1. Introduction

The purpose of this note is to prove Theorem 1.1 below, due to Jørgensen and Thurston and known to experts in the field; the earliest reference we found to this theorem and a sketch of its proof appears in Thurston’s notes [6, Chapter 5]. For definitions and notation see the next section. We study the triangulation of the thick part. For another geometric study of the triangulation of the thick part see Breslin’s [2].

We prove:

**Theorem 1.1** (Jørgensen, Thurston). Let $\mu > 0$ be a Margulis constant for $\mathbb{H}^3$. Then for any $d > 0$ there exists a constant $K > 0$, depending on $\mu$ and $d$, so that for any complete finite volume hyperbolic 3-manifold $M$, $N_d(M_{[\mu, \infty)})$ can be triangulated using at most $K \text{Vol}(M)$ tetrahedra.

The manifold $N_d(M_{[\mu, \infty)})$ is the closed $d$-neighborhood of the $\mu$-thick part of $M$ and is denoted by $X$ throughout this paper. By the Margulis Lemma, $M \setminus X$ consists of disjoint cusps and open solid tori, and each of these solid tori is a regular neighborhood of an embedded closed geodesic. We refer to removing such an open solid torus neighborhood of a geodesic $\gamma$ as *drilling out* $\gamma$. Thus $X$ is obtained from $M$ by drilling out short geodesics. Note that any preimage of any component of $M \setminus X$ in the universal cover $\mathbb{H}^3$ is convex (for an explicit description see Definition 4.1). In the next proposition we bring the basic facts about $X$; this proposition is independent of Theorem 1.1. For a complete Riemannian manifold $A$ and a point $a \in A$, we denote the radius of injectivity of $A$ at $a$ by $\text{inj}_A(a)$.

**Proposition 1.2.** Fix the notation of Theorem 1.1. Then the following hold:

1. There exists $R = R(\mu, d) > 0$, independent of $M$, so that for any $x \in X$, $\text{inj}_M(x) > R$.
2. $M_{[\mu, \infty)} \subset X \subset M$, and $X$ is obtained from $M$ by drilling out short geodesics and truncating cusps. In particular:
(a) If $\gamma \subset M$ is a geodesic of length less than $2R$ then $\gamma$ is drilled out.

(b) If $\gamma \subset M$ is a simple geodesic of length at least $2\mu$ then $\gamma$ is not drilled out.

**Proof of Proposition 1.2** It follows from the decay of radius of injectivity (see, for example, Proposition 4.19) that there exists $R > 0$, depending only on $\mu$ and $d$, so that for any $x \in X$, $\text{inj}_M(x) > R$. This establishes (1).

By construction $M_{[\mu, \infty)} \subset X \subset M$. Let $U$ be a component of $M_{(0, \mu)}$. The set of points removed from $U$ is:

$$\{x \in U | d(x, M \setminus U) > d\}.$$ 

When $U$ is a solid torus neighborhood of a closed short geodesic $\gamma$, the set of points removed is $\{x \in U | d(x, \gamma) \leq d(X, \gamma) - d\}$, and is either empty or an open solid torus neighborhood of $\gamma$. In the first case $U \subset X$ and in the second case we remove a neighborhood of $\gamma$. When $U$ is a cusp, $(M \setminus X) \cap U$ is isotopic to $U$. This establishes (2).

Let $\gamma \subset M$ be a geodesic of length less than $2R$. Then for every $p \in \gamma$, $\text{inj}_M(p) < R$. By (1) above $\gamma$ is drilled out. This establishes (2)(a).

Let $\gamma \subset M$ be a simple geodesic of length at least $2\mu$. It is clear from the definitions that if $\gamma$ is drilled out then $\gamma \subset M_{(0, \mu)}$. A geodesic is contained in $M_{(0, \mu)}$ if and only if it covers a short geodesic (that is, has the form $\delta^n$ for some geodesic $\delta$ with $l(\delta) < 2\mu$ and some $n > 0$). Such a geodesic is simple if and only if $n = 1$; we conclude that simple geodesics in $M_{(0, \mu)}$ are shorter than $2\mu$. Thus $\gamma \not\subset M_{(0, \mu)}$, and it is not drilled out. This establishes 2(b). \qed

We denote by $t_C(M)$ the minimal number of tetrahedra required to triangulate a link exterior in $M$, that is, the minimal number of tetrahedra required to triangulate $M \setminus \hat{N}(L)$, where the minimum is taken over all links $L \subset M$ (possibly, $L = \emptyset$) and all possible triangulations. Similarly we define $t_{HC}(M)$ to be the minimal number of tetrahedra necessary to triangulate $M \setminus \hat{N}(L)$, where $L \subset M$ ranges over all possible hyperbolic links. As a consequence of Theorem 1.1 we get the following corollary, showing that $\text{Vol}(M)$, $T_C(M)$, and $T_{HC}(M)$ are the same up-to linear equivalence:

**Corollary 1.3.** Let $\mu > 0$ be a Margulis constant for $\mathbb{H}^3$ and fix $d > 0$. Let $K > 0$ be the constant given in Theorem 1.1 and $v_3$ be the volume of a regular ideal tetrahedron in $\mathbb{H}^3$. Then for any complete finite volume hyperbolic 3-manifold $M$ we have:

$$t_C(M) \leq t_{HC}(M) \leq K\text{Vol}(M) \leq Kv_3 t_C(M) \leq Kv_3t_{HC}(M).$$

**Proof:** The first and last inequalities are obvious.

By Proposition 1.2(2), $X = N_d(M_{\geq \mu})$ is obtained from $M$ by drilling out geodesics; hence by Kojima it is a hyperbolic. Thus the second inequality follows directly for Theorem 1.1.

The proof of third inequality is well known (see, for example, Chapter C of [I]); we sketch its argument for the reader’s convenience. Let $L \subset M$ be a link and $\mathcal{T}$ a triangulation of $M \setminus \hat{N}(L)$ using $t_C(M)$
tetrahedra. Let \( \Delta \) be a 3-simplex in \( \mathbb{R}^3 \) and denote the characteristic maps of the tetrahedra in \( T \) by \( \delta_i : \Delta \to M \setminus \bar{L} \) (\( i = 1, \ldots, t_C(M) \)). Let \( f : M \setminus \bar{N}(L) \to M \) be the degree 1 map obtained by crushing each torus of \( \partial(M \setminus \bar{N}(L)) \) to a circle\(^1\). Denote \( f \circ \delta_i : \Delta \to M \) by \( f_i \) (\( i = 1, \ldots, t_C(M) \)).

With the notation as in the previous paragraph, we now prove the third inequality. Note that \( \Sigma_{i=1}^n f_i \) represents a generator of \( H_3(M) \cong \mathbb{Z} \). Let \( \tilde{f}_i \) be a lift of \( f_i \) to the universal cover \( \bar{H}^3 \). We construct a map \( \tilde{f}_i : \delta \to \bar{H}^3 \) by “pulling \( \tilde{f} \) tight”\(^2\). Note that \( \tilde{f}_i \) is homotopic to \( \bar{f}_i \); denote this homotopy by \( \bar{F}_{i,t}(p) \). Since \( \Sigma_{i=1}^n f_i \) defines an element of \( H_3(M) \), faces must cancel in pairs. Let \( F \) and \( F' \) be such a pair, that is, \( f_i(F) = -\bar{f}_j(F') \), and let \( p \in F \) and \( p' \in F' \) be corresponding points, that is, points with the same barycentric coordinates. By preforming the homotopy at constant speed we obtain, for any \( t \in [0, 1] \):

\[
\pi \circ \bar{F}_{i,t}(p) = \pi \circ \bar{F}_{j,t}(p').
\]

Here \( \pi \) is the universal cover projection. This implies that for any \( t \in [0, 1] \), \( \Sigma_{i=1}^{t_{C(M)}} \pi \circ \bar{F}_{i,t}(p) \) is homologous to \( \Sigma_{i=1}^n f_i \) and therefore represents a generator of \( H_3(M) \). For \( t = 1 \), we see that \( \Sigma_{i=1}^{t_{C(M)}} \pi \circ \bar{f}_i \) represents a generator for \( H_3(M) \); in particular, every point of \( M \) is in the image of at least one \( \pi \circ \bar{f}_i \). Hence the sum of the volumes of the images of \( \pi \circ \bar{f}_i \) is no less than \( \text{Vol}(M) \). Using this, the fact that volumes do not increase under \( \pi \), and the fact that the volume of a hyperbolic tetrahedron is less than \( v_3 \) we get:

\[
\text{Vol}(M) \leq \Sigma_{i=1}^{t_{C(M)}} \text{Vol}(\pi \circ \bar{f}_i) \\
\leq \Sigma_{i=1}^{t_{C(M)}} \text{Vol}(\bar{f}_i) \\
< t_{C(M)} v_3.
\]

The third inequality follows. \( \square \)

The proposition below is the key to the proof of Theorem \( \text{[1.1]} \) and is very useful in its own right. For this proposition we need the following notation, that we will use throughout this paper. Fix a Margulis constant \( \mu > 0 \) and \( d > 0 \), and let \( R > 0 \) be as in Proposition \( \text{[1.2]}(1) \). We define \( D = \min\{R, d\} \). A set \( A \) in a metric space is called \( D \)-separated if for any \( p, q \in A, p \neq q \), we have that \( d(p, q) > D \). Fix \( \{x_1, \ldots, x_N\} \subset X \) a generic set of \( N \) points (\( a \)-priori \( N \) may be infinite) fulfilling the following conditions:

1. \( \{x_1, \ldots, x_N\} \subset X \) is \( D \)-separated.
2. \( \{x_1, \ldots, x_N\} \) is maximal (with respect to inclusion) subject to this constraint.

Let \( V_1, \ldots, V_N \) be the Voronoi cells in \( M \) corresponding to \( \{x_1, \ldots, x_N\} \), that is,

\[
V_i = \{p \in M \mid d_M(p, x_i) \leq d_M(p, x_j) \text{ (} j = 1, \ldots, N) \}.
\]

\(^1\)That is, \( f \) is obtained as following: foliate each component of \( \partial M \setminus \bar{N}(L) \) by circles, where each leaf is isotopic to a meridian on \( M \). Then \( f \) is obtained by identifying each leaf to a point.

\(^2\)By “pulling \( \tilde{f} \) tight” we mean: for \( p \in \Delta \) a vertex, then \( \tilde{f}(p) = \tilde{f}(p) \). Next, for a general point \( p \in \Delta \), \( \tilde{f}(p) \) is the unique point of \( \bar{H}^3 \) that has the same barycentric coordinates as \( p \) (for more on barycentric coordinate see, for example, \( \text{[4]} \) Page 103)).
We emphasize that a-priori a Voronoi cell need not be “nice”; for example, it need not be a ball and may have infinite diameter. Consider the following simple example: given any metric space and a single point in it, the Voronoi cell corresponding to that point is the entire space.

**Proposition 1.4.** With the notation of Theorem 1.1 there exists a constant $C = C(\mu, d)$ so that the following holds:

1. $M$ is decomposed into $N \leq CV_{\text{ol}}(M)$ Voronoi cells.
2. $V_i \cap X$ is triangulated using at most $C$ tetrahedra ($i = 1, \ldots, N$) ($V_i \cap X$ may not be connected).
3. For any $i, i'$ ($i, i' = 1, \ldots, N$), the triangulations of $V_i \cap X$ and $V_i' \cap X$ given in (2) above coincide on $(V_i \cap X) \cap (V_i' \cap X)$.

We note that Theorem 1.1 follows easily from Proposition 1.4 by setting $K = C^2$.

**Structure of this paper.** In Section 2 we cover some basic preliminary notion. In Section 3 we describe the decomposition of $M$ into Voronoi cells. In an attempt to make this paper self-contained and accessible to all we provide proofs for many elementary facts about Voronoi cells. In Section 4 we study the intersection of the Voronoi cells with $X$. In Section 5 we prove Proposition 1.4.

**Strategy.** As mentioned, our approach is based on Thurston’s original work. However, as discussed in [1] pp 190–192, to make this work requires control over $V_i \cap X$. We now briefly explain our strategy for obtaining this control.

We first decompose $M$ into the $N$ Voronoi cells described above. An easy volume argument shows that $N$ is bounded above linearly in terms of the volume of $M$. We then show the following:

1. Every component of $V_i \cap X$ is a handlebody, as we will show it deformation retracts onto a surface contained in $\partial X$.
2. There is a universal bound on the number of components of $V_i \cap X$.
3. There is a universal bound on the genus of each component of $V_i \cap X$.

We obtain a certain cell decomposition of $V_i \cap X$.

Of course it is possible to triangulate $V_i \cap X$ with a bounded number of tetrahedra, but that is not quite enough: the triangulations must agree on intersection in order to yield a triangulation of $X$. (Consider a lens space: it is the union of two solid tori, but as there are infinitely many distinct lens spaces, they require arbitrarily many tetrahedra.) We triangulate $V_i \cap X$ in a way that agrees on intersections using the cell decomposition mentioned in the previous paragraph.

To get a bound on the number of tetrahedra, we observe that the faces of the cell decomposition mentioned above are totally geodesic. This is used to bound the number of vertices, which turns out to be the key for bounding the number of tetrahedra in our setting.

By contrast, when considering the cell decomposition of the lens space $L(p, q)$ obtained by taking two solid tori and a meridian disk for each, the number of vertices is not bounded; it equals the number of intersections between the disks, which is $p$. 
A note on notation. Objects in $\mathbb{H}^3$ are denoted using tilde (for example, $\tilde{s}$ or $\tilde{C}$) or using script lettering (for example, $\mathcal{A}$). Constants denoted by $C$ are universal (namely, $C$ as defined in Proposition 1.4, $C_3$ as define in Lemma 3.1, $C_2$ as define in Lemma 4.3, $C_1$ as define in Lemma 4.4, $C_0$ as define in Lemma 4.5, and $\bar{C}_0$ as defined in Section 5). Once defined they are fixed for the remainder of the paper. The constants $\mu$, $d$, $R$, and $D$ that were introduced in This section are fixed throughout this paper.

Acknowledgment. We have benefitted from conversations and correspondences about Theorem 1.1 with many experts and we are grateful to them all. In particular, we thank Colin Adams, Joseph Maher, and Sadayoshi Kojima. We thank the anonymous referee for a careful reading of this paper and many helpful remarks.

2. PRELIMINARIES

The notation of Section 1 is fixed for the remainder of this paper.

We assume familiarity with hyperbolic space $\mathbb{H}^3$ and its isometries, as well as the Margulis lemma. The model of $\mathbb{H}^3$ we use is upper half space. Given $\tilde{x}$, $\tilde{y} \in \mathbb{H}^3$, we denote the closed geodesic segment connecting them by $[\tilde{x}, \tilde{y}]$. All manifolds considered are assumed to be orientable. In a metric space, $N_d(\cdot)$ denotes the set of all points of distance at most $d$ from a given object. The ball of radius $r$ centered at $x$ is denoted $B(x, r)$. The volume of a ball of radius $r$ in $\mathbb{H}^3$ is denoted by $\text{Vol}(B(r))$. We use the notation $\text{int}(\cdot)$ and $\text{cl}(\cdot)$ for interior and closure.

We fix $\mu > 0$ a Margulis constant for $\mathbb{H}^3$. By hyperbolic manifold $M$ we mean a complete, finite volume Riemannian 3-manifold locally isometric to $\mathbb{H}^3$. The universal covering of a hyperbolic manifold $M$ is denoted $\pi : \mathbb{H}^3 \to M$; $\pi$ is called the universal cover projection, or simply the projection, from $\mathbb{H}^3$ to $M$. The thick part of $M$ is

$$M_{[\mu, \infty)} = \{ p \in M | \text{inj}_M(p) \geq \mu \}.$$ 

The thin part of $M$ is

$$M_{(0, \mu]} = \text{cl}\{ p \in M | \text{inj}_M(p) < \mu \} = \text{cl}(M \setminus M_{[\mu, \infty)}).$$

It is well known that $M = M_{(0, \mu]} \cup M_{[\mu, \infty)}$, $M_{(0, \mu]}$ is a disjoint union of closed solid torus neighborhood of short geodesics and closed cusps, and $M_{(0, \mu]} \cap M_{[\mu, \infty)}$ consists of tori.

3. VORONOI CELLS

Keep all notation as in the previous sections, and recall that $N$ was the number of Voronoi cells and $D = \min\{R, \bar{d}\}$. Since $\{x_1, \ldots, x_N\}$ was chosen generically, we may assume that the Voronoi cells $\{V_i\}$ are transverse to each other and to $\partial X$ (note that the Voronoi cells are a decomposition of $M$, not $X$, and $\partial X \subset \text{int}(M)$). In the remainder of the paper, all our constructions are generic and allow for small perturbation, and we always assume transversality (usually without explicit mention). We bound $N$ in term of the volume of $M$:
Lemma 3.1. There exists a constant $C_3$ so that $N \leq C_3 \text{Vol}(M)$.

Proof. For each $i$, $x_i \in X$, and hence by Proposition 1.2, $\text{inj}_M(x_i) > R \geq D$. Since $\{x_1,\ldots,x_N\}$ is $D$-separated, for $i \neq j$, $d(x_i,x_j) > D$. Hence $\{B(x_i,D/2)\}_{i=1}^N$ is a set of balls disjunctly embedded in $M$, each of volume $\text{Vol}(B(D/2))$. Thus $N \leq \text{Vol}(M)/\text{Vol}(B(D/2))$; the lemma follows by setting

$$C_3 = 1/\text{Vol}(B(D/2)).$$

The preimages of $\{x_1,\ldots,x_N\}$ in $\mathbb{H}^3$ gives rise to a Voronoi cell decomposition of $\mathbb{H}^3$ in a similar manner to the cells in $M$. It is convenient to fix one of these cells for each $i$:

Notation 3.2. 

1. For each $i$, fix a preimage of $x_i$, denoted $\tilde{x}_i$.
2. $\tilde{V}_i$ is the Voronoi cell corresponding to $\tilde{x}_i$, that is:

$$\tilde{V}_i = \{ \tilde{p} \in \mathbb{H}^3 | d(\tilde{p}, \tilde{x}_i) \leq d(\tilde{p}, \tilde{q}), \forall \tilde{q} \text{ so that } \pi(\tilde{q}) \in \{x_1,\ldots,x_N\}. \}$$

3. For each $i$, the components of $V_i \cap X$ are denoted by $V_{i,j}$ ($j = 1,\ldots,n_i$), where $n_i$ is the number of the components of $V_i \cap X$.
4. The preimage of $V_{i,j}$ in $\tilde{V}_i$ is denoted $\tilde{V}_{i,j}$, that is:

$$\tilde{V}_{i,j} = \{ \tilde{p} \in \tilde{V}_i | \pi(\tilde{p}) \in V_{i,j} \}.$$

Lemma 3.3. If $\tilde{p}, \tilde{p}' \in \tilde{V}_i$ project to the same point $p \in V_i$ then $d(\tilde{p}, \tilde{x}_i) = d(\tilde{p}', \tilde{x}_i)$.

Proof. Let $\tilde{p}, \tilde{p}'$ be points in $\tilde{V}_i$ that project to the same point and assume that $d(\tilde{x}_i, \tilde{p}) \neq d(\tilde{x}_i, \tilde{p}')$; say $d(\tilde{x}_i, \tilde{p}) < d(\tilde{x}_i, \tilde{p}')$. Since $\tilde{p}$ and $\tilde{p}'$ project to the same point, there is an isometry $\phi \in \pi_1(M)$ so that $\phi(\tilde{p}) = \tilde{p}'$. Let $\tilde{x}'_i = \phi(\tilde{x}_i)$, for some $\tilde{x}'_i \in \pi^{-1}(x_i)$. Since $\phi$ acts freely $\tilde{x}'_i \neq \tilde{x}_i$. We get: $d(\tilde{x}'_i, \tilde{p}') = d(\phi^{-1}(\tilde{x}'_i), \phi^{-1}(\tilde{p}')) = d(\tilde{x}_i, \tilde{p})$. Hence $\tilde{p}' \not\in \tilde{V}_i$, contradicting out assumption. The lemma follows.

In general, the distance between points in $V_i$ may be smaller than the distance between their preimages in $\tilde{V}_i$. However this is not the case when one of the points is $x_i$:

Lemma 3.4. For any $\tilde{V}_i$ and any $\tilde{p} \in \tilde{V}_i$, $d(\tilde{x}_i, \tilde{p}) = d(x_i, p)$ (here $p = \pi(\tilde{p})$).

Proof. Of all paths from $x_i$ to $p$ in $M$, let $\beta$ be one that minimizes length (note that $\beta$ need not be unique). First we claim that $\beta \subset V_i$. Suppose, for a contradiction, that this is not the case and let $q \in \beta$ be a point not in $V_i$. Then for some $j \neq i$, $d(q, x_j) < d(q, x_i)$. By connecting the shortest path from $p$ to $q$ to the shortest path from $q$ to $x_j$ we obtain a path strictly shorter than $\beta$, showing that $d(p, x_j) < l(\beta) = d(p, x_i)$. Thus $p \not\in V_i$, a contradiction. Hence $\beta \subset V_i$.

Let $\tilde{\beta}$ be the lift of $\beta$ to $\mathbb{H}^3$ starting at $\tilde{x}_i$. Then $\tilde{\beta}$ is a geodesic segment, say $[\tilde{x}_i, \tilde{p}']$, for some $\tilde{p}'$ that projects to $p$. Fix $\tilde{q} \in \pi^{-1}(x_1),\ldots,\pi^{-1}(x_n)$. Then $[\tilde{p}', \tilde{q}]$ projects to $\pi([\tilde{p}', \tilde{q}])$, a path that connects $p$ to some point of $\{x_1,\ldots,x_n\}$. By choice of $\beta$ (and since paths have the same length as their projections), $d(\tilde{x}_i, \tilde{p}') = l(\tilde{\beta}) = l(\beta) \leq l(\pi([\tilde{p}', \tilde{q}])) = l([\tilde{p}', \tilde{q}]) = d(\tilde{p}', \tilde{q})$. We conclude that $\tilde{p}' \in \tilde{V}_i$. 

□
We see that \( d(x_i, p) = l(\beta) = l(\tilde{\beta}) = d(\tilde{x}_i, \tilde{p}) \). Since \( \tilde{p}, \tilde{p}' \in \tilde{V}_i \), by Lemma 3.3 \( d(\tilde{x}_i, \tilde{p}) = d(\tilde{x}_i, \tilde{p}') \); the lemma follows.

A **convex polyhedron** is the intersection of half spaces in \( \mathbb{H}^3 \). Note that a convex polyhedron is not required to be of bounded diameter or finite sided (that is, the intersection of finitely many half spaces).

**Lemma 3.5.** \( \tilde{V}_i \) is a convex polyhedron that projects onto \( V_i \)

**Proof.** It is immediate that \( \tilde{V}_i \) is a convex polyhedron.

Given any \( \tilde{p} \in \tilde{V}_i \), \([\tilde{x}_i, \tilde{p}]\) is the shortest geodesic from \( \tilde{p} \) to any preimage of \( \{x_1, \ldots, x_N\} \). The projection of \([\tilde{x}_i, \tilde{p}]\) is the shortest geodesic from the projection of \( \tilde{p} \) to \( \{x_1, \ldots, x_N\} \). It follows easily that \( \pi(\tilde{p}) \in V_i \). As \( \tilde{p} \) was an arbitrary point of \( \tilde{V}_i \), we see that \( \tilde{V}_i \) projects into \( V_i \).

Conversely, given any \( p \in V_i \), let \( \beta \) be the shortest geodesic from \( \{x_1, \ldots, x_N\} \) to \( p \). Then \( \beta \) connects \( x_i \) to \( p \). Let \( \tilde{\beta} \) be the unique lift of \( \beta \) that starts at \( \tilde{x}_i \), and denote its terminal point by \( \tilde{p} \). Similar to the argument of the proof of Lemma 3.4 \( \tilde{\beta} \) is the shortest geodesic connecting any preimage of \( \{x_1, \ldots, x_N\} \) to \( \tilde{p} \), showing that \( \tilde{p} \in \tilde{V}_i \). Hence \( p \) is in the image is \( \tilde{V}_i \); As \( p \) was an arbitrary point of \( V_i \), we see that \( \tilde{V}_i \) projects onto \( V_i \).

**Decomposition of \( V_i \).** By Lemma 3.5 the boundary of \( \tilde{V}_i \) is decomposed into faces, edges and vertices. By the same lemma, it projects into \( V_i \). The images of this faces, edges and vertices from the decomposition of \( V_i \) that is the basis for our work in the next section. Note that some faces of \( \tilde{V}_i \) are identified, and the corresponding faces of \( V_i \) are contained in the interior, not boundary, of \( V_i \). (We will show in Lemma 4.9 (3) that faces in the interior of \( V_i \) are contained in \( M \setminus X \), and they will play no role in our construction.) We remark that this is not the final decomposition: in the next section we will add more faces, edges and vertices to the decomposition.

### 4. Decomposing \( X \)

We first define:

**Definition 4.1.** Fix \( r > 0 \) and a geodesic \( \tilde{\gamma} \subset \mathbb{H}^3 \). Let \( \tilde{C} = \{p \in \mathbb{H}^3 | d(p, \tilde{\gamma}) \leq r \} \). We call \( \tilde{C} \) a **cone** about \( \gamma \), or simply a **cone** and \( \tilde{\gamma} \) the **axis** of \( \tilde{C} \). The set \( \{p \in \mathbb{H}^3 | d(p, \gamma) \geq r \} \) is called the **exterior** of \( \tilde{C} \), denoted \( \tilde{E} \).

The reason we look at cones is that if \( V \) is a solid torus component of \( \cl(M \setminus X) \) and \( \gamma \) its core geodesic, then \( \pi^{-1}(V) \) is a cone and \( \pi^{-1}(\gamma) \) its axis. It can be seen directly that the intersection of a geodesic and a cone is (a possibly empty) connected set; hence cones are convex. If \( V \) is a cusp component of \( \cl(M \setminus X) \), then its preimage is a horoball which is also convex. Below, we often use the fact that the every component of the preimage of \( \cl(M \setminus X) \) is convex.\(^3\)

\(^3\)In the upper half space model, if \( \tilde{\gamma} \) is a Euclidean vertical straight ray from \( \tilde{p}_\infty \) in the \( xy \)-plane, then \( \tilde{C} \) is the cone of all Euclidean straight rays from \( \tilde{p}_\infty \) that form angle at most \( \alpha \) (for some \( \alpha \)) with \( \tilde{\gamma} \). If \( \tilde{\gamma} \) is a semicircle then \( \tilde{C} \) looks more like a banana.
Lemma 4.2. For any $i$, $V_i \cap \text{cl}(M \setminus X)$ is connected.

Proof. The number of times $V_i$ intersects $\text{cl}(M \setminus X)$ is at most the number of times $\widetilde{V}_i$ intersects the preimage of $\text{cl}(M \setminus X)$. Since $\widetilde{V}_i$ and any component of the preimage of $\text{cl}(M \setminus X)$ are both convex, their intersection is connected. Thus all we need to show is that $\widetilde{V}_i$ intersects only one component of the preimage of $\text{cl}(M \setminus X)$.

Suppose this is not the case, and let $\tilde{\alpha}$ be the shortest arc in $\widetilde{V}_i$ that connects distinct components of the preimage of $\text{cl}(M \setminus X)$. Since $\widetilde{V}_i$ is convex, $\tilde{\alpha}$ is a geodesic. Since $\tilde{\alpha}$ connects distinct components of the preimage of $\text{cl}(M \setminus X)$, some point on $\tilde{\alpha}$ projects into $M_{[\mu, \infty)}$. Let $\alpha = \pi(\tilde{\alpha})$. Recall that the distance from $\partial X$ to $M_{[\mu, \infty)}$ is $d$. We conclude that $l(\tilde{\alpha}) = l(\alpha) > 2d \geq 2D$. Thus the distance between the endpoints of $\tilde{\alpha}$ is greater than $2D$, and by the triangle inequality, for some point $\tilde{p} \in \tilde{\alpha}$, $d(\tilde{x}_i, \tilde{p}) > D$. Let $p$ be the image of $\tilde{p}$. By Lemma 3.4, $d(x_i, p) = d(\tilde{x}_i, \tilde{p})$ and by Lemma 3.5 $p \in V_i$. Hence by construction of the Voronoi cells, for any $j$, $d(x_j, p) \geq d(x_i, p) > D$. Thus $\{x_1, \ldots, x_N\} \subset X$ is a $D$-separated set, contradicting maximality of $\{x_1, \ldots, x_N\}$. The lemma follows. \hfill \Box

In the next lemma we bound the number of faces of $V_i$ that intersect $X$ and study that intersection:

Lemma 4.3. The following two conditions hold:

1. There exists a constant $C_2$ so that for every $i$, $1 \leq i \leq N$, the number of faces of $V_i$ that intersect $X$ is at most $C_2$.
2. For each $i$ and every face $F$ of $V_i$, $F \cap X$ is either empty, or a single annulus, or a collection of disks.

Proof. Let $V_i$ be a Voronoi cell, $F$ a face of $V_i$ so that $F \cap X \neq \emptyset$, and $p \in F \cap X$. Let $\tilde{p} \in \widetilde{V}_i$ be a preimage of $p$ ($\tilde{p}$ exists by Lemma 3.5) and let $\tilde{F}$ be a face of $\widetilde{V}_i$ containing $\tilde{p}$. Let $\tilde{x}$ be the preimage of $\{x_1, \ldots, x_N\}$ that is contained in the cell adjacent to $\tilde{F}$ on the opposite side from $\widetilde{V}_i$. By Lemma 3.4, $d(\tilde{x}_i, \tilde{p}) = d(x_i, p)$. Similar to the argument of the proof of Lemma 4.2, maximality of $\{x_1, \ldots, x_N\}$ implies that $d(x_i, p) < D$. We conclude that $d(\tilde{x}_i, \tilde{p}) < D$, and similarly $d(\tilde{x}, \tilde{p}) < D$. By the triangle inequality, $d(\tilde{x}_i, \tilde{x}) < 2D$.

For each face $F$ of $V_i$ with $F \cap X \neq \emptyset$, consider the cell adjacent to $\widetilde{V}_i$ along $\tilde{F}$ as constructed above. The balls of radius $D/2$ centered at the preimages of $\{x_1, \ldots, x_N\}$ in these cells are disjointly embedded and their centers are no further than $2D$ from $\tilde{x}_i$, so these balls are contained in $B(\tilde{x}_i, 2.5D)$. Thus (1) follows by setting

$$C_2 = \frac{\text{Vol}(B(2.5D))}{\text{Vol}(B(D/2))}.$$ 

For (2), fix $V_i$ and $F$ a face of $V_i$. Let $\tilde{F}$ be the face of $\widetilde{V}_i$ that projects to $F$. Since $\widetilde{V}_i$ is a convex polyhedron, $\tilde{F}$ is a totally geodesic convex polygon. By Lemma 4.2 $\tilde{F}$ intersects at most one component of the preimage of $M \setminus X$, and by convexity of that component and of $\tilde{F}$, the intersection is either empty or a disk. We see that one of the following holds:

1. When the intersection is empty: then the intersection of $\tilde{F}$ with the preimage of $X$ is $\tilde{F}$ (and hence a disk).
(2) When the intersection is a disk contained in \( \text{int}(\bar{F}) \): then the intersection of \( \bar{F} \) with the preimage of \( X \) is an annulus.

(3) When the intersection is a disk not contained in \( \text{int}(\bar{F}) \): then the intersection of \( \bar{F} \) with the preimage of \( X \) is a collection of disks.

We claim that the intersection of \( \bar{F} \) with the preimage of \( X \) projects homeomorphically onto its image. Otherwise, there are two points \( \bar{p}_1, \bar{p}_2 \in \bar{F} \) that project to the same point \( p \in F \cap X \). By maximality of \( \{x_1, \ldots, x_N\} \), \( d(x_i, p) < D \). By Lemma 3.4, \( d(\bar{x}_i, \bar{p}_1), d(\bar{x}_i, \bar{p}_2) = d(x_i, p) \). The shortest path from \( \bar{p}_1 \) to \( \bar{p}_2 \) that goes through \( \bar{x}_i \) projects to an essential closed path that contains \( x_i \) and has length less than \( 2D \). But then \( \text{inj}_M(x_i) < D \leq R \), contradicting Proposition 1.2 (1). Thus the intersection of the preimage of \( X \) with \( \bar{F} \) projects homeomorphically and (2) follows.

We consider the intersection of an edge \( e \) of \( V_i \) with \( X \). We call the components of \( e \cap X \) segments. In the next lemma we bound the number of segments:

**Lemma 4.4.** There exists a constant \( C_1 \) so that for every \( i, 1 \leq i \leq N \), the number of segments from the intersection of edges of \( V_i \) with \( X \) is at most \( C_1 \).

**Proof.** Fix \( i \) and \( e \) an edge of \( V_i \). We first show that \( e \) contributes at most two segment. If \( e \subset X \) then it contributes exactly one segment and if \( e \cap X = \emptyset \) then it contributes no segment. Otherwise, let \( \bar{e} \) be a lift of \( e \) that is in \( \bar{V}_i \). By Lemma 4.2, \( \bar{e} \) intersects at most one component of the preimage of \( \text{cl}(M \setminus X) \). Since \( \bar{e} \) and any component of the preimage of \( \text{cl}(M \setminus X) \) are both convex, their intersection is convex and hence connected. Thus the intersection of \( \bar{e} \) and the preimage of \( \text{cl}(M \setminus X) \) is connected, and projecting to \( M \) we see that the intersection of \( e \) and \( \text{cl}(M \setminus X) \) is connected as well. Thus \( e \) contributes at most 2 segments.

Since \( \bar{V}_i \) is convex, the intersection of 2 faces of \( \partial \bar{V}_i \) is at most one edge. Hence the number of edges is bounded above by the the number of pairs of faces, \( \frac{1}{2}C_2(C_2 - 1) \) (using Lemma 4.3). The number of edges of \( V_i \) is no larger; Lemma 4.4 follows by setting

\[
C_1 = C_2(C_2 - 1).
\]

**Lemma 4.5.** There exists a constant \( C_0 \) so that for every \( i, 1 \leq i \leq N \), the number of vertices of \( V_i \) that lie in \( X \) is at most \( C_0 \).

**Proof.** Each segment contributes at most 2 vertices. Lemma 4.5 follows by setting

\[
C_0 = 2C_1.
\]

**Definition 4.6.** Let \( \bar{C} \) be a cone, \( \bar{\gamma} \) its axis, and \( \bar{E} \) its exterior (recall Definition 4.1). Fix \( \bar{s} \in \bar{C} \). We say that a set \( \bar{K} \subset \bar{E} \) is \( \bar{s} \)-convex if for any \( \bar{p} \in \bar{K}, [\bar{p}, \bar{s}] \cap \bar{E} \) is contained in \( \bar{K} \).
Lemma 4.7. Let $\tilde{K} \subset \tilde{E}$ be an $\tilde{s}$-convex set (for some $\tilde{s} \in \tilde{C}$). Then there exists a deformation retract from $\tilde{K}$ onto $\tilde{K} \cap \partial \tilde{E}$.

Proof. Fix $\tilde{p} \in \tilde{K}$. Since $\tilde{C}$ is convex, $[\tilde{p}, \tilde{s}] \cap \tilde{C}$ is an interval, say $[\tilde{r}, \tilde{s}]$, and $\tilde{E} \cap [\tilde{r}, \tilde{s}] = [\tilde{p}, \tilde{r}]$. Since $\tilde{K}$ is $\tilde{s}$-convex, $[\tilde{p}, \tilde{r}]$ is $\tilde{K}$. We move $\tilde{p}$ along $[\tilde{p}, \tilde{r}]$ from its original position to $\tilde{r} \in \tilde{K} \cap \partial \tilde{E}$ in constant speed. It is easy to see that this is a deformation retract.

Notation 4.8. With the notation of the previous lemma, we define $f : \tilde{K} \rightarrow \tilde{K} \cap \partial \tilde{E}$ to be $f(\tilde{p}) = \tilde{r}$.

Recall the definition of $\tilde{V}_i, V_{i,j}$, and $\tilde{V}_{i,j}$ from Notation 3.2. Recall also that $n_i$ was the number of components of $V_i \cap X$ from Notation 3.2 (3).

Lemma 4.9. The following conditions hold:

1. For each $i$, $n_i \leq C_0$.
2. For each $i, j$, if $\tilde{V}_{i,j} \neq V_{i,j}$, there is a cone $\tilde{C}$, a component of the preimage of $\text{cl}(M \setminus X)$, so that $\tilde{V}_{i,j}$ is $\tilde{s}$ convex for any point $\tilde{s} \in \tilde{V}_{i,j} \cap \tilde{C}$.
3. The projection of $\bigcup_{j=1}^{n_i} \tilde{V}_{i,j}$ to $\bigcup_{j=1}^{n_i} V_{i,j}$ is a diffeomorphism.
4. For each $i, j$, $V_{i,j}$ is a handlebody.

Proof. Fix $i$. It is easy to see that each component of $V_i \cap X$ must contain a vertex of $V_i$. Applying Lemma 4.5, we see that there are at most $C_0$ such components. This establishes (1).

Any component of the preimage of $\text{cl}(M \setminus X)$ is a cone. Assuming that $\tilde{V}_{i,j} \neq V_{i,j}$, by Lemma 4.2, there exists a unique such component, say $\tilde{C}$, that intersects $\tilde{V}_i$. Fix a point $\tilde{s} \in \tilde{C} \cap \tilde{V}_i$. Fix $\tilde{p} \in \tilde{V}_{i,j}$. Convexity of $\tilde{V}_i$ implies that $[\tilde{p}, \tilde{s}] \subset \tilde{V}_i$. Convexity of $[\tilde{p}, \tilde{s}]$ and $\tilde{C}$ implies that $[\tilde{p}, \tilde{s}] \cap \tilde{C}$ is an interval, say $[\tilde{r}, \tilde{s}]$, and therefore $[\tilde{p}, \tilde{s}] \cap (\tilde{V}_i \setminus \text{int} \tilde{C}) = [\tilde{p}, \tilde{r}]$. Since $\tilde{V}_{i,j}$ is connected, $[\tilde{p}, \tilde{s}] \cap \tilde{V}_{i,j} = [\tilde{p}, \tilde{r}] \subset \tilde{V}_{i,j}$. This establishes (2).

For (3), it is easy to see that all we need to show is that the projection $\bigcup_{j=1}^{n_i} \tilde{V}_{i,j} \rightarrow \bigcup_{j=1}^{n_i} V_{i,j}$ is one-to-one. Assume not (this is similar to Lemma 4.3 (2)); then there exist $\tilde{p}_1, \tilde{p}_2 \in \bigcup_{j=1}^{n_i} \tilde{V}_{i,j}$ that project to the same point $p \in \bigcup_{j=1}^{n_i} V_{i,j}$. Then the shortest path from $\tilde{p}_1$ to $\tilde{p}_2$ that goes through $\tilde{x}_i$ projects to an essential closed path that contains $x_i$, and has length less than $2D$. But then $\text{inj}_M(x_i) < D < R/2$, contradicting Proposition 1.2 (1). This establishes (3).

If $V_i \subset X$ then $V_i \cap X$ is a ball and (4) follows. Otherwise, (4) follows from (2), Lemma 4.7, and (3).

We denote $V_{i,j} \cap \partial X$ by $P_{i,j}$. We bound $g(V_{i,j})$, the genus of $V_{i,j}$:

Lemma 4.10. $g(V_{i,j}) \leq C_1$.

Proof. If $V_{i,j} = V_i$ then it is a ball and there is nothing to show. Assume this is not the case. Then by Lemmas 4.9 and 4.7, $V_{i,j}$ deformation retracts onto $P_{i,j}$. Hence $\text{cl}(\partial V_{i,j} \setminus P_{i,j})$ is homeomorphic to $P_{i,j}$, and is a $g(V_{i,j})$-times punctured disk. The faces of $V_i$ induce a decomposition on $\text{cl}(\partial V_{i,j} \setminus P_{i,j})$. By Lemma 4.3 (2), each face of $\text{cl}(\partial V_{i,j} \setminus P_{i,j})$ is a disk or an annulus; in particular the Euler characteristic
of each such component is non-negative. Denote the faces of \( \text{cl}(\partial V_{i,j} \setminus P_{i,j}) \) by \( \{F_k\}_{k=1}^{k_0} \), the number of edges by \( e \), and the number of vertices by \( v \). Note further, that the edges of \( \text{cl}(\partial V_{i,j} \setminus P_{i,j}) \) come in two types, edges in the interior of \( \text{cl}(\partial V_{i,j} \setminus P_{i,j}) \) (say \( e_{\text{int}} \) of them) and edges on its boundary (say \( e_{\partial} \) of them).

Similarly, \( v_{\text{int}} \) (resp. \( v_{\partial} \)) denotes the number of vertices in the interior (resp. boundary) of \( \text{cl}(\partial V_{i,j} \setminus P_{i,j}) \).

Since the boundary of \( \text{cl}(\partial V_{i,j} \setminus P_{i,j}) \) consists of circles, \( e_{\partial} = v_{\partial} \). Since \( e_{\text{int}} \) is the number of segments on \( V_{i,j} \cap X \), by Lemma 4.4, \( e_{\text{int}} \leq C_1 \). An Euler characteristic calculation gives:

\[
1 - g(V_{i,j}) = \chi(\text{cl}(\partial V_{i,j} \setminus P_{i,j}))
= (\sum_{k=1}^{k_0} \chi(F_k)) - e + v
= (\sum_{k=1}^{k_0} \chi(F_k)) - (e_{\text{int}} + e_{\partial}) + (v_{\text{int}} + v_{\partial})
= (\sum_{k=1}^{k_0} \chi(F_k)) - e_{\text{int}} + v_{\text{int}}
\geq -e_{\text{int}} + 1
\geq -C_1 + 1.
\]

The lemma follows.

We use the notation \( \partial_{\infty} \) for the limit points at infinity. In particular, if \( \tilde{L} \) is a totally geodesic plane then \( \partial_{\infty} \tilde{L} \) is a simple closed curve and if \( \tilde{C} \) is a cone then \( \partial_{\infty} \tilde{C} \) consists of two points.

**Lemma 4.11.** Let \( \tilde{C} \) be a cone, \( \tilde{s} \in \tilde{C} \) a point not on the axis of \( \tilde{C} \). Let \( \tilde{B} \subset \partial\tilde{C} \) be an arc and \( \tilde{p} \in \tilde{B}, \tilde{q} \in \partial\tilde{C} \) points. Assume that the totally geodesic plane that contains \( \tilde{s} \) and \( \partial_{\infty}\tilde{C} \) intersects \( \tilde{B} \) in a finite set of points.

Then after an arbitrarily small perturbation of \( \tilde{p} \) in \( \tilde{B} \), there exists a closed arc \( \tilde{a} \subset \partial\tilde{C} \) connecting \( \tilde{p} \) and \( \tilde{q} \), so that \( \tilde{a} \) and \( \tilde{s} \) are contained in a totally geodesic plane.

**Remark.** The condition on \( \tilde{B} \) is generic, and since we allow for small perturbations in our construction we will always assume it holds.

**Proof.** Let \( \tilde{L} \) be a totally geodesic plane containing \( \tilde{p}, \tilde{q} \), and \( \tilde{s} \). We prove Lemma 4.11 in two cases:

**Case One.** \( \partial_{\infty}\tilde{C} \notin \partial_{\infty}\tilde{L} \). The reader can easily verify that in this case \( \tilde{L} \cap \partial\tilde{C} \) is connected. Then we take the arc \( \tilde{a} \) to be a component of \( \tilde{L} \cap \partial\tilde{C} \) that connects \( \tilde{p} \) to \( \tilde{q} \).

**Case Two.** \( \partial_{\infty}\tilde{C} \subset \partial_{\infty}\tilde{L} \). Equivalently, \( \tilde{\gamma} \subset \tilde{L} \), where \( \tilde{\gamma} \) denotes the axis of \( \tilde{C} \). Since \( \tilde{s} \notin \tilde{\gamma}, \tilde{L} \) is the unique totally geodesic plane that contains both \( \tilde{s} \) and \( \tilde{\gamma} \). By assumption, \( \tilde{B} \cap \tilde{L} \) is a finite set. By perturbing \( \tilde{p} \) slightly in \( \tilde{B} \) we reduce the problem to Case One. The lemma follows.

In the following lemma we construct the main tool we will use for cutting \( V_{i,j} \) into balls. In that lemma, \( \tilde{C} \) is a cone and \( \tilde{s} \in \tilde{C} \) a point. A collection of simple closed curves on \( \partial\tilde{C} \) is called **generic** if it intersects any totally geodesic plane containing \( \tilde{s} \) in a finite collection of points. As remarked after Lemma 4.11, we will always assume it holds.
Lemma 4.12. Let $\tilde{C}$ be a cone, $s \in \tilde{C}$ a point not on the axis of $\tilde{C}$, and $\mathcal{C} \subset \partial \tilde{C}$ a collection of $n + 1$ disjoint generic simple closed curves, for some $n \geq 0$.

Then there exists a graph $\mathcal{A} \subset \partial \tilde{C}$ with the following properties:

1. $\mathcal{A}$ has at most $2n - 1$ edges.
2. For every edge $e$ of $\mathcal{A}$, $e$ and $s$ are contained in a single totally geodesic plane.
3. $\mathcal{C} \cup \mathcal{A}$ is a connected trivalent graph.
4. The graph obtained by removing any edge of $\mathcal{A}$ from $\mathcal{A} \cup \mathcal{C}$ is disconnected.

Proof. We induct on $n$. If $n = 0$ there is nothing to prove.

Assume $n > 0$ and let $\tilde{c}$ be a component of $\mathcal{C}$. Let $\mathcal{C}' = \mathcal{C} \setminus \tilde{c}$. By the induction hypothesis, there exists a graph $\mathcal{A}' \subset \partial \tilde{C}$ with at most $2n - 3$ edges, so that every edge of $\mathcal{A}'$ is contained in a totally geodesic plane that contains $\tilde{s}$, and $\mathcal{C} \cup \mathcal{A}'$ is a connected trivalent graph.

**Case One.** $\tilde{c} \cap (\mathcal{C}' \cup \mathcal{A}') = \emptyset$. Fix $p \in \tilde{c}$ and $q \in \mathcal{C}' \cup \mathcal{A}'$ so that $\tilde{q}$ is not a vertex. Since $\tilde{c}$ is generic, by Lemma 4.11 after a small perturbation of $\tilde{p}$ in $\tilde{c}$, there exists an arc $\tilde{a}'$ connecting $\tilde{p}$ and $\tilde{q}$ so that $\tilde{a}'$ and $\tilde{s}$ are contained in a totally geodesic plane. Since the perturbation was generic we may assume that $\tilde{a}'$ is transverse to $\mathcal{C}' \cup \mathcal{A}' \cup \tilde{c}$.

Let $\tilde{a}$ be a component of $\tilde{a}'$ cut open along the points of $\tilde{c} \cap (\mathcal{C}' \cup \mathcal{A}')$ that connects $\tilde{c}$ to $\mathcal{C}' \cup \mathcal{A}'$. Since the perturbation was generic we may assume that the endpoint of $\tilde{a}$ on $\mathcal{A}' \cup \mathcal{C}'$ is not a vertex, so that $\mathcal{C} \cup \mathcal{A}' \cup \tilde{a}$ is a connected trivalent graph. The lemma follows in Case One by setting $\mathcal{A} = \mathcal{A}' \cup \tilde{a}$.

**Case Two.** $\tilde{c} \cap (\mathcal{C}' \cup \mathcal{A}') \neq \emptyset$. Let $\mathcal{A}''$ be the graph obtained from $\mathcal{A}'$ by adding a vertex at every point of $\mathcal{A}' \cap \tilde{c}$. Note that there is no bound on the number of edges of $\mathcal{A}''$, and the vertices of $\mathcal{C} \cup \mathcal{A}''$ have valence 3 or 4 (the vertices of valence 4 are $\mathcal{A}' \cap \tilde{c}$). Clearly, $\mathcal{C} \cup \mathcal{A}''$ is connected.

**Step One.** Let $\tilde{e}$ be an edge of $\mathcal{A}''$ so that the graph obtained by removing $\tilde{e}$ from $\mathcal{C} \cup \mathcal{A}''$ is connected. We remove $\tilde{e}$.

**Step Two.** Note that as after Step One we may have a vertex, say $\tilde{v}$, of valence 2. Let $\tilde{e}_1$ ($i = 1, 2$) be the other two edges incident to $\tilde{v}$; denote the endpoints of $\tilde{e}_i$ by $\tilde{v}_i$ and $\tilde{v}_i$. We remove $\tilde{e}_1$ and $\tilde{e}_2$. If the graph obtained is disconnected, then it consists of two components, one containing $\tilde{v}_1$ and one containing $\tilde{v}_2$. As in Case One, we construct an arc to connect the two components.

Step Two may produce a new vertex of valence 2. We iterate Step Two. This process reduces the number of edges and so terminates; when it does, we obtain a connected graph with no vertices of valence 2.

We now repeat Step One (if possible). After every application of Step One we repeat Step Two (if necessary). Step One also reduces the number of edges, so it terminates. When it does, we obtain a graph (still denoted $\mathcal{A}''$) so that $\mathcal{C} \cap \mathcal{A}''$ is connected, but removing any edge of $\mathcal{A}''$ disconnects it.

By construction, the vertices of $\mathcal{C} \cup \mathcal{A}''$ have valence 3 or 4. If a vertex has valence 4, we choose an edge adjacent to it from $\mathcal{A}''$. Perturbing the endpoint of this edge and applying Lemma 4.11 we obtain
two vertices of valence 3. We iterate this process. Since this process reduces the number of vertices of valence 4, it will terminate. The graph obtained is denoted $\mathcal{A}$.

By construction, conditions (2), (3) and (4) of Lemma 4.12 hold. All that remains is proving:

**Claim.** $\mathcal{A}$ has at most $2n - 1$ edges.

Proof of claim: Let $\Gamma$ be the graph obtained from $\mathcal{C} \cup \mathcal{A}$ by identifying every component of $\mathcal{C}$ to a single point. Note that the vertex set of $\Gamma$ has $n + 1$ vertices that correspond to the components of $\mathcal{C}$, and extra vertices from the vertices of $\mathcal{A}$ that are disjoint from $\mathcal{C}$; these vertices all have valence 3. The edges of $\Gamma$ are naturally in 1-1 correspondence with the edges of $\mathcal{A}$; thus to prove the claim all we need to show is that $\Gamma$ has at most $2n - 1$ edges.

It is easy to see that $\Gamma$ is connected because $\mathcal{C} \cup \mathcal{A}$ is. Moreover, if there is any edge $e$ of $\Gamma$ so that the graph obtained from $\Gamma$ by removing $e$ is connected, then the graph obtained from $\mathcal{C} \cup \mathcal{A}$ by removing the corresponding edge is connected as well; this contradict our construction. Hence $\Gamma$ is a tree, with $n + 1$ vertices of arbitrary valence, and all other vertices have valence 3. In particular, $\Gamma$ has at most $n + 1$ vertices of vertices of valence 1 or 2. We will use the following claim:

**Claim.** Let $G = (V, E)$ be a finite tree with vertex set $V$ and edge set $E$ and with $k \geq 2$ vertices of valence 1 or 2. Then $G$ has at most $2k - 3$ edges.

We prove the claim by induction on $k$. If $k = 2$, it is easy to see that $G$ is a single edge, and indeed $1 = 2 \cdot 2 - 3$. Assume from now on that $k > 2$.

It is well known that every finite tree has a leaf (that is, a vertex of valence 1). Let $v \in V$ be a leaf and $(v, v') \in E$ the only edge containing $v$. Consider $G' = (V - \{v\}, E - \{(v, v')\})$. There are four cases, depending on the valence of $v'$ as a vertex of $G'$:

1. The valence of $v'$ is zero: then $G$ is a single edge, contrary to our assumption.
2. The valence of $v'$ is one: that is, $v'$ is a leaf of $G'$. Note that in this case both $v$ and $v'$ have valence 1 or 2 in $G$, and we see that $G'$ has exactly $k - 1$ vertices of valence 1 or 2. By induction $G'$ has at most $2(k - 1) - 3 = 2k - 5$ edges. Since $G$ has exactly one more edge than $G'$, $G$ has at most $2k - 4$ edges in this case.
3. The valence of $v'$ is two: In this case, the number of vertices of valence 1 or 2 in $G'$ is exactly $k$ (note that $v'$ has valence 3 in $G$). Let $(v', v'')$ and $(v', v''')$ be the two edges adjacent to $v'$. Let $G''$ be the graph obtained from $G'$ by removing $v'$ from the vertex set and $(v', v''), (v', v''')$ from the edge set, and add the edge $(v'', v''')$. It is easy to see that $G''$ is a tree with exactly $k - 1$ vertices of valence 1 or 2. By induction $G''$ has at most $2(k - 1) - 3$ edges. Since $G'$ has one more edge than $G''$ and $G$ has one more edge than $G'$, $G$ has at most $2k - 3$ edges as desired.
4. The valence of $v'$ is at least three: then $G'$ has exactly $k - 1$ vertices of valence 1 or 2. Similar to the above, we see that $G$ has at most $2k - 4$ vertices in this case.
This proves the claim.

To establish (1), we use the claim and the fact that $\Gamma$ is a tree with at most $n + 1$ vertices of valence 1 or 2, and see that the number of edges in $\Gamma$ is at most $2(n + 1) - 3 = 2n - 1$.

This completes the proof of Lemma \[4.12\]

\[\square\]

Next, we prove the existence of totally geodesic disks that cut $V_{i,j}$ into balls. We note that the disks may not be disjoint. The precise statement is:

**Lemma 4.13.** For any $V_{i,j}$ there exists a 2-complex $K_{i,j} \subset V_{i,j}$ so that the following hold:

1. $V_{i,j}$ cut open along $K_{i,j}$ is a single ball.
2. The faces of $K_{i,j}$ are totally geodesic disks. The edges of $K_{i,j}$ have valence 3.
3. All the vertices are on $\partial V_{i,j}$.
4. $K_{i,j}$ has at most $2C_1 - 1$ faces and $4C_1 - 2$ edges in the interior of $V_{i,j}$.

**Proof.** If $V_{i,j} = V_i$ then it is a ball and there is nothing to prove. Assume this is not the case. Then $V_i \neq V_{i,j}$, and hence $V_i \cap \text{cl}(M \setminus X) \neq \emptyset$; by Lemma \[4.2\] $\tilde{V}_i$ intersects exactly one preimage of $\text{cl}(M \setminus X)$, say $\tilde{C}$. Recall that $\tilde{C}$ is a cone. We first establish conditions analogous to (1)–(4) for $\tilde{V}_{i,j}$.

Let $\tilde{P}_{i,j}$ denote $\tilde{V}_{i,j} \cap \partial \tilde{C}$. By Lemma \[4.7\] $\tilde{V}_{i,j}$ deformation retracts onto $\tilde{P}_{i,j}$; hence $\tilde{P}_{i,j} \subset \partial \tilde{C}$ is a connected, planar surface.

Let $\tilde{s} \in \tilde{V}_{i,j} \cap \tilde{C}$ be a point not on the axis of $\tilde{C}$. The Voronoi cells were constructed around generic points $\{x_i\}$. Therefore, after perturbing $\tilde{s}$ slightly if necessary, $\partial \tilde{P}_{i,j} \subset \partial \tilde{C}$ is a generic collection of circles, and Lemma \[4.12\] applies to give a graph, denoted $\mathcal{A}_{i,j}$, so that $\partial \tilde{P}_{i,j}$ and $\mathcal{A}_{i,j}$ fulfill the conditions of Lemma \[4.12\]. It follows easily from Lemma \[4.12\] (4) that $\mathcal{A}_{i,j} \subset \tilde{P}_{i,j}$.

Set $\mathcal{K}_{i,j}$ to be $f^{-1}(\mathcal{A}_{i,j})$, where the function $f$ is described in Notation \[4.8\]. By construction, for every edge $\tilde{e}$ of $\mathcal{A}_{i,j}$, $f^{-1}(\tilde{e})$ is the intersection of the totally geodesic plane containing $\tilde{e}$ and $\tilde{s}$ with $\tilde{V}_{i,j}$. Since $f$ is a deformation retract, $f^{-1}(\tilde{e})$ is a disk. These disks are the faces of $\mathcal{K}_{i,j}$; thus the faces of $\mathcal{K}_{i,j}$ are totally geodesic disks. By Lemma \[4.12\] $\partial \tilde{P}_{i,j} \cup \mathcal{A}_{i,j}$ is a trivalent graph. The edges of $\mathcal{K}_{i,j}$ correspond to the preimage of vertices of $\mathcal{A}_{i,j}$, and hence have valence 3. This establishes (2) for $\tilde{V}_{i,j}$.

There are 3 type of vertices: vertices of $\mathcal{K}_{i,j}$, intersection of edges of $\mathcal{K}_{i,j}$ with faces of $\tilde{V}_{i,j}$, and intersection of faces of $\mathcal{K}_{i,j}$ with edges of $\tilde{V}_{i,j}$. By construction, $\mathcal{K}_{i,j}$ has no vertices. By Lemma \[4.9\] (3) the faces and edges of $\tilde{V}_{i,j}$ are contained in its boundary. Condition (3) follows.

Denote the genus of $\tilde{V}_{i,j}$ by $n$. Then $|\partial \tilde{P}_{i,j}| = n + 1$. By Lemma \[4.12\] $\mathcal{A}_{i,j}$ has at most $2n - 1$ edges. By construction, each edge of $\mathcal{A}_{i,j}$ corresponds to exactly one face of $\mathcal{K}_{i,j}$. Hence $\mathcal{K}_{i,j}$ has at most $2n - 1$ faces. By Lemma \[4.10\] $n = g(\tilde{V}_{i,j}) \leq C_1$; thus $K_{i,j}$ has at most $2C_1 - 1$ faces. Similarly, every vertex of $\mathcal{A}_{i,j}$ corresponds to exactly one edge of $\mathcal{K}_{i,j}$ in the interior of $\tilde{V}_{i,j}$. Since the number of vertices of $\mathcal{A}_{i,j}$ is at most twice the number of its edges, we see that the number of edges of $\mathcal{K}_{i,j}$ in the interior of $\tilde{V}_{i,j}$ is at most $4C_1 - 2$. This establishes (4) for $\tilde{V}_{i,j}$. 
By construction, the components of $\tilde{V}_{i,j}$ cut open along $\mathcal{K}_{i,j}$ deformation retract onto $\tilde{P}_{i,j}$ cut open along $\mathcal{A}_{i,j}$. It follows from Lemma 4.12 (3) that $\mathcal{P}_{i,j}$ cut open along $\mathcal{A}_{i,j}$ consists of disks, and from Lemma 4.12 (4) that this is a single disk. We conclude that $\tilde{V}_{i,j}$ cut open along $\mathcal{K}_{i,j}$ is a single ball, establishing (1) for $\tilde{V}_{i,j}$.

By Lemma 4.9 (3) the projection of $\tilde{V}_{i,j}$ to $V_{i,j}$ is a diffeomorphism. Setting $K_{i,j}$ to be the image of $K_{i,j}$ under the universal covering projection we obtain a complex fulfilling the requirements of Lemma 4.13. □

5. Proof of Proposition 1.4

We use the notation of the previous sections.

We begin with the decomposition of $X$ given by $V_i \cap X = \{V_{i,j}\}_{j=1}^{n_i}$.

Fix one $V_{i,j}$ and consider its decomposition obtained by projecting the faces of $\tilde{V}_{i,j}$ to $V_{i,j}$ (as discussed in Lemma 3.5). Recall that all the faces of this decomposition are totally geodesic by construction. We decompose $V_{i,j}$ further using the faces of $K_{i,j}$, as described in Lemma 4.13. By Lemma 4.13 (2), these faces are totally geodesic as well. By Lemma 4.13 (4), all the vertices of this decomposition are on $\partial V_{i,j}$.

We first bound the number of these vertices:

**Claim.** There is a universal $\bar{C}_0$ so that the number of vertices in $V_{i,j}$ is at most $\bar{C}_0$.

**Proof of claim.** By Lemma 4.9 (3), the universal covering projection induces a diffeomorphism between $V_{i,j}$ and $\tilde{V}_{i,j}$. It follows that a totally geodesic disk and a geodesic segment in $V_{i,j}$ intersect at most once; this will be used below several times. By Lemma 4.13 (2) all the vertices are contained in $\partial V_{i,j}$.

We first bound the number of vertices that lie in the interior of $X$. There are three types of vertices:

- **The intersection of three faces of $\partial V_i$:** By Lemma 4.5 there are at most $C_0$ such vertices. (By transversality the intersection of more than three faces of $\partial V_i$ does not occur.)

- **The intersection of an edge of $\partial V_i$ with a face of $K_{i,j}$:** Since every face of $\partial V_{i,j}$ is totally geodesic and every edge of $K_{i,j}$ is a geodesic segment, every face meets every edge at most once. By Lemma 4.4 there are at most $C_1$ edges on $\partial V_{i,j}$, and by Lemma 4.4 there are at most $2C_1 - 1$ faces in $K_{i,j}$. It follows that there are at most $C_1(2C_1 - 1)$ vertices of this type.

- **The intersection of a face of $\partial V_i$ and an edge of $K_{i,j}$:** Since every edge of $\partial V_{i,j}$ is a geodesic segment and every face of $K_{i,j}$ is totally geodesic, every edge meets every face at most once. By Lemma 4.3 there are at most $C_2$ edges on $\partial V_{i,j}$. It is clear that we are discussing only edges of $K_{i,j}$ that lie in the interior of $V_{i,j}$. By Lemma 4.13 (4) there are at most $4C_1 - 2$ such edges. It follows that there are at most $C_2(4C_1 - 2)$ such vertices.

Next we bound the number of vertices on $\partial X$. There are two cases to consider.

- **An endpoint of an edge of $\partial V_i$:** Each such vertex is an endpoint of a segment (as defined before Lemma 4.4) and hence by that lemma there are at most $2C_1$ such vertices.
The intersection of a face of $K_{i,j}$ with $\partial(V_i \cap \partial X)$ and the intersection of an edge of $K_{i,j}$ with $\partial X$: Every face of $K_{i,j}$ contributes at most two such vertices. By Lemma $4.13$ (4), there are at most $8C_1 - 4$ such points.

The claim follows by setting (the different contributions are in brackets)

$$\bar{C} = [C_0] + [C_1 (2C_1 - 1)] + [C_2 (4C_1 - 2)] + [2C_1] + [8C_1 - 4].$$

$\square$

Next, we subdivide each face into triangles by adding edges (note that this does not require faces to be disks). This is done in $X$, so the subdivision agrees on adjacent cells (including a cell that is adjacent to itself). Note that the new edges have valence 2. Since the edges of $K_{i,j}$ have valence 3 and edges on the boundary have valence at most 3, all edges have valence at most 3.

By Lemma $4.13$ (1) $V_{i,j}$ cut open along $K_{i,j}$ is a single ball. Therefore there is a map from the closed ball $B$ onto $V_{i,j}$ that is obtained by identifying disks on $\partial B$ that correspond to the disks of $K_{i,j}$. Since edges have valence at most 3, no point of $V_{i,j}$ has more that 3 preimages. The preimages of the triangulated faces induce a triangulation of $\partial B$ with at most $3\bar{C}_0$ vertices. Denote the number of faces, edges, and vertices in this triangulation by $f$, $v$, and $e$, respectively. Note that $3f = 2e$, or $e = \frac{3}{2}f$. Euler characteristic gives: $2 = f - e + v = -\frac{1}{2}f + v$, or $f = 2v - 4$. Thus, $f \leq 6\bar{C}_0 - 4$. We obtain a triangulation of $B$ by adding a vertex in the center of $B$, and coning every vertex, edge, and triangle in $\partial B$. By construction there are exactly $f$ tetrahedra in this triangulation. The image of this triangulation gives a triangulation of $V_{i,j}$ that has at most $6\bar{C}_0 - 4$ tetrahedra. By Lemma $4.9$ (1), $n_i \leq C_0$. Since $\{V_{i,j}\}_{j=1}^{n_i}$ are mutually disjoint, by considering their union we obtain a triangulation of $V_i \cap X$ with at most $(6\bar{C}_0 - 4)C_0$ tetrahedra.

By construction the triangulation of $V_i \cap X$ agrees with that of $V_i' \cap X$ on $(V_i \cap X) \cap (V_i' \cap X)$.

Proposition $1.4$ follows from this and Lemma $4.5$ by setting $C = \max\{C_3, (6\bar{C}_0 - 4)C_0\}$.

**REFERENCES**

[1] Riccardo Benedetti and Carlo Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, Berlin, 1992. MR MR1219310 (94e:57015)

[2] Breslin, William, *Thick triangulations of hyperbolic n-manifolds*, Pacific J. Math. **241** (2009), no. 2, 215–225, MR 2507575 (2010b:30066)

[3] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, James Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, and Lei Ni, *The Ricci flow: techniques and applications. Part I*, Mathematical Surveys and Monographs, vol. 135, American Mathematical Society, Providence, RI, 2007. Geometric aspects. MR MR2302600 (2008f:53088)

[4] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354 (2002k:55001)

[5] Sadayoshi Kojima, *Isometry transformations of hyperbolic 3-manifolds*, Topology Appl. **29** (1988), no. 3, 297–307. MR MR953960 (90c:57033)

[6] William P Thurston, *The Geometry and Topology of Three-Manifolds*, [http://www.msri.org/publications/books/gt3m/](http://www.msri.org/publications/books/gt3m/) 1977.
TETRAHEDRAL NUMBER OF MANIFOLDS OF BOUNDED VOLUME

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, AR 72701
E-mail address: tsuyoshi@cc.nara-wu.ac.jp
E-mail address: yoav@uark.edu