Weakly smooth Langevin Monte Carlo using 

$p$-generalized Gaussian smoothing

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Abstract: Langevin Monte Carlo (LMC) is an iterative process for sampling from a distribution of interests. The nonasymptotic mixing time is studied mostly in the context of smooth (gradient-Lipschitz) log-densities, a significant constraint for its deployment in many sciences including computational statistics and artificial intelligence. In the original article, [5] eliminates this restriction and establishes polynomial-time convergence assurances for a variation of LMC in the context of weakly smooth log-concave distributions. Based on their approach, we generalize the Gaussian smoothing to $p$-generalized Gaussian perturbation process, while maintaining the induced bias and variance bounded. We also improve their nonasymptotic dependence on the dimension and strongly convex parameters.

Keywords and phrases: accelerated inexact gradient method, sequential Monte Carlo, partially observed Markov process model, parameter estimation.

1. Introduction

Over the last decade, Bayesian inference is one of the most prevalent inferring instruments for variety of disciplines including the computational statistics and artificial intelligence [4, 6, 14, 19, 21]. In general, Bayesian inference seeks to generate samples of the full posterior distribution over the parameters and perhaps latent variables which yields a mean to quantify uncertainty in the model and prevents it from over-fitting [17, 24]. A typical problem is aiming to sample up to a normalizing constant from a log-concave posterior distribution of the subsequent form:

$$\pi(x) = e^{-U(x)} / \int_{\mathbb{R}^d} e^{-U(y)} dy,$$
where the function $U(x)$, also known as the potential function, is convex. The most conventional approaches for sampling from a posterior distribution are random walks Metropolis Hasting [12, 15] where sampling is reduced to constructing a Markov kernel on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$-the Borel $\sigma$-field of $\mathbb{R}^d$, whose invariant probability distribution is $\pi$.

However, selecting a proper proposal distribution for the Metropolis Hastings algorithm is a complicated matter. As a result, it has been proposed to consider continuous dynamics which inherently leave the objective distribution $\pi$ invariant. Probably, one of the most well-known example of these continuous dynamics applications are the over-damped Langevin diffusion [18] associated with $U$, assumed to be continuously differentiable:

$$
\text{d}Y_t = -\nabla U(Y_t) \text{d}t + \sqrt{2d} \text{d}B_t, \quad (1.1)
$$

where $(B_t)_{t \geq 0}$ is a $d$-dimensional Brownian motion. Under suitable assumptions on $U$, this stochastic differential equations (SDE) possesses a unique strong solution $(Y_t)_{t \geq 0}$ and defines a strong Markov semigroup $(P_t)_{t \geq 0}$ which converges to $\pi$ in total variation [22], or Wasserstein distance [3]. Yet, simulating path solutions of such SDE is not feasible in most circumstances, and discretization of these equations are employed instead. Moreover, numerical solutions related to these approaches define Markov kernels for which $\pi$ is no longer stationary. Hence, measuring the error induced by these approximations is critical to justify their application to sample from the objective distribution $\pi$. We study in this paper the Euler-Maruyama discretization of Eq.(1.1) which defines the (possibly inhomogenous) Markov chain $(X_k)_{k \geq 0}$ given for all $k \geq 0$ by

$$
x_{k+1} = x_k - \eta_k \nabla U(x_k) + \sqrt{2\eta_k} \xi_k, \quad (1.2)
$$

where $(\eta_k)_{k \geq 1}$ is a sequence of step sizes which can be kept constant or decreasing to 0, and $\xi_k \sim \mathcal{N}(0, I_{d \times d})$ are independent Gaussian random vectors. Nonetheless, there is a critical gap in the theory of discretization of an underlying SDE to the potential broad spectrum of applications in statistical inference. In particular, the application of techniques from SDEs traditionally requires $U(x)$ to have Lipschitz-continuous gradients. This requirement prohibits many typical utilizations [11, 14, 16].

[5] has recently established an original approach to deal with weakly smooth (possibly non-smooth) potential problems directly. Their technique rests on results obtained
from the optimization community, perturbing a gradient evaluating point by a Gaussian. They do not demand strong assumptions such as the existence of proximal maps, composite structure [2, 11] or strong convexity [13]). In this paper, we show that Gaussian smoothing can be generalized to \( p \)-generalized Gaussian smoothing, which is novel in both Bayesian and optimization communities. In a more general context, this novel perturbation process provides additional features, comparable to Gaussian smoothing while preserving the same result when a \( p \)-generalized Gaussian is a Gaussian. In particular, the bias and variance are bounded for a general \( p \) while these bounds are equivalent to Gaussian smoothing for the case \( p = 2 \). In addition, we also improve the convergence result in Wasserstein distance by a simpler approach. We integrate this result with an LMC convergence result of stochastic gradient estimates [8], achieving a sharper nonasymptotic result for convergence in Wasserstein distance of LMC with stochastic gradients.

The paper is organized as follows. Section 2 sets out the notation and context necessary to give our core theorems, generalizing the results obtained from [5] for Gaussian smoothing. Section 3 broadens these outcomes of stochastic approximation of the potential (negative log-density) and composite structure of the potential, while Section 4 shows our conclusions.

2. Weakly smooth Langevin Monte Carlo using \( p \)-generalized Gaussian smoothing

The objective is to sample from a distribution \( \pi \propto \exp(-U(x)) \), where \( x \in \mathbb{R}^d \). We furnish the space \( \mathbb{R}^d \) with the regular Euclidean norm \( \| \cdot \| = \| \cdot \|_2 \) and use \( \langle \cdot, \cdot \rangle \) to specify inner products. While sampling from the exact distribution \( \pi(x) \) is generally computationally demanding, it is largely adequate to sample from an approximated distribution \( \tilde{\pi}(x) \) which is in the vicinity of \( \pi(x) \) by some distances. In this paper, we use Wasserstein distance and briefly define it in Appendix A.

Suppose the following conditions, identical to the first two assumptions of [5]:

**Assumption 1.** \( U \) is convex and sub-differentiable. Specifically, for all \( x \in \mathbb{R}^d \), there exists a sub-gradient of \( U \), \( \nabla U(x) \in \partial U(x) \), ensuring that \( \forall y \in \mathbb{R}^d : U(y) \geq U(x) + \)
\langle \nabla U(x), y - x \rangle.

**Assumption 2.** There exist \( L < \infty \) and \( \alpha \in [0, 1] \) so that \( \forall x, y \in \mathbb{R}^d \), we have

\[
\|\nabla U(x) - \nabla U(y)\|_2 \leq L \|x - y\|_2^\alpha,
\]

where \( \nabla U(x) \) represents an arbitrary sub-gradient of \( U \) at \( x \).

**Remark 1.** Condition 2 is known as \((L, \alpha)\)-weakly smoothness or Holder continuity of the (sub)gradients of \( U \). A feature that follows straightforwardly from Eq. (2.1) is that:

\[
U(y) \leq U(x) + \langle \nabla U(x), y - x \rangle + \frac{L}{1 + \alpha} \|y - x\|^{1+\alpha}, \ \forall x, y \in \mathbb{R}^d.
\]  

When \( \alpha = 1 \), it is equivalent to the standard smoothness (Lipschitz continuity of the gradients), whereas at the opposite extreme, when \( \alpha = 0 \), \( U \) is (possibly) non-smooth and Lipschitz-continuous.

[5] perturbs a gradient evaluating point by a Gaussian so that they can utilize weakly smooth Lipschitz potentials directly without any additional structure. Here, we generalize their approach to perturbation using \( p \)-generalize Gaussian smoothing. Particularly, consider \( \mu \geq 0 \), \( p \)-generalized Gaussian smoothing \( U_\mu \) of \( U \) is defined as:

\[
U_\mu(y) := \mathbb{E}_\xi [U(y + \mu \xi)],
\]

where \( \xi \sim N_p(0, I_{d \times d}) \) (the \( p \)-generalized Gaussian distribution), \( 1 \leq p \leq 2 \). The rationale for taking into account the \( p \)-generalized Gaussian smoothing \( U_\mu \) rather than \( U \) is that it typically benefits from superior smoothness properties. In particular, \( U_\mu \) is smooth albeit \( U \) is not. In addition, \( p \)-generalized Gaussian smoothing is more generalized than Gaussian smoothing in the sense that it contains all normal distribution when \( p = 2 \) and all Laplace distribution when \( p = 1 \). This family of distributions allows for tails that are either heavier or lighter than normal and in the limit, it contains all the continuous uniform distribution. It is possible to study \( p \in \mathbb{R}^+ \) but to simplify the proof, we are only interested in \( p \in [1, 2] \). Here we examine some primary features of \( U_\mu \) based on adapting those results of [5].

**Lemma 1.** Given that \( U : \mathbb{R}^d \rightarrow \mathbb{R} \) is a convex function which satisfies Eq. (2.1) for some \( L < \infty \) and \( \alpha \in [0, 1] \), then:
(i) \( \forall x \in \mathbb{R}^d : \left| U_\mu(x) - U(x) \right| = U_\mu(x) - U(x) \leq \frac{L_\mu^{1+\alpha} d^{\frac{1}{p}}}{1+\alpha} \).

(ii) \( \forall x, y \in \mathbb{R}^d : \| \nabla U_\mu(y) - \nabla U_\mu(x) \|_2 \leq \frac{L d^{\frac{1}{p}}}{\mu^{1-\alpha}(1+\alpha)^{1-\alpha}} \| y - x \|_2. \)

**Proof.** See Appendix B.

The subsequent lemma is a straightforward adaptation of the results of [5] and it establishes particular regularity conditions for \( p \)-generalized Gaussian smoothing that will be employed in our investigation. We demonstrate that \( p \)-generalized Gaussian smoothing maintains strong convexity in the following lemma.

**Lemma 2.** Given that \( \psi : \mathbb{R}^d \to \mathbb{R} \) is \( \lambda \)-strongly convex, then \( \psi_\mu \) is also \( \lambda \)-strongly convex.

**Proof.** See Appendix B.

### 3. Sampling for regularized potential

To study convergence of the continuous-time process (which involves strong convexity), we work with regularized potentials that have the following composite structure:

\[
\mathcal{U}(x) := U(x) + \psi(x),
\]

where \( \psi(\cdot) \) is \( m \)-smooth and \( \lambda \)-strongly convex. Observe that by the triangle inequality, we have:

\[
\| \nabla U(x) - \nabla U(y) \|_2 \leq \| \nabla U(x) - \nabla U(y) \|_2 + \| \nabla \psi(x) - \nabla \psi(y) \|_2 \\
\leq L \| x - y \|_2^m + m \| x - y \|_2.
\]

From which, by employing Lemma 1, we get:

\[
\| \nabla \mathcal{U}_\mu(x) - \nabla \mathcal{U}_\mu(y) \|_2 \leq \left( \frac{L d^{\frac{1}{p}}}{\mu^{1-\alpha}(1+\alpha)^{1-\alpha}} + m \right) \| x - y \|_2.
\]

In this case, \( U(\cdot) \) is \((L, \alpha)\)-weakly smooth (possibly with \( \alpha = 0 \), \( U \) is nonsmooth and Lipschitz-continuous). We now analyze LMC where we perturb the points at which gradients of \( \mathcal{U} \) are evaluated by a \( p \)-generalized Gaussian random variable. Remark
that it remains unclear whether it is possible to achieve such bounds for (LMC) without this perturbation. Recall that LMC in terms of the potential $\bar{U}$ can be specified as:

$$x_{k+1} = x_k - \eta \nabla \bar{U}(x_k) + \sqrt{2\eta} \xi_k,$$

(3.4)

where $\xi_k \sim N(0, I_{d \times d})$ are independent Gaussian random vectors. This method is actually the Euler-Mayurama discretization of the Langevin diffusion.

Rather than working with the algorithm specified by Eq. 3.4, we rectify the algorithm and inspect the following form:

$$y_{k+1} = y_k - \mu \omega - \eta \nabla \bar{U}(y_k - \mu \omega) + \sqrt{2\eta} \xi_k,$$

(3.5)

where $x_k = y_k + \mu \omega_k$ for every $k$ and $\{\omega\}$ is sequence of $p$-generalized Gaussian noise. We obtain the following result.

**Lemma 3.** For any $x \in \mathbb{R}^d$, and $z_1, z_2 \sim N(0, I_{d \times d})$, let

$$G(x, z_1, z_2) = \nabla \bar{U}(x + \mu z_1) - \frac{\mu}{\eta} z_1 - \frac{\mu}{\eta} z_2$$

denote a stochastic gradient of $\bar{U}_\mu$. Then $G(x, z_1, z_2)$ is an unbiased estimator of $\nabla \bar{U}_\mu$ whose (normalized) variance can be bounded as:

$$\sigma^2 = \frac{\mathbb{E}_{z_1, z_2}[\|\nabla \bar{U}_\mu(x) - G(x, z_1, z_2)\|_2^2]}{d} \leq 4d^{1-\alpha} \mu^2 \bar{L}^2 \left( \frac{p^2 \Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} \right)} \right)^\alpha + 4\mu^2 \Gamma \left( \frac{1}{p} \right) + \frac{8\mu^2 p^2 \Gamma \left( \frac{1}{p} \right)}{\eta^2 \Gamma \left( \frac{1}{p} \right)}.$$

**Proof.** See Appendix B. \qed

Let the distribution of the $k$th iterate $y_k$ be represented by $\pi_k$, and let $\pi_\mu \propto \exp(-\bar{U}_\mu)$ be the distribution with $\bar{U}_\mu$ as the potential. Our overall tactic for showing our core finding is as belows. First, we prove that the $p$-generalized Gaussian smoothing does not alter the objective distribution substantially in term of the Wasserstein distance, by bounding $W_2(\pi, \pi_\mu)$ (Lemma 4). Employing Lemma 3, we then deploy a result on mixing times of Langevin diffusion with stochastic gradients, which enables us to bound $W_2(\pi_k, \pi_\mu)$. 
Lemma 4. Assume that \( \pi \propto \exp(-U) \) and \( \pi_\mu \propto \exp(-U_\mu) \). We have the following bounds, for any \( \lambda \geq 0 \)

\[
W_2^2(\pi, \pi_\mu) \leq \frac{4(d + \lambda ||x^*||^2)}{\lambda} (a + e^\alpha - 1).
\]

where \( x^* \) is unique minimizer of \( U \), \( a = \frac{L_1 \mu_{1+a} d^{1+a}}{1+a} \). If, in addition, \( a \leq 0.1 \) and for sufficient small \( \lambda \), then

\[
W_2(\pi, \pi_\mu) \leq 3 \sqrt{\frac{da}{\lambda}}.
\]

Proof. See Appendix B.

Our primary outcome is reported in the subsequent theorem.

Theorem 1. Let the initial iterate \( y_0 \) be drawn from a probability distribution \( \pi_0 \). If the step size \( \eta \) satisfies \( \eta < \frac{2}{(M + m + \lambda)} \) and for sufficient small \( \lambda \), then:

\[
W_2(\pi_K, \pi) \leq (1 - \lambda \eta)^K W_2(\pi_0, \pi_\mu) + \frac{2(M + m)}{\lambda} (\eta \sqrt{d})^{1/2} + \frac{\sigma^2(\eta \sqrt{d})^{1/2}}{M + m + \lambda + \sigma \sqrt{\lambda}} + 3 \sqrt{\frac{da}{\lambda}},
\]

where \( M = \frac{Ld^{\mu_{1+a}(1-\alpha)}}{\mu_{1-a}(1+\alpha)^{1-a}} \), \( a = \frac{L_1 \mu_{1+a} d^{1+a}}{1+a} \), and \( \sigma^2 \leq 4d^2 \mu_{1-a} \mu_{2+a}^{1/2} \left( \frac{\mu_{1+a}^{\alpha/2} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{1}{p})} \right)^\alpha + 4 \mu^2 m^2 \left( \frac{\mu_{1+a}^{\alpha/2} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{1}{p})} \right) + \frac{8 \mu^2 \mu_{1+a}^{\alpha/2} \Gamma(\frac{\alpha}{2})}{ \eta \Gamma(\frac{1}{p})} \).

Proof. Combining Lemma 3 and Lemma 4, it follows straightforwardly from [5] Theorem 3.1.

Remark 2. Our bound is equivalent to \( O(\frac{\sqrt{d}}{\lambda} + d^{\alpha+0.5} + \frac{1+a}{\sqrt{\lambda}}) \). In comparison to the previous result of [5], which is of order \( O(d^{\alpha+0.5} + \frac{1+a}{\sqrt{\lambda}}) \), our approach has better dependences on both dimension \( d \) and strongly convex parameter \( \lambda \), and strictly better whenever \( O(d^{\alpha+0.5}) \) is not the dominant term. In addition, our method is more generalized than Gaussian smoothing, especially when we desire to explore the space by heavier tail distribution than normal.

4. Conclusion

We derive polynomial-time theoretical assurances for a Perturbed Langevin Monte Carlo (P-LMC) that uses \( p \)-generalized Gaussian smoothing and deploy to objective
distributions with weakly smooth log-densities. The algorithm we proposed, is perturbing the gradient evaluating points in Langevin Monte Carlo by a $p$-generalized Gaussian random variable, which is a generalization of the recent Gaussian smoothing LMC method.

It is potential to broaden our results to sampling from structured distributions with non-smooth and nonconvex negative log-densities [7]. Further, it is not hard to show that our results can be integrated seamlessly in a largely applicable derivative-free Langevin Monte Carlo algorithm.

There are several attractive directions for future research. For instance, as discussed in Remark 2, we speculate that the dependence on $d$ and $\lambda$ is not optimal and can be improved to match those obtained for the 2-Wasserstein distance using proximal assumption.

**Appendix A: Distance measures**

Define a transference plan $\zeta$, a distribution on $(\mathbb{R}^d \times \mathbb{R}^d, B(\mathbb{R}^d \times \mathbb{R}^d))$ where $B(\mathbb{R}^d \times \mathbb{R}^d)$ is the Borel $\sigma$-field of $(\mathbb{R}^d \times \mathbb{R}^d)$ so that $\zeta(A \times \mathbb{R}^d) = P(A)$ and $\zeta(\mathbb{R}^d \times A) = Q(A)$ for any $A \in B(\mathbb{R}^d)$. Let $\Gamma(P, Q)$ designate the set of all such transference plans. Then the 2-Wasserstein distance is formulated as:

$$W_2(P, Q) := \left( \inf_{\zeta \in \Gamma(P, Q)} \int_{x, y \in \mathbb{R}^d} \|x - y\|^2 \, d\zeta(x, y) \right)^{1/2}.$$  

**Appendix B: Proofs**

**Lemma 1.** Given that $U : \mathbb{R}^d \to \mathbb{R}$ is a convex function which satisfies Eq. 2.1 for some $L < \infty$ and $\alpha \in [0, 1]$, then:

(i) $\forall x \in \mathbb{R}^d : |U_\mu(x) - U(x)| = U_\mu(x) - U(x) \leq \frac{L \mu^{1+\alpha} d^{1+\alpha}}{1 + \alpha}$.

(ii) $\forall x, y \in \mathbb{R}^d : \|\nabla U_\mu(y) - \nabla U_\mu(x)\|_2 \leq \frac{L d \mu^{1-\alpha}}{\mu^{1-\alpha}(1 + \alpha)^{1-\alpha}} \|y - x\|_2$.

**Proof.** Part (i). Since $U_\mu(x) := E_\xi[U(x + \mu \xi)]$, $U(x) := E_\xi[U(x)]$ and $E_\xi(\xi) = 0$, which implies $E_\xi(\nabla U(x), \xi) = 0$, we have
\[ U_\mu(x) - U(x) = E[U(x + \mu \xi) - U(x) - \mu \langle \nabla U(x), \xi \rangle]. \]

First, if \( U \) is convex and \( \mu > 0 \), from A.1, \( U(x + \mu \xi) - U(x) - \mu \langle \nabla U(x), \xi \rangle \geq 0 \) for every \( \xi \) and \( x \), so \( U_\mu(x) \geq U(x) \), \( \forall x \). By the definition of the density of \( p \)-generalized Gaussian distribution \([1]\), we also have:

\[ U_\mu(x) - U(x) = \left( \frac{p^{1/\alpha}}{2\Gamma(\frac{1}{\alpha})} \right)^d \int_{\mathbb{R}^d} [U(x + \mu \xi) - U(x) - \mu \langle \nabla U(x), \xi \rangle] e^{-\|\xi\|^p_p / p} d\xi. \]

Applying Eq. 2.2 and previous inequality:

\[
|U_\mu(x) - U(x)| \leq \frac{L}{1 + \alpha} \mu^{1+\alpha} \left( \frac{p^{1/\alpha}}{2\Gamma(\frac{1}{\alpha})} \right)^d \int_{\mathbb{R}^d} \|\xi\|^{\frac{1+\alpha}{p}} \|\xi\|^{\frac{(1+\alpha)}{p}} e^{-\|\xi\|^p_p / p} d\xi
\]

\[
\leq \frac{L}{1 + \alpha} \mu^{1+\alpha} \left( \frac{p^{1/\alpha}}{2\Gamma(\frac{1}{\alpha})} \right)^d \int_{\mathbb{R}^d} \|\xi\|^p_p \|\xi\|^p_p e^{-\|\xi\|^p_p / p} d\xi
\]

\[
= \frac{L}{1 + \alpha} \mu^{1+\alpha} \left( \frac{p^{1/\alpha}}{2\Gamma(\frac{1}{\alpha})} \right)^d \int_{\mathbb{R}^d} \|\xi\|^p_p \|\xi\|^p_p e^{-\|\xi\|^p_p / p} d\xi.
\]

The second inequality follows from \( \|\xi\|_p \geq \|\xi\|_2 \) for any \( p \leq 2 \). Since \( 1 \leq p \) we have \( 1 + \alpha \leq 2p \) and \( x_\frac{1+\alpha}{2p} \) is concave. It follows that

\[
U_\mu(x) - U(x) \leq \frac{L}{1 + \alpha} \mu^{1+\alpha} \left[ \|\xi\|^{2p(1+\alpha)/(2p)}_p \right]^{1+\alpha/(2p)}
\]

\[
\leq \frac{L}{1 + \alpha} \mu^{1+\alpha} \left[ \Gamma\left( \frac{d}{2} \right) \right]^{1+\alpha/(2p)}
\]

\[
= \frac{L}{1 + \alpha} \mu^{1+\alpha} \left( \frac{2p}{\Gamma\left( \frac{d}{2} \right)} \right)^{(1+\alpha)/(2p)}
\]

\[
= \frac{L}{1 + \alpha} \mu^{1+\alpha} \left( \frac{2d (d + p)}{p} \right)^{(1+\alpha)/(2p)}
\]

Finally, for sufficiently large \( d \), we have

\[
|U_\mu(x) - U(x)| \leq \frac{L}{1 + \alpha} \mu^{1+\alpha} \frac{d^{1+\alpha}}{p}. 
\]
Part (ii). First, observe that, by Jensen’s inequality and Eq. 2.1:

\[
\| \nabla U_\mu(y) - \nabla U_\mu(x) \| \leq \left( \frac{p^{-1/2}}{2 \Gamma(\frac{1}{p})} \right)^d \int_{\mathbb{R}^d} \| \nabla U(y + \mu \xi) - \nabla U(x + \mu \xi) \|_p e^{-\|\xi\|_p^p / \mu} d\xi \\
\leq L \| y - x \|_2^{\alpha/2}.
\]

Further, by exchanging gradient and integral, the gradient of \( U_\mu \) can be expressed as:

\[
\nabla U_\mu(x) = \frac{1}{\mu} \left( \frac{p^{-1/2}}{2 \Gamma(\frac{1}{p})} \right)^d \int_{\mathbb{R}^d} U(x + \mu \xi) \|\xi\|_p^{p-1} e^{-\|\xi\|_p^p / \mu} d\xi.
\]

Thus, applying Jensen’s inequality, we also have:

\[
\| \nabla U_\mu(y) - \nabla U_\mu(x) \| \leq \left( \frac{p^{-1/2}}{2 \Gamma(\frac{1}{p})} \right)^d \int_{\mathbb{R}^d} |U(x + \mu \xi) - U(y + \mu \xi)| \|\xi\|_p^{p-1} e^{-\|\xi\|_p^p / \mu} d\xi.
\]

Using Eq. 2.2, we have that:

\[
|U(x + \mu \xi) - U(y + \mu \xi)| \leq \min \left\{ \langle \nabla U(y + \mu \xi), x - y \rangle + \frac{L}{1 + \alpha} \| x - y \|_2^{1 + \alpha}, \right. \\
\left. \langle \nabla U(x + \mu \xi), y - x \rangle + \frac{L}{1 + \alpha} \| x - y \|_2^{1 + \alpha} \right\}
\leq \frac{1}{2} \langle \nabla U(y + \mu \xi) - \nabla U(x + \mu \xi), x - y \rangle + \frac{L}{1 + \alpha} \| x - y \|_2^{1 + \alpha}
\leq \frac{L}{1 + \alpha} \| x - y \|_2^{1 + \alpha},
\]

where the second inequality comes from the minimum being smaller than the mean, and the last inequality is by convexity of \( U \) (which implies \( \langle \nabla U(x) - \nabla U(y), x - y \rangle \geq 0, \forall x, y \)). Thus, combining with Eq. B.2, we have:

\[
\| \nabla U_\mu(y) - \nabla U_\mu(x) \| \leq \frac{L}{\mu (1 + \alpha)} \| y - x \|_2^{1 + \alpha} \left( \frac{p^{-1/2}}{2 \Gamma(\frac{1}{p})} \right)^d \int_{\mathbb{R}^d} \|\xi\|_p^{p-1} e^{-\|\xi\|_p^p / \mu} d\xi \leq \frac{L}{\mu (1 + \alpha)} \| y - x \|_2^{1 + \alpha} \left[ \frac{p^{-1/2}}{2 \Gamma(\frac{1}{p})} \right] \left( \frac{n + p - 1}{p} \right) \frac{1}{\Gamma(\frac{2}{p})},
\]

(B.3)
From [20], we want to show that $E(\|\xi\|_{p}^{p-1}) \leq \left[ \frac{p^{\Gamma(d/p+1)}}{\Gamma(d/p)} \right]^{\frac{p-1}{p}}$. Since $\Gamma$ is log-convex, by Jensen’s inequality for any $p \geq 1$, we have

$$\frac{1}{p} \log \Gamma \left( \frac{d}{p} \right) + \frac{p-1}{p} \log \Gamma \left( \frac{d}{p} + 1 \right) \geq \log \Gamma \left( \frac{1}{p} \frac{d}{p} + \frac{p-1}{p} \left( \frac{d}{p} + 1 \right) \right),$$

$$\geq \log \Gamma \left( \frac{d+p-1}{p} \right) > 0.$$

Raising $e$ to the power of both sides, we get

$$\Gamma \left( \frac{d}{p} \right)^{\frac{1}{p}} \Gamma \left( \frac{d}{p} + 1 \right)^{\frac{p-1}{p}} \geq \Gamma \left( \frac{d+p-1}{p} \right),$$

which implies that

$$\left[ \frac{\Gamma \left( \frac{d}{p} + 1 \right)}{\Gamma \left( \frac{d}{p} \right)} \right]^{\frac{p-1}{p}} \geq \frac{\Gamma \left( \frac{d+p-1}{p} \right)}{\Gamma \left( \frac{d}{p} \right)}.$$

So

$$\|\nabla U(\mu)(y) - \nabla U(\mu)(x)\|_2 \leq \frac{L}{\mu(1+\alpha)} \|y-x\|_2^{1+\alpha} \frac{d^{\frac{p-1}{p}}}{\Gamma \left( \frac{d}{p} \right)}.$$

Finally, combining Eqs. B.1 and B.3:

$$\|\nabla U(\mu)(y) - \nabla U(\mu)(x)\|_2 = \|\nabla U(\mu)(y) - \nabla U(\mu)(x)\|_2^\alpha \cdot \|\nabla U(\mu)(y) - \nabla U(\mu)(x)\|_2^{1-\alpha}$$

$$\leq L^\alpha \left( \frac{Ld^{\frac{p-1}{p}}}{\mu(1+\alpha)} \right)^{1-\alpha} \|y-x\|_2$$

$$= \frac{Ld^{\frac{p-1}{p}(1-\alpha)}}{\mu^{1-\alpha}(1+\alpha)^{1-\alpha}} \|y-x\|_2,$$

as claimed.

**Lemma 2.** Given that $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ is $\lambda$-strongly convex, then $\psi_\mu$ is also $\lambda$-strongly convex.
\textbf{Proof.} Recall that a differentiable function $\psi$ is $\lambda$-strongly convex if, $\forall x, y \in \mathbb{R}^d$:

$$\psi(y) \geq \psi(x) + \langle \nabla \psi(x), y - x \rangle + \frac{\lambda}{2} \|y - x\|^2.$$ 

By the definition of a $p$-Gaussian smoothing, $\forall x, y \in \mathbb{R}^d$:

$$
\psi_{\mu}(y) - \psi_{\mu}(x) - \langle \nabla \psi_{\mu}(x), y - x \rangle \\
= \left( p^{1 - \frac{1}{p}} \right)^d \int_{\mathbb{R}^d} (\psi(y + \mu \xi) - \psi(x + \mu \xi) - \langle \nabla \psi(x + \mu \xi), y - x \rangle)e^{-\|\xi\|^2/p}d\xi \\
\geq \left( p^{1 - \frac{1}{p}} \right)^d \int_{\mathbb{R}^d} \lambda/2 \|y - x\|^2e^{-\|\xi\|^2/p}d\xi \\
= \frac{\lambda}{2} \|y - x\|^2,
$$

where we have used $\lambda$-strong convexity of $\psi$. \hfill \Box

\textbf{Lemma 3.} For any $x \in \mathbb{R}^d$, and $z_1, z_2 \sim N_p(0, I_{d \times d})$, let $G(x, z_1, z_2) := \nabla \mathcal{U}(x + \mu z_1) - \frac{\mu}{\eta} z_1 - \frac{\mu}{\eta} z_2$ denote a stochastic gradient of $\mathcal{U}_{\mu}$. Then $G(x, z_1, z_2)$ is an unbiased estimator of $\nabla \mathcal{U}_{\mu}$ whose (normalized) variance can be bounded as:

$$\sigma^2 := \frac{E_{z_1, z_2}[\|\nabla \mathcal{U}_{\mu}(x) - G(x, z_1, z_2)\|^2]}{d}$$

$$\leq 4d^{a-1} \mu^2 a L^2 \left( \frac{p^2 \Gamma \left( \frac{3}{p} \right)}{\Gamma \left( \frac{1}{p} \right)} \right)^\alpha + 4\mu^2 m^2 \left( \frac{p^2 \Gamma \left( \frac{3}{p} \right)}{\Gamma \left( \frac{1}{p} \right)} \right) + \frac{8\mu^2 p^2 \Gamma \left( \frac{3}{p} \right)}{\eta^2 \Gamma \left( \frac{1}{p} \right)}.$$

\textbf{Proof.} Recall that by definition of $\mathcal{U}_{\mu}$, we have $\nabla \mathcal{U}_{\mu}(x) = E_w[\nabla \mathcal{U}(x + \mu w)]$, where $w \sim N_p(0, I_{d \times d})$, and is independent of $z_1, z_2$. Clearly, $E_{z_1, z_2}[G(x, z_1, z_2)] = \nabla \mathcal{U}_{\mu}(x)$.

We now proceed to bound the variance of $G(x, z_1, z_2)$. First, using Young’s inequality (which implies $(a + b)^2 \leq 2(a^2 + b^2)$, $\forall a, b$) and that $z_1 \sim N_p(0, I_{d \times d})$, we have:

$$E_{z_1, z_2}[\|\nabla \mathcal{U}_{\mu}(x) - G(x, z_1, z_2)\|^2]$$

$$\leq 2E_{z_1}[\|E_{w}[\nabla \mathcal{U}(x + \mu w)] - \nabla \mathcal{U}(x + \mu z_1)\|^2] + \frac{4\mu^2 \mu^2 E_{z_1}[[z_1]]^2}{\eta^2} + \frac{4\mu^2 E_{z_1}[[z_2]]^2}{\eta^2}$$

$$= 2E_{z_1}[\|E_{w}[\nabla \mathcal{U}(x + \mu w)] - \nabla \mathcal{U}(x + \mu z_1)\|^2] + \frac{8\mu^2 \mu^2}{\eta^2 \text{tr}(\Sigma)}$$

$$= 2E_{z_1}[\|E_{w}[\nabla \mathcal{U}(x + \mu w)] - \nabla \mathcal{U}(x + \mu z_1)\|^2] + \frac{8\mu^2 p^2 \Gamma \left( \frac{3}{p} \right)}{\eta^2 \Gamma \left( \frac{1}{p} \right)}d.$$
where the last equation is by [1]. The rest of the proof is as follows. By Jensen’s inequality,
\[
E_{z_1} \left[ \| \nabla U(x + \mu w) \| - \nabla U(x + \mu z_1) \| \right]^2 \leq E_{z_1, w} \left[ \| \nabla U(x + \mu w) \| - \nabla U(x + \mu z_1) \| \right]^2.
\]
Hence, applying Young’s inequality \((a + b)^2 \leq 2(a^2 + b^2), \forall a, b\), we further have:
\[
E_{z_1, w} \left[ \| \nabla U(x + \mu w) \| - \nabla U(x + \mu z_1) \| \right]^2 \leq E_{z_1, w} \left[ (L\| \mu (w - z_1) \| \right)^2 + m\| \mu (w - z_1) \|_2^2 \right]
\]
\[
\leq 2L^2\mu^2 aE_{z_1, w} \left[ \| w - z_1 \|_2^2 \right] + 2m^2\mu^2 E_{z_1, w} \left[ \| w - z_1 \|_2^2 \right].
\]
Observe that \(f(y) = y^a\) is a concave function, \(\forall \alpha \in [0, 1]\). Hence, we have that \(E_{z_1, w} \left[ \| w - z_1 \|_2^2 \right] \leq (E_{z_1, w} \left[ \| w - z_1 \|_2^2 \right])^a\). As \(w\) and \(z_1\) are independent, \(w - z_1 \sim N_p(0, 2I_{d \times d})\).
Thus, we finally have:
\[
E_{z_1, z_2} \left[ \| \nabla U_\mu (x) - G(x, z_1, z_2) \|_2^2 \right]
\]
\[
\leq 4d^{a-1}\mu^2 aL^2 \left( \frac{p^2 \Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} \right)} \right)^a + 4\mu^2 m^2 \left( \frac{p^2 \Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} \right)} \right) + \frac{8\mu^2 p^2 \Gamma \left( \frac{1}{p} \right)}{\eta^2 \Gamma \left( \frac{1}{p} \right)}
\]
as claimed.

**Lemma 4.** Assume that \(\pi \propto \exp(-U)\) and \(\pi_\mu \propto \exp(-U_\mu)\). We have the following bounds, for any \(\lambda \geq 0\)
\[
W_2^2(\pi, \pi_\mu) \leq \frac{4(d + \lambda \| x^* \|_2^2)}{\lambda} \left( a + e^a - 1 \right)
\]
where \(x^*\) is unique minimizer of \(U\), \(a = \frac{L\Gamma^{1+\alpha}d^{1+\alpha}}{\Gamma^{1+\alpha}}\). If, in addition, for \(a \leq 0.1\) and for sufficient small \(\lambda\), then
\[
W_2(\pi, \pi_\mu) \leq 3 \sqrt{\frac{da}{\lambda}}.
\]

**Proof.** This proof adapts the technique of the proof of [9] Proposition 1. Without loss of generality we may assume that \(\int_{\mathbb{R}^d} \exp(-U(x))dx = 1\). We first give upper and lower bounds to the normalizing constant of \(\pi_\mu\), that is
\[
c_\mu \triangleq \int_{\mathbb{R}^d} \pi(x)e^{-|U_\mu(x) - U(x)|}dx.
\]
The constant \(c_\mu\) is an expectation with respect to the density \(\pi_\mu\), it can be trivially upper bounded by 1. From Lemma 1, \(|U_\mu(x) - U(x)| \leq a = \frac{L\Gamma^{1+\alpha}d^{1+\alpha}}{\Gamma^{1+\alpha}}\)
This fact yields

\[ e^{-a} \leq c_\lambda \leq 1. \]

Now we control the distance between densities \( \pi \) and \( \pi_\mu \) at any fixed \( x \in \mathbb{R}^d \):

\[
|\pi(x) - \pi_\mu(x)| = \pi(x) \left| 1 - \frac{e^{-|U_\mu(x) - U(x)|}}{c_\lambda} \right|
\leq \pi(x) \left\{ \left( 1 - e^{-|U_\mu(x) - U(x)|} \right) + e^{-|U_\mu(x) - U(x)|} \left( \frac{1}{c_\lambda} - 1 \right) \right\}
\leq \pi(x) \left( |U_\mu(x) - U(x)| + e^{|U_\mu(x) - U(x)|} - 1 \right).
\]

The second inequality is trivial while the last inequality follows from \( 1 - e^{-x} \leq x \) for any \( x \geq 0 \). To bound \( W_2 \), we use an inequality from [23](Theorem 6.15, page 115):

\[
W_2^2(\mu, \nu) \leq 2 \int_{\mathbb{R}^d} \|x\|_2^2 |\mu(x) - \nu(x)| dx.
\]

Combining this with the bound on \( |\pi(x) - \pi_\mu(x)| \) shown above, we have

\[
W_2^2(\pi, \pi_\mu) \leq 2 \int_{\mathbb{R}^d} \|x\|_2^2 \pi(x) (a + e^d - 1) dx.
\]

By [10], the following bound on the second moment, centered on the mode holds

\[
\int_{\mathbb{R}^d} \|x - x^*\|_2^2 \pi(x) dx \leq \frac{d}{\lambda}.
\]

In addition, by Young inequality

\[
\int_{\mathbb{R}^d} \|x\|_2^2 \pi(x) dx \leq 2 \int_{\mathbb{R}^d} \|x - x^*\|_2^2 \pi(x) dx + 2 \int_{\mathbb{R}^d} \|x^*\|_2^2 \pi(x) dx
\]
\[
\leq \frac{2d}{\lambda} + 2 \|x^*\|_2^2.
\]

Therefore

\[
W_2^2(\pi, \pi_\mu) \leq \frac{4(d + \lambda \|x^*\|_2^2)}{\lambda} (a + e^d - 1),
\]

which is the claim of the Lemma.
Finally, $a < 0.1$ ensures that $e^a - 1 \leq 1.06a$. Follow from previous inequality we have,

$$W_2(\pi, \pi_\mu) \leq \sqrt{\frac{8.24(d + \lambda \|x^*\|_2^2)a}{\lambda}}.$$ 

Treating $L, \mu, \|x^*\|_2^2$ as constants, choose $\lambda$ small enough so that $8.24\lambda \|x^*\|_2^2 < 0.76d$, then $W_2(\pi, \pi_\mu)$ is less than $3\sqrt{\frac{d}{\lambda}}$. $\square$
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References

[1] Arellano-Valle, R. B. and Richter, W.-D. (2012). On skewed continuous ln, p-symmetric distributions. Chilean Journal of Statistics, 3(2):193–212.

[2] Atchadé, Y. F. (2015). A Moreau-Yosida approximation scheme for a class of high-dimensional posterior distributions. arXiv preprint arXiv:1505.07072.

[3] Bolley, F., Gentil, I., and Guillin, A. (2012). Convergence to equilibrium in Wasserstein distance for Fokker–Planck equations. Journal of Functional Analysis, 263(8):2430–2457.

[4] Cesa-Bianchi, N. and Lugosi, G. (2006). Prediction, learning, and games. Cambridge University Press.

[5] Chatterji, N. S., Diakonikolas, J., Jordan, M. I., and Bartlett, P. L. (2019). Langevin Monte Carlo without smoothness. arXiv preprint arXiv:1905.13285.

[6] Chen, Y., Dwivedi, R., Wainwright, M. J., and Yu, B. (2018). Fast MCMC sampling algorithms on polytopes. The Journal of Machine Learning Research, 19(1):2146–2231.

[7] Cheng, X., Chatterji, N. S., Abbasi-Yadkori, Y., Bartlett, P. L., and Jordan, M. I. (2018). Sharp convergence rates for Langevin dynamics in the nonconvex setting. arXiv preprint arXiv:1805.01648.

[8] Dalalyan, A. S. and Karagulyan, A. (2019). User-friendly guarantees for the Langevin Monte Carlo with inaccurate gradient. Stochastic Processes and their Applications, 129(12):5278–5311.

[9] Dalalyan, A. S., Riou-Durand, L., and Karagulyan, A. (2019). Bounding the error of discretized Langevin algorithms for non-strongly log-concave targets. arXiv preprint arXiv:1906.08530.

[10] Durmus, A., Moulines, E., et al. (2019). High-dimensional Bayesian inference via the unadjusted Langevin algorithm. Bernoulli, 25(4A):2854–2882.
[11] Durmus, A., Moulines, E., and Pereyra, M. (2018). Efficient Bayesian computation by proximal Markov chain Monte Carlo: when Langevin meets Moreau. SIAM Journal on Imaging Sciences, 11(1):473–506.

[12] Dyer, M., Frieze, A., and Kannan, R. (1991). A random polynomial-time algorithm for approximating the volume of convex bodies. Journal of the ACM (JACM), 38(1):1–17.

[13] Hsieh, Y.-P., Kavis, A., Rolland, P., and Cevher, V. (2018). Mirrored Langevin dynamics. In Advances in Neural Information Processing Systems, pages 2878–2887.

[14] Kaipio, J. and Somersalo, E. (2006). Statistical and computational inverse problems, volume 160. Springer Science & Business Media.

[15] Lovász, L. and Vempala, S. (2007). The geometry of logconcave functions and sampling algorithms. Random Structures & Algorithms, 30(3):307–358.

[16] Marie-Caroline, C., Denis, K., Emilie, C., Jean-Yves, T., and Jean-Christophe, P. (2019). Preconditioned P-ULA for joint deconvolution-segmentation of ultrasound images. arXiv preprint arXiv:1903.08111.

[17] Neal, R. M. (1993). Bayesian learning via stochastic dynamics. In Advances in Neural Information Processing Systems, pages 475–482.

[18] Parisi, G. (1981). Correlation functions and computer simulations. Nuclear Physics B, 180(3):378–384.

[19] Rademacher, L. and Vempala, S. (2008). Dispersion of mass and the complexity of randomized geometric algorithms. Advances in Mathematics, 219(3):1037–1069.

[20] Richter, W.-D. (2007). Generalized spherical and simplicial coordinates. Journal of Mathematical Analysis and Applications, 336(2):1187–1202.

[21] Robert, C. and Casella, G. (2013). Monte Carlo statistical methods. Springer Science & Business Media.

[22] Roberts, G. O., Tweedie, R. L., et al. (1996). Exponential convergence of Langevin distributions and their discrete approximations. Bernoulli, 2(4):341–363.

[23] Villani, C. (2008). Optimal transport: old and new, volume 338. Springer Science & Business Media.

[24] Welling, M. and Teh, Y. W. (2011). Bayesian learning via stochastic gradient
Langevin dynamics. In *Proceedings of the 28th International Conference on Machine Learning (ICML-11)*, pages 681–688.