DISCRETE BAKER TRANSFORMATION AND CELLULAR AUTOMATA

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ABSTRACT. In this paper we propose a rule-independent description of applications of cellular automata rules for one-dimensional additive cellular automata on cylinders of finite sizes. This description is shown to be a useful tool for answering questions about automata’s state transition diagrams (STD). The approach is based on two transformations: one (called Baker transformation) acts on the $n$-dimensional Boolean cube $\mathbb{B}^n$ and the other (called index-baker transformation) acts on the cyclic group of power $n$. The single diagram of Baker transformation in $\mathbb{B}^n$ contains an important information about all automata on the cylinder of size $n$. Some of the results yielded by this approach can be viewed as a generalization and extension of certain results by O. Martin, A. Odlyzko, S. Wolfram [1]. Additionally, our approach leads to a convenient language for formulating properties, such as possession of cycles with certain lengths and given diagram heights, of automaton rules.

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1. INTRODUCTION

In this paper we consider the behavior of one-dimensional cellular automata acting on a finite cylinder of size $n$. The idea of our approach is as follows. Maximum and minimum of descriptive complexity of the rules produced by iterative applications of an arbitrary rule depend, in principle, on the cylinder size and the generating rule. It turns out, however, that in
the case of additive finite one-dimensional automata, the minimum complexity is reached at the second iteration regardless of the cylinder size and the generating rule. We show this by introducing a straight-forward rule-independent procedure that yields the results of the second iteration without applying the generating rule twice. This procedure is called *Discrete Baker Transformation* (DBT) and found to be a useful tool for answering a broad spectrum of questions on automata behavior.

In the following we will introduce the notation used in the rest of the paper. A cellular automaton is called additive if it is defined by additive rules \((X)\) acting on the cell and its right neighbors \([1, 2]\). In this paper we consider one-dimensional additive cellular automata (ACA) on a finite cylinder of size \(n\) with the states from boolean cube \(B^n\). Let us call an automaton state cyclic if it belongs to some cycle of the state diagram of the automaton. We denote the length of a string \(w\) by \(|w|\) and the parity of a binary string \(w\) by \(\nu(w)\). Pair \((n, X)\) where \(n \in \mathbb{N}\) and \(X\) is a finite 0,1-sequence of length \(m, m \leq n\), defines an additive cellular automaton (ACA) on the cylinder of size \(n\). We denote the automaton by \(A(n, X)\). The *standard length* of rule \(X\) for an automaton on the cylinder of size \(n\) is \(n\). Thus, the short notation \(X, |X| = m < n\) for a automaton rule on an \(n\)-cylinder means that \(X\) must be padded with zeroes at its right end until the length of \(n\). The rest of the paper uses the full notation by default.

The states of \(A(n, X)\) constitute a boolean cube \(\mathcal{B}^n\) of dimension \(n\). We write the strings as words in the alphabet \(\{0, 1\}\) or as vectors (e.g., \([i_1, i_2, \ldots, i_n]\)). For any given \(n\) two particular strings play an important role. They are \(0^n (= [0, \ldots, 0])\), \(1^n (= [1, \ldots, 1])\). We refer to them as 0, 1 correspondingly. Rule \(X\) acts as a linear operator in space of all states \(\mathcal{B}^n\) of \(A(n, X)\). We denote the operator by \(\hat{X}\). Accordingly, \(\hat{X} * s\) is the result of application of the rule \(X\) to state \(s\). Rules themselves (in the standard form) comprise the boolean cube \(\mathcal{B}^n\).

Any non-cyclic ACA state evolves into a cyclic state in finite time. The maximum time of this evolution is called the *height* of the ACA and is denoted by \(h^*\). It is defined to be 0 if the ACA has cyclic states only. To specify a concrete automaton \(A(n, X)\) we use \(h^*(n, X)\). Another important attribute of an ACA is the distribution of cycle lengths and, in particular, the maximum cycle length.

2. **Iterations of \(\hat{X}\)**

Clearly, the characteristics of STD s.t. maximal height or cycle length relate to lengths of the oriented chains of the correspondent graph. These chains of states in STD are produced by sequential actions of the same operator \(\hat{X}\), i.e. iterations of the action or operators which are the degrees
\(\hat{X}^i\) of given an operator \(\hat{X}\). Because we denote by \(\hat{I}\) the identical operator and accept \(\hat{X}^0 = \hat{I}\), the index of degree \(\hat{X}^i\) may run all natural numbers.

2.1. Kernels and images of \(\hat{X}\).

The following well known for all linear spaces and linear operators in them\(^1\) facts we apply below for operator \(\hat{X}\) and space \(B^n\).

**Proposition 1.** The following statements are well known:

1. The intersection of subspaces \(B^n\) is a subspace, and \(\{0\}\) is the least subspace of \(B^n\) with respect to inclusion.
2. Image \(\text{Im}(\hat{X})\) and kernel \(\ker(\hat{X})\) are linear subspaces of \(B^n\).
3. \(\forall i [\text{Im}(\hat{X}^{i+1}) \subseteq \text{Im}(\hat{X}^i) \& \ker(\hat{X}^{i+1}) \supseteq \ker(\hat{X}^i)]\).
4. \(\forall i [\text{Im}(\hat{X}^{i+1}) \subset \text{Im}(\hat{X}^i) \iff \ker(\hat{X}^{i+1}) \supseteq \ker(\hat{X}^i) \iff \text{Im}(\hat{X}^i) \cap \ker(\hat{X}^i) \neq \{0\}]\).
5. \(\forall i [\ker(\hat{X}^{i+1}) = \ker(\hat{X}^i) \iff \ker(\hat{X}^{i+2}) = \ker(\hat{X}^{i+1})]\).
6. \(\forall i [||\text{Im}(\hat{X}^{i+1})|| = \frac{|\text{Im}(\hat{X}^i)|}{|\ker(\hat{X}^i) \cap \text{Im}(\hat{X}^i)|}]\).

2.2. Composition of circulants.

Let \(L, L = [a_0, \ldots, a_{n-1}]\), is \(0, 1\)-sequence of length \(n\) and \(\sigma\) is the cyclic shift right of \(L\), i.e. \(\sigma * L = [a_{n-1}, a_0, \ldots, a_{n-2}]\). Also let \(s^\circ\) denote string \(s\) being read from the end to the beginning (i.e. in reverse order).

We call the square matrix which \(i\)-th row is \(\sigma^{i-1}(L)\) by circulant of \(L\) and denote as \(C(L)\). Since circulant is completely defined by its the first row, it has a sense to call the first row of a circulant leader row or leader. Let us define for any \(n\) a binary operation \(\boxtimes : B^n \times B^n \to B^n\) on leaders of circulants as following:

\[
(L \boxtimes M)_j = \sum_{k=0}^{n-1} L_k (M^{\circ})_{k-j-1(\mod n)}, j = 0, n-1.
\]

It can be noted that \((M^{\circ})_{k-j-1(\mod n)} = M_{j-k(\mod n)}\). We use expression \(M^{\circ}\) to present a ”geometric” structure of the operation: the second operand \(M\) in first becomes reversed and then cyclic shifted on one position right.

The next lemma shows the meaning the operation we introduced by \((\Box)\).

**Lemma 1.** Let \(|L| = |M| = n\). Then \(C(L)C(M) = C(Q)\) where \(Q = L \boxtimes M\).

\(^1\)Actually - for all commutative groups and their homomorphisms.
Proof. Let \( L^i \) be the \( i-th \) row of \( C(L) \) and \( T^j \) be the \( j-th \) column of \( C(M) \).

Since \( L^i = \sigma^i * L \) and \( T^j = \sigma^{j+1} * M^\sigma \), then
\[
(C(L)C(M))_{i,j} = \sum_{k=0}^{n-1} (\sigma^i L)_k (\sigma^{j+1} M^\sigma)_k = \sum_{k=0}^{n-1} L_{k-i \pmod n} (M^\sigma)_{k-j-1 \pmod n}.
\]

So \( Q \) is the first \((i = 0)\) row of the matrix product. From here we get
\[
(C(L)C(M))_{i+1,j+1} = \sum_{k=0}^{n-1} L_{k-i-1 \pmod n} M^\sigma_{k-j-2 \pmod n}.
\]

Setting \( k' = k - 1 \pmod n \) and taking into account that \( k, k' \) are tied variables running the same scope of numbers, we conclude that
\[
(C(L)C(M))_{i,j} = (C(L)C(M))_{i+1,j+1}.
\]

The last means that \( C(L)C(M) \) is the circulant of \( Q \). □

Thus, \( \boxtimes \) is just an image of matrix multiplication of circulants in space of their leaders. And because matrix multiplication is associative, \( \boxtimes \) is the associative operation too.

This is an useful property of the operation:

Lemma 2. Suppose \( |L| = |M| = qn' \) where \( q, n' \in \mathbb{N} \). Then
\[
\forall i[0 \leq i \leq qn' - 1 & q \nmid i \implies L_i = M_i = 0] \implies \forall i[0 \leq i \leq qn' - 1 & q \nmid i \implies (L \boxtimes M)_i = 0].
\]

Proof. Because if \( q \nmid k \) then \( L_k = 0 \), we can rewrite the definition of \( \boxtimes \) in the next form:
\[
(L \boxtimes M)_j = \sum_{i=0}^{n'-1} L_{iq} M_{j-iq \pmod n}, j = 0, n-1.
\]

Now, clearly \( q \nmid j \iff q \nmid (j - iq \pmod n) \) since \( q | n \). From here and the condition of the lemma for \( M \) we draw that if \( q \nmid j \) all \( M_{j-iq \pmod n} \) are equal 0, and \( (L \boxtimes M)_j = 0 \). □

This technical lemma has an important meaning for the theory. One application will be presented in section about the reduction of the problem to compute determinants (modulo 2) of automata rules.

2.3. Discrete baker transformation (DBT).

As we will see further, the case \( L \boxtimes L \) (“baker transformation”) presents the special interest.

In chaos theory the transformation \( x_{n+1} = 2\mu x_n \), where \( x \) is computed modulo 1 (for \( \mu = 1 \) see, [4,p.272]) is called baker transformation or map.
By this analogy we will call *discrete baker transformation* on the set \( \mathcal{B}^n \) of all boolean \( n \)-tuples the mapping \( b : \mathcal{B}^n \rightarrow \mathcal{B}^n \) acting according to the rule:

\[
(b \ast L)_i = \left\{ \begin{array}{ll}
\bigoplus_{j: \, 2j=i \pmod{n}} L_j, & \text{if } \{k|2k=i \pmod{n}\} \neq \emptyset, \\
0, & \text{else}.
\end{array} \right.
\]  

**Example 1.** Let \( n = 9, L = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8] \). Then \( b \ast L = [a_0, a_5, a_1, a_6, a_2, a_7, a_3, a_8, a_4] \). In case \( n = 8 \) we have \( b \ast [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7] = [a_0 + a_4, 0, a_1 + a_5, 0, a_2 + a_6, 0, a_3 + a_7, 0] \).

The next lemma proves the identity \( L \boxtimes L = b \ast L \) and therefore explains the meaning of the discrete baker transformation for us:

**Lemma 3.** \( C(L)C(L) = C(b \ast L) \).

**Proof.** Let \( |L| = n \). As we saw \( C(L)C(L) = C(Q) \) where \( Q_j = \sum_{k=0}^{n-1} L_k L_{k-j-1(\text{mod } n)}, j = 0, n-1 \). Because \( L_{k-j-1(\text{mod } n)} = L_{j-k(\text{mod } n)} \) we can write

\[
Q_j = \sum_{k=0}^{n-1} L_k L_{j-k(\text{mod } n)}.
\]

If we imagine string \( L \) in the form of a ring then we will see that \( k \)-th item in the sum is the product of \( k \)-th component of \( L \) counted from the 0-th component in positive direction (when numbers of component increase) and \( k \)-th component counted from \( j \)-th component in the opposite direction.

**Case 1: \( n \) is odd.** In this case for any \( j \) there exists only one component equidistant from 0-th and \( j \)-th components. Its number \( \gamma(j) \) is

\[
\gamma(j) = \left\{ \begin{array}{ll}
\frac{j}{2}, & \text{if } j \text{ is even}, \\
\frac{j + n-j}{2}, & \text{if } j \text{ is odd}.
\end{array} \right.
\]

Since all these products of different components occur in the sum twice, the sum (i.e. \( Q_j \)) is equal to \( L_{\gamma(j)} \). Since \( 2\gamma(j) = j \pmod{n} \) we have \( Q_j = (b \ast L)_j, j = 0, \ldots, n-1 \).

**Case 2: \( n \) is even.** This time either there exist exactly two components \( L_{\gamma(j)}, L_{\delta(j)} \) equidistant from 0-th and \( j \)-th components or no one at all. In the last subcase \( j \) is odd and the sum \( Q_j \) equals to 0 because every item in the sum occur twice. In the former subcase \( j \) is even and \( \gamma(j) = \frac{j}{2}, \delta(j) = \frac{n-j}{2} \). So we have

\[
Q_j = \left\{ \begin{array}{ll}
0, & \text{if } j \text{ is odd}, \\
L_{\frac{j}{2}} + L_{\frac{n-j}{2}}, & \text{if } j \text{ is even}.
\end{array} \right.
\]
Again the result coincides with \( b \ast L \). □

The next result plays an important role in decoding of the baker diagrams, see next sections.

**Theorem 1.** (i) **Conservation principle:** \( \forall X [\det_2(|X|, X) = \det_2(|X|, b \ast X)] \).

(ii) \( \forall X [\text{rank}(C(X)) \geq \text{rank}(C(b \ast X))] \).

**Proof.** (i) According to the lemma 3 \( C(L)C(L) = C(b \ast L) \). As it is well known \( \det(AB) = \det(A) \det(B) \). Therefore \( \det(AB) \) is odd iff the both of \( \det(A) \), \( \det(B) \) are odd.

(ii) This statement is the prompt consequence of \( b \) definition and proposition 1(3). □

Application a rule \( \hat{X} \) to a state \( s \) traces only step in complete trajectory of the state. Two steps produces the operator \( (\hat{X})^2 \) or \( \overset{\_}{b} \ast X \); four step result is produced by \( \overset{\_}{b}^2 \ast X \) and so forth, see the next theorem.

First, as usual we define \( b^{i+1} \ast X = b \ast (b^i \ast X) \).

**Theorem 2.** \( \overset{\_}{b}^i \ast X = (\hat{X})^{2^i} \).

**Proof.** For beginning we note that compositions of operators of kind \( \hat{X} \) is associative since these are linear operators.

Induction on \( i \). The basis \( i = 1 \) was proved in lemma 3. Now suppose the statement is true for \( i = k \) let’s prove it for \( i = k + 1 \). We have \( \overset{\_}{b}^{i+1} \ast X = (\overset{\_}{b}^i \ast X) \ast b \). (by the induction supposition)

\( (\overset{\_}{b}^i \ast X)^2 = (\overset{\_}{b}^{2i} X)^2 = (\overset{\_}{b}^{2i} X) \ast b \).

Despite the non-uniform scale, the result can play an important role in studying asymptotic behavior of the operators as \( \hat{X} \) and therefore in understanding the global structure of STD.

The amazing facts about behavior of \( b \) in \( \mathfrak{B}^n \) is described by the next two lemmas. To formulate and prove them it’s worth to select ”index projection” of the baker transformation. For given a number \( n \) it is the map \( \sharp : [0, \ldots, n - 1] \rightarrow [0, \ldots, n - 1] \) determined by the rule

\[ \sharp(i) = \text{rem}(2i, n), \]

(i.e. the remainder from division \( 2i \) by \( n \)). What we mean talking about index projection is that \( (b(L))_i = \bigoplus_{\sharp(j) = i} L_j \), see 3. From here the following follows.
Proposition 2. \( \forall k \forall X [\sharp^k([0, \ldots, |X| - 1]) = [0, \ldots, |X| - 1] \implies b^k(X) = X] \).

Proof. The condition \( \sharp^k([0, \ldots, |X| - 1]) = [0, \ldots, |X| - 1] \) implies that \( \sharp \) is a permutation on \([0, \ldots, |X| - 1]\). So \( b \) is a permutation of components of \( X \) (nothing to glue). Therefore the implication is true; yet the cycle of \( b \) could be even shorter: 0-vector is a good example. \( \square \)

As usual one can draw diagrams of the mapping \( \sharp \).

Example 2. The fig. 1 shows diagrams of \( \sharp \) on the segment \([0, \ldots, 32]\), \( n = 33 \). Note, \( n \) is odd, and the set of indices \( \{0, \ldots, 32\} \) is partitioned on 5 cycles: \( \{0\}, \{11, 22\}, \{1, 2, 4, 8, 16, 32, 31, 29, 25, 17\}, \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30\}, \{5, 7, 10, 13, 14, 19, 20, 23, 26, 28\} \) with lengths correspondingly 1,2,10,10,10.

![Figure 1](image-url)

**Figure 1.** The diagram of the mapping \( \sharp \) for \( n = 33 \).

Now we need a standard notion (see [6]) \( \text{ord}_m \ell \) that means the least integer number \( x \) s.t. \( \ell^x = 1 (\mod m) \).

Lemma 4. \( n \text{ odd} \implies (\forall i \in \{0, \ldots, n-1\}) \exists! j \in \{0, \ldots, n-1\})[\sharp(j) = i] \). So cycles of the mapping \( \sharp \) consist a partition of the set \( \{0, \ldots, n-1\} \). One of the cycles is \( \{0\} \). At last, the lengths of any of these cycles divide the number \( \text{ord}_n 2 \).

Proof. First let’s define set \( M(s) = \{\sharp^i(s) | i = 0, 1, 2, \ldots \} \) starting from arbitrary position number \( s \in \{0, \ldots, n-1\} \) of \( X \). It’s clear, all these sets are finite. For \( s = 0 \) we have \( \sharp(0) = 0 \) and so \( M(0) = \{0\} \). Set \( M(s), s > 0 \), is a cycle iff there exists \( i \) s.t. \( \sharp^i(s) = s \). To prove that let’s rewrite elements of \( M(s) \) in the natural order. We as before will denote by \( s \) the first element of this list.

One can note in first, that \( i \neq 2i (\mod n) \), \( i < n \), \( n \) is odd. This mean that no other loop (cycle with length 1) exists. Then, supposing the contrary (i.e. \( M(s) \) is not a cycle) we must conclude that for some \( m > 0 \) there exist at least two \( \sharp \)-prototypes \( a, b \in \{0, \ldots, n-1\} \), \( a \neq b \), of \( c = \sharp^m(s) \), one of them, say \( a \), is \( \sharp^{m-1}(s) \). Because \( 2b = c (\mod n) \) and \( b, c < n \) then \( 2b - c = n \). However because \( c > s \) the both of \( 2b, c \) are even numbers.
whereas \( n \) is odd. The contradiction proves that \( M(s), 0 \notin M(s) \), is a cycle because the unique opportunity is \( m = 0 \), since then \( s \) can be odd. Moreover we conclude that the least element of \( M(s) \) in case \( M(s) \neq \{0\} \) must be odd number.

Now, because \( 2^{\text{ord}_n 2} = 1 \mod(n) \) the sequence \( (2^i s \mod(n))_{i=0,1,2,...} \) is built from the recurrent pieces of length \( \text{ord}_n 2 \). In other words, \(|M(s)|\) divides \( \text{ord}_n 2 \). □

How many of the cycles of kind \( M(s) \) exist? At least two in not trivial case \( n > 1 \). The example shows 5 cycles for \( n = 33 \). In this case \( \text{ord}_n 2 = 10 \).

On the other side when we pass from the ”index projection” \( y \) to the baker transformation \( b \) we must keep in mind that the series of cycles in the set \( \{b^i \ast X| i = 0, 1, \ldots \} \), \(|X|\) is odd, could be different because of the distribution of ones in \( X \). Yet, it’s obvious that the lengths of these cycles divide lengths of the correspondent cycles for \( y \) and so must divide the same number \( \text{ord}_n X \). However we do not need the last statement to state the next corollary because by definition the baker transformation, proposition and from the previous lemma.

**Corollary 1.** If \( n \) is odd then \( \forall X \in \mathbb{B}^n [b^{\text{ord}_n 2} \ast X = X] \).

The cases of even \( n \) can be reduced to the previous cases in the following way.

**Lemma 5.** Let \(|X| = n = 2n'\). Then \( b \ast X = [y_0, 0, y_1, \ldots, y_{n'}, 0] \) for some boolean numbers \( y_0, \ldots, y_{n'} \) and \( b^2 \ast X = [z_0, 0, \ldots, z_{n'}, 0] \) where \([z_0, \ldots, z_{n'}] = b \ast [y_0, \ldots, y_{n'}]\).

**Proof.** First of all, by the definition only components of \( b \ast X \) with numbers \( 2i \mod(n), i = 0, n - 1 \) can be non-zero. Taking in account that all the numbers \( 2i \mod(n), i = 0, n - 1 \) are even (because \( n \) is even), we come to \( b \ast X = [y_0, 0, y_1, \ldots, y_{n'}, 0] \).

Then, when we apply \( b \) to \([y_0, 0, y_1, \ldots, y_{n'}, 0]\) its odd components play no role in forming components of the result, because all they are equal to \( 0 \). It’s important also that indices \( z(2i \mod(n)) \) are even. So the result \( b \ast [y_0, 0, y_1, \ldots, y_{n'}, 0] \) can be write as \([z_0, 0, \ldots, z_{n'}, 0]\).

It remains only to note that we can shorten vectors \([y_0, 0, y_1, \ldots, y_{n'}, 0], [z_0, 0, \ldots, z_{n'}, 0]\) to \([y_0, y_1, \ldots, y_{n'}], [z_0, z_1, \ldots, z_{n'}]\) because, as we said, the components of \([y_0, 0, y_1, \ldots, y_{n'}, 0], [z_0, 0, \ldots, z_{n'}, 0]\) with odd numbers play no role, and if \( k = 2k', t = 2t' \) then \( 2k = i \mod(n) \) & \( 2t = i \mod(n) \) \( \iff \) \( 2k' = \frac{i}{2} \mod(n') \) & \( 2t' = \frac{i}{2} \mod(n') \). □
Let \( \iota(a, b) \) be the maximal degree of \( a \) that divides \( b \) without a remainder and let’s call any boolean vector \( X \) as \( b \)-swept or baker-swept if \( \forall j [0 < j < |X| \land 2^{\iota(2, |X|)} \nmid j \implies X_j = 0] \). The prompt consequence of the previous lemma is

**Corollary 2.** \( \forall X [b^{\iota(2, |X|)} * X \text{ is } b \text{-swept}] \).

Now let us define a function \( c(n) \):

\[
c(n) = \begin{cases} 
\text{ord}_{n/2^{(2, n)}} 2, & \text{if } \frac{n}{2^{(2, n)}} > 1, \\
1, & \text{otherwise.}
\end{cases}
\]

**Theorem 3.** \( \forall n > 0 \forall X \in B^n [b^{\iota(2, n)} + c(n) * X = b^{\iota(2, n)} * X] \).

**Proof.** This results from lemma 5 about reduction, previous corollary 6 and lemma 4.

**Example 3.** The fig. 2 illustrate the theorem in case \( n = 2^3 \cdot 33 = 264 \). The diagram contains 5 cycles of lengths the same as in case \( n = 33 \). However, every vertex of every cycle is the root of the same tree of height 3.

**Figure 2.** The diagram of the mapping \( \natural \) for \( n = 264 \).

We call the number \( c(n) \) critical. The behavior of the function is shown by Fig. 5 for \( n \leq 200 \) (compare with fig. 4 from [6, p.85] and note, please, that their picture is the graph of \( \text{ord}_n 2 \) built only for odd numbers \( n \).)

**Corollary 3.** The numbers of the series

\[
\text{rank}(C(X)), \text{rank}(C(b * X)), \text{rank}(C(b^2 * X)), \ldots
\]

doesn’t increase and their minimum equals to \( \text{rank}(C(b^{\iota(2, |X|)} * X)) \).

**Proof.** Indeed, as we saw above \( b^{\iota(2, |X|)} * X \) belongs to the period (cycle) of the series \( X, b * X, b^2 * X, \ldots, \). Also according to theorem 1(ii) the series doesn’t increase.

The kneading ability of \( b \) is shown on the Fig. 4. There \( t \) is number of iteration of \( b \). The beginning is subsegment \([.1, .2]\) of the segment \([0, 1]\) consisting of the numbers of kind \( x = \sum_{i=1}^{13} a_i 2^{-i} \) where coefficients \( a_i, i = 1, 13 \) are boolean and \( .1 \leq x \leq .2 \).
FIGURE 3. Function $c(n)$ on segment $n = 1, 200$.

FIGURE 4. Pseudo-chaotic behavior $b$ on segment $[0, 1]$ for odd $n$, $n = 13$. 
Despite of this overt kneading, the period of length 12 is here. This is because of the finiteness of the mixed set: its power is $2^{13}$ or 8,192.

In case when set consists the rational numbers representing boolean sequences of length $2^m$ the picture is quite different as the fig. 5 shows.

![Figure 5](image)

**Figure 5.** Behavior $b$ on segment $[0, 1]$ for $n = 2^4$.

3. **Diagrams of $b$ and its Index Projection $\natural$**

As we saw the baker transformation is the useful tool and therefore it’s worth to gain more information about it as well as about its index projection $\natural$. The task become easier because we deal actually with the same transformations (as functions) for any space $B^n$. On the other side for given a number $n$ the diagrams of these mappings tell us something about all rules of additive automata on the cylinder of size $n$.

Clearly, $\natural$ acts on set of power $n$ whereas space of $b$ has power $2^n$. Therefore, for a fixed $n$ the $\natural$-diagram presents *rule*-independent data; whereas $b$-diagram differs individual rules $X \in B^n$. The problem arises, how to decode the information hidden in the diagram. In this paragraph we, in particular, give examples of the decoding.
3.1. $b$ as a linear operator in $\mathcal{B}^n$.

**Lemma 6.** For every $n$ the discrete baker transformation acts as a linear operator on $\mathcal{B}^n$.

**Proof.** It’s enough to check:

1. $b * 0 \cdot X = 0 \cdot b * X$.
2. $b * (X \oplus Y) = b * X \oplus b * Y$.

Point (1) follows $b * 0 = \vec{0}$ that is the prompt consequence of the definition (3).

Point (2) also is true because of

\[(b * (X \oplus Y))_i = \bigoplus_{j: 2j = i \text{ (mod } n)} (X \oplus Y)_j = \bigoplus_{j: 2j = i \text{ (mod } n)} (X)_j \oplus \bigoplus_{j: 2j = i \text{ (mod } n)} (Y)_j = (b * X)_i \oplus (b * Y)_i.\]

Because $b$ is the linear operator it can be represented by matrix $B_n$ for given a dimension $n$. As for operators $\hat{X}$ we accept that operator and the corresponding matrix act on vectors (strings) from left side. The matrixes for even $n$ essentially differs from the case of odd $n$.

In general for $i, j = 0, n - 1$

\[(B_n)_{i,j} = \begin{cases} 1, & \text{if } i = \text{rem}(2j, n), \\ 0, & \text{else}. \end{cases}\]

(Here, as before, $\text{rem}(a, b)$ denotes remainder of division $a$ with $b$.)

So despite of additivity $b$ differs from cellular automata as operators.

Now we present examples of $b$-diagrams.

**Example 4.** Generally speaking, when $n = 2^k$ the behavior of $b$ in $\mathcal{B}^n$ must be sufficiently predictable in view of theorem 3. We will discuss this later.

Fig. 6 represents the diagram of behavior $b$ in $\mathcal{B}^8$.

**Example 5.** The diagram for $\mathcal{B}^9$ is the collections of 8 cycles with length 1, 12 cycles with length 2, 8 cycles with length 3, and 76 cycles with length 6.

**Example 6.** Fig. 7 presents three kinds of connectivity components of the baker diagram in $\mathcal{B}^{10}$. The complete diagram includes 4 basins of cycles of length 1, 2 basins of cycles of length 2 and 6 basins with cycles of length 4.
3.2. ibble as a homomorphism of finite cyclic groups.

We denote by $\mathbb{C}_n$ the additive cyclic group $\langle \{0, 1, \ldots, n-1\}, +_n \rangle$ where $+_n$ is the addition w.r.t. modulo $n$ on $\{0, \ldots, n-1\}$.

**Lemma 7.** $ibble$ is a homomorphism of $\mathbb{C}_n$ on some its subgroup which is an isomorphism if $n$ is odd.

**Proof.** By the definition $ibble(i) = \text{rem}(2i, n)$; so $ibble(0) = 0$ and $ibble(i +_n j) = \text{rem}(2i +_n 2j, n) = \text{rem}(2i, n) +_n \text{rem}(2j, n) = ibble(i) +_n ibble(j)$. □

Thus the proposition is applicable here too, and this explains the fact that any diagram of $ibble$ consists several cycles, one of them is $\{0\}$; and every
vertex of every cycle is the root of the same (within isomorphism, of course) tree. Examples demonstrate the diagrams of $\hat{\pi}$ in $C_{33}, C_{264}$ correspondingly. The big advantage of $\hat{\pi}$ is that it acts in small space $C_n$ comparing with $B_n$ or with $2^n$ STD of $A(n, X)$ when $X$ run $B^n$. Indeed, it's impossible even to compute (to not say about visualizing) diagram for $B$ in $B_{264}$.

The other nice feature of $\hat{\pi}$-diagram is that shrinking factor is 2 for every level of the diagram apart from the level of cyclic vertexes.

4. Reading Baker diagrams

Here we start from upper estimates for height and cycle lengths. Then we show that these estimates are not improvable. And then we apply the results for decoding information given by baker and index-baker diagrams to characterize in whole the system of all ACA of the cylinder of size $n$.

Firstly, due to the conservation principle for $b$ all rules of the same connectivity component of the baker diagram in $B_n$ have the same determinant value. This, for example, means that if one rule of a connectivity component has $h^* = 0$ then the same is true for all others.

We say that $X \in B^n$ belong to the cycle of the baker transformation if there exists a number $t$ s.t. $b^t \ast X = X$.

**Lemma 8.** If $X$ belongs to a cycle of the baker transformation diagram in $B^n$ then $h^*(n, X) \leq 1$.

**Proof.** By the condition and theorem we have $\hat{X}^{2i^t} = \hat{X}$ for fixed a number $t$ and any $i, i > 0$. Therefore $\ker(\hat{X}) = \ker(\hat{X}^2)$ (see proposition)). This means that images of the operators $\hat{X}, \hat{X}^2$ also are the same and $\ker(\hat{X}) \cap \Im(\hat{X}) = \{0\}$. Hence two opportunity remain:
(i) $\ker(\hat{X}) = \{0\}$. If so then obviously $h^*(n, X) = 0$.
(ii) $\ker(\hat{X}) \neq \{0\}$. Then $2 \Im(\hat{X}^0) \supseteq \Im(\hat{X})$ and the correspondent shrinking factor $|\ker(\hat{X})| > 1$. So all states from $\Im(\hat{X})$ are in cycles and states from $\Im(\hat{X}^0) \setminus \Im(\hat{X})$ are not. So, $h^* = 1$. 

**Corollary 4.** If $n$ is odd then $h^*(n, X) = 1 - \det_2(C(X))$\(^3\).

**Proof.** This follows from the previous lemma and theorem above.

**Corollary 5.** (Lemma 12[1].) If $\kappa(X)$ is even and $n$ is odd then $h^*(n, X) = 1$.

---

\(^2\) $\hat{X}^0$ is the identical operator.

\(^3\) As we agreed in the beginning $\det_2(A)$ is the determinant modulo 2 of a matrix $A$. 
Proof. Indeed, if the parity of $X$ is 0 then $\text{det}_2(C(X)) = 0$ because sum of all column modulo 2 equals 0 and therefore the rank (in the boolean field) of matrix $C(X)$ isn’t $|X|$.

4.1. Upper height estimations.
Let’s start from height estimation. To estimate $h^*$ let’s note that the least effective value of shrinking factor can be 2 for all levels apart from level 0 consisting cyclic states only and where no shrinking exist. Indeed, if there exist non-cyclic states then the kernel of the rule has power bigger 1 and in-degree of any state is 0 for the dangling states and bigger 1 for the other, i.e. for states that have prototypes (see the proposition 1).

Therefore we can write $t + t(\sum_{i=1}^{h^*} 2^{i-1}) = |B^n|$, where $t$ is the number of cyclic states. The right part equals to $t2^{h^*}$. From here, setting the minimal value for $t$, $t = 1$, we get $h^*(n, X) \leq \log_2 |B^n| = n$. This estimate obtained by ”bare hands” can be improved as the following.

Theorem 4. $\forall n(\forall X \in B^n)[h^*(n, X) \leq 2^{\nu(2, n)}]$.

Proof. It is an easy consequence of the theorems 3 and 2. Indeed, from the theorem 3 it follows that $b^{\nu(2, n)} * X$ is a cyclic state. And then from theorem 2 we get that the superposition $\hat{X}^{2^{\nu(2, n)}}$ transfers any state $s$ into a cycle.

The number $\nu(2, n)$ is easy computable on $n$ and in the same time is the maximal height of index-baker (and baker diagrams too). On the other side, the baker diagram distributes rules $X \in B^n$ on levels with the same distance from attractors and therefore contains additional information about rules in comparison with index-baker diagram. So we can improve the estimate above for given a rule $X$. Let denote by $H(n, X)$ the distance from $X$ to the closest cyclic vertex in the baker diagram of $B^n$.

Theorem 5. $(\forall X \in B^n)[h^*(n, X) \leq 2^{H(n,X)}]$.

Proof. The proof actually the same as for the previous theorem because we use the analogous fact: if $H(n, X) = m$ then $b^m * X$ belong to a cycle. That mean $\hat{X}^{2^m} * s$ (see theorem 2) is cyclic state for any initial state $s \in B^n$. The last mean $h^*(n, X) \leq 2^m$.

For example, in case $n = 10$ this estimation looks as $h^* \leq 2$. And the value 2 is reached for $X = [1, 1, 1, 1, 1]$ as well as for $X' = [1, 1, 0, 1, 0, 1]$, see Fig. 8.

---

4Generally speaking to compute $H(n, X)$ one don’t need to compute complete baker diagram for $n$. However sometimes the complexity can be almost the same.
Example 7. The next Fig. 8 shows STD for $n = 10$, $X = [1, 1, 0, 1, 0, 1]$. Here $h^* = 2$ as it is estimated by theorem 5. So the upper estimate was reached for this rule $X$. The ”bare hands” estimate 10 is essentially bigger.

![Figure 8](image.png)

**Figure 8.** STD for $n = 10$, $X = [1, 1, 0, 1, 0, 1]$.

4.2. Cycle length estimation.

**Theorem 6.** For any given $n$ and any $X \in \mathfrak{B}^n$ lengths of cycles in STD of $\mathfrak{A}(n, X)$ must divide number $c^*(n) = 2^{(2^n)(2^{c(n)} - 1)}$.

**Proof.** According to theorems 3, 2 we have $\forall s[\hat{X}^{\exp_2(\epsilon(2^n) + \epsilon(n))} * s = \hat{X}^{\exp_2(\epsilon(2^n))} * s]$ where $\exp_a b = a^b$. Because state $s' = \hat{X}^{\exp_2(\epsilon(2^n))} * s$ belongs to a cycle we can inverse $\hat{X}^{\exp_2(\epsilon(2^n))}$ on $s'$ getting another state $s''$ from the cycle $s'' = \hat{X}^{-\exp_2(\epsilon(2^n))} * s'$, see fig. 10. Hence the relation is true

\[
\hat{X}^{\exp_2(\epsilon(2^n)(\epsilon(n)-1))} * s = s''.
\]

Then we can replace $s$ with $s''$ in (5) and having this done we come to the relation for the element $c''$ of the cycle

\[
\hat{X}^{\exp_2(\epsilon(2^n)) (\epsilon(n)-1)} * s'' = s''.
\]

This just mean that the number $c^*$ must be divided without any remainder by the length of the cycle. Also since $s$ was any initial state, it is possible to say this about any cycle of the diagram for the rule $X$. 

In the diagram of $\mathfrak{A}(8, [1, 1, 1])$ there are 2 cycles with length 1, 1 cycle of length 2, 3 - of length 4, and 30 cycles of length 8. So all possible for $n = 8$ lengths are realized because $c^*(8) = 2^3(2^1 - 1) = 8$.

However this happens not every time. Dimension $n$ constrains scope of possible lengths of cycles. The real collection of cycle lengths in STD of $\mathfrak{A}(n, X)$ for a fixed $n$ varies with $X$.

If $n = 10$ then $c(10) = 4, 2^4 - 1 = 15$, so lengths can run set of dividers of 30. For STD of $\mathfrak{A}(10, [1, 1, 0, 1, 0, 1])$ (see example before, Fig. 5) we have one fixed point 0, one cycle of length 15 and 8 cycles of length 30; $h^* = 2$. 

However for the rule $[1, 1, 1, 1]$ with $h^* = 1$ there are 24 cycles of length 10, three of length 5, and one fixed point; whereas for rule $[0, 0, 0, 0, 0, 1, 1, 1, 1]$ with $h^* = 1$ there exist 16 cycles of length 1 and 120 cycles of length 2.

At last for rule $[1, 0, 0, 1, 0, 1, 0, 1, 1]$ we have $h^* = 2$ and 1 cycle of length 1, 5 cycles of length 3, and 40 cycles of length 6.

One more example: $c(9) = 6$, $ι(2, 9) = 0$, so any cycle of $A(9, X)$ for any $X$ must be a divider of $2^6 - 1 = 63$. STD for $X = [1, 1, 0, 1, 0, 1, 0, 1, 0, 1]$ has cycles of lengths 7, STD for $X = [1, 1, 0, 1, 0, 1, 1, 1]$ has 6 cycles of length 21, and STD for $[1, 1]$ contains 4 cycles of length 63, but STD for $X = [1, 0, 0, 1, 0, 1, 1]$ - has no cycles with lengths that are multiple of 7.

As far we use rule independent computationally easy information about cycle lengths that we get from actually index projection $♮$ of the baker transformation. Again the baker diagram provides, in general, more accurate estimates depending on concrete rules $X$. True, this information is more costly in the sense of computability.

As before we use denotation $H(n, X)$ for the height of the rule $X$ in the diagram of $b$ in $B^n$. In addition, let’s denote by $C(n, X)$ the length of cycle in basin of which $X$ is for the diagram.

**Theorem 7.** For given $n$ and $X \in B^n$ lengths of cycles in STD of $A(n, X)$ must divide number $C^*(n, X) = 2^{H(n, X)}/(2^{C(n, X)} - 1)$.

The proof of the theorem repeats the proof in general case with the natural replacement $ι(2, n)$ and $c(n)$ with the numbers $H(n, X)$ and $C(n, X)$ that are specified for the considered rule $X$. (It is worth to note that $2^{H(n, X)}/(2^{C(n, X)} - 1)|2^{ι(2, n)}/(2^{c(n)} - 1)$, so no contradiction exists between the last two theorems.)

Now let’s apply the last result to the case $n = 10$. 
The rules \( X = [1, 1, 0, 1, 0, 1] \) and \( X = [1, 1, 1, 1] \) belong to basins of cycles with length 4 in the baker diagram. The both of them have \( H = 1 \). So the last theorem tell us nothing of new.

However the rule \( X = [0, 0, 0, 0, 0, 1, 1, 1] \) with \( H(10, X) = 1 \) belongs to basin of cycle with length 1. Therefore only cycles of lengths 1 and 2 can occur in STD of \( \mathcal{A}(10, X) \). And indeed, our computation brings 16 cycles of length 1, 120 cycles of length 2.

Then, rule \( [1, 0, 0, 1, 0, 1, 1, 1, 1] \) occurs in a basin of the cycle of length 2 in the baker diagram. Therefore in accordance with theorem 7 we have 1 cycle of length 1, 5 cycles of length 3, and 40 cycles of length 6.

At last, rule \( X = [1, 0, 0, 1, 0, 1, 0, 1, 1] \) belongs to the cycle of length 4 in the baker diagram. So \( H(10, X) = 0, \mathcal{C}(10, X) = 4 \). The theorem 7 states that the cycles of \( \mathcal{A}(10, X) \) must divide the number \( 2^6(2^4 - 1) = 15 \). And indeed, according to computations STD for \( \mathcal{A}(10, X) \) contains 1 fixed point and 51 cycles of length 5. Note, \( h^*(10, X) = 1 \).

The data are represented by the table, see fig. 10. We recall, that \( H^* = 1, c^* = 30 \); so the supposed on the base of index baker diagram collection of cycle lengths is 1, 2, 3, 5, 6, 10, 15, 30. Knowledge of the position of rule \( X \) (i.e. \( H(10, X) \)) and the power \( \mathcal{C}(10, X) \) of the attractor (=cycle) whose basin contains \( X \) in the baker diagram often provide more exact estimation.

| \( n \) | \( X \) | attractor power \( \mathcal{C}(10, X) \) | \( X \)-height \( H(10, X) \) | cycle lengths \( C^*(10, X) \) | \( h^*(10, X) \) |
|---|---|---|---|---|---|
| 1 | [0000001111] | 1 | 1 | 2 | 1,2 | 1 |
| 2 | [1001011011] | 2 | 1 | 6 | 1,3,6 | 2 |
| 3 | [1000101010] | 4 | 0 | 15 | 1,5 | 1 |
| 4 | [1101010000] | 4 | 1 | 30 | 1,15,30 | 2 |
| 5 | [1111000000] | 4 | 1 | 30 | 1,5,10 | 1 |

**Figure 10.** Cycle lengths for different rules, \( n = 10 \).

As another example of the theorem application we sum up the result relating to odd number \( n = 9 \) in the table of fig. 11. This time \( H^* = 0, \text{ord}_q 2 = 6, c^* = 2^6 - 1 = 63 \); so the supposed on the base of index baker diagram collection of cycle lengths is 1, 3, 7, 9, 21, 63. Now \( H(9, X) = 0 \) but the power \( \mathcal{C}(10, X) \) of the attractor (=cycle) whose basin contains \( X \) in the baker diagram indeed help getting more exact estimation, see the first 4 rows of the table.

The comparison of the theoretical estimates and experimental data done in these tables show that a rule position in baker diagram doesn’t provide complete information about the spectrum of cycle lengths and \( h^* \) value of STD for the rule.
This statement is confirmed by the following. Results about lower estimates of $h^*$ and maximal cycle length would be of great interest. Yet, the proposition:

$2^{H(n,X)-1} (1 - \det_2(n, X)) \leq h^*(n, X) \leq 2^{H(n,X)} (1 - \det_2(n, X))$

looks pretty naturally and has many supporting it examples. Nevertheless the example disproves it as the general statement.

**Example 8.** Let $n = 12$. The rules $X = [1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0]$ and $Y = [1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0]$ are in the basin of the rule $Z = [0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0]$. Moreover we have $b * X = Y, b * Y = Z, \mathcal{frak} b * Z = Z$. Despite $H(12, X) = 2 > 1 = H(12, Y) > 0 = H(12, Z)$ for all these rules $h^* = 1$. The difference takes place for cycle spectra. $X$ comes with cycle lengths 1(4), 2(6), and 4(60). For $Y$ there are numbers 1(16), 2(120). And for $Z$ we have 1(). It is a striking thing that $\ker \hat{X} = \ker \hat{Y} = \ker \hat{Z}$.

So, not trivial lower estimations of $h^*$ probably need more informative characteristics of a rule than $H$. One of the appropriate tools is obtaining below estimations by means of the special imbedding one diagram into another the generating rules of which are connected with each other by baker transformation. We will not develop this idea here. Instead further we check whether the language of DBT equalities and inequalities is able to give us more.

The results about upper estimations tell us that when we are moving from dangled vertexes of a baker diagram to its attractor then upper bounds of $h^*$ and maximal lengths of cycles monotonically decrease. But this is also true for the real maximal cycle lengths and $h^*$ (monotonicity principle).
5. **Equalities and Inequalities with b and ⊤**

This section continues the study how to interpret baker diagrams; now we use equations with the introduced operators.

5.1. **Equalities and inequalities with b.**

Application of $b$ to a rule $X$ in terms of a baker diagram $G$ mean a passage from vertex $X$ to the end of edge $(X, b \ast X)$ of the diagram. Therefore it's possible to express some relations and substructures of the graph $G$. In this way some conditions for rules can be set and solved. In general, expressive power of the first-order language with functions $b, \sharp$ on finite strings is not a simple problem.

If we restrict ourselves with non-quantified formulas, we come to systems of equalities and their negations. The use for our theme of equalities and inequalities with $b$ can be seen clearly on the next example.

**Lemma 9.** Three statement are equivalent:

1. $b^{c(|X|)} \ast X = X$;
2. $(X$ belongs to the cycle of the baker diagram in $\mathcal{B}^{[X]}$);
3. $\forall j [0 < j < |X| \& 2^{\epsilon(2, |X|)} \nmid j \implies X_j = 0]$.

**Proof.** Firstly let $|X|$ be odd. Then, according to theorem (1) is true for all $\mathcal{B}^{[X]}$ because $\epsilon(2, |X|) = 0$; the baker diagram consists of cycles only; and (3) is true trivially.

Now, let $|X|$ is even.

(1) $\implies$ (2) because in general $(Y, b \ast Y$) is the edge of the baker diagram and form (1) it follows that starting from $X$ by means of $c(|X|)$-edge path we come back to $X$.

To show (2) $\implies$ (3) let on the contrary there be a number $j, 0 < j < |X|, 2^{\epsilon(2, |X|)} \nmid j$, s.t. $j$-th component of $X$ is not 0. Since $\epsilon(|X|) > 0$ it's possible to pass to the result of any finite applications of $b$ to $X$ being inside the cycle. By the definition of $b$ $i$-th application of $b$ replaces all components of the argument, components whose numbers are of kind $2^{i-1}m, m$ odd, with 0 ("sweeps out" the components). So, after $\epsilon(2, |X|)$ subsequent applications of $b$ to $X$ the condition (3) must be true and never more any of these component can be 1. If so, then $X$ can not belong to the cycle, contradiction.

Now, let (3) is true. Then application $b$ to $X$ actually can be reduced to application of $b$ to the rule $X' \in \mathcal{B}^{[X]/\epsilon(2, |X|)}$ by subsequent applying lemma 5 Since $|X|/\epsilon(2, |X|)$ is odd number, it’s clear that $X$ is in a cycle.
It remains only to remind that length of any cycle of the baker diagram divides the number $c(|X|)$.\footnote{One have not to mix cycles of baker diagrams with cycles of automata STD. In the last case see theorems 6, 7.}

The condition (1) of the lemma is quite computable within complexity $O(|X|^3)$ since $c(|X|) \leq |X| - 1$ and the complexity of computing $b^* X$ does not exceed $|X|^2$. However the condition (3) can be checked for linear time relatively $|X|$.

Another remark is that despite the property of a rule $X$ to be in a baker-cycle generally looks as $\exists i [b^i X = X]$, in reality, as the lemma tells us, this quantifier is bounded.

Recall, (corollary 2), that we named a rule $X$ $b$-swept or baker-swept if $\forall j [0 < j < |X| & 2^{(2,|X|)} \mid j \Rightarrow X_j = 0]$. So the previous theorem states that $X$ belongs to a cycle of the baker diagram in $B(|X|)$ iff $X$-is $b$-compressed. Evidently, the property to be baker-compressed is easily checked. Therefore the next corollary has a sense.

**Corollary 6.** ($b^i X$ is a $b$-swept) $\implies h^*(|X|, X) \leq 2^i$.

The proof is evident due the theorems 5 and the previous lemma.

The sense of fixed point consideration for the baker transformation becomes clear in view of the next results.

**Theorem 8.** A rule $X$ is a fixed point of $b$ (i.e. $b^* X = X$) $\iff$ every attractor of STD for $A(|X|, X)$ is a fixed point of $\hat{X}$ and ($\hat{X} = \hat{I}$ or $h^* (|X|, X) = 1$).

**Proof.** $\Rightarrow$. As it follows from the previous lemma for $q = 1, r = 0$ the all cycle lengths of $A(|X|, X)$ must divide $2 - 1 = 1$, i.e. the attractor set of STD for $\hat{X}$ consists of fixed points only. Also $h^* \leq 1$. $\square$
Now, if $\det_2(|X|, X) = 1$ then, since every state is a fixed point of $\hat{X}$, we get $\hat{X} = I$, i.e. the identical operator. Otherwise, the determinant equals to 0 and $h^* = 1$.

$\iff \hat{X} = I$ means $X = [1, 0, \ldots, 0]$. Therefore $b \ast X = X$. Now, let $h^*(|X|, X) = 1$ and every attractor of $\mathcal{A}(|X|, X)$ is a fixed point. Then for any state $s$ the state $\hat{X} \ast s$ is a fixed point. So $\forall s[\hat{X}^2 \ast s = \hat{X} \ast s]$. This means $\hat{X}^2 = \hat{X}$ and therefore $b \ast X = X$.\footnote{It’s easy to find $s$ s.t. $\hat{X} \ast s \neq \hat{Y} \ast s$ if strings $X$ and $Y$ are not equal.}

When every attractor consists a single state, the quantity of the basins is a degree of 2 since it equals to $\frac{2^{|X|}}{|\ker X|}$.

Solution of the equations of the considered type doesn’t present big difficulties at least in case when the dimension of $X$ is given. For example, let $n = 9$ and $X = [a_0, a_1, \ldots, a_8]$. The equation looks as $[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8] = [a_0, a_5, a_1, a_6, a_2, a_7, a_3, a_8, a_4]$. So it’s not problem to write the general solution: $a_0 = a$, $a_3 = a_6 = c$, $a_1 = a_2 = a_4 = a_5 = a_7 = a_8 = b$ or the solution is $X_s = [a, b, b, c, b, b, c, b, b]$ where $a, b, c$ are arbitrary boolean numbers.

Let $a = 0$, $b = 1$, $c = 0$. The diagram is represented on Fig. 12.

\textbf{Figure 12.} The diagram for $[0, 1, 1, 0, 1, 1, 0, 1, 1]$.

Another example of lemma 10

\textbf{Corollary 7.} $b^2 \ast X = b \ast X \iff h^* \leq 2$ and lengths of cycles in STD of $\mathcal{A}(|X|, X)$ don’t exceed 2.

It’s interesting however that for odd $|X|$ the solutions are the same as for the equation $b \ast X = X$. Indeed, for odd $|X|$ the vector $X$ belongs to the cycle of the baker transformation, i.e. $X = b^t \ast X$ for some $t > 0$, see theorem $\exists t (2, |X|) = 0$. But because $b^2 \ast X = b \ast X$ we can transform $b^t \ast X$ into $b \ast X$. However it is not so for even $n$, in general.

\textbf{Proposition 3.} $b \ast X \neq X \& b^2 \ast X = \mathcal{B} \ast X \iff h^*(|X|, X) = 2$ or $\hat{X}$ has an attractor with length 2.

\textbf{Proof.} In fact, the maximal cycle length is equal or less than 2. Suppose it is 1. Then if $\det_2(|X|, X) = 1$ we have $X = I$ and hence $b \ast X = X$,
contradiction. So \( \det_2(|X|, X) = 0 \) and therefore \( h^*(|X|, X) \in \{1, 2\} \).

However the case \( h^* = 1 \) leads to the same contradiction in view of the theorem 8. □

The rule \([0, 0, 0, 0, 0, 1, 1, 1, 1]\) from the table 10 is an example: \( b \ast [0, 0, 0, 0, 0, 0, 1, 1, 1, 1] \neq [0, 0, 0, 0, 0, 0, 1, 1, 1, 1] \) but \( b^2 \ast [0, 0, 0, 0, 0, 0, 1, 1, 1, 1] = b \ast [0, 0, 0, 0, 0, 0, 1, 1, 1, 1] \).

The next two easy statements have a clear meaning for reading of the baker diagram. This is why we place them here.

**Theorem 9.** If \( X \) belongs to basin of zero in the baker diagram for \( B^n \) then \( \text{STD for } A(n, X) \) has the only attractor and it is 0.

**Proof.** By the condition \( \exists i [b^i \ast X = 0] \). This means that \( \exists i \forall s [(b^i \ast X) \ast s = 0] \), or every state \( s \) belongs to the basin of zero in STD for \( X \). □

What are those rules from \( B^n \) that belong to the basin of 0 of the baker diagram?

The next theorem answers a more general question:

**Theorem 10.** \( X \) belongs to the basin of a fixed point \( Y \) in baker diagram for \( B^n \) \iff \( b^{(2,n)} \ast X = Y \).

**Proof.** As we know (see, for example, corollary 6) \( Z = b^{(2,n)} \ast X \) belongs to a cycle of the baker diagram. And because the cycle contains only \( Y \) we get \( Z = Y \). □

**Example 9.** Let \( n = 6 \). Then \( \iota(2, n) = 1 \) and the equation \( b \ast X = 0 \) brings \( X_0 \oplus X_3 = 0, X_1 \oplus X_4 = 0, X_2 \oplus X_5 = 0 \). From here the general solution is \( X = [a, b, c, a, b, c] \), \( a, b, c \in \{0, 1\} \).

5.2. **Criteria for** \( h^* = 0 \) **and** \( h^* = 1 \). More complex formulas that include not only \( b \) but the binary operation \( \boxplus \), see definition 11 have more expressive power. For example, let’s write a criterion of \( h^*(|X|, X) = 0 \) (or to have non-zero determinant modulo 2, i.e. \( \det_2(n, X) = 1 \)).

**Theorem 11.** Let \( |X| = n \). Then

\[
(7) \quad h^*(n, X) = 0 \iff \bigotimes_{i=0}^{c(n)-1} b^{(2,n)+i} \ast X = I.
\]

**Proof.** First of all, one can reformulate \( h^*(n, X) = 0 \) as the proposition that any state \( s \) belong to a cycle of the STD.

Now, if \( s \) satisfies the equation

\[
(8) \quad \hat{X}^{c^*(n)} \ast s = s.
\]
then it belongs to a cycle. Conversely, if any state \( s \) belongs to a cycle of the STD then, taking into account the theorem 6, we come to the equation.

Since (8) true for any \( s \), we get

\[
\hat{X}^{c^*(n)} = \hat{I}.
\]

What remains is to transform (9) equivalently into the equation

\[
\bigotimes_{i=0}^{c(n)-1} b^{(2, n)+i} \ast X = I
\]

i.e. into the form given in the theorem condition.

For that, we need only to note that one can pass from operator product using the feature of operation \( \bigotimes \) to a \( \bigotimes \)-composition of vector notices of the rules. For example, from \( \hat{X} \hat{X} \) we can pass to \( \hat{X} \bigotimes X \). And then due to \( b^{(2, n)}(2^{c(n)} - 1) = \sum_{i=0}^{c(n)-1} 2^{(2, n)+i} \) we can represent the composition in the form we need by collection of segments of the composition into blocks of kind \( (b^i \ast X) \) according to theorem 2.

**Example 10.** Let \( n = 2^m \). Since \( \iota(2, 2^m) = m, c(2^m) = 1 \) the condition of the theorem looks as \( b^m \ast X = I \). Because \( b^m \ast X \) is swept \( b^m \ast X = [\sum_{i=0}^{m} X_i, 0, \ldots, 0] \). Therefore the solution of the equation is any rule \( X \) s.t. \( \sum_{i=0}^{m} X_i = 1 \). This means that there are just \( 2^{m-1} \) (one half of \( 2^m \)) rules with \( h^* = 0 \).

**Example 11.** Let \( n = 2^m 3 \). We have \( \iota(2, 2^m) = m, c(2^m 3) = c(3) = 2 \), and the condition of the theorem looks as

\[
(b^{m+1} \ast X) \bigotimes (b^m \ast X) = I.
\]

Because \( b^m \ast X \) is \( b \)-swept, it actually has 3 only non-zero components on positions \( 0, \cdot 2^m, 1 \cdot 2^m, 2 \cdot 2^m \). Let’s denote them as \( x, y, z \) correspondingly. Given with a concrete \( m \) we can easily (using the definition of \( b \)) write formulas expressing these variables in terms of the components of \( X \). In particular, when \( m = 1 \) (\( n = 6 \)) we get

\[
x = X_0 \oplus X_3, \quad y = X_1 \oplus X_4, \quad z = X_2 \oplus X_5,
\]

whereas for \( m = 2 \) (\( n = 12 \)) we have

\[
x = X_0 \oplus X_3 \oplus X_6 \oplus X_9, \\
y = X_1 \oplus X_2 \oplus X_4 \oplus X_7, \\
z = X_5 \oplus X_8 \oplus X_{10} \oplus X_{11}.
\]

Now, as it was said above, \( b^m \ast X = [x, 0, \ldots, 0, y, 0, \ldots, 0, z, 0, \ldots] \) and \( b^{m+1} \ast X = [x, 0, \ldots, 0, z, 0, \ldots, 0, y, 0, \ldots] \). Therefore \( (b^{m+1} \ast X) \bigotimes (b^m \ast X) = [x \oplus y \oplus z, 0, \ldots, 0, xz \oplus xy \oplus yz, 0, \ldots, 0, xz \oplus xy \oplus yz, 0, \ldots, 0] \).
Because the last vector must represent \( I \) we come to the system

\[
x \oplus y \oplus z = 1
\]
\[
xyz \oplus xy \oplus yz = 0,
\]
or, excluding \( x \) from the second equation on the base of the first,

\[
x = 1 \oplus y \oplus z
\]
\[
yz = 0.
\]

What remain is only to replace \( x, y, z \) with their "component meanings" according to \( (10) \). However for that we need to do \( n \) certain. We set \( n = 6 \), and transformed \( (11) \) into conditions for components of \( X \) to have \( h^* = 0 \):

\[
1 = X_0 \oplus X_1 \oplus X_2 \oplus X_3 \oplus X_4 \oplus X_5,
\]
\[
0 = (X_1 \oplus X_4)(X_2 \oplus X_5).
\]

The numbers of these rules are: 1, 2, 4, 8, 11, 13, 16, 19, 22, 25, 26, 31, 32, 37, 38, 41, 44, 47, 50, 52, 55, 59, 61, 62. Direct computations support this.

The next criterion needs not only equalities by inequalities too.

**Theorem 12.** Let \( |X| = n \). Then \( h^* = 1 \) is true if and only if

\[
\begin{bmatrix}
\bigotimes_{i=0}^{c(n)-1} b^{(2,n)+i} \ast X \\
\bigotimes_{i=0}^{c(n)-1} b^{(2,n)+i} \ast X
\end{bmatrix} X = X,
\]

\[
\begin{bmatrix}
\bigotimes_{i=0}^{c(n)-1} b^{(2,n)+i} \ast X \\
\bigotimes_{i=0}^{c(n)-1} b^{(2,n)+i} \ast X
\end{bmatrix} X \neq I.
\]

**Proof.** First of all we prove that \( h^*(|X|, X) = 1 \iff (\det_2(n, X) = 0 \) and

\[
\hat{X}^c(n) \hat{X} \ast s = \hat{X} \ast s
\]

is an identity relatively states \( s \).

\( \Rightarrow \). As theorem \( 6 \) tells us, for any given \( n \) and any \( X \in \mathfrak{B}^n \) lengths of cycles in STD of \( \mathfrak{A}(n, X) \) must divide number \( c^*(n) = 2^{(2,n)}(2^{c(n)} - 1) \). Now, if \( h^*(|X|, X) = 1 \) then the equation \( (13) \) is in fact an identity relatively \( s \). Indeed, this is clear not only for the states \( s \) that are not included in any cycle but for cyclic states too.

\( \Leftarrow \). Let \( X \) obeys the identity \( (13) \). Then every \( X(s) \) belongs to a cycle of the STD. Therefore the height of the STD for \( X \) can’t be bigger 1. And if \( \det_2(n, X) = 0 \) then the diagram height is not 0.

What remains is to transform equivalently the equation

\[
\hat{X}^c(n) \hat{X} = \hat{X}
\]

into the form given in the theorem condition. This can be done in the same way as in the previous theorem. □
Example 12. Let $n = 2^m$. Since $\iota(2, 2^m) = m$, $c(2^m) = 1$ the condition of the theorem looks as $b^m \star X \boxtimes X = X$. As we know (see the previous example also) $b^m \star X = [a, 0, \ldots, 0]$, $a = \sum_{i=0}^{m} X_i$. The inequality from the criterion enforces $\text{det}_2(2^m, X) = 0$; so $a = 0$. Thus we come to $0 \boxtimes X = X$. So the only rule $X$ for that $h^*(2^m, X) = 1$ is $X = 0$.

Because of the additional $X$ in the equation of the last theorem, the calculations become more complicated, but for $n = 6$ it is quite doable even by hands. In this way we found the list of all rules of length 6 that have $h^* = 1$: $0, 5, 10, 15, 17, 20, 21, 30, 34, 39, 40, 42, 51, 57, 60$. So the rules that do not occur in this list and the list of the example11 have the height 2, since according to our upper estimation 2 is the upper limit for $n = 6$.

Of course, this line of criteria can be continued, i.e. one can formulate analogously criteria for $h^*$ to be equal to given a number $k$. However, the complexity would increase and therefore the computational aspect of these expressions deserves a discussion.

Coming to the computational aspect of these results in general, let’s estimate the complexity of the computation setting by formula $\xi = \sum_{i=0}^{c(n)-1} b^i(2, n) \star X$. Since we deal with boolean strings, $b \star X$ can be calculated for $O(n)$ time, where $n = |X|$. Therefore in sum to compute all operands of $\boxtimes$ we need no more than $O(nc^*(n))$ of time. Also every $\boxtimes$ with $n$-long boolean strings needs no more than $O(n^2)$ time. Therefore we estimate the general time expenses as $O(c(n) \cdot n^2)$. This is comparable with the time one needs to calculate the rank of a $n \times n$-matrix with boolean elements. According to [1] the average value $c(n)$ of $c(n)$ grows as $o(n)$, i.e.

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{1 \leq i \leq n} c(n) = 0.$$  \hspace{1cm} (14)

This mean that the method, suggested in theorem11 to calculate our determinants modulo 2 is more effective in average than the method using rank computation.

Anyway, in case of small $c(n)$ the suggested method to compute the determinant modulo 2 has the good practical efficiency.

Therefore, the results presented by theorems1112 a certain theoretic and computational value.
5.3. Description of the basin of 0 in STD.

It is well known with given rule $X$, how to write the system of linear equations s.t. its solutions are pre-images of 0. However to describe the whole basin of $\{0\}$ we need to solve equation systems of kind $X^i \ast s = 0, i = 1, 2, \ldots$. The next result suggests a single equation, describing 0-basin of $X$. We recall that $\bigcirc$ is reversion of sequence, and $\sigma$ is the cyclic shift to right on one position.

**Theorem 13.** $s$ belongs to the basin of $\{0\}$ in STD for $A(n, X)$ if and only if $s \boxtimes (\sigma \ast (b^{(2, n)} \ast X)^\bigcirc) = 0$.

**Proof.** First of all, as we know, $h^*(n, X) \leq 2^{\ell(2, n)}$. Hence if $Z = b^{(2, n)} \ast X$ then ($s$ belongs to the basin of zero $\iff Z \ast s = 0$).

Now, all what remain to do is to write the circulant matrix equation $C(Z) \ast s = 0$ in terms of our operations $\boxtimes, \bigcirc, \sigma$. For that we pass firstly to $s \ast C^T(Z)$, where $C^T$ is the transposed circulant $C$, and then apply $\boxtimes$ instead of the right multiplication of the vector on the matrix. So $\sigma \ast (Z^\bigcirc)$ present the first column (row) of the circulant $C(Z)$ ($C^T(Z)$). $\square$

5.4. Determinant reduction.

For the next result it is convenient to introduce the operation of $b$-compression $X/b$ of given sequence $X$ as following. First of all we pass from $X = [x_0, \ldots, x_{n-1}]$ to $b^{(2, n)} \ast X = [z_0, 0, \ldots, z_1, 0, \ldots, z_{\ell(2, n)}, 0 \ldots]$. Here 0 occupy positions $j$ s.t. $2^{\ell(2, n)} \nmid j$. The positions $j$ which are multiple of $2^{\ell(2, n)}$ are occupied by $z_i$. At last, $X/b = [z_0, z_1, \ldots, z_{\ell(2, n)}]$.

**Theorem 14.** $\det_2(n, X) = \det_2(\frac{n}{\ell(2, n)}, X/b)$, i.e. in other words the determinant modulo 2 of the any rule $X$ coincides with the determinant modulo 2 of the result $b$-compression applied to $X$.

**Proof.** We suggest two proofs.
(1) Let’s start from the theorem 11. It can be reformulated as $(n = |X|)$:

\[
\det_2(n, X) = 1 \iff \bigotimes_{i=0}^{c(n)-1} b^{(2, n)+i} \ast X = I.
\]

Yet, $b^{(2, n)+i} \ast X = b^i \ast Y$ if $Y$ denotes $b^{\ell(2, n)} \ast X$. On the other side, as we know, all components of $Y$ which numbers are not divisible by $2^{\ell(2, n)}$ are equal to 0. The same is true for $b^i \ast Y$. And according to lemma all components with numbers not divisible by $2^{\ell(2, n)}$ of the left side

(15)

\[
\bigotimes_{i=0}^{c(n)-1} b^i \ast Y
\]

of the equation in the equivalence are 0.
The following discourse in essence repeats the reduction lemma \[5\]. Let’s pass from \(Y\) to a rule \(y\) by cancelling all components of \(Y\) that have numbers being not divisible by \(\iota(2, n)\). Clearly, \(y\) is \(b\)-compression of \(X\). Sure, \(y \in \mathfrak{B}^{n'}\) where \(n' = \frac{n}{\iota(2, n)}\). Due to lemma \[2\] if \(y\) is \(b\)-compression of \(X\) then \(b \ast Y\) also can be transformed into \(y'\) that is \(b\)-compression of \(b \ast X\), and \(y' = b \ast y\). Therefore the equality

\[
\bigotimes_{i=0}^{(n)-1} b^{\iota(2, n) + i} \ast X = I
\]

is true if and only if

\[
\bigotimes_{i=0}^{(n)-1} b^{i} \ast y = I
\]

is true. Of course, last \(I\) denote the sequence \([1, 0, \ldots] \in \mathfrak{B}^{n'}\).

By theorem \[11\] the last equality is equivalent \(\det_2(n', y) = 1\). So, \(\det_2(n, X) = 1 \iff \det_2(n', y) = 1\) if only \(y\) is \(b\)-compression of \(X\).

(II) The second proof is based on the classical formula for the determinant of a circulant matrix.

As it’s well known determinants of linear operator \(L\) and every its degree \(L^i, i \geq 1\), are equal or not equal to 0 simultaneously. For the boolean field and determinants modulo 2 this leads to the possibility to replace equality to 0 with equality. Therefore \(\det_2(|X|, X) = \det_2(|X|, b^m \ast X)\). If we set \(m = \iota(2, |X|)\) then we get \(b\)-compression \(X/b\) of \(X\). Now we write out the determinant \(\delta\) of the circulant matrix, produced by \(b^m = [z_0, z_{-1}, \ldots, z_{|X|}]\), using the formula (14.312) from [3, p.1068]:

\[
(16) \quad \delta = \prod_{j=1}^{|X|/m} \sum_{i=0}^{|X|/2m - 1} z_i w_j^i.
\]

Because, \(z_i = 0\) if \(2^m \nmid i\) we can write \(\delta = \prod_{j=1}^{|X|/m} \sum_{i=0}^{|X|/2m - 1} z_{i2^m} w_j^{i2^m}\). When \(i\) runs the list \(0, \ldots, |X|/m - 1\) the number \(w_j^{i2^m}\) runs subsequent roots \(e^{\frac{2\pi i}{2^m}}, i = 0, \ldots, |X|/2^m - 1\) of degree \(|X/b|\) of 1. So we come to \(\delta = \delta_i^{2^m}\) where \(\delta_i\) is the determinant of \(b\)-compression \(X/b\). Now the conclusion is obvious. \(\square\)

This theorem can help in case when the compression factor \(\frac{n}{\iota(2, n)}\) is sufficiently big comparing with \(n\). In particular, as we already know, \(\det_2(2^k, X) = X_0 \oplus X_1 \oplus \cdots \oplus X_{2^k - 1}\).
6. REFERENCES

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