TWO REPELLING RANDOM WALKS ON \( \mathbb{Z} \)

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Abstract. We consider two interacting random walks on \( \mathbb{Z} \) such that the transition probability of one walk in one direction decreases exponentially with the number of transitions of the other walk in that direction. The joint process may thus be seen as two random walks reinforced to repel each other. The strength of the repulsion is further modulated in our model by a parameter \( \beta \geq 0 \). When \( \beta = 0 \) both processes are independent symmetric random walks on \( \mathbb{Z} \), and hence recurrent. We show that both random walks are further recurrent if \( \beta \in (0, 1] \). We also show that these processes are transient and diverge in opposite directions if \( \beta > 2 \). The case \( \beta \in (1, 2] \) remains widely open. Our results are obtained by considering the dynamical system approach to stochastic approximations.

1. Introduction

We are concerned with the recurrence properties of two repelling random walks \( \{S^n_i; i = 1, 2, n \geq 0\} \) taking values on \( \mathbb{Z} \) in which the repulsion is determined by the full previous history of the joint process. Formally, assume that \( S^0_i, \ldots, S^n_i \in \mathbb{Z} \) are known for given but arbitrary \( n_0 \geq 1 \), and let \( \mathcal{F}_n = \sigma\{S^k_i, S^k_j : 0 \leq k \leq n\} \) be the natural filtration generated by both walks. The transition probability for each process is defined as

\[
\mathbb{P}(S^n_{i+1} = S^n_i + 1 \mid \mathcal{F}_n) = \psi((S^n_i - S^n_j)/n) = 1 - \mathbb{P}(S^n_{i+1} = S^n_i - 1 \mid \mathcal{F}_n),
\]

with \( i = 1, 2, j = 3 - i, n \geq n_0 \), and \( \psi : [-1, 1] \to [0, 1] \), defined by

\[
\psi(y) = \frac{1}{1 + \exp(\beta y)}, \quad \beta \geq 0.
\]

When \( \beta = 0 \), then \( \psi(y) = \frac{1}{2} \) for all \( y \in [-1, 1] \) and both \( S^n_1 \) and \( S^n_2 \) form two independent simple random walks on \( \mathbb{Z} \). To analyse the behaviour for \( \beta > 0 \), note that the quantity \( y = (S^n_j - S^n_i)/n \) represents the difference between the proportions of times the \( j \)-th walk made a right and a left transition up to time \( n \). Thus, if \( y \) is positive, then \( S^n_{i+1} \) transits with highest probability \( 1 - \psi(y) > \frac{1}{2} \) to the left. By contrast if \( y < 0 \), that is, if \( S^n_i \) has moved more to the left than to the right, then \( S^n_{i+1} \) moves to right with highest probability \( \psi(y) > \frac{1}{2} \). It is worth mentioning that \( \psi \) satisfies the following symmetry relation \( \psi(-y) = 1 - \psi(y) \), and hence it is not biased in any direction, left or right. The parameter \( \beta \) strengthens the repulsion between the walks: the larger the value of \( \beta \), the higher is the probability each walk goes in the direction less transited by the other walk. For given arbitrary initial conditions, the coordination of the walks towards a limiting direction, if any, is far from trivial.

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We regard a walk $S_i^n$ as recurrent (transient) if every vertex of $Z$ is visited by $S_i^n$ infinitely (only finitely) many times almost surely. Our main results are stated as follows.

**Theorem 1.** If $\beta > 2$, both random walks $S_1^n$ and $S_2^n$ are transient and
\[
\lim_{n \to \infty} S_1^n = -\lim_{n \to \infty} S_2^n = \pm \infty \text{ a.s.}
\]

**Theorem 2.** If $\beta \in [0, 1]$, then both $S_1^n$ and $S_2^n$ are recurrent.

**Remark 1.** The case $\beta = 0$ is trivial. Indeed, when $\beta = 0$, both $S_1^n$ and $S_2^n$ are two independent simple symmetric random walks and hence recurrent. The case $\beta \in (1, 2]$ remains widely open. The problem that arises in this case is mentioned the end of this article in Remark 2.

According to (1) and (2), the probability of a transition in a given direction decreases with the number of previous transitions made by the opponent walk in that direction. This allows to recognise the process studied throughout as being formed by two interacting reinforced random walks, namely one in which the reinforcement is set by repulsive behaviour of each walk. Self-attracting reinforced random walks were formally introduced in an unpublished paper by D. Coppersmith and P. Diaconis and have since been the subject of intense research, see for instance [Dav90], [Pem92], [Ben97], [Vol01], [Tar04], [MR09], [ACK14], and [CT17]. Self-repelling walks, have also deserved some attention, see [T95], [T01] and references therein. The recurrence properties of self-attracting walks have been considered among others by [Sel06], [MR09], [Sin14] and [CK14]. With the exception of [Che14], there are relatively few studies of interacting vertex reinforced random walks with ‘competition’ or ‘cooperation’. [Che14] considers two random walks that compete for the vertices of finite complete graphs and focuses on the asymptotic properties of the overlap of their vertex occupation measures.

In this article, we study the recurrence properties of $S_i^n$, $i = 1, 2$, by analysing the proportions of times each walk $i$ makes a left and a right transition up to time $n$. To do so, we identify the vector of empirical measures defined by these proportions with a stochastic approximation process. The latter have been quite effective while dealing with several reinforced processes such as vertex reinforced walks and generalized Pólya urns, see [Pem07] for a survey and further references. More precisely, we study the asymptotic behaviour of the involved stochastic approximation by considering the dynamical system approach described in [Ben96] and [Ben99]. The rest of this article is organised as follows. Section 2 shows that the vector of empirical measures of the times that each walk makes a left and a right transition forms a stochastic approximation process. This process is related to the flow induced by a smooth vector field defined on the product of two 1-simplices. It is therefore sufficient to consider the planar dynamics defined by the restriction of the field to the unit square. This together with the fact that the vector field has negative divergence suffices to show that the limit set of the stochastic approximation process corresponds to the set of equilibria of the vector field. Section 2 presents a characterisation of the equilibria in terms of the repulsion parameter $\beta$, and then shows that the stochastic approximation process converges to stable equilibria and does not converges to unstable equilibria. Section 3 finally presents the proof of Theorems 1 and 2. The proof of Theorem 1 is a straightforward application of the results in Section 2. By contrast, the proof of Theorem 2 is more involved. Beyond showing that the proportion of times each walk makes a left and a right transition...
converges toward $\frac{1}{2}$, the proof of Theorem 2 relies on an estimate for the speed of convergence, a zero-one law and a coupling argument.

2. The Dynamical System Approach

2.1. Stochastic approximations. For $n \geq 0$, $i = 1, 2$, define

$$\xi(n) = (\xi^n_1(n), \xi^n_2(n), \xi^n_3(n), \xi^n_4(n)), \quad \xi^n_i(n) = 1_{\{s^n_{i+1} - s^n_i = -1\}}, \quad \xi^n_i(n) = 1_{\{s^n_{i+1} - s^n_i = 1\}}, \quad (3)$$

and then let

$$X_n^i(n) = \frac{1}{n} \sum_{k=0}^{n-1} \xi^n_i(k), \quad X_n^i(n) = \frac{1}{n} \sum_{k=0}^{n-1} \xi^n_i(k), \quad (4)$$

be the proportion of left and right transitions of the $i$-th walk up to time $n$. Hereafter we denote by $X = \{X(n)\}_{n \geq 0}$ the process determined by $X(n) = (X_1^1(n), X_2^1(n), X_1^2(n), X_2^2(n))$, defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The process $X$ takes values on the set $\mathcal{D} = \Delta \times \Delta$, which equals the two-fold Cartesian product of the one-dimensional simplex $\Delta = \{x \in \mathbb{R}^2 \mid x_v \geq 0, \sum_v x_v = 1\}$. We will hereafter use $(x_1^i, x_2^i, x_3^i, x_4^i)$ to denote the coordinates of any point $x \in \mathcal{D}$ and also $x^i = (x_1^i, x_2^i)$ for $i = 1, 2$. Let $\mathcal{T}\mathcal{D} = \{(x_1^i, x_2^i) \in \mathbb{R}^{2 \times 2} \mid x_1^i + x_2^i = 0, i = 1, 2\}$ be the tangent space of $\mathcal{D}$. Now, let $\pi : \mathcal{D} \to \mathcal{T}\mathcal{D}$ be the map

$$x \mapsto \pi(x) = (\pi^1(x), \pi^2(x), \pi^3(x), \pi^4(x)) \quad (5)$$

where for $i = 1, 2$ and $v = l, r$,

$$\pi^i_v(x) = \psi(2x^i_v - 1), \quad j = 3 - i. \quad (6)$$

For further computations, it is also worth observing that, since $x^i \in \Delta$,

$$\pi^i_v(x) = \frac{e^{-\beta x^i_v}}{e^{-\beta x^i_v} + e^{-\beta x^i_v}}. \quad (7)$$

Lemma 1. The process $X = \{X(n)\}_{n \geq 0}$ satisfies the following recursion

$$X(n + 1) - X(n) = \gamma_n(F(X(n)) + U_n) \quad (8)$$

where

$$\gamma_n = \frac{1}{n + 1}, \quad U_n = \xi(n) - \mathbb{E}[\xi(n) \mid \mathcal{F}_n] \quad (9)$$

and $F : \mathcal{D} \to \mathcal{T}\mathcal{D}$ is the vector field $F = (F_1^1, F_1^2, F_2^1, F_2^2)$ defined by

$$F(X(n)) = -X(n) + \pi(X(n)). \quad (10)$$

The proof of Lemma 1 is presented in the Appendix. A discrete time process whose increments are recursively computed according to (8) is known as a stochastic approximation. Provided the random term $U_n$ can be damped by $\gamma_n$, (8) may be thought of as a Cauchy-Euler approximation scheme, $x(n + 1) - x(n) = \gamma_n F(x(n))$, for the numerical solution of the autonomous ODE

$$\dot{x} = F(x).$$

Under this perspective, a natural approach to determine the limit behaviour of the process $X$ consists in studying the asymptotic properties of the related ODE. This heuristic, known as the ODE method, has been rather effective while studying various reinforced stochastic processes.
Let \( x = (x_1, x_1, x_2, x_2) \) be a generic point of \( \mathcal{D} \). By (10), the ODE determined by the stochastic approximation in our case is given by the equation

\[
\dot{x} = F(x) = -x + \pi(x).
\]  

(11)

By using (7), equation (11) explicitly reads as

\[
\frac{d}{dt} x^1_v = -x^1_v + e^{-\beta x^2_v} + e^{-\beta x^2_1}, \quad v = l, r.
\]

(12)

\[
\frac{d}{dt} x^2_v = -x^2_v + e^{-\beta x^2_1} + e^{-\beta x^2_1}, \quad v = l, r.
\]

Because each \( x^1 \) and \( x^2 \) assume values on the one-dimensional simplex \( \Delta \), the system of four equations described by (12) can be reduced to two equations; for instance, those governing the evolution of \( x_1^1 \) and \( x_2^2 \). The dynamics of the ODE in (11) can therefore described on the unit square \([0,1]^2\) by identifying the field with its projection \( F \equiv (F^1_l, F^2_r) \). This observation allows to use the dynamical system approach to planar stochastic approximations described in [BH99] and [Ben99]. Theorem 3, stated below, is a consequence of this. It provides a crucial characterisation for the asymptotic behaviour of the process \( X \).

A point \( x \in \mathcal{D} \) is an equilibrium of \( F \) if \( F(x) = 0 \). The set of equilibria of \( F \) will hereafter be denoted by \( \mathcal{E} \).

**Theorem 3.** Let \( X = \{X(n)\}_{n \geq 0} \) be a process satisfying the recursion (8). For any \( \beta \in [0, \infty) \setminus \{2\} \), the process \( X \) converges almost surely toward an equilibrium of the vector field \( F \) defined in (10).

The proof of Theorem 3 is presented in Section 2.3. We present first a description of the equilibria of the vector field \( F \).

### 2.2. Equilibria

This section identifies the equilibria of vector field defined by (10) and further studies their stability depending on the repulsion parameter \( \beta \). For any point \( x \in \mathcal{D} \), let \( JF(x) \) be the Jacobian matrix of the vector field \( F \) at \( x \) and let \( \sigma(JF(x)) \) be the set of its eigenvalues. The equilibrium \( x \) is hyperbolic if all the eigenvalues of \( \sigma(JF(x)) \) have non-zero real parts. The hyperbolic equilibrium \( x \) is linearly stable if \( \sigma(JF(x)) \) contains only eigenvalues with negative real parts; otherwise \( x \) is said to be linearly unstable.

**Lemma 2.** Let \( g : [0,1] \rightarrow [0,1] \) be a strictly decreasing function and such that \( g(1-w) = 1 - g(w) \) for all \( w \in [0,1] \). Let \( E_1 \) and \( E_2 \) be the sets defined as

\[
E_1 = \left\{ x \in \mathcal{D} \mid x^i_v = g(x^j_r) \text{ for all } i, v \text{ and } j = 3 - i \right\},
\]

\[
E_2 = \left\{ x \in \mathcal{D} \mid x = (w, 1-w, 1-w, w) \text{ where } w = g(1-w) \right\}.
\]

Then \( E_1 = E_2 \). In particular, \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in E_2 \), and \((w, 1-w, 1-w, w) \in E_2 \) if and only if \((1-w, w, 1-w) \in E_2 \).

**Proof.** Assume \( x \in E_1 \). First we show that \( x^1_1 = x^2_2 \). Suppose, by contradiction, and without loss of generality, that \( x^2_1 < x^1_1 \). Since \( g \) is strictly decreasing, we would have that \( 1 - x^2_1 = x^1_1 = g(x^1_1) < g(x^2_1) = x^2_1 = 1 - x^1_1 \), contradicting the hypothesis that \( x^2_1 < x^1_1 \). Since \( x^1_1 = x^2_2 \), by setting \( w = x^1_1 = x^2_2 \), we have that \( x^1_1 = x^2_2 = (1-w) \). To conclude that \( x \in E_2 \), it sufficient to observe that \( w = g(1-w) \). Indeed,

\[
w = x^1_1 = g(x^2_1) = g(1 - x^2_1) = g(1-w).
\]
The second inequality holds because \( x \in E_1 \) and the third, because \( x_1^2 = 1 - x_2^2 \).

Conversely, assume that \( x \in E_2 \). Then \((x_1^1, x_2^1, x_3^1, x_4^1) = (w, 1 - w, 1 - w, w)\) for some \( w \) with \( w = g(1 - w) \). As an immediate consequence, we have that \( x_1^1 = g(x_1^2) \) and \( x_2^1 = g(x_2^2) \). To conclude, we show next that \( x_1^1 = g(x_1^2) \) and \( x_2^1 = g(x_2^2) \). Indeed,
\[
x_1^1 = x_2^1 = 1 - w = 1 - g(1 - w) = 1 - (1 - g(w)) = g(w) = g(x_1^2) = g(x_2^2).
\]
The third equality holds because \( x \in E_2 \) and hence \( w = g(1 - w) \). The fourth equality holds by hypothesis on \( g \), that is, \( g(1 - w) = 1 - g(w) \) for all \( w \). The last two equalities follow because \( w = x_1^1 = x_2^1 \).

**Lemma 3.** For \( \beta \in [0, 2] \), the point \( (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) is the only equilibrium for the vector field \( F \) given by (10). For any \( \beta > 2 \), the field has three equilibria,
\[
(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \quad (w, 1 - w, 1 - w, w) \quad \text{and} \quad (1 - w, w, w, 1 - w),
\]
where \( w \in (0, \frac{1}{2}) \) is uniquely determined by \( \beta \). The equilibrium \( (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) is linearly stable for \( \beta \in [0, 2) \) and linearly unstable for \( \beta > 2 \). The equilibria \((w, 1 - w, 1 - w, w)\) and \((1 - w, w, w, 1 - w)\) are linearly stable for \( \beta > 2 \).

**Proof.** Let \( \mathcal{E} \) be the set of equilibria of the vector field given by (10), and \( \psi \) be given as in (2). First we show that \( \mathcal{E} = E_2 \), where \( E_2 \) is defined as in Lemma 2 for \( g(w) = \psi(2w - 1) \).
To that end, note that \( x \in \mathcal{E} \) if and only if \( x_i^v = \pi_i(x) = \psi(2x_i^v - 1) = g(x_i^v) \) for all \( i \) and \( v \), where \( j = 3 - i \). This shows that \( \mathcal{E} = E_2 \). Next, we show that \( E_1 = E_2 \).

The previous equality is ensured by Lemma 2, provided that \( g \) is strictly decreasing and \( g(1 - w) = 1 - g(w) \) for all \( w \in [0, 1] \). These two assertions follow immediately by inspection on \( g(w) \), where
\[
g(w) = \frac{1}{1 + e^{2\beta w - \beta}}.
\]
This shows that \( \mathcal{E} = E_2 \) with \( g(w) = \psi(2w - 1) \). In particular, for all \( \beta \geq 0 \),
\[
\mathcal{E} = \left\{ x \in \mathcal{D} \mid x = (w, 1 - w, 1 - w, w), \text{ where } w = g(1 - w) \right\}.
\]

By Lemma 2, it follows that \( (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \mathcal{E} \) and \( (w, 1 - w, 1 - w, w) \in \mathcal{E} \) if and only if \( (1 - w, w, w, 1 - w) \in \mathcal{E} \). To conclude, it is sufficient to show two things. First, if \( \beta \in [0, 2] \), then there is no \( w \in [0, \frac{1}{2}) \) such that \( w = g(1 - w) \); and second, if \( \beta \in (2, \infty) \), then there is only one \( w \in [0, \frac{1}{2}) \) such that \( w = g(1 - w) \).

If \( \beta = 0 \), then the first assertion holds because \( g(1 - w) = \frac{1}{2} \) for all \( w \in [0, \frac{1}{2}] \). If \( \beta > 0 \), then both assertions hold because \( g(1 - w) \) is bounded from below by zero, increasing, strictly convex on \([0, \frac{1}{2})\), and such that
\[
\frac{\partial}{\partial w} g(1 - w) \bigg|_{w = \frac{1}{2}} > 1 \quad \text{if and only if} \quad \beta > 2.
\]

The stability of an isolated equilibrium point is determined by studying a linearization of the vector field provided by the Jacobian matrix at that point. For any \( x \in \mathcal{D} \) let \( JF(x) = [\partial F^i_k(x)/\partial x^j_k] \) for \( i = 1, 2, j = 1, 2, \) and \( k, s \in \{l, r\} \). For \( x_* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \mathcal{E} \), the Jacobian matrix is
\[
JF(x_*) = \begin{bmatrix}
-1 & 0 & -\frac{\beta}{4} & \frac{\beta}{4} \\
0 & -1 & \frac{\beta}{4} & -\frac{\beta}{4} \\
-\frac{\beta}{4} & \frac{\beta}{4} & -1 & 0 \\
\frac{\beta}{4} & -\frac{\beta}{4} & 0 & -1
\end{bmatrix}.
\]
The four eigenvalues of $JF(x_*)$ are easily computed and equal
\begin{equation}
-1, \quad -1, \quad -1 - \frac{\beta}{2}, \quad \text{and} \quad -1 + \frac{\beta}{2}.
\end{equation}
This shows that the equilibrium $x_* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is linearly stable if $\beta < 2$ and linearly unstable if $\beta > 2$.

Now, suppose that $\beta > 2$, and let $x_w = (w, 1 - w, 1 - w, w) \in \mathcal{E}$, where $w \in (0, \frac{1}{2})$. The Jacobian of the vector field at $x_w$ is given in this case by the matrix

$$
JF(x_w) = \begin{bmatrix}
-1 & 0 & -h(w, \beta) & h(w, \beta) \\
0 & -1 & h(w, \beta) & -h(w, \beta) \\
-h(w, \beta) & h(w, \beta) & 1 & 0 \\
h(w, \beta) & -h(w, \beta) & 0 & -1
\end{bmatrix},
$$

where

$$
h(w, \beta) = \frac{\beta}{2 + 2 \cosh(\beta - 2w\beta)}.
$$

Two eigenvalues of this matrix equal $-1$. The two other eigenvalues are $-1 \pm 2h(w, \beta)$. Simple analysis shows that the eigenvalue $-1 - 2h(w, \beta)$ is negative for any $\beta > 2$ and $w \in (0, \frac{1}{2})$. To conclude that $x_w$ is stable, it remains to show that $-1 + 2h(w, \beta)$ is negative. Note that, for $\beta > 2$ and $v \in [0, \frac{1}{2})$, the map $v \mapsto -1 + 2h(v, \beta)$ is increasing and equals 0 at a single value $w_*$ determined by

$$
w_* = \frac{\beta - \arccosh(\beta - 1)}{2\beta}.
$$

To conclude that $-1 + 2h(w, \beta)$ is negative, we show that $w < w_*$. A straightforward computation shows that $w_*$ is the unique solution to

$$
\frac{\partial}{\partial w_*} g(1 - w_*) = 1 \quad \text{for} \quad w_* \in (0, \frac{1}{2}).
$$

Since $x_w = (w, 1 - w, 1 - w, w) \in \mathcal{E}$ and $w \in (0, \frac{1}{2})$, we have that $g(1 - w) = w$, where $g$ is the map used in the definition of $\mathcal{E}$ in (14). Since $g(\frac{1}{2}) = \frac{1}{2}$, $w \mapsto g(1-w)$ is continuous, strictly increasing and strictly convex for $w \in [0, \frac{1}{2}]$, it follows that $w < w_*$. An analogous argument shows that the equilibrium $(1 - w, w, 1 - w)$ is stable when $\beta > 2$, because the Jacobian of the vector field at this point has the same spectrum as the Jacobian at $(w, 1 - w, 1 - w, w)$. \hfill \Box

2.3. Convergence to equilibria. This section presents the proof of Theorem 3. Its proof relies on Lemma 6 stated below. This lemma allows us to relate the limiting behaviour of the random process $X$ to the one of the flow induced by the vector field in (10). We will make use of the following terminology, mostly taken form [Ben99], to state this result.

A semi-flow on $\mathcal{D}$ is a continuous map $\phi : \mathbb{R}_+ \times \mathcal{D} \to \mathcal{D}$ such that $\phi_0$ is the identity on $\mathcal{D}$, and $\phi_{t+s} = \phi_t \circ \phi_s$ for any $t, s \geq 0$. To simplify notation we used $\phi_t(x)$ instead of $\phi(t, x)$. A subset $A \subset \mathcal{D}$ is said to be positively invariant if $\phi_t(A) \subset A$ for all $t \geq 0$. Let $F$ be a continuous Lipschitz vector field on $\mathcal{D}$. The semi-flow induced by $F$ is the unique smooth map $\Phi = \{\phi_t\}$ such that: 1. $\phi_0(x_0) = x_0$ for any $x_0 \in \mathcal{D}$, and 2. $\frac{d}{dt} \phi_t(x_0) = F(\phi_t(x_0))$ for all $t \geq 0$. 

A simple verification shows that vector field $F$ in (10) is Lipschitz continuous, hence the induced semi-flow $\Phi$ is uniquely determined by $F$. Moreover, the following lemma shows that $\mathcal{D}$ is positively invariant by $\Phi$.

**Lemma 4.** $\mathcal{D}$ is positively invariant for the semi-flow $\Phi$ induced by the vector field $F$ in (10).

**Proof.** Let $z = (z_1^1, z_1^2, z_1^3, z_1^4)$ be a generic point in $\mathcal{D}$. Suppose that $z \in \partial \mathcal{D}$ and hence, without loss of generality, that $z_1^1 = 0$ and $z_1^3 = 1$. Suppose $\phi_t$ is a solution of (11) with $\phi_0 = z$. For any $t \geq 0$, write $\phi(t, z) = \phi_t(z)$. Since $F(z) \in T\mathcal{D}$, it is sufficient to show that $\frac{d}{dt}\phi_0^1(t, z)|_{t=0} > 0$, in which case, it holds also that $\frac{d}{dt}\phi_0^1(t, z)|_{t=0} < 0$. By (5), (7), and (11), it follows that

$$\frac{d}{dt}\phi_t^1(t, z)|_{t=0} = \psi(2z_1^2 - 1) \geq \inf_y \psi(2y - 1) = \frac{1}{1 + e^\beta} > 0.$$ 

This shows that $F(z)$ points inwards whenever $z \in \partial \mathcal{D}$, and hence that $\phi_t \in \mathcal{D}$ for all $t > 0$ if $\phi_0 \in \mathcal{D}$. \hfill \Box

In terms of the semi-flow $\Phi$, a point $x \in \mathcal{D}$ is said to be an equilibrium if $\phi_t(x) = x$ for all $t \geq 0$. A point $x \in \mathcal{D}$ is periodic if $\phi_T(x) = x$ for some $T > 0$. The set $\gamma(x) = \{\phi_t(x) : t \geq 0\}$ is the orbit of $x$ by $\Phi$. A subset $\Gamma \subset \mathcal{D}$ is an orbit chain for $\Phi$ provided that for some natural number $k \geq 2$, $\Gamma$ can be expressed as the union $\Gamma = \{e_1, \ldots, e_k\} \cup \gamma_1 \cup \ldots \cup \gamma_{k-1}$ of equilibria $\{e_1, \ldots, e_k\}$ and nonsingular orbits $\gamma_1, \ldots, \gamma_{k-1}$ connecting them. If $e_1 = e_k$, $\Gamma$ is called a cyclic orbit chain.

Let $\delta > 0$, $T > 0$. A $(\delta, T)$-pseudo orbit from $x \in \mathcal{D}$ to $y \in \mathcal{D}$ is a finite sequence of partial orbits $\{\phi_t(y_i) : 0 \leq t \leq t_i\}$; $i = 0, \ldots, k - 1$; $t_i \geq T$ of the semi-flow $\Phi = \{\phi_t\}_{t \geq 0}$ such that

$$\|y_0 - x\| < \delta, \quad \|\phi_t(y_i) - y_{i+1}\| < \delta, \quad i = 0, \ldots, k - 1, \quad \text{and} \quad y_k = y.$$ 

A point $x \in \mathcal{D}$ is chain-recurrent if for every $\delta > 0$ and $T > 0$ there is a $(\delta, T)$-pseudo orbit from $x$ to itself. The set of chain-recurrent points of $\Phi$ is denoted by $\mathcal{R}(\Phi)$. The set $\mathcal{E}(\Phi)$ is closed, positively invariant by $\Phi$ and such that $\mathcal{E} \subset \mathcal{R}(\Phi)$.

Let $\mathcal{L}\{(X(n))\}$ be the limit set of the stochastic approximation process $X = \{X(n)\}_{n \geq 0}$. That is, for any point $\omega \in \Omega$, the value of $\mathcal{L}\{(X(n))\}$ at $\omega$ is given by the set of points $x \in \mathbb{R}^{md}$ for which $\lim_{k \to \infty} X(n_k, \omega) = x$, for some strictly increasing sequence of integers $\{n_k\}_{k \in \mathbb{N}}$.

Next we show that $\mathcal{L}\{(X(n))\}$ is almost surely connected and included in $\mathcal{R}(\Phi)$. This is the content of Lemma 6. To show Lemma 6, we use the following lemma.

**Lemma 5.** Let $X = \{X(n)\}_{n \geq 0}$ be a process satisfying the recursion in (8) such that $F$ defined by (10) is a continuous vector field with unique integral curves. Then

(i) $\{X(n)\}_{n \geq 0}$ is bounded,

(ii) $\lim_{n \to \infty} \gamma_n = 0$, $\sum_{n \geq 0} \gamma_n = \infty$,

(iii) for each $T > 0$, almost surely it holds that

$$\lim_{n \to \infty} \sup_{\{r: 0 \leq r_n < r \leq T\}} \left\| \sum_{k=n}^{r-1} \gamma_k U_k \right\| = 0,$$

where $\tau_0 = 0$ and $\tau_n = \sum_{k=0}^{n-1} \gamma_k$.

**Proof.** Item (i) follows by definition of $\{X(n)\}_{n \geq 0}$ in (4). Item (ii) is immediate by the form of $\gamma_n$ in (9). The proof of (iii) is presented in the Appendix. \hfill \Box
Lemma 6. Let $X = \{X(n)\}_{n \geq 0}$ be a process satisfying the recursion in (8) and $\mathcal{R}(\Phi)$, the chain-recurrent set of the semi-flow induced by the vector field $F$ in (10). Then, $\mathcal{L}(\{X(n)\})$ is almost surely connected and included in $\mathcal{R}(\Phi)$.

Proof. Since $X$ satisfies the properties (i)-(iii) in Lemma 5, the proof of the lemma follows from Theorem 1.2 in [Ben96]. □

We are now in the position to present the proof of Theorem 3.

Proof of Theorem 3. We show first that $\mathcal{R}(\Phi) \subset \mathcal{E}$. Let $\Phi = \{\phi_t\}_{t \geq 0}$ denote the planar semi-flow induced by the vector field $F \equiv (F_1, F_2)$, where $F_1$ and $F_2$ are two of the coordinate functions of the field defined by (10). By Lemma 3, we have that the field $F$ has isolated equilibria. It then follows from Theorem 6.12 in [Ben99] that for any point $p \in \mathcal{R}(\Phi)$ one of the following holds:

(a) $p$ is an equilibrium
(b) $p$ is periodic
(c) There exists a cyclic orbit chain $\Gamma \subset \mathcal{R}(\Phi)$ which contains $p$.

A simple computation shows that $\text{div} F$, the divergence of $F$, is negative, indeed $\text{div} F(x) = \partial F_1(x)/\partial x_1 + \partial F_2(x)/\partial x_2 = -2$. This implies that $\phi_t$ decreases area for $t > 0$. In this case, according to Theorem 6.15 in [Ben99], it follows that:

1. $\mathcal{R}(\Phi)$ is a connected set of equilibria which is nowhere dense and which does not separate the plane
2. If $\Phi$ has at most countably many equilibrium points, then $\mathcal{R}(\Phi)$ consists of a single stationary point.

Both options, (b) and (c), are therefore ruled out and hence $\mathcal{R}(\Phi) \subset \mathcal{E}$.

Observe now that, by Lemma 6, we have that $\mathcal{L}(\{X(n)\})$ is almost surely connected and included in $\mathcal{R}(\Phi)$. Since $\mathcal{R}(\Phi) \subset \mathcal{E}$ and $\mathcal{E}$ is formed by isolated points it follows that $X(n)$ converges almost surely towards a point of $\mathcal{E}$. □

2.4. Non-convergence to the unstable equilibrium. A step to characterise the asymptotic behaviour of the stochastic approximation $X$ consists in establishing that this process does not converges toward linearly unstable equilibria of $F$. This is accomplished here by using Theorem 1 in [Pem90], evoked in the proof of the following lemma.

Lemma 7. Let $X = \{X(n)\}_{n \geq 0}$ be a process satisfying the recursion in (8). Then, if $\beta > 2$,

$$\mathbb{P}\left( \lim_{n \to \infty} X(n) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \right) = 0.$$ 

Proof. Let $x_* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $U_n$ be defined as in Lemma 1. Throughout, $\|\cdot\|$ stands for the $L^1$ norm in $\mathbb{R}^4$. The proof follows from Theorem 1 in [Pem90], provided that the following conditions are satisfied:

(i) $x_*$ is a linearly unstable critical point of $F$,
(ii) $\|U_n\| \leq c_1$ for some postive constant $c_1$, and
(iii) For every $x \in B(x_*)$, $n > n_0$, and $\theta \in T\mathcal{D}$ with $\|\theta\| = 1$, there is a postive constant $c_2$ such that

$$\mathbb{E}\left[\max\{\langle \theta, U_n \rangle, 0\} \mid X(n) = x, \mathcal{F}_n\right] \geq c_2.$$ 

Condition (i) follows immediately from Lemma 3 and Condition (ii), by the definition of $U_n$ in (9). The rest of the proof concerns the verification of (iii).
Let $\mathcal{T}_2 = \{\theta \in \mathcal{T} : \|\theta\| = 1\}$ and $n_0$ be defined as in the first paragraph of the introduction. For $w \in \mathbb{R}$, let $w^+ = \max\{w, 0\}$. It is sufficient to show that, for all $n > n_0$, $x \in \mathcal{D}$, and $\theta \in \mathcal{T}_2$, we have that
\[
\mathbb{E}[\langle \theta, U_n \rangle^+ | X(n) = x, \mathcal{F}_n] \geq s(x)
\] (16)
where $s : \mathcal{D} \to \mathbb{R}$ is a continuous function with $s(x) > 0$.

Let
\[
s(x) = \frac{1}{2} \left( \min_{i,v} \pi^i_v(x) \right)^3.
\] (17)

Clearly $s$ is continuous because $\pi^i_v$ are continuous. Since $F(x) = -x + \pi(x)$ and since $F(x^*) = 0$, we have that $\pi(x^*) = x^* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and, therefore, $s(x^*) > 0$.

It remains to show (16). Let $\theta \in \mathcal{T}_2$. For each walk $i \in \{1, 2\}$, choose a vertex $v^i \in \{l, r\}$, such that
\[
\theta^i_{v^i} = \max_v \theta^i_v.
\]

Next, define the event $A = \bigcap_{i=1,2} \{ \xi^i_v(n) = 1 \}$, with $\xi$ as defined by (3). That is, $A$ is the event in which walk $i \in \{1, 2\}$ makes a transition to vertex $v^i$ at time $n + 1$. For all $n \geq n_0$ and $\theta \in \mathcal{T}_2$, we have that
\[
\mathbb{E}[\langle \theta, U_n \rangle^+ | X(n) = x, \mathcal{F}_n] \geq q(x, \theta)
\] (18)
where
\[
q(x, \theta) = \mathbb{E}[\langle \theta, U_n \rangle^+ | A, X(n) = x] \mathbb{P}(A | X(n) = x).
\] (19)

Note that the first equality follows because the distribution of $U_n$ is uniquely determined by $X(n)$ according to (9). The inequality in (18) holds because $\langle \theta, U_n \rangle^+$ is non-negative. Now, to show (16), it is sufficient to prove that for all $\theta \in \mathcal{T}_2$ and $x \in \mathcal{D}$
\[
q(x, \theta) \geq s(x).
\] (20)

Assume without loss of generality, that $v^i = l$, $i = 1, 2$. That is, $\theta \in \mathcal{T}_2$ is of the form $(\theta^1_l, \theta^1_r, \theta^2_l, \theta^2_r) = (a, -a, \frac{1}{2} - a, a - \frac{1}{2})$ for some $a \in [0, \frac{1}{2}]$. In that case, $A = \{ \xi^1_l(n) = 1, \xi^2_l(n) = 0, \xi^2_l(n) = 1, \xi^1_r(n) = 0 \}$. According to (9), we have $(U^i_n) = \xi^i_l(n) - \mathbb{E}[\xi^i_v(n) | \mathcal{F}_n] = \xi^i_l(n) - \pi^i_v(X(n))$. Using the previous equality and the particular form of $\theta$ and $A$, it follows by the definition of $q$ in (19) that
\[
q(x, \theta) = \left[ \sum_i \theta^i_l - \sum_{i,v} \theta^i_v \pi^i_v(x) \right]^+ \mathbb{P}(A | X(n) = x)
\]
\[
= \left[ \frac{1}{2} - \sum_{i,v} \theta^i_v \pi^i_v(x) \right]^+ \prod_{i=1}^2 \pi^i_l(x)
\]
\[
\geq \left[ \frac{1}{2} - \sum_{i,v} \theta^i_v \pi^i_v(x) \right]^+ \left( \min_{i,v} \pi^i_v(x) \right)^2,
\]

where the last equality uses the fact that the transitions of the walks are independent given the event $\{X(n) = x\}$ and, therefore, $\mathbb{P}(A | X(n) = x) = \prod_{i=1}^2 \pi^i_l(x)$.

To show (20), it is sufficient to show that $(\frac{1}{2} - \sum_{i,v} \theta^i_v \pi^i_v(x))^+ \geq \frac{1}{2} \min_{i,v} \pi^i_v(x)$. To simplify notation, set $\pi^i_v = \pi^i_v(x)$. Since $(\theta^1_l, \theta^1_r, \theta^2_l, \theta^2_r) = (a, -a, \frac{1}{2} - a, a - \frac{1}{2})$, it follows
that
\[
\frac{1}{2} - \sum_{i,v} \theta_{i,v}^v \pi_v^i = \frac{1}{2} - \left( a \pi_1^i + \left( \frac{1}{2} - a \right) \pi_2^i \right) + a \pi_1^i + \left( \frac{1}{2} - a \right) \pi_2^i
\]
\[
\geq \frac{1}{2} - \left( a \pi_1^i + \left( \frac{1}{2} - a \right) \pi_2^i \right) + a \min_{i,v} \pi_v^i + \left( \frac{1}{2} - a \right) \min_{i,v} \pi_v^i
\]
\[
\geq \frac{1}{2} \min_{i,v} \pi_v^i,
\]
where the last inequality uses the fact that \( a \in [0, \frac{1}{2}] \), \( \pi_1^i, \pi_2^i \in [0, 1] \), and, therefore, \( \frac{1}{2} - \left( a \pi_1^i + \left( \frac{1}{2} - a \right) \pi_2^i \right) \geq \frac{1}{2} - \left( a + \left( \frac{1}{2} - a \right) \right) = 0 \).

\[
3. \text{ PROOF OF THEOREMS 1 AND 2}
\]

This section presents the proof of the transience of both walks \( S_n^i \), \( i = 1, 2 \), when \( \beta \in (2, \infty) \), and the recurrence when \( \beta \in [0, 1] \). The problem that arises when \( \beta \in (1, 2) \) is mentioned in Remark 2 at the end of this section.

The transience will make use of the following lemma.

**Lemma 8.** There is a unique point \( x \in [0, 1] \), depending on \( \beta \), such that,
\[
\lim_{n \to \infty} \frac{1}{n} \left( S_n^1 - S_n^0, S_n^2 - S_n^0 \right) \in \left\{ (x, -x), (-x, x) \right\} \quad \text{a.s.}
\]
In addition, if \( 0 \leq \beta \leq 2 \), then \( x = 0 \), and if \( \beta > 2 \), then \( 0 < x < 1 \).

**Proof.** From Theorem 3, \( X \) converges almost surely towards one element of the set \( \mathcal{E} \), the set of equilibria of the vector field \( F \) in (10). This set is characterised by Lemma 3. Noting that
\[
(S_n^i - S_n^0)/n = 2X_r^i(n) - 1,
\]
(21) it follows by Lemma 3 that \( X_r^i(n) \to \frac{1}{2} \) a.s. for \( 0 \leq \beta < 2 \). As a consequence, when \( 0 \leq \beta < 2 \) we have that
\[
\frac{S_n^1 - S_n^0}{n} \to 0 \quad \text{a.s.}
\]

For the case \( \beta > 2 \), by Lemma 3 and Lemma 7 there is \( w \in [0, \frac{1}{2}] \) such that
\[
\lim_{n \to \infty} (X_r^1(n), X_r^2(n)) \in \left\{ (w, 1-w), (1-w, w) \right\} \quad \text{a.s.}
\]
Using (21) and setting \( x = 2w-1 \), it follows that \( \lim_{n \to \infty} (S_n^1 - S_n^0)/n \in \left\{ (x, -x), (-x, x) \right\} \) with \( 0 < x < 1 \). This concludes the proof of the lemma. □

**Proof of Theorem 1.** The proof of the theorem is an immediate consequence of Lemma 8. Indeed, if \( \beta > 2 \), by Lemma 8 it follows that \( (S_n^1/n, S_n^2/n) \) converges a.s. to \( (x, -x) \) or \( (-x, x) \) where \( x > 0 \). Therefore we have that either \( (S_n^1/n, S_n^2/n) \to (+\infty, -\infty) \) a.s. or \( (S_n^1/n, S_n^2/n) \to (-\infty, +\infty) \) a.s. □

The rest of this section is devoted to the proof of Theorem 2, that is, of the recurrence of \( S_n^1 \) and \( S_n^2 \) when \( \beta \in [0, 1] \). Observe that in this case, according to Lemma 3, the only equilibrium of \( F \) is the point \( x_* = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \). We argue that both walks \( S_n^i \) are recurrent provided the process \( X \) converges sufficiently fast towards \( x_* \). To this end, we will make use of several lemmas. The first of these provides a rate of convergence of \( X \) towards \( x_* \) when \( \beta \in [0, 1] \). This is obtained by considering the rate at which \( \phi_t(x) \) converges towards \( x_* \) and the rate for the almost sure convergence of \( X \) toward the
trajectories of $\Phi$. The latter relies on the shadowing techniques described in Section 8 of [Ben99]. The proof follows along the lines of the proof of Lemma 3.13 in [BRS13].

**Lemma 9.** If $\beta \in [0, 1]$, then
\[
\|X(n) - x_*\| = O\left(\frac{1}{\sqrt{n}}\right) \quad \text{a.s.}
\]

**Proof.** Lemma 3 shows that $x_*$ is the only equilibrium of $F$ when $\beta \in [0, 1]$. This lemma also shows that $x_*$ is hyperbolic and linearly stable. By Theorem 3 it then follows that a.s. $X(n) \to x_*$. Further, according to Theorem 5.1 in [Rob99], p. 153, we have that the equilibrium $x_*$ is exponentially attracting. More precisely, there is a neighbourhood $\mathcal{U} \subset \mathcal{D}$ of $x_*$ and two constants $C \geq 1$, $\zeta > 0$, such that for any initial condition $x \in \mathcal{U}$, the solution $\phi_t$ of (11) satisfies
\[
\|\phi_t(x) - x_*\| \leq Ce^{-t\zeta}\|x - x_*\| \quad \text{for all} \quad t \geq 0.
\] (22)

The constant $\zeta$ is such that $-\zeta$ is an upper bound for all the eigenvalues $\lambda$ of $JF(x_*)$, that is, $\Re(\lambda) \leq -\zeta < 0$. We observe that because of (15), here we have the explicit expression $\zeta = 1 - \beta/2$.

Let $\tau_n = \sum_{k=1}^n \gamma_k$ and let $Y : \mathbb{R}^+ \to \mathcal{D}$ be a continuous time piecewise affine process defined such that: (i) $Y(\tau_n) = X(n)$ and (ii) $Y$ is affine on $[\tau_n, \tau_{n+1}]$. $\{Y(t)\}_{t \geq 0}$ may be defined as the following linear interpolation of $X(n)$,
\[
Y(\tau_n + s) = X(n) + s \frac{X(n_1) - X(n)}{\tau_{n+1} - \tau_n}, \quad \text{for} \quad 0 \leq s \leq \gamma_{n+1}, \quad n \geq 0.
\]

By Proposition 8.3 in [Ben99], the interpolated process $\{Y(t)\}_{t \geq 0}$ is almost surely a $-\frac{1}{2}$-pseudotrajectory of $\Phi$, that is,
\[
\limsup_{t \to \infty} \frac{1}{t} \log \left( \sup_{0 \leq h \leq T} \|\phi_h(Y(t)) - Y(t + h)\| \right) \leq -\frac{1}{2}
\] (23)

for all $T > 0$.

In view of (22) and (23), by Lemma 8.7 in [Ben99], it follows that
\[
\limsup_{t \to \infty} \frac{1}{t} \log (\|Y(t) - x_*\|) \leq - \min \left\{ \frac{1}{2}, \zeta \right\}.
\]

This in turn implies that
\[
\|X(n) - x_*\| = O\left(n^{-\min \left\{ \frac{1}{2}, \zeta \right\}}\right)
\]

and hence concludes the proof because $\zeta \in [\frac{1}{2}, 1]$ when $\beta \in [0, 1]$. \qed

**Lemma 10.** If $\beta \in [0, 1]$, then, for any $i = 1, 2$, $v = r, l$,
\[
\left| \pi_v^i (X(n)) - \frac{1}{2} \right| = O\left(\frac{1}{\sqrt{n}}\right) \quad \text{a.s.}
\]

**Proof.** Let $x_* = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. By the definition of $\pi$, namely by equations (2) and (6), we have that $\|\nabla \pi_v^i (x_*)\|_\infty = \beta/2 \leq \frac{1}{2}$. By linearization of $\pi_v^i (X(n))$ at $x_*$ it follows that $\pi_v^i (X(n)) - \pi_v^i (x_*) = \langle \nabla \pi_v^i (x_*) , X(n) - x_* \rangle + R(X(n))$, where $R(X(n))$ is the error of the approximation. Therefore
\[
\left| \pi_v^i (X(n)) - \frac{1}{2} \right| = \left| \pi_v^i (X(n)) - \pi_v^i (x_*) \right|
\]
The proof is concluded by applying Lemma 9 and observing that \( \|R(X(n))\|/\|X(n) - x_*\| \) converges to zero as \( X(n) \) approaches \( x_* \).

\[ \text{Corollary 1. Let } P_n = \pi_i^i(X(n)) \text{ for some fixed } i \in \{1, 2\}. \text{ For each } \varepsilon > 0, \text{ there are sufficiently large } b \text{ and } m, \text{ depending on } \varepsilon, \text{ such that} \]
\[ \mathbb{P}(A) > 1 - \varepsilon, \text{ where } A = \left\{ P_n - \frac{1}{2} \leq \frac{b}{\sqrt{n}} \text{ for all } n > m \right\}. \]

\textbf{Proof.} Let \( \varepsilon > 0 \) be arbitrary. According to Lemma 10, there is a set \( \Omega^1 \subset \Omega \) with \( \mathbb{P}(\Omega^1) = 1 \), such that \( |P_n(\omega) - \frac{1}{2}| = O\left( \frac{1}{\sqrt{n}} \right) \) for each \( \omega \in \Omega^1 \). Therefore, for each \( \omega \in \Omega^1 \), there are well defined constants \( n(\omega) > 0 \) and \( b(\omega) > 0 \) such that
\[ |P_n(\omega) - \frac{1}{2}| \leq \frac{b(\omega)}{\sqrt{n}} \text{ for all } n > n(\omega). \]

Define \( \Omega_k = \{ \omega \in \Omega^1 | \max\{b(\omega), n(\omega)\} \leq k \}, k = 1, 2, 3, \ldots \) Note that \( (\Omega_k)_{k=1}^\infty \) is an increasing sequence of sets that converges to \( \Omega^1 \). Since \( \mathbb{P}(\Omega^1) = 1 \), by continuity of the probability measure, there is a sufficiently large \( k_* > 0 \) such that \( \mathbb{P}(\Omega_{k_*}) > 1 - \varepsilon \). Since \( A \supset \Omega_k \) provided that \( b > k_\varepsilon \) and \( m > k_\varepsilon \), it follows that \( \mathbb{P}(A) > 1 - \varepsilon \) for \( b > k_\varepsilon \) and \( m > k_\varepsilon \). \( \square \)

\textbf{Lemma 11.} Let \( b > 0 \) and \( m > 0 \), and define \( \{Z_n\}_{n \geq 0} \) as a non homogeneous random walk with independent increments, parametrized by \( b \) and \( m \), as follows. Set \( Z_n = Z_0 + \sum_{k=1}^n Y_k \), where \( Z_0 \in \mathbb{Z} \) and \( Y_1, Y_2, \ldots \) are independent random variables such that \( \mathbb{P}(Y_{n+1} = 1) = p_n = 1 - \mathbb{P}(Y_{n+1} = -1) \). Let \( c > 0 \) and \( \sigma_n = \text{Var}(Z_n)^{\frac{1}{2}} \). The following implications hold
\[ p_n = \begin{cases} 0, & \text{if } n \leq m \smallskip \frac{1}{2} - \min \left\{ \frac{1}{2}, \frac{b}{\sqrt{n}} \right\}, & \text{otherwise} \end{cases} \quad \Rightarrow \quad \mathbb{P}\left( \limsup_n \frac{Z_n}{\sigma_n} \geq c \right) = 1, \quad (24) \]
\[ p_n = \begin{cases} 1, & \text{if } n \leq m \smallskip \frac{1}{2} + \min \left\{ \frac{1}{2}, \frac{b}{\sqrt{n}} \right\}, & \text{otherwise} \end{cases} \quad \Rightarrow \quad \mathbb{P}\left( \liminf_n \frac{Z_n}{\sigma_n} \leq -c \right) = 1. \quad (25) \]

\textbf{Proof of Lemma 11.} We will only present the proof of (24). The proof of (25) goes analogously. Let \( A_c = \{ \limsup_n Z_n/\sigma_n \geq c \} \) and \( \mathcal{Z}_n = \sigma(\{Z_k : k \geq n\}) \). Define \( \mathcal{Z} = \bigcap_n \mathcal{Z}_n \), the tail sigma algebra generated by \( Z_n \). Observe that, by the definition of \( Z_n \), we have that \( \sigma_n = 2(\sum_{k=1}^n p_k(1 - p_k))^{\frac{1}{2}} \), thus the event \( A_c \) belongs to \( \mathcal{Z} \) because \( \sigma_n \to \infty \) as \( n \to \infty \). Therefore, in order to show that \( \mathbb{P}(A_c) = 1 \), by Kolmogorov’s zero-one law, it suffices to show that \( \mathbb{P}(A_c) > 0 \).
Since \( A_c = \{ \limsup_n Z_n / \sigma_n \geq c \} \supseteq \limsup_n \{ Z_n / \sigma_n \geq c \} \), we have that
\[
P(A_c) \geq P( \limsup_n \left\{ \frac{Z_n}{\sigma_n} \geq c \right\})
\geq \limsup_n P \left( \frac{Z_n}{\sigma_n} \geq c \right)
= \limsup_n P \left( \frac{Z_n - \mathbb{E}[Z_n]}{\sigma_n} \geq c - \mathbb{E}[Z_n] \right). \tag{26}
\]

Let \( Z \) be a standard normal random variable and let \( \stackrel{d}{\rightarrow} \) stand for convergence in distribution. Let us assume and argue later on that \( \ell = \lim_{n \to \infty} \mathbb{E}(Z_n) / \sigma_n \) exists and that \( |\ell| < \infty \). By observing that the random variables \( Y_n \) are uniformly bounded and that \( \sigma_n = \text{Var}(Z_n)^{1/2} \to \infty \) as \( n \to \infty \), it follows that Lindeberg’s conditions are met and thus the following central limit theorem holds
\[
\frac{Z_n - \mathbb{E}(Z_n)}{\sigma_n} \stackrel{d}{\rightarrow} Z.
\]

Combining the limit \( \ell \) with the bound in (26) gives
\[
P(A_c) \geq P(Z > c - \ell) > 0.
\]

To conclude the proof, it remains to verify that the limit \( \ell \) exists and is finite. By definition of \( Z_n \), it follows that
\[
\frac{\mathbb{E}(Z_n)}{\sigma_n} = \frac{Z_0/2 + \sum_{k=1}^{n} p_k - n/2}{\sqrt{\sum_{k=1}^{n} p_k(1 - p_k)}}, \tag{27}
\]
where \( p_k = \frac{1}{2} - b/\sqrt{k} \) for sufficiently large \( k \). A straightforward computation shows that the right hand-side of (27) converges to \(-4b\) as \( n \) goes to infinity. \( \square \)

We are ready for the proof of Theorem 2.

Proof of Theorem 2. Throughout the proof, \( i \in \{1, 2\} \) will be fixed. It is sufficient to show that \( P(\limsup_n \{ S_n^i = 0 \}) = 1 \). This will be achieved by proving that \( P(\limsup_n \{ S_n = 0 \}) > 1 - \epsilon \) for arbitrary \( \epsilon > 0 \). Recall that \( \pi^i(X(n)) \) is the probability of the event \( \{ S_n = S_n^i \} \) given \( X(n) \) for \( n \geq n_0 \), where \( n_0 \) is as defined in the Introduction. Let \( \{ U_n; n \geq 0 \} \) be a sequence of independent and identically distributed uniform random variables taking values on the open interval \((0, 1)\) and couple \( S_n^i \) with \( U_n \) such that
\[
S_{n+1}^i = S_n^i + 1 \quad \text{if and only if} \quad U_n \leq P_n \quad \text{for} \quad n \geq n_0,
\]
where \( P_n = \pi^i(X(n)) \).

Choose \( \epsilon > 0 \) arbitrarily. In accordance to Corollary 1, choose \( b > 0 \) and \( m > n_0 \) such that
\[
P(A) > 1 - \epsilon, \quad \text{where} \quad A = \left\{ \left| P_n - \frac{1}{2} \right| \leq \frac{b}{\sqrt{n}} \quad \text{for all} \quad n > m \right\}. \tag{28}
\]

Now, let \( Z_n \) be another walk with independent increments such that \( Z_0 = S_0^i \) and
\[
Z_{n+1} = Z_n + 1 \quad \text{if and only if} \quad U_n \leq p_n \quad \text{for all} \quad n \geq 0,
\]
where
\[
p_n = \begin{cases} 
0, & \text{if } 0 \leq n \leq m, \\
\frac{1}{2} - \min \left\{ \frac{1}{2}, \frac{1}{\sqrt{n}} \right\}, & \text{if } n > m.
\end{cases}
\]

Observe that the walks \( S^i_n \) and \( Z_n \) are coupled through \( U_n \) as follows. Given that \( p_n \leq P_n \), it follows that \( Z_{n+1} = Z_n + 1 \) implies that \( S^i_{n+1} = S^i_n + 1 \). Indeed, given that \( Z_{n+1} = Z_n + 1 \) and \( p_n \leq P_n \), we have that \( U_n \leq p_n \leq P_n \) and therefore \( S^i_{n+1} = S^i_n + 1 \). Since \( Z_0 = S^i_0 \) and \( p_n = 0 \) for all \( n = 0, 1, 2, \ldots, m \), it follows that \( S^i_n \geq Z_n \) for all \( n \geq 0 \), given the event \( B = \{ p_n \leq P_n \} \) for all \( n > m \}. \) As a consequence, we have that
\[
\mathbb{P} \left( \limsup_{n} \frac{S^i_n}{\sigma_n} \geq c \right) \geq \mathbb{P} \left( \limsup_{n} \frac{S^i_n}{\sigma_n} \geq c \bigg| B \right) \mathbb{P}(B) \\
\geq \mathbb{P} \left( \limsup_{n} \frac{Z_n}{\sigma_n} \geq c \bigg| B \right) \mathbb{P}(B) \\
= \mathbb{P}(B),
\]
where \( \sigma_n = \text{Var}(Z_n)^{\frac{1}{2}} \). The second inequality in (29) follows by the coupling of \( S_n \) and \( Z_n \). The equality in (29) follows by (24), because \( Z_n \) satisfies the hypotheses of Lemma 11, and hence \( \mathbb{P} \left( \limsup_{n} Z_n/\sigma_n \geq c \right) = 1 \) as well as \( \mathbb{P} \left( \limsup_{n} Z_n/\sigma_n \geq c \big| B \right) = 1 \).

Now, using (29) and observing that \( B \supseteq A \), we have that
\[
\mathbb{P} \left( \limsup_{n} \frac{S^i_n}{\sigma_n} \geq c \right) \geq \mathbb{P}(B) \geq \mathbb{P}(A) \geq 1 - \varepsilon, \tag{30}
\]
where the last inequality in (30) follows by definition of \( A \) in (28).

Since \( \varepsilon \) and \( c > 0 \) where arbitrarily chosen, we have, by (30), that \( \mathbb{P} \left( \limsup_{n} S^i_n/\sigma_n \geq c \right) = 1 \) for all \( c > 0 \). By using (25), we can show analogously that \( \mathbb{P} \left( \liminf_{n} S^i_n/\sigma_n \leq -c \right) = 1 \) for all \( c > 0 \). Using these two facts, and taking into account that \( \sigma_n \) converges to infinity as \( n \to \infty \), we conclude that
\[
\mathbb{P} \left( \limsup_{n} \{ S^i_n = 0 \} \right) \geq \mathbb{P} \left( \limsup_{n} \frac{S^i_n}{\sigma_n} = +\infty, \liminf_{n} \frac{S^i_n}{\sigma_n} = -\infty \right) \\
= \lim_{c \to \infty} \mathbb{P} \left( \limsup_{n} \frac{S^i_n}{\sigma_n} \geq c, \liminf_{n} \frac{S^i_n}{\sigma_n} \leq -c \right) = 1. \tag{29}
\]

Remark 2. The conclusion of Lemma 11, required in the demonstration of Theorem 2, relies on the fact that \( p_n \) converges sufficiently fast towards \( \frac{1}{2} \). The computations involved in Lemma 11 show that the rate of convergence must be such that \( p_n \leq \frac{1}{2} - bn - \rho \) for \( b > 0 \) and \( \rho = \frac{1}{2} \) for sufficiently large \( n \). The exponent \( \rho = \frac{1}{2} \) is critical in the sense that the conclusion of Lemma 11 does not holds if \( \rho < \frac{1}{2} \). According to Lemma 9, when \( \beta \in [0, 1] \) we have exactly the critical rate \( \rho = \frac{1}{2} \). The same arguments used throughout the proof of Lemma 9 also give the estimate \( \rho = \zeta \) for \( \zeta = 1 - \beta/2 \) when \( \beta \in (1, 2] \). This shows that the convergence of \( p_n \) towards \( \frac{1}{2} \) can be arbitrarily slow as \( \beta \neq 2 \), and in fact too slow for any \( \beta > 1 \). The question about the recurrence/transience of both random walks remains therefore open when \( \beta \in (1, 2] \).
Appendix

Proof of Lemma 1. By (4), it follows that

$$X_i(n + 1) - X_i(n) = \frac{1}{n + 1}(-X_i(n) + \xi_i(n)).$$

Likewise, an analogous expression for $X_i(n + 1) - X_i(n)$ can be derived in terms of $\xi_i(n)$ and $X_i(n)$. Hence, by using (9) and (10), it follows that

$$X(n + 1) - X(n) = \gamma_n \left\{ F(X(n)) + E[\xi(n) \mid \mathcal{F}_n] - \pi(X(n)) + U_n \right\}. \quad (31)$$

To conclude that (8) holds, we will show that $E[\xi(n) \mid \mathcal{F}_n] - \pi(X(n)) = 0$. By using the definition of the probabilities (1) and of $\xi(n)$ in (3), and further observing that $\psi$, defined in (2), satisfies $1 - \psi(y) = \psi(-y)$ for all $y$, we have that

$$E[\xi(n) \mid \mathcal{F}_n] = (\mathbb{P}(S_{n+1}^1 - S_n^1 = -1 \mid \mathcal{F}_n), \ldots, \mathbb{P}(S_{n+1}^2 - S_n^2 = 1 \mid \mathcal{F}_n))
= \left(\psi\left(\frac{S_{n+1}^2 - S_n^2}{n}\right), \psi\left(\frac{S_{n+1}^2 - S_n^2}{n}\right), \psi\left(\frac{S_{n+1}^2 - S_n^2}{n}\right)\right).$$

Now, since $(S_{n+1}^2 - S_n^2)/n = 2X_i(n) - 1$ and $(S_{n+1}^2 - S_n^2)/n = 2X_i(n) - 1$, we conclude, by using the definition of $\pi$, given by (5) and (6), that

$$E[\xi(n) \mid \mathcal{F}_n] = (\pi_1(X(n)), \pi_1(X(n)), \pi_1(X(n)), \pi_1(X(n))) = \pi(X(n)). \quad (32)$$

In view of (32), equation (31) reduces to (8).

Proof of Lemma 5.(iii). Let $M_n = \sum_{k=0}^n \gamma_k U_k$. The process $\{M_n\}_{n \geq 0}$ is a martingale with respect to $\{\mathcal{F}_n\}_{n \geq 0}$, that is

$$E[M_{n+1} \mid \mathcal{F}_{n+1}] = \sum_{k=0}^n \gamma_k U_k + \gamma_{n+1} E[U_{n+1} \mid \mathcal{F}_{n+1}] = M_n.$$  

Observe that

$$E[\|M_{n+1} - M_n\|^2 \mid \mathcal{F}_{n+1}] = \gamma_{n+1}^2 E[\|U_{n+1}\|^2 \mid \mathcal{F}_{n+1}]
\leq \gamma_{n+1}^2 \left(\sum_{i \in \{1, r\}, \ j \in \{1, 2\}} \xi_i^j(n + 1)\right)^2 \leq 16 \gamma_{n+1}^2.$$  

By using Doob’s decomposition for the sub-martingale $M_n^2$, consider the predictable increasing sequence $A_{n+1} = M_n^2 + M_n$ with $A_1 = 0$. The conditional variance formula for the increment $M_{n+1} - M_n$ gives

$$A_{n+2} - A_{n+1} = E[M_{n+1}^2 \mid \mathcal{F}_n] - M_n^2 = E[\|M_{n+1} - M_n\|^2 \mid \mathcal{F}_{n+1}],$$

and hence for any $n$,

$$A_{n+2} = \sum_{k=0}^n E[\|M_{k+1} - M_k\|^2 \mid \mathcal{F}_{n+1}] \leq 16 \sum_{k=0}^n \gamma_{n+1}^2.$$  

Passing to the limit $n \to \infty$ shows that almost surely $A_\infty < \infty$. According to Theorem 5.4.9 in [Dur10], p. 254, this in turn implies that $M_n$ converges almost surely to a finite limit in $\mathbb{R}^{2 \times 2}$ and hence that $\{M_n\}_{n \geq 0}$ is a Cauchy sequence. This is sufficient to conclude the proof. \[\square\]
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