Strong Coordination over Multi-hop Line Networks

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Abstract—We analyze the problem of strong coordination over a multi-hop line network where the node initiating the coordination is a terminal network node. We provide a characterization of the capacity region when the initiating node possesses unlimited local randomness and intermediate nodes operate under a functional regime. In this regime, next-hop messages are created only using common randomness and previous-hop incoming messages, i.e., local randomness at intermediate nodes is only used for generating actions.

I. INTRODUCTION

While communication is typically viewed as conveyance of data from one location to another over a network of channels subject to certain (reconstruction and/or security) constraints, it is also a means to establish jointly correlated behavior in a distributed fashion [1]. Using communications for correlating network behavior can be useful in distributed random variable generation [2], distributed sensing, and control systems.

Two modes of coordination have been studied in the literature – empirical coordination, where communication enables the generation of actions at various network nodes that have a prescribed target distribution, and strong coordination, where the actions sequences are collectively required to resemble the output of a jointly correlated source. The fundamental limits of several empirical and strong coordination problems can be found in [1]. However, most problems studied in [1] relate to two-terminal networks. The fundamental limits of generation of correlated random variables and channel simulations in two-terminal networks with multiple rounds of communication were considered in [3], [4]. In the more general three-terminal relay channel setting, inner and outer bounds of the required rate of communication were derived [5], [6]. While [5] considers only one-way communication, two-way communication with actions required only at the end terminals (and not the relay) is considered in [6].

In this work, we quantify the resources required for achieving strong correlation in multi-hop networks. This problem is closely related to those considered in [5], [7], [8]. In [8], the strong coordination rate region for two-hop (and multi-hop) line networks is characterized under the additional constraint that the joint statistics of the actions remain unchanged even when an eavesdropper observes the communication on links. This work does not consider this additional requirement. It presents a precise characterization of the tradeoffs between the required rates of communication, common randomness shared by all nodes, and local randomness at each node for establishing strong coordination in multi-hop line networks.

The remainder of this work is organized as follows. Section II presents the notation used in this work. Section III presents the formal definition of the strong coordination problem, and Section IV presents the main results of this work.

II. NOTATION

For $m, n \in \mathbb{N}$ with $m < n$, $\{m, n\} = \{m, m+1, \ldots, n\}$. Uppercase letters (e.g., $X$, $Y$) denote random variables (RVs), and the respective script versions (e.g., $\mathcal{X}$, $\mathcal{Y}$) denote their alphabets. In this work, all alphabets are assumed to be finite. Lowercase letters denote the realizations of random variables (e.g., $x$, $y$). Superscripts indicate the length of vectors. Single subscripts always indicate the node indices. In case of double subscripts, the first indicates the node index, and the next indicates the component index. Given a finite set $S$, $\text{unif}(S)$ denotes the uniform distribution on the set. Given a probability mass function (p.m.f.) $p_X$, $T^n_x[p_X]$ denotes the set of all $n$-letter typical sequences of length $n$ [9]. Given an event $E$, $\mathbb{P}(E)$ denotes the probability of occurrence of the event $E$. Given two probability mass functions $p$ and $q$ over the same set, $\mathbb{V}(p, q)$ denotes the variational distance between $p$ and $q$. Lastly, $p_{X_1, X_2, \ldots, X_k}^{\leq n}$ denotes the p.m.f. of $n$ i.i.d random variable $k$-tuples, with each $k$-tuple correlated according to p.m.f. $p_{X_1, X_2, \ldots, X_k}$.

III. PROBLEM DEFINITION

The line coordination problem is a multi-hop extension of the one studied in [7], and is depicted in Fig. 1. For the sake of completeness, the problem is formally defined here.

A line network with $h$ nodes (Nodes 1, ..., $h$) and $h-1$ links (modeled as noiseless bit pipes) that connect Node $i$ with Node $i+1$, $1 \leq i < h$ is given. Node 1 is specified an action sequence $\{X_1, i\}_{i \in \mathbb{N}}$, an i.i.d process with each $X_1, i$ distributed according to p.m.f. $Q_{X_1}$ over a finite set $X_1$. Nodes are assumed to possess local randomness, as well as common randomness shared by all $h$ nodes to enable strong coordination using block codes. A block code (of length $n$) uses $n$ symbols of the specified action (i.e., $X_1$), and common
and local randomness to generate actions $\hat{X}_i^n$ at Nodes $i, i > 2$ satisfying the following condition: the joint statistics of actions $(X_1^n, X_2^n, \ldots, X_h^n)$ and those of $n$ symbols output by discrete memoryless source $Q_{X_1^n, X_h^n}$ are nearly indistinguishable under the variational distance metric. The overall aim is to characterize the required rates of communication messages and randomness (both common and local) to achieve such strong coordination. The following definitions are now in order.

**Definition 1:** Given $Q_{X_1, \ldots, X_h}$ and $\varepsilon > 0$, a strong coordination $\varepsilon$-code of length $n$ at $(R_C, R_1, \ldots, R_h - 1, \rho_1, \ldots, \rho_h) \subseteq \mathbb{R}^{+h}$ is a collection of $h+1$ independent and uniform random variables $(M_c, M_{L_1}, \ldots, M_{L_h}), h-1$ message-generating functions \(\psi_j\) for $j = 1, \ldots, h$, and $h-1$ action-generating functions \(\phi_j\) for $j = 2, \ldots, h$ s.t.:  

- **Randomness constraints:**  
  \[
  \begin{align*}
  &\text{[Common]} \quad M_c \sim \text{unif}(\{1, 2^{R_C}\}), \\
  &\text{[Local]} \quad M_{L_i} \sim \text{unif}(\{1, 2^{R_{L_i}}\}), \quad i = 1, \ldots, h.
  \end{align*}
  \]  

- **Message-generation and action-generation constraints:**  
  \[
  \begin{align*}
  I_1 &\triangleq \psi_1(M_{L_1}, X_1^n, M_c) \in [1, 2^{R_{L_1}}], \\
  I_j &\triangleq \psi_j(M_{L_{j-1}}, I_{j-1}, M_{L_j}) \in [1, 2^{R_{L_j}}], \quad 2 \leq j < h, \\
  \hat{X}_j &\triangleq \phi_j(M_{L_j}, I_{j-1}, M_c), \quad 2 \leq j \leq h.
  \end{align*}
  \]

- **Strong coordination constraint:**  
  \[
  \mathcal{V}(Q_{X_1^n}^{\otimes n} | \hat{X}_2^n \cdots \hat{X}_h^n | X_1^n, Q_{X_2^n, \ldots, X_h^n}) \leq \varepsilon,  
  \]
  where $\hat{X}_2^n \cdots \hat{X}_h^n | X_1^n$ is the conditional p.m.f. induced by the code.

**Definition 2:** A tuple $\mathbf{R} \triangleq (R_C, R_1, \ldots, R_h - 1, \rho_1, \ldots, \rho_h)$ is said to be achievable for strong coordination of actions according to $Q_{X_1, \ldots, X_h}$ if for any $\varepsilon > 0$, there exists a strong coordination $\varepsilon$-code of some length $n \in \mathbb{N}$ at $\mathbf{R}$. Further, the $2h$-dimensional strong coordination capacity region is defined as the closure of the set of all achievable rate vectors.

![Fig. 2. Three possible encoder structures.](image)

Before we proceed to the results, we present a brief discussion on three modes of operation for intermediate nodes. These three modes highlight the options for message generation that at each intermediate node. In the first mode given in Fig. 2(a) (called functional message generation), the outgoing message is generated from only the incoming message and common randomness. In the second mode given in Fig. 2(b) (called action-dependent message generation), the intermediate node uses the incoming message, and local and common randomness to generate the node’s action. The outgoing message is then generated using the incoming message, common randomness, and the generated action. The third mode, which is termed as unrestricted message generation (see Fig. 2(c)) is where both the action and the next-hop message depend on the incoming message, local, and common randomness.

In theory, the set of rate vectors achievable using the unrestricted mode is a superset of those achievable using the action-dependent mode, which is in turn a superset of those achievable by the functional mode. We have been so far unable to develop achievable schemes that cleverly use intermediate actions or local randomness to generate next-hop messages. So it is, a priori, unclear if the inclusions are strict. In the remainder of the work, we present our results specifically for the functional mode of operation at intermediate nodes.

### IV. RESULTS

#### A. The Functional Message Generation Capacity Region

Consider a line network where Node 1 initiates coordination and has unlimited\(^1\) local randomness (i.e., $\rho_1 = \infty$). In this setting, the following capacity region characterization quantifies the tradeoffs between communication, local and common randomness rates in the functional message generation mode.

**Theorem 1:** Strong coordination of actions according to $Q_{X_1, \ldots, X_h}$ is achievable via functional message generation with common randomness rate $R_C$, message rates $\{R_i\}_{i=1}^{h-1}$, and local randomness rates $\{\rho_j\}_{j=2}^{h}$ iff there exist auxiliary RVs\(^2\) $\{A_j\}_{j=1}^{h-1}$ such that:

- For each $i = 1, \ldots, h - 1$,  
  \[X_{i+1} - \{A_k\}_{k=1}^{h-1} = (\{X_j : j \neq i + 1\}, \{A_i\}_{i=1}^{h-1})\]

- For each $i = 1, \ldots, h - 1$ and $S \subseteq [i, h]$,  
  
  \[R_i + R_c + \sum_{s \in S} \rho_s \geq I(\{X_j\}_{j=i+1}^{h} : \{A_j\}_{j=1}^{h-1} | X_S), \]

\[R_i \geq I(X_1 : \{A_j\}_{j=1}^{h-1})\]

**Proof of Achievability:** The proof combines ideas from the design of strong coordination codes using channel resolvability codes [1], [2], [11] and channel synthesis [12]. Let us ignore the problem at hand; consider an allied problem whose solution will be altered to design strong coordination codes.

The allied problem aims to distributely generate $n$ symbols of $h$ sources whose joint statistics is close to $Q_{X_1^n, X_h^n}$ under the variational distance metric, where (8) quantifies the closeness. This generation is to be achieved using $2h + 1$ independent and uniform messages via $2h$ codebooks as given in Fig. 3. To define the codebooks, pick a joint p.m.f. $Q$ with $2h + 1$ auxiliary RVs $A_0, \ldots, A_{h-1}$ and $B_1, \ldots, B_h$ such that:

- $Q_{X_1^n \cdots X_h} = Q_{X_1^n, X_2^n, \ldots, X_h^n}$; and (b) the following structure holds.

\[
Q_{A_0 \cdots A_{h-1} A_1 \cdots A_{h-1}} h-1 \prod_{j=0}^{h-1} Q_{B_{j+1} X_{j+1} | A_1 \cdots A_{h-1}}.
\]

Let the codebooks be constructed as follows.

**A1** Let $M_i \triangleq [1, 2^{R_i}]$ and $M_i \sim \text{unif}(M_i)$ for $0 \leq i \leq h$. Let $L_j \triangleq [1, 2^{R_j}]$ and $L_j \sim \text{unif}(L_j)$ for $1 \leq j \leq h$.

\(^1\)Only a sufficiently large rate of local randomness is needed (see Sec. IV-B).

\(^2\)The size of auxiliary RVs can be bounded by Carathéodory’s theorem [10].
A2 For each \((m_{h-1}, m_h) \in \mathcal{M}_{h-1} \times \mathcal{M}_h\), generate codeword \(A^n_{h-1}(m_{h-1}, m_h)\) randomly using \(Q_{A|h-1}\).

A3 For \((m_{h-1}, m_h) \in \mathcal{M}_{h-1} \times \mathcal{M}_h\) and \(l_h \in \mathcal{L}_h\), generate codeword \(B^n_{l_h}(m_{h-1}, m_h)\) randomly using \(Q_{B|A|h-1}\).

A4 For \(0 \leq k < h-1\), \((m_k, \ldots, m_h) \in \mathcal{M}_k \times \cdots \times \mathcal{M}_h\), and \(m_k \in \mathcal{M}_k\), generate codeword \(A^n_k(m_k, \ldots, m_h)\) randomly using \(Q_{A_k|A_{k+1} \ldots A_h}\).

A5 For \(k < h-1\), \((m_k, \ldots, m_h) \in \mathcal{M}_k \times \cdots \times \mathcal{M}_h\), and \(l_{k+1} \in \mathcal{L}_{k+1}\), generate codeword \(B^n_{l_{k+1}}(m_k, \ldots, m_h)\) randomly using \(Q_{B_{k+1}|A_k \ldots A_h}\).

A6 Given \((M_0, \ldots, M_h, L_1, \ldots, L_h) = (m_0, \ldots, m_h, l_1, \ldots, l_h)\), let for \(1 \leq i \leq h\), \(X^n_i\) be the output from the channel \(Q_{X_{i} | A_{i-1} \ldots A_h B_i}\) when the inputs are the codewords corresponding to the realized message indices (see Fig. 3).

Fig. 3. A scheme to generate \((X^n_1, \ldots, X^n_h)\) using \(2^h+1\) independent and uniformly distributed messages \((M_0, M_1, \ldots, M_h)\).

Now, fix \(\varepsilon > 0\). Suppose that we seek conditions on message rates \(\{R_i\}_{i=0}^h\) and \(\{r_j\}_{j=1}^h\) so that the following are met.

\[
E \left[ D_{KL}(\hat{Q}_{X^n_1 \ldots X^n_h} \| \hat{Q}_{X^n_1 \ldots X^n_h}^n) \right] \leq \varepsilon \\
E \left[ I(X^n_1; M_h) \right] \leq \varepsilon,
\]

where \(D_{KL}\) denotes the Kullback-Leibler divergence, and \(\hat{Q}\) is the joint p.m.f. induced by the code. The first constraint restricts the joint p.m.f. \(\hat{Q}\) to be near-i.i.d. The constraint in (9) ensures that message \(M_h\) and action \(X^n_1\) are nearly independent. This constraint is not demanded by the allied problem. It will be needed later to alter this scheme to build a strong coordination code where \(M_h\) will be identified with common randomness. To find the conditions on the rates, we proceed in a fashion similar to [5] and [7]. The notation and arguments for the manipulations in (10)-(12) are as follows:

- \(Y \triangleq (X_1, \ldots, X_h)\), and \(N \triangleq 2^{n(R_1 + \cdots + R_h + r_{i+1} + \cdots + r_h)}\).
- In (10), \(A^h \triangleq (A^n_0, \ldots, A^n_{h-1})\) and \(B^h \triangleq (B^n_0, \ldots, B^n_h)\) denote the codewords corresponding to the indices \((Y, m')\) \(\triangleq ((l'_1, \ldots, l'_h), (m'_0, \ldots, m'_{h-1}))\), and \(A'^h, B'^h\) are the codewords corresponding to the indices \((Y', m'')\);
- (11) follows by the use of the law of iterated expectations, where the inner expectation \(E_{res}\) is over all random codeword constructions conditioned on a particular selection for the codeword corresponding to the indices \((Y, m')\);
- (12) follows from Jensen’s inequality; and
- (13) follows from the following arguments. For a fixed \((Y', m'')\), consider the inner sum in (12). Partition this sum based on the values of \((Y', m'')\) using the following sets. For \(0 \leq i \leq h-1\), \(S \subseteq \llbracket i + 1, h \rrbracket\), let

\[
J_{i,S}(Y', m'') \triangleq \left\{ (m'_i=m'_{i} \ldots m'_{i}) \mid l'_i = l'_i, \forall i \in S \right\}, \quad i = 0
\]

\[
J_{i,S}(Y', m'') \triangleq \left\{ (m'_i=m'_{i} \ldots m'_{i}) \mid l'_i = l'_i, \forall i \in S \right\}, \quad i > 0
\]

\[
J_h(Y', m'') \triangleq \left\{ (m'_i=m'_{i} \ldots m'_{i}) \mid (m'_h-1, m_h) \neq (m'_{h-1}, m_h) \right\}.
\]

In the following equations, we drop the dependence of these sets on \((Y', m'')\) for the sake of clarity. We now compute the inner sum of (12) over these sets. First, note that when \((Y', m'') \in J_h(Y', m'')\), \((A', B')\) and \((A'', B'')\) are independent by construction (see A2-A5), and hence,

\[
\sum_{Y', m'' \in J_h} E_{res} \left[ \frac{Q_{Y|A'B'}(y|a''y''|b''y'')} {Q_{Y}(y)} \right] A'' = a'' B'' = b''
\]

\[
= \frac{E_{A''} \cdot B''} {Q_{Y}(y)} \left[ \frac{Q_{Y|A'B'}(y|a''y''|b''y'')} {Q_{Y}(y)} \right] A'' = a'' B'' = b''
\]

\[
= \sum_{Y', m'' \in J_h} E_{Y} \left[ \frac{Q_{Y|A'B'}(y|a''y''|b''y'')} {Q_{Y}(y)} \right]
\]

\[
= \sum_{Y', m'' \in J_h} \frac{1} {Q_{Y}(y)} \left[ \frac{Q_{Y|A'B'}(y|a''y''|b''y'')} {Q_{Y}(y)} \right]
\]

\[
\frac{1} {N} \frac{1} {Q_{Y}(y)} \leq 1,
\]

which is the first term in the logarithm in (13). For \(J_{i,S}\) when \(i \leq h-1\) and \(S \subseteq \llbracket i+1, h \rrbracket\), we proceed as follows.

\[
\sum_{Y', m'' \in J_{i,S}} E_{res} \left[ \frac{Q_{Y|A'B'}(y|a''y''|b''y'')} {Q_{Y}(y)} \right] A'' = a'' B'' = b''
\]

\[
= \sum_{Y', m'' \in J_{i,S}} E_{Y} \left[ \frac{Q_{Y|A'B'}(y|a''y''|b''y'')} {Q_{Y}(y)} \right]
\]

\[
= \sum_{Y', m'' \in J_{i,S}} \frac{1} {Q_{Y}(y)} \left[ \frac{Q_{Y|A'B'}(y|a''y''|b''y'')} {Q_{Y}(y)} \right]
\]

\[
\frac{1} {N} \frac{1} {Q_{Y}(y)} \leq 1,
\]

Noting that \(\log_2 |J_{i,S}| \leq n \left(R_1 + \cdots + R_i + 1 + \sum s \in S r_s \right)\), we obtain the other terms within the logarithm in (13).

Finally, we can split the outer sum in (13) depending on whether or not \((y^n, a^n, b^n) \in T^n_y[Q_{YAB}]\). As in [5], the sum for nonypical realizations in (13) is no more than

\[
P \left[ (y^n, a^n, b^n) \notin T^n_y[Q_{YAB}] \right] \log (1 + h^2 m_{yx}^{-1})\]

\[
= \min_{y \in \arg \max \{Q_{Y}(y) \} } \log (1 + h^2 m_{yx}^{-1})\]

This term goes to zero as \(n \to \infty\). The contribution from typical realizations can be made to vanish asymptotically, if for each \(0 \leq i \leq h-1\) and \(S \subseteq \llbracket i+1, h \rrbracket\) the codebook parameters satisfy:

\[
\sum_{k=i}^{h} R_k + \sum_{s \in S} r_s > I(Y; \{A_j\}_{j=1}^{h-1} | B_S)
\]

Now, to ensure \(X^n_1\) and \(M_h\) are near-independent, consider:

\[
E [I(X^n_1; M_h)] = E [D_{KL}(\hat{Q}_{X^n_1|M_h=1} \| \hat{Q}_{X^n_1})]
\]
Applying manipulations similar to those in (10)-(13) to the above terms, we see that \( E[I(X^n_1; M_h)] \) can be made vanishingly small, if additionally the following constraints hold:

\[
R_i + \cdots + R_{h-1} > I(X_1; \{A_j\}_{j=1}^{h-1}), \quad 0 \leq i < h, \quad (18)
\]

\[
R_0 + \cdots + R_{h-1} + r_1 > I(X_1; \{A_j\}_{j=1}^{h-1}; B_1). \quad (19)
\]

Thus, for sufficiently large \( n \), we can design a code to generate the \( h \) sources satisfying (8) and (9) if the parameters satisfy (15), (18), and (19). To design a strong coordination code, we now modify the code for the allied problem, and relate its parameters to those of Definition 1 using the following steps.

**B1** Pick joint p.m.f. \( Q \) satisfying (7) with: (a) \( A_0 \) and \( B_1 \) as constant RVs, \( M_0 = L_1 = 1 \); and (b) \( B_i = X_i \) for \( i > 1 \).

**B2** Pick \( \varepsilon > 0 \). Construct a code, say \( C \), that generates the \( h \) sources using the chosen p.m.f. \( Q \) in the above scheme.

**B3** Let \( M_{\varepsilon} \triangleq M_h \) be the common randomness, and for \( i > 1 \), let \( M_{L_i} \triangleq L_i \) be the local randomness at Node \( i \).

**B4** Given \( (X^n_1, M_h) = (x^n_1, m_h) \), Node 1 generates a realization \( (M_1, \ldots, M_{h-1}) \triangleq (m_1, \ldots, m_{h-1}) \) randomly using the conditional p.m.f. \( Q_{M_1,\ldots,M_{h-1}|X^n_1,M_h}(\cdot|x^n_1, m_h) \) that is derived from the chosen code \( C \). Node 1 then forwards \( l_1 \triangleq (m_1, \ldots, m_{h-1}) \) to Node 2.

**B5** For \( 1 < i \leq h \), Node \( i \) uses the received message indices \( l_{i-1} \triangleq (m_{i-1}, \ldots, m_{h-1}) \), common randomness \( M_i \triangleq M_h \), and local randomness \( M_{L_i} \triangleq L_i \) to identify \( A_{i-1}, \ldots, A_{h-1} \) and \( B_i \) codewords corresponding to these indices. By definition, \( X^n_i \) is the chosen \( B_i \)-codeword, and is a function of \( (l_{i-1}, M_i, M_{L_i}) \). Lastly, if \( i < h \), Node \( i \) sends \( l_i \triangleq (m_i, \ldots, m_{h-1}) \) to Node \( i+1 \).

For this strong coordination code, the following hold:

- \( R_0 = r_1 = 0 \), and for \( i \geq 1 \), the communication rate on the link between Node \( i \) and Node \( i+1 \) is \( R_i \triangleq \sum_{k=1}^{h-i} R_k \).
- The common randomness rate equals \( R_c \triangleq R_h \) and the local randomness required at Node \( i \), \( i > 1 \) is \( r_i \triangleq r_h \).
- The actions at the various nodes have a joint p.m.f. close to \( Q^n_{X_1,\ldots,X_h} \) w.r.t. the variational distance – a consequence of Pinsker’s inequality applied to (8).

Rewriting (15), (18), and (19) using the rates in Definition 1, we infer that a strong coordination code exists if there exists a joint p.m.f. \( Q \) satisfying (7) such that for any \( 1 \leq i < h \) and any \( S \subseteq [i+1, h] \),

\[
R_i + R_c + \sum_{s \in S} \rho_s > H(M_1, l_1, \{M_{L_i}\}_{i \in S})
\]

\[
= H(X^n_1) + \sum_{s \in S} \rho_s > H(X^n_1; \{A_j\}_{j=1}^{h-1} | \{B_j\}_{j=1}^{h-1}). \quad (20)
\]

Thus, for sufficiently large \( n \), we can design a code to generate the \( h \) sources satisfying (8) and (9) if the parameters satisfy (15), (18), and (19). To design a strong coordination code, we now modify the code for the allied problem, and relate its parameters to those of Definition 1 using the following steps.

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**B4** Given \( (X^n_1, M_h) = (x^n_1, m_h) \), Node 1 generates a realization \( (M_1, \ldots, M_{h-1}) \triangleq (m_1, \ldots, m_{h-1}) \) randomly using the conditional p.m.f. \( Q_{M_1,\ldots,M_{h-1}|X^n_1,M_h}(\cdot|x^n_1, m_h) \) that is derived from the chosen code \( C \). Node 1 then forwards \( l_1 \triangleq (m_1, \ldots, m_{h-1}) \) to Node 2.

**B5** For \( 1 < i \leq h \), Node \( i \) uses the received message indices \( l_{i-1} \triangleq (m_{i-1}, \ldots, m_{h-1}) \), common randomness \( M_i \triangleq M_h \), and local randomness \( M_{L_i} \triangleq L_i \) to identify \( A_{i-1}, \ldots, A_{h-1} \) and \( B_i \) codewords corresponding to these indices. By definition, \( X^n_i \) is the chosen \( B_i \)-codeword, and is a function of \( (l_{i-1}, M_i, M_{L_i}) \). Lastly, if \( i < h \), Node \( i \) sends \( l_i \triangleq (m_i, \ldots, m_{h-1}) \) to Node \( i+1 \).

For this strong coordination code, the following hold:

- \( R_0 = r_1 = 0 \), and for \( i \geq 1 \), the communication rate on the link between Node \( i \) and Node \( i+1 \) is \( R_i \triangleq \sum_{k=1}^{h-i} R_k \).
- The common randomness rate equals \( R_c \triangleq R_h \) and the local randomness required at Node \( i \), \( i > 1 \) is \( r_i \triangleq r_h \).
- The actions at the various nodes have a joint p.m.f. close to \( Q^n_{X_1,\ldots,X_h} \) w.r.t. the variational distance – a consequence of Pinsker’s inequality applied to (8).
Using the achievable scheme of Thm. 1 with auxiliary RVs \( A_j^{\eta+1} \) and \( B_j^{\beta+1} \), \( \eta, \beta \geq 0 \), where \( A_0, B_1 \) are constant RVs, a strong coordination \( \varepsilon \)-code can be constructed for any \( \varepsilon > 0 \) if (15) and (18) are met. Note that (19) is redundant when \( H(B_1) = 0 \). Further, if we also require decodability, we need (27) to hold when we set \( \nu_j \leq R_j, 1 \leq j \leq h-1 \). However, for this choice, both (18) and (27) cannot simultaneously hold.

To resolve this, for each realization \( (X_1^n, M_c) = (x_1^n, m_c) \), one can use a list decoder to identify a list of index tuples \( \mathcal{L}_{x_1^n, m_c} = \{ (m_1(k), \ldots, m_{n-1}(k), m_c) : k \in [1, |\mathcal{L}_{x_1^n, m_c}|] \} \) such that the \((A_1, \ldots, A_{h-1})\) codewords corresponding to each index tuple in the list are jointly typical with \( X_1^n = x_1^n \). Local randomness \( M_L \) can then be used to select an index tuple in the list based on the likelihood of obtaining \( X_1^n = x_1^n \) from the codewords corresponding to the index tuple. By carefully analyzing the list size and the likelihood values, it can be shown that the random selection in Step B4 can be achieved using local randomness of rate \( \rho_1 \) provided:

\[
\rho_1 > R_1 - I(X_1; A_1, \ldots, A_{h-1}).
\]

To derive a simple lower bound for \( \rho_1 \), we proceed as follows.

\[
H(l_1 | M_c) = I(X_1^n; M_L; l_1 | M_c) \\
\leq I(X_1^n; l_1 | M_c) + H(M_L | X_1^n M_c).
\]

Note that \( M_L \) is independent of \( (X_1^n, M_c) \). Hence,

\[
n_1 \rho_1 \geq H(M_L) \geq H(l_1 | M_c) - I(X_1^n; l_1 | M_c)
\]

\[
\geq H(l_1 | M_c) - n I(X_1^n; A_1, \ldots, A_{h-1}),
\]

where (30) uses (24)-(25). The bounds in (30) and (28) do not match. The bound in (30) suggests that schemes in which \( I(l_1; M_c) \neq 0 \) may require a lower rate of local randomness at Node 1. It is, however, unclear if this bound is achievable.

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