Centers of Mass and Rotational Kinematics for the Relativistic N-Body Problem in the Rest-Frame Instant Form.

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Abstract

In the Wigner-covariant rest-frame instant form of dynamics it is possible to develop a relativistic kinematics for the N-body problem which solves all the problems raised till now on this topic. The Wigner hyperplanes, orthogonal to the total timelike 4-momentum of any N-body configuration, define the intrinsic rest frame and realize the separation of the center-of-mass motion. The point chosen as origin of each Wigner hyperplane can be made to coincide with the covariant non-canonical Fokker-Pryce center of inertia. This is distinct from the canonical pseudo-vector describing the decoupled motion of the center of mass (having the same Euclidean covariance as the quantum
Newton-Wigner 3-position operator) and the non-canonical pseudo-vector for the Møller center of energy. These are the only external notions of relativistic center of mass, definable only in terms of the external Poincaré group realization. Inside the Wigner hyperplane, an internal unfaithful realization of the Poincaré group is defined while the analogous three concepts of center of mass weakly coincide due to the first class constraints defining the rest frame (vanishing of the internal 3-momentum). This unique internal center of mass is consequently a gauge variable which, through a gauge fixing, can be localized at the origin of the Wigner hyperplane. An adapted canonical basis of relative variables is found by means of the classical counterpart of the Gartenhaus-Schwartz transformation. The invariant mass of the N-body configuration is the Hamiltonian for the relative motions. In this framework we can introduce the same dynamical body frames, orientation-shape variables, spin frame and canonical spin bases for the rotational kinematics developed for the non-relativistic N-body problem.

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I. INTRODUCTION.

In the non-relativistic N-body problem the separation of the absolute translational motion of the center of mass from the relative motions can be easily carried out, due to the Abelian nature of the translation symmetry group. This implies that the associated Noether constants of motion (the conserved total 3-momentum) are in involution, so that the center-of-mass degrees of freedom decouple. Moreover, the fact that the non-relativistic kinetic energy of the relative motions is a quadratic form in the relative velocities allows the introduction of special sets of relative coordinates, the Jacobi normal relative coordinates, that diagonalize the quadratic form and correspond to different patterns of clustering of the centers of mass of the particles. Each set of Jacobi coordinates organizes the N particles into a hierarchy of clusters, in which each cluster of two or more particles has a mass given by an eigenvalue (reduced masses) of the quadratic form; the Jacobi normal coordinates join the centers of mass of cluster pairs.

On the other hand, the non-Abelian nature of the rotation symmetry group, whose associated Noether constants of motion (the conserved total angular momentum) are not in involution, prevents the possibility of a global separation of absolute rotations from the relative motions, so that there is no global definition of absolute vibrations. This has the consequence that an isolated deformable body can undergo rotations by changing its own shape (as shown by the falling cat and the diver). It was just to deal with these problems that the theory of the orientation-shape SO(3) principal bundle \( [1] \) has been developed in the context of molecular physics, emphasizing the gauge nature of a static (i.e. velocity-independent) definition of body frame for a deformable body. As a consequence, both the laboratory and body frame angular velocities as well as the orientational variables of the static body frame become unobservable gauge variables. This approach is associated with a set of point canonical transformations, which allow to define the body frame components of relative motions in a velocity-independent way.

In a previous paper \( [2] \) we showed that a more general class of non-point canonical transformations exists for \( N \geq 3 \), which allows to identify a family of canonical spin bases connected to the patterns of the possible clusterings of the spins associated with relative motions (namely the components of the center-of-mass angular momenta). The definition of these spin bases is independent of the use of Jacobi normal relative coordinates, just as the patterns of spin clustering are independent of the patterns of center-of-mass Jacobi clustering.

There exist two basic frames associated to each spin basis: the spin frame and the dynamical body frame. Their construction is guaranteed by the fact that, besides the existence on the relative phase space of a Hamiltonian symmetry left action of SO(3)\( [4] \) on the relative phase space, it is possible to define as many Hamiltonian non-symmetry right actions of

\[ ^1 \text{We adhere to the definitions used in Ref. [3]; in the mathematical literature our left action is a right action.} \]

\[ ^2 \text{The generators are the center-of-mass angular momentum, Noether constants of motion.} \]
SO(3) as the possible patterns of spin clustering. While for \( N = 3 \) the unique canonical spin basis coincides with a special class of global cross sections of the trivial orientation-shape SO(3) principal bundle, for \( N \geq 4 \) the existing spin bases and dynamical body frames turn out to be unrelated to the local cross sections of the static non-trivial orientation-shape SO(3) principal bundle, and evolve in a dynamical way dictated by the equations of motion. Both the orientation variables and the angular velocities become measurable quantities in each canonical spin basis.

In this way we get for each \( N \) a finite number of physically well-defined separations between rotational and vibrational degrees of freedom. The unique body frame of rigid bodies is replaced by a discrete number of evolving dynamical body frames and of spin canonical bases, both of which are grounded on patterns of spin couplings, direct analogues of the coupling of quantum angular momenta. These results might be useful in non-relativistic nuclear and molecular physics.

Besides translations and rotations, every isolated non-relativistic system admits the internal energy, the total mass and the Galilei boosts (which amounts essentially to the definition of the center of mass) as constants of the motion. Altogether, there are 11 constants of motion (one of them is a central charge) with which one gets a realization of the kinematical extended Galilei algebra [3,4].

The problem we want to tackle is what happens when we replace Galilean spacetime with Minkowski spacetime. Precisely what can be said in this case about the separation of the center of mass from the relative motions (the Abelian translation symmetry) and about the treatment of rotations (the non-Abelian rotational symmetry) already for the simplest system of \( N \) free scalar positive-energy particles?

The first immediate issue is how to describe a relativistic scalar particle. Among the various possibilities (see Refs. for a review of the various options) we will choose to start from the manifestly Lorentz covariant approach using Dirac’s first class constraints to identify free particles:

\[
p_i^2 - m_i^2 \approx 0. \tag{1.1}
\]

The associated Lagrangian description starts from the 4-vector positions \( x_i^\mu(\tau) \) and the action

\[
S = \int d\tau \left( -\epsilon \sum_i m_i \sqrt{\dot{x}_i^2(\tau)} \right),
\]

where \( \tau \) is a Lorentz scalar mathematical time parameter. Therefore Lorentz covariance implies the use of singular Lagrangians and of the associated Dirac’s theory of constraints for the Hamiltonian description. The time variables \( x_i^\mu(\tau) \) are the gauge variables associated to the mass-shell constraints, which have the two topologically disjoint solutions \( p_i^0 \approx \pm \sqrt{m_i^2 + \vec{p}_i^2} \). As discussed in Ref. [3,4] this implies that:

i) a combination of the time variables may be identified with the clock of one arbitrary observer labelling the evolution of the isolated system;

\[\text{The generators are not constants of motion.}\]

\[\text{We shall use } c = 1 \text{ everywhere and the convention } \eta^{\mu\nu} = \epsilon^{+---} \text{ for the Minkowski metric (with } \epsilon = \pm 1 \text{ according to the either particle physics or general relativity convention).}\]

\[\text{An affine parameter for the particle timelike worldlines.}\]
ii) the $N - 1$ relative times are connected with the observer freedom of looking at the $N$ particles either at the same time or with any prescribed delay among them.

The introduction of interactions in this picture without destroying the first class nature of the constraints is a difficult problem, which gave origin, in the two-particle case, to the DrozVincent-Komar-Todorov model. On the other hand, its extension to $N$ particles was never given in closed form.

When the particle is charged and interacts with a dynamical (non-external) electromagnetic field a problem of covariance reappears. The standard description of a charged scalar particle interacting with the electromagnetic field is based on the action

$$ S = -\epsilon m \int d\tau \sqrt{\epsilon \dot{x}^2(\tau)} - e \int d\tau \int d^4z \delta^4(z - x(\tau)) \dot{x}^\mu(\tau) A_\mu(z) - \frac{1}{4} \int d^4z F^{\mu\nu}(z) F_{\mu\nu}(z). $$

(1.2)

If we evaluate the canonical momenta of the isolated system charged particle plus electromagnetic field, we find two primary constraints:

$$ \chi(\tau) = \left( p - eA(x(\tau)) \right)^2 - \epsilon m^2 \approx 0, $$

$$ \pi^\alpha(z^\alpha, \vec{z}) \approx 0. $$

(1.3)

One realizes immediately that it is impossible to evaluate the Poisson bracket of the two constraints, because there is no concept of equal time. Also the Dirac Hamiltonian, which should be $H_D = H_c + \lambda(\tau) \chi(\tau) + \int d^3z \lambda^\alpha(z^\alpha, \vec{z}) \pi^\alpha(z^\alpha, \vec{z})$ does not make sense for the same reason. This problem is also present at the level of the Euler-Lagrange equations: precisely the formulation of a Cauchy problem for a system of coupled equations some of which are ordinary differential equations in the affine parameter $\tau$ along the particle worldline, while the others are partial differential equations depending on Minkowski coordinates $z^\mu$. Since the problem is due the absence of a covariant concept of equal time between the field and particle variables, a new formulation of the problem is needed.

In Ref. [7], after a discussion of the many time formalism, a solution of the previous covariance problem was found in a way suggested in the context of a description able to incorporate the gravitational field. There one considers an arbitrary 3+1 splitting of Minkowski space-time with spacelike hypersurfaces which is equivalent to a congruence of timelike accelerated observers. This is essentially Dirac’s reformulation of classical field theory (suitably extended to particles) on arbitrary spacelike hypersurfaces (equal time surfaces): it is also the classical basis of the Tomonaga-Schwinger formulation of quantum field theory. In this way, for each isolated system (containing any combination of particles, strings and fields) one gets its reformulation as a parametrized Minkowski theory, with the extra bonus of having the theory already prepared to the coupling to gravity in its ADM formulation, but with the price that the functions $z^\mu(\tau, \vec{\sigma})$ describing the embedding of the spacelike hypersurface

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6See Ref. [5] for the models with second class constraints corresponding to gauge fixings of the relative times.

7With $H_c$ the canonical Hamiltonian and with $\lambda(\tau), \lambda^\alpha(z^\alpha, \vec{z})$ Dirac’s multipliers.
in Minkowski spacetime become additional configuration variables in the action principle. Since the action is invariant under separate \( \tau \)-reparametrizations and space-diffeomorphisms, there are first class constraints ensuring the independence of the description from the choice of the 3+1 splitting: the embedding configuration variables \( z^\mu(\tau, \vec{\sigma}) \) are the gauge variables associated with this kind of general covariance.

Let us remark that, since the intersection of a timelike worldline with a spacelike hypersurface corresponding to a value \( \tau \) of the time parameter is identified by 3 numbers \( \vec{\eta}(\tau) \) and not by four, in parametrized Minkowski theories each particle must have a well defined sign of the energy: therefore we cannot simultaneously describe the two topologically disjoint branches of the mass hyperboloid as in the standard manifestly Lorentz-covariant theories. As a consequence, there are no more mass-shell constraints. Each particle with a definite sign of the energy is described by the canonical coordinates \( \vec{\eta}(\tau), \vec{\kappa}_i(\tau) \) with the derived 4-position of the particles given by \( x^\mu_i(\tau) = z^\mu(\tau, \vec{\eta}(\tau)) \). The derived 4-momenta \( p^\mu_i(\tau) \) are \( \vec{\kappa}_r \)-dependent solutions of \( p^2_i - \epsilon m_i^2 = 0 \) with the chosen sign of the energy.

In Minkowski spacetime, due to the independence of parametrized theories from the 3+1 splitting, we can restrict the foliation to have spacelike hyperplanes as leaves. In particular, for each configuration of the isolated system with timelike 4-momentum, we can restrict to the special foliation whose leaves are the hyperplanes orthogonal to the conserved total 4-momentum (Wigner hyperplanes). This special foliation is intrinsically determined by the configuration of the isolated system only. In this way \[7\] it is possible to define the Wigner-covariant rest-frame instant form of dynamics for every isolated system whose configurations have well defined and finite Poincaré generators with timelike total 4-momentum \[8\].

This formulation provides a clarification of the roles of the various relativistic centers of mass. This is a long standing problem which arose just after the foundation of special relativity in the first decade of the last century. In the next ninety years it became clear that the definition of a relativistic center of mass is highly non-trivial: no definition can enjoy all the properties of the non-relativistic center of mass. See Refs. \[11–16\] for a partial bibliography of all the existing attempts and Ref. \[17\] for reviews.

As shown in Appendix A, in the rest-frame instant form on Wigner hyperplanes only four first class constraints survive and the original configuration variables \( z^\mu(\tau, \vec{\sigma}) \), \( \vec{\eta}(\tau) \) and their conjugate momenta \( \rho^\mu(\tau, \vec{\sigma}), \vec{\kappa}_i(\tau) \) are reduced to:

i) a decoupled particle \( \tilde{x}_s^\mu(\tau), p^\mu_s \) (the only remnant of the spacelike hypersurface) with a positive mass \( \epsilon_s = \sqrt{\epsilon p^2_s} \) determined by the first class constraint \( \epsilon_s - M_{\text{sys}} \approx 0 \) \[9\] and with its rest-frame Lorentz scalar time \( T_s = \tilde{x}_s^\mu(\tau) \) put equal to the mathematical time as the gauge fixing \( T_s - \tau \approx 0 \) to the previous constraint. Here, \( \tilde{x}_s^\mu(\tau) \) is a non-covariant canonical variable for the external 4-center of mass. After the elimination of \( T_s \) and \( \epsilon_s \) with the previous pair of second class constraints, one remains with a decoupled free point (point

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8See Ref. \[10\] for the traditional forms of dynamics.

9See Appendix A for a review of parametrized Minkowski theories and of the rest-frame instant form of dynamics.

10\( M_{\text{sys}} \) being the invariant mass of the isolated system.
particle clock) of mass $M_{sys}$ and canonical 3-coordinates $\vec{z}_s = \epsilon_s (\vec{x}_s - \frac{\vec{p}_s}{p_s^0} \vec{x}_0)$, $\vec{k}_s = \frac{\vec{p}_s}{\epsilon_s}$. The non-covariant canonical $\vec{x}_s^\mu (\tau)$ must not be confused with the 4-vector $x_s^\mu (\tau) = z_s^\mu (\tau, \vec{\sigma} = 0)$ identifying the origin of the 3-coordinates $\vec{\sigma}$ inside the Wigner hyperplanes. The worldline $x_s^\mu (\tau)$ is arbitrary because it depends on $x_s^\mu (0)$ and its 4-velocity $\dot{x}_s^\mu (\tau)$ depends on the Dirac multipliers associated with the 4 first class constraints \[11\], as it will be shown in the next Section. The unit timelike 4-vector $u_s^\mu (p_s^0) = \frac{p_s^\mu}{\epsilon_s}$ is orthogonal to the Wigner hyperplanes and describes their orientation in the chosen inertial frame.

ii) the particle canonical variables $\vec{\eta}_i (\tau), \vec{\kappa}_i (\tau)$ inside the Wigner hyperplanes. They are restricted by the three first class constraints (the rest-frame conditions) $\vec{\kappa}_+ = \sum_{i=1}^N \vec{\kappa}_i \approx 0$.

Therefore, we need a doubling of the concepts:

1) there is the external viewpoint of an arbitrary inertial Lorentz observer, who describes the Wigner hyperplanes, as leaves of a foliation of Minkowski spacetime, determined by the timelike configurations of the isolated system. A change of inertial observer by means of a Lorentz transformation rotates the Wigner hyperplanes and induces a Wigner rotation of the 3-vectors inside each Wigner hyperplane. Every such hyperplane inherits an induced internal Euclidean structure while an external realization of the Poincaré group induces the internal Euclidean action.

As said above, an arbitrary worldline $x_s^\mu (\tau)$ is chosen as origin of the internal 3-coordinates on the Wigner hyperplanes; its velocity $\dot{x}_s^\mu (\tau)$ is determined only after the introduction of four gauge fixings for the four first class constraints (one of them is $T_s - \tau \approx 0$).

Three external concepts of 4-center of mass can be defined (each one of which there has an internal 3-location inside the Wigner hyperplanes):

a) the external non-covariant canonical 4-center of mass (also named center of spin \[15\]) $\vec{x}_s^\mu$ (with 3-location $\vec{\sigma}$),

b) the external non-covariant non-canonical Møller 4-center of energy \[13\] $R_s^\mu$ (with 3-location $\vec{\sigma}_R$),

c) the external covariant non-canonical Fokker-Pryce 4-center of inertia \[14,15\] $Y_s^\mu$ (with 3-location $\vec{\sigma}_Y$).

Only the canonical non-covariant center of mass $\vec{x}_s^\mu (\tau)$ is relevant in the Hamiltonian treatment with Dirac constraints, while only the Fokker-Pryce $Y_s^\mu$ is a 4-vector by construction.

2) there is the internal viewpoint inside the Wigner hyperplanes associated to a unfaithful internal realization of the Poincaré algebra: the internal 3-momentum $\vec{\kappa}_+$ vanishes due to

\[11\] $\vec{z}_s/\epsilon_s$ is the classical analogue of the Newton-Wigner 3-position operator \[12\] and, like it, is only covariant under the Euclidean subgroup of the Poincaré group only.

\[12\] Therefore this arbitrary worldline may be considered as an arbitrary centroid for the isolated system.

\[13\] They are Wigner spin-1 3-vectors, like the coordinates $\vec{\sigma}$. 

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\[11\]
\[12\]
\[13\]
the rest-frame conditions. The internal energy and angular momentum are the invariant mass \(M_{\text{sys}}\) and the spin (the angular momentum with respect to \(\vec{k}_s(\tau)\)) of the isolated system, respectively. Three internal 3-centers of mass: the internal canonical 3-center of mass can be correspondingly defined, the internal Møller 3-center of energy and the internal Fokker-Pryce 3-center of inertia. But, due to the rest-frame conditions, they coincide and become essentially the it gauge variable conjugate to \(\vec{r}_+\). As a natural gauge fixing for the rest-frame conditions \(\vec{r}_+ \approx 0\), we can add the vanishing of the internal Lorentz boosts: this is equivalent to locate the internal canonical 3-center of mass \(\vec{q}_+\) in \(\vec{\sigma} = 0\), i.e. in the origin \(x^\mu_s(\tau) = z^\mu(\tau, \vec{0})\). With these gauge fixings and with \(T_s - \tau \approx 0\), the worldline \(x^\mu_s(\tau)\) becomes uniquely determined except for the arbitrariness in the choice of \(x^\mu_s(0) [w^\mu(p_s) = p^\mu_s/\epsilon_s]\)

\[
x^\mu_s(\tau) = x^\mu_s(0) + u^\mu(p_s)T_s,
\]

and coincides with the external covariant non-canonical Fokker-Pryce 4-center of inertia, \(x^\mu_s(\tau) = x^\mu_s(0) + Y^\mu_s\).

This doubling of concepts replaces the separation of the non-relativistic 3-center of mass due to the Abelian translation symmetry. The non-relativistic conserved 3-momentum is replaced by the external \(\vec{p}_s = \epsilon_s \vec{k}_s\), while the internal 3-momentum vanishes, \(\vec{r}_+ \approx 0\), as a definition of rest frame.

In the final gauge we have \(\epsilon_s \equiv M_{\text{sys}}, T_s \equiv \tau\) and the canonical basis \(\vec{z}_s, \vec{k}_s, \vec{\eta}_s, \vec{\kappa}_s\) restricted by the three pairs of second class constraints \(\vec{r}_+ + \sum_{i=1}^N \vec{r}_i \approx 0, \vec{q}_+ \approx 0\), so that 6N canonical variables describe the N particles like in the non-relativistic case. We still need a canonical transformation \(\vec{\eta}_s, \vec{\kappa}_s \mapsto \vec{q}_+|0\rangle, \vec{r}_+|0\rangle, \vec{p}_a, \vec{\pi}_a (a = 1, \ldots, N - 1)\) in order to identify a set of relative canonical variables. The final 6N-dimensional canonical basis is \(\vec{z}_s, \vec{k}_s, \vec{p}_a, \vec{\pi}_a\). To get this result we need a highly non-linear canonical transformation, which can be obtained by exploiting the Gartenhaus-Schwartz singular transformation [18].

In the end, we obtain the Hamiltonian for relative motions as a sum of N square roots, each one containing a squared mass and a quadratic form in the relative momenta. This Hamiltonian goes into its non-relativistic counterpart in the limit \(c \to \infty\). This fact has the following implications:

a) if one tries to make the inverse Legendre transformation to find the associated Lagrangian, it turns out that, due to the presence of square roots, the Lagrangian is a hyperelliptic function of \(\vec{p}_a\) already in the free case. A closed form exists only for \(N=2, m_1 = m_2 = m\): \(L = -em\sqrt{4 - \hat{\rho}^2}\). This exceptional case already shows that the existence of the limiting velocity \(c\) (i.e. of the light-cone) forbids the traditional linear relation between the spin and the angular velocity.

b) the N quadratic forms in the relative momenta appearing in the relative Hamiltonian cannot be diagonalized simultaneously. In any case, the Hamiltonian is a sum of square roots, so that concepts like reduced masses, Jacobi normal relative coordinates and tensor of inertia cannot be extended to special relativity. As a consequence, the orientation-shape SO(3) principal bundle of Ref. [1] can be defined only by using unspecified relative coordinates.

c) the best way of studying rotational kinematics is by using the canonical spin bases of Ref. [2] with their spin frames and dynamical body frames: they can be build in the same way as in the non-relativistic case starting from the canonical basis \(\vec{p}_a, \vec{\pi}_a\).

Once these points are understood in the free case, the introduction of mutual action-at-a-distance interactions among the particles can be done without extra complications.
The paper is organized as follows. In Section II we review the rest-frame instant form on the Wigner hyperplane of N positive energy free scalar particles. In Section III we discuss the internal realization of the Poincaré algebra and we define the internal center-of-mass concepts. In Section IV we discuss the external realization of the Poincaré algebra and we define the external center-of-mass concepts. In Section V we construct the canonical relative variables associated with the canonical internal center of mass. In Section VI we analyze the relativistic rotational kinematics of relative motions inside the Wigner hyperplane using the same Hamiltonian methods for the construction of the spin bases of Ref. [2]. Some final comments on open problems are given in the Conclusions.

Appendix A contains a review of parametrized Minkowski theories and of the rest-frame instant form of dynamics. Some notations on spacelike hypersurfaces are listed in Appendix B. The results of Section V are extended to spinning particles in Appendix C. Some formulas for the Euler angles are reported in Appendix D. Finally, the treatment of the 3-body case is explicitly given in Appendix E.
II. THE REST-FRAME INSTANT FORM OF N FREE SCALAR RELATIVISTIC PARTICLES

Let us consider a system of N free scalar positive-energy particles in the framework of parametrized Minkowski theory (see Appendices A and B).

The configuration variables are a 3-vector $\mathbf{\eta}_i(\tau)$ for each particle $[x_i^\mu(\tau) = z_i^\mu(\tau, \mathbf{\eta}_i(\tau))]$. One has to choose the sign of the energy of each particle, because there are not mass-shell constraints (like $\epsilon p_i^2 - m_i^2 \approx 0$) at our disposal, due to the fact that one has only three degrees of freedom for particle, determining the intersection of a timelike trajectory and of the spacelike hypersurface $\Sigma_\tau$. For each choice of the sign of the energy of the N particles, one describes only one of the $2^N$ branches of the mass spectrum of the manifestly covariant approach based on the coordinates $x_i^\mu(\tau), p_i^\mu(\tau), i=1,..,N$, and on the constraints $\epsilon p_i^2 - m_i^2 \approx 0$ (in the free case). In this way, one gets a description of relativistic particles with a given sign of the energy with consistent couplings to fields.

The system of N free scalar and positive energy particles is described by the action

$$S = \int d\tau d^3\sigma \mathcal{L}(\tau, \mathbf{\eta}) = \int d\tau L(\tau),$$

$$\mathcal{L}(\tau, \mathbf{\eta}) = -\sum_{i=1}^{N} \frac{\delta(\mathbf{\eta}_i - \mathbf{\eta}_i(\tau)) m_i \sqrt{g_{\tau\tau}(\tau, \mathbf{\eta}) + 2g_{\tau\mathbf{\eta}}(\tau, \mathbf{\eta})\mathbf{\eta}_i(\tau) + g_{\mathbf{\eta}\mathbf{\eta}}(\tau, \mathbf{\eta})\mathbf{\eta}_i(\tau)}}{\sqrt{g_{\tau\tau}(\tau, \mathbf{\eta}) + 2g_{\tau\mathbf{\eta}}(\tau, \mathbf{\eta})\mathbf{\eta}_i(\tau) + g_{\mathbf{\eta}\mathbf{\eta}}(\tau, \mathbf{\eta})\mathbf{\eta}_i(\tau)}},$$

$$L(\tau) = -\sum_{i=1}^{N} m_i \sqrt{g_{\tau\tau}(\tau, \mathbf{\eta}_i) + 2g_{\tau\mathbf{\eta}}(\tau, \mathbf{\eta}_i)\mathbf{\eta}_i(\tau) + g_{\mathbf{\eta}\mathbf{\eta}}(\tau, \mathbf{\eta}_i)\mathbf{\eta}_i(\tau)}, \quad (2.1)$$

where the configuration variables are $z^\mu(\tau, \mathbf{\eta})$ and $\mathbf{\eta}_i(\tau)$, $i=1,..,N$. The action is invariant under separate $\tau$- and $\mathbf{\eta}$-reparametrizations.

The canonical momenta are

$$\rho_\mu(\tau, \mathbf{\eta}) = -\frac{\partial \mathcal{L}(\tau, \mathbf{\eta})}{\partial z_\mu^\nu(\tau, \mathbf{\eta})} = \sum_{i=1}^{N} \frac{\delta(\mathbf{\eta}_i - \mathbf{\eta}_i(\tau)) m_i}{\sqrt{g_{\tau\tau}(\tau, \mathbf{\eta}) + 2g_{\tau\mathbf{\eta}}(\tau, \mathbf{\eta})\mathbf{\eta}_i(\tau) + g_{\mathbf{\eta}\mathbf{\eta}}(\tau, \mathbf{\eta})\mathbf{\eta}_i(\tau)}} z_{\tau\mu}(\tau, \mathbf{\eta}) + z_{\tau\mathbf{\eta}}(\tau, \mathbf{\eta})\mathbf{\eta}_i(\tau)$$

$$= \left[(\rho_\mu z_\mu^\nu)(\tau, \mathbf{\eta})\gamma^\zeta z_\zeta(\tau, \mathbf{\eta})\right],$$

$^{14}$This is true for scalar positive-energy particles. For spinning positive-energy particles one has to add non-minimally some coupling of the spin to the electric field, which would be missed in the projection from the Lorentz covariant theory (with $2^N$ branches of the mass spectrum) to the theory describing only the branch, in which all the particles have positive energy. This can be done by performing an approximate Foldy-Wouthuysen transformation of the Lorentz covariant theory in presence of electromagnetic fields, because the electric field is the source of possible crossings of the deformed $2^N$ branches (classical counterpart of pair production). The additional spin-electric field coupling is the source of the spin-orbit term in the quantum electron Hamiltonian.
\[\kappa_{ir}(\tau) = -\frac{\partial L(\tau)}{\partial \eta^i_r(\tau)} = \]
\[m_i \frac{g_{\tau\tau}(\tau, \eta_i(\tau)) + g_{\tau\tilde{\kappa}}(\tau, \eta_i(\tau))\eta^\tilde{\kappa}(\tau)}{\sqrt{g_{\tau\tau}(\tau, \eta_i(\tau)) + 2g_{\tau\tau}(\tau, \eta_i(\tau))\eta^\tilde{\kappa}(\tau) + g_{\tau\tau}(\tau, \eta_i(\tau))\eta^\tilde{\kappa}(\tau)}}.\]
\[\{z^\mu(\tau, \tilde{\sigma}), \rho_\nu(\tau, \tilde{\sigma})\} = -\eta^\mu_i \delta^3(\tilde{\sigma} - \tilde{\sigma}^\prime),\]
\[\{\eta^\tilde{\kappa}_i(\tau), \kappa_{js}(\tau)\} = -\delta_{ij}\delta^\kappa_s.\]  

(2.2)

The canonical Hamiltonian \(H_c\) is zero, the Dirac Hamiltonian is given by Eq. (A3) [there are no other system-dependent primary constraints] and Eqs. (A2) become

\[\mathcal{H}_\mu(\tau, \tilde{\sigma}) = \rho_\mu(\tau, \tilde{\sigma}) - l_\mu(\tau, \tilde{\sigma}) \sum_{i=1}^{N} \delta^3(\tilde{\sigma} - \eta_i(\tau)) \sqrt{m_i^2 - \gamma^\tilde{\kappa}(\tau, \tilde{\sigma})\kappa_{ir}(\tau)\kappa_{is}(\tau)} - \]
\[z_{\tilde{\mu}}(\tau, \tilde{\sigma})\gamma^{\tilde{\kappa}}(\tau, \tilde{\sigma}) \sum_{i=1}^{N} \delta^3(\tilde{\sigma} - \eta_i(\tau))\kappa_{is} \approx 0.\]  

(2.3)

The conserved Poincaré generators are (the suffix “s” denotes the hypersurface \(\Sigma_\tau\))

\[p^\mu_s = \int d^3\sigma \rho^\mu(\tau, \tilde{\sigma}),\]
\[J^\mu\nu_s = \int d^3\sigma [z^\mu(\tau, \tilde{\sigma})\rho^\nu(\tau, \tilde{\sigma}) - z^\nu(\tau, \tilde{\sigma})\rho^\mu(\tau, \tilde{\sigma})].\]  

(2.4)

After the restriction to spacelike hyperplanes, the Dirac Hamiltonian is reduced to Eq. (A8) with the surviving ten constraints given by

\[\tilde{\mathcal{H}}^\mu(\tau) = \int d^3\sigma \mathcal{H}^\mu(\tau, \tilde{\sigma}) = p^\mu_s - l^\mu \sum_{i=1}^{N} \sqrt{m_i^2 + \kappa_i^2(\tau)} + b^\mu(\tau) \sum_{i=1}^{N} \kappa_{ir}(\tau) \approx 0,\]
\[\tilde{\mathcal{H}}^{\mu\nu}(\tau) = b^\mu_r(\tau) \int d^3\sigma \rho^\nu(\tau, \tilde{\sigma}) - b^\nu_r(\tau) \int d^3\sigma \rho^\mu(\tau, \tilde{\sigma}) = \]
\[= S^{\mu\nu}(\tau) - [b^\mu_r(\tau)b^\nu_r(\tau) - b^\nu_r(\tau)b^\mu_r(\tau)] \sum_{i=1}^{N} \eta^\mu_i(\tau) \sqrt{m_i^2 + \kappa_i^2(\tau)} - \]
\[= [b^\mu_r(\tau)b^\nu_r(\tau) - b^\nu_r(\tau)b^\mu_r(\tau)] \sum_{i=1}^{N} \eta^\mu_i(\tau) \kappa_{ir}(\tau) \approx 0.\]  

(2.5)

Here \(S^{\mu\nu}_s\) is the spin part of the Lorentz generators

\[J^{\mu\nu}_s = x_s^{\mu}s^{\nu} - x_s^{\nu}s^{\mu} + S^{\mu\nu}_s,\]
\[S^{\mu\nu}_s = b^\mu_r(\tau) \int d^3\sigma \rho^\nu(\tau, \tilde{\sigma}) - b^\nu_r(\tau) \int d^3\sigma \rho^\mu(\tau, \tilde{\sigma}).\]  

(2.6)

On the Wigner hyperplane we have the following constraints and Dirac Hamiltonian

\[\tilde{\mathcal{H}}^\mu(\tau) = p^\mu_s - u^\mu(p_s) \sum_{i=1}^{N} \sqrt{m_i^2 + \kappa_i^2} + \epsilon^\mu_s(u(p_s)) \sum_{i=1}^{N} \kappa_{ir} =\]
\[ u^\mu(p_s)[\epsilon_s - \sum_{i=1}^{N} \sqrt{m_i^2 + \kappa_i^2}] + \epsilon^\mu_r(u(p_s)) \sum_{i=1}^{N} \kappa_{ir} \approx 0, \]

or

\[ \epsilon_s - M_{\text{sys}} \approx 0, \quad M_{\text{sys}} = \sum_{i=1}^{N} \sqrt{m_i^2 + \kappa_i^2}, \]

\[ \tilde{\rho}_{\text{sys}} = \tilde{\kappa}_+ = \sum_{i=1}^{N} \tilde{\kappa}_i \approx 0, \]

\[ H_D = \lambda^\mu(\tau) \tilde{H}_\mu(\tau) = \lambda(\tau)[\epsilon_s - M_{\text{sys}}] - \tilde{\lambda}(\tau) \sum_{i=1}^{N} \tilde{\kappa}_i, \]

\[ \lambda(\tau) \approx -\hat{x}_{\mu}(\tau)u^\mu(p_s), \]

\[ \lambda_r(\tau) \approx -\hat{x}_{\mu}(\tau)\epsilon^\mu_r(u(p_s)), \]

\[ \hat{x}_{s}^\mu(\tau) = -\lambda(\tau)u^\mu(p_s), \]

\[ \hat{x}_{s}^\mu(\tau) \approx -\hat{\lambda}(\tau)u^\mu(p_s) + \epsilon^\mu_r(u(p_s))\lambda_r(\tau). \quad (2.7) \]

While the Dirac multiplier \( \lambda(\tau) \) is determined by the gauge fixing \( T_s - \tau \approx 0 \), the 3 Dirac’s multipliers \( \tilde{\lambda}(\tau) \) describe the classical zitterbewegung of the origin of the coordinates on the Wigner hyperplane: each gauge-fixing \( \tilde{\chi}(\tau) \approx 0 \) to the 3 first class constraints \( \tilde{\kappa}_+ \approx 0 \) (defining the internal rest-frame) gives a different determination of the multipliers \( \tilde{\lambda}(\tau) \) and therefore identifies a different worldline for the covariant non-canonical origin \( x_{s}^{(\tilde{\chi})\mu}(\tau) \) which induces the definition \( \tilde{\chi} \) of the internal 3-center of mass conjugate to \( \tilde{\kappa}_+ \).

The embedding describing Wigner hyperplanes is \( z^\mu(\tau, \tilde{\sigma}) = x_{s}^{\mu}(\tau) + \epsilon^\mu_r(u(p_s))\sigma^r \), with the \( \epsilon^\mu_r(u(p_s)) \) defined in Eqs. (3.13).

The various spin tensors and vectors are \[ J_{s}^{\mu\nu} = x_{s}^{\mu}p_{s}^{\nu} - x_{s}^{\nu}p_{s}^{\mu} + S_{s}^{\mu\nu} = \tilde{x}_{s}^{\mu}p_{s}^{\nu} - \tilde{x}_{s}^{\nu}p_{s}^{\mu} + \tilde{S}_{s}^{\mu\nu}, \]

\[ S_{s}^{\mu\nu} = [u^\mu(p_s)\epsilon^\nu(u(p_s)) - u^\nu(p_s)\epsilon^\mu(u(p_s))]\tilde{S}_s^{\tau\tau} + \epsilon^\mu(u(p_s))\epsilon^\nu(u(p_s))\tilde{S}_s^{\tau\tau} \equiv \]

\[ \equiv \left[ \epsilon^\mu_r(u(p_s))u^\nu(p_s) - \epsilon^\nu_r(u(p_s))u^\mu(p_s) \right] \sum_{i=1}^{N} \eta_i^r \sqrt{m_i^2 c^2 + \kappa_i^2} + \]

\[ + \left[ \epsilon^\mu_r(u(p_s))\epsilon^\nu_r(u(p_s)) - \epsilon^\nu_r(u(p_s))\epsilon^\mu_r(u(p_s)) \right] \sum_{i=1}^{N} \eta_i^r \kappa_i^r, \]

\[ \tilde{S}_{s}^{AB} = \epsilon^A_\mu(u(p_s))\epsilon^B_\nu(u(p_s))S_{s}^{\mu\nu}, \]

\[ \tilde{S}_s^{\tau\tau} \equiv \sum_{i=1}^{N} (\eta_i^r \kappa_i^s - \eta_i^s \kappa_i^r), \quad \tilde{S}_s^{\tau\tau} \equiv - \sum_{i=1}^{N} \eta_i^r \sqrt{m_i^2 c^2 + \kappa_i^2}; \]

\[ ^{15} \text{Naturally each choice } \tilde{\chi} \text{ leads to a different set of relative canonical conjugate variables.} \]
\[
\tilde{S}^{\mu\nu} = S^{\mu\nu} + \frac{1}{\sqrt{\epsilon p_s^2 (p^\rho_s + \sqrt{\epsilon p_s^2})}} [p_{\beta\delta} (S^{\beta\mu}_s p^\nu_s - S^{\beta\nu}_s p^\mu_s) + \sqrt{p_s^2} (S^{\alpha\mu}_s p^\nu_s - S^{\alpha\nu}_s p^\mu_s)],
\]

\[
\tilde{S}^{ij} = \delta^{ij} \tilde{S}_s, \quad \tilde{S}^{oi} = -\frac{\delta^{ir} S^r_s p^o_s}{p^o_s + \sqrt{\epsilon p_s^2}},
\]

\[
\tilde{S} \equiv \sum_{i=1}^{N} \vec{\eta}_i \times \vec{K}_i \approx \sum_{i=1}^{N} \vec{\eta}_i \times \vec{K}_i - \vec{\eta}_+ \times \vec{K}_+ = \sum_{a=1}^{N-1} \vec{\rho}_a \times \vec{\pi}_a. \tag{2.8}
\]

Let us remark that while \( L^{\mu\nu}_s = x^\mu_s p^\nu_s - x^\nu_s p^\mu_s \) and \( S^{\mu\nu}_s \) are not constants of the motion due to the classical zitterbewegung, both \( \tilde{L}^{\mu\nu}_s = \tilde{x}^\mu_s p^\nu_s - \tilde{x}^\nu_s p^\mu_s \) and \( \tilde{S}^{\mu\nu}_s \) are conserved.

The only remaining canonical variables describing the Wigner hyperplane in the final Dirac brackets are the non-covariant canonical coordinate \( \tilde{x}^\mu_s(\tau) \) and \( p^\mu_s \). The point with coordinates \( \tilde{x}^\mu_s(\tau) \) is the decoupled canonical external 4-center of mass, playing the role of a kinematical external 4-center of mass and of a decoupled observer with his parametrized clock (point particle clock). Its velocity \( \tilde{x}^\mu_s(\tau) \) is parallel to \( p^\mu_s \), so that it has no classical zitterbewegung.

The connection between \( x^\mu_s(\tau) \) and \( \tilde{x}^\mu_s(\tau) \) is given in Eq.(4.1) in Section IV. Let us remark that the constant \( x^\mu_s(0) \) [and \( \tilde{x}^\mu_s(0) \)] is arbitrary, reflecting the arbitrariness in the absolute location of the origin of the internal coordinates on each hyperplane in Minkowski spacetime.

After the separation of the relativistic canonical non-covariant external 4-center of mass \( \tilde{x}^\mu_s(\tau) \), on the Wigner hyperplane the \( N \) particles are described by the 6\( N \) Wigner spin-1 3-vectors \( \vec{\eta}_i(\tau), \vec{K}_i(\tau) \) restricted by the rest-frame condition \( \vec{K}_+ = \sum_{i=1}^{N} \vec{K}_i \approx 0 \).

The canonical variables \( \tilde{x}^\mu_s, p^\mu_s \) for the external 4-center of mass, may be replaced by the canonical pairs \( ^{16}\)

\[
T_s = \frac{p_s \cdot \tilde{x}_s}{\epsilon_s} = \frac{p_s \cdot x_s}{\epsilon_s},
\]

\[
\epsilon_s = \pm \sqrt{\epsilon p_s^2},
\]

\[
\vec{z}_s = \epsilon_s (\vec{x}_s - \frac{\vec{p}_s \cdot \vec{x}_s}{p_s^2}),
\]

\[
\vec{K}_s = \frac{\vec{p}_s}{\epsilon_s}, \tag{2.9}
\]

with the inverse transformation

\[
\tilde{x}_s^0 = \sqrt{1 + \vec{K}_s^2 (T_s + \frac{\vec{K}_s \cdot \vec{z}_s}{\epsilon_s})},
\]

\[
\tilde{x}_s = \frac{\tilde{z}_s}{\epsilon_s} + (T_s + \frac{\vec{K}_s \cdot \vec{z}_s}{\epsilon_s}) \vec{K}_s,
\]

\[
p_s^0 = \epsilon_s \sqrt{1 + \vec{K}_s^2},
\]

\[
\vec{p}_s = \epsilon_s \vec{K}_s. \tag{2.10}
\]

\(^{16}\)They make explicit the interpretation of \( \tilde{x}^\mu_s \) as a point particle clock.
This non-point canonical transformation in the rest-frame instant form can be summarized as \([\epsilon_s - M_{sys} \approx 0, \vec{k}_+ = \sum_{i=1}^{N} \vec{k}_i \approx 0]\)

\[
\begin{align*}
\tilde{x}_s^\mu & \rightarrow x_s^\mu, \\
\tilde{p}_s^\mu & \rightarrow \epsilon_s \tau, \vec{q}_s + \vec{k}_i
\end{align*}
\] (2.11)

The invariant mass \(M_{sys}\) of the system, which is also the internal energy of the isolated system, replaces the non-relativistic Hamiltonian \(H_{rel}\) for the relative degrees of freedom, after the addition of the gauge-fixing \(T_s - \tau \approx 0\) \footnote{Implying \(\lambda(\tau) = -\epsilon\) and identifying the time parameter \(\tau\), that labels the leaves of the foliation with the Lorentz scalar time of the center of mass in the rest frame, \(T_s = p_s \cdot \vec{x}_s / M_{sys}; \ M_{sys}\) generates the evolution in this time.} this reminds of the frozen Hamilton-Jacobi theory, in which the time evolution can be reintroduced by using the energy generator of the Poincaré group as Hamiltonian \footnote{See Refs. \[23\] for a different derivation of this result.}

After the gauge fixings \(T_s - \tau \approx 0\), the final Hamiltonian and the embedding of the Wigner hyperplane into Minkowski spacetime become

\[
H_D = M_{sys} - \lambda(\tau) \cdot \vec{k}_+,
\]

\[
z^\mu(\tau, \vec{\sigma}) = x_s^\mu(\tau) + \epsilon^\mu_r(u(p_s))\sigma_r = x_s^\mu(0) + u^\mu(p_s)\tau + \epsilon^\mu_r(u(p_s))\sigma_r,
\]

with

\[
\dot{x}_s^\mu(\tau) = \frac{d x_s^\mu(\tau)}{d\tau} + \{x_s^\mu(\tau), H_D\} = u^\mu(p_s) + \epsilon^\mu_r(u(p_s))\lambda_r(\tau),
\]

where \(x_s^\mu(0)\) is an arbitrary point and \(\epsilon^\mu_r(u(p_s)) = L^\mu_r(p_s, \tilde{p}_s)\). This equation visualizes the role of the Dirac multipliers as sources of the classical zittebewegung.

After the gauge fixings \(T_s - \tau \approx 0, \tilde{q}_+ \approx 0\), the embedding \(z^\mu(\tau, \vec{\sigma}) = x_s^\mu(\tau) + \epsilon^\mu_r(u(p_s))\sigma_r\) describing Wigner hyperplanes becomes \(z^\mu(\tau, \vec{\sigma}) = x_s^\mu(0) + u^\mu(p_s)\tau + \epsilon^\mu_r(u(p_s))\sigma_r\).

The particles’ worldlines in Minkowski spacetime and the associated momenta are

\[
\begin{align*}
x_s^\mu(\tau) &= z^\mu(\tau, \vec{\eta}_i(\tau)) = x_s^\mu(\tau) + \epsilon^\mu_r(u(p_s))\eta^\mu_r(\tau), \\
p_s^\mu(\tau) &= \sqrt{m_i^2 + \vec{k}_i(\tau)u^\mu(p_s) + \epsilon^\mu_r(u(p_s))\kappa_{ir}(\tau)} \Rightarrow \epsilon_\tau^2 = m_i^2.
\end{align*}
\] (2.13)

Inside the Wigner hyperplane three degrees of freedom of the isolated system \footnote{They describe an internal center-of-mass 3-variable \(\vec{\sigma}_{com}\) defined inside the Wigner hyperplane and conjugate to \(\vec{k}_+\); when the \(\vec{\sigma}_{com}\) are canonical variables they are denoted \(\vec{q}_+\).} become gauge variables. To eliminate the three first class constraints \(\vec{k}_+ \approx 0\) the natural gauge fixing is \(\vec{\chi} = \tilde{q}_+ \approx 0\) implying \(\lambda_r(\tau) = 0\): in this way the internal 3-center of mass gets
located in the origin \( x^\mu_s(\tau) = z^\mu(\tau, \bar{\sigma} = 0) \) of the Wigner hyperplane. The determination of \( \vec{q}_s \) for the \( N \) particle system will be done with the group theoretical methods of Ref. [24] in the next Section.

The same problem arises when one considers the rest-frame description of fields. A basis with a center of phase has already been found for a real Klein-Gordon field both in the covariant approach [25] and on spacelike hypersurfaces [26]. In this case also the internal center of mass has been found, but not yet a canonical basis containing it.

It turns out that the Wigner hyperplane is the natural setting for the study of the Dixon multipoles of extended relativistic systems [27] and for defining their canonical relative variables with respect to the center of mass. Also, the Wigner hyperplane with its natural Euclidean metric structure offers a natural solution to the problem of boost for lattice gauge theories and realizes explicitly the Machian view of dynamics according to which only relative motions are relevant.

The external rest-frame instant form realization of the Poincaré generators with non-fixed invariants \( \epsilon p^2_s = \epsilon^2_s \approx M^2_{sys} \), \( -\epsilon p^2_s \bar{S}^2_s \approx -\epsilon M^2_{sys} \bar{S}^2_s \), is obtained from Eq.(2.8):

\[
p^\mu_s, \quad J^{\mu \nu}_s = \vec{x}_s^\mu \vec{p}_s^\nu - \vec{z}_s^\nu \vec{p}_s^\mu + \bar{S}^\mu_\nu,
\]

\[
p^0_s = \sqrt{\epsilon^2_s + \vec{p}^2_s} = \epsilon_s \sqrt{1 + \vec{k}^2_s} \approx \sqrt{M^2_{sys} + \vec{p}^2_s} = M_{sys} \sqrt{1 + \vec{k}^2_s},
\]

\[
\vec{p}_s = \epsilon_s \vec{k}_s \approx M_{sys} \vec{k}_s,
\]

\[
J^{i j}_s = \vec{x}_s^i p^j_s - \vec{x}_s^j p^i_s + \delta^{i r} \delta^{j s} \sum_{i=1}^{N} (\eta^r_i \kappa^s_i - \eta^s_i \kappa^r_i) = z^i_s k^j_s - z^j_s k^i_s + \delta^{i r} \delta^{j s} \epsilon^{r s u} \bar{S}^u_s,
\]

\[
K^i_s = J^{0 i}_s = \vec{x}_s^0 p^i_s - \vec{x}_s^i \epsilon^2_s + \vec{p}^2_s - \frac{1}{\epsilon_s + \sqrt{\epsilon^2_s + \vec{p}^2_s}} \delta^{i r} p^s_s \sum_{i=1}^{N} (\eta^r_i \kappa^s_i - \eta^s_i \kappa^r_i) = \frac{\delta^{i r} p^s_s \epsilon^{r s u} \bar{S}^u_s}{M_{sys} + \sqrt{M^2_{sys} + \vec{p}^2_s}}.
\]

On the other hand, the internal realization of the Poincaré algebra is built inside the Wigner hyperplane by using the expression of \( S^{AB}_s \) given by Eq.(2.8) 23

\[
M_{sys} = H_M = \sum_{i=1}^{N} \sqrt{m^2_i + \bar{\kappa}^2_i},
\]

20In this paper there is a first treatment of the topics which will be treated in Sections III and IV

21In a next paper we will study Dixon’s multipoles for the \( N \)-body problem [28].

22There are four independent Hamiltonians \( p^0_s \) and \( J^{0 i}_s \) functions of the system invariant mass \( M_{sys} \); we give also the expression in the basis \( T_s, \epsilon_s, \vec{z}_s, \vec{k}_s \).

23This internal Poincaré algebra realization must not be confused with the previous external one based on \( S^{\mu \nu}_s \); \( \Pi \) and \( W^2 \) are the two non-fixed invariants of this realization.
\[ \vec{\kappa}_+ = \sum_{i=1}^{N} \vec{\kappa}_i \approx 0, \]
\[ \vec{J} = \sum_{i=1}^{N} \vec{\eta}_i \times \vec{\kappa}_i, \quad J^r = \vec{S}^r = \frac{1}{2} \epsilon^{rav} \vec{S}^{av} \equiv \vec{S}_s^r, \]
\[ \vec{K} = -\sum_{i=1}^{N} \sqrt{m_i^2 + \vec{\kappa}_i^2} \vec{\eta}_i, \quad K^r = J^{or} = \vec{S}_s^{rr}, \]
\[ \Pi = M_{sys}^2 - \vec{\kappa}_+^2 \approx M_{sys}^2 > 0, \]
\[ W^2 = -\epsilon(M_{sys}^2 - \vec{\kappa}_+^2) \vec{S}_s^2 \approx -\epsilon M_{sys}^2 \vec{S}_s^2. \] (2.15)

The meaning of the constraints \( \epsilon_s - M_{sys} \approx 0, \vec{\kappa}_+ \approx 0 \) is: i) the constraint \( \epsilon_s - M_{sys} \approx 0 \) is the bridge connecting the external and internal realizations \(^{24}\); ii) the constraints \( \vec{\kappa}_+ \approx 0 \), together with \( \vec{K} \approx 0 \) \(^{25}\) imply a unfaithful internal realization in which the only non-zero generators are the conserved energy and the spin of an isolated system.

For isolated systems the constraint manifold \(^{22}\) is a stratified manifold with each stratum corresponding to a type of Poincaré orbit. The main stratum (dense in the constraint manifold) corresponds to all configurations of the isolated system belonging to timelike Poincaré orbits with \( \epsilon p_+^2 \approx \epsilon M_{sys}^2 > 0 \). As said in Ref. \(^{30}\), this implies that the center-of-mass coordinates have been adapted to the co-adjoint orbits of the Poincaré group. But this canonical basis does not yet correspond to a typical form of the Poincaré group \(^{11}\) in its canonical action on the phase space of the isolated system, because the second Poincaré invariant \(^{26}\) does not appear among the canonical variables. In Ref. \(^{30}\) a canonical basis including both Poincaré invariants was found (all the coordinates are adapted to the co-adjoint action of the Poincaré group). As a consequence the new relative variables are adapted to the SO(3) group.

In Ref. \(^{16}\) a naive internal center-of-mass variable \( \vec{\eta}_+ = \frac{1}{N} \sum_{i=1}^{N} \vec{\eta}_i \) was introduced and there was the definition of relative variables \( \vec{\rho}_a, \vec{\pi}_a \) with the following point canonical transformation

\[ \begin{bmatrix} \vec{\eta}_i \\ \vec{\kappa}_i \end{bmatrix} \rightarrow \begin{bmatrix} \vec{\eta}_+ \\ \vec{\rho}_a \\ \vec{\pi}_a \end{bmatrix}, \quad a = 1, \ldots, N - 1 \]

\[ \vec{\eta}_i = \vec{\eta}_+ + \frac{1}{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\rho}_a, \]
\[ \vec{\kappa}_i = \frac{1}{N} \vec{\kappa}_+ + \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_a, \]

\(^{24}\) The external spin coincides with the internal angular momentum due to Eqs.\(^{A11}\).

\(^{25}\) As we shall see in the next Section \( \vec{K} \approx 0 \) is implied by the natural gauge fixing \( \vec{q}_+ \approx 0 \).

\(^{26}\) The Pauli-Lubanski invariant \( \vec{W}^2_s = -p_s^2 \vec{S}_s^2 \).

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\[ \tilde{\eta}_+ = \frac{1}{N} \sum_{i=1}^{N} \tilde{\eta}_i, \]
\[ \tilde{\kappa}_+ = \sum_{i=1}^{N} \tilde{\kappa}_i \approx 0, \]
\[ \tilde{\rho}_a = \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \tilde{\eta}_i, \]
\[ \tilde{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_{ai} \tilde{\kappa}_i, \]
\[ \{ \eta^+_i, \kappa^+_j \} = \delta_{ij} \delta^{rs}, \quad \{ \eta^+_+ , \kappa^+_\pm \} = \delta^{rs}, \quad \{ \rho^+_a, \pi^+_b \} = \delta_{ab} \delta^{rs}, \]
\[ \sum_{i=1}^{N} \gamma_{ai} = 0, \quad \sum_{a=1}^{N-1} \gamma_{ai} \gamma_{aj} = \delta_{ij} - \frac{1}{N}, \quad \sum_{i=1}^{N} \gamma_{ai} \gamma_{bi} = \delta_{ab}, \quad (2.16) \]

This is a family of canonical transformations depending on \( \frac{1}{2}(N-1)(N-2) \) free parameters (the independent parameters in the \( \gamma_{ai} \))

Let us see whether we can take \( \tilde{\sigma}_{sys} = \tilde{\eta}_+ \).

In the gauge \( T_s - \tau \approx 0, \quad (2.17) \)

the Hamiltonian and the rest-frame constraints are
\[ H_D = M_{sys} - \tilde{\lambda}(\tau) \cdot \tilde{\kappa}_+ , \]
\[ \tilde{\kappa}_+ \approx 0. \quad (2.18) \]

with the invariant mass given by
\[ M_{sys} = H_M = \]
\[ = \sum_{i=1}^{N} \sqrt{m_i^2 + \tilde{\kappa}_i^2} = \sum_{i=1}^{N} \sqrt{m_i^2 + \left[ \frac{1}{N} \tilde{\kappa}_+ + \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \tilde{\pi}_a \right]^2} \approx \]
\[ \approx \sum_{i=1}^{N} \sqrt{m_i^2 + N \sum_{a,b} \gamma_{ai} \gamma_{bi} \tilde{\pi}_a \cdot \tilde{\pi}_b}. \quad (2.19) \]

For the origin of coordinates we get
\[ x^\mu_s(T_s) = x^\mu_s(0) + u^\mu(p_s)T_s + \epsilon^\mu_s(u(p_s)) \int_{0}^{T_s} d\tau \lambda_s(\tau). \quad (2.20) \]

\[ ^{27} \text{It entails } 0 \approx \dot{T}_s - 1 = \dot{x}_s \cdot u(p_s) - 1 = -\lambda(\tau) - 1; \text{ after going to Dirac brackets we get } T_s \equiv \tau \text{ and } \epsilon_s \equiv \pm M_{sys}. \]
The Hamilton equations are \( \tau \equiv T_s \)

\[
\begin{align*}
\dot{\eta}_i(\tau) &\equiv \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2(\tau)}} - \vec{\lambda}(\tau), \\
\vec{k}_i(\tau) &\equiv 0,
\end{align*}
\]

\[
\Rightarrow \quad \vec{k}_i(\tau) \equiv m_i \frac{[\dot{\eta}_i + \vec{\lambda}](\tau)}{\sqrt{1 - (\ddot{\eta}_i(\tau) + \vec{\lambda}(\tau))^2}},
\]

\[
\sqrt{m_i^2 + \vec{k}_i^2(\tau)} \equiv \frac{m_i}{\sqrt{1 - (\ddot{\eta}_i(\tau) + \vec{\lambda}(\tau))^2}},
\]

\[
\Rightarrow \quad L_D = \sum_{i=1}^{N} \vec{k}_i \cdot \dot{\eta}_i - H_D = \sum_{i=1}^{N} \vec{k}_i \cdot (\dot{\eta}_i + \vec{\lambda}) - H_M =
\]

\[
= - \sum_{i=1}^{N} m_i \sqrt{1 - (\ddot{\eta}_i + \vec{\lambda})^2} =
\]

\[
= - \sum_{i=1}^{N} m_i \left[ \dot{\eta}_i + \vec{\lambda} + 1 \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \dot{\rho}_a \right]^2.
\]

(2.21)

with Euler-Lagrange equations for \( \vec{\eta}_i(\tau) \) and \( \vec{\lambda}(\tau) \)

\[
\frac{d}{d\tau} \left[ m_i \frac{\dot{\eta}_i(\tau) + \vec{\lambda}(\tau)}{\sqrt{1 - (\ddot{\eta}_i(\tau) + \vec{\lambda}(\tau))^2}} \right] \equiv 0,
\]

\[
\sum_{i=1}^{N} m_i \frac{\dot{\eta}_i(\tau) + \vec{\lambda}(\tau)}{\sqrt{1 - (\ddot{\eta}_i(\tau) + \vec{\lambda}(\tau))^2}} \equiv 0.
\]

(2.22)

If \( \vec{\lambda}(\tau) = \vec{g}(\tau) \), the solutions of the Hamilton and Euler-Lagrange equations are

\[
\vec{k}_i(\tau) \equiv \vec{\beta}_i, \quad \text{with} \quad \vec{k}_+ \equiv \sum_{i=1}^{N} \vec{\beta}_i = 0,
\]

\[
\vec{\eta}_i(\tau) + \vec{g}(\tau) \equiv \vec{\alpha}_i + \tau \frac{\vec{\beta}_i}{\sqrt{m_i^2 + \vec{\beta}_i^2}},
\]

\[
\Rightarrow \quad \vec{\eta}_+(\tau) + \vec{g}(\tau) \equiv \frac{1}{N} \sum_{i=1}^{N} \vec{\alpha}_i + \tau \frac{N}{N} \sum_{i=1}^{N} \frac{\vec{\beta}_i}{\sqrt{m_i^2 + \vec{\beta}_i^2}},
\]

\[
given \quad \vec{\eta}_+ \Rightarrow \vec{g},
\]

\[
\vec{\rho}_a(\tau) \equiv \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \vec{\alpha}_i + \tau \sqrt{N} \sum_{i=1}^{N} \frac{\gamma_{ai} \vec{\beta}_i}{\sqrt{m_i^2 + \vec{\beta}_i^2}},
\]

\[
\vec{\pi}_a(\tau) \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_{ai} \vec{\beta}_i.
\]

(2.23)

Let us add the gauge fixings \( \vec{\eta}_+ \approx 0 \): their time constancy imply
\[ \tilde{\eta}_+ \approx \{\tilde{\eta}_+, H_D\} = \frac{1}{N} \sum_{i=1}^{N} \frac{\kappa_i}{\sqrt{m_i^2 + \kappa_i^2}} - \tilde{\lambda}(\tau) \approx 0, \]

\[ \Rightarrow \quad \tilde{\lambda}(\tau) \approx \frac{1}{N} \sum_{i=1}^{N} \frac{\kappa_i}{\sqrt{m_i^2 + \kappa_i^2}} \neq 0, \]

\[ \Rightarrow \quad L_D|\tilde{\eta}_+ = 0 = -\sum_{i=1}^{N} m_i \sqrt{1 - \left[ \tilde{\lambda} + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \gamma_{ai} \tilde{\rho}_a \right]^2} \neq 0 \]

\[ \neq -\sum_{i=1}^{N} m_i \sqrt{1 - \frac{1}{N} \sum_{a,b} \gamma_{ai} \gamma_{bi} \tilde{\pi}_a \cdot \tilde{\pi}_b}. \quad (2.24) \]

Therefore \( \tilde{\eta}_+ \approx 0 \) is not the searched natural gauge fixing \( \tilde{q}_+ \approx 0 \) for the separation of the center-of-mass motion. The origin in this gauge is

\[ x_{s}(\tilde{\eta}_+)^\mu (T_s) = x_{s}^\mu (0) + \frac{1}{N} \sum_{i=1}^{N} \frac{\kappa_i}{\sqrt{m_i^2 + \kappa_i^2}} \epsilon_{r}^\mu (u(p_s)) T_s. \quad (2.25) \]

If we go to Dirac brackets with respect to \( \tilde{\eta}_+ \approx 0, \tilde{\kappa}_+ \approx 0 \), we get the following Hamiltonian for the relative variables

\[ H_M = M_{sys} = \sum_{i=1}^{N} \sqrt{m_i^2 + N \sum_{a,b} \gamma_{ai} \gamma_{bi} \tilde{\pi}_a \cdot \tilde{\pi}_b}. \quad (2.26) \]

However, it is practically impossible to get the associated Lagrangian \( L_R(\tilde{\rho}_a, \dot{\tilde{\rho}}_a) \) for the relative motions.

For \( \tilde{\eta}_+ = 0 \) one gets

\[ \vec{g} = \frac{1}{N} \sum_{i=1}^{N} \vec{\alpha}_i + \tau \sum_{i=1}^{N} \frac{\vec{\beta}_i}{\sqrt{m_i^2 + \beta_i^2}}, \quad \vec{\eta}_i = \vec{\alpha}_i - \frac{1}{N} \sum_{k=1}^{N} \vec{\alpha}_k + \tau \left( \frac{\vec{\beta}_i}{\sqrt{m_i^2 + \beta_i^2}} - \frac{1}{N} \sum_{k=1}^{N} \frac{\vec{\beta}_k}{\sqrt{m_k^2 + \beta_k^2}} \right). \]

In Section III we will find the natural canonical internal 3-center-of-mass variable \( \vec{q}_+ \) (replacing the naive \( \vec{\eta}_+ \)) whose vanishing implies \( \tilde{\lambda}(\tau) = 0 \). It will be seen that, unlike in the non-relativistic theory, \( \vec{q}_+ \) is not a linear combination of the \( \vec{\eta}_i \)'s with coefficients depending on the masses, but it is connected to the Møller internal 3-center of energy, in which the masses are replaced by the particle energies.
III. THE INTERNAL RELATIVISTIC CENTER-OF-MASS VARIABLES ON THE WIGNER HYPERPLANE

In this Section we will study the internal center of mass variables, while Section IV will be devoted to the external ones.

In the relativistic case of N free scalar particles with positive energy the Hamiltonian kinetic energy is not a quadratic form in the momenta and the Lagrangian form is unknown. The first problem is to separate the global translations: this is the old problem of the definition of a relativistic center of mass. As we have seen in Sections I and II, the rest-frame instant form of dynamics allows to clarify the problem provided one splits of the concept of relativistic center of mass into an external one (a pseudo-4-vector) and an internal one (a Wigner spin 1 3-vector).

The determination of the internal 3-center of mass can be done using the group theoretical methods of Ref. [24] (see also Ref. [11]): given a realization on the phase space of the ten Poincaré generators one can build three 3-position variables only in terms of them, which are:

i) a canonical internal center of mass \( \vec{q}_+ \);

ii) a non-canonical internal Møller center of energy \( \vec{R} + \); 

iii) a non-canonical internal Fokker-Pryce center of inertia \( \vec{y}_+ \).

On Wigner hyperplanes, due to \( \vec{κ}_+ \approx 0 \), we will see that they all coincide: \( \vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+ \).

Following Ref. [24] we will determine of \( \vec{R}_+ \), \( \vec{q}_+ \), \( \vec{y}_+ \) starting from the internal realization (2.13) of the Poincaré algebra. We get the following Wigner spin 1 3-vectors:

i) The internal Moller 3-center of energy and the associated spin vector

\[ \vec{R}_+ = -\frac{1}{M_{sys}} \vec{K} = \frac{\sum_{i=1}^{N} \sqrt{m_i^2 + κ_i^2}}{\sum_{k=1}^{N} \sqrt{m_k^2 + κ_k^2}} \vec{η}_i, \]

\[ \vec{S}_R = \vec{J} - \vec{R}_+ \times \vec{κ}_+, \]

\[ \{ R^r_+, κ^s_+ \} = δ^{rs}, \quad \{ R^r_+, M_{sys} \} = \frac{κ^r_+}{M_{sys}}, \]

\[ \{ R^r_+ , R^s_+ \} = -\frac{1}{M_{sys}^2} ε^{rsu} S^u_R, \]

\[ \{ S^r_R, S^s_R \} = ε^{rsu} (S^u_R - \frac{1}{M_{sys}^2} \vec{S}_R \cdot \vec{κ}_+ κ^u_+), \quad \{ S^r_R, M_{sys} \} = 0. \]

(3.1)

Let us notice that with the gauge fixing \( \vec{R}_+ \approx 0 \) we have

\[ \vec{R}_+ \approx 0 \Rightarrow \vec{R}_+ \circ \{ \vec{R}_+, H_D \} = \]

28No definition can retain all the properties of the non-relativistic center of mass.

29Or center of spin: the classical analogue [15,16] of the Newton-Wigner position operator [12].
\[
\tilde{q}_+ = \tilde{R}_+ - \frac{\tilde{J} \times \tilde{\Omega}}{\sqrt{M^2_{\text{sys}} - \tilde{\kappa}_+^2 (M_{\text{sys}} + \sqrt{M^2_{\text{sys}} - \tilde{\kappa}_+^2})}} = -\frac{\tilde{K}}{\sqrt{M^2_{\text{sys}} - \tilde{\kappa}_+^2}} + \frac{\tilde{J} \times \tilde{\kappa}_+}{\sqrt{M^2_{\text{sys}} - \tilde{\kappa}_+^2 (M_{\text{sys}} + \sqrt{M^2_{\text{sys}} - \tilde{\kappa}_+^2})}} + \frac{\tilde{K} \cdot \tilde{\kappa}_+}{M_{\text{sys}} \sqrt{M^2_{\text{sys}} - \tilde{\kappa}_+^2}} \approx \tilde{R}_+ \quad \text{for} \quad \tilde{\kappa}_+ \approx 0;
\]
\[
\tilde{S}_q = \tilde{J} - \tilde{q}_+ \times \tilde{\kappa}_+ = \frac{M_{\text{sys}} \tilde{J}}{\sqrt{M^2_{\text{sys}} - \tilde{\kappa}_+^2}} + \frac{\tilde{K} \times \tilde{\kappa}_+}{\sqrt{M^2_{\text{sys}} - \tilde{\kappa}_+^2}} - \frac{\tilde{J} \cdot \tilde{\kappa}_+}{\sqrt{M^2_{\text{sys}} - \tilde{\kappa}_+^2 (M_{\text{sys}} + \sqrt{M^2_{\text{sys}} - \tilde{\kappa}_+^2})}} \approx \tilde{S} = \tilde{J},
\]
\[
\{\tilde{S}_q, \tilde{\kappa}_+\} = \{\tilde{S}_q, \tilde{q}_+\} = 0, \quad \{S^r_q, S^u_q\} = e^{rsu} S^u_q.
\]

Moreover the internal boost generator of Eq. (2.13) may be rewritten as \(\tilde{K} = -M_{\text{sys}} \tilde{R}_+\), so that \(\tilde{R}_+ \approx 0\) implies \(\tilde{K} \approx 0\).

ii) The canonical internal 3-center of mass \(30\) and the associated spin vector

\[
\tilde{\kappa}_+ + \tilde{q}_+ = \frac{1}{\sqrt{m^2_k + \tilde{\kappa}_+^2}} \sum_{i=1}^N \sqrt{m^2_i + \tilde{\kappa}_i^2} \tilde{q}_i \times \tilde{\kappa}_i = \frac{1}{\sqrt{m^2_k + \tilde{\kappa}_+^2}} \sum_{i=1}^N \sqrt{m^2_i + \tilde{\kappa}_i^2} \tilde{q}_i \times \tilde{\kappa}_i = \tilde{S}_R = \sum_{i=1}^N \tilde{\eta}_i \times \tilde{\kappa}_i - \sum_{i=1}^N \sqrt{m^2_i + \tilde{\kappa}_i^2} \tilde{q}_i \times \tilde{\kappa}_i = \sum_{i=1}^N \sqrt{m^2_i + \tilde{\kappa}_i^2} \tilde{q}_i \times \tilde{\kappa}_i \approx \tilde{S} = \tilde{J},
\]
\[
\tilde{q}_+ = \tilde{R}_+ - \frac{\sum_{i=1}^N \tilde{\eta}_i \cdot (\tilde{\kappa}_i - \sqrt{m^2_i + \tilde{\kappa}_i^2} \tilde{\kappa}_i)}{\sqrt{m^2_k + \tilde{\kappa}_+^2} - \tilde{\kappa}_+^2 (\sum_{k=1}^N \sqrt{m^2_k + \tilde{\kappa}_+^2} + \sqrt{(\sum_{k=1}^N \sqrt{m^2_k + \tilde{\kappa}_+^2})^2 - \tilde{\kappa}_+^2})} \approx \tilde{q}_+.
\]

It satisfies \(\{q^+_+, q^+\}_+ = 0, \quad \{q^+_+, \kappa^+_\} = \delta^{rs}, \quad \{J^r, q^+_+\} = e^{rsu} q^+_u, \quad I_s \tilde{q}_+ = -\tilde{q}_+, \quad I_u \tilde{q}_+ = \tilde{q}_+, \)
\[ \approx \vec{R}_+, \]
\[ \vec{S}_q = \vec{J} + \vec{r}_+ \times \vec{R}_+ = \sum_{i=1}^{N} \vec{\eta}_i \times \vec{R}_i - \]
\[ \sum_{i=1}^{N} \left[ \vec{\eta}_i \times \vec{R}_i \cdot (\vec{R}_i - \frac{\sqrt{m_i^2 + \kappa_i^2}}{\sum_{k=1}^{N} \sqrt{m_k^2 + \kappa_k^2}} \vec{R}_+ ) - \vec{r}_+ \cdot \vec{\eta}_i \vec{R}_i \times \vec{R}_+ \right] \]
\[ = \frac{\sqrt{(\sum_{k=1}^{N} \sqrt{m_k^2 + \kappa_k^2})^2 - \kappa_+^2} \left( \sum_{k=1}^{N} \sqrt{m_k^2 + \kappa_k^2} \sqrt{(\sum_{k=1}^{N} \sqrt{m_k^2 + \kappa_k^2})^2 - \kappa_+^2} \right)} {\sqrt{M_{sys}^2 - \kappa_+^2}} = \vec{J} \]
\[ \approx \vec{S} = \vec{J}, \]
\[ M_{sys} = H = \sum_{i=1}^{N} \sqrt{m_i^2 + \kappa_i^2}, \]
\[ \sqrt{M_{sys}^2 - \kappa_+^2} = \sqrt{\left( \sum_{k=1}^{N} \sqrt{m_k^2 + \kappa_k^2} \right)^2 - \kappa_+^2}. \]

(3.4)

iii) Besides the internal canonical 3-center of mass \( \vec{q}_+ \) and the internal non-canonical Møller 3-center of energy, we can define an internal non-canonical Fokker-Pryce center of inertia \( \vec{y} \)
\[ \vec{y}_+ = \vec{q}_+ + \frac{\vec{S}_q \times \vec{R}_+}{\sqrt{M_{sys}^2 - \kappa_+^2} (M_{sys} + \sqrt{M_{sys}^2 - \kappa_+^2})} = \vec{R}_+ + \frac{\vec{S}_q \times \vec{R}_+}{M_{sys} \sqrt{M_{sys}^2 - \kappa_+^2}}, \]
\[ \vec{q}_+ = \vec{R}_+ + \frac{\vec{S}_q \times \vec{R}_+}{M_{sys} (M_{sys} + \sqrt{M_{sys}^2 - \kappa_+^2})} = \frac{M_{sys} \vec{R}_+ + \sqrt{M_{sys}^2 - \kappa_+^2} \vec{y}_+}{M_{sys} + \sqrt{M_{sys}^2 - \kappa_+^2}}, \]
\[ \{ y^r_+, y^s_+ \} = \frac{1}{M_{sys} \sqrt{M_{sys}^2 - \kappa_+^2}} e^{I_{s,u} } \left[ S^u_q + \frac{\vec{S}_q \cdot \vec{R}_+ \kappa_+^u}{\sqrt{M_{sys}^2 - \kappa_+^2} (M_{sys} + \sqrt{M_{sys}^2 - \kappa_+^2})} \right], \]
\[ \kappa_+ \approx 0 \Rightarrow \vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+. \]

(3.5)

Therefore the gauge fixings \( \vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+ \approx 0 \) imply \( \vec{\lambda}(\tau) \approx 0 \) and force the three internal collective variables to coincide with the origin of the coordinates, which now becomes
\[ x^{(q)}(T_s) = x^{(\mu)}(0) + u^\mu(p_s)T_s. \]

(3.6)

By adding the gauge fixings \( \vec{\chi} = \vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+ \approx 0 \), it can be shown that the origin \( x^{(\mu)}(\tau) \) becomes simultaneously the Dixon center of mass of an extended object \cite{22} and both the Pirani \cite{33} and Tulczyjew \cite{34} centroids (the Dixon multipoles for the N-body problem on the Wigner hyperplane will be studied in Ref. \cite{28}).

We need now a canonical transformation bringing from the basis \( \vec{R}_i, \vec{R}_i, \) to a new canonical basis \( \vec{q}_+, \vec{R}_+ (\approx 0), \vec{\rho}_{q,a}, \vec{\tau}_{q,a} \), in which \( \vec{S}_q = \sum_{a=1}^{N-1} \vec{\rho}_{q,a} \times \vec{\tau}_{q,a} \):
\[ \begin{array}{c}
\vec{R}_i \\
\vec{R}_i \\
\vec{R}_i
\end{array} \rightarrow \begin{array}{c}
\vec{q}_+ \\
\vec{\rho}_{q,a} \\
\vec{R}_+ \\
\vec{\tau}_{q,a}
\end{array} \]

(3.7)
Let us remark that this is not a point transformation: the relativistic internal center of mass \( \vec{q}_+ \), realizing the effective separation of the center of mass from the relative motions in the kinetic energy, is momentum dependent.

The canonical transformation (3.6) will be studied in Section V by using the method of Gartenhaus-Schwartz [18] as delineated in Ref. [35] (see Refs. [24,35,18] for the N=2 case).

Let us finally consider the non-relativistic limit of the Lagrangian of Eq. (2.21) \(^{31}\)

\[
L_D = -\sum_{i=1}^{N} m_i \sqrt{1 - (\vec{\eta}_i(\tau) + \vec{\lambda}(\tau))^2} + \sum_{i=1}^{N} m_i + L_{Dnr},
\]

\[
L_{Dnr}(t) = \sum_{i=1}^{N} \frac{m_i}{2} (\vec{\eta}_i + \vec{\lambda})^2(t), \quad S_{Dnr} = \int dt L_{Dnr}(t),
\]

\[
\vec{\kappa}_i(\tau) \mapsto_{c \to \infty} \vec{\kappa}'_i(t) + O(c^{-2}),
\]

\[
\vec{\eta}'_i(t) = \frac{\partial L_{Dnr}(t)}{\partial \vec{\eta}_i(t)} = m_i (\vec{\eta}_i + \vec{\lambda})(t),
\]

\[
\vec{\pi}_\lambda(t) = \frac{\partial L_{Dnr}(t)}{\partial \dot{\vec{\lambda}}(t)} = 0,
\]

\[
H_{c,nr} = \vec{\pi}_\lambda \cdot \dot{\vec{\lambda}} + \sum_{i=1}^{N} \vec{\kappa}'_i \cdot \vec{\eta}_i - L_{Dnr} = \sum_{i=1}^{N} \frac{\vec{\kappa}'_i^2}{2m_i} - \vec{\lambda} \cdot \vec{\kappa}'_+ + \vec{\kappa}'_+ \approx \sum_{i=1}^{N} \vec{\kappa}'_i,
\]

\[
H_{Dnr} = \sum_{i=1}^{N} \frac{\vec{\kappa}'_i^2(t)}{2m_i} - \vec{\lambda}(t) \cdot \vec{\kappa}'_+(t) + \vec{\mu}(t) \cdot \vec{\pi}_\lambda(t),
\]

\[
\vec{\pi}_\lambda(t) \approx \vec{\kappa}'_+ \approx 0, \quad (\text{non-relativistic rest frame}).
\]

The Lagrangian \( L_{Dnr} \) has been used in Ref. [4] [see its Eq.(2.1)] to describe the relative motions in the non-relativistic rest frame. In the non-relativistic limit \( \vec{q}_+ \) tends the the non-relativistic center of mass \( \vec{q}_{nr} = \sum_{i=1}^{N} m_i \vec{\eta}_i / \sum_{i=1}^{N} m_i \).

In conclusion, the non-relativistic Abelian translation symmetry generating the non-relativistic Noether constants \( \vec{P} = \text{const.} \) is splitted at the relativistic level into the two following symmetries: i) the external Abelian translation symmetry whose Noether constants of motion are \( \vec{p}_s = \epsilon_s \vec{k}_s \approx M_{sys} \vec{k}_s = \text{const.} \) (its conjugate variable is the external 3-center of mass \( \vec{z}_s \); ii) the internal Abelian gauge symmetry generating the three first class constraints \( \vec{\kappa}_+ \approx 0 \) (the rest-frame conditions) inside the Wigner hyperplane (its conjugate gauge variable is the internal 3-center of mass \( \vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+ \)), whose non-relativistic counterpart would be the non-relativistic rest-frame conditions \( \vec{P} \approx 0 \).

\(^{31}\)Here \( \tau = t \), with \( t \) the absolute Newton time; for the sake of simplicity we shall use the same notation for functions of \( \tau \) and \( t \); having \( c = 1 \) the non-relativistic limit \( c \to \infty \) is done by considering velocities \( << 1 \) and momenta \( << m \).
IV. THE EXTERNAL CENTER-OF-MASS VARIABLES ON THE WIGNER HYPERPLANE.

Let us study now the localization on the Wigner hyperplane of the external center-of-mass variables. Let us remember [7] that the external canonical non-covariant point of coordinates

$$\tilde{x}_s^\mu(\tau) = (\tilde{x}_s^\mu(\tau); \tilde{x}_s(\tau)) = z_s(\tau, \vec{\sigma}) = x_s^\mu(\tau) - \frac{1}{\epsilon_s(p_s^\mu + \epsilon_s)} \left[p_s^\mu S_s^\mu + \epsilon_s(S_s^\mu - S_s^{0\nu} P_{su}^\mu p_s^\nu \epsilon_s^2)\right],$$

(4.1)

lies in the Wigner hyperplane $z_s(\tau, \vec{\sigma}) = x_s^\mu(\tau) + \epsilon_s(u(p_s))\sigma^r$ at some 3-position $\vec{\sigma}$, like the true coordinate origin $x_s^\mu(\tau) = z_s(\tau, \vec{0})$, because $p_s \cdot \tilde{x}_s = p_s \cdot x_s$, see Ref. [7].

Like in Eqs. (3.1), (3.3) and (3.5) one can build [14] three external 3-variables, the canonical $\vec{q}_s$, the Moller $\vec{R}_s$ and the Fokker-Pryce $\vec{Y}_s$ by using the rest-frame realization of the Poincaré algebra given in Eqs. (2.14)

$$\vec{R}_s = -\frac{1}{p^2_s} \vec{K}_s = (\vec{x}_s - \frac{\vec{p}_s}{p^2_s} \vec{x}_s^0) - \frac{\vec{S}_s \times \vec{p}_s}{p_s^0 (p_s^0 + \epsilon_s)},$$

$$\vec{q}_s = \vec{x}_s - \frac{\vec{p}_s}{p^2_s} \vec{x}_s^0 = \frac{\vec{z}_s}{\epsilon_s} = \vec{R}_s + \frac{\vec{S}_s \times \vec{p}_s}{p_s^0 (p_s^0 + \epsilon_s)} = \frac{p_s^0 \vec{R}_s + \epsilon_s \vec{Y}_s}{p_s^0 + \epsilon_s},$$

$$\vec{Y}_s = \vec{q}_s + \frac{\vec{S}_s \times \vec{p}_s}{\epsilon_s (p_s^0 + \epsilon_s)} = \vec{R}_s + \frac{\vec{S}_s \times \vec{p}_s}{p_s^0 \epsilon_s},$$

$$\{R_s^\nu, R_s^\mu\} = -\frac{1}{(p_s^2)^2} \epsilon^{\nu\sigma} \Omega_s^{\mu\sigma}, \quad \vec{Q}_s = \vec{J}_s - \vec{R}_s \times \vec{p}_s,$$

$$\{q_s^\nu, q_s^\mu\} = 0,$$

$$\{Y_s^\nu, Y_s^\mu\} = \frac{1}{\epsilon_s p^2_s} \epsilon^{\nu\sigma} \left[\vec{S}_s^\nu + \frac{\vec{S}_s \cdot \vec{p}_s}{\epsilon_s (p_s^0 + \epsilon_s)} \vec{p}_s^\nu\right],$$

$$\vec{p}_s \cdot \vec{q}_s = \vec{p}_s \cdot \vec{R}_s = \vec{p}_s \cdot \vec{Y}_s = \vec{k}_s \cdot \vec{z}_s,$$

$$\vec{p}_s = 0 \Rightarrow \vec{q}_s = \vec{Y}_s = \vec{R}_s,$$

(4.2)

with the same velocity and coinciding in the Lorentz rest frame where $p_s^0 = \epsilon_s(1; 0)$.

We can now try to construct the following three external 4-positions (all located on the Wigner hyperplane):

i) the external canonical non-covariant 4-center of mass $\tilde{x}_s^\mu$;

ii) the external non-canonical and non-covariant Møller 4-center of energy $R_s^\mu$;

iii) the external covariant non-canonical Fokker-Pryce 4-center of inertia $Y_s^\mu$.

---

32When there are the gauge fixings $\vec{q}_s \approx 0$ it will be shown that it also coincides with the origin.
In Ref. [24] in a one-time framework without constraints and at a fixed time, it is shown that the 3-vector \( \bar{Y}_s \) (but not \( \tilde{q}_s \) and \( \bar{R}_s \)) satisfies the condition \( \{K^r_s, Y^s_r\} = \frac{1}{c^2} Y^r_s \{Y^s_r, p^o_s\} \) for being the space component of a 4-vector \( Y^s_\mu \). In the enlarged canonical treatment including time variables, it is not clear which are the time components to be added to \( \tilde{q}_s \), \( \bar{R}_s \), \( \bar{Y}_s \), to rebuild 4-dimensional quantities \( \tilde{x}^\mu_s \), \( R^o_s \), \( Y^\mu_s \), in an arbitrary Lorentz frame \( \Gamma \), in which the origin of the Wigner hyperplane is the 4-vector \( x^\mu_s = (x^0_s; \bar{x}_s) \). We have from Eq. (2.10)

\[
\tilde{x}^\mu_s(\tau) = (\tilde{x}^o_s(\tau); \bar{x}_s(\tau)) = x^\mu_s - \frac{1}{\epsilon_s(p^o_s + \bar{p}_s)} \left[ p^o_s S^\mu_o - \epsilon_s (S^\mu_o - S^o_o P^\mu_s) \right], \quad \bar{p}^\mu_s,
\]

\[
\bar{x}^o_s = \sqrt{1 + \tilde{k}^2_s(T_s + \tilde{k}_s \cdot \tilde{z}_s)} = \sqrt{1 + \tilde{k}^2_s(T_s + \tilde{k}_s \cdot \tilde{q}_s)} \neq x^o_s, \quad \bar{p}^o_s = \epsilon_s \sqrt{1 + \tilde{k}^2_s},
\]

\[
\bar{x}_s = \frac{\tilde{z}_s}{\epsilon_s} + (T_s + \tilde{k}_s \cdot \tilde{z}_s) \tilde{k}_s = \tilde{q}_s + (T_s + \tilde{k}_s \cdot \tilde{q}_s) \tilde{k}_s, \quad \bar{p}_s = \epsilon_s \tilde{k}_s. \tag{4.3}
\]

for the non-covariant (frame-dependent) canonical 4-center of mass and its conjugate momentum.

Each Wigner hyperplane intersects the worldline of the arbitrary origin 4-vector \( x^\mu_s(\tau) = z^\mu(\tau, \tilde{0}) \) in \( \tilde{0} = 0 \), the pseudo worldline of \( \tilde{x}^\mu_s(\tau) = z^\mu(\tau, \tilde{0}) \) in some \( \tilde{0} \) and the worldline of the Fokker–Pryce 4-vector \( Y^\mu_s(\tau) = z^\mu(\tau, \tilde{0}) \) in some \( \tilde{0} \); one also has \( R^o_s = z^\mu(\tau, \tilde{0}) \). Since we have \( T_s = u(p^o_s) \cdot x_s = u(p^o_s) \cdot \bar{x}_s = \equiv \tau \) on the Wigner hyperplane labelled by \( \tau \), we require that also \( Y^\mu_s \), \( R^o_s \) have time components such that they too satisfy \( u(p^o_s) \cdot Y_s = u(p^o_s) \cdot R_s = T_s \equiv \tau \). Therefore, it is reasonable to assume that \( \tilde{x}^\mu_s \), \( Y^\mu_s \) and \( R^o_s \) satisfy the following equations consistently with Eqs. (4.2), (4.3)

\[
\tilde{x}^\mu_s = (\tilde{x}^o_s; \tilde{x}_s) = (\tilde{x}^o_s, \tilde{q}_s, \tilde{p}^o_s) =
\]

\[
= (\tilde{x}^o_s, \tilde{z}_s, \epsilon_s + (T_s + \tilde{k}_s \cdot \tilde{z}_s) \tilde{k}_s) = x^\mu_s + \epsilon^\mu_s (u(p^o_s)) \tilde{\sigma}^u,
\]

\[
Y^\mu_s = (\tilde{y}^o_s; \tilde{y}_s) =
\]

\[
= (\tilde{y}^o_s, \frac{1}{\epsilon_s} \tilde{z}_s + (T_s + \tilde{k}_s \cdot \tilde{z}_s) \tilde{k}_s) = x^\mu_s + \epsilon^\mu_s (u(p^o_s)) \tilde{\sigma}^u,
\]

\[
R^o_s = (\tilde{r}^o_s; \tilde{r}_s) =
\]

\[
= (\tilde{r}^o_s, \frac{1}{\epsilon_s} \tilde{z}_s - (T_s + \tilde{k}_s \cdot \tilde{z}_s) \tilde{k}_s) = x^\mu_s + \epsilon^\mu_s (u(p^o_s)) \tilde{\sigma}^u.
\]
\[ = \tilde{x}_s^\mu - \eta_t^\mu \frac{(\tilde{S}_s \times \vec{p}_s)^\mu}{\epsilon_s u^\alpha(p_s)[1 + u^\alpha(p_s)]} = x_s^\mu + \delta_n^\mu(u(p_s))\sigma_n^R, \]

\[ T_s = u(p_s) \cdot x_s = u(p_s) \cdot \tilde{x}_s = u(p_s) \cdot Y_s = u(p_s) \cdot R_s, \]

\[ \tilde{\sigma}^r = \epsilon_r\mu(u(p_s))[x_s^\mu - \tilde{x}_s^\mu] = \epsilon_r\mu(u(p_s))[u_\nu(p_s)S_s^{\nu\mu} + S_s^{o\mu}] = \]

\[ = -\tilde{S}_s^{r\tau} + \frac{\tilde{S}_s^{r\tau} p_s^\mu}{\epsilon_s[1 + u^\alpha(p_s)]} = \epsilon_s R_s^r + \frac{\tilde{S}_s^{r\tau} u^s(p_s)}{1 + u^\alpha(p_s)} \approx \]

\[ \tilde{q}_s^{r\tau} = \epsilon_s q_s^r + \frac{\tilde{S}_s^{r\tau} u^s(p_s)}{1 + u^\alpha(p_s)} \approx \frac{\tilde{S}_s^{r\tau} u^s(p_s)}{1 + u^\alpha(p_s)}, \]

\[ \sigma_Y^r = \epsilon_r\mu(u(p_s))[x_s^\mu - Y_s^\mu] = \tilde{\sigma}^r - \epsilon_r\mu(u(p_s)) \frac{(\tilde{S}_s \times \vec{p}_s)^u}{\epsilon_s[1 + u^\alpha(p_s)]} = \]

\[ = \tilde{\sigma}^r + \frac{\tilde{S}_s^{r\tau} u^s(p_s)}{1 + u^\alpha(p_s)} \approx \epsilon_s q_s^r \tilde{q}_s^{r\tau} \approx 0, \]

\[ \Rightarrow x_s(\tilde{q}_s)^{r\tau}(\tau) = Y_s^\mu, \text{ when } \tilde{q}_s^{r\tau} \approx 0, \]

\[ \sigma_R^r = \epsilon_r\mu(u(p_s))[x_s^\mu - R_s^\mu] = \tilde{\sigma}^r + \epsilon_r\mu(u(p_s)) \frac{(\tilde{S}_s \times \vec{p}_s)^u}{\epsilon_s u^\alpha(p_s)[1 + u^\alpha(p_s)]} = \]

\[ = \tilde{\sigma}^r - \frac{\tilde{S}_s^{r\tau} u^s(p_s)}{u^\alpha(p_s)[1 + u^\alpha(p_s)]} \approx \epsilon_s R_s^r \frac{[1 - u^\alpha(p_s)]\tilde{S}_s^{r\tau} u^s(p_s)}{u^\alpha(p_s)[1 + u^\alpha(p_s)]} \approx \]

\[ \tilde{q}_s^{r\tau} \approx \frac{[1 - u^\alpha(p_s)]\tilde{S}_s^{r\tau} u^s(p_s)}{u^\alpha(p_s)[1 + u^\alpha(p_s)]}. \quad (4.4) \]

Therefore, the external Fokker-Pryce non-canonical center of inertia coincides with the origin \( x_s(\tilde{q}_s)^{r\tau}(\tau) \) carrying the internal center of mass.

Let us remember also that the origin \( x_s^\mu(\tau) \) corresponds to the unique special relativistic center-of-mass-like worldline of Refs. [33].

In each Lorentz frame one has different pseudo-worldlines describing \( R_s^\mu \) and \( \tilde{x}_s^\mu \); the 4-canonical center of mass \( \tilde{x}_s^\mu \) lies in between \( Y_s^\mu \) and \( R_s^\mu \) in every frame. If, in an arbitrary Lorentz frame, we consider the worldline \( Y_s^\mu \) of the covariant non-canonical Fokker-Pryce 4-center of inertia, the representation in this frame of all the pseudo-worldlines associated with \( \tilde{x}_s^\mu \) and \( R_s^\mu \) fill a worldtube [13] around \( Y_s^\mu \) whose invariant radius is \( \rho = \sqrt{-\epsilon W^2}/p^2 = |\vec{S}|/\sqrt{\epsilon p^2} \) \((W^2 = -\epsilon p^2 \vec{S}^2 \) is the Pauli-Lubanski Casimir when \( \epsilon p^2 > 0 \)). This is the classical intrinsic radius of the worldtube, in which the non-covariance effects (the pseudo-worldlines) of the canonical 4-center of mass \( \tilde{x}_s^\mu \) are located. See Ref. [33] for a discussion of the properties of the Möller radius. At the quantum level \( \rho \) becomes the Compton wavelength of the isolated system multiplied its spin eigenvalue \( \sqrt{s(s + 1)} \) , \( \rho \rightarrow \hat{\rho} = \sqrt{s(s + 1)}\hbar /M = \sqrt{s(s + 1)}\lambda_M \) with \( M = \sqrt{\epsilon p^2} \) the invariant mass and \( \lambda_M = \hbar /M \) its Compton wavelength. Therefore, the criticism to classical relativistic physics, based on quantum pair production, concerns
the testing of distances where, due to the Lorentz signature of spacetime, one has intrinsic classical covariance problems: it is impossible to localize the canonical 4-center of mass $\tilde{x}^\mu$ adapted to the first class constraints of the system (also named Pryce center of mass and having the same covariance of the Newton-Wigner position operator) in a frame independent way.

Let us remember [7] that $\rho$ is also a remnant of the energy conditions of general relativity in flat Minkowski spacetime: since the Møller non-canonical, non-covariant 4-center of energy $R^\mu$ has its non-covariance (its pseudo-worldlines) localized inside the same worldtube with radius $\rho$ (actually the latter was discovered in this way) [13], it turns out that for an extended relativistic system with the material radius smaller of its intrinsic radius $\rho$ one has: i) its peripheral rotation velocity can exceed the velocity of light; ii) its classical energy density cannot be positive definite everywhere in every frame.
V. THE INTERNAL RELATIVE VARIABLES FROM THE GARTENHAUS-SCHWARTZ TRANSFORMATION.

Given \( \vec{\eta}_i, \vec{\kappa}_i \), we must now find the canonical basis \( \vec{q}_+, \vec{\kappa}_+, \vec{\rho}_a, \vec{\pi}_a \) of Eq. (3.11).

We shall use the classical analog of the Gartenhaus-Schwartz singular transformation following the scheme used in Ref. [35] to find the center-of-mass subspace of phase space defined by \( \vec{\kappa}_+ = 0 \).

\[
U(\alpha) = e^{\alpha(\vec{q}_+ - \vec{\kappa}_+)} ,
\]

\[
\vec{q}_+ \cdot \vec{\kappa}_+ = -\frac{\sum_{k=1}^N \sqrt{m_k^2 + \kappa_k^2}}{\sum_{k=1}^N \sqrt{m_k^2 + \kappa_k^2}} \vec{n}_+ \cdot \vec{K}, \quad \vec{n}_+ = \frac{\vec{\kappa}_+}{|\vec{\kappa}_+|}, \quad \vec{K} = -\sum_{i=1}^N \sqrt{m_i^2 + \kappa_i^2} \vec{\eta}_i,
\]

\[
\vec{\kappa}_+(\alpha) = U(\alpha)\vec{\kappa}_+ = e^{-\alpha} \vec{\kappa}_+ \to \alpha \to \infty 0, \quad \vec{q}_+(\alpha) = U(\alpha)\vec{q}_+ = e^{\alpha} \vec{q}_+ \to \alpha \to \infty \infty, \quad U(-\alpha)\vec{q}_+ = e^{-\alpha} \vec{q}_+ \to \alpha \to \infty 0,
\]

\[
\Rightarrow \quad \vec{\kappa}_+(\alpha) \cdot \vec{q}_+(\alpha) = \vec{\kappa}_+ \cdot \vec{q}_+, \quad \vec{n}_+(\alpha) = \vec{n}_+ . \tag{5.1}
\]

Therefore, \( \lim_{\alpha \to \infty} U(\alpha) \) can only be applied to the set of functions on phase space which have vanishing Poisson bracket with \( \vec{\kappa}_+ \), namely to \( \vec{\kappa}_i \) [or \( \vec{\kappa}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^N \gamma_{ai} \vec{\kappa}_i \)] and to \( \vec{\rho}_a = \sqrt{N} \sum_{i=1}^N \gamma_{ai} \vec{\eta}_i \) of Eq. (2.14).

Since, for finite \( \alpha \), \( U(\alpha) \) is a canonical transformation, the Poisson brackets are preserved \( \{ \{ f(\alpha), g(\alpha) \} = U(\alpha)\{ f, g \} \} \) even in the limit \( \alpha \to \infty \).

Let \( f = f(\vec{\eta}_i, \vec{\kappa}_i) \) have zero Poisson bracket with \( \vec{\kappa}_+ \), \( \{ f, \vec{\kappa}_+ \} = 0 \), and let be \( f(\alpha) = U(\alpha)f \). Then, we have

\[
\{ \vec{\kappa}_+, f(\alpha) \} = e^{\alpha} \{ \vec{\kappa}_+(\alpha), f(\alpha) \} = e^{\alpha} \left( U(\alpha)\{ \vec{\kappa}_+, f \} \right) = 0 . \tag{5.2}
\]

Moreover, since the Jacobi identity \( \{ \vec{\kappa}_+, \{ \vec{q}_+, f \} \} + \{ f, \{ \vec{\kappa}_+, \vec{q}_+ \} \} + \{ \vec{q}_+, \{ f, \vec{\kappa}_+ \} \} \equiv 0 \) implies \( \{ \vec{\kappa}_+, \{ \vec{q}_+, f \} \} \equiv 0 \) [namely also \( \{ \vec{q}_+, f \} \) has zero Poisson bracket with \( \vec{\kappa}_+ \) if \( \{ f, \vec{\kappa}_+ \} = 0 \), so that \( U(\alpha)\{ \vec{q}_+, f \} \) has a well defined limit for \( \alpha \to \infty \)] one also has

\[
\{ \vec{q}_+, f(\alpha) \} = e^{-\alpha} \{ \vec{q}_+(\alpha), f(\alpha) \} = e^{-\alpha} \left( U(\alpha)\{ \vec{q}_+, f \} \right) \to _{\alpha \to \infty} 0 . \tag{5.3}
\]

Moreover, we have

\[
\frac{df(\alpha)}{d\alpha} = \{ f(\alpha), \vec{\kappa}_+ \cdot \vec{q}_+ \} = \{ f(\alpha), \vec{\kappa}_+(\alpha) \cdot \vec{q}_+(\alpha) \} . \tag{5.4}
\]

Therefore, the relative variables \( \vec{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^N \gamma_{ai} \vec{\kappa}_i \) and \( \vec{\rho}_a = \sqrt{N} \sum_{i=1}^N \gamma_{ai} \vec{\eta}_i \), which commute with \( \vec{\kappa}_+ \) [see Eq. (2.14)], satisfy

\[33\]The singular limit \( \alpha \to \infty \) is very similar to a contraction of a Lie algebra.
\[ \vec{\pi}(\alpha) = U(\alpha) \vec{\pi} \rightarrow_{\alpha \to \infty} \vec{\pi}(\infty) \overset{def}{=} \vec{\pi}_{qa}, \]
\[ \vec{\rho}(\alpha) = U(\alpha) \vec{\rho} \rightarrow_{\alpha \to \infty} \vec{\rho}(\infty) \overset{def}{=} \vec{\rho}_{qa}, \]

with \( \vec{\rho}_{qa}, \vec{\pi}_{qa} \), pairs of canonical variables having zero Poisson bracket with \( \vec{q}_+, \vec{\kappa}_+ \).

In this way one gets the searched canonical transformation (3.7).

Let us first evaluate \( \vec{\pi}_{qa} \) following the scheme of Ref. [35]. From Eq.(5.4) we get

\[ \frac{d\vec{\kappa}_i(\alpha)}{d\alpha} = \{ \vec{\kappa}_i(\alpha), \vec{\kappa}_+ \cdot \vec{q}_+(\alpha) \} = -\frac{\vec{\kappa}_+(\alpha)}{H_M(\alpha)} H_i(\alpha), \]

with the notations

\[ M_{sys} = H_M = \sum_{i=1}^{N} H_i, \quad H_M(\alpha) = \sum_{i=1}^{N} H_i(\alpha) \rightarrow_{\alpha \to \infty} H_M(\infty) \overset{def}{=} H_{(rel)}; \]
\[ H_i = \sqrt{m_i^2 + \vec{\kappa}_i^2}, \quad H_i(\alpha) = \sqrt{m_i^2 + \vec{\kappa}_i^2(\alpha)} \rightarrow_{\alpha \to \infty} H_i(\infty) \overset{def}{=} H_{(rel)i}; \]
\[ \Pi = H_M^2 - \vec{\kappa}_+^2 \approx H_M^2. \]

From \( m_i^2 = H_i^2(\alpha) - \vec{\kappa}_i^2(\alpha) \), we get

\[ \frac{dH_i(\alpha)}{d\alpha} H_i(\alpha) = \frac{d\vec{\kappa}_i(\alpha)}{d\alpha} \cdot \vec{\kappa}_i(\alpha), \]
\[ \Rightarrow \frac{dH_i(\alpha)}{d\alpha} = -\vec{\kappa}_i(\alpha) \cdot \frac{\vec{\kappa}_i(\alpha)}{H_M(\alpha)}; \]
\[ \Rightarrow \frac{dH_M(\alpha)}{d\alpha} = \sum_{i=1}^{N} \frac{dH_i(\alpha)}{d\alpha} = -\frac{\vec{\kappa}_+^2(\alpha)}{H_M(\alpha)}; \]
\[ \Rightarrow \Pi = H_M^2 - \vec{\kappa}_+^2 = H_M^2(\alpha) - \vec{\kappa}_+^2(\alpha) \rightarrow_{\alpha \to \infty} H_M^2(\infty) = H_{(rel)}^2; \]
\[ \text{or } \frac{d\Pi}{d\alpha} = 0. \]

Let us now introduce \( \theta(\alpha) \) such that \([\text{ch}^2 \theta(\alpha) - \text{sh}^2 \theta(\alpha) = 1 \text{ also for } \alpha \to \infty]\)

\[ \text{sh} \theta(\alpha) = \frac{|\vec{\kappa}_+| H_M(\alpha) - |\vec{\kappa}_+| H_M}{\Pi} \rightarrow_{\alpha \to \infty} \frac{|\vec{\kappa}_+|}{\sqrt{\Pi}}; \]
\[ \text{ch} \theta(\alpha) = \frac{H_M H_M(\alpha) - |\vec{\kappa}_+| |\vec{\kappa}_+|}{\Pi} \rightarrow_{\alpha \to \infty} \frac{H_M}{\sqrt{\Pi}}; \]
\[ \theta(\alpha) = \text{tanh}^{-1} \left( \frac{|\vec{\kappa}_+|}{H_M} \right) - \text{tanh}^{-1} \left( \frac{|\vec{\kappa}_+|}{H_M(\alpha)} \right) \rightarrow_{\alpha \to 0} 0, \quad \rightarrow_{\alpha \to \infty} \text{tanh}^{-1} \left( \frac{|\vec{\kappa}_+|}{H_M} \right). \]

Since we have

\[ \frac{d\theta(\alpha)}{d\alpha} = \frac{|\vec{\kappa}_+(\alpha)|}{H_M(\alpha)}, \quad \frac{d\vec{\kappa}_+(\alpha)}{d\alpha} = 0 \Rightarrow \vec{\kappa}_+(\alpha) = \vec{\kappa}_+, \]

we arrive at the coupled equations
\[
\frac{d\vec{\kappa}_i(\alpha)}{d\theta} = -H_i(\alpha)\vec{n}_+,
\]
\[
\frac{dH_i(\alpha)}{d\theta} = -\vec{\kappa}_i(\alpha) \cdot \vec{n}_+,
\]
whose integration gives
\[
\vec{\kappa}_i(\alpha) = \vec{\kappa}_i + \left( [ch \theta(\alpha) - 1] \vec{n}_+ \cdot \vec{\kappa}_i - sh \theta(\alpha) H_i \right) \vec{n}_+.
\]
\[
\Rightarrow \alpha \to \infty \quad \vec{\kappa}_i(\infty) = \vec{\kappa}_i + \left[ \frac{H_M}{\sqrt{\Pi}} - 1 \right] \vec{n}_+ \cdot \vec{\kappa}_i - \frac{|\vec{\kappa}_i|}{\sqrt{\Pi}} H_i \vec{n}_+ \approx \vec{\kappa}_i,
\]
\[
H_i(\alpha) = \sqrt{m_i^2 + \vec{\kappa}_i^2(\alpha)} = ch \theta(\alpha) H_i - sh \theta(\alpha) \vec{n}_+ \cdot \vec{\kappa}_i.
\]
\[
\Rightarrow \alpha \to \infty \quad H_i(\infty) = \sqrt{m_i^2 + \vec{\kappa}_i^2(\infty)} = \frac{1}{\sqrt{\Pi}} (H_M H_i - \vec{n}_+ \cdot \vec{\kappa}_i) \approx H_i,
\]
\[
\Rightarrow H_i = \sqrt{m_i^2 + \vec{\kappa}_i^2} = H_i(\alpha) ch \theta(\alpha) + \vec{n}_+ \cdot \vec{\kappa}_i(\alpha) sh \theta(\alpha) =
\]
\[
\frac{1}{\sqrt{\Pi}} [H_i(\infty) H_M + \vec{n}_+ \cdot \vec{\kappa}_i(\infty) |\vec{\kappa}_i|] \approx H_i(\infty),
\]
\[
\text{with } \sum_{i=1}^{N} H_i(\infty) = H_M(\infty) = \sqrt{\Pi} \text{ def } H_{(rel)},
\]
\[
\vec{\kappa}_i = \vec{\kappa}_i(\alpha) + [ch \theta(\alpha) - 1] \vec{n}_+ \cdot \vec{\kappa}_i(\alpha) \vec{n}_+ + sh \theta(\alpha) H_i(\alpha).
\]

Therefore, we get
\[
\vec{\pi}_a(\alpha) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_{ai} \vec{\kappa}_i(\alpha),
\]
\[
\vec{\pi}_{qa} \text{ def } \vec{\pi}_a(\infty) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \vec{\kappa}_i(\infty) =
\]
\[
\vec{\pi}_a + \frac{\vec{n}_+}{\sqrt{\Pi}} [H_M - \sqrt{\Pi} \vec{n}_+ \cdot \vec{\pi}_a - |\vec{\kappa}_i| H_a] =
\]
\[
\vec{\pi}_a - \frac{\vec{n}_+}{\sqrt{H_M^2 - \vec{\kappa}_i^2}} [H_a - \frac{H_M - \sqrt{H_M^2 - \vec{\kappa}_i^2}}{\vec{\kappa}_i^2} \vec{n}_+ \cdot \vec{\pi}_a] \approx \vec{\pi}_a,
\]
\[
H_a = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_{ai} H_i,
\]
\[
\vec{\kappa}_i(\infty) = \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_{qa},
\]
\[
H_{(rel)i} = H_i(\infty) = \sqrt{m_i^2 + \sum_{a=1}^{N-1} \gamma_{ai} \gamma_{ba} \vec{\pi}_{qa} \cdot \vec{\pi}_{qa}},
\]
\[
M_{sys} = H_M = \sum_{i=1}^{N} H_i = \sqrt{\Pi + \vec{\kappa}_i^2} \approx H_{(rel)} = H_M(\infty) = \sqrt{\Pi} =
\]
\[
\sum_{i=1}^{N} H_i(\infty) = \sum_{i=1}^{N} \sqrt{m_i^2 + N \sum_{ab} \gamma_{ai} \gamma_{bi} \vec{n}_q \cdot \vec{n}_{qh}}. \tag{5.13}
\]

Let us now evaluate \(\vec{\rho}_{qa}\).

Let us first remark that the following two quantities are invariant under the canonical transformation \(U(\alpha)\):

\[
I_i^{(1)} = H_M H_i - \vec{k}_+ \cdot \vec{r}_i = H_M(\alpha) H_i(\alpha) - \vec{k}_+(\alpha) \cdot \vec{r}_i(\alpha),
\]

\[
\Rightarrow \quad \frac{dI_i^{(1)}}{d\alpha} = 0,
\]

\[
I_i^{(2)} = \frac{\vec{k}_+ \cdot \vec{K}}{H_M} = -\sum_{i=1}^{N} |\vec{k}_+| (\vec{n}_+ \cdot \vec{\eta}_i H_i) = \frac{\vec{k}_+(\alpha) \cdot \vec{K}(\alpha)}{H_M(\alpha)},
\]

\[
\Rightarrow \quad \frac{dI_i^{(2)}}{d\alpha} = 0,
\tag{5.14}
\]

and that we have

\[
\frac{1}{H_M(\alpha)|\vec{k}_+(\alpha)|} = \frac{dJ_i^{(1)}(\alpha)}{d\alpha}, \quad \text{with} \quad J_i^{(1)}(\alpha) = \frac{\sinh \theta(\alpha)}{|\vec{k}_+(\alpha)| |\vec{k}_+|},
\]

\[
\frac{\vec{k}_+(\alpha)}{H_i^2(\alpha)} = \frac{dJ_i^{(2)}(\alpha)}{d\theta(\alpha)}, \quad \text{with} \quad J_i^{(2)}(\alpha) = \frac{\vec{k}_+(\alpha) \sinh \theta(\alpha)}{H_i H_i(\alpha)} + (\cosh \theta(\alpha) - 1) \frac{\vec{n}_+}{H_i},
\tag{5.15}
\]

Since we have also

\[
n^r_+ \frac{\partial}{\partial \vec{k}_i} n^s_+ = \frac{n^r_+}{|\vec{k}_+|} (\delta^{rs} - n^r_+ n^s_+) = 0,
\tag{5.16}
\]

we get preliminarily, for \(\vec{n}_+ \cdot \vec{\eta}_i(\alpha)\)

\[
\frac{d}{d\alpha} \vec{n}_+ \cdot \vec{\eta}_i(\alpha) = \{\vec{n}_+ \cdot \vec{\eta}_i(\alpha), \vec{k}_+(\alpha) \cdot \vec{q}_+(\alpha)\} =
\]

\[
= -n^r_+ \frac{\partial}{\partial k_i^r(\alpha)} \frac{\vec{n}_+ \cdot \vec{K}(\alpha) |\vec{k}_+(\alpha)|}{H_M(\alpha)} = -n^r_+ n^s_+ \frac{\partial}{\partial k_i^r(\alpha)} |\vec{k}_+(\alpha)| K^s(\alpha) \tag{5.17}
\]

Then, since

\[
n^r_+ \frac{\partial}{\partial k_i^r(\alpha)} |\vec{k}_+(\alpha)| = \frac{I_i^{(1)}}{H_M(\alpha) H_i(\alpha)},
\]

\[
\frac{\partial}{\partial k_i^r(\alpha)} K^s(\alpha) = -\frac{k_i^r(\alpha) \eta_i^s(\alpha)}{H_i(\alpha)},
\tag{5.18}
\]

we get

\[
\frac{d}{d\alpha} \vec{n}_+ \cdot \vec{\eta}_i(\alpha) = -\frac{I_i^{(2)} I_i^{(1)}}{H_M(\alpha) H_i(\alpha) |\vec{k}_+(\alpha)|} + \frac{\vec{n}_+ \cdot \vec{\eta}_i(\alpha) \vec{n}_+ \cdot \vec{k}_i(\alpha) |\vec{k}_+(\alpha)|}{H_i(\alpha)} =
\]

\[
= -\frac{\vec{n}_+ \cdot \vec{\eta}_i(\alpha)}{H_i(\alpha)} \frac{dH_i(\alpha)}{d\alpha} - \frac{I_i^{(1)} I_i^{(2)} dJ_i^{(1)}(\alpha)}{H_i(\alpha) d\alpha}. \tag{5.19}
\]
These equations have the solution

\[
\vec{n}_+ \cdot \vec{n}_i(\alpha) = \frac{H_i}{H_i(\alpha)} \vec{n}_+ \cdot \vec{n}_i - \frac{I_i^{(1)} I^{(2)}_i}{H_i(\alpha) \| \vec{k}_i^+ \|} \frac{sh \theta(\alpha)}{\| \vec{k}_i^+ \|} = \frac{H_i}{H_i(\alpha)} \vec{n}_+ \cdot \vec{n}_i - \frac{I^{(2)}_i}{\| \vec{k}_i^+ \|} (e^\alpha - \frac{H_i}{H_i(\alpha)}),
\]

(5.20)

For \( \vec{n}_i(\alpha) \) we have

\[
\frac{d\vec{n}_i^\alpha(\alpha)}{d\alpha} = \{ \vec{n}_i^\alpha(\alpha), \vec{k}_i^+(\alpha) \cdot \vec{q}_i^+(\alpha) \} = -n_i^\alpha \frac{\partial}{\partial k_i^+(\alpha)} \frac{\| \vec{k}_i^+ \| K^+(\alpha)}{H_M(\alpha)} = \vec{n}_+ \cdot \vec{n}_i(\alpha) \frac{\| \vec{k}_i^+ \| k_i^+(\alpha)}{H_i(\alpha) H_M(\alpha)} + \\
+ \sum_{j=1}^N H_j(\alpha) \vec{n}_+ \cdot \vec{n}_j(\alpha) \frac{\| \vec{k}_i^+ \| k_j^+(\alpha)}{H_i(\alpha) H_M(\alpha)} - n_j^+.
\]

(5.21)

By putting Eqs. (5.20) in Eq. (5.21) we get the equations determining \( \vec{n}_i(\alpha) \).

Instead of integrating these equations, let us study the equations for \( \vec{p}_a(\alpha) = \sqrt{N} \sum_{i=1}^N (\gamma a_i \vec{n}_i(\alpha) \), since for interactions depending on \( \vec{n}_i - \vec{n}_j \) we have \( \vec{n}_i(\alpha) - \vec{n}_j(\alpha) = \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj}) \vec{p}_a(\alpha) \). Eqs. (5.21) and (5.20) imply

\[
\frac{d\vec{p}_a(\alpha)}{d\alpha} = -\sum_{i,j=1}^N \sum_{b=1}^{N-1} \vec{n}_+ \cdot \vec{p}_b \frac{H_i H_j}{H_M} \gamma_{aj}(\gamma_{bi} - \gamma_{bj}) \frac{\vec{k}_j(\alpha)|\vec{k}_i^+(\alpha)|}{H_j(\alpha) H_M(\alpha)},
\]

\[
\downarrow
\frac{d\vec{p}_a(\alpha)}{d\theta(\alpha)} = -\sum_{i,j=1}^N \sum_{b=1}^{N-1} \vec{n}_+ \cdot \vec{p}_b \frac{H_i H_j}{H_M} \gamma_{aj}(\gamma_{bi} - \gamma_{bj}) \frac{d\vec{j}_j(\alpha)}{d\theta(\alpha)},
\]

(5.22)

whose solution is

\[
\vec{p}_a(\alpha) = \vec{p}_a - \sum_{i,j=1}^N \sum_{b=1}^{N-1} \vec{n}_+ \cdot \vec{p}_b \frac{H_i H_j}{H_M} \gamma_{aj}(\gamma_{bi} - \gamma_{bj}) \vec{j}_j(\alpha).
\]

(5.23)

For \( \alpha \to \infty \) we get

\[
\vec{p}_{qa} \overset{\text{def}}{=} \vec{p}_a(\infty) = \vec{p}_a - \sum_{i,j=1}^N \sum_{b=1}^{N-1} \gamma_{aj}(\gamma_{bi} - \gamma_{bj}) \frac{H_i}{H_M} \left[ \frac{\| \vec{k}_j(\infty) \|}{H_j(\infty)} \sqrt{\Pi} + (\frac{H_M}{\sqrt{\Pi}} - 1) \vec{n}_+ \cdot \vec{p}_b \right] = \\
= \vec{p}_a - \sum_{i,j=1}^N \sum_{b=1}^{N-1} \gamma_{aj}(\gamma_{bi} - \gamma_{bj}) \frac{H_i}{H_M} \frac{\vec{k}_j(\infty)}{H_j(\infty)} \sqrt{\Pi} \vec{n}_+ \cdot \vec{p}_b \approx \vec{p}_a.
\]

(5.24)

One can check that for \( N=2 \) and \( \gamma_1 = -\gamma_2 = 1/\sqrt{2} \) one reobtains the results of Ref. [35].

Let us now consider the spin vector \( \vec{S}_q = \vec{S}_s - \vec{q}_+ \times \vec{k}_+ = [\vec{n}_+ - \vec{q}_+] \times \vec{k}_+ + \sum_{a=1}^{N-1} \vec{p}_a \times \vec{n}_a \).

For arbitrary \( \alpha \) we have \( \vec{S}_q(\alpha) = \sum_{a=1}^{N-1} \vec{p}_a(\alpha) \times \vec{n}_a(\alpha) + [\vec{n}_+(\alpha) - \vec{q}_+(\alpha)] \times \vec{k}_+(\alpha) \) and, since \( \vec{q}_+(\alpha) \cdot \vec{k}_+(\alpha) \) is a scalar, \( \{ \vec{S}_q(\alpha), \vec{q}_+(\alpha) \cdot \vec{k}_+(\alpha) \} = 0 \). Since \( \lim_{\alpha \to \infty} \vec{k}_+(\alpha) = 0 \), we get
\[ \tilde{S}_q(\alpha) \to_{\alpha \to \infty} \tilde{S}_q = \sum_{a=1}^{N-1} \tilde{\rho}_{qa} \times \tilde{\pi}_{qa}, \] (5.25)

if we can show that \( \tilde{\eta}_+^{(\alpha)} - \tilde{q}_+^{(\alpha)} \to_{\alpha \to \infty} \) finite value. But, since the boost generator may be written as

\[ \tilde{K}(\alpha) = - \sum_{i=1}^{N} \tilde{\eta}_i(\alpha) H_i(\alpha) = -\tilde{\eta}_+^{(\alpha)} H_M(\alpha) + \sum_{a=1}^{N-1} \tilde{\rho}_a(\alpha) H_a(\alpha), \]

\[ H_a(\alpha) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_{ai} H_i(\alpha), \] (5.26)

we get

\[ \tilde{\eta}_+^{(\alpha)} - \tilde{q}_+^{(\alpha)} = \sum_{a=1}^{N-1} \tilde{\rho}_a(\alpha) H_a(\alpha) + \frac{\tilde{\kappa}_+^{(\alpha)} \times \left( \sum_{a=1}^{N-1} \tilde{\rho}_a(\alpha) \times \tilde{\pi}_a(\alpha) \right)}{\sqrt{\Pi} (\sqrt{\Pi} + H_M)} + \frac{\tilde{\kappa}_+^{(\alpha)} \times \left( \tilde{\kappa}_+^{(\alpha)} \times \sum_{a=1}^{N-1} \tilde{\rho}_a(\alpha) H_a(\alpha) \right)}{H_M(\alpha) \sqrt{\Pi} (\sqrt{\Pi} + H_M)} \to_{\alpha \to \infty} \]

\[ \to_{\alpha \to \infty} \sum_{a=1}^{N-1} \sqrt{\Pi} \sum_{i=1}^{N} \gamma_{ai} m_i^2 + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \tilde{\rho}_a(\alpha) \sum_{i=1}^{N} \gamma_{ai} m_i^2 \]

\[ \downarrow \]

\[ \tilde{q}_+ = \tilde{\eta}_+ - \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \tilde{\rho}_a(\alpha) \sum_{i=1}^{N} \gamma_{ai} m_i^2 + \tilde{\kappa}_+^{(\alpha)}, \]

(terms \( \to_{\alpha \to \infty} 0 \), i.e. \( \approx 0 \) due to \( \tilde{\kappa}_+ \approx 0 \)),

(5.27)

to be compared with Eq.(3.4).

In this way we have obtained the canonical transformation (3.7)

\[ \tilde{\eta}, \tilde{\kappa} \to \tilde{q}_-, \tilde{p}_q \]

(5.28)

even if it is not known how to get the inverse canonical transformation.

When we add the gauge fixings \( \tilde{q}_+ \approx 0 \) for \( \tilde{\kappa}_+ \approx 0 \) and we go to Dirac brackets, we get

\[ \tilde{\rho}_{qa} \equiv \tilde{\rho}_a, \quad \tilde{\pi}_{qa} \equiv \tilde{\pi}_a, \quad \tilde{S}_q = \sum_{a=1}^{N-1} \tilde{\rho}_{qa} \times \tilde{\pi}_{qa} \equiv \sum_{a=1}^{N-1} \tilde{\rho}_a \times \tilde{\pi}_a, \]

\[ H_{(rel)} = \sqrt{\Pi} = H_M(\infty) = \sum_{i=1}^{N} m_i^2 + N \sum_{ab}^1 \gamma_{ai} \gamma_{ba} \tilde{\pi}_{qa} \cdot \tilde{\pi}_{qb} \equiv M_{sys} = H_M. \] (5.29)

See Appendix C for the case of spinning particles.
VI. RELATIVISTIC ROTATIONAL KINEMATICS.

All the results of the previous Sections are needed to get the separation of the internal center-of-mass degrees of freedom in the relativistic theory: this has been accomplished by adding the gauge fixings $\vec{q}_+ \approx 0$ to the rest-frame constraints $\vec{\kappa}_+ \approx 0$ and by going to Dirac brackets. We are left with the relative canonical variables $\vec{\rho}_a \equiv \vec{\rho}_a$, $\vec{\pi}_a \equiv \vec{\pi}_a$ and the Hamiltonian

$$H_{(\text{rel})} = H_{M(\infty)} = \sum_{i=1}^{N} \sqrt{m_i^2 + N \sum_{ab} \gamma_{ai} \gamma_{bi} \vec{\pi}_{qa} \cdot \vec{\pi}_{qb}} \equiv M_{\text{sys}},$$

which replaces to the non-relativistic one $H_{\text{rel, nr}} = \frac{1}{2} \sum_{a,b=1}^{N-1} k^{-1}_{ab} [m_i, \gamma_{ai}] \vec{\pi}_{qa} \cdot \vec{\pi}_{qb}$ of Eq.(2.9) of Ref. [2] for $c \rightarrow \infty$.

Let us remark that in the Hamiltonian for the relative motions in the rest frame instant form, each square root identifies a diagonalization of these $N$ matrices. But this is impossible because

$$[K_{(i)}^{-1}, K_{(j)}^{-1}]_{ab} = G_{(ij)ab} = -G_{(ij)ba} = -G_{(ji)ab},$$

with $G_{(ij)ab} = -N[\gamma_{ai} \gamma_{bj} - \gamma_{aj} \gamma_{bi}]$. There are $\frac{1}{2} N(N-1)$ matrices $G_{ij}$, each one with $\frac{1}{2} (N-1)(N-2)$ independent elements. Therefore, the conditions $G_{(ij)ab} = 0$ are $\frac{1}{2} N(N-1)(N-2)$ conditions and we have only $\frac{1}{2} (N-1)(N-2)$ free parameters in the $\gamma_{ai}$.\[34\]

Therefore, it is impossible to diagonalize simultaneously the $N$ quadratic forms under the square roots: there are no relativistic normal Jacobi coordinates and no definition of reduced masses and tensors of inertia.

To find the analogue of $L_{\text{rel, nr}} = \frac{1}{2} \sum_{a,b=1}^{N-1} k^{-1}_{ab} [m_i, \gamma_{ai}] \vec{\rho}_a \cdot \vec{\rho}_b$ (Eq.(2.9) of Ref. [2]), we should perform an inverse Legendre transformation. The first half of Hamilton equations gives

$$\dot{\vec{\rho}}_{qa} \equiv \{\vec{\rho}_{qa}, H_{(\text{rel})}\} = \sum_{i=1}^{N} \sqrt{m_i^2 + N \sum_{ab} \gamma_{ai} \gamma_{bi} \vec{\pi}_{qa} \cdot \vec{\pi}_{qb}} \frac{N \gamma_{ai} \sum_{b=1}^{N-1} \gamma_{bi} \vec{\pi}_{qb}}{\sqrt{m_i^2 + N \sum_{ab} \gamma_{ai} \gamma_{bi} \vec{\pi}_{qa} \cdot \vec{\pi}_{qb}}},$$

\[\downarrow\]

$$\dot{\vec{\rho}}_{qa} \cdot \dot{\vec{\rho}}_{qb} = \sum_{i,j}^{1..N} \sqrt{m_i^2 + N \sum_{a1b1}^{1..N-1} \gamma_{ai} \gamma_{bi} \vec{\pi}_{qa1} \cdot \vec{\pi}_{qb1}} \frac{N \gamma_{bj} \sum_{f=1}^{N-1} \gamma_{jf} \vec{\pi}_{qf}}{\sqrt{m_j^2 + N \sum_{a2b2}^{1..N-1} \gamma_{aj} \gamma_{bj} \vec{\pi}_{qa2} \cdot \vec{\pi}_{qb2}}},$$

\[6.3\]

\[34\]At the non-relativistic level there is only one such matrix at the Hamiltonian level, i.e. $k^{-1}_{ab} = \sum_{i=1}^{N} \frac{1}{m_i} K^{-1}_{(i)ab}$, see Eq.(2.9) of Ref. [2].

\[35\]For $N=3$, there are 3 conditions and only 1 parameter; for $N=4$, 18 conditions and 3 parameters.
To get $\pi_{qa} \cdot \pi_{qb}$ in terms of $\dot{\rho}_{qa} \cdot \dot{\rho}_{qb}$, we should solve higher order algebraic equations. As already pointed out, this implies that $L_{rel}(\rho_{qa}, \dot{\rho}_{qa}) = \sum_{a=1}^{N-1} \pi_{qa} \cdot \dot{\rho}_{qa} - H$ is a hyperelliptic function already in the free case. This in turn means that, unlike the non-relativistic case, it is not possible to define either an Euclidean or a Riemannian metric on the space of velocities from the kinetic energy (see Ref. [2]). Therefore we cannot visualize the Lagrangian dynamics as in the non-relativistic case. The form of the canonical momenta

$$\pi_{qa}^r = \frac{\partial L_{rel}}{\partial \dot{\rho}_{qa}^r} = \sum_{b=1}^{N-1} f_{ab}(\dot{\rho}_{qc} \cdot \dot{\rho}_{qd}) \dot{\rho}_{qb}^r,$$

(6.4)

can only be given in implicit form.

Moreover, we cannot evaluate the restrictions on the relative velocities $\dot{\rho}_{qa}(\tau)$ resulting from the existence of the limiting light velocity $c$. If we try to follow the non-relativistic pattern of the static orientation-shape bundle approach (see Ref. [2]), we get

$$\rho_{qa}^r = R^{rs}(\theta^a) \dot{\rho}_{qa}^s(q),$$

$$\dot{\rho}_{qa}^r \overset{def}{=} R^{rs}(\theta^a) \dot{\rho}_{qa}^s, \quad \vec{v}_{qa} = \vec{\omega} \times \vec{\rho}_{qa} + \frac{\partial \vec{\rho}_{qa}}{\partial q^\mu} \dot{q}^\mu,$$

$$\dot{\rho}_{qa}^r = \{ \rho_{qa}, H_{rel} \},$$

$$\pi_{qa}^r = \sum_{b=1}^{N-1} f_{ab}(\dot{\rho}_{qa} \cdot \dot{\rho}_{qa}) \dot{\rho}_{qb}^r = R^{rs}(\theta^a) \tilde{\pi}_{qa}^s,$$

$$\tilde{\pi}_{qa}^r = \sum_{b=1}^{N-1} f_{ab}(\vec{v}_{qa} \cdot \vec{v}_{qa}) \vec{v}_{qb},$$

$$S_q^r = R^{rs}(\theta^a) \tilde{S}_q^s = \sum_{a=1}^{N-1} [\tilde{\rho}_{qa} \times \tilde{\pi}_{qa}]^r,$$

$$\tilde{S}_q^r = \sum_{a=1}^{N-1} \tilde{\rho}_{qa} \times \tilde{\pi}_{qa} =$$

$$\sum_{a=1}^{1...N-1} f_{ab} \left[ (\vec{\omega} \times \tilde{\rho}_{qc} + \frac{\partial \tilde{\rho}_{qc}}{\partial q^\mu} \dot{q}^\mu) \cdot (\vec{\omega} \times \tilde{\rho}_{qd} + \frac{\partial \tilde{\rho}_{qd}}{\partial q^\mu} \dot{q}^\mu) \right]$$

$$\tilde{\rho}_{qa} \times (\vec{\omega} \times \tilde{\rho}_{qb} + \frac{\partial \tilde{\rho}_{qb}}{\partial q^\mu} \dot{q}^\mu),$$

$$\quad \tilde{S}_q^r = \sum_{a=1}^{1...N-1} f_{ab} \left[ \tilde{I}^u_{(cd)}(q) \omega^u \omega^v + \vec{\omega} \cdot (\tilde{\rho}_{qc} \times \frac{\partial \tilde{\rho}_{qd}}{\partial q^\mu} + \tilde{\rho}_{qd} \times \frac{\partial \tilde{\rho}_{qc}}{\partial q^\mu}) \dot{q}^\mu + \frac{\partial \tilde{\rho}_{c}}{\partial q^\mu} \cdot \frac{\partial \tilde{\rho}_{d}}{\partial q^\nu} \dot{q}^\nu \dot{q}^\nu \right].$$

36 For the absolute velocities $\dot{\eta}_i(\tau)$ we have $|\dot{\eta}_i(\tau)| \leq c = 1$.

37 The first line defines the body frame components $\tilde{\rho}_{qa}^r$ of the vectors $\tilde{\rho}_{qa}$ in this approach ($\theta^a$ are Euler angles). The body frame components of the relative velocities are the $\vec{v}_{qa}^r$ of the second line, while those of the spin and of the angular velocity are $\tilde{S}_q^r$ and $\tilde{\omega}^r$ respectively.
\[
\left[ \dot{I}^{rs}_{(ab)}(q) \dot{\omega}^s + \ddot{a}^u_{(ab)\mu}(q) \dot{q}^\mu \right] = \\
= \sum_{ab}^{1..N-1} f_{ab} \left[ \dot{I}^{uv}_{(cd)}(q) \left( \ddot{\omega}^u + \ddot{A}^u_{(cd)\mu}(q) \dot{q}^\mu \right) \left( \ddot{\omega}^v + \ddot{A}^v_{(cd)\nu}(q) \dot{q}^\nu \right) \right] \\
\dot{I}^{rs}_{(ab)}(q) \left( \ddot{\omega}^s + \ddot{A}^s_{(ab)\mu}(q) \dot{q}^\mu \right),
\]

\[
\dot{I}^{rs}_{(ab)}(q) = \ddot{\rho}_{qa} \cdot \ddot{\rho}_{qb} \delta^{rs} - \frac{1}{2} \left( \ddot{\rho}_{qa} \ddot{\rho}_{ba} + \ddot{\rho}_{qb} \ddot{\rho}_{aq} \right),
\]

\[
\ddot{a}^u_{(ab)\mu}(q) = \frac{1}{2} \left[ \ddot{\rho}_{qa} - \frac{\partial \ddot{\rho}_{qa}}{\partial q^\mu} \right] = \dddot{I}^{uv}_{(ab)}(q) \dddot{A}^v_{(ab)\mu}(q),
\]

\[
\dddot{a}^u_{(ab)\mu}(q) = \frac{1}{2} \left[ \ddot{\rho}_{qa} \times \frac{\partial \ddot{\rho}_{qa}}{\partial q^\mu} + \ddot{\rho}_{qb} \times \frac{\partial \ddot{\rho}_{qa}}{\partial q^\mu} \right] = \dddot{I}^{uv}_{(ab)}(q) \dddot{A}^v_{(ab)\mu}(q),
\]

\[
\ddot{\omega}, q, m = \sum_{ab}^{1..N-1} f_{ab}(\ddot{v}_{qc} \cdot \ddot{v}_{qd}) \ddot{a}^u_{(ab)\mu}(q) = \dddot{I}^{uv}(\ddot{\omega}, q, m) \dddot{A}^v(\ddot{\omega}, q, m).
\] (6.5)

We see that there is no more a linear relation between the body frame spin and angular velocity. By expanding \( f_{ab}(x) \) in a power series around \( x = 0 \), we get that \( \dot{S}^r_\mu \) is an infinite series with all the powers of the body frame angular velocity. The lowest term is \( \dot{S}^r_{(q)} = \sum_{a,b}^{1..N-1} f_{ab}(0) \dot{I}^{rs}_{(ab)}(q) \left( \ddot{\omega}^s + \ddot{A}^s_{(ab)\mu}(q) \dot{q}^\mu \right) \) with \( f_{ab}(0) \) playing the role of the non-relativistic \( k_{ab} \).

Therefore, the tensor of inertia loses a clear identification: only its building blocks \( \dot{I}^{rs}_{(ab)} \), existing also in the non-relativistic theory, appear in the relativistic construction.

The N=2 case with equal masses \( m_1 = m_2 = m \) is the only case in which we can evaluate the relative Lagrangian. We get

\[
L_{rel}(\ddot{\rho}, \dot{\rho}) = -m \sqrt{4 - \ddot{\rho}^2}.
\] (6.6)

Therefore in this case the only existing relative velocity has the bound \( |\ddot{\rho}| \leq 2 \).

Let us write \( \ddot{\rho} = \rho \dot{\rho} \) with \( \rho = |\ddot{\rho}| \) and \( \dot{\rho} = \ddot{\rho} / |\ddot{\rho}| \). With only one relative variable the three Euler angles \( \theta^a \) are redundant: there are only two independent angles, those identifying the position of the unit 3-vector \( \dot{\rho} \) on \( \mathbb{S}^2 \). We shall use the following parametrization (Euler angles \( \theta^1 = \phi, \theta^2 = \theta, \theta^3 = 0 \))

\[
\ddot{\rho}^r = R^{rs}(\theta, \phi) \ddot{\rho}^s, \\
\ddot{\rho}_o = (0, 0, 1), \quad (\text{reference unit 3-vector}),
\]

\[
R^{rs}(\theta, \phi) = R_z(\theta) R_y(\phi) = \left( \begin{array}{ccc} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{array} \right),
\]

\[
\downarrow
\]

\[38\text{Recall that its diagonalization defines the Jacobi coordinates and the reduced masses.}\]
\[
\dot{R}^s = \begin{pmatrix}
-\sin \theta \cos \phi & -\cos \phi & -\sin \theta \cos \phi - \sin \theta \sin \phi \\
-\sin \theta \sin \phi + \cos \theta \cos \phi & \cos \phi & -\sin \theta \sin \phi + \sin \theta \cos \phi \\
-\cos \theta & 0 & -\sin \theta
\end{pmatrix},
\]
\[
R^T \dot{R} = \begin{pmatrix}
0 & -\cos \theta \dot{\phi} & \dot{\theta} \\
\cos \theta \dot{\phi} & 0 & \sin \theta \dot{\phi} \\
-\dot{\theta} & -\sin \theta \dot{\phi} & 0
\end{pmatrix}.
\]

(6.7)

Following the orientation-shape bundle approach, we get the following body frame velocity and angular velocity (\(\rho\) is the only shape variable in this case)

\[
\ddot{v}^r = R^{T_r s} \dot{\rho}^s = \rho(R^T \dot{R})^s \dot{\rho}^s + \dot{\rho} \ddot{v}_o = \rho \dot{\omega} \times \dot{\rho}_o + \dot{\rho} \ddot{v}_o =
\]

\[
\ddot{\omega} = (\ddot{\omega}^1 = -\sin \theta \dot{\phi}, \ddot{\omega}^2 = \dot{\theta}, 0),
\]

\[
\ddot{v}^2 = I(\rho) \ddot{\omega}^2 + \dot{\rho}^2,
\]

\[
\ddot{I} = \rho^2.
\]

(6.8)

The non-relativistic inertia tensor of the dipole \(I_n = \mu \rho^2\) is replaced by \(\ddot{I} = \dddot{I}/\mu = \rho^2\). The Lagrangian in anholonomic variables become

\[
\ddot{L}(\ddot{\omega}, \rho, \dot{\rho}) = -m \sqrt{4 - I(\rho) \ddot{\omega}^2 - \dot{\rho}^2}.
\]

(6.9)

It is clear that the bound \(|\dddot{\rho}| \leq 2\) put upper bounds on the kinetic energy of both the rotational and vibrational motions.

The canonical momenta are

\[
\dddot{\omega} = \frac{\partial \dot{L}}{\partial \ddot{\omega}} = \frac{m \ddot{I}(\rho) \ddot{\omega}}{\sqrt{4 - \ddot{I}(\rho) \ddot{\omega}^2 - \dot{\rho}^2}},
\]

\[
\dddot{\rho} = \frac{\partial \dot{L}}{\partial \dot{\rho}} = \frac{m \dot{\rho}}{\sqrt{4 - \ddot{I}(\rho) \ddot{\omega}^2 - \dot{\rho}^2}}.
\]

(6.10)

The body frame spin is not linear in the body frame angular velocity (only approximately for slow rotations). When \(|\dddot{\rho}|\) varies between 0 and 2 the momenta vary between 0 and \(\infty\), namely in phase space there is no bound coming from the limiting light velocity. This shows once more that in special relativity it is convenient to work in the Hamiltonian framework avoiding relative and angular velocities.

Since we have \(\sqrt{4 - \ddot{I}(\rho) \ddot{\omega}^2 - \dot{\rho}^2} = \frac{2m}{\sqrt{m^2 + I^{-1}(\rho) \ddot{\omega}^2 + \dot{\rho}^2}}\), the inversion formulas are

\[\mu = \frac{m_1 m_2}{m_1 + m_2}\] is the reduced mass; see Ref. [3].
\[ \vec{\omega} = \frac{\vec{S}}{m I(\rho)} \sqrt{4 - \vec{I} S^2} - \vec{\rho}^2 = \frac{2 I^{-1}(\rho) \vec{S}}{\sqrt{m^2 + I^{-1}(\rho) \vec{S}^2 + \pi^2}}, \]
\[ \dot{\vec{\rho}} = \frac{\pi}{m} \sqrt{4 - \vec{I} S^2} - \dot{\vec{\rho}}^2 = \frac{2 \pi}{\sqrt{m^2 + I^{-1}(\rho) \vec{S}^2 + \pi^2}}. \] (6.11)

On the other hand, the Hamiltonian becomes
\[ \tilde{H} = \pi \dot{\vec{\rho}} + \vec{S} \cdot \vec{\omega} - \tilde{L} = 2 \sqrt{m^2 + I^{-1}(\rho) \vec{S}^2 + \pi^2}. \] (6.12)

Therefore, this special case identifies the following point-canonical transformation followed by a transformation to an anholonomic basis like in the non-relativistic framework
\[
\begin{pmatrix}
\rho \\
\pi_q
\end{pmatrix} \quad \rightarrow \quad \begin{pmatrix}
\theta \\
\phi \\
\rho
\end{pmatrix}
\quad \text{non can.} \quad \begin{pmatrix}
\theta \\
\phi \\
\rho
\end{pmatrix}
\]

\[ \rho^s = \rho R^{rs}(\theta, \phi) \hat{\rho}_o^s = \rho (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \]

\[ \pi^s_q = R^{rs}(\theta, \phi) \hat{\pi}_o^s = R^{rs}(\theta, \phi) \frac{m \vec{\nu}^s}{\sqrt{4 - \vec{\nu}^2}} = R^{rs}(\theta, \phi) \frac{m [\rho \vec{\omega} \times \hat{\rho}_o + \rho \dot{\hat{\rho}}_o]_s}{\sqrt{4 - \vec{I}(\rho) \vec{\omega}^2 - \dot{\rho}^2}} = R^{rs}(\theta, \phi) \frac{m [\rho \vec{\omega} \times \hat{\rho}_o + \rho \dot{\hat{\rho}}_o]_s}{\sqrt{4 - \vec{I}(\rho) \vec{\omega}^2 - \dot{\rho}^2}}. \] (6.13)

In conclusion, due to the absence of a workable Lagrangian approach, we are forced to try to define everything at the Hamiltonian level. In order to get an extension of this results for arbitrary N, we will abandon the static orientation-shape bundle approach and we shall investigate the canonical spin bases as in the non-relativistic case of Ref. [2]. In this approach we have to guess in some way a set of non-point canonical transformations from the canonical variables \( \vec{\rho}_{qa}, \vec{\pi}_{qa} \) to a basis which generalizes the previous result for the N=2 equal mass case.

The non-relativistic non-Abelian rotational symmetry generating the Noether constants of motion \( \vec{S} = \text{constant} \) is replaced by the internal non-Abelian rotational symmetry generating the constants of motion \( \vec{S}_q \) inside the Wigner hyperplane with the rest-frame conditions \( \vec{\kappa}_+ \approx 0 \).
VII. CANONICAL SPIN BASES

In this Section we show that the construction of the canonical spin bases with the associated spin frame and evolving dynamical body frames in the relativistic case starting from the relative canonical variables $\vec{\rho}_q, \vec{\pi}_q$, $a = 1,..,N - 1$, is identical to that proposed in Ref. [2] for the non-relativistic case. This happens because the total conserved rest-frame spin is $\vec{S}_q = \sum_{a=1}^{N-1} \vec{\rho}_q \times \vec{\pi}_q = \sum_{a=1}^{N-1} \vec{S}_q$ like in the non-relativistic case and because the construction is based only on the possible spin clusterings which can be obtained from the individual $\vec{S}_q$. Only the Hamiltonian for relative motions is different.

We shall sketch the construction for $N = 2, 3$ and $N \geq 4$ by referring to Ref. [2] for the relevant calculations.

A. 2-Body Systems.

Let us start from the case $N=2, i = 1,2$

$$
\begin{align*}
\vec{q}_1 & \rightarrow \alpha \beta \\
\vec{q}_2 & \rightarrow \vec{q}_3 \quad (7.1)
\end{align*}
$$

After the elimination of the internal center-of-mass degrees of freedom with the gauge fixings $\vec{q}_+ \approx 0$, the rest-frame dynamics of the relative motions is governed by the Hamiltonian

$$
H_{(rel)} = \sqrt{m_1^2 + \vec{\pi}_q^2} + \sqrt{m_2^2 + \vec{\pi}_q^2} \equiv M_{sys}. 
$$

The spin is $\vec{S}_q = \vec{\rho}_q \times \vec{\pi}_q \Rightarrow [S_i, S_j] = \epsilon_{ijk} S_k, \{\hat{R}, S_q\} = 0, \{\hat{R}, \vec{S}_q\} = \epsilon_{ijk} \hat{R}^k$. The vectors $\vec{S}_q$ and $\hat{R}$ are the generators of an E(3) group containing SO(3) as a subgroup.

Let us consider the following canonical transformation adapted to the spin

$$
\begin{align*}
\vec{F}_q & \rightarrow \alpha \beta \\
\vec{F}_q & \rightarrow \vec{S}_q \quad (7.4)
\end{align*}
$$

where
\[ \alpha = \tan^{-1} \frac{1}{S_q} \left( \tilde{\rho}_q \cdot \tilde{\pi}_q - \frac{(\rho_q)^2}{\rho_q^3} \pi_q^3 \right), \]

\[ \beta = \tan^{-1} \frac{S_q^2}{S_q} \sqrt{(S_q)^2 - (S_q^3)^2}, \]

\[ \sin \beta = \frac{S_q^2}{\sqrt{(S_q)^2 - (S_q^3)^2}}, \quad \cos \beta = \frac{S_q^1}{\sqrt{(S_q)^2 - (S_q^3)^2}} \] (7.5)

We have

\[ S_q^1 = \sqrt{(S_q)^2 - (S_q^3)^2} \cos \beta, \]

\[ S_q^2 = \sqrt{(S_q)^2 - (S_q^3)^2} \sin \beta, \]

\[ S_q^3 , \] (7.6)

\[ \hat{R}^1 = \hat{\rho}^1 = \sin \theta \cos \varphi = \sin \beta \sin \alpha - \frac{S_q^3}{S_q} \cos \beta \cos \alpha, \]

\[ \hat{R}^2 = \hat{\rho}^2 = \sin \theta \sin \varphi = -\cos \beta \sin \alpha - \frac{S_q^3}{S_q} \sin \beta \cos \alpha, \]

\[ \hat{R}^3 = \hat{\rho}^3 = \cos \theta = \frac{1}{S_q} \sqrt{(S_q)^2 - (S_q^3)^2} \cos \alpha, \]

\[ (\hat{S}_q \times \hat{R})^1 = \hat{S}_q^2 \hat{R}^3 - \hat{S}_q^3 \hat{R}^2 = \sin \beta \cos \alpha + \frac{S_q^3}{S_q} \cos \beta \sin \alpha, \]

\[ (\hat{S}_q \times \hat{R})^2 = \hat{S}_q^3 \hat{R}^1 - \hat{S}_q^1 \hat{R}^3 = -\cos \beta \cos \alpha + \frac{S_q^3}{S_q} \sin \beta \sin \alpha, \]

\[ (\hat{S}_q \times \hat{R})^3 = -\hat{S}_q^1 \hat{R}^2 - \hat{S}_q^2 \hat{R}^1 = \frac{1}{S_q} \sqrt{(S_q)^2 - (S_q^3)^2} \sin \alpha. \] (7.7)

Then we have the following inverse canonical transformation

\[ \tilde{\rho}_q = \rho_q \hat{R}(\alpha, \beta, S_q, S_q^3), \]

\[ \tilde{\pi}_q = \tilde{\pi}_q \hat{R}(\alpha, \beta, S_q, S_q^3) - \frac{S_q}{\rho_q} \hat{R}(\alpha, \beta, S_q, S_q^3) \times \hat{S}_q(\beta, S_q, S_q^3), \]

\[ \Rightarrow \quad \tilde{\pi}_q^2 = \tilde{\pi}_q^2 + \frac{S_q^2}{\rho_q^2}, \]

\[ \hat{S}_q \times \hat{R}(\alpha) = \frac{\partial \hat{R}(\alpha)}{\partial \alpha} = \hat{R}(\alpha + \frac{\pi}{2}), \]

\[ \Rightarrow \quad \alpha = -\tan^{-1} \frac{(\hat{S}_q \times \hat{R})^3}{[\hat{S}_q \times (\hat{S}_q \times \hat{R})]^3}. \] (7.8)

From the last line of this equation we see that the angle \( \alpha \) can be expressed in terms of \( \hat{S}_q \) and \( \hat{R} \).

The conjugate variables \( \rho_q, \tilde{\pi}_q \) can be called \textit{dynamical shape variables} : they describe the vibration of the dipole.
Then the rest-frame Hamiltonian for the relative motion becomes

\[
H_{(rel)} = H_M(\infty) = \sqrt{m_1^2 + \tilde{\pi}_q^2} + \sqrt{m_2^2 + \tilde{\pi}_q^2} = \\
\sqrt{m_1^2 + \frac{1}{I}(S_q)^2 + \tilde{\pi}_q^2} + \sqrt{m_2^2 + \frac{1}{I}(S_q)^2 + \tilde{\pi}_q^2},
\]

\[
\Rightarrow \quad \tilde{\omega} = \frac{\partial H_{(rel)}}{\partial \tilde{S}_q} = \frac{S_q}{I} \left( \sqrt{m_1^2 + \frac{1}{I}(S_q)^2 + \tilde{\pi}_q^2} + \sqrt{m_2^2 + \frac{1}{I}(S_q)^2 + \tilde{\pi}_q^2} \right),
\]

\[
\hat{\rho}_q = \frac{\partial H_{(rel)}}{\partial \tilde{\rho}_q} = \tilde{\pi}_q \left( \frac{1}{\sqrt{m_1^2 + \frac{1}{I}(S_q)^2 + \tilde{\pi}_q^2}} + \frac{1}{\sqrt{m_2^2 + \frac{1}{I}(S_q)^2 + \tilde{\pi}_q^2}} \right),
\]

\[
\tilde{\pi}_q|_{\tilde{\rho}_q=0} = 0,
\]

\[
H^{(S)}_{(rel)} = H_{(rel)}|_{\tilde{\rho}_q=0} = \sqrt{m_1^2 + \frac{1}{I}(S_q)^2 + \sqrt{m_2^2 + \frac{1}{I}(S_q)^2}},
\]

\[
H^{(S=0)}_{(rel)} = H_{(rel)}|_{S_q=0} = \sqrt{m_1^2 + \tilde{\pi}_q^2} + \sqrt{m_2^2 + \tilde{\pi}_q^2},
\]

(7.9)

where \( \tilde{I} = \rho^2 \equiv \tilde{I}_{nr}/\mu \) is the non-relativistic baricentric inertia tensor \( \tilde{I}_{nr} \) of the dipole divided by the reduced mass \( \mu = \frac{m_1 m_2}{m_1 + m_2} \). The quantities \( H^{(S)}_{(rel)} \) and \( H^{(S=0)}_{(rel)} \) are the purely rotational and purely vibrational Hamiltonians, respectively.

For equal masses we get formally the same results of the previous Section but in a different canonical basis, if we make the identifications \( \pi = \tilde{\pi}_q, \rho = \rho_q \) and \( \tilde{R}^r = R^{rs}(\theta, \phi)\tilde{R}^s_0 \).

Only after a non-point transformation \( \alpha, S_q, \beta, S_3^q \mapsto \theta, \pi_\theta, \phi, \pi_\phi \) [i.e. from Eq.(7.7) to Eq.(7.4)], Eqs.(7.11) become Eqs.(5.13).

B. 3-Body Systems.

In the case \( N=3 \) the range of the indices is \( i = 1, 2, 3, a = 1, 2 \). The spin is \( \tilde{S}_q = \sum_{a=1}^2 \tilde{\rho}_{qa} \times \tilde{\pi}_{qa} = \sum_{a=1}^2 \tilde{S}_{qa} \) after the canonical transformation which separates the internal center of mass

\[
\begin{pmatrix}
\tilde{q}_1 \\
\tilde{k}
\end{pmatrix} \mapsto \begin{pmatrix}
\tilde{q}_+ \\
\tilde{k} + \tilde{\rho}_{qa} \\
\tilde{\pi}_{qa}
\end{pmatrix}
\]

(7.10)

After the gauge fixings \( \tilde{q}_+ \approx 0 \), the relative motions in the rest-frame instant form are governed by the Hamiltonian

\[
H_{(rel)} = H_M(\infty) = \sum_{i=1}^3 \sqrt{m_i^2 + 3 \sum_{a,b=1}^2 \gamma_{ab} \gamma_{ba} \tilde{\pi}_{qa} \cdot \tilde{\pi}_{qb} \equiv M_{sys}.}
\]

(7.11)

We shall assume \( \tilde{S}_q \neq 0 \), because the exceptional SO(3) orbit \( S_q = 0 \) has to be studied separately. This is done by adding \( S_q \approx 0 \) as a first class constraint and studying separately the following three disjoint strata (they have a different number of first class constraints):

a) \( \tilde{S}_q \approx 0 \), but \( \tilde{S}_1 \approx -\tilde{S}_2 \neq 0 \); b) \( \tilde{S}_{qa} \approx 0, a = 1, 2 \) (in this case we have \( \tilde{\rho}_{qa} - k_a \tilde{\pi}_{qa} \approx 0 \)).
For each value of $a = 1, 2$, we can consider the canonical transformation (7.11)

\[
\tilde{\rho}_{qa} \quad \tilde{\pi}_{qa} \rightarrow \begin{array}{ccc}
\alpha_a & \beta_a & \rho_{qa} \\
S_{qa} & S_{qa} & \tilde{\pi}_{qa}
\end{array}
\] (7.12)

where

\[
\alpha_a = \tan^{-1} \frac{1}{S_{qa}} (\tilde{\rho}_{qa} \cdot \tilde{\pi}_{qa} - \frac{(\rho_{qa})^2}{\rho_{3}^2 \tilde{\pi}_{qa}}),
\]

\[
\beta_a = \tan^{-1} \frac{S_{qa}^2}{S_{qa}^1}, \quad \sin \beta_a = \frac{S_{qa}^2}{\sqrt{(S_{qa}^2)^2 - (S_{qa}^3)^2}}, \quad \cos \beta_a = \frac{S_{qa}^1}{\sqrt{(S_{qa}^2)^2 - (S_{qa}^3)^2}}.
\] (7.13)

\[
\tilde{\rho}_{qa} = \rho_{qa} \hat{R}_a, \quad \rho_{qa} = \sqrt{\tilde{\rho}_{qa}^2}, \quad \hat{R}_a = \frac{\tilde{\rho}_{qa}}{\rho_{qa}} = \hat{\tilde{\rho}}_{qa}, \quad \hat{R}_a^2 = 1,
\]

\[
\tilde{\pi}_{qa} = \tilde{\pi}_{qa} \hat{R}_a - \frac{S_{qa}}{\rho_{qa}} \hat{R}_a \times \hat{\tilde{S}}_{qa}, \quad \tilde{\pi}_{qa} = \tilde{\pi}_{qa} \cdot \hat{R}_a.
\] (7.14)

\[
\tilde{\rho}_{qa} = \rho_{qa} \hat{R}_a (\alpha_a, \beta_a, S_{qa}, S_{qa}^3),
\]

\[
\tilde{\pi}_{qa} = \tilde{\pi}_{qa} \rho_{qa} (\alpha_a, \beta_a, S_{qa}, S_{qa}^3) - \frac{S_{qa}}{\rho_{qa}} \rho_{qa} (\alpha_a, \beta_a, S_{qa}, S_{qa}^3) \times \hat{\tilde{S}}_{qa} (\beta_a, S_{qa}, S_{qa}^3) = \]

\[
\tilde{\pi}_{qa} \hat{R}_a (\alpha_a, \beta_a, S_{qa}, S_{qa}^3) - \frac{S_{qa}}{\rho_{qa}} \hat{R}_a (\alpha_a, \beta_a, S_{qa}, S_{qa}^3) \times \hat{\tilde{S}}_{qa} (\beta_a, S_{qa}, S_{qa}^3).
\] (7.15)

We have now two unit vectors $\hat{R}_a$ and two E(3) realizations generated by $\tilde{S}_{qa}$, $\hat{R}_a$ respectively and fixed invariants $\hat{R}_a^2 = 1$, $\tilde{S}_{qa} \cdot \hat{R}_a = 0$ (non-irreducible, type 2 [30]).

In order to implement a SO(3) Hamiltonian right action in analogy with the rigid body theory, we must construct an orthonormal triad or body frame $\hat{N}$, $\hat{\chi}$, $\hat{\xi}$, $\hat{N} \times \hat{\chi}$. The decomposition $\tilde{S} = \tilde{S}^1 \hat{\chi} + \tilde{S}^2 \hat{N} \times \hat{\chi} + \tilde{S}^3 \hat{N}$ identifies the SO(3) scalar generators $\tilde{S}^r$ of the right action provided they satisfy $\{\tilde{S}^r, \tilde{S}^s\} = -\epsilon^{rsu} \tilde{S}^u$. This latter condition together with the obvious requirement that $\hat{N}$, $\hat{\chi}$, $\hat{\xi}$ be SO(3) vectors $\{\tilde{N}^r, \tilde{S}^s\} = \epsilon^{rsu} \hat{\tilde{N}}^u$, $\{\hat{\chi}^r, \tilde{S}^s\} = \epsilon^{rsu} \tilde{\chi}^u$, $\{\hat{\xi}^r, \tilde{S}^s\} = \epsilon^{rsu} \hat{\xi}^u$ entails the equations [40] $\{\tilde{N}^r, \tilde{\xi}^u\} = \{\tilde{N}^r, \tilde{\chi}^u\} = \{\tilde{\chi}^r, \tilde{\chi}^u\} = 0$.

To each solution of these equations is associated a couple of canonical realizations of the E(3) group (type 2, non-irreducible): one with generators $\tilde{S}$, $\tilde{N}$ and non-fixed invariants $\tilde{S}^3 = \tilde{S} \cdot \tilde{N}$ and $|\tilde{N}|$; another with generators $\tilde{S}$, $\tilde{\chi}$ and non-fixed invariants $\tilde{S}^3 = \tilde{S} \cdot \tilde{\chi}$ and $|\tilde{\chi}|$. These latter contain the relevant information for constructing the angle $\alpha$ and the new canonical pair $\tilde{S}^3$, $\gamma = \tan^{-1} \frac{S_{qa}^2}{S_{qa}^1}$ of SO(3) scalars. Since $\{\alpha, \tilde{S}^3\} = \{\alpha, \gamma\} = 0$ must hold, it follows that the vector $\tilde{N}$ necessarily belongs to the $\tilde{S} \cdot \hat{R}$ plane. The three canonical pairs $S$, $\alpha$, $S^3$, $\beta$, $\tilde{S}^3$, $\gamma$ will describe the orientational variables of our Darboux basis, while

\[\text{With } \tilde{S}^r = \tilde{S} \cdot \hat{e}_r, \text{ the conditions } \{\tilde{S}^r, \tilde{S}^s\} = -\epsilon^{rsu} \hat{\tilde{S}}^u \text{ imply the equations } \tilde{S} \cdot \hat{e}_r \times \hat{e}_s + S^i S^j \{\hat{e}_r, \hat{e}_s\} = \epsilon_{rsu} S^k \hat{e}^u_k, \text{ hence the quoted result.}\]
$|\vec{N}|$ and $|\vec{\chi}|$ will belong to the *shape* variables. Alternatively, an anholonomic basis can be constructed by replacing the above six variables by $\vec{S}^r$ and three uniquely determined Euler angles $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$.

In the $N=3$ case it turns out that a solution of the previous equation corresponding to a *body frame* determined by the 3-body system configuration only, as in the *rigid body* case, is completely individuated once two orthonormal vectors $\vec{N}$ and $\vec{\chi}$, functions of the relative coordinates and independent of the momenta, are found such that $\vec{N}$ lies in the $\vec{S}$ - $\vec{R}$ plane$^{[41]}$. We do not known whether in the case $N=3$ other solutions of the previous equations exist leading to momentum dependent body frames$^{[42]}$.

The *simplest choice* for the orthonormal vectors $\vec{N}$ and $\vec{\chi}$ functions only of the coordinates is

$$\vec{N} = \frac{1}{2}(\hat{R}_1 + \hat{R}_2) = \frac{1}{2}(\hat{\rho}_{q1} + \hat{\rho}_{q2}), \quad \hat{N} = \frac{\vec{N}}{|\vec{N}|}, \quad |\vec{N}| = \sqrt{\frac{1 + \hat{\rho}_{q1} \cdot \hat{\rho}_{q2}}{2}},$$

$$\vec{\chi} = \frac{1}{2}(\hat{R}_1 - \hat{R}_2) = \frac{1}{2}(\hat{\rho}_{q1} - \hat{\rho}_{q2}), \quad \hat{\chi} = \frac{\vec{\chi}}{|\vec{\chi}|}, \quad |\vec{\chi}| = \sqrt{\frac{1 - \hat{\rho}_{q1} \cdot \hat{\rho}_{q2}}{2}} = \sqrt{1 - \hat{N}^2},$$

$$\vec{N} \times \vec{\chi} = \frac{-1}{2} \hat{\rho}_{q1} \times \hat{\rho}_{q2}, \quad |\vec{N} \times \vec{\chi}| = |\vec{N}| |\vec{\chi}| = \frac{1}{2} \sqrt{1 - (\hat{\rho}_{q1} \cdot \hat{\rho}_{q2})^2},$$

$$\vec{N} \cdot \vec{\chi} = 0, \quad \{N^r, N^s\} = \{\chi^r, \chi^s\} = \{N^r, \chi^s\} = 0,$$

$$\hat{R}_1 = \hat{\rho}_{q1} = \vec{N} + \vec{\chi}, \quad \hat{R}_2 = \hat{\rho}_{q2} = \vec{N} - \vec{\chi}, \quad \hat{R}_1 \cdot \hat{R}_2 = \hat{\rho}_{q1} \cdot \hat{\rho}_{q2} = \hat{N}^2 - \hat{\chi}^2. \quad (7.16)$$

Likewise, we have for the spins

$$\vec{S}_q = \vec{S}_{q1} + \vec{S}_{q2},$$

$$\vec{W}_q = \vec{S}_{q1} - \vec{S}_{q2},$$

$$\vec{S}_{q1} = \frac{1}{2}(\vec{S}_q + \vec{W}_q), \quad \vec{S}_{q2} = \frac{1}{2}(\vec{S}_q - \vec{W}_q),$$

$$\{W_q^r, W_q^s\} = e^{r\lambda\mu}\vec{S}_q^\mu. \quad (7.17)$$

We therefore succeeded in constructing an orthonormal triad (the *dynamical body frame*) and two $E(3)$ realizations (non-irreducible, type 3): one with generators $\vec{S}_q$, $\vec{N}$ and non-fixed invariants $|\vec{N}|$ and $\vec{S} \cdot \vec{N}$, the other with generators $\vec{S}_q$ and $\vec{\chi}$ and non-fixed invariants

$^{[41]}$Let us remark that any pair of orthonormal vectors $\vec{N}$, $\vec{\chi}$ function only of the relative coordinates can be used to build a body frame. This freedom is connected to the possibility of redefining a body frame by using a configuration-dependent arbitrary rotation, which leaves $\vec{N}$ in the $\vec{S}$-$\vec{R}$ plane.

$^{[42]}$Anyway, our constructive method necessarily leads to momentum-dependent solutions of the previous equations for $N \geq 4$ and therefore to momentum-dependent or *dynamical body frames*. See the following Subsection C.
|\chi| and \vec{S}_q \cdot \hat{\chi}. As said in Ref. \[2\] this is equivalent to the determination of the non-conserved generators \vec{S}_q^r of a Hamiltonian right action of SO(3): \vec{S}_q^1 = \vec{S}_q \cdot \hat{\chi} = \vec{S}_q \cdot \hat{e}_1, \vec{S}_q^2 = \vec{S}_q \cdot \hat{N} \times \hat{\chi} = \vec{S}_q \cdot \hat{e}_2, \vec{S}_q^3 = \vec{S}_q \cdot \hat{N} = \vec{S}_q \cdot \hat{e}_3.

The realization of the E(3) group with generators \vec{S}_q, \vec{N} and non-fixed invariants \mathcal{I}_1 = \vec{N}^2, \mathcal{I}_2 = \vec{S}_q \cdot \vec{N} leads to the final canonical transformation

\[
\begin{align*}
\begin{array}{cccc}
\bar{\rho}_{qa} & \rightarrow & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \rho_{qa} \\
|\vec{I}| & \rightarrow & \alpha & \beta & \gamma & |\vec{N}| & \rho_{qa}
\end{array}
\end{align*}
\]

(7.18)

where

\[
|\vec{N}| = \sqrt{\frac{1 + \rho_{q1} \cdot \rho_{q2}}{2}},
\]

\[
\vec{S}_q^3 = \vec{S}_q \cdot \hat{N} = \frac{1}{\sqrt{2}} \sum_{a=1}^{2} \bar{\rho}_{qa} \times \bar{\pi}_{qa} \cdot \frac{\hat{\rho}_{q1} + \hat{\rho}_{q2}}{\sqrt{1 + \rho_{q1} \cdot \rho_{q2}}} \approx S_q \cos \psi,
\]

\[
\cos \psi = \hat{\vec{S}}_q \cdot \hat{\vec{N}} = \frac{\vec{S}_q^3}{S_q}, \quad \sin \psi = \frac{1}{S_q} \sqrt{(S_q^2)^2 - (\vec{S}_q^3)^2},
\]

\[
S_q = \hat{\vec{S}}_q = \sqrt{\sum_{a=1}^{2} (\bar{\rho}_{qa} \times \bar{\pi}_{qa})^2},
\]

\[
S_q^3 = \sum_{a=1}^{2} (\bar{\rho}_{qa} \times \bar{\pi}_{qa})^3,
\]

\[
\begin{align*}
\alpha &= -tg^{-1} \left(\frac{(\vec{S}_q \times \vec{N})^3}{[\vec{S}_q \times (\vec{S}_q \times \vec{N})]^3}\right) \\
\beta &= tg^{-1} \frac{[\vec{S}_q \times (\hat{\rho}_{q1} + \hat{\rho}_{q2})]^3}{[\vec{S}_q \times [(\hat{\rho}_{q1} + \hat{\rho}_{q2})]^3]}, \\
\gamma &= tg^{-1} \frac{\vec{S}_q^3 \cdot \hat{\vec{N}} \cdot \hat{\chi}}{\vec{S}_q^3 \cdot \hat{\vec{N}} \cdot \hat{\chi}} = tg^{-1} \frac{\vec{S}_q^2}{\vec{S}_q^3},
\end{align*}
\]

\[
\Rightarrow \quad \sin \gamma = \frac{\vec{S}_q^2}{\sqrt{(S_q^2)^2 - (\vec{S}_q^3)^2}}, \quad \cos \gamma = \frac{\vec{S}_q^3}{\sqrt{(S_q^2)^2 - (\vec{S}_q^3)^2}},
\]

\[
= tg^{-1} \frac{\sqrt{2} \vec{S}_q \cdot \hat{\rho}_{q2} \times \hat{\rho}_{q1}}{\sqrt{1 + \hat{\rho}_{q1} \cdot \hat{\rho}_{q2}} \vec{S}_q \cdot \hat{\rho}_{q2} + \hat{\rho}_{q1} \cdot \hat{\rho}_{q2}},
\]

\[
\xi = \vec{W}_q \cdot \hat{N} \times \hat{\chi} = \sqrt{1 - \vec{N}^2} = \sqrt{2} \sum_{a=1}^{2} (-a)^{a+1} \bar{\rho}_{qa} \times \bar{\pi}_{qa} \cdot (\hat{\rho}_{q2} \times \hat{\rho}_{q1}) / [1 - \hat{\rho}_{q1} \cdot \hat{\rho}_{q2}] \sqrt{1 + \hat{\rho}_{q1} \cdot \hat{\rho}_{q2}}.
\]
For $N=3$ the dynamical shape variables, functions of the relative coordinates $\bar{\rho}_{qa}$ only, are $|\bar{N}|$, $\bar{\rho}_{qa}$, while the conjugate shape momenta are $\xi$, $\bar{\pi}_{qa}$.

The final array (7.18) is nothing else than a scheme $B$ [31] of a realization of an $E(3)$ group with generators $\bar{S}_q$, $\bar{N}$ (non-irreducible, type 3). In particular, the two canonical pairs $S^3_q$, $\beta$, $S_q$, $\alpha$, constitute the irreducible kernel of the $E(3)$ scheme $A$, whose invariants are $\bar{S}^3_q$, $|\bar{N}|$; $\gamma$ and $\xi$ are the so-called supplementary variables conjugated to the invariants; finally, the two pairs $\rho_{qa}$, $\bar{\pi}_{qa}$ are so-called inessential variables. Let us remark that $S^3_q$, $\beta$, $S_q$, $\alpha$, $\gamma$, $\xi$, are a local coordinatization of every $E(3)$ coadjoint orbit with $\bar{S}^3_q = \text{const.}$, $|\bar{N}| = \text{const.}$ and fixed values of the inessential variables, present in the 3-body phase space.

We can now reconstruct $\bar{S}_q$ and define a new unit vector $\hat{R}$ orthogonal to $\bar{S}_q$ by adopting the prescription of Eq.(7.10) as follows

$$\hat{S}^1_q = \frac{1}{S_q} \sqrt{(S_q)^2 - (S^3_q)^2} \cos \beta,$$
$$\hat{S}^2_q = \frac{1}{S_q} \sqrt{(S_q)^2 - (S^3_q)^2} \sin \beta,$$
$$\hat{S}^3_q = \frac{S^3_q}{S_q},$$

$$\hat{R}^1 = \sin \beta \sin \alpha - \frac{S^3_q}{S_q} \cos \beta \cos \alpha,$$
$$\hat{R}^2 = -\cos \beta \sin \alpha - \frac{S^3_q}{S_q} \sin \beta \cos \alpha,$$
$$\hat{R}^3 = \frac{1}{S_q} \sqrt{(S_q)^2 - (S^3_q)^2} \cos \alpha,$$

$$\hat{R}^2 = 1, \quad \hat{R} \cdot \hat{S}_q = 0, \quad \{\hat{R}^r, \hat{R}^s\} = 0,$$

$$(\hat{S}_q \times \hat{R})^1 = \hat{S}^2_q \hat{R}^3 - \hat{S}^3_q \hat{R}^2 = \sin \beta \cos \alpha + \frac{S^3_q}{S_q} \cos \beta \sin \alpha,$$
$$\hat{R}^2 = \hat{S}^3_q \hat{R}^1 - \hat{S}^1_q \hat{R}^3 = -\cos \beta \cos \alpha + \frac{S^3_q}{S_q} \sin \beta \sin \alpha,$$

$$\hat{R}^3 = \hat{S}^3_q \hat{R}^2 - \hat{S}^1_q \hat{R}^1 = \frac{1}{S_q} \sqrt{(S_q)^2 - (S^3_q)^2} \sin \alpha,$$

(7.20)

The vectors $\hat{S}_q$, $\hat{R}$, $\hat{S}_q \times \hat{R}$ build up the spin frame for $N=3$. The angle $\alpha$ conjugate to $S_q$ is explicitly given by

$$\alpha = -tg^{-1} \frac{(\hat{S}_q \times \hat{N})^3}{[\hat{S}_q \times (\hat{S}_q \times \hat{N})]^3} = -tg^{-1} \frac{(\hat{S}_q \times \hat{R})^3}{[\hat{S}_q \times (\hat{S}_q \times \hat{R})]^3}. \quad (7.21)$$

43 The two expressions of $\alpha$ given above are consistent with the fact that $\hat{S}_q$, $\hat{R}$ and $\hat{N}$ are coplanar, so that $\hat{R}$ and $\hat{N}$ differ by a term in $\hat{S}_q$. 

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As a consequence of this definition of $\hat{R}$, we get the following expressions for the dynamical body frame $\hat{N}$, $\hat{\chi}$, $\hat{N} \times \hat{\chi}$ in terms of the final canonical variables

\[
\hat{N} = \cos \psi \hat{S}_q + \sin \psi \hat{R} = \frac{\hat{S}_q^3}{S_q} \hat{S}_q + \frac{1}{S_q} \sqrt{(S_q)^2 - (\hat{S}_q^3)^2} \hat{R} = \\
= \hat{N}[S_q, \alpha; S_q^3, \beta; \hat{S}_q^3, \gamma],
\]

\[
\hat{\chi} = \sin \psi \cos \gamma \hat{S}_q - \cos \psi \cos \gamma \hat{R} + \sin \gamma \hat{S}_q \times \hat{R} = \\
= \frac{1}{S_q} \sqrt{(S_q)^2 - (\hat{S}_q^3)^2} \cos \gamma \hat{S}_q - \frac{\hat{S}_q^3}{S_q} \cos \gamma \hat{R} + \sin \gamma \hat{S}_q \times \hat{R} = \\
= \frac{\hat{S}_q^3}{S_q} \hat{S}_q^3 \hat{S}_q + \frac{\hat{S}_q^2}{S_q} \hat{S}_q \times \hat{R} = \\
= \hat{\chi}[S_q, \alpha; S_q^3, \beta; \hat{S}_q^3, \gamma],
\]

\[
\hat{N} \times \hat{\chi} = \sin \psi \sin \gamma \hat{S}_q - \cos \psi \sin \gamma \hat{R} - \cos \gamma \hat{S}_q \times \hat{R} = \\
= \frac{1}{S_q} \sqrt{(S_q)^2 - (\hat{S}_q^3)^2} \sin \gamma \hat{S}_q - \frac{\hat{S}_q^3}{S_q} \sin \gamma \hat{R} - \cos \gamma \hat{S}_q \times \hat{R} = \\
= \frac{\hat{S}_q^3}{S_q} \hat{S}_q^3 \hat{R} - \frac{\hat{S}_q^2}{S_q} \hat{S}_q \times \hat{R} = \\
= (\hat{N} \times \hat{\chi})[S_q, \alpha; S_q^3, \beta; \hat{S}_q^3, \gamma],
\]

\[
\hat{S}_q = \sin \psi \cos \gamma \hat{\chi} + \sin \psi \sin \gamma \hat{N} \times \hat{\chi} + \cos \psi \hat{N} = \\
= \frac{1}{S_q} \left[ \hat{S}_q^3 \hat{\chi} + \hat{S}_q^2 \hat{N} \times \hat{\chi} + \hat{S}_q^3 \hat{N} \right],
\]

\[
\hat{R} = -\cos \psi \cos \gamma \hat{\chi} - \cos \psi \sin \gamma \hat{N} \times \hat{\chi} + \sin \psi \hat{N},
\]

\[
\hat{R} \times \hat{S}_q = -\sin \gamma \hat{\chi} + \cos \gamma \hat{N} \times \hat{\chi}.
\]

(7.22)

While $\psi$ is the angle between $\hat{S}_q$ and $\hat{N}$, $\gamma$ is the angle between the plane $\hat{N} - \hat{\chi}$ and the plane $\hat{S}_q - \hat{\hat{N}}$. As in the case $N=2$, $\alpha$ is the angle between the plane $\hat{S}_q - \hat{\hat{f}}_3$ and the plane $\hat{S}_q - \hat{\hat{R}}$, while $\beta$ is the angle between the plane $\hat{S}_q - \hat{\hat{f}}_3$ and the plane $\hat{f}_3 - \hat{\hat{f}}_1$.

Owing to the results of Appendix C of Ref. [2], which allow to reexpress $S_{qa} = |\tilde{S}_{qa}|$, $S_{qa}^3$, $\beta_a = t^{-1}S_{qa}^3$, and $\alpha_a = -t^{-1}(S_{qa} \times R_{qa})^3$ in terms of the final canonical variables, we can reconstruct the inverse canonical transformation.

The existence of the spin frame and of the dynamical body frame allows to define two decompositions of the relative variables, which make explicit the inverse canonical transformation. For the relative coordinates we get from Eqs. (7.16) and Appendix C of I

\[
\hat{\rho}_{qa} = \rho_{qa} \hat{R}_a = \rho_{qa}[\hat{N} + (-)^{a+1} \hat{\chi}] = \rho_{qa}[|\hat{N}|\hat{N} + (-)^{a+1} \sqrt{1 - \hat{N}^2} \hat{\chi}] =
\]

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Finally, the results in Appendix D allow to perform a sequence of a canonical transformation to Euler angles $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ with their conjugate momenta, followed by a transition to the anholonomic basis used in the orientation-shape bundle approach 4.

\[
\tilde{\alpha} = \arctg \frac{p_{\tilde{\beta}} - p_{\tilde{\gamma}}}{p_{\tilde{\alpha}}}, \\
\tilde{\beta} = \beta, \\
\tilde{\gamma} = \gamma.
\]

The results of Appendix C of Ref. [3] give the analogous formulas for the relative momenta

\[
\pi_{qa} = \pi_{qa} \hat{R}_a + \frac{S_q}{\rho_{qa}} \hat{S}_q \times \hat{R}_a = \pi_{qa} \hat{R}_a + \frac{S_q}{\rho_{qa}} \hat{S}_q \times \hat{p}_a = \\
= [\pi_{qa} \cdot \hat{N}] \hat{N} + [\pi_{qa} \cdot \hat{\chi}] \hat{\chi} + [\pi_{qa} \cdot \hat{\chi}_q] \hat{\chi}_q = \\
= \frac{1}{S_q} \left[ ((\pi_{qa} \cdot \hat{N})\hat{S}_q^3 + (\pi_{qa} \cdot \hat{\chi})\hat{S}_q^1 + (\pi_{qa} \cdot \hat{\chi}_q)\hat{S}_q^2) \hat{S}_q + \\
+ ((\pi_{qa} \cdot \hat{N})\sqrt{(S_q)^2 - (\hat{S}_q^3)^2} - [(\pi_{qa} \cdot \hat{\chi}_q)\hat{S}_q^2 + \frac{S_q}{\hat{S}_q^3}(\hat{S}_q^1\hat{S}_q^3)\hat{S}_q^2,\hat{S}_q^3]} \hat{\chi}_q \right] = \\
= \pi_{qa}[S_q, \alpha; S_q^3, \beta; S_q^3, \gamma; \rho_{qa}, |\hat{N}|], \xi; |\hat{N}|], \xi; |\hat{N}|].
\]

\[
(7.23)
\]

Finally, the results in Appendix D allow to perform a sequence of a canonical transformation to Euler angles $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ with their conjugate momenta, followed by a transition to the anholonomic basis used in the orientation-shape bundle approach 4.

\[
\pi_{qa} = \pi_{qa} \hat{R}_a + \frac{S_q}{\rho_{qa}} \hat{S}_q \times \hat{R}_a = \pi_{qa} \hat{R}_a + \frac{S_q}{\rho_{qa}} \hat{S}_q \times \hat{p}_a = \\
= [\pi_{qa} \cdot \hat{N}] \hat{N} + [\pi_{qa} \cdot \hat{\chi}] \hat{\chi} + [\pi_{qa} \cdot \hat{\chi}_q] \hat{\chi}_q = \\
= \frac{1}{S_q} \left[ ((\pi_{qa} \cdot \hat{N})\hat{S}_q^3 + (\pi_{qa} \cdot \hat{\chi})\hat{S}_q^1 + (\pi_{qa} \cdot \hat{\chi}_q)\hat{S}_q^2) \hat{S}_q + \\
+ ((\pi_{qa} \cdot \hat{N})\sqrt{(S_q)^2 - (\hat{S}_q^3)^2} - [(\pi_{qa} \cdot \hat{\chi}_q)\hat{S}_q^2 + \frac{S_q}{\hat{S}_q^3}(\hat{S}_q^1\hat{S}_q^3)\hat{S}_q^2,\hat{S}_q^3]} \hat{\chi}_q \right] = \\
= \pi_{qa}[S_q, \alpha; S_q^3, \beta; S_q^3, \gamma; |\hat{N}|], \xi; |\hat{N}|], \xi; |\hat{N}|].
\]

\[
(7.24)
\]

See Appendix C of Ref. [3] for the expression of the body frame components of $\pi_{qa}$. 

44See Appendix C of Ref. [3] for the expression of the body frame components of $\pi_{qa}$. 

47
\[
\gamma = \frac{\pi}{2} - \tilde{\gamma} - \arctg \frac{ctg \tilde{\beta} p_{\tilde{\gamma}} - \frac{p_{\tilde{\gamma}}}{\sin \tilde{\beta}}}{p_{\tilde{\beta}}},
\]
\[
\beta = \tilde{\alpha} + \arctg \frac{ctg \tilde{\beta} p_{\tilde{\alpha}} - \frac{p_{\tilde{\alpha}}}{\sin \tilde{\beta}}}{p_{\tilde{\beta}}} = \frac{\pi}{2},
\]
\[
(7.25)
\]
Here \( p_{\tilde{\alpha}}, p_{\tilde{\beta}}, p_{\tilde{\gamma}} \) are the functions of \( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\gamma}, \tilde{\gamma} \), given in Eqs. (D3). Eqs. (D3), (7.25), (7.22) and \( \tilde{S}_q^2 = \tilde{S}_q \cdot \tilde{N} \times \tilde{\chi} \) lead to the determination of the dynamical orientation variables \( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \) in terms of \( \tilde{\rho}_{qa}, \tilde{\pi}_{qa} \). Let us stress that, while in the orientation-shape bundle approach the orientation variables \( \theta^\alpha \) are gauge variables, the Euler angles \( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \) are uniquely determined in terms of the original variables.

In conclusion the complete transition to the anholonomic basis used in the static theory of the orientation-shape bundle is

\[
\begin{array}{cccc}
\alpha & \beta & \gamma & |N| \\
\tilde{S}_q = & S_q & S_q^3 & S_q^3 \\
\rho_{qa} & \pi_{qa} & \xi & \tilde{\pi}_{qa} \\
\end{array}
\]
\[
\rightarrow
\begin{array}{cccc}
\tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} & |\tilde{N}| \\
\tilde{S}_q^1 & \tilde{S}_q^2 & \tilde{S}_q^3 & \xi \\
\rho_{qa} & \pi_{qa} & \tilde{\pi}_{qa} \\
\end{array}
\]
\[
(7.26)
\]
with the 3 pairs of conjugate canonical dynamical shape variables: \( \rho_{qa}, \pi_{qa}, |\tilde{N}|, \xi \).

Eqs. (7.23), (7.26), (7.21) and (D2) imply

\[
\rho_{qa}^r = R^r_s(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \rho_{qa}^s(q), \quad \text{with}
\]
\[
\rho_{qa}^1(q) = (-)^{a+1} \rho_{qa} \sqrt{1 - \tilde{N}^2}, \quad \rho_{qa}^2(q) = 0, \quad \rho_{qa}^3(q) = \rho_{qa} |\tilde{N}|,
\]
and

\[
S_q^r = R^r_s(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \tilde{S}_q^s;
\]
\[
(7.27)
\]
so that the final visualization of our sequence of transformations is

\[
\begin{array}{cccc}
\tilde{\rho}_{qa} & \tilde{\pi}_{qa} \\
\rho_{qa} & \pi_{qa} \\
\end{array}
\]
\[
\rightarrow
\begin{array}{cccc}
\tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} & q^\mu(\tilde{\rho}_{qa}) \\
\tilde{S}_q^1 & \tilde{S}_q^2 & \tilde{S}_q^3 & p_\mu(\tilde{\rho}_{qa}, \tilde{\pi}_{qa}) \\
\end{array}
\]
\[
(7.28)
\]
Note furthermore that we get \( \tilde{\rho}_{qa}^2 = \tilde{\rho}_{qa} \cdot \tilde{N} \times \tilde{\chi} = 0 \) by construction and this entails that using our dynamical body frame is equivalent to a convention (xxzz gauge) about the body frame of the type of xzzz and similar gauges quoted in Ref. [1].

Finally, we can give the expression of the Hamiltonian for the relative motions in terms of the anholonomic Darboux basis (7.26). By using Eqs. (E12) and (E13) we get

\[45\]The Hamiltonian in the basis (7.18) can be obtained with the following replacements \( \tilde{S}_q^1 = \sqrt{(S_q)^2 - (\tilde{S}_q^3)^2} \cos \gamma \) and \( \tilde{S}_q^1 = \sqrt{(S_q)^2 - (\tilde{S}_q^3)^2} \sin \gamma \).
\[ H_{(rel)} \equiv M_{sys} = H_M(\infty) = \sum_{i=1}^{3} m_i^2 + 3 \sum_{a,b=1}^{2} \gamma_{ai} \gamma_{bi} \hat{p}_{qa} \cdot \hat{p}_{qb} = \]
\[ = \sum_{i=1}^{3} \left( m_i^2 + \frac{3}{N^2} \left[ \frac{(\gamma_{1i})^2}{2\rho_{q1}^2} + \frac{(\gamma_{2i})^2}{2\rho_{q2}^2} + \frac{\gamma_{1i}\gamma_{2i}}{\rho_{q1}\rho_{q2}} \right] (\hat{S}^1_i)^2 + \right. \]
\[ + 3 \left[ \frac{(\gamma_{1i})^2}{2\rho_{q1}^2} + \frac{(\gamma_{2i})^2}{2\rho_{q2}^2} + \frac{\gamma_{1i}\gamma_{2i}(2N^2 - 1)}{\rho_{q1}\rho_{q2}} \right] (\hat{S}^2_i)^2 + \right. \]
\[ + \left. \frac{3}{1 - N^2} \left[ \frac{(\gamma_{1i})^2}{2\rho_{q1}^2} + \frac{(\gamma_{2i})^2}{2\rho_{q2}^2} - \frac{\gamma_{1i}\gamma_{2i}}{\rho_{q1}\rho_{q2}} \right] (\hat{S}^3_i)^2 + \right. \]
\[ + 3\sqrt{1 - N^2} \left[ \frac{\gamma_{1i}^2}{4\rho_{q1}^2} - \frac{(\gamma_{2i})^2}{4\rho_{q2}^2} \right] + 4\gamma_{1i}\gamma_{2i} |\hat{N}| \sqrt{1 - N^2} \left( \hat{\pi}_{q1} - \hat{\pi}_{q2} \right) \right) \hat{S}^2_i - \]
\[ - \frac{3}{|\hat{N}| \sqrt{1 - N^2}} \left[ \frac{(\gamma_{1i})^2}{2\rho_{q1}^2} - \frac{(\gamma_{2i})^2}{2\rho_{q2}^2} \right] \hat{S}^1_i \hat{S}^3_i + \]
\[ + 6(\gamma_{1i})^2 \left( \frac{\hat{\pi}_{q1}^2 + \frac{\xi^2(1 - N^2)}{4\rho_{q1}^2}}{4\rho_{q1}^2} \right) + 6(\gamma_{2i})^2 \left( \frac{\hat{\pi}_{q2}^2 + \frac{\xi^2(1 - N^2)}{4\rho_{q2}^2}}{4\rho_{q2}^2} \right) \right) + \]
\[ + 12\gamma_{1i}\gamma_{2i} \left[ \frac{(2N^2 - 1)\hat{\pi}_{q1} \hat{\pi}_{q2} - |\hat{N}|(1 - N^2)\hat{\xi}(\frac{\hat{\pi}_{q1}}{\rho_{q2}} + \frac{\hat{\pi}_{q2}}{\rho_{q1}}) + \right] \right. \]
\[ + \frac{\xi^2(1 - N^2)(2N^2 - 1)}{4\rho_{q1}\rho_{q2}} \right) \right)^{1/2} = \]
\[ = \sum_{i=1}^{3} H_{(rel)i}, \quad (7.29) \]

where \( q^\mu = (\rho_{q1}, \rho_{q2}, |\hat{N}|) \), \( p^\mu = (\hat{\pi}_{q1}, \hat{\pi}_{q2}, \hat{\xi}) \) are the dynamical shape variables.

By using the results of Appendix E, Eq (E4), we can put the Hamiltonian in a form reminiscent of the non-relativistic orientation-shape bundle approach \(^{46}\)

\[ H_{(rel)} = \sum_{i=1}^{3} H_{(rel)i} = \]
\[ = \sum_{i=1}^{3} \sqrt{m_i^2 + \tilde{T}_i^{-1rs}(q) \tilde{S}_q^r \tilde{S}_q^s + \tilde{u}_{i}^{\mu\nu}(q) (p_\mu - \tilde{S}_q \cdot \tilde{A}_i \mu(q)) \left( p_\nu - \tilde{S}_q \cdot \tilde{A}_i \nu(q) \right) \left( p_\mu - \tilde{S}_q \cdot \tilde{A}_i \mu(q) \right) \left( p_\nu - \tilde{S}_q \cdot \tilde{A}_i \nu(q) \right) \right) \quad (7.30) \]

A purely rotational (vertical) Hamiltonian is \( H_{(rel)}^{(rot)} = H_{(rel)} \big|_{q=0} = H_{(rel)} \big|_{p_\mu = \tilde{S}_q \cdot \tilde{A}_i \mu(q)} \).

\(^{46}\)The N=3 non-relativistic Hamiltonian is \( H_{rel} = \frac{1}{2} \left[ \left( \tilde{T}^{-1}(q) \right)^{rs} \tilde{S}_q^r \tilde{S}_q^s + \tilde{g}^{\mu\nu}(q) (p_\mu - \tilde{S}_q \cdot \tilde{A}_i \mu(q)) (p_\nu - \tilde{S}_q \cdot \tilde{A}_i \nu(q)) \right] \), with the quantities \( \tilde{A}_i \mu(q) \), \( \tilde{g}^{\mu\nu}(q) \), \( \tilde{T}^{-1}(q) \) evaluated in Appendix E of Ref. [2]. While the purely rotational Hamiltonian (defined by \( \hat{q}^\mu = 0 \) implying \( p_\mu = \tilde{S}_q \cdot \tilde{A}_i \mu(q) \)) is \( H_{rel}^{(rot)} = \frac{1}{2} \left( \tilde{T}^{-1}(q) \right)^{rs} \tilde{S}_q^r \tilde{S}_q^s \), the purely vibrational Hamiltonian \( H_{rel}^{(vib)} \) is defined in our approach by the requirement \( \tilde{\omega} = 0 \). Since \( \tilde{S}_q \big|_{\tilde{\omega} = 0} \neq 0 \), unlike in the static orientation-shape bundle approach we have \( H_{rel} \neq H_{rel}^{(rot)} + H_{rel}^{(vib)} \).
since under each square root there is a different gauge potential \( \tilde{\mathcal{A}}_{\mu}(q) \) we get that under each square root there is the mass term plus a quadratic expression in the body spin but with coefficients depending on the shape variable and on \( \tilde{S}_q \) itself, since \([p_\mu - \tilde{S}_q \cdot \tilde{\mathcal{A}}_{\mu}(q)]|_{q=0} = \tilde{S}_q : [\tilde{\mathcal{C}}_{\mu}(\tilde{S}_q, q) - \tilde{\mathcal{A}}_{\mu}(q)]\). Therefore, we have

\[
H^{(rot)}_{\text{(rel)}} = H^{(rot)}_{\text{(rel)}}(\tilde{S}_q, q) = \sum_{i=1}^{N} \sqrt{m_i^2 + \left[ \mathcal{T}_{i}^{-1rs}(q) + \tilde{v}_i^{\mu\nu}(q) \left( \tilde{\mathcal{C}}_{\mu}^r(\tilde{S}_q, q) - \tilde{\mathcal{A}}_{\mu}^r(q) \right) \left( \tilde{\mathcal{C}}_{\nu}^s(\tilde{S}_q, q) - \tilde{\mathcal{A}}_{\nu}^s(q) \right) \right] \tilde{S}_q^r \tilde{S}_q^s}.
\]

(7.31)

In the non-relativistic limit (where \( p_\mu = \tilde{S}_q \cdot \tilde{\mathcal{A}}_{\mu}(q) \) with \( \tilde{\mathcal{A}}_{\mu}(q) \) the non-relativistic gauge potential) we get

\[
H^{(S)}_{\text{(rel)}} \rightarrow c^{-\infty} \sum_{i=1}^{N} m_i c^2 + \frac{1}{2} \sum_{i=1}^{N} \frac{m_i}{m} \lim_{c \rightarrow \infty} \left[ \mathcal{T}_{i}^{-1rs}(q) + \tilde{v}_i^{\mu\nu}(q) \left( \tilde{\mathcal{C}}_{\mu}^r(\tilde{S}_q, q) - \tilde{\mathcal{A}}_{\mu}^r(q) \right) \left( \tilde{\mathcal{C}}_{\nu}^s(\tilde{S}_q, q) - \tilde{\mathcal{A}}_{\nu}^s(q) \right) \right] \tilde{S}_q^r \tilde{S}_q^s + O(1/c),
\]

(7.32)

so that the non-relativistic inverse tensor of inertia is recovered as

\[
\mathcal{I}^{-1rs}(q) = \lim_{c \rightarrow \infty} \left( \sum_{i=1}^{N} \frac{1}{m_i} \left[ \mathcal{T}_{i}^{-1rs}(q) + \tilde{v}_i^{\mu\nu}(q) \left( \tilde{\mathcal{C}}_{\mu}^r(\tilde{S}_q, q) - \tilde{\mathcal{A}}_{\mu}^r(q) \right) \left( \tilde{\mathcal{C}}_{\nu}^s(\tilde{S}_q, q) - \tilde{\mathcal{A}}_{\nu}^s(q) \right) \right] \right).
\]

(7.33)

These results together with

\[
H_{\text{(rel)}} - \sum_{i=1}^{3} m_i \rightarrow c^{-\infty} \frac{1}{2} \sum_{i=1}^{N} \left( \mathcal{T}_{i}^{-1rs}(q) \right) \tilde{S}_q^r \tilde{S}_q^s + \frac{\tilde{v}_i^{\mu\nu}(q)}{m_i} \left( p_\mu - \tilde{S}_q \cdot \tilde{\mathcal{A}}_{\mu}(q) \right) \left( p_\nu - \tilde{S}_q \cdot \tilde{\mathcal{A}}_{\nu}(q) \right) = \frac{1}{2} \left( (\mathcal{I}^{-1}(q))^{rs} \tilde{S}_q^r \tilde{S}_q^s + g^{\mu\nu}(q) (p_\mu - \tilde{S}_q \cdot \tilde{\mathcal{A}}_{\mu}(q)) (p_\nu - \tilde{S}_q \cdot \tilde{\mathcal{A}}_{\nu}(q)) \right).
\]

(7.34)

imply that the non-relativistic gauge potential and metric are

\[
\tilde{\mathcal{A}}_{\mu}(q) = \lim_{c \rightarrow \infty} \tilde{\mathcal{C}}_{\mu}(\tilde{S}_q, q),
\]

\[
g^{\mu\nu}(q) = \lim_{c \rightarrow \infty} \sum_{i=1}^{N} \frac{\tilde{v}_i^{\mu\nu}(q)}{m_i}.
\]

(7.35)

A purely vibrational Hamiltonian \( H^{(vib)}_{\text{(rel)}} \) can be defined by requiring the vanishing of the (now measurable) body frame components of the angular velocity \( \tilde{\omega}^r = 0 \). These conditions
transform Eq. (E9) in equations for the determination of $\hat{S}_q^r|_{\omega^*=0}$: if we put their solution into Eq. (7.30), we get $H_{\text{(rel)}}$. 

On the other hand, the orientation-shape bundle approach privileges the gauge choice $\hat{S}_q^r = 0$ (special connection C quoted in Ref. [2]) to define a purely vibrational (C-horizontal) Hamiltonian

$$H_{\text{(rel)}}^{(S=0)} = H_{\text{(rel)}}|_{\hat{S}_q^r} = \sum_{i=1}^{N} \sqrt{m_i^2 + \tilde{v}_i^{\mu\nu}(q)p_{\mu}p_{\nu}},$$

(7.36)

with the non-relativistic limit

$$H_{\text{(rel)}}^{(S=0)} \to_{c \to \infty} \sum_{i=1}^{N} m_i c^2 + \frac{1}{2} \sum_{i=1}^{N} \tilde{v}_i^{\mu\nu}(q)p_{\mu}p_{\nu} + O(1/c) = \sum_{i=1}^{N} m_i c^2 + \frac{1}{2} \tilde{\Sigma}^{\mu\nu}(q)p_{\mu}p_{\nu} + O(1/c).$$

(7.37)

However, we cannot use this definition because our canonical construction is valid only if $S_q^r \neq 0$.

### C. N-Body Systems.

Let us now consider the general case with $N \geq 4$. Instead of coupling the centers of mass of particle clusters as it is done with Jacobi coordinates (center-of-mass clusters), the canonical spin bases will be obtained by coupling the spins of the 2-body subsystems (relative particles) $\tilde{\rho}_{qa}, \tilde{\pi}_{qa}$, $a = 1, .., N - 1$, defined in Eqs. (3.7), in all possible ways (spin clusters from the addition of angular momenta). Let us stress that we can build a spin basis with a pattern of spin clusters which is completely unrelated to a possible pre-existing center-of-mass clustering.

Let us consider the case $N = 4$ as a prototype of the general construction. We have now three relative variables $\tilde{\rho}_{q1}, \tilde{\rho}_{q2}, \tilde{\rho}_{q3}$ and related momenta $\tilde{\pi}_{q1}, \tilde{\pi}_{q2}, \tilde{\pi}_{q3}$. In the following formulas we use the convention that the subscripts $a, b, c$ mean any permutation of 1, 2, 3.

As in Ref. [3], we define the following sequence of canonical transformations (we assume $S_q \neq 0$; $S_qA \neq 0$, $A = a, b, c$) corresponding to the spin clustering pattern $abc \leftrightarrow (ab)c \leftrightarrow ((ab)c)$ [build first the spin cluster $(ab)$, then the spin cluster $((ab)c)$]:
See Appendix F of Ref. [2] for the explicit construction of the canonical transformations.

The first non-point canonical transformation is based on the existence of the three unit vectors \( \hat{R}_A, A = a, b, c \), and of three \( \text{E}(3) \) groups with fixed values \( (R_A^2 = 1, \mathbf{S}_A \cdot \hat{R}_A = 0) \) of their invariants. One uses Eqs. (7.13), (7.14) and (7.15).

In the next canonical transformation the spins of the relative particles \( a \) and \( b \) are coupled to form the spin cluster \( (ab) \), leaving the relative particle \( c \) as a spectator. We use the definitions \( \tilde{N}_{(ab)} = \frac{1}{2}(\tilde{R}_a + \tilde{R}_b), \tilde{\chi}_{(ab)} = \frac{1}{2}(\tilde{R}_a - \tilde{R}_b), \mathbf{S}_{(ab)} = \tilde{S}_{qa} + \tilde{S}_{qb}, \tilde{W}_{q(ab)} = \tilde{S}_{qa} - \tilde{S}_{qb} \). We get \( \tilde{N}_{(ab)} \cdot \tilde{\chi}_{(ab)} = 0 \), \( \{N^r_{(ab)}, N^s_{(ab)}\} = \{N^r_{(ab)} \cdot \chi^s_{(ab)}\} = \{\chi^r_{(ab)}, \chi^s_{(ab)}\} = 0 \) and a new \( \text{E}(3) \) realization generated by \( \mathbf{S}_{(ab)} \) and \( \tilde{N}_{(ab)} \), with non-fixed invariants \( |\tilde{N}_{(ab)}|, \tilde{S}_{(ab)} \cdot \tilde{N}_{(ab)} = \Omega_{(ab)} \).

From Eqs. (7.23) we get

\[
\tilde{\rho}_{qa} = \rho_{qa} \left| \tilde{N}_{(ab)} \right| \tilde{N}_{(ab)} + \sqrt{1 - \tilde{N}_{(ab)}^2} \tilde{\chi}_{(ab)} ,
\tilde{\rho}_{qb} = \rho_{qb} \left| \tilde{N}_{(ab)} \right| \tilde{N}_{(ab)} - \sqrt{1 - \tilde{N}_{(ab)}^2} \tilde{\chi}_{(ab)} ,
\tilde{\rho}_{qc} = \rho_{qc} \hat{R}_c.
\]

Eq. (7.19) defines \( \alpha_{(ab)} \) and \( \beta_{(ab)} \), so that Eq. (7.20) defines a unit vector \( \hat{R}_{(ab)} \) with \( \mathbf{S}_{(ab)} \cdot \hat{R}_{(ab)} = 0, \{\hat{R}_{(ab)}^r, \hat{R}_{(ab)}^s\} = 0 \). This unit vector identifies the spin cluster \( (ab) \) in the same way as the unit vectors \( R_A = \hat{\rho}_{qA} \) identify the relative particles \( A \).

The next step is the coupling of the spin cluster \( (ab) \) with unit vector \( \hat{R}_{(ab)} \) [described by the canonical variables \( \alpha_{(ab)}, S_{(ab)}, \beta_{(ab)}, S^3_{(ab)} \)] with the relative particle \( c \) with unit vector \( \hat{R}_c \) and described by \( \alpha_c, S_{qc}, \beta_c, S^3_{qc} \); this builds the spin cluster \( ((ab)c) \). Again Eq. (7.16) allows to define \( \tilde{N}_{((ab)c)} = \frac{1}{2}(\tilde{R}_{(ab)} + \tilde{R}_c), \tilde{\chi}_{((ab)c)} = \frac{1}{2}(\tilde{R}_{(ab)} - \tilde{R}_c), \mathbf{S}_{((ab)c)} = \tilde{S}_{q(ab)c} + \tilde{S}_{qc}, \tilde{W}_{q((ab)c)} = \tilde{S}_{qa((ab)c)} - \tilde{S}_{qc} \). Since we have \( \tilde{N}_{((ab)c)} \cdot \tilde{\chi}_{((ab)c)} = 0 \) and \( \{N^r_{((ab)c)}, N^s_{((ab)c)}\} = \{\chi^r_{((ab)c)}, \chi^s_{((ab)c)}\} = 0 \) due to \( \{\hat{R}_{(ab)}^r, \hat{R}_c^s\} = 0 \), a new \( \text{E}(3) \) group generated by \( \tilde{S}_q \) and \( \tilde{N}_{((ab)c)} \) with non-fixed invariants \( |\tilde{N}_{((ab)c)}|, \tilde{S}_q \cdot \tilde{N}_{((ab)c)} = \tilde{S}_q^3 |\tilde{N}_{((ab)c)}| \) emerges.

Eq. (7.19) defines \( \alpha_{((ab)c)} \) and \( \beta_{((ab)c)} \), so that Eq. (7.20) allows to identify a final unit vector \( \hat{R}_{((ab)c)} \) with \( \tilde{S}_q \cdot \hat{R}_{((ab)c)} = 0 \) and \( \{\hat{R}_{((ab)c)}^r, \hat{R}_{((ab)c)}^s\} = 0 \).

In conclusion, when \( S_q \neq 0 \), we find both a dynamical body frame \( \tilde{\chi}_{((ab)c)}, \hat{N}_{((ab)c)} \times \tilde{\chi}_{((ab)c)}, \hat{N}_{((ab)c)} \), and a spin frame \( \tilde{S}_q, \hat{R}_{((ab)c)}, \hat{R}_{((ab)c)} \times \tilde{S}_q \) like in the 3-body case. There is an important difference, however: the orthonormal vectors \( \tilde{N}_{((ab)c)} \) and \( \tilde{\chi}_{((ab)c)} \) depend on the momenta of the relative particles \( a \) and \( b \) through \( \tilde{R}_{(ab)} \), so that our results do not share any relation with the N=4 non-trivial SO(3) principal bundle of the orientation-shape bundle approach.

The final 6 dynamical shape variables are \( q^\mu = \{ |\tilde{N}_{((ab)c)}|, \gamma_{(ab)}, |\tilde{N}_{(ab)}|, \rho_{qa}, \rho_{qb}, \rho_{qc} \} \). While the last four depend only on the original relative coordinates \( \tilde{\rho}_{qA} \), \( A = a, b, c \), the first two depend also on the original momenta \( \tilde{\rho}_{qA} \); therefore they are generalized shape variables. By using Appendix D, we obtain

\[
N_e \to \begin{pmatrix} \hat{\alpha} \ 
\hat{\beta} \ 
\hat{\gamma} \ | \tilde{N}((ab)c) \ | \ 
\gamma_{(ab)} \ | \tilde{N}_{(ab)} \ | \rho_{qa} \ 
\rho_{qb} \ 
\rho_{qc} \ 
\end{pmatrix}.
\]

(7.38)
\[
\rho_{qA}^r = \mathcal{R}^r_s(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \rho_{qA}^s(q^\mu, p_\mu, \tilde{S}_q^r), \quad A = a, b, c,
\]

(7.40)

This means that for N=4 the dynamical body frame components \(\rho_{qA}^r\) depend also on the dynamical shape momenta and on the dynamical body frame components of the spin. It is clear that this result stands outside the orientation-shape bundle approach completely.

As shown in Appendix F of Ref. [2], starting from the Hamiltonian \(H_{\text{rel}}((ab)(cd))\) in the final variables, we can define a rotational Hamiltonian \(H_{\text{rot}}((ab)(cd))\) (for \(\dot{q}^\mu = 0\), see Eqs.(F18) of Ref. [2]) and a vibrational Hamiltonian \(H_{\text{vib}}((ab)(cd))\) (vanishing of the physical dynamical angular velocity \(\dot{\omega}_{(ab)(cd)}^r = 0\), see Eqs.(F21) of Ref. [2]), but \(H_{\text{rel}}((ab)(cd))\) is not the sum of these two Hamiltonians. In the rotational Hamiltonian and in the spin-angular velocity relation we find two inertia-like tensors depending only on the dynamical shape variables.

The price to be paid for the existence of 3 global dynamical body frames for N=4 is a more complicated form of the Hamiltonian kinetic energy. On the other hand, dynamical vibrations and dynamical angular velocity are measurable quantities in each dynamical body frame.

For N=5 we can repeat the previous construction either with the sequence of spin clusterings \(abcd \mapsto (ab)cd \mapsto ((ab)c)d \mapsto (((ab)c)d)\) or with the sequence \(abcd \mapsto (ab)(cd) \mapsto ((ab)(cd))\) [a, b, c, d any permutation of 1, 2, 3, 4] as said in the Introduction. Each spin cluster (...) will be identified by the unit vector \(\vec{R}_{(\ldots)}\), axis of the spin frame of the cluster. All the final dynamical body frames built with this construction will have their axes depending on both the original configurations and momenta.

This construction is trivially generalized to any N: we have only to classify all the possible spin clustering patterns.

Therefore, for \(N \geq 4\) our sequence of canonical and non-canonical transformations leads to the following result to be compared with Eq.(7.23) of the 3-body case

\[
\begin{array}{cccccc}
\dot{\rho}_{qA} & \dot{\pi}_{qA} \\
\rho_{qA} & \pi_{qA} \\
\end{array}
\xrightarrow{\text{non-con.}}
\begin{array}{cccc}
\tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} & q^\mu(\rho_{qA}, \pi_{qA}) \\
S_q^1 & S_q^2 & S_q^3 & p_\mu(\rho_{qA}, \pi_{qA}) \\
\end{array}
\]

(7.41)

This state of affairs suggests that for \(N \geq 4\) and with \(S_q \neq 0\), \(S_{qA} \neq 0\), \(A = a, b, c\), viz. when the standard (3N-3)-dimensional orientation-shape bundle is not trivial, the original (6N-6)-dimensional relative phase space admits the definition of as many dynamical body frames as spin canonical bases [47] which are globally defined (apart isolated coordinate singularities) for the non-singular N-body configurations with \(\tilde{S}_q \neq 0\) (and with non-zero spin for each spin subcluster).

These dynamical body frames do not correspond to local cross sections of the static non-trivial orientation-shape SO(3) principal bundle and the spin canonical bases do not coincide with the canonical bases associated with the static theory. Therefore, we do not get gauge potentials and all the other quantities evaluated in Appendix E for N=3.

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[47] Recall that a different Hamiltonian right SO(3) action on the relative phase space corresponds to each of them.
VIII. THE CASE OF INTERACTING PARTICLES.

As shown in Ref. \[20\] and in its bibliography, the action-at-a-distance interactions inside the Wigner hyperplane may be introduced either under the square roots (scalar and vector potentials) or outside (scalar potential like the Coulomb one) appearing in the free Hamiltonian (2.19) or (6.1).

In the rest-frame instant form the most general Hamiltonian with only action-at-a-distance interactions is

\[
H = \sum_{i=1}^{N} \sqrt{m_i^2 + U_i + [\vec{k}_i - \vec{V}_i]^2} + V, \tag{8.1}
\]

with

\[
U_i = U(\vec{k}_i, \vec{r}_i - \vec{r}_k), \quad \vec{V}_i = \vec{V}_i(\vec{k}_j \neq \vec{k}_i, \vec{r}_i - \vec{r}_j), \quad V = V_o(|\vec{r}_i - \vec{r}_j|) + V'(\vec{k}_i, \vec{r}_i - \vec{r}_j).
\]

In the rest frame the Hamiltonian for the relative motion becomes

\[
H_{(rel)} = \sum_{i=1}^{N} \left[ m_i^2 + \tilde{U}_i + \left[ \sqrt{n} \sum_{a=1}^{N-1} \gamma_{ai} \tilde{r}_{qa} - \tilde{V}_i \right]^2 \right] + \tilde{V}, \tag{8.2}
\]

with

\[
\tilde{U}_i = U \left( \sqrt{n} \sum_{a=1}^{N-1} \gamma_{ak} \tilde{r}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ah} - \gamma_{ak}) \tilde{p}_{qa} \right),
\]

\[
\tilde{V}_i = \tilde{V}_i \left( \left[ \sqrt{n} \sum_{a=1}^{N-1} \gamma_{aj} \tilde{r}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj}) \tilde{p}_{qa} \right] \right),
\]

\[
\tilde{V} = V_o \left( \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj}) \tilde{p}_{qa} \right) + V' \left( \left[ \sqrt{n} \sum_{a=1}^{N-1} \gamma_{ai} \tilde{r}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj}) \tilde{p}_{qa} \right] \right). \tag{8.3}
\]

The prices for the existence of 3 possible global dynamical body frames for N=4 are:

i) a more complicated form of the Hamiltonian kinetic energy but with a definition of measurable dynamical vibrations and dynamical angular velocity in each dynamical body frame;

ii) the fact that a potential \( V(\vec{r}_{ij} \cdot \vec{r}_{hk}) \) with \( \vec{r}_{ij} = \vec{r}_i - \vec{r}_j \) becomes dependent also on the shape momenta, since we have

\[
V(\vec{r}_{ij} \cdot \vec{r}_{hk}) = V \left( \frac{1}{N} \sum_{a,b}^{N-1} (\Gamma_{ai} - \Gamma_{aj})(\Gamma_{bh} - \Gamma_{bk}) \tilde{p}_{qa} \cdot \tilde{p}_{qb} \right). \tag{8.4}
\]

For N=4, due to Eq.(E6), in the pattern ((ab)c) we have

\[
V = \tilde{V}_{((ab)c)} \left[ \tilde{p}_{qa}, \tilde{p}_{qb}, |N_{((ab)c)}|, \gamma_{(ab)}, |\tilde{N}_{(ab)c)}|, \xi_{((ab)c)}, \Omega_{(ab)}; \tilde{S}_{q} \right]. \tag{8.5}
\]

For more general potentials \( V(\vec{r}_{ij} \cdot \vec{r}_{hk}, \vec{k}_i \cdot \vec{r}_{hk}, \vec{k}_i \cdot \vec{k}_j) \), like the non-relativistic limit of the relativistic Darwin potential of Ref. \[20\], more complicated expressions are obtained.
IX. CONCLUSIONS.

In this paper we have explored the relativistic kinematics of a system of N scalar positive-energy particles. In the framework of the rest-frame instant form of dynamics it is possible to find the relativistic extension of the Abelian translational and non-Abelian rotational symmetries whose associated Noether constants of motion are fundamental for the study of isolated systems. In the relativistic case the rest-frame description on the Wigner hyperplanes allows to clarify all the problems by virtue of a doubling of all the concepts: they can be either external (namely observed by an arbitrary inertial Lorentz frame) or internal (namely observed by an inertial observer at rest inside the Wigner hyperplane). Correspondingly two realizations of the Poincaré algebra are naturally defined.

After a clarification of the possible external and internal definitions of relativistic center of mass, we have shown that it is possible to define a family of canonical transformations for the definition of canonical relative variables. The Hamiltonian in the rest frame can be expressed in terms of these variables. It turns out that, due to the presence of the square roots in the Hamiltonian, the non-relativistic concepts of Jacobi normal relative coordinates, reduced masses and barycentric tensor of inertia cannot be extended to the relativistic formulation.

On the other hand, the rest-frame description with the Wigner hyperplanes allows to use the non-relativistic formalism developed in Ref. [2] for the study of the rotational kinematics, since it it is independent of Jacobi coordinates. Therefore, we can extend the concepts of canonical spin bases, spin frames and dynamical body frames to the relativistic level and we find again that, due to the non-Abelian nature of rotations, a global separation of rotations from vibrations is not possible.

In a future paper [28] we will conclude the study of relativistic kinematics by defining Dixon’s multipoles [27] for the relativistic N-body problem in the rest-frame instant form of dynamics. It will be shown that, in this framework, we can recover concepts like the tensor of inertia by using the quadrupole moment.

The final task should be the extension of all these results to relativistic extended (continua) isolated systems.
APPENDIX A: PARAMETRIZED MINKOWSKI THEORIES.

In this Appendix we review the main aspects of parametrized Minkowski theories and of the canonical reduction of gauge systems, following Refs. [7,20,29], where a complete treatment of N scalar charged positive energy particles plus the electromagnetic field is given.

The starting point was Dirac’s [9] reformulation of classical field theory on spacelike hypersurfaces foliating Minkowski spacetime $M^4$. The foliation is defined by an embedding $R \times \Sigma \rightarrow M^4$, $(\tau, \vec{\sigma}) \mapsto z^\mu(\tau, \vec{\sigma})$, with $\Sigma$ an abstract 3-surface diffeomorphic to $R^3$. In this way one gets a parametrized field theory with a covariant 3+1 splitting of flat spacetime, which is already in a form suited to the transition to general relativity in its ADM canonical formulation.

The price to be paid is that one has to add the embeddings $z^\mu(\tau, \vec{\sigma})$ identifying the points of the spacelike hypersurface $\Sigma_\tau$ as new configuration variables and then to redefine the fields on $\Sigma_\tau$ in such a way they know the whole hypersurface $\Sigma_\tau$ of $\tau$-simultaneity.

Then one rewrites the Lagrangian of the given isolated system in the form required by the coupling to an external gravitational field, makes the 3+1 splitting of Minkowski spacetime and replaces all the fields of the system by the new fields on $\Sigma_\tau$. Instead of considering the 4-metric as describing a gravitational field, here one replaces the 4-metric with the induced metric $g^{AB}[z] = z^A_{\mu} \eta_{\mu\nu} z^B_{\nu}$ on $\Sigma_\tau$, a functional of $z^\mu$, and considers the embedding coordinates $z^\mu(\tau, \vec{\sigma})$ as independent fields. We use the notation $\sigma^A = (\tau, \sigma^i)$ of Refs. [7,20].

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The evolution vector is given by $z^\mu_\tau = N[z](\text{flat}) l^\mu + N[z](\text{flat}) z^\mu_\sigma$, where $l^\mu(\tau, \vec{\sigma})$ is the normal to $\Sigma_\tau$ in $z^\mu(\tau, \vec{\sigma})$ and

\begin{align}
N[z](\text{flat}) \tau(\tau, \vec{\sigma}) &= \sqrt{4 g_{\tau\tau} - 3 g_{\tau i} g_{\tau i}^4 g_{\tau s}^4} = \sqrt{4 g^{3/2}}, \\
N[z](\text{flat}) \sigma(\tau, \vec{\sigma}) &= 3 g_{\tau i}(\tau, \vec{\sigma}) N[z](\text{flat}) (\tau, \vec{\sigma}) = 4 g_{\tau\tau}, \quad \text{(A1)}
\end{align}

48 The only ones carrying Lorentz indices; the scalar parameter $\tau$ labels the leaves of the foliation and $\vec{\sigma}$ are curvilinear coordinates on $\Sigma_\tau$.

49 For a Klein-Gordon field $\phi(x)$, this new field is $\tilde{\phi}(\tau, \vec{\sigma}) = \phi(z(\tau, \vec{\sigma}))$: it contains the nonlocal information about the embedding and the associated notion of equal time.

50 These are Lorentz scalars, having only surface indices.

51 Namely as an independent field like in metric gravity, where one adds the Hilbert action to the action for the matter fields.

52 I.e. $\eta^{\mu\nu} = z^A_{\mu} g^{AB} z^\nu_B$ with $g^{AB}$ the inverse of $g_{AB}$.

53 Note that in metric gravity the $z^\mu_A \neq \partial z^\mu / \partial \sigma^A$ are not cotetrad fields since no holonomic coordinates $z^\mu(\sigma)$ exist.
are the flat lapse and shift functions defined through the metric like in metric gravity (here $3g^{\rho\pi}4g_{\delta\delta} = \delta_4^\delta$); however, in Minkowski spacetime they are functionals of $z^\mu(\tau, \vec{\sigma})$ instead of being independent variables. See Appendix B for notations on spacelike hypersurfaces.

From this Lagrangian for the isolated system we have that: i) the possible constraints of the system are Lorentz scalars; ii) four primary first class constraints are added which imply the independence of the description from the choice of the foliation with spacelike hypersurfaces:

$$H_\mu(\tau, \vec{\sigma}) = \rho_\mu(\tau, \vec{\sigma}) - l_\mu(\tau, \vec{\sigma})T^r_{\text{system}}(\tau, \vec{\sigma}) - z^\gamma_{\rho\mu}(\tau, \vec{\sigma})T^r_{\text{system}}(\tau, \vec{\sigma}) \approx 0. \ (A2)$$

Here $T^r_{\text{system}}(\tau, \vec{\sigma})$, $T^r_{\text{system}}(\tau, \vec{\sigma})$, are the components of the energy-momentum tensor in the holonomic coordinate system on $\Sigma$ corresponding to the energy- and momentum-density of the isolated system. These four constraints satisfy an Abelian Poisson algebra being solved in 4-momenta $\rho_\mu(\tau, \vec{\sigma})$ conjugate to the embedding variables $z^\mu(\tau, \vec{\sigma})$: $\{H_\mu(\tau, \vec{\sigma}), H_\nu(\tau, \vec{\sigma})\} = 0$.

We see that the embedding fields $z^\mu(\tau, \vec{\sigma})$ are the gauge variables associated with this kind of general covariance.

The Dirac Hamiltonian is

$$H_D = H_{(c)} + \int d^3\sigma\lambda^\mu(\tau, \vec{\sigma})H_\mu(\tau, \vec{\sigma}) + (\text{system-dependent primary constraints}), \ (A3)$$

with $\lambda^\mu(\tau, \vec{\sigma})$ arbitrary Dirac multipliers. By using $^4\eta^{\mu\nu} = [l^\mu l^\nu - z^\mu g^{rs}z^\nu](\tau, \vec{\sigma})$ we can write

$$\lambda_\mu(\tau, \vec{\sigma})H^\mu(\tau, \vec{\sigma}) = [(\lambda_\mu l^\mu)(l_\nu H^\nu) - (\lambda_\mu z^\mu g^{rs}z^\nu)](\tau, \vec{\sigma}) = \left[\left(\lambda_{\nu}l^{\nu}\right)(l_\mu H^\mu)(\tau, \vec{\sigma}) - N(\text{flat})_{\nu}(\tau, \vec{\sigma})(3g^{rs}z_\nu H^\nu)\right](\tau, \vec{\sigma}), \ (A4)$$

with the (non-holonomic form of the) constraints $\check{H}(\tau, \vec{\sigma}) = (l_\mu H^\mu)(\tau, \vec{\sigma}) \approx 0, \ \check{H}_\nu(\tau, \vec{\sigma}) = (z^{\gamma}_{\rho\mu}H^\mu)(\tau, \vec{\sigma}) \approx 0$, satisfying the universal Dirac algebra

$$\{\check{H}_r(\tau, \vec{\sigma}), \check{H}_s(\tau, \vec{\sigma}')\} = \check{H}_r(\tau, \vec{\sigma}')\frac{\partial\delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial\sigma^r} + \check{H}_s(\tau, \vec{\sigma})\frac{\partial\delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial\sigma^r},$$

$$\{\check{H}(\tau, \vec{\sigma}), \check{H}_r(\tau, \vec{\sigma}')\} = \check{H}(\tau, \vec{\sigma})\frac{\partial\delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial\sigma^r},$$

$$\{\check{H}(\tau, \vec{\sigma}), \check{H}(\tau, \vec{\sigma}')\} = [3g^{rs}(\tau, \vec{\sigma})\check{H}_s(\tau, \vec{\sigma}) + 3g^{rs}(\tau, \vec{\sigma}')\check{H}_s(\tau, \vec{\sigma}')\frac{\partial\delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial\sigma^r}. \ (A5)$$

In this way we have defined new flat lapse and shift functions.

---

54 See for instance Eq. (2.3) for the case of $N$ free scalar particles.

55 $H_{(c)}$ is the canonical part: it is either zero or weakly vanishing due to system-dependent secondary constraints.

57
\[
N_{(\text{flat})}(\tau, \vec{\sigma}) = \lambda_\mu(\tau, \vec{\sigma})l^\mu(\tau, \vec{\sigma}), \\
N_{(\text{flat})}\flat(\tau, \vec{\sigma}) = \lambda_\mu(\tau, \vec{\sigma})z^\mu(\tau, \vec{\sigma}).
\]

which have the same content of the arbitrary Dirac multipliers \(\lambda_\mu(\tau, \vec{\sigma})\), namely they multiply primary first class constraints satisfying the Dirac algebra. In Minkowski spacetime they are quite distinct from the previous lapse and shift functions \(N_{[z]}(\text{flat})\), \(N_{[z]}(\text{flat})\flat\), defined starting from the metric. Only by using the Hamilton equations \(z^\mu(\tau, \vec{\sigma}) = \{z^\mu(\tau, \vec{\sigma}), H_D\} = \lambda_\mu(\tau, \vec{\sigma})\) we get \(N_{[z]}(\text{flat})(\tau, \vec{\sigma}) \overset{\circ}{=} N_{(\text{flat})}(\tau, \vec{\sigma}), N_{[z]}(\text{flat})\flat(\tau, \vec{\sigma}) \overset{\circ}{=} N_{(\text{flat})}\flat(\tau, \vec{\sigma}).\)

Therefore, when we consider arbitrary 3+1 splittings of spacetime with arbitrary space-time duration variables, the descriptions of metric gravity plus matter and the parametrized Minkowski description of the same matter do not seem to follow the same pattern. The situation changes however if the allowed 3+1 splittings of spacetime in ADM metric gravity are restricted to have the leaves approaching Minkowski spacelike hyperplanes at spatial infinity and if parametrized Minkowski theories are restricted to spacelike hyperplanes.

The restriction of parametrized Minkowski theories to flat hyperplanes in Minkowski spacetime is done by adding the gauge-fixings \[\]
\[
z^\mu(\tau, \vec{\sigma}) - x^\mu(x) - b^\mu_\flat(\tau)\sigma^\flat \approx 0.
\]

(A7)

Here \(x^\mu_s(\tau)\) denotes a point on the hyperplane \(\Sigma_r\) chosen as an arbitrary origin; the \(b^\mu_\flat(\tau)\)'s form an orthonormal triad at \(x^\mu_s(\tau)\) and the \(\tau\)-independent normal to the family of spacelike hyperplanes is \(b^\mu = b^\mu_\flat = \epsilon^\mu_{\alpha\beta}b^\nu_\flat b^\beta_{\flat}(\tau)b^\alpha_{\flat}(\tau)\). Each hyperplane is described by 10 configuration variables, \(x^\mu_s(\tau)\) and the 6 independent degrees of freedom contained in the triad \(b^\mu_\flat(\tau)\), and by the 10 conjugate momenta: \(p^\mu_s = p^\mu_\flat, J^\mu_\nu = x^\mu_s p^\nu_\flat - x^\mu_\flat p^\nu_s + S^\mu_\nu\).

With these 20 canonical variables it is possible to build 10 Poincaré generators \(\bar{p}^\mu_s = p^\mu_s, J^\mu_\nu = x^\mu_s p^\nu_\flat - x^\nu_\flat p^\mu_s + S^\mu_\nu\).

After the restriction to spacelike hyperplanes, the piece \(\int d^3\sigma \lambda^\mu(\tau, \vec{\sigma})\mathcal{H}_\mu(\tau, \vec{\sigma})\) of the Dirac Hamiltonian is reduced to

\[
\tilde{\lambda}^\mu(\tau)\mathcal{H}_\mu(\tau) - \frac{1}{2} \tilde{\lambda}^{\mu\nu}(\tau)\mathcal{H}_{\mu\nu}(\tau),
\]

(A8)

because the time constancy of the gauge-fixings \(z^\mu(\tau, \vec{\sigma}) - x^\mu_s(\tau) - b^\mu_\flat(\tau)\sigma^\flat \approx 0\) implies \(\dot{\lambda}_\mu(\tau, \vec{\sigma}) = \tilde{\lambda}_\mu(\tau, \vec{\sigma}) + \tilde{\lambda}_{\mu\nu}(\tau)b^\nu_\flat\sigma^\flat\).

with

\[
\tilde{\lambda}^\mu(\tau) = -\dot{x}^\mu_s(\tau), \quad \tilde{\lambda}^{\mu\nu}(\tau) = -\dot{\tilde{\lambda}}^{\mu\nu}(\tau) = \frac{1}{2} \sum_\nu \dot{b}^\mu_\flat b^\nu_\flat - b^\nu_\flat \dot{b}^\mu_\flat(\tau).
\]

(A9)

Since at this stage we have \(z^\mu(\tau, \vec{\sigma}) \approx b^\mu_\flat(\tau)\), we get

\[
\begin{align*}
N_{[z]}(\text{flat})(\tau, \vec{\sigma}) &\approx N_{[z]}(\text{flat})(\tau, \vec{\sigma})l^\mu(\tau, \vec{\sigma}) + N_{[z]}(\text{flat})(\tau, \vec{\sigma})b^\mu_\flat(\tau, \vec{\sigma}) \\
&\approx \dot{x}^\mu_s(\tau) + \dot{b}^\mu_\flat(\tau)\sigma^\flat = -\tilde{\lambda}^\mu(\tau) - \tilde{\lambda}^{\mu\nu}(\tau)b_{\nu\tau}(\tau)\sigma^\flat.
\end{align*}
\]

(A10)

Only now we get the coincidence of the two definitions of flat lapse and shift functions independently from the equations of motion, i.e.
\[ N_{z(\text{flat})}(\tau, \vec{\sigma}) \approx N_{(\text{flat})}(\tau, \vec{\sigma}), \quad N_{z(\text{flat})r}(\tau, \vec{\sigma}) \approx N_{(\text{flat})\hat{r}}(\tau, \vec{\sigma}). \quad \text{(A11)} \]

The description on arbitrary foliations with spacelike hyperplanes is independent from the choice of the foliation, due to the remaining 10 first class constraints

\[
\begin{align*}
\mathcal{H}^\mu(\tau) &= \int d^3 \sigma \mathcal{H}^\mu(\tau, \vec{\sigma}) = p^\mu_s - p^\mu_{\text{sys}} = p^\mu_s - \\
&\quad - [\text{total momentum of the system inside the hyperplane}]^\mu \approx 0, \\
\tilde{\mathcal{H}}^{\mu\nu}(\tau) &= b^\mu_s(\tau) \int d^3 \sigma \sigma^\nu \mathcal{H}^\mu(\tau, \vec{\sigma}) - b^\nu_s(\tau) \int d^3 \sigma \sigma^\mu \mathcal{H}^\nu(\tau, \vec{\sigma}) = S^\mu_{s\nu} - S^{\mu\nu}_{\text{sys}} = \\
&\quad = S^\mu_{s\nu} - [\text{intrinsic angular momentum of the system inside the hyperplane}]^{\mu\nu} = S^\mu_{s\nu} - \\
&\quad - (b^\mu_s(\tau)l^\nu - b^\nu_s(\tau)l^\mu)[\text{boost part of system's angular momentum}]^{\mu\nu} - \\
&\quad - (b^\mu_s(\tau)b^\nu_s(\tau) - b^\nu_s(\tau)b^\mu_s(\tau)) [\text{spin part of system's angular momentum}]^{\mu\nu} \approx 0. \quad \text{(A12)}
\end{align*}
\]

Therefore, on spacelike hyperplanes in Minkowski spacetime we have

\[
N_{(\text{flat})}(\tau, \vec{\sigma}) = \lambda_\mu(\tau, \vec{\sigma})l^\mu(\tau, \vec{\sigma}) \mapsto \\
\mapsto N_{(\text{flat})}(\tau, \vec{\sigma}) = N_{z(\text{flat})}(\tau, \vec{\sigma}) = \\
= -\tilde{\lambda}_\mu(\tau)l^\mu - l^\mu \tilde{\lambda}_{\mu\nu}(\tau)b^\nu_s(\tau)\sigma^s = -\lambda(\tau) - \frac{1}{2}\lambda_{\tau^s}(\tau)\sigma^s, \\
N_{(\text{flat})r}(\tau, \vec{\sigma}) = \lambda_\mu(\tau, \vec{\sigma})z^\mu_r(\tau, \vec{\sigma}) \mapsto \\
\mapsto N_{(\text{flat})r}(\tau, \vec{\sigma}) = N_{z(\text{flat})r}(\tau, \vec{\sigma}) = \\
= -\tilde{\lambda}_\mu(\tau)b^\mu_r - b^\mu_r(\tau)\tilde{\lambda}_{\mu\nu}(\nu)(\tau)b^\nu_s(\tau)\sigma^s = -\lambda(\tau) - \frac{1}{2}\lambda_{\tau^s}(\tau)\sigma^s, \\
\lambda_A(\tau) = \tilde{\lambda}_\mu(\tau)b^\mu_A(\tau), \quad \tilde{\lambda}_\mu(\tau) = b^\mu_A(\tau)\lambda_A(\tau), \\
\lambda_{AB}(\tau) = \tilde{\lambda}_{\mu\nu}(\tau)[b^\mu_{A\nu}b^\nu_A - b^\mu_{B\nu}b^\nu_B](\tau) = 2[\tilde{\lambda}_{\mu\nu}b^\mu_Ab^\nu_B](\tau), \\
\tilde{\lambda}_{\mu\nu}(\tau) = \frac{1}{4}[b^\mu_Ab^\nu_B - b^\mu_Bb^\nu_A](\tau)\lambda_{AB}(\tau) = \\
= \frac{1}{2}[b^\mu_Ab^\nu_B\lambda_{AB}](\tau). \quad \text{(A13)}
\]

This is the main difference of the present approach from the treatment of parametrized Minkowski theories given in standard references: there, no configuration action is defined but only a phase space action, in which people use, wrongly, \(N_{z(\text{flat})}, N_{z(\text{flat})r}\) instead of \(N_{(\text{flat})}\), \(N_{(\text{flat})r}\) not only on spacelike hyperplanes but also on arbitrary spacelike hypersurfaces.

At this stage the embedding canonical variables \(z^\mu(\tau, \vec{\sigma}), \rho_\mu(\tau, \vec{\sigma})\) are reduced to:

i) \(x_s^\mu(\tau), p_s^\mu\) with \(\{x_s^\mu, p_s^\mu\} = -4\eta^\mu\nu\), parametrizing the arbitrary origin of the coordinates on the family of spacelike hyperplanes. The four constraints \(\mathcal{H}^\mu(\tau) \approx p_s^\mu - p^\mu_{\text{sys}} \approx 0\) mean
that \( p_s^\mu \) is determined by the 4-momentum of the isolated system.

ii) \( b_A^\mu(\tau) \) \( \square \) and \( S_s^{\mu\nu} = -S_s^{\nu\mu} \), with the orthonormality constraints \( b_A^\mu 4\eta_{\mu\nu}b_B^\nu = 4\eta_{AB} \). The non-vanishing Dirac brackets enforcing the orthonormality constraints \( \square \) \( \square \) for the \( b_A^\mu \)'s are

\[
\begin{align*}
\{ b_A^\mu, S_s^{\mu\nu} \} &= 4\eta^{\rho\mu}b_A^\nu - 4\eta^{\rho\nu}b_A^\mu, \\
\{ S_s^{\mu\nu}, S_s^{\alpha\beta} \} &= C^{\mu\nu\alpha\beta}_s S_s^{\gamma\delta},
\end{align*}
\]

(A14)

with \( C^{\mu\nu\alpha\beta}_s \) the structure constants of the Lorentz algebra.

Then one has that \( p_s^\mu, J_s^{\mu\nu} = x_s^\mu p_s^\nu - x_s^\nu p_s^\mu + S_s^{\mu\nu}, \) satisfy the algebra of the Poincaré group, with \( S_s^{\mu\nu} \) playing the role of the spin tensor. The other six constraints \( \mathcal{H}^{\mu\nu}(\tau) \approx S_s^{\mu\nu} - S_{sys}^{\mu\nu} \approx 0 \) mean that \( S_s^{\mu\nu} \) coincides with the spin tensor of the isolated system.

For the velocity of the origin \( x_s^\mu(\tau) \) we get

\[
\begin{align*}
\dot{x}_s^\mu(\tau) &\overset{\Delta}{=} \{ x_s^\mu(\tau), H_D \} = \tilde{\lambda}_\nu(\tau) \{ x_s^\mu(\tau), \mathcal{H}^{\nu\mu}(\tau) \} = \\
&= -\tilde{\lambda}^\mu(\tau) = [u^\mu(p_s)u^{\nu}(p_s) - \epsilon^\nu_r(u(p_s))\epsilon^\nu_r(u(p_s))]\dot{x}_s^{\nu}(\tau) = \\
&= -u^\mu(p_s)\lambda(\tau) + \epsilon^\mu_r(u(p_s))\lambda_r(\tau),
\end{align*}
\]

\( \dot{x}_s^2(\tau) = \lambda^2(\tau) - \tilde{\lambda}^2(\tau) > 0, \quad \dot{x}_s \cdot u(p_s) = -\lambda(\tau), \)

\[
\begin{align*}
U_s^\mu(\tau) &= \frac{\dot{x}_s^\mu(\tau)}{\sqrt{\dot{x}_s^2(\tau)}} = -\lambda(\tau)u^\mu(p_s) + \lambda_r(\tau)\epsilon^\mu_r(u(p_s)),
\end{align*}
\]

\( \Rightarrow \)

\[
\dot{x}_s^\mu(\tau) = x_s^\mu(0) - u^\mu(p_s)\int_0^\tau d\tau_1\lambda(\tau_1) + \epsilon^\mu_r(u(p_s))\int_0^\tau d\tau_1\lambda_r(\tau_1).
\]

(A15)

Let us remark that, for each configuration of an isolated system with timelike total 4-momentum there is a privileged family of hyperplanes (the Wigner hyperplanes orthogonal to \( p_s^\mu \), existing when \( \epsilon p_s^2 > 0 \) corresponding to the \textit{intrinsic rest-frame} of the isolated system. If we choose these hyperplanes with suitable gauge fixings for the constraints \( \mathcal{H}^{\mu\nu}(\tau) \approx 0 \), we are left with the four constraints \( \mathcal{H}^{\mu\nu}(\tau) \approx 0 \), which can be rewritten as

\[
\epsilon_s = \sqrt{\epsilon p_s^2} \approx \text{[invariant mass of the isolated system under investigation]} = M_{sys};
\]

\[
\bar{p}_{sys} = [3 - \text{momentum of the isolated system inside the Wigner hyperplane}] \approx 0,
\]

\[
H_D = H_{(c)} + \tilde{\lambda}^{\mu}(\tau)\mathcal{H}_{\mu}(\tau) + (\text{system - dependent primary constraints}) = \\
= H_{(c)} + \lambda(\tau)[\epsilon_s - M_{sys}] - \tilde{\lambda}(\tau) \cdot \bar{p}_{sys} + \\
+ (\text{system - dependent primary constraints}).
\]

(A16)

There is no more a restriction on \( p_s^\mu \), because \( u^\mu(p_s) = p_s^\mu/\sqrt{\epsilon p_s^2} \) gives the orientation of the Wigner hyperplanes containing the isolated system, with respect to an arbitrary given external observer.

\footnote{With the \( b_A^\mu(\tau) \)'s being three orthogonal spacelike unit vectors generating the fixed \( \tau \)-independent timelike unit normal \( b^\mu_\tau = l^\mu \) to the hyperplanes.}
In this special gauge, after having gone to Dirac brackets we have 
\[ b^\mu_A \equiv L^\mu_A(p_s, \hat{p}_s) \] (the standard Wigner boost for timelike Poincaré orbits), 
\[ S^{\mu\nu}_s \equiv S^{\mu\nu}_{\text{sys}}, \lambda_{\mu\nu}(\tau) \equiv 0. \] The origin \( x^\mu_s(\tau) \) does not belong any more to the canonical basis for these Dirac brackets and is replaced by the non-covariant canonical variable \[ \tilde{x}_s^\mu(\tau) = x^\mu_s(\tau) - \frac{1}{\epsilon_s(p_s^2 + \epsilon_s)} \left[ p_s^{\nu} S^{\nu\mu}_s + \epsilon_s(S^{\mu\nu}_s - S^{\nu\mu}_s \frac{p_s^\mu p_s^\nu}{\epsilon_s}) \right]. \]

In general, we have the problem that in the gauges where \( \tilde{\lambda}_{\mu\nu}(\tau) \) or \( \tilde{\lambda}_{AB}(\tau) \) are different from zero, the foliations with leaves \( \Sigma_\tau \) associated to arbitrary 3+1 splittings of Minkowski spacetime are geometrically ill-defined at spatial infinity so that the variational principle describing the isolated system could make sense only for those 3+1 splittings having these part of the Dirac’s multipliers vanishing. The problem is that, since on hyperplanes \( \dot{l}^\mu = 0 \) and \( l^\mu b_{\tau\mu}(\tau) = 0 \) imply \( \dot{l}^\mu b_{\tau\mu}(\tau) = 0 \), Eqs.(A10) implies \( \lambda_{\tau\tau}(\tau) = 0 \) (i.e. only three \( \tilde{\lambda}_{\mu\nu}(\tau) \) are independent) on spacelike hyperplane, because otherwise Lorentz boosts could generate crossing of the foliation leaves. To avoid inconsistencies this suggests to make the reduction from arbitrary spacelike hypersurfaces either directly to the Wigner hyperplanes or to spacelike hypersurfaces approaching Wigner hyperplanes asymptotically.

Till now, therefore, the 3+1 splittings of Minkowski spacetime whose leaves are Wigner hyperplanes are the only ones for which the foliation is well defined at spatial infinity: both the induced proper time interval and shift functions are finite there.

One obtains in this way a new kind of instant form of the dynamics, the Wigner-covariant 1-time rest-frame instant form \[ [7,29]. \] For any isolated system all the variables become Wigner covariant except for the external canonical center of mass \( \tilde{x}_s^\mu \), which loses even Lorentz covariance. This does not matter, however, since it is a completely decoupled variable. This is the special relativistic generalization of the non-relativistic separation of the center of mass from the relative motion \[ H = \frac{\hat{p}_s^2}{2M} + H_{\text{rel}}. \] The role of the center of mass is taken by the Wigner hyperplane, identified by a point \( \tilde{x}^\mu(\tau) \) and its normal \( p_s^\mu \).

\[ ^{57} \text{Asymptotically, we must fix the gauge freedom generated by the spin part of Lorentz boosts, see Eq.}(A9); \text{how this can be done before the restriction to spacelike hyperplanes has still to be studied.} \]
APPENDIX B: NOTATIONS ON SPACELIKE HYPERSURFACES.

Let us first review some preliminary results from Refs. \[1\] needed in the description of physical systems on spacelike hypersurfaces.

Let \( \{ \Sigma_\tau \} \) be a one-parameter family of spacelike hypersurfaces foliating Minkowski spacetime \( M^4 \) with 4-metric \( \eta_{\mu\nu} = \epsilon(+-+-) \), \( \epsilon = \pm 1 \) and giving a 3+1 decomposition of it. At fixed \( \tau \), let \( z^\mu(\tau, \vec{\sigma}) \) be the coordinates of the points on \( \Sigma_\tau \) in \( M^4 \), \{ \vec{\sigma} \} a system of coordinates on \( \Sigma_\tau \). If \( \sigma^A = (\sigma^\tau = \tau; \vec{\sigma} = \{ \sigma^f \} \) \[2\] and \( \partial_A = \partial/\partial\sigma^A \), one can define the cotetrads

\[
  z^A_\mu(\tau, \vec{\sigma}) = \partial_A z^\mu(\tau, \vec{\sigma}), \quad \partial_B z^A_\mu - \partial_A z^B_\mu = 0, \tag{B1}
\]

so that the metric on \( \Sigma_\tau \) is

\[
  g_{AB}(\tau, \vec{\sigma}) = z^A_\mu(\tau, \vec{\sigma})\eta_{\mu\nu}z^B_\nu(\tau, \vec{\sigma}), \quad \epsilon g_{\tau\tau}(\tau, \vec{\sigma}) > 0,
\]

\[
  g(\tau, \vec{\sigma}) = -\det g_{AB}(\tau, \vec{\sigma}) = (\det g^A_\mu(\tau, \vec{\sigma}))^2, \quad \gamma(\tau, \vec{\sigma}) = -\det g^{s\bar{s}}(\tau, \vec{\sigma}) = \det g^{s\bar{s}}(\tau, \vec{\sigma}), \tag{B2}
\]

where \( g^{s\bar{s}} = -\epsilon^3 g^{s\bar{s}} \) with \( 3 g^{s\bar{s}} \) having positive signature \((+++)\).

If \( \gamma^{s\bar{s}}(\tau, \vec{\sigma}) = -\epsilon^3 g^{s\bar{s}} \) is the inverse of the 3-metric \( g^{s\bar{s}}(\tau, \vec{\sigma}) = \delta_{\bar{s}}^s \), the inverse \( g^{AB}(\tau, \vec{\sigma}) g_{AB}(\tau, \vec{\sigma}) = \delta^A_B \) is given by

\[
  g^{\tau\tau}(\tau, \vec{\sigma}) = \gamma(\tau, \vec{\sigma}),
\]

\[
  g^{\tau\bar{f}}(\tau, \vec{\sigma}) = -\gamma g_{\bar{r}u}^{\bar{r}u}(\tau, \vec{\sigma}) = \epsilon g^{\tau\bar{r}} g^{\bar{r}u}(\tau, \vec{\sigma}),
\]

\[
  g^{s\bar{s}}(\tau, \vec{\sigma}) = \gamma^{s\bar{s}}(\tau, \vec{\sigma}) + \frac{\gamma}{g} g_{\bar{r}u} g_{\bar{r}v} \gamma^{\bar{r}u} \gamma^{\bar{r}v}(\tau, \vec{\sigma}) =
\]

\[
  = -\epsilon^3 g^{s\bar{s}}(\tau, \vec{\sigma}) + \frac{\gamma}{g} g_{\bar{r}u} g_{\bar{r}v} \gamma^{\bar{r}u} \gamma^{\bar{r}v}(\tau, \vec{\sigma}), \tag{B3}
\]

so that \( 1 = g^{C\bar{C}}(\tau, \vec{\sigma}) g_{C\tau}(\tau, \vec{\sigma}) \) is equivalent to

\[
  \frac{g(\tau, \vec{\sigma})}{\gamma(\tau, \vec{\sigma})} = g_{\tau\tau}(\tau, \vec{\sigma}) - \gamma^{s\bar{s}}(\tau, \vec{\sigma}) g_{\tau\bar{f}}(\tau, \vec{\sigma}) g_{s\bar{s}}(\tau, \vec{\sigma}). \tag{B4}
\]

We have

\[
  z^\mu_\tau(\tau, \vec{\sigma}) = (\sqrt{g^\mu_\gamma} + g_{\tau\tau} \gamma^{s\bar{s}} z^\mu_\bar{s})(\tau, \vec{\sigma}), \tag{B5}
\]

and

\[58\] \( \epsilon = +1 \) is the particle physics convention; \( \epsilon = -1 \) the general relativity one.

\[59\] The notation \( \bar{A} = (\tau, \bar{r}) \) with \( \bar{r} = 1, 2, 3 \) will be used; note that \( \bar{A} = \tau \) and \( \bar{A} = \bar{r} = 1, 2, 3 \) are Lorentz-scalar indices.
where

\[ l^\mu(\tau, \bar{\sigma}) = \left( \frac{1}{\sqrt{\gamma}} \epsilon^\mu_{\alpha \beta \gamma} z_1^\alpha z_2^\beta z_3^\gamma \right)(\tau, \bar{\sigma}), \]

\[ l^2(\tau, \bar{\sigma}) = 1, \quad l_\mu(\tau, \bar{\sigma}) z_\nu^\nu(\tau, \bar{\sigma}) = 0, \tag{B7} \]

is the unit (future pointing) normal to \( \Sigma_\tau \) at \( z^\mu(\tau, \bar{\sigma}) \).

For the volume element in Minkowski spacetime we have

\[ d^4z = z^\mu(\tau, \bar{\sigma}) dt d^3\Sigma_{\mu} = dt [z^\mu(\tau, \bar{\sigma}) l_\mu(\tau, \bar{\sigma})] \sqrt{\gamma(\tau, \bar{\sigma})} d^3\sigma = \]

\[ = \sqrt{g(\tau, \bar{\sigma})} dt d^3\sigma. \tag{B8} \]

Let us remark that, according to the geometric approach of Ref. [44], one can write

\[ z^\nu_{\tau}(\tau, \bar{\sigma}) = N(\tau, \bar{\sigma}) l^\mu(\tau, \bar{\sigma}) + N^\nu(\tau, \bar{\sigma}) z^\nu(\tau, \bar{\sigma}), \tag{B9} \]

where \( N = \sqrt{g/\gamma} = \sqrt{g_{\tau\tau} - \gamma g_{\tau\bar{\sigma}} g_{\bar{\sigma} \bar{\sigma}}} = \sqrt{g_{\tau\tau} + \epsilon g_{\tau\bar{\sigma}} g_{\bar{\sigma} \bar{\sigma}}} \) and \( N^\nu = g_{\tau\bar{\sigma}} \gamma_{\bar{\sigma}} = -\epsilon g_{\tau\bar{\sigma}} \gamma_{\bar{\sigma}} \)

are the standard lapse and shift functions \( N_{[\tau]}^{[flat]}, N_\nu^{[flat]} \) of Appendix A, so that

\[ g_{\tau\tau} = \epsilon N^2 + g_{\tau\bar{\sigma}} N_{\bar{\sigma}} = \epsilon [N^2 - 3 g_{\tau\bar{\sigma}} N_{\bar{\sigma}} N_{\bar{\sigma}}], \]

\[ g_{\tau\bar{\sigma}} = g_{\tau\bar{\sigma}} N_{\bar{\sigma}} = -\epsilon^3 g_{\tau\bar{\sigma}} N_{\bar{\sigma}}, \]

\[ g_{\bar{\sigma}} = \epsilon N^{-2}, \]

\[ g_{\bar{\sigma} \bar{\sigma}} = -\epsilon^3 g_{\tau\bar{\sigma}} N_{\bar{\sigma}}, \]

\[ g_{\tau\bar{\sigma}} = -\epsilon g_{\tau\bar{\sigma}}/N^2, \]

\[ g_{\tau\bar{\sigma}} = \gamma_{\bar{\sigma}} + \epsilon \frac{N_{\bar{\sigma}} N_{\bar{\sigma}}}{N^2} = -\epsilon^3 g_{\tau\bar{\sigma}} N_{\bar{\sigma}} N_{\bar{\sigma}}, \]

\[ \frac{\partial}{\partial z^\mu_{\tau}} = l_\mu \frac{\partial}{\partial N} + z^\nu_{\nu} \gamma^\nu_{\bar{\sigma}} \frac{\partial}{\partial N^\nu} = l_\mu \frac{\partial}{\partial N} - \epsilon z^\nu_{\nu} \gamma^\nu_{\bar{\sigma}} \frac{\partial}{\partial N^\nu}, \]

\[ d^4z = N \sqrt{g} dt d^3\sigma. \tag{B10} \]

The rest frame form of a timelike four-vector \( p^\mu \) is \( \tilde{p}^\mu = \eta \sqrt{\epsilon p^2}(1; \tilde{0}) = \eta^\mu_{\nu} \eta \sqrt{\epsilon p^2}, \tilde{p}^2 = p^2 \), where \( \eta = \text{sign} p^\nu \). The standard Wigner boost transforming \( \tilde{p}^\mu \) into \( p^\mu \) is

\[ L^\mu_{\nu}(p, \tilde{p}) = \epsilon^\mu_{\nu}(u(p)) = \]

\[ = \eta^\mu_{\nu} + 2 P^\mu P_{\nu} / \epsilon p^2 - (p^\mu + \tilde{p}^\mu)(p_{\nu} + \tilde{p}_{\nu}) = \]

\[ = \eta^\mu_{\nu} + 2 u^\mu(p) u_{\nu}(\tilde{p}) - (u^\mu(p) + u^\mu(\tilde{p}))(u_{\nu}(p) + u_{\nu}(\tilde{p})) / 1 + w^\nu(p), \]

\[ \nu = 0 \quad \epsilon^\mu_{\nu}(u(p)) = u^\mu(p) = p^\mu / \eta \sqrt{\epsilon p^2}, \]

\[ \nu = r \quad \epsilon^\mu_{\nu}(u(p)) = (-u_r(p); \delta^\nu_r - u^i(p) u_r(p) / 1 + w^o(p)). \tag{B11} \]
The inverse of $L^\mu_\nu(p,\hat{p})$ is $L^\mu_\nu(\hat{p},p)$, the standard boost to the rest frame, defined by

$$L^\mu_\nu(\hat{p},p) = L^\mu_\nu(p,\hat{p})|_{\vec{p} \to -\vec{p}}. \quad (B12)$$

Therefore, we can define the following cotetrads and tetrads

$$\epsilon^\mu_A(u(p)) = L^\mu_A(p,\hat{p}),$$
$$\epsilon^\mu_A(u(p)) = L^A_\mu(p,\hat{p}) = \eta^{AB} \eta_{\mu\nu} \epsilon^\nu_B(u(p)), $$
$$\epsilon^\mu_B(u(p)) = \eta_{\mu\nu} \epsilon^\nu_A(u(p)) = u_\mu(p),$$
$$\epsilon^\mu_B(u(p)) = -\delta^{rs} \eta_{\mu\nu} \epsilon^\nu_r(u(p)) = (\delta^{rs} u_\mu(p); \delta^r_j - \delta^r_s \delta_{jh} u_h(p)) \frac{u^h(p)}{1 + u^o(p)}),$$
$$\epsilon^A_\alpha(u(p)) = u_A(p), \quad \epsilon^A_\alpha(u(p)) = \eta_{\mu\nu} \epsilon^\nu_B(u(p)) = u_{\alpha}(p),$$

which satisfy

$$\epsilon^\mu_A(u(p)) \epsilon^\nu_A(u(p)) = \eta^\mu\nu,$$
$$\epsilon^\mu_A(u(p)) \epsilon^\nu_B(u(p)) = \eta^A_B,$$
$$\eta^{\mu\nu} = \epsilon^\mu_A(u(p)) \eta^{AB} \epsilon^\nu_B(u(p)) = u^\mu(p) u^\nu(p) - \sum_{r=1}^3 \epsilon^\mu_r(u(p)) \epsilon^\nu_r(u(p)),$$
$$\eta_{AB} = \epsilon^\mu_A(u(p)) \eta_{\mu\nu} \epsilon^\nu_B(u(p)),$$
$$p_\alpha \frac{\partial}{\partial p_\alpha} \epsilon^\mu_A(u(p)) = p_\alpha \frac{\partial}{\partial p_\alpha} \epsilon^\mu_B(u(p)) = 0. \quad (B13)$$

The Wigner rotation corresponding to the Lorentz transformation $\Lambda$ is

$$R^\mu_\nu(\Lambda, p) = [L(\hat{p}, p) \Lambda^{-1} L(\Lambda p, \hat{p})]^{\mu}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i_j(\Lambda, p) \end{pmatrix},$$

$$R^i_j(\Lambda, p) = (\Lambda^{-1})^i_j - \frac{(\Lambda^{-1})^i_o p_\beta (\Lambda^{-1})^\beta_j}{p_\rho (\Lambda^{-1})^\rho_o + \eta \sqrt{cp^2}} - \frac{p^i}{p^o + \eta \sqrt{cp^2}} [(\Lambda^{-1})^i_o - \frac{(\Lambda^{-1})^o_o - 1)}{p_\rho (\Lambda^{-1})^\rho_o + \eta \sqrt{cp^2}}. \quad (B15)$$

The polarization vectors transform under the Poincaré transformations $(\alpha, \Lambda)$ in the following way

$$\epsilon^\mu_r(u(\Lambda p)) = (R^{-1})^r_s \Lambda^\mu_\nu \epsilon^\nu_s(u(p)). \quad (B16)$$

---

60The $\epsilon^\mu_r(u(p))$'s are also called polarization vectors; the indices $r, s$ will be used for $A=1,2,3$ and $\bar{o}$ for $A = o$. 
APPENDIX C: THE GARTENHAUS-SCHWARTZ TRANSFORMATION FOR SPINNING PARTICLES.

In Ref. [38] there is the rest-frame instant form description of a system of N spinning positive-energy particles with the intrinsic spin described by Grassmann variables.

On the Wigner hyperplane the N spinning particles are described by a canonical basis containing the center-of-mass variables $\tilde{x}_\mu^s$, $p_\mu^s$, the pairs $\vec{\eta}_i$, $\vec{\kappa}_i$, $i = 1, \ldots, N$, of Eq.(2.16) and three Grassmann variables for each spin, $\xi_i^r \equiv \epsilon^r_\mu(u(p_s))\xi^\mu_i[38]$ satisfying $\{\xi_i^r, \xi_j^s\} = -i\delta^{rs}\delta_{ij}$ and having vanishing Poisson bracket with all the other variables.

The rest-frame external realization of the Poincaré algebra is built in analogy to Eq.(2.14) but with a modified spin tensor $\bar{S}_{\mu\nu}^s$

$$\bar{S}_s = \sum_{i=1}^{N} \left( \vec{\eta}_i \times \vec{\kappa}_i + \bar{S}_i^s \right),$$

$$\bar{S}_{i\xi}^s = -\frac{i}{2} \epsilon^{rsv} \xi^r_i \xi^s_i, \quad \{\bar{S}_{i\xi}^r, \bar{S}_{j\xi}^s\} = \delta_{ij}\epsilon^{rsv}\bar{S}_{i\xi}^u.$$

(C1)

In absence of interactions Eqs.(2.7), (2.8), (2.16) remain valid.

By using the expression of $\bar{S}_{\mu\nu}^s$ on the Wigner hyperplane given in Ref. [38] and the methodology of Ref. [24], the internal realization (2.15) of the Poincaré algebra becomes

$$H_M = M_{sys} = \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{\kappa}_i^2} = \sum_{i=1}^{N} H_i, \quad H_i = \sqrt{m_i^2 + \vec{\kappa}_i^2},$$

$$\vec{\kappa}_+ = \sum_{i=1}^{N} \vec{\kappa}_i \quad (\approx 0),$$

$$\vec{J} = \bar{S}_s = \sum_{i=1}^{N} \left( \vec{\eta}_i \times \vec{\kappa}_i + \bar{S}_i^s \right) = \vec{J}_B + \vec{J}_S, \quad \vec{J}_S = \sum_{i=1}^{N} \vec{S}_i^s,$$

$$\vec{K} = -\sum_{i=1}^{N} \vec{\eta}_i H_i + \sum_{i=1}^{N} \frac{\vec{S}_i^s \times \vec{\kappa}_i}{m_i + H_i} = \vec{K}_B + \vec{K}_S, \quad \vec{K}_B = -\sum_{i=1}^{N} \vec{\eta}_i H_i.$$

(C2)

Following Ref. [24], if we put

$$\vec{K} = -H_M \vec{q}_+ + \frac{\vec{S}_q \times \vec{\kappa}_+}{\sqrt{\Pi} + H_M}, \quad \Pi = H_M^2 - \vec{\kappa}_+^2,$$

$$\vec{S}_q = \vec{J} - \vec{q}_+ \times \vec{\kappa}_+,$$

(C3)

we get consistently the expression of the internal canonical center of mass $\vec{q}_+$ given in Eq.(1.4), with $\vec{S}_q$ the associated spin vector.

As in Section VI, let us apply the Gartenhaus-Schwartz transformation to go from the internal canonical basis $\vec{\eta}_i$, $\vec{\kappa}_i$, $\vec{\xi}_i$ (with $\vec{\kappa}_+ \approx 0$) to the center-of-mass basis $\vec{q}_+$, $\vec{\kappa}_+$, $\vec{\rho}_q$, $\vec{\pi}_q$, $\vec{\xi}_q$ (again with $\vec{\kappa}_+ \approx 0$) with $\vec{S}_{i\xi} \rightarrow \vec{S}_{q\xi}$. By using Eqs.(5.1) and (5.4) with $\vec{K} = \vec{K}_B + \vec{K}_S$ of Eq.(C2), we can find the differential equations for the $\alpha$-dependence of the various quantities.

Since $\{\vec{\pi}_a, \vec{K}_S\} = 0$, $\vec{\pi}_q = \lim_{\alpha \rightarrow \infty} \vec{\pi}_q(\alpha)$ is the same as in the spinless case; the $\vec{\pi}_q$'s are given in Eqs.(5.13). As in the spinless case $\Pi$ is an invariant and we have $\sqrt{\Pi} = H_M(\infty) = \ldots$
\[ H_{(red)} = \sum_{i=1}^{N} H_i(\infty) \] with \( H_i(\infty) = (H_M H_i - \vec{\kappa}_i \cdot \vec{\kappa}_i)/\sqrt{\Pi} \). Moreover, \( \vec{n}_+ = \vec{\kappa}_+ / |\vec{\kappa}_+| \) is invariant.

For the spin variables \( \vec{S}_{i\xi} \) we get
\[
\frac{d\vec{S}_{i\xi}(\alpha)}{d\alpha} = \{ \vec{S}_{i\xi}(\alpha), \vec{q}_+(\alpha) \cdot \vec{\kappa}_+(\alpha) \} = \left\{ \frac{\vec{K}(\alpha) \cdot \vec{\kappa}_+(\alpha)}{H_M(\alpha)}, \vec{S}_{i\xi}^r \right\} =
\]
\[
= \frac{\kappa_+^s(\alpha)}{H_M(\alpha)} \left\{ K^s(\alpha), \vec{S}_{i\xi}^r(\alpha) \right\} = \frac{\left[ (\vec{\kappa}_+(\alpha) \times \vec{\kappa}_i(\alpha)) \times \vec{S}_{i\xi}(\alpha) \right]^r}{H_M(\alpha)(m_i + H_i(\alpha))},
\]
\[
\Rightarrow \frac{d\vec{S}_{i\xi}(\alpha)}{d\theta(\alpha)} = \frac{(\vec{n}_+ \times \vec{\kappa}_i(\alpha)) \times \vec{S}_{i\xi}(\alpha)}{m_i + H_i(\alpha)}. \quad (C4)
\]

This equation coincides with Eq.(3.10) of Ref. [35]. By using Eq.(3.11) of Ref. [35], its integration provides Thomas precession of the spin variable \( \vec{S}_{i\xi} \) about an axis \( \vec{\kappa}_i \times \vec{n}_+ \) in the instantaneous center-of-mass frame [39]:
\[
\vec{S}_{i\xi}(\alpha) = \cos \gamma(\alpha) \vec{S}_{i\xi} + [1 - \cos \gamma(\alpha)](\vec{v}_i \cdot \vec{S}_{i\xi})\vec{v}_i - \sin \gamma(\alpha)\vec{v}_i \times \vec{S}_{i\xi},
\]
\[
\vec{v}_i = \frac{\vec{\kappa}_i \times \vec{n}_+}{|\vec{\kappa}_i \times \vec{n}_+|}.
\]
\[
tg \frac{\gamma(\alpha)}{2} = \frac{1 - \cos \gamma(\alpha)}{\sin \gamma(\alpha)} = \frac{|\vec{\kappa}_i \times \vec{n}_+|}{(m_i + H_i)ctgh \frac{\gamma(\alpha)}{2} - \vec{\kappa}_i \cdot \vec{n}_+}
\]
\[
\rightarrow_{\alpha \rightarrow \infty} \frac{|\vec{\kappa}_i \times \vec{n}_+|}{(m_i + H_i)(H_M + \sqrt{\Pi}) - \vec{\kappa}_i \cdot \vec{n}_+} \approx 0, \quad \left[ tg \frac{\gamma(\infty)}{2} = \frac{H_M + \sqrt{\Pi}}{|\vec{\kappa}_+|} \right],
\]
\[
sin \gamma(\alpha) = 2\frac{(m_i + H_i)ctgh \frac{\gamma(\alpha)}{2} - \vec{\kappa}_i \cdot \vec{n}_+}{|\vec{\kappa}_i \times \vec{n}_+|^2 + [(m_i + H_i)ctgh \frac{\gamma(\alpha)}{2} - \vec{\kappa}_i \cdot \vec{n}_+]^2}|\vec{\kappa}_i \times \vec{n}_+|,
\]
\[
cos \gamma(\alpha) = 1 - 2\frac{|\vec{\kappa}_i \times \vec{n}_+|^2}{|\vec{\kappa}_i \times \vec{n}_+|^2 + [(m_i + H_i)ctgh \frac{\gamma(\alpha)}{2} - \vec{\kappa}_i \cdot \vec{n}_+]^2}. \quad (C5)
\]

Therefore, we obtain
\[
\vec{S}_{q\xi} = \lim_{\alpha \rightarrow \infty} \vec{S}_{i\xi}(\alpha) = \left[ 1 - \left( \frac{|\vec{\kappa}_i \times \vec{n}_+|^2}{(m_i + H_i)(m_i + H_i(\infty))(H_M + \sqrt{\Pi})\sqrt{\Pi}} \right) \vec{S}_{i\xi} +
\right.
\]
\[
+ \frac{\vec{\kappa}_i \times \vec{n}_+}{(m_i + H_i)(m_i + H_i(\infty))(H_M + \sqrt{\Pi})\sqrt{\Pi}} \vec{S}_{i\xi} \vec{K}_i \vec{\kappa}_i \times \vec{\kappa}_+
\]
\[
- \frac{(m_i + H_i)(H_M + \sqrt{\Pi}) - \vec{\kappa}_i \cdot \vec{\kappa}_+}{(m_i + H_i)(m_i + H_i(\infty))(H_M + \sqrt{\Pi})\sqrt{\Pi}} (\vec{\kappa}_i \times \vec{\kappa}_+). \quad \vec{S}_{i\xi} \approx \vec{S}_{i\xi}. \quad (C6)
\]

For the Grassmann variables \( \vec{\xi}_i \), we get the same differential equation
\[
\frac{d\xi_i^r(\alpha)}{d\alpha} = \{ \xi_i^r(\alpha), \vec{q}_+(\alpha) \cdot \vec{\kappa}_+(\alpha) \} = \frac{\left[ (\vec{\kappa}_+(\alpha) \times \vec{\kappa}_i(\alpha)) \times \vec{\xi}_i \right]^r}{H_M(\alpha)(m_i + H_i(\alpha))},
\]
\[
\frac{d\vec{\xi}(\alpha)}{d\theta(\alpha)} = \left(\vec{n}_+ \times \vec{\xi}(\alpha)\right) \times \vec{\xi}(\alpha) \over m_i + H_i(\alpha), \quad \downarrow
\]
\[
\xi_{qi} = \lim_{\alpha \to \infty} \vec{\xi}(\alpha) = \left[1 - \left(\frac{1}{\sqrt{(m_i + H_i)(m_i + H_i(\infty))(H_M + \sqrt{\Pi})\sqrt{\Pi}}}ight)^2\right] \vec{\xi}_i + \frac{\vec{\xi}_i \times \vec{\xi}_+ \cdot \vec{\xi}_i \times \vec{\xi}_+}{(m_i + H_i)(m_i + H_i(\infty))(H_M + \sqrt{\Pi})\sqrt{\Pi}} - \frac{\vec{\xi}_i \times \vec{\xi}_+ \cdot \vec{\xi}_i \times \vec{\xi}_+}{(m_i + H_i)(m_i + H_i(\infty))(H_M + \sqrt{\Pi})\sqrt{\Pi}} \vec{\xi}_i \approx \vec{\xi}_i. \tag{C7}
\]

Let us remark that, now, as one can easily check, besides the invariants \(I_i^{(1)}\) and \(I_i^{(2)}\) of Eqs. (5.14) of the spinless case (we have \(I_i^{(2)} = I_B^{(2)} + I_S^{(2)}\) since \(\vec{K} = \vec{K}_B + \vec{K}_S\)) there are also the following invariants
\[
I_i^{(3)} = (\vec{n}_i \times \vec{n}_+) \cdot \vec{S}_{ik}.
\]
\[
I_i^{(2)} = I_B^{(2)} + I_S^{(2)}, \quad I_i^{(3)} = \frac{\vec{K}_i}{H_M} \sum_{i=1}^{N} \frac{I_i^{(3)}}{m_i + H_i}. \tag{C8}
\]

Let us now consider the position vectors. Like in the spinless case the preliminary calculations for Eq. (5.13), now give
\[
\frac{d}{d\alpha} \vec{n}_+ \cdot \vec{n}_i(\alpha) = -\vec{n}_+ \cdot \vec{n}_i(\alpha) \frac{dH_i(\alpha)}{H_i(\alpha)} - \frac{I_i^{(1)}(\alpha)}{H_i(\alpha)} \frac{dI_i^{(1)}(\alpha)}{d\alpha} - \frac{1}{H_i(\alpha)} \frac{dH_i(\alpha)}{d\alpha} \frac{I_i^{(3)}(\alpha)}{(m_i + H_i(\alpha))^2}. \tag{C9}
\]

These equations have the solution
\[
\vec{n}_+ \cdot \vec{n}_i(\alpha) = \frac{H_i(\alpha)}{H_i(\alpha)} \vec{n}_+ \cdot \vec{n}_i - \frac{I_i^{(2)}(\alpha)}{|\vec{K}_+|} (e^{\alpha} - \frac{H_i(\alpha)}{H_i(\alpha)}) + \frac{I_i^{(3)}(\alpha)}{H_i(\alpha)} \left(\frac{1}{m_i + H_i(\alpha)} - \frac{1}{m_i + H_i(\alpha)}\right). \tag{C10}
\]

For \(\vec{n}_i(\alpha)\) we have
\[
\frac{d\vec{n}_i(\alpha)}{d\alpha} = \{\vec{n}_i(\alpha), \vec{n}_i(\alpha) \cdot \vec{q}_i(\alpha)\} = -n^s_+ \frac{\partial}{\partial k_i(\alpha)} \frac{|\vec{K}_+| K_i(\alpha)}{H_M(\alpha)} =
\]
\[
= \vec{n}_+ \cdot \vec{n}_i(\alpha) \frac{|\vec{K}_+| K_i(\alpha)}{H_i(\alpha)H_M(\alpha)} + \sum_{j=1}^{N} \frac{H_j(\alpha)\vec{n}_+ \cdot \vec{n}_j(\alpha)}{H_M(\alpha)} \left(\frac{|\vec{K}_+| K_j(\alpha)}{H_i(\alpha)H_M(\alpha)} - \vec{n}_+ \right) +
\]
The equations for \( \vec{\rho} \) are [see Eq.(3.21) of Ref. [38]]

\[
\frac{d\vec{\rho}^a(\alpha)}{d\alpha} = \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \frac{d\vec{\eta}_{i}(\alpha)}{d\alpha} =
\]

\[
\begin{align*}
&= \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \left( \frac{H_i |\vec{k}_+(\alpha)| \vec{k}_i(\alpha)}{H_i^2(\alpha)} \left( \vec{n}_+ \cdot \vec{\eta}_i + \frac{I^{(2)}_i}{|\vec{k}_+|} \right) + \\
&+ \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \left( \frac{|\vec{k}_+(\alpha)| \vec{n}_+ \cdot (\vec{S}_{i\xi} \times \vec{k}_i(\alpha))}{H_i(\alpha) (m_i + H_i(\alpha))^2} \right) + \\
&+ \frac{I^{(3)}_i}{H_i H_i(\alpha)} \left( \frac{1}{m_i + H_i(\alpha)} - \frac{1}{m_i + H_i} \right) \right) + \\
&+ \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \left( \frac{|\vec{k}_+(\alpha)| \vec{n}_+ \times \vec{S}_{i\xi}(\alpha)}{H_i(\alpha)(m_i + H_i(\alpha))} \right).
\end{align*}
\]

By using the results contained in Ref. [38], this equation can be integrated with the final result

\[
\vec{\rho}_a(\alpha) = \vec{\rho}_a - \sum_{i,j=1}^{N-1} \gamma_{aj}(\gamma_{bi} - \gamma_{bj}) \frac{H_i H_j}{H_M} \tilde{J}^{(2)}_j(\alpha) \vec{n}_+ \cdot \vec{\eta}_b + \\
+ \sqrt{N} \sum_{i=1}^{N} \frac{I^{(3)}_i}{H_i H_i(\alpha)} \sum_{j=1}^{N} \gamma_{aj} \frac{k_j(\alpha)}{H_j(\alpha)} \sin\theta(\alpha) + \\
+ \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \left( \frac{I^{(3)}_i (\vec{k}_i - |\vec{n}_+ \cdot \vec{\eta}_i| \vec{n}_+ \sin\theta(\alpha))}{H_i(\alpha)} + \\
+ \frac{I^{(3)}_i (H_i - H_i(\alpha)) \vec{n}_+}{H_i(\alpha)} \right) \right] + \left[ \sin\theta(\alpha) - 1 \right] \vec{n}_+ \cdot \vec{S}_{i\xi} \vec{k}_i \times \vec{n}_+ - \\
- \sin\theta(\alpha) \left( \frac{(m_i + H_i) \left[ \sin\theta(\alpha) + 1 \right] - \vec{n}_+ \cdot \vec{k}_i \sin\theta(\alpha)}{\sin\theta(\alpha) + 1} \right).
\]

Then we get

\[
\vec{\rho}_{qa} = \lim_{\alpha \to \infty} \vec{\rho}_a(\alpha) = \vec{\rho}_a -
\]
\[
\begin{align*}
- \sum_{i,j=1}^{N-1} \sum_{b=1}^{N-1} \gamma_{aj}(\gamma_{bi} - \gamma_{bj}) \frac{H_i}{H_M} \left[ \left| \vec{\kappa}_j(\infty) \right| \vec{\kappa}_j(\infty) \right] + \left( \frac{H_M}{\sqrt{\Pi}} - 1 \right) \vec{n}_+ \cdot \vec{\rho}_b + \\
+ \sqrt{N} \sum_{i=1}^{N} \frac{I_i^{(3)}}{H_M(m_i + H_i)} \sum_{j=1}^{N} \gamma_{aj} \frac{\left| \vec{\kappa}_i(\infty) \right|}{H_j(\infty)} \sqrt{\Pi} + \\
+ \sqrt{N} \sum_{i=1}^{N} \frac{\gamma_{ai}}{(m_i + H_i)(m_i + H_i(\infty))} \left[ \left| \vec{\kappa}_i(\infty) \right| \vec{\kappa}_i(\infty) \right] - \vec{n}_+ \cdot \vec{\kappa}_i \vec{n}_+ - \\
- \frac{\left| \vec{\kappa}_i(\infty) \right|}{\sqrt{\Pi}} \frac{\left| \vec{\kappa}_i(\infty) \right|}{H_M - \sqrt{\Pi}} 
\end{align*}
\]

\(\approx \vec{\rho}_a.\) (C14)

In the same way as in the spinless case we obtain

\[
\vec{J}_S(\alpha) = \sum_{i=1}^{N} \left[ \vec{\eta}_i(\alpha) \times \vec{\kappa}_i(\alpha) + \vec{S}_{i\xi}(\alpha) \right] \\
\rightarrow_{\alpha \to \infty} \vec{S}_q = \sum_{a=1}^{N-1} \vec{\rho}_qa \times \vec{\pi}_qa + \sum_{i=1}^{N} \vec{S}_{qi\xi}. \quad \text{(C15)}
\]
APPENDIX D: EULER ANGLES.

Let us denote by $\alpha$, $\beta$, $\gamma$ the Euler angles chosen as orientation variables $\theta^a$.

Let $\hat{f}_1 = \hat{i}$, $\hat{f}_2 = \hat{j}$, $\hat{f}_3 = \hat{k}$ be the unit 3-vectors along the axes of the space frame and $\hat{e}_1 = \hat{\chi}$, $\hat{e}_2 = \hat{N} \times \hat{\chi}$, $\hat{e}_3 = \hat{N}$, the unit 3-vectors along the axes of a body frame. Then we have

$$
\begin{align*}
\tilde{S}_q &= S_q^{r^e} f_r = R^{rs}(\alpha, \beta, \gamma) \tilde{S}_q^s f_r = \tilde{S}_q^s \hat{e}_s, \\
\hat{e}_s &= (R^T)^{sr}(\alpha, \beta, \gamma) \tilde{f}_r = R^s_r (\alpha, \beta, \gamma) \tilde{f}_r.
\end{align*}
$$

(D1)

There are two main conventions for the definition of the Euler angles $\alpha$, $\beta$, $\gamma$.

A) The $y$-convention (see Refs. [10] (Appendix B) and [11]):

i) perform a first rotation of an angle $\alpha$ around $\hat{f}_3$ [$\hat{f}_1 \mapsto \hat{e}_1 = \cos \alpha \hat{f}_1 + \sin \alpha \hat{f}_2$, $\hat{f}_2 \mapsto \hat{e}_2 = -\sin \alpha \hat{f}_1 + \cos \alpha \hat{f}_2$, $\hat{f}_3 \mapsto \hat{e}_3 = \hat{f}_3$];

ii) perform a second rotation of an angle $\beta$ around $\hat{e}_2$ [$\hat{e}_1 \mapsto \hat{e}_1 = \cos \beta \hat{e}_1 - \sin \beta \hat{e}_3$, $\hat{e}_2 \mapsto \hat{e}_2 = \hat{e}_2$, $\hat{e}_3 \mapsto \hat{e}_3 = \sin \beta \hat{e}_1 + \cos \beta \hat{e}_2$];

iii) perform a third rotation of an angle $\gamma$ around $\hat{e}_3$ [$\hat{e}_1 \mapsto \hat{e}_1 = \cos \gamma \hat{e}_1 + \sin \gamma \hat{e}_2$, $\hat{e}_2 \mapsto \hat{e}_2 = -\sin \gamma \hat{e}_1 + \cos \gamma \hat{e}_2$].

In this way one gets

$$
\begin{pmatrix}
\hat{\chi} \\
\hat{N} \times \hat{\chi} \\
\hat{\chi}
\end{pmatrix} 
= 
R(\alpha, \beta, \gamma) 
\begin{pmatrix}
\hat{f}_1 \\
\hat{f}_2 \\
\hat{f}_3
\end{pmatrix},
$$

with

$$
tg \alpha = \frac{\hat{N}^2}{\hat{N}^3},
$$

$$
\cos \beta = \hat{N}^3,
$$

$$
tg \gamma = -\frac{\hat{\chi}^3}{(\hat{N} \times \hat{\chi})^3}.
$$

(D2)

Since $\hat{N}$ and $\hat{\chi}$ are functions of $\tilde{p}_{qa}$ only, see Eq. (15), it follows $\{\alpha, \beta\} = \{\beta, \gamma\} = \{\gamma, \alpha\} = 0$.

B) The $x$-convention (see Refs. [12], [10] (in the text) and [13]): the Euler angles $\theta$, $\varphi$ and $\psi$ are:

i) $\theta = \tilde{\beta}$; ii) $\cos \varphi = -\sin \alpha$, $\sin \varphi = \cos \alpha$; iii) $\cos \psi = \sin \tilde{\gamma}$, $\sin \psi = -\cos \tilde{\gamma}$.

We use the $y$-convention. Following Ref. [13], let us introduce the canonical momenta $p_\alpha$, $p_\beta$, $p_\gamma$ conjugated to $\alpha$, $\beta$, $\gamma$: $\{\alpha, p_\alpha\} = \{\beta, p_\beta\} = \{\gamma, p_\gamma\} = 1$ (note that this Darboux chart does not exist globally). Then, the results of Ref. [13] imply
\[ S_{q}^{1} = -\sin \tilde{\alpha} p_{\beta} + \frac{\cos \tilde{\alpha}}{\sin \beta} p_{\gamma} - \cos \tilde{\alpha} \cot \beta p_{\tilde{\alpha}}, \]
\[ S_{q}^{2} = \cos \tilde{\alpha} p_{\beta} + \frac{\sin \tilde{\alpha}}{\sin \beta} p_{\gamma} - \sin \tilde{\alpha} \cot \beta p_{\tilde{\alpha}}, \]
\[ S_{q}^{3} = p_{\tilde{\alpha}}, \]

\[ \tilde{S}_{q}^{1} = \sin \tilde{\gamma} p_{\beta} - \frac{\cos \tilde{\gamma}}{\sin \beta} p_{\tilde{\alpha}} + \cos \tilde{\gamma} \cot \beta p_{\tilde{\gamma}}, \]
\[ \tilde{S}_{q}^{2} = \cos \tilde{\gamma} p_{\beta} + \frac{\sin \tilde{\gamma}}{\sin \beta} p_{\tilde{\alpha}} - \sin \tilde{\gamma} \cot \beta p_{\tilde{\gamma}}, \]
\[ \tilde{S}_{q}^{3} = p_{\tilde{\gamma}}, \]

\[ p_{\tilde{\alpha}} = S_{q}^{3} = -\sin \tilde{\beta} \cos \tilde{\gamma} \tilde{S}_{q}^{1} + \sin \tilde{\beta} \sin \tilde{\gamma} \tilde{S}_{q}^{2} + \cos \tilde{\beta} \tilde{S}_{q}^{3}, \]
\[ p_{\tilde{\beta}} = -\sin \tilde{\alpha} S_{q}^{1} + \cos \tilde{\alpha} S_{q}^{2} = \sin \tilde{\gamma} \tilde{S}_{q}^{1} - \cos \tilde{\gamma} \tilde{S}_{q}^{2}, \]
\[ p_{\tilde{\gamma}} = \tilde{S}_{q}^{3} = \cos \tilde{\alpha} \sin \tilde{\beta} S_{q}^{1} + \sin \tilde{\alpha} \sin \tilde{\beta} S_{q}^{2} + \cos \tilde{\beta} S_{q}^{3}. \]

(D3)
APPENDIX E: THE 3-BODY CASE.

Let us try to rewrite the 3-body Hamiltonian (E.29) in a form reminiscent of the non-relativistic Eq.(3.31) of Ref. [2], since this is the form used in the static orientation-shape bundle approach [1]. We shall use the notation \( q^\mu \) for the 3 generalized shape variables (\( \rho_{qa}, |\vec{N}| \)), and \( p_\mu \) for the conjugate momenta (\( \tilde{\pi}_{qa}, \xi \)).

In the non-relativistic orientation-shape bundle approach [1] one adopts the condition \( \bar{S}_q = 0 \) for the definition of C-horizontal (corresponding to a gauge convention on the definition of vibrations); on the other hand, the intrinsic concept of vertical (corresponding to pure rotational motion) is defined by the condition of vanishing shape velocities \( \dot{q}^\mu = 0 \). If one would naively follow the non-relativistic formulation, the following decomposition of the set \((\bar{S}_q, p_\mu)\) into a vertical () part and a C-horizontal ()\(_{ch}\) part

\[
(\bar{S}_q, p_\mu) = (\bar{S}_q, p_\mu)|_{\dot{q}=0} \overset{def}{=} \bar{S}_q \cdot \bar{C}_\mu(\bar{S}_q, q, m_i, \gamma_{ai}))_v + \]
\[+ (\vec{0}, \ p_\mu)|_{\dot{q}=0} \overset{def}{=} p_\mu - \bar{S}_q \cdot \bar{C}_\mu(\bar{S}_q, q, m_i, \gamma_{ai}))_{ch}, \tag{E1}
\]

would be expected. This separation would identify the gauge potential \( \bar{C}_\mu(\bar{S}_q, q, m_i, \gamma_{ai}) \), which could also be spin-dependent.

But in the relativistic case, since \( H_{(rel)} = \sum_{i=1}^3 H_{(rel)i} \) with each term being a square root, the shape velocities \( \dot{q}^\mu \), evaluated by means of the first half of Hamilton equations, have to be written as the sum of 3 terms \( \dot{q}^\mu_i, i = 1, 2, 3, \)

\[
\dot{q}^\mu = \frac{\partial H_{(rel)}}{\partial p_\mu} = \sum_{i=1}^3 \frac{1}{2H_{(rel)i}} \frac{\partial H^2_{(rel)i}}{\partial p_\mu} \overset{def}{=} \sum_{i=1}^3 \dot{q}^\mu_i, \quad \dot{q}^\mu_i = \frac{1}{2H_{(rel)i}} \frac{\partial H^2_{(rel)i}}{\partial p_\mu},
\]
\[
\dot{q}_1 = \frac{\partial H_{(rel)}}{\partial \tilde{\pi}_{q1}} = \sum_{i=1}^3 \frac{1}{2H_{(rel)i}} \frac{\partial H^2_{(rel)i}}{\partial \tilde{\pi}_{q1}} \overset{def}{=} \sum_{i=1}^3 \dot{q}_{1i},
\]
\[
\dot{q}_2 = \frac{\partial H_{(rel)}}{\partial \tilde{\pi}_{q2}} = \sum_{i=1}^3 \frac{1}{2H_{(rel)i}} \frac{\partial H^2_{(rel)i}}{\partial \tilde{\pi}_{q2}} \overset{def}{=} \sum_{i=1}^3 \dot{q}_{2i},
\]
\[
|\vec{N}| = \frac{\partial H_{(rel)}}{\partial \xi} = \sum_{i=1}^3 \frac{1}{2H_{(rel)i}} \frac{\partial H^2_{(rel)i}}{\partial \xi} \overset{def}{=} \sum_{i=1}^3 |\vec{N}|_i. \tag{E2}
\]

Therefore, the presence of the 3 square roots \( H_{(rel)i} \) with \( H_{(rel)} = \sum_{i=1}^3 H_{(rel)i} \) allows to introduce 3 concepts of i-vertical \( [\dot{q}^\mu_i = 0] \). As a consequence, now 3 concepts of Ch – i-horizontal (one for each particle) can be introduced, each one defining a decomposition of the type:

61 It corresponds to the choice of a special connection \( C \) on the SO(3) principal bundle determined by the Euclidean metric in the non-relativistic kinetic energy.

62 Remnants of the positive energy branch of the mass-shell conditions \( p_i^2 = m_i^2 \), which are characteristic of Lorentz signature.
with Eq. (7.31) giving its purely rotational content. This implies the presence of 3 different particle gauge potentials \( \vec{A}_\mu(q) \) (one for each particle) to be contrasted with the global but spin-dependent gauge potential \( \vec{C}_\mu(q, m_i, \gamma_{ai}) \) appearing in the vertical component of the momenta still given by the first part of Eq. (E1).

Therefore in the dynamical body frame approach [2] we could introduce 3 concepts of wave functionals characteristic of independent wave functions: since, as we shall see, even the angular velocity has the form 
\[ \vec{\omega} = \sum_{i=1}^{3} \vec{\omega}_i, \]
we could require to have \( \dot{\vec{\omega}}_r = 0 \) separately.

On the other hand, since in this approach the angular velocity is a measurable quantity, the dynamical vibrations are defined by the requirement \( \dot{\vec{\omega}}_r = 0 \).

This suggests to write the Hamiltonian for relative motions in the form
\[
H_{(rel)} = \sum_{i=1}^{3} H_{(rel)i} = \sum_{i=1}^{3} \sqrt{m_i^2 + \vec{T}_i^{-1rs}(q) \tilde{S}_{q}^r \tilde{S}_{q}^s + \tilde{\gamma}_{i}^{\mu\nu}(q) \left( \frac{p_\mu - \tilde{S}_{q}^\mu \cdot \vec{A}_{\mu}(q)}{\tilde{S}_{q}^\nu - \tilde{S}_{q}^\nu \cdot \vec{A}_{\nu}(q)} \right)},
\]
with Eq. (7.31) giving its purely rotational content.

It is clear that the generalized shape coordinates are not normal coordinates for the Hamiltonian. Now there are 3 inverse metrics \( \tilde{S}_{q}^r \cdot \tilde{S}_{q}^s \). There is no concept of inertia tensor and of reduced masses. Instead, there are 3 mass-independent particle tensors \( \vec{T}_i^{-1rs}(q) \) replacing the inverse of the non-relativistic inertia tensor \( \tilde{T}_i^{-1rs}(q, m) \) of Eq. (F18) of Ref. [2].

Let us see how it is possible to find \( \vec{C}(\tilde{\Sigma}, q, m_i, \gamma_{ai}) \), \( \vec{A}_\mu(q) \), \( \tilde{\gamma}_{i}^{\mu\nu}(q) \), \( \vec{T}_i^{-1rs}(q) \) starting from our choice of variables.

The 3 equations (E2) can be inverted to get \( p_\mu \) in terms of \( q^\mu \), \( \dot{q}^\mu \), \( \tilde{S}_{q}^i \), \( m_i \), \( \gamma_{ai} \): this is as difficult as finding the Lagrangian for the relative motion. Then, by definition we have
\[
\tilde{S}_{q}^i \cdot \vec{C}_{\mu}(\tilde{\Sigma}, q) = p_\mu|_{\dot{q}=0}, \quad \text{namely}
\]
\[
\tilde{S}_{q}^i \cdot \vec{C}_{\mu}(\tilde{\Sigma}, q, m_i, \gamma_{ai}) = \tilde{\pi}_{i} \frac{\dot{\Sigma}_{q}^i}{\rho_{q^i}},
\]
\[
\tilde{S}_{q}^i \cdot \vec{C}_{\mu}(\tilde{\Sigma}, q, m_i, \gamma_{ai}) = \tilde{\pi}_{2} \frac{\dot{\Sigma}_{q}^2}{\rho_{q^2}},
\]
\[
\tilde{S}_{q}^i \cdot \vec{C}_{\mu}(\tilde{\Sigma}, q, m_i, \gamma_{ai}) = \tilde{\pi}_{3} \frac{\dot{\Sigma}_{q}^3}{\rho_{q^3}},
\]

From Eqs. (E2) we have the following form for the components \( q^\mu_i \)
\[
\dot{\rho}_{q^1} = \frac{1}{2 H_{(rel)\iota}} \left[ 2(\gamma_{1\iota})^2 \tilde{\pi}_{q^1} + 2 \gamma_{1\iota} \gamma_{2\iota} \left( (2\tilde{N}^2 - 1) \tilde{\pi}_{q^2} + |\tilde{N}|(1 - \tilde{N}^2) \frac{\tilde{S}_{q}^2 - \xi}{\rho_{q^2}} \right) \right],
\]
\[
\dot{\rho}_{q^2} = \frac{1}{2 H_{(rel)\iota}} \left[ 2(\gamma_{2\iota})^2 \tilde{\pi}_{q^2} + 2 \gamma_{2\iota} \gamma_{1\iota} \left( (2\tilde{N}^2 - 1) \tilde{\pi}_{q^1} - |\tilde{N}|(1 - \tilde{N}^2) \frac{\tilde{S}_{q}^1 + \xi}{\rho_{q^1}} \right) \right],
\]
\[
|\tilde{N}|\iota = \frac{1}{2 H_{(rel)\iota}} \left[ (1 - \tilde{N}^2) \left( \frac{(\gamma_{1\iota})^2}{2\rho_{q^1}} + \frac{(\gamma_{2\iota})^2}{2\rho_{q^2}} \right) \xi - 2 \gamma_{1\iota} \gamma_{2\iota} |\tilde{N}|(1 - \tilde{N}^2) \left( \frac{\tilde{\pi}_{q^1}}{\rho_{q^1}} + \frac{\tilde{\pi}_{q^2}}{\rho_{q^2}} \right) + 2\sqrt{1 - \tilde{N}^2} \left( \frac{(\gamma_{1\iota})^2}{4\rho_{q^1}} - \frac{(\gamma_{2\iota})^2}{4\rho_{q^2}} \right) \tilde{S}_{q}^i \right].
\]
Since \( \dot{q}_i^\mu = 0 \) implies \( \frac{\partial H_{(rel)i}}{\partial p_\mu} = 0 \), we get \( \ddot{S}_q \cdot \ddot{A}_{\mu i}(q) = p_{\mu} |_{\dot{q}_i=0} \). Then, from these equations we can find \( \ddot{A}_{\mu i}(q) \). Using the shape variables of our canonical basis we find

\[
\ddot{A}_{\mu i}(q) = \ddot{A}_{i\mu}(q) = 0,
\]

\[
\ddot{A}_{i\xi}(q) = -\frac{\sqrt{1 - \tilde{N}^2}}{\gamma_{11}[(1 - \tilde{N}^2) \rho_{q2} + (2\tilde{N}^2 - 1) \rho_{q1} + (\gamma_{2i} / \rho_{q1}) - (2\tilde{N}^2 - 1) \gamma_{1i} \gamma_{2i}] \ddot{A}_{\xi}(q)},
\]

\[
\ddot{A}_{i\rho_1}(q) = -\frac{\sqrt{1 - \tilde{N}^2}}{\gamma_{11}[(1 - \tilde{N}^2) \rho_{q2} + (2\tilde{N}^2 - 1) \rho_{q1} + (\gamma_{2i} / \rho_{q1}) - (2\tilde{N}^2 - 1) \gamma_{1i} \gamma_{2i}] \ddot{A}_{\rho_1}(q)},
\]

\[
\ddot{A}_{i\rho_2}(q) = +\frac{\sqrt{1 - \tilde{N}^2}}{\gamma_{11}[(1 - \tilde{N}^2) \rho_{q2} + (2\tilde{N}^2 - 1) \rho_{q1} + (\gamma_{2i} / \rho_{q1}) - (2\tilde{N}^2 - 1) \gamma_{1i} \gamma_{2i}] \ddot{A}_{\rho_2}(q)}.
\]

The term in \( H_{(rel)i} \) quadratic in the \( p_\mu \)'s identifies the \( \dddot{\nu}_{i\mu}^\nu(q) \) so that

\[
\dddot{\nu}_{i\mu}^\nu(q) = \begin{pmatrix}
\gamma_{1i}^2 & (2\tilde{N}^2 - 1) \gamma_{1i} \gamma_{2i} & -\frac{|\tilde{N}|(1 - \tilde{N}^2)}{\rho_{q2}} \gamma_{1i} \gamma_{2i} \\
(2\tilde{N}^2 - 1) \gamma_{1i} \gamma_{2i} & \gamma_{2i}^2 & -\frac{|\tilde{N}|(1 - \tilde{N}^2)}{\rho_{q1}} \gamma_{1i} \gamma_{2i} \\
-\frac{|\tilde{N}|(1 - \tilde{N}^2)}{\rho_{q2}} \gamma_{1i} \gamma_{2i} & -\frac{|\tilde{N}|(1 - \tilde{N}^2)}{\rho_{q1}} \gamma_{1i} \gamma_{2i} & 1 - \tilde{N}^2 - \frac{\gamma_{1i}^2}{\rho_{q1}} + \frac{\gamma_{2i}^2}{\rho_{q2}} + (2\tilde{N}^2 - 1) \gamma_{1i} \gamma_{2i}
\end{pmatrix}.
\]

The body frame angular velocity results

\[
\dddot{\omega}_r = \frac{1}{H_{(rel)i}} \dddot{S}_q = \sum_{i=1}^{N} \frac{1}{2H_{(rel)i}} \dddot{H}_{(rel)i} = \sum_{i=1}^{N} \dddot{\omega}_r,
\]

\[
\dddot{\omega}_i = \frac{1}{H_{(rel)i}} \dddot{S}_q = \dddot{A}_{\mu i}(q) \dddot{\nu}_{i\mu}^\nu(q) \left( p_\nu - \dddot{S}_q \cdot \dddot{A}_{\nu}(q) \right),
\]

\[
\dddot{\omega}_1 |_{\dddot{S}_q = 0} = \frac{1}{2H_{(rel)i}} \left[ \frac{2}{\tilde{N}^2} \left( \frac{\gamma_{1i}^2}{4\rho_{q1}^2} + \frac{\gamma_{2i}^2}{4\rho_{q2}^2} + \frac{\gamma_{1i} \gamma_{2i}}{2\rho_{q1} \rho_{q2}} \right) \ddot{S}_1 - \frac{2}{\tilde{N}^2} \left( \frac{\gamma_{1i}^2}{4\rho_{q1}^2} - \frac{\gamma_{2i}^2}{4\rho_{q2}^2} \right) \ddot{S}_q \right],
\]

\[
\dddot{\omega}_2 |_{\dddot{S}_q = 0} = \frac{1}{2H_{(rel)i}} \left[ 2 \left( \frac{\gamma_{1i}^2}{4\rho_{q1}^2} + \frac{\gamma_{2i}^2}{4\rho_{q2}^2} + \frac{\gamma_{1i} \gamma_{2i}}{2\rho_{q1} \rho_{q2}} \right) \ddot{S}_1 + \frac{\sqrt{1 - \tilde{N}^2}}{2 \rho_{q2}} \left( \frac{\gamma_{1i}^2}{4\rho_{q1}^2} - \frac{\gamma_{2i}^2}{4\rho_{q2}^2} \right) \xi + 2 \gamma_{1i} \gamma_{2i} \ddot{S}_1 \right],
\]

\[
\dddot{\omega}_3 |_{\dddot{S}_q = 0} = \frac{1}{2H_{(rel)i}} \left[ \frac{2}{\sqrt{1 - \tilde{N}^2}} \left( \frac{\gamma_{1i}^2}{4\rho_{q1}^2} + \frac{\gamma_{2i}^2}{4\rho_{q2}^2} - \frac{\gamma_{1i} \gamma_{2i}}{2\rho_{q1} \rho_{q2}} \right) \ddot{S}_1 - \frac{2}{\sqrt{1 - \tilde{N}^2}} \left( \frac{\gamma_{1i}^2}{4\rho_{q1}^2} - \frac{\gamma_{2i}^2}{4\rho_{q2}^2} \right) \ddot{S}_q \right],
\]

\[
\text{or } \dddot{\omega}_i |_{\dddot{S}_q = 0} = -\frac{1}{2H_{(rel)i}} \dddot{S}_q = 0 \dddot{A}_{\mu i}(q) \dddot{\nu}_{i\mu}^\nu(q) p_\nu.
\]
This allows to determine the functions $\tilde{T}_i^{-1rs}(q)$ once $\tilde{A}_{i\mu}(q)$ and $\tilde{v}_i^{\mu\nu}(q)$ are known. From the equalities
\begin{align}
H_{(rel)}\bar{\omega}_i^1 &= \tilde{T}_i^{-11s}(q)\tilde{S}_q^s,
H_{(rel)}\bar{\omega}_i^2 &= \tilde{T}_i^{-12s}(q)\tilde{S}_q^s - \tilde{A}_{i\mu}^2(q)\tilde{v}_i^{\mu\nu}(q)\left(p_\nu - \tilde{S}_q^2\tilde{A}_{i\nu}(q)\right),
H_{(rel)}\bar{\omega}_i^3 &= \tilde{T}_i^{-13s}(q)\tilde{S}_q^s,
\end{align}
we find the following non-zero components
\begin{align}
\tilde{T}_i^{-111}(q) &= \frac{1}{N^2}\left(\frac{(\gamma_{1i})^2}{4\rho_{q_1}^2} + \frac{(\gamma_{2i})^2}{4\rho_{q_2}^2} + \frac{\gamma_{1i}\gamma_{2i}}{2\rho_{q_1}\rho_{q_2}}\right),
\tilde{T}_i^{-113}(q) &= -\frac{1}{|N|\sqrt{1 - N^2}}\left(\frac{(\gamma_{11})^2}{4\rho_{q_1}^2} - \frac{(\gamma_{21})^2}{4\rho_{q_2}^2}\right),
\tilde{T}_i^{-122}(q) &= \frac{1}{2}\left[\frac{\gamma_{11}^2}{\rho_{q_1}^2} + \frac{\gamma_{21}^2}{\rho_{q_2}^2} + \frac{2(\tilde{N}^2 - 1)\gamma_{11}\gamma_{21}}{\rho_{q_1}\rho_{q_2}}\right] - \\
&- \frac{|N|(1 - \tilde{N}^2)\gamma_{11}\gamma_{21}}{\rho_{q_1}\rho_{q_2}}\left(\rho_{q_1}\tilde{A}_{i\mu_1}^2(q) - \rho_{q_2}\tilde{A}_{i\mu_2}^2(q)\right) - \frac{\sqrt{1 - \tilde{N}^2}}{2}\left(\frac{\gamma_{11}^2}{\rho_{q_1}^2} - \frac{\gamma_{21}^2}{\rho_{q_2}^2}\right)\tilde{A}_{i\xi}^2(q),
\tilde{T}_i^{-133}(q) &= \frac{1}{\sqrt{1 - \tilde{N}^2}}\left(\frac{(\gamma_{1i})^2}{4\rho_{q_1}^2} + \frac{(\gamma_{2i})^2}{4\rho_{q_2}^2} - \frac{\gamma_{1i}\gamma_{2i}}{2\rho_{q_1}\rho_{q_2}}\right). \tag{E11}
\end{align}

Finally, let us recall the following results of Appendix C of Ref. [2]
\begin{align}
\bar{\pi}_{q_1}^2 &= \tilde{\pi}_{q_1}^2 + \frac{1}{4\rho_{q_1}^2}\left[\xi^2(1 - \tilde{N}^2) + (\tilde{S}_q^2)^2 + \frac{1}{N^2}(\tilde{S}_q^1)^2 + \\
&+ \frac{(\tilde{S}_q^3)^2}{1 - \tilde{N}^2} + 2(\xi\sqrt{1 - \tilde{N}^2}\tilde{S}_q^2 - \tilde{S}_q^1\tilde{S}_q^3)\right],
\bar{\pi}_{q_2}^2 &= \tilde{\pi}_{q_2}^2 + \frac{1}{4\rho_{q_2}^2}\left[\xi^2(1 - \tilde{N}^2) + (\tilde{S}_q^2)^2 + \frac{1}{N^2}(\tilde{S}_q^1)^2 + \\
&+ \frac{(\tilde{S}_q^3)^2}{1 - \tilde{N}^2} - 2(\xi\sqrt{1 - \tilde{N}^2}\tilde{S}_q^2 - \tilde{S}_q^1\tilde{S}_q^3)\right],
\bar{\pi}_{q_1} \cdot \bar{\pi}_{q_2} &= (2\tilde{N}^2 - 1)\tilde{\pi}_{q_1}\tilde{\pi}_{q_2} + \\
&+ |N|\sqrt{1 - \tilde{N}^2}\left[\left(\frac{\tilde{\pi}_{q_1}}{\rho_{q_2}} - \frac{\tilde{\pi}_{q_2}}{\rho_{q_1}}\right)\tilde{S}_q^2 - \left(\frac{\tilde{\pi}_{q_1}}{\rho_{q_2}} + \frac{\tilde{\pi}_{q_2}}{\rho_{q_1}}\right)\xi\sqrt{1 - \tilde{N}^2}\right] + \\
&+ \frac{1}{4\rho_{q_1}\rho_{q_2}}\left[(2\tilde{N}^2 - 1)(\tilde{S}_q^2)^2 + \frac{1}{N^2}(\tilde{S}_q^1)^2 - \frac{(\tilde{S}_q^3)^2}{1 - \tilde{N}^2}\right] + \\
&+ (1 - \tilde{N}^2)(2\tilde{N}^2 - 1)\xi^2, \tag{E12}
\end{align}
\begin{align}
\bar{\pi}_{q_1}^2 &= \tilde{\pi}_{q_1}^2 + \frac{1}{4\rho_{q_1}^2}\left[\xi^2(1 - \tilde{N}^2) + (\tilde{S}_q^2)^2 + \frac{1}{N^2}(\tilde{S}_q^1)^2 + \\
&+ \frac{(\tilde{S}_q^3)^2}{1 - \tilde{N}^2} - 2(\xi\sqrt{1 - \tilde{N}^2}\tilde{S}_q^2 - \tilde{S}_q^1\tilde{S}_q^3)\right],
\bar{\pi}_{q_2}^2 &= \tilde{\pi}_{q_2}^2 + \frac{1}{4\rho_{q_2}^2}\left[\xi^2(1 - \tilde{N}^2) + (\tilde{S}_q^2)^2 + \frac{1}{N^2}(\tilde{S}_q^1)^2 + \\
&+ \frac{(\tilde{S}_q^3)^2}{1 - \tilde{N}^2} + 2(\xi\sqrt{1 - \tilde{N}^2}\tilde{S}_q^2 - \tilde{S}_q^1\tilde{S}_q^3)\right],
\bar{\pi}_{q_1} \cdot \bar{\pi}_{q_2} &= (2\tilde{N}^2 - 1)\tilde{\pi}_{q_1}\tilde{\pi}_{q_2} + \\
&+ |N|\sqrt{1 - \tilde{N}^2}\left[\left(\frac{\tilde{\pi}_{q_1}}{\rho_{q_2}} - \frac{\tilde{\pi}_{q_2}}{\rho_{q_1}}\right)\tilde{S}_q^2 - \left(\frac{\tilde{\pi}_{q_1}}{\rho_{q_2}} + \frac{\tilde{\pi}_{q_2}}{\rho_{q_1}}\right)\xi\sqrt{1 - \tilde{N}^2}\right] + \\
&+ \frac{1}{4\rho_{q_1}\rho_{q_2}}\left[(2\tilde{N}^2 - 1)(\tilde{S}_q^2)^2 + \frac{1}{N^2}(\tilde{S}_q^1)^2 - \frac{(\tilde{S}_q^3)^2}{1 - \tilde{N}^2}\right] + \\
&+ (1 - \tilde{N}^2)(2\tilde{N}^2 - 1)\xi^2, \tag{E12}
\end{align}
\[ + \left( \frac{\tilde{S}_q^3}{1 - \tilde{N}^2} + 2(\xi \sqrt{1 - \tilde{N}^2} \tilde{S}_q^2 - \frac{\tilde{S}_q^1 \tilde{S}_q^3}{|\tilde{N}| \sqrt{1 - \tilde{N}^2}}) \right), \]

\[ \tilde{\pi}_{q_2} = \tilde{\pi}_{q_2}^2 + \frac{1}{4\rho_{q_2}^2} \left[ \xi^2(1 - \tilde{N}^2) + (\tilde{S}_q^2)^2 + \frac{1}{\tilde{N}^2} (\tilde{S}_q^1)^2 + \frac{(\tilde{S}_q^3)^2}{1 - \tilde{N}^2} - 2(\xi \sqrt{1 - \tilde{N}^2} \tilde{S}_q^2 - \frac{\tilde{S}_q^1 \tilde{S}_q^3}{|\tilde{N}| \sqrt{1 - \tilde{N}^2}}) \right], \]

\[ \tilde{\pi}_{q_1} \cdot \tilde{\pi}_{q_2} = (2\tilde{N}^2 - 1)\tilde{\pi}_{q_1} \tilde{\pi}_{q_2} + \]

\[ + |\tilde{N}| \sqrt{1 - \tilde{N}^2} \left[ (\tilde{\pi}_{q_1} - \tilde{\pi}_{q_2}) \tilde{S}_q^2 - (\tilde{\pi}_{q_1} + \tilde{\pi}_{q_2}) \xi \sqrt{1 - \tilde{N}^2} \right] + \]

\[ + \frac{1}{4\rho_{q_1} \rho_{q_2}} \left[ (2\tilde{N}^2 - 1)(\tilde{S}_q^2)^2 + \frac{1}{\tilde{N}^2} (\tilde{S}_q^1)^2 - \frac{(\tilde{S}_q^3)^2}{1 - \tilde{N}^2} + \xi^2(1 - \tilde{N}^2)(2\tilde{N}^2 - 1) \right]. \] (E13)
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