A note on the metastability in three modifications of the standard Ising model

K. Bashiri

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Abstract

We consider three extensions of the standard 2D Ising model with Glauber dynamics on a finite torus at low temperature. The first model (see Chapter 2) is an anisotropic version, where the interaction energy takes different values on vertical and on horizontal bonds. The second model (Chapter 3) adds next-nearest-neighbor attraction to the standard Ising model. And the third model (Chapter 4) associates different alternating signs for the magnetic fields on even and odd rows. All these models have already been studied, and results concerning metastability have been established using the so-called pathwise approach (see [7],[8],[9]). In this text, we extend these earlier results, and apply the potential-theoretic approach to metastability to obtain more precise asymptotic information on the transition time from the metastable phase to the stable phase.

1 Introduction

Section 1.1 provides basic background and motivation for metastability. In Section 1.2 we give a rough overview of this text, and Section 1.3 introduces quickly the main method that is used in this text in a general context. In Section 1.4 we add some further definitions that will be used for the remaining part of this work.

1.1 Background

In many physical, biological or chemical evolutions, one can observe a phenomenon called metastability. If the states of the system are associated to an energy functional, this phenomenon can be described as follows. For a relatively long time, the state of the system resides around a local minimum of the energy landscape, which is not the global minimum. This state is called the metastable state. However, under thermal fluctuations and after many unsuccessful attempts the system can finally free itself from this valley in the energy landscape and it manages the crossover to the global minimum, which is called the stable state.
Often, this crossover is triggered by reaching a critical state in the system. An example is over-saturated water vapour, where below critical temperature, the formation of a critical droplet is needed to achieve the transition from the gas-phase to the liquid-phase. An analogue situation holds for over-cooled liquids and for magnetic hysteresis.

Through the last decades many mathematical models have been built to study this phenomenon. One is mostly interested in three topics:

i) The average transition time from the metastable to the stable state,

ii) The exponential distribution of this transition time,

iii) The typical paths for the transition from the metastable to the stable state.

Mainly two methods have been crystallized to be very fruitful to tackle these problems. The first one is the pathwise approach, initiated by Casanard, Galves, Olivieri and Vares in [6]. Motivated by the Freidlin and Wentzell theory, one uses large deviation estimates on the path space to identify the most likely paths of the system for the transition from the metastable to the stable state. The advantage is a very detailed description of the tube of typical paths for the transition, but at the same time the average transition time can only be computed up to a multiplicative factor of the order $e^{\varepsilon \beta}$ as $\beta \to \infty$, where $\beta$ is the inverse temperature and $\varepsilon > 0$ is independent of $\beta$ and can be chosen arbitrary small. For an extensive discussion on the pathwise approach to metastability, the reader is referred to the monograph [11] by Olivieri and Vares.

The second method is the potential-theoretic approach to metastability, which was initiated by Bovier, Eckhoff, Klein and Gayrard in [2]. Here, one uses potential theory to rewrite the average transition time in terms of quantities from electric networks, namely capacities. Now, using variational principles for the capacity, the average transition time can be computed up to a multiplicative error that tends to one as $\beta \to \infty$, which provides a sharp estimate. This method is also the basis of this text, and in Section 1.3 we will shortly review a general recipe to obtain metastability results for a stochastic process on a finite graph, whose dynamics are given through a metropolis algorithm. For a more detailed overview, we refer to the 2015 monograph [4] by Bovier and den Hollander (especially Chapter 16).

Probably the easiest application of this methods, where one can rigorously investigate metastable behavior, is the two-dimensional standard Ising model on a finite torus in the low temperature regime. Neves and Schonmann applied 1991 in [10] the pathwise approach to this model, and in the year 2002 the potential-theoretic approach was used in [5] by Bovier and Manzo. In this paper, we study three modifications of the Ising model that are defined in chapters 2–4. The pathwise approach has already been applied to all these models in [7], [8] and [9], respectively. In Chapters 7.7 – 7.10 of [11], a brief overview on these three papers is given. Here, we complement these results and apply the potential-theoretic approach to obtain more precise information on topic i).

All three models differ from the standard Ising model mainly in the fact that we lose the applicability of isoperimetrical inequalities. Namely, in the Ising case, for a given number of up-spins, the configuration with minimal energy is a droplet of up-spins with its shape being a square (or a quasi-square) with a possible bar of up-spins attached to one of its (longer) sides. Here, we do not have this property. Instead we need to look at the stability of certain classes of configurations separately in order to specify the metastable and the critical state rigorously.
1.2 Outline

We first give a structural outline of this work.

The next subsection is a quick overview of the setting and the results of Chapter 16 in [4]. A dynamical spin-flip model on the two-dimensional lattice is introduced, which is driven by a general energy function. Also definitions concerning the geometrical properties of the energy landscape are given, which is crucial for the study of metastability. At the end of that section, we state the so-called metastability theorems that cover the topics i) and ii) of those listed in Section 1.1. Topic iii) has already been solved in [7], [9] and [8], respectively. Hence, in order to apply the potential-theoretic approach to the three above mentioned models with their specific energy functionals, we need to verify the conditions of the metastability theorems and to compute the model-dependent parameters in that cases. This is the content of chapters 2–4. Finally, Section 1.4 introduces some additional objects that will be needed for all three cases.

In Chapter 2 we look at the same model as in [7], where the interaction between neighboring spins is anisotropic in the sense that the attraction on horizontal bonds is stronger than on vertical bonds.

In Chapter 3 we allow next-nearest-neighbor attraction, i.e. two spins that have euclidean distance of $\sqrt{2}$ feel an interaction energy, which is strictly less than the interaction energy between nearest-neighbor bonds. This has the physical intuition that next-nearest-neighbor attraction is seen as a perturbation of nearest-neighbor attraction. An interesting fact is that the local minima of the energy landscape are given by droplets of octogonal shape. For the pathwise approach to this model we refer to [8].

Chapter 4 modifies the standard Ising model by allowing the magnetic field to take alternating signs and absolute values on even and on odd rows. The pathwise approach has been applied to this model in [9].

1.3 Potential-theoretic approach to metastability

Let $\Lambda \subset \mathbb{Z}^2$ be a finite, square box with periodic boundary conditions, centered at the origin. $S = \{-1, 1\}^\Lambda$ will be called the configuration space. An element $\sigma \in S$ is called configuration, and at each site $x \in \Lambda$, $\sigma(x) \in \{-1, 1\}$ is called the spin-value at $x$. By abuse of notation, we often identify each configuration $\sigma \in S$ with the sites that have spin +1, i.e.

$$\sigma \equiv \{x \in \Lambda \mid \sigma(x) = +1\}. \quad (1.1)$$

Moreover, we represent $\sigma$ geometrically by identifying to each $x \in \sigma$ a unit square centered at $x$. See Figure 1 for an example.

The energy of the system is given by a Hamiltonian $H : S \to \mathbb{R}$. If $\beta > 0$ is the inverse temperature, the Gibbs measure associated with $H$ and $\beta$ is given by

$$\mu_\beta(\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)}, \quad \text{for } \sigma \in S, \quad (1.2)$$

where $Z_\beta$ is a normalization constant called partition function.

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1 The setting in [4] is more general, but to keep it as simple as possible, we restrict to our situation of a dynamical spin-flip model on the two-dimensional lattice.
For $\sigma \in S$ and $x \in \Lambda$, we define $\sigma^x \in S$ by
\[
\sigma^x(y) = \begin{cases} 
\sigma(y) & : y \neq x, \\
-\sigma(x) & : y = x.
\end{cases}
\tag{1.3}
\]
For all $\sigma, \sigma' \in S$, we write $\sigma \sim \sigma'$, if there exists $x \in \Lambda$ such that $\sigma^x = \sigma'$. This induces a graph structure on $S$ by defining an edge between each $\sigma, \sigma' \in S$, whenever $\sigma \sim \sigma'$.

The dynamics of the system is given by the continuous time Markov Chain $(\sigma_t)_{t \geq 0}$ on $S$, whose generator $L_\beta$ is given by
\[
(L_\beta f)(\sigma) = \sum_{x \in \Lambda} c_\beta(\sigma, \sigma^x)(f(\sigma^x) - f(\sigma)),
\tag{1.4}
\]
where $f : S \to \mathbb{R}$ and
\[
c_\beta(\sigma, \sigma') = \begin{cases} 
\exp(-\beta \max\{0, H(\sigma') - H(\sigma)\}) & : \sigma \sim \sigma', \\
0 & : \text{else}.
\end{cases}
\tag{1.5}
\]
Notice that for $\beta = \infty$, only moves to configurations with lower or equal energy are permitted. Moreover, one can immediately see that the following detailed balance condition holds:
\[
\mu_\beta(\sigma)c_\beta(\sigma, \sigma') = \mu_\beta(\sigma')c_\beta(\sigma', \sigma) \quad \forall \sigma, \sigma' \in \mathcal{X}_n(\beta).
\tag{1.6}
\]
Hence, the dynamics is reversible with respect to the Gibbs measure. The law of $(\sigma_t)_{t \geq 0}$ given that $\sigma_0 = \sigma \in S$ will be denoted by $\mathbb{P}_\sigma$, and for a set $A \subset S$, we denote its first hitting time after the starting configuration has been left by $\tau_A$, i.e.
\[
\tau_A = \inf\{t > 0 | \sigma_t \in A, \exists 0 < s < t : \sigma_s \neq \sigma_0\}.
\tag{1.7}
\]
A few definitions are needed to describe the geometry of the energy landscape of the system.

**Definition 1.1**  

i) Let $\sigma, \sigma' \in S$. The communication height between $\sigma$ and $\sigma'$ is defined by
\[
\Phi(\sigma, \sigma') = \min_{\gamma: \sigma \rightarrow \sigma'} \max_{\eta \in \gamma} H(\eta),
\tag{1.8}
\]
where the minimum is taken over all finite paths $\gamma$ of allowed moves in $S$ going from $\sigma$ to $\sigma'$. 

\[\]
ii) Let $\sigma, \sigma' \in S$. A finite path $\gamma : \sigma \to \sigma'$ is called optimal path between $\sigma$ and $\sigma'$, if
\[
\Phi(\sigma, \sigma') = \max_{\eta \in \gamma} H(\eta). \tag{1.9}
\]
The set of all optimal paths between $\sigma$ and $\sigma'$ is denoted by $(\sigma \to \sigma')_{opt}$.

iii) Let $\sigma \in S$. The stability level of $\sigma$ is defined by
\[
V_{\sigma} = \min_{\eta \in S : H(\eta) < H(\sigma)} \Phi(\sigma, \eta) - H(\sigma). \tag{1.10}
\]

iv) The set of stable states in $S$ is defined by:
\[
S_{stab} = \{ \sigma \in S \mid H(\sigma) = \min_{\eta \in S} H(\eta) \}. \tag{1.11}
\]
v) The set of metastable states in $S$ is defined by:
\[
S_{meta} = \{ \sigma \in S \mid V_{\sigma} = \max_{\eta \in S \setminus S_{stab}} V_{\eta} \}. \tag{1.12}
\]

According to the motivation in Section 1.1 to study metastable behavior, it will be crucial to define the notion of a critical state in a mathematically precise way. This is done in the following

**Definition 1.2** Let $(m, s) \in S_{meta} \times S_{stab}$ and let
\[
\Gamma^* = \Phi(m, s) - H(m). \tag{1.13}
\]
Then $(P^*(m, s), C^*(m, s))$ is defined as the maximal subset of $S \times S$ such that
1.) $\forall \sigma \in P^*(m, s) \exists \sigma' \in C^*(m, s) : \sigma \sim \sigma'$, and
   $\forall \sigma' \in C^*(m, s) \exists \sigma \in P^*(m, s) : \sigma \sim \sigma'$,
2.) $\forall \sigma \in P^*(m, s) : \Phi(m, \sigma) < \Phi(\sigma, s)$,
3.) $\forall \sigma' \in C^*(m, s) \exists \gamma : \sigma' \to s : \max_{\eta \in \gamma} H(\eta) - H(m) \leq \Gamma^*, \forall \eta \in \gamma : \Phi(m, \eta) \geq \Phi(\eta, s)$.
We call $P^*(m, s)$ the set of protocritical states, and $C^*(m, s)$ the set of critical states.

We are now in the position to formulate the metastability theorems (see Theorem 16.4 – 16.6 in [4]). These will hold subject to the following hypothesis

(H1) $S_{meta} = \{ m \}$ and $S_{stab} = \{ s \}$,

where $m, s \in S$. One challenge is to verify this hypothesis in the Chapters 2–4 for the three models. Under (H1), it would not lead to confusions, if we abbreviate $P^* = P^*(m, s)$ and $C^* = C^*(m, s)$.

**Theorem 1.3** Subject to (H1), it holds that

a) $\lim_{\beta \to \infty} \mathbb{P}_m[\tau_{C^*} < \tau_s \mid \tau_s < \tau_m] = 1$,
b) If, moreover, the following assumption holds

\[(H2) \quad \sigma' \rightarrow \{\sigma \in \mathcal{P}^*(m, s) : \sigma \sim \sigma'\} \text{ is constant on } C^*, \]

then for all \(\chi \in C^*\), it holds: \(\lim_{\beta \to \infty} P_m[\sigma_{\tau_{C^*}} = \chi] = \frac{1}{|C^*|}.\)

**Theorem 1.4** Subject to \((H1)\), it holds that

a) \(\lim_{\beta \to \infty} \lambda_{\beta} E_m[\tau_s] = 1\), where \(\lambda_{\beta}\) is the second largest eigenvalue of \(-L_{\beta}\).

b) \(\lim_{\beta \to \infty} P_m[\tau_{\sigma} > t \cdot E_m[\tau_s]] = e^{-t}\) for all \(t \geq 0\).

**Theorem 1.5** Subject to \((H1)\), it holds that

a) There exists a constant \(K \in (0, \infty)\) such that \(\lim_{\beta \to \infty} e^{-\beta \Gamma^*} E_m[\tau_s] = K\),

b) Let

\[- S^* \subset S \text{ be the subgraph obtained by removing all vertices } \eta \text{ with } H(\eta) > \Gamma^* + H(m) \]
\[- S^{**} \subset S^* \text{ be the subgraph obtained by removing all vertices } \eta \text{ with } H(\eta) = \Gamma^* + H(m) \]
\[- S_m = \{\eta \in S \mid \Phi(m, \eta) < \Phi(\eta, s) = \Gamma^* + H(m)\}, \]
\[- S_s = \{\eta \in S \mid \Phi(\eta, s) < \Phi(m, \eta) = \Gamma^* + H(m)\}, \]
\[- S_1, \ldots, S_I \subset S^{**} \text{ be such that } S^{**} \backslash (S_m \cup S_m) = \bigcup_{i=1}^I S_i \text{ and each } S_i \text{ is a maximal set of communicating configurations.} \]

Then:

\[ \frac{1}{K} = \min_{C_1, \ldots, C_I \in [0, 1]} \min_{h:S^* \to [0, 1]} \min_{h|S_m=1,h|S_s=0,h|S_i=C_i} \frac{1}{2} \sum_{\eta, \eta' \in S^*} 1_{\{\eta \sim \eta'\}} [h(\eta) - h(\eta')]^2. \]  

(1.14)

Theorem 1.3 says that the set of critical states is a gate for the transition, i.e. that \(C^*\) has to be reached in order to cross over from the metastable to the stable state. If the additional assumption holds, then part b) of Theorem 1.3 says that the entrance into \(C^*\) is uniformly distributed. Theorem 1.4 represents the average transition time of the system in terms of the spectrum of its generator and part b) covers topic ii) of section 1.1. Finally, Theorem 1.5 covers topic i) and gives a variational formula to compute the prefactor.

### 1.4 Further definitions

We conclude this chapter with some definitions that are used in all three situations in Chapters 2–4.

- For \(x \in \mathbb{R}\), \([x]\) denotes the smallest integer greater than \(x\).
Hamiltonian

In the same setting as in Section 1.3 let the 2 Anisotropic Ising model \( \sigma \) where \( \sigma \) representation of \( \Lambda \) and \( \Lambda^\star \) equal to its vertical length \( l \).

Two droplets on the torus are called isolated, if their Euclidean distance is greater or equal to \( \sqrt{2} \).

Let \( \sigma \in S \) and \( x \in \Lambda \) be such that \( \sigma(x) = +1 \). Then \( x \) is called protuberance, if \( \sum_{y \in \Lambda : |y-x|=1} \sigma(y) = -2 \).

If \( \sigma \in S \) consists of a single, connected droplet, then \( R(\sigma) \) is the smallest rectangle that contains \( \sigma \).

A row or column of a connected configuration \( \sigma \in S \) is defined as the intersection of a row or a column of \( \Lambda \) with \( \sigma \).

For \( \sigma \in S \), let \( |\sigma| \) be the area of \( \sigma \), i.e. its number of \((+1)\)-spins. Further, \( \partial(\sigma) \) is the Euclidean boundary of \( \sigma \) in its geometric representation and \( |\partial(\sigma)| \) denotes the perimeter, i.e. the length of \( \partial(\sigma) \).

For \( A \subset S \), let \( \partial^+A = \{ \sigma \in S \setminus A : \exists \sigma' \in S : \sigma \sim \sigma' \} \) denote the outer boundary \( A \). We also define \( A^+ = A \cup \partial^+A \). Moreover, if \( \eta \in S \), then \( A \sim \eta = \{ \sigma \in A : \sigma \sim \eta \} \).

In all three situations in the following chapters, we will have to show that

\[
\begin{align*}
m &= \square, \quad \text{and} \quad s = \Box,
\end{align*}
\]

where \( \square \in S \) is the configuration, where all spin values are \(-1\) and \( \Box \) is the configuration with all spin values being \(+1\).

2 Anisotropic Ising model

In the same setting as in Section 1.3 let the Hamiltonian be given by

\[
H^\Lambda(\sigma) = -\frac{J_H}{2} \sum_{(x,y) \in \Lambda^*_H} \sigma(x)\sigma(y) - \frac{J_V}{2} \sum_{(x,y) \in \Lambda^*_V} \sigma(x)\sigma(y) - \frac{h}{2} \sum_{x \in \Lambda} \sigma(x),
\]

where \( \sigma \in S \), \( J_H, J_V, h > 0 \), \( \Lambda^*_H \) is the set of unordered horizontal nearest-neighbor bonds in \( \Lambda \) and \( \Lambda^*_V \) is the set of unordered vertical nearest-neighbor bonds in \( \Lambda \). Using the geometric representation of \( \sigma \), one can rewrite \( H^\Lambda(\sigma) \) as

\[
H^\Lambda(\sigma) = H^\Lambda(\square) - h|\sigma| + J_H|\partial_V(\sigma)| + J_V|\partial_H(\sigma)|,
\]
where $|\partial_V(\sigma)|$ is the length of the vertical part of $\partial(\sigma)$ and $|\partial_H(\sigma)|$ is the length of the horizontal part of $\partial(\sigma)$. In Figure 1 we get that $|\partial_V(\sigma)| = 34$ and $|\partial_H(\sigma)| = 40$.

The critical length in this model will be given by

$$L^*_V = \left\lceil \frac{2J_V}{h} \right\rceil.$$  \hspace{1cm} (2.3)

The following assumptions will be made for this chapter.

**Assumption 2.1**

a) $J_H > J_V$,  
b) $2J_V > h$,  
c) $\frac{2J_V}{h} \notin \mathbb{N}$,  
d) $|\Lambda| > \left( \max\left\{ \frac{2J_H}{2L^*_V}, \frac{2J_H(L^*_V-1)}{2L^*_V-h(L^*_V-1)} + L^*_V \right\} \right)^2$.

By symmetry, assumption a) could have also been chosen the other way around. Assumption c) is made to avoid degeneracy in the arguments, and b) induces that the dynamics prefers aligned neighboring spins than $(+1)$-spins. In that way the dynamics, starting from $\Box$, will have to increase the energy in order to obtain $(+1)$-spins and to reach $\blacksquare$, since it needs to break bonds in $\Box$. This is essential to obtain the metastable behavior of the system. Assumption d) implies that after the critical state has reached for an optimal path, it can avoid to reach that energy level again and that it is not profitable to enlarge a droplet such that one side is subcritical and the other side wraps around the torus. Moreover, d) assures that the torus is large enough to contain a critical droplet. More details on these facts will be seen later. It immediately follows from Assumption 2.1 c) that

$$(L^*_V - 1)h < 2J_V < L^*_V h. \hspace{1cm} (2.4)$$

Before stating the main result of this chapter, we need the following definition.

**Definition 2.2**

- Let $R(L^*_V-1,L^*_V)^{1pr}$ denote the set of all configurations consisting only of a rectangle from $R(L^*_V-1,L^*_V)$ and with an additional protuberance attached to one of its longer sides. The right droplet in Figure 2 provides an example.
- Let $R(L^*_V-1,L^*_V)^{2pr}$ be the set of all configurations that are obtained from a configuration in $R(L^*_V-1,L^*_V)^{1pr}$ by adding a second $(+1)$-spin next to the protuberance.

Now we can formulate the main result of this chapter.

**Theorem 2.3**

Under Assumption 2.1, the pair $(\Box, \blacksquare)$ satisfies (H1) and (H2) so that Theorems 1.3–1.5 hold for the anisotropic Ising model.

Moreover, the quadruple $(\mathcal{P}^*, C^*, \Gamma^*, K)$ is given by

- $\mathcal{P}^* = R(L^*_V-1,L^*_V)$,
- $C^* = R(L^*_V-1,L^*_V)^{1pr}$,
- $\Phi(\Box, \blacksquare) - H^A(\Box) = 2L^*_V(J_H + J_V) - h(1 + (L^*_V - 1)L^*_V) =: \Gamma_A =: E_A^* - H^A(\Box)$,
- $K^{-1} = \frac{4(2L^*_V-1)}{3} |\Lambda|$.

**Proof.** The proof is divided into the Sections 2.1–2.6. \hfill $\square$
2.1 Proof of $\Phi(\square,\square) - H^A(\square) \leq \Gamma^*_A$

It will be enough to construct a path $\gamma = (\gamma(n))_{n \geq 0} : \square \to \square$ such that

$$\max_{\eta \in \gamma} H^A(\eta) \leq H^A(\square) + \Gamma^*_A = E^*_A. \tag{2.5}$$

This path will be called reference path.

Construction of $\gamma$. Let $\gamma(0) = \square$. In the first step an arbitrary spin is flipped from $-1$ to $+1$. Then $\gamma$ will first pass through a sequence of squares and quasi-squares as follows. If at some step $i$, $\gamma(i)$ is a square, then a $+1$–spin is added somewhere above the droplet. Afterwards, this row is filled by successively flipping in this row adjacent $(-1)$–spins until the droplet has the shape of a quasi-square. This procedure is stopped, when $R((L^*_V - 1) \times L^*_V)$ is reached. Now the same adding structure is continued but solely on the right side of the droplet. If the droplet winds around the torus, then one adds a $+1$–spin above the droplet and fills this row until it also winds around the torus. This is repeated until $\square$ is reached.

Inequality (2.5) holds. Let $k^*$ be such that $\gamma(k^*) \in R((L^*_V - 1) \times L^*_V)$. Then $H^A(\gamma(k^*)) = E^*_A - 2J_V + h < E^*_A$. If we go backwards in the path from that point on, then we will have to cut the top row of $R((L^*_V - 1) \times L^*_V)$, which has the length $L^*_V - 1$. This is an increase of the energy in each step until the top row turns into a protuberance. At this point the energy equals to

$$H^A(\gamma(k^* - (L^*_V - 2))) = E^*_A - 2J_V + (L^*_V - 1)h < E^*_A \tag{2.6}$$

by (2.4). Cutting the last protuberance decreases the energy even more. With the same reasoning, if we keep on going backwards in the path of $\gamma$, we will always stay below $E^*_A$, since the size of the above and right bars of the droplets will be at most $L^*_V - 1$. Hence, we get that

$$\max_{i=1,\ldots,k^*} H^A(\gamma(i)) < E^*_A. \tag{2.7}$$

Let us look at the remaining path of $\gamma$ after the step $k^* + 2$. It holds that $H^A(\gamma(k^* + 2)) = E^*_A - h < E^*_A$. Until the right column is filled, the energy is decreased in every step. Afterwards, a protuberance is added on the right side and the energy increases by $2J_V - h$. Again by (2.4), we get

$$H^A(\gamma(k^* + (L^*_V + 1))) = E^*_A + 2J_V - L^*_V h < E^*_A \tag{2.8}$$

Repeating this until the droplet wraps around the torus, the following energy level is reached

$$E^*_A - (hL^*_V - 2J_V)(\sqrt{|\Lambda|} - L^*_V) - h(L^*_V - 1) - 2J_V L^*_V. \tag{2.9}$$
Now we add a protuberance above the droplet and the energy increases by $2J_H - h$. Assumption 2.1(d) and (2.3) imply
\[
E^*_A - (hL^*_V - 2J_V)(\sqrt{\mid \Lambda \mid - L^*_V}) - h(L^*_V - 1) - 2J_V L^*_V + 2J_H - h \\
\leq E^*_A + (hL^*_V - 2J_V)L^*_V - hL^*_V - 2J_V L^*_V \\
< E^*_A - 2J_V L^*_V. 
\]
Filling this row, decreases the energy by $(\sqrt{\mid \Lambda \mid - 1})h + 2J_V$. In the same way, one can show that the remaining part of the path stays below $E^*_A$. Together with (2.7), we infer (2.5).

2.2 Proof of $\Phi(\square, \blacksquare) - H^A(\square) \geq \Gamma^*_A$

It is enough to show that every optimal path from $\square$ to $\blacksquare$ has to pass through $R(L^*_V - 1, L^*_V)^{1pr}$. The following observations will be very useful.

**Lemma 2.4** Let $\sigma \in S$ be a local minimum of the energy $H^A$, i.e. $H^A(\sigma^x) > H^A(\sigma)$ for all $x \in \Lambda$. Then $\sigma$ is a union of isolated rectangles.

**Proof.** Assume $\sigma$ has a connected component $\sigma_1$ that is not a rectangle. Consider a connected component $\gamma_1$ of $R(\sigma_1) \cap (Z^2 \setminus \sigma_1)$. Let $l_1$ be the maximal component of the boundary of $\gamma_1$ that does not belong to the boundary of $R(\sigma_1)$. An example would be:

![Example of boundary](image)

Then, since $\sigma_1$ is connected and $l_1$ lies inside $R(\sigma_1)$, $l_1$ has both a horizontal part and a vertical part. Let $x \in \Lambda$ be a site with $\sigma(x) = -1$, where both such parts come together. Obviously, $\sigma^x$ has strictly lower energy than $\sigma$, since $x$ has at least two nearest-neighbor (+1)-spins. □

As an immediate consequence, we get:

**Corollary 2.5** Assume that $\sigma \in S$ consists of only one connected component. Then
\[
H^A(\sigma) \geq H^A(R(\sigma)), \tag{2.11}
\]
and equality holds, if and only if $\sigma = R(\sigma)$.

We first show that every optimal path has to cross $R(L^*_V - 1, L^*_V)$.

**Lemma 2.6** Let $\gamma \in (\square, \blacksquare)_{opt}$. Then $\gamma$ has to cross $R(L^*_V - 1, L^*_V)$.

**Proof.** Assume the contrary, i.e. $\gamma \cap R(L^*_V - 1, L^*_V) = \emptyset$. Let us first assume that throughout its whole path $\gamma$ consists of a single connected component. On its way to $\blacksquare$, $\gamma$ has to cross a
configuration, whose rectangular envelope has both horizontal and vertical length greater or equal to $L_V$. Let

$$t = \min \{ l \geq 0 \mid P_H R(\gamma(l)), P_V R(\gamma(l)) \geq L_V^* \}.$$  \hfill (2.12)

Since $\gamma$ is assumed to be connected, we observe that either $P_H R(\gamma(t-1)) = L_V^* - 1$ holds or $P_V R(\gamma(t-1)) = L_V^* - 1$. In the following we analyze both cases and show that the assumption $\gamma \cap R(L_V^* - 1, L_V^*) = \emptyset$ leads to a contradiction.

- **Case 1**: $[P_V R(\gamma(t-1)) = L_V^* - 1].$
  From the definition of $t$, it is clear that $R(\gamma(t-1)) \in R((L_V^* + m) \times (L_V^* - 1))$ for some $m \geq 0$.

  - **Case 1.1**: $[m = 0].$
    Since $\gamma$ does not cross $R(L_V^* \times (L_V^* - 1))$, we have by Corollary 2.5 that
    $$H^A(\gamma(t-1)) > H^A(L_V^* \times (L_V^* - 1)) = H^A(\square) + \Gamma_A - 2J_H + h = E_A^* - 2J_H + h.$$  \hfill (2.13)
    The minimal increase of energy to enlarge the vertical length of the rectangular envelope of a configuration is $2J_H - h$. Hence,
    $$H^A(\gamma(t)) \geq H^A(\gamma(t-1)) + 2J_H - h > E_A^*.$$  \hfill (2.14)
    This contradicts $\gamma \in (\square, \square)_{opt}$, since we already know from Section 2.1 that $\Phi(\square, \square) \leq E_A^*$.

  - **Case 1.2**: $[m \in [1, \sqrt{\lambda} - L_V^*]].$
    Again, by Corollary 2.5 we have that
    $$H^A(\gamma(t-1)) \geq H^A((L_V^* + m) \times (L_V^* - 1))$$
    $$= H^A(L_V^* \times (L_V^* - 1)) + m(2J_V - h(L_V^* - 1))$$
    $$> E_A^* - 2J_H + h,$$  \hfill (2.15)
    where we used inequality [2.4] in the last step. As before, this leads to a contradiction, since
    $$H^A(\gamma(t)) \geq H^A(\gamma(t-1)) + 2J_H - h > E_A^*.$$  \hfill (2.16)

  - **Case 1.3**: $[m = \sqrt{\lambda} - L_V^*].$
    In this case, $\gamma(t-1)$ wraps around the torus. Using Assumption 2.1 d), we infer that
    $$H^A(\gamma(t-1)) \geq H^A(\sqrt{\lambda} \times (L_V^* - 1))$$
    $$= H^A(L_V^* \times (L_V^* - 1)) + (\sqrt{\lambda} - L_V^*)(2J_V - h(L_V^* - 1)) - 2J_V(L_V^* - 1)$$
    $$> H^A(L_V^* \times (L_V^* - 1)) = E_A^* - 2J_H + h.$$  \hfill (2.17)
    Finally,
    $$H^A(\gamma(t)) \geq H^A(\gamma(t-1)) + 2J_H - h > E_A^*,$$  \hfill (2.18)
    which is a contradiction.

  - **Case 2**: $[P_H R(\gamma(t-1)) = L_V^* - 1].$
    Here we have that $R(\gamma(t-1)) \in R((L_V^* - 1) \times (L_V^* + m'))$ for some $m' \geq 0.$
• Case 2.1: \([m' = 0]\).
Since \(\gamma\) does not cross \(R((L^*_V - 1) \times L^*_V)\), we have by Corollary 2.5 that
\[
H^A(\gamma(t - 1)) > H^A((L^*_V - 1) \times L^*_V) = E_A' - 2J_V + h. \tag{2.19}
\]
The minimal increase of energy to enlarge the horizontal length of the rectangular envelope of a configuration is \(2J_V - h\). Hence,
\[
H^A(\gamma(t)) \geq H^A(\gamma(t - 1)) + 2J_V - h > E_A'. \tag{2.20}
\]
As before, this contradicts \(\gamma \in (\mathfrak{D}, \mathfrak{D}_\text{opt})\).

• Case 2.2: \([m' \in [1, \sqrt{\mid\Lambda\mid} - L^*_V]\)].
As before, this case also leads to a contradiction, since
\[
H^A(\gamma(t)) \geq H^A(\gamma(t - 1)) + 2J_V - h \geq H^A((L^*_V - 1) \times (L^*_V + m')) + 2J_V - h
\]
\[
= H^A((L^*_V - 1) \times L^*_V) + m'(2J_H - h(L^*_V - 1)) + 2J_V - h
\]
\[
> E_A'. \tag{2.21}
\]
where we have used inequality (2.4) and Assumption 2.1 a) in the last step.

• Case 2.3: \([m' = \sqrt{\mid\Lambda\mid} - L^*_V]\).
Using Assumption 2.1 d), we infer that
\[
H^A(\gamma(t - 1)) \geq H^A((L^*_V - 1) \times \sqrt{\mid\Lambda\mid})
\]
\[
= H^A((L^*_V - 1) \times L^*_V) + (\sqrt{\mid\Lambda\mid} - L^*_V)(2J_H - h(L^*_V - 1)) - 2J_H(L^*_V - 1)
\]
\[
> H^A((L^*_V - 1) \times L^*_V) = E_A' - 2J_V + h. \tag{2.22}
\]
Finally,
\[
H^A(\gamma(t)) \geq H^A(\gamma(t - 1)) + 2J_V - h > E_A', \tag{2.23}
\]
which is a contradiction.

Now assume that \(\gamma\) can consist of several connected components. Let
\[
\ell = \min\{j \in \mathbb{N} \mid \exists \sigma \subset \gamma(j) : \sigma \text{ is connected and } P_V(\sigma), P_H(\sigma) \geq L^*_V\}. \tag{2.24}
\]
There are three possible cases.

(i) Let \(\gamma\) consist of a single connected component at the steps \(\ell\) and \(\ell - 1\). Then it can be seen easily that the above proof can be carried over to this case.

(ii) Let \(\gamma\) consist of several isolated droplets \(\gamma^1, \ldots, \gamma^n\) for some \(n \geq 2\) at the steps \(\ell\) and \(\ell - 1\). Let \(\gamma^i\) be the component that reaches at time \(\ell\) the configuration, whose rectangular envelope has both horizontal and vertical length greater or equal to \(L^*_V\). One immediately observes that \(H^A(\gamma^i(k)) \leq H^A(\gamma(k))\) for all \(k \leq \ell\), since all other components contribute nonnegative energy, which follows from Assumption 2.1 and the definition of \(\ell\) by an easy computation. Hence, applying the same arguments from above to \(\gamma^i\) concludes the proof in this case.
(iii) Let $\gamma$ be such that at the steps $\ell-1$ and $\ell$, there are two connected components that touch each other at their corners. Of course, this case can only differ from the previous cases, if $\gamma(\ell)$ is obtained from $\gamma(\ell-1)$ by flipping a $(-1)$-spin at these corners. However, also here it can easily be seen from the arguments above that this contradicts $\gamma \in (\square, \square)_{\text{opt}}$, since the resulting droplet will have horizontal and vertical length greater or equal to $L^*_V$, but a large part in the rectangular envelope of the droplet are $(-1)$-spins.

This concludes the proof. \hfill \Box

As a byproduct, we obtain the following lemma that concludes the proof of $\Phi(\square, \square) - H^A(\Xi) \geq \Gamma^*_A$.

**Lemma 2.7** Let $\gamma \in (\square, \square)_{\text{opt}}$. In order to cross a configuration whose rectangular envelope has both vertical and horizontal length greater or equal to $L^*_V$, $\gamma$ has to pass through $R(L^*_V - 1, L^*_V)$ and $R(L^*_V - 1, L^*_V)^{\text{1pr}}$. In particular, each optimal path between $\square$ and $\square$ has to cross $R(L^*_V - 1, L^*_V)^{\text{1pr}}$.

**Proof.** Consider the time step $t$ defined in the proof of Lemma 2.6. It was shown there that necessarily $\gamma(t-1)$ needs to belong to $R(L^*_V - 1, L^*_V)$. Since $P_V R(\gamma(t)), P_H R(\gamma(t)) \geq L^*_V$, $\gamma(t)$ must be obtained from $\gamma(t-1)$ by flipping a $(-1)$-spin at a site that is attached to a longer side of the droplet. This implies that $\gamma(t)$ needs to belong to $R(L^*_V - 1, L^*_V)^{\text{1pr}}$. \hfill \Box

### 2.3 Identification of $\mathcal{P}^*$ and $\mathcal{C}^*$

From Section 2.1, it is clear that $R(L^*_V - 1, L^*_V) \subset \mathcal{P}^*$. Now let $\sigma \in \mathcal{P}^*$ and $x \in \Lambda$ be such that $\sigma^x \in \mathcal{C}^*$. It follows from the definition of $\mathcal{P}^*$ and $\mathcal{C}^*$ that there exists $\gamma \in (\square, \square)_{\text{opt}}$ and $\ell \in \mathbb{N}$ such that

(i) $\gamma(\ell) = \sigma$ and $\gamma(\ell+1) = \sigma^x$,
(ii) $H^A(\gamma(k)) \leq E^*_A$ for all $k \in \{0, \ldots, \ell\}$,
(iii) $\Phi(\square, \gamma(k)) \geq \Phi(\gamma(k), \square)$ for all $k \geq \ell + 1$.

By Lemma 2.7 (ii) implies that $\min(P_H R(\sigma), P_V R(\sigma^x)) \leq L^*_V - 1$, since otherwise the energy level $E^*_A$ would have been reached. There are two possible cases.

- **Case 1:** $[P_H R(\sigma^x), P_V R(\sigma^x)] \geq L^*_V$.

  Lemma 2.7 implies that we necessarily have that $\sigma \in R(L^*_V - 1, L^*_V)$ and $\sigma^x \in R(L^*_V - 1, L^*_V)^{\text{1pr}}$.

- **Case 2:** $[\min(P_H R(\sigma^x), P_V R(\sigma^x)) \leq L^*_V - 1]$.

  Also by Lemma 2.7 there must exist some $k^* \geq \ell + 2$ such that $\gamma(k^*) \in R(L^*_V - 1, L^*_V)$. But this contradicts (iii), since $\Phi(\square, \gamma(k^*)) < \Phi(\gamma(k^*), \square) = E^*_A$. Therefore, such a path $\gamma$ can not exist, which contradicts the fact $\sigma^x \in \mathcal{C}^*$.

Hence, only Case 1 can hold true. We conclude that $\mathcal{P}^* = R(L^*_V - 1, L^*_V)$ and $\mathcal{C}^* = R(L^*_V - 1, L^*_V)^{\text{1pr}}$. 

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2.4 Verification of (H1)

Obviously, \( S_{\text{stab}} = \{ \boxplus \} \), since \( \boxplus \) minimizes all three sums in (2.1). It remains to show that \( S_{\text{meta}} = \{ \boxminus \} \).

Let \( \sigma \in S \setminus \{ \boxplus, \boxminus \} \). We have to show that \( V_\sigma < \Gamma_A^* \), i.e. there exists \( \sigma' \in S \) such that \( H^A(\sigma') < H^A(\sigma) \) and \( \Phi(\sigma, \sigma') - H^A(\sigma) < \Gamma_A^* \). There are four possible cases.

- Case 1: \( \sigma \) contains a connected component, which is not a rectangle.
  
  Let \( \sigma \) be obtained from \( A \sigma, \sigma \) \( \Gamma \). Moreover, \( \Phi(\sigma, \sigma') = 0 < \Gamma_A^* \).

- Case 2: \( \sigma \) contains a connected component, which is a rectangle \( R = l_1 \times l_2 \) with \( l_2 \geq L_1^* \) and \( l_1 < \sqrt{|\Lambda|} \).
  
  Let \( \sigma' \) be obtained from \( \sigma \) by attaching on the right of \( R \) a new column of length \( l_2 \). Then:
  
  \[
  H^A(\sigma') \leq H^A(\sigma) + 2J_V - l_2h < H^A(\sigma), \quad \text{and} \quad \Phi(\sigma, \sigma') - H^A(\sigma) = 2J_V - h < \Gamma_A^*. \tag{2.25}
  \]

- Case 3: \( \sigma \) contains a connected component, which is a rectangle \( R = l_1 \times l_2 \) with \( l_2 \leq L_1^* - 1 \) and \( l_1 < \sqrt{|\Lambda|} \).
  
  Let \( \sigma' \) be obtained from \( \sigma \) by cutting the right column of \( R \). Then:
  
  \[
  H^A(\sigma') = H^A(\sigma) - 2J_V + l_2h < H^A(\sigma) - 2J_V + (L_1^* - 1)h < H^A(\sigma), \quad \text{and} \quad \Phi(\sigma, \sigma') - H^A(\sigma) = (l_2 - 1)h < \Gamma_A^*. \tag{2.26}
  \]

- Case 4: \( \sigma \) contains a connected component, which is a rectangle \( R = l_1 \times l_2 \) with \( l_1 = \sqrt{|\Lambda|} \).
  
  Let \( \sigma' \) be obtained from \( \sigma \) by attaching above \( R \) a row that also wraps around the torus. Then, by Assumption \( \text{[2.1]} \) d):
  
  \[
  H^A(\sigma') = H^A(\sigma) + 2J_H - l_1h < H^A(\sigma), \quad \text{and} \quad \Phi(\sigma, \sigma') - H^A(\sigma) = 2J_H - h < \Gamma_A^*. \tag{2.27}
  \]

We conclude that \( S_{\text{meta}} = \{ \boxminus \} \).

2.5 Verification of (H2)

Obviously, \( |\{ \sigma \in \mathcal{P}^* | \sigma \sim \sigma' \}| = 1 \) for all \( \sigma' \in \mathcal{C}^* \). Therefore, (H2) holds.

2.6 Computation of K

We start the computation of \( K^{-1} \) from the variational formula given in Theorem \( \text{[1.5]} \).

- Lower bound. Since the sum in the variational formula of Theorem \( \text{[1.5]} \) has only nonnegative summands, we can bound \( K^{-1} \) from below by
  
  \[
  \frac{1}{K} \geq \min_{C_1, \ldots, C_J \in [0,1]} \min_{h, S' \rightarrow [0,1]} \min_{h|_{S_1} = 1, h|_{S_2} = 0, h|_{S_i} = C_i, \forall i} \frac{1}{2} \sum_{h, h', \eta, \eta' \in (\mathcal{C}^*)^+} 1_{\{h, \eta \sim h', \eta' \}} |h(\eta) - h(\eta')|^2. \tag{2.28}
  \]
Obviously, it holds that $\partial^+ C^* \cap S^* = R(L_V^* - 1, L_V^*) \cup R(L_V^* - 1, L_V^*)^{2pr}$. Moreover, similar computations as in Section 2.1 show that $R(L_V^* - 1, L_V^*) \subset S_{\Theta}$ and $R(L_V^* - 1, L_V^*)^{2pr} \subset S_{\Theta}$. This leads to

$$
\frac{1}{K} \geq \min_{h : C^* \to [0,1]} \sum_{\eta \in C^*} \left( \sum_{\eta' \in R(L_V^* - 1, L_V^*), \eta' \sim \eta} (1 - h(\eta'))^2 + \sum_{\eta' \in R(L_V^* - 1, L_V^*)^{2pr}, \eta' \sim \eta} h(\eta')^2 \right)
$$

$$=
\sum_{\eta \in C^*} \min_{h \in [0,1]} \left( |R(L_V^* - 1, L_V^*) \sim \eta| (1 - h)^2 + |R(L_V^* - 1, L_V^*)^{2pr} \sim \eta| h^2 \right) \tag{2.29}
$$

For all $\eta \in C^*$ we have that $|R(L_V^* - 1, L_V^* \sim \eta| = 1$. If the protuberance in $\eta$ is attached at a corner of $(L_V^* - 1) \times L_V^*$, then $|R(L_V^* - 1, L_V^*)^{2pr} \sim \eta| = 1$, otherwise $|R(L_V^* - 1, L_V^*)^{2pr} \sim \eta| = 2$. Taking into account that there are $|\Lambda|$ possible locations for each shape of a critical droplet and 2 possible rotations, we obtain:

$$
\frac{1}{K} \geq \left( 2(L_V^* - 2)^2 \frac{2}{3} + 4 \frac{1}{2} \right) 2|\Lambda| = \frac{4(2L_V^* - 1)}{3} |\Lambda|. \tag{2.30}
$$

**Upper bound.** Let

$R_0^- = \{ \sigma \in S^* \mid \sigma \text{ is not connected and } \min(P_H R(\eta), P_V R(\eta)) \leq L_V^* - 1 \}$

for all its connected components $\eta$,

$R_0^+ = \{ \sigma \in S^* \mid \sigma \text{ is not connected and } P_H R(\eta), P_V R(\eta) \geq L_V^* \}$

for at least one of its connected components $\eta$,

$R_1^- = \{ \sigma \in S^* \mid \sigma \text{ is connected and } \min(P_H R(\sigma), P_V R(\sigma)) \leq L_V^* - 1 \}$,

$R_1^+ = \{ \sigma \in S^* \mid \sigma \text{ is connected and } P_V R(\sigma), P_H R(\sigma) \geq L_V^* \}$.

Set $R^- = R_0^- \cup R_1^-$ and $R^+ = R_0^+ \cup R_1^+$. The following lemma is very important.

**Lemma 2.8** Let $\sigma \in R^-$ and $\sigma' \in R^+$. Then $\sigma \sim \sigma'$, if and only if $\sigma \in P^{*}$ and $\sigma' \in C^{*}$.

**Proof.** We skip the details of this proof, since they walk along the same lines as the proof of Lemma 2.6. One should only notice that, if $\sigma$ is connected and $\sigma'$ is not connected or vice versa, then $|H^A(\sigma') - H^A(\sigma)| = 2J_H + 2J_V - h$. \hfill $\Box$

Finally, notice that for all $i$ either $S_i \subset R^-$ holds or $S_i \subset R^+$, since the $S_i$ are connected. For the same reason and by Section 2.1, $S_{\Theta} \subset R^-$ and $S_{\Theta} \subset R^+$ holds. Therefore,

$$
\frac{1}{K} \leq \min_{h : S^{*} \to [0,1], h|_{R^*} = 1, h|_{R^*+} = 0} \frac{1}{2} \sum_{\eta \eta' \in S^*} \mathbf{1}_{\{\eta \sim \eta'\}} (h(\eta) - h(\eta'))^2
$$

$$=
\min_{h : (C^*)^{*} \to [0,1], h|_{R^*} = 1, h|_{R^*+} = 0} \frac{1}{2} \sum_{\eta \eta' \in (C^*)^{+}} \mathbf{1}_{\{\eta \sim \eta'\}} (h(\eta) - h(\eta'))^2 \tag{2.32}
$$

$$=
\min_{h : C^* \to [0,1]} \sum_{\eta \in C^*} \left( \sum_{\eta' \in R(L_V^* - 1, L_V^*) \sim \eta} (1 - h(\eta'))^2 + \sum_{\eta' \in R(L_V^* - 1, L_V^*)^{2pr}, \eta' \sim \eta} h(\eta')^2 \right)
$$

$$= \frac{4(2L_V^* - 1)}{3} |\Lambda|. \tag{2.33}
$$
3 Ising model with next-nearest-neighbor attraction

In this chapter, let the Hamiltonian be given by

$$H_{\text{NN}}(\sigma) = \frac{\tilde{J}}{2} \sum_{(x,y) \in \Lambda^*} \sigma(x)\sigma(y) - \frac{K}{2} \sum_{(x,y) \in \Lambda^{**}} \sigma(x)\sigma(y) - \frac{h}{2} \sum_{x \in \Lambda} \sigma(x), \quad (3.1)$$

where $\sigma \in S$, $\tilde{J}, K, h > 0$, $\Lambda^*$ is the set of unordered nearest-neighbor bonds in $\Lambda$ and $\Lambda^{**}$ is the set of unordered next-nearest-neighbor bonds in $\Lambda$, i.e

$$\Lambda^{**} = \{ \{x, y\} \in \Lambda^2 \mid |x - y| = \sqrt{2} \}. \quad (3.2)$$

Using the geometric representation of $\sigma$, one can rewrite $H_{\text{NN}}(\sigma)$ as

$$H_{\text{NN}}(\sigma) = H_{\text{NN}}(\Xi) - h|\sigma| + J|\partial(\sigma)| - K|A(\sigma)|, \quad (3.3)$$

where $J = \tilde{J} + 2K$ and $|A(\sigma)|$ is the number of corners (or right angles) of $\sigma$. Note that in the following situation, we count 4 corners.

The critical length in this model will be given by

$$\ell^* = \left\lceil \frac{2K}{h} \right\rceil \quad \text{and} \quad D^* = \left\lceil \frac{2J}{h} \right\rceil \quad \text{and} \quad L^* = D^* - 2(\ell^* - 1). \quad (3.4)$$

The following assumptions will be made for this chapter.

**Assumption 3.1**

a) $2K > h$,

b) $\tilde{J} \geq 2K + h$,

c) $\frac{2J}{h} \notin \mathbb{N}$, $\frac{2K}{h} \notin \mathbb{N}$,

d) $|\Lambda| > \left( \frac{2J(D^* - 1)}{2J-h(D^*-1)} + D^* \right)^2$.

Similar as in Chapter 2, a) and b) induce a hierarchy in the sense that for the system it is most important to align nearest-neighbors-neighbors, then next-nearest-neighbors and then to align the spin values with the sign of the magnetic field. Assumption c) is made for non-degeneracy reasons. Assumption d) implies that it is not profitable to enlarge a droplet such that one side is subcritical and the other side wraps around the torus. This will become clear later in Lemma 3.6. Moreover, d) assures that the torus is large enough to contain a critical droplet. It immediately follows from Assumption 3.1 c) that

$$(\ell^* - 1)h < 2K < \ell^*h \quad \text{and} \quad (D^* - 1)h < 2J < D^*h. \quad (3.5)$$

We need a few definitions that are mostly carried over from [8].
Definition 3.2 • $A \subset \mathbb{Z}^2$ is called an oblique bar, if $A = \{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$ and it holds that either $x_i = x_{i-1} + (1, 1)^T$ for all $i \leq n$ or $x_i = x_{i-1} + (1, -1)^T$ for all $i \leq n$.
• $\sigma \in S$ is called octagon of side lengths $D_n, D_w \in \mathbb{N} \cap [1, \sqrt{N} - 1]$ and oblique edge lengths $\ell_{ne}, \ell_{nw}, \ell_{sw}, \ell_{se} \in \mathbb{N}$, if $R(\sigma) \in R(D_n, D_w)$ and if $\sigma$ is obtained from $R(\sigma)$ by cutting $\ell_{ne} - 1$ oblique bars from the upper right corner of $R(\sigma)$, $\ell_{nw} - 1$ oblique bars from the upper left corner, $\ell_{sw} - 1$ oblique bars from the down left corner, $\ell_{se} - 1$ oblique bars from the down right corner. The set of all such octagons and their rotations is denoted by $Q(D_n, D_w; \ell_{ne}, \ell_{nw}, \ell_{sw}, \ell_{se})$. We often abuse the notation by denoting configurations in this set in the same way.

• Any $Q \in Q(D_n, D_w; \ell_{ne}, \ell_{nw}, \ell_{sw}, \ell_{se})$ has 8 edges. The upper right edge of length $\ell_{ne}$ is called NE-edge, the upper left edge of length $\ell_{nw}$ NW-edge, the down left edge of length $\ell_{sw}$ SW-edge and the down right edge of length $\ell_{se}$ is called SE-edge. These four edges are also called oblique edges. The four remaining horizontal or vertical edges are called coordinate edges. We call the upper coordinate edge N-edge, the left one W-edge, the bottom one S-edge and the right coordinate edge E-edge. An example with $D_n = 15$, $D_w = 12$, $\ell_{ne} = 5$, $\ell_{nw} = 6$, $\ell_{sw} = 4$, $\ell_{se} = 3$ is given by

\[ \begin{array}{cccc}
N & NW & NE & S \\
W & & & \\
SW & & & SE \\
SW & & & SE \\
W & & & \\
\end{array} \]

• The length of the N-edge is $D_n - (\ell_{ne} - 1) - (\ell_{nw} - 1)$ and will be denoted by $L_n$. In the same way, we define $L_w, L_s$ and $L_e$ as the lengths of the W-edge, S-edge and E-edge, respectively.

• $Q(D_n, D_w; \ell_{ne}, \ell_{nw}, \ell_{sw}, \ell_{se})$ is called stable octagon, if $L_n, L_w, L_s, L_e, \ell_{ne}, \ell_{nw}, \ell_{sw}, \ell_{se} \geq 2$.

• If $\ell_{ne} = \ell_{nw} = \ell_{sw} = \ell_{se} = \ell$, we write $Q(D_n, D_w; \ell_{ne}, \ell_{nw}, \ell_{sw}, \ell_{se}) = Q(D_n, D_w; \ell)$.

• If $\ell = \ell^*$, we write $Q(D_n, D_w; \ell^*) = Q(D_n, D_w)$.

• If $L_n = L_w = L_s = L_e = \ell_{ne} = \ell_{nw} = \ell_{sw} = \ell_{se} = \ell$, we write $Q(3\ell - 2, 3\ell - 2, \ell) = Q(\ell)$.

• $Q(D_n, D_w; \ell)^{1pr}$ denotes the set of all configurations consisting only of an octagon from $Q(D_n, D_w; \ell)$ and with an additional protuberance attached at the interior of one of its longest coordinate edges. Here, the interior of a coordinate edge are all sites except of the two sites at both ends of the edge. The right droplet in Figure 3 provides an example.

• $Q(D_n, D_w; \ell)^{2pr}$ denotes the set of all configurations that are obtained from a configuration in $Q(D_n, D_w; \ell)^{1pr}$ by adding a second (+1)–spin adjacent to the protuberance at the interior of the coordinate edge.

Note that the energy of an octagon $Q \in Q(D_n, D_w; \ell_{ne}, \ell_{nw}, \ell_{sw}, \ell_{se})$ is given by

\[ H_{NN}(Q) = H_{NN}(\emptyset) - hD_nD_w + 2J(D_n + D_w) + \sum_{a \in \{ne, nw, sw, se\}} F(\ell_a), \quad (3.6) \]

where $F(\ell) = -K(2\ell - 1) + \frac{1}{2}h(\ell - 1)\ell$.

Now we can formulate the main result of this chapter.
Theorem 3.3 Under Assumption 3.1, the pair $(\bigotimes, \bigoplus)$ satisfies (H1) and (H2) so that Theorems 1.3–1.5 hold for the Ising model with next-nearest-neighbor attraction. Moreover, the quadruple $(\mathcal{P}^*, \mathcal{C}^*, \Gamma^*, K)$ is given by

- $\mathcal{P}^* = Q(D^* - 1, D^*)$,
- $\mathcal{C}^* = Q(D^* - 1, D^*)^{1\text{st}}$,
- $\Phi(\bigotimes, \bigoplus) - H^{\text{NN}}(\bigotimes) = H^{\text{NN}}(Q(D^* - 1, D^*)) + 2J - 4K - h =: \Gamma_{\text{NN}}^* =: E_{\text{NN}}^* - H^{\text{NN}}(\bigotimes)$,
- $K^{-1} = \frac{4(2L^* - 5)}{3} |\Lambda|$.

Proof. The proof is divided into the Sections 3.1–3.6. □

Figure 3: Configurations in $\mathcal{P}^*$ and $\mathcal{C}^*$ with $\ell^* = 5$ and $D^* = 14$

3.1 Proof of $\Phi(\bigotimes, \bigoplus) - H^{\text{NN}}(\bigotimes) \leq \Gamma_{\text{NN}}^*$

As in Section 2.1, we need to construct a reference path $\gamma : \bigotimes \to \bigoplus$ such that

$$\max_{\eta \in \gamma} H^{\text{NN}}(\eta) \leq H^{\text{NN}}(\bigotimes) + \Gamma_{\text{NN}}^* = E_{\text{NN}}^*.$$

Construction of $\gamma$. The construction of $\gamma$ will be similar to [8] but simplified, since we are mainly interested in the part of the path around the critical state. We quickly sketch this critical part of the path and give the exact reference for the remaining steps.

- [From $\bigotimes$ to $Q(2).$]
  See Scheme 5.1 of [8].
- [From $Q(\ell)$ to $Q(\ell + 1)$ for all $\ell = 2, \ldots, \ell^* - 1.$]
  See Scheme 5.2 of [8].
- [From $Q(D, D)$ to $Q(D + 1, D + 1)$ for all $D = \ell^*, \ldots, \sqrt{|\Lambda|} - 2.$]
  This is Scheme 5.5 of [8], and it goes as follows. A (+1)–spin is added somewhere at the interior of the E-edge of $Q(D, D)$. Afterwards, this row is filled by successively flipping in this column adjacent (-1)–spins until $Q(D, D + 1; \ell^* + 1, \ell^*, \ell^*, \ell^* + 1)$ is reached. Then a (-1)–spin is flipped at the upper end of the SE-edge. Now (-1)–spins are flipped until $Q(D, D + 1; \ell^* + 1, \ell^*, \ell^*, \ell^* + 1)$ is reached. Next, the same this is done at the NE-edge such that $Q(D, D + 1; \ell^*, \ell^*, \ell^*, \ell^*) = Q(D, D + 1)$ is reached. Then this procedure is repeated below $Q(D, D + 1)$, i.e. first (-1)–spins are flipped at the S-edge until $Q(D + 1, D + 1; \ell^*, \ell^*, \ell^*, \ell^* + 1)$ is reached, then an oblique bar is added at the SW-edge to reach $Q(D + 1, D + 1; \ell^*, \ell^*, \ell^*, \ell^* + 1)$, and finally, (-1)–spins are flipped at the SE-edge, until we arrive at $Q(D + 1, D + 1)$. 

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Lastly, flip all remaining \((-1)\)–spins outside of \(Q(\sqrt{|\Lambda|} - 1, \sqrt{|\Lambda|} - 1)\) until \(\boxplus\) is reached.

Inequality (3.7) holds. Let \(k^*\) be such that \(\gamma(k^*) \in Q(D^* - 1, D^*)\). Then \(H^{\text{NN}}(\gamma(k^*)) = E^*_\text{NN} - 2\tilde{J} + h < E^*_\text{NN}\). If we go backwards in the path from that point on, then we will have to cut the NE-edge of \(Q(D^* - 1, D^*)\). This is an increase of the energy in each step until only one \((+1)\)–spin remains on this edge. At this point the energy equals to

\[
E^*_\text{NN} - 2\tilde{J} + \ell^* h < E^*_\text{NN},
\]

(3.8)

where we have used Assumption 3.1 b) and (3.5). Cutting the last \((+1)\)–spin on this edge lowers the energy even more. Next we do the same thing on the SE-edge, i.e. we cut all but one \((+1)\)–spins on this edge and arrive at the energy

\[
E^*_\text{NN} - 2\tilde{J} - 2K + 2\ell^* h < E^*_\text{NN}.
\]

(3.9)

Cutting the last \((+1)\)–spin on this edge, we arrive at \(E^*_\text{NN} - 2J + (2\ell^* + 1)h\). Finally, we need to cut all but one \((+1)\)–spins on the E-edge, which leads to the energy level

\[
E^*_\text{NN} - 2J + (D^* - 1)h < E^*_\text{NN},
\]

(3.10)

and cutting the last protuberance, we arrive at the energy \(E^*_\text{NN} - 4J + 4K + D^* h\). With the same reasoning, if we keep on going backwards in the path of \(\gamma\), we will always stay below \(E^*_\text{NN}\), since the size of the cutted sides of the octagon will be at most \(D^* - 1\). Hence, we get that

\[
\max_{i=1,...,k^*} H^{\text{NN}}(\gamma(i)) < E^*_\text{NN}.
\]

(3.11)

Let us look at the remaining path of \(\gamma\) after the step \(k^* + 2\). It holds that \(H^{\text{NN}}(\gamma(k^* + 2)) = E^*_\text{NN} - h < E^*_\text{NN}\). First, \(L^* - 4\) \((+1)\)–spins are attached at the interior of the S-edge. The energy is decreased to \(E^*_\text{NN} - (L^* - 3)h\). Afterwards, a \((+1)\)–spin is added at the SW-edge, which decreases the energy to

\[
E^*_\text{NN} + 2K - (L^* - 2)h < E^*_\text{NN},
\]

(3.12)

where we have used the inequality \(L^* \geq 2\ell^* + 1\), which follows immediately from Assumption 3.1 b). Filling the SW-edge decreases the energy by \((\ell^* - 1)h\). Then we do the same things for the SE-edge by attaching first a \((+1)\)–spin on this edge, which increases the energy to

\[
E^*_\text{NN} + 4K - (L^* + \ell^* - 2)h < E^*_\text{NN},
\]

(3.13)

and then filling up this edge, which decreases the energy to \(E^*_\text{NN} + 4K - (D^* - 1)h\). Next, a protuberance is added at the interior of the E-edge. We arrive at the energy level

\[
E^*_\text{NN} + 2J - D^* h < E^*_\text{NN}.
\]

(3.14)

This configuration is now “over the hill”, since, if we keep on following the path of \(\gamma\), we will always stay below \(E^*_\text{NN}\), since the size of the added sides to the octagon will be at least \(D^*\). Together with (3.11), we infer (3.7).
3.2 Proof of $\Phi(\square, \boxplus) - H^{\text{NN}}(\square) \geq \Gamma^{*}_{\text{NN}}$

We first list a few observations taken from [8].

Lemma 3.4 Let $\sigma \in S$ be a local minimum of the energy $H^{\text{NN}}$, i.e. $H^{\text{NN}}(\sigma^x) > H^{\text{NN}}(\sigma)$ for all $x \in \Lambda$. Then $\sigma$ is either a union of isolated and stable octagons or $\sigma$ is a rectangle that wraps around the torus.

Proof. The first fact was proven in Lemma 2.1 of [8]. Now assume that $\sigma$ wraps around the torus. Then a straightforward adaptation of the proof of Lemma 2.4 yields the claim. □

Lemma 3.5 Assume that $\sigma \in S$ consists of only one connected component that does not wrap around the torus. Let $R(\sigma) \in R(D_n, D_w)$ with $D_n \geq D_w$.

- If $D_w \geq 2\ell^* - 1$, then
  \[ H^{\text{NN}}(\sigma) \geq H^{\text{NN}}(Q(D_n, D_w)), \]  
  and equality holds, if and only if $\sigma = Q(D_n, D_w)$.
- If $D_w < 2\ell^* - 1$ and $D_w$ is odd, then
  \[ H^{\text{NN}}(\sigma) \geq H^{\text{NN}}(Q(D_n, D_w; \frac{1}{2}(D_w + 1))), \]  
  and equality holds, if and only if $\sigma = Q(D_n, D_w; \frac{1}{2}(D_w + 1))$.
- If $D_w < 2\ell^* - 1$ and $D_w$ is even, then
  \[ H^{\text{NN}}(\sigma) \geq H^{\text{NN}}(Q(D_n, D_w; \frac{1}{2}D_w, \frac{1}{2}D_w, \frac{1}{2}D_w + 1, \frac{1}{2}D_w + 1)), \]  
  and equality holds, if and only if $\sigma = Q(D_n, D_w; \frac{1}{2}D_w, \frac{1}{2}D_w, \frac{1}{2}D_w + 1, \frac{1}{2}D_w + 1)$.

Proof. See Lemma 3.2 and the proof of Lemma 4.1A in [8]. The main step is to notice that the function $t \mapsto F(t)$ is minimized in $\ell^*$. □

In the following lemma we show that every optimal path has to cross $Q(D^* - 1, D^*)$.

Lemma 3.6 Let $\gamma \in (\square, \boxplus)_{\text{opt}}$. Then $\gamma$ has to cross $Q(D^* - 1, D^*)$.

Proof. Assume the contrary, i.e. $\gamma \cap Q(D^* - 1, D^*) = \emptyset$. Using the same arguments as in the end of the proof of Lemma 2.6, we can restrict ourselves to the case that throughout its whole path $\gamma$ consists only of a single connected component. On its way to $\boxplus$, $\gamma$ has to cross a configuration, whose rectangular envelope has both horizontal and vertical length greater or equal to $D^*$. Let

\[ t = \min\{t \geq 0 \mid P_H R(\gamma(t)), P_V R(\gamma(t)) \geq D^*\}. \]  

(3.18)

Notice that from the definition of $t$, it is clear that $R(\gamma(t - 1)) \in R(D^* + m, D^* - 1)$ for some $m \geq 0$.

- Case 1: $|m = 0|$.
  Obviously, $D^* \geq 2\ell^* - 1$. Hence, by Lemma 3.5 and since $\gamma$ does not cross $Q(D^* - 1, D^*)$, we have that
  \[ H^{\text{NN}}(\gamma(t - 1)) > H^{\text{NN}}(Q(D^* - 1, D^*)) = E^{*}_{\text{NN}} - 2\bar{t} + h. \]  

(3.19)
The minimal increase of energy to enlarge the rectangular envelope of a configuration is $2\tilde{J} - h$. Hence,

$$H_{NN}(\gamma(t)) \geq H_{NN}(\gamma(t-1)) + 2\tilde{J} - h > E_{NN}^*.$$  \hspace{1cm} (3.20)

This contradicts $\gamma \in (\square, \square)_{\text{opt}}$, since we already know from Section 3.1 that $\Phi(\square, \square) \leq E_{NN}^*$.

**Case 2: \([m \in [1, \sqrt{|A|} - D^*]]\).**

Again, by Corollary 3.5 we have that

$$H_{NN}(\gamma(t-1)) \geq H_{NN}(Q(D^* + m, D^* - 1))$$

$$= H_{NN}(Q(D^*, D^* - 1)) + m(2J - h(D^* - 1))$$

$$> E_{NN}^* - 2\tilde{J} + h.$$  \hspace{1cm} (3.21)

As before, this leads to a contradiction, since

$$H_{NN}(\gamma(t)) \geq H_{NN}(\gamma(t-1)) + 2\tilde{J} - h > E_{NN}^*.$$  \hspace{1cm} (3.22)

**Case 3: \([m = \sqrt{|A|} - D^*]\).**

In this case, $\gamma(t-1)$ wraps around the torus. One can easily observe that $H_{NN}(\gamma(t-1)) \geq H_{NN}(R(\sqrt{|A|}, D^* - 1))$. We infer that

$$H_{NN}(\gamma(t-1)) \geq H_{NN}(R(\sqrt{|A|}, D^* - 1))$$

$$= H_{NN}(Q(D^*, D^* - 1)) + (\sqrt{|A|} - D^*)(2J - h(D^* - 1)) - 2J(D^* - 1) - 4F(\ell^*)$$

$$> H_{NN}(Q(D^*, D^* - 1)) = E_{NN}^* - 2\tilde{J} + h,$$  \hspace{1cm} (3.23)

where we have used $F(\ell^*) < 0$ and Assumption 3.1 d). Finally,

$$H_{NN}(\gamma(t)) \geq H_{NN}(\gamma(t-1)) + 2\tilde{J} - h > E_{NN}^*.$$  \hspace{1cm} (3.24)

This concludes the proof. \hfill \square

As a byproduct, we obtain the following lemma that concludes the proof of $\Phi(\square, \square) - H_{NN}(\square) \geq \Gamma_{NN}$.

**Lemma 3.7** Let $\gamma \in (\square, \square)_{\text{opt}}$. In order to cross a configuration whose rectangular envelope has both vertical and horizontal length greater or equal to $D^*$, $\gamma$ has to pass through $Q(D^* - 1, D^*)$ and $Q(D^* - 1, D^*)^\text{1pr}$. In particular, each optimal path between $\square$ and $\square$ has to cross $Q(D^* - 1, D^*)^\text{1pr}$.

**Proof.** Consider the time step $t$ defined in the proof of Lemma 3.6. It was shown there that necessarily $\gamma(t-1)$ needs to belong to $Q(D^* - 1, D^*)$. Since $P_V R(\gamma(t)), P_H R(\gamma(t)) \geq D^*$, $\gamma(t)$ must be obtained from $\gamma(t-1)$ by flipping a $(-1)$–spin at a site that is attached at the coordinate edge of a longer side of the droplet. If it would not attach at the interior of the coordinate edge, then the energy level $E_{NN}^* + 2K$ would be reached. Hence, the protuberance must be added at the interior of the coordinate edge, which implies that $\gamma(t)$ needs to belong to $Q(D^* - 1, D^*)^\text{1pr}$. \hfill \square
Lemma 3.7 implies that necessarily \( \sigma \cdot H \cdot \sigma \). Hence, only Case 1 can hold true. We conclude that 

\[
\sigma \cdot S \cdot H
\]

Obviously, \( S_{\text{stab}} = \{ \square \} \), since \( \square \) minimizes all three sums in (3.1). It remains to show that \( S_{\text{meta}} = \{ \Box \} \).

Let \( \sigma \in S \setminus \{ \square, \Box \} \). As in Section 2.4, we have to show that there exists \( \sigma' \in S \) such that 

\[
H^{\text{NN}}(\sigma') < H^{\text{NN}}(\sigma) \quad \text{and} \quad \Phi(\sigma, \sigma') - H^{\text{NN}}(\sigma) < 0 < \Gamma_{\text{NN}}^*.
\]

- **Case 1:** \( [\sigma \text{ contains a connected component, which is not a stable octagon and not a rectangle that wraps around the torus}]

  Lemma 3.4 implies that \( \sigma \) is not a local minimum, i.e., there exists \( x \in \Lambda \) such that 

  \[
  H^{\text{NN}}(\sigma^x) < H^{\text{NN}}(\sigma) \quad \text{and} \quad \Phi(\sigma, \sigma^x) - H^{\text{NN}}(\sigma) = 0 < \Gamma_{\text{NN}}^*.
  \]

- **Case 2:** \( [\sigma \text{ contains a connected component } Q, \text{ which is a stable octagon with } P_V R(Q) \geq D^*] \)

  Let \( \sigma' \) be obtained from \( \sigma \) by attaching at \( Q \) an oblique bar at its NE-edge and its SE-edge respectively, and a vertical bar at its E-edge in the same way that was described in the third step of the construction of \( \gamma \) given in Section 3.1. Then we obtain:

  \[
  H^{\text{NN}}(\sigma') - H^{\text{NN}}(\sigma) \leq 2J - P_V R(Q) h < 2J - D^* h < 0, \quad \text{and} \quad \Phi(\sigma, \sigma') - H^{\text{NN}}(\sigma) \leq 2J - h < \Gamma_{\text{NN}}^*. \quad (3.25)
  \]

- **Case 3:** \( [\sigma \text{ contains a connected component } Q, \text{ which is a stable octagon with } P_V R(Q) \leq D^* - 1] \)

  Let \( \sigma' \) be obtained from \( \sigma \) by detaching the NE-edge, the SE-edge and the E-edge of \( Q \) respectively, similar as in the construction of \( \gamma \) given in Section 3.1, but in the reverse way. Then:

  \[
  H^{\text{NN}}(\sigma') - H^{\text{NN}}(\sigma) = -2J + P_V R(Q) h \leq -2J + (D^* - 1) h < 0, \quad \text{and} \quad \Phi(\sigma, \sigma') - H^{\text{NN}}(\sigma) \leq (P_V R(Q) - 1) h < \Gamma_{\text{NN}}^*. \quad (3.26)
  \]
Case 4: $|\sigma|$ contains a connected component $R$ that is a rectangle that wraps around the torus. Let $\sigma'$ be obtained from $\sigma$ by attaching at $R$ a bar that also wraps around the torus. Then, by Assumption 2.1 d):

\[
H^{NN}(\sigma') - H^{NN}(\sigma) = 2\tilde{J} - \sqrt{|\Lambda|}h < 0, \quad \text{and}
\]

\[
\Phi(\sigma, \sigma') - H^{NN}(\sigma) = 2\tilde{J} - h < \Gamma^{*}_{NN}.
\]  

(3.27)

We conclude that $S_{\text{meta}} = \{\square\}$.

3.5 Verification of (H2)

Obviously, $|\{\sigma \in P^* | \sigma \sim \sigma'\}| = 1$ for all $\sigma' \in C^*$. Therefore, (H2) holds.

3.6 Computation of $K$

The computation of $K^{-1}$ here will be done analogously to Section 2.6.

- Lower bound. Note that $\partial^+ C^* \cap S^* = Q(D^* - 1, D^*) \cup Q(D^* - 1, D^*)^{2pr}, Q(D^* - 1, D^*) \subset S_\Xi$ and $Q(D^* - 1, D^*)^{2pr} \subset S_\Xi$. Hence,

\[
\frac{1}{K} \geq \min_{c_1, \ldots, c_{\ell} \in [0,1]} \min_{h, S_i \in [0,1], \eta|_{S_i} = 0, \eta|_{S_i} = C_i \forall i} \sum_{\eta, \eta' \in \{C^*\}} \left\{ \sum_{\eta' \in Q(D^* - 1, D^*)^{2pr}, \eta' \sim \eta} \left[ 1 - h(\eta) \right]^2 + \sum_{\eta' \in Q(D^* - 1, D^*)^{2pr}, \eta' \sim \eta} h(\eta') \right\}
\]

\[
= \min_{h, \eta \in C^*} \left( \sum_{\eta' \in Q(D^* - 1, D^*)} \left[ 1 - h(\eta) \right]^2 + \sum_{\eta' \in Q(D^* - 1, D^*)^{2pr}, \eta' \sim \eta} h(\eta') \right)
\]

\[
= \sum_{\eta \in C^*} \left| Q(D^* - 1, D^*) \sim \eta \right| \left| Q(D^* - 1, D^*)^{2pr} \sim \eta \right| \left| Q(D^* - 1, D^*)^{2pr} \sim \eta \right|.
\]

For all $\eta \in C^*$ we have that $|Q(D^* - 1, D^*) \sim \eta| = 1$. Notice that there are 4 possible seats at a longer coordinate edge of a critical droplet such that $|Q(D^* - 1, D^*)^{2pr} \sim \eta| = 1$, and $2(L^* - 4)$ possible seats such that $|Q(D^* - 1, D^*)^{2pr} \sim \eta| = 2$. Moreover, we observe that there are $|\Lambda|$ possible locations for a configuration in $C^*$, and that there are two analogue rotations for each critical droplet. Therefore, we obtain

\[
\frac{1}{K} \geq \left( 2(L^* - 4) \frac{2}{3} + 4 \frac{1}{2} \right) 2|\Lambda| = \frac{4(2L^* - 5)}{3}|\Lambda|.
\]

(3.29)

- Upper bound. Define

\[
R_0^- = \{ \sigma \in S^* | \sigma \text{ is not connected and } \min(P_H(\eta), P_V(\eta)) \leq D^* - 1 \text{ for all its connected components } \eta \},
\]

\[
R_0^+ = \{ \sigma \in S^* | \sigma \text{ is not connected and } P_H(\eta), P_V(\eta) \geq D^* \text{ for at least one of its connected components } \eta \},
\]

\[
R_1^- = \{ \sigma \in S^* | \sigma \text{ is connected and } \min(P_H(\sigma), P_V(\sigma)) \leq D^* - 1 \},
\]

\[
R_1^+ = \{ \sigma \in S^* | \sigma \text{ is connected and } P_V(\sigma), P_H(\sigma) \geq D^* \}.
\]

Set $R^- = R_0^- \cup R_1^-$ and $R^+ = R_0^+ \cup R_1^+$. The following lemma is very important.
Finally, notice that for all \(i\) one can rewrite \(\Lambda\) instead of \(\ell\) in definition \(\Lambda\). One should only notice that, if \(\sigma\) is connected and \(\sigma'\) is not connected or vice versa, then \(|H_{\text{NN}}(\sigma') - H_{\text{NN}}(\sigma)| \geq 4\bar{J} + 2K - h\).

\[|H_{\text{NN}}(\sigma') - H_{\text{NN}}(\sigma)| \geq 4\bar{J} + 2K - h. \]

Finally, notice that for all \(i\) either \(S_i \subset R^-\) holds or \(S_i \subset R^+\), since the \(S_i\) are connected. For the same reason and by Section \([3.1]\), \(S_{\emptyset} \subset R^-\) and \(S_{\emptyset} \subset R^+\) holds. Therefore,

\[
\frac{1}{K} \leq \min_{h: S \to [0,1], h|_{|R^- = 1, h|_{|R^+ \cap C*,=0}} = 0} \frac{1}{2} \sum_{\eta, \eta'} 1_{[\eta \sim \eta']} [h(\eta) - h(\eta')]^2
\]

\[
= \min_{h: (C^*)^1 \to [0,1], h|_{|R^- = 1, h|_{|R^+ \cap C*,=0}} = 0} \frac{1}{2} \sum_{\eta, \eta'} 1_{[\eta \sim \eta']} [h(\eta) - h(\eta')]^2
\]

\[
= \min_{h: C^* \to [0,1], \eta \in C^*} \left( \sum_{\eta' \in Q(D^* - 1, D^*)^1, \eta' \sim \eta} [1 - h(\eta')]^2 + \sum_{\eta' \in Q(D^* - 1, D^*)^2, \eta' \sim \eta} h(\eta')^2 \right)
\]

\[
= \frac{4(2L^* - 5)}{3} |\Lambda|.
\]

4 Ising model with alternating magnetic field

We adapt the same strategy as in the Chapters 2 and 3 to a third modification of the Ising model. Here, the Hamiltonian is given by

\[
H^\pm(\sigma) = -\frac{J}{2} \sum_{(x,y) \in \Lambda^*} \sigma(x)\sigma(y) + \frac{h_{\text{odd}}}{2} \sum_{x \in \Lambda_{\text{odd}}} \sigma(x) - \frac{h_{\text{even}}}{2} \sum_{x \in \Lambda_{\text{even}}} \sigma(x),
\]

where \(\sigma \in S\), \(J, h_{\text{odd}}, h_{\text{even}} > 0\), \(\Lambda_{\text{odd}} = \{(x_1, x_2) \in \Lambda \mid x_2 \text{ is odd}\}\) are the odd rows in \(\Lambda\), \(\Lambda_{\text{even}} = \Lambda \setminus \Lambda_{\text{odd}}\) are the even rows and \(\Lambda^*\) is the set of unordered nearest-neighbor bonds in \(\Lambda\). One can rewrite \(H^\pm(\sigma)\) geometrically as

\[
H^\pm(\sigma) = H^\pm(\emptyset) + h_{\text{odd}}|\sigma \cap \Lambda_{\text{odd}}| - h_{\text{even}}|\sigma \cap \Lambda_{\text{even}}| + J|\partial(\sigma)|.
\]

Under the assumptions below, the critical lengths in this model will be given by

\[
l_b^* = \left\lfloor \frac{\mu}{\varepsilon} \right\rfloor \quad \text{and} \quad l_h^* = 2l_b^* - 1, \]

where

\[
\varepsilon = h_{\text{even}} - h_{\text{odd}}, \quad \text{and} \quad \mu = 2J - h_{\text{odd}}.
\]

\(l_b^*\) will be the length of the basis of the critical droplet, and \(l_h^*\) will be its height. The following assumptions will be made for this chapter.

**Assumption 4.1**

a) \(h_{\text{even}} > h_{\text{odd}}\).

b) \(J > h_{\text{even}}\).
c) $\mu \notin \mathbb{N}$,

d) $|\Lambda| > \left(2 \left[ \frac{2J(l_b^* - 1) + h_{\text{odd}}}{4J - \varepsilon(l_b^* - 1)} \right] + l_b^* \right)^2$.

Assumption a) ensures that $\boxplus$ is the stable state in this system. Assumptions b), c) and d) are made due to similar reasons as in the Chapters 2 and 3. Assumption b) can also be modified in various ways. E.g. one can take $J < h_{\text{even}} < 2J$. We refer to [9], page 10, where several other regimes are listed. In contrast to [9], in this text, we only consider the regime given in Assumption 4.1 since all other regimes can be handled in a similar way without using new ideas. It immediately follows from Assumption 4.1 c) that

$$(l_b^* - 1)\varepsilon < \mu < l_b^*\varepsilon. \quad (4.5)$$

The protocritical and the critical set in this model are given through the following sets. Figure 4 below provides an example.

**Definition 4.2**

- $P_1 = \text{the set of all configurations consisting only of a rectangle from } R((l_b^* - 1) \times l_b^*) \text{ that start and end in } \Lambda_{\text{even}} \text{ (i.e. the bottom and the top row belong to } \Lambda_{\text{even}} \text{) and with an additional protuberance attached at one of its longer sides on a row in } \Lambda_{\text{even}}$,}
- $C_1 = \text{the set of all configurations that are built from a configuration in } P_1 \text{ by adding a second } (+1)\text{-spin in } \Lambda_{\text{odd}} \text{ adjacent to the protuberance}$,
- $P_2 = \text{the set of all configurations consisting only of a rectangle from } R(l_b^* \times (l_b^* - 2)) \text{ that start and end in } \Lambda_{\text{even}} \text{ and with an additional vertical or horizontal bar of length 2 attached above or below the droplet}$,
- $C_2 = \text{the set of all configurations that are built from a configuration in } P_2 \text{ by adding a third } (+1)\text{-spin above or next to the bar of length 2 in } P_2, \text{ i.e. above or below } R(l_b^* \times (l_b^* - 2)) \text{ there is a vertical bar of length 2 with a } (+1)\text{-spin next to it in an odd row}$.

We now state the main result of this chapter.

**Theorem 4.3** Under Assumption 4.1, the pair $(\Box, \bigotimes)$ satisfies (H1) so that Theorem 1.3 a), Theorem 1.4 and Theorem 1.5 hold for the Ising model with alternating magnetic field.

Moreover, the quadruple $(P^*, C^*, \Gamma^*, K)$ is given by

- $P^* = P_1 \cup P_2$,
- $C^* = C_1 \cup C_2$,
- $\Phi(\Box, \bigotimes) - H^\pm(\Box) = 4J l_b^* + \mu(l_b^* - 1) - \varepsilon(l_b^*(l_b^* - 1) + 1) =: \Gamma^*_\pm =: E^*_\pm - H^\pm(\Box)$,
- $K^{-1} = \frac{14(l_b^* - 1)}{3} |\Lambda|$.

**Proof.** The proof is divided into the Sections 4.1–4.5. \qed
4.1 Proof of $\Phi(\Box, \boxplus) - H^\pm(\Box) \leq \Gamma_\pm^*$

As in Chapters 2 and 3, we construct a reference path $\gamma : \Box \rightarrow \boxplus$ such that

$$\max_{\eta \in \gamma} H^\pm(\eta) \leq H^\pm(\Box) + \Gamma_\pm^* = E_\pm^*.$$ \hspace{1cm} (4.6)

Construction of $\gamma$. $\gamma$ is given through the following scheme.

- Let $\gamma(0) = \Box$.
- In the first step an arbitrary spin in $\Lambda_{\text{even}}$ is flipped from $(-1)$ to $(+1)$.
- From $R(l \times (2l - 1))$ to $R((l + 1) \times (2(l + 1) - 1))$ for $l \leq l_b^* - 1$.

A protuberance is added on the right of the droplet at a row that belongs to $\Lambda_{\text{even}}$. Then successively adjacent $(-1)$–spins are flipped until the droplet belongs to $R((l + 1) \times (2l - 1))$. Now a protuberance is added in the row above the droplet, which is an odd row. Afterwards, a second $(+1)$–spin is added above the protuberance on the even row. Then one adds successively vertical bars of length 2 above the droplet, until $R((l + 1) \times (2(l + 1) - 1))$ is reached.

- From $R(l \times l_b^*)$ to $R((l + 1) \times l_b^*)$ for $l \geq l_b^*$.

A protuberance is added on the right of the droplet at a row that belongs to $\Lambda_{\text{even}}$, and successively adjacent $(-1)$–spins are flipped until the droplet belongs to $R((l + 1) \times l_b^*)$. This procedure is repeated until the droplet wraps around the torus.

- From $R(\sqrt{\vert \Lambda \vert} \times l_b^*)$ to $\boxplus$.

A protuberance is added in the odd row above the droplet. Afterwards, a second $(+1)$–spin is added above the protuberance on the even row. Then one adds successively vertical bars of length 2 above the droplet, until a configuration in $R(\sqrt{\vert \Lambda \vert} \times (l_b^* + 2))$ is reached. This is repeated until $\boxplus$ is reached.

Inequality (4.6) holds. Let $k^*$ be such that $\gamma(k^*) \in R(l_b^* \times (l_b^* - 2))$. Then $H^\pm(\gamma(k^*)) = E_\pm^* - 4J + \varepsilon - h_{\text{odd}} < E_\pm^*$. If we go backwards in the path from that point on, then we will have to cut the right bar of the droplet. Doing that, the highest energy level is reached when only two adjacent $(+1)$–spins remain, one in $\Lambda_{\text{even}}$ and one in $\Lambda_{\text{odd}}$. Indeed, at that point the energy equals

$$E_\pm^* - 4J + \varepsilon(l_b^* - 1) < E_\pm^*.$$ \hspace{1cm} (4.7)

and cutting the last $(+1)$–spins, we reach at $R((l_b^* - 1) \times (l_b^* - 2))$ and the energy decreases to $E_\pm^* - 6J + \varepsilon l_b^*$. Next, we have to cut the above two rows by successively cutting vertical bars of length 2. Doing that, the highest energy point is the stage, where only one vertical bar of
length 2 and a single $(+1)$–spin in $\Lambda_{\text{odd}}$ next to it have remained. At this point the energy equals to

\[ E^*_\pm - 6J + 2\varepsilon(l^*_b - 1) + h_{\text{even}} < E^*_\pm. \]  

(4.8)

With the same reasoning, if we keep on going backwards in the path of $\gamma$, we will always stay below $E^*_\pm$, since the sizes of the cutted columns and rows further decreases. Hence, we get that

\[ \max_{i=1,\ldots,k^*} H^\pm(\gamma(i)) < E^*_\pm. \]  

(4.9)

Let us look at the path of $\gamma$ after the step $k^* + 2$. First, one has to fill the two rows above the droplet. This lowers the energy to $E^*_\pm - \varepsilon(l^*_b - 1) - h_{\text{odd}}$. Afterwards, a protuberance is attached on the right of the droplet in a row that belongs to $\Lambda_{\text{even}}$. The energy increases to

\[ E^*_\pm + \mu - \varepsilon l^*_b - h_{\text{odd}} < E^*_\pm. \]  

(4.10)

Adding a second $(+1)$–spin adjacent to the protuberance further increases the energy by $h_{\text{odd}}$. But by (4.5), we still get

\[ E^*_\pm + \mu - \varepsilon l^*_b < E^*_\pm. \]  

(4.11)

If we fill this column, we further decrease the energy so that the energy still remains below $E^*_\pm$. In the following, in the same way, columns are added successively on the right of the droplet and each column decreases the energy by $\mu - \varepsilon l^*_b$. This is repeated until the droplet wraps around the torus. It is easy to see that the remaining part of $\gamma$ stays below $E^*_\pm$. Together with (4.9), we conclude (4.6).

### 4.2 Proof of $\Phi(\Box, \Box) - H^\pm(\Box) \geq \Gamma^*_\pm$

We first list two useful facts that were already proven in [9]. The first one characterizes all $h_{\text{odd}}$–stable configurations, i.e. all configurations $\sigma \in S$ such that there exists $\sigma' \in S$ with

- $H^\pm(\sigma') < H^\pm(\sigma)$, and
- $\Phi(\sigma', \sigma) - H^\pm(\sigma) \leq h_{\text{odd}}$.

**Lemma 4.4** $\sigma \in S$ is $h_{\text{odd}}$–stable, if and only if $\sigma$ is a union of isolated rectangles of the form $R(l_1 \times l_2)$ that start and end in $\Lambda_{\text{even}}$ and where $l_1 \geq 2$, $l_2 \geq 3$, $l_2$ is odd and the rectangle can possibly wrap around the torus. Such rectangles are called stable rectangles.

The second fact is the analogue of Corollary 2.5 for this model.

**Lemma 4.5** Let $\sigma \in S$ be such that $R(\sigma)$ is a stable rectangle. Then

\[ H^\pm(\sigma) \geq H^\pm(R(\sigma)), \]  

(4.12)

and equality holds, if and only if $\sigma = R(\sigma)$.

---

2The first one is Proposition 3.1 in [9] and the comment after it, and the second one is Lemma 3.3 from [9].
For a rectangle \( R \in R(l_1 \times l_2) \), we write \( R_{\text{odd}} \), if it starts in \( \Lambda_{\text{odd}} \), i.e. its bottom row belongs to \( \Lambda_{\text{odd}} \), otherwise we still write \( R \). Note that \( H^\pm(R_{\text{odd}}) \geq H^\pm(R) \). We will use this fact tacitly several times to prove the following lemma.

**Lemma 4.6** Let \( \gamma \in (\boxdot, \boxtimes)_{\text{opt}} \). Then \( \gamma \) has to cross \( P_1 \cup P_2 \).

**Proof.** Assume the contrary, i.e. \( \gamma \cap \{P_1 \cup P_2\} = \emptyset \). By the same arguments as in the end of the proof of Lemma 2.6, we can assume without restriction that throughout its whole path \( \gamma \) has only one connected component. Since \( \gamma \) leads to \( \boxtimes \), there exists some time \( t \) such that \( P_V R(\gamma(j)) \geq l_h^* \) and \( P_H R(\gamma(j)) \geq l_b^* \) for all \( j \geq t \), i.e.

\[
 t - 1 = \max \{ j \geq 0 \mid P_V R(\gamma(j)) < l_h^* \text{ or } P_H R(\gamma(j)) < l_b^* \}. 
\] (4.13)

Note that \( \gamma(t-1) \) has to satisfy either

1.) \( P_H R(\gamma(t-1)) = l_b^* - 1 \) and \( P_V R(\gamma(t-1)) = l_h^* + n \) for some \( n \geq 0 \), or

2.) \( P_V R(\gamma(t-1)) = l_h^* - 1 \) and \( P_H R(\gamma(t-1)) = l_b^* + m \) for some \( m \geq 0 \).

- **Case 1:** \( [P_H R(\gamma(t-1)) = l_b^* - 1 \) and \( P_V R(\gamma(t-1)) = l_h^* + n \) for some \( n \geq 0 \)].

  - **Case 1.1:** \( [n = 0] \).

Let \( \tau \) be the first time that a second \((+1)\)-spin is added outside of \( R(\gamma(t-1)) = R((l_b^* - 1) \times l_h^*) \), i.e.

\[
 \tau = \min \{ j \geq t + 1 \mid |\gamma(j) \setminus R(\gamma(t-1))| = 2 \} 
\] (4.14)

Note that \( |\gamma(\tau - 1) \setminus R(\gamma(t-1))| = 1 \) and that this protuberance is either on the right or on the left of \( R(\gamma(t-1)) \), since \( \gamma(t-1) \) was the last configuration with the property \( P_H R(\gamma(t-1)) = l_b^* - 1 \). Analogously, \( P_V R(\gamma(t-1)) = l_h^* \), otherwise, this would also contradict the definition of \( t - 1 \). Now if \( \gamma(\tau - 1) \setminus R(\gamma(t-1)) \in \Lambda_{\text{odd}} \), we have that

\[
 H^\pm(\gamma(\tau - 1)) \geq H^\pm((l_b^* - 1) \times l_h^*) + 2J + h_{\text{odd}} = E_{\pm}^* + h_{\text{even}} > E_+^*. 
\] (4.15)

This contradicts \( \gamma \in (\boxdot, \boxtimes)_{\text{opt}} \), since we already know from Section 4.1 that \( \Phi(\boxdot, \boxtimes) \leq E_+^* \). But if \( \gamma(\tau - 1) \setminus R(\gamma(t-1)) \in \Lambda_{\text{even}} \), then, since \( \gamma \) does not cross \( P_1 \) and since the minimal increase of energy to enlarge the rectangular envelope is \( 2J - h_{\text{even}} \), we have by Lemma 4.5 that

\[
 H^\pm(\gamma(\tau - 1)) > H^\pm((l_b^* - 1) \times l_h^*) + 2J - h_{\text{even}} = E_{\pm}^* - h_{\text{odd}}. 
\] (4.16)

\( \gamma(\tau) \) is obtained from \( \gamma(\tau - 1) \) by flipping a \((-1)\)-spin outside of \( R(\gamma(t-1)) \). One can easily see that the most profitable way is to flip a \((-1)\)-spin at a site that is adjacent to the protuberance of \( \gamma(\tau - 1) \), which consequently must belong to \( \Lambda_{\text{odd}} \). Hence,

\[
 H^\pm(\gamma(\tau)) > H^\pm(\gamma(k - 1)) + h_{\text{odd}} > E_+^*, 
\] (4.17)

which leads to a contradiction.\[3\]

\[3\]Two droplets that touch each other at a corner are not connected.
• Case 1.2: \( n = 2k \) for some \( k > 1 \).

According to Lemma 4.5, we have that
\[
H^\pm(\gamma(t)) \geq H^\pm(\gamma(t-1)) + 2J - h_{\text{even}} \geq H^\pm((l^*_h - 1) \times (l^*_h + 2k)) + 2J - h_{\text{even}}
\]
\[
= H^\pm((l^*_h - 1) \times l^*_h) + k(4J - \varepsilon(l^*_h - 1)) + 2J - h_{\text{even}}
\]
\[
> E^*_\pm.
\]

As before, this leads to a contradiction.

• Case 1.3: \( n = 2k + 1 \) for some \( k \geq 0 \).

It holds that either the top or bottom row of \( \gamma(t) \) must belong to \( \Lambda_{\text{odd}} \). Similar to Case 1.2, we obtain a contradiction, since
\[
H^\pm(\gamma(t)) \geq H^\pm(\gamma(t-1)) + 2J - h_{\text{even}} \geq H^\pm((l^*_h - 1) \times (l^*_h + 2k)) + 4J + h_{\text{odd}} - h_{\text{even}}
\]
\[
\geq H^\pm((l^*_h - 1) \times l^*_h) + 4J + h_{\text{odd}} - h_{\text{even}}
\]
\[
> E^*_\pm.
\]

• Case 1.4: \( \left| P_H R(\gamma(t-1)) \right| = \sqrt{|\Lambda|} \).

Using Assumption 4.1(d), we observe that
\[
H^\pm(\gamma(t-1)) \geq H^\pm((l^*_h - 1) \times \sqrt{|\Lambda|})
\]
\[
\geq H^\pm((l^*_h - 1) \times l^*_h) + [\sqrt{|\Lambda|} - l^*_h]/2)(4J - \varepsilon(l^*_h - 1)) - 2J(l^*_h - 1)
\]
\[
> H^\pm((l^*_h - 1) \times l^*_h) + h_{\text{odd}}.
\]

This leads to a contradiction, since
\[
H^\pm(\gamma(t)) \geq H^\pm(\gamma(t-1)) + 2J - h_{\text{even}}
\]
\[
> H^\pm((l^*_h - 1) \times l^*_h) + h_{\text{odd}} + 2J - h_{\text{even}} = E^*_\pm.
\]

• Case 2: \( \left| P_H R(\gamma(t-1)) \right| = l^*_h - 1 \) and \( P_H R(\gamma(t-1)) = l^*_h + m \) for some \( m \geq 0 \).

Assume first that \( \gamma(t) \) starts in \( \Lambda_{\text{odd}} \). Then by cutting the top and the bottom row of \( \gamma(t) \), which belong to \( \Lambda_{\text{odd}} \), we easily obtain a contradiction, since
\[
H^\pm(\gamma(t)) \geq H^\pm((l^*_h + m) \times (l^*_h - 2)) + 4J + 2h_{\text{odd}}
\]
\[
= H^\pm(l^*_h \times (l^*_h - 2)) + 2m(\mu - \varepsilon(L^*_h - 1)) + 4J + 2h_{\text{odd}} > E^*_\pm.
\]

So from now on assume that \( \gamma(t) \) starts in \( \Lambda_{\text{even}} \). Moreover, \( \gamma(t) \) is obtained from \( \gamma(t-1) \) either by flipping a \(-1\)-spin above \( R(\gamma(t-1)) \) or below. Without restriction, we assume that above \( R(\gamma(t-1)) \) a \((+1)\)-spin is added. Finally, let \( P_H R(\gamma(t)) \times (l^*_h - 2) \) denote the rectangle of horizontal length \( P_H R(\gamma(t)) \) and vertical length \( l^*_h - 2 \) that start from the same row as the bottom of \( R(\gamma(t)) \).

• Case 2.1: \( \left| \{\gamma(t) \setminus (P_H R(\gamma(t)) \times (l^*_h - 2)\} \right| > 2 \).

Note that we necessarily have that \( \gamma(t-1) \) has at least two \((+1)\)-spins in its uppermost row. If \( m = 0 \), then, since \( \gamma \) does not cross \( \mathcal{P}_2 \), we have by Lemma 4.5 that
\[
H^\pm(\gamma(t-1)) = H^\pm((l^*_h \times (l^*_h - 2)) + 2J + 2h_{\text{odd}} = E^*_\pm - 2J + h_{\text{even}}.
\]

This leads to a contradiction, since
\[
H^\pm(\gamma(t)) \geq H^\pm(\gamma(t-1)) + 2J - h_{\text{even}} > E^*_\pm.
\]
If $m > 0$ and $P_H R(\gamma(t - 1)) < \sqrt{|\mathcal{A}|}$, then similarly, we observe
\[
H^\pm(\gamma(t)) \geq H^\pm(\gamma(t - 1)) + 2J - h_{\text{even}}
\]
\[
\geq H^\pm(l_h^* + m) \times (l_h^* - 2) + 2J + 2h_{\text{odd}} + 2J - h_{\text{even}}
\]
\[
= H^\pm(l_h^* \times (l_h^* - 2)) + m(\mu - \epsilon(l_h^* - 1)) + 4J + 2h_{\text{odd}} - h_{\text{even}}
\]
\[
> E^*_\pm,
\]
which is a contradiction. Finally, if $P_H R(\gamma(t - 1)) = \sqrt{|\mathcal{A}|}$, it holds that
\[
H^\pm(\gamma(t)) \geq H^\pm(\gamma(t - 1)) + 2J - h_{\text{even}}
\]
\[
\geq H^\pm(\sqrt{|\mathcal{A}|} \times (l_h^* - 2)) + 2J + 2h_{\text{odd}} - 2J + 2J - h_{\text{even}}
\]
\[
= H^\pm(l_h^* \times (l_h^* - 2)) + (\sqrt{|\mathcal{A}|} - l_h^*) (\mu - \epsilon(l_h^* - 1)) - 2J(l_h^* - 1) + 4J + 2h_{\text{odd}} - h_{\text{even}}
\]
\[
> E^*_\pm,
\]

\textbf{Case 2.2: } $|\{P_H R(\gamma(t)) \times (l_h^* - 2)\}| = 2$.

Define
\[
T = \max \left\{ j \geq t \mid |\gamma(j) \setminus \{P_H R(\gamma(t)) \times (l_h^* - 2)\}| \leq 2 \right\}
\]
\[\text{Lemma 4.5 implies that} \]
\[
H^\pm(\gamma(T)) > H^\pm(l_h^* \times (l_h^* - 2)) + 4J - h_{\text{even}} + h_{\text{odd}} = E^*_\pm - h_{\text{odd}},
\]
and therefore
\[
H^\pm(\gamma(T + 1)) \geq H^\pm(\gamma(T)) + h_{\text{odd}} > E^*_\pm.
\]

If $m' > 0$ and $P_H R(\gamma(T)) < \sqrt{|\mathcal{A}|}$, then
\[
H^\pm(\gamma(T + 1)) \geq H^\pm(\gamma(T)) + h_{\text{odd}}
\]
\[
\geq H^\pm(l_h^* + m) \times (l_h^* - 2) + 4J - h_{\text{even}} + h_{\text{odd}} + h_{\text{odd}}
\]
\[
> E^*_\pm.
\]

Finally, if $P_H R(\gamma(T)) = \sqrt{|\mathcal{A}|}$, it holds that
\[
H^\pm(\gamma(T + 1)) \geq H^\pm(\gamma(T)) + h_{\text{odd}}
\]
\[
\geq H^\pm(\sqrt{|\mathcal{A}|} \times (l_h^* - 2)) + 4J - h_{\text{even}} + 2h_{\text{odd}}
\]
\[
> E^*_\pm.
\]

This concludes the proof. \hfill \Box

The following observation concludes the proof of $\Phi(\Box, \Box) - H^\pm(\Box) \geq \Gamma^*_{\pm}$.

\textbf{Lemma 4.7} Let $\gamma \in (\Box, \Box)_{\text{opt}}$. In order to cross at a time $t$ a configuration $\gamma(t)$ such that $P_V R(\gamma(j)) \geq l_h^*$ and $P_H R(\gamma(j)) \geq l_h^*$ for all $j \geq t$, $\gamma$ has to pass through $\mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{C}_1 \cup \mathcal{C}_2$ at some time $t' \geq t - 1$. In particular, every optimal path between $\Box$ and $\Box$ has to cross $\mathcal{C}_1 \cup \mathcal{C}_2$.
Proof. Consider the time step \( t \) defined in the proof of Lemma [1.6]. It was shown that there necessarily needs to exist a time \( t' \geq t-1 \) such that \( \gamma(t') \in \mathcal{P}_1 \cup \mathcal{P}_2 \), otherwise \( \gamma \in (\varnothing, \boxplus)_{\text{opt}} \) can not hold true. By the definition of \( t \), \( P_V R(\gamma(t'+1)) \geq l_h^* \) and \( P_H R(\gamma(t'+1)) \geq l_h^* \). Hence, by a straightforward analysis of all the possible cases, we observe that \( \gamma(t'+1) \) must belong to \( \mathcal{C}_1 \cup \mathcal{C}_2 \), since otherwise the energy level would exceed \( E_{\pm}^* \), which violates the fact that \( \gamma \in (\varnothing, \boxplus)_{\text{opt}} \). This proves the lemma. \( \square \)

4.3 Identification of \( \mathcal{P}^* \) and \( \mathcal{C}^* \)

Repeating similar computations as in Section 4.1, it is clear that \( \mathcal{P}_1 \cup \mathcal{P}_2 \subset \mathcal{P}^* \). Now let \( \sigma \in \mathcal{P}^* \) and \( x \in \Lambda \) be such that \( \sigma^x \in \mathcal{C}^* \). Then there exists \( \gamma \in (\varnothing, \boxplus)_{\text{opt}} \) and \( \ell \in \mathbb{N} \) such that

(i) \( \gamma(\ell) = \sigma \) and \( \gamma(\ell + 1) = \sigma^x \),

(ii) \( H^\pm(\gamma(k)) < E^*_\pm \) for all \( k \in \{0, \ldots, \ell\} \),

(iii) \( \Phi(\varnothing, \gamma(k)) \geq \Phi(\gamma(k), \boxplus) \) for all \( k \geq \ell + 1 \).

As in the proof of Lemma [4.6] and in Lemma [4.7], let

\[
\max\{ j \geq 0 \mid P_V R(\gamma(j)) < l_h^* \text{ or } P_H R(\gamma(j)) < l_h^* \}. \tag{4.32}
\]

We know from Lemma [4.7] that there exists \( t' \geq t \) such that \( \gamma(t') \in \mathcal{P}_1 \cup \mathcal{P}_2 \) and \( \gamma(t'+1) \in \mathcal{C}_1 \cup \mathcal{C}_2 \). We get from fact (ii) that \( \ell \leq t' \).

- If \( \ell = t' \), then we have that \( \sigma \in \mathcal{P}_1 \cup \mathcal{P}_2 \) and \( \sigma^x \in \mathcal{C}_1 \cup \mathcal{C}_2 \).
- If \( \ell < t' \), then fact (iii) is violated, since \( \Phi(\varnothing, \gamma(t')) < \Phi(\gamma(t'), \boxplus) = E^*_\pm \). Therefore, such a path \( \gamma \) can not exist, which contradicts the fact \( \sigma^x \in \mathcal{C}^* \).

Hence, it must be the case that \( \ell = t' \). We conclude that \( \mathcal{P}^* = \mathcal{P}_1 \cup \mathcal{P}_2 \) and \( \mathcal{C}^* = \mathcal{C}_1 \cup \mathcal{C}_2 \).

4.4 Verification of (H1)

Obviously, \( S_{\text{stab}} = \{\boxplus\} \), since \( h_{\text{even}} > h_{\text{odd}} \). It remains to show that \( S_{\text{meta}} = \{\boxplus\} \). Let \( \sigma \in S \). There are four cases.

- Case 1: \( \sigma \) contains a component, which is not a stable rectangle.

Lemma [4.4] implies that \( \sigma \) is not \( h_{\text{odd}} \)-stable, i.e. there exists \( \sigma' \in S \) such that \( H^\pm(\sigma') < H^\pm(\sigma) \) and \( \Phi(\sigma, \sigma') - H^\pm(\sigma) \leq h_{\text{odd}} < \Gamma_{\pm}^* \).

- Case 2: \( \sigma \) contains a connected component \( R \), which is a stable rectangle with \( P_V R \geq l_h^* \) and \( P_H R < \sqrt{|\Lambda|} \).

Let \( \sigma' \) be obtained from \( \sigma \) by attaching on the right of \( R \) a column of length \( P_V R \). We start to attach on an even row on the right of \( R \) and then successively flip adjacent spins until the column is filled. Then

\[
H^\pm(\sigma') \leq H^\pm(\sigma) + \mu - \frac{P_V R + 1}{2} \varepsilon \leq H^\pm(\sigma) + \mu - l_h^* \varepsilon < H^\pm(\sigma), \tag{4.33}
\]

and

\[
\Phi(\sigma, \sigma') - H^\pm(\sigma) \leq 2J - h_{\text{even}} < \Gamma_{\pm}^*. \tag{4.34}
\]
• Case 3: \([\sigma] contains a connected component \(R\) which is a stable rectangle with \(P_V R \leq l_h^* - 2\) and \(P_H R < \sqrt{|\Lambda|}\).
Let \(\sigma'\) be obtained from \(\sigma\) by cutting the right column of \(R\). Then
\[
H^\pm(\sigma') = H^\pm(\sigma) - \mu + \frac{P_V R + 1}{2} \varepsilon \leq H^\pm(\sigma) - \mu + \frac{(l_h^* - 1)}{2} \varepsilon < H^\pm(\sigma),
\]
and
\[
\Phi(\sigma, \sigma') - H^\pm(\sigma) \leq \frac{P_V R - 1}{2} \varepsilon + h_{\text{odd}} < \Gamma^*_\pm.
\]

• Case 4: \([\sigma] contains a connected component \(R\), which is a stable rectangle with \(P_H R = \sqrt{|\Lambda|}\).\)
Let \(\sigma'\) be obtained from \(\sigma\) by attaching above \(R\) successively vertical bars of length 2 until the two rows above \(R\) wrap around the torus. Then:
\[
H^\pm(\sigma') = H^\pm(\sigma) + 4J - l_1 \varepsilon < H^\pm(\sigma), \quad \text{and}
\]
\[
\Phi(\sigma, \sigma') - H^\pm(\sigma) \leq 4J - \varepsilon < \Gamma^*_\pm.
\]
This proves that \(S_{\text{meta}} = \{\Xi\}\).

4.5 Computation of \(K\)

Before estimating \(K^{-1}\) from below and above, we define \(\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2\), where

• \(\tilde{C}_1\) is the set of all configurations \(\sigma\) that are obtained from a configuration \(\sigma' \in C_1\) as follows. There is a column in \(\sigma'\) that has length 2. \(\sigma\) is obtained from \(\sigma'\) by adding a third (+1)-spin on the even row adjacent to this column, and

• \(\tilde{C}_2\) is the set of all configurations \(\sigma\) that are obtained from a configuration \(\sigma' \in C_2\) as follows. There is a component of three (+1)-spins above or below the \(l_h^* \times (l_h^* - 2)\)-rectangle in \(\sigma'\). \(\sigma\) is obtained from \(\sigma'\) by adding a (+1)-spin such that this component becomes a \(2 \times 2\)-square.

![Figure 5: An example of an element in \(\tilde{C}_1\) and \(\tilde{C}_2\)](image)

Note that \(\partial^+ C^* \cap S^* = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \tilde{C}_1 \cup \tilde{C}_2 = \mathcal{P}^* \cup \tilde{C} \) and \(\mathcal{P}^* \subset S_{\Xi}\) and \(\tilde{C} \subset S_{\Xi}\).
• **Lower bound.** Using these definitions and facts, we can estimate $K^{-1}$ as follows.

\[
\frac{1}{K} \geq \min_{C_1,...,C_l \in [0,1]} \frac{1}{h_{s_{\mathbb{R}}}=0,h_{s_{\mathbb{M}}}=0} \frac{1}{2} \sum_{\eta,\eta' \in (C^*)^+} \mathbb{1}_{\eta \sim \eta'} [h(\eta) - h(\eta')]^2
\]

\[
= \min_{h_{c^*} \rightarrow [0,1]} \sum_{\eta \in C^*} \left( \sum_{\eta' \in \mathcal{P}^* \cap [0,1]} \left[ 1 - h(\eta) \right]^2 + \sum_{\eta' \in \mathcal{C} \cap (C^*)^+} h(\eta) \right)
\]

\[
= \sum_{\eta \in \mathcal{C}_1} |\mathcal{P}^* \sim \eta| \cdot |\mathcal{C} \sim \eta| + \sum_{\eta \in \mathcal{C}_2} |\mathcal{P}^* \sim \eta| \cdot |\mathcal{C} \sim \eta|.
\]

For all $\eta \in \mathcal{C}_1$ we have that $|\mathcal{P}^* \sim \eta| = 1$, whereas for all $\eta \in \mathcal{C}_2$ we have that $|\mathcal{P}^* \sim \eta| = 2$. Moreover, $|\mathcal{C} \sim \eta| = 1$ for all $\eta \in \mathcal{C}^*$. Finally, it can be seen easily that $|\mathcal{C}_1| = |\mathcal{C}_2| = 4|\Lambda|/(l_0^* - 1)$.

Hence,

\[
\frac{1}{K} \geq |\mathcal{C}_1| \frac{1}{2} + |\mathcal{C}_2| \frac{2}{3} = \frac{14(l_0^* - 1)}{3} |\Lambda|.
\]

• **Upper bound.** We say that a row or a column of a configuration is a singleton, if it consists only of a single (+1)–spin. Define the following subsets of $S^*$

\[
R_0^- = \{ \sigma \in S^* | \sigma \text{ is not connected and either } P_V(R(\eta)) < l_0^* \text{ or } P_H(R(\eta)) < l_0^* \}
\]

for all its connected components $\eta$,

\[
R_0^+ = \{ \sigma \in S^* | \sigma \text{ is connected and } P_V(R(\eta)) \geq l_0^* \text{ and } P_H(R(\eta)) \geq l_0^* \}
\]

for at least one of its connected components $\eta$,

\[
R_1^- = \{ \sigma \in S^* | \sigma \text{ is connected and } P_V(R(\sigma)) < l_0^* \}
\]

\[
R_2^- = \{ \sigma \in S^* | \sigma \text{ is connected and } P_H(R(\sigma)) < l_0^* \}
\]

\[
R_3^- = \{ \sigma \in S^* | \sigma \text{ is connected and } P_V(R(\sigma)) \geq l_0^* \text{ and } P_H(R(\sigma)) = l_0^* \}
\]

and at least two rows of $\sigma$ are singletons,

\[
R_4^- = \{ \sigma \in S^* | \sigma \text{ is connected and } P_H(R(\sigma)) = l_0^* \text{ and } P_V(R(\sigma)) = l_0^* \}
\]

and at least one column of $\sigma$ is a singleton,

\[
R_1^+ = \{ \sigma \in S^* | \sigma \text{ is connected and } P_V(R(\sigma)) = l_0^* \text{ and } P_H(R(\sigma)) = l_0^* \}
\]

and no column of $\sigma$ is a singleton

and at most one row of $\sigma$ is a singleton,

\[
R_2^+ = \{ \sigma \in S^* | \sigma \text{ is connected and } P_H(R(\sigma)) > l_0^* \text{ and } P_V(R(\sigma)) = l_0^* \}
\]

and at most one row of $\sigma$ is a singleton,

\[
R_3^+ = \{ \sigma \in S^* | \sigma \text{ is connected and } P_H(R(\sigma)) = l_0^* \text{ and } P_V(R(\sigma)) > l_0^* \}
\]

\[
R_4^+ = \{ \sigma \in S^* | \sigma \text{ is connected and } P_H(R(\sigma)) > l_0^* \text{ and } P_V(R(\sigma)) > l_0^* \}
\]

Moreover, let $R^- = R_0^- \cup R_1^- \cup R_2^- \cup R_3^- \cup R_4^-$ and $R^+ = R_0^+ \cup R_1^+ \cup R_2^+ \cup R_3^+ \cup R_4^+$. Notice that $S^* = R^- \cup R^+$, $R^- \cap R^+ = \emptyset$, $\mathcal{P}^* \subset R^-$ and $\mathcal{C}^* \subset R^+$.

**Lemma 4.8** Let $\sigma \in R^-$ and $\sigma' \in R^+$. Then $\sigma \sim \sigma'$, if and only if $\sigma \notin \mathcal{P}^*$ and $\sigma' \notin \mathcal{C}^*$.

**Proof.** We show for each case separately that, if $\sigma \notin \mathcal{P}^*$ or $\sigma' \notin \mathcal{C}^*$, then $\sigma, \sigma' \in S^*$ is contradicted.
• Case 0: \( \{\sigma \in R_0^- \text{ or } \sigma' \in R_0^\pm\} \).
  We only treat the case \( \sigma \in R_0^- \) and \( \sigma' \in R_0^+ \). The other case is analogue. There must be a component \( \eta \subset \sigma \) such that either (i) \( P_H R(\eta) = l_1^b + m \) for some \( m \geq 0 \) and \( P_H R(\eta) = l_1^b - 1 \) or (ii) \( P_V R(\eta) = l_1^b - 1 \) and \( P_H R(\eta) = l_1^b + m \) for some \( m \geq 0 \), and a component \( \eta' \subset \sigma' \) such that \( P_V R(\eta') \geq l_1^b \) and \( P_H R(\eta') \geq l_1^b \) and \( \eta \sim \eta' \). We only consider case (ii), because the other one is analogue. It is easy to see that

\[
H^\pm(\sigma) \geq H^\pm(\eta) + 4J - h_{\text{even}}. \tag{4.41}
\]

Then

\[
H^\pm(\sigma) \geq H^\pm(\eta) + 4J - h_{\text{even}} \geq H^\pm((l_1^b + m) \times (l_1^b - 2)) + 6J - \varepsilon = E_{\pm}^* + m(\mu - \varepsilon(l_1^b - 1)) + 2J - h_{\text{odd}} > E_{\pm}^*.
\]

This contradicts \( \sigma \in S^* \). Hence, this case is not possible.

• Cases 1, 2, 3, 4, 5: \( \{(\sigma \in R_1^-, \sigma' \in R_1^+) \text{ or } (\sigma \in R_1^+, \sigma' \in R_1^-) \} \text{ or } (\sigma \in R_2^-, \sigma' \in R_2^+) \text{ or } (\sigma \in R_2^+, \sigma' \in R_2^-) \).

Due to the fact that both configurations are connected, all these cases are not possible.

• Cases 6, 7: \( \{(\sigma \in R_1^- \setminus P^*, \sigma' \in R_1^+ \setminus C^*) \text{ or } (\sigma \in R_1^-, \sigma' \in R_2^+) \} \).

Necessarily, each row of \( \sigma \) needs to have more than two \((+1)\)-spins, since to increase \( P_V R(\sigma) \), a row must be added to \( \sigma \) that consists of a single \((+1)\)-spin only. Now the same computations as in Case 2.1 from the proof of Lemma 4.6 show that both cases are not possible.

• Case 8: \( \{\sigma \in R_2^- \text{ and } \sigma' \in R_1^+\} \).

To increase \( P_H R(\sigma) \), it is necessary to add a column to \( \sigma \) that consists of a single \((+1)\)-spin only. But since no column of \( \sigma' \) is a singleton, \( \sigma \sim \sigma' \) can not hold true, which implies that this case is not possible.

• Case 9: \( \{\sigma \in R_2^- \text{ and } \sigma' \in R_3^-\} \).

The same computations as in the Cases 1.2, 1.3 and 1.4 from the proof of Lemma 4.6 show that this case is not possible.

• Cases 10, 11: \( \{(\sigma \in R_3^- \setminus P^*, \sigma' \in R_3^+ \setminus C^*) \text{ or } (\sigma \in R_3^-, \sigma' \in R_3^+) \} \).

\( \sigma' \) is obtained from \( \sigma \) by adding a \((+1)\)-spin to one of the two rows of \( \sigma \) that are singletons. The same computations as in the Case 2.2 from the proof of Lemma 4.6 show that this case is not possible.

• Cases 12, 13: \( \{(\sigma \in R_3^+, \sigma' \in R_3^-) \text{ or } (\sigma \in R_3^-, \sigma' \in R_3^+) \} \).

\( \sigma' \) is obtained from \( \sigma \) by attaching a protuberance above or below of \( \sigma \). Let \( P_H R(\sigma) = l_1^b + m \) for some \( m \geq 0 \). Obviously, \( H^\pm(\sigma) \geq H^\pm((l_1^b + m) \times (l_1^b - 2)) + 4J - \varepsilon \). This implies that

\[
H^\pm(\sigma') \geq H^\pm(\sigma) + 2J - h_{\text{even}} \geq H^\pm((l_1^b + m) \times (l_1^b - 2)) + 6J - \varepsilon - h_{\text{even}} = E_{\pm}^* + m(\mu - \varepsilon(l_1^b - 1)) + 2J - h_{\text{odd}} - h_{\text{even}} > E_{\pm}^*.
\]

Hence, this case is not possible.

• Case 14: \( \{\sigma \in R_1^- \setminus P^* \text{ and } \sigma' \in R_1^+ \setminus C^*\} \).

The same computations as in the Case 1.1 from the proof of Lemma 4.6 show that this case is not possible.

• Cases 15, 16: \( \{(\sigma \in R_1^-, \sigma' \in R_1^+) \text{ or } (\sigma \in R_4^-, \sigma' \in R_4^+) \} \).

\( \sigma' \) is obtained from \( \sigma \) by attaching a protuberance outside of \( \sigma \). Obviously, \( H^\pm(\sigma) \geq H^\pm((l_1^b - 1) \times (l_1^b - 2)) + 2J - h_{\text{even}} \). This implies that

\[
H^\pm(\sigma') \geq H^\pm(\sigma) + 2J - h_{\text{even}} \geq H^\pm((l_1^b - 1) \times (l_1^b - 2)) + 4J - 2h_{\text{even}} = E_{\pm}^* + 2J - h_{\text{odd}} - h_{\text{even}} > E_{\pm}^*.
\]

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Hence, this case is not possible. □

Moreover, we have that for all $i$ either $S_i \subset \mathbb{R}^-$ holds or $S_i \subset \mathbb{R}^+$, since the $S_i$ are connected. For the same reason and by Section 4.1, $S_{\ominus} \subset \mathbb{R}^-$ and $S_{\oplus} \subset \mathbb{R}^+$ holds. Therefore,

\[
\frac{1}{K} \leq \min_{h:S^* \to [0,1]} \left( \frac{1}{2} \sum_{\eta, \eta' \in S^*} I_{\{\eta \sim \eta'\}} [h(\eta) - h(\eta')]^2 \right)
\]

\[
= \min_{h:(C^*)^+ \to [0,1]} \left( \frac{1}{2} \sum_{\eta, \eta' \in (C^*)^+} I_{\{\eta \sim \eta'\}} [h(\eta) - h(\eta')]^2 \right)
\]

\[
= \min_{h:C^* \to [0,1]} \sum_{\eta \in C^*} \left( \sum_{\eta' \in P^*, \eta' \sim \eta} [1 - h(\eta)]^2 + \sum_{\eta' \in \bar{C}, \eta' \sim \eta} h(\eta)^2 \right)
\]

\[
= \frac{14 (l^*_b - 1)}{3} |\Lambda|.
\]

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