Painlevé analysis for the cosmological field equations in Weyl Integrable Spacetime

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We apply singularity analysis to investigate the integrability properties of the gravitational field equations in Weyl Integrable Spacetime for a spatially flat Friedmann–Lemaître–Robertson–Walker background spacetime induced with an ideal gas. We find that the field equations possess the Painlevé property in the presence of the cosmological constant and the analytic solution is given by a left Laurent expansion.

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1. INTRODUCTION

The Weyl Integrable Spacetime (WIS) is a natural way to extend Einstein’s General Relativity, in which a scalar field is introduced in the natural space by geometrical degrees of freedom [1]. Scalar fields play an important role in the description of gravitational phenomena at large scales [2, 3]. Indeed, the late time or early time acceleration phases of the universe have been proposed to be attributed to scalar fields [4–8]. Another novelty of WIS is that interaction is introduced geometrically between the matter components [9–11]. Cosmological models with interaction in the dark sector of the universe have been widely studied before. Furthermore, that interacting models survive the constraints follow by the cosmological observations [12–14]. Specifically, the interaction in the dark sector of the universe is a mechanism to solve the cosmic coincidence problem and it has been used for the

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explanation of the discrepancy in the cosmological constant [15–19]. In addition, WIS is related with the chameleon cosmology [20].

There are various studies in the literature on cosmological models on WIS. Multi-dimensional spacetimes were investigated in [21], while lower-dimensional gravitational models were studied in [22]. Inflation is WIS investigated in [23, 24], where it was found that nonsingular inflationary solutions are provided by the specific theory. In [25–27] WIS has been considered as the dark energy mechanism which drives the present acceleration of the universe, while recently an extension of WIS in Einstein-Aether theory was studied in [28]. The dynamical analysis of the field equations with different fluid sources for the construction of the cosmological history was the subject of study in [29].

The field equations in WIS are nonlinear ordinary differential equations of second-order. Because of the new degrees of freedom provided by the scalar field there are not many known solutions in the literature. Note that the cosmological field equations in WIS admit a minisuperspace description the Noether symmetry analysis for which was applied in [30] for the construction of conservation laws and the derivation of analytic solutions. The Noether symmetry analysis is a powerful and systematic approach for the study of the integrability properties of dynamical systems which follow from a variational principle with many applications in cosmological studies [31–36]. For the cosmological model in WIS in which the matter source is an ideal gas it was found that the introduction of the cosmological constant term into the gravitational Action Integral leads to a dynamical system with fewer Noetherian conservation laws from which we are not able to infer the integrability properties. In this work we investigate the same problem with the use of another important tool of analysis for the construction of conservation laws and analytic solutions, known as Painlevé analysis or singularity analysis [37]. The singularity analysis for the analysis of the field equations in gravity covers a wide range of applications; see for instance [38–45]. The determination of the integrability properties of dynamical systems is essential in physics and in all areas of applied mathematics. The novelty when a physical system is described by an integrable dynamical system is that we know that an actual solution exists when we apply numerical methods for the study of the system, or there exists a closed-form function which solves the dynamical system. For a recent discussion on the integrable cosmological models we refer the reader to [47]. In our study the singularity analysis leads to the construction of analytic solutions expressed as Laurent expansions around a movable singularity. The plan
of the paper is as follows.

In Section 2 we present the cosmological model of our consideration which is that of WIS in an homogeneous and isotropic universe in which the matter source is that of an ideal gas. In Section 3 we review previous results wherein we show the existence of additional conservation laws for the field equations in the absence of the cosmological constant term. Section 4 includes the main analysis of this study, in which the singularity analysis is applied for the study of the integrability properties for the field equations. We find that the field equations possess the Painlevé property with or without a cosmological constant term and the analytic solution is expressed by Laurent expansions. Finally, in Section 5 we summarize our results and we draw our conclusions.

2. FIELD EQUATIONS

In WIS or Weyl Integrable Geometry (WIG), the modified Einstein-Hilbert Action is expressed as

\[ S_W = \int dx^4 \sqrt{-g} \left( \hat{R} + \xi \left( \hat{\nabla}_\nu \left( \hat{\nabla}_\mu \phi \right) \right) g^{\mu\nu} - \Lambda + L_m \right), \tag{1} \]

in which \( g_{\mu\nu} \) is the metric tensor for the physical space, \( \hat{\nabla}_\mu \) denotes covariant derivative defined by the symbols \( \hat{\Gamma}^{\kappa}_{\mu\nu} \), where \( \hat{\Gamma}^{\kappa}_{\mu\nu} \) are the Christoffel symbols for the conformally related metric \( \hat{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu} \). The scalar field \( \phi \) is the coupling function, \( \Lambda \) is the cosmological constant term, the parameter \( \xi \) is an arbitrary coupling constant and \( L_m \) is the Lagrangian function for the matter source.

When \( L_m \) describes a perfect fluid with energy density \( \rho \) and pressure component \( p \), the field equations in the Einstein-Weyl theory are derived to be

\[ \hat{G}_{\mu\nu} + \hat{\nabla}_\nu \left( \hat{\nabla}_\mu \phi \right) - (2\xi - 1) \left( \hat{\nabla}_\mu \phi \right) \left( \hat{\nabla}_\nu \phi \right) + \xi g_{\mu\nu} g^{\kappa\lambda} \left( \hat{\nabla}_\kappa \phi \right) \left( \hat{\nabla}_\lambda \phi \right) - \Lambda g_{\mu\nu} = - (\hat{\rho} + \hat{p}) u^\mu u^\nu - \hat{p} g_{\mu\nu}, \tag{2} \]

where \( \hat{G}_{\mu\nu} \) is the Einstein tensor with respect the metric \( \hat{g}_{\mu\nu} \) and \( u^\mu \) is the comoving observer. Parameters \( \hat{\rho}, \hat{p} \) are the energy density and pressure components for the matter source multiplied by the factor \( e^{-\phi} \).

The field equations (2) can be written in the equivalent form

\[ G_{\mu\nu} - \lambda \left( \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi^{,\kappa} \phi_{,\kappa} \right) - \Lambda g_{\mu\nu} = - (\hat{\rho} + \hat{p}) u^\mu u^\nu - \hat{p} g_{\mu\nu}, \tag{3} \]
where $G_{\mu\nu}$ is the Einstein tensor for the background space $g_{\mu\nu}$ and $\lambda = 2\xi - \frac{3}{2}$.

For the homogeneous and isotropic spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime,

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right),$$  

the modified Friedmann’s equations are as follow

$$3H^2 - \frac{\lambda}{2N^2}\dot{\phi}^2 - \Lambda - e^{-\phi^2} \rho = 0,$$

$$\dot{H} + H^2 + \frac{1}{6} e^{-\phi^2} (\rho + 3p) + \frac{\lambda}{3N^2}\dot{\phi}^2 - \frac{\Lambda}{3} = 0,$$

$$\ddot{\phi} - \frac{\dot{N}}{N}\dot{\phi} + 3H\dot{\phi} + \frac{1}{2\lambda} N^2 e^{-\phi^2} \rho = 0$$

and

$$\dot{\rho} + 3NH (\rho + p) - \rho \dot{\phi} = 0$$

in which we have considered the comoving observer, $u^\mu = \frac{1}{N}\delta^\mu_t$, and $H = \frac{1}{N} \frac{\dot{a}}{a}$ is the Hubble function.

In this study we consider that the matter source is that of an ideal gas, that is $p = (\gamma - 1) \rho$, $\gamma < 2$. For $\gamma = 2$ a two scalar field model is recovered [48]. From equation (8) it follows that $\rho = \rho_0 a^{-3\gamma} e^{\phi}$. Thus we end with the set of differential equations (5), (6) and (7).

The cosmological history for this specific cosmological model investigated before in [29]. From the analysis of the dynamics it was found that the cosmological history is consisted by two matter epochs and two acceleration phases. The matter epochs correspond to unstable asymptotic solutions while for the two accelerated phases the one is always unstable which can be corresponded to the early acceleration phase of the universe, and the second accelerated asymptotic solution describes the future de Sitter attractor.

### 3. LAGRANGIAN DESCRIPTION

In the case of an ideal gas, the cosmological field equations (5), (6) and (7) can be reproduced by the variation of the point-like Lagrangian

$$\mathcal{L} \left(N, a, \dot{a}, \phi, \dot{\phi}\right) = \frac{1}{N} \left(-3a\dot{a}^2 + \frac{\lambda}{2} a^3\dot{\phi}^2\right) - N \left(a^3 \Lambda + \rho_{m0} e^{\phi} a^{3-3\gamma}\right).$$
The Lagrangian function (9) is a singular Lagrangian because \( \frac{∂L}{∂\dot{N}} = 0 \). The constraint equation (5) is the Euler-Lagrange equation with respect to the lapse function \( N(t) \), i.e. \( \frac{∂L}{∂N} = 0 \), while the second-order differential equations, (6) and (7), are the Euler-Lagrange equations with respect to the variables \( a(t) \) and \( \phi(t) \) respectively, i.e. 
\[
\frac{d}{dt} \left( \frac{∂L}{∂\dot{a}} \right) - \frac{∂L}{∂a} = 0; \quad \frac{d}{dt} \left( \frac{∂L}{∂\dot{\phi}} \right) - \frac{∂L}{∂\phi} = 0.
\]

The existence of the singular Lagrangian (9) and the minisuperspace description of the gravitational model are important characteristics for the study of the integrability properties of the field equations. Without loss of generality the lapse function can be assumed to be \( N(t) = N(a(t), \phi(t)) \). In such a consideration, the Lagrangian (9) is regular while equation (5) is the conservation law of “energy”, that is, the Hamiltonian of the dynamical system.

### 3.1. Normal coordinates

We follow [29] and without loss of generality we select \( N(t) = a(t)^3 \). Furthermore, we define the new field \( \Phi \),
\[
\phi = \frac{1}{\lambda (2 - \gamma)} (2\lambda \ln \Phi - \ln a).
\]
In the new variables \( \{a, \Phi\} \) the Lagrangian of the field equations is
\[
L(a, \dot{a}, \Phi, \dot{\Phi}) = \frac{1}{2} \left( 6 - \frac{1}{(\gamma - 2)^2 \lambda} \right) \left( \frac{\dot{a}}{a} \right)^2 + \frac{2}{(\gamma - 2)^2 a \Phi} - \frac{2\lambda}{(\gamma - 2)^2} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 + \Lambda a^6 + a^{3(2 - \gamma) - \frac{1}{2\lambda(2 - \gamma)}} \Phi^{\frac{1}{2 - \gamma}}.
\]

The field equations are
\[
\frac{1}{2} \left( 6 - \frac{1}{(\gamma - 2)^2 \lambda} \right) \left( \frac{\ddot{a}}{a} \right)^2 + \frac{2}{(\gamma - 2)^2 a \Phi} - \frac{2\lambda}{(\gamma - 2)^2} \left( \frac{\ddot{\Phi}}{\Phi} \right)^2 - \Lambda a^6 - a^{3(2 - \gamma) - \frac{1}{2\lambda(2 - \gamma)}} \Phi^{\frac{1}{2 - \gamma}} = 0,
\]
(12)
\[
\dot{a} + \frac{1}{2} a^{7 - 3\gamma - \frac{1}{2\lambda(2 - \gamma)}} \Phi^{\frac{1}{2 - \gamma}} \dot{\Phi}^2 - \frac{\dot{\Phi}^2}{\Phi} - \Lambda a^7 = 0
\]
(13)
and
\[
\dot{\Phi} - \frac{\ddot{\Phi}^2}{\Phi} - \frac{\Lambda}{2\lambda} a^6 \Phi = 0.
\]
(14)

When \( \Lambda = 0 \), equation (14) provides the conservation law \( I_0 = \frac{d}{dt} \ln \Phi \). This conservation law was derived before in [29, 30]. The set of variables \( \{a, \Phi\} \) constitutes the normal coordinates for the field equations. However, in the presence of the cosmological constant the conservation law does not exist. The existence of the second conservation law for the
field equations is essential in order to be able to conclude about the integrability of the field equations. Indeed, in [30] the Hamilton-Jacobi equation was solved and the field equations were reduced into a system of two first-order ordinary differential equation.

In the presence of $\Lambda$, we applied the symmetry analysis for point and contact symmetries and we find that the field equations do not admit any Noether symmetry. Furthermore, we considered polynomial functions of $I(a, \dot{a}, \Phi, \dot{\Phi})$ to be a conservation law. However, we were not able to determine any function $I$ with this specific requirement.

The concept of integrability is not limited in the existence of invariant functions and conservation laws. According to the Painlevé approach, a dynamical system is integrable if admits a movable pole and its solution is expressed in terms of a Laurent expansion around the movable pole. In the following we apply the singularity analysis in order to investigate the integrability properties for the dynamical system of our consideration.

4. SINGULARITY ANALYSIS

The modern treatment of singularity analysis is described by the ARS algorithm. The algorithm has three main steps. They are (a) derivation of the leading-order behaviour, (b) derivation of the resonances and (c) the consistency test. For more details and examples on the application of the ARS algorithm we refer the reader to [37]. In the first step of the algorithm we should show that there exists a movable singularity for the dynamical system at which the solution is approximated by a singular expression. For instance, for power-law expressions $(\tau - \tau_0)^p$, the exponent $p$ should be negative number. However, it has been shown that $p$ can be also a rational number. The resonances should be the same in number as the degrees of freedom of the problem and one of the resonances has to be $-1$, while the resonances should be rational numbers. If the requirements of the steps (a) and (b) are satisfied, we write the solution in a Laurent expansion, in our case in Puiseux series, and we replace in the original equation to check if it is an actual solution for the original dynamical system.

We follow [46] and we apply the singularity analysis for the equivalent dynamical system in the dimensionless variables. Indeed, we follow the $H$—normalization approach and we
define the new variables

\[ x = \sqrt{\frac{1}{6H}} \phi \] \quad \Omega_A = \frac{\Lambda}{3H^2} \quad \Omega_m = \frac{\rho_m}{3H^2} e^{-\frac{\phi}{2}} \quad d\tau = NHdt. \quad (15) \]

where the latter parameter \( \tau \) is the new independent variable.

Consequently, the field equations (5)-(8) for the ideal gas are written as the following equivalent algebraic-differential system

\[
\frac{d\Omega_A}{d\tau} = -\Omega_A \left( (\gamma - 2) \lambda x^2 + \gamma (\Omega_A - 1) \right) \quad (16)
\]

and

\[
\frac{dx}{d\tau} = \frac{1}{12\lambda} \left( (\lambda x^2 - 1) \left( \sqrt{6} - 6(\gamma - 2)\lambda x \right) + \left( \sqrt{6} - 6\gamma \lambda x \right) \Omega_A \right) \quad (17)
\]

with constraint equation

\[ \Omega_m = 1 - \lambda x^2 - \Omega_A. \quad (18) \]

Therefore we continue with the investigation of the singularity analysis for the system of first-order ordinary differential equations (16) and (17).

4.1. Case \( \Lambda = 0 \)

We focus now on the case \( \Lambda = 0 \). Equation (17) reduces to the simple form

\[
\frac{dx}{d\tau} = \frac{1}{12\lambda} \left( (\lambda x^2 - 1) \left( \sqrt{6} - 6(\gamma - 2)\lambda x \right) \right). \quad (19)
\]

The closed-form solution of the this equation is

\[
\tau - \tau_0 = \frac{6\lambda (\gamma - 2)}{1 - 6\lambda (\gamma - 2)^2} \ln \frac{(\lambda x^2 - 1)}{(\sqrt{6} - 6\lambda (\gamma - 2)x)^2} - \frac{2\sqrt{\lambda}}{1 - 6\lambda (\gamma - 2)^2} \arctan h (\lambda x). \quad (20)
\]

However, we are interested to investigate if equation (19) possesses the Painlevé property and if the analytic solution can be expressed in a Laurent expansion around a movable pole.

According to the ARS algorithm, we replace \( x(t) = x_0 (\tau - \tau_0)^p \) in (19), that is,

\[
p x_0 (\tau - \tau_0)^{-1+p} = \frac{1}{12\lambda} \left( (\lambda x_0^2 (\tau - \tau_0)^{2p} - 1) \left( \sqrt{6} - 6(\gamma - 2)\lambda x_0 (\tau - \tau_0)^p \right) \right). \quad (21)
\]

Hence, the dominant terms provides the algebraic equation \(-1 + p = 3p\), that is, \( p = -\frac{1}{2} \), and \( \lambda (\gamma - 2) x_0^2 - 1 = 0 \).
For the determination of the resonances we substitute
\[ x(t) = x_0 \left( \tau - \tau_0 \right)^{-\frac{1}{2}} + m (\tau - \tau_0)^{-\frac{1}{2} + S} , \quad \lambda x_0^2 + 1 = 0 , \tag{22} \]
into (19) and we expand around \( m^2 \to 0 \). Hence, we end with the algebraic equation \( S + 1 = 0 \), which means that \( S = -1 \). That is in agreement with the singularity analysis, because one of the resonances should be \(-1\). This resonance indicates that the leading-order behaviour describes a movable singularity and \( t_0 \) is an integration constant.

For the third step of the ARS algorithm, known as the consistency test, we substitute
\[ x(\tau) = x_0 \left( \tau - \tau_0 \right)^{-\frac{1}{2}} + x_1 (\tau - \tau_0)^0 + x_2 (\tau - \tau_0)^{\frac{1}{2}} + x_3 (\tau - \tau_0) + ... , \tag{23} \]
into (19) where we find
\[ x_1 = x_0^2 \frac{x_0^2}{3\sqrt{6}} , \quad x_2 = x_0 (x_0^2 + 18 (\gamma - 2)) \frac{1}{72} , \quad ... \]

We conclude that equation (19) possesses the Painlevé property and the analytic solution is described by the Puiseux expansion.

\[ \frac{\Omega_\Lambda (\tau)}{\Omega_{\Lambda 0}} = \Omega_{\Lambda 0} (\tau - \tau_0)^q , \tag{24} \]
\[ x(\tau) = x_0 (\tau - \tau_0)^p \tag{25} \]
and as above we determine \( p = -\frac{1}{2} \) and \( p = -1 \) with constraint equation \( \lambda (\gamma - 2) x_0^2 - 1 = \gamma \Omega_{\Lambda 0} \). Easily we see that for \( \Omega_{\Lambda} = 0 \), the previous leading-order term is recovered.

For the resonances we replace
\[ \Omega_\Lambda (\tau) = \Omega_{\Lambda 0} (\tau - \tau_0)^{-1} + n (\tau - \tau_0)^{-1 + S} , \tag{26} \]
\[ x(\tau) = x_0 \left( \tau - \tau_0 \right)^{-\frac{1}{2}} + m (\tau - \tau_0)^{-\frac{1}{2} + S} \tag{27} \]
in (16), (17).
We expand around $m^2 \to 0$, $n^2 \to 0$ and $mn \to 0$. The first-order perturbed terms provide the matrix

$$A = \begin{pmatrix}
\frac{-2}{x_0} (\Omega_{A0} (\gamma \Omega_{A0} - 1)) - (S + \gamma \Omega_{A0}) \\
(1 + S - \gamma \Omega_{A0}) & \frac{-1}{2} x_0 \gamma
\end{pmatrix}, \tag{28}
$$

where the zeros of the determinant $\det(A) = 0$, give the resonances; they are

$$S_1 = -1, \quad S_2 = 0. \tag{29}$$

The resonances indicate that there exists a movable singularity and that one of the coefficients, $x_0$ or $\Omega_{A0}$, is arbitrary and is the second integration constant of the dynamical system.

We write the Puiseux series

$$\Omega_A (\tau) = \Omega_{A0} (\tau - \tau_0)^{-1} + \Omega_{A1} (\tau - \tau_0)^{-\frac{1}{2}} + \Omega_{A2} (\tau - \tau_0)^{0} + \Omega_{A3} (\tau - \tau_0)^{\frac{1}{2}} + \ldots, \tag{30}
$$

$$x (\tau) = x_0 (\tau - \tau_0)^{-\frac{1}{2}} + x_1 (\tau - \tau_0)^{0} + x_2 (\tau - \tau_0)^{\frac{1}{2}} + x_3 (\tau - \tau_0) + \ldots \tag{31}
$$

which we replace in (16), (17). Hence, the latter Puiseux series solves the dynamical system when

$$\Omega_{A1} = \frac{4x_1 \Omega_{A0} (\gamma \Omega_{A0} - 1)}{x_0 (1 + 2 \gamma \Omega_{A0})}, \quad \Omega_{A1} = \frac{x_0 (\sqrt{6} x_0 (1 - 2 \Omega_{A0}) + 6 \gamma \Omega_{A1} (\gamma \Omega_{A0} - 1))}{6 (\gamma \Omega_{A0} - 1) (2 \gamma \Omega_{A0} - 3)}, \ldots. \tag{32}
$$

We summarize our results in the following proposition.

**Proposition 1:** The cosmological field equations in Weyl Integrable theory for a spatially flat FLRW background geometry induced with a cosmological constant term and an ideal gas form a dynamical system which possesses the Painlevé property that is, the field equations are integrable and the analytic solution is expressed by the Puiseux expansions (30) and (31).

5. CONCLUSIONS

In this piece of study we investigated the integrability properties of the field equations in Weyl Integrable theory. In particular we studied the existence of analytic solutions of the cosmological scenario for a universe with matter source that of an ideal gas in a spatially flat FLRW geometry with a nonzero cosmological constant term. For our analysis we applied the singularity analysis which has been widely applied in gravitational physics with many interesting results.
The field equations for the model of our consideration have the property to admit a
minisuperspace. That means that the field equations follow from the variation for the
dynamical variables of a point-like Lagrangian. For the case in which there is no cosmological
constant term, from previous results we know that there exists a second conservation law
which indicates the integrability properties for the dynamical system. However, in the
presence of the cosmological constant, this conservation law does not exist. Thus we were
not able to infer about the integrability of the field equations.

For simplicity of our analysis we wrote the field equations into the equivalent system
with the use of dimensionless variables. For the equivalent system we applied the singularity
analysis and we found that the field equations possess the Painlevé property, that is, the
cosmological model of our analysis is integrable with or without the cosmological constant
term. This is a very interesting result because after the loss of the additional integration
constant provided by Noether’s analysis we were not able to make an inference about the
integrability and someone could infer that the introduction of the cosmological constant into
the field equations may violate the integrability property.

As we have discussed before, this cosmological model describes important eras of the
cosmological history. The existence of the analytic solution indicates that there exist an
actual real solutions which correspond to the numerical simulations of the field equations.
This is an important feature because we can understand the evolution of the dynamical
system according to the free parameters. In addition the above analysis can be used for the
analytic reconstruction of the cosmographic parameters [49, 50] and the relation of these
parameters with the initial value problem.

This work contributes to the subject of the derivation of exact and analytic solutions in
gravitational physics, while it shows that Noether symmetry analysis and the singularity
analysis are two complementary methods which can provide interesting results.

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