Very weak solution for the exterior stationary Stokes equations with Navier slip boundary condition

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Abstract

In some problems of fluid mechanics, it is possible to be confronted with data that are not regular, that is why we are interested here in the search for the so-called very weak solutions for the stationary Stokes problem with Navier-type boundary conditions in a three-dimensional exterior domain. The problem describes the flow of a viscous and incompressible fluid past an obstacle where we assume that the fluid may slip on the boundary of the obstacle. Because the flow domain is unbounded, we set the problem in weighted Sobolev spaces in order to control the behavior at infinity of the solutions. Our purpose is to prove the existence and the uniqueness of a very weak solution in a Hilbertian framework.

Keywords: Stokes equations, Navier boundary condition, exterior domain, weighted spaces, strong solutions, very weak solutions.

1 Introduction

We are interested in this paper on the existence of very weak solutions for the (linear stationary) Stokes problem in a domain $\Omega$ of $\mathbb{R}^3$. The set $\Omega$ is simply connected exterior domain, namely the complement of a simply connected bounded domain which represents the obstacle. The system of stationary Stokes be written as follows:

$$-\Delta u + \nabla \pi = f \quad \text{and} \quad \text{div} \, u = 0 \quad \text{in} \Omega,$$

where the unknowns are $u$ the velocity of the fluid and its pressure $\pi$ and $f$ the external forces acting on the fluid. To these equations, we supplement the following Navier’s type slip boundary conditions:

$$u \cdot n = 0, \quad 2[D(u)n]_\tau + \alpha u_\tau = 0 \quad \text{on} \quad \Gamma,$$
where $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ denotes the rate-of-strain tensor field, $\alpha$ is a scalar friction function, $n$ is the unit normal vector to $\Gamma$ and the notation $[\cdot]_r$ denotes the tangential component of a vector on $\Gamma$. The boundary condition (1.2), proposed by H. Navier in 1827 [33]. The first condition in (1.2) is the no-penetration condition and the second condition expresses the fact that the tangential velocity is proportional to the tangential stress. The Navier slip conditions have been extensively studied, see for instance [2, 3, 15, 18, 21, 25, 26, 36] and references therein.

The purpose of this paper is to study the exterior problem composed by the Stokes equations (1.1) and the Navier slip boundary conditions (1.2), with a positive friction function $\alpha$ and where we also include the non homogeneous case. Although the Stokes problem set in bounded domains with conditions (1.2) has been well studied by various authors (see for instance [12, 16, 1] or [13, 36] for the case $\alpha = 0$ and references therein), to the best of our knowledge, it is not the case when the domain is unbounded and $\alpha > 0$. We can just mention [34] where (1.2) was used for the stationary Navier-Stokes equations in exterior domains and [19] for the stationary exterior Stokes equations, the authors in their articles they studied some results of existence and uniqueness for different types of solutions, including the variational solution $W^{\alpha,2}(\Omega) \times L^2(\Omega)$ and the strong solutions $W^{2,2}_{k+1}(\Omega) \times W^{1,2}_{k+1}(\Omega)$ for $k \in \mathbb{Z}$. For the case of Navier boundary conditions without friction ($\alpha = 0$), let us mention [8, 30], where the following boundary conditions (also known as Hodge boundary conditions) were used:

$$u \cdot n = 0, \quad \text{curl } u \times n = 0 \quad \text{on} \quad \Gamma, \quad (1.3)$$

where $\text{curl } u$ is the vorticity field. These conditions coincide with (1.2) on flat boundaries when $\alpha = 0$. They were also used in [17] for the study of the non stationary Navier-Stokes equations in half-spaces of $\mathbb{R}^2$. We finally refer to [31, 32] for the study of the non stationary problem of Navier-Stokes with mixed boundary conditions that include (1.2) without friction.

The notion of very weak solution for the stationary Stokes or Navier-Stokes equations, corresponding to very irregular data, has been developed in the last years by Giga [22] and also by Lions and Magenes [28] for the Laplace's equation. The Stokes problem when $\Omega$ is bounded with Dirichlet boundary condition has been studied by a large number of authors, from different points of view, here so we give some examples: Giga and Sohr [23], Amrouche and Girault [5] Galdi, Simader and Sohr [20], K. Schumacher [35], Amrouche and Rodriguez-Bellido [14], for the exterior domain we can mention Amrouche and Meslameni [9]. The very weak solutions of the problem (1.1)–(1.2) has been studied for a bounded domain in [12] for $\alpha = 0$, in [30] for the exterior domain for the case $\alpha = 0$ and in [10] for the half-space.

The aim of this work is to study some results of existence, uniqueness of very weak solution for the stationary Stokes problem (1.1) with the boundary condition (1.2). The study is based on a $L^2$-theory and, because the domain $\Omega$ is unbounded, we choose to set the problem in weighted spaces. The weight functions are polynomials and enable to describe the growth or the decay of functions at infinity which allows to look for solutions of (1.1)–(1.2) with various behavior at infinity and this is one of the main advantages of the weighted spaces. One important question is to define rigorously the traces of the vector functions which are living in subspaces of $W^{0,2}_{-k-1}(\Omega)$ (see Lemma 2.5). We prove existence and uniqueness of very weak solutions $(u, \pi)$ belongs to $W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$ (see Definition 3.4). The main idea here relies on the use of a duality argument using the strong solutions obtained in [19].
The paper is organized as follows. In Section 2, we introduce the Notations, the functional framework based on weighted Hilbert spaces. We recall the definitions of some spaces and their respective norms, besides some density results, characterization of dual space, we shall precise in which sense, the Navier slip boundary conditions are taken. We recall the main results on Stokes problem that we shall use. The main results of this paper are presented in Theorems 3.6 – 3.8 which proves the existence and uniqueness of very weak solution $(u, \pi)$ belongs to $W_{−k−1}^{0,2}(\Omega) \times W_{−k−1}^{−1,2}(\Omega)$, for $k \in \mathbb{Z}$.

2 Notations and preliminaries

2.1 Notations

Throughout this paper we assume that $\Omega'$ denotes a bounded open in $\mathbb{R}^3$ of class $\mathcal{C}^{2,1}$, simply connected bounded and with a connected boundary $\partial\Omega' = \Gamma$, representing an obstacle. Let $\Omega$ be the complement of $\Omega'$ in $\mathbb{R}^3$, in other words an exterior domain. We will use bold characters for vector and matrix fields. Let $\mathbb{N}$ denote the set of set of non-negative integers and $\mathbb{Z}$ the set of all integers. For any multi-index $\lambda \in \mathbb{N}^3$, we denote by $\partial^\lambda$ the differential operator of order $\lambda$,

$$\partial^\lambda = \frac{\partial^{|\lambda|}}{\partial_1^{\lambda_1} \partial_2^{\lambda_2} \partial_3^{\lambda_3}}, \quad |\lambda| = \lambda_1 + \lambda_2 + \lambda_3.$$

For any $k \in \mathbb{Z}$, $\mathcal{P}_k$ stands for the space of polynomials of degree less than or equal to $k$ and $\mathcal{P}_k^\Delta$ the harmonic polynomials of $\mathcal{P}_k$. If $k$ is a negative integer, we set by convention $\mathcal{P}_k = \{0\}$. We denote by $\mathcal{D}(\Omega)$ the space of $\mathcal{C}^{\infty}$ functions with compact support in $\Omega$, $\mathcal{D}(\mathbb{R}^3)$ the restriction to $\Omega$ of functions belonging to $\mathcal{D}(\mathbb{R}^3)$. We recall that $\mathcal{D}'(\Omega)$ is the well-known space of distributions defined on $\Omega$. We recall that $L^2(\Omega)$ is the well-known Lebesgue real space and for $m \geq 1$, we recall that $H^m(\Omega)$ is the well-known Hilbert space $W^{m,2}(\Omega)$. We shall write $u \in H^m_0(\Omega)$ to mean that $u \in H^m(\Omega)$, for any bounded domain $\Omega$, with $\Omega \in \mathbb{R}^3$. For any positive real number $R$, let $B_R$ denote the open ball centered at the origin, with radius $R$ and assuming that $R$ is sufficiently large for $\overline{\Omega'} \subset B_R$, we denote by $\Omega_R$ the intersection $\Omega \cap B_R$. The notation $\langle \cdot , \cdot \rangle$ will denote adequate duality pairing and will be specified when needed. If not specified, $\langle , \cdot \rangle$ will denote the duality pairing between the space $H^{-1/2}(\Gamma)$ and its dual space $H^{1/2}(\Gamma)$. Given a Banach space $X$, with dual space $X'$ and a closed subspace $Y$ of $X$, we denote by $X' \perp Y$ the subspace of $X'$ orthogonal to $Y$, i.e.

$$X' \perp Y = \{ f \in X'; \langle f , u \rangle = 0 \quad \forall u \in Y \} = (X/Y)' .$$

The space $X' \perp Y$ is also called the polar space of $Y$ in $X'$. Given $A$ and $B$ two matrix fields, such that $A = (a_{ij})_{1 \leq i, j \leq 3}$ and $B = (b_{ij})_{1 \leq i, j \leq 3}$, then we define $A : B = (a_{ij}b_{ij})_{1 \leq i, j \leq 3}$. Finally, as usual, $C > 0$ denotes a generic constant the value of which may change from line to line and even at the same line.

2.2 Weighted Hilbert spaces

Let $x = (x_1, x_2, x_3)$ be a typical point in $\mathbb{R}^3$ and let $r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ denotes its distance to the origin. In order to control the behaviour at infinity of our functions and distributions we use for basic weights the quantity $\rho(x) = (1 + r^2)^{1/2}$ which is equivalent to $r$ at infinity. For $k \in \mathbb{Z}$, we introduce

$$W_{k}^{0,2}(\Omega) = \left\{ u \in \mathcal{D}'(\Omega), \rho^k u \in L^2(\Omega) \right\} ,$$
which is a Hilbert space equipped with the norm:
\[ \|u\|_{W^{0,2}_k(\Omega)} = \|\rho^k u\|_{L^2(\Omega)}. \]

Let \( m \geq 1 \) be an integer. We define the weighted Hilbert space:
\[ W^{m,2}_k(\Omega) = \left\{ u \in \mathcal{D}'(\Omega); \forall \lambda \in \mathbb{N}^3 : 0 \leq |\lambda| \leq m, \rho^{k-m+|\lambda|} \partial^\lambda u \in L^2(\Omega) \right\}, \]
equipped with the norm
\[ \|u\|_{W^{m,2}_k(\Omega)} = \left( \sum_{|\lambda| \leq m} \|\rho^{k-m+|\lambda|} \partial^\lambda u\|_{L^2(\Omega)}^2 \right)^{1/2}. \]

We define the semi-norm
\[ |u|_{W^{m,2}_k(\Omega)} = \left( \sum_{|\lambda| = m} \|\rho^k \partial^\lambda u\|_{L^2(\Omega)}^2 \right)^{1/2}. \]

Let us give an examples of such spaces that will be often used in the remaining of the paper.
For \( m = 1 \), we have
\[ W^{1,2}_k(\Omega) = \left\{ u \in \mathcal{D}'(\Omega), \rho^{k-1} u \in L^2(\Omega), \rho^k \nabla u \in L^2(\Omega) \right\}. \]

For \( m = 2 \), we have
\[ W^{2,2}_{k+1}(\Omega) := \left\{ u \in W^{1,2}_k(\Omega), \rho^{k+1} \nabla^2 u \in L^2(\Omega) \right\}. \]

For the sake of simplicity, we have defined these spaces with integer exponents on the weight function. But naturally, these definitions can be extended to real number exponents with eventually some slight modifications (see [6] for more details).

We shall now give some basic properties of those spaces:

**Properties 2.1.**

1. The space \( \mathcal{D}(\overline{\Omega}) \) is dense in \( W^{m,2}_k(\Omega) \).

2. For any \( \lambda \in \mathbb{N}^3 \), the mapping
\[ u \in W^{m,2}_k(\Omega) \rightarrow \partial^\lambda u \in W^{m-|\lambda|,2}_k(\Omega) \] (2.1)
is continuous.

3. For \( m \in \mathbb{N} \setminus \{0\} \) and \( k \in \mathbb{Z} \), we have the following continuous imbedding:
\[ W^{m,2}_k(\Omega) \hookrightarrow W^{m-1,2}_{k-1}(\Omega). \] (2.2)

4. The space \( \mathcal{D}_{m-2-k} \) is the space of all polynomials included in \( W^{m,2}_k(\Omega) \) and the following Poincaré-type inequality holds:
\[ \forall u \in W^{m,2}_k(\Omega), \quad \inf_{\mu \in \mathcal{P}_{j'}} \|u + \mu\|_{W^{m,2}_k(\Omega)} \leq C|u|_{W^{m,2}_k(\Omega)}, \] (2.3)
where \( j' = \min(m-2-k, m-1) \). In other words the semi-norm \( |\cdot|_{W^{m,2}_k(\Omega)} \) is a norm on \( W^{m,2}_k(\Omega)/\mathcal{D}_{j'} \). In particular \( |\cdot|_{W^{1,2}_0(\Omega)} \) is a norm on \( W^{1,2}_0(\Omega) \).
For more details on the above properties and the ones that we shall present in the following, the reader can refer to [24, 27, 6, 7] and references therein.

Note that all the local properties of the space \( W^{m,2}_k(\Omega) \) coincide with those of the standard Hilbert spaces \( H^{m,2}(\Omega) \). Hence, it also satisfies the usual trace theorems on the boundary \( \Gamma \).

Therefore, we can define the space

\[
\dot{W}^{m,2}_k(\Omega) = \left\{ u \in W^{m,2}_k(\Omega), \gamma_0 u = 0, \gamma_1 u = 0, \cdots, \gamma_{m-1} u = 0 \text{ on } \Gamma \right\}.
\]

If \( \Omega \) is the whole space \( \mathbb{R}^3 \), then the spaces \( \dot{W}^{m,2}_k(\mathbb{R}^3) \) and \( W^{m,2}_k(\mathbb{R}^3) \) coincide. The space \( \mathcal{D}(\Omega) \) is dense in \( \dot{W}^{m,2}_k(\Omega) \). Therefore, the dual space of \( \dot{W}^{m,2}_k(\Omega) \), denoted by \( W^{-m,2}_k(\Omega) \) is a space of distributions. For \( m \in \mathbb{N} \) and \( k \in \mathbb{Z} \), we have the continuous imbedding

\[
W^{-m,2}_k(\Omega) \subset W^{-m-1,2}_{k-1}(\Omega). \tag{2.4}
\]

Moreover, we have the following Poincaré-type inequality:

\[
\forall u \in \dot{W}^{m,2}_k(\Omega), \quad \| u \|_{W^{m,2}_k(\Omega)} \leq C |u|_{W^{m,2}_k(\Omega)}. \tag{2.5}
\]

We now introduce some weighted Hilbert spaces that are specific for the study of the Stokes problem (1.1) with the Navier boundary conditions (1.2). We start by introducing the following space for \( k \in \mathbb{Z} \):

\[
H_k(\text{div}; \Omega) = \left\{ v \in W^{0,2}_k(\Omega), \text{div } v \in W^{0,2}_{k+1}(\Omega) \right\}.
\]

This space is endowed with the norm

\[
\| v \|_{H_k(\text{div}; \Omega)} = \left( \| v \|^2_{W^{0,2}_k(\Omega)} + \| \text{div } v \|^2_{W^{0,2}_{k+1}(\Omega)} \right)^{1/2}.
\]

Observe that \( \mathcal{D}(\overline{\Omega}) \) is dense in \( H_k(\text{div}; \Omega) \). For the proof, one can use the same arguments as for the proof of the density of \( \mathcal{D}(\overline{\Omega}) \) in \( W^{m,2}_k(\Omega) \) (see [24]). Therefore, denoting by \( n \) the unit normal vector to the boundary \( \Gamma \) pointing outside \( \Omega \), if \( v \) belongs to \( H_k(\text{div}; \Omega) \), then \( v \) has a normal trace \( v \cdot n \) in \( H^{-1/2}(\Gamma) \) and there exists \( C > 0 \) such that

\[
\forall v \in H_k(\text{div}; \Omega), \quad \| v \cdot n \|_{H^{-1/2}(\Gamma)} \leq C \| v \|_{H_k(\text{div}; \Omega)}. \tag{2.6}
\]

Moreover the below Green’s formula holds. For any \( v \in H_k(\text{div}; \Omega) \) and \( \varphi \in W^{-1,2}_{k-1}(\Omega) \), we have

\[
\langle v \cdot n, \varphi \rangle_{\Gamma} = \int_{\Omega} v \cdot \nabla \varphi \, dx + \int_{\Omega} \varphi \, \text{div } v \, dx. \tag{2.7}
\]

The closure of \( \mathcal{D}(\Omega) \) in \( H_k(\text{div}; \Omega) \) is denoted by \( \dot{H}_k(\text{div}; \Omega) \) and can be characterized by

\[
\dot{H}_k(\text{div}; \Omega) = \left\{ v \in H_k(\text{div}; \Omega), v \cdot n = 0 \text{ on } \Gamma \right\}.
\]

Its dual space is denoted by \( H^{-1}_k(\text{div}; \Omega) \) and is characterized by the below proposition.

**Proposition 2.2.** Assume that \( \Omega \) is of class \( \mathcal{C}^{1,1} \) and let \( k \in \mathbb{Z} \). A distribution \( f \) belongs to \( H^{-1}_k(\text{div}; \Omega) \) if and only if there exist \( \psi \in W^{0,2}_k(\Omega) \) and \( \chi \in W^{0,2}_{k-1}(\Omega) \), such that \( f = \psi + \nabla \chi \).

Moreover

\[
\| \psi \|_{W^{0,2}_k(\Omega)} + \| \chi \|_{W^{0,2}_{k-1}(\Omega)} \leq C \| f \|_{H^{-1}_k(\text{div}; \Omega)}.
\]

The proof of Proposition 2.2 can be found in [8, Proposition 1.3]. A consequence of this proposition and the imbedding (2.4) is that, for any \( k \in \mathbb{Z} \), we have the imbedding

\[
H^{-1}_k(\text{div}; \Omega) \subset W^{-1,2}_{k-1}(\Omega). \tag{2.8}
\]
2.3 Density and trace results

In this work, we are going to study the existence and uniqueness of very weak solutions for the stationary Stokes problem (1.1)–(1.2). Let us recall some notations related to the boundary condition. First, for any vector field $v$ on $\Gamma$, we can write

$$ v = v_\tau + (v \cdot n) n, \quad \text{(2.9)} $$

where $v_\tau$ is the projection of $v$ on the tangent hyper-plan to $\Gamma$. Next, for any point $x$ on $\Gamma$, one may choose an open neighbourhood $W$ of $x$ in $\Gamma$ small enough to allow the existence of two families of $C^2$ curves on $W$ and where the lengths $s_1$ and $s_2$ along each family of curves are possible system of coordinates. Denoting by $\tau_1, \tau_2$ the unit tangent vectors to each family of curves, we have

$$ v_\tau = (v \cdot \tau_1) \tau_1 + (v \cdot \tau_2) \tau_2. \quad \text{(2.10)} $$

As a result for any $v \in \mathcal{D}(\Omega)$ the following formula holds (see [12, Lemma 2.1])

$$ 2[D(v)n]_\tau = \nabla_\tau (v \cdot n) - \Lambda v \quad \text{on} \quad \Gamma, \quad \text{(2.11)} $$

where

$$ \Lambda v = \sum_{k=1}^2 (v_\tau \cdot \frac{\partial n}{\partial s_k}) \tau_k. $$

Now, we need to introduce some functional spaces. Otherwise, it is necessary to establish some Green formulas which are deduce from density lemmas. We start by the following space:

$$ T_{k+1}^1(\Omega) = \left\{ v \in H_{k-1}^1(\text{div}, \Omega); \ \text{div} v \in \dot{W}^{1,2}_{k+1}(\Omega) \right\}, $$

which is a Hilbert space equipped with the following norm

$$ \| v \|_{T_{k+1}^1(\Omega)} = \| v \|_{H_{k-1}^{0,2}(\Omega)} + \| \text{div} v \|_{\dot{W}^{1,2}_{k+1}(\Omega)} $$

The following Lemma state some properties related to the space $T_{k+1}(\Omega)$.

**Lemma 2.3.** Let $k \in \mathbb{Z}$, the following properties hold:

1. A distribution $f$ belongs to $(T_{k+1}(\Omega))'$ if and only if there exist $\phi \in W^{0,2}_{-k+1}(\Omega)$ and $f_0 \in W^{-1,2}_{-k-1}(\Omega)$, such that

$$ f = \phi + \nabla f_0 $$

2. The space $\mathcal{D}(\Omega)$ is dense in $T_{k+1}(\Omega)$.

3. For any $\chi \in W^{-1,2}_{-k-1}(\Omega)$ and $v \in T_{k+1}(\Omega)$, we have

$$ \langle \nabla \chi, v \rangle_{(T_{k+1}(\Omega))' \times T_{k+1}(\Omega)} = -\langle \chi, \text{div} v \rangle_{W^{-1,2}_{-k-1}(\Omega) \times \dot{W}^{1,2}_{k+1}(\Omega)}. \quad \text{(2.11)} $$
The goal is to prove that the mapping 

\[ H_{-k-1}(\Delta; \Omega) = \{ v \in W_{-k-1}^{0,2}(\Omega) ; \Delta v \in (T_{k+1}(\Omega))' \} , \]

which has the following norm:

\[ \| v \|_{H_{-k-1}(\Delta; \Omega)} = \| v \|_{W_{-k-1}^{0,2}(\Omega)} + \| \Delta v \|_{(T_{k+1}(\Omega))'} \]

**Lemma 2.4.** The space \( \mathcal{D}(\overline{\Omega}) \) is dense in \( H_{-k-1}(\Delta; \Omega) \).

The proofs of Lemma 2.3 and Lemma 2.4 can be found in [30]. Finally, in order to write a Green formula, we define for \( k \in \mathbb{Z} \):

\[ S_{k+1}(\Omega) = \{ v \in W_{k+1}^{2,2}(\Omega) ; \text{div} v = 0, \text{v} \cdot n = 0, 2[D(v)n]_\tau + \alpha v_\tau = 0 \text{ on } \Gamma \} \]

The next lemma will help us to prove a trace result.

**Lemma 2.5.** Let \( \alpha \) belongs to \( W^{1,\infty}(\Gamma) \). The linear mapping \( \Theta : u \mapsto 2[D(u)n]_\tau + \alpha u_\tau \) defined on \( \mathcal{D}(\overline{\Omega}) \) can be extended to a linear and continuous mapping still denoted by \( \Theta \), from \( H_{-k-1}(\Delta; \Omega) \) into \( H^{-3/2}(\Gamma) \) and we have the following Green's formula: for any \( u \in H_{-k-1}(\Delta; \Omega) \) and \( \varphi \in S_{k+1}(\Omega) \),

\[ \langle \Delta u, \varphi \rangle_{(T_{k+1}(\Omega))' \times T_{k+1}^2(\Omega)} = \int_\Omega u \cdot \nabla \varphi \, dx + \int_\Gamma (2[D(u)n]_\tau + \alpha u_\tau, \varphi)_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} - \langle 2[D(\varphi)n] \cdot n, u - \varphi \rangle_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)} \]  

(2.12)

**Proof.** The goal is to prove that the mapping \( \Theta \) defined on \( \mathcal{D}(\overline{\Omega}) \) is continuous for the norm of \( H_{-k-1}(\Delta; \Omega) \). Let \( u \in \mathcal{D}(\overline{\Omega}) \) and \( \varphi \in S_{k+1}(\Omega) \). Then thanks to the following identity

\[ \Delta u = 2 \text{div} D(u) - \nabla \text{div} D(u) \]  

(2.13)

and the Green formula (2.7), we have

\[ \langle \Delta u, \varphi \rangle_{(T_{k+1}(\Omega))' \times T_{k+1}^2(\Omega)} = -2 \int_\Omega D(u) : \nabla \varphi \, dx + \int_\Omega \text{div} u \text{div} \varphi \, dx + 2 \langle D(u)n, \varphi \rangle_\Gamma \]

Next, the fact that \( \varphi \cdot n = 0 \) on \( \Gamma \) also implies that

\[ \langle D(u)n, \varphi \rangle_\Gamma = \langle ([D(u)n] \cdot n)n + [D(u)n]_\tau, \varphi \rangle_\Gamma = \langle [D(u)n]_\tau, \varphi \rangle_\Gamma. \]

It follows that, for any \( u \in \mathcal{D}(\overline{\Omega}) \) and \( \varphi \in S_{k+1}(\Omega) \), we have

\[ -\langle \Delta u, \varphi \rangle_\Omega = -2 \int_\Omega D(u) : \nabla \varphi \, dx - 2 \langle [D(u)n]_\tau, \varphi \rangle_\Gamma. \]

Since we have

\[ \int_\Omega D(u) : \nabla \varphi \, dx = \int_\Omega D(u) : D(\varphi) \, dx, \]

Then, we obtain

\[ \langle \Delta u, \varphi \rangle_{(T_{k+1}(\Omega))' \times T_{k+1}^2(\Omega)} = \int_\Omega u \cdot \Delta \varphi \, dx + \langle 2[D(u)n]_\tau, \varphi \rangle_\Gamma - \langle 2[D(\varphi)n, u]_\tau \rangle_\Gamma. \]  

(2.14)
Now, since $2[D(\phi)n]_\tau = -\alpha \phi_\tau$ on $\Gamma$, we have
\[-2\langle D(\phi)n, u \rangle = \langle \alpha \phi_\tau, u_\tau \rangle_\Gamma - 2\langle [D(\phi)n] \cdot n, u \cdot n \rangle_\Gamma\]
Plunging this in (2.14), for any $u \in \mathcal{D}(\Omega)$ and $\phi \in S_{k+1}(\Omega)$ we have
\[
\langle \Delta u, \phi \rangle_{T^2_{k+1}(\Omega) \times T^2_{k+1}(\Omega)} = \int \Omega \cdot \Delta \phi \, dx + \langle 2[D(\phi)n]_\tau + \alpha u_\tau, \phi \rangle_{H^{3/2}(\Gamma) \times H^{3/2}(\Omega)} - 2\langle [D(\phi)n] \cdot n, u \cdot n \rangle_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)}.
\]
Let now $\mu$ be any element of $H^{3/2}(\Gamma)$. Since $\alpha \in W^{1,\infty}(\Gamma)$, then $\alpha \mu_\tau \in H^{1/2}(\Gamma)$. So there exists an element $\phi$ in $W^{2,2}_{k+1}(\Omega)$ such that
\[
\phi = \mu_\tau \quad \text{and} \quad \frac{\partial \phi}{\partial n} = \Lambda \mu - n \text{div}_\Gamma \mu_\tau - \alpha \mu_\tau \quad \text{on} \, \Gamma.
\]
Where
\[
\Lambda \mu = \sum_{k=1}^2 \left( \mu_\tau \cdot \frac{\partial n}{\partial s_k} \right) r_k.
\]
As $\Lambda \mu \cdot n = 0$ on $\Gamma$, we have
\[
\frac{\partial \phi}{\partial n} \cdot n = -\text{div}_\Gamma \mu_\tau \quad \text{on} \, \Gamma
\]
and we recall the following formula (see [5]):
\[
\text{div}v = \text{div}_\Gamma \nu_\tau + \beta v \cdot n + \frac{\partial v}{\partial n} \cdot n \quad \text{on} \, \Gamma,
\]
where $\beta$ denotes the mean curvature of $\Gamma$, $\text{div}_\Gamma$ is the surface divergence. We deduce that
\[
\text{div} \phi = 0 \quad \text{on} \, \Gamma.
\]
Using relations (2.10), we have
\[
2[D(\phi)n]_\tau + \alpha \phi_\tau = \left( \frac{\partial \phi}{\partial n} \right)_\tau - \Lambda \phi + \alpha \phi_\tau.
\]
Since $\frac{\partial \phi}{\partial n} = \Lambda \mu - n \text{div}_\Gamma \mu_\tau - \alpha \mu_\tau$ on $\Gamma$ and $\phi = \mu_\tau$ on $\Gamma$, then we have
\[
\left( \frac{\partial \phi}{\partial n} \right)_\tau = (\Lambda \mu)_\tau - \alpha \mu_\tau
\]
We deduce that $2[D(\phi)n]_\tau + \alpha \phi_\tau = 0$ on $\Gamma$. Which implies that the function $\phi$ belongs to $S_{k+1}(\Omega)$ and satisfies:
\[
\begin{align*}
\phi &= \mu_\tau \quad \text{on} \, \Gamma, \\
\nabla \phi \cdot n &= \Lambda \mu - n \text{div}_\Gamma \mu_\tau - \alpha \mu_\tau \quad \text{on} \, \Gamma.
\end{align*}
\]
In addition, we have the following estimate
\[
\|\phi\|_{W^{2,2}_{k+1}(\Omega)} \leq C \|\mu\|_{H^{3/2}(\Gamma)} \leq C \|\mu\|_{H^{3/2}(\Gamma)}.
\]
Consequently,
\[
\left| \langle 2[D(u)n]_\tau + \alpha u_\tau, \mu \rangle \right| = \left| \langle 2[D(u)n]_\tau + \alpha u_\tau, \varphi \rangle \right| = \left| \langle 2[D(u)n]_\tau + \alpha u_\tau, \varphi \rangle \right| \\
\leq \left| \langle \Delta u, \varphi \rangle \right| + \left| \int _\Omega u \cdot \Delta \varphi \, dx - 2 \langle [D(\varphi)n]_\tau, u \cdot n \rangle_\Gamma \right| \\
\leq \| \Delta u \| _{(T_{k+1}(\Omega))'} \| \varphi \| _{T_{k+1}(\Omega)} + \| u \| _{W^{0,2}_k(\Omega)} \| \varphi \| _{W^{2,2}_{k+1}(\Omega)} \\
\leq C \| u \| _{H_{k-1}(\Delta; \Omega)} \| \varphi \| _{W^{2,2}_{k+1}(\Omega)}
\]

Thus, using (2.16), we obtain that for any \( u \in \mathcal{D}(\Omega) \):
\[
\| 2[D(u)n]_\tau + \alpha u_\tau \| _{H^{-3/2}(\Gamma)} \leq C \| u \| _{H_{k-1}(\Delta; \Omega)}.
\]

Therefore, the linear mapping \( \Theta : u \to 2[D(u)n]_\tau + \alpha u_\tau \) defined in \( \mathcal{D}(\Omega) \) is continuous for the norm of \( H_{k-1}(\Delta; \Omega) \). Finally, by density of \( \mathcal{D}(\Omega) \) in \( H_{k-1}(\Delta; \Omega) \), we can extend this mapping from \( H_{k-1}(\Delta; \Omega) \) into \( H^{-3/2}(\Gamma) \) and formula (2.12) holds. \( \square \)

3 Very weak solution

In this section, we consider the Stokes problem with Navier slip boundary conditions:
\[
\begin{cases} 
-\Delta u + \nabla \pi = f & \text{and} \quad \text{div} \, u = \chi \quad \text{in} \quad \Omega, \\
\quad u \cdot n = g \quad \text{and} \quad 2[D(u)n]_\tau + \alpha u_\tau = h \quad \text{on} \quad \Gamma.
\end{cases}
\]

Our aim is to investigate the existence and the uniqueness of a very weak solutions belongs to \( W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega) \), where \( k \in \mathbb{Z} \) for the Stokes problem \((\mathcal{S}_T)\). The main idea consists in the use of a duality argument with the strong solutions obtained in [19]. We assume that \( \alpha \) is a positive function that belongs to \( W^{1,\infty}(\Gamma) \).

Before stating the theorem of the existence and the uniqueness of the very weak solution for Stokes problem, we need to introduce the following null spaces for \( s \in \{1, 2\} \) and \( k \in \mathbb{Z} \):
\[
\mathcal{N}^{s,2}_k(\Omega) = \left\{ (u, \pi) \in W^{s,2}_k(\Omega) \times W^{s-1,2}_k(\Omega); \ T(u, \pi) = (0, 0) \quad \text{in} \quad \Omega \quad \text{and} \quad u \cdot n = 0, \quad 2[D(u)n]_\tau + \alpha u_\tau = 0 \quad \text{on} \quad \Gamma \right\}
\]

with
\[
T(u, \pi) = (-\Delta u + \nabla \pi, \text{div} \, u).
\]

For all \( k \in \mathbb{Z} \), we introduce the following spaces:
\[
N_k = \left\{ (\lambda, \mu) \in \mathcal{R}_k \times \mathcal{R}^\Delta_{k-1}; \ -\Delta \lambda + \nabla \mu = 0 \quad \text{and} \quad \text{div} \, \lambda = 0 \right\}
\]

that is the null space of the Stokes operator in the whole space \( \mathbb{R}^3 \), we recall that by agreement on the notation \( \mathcal{R}_k \), the space \( N_k = \{(0, 0)\} \) when \( k < 0 \) and \( N_0 = \mathcal{R}_0 \times \{0\} \).

The next proposition characterizes the kernel of \( \mathcal{N}_k^{1,2}(\Omega) \):

**Proposition 3.1.** Suppose that \( \Omega \) is of class \( \mathcal{C}^{1,1} \) and assume \( k \in \mathbb{Z} \).

- If \( k \geq 0 \), then \( \mathcal{N}_k^{1,2}(\Omega) = \{(0, 0)\} \).
• If \( k < 0 \), then \( \mathcal{N}^{1,2}_k(\Omega) = \{ (v - \lambda, \theta - \mu); (\lambda, \mu) \in N_{-k-1} \} \), where \((v, \theta) \in W^{1,2}_0(\Omega) \times L^2(\Omega)\) is the unique solution of the following problem:

\[
\begin{cases}
-\Delta v + \nabla \theta = 0 & \text{and } \text{div} \, v = 0 \quad \text{in } \Omega, \\
v \cdot n = \lambda \cdot n & \text{and } 2[D(v)n]_\Gamma + \alpha v = 2[D(\lambda)n]_\Gamma + \alpha \lambda & \text{on } \Gamma.
\end{cases}
\]  

(3.1)

The proof of this Proposition can be found in [19].

The next theorem states an existence, uniqueness and regularity result for problem \((\mathcal{S}_T)\) (for instance see [19]).

**Theorem 3.2.** Suppose that \( \Omega \) is of class \( \mathcal{C}^{2,1} \) and let \( k \in \mathbb{Z} \). Assume that \( f \in W^{0,2}_{k+1}(\Omega) \), \( g \in H^{3/2}(\Gamma) \), \( \chi \in W^{1,2}_{k+1}(\Omega) \) and \( h \in H^{1/2}(\Gamma) \) satisfying \( \mathbf{h} \cdot \mathbf{n} = 0 \) on \( \Gamma \). Assume moreover that the following compatibility condition is satisfied

\[
\forall (\xi, \eta) \in \mathcal{N}^{1,2}_{-k}(\Omega), \quad \int_\Omega f \cdot \xi d\mathbf{x} - \int_\Omega \chi \eta d\mathbf{x} = \langle g, 2[D(\xi)n] \cdot \eta - \eta \rangle_\Gamma - \langle h, \xi \rangle_\Gamma.
\]  

(3.2)

Then, the Stokes problem \((\mathcal{S}_T)\) has a solution \((u, \pi) \in W^{2,2}_{k+1}(\Omega) \times W^{1,2}_{k+1}(\Omega)\) unique up to an element of \( \mathcal{N}^{1,2}_k(\Omega) \) and we have:

\[
\inf_{(\lambda, \mu) \in \mathcal{N}^{1,2}_k(\Omega)} \left( \| u + \lambda \|_{W^{2,2}_{k+1}(\Omega)} + \| \pi + \mu \|_{W^{1,2}_{k+1}(\Omega)} \right) \leq C \left( \| f \|_{W^{0,2}_{k+1}(\Omega)} + \| \chi \|_{W^{1,2}_{k+1}(\Omega)} + \| g \|_{H^{3/2}(\Gamma)} + \| h \|_{H^{1/2}(\Gamma)} \right).
\]

The following lemma proves the identity between some null spaces.

**Lemma 3.3.** Assume that \( \Omega \) is of class \( \mathcal{C}^{2,1} \), \( \alpha \in W^{1,\infty}(\Gamma) \) and \( k \in \mathbb{Z} \) then we have the following identity:

\[
\mathcal{N}^{2,2}_{-k+1}(\Omega) = \mathcal{N}^{1,2}_{-k}(\Omega).
\]

**Proof.** For the proof of this lemma, we shall apply a technique used in [9] for the Stokes equation with the Dirichlet boundary conditions. The proof falls into two parts:

• First inclusion: Using the imbedding (2.2), then we have \( \mathcal{N}^{2,2}_{-k+1}(\Omega) \subset \mathcal{N}^{1,2}_{-k}(\Omega) \).

• Second inclusion: Let \((u, \pi) \in W^{1,2}_{-k}(\Omega) \times W^{0,2}_{-k}(\Omega)\) such that

\[
\begin{cases}
-\Delta u + \nabla \pi = 0 & \text{and } \text{div} \, u = 0 \quad \text{in } \Omega, \\
u \cdot n = 0 & \text{and } 2[D(u)n]_\Gamma + \alpha u = \mathbf{0} \quad \text{on } \Gamma.
\end{cases}
\]

Note that if \( u \in W^{1,2}_{-k}(\Omega) \) and \( -\Delta u + \nabla \pi = \mathbf{0} \) in \( \Omega \) with \( \pi \in W^{0,2}_{-k}(\Omega) \), we obtain \( u \in H^2_{-k}(\Omega) \) then thanks [19, Lemma 2.5] we have \( u \rightarrow 2[D(u)n]_\Gamma + \alpha u \) belongs to \( H^{-1/2}(\Gamma) \), and if \( \text{div} \, u = 0 \) in \( \Omega \), then \( u \cdot n \in H^{1/2}(\Gamma) \). That means that the boundary conditions makes sense.

Now, let us introduce the following partition of unity:

\[
\varphi, \psi \in \mathcal{C}^{\infty}(\mathbb{R}^3), \quad 0 \leq \varphi, \psi \leq 1, \quad \varphi + \psi = 1 \quad \text{in } \mathbb{R}^3,
\]

\[
\varphi = 1 \quad \text{in } B_R, \quad \text{supp} \, \varphi \subset B_{R+1}.
\]

(3.3)
Let $\Omega_{R+1}$ denote the intersection $\Omega \cap B_{R+1}$. Then, we can write

$$u = \varphi u + \psi, \quad \pi = \varphi \pi + \psi \pi.$$

The pair $(u, \pi)$ has an extension $(\tilde{u}, \tilde{\pi})$ that belongs to $W^{1,2}_{-k}(\mathbb{R}^3) \times W^{0,2}_{-k}(\mathbb{R}^3)$. It suffices to prove that $(\varphi u, \varphi \pi)$ belongs to $H^2(\Omega_{R+1}) \times H^1(\Omega_{R+1})$ and that $(\psi \tilde{u}, \psi \tilde{\pi})$ belongs to $W^{2,2}_{-k+1}(\mathbb{R}^3) \times W^{1,2}_{-k+1}(\mathbb{R}^3)$.

To that end, consider first

$$-\Delta (\psi \tilde{u}) + \nabla (\psi \tilde{\pi}) = f_1 \quad \text{and} \quad \text{div}(\psi \tilde{u}) = \chi_1 \quad \text{in} \quad \mathbb{R}^3,$$

where

$$f_1 = -(2 \nabla \tilde{u} \nabla \psi + u \Delta \psi) + \tilde{\pi} \nabla \psi \quad \text{and} \quad \chi_1 = \tilde{u} \cdot \nabla \psi.$$

We easily see that $f_1$ and $\chi_1$ have bounded supports and belong to $L^2_{loc}(\mathbb{R}^3) \times W^{1,2}_{loc}(\mathbb{R}^3)$. Consequently, $(f_1, \chi_1)$ belongs to $W^{0,2}_{-k+1}(\mathbb{R}^3) \times W^{1,2}_{-k+1}(\mathbb{R}^3)$, we need to show that $f_1$ and $\chi_1$ satisfy the following compatibility condition:

$$\forall (\Lambda, \mu) \in N_{-k-1}, \quad \langle f_1, \Lambda \rangle_{W^{0,2}_{-k+1}(\mathbb{R}^3) \times W^{0,2}_{-k+1}(\mathbb{R}^3)} - \langle \chi_1, \mu \rangle_{W^{1,2}_{-k+1}(\mathbb{R}^3) \times W^{1,2}_{-k+1}(\mathbb{R}^3)} = 0,$$

(3.5)

For this, using the same calculation in the proof of [19, Theorem 3.7]. We obtain (3.5).

Therefore, it follows from [4, Theorem 3.9], that there exists a unique solution $(\tilde{v}, \tilde{q}) \in (W^{2,2}_{-k+1}(\mathbb{R}^3) \times W^{1,2}_{-k+1}(\mathbb{R}^3))$ satisfying the following Stokes problem:

$$-\Delta \tilde{v} + \nabla \tilde{q} = f_1 \quad \text{and} \quad \text{div} \tilde{v} = \chi_1 \quad \text{in} \quad \mathbb{R}^3.$$

It follows that $(\tilde{v} - \psi \tilde{u}, \tilde{q} - \psi \tilde{\pi})$ belongs to $N_{k-1}$. Since $N_{k-1} \subset W^{2,2}_{-k+1}(\mathbb{R}^3) \times W^{1,2}_{-k+1}(\mathbb{R}^3)$, then there exist $(P, Q)$ belongs to $W^{2,2}_{-k+1}(\mathbb{R}^3) \times W^{1,2}_{-k+1}(\mathbb{R}^3)$ such that $(\tilde{v} - \psi \tilde{u}, \tilde{q} - \psi \tilde{\pi}) = (P, Q)$. Consequently, $(\psi \tilde{u}, \psi \tilde{\pi})$ belongs to $W^{2,2}_{-k+1}(\mathbb{R}^3) \times W^{1,2}_{-k+1}(\mathbb{R}^3)$.

Consider now the system

$$-\Delta (\varphi \tilde{u}) + \nabla (\varphi \tilde{\pi}) = f_2 \quad \text{and} \quad \text{div}(\varphi \tilde{u}) = \chi_2,$$

where $f_2$ and $\chi_2$ have similar expressions as $f_1$ and $\chi_1$ with $\psi$ replaced by $\varphi$. It is easy to check that $(f_2, \chi_2)$ belongs to $L^2(\Omega_{R+1}) \times H^1(\Omega_{R+1})$. Then the regularity results for the Stokes problem with Navier boundary conditions in a bounded domain of class $C^{2,1}$ (see [11]), allow to prove that the following Stokes problem:

$$\begin{cases}
-\Delta \varphi u + \nabla \varphi \pi = f_2 \quad \text{and} \quad \text{div} \varphi u = \chi_2 \quad \text{in} \quad \Omega_{R+1}, \\
\varphi u \cdot n = 0 \quad \text{and} \quad 2|D(\varphi u)n| + \alpha(\varphi u) = 0 \quad \text{on} \quad \Gamma. \\
\varphi u \cdot n = 0 \quad \text{and} \quad 2|D(\varphi u)n| + \alpha(\varphi u) = 0 \quad \text{on} \quad \partial B_{R+1},
\end{cases}$$

has a solution $(\varphi \tilde{u}, \varphi \tilde{\pi})$ belongs to $H^2(\Omega_{R+1}) \times H^1(\Omega_{R+1})$ which also implies that $(\varphi \tilde{u}, \varphi \tilde{\pi})$ belongs to $W^{2,2}_{-k+1}(\mathbb{R}^3) \times W^{1,2}_{-k+1}(\mathbb{R}^3)$.

Consequently, the pair $(u, \pi)$ belongs to $W^{2,2}_{-k+1}(\Omega) \times W^{1,2}_{-k+1}(\Omega)$ and thus $\mathcal{N}^{1,2}_{-k}(\Omega) \subset \mathcal{N}^{2,2}_{-k+1}(\Omega)$.
Now, we begin by giving the definition of very weak solutions.

**Definition 3.4.** Given \( f, \chi, g \) and \( h \) with
\[
\begin{align*}
    f & \in (T_{k+1}^2(\Omega))^\prime, \\
    \chi & \in W^{-0,2}_{-k}(\Omega), \\
    g & \in H^{-1/2}(\Gamma), \\
    h & \in H^{-3/2}(\Gamma)
\end{align*}
\]
such that \( h \cdot n = 0 \) on \( \Gamma \), a pair \((u, \pi) \in W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)\) is called very weak solution of \((\mathcal{S}_T)\) if and only if; for any \((\varphi, q) \in S_{k+1}^2(\Omega) \times W_{k+1}^{1,2}(\Omega)\), the relations
\[
-\int_\Omega u \cdot \Delta \varphi dx = \langle \pi, \text{div} \varphi \rangle_{W^{-1,2}_{-k-1}(\Omega) \times \dot{W}_{k+1}^{1,2}(\Omega)}
\]
\[
= \langle f, \varphi \rangle_{{(T_{k+1}^2(\Omega))}^\prime \times T_{k+1}^2(\Omega)} + \langle h, \varphi \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} - \langle g, 2[D(\varphi)n] \cdot n \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}
\]
and
\[
\int_\Omega u \cdot \nabla q dx = -\int_\Omega \chi q dx + \langle g, q \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}
\]
are satisfied.

Note that \( S_{k+1}(\Omega) \) is included in \( T_{k+1}(\Omega) \), which insures that all brackets are well defined.

**Proposition 3.5.** Under the assumptions of Definition 3.4, the following two statements are equivalent:

i) \((u, \pi) \in W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)\) is a very weak solution of \((\mathcal{S}_T)\).

ii) \((u, \pi) \in W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)\) satisfies the problem \((\mathcal{S}_T)\) in the sense of distributions.

**Proof.**

i) \(\Rightarrow\) ii) Let \((u, \pi) \in W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)\) a very weak solution of \((\mathcal{S}_T)\). Therefore, it follows immediately from (3.6) and (3.7), that for any \(\varphi \in \mathcal{D}(\Omega)\), we have
\[
-\Delta u + \nabla \pi = f \quad \text{and} \quad \text{div} u = \chi \quad \text{in} \ \Omega.
\]

It remains now to prove that the Navier boundary conditions are satisfied. Let us first prove that \(u \cdot n = g\) on \(\Gamma\). For this, we consider the equation \(\text{div} u = \chi\) in \(\Omega\). We multiply this equation by \(q \in W^{1,2}_{k+1}(\Omega)\) and compare with (3.7). We get
\[
\langle u \cdot n, q \rangle_{H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)} = \langle g, q \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}.
\]

Which yields \(u \cdot n = g \in H^{-1/2}(\Gamma)\).

Next, writing
\[
-\Delta u = f - \nabla \pi.
\]

Using Lemma 2.3, we deduce that \(u\) belongs to \(H_{-k-1}(\Delta; \Omega)\). It follows from Lemma 2.5 that \(2[D(u)n]_\tau + \alpha u_\tau \in H^{-3/2}(\Gamma)\). Applying (2.12) and (2.11) enables us to write: for any \(\varphi \in S_{k+1}^2(\Omega)\),
\[
\begin{align*}
-\int_\Omega u \cdot \Delta \varphi dx - \langle 2[D(u)n]_\tau + \alpha u_\tau, \varphi \rangle_{H^{3/2}(\Gamma) \times H^{3/2}(\Gamma)}
\end{align*}
\]
\[
+ \langle 2[D(\varphi)n] \cdot n, u \cdot n \rangle_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)} = \langle f, \varphi \rangle_{{(T_{k+1}^2(\Omega))}^\prime \times T_{k+1}^2(\Omega)} + \langle \pi, \text{div} \varphi \rangle_{W^{-1,2}_{-k-1}(\Omega) \times \dot{W}_{k+1}^{1,2}(\Omega)}.
\]
Comparing (3.6) with (3.8), we get, for any $\varphi \in S_{k+1}(\Omega)$

$$
\langle 2[D(u)n]_\tau + \alpha u_\tau,\varphi \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} = \langle h,\varphi \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)}.
$$

Let $\mu \in H^{3/2}(\Gamma)$, there exists $\varphi \in S_{k+1}(\Omega)$ such that $\varphi = \mu_\tau$ on $\Gamma$ (see proof of Lemma 2.5). Consequently,

$$
\langle 2[D(u)n]_\tau + \alpha u_\tau,\mu \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} = \langle h,\mu \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)}
$$

and we deduce that $2[D(u)n]_\tau + \alpha u_\tau = h$ on $\Gamma$.

ii)$\Rightarrow$i) Suppose that $(u,\pi) \in W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$ satisfies the problem $(S_\pi)$ in the sense of distributions. Then, we have

$$
-\Delta u + \nabla \pi = f \quad \text{and} \quad \text{div } u = \chi \quad \text{in } S'(\Omega),
$$

which implies that, for any $\varphi \in \mathscr{D}(\Omega)$, we have

$$
\langle -\Delta u + \nabla \pi, \varphi \rangle_{\mathscr{D}'(\Omega) \times \mathscr{D}(\Omega)} = \langle f, \varphi \rangle_{\mathscr{D}'(\Omega) \times \mathscr{D}(\Omega)}.
$$

(3.9)

Since $\mathscr{D}(\Omega)$ is dense in $T_{k+1}(\Omega)$, then (3.9) is still valid for any $\varphi \in T_{k+1}(\Omega)$. In particular, (3.9) is still valid for any $\varphi \in S_{k+1}(\Omega) \subset T_{k+1}(\Omega)$. Since $\nabla \pi \in (T_{k+1}(\Omega))'$ and $-\Delta u = f - \nabla \pi \in (T_{k+1}(\Omega))'$. Then according to (2.11) and (2.12), it is clear that (3.6) holds.

Now from the equation $\text{div } u = \chi$ in $\Omega$, we can deduce that for any $q \in \mathscr{D}(\Omega)$

$$
\int_\Omega \text{div } u q \, dx = \int_\Omega \chi q \, dx.
$$

Using the Green’s formula (2.7), we can deduce (3.7).

Now, we are in position to prove the existence and uniqueness of a very weak solution for Stokes problem $(S_\pi)$.

**Theorem 3.6.** Suppose that $g = 0$. Given any $f, \chi$ and $h$ with

$$
f \in (T_{k+1}(\Omega))', \quad \chi \in W^{0,2}_{-k}(\Omega), \quad h \in H^{-3/2}(\Gamma),
$$

such that $h \cdot n = 0$ on $\Gamma$ and for any $(\xi, \eta) \in A^{2,2}_{k+1}(\Omega)$,

$$
\langle f, \xi \rangle_{(T_{k+1}(\Omega))' \times T_{k+1}(\Omega)} - \int_\Omega \chi \eta \, dx + \langle h, \xi \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} = 0,
$$

(3.10)

then, problem $(S_\pi)$ has a solution $(u,\pi) \in W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$ unique up to an element of $A^{2,2}_{-k-1}(\Omega)$.

**Proof.** The proof of the following theorem is similar to [30, Theorem 7]. Observe that, in view of Proposition 3.5, if the pair $(u,\pi)$ that belongs to $W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$ is a solution of $(S_\pi)$, then for any $(\varphi,q) \in S_{k+1}(\Omega) \times W^{1,2}_{k+1}(\Omega)$, adding (3.6) and (3.7), we have

$$
\int_\Omega u \cdot (-\Delta \varphi + \nabla q) \, dx - \langle \pi, \text{div } \varphi \rangle_{W^{-1,2}_{-k-1}(\Omega) \times W^{1,2}_{k+1}(\Omega)} = \langle f, \varphi \rangle_{(T_{k+1}(\Omega))' \times T_{k+1}(\Omega)}
$$

$$
+ \langle h, \varphi \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} - \int_\Omega \chi q \, dx + \langle g, q - 2[D(\varphi)n] \cdot n \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)},
$$

(3.11)}
In particular, if \( \forall (\varphi, q) \in \mathcal{N}_{k+1}^{2,2}(\Omega) \) we obtain (3.10).

It remains now to look for \( (u, \pi) \) belongs to \( W_{-k-1}^{0,2}(\Omega) \times W_{-k-1}^{-1,2}(\Omega) \) satisfying (3.11). To that end, let \( T \) be the linear form defined from \( W_{-k-1}^{0,2}(\Omega) \times W_{-k-1}^{-1,2}(\Omega) \perp \mathcal{N}_{k+1}^{2,2}(\Omega) \) onto \( \mathbb{R} \) by

\[
T(F, \theta) = \langle f, \varphi \rangle_{(T_{k+1}(\Omega))'} \times T_{k+1}(\Omega) + \langle h, \varphi \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} - \int_{\Omega} \chi q \, dx,
\]

where the pair \( (\varphi, q) \in W_{k+1}^{2,2}(\Omega) \times W_{k+1}^{-1,2}(\Omega) \) is a solution of the following problem (see Theorem 3.2):

\[
\begin{aligned}
-\Delta \varphi + \nabla q &= F \quad \text{and} \quad \text{div} \varphi = -\theta \quad \text{in} \ \Omega, \\
\varphi \cdot n &= 0 \quad \text{and} \quad 2|\mathcal{D}(\varphi) n| + \alpha \varphi_r = 0 \quad \text{on} \ \Gamma
\end{aligned}
\]

and satisfying the following estimate:

\[
\inf_{(\lambda, \mu) \in \mathcal{N}_{k+1}^{2,2}(\Omega)} \left( \|\varphi + \lambda\|_{W_{k+1}^{2,2}(\Omega)} + \|q + \mu\|_{W_{k+1}^{-1,2}(\Omega)} \right) \leq C \left( \|F\|_{W_{-k-1}^{0,2}(\Omega)} + \|\theta\|_{W_{k+1}^{-1,2}(\Omega)} \right).
\]

(3.12)

Then for any pair \( (F, \theta) \in W_{k+1}^{0,2}(\Omega) \times W_{k+1}^{-1,2}(\Omega) \) and for any \( (\lambda, \mu) \in \mathcal{N}_{k+1}^{2,2}(\Omega) \), we can write

\[
|T(F, \theta)| = \left| \left( f, \varphi \right)_{(T_{k+1}(\Omega))'} \times T_{k+1}(\Omega) + \left( h, \varphi \right)_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} - \int_{\Omega} \chi q \, dx \right|
\]

\[
= \left| \left( f, \varphi + \lambda \right)_{(T_{k+1}(\Omega))'} \times T_{k+1}(\Omega) + \left( h, \varphi + \lambda \right)_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} - \int_{\Omega} \chi (q + \mu) \, dx \right|
\]

\[
\leq C \left( \|f\|_{T_{k+1}(\Omega)} + \|\lambda\|_{W_{k+1}^{0,2}(\Omega)} + \|h\|_{H^{-3/2}(\Gamma)} \right) \left( \|\varphi + \lambda\|_{W_{k+1}^{2,2}(\Omega)} + \|q + \mu\|_{W_{k+1}^{-1,2}(\Omega)} \right).
\]

Using (3.12), we have

\[
|T(F, \theta)| \leq C \inf_{(\lambda, \mu) \in \mathcal{N}_{k+1}^{2,2}(\Omega)} \left( \|\varphi + \lambda\|_{W_{k+1}^{2,2}(\Omega)} + \|q + \mu\|_{W_{k+1}^{-1,2}(\Omega)} \right)
\]

\[
\leq C \left( \|F\|_{W_{k+1}^{0,2}(\Omega)} + \|\theta\|_{W_{k+1}^{-1,2}(\Omega)} \right)
\]

From this, we can deduce that the linear form \( T \) is continuous on the space \( W_{k+1}^{0,2}(\Omega) \times W_{k+1}^{-1,2}(\Omega) \perp \mathcal{N}_{k+1}^{2,2}(\Omega) \). Since the dual space of \( W_{k+1}^{0,2}(\Omega) \times W_{k+1}^{-1,2}(\Omega) \perp \mathcal{N}_{k+1}^{2,2}(\Omega) \) is \( W_{-k-1}^{0,2}(\Omega) \times W_{-k-1}^{-1,2}(\Omega) \perp \mathcal{N}_{k+1}^{2,2}(\Omega) \), From Riesz’s Representation theorem, we deduce that there exists a unique \((u, \pi)\) belongs to \( W_{k+1}^{0,2}(\Omega) \times W_{-k-1}^{-1,2}(\Omega) \perp \mathcal{N}_{k+1}^{2,2}(\Omega) \) satisfying (3.11).

Now, we will finish with the case that \( g \) is not vanish. For this reason we need the following lemma concerning the existence and uniqueness for the exterior Neumann problem.

**Lemma 3.7.** For any \( f \) in \( L^2(\Omega) \) and \( g \) in \( H^{-1/2}(\Gamma) \). Then, the problem:

\[
-\Delta u = f \quad \text{in} \ \Omega, \quad \frac{\partial u}{\partial n} = g \quad \text{on} \ \Gamma,
\]

(3.13)

has a unique solution \( u \in W_{-1}^{1,2}(\Omega) \) and we have the following estimate:

\[
\|u\|_{W_{-1}^{1,2}(\Omega)} \leq C \left( \|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma)} \right)
\]

(3.14)

**Proof.** Let us extend \( f \) by zero in \( \Omega' \) and let \( \tilde{f} \) denote the extended function. Then \( \tilde{f} \) belongs to \( L^2(\mathbb{R}^3) \). Thanks [6, Theorem 3.9], there exists a unique function \( \tilde{v} \in W_{0}^{2,2}(\mathbb{R}^3) \) such that

\[
-\Delta \tilde{v} = \tilde{f} \quad \text{in} \ \mathbb{R}^3.
\]

(3.15)
Then $\nabla \bar{v} \cdot n$ belongs to $H^{1/2}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)$. It follows from [29, Theorem 2.7], that the following problem:

$$
\Delta w = 0 \quad \text{in } \Omega \quad \text{and} \quad \nabla w \cdot n = g - \nabla \bar{v} \cdot n \quad \text{on } \Gamma,
$$

has a unique solution $w \in W^{-1,2}_0(\Omega)$. Thus $u = \bar{v}|_\Omega + w \in W^{-1,2}_0(\Omega)$ is the required solution of (3.13).

**Theorem 3.8.** Given any $f, \chi, g$ and $h$ with

$$
f \in (T^2_{k+1}(\Omega))^t, \quad \chi \in W^{-0,2}_k(\Omega), \quad g \in H^{-1/2}(\Gamma), \quad h \in H^{-3/2}(\Gamma),
$$
such that $h \cdot n = 0$ on $\Gamma$ and for any $(\xi, \eta) \in \mathcal{N}^{2,2}_{k+1}(\Omega),$

$$
\langle f, \xi \rangle_{(T^2_{k+1}(\Omega))^t \times T^2_{k+1}(\Omega)} - \int_\Omega \chi \eta dx = \langle g, 2[D(\xi)n] \cdot n - \eta \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} - \langle h, \xi \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)},
$$

then, problem $(\mathcal{S}_T)$ has a solution $(u, \pi) \in W^{-0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$ unique up to an element of $\mathcal{N}^{2,2}_{-k+1}(\Omega)$.

**Proof.** Let $g \in H^{-1/2}(\Gamma)$. Consider the following exterior Neumann problem

$$
\Delta v = 0 \quad \text{in } \Omega \quad \text{and} \quad \nabla v \cdot n = g \quad \text{on } \Gamma.
$$

Then thanks to Lemma 3.7, this problem has a solution $v$ that belongs to $W^{-1,2}_{-k-1}(\Omega)$. Let $R$ be a positive real number large enough so that $\overline{\Omega^c} \subset B_R$ and let $\psi \in D(\Delta)$ such that $0 \leq \psi \leq 1$, supp $\psi \subset \overline{\Omega}_{R+1}$ and $\psi = 1$ in $\overline{\Omega}_R$. Set now $w = v\psi$, then $w$ has obviously a compact support and thus $w$ belongs to $W^{-1,2}_{-k-1}(\Omega)$. Next, $\Delta w = v\Delta \psi + 2\nabla v \nabla \psi$ has a compact support and thus belongs to $W^{0,2}_{-k}(\Omega)$.

Next we look for a pair $(z, \pi) \in W^{-0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$ satisfying

$$
\begin{cases}
-\Delta z + \nabla \pi = f + \Delta (\nabla w) \quad \text{and} \quad \text{div} z = \chi - \Delta w \quad \text{in } \Omega, \\
z \cdot n = 0 \quad \text{and} \quad 2[D(z)n]_\tau + az_\tau = h' \quad \text{on } \Gamma,
\end{cases}
$$

where $h' = h - 2[D(\nabla w)n]_\tau - \alpha(\nabla w)_\tau$.

Observe that in view of the assumption on $f$ and Lemma 2.3, $f + \Delta (\nabla w)$ belongs to $(T^2_{k+1}(\Omega))^t$. Besides, as $w$ belongs to $W^{-1,2}_{-k-1}(\Omega)$, then $\nabla w$ belongs to $W^{0,2}_{-k-1}(\Omega)$ and $\Delta (\nabla w)$ belongs to $(T^2_{k+1}(\Omega))^t$. As a consequence, $h'$ belongs to $H^{-3/2}(\Gamma)$ and clearly satisfies $h' \cdot n = 0$ on $\Gamma$. Thus, due to the Theorem 3.6, problem (3.19) has a solution $(z, \pi) \in W^{-0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$ if the following condition is satisfied.

$$
\forall (\xi, \eta) \in \mathcal{N}^{2,2}_{k+1}(\Omega), \quad \langle f + \Delta (\nabla w), \xi \rangle_{(T^2_{k+1}(\Omega))^t \times T^2_{k+1}(\Omega)} - \int_\Omega (\chi - \Delta w) \eta dx + \langle h', \xi \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} = 0.
$$

Let $(\xi, \eta)$ be in $\mathcal{N}^{2,2}_{k+1}(\Omega)$. For any $(\phi, \psi) \in D(\overline{\Omega}) \times D(\overline{\Omega})$, we have

$$
\int_\Omega \left[ (\Delta \phi + \nabla \psi) \cdot \xi - \eta \text{div} \phi \right] dx = \langle \phi \cdot n, 2[D(\xi)n] \cdot n - \eta \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} - \langle 2[D(\phi)n]_\tau + a \phi_\tau, \xi_\tau \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)}.
$$
In particular, the Green’s formula (3.21) holds for any $\varphi = \nabla w \in W_{-k-1}^{0,2}(\Omega)$ and $\psi = 0$. We obtain

$$
\langle -\Delta (\nabla w), \xi \rangle_{(T^2_{k+1}(\Omega))^\prime \times T^2_{k+1}(\Omega)} - \int_\Omega \Delta w \eta dx
= \langle g, 2[D(\xi)\mathbf{n} \cdot \mathbf{n} - \eta] \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} - \langle h - h', \xi \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} = 0.
$$

Combining (3.17) and (3.22) allows to obtain (3.20). Thus setting $u = z + \nabla w$, the pair $(u, \pi) \in W_{-k-1}^{0,2}(\Omega) \times W_{-k-1}^{-1,2}(\Omega)$ is the solution of ($\mathcal{F}_T$).

□

Conclusion

In this paper, we solved the exterior Stokes problem with the Navier slip boundary conditions with a positive friction term. To prescribe the growth or decay of functions at infinity, we set the problem in weighted Sobolev spaces. Our study is based on an $L^2$-theory. We established the existence and the uniqueness of the very weak solution to the problem. The main idea consists in the use of a duality argument with the strong solutions obtained in [19]. In a forthcoming work, we will study the very weak solution to the problem in weighted $L^p$ spaces.

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