(S, \varphi^! R)) = \text{Hom}_S(S, \varphi^! R).
\]

Moreover, since \( \varphi \) is étale, we know that the functor \( \varphi^! \) coincides with the pullback functor \( \varphi^* \) because the relative canonical bundle of an étale map is trivial (see the definition of \( f^! \) for a smooth map given on [Har66, Chapter VII, §4, Pg 388]). Thus, we have isomorphisms of \( S \)-modules

\[
\text{Hom}_R(S, R) \cong \text{Hom}_S(S, \varphi^! R) \cong \text{Hom}_S(S, \varphi^* R) = \text{Hom}_S(S, S) \cong S.
\]

In particular, we then get a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\tilde{\Psi}_g} & S \\
\downarrow{g} & & \downarrow{\tilde{\text{ev}}_1} \\
R & & R
\end{array}
\]

where \( \tilde{\Psi}_g \) is \( S \)-linear and \( \tilde{\text{ev}}_1 \) is \( R \)-linear. We claim that

\[
\tilde{\Psi}_g(1) \notin n.
\]

Indeed, if \( \tilde{\Psi}_g(1) \in n = mS \), then by \( R \)-linearity of \( \tilde{\text{ev}}_1 \), we get

\[
1 = g(1) = \tilde{\text{ev}}_1(\tilde{\Psi}_g(1)) \in \tilde{\text{ev}}_1(mS) \subseteq m,
\]

which is a contradiction. Thus, \( \tilde{\Psi}_g(1) \notin n \), which shows that \( \tilde{\Psi}_g \) is an \( S \)-linear map that sends \( 1 \in T \) to a unit in \( S \). But this is precisely what we wanted to show. \( \square \)

**Corollary 3.3.3.** Let \( (R, m) \) be a Noetherian local ring. Then \( R \) is a splinter if and only if its strict henselization \( R^{sh} \) is a splinter.

**Proof.** If \( R^{sh} \) is a splinter, so too is \( R \) by faithfully flat descent. Recall that \( R^{sh} \) is a filtered colimit of pairs \( (S, n) \), where \( (R, m) \to (S, n) \) is an étale homomorphism of Noetherian local rings and \( \kappa(n) \) is contained in a fixed choice of a separable algebraic closure of \( \kappa(m) \). Let \( R^{sh} \to T \) be a finite map such that the induced map Spec\( (T) \to \text{Spec}(R^{sh}) \) is surjective.
Since $T$ is a finitely presented $R^{sh}$-algebra, there exists a model $T_S$ of $T$ over one of the pairs $(S, n)$ and a fibered square

$$
\begin{array}{c}
\text{Spec}(T) \\
\downarrow \\
\text{Spec}(R^{sh})
\end{array} \quad \begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow
\end{array} \quad 
\begin{array}{c}
\text{Spec}(T_S) \\
\downarrow \\
\text{Spec}(S)
\end{array}
$$

Note that the map $(S, n) \to (R^{sh}, m)$ is faithfully flat since $R^{sh}$ is also the strict henselization of $S$. Thus, by faithfully flat descent of finite generation [Bou89, Chapter 1, §3.6, Proposition 11], we have that $T_S$ is a finitely generated $S$-module. Moreover, by commutativity of the above diagram, Spec($T_S$) → Spec($S$) is surjective. Since $R$ is a splinter and $S$ is an essentially étale extension of $R$, Theorem A implies $S$ is also a splinter. Therefore $S \to T_S$ splits, and so by base change, $R^{sh} \to T$ splits.

□

Remark 3.3.4. Theorem $\text{A}$ $\Rightarrow$ Corollary 3.3.3 while faithfully flat descent of the splinter property shows that Corollary 3.3.3 $\Rightarrow$ Theorem $\text{B}$ using the faithfully flat map $R^{sh} \to R^{sh}$. Thus, Theorem $\text{A}$ implies Theorem $\text{B}$. Since Theorem $\text{A}$ is deduced as a consequence of Theorems $\text{A}$ and $\text{B}$ are equivalent.

The derived version of Theorem $\text{A}$ also holds with some caveats.

Corollary 3.3.5. Let $\varphi : (R, m) \to (S, n)$ be an étale homomorphism of Noetherian local rings such that $R$ is a derived splinter.

1. If $R$ has prime characteristic $p > 0$ or has mixed characteristic, then $S$ is a derived splinter.

2. If $R$ is essentially of finite type over a field $k$ of characteristic 0, then $S$ is a derived splinter.

Proof. Since $\varphi$ is an étale homomorphism of local rings, $R$ and $S$ both have the same characteristic.

(1) follows in prime characteristic $p > 0$ by [Bha12, Theorem 1.4] and Theorem $\text{A}$ since splinters and derived splinters coincide in prime characteristic. The same is true in mixed characteristic by forthcoming work of Bhatt [Bha].

(2) Since $R$ being a derived a splinter is equivalent to it having rational singularities ([Kov00] and [Bha12, Theorem 2.12]), it suffices for us to show that rational singularities ascend under an étale homomorphism of local rings. By definition of rational singularities, there exists a proper birational map $f : X \to \text{Spec}(R)$ with $X$ regular over $k$ such that $\mathcal{O}_{\text{Spec}(R)} \simeq f_*(\mathcal{O}_X)$ and $R^i f_*(\mathcal{O}_X) = 0$ for all $i > 0$. Then the base change map $f_S : X_S \to \text{Spec}(S)$ is also proper birational. Moreover, $X_S$ is regular over $k$ since $X_S \to X$ is an essentially étale morphism (it is a base change of the essentially étale map $\text{Spec}(S) \to \text{Spec}(R)$) and since regularity ascends under essentially étale morphisms [Sta19, Tag 025N]. Finally, by flat base change of higher direct images [Har77, Chapter 3, Proposition 9.3], we have

$$R^i(f_S)_*(\mathcal{O}_{X_S}) = \text{Spec}(\varphi)^\vee R^i f_*(\mathcal{O}_X),$$
for all $i \geq 0$. In particular, this implies
\[(f_S)_*(\mathcal{O}_{X_s}) = \text{Spec}(\varphi)^* f_* (\mathcal{O}_X) \simeq \text{Spec}(\varphi)^* \mathcal{O}_{\text{Spec}(R)} = \mathcal{O}_{\text{Spec}(S)} , \]
and $R^i (f_S)_*(\mathcal{O}_{X_s}) = 0$, for $i > 0$. Thus, $S$ has rational singularities, and consequently, it is a derived splinter. □

Remark 3.3.6. Recently Kovács has generalized the notion of a rational singularity to arbitrary characteristic for excellent normal Cohen–Macaulay schemes that admit a dualizing complex [Kov18, Definition 1.3]. He shows that if $X$ is such a scheme that is also a derived splinter, then $X$ has rational singularities in this more general sense [Kov18, Theorem 8.7]. In fact, $X$ is also pseudorational in the sense of Lipman and Teissier [LT81] by [Kov18, Corollary 9.14]. However, the converse is false, that is, a rational singularity is not always a derived splinter. For example, in prime characteristic there exist finite type graded rings over fields with rational singularities that are not $F$-rational [HW96, Example (2.11)]. Any such ring cannot be a splinter by Remark 2.4.1(4), hence also not a derived splinter by [Bha12, Theorem 1.4].

4. SOME OPEN QUESTIONS

We conclude this paper with some questions that we believe are open for splinters. The first, is the generalization of Theorem A mentioned in the introduction:

**Question 1:** Suppose $\varphi : R \to S$ is a flat homomorphism of Noetherian rings with geometrically regular fibers. If $R$ is a splinter, is $S$ a splinter?

**Remark 4.0.1.**

(1) Question 1 reduces to the following special case –

**Question 1′:** If $R$ is a splinter, is $R[x]$ a splinter?

Indeed, if $R \to S$ is a regular map, then Néron-Popescu desingularization implies that $S$ can be written as a filtered colimit of smooth $R$-algebras [Pop90, Swa98]. Note, however, that the smooth $R$-algebras in the colimit are not necessarily $R$-subalgebras of $S$. If $S \to T$ is a finite ring map that is surjective on $\text{Spec}$, then by descent of properties of morphisms over projective limits of schemes [Sta19, Tag 01ZM, Tag 01ZO, and Tag 07RR], there exists a model $T_{S'}$ of $T$ over a smooth $R$-algebra $S'$ such that $S' \to T_{S'}$ is finite and induces a surjection on $\text{Spec}$. To get a splitting of $S \to T$ it suffices to show that $S' \to T_{S'}$ splits. Therefore to answer Question 1, we may assume $R \to S$ is a smooth ring map. However, given $q \in \text{Spec}(S)$, in a suitable affine open neighborhood $U$ of $q$, the map $U \to \text{Spec}(R)$ factors as an étale map $U \to \mathbb{A}_R^n$ followed by the canonical projection $\mathbb{A}_R^n \to \text{Spec}(R)$ [BLR90, Chapter 2, Remark 12]. Hence by Theorem A and induction on $n$, it further suffices to show that a polynomial ring over a splinter remains a splinter.

(2) Question 1′ implies the direct summand theorem. In fact, a proof of the direct summand theorem follows if we can answer Question 1′ when $R$ is an excellent and Gorenstein. Indeed, by a reduction due to Hochster [Hoc83, Theorem (6.1)], it suffices to prove that complete, unramified regular local rings are splinters. Any such
regular local ring is of the form $V[[x_1, \ldots, x_n]]$, where $V$ is a $p$-adically complete (hence also excellent) DVR of mixed characteristic $(0, p)$.

Consider the factorization

$$V \to V[x_1, \ldots, x_n]_{(p,x_1,\ldots,x_n)} \xrightarrow{(p,x_1,\ldots,x_n)-\text{adic completion}} V[[x_1, \ldots, x_n]].$$

If we can show that $V[x_1, \ldots, x_n]$ is a splinter (which follows if Question 1 is true), then the localization $V[x_1, \ldots, x_n]_{(p,x_1,\ldots,x_n)}$ is also a splinter (Lemma 2.1.3). But $V[x_1, \ldots, x_n]_{(p,x_1,\ldots,x_n)}$ is excellent because $V$ is complete, and so, Theorem C will then imply that its $(p, x_1, \ldots, x_n)$-adic completion $V[[x_1, \ldots, x_n]]$ is also a splinter.

(3) Suppose $R$ is a Gorenstein ring of prime characteristic $p > 0$ that is also a $G$-ring. If $R$ is a splinter, then so is $R[x]$. First note that $R[x]$ is also a $G$-ring since the property of being a $G$-ring is preserved under essentially finite type maps [EGAIV] Théorème (7.4.4). Let $\mathfrak{m}$ be a maximal ideal of $R[x]$ and $p$ denote its contraction to $R$. Since the splinter condition can be checked locally at the closed points (Lemma 2.1.1), it suffices to show $R[x]_{\mathfrak{m}}$ is a splinter. Note $R_p$ is weakly $F$-regular by Corollary 3.2.4 since $R_p$ is a splinter and a $G$-ring. Moreover, the local homomorphism

$$R_p \to R[x]_{\mathfrak{m}}$$

is regular, $R[x]_{\mathfrak{m}}$ is a $G$-ring, and the singular locus of $R[x]_{\mathfrak{m}}$ is closed by [ILO14, Exposé 1, Proposition 5.5.1(i)]. Thus, $R[x]_{\mathfrak{m}}$ is also weakly $F$-regular by [HH94(a), Theorem (7.24)]. But a weakly $F$-regular ring is always a splinter by Remark 2.4.1(2), hence $R[x]_{\mathfrak{m}}$ is a splinter as desired.

If $R$ is a locally excellent $\mathbb{Q}$-Gorenstein splinter of prime characteristic $p > 0$, then $R[x]$ is also a splinter. Indeed, $R$ is then $F$-regular by [Sin99 Theorem 1.1] (i.e. all localizations of $R$ are weakly $F$-regular) and a $G$-ring since it is locally excellent. Therefore, by the same reasoning as in the previous paragraph, $R[x]$ is a splinter.

**Question 2:** Let $R$ be an excellent ring. Is the locus of primes $p \in \text{Spec}(R)$ such that $R_p$ is a splinter open?

If $R$ is locally excellent, but not excellent, then Question 2 has a negative answer by the following general result of Hochster:

**Theorem 4.0.2.** [Hoc73(b), Proposition 2] Let $\mathcal{P}$ be a property of Noetherian local rings. Let $k$ be an algebraically closed field, and let $(R, \mathfrak{m})$ be essentially of finite type over $k$ such that

1. $R$ is a domain,
2. $R/\mathfrak{m} = k$, and
3. for every field extension $L/k$, the ring $(R \otimes_k L)_{\mathfrak{m}}$ fails to satisfy $\mathcal{P}$.

Additionally, suppose any field satisfies $\mathcal{P}$. For all $n \in \mathbb{N}$, let $R_n$ be a copy of $R$ with maximal ideal $\mathfrak{m}_n = \mathfrak{m}$. Let $R' := \bigotimes_{n \in \mathbb{N}} R_n$, where the infinite tensor product is taken over $k$. Then each $\mathfrak{m}_n R'$ is a prime ideal of $R'$. Moreover, if $S = R' \setminus (\bigcup_m \mathfrak{m}_n R')$, then

$$T := S^{-1} R'.$$
is a Noetherian domain whose locus of primes that satisfy $P$ is not open in $\text{Spec}(T)$. Furthermore, each local ring of $T$ is essentially of finite type over $k$, hence $T$ is locally excellent.

**Example 4.0.3.** Let $P$ be the property of being a splinter. Note that all fields are splinters. Let $(R, m)$ be the local ring of a closed point of an algebraic variety over $k = \overline{k}$ such that such that $R$ is not a splinter (for example, choose a non-normal $R$). Then for any field extension $L/k$, $(R \otimes_k L)_m$ cannot be a splinter because $(R \otimes_k L)_m$ is a faithfully flat extension of $R$, and splinters descend under faithfully flat maps (Proposition 2.2.8). Thus, $R$ satisfies requirements (1) – (3) of Theorem 4.0.2 and so, the theorem gives us a locally-excellent ring whose splinter locus is not open.

**Question 3:** Are derived splinters of mixed characteristic Cohen–Macaulay?

Question 3 is false in characteristic 0 if derived splinters are replaced by splinters, because normal rings are not necessarily Cohen–Macaulay. As observed in Corollary 3.2.4, if $R$ is a Noetherian splinter of prime characteristic that is also a $G$-ring, then $R$ is Cohen–Macaulay. On the other hand, Question 3 seems to be completely open in mixed characteristic. One cannot use Kovács’s characteristic independent definition of rational singularities to show that derived splinters are Cohen–Macaulay because the Cohen–Macaulay property is built into his definition [Kov18, Definition 1.3]. At least for rings possessing dualizing complexes, one obstruction seems to be that we do not know whether Grauert-Riemenschneider type vanishing theorems hold in mixed characteristic. Translated into a question about local commutative algebra, Cohen–Macaulayness of local derived splinters $(R, m)$ in mixed characteristic will follow if one can show that $H^i_m(R^+) = 0$ for $i < \dim(R)$. This is because when $R \to R^+$ is a pure map (which it is when $R$ is a splinter), $H^i_m(R)$ injects into $H^i_m(R^+)$ by [HR74, Corollary 6.6].

5. **Acknowledgments**

We thank Takumi Murayama, Thomas Polstra, Karl Schwede and Anurag Singh for helpful conversations and their interest. We especially thank Bhargav Bhatt, Mel Hochster, Linquan Ma and Karen Smith for patiently answering our many questions and for comments on a draft. We first became aware of some of the questions addressed in this paper during a lecture series taught by Linquan at the University of Illinois at Chicago. Part of this research was carried out at the birthday conference in honor of Bernd Ulrich at the University of Notre Dame, and we thank the conference organizers for providing a stimulating environment.

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