ON IMAGINARY PLANE CURVES AND SPIN QUOTIENTS
OF COMPLEX SURFACES BY COMPLEX CONJUGATION

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Abstract. It is proven that for any topological or analytical types of isolated singular points of plane curves, there exists a non-real irreducible plane algebraic curve of degree $d$ which goes through $d^2$ real distinct points and has imaginary singular points of the given types. This result is used to construct a series of examples of complex algebraic surfaces $X$ defined over $\mathbb{R}$ whose quotients $Y = X/\text{conj}$ by the complex conjugation $\text{conj}$ are Spin simply connected 4-manifolds with signature $16k$, for arbitrary integer $k > 0$. In the previously known examples the signature of Spin and simply connected quotients $Y$ was zero.

§1. Introduction

Given a complex algebraic surface $X$ defined over $\mathbb{R}$, we denote by $X_\mathbb{R}$ the fixed point set of the complex conjugation $\text{conj}: X \to X$, and by $Y = X/\text{conj}$ the quotient space. For nonsingular $X$ its quotient $Y$ is a closed 4-manifold, which inherits from $X$ an orientation and smooth structure making the quotient map, $q: X \to Y$, an orientation preserving smooth double covering branched along $X_\mathbb{R}$. The natural question is to describe the differential topology types of 4-manifolds which can arise as the quotients $Y$ (cf. [D]).

In a plenty of examples the condition that $Y$ is simply connected (which is the case for example if $X$ is simply connected and $X_\mathbb{R} \neq \emptyset$) implies that $Y$ splits into a connected sum of copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$. This property for $K3$ surfaces was noticed by S. Donaldson [D], S. Akbulut [Ak] set up it for a family of double planes branched along real curves. In [F1] the family of such double planes was essentially extended and in [F2] this decomposability property was set up for the complete intersection surfaces which are constructed by a small perturbation method (this is a straightforward method which produces however a plenty of examples of arbitrary multi-degree). Surprisingly, no examples of complex algebraic surfaces were known which had Spin and simply connected quotients $Y$ with non-vanishing signature. On the other hand, in certain cases (for example for real double planes $X \to \mathbb{C}P^2$) it is not difficult to show that if the quotient $Y$ is Spin and simply connected then its signature vanish.

Recently R. Gompf informed one of the authors about the construction of similar examples in symplectic category. Namely, by taking fiber sums of elliptic surfaces he constructed a symplectic 4-manifold $X$ with an anti-symplectic involution $\text{conj}: X \to X$, which has simply connected and Spin quotient $Y = X/\text{conj}$ with

1991 Mathematics Subject Classification. Primary: 14P25, 57N13.

Key words and phrases. Imaginary Singularities, Quotients by Complex Conjugation.

The second author was partially supported by the grant no. 6836-1-9 of the Ministry of Science and Technology, Israel.
signature \( \sigma(Y) < 0 \). The method of Gompf, probably, can be developed to produce examples of elliptic surfaces \( X \) with the same properties of the quotients.

In the present paper we construct examples of a different nature, when \( X \) are algebraic surfaces of general type and their quotients \( Y \) have positive signature.

**Theorem 1.1.** There exist nonsingular complex algebraic surfaces \( X \) of general type defined over \( \mathbb{R} \) which have simply connected Spin quotients \( Y = X/\text{conj} \) with signature \( \sigma(Y) = 16k \) for arbitrary integer \( k \geq 1 \).

These surfaces appear as double coverings over \( \mathbb{C}P^2 \) blown up at \( k \) pairs of complex conjugated imaginary points, \( z_i, \bar{z}_i, i = 1, \ldots, k \). The branching locus of such a covering is the proper image of a plane real curve, \( A \subset \mathbb{C}P^2 \), of even degree, \( 2d \), which has non-degenerate singularities of multiplicity 6 at the above points \( z_i, \bar{z}_i \) and the real part \( A_R = A \cap \mathbb{R}P^2 \) consisting of \( d^2 \) ovals lying separately on \( \mathbb{R}P^2 \). The curve \( A \) is obtained by a perturbation of the curve \( B \cup \text{conj}(B) \), where \( B \) is a plane imaginary (that is not invariant under the complex conjugation) curve of degree \( d \) which intersects \( \mathbb{R}P^2 \) precisely at \( d^2 \) points and has non-degenerate singularities of multiplicity 6 at \( z_i, i = 1, \ldots, k \) as its only singularities.

Existence of such a curve \( B \) follows from the following general result.

**Theorem 1.2.** For any given topological or analytic types of isolated singular points of plane algebraic curves, there exists an imaginary algebraic curve in \( \mathbb{C}P^2 \) of an arbitrarily large degree \( d \) which has prescribed numbers of imaginary singular points of the given types as its only singularities and exactly \( d^2 \) real points.

We prove Theorem 1.2 in §2 and Theorem 1.1 in §3.

We thank Prof. A. Libgober (Chicago) for consulting us on the topology of singular curves on algebraic surfaces.

**§2. Singular plane imaginary curves with maximal number of real points**

For the sake of simplicity we denote plane curves and polynomials defining them by the same symbol.

In this section we prove the following

**Theorem 2.1.** For any \( k \geq 1 \), \( n \geq 0 \) there exists a curve in \( \mathbb{C}P^2 \) of degree \( d = 3 \cdot 2^{3k+n} \) which has \( 2^2n \) imaginary non-degenerate singular points of multiplicity \( m = 2k+1 \) and exactly \( d^2 \) distinct real points.

This statement immediately implies that the curve in Theorem 2.1 is irreducible and imaginary, and at any its real point this curve is non-singular and transversal to \( \mathbb{R}P^2 \). This is because a curve \( B \subset \mathbb{C}P^2 \) cannot have more then \((\deg B)^2\) real points unless it has a real irreducible component, which follows from Bezout’s theorem applied to \( B \) and \( \bar{B} \). On the other hand, for each real component \( F \) of \( B \) the intersection \( F \cap \mathbb{R}P^2 \) contains either infinite number of points, or only isolated (hence singular) points, which number is bounded from above by \( p = \frac{1}{2}(\deg F - 1)(\deg F - 2) \) (arithmetic genus of the curve).

Let us first derive Theorem 1.2 from Theorem 2.1.

**Proof of Theorem 1.2.** As it is mentioned above, the real points of the curve \( C \) from Theorem 2.1 are transversal to \( \mathbb{R}P^2 \) and therefore persist under small deformations.
On the other hand, any topological or analytic singularity is determined by the $(\mu + 1)$-jet of the local Taylor series at a singular point, where $\mu$ is Milnor number. It is known that under the condition (see, for instance [DG])

\[(m_1 + 1) + \ldots + (m_r + 1) \leq d + 1,\]

for polynomials of degree $d$, the $m_i$-jets, $i = 1, \ldots, r$, of their local Taylor series at any $r$ distinct points can be prescribed independently. Hence, given certain singularities, one takes the curve $C$ from Theorem 2.1 with sufficiently big $k, n$, and then by a small variation of $C$ in the space of curves of degree $d$, describes the jets, defining these singularities at some non-degenerate singular points of $C$. Assume that the curve $\tilde{C}$, obtained in the latter step has extra singular points $w_1, \ldots, w_p$. Using independence of jets mentioned above, one deforms $C$ into a curve $C_1$, which has the same jets at the points giving prescribed singularities and which does not pass through $w_i$, $(1 \leq i \leq p)$. Then by Bertini’s theorem, one kills all extra singularities by variation in the linear system

$$\lambda \tilde{C} + \lambda_1 \tilde{C}_1 + \ldots + \lambda_p \tilde{C}_p, \quad \lambda, \lambda_1, \ldots, \lambda_p \in \mathbb{R} \quad \square$$

Proof of Theorem 2.1. We proceed by induction on $k$ in the case $n = 0$, and then by induction on $n$ with a fixed $k \geq 1$.

**Step 1.** Let $n = 0$, $k = 1$, $d = 3$, $m = 2$. Let $C_3$ be a real cuspidal cubic curve and $C_3'$ be a cubic curve, meeting $C_3$ at 9 distinct real points (for instance, we can take $C_3'$ to be the union of three real straight lines intersecting $C_3$ each at three real points). The pencil

$$\Lambda = \{ \alpha C_3 + \beta C_3' \mid [\alpha : \beta] \in \mathbb{RP}^1 \}$$

intersects the discriminant in the space $\mathbb{RP}^9$ of real cubics at $C_3$ with multiplicity 2, because cuspidal curves form a cuspidal edge of the discriminant (see, for instance [Gu, §2]). Since a small variation of the pencil $\Lambda$ is a pencil with 9 real base points as well, we can move $\Lambda$ in such a position that its complexification will meet the discriminant at two imaginary points near $C_3$, which correspond to imaginary nodal cubics with 9 real points.

**Step 2.** The procedure in the case $n = 0$ will be as follows: given an imaginary curve of degree $d$ with a non-degenerate point of multiplicity $m < d$ and $d^2$ real points, we construct a curve of degree 8$d$ with a non-degenerate point of multiplicity 2$m$ and 64$d^2$ real points.

Let $C_d$ be an imaginary curve of degree $d$ with an imaginary non-degenerate singular point $z$ of multiplicity $m$, having exactly $d^2$ real points, $w_i, i = 1, \ldots, d^2$. It is easily seen, say, by (2.2), that for polynomials of degree $d$, the $m$-jet at $z$ and 0-jet at any point $z' \neq z$ are independent. Hence, varying a little the curve $C_d$ we can provide that none of the points $w_i, 1 \leq i \leq d^2$, lies on the real straight line $L$ through $z$, and $L$ is not tangent to $C_d$ at $z$. Let us choose real projective coordinates $[x_0 : x_1 : x_2]$ so that $L = \{ x_0 = 0 \}$, and

$$w_i = [1 : x_i : y_i], \quad x_i > 0, \quad y_i > 0, \quad i = 1, \ldots, d^2.$$  

The transformation $[x_0 : x_1 : x_2] \mapsto [x_0^2 : x_1^2 : x_2^2]$ turns the points $w_1, \ldots, w_{d^2}$ into 4$d^2$ real points of the curve

\[(2.3) \quad C_{2d}(x_0, x_1, x_2) \overset{\text{def}}{=} C_d(x_0^2, x_1^2, x_2^2)\]
of degree $2d$. The point $z$ will turn into two imaginary singular points $z_1, z_2$ of $C_{2d}$, such that $z_i, i = 1, 2$, is a center of $m$ non-singular branches quadratically tangent to the straight line $L_i$ through $z_i$ and $[1:0:0]$; indeed, in affine coordinates $x = x_0/x_2, y = x_1/x_2$ a local branch of $C_d$ centered at $z = (0, a), a \neq 0$, has equation

$$ y = a + bx + O(x^2), \quad b \neq 0, $$

that is transformed into

$$ y^2 = a + bx^2 + O(x^4), $$

which defines two non-singular branches

$$ y = \pm \sqrt{a} \left(1 + \frac{b}{2a}x^2\right) + O(x^3), $$

centered at the points $z_{1,2} = (0, \pm \sqrt{a})$ and quadratically tangent to the lines $y = \pm \sqrt{a}$, respectively.

Step 3. Real conics tangent to $L_1$ at $z_1$, form a pencil. Let $K$ be a generic conic in this pencil, having a real oval. We choose on $K$ three generic real points and change coordinates in such a way that these points become $[1:0:0], [0:1:0], [0:0:1]$, respectively. The Cremona transformation (see detail in [Wa])

$$ (2.4) \quad [x_0:x_1:x_2] \mapsto [x_1x_2:x_0x_2:x_0x_1] $$

takes the conic $K$ to a real straight line $L^*$, and the curve $C_{2d}$ to a curve $C_{4d}$ of degree $4d$ having $4d^2$ real non-singular points $v_i, i = 1, \ldots, 4d^2$, three non-degenerate singularities of multiplicity $2d$ at the real points $[1:0:0], [0:1:0], [0:0:1]$, and an imaginary singular point $z^*_1 \in L^*$ (coming from $z_1$), which is a center of $m$ non-singular local branches quadratically tangent to $L^*$.

Step 4. Keeping singularity at $z^*_1$, we shall deform the ordinary singular points $[1:0:0], [0:1:0], [0:0:1]$ of $C_{4d}$ in order to get $12d^2$ more real non-singular points. We apply a modified Viro method (see the original Viro method in [Vi1, Vi2, Vi3, Ri] and [GKZ, Chapter 11], and relevant modifications in [Sh]).

In the affine coordinates $x = x_1/x_0, y = x_2/x_0$ the curve $C_{4d}$ is defined by a polynomial $F_0(x, y)$ with Newton triangle

$$ \Delta_0 = \text{conv}((2d,0), (0,2d), (2d,2d)). $$

Consider the polynomials

$$ F_1(x, y) = x^{2d}y^{2d}F_0(y^{-1}, x^{-1}), $$

$$ F_2(x, y) = x^{-2d}y^{4d}F_0(x, xy^{-1}), $$

$$ F_3(x, y) = x^{4d}y^{-2d}F_0(x^{-1}y, y). $$

It is easily seen that

(i) these polynomials have Newton triangles

$$ \Delta_1 = \text{conv}((0,0), (2d,0), (0,2d)), $$

$$ \Delta_2 = \text{conv}((0,2d), (0,4d), (2d,2d)), $$

$$ \Delta_3 = \text{conv}((2d,0), (4d,0), (2d,2d)), $$
respectively, which together with \( \Delta_0 \) form a subdivision of the Newton triangle \( T_{4d} = \text{conv}((0,0), (4d,0), (0,4d)) \) of a generic polynomial of degree \( 4d \).

(ii) each curve \( F_i \) has exactly \( 4d^2 \) real points in \( (\mathbb{R}^*)^2 \), \( i = 0, 1, 2, 3 \),

(iii) the coefficients of any common monomial in \( F_i, F_j \), \( i \neq j \), coincide.

In particular, the polynomials \( F_0, F_1, F_2, F_3 \) uniquely define a set of coefficients \( a_{ij} \) of monomials \( x^iy^j \), \( i + j \leq 4d \). Let \( \nu : T_{4d} \to \mathbb{R} \) be a convex piecewise-linear function, whose linearity domains are just \( \Delta_0, \Delta_1, \Delta_2, \Delta_3 \), which vanishes on \( \Delta_0 \) and is integral-valued at integral points. We will look for the desired deformation in the Viro-type family

\[
\Phi_t(x,y) = \sum_{i+j \leq 4d} A_{ij}(t)x^iy^jt^{\nu(i,j)}, \quad t > 0,
\]

where \( A_{ij}(t) = a_{ij} + O(t) \), \( i + j \leq 4d \), are smooth functions. We find out \( A_{ij}(t) \) assuming that the curves \( \Phi_t \) have \( m \) smooth branches at \( z_1^* \) tangent to \( L^* \). This means that in suitable coordinates in a neighborhood of \( z_1^* \) coefficients of monomials lying under the Newton diagram \( [(0,m), (2m,0)] \) vanish, that imposes \( m(m+1) \) linear conditions on coefficients of the polynomial \( \Phi_t(x,y) \):

\[
\sum_{i+j \leq 4d} \alpha_{ij}^{(k)} A_{ij}(t)t^{\nu(i,j)} = 0, \quad k = 1, ..., m(m+1).
\]

These \( m(m+1) \) linear equations, imposed by singularity at \( z_1^* \), are independent in the space of polynomials with Newton triangle \( \Delta_0 \), because the Cremona transformation (2.4) reduces this to the evident independence in the space of polynomials of degree \( 2d \). In particular, there exists a set

\[
\Lambda \subset \Delta_0 \cap \mathbb{Z}^2, \quad \text{card}(\Lambda) = m(m+1),
\]

such that

\[
\det \left( \alpha_{ij}^{(k)} \right)_{k=1,...,m(m+1)} \neq 0.
\]

Since \( \nu \big|_{\Delta_0} = 0 \) and \( A_{ij} = a_{ij}, i + j \leq 4d, t = 0 \) is a solution of (2.6), the latter inequality by the implicit function theorem provides existence of a solution \( A_{ij}(t) = a_{ij} + O(t), i + j \leq 4d \), of (2.6) for small \( t > 0 \).

**Step 5.** Now we shall show that the curve \( \Phi_t \) has \( 16d^2 \) real points. Let \( Q \subset (\mathbb{R}^*)^2 \) be a compact neighborhood of all the real points of the curves \( F_i, i = 0, 1, 2, 3 \). By construction \( \Phi_t(x,y) = F_0(x,y) + O(t) \); hence for a sufficiently small \( t > 0 \), the curve \( \Phi_t \) has in \( Q \) at least \( 4d^2 \) real points close to these of the curve \( F_0 \). Let \( l_k(i,j) = \gamma_0\kappa + \gamma_1\kappa^i + \gamma_2\kappa^j \) be the linear function equal to \( \nu(i,j) \) on \( \Delta_k, 1 \leq k \leq 3 \).

The substitution of \( \nu_k = \nu - l_k \) for \( \nu \) in (2.5) gives us the polynomial

\[
\Phi_{t,k}(x,y) = \sum_{i+j \leq 4d} A_{ij}(t)x^iy^jt^{\nu_k(i,j)} = F_k(x,y) + O(t);
\]

hence for a sufficiently small \( t > 0 \) the curve \( \Phi_{t,k} \) has in \( Q \) at least \( 4d^2 \) real points close to these for the curve \( F_k \). On the other hand, substitution of \( \nu_k \) for \( \nu \) is equivalent to multiplication of the polynomial by a positive number and the coordinate change

\[
T_k(x,y) = (xt^{-\gamma_1\kappa}, yt^{-\gamma_2\kappa}).
\]
Therefore the curve $\Phi_t$ has at least $4d^2$ real points in $T_k^{-1}(Q), k = 1, 2, 3$. Observing that $Q, T_1^{-1}(Q), T_2^{-1}(Q), T_3^{-1}(Q)$ are disjoint for small $t > 0$, and that $16d^2$ is the maximal possible number of real points for an irreducible imaginary curve of degree $4d$, one concludes that the curve $\Phi_t$ has exactly $16d^2$ real points.

**Step 6.** Now let us make a real coordinate change in the projective plane such that $L^* = \{x_0 = 0\}$ and all $16d^2$ real points of the projective closure $\tilde{\Phi}_t$ of the curve $\Phi_t$ lie in the domain $x_0 = 1, x_1 > 0, x_2 > 0$. Consider the curve

$$C_{8d}(x_0, x_1, x_2) \overset{\text{def}}{=} \Phi_t(x_0^2, x_1^2, x_2^2)$$

of degree $8d$. Each real point of the curve $\tilde{\Phi}_t$ turns into $4$ real points of the curve $C_{8d}$, and the singular point $z_1^*$ of $\tilde{\Phi}_t$ turns into two ordinary singular points of multiplicity $2m$: indeed, in the coordinates $x = x_1/x_2, y = x_0/x_2$ the equation

$$y = b(x - a)^2 + O((x - a)^3), \quad a, b \neq 0,$$

of a local branch of $\tilde{\Phi}_t$ centered at $z_1 = (a, 0)$, turns into the equation

$$y^2 = b(x^2 - a)^2 + O((x^2 - a)^3),$$

which defines two pairs of non-singular transversal branches centered at the points $(\sqrt{a}, 0), (-\sqrt{a}, 0)$.

Hence $C_{8d}$ has $64d^2 = (8d)^2$ real singular points and an imaginary singular point of multiplicity $2m$. This completes the induction step, and thereby the proof of Theorem 2.1 for $n = 0$.

**Step 7.** Now we use induction on $n$, applying the transformation (2.3), so that each time the coordinate system is chosen in order to provide $x_1/x_0 > 0, x_2/x_0 > 0$ for all real points of $C_d$, and $x_0x_1x_2 \neq 0$ for all imaginary non-degenerate singular points of $C_d$. □

§3. **Proof of Theorem 1.1**

Consider an imaginary curve $B \subset \mathbb{CP}^2$ of degree $d$ which intersects $\mathbb{RP}^2$ in $d^2$ points, $p_1, \ldots, p_{d^2}$, and has non-degenerated singularities of multiplicity $m = 6$ at $z_i, \in \mathbb{CP}^2 - \mathbb{RP}^2$, $i = 1, \ldots, k$, as its only singularities, (such $B$ exists by Theorem 1.2). Then $A_0 = B \cup \text{conj}(B)$ is a degree $2d$ curve, which has in addition to the singularities at $z_i$ and $\overline{z}_i = \text{conj}(z_i), i = 1, \ldots, k$, nodes at $p_j, j = 1, \ldots, d^2$. We can perturb $A_0$ preserving its singularities at $z_i, \overline{z}_i$ and making the nodes smooth so that each node gives rise to an oval. If $B$ is defined by equation $f = 0$, then $A_0$ is defined by $f \circ \tilde{f} = 0$, where $\tilde{f}$ is the polynomial conjugated to $f$, and the perturbed curve, which we denote by $A_\varepsilon$, is defined by the polynomial $g_\varepsilon = f \cdot \tilde{f} - \varepsilon h^2$, where $\varepsilon > 0$ is a sufficiently small real number and $h$ is a degree $d$ real homogeneous polynomial which defines a curve not containing the points $p_i, i = 1, \ldots, d^2$, and containing $z_i, \overline{z}_i, i = 1, \ldots, k$, with multiplicity $> 3$ (such $h$ can be easily constructed since $d >> k$).

Consider blow up $P \rightarrow \mathbb{CP}^2$ centered at points $z_i, \overline{z}_i, i = 1, \ldots, k$, and denote by $\tilde{A}_\varepsilon \subset P$ the proper image of $A_\varepsilon$ and by $\tilde{B}$ the proper image of $B$. $\tilde{A}_\varepsilon$ is a nonsingular divisible by $2$ divisor in $P$; therefore there exists an algebraic surface $X$ with a double covering $p: X \rightarrow P$ branched along $\tilde{A}_\varepsilon$. It is well known [A] that
there exist two liftings to $X$ of the complex conjugation $\text{conj}$, induced on $P$ from $\mathbb{C}\mathbb{P}^2$. These liftings are anti-holomorphic involutions, which commute and give in product the covering transform of $p$. The fixed point set of one of these involution consists of $d^2$ sphere components, which are projected by $p$ onto the union $W^+$ of $d^2$ discs bounded in $\mathbb{R}\mathbb{P}^2$ by the ovals of $A_\mathbb{R} = A \cap \mathbb{R}\mathbb{P}^2$. We choose and denote by $\text{conj}_X$ the other involution, whose fixed point set, $X_\mathbb{R} = \text{Fix}(\text{conj}_X)$, is projected into the complementary part, $W^- = \mathbb{R}\mathbb{P}^2 - \text{Int}(W^+)$. Denote by $Q = P/\text{conj}$, $Y = X/\text{conj}_X$ the quotients and by $q: P \to Q$, $q': X \to Y$ the quotient maps.

Arnold noticed in [A] (in a different but analogous case of double planes) that factorization by $\mathbb{Z}/2 \times \mathbb{Z}/2$ in different order gives the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{q'} & Y \\
p & & \downarrow p' \\
P & \xrightarrow{q} & Q,
\end{array}
$$

where the map $p'$ turns out to be the double covering branched along the surface $\mathcal{A}_\varepsilon = q(W^+) \cup q(\tilde{\mathcal{A}}_\varepsilon)$ called the Arnold surface.

Recall now the well known diffeomorphism $\mathbb{CP}^2/\text{conj} \cong S^4$ (see, e.g., [M]) which implies $Q \cong \#_k \mathbb{CP}^2$. Denote by $E'_i, E''_i \subset P$ the exceptional curves over $z_i, \overline{z}_i$, $i = 1, \ldots, k$, and take as the generators of $H_2(Q)$ the fundamental classes $[E_i]$ of $E_i = q(E_1) = q(E_2)$ with the orientations coming from $E_i$.

Further, note that varying $\varepsilon \to 0$ we get an isotopy in $Q$ between $A_\varepsilon$ and $q(\tilde{B})$ (which pinch the ovals of $\mathcal{A}_\varepsilon$), cf. [F1]. This implies that $\mathcal{A}_\varepsilon$ endowed with the orientation coming from $B$ realizes class $[\mathcal{A}_\varepsilon] = 6 \sum_{i=1}^{k} [E_i] \in H_2(Q)$. Applying Hirzebruch signature formula to the branched covering $Y \to Q$ we obtain

$$
\sigma(Y) = 2\sigma(Q) - \frac{1}{2} \mathcal{A}_\varepsilon \circ \mathcal{A}_\varepsilon = -2k + \frac{1}{2} 6^2 k = 16k
$$

The formula for the Stiefel–Whitney class $w_2$ of the branched covering gives

$$
w_2(Y) = p'^*(w_2(Q)) + p''^*(\frac{1}{2} [\mathcal{A}_\varepsilon]_2) = p'^*(\sum_{i=1}^{k} [E_i]_2) + p''^*(3 \sum_{i=1}^{k} [E_i]_2) = 0,
$$

where subscript “2” denotes the reduction of an integer class modulo 2.

To finish the proof of properties of $Y$ we need only to notice that it is simply connected. To see it note first that the fundamental group $\pi_1(P - \tilde{A})$ is abelian, which follows for instance from the Nori’s theorem [N, Theorem II]. Thus it is cyclic with the generator represented by a loop around $\tilde{A}$, which follows from the following diagram

$$
\begin{array}{ccc}
H^2(\tilde{A}) & \xrightarrow{\cong} & \mathbb{Z} \\
\downarrow & & \downarrow \\
H_1(P - \tilde{A}) & \xrightarrow{\cong} & H^3(P, \tilde{A}) \\
\downarrow & & \downarrow \\
H^3(P) & \xrightarrow{\cong} & 0
\end{array}
$$
Therefore $\pi_1(X - p^{-1}(\tilde{A}))$ is also cyclic with the generator represented by a loop around $p^{-1}(\tilde{A})$. Thus $X$ is simply connected and, therefore $Y$ is simply connected, since $X_R \neq \emptyset$.

Finally, note that involution $\text{conj}_X$ is the involution of complex conjugation for some embedding $X \to \mathbb{CP}^N$, because as it was noticed by Comessatti (cf. [CP]), any anti-holomorphic involution on a complex algebraic variety $X$ is the restriction of the complex conjugation on $\mathbb{CP}^N$ for some embedding $X \to \mathbb{CP}^N$ (such an embedding is defined by the line bundle $L \otimes \text{conj}_X^*(L)$ for any very ample line bundle $L$ on $X$). □

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