HIGHER ORDER JORDAN OSSERMAN
PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. We study the higher order Jacobi operator in pseudo-Riemannian geometry. We exhibit a family of manifolds so that this operator has constant Jordan normal form on the Grassmannian of subspaces of signature \((r, s)\) for certain values of \((r, s)\). These pseudo-Riemannian manifolds are new and non-trivial examples of higher order Osserman manifolds.

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1. Introduction

Let \((M, g)\) be a pseudo-Riemannian manifold of signature \((p, q)\) and dimension \(m = p + q\). Let \(R(\cdot, \cdot)\) be the associated Riemann curvature operator. The Jacobi operator \(J(X) : Y \rightarrow R(Y, X)X\) is a self-adjoint operator which plays an important role in the study of geodesic variations and in many other applications.

We say that \((M, g)\) is Riemannian if \(p = 0\) and Lorentzian if \(p = 1\). Osserman observed that if \((M, g)\) is a local 2 point homogeneous Riemannian manifold, then the eigenvalues of the Jacobi operator are constant on the unit sphere bundle of \(M\). He wondered if the converse held, i.e. if the eigenvalues of the Jacobi operator are constant on the unit sphere bundle, does this imply that \((M, g)\) is a local 2 point homogeneous space (or equivalently, that \((M, g)\) is either flat or is a rank 1 symmetric space). This has been shown to be true if \(m \neq 16\) by Chi and Nikolayevsky. The case \(m = 16\) is still open.

It is natural to pose this question in the pseudo-Riemannian setting as well. Let \(S^\pm(M, g)\) be the pseudo-sphere bundles of unit spacelike (+) and timelike (−) vectors for a pseudo-Riemannian manifold \((M, g)\) of signature \((p, q)\). Then \((M, g)\) is said to be spacelike Osserman (resp. timelike Osserman) if the the eigenvalues of \(J(\cdot)\) are constant on \(S^+(M, g)\) (resp. on \(S^-(M, g)\)). The notions spacelike Osserman and timelike Osserman are equivalent and if \((M, g)\) is either of them, then \((M, g)\) is said to be Osserman. It is known that any Lorentzian Osserman manifold has constant sectional curvature [1, 5]. On the other hand, if \(p \geq 2\) and \(q \geq 2\), then there exist Osserman pseudo-Riemannian manifolds of signature \((p, q)\) which are not locally homogeneous [2, 3].

In the higher signature setting, unlike in the Riemannian setting, the eigenvalue structure does not determine the Jordan normal form of a symmetric linear operator. We say that \((M, g)\) is spacelike Jordan Osserman (resp. timelike Jordan Osserman) if the Jordan normal form (i.e. the conjugacy class) of \(J(\cdot)\) is constant on \(S^+(M, g)\) (resp. on \(S^-(M, g)\)). The notions spacelike Jordan Osserman and timelike Jordan Osserman are distinct. While spacelike Jordan Osserman or timelike Jordan Osserman implies Osserman, the reverse implication fails in general. There is an extensive literature on this question; see [2, 3, 4, 5] for further details.
This paper focuses on the higher order Jacobi operator, which was first defined by Stanilov and Videv [14] in the Riemannian setting. We consider it in the pseudo-Riemannian setting. Let $\text{Gr}_{r,s}(M,g)$ be the Grassmannian bundle of non-degenerate subspaces of signature $(r, s)$. We say that a pair $(r, s)$ is admissible if $\text{Gr}_{r,s}(M,g)$ is non-empty and does not consist of a single point, i.e.,

$$0 \leq r \leq p, \quad 0 \leq s \leq q, \quad \text{and} \quad 1 \leq r + s \leq m - 1.$$ 

Let $(r, s)$ be admissible and let $\{e_1, \ldots, e_{r+s}\}$ be an orthonormal basis for a subspace $\pi \in \text{Gr}_{r,s}(M,g)$. The higher order Jacobi operator is defined [12] in the pseudo-Riemannian setting by:

$$J(\pi) := \sum_{1 \leq i \leq r+s} (e_i, e_i) J(e_i).$$

The operator $J(\pi)$ is independent of the particular orthonormal basis chosen. One says that $(M, g)$ is \textit{Osserman} of type $(r, s)$ if the eigenvalues of $J(\cdot)$ are constant on $\text{Gr}_{r,s}(M,g)$. Note that $(M, g)$ is Osserman of type $(0, 1)$ or $(1, 0)$ if and only if $(M, g)$ is Osserman. Similarly, one says that $(M, g)$ is \textit{Jordan Osserman} of type $(r, s)$ if the Jordan normal form of $J(\cdot)$ is constant on $\text{Gr}_{r,s}(M,g)$; $(M, g)$ is Jordan Osserman of type $(0, 1)$ (resp. $(1, 0)$) if and only if $(M, g)$ is spacelike (resp. timelike) Jordan Osserman.

We shall use the following basic duality result [10, 12]:

\textbf{Theorem 1.1.} Let $(M, g)$ be a pseudo-Riemannian manifold of signature $(p, q)$. Let $(r, s)$ be admissible.

1. If $(M, g)$ is Osserman of type $(r, s)$, then $(M, g)$ is Osserman of type $(\bar{r}, \bar{s})$ for every admissible $(\bar{r}, \bar{s})$ with $\bar{r} + \bar{s} = r + s$ or $\bar{r} + \bar{s} = m - r - s$.

2. If $(M, g)$ is Jordan Osserman of type $(r, s)$, then $(M, g)$ is Jordan Osserman of type $(p - r, q - s)$.

In view of Theorem 1.1, we say that $(M, g)$ is \textit{k Osserman} if $(M, g)$ is Osserman of type $(r, s)$ for any (and hence for all) admissible $(r, s)$ with $r + s = k$ or $r + s = m - k$.

It is known that in either the Riemannian or the Lorentzian settings, if $(M, g)$ is \textit{k Osserman} for some $k$ with $2 \leq k \leq m - 2$, then $(M, g)$ has constant sectional curvature [8, 13]. Thus, we shall consider the higher signature setting and assume $p \geq 2$ and $q \geq 2$ henceforth.

Let $M = \mathbb{R}^{2p}$ with the usual coordinates $(x, y) := (x_1, \ldots, x_p, y_1, \ldots, y_p)$. Let

$$\mathcal{X} := \text{span}_{1 \leq i \leq p} \{\partial^x_i\} \quad \text{and} \quad \mathcal{Y} := \text{span}_{1 \leq i \leq p} \{\partial^y_i\}$$

define two distributions of $TM$. The splitting $TM = \mathcal{X} \oplus \mathcal{Y}$ is, of course, just the usual splitting $T(\mathbb{R}^{2p}) = T(\mathbb{R}^p) \oplus T(\mathbb{R}^p)$. This defines a projection

$$\rho_\mathcal{X} : T(\mathbb{R}^{2p}) \to \mathcal{X}.$$ 

\textbf{Definition 1.2.} Let $S^2(T\mathbb{R}^p)$ be the bundle of symmetric bilinear forms on $T\mathbb{R}^p$. Let $\psi \in C^\infty(S^2(T\mathbb{R}^p))$ be a symmetric 2 tensor with components $\psi_{ij} := \psi(\partial^x_i, \partial^x_j)$. Define a pseudo-Riemannian metric of balanced (or neutral) signature $(p, p)$ on $\mathbb{R}^{2p}$ by setting:

$$g_\psi(x, y)(\partial^x_i, \partial^x_j) = \delta_{ij}, \quad g_\psi(x, y)(\partial^y_i, \partial^y_j) = 0, \quad g_\psi(x, y)(\partial^x_i, \partial^y_j) := \psi_{ij}(x).$$

The coefficients of $g_\psi$ depend on $x$ but not on $y$. Furthermore, the distribution $\mathcal{Y}$ is totally isotropic with respect to $g_\psi$. In Section 2 we will show that:
Lemma 1.3. Let $\psi_{ij/kl} := \partial^2 f_{ij}/\partial x_k \partial x_l$. Let $Z_\nu$ be vector fields on $(M,g_\psi)$. Then:
1. $R(Z_1, Z_2)Z_3 = 0$ if $Z_1 \in \mathcal{X}$, or if $Z_2 \in \mathcal{X}$, or if $Z_3 \in \mathcal{X}$.
2. $R(\partial^2 f_i, \partial^2 f_j)\partial^2 f_k = -\frac{1}{2} \frac{\partial}{\partial x^l}(\psi_{ik/lk} + \psi_{jk/lk} - \psi_{lk/ik} - \psi_{lk/jk})\partial_{ij/kl}$.

If $A$ is a self-adjoint linear operator, then we may use the inner product to define an associated symmetric bilinear form $\mathcal{A}$ by setting
$$\mathcal{A}(Z_1, Z_2) := (AZ_1, Z_2).$$

Conversely, every symmetric bilinear form arises in this way. A self-adjoint operator $A$ is said to be positive semi-definite if and only if the associated quadratic form $\mathcal{A}$ is positive semi-definite. The Jacobi operator and the associated bilinear form which are defined by $(M,g_\psi)$ are supported on $\mathcal{X}$. If we set $X_\nu := \rho_X Z_\nu$, then:
$$J(Z_1)Z_2 = J(X_1)X_2, \quad \text{and} \quad (J(Z_1)Z_2, Z_3) = (J(X_1)X_2, X_3).$$

Since $J(X_1)X_1 = 0$, rank $(J(Z_1)) \leq p - 1$.

Definition 1.4.
1. If $\psi \in C^\infty(S^2(T\mathbb{R}^p))$ and if $K$ is a compact subset of $\mathbb{R}^p$, define the semi-norm $|\psi|_K = \max_{x \in K; 1 \leq i,k,l \leq p} |\psi_{ij/kl}|(x)$.
2. Let $f \in C^\infty(\mathbb{R}^p)$. Define a symmetric 2 tensor field $\psi_f$ on $T\mathbb{R}^p$ by setting $\psi_{f,ij} = \partial^2 f \cdot \partial^2 f$.
3. Let $\Psi$ be the set of all $\psi \in C^\infty(S^2(T\mathbb{R}^p))$ so that $J(X)$ is positive semi-definite of rank $p - 1$ for every $0 \neq X \in T\mathbb{R}^p$.

The metrics $g_\psi$ for $\psi \in \Psi$ will be important in our discussion. We show that this class of metrics is non-trivial by establishing the following result in Section 3.

Lemma 1.5. Let $p \geq 2$.
1. If $f \in C^\infty(\mathbb{R}^p)$ and if $H(f)$ is positive definite, then $\psi_f \in \Psi$.
2. $\Psi$ is convex, i.e. if $0 < a \in \mathbb{R}$ and if $\psi \in \Psi$, then $a \psi \in \Psi$.
3. $\Psi$ is convex, i.e. if $t \in (0,1]$ and if $\psi_1, \psi_2 \in \Psi$, then $t \psi_1 + (1-t) \psi_2 \in \Psi$.
4. Let $K$ be a compact subset of $\mathbb{R}^p$, let $\phi \in C^\infty(K)$, and let $\psi_0 \in \Psi$. There exists $\varepsilon = \varepsilon(\phi, \psi_0) > 0$ so that if $\psi_1 \in C^\infty(S^2(T\mathbb{R}^p))$ and $|\psi_1|_K < \varepsilon$, then $\psi_0 + \phi \psi_1 \in \Psi$.

If $(M,g)$ has constant sectional curvature, then $(M,g)$ is Jordan Osserman of type $(r,s)$ for any admissible $(r,s)$. There are a number of spacelike Jordan Osserman and timelike Jordan Osserman manifolds which are known \cite{4,5,6}. Higher order Osserman tensors have been constructed \cite{7,8,9,10}. Bonome, Castro, and Garcia-Rio have exhibited higher order Osserman manifolds \cite{11} in signature $(2,2)$. However, no examples of higher order Jordan Osserman manifolds were known previously which did not have constant sectional curvature. In Section 3 we will demonstrate examples of pseudo-Riemannian manifolds which are Jordan Osserman of type $(r,s)$ for certain, but not for all values of $(r,s)$:

Theorem 1.6. Let $p \geq 2$.
1. $(M,g_\psi)$ is k Osserman for every admissible k.
2. If $\psi \in \Psi$, then $(M,g_\psi)$ is:
   (a) Jordan Osserman of types $(0,r)$, $(p,p-r)$, $(r,0)$, $(p-r,p)$ if $0 < r \leq p$;
   (b) not Jordan Osserman of type $(r,s)$ if $0 < r < p$ and $0 < s < p$. 

Thus we use equation (2.1) to see that (\ref{2.1}) inner product is non-degenerate, this implies that Consequently, the condition \( R(\bar{p}, \bar{q}) > 0 \) of Theorem 1.5 (1) is simply the condition that \( S_f \) is concave. By using assertions (2) and (3) of Theorem 1.5, we may perturb these metrics to construct examples of higher order Jordan Osserman manifolds which are not hypersurfaces. Therefore, Lemma 1.3 shows that there are manifolds described by Theorems 1.4 and 1.7 which are neither locally homogeneous nor of constant sectional curvature for generic \( \psi \in \Psi \).

2. The Curvature Tensor of \((M, g_\psi)\)

If \( Z_1, Z_2, \) and \( Z_3 \) are coordinate vector fields, then we have:

\[
(\nabla_{Z_i} Z_j, Z_3) = \frac{1}{2}\{Z_1(Z_2, Z_3) + Z_2(Z_1, Z_3) - Z_3(Z_1, Z_2)\}.
\]

We use equation (2.1) to see that \((\nabla_{Z_i} \partial^p_r, Z_3) = 0\) for all \( Z_1 \) and \( Z_3 \). Since the inner product is non-degenerate, this implies that \( \nabla \partial^p_r = 0 \). Consequently,

\[
R(Z_1, Z_2) \partial^p_r = (\nabla_{Z_1} \nabla_{Z_2} - \nabla_{Z_2} \nabla_{Z_1} - \nabla_{[Z_1, Z_2]})(\partial^p_r = 0).
\]

Thus \( R(Z_1, Z_2, Z_3, Z_4) = 0 \) if \( Z_3 \in \mathcal{Y} \). We use the curvature symmetries

\[
R(Z_1, Z_2, Z_3, Z_4) = R(Z_3, Z_4, Z_1, Z_2) = -R(Z_4, Z_3, Z_1, Z_2) = -R(Z_1, Z_2, Z_4, Z_3)
\]
to see that \( R(Z_1, Z_2, Z_3, Z_4) = 0 \) if any of the \( Z_i \in \mathcal{Y} \); this proves Lemma 1.3 (1).

By equation (2.1), \((\nabla_{\partial^p_r} \partial^q_s, \partial^p_r) = \frac{1}{2}(\psi_{ij/k} + \psi_{jk/i} - \psi_{ij/k})\) and \((\nabla_{\partial^p_r} \partial^q_s, \partial^q_s) = 0\). Consequently,

\[
\nabla_{\partial^p_r} \partial^q_s = \frac{1}{2} \sum_k (\psi_{ik/j} + \psi_{jk/i} - \psi_{ij/k}) \partial^q_k.
\]

We complete the proof of Lemma 1.3 by computing:

\[
R(\partial^p_r, \partial^q_s) \partial^q_k = (\nabla_{\partial^p_r} \nabla_{\partial^q_s} - \nabla_{\partial^q_s} \nabla_{\partial^p_r}) \partial^q_k
\]

\[
= \frac{1}{2} \nabla_{\partial^q_s} \{ \psi_{ij/k} + \psi_{kl/j} - \psi_{jk/i} \} \partial^q_k - \frac{1}{2} \nabla_{\partial^p_r} \{ \psi_{il/k} + \psi_{kl/i} - \psi_{ik/l} \} \partial^q_k
\]

\[
= \frac{1}{2} \partial_i \{ \psi_{ij/k} + \psi_{kl/j} - \psi_{jk/i} \} - \partial_j \{ \psi_{il/k} + \psi_{kl/i} - \psi_{ik/l} \} \partial^q_k
\]

\[
= \frac{1}{2} (\psi_{ij/k} + \psi_{kl/j} - \psi_{jk/i}) - \psi_{il/k} - \psi_{kl/i} + \psi_{ik/l} \partial^q_k.
\]
We now establish the assertions of Lemma 1.5. Let $f \in C^\infty(\mathbb{R}^p)$. We then set $H_{ij} = \partial_j f \partial_i f$ and $\psi_{ij} := \partial^2 f \partial_i \partial_j f$. By Lemma 1.3,

\begin{align*}
(R(\partial_i^x, \partial_j^y)\partial_k^z, \partial_l^w) &= -\frac{1}{2}(\psi_{jk/l} + \psi_{il/jk} - \psi_{lk/ji} - \psi_{jl/ik}) \\
&= H_{ik}H_{jk} - H_{hk}H_{ji}, \text{ so so}
\end{align*}

\begin{equation}
(2.2) \quad (J(X_1)X_2, X_3) = H(X_1, X_1)H(X_2, X_3) - H(X_1, X_2)H(X_1, X_3).
\end{equation}

Suppose $H$ is positive definite. If $0 \neq X_1 \in \mathcal{X}$, let

$$Z = \{x_2 \in \mathcal{X} : H(X_1, X_2) = 0\}.$$ 

By equation (2.2), $(J(X_1), \cdot)$ is positive definite on $Z$ because

$$(J(X_1)X_2, X_3) = H(X_1, X_1)H(X_2, X_3) \quad \text{for} \quad Z_2, Z_3 \in Z.$$ 

Since rank $(J(X_1)) \leq p - 1$, we have that $J(X_1)$ is positive semi-definite of rank $p - 1$. This proves Lemma 1.5 (1). Assertion (2) of Lemma 1.5 follows from Lemma 1.3 as

$$J_{a\psi} = aJ_\psi.$$ 

Let $\psi_1, \psi_2 \in \Psi$, let $a_1 > 0$, and let $a_2 > 0$. By Lemma 1.3,

$$R_{a_1 \psi_1 + a_2 \psi_2}(\partial_i^x, \partial_j^y)\partial_k^z = a_1 R_{\psi_1}(\partial_i^x, \partial_j^y)\partial_k^z + a_2 R_{\psi_2}(\partial_i^x, \partial_j^y)\partial_k^z,$$

$$J_{a_1 \psi_1 + a_2 \psi_2}(X_1) = a_1 J_{\psi_1}(X_1) + a_2 J_{\psi_2}(X_1).$$

Thus $J_{a_1 \psi_1 + a_2 \psi_2}(X_1)$ is positive semi-definite of rank at least $p - 1$ for $0 \neq X_1 \in \mathcal{X}$. Since rank $(J_\psi(X_1)) \leq p - 1$ for any $0 \neq X_1 \in \mathcal{X}$, rank $(J_{a_1 \psi_1 + a_2 \psi_2}(X_1)) = p - 1$ so $a_1 \psi_1 + a_2 \psi_2 \in \Psi$. This establishes Lemma 1.3 (3).

We complete the proof of Lemma 1.3 by studying small compactly supported perturbations. Previously, we have suppressed the point $(x, y)$ of $M$ from the notation. It is now necessary to introduce it explicitly. Thus we will denote the Jacobi operator by $J(x, y; Z)$ for $(x, y) \in M$ and $Z \in \mathbb{R}^p$. By Lemma 1.3, $J(x, y; Z) = J(x; X)$ for $X = \rho_X Z$.

We must introduce an auxiliary positive definite quadratic form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^p$. Let $S^{p-1}$ be the associated unit sphere. Let $SK := K \times S^{p-1}$. Then

$$T(SK) = \{(x, X, X_3) \in (\mathbb{R}^p)^3 : x \in K, \langle X_2, X_2 \rangle = 1, \langle X_2, X_3 \rangle = 0\}.$$ 

Let $\psi \in \Psi$. Then $J_\psi(x; X_2)$ defines a bilinear symmetric quadratic form $J_\psi(x; X_2)$ on $T(x, X_2)(SK)$: $J_\psi(x; X_2)$ is positive semi-definite of rank $p - 1$ if and only if $J_\psi(x; X_2)$ is positive definite. Since $SK$ is compact and since we are considering compact perturbations, Lemma 1.3 now follows from Lemma 1.3. \qed 

3. The Higher Order Jacobi Operator of $(M, g_\psi)$

By Lemma 1.3, $J(Z_1)Z_3 = R(Z_3, Z_1)Z_1 = 0$ if $Z_1 \in \mathcal{Y}$; thus $\mathcal{Y} \subset \ker(J(Z_1))$. Furthermore, range $(J(Z_2)) \subset \text{span} \{R(\partial_i^x, \partial_j^y)\partial_k^z \} \subset \mathcal{Y}$. Thus

\begin{equation}
(3.1) \quad J(Z_1)J(Z_2) = 0.
\end{equation}

Let $\{e_i\}$ be an orthonormal basis for $\pi \in \text{Gr}_{r,s}(M, g_\psi)$. We then have

$$J(\pi)^2 = \sum_{i,j} \langle e_i, e_i \rangle \langle e_j, e_j \rangle J(e_i)J(e_j) = 0.$$ 

Therefore, Spec $(J(\pi)) = \{0\}$ so $(M, g_\psi)$ is Osserman of type $(r, s)$ and hence is $k$ Osserman for any admissible $k$. This proves Theorem 1.4 (1).
Recall from equation (1.1) that $\rho_X$ is projection on $X$. We begin the proof of Theorem 1.6 (2) with a technical Lemma:

**Lemma 3.1.** Let $\psi \in \Psi$, let $\pi := \{Z_1, \ldots, Z_r\}$ and let $J := J(Z_1) + \ldots + J(Z_r)$.

1. If $\dim(\rho_X \pi) = 0$, then $\ker(J) = 0$.
2. If $\dim(\rho_X \pi) = 1$, then $\ker(J) = p - 1$.
3. If $\dim(\rho_X \pi) > 1$, then $J$ is positive semi-definite of rank $p$.

**Proof.** Let $X_i = \rho_X(Z_i); J(Z_i) = J(X_i)$. Thus if $X_i = 0$, then $J(Z_i) = 0$. Suppose $X_i \neq 0$. Since $\psi \in \Psi$, we have that $\ker(J(X_i)) = \rho \cdot X_i \oplus \mathcal{Y}$ if $X_i \neq 0$.

If $\dim(\rho_X \pi) = 0$, then $X_i = 0$ for all $i$ and $J = 0$. Suppose $\dim(\rho_X \pi) = 1$. We may suppose without loss of generality that $X_1 \neq 0$ and let $X_i = c_iX_1$ for $i \geq 1$. Then $J = (\sum_i c_i^2)J(X_1)$ has rank $p - 1$.

Suppose $\dim(\rho_X \pi) \geq 2$. We suppose, without loss of generality, that $\{X_1, X_2\}$ is a linearly independent set. By Lemma 1.3, $\ker(J) \subseteq \mathcal{Y}$. Therefore, we have $\ker(J) \leq \dim(\mathcal{Y}) = p$. Conversely, $J(X_i)$ is positive semi-definite as

$$J(Z, Z) = (J(X_1)Z, Z) + \ldots + (J(X_r)Z, Z) \geq 0.$$ 

Furthermore, equality holds if and only if $(J(X_1)\rho_X Z, \rho_X Z) = 0$ for $1 \leq i \leq p$. By equation (2.2), this means that $\rho_X Z$ is a multiple of $X_i$ for $i = 1, 2$. Consequently, $\rho_X Z = 0$ so $Z \in \mathcal{Y}$. Therefore, $\ker(J) = p$.

Let $\psi \in \Psi$. Let $\{Z_1, \ldots, Z_r\}$ be an orthonormal basis for a spacelike subspace $\pi \in \text{Gr}_{0, r}(M, g)$. Let $X_1 := \rho_X Z_1$. Since $\pi$ is spacelike, $\pi \cap \mathcal{Y} = \{0\}$. Since $\ker \rho_X = \mathcal{Y}$, $\dim(\rho_X \pi) = r$. We have

$$\ker(J) = J(X_1) + \ldots + J(X_r).$$

Since $(J(\pi))^2 = 0$, the Jordan normal form of $J(\pi)$ is determined by the rank. We apply Lemma 1.1 to see that the rank is $p - 1$ if $r = 1$ and that the rank is $p$ if $r > 1$. This shows $(M, g_\psi)$ is Jordan Osserman of type $(0, r)$. One shows similarly that $(M, g_\psi)$ is Jordan Osserman of type $(r, 0)$. Lemma 1.3 then shows that $(M, g_\psi)$ is Jordan Osserman of types $(p, p - r)$ and $(p - r, p)$. Thus Theorem 1.6 (2a) is established.

To prove Theorem 1.6 (2b), it is convenient to define the elements:

$$\tilde{X}_i := \partial_{\tilde{e}} - \frac{1}{2} \sum_j \psi_{ij} \partial_{\tilde{y}^j}.$$

Note that $\rho_X \tilde{X}_i = \partial_{\tilde{e}}$ and that $\{\tilde{X}_1, \ldots, \tilde{X}_p, \partial_{\tilde{y}^1}, \ldots, \partial_{\tilde{y}^p}\}$ is a hyperbolic basis, i.e.

$$\langle \tilde{X}_i, \tilde{X}_j \rangle = (\partial_{\tilde{y}^i}, \partial_{\tilde{y}^j}) = 0 \quad \text{and} \quad \langle \tilde{X}_i, \partial_{\tilde{y}^j} \rangle = \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq p.$$

Let $0 < r < p$ and $0 < s < p$. We must show that $(M, g_\psi)$ is not Jordan Osserman of type $(r, s)$. By Theorem 1.1 (2), we may assume $r + s \leq p$. We assume $0 < r \leq s < p$ as the situation when $0 < s \leq r < p$ is similar. We distinguish two cases:

1) Suppose $s = 1$. Then $r = 1$. We use equation (3.3) to define the following subspaces of type $(1, 1)$ with the indicated orthonormal bases and Jacobi operators:

$$\pi_1 := \text{span} \{\tilde{X}_1 - \epsilon \partial_{\tilde{y}^1}, \tilde{X}_1 + \frac{1}{2} \partial_{\tilde{y}^1}\}, \quad J(\pi_1) = -J(\tilde{X}_1) + J(\tilde{X}_1),$$

$$\pi_2 := \text{span} \{\epsilon \tilde{X}_1 - \frac{1}{2} \epsilon^{-1} \partial_{\tilde{y}^1}, \tilde{X}_2 + \frac{1}{2} \partial_{\tilde{y}^1}\}, \quad J(\pi_2) = -\epsilon J(\tilde{X}_1) + J(\tilde{X}_2).$$
As $J(\pi_1) = 0$, rank $(J(\pi_1)) = 0$. Since rank $(J(\tilde{X}_2)) = \text{rank}(J(\partial_x^2)) = p - 1$, rank $(J(\pi_2)) \geq \text{rank}(J(\tilde{X}_2)) \geq p - 1$ if $\varepsilon$ is small. Consequently, $(M, g_\varepsilon)$ is not Jordan Osserman of type $(1, 1)$.

2) Suppose $1 \leq r \leq s < p$, $s \geq 2$, and $r + s \leq p$; necessarily $p \geq 3$. For $\alpha \neq 0$ and $\beta \neq 0$, we use equation (3.3) to define timelike subspaces $\pi^-(\alpha)$ and spacelike subspaces $\pi^+(\beta)$ with the indicated orthonormal bases and Jacobi operators:

$$
\pi^-(\alpha) := \text{span}\{\alpha \tilde{x}_1 - \frac{1}{2} \alpha^{-1} \partial_1^\mu, ..., \alpha \tilde{x}_r - \frac{1}{2} \alpha^{-1} \partial_r^\mu\},
$$

$$
J^\alpha := -\alpha\{J(\partial_1^\mu) + ... + J(\partial_r^\mu)\},
$$

$$
\pi^+(\beta) := \text{span}\{\beta \tilde{x}_{r+1} + \frac{1}{2} \beta^{-1} \partial_{r+1}^\mu, ..., \beta \tilde{x}_{r+s} + \frac{1}{2} \beta^{-1} \partial_{r+s}^\mu\},
$$

$$
J^\beta := \beta\{J(\partial_{r+1}^\mu) + ... + J(\partial_{r+s}^\mu)\}.
$$

Let $\pi(\alpha, \beta) := \pi^-(\alpha) \oplus \pi^+(\beta) \in G_{r,s}(M, g)$. The associated Jacobi operator $J(\pi(\alpha, \beta)) = J^\alpha + J^\beta$. Since range $(J) \subset \mathcal{V}$, one sees that

$$
\text{rank}(J(\pi(\alpha, \beta))) \leq p \quad \text{for all} \quad \alpha, \beta.
$$

We have $\rho_x(\tilde{x}_i) = \partial_x^i$. We apply Lemma 3.1: $J(\cdot)$ is supported on $\mathcal{X}$. Since $s \geq 2$, $J^\beta$ is positive semi-definite of rank $p$. Thus rank $(\{J(1, \beta)\}) = p$ for $\beta$ large. Note that $J^\alpha$ is negative semi-definite of rank at least $p - 1$. Thus for $\alpha$ large, $J(\alpha, 1)$ determines a quadratic form of signature $(u, v)$ for $u \geq p - 1$ and $u + v \leq p$. Thus by continuity, there must exist $(\alpha, \beta)$ with $\alpha \neq 0$ and $\beta \neq 0$ so $J(\pi(\alpha, \beta))$ determines a degenerate quadratic form on $\mathcal{X}$. For such values of $(\alpha, \beta)$, rank $(\{J(\pi(\alpha, \beta))\}) < p$. This shows that rank $(\{J(\pi(\alpha) \oplus \pi(\beta))\})$ is not constant and hence $(M, g_\varepsilon)$ is not Jordan Osserman of type $(r, s)$. The proof of Theorem 1.6 is now complete. $\square$

4. The Higher Order Jacobi Operator of $(N, g_N)$

Let $N := M \times \mathbb{R}^{(u,v)}$ with the product metric. Decompose $TN = \mathcal{X} \oplus \mathcal{Y} \oplus T\mathbb{R}^{(u,v)}$ and let $\rho_X : TN \to \mathcal{X}$ be the associated projection. Then

$$
J(U_1)U_2 = J(\rho_X U_1)\rho_X U_2 \quad \text{for any} \quad U_1, U_2 \in TN.
$$

Let $\{U_\nu\}$ be an orthonormal basis for a non-degenerate subspace $\pi \in G_{r,s}(N)$. Let $\varepsilon_\nu := (U_\nu, U_\nu) = \pm 1$. Then $J(\pi) = \sum_\nu \varepsilon_\nu J(\rho_X(U_\nu))$ so equation (3.3) yields:

$$
J(\pi)^2 = \sum_\mu \sum_\nu \varepsilon_\nu \varepsilon_\mu J(\rho_X(U_\nu))J(\rho_X(U_\mu)) = 0.
$$

This shows that $(N, g_N)$ is Osserman of type $(r, s)$ for every admissible $(r, s)$. Therefore $(N, g_N)$ is $k$ Osserman for every admissible $k$ which completes the proof of Theorem 1.7 (1).

Since $J(\pi)^2 = 0$, $(N, g_N)$ will be Jordan Osserman of type $(r, s)$ if and only if rank $(J(\pi))$ is constant on $G_{r,s}(N, g_N)$. We use this observation to prove Theorem 1.7 (2). Let $\{U_1, ..., U_r\}$ be an orthonormal basis for a timelike subspace $\pi$ in $G_{r,0}(N, g_N)$. We have

$$
\rho_X(\pi) = \text{span}\{\rho_X(U_1), ..., \rho_X(U_r)\}, \quad \text{and}
$$

$$
J(\pi) = -J(\rho_X(U_1)) - ... - J(\rho_X(U_r)).
$$

Lemma 3.1 implies:

$$
\text{rank}(J(\pi)) = \begin{cases} 0 & \text{if} \ dim \rho_X(\pi) = 0, \\
 p - 1 & \text{if} \ dim \rho_X(\pi) = 1, \\
 p & \text{if} \ dim \rho_X(\pi) \geq 2. \end{cases}
$$
Suppose that \( u = 0 \). Then \( \dim(\rho_X(\pi)) = r \) is independent of \( \pi \) and, by equation (4.1), \((N,g_N)\) is Jordan Osserman of type \((r,0)\). Dually, by Theorem \( \ref{t7} \), \((N,g_N)\) is Jordan Osserman of type \((p-r,p+v)\). This proves Theorem \( \ref{t7} \) (2a-i).

Suppose \( r \geq u + 2 \) and \( u > 0 \). Then \( \dim \rho_X(\pi) \geq r - u \geq 2 \) and hence \( \text{rank}(J(\pi)) = p \). Thus \((N,g_N)\) is Jordan Osserman of type \((r,0)\) and dually, it is Jordan Osserman of type \((p-r,v+p)\). This completes the proof of assertion (2a-iii). Assertions (2a-ii) and (2a-iv) are proved similarly.

The proof of assertion (3) decomposes into several cases. Suppose \( u > 0 \) and \( r \leq u + 1 \). We must show that \((N,g_N)\) is not Jordan Osserman of type \((r,0)\); we may then use duality to see that \((N,g_N)\) is not Jordan Osserman of type \((\bar{p} - r, q)\). Let \( \{V_1^{-},...,V_u^{-}\} \subset TN \) and \( \{Z_1^{-},...,Z_p^{-}\} \subset TM \) be orthonormal sets of timelike vectors. Let

\[
\pi(a,b) := \text{span}\{Z_1^{-},...,Z_a^{-},V_1^{-},...,V_b^{-}\}.
\]

Suppose first that \( 0 < r \leq u \). Then

\[
\begin{align*}
\dim \rho_X(\pi(1,r-1)) &= 1 \quad \text{so} \quad \text{rank}(J(\pi(1,r-1))) = p - 1, \\
\dim \rho_X(\pi(0,r)) &= 0 \quad \text{so} \quad \text{rank}(J(\pi(0,r))) = 0.
\end{align*}
\]

Therefore \((N,g_N)\) is not Jordan Osserman of type \((0,r)\). Suppose next \( r = u + 1 \). We then have \( r \geq 2 \) and

\[
\begin{align*}
\dim \rho_X(\pi(2,u-1)) &= 2 \quad \text{so} \quad \text{rank}(J(\pi(2,u-1))) = p, \\
\dim \rho_X(\pi(1,u)) &= 1 \quad \text{so} \quad \text{rank}(J(\pi(1,u))) = p - 1.
\end{align*}
\]

Thus \((N,g_N)\) is not Jordan Osserman of type \((0,r)\). The case \( v > 0 \) and \( r < v + 2 \) is similar and is omitted in the interests of brevity.

Finally, suppose \( 1 \leq r \leq \bar{p} - 1 \) and \( 1 \leq s \leq \bar{q} - 1 \). Let

\[
\{V_1^{-},...,V_u^{-},V_1^{+},...,V_v^{+}\}
\]

be an orthonormal basis for \( \mathbb{R}^{(u,v)} \). We define maps

\[
T_{a,b} \pi := \pi \oplus \text{span}\{V_1^{-},...,V_a^{-},V_1^{+},...,V_b^{+}\}
\]

from \( \text{Gr}_{\alpha,\beta}(M,g_\psi) \) to \( \text{Gr}_{\alpha+a,\beta+b}(N,g_N) \). We then have

\[
J_N(T_{a,b} \pi) = J_M(\pi) \quad \text{for all} \quad \pi \in \text{Gr}_{\alpha,\beta}(M,g_\psi).
\]

Suppose \((N,g_N)\) is Jordan Osserman of type \((r,s)\). Expand

\[
\begin{align*}
r &= \alpha + a, \quad \text{where} \quad 1 \leq \alpha \leq p - 1 \quad \text{and} \quad 0 \leq a \leq u; \\
s &= \beta + b, \quad \text{where} \quad 1 \leq \beta \leq p - 1 \quad \text{and} \quad 0 \leq b \leq v.
\end{align*}
\]

If \((N,g_N)\) is Jordan Osserman of type \((r,s)\), then we may use equation (4.2) to see that \((M,g_\psi)\) is Jordan Osserman of type \((\alpha,\beta)\). This contradicts Theorem \( \ref{t6} \) and thereby completes the proof of Theorem \( \ref{t7} \). \( \square \)

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REFERENCES

[1] N. Blažic, N. Bokan, and P. Gilkey, A note on Osserman Lorentzian manifolds, *Bulletin of the London Math Society*, 29, (1997), 227–230.
[2] N. Blažić, N. Bokan, P. Gilkey and Z. Rakić, Pseudo-Riemannian Osserman manifolds, *J. Balkan Soc. of Geometers*, 12, (1997), 1–12.
[3] A. Bonome, P. Castro, E. Garcia-Rio, Four-Dimensional Generalized Osserman Manifolds, *Classical and Quantum Gravity*, 18 (2001), 4813–4822.
[4] Q.-S. Chi, A curvature characterization of certain locally rank-one symmetric spaces, *J. Differ. Geom.*, 28, (1988), 187–202.
[5] E. García-Río, D. Kupeli, and M. Vázquez-Abal, On a problem of Osserman in Lorentzian geometry, *Diff. Geom. Appl.*, 7, (1997), 85–100.
[6] E. García-Rió, M. E. Vázquez-Abal and R. Vázquez-Lorenzo, Nonsymmetric Osserman pseudo-Riemannian manifolds, *Proc. Amer. Math. Soc.*, 126, (1998), 2771–2778.
[7] E. García-Rió, D. Kupeli, and R. Vázquez-Lorenzo, *Osserman Manifolds in Semi-Riemannian Geometry*, Lecture notes in Mathematics, Springer Verlag, (2002), ISBN 3-540-43144-6.
[8] P. Gilkey, Algebraic curvature tensors which are p Osserman, *Diff. Geometry and Appl.*, 14, (2001), 297–311.
[9] —, *Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor*, World Scientific Press (2001), ISBN 981-02-04752-4.
[10] P. Gilkey and R. Ivanova, The Jordan normal form of higher order Osserman algebraic curvature tensors, *Comment. Math. Univ. Carolinae*, 43, (2002) 231–242.
[11] P. Gilkey, R. Ivanova, and T. Zhang, Szabó Osserman IP Pseuod-Riemannian manifolds, preprint: [http://arXiv.org/abs/math.DG/0205085](http://arXiv.org/abs/math.DG/0205085).
[12] P. Gilkey, G. Stanilov and V. Videv, Pseudo-Riemannian manifolds whose generalized Jacobi operator has constant characteristic polynomial, *J. Geom.*, 62, (1998), 144–153.
[13] P. Gilkey and I. Stavrov, Curvature tensors whose Jacobi or Szabó operator is nilpotent on null vectors, *Bull. London Math. Soc.*, to appear.
[14] Y. Nikolayevsky, Osserman Conjecture in dimension n ≠ 16, preprint: [http://arXiv.org/abs/math.DG/0204258](http://arXiv.org/abs/math.DG/0204258).
[15] R. Osserman, Curvature in the eighties, *Amer. Math. Monthly*, 97, (1990), 731–756.
[16] G. Stanilov and V. Videv, Four dimensional pointwise Osserman manifolds, *Abh. Math. Sem. Univ. Hamburg*, 68, (1998), 1–6.
[17] I. Stavrov, Ph. D. Thesis, University of Oregon (2003).

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