A simple analytical description for the cross-tie domain wall structure.

Konstantin L. Metlov
Institute of Physics ASCR, Na Slovance 2, Prague 8, 182 21, Czech Republic
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A closed form analytical expression for the magnetization vector distribution within the cross-tie domain wall in an isotropic ferromagnetic thin film is given. The expression minimizes the exchange energy functional exactly, and the magnetostatic energy by means of an adjustable parameter (wall width). The equilibrium value of the wall width and the film thickness corresponding to the transition between the Néel and the cross-tie walls are calculated. The results are compared with the recent experiments and are in good qualitative agreement.

The structure of magnetic domain walls (transition regions separating magnetic domains) was in a focus of extensive research in 1930s – 60s and during the time a lot of useful results on the static structures of various wall types as well as their dynamic properties were obtained. Many of the results on the theory of the domain walls can be found in books [1, 2]. In particular, there are analytical expressions for the distribution of the magnetization vector inside of the one-dimensional Bloch and Néel domain walls, which are the starting point for calculating their static and dynamic properties.

The cross-tie domain wall was also observed in a number of experiments a that time (see Ref. 3 and references therein) and also with modern high resolution techniques [4, 5, 6], and are usual in thin and ultra-thin magnetic films important for modern applications [7]. However, there is no [4] (also see p. 163 in Ref. 2) closed form analytical expression for the cross-tie domain wall structure. This is, probably, due to the fact that magnetization distribution even in straight cross-tie domain wall is two-dimensional (unlike one-dimensional ones of Bloch and Néel walls). It means, the structure of such a wall is defined by a system of non-linear integral (due to the long-range dipolar interactions) partial differential equations, and there is no way to reduce this system to a single equation (as it was done for the Néel wall [8]).

Consider a thin film having the thickness \( h \) made of soft (isotropic) magnetic material, in the Cartesian coordinate system \( X, Y, Z \) chosen in such a way that \( 0Z \) axis is perpendicular to the film plane. The parameters of the material entering the calculation are the exchange constant \( C \), and the saturation magnetization constant \( M_s \). If the film is thin enough, so that \( h \) is of the order of a few exchange lengths \( L_E = \sqrt{C/M_s} \), the dependence of the magnetization distribution on \( Z \) can be neglected and the task becomes essentially two-dimensional. If we also neglect the dipolar interaction for a while, the magnetization distribution \( \vec{M}(\vec{r}) = M_s \vec{m}(\vec{r}) \) where \(|\vec{m}| = 1\) is defined by the minimum of the exchange energy functional:

\[
e^{ex} = \frac{C}{2} h \int \int \sum_{i=x,y,z} (\vec{\nabla} m_i)^2 \; d^2 \vec{r} = Ch \int \int \frac{4}{(1+u^2w)^2} \left( \frac{\partial w}{\partial z} \frac{\partial m}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial m}{\partial z} \right) \; d^2 \vec{r},
\]

where the integration runs over all \( X-Y \) plane. The last expression is given using the parametrization of the magnetization vector field by a complex function \( w(z, \tau) \) of a complex variable \( z = X+iY, 1 = \sqrt{-1} \), line over a variable means complex conjugation, so that \( m_x + i m_y = 2w/(1+w\bar{w}) \) and \( m_z = (1-w\bar{w})/(1+w\bar{w}) \), \( \partial/\partial z = (\partial/\partial X - i\partial/\partial Y)/2 \). It is easy to see that the energy functional (1) is scale-invariant, and the exchange energy of the magnetization distribution \( \vec{m}(\vec{r}/c) \) or \( w(z/c, \tau/c) \), where \( c \) is real constant, is independent on \( c \). If we include the dipolar interaction back into the picture this scale-invariance breaks. The dipolar interaction will be considered in this paper to determine the scale of the magnetization distribution obtained by minimizing (1).

The system of Euler equations for the function \( w(z, \tau) \) giving extremum to the functional (1) is:

\[
\frac{\partial}{\partial z} \left( \frac{\partial w}{\partial \tau} \right) = \frac{2\bar{w}}{1+w\bar{w}} \frac{\partial w}{\partial \tau} \frac{\partial w}{\partial \tau},
\]

It is easy to see that any analytic (in a sense that the Cauchy-Riemann conditions \( \partial w/\partial \tau = 0 \) are satisfied) function of a complex variable satisfies the equation (2). The partial solution of this class was first written as a rational polynomial in \( z \) by Belavin and Polyakov [9] and can be thought as a superposition of vortices. There are other solutions whose applications to the magnetism of small cylindrical particles are given elsewhere [10].

*Electronic address: metlov@fzu.cz
Consider the function
\[ w_{C-T}(z) = i \tan(z/c), \]  
while the Cauchy-Riemann conditions are satisfied for this function, and, therefore, it minimizes the functional (1) exactly, it is not a Belavin-Polyakov (BP) soliton because it can not be represented as a rational polynomial in \( z \) and, consequently, has infinite energy when integrated over the whole space (while the energy of BP solitons is always finite). Even though the total exchange energy of the magnetization distribution \( w_{C-T} \) is infinite, it is extremal, because \( w_{C-T} \) satisfies (3). Also, the corresponding \( \vec{m} = \{m_x, m_y, m_z\} = \{-\tanh(2Y/c), \sin(2X/c)/\cosh(2Y/c), \cos(2X/c)/\cosh(2Y/c)\} \) (compare [11]) plotted in Fig. 1 has the topology of the magnetization distribution in the cross-tie domain wall [4].

The exchange energy per unit of the wall area is finite
\[ \gamma_{ex} = \frac{e_{ex}}{c\pi h} = \frac{M_s^2 L_E}{c\pi} \int_{-c\pi/2}^{c\pi/2} dX \int_{-\infty}^{\infty} dY \frac{4}{c^2 \cosh^2(2Y/c)} = M_s^2 h \lambda^2 \frac{4\pi}{\zeta}, \]  
where dimensionless variables \( \lambda = L_E/h \) and \( \zeta = c\pi/h \) were introduced. This energy term (4) forces the wall to expand infinitely as it has the smallest value 0 only for \( c, \zeta \to \infty \).

The evaluation of the magnetostatic energy is straightforward, but a little bit voluminous, it was done by calculating the density of magnetic charges (having both volume and surface contributions), solving the Poisson equation for the scalar magnetic potential in the Fourier representation and performing the convolution of the density of charges and the potential to obtain the energy. The final result can be represented as
\[ \gamma_{dip} = M_s^2 h f(\zeta), \]  
\[ f(\zeta) = \frac{\zeta^2}{\pi} \int_0^\infty du \int_0^\infty dv \frac{(u^2 + v^2) \sin^2(\pi u/\zeta)}{u^2 \cosh^2(\pi u/2)(1 + u^2 + v^2)}. \]  
One of integrals in \( f(\zeta) \) can be taken analytically, the other is rapidly converging and easy to take numerically. The representation (6) was chosen for its compactness. It is also possible to build a simple analytical approximation
FIG. 2: The normalized energy per unit wall area (solid lines) of the equilibrium cross-tie, Néel and Bloch domain walls as a function of film thickness. The dashed line shows the normalized width of the cross-tie wall. The dotted lines show the equilibrium cross-tie domain wall energy and width calculated using the approximate magnetostatic function.

for $f(\zeta)$

$$f(\zeta) = C_1 \zeta + C_2 (1 - e^{-2\pi/\zeta}) \zeta^2$$

(7)

$$C_1 = \int_0^\infty du \frac{\pi u}{(1 + u^2)(1 + \cosh \pi u)} = 0.17753 \ldots$$

(8)

$$C_2 = \int_0^\infty du \frac{1}{4(1 + u^2)^{3/2} \cosh^2(\pi u/2)} = 0.12119 \ldots$$

(9)

which is asymptotically correct for $\zeta \ll 1$ and produces the worst case error at $\zeta \to \infty$ around 5%.

The total energy per unit wall area $\gamma = \gamma^\text{ex} + \gamma^\text{disp}$ was minimized numerically with respect to the parameter $\zeta$, and the result is shown in Fig. 2 in comparison to the energy per unit area of the Néel and Bloch domain walls calculated according to the model of Dietze and Thomas [12]. As it could be expected for films thicker than a certain value $h \approx 2.3 L_E$ the cross-tie domain wall takes over the Néel one as the lowest energy configuration. Another interesting fact is that in the shown region of film thicknesses (where roughly the assumption of uniformity of the magnetization distribution along the thickness, $\partial \vec{M}/\partial Z = 0$, can be expected to hold) the energy of the cross-tie domain wall is lower than the energy of Bloch wall. This suggests that the transition from the cross-tie wall to the Bloch one takes place at larger thicknesses, where $\partial \vec{M}/\partial Z = 0$ can not be safely assumed anymore.

As it is visible from Fig. 2 the approximate expression for magnetostatic function (7) after the energy minimization produces results almost indistinguishable from the exact ones and having only a slight deviations for extremely small film thicknesses. Thus, the expression (7) is safe to use instead of (9) for most practical estimations of the cross-tie wall magnetostatic energy.

The qualitative validity of the presented model is supported by the good fit of the image signal (roughly proportional to the magnetization vector component parallel to the wall plane) presented in Fig. 10 of Ref. 4 to the empirically chosen $\tanh(Y/\delta_w)$ function. In fact, $m_y(X,Y) = - \tanh(2Y/c)$ exactly for the solution (4). Thus, the cross-tie wall width, as defined in Ref. 4, is $\delta_w = c/2$. The permalloy film in that experiment had roughly $h/L_E = 60 \text{nm}/18.2 \text{nm} \approx$...
3. Reading from Fig. 2 we get the theoretical width of the cross-tie wall $\delta_w = 4.8 L_E/(2\pi) \approx 14 \text{nm}$, which is of the order of the measured value of $42 \pm 10 \text{nm}$. It is expected that better quantitative correspondence can be obtained for thinner films and also by including the anisotropy energy (which is small for permalloy) into the minimization.

The presented simple model for the magnetization distribution in the cross-tie domain wall can be a starting point for analytical estimations of its static and dynamic properties, especially for short wavelength excitations on the cross-tie wall background, where the fact that minimizes the exchange energy functional exactly can lead to significant simplifications. It can be also useful for interpretation of MFM images, and for supplying initial conditions for numerical finite-element micromagnetic simulations.

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