IN WHICH DIMENSIONS IS THE BALL RELATIVELY WORST COVERING?

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Abstract. We consider the problem of identifying the worst centrally-symmetric shape for covering $n$-dimensional space with a lattice. Here we focus on the dimensions where the thinnest lattice covering with balls is known and ask whether the ball is relatively worst at covering in these dimensions compared to all shapes of arbitrarily small asphericity. We find that the ball is relatively worst covering in 3 dimensions, but not so for 4 and 5 dimensions.

1. Introduction

The lattice sphere packing and covering problems can be stated in similar ways: in both problems we look for an optimal arrangement of equal-sized spheres centered at points of a lattice; whereas in the packing problem we must have no overlap between spheres and we must minimize the amount of uncovered volume, in the covering problem we must have no uncovered volume and we must minimize the amount of overlap between spheres. The lattice sphere packing problem has attained great importance partly because many of the lattices that give good packing fractions in low dimensions are related to objects of exceptional symmetry in geometry [2]. By contrast, the lattices which give good covering fractions in low dimensions do not seem to be imbued with symmetry to the same extent. In fact, many highly symmetric lattices seem instead to be locally worst at covering with spheres [9]. Instead of lattices that are worst at covering with spheres relative to similar lattices, in this article we are interested in centrally-symmetric shapes that are worst for covering relative to similar centrally-symmetric shapes.

In two and three dimensions it has been shown that the rounded octagon and the ball respectively are worst at packing relative to similar shapes [5, 6], and in both cases these shapes have been conjectured to be absolutely worst [3, 7]. Reinhardt’s conjecture about the rounded octagon has been an open problem since 1934. In the case of covering, it has been shown by L. Fejes Tóth that the disk is the worst shape...
for covering the plane [11]. In this article we show that also in three
dimensions, the ball is relatively worst at covering, but is not so in
four and five dimensions. We limit our attention to these dimensions
because those are the only dimensions where the lattice sphere covering
problem is solved. In all these dimensions, $A^*_n$ is known to be the
optimal sphere covering lattice [2]. Nevertheless, we establish results
that would make it easy to determine whether the $n$-dimensional ball
is relatively worst at covering if the optimal sphere covering lattice in
that dimension were to be known.

The investigation in this article follows similar lines to the one in Ref.
[5]. While many of the concepts and the results in the case of covering
are analogous to those in the case of packing and require only minor
changes, others require gross changes. With this in mind, we have set
out to write this article so as to be self-contained. To this end, much
of the content of this paper is closely similar (sometimes nearly verba-
tim) to the content of Ref. [5]. We omit a result and cite Ref. [5] only
when the result is nearly identical, not merely analogous. Many of the
concepts have identical names to the analogous concepts in the case of
packing, except with the addition of the modifier “covering”. For conve-
nience we usually drop this modifier, implicitly referring throughout
the paper out of two analogous concepts to the one that deals with
covering.

2. Convex Bodies and Lattices

An $n$-dimensional convex body is a convex, compact subset of $\mathbb{R}^n$
with a nonempty interior. A body is symmetric about the origin (or
origin-symmetric) if $-K = K$. In this article we discuss only such
bodies, and we will implicitly assume that every body mentioned is
symmetric about the origin. We denote by $B^n$ the Euclidean unit
ball of $\mathbb{R}^n$. The space of origin-symmetric convex bodies $K_0^n$
in $\mathbb{R}^n$ is a metric space equipped with the Hausdorff metric $\delta_H(K, K') = \min\{\varepsilon: K \subseteq K' + \varepsilon B^n, K' \subseteq K + \varepsilon B^n\}$. The set of bodies $K$ satisfying $aB^n \subseteq K \subseteq bB^n$ for $b > a > 0$ is compact [4].

Let $S^{n-1} = \partial B^n$ be the unit sphere. The radial distance of a body
in the direction $x \in S^{n-1}$ is given by $r_K(x) = \max\{\lambda: \lambda x \in K\}$. A
body is uniquely determined by its radial distance function since $K = \bigcup_{x \in S^{n-1}} [0, r_K(x)]x$. For origin-symmetric bodies, the radial distance is
an even functions.

An $n$-dimensional lattice is the image of the integer lattice $\mathbb{Z}^n$ under
some non-singular linear map $T$. The determinant $d(\Lambda)$ of a lattice
$\Lambda = T\mathbb{Z}^n$ is the volume of the image of the unit cube under $T$ and is
given by $d(\Lambda) = |\det T|$. The space $\mathcal{L}^n$ of $n$-dimensional lattices can be equipped with the metric $\delta(\Lambda, \Lambda') = \min\{||T - T'|| : \Lambda = T\mathbb{Z}^n, \Lambda' = T'\mathbb{Z}^n\}$, where $|| \cdot ||$ is the Hilbert-Schmidt norm.

We call $\Lambda$ a covering lattice for $K$ if for any point $x \in \mathbb{R}^n$, there is a lattice point $l \in \Lambda$ such that $x \in K + l$, i.e. $\{K + l : l \in \Lambda\}$ is a covering of $\mathbb{R}^n$. The density of this covering is given by $\text{vol } K / d(\Lambda)$, and must be greater than or equal to one. The set of covering lattices for some body $K$ and of determinant at least some value is compact [4].

The critical (covering) determinant $d_K$ is the maximum, necessarily attained due to compactness, of all determinants of covering lattices for $K$. A lattice attaining this maximum is called a critical (covering) lattice of $K$. If a covering lattice of $K$ locally maximizes the determinant amongst covering lattices of $K$, it is called an extreme (covering) lattice of $K$. Clearly, if $K' \supseteq K$, then $d_{K'} \geq d_K$. If this inequality is strict whenever $K'$ is a proper superset of $K$, we say that $K$ is an inextensible body. The optimal covering fraction for $K$ is $\vartheta(K) = \text{vol } K / d_K$.

Note that $\vartheta(TK) = \vartheta(K)$ for any nonsingular linear transformation $T$. Therefore, we may define $\vartheta$ as a function over the space of linear classes of $n$-dimensional bodies, equipped with the Banach-Mazur distance $\delta_{BM}([K], [L]) = \min\{t : L' \subseteq K' \subseteq e^t L', K' \in [K], L' \in [L]\}$. Since this space is compact, there must be a body $K$ with the highest possible optimal covering fraction amongst all $n$-dimensional bodies. We call this an absolutely worst covering body. If a body belongs to a class which is a local minimum of $\vartheta$ in this space, we say it is relatively worst covering. A relatively worst covering body is necessarily inextensible, but the converse is not necessarily true.

Below we show that the unit ball is relatively worst packing for $n = 3$, and extensible for $n = 4$ and 5.

3. Primitive simplices and semi-eutaxy

The Voronoi polytope $P_I$ of a lattice point $I \in \Lambda$ is the set of all points for which $I$ is the closest lattice point, that is, $P_I = \{x \in \mathbb{R}^n : ||x - I|| \leq ||x - I'||$ for all $I' \in \Lambda\}$. Note that $P_I = P_0 + I$. The Voronoi polytopes of the lattice points of $\Lambda$ form the cells of a $\Lambda$-periodic honeycomb, which we call the Voronoi honeycomb of $\Lambda$. If the combinatorial type of the Voronoi polytope $P_0$ (equivalently, the combinatorial type of the Voronoi honeycomb) is the same as for all lattices in a neighborhood of $\Lambda$, we say that $\Lambda$ is generic. If $\Lambda$ is generic, then each vertex of the Voronoi polytope lies at the intersection of exactly $n$ facets [1]. Similarly, if $\Lambda$ is generic, then each vertex of the Voronoi honeycomb lies at the intersection of $n + 1$ cells. Therefore, modulo translations...
by vectors of \( \Lambda \), each vertex of the Voronoi polytope \( P_0 \) is equivalent to exactly \( n \) others and all equivalent vertices are equidistant from the origin. Therefore, the Voronoi polytope can be described as the convex hull of simplices, each with a circumscribing sphere centered at the origin. We call these simplices the primitive simplices of \( \Lambda \). Note that \(-S\) is a primitive simplex of \( \Lambda \) whenever \( S \) is.

A Delone simplex of \( \Lambda \) whose circumcenter is at some vertex \( x \) of \( P_0 \), when translated by \(-x\), is simply \(-S\), where \( S \) is the unique primitive simplex with vertex \( x \). Since the Delone triangulation retains its combinatorics for nearby lattices \( T\Lambda \), where \(||T-Id||\) is small enough, the primitive simplices of \( T\Lambda \) are simply translates of the images under \( T \) of the primitive simplices of \( \Lambda \).

A lattice \( \Lambda \) is a covering lattice for the ball of radius \( r \) if and only if \( r \geq \mu(\Lambda) = \max_S \text{cr}(S) \), where the maximum runs over all primitive simplices \( S \) of \( \Lambda \), and \( \text{cr}(S) \) denotes the circumradius of \( S \). We call \( \mu(\Lambda) \) the covering radius of \( \Lambda \), and we refer to the primitive simplices of \( \Lambda \) attaining \( \mu(\Lambda) \) as the maximal primitive simplices. We denote the set of maximal primitive simplices of \( \Lambda \) by \( X(\Lambda) \). In the following lemma and subsequently we use the symbol \( \triangleleft \) to denote inclusion up to translation, that is, \( A \triangleleft B \) if and only if there exists \( t \) such that \( A + t \subseteq B \).

**Lemma 1.** Let \( \Lambda \) be a generic lattice of covering radius 1. Let \( K \) be a nearly spherical body in the sense that \((1 - \varepsilon)B^n \subseteq K \subseteq (1 + \varepsilon)B^n\). If \( \varepsilon \) is small enough then \( \Lambda \) is a covering lattice of \( K \) if and only if \( S \triangleleft K \) for all \( S \in X(\Lambda) \).

**Proof.** First let us assume that \( \Lambda \) is a covering lattice of \( K \). Note that for each simplex \( S \in X(\Lambda) \), \(-S\) is a translate of a Delone simplex, and that the bodies \( K + 1 \), where \( 1 \) runs over the vertices of the Delone simplex, must cover the Delone simplex. There must be a point \( x \) common to all \( n + 1 \) bodies: \( x \in K + 1 \) for all vertices \( 1 \) of the Delone simplex. Therefore, the points \( x - 1 \) are in \( K \), and their convex hull, which is a translate of \( S \) is contained in \( K \).

Now let us assume that \( S + t_S \subseteq K \) for all \( S \in X(\Lambda) \). By the fact that \( K \) is nearly spherical we also have that \( K \) contains all the non-maximal primitive simplices of \( \Lambda \) and that \( \max_S ||t_S|| \) is arbitrarily small for arbitrarily small \( \varepsilon \). We will show that \( \Lambda \) is a covering lattice for \( P' \) the convex hull of \( S + t_S \) where \( S \) runs over all primitive simplices of \( \Lambda \) and \( t_S = 0 \) for non-maximal \( S \).

Consider the Voronoi polytope \( P_0 \) and form a triangulation of it, i.e. a subdivision of \( P_0 \) into simplices with no new vertices and such that
any two simplices intersect at a common face or not at all. When repeated for all $P_I$ this is a $\Lambda$-periodic triangulation of $\mathbb{R}^n$. Now, leaving the combinatorics of the triangulation unchanged, let us translate each of its vertices by $t_S$ whenever the vertex is equivalent to a vertex of $S$ modulo translation by vectors of $\Lambda$. If $\varepsilon$ is small enough, the result is still a triangulation of $\mathbb{R}^n$. The cells obtained by the union of the simplices whose union previously gave the cells of the Voronoi honeycomb also form a $\Lambda$-periodic subdivision of space. These cells are in general not convex, but their convex hulls are lattice translates of $P'$. Therefore, the lattice translates of $P'$ cover $\mathbb{R}^n$ and $\Lambda$ is in fact a covering lattice for $P'$ and for $K \supseteq P'$.

Let $S$ be a primitive simplex of a generic lattice $\Lambda$ and let $x_1, \ldots, x_{n+1}$ be its vertices. We define a symmetric linear map associated with $S$ as follows:

$$Q_S(\cdot) = \sum_{j=1}^{n+1} \alpha_j \langle x_j, \cdot \rangle x_j$$

where $\alpha_j$ are the barycentric coordinates of the circumcenter of $S$:

$$\sum_{j=1}^{n+1} \alpha_j x_j = 0 \quad \text{and} \quad \sum_{j=1}^{n+1} \alpha_j = 1.$$  

Note that $Q_S = Q_{-S}$. The importance of $Q_S$ can be seen for example in the following lemma. Here we work with the linear space $\text{Sym}^n$ of symmetric linear maps $\mathbb{R}^n \to \mathbb{R}^n$ equipped with the inner product $\langle Q, Q' \rangle = \text{trace} QQ'$.

**Lemma 2.** Let $S$ be a simplex inscribed in the unit sphere centered at the origin and let $T$ be a nonsingular linear map. Then

$$\text{cr}(TS)^2 = 1 + \langle M, Q_S \rangle + O(||M||^2),$$

where $M = TT - \text{Id}$ and the error term is non-negative.

**Proof.** Let $x_i, i = 1, \ldots, n+1$, be the vertices of $S$. The center $y$ and radius $R = \sqrt{1+a}$ of the circumsphere of $TS$ are determined by the $n+1$ equations $||T(x_i) - y||^2 = 1 + a$, $i = 1, \ldots, n+1$. Defining the $n+1$-element vector $y'$ whose first $n$ elements give $y$ and its last elements is $\frac{1}{2}(a - ||y||^2)$, we can write the system of equations as a linear one: $2A(T \oplus 1)y' = b$ where

$$A = \begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1n} & 1 \\
  x_{21} & x_{22} & \cdots & x_{2n} & 1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  x_{(n+1)1} & x_{(n+1)2} & \cdots & x_{(n+1)n} & 1
\end{pmatrix},$$
\[ b_i = \left( ||T(x_i)||^2 - 1 \right) = \langle x_i, Mx_i \rangle, \]

and \( T \oplus 1 \) is the direct sum of \( T \) with the \( 1 \times 1 \) unit matrix. We can easily recover the circumcenter and radius by inverting the linear system:

\[ y' = (1/2)(T \oplus 1)^{-1}A^{-1}b. \]

Clearly then,

\[ a = (a - ||y||^2) + ||y||^2 = \langle c, b \rangle + O(\varepsilon^2), \]

where \( c \) is the bottom row of \( A^{-1} \). By definition, the elements of \( c \) satisfy \( \sum_{i=1}^{n+1} c_i x_{ij} = 0 \) for \( j = 1, \ldots, n \) and \( \sum_{i=1}^{n+1} c_i = 1 \), so these are the same coefficients \( \alpha_i \) of (1). In summary, we have that

\[ R^2 = 1 + a = 1 + \sum_{i=1}^{n+1} \alpha_i \langle x_i, Mx_i \rangle + O(||M||^2) \]

\[ = 1 + \langle M, Q_S \rangle + O(||M||^2), \]

and the error term, given by \( ||y||^2 \), is non-negative. \( \square \)

A finite set of symmetric maps \( \{Q_1, \ldots, Q_m\} \) is said to be \emph{semi-eutactic} if there are non-negative coefficients (called eutaxy coefficients) \( v_1, \ldots, v_m \) such that \( \text{Id} = \sum_{i=1}^{m} v_i Q_i \). Similarly, a set of simplices is said to be semi-eutactic if the associated set of symmetric maps is semi-eutactic. We say that a set of simplices \( S \) is redundantly semi-eutactic if the set \( X \setminus \{S, -S\} \) is semi-eutactic for all \( S \in X \). If a set of simplices is semi-eutactic but for all \( S \in X \) the set \( X \setminus \{S, -S\} \) is not semi-eutactic, we say it is critically semi-eutactic. \( X \) is critically semi-eutactic if and only if its eutaxy coefficients are unique and positive.

We can now prove three results relating these eutaxy properties with the existence or non-existence of certain covering lattices for certain bodies. The first is a sufficient condition (and necessary under the assumption of genericity), originally proved by Barnes and Dickson, for a lattice to be extreme for \( B^n \).

**Theorem 1.** (Barnes and Dickson [1]) Let \( \Lambda \) be a generic lattice such that the circumradius of its maximal primitive simplices is 1. The following are equivalent:

1. \( \Lambda \) is extreme for \( B^n \);
2. \( X(\Lambda) \) is semi-eutactic.

**Proof.** Suppose first that \( X(\Lambda) \) is not semi-eutactic. By the fundamental theorem of linear algebra and the fact that a subspace of \( \mathbb{R}^m \) does not contain a non-zero non-negative vector if and only if its orthogonal complement contains a positive vector (sometimes known as Farkas’s Lemma), we conclude that there exists a symmetric map \( M \) such that \( \langle M, Q_S \rangle < 0 \) for all \( S \in X(\Lambda) \) and trace \( M > 0 \). Let \( T_\varepsilon = \sqrt{\text{Id} + \varepsilon M}, \)
where the square root $\sqrt{A}$ of a positive definite symmetric map $A$ is meant to denote the unique positive definite map $B$ such that $B^2 = A$. From Lemma 2, as long as $\varepsilon$ is small enough, $cr(T_\varepsilon S) < 1$ for all $S \in X(\Lambda)$. Therefore $S < T_\varepsilon^{-1}B^n$ for all $S \in X(\Lambda)$ and so $\Lambda$ is a covering lattice of $T_\varepsilon^{-1}B^n$. Equivalently, $T_\varepsilon\Lambda$ is a covering lattice of $B^n$. Also, trace $M > 0$ implies that $det T_\varepsilon > 1$ for small enough $\varepsilon$, so $\Lambda$ is not extreme.

Conversely, suppose that $\Lambda$ is not extreme. Then for arbitrarily small $\varepsilon$ there exists a map $T$ satisfying $||T - Id|| < \varepsilon$, $det T > 1$, and $cr(S) \leq 1$ for all primitive simplices $S$ of $T\Lambda$. Since the primitive simplices of $T\Lambda$ are just the images under $T$ of the primitive simplices of $\Lambda$, we have from Lemma 2 that $\langle M, Q_S \rangle \leq 0$ for all $S \in X(\Lambda)$, where $M = TT^T - Id$. Moreover, since $det T > 1$, we have that trace $M > 0$.

Note that the map $M' = M - \frac{\text{trace } M}{2n} Id$ satisfies trace $M' > 0$ and $\langle M, Q_S \rangle < 0$ for all $S \in X(\Lambda)$. Again, from Farkas’s Lemma, the existence of such a map $M'$ implies that $X(\Lambda)$ is not semi-eutactic. □

In cases where the critical covering lattice for $B^n$ is unique up to rotations and generic, which is the case for $n = 2, 3, 4,$ and 5, we can prove the following necessary and sufficient condition for $B^n$ to be extensible.

**Theorem 2.** Let $\Lambda_0$ be the unique (up to rotation) critical covering lattice of $B^n$, and let $\Lambda_0$ be generic. Then the following are equivalent:

1. $B^n$ is extensible;
2. $X(\Lambda_0)$ is redundantly semi-eutactic.

**Proof.** First let us assume that $X(\Lambda_0)$ is not redundantly semi-eutactic. That is, we assume that there is a maximal primitive simplex $S_0 \in X(\Lambda_0)$ such that $X(\Lambda_0) \setminus \{\pm S_0\}$ is not semi-eutactic. Consider the $\varepsilon$-symmetrically augmented ball $B_\varepsilon = \text{conv}(B^n, \pm (1+\varepsilon)p)$, where $p \in S^{n-1}$ is some arbitrarily chosen “north pole”. We are free to assume that $\Lambda_0$ is rotated so that $p$ is one of the vertices of $S_0$. By exactly the same argument as in the proof of Thm. □ we conclude that there exists a linear map $T$ such that $det T > 1$, $cr(TS) < 1$ for all $S \in X(\Lambda_0) \setminus \{\pm S_0\}$, and $||T - Id||$ is arbitrarily small. In fact, if $||T - Id||$ is small enough, then $TS_0$ can be translated so as to lie within $B_\varepsilon$ and therefore $T\Lambda_0$ is a covering lattice of $B_\varepsilon$ by Lemma □. Since $d(T\Lambda_0) > d(\Lambda_0)$, and since for each proper $K \supset B^n$, there exists $\varepsilon > 0$ such that $K \supset B_\varepsilon \supset B^n$, it follows that $B^n$ is inextensible.

From uniqueness of $\Lambda_0$ and compactness of the set of covering lattices of $B^n$ of determinant greater than some value, we also have a stability
result about nearly-optimal covering lattices of $B^n$: for each $\varepsilon > 0$, there exists some $\varepsilon' > 0$ such that if $\Lambda$ is a covering lattice for $B^n$ and $d(\Lambda) > d(\Lambda_0) - \varepsilon'$ then there exists a rotation $U\Lambda_0$ of $\Lambda_0$ such that $\delta(\Lambda, U\Lambda_0) < \varepsilon$. Similarly, if $\Lambda$ is a critical lattice for a nearly spherical body $K$ satisfying $(1 - \varepsilon')B^n \subseteq K \subseteq (1 + \varepsilon')B^n$, then again there exists a rotation $U\Lambda_0$ of $\Lambda_0$ such that $\delta(\Lambda, U\Lambda_0) < \varepsilon$.

Now suppose that $X(\Lambda_0)$ is redundantly semi-eutactic. Let $T\Lambda_0$ be a critical lattice of $B_\varepsilon$. $||T - \text{Id}||$ can be made arbitrarily small by choosing $\varepsilon$ sufficiently small and appropriately rotating $\Lambda_0$. Since $T\Lambda_0$ is a covering lattice of $B_\varepsilon$ then by Lemma[1] $TS \triangleleft B_\varepsilon$ for all $S \in X(\Lambda_0)$, and if $\varepsilon$ is small enough then $TS \triangleleft B^n$ for all but one pair $S = \pm S_0$. Since $\Lambda_0$ is redundantly semi-eutactic, the requirement that $\text{cr}(TS) \leq 1$ whenever $S \in X(\Lambda_0) \setminus \{\pm S_0\}$, necessarily implies, when $||T - \text{Id}||$ is small enough, that $\det T \leq 1$. Of course, since $d_{B_\varepsilon} \geq d_{B^n}$, we must have $d_{B_\varepsilon} = d_{B^n}$, and $B^n$ is extensible.

**Corollary 1.** $B^n$ is extensible for $n = 4$ and 5, and inextensible for $n = 2$ and 3.

**Proof.** The Voronoi polytope of $A_n^*$ is the permutohedron. The $n!$ primitive simplices of $A_n^*$, and their $n!/2$ associated symmetric maps and eutaxy coefficients can easily be calculated. One can therefore verify that for $n = 2$ and $n = 3$ the eutaxy coefficients are positive and unique and that for $n \geq 4$ the eutaxy coefficient of any single symmetric map can be set to zero. Therefore $X(\Lambda^*_n)$ is redundantly semi-eutactic for all $n \geq 4$ but not for $n = 2, 3$.\[\square\]

We now focus on the case where $B^n$ is inextensible. Particularly, we will assume that the critical lattice $\Lambda_0$ is unique, generic, and its set of maximal primitive simplices is critically semi-eutactic. This is the case for $n = 2$ and 3.

**Theorem 3.** Let $\Lambda_0$ be the unique critical covering lattice of $B^n$, and let $\Lambda_0$ be generic and $X(\Lambda_0) = \{S_1, \ldots, S_{2m}\}$ be critically semi-eutactic with eutaxy coefficients $v_i$ such that $\sum_{i=1}^{2m} v_i Q_i S_i = \text{Id}$. For each simplex $S_i$, denote by $x_{ij}$, $j = 1, \ldots, n + 1$, its vertices and by $\alpha_{ij}$ the corresponding barycentric coordinates of the circumcenter of $S_i$ (see [1]). Let $K$ be a nearly spherical body $(1 - \varepsilon)B^n \subseteq K \subseteq (1 + \varepsilon)B^n$, and let $r_{ij} = 1 + \rho_{ij}$ be the values of the radial distance function of $K$ evaluated at the directions $x_{ij}$, $i = 1, \ldots, 2m$, $j = 1, \ldots, n + 1$. There exists a covering lattice $\Lambda'$ of $K$ whose determinant is bounded as follows:

$$\frac{d(\Lambda')}{d(\Lambda_0)} \geq 1 + \sum_{i=1,j=1}^{n+2m} v_i \alpha_{ij} \rho_{ij} - \varepsilon' \sum_{i=1,j=1}^{n+2m} |\rho_{ij}|,$$
where $\varepsilon'$ depends on $\varepsilon$ and becomes arbitrarily small as $\varepsilon \to 0$.

Proof. We first prove the existence of a symmetric map $M$ and translation vectors $t_i$, $i = 1, \ldots, 2m$ satisfying trace $M = \sum_{i=1,j=1}^{n+1,2m} v_i \alpha_{ij} \rho_{ij}$, and

$$\langle x_{ij}, Mx_{ij} + t_i \rangle = \rho_{ij} \text{ for all } i = 1, \ldots, 2m \text{ and } j = 1, \ldots, n + 1. \tag{2}$$

Taking the sum $\sum_{j=1}^{n+1} \alpha_{ij} \langle \cdot, M \rangle$ of both sides of (2), we obtain

$$\sum_{j=1}^{n+1} \alpha_{ij} \langle x_{ij}, Mx_{ij} \rangle = \sum_{j=1}^{n+1} \alpha_{ij} \rho_{ij}. \tag{3}$$

Therefore, by the affine independence of the vertices of the simplex $S_i$, for fixed $i$ and $M$, a vector $t_i$ satisfying (2) for all $j = 1, \ldots, n + 1$ exists if and only if $\sum_{j=1}^{n+1} \alpha_{ij} \rho_{ij} = \langle M, Q_{S_i} \rangle$. Let us denote $\rho_i = \sum_{j=1}^{n+1} \alpha_{ij} \rho_{ij}$.

All that is left to do is to find a map $M$ such that $\langle M, Q_{S_i} \rangle = \rho_i$ for all $i = 1, \ldots, 2m$, and trace $M = \sum_{i=1}^{2m} v_i \rho_i$. From the fact that the eutaxy coefficients are unique (modulo the trivial degeneracy associated with the fact that $Q_S = Q_{-S}$) and the fundamental theorem of linear algebra, it is easy to see that such a map must exist regardless of the values of $\rho_{ij}$. Moreover, the map $M$ and translations vectors $t_i$ can be chosen consistently so as to depend linearly on $\rho_{ij}$.

We now wish to find a contraction factor $1 - \delta$ such that $|||1 - \delta y_{ij}||| \leq r_K(y_{ij}/||y_{ij}||)$ for all $i, j$, where $y_{ij} = (\text{Id} + M)x_{ij} + t_i$. Therefore, for all $i, j$ we must have

$$\delta \geq \delta_{ij} = \frac{||y_{ij}|| - r_K(y_{ij}/||y_{ij}||)}{||y_{ij}||}. \tag{4}$$

We wish to bound the values of $\delta_{ij}$ using only the values of $r_K$ evaluated at $x_{ij}$ (not $y_{ij}$) and the fact that it is everywhere bounded between $1 - \varepsilon$ and $1 + \varepsilon$. We do this as illustrated in Figure 1. In the plane containing the origin $O$, the point $(1 + \rho_{ij})x_{ij}$ (denoted $A$ in the figure), and the point $y_{ij}$ (denoted $B$), draw the tangent $AX$ from $A$ to the circle of radius $1 - \varepsilon$ about the origin in the direction toward $B$. Note that $B$ lies on the line through $A$ perpendicular to $OA$. Since $\rho_{ij} < \varepsilon$, the angle $\beta = \overrightarrow{AOX}$ satisfies $\beta \leq \cos^{-1} \frac{1 - \varepsilon}{1 + \varepsilon} \leq 2\sqrt{\varepsilon}$. By convexity, the segment $AX$ must lie in $K$. We mark the intersection of the tangent $AX$ and the ray $OB$ as $C$. Then either $\delta_{ij} \leq 0$, or the boundary of $K$ intersects the ray $OB$ between $C$ and $B$. Since $y_{ij} - x_{ij}$ depends linearly on the values $\rho_{ij}$, the angle $\gamma = \overrightarrow{AOB}$ satisfies $\gamma \leq C \sum_{i,j} \rho_{ij}$ for some
constant $C$. By the law of sines we have

$$|BC| = \frac{|AB| \sin(\beta)}{\cos(\beta - \gamma)} \leq \frac{(1 + \epsilon)\gamma\beta}{1 - \frac{1}{2}(\beta - \gamma)^2} \leq (1 + \epsilon)\epsilon' \sum_{ij} |\rho_{ij}|,$$

where $\epsilon'$ depends on $\epsilon$ and becomes arbitrarily small as $\epsilon \to 0$. Therefore, if we let $\delta = \epsilon' \sum_{ij} |\rho_{ij}|$, then $\delta_{ij} \leq \delta$ for all $i$ and $j$, and for each simplex $S \in X(\Lambda_0)$, we now have that $(1 - \delta)(\text{Id} + M)S \ll K$. Therefore, $\Lambda' = (1 - \delta)(\text{Id} + M)\Lambda_0$ is a covering lattice for $K$. 

**Figure 1.** Illustration of the construction given in the proof of Thm. 3 to bound the contraction factor needed to ensure that the original point $B$ when contracted to $C$ lies inside the body $K$.
The determinant of the lattice $\Lambda'$ is given by
\[
\frac{d(\Lambda')}{d(\Lambda_0)} = (1 - \delta)^n \det(\text{Id} + M)
\]
\[
\geq \left( 1 + \sum_{i=1,j=1}^{2m,n+1} v_i \delta_{ij} \rho_{ij} - C \sum_{ij} |\rho_{ij}|^2 \right)^n \left( 1 - \varepsilon' \sum_{ij} |\rho_{ij}| \right)^n
\]
\[
\geq 1 + \sum_{i=1,j=1}^{n+1,2m} v_i \delta_{ij} \rho_{ij} - \varepsilon'' \sum_{i=1,j=1}^{n+1,2m} |\rho_{ij}|,
\]
where the quadratic and higher order terms have been absorbed into the last term. \hfill \Box

4. The case $n = 3$

We now turn to prove the main result, which is that the 3-dimensional ball is relatively worst covering. Given Theorem 3 the proof proceeds much as the proof of Theorem 5 of Ref. [5] does. As in Ref. [5], we start with three lemmas, of which we will only prove the first here, since it is the only one which varies significantly from its analog in Ref. [5].

Lemma 3. Let
\[
c_l = P_l(1) + 3P_l(\frac{2}{5}) + P_l(\frac{4}{5}) + 4P_l(\frac{2}{5}) + 2P_l(\frac{4}{5}) + P_l(0),
\]
where $P_l(t)$ is the Legendre polynomial of degree $l$. Then $c_l = 0$ if and only if $l = 2$. Moreover, $|c_l - 1| < C/2$ for some constant $C$.

Proof. We introduce the following rescaled Legendre polynomials: $Q_l(t) = 5^l l! P_l(t)$. From their recurrence relation—given by $Q_{l+1}(t) = (2l + 1)(5t)Q_l(t) - 25l^2 Q_{l-1}(t)$—and the base cases—$Q_0(t) = 1$ and $Q_1(t) = 5t$—it is clear that the values of $Q_l(t)$ at $t = k/5$ for $k = 0, \ldots, 5$ are integers. We are interested in residues of these integers modulo 16. If $Q_l(k/5) \equiv Q_{l+1}(k/5) \equiv 0 \pmod{16}$ for some $k$ and $l$ then for all $l' \geq l$ we also have $Q_{l'}(k/5) \equiv 0 \pmod{16}$. This is in fact the case, as can be easily checked, for $k = 1, 3, 5$ and $l = 6$. For $k = 0, 2$, or 4 it is easy to show by induction that the residue of $Q_l(k/5)$ modulo 16 depends only on $k$ and the residue of $l$ modulo 8 and takes the following values:

\[
Q_l(0) \equiv 1, 0, 7, 0, 9, 0, 7, 0 \pmod{16}
\]
\[
Q_l(2/5) \equiv 4, 8, 12, 8, 4, 8, 12, 8 \pmod{16}
\]
\[
Q_l(4/5) \equiv 3, 12, 5, 4, 11, 12, 5, 4 \pmod{16}
\]
resp. for $l \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{8}$.
Therefore, it is easy to verify that regardless of the residue of \( l \) modulo 8, the quantity \( 5^l l! c_l = Q_l(1) + 3Q_l(\frac{3}{2}) + Q_l(\frac{5}{2}) + 4Q_l(\frac{7}{2}) + 2Q_l(\frac{9}{2}) + Q_l(0) \) is an integer of non-zero residue modulo 16 for \( l \geq 6 \) and therefore \( c_l \) does not vanish. The cases \( l < 6 \) are easily checked by hand. The second part of the lemma follows from the bound \(|P_l(t)| < (\pi l \sqrt{1 - t^2})^{-1/2} \). \[\square\]

Fixing some arbitrary pole \( p \in S^2 \), we define a zonal measure (function) to be a measure (function) on \( S^2 \) which is invariant with respect to rotations that preserve \( p \). A convolution of a function \( f \) with a zonal measure \( \mu \) is given by \((\mu * f)(y) = \int f(x) d\mu(U_y(x)), \) where \( U_y \) is any rotation which takes \( y \) to \( p \). Convolution of \( f \) with a zonal measure acts as multiplier transformation on the harmonic expansion \( f \). That is, if \( f(x) = \sum_{l=0}^{\infty} f_l(x) \), where \( f_l(x) \) is a spherical harmonic of degree \( l \), then \((\mu * f)(x) = \sum_{l=0}^{\infty} c_l f_l(x) \).

Consider the 24 vertices of the Voronoi polytope of \( A_3^2 \) (the Archimedean truncated octahedron) rotated in such a way that one of them is at \( p \), and denote them as \( x_i, i = 1, \ldots, 24 \). There is a unique zonal measure \( \mu \) such that for every continuous zonal function \( f \)

\[
\int_{S^2} f(y) d\mu(y) = \frac{1}{2} \sum_{i=1}^{24} f(x_i).
\]

From the values \( \langle p, x_i \rangle, i = 1, \ldots, 24 \), the multiplier coefficients associated with convolution with this measure can be easily calculated (see Ref. \[8\]). It can be easily shown that these coefficients vanish for odd \( l \) and are equal to the coefficients \( c_l \) of Lemma 3 for even \( l \). The proof of the following two lemmas is identical to the proofs of Lemmas 3 and 1 of Ref. \[5\] respectively. We denote by \( \sigma \) the Lebesgue measure on \( S^2 \) normalized such that \( \sigma(S^2) = 1 \).

**Lemma 4.** Let \( \mu \) be the zonal measure described above, let \( \Phi \) be the operator of convolution with \( \mu \), and let \( Z \) be the space, equipped with the \( L^1(\sigma) \) norm, of even functions on \( S^2 \) for which \( f_2 = 0 \). Then \( \Phi \) maps \( Z \) to \( Z \), and as an operator \( Z \to Z \) it is one-to-one, bounded, and has a bounded inverse.

**Lemma 5.** Given \( \varepsilon > 0 \), there exists \( \varepsilon' > 0 \) such that if a convex body \( K \) satisfies \( (1 - \varepsilon') B^3 \subseteq K \subseteq (1 + \varepsilon') B^3 \), then \( K \) has a linear image \( K' = TK \) that satisfies \( (1 - \varepsilon) B^3 \subseteq K' \subseteq (1 + \varepsilon) B^3 \) and whose radial function has mean 1 and vanishing second spherical harmonic component.

**Theorem 4.** There exists \( \varepsilon > 0 \) such that if a convex body \( K \) is a non-ellipsoidal origin-symmetric convex body and \( (1 - \varepsilon) B^3 \subseteq K \subseteq
(1 + \varepsilon)B^3$, then \vartheta(K) < \vartheta(B^3). In other words, $B^3$ is relatively worst covering.

Proof. Given Lemma 5 and the fact that \vartheta is invariant under linear transformations, we may assume without loss of generality that $K$ is a non-spherical body whose radial function has an expansion in spherical harmonics of the form

$$r_K(x) = 1 + \rho(x) = 1 + \sum_{l \text{ even}, \, l \geq 4} \rho_l(x).$$

The volume of $K$ satisfies

$$\text{vol } K = \frac{4\pi}{3} \int S^2 r_K^3(x) d\sigma \leq \frac{4\pi}{3} + \varepsilon'' ||\rho||_1,$$

where $\varepsilon''$ is arbitrarily small for arbitrarily small $\varepsilon$.

We consider all the rotations $U(K)$ of the body $K$ and the determinant of the covering lattice obtained when the construction of Theorem 3 is applied to $U(K)$. Note that the determinant obtained depends only on $\rho_{ij} = r_{U(K)}(x_{ij}) - 1 = \rho(U^{-1}(x_{ij}))$, where $x_{ij}$ run over all 24 vertices of the three dimensional permutohedron. Let us define $\Delta_K = 1 - \frac{\vartheta(K)^{-1}}{\vartheta(B^3)^{-1}}$. Combining (3) with Theorem 3 we get

$$\Delta_K \leq \min_{U \in SO(3)} \left[ -\frac{1}{8} \sum_{i,j=1}^{6,4} \rho_{ij} + \varepsilon' \sum_{ij} |\rho_{ij}| + \varepsilon'' ||\rho||_1 \right].$$

We may pick a single point, say $x_{11}$, and decompose $SO(3)$ into subsets $U_y$ of all rotations such that $U^{-1}(x_{11}) = y$. In each subset $U_y$ the minimum on the right hand side of (4) is no larger than the average value over $U_y$ (with respect to the obvious uniform measure). This averaging procedure transforms (4) into

$$\Delta_K \leq \min_{y \in S^2} \left[ -\frac{1}{8} \Phi[\rho](y) + \varepsilon' \Phi[|\rho|](y) + \varepsilon'' ||\rho||_1 \right],$$

where $\Phi$ is the convolution operator in Lemma 4. Since $\int \Phi[\rho] d\sigma = 0$ and $\Phi[|\rho|]$ is non-negative, we have that $\min(-\frac{1}{8} \Phi[\rho] + \varepsilon' \Phi[|\rho|]) \leq -\frac{1}{16} ||\Phi[\rho]||_1 + \varepsilon' ||\Phi||_1 + \varepsilon'' ||\rho||_1$, and so

$$\Delta_K \leq -\frac{1}{16} ||\Phi^{-1}||^{-1} \cdot ||\rho||_1 + (\varepsilon' ||\Phi|| + \varepsilon'') ||\rho||_1.$$

Therefore, we conclude that there is a coefficient $c > 0$ such that $\Delta_K < -c ||\rho||_1$. \qed
1. E. S. Barnes and T. J. Dickson, *Extreme coverings of n-space by spheres*, J. Austral. Math. Soc 7 (1967), no. 11, 5127.

2. J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices, and groups (third edition)*, Springer, New York, 1998.

3. M. Gardner, *New mathematical diversions (revised edition)*, Math. Assoc. Amer., Washington, 1995.

4. P. M. Gruber, *Convex and discrete geometry*, Springer, New York, 2007.

5. Y. Kallus and F. Nazarov, *In which dimensions is the ball relatively worst packing?,* arXiv:1212.2551.

6. F. L. Nazarov, *On the reinhardt problem of lattice packings of convex regions, local extremality of the reinhardt octagon*, J. Soviet Math. 43 (1988), 2687.

7. K. Reinhardt, *Über die dichteste gitterförmige lagerung kongruente bereiche in der ebene und eine besondere art konvexer kurven*, Abh. Math. Sem., Hamburg, Hansischer Universität, Hamburg 10 (1934), 216.

8. F. E. Schuster, *Convolutions and multiplier transformations of convex bodies*, Transact. Amer. Math. Soc. 359 (2007), 5567.

9. M. Dutour Sikirić, A. Schürmann, and F. Vallentin, *Inhomogeneous extreme forms*, Annales de l’Institut Fourier, to appear (arXiv:1008.4751).

10. G. Szego, *Orthogonal polynomials*, American Mathematical Society, Providence, 2003.

11. L. Fejes Tóth, *Lagerungen in der ebene, auf der kugel und im raum*, Springer, 1972.

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