THE REGULARITY OF ALMOST-COMMUTING
GROTHENDIECK-SPRINGER RESOLUTIONS AND BOREL ANALOGS OF
CALOGERO-MOSER VARIETIES

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Abstract. Consider the moment map $\mu: T^*(b \times \mathbb{C}^n) \to b^*$. We prove that the preimage of 0 under $\mu$ is a complete intersection. This allows us to study nearby fibers of $\mu$ as they are equidimensional, and one may also construct GIT quotients $\mu^{-1}(0)/\chi B$ by varying the stability condition $\chi$. We also give an explicit description of the irreducible components for completeness. Finally, we study a variety analogous to the scheme with connections to Calogero-Moser phase space, studied by Wilson.

1. Introduction

The classical Springer resolution and the Grothendieck-Springer resolution are foundational and important schemes in algebraic geometry and representation theory. The Springer resolution is a desingularization of the variety of nilpotent elements in a semisimple Lie algebra (see, e.g., [CG10, DG84]), while the Grothendieck-Springer resolution is the minimal, symplectic resolution of the variety of the semisimple Lie algebra, which contains the Springer resolution (see, e.g., [CG10, Im18]).

We consider the case for $G = \text{GL}_n(\mathbb{C})$, the general linear group of invertible complex $n \times n$ matrices, and let $\mathfrak{g} = \text{Lie}(G)$, the Lie algebra of $G$ of all complex $n \times n$ matrices. Under the adjoint action of $G$ on $\mathfrak{g}$ (i.e., $G$ acts by conjugation), we have the natural adjoint quotient map $\rho: \mathfrak{g} \rightarrow \mathfrak{g}/G \cong \mathbb{C}^n$, which sends $r$ to the tuple of coefficients of its characteristic polynomial $\chi_r(t)$. So we have that $r \mapsto (\text{tr}(r), \ldots, \det(r))$, and these polynomials are invariant under the adjoint action. Since $\rho^{-1}(0)$ consists of those $r \in \mathfrak{g}$ such that $\chi_r(t) = t^n$, the preimage of $\rho$ is precisely the nilpotent elements in $\mathfrak{g}$. We denote $\mathcal{N} := \rho^{-1}(0)$, the nilpotent cone of $\mathfrak{g}$.

Now let $B$ be the standard Borel subgroup of $G$, which are the invertible upper triangular matrices, and let $\mathfrak{b} = \text{Lie}(B)$, the Lie algebra of $B$, consisting of all upper triangular matrices. Define $G/B$ to be the full flag (projective) variety parameterizing Borel subalgebras in $\mathfrak{g}$. Let

$$\tilde{\mathfrak{g}} := G \times_B \mathfrak{b} = \{(x, b) \in \mathfrak{g} \times G/B : x \in \mathfrak{b}\},$$
$$\tilde{\mathcal{N}} := G \times_B \mathfrak{n} = \{(x, b) \in \mathcal{N} \times G/B : x \in \mathfrak{b}\}.$$

Let $p: \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ be the Springer resolution ([CG10, Chapter 3] and [DG84]) and $\pi: \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ be the Grothendieck-Springer resolution ([Spr76, Ste74, Ste76]). Then we have an inclusion $\iota: \tilde{\mathcal{N}} \rightarrow \tilde{\mathfrak{g}}$ such that $\pi \circ \iota = \gamma \circ p$, where $\gamma: \mathcal{N} \rightarrow \mathfrak{g}$ is the natural inclusion. The Springer resolution is a symplectic resolution of the singular symplectic cone $\mathcal{N}$, while $\pi$ is a versal Poisson deformation of $p$.

In this manuscript, we consider a moment map associated to the Grothendieck-Springer resolution and show that the preimage of 0 under this map is a complete intersection. Such property is important and interesting since it provides the exact number of irreducible components in the special fiber. Furthermore, one can also construct topological fibers, analogous to the topological Springer resolution.
fibers of type $A$ corresponding to nilpotent endomorphisms with two equally-sized Jordan blocks studied in [Kho02, Kho04], and study the topology and intersection of the cotangent bundle of the Grothendieck-Springer fibers. Next, we will give a construction of this moment map.

Using the trace pairing, we have the identifications $g \simeq g^*$ and $b^* \simeq g/u$, where $u$ is the maximal nilpotent subalgebra of $b$ (i.e., strictly upper triangular matrices). The latter pairing is given by $b \times g \to \mathbb{C}$, $(r, s) \mapsto \text{tr}(rs)$, where this map factors through the bilinear, non-degenerate pairing $b \times g/u \to \mathbb{C}$. Let $B$ act on the space $b \times \mathbb{C}^n$ via $b.(r, i) = (brb^{-1}, bi)$. We differentiate the $B$-action to obtain the comoment map

$$b = \text{Lie}(B) \overset{\delta}{\to} \Gamma(T_{b \times \mathbb{C}^n}) \subseteq \mathbb{C}[T^*(b \times \mathbb{C}^n)],$$

where

$$a(v)(r, i) = \frac{d}{dt} \left( \exp(tv).(r, i) \right) \bigg|_{t=0} = ([v, r], vi).$$

Now, we dualize $a$ to obtain the moment map

$$\mu: T^*(b \times \mathbb{C}^n) \simeq b \times g/u \times C^n \times (\mathbb{C}^n)^* \to b^*,$$

where $(r, \pi, i, j) \mapsto \text{ad}^*_r(s) + a^*(ij)$

and $a: g \to \text{End}(\mathbb{C}^n)$ is the natural $g$-representation on $\mathbb{C}^n$. Since $g = \text{Lie}(G)$, we pull back $a$ to obtain $a^*: \text{End}(\mathbb{C}^n)^* \to g^*$, where $a^*(ij) = ij$. So the map for $\mu$ can be rewritten as $(r, s, i, j) \mapsto [r, s] + ij$.

Note that since $B$ acts on $b \times \mathbb{C}^n$ as $b.(r, i) = (brb^{-1}, bi)$, the $B$-action is induced onto $T^*(b \times \mathbb{C}^n)$ as

$$(g', r, i) = (g'g^{-1}, r, gi),$$

which induces the moment map

$$\mu_G: T^*(G \times b \times \mathbb{C}^n) \equiv G \times g^* \times b \times \mathbb{C}^n \times (\mathbb{C}^n)^* \to g^*, \quad (g, \theta, r, s, i, j) \mapsto -\theta + a^*(ij).$$

We also have a natural $B$-action on the second and the third factors of $G \times b \times \mathbb{C}^n$, given by $b.(g', r, i) = (g'g^{-1}, r, gi)$. This action is induced onto the cotangent bundle, with its moment map given by $\mu_B: T^*(G \times b \times \mathbb{C}^n) \to b^*$, where $(g, \theta, r, s, i, j) \mapsto \text{Ad}_g(\theta) + \text{ad}^*_r(s)$. Thus there is a natural $G \times B$-action on $G \times b \times \mathbb{C}^n$. We combine the two maps $\mu_G$ and $\mu_B$ to obtain the moment map

$$\mu_{G \times B}: T^*(G \times b \times \mathbb{C}^n) \equiv G \times g^* \times b \times \mathbb{C}^n \times (\mathbb{C}^n)^* \to (g \times b)^* \equiv g^* \times b^*,$$

which is given by

$$(g, \theta, r, s, i, j) \mapsto (-\theta + a^*(ij), \text{Ad}_g(\theta) + \text{ad}^*_r(s)).$$

Now consider

$$\mu_{G \times B}^{-1}(0, \bar{0}) = \{ (g, \theta, r, s, i, j) \in T^*(G \times b \times \mathbb{C}^n) : \theta = ij, g\theta g^{-1} = -\text{ad}^*_r(s) \}.$$

We may set $g = 1$ since $g \in G$ is a free variable. Then there is a bijection between $B$-orbits on $\mu_{G \times B}^{-1}(0, \bar{0})$ and $G \times B$-orbits on $\mu_{G \times B}^{-1}(0, \bar{0})$, giving us an isomorphism between the quotient stacks $\mu_{G \times B}^{-1}(0, \bar{0})/B$ and $T^*(\bar{g} \times \mathbb{C}^n/G)$ (see [Im18, Prop. 1.1] and [Nev11, Cor. 3.3]). This gives us a connection between the Hamiltonian reduction of the enhanced $B$-moment map and the Grothendieck-Springer resolution. That is, studying the $G$-orbits on $T^*(\bar{g} \times \mathbb{C}^n)$ is equivalent to studying the $B$-orbits on $T^*(b \times \mathbb{C}^n)$.

In [Nev11], Nevins finds at least $2^n$ irreducible components in the locus $\mu^{-1}(0)$, which is defined by $n(n + 1)/2$ equations, and conjectures that $\mu^{-1}(0)$ is a complete intersection. The first author has proved this conjecture when $\mu$ is restricted to its semisimple locus in [Im14, Im18]. In this manuscript, we prove Nevins’ conjecture to the entire locus $\mu^{-1}(0)$.

**Theorem 1.1.** The components of $\mu$ form a complete intersection.

Next, we give a description of the irreducible components, which had previously appeared in [Nev11] under the assumption that $\mu^{-1}(0)$ was a complete intersection. Let

$$\mathcal{M} := \{ (r, s, i, j) \in b \times b^* \times \mathbb{C}^n \times (\mathbb{C}^n)^* : [r, s] + ij = 0 \} = \mu^{-1}(0),$$
and let $\mathcal{D}$ be the elements $(r, s, i, j) \in \mathcal{M}$ such that

(a) $r = (r_{pq})_{p,q=1}^n$, with pairwise distinct eigenvalues $r_1, \ldots, r_n$ (note $r_{pp} = \rho_p$ for all $1 \leq p \leq n$),
(b) $i = \sum_{a \in \mathcal{A}} e_a$ and $j = \sum_{a' \in \mathcal{A}} e_{a'}$ for some $\mathcal{A}, \mathcal{A}' \subseteq \{1, \ldots, n\}$ such that $\mathcal{A} \cap \mathcal{A}' = \emptyset$, where $e_a$ is the unit vector with 1 in the $a$-th entry and 0 otherwise,
(c) $s = (s_{pq})_{p,q=1}^n$ is given by

$$s_{pq} = \begin{cases} \sigma_p & \text{if } p = q, \\ \rho_p - \rho_q & \text{if } p \in \mathcal{A}, q \in \mathcal{A}', \text{ and } p > q, \\ 0 & \text{otherwise}, \end{cases}$$

for some $s_1, \ldots, s_n \in \mathbb{C}$.

Theorem 1.2. The irreducible components of $\mathcal{M}$ are the closures of

$$\mathcal{M}_a : = \left\{ O_{(r,s,i,j)} : (r, s, i, j) \in \mathcal{D} \text{ with } \supp(i) = \mathcal{A}, \supp(j) \cup \mathcal{A} = \{1, \ldots, n\} \right\}$$

for all $\mathcal{A} \subseteq \{1, \ldots, n\}$. Moreover, $\mathcal{M}$ is reduced and equidimensional with $\dim \mathcal{M} = \binom{n}{2} + 2n$.

The proof that we give of Theorem 1.2 is different from the one given in [Nev11], where instead we follow the analogous proof from [GG06] instead of lifting up to the $\mathfrak{g}$ setting from $\mathfrak{b}$.

We note that Theorem 1.1 allows one to construct affine and geometric invariant theory (GIT) quotients (see, e.g., [Kin94, New09]), and it would be interesting to show their connections to the Hilbert scheme $\text{Hilb}^n(\mathbb{C}^2)$ of $n$ points on a complex plane (see, e.g., [GG06, Nak99]), which is described as

$$\text{Hilb}^n(\mathbb{C}^2) = \{ I \subseteq \mathbb{C}[r, s] : I \text{ is an ideal satisfying } \text{length}(V(I)) = \text{dim}_\mathbb{C}(\mathbb{C}[r, s]/I) = n \},$$

and the isospectral Hilbert scheme [Hai01, Section 3.2], which is the reduced fiber product

$$(\text{Hilb}^n(\mathbb{C}^2) \times_{\text{S}^n(\mathbb{C}^2)} (\mathbb{C}^2)^n)^\text{red},$$

where $\text{S}^n(\mathbb{C}^2) = (\mathbb{C}^2)^n/\text{S}_n$, unordered $n$-tuples of points in $\mathbb{C}^2$. For notational purposes, we identify the closed points of $\text{Hilb}^n(\mathbb{C}^2)$ with ideals $I \subseteq \mathbb{C}[r, s]$ satisfying $\text{dim}_\mathbb{C}(\mathbb{C}[r, s]/I) = n$. That is, the Hilbert scheme $\text{Hilb}^n(\mathbb{C}^2)$ parameterizes finite closed subschemes of length $n$, while the reduced closed points of the isospectral Hilbert scheme are the tuples $(I, p_1, \ldots, p_n) \in \text{Hilb}^n(\mathbb{C}^2) \times (\mathbb{C}^2)^n$ such that $\pi(I) = \{ [p_1, \ldots, p_n] \}$, where $\pi : \text{Hilb}^n(\mathbb{C}^2) \to \text{S}^n(\mathbb{C}^2)$ is the Hilbert-Chow morphism associating a closed subscheme with its corresponding cycle. It appears that there may be a natural morphism between the Hamiltonian reduction of our Borel moment map and the isospectral Hilbert scheme since the points $(r_1, s_1), \ldots, (r_n, s_n) \in \mathbb{C}^2$ obtained from the diagonal entries of $r$ and $s$ do not commute under the $\mathfrak{B}$-action, and thus are distinguishable, as in the case of the isospectral Hilbert scheme.

It would also be interesting to show (rational) morphisms between the Hamiltonian reduction of our Borel moment map and the flag Hilbert scheme on a complex plane

$$\text{FHilb}^n(\mathbb{C}^2) = \{ I_n \subseteq \ldots \subseteq I_1 \subseteq I_0 = \mathbb{C}[r, s] : \text{dim}_\mathbb{C}(\mathbb{C}[r, s]/I_i = i) \},$$

which also may be considered in terms of lower triangular matrices $B^-$ in $G$, i.e., given $\mathfrak{b}^- = \text{Lie}(B^-)$ with $\mathfrak{u}^-$ as the maximal nilpotent subalgebra of $\mathfrak{b}^-$,

$$\text{FHilb}^n(\mathbb{C}^2) = \{ (r, s, i) \in \mathfrak{b}^- \times \mathfrak{b}^- \times \mathfrak{u}^n : [r, s] = 0, r^{a^i} \in \text{span } \mathbb{C}^n \}/B^-,$$

especially since both constructions involve $\mathfrak{B}$ (or $\mathfrak{B}^-$)-conjugation action on its subalgebra. Flag Hilbert schemes are fascinating in their own right, and are of great interest in categorical representation theory and quantum topology since they give correspondence between Koszul complexes of
the torus fixed points on FHilb$^n(\mathbb{C}^2)$ and idempotents in the category of Soergel bimodules (see, e.g., [GNR16]).

Next, we give constructions of varieties analogous to the ones in [Wil98]. Indeed, let

$$\mathcal{C}_n = \{(r, s) \in b \times b^* : \text{rk}(\lambda) = 1\},$$

where $I_n$ is the $n \times n$ identity matrix. Let $C_n = \mathcal{C}_n/B$, where $B$ acts on the pair $(r, s)$ by a simultaneous conjugation. Let $C'_n = \{(r, s) \in C_n : r$ is diagonalizable}, which is a subspace of $C_n$.

We show that $r \in C'_n$ implies that the eigenvalues of $r$ must be pairwise distinct (see Lemma 6.4). Now let $\mathcal{B} := b \times \mathbb{C}^n$, and for simplicity, we omit the bar over the elements in $b^*$. Consider the following subvariety

$$\tilde{C}_n = \{(r, s, i, j) \in T^* \mathcal{B} : [r, s] + I_n = -ij\}.$$  

We see that $\tilde{C}_n = \mu^{-1}(-I_n)$, where $\mu$ is the moment map (1).

**Remark 1.3.** Our moment map and subvariety $\tilde{C}_n$ differs from the moment map and subvariety (after replacing $B$ with $G$) considered in [Wil98] by $i \mapsto -i$. We do this to match the moment map from [GG06, Nev11] (see (1) above). Note that this sign difference comes from the fact that [Wil98, page 9, lines 8-9] uses $g \times (\mathbb{C}^\ast)^\ast$ in his study of Calogero-Moser systems instead of $g \times \mathbb{C}^\ast$ as in [GG06].

Note that for fixed $(r, s) \in b \times b^*$, there exist vectors $i$ and $j$ satisfying (4) if and only if $(r, s) \in \mathcal{C}_n$. That is, we claim that every rank 1 matrix is of the form $ij$ for some column vector $i$ and row vector $j$.

Since $\text{tr}(\lambda) = n$, we must have $1 \leq \text{rk}(\lambda) \leq 1$ from (4). Thus we have $\text{rk}(\lambda) = 1$, and we compute $i$ and $j$ up to a common scalar multiple $\lambda$ by $i \mapsto \lambda i$ and $j \mapsto \lambda^{-1} j$. Therefore, we can identify $\mathcal{C}_n$ with the quotient of $\tilde{C}_n$ by the scalar matrices in $B$ under the action (2), and we obtain the same space $C_n$ as a quotient of $\tilde{C}_n$ or of $\mathcal{C}_n$, i.e., since the $B$-action given by (2) above preserves $\tilde{C}_n$, we will also write $C_n = \tilde{C}_n/B$.

**Theorem 1.4.** The scheme $C_n$ is a smooth irreducible affine algebraic variety of dimension $2n$.

This paper is organized as follows. In Section 2, we give some background on $B$-invariant functions on $b$ and on $b^*$. In Section 3, we give a notion of Jordan $B$-canonical form for matrices, which are crucial in the proof for Theorem 1.1. We prove Theorem 1.1 in Section 4, and we prove Theorem 1.2 in Section 5. We prove Theorem 1.4 in Section 6.

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**2. Preliminaries**

Throughout this paper, we consider all groups and (Lie) algebras to be over $\mathbb{C}$ unless otherwise stated. Furthermore, let $b$ be the set of upper triangular matrices, which is the standard Borel subalgebra of $g = \mathfrak{gl}_n$. Let $u$ be the set of strictly upper triangular matrices, which is the upper nilpotent part of $g$. Let $\mathcal{B} := b \times \mathbb{C}^n$. We will identify the cotangent bundle of $\mathcal{B}$ as $T^* \mathcal{B} \simeq b \times b^* \times \mathbb{C}^n \times (\mathbb{C}^n)^\ast$, where $b^* = g/u$, and $(\mathbb{C}^n)^\ast$ is dual to $\mathbb{C}^n$. For an element $x \in g$, we write the corresponding element $x \in g/u$. Let $\mathbb{N}$ be the set of all nonnegative integers (in particular, $0 \in \mathbb{N}$).
There is an induced $B$-action on $T^*\mathcal{B}$ given by
\begin{equation}
    b.(r,s,i,j) = (brb^{-1}, bsb^{-1}, bi, jb^{-1}),
\end{equation}
which is a Hamiltonian action with moment map
\begin{equation}
    \mu: T^*\mathcal{B} \to \mathfrak{b}^*,
    (r,s,i,j) \mapsto [r,s] + ij.
\end{equation}
There is an $\mathfrak{b}$-action on $T^*\mathcal{B}$ given by
\[ q \cdot (r,s,i,j) = ([q,r], [s,q], qi, -qj) \]
Consider the scheme
\[ \mathcal{M} := \{(r,s,i,j) \in \mathfrak{b} \times \mathfrak{b}^* \times \mathbb{C}^n \times (\mathbb{C}^n)^* : [r,s] + ij = 0 \}. \]
Then $\mathcal{M}$ is equal to $\mu^{-1}(0) \subseteq T^*\mathcal{B}$. Note that the $B$-action preserves $\mathcal{M}$:
\begin{equation*}
    [brb^{-1}, b^{-1}sb] + bijb^{-1} = b([rs] + ij)b^{-1} = 0.
\end{equation*}
\begin{lemma}
    For the adjoint action of $B$ on $\mathfrak{b}$, we have $\mathfrak{b}/B \cong \text{Spec}(\mathbb{C}[\text{diag}(r)]) \cong \mathbb{C}^n$.
    \begin{proof}
        Recall that the adjoint action is given by $b.r = brb^{-1}$, where $b \in B$ and $r \in \mathfrak{b}$. For $1 \leq \ell \leq n$, define a map $f_{\ell}: \mathfrak{b} \to \mathbb{C}$ defined by $f_{\ell}(r) = r_{\ell\ell}$. For $b \in B$, we have $b.f_{\ell}(r) = f_{\ell}(b^{-1}rb) = r_{\ell\ell}$. So $\mathbb{C}[f_1, \ldots, f_n] \subseteq \mathbb{C}[\mathfrak{b}]^B$.
        Now, let $\lambda: \mathfrak{c}^* \to \mathfrak{b}$ be a 1-parameter subgroup defined as
        \begin{equation}
            (\lambda(t)r)_{\ell\gamma} = \begin{cases}
                t^{n-a} & \text{if } \ell = \gamma = a, \\
                \delta_{\ell\gamma} & \text{otherwise,}
            \end{cases}
        \end{equation}
        where $\delta_{\ell\gamma}$ is the Kronecker delta. Then
        \begin{equation*}
            (\lambda(t)r)_{\ell\gamma} = (\lambda(t)r\lambda(t)^{-1})_{\ell\gamma} = \begin{cases}
                r_{\ell\ell} & \text{if } \ell = \gamma, \\
                r_{\ell\gamma}t^{\gamma-\ell} & \text{if } \ell < \gamma, \\
                0 & \text{if } \ell > \gamma.
            \end{cases}
        \end{equation*}
        So
        \begin{equation*}
            \lim_{t \to 0}(\lambda(t)r)_{\ell\gamma} = \begin{cases}
                r_{\ell\ell} & \text{if } \ell = \gamma, \\
                0 & \text{if } \ell \neq \gamma.
            \end{cases}
        \end{equation*}
        Since the off-diagonal entries of $r$ are zero, a $B$-invariant polynomial is independent of the off-diagonal coordinate functions. Thus $\mathbb{C}[\mathfrak{b}]^B \cong \mathbb{C}[\text{diag}(r)]$. \hfill \square
    \end{proof}
\end{lemma}
\begin{lemma}
    For the coadjoint action of $B$ on $\mathfrak{b}^*$, we have $\mathfrak{b}^*/B \cong \text{Spec}(\mathbb{C}[\text{tr}(s)]) \cong \mathbb{C}$.
    \begin{proof}
        We have $\mathbb{C}[\text{tr}(s)] \subseteq \mathbb{C}[\mathfrak{b}^*]^B$ since $F(\text{tr}(bsb^{-1})) = F(\text{tr}(sb^{-1}b)) = F(\text{tr}(s))$ for any $s \in \mathfrak{b}^*$ and $b \in B$.
        Now suppose $F \in \mathbb{C}[\mathfrak{b}^*]^B$, and let $s \in \mathfrak{b}^* = \mathfrak{g}/\mathfrak{u}$. Then for a 1-parameter subgroup $\lambda_1(t)$ with coordinates
        \begin{equation*}
            (\lambda_1(t)s)_{\ell\gamma} = \begin{cases}
                t^{\ell-1} & \text{if } \ell = \gamma, \\
                0 & \text{if } \ell \neq \gamma,
            \end{cases}
        \end{equation*}
        we have
        \begin{equation*}
            (\lambda_1(t)s)_{\ell\gamma} = (\lambda_1(t)s(\lambda_1(t)^{-1}))_{\ell\gamma} = \begin{cases}
                * & \text{if } \ell < \gamma, \\
                s_{\ell\ell} & \text{if } \ell = \gamma, \\
                t^{\ell-\gamma}s_{\ell\gamma} & \text{if } \ell > \gamma.
            \end{cases}
        \end{equation*}
    \end{proof}
\end{lemma}
Taking the limit as \( t \to 0 \), we have

\[
\lim_{t \to 0} (\lambda_1(t).s)_{i\gamma} = \begin{cases} 
* & \text{if } i < \gamma, \\
s_{ii} & \text{if } i = \gamma, \\
0 & \text{if } i > \gamma.
\end{cases}
\]

Since the entries below the main diagonal of \( s \) are zero, our \( B \)-invariant polynomial \( F \) is independent of the coordinates \( \{s_{i\gamma}\}_{i > \gamma} \). Now consider another 1-parameter subgroup \( \lambda_2(t) \), where

\[
\lambda_2(t)_{i\gamma} = \begin{cases} 
t^{i-1} & \text{if } i \leq \gamma, \\
0 & \text{if } i > \gamma.
\end{cases}
\]

Then

\[
(\lambda_2(t).s)_{i\gamma} = \begin{cases} 
t^{i-\gamma} \left( \sum_{k=1}^{n} s_{k\gamma} - \sum_{k=i}^{n} s_{k,\gamma-1} \right) & \text{if } i < \gamma, \\
t^{i-\gamma} \left( \sum_{k=i}^{n} s_{k\gamma} - \sum_{k=i+1}^{n} s_{k,\gamma-1} \right) & \text{if } i \geq \gamma,
\end{cases}
\]

with

\[
\begin{align*}
\lambda_2(t)_{i\gamma} &= \begin{cases} 
t^{i-1} & \text{if } i \leq \gamma, \\
0 & \text{if } i > \gamma.
\end{cases}
\end{align*}
\]

Since \( F \) is \( B \)-invariant, \( F \) must take the same value on any orbit closure, i.e. \( F(s) = F(s') \) for any \( s' \in \overline{B.s} \). Thus \( \lim_{t \to 0} F(\lambda_1(t).s) = \lim_{t \to 0} F(\lambda_2(t).s) \) must hold for any values of \( \{s_{i\beta}\}_{i > \beta} \). Now for each \( 1 \leq i < n \) (starting with \( i = 1 \) in ascending order), choose \( \{s_{ki}\}_{i \leq n} \) in \( \sum_{i \leq k \leq n} s_{ki} \) such that

\[
(9) \quad - \sum_{k=i+1}^{n} s_{ki} = s_{ii} - \sum_{k=i}^{n} s_{k,i-1}.
\]

By solving for \( s_{ii} \) in (9) and summing over all \( 1 \leq i \leq n - 1 \), we obtain

\[
\sum_{i=1}^{n-1} \left( \sum_{k=i}^{n} s_{k,i-1} - \sum_{k=i+1}^{n} s_{ki} \right) = \sum_{k=1}^{n} s_{kk} + \sum_{i=2}^{n} \sum_{k=i}^{n} s_{k,i-1} - \sum_{i=1}^{n} \sum_{k=i+1}^{n} s_{ki} - \sum_{k=n}^{n} s_{k,n-1}
\]

or

\[
(10) \quad \sum_{k=1}^{n-1} s_{k,n-1} = \sum_{i=1}^{n-1} s_{ii}.
\]

By (8) and by choosing appropriate choices for \( \{s_{i\beta}\}_{i > \beta} \) in (9), we have \( (\lambda_2(t).s)_{ii} = 0 \) for each \( 1 \leq i < n \), whereas

\[
(\lambda_2(t).s)_{nn} = s_{nn} - \sum_{n-1 < k \leq n} s_{k,n-1} = \text{tr}(s)
\]

by (10). This means all coordinate entries are zero except the \((n, n)\)-entry, which is \( \text{tr}(s) \). Thus for \( F \in \mathbb{C}[b^*]^B \), we have \( F \in \mathbb{C}[\text{tr}(s)] \), and so \( \mathbb{C}[b^*]^B \subseteq \mathbb{C}[\text{tr}(s)] \). Hence, we have \( \mathbb{C}[b^*]^B = \mathbb{C}[\text{tr}(s)] \) and the claim follows.

**3. The Jordan \( B \)-Canonical Form**

In this section, we will construct an analog of the Jordan canonical form for a matrix under the \( B \)-action.

Let us first consider the \( n = 2 \) case. So take

\[
r = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} \in b \quad \text{and} \quad b = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} \in B.
\]
Then we have

\[ brb^{-1} = \begin{pmatrix} r_{11} & \frac{b_{12}(r_{22}-r_{11})+b_{11}r_{12}}{b_{22}} \\ 0 & r_{22} \end{pmatrix}. \]

We see that the diagonal entries \( \{r_{11}, r_{22}\} \) are fixed under the \( B \)-conjugation action. Now, let us examine the upper right entry, and see how we can obtain

\[ b_{12}(r_{22}-r_{11}) + b_{11}r_{12} = 0. \]

Note that we have multiplied the upper right entry by \( b_{22} \) (recall that \( b_{22} \neq 0 \) because \( b_{22} \) is a diagonal entry of \( b \in B \)). Therefore, if \( r_{22} \neq r_{11} \), we can take \( b_{12} = -b_{11}r_{12}/(r_{22}-r_{11}) \) (note we can have \( b_{12} = 0 \) since it is an off-diagonal entry). If \( r_{11} = r_{22} \), then we must have \( r_{12} = 0 \) since \( b_{11} \neq 0 \). Therefore, we can classify the orbits \( O_r = BrB^{-1} \) into the following 3 types:

**Lemma 3.1.** Let \( n = 2 \) and \( r = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} \in \mathfrak{b} \). Consider a \( B \)-orbit \( O_r \). Then \( r \) is in one of the following distinct cases:

- **distinct eigenvalues:** \( r_{11} \neq r_{22} \),
- **two Jordan blocks:** \( r_{11} = r_{22} \) and \( r_{12} = 0 \),
- **one Jordan block:** \( r_{11} = r_{22} \) and \( r_{12} \neq 0 \).

We note that the order of the eigenvalues in \( \text{diag}(r) \) cannot be permuted under the Borel action.

To generalize this to arbitrary \( n \), we need to weaken the classical notion of Jordan canonical form for \( B \)-orbits. We construct an embedding \( \xi_\alpha: \mathfrak{gl}_n \to \mathfrak{gl}_n \) by considering an (ordered) subset \( \alpha = \{a_1 < a_2 < \ldots < a_m\} \subseteq \{1, 2, \ldots, n\} \) and mapping a matrix \( (x_{pq})_{p,q=1}^n \) to \( (x'_{uv})_{u,v=1}^n \) by

\[ x'_{uv} = \begin{cases} x_{pq} & \text{if } a_p = u \text{ and } a_q = v \text{ for some } 1 \leq p \leq q \leq n, \\ 1 & \text{if } u = v \text{ and there does not exists } p \text{ such that } a_p = u, \\ 0 & \text{otherwise}. \end{cases} \]

With a slight abuse of a notation, we will also write \( \xi_\alpha \) for the embedding \( \xi_\alpha: \text{GL}_m \to \text{GL}_n \) of Lie groups, which naturally restricts to an embedding of the Borel subgroup of \( \text{GL}_m \) into the Borel subgroup of \( \text{GL}_n \).

**Remark 3.2.** The embedding \( \xi_\alpha \) corresponds to embedding \( \text{GL}_m \) along the roots

\[ \alpha_{a_1,a_2}, \alpha_{a_2,a_3}, \ldots, \alpha_{a_{m-1},a_m}, \]

where \( \alpha_{p,q} = \alpha_p + \alpha_{p+1} + \ldots + \alpha_q \).

Next, define the **Jordan \( B \)-canonical form** to be the matrix such that there exists a partition \( \alpha^{(1)} \cup \alpha^{(2)} \sqcup \ldots \cup \alpha^{(\ell)} = \{1, 2, \ldots, n\} \), where \( \alpha^{(j)} \neq \emptyset \) for each \( j \), such that the preimage of \( \xi_{\alpha^{(j)}} \) is a Jordan block for all \( 1 \leq j \leq \ell \). We call the submatrix for any such \( \alpha^{(j)} \) a **Jordan \( B \)-block**.

**Lemma 3.3.** For any upper-triangular matrix \( r \in \mathfrak{b} \), there exists a unique matrix in Jordan \( B \)-canonical form in its orbit \( \{brb^{-1} : b \in B\} \).

**Proof.** Since the order of the eigenvalues cannot change under the \( B \)-action, the uniqueness is clear. To show existence, start by considering a pair \( 1 \leq p < q \leq n \), and consider the Borel subgroup of the corresponding copy of \( \text{GL}_2 \) (i.e., along the root \( \alpha_{p,q} \)). Let \( b = \xi_{\{p,q\}} \begin{pmatrix} b_{pp} & b_{pq} \\ b_{qp} & b_{qq} \end{pmatrix} \). Then the
only entries of \((r'_{xy})^n_{x,y=1} = brb^{-1}\) that are not simply scaled are
\[
\begin{align*}
    r'_{pq} &= \frac{b_{pq}(r_{q} - r_p) + b_{pp}r_{pq}}{b_{qq}}, \\
    r'_{kj} &= \frac{(b_{pp}r_{kj} - b_{pq}r_{kp})b_{kk}}{b_{pp}b_{qq}} \quad \text{for } k < p, \\
    r'_{jk} &= \frac{(b_{pp}r_{jk} + b_{pq}r_{qk})b_{kk}}{b_{pp}b_{qq}} \quad \text{for } k > q,
\end{align*}
\]
where \(b_{kk} = 1\) if \(k \neq p, q\). By applying Lemma 3.1, we can set \(b_{pq} = 0\) whenever \(r_{pp} \neq r_{qq}\). Hence, proceeding by induction on diagonals \(q - p\), we can reduce this to the case when \(r\) is the image under \(\xi_{\alpha(i)}\) of the matrices
\[
\Lambda^{(j)} := \begin{pmatrix}
    \lambda^{(j)} & r_{12} & r_{13} & \cdots & r_{1m} \\
    & \lambda^{(j)} & r_{23} & \cdots & r_{2m} \\
    & & \cdots & \cdots & \cdots \\
    & & & \lambda^{(j)} & r_{m-1,m} \\
    & & & & \lambda^{(j)}
\end{pmatrix}
\]
for all \(1 \leq j \leq n\).

Next, we need to consider the specific case of a single such matrix with \(m = 3\), where we have
\[
b\Lambda^{(j)}b^{-1} = \begin{pmatrix}
    \lambda^{(j)} & r'_{12} & r'_{13} & r'_{14} \\
    & \lambda^{(j)} & r'_{23} & r'_{24} \\
    & & \cdots & \cdots \\
    & & & \lambda^{(j)}
\end{pmatrix}.
\]
From the above, we can assume without loss of generality that \(r_{pq} = 0\) for all \(p < q\). If \(r_{12} = r_{13} = r_{23} = 0\), then we have three Jordan blocks of size 1. If \(r_{12} = r_{13} = 0\) and \(r_{23} \neq 0\), then we have two Jordan blocks of size 1 and 2, and similar for \(r_{13} = r_{23} = 0\) and \(r_{12} \neq 0\). If \(r_{13} = 0\) and \(r_{12}, r_{23} \neq 0\), then we have one Jordan block of size 3. If \(r_{12}, r_{13}, r_{23} \neq 0\), then we obtain a Jordan block of size 3 by choosing
\[
b_{11} = b_{22} = b_{33}, \quad b_{12} - b_{23} = -b_{11}.
\]
If \(r_{12}, r_{13} \neq 0\) and \(r_{23} = 0\), then we obtain two Jordan blocks of sizes 2 and 1 by choosing \(b_{11} = b_{22} = b_{33}\), and case \(r_{13}, r_{23} \neq 0\) and \(r_{12} = 0\) is similar. If \(r_{12} = r_{23} = 0\) and \(r_{13} \neq 0\), then we are in Jordan B-canonical form and cannot reduce further as \(r'_{13} = \frac{b_{13}}{b_{23}}r_{13}\).

Fix some \(p = \{p_1 < p_2 < p_3\}\), and let \(b \in \xi_pB_3\), where \(B_3\) is the Borel subgroup of \(GL_3\). From a straightforward computation, we can see that the only entries in \((r'_{xy})^n_{x,y=1} = brb^{-1}\) that are not simply scaled are, for some \(p, q \in p\) with \(p < q\), in the coordinates \((k, q)\) for \(k \leq p\) and \((p, k)\) for \(k \geq q\). Therefore, by applying the GL3 case inducting on \(p_3 - p_1\), the claim follows. \(\square\)

4. PROOF OF THEOREM 1.1

Our proof of the complete intersection of the irreducible components of the preimage of 0 under the Borel moment map closely follows the proof of [GG06, Thm. 1.1].

To begin, we show the analog of [GG06, Lemma 2.1].

Lemma 4.1. For any \(r \in b\), the centralizer (equivalently, the stabilizer under the conjugation action) group \(B^r := \{b \in B : br = rb\}\) acts on \(\mathbb{C}^n\) with finitely many orbits.

Proof. Without loss of generality, assume \(r\) is in Jordan B-canonical form by Lemma 3.3. If \(r\) is a Jordan block with eigenvalue \(\lambda\), then the nonzero orbits of \(B^r\) in \(\mathbb{C}^n\) are the generalized eigenspaces
\[
\mathcal{O}_k := \{i \in \mathbb{C}^n : (r - \lambda)^k i \neq 0, (r - \lambda)^{k+1} i = 0\}
\]
for all \( k = 0, 1, \ldots, n - 1 \). If \( r \) has \( \ell \) Jordan \( B \)-blocks given by the indices \( a^{(p)} := \{a_1^{(p)}, \ldots, a_m^{(p)} \} \) (recall that \( \bigcup_{p=1}^{\ell} a^{(p)} = \{1, 2, \ldots, n\} \)), then we decompose \( C^n = \bigoplus_{p=1}^{\ell} V^{(p)} \), where
\[
V^{(p)} := \text{span}_C \{e_{a_1}, \ldots, e_{a_m} \},
\]
\( a_q := a_q^{(p)} \), and \( m := m^{(p)} \). Here, the subspaces \( V^{(p)} \) follow the decomposition given by the indices \( a^{(p)} \). Now consider the subgroup of \( B \) that respects the \( B \)-block decomposition:
\[
\mathcal{B}' := B^r \cap \left( \prod_{p=1}^{\ell} \text{GL} \left( V^{(p)} \right) \right).
\]
It is clear from the above that \( \mathcal{B}' \) has finitely many orbits, and since increasing the size of the group cannot increase the number of orbits, we have that \( \mathcal{B}' \) has finitely many orbits in \( C^n \).

Let \( O_r \) be a conjugacy class of \( r \) in \( b \), and define the reduced subscheme \( \mathfrak{M}(O_r) := \{(r', s', i', j') \in \mathfrak{M} : r' \in O_r \} \). We give the analog of [GG06, Prop. 2.4].

**Proposition 4.2.** Given a conjugacy class \( O_r \subseteq b \) of \( r \in b \), the subscheme \( \mathfrak{M}(O_r) \) in \( T^* \mathcal{B} \) is Lagrangian.

**Proof.** Assume that \( \mathcal{B} \) is an arbitrary smooth \( B \)-variety. Let \( \text{Orb} \) denote the set of all \( B \)-orbits of \( \mathfrak{M} \), and for each orbit \( O \in \text{Orb} \), let \( T^*_O \mathcal{B} \subseteq T^* \mathcal{B} \) denote the conormal bundle to \( O \). The natural \( B \)-action on \( T^* \mathcal{B} \) is Hamiltonian with moment map \( \mu : T^* \mathcal{B} \to b^* \), and we have
\[
\mu^{-1}(0) = \bigcup_{O \in \text{Orb}} T^*_O \mathcal{B}
\]
(12)
since the conormal bundle to \( O \) is a subbundle of \( T^* \mathcal{B} \) consisting of cotangent vectors to \( \mathcal{B} \) that are zero on the cotangent bundle to \( O \). Since \( O \subseteq \mathfrak{M} \), it follows that \( T^*_O \mathcal{B} \subseteq \mu^{-1}(0) \). Since \( O \in \text{Orb} \), Equation (12) follows.

Now, assume that \( \mathcal{B} = b \times C^n \). Since we can write \( \mu^{-1}(0) \) as a union of conormal bundles to \( O \) as in (12), given any point \((r', s', i', j') \in \mathfrak{M}(O_r)\) and any pair of tangent vectors \( X, Y \) in the tangent space \( T_{(r', s', i', j')} \mathfrak{M}(O) \), we have \( \omega_{(r', s', i', j')}(X, Y) = 0 \). That is, given a canonical symplectic form \( \omega = \sum_{n \geq 0} dsu^\alpha \wedge dr\nu \), the restricted symplectic form \( \omega|_{\mathfrak{M}(O)} \) vanishes. Finally, given the projection map \( \pi : b \times (C^n)^* \to b \) onto the first factor and a conjugacy class \( O_r \subseteq b \), the set \( \pi^{-1}(O_r) \) is a finite union of \( B \)-orbits from Lemma 4.1. So \( \mathfrak{M}(O_r) \) is a finite union of \( T^*_O \mathcal{B} \), where \( O \in \text{Orb} \). Let us enumerate these orbits as \( O_\alpha \in \text{Orb} \), where \( 1 \leq \alpha \leq \ell \). Since \( \dim \mathfrak{M}(O_r) = \dim T^*_O \mathcal{B} = \frac{1}{2} \dim T^* \mathcal{B} \), it follows that \( \mathfrak{M}(O_r) \) is Lagrangian.

Next, we have an analog of [GG06, Prop. 2.5], but since \( b/\mathbb{B} \cong C \) (Lemma 2.2), we will work with \( b/\mathbb{B} \cong C^n \) (Lemma 2.1).

**Proposition 4.3.** Consider the map \( \Lambda : T^* \mathcal{B} \to b/\mathbb{B} \cong C^n \) given by
\[
(r, s, i, j) \mapsto \text{diag}(r) = (r_{11}, \ldots, r_{nn}),
\]
where \( r = (r_{pq})_{p,q=1}^n \in b \). Then the morphism
\[
\mu \times \Lambda : T^* \mathcal{B} \to b^*/(b/\mathbb{B}) \text{ given by } (r, s, i, j) \mapsto ([r, s] + \mathbb{I}, \text{diag}(r))
\]
is flat. Moreover, all nonempty (scheme-theoretic) fibers of this morphism have dimension \( \binom{n}{2} + n \).

**Proof.** For any tuple \( r' = (r_1, r_2, \ldots, r_n) \in C^n \), the set of all elements \( r \in b \) such that \( \Lambda(r, s, i, j) = r' \) (for fixed \( s, i, \) and \( j \)) is a finite union of \( B \)-conjugacy classes by considering the possible Jordan \( B \)-block decompositions of \( r \) (in particular, the number of conjugacy classes is a product of Bell numbers\(^1\)). Therefore, the zero fiber of \( \mu \times \Lambda \), denoted by \( \xi \), is equal to a finite union of Lagrangian

\(^1\)Recall that the \( N \)-th Bell number counts the number of partitions of a set of size \( N \).
subschemes \( \mathcal{M}(\mathcal{O}_r) \) from Proposition 4.2, where \( \mathcal{O}_r \) is a conjugacy class of a nilpotent matrix \( r \in \mathfrak{b} \). Hence, we have
\[
\dim \xi \leq \frac{1}{2} \dim T^*\mathcal{B} = \left( \begin{array}{c} n \\ 2 \end{array} \right) + n = \dim T^*\mathcal{B} - \dim \mathfrak{b} \times \mathbb{C}^n
\]
since Lagrangian subschemes can have dimension at most \( \frac{1}{2} \dim T^*\mathcal{B} \).

We define a \( C^* \)-action on \( T^*\mathcal{B} \) by scalar multiplication, i.e., we have \( \alpha \cdot (r, s, i, j) = (\alpha r, \alpha s, \alpha i, \alpha j) \). Next let \( C^* \) act on \( \mathfrak{b}^* \times \mathbb{C}^n \) by \( \alpha \cdot (s, i) := (\alpha^2 s, \alpha i) \). Then the map \( \mu \times \Lambda \) is a \( C^* \)-equivariant morphism since
\[
(\mu \times \Lambda) (\alpha \cdot (r, s, i, j)) = (\mu \times \Lambda)(\alpha r, \alpha s, \alpha i, \alpha j) = \alpha \cdot \big( (\alpha r, \alpha s, \alpha i, \alpha j) \big) = \alpha \cdot (\mu \times \Lambda)(r, s, i, j).
\]

Let \( \mathbb{C}[T^*\mathcal{B}]_{\leq i} \) be the set of polynomials of degree less than or equal to \( i \). This forms an \( \mathbb{N} \)-filtration on \( \mathbb{C}[T^*\mathcal{B}] \). Let \( \mathbb{C}[T^*\mathcal{B}]_\leq \) be the set of all finite expressions of the form \( \sum_i b_i \alpha^i \) with \( b_i \in \mathbb{C}[T^*\mathcal{B}]_{\leq i} \). Since \( C^* \)-equivariance is well-defined under the limit as \( \alpha \to 0 \), we have a natural ring embedding \( \mathbb{C}[\alpha] \hookrightarrow \mathbb{C}[T^*\mathcal{B}] \cong \sum_{i \in \mathbb{N}} \mathbb{C}[T^*\mathcal{B}]_{\leq i} \alpha^i \subseteq \mathbb{C}[T^*\mathcal{B}]\{\alpha\} \), which gives a surjective morphism of algebraic varieties \( \mathbb{C}(\mathbb{C}[T^*\mathcal{B}]) \to \mathbb{C} \).

Since \( \alpha \) is not a zero-divisor, \( \mathbb{C}[T^*\mathcal{B}] \) is a torsion-free \( \mathbb{C}\{\alpha\} \)-module. So \( \mathbb{C}[T^*\mathcal{B}]_\leq \) is flat. Thus \( \mu \times \Lambda \) is a flat morphism, and the dimension of any fiber of the map \( \mu \times \Lambda \) is \( \left( \begin{array}{c} n \\ 2 \end{array} \right) + n \) (also see [CG10, Section 2.3.9]). \( \square \)

Finally, we obtain analogs of [GG06, Cor. 2.6] and [GG06, Cor. 2.7]. Let \( \overline{\Lambda} := \Lambda|_{\mathfrak{m}} \) be the restriction of \( \Lambda \) to the closed subscheme \( \mathfrak{m} \).

**Corollary 4.4.** The moment map \( \mu \) is flat.

**Proof.** The moment map \( \mu \) is the projection of \( \mu \times \Lambda \) onto the first factor \( \mathfrak{p}_1: \mathfrak{b}^* \times \mathbb{C}^n \to \mathfrak{b}^* \). Since the composition \( \mathfrak{p}_1 \circ (\mu \times \Lambda) \) is flat, \( \mu \) is flat. \( \square \)

**Corollary 4.5.** The scheme \( \mathfrak{m} \) is a complete intersection in \( T^*\mathcal{B} \) with \( \dim \mathfrak{m} = \left( \begin{array}{c} n \\ 2 \end{array} \right) + 2n \). Moreover, the map \( \overline{\Lambda}: \mathfrak{m} \to \mathbb{C}^n \) from Proposition 4.3 is a flat morphism with fibers of dimension \( \left( \begin{array}{c} n \\ 2 \end{array} \right) + n \) that are Lagrangian subschemes in \( T^*\mathcal{B} \).

**Proof.** The restriction \( \overline{\Lambda} \) is surjective since clearly \( (r, 0, 0, 0) \in \mu^{-1}(0) \) for any \( r \in \mathfrak{b} \). Thus all fibers of \( \overline{\Lambda} \) are nonempty.

Taking a flat base change with respect to the embedding \( \{0\} \times \mathbb{C}^n \to \mathfrak{b}^* \times \mathbb{C}^n \) yields that the scheme \( \mathfrak{m} = \mu^{-1}(0) = (\mu \times \Lambda)^{-1}(\{0\} \times \mathbb{C}^n) \) is a complete intersection in \( T^*\mathcal{B} \) and that \( \overline{\Lambda}: \mathfrak{m} \to \mathbb{C}^n \) is flat. This implies that the dimension of any irreducible component of any fiber of this morphism \( \overline{\Lambda} \) is \( \dim T^*\mathcal{B} - \dim (\mathfrak{b} \times \mathbb{C}^n) = \left( \begin{array}{c} n \\ 2 \end{array} \right) + n \).

Consider a fiber \( \xi = \overline{\Lambda}^{-1}(\text{diag}(r)) \), and so for \( (r, s, i, j) \in \xi \), the diagonal entries of \( r \) are \( \text{diag}(r) \). Recall that every conjugacy class \( \mathcal{O}_r \) in \( \mathfrak{b} \) is determined by its Jordan \( B \)-canonical form. Since there exists only a finite number of Jordan \( B \)-canonical forms for an \( n \times n \) matrix with a fixed diagonal, \( \xi \) is a finite union of \( \mathcal{M}(\mathcal{O}_r) \), where \( \mathcal{O}_r \) is a conjugacy class in \( \mathfrak{b} \). By Proposition 4.2, \( \mathcal{M}(\mathcal{O}_r) \) is a Lagrangian subscheme. Next, any irreducible component of the corresponding scheme-theoretic fiber must be the closure of an irreducible component of \( \mathcal{M}(\mathcal{O}_r) \) for some conjugacy class \( \mathcal{O}_r \) of \( \mathfrak{b} \), and hence a Lagrangian subscheme, since every irreducible component of the scheme-theoretic fiber has dimension \( \dim \mathcal{M}(\mathcal{O}_r) \).

We conclude that Corollary 4.5 yields Theorem 1.1. \( \square \)
5. Proof of Theorem 1.2

The results in this section are in [Nev11] (with assuming Theorem 1.1). We give a slightly different proof that instead closely follows [GG06] rather than lifting up to the $g$ setting.

We give the analog of [GG06, Lemma 2.8]. Note that we have to give an upper triangular version of [GG06, Lemma 2.8], so we have interchanged the roles of $i \leftrightarrow j$. This is [Nev11, Eq. (4.1)]. Since this does not require the lift from [Nev11, Lemma 4.1], we do not require [GG06, Lemma 2.3]. Thus, in practice, ours is distinct as we do not lift to the $g$ and $G$ setting and instead work directly with the Borel subalgebra and the Borel subgroup.

**Lemma 5.1.** Let $(r, s, i, j) \in \mathfrak{M}$ such that $r_{pp} \neq r_{pq}$ for all $1 \leq p < q \leq n$ (i.e., the eigenvalues of $r$ are pairwise distinct). Then the $B$-orbit of $(r, s, i, j)$ contains a representative such that

(a) $r = (r_{pq})_{p,q=1}^n$ is diagonal with eigenvalues $\rho_1, \ldots, \rho_n$ (note $r_{pp} = \rho_p$ for all $1 \leq p \leq n$),

(b) $i = \sum_{a \in \mathfrak{a}} e_a$ and $j = \sum_{a' \in \mathfrak{a}'} e_{a'}$ for some $\mathfrak{a}, \mathfrak{a}' \subseteq \{1, \ldots, n\}$ such that $\mathfrak{a} \cap \mathfrak{a}' = \emptyset$,

(c) $s = (s_{pq})_{p,q=1}^n$ is given by

$$s_{pq} = \begin{cases} 
\sigma_p & \text{if } p = q, \\
\frac{1}{\rho_p - \rho_q} & \text{if } p \in \mathfrak{a}, q \in \mathfrak{a}', \text{ and } p > q, \\
0 & \text{otherwise,}
\end{cases}$$

for some $s_1, \ldots, s_n \in \mathbb{C}$.

Conversely, given $(r, s, i, j)$ that satisfy these conditions for any choice of $\rho_1, \ldots, \rho_n$, $\sigma_1, \ldots, \sigma_n$, $\mathfrak{a}$, $\mathfrak{a}'$ (i.e., with $\rho_1, \ldots, \rho_n$, $\sigma_1, \ldots, \sigma_n$, $\mathfrak{a}$, $\mathfrak{a}'$ pairwise distinct and $\mathfrak{a} \cap \mathfrak{a}' = \emptyset$), we have $(r, s, i, j) \in \mathfrak{M}$ with $\text{supp}(i) = \mathfrak{a}$, and $\text{supp}(j) = \mathfrak{a}'$.

Note that $s_{pp} = \sigma_p$ in Lemma 5.1, but $\sigma_p$ are considered as free variables for the converse.

**Proof.** From Lemma 3.3, we can assume $r$ is diagonal, so we obtain (a).

Next, we need to solve

$$[r, s] + ij + u = \begin{pmatrix}
0 \\
(r_{22} - r_{11})s_{21} & 0 \\
\vdots & \ddots \\
(r_{nn} - r_{n-1,n-1})s_{n,1} & \cdots & (r_{nn} - r_{n-1,n-1})s_{n,n-1} & 0
\end{pmatrix} + ij + u = 0 + u.$$

Let $\mathfrak{a} = \text{supp}(i)$ and $\mathfrak{a}' = \text{supp}(j)$. Note that $ij$ is the matrix with a row $p$ given by $j$ scaled by $i_j$ if $p \in \mathfrak{a}$ and is 0 otherwise. However, to have a solution to (13), we must have $\mathfrak{a} \cap \mathfrak{a}' = \emptyset$ as otherwise there will be a nonzero entry along the diagonal of $ij$.

Next, by computing $br - rb = 0$, we see that the centralizer of $r$ is given by the diagonal matrices. Therefore, we can assume $i$ is a vector with entries $\{0, 1\}$. Since $\text{supp}(i) \cap \text{supp}(j) = \emptyset$, we can (independently) scale all nonzero entries of $j$ to be 1. Since we are working in $\mathfrak{g}/\mathfrak{u}$, we do not care about the strictly upper diagonal portion in (13). Therefore, we have (b). Next, solving for $(s_{pq})_{p,q=1}^n$ yields (c). Note that we have also shown the converse statement of the lemma.

We give the analog of [GG06, Lemma 2.9], which we split into the following two lemmas. We note that the condition that $\text{supp}(i)$ and $\text{supp}(j)$ are disjoint is redundant from (13) as noted above, but we have included it for clarity.

Let $\Delta = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_i = x_j \text{ for some } i \neq j\}$, the big diagonal in $\mathbb{C}^n$.

**Lemma 5.2.** Fix some $\mathfrak{a} \subseteq \{1, \ldots, n\}$. Recall from Theorem 1.2 that

$$\mathfrak{M}_\mathfrak{a} = \left\{ (r, s, i, j) \in \mathfrak{M} : \begin{array}{ll}
\text{supp}(i) = \mathfrak{a}, \\
\text{supp}(j) \sqcup \mathfrak{a} = \{1, \ldots, n\}
\end{array} \right\}.$$
Then $\mathcal{M}'_a$ is connected of $\dim \mathcal{M}'_a = \binom{n}{2} + 2n$, and both of the actions of $B$ and $b$ on $\mathcal{M}'_a$ are free.

Proof. Let $(r, s, i, j) \in \mathcal{M}'_a$ be a representative from Lemma 5.1 with $\text{supp}(i) = a$. To see that the isotropy (or stabilizer) group $\{ b \in B : b(r, i, j) = (r, i, j) \}$ is trivial, note that $brb^{-1} = r$ (equivalently $[b, r] = 0$) implies that $b$ is a diagonal matrix and $bi = i$ implies $b$ is the identity matrix. Similarly, the isotropy Lie algebra $\{ b \in B : b(r, i, j) = 0 \}$ is trivial. Therefore, the same holds for $(r, s, i, j)$. Note that the representative is unique since we cannot permute the diagonal entries of $r$ by the $B$-action, nor can we permute $i$ and $j$ by elements of the centralizer of $r$ (recall, this is the group of diagonal matrices). Hence, we obtain

$$\mathcal{M}'_a \simeq B \times \left( (\mathbb{C}^n \setminus \Delta) \times \mathbb{C}^n \right),$$

and the claim follows. □

Lemma 5.3. Choose some $a \subseteq \{1, \ldots, n\}$. Let

$$\mathcal{M}''_a := \left\{ \mathcal{O}(r, s, i, j) : (r, s, i, j) \in \mathfrak{D} \text{ with } \begin{array}{l} \text{supp}(i) \subseteq a, \ a \cap \text{supp}(j) = \emptyset, \\ \text{supp}(i) \cup \text{supp}(j) \neq \{1, \ldots, n\} \end{array} \right\}.$$ Then $\dim \mathcal{M}''_a < \binom{n}{2} + 2n$.

Proof. Let $(r, s, i, j) \in \mathcal{M}''_a$ be a representative from Lemma 5.1 with $\text{supp}(i) \subseteq a$. Let $\Sigma = \text{supp}(i) \cup \text{supp}(j)$. Consider the subgroup $B'' \subseteq B$ given by the diagonal matrices $(b_{pp})_{p=1}^n$ with

$$b_{pp} = \begin{cases} 1 & \text{if } p \in a \cup a', \\ a_{aa'} & \text{if } a'' \in \{1, \ldots, n\} \setminus \Sigma. \end{cases}$$

Note that $\{1, \ldots, n\} \setminus \Sigma \neq \emptyset$ by assumption. Hence, the subgroup $B''$ is of strictly positive dimension that acts trivially on $(r, s, i, j)$. Hence, we must have $\dim \mathcal{M}''_a < \dim \mathcal{M}'_a = \dim \mathcal{M} = \binom{n}{2} + 2n$ (the last equality is by Corollary 4.5). □

Proof of Theorem 1.2. Since the diagonal entries under the $B$-action cannot change, it is straightforward to see that $\mathcal{M}$ can be written as

(14) $$\mathcal{M} = \left( \bigcup_{a \subseteq \{1, \ldots, n\}} \mathcal{M}'_a \right) \sqcup \left( \bigcup_{a \subseteq \{1, \ldots, n\}} \mathcal{M}''_a \right) \sqcup \overline{\Lambda}^{-1}(\Delta).$$

Next, note that $\dim \Delta < n$, and thus Corollary 4.5 implies that $\dim \overline{\Lambda}^{-1}(\Delta) < n + \binom{n}{2} + n = \binom{n}{2} + 2n$. Also from Corollary 4.5, we have that $\mathcal{M}$ is a complete intersection, and so every irreducible component must have dimension $\binom{n}{2} + 2n$. Hence the closures of $\mathcal{M}_a''$ (from Lemma 5.3) and $\pi^{-1}(\Delta)$ cannot be irreducible components. The claim that the closure of $\mathcal{M}_a'$ is an irreducible component follows from Lemma 5.2, and thus we have obtained all irreducible components from the decomposition (14).

Since there are no fixed points under the $B$-action on $\mathcal{M}_a'$, the map $\mu$ is a submersion at generic points of the scheme $\mathcal{M}$. Thus $\mathcal{M}$ is generically reduced. We have that $\mathcal{M}$ is Cohen-Macaulay since it is a complete intersection. We conclude that $\mathcal{M}$ is reduced (see, e.g., [Eis95, Ex. 18.9] or [CG10, Thm. 2.2.11]). □

6. Proof of Theorem 1.4

In this section, we will work with $\tilde{C}_n$, proving Theorem 1.4. Our proofs closely follow those of the analogous statements from [Wil98]. We first prove a lemma analogous to [Wil98, Cor. 1.4].

Lemma 6.1. Fix $(r, s, i, j) \in \tilde{C}_n$. Suppose $b \in B$ is a matrix that commutes with both $r$ and $s$. Then $b = \lambda I_n$ for some $\lambda \in \mathbb{C}$ (i.e., $b$ is a scalar matrix).
Proof. Consider some representative \( \tilde{s} \in \mathfrak{g} \) of \( s \). Note that \( (r, \tilde{s}, i, j) \) is an element of the variety defined by [Wil98, Eq. (1.1)]. For any matrix \( b \) that commutes with both \( r \) and \( \tilde{s} \), any eigenspace of \( b \) is a common invariant subspace for \( r \) and \( \tilde{s} \). Therefore any eigenspace of \( b \) must be \( \mathbb{C}^n \) by [Wil98, Lemma 1.3]. Since this holds for every representative of \( s \), we must have \( b = \lambda I_n \) for some \( \lambda \in \mathbb{C} \) as desired. \( \square \)

The following corollary is a generalization of [Wil98, Cor. 1.5].

**Corollary 6.2.** The group \( B \) acts freely on \( \tilde{C}_n \).

**Proof.** If \( (brb^{-1}, bsb^{-1}, bi, jb^{-1}) = (r, s, i, j) \), for some \( b \in B \), we have \( br = rb \) and \( bs = sb \). Therefore, \( b \) commutes with \( r \) and \( s \), and so \( b = \lambda I_n \) by Lemma 6.1. Since we have \( bi = \lambda i = i \), we have \( \lambda = 1 \) and \( b \) the identity matrix. \( \square \)

We follow [Wil98, Prop. 1.7] to obtain:

**Proposition 6.3.** The differential of \( \mu \) is surjective at every point of \( \tilde{C}_n \).

**Proof.** The differential of \( \mu \) at the point \( (r, s, i, j) \in T^* \mathfrak{B} \) is

\[
d\mu(P, Q, a, b) = [r, Q] + [P, s] + ib - aj.
\]

The annihilator of the image of this map with respect to the nondegenerate bilinear form \( (r, s) \mapsto \text{tr}(rs) \) consists of all matrices \( R \) such that

\[
[R, r] = [R, s] = Ri = jR = 0.
\]

Thus \( d\mu \) is surjective at \( (r, s, i, j) \) if and only if \( R = 0 \) is the only solution to these equations. Yet if \( (r, s, i, j) \in \tilde{C}_n \), then by Lemma 6.1, the first two equations imply that \( R \) is a scalar. The last two equations show that \( R = 0 \). \( \square \)

Since \( \tilde{C}_n = \mu^{-1}(-I_n) \) for the point \(-I_n \in \mathfrak{b}^*\), it follows from Proposition 6.3 and the implicit function theorem that \( \tilde{C}_n \) is a smooth subvariety of \( T^* \mathfrak{B} \) with every component of dimension \( \left( \frac{n}{2} \right) + 2n \).

It follows from Corollary 6.2 that the quotient space \( C_n \) is a smooth affine variety of dimension \( 2n \).

For the remainder of this section, we aim to prove that \( C_n \) is irreducible. Since \( C_n \) is smooth, it is equivalent to proving that it is connected (here, we may work with either the classical complex or Zariski topology).

Let \( \pi : C_n \to \mathbb{C}^n \) be defined by \( (r, s, i, j) \mapsto \text{diag}(r) \). It is clear that \( \pi \) is surjective.

The following is the analog of [Wil98, Prop. 1.10].

**Lemma 6.4.** Let \( (r, s, i, j) \in \tilde{C}_n \) such that \( r \) is diagonalizable. Then the eigenvalues of \( r \) are distinct, and the \( B \)-orbit of \( (r, s, i, j) \) contains a unique element such that

\( (a) \ r = \text{diag}(-\rho_1, -\rho_2, \ldots, -\rho_n) \) is diagonal,

\( (b) \) the entries in the vectors \( i \) (resp. \( j \)) are equal to \(-1 \) (resp. \( 1 \)),

\( (c) \ s \) is a lower Calogero-Moser matrix: it has entries

\[
s_{pq} = \begin{cases} 
0 & \text{if } p < q, \\
\sigma_p & \text{if } p = q, \\
(\rho_p - \rho_q)^{-1} & \text{if } p > q,
\end{cases}
\]

for some \( \sigma_p \in \mathbb{C}^n \).

**Proof.** From the Jordan \( B \)-canonical form (Lemma 3.3), we can consider \( r \) to be a diagonal matrix with diagonal \( (-\rho_1, -\rho_2, \ldots, -\rho_n) \). Note that the diagonal of \( [r, s] \) is \( (0, \ldots, 0) \), and so if we consider the diagonal entries \( [r, s] + I_n = -ij \) in (4), we obtain \( -i_p j_p = 1 \) for all \( 1 \leq p \leq n \). Thus, \(-ij\) is
the matrix with every entry being 1. For all \( p > q \), the \((p, q)\)-entry of \([r, s] + I_n\) is \( s_{pq}(\rho_p - \rho_q) \). Therefore, if a diagonal entry of \( r \) repeats, then for some \( p \) and \( q \), \( s_{pq}(\rho_p - \rho_q) = 0 \neq 1 \), which is a contradiction. Thus, we have \( \rho_p \neq \rho_q \) for \( p \neq q \). Note that the centralizer (equivalently stabilizer) of \( r \) is given by diagonal matrices, so we can act by diagonal matrices without changing \( r \). Acting by the diagonal matrices, we can obtain \( i = (-1, -1, \ldots, -1) \), and so subsequently we have \( j = (1, 1, \ldots, 1) \). However, by fixing \( i \) and \( j \) as in (b) have no longer have any choice of element in the orbit, so such an element is unique. Since \( 1 = s_{pq}(\rho_p - \rho_q) \) for all \( p > q \) and \( s \) is lower triangular, we obtain that \( s \) is a lower Calogero-Moser matrix. \( \square \)

We remark that since the eigenvalues of \( r \) are pairwise distinct if \( r \) is diagonalizable, not every matrix \( r \) occurs as a part of the quadruple \((r, s, i, j)\) in \( \widetilde{C}_n \).

A consequence of Lemma 6.4 is that

\[ C^d_n := \{(r, s, i, j) \in C_n : r \text{ is diagonalizable}\} = \pi^{-1}(\mathbb{C}^n \setminus \Delta). \]

Furthermore, note that in Lemma 6.4, we cannot permute either the \( \rho_i \) or the \( \sigma_i \) using the Borel group \( B \). So such set of points is fixed and ordered under the \( B \)-action. Therefore the parameters \((\rho_p, \sigma_p)\) define an isomorphism \( C^d_n \cong (\mathbb{C}^n \setminus \Delta) \times \mathbb{C}^n \), with \( \pi \) corresponding to the projection onto the first factor. Thus we have the following analog of [Wil98, Lemma 1.9].

**Corollary 6.5.** We have that \( \pi^{-1}(\mathbb{C}^n \setminus \Delta) \) is connected.

Next, we need the following lemma, which is similar to [Wil98, Lemma 1.8].

**Lemma 6.6.** The fibers of \( \pi \) have dimension at most \( n \).

In order to prove Lemma 6.6, we first study the restriction to \( \widetilde{C}_n \) of the projection \( p : T^* \mathfrak{B} \to \mathbb{C}^{2n} \) onto the last two factors \((i, j)\). The following generalizes [Wil98, Lemma 1.11].

**Lemma 6.7.** Fix some \( r_0 \in \mathfrak{b} \), and define

\[ \widetilde{C}_n(r_0) = \{(r, s, i, j) \in \widetilde{C}_n : r = r_0\}. \]

Then \( p(\widetilde{C}_n(r_0)) \) is contained in an \( n \)-dimensional subvariety of \( \mathbb{C}^{2n} \).

**Proof.** Since \( p \) is \( B \)-invariant, assume that \( r_0 \) is in Jordan \( B \)-canonical form.

We first consider the case when \( r_0 \) consists of a single Jordan \( B \)-block, \( i.e., \)

\[ (r_0)_{pq} = \begin{cases} 
\lambda & \text{if } p = q, \\
1 & \text{if } p = q + 1, \\
0 & \text{otherwise}.
\end{cases} \]

Recall that for a matrix \( A \) to have rank 1, for the minimal \( d \) such that \( A_{p-d,p} \neq 0 \) for some \( p \), there exists a unique such \( p \). In other words, the lowest diagonal that is not 0 has exactly one nonzero entry. Next in \( B = [r_0, s] + I_n \), we have

\[ \sum_{p=1}^{n-d} B_{p-d,p} = 0 \]

for all \( d < 0 \). Therefore, \( B \) is upper triangular since \( \text{rk}(B) = \text{rk}(-ij) = 1 \). Moreover, there exists a unique nonzero entry on the main diagonal of \( B \), which is equal to

\[ \text{tr}(B) = \text{tr}([r_0, s]) + \text{tr}(I_n) = 0 + n = n. \]

Next, suppose the first nonzero entry in \( j \) occurs at position \( p \), and so the last nonzero entry in \( -i \) also occurs in \( p \). Note that \( -i_p j_p = \text{tr}(B) \) from above. Therefore, for each of the pairs \((i, j)\), there

\(^2\text{Recall that } s \in \mathfrak{b}^* \text{ is a lower triangular matrix.} \)
are $n$ families corresponding to the choice of entry $p$. Every such family has dimension $n$ since there are $n - p + 1$ parameters in $j$ and $p$ parameters in $i$ with one relation: $-i_p j_p = n$. Hence, the claim holds when $r_0$ has a single Jordan $B$-block.

Now, assume $r_0 = \bigoplus a \cdot r_a$, the direct sum of several Jordan $B$-blocks of sizes $n_a$, and we write $s_{ab}$, $(i_a, j_a)$ for the corresponding $B$-block decompositions of $s$, $i$, and $j$, respectively. Then taking the $(a, a)$-block in (4) gives $[r_{aa}, s_{aa}] + I_{n_a} = -i_a j_a$. By the above, there are only at most $n_a$ parameters in $(i_a, j_a)$, and so the claim follows. 

Next, we obtain a result analogous to [Wil98, Cor. 1.12].

**Corollary 6.8.** Fix a conjugacy class $O_r$ in $b$, and define

$$C_n(O_r) := \{(r', s, i, j) \in C_n : r' \in O_r\}.$$

Then $\dim C_n(O_r) \leq n$.

**Proof.** Fix $r_0 \in O_r$ such that $C_n(O_r) = \overline{C}_n(r_0)/B^{r_0}$ (recall that $B^{r_0}$ denotes the centralizer of $r_0$ in $B$). From (4), it is clear that the part of $\overline{C}_n(r_0)$ lying over a fixed $(i, j) \in \mathbb{C}^{2n}$ is parametrized by the Lie algebra of $B^{r_0}$. We have $\dim \overline{C}_n(r_0) \leq n + \dim B^{r_0}$ by Lemma 6.7. Since the $B^{r_0}$-action on $\overline{C}_n(r_0)$ is free by Corollary 6.2, the claim follows.

**Proof of Lemma 6.6.** Note that the conjugacy classes of $b$ are parameterized by the diagonal entries (i.e., the image under $\pi$) and the Jordan $B$-blocks decomposition (Lemma 3.3). Recall that for a fixed diagonal, there are only a finite number of ways to decompose $r$ into Jordan $B$-blocks. Therefore, each fiber of $\pi$ is a union of sets $C_n(O_r)$ for a finite number of orbits $O_r$. The claim follows from Corollary 6.8. 

**Theorem 6.9.** The variety $C_n$ is connected. Moreover, $C_n$ is irreducible.

**Proof.** We have $\dim \pi^{-1}(\Delta) \leq 2n - 1$ by Lemma 6.6 and because $\Delta$ is a reducible subvariety of $\mathbb{C}^n$ of dimension $n - 1$. Let $X = C_n \setminus \pi^{-1}(\Delta) = \pi^{-1}(\mathbb{C}^n \setminus \Delta)$, the complement of $\pi^{-1}(\Delta)$, and $X$ is a connected open subset of $C_n$ by Corollary 6.5. Let $Y$ denote the connected component containing $X$. If $\pi^{-1}(\Delta)$ is not contained in $Y$, then $C_n$ would have a connected component of dimension less than $2n$, which cannot happen since $\dim(C_n) = 2n$. Hence, we have

$$Y = \pi^{-1}(\mathbb{C}^n \setminus \Delta) \cup \pi^{-1}(\Delta) = \pi^{-1}(\mathbb{C}^n) = C_n,$$

and so $C_n$ is connected. 

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