Formation of a disc gap induced by a planet: Effect of the deviation from Keplerian disc rotation

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ABSTRACT

The gap formation induced by a giant planet is important in the evolution of the planet and the protoplanetary disc. We examine the gap formation by a planet with a new formulation of one-dimensional viscous discs which takes into account the deviation from Keplerian disc rotation due to the steep gradient of the surface density. This formulation enables us to naturally include the Rayleigh stable condition for the disc rotation. It is found that the derivation from Keplerian disc rotation promotes the radial angular momentum transfer and makes the gap shallower than in the Keplerian case. For deep gaps, this shallowing effect becomes significant due to the Rayleigh condition. In our model, we also take into account the propagation of the density waves excited by the planet, which widens the range of the angular momentum deposition to the disc. The effect of the wave propagation makes the gap wider and shallower than the case with instantaneous wave damping. With these shallowing effects, our one-dimensional gap model is consistent with the recent hydrodynamic simulations.

Key words: accretion, accretion discs, protoplanetary discs, planets and satellites: formation

1 INTRODUCTION

A planet in a protoplanetary disc gravitationally interacts with the disc and exerts a torque on it. The torque exerted by the planet disperses the surrounding gas and forms a disc gap along the orbit of the planet (Lin & Papaloizou 1979; Goldreich & Tremaine 1980). However, a gas flow into the gap is also caused by viscous diffusion and hence the gap depth is determined by the balance between the planetary torque and the viscous diffusion. Accordingly, only a large planet can create a deep gap (Lin & Papaloizou 1993; Takeuchi et al. 1996; Ward 1997; Rafikov 2002; Crida et al. 2006).

The gap formation strongly influences the evolution of both the planet and the protoplanetary disc in various ways. For example, a deep gap prevents disc gas from accreting onto the planet and slows down the planet growth (D’Angelo et al. 2002; Bate et al. 2003; Tanigawa & Ikoma 2003), and also changes the planetary migration from the type I to the slower type II (Lin & Papaloizou 1986; Ward 1997). Furthermore, a sufficiently deep gap inhibits gas flow across the gap (Artymowicz & Lubow 1996; Lubow et al. 1999; Kley 1999), which is a possible mechanism for forming an inner hole in the disc (Zhu et al. 2013; Dodson-Robinson & Saly 2013).

Because of their importance, disc gaps induced by planets have been studied by many authors, using simple one-dimensional disc models (e.g., Takeuchi et al. 1996; Ward 1997; Crida et al. 2006; Lubow & D’Angelo 2006) and numerical hydrodynamic simulations (Artymowicz & Lubow 1994; Kley 1999; Varni`ere et al. 2004; Duffell & MacFadyen 2013; Fung et al. 2014). One-dimensional disc models predict an exponential dependence of the gap depth. That is, the minimum surface density at the gap bottom is proportional to \(\exp\left(-A(M_p/M_\star)^2\right)\), where \(M_p\) and \(M_\star\) are the masses of the planet and the central star, and \(A\) is a non-dimensional parameter (see also eq. 65). On the other hand, recent high-resolution hydrodynamic simulations done by Duffell & MacFadyen (2013, hereafter DM13) show that the gap is much shallower for a massive planet than the prediction of one-dimensional models. According to their results, the minimum surface density at the gap is proportional to \((M_p/M_\star)^{-2}\). Varni`ere et al. (2004) and Fung et al. (2014) obtained similar results from their hydrodynamic simulations. Its origin has not yet been clarified by the one-dimensional disc model. Fung et al. (2014) also estimated
the gap depth with a “zero-dimensional” analytic model, by simply assuming that the planetary gravitational torque is produced only at the gap bottom. Their simple model succeeds in explaining the dependence of the minimum surface density produced only at the gap bottom. Fung et al. (2014) observed that the gap becomes shallow due to the effects of the deviation from the Kepler rotation and the wave propagation, respectively. We find that the zero-dimensional model proposed by Fung et al. (2014) is the assumption of the Keplerian rotational speed. The disc–planet interaction would be weak because the disc gas around the planet decreases over a wide region. Hence, we cannot neglect the effect of wave propagation on the gap formation.

Another simplification is in the wave propagation at the disc–planet interaction. The density waves excited by planets radially propagate in the disc and the angular momenta of the waves are deposited on the disc by damping. This angular momentum deposition is the direct cause of the gap formation. Most previous studies simply assume instantaneous damping of the density waves after their excitation. For a disc with no gap, the order of magnitude of the non-dimensional parameter $\eta$ is $O(h^2/R^2)$, because the term in parentheses in equation (3) is comparable to $\sim 1/R$. On the other hand, if a planet opens a deep gap with a width of $\sim h$, the steep gradient of the surface density increases $\eta$ to $O(h/R)$. Hence, it also enhances the deviation of the disc rotation from the Kepler rotation in the deep gap. We neglect the term $\partial c/\partial R$ in equation (3) because the temperature gradient would be small. Neglecting the smaller terms of $O(h/R)$, we approximately obtain $\partial \Omega/\partial R$ as

$$\frac{\partial \Omega}{\partial R} = -\frac{3\Omega_K}{2R^2} \left[ 1 - \frac{h^2}{3} \frac{\partial^2 \ln \Sigma}{\partial R^2} \right].$$

Note that the second term in the parentheses is of order unity since $d^2 \ln \Sigma/dx^2 \sim 1/h^2$ in a deep gap. Therefore, it is found that $\partial \Omega/\partial R$ is significantly altered from the Keplerian value due to the steep gradient of the surface density, though the deviation of $\Omega$ is small ($\sim h/R$). As shown later, this deviation promotes radial viscous transfer of angular momentum and makes a gap shallower.
2.2 Basic equations describing a disc gap around a planet

The equations for conservation of mass and angular momentum are given by

\[ \frac{\partial \Sigma}{\partial t} + \frac{1}{2\pi R} \frac{\partial F_M}{\partial R} = S_M, \]  

and

\[ \frac{\partial}{\partial t} (\Sigma j) + \frac{1}{2\pi R} \frac{\partial F_j}{\partial R} = j S_M + \frac{1}{2\pi R} \Lambda_d, \]  

where \( F_M \) and \( F_j \) are the radial fluxes of mass and angular momentum, and \( j (= R^2 \Omega) \) is the specific angular momentum. In equation (5), the source term \( S_M \) represents the mass accretion rate onto a unit surface area of the disc. The accretion of disc gas onto the planet can be included in \( S_M \) as a negative term. In equation (6), \( \Lambda_d(R) \) represents the deposition rate of the angular momentum from the planet on the ring region with radius \( R \).

To describe the deposition rate \( \Lambda_d \), we consider the angular momentum transfer from the planet to the disc. This transfer process can be divided into two steps. First the planet excites a density wave by the gravitational interaction with the disc (e.g., Goldreich & Tremaine 1980). Second, the density waves are gradually damped due to the disc viscosity or a nonlinear effect (Takeuchi et al. 1996; Goodman & Rafikov 2001). As a result of the wave damping, the angular momenta of the waves are deposited on the disc. If instantaneous wave damping is assumed, the deposition rate \( \Lambda_d \) is determined only by the wave excitation. In Section 2.4.1, we will describe the deposition rate for the case with instantaneous wave damping. In Section 2.4.2, we will give a simple model of \( \Lambda_d \) for the case of gradual wave damping.

The radial angular momentum flux \( F_j \) is given by (e.g., Lynden-Bell & Pringle 1974)

\[ F_j = j F_M - 2\pi R^3 \nu \Sigma \frac{\partial \Omega}{\partial R}. \]  

The first term is the advection transport by the disc radial mass flow, \( F_M \), and the second term represents the viscous transport. For the kinetic viscosity, we adopt the \( a \) prescription, i.e., \( \nu = a \epsilon h \) (Shakura & Sunyaev 1973). Note that \( F_j \) does not include the angular momentum transport by the density waves in our formulation.

Equations (5)–(7) describe the time evolution of the three variables \( \Sigma, F_M, \) and \( F_j \) with the given mass source term \( S_M \), the angular momentum deposition rate from a planet \( \Lambda_d \) and the disc angular velocity \( \Omega \). Note that \( \Omega \) depends on \( \partial \Sigma / \partial R \), as in equations (2) and (4).

Next we consider the disc gap in a steady state (\( \partial / \partial t = 0 \)). The time scale for the formation of a steady gap is approximately equal to the diffusion time within the gap width, \( t_{\text{diff}} = h^2 / \nu \). For a nominal value of \( \alpha \) (\( \sim 10^{-3} \)), the diffusion time is roughly given by \( 10^3 \) Keplerian periods, which is shorter than the growth time of planets (\( 10^-7 \) yr) (Kokubo & Ida 2000, 2002) or the life time of protoplanetary discs (\( 10^6 \) yr) (Haisch et al. 2001). Hence the assumption of a steady gap would be valid. In this case, we assume \( \Sigma_0 = S_M = 0 \). Although gas accretion onto the planet occurs for \( M_p \gtrsim 10 M_\oplus \) (Mizuno 1984; Kanagawa & Fujimoto 2013), the assumption of \( S_M = 0 \) would be valid if the accretion rate onto the planet is smaller than the radial disc accretion rate, \( F_M \).

Under these assumptions, equation (5) shows that \( F_M \) is constant. Equation (6) yields

\[ F_j = F_j(\infty) - \int_R^\infty \Lambda_d dR', \]  

where \( F_j(\infty) \) is the angular momentum flux without the planet. From equations (7) and (8), we obtain

\[ j F_M - 2\pi R^3 \nu \Sigma \frac{\partial \Omega}{\partial R} = F_j(\infty) - \int_R^\infty \Lambda_d dR'. \]  

Equation (9) with a constant mass accretion rate describes a steady disc gap around a planet for a given \( \Lambda_d \). Since \( d\Omega / dR \) is given by equation (1), equation (9) is the second-order differential-integral equation. Note that equation (9) is derived from equation (8) and indicates the angular momentum conservation.

By differentiating equation (9), we obtain a rather familiar expression for the mass flux:

\[ F_M = \left( \frac{dj}{dR} \right)^{-1} \left[ \frac{d}{dR} \left( 2\pi R^3 \nu \Sigma \frac{\partial \Omega}{\partial R} \right) + \Lambda_d \right]. \]  

Note that this expression is valid only for the steady state. In a time-dependent case, equation (9) should include the term \(-2\pi R \Sigma (\partial j / \partial t) \) in the parentheses. As a boundary condition, the disc surface density should approach its unperturbed values at both sides of the gap far from the planet.

Here, we also consider the unperturbed surface density. In the unperturbed state, \( \Omega \) can be replaced by \( \Omega_K \), by neglecting the smaller term \( O(h^2 / R^2) \) (see equations (2) and (3)). Furthermore, setting \( \Lambda_d = 0 \) in equation (9), we obtain the unperturbed surface density \( \Sigma_0 \) as

\[ 3\pi R^2 \nu \Omega_K \Sigma_0 (R) = -R^2 \Omega_K F_M + F_j(\infty). \]  

Thus \( \Sigma_0 \) is given by

\[ \Sigma_0 = -\frac{F_M}{3\pi \nu} \left( 1 - \frac{F_j(\infty)}{R^2 \Omega_K F_M} \right). \]  

This agrees with the well-known solution for steady viscous accretion discs (e.g., Lynden-Bell & Pringle 1974).

2.3 Rayleigh condition

For a deep gap around a large planet, the derivative of the angular velocity deviates significantly from the Keplerian velocity, as shown in Section 2.2.1. A sufficiently large deviation in \( \Omega \) violates the so-called Rayleigh stable condition of \( dj / dR \geq 0 \) (see Chandrasekhar 1961). Such a steep gap is dynamically unstable, which would cause a strong angular momentum transfer, lessening the steepness of the gap. This would make the unstable region marginally stable (i.e., \( dj / dR = 0 \)).

Using equation (10), we give \( dj / dR \) as

\[ \frac{dj}{dR} = \frac{1}{2} R_p \Omega_K (1 + h_R^2 \frac{d^2 \ln \Sigma}{dR^2}), \]  

where the suffix \( p \) indicates the value at \( R = R_p \); this suffix is also used for other quantities. Hence, using the second-derivative of the surface density, the marginally stable condition \( dj / dR = 0 \) can be rewritten as (Tanigawa & Ikoma)
the marginally stable condition. Furthermore, by using the angular momentum deposition rate \( \Lambda \), the angular momentum flux in the unstable region.

In the disc–planet interaction, a planet excites density waves and the angular momenta of the waves are deposited on the disc through their damping. The angular momentum deposition occurs at a different site from the wave damping. Although the wave damping has an effective viscosity because of the shift of the Lindblad resonances. For simplicity, however, we ignore this effect in the present paper. Hence, in our model, the excitation torque density \( \Lambda_{ex} \) is simply proportional to the disc surface density at \( R, \Sigma(R) \), and is independent of the surface density gradient even for deep gaps. For a large planet with a mass of \( M_p/M_* \gtrsim (h_p/R_p)^3 \); furthermore, the non-linear effect would not be negligible for wave excitation (Warren et al. 2002; Muto & Inutsuka 2009). Then we obtain a simple model of angular momentum deposition, described below.

2.4 Angular momentum deposition from a planet

In the disc–planet interaction, a planet excites density waves and the angular momenta of the waves are deposited on the disc through their damping. The angular momentum deposition rate \( \Lambda_d \) is determined by the later process. Firstly, we will consider the deposition rate \( \Lambda_d \) in the case with instantaneous wave damping. In this case, the deposition rate is governed only by the wave excitation. Next, taking into account the wave propagation before damping, we will model the deposition rate in a simple form.

2.4.1 Case with instantaneous wave damping

Under the assumption of instantaneous wave damping, the angular momentum deposition rate \( \Lambda_d(R) \) is equal to the excitation torque density \( \Lambda_{ex}(R) \), which is the rate at which a planet adds angular momenta to density waves per unit radial distance at \( R \). That is,

\[
\Lambda_d = \Lambda_{ex}. \tag{16}
\]

At a position far from the planet, the excitation torque density is given by the WKB formula (e.g., Ward 1986) as

\[
\Lambda_{ex}^{WKB} = \pm C \frac{\pi}{R_p^2} \Sigma L M_p \frac{\partial^2}{\partial z^2} \left( R_p \Omega_{Kp} \right)^2 \left( \frac{R_p}{R - R_p} \right)^4, \tag{17}
\]

where \( C = (2^3/3^2)[2K_0(2/3) + K_1(2/3)]^2/\pi \approx 0.798 \) and \( K_i \) denote the modified Bessel functions. The sign of equation (17) is positive for \( R > R_p \) or negative for \( R \leq R_p \). In the close vicinity of the planet, \( |R - R_p| \lesssim h_p \), on the other hand, the WKB formula is overestimated. Thus, we model the excitation torque density \( \Lambda_{ex} \) with a simple cutoff as

\[
\Lambda_{ex} = \begin{cases} 
\Lambda_{ex}^{WKB} \text{ for } |R - R_p| > h_p \Delta, \\
0 \text{ for } |R - R_p| \leq h_p \Delta. 
\end{cases} \tag{18}
\]

The cut-off length \( h_p \Delta \) is determined so that the one-sided torque \( T = \int_{R_p}^\infty \Lambda_{ex} dR \) agrees with the result of the linear theory for realistic discs (Takeuchi & Miyama 1998; Tanaka et al. 2002; Muto & Inutsuka 2009). Then we obtain \( \Delta = 1.3 \).

Note that the WKB formula is derived for discs with no gap. Petrovich & Rafikov (2014) reported that the torque density is altered by the steep gradient of the surface density because of the shift of the Lindblad resonances. For simplicity, however, we ignore this effect in the present paper. Hence, in our model, the excitation torque density \( \Lambda_{ex} \) is simply proportional to the disc surface density at \( R, \Sigma(R) \), and is independent of the surface density gradient even for deep gaps. For a large planet with a mass of \( M_p/M_* \gtrsim (h_p/R_p)^3 \); furthermore, the non-linear effect would not be negligible for wave excitation (Warren et al. 2002; Miyoshi et al. 1999). This non-linear effect is also neglected in our simple model.

2.4.2 Case with wave propagation

When wave propagation is included, the angular momentum deposition occurs at a different site from the wave excitation and equation (15) is not valid. In this case, the angular momentum deposition is also governed by the damping of the waves. Although the wave damping has been examined in previous studies (e.g., Takeuchi et al. 1998; Korycansky & Papaloizou 1996; Goodman & Rafikov 2001), it is not clear yet how the density waves are damped in a disc with deep gaps. In the present study, therefore, we adopt a simple model of angular momentum deposition, described below.

Since the waves are eventually damped in the disc, the one-sided torque (i.e., the total angular momentum of the waves excited at the outer disc in unit time) is equal to the total deposition rate in the steady state. That is,

\[
T = \int_{R_p}^\infty \Lambda_{ex} dR' = \int_{R_p}^\infty \Lambda_d dR'. \tag{19}
\]

Using the one-sided torque, the angular momentum deposition rate can be expressed by

\[
\Lambda_d = \pm T f(R), \tag{20}
\]

where the distribution function \( f(R) \) satisfies \( \int_{R_p}^\infty f(R) dR = 1 \), and the sign is the same as in equation (17). As a simple model, we assume a distribution function \( f(R) \) given by

\[
f(R) = \begin{cases} 
\frac{1}{w_{d}} \text{ for } x_d h_p - \frac{R_p}{w_d} < |R - R_p| < x_d h_p + \frac{R_p}{w_d}, \\
0 \text{ otherwise},
\end{cases} \tag{21}
\]

where \( w_{d} \) is the dissipation of the waves, \( x_d \) is the damping length of the waves, and \( \nu_d \) is the damping rate of the waves.
In this simple model, the non-dimensional parameter \( x_d \) determines the position of the angular momentum deposition and the parameter \( w_d \) represents the radial width of the deposition site. The waves propagate from the excitation site to the deposition site around \( |x| = x_d \). Since the density waves propagate away from the planet, the deposition site is farther from the planet than the excitation site. The parameter \( x_d \) should be consistent with this condition.

In the case with wave propagation, we use equations (20) and (21) to obtain the gap structure with equation (19). It should be noted that \( T \) in equation (20) depends on the surface density distribution through the definition of equation (19), because \( \lambda_x \) is proportional to \( \Sigma \). These coupled equations are solved as follows. First, we obtain the surface density distribution with equation (9) for a given \( T \). Next, we determine the corresponding mass of the planet from equation (10), using the obtained surface density.

2.5 Local approximation and non-dimensional equations

The typical width of a disc gap is comparable to the disc scale height and much smaller than the orbital radius of the planet. Thus it is convenient to use the local coordinate defined by

\[
x = \frac{R - R_p}{h_p}.
\]

Note that the suffix \( p \) indicates the value at \( R = R_p \).

We adopt a local approximation in which terms proportional to \( h_p/R_p \) and higher order terms are neglected. From equations (2) and (3), the deviation in \( \Omega \) from \( \Omega_K \) is given by

\[
\Omega - \Omega_K = \frac{h_p \Omega_K d \ln \Sigma}{2 R_p},
\]

and is proportional to \( h_p/R_p \). Thus, the disc angular velocity \( \Omega \) is replaced by the angular velocity of the planet \( \Omega_K \) under the local approximation, and the specific angular momentum \( j \) also is given by \( R_p^2 \Omega_K \). As for the derivative \( d\Omega/dR \), we cannot neglect the deviation from the Keplerian value. Equation (4) yields

\[
\frac{d\Omega}{dR} = -\frac{3 \Omega_K}{2 R_p} \left( 1 - \frac{1}{3} \frac{d^2 \ln \Sigma}{dx^2} \right).
\]

Equation (2) can be rewritten in the local approximation as

\[
R^2 \Omega_K F_M + 3 \pi R_p^2 \nu_p \Sigma \Omega_K \left( 1 - \frac{1}{3} \frac{d^2 \ln \Sigma}{dx^2} \right) = F_1(\infty) - \int_{x'}^{\infty} \lambda_d h_p dx',
\]

Because of the local approximation, equation (21) cannot be applied for the wide gap formation. If the half width of gap is narrower than about 1/3\( R_p \), equation (21) would be valid.

Here we introduce the non-dimensional surface density, \( s \), defined by

\[
s = \frac{\Sigma}{\Sigma_0(R_p)},
\]

where \( \Sigma_0(R_p) \) is the unperturbed surface density at \( R = R_p \).
in previous studies. Neglecting the deviation in \( d\Omega/dR \) from the Keplerian (i.e., the term of \( d^2 \ln s/dx^2 \)) in equation (27), we have

\[
s = 1 - \frac{1}{3} \int_{s}^{\infty} \lambda_d dx' \quad (35)
\]

Here we also assume instantaneous wave damping and adopt \( \lambda_d = \lambda_{\infty} \) (eq. (32)). Differentiating equation (35), we obtain

\[
\frac{d\ln s}{dx} = \begin{cases} 
\pm \frac{C}{3\Delta^2} K & \text{for } |x| > \Delta, \\
0 & \text{for } |x| \leq \Delta.
\end{cases} \quad (36)
\]

Hence, we obtain the surface density in the Keplerian discs with the instantaneous wave damping as

\[
s(x) = \begin{cases} 
\exp\left(-\frac{C}{9|x|^2} K\right) & \text{for } |x| > \Delta, \\
\exp\left(-\frac{C}{9\Delta^2} K\right) & \text{for } |x| \leq \Delta.
\end{cases} \quad (37)
\]

Using equation (31), \( C = 0.798 \) and \( \Delta = 1.3 \), the minimum surface density, \( s_{\min} \), is

\[
s_{\min} = \exp\left[-0.040\alpha^{-1}\left(\frac{R_p}{h_p}\right)^{0.5}\left(\frac{M_\ast}{M_\odot}\right)^2\right]. \quad (38)
\]

This solution is almost the same as that in the previous one-dimensional gap model (e.g., Lubow & D’Angelo 2006).

For a very large \( K \), the Rayleigh condition is violated and equations (40) and (41) are invalid. Tanigawa & Ikoma (2007) obtained the gap structure in Keplerian discs, including the Rayleigh condition. Their solution is described in Appendix A. In Appendix A, we also derive gap solutions in Keplerian discs, taking into account the wave propagation with the simple model of equation (20) and (21).

### 3.2 Case of the wide-limit gap

Next, we consider a situation implied by the zero-dimension analysis done by Fung et al. (2014), which assumes that the wave excitation occurs only at a gap bottom. This assumption would be valid if the gap bottom region is wide enough. Hence, we call this situation ‘wide-limit gap’ case. Since the density waves are excited at the gap bottom with \( s \approx s_{\min} \), the one-sided torque of equation (33) is simply given by

\[
T = \frac{C}{3\Delta^3} K s_{\min} \approx 0.121 K s_{\min}. \quad (39)
\]

Using equation (27), we can estimate \( s_{\min} \) of the wide-limit gap. The right-hand side of equation (27) can be rewritten as \( 1 - \bar{T}/3 \) at \( x = 0 \). In the left-hand side of equation (27), moreover, we can neglect the term \( d^2 \ln s/dx^2 \) when a flat-bottom gap is assumed. Then, the relation between \( s_{\min} \) and \( \bar{T} \) is obtained as

\[
s_{\min} = 1 - \frac{\bar{T}}{3}. \quad (40)
\]

Equations (39) and (40) yield

\[
s_{\min} = \frac{1}{1 + 0.040K}. \quad (41)
\]

For a large \( K \), \( s_{\min} \) given by equation (41) is proportional to \( 1/K \). This result agrees with the zero-dimensional model by Fung et al. (2014). In the zero-dimensional model, the minimum surface density is estimated from a balance between the planetary torque and the viscous angular momentum flux outside the gap. Such a balance is also seen from equation (40) (and eq. (27)). The first and second terms in the right-hand side of equation (40) correspond to the viscous angular momentum flux outside the gap and the planetary torque and the left-hand side is negligibly small for a large \( K \).

With their hydrodynamic simulations for \( K \lesssim 10^4 \), DM13 derived a similar result:

\[
s_{\min} = \frac{29}{29 + K} = \frac{1}{1 + 0.034K}. \quad (42)
\]

It is found that equations (11) and (42) are consistent with each other. Note that these minimum surface densities are much larger than that of the Keplerian disc (eq. (35)) for a large \( K \) because equation (39) is not accepted in the Keplerian solution. The wide-limit gaps assume that all the waves are excited in the bottom region with \( s \approx s_{\min} \), i.e., equation (59). In Sections 4 and 5, we will check whether or not this assumption is valid, by comparing it with our one-dimensional solutions.

### 4 GAP STRUCTURE IN THE CASE WITH INSTANTANEOUS WAVE DAMPING

#### 4.1 Linear solutions for shallow gaps

Here we present the numerical solution of the gap in the case with instantaneous wave damping (i.e., \( \lambda_d = \lambda_{\infty} \)).

First, we consider the case with a small \( K \) in equation (27), in which \( \lambda_d \) is proportional to \( K \). This case corresponds to a shallow gap around a small planet. Since \( |s - 1| \) is small, it is useful to express the solution as

\[
s = \exp(Ky), \quad (43)
\]

or \( s = 1 + Ky \). As seen in the next subsection, the former expression is better for an intermediate \( K \) (\( \sim 10 \)). Substituting equation (43) into equation (27) with equation (16), we can expand it into a power series of \( Ky \). The first order terms give the linear equation of \( y \):

\[
\frac{d^2y}{dx^2} - 3y = \begin{cases} 
\frac{C}{3|x|^3} & \text{for } |x| > \Delta, \\
\frac{C}{3\Delta^3} & \text{otherwise},
\end{cases} \quad (44)
\]

where the sign in the right-hand side is negative for \( x > 0 \) and positive for \( x \leq 0 \). Equation (44) is an inhomogeneous linear differential equation, and can be integrated with the boundary conditions of equation (41). We do not need to take care of the Rayleigh condition in the shallow gaps. A detailed derivation of the linear solution is described in Appendix C.

Fig. 1a shows \( y \), which can be converted into the surface density \( s \) by equation (43). In these shallow gaps, the gap depth is almost the same as for the Keplerian case, though our model gives a smooth surface density distribution.

1 In the notation of Fung et al. (2014), \( K \) is given by \( q^2/(\alpha h/r)^2 \).

2 In the notation of DM13, \( K \) is given by \( M^{-1}(M_{sh}/M_p)^2\alpha^{-1} \).
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Fig. 1. Linear solution for $y$ (a) and $dy/dx$ (b). The surface density and angular velocity are given by $y$ and $dy/dx$ with equations (43) and (45), respectively. The dashed line is the solution for the Keplerian disc.

$y = \frac{R - R_p}{h_p}$

$\frac{dy}{dx} = \frac{\Omega h_p dy}{2R_p}$

Fig. 2. Surface density distributions for $K = 50$ (a) and 200 (b). The red line is the exact solution (see text). The chain line is the linear solution given by equation (43) and the dashed line is the solution for the Keplerian case (eq. [37]). The dotted line represents the minimum surface densities for the wide-limit gap given by equation (41).

$\frac{d\Omega}{dx} = \frac{d\Omega_K}{dx} \left( 1 - \frac{K d^2 y}{3 \ dx^2} \right)$

$\Delta\Omega = \Omega - \Omega_K = k \frac{h_p \Omega K_p \ dy}{2R_p}$

At regions far from the planet, the surface density perturbation is rather small and the linear approximation is valid. Thus, we adopt a linear solution at $|x| > 10$. Note that this linear solution has different coefficients for the homogeneous terms from those in Section 4.1 (see Appendix C). The coefficients of the homogeneous solution are given to satisfy the boundary conditions of equation (34). At $|x| \leq 10$, we integrate equation (27) with the fourth-order Runge-Kutta integrator. In the Rayleigh unstable region, the surface density is governed by the marginally stable condition (eq. [29]), instead of equation (27).

Fig. 3 illustrates the angular velocities (a) and specific angular momenta (b) for the exact solutions for $K = 50$ and 200. Similar to the linear solution in Fig. 1, the shear motion is enhanced at $|x| \geq 1.4$. This enhancement of the shear mo-
Figure 3. (a) Deviation from Keplerian disc rotation and (b) specific angular momentum, for $K = 50$ (dashed) and 200 (solid). The filled circles indicate the edge of the marginally stable region for the Rayleigh condition.

Figure 4. Shear of exact solutions for $K = 50$ (dashed) and 200 (solid).

Figure 5. Excitation torque density given by equation (30) for $K = 200$. The two vertical lines indicate the positions with $s = 10s_{\text{min}}$.

In the wide-limit gap, it is assumed that the density waves are excited only at the gap bottom with $s \approx s_{\text{min}}$. Fig. 5 shows the excitation torque density given by equation (18) for the exact solution with $K = 200$. This torque density indicates that the waves are excited mainly in the region with $s > 10s_{\text{min}}$. Thus the assumption of wave excitation at the gap bottom is not valid in this case. Since wave excitation with a larger $s$ increases the one-sided torque, this can explain why the gap of the exact solution is much deeper than the wide-limit gap in Fig. 2. Note that this result for the wave excitation is obtained in the case of instantaneous wave damping. The effect of the wave propagation can change the gap width and the mode of wave excitation, as seen in the next section.

4.3 Effect of the Rayleigh condition

We further examine the effect of the Rayleigh condition on the gap structure. Fig. 6 shows the surface densities (a) and specific angular momenta (b) for the exact solution and the solution without the Rayleigh condition. The solution without the Rayleigh condition has unstable regions with $dj/dx < 0$ (i.e., $1.4 < |x| < 3.1$). This comparison between these two solutions directly shows how the Rayleigh condition changes the gap structure. The Rayleigh condition increases $s_{\text{min}}$ by a factor 6 for $K = 200$. This is because the marginal condition of $d^2 \ln s/dx^2 \geq -1$ keeps the surface density gradient less steep and makes the gap shallow.

It can be considered that the marginally stable state is maintained by $\nu_{\text{eff}}$ of equation (15). The non-dimensional form of equation (15) is given by

$$\nu_{\text{eff}} = 3 - \int_{s_{\text{min}}}^{\infty} \lambda dx'. \quad (47)$$

Fig. 7 shows $\nu_{\text{eff}}$ in the unstable region for $K = 200$. The effective viscosity is twice as large as the original value at $x = 1.8$. This enhancement of the effective viscosity causes the shallowing effect in Fig. 6.

In Fig. 6, we also plot the surface density distribution given by Tanigawa & Ikoma (2007) (hereafter TI07), in
Formation of a disc gap induced by a planet

0.001
0.01
0.1
1

-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0

-4
-3
-2
-1
0
1
2
3
4

Figure 6. (a) Surface density distribution and (b) specific angular momentum distribution, for $K = 200$. The red line indicates the exact solution. The green line is the solution without the Rayleigh condition (see text). The chain line in (a) denotes the surface density distribution given by the model of Tanigawa & Ikoma (2007) (eq. [A3], TI07).

Figure 7. Effective viscosity $\nu_{\text{eff}}$ of the exact solution with $K = 200$.

which the Rayleigh condition is taken into account (for details, see Appendix A). Their model gives a shallower gap than our exact solution. This is because a very steep surface density gradient in the Keplerian solution is suppressed by the Rayleigh condition to a greater extent than in our model.

Figure 8. Minimum surface densities, $s_{\text{min}}$, for the exact solution (red line) and the solution without the Rayleigh condition (green line). The dashed line is $s_{\text{min}}$ in the Keplerian case. The chain, dotted and solid lines denote $s_{\text{min}}$ given by the model of TI07, the wide-limit gap (eq. [41]) and the empirical relation of DM13 (eq. [42]) respectively.

We also show that the Keplerian solution by TI07 does not satisfy the angular momentum conservation. The Keplerian solution without the Rayleigh condition (eq. [37]) is derived just from equation [35] (or eq. [27]), which is originated from equation [9]. In this solution, thus, the angular momentum conservation is satisfied. However, when the Rayleigh condition is violated, the marginal stable condition (eq. [29]) is used instead of equation [37]. Because of this, the surface density at the flat bottom of TI07’s solution does not satisfy equation [35] or the angular momentum conservation, either. This violation is resolved in our formulation because our exact solution always satisfies equation [27] outside the Rayleigh unstable region.

4.4 Gap depth

Fig. 8 shows the minimum surface densities, $s_{\text{min}}$, as a function of $K$ for the exact solutions. For comparison, we also plot $s_{\text{min}}$ for the solutions without the Rayleigh condition and the Keplerian solutions. These solutions give deeper gaps than the exact solution, similar to the result of Section 4.2. It is found that the shallowing effect due to the Rayleigh condition becomes significant with an increase in $K$. This is because the Rayleigh condition is violated more strongly for large $K$.

In Fig. 8 on the other hand, the exact solution is much deeper than DM13’s results and the wide-limit gap, though the latter two cases agree well with each other. The model of TI07 also gives much deeper gaps than DM13. These comparisons indicate that in the case with instantaneous wave damping, our exact solution cannot reproduce the hydrodynamic simulations of DM13. This difference in the gap depth

3 By introducing the effective viscosity of equation [47] and multiplying the LHS of equation [27] by $\nu_{\text{eff}}/\nu_{0}$, equation [27] is recovered in the Rayleigh unstable region.
from DM13 is likely to be due to the fact that the assumption of the wide-limit gap is not satisfied in the case with instantaneous wave damping (see Fig. 5). In the next section, we will see that the effect of wave propagation widens the gap and makes the assumption of the wide-limit gap valid.

5 EFFECT OF DENSITY WAVE PROPAGATION

In this section, we consider the effect of wave propagation. Wave propagation changes the radial distribution of the angular momentum deposition. A simple model of angular momentum deposition rate altered by wave propagation is described in Section 2.4.2. Using this simple model, we solve equation (27) with the Rayleigh condition in the similar way to the previous section. At the region far from the planet (i.e., |x| > 10), we use the linear solution to equation (C1) with g(x) = 0 in this case.

5.1 Gap structure for \( K = 200 \)

Fig. 9 illustrates the surface densities (a) and the excitation torque densities (b) of the exact solutions in the case with wave propagation. The angular momenta of the excited waves are deposited around |x| = \( x_d \) in our model. A large \( x_d \) indicates a long propagation length between the excitation and the damping. The parameter \( K \) is set to 200. For an increasing \( x_d \), the gap becomes wider and shallower. The gap width is directly governed by the position of the angular momentum deposition. For \( x_d = 3 \) and 4, the gap depths are consistent with the wide-limit gap (and also DM13). For \( x_d = 4 \), the density waves are excited mainly at the bottom region with \( s \simeq s_{\text{min}} \) as seen in Fig. 9a). Moreover, for \( x_d = 3 \), a major part of the wave excitation occurs at the bottom. That is, the assumption of the wide-limit gap is almost satisfied for the solutions with \( x_d = 3 \) and 4. This explains why the gap depths are consistent with the wide-limit gap for these large \( x_d \).

It is also valuable to compare the gap width with hydrodynamic simulations. DM13 performed a simulation for the case of \( M_p = 1/4 M_f \) (2\( M_m \) in their notation), \( \alpha = 10^{-3} \) and \( h_p/R_p = 0.05 \). This case corresponds to \( K = 200 \). In this simulation, they found that the gap width is about 6\( h_p \), assuming that these gap edges are located at the position with \( \Sigma = (1/3)\Sigma_0(R_p) \) (i.e., \( s = 1/3 \)). If we adopt the same definition of the gap edge, the gap widths of our exact solutions with \( x_d = 3 \) and 4 are 6.1\( h_p \) and 7.7\( h_p \), respectively. Hence, if we take into account the wave propagation and adopt \( x_d = 3 \)–4, our exact solution can almost reproduce both of the gap width and depth of the hydrodynamic simulations by DM13, for \( K = 200 \).

It should be also noted that, for \( x_d = 2 \), the wave excitation mainly occurs at |x| > \( x_d \) (80% of the excitation torques come from this region). However, the deposition site should be farther from the planet than the excitation site because the density waves propagate away from the planet. Thus, the case with \( x_d = 2 \) does not represent a realistic wave propagation. From now on, we judge that our simple model for the wave propagation is valid if more than half of the one-sided torque arises from the excitation at |x| < \( x_d \). In the case with \( x_d = 3 \) or 4, the excitation at |x| < \( x_d \) contributes 55% or 78% of the one-sided torque, respectively.

In Fig. 10, we check the effect of the width of the deposition site, \( w_d \), for \( x_d = 3 \) and \( K = 200 \). It is found that the width \( w_d \) has only a small influence on the gap structure.

We show that the deviation from the Keplerian rotation is also important in the case with wave propagation. In Fig. 11, we plot the solution with the Keplerian rotation and our exact solution. The Keplerian solution is derived from equation (43) with the angular momentum deposition model (eqs. 20 and 21). When the Rayleigh condition is violated, the marginal stable condition (eq. (29)) is used. A detail derivation of this solution is described in Appendix B. In the Keplerian solution of Fig. 11, the Rayleigh condition is violated over the whole region of the angular momentum deposition. Then the minimum surface density is given by equation (13), which is much larger than our solution and equation (11). Because equation (13) does not satisfy equation (5), the Keplerian solution does not satisfy the angular momentum conservation, as pointed out in Section 4.3. On the other hand, in the zero-dimension analysis by Fung et al. (2014) (or in eq. (11)), \( s_{\text{min}} \) is estimated from a balance be-
between the planetary torque and the viscous angular momentum flux (i.e., from the angular momentum conservation). Because of this difference, the Keplerian solution gives a much shallower gap than the estimation in equation (41). Because of this difference, the Keplerian solution gives a much shallower gap than the estimation in equation (41). Because of this difference, the Keplerian solution gives a much shallower gap than the estimation in equation (41). Because of this difference, the Keplerian solution gives a much shallower gap than the estimation in equation (41).

5.2 Dependences of the gap depth and width on $K$

Fig. 12 shows the minimum surface densities $s_{\text{min}}$ of the exact solutions in the case with the wave propagation for $x_d = 3$ (blue), $4$ (red) and $6$ (green). The parameter $w_d$ is $h_p$, $K$ is given by equation (42). For the values of $s_{\text{min}}$, our exact solutions are given by equation (41) for $K > 30$, $s_{\text{min}}$ is given by equation (43) and independent of $K$ because of the Rayleigh condition, as seen in Fig. 9. This unrealistic result in the Keplerian solution is related with the violation of the angular momentum conservation, as pointed out in Section 4.3 (and see also Appendix B).

To check which $x_d$ is preferable, we also compare the gap width with the hydrodynamic simulations. Fig. 13 shows the gap width of the exact solutions as a function of $K$. Similar to DM13, the gap edge is defined by the position with $s = 1/3$. In this definition, the gap width is roughly given by twice $x_d h_p$ for our exact solutions with $K > 50$. Note that this definition is useless for $K < 50$ because of shallow gaps with $s_{\text{min}} > 1/3$. The dashed lines represent the cases of unrealistic wave propagation, similar to Fig. 12. The results of DM13 and Varni`ere et al. (2004) are also plotted in Fig. 13. Varni`ere et al. (2004) also performed hydrodynamic simulations of gap formation for $M_p/M = 10^4 - 2 \times 10^5$, $K = 6 \times 10^{-2} - 6 \times 10^{-3}$ and $h_p/R_p = 0.04$ (i.e., $K = 600 - 6 \times 10^3$). Their gap depths almost agree with DM13’s relation. For $K < 300$, our exact solutions with $x_d = 3$ and $4$ agree with the results of DM13 and Varni`ere et al. (2004), respectively. For $K > 300$, on the other hand, the widths obtained by Varni`ere et al. (2004) are wider than those given by DM13.

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**Figure 10.** Gap structures for $w_d = 0.1 h_p$ (green), $h_p$ (blue) and $1.5 h_p$ (light-blue). The parameters $K$ and $x_d$ are set to 200 and 3, respectively. The gray dashed line indicates the solution in the instantaneous damping case and the dotted line is the minimum surface density of the wide-limit gap (eq. [41]).

**Figure 11.** The Keplerian solution in the case of wave propagation for $K = 200$, $x_d = 4$ and $w_d = h_p$ (blue line). For comparison, the exact solution (red line) is also plotted.

**Figure 12.** Minimum surface densities, $s_{\text{min}}$, of the exact solutions in the case with the wave propagation for $x_d = 3$ (blue), $4$ (red) and $6$ (green). The parameter $w_d$ is $h_p$. The plot also shows the Keplerian solution with the wave propagation for $x_d = 3$ (blue), $4$ (red) and $6$ (green). The parameter $w_d$ is $h_p$. We also plot results by DM13 (eq. (42), solid line) and the wide-limit gap (eq. (41), dotted line) and the exact solution with instantaneous damping (gray line). The dashed lines indicate exact solutions with unrealistic wave propagation.
Our exact solution with $x_d = 6$ agrees with the widths of Varni`ere et al. (2004), while widths of DM13 correspond to our solutions of unrealistic wave propagation. The preferable $x_d$ cannot be determined only by this comparison, and we still have a large uncertainty in the preferable value of $x_d$.

The difference of widths between DM13 and Varni`ere et al. (2004) would be caused by different parameters in their simulations (e.g., the disc viscosity, spatial resolution and width of a computation domain). However, the origin of the difference is still unclear. Note that the results of Kley & Dirksen (2006) and Fung et al. (2014) may support the wide gap formation of Varni`ere et al. (2004). Kley & Dirksen (2006) also showed that the disc rotation has some eccentricity when the gap is extended to the 1:2 Lindblad resonance. The eccentric gaps are formed for a large $K$ ($\geq 10^4$) (e.g., Kley & Dirksen (2006); Fung et al. 2014). Such wide gaps by massive giant planets are beyond the scope of our one-dimensional disc model adopting the local approximation.

In the above, we found that a larger $x_d$ is required for a larger $K$ (i.e., a massive planet) in order to reproduce the minimum surface densities derived by DM13 and Varni`ere et al. (2004). It should be noted that the propagation distance is not proportional to the parameter $x_d$. The propagating distance of waves is defined by the distance from the wave excitation site to the angular momentum deposition site (i.e., $x_d$). Since the one-sided torque is radially distributed (see Fig. 10), the wave excitation site can be approximately given by the median of the distribution, i.e., the point within the half of the one-sided torque arises. Such a excitation site shifts away from the planet with an increase of $K$ (see Fig. 12). D’Angelo & Lubow (2010) also showed this tendency that the peak of the excitation torque density shifts away from the planet as a deep gap is formed (Fig. 15 in that paper), using hydrodynamic simulations. Hence, because of this shift of the excitation site, the propagating distance does not increase much as $x_d$ with $K$. Goodman & Rafikov (2001) showed that the propagating distance of waves decreases with an increase of the planet mass due to the non-linear wave damping. Hence, further studies are needed in order to confirm whether this results given by Goodman & Rafikov (2001) conflicts with ours because of the shift of the excitation site. Furthermore, the non-linear wave damping would be weakened by the steep surface density gradient at the gap edge, as pointed out by Petrovich & Rafikov (2012). In order to fix the parameter $x_d$, such a wave damping effect in the gap should be taken into account in future work.

### 6 SUMMARY AND DISCUSSION

We re-examined the gap formation in viscous one-dimensional discs with a new formulation. In our formulation, we took into account the deviation from Keplerian disc rotation and included the Rayleigh stable condition, consistently. We also examined the effect of wave propagation. Our results are summarized as follows.

(i) The deviation from the Keplerian disc rotation makes the gap shallow. This is because of the enhancement of the shear motion and the viscous angular momentum transfer at the gap edges (see Fig. 1).

(ii) For deep gaps, the deviation from the Keplerian disc rotation is so large that the Rayleigh stable condition is violated. An enhanced viscosity dissolves such unstable rotation and makes it marginally stable (see Fig. 7). This effect also makes the gap shallower (see Fig. 9).

(iii) To include the effect of wave propagation, we adopted a simple model where the position of the angular momentum deposition is parameterized by $x_d$. A large $x_d$ indicates a long propagation length. The effect of wave propagation makes the gap wider and shallower (Fig. 9). In a wide gap, the waves are mainly excited at the flat bottom, which reduces the one-sided torque and the gap depth. For a sufficiently large $x_d$, the gap depth of our exact solution agrees well with the wide-limit gap and with the results of hydrodynamic simulations. At $K = 1000$, our model requires $x_d > 6$ for the agreement (Fig. 12). In the case of instantaneous wave damping, on the other hand, our exact solution gives much deeper gaps than those of hydrostatic simulations.

(iv) To check the validity of the large $x_d$, the gap width of our exact solution is compared with results of hydrodynamic simulations. For $K = 1000$, our exact solution with $x_d > 6$ has a gap width of $12 h_p$, which is larger than those of DM13 ($\sim 8 h_p$). The gap widths of Varni`ere et al. (2004).

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4 In Fig. 12 the exact solution have transition points from the realistic wave propagation (solid lines) to the unrealistic one (dashed lines) for each $x_d$. At the transition point of $K$, the excitation site defined by the median is equation to $x_d$. Fig. 12 shows that the excitation site moves away from the planet with an increasing $K$ since the transition point of $K$ increases with $x_d$. 

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**Figure 13.** Gap widths of our solutions for $x_d = 3$ (blue), 4 (red) and 6 (green). The edge of the gap is defined by the position with $s = 1/3$, in the same way as defined by DM13. The parameter $w_d$ is set to $h_p$. The dashed lines represent solutions with unrealistic wave propagation, similar to Fig. 12. The circles indicate the gap widths obtained by DM13 (Fig. 6 of their paper), and the triangles show the gap widths by Varni`ere et al. (2004) (twice $\Delta_{1000}$ in table 1 of Varni`ere et al.).
on the other hand, are almost consistent with our exact solutions. Because of this uncertainty in the gap width of hydrodynamic simulations, it is difficult to fix the preferable $x_d$ by this comparison.

(v) When the Rayleigh condition is taken into account, the deviation from the Keplerian rotation should also be included in order to keep the angular momentum conservation. The Keplerian solutions with the Rayleigh condition give much shallower gaps, as shown in figures 8 and 12.

In future works, we need to determine the preferable value of $x_d$. Previous studies (e.g., Takeuchi et al. 1996; Korycansky & Papaloizou 1996; Goodman & Rafikov 2001; Dong et al. 2011) have investigated the wave propagation with no gap. As pointed out by Petrovich & Rafikov (2012), however, the gap structure can affect the wave damping. Since our result shows that the wave damping significantly affects both the gap depth and width, the wave damping should be treated accurately in both one-dimensional models and hydrodynamic simulations for gap formation.

Our simple model does not include the effect of the deviation from Keplerian disc rotation on the wave excitation. Petrovich & Rafikov (2012) showed that a steep surface density gradient modifies the excitation torque. Such an effect on the wave excitation should be included in future studies on the gap formation. Nevertheless, it is also considered that when the waves are mainly excited at the flat-bottom, such as for the wide-limit gap, the deviation of the disc rotation would not affect the wave excitation significantly.

We also neglect the non-linearity of wave excitation, whereas the non-linearity cannot be neglected for large planets as $M_p/M_* \gtrsim (h_p/R_p)^3$. According to Miyoshi et al. (1999), the non-linearity makes the excitation torque small compared to the value for linear theory. This possibly leads to an additional shallowing effect. However, this effect would not significantly influence the gap depth since $s_{\text{min}}$ is scaled by only $K$ in DM13’s relation (eq. 12).

The Rossby wave instability may be essential for the gap formation. In present study, we included only the Rayleigh condition. A more detail investigation including both the Rayleigh condition and the Rossby wave instability should be done in future works.

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APPENDIX A: GAP MODEL BY TANIGAWA & IKOMA (2007)

Following Tanigawa & Ikoma (2007), we describe surface density structures of gap in a disc with the Keplerian rotation. In this section, we consider the gap structure in the case of instantaneous wave damping. The gap structure in the case with wave propagation is discussed in Appendix B.

In the case of instantaneous wave damping, the angular momentum deposition rate is given by $\lambda_{\text{ex}}$ given by equation (50). At far from a planet, the Rayleigh condition is always satisfied because the planetary gravity is weak. Hence, the gradient of surface density is given by equation (59) and the second derivative of surface density is given by

$$\frac{d^2 \ln s}{dx^2} = \pm \frac{4KC}{3x^5}.$$  \hspace{1cm} (A1)

The stability of the Rayleigh condition is checked by equation (A1). Namely, when $d^2 \ln s/dx^2 < -1$ in equation (A1), the surface density is described by the marginal Rayleigh stable state, instead of equation (57). We should point out that the second derivative given by equation (A1) is used to determine the stability of the Rayleigh condition and does not affect the surface density distribution. Because of this, the surface density gradient is steeper and the shallowing effect of the Rayleigh condition is much higher than in our model. Using equation (50), we obtain the outer edge of the marginal Rayleigh stable region, $x = x_m$, as

$$x_m = \left(\frac{4}{3}CK\right)^{1/5}.$$  \hspace{1cm} (A2)

In consideration of the continuity of the surface density distribution, the surface density in the marginal Rayleigh stable region ($x < x_m$) is given by

$$\ln s = -\frac{5}{6}s_m^2 + \frac{5}{7}s_m|x| - \frac{1}{2}x^2,$$

$$= -0.854K^{2/5} + 1.266K^{1/5}|x| - 0.5x^2,$$  \hspace{1cm} (A3)

and the surface density for $x > x_m$ is given by equation (57). Since there is no torque density for $x > 1.3$ in this case, the minimum surface density $s_{\text{min}}$ is given by $s(x = 1.3)$. Thus, we give $s_{\text{min}}$ as

$$s_{\text{min}} = \exp\left(-0.854K^{2/5} + 1.645K^{1/5} - 0.845\right).$$  \hspace{1cm} (A4)

Note that if $x_m < 1.3$, the whole region of the gap is the Keplerian rotating part and $s_{\text{min}}$ is given by equation (57) with $x = 1.3$. It should be also noticed that the angular momentum conservation is not satisfied at the flat bottom region with $s = s_{\text{min}}$ of equation (A4), as explained in the subsection 4.3 (and see also, Appendix B).

APPENDIX B: GAP MODEL IN A KEPLERIAN ROTATING DISC WITH WAVE PROPAGATION

Here, by assuming the Keplerian disc rotation, we derive gap solutions in the case with wave propagation. Following Tanigawa & Ikoma (2007), we consider the marginal condition when the Rayleigh condition is violated. The angular momentum deposition rate is given by equation (52). Ignoring deviation in $d\Omega/dR$ form the Keplerian and differentiating, we give

$$\frac{ds}{dx} = \begin{cases} \frac{f}{x} \frac{h_p}{w_d} & \text{for } x_d - \frac{w_d}{2h_p} < |x| < x_d + \frac{w_d}{2h_p}, \\ 0 & \text{otherwise}, \end{cases}$$  \hspace{1cm} (B1)
Integrating equation (B1), we obtain the surface density without the Rayleigh condition as

\[
s = \begin{cases} 
1 & \text{for } |x| > x_d + \frac{u_d}{2h_0}, \\
1 - \frac{T h_0}{4w_0} \left( x_d + \frac{u_d}{2h_0} - x \right) & \text{for } x_d - \frac{w_d}{2h_0} < |x| < x_d + \frac{u_d}{2h_0}, \\
1 - \frac{1}{T} & \text{for } |x| < x_d - \frac{w_d}{2h_0}.
\end{cases}
\]

(B2)

Equation (B2) does not satisfy the Rayleigh condition, especially for a large \( K \) (or a large \( T \)). First, the Rayleigh condition is violated near \( |x| = x_d + w_d/2h_0 \) because \( ds/dx \) of equation (B1) is not continuous there. In order to make \( ds/dx \) continuous, the marginally stable condition (eq. (29)) is used instead of equation (B1) in the region where \( x_d + w_d/(3w_0) < |x| < x_d + w_d/2h_0 \). In addition, the marginally stable condition should also be used near \( |x| = x_d - w_d/2h_0 \), for a large \( T \). From equation (B1), we find that the Rayleigh condition is violated from \( |x| = x_d - w_d/2h_0 \) to \( x_d + (w_d/2h_0)(1 - 6/T) + 1 \). These two Rayleigh unstable regions are merged for \( T > 3(w_d/h_0) \). In such large-\( K \) cases, since the marginally stable condition is used in the whole region of the angular momentum deposition, the minimum surface density is given by

\[
s_{\text{min}} = \exp \left[ -\frac{1}{2} \left( 1 - \frac{h_0}{w_0} \right) \right].
\]

(B3)

which is independent of \( K \). This unrealistic minimum surface density does not satisfy equation (B5) which is originated by the angular momentum conservation. As indicated by Fung et al. (2014), the planetary torque should balance with the viscous angular momentum flux outside the gap for the angular momentum conservation. In our formulation, two terms in the right-hand side of equation (27) balance with each other in the bottom region (the left-hand side is negligibly small). However, the minimum surface density given by equation (B3) independent of \( K \) breaks down such a balance. Hence, equation (B3) also violates the angular momentum conservation.

For a small \( K \), equation (B2) is approximately valid because the Rayleigh unstable region near \( |x| = x_d + w_d/2h_0 \) does not significantly affect \( s_{\text{min}} \). Substituting equation (B2) into equation (B3), we give \( T \) as

\[
T = 3 \left[ \frac{9\Delta^3}{2KC^3} \left( 1 - \frac{KC x_d}{9 \left[ x_d^2 - (w_d/2h_0)^2 \right]^2} \right) + 1 \right]^{-1}.
\]

(B4)

Substituting equation (B2) into equation (B4), we obtain \( s_{\text{min}} \) for a small \( K \) as

\[
s_{\text{min}} = \frac{C K}{9\Delta^3} \left( 1 - \frac{KC x_d}{9 \left[ x_d^2 - (w_d/2h_0)^2 \right]^2} \right)^{-1} + 1.
\]

(B5)

APPENDIX C: SOLUTIONS FOR THE LINEARIZED EQUATION OF GAPS

Here, we consider solutions to the following differential equation:

\[
\frac{d^2 y}{dx^2} - 3y = g(x),
\]

(C1)

where \( g(x) \) is an arbitrary odd function of \( x \). A general solution of this equation is given by a combination of homogeneous solutions, \( e^{\pm \sqrt{3}x} \), and a particular solution. We seek a solution which satisfies the boundary conditions of \( y = 0 \) at \( x = \pm \infty \). Since \( g(x) \) is odd in this case, the solution is an even function of \( x \). For simplicity, we consider the solution for \( x > 0 \). Because of the symmetry of the equation, the solution of \( x < 0 \) can be obtained by inverting the sign of \( x \) in the solution for \( x > 0 \).

A particular solution \( y_p(x) \) of equation (C1) can be given by

\[
y_p(x) = \frac{1}{2\sqrt{3}} \left[ e^{\sqrt{3}x} \int_x^\infty g(x')e^{-\sqrt{3}x'} dx' - e^{-\sqrt{3}x} \int_x^\infty g(x')e^{\sqrt{3}x'} dx' \right].
\]

(C2)

In the case with instantaneous wave damping (see eq. [44]), \( g(x) \) is given by

\[
g(x) = - \begin{cases} 
\frac{C}{3|\Delta|^3} & \text{for } |x| > \Delta, \\
\frac{C}{3\Delta^3} & \text{otherwise}.
\end{cases}
\]

(C3)

Substituting equation (C3) into equation (C2), the particular solution for \( x > \Delta \) is obtained as

\[
y_p(x) = \sqrt{\frac{3C}{18}} \left[ \frac{\sqrt{3}}{x} - \frac{3}{2} \left( e^{-\sqrt{3}x} \text{Ei}(\sqrt{3}x) - e^{\sqrt{3}x} \text{Ei}(-\sqrt{3}x) \right) \right],
\]

(C4)

where \( \text{Ei}(ax) \) denotes the exponential integral function (e.g., Abramowitz & Stegun (1965)). For \( x \leq \Delta \), the particular solution is given by

\[
y_p(x) = a_+ e^{\sqrt{3}x} + a_- e^{-\sqrt{3}x} + \frac{C}{9\Delta^3},
\]

(C5)

where \( a_+ \) and \( a_- \) are defined by

\[
a_\pm = \frac{C}{6} \left[ e^{\mp \Delta(\mp \sqrt{3}\Delta)} - \frac{1}{\Delta} \left( \frac{1}{\Delta^2} + \frac{1}{2} \right) e^{\mp \sqrt{3}\Delta} \right].
\]

(C6)

Using this particular solution given by equations (C4) and (C5), we can obtain the general solution of equation (41). Since this particular solution vanishes at \( x \to \infty \), the coefficient of the homogeneous solution \( e^{\sqrt{3}x} \) is zero. The coefficient of \( e^{-\sqrt{3}x} \), \( B \), is obtained as

\[
B = \frac{1}{\sqrt{3}} \left. \frac{dy}{dx} \right|_{x=0}.
\]

(C7)

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