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On a conjecture by Pierre Cartier about a group of associators

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Abstract
In [8], Pierre Cartier conjectured that for any non commutative formal power series $\Phi$ on $X = \{x_0, x_1\}$ with coefficients in a $\mathbb{Q}$-extension, $A$, subjected to some suitable conditions, there exists an unique algebra homomorphism $\varphi$ from the $\mathbb{Q}$-algebra generated by the convergent polyzétas to $A$ such that $\Phi$ is computed from $\Phi_{KZ}$ Drinfel’d associator by applying $\varphi$ to each coefficient. We prove $\varphi$ exists and it is a free Lie exponential over $X$. Moreover, we give a complete description of the kernel of polyzéta and draw some consequences about a structure of the algebra of convergent polyzétas and about the arithmetical nature of the Euler constant.

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1 Introduction

1.1 Drinfel’d associator and polyzetas

In 1986, in order to study the linear representation of the braid group $B_n$ coming from the monodromy of the Knizhnik-Zamolodchikov differential equations over $\mathbb{C}^n = \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$ [14]:

$$dF(z) = \Omega_n(\mathbf{z}) F(z) \quad \text{with} \quad \Omega_n(z) = \frac{1}{2i\pi} \sum_{1 \leq i < j \leq n} t_{ij} \frac{d(z_i - z_j)}{z_i - z_j}, \quad (1)$$

and $\{t_{ij}\}_{i,j \geq 1}$ are noncommutative variables, Drinfel’d introduced a class of formal power series $\Phi$ on noncommutative variables over the finite alphabet $X = \{x_0, x_1\}$. Such a power series $\Phi$ is called an associator.
Since the system (1) is completely integrable then $d\Omega_n - \Omega_n \wedge \Omega_n = 0$ [7, 14]. It is equivalent to the fact the $\{t_{i,j}\}_{i,j \geq 1}$ satisfy the infinitesimal braid relations:

\begin{align}
t_{i,j} &= 0 \quad \text{for } i = j, \quad (2) \\
t_{i,j} &= t_{j,i} \quad \text{for } i \neq j, \quad (3) \\
[t_{i,j}, t_{i,k} + t_{j,k}] &= 0 \quad \text{for distinct } i,j,k, \quad (4) \\
[t_{i,j}, t_{k,l}] &= 0 \quad \text{for distinct } i,j,k,l. \quad (5)
\end{align}

Example 1. \quad • $\mathcal{T}_2 = \{t_{1,2}\}$.

\[
\Omega_2(z_1, z_2) = \frac{t_{1,2}}{2\pi i} \frac{d(z_1 - z_2)}{z_1 - z_2} \quad \text{with} \quad F(z_1, z_2) = (z_1 - z_2)^{t_{1,2}/2\pi i}.
\]

• $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$, $[t_{1,3}, t_{1,2} + t_{2,3}] = [t_{2,3}, t_{1,2} + t_{1,3}] = 0$.

\[
\Omega_3(z_1, z_2, z_3) = \frac{1}{2\pi i} \left[ t_{1,2} \frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3} \frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3} \frac{d(z_2 - z_3)}{z_2 - z_3} \right].
\]

\[
F(z_1, z_2, z_3) = G \left( \frac{z_1 - z_2}{z_1 - z_3} \right) (z_1 - z_3)^{(t_{1,2} + t_{1,3} + t_{2,3})/2\pi i},
\]

where $G$ satisfies the following fuchsian differential equation with three regular singularities at $0, 1$ and $\infty$:

\[
(\text{DE}) \quad dG(z) = [x_0 \omega_0(z) + x_1 \omega_1(z)]G(z),
\]

with

\[
x_0 := \frac{t_{1,2}}{2\pi i} \quad \text{and} \quad \omega_0(z) := \frac{dz}{z},
\]

\[
x_1 := -\frac{t_{2,3}}{2\pi i} \quad \text{and} \quad \omega_1(z) := \frac{dz}{1 - z}.
\]

As already shown by Drinfel’d, the equation $(\text{DE})$ admits, on the simply connected domain $\mathbb{C} - \{ -\infty, 0 \} \cup \{1, +\infty\}$, two specific solutions

\[
G_0(z) \overset{z \to 0}{\longrightarrow} \exp[x_0 \log(z)] \quad \text{and} \quad G_1(z) \overset{z \to 1}{\longrightarrow} \exp[-x_1 \log(1 - z)]. \quad (6)
\]

He also proved there exists the associator $\Phi_{KZ}$ such that $G_1^{-1}(z)G_0(z) = \Phi_{KZ}$.

After that, Lê and Murakami expressed the coefficients of the Drinfel’d associator $\Phi_{KZ}$ in terms of convergent polyzetas [22], i.e. for $r_1 > 1$,

\[
\zeta(r_1, \ldots, r_k) = \sum_{n_1 > \ldots > n_k > 0} \frac{1}{n_1^{r_1} \cdots n_k^{r_k}}. \quad (7)
\]

In [22], the authors also expressed the divergent coefficients as linear combinations of convergent polyzetas via a regularization process (see also [38]). This process is one of many ways to regularize the divergent terms.
1.2 Group of associators and regularized Chen generating series

The algebraic aspects of our regularization process based essentially on various products among polylog\(\zeta\)tas (see [33]) and its analytical aspects will be described, in Section 3.1, as the finite part, of the asymptotic expansions in different scales of comparison\(^2\) [5]. It will be seen also, in Section 3.2, as the action of the differential Galois group of the polylogarithms\(^3\) (recalled in Section 2.1.2)

\[\text{Li}_{r_1,\ldots,r_k}(z) = \sum_{n_1 > \ldots > n_k > 0} \frac{z^{n_1}}{n_1^{r_1} \ldots n_k^{r_k}}.\]  

(8)

on the asymptotic expansion of polylogarithms, at \(z = 1\) and in the comparison scale \([(1 - z)^a \log^b(1 - z)]_{a \in \mathbb{Z}, b \in \mathbb{N}}\), and the same action on the asymptotic expansions, at \(+\infty\) and in the comparison scales \(\{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}\) and \(\{n^a \varphi_1^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}\) of the harmonic sums (recalled in Section 2.1.1)

\[H_{r_1,\ldots,r_k}(N) = \sum_{n_1 > \ldots > n_k > 0} \frac{1}{n_1^{r_1} \ldots n_k^{r_k}}.\]

(9)

This action leads then to a conjecture by Pierre Cartier ([8], conjecture C3) and to the description of the group of associators yielding the ideal of polynomial relations among coefficients of associators (theorems 13 and 14). This group is fact, closely linked to the group of the Chen generating series studied by K.T. Chen to describe the solutions of differential equations [10] and it turns out that each associator regularizes a Chen generating series of the differential forms \(\omega_0\) and \(\omega_1\) along the integration path on the simply connected domain \(\mathbb{C} = (-\infty, 0] \cup [1, +\infty[)\).

1.3 Global renormalization and global regularization

In fact, our regularization process based essentially on two noncommutative generating series over \(Y = \{y_i\}_{i \geq 1}\), which encodes the multi-indices \((r_1,\ldots,r_k)\) by the words \(y_{r_1} \ldots y_{r_k}\) over the monoid generated by \(Y\), denoted by \(Y^*\), of polylogarithms and of harmonic sums (recalled in Section 2.2.1)

\[\Lambda(z) = \sum_{w \in Y^*} \text{Li}_w(z) \ w \ \text{and} \ H(N) = \sum_{w \in Y^*} H_w(N) \ w.\]

(10)

Through the algebraic combinatorial aspects\(^4\) [45] and the topological aspects [2] of formal power series in noncommutative variables, we have already

---

1First source of ambiguity leading to the problem of rewriting expressions of poly\(\zeta\)tas in a canonical form using irreducible Lyndon words (see [34, 32]).

2Second source of ambiguity leading to the problem to determine the value of regularized poly\(\zeta\)tas and its analytical meaning (see [33, 37]).

3Third source of ambiguity leading to the problem of fixing the integration path to solve \((DE)\) and its monodromy group (see [35]) or its differential Galois group (see [38]).

4See [45] to get an idea of these aspects of combinatorial Hopf algebra of the shuffle product,
showed the existence of noncommutative formal series over $Y$, $Z_1$ and $Z_2$ with constant terms, such that [39]

$$\lim_{z \to 1} \exp \left( y_1 \log \frac{1}{1-z} \right) \Lambda(z) = Z_1,$$  \hfill (11)

$$\lim_{N \to \infty} \exp \left( \sum_{k \geq 1} H_{y_k}(N) \frac{(-y_1)^k}{k} \right) H(N) = Z_2.$$  \hfill (12)

Moreover, $Z_1$ and $Z_2$ are equal and stand for the noncommutative generating series of all convergent polyzêtas $\{\zeta(w)\}_{w \in Y^* - y_1 Y^*}$ as shown by the factorized form indexed by Lyndon words (recalled in Section 2.2). This theorem enables, in particular, to explicit the counter-terms eliminating the divergence of the polylogarithms $\{\text{Li}_w(z)\}_{w \in Y^* Y^*}$, for $z \to 1$, and of the harmonic sums $\{H_w(N)\}_{w \in Y^* Y^*}$, for $N \to \infty$, and to calculate the Euler-Mac Laurin constants associated to the divergent polyzêtas $\{\zeta(w)\}_{w \in Y^* Y^*}$ (see Corollary 4). It allows also to give, in Section 3.3 and via identification of locale coordinates in infinite dimension, a complete description of the kernel by its generators, of the following algebra homomorphism$^5$

$$\zeta : (A \oplus (Y - y_1)A(Y), \mathfrak{w}) \longrightarrow (\mathbb{R},.)$$  \hfill (13)

$$y_{r_1} \ldots y_{r_k} \longmapsto \sum_{n_1 > \ldots > n_k > 0} \frac{1}{n_1 \ldots n_k},$$  \hfill (14)

and the set of $A$-irreducible polyzêtas forming a transcendence basis of the image of $\zeta$, with $A = \mathbb{Q}[\pi]$ (see Corollary 10).

Finally, via the indiscernability (recalled in Section 2.3) over the group of associators, this study makes precise the structure of the $A$-algebra generated by the convergent polyzêtas (see Theorem 19) and concludes the main challenge of the polynomial relations among polyzêtas indexed by convergent Lyndon words which are algebraically independent on the Euler constant and motivated [32, 34, 3, 49]. In particular, the $A$-algebra generated by the convergent polyzêtas was conjectured to be free [32, 34] and it will be proved, thanks to the propositions 15, 16 and 17. Moreover, this free $A$-algebra is graded by weight meaning there is no linear relation among convergent polyzêtas of different weight (see Theorem 19).

$^5$Here, $\epsilon$ stands for the empty word over $Y$. 

denoted by $\mathfrak{w}$, and its co-product, denoted by $\Delta_{\mathfrak{w}}$. For the quasi-shuffle product, denoted by $\otimes$, and its co-product, denoted by $\Delta_{\otimes}$, see Annexe A.

In our works, recalled in Annexe B, these algebraic combinatorial aspects were explored systematically to expand the outputs of nonlinear controlled dynamical system with singular inputs (Corollary 17) on polylogarithmic functional basis [26, 27, 31]. In this way [39], polyzêtas do appear then as fundamental arithmetical constant for the asymptotic analysis and for the renormalization of the outputs and their successive derivations (Corollary 18) via the extended Fliess fundamental formula (Theorem 23).
2 Background: structures and analytical studies of harmonic sums and of polylogarithms

2.1 Structures of harmonic sums and of polylogarithms

2.1.1 Quasi-symmetric functions and harmonic sums

Let \( \{ t_i \}_{i \in \mathbb{N}_+} \) be an infinite set of variables. The elementary symmetric functions \( \eta_k \) and the power sums \( \psi_k \) are defined by (see [45])

\[
\eta_k(t) = \sum_{n_1 > \ldots > n_k > 0} t_{n_1} \ldots t_{n_k} \quad \text{and} \quad \psi_k(t) = \sum_{n > 0} t_n^k.
\]

They are respectively coefficients of the following generating functions

\[
\eta(t | z) = \prod_{i \geq 1} (1 + t_i z) \quad \text{and} \quad \psi(t | z) = \sum_{i \geq 1} t_i z \frac{1}{1 - t_i z}.
\]

These generating functions satisfy a Newton identity

\[
z \frac{d}{dz} \log \eta(t | z) = \psi(t | -z). \tag{17}
\]

The fundamental theorem from symmetric functions theory asserts that \( \{ \eta_k \}_{k \geq 0} \) are linearly independent, and provides remarkable identities like (with \( \eta_0 = 1 \)):

\[
\eta_k = \frac{(-1)^k}{k!} \sum_{s_1, \ldots, s_k \geq 0, s_1 + \ldots + s_k = k+1} \left( -\psi_1 \right)^{s_1} \ldots \left( -\psi_k \right)^{s_k} \left( \frac{1}{s_k} \right). \tag{18}
\]

Let \( Y \) be the infinite alphabet \( \{ y_i \}_{i \geq 1} \) equipped with the order \( y_1 > y_2 > y_3 > \ldots \) and let \( \mathcal{L} Y \) be the set of Lyndon words over \( Y \). The length of \( w = y_{s_1} \ldots y_{s_r} \in \mathcal{L} Y \) is denoted by \( |w| \) and its degree equals to \( s_1 + \ldots + s_r \).

The quasi-symmetric function \( F_w \), of depth \( r = |w| \) and of degree (or weight) \( s_1 + \ldots + s_r \), is defined by

\[
F_w(t) = \sum_{n_1 > \ldots > n_r > 0} t_{n_1}^{s_1} \ldots t_{n_r}^{s_r}. \tag{19}
\]

In particular, \( F_y^k = \eta_k \) and \( F_{y_k} = \psi_k \). The functions \( \{ F_{y_i^j} \}_{k \geq 0} \) are linearly independent and integrating differential equation (17) shows that functions \( F_{y_i^j} \) and \( F_{y_k} \) are linked by the formula

\[
\sum_{k \geq 0} F_{y_i^j} z^k = \exp \left( - \sum_{k \geq 1} F_{y_k} \frac{(-z)^k}{k} \right). \tag{20}
\]

Every \( H_w(N) \) can be obtained by specializing, in the quasi-symmetric function \( F_w \), the variables \( \{ t_i \}_{i \geq 1} \) as follows [41]

\[
\forall N \geq i \geq 1, t_i = 1/i \quad \text{and} \quad \forall i > N, t_i = 0. \tag{21}
\]
In the same way, for \( w \in Y^* - y_1 Y^* \), the convergent polyzéta \( \zeta(w) \) can be obtained by specializing, in \( F_w \), the variables \( \{ t_i \}_{i \geq 1} \) as follows [41]

\[
\forall N \geq i \geq 1, \quad t_i = 1/i.
\]

The notation \( F_w \) is extended by linearity to all polynomials over \( Y \).

If \( u, v \in Y^* \), of length \( r, s \) and of weight \(^6p, q \) respectively, \( F_{u \# Y v} \) is a quasisymmetric function of depth \( r + s \) and of weight \( p + q \), and \( F_{u \# Y v} = F_u F_v \), where \( \# \) is the quasi-shuffle product (41). Hence,

\[
\forall u, v \in Y^* \Rightarrow \forall u, v \in Y^* - y_1 Y^*, \quad \zeta(u \# Y v) = \zeta(u) \zeta(v).
\]

Remarkable identity (18) can be then seen as

\[
y_i^k = \frac{(-1)^k}{k!} \sum_{s_1, \ldots, s_{k+1} \geq 0, \: s_1 + \cdots + s_{k+1} = k+1} \left( \frac{k!}{s_1! \cdots s_k!} (-y_1)^{s_1} \cdots (-y_k)^{s_k} \right) (1+z)^{-s_1} \cdots (1+z)^{-s_k}.
\]

### 2.1.2 Iterated integrals and polylogarithms

Let \( X \) be the finite alphabet \( \{ x_0, x_1 \} \) equipped with the order \( x_0 < x_1 \). Let

\[
\mathcal{C} := \mathbb{C} \left[ \frac{1}{z}, \frac{1}{1-z} \right] \quad \text{and} \quad \mathcal{G} := \left\{ \frac{1}{z}, \frac{z}{1-z}, \frac{1}{1-z}, 1-z \right\}.
\]

This ring \( \mathcal{C} \) is invariant under differentiation and under the homographic transformations belonging to the group \( \mathcal{G} \) whose elements commute the singularities \( \{0, 1, +\infty\} \).

The iterated integral over \( \omega_1, \omega_2 \) associated to the word \( w = x_{i_1} \cdots x_{i_k} \) over \( Y^* \) (the monoid generated by \( X \)) and along the integration path \( z_0 \Rightarrow z \) is the following multiple integral defined by

\[
\int_{z_0 \Rightarrow z} \omega_{i_1} \cdots \omega_{i_k} = \int_{z_0}^{z} \omega_{i_1} (t_1) \int_{z_0}^{t_1} \omega_{i_2} (t_2) \cdots \int_{z_0}^{t_{r-1}} \omega_{i_r} (t_r) \omega_{i_r} (t_r) \int_{z_0}^{t_{r-1}} \omega_{i_r} (t_r),
\]

where \( t_1 \cdots t_{r-1} \) is a subdivision of the path \( z_0 \Rightarrow z \). In a shortened notation, we denote this integral by \( \alpha^z_{x_0} (w) \) and \(^8\alpha^z_{x_0} (\varepsilon) = 1 \). One can check that the polylogarithm \( \text{Li}_{s_1, \ldots, s_r} \) is also the value of the iterated integral over \( \omega_0, \omega_1 \) and along the integration path \( 0 \Rightarrow z \) [29, 31] :

\[
\text{Li}_w (z) = \alpha^z_{x_0} (x^{s_1-1}_0 x_1 \cdots x^{s_r-1}_0 x_1).
\]

The definition of polylogarithms is extended over the words \( w \in Y^* \) by putting \( \text{Li}_{x_0} (z) := \log z \). The \( \{ \text{Li}_w \}_{w \in X^*} \) are \( \mathcal{C} \)-linearly independent [35, 32]. The

\(^6\)The weight is as in Equation (19).

\(^7\)See Annexe A, for the study of the Hopf algebra of \( \# \) which is not included in [45].

\(^8\)In all the sequel, we follow the notations of [2, 45].

\(^9\)Here, \( \varepsilon \) stands for the empty word over \( X \).
functions $P_w(z) := (1 - z)^{-1} \text{Li}_w(z)$, $w \in X^*$, are also $\mathbb{C}$-linearly independent. Since, for $w \in Y^*$, $P_w$ is the ordinary generating function of $\{H_w(N)\}_{N \geq 0}$ [37]:

$$P_w(z) = \sum_{N \geq 0} H_w(N) z^N$$  \hspace{1cm} (29)

then, as a consequence of the classical isomorphism between convergent Taylor series and their associated sums, the harmonic sums $\{H_w(N)\}_{w \in Y^*}$ are also $\mathbb{C}$-linearly independent. Firstly, $\ker P = \{0\}$ and $\ker H = \{0\}$, and secondly, $P$ is a morphism for the Hadamard product:

$$P_u(z) \odot P_v(z) = \sum_{N \geq 0} H_u(N) H_v(N) z^N = \sum_{N \geq 0} H_u \otimes H_v(N) z^N = P_u \otimes P_v(z).$$  \hspace{1cm} (30)

**Proposition 1** ([37]). Extended by linearity, the following maps are isomorphism of algebras

$$P : (\mathbb{C}(Y), \mathcal{W}) \rightarrow (\mathbb{C}[P_w]_{w \in Y^*}, \odot), \quad u \mapsto P_u,$$

$$H : (\mathbb{C}(Y), \mathcal{W}) \rightarrow (\mathbb{C}[H_w]_{w \in Y^*}), \quad u \mapsto H_u = \{H_u(N)\}_{N \geq 0}.$$  

Studying the equivalence between action of $\{(1 - z)^j\}_{j \in \mathbb{Z}}$ over $P_w(z)_{w \in Y^*}$ and that of $\{N^k\}_{k \in \mathbb{Z}}$ over $\{H_w(N)\}_{w \in Y^*}$ (see [12]), we have

**Theorem 1** ([39]). The Hadamard $\mathbb{C}$-algebra of $\{P_w\}_{w \in Y^*}$ can be identified with that of $\{P_l\}_{l \in \text{Lyon}Y}$. In the same way, the algebra of harmonic sums $\{H_w\}_{w \in Y^*}$ with polynomial coefficients can be identified with that of $\{H_l\}_{l \in \text{Lyon}Y}$.

By Identity (25) and by applying the isomorphism $H$ on the set of Lyndon words $\{y_t\}_{1 \leq t \leq k}$, we obtain $H_{y_t}$ as polynomials in $\{H_{y_t}\}_{1 \leq t \leq k}$ (which are algebraically independent), and

$$H_{y_t} = \sum_{s_1 + \ldots + s_k = k + 1} \frac{(-1)^k}{s_1! \ldots s_k!} \left( - \frac{H_{y_{s_1}}}{1} \right)^{s_1} \ldots \left( - \frac{H_{y_{s_k}}}{k} \right)^{s_k}.$$  \hspace{1cm} (31)

### 2.2 Results à la Abel for generating series of harmonic sums and of polylogarithms

#### 2.2.1 Generating series of harmonic sums and of polylogarithms

Let $H(N)$ be the noncommutative generating series of $\{H_w(N)\}_{w \in Y^*}$ [37]:

$$H(N) := \sum_{w \in Y^*} H_w(N) w.$$  \hspace{1cm} (32)

Let $\{\Sigma_w\}_{w \in Y^*}$, $\{\Sigma_l\}_{l \in \text{Lyon}Y}$ be respectively a PBW basis of the enveloping algebra $U(Lie_\mathbb{Q}(Y))$ and the quasi-shuffle algebra $(\mathbb{Q}(Y), \mathcal{W})$ (viewed as a $\mathbb{Q}$-module) on duality such that $\{\Sigma_l\}_{l \in \text{Lyon}X}$, $\{\Sigma_t\}_{l \in \text{Lyon}X}$ are respectively a basis of $Lie_\mathbb{Q}(Y)$ and a transcendence basis of $(\mathbb{Q}(Y), \mathcal{W})$ (see Annexe A).
Theorem 2 (Factorization of $H$). Let

$$H_{\text{reg}}(N) := \prod_{l \in \mathcal{L} \setminus \{y_1\}} e^{H_{\Sigma_l}(N) \Sigma_l}.$$ 

Then $H(N) = e^{H_{\Sigma_1}(N) y_1 H_{\text{reg}}(N)}$.

Proof. See Annexe A. \hfill \Box

For $l \in \mathcal{L} \setminus \{y_1\}$, the polynomial $\Sigma_l$ is a finite combination of words in $Y^* - y_1 Y^*$. Then we can state the following

Definition 1. We set $Z := H_{\text{reg}}(\infty)$.

The noncommutative generating series of polylogarithms \cite{35, 32}

$$L := \sum_{w \in X^*} \text{Li}_w w \tag{33}$$

satisfies Drinfel’d’s differential equation (DE) of Example 1

$$dL = (x_0 \omega_0 + x_1 \omega_1)L \tag{34}$$

with boundary condition \cite{15, 16}

$$L(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0^+} e^{x_0 \log \varepsilon}. \tag{35}$$

This enables us to prove that $L$ is the exponential of a Lie series\textsuperscript{10} \cite{35, 32}. Hence,

Proposition 2 (Logarithm of $L$, \cite{38}). Let $\pi_1(w)$ is the following Lie series

$$\pi_1(w) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in X^+} \langle w \mid u_1 \cdots u_k \rangle u_1 \cdots u_k. \tag{38}$$

Then

$$\log L(z) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in X^+} \text{Li}_{u_1 \cdots u_k}(z) u_1 \cdots u_k \tag{39}$$

$$= \sum_{w \in X^*} \text{Li}_w(z) \pi_1(w). \tag{40}$$

Applying a theorem of Ree \cite{44, 45}, $L$ satisfies Friedrichs criterion \cite{35, 32} :

$$\forall u, v \in X^*, \quad \text{Li}_{u \cdots v} = \text{Li}_u \text{Li}_v, \tag{36}$$

$$\Rightarrow \forall u, v \in x_0 X^* x_1, \quad \zeta(u \cdots v) = \zeta(u) \zeta(v). \tag{37}$$

\textsuperscript{10} i.e., $L$ is group-like for the co-product $\Delta_{\text{gg}}: \Delta_{\text{gg}}(L) = L \otimes L.$
Proposition 3 (Successive differentiation of $L$, [38]). For any $l \in \mathbb{N}$, let

$$P_l(z) = \sum_{w_{gt}(r)=l} \sum_{w \in X^{\deg(r)}} \prod_{j=1}^{\deg(r)} \left( \sum_{i=1}^{r_j} r_i + j - 1 \right) \tau_r(w) \in \mathcal{C}(X),$$

where, for any $w = x_{i_1} \cdots x_{i_k}$ and $r = (r_1, \ldots, r_k)$ of degree $\deg(r) = k$ and of weight $w_{gt}(r) = k + r_1 + \cdots + r_k$, the polynomial $\tau_r(w) = \tau_{r_1}(x_{i_1}) \cdots \tau_{r_k}(x_{i_k})$ is defined by

$$\forall r \in \mathbb{N}, \quad \tau_r(x_0) = \partial_r \frac{x_0}{z} = \frac{-r!x_0}{(-z)^{r+1}} \quad \text{and} \quad \tau_r(x_1) = \partial_r \frac{x_1}{1-z} = \frac{r!x_1}{(1-z)^{r+1}}.$$

Denoting $\partial = d/dz$, we have $\partial^l L(z) = P_l(z)L(z)$.

Let $\{\tilde{S}_l\}_{l \in \mathbb{Z}^{\text{ynX}}}$ be the transcendence basis of the shuffle algebra $(\mathbb{Q}(X), \shuffle)$ and $\{S_w\}_{w \in X^\ast}$ be the associated completed basis of the shuffle algebra $(\mathbb{Q}(X), \shuffle)$ (viewed as a $\mathbb{Q}$-module). They are defined as follows [45]

$$\tilde{S}_1 = 1 \quad \text{for } l = 1_X,$$

$$\tilde{S}_l = x \tilde{S}_u, \quad \text{for } l = xu \in \mathbb{Z}^{\text{ynX}},$$

$$\tilde{S}_w = \frac{\tilde{S}_{1_{l_1}^{i_1}}} {i_1!} \cdots \frac{\tilde{S}_{1_{l_k}^{i_k}}} {i_k!} \quad \text{for } w = l_1^{i_1} \cdots l_k^{i_k}, l_1 > \cdots > l_k. \quad (39)$$

Let $\{S_w\}_{w \in Y^\ast}$ be the PBW basis of the envelopping algebra $\mathcal{U}(\text{Lie}\mathbb{Q}(X))$ in duality with the basis $\{\tilde{S}_w\}_{w \in Y^\ast}$ and $\{\tilde{S}_l\}_{l \in \mathbb{Z}^{\text{ynX}}}$ is then the basis of the Lie algebra $\text{Lie}\mathbb{Q}(X)$ [45].

Theorem 3 (Factorization of $L$, [35, 32]). Let

$$L_{\text{reg}} := \prod_{l \in \mathbb{Z}^{\text{ynX}} - X} e^{1_{l_{i_1}}} \tilde{S}_i.$$

Then $L(z) = e^{-x_1 \log(1-z)} L_{\text{reg}}(z) e^{x_0 \log z}$.

For $l \in \mathbb{Z}^{\text{ynX}} - X$, the polynomial $S_l$ is a finite combination of words in $x_0 X^\ast x_1$. Then we can state the following

Definition 2 ([35, 32]). We set $Z_{\shuffle} := L_{\text{reg}}(1)$.

In the definitions 1 and 2 only \textit{convergent} polyzetas arise and these noncommutative generating series will induce, in Section 3.1, two algebra morphisms of regularization as shown in the theorems 8 and 9 respectively. Hence, these power series are quite different of those given in [22] or in [43] (the last is based on [4], see [8]) needing a regularization process.
2.2.2 Asymptotic expansions by noncommutative generating series and regularized Chen generating series

Let \( \rho_{1-z}, \rho_{1-z}^{-1} \) and \( \rho_z \) [36, 32] be three monoid morphisms verifying

\[
\rho_{1-z}(x_0) = -x_1 \quad \text{and} \quad \rho_{1-z}(x_1) = -x_0, \quad (41)
\]
\[
\rho_{1-1/z}(x_0) = -x_0 + x_1 \quad \text{and} \quad \rho_{1-1/z}(x_1) = -x_0, \quad (42)
\]
\[
\rho_{1/z}(x_0) = -x_0 + x_1 \quad \text{and} \quad \rho_{1/z}(x_1) = x_1. \quad (43)
\]

Using homographic transformations belonging to the group \( \mathcal{G} \), one has [36, 32]

\[
\begin{align*}
L(1 - z) &= e^{x_0 \log(1 - z)} \rho_{1-z}^{\mathit{reg}}(z) e^{-x_1 \log z} Z_{\omega z}, \\
L(1 - 1/z) &= e^{x_0 \log(1 - z)} \rho_{1-1/z}^{\mathit{reg}}(z) e^{-x_1 \log z} \rho_{1-1/z}^{-1}(z) e^{i\pi x_0} \\
L(1/z) &= e^{-x_1 \log(1 - z)} \rho_{1/z}^{\mathit{reg}}(z) e^{-x_0 + x_1} \log z \rho_{1/z}^{-1}(z) e^{i\pi x_1} Z_{\omega z}.
\end{align*}
\]

Thus, (35) and (44) yield [36, 32]

\[
L(z) \xrightarrow{z \to 0} \exp(x_0 \log z) \quad \text{and} \quad L(z) \xrightarrow{z \to 1} \exp(-x_1 \log(1 - z)) Z_{\omega z}. \quad (47)
\]

Let us call \( \mathbb{L}_C \) the smallest algebra containing \( C \), closed under derivation and under integration with respect to \( \omega_0 \) and \( \omega_1 \). It is the \( \mathcal{C} \)-module generated by the polylogarithms \( \{\mathbb{L}_w\}_{w \in X^*} \).

Let \( \pi_Y : \mathbb{L}_C \langle X \rangle \to \mathbb{L}_C \langle Y \rangle \) be a projector such that for any \( f \in \mathbb{L}_C \) and \( w \in X^* \), \( \pi_Y(f w x_0) = 0 \). Then [39]

\[
\Lambda(z) = \pi_Y L(z) \xrightarrow{z \to 1} \exp \left( y_1 \log \frac{1}{1 - z} \right) \pi_Y Z_{\omega z}. \quad (48)
\]

Since the coefficient of \( z^N \) in the ordinary Taylor expansion of \( P_{y_1}^t(N) \) is \( H_{y_1}^t(N) \) then let

\[
\begin{align*}
\text{Mono}(z) &:= e^{-(x_1+1) \log(1 - z)} = \sum_{k \geq 0} P_{y_1}^t(z) y_1^k, \\
\text{Const} &:= \sum_{k \geq 0} H_{y_1}^t y_1^k = \exp \left( - \sum_{k \geq 1} H_{y_1} \frac{(-y_1)^k}{k} \right).
\end{align*}
\]

Proposition 4 ([39]). We have

\[
\pi_Y P(z) \xrightarrow{z \to 1} \text{Mono}(z) \pi_Y Z_{\omega z} \quad \text{and} \quad H(N) \xrightarrow{N \to \infty} \text{Const}(N) \pi_Y Z_{\omega z}.
\]

Proof. Let \( \mu \) be the morphism verifying \( \mu(x_0) = x_1 \) and \( \mu(x_1) = x_0 \). Then, by Theorem 3, the noncommutative generating series of \( \{P_w\}_{w \in X^*} \) is given by

\[
\begin{align*}
P(z) &= (1 - z)^{-1} L(z) \\
&= e^{(x_1+1) \log(1 - z)} \rho_{1-z}^{\mathit{reg}}(z) e^{x_0 \log z} \\
&= e^{x_0 \log z} \mu^{\mathit{reg}}(1 - z) e^{(x_1+1) \log(1 - z)} Z_{\omega z} \\
&= e^{x_0 \log z} \mu^{\mathit{reg}}(1 - z) \text{Mono}(z) Z_{\omega z}.
\end{align*}
\]

Thus, \( P(z) \xrightarrow{z \to 0} e^{x_0 \log z} \) and \( P(z) \xrightarrow{z \to 1} \text{Mono}(z) Z_{\omega z} \), yielding the expected results.
As consequence of (48)-(50) and of Proposition 4, one gets

**Theorem 4** (*â la Abel, [39]).

\[
\lim_{z \to 1} \exp\left( \frac{1}{1-z} \right) \Lambda(z) = \lim_{N \to \infty} \exp \left( \sum_{k \geq 1} H_{y_k}(N) \frac{(-y_1)^k}{k} \right) H(N) = \pi Y Z_{\omega}. 
\]

Therefore, the knowledge of the ordinary Taylor expansion at 0 of the polylogarithmic functions \( \{P_w(1-z)\}_{w \in X^*} \) gives

**Theorem 5** ([12]). For all \( g \in C\{P_w\}_{w \in Y^*} \), there exists algorithmically computable \( c_j \in \mathbb{Z}, \alpha_j \in \mathbb{Z}, \beta_j \in \mathbb{N} \) and \( b_i \in \mathbb{Z}, \kappa_i \in \mathbb{N}, \eta_i \in \mathbb{Z} \) such that

\[
g(z) \sim +\infty \sum_{j=0}^{+\infty} c_j (1-z)^{\alpha_j} \log^{\beta_j}(1-z) \quad \text{and} \quad [z^n]g(z) \sim +\infty \sum_{i=0}^{+\infty} b_i n^\eta_i \log^{\kappa_i}(n). 
\]

**Definition 3.** Let \( Z \) be the \( \mathbb{Q} \)-algebra generated by convergent polyzetas and let \( Z' \) be the\(^{11}\) \( \mathbb{Q}[\gamma] \)-algebra generated by \( Z \).

**Corollary 1** ([12]). There exists algorithmically computable \( c_j \in \mathbb{Z}, \alpha_j \in \mathbb{Z}, \beta_j \in \mathbb{N} \) and \( b_i \in \mathbb{Z}, \kappa_i \in \mathbb{N}, \eta_i \in \mathbb{Z} \) such that

\[
\forall w \in Y^*, \quad P_w(z) \sim +\infty \sum_{j=0}^{+\infty} c_j (1-z)^{\alpha_j} \log^{\beta_j}(1-z) \quad \text{for} \quad z \to 1, \\
\forall w \in Y^*, \quad H_w(N) \sim +\infty \sum_{i=0}^{+\infty} b_i N^\eta_i \log^{\kappa_i}(N) \quad \text{for} \quad N \to +\infty. 
\]

The Chen generating series along the path \( z_0 \leadsto z \), associated to \( \omega_0, \omega_1 \) is the following

\[
S_{z_0 \leadsto z} := \sum_{w \in X^*} \langle S | w \rangle \quad \text{with} \quad \langle S | w \rangle = \alpha_{z_0}^w(w) \quad (51)
\]

which solves the differential equation (34) with the initial condition \( S_{z_0 \leadsto z_0} = 1 \). Thus, \( S_{z_0 \leadsto z} \) and \( L(z)L(z_0)^{-1} \) satisfy the same differential equation taking the same value at \( z_0 \) and

\[
S_{z_0 \leadsto z} = L(z)L(z_0)^{-1}. \quad (52)
\]

Any Chen generating series \( S_{z_0 \leadsto z} \) is group like [44] and depends only on the homotopy class of \( z_0 \leadsto z \) [10]. The product of \( S_{z_1 \leadsto z_2} \) and \( S_{z_0 \leadsto z_1} \) is the Chen generating series

\[
S_{z_0 \leadsto z_2} = S_{z_1 \leadsto z_2} S_{z_0 \leadsto z_1}. \quad (53)
\]

\(^{11}\) Here, \( \gamma \) stands for the Euler constant \( \gamma = .57721566490153286060651209008240243 \ldots \).
Proposition 6. Let \( \varepsilon \in ]0, 1[ \) and let \( z_i = \varepsilon \exp(i\theta_i) \), for \( i = 0 \) or 1. We set \( \theta = \theta_1 - \theta_0 \). Let \( \Gamma_0(\varepsilon, \theta) \) (resp. \( \Gamma_1(\varepsilon, \theta) \)) be the path turning around 0 (resp. 1) in the positive direction from \( z_0 \) to \( z_1 \). By induction on the length of \( w \), one has

\[
| \langle \mathcal{S}_{\Gamma_i(\varepsilon, \theta)} \mid w \rangle | = (2\varepsilon)^{|w|/2} / |w|!
\]

where, \( |w| \) denotes the length of \( w \) and \( |w|_{x_i} \) denotes the number of occurrences of letter \( x_i \) in \( w \), for \( i = 0, 1 \). For \( \varepsilon \to 0^+ \), these estimations yield

\[
\mathcal{S}_{\Gamma_i(\varepsilon, \theta)} = e^{i\theta x_i} + o(\varepsilon).
\]

In particular, if \( \Gamma_0(\varepsilon) \) (resp. \( \Gamma_1(\varepsilon) \)) is a circular path of radius \( \varepsilon \) turning around 0 (resp. 1) in the positive direction, starting at \( z = \varepsilon \) (resp. \( 1 - \varepsilon \)), then, by the noncommutative residu theorem [35, 32], we get

\[
\mathcal{S}_{\Gamma_0(\varepsilon)} = e^{2i\pi x_0} + o(\varepsilon) \quad \text{and} \quad \mathcal{S}_{\Gamma_1(\varepsilon)} = e^{-2i\pi x_1} + o(\varepsilon).
\]

Finally, the asymptotic behaviors of \( L \) on (47) give

**Proposition 5** ([32, 35]). We have \( \mathcal{S}_{\Gamma_{-1-\varepsilon}} = -e^{-x_1 \log \varepsilon} e^{-x_0 \log \varepsilon} \).

In other terms, \( Z_{\underline{\mu}} \) is the regularized Chen generating series \( \mathcal{S}_{\Gamma_{-1-\varepsilon}} \) of differential forms \( \omega_0 \) and \( \omega_1 : Z_{\underline{\mu}} \) is the noncommutative generating series of the finite parts of the coefficients of the Chen generating series \( e^{x_1 \log \varepsilon} \mathcal{S}_{\Gamma_{-1-\varepsilon}} e^{x_0 \log \varepsilon} \); the concatenation of \( e^{x_0 \log \varepsilon} \) and then \( \mathcal{S}_{\Gamma_{-1-\varepsilon}} \) and finally, \( e^{x_1 \log \varepsilon} \).

**Proposition 6.** Let \( \rho_{1-1/z} \) be the morphism is given in Section 2.2.2. We have

\[
\prod_{i \in \mathbb{R}_+ \times X} e^{\zeta(i)l} = e^{\pi x_0} \prod_{i \in \mathbb{R}_+ \times X} e^{\zeta(i)\rho_{1-1/z}(l)} e^{i\pi (x_0 + x_1)} \prod_{i \in \mathbb{R}_+ \times X} e^{\zeta(i)\rho_{1-1/z}(l)} e^{-i\pi x_1}.
\]

**Proof.** Following the hexagonal path given in Figure 1, one has [36, 32]

\[
(S_{\varepsilon \to \varepsilon} e^{i\pi x_0}) \rho_{1-1/z}(S_{\varepsilon \to \varepsilon} e^{i\pi x_0}) \rho_{1-1/z}^2(S_{\varepsilon \to \varepsilon} e^{i\pi x_0}) = 1 + O(\sqrt{\varepsilon}).
\]

By Proposition 5, it follows the hexagonal relation [15, 16, 36, 32] which is

\[
Z_{\underline{\mu}} e^{i\pi x_0} \rho_{1-1/z}(Z_{\underline{\mu}}) e^{i\pi (x_0 + x_1)} \rho_{1-1/z}^2(Z_{\underline{\mu}}) e^{-i\pi x_1} = 1,
\]

\[
\iff e^{i\pi x_0} \rho_{1-1/z}(Z_{\underline{\mu}}) e^{i\pi (x_0 + x_1)} \rho_{1-1/z}^2(Z_{\underline{\mu}}) e^{-i\pi x_1} = Z_{\underline{\mu}}^{-1}.
\]

It follows then the expected result.

\[
\square
\]
2.3 Indiscernability over a class of formal power series

2.3.1 Residual calculus and representative series

**Definition 4.** Let $S \in \mathbb{Q}[\langle X \rangle]$ and let $P \in \mathbb{Q}[X]$. The **left residual** (resp. **right residual**) of $S$ by $P$, is the formal power series $P \triangleleft S$ (resp. $S \triangleright P$) in $\mathbb{Q}[\langle X \rangle]$ defined by:

$$\langle P \triangleleft S \mid w \rangle = \langle S \mid wP \rangle \quad (\text{resp. } \langle S \triangleright P \mid w \rangle = \langle S \mid PW \rangle).$$

We straightforwardly get, for any $P, Q \in \mathbb{Q}[X]$:

$$P \triangleleft (Q \triangleright S) = PQ \triangleleft S,$$

$$(S \triangleright P) \triangleright Q = S \triangleright PQ,$$

$$(P \triangleleft S) \triangleright Q = P \triangleleft (S \triangleright Q). \quad (57)$$

In case $x, y \in X$ and $w \in X^*$, we get $x \triangleleft (wy) = \delta_{x,y}w$ and $xw \triangleright y = \delta_{x,y}w$.

**Lemma 1** (Reconstruction lemma). Let $S \in \mathbb{Q}[\langle X \rangle]$. Then

$$S = \langle S \mid e \rangle + \sum_{x \in X} x(S \triangleright x) = \langle S \mid e \rangle + \sum_{x \in X} (x \triangleleft S)x.$$

**Lemma 2.** The left and right residuals by a letter $x$ are derivations in $(\mathbb{Q}[\langle X \rangle], \shuffle)$:

$$x \triangleleft (u \shuffle v) = (x \triangleleft u) \shuffle v + u \shuffle (x \triangleleft v), \quad (u \shuffle v) \triangleright x = (u \triangleright x) \shuffle v + u \shuffle (v \triangleright x).$$

**Proof.** Use the recursive definitions of the shuffle product. \(\square\)

**Lemma 3.** For any Lie polynomial $Q \in \text{Lie}_\mathbb{Q}(X)$, the linear maps “$Q \triangleleft$” and “$\triangleright Q$” are derivations on $(\mathbb{Q}[\text{Lyn}X], \shuffle)$.

**Proof.** For any $l, l_1, l_2 \in \text{Lyn}X$, we have

$$\hat{l} \triangleleft (l_1 \shuffle l_2) = l_1 \shuffle (\hat{l} \triangleleft l_2) + (\hat{l} \triangleleft l_1) \shuffle l_2 = l_1 \delta_{l_2 \triangleleft l} + \delta_{l_1 \triangleright l_2},$$
$$\quad (l_1 \shuffle l_2) \triangleright \hat{l} = l_1 \shuffle (l_2 \triangleright \hat{l}) + (l_1 \triangleright \hat{l}) \shuffle l_2 = l_1 \delta_{l_2 \triangleright l} + \delta_{l_1 \triangleleft l_2}.$$ \(\square\)

**Lemma 4.** For any Lyndon word $l \in \text{Lyn}X$ and $\hat{S}_l$ defined as in (39), one has

$$x_1 \triangleleft l \triangleright x_0 = 0 \quad \text{and} \quad x_1 \triangleleft \hat{S}_l = \hat{S}_l \triangleright x_0 = 0.$$

**Proof.** Since $x_1 \triangleleft$ and $\triangleright x_0$ are derivations and for any $l \in \text{Lyn}X - X$, the polynomial $\hat{S}_l$ belongs to $x_0 \mathbb{Q}(X)x_1$ then it follows the expected results. \(\square\)

**Theorem 6** (On representative series). The following properties are equivalent for any series $S \in \mathbb{Q}[\langle X \rangle]$:

1. The left $\mathbb{C}$-module $\text{Res}_g(S) = \text{span}\{w \triangleleft S \mid w \in X^*\}$ is finite dimensional.
2. The right $\mathbb{C}$-module $\text{Res}_g(S) = \text{span}\{S \triangleright w \mid w \in X^*\}$ is finite dimensional.
3. There are matrices $\lambda \in M_{1,n}(\mathbb{Q})$, $\eta \in M_{n,1}(\mathbb{Q})$ and a representation of $X^*$ in $M_{n,n}$, such that

$$S = \sum_{w \in X^*} [\lambda \mu] w = \lambda \left( \prod_{l \in \gamma} e^{\mu(S_l) \delta_l} \right) \eta.$$ 

A series that satisfies the items of this theorem will be called representative series. This concept can be found in [1, 42, 13]. The two first items are in [18, 24]. The third item can be deduced from [9, 11] for example and it was used to factorize first time, by Lyndon words, the output of bilinear and analytical dynamical systems respectively in [26, 27] and to study polylogarithms, hypergeometric functions and associated functions in [29, 31, 38]. The dimension of $\text{Res}_{\mathbb{Q}}(S)$ is equal to that of $\text{Res}_{\mathbb{Q}}(S)$, and to the minimal dimension of a representation satisfying the third point of Theorem 6. This rank is then equal to the rank of the Hankel matrix of $S$, that is the infinite matrix $\langle S | uv \rangle_{u,v \in X}$ indexed by $X^* \times X^*$ and is also called Hankel rank of $S$ [18, 24]:

**Definition 5** ([18, 24]). The Hankel rank of a formal power series $S \in \mathbb{C} \langle \langle X \rangle \rangle$ is the dimension of the vector space

$$\{ S \triangleright \Pi \mid \Pi \in \mathbb{C}(X) \}, \quad \text{(resp. } \{ \Pi \triangleleft S \mid \Pi \in \mathbb{C}(X) \}).$$

The triplet $(\lambda, \mu, \eta)$ is called a linear representation of $S$. We define the minimal representation$^{12}$ of $S$ as being a representation of $S$ of minimal dimension.

For any proper series $S$, the following power series is called “star of $S$”

$$S^* = 1 + S + S^2 + \ldots + S^n + \ldots.$$  \hfill (58)

**Definition 6** ([2, 47]). A series $S$ is called rational if it belongs to the closure in $\mathbb{Q} \langle \langle X \rangle \rangle$ of the noncommutative polynomial algebra by sum, product and star operation of proper$^{13}$ elements. The set of rational power series will be denoted by $\mathbb{Q}^{\text{rat}} \langle \langle X \rangle \rangle$.

**Lemma 5.** For any noncommutative rational series (resp. polynomial) $R$ and for any polynomial $P$, the left and right residuals of $R$ by $P$ are rational (resp. polynomial).

**Theorem 7** (Schützenberger, [2, 47]). Any noncommutative power series is representative if and only if it is rational.

### 2.3.2 Continuity and indiscernability

**Definition 7** ([25, 39]). Let $\mathcal{H}$ be a class of formal power series over $X$ and let $S \in \mathbb{C} \langle \langle X \rangle \rangle$.

$^{12}$ It can be shown that all minimal representations are isomorphics (see [2]).

$^{13}$ A series $S$ is said to be proper if $\langle S | e \rangle = 0$. 
1. $S$ is said to be continuous\textsuperscript{14} over $\mathcal{H}$ if for any $\Phi \in \mathcal{H}$, the following sum, denoted $\langle S \parallel \Phi \rangle$, is convergent in norm

$$\sum_{w \in X^*} \langle S \mid w \rangle \langle \Phi \mid w \rangle.$$

The set of continuous power series over $\mathcal{H}$ will be denoted by $\mathbb{C}^{\text{cont}}\langle \langle X \rangle \rangle$.

2. $S$ is said to be indiscernable\textsuperscript{15} over $\mathcal{H}$ if and only if

$$\forall \Phi \in \mathcal{H}, \quad \langle S \mid \Phi \rangle = 0.$$

Let $\rho$ be the monoid morphism verifying $\rho(x_0) = x_1$ and $\rho(x_1) = x_0$ and let $\tilde{w} = \rho(\bar{w})$, where $\bar{w}$ is the mirror of $w$.

**Lemma 6.** Let $S \in \mathbb{C}^{\text{cont}}\langle \langle X \rangle \rangle$. If $\langle S \parallel Z_{\sqcup \sqcup} \rangle = 0$ then $\langle \hat{S} \mid Z_{\sqcup \sqcup} \rangle = 0$, where

$$\hat{S} := \sum_{w \in X^*} \langle S \mid w \rangle \tilde{w}.$$

**Proof.** For any $w \in x_0 X^* x_1$, by “duality relation”, one has (see [40, 50, 36])

$$\zeta(\bar{w}) = \zeta(w), \quad \text{or equivalently} \quad Z_{\sqcup \sqcup} = \hat{Z}_{\sqcup \sqcup} := \sum_{w \in X^*} \langle Z_{\sqcup \sqcup} \mid w \rangle \tilde{w}.$$

Using the fact

$$\langle \hat{S} \mid Z_{\sqcup \sqcup} \rangle = \sum_{\bar{w} \in X^*} \langle S \mid \bar{w} \rangle \langle Z_{\sqcup \sqcup} \mid \bar{w} \rangle = \sum_{w \in X^*} \langle S \mid w \rangle \langle Z_{\sqcup \sqcup} \mid w \rangle,$$

one gets finally the expected result. \qed

**Lemma 7.** Let $\mathcal{H}$ be a monoid containing $\{e^{tx} : t \in \mathbb{C}\}$. Let $S \in \mathbb{C}^{\text{cont}}\langle \langle X \rangle \rangle$ being indiscernable over $\mathcal{H}$. Then for any $x \in X$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbb{C}^{\text{cont}}\langle \langle X \rangle \rangle$ and they are indiscernable over $\mathcal{H}$.

**Proof.** Let us calculate $\langle x \triangleleft S \mid \Phi \rangle = \langle S \mid \Phi x \rangle$ and $\langle S \triangleright x \mid \Phi \rangle = \langle S \mid x \Phi \rangle$. Since

$$\lim_{t \to 0} e^{tx} - 1 = x \quad \text{and} \quad \lim_{t \to 0} \frac{e^{tx} - 1}{t} = x,$$

then, for any $\Phi \in \mathcal{H}$, by uniform convergence, one has

$$\langle S \mid \Phi x \rangle = \langle S \mid \lim_{t \to 0} \Phi \frac{e^{tx} - 1}{t} \rangle = \lim_{t \to 0} \langle S \mid \Phi \frac{e^{tx} - 1}{t} \rangle,$$

$$\langle S \mid x \Phi \rangle = \langle S \mid \lim_{t \to 0} \frac{e^{tx} - 1}{t} \Phi \rangle = \lim_{t \to 0} \langle S \mid \frac{e^{tx} - 1}{t} \Phi \rangle.$$

\textsuperscript{14}See [25, 39], for a convergence criterion and an example of continuous generating series.

\textsuperscript{15}Here, we adapt this notion developed in [25] via the residual calculus.
Since $S$ is indiscernable over $\mathcal{H}$ then
\[
\langle S \parallel \Phi x \rangle = \lim_{t \to 0} \frac{1}{t} \langle S \parallel \Phi e^{tx} \rangle - \lim_{t \to 0} \frac{1}{t} \langle S \parallel \Phi \rangle = 0,
\]
\[
\langle S \parallel x \Phi \rangle = \lim_{t \to 0} \frac{1}{t} \langle S \parallel e^{tx} \Phi \rangle - \lim_{t \to 0} \frac{1}{t} \langle S \parallel \Phi \rangle = 0.
\]

**Proposition 7.** Let $\mathcal{H}$ be a monoid containing $\{e^{tx}\}_{t \in \mathbb{C}}$. The formal power series $S \in \mathbb{C}^{\langle \langle X \rangle \rangle}$ is indiscernable over $\mathcal{H}$ if and only if $S = 0$.

**Proof.** If $S = 0$ then it is immediate that $S$ is indiscernable over $\mathcal{H}$. Conversely, if $S$ is indiscernable over $\mathcal{H}$ then by Lemma 7, for any word $w \in X^*$, by induction on the length of $w$, $w \triangleleft S$ is indiscernable over $\mathcal{H}$ and then in particular,
\[
\langle w \triangleleft S \parallel \text{Id}_{\mathcal{H}} \rangle = \langle S \mid w \rangle = 0.
\]
In other words, $S = 0$. □

### 3 Group of associators : polynomial relations among convergent polyzêtas and identification of local coordinates

#### 3.1 Generalized Euler constants and global regularization of polyzêtas

**Theorem 8 ([33]).** Let $\zeta : (\mathbb{Q} \langle Y \rangle, \triangleright \triangleright) \to (\mathbb{R}, \cdot)$ be the morphism verifying the following properties
\[
\begin{align*}
&\text{for } u, v \in Y^*, \zeta(u \triangleright v) = \zeta(u)\zeta(v), \\
&\text{for all convergent word } w \in Y^* - y_1Y^*, \zeta(w) = \zeta(w).
\end{align*}
\]
Then
\[
\sum_{w \in X^*} \zeta(w) \cdot w = Z_{\triangleright \triangleright}.
\]

**Corollary 2 ([33]).** For any $w \in X^*$, $\zeta(w)$ belongs to the algebra $\mathbb{Z}$.

**Theorem 9 ([33]).** Let $\zeta : (\mathbb{Q} \langle X \rangle, \triangleright \triangleright) \to (\mathbb{R}, \cdot)$ be the morphism verifying the following properties
\[
\begin{align*}
&\text{for } u, v \in X^*, \zeta(u \triangleright v) = \zeta(u)\zeta(v),
\end{align*}
\]
• for all convergent word \( w \in x_0X^*x_1, \zeta_{\pm}(w) = \zeta(w) \),
• \( \zeta_{\pm}(x_0) = \zeta_{\pm}(x_1) = 0 \).

Then

\[
\sum_{w \in X^*} \zeta_{\pm}(w) w = Z_{\pm}.
\]

**Corollary 3 ([33]).** For any \( w \in Y^* \), \( \zeta_{\pm}(w) \) belongs to the algebra \( Z \).

**Definition 8.** For any \( w \in Y^* \), let \( \gamma_w \) be the constant part\(^{16}\) of the asymptotic expansion, on the comparison scale \( \{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}} \), of \( H_w(n) \).

Let \( Z_\gamma \) be the noncommutative generating series of \( \{\gamma_w\}_{w \in Y^*} : \)

\[
Z_\gamma := \sum_{w \in Y^*} \gamma_w w.
\]

**Definition 9.** We set

\[
B(y_1) := \exp(-\sum_{k \geq 1} \gamma_{y_1} (-y_1)^k/k) \quad \text{and} \quad B'(y_1) := e^{-\gamma y_1} B(y_1).
\]

The power series \( B'(y_1) \) corresponds in fact to the mould\(^{17}\) Mono in [17] and to the \( \Phi_{\text{corr}} \) in [43] (see also [4, 8]). While the power series \( B(y_1) \) corresponds to the Gamma Euler function with its product expansion,

\[
B(y_1) = \Gamma(y_1 + 1), \quad \frac{1}{\Gamma(y_1 + 1)} = e^{\gamma y_1} \prod_{n \geq 1} \left(1 + \frac{y_1}{n}\right) e^{-\gamma/n}.
\]

**Lemma 8 ([39]).** Let \( b_{n,k}(t_1, \ldots, t_{n-k+1}) \) be the (exponential) partial Bell polynomials in the variables \( \{t_l\}_{l \geq 1} \) given by the exponential generating series

\[
\exp\left(u \sum_{l=0}^{\infty} \frac{t_l}{l!}\right) = \sum_{n,k=0}^{\infty} b_{n,k}(t_1, \ldots, t_{n-k+1}) \frac{u^n}{n!}.
\]

For any \( m \geq 1 \), let \( t_m = (-1)^m (m-1)! \gamma_{y_1} \). Then

\[
B(y_1) = 1 + \sum_{n \geq 1} \left(\sum_{k=1}^{n} b_{n,k}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right) \frac{(-y_1)^n}{n!}.
\]

Since the ordinary generating series of the finite parts of coefficients of \( \text{Const}(N) \) is nothing else but the power series \( B(y_1) \), taking the constant part on either side of \( H(N) \) \( \overset{N \to \infty}{\longrightarrow} \text{Const}(N) \pi Y Z_{\pm} \) (see Proposition 4), yields

**Theorem 10 ([39]).** We have \( Z_\gamma = B(y_1) \pi Y Z_{\pm} \).

\(^{16}\) i.e. \( \gamma_w \) is the Euler-Mac Laurin constant of \( H_w(n) \).
\(^{17}\) The readers can see why we have introduced the power series Mono\((z)\) in Proposition 4.
Identifying the coefficients of \( y_t^k w \) on either side using the identity\(^{18} \) [33]

\[
\forall u \in X^* x_1, \quad x_t^k x_0 u = \sum_{l=0}^{k} \frac{x_1^l u}{l!} (x_0 (-x_1)^{k-l} \omega u)
\]

(60)

and applying the morphism \( \zeta_\omega \) given in Theorem 9, we get [33]

\[
\forall u \in X^* x_1, \quad \zeta_\omega (x_t^k x_0 u) = \zeta(x_0 (-x_1)^k \omega u).
\]

(61)

**Corollary 4** ([39]). For \( w \in x_0 X^* x_1 \), i.e. \( w = x_0 u \) and \( \pi_Y w \in Y^* - y_1 Y^* \), and for \( k \geq 0 \), the constant \( \gamma_{\omega} (x_t^k w) \) associated to the divergent polyzeta \( \zeta(x_t^k w) \) is a polynomial of degree \( k \) in \( \gamma \) and with coefficients in \( Z \):

\[
\gamma_{x_t^k w} = \sum_{l=0}^{k} \frac{\zeta(x_0 (-x_1)^{k-l} \omega u)}{l!} \left( \sum_{j=1}^{k} b_{l,j} (\gamma, -\zeta(2), 2\zeta(3), \ldots) \right).
\]

Moreover, for \( l = 0, \ldots, k \), the coefficient of \( \gamma^l \) is of weight \( |w| + k - l \).

In particular, for \( s > 1 \), the constant \( \gamma_{y_1 y_s} \) associated to \( \zeta(y_1 y_s) \) is linear in \( \gamma \) and with coefficients in \( \mathbb{Q}[\zeta(2), \zeta(2i+1)]_{0 < i \leq (s-1)/2} \).

**Corollary 5** ([39]). The constant \( \gamma_{x_t^k} \) associated to the divergent polyzeta \( \zeta(x_t^k) \) is a polynomial of degree \( k \) in \( \gamma \) with coefficients in \( \mathbb{Q}[\zeta(2), \zeta(2i+1)]_{0 < i \leq (k-1)/2} \):

\[
\gamma_{x_t^k} = \sum_{s_1 + \cdots + s_k = k} \frac{(-1)^k}{s_1! \cdots s_k!} (-\gamma)^{s_1} \left( -\frac{\zeta(2)}{2} \right)^{s_2} \cdots \left( -\frac{\zeta(k)}{k} \right)^{s_k}.
\]

Moreover, for \( l = 0, \ldots, k \), the coefficient of \( \gamma^l \) is of weight \( k - l \).

We thereby obtain the following algebra morphism, denoted by \( \gamma_\circ \), for the regularization to \( \gamma \) with respect to the quasi-shuffle product independently to the regularization with respect to the shuffle product\(^{19} \) and then by applying the tensor product of morphisms \( \gamma_\circ \otimes \text{Id} \) on the diagonal series, over \( Y \), we get (see Annexe A)

**Theorem 11.** The mapping \( \gamma_\circ \) realizes the morphism from \( (\mathbb{Q}[Y], \omega) \) to \( (\mathbb{R}, \cdot) \) verifying the following properties

\(^{18}\)By the Convolution Theorem [28], this is equivalent to

\[
\forall u \in X^*, \quad \alpha_0^k(x_t^k x_0 u) = \int_0^s \frac{[\log(1-s) - \log(1-z)]^k}{k!} \alpha_0^k(u) \frac{ds}{s}
\]

\[
= \sum_{l=0}^{k} \frac{(-\log(1-z))^l}{l!} \int_0^s \frac{\log^k(1-s)}{(k-l)!} \alpha_0^k(u) \frac{ds}{s}
\]

This theorem induces de facto the algebra morphism of regularization to 0 with respect to the shuffle product, as shown the Theorem 9.

\(^{19}\)In [4, 8, 21, 48], the authors suggest the simultaneous regularizations, with respect to the shuffle and the quasi-shuffle product, to \( T \) and then to set \( T = 0 \).
• for any word $u, v \in Y^*$, $\gamma_u \smalldot v = \gamma_u \gamma v$,
• for any convergent word $w \in Y^* - y_1Y^*$, $\gamma_w = \zeta(w)$

Then $Z_\gamma = e^{\gamma y_1} Z_Y$.

### 3.1.2 Identities of noncommutative generating series of polyzêtas

**Corollary 6.** With the notations of Definition 9, we have

$$Z_\gamma = B(y_1)\pi_Y Z_{\omega_\gamma} \iff Z_{\delta} = B'(y_1)\pi_Y Z_{\omega_\gamma},$$

$$\pi_Y Z_{\omega_\gamma} = B^{-1}(x_1)Z_\gamma \iff Z_{\omega_\gamma} = B^{-1}(x_1)\pi_X Z_{\delta}.$$ 

Roughly speaking, for the quasi-shuffle product, the regularization to $\gamma$ is “equivalent” to the regularization to 0.

Note also that the constant $\gamma_{y_1} = \gamma$ is obtained as the finite part of the asymptotic expansion of $H_1(n)$ in the comparison scale $\{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$.

In the same way, since $n$ and $H_1(n)$ are algebraically independent, as arithmetical functions (see Proposition 1), then $\{n^a H^b_1(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$ constitutes a new comparison scale for asymptotic expansions.

Hence, the constants $\zeta_{\omega_\gamma}(x_1) = 0$ and $\zeta_{\delta}(y_1) = 0$ can be interpreted as the finite part of the asymptotic expansions of $L_1(z)$ and $H_1(n)$ respectively in the comparison scales $\{(1-z)^a \log(1-z)^b\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$ and $\{n^a \log^b_1(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$.

**Definition 10 ([33]).** Let $C_1 := \mathbb{Q} \oplus \mathbb{Q}(X)x_1, C_2 := \mathbb{Q} \oplus (Y - \{y_1\})\mathbb{Q}(Y)$.

**Lemma 9 ([32, 33]).** We get $(C_1, \omega_\gamma) \cong (C_2, \delta)$.

Using a theorem of Radford [45] and its analogous over $Y$ (see Annexe A), we get

**Proposition 8 ([32, 33]).**

$$\begin{align*}
(Q(X), \omega_\gamma) & \cong (Q[\mathcal{L}ynX], \omega_\gamma) = C_1[x_0, x_1], \\
(Q(Y), \delta) & \cong (Q[\mathcal{L}ynY], \delta) = C_2[y_1].
\end{align*}$$

This insures the effective way to get the finite part of the asymptotic expansions, in the comparison scales $\{(1-z)^a \log(1-z)^b\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$ and $\{n^a \log^b_1(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$, of $\{L_{\omega_\gamma}(z)\}_{w \in Y^*}$ and $\{H_{\omega_\gamma}(N)\}_{w \in Y^*}$ respectively.

**Proposition 9 ([32, 33]).** The restrictions of $\zeta_{\omega_\gamma}$ and $\zeta_{\delta}$ over $(C_1, \omega_\gamma)$ and $(C_2, \delta)$ respectively coincide with the following surjective algebra morphism

$$\begin{align*}
\zeta : & (C_2, \delta) \to (\mathbb{R},) \\
y_{r_1} \cdots y_{r_k} & \mapsto x_0^{r_1-1} \cdots x_0^{r_k-1} \sum_{n_1 > \cdots > n_k > 0} \frac{1}{n_1^{r_1} \cdots n_k^{r_k}}.
\end{align*}$$
In Section 3.3 we will give the complete description of the kernel \( \ker \zeta \).
With the double regularization\(^{20}\) to zero \([4, 8, 33, 43]\), the Drinfel’d associator \( \Phi_{KZ} \) corresponds then to \( Z_{\sqcup \sqcup} \) (obtained with only convergent polyzêtas) as being the unique group-like element satisfying \([35, 32]\)
\[
\langle Z_{\sqcup \sqcup} | x_0 \rangle = \langle Z_{\sqcup \sqcup} | x_1 \rangle = 0 \quad \text{and} \quad \forall x \in x_0 X^* x_1, \quad \langle Z_{\sqcup \sqcup} | w \rangle = \zeta(w). \quad (62)
\]
As consequence of Proposition 2, one has

**Proposition 10 ([38]).**

\[
\log Z_{\sqcup \sqcup} = \sum_{w \in X^*} \zeta_{\sqcup \sqcup}(w) \pi_1(w),
\]

\[
= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in X^*-\{e\}} \zeta_{\sqcup \sqcup}(u_1 \ldots u_k) u_1 \ldots u_k.
\]

The associator \( \Phi_{KZ} \) can be also graded in the adjoint basis of \( U(Lie Q(\langle X \rangle)) \) as follows

**Proposition 11 ([38]).** For any \( l \in \mathbb{N} \) and \( P \in \mathbb{C}(\langle X \rangle) \), let \( \circ \) denotes the composite operation defined by \( x_1 x_0^l \circ P = x_1 (x_0^l \circ P) \). Then

\[
Z_{\sqcup \sqcup} = \sum_{k \geq 0} \sum_{l_1, \ldots, l_k \geq 0} \zeta_{\sqcup \sqcup}(x_1 x_0^{l_1} \circ \ldots \circ x_1 x_0^{l_k}) \prod_{i=0}^{k} \text{ad}_{x_0}^{l_i} x_1,
\]

where \( \text{ad}_{x_0}^{l_i} x_1 \) is iterated Lie bracket \( \text{ad}_{x_0}^{l_i} x_1 = [x_0, \text{ad}_{x_0}^{l_i-1} x_1] \) and \( \text{ad}_{x_0}^{0} x_1 = x_1 \).

Using the following expansion \([6]\)

\[
\text{ad}_{x_0}^{n} x_1 = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) x_0^{n-i} x_1 x_0^{i},
\]

one deduces then, via the regularization process of Theorem 9, the expression of the Drinfel’d associator \( \Phi_{KZ} \) given by Lê and Murakami \([22]\).

### 3.2 Action of differential Galois group of polylogarithms on their asymptotic expansions

#### 3.2.1 Group of associators theorem

Let \( A \) be a commutative \( \mathbb{Q} \)-algebra.

Since the polyzêtas satisfy \((36)\), then by the Friedrichs criterion we can state the following

---

\(^{20}\)This double regularization is deduced from of the noncommutative generating series \( Z_{\sqcup \sqcup} \) and \( Z_{\sqcup \sqcup} \) on the definitions 1 and 2 (see the theorems 8 and 9).
Definition 11. Let $dm(A)$ be the set of $\Phi \in A\langle X \rangle$ such that\(^{21}\) $\langle \Phi \mid e \rangle = 1$, $\langle \Phi \mid x_0 \rangle = \langle \Phi \mid x_1 \rangle = 0$, $\Delta \Phi = \Phi \otimes \Phi$ and such that, for
\[
\Psi = B^\dagger(y) \pi Y \Phi \in A\langle Y \rangle
\]
then\(^{22}\) $\Delta \Psi = \Psi \otimes \Psi$.

Proposition 12 ([38]). If $G(z)$ and $H(z)$ are exponential solutions of (DE) then there exists a Lie series $C \in \text{Lie}_C\langle X \rangle$ such that $G(z) = H(z) \exp(C)$.

Proof. Since $H(z)H(z)^{-1} = 1$ then by differentiating, we have
\[
d[H(z)]H(z)^{-1} = -H(z)d[H(z)]^{-1}.
\]
Therefore if $H(z)$ is solution of Drinfel’d equation then
\[
d[H(z)]^{-1} = -H(z)^{-1}[dH(z)]H(z)^{-1} = -H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)],
\]
\[
d[H(z)]^{-1}G(z) = H(z)^{-1}[dG(z)] + [dH(z)]^{-1}G(z) = H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)]G(z)
- H(z)^{-1} [x_0\omega_0(z) + x_1\omega_1(z)]G(z).
\]
By simplification, we deduce then $H(z)^{-1}G(z)$ is a constant formal power series. Since the inverse and the product of group like elements is group like then we get the expected result.

The differential $\mathcal{C}$-module $\mathcal{C}\{\text{Li}_w\}_{w \in X^*}$ is the universal Picard-Vessiot extension of every linear differential equations, with coefficients in $\mathcal{C}$ and admitting \{0, 1, $\infty$\} as regular singularities. The universal differential Galois group, noted by $\text{Gal}(\mathcal{L}_C)$, is the set of differential $\mathcal{C}$-automorphisms of $\mathcal{C}\{\text{Li}_w\}_{w \in X^*}$ (i.e the automorphisms of $\mathcal{C}\{\text{Li}_w\}_{w \in X^*}$ that let $\mathcal{C}$ point-wise fixed and that commute with derivation). The action of an automorphism of $\text{Gal}(\mathcal{L}_C)$ can be determined by its action on $\text{Li}_w$, for $w \in X^*$. It can be resumed as its action on the noncommutative generating series $L[38]$

Let $\sigma \in \text{Gal}(\mathcal{L}_C)$. Then
\[
\sum_{w \in X^*} \sigma \text{Li}_w w = \prod_{i \in \text{Dyn}X} e^{\sigma \text{Li}_{S_i} S_i}. \tag{64}
\]

Since $d\sigma \text{Li}_{S_i} = \sigma d\text{Li}_{S_i} = \omega_i$ then by integrating the two members, we obtain $\sigma \text{Li}_{S_i} = \text{Li}_{S_i} + c_{S_i}$, where $c_{S_i}$ is a constant of integration. More generally, for any Lyndon word $l = x_{i_1}^1 \cdots x_{i_k}^k$ with $i_1 > \cdots > i_k$, one has
\[
\sigma \text{Li}_{S_i} = \int \omega_{x_i} \frac{\sigma \text{Li}_{S_{i_1}}^i}{i_1!} \cdots \frac{\sigma \text{Li}_{S_{i_k}}^i}{i_k!} + c_{S_i}. \tag{65}
\]

\(^{21}\Delta \Phi\) denotes the co-product of the shuffle product.

\(^{22}\Delta \Psi\) denotes the co-product of the quasi-shuffle product.
where \(c_{S_i}\) is a constant of integration. For example,

\[
\begin{align*}
\sigma \text{Li}_{x_0x_1} & = \text{Li}_{x_0x_1} + c_{x_1} \text{Li}_{x_0} + c_{x_0x_1}, \\
\sigma \text{Li}_{x_0^2x_1} & = \text{Li}_{x_0^2x_1} + \frac{c_{x_1}}{2} \text{Li}_{x_0^2} + c_{x_0x_1} \text{Li}_{x_0} + c_{x_0^2x_1}, \\
\sigma \text{Li}_{x_0x_1^2} & = \text{Li}_{x_0x_1^2} + c_{x_1} \text{Li}_{x_0x_1} + \frac{c_{x_1}}{2} \text{Li}_{x_0} + c_{x_0x_1^2}.
\end{align*}
\]

Consequently,

\[
\sum_{w \in \mathcal{X}} \sigma \text{Li}_w w = e^{C_{\sigma}} = \prod_{l \in \mathcal{L}} e^{c_{S_l}}. 
\]

The action of \(\sigma \in \text{Gal}(\mathbb{L}_C)\) over \(\{\text{Li}_w\}_{w \in \mathcal{X}}\) is then equivalent to the action of the Lie exponential \(e^{C_{\sigma}} \in \text{Gal}(\mathcal{D}E)\) over the exponential solution \(\mathcal{L}\). So,

**Theorem 12** ([38]). We have \(\text{Gal}(\mathbb{L}_C) = \{e^{C} \mid C \in \text{Lie}_{\mathbb{C}}(\mathcal{L})\}\).

Typically, since \(\mathcal{L}(z_0)^{-1}\) is group-like then \(S_{z_0 \rightarrow z} = \mathcal{L}(z)\mathcal{L}(z_0)^{-1}\) is an other solution of (34) as already saw in (52).

**Theorem 13** (Group of associators theorem). Let \(\Phi \in A(\mathcal{X})\) and \(\Psi \in A(\mathcal{Y})\) be group-like elements, for the co-products \(\Delta_{\mathcal{L}}, \Delta_{\mathcal{W}}\) respectively, such that \(\Psi = B(y_1)\pi_Y \Phi\). There exists an unique \(C \in \text{Lie}_{\mathbb{C}}(\mathcal{X})\) such that \(\Phi = Z_{\mathcal{L}, \mathcal{W}} e^{C}\) and \(\Psi = B(y_1)\pi_Y (Z_{\mathcal{L}, \mathcal{W}} e^{C})\).

**Proof.** If \(C \in \text{Lie}_{\mathbb{C}}(\mathcal{X})\) then \(L' = e^{C}\) is group-like, for the co-product \(\Delta_{\mathcal{L}}\), and \(e^{C} \in \text{Gal}(\mathcal{D}E)\). Let \(H'\) be the noncommutative generating series of the Taylor coefficients, belonging to the harmonic algebra, of \(\{(1-z)^{-1} (L' | w)\}_{w \in \mathcal{Y}}\). Then \(H'(N)\) is also group-like, for the co-product \(\Delta_{\mathcal{W}}\). By the asymptotic expansion of \(\mathcal{L}\), we have \(L'(z) \underset{z \rightarrow 1}{\sim} e^{-x_1 \log(1-z)} Z_{\mathcal{L}, \mathcal{W}} e^{C}\) [36, 32]. We put then \(\Phi := Z_{\mathcal{L}, \mathcal{W}} e^{C}\) and we deduce that

\[
\frac{L'(z)}{1-z} \underset{z \rightarrow 1}{\sim} \text{Mono}(z) \Phi \quad \text{and} \quad H'(N) \underset{N \rightarrow \infty}{\sim} \text{Const}(N)\pi_Y \Phi,
\]

where the expressions of \(\text{Mono}(z)\) and \(\text{Const}(N)\) are given on (49) and (50) respectively. Let \(\kappa_w\) be the constant part of \(H'_w(N)\). Then

\[
\sum_{w \in \mathcal{Y}} \kappa_w w = B(y_1)\pi_Y \Phi.
\]

We put then \(\Psi := B(y_1)\pi_Y \Phi\) (and also \(\Psi' := B'(y_1)\pi_Y \Phi\)).

**Corollary 7.** We have

\[
dm(A) = \{Z_{\mathcal{L}, \mathcal{W}} e^{C} \mid C \in \text{Lie}_{\mathbb{C}}(\mathcal{X})\} \quad \text{and} \quad \langle e^C \mid e \rangle = 1, \langle e^C \mid x_0 \rangle = 0, \langle e^C \mid x_1 \rangle = 0. 
\]

**Proof.** On the one hand, \(\langle \Phi \mid x_0 \rangle = \langle Z_{\mathcal{L}, \mathcal{W}} \mid x_0 \rangle = 0, \langle \Phi \mid x_1 \rangle = \langle Z_{\mathcal{L}, \mathcal{W}} \mid x_1 \rangle = 0\) and on the other, \(\langle \Phi \mid e \rangle = \langle Z_{\mathcal{L}, \mathcal{W}} \mid e \rangle = 1\), the result follows. 

\(\Box\)
Note also that if \( Z \subseteq A \) then \( dm(A) \) forms a group and with the notations of Corollary 6, we obtain

**Corollary 8.** For any associator \( \Phi = Z e^G \in dm(A) \), let \( \Psi = B(y_1) \pi_Y \Phi \) and let \( \Psi' = B'(y_1) \pi_Y \Phi \). Then

\[
\Psi = B(y_1) \pi_Y \Phi \iff \Psi' = B'(y_1) \pi_Y \Phi.
\]

**Proof.** Since \( \Psi \) is group like and since \( \langle \Phi \mid x_1 \rangle = \langle \Psi' \mid y_1 \rangle = 0 \) and \( \langle \Psi \mid y_1 \rangle = \gamma \), using the factorization by Lyndon words, we get the expected result. □

**Lemma 10.** Let \( \Phi = Z e^G \in dm(A) \) and let \( \Psi = B(y_1) \pi_Y (Z e^G) \). The local coordinates (of second kind) of \( \Phi \) (resp. \( \Psi \)) are polynomials on \( \langle \zeta \rangle \{ \zeta \} \in \mathcal{L} \) (resp. \( \{ \zeta \} \in \mathcal{L} \)) of \( Z \) (resp. \( Z' \)). While \( C \) describes \( \mathcal{L} \alpha \{ \langle X \rangle \} \), these coordinates describe \( A[[\zeta \rangle \{ \zeta \} \in \mathcal{L}] \) (resp. \( A[\{ \zeta \} \in \mathcal{L}] \)).

**Proof.** Let \( \Phi \in dm(A) \). By Corollary 7, there exists \( P \in \mathcal{L} \alpha \{ \langle X \rangle \} \) verifying \( \langle e^P \mid e \rangle = 1 \), \( \langle e^P \mid x_0 \rangle = \langle e^P \mid x_1 \rangle = 0 \) such that \( \Phi = Z e^P \). Using the factorization forms by Lyndon words, we get

\[
\prod_{l \in \mathcal{L} \alpha \{ X - X \}} e^{\phi(\hat{S}_l)} s_l = \left( \prod_{l \in \mathcal{L} \alpha \{ X - X \}} e^{\zeta(\hat{S}_l)} s_l \right) \left( \prod_{l \in \mathcal{L} \alpha \{ X - X \}} e^{p_{\hat{S}_l} \hat{S}_l} \right).
\]

Expanding the Hausdorff product and identifying the local coordinates in the PBW-Lyndon basis there exists \( I_l \subseteq \{ \lambda \in \mathcal{L} \alpha \{ X - X \} \} \) s.t. \( |\lambda| \leq |l| \), for \( l \in \mathcal{L} \alpha \{ X - X \} \), and the coefficients \( \{ p_{\hat{S}_l} \}_{u \in I_l} \) belonging to \( A \) such that

\[
\phi(\hat{S}_l) = \sum_{u \in I_l} p_{\hat{S}_u} \zeta(\hat{S}_u).
\]

This belongs to \( A[[\zeta \rangle \{ \zeta \} \in \mathcal{L}] \) and holds for any \( P \in \mathcal{L} \alpha \{ \langle X \rangle \} \). □

With the notations of Definition 9 and by Corollary 8, we get in particular

**Lemma 11.** For any \( \Phi \in dm(A) \), by identifying the local coordinates (of second kind) on two members of the identities \( \Psi = B(y_1) \pi_Y \Phi \), or equivalently of \( \Psi' = B'(y_1) \pi_Y \Phi \), we get polynomial relations, of coefficients in \( A \), among generators of the \( A \)-algebra of convergent polyzêtas.

Therefore,

**Theorem 14.** While \( \Phi \) describes \( dm(A) \), the identities \( \Psi = B(y_1) \pi_Y \Phi \) describe the ideal of polynomial relations, of coefficients in \( A \), among generators of the \( A \)-algebra of convergent polyzêtas. Moreover, if the Euler constant, \( \gamma \), does not belong to \( A \) then these relations are algebraically independent on \( \gamma \).

Simplified computations on Section 3.3 is an example of such identities. Some consequences of Theorem 14 will be drawn in Section 4.2.
3.2.2 Concatenation of Chen generating series

As an example of the action of the differential Galois group of polylogarithms on their asymptotic expansions, we are interested on the action of their monodromy group which is contained in Gal(ΔE).

The monodromies at 0 and 1 of L are given respectively by [35, 32]

\[ M_0L = Le^{2\pi i m_0} \quad \text{and} \quad M_1L = LZ_{\text{uni}}^{-1}e^{-2\pi i x_1}Z_{\text{uni}} = Le^{2\pi i m_1}, \quad (70) \]

where \( m_0 = x_0 \) and \( m_1 = \prod_{l \in \mathcal{L}} \exp(-2\pi i \int_{S_l} \frac{\zeta(S_l)}{adS_l}(-x_1)), \) \( (71) \)

• If \( C = 2\pi i m_0 \) then

\[ \Phi = Z_{\text{uni}}e^{2\pi i x_0}, \quad (72) \]
\[ \Psi = \exp\left(\gamma y_1 - \sum_{k \geq 2} \zeta(k)\frac{(-y_1)^k}{k}\right)\pi Y Z_{\text{uni}} \quad (73) \]
\[ = Z_{\text{uni}}. \quad (74) \]

The monodromy at 0 consists in the multiplication on the right of \( Z_{\text{uni}} \) by \( e^{2\pi i x_0} \) and does not modify \( Z_{\text{uni}} \).

• If \( C = 2\pi i m_1 \) then

\[ \Phi = e^{-2\pi i x_1}Z_{\text{uni}}, \quad (75) \]
\[ \Psi = \exp\left(\gamma y_1 - \sum_{k \geq 2} \zeta(k)\frac{(-y_1)^k}{k}\right)\pi Y Z_{\text{uni}} \quad (76) \]
\[ = e^{-2\pi i y_1}Z_{\text{uni}}. \quad (77) \]

The monodromy at 1 consists in the multiplication on left of \( Z_{\text{uni}} \) and of \( Z_{\text{uni}} \) by \( e^{-2\pi i x_1} \) and \( e^{-2\pi i y_1} \) respectively.

**Remark 1.**

1. The monodromies around singularities of \( L \) could not allow, in this case, neither to introduce the factor \( e^{\gamma x_1} \) on the left of \( Z_{\text{uni}} \) nor to eliminate the left factor \( e^{\gamma y_1} \) in \( Z_{\gamma} \) (by putting\(^{23}\) \( T = 0 \), for example).

2. By Proposition 5, we already saw that \( Z_{\text{uni}} \) is the concatenation of Chen generating series [10] \( e^{2\pi i \log \varepsilon} \) and then \( S_{\varepsilon \rightarrow 1-\varepsilon} \) and finally, \( e^{x_1 \log \varepsilon} \).

\[ Z_{\text{uni}} \xrightarrow{\varepsilon \rightarrow 0^+} e^{x_1 \log \varepsilon} S_{\varepsilon \rightarrow 1-\varepsilon} e^{x_0 \log \varepsilon}. \quad (78) \]

From (72) and (75), the action of the monodromy group gives

\[ e^{x_1} e^{2k_1 i\pi} Z_{\text{uni}} e^{x_0} e^{2k_0 i\pi} \xrightarrow{\varepsilon \rightarrow 0^+} e^{x_1 (\log \varepsilon + 2k_1 i\pi)} S_{\varepsilon \rightarrow 1-\varepsilon} e^{x_0 (\log \varepsilon + 2k_0 i\pi)}. \quad (79) \]

\(^{23}\) Why?
as being the concatenation of the Chen generating series \( e^{x_0(\log \varepsilon + 2k_0 \pi i)} \) (along circular path turning \( k_0 \) times around 0), then the Chen generating series \( S_{\varepsilon \rightarrow 1 - \varepsilon} \) and finally, the Chen generating series \( e^{x_1(\log \varepsilon + 2k_1 \pi i)} \) (along circular path turning \( k_1 \) times around 1).

3. More generally, by Corollary 7, the action of the Galois differential group of polylogarithms states, for any Lie series \( C \), the associator \( \Phi = Z \circ \varepsilon \) is the concatenation of some Chen generating series \( e^C \) and \( e^{x_0 \log \varepsilon} \) and then the Chen generating series \( S_{\varepsilon \rightarrow 1 - \varepsilon} \) and finally, \( e^{x_1 \log \varepsilon} \):

\[
\Phi \xrightarrow{\varepsilon \rightarrow 0^+} e^{x_1 \log \varepsilon} S_{\varepsilon \rightarrow 1 - \varepsilon} e^{x_0 \log \varepsilon} e^C.
\]

By construction (see Theorem 13) the associator \( \Phi \) is then the noncommutative generating series of the finite parts of the coefficients of the Chen generating series \( S_{z_0 \rightarrow 1 - z_0} e^C \), for \( z_0 = \varepsilon \rightarrow 0^+ \). Hence,

**Corollary 9.** Let \( \Phi \in dm(A) \). For any differential produced formal power series \( S \) over \( X \), there exists a differential representation \((A, f)\) such that:

\[
\langle \Phi \mid S \rangle = \sum_{w \in X^*} \langle \Phi \mid w \rangle A(w) \circ f_0 = \prod_{l \in \mathbb{L} \in X^* X} e^{\langle \Phi \mid \hat{S}_l \rangle} A(S_l) \circ f_0.
\]

### 3.3 Algebraic combinatorial studies of polynomial relation among polyzêta via a group of associators

Here, \( \bar{Y} = \{y_1\} \cup \{\bar{y}_k\}_{k \geq 2} \). With the factorization of the monoids \( X^* \) and \( \bar{Y}^* \) by Lyndon words, let \( \{\hat{l}\}_{l \in \mathbb{L} \in X^* X} \) and \( \{\hat{\bar{l}}\}_{l \in \mathbb{L} \in \bar{Y}^* \bar{Y}} \) be the dual of the Lyndon basis over \( X \) and \( \bar{Y} \).

#### 3.3.1 Preliminary study

As in Definition 10, let

\[
A_1 = Ae \oplus x_0 A(X) x_1 \quad \text{and} \quad A_2 = Ae \oplus (\bar{Y} - \{y_1\}) A(\bar{Y}).
\]

For \( \Phi \in dm(A) \), let \( \Psi = B'(y_1) \pi_{Y} \Phi \). Let us introduce two algebra morphisms

\[
\phi : (A_1, \underline{u}) \quad \mapsto \quad (A, \underline{v}), \quad \psi : (A_2, \underline{u}) \quad \mapsto \quad (A, \underline{v}),
\]

verifying respectively \( \phi(e) = 1, \phi(x_0) = \phi(x_1) = 0 \) and \( \psi(e) = 1, \psi(y_1) = 0 \).

**Lemma 12.** For any \( \Phi \in dm(A) \), let \( \Psi = B'(y_1) \pi_{Y} \Phi \). Then

\[
\forall w \in \bar{Y}^* - y_1 \bar{Y}^*, \quad \psi(w) = \phi(\pi_X w),
\]

or equivalently,

\[
\forall w \in x_0 X^* x_1, \quad \phi(w) = \psi(\pi_Y w).
\]

---

\[\text{24} \text{See Corollary } 22 \text{ of Annexe } B.\]
Lemma 13. We have

\[ \Phi = \sum_{u \in X^*} \phi(u) u = \prod_{l \in \mathcal{L}y_nX - X} e^{\phi(l)} e \quad \text{and} \quad \Psi = \sum_{v \in \bar{Y}^*} \psi(u) u = \prod_{l \in \mathcal{L}yn\bar{Y} - \{y_1\}} e^{\psi(l)} e. \]

With the notations in Lemma 13, we can state the following

Definition 12. We put

\[ R = \bigcap_{\Phi \in \text{dm}(A)} \ker \phi \quad (\text{resp.} \quad \bigcap_{\Phi \in \text{dm}(A)} \ker \psi). \]

Lemma 14. For any \( \Phi \in \text{dm}(A) \), let \( \Psi = B'(y_1) \pi \bar{Y} \Phi \). Let \( Q \in \mathbb{Q}[\mathcal{L}ynX] \) (resp. \( \mathbb{Q}[\mathcal{L}yn\bar{Y}] \)). Then

\[ \langle Q | \Phi \rangle = 0 \iff Q \in \ker \phi \quad (\text{resp.} \quad \langle Q | \Psi \rangle = 0 \iff Q \in \ker \psi). \]

Or equivalently (see Definition 7),

\[ Q \in R \iff Q \text{ is indiscernable over } \text{dm}(A). \]

Let \( \Phi_1, \Phi_2 \in \text{dm}(A) \). By Corollary 7, for \( i = 1 \) or 2, there exists an unique \( P_i \in \mathcal{L}ie_A(\langle X \rangle) \) such that \( e^{-P_i} \) is well defined and

\[ \Phi_i = Z_i e^{P_i}, \quad \text{or equivalently,} \quad Z_{ai} = \Phi_1 e^{-P_1} = \Phi_2 e^{-P_2}. \quad (83) \]

Then, we get \( \Phi_1 = \Phi_2 e^{P_1 - P_2} \) and \( \Phi_2 = \Phi_1 e^{P_2 - P_1} \). By Lemma 10, it follows

Lemma 15. Let \( \Phi_1 \) and \( \Phi_2 \in \text{dm}(A) \). For any convergent Lyndon word, \( l \), there exists a finite set \( I_l \subset \{ \lambda \in \mathcal{L}ynX - X \mid |\lambda| \leq |l| \} \) and the coefficients \( \{p'_{i,u}\}_{u \in I_l} \) and \( \{p''_{i,u}\}_{u \in I_l} \), for \( i = 1 \) or 2, belonging to \( A \) such that

\[ \phi_i(l) = \sum_{u \in I_l} p'_{i,u} \zeta(u), \quad \text{or equivalently,} \quad \zeta(l) = \sum_{u \in I_l} p''_{i,u} \phi_i(u). \]

There also exists the coefficients \( \{p''_{i,u}\}_{u \in I_l} \) and \( \{p''_{i,u}\}_{u \in I_l} \) belonging to \( A \) such that

\[ \phi_1(l) = \sum_{u \in I_l} p'_{i,u} \phi_2(u), \quad \text{or equivalently,} \quad \phi_2(l) = \sum_{u \in I_l} p''_{i,u} \phi_1(u). \]

Therefore, the \( \{\phi_i(l)\}_{l \in \mathcal{L}ynX - X} \) (resp. \( \{\psi_i(l)\}_{l \in \mathcal{L}yn\bar{Y} - \{y_1\}} \), for \( i = 1 \) or 2, are also generators of the \( A \)-algebra generated by convergent polyzéitas.

3.3.2 Description of polynomial relations among coefficients of associator and irreducible polyzéitas

Since the identities of Corollary 8 (see also Corollary 6) hold for any pair of bases, in duality, compatible with factorization of the monoid \( X^* \) (resp. \( \bar{Y}^* \)) then, by Corollary 8, one gets
Theorem 15. For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\pi_Y\Phi$. We have
\[
\prod_{\ell \in \mathcal{L}Y - y_1} e^{\psi(\ell)\hat{i}} = \exp \left( \sum_{k \geq 2} \zeta(k) \frac{(-y_1)^k}{k} \right) \pi_Y \prod_{\ell \in \mathcal{L}X - X} e^{\phi(\ell)\hat{i}}.
\]

If $\Phi = Z_{irr}$ and $\Psi = Z_{\nu}$ then, for $\ell \in \mathcal{L}Y - X$ (resp. $\mathcal{L}Y - y_1$), one has $\zeta(l) = \phi(l)$ (resp. $\psi(l)$). Hence, one obtains (see also Corollary 6)

Theorem 16 (Bis repetita).
\[
\prod_{\ell \in \mathcal{L}Y - y_1} e^{\zeta(\ell)\hat{i}} = \exp \left( \sum_{k \geq 2} \zeta(k) \frac{(-y_1)^k}{k} \right) \pi_Y \prod_{\ell \in \mathcal{L}X - X} e^{\zeta(\ell)\hat{i}}.
\]

Corollary 10. For any $\ell \in \mathcal{L}Y - y_1$ (resp. $\mathcal{L}X - X$), let $P_\ell \in \mathcal{U}(\mathcal{L}Y)$ (resp. $\mathcal{U}(\mathcal{L}X)$) be the decomposition of the polynomial $\pi_X \ell \in \mathcal{Q}(X)$ (resp. $\pi_Y \ell \in \mathcal{Q}(Y)$) in the PBW basis, induced by $\{\ell\} \in \mathcal{L}Y$ (resp. $\{\ell\} \in \mathcal{L}X$), and let $P_\ell \in \mathcal{Q}[\mathcal{L}Y - X]$ (resp. $\mathcal{Q}[\mathcal{L}X - X]$) be its dual. Then one obtains
\[
\pi_X \ell - P_\ell \in \ker \phi \quad (\text{resp. } \pi_Y \ell - P_\ell \in \ker \psi).
\]

In particular, for $\phi = \zeta$ (resp. $\psi = \zeta$) then one also obtains
\[
\pi_X \ell - P_\ell \in \ker \zeta \quad (\text{resp. } \pi_Y \ell - P_\ell \in \ker \zeta).
\]

Moreover, for any $\ell \in \mathcal{L}Y - y_1$ (resp. $\mathcal{L}X - X$), the homogeneous polynomial $\pi_X \ell - P_\ell \in \mathcal{Q}(X)$ (resp. $\mathcal{Q}(Y)$) is of degree equal $|\ell| \geq 2$.

Proof. Since
\[
\ell \in \mathcal{L}Y \iff \pi_X \ell \in \mathcal{L}X - \{x_0\}
\]
then identifying the local coordinates (of second kind) on the two members of each identity in Theorem 15, one obtains
\[
\forall \ell \in \mathcal{L}Y - y_1 \subset Y^* - y_1Y^*, \quad \psi(\ell) = \phi(P_\ell),
\]

(resp. $\forall \ell \in \mathcal{L}X - X \subset x_0X^*x_1$, $\phi(\ell) = \psi(P_\ell)$).

By Lemma 12, we get the expected result. \hfill \square

With the notations of Corollary 10, we get the following

Definition 13. Let $Q_\ell$ be the decomposition of the proper polynomial $\pi_Y \ell - P_\ell$ (resp. $\pi_X \ell - P_\ell$) in $\mathcal{L}Y$ (resp. $\mathcal{L}X$). Let
\[
\mathcal{R}_Y := \{Q_\ell\}_{\ell \in \mathcal{L}Y - y_1}, \quad \mathcal{R}_X := \{Q_\ell\}_{\ell \in \mathcal{L}X - X},
\]

$\mathcal{L}_{irr}Y := \{\ell \in \mathcal{L}Y - y_1 \mid Q_\ell = 0\}$ and $\mathcal{L}_{irr}X := \{\ell \in \mathcal{L}X - X \mid Q_\ell = 0\}$.

It follows that
Lemma 16. We have

\[
\begin{align*}
\langle \mathbb{Q}[\text{Lyn}Y - y_1], \text{te} \rangle &= \langle \mathbb{Q}[\text{Lyn}X - X], \omega \rangle = \langle \mathbb{Q}[\text{Lyn}Y], \text{te} \rangle, \\
\langle \mathbb{Q}[\text{Lyn}X - X], \omega \rangle &= \langle \mathbb{Q}[L_{\text{irr}}X], \omega \rangle.
\end{align*}
\]

Then we can state the following

Definition 14. Any word \( w \) is said to be irreducible if and only if \( w \) belongs to \( L_{\text{irr}}Y \) (resp. \( L_{\text{irr}}X \)). In this case, the polyglot \( \zeta(w) \) est said to be \( \mathbb{Q} \)-irreducible.

For any \( P \in \mathbb{Q}[L_{\text{irr}}X] \), there exists a differential representation \( (A, f) \) such that \( P \) can be finitely factorized (see also Corollary 9):

\[
P = \sigma_f \circ \sum_{w \in X^*_{\text{irr}}} A(w) \circ f \circ w = \prod_{\ell \in L_{\text{irr}}X, \text{finite}} e^{A(\ell)^t \circ f},
\]

where \( X^*_{\text{irr}} \) denotes the set of words obtaining by shuffling on \( L_{\text{irr}}X \).

Lemma 17. Any proper polynomial \( P \in \langle \mathbb{Q}[L_{\text{irr}}X], \omega \rangle \) (resp. \( \langle \mathbb{Q}[L_{\text{irr}}Y], \text{te} \rangle \)) is indiscernable over Chen generating series \( \{e^{t \cdot x}\}_{x \in X} \).

\[
\langle P \mid e^{t \cdot x} \rangle = \langle P \mid e^{t \cdot z_1} \rangle = 0 \quad \text{(resp.} \quad \langle P \mid e^{t \cdot y_1} \rangle = 0).
\]

Proof. By construction, \( x_0 \) and \( x_1 \notin L_{\text{irr}}X \) (resp. \( y_1 \notin L_{\text{irr}}X \)). For any \( n > 1, x^n_0 \) and \( x^n_1 \) (resp. \( y^n_1 \)) are not Lyndon words then they do not belong to \( L_{\text{irr}}X \) (resp. \( L_{\text{irr}}X \)). Therefore, for any \( n \geq 0 \), one has \( \langle P \mid x^n_0 \rangle = \langle P \mid x^n_1 \rangle = 0 \) (resp. \( \langle P \mid y^n_1 \rangle = 0 \)). Using the expansion of the exponential, we find the expected result.

Lemma 18. Let \( \Phi \in dm(A) \) and let \( t \in C, x \in X \). For any proper polynomial \( P \in \langle \mathbb{Q}[L_{\text{irr}}X], \omega \rangle \), if \( \langle P \mid \Phi \rangle = 0 \) then \( \langle P \mid \Phi e^{t \cdot x} \rangle = 0 \) and \( \langle P \mid e^{t \cdot z_1} \Phi \rangle = 0 \).

Proof. Since \( P \in \langle \mathbb{Q}[L_{\text{irr}}X], \omega \rangle \) and \( P \) is proper then, by Lemma 17, for any \( t \in C \) and for any \( x \in X \), we have \( \langle P \mid e^{t \cdot x} \rangle = 0 \) and then \( \langle P \mid \Phi e^{t \cdot x} \rangle = 0 \).

Since \( \supp(P) \subseteq x^n_0X^n_1 \) then \( \langle P \mid e^{t \cdot x_1 \Phi} \rangle = \langle P \circ e^{t \cdot x_0} \Phi \rangle = 0 \).

Next, for \( \Phi \in dm(A) \), there exists \( C \) such that \( e^{t \cdot z_1 \Phi} = e^{t \cdot z_1 \cdot Z_{\text{irr}} \circ C} \) and, by Proposition 5, we get

\[
e^{t \cdot z_1 \Phi} \xrightarrow{\epsilon \to 0^+} e^{t(1 + t \log \epsilon)} \cdot \epsilon^{x_0 \log \epsilon} \cdot e^C.
\]

Hence, there exists a Chen generating series \( C_{x_1}^{z_1} - z_0 \) and \( S_{z_0}^{z_1} - z_0 \) such that we get the following asymptotic behaviour (see Section 3.2.2)

\[
e^{t \cdot z_1 \Phi} \xrightarrow{\epsilon \to 0^+} C_{z_1}^{z_1 - z_0} \cdot S_{z_0}^{z_1 - z_0} \cdot e^C.
\]

and the following concatenation holds [10] (see Formula (53))

\[
\iff C_{z_1}^{z_1 - z_0} S_{z_0}^{z_1 - z_0} = S_{z_0}^{z_1 - z_0} = S_{z_0}^{z_1 - z_0} e^C.
\]

\[25\text{See Corollary 22 of Annexe B.}\]
Since $P \in \mathbb{Q}[\mathcal{L}_{irr} X]$ then by (84), applying $\langle \sigma f_{\alpha} \parallel \bullet \rangle$ to the two sides of the previous equality, one has
\[
\langle \sigma f_{\alpha} \parallel C_{z \rightarrow 1 - z_0} S_{z_0 \rightarrow z} e^C \rangle = \langle \sigma f_{\alpha} \parallel S_{z_0 \rightarrow 1 - z_0} e^C \rangle.
\]
Thus, for $z_0 = \varepsilon \rightarrow 0^+$, one obtains
\[
\langle \sigma f_{\alpha} \parallel e^{x_1} \Phi \rangle \underset{\varepsilon \rightarrow 0^+}{\longrightarrow} \langle \sigma f_{\alpha} \parallel \Phi \rangle.
\]
Since $\langle \sigma f_{\alpha} \parallel \Phi \rangle = \langle P \parallel \Phi \rangle = 0$ then we get the expected result. \hfill \Box

**Lemma 19.** For any $\Phi \in dm(A)$, let $\Psi = B'(y_1) \pi_{Y} \Phi$. We have $\mathcal{R}_Y \subseteq \ker \psi$ and $\mathcal{R}_X \subseteq \ker \phi$. In particular, $\mathcal{R}_Y \subseteq \ker \zeta$ and $\mathcal{R}_X \subseteq \ker \zeta$. 

**Proposition 13.** We have $\mathcal{R}_X \subseteq \mathcal{R}$ (resp. $\mathcal{R}_Y \subseteq \mathcal{R}$).

**Proposition 14.** For any proper polynomial $Q \in \mathbb{Q}[\mathcal{L}_{irr} X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr} Y]$),
\[
Q \in \mathcal{R} \iff Q = 0.
\]

**Proof.** If $Q = 0$ then since, for $\Phi \in dm(A)$, $\phi$ is an algebra homomorphism then $\phi(Q) = 0$. Hence, $Q \in \ker \phi$ and then $Q \in \mathcal{R}$.

Conversely, if $Q \in \mathcal{R}$ then, for $\Phi \in dm(A)$, we get $\langle Q \parallel \Phi \rangle = 0$. That means $Q$ is indiscernable over $dm(A)$. Let $\mathcal{H}$ be the monoid generated by $dm(A)$ and by the Chen generating series $\{e^{x_1} \}_{x \in X}$. By Lemma 25, $Q$ is continuous over $\mathcal{H}$ and by Lemma 18, it is indiscernable over $\mathcal{H}$. By Proposition 7, the expected result follows. \hfill \Box

Therefore, by the propositions 13 and 14, we obtain

**Theorem 17.** We have $\mathcal{R} = \mathcal{R}_X$ (resp. $\mathcal{R}_Y$).

**Proposition 15.** For any $\Phi \in dm(A)$, let $\Psi = B'(y_1) \pi_{Y} \Phi$. Let $Q \in (\mathbb{Q}[\mathcal{L}_{irr} X], \cup)$ (resp. $(\mathbb{Q}[\mathcal{L}_{irr} Y], \sqcup)$) such that $\langle \Phi \parallel Q \rangle = 0$ (resp. $\langle \Psi \parallel Q \rangle = 0$). Then $Q = 0$.

**Proof.** Let $\mathcal{H}$ defined as being the monoid generated by $\Phi$ and by Chen generating series $\{e^{x_1} \}_{x \in X}$. By assumption, $\langle \Phi \parallel Q \rangle = 0$ and by Lemma 18, $Q$ is then indiscernable over $\mathcal{H}$. Finally, by Proposition 7, it follows that $Q = 0$. \hfill \Box

**Proposition 16.** For any $\Phi \in dm(A)$, let $\Psi = B'(y_1) \pi_{Y} \Phi$. We get $\ker \phi = \mathcal{R}_X$ (resp. $\ker \psi = \mathcal{R}_Y$). In particular, $\ker \zeta = \mathcal{R}_X$ (resp. $\ker \zeta = \mathcal{R}_Y$).

**Proof.** By Lemma 19, $\mathcal{R}_X$ and $\mathcal{R}_Y$ are included in $\ker \phi$ and $\ker \psi$ respectively. Conversely, two cases can occur (see Lemma 16) :

1. Case $Q \notin \mathbb{Q}[\mathcal{L}_{irr} X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr} Y]$). By Lemma 16, $Q \equiv_{\mathcal{R}_X} Q_1$ (resp. $Q \equiv_{\mathcal{R}_Y} Q_1$) such that $Q_1 \in \mathbb{Q}[\mathcal{L}_{irr} X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr} Y]$) and $\phi(Q_1) = 0$ (resp. $\psi(Q_1) = 0$). This case is then reduced to the following

2. Case $Q \in \mathbb{Q}[\mathcal{L}_{irr} X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr} Y]$). Using Proposition 15, we have $Q \equiv_{\mathcal{R}_X} 0$ (resp. $Q \equiv_{\mathcal{R}_Y} 0$).
Then, $R_X$ (resp. $R_Y$) contains $\ker \phi$ (resp. $\ker \psi$).

For any $Q \in (\mathbb{Q}[L_{irr}X], \uplus)$ (resp. $(\mathbb{Q}[L_{irr}Y], \uplus)$), $\zeta(Q)$ is then a polynomial on $A$-irreducible polyzêtas (see Definition 14). Moreover,

**Proposition 17.** The $\mathbb{Q}$-algebra $Z$ is generated by the family of $A$-irreducible polyzêtas $\{\zeta(\ell)\}_{\ell \in L_{irr}Y}$ (resp. $\{\zeta(\ell)\}_{\ell \in L_{irr}X}$).

**Proof.** By Radford’s theorem [45], one just needs to prove for Lyndon words:

Let $\ell \in \mathcal{L}_{yn}Y - y_1$. If $\pi_X \ell = \check{P}_\ell$ then the result follows else one has $\pi_X \ell - \check{P}_\ell \in \ker \zeta$. Hence, $\zeta(\ell) = \zeta(\check{P}_\ell)$.

Since $\check{P}_\ell \in \mathbb{Q}[\mathcal{L}_{yn}X] - X$ then $\check{P}_\ell$ is polynomial on Lyndon words, over $X$, of degree less or equal $|\ell|$. For each Lyndon word does appear in this decomposition of $\check{P}_\ell$, after applying $\pi_Y$, one uses the same recurcive procedure until getting Lyndon words in $L_{irr}Y$.

The same treatment works for any $\ell' \in \mathcal{L}_{yn}X - X$.

For any $\Phi \in dm(A)$, by Proposition 16, one also has $\ker \phi = \ker \zeta = R_X$. That means, for any irreducible Lyndon words $l \neq l'$,

$$\phi(l) = \phi(l') \iff \zeta(l) = \zeta(l').$$

(85)

Let us state then the following

**Lemma 20.** Let $\Phi \in dm(A)$. Let us define the map

$$\varphi : Z \longrightarrow A$$

as follows

$$\forall \ell \in L_{irr}X, \quad \varphi(\zeta(\ell)) := \phi(l).$$

Then $\varphi$ is an algebra homomorphism and $\{\varphi(\zeta(\ell))\}_{\ell \in L_{irr}X}$ are generators of $A$.

Thus, for any $\theta \in Z$ there exist the coefficients $\{\alpha_{l_1, \ldots, l_n}\}_{n \in \mathbb{N}} \in L_{irr}X$ in $A$ such that (see Proposition 17 and Lemma 20)

$$\varphi(\theta) = \sum_{n \geq 0} \sum_{l_1, \ldots, l_n \in L_{irr}X} \alpha_{l_1, \ldots, l_n} \varphi(\zeta(l_1)) \cdots \varphi(\zeta(l_n)).$$

In particular, since for any $w \in X^*$, $\zeta_{\uplus}(w)$ belongs to $Z$ (see Corollary 3) then $\varphi(\zeta_{\uplus}(w))$ is well defined and $\varphi(\zeta_{\uplus}(w))$ can be expressed as polynomial on convergent polyzêtas with coefficients in $A$ :

**Lemma 21.** With the notations in Lemma 20, one has

$$\forall w \in X^*, \quad \varphi(\zeta_{\uplus}(w)) = \sum_{u, v \in X^*} \langle e^C | v \rangle \cdot \zeta_{\uplus}(u).$$

**Proof.** The expected result follows by identifying coefficients in $\Phi = Z_{\uplus} e^C$. 

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Finally, we can state the following

**Theorem 18.** For any $\Phi \in dm(A)$, there exists an unique algebra homomorphism

$$\varphi : Z \rightarrow A$$

such that $\Phi$ is computed from $Z_{\omega i}$ by applying $\varphi$ to each coefficient:

$$\Phi = \sum_{w \in X^*} \varphi(\zeta_{\omega i}(w)) w = \prod_{l \in \mathcal{L}_0nX} e^{\varphi(\zeta(l)) l}.$$

**Remark 2.**

1. In this work, neither the question deciding any real number belongs to $Z$ or not nor the question expliciting $\{\alpha_{l_1, \ldots, l_n}\}_{l_1, \ldots, l_n \in \mathcal{L}_{\text{irr}}}^{n}$ in (86), are considered.

2. Now, by considering the commutative indeterminates $t_1, t_2, t_3, \ldots$, let $A$ be the $\mathbb{Q}$-algebra obtained by specializing $\mathbb{Q}[t_1, t_2, t_3, \ldots]$ at $t_1 = i\pi$:

$$A = \mathbb{Q}[i\pi][t_2, t_3, \ldots].$$

Neither the Lie exponential series $e^{i\pi x}$ nor $e^{i\pi x_1}$ does belong to $dm(A)$ but it belongs to $\text{Gal}(DE)$. In particular, it figures in the modromies (see Section 3.2.2) or in the functional relations (see (45) and (46)) of polylogarithms and in the hexagonal relation of polyzetas (see Proposition 6).

3. Applying Baker–Campbell–Hausdorff formula [6] to Proposition 6 we get, at orders 2 and 3 as examples, the famous Euler’s formula saying $\zeta(2)$ is an algebraic number over $A = \mathbb{Q}[i\pi]$:

$$\zeta(2) + \frac{(i\pi)^2}{6} = 0 \quad \text{(order 2)},$$

$$\zeta(3) - \zeta(2,1) = 0 \quad \text{(order 3, imaginary part).}$$

Therefore, the first coming in mind homomorphism

$$\varphi : Z \rightarrow A$$

maps, at least $\zeta(2)$ to $\varphi(\zeta(2)) = \pi^2/6$.

4. For this reason, in [30], we have to consider the $\mathbb{Q}$-algebra generated by $i\pi$ and by other $A$-irreducible polyzetas obtained in [32, 34, 3, 49] (and such algebra is denoted in this work by $A$). This algebra came up from the studies of monodromies [32, 35], as already shown in (70), and the Kummer type functional equations of polylogarithms [32, 36], as already shown in (44)–(46). In particular, by (46), we get for example [36, 32],

$$\text{Li}_{2,1} \frac{1}{t} = -\frac{(i\pi)^2}{2} \log t + i\pi(\zeta(2) - \frac{\log^2 t}{2} - \text{Li}_2 t)$$

$$- \text{Li}_{2,1} t + \text{Li}_3 t - \log t \text{Li}_2 t + \zeta(3) - \frac{\log^3 t}{6}. \quad (90)$$

specializing $t = 1$, the real part of this leads again to the Euler’s indentity (89).
4 Concluding remarks: complete description of \( \ker \zeta \) and structure of polyzêtas

For the same raison as already said in Remark 2(2), let us consider now the commutative indeterminates \( t_1, t_2, t_3, \ldots \).

Let \( A \) be the \( \mathbb{Q} \)-algebra obtained by specializing \( \mathbb{Q}[t_1, t_2, t_3, \ldots] \) at \( t_1 = i\pi \):

\[
A = \mathbb{Q}[i\pi][t_2, t_3, \ldots].
\]

4.1 A conjecture by Pierre Cartier

**Definition 15** ([8, 43]). Let \( DM(A) \) denotes the set of \( \Phi \in A[[X]] \) such that

\[
\langle \Phi \mid \epsilon \rangle = 1, \quad \langle \Phi \mid x_0 \rangle = \langle \Phi \mid x_1 \rangle = 0, \quad \Delta_{\shuffle} \Phi = \Phi \otimes \Phi
\]

and such that, for

\[
\bar{\Psi} = \exp \left( - \sum_{n \geq 2} \langle \pi_Y \Phi \mid y_n \rangle \frac{(-y_1)^n}{n} \right) \pi_Y \Phi \in A[[Y]],
\]

then \( \Delta_{\shuffle} \bar{\Psi} = \bar{\Psi} \otimes \bar{\Psi} \).

Since \( DM(A) \) contains already \( Z_{\shuffle} \), then for \( \Phi \in DM(A) \), by Theorem 13, there exists \( C \in \text{Lie}_A[[X]] \) verifying

\[
\langle e^C \mid \epsilon \rangle = 1 \quad \text{and} \quad \langle e^C \mid x_0 \rangle = \langle e^C \mid x_1 \rangle = 0
\]

such that

\[
\Phi = Z_{\shuffle} e^C
\]

and such that

\[
\Psi = B'(y_1)\pi_Y \Phi = \exp \left( - \sum_{k \geq 2} \zeta(k) \frac{(-y_1)^k}{k} \right) \pi_Y \Phi,
\]

\[
\bar{\Psi} = \exp \left( - \sum_{n \geq 2} \langle \pi_Y \Phi \mid y_n \rangle \frac{(-y_1)^n}{n} \right) \pi_Y \Phi.
\]

By construction (see Definition 11 and Theorem 13), such \( \Phi \) and \( \Psi \) are group-like (for the co-products \( \Delta_{\shuffle} \) and \( \Delta_{\shuffle} \) respectively) and here, \( \bar{\Psi} \) must be also group-like (for the co-product \( \Delta_{\shuffle} \)). If such a Lie series \( C \) exists then it is unique, due to the fact that \( e^C = \Phi Z_{\shuffle}^{-1} \), and it is group-like (for the co-product \( \Delta_{\shuffle} \)).

\*\*\*We do not consider in any case \( A = \mathbb{Q} \) as in previous versions.
Corollary 11 (conjectured by Cartier, [8]). For any $\Phi \in DM(A)$, there exists an unique algebra homomorphism\textsuperscript{27}

$$\bar{\varphi} : Z \longrightarrow A$$

such that $\Phi$ is computed from $Z_{\bar{\varphi}}$ by applying $\bar{\varphi}$ to each coefficient.

Proof. By Theorem 18, use the fact

$$DM(Q) \subseteq DM(A) \subseteq dm(A).$$

\hfill \square

4.2 Arithmetical nature of $\gamma$

By Theorem 14, under the assumption that the Euler constant, $\gamma$, does not belong to a commutative $Q$-algebra $A$ then $\gamma$ does not verify any polynomial with coefficients in $A$ among the convergent polyzêtas. It follows then,

Corollary 12. If $\gamma \notin A$ then it is transcendental over the $A$-algebra generated by the convergent polyzêtas.

Or equivalently, by contraposition,

Corollary 13. If there exists a polynomial relation with coefficients in $A$ among the Euler constant, $\gamma$, and the convergent polyzêtas then $\gamma \in A$.

Therefore,

Corollary 14. If the Euler constant, $\gamma$, does not belong to $A$ then $\gamma$ is not algebraic over $A$.

Remark 3. 1. In the same spirit of Theorem 11, let $\zeta_{T_{\bar{\varphi}}}$ be the regularization morphism\textsuperscript{28} from $(Q(Y), \langle \cdot \rangle)$ to $(R, _.)$ mapping $y_1$ to $T$. Let $Z_{T_{\bar{\varphi}}}$ be the noncommutative generating series of polyzêtas regularized with respect to $\zeta_{T_{\bar{\varphi}}}$. Thus, as in Theorem 11 and by infinite factorization by Lyndon words, we also get

$$Z_{T_{\bar{\varphi}}} := \sum_{w \in X^*} \zeta_{T_{\bar{\varphi}}} (w) \, w = e^{Ty_1} Z_{\bar{\varphi}}. \quad (95)$$

2. Now let us consider the regularization, for $N \rightarrow +\infty$ and with respect to $\zeta_{T_{\bar{\varphi}}}$, of the power series $\text{Const}(N)$ given in (50) as

$$B^T(y_1) = e^{Ty_1} B'(y_1) \quad (96)$$

\textsuperscript{27}See Remark 2(3) to have an example of $\bar{\varphi}$.

\textsuperscript{28}This is a symbolic regularization and does not yet have an analytical justification as it is done, separately, for $\zeta_{\bar{\varphi}}$ and $\zeta_{\bar{\varphi}}$ in Section 3.1.2 as finite parts of the asymptotic expansions, in different scales of comparison, of $Li_{\alpha_1}(z)$, for $z \rightarrow 1$, and $H_{\eta_1}(N)$, for $N \rightarrow \infty$, respectively.
As in Corollary 6, we always get
\[ Z_T = B^T(y_1)\pi Y Z_{\omega_i} \iff Z_{\omega_i} = B'(y_1)\pi Y Z_{\omega_i}. \] (97)

Hence, roughly speaking, for the quasi-shuffle product, the symbolic regularization to \( T \) is also "equivalent" to the regularization to 0.

3. Again, as in Corollary 12, if \( T \notin A \) then \( T \notin \bar{A} \).

A contrario, as in Corollary 13, if there exists a polynomial relation with coefficients in \( A \) among \( T \) and convergent polyzêtas then \( T \in A \).

4.3 Structure and arithmetical nature of polyzêtas

Once again, let us consider
\[
(A_1, \omega_i) = (A \oplus x_0A(X)x_1, \omega_i) \equiv A[\mathcal{L}ynX - X], \omega_i
\]
\[
= (\mathcal{R}_X, \omega_i) \oplus (A[\mathcal{L}_{irr}X], \omega_i),
\]
\[
(A_2, \omega_i) = (A \oplus (\bar{Y} - \{y_1\}) A(\bar{Y}), \omega_i)
\]
\[
= (A[\mathcal{L}yn\bar{Y} - y_1], \omega_i)
\]
\[
= (\mathcal{R}_{\bar{Y}}, \omega_i) \oplus (A[\mathcal{L}_{irr}\bar{Y}], \omega_i)
\]
\[
(\text{see Definition 10, Lemma 16, Definition 13). Then } (A_1, \omega_i) \equiv (A_2, \omega_i) \text{ [32, 33].}
\]

Let us consider again the following algebra morphism (see Proposition 9)
\[
\zeta : (A_2, \omega_i) \rightarrow (\mathcal{R},)
\]
\[
y_r_1 \ldots y_r_k \\
x_0x_1^{r_1-1} \ldots x_0x_k^{r_k-1} \mapsto \sum_{n_1 > \ldots > n_k > 0} \frac{1}{n_1 \ldots n_k}. \]
\]

**Lemma 22.** The image of the algebra morphism \( \zeta \) is \( Z \).

Let us make precise the structure of \( Z \) and the arithmetical nature of polyzêtas:

As consequences of the propositions 15, 16 and 17, by taking \( \Phi = Z_{\omega_i} \), we have
\[
\text{Im } \zeta = \zeta(A[\mathcal{L}_{irr}\bar{Y}]) \quad \text{and} \quad \text{ker } \zeta = R_{\bar{Y}}
\]
\[
(\text{resp. Im } \zeta = \zeta(A[\mathcal{L}_{irr}X]) \quad \text{and} \quad \text{ker } \zeta = R_X).
\]

By Corollary 10, ker \( \zeta \) is an ideal generated by the homogenous polynomials of degree \( \geq 2 \). Hence, the quotient \( A_1/R_X \) or \( A_2/R_Y \) (the source by the kernel of \( \zeta \)) is graded [6] and it is isomorphic to \( \text{Im } \zeta \).

Therefore, by Lemma 22 and Proposition 17, we obtain respectively the following direct consequences

**Theorem 19** (Structure of polyzêtas). The \( A \)-algebra \( Z \) is
1. isomorphic to the graded algebra \((A_1/\mathcal{R}_X, \ldots)\), or equivalently, \((A_2/\mathcal{R}_Y, \mathcal{W})\).

2. freely generated by the \(A\)-irreducible polyzetas \(\{\zeta(l)\}_{l \in \mathcal{L}, Y}\) (resp. \(\{\zeta(l)\}_{l \in \mathcal{L}, X}\)).

For any \(p \geq 2\), let

\[
Z_p = \text{span}_Q \{\zeta(w) \mid w \in x_0 X^*x_1, |w| = p\}. \tag{108}
\]

By definition of graded algebra [6], Theorem 19 means also that

\[
Z = A \oplus \bigoplus_{p \geq 2} Z_p \tag{109}
\]

and there is no linear relation among elements of different \(Z_p\) ([8] conjecture C1, [48]).

Thus, if \(\theta\) is a (\(A\)-irreducible) polyzêta verifying the following algebraic equation

\[
\theta^n + a_{n-1}\theta^{n-1} + \ldots + a_0 = 0 \tag{110}
\]

then \(\theta = 0\) because \(Z_p, Z_{p_2} \subset Z_{p_1+p_2}\) for \(p_1, p_2 \geq 2\), and each monomial in (110) is then of different weight. By consequence,

**Corollary 15.** Any (\(A\)-irreducible) polyzêta \(\theta\) is a transcendental over \(Q\).

**Remark 4.** In this work, neither the study of \(\dim Z_p\) [50] (see also [8], conjecture C2) nor the estimate of the number of \(A\)-irreducible polyzetas generating \(Z_p\), are discussed knowing the \(A\)-irreducible polyzetas form transcendence basis of the \(A\)-algebra \(Z\).

## 5 Annexe A : pair of bases in duality and proof of Theorem 2

### 5.1 Preliminary results

Let \(Q\langle Y \rangle\) be equipped the concatenation and the quasi-shuffle, \(\mathcal{W}\), defined by

\[
\forall y_i, y_j \in Y = \{y_i\}_{i \geq 1}, \forall u, v \in Y^*, \quad y_i \mathcal{W} y_j v = y_i (u \mathcal{W} y_j v) + y_{i+j} (y_i \mathcal{W} v),
\]

or by its associated co-product, \(\Delta_{\mathcal{W}}\), defined by

\[
\forall y_k \in Y, \quad \Delta_{\mathcal{W}}(y_k) = y_k \otimes 1 + 1 \otimes y_k + \sum_{i+j=k} y_i \otimes y_j.
\]

Satisfying, for any \(u, v, w \in Y^*, \langle u \otimes v \mid \Delta_{\mathcal{W}}(w) \rangle = \langle u \mathcal{W} v \mid w \rangle\).

**Lemma 23.** Let \(S_1, \ldots, S_n\) be proper formal power series in \(Q\langle Y \rangle\). Let \(P_1, \ldots, P_m\) be primitive elements in \(Q\langle Y \rangle\), for the co-product \(\mathcal{W}\).

---

\(^{29}\) i.e., for any \(i = 1, \ldots, m, \Delta_{\mathcal{W}}(P_i) = 1 \otimes P_i + P_i \otimes 1\).
1. If \( n > m \) then \( \langle S_1 \ldots S_n \mid P_1 \ldots P_m \rangle = 0 \).

2. If \( n = m \) then

\[
\langle S_1 \ldots S_n \mid P_1 \ldots P_n \rangle = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} \langle S_i \mid P_{\sigma(i)} \rangle.
\]

3. If \( n < m \) then, by considering the language \( \mathcal{M} \) over \( A = \{ P_1, \ldots, P_m \} \)

\[
\mathcal{M} = \{ w \in A^* \mid w = P_{j_1} \ldots P_{j_{|w|}}, j_1 < \ldots < j_{|w|}, |w| \geq 1 \}
\]

and the morphism \( \mu : Q(A) \rightarrow Q(Y) \), one has :

\[
\langle S_1 \ldots S_n \mid P_1 \ldots P_m \rangle = \sum_{w_1, \ldots, w_m \in \mathcal{M}} \prod_{i=1}^{n} \langle S_i \mid \mu(w_i) \rangle.
\]

Proof. On one hand, since the \( P_i \)'s are primitive then

\[
\Delta^{(n-1)}(P_i) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 1^p \otimes P_i \otimes 1^q.
\]

On the other hand, \( \langle S_1 \ldots S_n \mid P_1 \ldots P_m \rangle = \langle S_1 \otimes \ldots \otimes S_n \mid \Delta^{(n-1)}(P_1 \ldots P_m) \rangle = \Delta^{(n-1)}(P_1 \ldots P_m) \).

Hence,

\[
\langle S_1 \ldots S_n \mid P_1 \ldots P_m \rangle = \bigotimes_{i=1}^{n} S_i \otimes \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 1^p \otimes P_i \otimes 1^q.
\]

1. For \( n > m \), by expanding \( \Delta^{(n-1)}(P_1) \ldots \Delta^{(n-1)}(P_m) \), one obtains a sum of tensors containing at least one factor equal to 1. For \( j = 1, \ldots, n \), the formal power series \( S_j \) is proper and the result follows immediately.

2. For \( n = m \), since

\[
\prod_{i=1}^{n} \Delta^{(n-1)}(P_i) = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} P_{\sigma(i)} + Q,
\]

where \( Q \) is sum of tensors containing at least one factor equal to 1 and the \( S_j \)'s are proper then \( \langle S_1 \otimes \ldots \otimes S_n \mid Q \rangle = 0 \). Thus, the result follows.

3. For \( n < m \), en tenant compte que, for \( j = 1, \ldots, n \), formal power series \( S_j \) is proper, the expected follows by expanding the product

\[
\prod_{i=1}^{m} \Delta^{(n-1)}(P_i) = \prod_{i=1}^{m} \sum_{p+q=n-1} 1^p \otimes P_i \otimes 1^q.
\]
Proposition 18. 1. We have

$$\log \left( \sum_{w \in \mathcal{Y}^*} w \otimes w \right) = \sum_{w \in \mathcal{Y}^+} w \otimes \pi_1(w) = \sum_{w \in \mathcal{Y}^+} \pi_1^*(w) \otimes w,$$

where $\pi_1^*$ is the adjoint of $\pi_1$ and they are given by

$$\pi_1(w) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in \mathcal{Y}^+} \langle w | u_1 \ldots u_k \rangle u_1 \ldots u_k,$$

$$\pi_1^*(w) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in \mathcal{Y}^+} \langle w | u_1 \ldots u_k \rangle u_1 \ldots u_k.$$

In particular, for any $y_k \in \mathcal{Y}$, one has

$$\pi_1(y_k) = y_k + \sum_{l \geq 2} \frac{(-1)^{l-1}}{l} \sum_{j_1, \ldots, j_l \geq 1 \atop j_1 + \ldots + j_l = k} y_{j_1} \ldots y_{j_l} \text{ and } \pi_1^*(y_k) = y_k.$$

2. For any $w \in \mathcal{Y}^+$, we have

$$w = \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \ldots, u_k \in \mathcal{Y}^*} \langle w | u_1 \ldots u_k \rangle \pi_1(u_1) \ldots \pi_1(u_k),$$

$$= \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \ldots, u_k \in \mathcal{Y}^*} \langle w | u_1 \ldots u_k \rangle \pi_1^*(u_1) \ldots \pi_1^*(u_k).$$

Proof. 1. Expanding the logarithm, we have

$$\log \left( \sum_{w \in \mathcal{Y}^*} w \otimes w \right) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \left( \sum_{w \in \mathcal{Y}^+} w \otimes w \right)^k$$

$$= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in \mathcal{Y}^+} (u_1 \ldots u_k) \otimes u_1 \ldots u_k$$

$$= \sum_{w \in \mathcal{Y}^+} w \otimes \left( \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in \mathcal{Y}^+} \langle w | u_1 \ldots u_k \rangle u_1 \ldots u_k \right).$$

In the same way,

$$\log \left( \sum_{w \in \mathcal{Y}^*} w \otimes w \right) = \sum_{w \in \mathcal{Y}^+} \left( \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in \mathcal{Y}^+} \langle w | u_1 \ldots u_k \rangle u_1 \ldots u_k \right) \otimes w.$$

Thus, the expressions of $\pi_1(w)$ and $\pi_1^*(w)$ follow immediatly.
2. Since exp and log are mutually inverse then, by the previous results, one has

$$\sum_{w \in Y^*} w \otimes w = \sum_{k \geq 0} \frac{1}{k!} \left( \sum_{w \in Y^+} w \otimes \pi_1(w) \right)^k$$

$$= \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \ldots, u_k \in Y^+} (u_1 \shuffle \ldots \shuffle u_k) \otimes (\pi_1(u_1) \ldots \pi_1(u_k))$$

$$= \sum_{w \in Y^+} w \otimes \left( \sum_{k \geq 1} \frac{1}{k!} \sum_{w_1, \ldots, w_k \in Y^+} \langle w | u_1 \shuffle \ldots \shuffle u_k \rangle \pi_1(u_1) \ldots \pi_1(u_k) \right).$$

In the same way,

$$\sum_{w \in Y^*} w \otimes w = \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \ldots, u_k \in Y^+} (\pi_1^*(u_1) \shuffle \ldots \shuffle \pi_1^*(u_k)) \otimes (u_1 \ldots u_k)$$

$$= \sum_{w \in Y^+} \left( \sum_{k \geq 1} \frac{1}{k!} \sum_{w_1, \ldots, w_k \in Y^+} \langle w | u_1 \ldots u_k \rangle \pi_1^*(u_1) \shuffle \ldots \shuffle \pi_1^*(u_k) \right) \otimes w.$$

It follows then the expected result.

5.2 Pair of bases in duality

Definition 16. Let \{\Sigma_l\}_{l \in \text{Lyn}Y} be the family of \text{Lie}_Q\langle Y\rangle obtained as follows

$$\Sigma_{y_k} = \pi_1(y_k) \quad \text{for} \quad k \geq 1,$$

$$\Sigma_l = [\Sigma_s, \Sigma_r] \quad \text{for} \quad l \in \text{Lyn}X, \text{ standard factorization of } l = (s,r),$$

and the family \{\Sigma_w\}_{w \in Y^*} of \text{U}(\text{Lie}_Q\langle Y\rangle) (viewed as a Q-module) obtained as follows

$$\Sigma_l = 1 \quad \text{for} \quad l = 1_{Y^*},$$

$$\Sigma_w = \Sigma_{l_1}^{i_1} \ldots \Sigma_{l_k}^{i_k} \quad \text{for} \quad w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k, l_1, \ldots, l_k \in \text{Lyn}Y.$$

Let \{\Sigma_w\}_{w \in Y^*} be the family of the quasi-shuffle algebra (viewed as a Q-module) obtained by duality with \{\Sigma_w\}_{w \in Y^*}:

$$\forall u, v \in Y^*, \quad \langle \Sigma_v | \Sigma_u \rangle = \delta_{u,v}.$$

Proposition 19. 1. For \(l \in \text{Lyn}Y\), the polynomial \(\Sigma_l\) is upper triangular:

$$\Sigma_l = l + \sum_{v > w, (v) = (l)} c_{v,u}.$$
2. The families \( \{ \Sigma_w \}_{w \in Y^*} \) and \( \{ \bar{\Sigma}_w \}_{w \in Y^*} \) are upper and lower triangular respectively. On other words, for any \( w \in Y^+ \), one has

\[
\Sigma_w = w + \sum_{v > w, (v) = (w)} c_v v \quad \text{and} \quad \bar{\Sigma}_w = w + \sum_{v < w, (v) = (w)} d_v v,
\]

where, for any \( y_k \in Y \) and \( w \in Y^* \), \((w)\) denotes the degree of \( w \) and \((y_k) = \deg(y_k) = k\).

**Proof.**

1. Let us prove it by induction on the length of \( l \):

   - The result is immediat for \( l \in Y \).
   - The result is suppose verified for any \( l \in \mathcal{L}Y \cap Y^k \) and \( 0 \leq k \leq N \).
   - At \( N + 1 \), by the standard factorization \( (l_1, l_2) \) of \( l \), one has, by definition, \( \Sigma_l = [\Sigma_{l_1}, \Sigma_{l_2}] \) and \( l_2l_1 > l_1l_2 \). By induction hypothesis,

\[
\Sigma_{l_1} = l_1 + \sum_{v > l_1, (v) = (l_1)} c_v v \quad \text{and} \quad \Sigma_{l_2} = l_2 + \sum_{v > l_2, (v) = (l_2)} d_v v,
\]

\[
\Rightarrow \quad \Sigma_l = l + \sum_{(w) = (l)} e_w w,
\]

getting \( e_w \)'s from \( c_v \)'s and \( d_u \)'s. Actually, the Lie bracket gives

\[
\Sigma_l = [l_1, l_2] + \sum_{w > l_1, (w) = (l_1)} d_w l_1 u + \sum_{v > l_2, (v) = (l_2)} c_v v u - \sum_{w > l_1, (w) = (l_1)} c_w l_2 v - \sum_{v > l_2, (v) = (l_2)} c_v d_w v u = [l_1, l_2] + \sum_{w > l_1, (w) = (l_1)} c'_w v - \sum_{w > l_1, (w) = (l_1)} c_v d_u v u
\]

\[
= [l_1, l_2] + \sum_{w > l_1, (w) = (l_1)} c'_w v - \sum_{w > l_1, (w) = (l_1)} c_v d_w v u - \sum_{w > l_2, (w) = (l_2)} d'_w u + \sum_{v > l_2, (v) = (l_2)} c_v d_w v u
\]

\[
= [l_1, l_2] + \sum_{w > l_1, (w) = (l_1)} c'_w v - \sum_{w > l_1, (w) = (l_1)} c_v d_w v u - \sum_{w > l_2, (w) = (l_2)} d'_w u + \sum_{v > l_2, (v) = (l_2)} c_v d_w v u
\]

Hence, the conclusion follows.

2. Let \( w = l_1 \ldots l_k \), with \( l_1 > \ldots > l_k \) and \( l_1, \ldots, l_k \in \mathcal{L}Y \). By (ii),

\[
\Sigma_{l_i} = l_i + \sum_{w > l_i, (w) = (l_i)} c_v v \quad \text{and} \quad \Sigma_w = l_1 \ldots l_k + \sum_{w > l_i, (w) = (l_i)} d_v v,
\]
where the $d_n$'s are obtained from the $c_{i,v}$'s. Hence, the family $\{\Sigma_w\}_{w \in Y^*}$ is upper triangular and, by duality, the family $\{\tilde{\Sigma}_w\}_{w \in Y^*}$ is lower triangular.

**Theorem 20.**
1. The family $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$ forms a basis of the free Lie algebra.
2. The family $\{\Sigma_w\}_{w \in Y^*}$ forms a basis of the free associative algebra $\mathbb{Q}\langle Y \rangle$.
3. The family $\{\tilde{\Sigma}_w\}_{w \in Y^*}$ generate freely the quasi-shuffle algebra.
4. The family $\{\tilde{\Sigma}_l\}_{l \in \mathcal{L}ynY}$ forms a transcendence basis of the quasi-shuffle algebra.

**Proof.** The family $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$ of upper triangular polynomials is free and then, by a theorem of Viennot, we get the first result. The second is a direct consequence of the Poincaré-Birkhoff-Witt theorem. By the Cartier-Quillen-Milnor-Moore theorem, we get the third one and the last one is also obtained as consequence of the constructions of the families $\{\tilde{\Sigma}_l\}_{l \in \mathcal{L}ynY}$ and $\{\tilde{\Sigma}_w\}_{w \in Y^*}$ of lower triangular polynomials.

Now, let us clarify the basis $\{\tilde{\Sigma}_w\}_{w \in Y^*}$ and then the transcendence basis $\{\tilde{\Sigma}_l\}_{l \in \mathcal{L}ynY}$ of the quasi-shuffle algebra $(\mathbb{Q}\langle Y \rangle, \mathfrak{m})$ as follows.

**Theorem 21.** We have
1. For $w = 1_{Y^*}$, $\tilde{\Sigma}_w = 1$.
2. For any $w = l_1^{i_1} \cdots l_k^{i_k}$, with $l_1, \ldots, l_k \in \mathcal{L}ynY$ and $l_1 > \ldots > l_k$,
   $$\tilde{\Sigma}_w = \frac{\tilde{\Sigma}_{l_1^{i_1}} \cdots \tilde{\Sigma}_{l_k^{i_k}}}{i_1! \cdots i_k!}.$$
3. For any $y \in Y$, $\tilde{\Sigma}_y = \pi_1^*(y)$.
4. For any $l = yu \in \mathcal{L}ynY - Y$, $\tilde{\Sigma}_l = y \tilde{\Sigma}_u$.

**Proof.**
1. Since $\Sigma_{1_{Y^*}} = 1$ then $\Sigma_{1_{Y^*}} = 1$.
2. Let $u = u_1 \ldots u_n = l_1^{i_1} \cdots l_k^{i_k}, v = v_1 \ldots v_m = h_1^{j_1} \cdots h_p^{j_p}$ with $l_1, \ldots, l_k, h_1, \ldots, h_p, u_1, \ldots, u_n, v_1, \ldots, v_m \in \mathcal{L}ynY, l_1 > \ldots > l_k, h_1 > \ldots > h_p$, $u_1 \geq \ldots \geq u_n, v_1 \geq \ldots \geq v_m$ and $i_1 + \ldots + i_k = n, j_1 + \ldots + j_p = m$. Hence, if $m \geq 2$ (resp. $n \geq 2$) then $v \notin \mathcal{L}ynY$ (resp. $u \notin \mathcal{L}ynY$).

Since
$$\langle \tilde{\Sigma}_{u_1} \cdots \tilde{\Sigma}_{u_n} | \prod_{i=1}^n \Sigma_{u_i} \rangle = \langle \tilde{\Sigma}_{u_1} \otimes \cdots \otimes \Sigma_{u_n} | \Delta_{\mathfrak{m}}^{(n-1)}(\Sigma_{v_1} \cdots \Sigma_{v_m}) \rangle$$
then many cases occur:
(a) Case \( n > m \). By Lemma 23(1), \((\tilde{\Sigma}_u, \tilde{\Sigma}_v, \ldots, \tilde{\Sigma}_n | \Sigma_{v_1} \ldots \Sigma_{v_m}) = 0\).

(b) Case \( n = m \). By Lemma 23(2), one has

\[
\langle \tilde{\Sigma}_u | \Sigma_{v_1} \ldots \Sigma_{v_m} \rangle = \sum_{\sigma \in \tilde{\Sigma}_n} \prod_{i=1}^n \delta_{\Sigma_{u_i}, \Sigma_{v_{\sigma(i)}}}.
\]

Thus, if \( u \neq v \) then \((u_1, \ldots, u_n) \neq (v_1, \ldots, v_n)\) then the second member is vanishing else, i.e. \( u = v \), the second member equals 1 because the factorization by Lyndon words is unique.

(c) Case \( n < m \). By Lemma 23(3), let us consider the following language over the alphabet \( A = \{\Sigma_{v_1}, \ldots, \Sigma_{v_m}\} \):

\[
\mathcal{M} = \{ w \in \mathcal{A}^* | w = \Sigma_{v_{j_1}} \ldots \Sigma_{v_{j_{|w|}}}, j_1 < \ldots < j_{|w|}, |w| \geq 1 \},
\]

and the morphism \( \mu : \mathbb{Q}(\mathcal{A}) \rightarrow \mathbb{Q}(Y) \). We get:

\[
\langle \tilde{\Sigma}_i \tilde{\Sigma}_u \ldots \tilde{\Sigma}_n | \prod_{i=1}^n \Sigma_{u_i} \rangle = \sum_{w_1, \ldots, w_m \in \mathcal{M}} \prod_{i=1}^n \langle \tilde{\Sigma}_u_i | \mu(w_i) \rangle = 0.
\]

Because in this product, on one hand, there exists at least one \( w_i \in \mathcal{M}, |w_i| \geq 2 \), corresponding to \( \Sigma_{v_{j_1}} \ldots \Sigma_{v_{j_{|w_i|}}} = \mu(w_i) \) such that \( v_{j_1} \geq \ldots \geq v_{j_{|w_i|}} \) and on other hand, \( \nu_i := v_{j_1} \ldots v_{j_{|w_i|}} \notin \text{Lyn}Y \) and \( u_i \in \text{Lyn}Y \).

By consequent,

\[
\langle \tilde{\Sigma}_u | \Sigma_{v} \rangle = \langle \tilde{\Sigma}_{j_1} \tilde{\Sigma}_{j_2} \ldots \tilde{\Sigma}_{j_k} | \Sigma_{h_1} \ldots \Sigma_{h_p} \rangle = \delta_{u,v}.
\]

3. For any \( l \in Y, \Sigma_l = \pi_1(l), \tilde{\Sigma}_l = \pi_1^*(l) \) and \( \pi_1, \pi_1^* \) are mutually adjoint.

4. By decomposing \( bu \) in the PBW-Lyndon basis, we get on one hand,

\[
bu = \sum_{w \in Y^*} \langle \tilde{\Sigma}_w | bu \rangle \Sigma_w + \sum_{aw \in \text{Lyn}Y} \langle \tilde{\Sigma}_{aw} | bu \rangle \Sigma_{aw} + \text{sum of decreasing products of length } \geq 2, \text{ of Lie polynomials}.
\]

Since for any \( aw = l_1^{i_1} \ldots l_k^{i_k} \notin \text{Lyn}Y \) \((k \geq 2)\) with \( l_1, \ldots, l_k \in \text{Lyn}Y \) and \( l_1 > \ldots > l_k \), then \( \Sigma_{aw} = \Sigma_{l_1} \ldots \Sigma_{l_k} \). Since (see Proposition 19)

\[
\Sigma_w = w + \sum_{v > w, (v) = (w)} c_v v \text{ and } \Sigma_{bw} = bw + \sum_{v > bw, (v) = (bw)} d_v v
\]

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then, by decomposing \( u \) in the PBW-Lyndon basis and then by multiplying by \( b \), we get on other hand,

\[
bu = \sum_{w \in Y^*} \langle \Sigma_w | u \rangle b \Sigma_w
\]

\[
= \sum_{w \in Y^*} \langle \Sigma_w | u \rangle \left( \Sigma_{bw} - \sum_{v > bw, (v) = (bw)} d_v \sum_{w' \in Y^*} \langle \Sigma_{w'} | v \rangle \Sigma_{w'} \right)
\]

(by decomposing \( v \) in PBW-Lyndon basis)

\[
= \sum_{w \in Y^*} \langle \Sigma_w | u \rangle \Sigma_{bw} - \sum_{w \in Y^*} \sum_{v > bw, (v) = (bw)} \sum_{w' \in Y^*} d_v \langle \Sigma_{w'} | v \rangle \Sigma_{w'}
\]

+ sum of decreasing products, of length \( \geq 2 \), of Lie polynomials.

After splitting these two sums on two disjoint supports, one has

- for any \( bw = l_1^{i_1} \ldots l_n^{i_n} \notin L_{yn}Y \) (\( n \geq 2 \)) with \( l_1 > \ldots > l_n \) and \( \Sigma_{aw} = \Sigma_{l_1}^{i_1} \ldots \Sigma_{l_n}^{i_n} \).
- for any \( w' = \lambda_1^{j_1} \ldots \lambda_m^{j_m} \notin L_{yn}Y \) (\( m \geq 2 \)) with \( \lambda_1 > \ldots > \lambda_m \) and \( \Sigma_{aw'} = \Sigma_{\lambda_1}^{j_1} \ldots \Sigma_{\lambda_m}^{j_m} \).

In the second sum, since each word \( v \) is greater than the Lyndon word \( bw \), then the Lie polynomial \( \Sigma_{bw} \) does not appear in the decomposition, in the PBW-Lyndon basis, of \( v \). More precisely, (see Proposition 19)

\[
\Sigma_{w'} = w' + \sum_{w' > v', (w') = (v')} e_{w'} v' \quad \text{with} \quad e_{w'} \geq 0,
\]

\[
\Rightarrow \langle \Sigma_{w'} | v \rangle = \langle w' | v \rangle + \sum_{w' > v', (w') = (v')} e_{v'} \langle v' | v \rangle.
\]

In particular (for \( w' = bw \in L_{yn}Y \)), the coefficient of the Lie polynomial \( \Sigma_{bw} \) in the decomposition of \( v \) (\( > bw \)) is vanishing:

\[
\langle \Sigma_{bw} | v \rangle = \langle bw | v \rangle + \sum_{v > bw, (v) = (bw)} e_{v'} \langle v' | v \rangle = 0.
\]

Thus, by identifying the coefficients in these two expressions of Lyndon word \( bu \), one has \( \langle \Sigma_{aw} | bu \rangle = \delta_{a,b} \langle \Sigma_w | u \rangle \). In other words, \( \Sigma_{bw} = b \Sigma_w \).

Corollary 16. 1. For \( w \in Y^+ \), the polynomial \( \Sigma_w \) is proper and homogeneous of degree \( |w| \), for \( \deg(y_i) = i \), and of rational positive coefficients.
\[ \sum_{w \in Y^*} w \otimes w = \sum_{w \in Y^*} \tilde{\Sigma}_w \otimes \Sigma_w = \prod_{l \in \mathcal{L}ynY} \exp(\tilde{\Sigma}_l \otimes \Sigma_l). \]

3. The family \( \mathcal{L}ynY \) forms a transcendence basis\(^{30}\) of the quasi-shuffle algebra and the family of proper polynomials of rational positive coefficients defined by, for any \( w = l_1^{i_1} \ldots l_k^{i_k} \) with \( l_1 > \ldots > l_k \) and \( l_1, \ldots, l_k \in \mathcal{L}ynY \),

\[ \chi_w = \frac{l_1^{i_1} \ldots l_k^{i_k}}{i_1! \ldots i_k!} \]

forms a basis of the quasi-shuffle algebra.

4. Let \( \{ \xi_{w} \}_{w \in Y^*} \) be the basis of the envelopping algebra \( \mathcal{U}(\mathfrak{L}ie_{\mathbb{Q}} \langle X \rangle) \) obtained by duality with the basis \( \{ \chi_{w} \}_{w \in Y^*} : \)

\[ \forall u, v \in Y^*, \quad \langle \chi_v \ | \ \xi_u \rangle = \delta_{u,v}. \]

Then the family \( \{ \Omega_l \}_{l \in \mathcal{L}ynY} \) forms a basis of the free Lie algebra \( \mathfrak{L}ie_{\mathbb{Q}} \langle Y \rangle \).

**Proof.**

1. The proof can be done by induction on the length of \( w \) using the fact that the product \( \otimes \) conserve the property, \( l \) homogeneity and rational positivity of the coefficients.

2. Expressing \( w \) in the basis \( \{ \tilde{\Sigma}_w \}_{w \in Y^*} \) of the quasi-shuffle algebra and then in the basis \( \{ \Sigma_w \}_{w \in Y^*} \) of the envelopping algebra, we obtain successively

\[ \sum_{w \in Y^*} w \otimes w = \sum_{w \in Y^*} \left( \sum_{u \in X^*} \langle \Sigma_u \ | \ w \rangle \Sigma_u \right) \otimes w \]
\[ = \sum_{u \in X^*} \tilde{\Sigma}_u \otimes \left( \sum_{w \in X^*} \langle \Sigma_u \ | \ w \rangle w \right) \]
\[ = \sum_{u \in Y^*} \tilde{\Sigma}_u \otimes \Sigma_u \]
\[ = \sum_{i_1 > \ldots > i_k} \frac{\tilde{\Sigma}_{i_1^{l_1} \ldots i_k^{l_k}}}{i_1! \ldots i_k!} \otimes \Sigma_{i_1} \ldots \Sigma_{i_k} \]
\[ = \prod_{l \in \mathcal{L}ynY} \sum_{i \geq 0} \frac{\tilde{\Sigma}_l^{i}}{i!} \otimes \Sigma_l^{i} \]
\[ = \prod_{l \in \mathcal{L}ynY} \exp(\tilde{\Sigma}_l \otimes \Sigma_l). \]

\(^{30}\)This result is an analogous of a Radford theorem (see [45]). Thus the bases \( \mathcal{L}ynY \) and \( \{ \tilde{\Sigma}_l \}_{l \in \mathcal{L}ynY} \) belong to the class of Radford bases, i.e. the class of transcendence bases, of the quasi-shuffle algebra, as well as the bases \( \mathcal{L}ynX \) and \( \{ S_l \}_{l \in \mathcal{L}ynX} \) belong to the class of Radford bases of the shuffle algebra.
3. For \( w = l_1^i \ldots l_k^i \) with \( l_1, \ldots, l_k \in \mathcal{L} \mathcal{y} \mathcal{n} \mathcal{Y} \) and \( l_1 > \ldots > l_k \), by Proposition 19, the polynomial of rational positive coefficients \( \hat{\Sigma}_w \) is lower triangular:

\[
\hat{\Sigma}_w = \frac{\hat{\Sigma}_{i_1 l_1} \ldots \hat{\Sigma}_{i_k l_k}}{i_1! \ldots i_k!} = w + \sum_{v < w, (v) = (w)} c_v v.
\]

In particular, for any \( l_j \in \mathcal{L} \mathcal{y} \mathcal{n} \mathcal{Y} \), \( \hat{\Sigma}_{l_j} \) is lower triangular:

\[
\hat{\Sigma}_{l_j} = l_j + \sum_{v < l_j, (v) = (l_j)} c_v v.
\]

Hence, \( \Sigma_w = \chi_w + \chi'_w \), where \( \chi'_w \) is a proper polynomial of \( \mathbb{Q}\langle \mathcal{Y} \rangle \) of rational positive coefficients. We deduce then the support of \( \chi_w \) contains words which are less than \( w \) and \( \langle \chi_w | w \rangle = 1 \). Thus, the proper polynomial \( \chi_w \) of rational positive coefficients is lower triangular:

\[
\chi_w = w + \sum_{v < w, (v) = (w)} c_v v,
\]

\[
\Rightarrow \forall l \in \mathcal{L} \mathcal{y} \mathcal{n} \mathcal{Y}, \quad \chi_l = l + \sum_{v < l, (v) = (l)} c_v v.
\]

It follows then expected results.

4. By duality, for \( w \in \mathcal{Y}^* \), the proper polynomial \( \xi_w \) is upper triangular. In particular, for any \( l \in \mathcal{L} \mathcal{y} \mathcal{n} \mathcal{Y} \), the proper polynomial \( \xi_l \) is upper triangular:

\[
\xi_l = l + \sum_{v > l, (v) = (l)} d_v v.
\]

Hence, the family \( \{\xi_l\}_{l \in \mathcal{L} \mathcal{y} \mathcal{n} \mathcal{Y}} \) is free and its elements verify an analogous of the generalized criterion of Friedrichs:

- for \( w \in \mathcal{L} \mathcal{y} \mathcal{n} \mathcal{Y} \), one has \( \langle \chi_w | \xi_l \rangle = \delta_{w,l} \),
- for \( w \notin \mathcal{L} \mathcal{y} \mathcal{n} \mathcal{Y} \), \( w = l_1 \ldots l_n \) with \( l_1, \ldots, l_n \in \mathcal{L} \mathcal{y} \mathcal{n} \mathcal{Y} \) and \( l_1 > \ldots > l_n \), one has \( \langle \chi_w | \xi_l \rangle = \langle \chi_{l_1} \xi_{l_2} \ldots \xi_{l_n} | \xi_l \rangle = 0 \).

Moreover, the polynomials \( \xi_l \)'s are primitive: by Corollary 16(3), one has

\[
\Delta \langle \xi_l \rangle = \sum_{u,v \in \mathcal{Y}^*} \langle u \otimes v | \xi_l \rangle u \otimes v
\]

\[
= \sum_{u \in \mathcal{Y}^*} \langle u | 1_{\mathcal{Y}^*} \otimes \xi_l \rangle u \otimes 1_{\mathcal{Y}^*} + \sum_{v \in \mathcal{Y}^*} \langle 1_{\mathcal{Y}^*} \otimes v | \xi_l \rangle 1_{\mathcal{Y}^*} \otimes v
\]

\[
+ \sum_{u,v \in \mathcal{Y}^+} \langle u \otimes v | \xi_l \rangle u \otimes v + \langle 1_{\mathcal{Y}^*} \otimes 1_{\mathcal{Y}^*} | \xi_l \rangle 1_{\mathcal{Y}^*} \otimes 1_{\mathcal{Y}^*}
\]

\[
= \xi_l \otimes 1_{\mathcal{Y}^*} + 1_{\mathcal{Y}^*} \otimes \xi_l.
\]

Because, after decomposing \( u \) and \( v \) on the basis \( \{\chi_l\}_{l \in \mathcal{L} \mathcal{y} \mathcal{n} \mathcal{Y}} \) and by the previous criterion, the third term is vanishing. The last one is also vanishing since the \( \xi_l \)'s are proper. By a theorem of Viennot, we obtain then the expected result.
5.3 Proof of Theorem 2

Applying the tensor product of isomorphisms $H \otimes \text{Id}$ (Proposition 1) on the diagonal series (Corollary 16(ii)), the infinite factorization, by Lyndon words, of the noncommutative generating series of harmonic sums follows$^{31}$:

$$H(N) = \sum_{w \in Y^*} H_w(N)w = \prod_{l \in \text{Lyn}Y} \exp(H_{\Sigma_l}(N)\Sigma_l).$$

(111)

6 Annex B : differential realization

To facilitate reading, the following results are placed in this Annex which can be skipped by readers already familiar with the techniques developed by Fliess (and adapted by us for studies in this paper).

6.1 Polysystem and convergence criterion

6.1.1 Serial estimates from above

Here, generalizing a little, $K$ is supposed a $\mathbb{C}$-algebra and a complete normed vector space equipped with a norm denoted by $\|\|$. For any $n \in \mathbb{N}$, $X^{\geq n}$ denotes the set of words over $X$ of length greater than or equal to $n$. The set of formal power series (resp. polynomials) on $X$, is denoted by $K[[X]]$ (resp. $K[X]$).

**Definition 17 ([25, 39]).** Let $\xi, \chi$ be real positive functions over $X^*$. Let $S \in K[[X]]$.

1. $S$ will be said $\xi$–exponentially bounded from above if it verifies

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \quad \|\langle S \mid w\rangle\| \leq K\xi(w)/|w|!.$$

We denote by $K^{\xi-\text{em}}[[X]]$ the set of formal power series in $K[[X]]$ which are $\xi$–exponentially bounded from above.

2. $S$ verifies the $\chi$–growth condition if it satisfies

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \quad \|\langle S \mid w\rangle\| \leq K\chi(w)|w|!.$$

We denote by $K^{\chi-\text{gc}}[[X]]$ the set of formal power series in $K[[X]]$ verifying the $\chi$–growth condition.

$^{31}$This proof omitted in previous versions uses mainly the results presented in this annexe that have not been published earlier but have already been presented at various workshops. It is an analogous way to obtain the infinite factorization, by Lyndon words over the alphabet $X$, of the noncommutative generating series of polylogarithms (see Theorem 3) by applying the tensor product of isomorphisms $Li \otimes \text{Id}$ (see Proposition 1) on the diagonal series, over $X$. 

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Lemma 24. We have
\[ R = \sum_{w \in X^*} |w|! |w| \Rightarrow \langle R^{1/2} | w \rangle = \sum_{u, v \in X^* \text{ supp}(u \cup v) \ni w} |u|! |v|! \leq 2^{|w|} |w|!. \]

Proof. One has
\[
\sum_{u, v \in X^* \text{ supp}(u \cup v) \ni w}|u|! |v|! = \sum_{k=0}^{|w|}\sum_{|u|+|v|=|w|\neg k} k!(|w|\neg k)!
\]
\[
= \sum_{k=0}^{|w|}\binom{|w|}{k} k!(|w|\neg k)!
\]
\[
= \sum_{k=0}^{|w|} |w|!
\]
\[
= (1+|w|) |w|!.
\]

By induction on the length of \( w \), one has \( 1 + |w| \leq 2^{|w|} \). It follows the expected result.

Proposition 20. Let \( S_1 \) and \( S_2 \) verifying the growth condition. Then \( S_1 + S_2 \) and \( S_1 \uplus S_2 \) also verifies the growth condition.

Proof. The proof for \( S_1 + S_2 \) is immediate.

Next, since \( \| \langle S_i | w \rangle \| \leq K_i \chi_i(w) |w|! \), for \( i = 1 \) or \( 2 \) and for \( w \in X^* \), then\(^{32}\)
\[
\langle S_1 \uplus S_2 | w \rangle = \sum_{u, v \in X^* \text{ supp}(u \uplus v) \ni w} \langle S_1 | u \rangle \langle S_2 | v \rangle,
\]
\[
\Rightarrow \| \langle S_1 \uplus S_2 | w \rangle \| \leq K_1 K_2 \sum_{u, v \in X^* \text{ supp}(u \uplus v) \ni w} (\chi_1(u) |u|!)(\chi_2(v) |v|!).
\]

Let \( K = K_1 K_2 \) and let \( \chi \) be a real positive function over \( X^* \) such that
\[
\forall w \in X^*, \quad \chi(w) = \max\{\chi_1(u)\chi_2(v) \mid u, v \in X^* \text{ and supp}(u \uplus v) \ni w\}.
\]

With the notations in Lemma 24, we get
\[
\| \langle S_1 \uplus S_2 | w \rangle \| \leq K \chi(w) (R^{1/2} | w \rangle.
\]

Hence, \( S_1 \uplus S_2 \) verifies the \( \chi' \)-growth condition with \( \chi' \) defined as \( \chi'(w) = 2^{|w|} \chi(w) \).

Definition 18 ([25, 39]). Let \( \xi \) be a real positive function defined over \( X^* \), \( S \) will be said \( \xi \)-exponentially continuous if it is continuous over \( K^\xi-\text{em}\langle \langle X \rangle \rangle \). The set of formal power series which are \( \xi \)-exponentially continuous is denoted by \( K^\xi-\text{ec}\langle \langle X \rangle \rangle \).

\(^{32}\langle S_1 \uplus S_2 | w \rangle \) is the coefficient of the word \( w \) in the power series \( S_1 \uplus S_2 \).
Lemma 25 ([25, 39]). For any real positive function $\xi$ defined over $X^*$, we have $\mathbb{K}\langle X \rangle \subset \mathbb{K}^{\xi-ec}\langle X \rangle$. Otherwise, for $\xi = 0$, we get $\mathbb{K}\langle X \rangle = \mathbb{K}^{0-ec}\langle X \rangle$. Hence, any polynomial is $0-$exponentially continuous.

Proposition 21 ([25, 39]). Let $\xi, \chi$ be a real positive functions over $X^*$ and let $P \in \mathbb{K}\langle X \rangle$.

1. Let $S \in \mathbb{K}^{\xi-em}\langle X \rangle$. The right residual of $S$ by $P$ belongs to $\mathbb{K}^{\xi-em}\langle X \rangle$.

2. Let $R \in \mathbb{K}^{\chi-sc}\langle X \rangle$. The concatenation $SR$ belongs to $\mathbb{K}^{\chi-sc}\langle X \rangle$.

Proof. 1. Since $S \in \mathbb{K}^{\xi-em}\langle X \rangle$ then

$$\exists K \in \mathbb{R}^+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \quad \|S \mid w\| \leq K\xi(w)/|w|!.$$ 

If $u \in \text{supp}(P) := \{w \in X^* \mid \langle P \mid w \rangle \neq 0\}$ then, for any $w \in X^*$, one has $\langle S \triangleright u \mid w \rangle = \langle S \mid uw \rangle$ and $S \triangleright u$ belongs to $\mathbb{K}^{\xi-em}\langle X \rangle$ :

$$\exists K \in \mathbb{R}^+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \quad \|S \triangleright u \mid w\| \leq |K\xi(u)|\xi(w)/|w|!.$$ 

It follows then $S \triangleright P$ is $\mathbb{K}^{\xi-em}\langle X \rangle$ by taking $K_1 = K \max_{u \in \text{supp}(P)} \xi(u)$.

2. Since $R \in \mathbb{K}^{\chi-sc}\langle X \rangle$ then

$$\exists K \in \mathbb{R}^+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \quad \|R \mid w\| \leq K\chi(w)/|w|!.$$ 

Let $v \in \text{supp}(P)$ such that $v \neq e$. Since, for any $w \in X^*$, $Rv$ belongs to $\mathbb{K}^{\chi-sc}\langle X \rangle$ and one has $\langle Rv \mid w \rangle = \langle R \mid v \triangleright w \rangle$ :

$$\exists K \in \mathbb{R}^+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \quad \|R \mid v \triangleright w\| \leq K\chi(v \triangleright w)(|w| - |v|)! \leq K |w| \chi(w)/\chi(v).$$ 

Note that if $v \triangleleft w = 0$ then $\langle Rv \mid w \rangle = 0$ and the previous conclusion holds. It follows then $RP$ is $\mathbb{K}^{\chi-sc}\langle X \rangle$ by taking $K_2 = K \min_{v \in \text{supp}(P)} \chi(v)^{-1}$.

\[\square\]

Proposition 22 ([25, 39]). Two real positive morphisms over $X^*$, $\xi$ and $\chi$ are assumed to verify the condition

$$\sum_{x \in X} \chi(x)\xi(x) < 1.$$ 

Then for any $F \in \mathbb{K}^{\chi-sc}\langle X \rangle$, $F$ is continuous over $\mathbb{K}^{\xi-em}\langle X \rangle$.

Proof. If $\xi, \chi$ verify the upper bound condition then the following power series

$$\sum_{w \in X^*} \chi(w)\xi(w) = \left(\sum_{x \in X} \chi(x)\xi(x)\right)^{\ast}$$
is well defined. If \( F \in \mathbb{K}^{\mathfrak{X}^{sc}} \langle X \rangle \) and \( C \in \mathbb{K}^{\mathfrak{X}^{em}} \langle X \rangle \) then there exists \( K_i \in \mathbb{R}^+ \) and \( n_i \in \mathbb{N} \) such that for any \( w \in X^{\geq n_i}, \ i = 1, 2 \), one has

\[
\| F \cdot w \| \leq K_1 \chi(w) \| w \| \quad \text{and} \quad \| C \cdot w \| \leq K_2 \chi(w) \| w \|.
\]

Hence,

\[
\forall w \in X^*, \ |w| \geq \max\{n_1, n_2\}, \quad \| F \cdot w \| \langle C \cdot w \| \leq K_1 K_2 \chi(w) \xi(w),
\]

\[
\Rightarrow \quad \sum_{w \in X^*} \| (F \cdot w) \langle C \cdot w \| \leq K_1 K_2 \sum_{w \in X^*} \chi(w) \xi(w) = K_1 K_2 \left( \sum_{x \in X} \chi(x) \xi(x) \right)^\ast.
\]

\[\Box\]

6.1.2 Upper bounds à la Cauchy

Let \( q_1, \ldots, q_n \) be commutative indeterminates over \( \mathbb{C} \). The algebra of formal power series (resp. polynomials) over \( \{q_1, \ldots, q_n\} \) with coefficients in \( \mathbb{C} \) is denoted by \( \mathbb{C}[q_1, \ldots, q_n] \) (resp. \( \mathbb{C}[q_1, \ldots, q_n] \)).

**Definition 19** ([25, 39]). Let

\[
f = \sum_{i_1, \ldots, i_n \geq 0} f_{i_1, \ldots, i_n} q_1^{i_1} \cdots q_n^{i_n} \in \mathbb{C}[q_1, \ldots, q_n].
\]

We set

\[
E(f) := \{ \rho \in \mathbb{R}^n : \exists C_f \in \mathbb{R}^+ \text{ s.t. } \forall i_1, \ldots, i_n \geq 0, \ |f_{i_1, \ldots, i_n}| \rho_{i_1}^{\ast} \cdots \rho_{i_n}^{\ast} \leq C_f \}
\]

\[
\tilde{E}(f) := \text{interior of } E(f) \text{ in } \mathbb{R}^n.
\]

\[
CV(f) := \{ q \in \mathbb{C}^n : \langle |q_1|, \ldots, |q_n| \rangle \in \tilde{E}(f) \} : \text{convergence domain of } f.
\]

The power series \( f \) is to be said convergent if \( CV(f) \neq \emptyset \). Let \( \mathcal{U} \) be an open domain in \( \mathbb{C}^n \) and let \( q \in \mathbb{C}^n \). The power series \( f \) is to be said convergent on \( q \) (resp. over \( U \)) if \( q \in CV(f) \) (resp. \( U \subset CV(f) \)). We set

\[
\mathbb{C}^{CV} [q_1, \ldots, q_n] = \{ f \in \mathbb{C}[q_1, \ldots, q_n] : CV(f) \neq \emptyset \}.
\]

Let \( q \in CV(f) \). There exists the constants \( C_f, \rho \) and \( \tilde{\rho} \) such that \( |q_1| < \tilde{\rho} < \rho, \ldots, |q_n| < \tilde{\rho} < \rho \) and, for \( i_1, \ldots, i_n \geq 0, \ |f_{i_1, \ldots, i_n}| \rho_{i_1}^{\ast} \cdots \rho_{i_n}^{\ast} \leq C_f \). The convergence modulus of \( f \) at \( q \) is \( (C_f, \rho, \tilde{\rho}) \).

Suppose that \( CV(f) \neq \emptyset \) and let \( q \in CV(f) \). If \( (C_f, \rho, \tilde{\rho}) \) is a convergence modulus of \( f \) at \( q \) then \( |f_{i_1, \ldots, i_n} q_1^{i_1} \cdots q_n^{i_n}| \leq C_f (\tilde{\rho} / \rho_1)^{i_1} \cdots (\tilde{\rho} / \rho_1)^{i_n} \). Hence, at \( q \), the power series \( f \) is majorized termwise by

\[
C_f \prod_{k=0}^{m} \left( 1 - \frac{\tilde{\rho}_k}{\rho_k} \right)^{-1}.
\]

(112)
Hence, \( f \) is uniformly absolutely convergent in \( \{ q \in \mathbb{C}^n : |q_1| < \rho, \ldots, |q_n| < \rho \} \) which is an open domain in \( \mathbb{C}^n \). Thus, \( \text{CV}(f) \) is an open domain in \( \mathbb{C}^n \). Since the partial derivation of order \( j_1, \ldots, j_n \geq 0 \) of \( f \) is estimated by

\[
\|D_{i_1}^{j_1} \cdots D_{i_n}^{j_n} f\| \leq C_f \frac{\partial^{j_1 + \cdots + j_n}}{\partial \rho^{j_1 + \cdots + j_n}} \prod_{k=0}^{m} \left( 1 - \frac{\rho_k}{\rho} \right)^{-1} .
\] (113)

**Proposition 23** ([25]). We have \( \text{CV}(f) \subset \text{CV}(D_{i_1}^{j_1} \cdots D_{i_n}^{j_n} f) \).

Let \( f \in \mathbb{C}^\text{cv}[q_1, \ldots, q_n] \) and let \( \{A_i\}_{i=0,1} \) be a polysystem defined as follows

\[
A_i(q) = \sum_{j=1}^{n} \frac{\partial}{\partial q_j} A_i^j(q), \quad \text{with} \quad A_i^j(q) \in \mathbb{C}^\text{cv}[q_1, \ldots, q_n], \quad j = 1, \ldots, n .
\] (114)

**Lemma 26** ([20]). For \( i = 0, 1 \) and \( j = 1, \ldots, n \), one has \( A_i \circ q_j = A_i^j(q) \). Thus,

\[
\forall i = 0, 1, \quad A_i(q) = \sum_{j=1}^{n} (A_i \circ q_j) \frac{\partial}{\partial q_j} .
\]

Let \( (\rho, \tilde{\rho}, C_f), \{(\rho, \tilde{\rho}, C_i)\}_{i=0,1} \) be respectively the convergence modulus at \( q \in \text{CV}(f) \cap \bigcap_{j=1}^{n} \text{CV}(A_i^j) \) (115)
of \( f \) and \( \{A_i^j\}_{j=1, \ldots, n} \). Let us consider the following monoid morphisms

\[
\forall w = v x_1 x_i \in X, v \in X^*, \quad A(w) = A(v)A_i \quad \text{and} \quad C(w) = C(v)C_i .
\] (117)

**Lemma 27** ([19]). For any word \( w \), \( A(w) \) is continuous over \( \mathbb{C}^\text{cv}[q_1, \ldots, q_n] \) and, for any \( f, g \in \mathbb{C}^\text{cv}[q_1, \ldots, q_n] \), one has

\[
A(w) \circ (f g) = \sum_{u, v \in X^*} (u \cdot w | v \cdot w) (A(u) \circ f)(A(v) \circ g) .
\]

These notations are extended, by linearity, to \( \mathbb{K}(X) \) and we will denote \( A(w) \circ f_q \) the evaluation of \( A(w) \circ f \) at \( q \).

**Definition 20** ([19]). Let \( f \in \mathbb{C}^\text{cv}[q_1, \ldots, q_n] \). The generating series of the polysystem \( \{A_i\}_{i=0,1} \) and of the observation \( f \) is given by

\[
\sigma f := \sum_{w \in X^*} A(w) \circ f w \in \mathbb{C}^\text{cv}[q_1, \ldots, q_n] \langle X \rangle .
\]

\[
\sigma f_q := \sum_{w \in X^*} A(w) \circ f_{w|q} w \in \mathbb{C}\langle X \rangle .
\]

The last generating series is called Fließ generating series of the polysystem \( \{A_i\}_{i=0,1} \) and of the observation \( f \) at \( q \).
Lemma 28 ([19]). Let \( \{A_i\}_{i=0,1} \) be a polysystem. Then, the map
\[
\sigma : (C^\infty[q_1, \ldots, q_n], \cdot) \rightarrow (C^\infty[q_1, \ldots, q_n] \langle X \rangle, \cdot),
\]
is an algebra morphism, i.e. for any \( f, g \in C^\infty[q_1, \ldots, q_n] \) and \( \mu, \nu \in \mathbb{C} \), one has \( \sigma(\nu f + \mu g) = \nu \sigma f + \mu \sigma g \) and \( \sigma(fg) = \sigma f \cdot \sigma g \).

Lemma 29 ([20]). Let \( \{A_i\}_{i=0,1} \) be a polysystem and \( f \in C^\infty[q_1, \ldots, q_n] \). Then
\[
\forall x_i \in X, \quad \sigma(A_i \circ f) = x_i \circ \sigma f \in C^\infty[q_1, \ldots, q_n] \langle X \rangle
\]
\[
\forall w \in X^+, \quad \sigma(A(w) \circ f) = w \circ \sigma f \in C^\infty[q_1, \ldots, q_n] \langle X \rangle.
\]

Lemma 30 ([25]). Let \( \tau = \min_{1 \leq k \leq n} \rho_k \) and \( r = \max_{1 \leq k \leq n} \hat{\rho}_k / \rho_k \). We have
\[
\|A(w) \circ f\| \leq C_f \frac{(n+1) C(w) |w|!}{(1 - r)^n} \left[ \frac{n}{\tau (1 - r)^{n+1}} \right]^{\rho|w|}
\]
\[
\leq C_f \frac{(n+1) C(w) |w|!}{(1 - r)^n} \left[ \frac{n}{\tau (1 - r)^{n+1}} \right]^{\rho|w|}.
\]

Theorem 22 ([25]). Let \( K = C_f (n+1)(1 - r)^{-n} \) and let \( \chi \) be the real positive function defined over \( X^+ \) by
\[
\forall i = 0, 1, \quad \chi(x_i) = \frac{C_i n}{\tau (1 - r)^{(n+1)}},
\]
Then the generating series \( \sigma f \) of the polysystem \( \{A_i\}_{i=0,1} \) and of the observation \( f \) satisfies the \( \chi \)-growth condition.

It is the same for the Fliess generating series \( \sigma f_{pq} \) of the polysystem \( \{A_i\}_{i=0,1} \) and of the observation \( f \) at \( q \).

6.2 Polysystems and nonlinear differential equation

6.2.1 Nonlinear differential equation (with three singularities)

Let us consider the following singular inputs\(^{33}\)
\[
u_0(z) := z^{-1} \quad \text{and} \quad u_1(z) := (1 - z)^{-1}, \quad (118)
\]
and the following nonlinear dynamical system\(^{34}\)
\[
\begin{aligned}
g(z) &= f(q(z)), \\
q(z) &= A_0(q) u_0(z) + A_1(q) u_1(z), \\
q(z_0) &= q_0,
\end{aligned}
\quad (119)
\]
\(^{33}\)These singular inputs are not included in the studies of Fliess motivated, in particular, by the renormalization of \( y(z) \) at \( +\infty \) [19, 20].
\(^{34}\)Any differential equation with singularities in \( \{a, b, c\} \), via homographic transformation \((z - a)(c - b)(z - b)^{-1}(c - a)^{-1}, \) can be changed into a differential equation with singularities in \( \{0, 1, +\infty\} \) (the singularities of homographic transformations belonging to the group \( \mathbb{G} \)).
where, the state $q = (q_1, \ldots, q_n)$ belongs to the complex analytic manifold of dimension $n$, $q_0$ is the initial state, the observation $f$ belongs to $C^\infty[q_1, \ldots, q_n]$ and $\{A_i\}_{i=0,1}$ is the polysystem defined on (114).

**Definition 21** ([27]). The following power series is called transport operator of the polysystem $\{A_i\}_{i=0,1}$ and of the observation $f$

$$T := \sum_{w \in X^*} \alpha^z_w(w) A(w).$$

By the factorization of the monoid by Lyndon words, we have [27]

$$T = (\alpha^z_{w_0} \otimes A) \left( \sum_{w \in X^*} w \otimes w \right) = \prod_{l \in \text{Lym} X} \exp[\alpha^z_{w_l}(S_l) A(\tilde{S}_l)]. \quad (120)$$

Let us consider again the Chen generating series $S_{z_0 \rightarrow z}$ given in (51) of the differential forms involed in (DE) of Example 1, i.e. $\omega_0(z) = u_0(z) \, dz$ and $\omega_1(z) = u_1(z) \, dz$, verifying the upper bound conditions given on (56).

6.2.2 Asymptotic behaviour of the successive differentiation of the output via extended Fliess fundamental formula

**Theorem 23** ([39]). The Fliess fundamental formula can be extended as follows

$$y(z) = T \circ f_{|q_0} = \sum_{w \in X^*} \langle S_{z_0 \rightarrow z} \mid w \rangle \langle A(w) \circ f_{|q_0} \mid w \rangle = \langle \sigma f_{|q_0} \mid S_{z_0 \rightarrow z} \rangle.$$ 

By the factorization of the Lie exponential series $L$, it follows the expansions of the output $y$ of nonlinear dynamical system with singular inputs,

**Corollary 17** ([39]).

$$y(z) = \sum_{w \in X^*} g_w(z) A(w) \circ f_{|q_0},$$

$$= \sum_{k \geq 0, n_1, 0, n_2 \geq 0} g_{x_0^n y_{z_1} x_0^n y_{z_2}}(z) \text{ ad}^{n_1}_{A_0} A_1 \ldots \text{ ad}^{n_2}_{A_0} A_1 e^{\log z A_0} \circ f_{|q_0},$$

$$= \prod_{l \in \text{Lym} X} \exp \left( g_{S_l}(z) A(\tilde{S}_l) \circ f_{|q_0} \right),$$

$$= \exp \left( \sum_{w \in X^*} g_w(z) A(\pi_1(w)) \circ f_{|q_0} \right).$$

where, for any word $w$ in $X^*$, $g_w$ belongs to the polylogarithm algebra.

Since $S_{z_0 \rightarrow z} = L(z)L(z_0)^{-1}$ and since $\sigma f_{|q_0}$ and $L(z_0)^{-1}$ are invariant by $\partial = d/dz$ then $\partial^l y(z) = (\sigma f_{|q_0} \mid \partial^l S_{z_0 \rightarrow z}) = (\sigma f_{|q_0} \mid \partial^l L(z)L(z_0)^{-1})$, for $l \geq 0$.

With the notations of Proposition 3, we get

$$\partial^l y(z) = \langle \sigma f_{|q_0} \mid [P_l(z)L(z)]L(z_0)^{-1} \rangle = \langle \sigma f_{|q_0} \triangleright P_l(z) \mid L(z)L(z_0)^{-1} \rangle. \quad (121)$$
For \( z_0 = \varepsilon \to 0^+ \), the asymptotic behaviour and the renormalization at \( z = 1 \) of \( \partial^l y(z) \) (or the asymptotic expansion and the renormalization of its Taylor coefficients at \(+\infty\)) are deduced from Proposition 5 and extend a little bit the results of [39] as follows

**Corollary 18.** For any integer \( l \), we have

\[
\partial^l y(1) \overset{\varepsilon \to 0^+}{\sim} \langle \sigma f_{\varepsilon_0} \circ P_1(1 - \varepsilon) \rangle e^{-x_1 \log \varepsilon} Z_{\varepsilon_0} e^{-x_0 \log \varepsilon} = \sum_{w \in X^*} (\mathcal{A}(w) \circ f_{\varepsilon_0} | u) \langle P_1(1 - \varepsilon) e^{-x_1 \log \varepsilon} Z_{\varepsilon_0} e^{-x_0 \log \varepsilon} | u \rangle.
\]

**Corollary 19.** The differentiation of order \( l \in \mathbb{N} \) of the output \( y \) of the dynamical system (119) is a \( C \)-combination of the elements \( g \) belonging to the polylogarithm algebra. If its ordinary Taylor expansion exists the coefficients of this expansion belong to the algebra of harmonic sums and there exists algorithmically computable coefficients \( a_i \in \mathbb{Z}, b_i \in \mathbb{N} \) and \( c_i \) belong to the \( C \)-algebra generated by \( \mathbb{Z} \) and by the Euler’s \( \gamma \) constant, such that

\[
\partial^l y(z) = \sum_{n \geq 0} y_n^{(l)} z^n, \quad y_n^{(l)} \overset{n \to \infty}{\sim} \sum_{i \geq 0} c_i n^{a_i} \log^{b_i} n.
\]

### 6.3 Differential realization

#### 6.3.1 Differential realization

**Definition 22.** The Lie rank of a formal power series \( S \in \mathbb{K} \langle X \rangle \) is the dimension of the vector space generated by \( \{ S \gg \Pi \mid \Pi \in \text{Lie}_\mathbb{K}(X) \} \), or by \( \{ \Pi \ll S \mid \Pi \in \text{Lie}_\mathbb{K}(X) \} \).

**Definition 23.** Let \( S \in \mathbb{K} \langle X \rangle \) and let us put

\[
\text{Ann}(S) := \{ \Pi \in \text{Lie}_\mathbb{K}(X) \mid S \gg \Pi = 0 \},
\]

\[
\text{Ann}^+(S) := \{ Q \in (\mathbb{K} \langle X \rangle, \omega) \mid Q \gg \text{Ann}(S) = 0 \}.
\]

It is immediate that \( \text{Ann}^+(S) \ni S \) and it follows that (see [20, 46])

**Lemma 31.** Let \( S \in \mathbb{K} \langle X \rangle \). If \( S \) is of finite Lie rank, \( d \), then the dimension of \( \text{Ann}^+(S) \) equals \( d \).

By Lemma 3, the residuals are derivations for shuffle product. Then,

**Lemma 32.** Let \( S \in \mathbb{K} \langle X \rangle \). Then :

1. For any \( Q_1 \) and \( Q_2 \in \text{Ann}^+(S) \), one has \( Q_1 \omega Q_2 \in \text{Ann}^+(S) \).
2. For any \( P \in \mathbb{K} \langle X \rangle \) and \( Q_1 \in \text{Ann}^+(S) \), one has \( P \triangleleft Q_1 \in \text{Ann}^+(S) \).

**Definition 24** ([20]). The formal power series \( S \in \mathbb{K} \langle X \rangle \) is differentially produced if there exists
• an integer $d$,
• a power series $f \in \mathbb{K}[\bar{q}_1, \ldots, \bar{q}_d]$,
• a homomorphism $A$ from $X^*$ maps to the algebra of differential operators generated by

$$A(x_i) = \sum_{j=1}^{d} A_1^j(\bar{q}_1, \ldots, \bar{q}_d) \frac{\partial}{\partial \bar{q}_j}, \quad A_1^j(\bar{q}_1, \ldots, \bar{q}_d) \in \mathbb{K}[\bar{q}_1, \ldots, \bar{q}_d], j = 1, \ldots, d,$$

such that, for any $w \in X^*$, $\langle S \mid w \rangle = A(w) \circ f|_w$.

The couple $(A, f)$ is called differential representation of $S$ of dimension $d$.

**Proposition 24 ([46])**. Let $S \in \mathbb{K}\langle \langle X \rangle \rangle$. If $S$ is differentially produced then it verifies the growth condition and its Lie rank is finite.

**Proof.** Let $(A, f)$ be a differential representation of $S$ of dimension $d$. Then, by the notations of Definition 20, we get

$$\sigma f|_w = S = \sum_{w \in X^*} (A(w) \circ f)|_w w.$$

For any $j = 1, \ldots, d$, we put

$$T_j = \sum_{w \in X^*} \frac{\partial (A(w) \circ f)}{\partial \bar{q}_j} w$$

$$\iff \forall w \in X^*, \quad \langle T_j \mid w \rangle = \frac{\partial (A(w) \circ f)}{\partial \bar{q}_j}.$$

Firstly, by Theorem 22, the generating series $\sigma f$ verifies the growth condition. Secondly, for any $\Pi \in \text{Lie}_\mathbb{K}\langle X \rangle$ and for any $w \in X^*$, one has

$$\langle \sigma f \triangleright \Pi \mid w \rangle = \langle \sigma f \mid \Pi w \rangle = A(\Pi w) \circ f = A(\Pi) \circ (A(w) \circ f).$$

Since $A(\Pi)$ is a derivation over $\mathbb{K}[\bar{q}_1, \ldots, \bar{q}_d]$

$$A(\Pi) = \sum_{j=1}^{d} (A(\Pi) \circ \bar{q}_j) \frac{\partial}{\partial \bar{q}_j},$$

$$\Rightarrow A(\Pi) \circ (A(w) \circ f) = \sum_{j=1}^{d} (A(\Pi) \circ \bar{q}_j) \frac{\partial (A(w) \circ f)}{\partial \bar{q}_j}$$

then we deduce that

$$\forall w \in X^*, \quad \langle \sigma f \triangleright \Pi \mid w \rangle = \sum_{j=1}^{d} (A(\Pi) \circ \bar{q}_j) \langle T_j \mid w \rangle,$$

$$\iff \sigma f \triangleright \Pi = \sum_{j=1}^{d} (A(\Pi) \circ \bar{q}_j) T_j$$

That means $\sigma f \triangleright \Pi$ is $\mathbb{K}$-linear combination of $\{T_j\}_{j=1, \ldots, d}$ and the dimension of the vector space span$\{\sigma f \triangleright \Pi \mid \Pi \in \text{Lie}_\mathbb{K}\langle X \rangle\}$ is less than or equal to $d$. □
6.3.2 Fliess’ local realization theorem

**Proposition 25** ([46]). Let $S \in \mathbb{K}\langle X \rangle$ such that its Lie rank equals $d$. Then there exists a basis $S_1, \ldots, S_d \in \mathbb{K}\langle X \rangle$ of $(\text{Ann}^+(S), \omega) \cong (\mathbb{K}[S_1, \ldots, S_d], \omega)$ such that the $S_i$’s are proper and for any $R \in \text{Ann}^+(S)$, one has

$$R = \sum_{i_1, \ldots, i_d \geq 0} \frac{r_{i_1, \ldots, i_d}}{i_1! \cdots i_d!} S_1^{i_1} \cdots S_d^{i_d},$$

where the coefficients $\{r_{i_1, \ldots, i_d}\}_{i_1, \ldots, i_d \geq 0}$ belong to $\mathbb{K}$ and $r_{0, \ldots, 0} = \langle R | e \rangle$.

**Proof.** By Lemma 31, a such basis exists. More precisely, since the Lie rank of $S$ is $d$ then there exists $P_1, \ldots, P_d \in \text{Lie}_\mathbb{K}(X)$ such that $S \triangleright P_1, \ldots, S \triangleright P_d \in (\mathbb{K}\langle X \rangle, \omega)$ are $\mathbb{K}$-linearly independent. By duality, their exists $S_1, \ldots, S_d \in (\mathbb{K}\langle X \rangle, \omega)$ such that

$$\forall i, j = 1, \ldots, d, \quad \langle S_i | P_j \rangle = \delta_{i,j}, \quad \text{and} \quad R = \prod_{i=1}^d \exp(S_i P_i).$$

Expending this product, one obtains, via Poincaré-Birkhoff-Witt theorem, the expected expression for the coefficients $r_{i_1, \ldots, i_d} = \langle R | P_1^{i_1} \cdots P_d^{i_d} \rangle$. Hence, $(\text{Ann}^+(S), \omega)$ is generated by $S_1, \ldots, S_d$.

With the notations of Proposition 25, one has respectively

**Corollary 20.** If $S \in \mathbb{K}[S_1, \ldots, S_d]$ then, for any $i = 0, 1$ and $j = 1, \ldots, d$, one has $x_i \triangleleft S \in \text{Ann}^+(S) = \mathbb{K}[S_1, \ldots, S_d]$.

**Corollary 21.** The power series $S$ verifies the growth condition if and only if, for any $i = 1, \ldots, d$, $S_i$ also verifies the growth condition.

**Proof.** Assume their exists $j \in [1, \ldots, d]$ such that $S_j$ does not verify the growth condition. Since $S \in \text{Ann}^+(S)$ then using the decomposition of $S$ on $S_1, \ldots, S_d$, one obtains a contradiction with the fact that $S$ verifies the growth condition.

Conversely, using Proposition 20, we get the expected results.

**Theorem 24** ([20]). The formal power series $S \in \mathbb{K}\langle X \rangle$ is differentially produced if and only if its Lie rank is finite and if it verifies the $\chi$-growth condition.

**Proof.** By Proposition 24, one gets a direct proof.

Conversely, since the Lie rank of $S$ equals $d$ then by Proposition 25, by putting $\sigma f_{\bar{o}} = S$ and, for any $j = 1, \ldots, d$, $\sigma \bar{q}_j = S_j$,

1. we choose the observation $f$ as follows

$$f(q_1, \ldots, q_d) = \sum_{i_1, \ldots, i_d \geq 0} \frac{r_{i_1, \ldots, i_d}}{i_1! \cdots i_d!} q_1^{i_1} \cdots q_d^{i_d} \in \mathbb{K}[q_1, \ldots, q_d],$$

such that

$$\sigma f_{\bar{o}}(q_1, \ldots, q_d) = \sum_{i_1, \ldots, i_d \geq 0} \frac{r_{i_1, \ldots, i_d}}{i_1! \cdots i_d!} (\sigma q_1)^{i_1} \cdots (\sigma q_d)^{i_d},$$

55
2. it follows that, for $i = 0, 1$ and for $j = 1, \ldots, d$, the residuals $x_i \triangleleft \sigma q_j$ belongs to $\text{Ann}^\perp(\sigma f_{i\alpha})$ (see also Lemma 32),

3. since $\sigma f$ verifies the $\chi$-growth condition then, by Corollary 21, the generating series $\sigma q_j$ and $x_i \triangleleft \sigma q_j$ (for $i = 0, 1$ and for $j = 1, \ldots, d$) verify also the growth condition. We then take (see Lemma 29)

\[ \forall i = 0, 1, \forall j = 1, \ldots, d, \sigma A_i^j(q_1, \ldots, q_d) = x_i \triangleleft \sigma q_j, \]

by expressing $\sigma A_i^j$ on the basis $\{\sigma q_i\}_{i=1, \ldots, d}$ of $\text{Ann}^\perp(\sigma f_{i\alpha})$,

4. the homomorphism $A$ is then determined as follows

\[ \forall i = 0, 1, A(x_i) = \sum_{j=0}^{d} A_i^j(q_1, \ldots, q_d) \frac{\partial}{\partial q_j}, \]

where, for $i = 0, 1, j = 1, \ldots, d, A_i^j(q_1, \ldots, q_d) = A(x_i) \circ q_j$ (see Lemma 26).

Thus, $(A, f)$ provides a differential representation\(^{35}\) of dimension $d$ of $S$. \hfill \Box

Moreover, one also has the following

**Theorem 25** ([20]). Let $S \in \mathbb{K}[[X]]$ supposed to be a differentially produced formal power series. If $(A, f)$ and $(A', f')$ are two differential representations of dimension $n$ of $S$ then there exists a continuous and convergent automorphism $h$ of $\mathbb{K}$ such that, for $w \in X^*$, $g \in \mathbb{K}, h(A(w) \circ g) = A'(w) \circ (h(g))$ and $f' = h(f)$.

Since any rational power series (resp. polynomial), verifies the growth condition and its Lie rank is less or equal to its Hankel rank which is finite [20] then

**Corollary 22.** Any rational power series and any polynomial over $X$ with coefficients in $\mathbb{K}$ are differentially produced.

**Remark 5.** 1. By Corollary 20, if $S$ is polynomial then for any $j = 1, \ldots, d$, $S_j$ is polynomial. Therefore, for $i = 0, 1$ and $j = 1, \ldots, d, x_i \triangleleft S_j$ is also polynomial over $X$. In this case, let $(A, f)$ be a differential representation of $S$ of dimension $d$. Then $f$ and $\{A_i^j\}_{i=0,1}^{j=1,\ldots,d}$ are obviously polynomial on $q_1, \ldots, q_d$ and the Lie algebra generated by $\{A_i^j(x_{i\alpha})\}_{i=0,1}^{j=1,\ldots,d}$ is nilpotent.

2. Note also that, by Theorem 6, if $S$ is rational over $X$ of linear representation $(\lambda, \mu, \eta)$ then the observation $f(q_1, \ldots, q_n)$ equals $\lambda_1 q_1 + \ldots + \lambda_n q_n$ and the polysystem $\{A(x)\}_{x \in X}$ is obtained by putting

\[ \forall x_i \in X, A(x_i) = \sum_{j=1}^{n} (\mu(x_i))^j \frac{\partial}{\partial q_j}, \]

yields linear representation not necessarily of minimal dimension [20].

\(^{35}\)In [20, 46], the reader can found the discussion on the minimal differential representation.
3. Assume \( S \in \mathbb{K} \oplus x_0 \mathbb{K}[\langle X \rangle] x_1 \) and \( S \) is a differentially produced. If there exists a basis \( S_1, \ldots, S_d \) of \( (\text{Ann}^+(S_1, \ldots, S_d), \mathbb{K}[S_1, \ldots, S_d]) \), such that

\[
S = \sum_{i_1, \ldots, i_d \geq 0} \sum_{i_1, \ldots, i_n} \frac{S_{i_1}^{i_1}}{i_1!} \cdots \frac{S_{i_d}^{i_d}}{i_d!} \in (\mathbb{K}[S_1, \ldots, S_i], \mathbb{K}[S_1, \ldots, S_d]).
\]

We put \( \Sigma_i := \pi_Y S_i \), for \( i = 1, \ldots, d \) and then

\[
\Sigma := \sum_{i_1, \ldots, i_n \geq 0} \sum_{i_1, \ldots, i_n} \frac{\Sigma_{i_1}^{i_1}}{i_1!} \cdots \frac{\Sigma_{i_d}^{i_d}}{i_d!} \in (\mathbb{K}[\Sigma_1, \ldots, \Sigma_d], \mathbb{K}).
\]

It is a generalization of a Radford’s theorem because [29, 30] :

- If \( S \in \mathbb{Q}(X) \) then (122), (123) are decompositions on Radford bases.
- If \( S \) is rational then these are noncommutative partial decompositions. In general one has \( \pi_Y S \neq \Sigma \) but \( \zeta(S_i) = \zeta(\Sigma_i) \) and

\[
\zeta(S) = \zeta(\Sigma) = \sum_{i_1, \ldots, i_n \geq 0} \frac{\zeta(S_1)^{i_1}}{i_1!} \cdots \frac{\zeta(S_d)^{i_d}}{i_d!}.
\]

Thus, these yield also identities on polyzetas at arbitrary weight [37].

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