CONVERGENCE OF QUASILINEAR PARABOLIC EQUATIONS TO SEMILINEAR EQUATIONS

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ABSTRACT. In this work we consider a family of reaction-diffusion equations with variable exponents reaching as a limit problem a semilinear equation. We provide uniform estimates for the solutions and we prove that the solutions of the family of quasilinear equations with variable exponents converge to the solution of a limit semilinear equation when the exponents go to 2. Moreover, the robustness of the global attractors is also studied.

1. Introduction. In this paper we improve the results presented in our previous work [3]. We proved in [3] a weak upper semicontinuity of the global attractors for problem (2) as \( j \to +\infty \). In this work we will prove that in fact we can obtain more, i.e., we can obtain upper semicontinuity of the global attractors, but for this, we have to revisit the uniform estimates of the solutions and to prove the continuity of the flow in another way.

Let us consider the semilinear equation

\[
\begin{aligned}
\frac{\partial_t u - \Delta u + u = b(u), \quad x \in \Omega, \ t > 0,}{}
\frac{\partial u}{\partial \vec{n}} = 0, \quad x \in \partial \Omega, \ t \geq 0, \\
u(0) = u_0 \in L^2(\Omega),
\end{aligned}
\]

where \( \Omega \) is a bounded domain with sufficiently smooth boundary in \( \mathbb{R}^N, \ N \geq 1 \) and \( b : \mathbb{R} \to \mathbb{R} \) is a continuously differentiable globally Lipschitz function and consider \( B : L^2(\Omega) \to L^2(\Omega) \) its associated Nemytskii operator, that is, for any \( u : [0, \infty) \to L^2(\Omega) \) and \( x \in \Omega, \ B(u(t))(x) := b(u(x,t)) \). It is easy to see that \( B \) is

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globally Lipschitz with the same Lipschitz constant as \( b \). The problem (1) can be seen as limit problem for

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta_{p_j(x)} u + |u|^{p_j(x)-2} u = b(u), & \quad x \in \Omega, \; t > 0, \\
\frac{\partial u}{\partial n} = 0, & \quad x \in \partial \Omega, \; t \geq 0, \\
u(0) = u_0 \in L^2(\Omega)
\end{aligned}
\]

when \( p_j \to 2 \) in \( L^\infty(\Omega) \) as \( j \to +\infty \).

Suppose \( (p_j)_{j \in \mathbb{N}} \) is a sequence of continuous functions \( p_j : \Omega \to \mathbb{R} \) such that \( p_j^- := \inf p_j \geq 2 \) and \( (p_j)_{j \in \mathbb{N}} \) converges to 2 in \( L^\infty(\Omega) \) as \( j \) goes to +\( \infty \). We denote \( p_j^+ := \sup p_j \). We are supposing \( 2 \leq p_j(x) \leq p_1(x) \leq p_0 \) for all \( x \in \Omega \).

The operator \( A_{j,1} : W^{1,p_j(\cdot)}(\Omega) \to [W^{1,p_j(\cdot)}(\Omega)]^* \) given by

\[
A_{j,1}u(v) = \int_\Omega |\nabla u(x)|^{p_j(x)-2} \nabla u(x) \cdot \nabla v(x) dx + \int_\Omega |u(x)|^{p_j(x)-2} u(x)v(x) dx
\]

for any \( u \in W^{1,p_j(\cdot)}(\Omega) \) and \( v \in W^{1,p_j(\cdot)}(\Omega) \) is a maximal monotone operator and consequently the operator \( A_j \), the realization of \( A_{j,1} \) at \( L^2(\Omega) \), is maximal monotone in \( L^2(\Omega) \). We denote

\[
A_j u = -\Delta_{p_j(x)} u + |u|^{p_j(x)-2} u = -\text{div} \left( |\nabla u|^{p_j(x)-2} \nabla u \right) + |u|^{p_j(x)-2} u,
\]

for \( j \in \mathbb{N} \). Moreover, the operator \( A_j \) is the subdifferential \( \partial \varphi_j \) of the convex, proper and lower semicontinuous map \( \varphi_j : L^2(\Omega) \to \mathbb{R} \cup \{ \infty \} \) given by

\[
\varphi_j(u) = \begin{cases}
\int_\Omega \frac{1}{p_j(x)} |\nabla u(x)|^{p_j(x)} + \int_\Omega \frac{1}{p_j(x)} |u(x)|^{p_j(x)} dx; & \text{if } u \in W^{1,p_j(\cdot)}(\Omega) \\
+\infty, & \text{otherwise}.
\end{cases}
\]

We refer the reader to [5, 6] for more details on the Sobolev space with variable exponents and for the \( p(x) \)-Laplacian operator.

Problem (2) can be written abstractly as

\[
\begin{aligned}
\frac{du}{dt} + A_j u = B(u), & \quad t > 0, \\
u(0) = u_0 \in L^2(\Omega),
\end{aligned}
\]

with \( B : L^2(\Omega) \to L^2(\Omega) \) globally Lipschitz. It is well-known that the problem (3) has a unique strong global solution \( u_j \) and has a global attractor in \( L^2(\Omega) \), which we will denote \( A_j \), see for instance [7, 8].

Define the operator \( A : D(A) \subset L^2(\Omega) \to L^2(\Omega) \) by

\[
D(A) := \left\{ u \in H^2(\Omega); \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \right\},
\]

and

\[
Au := -\Delta u + u
\]

for all \( u \in D(A) \).

It is well-known that \( A \) is a positive self-adjoint operator in \( L^2(\Omega) \) with compact resolvent and therefore \(-A\) generates a compact analytic semigroup on \( L^2(\Omega) \) (that is, \( A \) is a sectorial operator in the sense of Henry [4]). If \( A \) is as above, consider in the space \( L^2(\Omega) \) the following semilinear autonomous parabolic problem

\[
\begin{aligned}
\frac{du}{dt} + Au = B(u), & \quad t > 0, \\
u(0) = u_0 \in L^2(\Omega).
\end{aligned}
\]
Let \( \{S(t)u_0 := u(t, u_0)\}_{t \geq 0} \) be the unique mild solution of (4) in \( L^2(\Omega) \) (see [3] for details).

**Theorem 1.1.** [3] The problem (4) has a global attractor in \( L^2(\Omega) \), which we will denote by \( A_\infty \).

2. Uniform estimates on the solutions. We can rewrite the Proposition 1 in [3] in the following way

**Proposition 1.** Consider \( b \) and \( B \) with Lipschitz constant \( L < \frac{1}{2p_j^+ (|\Omega|+1)^2} \) and let \( p_j^- , p_j^+ \in [2, p_0] , \) \( p_0 > 2 \), for any \( j \). Then there exists a ball \( M_0 \) of radius \( R_0 \), which does not depend on \( j \), such that for any bounded set \( D \) in \( L^2(\Omega) \) there exists a time \( T = T(D) \) such that \( u_j(t) \in M_0 \) if \( t \geq T \) for any \( j \in \mathbb{N} \) and \( u_j(0) = u_{0j} \in D \).

Also, for any bounded set \( D \) there exists a constant \( R = R(D) \) such that
\[
\|u_j(t)\|_{L^2(\Omega)} \leq R , \text{ for all } t \geq 0 , \; j \in \mathbb{N} \; \text{and} \; u_j(0) = u_{0j} \in D .
\]

**Sketch of the proof:** Let \( \tau > 0 \). Multiplying the equation in (3) by \( u_j(\tau) \) we obtain that
\[
\frac{1}{2} \frac{d}{d\tau} \|u_j(\tau)\|_{L^2(\Omega)}^2 + \int_\Omega \|\nabla u_j(\tau)|_{p_j} dx + \int_\Omega \|u_j(\tau)|_{p_j} dx \\
\leq L\|u_j(\tau)\|_{L^2(\Omega)}^2 + \|B(\tau)\|_{L^2(\Omega)} \|u_j(\tau)\|_{L^2(\Omega)} .
\]

(5)

If \( \|u_j(\tau)\|_{W^{1,p_j}\Omega} \geq 1 \), then using (5) and estimates on the operator like in Lemma 3.1 in [1], we obtain that
\[
\frac{1}{2} \frac{d}{d\tau} \|u_j(\tau)\|_{L^2(\Omega)}^2 - \frac{1}{2p_j^+} \|u_j(\tau)\|_{W^{1,p_j}\Omega}^2 + L\|u_j(\tau)\|_{L^2(\Omega)}^2 + C_0 \|u_j(\tau)\|_{L^2(\Omega)}^2 ,
\]

(6)

where \( C_0 := \|B(0)\|_{L^2(\Omega)} \geq 0 \).

Since \( -\|u_j(\tau)\|_{W^{1,p_j}\Omega}^2 \leq -\|u_j(\tau)\|_{W^{1,p_j}\Omega}^2 \) then from (6)
\[
\frac{1}{2} \frac{d}{d\tau} \|u_j(\tau)\|_{L^2(\Omega)}^2 \leq -\frac{1}{2p_j^+} \|u_j(\tau)\|_{W^{1,p_j}\Omega}^2 + L\|u_j(\tau)\|_{L^2(\Omega)}^2 + C_0 \|u_j(\tau)\|_{L^2(\Omega)}^2
\]
\[
\leq \left(-\frac{1}{2p_j^+ (|\Omega|+1)^2} + L + \frac{1}{\epsilon^2} \right) \|u_j(\tau)\|_{L^2(\Omega)}^2 + \frac{1}{2} \left( \frac{C_0}{\epsilon} \right)^2 .
\]

Now, choosing \( \epsilon = \epsilon_1 > 0 \) sufficiently small we obtain
\[
\frac{1}{2} \frac{d}{d\tau} \|u_j(\tau)\|_{L^2(\Omega)}^2 \leq -\alpha \|u_j(\tau)\|_{L^2(\Omega)}^2 + C_1 ,
\]

where \( \alpha := \frac{1}{2p_j^+ + 1 (|\Omega|+1)^2} - \frac{1}{2} L > 0 \) and \( C_1 := \left( \frac{C_0}{\epsilon} \right)^2 \). Then
\[
\frac{d}{d\tau} \left[ \|u_j(\tau)\|_{L^2(\Omega)}^2 \right] e^{2\alpha \tau} + 2\alpha \|u_j(\tau)\|_{L^2(\Omega)}^2 e^{2\alpha \tau} \leq 2C_1 e^{2\alpha \tau} .
\]

(7)

If \( \|u_j(\tau)\|_{W^{1,p_j}\Omega} < 1 \), then
\[
\|u_j(\tau)\|_{L^2(\Omega)} \leq (|\Omega|+1) \|u_j(\tau)\|_{W^{1,p_j}\Omega} \leq |\Omega|+1 .
\]
Thus, by (5) and estimates on the operator like in Lemma 3.1 in [1] we have
\[
\frac{1}{2} \frac{d}{dt} \| u_j(t) \|_{L^2(\Omega)}^2 \leq - \frac{1}{2p_j^+} \| u_j(t) \|_{W^{1,p_j^+}(\Omega)}^2 + L \| u_j(t) \|_{L^2(\Omega)}^2 + C_0 \| u_j(t) \|_{L^2(\Omega)}^2 \\
\leq L(\| \Omega \| + 1)^2 + C_0(\| \Omega \| + 1),
\]
where \( C_0 := \| B(0) \|_{L^2(\Omega)} \geq 0. \)
Considering \( y_j(\tau) := \| u_j(\tau) \|_{L^2(\Omega)}^2 \) we obtain from (7) and (8) that for a suitable constant \( C_3 > 0 \)
\[
\frac{d}{dt} [y_j(\tau)e^{2\alpha\tau}] \leq C_3 e^{2\alpha\tau}, \text{ for all } \tau > 0.
\]
Integrating from 0 to \( t \), we have
\[
y_j(t)e^{2\alpha t} \leq y_j(0) + \frac{C_3}{2\alpha} e^{2\alpha t} - \frac{C_3}{2\alpha} \leq \| u_{0j} \|^2 + \frac{C_3}{2\alpha} e^{2\alpha t}.
\]
Multiplying by \( e^{-2\alpha t} \), we obtain
\[
\| u_j(t) \|^2_{L^2(\Omega)} = y_j(t) \leq \| u_{0j} \|^2_{L^2(\Omega)} e^{-2\alpha t} + \frac{C_3}{2\alpha} \leq \| u_{0j} \|^2_{L^2(\Omega)} e^0 + \frac{C_3}{2\alpha}, \text{ for all } t \geq 0.
\]
Since \( u_{0j} \in D \) and \( D \) is bounded, there exists \( r_0 \geq 0 \) such that \( \| u_{0j} \|_{L^2(\Omega)} \leq r_0 \) for all \( j \in \mathbb{N} \). Thus,
\[
\| u_j(t) \|_{L^2(\Omega)} \leq R := \left( r_0^2 + \frac{C_3}{2\alpha} \right)^{1/2}, \text{ for all } t \geq 0 \text{ and } j \in \mathbb{N}.
\]
\[\square\]

Let us now provide estimates for the solutions in a more regular space.

**Theorem 2.1.** Consider \( b \) and \( B \) with Lipschitz constant \( L < \frac{1}{2^{p_j^+} (\| \Omega \| + 1)^2} \). Let \( p_j^+, p_j^{-} \in [2, p_0], \) \( p_0 > 2, \) for any \( j \). Then there exists constants \( r_1, R_1 > 0 \) and a compact set \( K_0 \) in \( L^2(\Omega) \), which do not depend on \( j \), such that for any bounded set \( D \) in \( L^2(\Omega) \) there exists a time \( T = T(D) \) such that
\[
\varphi_j(u_j(t)) \leq r_1, \\
\| u_j(t) \|_{W^{1,p_j^+}(\Omega)} \leq R_1, \\
u_j(t) \in K_0,
\]
for \( t \geq T \) and any solution \( u_j \) of (2) with \( u_{0j} \in D \).

Also, for any bounded set \( D \) in \( L^2(\Omega) \) and any \( r > 0 \) there exists \( r_2(D, r), R_2(D, r) > 0 \) such that
\[
\varphi_j(u_j(t)) \leq r_2, \\
\| u_j(t) \|_{W^{1,p_j^+}(\Omega)} \leq R_2,
\]
for \( t \geq r \) and any solution \( u_j \) of (2) with \( u_{0j} \in D \).

Moreover, if \( \varphi_j(u_{0j}) \leq r_0, \) for any \( u_{0j} \in D \) and \( j \in \mathbb{N} \), then there exists a constant \( \kappa = \kappa(r_0, D) \) such that \( \| u_j(t) \|_{W^{1,p_j^+}(\Omega)} \leq \kappa, \) for all \( t \geq 0 \) and any solution \( u_j \) of (2).
Proof. Using Proposition 1 we can obtain the following two inequalities
\[
\frac{d}{dt} \varphi_j(u_j(t)) \leq \frac{1}{2} \left( LR + ||B(0)||_{L^2(\Omega)} \right)^2 =: \frac{1}{2} K_1^2, \tag{14}
\]
for each \( t \geq T \) and for all \( j \in \mathbb{N} \), and
\[
\frac{1}{2} \frac{d}{dt} ||u_j(t)||_{L^2(\Omega)}^2 + \varphi_j(u_j(t)) \leq K_1 R, \tag{15}
\]
for each \( t \geq T(D) \) and for all \( j \in \mathbb{N} \). Indeed, using the equality (see [2, p.189])
\[
\frac{d}{dt} \varphi_j(u(t)) = \left\langle \partial \varphi_j(u(t)), \frac{du}{dt}(t) \right\rangle, \tag{16}
\]
we have that
\[
\frac{d}{dt} \varphi_j(u_j(t)) = \left\langle B(u_j(t)) - \frac{du_j}{dt}(t), \frac{du_j}{dt}(t) \right\rangle \leq -||B(u_j(t)) - \frac{du_j}{dt}(t)||_{L^2(\Omega)}^2 + ||B(u_j(t)) - \frac{du_j}{dt}(t)||_{L^2(\Omega)} ||B(u_j(t))||_{L^2(\Omega)}.
\]
So, by using the Young inequality we obtain
\[
\frac{d}{dt} \varphi_j(u_j(t)) + \frac{1}{2} ||B(u_j(t)) - \frac{du_j}{dt}(t)||_{L^2(\Omega)}^2 \leq \frac{1}{2} ||B(u_j(t))||_{L^2(\Omega)}^2
\]
and by Proposition 1 it follows that
\[
\frac{d}{dt} \varphi_j(u_j(t)) \leq \frac{1}{2} \left( ||B(u_j(t)) - B(0)||_{L^2(\Omega)} + ||B(0)||_{L^2(\Omega)} \right)^2 \leq \frac{1}{2} \left( LR + ||B(0)||_{L^2(\Omega)} \right)^2 =: \frac{1}{2} K_1^2,
\]
for each \( t \geq T \) and for all \( j \in \mathbb{N} \).

Now, using the definition of subdifferential we have
\[
\varphi_j(u_j(t)) \leq \langle \partial \varphi_j(u_j(t)), u_j(t) \rangle.
\]
Then
\[
\frac{1}{2} \frac{d}{dt} ||u_j(t)||_{L^2(\Omega)}^2 + \varphi_j(u_j(t)) \leq \left\langle \frac{du_j}{dt}(t), u_j(t) \right\rangle + \langle \partial \varphi_j(u_j(t)), u_j(t) \rangle = \langle B(u_j(t)), u_j(t) \rangle \leq ||B(u_j(t))||_{L^2(\Omega)} ||u_j(t)||_{L^2(\Omega)} \leq K_1 R,
\]
for each \( t \geq T(D) \) and for all \( j \in \mathbb{N} \).

Fixing \( r > 0 \) and integrating both sides of (15) over \((t, t + r)\) where \( t \geq T(D) \) we get
\[
\int_t^{t+r} \varphi_j(u_j(s)) ds \leq \frac{1}{2} ||u_j(t)||_{L^2(\Omega)}^2 + \int_t^{t+r} K_1 R_0 ds \leq \frac{1}{2} R_0^2 + K_1 R_0 r.
\]
For \( s \in (t, t + r) \) we integrate (14) over the interval \((s, t + r)\) and obtain that
\[
\varphi_j(u_j(t + r)) \leq \varphi_j(u_j(s)) + r \frac{K_1^2}{2}.
\]
Integrating again but now over \((t, t + r)\) we have
\[
\varphi_j(u_j(t + r)) \leq \int_t^{t+r} \varphi_j(u_j(s)) ds + r^2 \frac{K_1^2}{2} \\
\leq \frac{1}{2} R_0^2 + K_1 R_0 r + r^2 \frac{K_1^2}{2}.
\]
Thus
\[
\varphi_j(u_j(t + r)) \leq \frac{1}{2r} R_0^2 + K_1 R_0 + r \frac{K_1^2}{2} =: r_1,
\]
if \(t \geq T(D)\).

Therefore,
\[
\int_\Omega \frac{1}{p_j(x)} \left| \nabla u_j(t, x) \right|^{p_j(x)} dx + \int_\Omega \frac{1}{p_j(x)} \left| u_j(t, x) \right|^{p_j(x)} dx \leq r_1,
\]
if \(t \geq T(D) = T(D) + r\). Then
\[
\frac{1}{p_j} \left| \rho_j (\nabla u_j(t)) + \rho_j (u_j(t)) \right| \leq r_1
\]
if \(t \geq T(B)\), and hence,
\[
\rho_j (\nabla u_j(t)) + \rho_j (u_j(t)) \leq p_j r_1,
\]
if \(t \geq T(B)\).

From this it follows that there exists \(R > 0\) such that
\[
\|u_j(t)\|_{W^{1,p_j(\cdot)}(\Omega)} \leq R, \text{ if } t \geq T(D).
\]
Taking into account that there exists \(K(\Omega)\) such that
\[
\|u\|_{H^1(\Omega)} \leq K(\Omega) \|u\|_{W^{1,p_j(\cdot)}(\Omega)},
\]
for any \(u \in W^{1,p_j(\cdot)}(\Omega)\), we obtain that the ball \(K_0\) of radius \(\tilde{R} = K(\Omega) R\) in the space \(H^1(\Omega)\) defined by
\[
K_0 = \{ y \in H^1(\Omega) : \|y\|_{H^1(\Omega)} \leq \tilde{R} \}
\]
is absorbing for any \(j\). Since the embedding \(H^1(\Omega) \subset L^2(\Omega)\) is compact, we obtain that \(K_0\) is a compact set of \(L^2(\Omega)\).

If we replace in the previous arguments the constants \(R_0\) by the constant \(R(D)\) from Proposition 1 and \(T(B)\) by 0, then we obtain (12)-(13).

Let us prove now the last statement. As \(D\) is bounded in \(L^2(\Omega)\), there exists \(T(D)\) such that (18) holds for any solution \(u_j\) of (2) with \(u_{0j} \in D\). Integrating (14) over the interval \((0, t)\) with \(t \leq T(D)\) and using that \(\varphi_j(u_{0j}) \leq r_0\), for any \(u_{0j} \in D\) and \(j \in \mathbb{N}\), we obtain
\[
\varphi_j(u_j(t)) \leq \varphi_j(u_j(0)) + \frac{K_1^2}{2} T(D)
\]
\[
\leq r_0 + \frac{K_1^2}{2} T(D) = \gamma(r_0, D), \forall t \in [0, T(D)].
\]
Arguing as in the proof of the first statement we obtain the existence of a constant \(\kappa_1 = \kappa_1(r_0, D)\) such that
\[
\|u_j(t)\|_{W^{1,p_j(\cdot)}(\Omega)} \leq \kappa_1, \text{ for all } t \in [0, T(D)].
\]
Hence, using (18) we have
\[
\|u(t)\|_{W^{1,p_j(\cdot)}(\Omega)} \leq \kappa, \text{ for all } t \geq 0,
\]
where $\kappa = \kappa(r_0, D) = \max\{\kappa_1, R\}$. \hfill \lozenge

**Corollary 1.** The set $\bigcup_{j \in \mathbb{N}} \mathcal{A}_j$ is compact in the space $L^2(\Omega)$.

**Proof.** By Theorem 2.1 the set $K_0$ (compact in $L^2(\Omega)$) attracts bounded sets of $L^2(\Omega)$. Since the pullback attractor by definition is the minimal closed pullback attracting set, we obtain that $\mathcal{A}_j \subset K_0$ for every $j \in \mathbb{N}$. Then $\bigcup_{j \in \mathbb{N}} \mathcal{A}_j \subset K_0$ is a compact set in $L^2(\Omega)$. \hfill \lozenge

3. **Continuity of the flow and upper semicontinuity of the attractors.** Our objective in this section is to prove that the limit problem (3) as $j \to \infty$ is described by the parabolic semilinear equation in (4). We will prove continuity of the flow and upper semicontinuity of the family of global attractors as $j$ goes to $+\infty$ for the problem (2) with respect to the parameters $p_j$.

The next result guarantees that (4) is in fact the limit problem for (3), as $j \to \infty$.

**Theorem 3.1.** Let $u_j$ be the solution of (3) with $u_j(0) = u_{0j} \to u_0$ in $L^2(\Omega)$ as $j \to \infty$ and $\varphi_j(u_{0j}) \leq r_0$ for all $j$. Then there exist a solution $u$ of problem (4) with $u(0) = u_0$ and a subsequence $\{u_{0n}\}_n$ of $\{u_j\}_j$ such that, for each $T > 0$, $u_{0n} \to u$ in $C([0, T]; L^2(\Omega))$ as $n \to +\infty$.

**Proof.** Let $T > 0$ be fixed. Let $u_j$ be the solution of (3) with $u_j(0) = u_{0j} \to u_0$ in $L^2(\Omega)$ as $j \to \infty$. Then

$$\frac{du_j}{dt} + \partial \varphi_j(u_j) = B(u_j), \quad \text{for a.a. } t \in (0, T).$$

Take an arbitrary $\varepsilon > 0$. Taking the scalar product with $\frac{du_j}{dt}$ and integrating over $(\varepsilon, T)$ and using the equality [2, p.189]

$$\left\langle \partial \varphi_j(u_j(t)), \frac{du_j}{dt}(t) \right\rangle = \frac{d}{dt} \varphi_j(u_j(t)),$$

we obtain

$$\frac{1}{2} \int_{\varepsilon}^{T} \left\| \frac{du_j}{dt} \right\|_{L^2(\Omega)}^2 \, dt + \varphi_j(u_j(T)) \leq \varphi_j(u_j(\varepsilon)) + \frac{1}{2} \int_{\varepsilon}^{T} \|B(u_j(t))\|_{L^2(\Omega)}^2 \, dt.$$

Further, by (12) there exists an $\varepsilon$-depending constant $c_\varepsilon$ and by Proposition 1 there exists a constant $C$ such that

$$\int_{\varepsilon}^{T} \left\| \frac{du_j}{dt} \right\|_{L^2(\Omega)}^2 \, dt \leq c_\varepsilon + CT, \text{ for all } t \in [\varepsilon, T].$$

Thus, $\frac{du_j}{dt}$ is bounded in $L^2(\varepsilon, T; L^2(\Omega))$. Also, $u_j$ is bounded in $L^\infty(0, T; L^2(\Omega))$, so there exists a subsequence $\{u_{jn}\}$ and a function $u$ such that

$$u_{jn} \to u \text{ weakly star in } L^\infty(0, T; L^2(\Omega)),
$$

$$u_{jn} \to u \text{ weakly in } L^2(0, T; L^2(\Omega)),
$$

$$\frac{du_{jn}}{dt} \to \frac{du}{dt} \text{ weakly in } L^2(\varepsilon, T; L^2(\Omega)).$$

So, we guess that this $u$ is our candidate of the solution of the limit problem that we are looking for. Let us rigorously prove it in the sequel. By (13) there is an $\varepsilon$-depending constant $R_\varepsilon$ such that

$$\|u_j(t)\|_{H^1(\Omega)} \leq K \|u_j(t)\|_{W^{1, p_j}(\Omega)} \leq R_\varepsilon, \text{ for all } t \in [\varepsilon, T].$$
Hence, the compact embedding $H^1(\Omega) \subset L^2(\Omega)$ implies that for any $t \in [\varepsilon, T]$ the sequence $u_{j_n}(t)$ is relatively compact in $L^2(\Omega)$. Thus, by Ascoli-Arzelà theorem there exists a subsequence (labeled the same) such that

$$ u_{j_n} \to u \text{ in } C([\varepsilon, T], L^2(\Omega)). $$

Also, the function $u$ is absolutely continuous in $[\varepsilon, T]$.

We will prove that $u$ is an integral solution (and then a strong solution) of the limit problem (4) in the interval $(\varepsilon, T)$.

We have

$$ \frac{1}{2} \| u_{j_n}(t) - \phi \|^2_{L^2(\Omega)} \leq \frac{1}{2} \| u_{j_n}(s) - \phi \|^2_{L^2(\Omega)} + \int_s^t \langle B(u_{j_n}(\tau)) + \Delta p_{j_n}(\varepsilon)(\phi), u_{j_n}(\tau) - \phi \rangle d\tau $$

for every $\phi \in D(A_{j_n})$ and $0 \leq s \leq t \leq T$.

Now, the idea is to take the limit as $n \to \infty$ ($p_{j_n} \to 2$) on the last inequality.

As $L^2(\varepsilon, T; L^2(\Omega))$ is a reflexive Banach space there is $w \in L^2(\varepsilon, T; L^2(\Omega))$ such that $B \circ u_{j_n} \to w$ in $L^2(\varepsilon, T; L^2(\Omega))$. Since $B(u_{j_n}(t)) \to B(u(t))$ in $L^2(\Omega)$ it follows that $w = B \circ u$. Thus, $B \circ u_{j_n} \to B \circ u$ in $L^2(\varepsilon, T; L^2(\Omega))$.

Now observe that since $B \circ u_{j_n} \to B \circ u$ in $L^2(\varepsilon, T; L^2(\Omega))$ implies that $u_{j_n} \to u$ in $C(\varepsilon, T; L^2(\Omega))$ and consequently

$$ u_{j_n} \to u \text{ in } L^2(s, t; L^2(\Omega)), \forall \varepsilon \leq s \leq t \leq T; $$

then

$$ \langle B \circ u_{j_n} - h, u_{j_n} - \theta \rangle_{L^2(s, t; L^2(\Omega))} \to \langle B \circ u - h, u - \theta \rangle_{L^2(s, t; L^2(\Omega))} $$

for all $\theta, h \in L^2(\Omega)$.

Now consider $\Phi \in C^\infty_0(\Omega) \subset D(A_{j_n}) \subset L^2(\Omega)$ and let be $\Phi := A(\Phi) \in L^2(\Omega)$. We already knows from (19) that holds

$$ \frac{1}{2} \| u_{j_n}(t) - \Phi \|^2_{L^2(\Omega)} \leq \frac{1}{2} \| u_{j_n}(s) - \Phi \|^2_{L^2(\Omega)} + \int_s^t \langle B(u_{j_n}(\tau)) - A_{j_n}(\Phi), u_{j_n}(\tau) - \Phi \rangle d\tau $$

$$ = \frac{1}{2} \| u_{j_n}(s) - \Phi \|^2_{L^2(\Omega)} + \int_s^t \langle B(u_{j_n}(\tau)) - h, u_{j_n}(\tau) - \Phi \rangle d\tau $$

$$ + \int_s^t \langle B \circ u_{j_n} - h, u_{j_n}(\tau) - \Phi \rangle d\tau. $$

We claim that $f_s^t (\Phi - A_{j_n}(\Phi), u_{j_n}(\tau) - \Phi) d\tau \to 0$ as $n \to +\infty$. In fact, for each $\tau > \varepsilon$

$$ |\langle B \circ u_{j_n} - h, u_{j_n}(\tau) - \Phi \rangle| \leq \int_\Omega \left( \left| \nabla \Phi \right| - \left| \nabla \Phi \right|^{p_{j_n}(\varepsilon)-1} \right) \left| \nabla u_{j_n}(\tau) \right| dx $$

$$ + \int_\Omega \left| \nabla \Phi \right|^2 - \left| \nabla u_{j_n}(\tau) \right|^{p_{j_n}(\varepsilon)} dx $$

$$ + \int_\Omega \left( \left| \Phi \right| - \left| \Phi \right|^{p_{j_n}(\varepsilon)-1} \right) \left| u_{j_n}(\tau) \right| dx $$

$$ + \int_\Omega \left| \Phi \right|^2 - \left| u_{j_n}(\tau) \right|^{p_{j_n}(\varepsilon)} dx.$$
Since \( p_{j_n}(x) \rightarrow 2 \) for all \( x \in \Omega \) it follows by Dominated Convergence Theorem that
\[
\int_{\Omega} |\nabla \theta|^2 - |\nabla \theta|^{p_{j_n}(x)} \, dx \rightarrow 0 \quad \text{and} \quad \int_{\Omega} |\theta|^2 - |\theta|^{p_{j_n}(x)} \, dx \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

On the other hand, using the Mean Value Theorem we obtain
\[
\int_{\Omega} \left( |\nabla \theta| - |\nabla \theta|^{p_{j_n}(x)-1} \right) |\nabla u_{j_n}(\tau)| \, dx
\leq \int_{\Omega} \left( |\nabla \theta| - |\nabla \theta|^{p_{j_n}(x)-1} \right) \, dx
\leq \|p_{j_n} - 2\| \int_{\Omega} |\nabla \theta|^{\tau(n,x)+1} |\nabla u_{j_n}(\tau)| \, dx
\leq \|p_{j_n} - 2\| \left[ \int_{\Omega} \frac{1}{g_{j_n}(x)} |\nabla \theta|^{\tau(n,x)+1} q_{j_n}(x) \, dx + \int_{\Omega} \frac{1}{p_{j_n}(x)} |\nabla u_{j_n}(\tau)|^{p_{j_n}(x)} \, dx \right]
\leq \|p_{j_n} - 2\| \left[ \int_{\Omega} |\nabla \theta|^{\tau(n,x)+1} q_{j_n}(x) \, dx + \frac{1}{2} \int_{\Omega} |\nabla u_{j_n}(\tau)|^{p_{j_n}(x)} \, dx \right]
\]

Using the Proposition 2.1 we obtain that there exists a constant \( C > 0 \) such that
\[
\int_{\Omega} |\nabla u_{j_n}(\tau)|^{p_{j_n}(x)} \, dx \leq C \quad \text{for every} \quad \tau \in (s, t) \quad \text{and} \quad n \in \mathbb{N}.
\]

On the other hand, as \( 3 < \tau(n,x) + 1 < p_{j_n}(x) + 1 < p_0 + 1 \) and \( 1 < q_{j_n}(x) < 2 \) we obtain
\[
\int_{\Omega} |\nabla \theta|^{\tau(n,x)+1} q_{j_n}(x) \, dx \leq \int_{\Omega} |\nabla \theta|^3 \, dx + \int_{\Omega} |\nabla \theta|^{2(p_0+1)} \, dx \leq \tilde{C} < \infty, \forall \ n \in \mathbb{N}.
\]

Thus, considering \( \tilde{C} := C + \tilde{C} > 0 \), we have
\[
\int_{\Omega} \left( |\nabla \theta| - |\nabla \theta|^{p_{j_n}(x)-1} \right) |\nabla u_{j_n}(\tau)| \, dx \leq \|p_{j_n} - 2\| \infty \tilde{C} \rightarrow 0
\]
as \( n \rightarrow \infty \) for all \( \bar{\theta} \in C_0^\infty(\Omega) \).

Thus, we conclude that
\[
\int_{\Omega} \left( |\nabla \theta| - |\nabla \theta|^{p_{j_n}(x)-1} \right) |\nabla u_{j_n}(\tau)| \, dx \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty.
\]

Similarly, we prove that
\[
\int_{\Omega} \left( |\theta| - |\theta|^{p_{j_n}(x)-1} \right) |u_{j_n}(\tau)| \, dx \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty.
\]

So, taking the limit in (20) as \( n \rightarrow \infty \), we obtain
\[
\frac{1}{2} \|u(t) - \bar{\theta}\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|u(s) - \bar{\theta}\|_{L^2(\Omega)}^2 + \int_s^t \langle B(u(\tau)) - A(\bar{\theta}), u(\tau) - \bar{\theta} \rangle \, d\tau \tag{22}
\]

for every \( \bar{\theta} \in C_0^\infty(I) \) and \( \varepsilon \leq s \leq t \leq T \).

Now, we use a density argument to conclude that \( u \) is a solution of the limit problem (4) in the interval \((\varepsilon, T)\). Let \( \bar{\theta} \in \mathcal{D}(A) \). So, there exists a sequence
\{\overline{\theta}_j\}_{j \in \mathbb{N}} \subset C_0^\infty(\Omega) \text{ such that } \|\overline{\theta}_j - \overline{\theta}\|_{W^{1,2}(\Omega)} \to 0 \text{ as } j \to \infty \text{ and consequently } \|\overline{\theta}_j - \overline{\theta}\|_{L^2(\Omega)} \to 0 \text{ as } j \to \infty. \quad \text{By (22)},
\frac{1}{2}\|u(t) - \overline{\theta}_j\|_{L^2(\Omega)}^2 \leq \frac{1}{2}\|u(s) - \overline{\theta}_j\|_{L^2(\Omega)}^2 + \int_s^t \langle B(u(t')) - A(\overline{\theta}_j), u(t') - \overline{\theta}_j \rangle \, dt \quad (23)
\]
for every \( j \in \mathbb{N} \) and \( \varepsilon \leq s \leq t \leq T \). Obviously, \( \frac{1}{2}\|u(t) - \overline{\theta}_j\|_{L^2(\Omega)}^2 \to \frac{1}{2}\|u(t) - \overline{\theta}\|_{L^2(\Omega)}^2 \) as \( j \to \infty \) and \( \frac{1}{2}\|u(s) - \overline{\theta}_j\|_{L^2(\Omega)}^2 \to \frac{1}{2}\|u(s) - \overline{\theta}\|_{L^2(\Omega)}^2 \) as \( j \to \infty \). Using the Dominated Convergence Theorem we obtain
\[
\int_s^t \langle B(u(t)) - A(\overline{\theta}_j), u(t) - \overline{\theta}\rangle \, dt \to \int_s^t \langle B(u(t)) - A(\overline{\theta}), u(t) - \overline{\theta}\rangle \, dt
\]
as \( j \to \infty \). So, we get
\[
\frac{1}{2}\|u(t) - \overline{\theta}\|_{L^2(\Omega)}^2 \leq \frac{1}{2}\|u(s) - \overline{\theta}\|_{L^2(\Omega)}^2 + \int_s^t \langle B(u(t)) - A(\overline{\theta}), u(t) - \overline{\theta}\rangle \, dt \quad (24)
\]
for every \( \overline{\theta} \in \mathcal{D}(A) \) and \( \varepsilon \leq s \leq t \leq T \) and \( u \) is a solution of (4).

In order to check that \( u(\cdot) \) is a strong solution of problem (4) in the interval \([0, T]\), it remains to prove that \( u(t) \to u_0 \) as \( t \to 0^+ \). Let \( v(t) = e^{-\lambda t}u_0 \). The difference \( v_{j_n}(t) = u_{j_n}(t) - v(t) \) satisfies the equality
\[
d\frac{d}{dt}v_{j_n}(t) = -A_{j_n}u_{j_n} + \lambda v + B(u_{j_n}(t)) = -A_{j_n}u_{j_n} + A_{j_n}v + \lambda v - A_{j_n}v + B(u_{j_n}) = -A_{j_n}u_{j_n} + A_{j_n}v + \lambda v - \lambda |v_{j_n}(x)|^2 v + B(u_{j_n}).
\]

Multiplying it by \( v_{j_n} \) and using that the operators \( A_{j_n} \) are monotone we get
\[
\frac{d}{dt}\|v_{j_n}\|_{L^2(\Omega)}^2 \leq 2\lambda\|v(t)\|_{L^2(\Omega)}\|v_{j_n}(t)\|_{L^2(\Omega)} + 2\lambda^2 \int_\Omega |v(t)|^{2p_{j_n}(x)-2} \, dx \|v_{j_n}(t)\|_{L^2(\Omega)} + \langle B(u_{j_n}), v_{j_n} \rangle \leq K.
\]

Integration over the interval \([0, t]\) gives
\[
\|v_{j_n}(t)\|_{L^2(\Omega)}^2 \leq \|v_{j_n}(0)\|_{L^2(\Omega)}^2 + Kt.
\]
As
\[
\|u(t) - v(t)\|_{L^2(\Omega)} = \lim_{n \to \infty} \|v_{j_n}(t)\|_{L^2(\Omega)} \leq \sqrt{Kt}, \text{ for } t > 0,
\]
we obtain that
\[
\|u(t) - u_0\|_{L^2(\Omega)} \leq \|u(t) - v(t)\|_{L^2(\Omega)} + \|v(t) - u_0\|_{L^2(\Omega)} < \delta
\]
if \( 0 < t < \eta(\delta) \). Thus, \( u(t) \to u_0 \) as \( t \to 0^+ \).

Finally, we need to establish that \( u_{j_n} \to u \) in \( C([0, T], L^2(\Omega)) \). In light of the previous convergences, for this aim it is enough to prove that for any sequence of times \( t_n \to 0^+ \) we have \( u_{j_n}(t_n) \to u_0 \). Indeed,
\[
\|u_{j_n}(t_n) - u_0\|_{L^2(\Omega)} \leq \|u_{j_n}(t_n) - v(t_n)\|_{L^2(\Omega)} + \|v(t_n) - u_0\|_{L^2(\Omega)} \leq \sqrt{\|v_{j_n}(0)\|_{L^2(\Omega)}^2 + Kt_n + \|v(t_n) - u_0\|_{L^2(\Omega)}},
\]

Now we are able to establish the upper semicontinuity of the global attractors.
Theorem 3.2. The family of global attractors \( \{ A_j ; j \in \mathbb{N} \} \) is upper semicontinuous on \( j \) at infinity, in the topology of \( L^2(\Omega) \), that is, \( \text{dist}(A_j, A_\infty) \to 0 \) as \( j \to \infty \).

Proof. If this is not true, then there exist \( \epsilon > 0 \) and a sequence \( y_n \in A_{j_n}, j_n \to \infty \), such that
\[
\text{dist}(y_n, A_\infty) \geq \epsilon, \quad \forall n.
\]
Let \( K = \bigcup_j A_j \). By Corollary 1, \( K \) is a compact set in \( L^2(\Omega) \). Then, there exists \( T_1 \) such that
\[
\text{dist}(S(t,K), A_\infty) < \frac{\epsilon}{2}, \quad \forall t \geq T_1.
\]
By the invariance of the attractors there exists \( \psi_n \in A_{j_n} \subset K \) such that \( y_n = u_{j_n}(T_1, \psi_n) \). So, \( \psi_n \to \psi_0 \) for some \( \psi_0 \in K \) (at least for a subsequence). Using again the invariance of the attractors and (9) we have that \( \varphi_{j_n}(\psi_n) \) is uniformly bounded, so by Theorem 3.1, there exists \( n_0(T_1) \) such that
\[
\|u_{j_n}(T_1, \psi_n) - u(T_1, \psi_0)\|_{L^2(\Omega)} < \frac{\epsilon}{2},
\]
if \( n > n_0 \). Then we obtain
\[
\text{dist}(y_n, A_\infty) \leq \|u_{j_n}(T_1, \psi_n) - u(T_1, \psi_0)\|_{L^2(\Omega)} + \text{dist}(u(T_1, \psi_0), A_\infty)
< \frac{\epsilon}{2} + \text{dist}(S(T_1, K), A_\infty) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]
if \( n > n_0 \), which is a contradiction. \( \square \)

4. Final remarks. In the way that we proved the continuity of the flow in the previous work [3] it was necessary to use
\[
|\nabla u(x,t)| + |u(x,t)| \leq C, \quad \text{for a.e. } x \in \Omega,
\]
for any \( t \in [0, T] \), for some constant \( C = C(T) > 0 \) (see page 3917 in [3]). So, we have to express our thanks to Yu Yangyang and Rong-Nian Wang for pointing out a problem in the proof of Theorem 3 in [3] and here we give a corrigendum to Theorem 3 in [3], because it is only true for \( N = 1 \) : Since \( \mathcal{H}^\alpha \hookrightarrow L^\infty(\Omega) \) for \( \alpha > \frac{N}{4} \) (see [4]), where \( \mathcal{H}^\alpha \), for \( 0 \leq \alpha \leq 1 \) means the fractional power spaces associated with the operator \( Au := -\Delta u + u \); that is, \( \mathcal{H}^\alpha = D(A^\alpha) \) endowed with the graph norm \( \| \cdot \|_{\mathcal{H}^\alpha} = \| A^\alpha \cdot \|_{\mathcal{H}} \) (with the particular case \( \mathcal{H}^1 = D(A), \mathcal{H}^{\frac{1}{2}} = H^1(\Omega) \) and \( \mathcal{H}^0 = L^2(\Omega) \)), then for some constant \( c > 0 \) we have \( \| u(t) \|_{L^\infty(\Omega)} \leq c \| u(t) \|_{\mathcal{H}^\alpha} \), for any \( t \in [0, T] \). So, using Theorem 2 in [3], we have
\[
|u(x,t)| \leq \sup_{t \in [0,T]} \| u(t) \|_{L^{\infty}(\Omega)} \leq c \sup_{t \in [0,T]} \| u(t) \|_{\mathcal{H}^\alpha} \leq c_0 \sup_{t \in [0,T]} \| u(t) \|_{\mathcal{H}^{\frac{1}{2}}} < \infty
\]
for a.e. \( x \in \Omega \), for \( t \in [0, T] \), for any \( \frac{N}{4} < \alpha < 1 \). Now, since \( \mathcal{H}^{\frac{1}{2}} \hookrightarrow L^\infty(\Omega) \) and \( \nabla u \in \mathcal{H}^{\frac{1}{2}} \) (because \( u \in \mathcal{H}^1 \)), we have the following
\[
\sup_{t \in [0,T]} \| \nabla u(t) \|_{L^{\infty}(\Omega)} \leq \sup_{t \in [0,T]} \| \nabla u(t) \|_{\mathcal{H}^{\frac{1}{2}}(\Omega)} = \sup_{t \in [0,T]} \| u(t) \|_{H^1(\Omega)} < \infty.
\]
It is probably also possible to obtain the estimate (25) for \( N > 1 \), but maybe restrictions on the data would appear and different techniques have to be used. In fact, to obtain estimates like (25) for parabolic PDEs is, in general, not that easy at all. For problems with variable exponents we only know the work [9] which is only for one-dimensional problems and assume restrictions on the initial data.
The good point is that with this new proof we presented in this work we do not need (25) anymore and our result is even better, i.e., now we obtained upper semicontinuity of the global attractors instead of only a weak upper semicontinuity.

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