AVERAGING PRINCIPLE FOR ONE DIMENSIONAL STOCHASTIC BURGERS EQUATION

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Abstract. In this paper, we consider the averaging principle for one dimensional stochastic Burgers equation with slow and fast time-scales. Under some suitable conditions, we show that the slow component strongly converges to the solution of the corresponding averaged equation. Meanwhile, when there is no noise in the slow component equation, we also prove that the slow component weakly converges to the solution of the corresponding averaged equation with the order of convergence \(1 - r\), for any \(0 < r < 1\).

1. Introduction

Many multiscale problems arise from material sciences, chemistry, fluids dynamics, biology, ecology, climate dynamics and other application areas, see, e.g., \([1, 12, 19, 23, 27, 29, 36]\) and references therein. E and Engquist \([12]\) pointed out “Problems in these areas are often multiphysics in nature; namely, the processes at different scales are governed by physical laws of different character: for example, quantum mechanics at one scale and classical mechanics at another.” For instance, dynamics of chemical reaction networks often take place on notably different times scales, from the order of nanoseconds \((10^{-9} \text{ s})\) to the order of several days, the use of two-time or multi-time scales is common. Another example with multiple time scales is that of protein folding. While the time scale for the vibration of the covalent bonds is on the order of femtoseconds \((10^{-15} \text{ s})\), folding time for the proteins may very well be on the order of seconds.

Many two-time scale/slow-fast systems can be formally written as

\[
\begin{align*}
\dot{X}_t^\varepsilon &= b_1(X_t^\varepsilon, Y_t^\varepsilon)\,dt + \sigma_1(X_t^\varepsilon, Y_t^\varepsilon)\,dW_1^t, \quad X_0^\varepsilon = x, \\
\dot{Y}_t^\varepsilon &= \frac{1}{\varepsilon} b_2(X_t^\varepsilon, Y_t^\varepsilon)\,dt + \frac{1}{\sqrt{\varepsilon}} \sigma_2(X_t^\varepsilon, Y_t^\varepsilon)\,dW_2^t, \quad Y_0^\varepsilon = y,
\end{align*}
\]

(1.1)

where \(W_1^t, W_2^t\) are independent Wiener processes, and the small parameter \(\varepsilon\) quantifies the ratio of the \(X^\varepsilon\) and \(Y^\varepsilon\) time scales. For many practical problems, it is of interest to study the behavior of the system (1.1) for \(\varepsilon \ll 1\), and how dynamics of this system depends on \(\varepsilon\) as \(\varepsilon \to 0\). However, since \(\varepsilon \ll 1\), it is often very difficult to directly calculate \(X^\varepsilon\), and systems of this type are problematic for computer simulations. The averaging principle can be applied to solve these problems of this type. Roughly speaking, if the dynamics for \(Y^\varepsilon\)
with $X^\varepsilon = x$ fixed has an invariant probability measure $\mu^x(dy)$ and the following integrals exist
\[ \bar{b}_1(x) := \int b_1(x,y)\mu^x(dy), \quad \bar{\sigma}_1(x) := \int \sigma_1(x,y)\mu^x(dy), \]
then under appropriate assumptions on all coefficients in the system (1.1), the effective dynamics for $X^\varepsilon$ in the limit of $\varepsilon \to 0$ is a stochastic differential equation:
\[ d\bar{X}_t = \bar{b}_1(\bar{X}_t)dt + \bar{\sigma}_1(\bar{X}_t)dW^1_t, \quad \bar{X}_0 = x. \]

The theory of averaging principle has a long history and rich results. Bogoliubov and Mitropolsky [2] first studied the averaging principle for the deterministic systems. Later on, the theory of averaging principle for stochastic differential equations was firstly established by Khasminskii [21]. Since then, averaging principle for stochastic reaction-diffusion systems has become an active research area which attracted much attention. For example, Cerrai and Freidlin [5] proved the averaging principle for a general class of stochastic reaction-diffusion systems, which extended the classical Khasminskii-type averaging principle for finite dimensional systems to infinite dimensional systems. Recently, based on the averaging principle, the fast flow asymptotics for a stochastic reaction-diffusion-advection equation are obtained by Cerrai and Freidlin [6]. We refer to [3, 4, 13, 15, 16, 17, 18, 22, 24, 33, 34, 35] and references therein for more interesting results on this topic.

To the best of our knowledge, there are rarely studies to deal with highly nonlinear term on this topic. In this paper, we are interested in studying the averaging principle for one dimensional stochastic Burgers, i.e., considering the following stochastic slow-fast system on the interval $[0,1]$:

\[
\begin{aligned}
&\frac{\partial X^\varepsilon_t(\xi)}{\partial t} = \left[ \Delta X^\varepsilon_t(\xi) + \frac{1}{2} \frac{\partial}{\partial \xi}(X^\varepsilon_t(\xi))^2 + f(X^\varepsilon_t(\xi), Y^\varepsilon_t(\xi)) \right] + \frac{\partial W^Q_1}{\partial t}(t, \xi), \quad X^\varepsilon_0 = x \\
&\frac{\partial Y^\varepsilon_t(\xi)}{\partial t} = \frac{1}{\varepsilon} \left[ \Delta Y^\varepsilon_t(\xi) + g(X^\varepsilon_t(\xi), Y^\varepsilon_t(\xi)) \right] + \frac{1}{\sqrt{\varepsilon}} \frac{\partial W^Q_2}{\partial t}(t, \xi), \quad Y^\varepsilon_0 = y \\
&X^\varepsilon_t(0) = X^\varepsilon_t(1) = Y^\varepsilon_t(0) = Y^\varepsilon_t(1) = 0,
\end{aligned}
\]

where $\varepsilon > 0$ is a small parameter describing the ratio of time scales between the slow component $X^\varepsilon_t$ and fast component $Y^\varepsilon_t$. The coefficients $f$ and $g$ satisfy some suitable conditions. \{W^Q_1\}_{t \geq 0}$ and \{W^Q_2\}_{t \geq 0} are $L^2(0,1)$-valued mutually independent $Q_1$ and $Q_2$-Wiener processes.

For any $t \in [0, T]$, as $\varepsilon \to 0$, the slow component $X^\varepsilon_t$ in (1.2) converges to $\bar{X}_t$, which is the solution of the averaged equation:

\[
\begin{aligned}
&d\bar{X}_t = \Delta \bar{X}_t dt + \frac{1}{2} \frac{\partial}{\partial \xi}(\bar{X}_t)^2 dt + \bar{f}(\bar{X}_t)dt + dW^Q_1(t), \\
&\bar{X}_0 = x.
\end{aligned}
\]

with the average
\[ \bar{f}(x) = \int_{L^2(0,1)} f(x,y)\mu^x(dy), \]
where $\mu^x$ denotes the unique invariant measure for the fast component equation with frozen slow component variable $x$ (see the equation (3.25) for details).

We aim to study the rate of convergences of the process $X^\varepsilon$ to $\bar{X}$, both in the strong convergence sense and in the weak convergence sense. Under some appropriate conditions, the result of strong convergence is stated as follows:
• For any $x \in H^\alpha(0, 1)$ with $\alpha \in (1, \frac{3}{2}]$ and $y \in L^2(0, 1)$, $p > 0$, $T > 0$, there exists a positive constant $C$ which is independent of $\varepsilon$ such that
\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} \|X_t^\varepsilon - \tilde{X}_t\|^{2p}\right) \leq C\left(\frac{1}{-\log \varepsilon}\right)^{\frac{1}{2p}} \to 0, \quad \text{as} \quad \varepsilon \to 0. \tag{1.4}
\]
Here we denote by $\| \cdot \|$ the norm of $L^2(0, 1)$.

If $Q_1 = 0$ in the system (1.2), then under some conditions, the result of weak convergence is stated as follows:

• For any $x \in H^\theta(0, 1)$ with $\theta \in (0, 1]$, $y \in L^2(0, 1)$, $\phi \in C^2_b(L^2(0, 1))$, $r \in (0, 1)$, $\delta \in (0, \frac{1}{2})$, $t \in (0, T]$, there exists a positive constant $C'$ which is independent of $\varepsilon$ such that
\[
\left|\mathbb{E}\phi(X_t^\varepsilon) - \mathbb{E}\phi(\bar{X}_t)\right| \leq C(1 + t^{-\theta} + \frac{t^2}{r^2\varepsilon^{1-r}}). \tag{1.5}
\]

Comparing with the strong convergence, while the requirement on the regularity of initial value $x$ in weak convergence is weaker, the rate of the convergence is pleasant in this case. The idea of the proof follows a procedure inspired by [3], in which the authors considered a relative simple framework (without the nonlinear term and with $f$ being bounded). In our case, to deal with the nonlinear term and unbounded $f$ is a nontrivial task.

The proof of (1.4) is based on the Khasminskii argument introduced in [21], but it is clearly more involved than in [21], as it concerns the nonlinear term in the Burgers' equation and unbounded $f$. To be precise, we split the interval $[0, T]$ into some subintervals of size $\delta > 0$, then on each interval $[k\delta, (k+1)\delta)$, $k \geq 0$, we construct an auxiliary process $(\tilde{X}_t^\varepsilon, \tilde{Y}_t^\varepsilon)$, $t \in [k\delta, (k+1)\delta)$, associated with the system (1.2). Based on the exponential ergodicity of the fast component equation with frozen slow component $x$ in the system (1.2), by controlling the error between $\tilde{X}_t^\varepsilon$ and $X_t^\varepsilon$, allows us to deduce (1.4). The biggest challenge in studying the strong convergence (1.4) is to deal with the nonlinear term. To overcome this difficulty, we first give some estimates of the slow component $X_t^\varepsilon$ and fast component $Y_t^\varepsilon$ in $L^2(0, 1)$. Secondly, by using the smoothness of semigroup $e^{t\Delta}$ and the interpolation inequality, we can further obtain $\sup_{\varepsilon \in (0, 1)} \mathbb{E}\sup_{t \in [0, T]} |X_t^\varepsilon|_\alpha \leq C_{p, T}$, which is a key step for proving (1.4), where $| \cdot |_\alpha$ is the Sobolev norm. Finally, we obtain the result by applying the skill of stopping time and following the procedure inspired by [15].

To obtain the weak convergence (1.5), we use the asymptotic expansion with respect to $\varepsilon$ of the solution to the Kolmogorov equation corresponding to the system (1.2). However, some problems appear since the operator $\Delta$ is unbounded in the Kolmogorov equation. To overcome this difficulty, following the approach used in [3], we first use the Galerkin approximation to reduce the infinite dimensional problem to a finite dimensional one, then the remaining part is to establish the rate of convergence with some bounds which is independent of the dimension. Note that instead of using the asymptotic expansion of the solution to the Kolmogorov equation, an alternative martingale approach was applied to prove the weak convergence for stochastic reaction-diffusion equations with unbounded multiplicative noise [4].

Finally, we refer that, in recent years, there are many interesting results for stochastic Burger's equation [7, 8, 9, 10, 11, 14, 20, 25, 26, 30, 31, 32].

The rest of the paper is organized as follows. In Section 2, under some suitable assumptions, we formulate our main results. Section 3 and Section 4 are devoted to proving the
strong convergence and weak convergence, respectively. In the Appendix 5, we recall some useful inequalities.

Throughout the paper, $C$, $C_p$ and $C_{p,T}$ will denote positive constants which may change from line to line, where $C_p$ depends on $p$, $C_{p,T}$ depends on $p, T$.

2. Notations and main results

Let $L^2 := L^2(0, 1)$ be the space of square integrable real-valued functions on the interval $[0, 1]$. The norm and the inner product on $L^2$ are denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. The space $C^2_b(L^2)$ is the functions from $L^2$ to $\mathbb{R}$, which are twice continuously differentiable with first and second bounded derivative. For $k \in \mathbb{N}$, $W^{k,2}(0, 1)$ is the Sobolev space of all functions in $L^2$ whose differentials belong to $L^2$ up to the order $k$. The usual Sobolev space $W^{k,2}(0, 1)$ can be extended to the $W^{s,2}(0, 1)$, for $s \in \mathbb{R}$. Set $H^k \triangleq W^{k,2}(0, 1)$ and denote by $H^k_0$ the subspace of $H^1$ of all functions whose trace at $0$ and $1$ vanishes. The Laplacian operator $\Delta$ is given by

$$Ax = \Delta x = \frac{\partial^2}{\partial \xi^2} x, \quad x \in \mathcal{D}(A) = H^2 \cap H^1_0.$$ 

It is well known that $\Delta$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{t\Delta}\}_{t \geq 0}$. The eigenfunctions of $\Delta$ are given by $e_k(\xi) = \sqrt{2} \sin(k\pi \xi)$, $\xi \in [0, 1], k \in \mathbb{N}$, with the corresponding eigenvalues $\lambda_k = -k^2\pi^2$. The operator $\Delta$ satisfies the smoothing property: for any $s_1, s_2 \in \mathbb{R}$ with $s_1 \leq s_2$,

$$|e^{t\Delta}z|_{H^{s_2}} \leq C \left(1 + t^{(s_1 - s_2)/2}\right) |z|_{H^{s_1}}, \quad \text{for } z \in H^{s_1}. \quad (2.1)$$

For any $\alpha \in \mathbb{R}$, $(-A)^\alpha$ is the power of the operator $-A$, and $| \cdot |_\alpha$ is the norm of $\mathcal{D}((-A)^{\alpha/2})$ which is equivalent to the norm of $H^\alpha$.

Define the bilinear operator $B(x, y) : L^2 \times H^1_0 \to H^{-1}_0$ by

$$B(x, y) = x \cdot \partial \xi y,$$

and the trilinear operator $b(x, y, z) : L^2 \times H^1_0 \times L^2 \to \mathbb{R}$ by

$$b(x, y, z) = \int_0^1 x(\xi) \partial \xi y(\xi) z(\xi) d\xi.$$ 

For convenience, set $B(x) = B(x, x)$, for $x \in H^1_0$.

With the above notations, the system (1.2) can be rewritten as:

$$\begin{cases}
\d X_t^\varepsilon = [AX_t^\varepsilon + B(X_t^\varepsilon) + f(X_t^\varepsilon, Y_t^\varepsilon)]dt + dW_t^{Q_1}, & X_0^\varepsilon = x \\
\d Y_t^\varepsilon = \frac{1}{\varepsilon}[AY_t^\varepsilon + g(X_t^\varepsilon, Y_t^\varepsilon)]dt + \frac{1}{\sqrt{\varepsilon}}dW_t^{Q_2}, & Y_0^\varepsilon = y \\
X_t^\varepsilon(0) = X_t^\varepsilon(1) = Y_t^\varepsilon(0) = Y_t^\varepsilon(1) = 0.
\end{cases} \quad (2.2)$$

Here, the $Q_1$-Wiener process $W_t^{Q_1}$ is given by

$$W_t^{Q_1} = \sum_{k=1}^{\infty} \sqrt{\alpha_k/\beta k^2} e_k, \quad t \geq 0, \quad (2.3)$$

where $\alpha_k \geq 0$ satisfies $TrQ_1 := \sum_{k=1}^{\infty} \alpha_k < +\infty$, and $\{\beta_k\}_{k \in \mathbb{N}}$ is a sequence of mutually independent standard Brownian motions. Throughout this paper, we assume that $W_t^{Q_2}$ also has a similar decomposition as in (2.3) with $TrQ_2 < \infty$. Note that $W_t^{Q_1}$ and $W_t^{Q_2}$ are independent.

We impose the global Lipschitz condition on the functions $f, g : L^2 \times L^2 \to L^2$ in (2.2).
\textbf{A1.} There exist two constants }L_f, L_g > 0 \text{ such that for any } x_1, x_2, y_1, y_2 \in L^2, \\
\|f(x_1, y_1) - f(x_2, y_2)\| \leq L_f(\|x_1 - x_2\| + \|y_1 - y_2\|), \\
\|g(x_1, y_1) - g(x_2, y_2)\| \leq L_g(\|x_1 - x_2\| + \|y_1 - y_2\|).

Following the standard approach developed in [9], one can verify that under the condition \textbf{A1}, there exists a unique mild solution to the system (2.2). More specifically, for any given initial value } x, y \in L^2, \text{ and } T > 0, \text{ there exist a unique } X^\varepsilon \in C([0, T]; L^2) \cap L^2(0, T; H_0^1) \text{ and a unique } Y^\varepsilon \in C([0, T]; L^2) \cap L^2(0, T; H_0^1) \text{ satisfying}

\begin{equation}
X^\varepsilon_t = e^{tA}x + \int_0^t e^{(t-s)A}B(X^\varepsilon_s)ds + \int_0^t e^{(t-s)A}f(X^\varepsilon_s, Y^\varepsilon_s)ds + \int_0^t e^{(t-s)A}dW^Q_1, \\
Y^\varepsilon_t = e^{tA/\varepsilon}y + \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}g(X^\varepsilon_s, Y^\varepsilon_s)ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t e^{(t-s)A/\varepsilon}dW^Q_2.
\end{equation}

\textbf{A2. } \eta := \lambda_1 - L_g > 0.

The following condition on the } Q_1\text{-Wiener process } W^Q_1 \text{ is used to establish the strong convergence of } X^\varepsilon \text{ to } \bar{X}.

\textbf{A3.} There exist constants } \alpha \in (1, \frac{3}{2}) \text{ and } \beta \in (0, \frac{1}{2}) \text{ such that}

\[ \sum_{k=1}^{\infty} \alpha_k \lambda_k^{\alpha + 2\beta - 1} < +\infty. \]

For any } x \in L^2, \text{ denote by } D\varphi(x), D^2\varphi(x) \text{ the first and the second Fréchet derivatives of the function } \varphi : L^2 \to \mathbb{R}, \text{ respectively. By Riesz representation theorem, we have}

\[ D\varphi(x) \cdot h = \langle D\varphi(x), h \rangle, \quad D^2\varphi(x) \cdot (h, k) = \langle D^2\varphi(x)h, k \rangle, \quad h, k \in L^2. \]

\textbf{A4.} Assume that } f \text{ and } g \text{ are twice differentiable with respect to the first and the second variable, respectively, and that there exists a constant } C > 0 \text{ such that for any } x, y, h, k \in L^2, \text{ the following inequalities hold:}

\begin{align}
\|D^2_{xx}f(x, y)(h, k)\| &\leq C\|h\|\|k\|; \\
\|D_yg(x, y) \cdot h\| &\leq C\|h\|, \quad |D^2_{yy}g(x, y)(h, k)| \leq C\|h\|\|k\|; \\
|\langle f(x, y), x \rangle| &\leq C(1 + \|x\|^2).
\end{align}

Now we are going to formulate our main results. The first result gives the convergence rate in the sense of the trajectory distance between the slow component } X^\varepsilon_t \text{ and the averaged component } \bar{X}_t, \text{ as } \varepsilon \to 0, \text{ uniformly with respect to } t \in [0, T].

\textbf{Theorem 2.1} (Strong convergence). \textit{Assume the conditions A1, A2 and A3 hold. Then, for any } x \in H^\alpha \text{ with } \alpha \text{ given in A3, } y \in L^2, \text{ } p > 0 \text{ and } T > 0, \text{ there exists a constant } C := C_{x, y, T, p, \alpha} > 0 \text{ such that}

\[ \mathbb{E}\left( \sup_{0 \leq t \leq T} \|X^\varepsilon_t - \bar{X}_t\|^{2p} \right) \leq C\left( \frac{1}{-\log \varepsilon} \right)^{\frac{1}{4p}} \to 0, \quad \text{as } \varepsilon \to 0. \]
With some regularity for the initial value \( x \), the following result describes the convergence rate in the sense of the law distance between the slow component \( X_t^\varepsilon \) and the averaged component \( \tilde{X}_t \), as \( \varepsilon \to 0 \).

**Theorem 2.2** (Weak convergence 1). Assume the conditions \( A_1, A_2 \) and \( A_4 \) hold, and that \( Q_1 = 0 \) in (2.2). Then for any \( x \in H^0 \) with \( \theta \in (0,1] \), \( y \in L^2 \), \( \phi \in C_0^\delta (L^2) \), \( t \in (0,T] \), \( \delta \in (0,\frac{1}{2}) \), there exists a constant \( C := C_{x,y,T,\theta,\delta} > 0 \) such that for any \( \varepsilon \in (0,1) \), we have

\[
|\mathbb{E}[\phi(X_t^\varepsilon)] - \mathbb{E}[\phi(\tilde{X}_t)]| \leq C(1 + t^{-\theta - \frac{\phi^2}{1+r}})\varepsilon^{1-r}.
\] (2.8)

The constant appeared in (2.8) can become quite terse at the cost of the higher regularity condition on the initial value \( x \). More precisely, we have the following result:

**Theorem 2.3** (Weak convergence 2). Assume the conditions of Theorem 2.2 hold. Then, for any \( x \in H^0 \) with \( \theta \in (1,\frac{3}{2}) \), \( y \in L^2 \), \( t \in (0,T] \) and \( r \in (0,1) \), there exists a constant \( C := C_{x,y,T,\theta} > 0 \) such that

\[
|\mathbb{E}[\phi(X_t^\varepsilon)] - \mathbb{E}[\phi(\tilde{X}_t)]| \leq C\varepsilon^{1-r}.
\]

**Remark 2.4.** In Theorems 2.2 and 2.3, if we don’t impose the condition \( Q_1 = 0 \) in the system (2.2), there exist some essential difficulties. For instance, it is not clear how to establish Lemma 4.8, which plays a crucial role in proving Theorems 2.2 and 2.3.

### 3. Proof of Theorem 2.1

In this section, we are devoted to proving Theorem 2.1. The proof consists of the following several steps. In the first step, we give some priori estimates of the solution \( (X_t^\varepsilon, Y_t^\varepsilon) \) to the system (2.2). In the second step, following the idea inspired by Khasminskii in [21], we introduce an auxiliary process \( (\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon) \in L^2 \times L^2 \) and also give the uniform bounds. Meanwhile, we deduce an estimate of the process \( X_t^\varepsilon - \hat{X}_t^\varepsilon \) in the space \( L^{2p}(\Omega, C([0,T], L^2)) \). In the last step, based on the ergodicity property of the averaged equation (see (3.25)), we make use of the skill of the stopping time and some approximation techniques to give a control of \( |\hat{X}_t^\varepsilon - \tilde{X}_t^\varepsilon|_{L^{2p}(\Omega, C([0,T], L^2))} \). Consequently, we deduce the convergence rate in Theorem 2.1.

Recalling that \( V := H_0^1 \) is continuously and densely embedded in \( L^2 \), consider the Gelfand triple: \( V \subset L^2 \subset V^* \), where \( V^* \) is the dual space of \( V \). According to the Poincaré inequality, we have that for any \( x \in V \),

\[
\langle V^*, \langle Ax, x \rangle_V \rangle = -\|\nabla x\|^2 \leq -\lambda_1\|x\|^2,
\] (3.1)

where \( \langle \cdot, \cdot \rangle_V \) denotes the dualization between \( V^* \) and \( V \).

#### 3.1. Some priori estimates of \( (X_t^\varepsilon, Y_t^\varepsilon) \)

We first prove the uniform bounds, with respect to \( \varepsilon \in (0,1) \) and \( t \in [0,T] \), for \( p \)-moments of the solutions to the system (2.2). The main proof follows the techniques in [28] and [25, 26], where the authors deal with the 2D stochastic Navier-Stokes equation and 1D stochastic Burgers’ equation, respectively.

**Lemma 3.1.** Under conditions \( A_1 \) and \( A_2 \), for any \( x, y \in L^2 \), \( p \geq 2 \) and \( T > 0 \), there exists a constant \( C_{p,T} > 0 \) such that

\[
\sup_{\varepsilon \in (0,1)} \sup_{0 \leq t \leq T} \mathbb{E}\|X_t^\varepsilon\|^{2p} \leq C_{p,T}(1 + \|x\|^{2p} + \|y\|^{2p}),
\] (3.2)

\[
\sup_{\varepsilon \in (0,1)} \sup_{0 \leq t \leq T} \mathbb{E}\|Y_t^\varepsilon\|^{2p} \leq C_{p,T}(1 + \|x\|^{2p} + \|y\|^{2p}).
\] (3.3)
Proof. According to Itô’s formula, we have
\[ \frac{d}{dt} \mathbb{E}\|Y_t^\varepsilon\|^{2p} = \frac{2p\lambda_1}{\varepsilon} \mathbb{E}\|\|Y_t^\varepsilon\|^{2p-2}(-\|Y_t^\varepsilon\|)\| + \frac{2p}{\varepsilon} \mathbb{E}\left[ \|Y_t^\varepsilon\|^{2p-2}\langle g(X_t^\varepsilon, Y_t^\varepsilon), Y_t^\varepsilon \rangle \right] \]
\[ + \frac{p}{\varepsilon} \mathbb{E}\|Y_t^\varepsilon\|^{2p-2}\text{Tr}Q_2 + \frac{2p(p-1)}{\varepsilon} \mathbb{E}\|Y_t^\varepsilon\|^{2p-2}\text{Tr}Q_2, \tag{3.4} \]
where Itô’s formula can be understood in the way that we first use the Galerkin approximation to get (3.4) in the finite dimensional setting, then we take the limit of the dimension to obtain (3.4) in the infinite dimensional setting. Using (3.1) and condition A1, it follows from (3.4) that
\[ \frac{d}{dt} \mathbb{E}\|Y_t^\varepsilon\|^{2p} \leq -\frac{2p\lambda_1}{\varepsilon} \mathbb{E}\|Y_t^\varepsilon\|^{2p} + \frac{2p}{\varepsilon} \mathbb{E}\left\{ \|Y_t^\varepsilon\|^{2p-2} \left[ C\|Y_t^\varepsilon\| + L_g\|X_t^\varepsilon\|\|Y_t^\varepsilon\| + \|Y_t^\varepsilon\|\right]\right\} \]
\[ + \frac{p}{\varepsilon} \mathbb{E}\|Y_t^\varepsilon\|^{2p-2}\text{Tr}Q_2 + \frac{2p(p-1)}{\varepsilon} \mathbb{E}\|Y_t^\varepsilon\|^{2p-2}\text{Tr}Q_2. \tag{3.5} \]
From (3.5), using condition A2 and the Young inequality, we deduce that there exists a constant \( \gamma > 0 \) such that
\[ \frac{d}{dt} \mathbb{E}\|Y_t^\varepsilon\|^{2p} \leq -\frac{p\gamma}{\varepsilon} \mathbb{E}\|Y_t^\varepsilon\|^{2p} + \frac{C_p}{\varepsilon} \mathbb{E}\|X_t^\varepsilon\|^{2p} + \frac{C_p}{\varepsilon}. \tag{3.6} \]
Applying the comparison theorem gives
\[ \mathbb{E}\|Y_t^\varepsilon\|^{2p} \leq \|y\|^{2p}e^{-\frac{p\gamma}{\varepsilon}t} + \frac{C_p}{\varepsilon} \int_0^t e^{-\frac{p\gamma}{\varepsilon}(t-s)} \left( 1 + \mathbb{E}\|X_s^\varepsilon\|^{2p} \right) ds. \tag{3.7} \]
For \( X_t^\varepsilon \), note that by Lemma 5.1, \( b(x, y, y) = 0 \) for any \( x, y \in H^1_0 \). Similarly to (3.4), applying Itô’s formula, we have
\[ \frac{d}{dt} \mathbb{E}\|X_t^\varepsilon\|^{2p} = 2p\lambda_1 \mathbb{E}\left[ \|X_t^\varepsilon\|^{2p-2}\|X_t^\varepsilon\|^2 \right] + 2p\mathbb{E}\left[ \|X_t^\varepsilon\|^{2p-2}\langle f(X_t^\varepsilon, Y_t^\varepsilon), X_t^\varepsilon \rangle \right] \]
\[ + p\mathbb{E}\|X_t^\varepsilon\|^{2p-2}\text{Tr}Q_1 + 2p(p-1)\mathbb{E}\left\{ \|X_t^\varepsilon\|^{2p-4}\|\sqrt{Q}X_t^\varepsilon\|^2 \right\}. \]
Using (3.1), we obtain
\[ \frac{d}{dt} \mathbb{E}\|X_t^\varepsilon\|^{2p} \leq -2p\lambda_1 \mathbb{E}\|X_t^\varepsilon\|^{2p} + 2p\mathbb{E}\left[ \|X_t^\varepsilon\|^{2p-2}\langle f(X_t^\varepsilon, Y_t^\varepsilon), X_t^\varepsilon \rangle \right] \]
\[ + p\mathbb{E}\|X_t^\varepsilon\|^{2p-2}\text{Tr}Q_1 + 2p(p-1)\mathbb{E}\left\{ \|X_t^\varepsilon\|^{2p-4}\|\sqrt{Q}X_t^\varepsilon\|^2 \right\}. \]
In the same way as in (3.6) and (3.7), one can verify that
\[ \mathbb{E}\|X_t^\varepsilon\|^{2p} \leq \|x\|^{2p}e^{C_p t} + C_p \int_0^t e^{-\frac{p\gamma}{\varepsilon}(t-s)} \left( 1 + \mathbb{E}\|Y_s^\varepsilon\|^{2p} \right) ds. \tag{3.8} \]
Combining (3.7) and (3.8), we get that, for any \( t \in [0, T] \),
\[ \mathbb{E}\|Y_t^\varepsilon\|^{2p} \leq C_{p,T}(1 + \|x\|^{2p} + \|y\|^{2p}) + \frac{C_p}{\varepsilon} \int_0^t e^{-\frac{p\gamma}{\varepsilon}(t-s)} \int_s^t \mathbb{E}\|Y_r^\varepsilon\|^{2p} dr ds + \frac{C_p}{\varepsilon} \int_0^t e^{-\frac{p\gamma}{\varepsilon}(t-s)} ds, \]
which implies
\[ \mathbb{E}\|Y_t^\varepsilon\|^{2p} \leq C_{p,T}(1 + \|x\|^{2p} + \|y\|^{2p}) + C_p \int_0^t \mathbb{E}\|Y_r^\varepsilon\|^{2p} dr. \]
Using Gronwall’s inequality, we get (3.3). The inequality (3.2) follows by combining (3.3) and (3.8). The proof is complete. \( \square \)
In order to estimate the high-order norm of $|X_t^\alpha|$, with $\alpha \in (1, \frac{3}{2})$, we first give a control of the stochastic convolution $W_A(t) := \int_0^t e^{(t-s)A}dW_s^{Q_1}$.

**Lemma 3.2.** Under the condition \textbf{A3}, for any $p, T > 0$ and $\alpha \in (1, \frac{3}{2})$ which is given in \textbf{A3}, there exists a positive constant $C_{p,T}$ such that

$$
\mathbb{E} \sup_{0 \leq t \leq T} |W_A(t)|_{\alpha}^{2p} \leq C_{p,T}.
$$

(3.9)

**Proof.** By the Hölder inequality, it suffices to prove (3.9) for large enough $p$. Using the factorization method, for $\beta \in (0, \frac{1}{2})$ in \textbf{A3}, we write

$$
W_A(t) = \frac{\sin \pi \beta}{\pi} \int_0^t e^{(t-s)A} (t-s)^{-\beta} Z_s ds,
$$

where

$$
Z_s = \int_0^s e^{(s-r)A} (s-r)^{-\beta} dW_r^{Q_1}.
$$

Choosing $p > 1$ large enough such that $2p(1-\beta)/2p-1 < 1$, we get

$$
|W_A(t)|_{\alpha} \leq C \left( \int_0^t (t-s)^{-2p(1-\beta)/2p-1} ds \right)^{2p-1} |Z|_{L^{2p}(0,T;H^\alpha)} \leq C_p t^{\beta-1} |Z|_{L^{2p}(0,T;H^\alpha)},
$$

which implies

$$
\sup_{0 \leq t \leq T} |W_A(t)|_{\alpha}^{2p} \leq C_{p,T} |Z|_{L^{2p}(0,T;H^\alpha)}^{2p}.
$$

(3.10)

Notice that $(-A)^{\alpha/2}Z_s \sim N(0, \tilde{Q}_s)$, which is a Gaussian random variable with mean zero and the covariance operator given by

$$
\tilde{Q}_s x = \int_0^s r^{-2\beta} e^{rA} (-A)^{\alpha} Q_1 e^{rA^*} x dr.
$$

For any $p \geq 1$, $s > 0$, we follow the proof of [8, Corollary 2.17] to obtain

$$
\mathbb{E}|(-A)^{\alpha/2}Z_s|^{2p} \leq C_p [\text{Tr}(\tilde{Q}_s)]^p = C_p \left( \sum_{k=1}^{\infty} \int_0^s r^{-2\beta} e^{-2r\lambda_k} \lambda_k^{\alpha} \alpha_k dr \right)^p
$$

$$
= C_p \left( \sum_{k=1}^{\infty} \lambda_k^{\alpha+2\beta-1} \alpha_k \int_0^{2\lambda_k} r^{-2\beta} e^{-r} dr \right)^p \leq C_p \left( \sum_{k=1}^{\infty} \lambda_k^{\alpha+2\beta-1} \alpha_k \right)^p < \infty,
$$

(3.11)

where in the last two inequalities we use the fact $\int_0^{2\lambda_k} r^{-2\beta} e^{-r} dr \leq \int_0^\infty r^{-2\beta} e^{-r} dr$, and the condition \textbf{A3}. We conclude the proof of Lemma 3.2 by combining (3.10) and (3.11). \hfill \Box

**Lemma 3.3.** Under the conditions \textbf{A1-A3}, for $T > 0$ and $p > 0$, there exists a constant $C_{p,T} > 0$ such that for any $x \in H^\alpha$ with $\alpha$ given in \textbf{A3}, and for any $y \in L^2$, we have

$$
\sup_{\varepsilon \in (0,1)} \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon|_{\alpha}^{2p} \right) \leq C_{p,T} (1 + |x|_{\alpha}^{2p} + \|y\|_{2p}^{2p}).
$$

(3.12)

**Proof.** Using the Hölder inequality, it suffices to prove (3.12) for large enough $p$. Recall that

$$
X_t^\varepsilon = e^{tA}x + \int_0^t e^{(t-s)A}B(X_s^\varepsilon)ds + \int_0^t e^{(t-s)A}f(X_s^\varepsilon, Y_s^\varepsilon)ds + \int_0^t e^{(t-s)A}dW_s^{Q_1}.
$$
For the first term, it is clear that $|e^{At}x|^2 \leq |x|^2$. For the second term, according to (2.1) and Lemma 5.2, we have

$$\left| \int_0^t e^{(t-s)A}B(X^\varepsilon_s)ds \right|_\alpha \leq \int_0^t \left| e^{(t-s)A}B(X^\varepsilon_s) \right|_\alpha ds \leq C \int_0^t (1 + (t - s)^{-\frac{\alpha_3 - \alpha}{2}}) |B(X^\varepsilon_s)|_{-\alpha_3} ds \leq C \int_0^t (1 + (t - s)^{-\frac{\alpha_3 - \alpha}{2}}) |X^\varepsilon_s|_{\alpha_1} |X^\varepsilon_s|_{\alpha_2+1} ds,$$

where $\alpha_1 + \alpha_2 + \alpha_3 > \frac{1}{2}$ and $\alpha_i > 0, i = 1, 2, 3$. Using the interpolation inequality, we have that

$$|X^\varepsilon_s|_{\alpha_1} \leq C \|X^\varepsilon_s\|^{\frac{\alpha - \alpha_1}{\alpha}} |X^\varepsilon_s|^{\frac{\alpha_1}{\alpha}}; \quad (3.13)$$

for any $0 < \alpha_1 < \alpha$, and that

$$|X^\varepsilon_s|_{\alpha_2+1} \leq C \|X^\varepsilon_s\|^{\frac{\alpha - \alpha_2 - 1}{\alpha}} |X^\varepsilon_s|^{\frac{\alpha_2+1}{\alpha}}; \quad (3.14)$$

for any $0 < \alpha_2 + 1 < \alpha$. Let $\alpha_1$ and $\alpha_2$ be small enough such that $1 + \alpha_1 + \alpha_2 \in (1, \alpha)$. It follows from (3.13) and (3.14) that

$$\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t e^{(t-s)A}B(X^\varepsilon_s)ds \right|^{2p}_\alpha \leq C \mathbb{E} \left( \sup_{t \in [0,T]} \int_0^t (1 + (t - s)^{-\frac{\alpha_3 - \alpha}{2}}) \|X^\varepsilon_s\|^{\frac{2\alpha - \alpha_1 - \alpha_2 - 1}{\alpha}} |X^\varepsilon_s|^{\frac{\alpha_1 + \alpha_2 + 1}{\alpha}} ds \right)^{2p} \leq C_{p,T} \mathbb{E} \left( \left( \sup_{t \in [0,T]} \int_0^t (1 + (t - s)^{-\frac{\alpha_3 - \alpha}{2}}) \|X^\varepsilon_s\|^{\frac{2p}{\alpha}} |X^\varepsilon_s|^{\frac{\alpha_2 + \alpha_1 + 1}{\alpha}} ds \right) \right)^{2p-1} \times \left( \sup_{t \in [0,T]} \int_0^t \|X^\varepsilon_s\|^{\frac{2p}{\alpha}} |X^\varepsilon_s|^{\frac{\alpha_2 + \alpha_1 + 1}{\alpha}} ds \right) \right)^{2p-1} \left( \int_0^T \mathbb{E} \|X^\varepsilon_s\|^{\frac{2p}{\alpha}} ds \right)^{2p-1} \left( \int_0^T \mathbb{E} |X^\varepsilon_s|^{2p} ds \right) \cdot (3.15)$$

By Lemma 3.1, we obtain

$$\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t e^{(t-s)A}B(X^\varepsilon_s)ds \right|^{2p}_\alpha \leq C_{p,T} \left( 1 + \int_0^T s^{-\frac{\alpha_1 + \alpha_2}{2}} \frac{2p}{2p-1} ds \right)^{2p-1} \left( 1 + \int_0^T \mathbb{E} |X^\varepsilon_s|^{2p} ds \right). \quad (3.15)$$

We choose positive constants $\alpha_1, \alpha_2, \alpha_3$ such that $0 < 1 + \alpha_1 + \alpha_2 < \alpha$ and $\alpha_1 + \alpha_2 + \alpha_3 > \frac{1}{2}$. Let $p$ be large enough such that $\frac{\alpha_1 + \alpha_2}{2} \cdot \frac{2p}{2p-1} < 1$ (for instance, $\alpha_3 = \frac{1}{2}, \alpha_1 = \alpha_2 = \frac{\alpha - 1}{4}$, and $p > \frac{2}{3-2\alpha}$). Consequently, from (3.15), we get

$$\mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^t e^{(t-s)A}B(X^\varepsilon_s)ds \right|^{2p}_\alpha \right) \leq C_{p,T} \left( 1 + \int_0^T \mathbb{E} |X^\varepsilon_s|^{2p} ds \right). \quad (3.16)$$
For the third term, according to (2.1), we obtain
\[\mathbb{E}\left(\sup_{t \in [0,T]} \left| \int_0^t e^{(t-s)\alpha} f(X_s^\varepsilon, Y_s^\varepsilon) ds \right|^{2p}\right)\]
\[\leq C\mathbb{E}\left[ \sup_{t \in [0,T]} \left( 1 + (t-s)^{-\frac{\alpha}{2}} \right) \|X_s^\varepsilon\| + \|Y_s^\varepsilon\| \right]^{2p} \int_0^t \left( 1 + \mathbb{E}\|X_s^\varepsilon\|^{2p} + \mathbb{E}\|Y_s^\varepsilon\|^{2p} \right) ds.\]
Taking \( p \) large enough such that \( \frac{\alpha p}{2p-1} < 1 \), it follows from Lemma 3.1 that
\[\mathbb{E}\left( \sup_{t \in [0,T]} \left| \int_0^t e^{(t-s)\alpha} f(X_s^\varepsilon, Y_s^\varepsilon) ds \right|^{2p}\right) \leq C_{p,T}(1 + \|x\|^{2p} + \|y\|^{2p}). \quad (3.17)\]
We conclude the proof of Lemma 3.3 by combining (3.16), (3.17), Lemma 3.2 and Gronwall’s inequality.

Now we are equipped to prove the Hölder continuity of \( t \mapsto X_t^\varepsilon \), which holds uniformly with respect to \( \varepsilon \in (0, 1) \).

**Lemma 3.4.** Under the conditions **A1-A3**, for any \( x \in H^\alpha, y \in L^2, T > 0, 0 < t \leq t + h \leq T \), there exists a constant \( C_{p,T} > 0 \) such that
\[\sup_{\varepsilon \in (0,1)} \mathbb{E}\|X_{t+h}^\varepsilon - X_t^\varepsilon\|^{2p} \leq C_{p,T}h^p(1 + |x|^{4p}_\alpha + \|y\|^{4p}).\]

**Proof.** After simple calculations, we have
\[X_{t+h}^\varepsilon - X_t^\varepsilon = (e^{Ah} - I)X_t^\varepsilon + \int_t^{t+h} e^{(t+s-h)A} B(X_s^\varepsilon) ds + \int_t^{t+h} e^{(t+s-h)A} f(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_t^{t+h} e^{(t+s-h)A} dW_s^{Q_1},\]
\[=: I_1 + I_2 + I_3 + I_4.\]
For \( I_1 \), note that for \( \alpha \) given in **A3**, there exists a constant \( C_\alpha > 0 \) such that for any \( x \in D((-A)^{\alpha}), \|e^{Ah}x - x\| \leq C_\alpha h^\frac{\alpha}{4} |x|_\alpha \). Then using Lemma 3.3, we get
\[\mathbb{E}\|I_1\|^{2p} \leq C_\alpha h^{2\alpha p} \mathbb{E}\|X_t^\varepsilon\|^{2p} \leq C_{p,T}h^{2p}(1 + |x|^{2p}_\alpha + \|y\|^{2p}). \quad (3.18)\]
For \( I_2 \), using the contractive property of the semigroup \( e^{tA} \), Corollary 5.3 and Lemma 3.3, we obtain
\[\mathbb{E}\|I_2\|^{2p} \leq \mathbb{E}\left( \int_t^{t+h} \|B(X_s^\varepsilon)\| ds \right)^{2p} \leq C\mathbb{E}\left( \int_t^{t+h} |X_s^\varepsilon|^{2p} ds \right)^{2p} \leq C_{p,T}h^{2p} \mathbb{E}\sup_{s \in [0,T]} |X_s^\varepsilon|^{4p} \leq C_{p,T}h^{2p}(1 + |x|^{4p}_\alpha + \|y\|^{4p}). \quad (3.19)\]
For \( I_3 \), applying condition **A1** and Lemma 3.1, we get
\[\mathbb{E}\|I_3\|^{2p} \leq h^{2p-1} \mathbb{E}\int_t^{t+h} \|f(X_s^\varepsilon, Y_s^\varepsilon)\|^{2p} ds \]
\[\leq Ch^{2p-1} \mathbb{E}\int_t^{t+h} (1 + \|X_s^\varepsilon\| + \|Y_s^\varepsilon\|)^{2p} ds \leq C_{p,T}h^{2p}(1 + |x|^{2p} + \|y\|^{2p}). \quad (3.20)\]
For $I_4$, note that $I_4$ is the centered Gaussian random variable with the variance given by $S_h x = \int_0^h e^{(h-r)A} Q_1 e^{(h-r)A^*} x dr$. Then, for any $p \geq 1$, we get

$$
\mathbb{E}\|I_4\|^{2p} \leq C_p[\text{Tr}(S_h)]^p = C_p\left(\sum_{k=1}^{\infty} \int_0^h e^{-2(h-r)\lambda_k} \alpha_k dr\right)^p \leq C_p(\sum_{k=1}^{\infty} \alpha_k)^p h^p.
$$

(3.21)

Putting (3.18)-(3.21) together, the result follows.

\[ \square \]

3.2. Estimates of the auxiliary process $(\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon)$. Following the idea inspired by Khasminskii [21], we introduce an auxiliary process $(\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon) \in L^2 \times L^2$. Specifically, we split the interval $[0, T]$ into some subintervals of size $\delta > 0$. With the initial value $\hat{Y}_0^\varepsilon = Y_0^\varepsilon = y$, for any $t \in [k\delta, \min((k+1)\delta, T)]$, $k \in \mathbb{N}$, we construct the process $\hat{Y}_t^\varepsilon$ as follows:

$$
\hat{Y}_t^\varepsilon = Y_{k\delta}^\varepsilon + \frac{1}{\varepsilon} \int_{k\delta}^t A\hat{Y}_s^\varepsilon ds + \frac{1}{\varepsilon} \int_{k\delta}^t g(X_s^\varepsilon, \hat{Y}_s^\varepsilon) ds + \frac{1}{\varepsilon} \int_{k\delta}^t dW_s^{Q_2},
$$

(3.22)

where $(X_s^\varepsilon, Y_s^\varepsilon)$ is the solution to the system (2.2). Then, for any $t \in [0, T]$, we construct the process $\hat{X}_t^\varepsilon$ as follows:

$$
\hat{X}_t^\varepsilon = x + \int_0^t A\hat{X}_s^\varepsilon ds + \int_0^t B(X_s^\varepsilon) ds + \int_0^t f(X_s^\varepsilon, \hat{Y}_s^\varepsilon) ds + W_t^{Q_1},
$$

(3.23)

where $s(\delta) = \lfloor \frac{t}{\delta} \rfloor \delta$ is the nearest breakpoint proceeding $s$. Note that for any $t \in [k\delta, (k+1)\delta)$, the fast component $\hat{Y}_t^\varepsilon$ does not depend on the slow component $\hat{X}_t^\varepsilon$. The following result gives a control of the auxiliary process $(\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon)$.

**Lemma 3.5.** Under the conditions **A1-A3**, for any $x, y \in L^2$, $p \geq 2$ and $T > 0$, there exists a constant $C_{p,T} > 0$ such that

$$
\sup_{\varepsilon \in (0, 1)} \sup_{t \in [0, T]} \mathbb{E}\|\hat{Y}_t^\varepsilon\|^{2p} \leq C_{p,T}(1 + \|x\|^{2p} + \|y\|^{2p}).
$$

In addition, for any $x \in H^\alpha$, $y \in L^2$, $p \geq 2$ and $T > 0$, we have

$$
\sup_{\varepsilon \in (0, 1)} \mathbb{E}\left(\sup_{t \in [0, T]} |\hat{X}_t^\varepsilon|_{\alpha}^{2p}\right) \leq C_{p,T}(1 + |x|_{\alpha}^{2p} + \|y\|^{2p}).
$$

(3.24)

From the construction of $(\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon)$, since the proof of Lemma 3.5 can be carried out in the same way as in the proof of Lemmas 3.1 and 3.3, we omit the details.

We now give a control of $Y_t^\varepsilon - \hat{Y}_t^\varepsilon$.

**Lemma 3.6.** Under the conditions **A1-A3**, for any $x \in H^\alpha, y \in L^2$, $p \geq 2$, $T > 0$ and $\varepsilon \in (0, 1)$, there exists a constant $C_{p,T} > 0$ such that

$$
\sup_{0 \leq t \leq T} \mathbb{E}\|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^{2p} \leq C_{p,T}(1 + |x|_{\alpha}^{4p} + \|y\|^{4p}) \frac{\delta^{p+1}}{\varepsilon}.
$$
Proof. For $t \in [0, T]$ with $t \in [k\delta, (k + 1)\delta)$, by Itô’s formula and Lemma 3.4, similarly to (3.4), we have

$$
\frac{d}{dt} \mathbb{E}\|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^{2p} = \frac{2p}{\varepsilon} \mathbb{E}\left[\|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^{2p-2}(-|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|_1)\right] + \frac{2p}{\varepsilon} \mathbb{E}\left[\|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^{2p-2}(g(X_t^\varepsilon, Y_t^\varepsilon) - g(X_{k\delta}^\varepsilon, \hat{Y}_t^\varepsilon), (Y_t^\varepsilon - \hat{Y}_t^\varepsilon))\right]
\leq -\frac{2p}{\varepsilon}(\lambda_1 - L_g) \mathbb{E}\left[\|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^{2p}\right] + \frac{p}{\varepsilon}(\lambda_1 - L_g) \mathbb{E}\left[\|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^{2p}\right] + \frac{C_p}{\varepsilon} \mathbb{E}\left[\|X_t^\varepsilon - X_{k\delta}^\varepsilon\|^{2p}\right]
\leq -\frac{p}{\varepsilon}(\lambda_1 - L_g) \mathbb{E}\left[\|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^{2p} + C_{p,T}(1 + |x|_4^4 + \|y\|_4^4) \frac{(t - k\delta)^p}{\varepsilon}\right].
$$

Gronwall’s inequality yields that

$$
\mathbb{E}\|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^{2p} \leq \frac{C_{p,T}}{\varepsilon}(1 + |x|_4^4 + \|y\|_4^4) \int_{k\delta}^t e^{-\frac{p}{\varepsilon}(\lambda_1 - L_g)(t-s)}(s - k\delta)^p ds
\leq C_{p,T}(1 + |x|_4^4 + \|y\|_4^4) \frac{\delta^{p+1}}{\varepsilon},
$$

which ends the proof.

**Lemma 3.7.** Under the conditions A1-A3, for any $x \in H^\alpha, y \in L^2$, $p \geq 2$, $T > 0$ and $\varepsilon \in (0, 1)$, there exists a constant $C_{p,T} > 0$ such that

$$
\mathbb{E}\left(\sup_{0 \leq t \leq T}\|X_t^\varepsilon - \hat{X}_t^\varepsilon\|^{2p}\right) \leq C_{p,T}(\delta^p + \frac{\delta^{p+1}}{\varepsilon})(1 + |x|_6^6 + \|y\|_6^6).
$$

Proof. In view of (2.4) and (3.23), we write

$$
X_t^\varepsilon - \hat{X}_t^\varepsilon = \int_0^t e^{(t-s)A}[B(X_s^\varepsilon) - B(X_{s(\delta)}^\varepsilon)] ds + \int_0^t e^{(t-s)A}[f(X_s^\varepsilon, Y_s^\varepsilon) - f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon)] ds.
$$

Using (2.1), condition A1 and Lemma 5.4, we get

$$
\|X_t^\varepsilon - \hat{X}_t^\varepsilon\|^{2p} \leq C_p \left\{ \int_0^t \left[1 + (t-s)^{-\frac{1}{2}}\right] \left|B(X_s^\varepsilon) - B(X_{s(\delta)}^\varepsilon)\right| ds \right\}^{2p}
\leq C_p \left\{ \int_0^t \left[1 + (t-s)^{-\frac{1}{2}}\right] \|X_s^\varepsilon - X_{s(\delta)}^\varepsilon\| ds \right\}^{2p}
\leq C_p \left\{ \int_0^t \left[1 + (t-s)^{-\frac{1}{2}}\right] \|X_s^\varepsilon - X_{s(\delta)}^\varepsilon\| (|X_s^\varepsilon|_1 + |X_{s(\delta)}^\varepsilon|_1) ds \right\}^{2p}
+ C_{p,T} \int_0^t \left[\|X_s^\varepsilon - X_{s(\delta)}^\varepsilon\|^{2p} + \|Y_s^\varepsilon - \hat{Y}_s^\varepsilon\|^{2p}\right] ds
\leq C_p \left\{ \int_0^t \left[1 + (t-s)^{-\frac{1}{2}}\right]^{2p} ds \right\}^{2p-1} \left\{ \int_0^t \|X_s^\varepsilon - X_{s(\delta)}^\varepsilon\|^{4p} ds \right\}^{\frac{1}{2}}
\times \left( \int_0^t (|X_s^\varepsilon|_1 + |X_{s(\delta)}^\varepsilon|_1)^{4p} ds \right)^{\frac{1}{2}}
+ C_{p,T} \int_0^t \left[\|X_s^\varepsilon - X_{s(\delta)}^\varepsilon\|^{2p} + \|Y_s^\varepsilon - \hat{Y}_s^\varepsilon\|^{2p}\right] ds.
$$
According to Lemmas 3.3, 3.4 and 3.6, we obtain
\[
E \left( \sup_{0 \leq t \leq T} \| X_t^\varepsilon - \bar{X}_t \|^{2p} \right) \leq C_{p,T} \left( \int_0^T E \left\| X_s^\varepsilon - X_{s(\delta)}^\varepsilon \right\|^{4p} ds \right)^{\frac{1}{2}}
\times \left( \int_0^T \left( E \| X_s^\varepsilon \|^{4p} + E \| X_{s(\delta)}^\varepsilon \|^{4p} \right) ds \right)^{\frac{1}{2}}
+ C_{p,T} \int_0^T E \| X_s^\varepsilon - X_{s(\delta)}^\varepsilon \|^{2p} + E \| Y_s^\varepsilon - \bar{Y}_s \|^{2p} ds
\leq C_{p,T} (\delta^p + \frac{\delta^{p+1}}{\varepsilon})(1 + |x|^{6p} + \|y\|^{6p}).
\]

The proof is complete. \( \Box \)

3.3. The averaged equation. For any fixed \( x \in L^2 \), we consider the following frozen equation associated with the fast component:
\[
\begin{cases}
\frac{\partial Y_t(\xi)}{\partial t} = AY_t(\xi) + g(x, Y_t(\xi)) + \frac{\partial W^{Q_2}}{\partial t}(t, \xi), & Y_0(\xi) = y,
\end{cases}
\]
\( t \in [0, \infty) \). (3.25)

Since \( g(x, \cdot) \) is Lipshcitz continuous, it is easy to prove that for any fixed \( x, y \in L^2 \), the equation (3.25) has a unique mild solution denoted by \( Y_t^{x,y} \). For any \( x \in L^2 \), let \( P^x_t \) be the transition semigroup of \( Y_t^{x,y} \), that is, for any bounded measurable function \( \varphi \) on \( L^2 \) and \( t \geq 0 \),
\[
P^x_t \varphi(y) = E \varphi(Y_t^{x,y}), \quad y \in L^2.
\]

The asymptotic behavior of \( P^x_t \) has been studied in many literatures. The following result shows the existence and uniqueness of the invariant measure and gives the exponential convergence to the equilibrium (see [5, Theorem 3.5]).

**Proposition 3.8.** For any \( x, y \in L^2 \), there exists a unique invariant measure \( \mu^x \) for (3.25). Moreover, there exists \( C > 0 \) such that for any bounded measurable function \( \varphi : L^2 \to \mathbb{R} \),
\[
\left| P^x_t \varphi(y) - \int_{L^2} \varphi(z) \mu^x(dz) \right| \leq C(1 + \|x\| + \|y\|) e^{-\frac{(\lambda_1 - L_0)|t|}{2} (t \wedge 1)^{-1/2}} |\varphi|_{\infty},
\]
where \( |\varphi|_{\infty} = \sup_{x \in L^2} |\varphi(x)| \).

Furthermore, we have the following result, whose proof can refer to [5, Remark 3.6].

**Proposition 3.9.** For any \( x, y \in L^2 \), there exists \( C > 0 \) such that for any Lipschitz function \( \varphi : L^2 \to \mathbb{R} \),
\[
\left| P^x_t \varphi(y) - \int_{L^2} \varphi(z) \mu^x(dz) \right| \leq C(1 + \|x\| + \|y\|) e^{-\frac{(\lambda_1 - L_0)|t|}{2} |\varphi|_{Lip}},
\]
where \( |\varphi|_{Lip} = \sup_{x,y \in L^2, x \neq y} |\varphi(x) - \varphi(y)| / \|x-y\| \).

In the sequel we shall prove that the slow component \( X_t^\varepsilon \) in the system (2.2) converges strongly to \( \bar{X}_t \), which is the solution of the averaged equation:
\[
\begin{cases}
d\bar{X}_t = A\bar{X}_t dt + B(\bar{X}_t)dt + \tilde{f}(\bar{X}_t)dt + dW^{\xi}, \\
\bar{X}_0 = x.
\end{cases}
\]
(3.26)
where
\[
\tilde{f}(x) = \int_{L^2} f(x, y) \mu^x(dy), \quad x \in L^2.
\]

The following result gives a control of $|\hat{X}_t^\varepsilon - \check{X}_t|$.

**Lemma 3.10.** Under the conditions **A1-A3**, for any $x \in H^\alpha$, $y \in L^2$, $p \geq 1$, $T > 0$ and $\varepsilon \in (0, 1)$, there exists a constant $C_{p,T} > 0$ such that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \|\hat{X}_t^\varepsilon - \check{X}_t\|^{2p} \leq C_{p,T}(1 + |x|_{16p}^{6p} + \|y\|^{6p}) \left(\frac{1}{- \log \varepsilon}\right)^{\frac{1}{p}}.
\]

**Proof.** From (3.23) and (3.26), we have
\[
\hat{X}_t^\varepsilon - \check{X}_t = \int_0^t e^{(t-s)A} \left[ B(X_s^\varepsilon) - B(X_s) \right] ds + \int_0^t e^{(t-s)A} \left[ B(\hat{X}_s^\varepsilon) - B(\check{X}_s) \right] ds \\
+ \int_0^t e^{(t-s)A} \left[ B(\check{X}_s^\varepsilon) - B(\check{X}_s) \right] ds + \int_0^t e^{(t-s)A} \left[ f(X_s^\varepsilon, \hat{Y}_s^\varepsilon) - \check{f}(X_s^\varepsilon) \right] ds \\
+ \int_0^t e^{(t-s)A} \left[ \hat{f}(X_s^\varepsilon) - \check{f}(X_s^\varepsilon) \right] ds + \int_0^t e^{(t-s)A} \left[ \hat{f}(X_s^\varepsilon) - \check{f}(X_s) \right] ds \\
:= \sum_{k=1}^6 J_k(t).
\]

For $J_1(t)$, in the same way as in the proof of Lemma 3.7, we get
\[
\mathbb{E} \sup_{0 \leq t \leq T} \|J_1(t)\|^{2p} \leq C_{p,T} \left[ \int_0^T \mathbb{E} \left\| X_s^\varepsilon - X_s \right\|^{4p} ds \right]^{\frac{1}{2}} \left[ \int_0^T \left( \mathbb{E} |X_s^\varepsilon| + \mathbb{E} |X_s| \right)^{4p} ds \right]^{\frac{1}{2}} \\
\leq C_{p,T} \delta^p (1 + |x|_{16p}^{6p} + \|y\|^{6p}). \tag{3.27}
\]

For $J_2(t)$, using Lemmas 3.3, 3.5 and 3.7 gives
\[
\mathbb{E} \sup_{0 \leq t \leq T} \|J_2(t)\|^{2p} \leq C_{p,T} \left[ \int_0^T \mathbb{E} \left\| X_s^\varepsilon - \hat{X}_s^\varepsilon \right\|^{4p} ds \right]^{\frac{1}{2}} \left[ \int_0^T \left( \mathbb{E} |X_s^\varepsilon|^{1p} + \mathbb{E} |\hat{X}_s^\varepsilon|^{1p} \right) ds \right]^{\frac{1}{2}} \\
\leq C_{p,T} \left( \delta^p + \frac{\delta^{p+1} \pi}{\sqrt{\varepsilon}} \right)(1 + |x|_{16p}^{6p} + \|y\|^{6p}). \tag{3.28}
\]

For $J_3(t)$, according to (2.1) and Lemma 5.4, we have
\[
\sup_{0 \leq t \leq T} \|J_3(t)\|^{2p} \leq C_p \left\{ \sup_{0 \leq t \leq T} \left[ \int_0^t \left[ \left( 1 + (t-s)^{-\frac{1}{2}} \right) \left| B(\hat{X}_s^\varepsilon) - B(\check{X}_s) \right| \right] ds \right]^{2p} \right\} \\
\leq C_p \left\{ \sup_{0 \leq t \leq T} \left[ \int_0^t \left[ \left( 1 + (t-s)^{-\frac{1}{2}} \right) \left| \hat{X}_s^\varepsilon - \check{X}_s \right| \right] \left( |\hat{X}_s^\varepsilon| + |\check{X}_s| \right) ds \right]^{2p} \right\}. \tag{3.29}
\]

In order to control $J_3(t)$, we make a use of the skill of stopping times. For any fixed $n \geq 1$ and $\varepsilon > 0$, define the stopping time:
\[
\tau_n^\varepsilon = \inf \left\{ t > 0 : |\hat{X}_t^\varepsilon| + |\check{X}_t| > n \right\}. \tag{3.30}
\]
It follows from (3.29) and (3.30) that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| J_3(t) \|^{2p} \leq C_p \mathbb{E} \left( \sup_{0 \leq t \leq T} \int_0^t (1 + (t - r)^{-\frac{1}{4}}) \left\| \hat{X}_r - \tilde{X}_r \right\| \left( |\hat{X}_r|_1 + |\tilde{X}_r|_1 \right) dr \right)^{2p} \\
\leq C_p n^{2p} \mathbb{E} \left( \sup_{0 \leq t \leq T} \int_0^t (1 + (t - r)^{-\frac{1}{4}}) \left\| \hat{X}_r - \tilde{X}_r \right\| dr \right)^{2p} \\
\leq C_p n^{2p} \left( \sup_{0 \leq t \leq T} \int_0^t (1 + (t - r)^{-\frac{1}{4}}) \left( \frac{2p}{2p - 1} \right) \mathbb{E} \int_{T \wedge \tau_n}^T \left\| \hat{X}_r - \tilde{X}_r \right\|^{2p} dr \right)^{2p - 1} \\
\leq C_p n^{2p} \mathbb{E} \sup_{0 \leq r \leq T} \left\| \hat{X}_r - \tilde{X}_r \right\|^{2p} dr.
\]
(3.31)

For $J_5(t)$, using the contractive property of the semigroup $e^{tA}$, $t \geq 0$, Lipschitz continuity of $\tilde{f}$, and Lemma 3.7, we obtain
\[
\mathbb{E} \sup_{0 \leq t \leq T} |J_5(t)|^{2p} \leq C_{p,T} \mathbb{E} \int_0^T \| X^{\varepsilon}_s - \tilde{X}_s \|^{2p} ds \leq C_{p,T}(\delta^p + \frac{\delta^{p+1}}{\varepsilon})(1 + |x|^{2p} + \|y\|^{2p}).
\]
(3.32)

For $J_6(t)$, similarly to the estimate of $J_5(t)$, we get
\[
\mathbb{E} \sup_{0 \leq t \leq T} |J_6(t)|^{2p} \leq C_{p,T} \mathbb{E} \int_0^T \left\| \hat{X}_r - \tilde{X}_r \right\|^{2p} ds.
\]
(3.33)

For $J_4(t)$, set $n_t = \left[ \frac{1}{\delta} \right]$, where $t \in [0, T)$ and $\delta > 0$. We write
\[
J_4(t) = J_4^1(t) + J_4^2(t) + J_4^3(t),
\]
where
\[
J_4^1(t) = \sum_{k=0}^{n_t-1} \int_{k\delta}^{(k+1)\delta} e^{(t-s)A} \left[ f(X^{\varepsilon}_{k\delta}, \hat{Y}^{\varepsilon}_{s}) - \tilde{f}(X^{\varepsilon}_{k\delta}) \right] ds,
\]
\[
J_4^2(t) = \sum_{k=0}^{n_t-1} \int_{k\delta}^{(k+1)\delta} e^{(t-s)A} \left( \tilde{f}(X^{\varepsilon}_{k\delta}) - \tilde{f}(X^{\varepsilon}_s) \right) ds,
\]
\[
J_4^3(t) = \int_{n_t\delta}^t e^{(t-s)A} \left[ f(X^{\varepsilon}_{n_t\delta}, \hat{Y}^{\varepsilon}_s) - \tilde{f}(X^{\varepsilon}_s) \right] ds.
\]

For $J_4^2(t)$, we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| J_4^2(t) \|^{2p} \leq C_{p,T} \int_0^T \mathbb{E} \left[ \| X^{\varepsilon}_{s(\delta)} \|^{2p} - X^{\varepsilon}_s \right]^{2p} ds \leq C_{p,T}\delta^p(1 + |x|^{2p} + \|y\|^{2p}).
\]
(3.34)

For $J_4^3(t)$, it follows from Lemmas 3.1 and 3.5 that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| J_4^3(t) \|^{2p} \leq C_p \delta^{2p-1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_{n_t\delta}^t \left( 1 + \|X^{\varepsilon}_{n_t\delta}\|^{2p} + \|\hat{Y}^{\varepsilon}_s\|^{2p} + \|X^{\varepsilon}_s\|^{2p} \right) ds \right] \\
\leq C_{p,T}\delta^{2p-1}(1 + |x|^{2p} + \|y\|^{2p}).
\]
(3.35)

For $J_4^1(t)$, from the construction of $\hat{Y}^{\varepsilon}_t$, we obtain that, for any $k$ and $s \in [0, \delta)$,
\[
\hat{Y}^{\varepsilon}_{s+k\delta} = Y^{\varepsilon}_{k\delta} + \frac{1}{\varepsilon} \int_0^s A\hat{Y}^{\varepsilon}_{r+k\delta} dr + \frac{1}{\varepsilon} \int_0^s g(X^{\varepsilon}_{k\delta}, \hat{Y}^{\varepsilon}_{r+k\delta}) dr + \frac{1}{\sqrt{\varepsilon}} \int_0^s d\hat{W}^{Q_2}(r),
\]
where \( \tilde{W}^{Q_2}(t) := W^{Q_2}(t + k\delta) - W^{Q_2}(k\delta) \) is the shift version of \( W^{Q_2}(t) \). Let \( \tilde{W}^{Q_2}(t) \) be a \( Q_2 \)-Wiener process which is independent of \( W^{Q_1}(t) \) and \( W^{Q_2}(t) \). Denote by \( \tilde{W}^{Q_2}(t) = \sqrt{\varepsilon}W^{Q_2}(t) \). We construct a process \( Y^{X_{k\delta},Y_{k\delta}} \) by means of

\[
Y^{X_{k\delta},Y_{k\delta}} = Y_{k\delta} + \int_{0}^{\varepsilon} \tilde{W}^{X_{k\delta},Y_{k\delta}} dr + \int_{0}^{\varepsilon} g(X_{k\delta}, Y_{k\delta}) dr + \int_{0}^{\varepsilon} dW^{Q_2}(r)
\]

This, together with the uniqueness of the solution to the equation (3.22), implies that the distribution of \( (X_{k\delta}, \hat{Y}_{k\delta+k\delta}) \) coincides with the distribution of \( (X_{k\delta}, Y^{X_{k\delta},Y_{k\delta}}) \).

In order to estimate \( \mathbb{E} \sup_{0 \leq t \leq T} \| J_4^1(t) \|^{2p} \), we first give a control of \( \mathbb{E} \sup_{0 \leq t \leq T} \| J_4^1(t) \|^{2} \):

\[
\mathbb{E} \sup_{0 \leq t \leq T} \| J_4^1(t) \|^{2} = \mathbb{E} \sup_{0 \leq t \leq T} \left\| \sum_{k=0}^{n_t-1} e^{(t-(k+1)\delta)} A \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta-s)A} \left[ f(X_{k\delta}, \hat{Y}_{s}) - \hat{f}(X_{k\delta}) \right] ds \right\|^{2}
\]

\[
\leq \mathbb{E} \sup_{0 \leq t \leq T} \left\{ \sum_{k=0}^{n_t-1} \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta-s)A} \left[ f(X_{k\delta}, \hat{Y}_{s}) - \hat{f}(X_{k\delta}) \right] ds \right\}^{2}
\]

\[
\leq C_T \frac{\varepsilon^2}{\delta^2} \max_{0 \leq k \leq \frac{T}{\varepsilon} - 1} \mathbb{E} \left\| \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta-s)A} \left[ f(X_{k\delta}, \hat{Y}_{s}) - \hat{f}(X_{k\delta}) \right] ds \right\|^{2}
\]

\[
= C_T \frac{\varepsilon^2}{\delta^2} \max_{0 \leq k \leq \frac{T}{\varepsilon} - 1} \int_{0}^{\frac{\delta}{\varepsilon}} \int_{0}^{\frac{\delta}{\varepsilon}} \Psi_k(s, \tau) ds d\tau,
\]

where

\[
\Psi_k(s, \tau) = \mathbb{E} \left( e^{(\delta-s)A} \left( f(X_{k\delta}, \hat{Y}_{s+k\delta}) - \hat{f}(X_{k\delta}) \right) e^{(\delta-\tau)eA} \left( f(X_{k\delta}, \hat{Y}_{\tau+k\delta}) - \hat{f}(X_{k\delta}) \right) \right)
\]

Similar as the argument in [15, appendix A], using Lemma 3.1, one can verify that

\[
\Psi_k(s, \tau) \leq C T \mathbb{E} \left( 1 + \| X_{k\delta} \|^2 + \| Y_{k\delta} \|^2 \right) e^{-\frac{1}{2}(s-\tau)^2} \leq C T \left( 1 + \| x \|^2 + \| y \|^2 \right) e^{-\frac{1}{2}(s-\tau)^2}.
\]

Combining (3.36) and (3.37), we get that for any \( \varepsilon \in (0, 1) \)

\[
\mathbb{E} \sup_{0 \leq t \leq T} \| J_4^1(t) \|^{2} \leq C_T \frac{\varepsilon}{\delta} (1 + \| x \|^2 + \| y \|^2).
\]
By Lemmas 3.3 and 3.5, we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| J(t) \|^{2p} \leq \mathbb{E} \left( \int_0^T |f(X_s^\varepsilon, \hat{Y}_s^\varepsilon)| + f(X_s^\varepsilon) ds \right)^{2p}
\]
\[
\leq C_pT \left[ 1 + \sup_{s \in [0,T]} \mathbb{E} \left( \| X_s^\varepsilon \|^{2p} \right) + \sup_{s \in [0,T]} \mathbb{E} \left( \| \hat{Y}_s^\varepsilon \|^{2p} \right) \right]
\]
\[
\leq C_pT(1 + \| x \|^{2p} + \| y \|^{2p}).
\] (3.39)

This, together with (3.38), implies
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| J(t) \|^{2p} \leq \left( \mathbb{E} \sup_{0 \leq t \leq T} \| J(t) \|^{2(2p-1)} \right)^{\frac{1}{2p}} \left( \mathbb{E} \sup_{0 \leq t \leq T} \| J(t) \|^2 \right)^{\frac{1}{2}}
\]
\[
\leq C_pT(1 + \| x \|^{2p} + \| y \|^{2p}) \frac{\varepsilon}{\delta}. \] (3.40)

Consequently, combining (3.34), (3.35) and (3.40), we get
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| J(t) \|^{2p} \leq C_pT(1 + \| x \|^{2p} + \| y \|^{2p}) \left( \delta^p + \delta^{2p-1} + \frac{\varepsilon}{\delta} \right). \] (3.41)

According to the estimates (3.27)-(3.28), (3.31)-(3.33), (3.41), we obtain
\[
\mathbb{E} \left( \sup_{0 \leq r \leq s \leq \tau_\varepsilon^T} \| \hat{X}_r^\varepsilon - \hat{X}_s^\varepsilon \|^{2p} \right) \leq C_pT(1 + \| x \|^{6p} + \| y \|^{6p}) \left( \delta^p + \delta^{p+1} \frac{1}{\sqrt{\varepsilon}} + \delta^{p+1} \frac{1}{\varepsilon} + \delta^{2p-1} + \frac{\varepsilon}{\delta} \right)
\]
\[
+ C_pTn^{2p} \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \leq \tau_\varepsilon^T} \| \hat{X}_r^\varepsilon - \hat{X}_s^\varepsilon \|^{2p} ds.
\]

Using Gronwall’s inequality, we get
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \| \hat{X}_t^\varepsilon - \hat{X}_t \|^{2p} \right)
\]
\[
\leq C_pT(1 + \| x \|^{6p} + \| y \|^{6p}) \left( \delta^p + \delta^{p+1} \frac{1}{\sqrt{\varepsilon}} + \delta^{p+1} \frac{1}{\varepsilon} + \delta^{2p-1} + \frac{\varepsilon}{\delta} \right) e^{C_pTn^{2p}},
\]

which implies
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \| \hat{X}_t^\varepsilon - \hat{X}_t \|^{2p} \cdot 1_{\{T < \tau_\varepsilon^T\}} \right)
\]
\[
\leq C_pT(1 + \| x \|^{2p} + \| y \|^{2p}) \left( \delta^p + \delta^{p+1} \frac{1}{\sqrt{\varepsilon}} + \delta^{p+1} \frac{1}{\varepsilon} + \delta^{2p-1} + \frac{\varepsilon}{\delta} \right) e^{C_pTn^{2p}}.
\]

Taking \( n = \sqrt{\frac{1}{8C_pT}} \log \varepsilon, \delta = \varepsilon^{\frac{1}{2}}, \) we get
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \| \hat{X}_t^\varepsilon - \hat{X}_t \|^{2p} \cdot 1_{\{T \leq \tau_\varepsilon^T\}} \right) \leq C_pT\varepsilon^{\frac{p}{2}}(1 + \| x \|^{6p} + \| y \|^{6p}). \] (3.42)
Note that, similarly to the proof of Lemma 3.3, one can check that uniformly in \( \varepsilon \in (0, 1) \),
\[ \mathbb{E}(\sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon|_1) \leq C_T(1 + |x|_\alpha). \]
Combining this with (3.24), we deduce that
\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} \|\hat{X}_t^\varepsilon - \tilde{X}_t\|^{2p} \cdot 1_{\{T > t_0^\varepsilon\}}\right) \leq \left(\mathbb{E}\sup_{0 \leq t \leq T} \|\hat{X}_t^\varepsilon - \tilde{X}_t\|^{4p}\right)^{\frac{1}{2}} \cdot \left[\mathbb{P}(T > t_0^\varepsilon)\right]^{\frac{1}{2}}
\leq C_p \left(\mathbb{E}\sup_{0 \leq t \leq T} \|\hat{X}_t\|^{4p} + \mathbb{E}\sup_{0 \leq t \leq T} \|\hat{X}_t^\varepsilon\|^{4p}\right)^{\frac{1}{2}} \frac{1}{\sqrt{n}} \left(\sup_{\varepsilon \in (0, 1)} \mathbb{E}\sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon|_1 + \mathbb{E}\sup_{0 \leq t \leq T} |\hat{X}_t|_1\right)^{\frac{1}{2}}
\leq \frac{C_{p,T}}{\sqrt{-\log \varepsilon}} \left(1 + |x|_\alpha^{2p + \frac{1}{2}} + \|y\|^{2p + \frac{1}{2}}\right).
\]
Putting together (3.42) and (3.43), we obtain
\[
\mathbb{E}\sup_{0 \leq t \leq T} \|\hat{X}_t^\varepsilon - \tilde{X}_t\|^{2p} \leq C_{p,T}(1 + |x|_\alpha^{6p} + \|y\|^{6p}) \left(\frac{1}{\log \varepsilon}\right)^{\frac{1}{3p}}.
\]
The proof is complete. \( \square \)

3.4. Proof of Theorem 2.1. Taking \( \delta = \varepsilon^{\frac{1}{2}} \), Lemma 3.7 implies
\[
\mathbb{E}\sup_{0 \leq t \leq T} \|X_t^\varepsilon - \hat{X}_t\|^{2p} \leq C_{p,T}(1 + |x|_\alpha^{2p} + \|y\|^{2p})\varepsilon^{\frac{1}{3} - \frac{1}{2p}}.
\]
Combining this with Lemma 3.10, we obtain
\[
\mathbb{E}\sup_{0 \leq t \leq T} \|X_t^\varepsilon - \tilde{X}_t\|^{2p} \leq \mathbb{E}\sup_{0 \leq t \leq T} \|X_t^\varepsilon - \hat{X}_t\|^{2p} + \mathbb{E}\sup_{0 \leq t \leq T} \|\hat{X}_t - \tilde{X}_t\|^{2p}
\leq C_{p,T}(1 + |x|_\alpha^{6p} + \|y\|^{6p}) \left(\frac{1}{\log \varepsilon}\right)^{\frac{1}{3p}} \rightarrow 0 \quad (\varepsilon \rightarrow 0),
\]
which concludes the proof of Theorem 2.1. \( \square \)

4. Proofs of Theorems 2.2 and 2.3

This section is devoted to proving Theorems 2.2 and 2.3. Following the procedure inspired by [3], the proofs are based on the Galerkin approximation and the asymptotic expansion with respect to \( \varepsilon \) of the solution to the Kolmogorov equation corresponding to (2.2) with \( Q_1 = 0 \). Since the proofs are tediously long and technical, we first give a brief summary of the main ideas and steps in the proofs of Theorems 2.2 and 2.3.

Step 1. Due to the unboundedness of operator \( \Delta \), we use the Galerkin approximation to reduce the infinite dimensional problem to a finite dimensional one as follows.

Let \( H_N = \text{span}\{e_k; 1 \leq k \leq N\} \). Denote by \( P_N \) the orthogonal projection of \( L^2 \) onto \( H_N \). Set
\[ f_N(x, y) = P_N(f(x, y)), \quad g_N(x, y) = P_N(g(x, y)), \quad B_N(x) = P_N(B(x)), \quad W^{Q_2}(t) = P_NW^{Q_2}(t) \]
for \( x, y \in H_N \). The following equation is the finite dimensional projection of the system (2.2) with \( Q_1 = 0 \):
\[
\begin{cases}
\frac{dX_N^\varepsilon(t)}{dt} = [AX_N^\varepsilon(t) + B_N(X_N^\varepsilon(t)) + f_N(X_N^\varepsilon(t), Y_N^\varepsilon(t))]dt, & X_N^\varepsilon(0) = P_Nx, \\
\frac{dY_N^\varepsilon(t)}{dt} = \frac{1}{\varepsilon}[AY_N^\varepsilon(t) + g_N(X_N^\varepsilon(t), Y_N^\varepsilon(t))]dt + \frac{1}{\sqrt{\varepsilon}}dW_N^{Q_2}(t), & Y_N^\varepsilon(0) = P_Ny.
\end{cases}
\]
(4.1)

Similarly, we consider the finite dimensional projection of the equation (1.3) with \( Q_1 = 0 \):
\[
\begin{cases}
\frac{d\tilde{X}_N(t)}{dt} = [A\tilde{X}_N(t) + B_N(\tilde{X}_N(t)) + \tilde{f}_N(\tilde{X}_N(t))]dt, \\
\tilde{X}_N(0) = P_Nx,
\end{cases}
\]
(4.2)
where \( \bar{f}(x) = \int_{H_N} P_N f(x,y) \mu_N^\varepsilon(dy) \), and \( \mu_N^\varepsilon(dy) \) is the unique invariant measure for
\[
dY_N(t) = \left[AY_N(t) + g_N(x,Y_N(t))\right]dt + dW_N^{Q^2}(t).
\]
For the test function \( \phi \in C_b^2(L^2) \), we have
\[
\mathbb{E}\left[\phi\left(X^\varepsilon(t)\right)\right] - \phi(\bar{X}(t))
= \left\{ \mathbb{E}\left[\phi\left(X^\varepsilon(t)\right)\right] - \mathbb{E}\left[\phi\left(X_N^\varepsilon(t)\right)\right] \right\} + \left\{ \mathbb{E}\left[\phi\left(X_N^\varepsilon(t)\right)\right] - \phi(\bar{X}_N(t)) \right\} + \left\{ \phi(\bar{X}_N(t)) - \phi(\bar{X}(t)) \right\}.
\]
(4.3)
It is not difficult to show that the first term and the third term in (4.3) converge to 0, as \( N \to \infty \). Therefore, in order to establish Theorems 2.2 and 2.3, it remains to show that the second term in (4.3) converges to 0 as \( N \to \infty \). We will give the main idea in the next step.

**Step 2.** Inspired by [3], we construct an asymptotic expansion of \( \mathbb{E}\left[\phi\left(X_N^\varepsilon(t)\right)\right] \). Roughly speaking, it has an expansion with respect to the small parameter \( \varepsilon \):
\[
\mathbb{E}\left[\phi\left(X_N^\varepsilon(t)\right)\right] = \phi(\bar{X}_N(t)) + \varepsilon u_1 + v^\varepsilon.
\]
In order to control the second term in (4.3), the main task is to analyze \( u_1 \) and \( v^\varepsilon \). Almost all the work in this section is to deal with this step.

This section is organized as follows. Subsections 4.1 and 4.2 are to establish some properties of \( \bar{X}_N \) and \( (X_N^\varepsilon, Y_N^\varepsilon) \) respectively. The asymptotic expansion of \( \mathbb{E}\left[\phi\left(X_N^\varepsilon(t)\right)\right] \) will be given in Subsection 4.3. Based on some results obtained in subsections 4.1 and 4.2, subsection 4.4 is to study properties of \( u_1 \) and \( v^\varepsilon \). Finally, we prove Theorems 2.2 and 2.3 in subsections 4.5 and 4.6, respectively.

4.1. **Properties of \( \bar{X}_N \).** This subsection is to establish some properties of \( \bar{X}_N \). For simplicity, we omit the index \( N \).

**Lemma 4.1.** Assume the conditions A1, A2 and A4 hold.

(1) For any \( x \in L^2 \), \( T > 0 \), there exists a constant \( C > 0 \) such that
\[
\sup_{0 \leq t \leq T} \|\bar{X}_t\| \leq C(1 + \|x\|).
\]
(4.4)

(2) Furthermore, for any \( x \in H^\theta \) with \( \theta \in (0,1) \), \( \gamma \in (1, \frac{2}{\omega}) \), \( t \in (0,T) \), there exist \( k \in \mathbb{N} \) and a constant \( C = C_{\gamma, \theta, T} > 0 \) such that
\[
|\bar{X}_t|_\gamma \leq C(\|x\|_\theta + 1)t^{-\frac{\omega-\theta}{2\theta}}e^{C\|x\|_k}.
\]
(4.5)

**Proof.** Multiplying both sides of the equation (4.2) by \( 2\bar{X}_t \) and integrating with respect to \( \xi \), we get
\[
\frac{d}{dt} \|\bar{X}_t\|^2 = 2\langle A\bar{X}_t, \bar{X}_t \rangle + 2\langle \bar{f}(\bar{X}_t), \bar{X}_t \rangle \leq C(1 + \|\bar{X}_t\|^2),
\]
which implies (4.4) by applying Gronwall’s inequality.

To prove (4.5), note that
\[
\bar{X}_t = e^{tA}x + \int_0^t e^{(t-s)A}B(\bar{X}_s)ds + \int_0^t e^{(t-s)A}\bar{f}(\bar{X}_s)ds.
\]
For the first term, we have
\[
|e^{tA}x|_\gamma \leq Ct^{-\frac{\omega-\theta}{2\theta}}|x|_\theta.
\]
(4.6)
For the second term, it follows from (2.1) and Lemma 5.2 that
\[
\left| \int_0^t e^{(t-s)A} B(\bar{X}_s) ds \right|_\gamma \leq C \int_0^t \left[ 1 + (t-s)^{-\frac{1+\varepsilon}{2}} \right] |B(\bar{X}_s)| \frac{1}{\gamma} ds \\
\leq C \int_0^t (t-s)^{-\frac{1+\varepsilon}{2}} \| \bar{X}_s \| \| \bar{X}_s \|_\gamma ds \\
\leq C \int_0^t (t-s)^{-\frac{1+\varepsilon}{2}} (1 + \| x \|) |\bar{X}_s|_\gamma ds.
\] (4.7)

For the last term, using (2.1) and (4.4), we get
\[
\left| \int_0^t e^{(t-s)A} \bar{f}(\bar{X}_s) ds \right|_\gamma \leq C \int_0^t \left[ 1 + (t-s)^{-\frac{1}{2}} \right] (1 + \| \bar{X}_s \|) ds \leq C(1 + \| x \|).
\] (4.8)

Consequently, combining (4.6)-(4.8), we conclude the proof of (4.5) by using Lemma 5.5. □

Note that from Lemma 4.1, using the interpolation inequality, we get that for any \( \gamma \in (0, 1] \), \( \theta \in (0, 1) \), \( \delta \in (0, \frac{1}{2}) \), \( t \in (0, T] \), there exist \( k \in \mathbb{N} \) and a constant \( C = C_{\theta, \delta, T} > 0 \) such that
\[
|\bar{X}_t|_\gamma \leq C \| \bar{X}_t \|^{\frac{1+\delta-\gamma}{1+\delta}} |\bar{X}_t|_\gamma^{\frac{\delta}{1+\delta}} \leq C t^{-\frac{1+\delta-\theta}{2}} \gamma \left( \| x \|_\theta + 1 \right) e^{C\| x \|^k}.
\] (4.9)

**Lemma 4.2.** Under the conditions A1, A2 and A4, for any \( \theta \in (0, 1) \), \( \alpha \in (0, \frac{1}{2}) \), \( x \in H^\theta \), \( 0 < s < t \leq T \), there exist \( k \in \mathbb{N} \) and a constant \( C = C_{\theta, \alpha, T} > 0 \) such that
\[
|\bar{X}(t, x) - \bar{X}(s, x)|_1 \leq C(t-s)^{\frac{\alpha}{2}} s^{-\frac{1+\alpha-\theta}{2}} \left( \| x \|_\theta + 1 \right) e^{C\| x \|^k}.
\]

**Proof.** In view of (4.2), we write
\[
\bar{X}(t, x) - \bar{X}(s, x) = \left( e^{A(t-s)} - I \right) \bar{X}(s, x) + \int_s^t e^{(t-r)A} B(\bar{X}(r, x)) dr + \int_s^t e^{(t-r)A} \bar{f}(\bar{X}(r, x)) dr.
\] (4.10)

For the first term, using the property \( \| (e^{tA} - I)x \| \leq Ct^{\frac{\alpha}{2}} \| x \|_\alpha \) and Lemma 4.1, we get that there exists \( k \in \mathbb{N} \) such that
\[
\left| \left( e^{A(t-s)} - I \right) \bar{X}(s, x) \right|_1 \leq C(t-s)^{\frac{\alpha}{2}} \| \bar{X}(s, x) \|_{1+\alpha} \\
\leq C(t-s)^{\frac{\alpha}{2}} s^{-\frac{1+\alpha-\theta}{2}} \left( \| x \|_\theta + 1 \right) e^{C\| x \|^k}.
\] (4.11)

For the second term, according to Lemma 4.1, there exists some \( k \in \mathbb{N} \) such that
\[
\left| \int_s^t e^{(t-r)A} B(\bar{X}(r, x)) dr \right|_1 \leq C \int_s^t \left[ 1 + (t-r)^{-\frac{1}{2}} \right] \| B(\bar{X}(r, x)) \| \frac{1}{\frac{1}{2}} dr \\
\leq C \int_s^t \left[ 1 + (t-r)^{-\frac{1}{2}} \right] \| \bar{X}(r, x) \| \| \bar{X}(r, x) \|_{1+\alpha} dr \\
\leq C \int_s^t \left[ 1 + (t-r)^{-\frac{1}{2}} \right] r^{-\frac{1+\alpha-\theta}{2}} \left( \| x \|_\theta + 1 \right) e^{C\| x \|^k} dr \\
\leq C(t-s)^{\frac{\alpha}{2}} s^{-\frac{1+\alpha-\theta}{2}} \left( \| x \|_\theta + 1 \right) e^{C\| x \|^k}.
\] (4.12)
For the third term, using Lemma 4.1 again, we obtain
\[
\left| \int_s^t e^{(t-r)A} f(\bar{X}(r,x)) dr \right| \leq C \int_s^t [1 + (t-r)^{-\frac{1}{2}}] \| \bar{f}(\bar{X}(r,x)) \| dr
\leq C \int_s^t [1 + (t-r)^{-\frac{1}{2}}] (1 + \| \bar{X}(r,x) \|) dr
\leq C (t-s)^{\frac{3}{2}} (1 + \| x \|).
\] (4.13)

The result follows by combining (4.10)-(4.13).

\end{proof}

\textbf{Lemma 4.3.} \textit{Under the conditions A1, A2 and A4, for any } x \in H^0 \text{ with } \theta \in (0, 1), 0 \leq t \leq T, \text{ there exist } k \in \mathbb{N} \text{ and a constant } C = C_{\theta,T} > 0 \text{ such that}
\[
\| \frac{d}{dt} \bar{X}(t,x) \| \leq C t^{-1 + \frac{\theta}{2}} (|x|^2 + 1)e^{C\|x\|^k}.
\]

\begin{proof}
Recall that
\[
\frac{d}{dt} \bar{X}(t,x) = A\bar{X}(t,x) + B(\bar{X}(t,x)) + \bar{f}(\bar{X}(t,x)).
\] (4.14)

Taking } \delta = \frac{1}{4} \text{ in (4.9), we get } ||B(\bar{X}(t,x))|| \leq C ||\bar{X}(t,x)||^2 \leq C t^{-1 + \frac{\theta}{2}} (|x|^2 + 1)e^{C\|x\|^k}. \text{ It is easy to see that } \| \bar{f}(\bar{X}(t,x)) \| \leq C(1 + ||x||). \text{ Hence, to prove Lemma 4.3, it remains to control the first term in (4.14). From (4.14), we write}
\[
\bar{X}(t,x) = e^{tA}x + \int_0^t e^{(t-s)A} B(\bar{X}(t,x)) ds + \int_0^t e^{(t-s)A} \left[ B(\bar{X}(s,x)) - B(\bar{X}(t,x)) \right] ds
+ \int_0^t e^{(t-s)A} \bar{f}(\bar{X}(t,x)) ds + \int_0^t e^{(t-s)A} \left[ \bar{f}(\bar{X}(s,x)) - \bar{f}(\bar{X}(t,x)) \right] ds
:= I_1 + I_2 + I_3 + I_4 + I_5.
\]

For } I_1, \text{ using (2.1), we have}
\[
\| AI_1 \| \leq C t^{-1 + \frac{\theta}{2}} |x|_\theta.
\]

For } I_2, \text{ we deduce from Corollary 5.3 and (4.9) that}
\[
\| AI_2 \| \leq \left\| (e^{tA} - I) B(\bar{X}(t,x)) \right\| \leq 2 \left\| B(\bar{X}(t,x)) \right\|
\leq 2 \left\| \bar{X}(t,x) \right\|_1^2 \leq C t^{-1 + \frac{\theta}{2}} (|x|^2 + 1)e^{C\|x\|^k}.
\]

For } I_3, \text{ according to Lemma 4.2 and (4.9), we get}
\[
\| AI_3 \| \leq C \int_0^t \frac{1}{t-s} \left\| B(\bar{X}(t,x)) - B(\bar{X}(s,x)) \right\| ds
\leq C \int_0^t \frac{1}{t-s} \left\| \bar{X}(t,x) - \bar{X}(s,x) \right\|_1 (|\bar{X}(t,x)|_1 + |\bar{X}(s,x)|_1) ds
\leq C \int_0^t \frac{1}{t-s} (t-s)^{-\frac{1+\theta}{2}} s^{-\frac{1+\theta}{2}} (t^{-\frac{1}{2} + \frac{\theta}{2}} + s^{-\frac{1}{2} + \frac{\theta}{2}}) (|x|^2 + 1)e^{C\|x\|^k} ds
\leq C t^{-1 + \frac{\theta}{2}} (|x|^2 + 1)e^{C\|x\|^k}.
\]

For } I_4, \text{ it follows from Lemma 4.1 that}
\[
\| AI_4 \| = \left\| (e^{tA} - I) \bar{f}(\bar{X}(t,x)) \right\| \leq C(1 + \| \bar{X}(t,x) \|) \leq C(1 + \| x \|).
\]
For $I_5$, using Lemma 4.2 gives
\[
\|A_5\| \leq C \int_0^t \frac{1}{t-s} \|\bar{X}(t, x) - \bar{X}(s, x)\| \, ds \leq Ct^{-\frac{1+\theta}{2}}(\|x\|_\theta + 1)e^{C\|x\|^k}. \]

The conclusion follows by the above estimates. \qed

Denote by $\eta^h(t, x)$ the derivative of $\bar{X}(t, x)$ with respect to $x$ in the direction $h$. $\eta^h(t, x)$ satisfies the following equation
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d\eta^h(t, x)}{dt} = A\eta^h(t, x) + D\bar{f}(\bar{X}(t, x)) \cdot \eta^h(t, x) + D\xi \left[ \bar{X}(t, x)\eta^h(t, x) \right] \\
\eta^h(0, x) = h.
\end{array} \right.
\end{align*}
\] (4.15)

The following three Lemmas give some bounds for $\eta^h(t, x)$.

**Lemma 4.4.** Assume the conditions A1, A2 and A4 hold.

(1) For any $t \in (0, T]$, $h \in L^2$, there exists a constant $C > 0$ such that
\[
\|\eta^h(t, x)\|^2 + \int_0^t |\eta^h(s, x)|^2 \, ds \leq Ce^{C\|x\|^k}\|h\|^2.
\] (4.16)

(2) For any $x \in H^\theta$ with $\theta \in (0, 1)$, $h \in L^2$, $\gamma \in (1, \frac{3}{2})$, $t \in (0, T]$, there exist $k \in \mathbb{N}$ and a constant $C = C_{\gamma, \theta, T}$ such that
\[
|\eta^h(t, x)| \leq Ct^{-\frac{\theta}{2}}(\|x\|_\theta + 1)e^{C\|x\|^k}\|h\|.
\] (4.17)

**Proof.** Multiplying both sides of the equation (4.15) by $\eta^h(t, x)$ and integrating with respect to $\xi$, we obtain
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\eta^h(t, x)\|^2 + |\eta^h(t, x)|^2 &= \int_0^t \left[ D\bar{f}(\bar{X}(t, x))\eta^h(t, x) \right] \eta^h(t, x) \, d\xi \\
&\quad + \int_0^t D\xi \left[ \bar{X}(t, x)\eta^h(t, x) \right] \eta^h(t, x) \, d\xi \\
&\leq C\|\eta^h(t, x)\|^2 - \int_0^t \bar{X}(t, x)\eta^h(t, x)D\xi \eta^h(t, x) \, d\xi \\
&= C\|\eta^h(t, x)\|^2 - \int_0^t \bar{X}(t, x)\eta^h(t, x) \, d\xi.
\end{align*}
\]

According to Lemma 5.2 and the interpolation inequality, it follows that
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\eta^h(t, x)\|^2 + |\eta^h(t, x)|^2 &\leq C\|\eta^h(t, x)\|^2 + \|\bar{X}(t, x)\|\|\eta^h(t, x)\| |\eta^h(t, x)|_\frac{3}{2} \\
&\leq C\|\eta^h(t, x)\|^2 + \|\bar{X}(t, x)\|\|\eta^h(t, x)\|^\frac{3}{2} |\eta^h(t, x)|_\frac{3}{2} \\
&\leq C\|\eta^h(t, x)\|^2 + \frac{1}{2} |\eta^h(t, x)|^2 + C\|\bar{X}(t, x)\|^5 |\eta^h(t, x)|^2.
\end{align*}
\]

This implies
\[
\|\eta^h(t, x)\|^2 + \int_0^t |\eta^h(s, x)|^2 \, ds \leq \|h\|^2 + C\int_0^t \left( 1 + \|\bar{X}(s, x)\|^5 \right) |\eta^h(s, x)|^2 \, ds.
\]

Then, (4.16) follows by using Gronwall’s inequality and Lemma 4.1.

To show (4.17), notice that
\[
\eta^h(t, x) = e^{tA}h + \int_0^t e^{(t-s)A}D\bar{f}(\bar{X}(s, x)) \cdot \eta^h(s, x) \, ds + \int_0^t e^{(t-s)A}D\xi \left[ \bar{X}(s, x)\eta^h(s, x) \right] \, ds.
\] (4.18)
Using (4.16), Lemmas 5.2 and 4.1, we obtain that for $\gamma \in (1, \frac{3}{2})$,
\[
|\eta^h(t, x)| \leq Ct^{-\frac{7}{2}}\|h\| + C \int_0^t (t-s)^{-\frac{7}{4}} \left| D\tilde{f}(\bar{X}(s, x)) \cdot \eta^h(s, x) \right| ds \\
+ C \int_0^t (t-s)^{-\frac{2s+1}{4}} \left| D\xi \left[ \bar{X}(s, x)\eta^h(s, x) \right] \right| ds \\
\leq Ct^{-\frac{7}{2}}\|h\| + C \int_0^t (t-s)^{-\frac{7}{2}}\|\eta^h(s, x)\| ds \\
+ C \int_0^t (t-s)^{-\frac{2s+1}{4}} \left[ |\eta^h(s, x)||\bar{X}(s, x)|_0 + \|\bar{X}(s, x)||\eta^h(s, x)|_{\gamma} \right] ds \\
\leq Ct^{-\frac{7}{2}}\|h\| + Ce^{C|x|^k}\|h\| \\
+ C \int_0^t (t-s)^{-\frac{2s+1}{4}}|\eta^h(s, x)|_{\gamma} (1 + \|x\|) ds.
\]
Consequently, by Lemma 5.5, we get (4.17). The proof is complete.

Note that from Lemma 4.4, using the interpolation inequality, we deduce that for any $\gamma \in (0, 1], \theta \in (0, 1), t \in (0, T]$, there exist $k \in \mathbb{N}$ and a constant $C = C_{\theta, T} > 0$ such that
\[
|\eta^h(t, x)|_{\gamma} \leq Ct^{-\frac{7}{2}}(1 + |x|_\theta) e^{C|x|^k}\|h\|.
\]

**Lemma 4.5.** Under the conditions A1, A2 and A4, for any $x \in H^0$ with $\theta \in (0, 1)$, $\alpha \in (0, \frac{1}{2})$, $0 < s < t \leq T$, there exist $k \in \mathbb{N}$ and a constant $C = C_{\alpha, \theta, T}$ such that
\[
|\eta^h(t, x) - \eta^h(s, x)|_1 \leq C(t-s)^{\frac{7}{2}} s^{-\frac{1+\alpha}{2}} (1 + |x|_{\theta}) e^{C|x|^k}\|h\|.
\]

**Proof.** From (4.15), we write
\[
\eta^h(t, x) - \eta^h(s, x) = (e^{(t-s)A} - I)\eta^h(s, x) + \int_s^t e^{(t-r)A}D\tilde{f}(\bar{X}(r, x)) \cdot \eta^h(r, x) dr \\
+ \int_s^t e^{(t-r)A}D\xi \left[ \bar{X}(r, x)\eta^h(r, x) \right] dr \\
= I_1 + I_2 + I_3.
\]

For $I_1$, using (2.1) and (4.17), we have
\[
|I_1| \leq C(t-s)^{\frac{7}{2}} s^{-\frac{1+\alpha}{2}} (1 + |x|_\theta) e^{C|x|^k}\|h\|.
\]

For $I_2$, using (4.16), we get
\[
|I_2| \leq \int_s^t (t-r)^{-\frac{7}{2}} \left| D\tilde{f}(\bar{X}(r, x)) \cdot \eta^h(r, x) \right| dr \leq C(t-s)^{\frac{1}{2}} e^{C|x|^k}\|h\|.
\]

For $I_3$, using (2.1), Lemmas 4.1 and 4.4, we obtain
\[
|I_3| \leq C \int_s^t \left[ 1 + (t-r)^{-\frac{7}{4}} \right] \left| D\xi \left[ \bar{X}(r, x)\eta^h(r, x) \right] \right| ds \\
\leq C \int_s^t \left[ 1 + (t-r)^{-\frac{7}{4}} \right] \left| B(\bar{X}(r, x), \eta^h(r, x)) + B(\eta^h(r, x), \bar{X}(r, x)) \right| ds \\
\leq C \int_s^t \left[ 1 + (t-r)^{-\frac{7}{4}} \right] \left( \left| \bar{X}(r, x) \right| \left| \eta^h(r, x) \right|_{1+\alpha} + \left| \bar{X}(r, x) \right|_{1+\alpha} \left| \eta^h(r, x) \right| \right) dr \\
\leq C(t-s)^{\frac{1}{2}} s^{-\frac{1+\alpha}{2}} (1 + |x|_\theta) e^{C|x|^k}\|h\|.
\]
We conclude the proof by combining (4.20)-(4.23).

**Lemma 4.6.** Under the conditions A1, A2 and A4, for any \( x \in H^\theta \) with \( \theta \in (0, 1) \), \( h \in L^2 \), \( 0 < s < t \leq T \), there exist \( k \in \mathbb{N} \) and a constant \( C = C_{\theta, T} \) such that

\[
\left\| \frac{d}{dt} \eta^h(t, x) \right\| \leq Ct^{-1}(|x|^2 + 1)e^{C\|x\|^k} \|h\|.
\]

**Proof.** We first control the first term in (4.15). Notice that

\[
\begin{align*}
\eta^h(t, x) &= e^{tA}h + \int_0^t e^{(t-s)A}D\tilde{f}(\bar{X}(t, x)) \cdot \eta^h(t, x)ds \\
&+ \int_0^t e^{(t-s)A} [D\tilde{f}(\bar{X}(s, x)) \cdot \eta^h(s, x) - D\tilde{f}(\bar{X}(t, x)) \cdot \eta^h(t, x)] ds \\
&+ \int_0^t e^{(t-s)A} D_\xi [\bar{X}(t, x)\eta^h(t, x)] ds \\
&+ \int_0^t e^{(t-s)A} \{D_\xi [\bar{X}(s, x)\eta^h(s, x)] - D_\xi [\bar{X}(t, x)\eta^h(t, x)]\} ds \\
= I_1 + I_2 + I_3 + I_4 + I_5.
\end{align*}
\]

For \( I_1 \), we have

\[
\|AI_1\| \leq C t^{-1} \|h\|. \tag{4.24}
\]

For \( I_2 \), it follows from condition A4 that

\[
\|AI_2\| = \|(e^{tA} - I)D\tilde{f}(\bar{X}(t, x)) \cdot \eta^h(t, x)\| \leq 2 \|D\tilde{f}(\bar{X}(t, x)) \cdot \eta^h(t, x)\| \leq Ce^{C\|x\|^k} \|h\|. \tag{4.25}
\]

For \( I_3 \), according to Lemmas 4.2 and 4.5, we obtain

\[
\begin{align*}
\|AI_3\| &\leq C \int_0^t \frac{1}{t-s} \left\| D\tilde{f}(\bar{X}(t, x)) \cdot \eta^h(t, x) - D\tilde{f}(\bar{X}(s, x)) \cdot \eta^h(s, x) \right\| ds \\
&\leq C \int_0^t \frac{1}{t-s} \left\| [D\tilde{f}(\bar{X}(t, x)) - D\tilde{f}(\bar{X}(s, x))] \cdot \eta^h(t, x) \right\| ds \\
&+ C \int_0^t \frac{1}{t-s} \| D\tilde{f}(\bar{X}(s, x)) \cdot (\eta^h(t, x) - \eta^h(s, x)) \| ds \\
&\leq C \int_0^t \frac{1}{t-s} \|\bar{X}(t, x) - \bar{X}(s, x)\| e^{C\|x\|^k} \|h\| ds + C \int_0^t \frac{1}{t-s} \|\eta^h(t, x) - \eta^h(s, x)\| ds \\
&\leq Ct^{-\frac{s}{2}}(1 + |x|^2)e^{C\|x\|^k} \|h\|. \tag{4.26}
\end{align*}
\]

For \( I_4 \), we deduce from Lemma 5.2, (4.9) and (4.19) that

\[
\begin{align*}
\|AI_4\| &= \|(e^{tA} - I)D_\xi [\bar{X}(t, x)\eta^h(t, x)]\| \\
&\leq 2 \|B(\bar{X}(t, x), \eta^h(t, x)) + B(\eta^h(t, x), \bar{X}(t, x))\| \\
&\leq C \|\bar{X}(t, x)\|_1 \|\eta^h(t, x)\|_1 \leq Ct^{-1+\frac{\alpha}{2}}(|x|^2 + 1)e^{C\|x\|^k} \|h\|. \tag{4.27}
\end{align*}
\]
For $I_5$, using Lemmas 4.2 and 4.5, (4.9) and (4.19), we get
\[
\|AI_5\| \leq C \int_0^t \frac{1}{t-s} \left\| B \left( \bar{X}(s, x), \eta^h(t, x) - \eta^h(s, x) \right) + B \left( \eta^h(t, x), \bar{X}(s, x) \right) \right\| ds 
+ C \int_0^t \frac{1}{t-s} \left( |\bar{X}(s, x)| \right)_1 \|\eta^h(t, x) - \eta^h(s, x)\|_1 + \|\eta^h(t, x) - \eta^h(s, x)\|_1 \right) ds 
\leq C t^{-1+\frac{2\beta}{\gamma}} (|x|^2 + 1)e^{C\|x\|^k} \|h\|. 
\] (4.28)

Putting (4.24)-(4.28) together, we get
\[
\|A\eta^h(t, x)\| \leq C t^{-1}(|x|^2 + 1)e^{C\|x\|^k} \|h\|. 
\] (4.29)

For the second term in (4.15), we have
\[
\|D\hat{f}(\bar{X}(t, x)) \cdot \eta^h(t, x)\| \leq Ce^{C\|x\|^k} \|h\|. 
\] (4.30)

For the third term in (4.15), it follows by (4.27) that
\[
\left\| D_{\xi} \left[ \bar{X}(t, x) \eta^h(t, x) \right] \right\| \leq C t^{-1+\frac{2\beta}{\gamma}} (|x|^2 + 1)e^{C\|x\|^k} \|h\|. 
\] (4.31)

The result follows by combining (4.29)-(4.31).

Denote by $\zeta^{h,k}(t, x)$ the second derivative of $\bar{X}(t, x)$ with respect to $x$ towards the directions $h, k \in L^2$. Then, $\zeta^{h,k}(t, x)$ satisfies
\[
\frac{d\zeta^{h,k}(t, x)}{dt} = A\zeta^{h,k}(t, x) + D^2\hat{f}(\bar{X}(t, x)) \cdot (\eta^h(t, x), \eta^k(t, x)) + D\hat{f}(\bar{X}(t, x)) \cdot \zeta^{h,k}(t, x) 
+ D_{\xi} \left[ \eta^k(t, x)\eta^h(t, x) \right] + D_{\xi} \left[ \bar{X}(t, x) \zeta^{h,k}(t, x) \right]. 
\] (4.32)

The following lemma gives a control of $\zeta^{h,k}(t, x)$.

**Lemma 4.7.** Under the conditions **A1, A2 and A4**, for any $x \in H^\theta$ with $\theta \in (0, 1)$, $h, k \in L^2$, $t \in (0, T]$, there exists a constant $C > 0$ such that
\[
\left\| \zeta^{h,k}(t, x) \right\| \leq C e^{C\|x\|^k} \|h\| \|k\|. 
\]

**Proof.** Multiply both sides of the equation (4.32) by $\zeta^{h,k}(t, x)$, integrate with respect to $\xi$, then we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\zeta^{h,k}(t, x)\|^2 + |\zeta^{h,k}(t, x)|_1^2 = \int_0^1 \left[ D^2\hat{f}(\bar{X}(t, x)) \cdot (\eta^h(t, x), \eta^k(t, x)) \right] \zeta^{h,k}(t, x) d\xi 
+ \int_0^1 \left[ D\hat{f}(\bar{X}(t, x)) \cdot \zeta^{h,k}(t, x) \right] \zeta^{h,k}(t, x) d\xi 
+ \int_0^1 \left\{ D_{\xi} \left[ \eta^k(t, x)\eta^h(t, x) \right] \right\} \zeta^{h,k}(t, x) d\xi 
+ \int_0^1 \left\{ D_{\xi} \left[ \bar{X}(t, x)\zeta^{h,k}(t, x) \right] \right\} \zeta^{h,k}(t, x) d\xi 
= I_1 + I_2 + I_3 + I_4. 
\] (4.33)

For $I_1$, using (4.16), we get
\[
\|I_1\| \leq C e^{C\|x\|^k} \|h\|^2 \|k\|^2 + \|\zeta^{h,k}(t, x)\|^2. 
\] (4.34)
For $I_2$, we have
\[ \|I_2\| \leq C \|\zeta^{h,k}(t,x)\|^2. \] (4.35)

For $I_3$, according to Lemma 5.2, we get
\[ \|I_3\| \leq \|b(\eta^k(t,x), \eta^h(t,x), \zeta^{h,k}(t,x)) + \eta^h(t,x), \eta^k(t,x), \zeta^{h,k}(t,x)\| \leq \left( \|\eta^k(t,x)\| \|\eta^h(t,x)\|_1 + \|\eta^h(t,x)\| \|\eta^k(t,x)\|_1 + \|\eta^k(t,x)\| \|\eta^h(t,x)\|_1 \right) \|\zeta^{h,k}(t,x)\|_1 \leq C \left( \|\eta^k(t,x)\| \|\eta^h(t,x)\|_1 + \|\eta^h(t,x)\| \|\eta^k(t,x)\|_1 \right)^2 + \frac{1}{4} \|\zeta^{h,k}(t,x)\|^2. \] (4.36)

For $I_4$, using Lemma 5.2 and the interpolation inequality, we obtain
\[ \|I_4\| = \left| \int_0^1 \left[ \ddot{X}(t,x)\zeta^{h,k}(t,x) \right] D_x \zeta^{h,k}(t,x) \right| \leq \left| b(\ddot{X}(t,x), \zeta^{h,k}(t,x), \zeta^{h,k}(t,x)) \right| \leq C \|\ddot{X}(t,x)\| \|\zeta^{h,k}(t,x)\|_1 \|\zeta^{h,k}(t,x)\|_2 \leq C \left( 1 + \|x\|^5 \right) \|\zeta^{h,k}(t,x)\|^2 + \frac{1}{4} \|\zeta^{h,k}(t,x)\|^2. \] (4.37)

Then, combining (4.33)-(4.37), using Lemma 4.4, we get
\[ \|\zeta^{h,k}(t,x)\|^2 \leq Ce^{C\|x\|^5} \|h\|^2 \|k\|^2 + C \sup_{0 \leq s \leq T} \|\eta^k(s,x)\|^2 \int_0^t \|\eta^h(s,x)\|^2 ds \quad + C \sup_{0 \leq s \leq T} \|\eta^h(s,x)\|^2 \int_0^t \|\eta^k(s,x)\|_2^2 ds + C \int_0^t (1 + \|x\|^5) \|\zeta^{h,k}(s,x)\|^2 ds \leq Ce^{C\|x\|^5} \|h\|^2 \|k\|^2 + C \int_0^t (1 + \|x\|^5) \|\zeta^{h,k}(s,x)\|^2 ds. \]

The desired result follows by using Gronwall’s inequality.

\[ \square \]

4.2. Properties of $(X_N^\varepsilon, Y_N^\varepsilon)$. This subsection is to establish some properties of $(X_N^\varepsilon, Y_N^\varepsilon)$. For simplicity, we omit the index $N$.

Lemma 4.8. Assume the conditions A1, A2 and A4 hold.
(1) For any $x \in L^2$, $t \in [0,T]$, there exists a constant $C > 0$ such that
\[ \|X_1^\varepsilon\| \leq C(1 + \|x\|). \] (4.38)

(2) For any $x \in H^\theta$ with $\theta \in (0,1)$, $y \in L^2$, $\gamma \in (1, \frac{3}{2})$, $p \geq 1$, $t \in (0,T]$, there exist $k \in \mathbb{N}$ and a constant $C = C_{p,\theta,\gamma,T}$ such that
\[ \mathbb{E}\|X_1^\varepsilon\|_2^p \leq Ct \frac{p(\gamma - \theta)}{2} (1 + |x|_p^p + \|y\|^p) e^{C\|x\|^k}. \] (4.39)

Proof. Recall that
\[ \frac{d}{dt} X_1^\varepsilon = AX_1^\varepsilon + B(X_1^\varepsilon) + f(X_1^\varepsilon, Y_1^\varepsilon). \]

Multiplying both sides of the above equation by $2X_1^\varepsilon$ and using (2.7) in A4, we obtain
\[ \frac{1}{2} \frac{d}{dt} \|X_1^\varepsilon\|^2 = -\|X_1^\varepsilon\|^2 + \langle f(X_1^\varepsilon, Y_1^\varepsilon), X_1^\varepsilon \rangle \leq C(1 + \|X_1^\varepsilon\|^2). \]
Then, (4.38) follows by using Gronwall’s inequality.

To show (4.39), recall that
\[ X_t^\varepsilon = e^{tA}x + \int_0^t e^{(t-s)A}B(X_s^\varepsilon)ds + \int_0^t e^{(t-s)A}f(X_s^\varepsilon, Y_s^\varepsilon)ds := I_1 + I_2 + I_3. \] (4.40)

For \( I_1 \), (2.1) implies
\[ |e^{tA}x|_\gamma \leq Ct^{-\frac{\theta}{2}}|x|_\theta. \] (4.41)

For \( I_2 \), similarly as the proof of Lemma 4.1, using (4.38), we obtain
\[ \left| \int_0^t e^{(t-s)A}B(X_s^\varepsilon)ds \right|_\gamma \leq C \int_0^t (t-s)^{-\frac{1+2\gamma}{4}}(1 + \|x\|)\|X_s^\varepsilon\|_\gamma ds. \] (4.42)

For the last term, we get
\[ \left| \int_0^t e^{(t-s)A}f(X_s^\varepsilon, Y_s^\varepsilon)ds \right|_\gamma \leq C \int_0^t (t-s)^{-\frac{\gamma}{2}}(1 + \|X_s^\varepsilon\| + \|Y_s^\varepsilon\|)ds. \] (4.43)

Combining (4.40)-(4.43), it follows from the Minkowski inequality that for any \( p > 1 \),
\[ \left[ \mathbb{E}|X_t^\varepsilon|_{\gamma}^p \right]^{1/p} \leq Ct^{-\frac{\theta}{2}}|x|_\theta + C \int_0^t (t-s)^{-\frac{1+2\gamma}{4}}(1 + \|x\|) \left[ \mathbb{E}|X_s^\varepsilon|_{\gamma}^p \right]^{1/p} ds + (1 + \|x\| + \|y\|). \]

Using Lemma 5.5, we get (4.39). \( \square \)

Similarly to (4.9), for any \( x \in H^\theta \) with \( \delta \in (0, \frac{1}{4}) \), \( \gamma \in (0, 1], p \geq 1, t \in (0, T] \), there exist \( k \in \mathbb{N} \) and a constant \( C = C_{\theta, \alpha, T} \) such that
\[ \mathbb{E}|X_t^\varepsilon|_{\gamma}^p \leq Ct^{-\frac{\theta}{2}\left(1+\frac{\delta}{1+\delta}\right)}\left(1 + |x|_\theta^p + \|y\|^p\right)e^{C\|x\|^k}. \] (4.44)

**Lemma 4.9.** Assume the conditions A1, A2 and A4 hold.

(1) For any \( x \in H^\theta \) with \( \theta \in (0, 1), y \in L^2, \alpha \in (0, \frac{1}{4}), 0 < s < t \leq T \), there exist \( k \in \mathbb{N} \) and a constant \( C = C_{\theta, \alpha, T} \) such that
\[ \left[ \mathbb{E}|X_t^\varepsilon - X_s^\varepsilon|_1 \right]^{1/4} \leq C(t-s)^{\frac{\alpha}{2} - \frac{\theta}{2}}\left(\|x\| + \|y\| + 1\right)e^{C\|x\|^k}. \]

(2) For any \( x, y \in L^2, \alpha \in (0, \frac{1}{4}), 0 < s < t \leq T \), there exists a constant \( C > 0 \) such that
\[ \mathbb{E}\|Y_t^\varepsilon - Y_s^\varepsilon\|^2 \leq C(\|x\|^2 + \|y\|^2 + 1)\left(s^{-2\alpha} + \varepsilon^{-2\alpha}\right)(t-s)^{2\alpha}. \]

**Proof.** The proof of (1) can be carried out in the same way as in Lemma 4.2. For the proof of (2), since the approach here is almost the same as the one in [3], we refer to [3, Proposition A.4] for details. \( \square \)

**Lemma 4.10.** Under the conditions A1, A2 and A4, for any \( x \in H^\theta \) with \( \theta \in (0, 1), \alpha \in (0, \frac{1}{4}), 0 < t \leq T \), there exist \( k \in \mathbb{N} \) and a constant \( C = C_{\theta, \alpha, T} \) such that
\[ \left[ \mathbb{E}\|AX_t^\varepsilon\|^2 \right]^{\frac{1}{2}} \leq Ct^{-\frac{\theta}{2}}(|x|_\theta^2 + \|y\|^2 + 1)e^{C\|x\|^k} + C\varepsilon^{-\alpha}. \]

**Proof.** For \( 0 < s < t \), we write
\[ X_t^\varepsilon = e^{tA}x + \int_0^t e^{(t-s)A}B(X_s^\varepsilon)ds + \int_0^t e^{(t-s)A}[B(X_s^\varepsilon) - B(X_t^\varepsilon)]ds \]
\[ + \int_0^t e^{(t-s)A}f(X_s^\varepsilon, Y_s^\varepsilon)ds + \int_0^t e^{(t-s)A}[f(X_s^\varepsilon, Y_s^\varepsilon) - f(X_t^\varepsilon, Y_t^\varepsilon)]ds \]
\[ = I_1 + I_2 + I_3 + I_4 + I_5. \]
For $I_1$, using (2.1), we have
\[
\|Ae^{tA}x\| \leq Ct^{-1+\frac{\delta}{2}}|x|_\theta^2. \tag{4.45}
\]

For $I_2$, taking $p = 4$, $\gamma = 1$ and $\delta = 1/4$ in (4.44), we get
\[
\left[\mathbb{E}\|AI_2\|^2\right]^{1/2} = \mathbb{E}\left\|(e^{tA} - I)B(X^\varepsilon_t)\right\|^2 \leq C \mathbb{E}|X^\varepsilon_t|_1^4 \leq C t^{-1+\frac{4\delta}{5}}(1 + |x|_\theta^2 + \|y\|^2)C|x||^k. \tag{4.46}
\]

For $I_3$, using Lemma 5.4, we obtain
\[
|AI_3| \leq C \int_0^t \frac{1}{t-s} |B(X^\varepsilon_s) - B(X^\varepsilon_t)| ds \leq C \int_0^t \frac{1}{t-s} |X^\varepsilon_t - X^\varepsilon_s|_1(|X^\varepsilon_t|_1 + |X^\varepsilon_s|_1) ds.
\]

According to the Minkowski inequality, by Lemmas 4.8 and 4.9, it follows that
\[
\left[\mathbb{E}\|AI_3\|^2\right]^{1/2} \leq C \mathbb{E} \int_0^t \frac{1}{t-s} |X^\varepsilon_t - X^\varepsilon_s|_1(|X^\varepsilon_t|_1 + |X^\varepsilon_s|_1) ds
\]
\[
\leq C \int_0^t \frac{1}{t-s} \left(\mathbb{E} |X^\varepsilon_t - X^\varepsilon_s|_1(|X^\varepsilon_t|_1 + |X^\varepsilon_s|_1)\right)^{1/2} ds
\]
\[
\leq C \int_0^t \frac{1}{t-s} \left(\mathbb{E} |X^\varepsilon_t - X^\varepsilon_s|^4_1\right)^{1/4} ds
\]
\[
\leq C t^{-1+\frac{4\delta}{5}}(|x|_\theta^2 + \|y\|^2 + 1)e^{C|x||^k}. \tag{4.47}
\]

For $I_4$, we have
\[
\mathbb{E}\|AI_4\| = \mathbb{E}\left\|(e^{tA} - I)f(X^\varepsilon_t, Y^\varepsilon_t)\right\| \leq C(1 + \mathbb{E}|X^\varepsilon_t| + \mathbb{E}|Y^\varepsilon_t|) \leq C(1 + \|x\| + \|y\|). \tag{4.48}
\]

For $I_5$, using the Minkowski inequality and Lemma 4.9, we obtain
\[
\left[\mathbb{E}\|AI_5\|^2\right]^{1/2} \leq C \int_0^t \frac{1}{t-s} \left[\left(\mathbb{E} |X^\varepsilon_t - X^\varepsilon_s|^2\right)^{1/2} + \left(\mathbb{E} |Y^\varepsilon_t - Y^\varepsilon_s|^2\right)^{1/2}\right] ds
\]
\[
\leq C t^{-\frac{1+\frac{4\delta}{5}}{2}}(|x|_\theta + \|y\| + 1)e^{C|x||^k} + C\varepsilon^{-\alpha}. \tag{4.49}
\]

Combining (4.45)-(4.49) yields the desired result. \qed

**Lemma 4.11.** Under the conditions $A_1$, $A_2$ and $A_4$, for any $x \in H^\theta$ with $\theta \in (0, 1)$, $\alpha \in (0, \frac{1}{2})$, $0 < t \leq T$, there exist $k \in \mathbb{N}$ and a constant $C = C_{\theta, \alpha, T}$ such that
\[
\mathbb{E}\left\|\frac{d}{dt}X^\varepsilon_t\right\| \leq Ct^{-1+\frac{\delta}{2}}(|x|_\theta^2 + \|y\|^2 + 1)e^{C|x||^k} + C\varepsilon^{-\alpha}.
\]

**Proof.** Recall that
\[
\frac{d}{dt}X^\varepsilon_t = AX^\varepsilon_t + B(X^\varepsilon_t) + f(X^\varepsilon_t, Y^\varepsilon_t).
\]

Choosing $p = 2$, $\gamma = 1$ and $\delta = \frac{1}{4}$ in (4.44), we get $\mathbb{E}\|B(X^\varepsilon_t)\| \leq C \mathbb{E}|X^\varepsilon_t|_1^2 \leq Ct^{-1+\frac{4\delta}{5}}(|x|_\theta^2 + \|y\|^2 + 1)$. It is clear that $\mathbb{E}\|f(X^\varepsilon_t, Y^\varepsilon_t)\| \leq C(1 + \|x\| + \|y\|)$. By Lemma 4.10, we have
\[
\mathbb{E}\|AX^\varepsilon_t\| \leq \left[\mathbb{E}\|AX^\varepsilon_t\|^2\right]^{1/2} \leq Ct^{-1+\frac{\delta}{2}}(|x|_\theta^2 + \|y\|^2 + 1)e^{C|x||^k} + C\varepsilon^{-\alpha}.
\]

The proof is complete. \qed
4.3. The asymptotic expansion of $\mathbb{E} [\phi(X_N^\varepsilon(t))]$. For any $x, y \in L^2$ and $t \geq 0$, set

$$u_N^\varepsilon(t, x, y) = \mathbb{E} [\phi(X_N^\varepsilon(t, x, y))]$$

and

$$\bar{u}_N(t, x) = \phi(X_N(t, x)).$$

In this subsection, we shall find an asymptotic expansion of $u_N^\varepsilon$ with respect to $\varepsilon$:

$$u_N^\varepsilon = u_0^N + \varepsilon u_1^N + v_N^\varepsilon,$$

where $v_N^\varepsilon$ is a residual term, $u_0^N$ and $u_1^N$ will be constructed below.

For any $\psi(x, y) : L^2 \times L^2 \to \mathbb{R}$ with $\psi \in C^2$, we introduce the differential operators:

$$L_1^N \psi(x, y) = \langle A_N x + B_N(x) + f_N(x, y), D_x \psi(x, y) \rangle,$$

$$L_2^N \psi(x, y) = \langle A_N x + g_N(x, y), D_y \psi(x, y) \rangle + \frac{1}{2} \text{Tr} \left(D_y^2 \psi(x, y) \right).$$

For any $\psi : L^2 \to \mathbb{R}$ with $\psi \in C^1$, denote by

$$\bar{L}^N \psi(x) = \langle A_N x + B_N(x) + \bar{f}_N(x), D_x \psi(x) \rangle.$$

Set

$$L_N^\varepsilon = L_1^N \varepsilon + \frac{1}{\varepsilon} L_2^N.$$  

For simplicity, we omit the index $N$. Notice that $\bar{u}$ does not depend on $y$. It is well known that $u^\varepsilon$ and $\bar{u}$ satisfy the following Kolmogorov equations:

$$\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u^\varepsilon(t, x, y)}{\partial t} = L^\varepsilon u^\varepsilon(t, x, y) \\
u^\varepsilon(0, x, y) = \phi(x),
\end{array} \right.
\end{align*}$$

and

$$\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial \bar{u}(t, x)}{\partial t} = \bar{L} \bar{u}(t, x) \\
\bar{u}(0, x) = \phi(x),
\end{array} \right.
\end{align*}$$

respectively. Using (4.52)-(4.54), we have

$$\frac{\partial u_0}{\partial t} + \varepsilon \frac{\partial u_1}{\partial t} + \frac{\partial v^\varepsilon}{\partial t} = L_1 u_0 + \frac{1}{\varepsilon} L_2 u_0 + \varepsilon L_1 u_1 + L_2 u_1 + L_1 v^\varepsilon + \frac{1}{\varepsilon} L_2 v^\varepsilon,$$

which implies

$$\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u_0}{\partial t} = 0 \\
\frac{\partial u_1}{\partial t} = L_1 u_0 + L_2 u_1 \\
\frac{\partial v^\varepsilon}{\partial t} = L^\varepsilon v^\varepsilon + \varepsilon (L_1 u_1 - \frac{\partial u_1}{\partial t}).
\end{array} \right.
\end{align*}$$

To obtain $u_0$ and $u_1$, we need the following lemma which is similar to [3, Lemma 4.3]. However, instead of imposing the condition that $f$ is bounded, the coefficient $f$ in our paper is Lipschitz.

**Lemma 4.12.** Assume the conditions A1, A2 and A4 hold. Fix $x \in L^2$.

1. If $\Psi$ is a Lipschitz continuous function and $\Phi$ is a function of class $C^2$ satisfying $L_2 \Phi = -\Psi$, then for any $y \in L^2$, we have

$$\Phi(y) = \int_{L^2} \Phi(z) \mu^\varepsilon(dz) + \int_0^{+\infty} \mathbb{E} [\Psi(Y_{s,x}^y)] ds.$$
(2) Suppose that $\Psi$ is a Lipschitz continuous function of class $C^2$ such that $\int_{L^2} \Psi(y)\mu^x(dy) = 0$. Let $\Phi(y) = \int_0^{\infty} \mathbb{E}[\Psi(Y_s^{x,y})] ds$. Then, $\Phi$ is of class $C^2$ and satisfies $L_2\Phi = -\Psi$. Moreover, there exists a constant $C$ which is independent of $N$ such that for any $y \in L^2$

$$|\Phi(y)| \leq C(1 + \|x\| + \|y\|)|\Psi|_{\text{Lip}}.$$  \hspace{1cm} (4.57)

Proof. Since the proof of (1) and the proof of the first part of (2) are similar to [3, Lemma 4.3], we omit the details. Here we only give a proof of (4.57).

For any $y \in L^2$, Proposition 3.9 implies

$$\left| \mathbb{E}[\Psi(Y_s^{x,y})] - \int_{L^2} \Psi(z)\mu^x(dz) \right| \leq C(1 + \|x\| + \|y\|)e^{-\frac{(s-L_2)^2}{2}}|\Psi|_{\text{Lip}}.$$ \hspace{1cm} (4.58)

Noting that $\int_{L^2} \Psi(y)\mu^x(dy) = 0$, we obtain (4.57) by integrating (4.58) with respect to $s$. The proof is complete. \hfill \Box

It follows from (4.56) that the function $u_0$ is independent of $y$, thus we can write $u_0(t, x, y) = u_0(t, x)$. We also choose the initial condition $u_0(0, x) = \phi(x)$. In view of the second equation in (4.56) and noting that $\int_{L^2} L_2u_1(t, x, y)\mu^x(dy) = 0$, we have

$$\frac{\partial u_0}{\partial t}(t, x) = \int_{L^2} \frac{\partial u_0}{\partial t}(t, x)\mu^x(dy)$$

$$= \int_{L^2} L_1 u_0(t, x)\mu^x(dy) + \int_{L^2} L_2 u_1(t, x, y)\mu^x(dy)$$

$$= \langle Ax + B(x) + \int_{L^2} f(x, y)\mu^x(dy), D_x u_0(t, x) \rangle$$

$$= \bar{L} u_0(t, x).$$

This, together with the uniqueness of the solution to (4.55), implies that $u_0 = \bar{u}$.

From $\bar{L} u_0 = L_1 u_0 + L_2 u_1$ and the definitions of $\bar{L}$ and $L_1$, we deduce that

$$L_2 u_1(t, x, y) = (\bar{f}(x) - f(x, y), D_x u_0(t, x)) = -\chi(t, x, y),$$

where $\chi$ is of class $C^2_b$ with respect to $y$, and satisfies that for any $t \geq 0$ and $x \in L^2$,

$$\int_{L^2} \chi(t, x, y)\mu^x(dy) = 0.$$  \hspace{1cm} (4.59)

According to Lemma 4.12, we obtain

$$u_1(t, x, y) = \int_0^{+\infty} \mathbb{E}[\chi(t, x, Y_{s,x}^{x,y})] ds.$$ \hspace{1cm} (4.60)

In what follows, we are going to show the regularity of $u_1$ with respect to $t$ and $x, y$. In order to avoid the non-integrability at $t = 0$, we introduce a parameter $\rho(\varepsilon) = \varepsilon^{-\frac{1}{2}}, 0 < a \leq \frac{\alpha}{2}$. By the third equation of (4.56) and Itô’s formula, we have

$$v^\varepsilon(t, x, y) = \mathbb{E}[v^\varepsilon(\rho(\varepsilon), X^\varepsilon(t - \rho(\varepsilon), x, y), Y^\varepsilon(t - \rho(\varepsilon), x, y))]$$

$$+ \varepsilon \mathbb{E} \left[ \int_{\rho(\varepsilon)}^t \left( L_1 u_1 - \frac{\partial u_1}{\partial s}(s, X^\varepsilon(t - s, x, y), Y^\varepsilon(t - s, x, y)) \right) ds \right].$$

Using the expansion (4.52) and the fact $u_0 = \bar{u}$, we get

$$u^\varepsilon(t, x, y) - \bar{u}(t, x, y)$$

$$= \varepsilon u_1(t, x, y) + \mathbb{E}[v^\varepsilon(\rho(\varepsilon), X^\varepsilon(t - \rho(\varepsilon), x, y), Y^\varepsilon(t - \rho(\varepsilon), x, y))]$$

$$+ \varepsilon \mathbb{E} \left[ \int_{\rho(\varepsilon)}^t \left( L_1 u_1 - \frac{\partial u_1}{\partial s}(s, X^\varepsilon(t - s, x, y), Y^\varepsilon(t - s, x, y)) \right) ds \right].$$ \hspace{1cm} (4.60)
Hence, it remains to control each terms in (4.60), which will be shown in the next subsection.

4.4. Estimates of $u_1, v^\varepsilon, L_1u_1$ and $\frac{\partial u_1}{\partial t}$. Note that the index $N$ is omitted in the equations (4.1) and (4.2).

**Lemma 4.13.** Under the conditions $A_1, A_2$ and $A_4$, there exists a constant $C > 0$ such that for any $0 \leq t \leq T$, $x, y \in L^2$, we have

$$|u_1(t, x, y)| \leq Ce^{C|x|^5}(1 + \|y\|).$$

**Proof.** By (4.59) and Lemma 4.12, we have

$$|u_1(t, x, y)| \leq C(1 + \|x\| + \|y\|)|y \mapsto \chi(t, x, y)|_{\operatorname{Lip}},$$

where

$$\chi(t, x, y) = \langle f(x, y) - \bar{f}(x), D_xu_0(t, x) \rangle.$$

Noting that $f$ is Lipschitz, we get that for any $y_1, y_2 \in L^2$,

$$|\chi(t, x, y_1) - \chi(t, x, y_2)| \leq C\|y_1 - y_2\||D_xu_0(t, x)||.$$

To bound $\|D_xu_0(t, x)\|$, recalling that $u_0 = \bar{u}$ and $\bar{u}(t, x) = \phi(\bar{X}(t, x))$, we have

$$D_xu_0(t, x) = D\phi(\bar{X}(t, x)) \cdot \eta^h(t, x),$$

where $\eta^h(t, x) = D_x\bar{X}(t, x) \cdot h$. Finally the conclusion follows by using Lemma 4.4. \qed

**Lemma 4.14.** Under the conditions $A_1, A_2$ and $A_4$, for any $x \in H^\theta$ with $\theta \in (0, 1]$, $y \in L^2$, $0 \leq t \leq T$, there exist $k \in \mathbb{N}$ and a constant $C = C_0, T$ such that

$$\left|\frac{\partial u_1}{\partial t}(t, x, y)\right| \leq Ct^{-1}(1 + |x|^{2\theta})e^{C\|x\|^k}(1 + \|y\|).$$

**Proof.** By the definition of $u_1$, we have

$$\frac{\partial u_1}{\partial t}(t, x, y) = \int_0^{+\infty} \mathbb{E}\left[\frac{\partial \chi}{\partial t}(t, x, Y_s^x, y)\right]ds,$$

where

$$\frac{\partial \chi}{\partial t}(t, x, y) = \langle f(x, y) - \bar{f}(x), \frac{\partial}{\partial t}D_xu_0(t, x) \rangle.$$

Using Lemma 4.12, we get

$$\left|\frac{\partial u_1}{\partial t}(t, x, y)\right| \leq C(1 + \|x\| + \|y\|)|y \mapsto \frac{\partial \chi}{\partial t}(t, x, y)|_{\operatorname{Lip}}.$$ (4.61)

It follows that for any $y_1, y_2 \in L^2$,

$$\left|\frac{\partial \chi}{\partial t}(t, x, y_1) - \frac{\partial \chi}{\partial t}(t, x, y_2)\right| \leq \|f(x, y_1) - f(x, y_2)\|\left|\frac{\partial}{\partial t}D_xu_0(t, x)\right|$$

$$\leq C\|y_1 - y_2\|\left|\frac{\partial}{\partial t}D_xu_0(t, x)\right|. \quad (4.62)$$
According to Lemmas 4.3, 4.4 and 4.6, for any $h \in L^2$, there exist $k \in \mathbb{N}$ and a constant $C = C_{\theta,T}$ such that

$$
\left| \frac{\partial}{\partial t} D_x u_0(t,x,h) \right| = \left| \frac{\partial}{\partial t} \left[ D\phi(\bar{X}(t,x)) \cdot \eta^h(t,x) \right] \right| \\
\leq \left| \partial^2_x \phi(\bar{X}(t,x)) \left( \eta^h(t,x), \frac{d}{dt} \bar{X}(t,x) \right) \right| + \left| D\phi(\bar{X}(t,x)) \cdot \frac{d}{dt} \eta^h(t,x) \right| \\
\leq C t^{-1+\frac{4}{T}} \|1 + |x|^2\| \|\eta^h(t,x)\| + C t^{-1} \| \|x|^2\| \|\eta^h(t,x)\|,
$$

Hence, we obtain

$$
\left| \frac{\partial}{\partial t} D_x u_0(t,x) \right| \leq C t^{-1} \| \|x|^2\| \|\eta^h(t,x)\|.
$$

The result follows by combining (4.61)-(4.63).

Lemma 4.15. Under the conditions A1, A2 and A4, there exists a positive constant $C$ such that for any $0 \leq t \leq T$, $x \in H^2$, $y \in L^2$,

$$
|L_1 u_1(t,x,y)| \leq C e^{|x||y|^5} (1 + \|y\|)(1 + |x|^2 + \|y\| + \|Ax\|).
$$

Proof. By the definition of $L_1$, we have

$$
L_1 u_1(t,x,y) = \langle Ax + B(x) + f(x,y), D_x u_1(t,x,y) \rangle.
$$

It is easy to see that

$$
|Ax + B(x) + f(x,y)| \leq C (1 + |x|^2 + \|y\| + \|Ax\|).
$$

Thus it remains to estimate $D_x u_1(t,x,y)$. Recall that $u_1$ satisfies

$$
L_2 u_1(t,x,y) = -\chi(t,x,y).
$$

For fixed $r > 0$, $h \in L^2$, define $\tilde{u}(t,x,y) := \frac{u_1(t,x+rh,y)-u_1(t,x,y)}{r}$. It follows that

$$
L_2 \tilde{u}(t,x,y) = -\frac{\chi(t,x+rh,y)-\chi(t,x,y)}{r} - \frac{\langle g(x+rh,y) - g(x,y), D_y u_1(t,x+rh,y) \rangle}{r} := -\Gamma(t,x,y,h,r).
$$

According to Lemma 4.12, we obtain

$$
\frac{u_1(t,x+rh,y)-u_1(t,x,y)}{r} = \int_H \frac{u_1(t,x+rh,y)-u_1(t,x,y)}{r} \mu(x) (dy) \\
= \int_0^{+\infty} E[\Gamma(t,x,Y_{s}^{x,y},h,r)] ds.
$$

In the same way as in the argument in [3, Section 5.3], as $r \to 0$, we deduce that

$$
\lim_{r \to 0} \int_{L^2} \frac{u_1(t,x+rh,y)-u_1(t,x,y)}{r} \mu(x) (dy) \leq C (1 + \|x\|) \|h\|
$$

and

$$
\lim_{r \to 0} \Gamma(t,x,y,h,r) = \Theta(t,x,y) \cdot h := D_x \chi(t,x,y) \cdot h + \langle D_x g(x,y) \cdot h, D_y u_1(t,x,y) \rangle.
$$
On one hand, by the definition of $\chi$, we have

$$D_x \chi(t, x, y) \cdot h = \langle (D_x f(x, y) - D_x \bar{f}(x)) \cdot h, D_x u_0(t, x) \rangle + D_{xx}^2 u_0(t, x) \cdot (h, f(x, y)).$$

Using condition A4, Lemmas 4.4 and 4.7, it follows that

$$|\langle (D_x f(x, y) - D_x \bar{f}(x)) \cdot h, D_x u_0(t, x) \rangle| \leq Ce^{C|x|^5} \|h\|$$

and

$$|D_{xx}^2 u_0(t, x) \cdot (h, k)| \leq Ce^{C|x|^5} \|h\| \|k\|.$$ Then we obtain

$$|D_x \chi(t, x, y) \cdot h| \leq Ce^{C|x|^5} \|h\| + Ce^{C|x|^5} \|h\| \cdot |f(x, y)| \leq Ce^{C|x|^5} (1 + \|y\|) \|h\|. \quad (4.66)$$

On the other hand, using (2.7) and following the argument in [3, Lemma 4.3], we have

$$|\langle D_x g(x, y) \cdot h, D_y u_1(t, x, y) \rangle| \leq C \|h\| \|D_y u_1(t, x, y)\| \leq C (1 + \|y\|^2) \|h\|. \quad (4.67)$$

Therefore, putting (4.66) and (4.67) together, we get

$$|\Theta(t, x, y) \cdot h| \leq Ce^{C|x|^5} (1 + \|y\|^2) \|h\|. \quad (4.68)$$

Notice that for any $t, x, r, h$, by the definition of $\Gamma$, we have $\int_{L_2} \Gamma(t, x, y, h) \mu^x(dy) = 0$, which implies $\int_{L_2} \Theta(t, x, y) \cdot h \mu^x(dy) = 0$ by using the dominated convergence theorem. Consequently, we obtain

$$|D_x u_1(t, x, y) \cdot h| = \lim_{r \to 0} \int_{L_2} \frac{u_1(t, x + rh, y) - u_1(t, x, y)}{r} \mu^x(dy)$$

$$+ \int_0^\infty E[\Theta(t, x, Y^x_t, h)] ds. \quad (4.69)$$

One can verify that Lemma 4.12 can be extended to the case where $\Psi$ has the form of quadratic growth as in (4.68). Applying Lemma 4.12 to $\Theta$, from (4.69), we get

$$|D_x u_1(t, x, y) \cdot h| \leq Ce^{C|x|^5} (1 + \|y\|) \|h\|. \quad (4.70)$$

and therefore

$$|L_1 u_1(t, x, y)| \leq Ce^{C|x|^5} (1 + \|y\|)(1 + |x|^2 + \|y\| + \|Ax\|). \quad (4.70)$$

The result follows by combining (4.64)-(4.65) and (4.70).

**Lemma 4.16.** Under the conditions A1, A2 and A4, for any $x \in H^\theta$ with $\theta \in (0, 1)$, $y \in L^2$, $\alpha \in (0, 1/4)$, $0 \leq t \leq T$, there exist $k \in \mathbb{N}$ and a constant $C = C_{\alpha, \theta, T}$ such that

$$|v^\epsilon(\rho(\epsilon), x, y)| \leq C \frac{\rho(\epsilon)^2}{\theta} (1 + |x|^2 + \|y\|^2)e^{C|x|^k} + Ce^{C|x|^5} (1 + \|y\|) + C\rho(\epsilon) \epsilon^{-\alpha}.$$

**Proof.** Using (4.52) and noting that $u_0$ is independent of $y$, we write

$$v^\epsilon(\rho(\epsilon), x, y) = [u^\epsilon(\rho(\epsilon), x, y) - u^\epsilon(0, x, y)] - [u_0(\rho(\epsilon), x) - u_0(0, x)] - \epsilon u_1(\rho(\epsilon), x, y)$$

$$:= I_1 - I_2 - I_3. \quad (4.71)$$
For $I_1$, by Lemma 4.11, we have
\[
\|I_1\| = \left\| \int_0^{\rho(\varepsilon)} \frac{d}{dt} u^\varepsilon(t, x, y) dt \right\| = \left\| \int_0^{\rho(\varepsilon)} \frac{d}{dt} \mathbb{E}[\phi(X^\varepsilon(t, x, y))] dt \right\|
\leq C \int_0^{\rho(\varepsilon)} \mathbb{E} \left\| \frac{d}{dt} X^\varepsilon(t, x, y) \right\| dt
\leq C \rho(\varepsilon)^{\frac{2}{\theta}} (|x|^2 + \|y\|^2 + 1) e^{C\|x\|^k} + C \rho(\varepsilon) \varepsilon^{-\alpha}. \tag{4.72}
\]

For $I_2$, recalling that $u_0 = \bar{u}$, by Lemma 4.3, we get
\[
\|I_2\| = \left\| \int_0^{\rho(\varepsilon)} \frac{d}{dt} u_0(t, x) dt \right\| = \left\| \int_0^{\rho(\varepsilon)} \frac{d}{dt} \phi(\hat{X}(t, x)) dt \right\|
\leq C \int_0^{\rho(\varepsilon)} \left\| \frac{d}{dt} \hat{X}(t, x) \right\| dt
\leq \frac{C \rho(\varepsilon)^{\frac{2}{\theta}}}{\rho(\varepsilon)^{\frac{2}{\theta}}} (1 + |x|^2) e^{C\|x\|^k}. \tag{4.73}
\]

For $I_3$, it follows from Lemma 4.13 that
\[
\|I_3\| \leq C \varepsilon e^{C\|x\|^5} (1 + \|y\|). \tag{4.74}
\]
Putting together (4.71)-(4.74), the result follows.

4.5. Proof of Theorem 2.2. From (4.50)-(4.51) and (4.60), we get
\[
\mathbb{E}[\phi(X^\varepsilon_N(t)) - \mathbb{E}[\phi(\hat{X}_N(t))] = \varepsilon u_1(t, x, y) + \mathbb{E}[v^\varepsilon(\rho(\varepsilon), X^\varepsilon(t - \rho(\varepsilon), x, y), Y^\varepsilon(t - \rho(\varepsilon), x, y))]
+ \varepsilon \mathbb{E} \left[ \int_{\rho(\varepsilon)}^{t} (L_1 u_1 - \frac{\partial u_1}{\partial s})(s, X^\varepsilon(t - s, x, y), Y^\varepsilon(t - s, x, y)) ds \right].
\]
Using Lemmas 4.13-4.16 and 4.10, (4.44), (4.8), the Hölder inequality, and the fact that $\rho(\varepsilon) = \varepsilon^{\frac{1}{a}}, 0 < a \leq \frac{\theta}{2}$, we obtain
\[
\|\mathbb{E}[\phi(X^\varepsilon_N(t)) - \mathbb{E}[\phi(\hat{X}_N(t))]\| \leq C \varepsilon e^{C\|x\|^5} (1 + \|y\|) + C \rho(\varepsilon)^{\frac{2}{\theta}} \mathbb{E} \left[ (1 + |X^\varepsilon(t - \rho(\varepsilon))|^{\frac{2}{\theta}} + \|Y^\varepsilon(t - \rho(\varepsilon))\|^2) e^{C\|X^\varepsilon(t - \rho(\varepsilon))\|^k} \right]
+ C \rho(\varepsilon)^{\frac{2}{\theta}} \mathbb{E} \left[ e^{C\|X^\varepsilon(t - \rho(\varepsilon))\|^5} \left( 1 + \|Y^\varepsilon(t - \rho(\varepsilon))\|^2 \right) \right] + C \rho(\varepsilon)^{-\alpha}
+ C \varepsilon \int_{\rho(\varepsilon)}^{t} \mathbb{E} \left[ e^{C\|X^\varepsilon(t - s)\|^5} (1 + \|Y^\varepsilon(t - s)\|)(1 + |X^\varepsilon(t - s)|^{\frac{2}{\theta}} + \|Y^\varepsilon(t - s)\|^2 + \|AX^\varepsilon(t - s)\|) ds \right]
+ C \varepsilon \int_{\rho(\varepsilon)}^{t} s^{\frac{1}{2}} \mathbb{E} \left[ \left( |X^\varepsilon(t - s)|^{\frac{2}{\theta}} + 1 \right) e^{C\|X^\varepsilon(t - s)\|^k} \right] ds
\leq C \varepsilon + C t^{\theta + \frac{2}{1 + \theta}} \rho(\varepsilon)^{\frac{2}{\theta}} - C \varepsilon \log \varepsilon \leq C \varepsilon t^{\theta + \frac{2}{1 + \theta}} - C \varepsilon \log \varepsilon, \tag{4.75}
\]
where the constant $C$ is independent of the dimension $N$. Then, letting $N \to +\infty$ in (4.75), in view of (4.3), the desired result follows.
4.6. Proof of Theorem 2.3. Since the proof of Theorem 2.3 is similar to that of Theorem 2.2, we only sketch the difference here. Benefited from a higher regularity of initial value, following the proof of (4.5) in Lemma 4.1, one can verify that if \( \theta \in (1, \frac{3}{2}) \), then
\[
\sup_{t \in [0,T]} |\tilde{X}_t|_{\theta} \leq C|x|_{\theta} e^{C\|x\|}. \tag{4.76}
\]
Similarly to the proof of (4.39) in Lemma 4.8, we show that if \( \theta \in (1, \frac{3}{2}) \), then for any \( p \geq 1 \), we have
\[
\sup_{t \in [0,T]} \mathbb{E}|X^\varepsilon_t|_{\theta}^p \leq C(1 + \|x\|_{\theta}^p + \|y\|_\theta^p). \tag{4.77}
\]
Consequently, based on (4.76) and (4.77), one can improve the corresponding results in Lemmas 4.13-4.16. Then we can obtain Theorem 2.3 in an analogous way as in the proof of Theorem 2.2. \( \square \)

5. Appendix

The following properties of \( b(\cdot, \cdot, \cdot) \) and \( B(\cdot, \cdot) \) are well-known (for example see [11]).

Lemma 5.1. For any \( x, y \in H^1_0 \), it holds that
\[
b(x, x, y) = -\frac{1}{2}b(x, y, x), \quad b(x, y, y) = 0.
\]

Lemma 5.2. Suppose that \( \alpha_i \geq 0 \) (\( i = 1, 2, 3 \)) satisfies one of the following conditions

(1) \( \alpha_i \neq \frac{1}{2} (i = 1, 2, 3) , \alpha_1 + \alpha_2 + \alpha_3 \geq \frac{1}{2} \),

(2) \( \alpha_i = \frac{1}{2} \) for some \( i \), \( \alpha_1 + \alpha_2 + \alpha_3 > \frac{3}{2} \),

then \( b \) is continuous from \( H^{\alpha_1} \times H^{\alpha_2+1} \times H^{\alpha_3} \) to \( \mathbb{R} \), i.e.
\[
|b(x, y, z)| \leq C|x|_{\alpha_1}|y|_{\alpha_2+1}|z|_{\alpha_3}.
\]

The following inequalities can be derived by the above lemma.

Corollary 5.3. For any \( x \in H^1_0 \), we have

(1) \( \|B(x)\| \leq C|x|^2 \),

(2) \( \|B(x)\|_{-1} \leq C\|x\| \cdot |x| \).

Lemma 5.4. For any \( x, y \in H^1_0 \), we have

(1) \( \|B(x) - B(y)\| \leq C|x - y|_1 (|x|_1 + |y|_1) \).

(2) \( \|B(x) - B(y)\|_{-1} \leq C\|x - y\| (|x|_1 + |y|_1) \).

Similar as the argument in the proof of [25, Theorem 2.6], we can easily obtain the following Gronwall-type inequality, whose proof is based on iteration and we omit the details.

Lemma 5.5 (Gronwall-type inequality). Let \( f(t) \) be a non-negative real-valued integrable function on \([0, T]\). For any given \( \alpha, \beta \in (0, 1) \), if there exist two positive constants \( C_1, C_2 \) such that
\[
f(t) \leq C_1 t^{-\alpha} + C_2 \int_0^t (t - s)^{-\beta} f(s) ds, \quad \forall t \in [0, T],
\]
then there exists some \( k \in \mathbb{N} \) and a positive constant \( C := C_{\alpha, \beta, T} \) such that
\[
f(t) \leq CC_1 t^{-\alpha} e^{CC_2 t}, \quad \forall t \in [0, T].
\]
Acknowledgment. This paper is partially supported by Key Laboratory of Random Complex Structures and Data Science, PCSDS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences (No: 2008DP173182), by the Priority Academic Program Development of Jiangsu Higher Education Institutions, by Natural Science Foundation of the Higher Education Institutions of Jiangsu Province (No:16KJB110006), by Key Research Program of Frontier Sciences, CAS (No: QYZDB-SSW-SYS009) and the Fundamental Research Funds for the Central Universities (No. WK 3470000008), by NSFC (No: 11271356, No: 11371041, No: 11431014, No: 11271169, No: 11601196, No: 11771187, No. 11671372, No. 11721101).

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