NECESSARY CONDITIONS AND TIGHT TWO-LEVEL CONVERGENCE BOUNDS FOR PARAREAL AND MULTIGRID REDUCTION IN TIME*  

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Abstract. Parareal and multigrid reduction in time (MGRiT) are two of the most popular parallel-in-time methods. The basic idea is to treat time integration in a parallel context by using a multigrid method in time. If Φ is the (fine-grid) time-stepping scheme of interest, such as RK4, then let Ψ denote a “coarse-grid” time-stepping scheme chosen to approximate k steps of Φ, where k ≥ 1. In particular, Ψ defines the coarse-grid correction, and evaluating Ψ should be (significantly) cheaper than evaluating Φk. Parareal is a two-level method with a fixed relaxation scheme, and MGRiT is a generalization to the multilevel setting, with the additional option of a modified, stronger relaxation scheme.

A number of papers have studied the convergence of Parareal and MGRiT. However, there have yet to be general conditions developed on the convergence of Parareal or MGRiT that answer simple questions such as, (i) for a given Φ and k, what is the best Ψ, or (ii) can Parareal/MGRiT converge for my problem? This work derives necessary and sufficient conditions for the convergence of Parareal and MGRiT applied to linear problems, along with tight two-level convergence bounds, under minimal additional assumptions on Φ and Ψ. Results all rest on the introduction of a temporal approximation property (TAP) that indicates how Φk must approximate the action of Ψ on different vectors. Loosely, for unitarily diagonalizable operators, the TAP indicates that the fine-grid and coarse-grid time integration schemes must integrate geometrically smooth spatial components similarly, and less so for geometrically high frequency. In the (non-unitarily) diagonalizable setting, the conditioning of each eigenvector, v, must also be reflected in how well Ψv ≈ Φkθv. In general, worst-case convergence bounds are exactly given by min ε < 1 such that an inequality along the lines of ||(Ψ − Φk)1v|| ≤ ε||I − Ψv|| holds for all v. Such inequalities are formalized as different realizations of the TAP in Section 2, and form the basis for convergence of Parareal and MGRiT applied to linear problems.

1. Introduction. Efficiently distributing computational work over many processors, or parallelizing, is fundamental to running large-scale numerical simulations. In the case of partial differential equations (PDEs) in space and time, problems are at least 3-4 dimensional, and as many as seven or more for problems such as radiative transport. Additionally accounting for multiple variables that may have to be solved for, even a moderate number of points in each dimension requires a massive number of unknowns, as well as a high level of parallelism, to obtain an accurate solution. Furthermore, computational power is largely increasing in the number of processors available and less in the power of individual processors, making increased parallelism an important area of research.

Steady state PDEs (non-time-dependent) are typically posed as boundary value problems (BVPs), which provide a natural mechanism to parallelize in space. When time derivates are introduced, adding parallelism in the time dimension is more complicated. In particular, for time-dependent PDEs, it is often the case that only an initial value in time is given. To that end, propagating information through the temporal domain appears to be an inherently sequential process because the initial information can only be propagated one direction, namely forward in time. This is how most time-dependent PDEs are solved – given some initial value in time, a BVP is formulated and discretized in the spatial domain. The IVP is then propagated forward one time step by solving the BVP and applying some time integration routine, and the process repeats based on a new “initial value” in time. However, as the number of processors available to run numerical simulations has increased, so has the interest in so-called parallel-in-time (PiT) methods, which are designed to parallelize the process of integrating forward in time.

Because time integration typically involves solving for a solution at a set of discrete “time points,” this can be represented in block matrix form, where

\begin{equation}
Au = \begin{bmatrix}
    I & 0 & 0 \\
    -\Phi & I & 0 \\
    -\Phi & -\Phi & I \\
    \vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix} = f.
\end{equation}

Here, u_i is the solution at the ith time point, f the right-hand side, and Φ some invertible operator that progresses the solution from time t_i to t_{i+1}. In this setting, classical (sequential) time integration can be seen as a direct (forward) solve of (1). Parallel-in-time methods can typically be posed as some form of

*This work was performed under the auspices of the U.S. Department of Energy under grant number (NNSA) DE-NA0002376 and Lawrence Livermore National Laboratory under contracts B614452 and B627942.

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for convergence. Theory is based on building error-propagation operators and appealing to block-Toeplitz
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for arbitrary time-integration schemes. For the most general results, the only assumption is that the problem
spatial discretization is symmetric and definite as in [35, 36], results here provide exact bounds on convergence
time-integration operators being diagonalizable and unitarily diagonalizable. Indeed, if we assume that the
leads to necessary and sufficient conditions for convergence of error propagation in the
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temporal approximation property
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accurate estimates of convergence over traditional techniques.

Many parallel-in-time methods have been proposed, with varying levels of success. Some of the prominent
methods include full space-time multigrid [7, 17], parallel full approximation scheme in space and time
(PFASST) [6], Parareal [19], and multigrid reduction in time (MGRiT) [8]. Parareal is perhaps the most
well-known and one of the original ideas for parallel-in-time integration. Parareal is effectively a two-
level multigrid method. Time points are partitioned into C-points and F-points, and relaxation consists of
integrating each C-point \(k-1\) time steps forward, based on its current solution (that is, to the final
F-point preceding the next C-point); here \(k\) denotes the coarsening factor. This is followed by a coarse-grid
correction, which approximately inverts the Schur complement of (1). In particular, \(k\) steps on the fine grid,
\(\Phi^k\), are approximated by some operator \(\Psi\) that is cheaper to evaluate. The simplest example is letting \(\Psi\) be
the same time-integration scheme as \(\Phi\), using time steps that are \(k\) times larger. The multigrid reduction in
time (MGRiT) algorithm generalizes this to the multilevel setting, by recursively coarsening the temporal
grid until it is sufficiently small to solve directly at minimal cost.

Analysis of Parareal dates back to [1], where Parareal is analyzed from a time-integration perspective,
looking at the stability and error of Parareal as a time-integration routine with respect to the continuous
problem. In [15], the connection between Parareal and a two-level multigrid algorithm with F-relaxation is
discussed, and initial bounds developed for Parareal that are, in some cases, sharp. An analysis of the
nonlinear case is developed in [13], largely demonstrating that Parareal is applicable to nonlinear problems,
and the convergence of Parareal applied to (nonlinear) turbulent plasma simulations is discussed in [26].
More recently, [35, 36] analyze several specific time integration schemes applied to problems of the form
\(u_t = Au + g\), where \(A\) is symmetric negative definite. Some of the results are tight, but are indeed limited to
specific time-integration schemes and symmetric negative definite finite spatial discretizations. A detailed analysis
of error propagation of two-level MGRiT and Parareal is developed in [5], under the assumption that fine-
and coarse-grid time-stepping operators commute and are diagonalizable. Results in [5] are, to some extent,
a generalization of [15], and also introduce FCF-relaxation to the analysis, a variation in MGRiT that is
not used in Parareal, but that can be important for convergence on difficult problems. Numerical results in [5]
demonstrate the derived bounds appear to be tight when applied to a number of model problems. In fact, the framework developed in this paper is a substantial generalization of that in [5]. One corollary
proven here is that, for a certain class of problems, the bounds developed in [5] are indeed exact to \(O(1/N_c)\),
where \(N_c\) is the number of time points on the coarse grid. Relaxation was generalized further in the recent
paper [14] using an algebraic perspective, similar to the framework used in this paper. Finally, in [11], local
mode analysis techniques are generalized and applied to parabolic parallel-in-time problems, providing more
accurate estimates of convergence over traditional techniques.

Despite a number of works analyzing Parareal and MGRiT, there remains a gap in the literature in
answering the fundamental question, for a general problem, what must a coarse-grid time-stepping scheme
satisfy to see convergent behavior? The main contribution of this paper is the development of, under
minimal assumptions, necessary and sufficient conditions for the convergence in norm of linear Parareal
and two-level MGRiT. A simple \textit{temporal approximation property} (TAP) is introduced that measures how
accurately the fine-grid time-integration scheme approximates the coarse-grid integration scheme. This
leads to necessary and sufficient conditions for convergence of error propagation in the \(\ell^2\) and \(A^*A\)-norms,
including tight bounds on convergence. Additional results are established under further assumptions on the
time-integration operators being diagonalizable and unitarily diagonalizable. Indeed, if we assume that the
spatial discretization is symmetric and definite as in [35, 36], results here provide exact bounds on convergence
for arbitrary time-integration schemes. For the most general results, the only assumption is that the problem
is linear. Most results also require that the same operator \(\Phi\) integrates the solution at all time-points (for
example, there can be no time-dependent differential spatial coefficients). In all cases, some variation on
the TAP provides a simple and relatively intuitive explanation of exactly how the coarse-grid operator must
approximate the fine-grid operator for convergence. Given a time-integration scheme of interest, it can easily
be plugged into results here for a more problem-specific description of necessary and sufficient conditions
for convergence. Theory is based on building error-propagation operators and appealing to block-Toeplitz
matrix theory; it is interesting to point out that a similar approach as used here likely provides a general
convergence framework for the recent idea of using circulant preconditioners in time to solve the space-time system in parallel [16, 23].

The rest of this paper proceeds as follows. Section 2 presents the main theoretical contributions in a concise and accessible manner. Proofs of these results are then established in the following sections. Section 3 discusses convergence of reduction-based multigrid-type methods and derives analytic formulas for two-level error- and residual-propagation operators of MGRiT. The most general theorems are then derived in Section 4, and further analysis based on additional assumptions is given in Section 5. Some of the analysis can be extended to the time-dependent case, and a discussion on that topic is given in Section 6. There are many applications for the new theorems, such as finding the “best” coarse-grid time-stepping scheme for a given problem, better understanding why hyperbolic problems tend to be difficult for parallel-in-time solvers [29], and understanding the effect of spatial coarsening on the convergence of Parareal/MGRiT [18, 28], among others. A brief discussion on implications of results derived here is given in Section 7, and a detailed study is the topic of a forthcoming paper.

2. Statement of results. This section presents the theoretical contributions of this paper; proofs are derived in the sections that follow. The underlying idea is that Parareal and MGRiT are iterative methods to solve a discrete linear system (1), of which the exact solution is simply the space-time vector achieved through sequential time-stepping. Here, convergence theory is derived to provide, under certain assumptions, necessary conditions and sufficient conditions for two-level MGRiT/Parareal to converge to the sequential solution in the \( \ell^2 \)- and \( A^*A \)-norms. In certain cases, tight bounds in the \( \ell^2 \)- and \( A^*A \)-norms can also be put on the convergence factor of each iteration.

Let \( \Phi \) be a fine-grid time-stepping operator and \( \Psi \) a coarse-grid time-stepping operator, and suppose we coarsen in time by a factor of \( k \). The primary results rest on three assumptions:

1. \( \Phi \) and \( \Psi \) are linear.
2. \( \| \Phi \|, \| \Psi \| < 1 \).
3. \( \Phi \) and \( \Psi \) are independent of time; that is, the same operator propagates the solution from time \( t_i \) to \( t_{i+1} \) and from \( t_j \) to \( t_{j+1} \), for all \( i, j \).

Assumption (1) restricts our attention to the linear case, which, as in many problems, allows for a more detailed analysis. The second assumption is generally equivalent to \( \Phi \) and \( \Psi \) being stable time-stepping operators, which is a natural and reasonable thing to assume. The third assumption is the strongest one, yet it still encompasses all problems for which variables do not have time-dependent coefficients, which consists of a large class of space-time PDEs, among other problems. Some of the theory developed here applies to the time-dependent case as well. In particular, sufficient conditions can be derived (see Theorem 31) for convergence if \( \Phi \) and \( \Psi \) are time-dependent, but simultaneously diagonalizable for all time steps, which occurs, for example, in the case of time-dependent reaction terms, or for adaptive time-stepping. Some other results hold in the time-dependent case as well, which are discussed in Section 6.

Further assumptions that come up (yielding stronger convergence results) are:

4. Assume that \( (\Psi - \Phi^k) \) is invertible.
5. Assume that \( \Phi \) and \( \Psi \) commute.
   (a) Assume that \( \Phi \) and \( \Psi \) are diagonalizable.
   (b) Assume that \( \Phi \) and \( \Psi \) are normal (unitarily diagonalizable).

A discussion on these assumptions is provided in Section 2.4; however, it is worth noting that all of them are plausible assumptions for many problems of interest.

Most results here are asymptotic in the sense that certain approximation properties and bounds are given up to order \( O(1/N_c) \), where \( N_c \) is the number of coarse-grid time steps. However, the leading constants in the \( O(1/N_c) \) terms are also generally quite small. Furthermore, because parallel-in-time is only interesting when the number of time steps is relatively large, in practice these terms can generally be considered negligible.

2.1. Tight convergence of \( \ell^2 \)-residual and \( A^*A \)-error. This section introduces necessary and sufficient conditions for convergence of residual in the \( \ell^2 \)-norm and, equivalently, error in the \( A^*A \)-norm (see (14)), including tight bounds in norm. In the case that \( \Phi \) and \( \Psi \) commute, these results hold for error in the \( \ell^2 \)-norm as well (Proposition 4). To start, a new temporal approximation property (TAP) is introduced, which is the fundamental assumption leading to convergence bounds.

**Definition 1** (Temporal approximation property). Let \( \Phi \) denote a fine-grid time-stepping operator and \( \Psi \) denote a coarse-grid time-stepping operator, for all time points, with coarsening factor \( k \). Then, \( \Phi \) satisfies
an $F$-relaxation temporal approximation property with power $p$ ($F$-TAP), with respect to $Ψ$, with constant $ϕ_{F,p}$, if, for all vectors $v$,

\[
\| (Ψ - Φ^k)^p v \| \leq ϕ_{F,p} \left[ \min_{x \in [0,2π]} \| (I - e^{ix}Ψ)^p v \| \right].
\]

Similarly, $Φ$ satisfies an FCF-relaxation temporal approximation property with power $p$ ($FCF$-TAP), with respect to $Ψ$, with constant $ϕ_{FCF,p}$, if, for all vectors $v$,

\[
\| (Ψ - Φ^k)^p v \| \leq ϕ_{FCF,p} \left[ \min_{x \in [0,2π]} \| (Φ^{-k}(I - e^{ix}Ψ))^p v \| \right].
\]

Note that in the case of real-valued operators, the $\min_{x \in [0,2π]}$ is achieved at $x = 0$, in which case $e^{ix}Ψ = Ψ$, and the TAP takes a slightly simpler form.

Necessary and sufficient conditions for convergence of MGRiT and Parareal under various further assumptions are all based on satisfying one of the above approximation properties with a nicely bounded constant, typically less than one. The two variations on a TAP are conceptually simple and can be presented in a more intuitive manner as follows. An F-TAP requires that $Φ^k$ approximates the action of $Ψ$ very accurately for vectors $v \approx Ψv$, and less accurately for $v$ that differs significantly from $Ψv$. If $Φ$ and $Ψ$ commute and have an orthogonal eigenvector basis, this means that $Φ^k must approximate $Ψ$ very accurately for eigenvectors of $Ψ$ with associated eigenvalue close to one in magnitude, and less accurately for smaller eigenvalues. In the context of PDEs, order-one eigenmodes of $Ψ$ typically correspond to the smallest eigenvalues. In the context of PDEs, order-one eigenmodes of $Ψ$ typically correspond to the smallest eigenvalues. However, for vectors $v \approx Ψv$, and less accurately for $v$ that differs significantly from $Ψv$. If $Φ$ and $Ψ$ commute and have an orthogonal eigenvector basis, this means that $Φ^k must approximate $Ψ$ very accurately for eigenvectors of $Ψ$ with associated eigenvalue close to one in magnitude, and less accurately for smaller eigenvalues. In the context of PDEs, order-one eigenmodes of $Ψ$ typically correspond to the smallest eigenvalues. In the context of PDEs, order-one eigenmodes of $Ψ$ typically correspond to the smallest eigenvalues. However, the fine-grid and coarse-grid time-stepping operators must propagate "smooth" modes in the spatial domain (corresponding to small eigenvalues) very similarly. In the case of an FCF-TAP, the additional term $Φ^{-k}$ makes the TAP easier to satisfy because $\| Φ^{-k}v \| > 1$ for all $v$. Think of this as an extra fudge factor to help convergence (at the added expense of FCF-relaxation). When $Ψ$ is not diagonalizable, the eigenvectors do not form a basis, and when $\| (I - e^{ix}Ψ)v \| \approx 0 is a more complicated question. Further analysis of that case, particularly for hyperbolic problems, is ongoing work.

Necessary and sufficient conditions for convergence of MGRiT and Parareal are now presented with respect to the TAP. Let $C^F_ρ$ denote the worst-case convergence factor of residual in the $ℓ^2$-norm and (equivalently) error in the $A^∗A$-norm for all but the first iteration of MGRiT with $F$-relaxation, and likewise for $C^FCF_ρ$ and FCF-relaxation. For example, successive error and residual vectors of MGRiT with $F$-relaxation are bounded via

\[
\| e_{i+1} \|_{A^∗A} \leq C^F_ρ \| e_i \|_{A^∗A}, \quad \| r_{i+1} \| \leq C^F_ρ \| r_i \|,
\]

for all $i \neq 0$.

**Theorem 2 (Necessary and sufficient conditions – $F$-relaxation).** Let $Φ$ denote the fine-grid time-stepping operator and $Ψ$ denote the coarse-grid time-stepping operator, for all time points, with coarsening factor $k$, and $N_c$ time points on the coarse grid. Suppose that $Φ$ satisfies an $F$-TAP with respect to $Ψ$, with constant $ϕ_F$. Then, residual and error propagation for the first iteration of Parareal/MGRiT with $F$-relaxation are bounded in the $ℓ^2$-norm and $A^∗A$-norm, respectively, by $F$. Moreover, all other iterations are bounded by

\[
C^F_ρ < ϕ_F \left( 1 + \| Ψ^{N_c} \| \right),
\]

and satisfying $ϕ_F \left( 1 + \| Ψ^{N_c} \| \right) < 1$ is a sufficient condition for convergence on every iteration but one. Additionally, assume that $(Ψ - Φ^k)$ is invertible. Then,

\[
C^F_ρ \geq \frac{ϕ_F}{1 + O(1/√N_c)},
\]

and satisfying $ϕ_F < 1 + O(1/√N_c)$ is a necessary condition for convergence on every iteration but one.

Finally, assume that $Φ$ and $Ψ$ commute and either, (i) $(Ψ - Φ^k)$ is invertible, or (ii) $Φ$ and $Ψ$ are diagonalizable. Then, $Φ$ satisfying an $F$-TAP, for power $p \geq 1$, with respect to $Ψ$, with $ϕ_{F,p} < (1 + O(1/√N_c))$, is a necessary condition to see convergent behavior of Parareal and two-level MGRiT with $F$-relaxation, after $p$ iterations, with respect to residual in the $ℓ^2$-norm and error in the $A^∗A$-norm.
THEOREM 3 (Necessary and sufficient conditions – FCF-relaxation). Let \( \Phi \) denote the fine-grid time-stepping operator and \( \Psi \) denote the coarse-grid time-stepping operator, for all time points, with coarsening factor \( k \), and \( N_c \) time points on the coarse grid. Suppose that \( \Phi \) satisfies an FCF-TAP with respect to \( \Psi \), with constant \( \varphi_{FCF} \). Then, residual and error propagation for the first iteration of MGRiT with FCF-relaxation are bounded in the \( \ell^2 \)-norm and \( A^*A \)-norm, respectively, by \( \sqrt{k} \). Moreover, all other iterations are bounded by

\[
C^\rho FCF < \varphi_{FCF} \left( 1 + \|\Phi^{-k}\Psi N_c\Phi^k\| \right),
\]

and satisfying \( \varphi_{FCF} \left( 1 + \|\Phi^{-k}\Psi N_c\Phi^k\| \right) < 1 \) is a sufficient condition for convergence on every iteration but one. Additionally, assume that \( (\Psi - \Phi^k) \) is invertible. Then,

\[
C^\rho FCF \geq \frac{\varphi_{FCF}}{1 + O(1/\sqrt{N_c})},
\]

and satisfying \( \varphi_{FCF} < 1 + O(1/\sqrt{N_c}) \) is a necessary condition for convergence on every iteration but one.

Finally, assume that \( \Phi \) and \( \Psi \) commute and either, (i) \( (\Psi - \Phi^k) \) is invertible, or (ii) \( \Phi \) and \( \Psi \) are diagonalizable. Then, \( \Phi \) satisfies an FCF-TAP, for power \( p \geq 1 \), with respect to \( \Psi \), with \( \varphi_{FCF,p} < (1+O(1/\sqrt{N_c})) \), is a necessary condition to see convergent behavior of two-level MGRiT with FCF-relaxation, after \( p \) iterations, with respect to residual in the \( \ell^2 \)-norm and error in the \( A^*A \)-norm.

Theorems 2 and 3 present necessary conditions for convergence of Parareal and MGRiT with minimal assumptions. The first statements in each theorem provide necessary and sufficient conditions that every iteration is convergent in the \( \ell^2 \)-norm for residual and \( A^*A \)-norm for error. Note that (4) and (6) indicate that the F-TAP and FCF-TAP provide fairly tight bounds on convergence of a single iteration. That is, for moderately large \( N_c \), \( \|\Psi N_c\| \approx 0 \) and \( C^\rho \approx \varphi_F \), and likewise for \( C^\rho FCF \). In considering convergence over many iterations, however, this gap can be non-trivial; that is, in some cases, the convergence after \( p \) iterations can be \( \ll \varphi^p \). In fact, in some cases it is possible to see divergent behavior on initial iterations, but eventual convergence (because for some operators, \( \|A^k\| \ll \|A\|^k \)). Under the additional assumption that \( \Phi \) and \( \Psi \) commute (which holds, for example, in using arbitrary single-step multi-stage integration schemes for \( \Phi \) and \( \Psi \) (Section 2.4)), the second statement in each theorem provides necessary conditions to see convergence of residual in the \( \ell^2 \)-norm and error in the \( A^*A \)-norm after an arbitrary number of iterations.

2.2. Tight convergence of \( \ell^2 \)-error. Section 2.1 developed necessary and sufficient conditions for convergence of residual in the \( \ell^2 \)-norm and error in the \( A^*A \)-norm. In certain cases, considering convergence of error in the \( \ell^2 \)-norm requires a slightly different approach. First, Lemma 4 gives additional assumptions such that equivalent results as Theorems 2 and 3 follow immediately for error in the \( \ell^2 \)-norm.

PROPOSITION 4 (Extension to error in the \( \ell^2 \)-norm). If \( \Phi \) and \( \Psi \) commute, then conditions and bounds in Theorems 2 and 3 hold for convergence of error in the \( \ell^2 \)-norm, on all but the last iteration (as opposed to the first).

As will be discussed in Section 2.4, in most cases for which the spatial problem is not re-discretized on the coarse-grid, \( \Phi \) and \( \Psi \) commute. However, if \( \Phi \) and \( \Psi \) do not commute, we need to introduce a modified inverse temporal approximation property to study convergence of error in the \( \ell^2 \)-norm.

DEFINITION 5 (Inverse temporal approximation property). Let \( \Phi \) denote a fine-grid time-stepping operator and \( \Psi \) denote a coarse-grid time-stepping operator, for all time points, with coarsening factor \( k \). Then, \( \Phi \) satisfies an F-relaxation inverse temporal approximation property (F-ITAP), with respect to \( \Psi \), with constant \( \varphi_F \), if, for all vectors \( \mathbf{v} \),

\[
\max_{x \in [0,2\pi]} \| (I - e^{ix}\Psi)^{-1}(\Psi - \Phi^k)\mathbf{v} \| \leq \varphi_F \|\mathbf{v}\|.
\]

Similarly, \( \Phi \) satisfies an FCF-relaxation inverse temporal approximation property (FCF-ITAP), with respect to \( \Psi \), with constant \( \varphi_{FCF} \), if, for all vectors \( \mathbf{v} \),

\[
\max_{x \in [0,2\pi]} \| (I - e^{ix}\Psi)^{-1}(\Psi - \Phi^k)\mathbf{v} \| \leq \varphi_{FCF} \|\Phi^{-k}\mathbf{v}\|.
\]
In the case that \((\Phi - \Psi^k)\) is invertible, \((\Psi - \Phi^k)\) can be moved to the right-hand side. For example, the F-ITAP can be expressed as
\[
\max_{x \in [0, 2\pi]} \| (I - e^{ix}\Psi)^{-1}v \| \leq \hat{\varphi}_F \| (\Psi - \Phi^k)^{-1}v \|,
\]
for all \(v\). Note the ITAP is not considered with respect to powers \(p\). This is because derived results based on powers also assume that \(\Phi\) and \(\Psi\) commute (see Theorems 2 and 3), in which case the results from Section 2.1 hold for error in the \(\ell^2\)-norm, and the ITAP is not necessary.

Similar to Section 2.1, let \(C^F_\rho\) denote the worst-case convergence factor of error in the \(\ell^2\)-norm for all but the last iteration of MGRiT with F-relaxation, and likewise for \(\tilde{C}^{FCF}_\rho\) and FCF-relaxation. For example,
\[
\| e_{i+1} \| \leq C^F_\rho \| e_i \|
\]
for \(i = 0, 1, \ldots, N - 2\), where \(N\) is the number of time steps. The following theorems provide necessary and sufficient conditions for the convergence of error in the \(\ell^2\)-norm.

**Theorem 6 (Necessary and sufficient conditions – F-relaxation).** Let \(\Phi\) denote the fine-grid time-stepping operator and \(\Psi\) denote the coarse-grid time-stepping operator, for all time points, with coarsening factor \(k\), and \(N_c\) time points on the coarse grid. Suppose that \(\Phi\) satisfies an F-ITAP with respect to \(\Psi\), with constant \(\hat{\varphi}_F\). Then, error propagation for the last iteration of Parareal/MGRiT with F-relaxation is bounded in the \(\ell^2\)-norm by \(\sqrt{k}\). Moreover, all other iterations are bounded by
\[
\tilde{C}^F_\rho < \hat{\varphi}_F (1 + \| \Psi^{N_c} \|),
\]
and satisfying \(\hat{\varphi}_F (1 + \| \Psi^{N_c} \|) < 1\) is a sufficient condition for convergence on every iteration but one. Additionally, assume that \((\Psi - \Phi^k)\) is invertible. Then,
\[
\tilde{C}^F_\rho \geq \frac{\hat{\varphi}_F}{1 + \mathcal{O}(1/\sqrt{N_c})},
\]
and satisfying \(\hat{\varphi}_F < 1 + \mathcal{O}(1/\sqrt{N_c})\) is a necessary condition for convergence on every iteration but one.

**Theorem 7 (Necessary and sufficient conditions – FCF-relaxation).** Let \(\Phi\) denote the fine-grid time-stepping operator and \(\Psi\) denote the coarse-grid time-stepping operator, for all time points, with coarsening factor \(k\), and \(N_c\) time points on the coarse grid. Suppose that \(\Phi\) satisfies an FCF-TAP with respect to \(\Psi\), with constant \(\hat{\varphi}_{FCF}\). Then, error propagation for the last iteration of MGRiT with FCF-relaxation is bounded in the \(\ell^2\)-norm by \(\sqrt{k}\). Moreover, all other iterations are bounded by
\[
\tilde{C}^{FCF}_\rho < \hat{\varphi}_{FCF} (1 + \| \Psi^{N_c} \|),
\]
and satisfying \(\hat{\varphi}_{FCF} (1 + \| \Psi^{N_c} \|) < 1\) is a sufficient condition for convergence on every iteration but one. Additionally, assume that \((\Psi - \Phi^k)\) is invertible. Then,
\[
\tilde{C}^{FCF}_\rho \geq \frac{\hat{\varphi}_{FCF}}{1 + \mathcal{O}(1/\sqrt{N_c})},
\]
and satisfying \(\hat{\varphi}_{FCF} < 1 + \mathcal{O}(1/\sqrt{N_c})\) is a necessary condition for convergence on every iteration but one.

As in Section 2.1, for reasonably large \(N_c\), these bounds are tight.

**2.3. Additional results for commuting diagonalizable operators.** As it turns out, results from above can be strengthened in some sense under the additional assumption that \(\Phi\) and \(\Psi\) commute and are diagonalizable. This leads to exact bounds on convergence in a modified norm, that are fairly tight for a large number of iterations, \(p\), as well. By norm equivalence in finite-dimensional spaces, convergence in the modified norm is also necessary and sufficient for (asymptotic) convergence in the \(\ell^2\)- and \(A^*A\)-norms. The constants in norm equivalence depend on the conditioning of the eigenvectors. First, we introduce a less general approximation property based on the assumption that \(\Phi\) and \(\Psi\) commute and are diagonalizable.
DEFINITION 8 (Temporal eigenvalue approximation property). Let $\Phi$ denote a fine-grid time-stepping operator and $\Psi$ denote a coarse-grid time-stepping operator, of size $N_x \times N_x$, for all time points, with coarsening factor $k$. Suppose that $\Phi$ and $\Psi$ commute and are diagonalizable, with eigenvectors given by $\{\lambda_k\}$ and $\{\mu_\ell\}$, respectively. Then, $\Phi$ satisfies an F-relaxation temporal eigenvalue approximation property (F-TEAP), with respect to $\Psi$, with constant $\varphi_F$, if, for $\ell = 0, \ldots, N_x - 1$,

$$|\mu_\ell - \lambda_k| \leq \varphi_F (1 - |\mu_\ell|).$$

Similarly, $\Phi$ satisfies an FCF-relaxation temporal approximation property (FCF-TEAP), with respect to $\Psi$, with constant $\varphi_{FCF}$, if, for $\ell = 0, \ldots, N_x - 1$,

$$|\mu_\ell - \lambda_k| \leq \varphi_{FCF} \frac{1 - |\mu_\ell|}{|\lambda_k|}.$$  

Note that for the TEAP, there is no distinction between powers, because scalars commute.

THEOREM 9 (The diagonalizable case – F-relaxation). Let $\Phi$ denote the fine-grid time-stepping operator and $\Psi$ denote the coarse-grid time-stepping operator, for all time points, with coarsening factor $k$, and $N_x$ time points on the coarse grid. Assume that $\Phi$ and $\Psi$ commute and are diagonalizable, with eigenvectors given as columns of $U$, and that $\Phi$ satisfies an F-TEAP with respect to $\Psi$, with constant $\varphi_F < 1$. Let $e_{p+1}$ denote the error vector of Parareal/MGRiT with F-relaxation after $p+1$ iterations. Then, $\|e_1\|_{(UU^*)^{-1}} \leq k\|e_0\|_{(UU^*)^{-1}}$, and

$$\|e_{p+1}\|_{(UU^*)^{-1}}^2 \leq (\varphi_F^2 - O(1/N_x^2)) \|e_1\|_{(UU^*)^{-1}}^2.$$  

Furthermore, this bound is tight; that is, there exists an initial error $e_0$ such that (12) holds with equality, to $O(1/N_x^2)$.

This also provides necessary and sufficient (asymptotic) conditions for convergence in the $\ell^2$- and $A^*A$-norms. That is, iterations may diverge at first, but will eventually converge in the $\ell^2$- and $A^*A$-norms.

THEOREM 10 (The diagonalizable case – FCF-relaxation). Let $\Phi$ denote the fine-grid time-stepping operator and $\Psi$ denote the coarse-grid time-stepping operator, for all time points, with coarsening factor $k$, and $N_x$ time points on the coarse grid. Assume that $\Phi$ and $\Psi$ commute and are diagonalizable, with eigenvectors given as columns of $U$, and that $\Phi$ satisfies an FCF-TEAP with respect to $\Psi$, with constant $\varphi_{FCF} < 1$. Let $e_{p+1}$ denote the error vector of MGRiT with FCF relaxation after $p + 1$ iterations. Then, $\|e_1\|_{(UU^*)^{-1}} \leq k\|e_0\|_{(UU^*)^{-1}}$, and

$$\|e_{p+1}\|_{(UU^*)^{-1}}^2 \leq \left(\varphi_{FCF}^2 - O(1/N_x^2)\right) \|e_1\|_{(UU^*)^{-1}}^2.$$  

Furthermore, this bound is tight; that is, there exists an initial error $e_0$ such that (13) holds with equality, to $O(1/N_x^2)$.

This also provides necessary and sufficient (asymptotic) conditions for convergence in the $\ell^2$- and $A^*A$-norms. That is, iterations may diverge at first, but will eventually converge in the $\ell^2$- and $A^*A$-norms.

Note that in the case of normal matrices, $U^{-1} = U^*$, and we have that the $(UU^*)^{-1}$-norm is exactly equal to the $\ell^2$-norm. In that case, the TEAP and TAP are equivalent, and we have an exact bound on Parareal and two-level MGRiT convergence of residual in the $\ell^2$-norm and error in the $\ell^2$- and $A^*A$-norms. In the commuting and diagonalizable case, these results are an extension of the upper bounds developed in [5].

Some of these results can be extended to the time-dependent case as well, such as when there are time-dependent reaction terms in a PDE or variable time-step size. Such scenarios are discussion in Section 6.

Remark 11 (Convergence bounds observed in practice). It is worth pointing out that the TAP and Theorems 2, 3, 6, 7, 9, and 10 not only define worst-observable convergence factors of Parareal and two-level MGRiT, but such convergence factors are likely to be observed in practice. In the theoretical derivations that follow, convergence bounds are derived based on minimum and maximum eigenvalues or singular values of block-Toeplitz matrices. In many cases, it can be shown that there are clusters of singular modes or
then Φ

\[ \Phi = I + 4\delta t L^2 + \frac{5\delta t}{6} L^3 + \frac{4\delta t^3}{72} L^4, \]

\[ \Psi = I + 25\delta t L^2 + 25\delta t L^3 + \frac{44\delta t^2}{6} L^4 + \frac{58\delta t^3}{3} L^4 + \frac{26\delta t^4}{3} L^4 \]

\[ = \Phi^2 - \left[ \frac{45}{4} L^5 + \frac{54\delta t}{15} L^6 + \frac{47\delta t^2}{72} L^7 + \frac{4\delta t^3}{170} L^8 \right] \]

\[ = \Phi^2 - \frac{45}{4} L^5 \left[ I + \frac{54\delta t}{15} L + \frac{47\delta t^2}{72} L^2 + \frac{4\delta t^3}{170} L^3 \right]. \]

Thus as long as there does not exist an eigenvalue \( \lambda_i \) of \( L \) that is one of the three roots of

\[ p(\lambda) = 1 + \frac{5\delta t}{15} \lambda + \frac{4\delta t^2}{72} \lambda^2 + \frac{4\delta t^3}{170} \lambda^3, \]

then \( \Phi - \Psi^k \) is invertible. If such an eigenvalue does exist, a slight perturbation to \( \delta t \) would likely nullify that property. In the case that \( \Phi \) and \( \Psi \) are based on different operators (for example, if spatial coarsening is used [18]), the analysis is more difficult, but it is generally not likely for \( \Phi^k \) to exactly preserve an eigenmode of \( \Psi \).

3. Convergence theory framework.

3.1. Error and residual propagation. This work derives necessary and sufficient conditions for convergence of Parareal and MGRiT using a linear-algebra framework. In particular, we pose these algorithms as iterative methods, and seek to pose bounds on how they propagate error and residual corresponding to the linear system in (1). Let \( \hat{x} \) be the exact solution to \( Ax = b \) and \( x_i \) the approximate solution at the \( i \)th iterate. Then the error and residual at iteration \( k \), respectively, are given, by

\[ e_i = \hat{x} - x_i, \]

\[ r_i = b - Ax_i = A(\hat{x} - x_i) = A e_i. \]

Note that, assuming \( A \) is nonsingular, \( e_i = 0 \) if and only if \( r_i = 0 \). Although in practice users typically want the error to be zero, the error cannot be easily measured in practice, while the residual can. Thus, bounding error and residual propagation are both of interest, and asymptotically (in convergence) equivalent.

Let \( \mathcal{E} \) denote the error-propagation operator and \( \mathcal{R} \) the residual-propagation operator. For fixed-point iterative methods, error propagation takes the form \( \mathcal{E} = I - M^{-1} A \), where \( M \) is some approximation of \( A \). From above, observe that

\[ e_i = \mathcal{E}^i e_0 \iff A^{-1} r_i = \mathcal{E}^i A^{-1} r_0 \iff r_i = (A \mathcal{E} A^{-1})^i r_0. \]

Thus, residual propagation is formally similar to error propagation, where \( \mathcal{R} = A \mathcal{E} A^{-1} = I - AM^{-1} \). In this form, error propagation is a measure of \( M \) as a left approximate inverse of \( A \) and residual propagation a
A simple two-level reduction-based multigrid method is given by letting

\begin{equation}
\|R\|^2 = \sup_{x \neq 0} \frac{\langle Rx, Rx \rangle}{\langle x, x \rangle} = \sup_{x \neq 0} \frac{\langle A^* A^{-1} x, A^* A^{-1} x \rangle}{\langle x, x \rangle} = \sup_{y \neq 0} \frac{\langle A^* A y, \tilde{E} y \rangle}{\langle A^* A y, y \rangle} = \|E\|^2_{A^* A};
\end{equation}

that is, residual propagation in the ℓ²-norm is equivalent to error propagation in the $A^* A$-norm. In general, the $A^* A$-norm is considered a stronger norm, also consistent with the result that one can have an arbitrarily accurate left approximate inverse that makes for a poor right approximate inverse [24].

Notationally, let $\Phi$ denote the fine-grid time stepping operator and $\Psi$ the coarse-grid time-stepping operator. Ideal interpolation is defined through

\begin{equation}
Z\Phi = \Psi Z,
\end{equation}

for symmetric positive definite matrices, where $Z$ is referred to as restriction by injection, followed by an exact solve on F-points, and matrices $A, P, \text{and } R$ take the following block forms:

\begin{equation}
A = \begin{bmatrix}
A_{ff} & A_{fc} \\
A_{cf} & A_{cc}
\end{bmatrix}, \quad P = \begin{bmatrix} W \\ I \end{bmatrix}, \quad R = \begin{bmatrix} Z & I \end{bmatrix}.
\end{equation}

A simple two-level reduction-based multigrid method is given by letting $Z = -A_{cf} A_{ff}^{-1}$ and $W = 0$. In this case, coarse-grid correction yields zero error at C-points [21]. The restriction operator defined through $Z$ is referred to as “ideal restriction,” where it is ideal in being the unique restriction operator that yields an exact coarse-grid correction at C-points. Following this with an exact solve on F-points as a relaxation scheme then yields an exact solution at F-points, without modifying the solution at C-points [21, 22]. Thus, the solution is exact and we have a two-grid reduction, where solving $A x = b$ is reduced to solving one system based on $A_{ff}$ and one system based on $R A P$.

MGRiT is also a reduction-based multigrid method, instead using the so-called “ideal interpolation” operator. Ideal interpolation is defined through $W = -A_{cf}^{-1} A_{fc}$. For symmetric positive definite matrices, ideal interpolation is ideal in a certain theoretical sense [10, 37]. In the nonsymmetric setting, ideal interpolation is ideal as the unique interpolation operator that eliminates the contribution of F-point residuals to the coarse-grid right-hand side [8, 22]. When coupled with $R = \begin{bmatrix} 0 & I \end{bmatrix}$, referred to as restriction by injection, coarse-grid correction then yields zero residual at C-points. Note that an exact solve on F-points yields zero residual at F-points. Thus, coarse-grid correction with $P_{\text{ideal}}$ and restriction by injection, preceded by an exact solve on F-points, also yields an exact two-level reduction [22].

In the algebraic setting, $A_{ff}^{-1}$ is often not easily computed, so approximations are made, such as in AMG methods based on an approximate ideal restriction (AIR) [21, 22]. MGRiT and the system in (1) are unique in that the action of $A_{ff}^{-1}$ can be computed, so ideal interpolation and exact F-relaxation are feasible choices. In this case, assuming a block form as in (15), $R A P_{\text{ideal}} = S_A$, where $S_A$ is the Schur complement.

\footnote{Note that the ordering is important: coarse-grid correction with $P_{\text{ideal}}$ and restriction by injection, followed by an exact solve on F-points, does not yield a two-grid reduction [22].}
independent of \( R \) [22]. Although we can express a closed form for \( S_A \) (see (27)), \( S_A \) is not amenable to a recursive multilevel algorithm. In particular, one time step on the Schur-complement coarse grid simply consists of taking \( k \) steps on the fine grid, which is no cheaper than solving the fine grid problem directly. Because of this, MGRiT is based on a non-Galerkin coarse grid, where we approximate \( \Phi^k \) with some other operator \( \Psi \). Usually, this is accomplished by approximating \( k \) steps of size \( \delta t \) with one step of size \( k \delta t \).

The following section derives error and residual-propagation operators for MGRiT. Further details on reduction-based multigrid methods can be found in [20, 21, 22, 27], and further details on the MGRiT algorithm can be found in, for example, [5, 8, 9, 12].

3.3. Error and residual-propagation operators. Consider residual and error propagation of two-level MGRiT, with a non-Galerkin coarse-grid operator, \( B_\Delta^{-1} \), to approximate the Schur complement, \( A_\Delta := S_A^{-1} \) (\( A_\Delta \) is used to be consistent with previous works [5, 8]). Because MGRiT is based on ideal interpolation, here we use a pre-relaxation scheme of F-relaxation or FCF-relaxation [22]. It is important to note that, in the case of the MGRiT algorithm, F-relaxation and C-relaxation refer to an exact solve on F- and C-points, respectively.

Recall that error propagation of relaxation and coarse-grid correction each take the form of a classic fixed-point method, \( I - M^{-1}A \). The approximate inverses for an exact solve on F-points, an exact solve on C-points, and coarse-grid correction are, given, respectively, by

\[
M_F^{-1} = \begin{bmatrix} A_{ff} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_C^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & A_{cc}^{-1} \end{bmatrix},
\]

\[
M_{cgc}^{-1} = \begin{bmatrix} -A_{ff}^{-1}A_{fc} \\ I \end{bmatrix} B_\Delta^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} 0 & -A_{ff}^{-1}A_{fc}B_\Delta^{-1} \\ 0 & B_\Delta^{-1} \end{bmatrix}.
\]

Then, error and residual propagation of two-level MGRiT with pre F-relaxation are given by \( \mathcal{E}_F = I - (M_F^{-1} + M_{cgc}^{-1}AM_F^{-1})A \) and \( \mathcal{R}_F = I - A(M_F^{-1} + M_{cgc}^{-1}AM_F^{-1}) \), respectively. Note that

\[
M_{cgc}^{-1}AM_F^{-1} = \begin{bmatrix} 0 & -A_{ff}^{-1}A_{fc}B_\Delta^{-1} \\ 0 & B_\Delta^{-1} \end{bmatrix} \begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix} \begin{bmatrix} A_{ff}^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -A_{ff}^{-1}A_{fc}B_\Delta^{-1}A_{cf}A_{ff}^{-1} \\ B_\Delta^{-1}A_{cf}A_{ff}^{-1} \end{bmatrix}.
\]

Combining,

\[
\mathcal{R}_F = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix} \begin{bmatrix} A_{ff}^{-1} + A_{ff}^{-1}A_{fc}B_\Delta^{-1}A_{cf}A_{ff}^{-1} - A_{ff}^{-1}A_{fc}B_\Delta^{-1} \\ -B_\Delta^{-1}A_{fc}A_{ff}^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I - A_\Delta B_\Delta^{-1} \end{bmatrix} R_{\text{ideal}} \tag{16}
\]

\[
\mathcal{E}_F = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A_{ff}^{-1} + A_{ff}^{-1}A_{fc}B_\Delta^{-1}A_{cf}A_{ff}^{-1} - A_{ff}^{-1}A_{fc}B_\Delta^{-1} \\ -B_\Delta^{-1}A_{fc}A_{ff}^{-1} \end{bmatrix} \begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix} = \begin{bmatrix} 0 & -A_{ff}^{-1}A_{fc}(I - B_\Delta^{-1}A_\Delta) \\ 0 & I - B_\Delta^{-1}A_\Delta \end{bmatrix} = P_{\text{ideal}} \begin{bmatrix} 0 & 0 \\ I - B_\Delta^{-1}A_\Delta \end{bmatrix} \tag{17}
\]

To consider FCF-relaxation, note that MGRiT residual and error propagation for FCF-relaxation is equivalent to multiplying \( \mathcal{R}_F \) and \( \mathcal{E}_F \) by residual and error propagation for FC-relaxation, which are respectively given by

\[
I - A(M_F^{-1} + M_{cgc}^{-1}AM_F^{-1}) = I - \begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix} \begin{bmatrix} A_{ff}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ A_{cf}^{-1}A_{fc}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.
\]
It follows that residual and error propagation of two-level MGRiT with pre FCF-relaxation are given by

\[
R_{FCF} = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]

It follows that residual and error propagation of two-level MGRiT with pre FCF-relaxation are given by

\[
R_{FCF} = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]

Note from (14) that \(\|E^p\|_{A^*A} = \|R^p_F\|\) and \(\|E^p_{FCF}\|_{A^*A} = \|R^p_{FCF}\|\), for \(p \geq 1\).

3.4. MGRiT matrices. So far, derivations have been purely algebraic and assumed no structure to the linear system. Focusing on the MGRiT framework, consider the MGRiT system (1) and suppose we coarsen in time by a factor of \(k\). This corresponds to partitioning time points into C-points and F-points, such that for every \(k\) points, \(k - 1\) are F-points. For convenience, assume that the first and last time points are C-points, in which case the total number of C-points is given by \(N_c = 1 + \frac{N - 1}{k}\), where \(N\) is the total number of time points. Then, blocks in an FC-partitioning of the matrix \(A\) (15) take the following form:

\[
A_{ff} = \begin{bmatrix}
I & -\Phi & I & & & & \\
& \ddots & & -\Phi & I & & \\
& & & \ddots & & \vdots & \\
& & & & \ddots & -\Phi & I \\
& & & & & \ddots & \vdots \\
& & & & & & \vdots \\
& & & & & & I \\
\end{bmatrix}, \quad A_{fc} = \begin{bmatrix}
-\Phi & & & & & & 0 \\
0 & & & & & & \vdots \\
& \vdots & & & \vdots & & \vdots \\
& & \vdots & & \vdots & & \vdots \\
0 & & & & & & \vdots \\
& & & \vdots & & \vdots & \vdots \\
& & & & \vdots & \vdots & \vdots \\
& & & & & 0 & \vdots \\
\end{bmatrix}
\]

\[
A_{cf} = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}, \quad A_{cc} = \begin{bmatrix}
I & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & 0 & 0 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & 0 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 & 0 & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & 0 & 0 & \vdots \\
\end{bmatrix}
\]

Dotted lines are used to highlight the block nature, where each group of \(k - 1\) F-points are adjacent in the time domain, while all C-points are disconnected. Next, further matrix forms that arise in residual and error
propagation are derived:

\[
(A_{ff})^{-1} = \begin{bmatrix}
I & \Phi & I \\
\Phi^k & I & \Phi \\
\vdots & \ddots & \ddots \\
\Phi & \cdots & \Phi & I \\
\end{bmatrix},
\]

(24)

\[
-A_c(A_{ff})^{-1} = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & 0 \\
\Phi^k & \cdots & \cdots & \cdots & \Phi \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
\end{bmatrix},
\]

(25)

\[
-(A_{ff})^{-1}A_{fc} = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & 0 \\
\Phi^k & \cdots & \cdots & \cdots & \Phi \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
\end{bmatrix}.
\]

(26)

Note from (26) that the action of \(X A_c(A_{ff})^{-1} A_{fc}\) shifts all columns of \(X\) to the left, and scales all entries by \(\Phi^k\). This will be useful in future derivations.

Recall that \(RAP_{ideal}\) is given by the Schur complement of \(A\), independent of \(R\). To be consistent with \([5, 8, 11]\), denote \(A_\Delta := S_A = RAP_{ideal}\). From above, it follows that

\[
A_\Delta = A_{cc} - A_c(A_{ff})^{-1} A_{fc}
\]

(27)

Observe that the coarse-grid operator consists of taking \(k\) time steps with the time-stepping function \(\Phi\). Because this is no cheaper to evaluate than \(k\) individual steps of \(\Phi\) – that is, propagating \(k\) steps on the fine grid – a non-Galerkin coarse-grid is used. Instead of taking \(k\) time steps of size \(\delta t\), corresponding to \(\Phi^k\), \(k\) steps are approximated by some operator \(\Psi\),

\[
A_\Delta B_\Delta^{-1} = \begin{bmatrix}
I & \Phi^k & I \\
\Psi^2 & \Psi & I \\
\vdots & \cdots & \cdots & \cdots & I \\
\Psi^{N_c-1} & \Psi^{N_c-2} & \cdots & \cdots & I \\
\end{bmatrix}
\]

(28)

\[
(\Psi - \Phi^k) \Psi^{N_c-2} \begin{bmatrix}
I \\
\Psi - \Phi^k \\
\vdots \\
(\Psi - \Phi^k)^{N_c-3} \end{bmatrix} \Psi - \Phi^k I.
\]
the Toeplitz matrix corresponds with a Fourier generator function,

\[ I - A_\Delta B_\Delta^{-1} = \text{diag}(\Psi - \Phi^k) \begin{bmatrix} 0 & I & 0 & \ldots & \Psi^{N_c-2} & \Psi^{N_c-3} & \ldots & I & 0 \end{bmatrix}, \]

(29)

\[ I - B_\Delta^{-1} A_\Delta = \begin{bmatrix} 0 & I & 0 & \ldots & \Psi^{N_c-2} & \Psi^{N_c-3} & \ldots & I & 0 \end{bmatrix} \text{diag}(\Psi - \Phi^k). \]

(30)

Note that if \( \Phi \) and \( \Psi \) commute, then \( I - B_\Delta^{-1} A_\Delta = I - A_\Delta B_\Delta^{-1} \).

4. The general case. This section derives a sequence of linear algebra lemmas, which are then used to present and prove a more precise version of Theorems 2 and 3. The underlying idea is that the \( \ell^2 \)-norm of an operator is given by the largest singular value, which is also equivalent to one divided by the smallest nonzero singular value of the respective pseudoinverse. Here, we rely on block-Toeplitz matrix theory to place tight bounds on the maximum and minimum singular values of operators related to error and residual propagation.

From Section 3.3 and (17), (19), (21), and (23), error- and residual-propagation operators for \( p \) iterations of two-level MGRiT, with F-relaxation and FCF-relaxation, take the following forms:

\[ E_F^p = P_{\text{ideal}}(I - B_\Delta^{-1} A_\Delta)^p, \quad E_{FCF}^p = P_{\text{ideal}} \left( (I - B_\Delta^{-1} A_\Delta) A_{cf} A_{ff}^{-1} A_{fc} \right)^p, \]

(31)

\[ R_F^p = (I - A_\Delta B_\Delta^{-1})^p R_{\text{ideal}}, \quad R_{FCF}^p = (I - A_\Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc} \]

(32)

where matrices are as in Section 3.4. Notice that convergence over \( p > 1 \) iterations in all cases is determined by bounding either \( \|(I - A_\Delta B_\Delta^{-1})^p\| < 1 \) for F-relaxation or \( \|(I - A_\Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc}^p\| < 1 \) for FCF-relaxation. The leading (trailing) factor of \( R_{\text{ideal}} (P_{\text{ideal}}) \) accounts for the single iteration in Theorems 2 and 3 on which convergence may not be observed. The following Lemma proves that \( \|R_{\text{ideal}}\|, \|P_{\text{ideal}}\| < \sqrt{k} \).

**Lemma 12** (Bounds on \( \|R_{\text{ideal}}\|, \|P_{\text{ideal}}\| \)). Let \( \|\Phi\| < 1 \). Then,

\[ \|R_{\text{ideal}}\| = \|P_{\text{ideal}}\| < \sqrt{k}. \]

**Proof.** From (25), note that \( \|R_{\text{ideal}}\| = \sigma_{\max}(R_{\text{ideal}}) = \sqrt{\lambda_{\max}(R_{\text{ideal}} R_{\text{ideal}}^*)} \), where \( R_{\text{ideal}} R_{\text{ideal}}^* \) is block diagonal, with an identity in the first block, and the rest given by \( \sum_{i=0}^{k-1} \Phi^i (\Phi^e)^* \). Then,

\[ \|R_{\text{ideal}}\| = \sqrt{\sum_{i=0}^{k-1} \Phi^i (\Phi^e)^*} \leq \sum_{i=0}^{k-1} \|\Phi^i (\Phi^e)^*\| = \sum_{i=0}^{k-1} \|\Phi^e\|^2 < \sqrt{k}. \]

An analogous derivation confirms that \( \|P_{\text{ideal}}\| < \sqrt{k} \).

Note that Lemma 12 is not necessarily tight, but sufficient to prove that error cannot diverge significantly in the first/last iteration.

4.1. Residual-propagation and \( I - A_\Delta B_\Delta^{-1} \). Now, we consider the maximum singular value of \( I - A_\Delta B_\Delta^{-1} \) and \( (I - A_\Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc} \), which arises in residual propagation. From (26) and (29), it is clear that both of these operators are block-Toeplitz matrices. Appealing to block-Toeplitz matrix theory, asymptotically (in \( N_c \)) tight bounds can be placed on the maximum singular value by way of considering the operator’s generator function. Let \( \alpha_i \) denote the (potentially matrix-valued) Toeplitz coefficient for the \( i \)th diagonal of a (block) Toeplitz matrix, where \( \alpha_0 \) is the diagonal, \( \alpha_{-1} \) the first subdiagonal, and so on. Then the Toeplitz matrix corresponds with a Fourier generator function,

\[ F(x) = \sum_{i=-\infty}^{\infty} \alpha_i e^{-ix}. \]
The following theorems introduce specific results from the field of block-Toeplitz operator theory, which prove important in further derivations.

**Theorem 13** (Minimum eigenvalue of Hermitian block-Toeplitz operators [25, 30, 32]). Let $T_N(F)$ be an $N \times N$ Hermitian block-Toeplitz matrix, with self-adjoint, continuous generating function $F(x) : [0, 2\pi] \to \mathbb{C}^{m \times m}$, where $F(x) = F(x)^*$, and the minimum eigenvalue of $F(x)$ is not constant. Then, the smallest eigenvalue of $T_N(F)$ is given by

$$\lambda_{\text{min}}(T_N) = \min_{x \in [0, 2\pi]} \lambda_{\text{min}}(F(x)) + O(N^{-\alpha}),$$

where $\alpha > 0$ is the order of the highest-order zero in $x$ of

$$\lambda_{\text{min}}(F(x)) = \left[ \min_{y \in [0, 2\pi]} \lambda_{\text{min}}(F(y)) \right].$$

**Theorem 14** (Maximum singular value of block-Toeplitz operators [34]). Let $T_N(F)$ be an $N \times N$ block-Toeplitz matrix, with continuous generating function $F(x) : [0, 2\pi] \to \mathbb{C}^{m \times m}$. Then, the maximum singular value is bounded above by

$$\sigma_{\text{max}}(T_N(F)) \leq \max_{x \in [0, 2\pi]} \sigma_{\text{max}}(F(x)),$$

for all $N \in \mathbb{N}$.

Theorem 14 is now used in the following theorem to derive upper bounds on the maximum singular values of interest.

**Theorem 15** (Sufficient conditions). Let $\Phi$ denote the fine-grid time-stepping operator and $\Psi$ denote the coarse-grid time-stepping operator, with coarsening factor $k$, and $N_c$ coarse-grid time points. Assume that $\Phi$ satisfies an F-TAP with respect to $\Psi$, with constant $\varphi_{F,1}$. Then,

$$\|I - A_{\Delta}B_{\Delta}^{-1}\| \leq \varphi_{F,1} (1 + \|\Psi N_c\|).$$

Similarly, assume that $\Phi$ satisfies an FCF-TAP with respect to $\Psi$, with constant $\varphi_{FCF,1}$. Then,

$$\| (I - A_{\Delta}B_{\Delta}^{-1})A_{cf}A_{ff}^{-1}A_{fc}^{-1} \| \leq \varphi_{FCF,1} (1 + \|\Phi^{-k}\Psi N_c\Phi^k\|).$$

**Proof.** Notice from (29) that $I - A_{\Delta}B_{\Delta}^{-1}$ is a block-Toeplitz matrix with generating coefficients $\alpha_i = (\Psi - \Phi^k)\Psi^{-(1+i)}$ for $i = -1, \ldots, -N_c$ and $\alpha_i = 0$ for $i \geq 0$. Let $F_F(x)$ denote this generating function. Under the assumption that $\|\Phi\| < 1$ (to apply a geometric series argument to $\sum \Psi^i$),

$$F_F(x) = (\Psi - \Phi^k)^N \sum_{r=0}^{N-1} \Psi^r e^{ix}$$

which is again a block-Toeplitz matrix. By properties of block-Toeplitz matrices, we can write

$$F_F(x) = e^{ix}(\Psi - \Phi^k)(I - e^{inx}\Psi N_c)(I - e^{ix}\Psi)^{-1}.$$ 

Recall, under the assumption of an F-TAP, $\| (\Psi - \Phi^k) \| \leq \varphi_{F,1} \left[ \min_{x \in [0, 2\pi]} \| (I - e^{ix}\Psi) \| \right]$ for all $\mathbf{v}$.
Theorem 14 then yields
\[
\|I - A_\Delta B_\Delta^{-1}\| \leq \max_{\mathbf{x} \in [0,2\pi]} \sigma_{\text{max}}(F_{cF}(x))
\]
\[
= \max_{\mathbf{x} \neq 0, \mathbf{v} \neq 0} \frac{\|[(\Psi - \Phi^k)(I - e^{iN\mathbf{x}}\Psi^N)(I - e^{i\mathbf{x}})\Psi^{-1}]\mathbf{v}\|}{\|\mathbf{v}\|}
\]
\[
\leq \max_{\mathbf{v} \neq 0} \frac{\|[(\Psi - \Phi^k) + (\Psi - \Phi^k)\Psi^N]\mathbf{v}\|}{\min_{\mathbf{x} \in [0,2\pi]} \|[(I - e^{i\mathbf{x}})\Psi]\mathbf{v}\|}
\]
\[
\leq \max_{\mathbf{v} \neq 0} \frac{\|[(\Psi - \Phi^k)\Psi^N]\mathbf{v}\|}{\min_{\mathbf{x} \in [0,2\pi]} \|[(I - e^{i\mathbf{x}})\Psi]\mathbf{v}\|}
\]
\[
= \varphi_{F,1}(1 + \|\Psi^N\|).
\]

A similar proof follows for the case of FCF-relaxation, where the generator function, \(F_{\text{FCF}}\), has coefficients \(\alpha_i = (\Psi - \Phi^k)\Psi^{-1} + \Phi^k\) for \(i = -1, \ldots, -N_c\) and \(\alpha_i = 0\) for \(i \geq 0\). Then, by assumption of an FCF-TAP1 with constant \(\varphi_{\text{FCF},1}\),
\[
\|I - A_\Delta B_\Delta^{-1}A_{cf}A_{ff}^{-1}A_{f}\| \leq \max_{\mathbf{x} \in [0,2\pi]} \sigma_{\text{max}}(F_{\text{FCF}}(x))
\]
\[
= \max_{\mathbf{x} \neq 0, \mathbf{v} \neq 0} \frac{\|[(\Psi - \Phi^k)(I - e^{iN\mathbf{x}}\Psi^N)(I - e^{i\mathbf{x}})\Phi^{k}\mathbf{v}\|}{\|\mathbf{v}\|}
\]
\[
\leq \max_{\mathbf{v} \neq 0} \frac{\|[(\Psi - \Phi^k)\Psi^N]\mathbf{v}\|}{\min_{\mathbf{x} \in [0,2\pi]} \|[(I - e^{i\mathbf{x}})\Psi]\mathbf{v}\|}
\]
\[
\leq \max_{\mathbf{v} \neq 0} \frac{\|[(\Psi - \Phi^k)\Psi^N]\mathbf{v}\|}{\min_{\mathbf{x} \in [0,2\pi]} \|[(I - e^{i\mathbf{x}})\Psi]\mathbf{v}\|}
\]
\[
= \varphi_{\text{FCF},1}(1 + \|\Phi^{-k}\Psi^N\Phi^k\|).
\]

Next, a more technical path is pursued, where the maximum singular values of \((I - A_\Delta B_\Delta^{-1})^p\) and \(((I - A_\Delta B_\Delta^{-1})A_{cf}A_{ff}^{-1}A_{f})^p\) are analyzed by means of the minimum singular value of the respective pseudoinverses. First, a pseudoinverse is derived for operators of the form \((I - A_\Delta B_\Delta^{-1})^p\) and \(((I - A_\Delta B_\Delta^{-1})A_{cf}A_{ff}^{-1}A_{f})^p\), for \(p \geq 1\). These pseudoinverses almost take the form of finite block Toeplitz matrices, and we appeal again to Toeplitz matrix theory to bound the smallest nonzero singular value from above.

First, some general pseudoinverses and their properties are derived. Let \(f, g, \) and \(h\) be invertible scalars or operators and define
\[
A_0 = \begin{bmatrix}
g \quad g \\
g \quad g \\
\end{bmatrix} = \begin{bmatrix}
0 \quad I \quad 0 \\
f^2 \quad f \quad I \quad 0 \\
\vdots \quad \vdots \quad \vdots \quad \ddots \\
gf^{2h} \quad gf \quad gh \quad 0 \\
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\begin{bmatrix}
0 \\
gfh \\
gfh \\
\end{bmatrix} \\
\begin{bmatrix}
gf \quad 0 \\
gf \quad 0 \\
\vdots \quad \vdots \\
gh \quad 0 \\
\end{bmatrix} \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\begin{bmatrix}
0 \\
gfh \\
gfh \\
\end{bmatrix} \\
\begin{bmatrix}
\begin{bmatrix}
gf \quad 0 \\
gf \quad 0 \\
\vdots \quad \vdots \\
gh \quad 0 \\
\end{bmatrix} \\
\end{bmatrix} \\
\end{bmatrix}
\]

\]
(35) \[ A_1 = \begin{bmatrix} g & g & \cdots & g \\ g & g & \cdots & g \\ \vdots & \vdots & \ddots & \vdots \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix} \begin{bmatrix} I & I & \cdots & I \\ f & f & \cdots & f \\ f^2 & f^2 & \cdots & f^2 \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} h \\ h \\ \vdots \end{bmatrix} \] 

Note that (34) and (35) are general matrix forms, which encompass the coarse-grid correction and diagonal blocks of two-grid residual and error propagation for MGRIT, with F- and FCF-relaxation; in particular, \( I - A_{\Delta}B_{\Delta}^{-1} \) (29) takes the form of (34) and \( (I - A_{\Delta}B_{\Delta}^{-1})A_{c}fA_{f}^{-1}A_{f} \) (26, 29) takes the form of (35), with an additional zero row due to FCF-relaxation (26). To construct pseudoinverses for operators of these forms, recall the four properties that define a pseudoinverse: \( AA^\dagger A = A \), \( A^\dagger AA^\dagger = A^\dagger \), \( (AA^\dagger)^* = AA^\dagger \), and \( (A^\dagger A)^* = A^\dagger A \). The subtle part of constructing a pseudoinverse is preserving the (not full rank) image and kernel of \( A \). However, matrices in (34) and (35) have the nice property that they are full rank in the first \( n - 1 \) or \( n - 2 \) rows and columns, respectively. In the case of (34), note that

\[
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ \end{bmatrix} = \begin{bmatrix} 0 \\ I \\ \vdots \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} \]

Then, if we can build a matrix \( \hat{A}_0^\dagger \) such that

(36) \[ A_0\hat{A}_0^\dagger = \begin{bmatrix} 0 & I \\ I & I \\ \vdots & \vdots \end{bmatrix}, \quad \hat{A}_0^\dagger A_0 = \begin{bmatrix} I & 0 \\ I & 0 \\ \vdots & \vdots \end{bmatrix}, \quad \hat{A}_0^\dagger \hat{A}_0 = \begin{bmatrix} 0 & I \\ I & I \\ \vdots & \vdots \end{bmatrix} = \hat{A}_0^\dagger, \]

it follows that all four properties of a pseudoinverse are satisfied. A similar result holds for \( A_1 \). We now have all the tools needed to construct a pseudoinverse of \( A_0 \) and \( A_1 \), which is summarized in the following Lemma.

**Lemma 16.** Let \( f \), \( g \), and \( h \) be invertible operators and \( A_0 \) and \( A_1 \) be matrices defined as in (34) and (35), respectively. Then, the unique pseudoinverses of \( A_0 \) and \( A_1 \) are given by

(37) \[ A_0^\dagger = \begin{bmatrix} g & 0 & \cdots & 0 \\ g & g & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g & g & \cdots & 0 \\ \end{bmatrix}^\dagger = \begin{bmatrix} h^{-1}g^{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A_1^\dagger = \begin{bmatrix} h^{-1}g^{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]
\[
A_l^i = \begin{bmatrix}
0 & 0 & \cdots \\
0 & 0 & \cdots \\
gf & 0 & \cdots \\
gf^2 & gf & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
gf & 0 & \cdots & \cdots & \cdots \\
gfh & 0 & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots 
\end{bmatrix}^+ = \begin{bmatrix}
0 & 0 & h^{-1}g^{-1} & \cdots & \cdots \\
0 & 0 & -h^{-1}fg^{-1} & h^{-1}g^{-1} & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

Proof. The third relation in (36) simply requires that the first column of \(A_0^1\) is zero. Working through the system of equations established by the first two relations (and similar relations for \(A_1\)) yields the operators in (37) and (38).

Notice that the pseudoinverse of \(A_1\) is effectively equivalent to that of \(A_0\), except with an additional zero row and column. The only difference this leads to in the final results is an \(O(1/\sqrt{N_e} - 1)\) perturbation. For large \(N_e\), this difference is arbitrary and, for simplicity’s sake, F-relaxation and FCF-relaxation are both treated in the form \(A_0\) moving forward.

The following Lemma generalizes this result, deriving the pseudoinverse for general operators of the form in (34) raised to powers.

Lemma 17 (Pseudoinverse for matrix powers). Let \(A_0\) be as in (34) and define the Toeplitz matrix

\[
\mathcal{T}_0 := \begin{bmatrix}
-h^{-1}fg^{-1} & h^{-1}g^{-1} & \cdots \\
h^{-1}fg^{-1} & h^{-1}g^{-1} & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
h^{-1}fg^{-1} & h^{-1}g^{-1} & \cdots \\
h^{-1}fg^{-1} & h^{-1}g^{-1} & \cdots \\
\end{bmatrix}.
\]

Then, for \(p \geq 1\),

\[
(A_0^p)^+ = \begin{bmatrix} I & 0_{p \times p} \end{bmatrix} \mathcal{T}_0^p \begin{bmatrix} 0_{p \times p} \ I \end{bmatrix}.
\]

Proof. The case of \(p = 1\) was proven in Lemma 16, with pseudoinverse denoted by \(A_0^1\). Now let \(A_0^1\) and \(A_1^1\) be tentative left and right pseudoinverses for \(A_0^p\), \(p > 1\). We start the proof by building \(A_0^1\) and \(A_1^1\) to satisfy certain properties of the pseudoinverse, and conclude by merging them in a certain way to derive \((A_0^p)^+\).

First, note that \(A_0^p\) is a strictly lower triangular matrix, with zeros on the diagonal and first \(p - 1\) subdiagonals. Building on the proof of Lemma 16 let us build \(A_0^1\) such that \(A_0^1A_0^p\) is diagonal, with zeros on the last \(p\) entries, and the identity elsewhere (similar to (36)). This immediately satisfies two properties of a pseudoinverse, \((A_0^1A_0^p)^* = A_0^1A_0^p\) and \(A_0^pA_1^1A_0^p = A_0^p\). To do so, let us start by considering \((A_0^1)^p\) as a naive attempt at a pseudoinverse for \(A_0^p\):

\[
(A_0^1)^pA_0^p = (A_0^p)^{p-1} \begin{bmatrix} I & 0 \end{bmatrix} A_0A_0^{p-2}.
\]

Here, \(\begin{bmatrix} I & 0 \end{bmatrix} A_0\) simply eliminates the final row of \(A_0\). Note from (37) that only the second-to-last row of \(A_0^1\) depends on the final row of \(A_0\). If we then apply \(A_0^1\) again,

\[
(A_0^1)^pA_0^p = (A_0^p)^{p-2} \begin{bmatrix} I & 0 & e_1 \\
e_0 & 0 & 0 \end{bmatrix} A_0A_0^{p-3},
\]

where \(e_0\) and \(e_1\) are an error vector and scalar. Now, only the second-to-last and last rows of \(A_0^1\) depend on the final two rows of \(A_0\). Continuing this process to the power of \(p\), \((A_0^1)^pA_0^p\) is given by an identity in the upper left block, zeros in the upper right, and \(p\) rows of error. To that end, define

\[
A_0^1A_0^p := \begin{bmatrix} I & 0_{p \times p} \end{bmatrix} (A_0^1)^p A_0^p = \begin{bmatrix} I & 0_{p \times p} \end{bmatrix}.
\]
Note that we have defined $A_{1}^\dagger$ by forming $(A_{0}^{\dagger})^p$ and eliminating the last $p$ rows.

In an analogous process, define $A_{1}^\dagger$ as $(A_{0}^{\dagger})^p$ with the first $p$ columns set to zero. Following similar steps as above, we arrive at

\begin{equation}
A_{0}^{p}A_{1}^\dagger := A_{0}^{p} \begin{bmatrix} 0_{p \times p} & I \end{bmatrix} = \begin{bmatrix} 0_{p \times p} & I \end{bmatrix}.
\end{equation}

Now, recalling that $A_{0}^{p}$ is zero in the first $p$ rows and last $p$ columns, (39) and (40) yield

\begin{align*}
A_{1}^\dagger A_{0}^{p} = A_{0}^{p} \begin{bmatrix} 0_{p \times p} & I \end{bmatrix} A_{0}^{p} &= \begin{bmatrix} I & 0_{p \times p} \end{bmatrix} (A_{0}^{\dagger})^p \begin{bmatrix} 0_{p \times p} & I \end{bmatrix} A_{0}^{p} = \begin{bmatrix} I & 0_{p \times p} \end{bmatrix},
A_{0}^{p} A_{0}^{\dagger} = \begin{bmatrix} I & 0_{p \times p} \end{bmatrix} A_{0}^{\dagger} = \begin{bmatrix} I & 0_{p \times p} \end{bmatrix} (A_{0}^{\dagger})^p \begin{bmatrix} 0_{p \times p} & I \end{bmatrix} = \begin{bmatrix} 0_{p \times p} & I \end{bmatrix}.
\end{align*}

Defining

\begin{equation}
A_{0}^{\dagger} := \begin{bmatrix} I & 0_{p \times p} \end{bmatrix} (A_{0}^{\dagger})^p \begin{bmatrix} 0_{p \times p} & I \end{bmatrix} = \begin{bmatrix} I & 0_{p \times p} \end{bmatrix} T_{0}^{p} \begin{bmatrix} 0_{p \times p} & I \end{bmatrix},
\end{equation}

it immediately follows that $A_{0}^{\dagger}$ satisfies the four properties of a pseudoinverse: $(A_{0}^{\dagger}A_{0})^* = A_{0}^{\dagger}A_{0}$, $(A_{0}A_{0}^{\dagger})^* = A_{0}A_{0}^{\dagger}$, $A_{0}A_{0}^{\dagger}A_{0} = A_{0}$, and $A_{0}^{\dagger}A_{0}A_{0}^{\dagger} = A_{0}^{\dagger}$.

Now, we introduce three Lemmas on Toeplitz matrices, related to the pseudoinverse derived in Lemma 17.

**Lemma 18.** Consider a matrix of the form

\begin{equation}
\mathcal{T} = \begin{bmatrix}
-a & b \\
 & -a & \ddots \\
 & \ddots & -a & b \\
 & \ddots & \ddots & -a \\
\end{bmatrix},
\end{equation}

where $\mathcal{T}$ is $n \times n$ and $a$ and $b$ some invertible coefficients or operators. Then, $\mathcal{T}^p$ is (block) Toeplitz, for \( p \in \mathbb{N}, \ p < [n/2], \) with generating coefficients given by \( \{\alpha_\ell\} \), where

\begin{equation}
\alpha_\ell = \binom{p}{\ell} (-a)^{p-\ell} b^\ell.
\end{equation}

**Proof.** Because $\mathcal{T}$ is upper triangular, $\mathcal{T}^p$ is upper triangular for $p \geq 0$; furthermore, the stencil expands one super-diagonal with each matrix multiplication, so $\mathcal{T}^p$ has exactly $p + 1$ nonzero diagonals. Then, the defining Toeplitz coefficients of $\mathcal{T}^p$ are given by the $p+1$ nonzero entries in the first row of $\mathcal{T}^p$, or, equivalently, the $p + 1$ nonzero elements of $\mathbf{e}_0 \mathcal{T}^p$, where $\mathbf{e}_0$ is the first canonical (block) basis vector, $(I, 0, \ldots, 0)$.

Now, consider a linear algebra framework to represent polynomials, where columns of $\mathcal{T}$ represent powers of $x$. Then, given some coefficient vector $\mathbf{v}$, $\mathbf{v} \mathcal{T} = \mathbf{w}$, where $\mathbf{w}$ represents some polynomial $p(x) \sim \mathbf{w}$, and $\mathbf{w}_i$ corresponds to the polynomial coefficient of $x^i$. Then, for example, $\mathbf{e}_0 T \sim -a + bx \leftrightarrow -a \mathbf{e}_0 + b \mathbf{e}_1$.

Continuing,

\[ e_0 T^2 = (-a e_0 + b e_1) T \]
\[ \sim -a (-a + bx) + b (-ax + bx^2) \]
\[ = (-a + bx)^2 \]
\[ \leftrightarrow a^2 e_0 - 2 a b e_1 + b^2 e_2. \]

Here, the $i$th term in $(-a + bx)^2$ is given by the binomial formula and corresponds to the $i$th basis vector, $e_i$, $i = 0, 1, 2$. By an inductive argument,

\[ e_0 T^p = \left( \sum_{\ell=0}^{p-1} \binom{p-1}{\ell} (-a)^{p-1-\ell} b^\ell e_\ell \right) \mathcal{T} \]
Combining the terms of \(-\ell\) adjoint in the case of a block matrix. The last equality follows from shifting the second term in the series \((upper p<for that is, by setting the first \(p\) from Lemma 17, \(A\) Theorem 10.4.11]. In particular, let \(\hat{A}\)

\[\sum_{\ell=0}^{p-1} \binom{p-1}{\ell} (-a)^{p-1-\ell} b^\ell x^\ell \] \((-a + bx)\)

\[= (-a + bx)^p.\]

To represent in vector form, the binomial formula is applied to \((-a + bx)^p\), yielding the coefficients in (42).

**Lemma 19.** Consider an \(n \times n\) matrix \(T\) as in (41), and define

\[\hat{T}_p := \begin{bmatrix} I_{(n-p) \times (n-p)} & 0_{p \times p} \end{bmatrix} T^p,\]

for \(p < \lfloor n/2 \rfloor\). That is, \(\hat{T}_p\) corresponds to the last \(p\) rows eliminated from \(T^p\). Then, \(\hat{T}_p \hat{T}_p^*\) is Toeplitz in the upper \((n-p) \times (n-p)\) block and zero elsewhere, with Toeplitz generating coefficients given by

\[\alpha_{\ell} = (-1)^\ell \sum_{j=0}^{p-\ell} \binom{p}{j} \binom{p}{j+\ell} a^{p-j-b} b^j (a^*)^{j+\ell} (a^*)^{p-j-\ell},\]

for \(\ell = 0, \ldots, p\), and \(\alpha_{-\ell} = \alpha_\ell^*\).

**Proof.** From Lemma 18, \(T^p\) is Toeplitz and upper triangular with \(p\) nonzero super-diagonals. Eliminating the final \(p\) rows of \(T^p\), it is straightforward to confirm that \(\hat{T}_p \hat{T}_p^*\) is self-adjoint, Toeplitz in the upper left \((n-p) \times (n-p)\) block, and zero in the final \(p\) rows and columns. By self-adjointness, the generating coefficients of \(\hat{T}_p \hat{T}_p^*\) can be found by considering the \(p+1\) nonzero entries in the first row of \(\hat{T}_p \hat{T}_p^*\) (and their adjoints will be coefficients for the first \(p + 1\) rows). Let \(e_\ell\) denote the \(\ell\)th canonical basis vector. Then, for \(\ell = 0, \ldots, p\),

\[\alpha_{\ell} = \sum_{j=0}^{p-\ell} \binom{p}{j} \binom{p}{j+\ell} (-1)^j (-1)^{j+\ell} a^{p-j-b} b^j (a^*)^{j+\ell} (a^*)^{p-j-\ell},\]

where \(\hat{T}_p\) denotes the adjoint of operator entries, either the conjugate in the case of a scalar matrix, or block adjoint in the case of a block matrix. The last equality follows from shifting the second term in the series up \(\ell\) rows (because the operator is Toeplitz), and noting that \(\hat{T}_p\) is only nonzero for the diagonal and first \(p\) super-diagonals. Plugging in coefficients of \(\hat{T}_p\) from Lemma 18 yields

\[\alpha_{\ell} = \sum_{j=0}^{p-\ell} \binom{p}{j} \binom{p}{j+\ell} (-1)^j (-1)^{j+\ell} a^{p-j-b} b^j (a^*)^{j+\ell} (a^*)^{p-j-\ell}.\]

Combining the terms of \(-1\) yields the result in (44).

**Lemma 20.** Define \(\hat{T}_p\) as in Lemma 19 (43) and define \(\hat{A}_p\) similarly, in the form of the pseudoinverse from Lemma 17,

\[\hat{A}_p = \begin{bmatrix} I_{(n-p) \times (n-p)} & 0_{p \times p} \end{bmatrix} T^p \begin{bmatrix} 0_{p \times p} & I_{(n-p) \times (n-p)} \end{bmatrix},\]

that is, by setting the first \(p\) columns and last \(p\) rows of \(T^p\) equal to zero. Then,

\[\sigma_{\text{min}}(\hat{A}_p) \leq \sigma_{\text{min}}(\hat{T}_p),\]

where \(\sigma_{\text{min}}\) denotes the minimum nonzero singular value.

**Proof.** This proof follows from a generalization of the monotonicity theorem or Weyl’s inequality [2, Theorem 10.4.11]. In particular, let \(A\) and \(B\) be Hermitian matrices. Then,

\[\lambda_{\text{min}}(A) + \lambda_{\text{min}}(B) \leq \lambda_{\text{min}}(A + B) \leq \lambda_{\text{min}}(A) + \lambda_{\text{max}}(B).\]
In this case, \( \hat{T}_p = \hat{A}_p + \varepsilon \), where \( \varepsilon \) is only nonzero in the upper left \( p \times p \) block, where it corresponds to that of \( \hat{T}_p \). Then,

\[
\hat{T}_p \hat{T}_p^* = (\hat{A}_p + \varepsilon)(\hat{A}_p + \varepsilon)^* = \hat{A}_p \hat{A}_p^* + \hat{A}_p \varepsilon^* + \varepsilon \hat{A}_p^* + \varepsilon \varepsilon^* = \hat{A}_p \hat{A}_p^* + \varepsilon \varepsilon^*.
\]

The last equality follows from noting that, because \( \varepsilon \) is only nonzero in the upper left \( p \times p \) block and \( \hat{A}_p \) is zero in the first \( p \) columns and last \( p \) rows, \( \hat{A}_p \varepsilon^* = \varepsilon \hat{A}_p^* = 0 \). Because \( \varepsilon \varepsilon^* \) is symmetric positive semi-definite (SPSD), it follows that \( \hat{T}_p \hat{T}_p^* - \hat{A}_p \hat{A}_p^* \geq 0 \), in a positive-semi-definite sense. By (45),

\[
\lambda_{\min} (\hat{A}_p \hat{A}_p^*) \leq \lambda_{\min} (\hat{T}_p \hat{T}_p^*) - \lambda_{\min} (\varepsilon \varepsilon^*) = \lambda_{\min} (T_p T_p^*).
\]

To that end, the minimum nonzero eigenvalue of \( \hat{T}_p \hat{T}_p^* \) provides an upper bound on the minimum nonzero eigenvalue of \( \hat{A}_p \hat{A}_p^* \), and the result follows because the singular values of a matrix \( M \) are given by the square root of the eigenvalues of \( MM^* \).

By (45),

\[
\lambda_{\min} (\hat{A}_p \hat{A}_p^*) \leq \lambda_{\min} (\hat{T}_p \hat{T}_p^*) - \lambda_{\min} (\varepsilon \varepsilon^*) = \lambda_{\min} (T_p T_p^*).
\]

Building on the above machinery, recall that (block) Toeplitz matrices are associated with a (matrix-valued) Fourier generating function, where the defining Toeplitz coefficients correspond to the Fourier coefficients of the generating function. The following lemma derives the generating function for \( \hat{T}_p \hat{T}_p^* \).

**Lemma 21.** Let \( \hat{T}_p \) be as in Lemma 19. Then the real-valued Fourier generating function for the nonzero Toeplitz block in \( \hat{T}_p \hat{T}_p^* \) is given by

\[
F_p(x) = (e^{ix} a - b)^p \left[(e^{ix} a - b)^p\right]^*.
\]

**Proof.** From the binomial formula,

\[
(e^{ix} a - b)^p = \sum_{j=0}^{p} (-1)^j \binom{p}{j} e^{i(p-j)x} a^{p-j} b^j.
\]

Then,

\[
[e^{ix} a - b)^p]^* = \sum_{j=0}^{p} (-1)^j \binom{p}{j} e^{-i(p-j)x} (b^*)^{j} (a^*)^{p-j}.
\]

Recall the Fourier generating function for a Toeplitz matrix takes the form of a Fourier series, \( \sum_{\ell=-\infty}^{\infty} \alpha_\ell e^{-i\ell x} \). To prove that \( F_p(x) = (e^{ix} a - b)^p \left[(e^{ix} a - b)^p\right]^* \) is indeed the generating function for the nonzero Toeplitz block in \( \hat{T}_p \hat{T}_p^* \), we must show that for each power \( e^{-i\ell x} \), \( \ell = -p, ..., p \), the coefficient in (47) matches that of \( \alpha_\ell \) in (44). To that end, let \( k = k - j \) and substitute \( k = \ell + j \) in (47). Then, the coefficient of \( e^{i\ell x} \) in \( F_p(x) \) is given by

\[
e^{i\ell x} (-1)^{p-\ell} \sum_{j=0}^{p-\ell} \binom{p}{j} \binom{p}{\ell+j} e^{i(k-j)x} a^{p-j} b^j (b^*)^{j+\ell} (a^*)^{p-j-\ell}.
\]

Note, the series limits changed to \([0, p - \ell] \) because for \( \binom{p}{j} = 0 \) for \( j > p - \ell \). Indeed, this matches the Toeplitz generating coefficients of \( \hat{T}_p \hat{T}_p^* \) (44), which completes the proof.
We now have all the necessary tools to prove necessary conditions for convergence of MGRiT and Parareal.

**Theorem 22 (Necessary conditions).** Let $\Phi$ denote the fine-grid time-stepping operator and $\Psi$ denote the coarse-grid time-stepping operator, with coarsening factor $k$, and $N_c$ coarse-grid time points. Assume that $(\Psi - \Phi^k)$ is invertible and that $\Phi$ satisfies an F-TAP with respect to $\Psi$, with constant $\varphi_{F,1}$. Then,

$$\|I - A\Delta B_\Delta^{-1}\| \geq \frac{\varphi_{F,1}}{1 + O(1/\sqrt{N_c})}.$$  

If we further assume that $\Phi$ and $\Psi$ commute, that is, $\Phi\Psi = \Psi\Phi$, and that $\Phi$ satisfies an F-TAP with respect to $\Psi$, with constant $\varphi_{F,p}$, then

$$\|(I - A\Delta B_\Delta^{-1})^p\| \geq \frac{\varphi_{F,p}}{1 + O(1/\sqrt{N_c})}.$$  

Similarly, assume that $(\Psi - \Phi^k)$ is invertible and that $\Phi$ satisfies an FCF-TAP with respect to $\Psi$, with constant $\varphi_{FCF,1}$. Then

$$\|(I - A\Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc}\| \geq \frac{\varphi_{FCF,1}}{1 + O(1/\sqrt{N_c})}.$$  

If we further assume that $\Phi$ and $\Psi$ commute, that is, $\Phi\Psi = \Psi\Phi$, and that $\Phi$ satisfies an FCF-TAP with respect to $\Psi$, with constant $\varphi_{FCF,p}$, then

$$\left\|\left((I - A\Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc}\right)^p\right\| \geq \frac{\varphi_{FCF,p}}{1 + O(1/\sqrt{N_c})}.$$  

**Proof.** To bound $(I - A\Delta B_\Delta^{-1})^p$ and $\left((I - A\Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc}\right)^p$ in norm, we note that the $\ell^2$-norm of an operator is given by its largest singular value, which is equal to one over the smallest nonzero singular value of the operator’s inverse or pseudoinverse.

Now, notice that these operators exactly take the form of $A_0$ (34) and $A_1$ (35). For F-relaxation, $f = \Psi$, $g = (\Psi - \Phi^k)$, and $h = I$, and for FCF relaxation, $f = \Psi, g = (\Psi - \Phi^k)$, and $h = \Phi^k$. Lemma 17 gives an exact pseudoinverse for powers of such operators, and Lemma 20 proves the minimum nonzero singular value of this pseudoinverse is bounded above by the minimum singular value of the Toeplitz operator $\tilde{T}_p$ (43), with $a = h^{-1}g$ and $b = h^{-1}g^{-1}$ (41). Equivalently, we can consider the minimum nonzero eigenvalue of the corresponding normal equations. Appealing to Lemma 21, the Fourier generating functions for the nonzero Toeplitz block in these operators are given by

\begin{align}
F_F(x,p) &= \left(e^{ix}\Psi(\Psi - \Phi^k)^{-1} - (\Psi - \Phi^k)^{-1}\right)^p \left(e^{ix}\Psi(\Psi - \Phi^k)^{-1} - (\Psi - \Phi^k)^{-1}\right)^{x^p}, \\
F_{FCF}(x,p) &= \left(e^{ix}\Phi^{-k}\Psi(\Psi - \Phi^k)^{-1} - \Phi^{-k}(\Psi - \Phi^k)^{-1}\right)^p \left(e^{ix}\Phi^{-k}\Psi(\Psi - \Phi^k)^{-1} - \Phi^{-k}(\Psi - \Phi^k)^{-1}\right)^{x^p},
\end{align}

where $F_F(x,p)$ will lead to a bound on $(I - A\Delta B_\Delta^{-1})^p$ (F-relaxation) and $F_{FCF}(x,p)$ will lead to a bound on $(I - A\Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc}^p$ (FCF relaxation).

By Theorem 13 we seek the infimum over $x$ of the minimum nonzero eigenvalue of $F_F(x,p)$ and $F_{FCF}(x,p)$. Let $\lambda_k(A)$ and $\sigma_k(A)$ denote the $k$th eigenvalue and singular value of some operator, $A$ and consider the case of $p = 1$ for $F_F(x,1)$:

$$\min_{x \in [0,2\pi]} \lambda_k(F_F(x,1)) \geq \min_{x \in [0,2\pi], k} \sigma_k\left((e^{ix}\Psi - I)(\Psi - \Phi^k)\right)^2$$

$$= \min_{x \in [0,2\pi], \psi \neq 0} \frac{\|(e^{ix}\Psi - I)(\Psi - \Phi^k)(\psi)^{-1}\|}{\|\psi\|^2}$$

$$= \min_{x \in [0,2\pi], \psi \neq 0} \frac{\|(e^{ix}\Psi - I)(\psi\|}{\|(\Psi - \Phi^k)\psi\|^2}.$$
Appealing to Theorem 13,
\[
\|I - A_\Delta B_\Delta^{-1}\| \geq \frac{1}{\sqrt{\min_{x \in [0,2\pi]} \frac{\|(e^{ix} \Psi - I)v\|^2}{\|(e^{ix} \Phi - \Psi)v\|^2} + O(1/N_c)}} = \frac{1}{\sqrt{\min_{x \in [0,2\pi]} \frac{\|(e^{ix} \Psi - I)v\|^2}{\|(e^{ix} \Phi - \Psi)v\|^2} + O(1/\sqrt{N_c})}}.
\]
(50)

By assumption of an F-TAP with constant \( \varphi_{F,1} \)
\[
\|(\Psi - \Phi^k)v\| \leq \varphi_{F,1} \left[ \min_{x \in [0,2\pi]} \|(I - e^{ix}\Psi)v\| \right] = \frac{\varphi_{F,1}}{1 + O(1/\sqrt{N_c})} \left[ \min_{x \in [0,2\pi]} \|(I - e^{ix}\Psi)v\| + O(1/\sqrt{N_c}) \right].
\]
(51)

Assuming that \( \varphi_{F,1} \) is a tight bound, there exists some \( v \) such that (51) holds with equality. Then, plugging (51) into (50),
\[
\|I - A_\Delta B_\Delta^{-1}\| \geq \frac{\varphi_{F,1}}{1 + O(1/\sqrt{N_c})}.
\]
A similar derivation based on the assumption of an FCF-TAP with constant \( \varphi_{FCF,1} \) follows to bound
\[
\left\| (I - A_\Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc} \right\| \geq \frac{\varphi_{FCF,1}}{1 + O(1/\sqrt{N_c})}.
\]

Finally, if \( \Phi \) and \( \Psi \) commute, then, for example,
\[
\left( e^{ix}(\Psi - \Phi^k)^{-1} - (\Psi - \Phi^k)^{-1} \right)^p = (e^{ix}(\Psi - I)^p(\Psi - \Phi^k)^{-p}.
\]
Under the assumption of an F-TAP and FCF-TAP with constants \( \varphi_{F,p} \) and \( \varphi_{FCF,p} \), respectively, for \( p \geq 1 \), an analogous derivation as used for \( p = 1 \) yields the bounds
\[
\left\| (I - A_\Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc} \right\|^p \geq \frac{\varphi_{F,p}}{1 + O(1/\sqrt{N_c})},
\]
and similarly for the case of FCF-relaxation.

Coupling Theorems 15 and 22 with the operator form of residual propagation for \( p \) iterations (32) and the equivalence of \( \|R\| = \|\mathcal{E}\|_{A^\ast A} \) (14) completes the proofs of Theorems 2 and 3. Recall from (29) and (30) that if \( \Phi \) and \( \Psi \) commute, then \( I - B_\Delta^{-1} A_\Delta = I - A_\Delta B_\Delta^{-1} \), which proves Proposition 4.

4.2. Error-propagation and \( I - B_\Delta^{-1} A_\Delta \). This section provides proofs of Theorems 6 and 7. The framework developed in previous sections allows for a streamlined presentation. First, sufficient conditions for convergence of error in the \( \ell^2 \)-norm (based on \( B_\Delta^{-1} \) as a left approximate inverse) are presented.

**Theorem 23 (Sufficient conditions (\( \ell^2 \)-error)).** Let \( \Phi \) denote the fine-grid time-stepping operator and \( \Psi \) denote the coarse-grid time-stepping operator, with coarsening factor \( k \), and \( N_c \) coarse-grid time points. Assume that \( \Phi \) satisfies an F-ITAP with respect to \( \Psi \), with constant \( \varphi_F \). Then,
\[
\|I - B_\Delta^{-1} A_\Delta\| \leq \varphi_F \left( 1 + \|\Psi N_c\| \right).
\]
Similarly, assume that \( \Phi \) satisfies an FCF-ITAP with respect to \( \Psi \), with constant \( \varphi_{FCF} \). Then,
\[
\left\| (I - B_\Delta^{-1} A_\Delta) A_{cf} A_{ff}^{-1} A_{fc} \right\| \leq \varphi_{FCF} \left( 1 + \|\Psi N_c\| \right).
\]

\(^3\)Note that in (51), the leading constant in the \( O(1/\sqrt{N_c}) \) terms changes; however, the change in constant is marginal for any moderate \( N_c \gg O(1) \).
Proof. The proof is analogous to that of Theorem 15. Notice from (30) that $I - B^{-1}_\Delta A_\Delta$ is a block-Toeplitz matrix with generating coefficients $\alpha_i = \Psi^{-(1+i)}(\Psi - \Phi^k)$ for $i = -1, ..., -N_c$ and $\alpha_i = 0$ for $i \geq 0$. Following Theorem 15, the generating function is given by

$$F_F(x) = e^{ix}(I - e^{iN_c x} \Psi^N)(I - e^{ix} \Psi)^{-1}(\Psi - \Phi^k).$$

Recall, under the assumption of an F-ITAP, $\tilde{\varphi}_F \|v\| \leq \left[\max_{x \in [0,2\pi]} \| (I - e^{ix} \Psi)^{-1}(\Psi - \Phi^k)v \| \right]$ for all $v$, with equality for some $v$. Theorem 14 then yields

$$\|I - B^{-1}_\Delta A_\Delta\| \leq \max_{x \in [0,2\pi]} \sigma_{\max}(F_F(x))$$

$$= \max_{x \in [0,2\pi], v \neq 0} \| (I - e^{iN_c x} \Psi^N)(I - e^{ix} \Psi)^{-1}(\Psi - \Phi^k)v \|$$

$$\leq \max_{x \in [0,2\pi], v \neq 0} \left(1 + \|\Psi^N_c\|\right) \| (I - e^{ix} \Psi)^{-1}(\Psi - \Phi^k)v \|$$

$$= \tilde{\varphi}_F \left(1 + \|\Psi^N_c\|\right).$$

A similar proof follows for the case of FCF-relaxation, where the generator function, $F_{FCF}$, has coefficients $\alpha_i = \Psi^{-(1+i)}(\Psi - \Phi^k)\Phi^k$ for $i = -1, ..., -N_c$ and $\alpha_i = 0$ for $i \geq 0$. By assumption of an FCF-ITAP with constant $\tilde{\varphi}_{FCF}$, the result follows.

Now, we present a similar result to Theorem 22, which provides necessary conditions for convergence of error in the $\ell^2$-norm.

**Theorem 24 (Necessary conditions ($\ell^2$-error)).** Let $\Phi$ denote the fine-grid time-stepping operator and $\Psi$ denote the coarse-grid time-stepping operator, with coarsening factor $k$, and $N_c$ coarse-grid time points. Assume that $(\Psi - \Phi^k)$ is invertible and that $\Phi$ satisfies an F-ITAP with respect to $\Psi$, with constant $\tilde{\varphi}_F$. Then,

$$\|I - B^{-1}_\Delta A_\Delta\| \geq \frac{\tilde{\varphi}_F}{1 + O(1/\sqrt{N_c})}.$$ 

Similarly, assume that $(\Psi - \Phi^k)$ is invertible and that $\Phi$ satisfies an FCF-ITAP with respect to $\Psi$, with constant $\tilde{\varphi}_{FCF}$. Then

$$\left\| (I - B^{-1}_\Delta A_\Delta)A_{cf}^1A_{cf}^1 \right\| \geq \frac{\tilde{\varphi}_{FCF}}{1 + O(1/\sqrt{N_c})}.$$ 

Proof. This proof is analogous to that of Theorem 22, this time with $g = I, f = \Psi$, and $h = \Psi - \Phi^k$ for F-relaxation, and $h = (\Psi - \Phi^k)\Phi^k$ for FCF-relaxation. Similar to (48) and (49), the Fourier generating functions of interest are then given by

$$F_F(x) = \left(e^{ix}(\Psi - \Phi^k)^{-1}\Psi - (\Psi - \Phi^k)^{-1}\right)\left(e^{ix}(\Psi - \Phi^k)^{-1}\Psi - (\Psi - \Phi^k)^{-1}\right)^*,$$

$$F_{FCF}(x) = \left(e^{ix}(\Psi - \Phi^k)^{-1}\Phi^{-k} - \Phi^{-k}(\Psi - \Phi^k)\right)\left(e^{ix}(\Psi - \Phi^k)^{-1}\Phi^{-k} - \Phi^{-k}(\Psi - \Phi^k)\right)^*.$$ 

Following the algebraic steps in the proof of Theorem 22 completes the proof.

5. **The diagonalizable case.** So far results have been derived in terms of the time-stepping operators $\Phi$ and $\Psi$. In this section, we assume that $\Phi$ and $\Psi$ commute and are diagonalizable. In general this corresponds to the spatial operator being diagonalizable, which holds for many parabolic-type problems, among others. The purposes of this section are:

1. Derive exact bounds on convergence for diagonalizable operators.
2. Show that theory developed in this paper is, in some sense, a strengthening and generalization of previous results in [5].
If $\Phi$ and $\Psi$ commute and are diagonalizable, this means that they are diagonalizable with the same eigenvectors, or simultaneously diagonalizable. Under the assumption of simultaneous diagonalizability, certain bounds can be derived on functions of $\Phi$ and $\Psi$. Let $\Phi = U\Lambda U^{-1}$, where $\lambda_i = \lambda^i$, for $i = 1, \ldots, n$, is a diagonal matrix consisting of the eigenvalues of $\Phi$, and columns of $U$ are the corresponding eigenvectors. Similarly, let $\Psi = U\Xi U^{-1}$, for diagonal matrix $\Xi$, where $\Xi_{ii} = \mu_i$ are the eigenvalues of $\Psi$. Now let $\mathcal{A}$ be some matrix operator, where each entry is a rational function of $\Phi$ and $\Psi$.

\[
\mathcal{A}(\Phi) = \begin{bmatrix}
a_{00}(\Phi, \Psi) & a_{01}(\Phi, \Psi) & \cdots \\
a_{10}(\Phi, \Psi) & a_{11}(\Phi, \Psi) & \cdots \\
\vdots & \vdots & \ddots \\
\end{bmatrix} = \begin{bmatrix}
U & & \\
& U & \\
\vdots & & \ddots \\
\end{bmatrix}
\begin{bmatrix}
a_{00}(\Lambda, \Xi) & a_{01}(\Lambda, \Xi) & \cdots \\
a_{10}(\Lambda, \Xi) & a_{11}(\Lambda, \Xi) & \cdots \\
\vdots & \vdots & \ddots \\
\end{bmatrix} \begin{bmatrix}
U^{-1} & & \\
& U^{-1} & \\
\vdots & \vdots & \ddots \\
\end{bmatrix}.
\]

Denote $U$ as the block diagonal matrix of eigenvectors, $U$, in (52), define $\mathcal{P}$ as the orthogonal permutation matrix such that $\mathcal{P}\mathcal{A}(\Lambda, \Xi)\mathcal{P}^T$ is block diagonal, with blocks given by $\mathcal{A}(\lambda_i, \mu_i)$, and let $\tilde{U} = U\mathcal{P}^T$. Then,

\[
\|\mathcal{A}(\Phi, \Psi)\|_{(\tilde{U}\tilde{U}^*)^{-1}} = \sup_{x \neq 0} \frac{\|\tilde{U}^{-1}\mathcal{P}\mathcal{A}(\Lambda, \Xi)\mathcal{P}^T U^{-1}x\|}{\|U^{-1}x\|^2} = \sup_{x \neq 0} \frac{\|\mathcal{P}\mathcal{A}(\Lambda, \Xi)\mathcal{P}^T x\|}{\|x\|^2} = \sup_{i} \|\mathcal{A}(\lambda_i, \mu_i)\|.
\]

Thus, the $(\tilde{U}\tilde{U}^*)^{-1}$-norm of $\mathcal{A}(\Phi, \Psi)$ can be computed by maximizing the norm of $\mathcal{A}$ over eigenvalue indices of $\Phi$ and $\Psi$. In the case that $\Phi$ and $\Psi$ are normal matrices, $U$ is unitary and the $(\tilde{U}\tilde{U}^*)^{-1}$-norm reduces to the standard Euclidean 2-norm. More generally, we have the relation

\[
\frac{1}{\kappa(U)} \left( \sup_{i} \|\mathcal{A}(\lambda_i, \mu_i)^k\| \right) \leq \|\mathcal{A}(\Phi, \Psi)^k\| \leq \kappa(U) \left( \sup_{i} \|\mathcal{A}(\lambda_i, \mu_i)^k\| \right),
\]

where $\kappa(U)$ denotes the matrix condition number of $U$.

Here, we are interested in $\mathcal{A}$ corresponding to the error- and residual-propagation operators of MGRiT, $\mathcal{R}$ and $\mathcal{E}$ (see (16), (18), (20), and (22)). For notation, let, for example, $\mathcal{R}_p(\lambda_i, \mu_i)$ denote residual propagation for F-relaxation (16) operating on eigenvalues $\lambda_i$ and $\mu_i$ as opposed to operators $\Phi$ and $\Psi$. Convergence of MGRiT requires $\|\mathcal{R}^p\|, \|\mathcal{E}^p\| \to 0$ with iteration $p$; to that end, bounding $\sup_{i} \|\mathcal{R}(\lambda_i, \mu_i)^p\| < 1$ for all $i$ provides necessary and sufficient conditions for $\|\mathcal{R}(\Phi, \Psi)^p\| \to 0$ with $p$, and similarly for $\mathcal{E}(\lambda_i, \mu_i)$.

### 5.1. Necessary conditions

First, let us extend the necessary conditions for $p$ iterations (Theorems 9, 10, and 22) to the diagonalizable case. For notation, let, for example, $[I - A^{\Delta}B^{-\Delta}_{\chi}]$ denote the $N_x \times N_x$ matrix of $I - A^{\Delta}B_{\chi}^{-1}$ evaluated at the $i$th eigenmode of $\Phi$ and $\Psi$, where $\Phi$ and $\Psi$ are $N_x \times N_x$. Then, by assumption of simultaneous diagonalizability and the fact that $I - A^{\Delta}B_{\chi}^{-1} = I - B_{\chi}^{-1}A^{\Delta}$ when $\Phi$ and $\Psi$ commute,

\[
\| (I - B_{\chi}^{-1}A^{\Delta})^\ell \|_{(UU^*)^{-1}} = \| (I - A^{\Delta}B_{\chi}^{-1})^\ell \|_{(UU^*)^{-1}} = \sup_i \left\| (I - A^{\Delta}B_{\chi}^{-1})^\ell \right\|,
\]

\[
\| (I - B_{\chi}^{-1}A^{\Delta})A_{cf}A_{f1}^{-1}A_{fc}^\ell \|_{(UU^*)^{-1}} = \| (I - A^{\Delta}B_{\chi}^{-1})A_{cf}A_{f1}^{-1}A_{fc}^\ell \|_{(UU^*)^{-1}} = \sup_i \left\| (I - A^{\Delta}B_{\chi}^{-1})A_{cf}A_{f1}^{-1}A_{fc}^\ell \right\|.
\]

Now, we can follow the derivation in Section 4. For a pseudoinverse (Lemmas 16 and 17), we have $f = \mu_i$, $g = \mu_i - \lambda_i^k$, and, for F-relaxation, $h = 1$, while for FCF-relaxation, $h = \lambda_i^k$. Because $f, g$, and $h$ now commute, we can remove a leading factor of $g^{-1}h^{-1}$, and we are interested in the smallest nonzero singular

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4A similar modified norm also occurs in the case of integrating in time with a mass matrix [5].
The first term in (57) has real roots given by
\( F\) or, equivalently, \( \sin(x) \).

Noting that \( \Theta \), led to convergence of eigenvalues to the infimum over \( F(x) \) in the diagonalizable case, \( O(1/N_c^2) \), compared with the general case in Section 4, where the first-order root in \( x \) led to convergence \( O(1/N_c) \).

This discussion is summarized in the following theorem on necessary conditions for convergence.

**Theorem 25** (Necessary conditions – the diagonalizable case). Let \( \Phi \) denote the fine-grid time-stepping operator and \( \Psi \) denote the coarse-grid time-stepping operator, with coarsening factor \( k \), and \( N_c \) coarse-grid value of (Lemma 17)

\[
(A_0^p)^\dagger = \frac{1}{\mu_i(\mu_i - \lambda_i^2)} \begin{bmatrix} I & 0_{p \times p} \end{bmatrix} T_0^p \begin{bmatrix} 0_{p \times p} & I \end{bmatrix}, \quad \text{where} \quad T_0 = \begin{bmatrix} -\mu_i & 1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\mu_i & \cdots & \cdots & \cdots & -\mu_i \end{bmatrix}.
\]

Following the further derivations in Section 4, the minimum nonzero singular value of \((A_0^p)^\dagger\) is bounded above by the minimum eigenvalue of \( \hat{T}_p \hat{T}_p^\ast \), where \( a = \mu_i \) and \( b = 1 \) (see (41), (43), and Lemmas 18, 19, and 20). The Fourier generating function for \( \hat{T}_p \hat{T}_p^\ast \) from Lemma 21 (46) now takes the form

\[
F_p(x) = (1 + |\mu_i|^2 - (\mu_i e^{-ix} + \mu_i e^{ix}))^p
= \left(1 + \Re(\mu_i)^2 + \Im(\mu_i)^2 - 2\Re(\mu_i)\cos(x) - 2\Im(\mu_i)\sin(x)\right)^p.
\]

To obtain the minimum of \( F_p(x) \), note that

\[
F_p'(x) = p\left(2\Re(\mu_i)^2 - (\mu_i e^{-ix} + \mu_i e^{ix})\right)^{p-1}
= p\left(2\Re(\mu_i)\sin(x) - 2\Im(\mu_i)\cos(x)\right)^p
= p\left((\sin(x) - \Im(\mu_i))^2 + (\cos(x) - \Re(\mu_i))^2\right).
\]

The first term in (57) has real roots given by \( n\pi + \arctan(\Im(\mu_i)/\Re(\mu_i)) \), for \( n \in \mathbb{Z} \). Any other roots of \( F_p'(x) \) must satisfy

\[
(\sin(x) - \Im(\mu_i))^2 + (\cos(x) - \Re(\mu_i))^2 = 0,
\]
or, equivalently, \( \sin(x) = \Im(\mu_i) \) and \( \cos(x) = \Re(\mu_i) \). Suppose this holds. Solving for \( \tilde{x} = \arcsin(\Im(\mu_i)) \), we must also have \( \cos(\tilde{x}) - \Re(\mu_i) = 0 \), which implies \( \Re(\mu_i) = \sqrt{1 - \Im(\mu_i)^2} \). However, then \( |\mu_i| = 1 \), which violates the assumption that \( |\mu_i| < 1 \). It follows that the only real roots of \( F_p'(x) \) are given by \( \tilde{x} = n\pi + \arctan(\Im(\mu_i)/\Re(\mu_i)) \), for \( n \in \mathbb{Z} \).

Given \( F_p(x) \) is continuous and \( 2\pi \)-periodic, the infimum of \( F_p(x) \) over \( x \in \mathbb{R} \) is attained at one of the roots of \( F_p'(x) \). It is easily confirmed that the infimum is achieved for even \( n, \tilde{x}_{\min} := 2n\pi + \arctan(\Im(\mu_i)/\Re(\mu_i)) \), where \( n \in \mathbb{Z} \), which yields

\[
\inf_{x \in \mathbb{R}} F_p(x) = \left(1 + \Re(\mu_i)^2 + \Im(\mu_i)^2 - 2\Re(\mu_i)\cos(\tilde{x}) - 2\Im(\mu_i)\sin(\tilde{x})\right)^p
= \left(1 + \Re(\mu_i)^2 + \Im(\mu_i)^2 - \frac{\Re(\mu_i)}{\sqrt{1 + \frac{\Im(\mu_i)^2}{\Re(\mu_i)^2}}} - 2\frac{\Im(\mu_i)^2}{\sqrt{1 + \frac{\Im(\mu_i)^2}{\Re(\mu_i)^2}}}\right)^p
= \left(1 + |\mu_i|^2 - 2|\mu_i|\right)^p
= (1 - |\mu_i|)^{2p}.
\]

Noting that \( F_p(x) - (1 - |\mu_i|)^{2p} \) has a zero of order two at \( \tilde{x}_{\min} \), it follows from Theorem 13 that \( \lambda_{\min}(\hat{T}_p \hat{T}_p^\ast) = (1 - |\mu_i|)^{2p} + O(1/N_c^2) \). Note the faster convergence in \( N_c \) of eigenvalues to the infimum over \( F(x) \) in the diagonalizable case, \( O(1/N_c^2) \), compared with the general case in Section 4, where the first-order root in \( x \) led to convergence \( O(1/N_c) \).
time points. Assume that $\Phi$ and $\Psi$ commute and are diagonalizable, with eigenvectors as columns of $U$. Then,

$$\| (I - A \Delta B_\Delta^{-1})^p \|_{(U^*)^{-1}} \geq \frac{|\mu_i - \lambda^k_i|^p}{\sqrt{(1 - |\mu_i|)^{2p} + O(1/N_c^2)}} \geq \frac{|\mu_i - \lambda^k_i|^p}{(1 - |\mu_i|)^p + O(1/N_c^2)},$$

$$\| (I - A \Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc} \|_{(U^*)^{-1}} \geq \frac{|\lambda^k_i|^p |\mu_i - \lambda^k_i|^p}{\sqrt{(1 - |\mu_i|)^{2p} + O(1/N_c^2)}} \geq \frac{|\lambda^k_i|^p |\mu_i - \lambda^k_i|^p}{(1 - |\mu_i|)^p + O(1/N_c^2)}.$$ 

Proof. The proof follows from (55), Theorem 13, the minimum derived for $F_p(x)$, and the fact that $x + y > \sqrt{x^2 + y^2}$ for $x, y > 0$. 

5.2. Sufficient conditions. Now consider sufficient conditions for convergence under the assumption that $\Phi$ and $\Psi$ commute and are diagonalizable. To do so, we consider the minimum nonzero singular value of $(I - A \Delta B_\Delta^{-1})^t$. As in Section 5.1 and (56), we can pull out leading constants, form the normal equations with the remaining block, and reduce the problem to finding the minimum nonzero singular value of the following symmetric positive semi-definite matrix

$$\begin{bmatrix}
0 & 0 & 0 & \cdots \\
0 & 1 + |\mu_i|^2 & -\pi_i & \\
0 & -\pi_i & \ddots & \ddots \\
\vdots & \ddots & 1 + |\mu_i|^2 & -\pi_i \\
& & -\pi_i & 1
\end{bmatrix}.$$ 

(58)

The nonzero block is a single-entry perturbation to a symmetric tridiagonal Toeplitz matrix, for which we can place tight bounds on the minimum nonzero eigenvalue (see Appendix, Lemma 32 and (70)). Using the bounds derived in (70) leads to the following theorem.

Theorem 26 (Tight bounds – the diagonalizable case). Let $\Phi$ denote the fine-grid time-stepping operator and $\Psi$ denote the coarse-grid time-stepping operator, with coarsening factor $k$, and $N_c$ coarse-grid time points. Assume that $\Phi$ and $\Psi$ commute and are diagonalizable, with eigenvectors as columns of $U$. Then,

$$\sup_i \frac{|\mu_i - \lambda^k_i|}{\sqrt{(1 - |\mu_i|)^2 + \pi^2/|\mu_i|^2 N_c^2}} \leq \| I - A \Delta B_\Delta^{-1} \|_{(U^*)^{-1}} \leq \sup_i \frac{|\mu_i - \lambda^k_i|}{\sqrt{(1 - |\mu_i|)^2 + \pi^2/|\mu_i|^2 N_c^2}}.$$ 

(59)

$$\sup_i \frac{|\lambda^k_i|^p |\mu_i - \lambda^k_i|^p}{\sqrt{(1 - |\mu_i|)^2 + \pi^2/|\mu_i|^2 N_c^2}} \leq \| (I - A \Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc} \|_{(U^*)^{-1}} \leq \sup_i \frac{|\lambda^k_i|^p |\mu_i - \lambda^k_i|^p}{\sqrt{(1 - |\mu_i|)^2 + \pi^2/|\mu_i|^2 N_c^2}}.$$ 

Furthermore, for $p \geq 1$

$$\| (I - A \Delta B_\Delta^{-1})^p \|_{(U^*)^{-1}} = \sup_i \frac{|\mu_i - \lambda^k_i|^p}{\sqrt{(1 - |\mu_i|)^2 p + O(1/N_c^2)}},$$

(60)

$$\| (I - A \Delta B_\Delta^{-1}) A_{cf} A_{ff}^{-1} A_{fc} \|_{(U^*)^{-1}} = \sup_i \frac{|\lambda^k_i|^p |\mu_i - \lambda^k_i|^p}{\sqrt{(1 - |\mu_i|)^2 p + O(1/N_c^2)}}.$$ 

Proof. The single-iteration bounds follow immediately from (55) and Lemma 32 (70)). Applying the sub-multiplicative norm property to (59) yields an upper bound on $p$ iterations, and Theorem 25 yields lower bounds, each of which take the form, for example, with $F$-relaxation,

$$\sup_i \frac{|\mu_i - \lambda^k_i|^p}{\sqrt{(1 - |\mu_i|)^2 + O(1/N_c^2)}}.$$ 

This completes the proof.
5.3. Relation to the TAP. Returning to the general theoretical framework, suppose that $\Phi$ and $\Psi$ commute and are diagonalizable, $\Phi = U A U^{-1}$ and $\Psi = U \Xi U^{-1}$. Further suppose that $\Phi$ satisfies an F-TAP with respect to $\Psi$, with constant $\varphi_{F,P}$, in the $(U U^*)^{-1}$-norm. This is equivalent to saying that there exists a constant $\varphi_{F,P}$ such that for all $v$,

$$
\| (\Psi - \Phi^k)^p v \|_{(U U^*)^{-1}} \leq \varphi_{F,P} \left[ \min_{x \in [0,2\pi]} \| (I - e^{ix}\Psi)^p v \|_{(U U^*)^{-1}} \right],
$$

(61) \quad \iff \quad \| (\Xi - \Lambda^k)^p U^{-1} v \| \leq \varphi_{F,P} \left[ \min_{x \in [0,2\pi]} \| (I - e^{ix}\Xi)^p U^{-1} v \| \right].

Now note that if $\Phi$ and $\Psi$ are diagonalizable, the eigenvectors form a basis, and any vector $v$ can be written as a linear combination of eigenvectors of $\Phi, \Psi$, where $v = \sum_{\ell=1}^{N_c} \alpha_{\ell} u_{\ell}$. Then, because $U^{-1} u_{\ell} = e_{\ell}$, where $e_{\ell}$ is the $\ell$th canonical basis vector, (61) reduces to

$$
\sum_{\ell=0}^{N_c-1} \alpha_{\ell} |\mu_{\ell} - \lambda_{k,\ell}|^p \leq \varphi_{F,P} \sum_{\ell=0}^{N_c-1} (1 - |\mu_{\ell}|)^p.
$$

Note that this is only satisfied for all $v$ if, for every eigenvalue index $\ell$,

$$
|\mu_{\ell} - \lambda_{k,\ell}|^p \leq \varphi_{F,P} (1 - |\mu_{\ell}|)^p \quad \iff \quad |\mu_{\ell} - \lambda_{k,\ell}| \leq \varphi_{F,P} (1 - |\mu_{\ell}|).
$$

Indeed, this is exactly the F-TEAP introduced in Section 2.1. A similar property holds for FCF-relaxation, which is summarized in the following proposition.

**Proposition 27** (Equivalent approximation properties). The F-TEAP is the same as the F-TAP in the $(U U^*)^{-1}$-norm, for arbitrary $p$, and the FCF-TEAP is the same as the FCF-TAP in the $(U U^*)^{-1}$-norm, for arbitrary $p$. If $\Phi$ and $\Psi$ are normal, the two types of approximation property are identical.

**Proof.** The proof follows from the above discussion. \qed

We are now ready to present the final result.

**Theorem 28** (Tight bounds – multiple iterations). Let $\Phi$ denote the fine-grid time-stepping operator and $\Psi$ denote the coarse-grid time-stepping operator, with coarsening factor $k$, and $N_c$ coarse-grid time points. Assume that $\Phi$ and $\Psi$ commute and are diagonalizable, with eigenvectors given as columns of $U$. Suppose that $\Phi$ satisfies an F-TEAP with respect to $\Psi$, with constant $\varphi_F$. Then, for $p \geq 1$,

$$
\| (I - A_{\Delta} B_{\Delta}^{-1})^p \|_{(U U^*)^{-1}} = \varphi_F^{2p} - O(1/N_c^2).
$$

Similarly, suppose that $\Phi$ satisfies an FCF-TEAP with respect to $\Psi$, with constant $\varphi_{FCF}$. Then, for $p \geq 1$,

$$
\| (I - A_{\Delta} B_{\Delta}^{-1}) A_{\tau} f A_{\tau}^\dagger A_{\tau} f \|_{(U U^*)^{-1}} = \varphi_{FCF}^{2p} - O(1/N_c^2).
$$

**Proof.** By assumption of the T-FEAP and Theorem 26, to order $O(1/N_c^2)$,

$$
\| (I - A_{\Delta} B_{\Delta}^{-1})^p \|_{(U U^*)^{-1}} = \sup_i \left[ \frac{1}{1 - |\mu_i|^2} \right] \left[ \frac{\varphi_{FCF}^{2p}(1 - |\mu_{i_{\text{max}}}|)^{2p}}{\varphi_F^{2p}(1 - |\mu_i|^2)^{2p} + O(1/N_c^2)} \right].
$$

Note, equality holds in the second relation because there exists some $i = i_{\text{max}}$ such that $\varphi_F$ is tight. A simple Taylor/Laurent series argument about $N_c = \infty$ confirms that for $N_c > 1/(1 - |\mu_i|)^p$,

$$
\frac{1}{(1 - |\mu_i|^2)^{2p} + O(1/N_c^2)} = \frac{1}{(1 - |\mu_i|^2)^{2p}} - \frac{1}{N_c^2 (1 - |\mu_i|^2)^{2p} + O(1/N_c^2)} + \frac{1}{N_c^2 (1 - |\mu_i|)^{2p}} - \cdots
$$

This yields $\| (I - A_{\Delta} B_{\Delta}^{-1})^p \|_{(U U^*)^{-1}} = \varphi_F^{2p} - O(1/N_c^2)$. An analogous proof confirms the result for FCF-relaxation. \qed
Note that in practice, it is typically not eigenvalues $|\mu_i| \approx 1$ for which the maximum $\varphi_F$ is obtained for a TEAP [5]. To that end, the $O(1/N^2)$ in Theorem 28 will generally be fairly small, except for potentially in the case of very large $p$. How tight the bounds are clearly depends on the size of $p$ and $N_c$; in practice, however, Theorem 28 indicates that the upper bound on convergence in the $(UU^*)^{-1}$-norm will generally not improve in later iterations, that is, $\|E^p\|_{(UU^*)^{-1}} \approx \|E\|^p_{(UU^*)^{-1}}$.

These results also lead to a corollary, which proves that, in some cases, the bounds derived in [5] are asymptotically exact in $N_c$, in a single-iteration sense.

**Corollary 29 (Sharp matrix inequalities).** The matrix norm inequality $\|M\|^2_2 \leq \|M\|_1 \|M\|_\infty$ [2, Fact 11.9.27] is asymptotically exact, for $[I - A\Delta B_{\Delta}^{-1}]$, and $[(I - A\Delta B_{\Delta}^{-1})A_{cf}A_{fj}^1A_{fc}]$; that is,

$$\lim_{N_c \to \infty} \left\| [I - A\Delta B_{\Delta}^{-1}] \right\|_2 = \left\| [I - A\Delta B_{\Delta}^{-1}] \right\|_1 \lim_{N_c \to \infty} \left\| [I - A\Delta B_{\Delta}^{-1}] \right\|_\infty,$$

and likewise for $\left\| [(I - A\Delta B_{\Delta}^{-1})A_{cf}A_{fj}^1A_{fc}] \right\|^2_2$.

Moreover,

$$\lim_{N_c \to \infty} \left\| (I - A\Delta B_{\Delta}^{-1})^p \right\|_{(UU^*)^{-1}} = \lim_{N_c \to \infty} \left\| (I - A\Delta B_{\Delta}^{-1})^p \right\|_{(UU^*)^{-1}} = \left( \sup_i |\mu_i - \lambda_i^p| \right)^p,$$

$$\lim_{N_c \to \infty} \left\| [(I - A\Delta B_{\Delta}^{-1})A_{cf}A_{fj}^1A_{fc}] \right\|^p_{(UU^*)^{-1}} = \lim_{N_c \to \infty} \left\| [(I - A\Delta B_{\Delta}^{-1})A_{cf}A_{fj}^1A_{fc}] \right\|^p_{(UU^*)^{-1}} = \left( \sup_i |\lambda_i^p| \right)^p.$$

**Proof.** The inequality in (62) was used in [5] to establish bounds

$$\left\| [I - A\Delta B_{\Delta}^{-1}] \right\| \leq \frac{|\mu_i - \lambda_i^k|(1 - |\mu_i|^{N_c-1})}{1 - |\mu_i|},$$

$$\left\| [(I - A\Delta B_{\Delta}^{-1})A_{cf}A_{fj}^1A_{fc}] \right\| \leq \frac{|\lambda_i^k| |\mu_i - \lambda_i^k|(1 - |\mu_i|^{N_c-2})}{1 - |\mu_i|}.$$  

It is easily verified that as $N_c \to \infty$, these bounds asymptote as in (59). The second results follows from a limiting argument applied to Theorem 28.

### 6. The time-dependent case

The previous section focused on the specific case of commuting, diagonalizable time-stepping operators. This section moves to the more general setting of (almost) arbitrary, linear time-stepping operators. In particular, we drop the assumption that $\Phi$ and $\Psi$ are fixed for all time points, allowing for $\Phi$ and $\Psi$ to be time-dependent operators. Much of the theory so far has, on some level, been based on Toeplitz matrix theory. Allowing for time-dependent operators leads to non-Toeplitz block matrices, and such theory does not apply. Indeed, without some further assumptions or knowledge of $\Phi$ and $\Psi$, in general results cannot be extended to the time-dependent setting. However, this section shows that the pseudoinverse of $I - A\Delta B_{\Delta}^{-1}$ derived in Lemma 16 can indeed be extended to the time-dependent setting. Although bounds for its minimum singular value are not clear, the resulting bi-diagonal matrix is still more amenable to analysis than the dense lower-triangular matrix of $I - A\Delta B_{\Delta}^{-1}$ (29).

Note, in the time-dependent setting, $I - B_{\Delta}^{-1}A_{\Delta} \neq I - A\Delta B_{\Delta}^{-1}$, even if $\Phi$ and $\Psi$ commute. To that end, this section considers $I - A\Delta B_{\Delta}^{-1}$, corresponding to residual propagation in the $\ell^2$-norm and error propagation in the $A^*A$-norm. However, similar results can be derived for error in the $\ell^2$-norm based on analogous derivations applied to $I - B_{\Delta}^{-1}A_{\Delta}$.

For preliminary notation, assume we are considering $N$ time points and a coarse grid of $N_c$ time points.
Then the linear system corresponding to time integration (1) takes the generalized form of

\[
A\mathbf{u} = \begin{bmatrix}
I & -\Phi_1 & I \\
-\Phi_1 & I & -\Phi_2 & I \\
& \ddots & \ddots & \ddots \\
& & -\Phi_{N-1} & I \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_0 \\
\mathbf{u}_1 \\
\vdots \\
\mathbf{u}_{N-1}
\end{bmatrix} = \mathbf{f},
\]

where \(\Phi_i\) denotes \(\Phi\) evaluated at time point \(t_i\). As in the time-independent case, there is a closed form for inverses with the form of (63), which will prove useful for further derivations:

\[
\begin{bmatrix}
I & -\Phi_1 & I \\
-\Phi_1 & I & -\Phi_2 & I \\
& \ddots & \ddots & \ddots \\
& & -\Phi_{N-1} & I \\
\end{bmatrix}^{-1}
= \begin{bmatrix}
I & \Phi_1 & \Phi_2 & \Phi_3 & \cdots & \Phi_{N-1} \\
\Phi_1 & \Phi_2 & \Phi_3 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\
\Phi_{N-1} & \Phi_{N-1} & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}.
\]

Excusing the slight abuse of notation, define \(\Phi^j_i := \Phi_i \Phi_{i-1} \cdots \Phi_j\). Then, using the inverse in (64) and analogous matrix derivations as in Section 3.4, leads to a Schur complement coarse grid given by

\[
A_\Delta = \begin{bmatrix}
I & -\Phi_1 & I & -\Phi_{k+1} & I \\
-\Phi_1 & I & -\Phi_2 & I & -\Phi_{2k+1} & I \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & -\Phi_{N-1} & I & -\Phi_{(N_c-2)k+1/2k} & I \\
\end{bmatrix}.
\]

Notice that, despite the more complicated notation, the Schur-complement coarse-grid operator in the time-dependent case does exactly what it does in the time-independent case (27): it takes exactly \(k\) steps on the fine grid, in this case using the appropriate sequence of time-dependent operators. Let \(\Psi_i\) denote the non-Galerkin approximation to \(\Phi^{(i-1)k+1}_i\). Then, the operator we are primarily interested in for error and residual propagation, \(I - A_\Delta B_\Delta^{-1}\), is given by

\[
I - A_\Delta B_\Delta^{-1} = \begin{bmatrix}
\Phi^1 - \Psi_1 \\
(\Phi^2_{2k+1} - \Psi_2)\Psi_1 \\
(\Phi^3_{4k+1} - \Psi_3)\Psi_2 \\
\vdots \\
(\Phi^{(N_c-2)k+1}_{(N_c-1)k} - \Psi_{N_c-1})\Psi_{N_c-2} \\
\end{bmatrix} \begin{bmatrix}
\Phi^{(N_c-2)k+1}_{(N_c-1)k} - \Psi_{N_c-1} \\
\vdots \\
0 \\
\end{bmatrix}
\]

Moreover, the pseudo-inverses derived in Lemma 16 can be extended to the time-dependent case.

**Lemma 30 (Time-dependent pseudo-inverse).** Let \(\{\Phi_i\}_{i=1}^{N-1}\) and \(\{\Psi_i\}_{i=1}^{N_c-1}\) denote two sets of operators and, for notation, define \(\Phi^j_i := \Phi_i \Phi_{i-1} \cdots \Phi_j\). Assume that \(\Phi^{(i-1)k+1}_i - \Psi_j\) is invertible, for \(i = 1, \ldots, N_c-1\), and define \(I - A_\Delta B_\Delta^{-1}\) as in (65). Then,

\[
(I - A_\Delta B_\Delta^{-1})^{-1} =
\begin{bmatrix}
\Phi^1_i - \Psi_1 & 0 \\
0 & -\Psi_1(\Phi^1_i - \Psi_1)^{-1} \\
& \ddots \\
0 & -\Psi_{N_c-2}(\Phi^{(N_c-3)k+1}_{(N_c-2)k} - \Psi_{N_c-2})^{-1} \\
\end{bmatrix}.
\]

**Proof.** Following from the proof of Lemma 16, it is easy to confirm from (65) and (66) that \((I - A_\Delta B_\Delta^{-1})^{-1}\) satisfies the four properties of a pseudo-inverse.
As in the time-dependent case, we seek the minimum nonzero singular value of \((I - A_\Delta B_\Delta^{-1})^\dagger\), which is equivalent to the maximum singular value (and \(\ell^2\)-norm) of \(I - A_\Delta B_\Delta^{-1}\). This can be expressed as a minimization over a linear combination of operators as follows:

\[
\sigma_{\text{min}} \left( (I - A_\Delta B_\Delta^{-1})^\dagger \right)^2 = \min_{v \not\in \ker((I - A_\Delta B_\Delta^{-1})^\dagger)} \frac{\|I - A_\Delta B_\Delta^{-1})^\dagger v\|^2}{\|v\|^2}
\]

\[
= \min_{v_i, i=1, \ldots, (N_c-1)} \frac{\|v_i\|^2 + \sum_{k=1}^{N_c-2} \left( (\Phi_{ik}^k - \Psi_k) - \psi v_i \right) \left( \Phi_{ik}^k - \Psi_k \right) v_i}{\|v_i\|^2}
\]

Further analysis likely requires some knowledge on \(\{\Phi_i\}\) and \(\{\Psi_i\}\). In particular, now we are letting \(\{\Phi_i\}\) and \(\{\Psi_i\}\) be completely arbitrary operators. In practice, there is typically some continuity in how operators change between time steps, that is, \(\Phi_i\) and \(\Phi_{i+1}\) are similar in some sense.

### 6.1. Time-dependent and diagonalizable

Finally, suppose that \(\Phi\) and \(\Psi\) are time-dependent, but simultaneously diagonalizable for all times, \(t_i\). In particular this allows for time-dependent reaction terms in the spatial operator, \(L\), and for variable time-step size, as occurs in, for example, adaptive time-stepping.

To that end,

\[
\|I - A_\Delta B_\Delta^{-1}\|_{(UU^*)^{-1}} = \max_i \left\| [I - A_\Delta B_\Delta^{-1}]_i \right\| = \frac{1}{\min_i \left\| [I - A_\Delta B_\Delta^{-1}]_i \right\|} = \frac{1}{\min_i \sigma_{\text{max}} \left( [I - A_\Delta B_\Delta^{-1}]_i \right)}
\]

where, recall, \([I - A_\Delta B_\Delta^{-1}]_i\) denotes (66) evaluated at eigenvalues of \(\Phi\) and \(\Psi\) as opposed to the actual operators. As previously, \(\sigma_{\text{max}} \left( [I - A_\Delta B_\Delta^{-1}]_i \right)\) is given by the square root of the minimum nonzero eigenvalue of the corresponding normal residual equations (that is, \(MM^*\) as opposed to \(M^*M\)). Eliminating the final zero-row and zero-column (corresponding to the zero eigenvalue), this is equivalent to the minimum eigenvalue of the tridiagonal matrix

\[
\begin{bmatrix}
\frac{1}{|\lambda_k^1 - \mu_1|^2} & \frac{1}{|\lambda_k^2 - \mu_2|^2} & \cdots \\
\frac{1}{|\lambda_k^1 - \mu_1|^2} & \frac{1}{|\lambda_k^2 - \mu_2|^2} & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix}
\]

Note the change in notation - here, for example, \(\lambda_k^1\) denotes the product of the \(i\)th eigenvalue of \(\Phi\) at times \(t_1, \ldots, t_k\), that is, \(\lambda_k^1 = \lambda_i(t_k) \lambda_i(t_{k-1}) \ldots \lambda_i(t_1)\), and \(\mu_1\) denotes the \(i\)th eigenvalue of \(\Psi\) evaluated at time \(t_1\). The \(i\) is dropped from eigenvalues to limit subscript/superscript notation.

This leads to the final result on convergence in the time-dependent case:

**Theorem 31.** Let \(\{\lambda_j^{(j)}\}\) and \(\{\mu_j^{(j)}\}\) denote the sets of the \(i\)th eigenvalue of \(\Phi\) and \(\Psi\), respectively, evaluated at time indices, \(j = 1, \ldots, N_c-1\). Let \(\sigma_{\text{min}}\) denote the minimum nonzero eigenvalue of (68). Then,

\[
\|I - A_\Delta B_\Delta^{-1}\|_{(UU^*)^{-1}} = \frac{1}{\sqrt{\sigma_{\text{min}}}.
\]

Now, assume that for each eigenvalue index \(i\), a TEAP-like approximation property holds, where, for all \(j = 1, \ldots, N_c-1\),

\[
|\lambda_{jk}^{(j)} - \mu_j|^2 \leq \phi_j^{(i)} |1 - \mu_j|.
\]
Then,
\[
\|I - A_{\Delta}B_{\Delta}^{-1}\|^2_{(U,U^*)}^{-1} \leq \max_i \max_{j=1,\ldots,N_r-2} \frac{\varphi_j(i) \varphi_{j+1}(i)}{\varphi_j(i) + \mu_j \varphi_{j+1}(i)}.
\]

Moreover, sufficient conditions for convergence are that for all \(i\) and for all \(j\),
\[
\frac{\varphi_j(i) \varphi_{j+1}(i)}{\varphi_j(i) + \mu_j \varphi_{j+1}(i)} < 1,
\]
and, in addition, that for all \(i\), \(\hat{\varphi}_1(i) < 1\).

Proof. The proof follows by applying the Gershgorin circles theorem to (68) to bound the minimum eigenvalue from below, and using this to bound the maximum singular value of \([I - A_{\Delta}B_{\Delta}^{-1}]_i\) from above. For example, forming the Gershgorin disc for row one of (68) yields a lower bound
\[
\frac{1 - |\mu_2|}{|\lambda_{k+1}^i - \mu_2|^2} + \frac{|\mu_1| (1 - |\mu_1|)}{|\lambda_k^i - \mu_1|^2} \geq \frac{1}{\varphi_1(i)} + \frac{|\mu_1|}{\varphi_2(i)}.
\]
Repeating for all rows and inverting yields the result. \(\square\)

Note from Section 5 that in the diagonalizable case, Gershgorin is indeed asymptotically tight on the non-boundary rows. The Gershgorin disc for row \(i > 1\) of (58) bounds the minimum eigenvalue below by \(1 + |\mu_i|^2 - 2|\mu_i| (1 - |\mu_i|)^2\), which is exactly the minimum eigenvalue of (58) to \(O(1/N_r^2)\). In practice, it is likely that the bound in Theorem 31 is not tight, in particular the boundary term \(\hat{\varphi}_1(i)\), and that convergence will actually resemble \(\hat{\varphi}_j(i) \hat{\varphi}_{j+1}(i)\).

Fortunately, although we cannot derive a closed form for eigenvalues of (68), a simple estimate of the convex hull allows us to compute exact convergence bounds on two-level Parareal and MGRiT by solving a tridiagonal eigenvalue problem, which is a computationally tractable task. This result allows for rigorous, problem-specific analysis in some time-dependent cases, such as adaptive time-stepping.

7. Conclusion. This paper derives necessary and sufficient conditions for the convergence of Parareal and MGRiT, assuming that time-stepping operators \(\Phi\) and \(\Psi\) are linear and not time-dependent, and sufficient conditions for a subset of the general linear (time-dependent) case. This is accomplished by introducing a temporal approximation property (TAP), which gives a measure of how accurately \(\Phi^k\) approximates the action of \(\Psi\), for any vector \(v\). How accurately the TAP is satisfied then defines the \(\ell^2\) and \(A^*A\)-norm of error reduction over successive iterates. With further assumptions on the diagonalizability of \(\Phi\) and \(\Psi\), these results are strengthened to give tight bounds on an arbitrary number of iterations.

For space-time PDEs with a symmetric positive semi-definite (or symmetric negative semi-definite) spatial component, the real eigenvalues of \(\Phi\) and \(\Psi\) can be explicitly computed as a function of time-step size, \(\delta t\), and eigenvalues of the spatial operator. With a simple estimate of the minimum and maximum eigenvalue of the spatial operator, exact bounds on the convergence of Parareal and MGRiT can be easily computed by evaluating the TEAP over the range of eigenvalues of \(\Phi\) and \(\Psi\). In the general case, for example, that arises in hyperbolic PDEs, the eigenvectors no longer form an orthogonal basis and the TAP does not reduce to just considering eigenvalues. However, for most time-stepping schemes applied to some operator \(L\), it is straightforward to expand \(\Phi\) and \(\Psi\) in terms of \(L\). This permits a robust method to derive the expected convergence of Parareal and MGRiT applied to a problem of the form \(u = Lu + g\), for arbitrary \(L\).

Further research regarding the optimal \(\Psi\) with respect to \(\Phi\), the difficulties in solving hyperbolic problems, and the more general time-dependent and nonlinear cases, are ongoing work.

Acknowledgments. The author would like to thank Andreas Hessenthaler for bringing up convergence of MGRiT as a research topic and providing useful numerical comparisons, Professor Tom Manteuffel for his helpful comments and discussions, and Professor Stefano Serra-Capizzano for generously sharing his insight and expertise on block-Toeplitz matrix theory.
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Appendix A.

Lemma 32 (Minimum eigenvalue of tridiagonal Toeplitz perturbation). Define the \( n \times n \) tridiagonal matrix

\[
D_i = \begin{bmatrix}
1 + |\mu_i|^2 & -\mu_i \\
-\mu_i & \ddots & \ddots \\
& \ddots & 1 + |\mu_i|^2 & -\mu_i \\
& & -\mu_i & 1
\end{bmatrix}.
\]

The minimum eigenvalue of \( D_i \), denoted \( \lambda_n \), is bounded by

\[
(1 - |\mu_i|)^2 + \frac{\pi^2|\mu_i|}{6n^2} \leq 1 + |\mu_i|^2 + 2|\mu_i| \cos \left( \frac{n\pi}{n+1/2} \right) \leq \lambda_n \leq 1 + |\mu_i|^2 + 2|\mu_i| \cos \left( \frac{n\pi}{n+1} \right) \leq (1 - |\mu_i|)^2 + \frac{\pi^2|\mu_i|}{n^2}.
\]

Proof. Let \( \hat{D}_i \) denote the self-adjoint, tridiagonal, Toeplitz matrix for which \( D_i \) is a rank-one perturbation. In the scalar setting, there is a closed form for eigenvalues of a tridiagonal Toeplitz matrix of size
Given a $n \times n$ matrix $\hat{D}_i$, the characteristic equation for the eigenvalues $\lambda_i$ is given by

$$
\lambda(\hat{D}_i) = \left\{ 1 + |\mu_i|^2 + 2|\mu_i| \cos \left( \frac{\ell \pi}{n+1} \right) \mid \ell = 1, \ldots, n \right\}.
$$

Returning to (69), consider the rank-one perturbation in $D_i$. Following from [38], the spectrum of a tridiagonal Toeplitz matrix, and single-entry perturbations, is derived by building and solving a three-term recursion relation. One of the general results in [38], Eq. (7), states that all eigenvalues of the matrices in (69) take the form

$$
\lambda = 1 + |\mu_i|^2 + 2|\mu_i| \cos(\theta),
$$

for a given $\theta \neq m\pi$, $m \in \mathbb{Z}$. In the case of a Toeplitz matrix, the necessary conditions on $\theta$ are $\sin ((n+1)\theta) = 0$, which is satisfied for $\theta = \frac{\ell \pi}{n+1}$, $\ell = 1, \ldots, n$, yielding the result in (71). For the perturbation in $D_i$, necessary conditions on $\theta$ are that (Eq. (6), [38])

$$
T(\theta) := \sin ((n+1)\theta) + |\mu_i| \sin(n\theta) = 0.
$$

Unfortunately, (73) does not have a closed-form solution as found in the Toeplitz case and several other perturbations with closed form spectrum, introduced in [4, 38]. However, each eigenvalue of $D_i$ can be shown to be a small perturbation to eigenvalues of $\hat{D}_i$. Denote $\{\lambda_i\}_{i=1}^n$ as the eigenvalues of $\hat{D}_i$, with corresponding $\theta$-values $\{\theta_i\}_{i=1}^n$, and $\{\lambda_i\}_{i=1}^n$ the eigenvalues of $D_i$, with corresponding $\theta$-values $\{\theta_i\}_{i=1}^n$. Consider $\theta_i = \frac{\ell \pi}{n+1}$, $\ell = 1, \ldots, n$, which yields all $n$ eigenvalues of $\hat{D}_i$, in the context of necessary conditions for $D_i$ (73):

$$
T(\hat{\theta}_i) = |\mu_i| \sin \left( \frac{n}{n+1} \frac{\ell \pi}{2} \right) \rightarrow \begin{cases} < 0 & 2|\ell| < 0 \ 2 \not| \ell. \end{cases}
$$

Now, define $\hat{\theta}_i = \frac{\ell \pi}{n+1}$ for $\ell = 1, \ldots, n$. Then, under the assumption that $|\mu_i| < 1$,

$$
T(\hat{\theta}_i) = \sin \left( \frac{n+1}{n+1} \frac{\ell \pi}{2} \right) + |\mu_i| \sin \left( \frac{n}{n+1} \frac{\ell \pi}{2} \right)
$$

$$
= - \sin \left( \frac{n}{n+1} \frac{\ell \pi}{2} \right) + |\mu_i| \sin \left( \frac{n}{n+1} \frac{\ell \pi}{2} \right)
$$

$$
= -(1 - |\mu_i|) \sin \left( \frac{n}{n+1} \frac{\ell \pi}{2} \right)
$$

$$
\rightarrow \begin{cases} < 0 & 2|\ell| > 0 \ 2 \not| \ell. \end{cases}
$$

From (74), (75), and the continuity of $T(\theta)$, it follows that there exists $\theta_i \in \left( \frac{\ell \pi}{n+1}, \frac{\ell \pi}{n+1} \right)$, $\theta_i \neq m\pi, m \in \mathbb{Z}$, such that $T(\theta_i) = 0$, for $\ell = 1, \ldots, n$. Following from (72), eigenvalues of $D_i$ take the form

$$
\lambda_i = 1 + |\mu_i|^2 + 2|\mu_i| \cos(\theta_i).
$$

The smallest nonzero eigenvalue of $D_i$ is given by $\lambda_n = 1 + |\mu_i|^2 + 2|\mu_i| \cos(\theta_n)$, where $\cos(\theta_i) \rightarrow -1$ as $\ell \rightarrow \infty$. Given $\frac{n\pi}{n+1} \leq \theta_i \leq \frac{n\pi}{n+1}$, $\lambda_n$ can then be bounded by

$$
1 + |\mu_i|^2 + 2|\mu_i| \cos \left( \frac{n\pi}{n+1/2} \right) \leq \lambda_n \leq 1 + |\mu_i|^2 + 2|\mu_i| \cos \left( \frac{n\pi}{n+1} \right).
$$

With a little extra work, we can show that $\lambda_n = 1 + |\mu_i|^2 + O(1/n^2)$, which leads to necessary and sufficient conditions for convergence.
Consider the term $\cos\left(\frac{n\pi}{n+\frac{1}{2}}\right)$. As $n \to \infty$, $\cos\left(\frac{n\pi}{n+\frac{1}{2}}\right) \to +1$, that is, from above. Consider a series expansion of $f(n) = \cos\left(\frac{n\pi}{n+\frac{1}{2}}\right)$ about $n = \infty$. To accomplish this, apply the change of variable $n = \frac{1}{w}$ to get $f(w) = \cos\left(\frac{\pi}{1 + \frac{1}{w}}\right)$, and expand about $w = 0$. Formally, this would be expanded as a Laurent series, but recognizing that $w^k f(w) = w^k \cos\left(\frac{\pi}{1 + \frac{1}{w}}\right)$ is holomorphic about $w = 0$ for $k = 0, 1, \ldots$, negative coefficients in the Laurent series are zero by the Cauchy Integral Theorem. Our expansion reduces to a Taylor expansion for $\cos\left(\frac{\pi}{1 + \frac{1}{w}}\right)$ about $w = 0$,

$$\cos\left(\frac{\pi}{1 + \frac{1}{w}}\right) = -1 + \frac{\pi^2 w^2}{8} - \frac{\pi^3 w^3}{8} + O(w^4).$$

In fact, truncating the Taylor expansion at $k = 1$ leads to a remainder term

$$\cos\left(\frac{\pi}{1 + \frac{1}{w}}\right) = -1 + \int_0^w (w - t) \frac{\partial}{\partial t}\left[\cos\left(\frac{\pi}{1 + \frac{1}{t}}\right)\right] dt$$

(77)

$$= -1 + \int_0^w (w - t) \frac{-4\pi \left[\pi \cos\left(\frac{\pi}{1 + \frac{1}{t}}\right) + (2 + t) \sin\left(\frac{\pi}{1 + \frac{1}{t}}\right)\right]}{(2 + t)^4} dt.$$

Note that

$$0 < \frac{\pi^2}{6} < \frac{-4\pi \left[\pi \cos\left(\frac{\pi}{1 + \frac{1}{t}}\right) + (2 + t) \sin\left(\frac{\pi}{1 + \frac{1}{t}}\right)\right]}{(2 + t)^4} \leq \frac{\pi^2}{4},$$

for all $t \in [0, 1/10]$ (this range is not tight, just sufficient for our purposes). By positivity of the two terms being integrated in the remainder (77), the remainder can be bounded above and below. For $w \in [0, 1/10]$, substituting in for $n \geq 10$ yields

$$-1 + \frac{\pi^2}{12n^2} \leq \cos\left(\frac{n\pi}{n+\frac{1}{2}}\right) \leq -1 + \frac{\pi^2}{8n^2}.$$  

(78)

A similar expansion on $\cos\left(\frac{n\pi}{n+1}\right)$ yields a truncated Taylor expansion and remainder given by

$$\cos\left(\frac{n\pi}{n+1}\right) = -1 + \int_0^w (w - t) -\frac{\pi \left[\pi \cos\left(\frac{\pi}{1 + \frac{1}{t}}\right) + 2(1 + t) \sin\left(\frac{\pi}{1 + \frac{1}{t}}\right)\right]}{(1 + t)^4} dt.$$  

Here, note that

$$0 < \frac{\pi^2}{2} < \frac{-\pi \left[\pi \cos\left(\frac{\pi}{1 + \frac{1}{t}}\right) + 2(1 + t) \sin\left(\frac{\pi}{1 + \frac{1}{t}}\right)\right]}{(1 + t)^4} \leq \pi^2,$$

(79)

for $t \in [0, 1/10]$. Integrating the product of two positive functions, the bounds in (79) can be pulled out and, for $w \in [0, 1/10]$, substituting in for $n \geq 10$ yields

$$-1 + \frac{\pi^2}{4n^2} \leq \cos\left(\frac{n\pi}{n+\frac{1}{2}}\right) \leq -1 + \frac{\pi^2}{2n^2}.$$  

Returning to the upper and lower bounds on $\lambda_n$ in (76) completes the proof.