Eigenvalue spectrum of massless Dirac operators on the lattice∗

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Abstract

We present a detailed study of the interplay between chiral symmetry and spectral properties of the Dirac operator in lattice gauge theories. We consider, in the framework of the Schwinger model, the fixed point action and a fermion action recently proposed by Neuberger. Both actions show the remnant of chiral symmetry on the lattice as formulated in the Ginsparg-Wilson relation. We check this issue for practical implementations, also evaluating the fermion condensate in a finite volume by a subtraction procedure. Moreover, we investigate the distribution of the eigenvalues of a properly defined anti-hermitian lattice Dirac operator, studying the statistical properties at the low lying edge of the spectrum. The comparison with the predictions of chiral Random Matrix Theory enables us to obtain an estimate of the infinite volume fermion condensate.

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1 Introduction

A local lattice theory describing the physical degrees of freedom of the corresponding continuum theory necessarily breaks chiral symmetry [1]. In a usual discretization, like the Wilson action, this breaking is so violent, that no trace of the chiral properties of the continuum is kept in the lattice theory. This involves many inconveniences – in particular for lattice QCD. At the classical level, for fixed gauge field configurations, zero modes of the (Euclidean) Dirac operator have no definite chirality, and the Atiyah-Singer theorem of the continuum theory [2], relating the index of the Dirac operator to the topological charge of the background gauge configuration, finds no strict correspondence on the lattice. At the quantum level, the chirality-breaking terms – when combined with the ultraviolet divergences of the theory – give raise to non-universal finite renormalizations. Examples are the additive renormalization of the bare quark mass $m_q$ and the finite multiplicative renormalizations of the chiral currents, which spoil the usual current algebra. Since the chirality is explicitly broken, the definition of an order parameter for the spontaneous breaking of the chiral symmetry is not straightforward.

In an early paper [3] of lattice quantum field theory, Ginsparg and Wilson made definite the concept of a chiral limit in the framework of a lattice theory breaking explicitly chiral symmetry, providing a general condition – we will refer to it as the Ginsparg-Wilson Condition (GWC) – for the fermion matrix of the lattice theory, i.e. the lattice Dirac operator. For a long time Ginsparg and Wilson’s ideas remained academic, since no acceptable solution of the GWC was found in the case of dynamical gauge fields, locality of the lattice action being the real bottle-neck in this matter.

New attention on the GWC was raised by Hasenfratz [4], who pointed out that the fixed point (FP) action, which is local by construction, satisfies this relation. This result is natural, since the FP action is a classically perfect action.

Chiral symmetry on the lattice is made explicit in the overlap formalism [5]; recently Neuberger [6] found a form of the corresponding Dirac operator, which also satisfies the GWC [7].

In a series of papers, the GWC was theoretically analyzed, showing that it is a sufficient condition for the restoration of the main features of the continuum (symmetric) theory. In the following we refer to actions/fermions satisfying the GWC as to GW actions/fermions. At the classical level, for FP actions, the Atiyah-Singer theorem finds correspondence on the lattice [4, 8].
at the quantum level, no fine tuning, mixing and current renormalization occur, and a natural definition for an order parameter of the spontaneous breaking of the chiral symmetry is possible \([11]\). The explicit form of the lattice symmetry corresponding to the chiral symmetry of the continuum has been identified \([10]\). On the basis of this symmetry it has been shown \([11]\) that the low-energy mechanisms of the continuum QCD (e.g. Goldstone’s theorem for pions, solution of the U(1) problem) emerge also in the theory with finite cut-off.

Monte Carlo calculations require actions with finite number of couplings (ultra-local actions), while any solution of the GWC – even when local – is expected \([12]\) to be extended all over the lattice. In the case of the FP action, the renormalization group theory ensures exponential damping of the couplings with the distance, and a parametrization procedure is viable in principle; the point to be verified is to what extent an approximation of the FP action in terms of a restricted set of couplings is able to reproduce the nice theoretical properties of the ‘ideal’ FP action.

In this paper we check these issues in the case of the Schwinger model \((d = 2\) QED for massless fermions), testing a parametrization \([13]\) of the FP action for the non-overlapping block-spin transformation. We focus mainly on the interplay between chiral symmetry and spectral properties of the Dirac operator. Preliminary results were presented in \([13]\) and \([15]\). We also test Neuberger’s proposal \([6]\). Previous investigations on this Dirac operator in the framework of the Schwinger model were accomplished in \([16]\) and \([17]\) (for an application of the standard overlap formalism, see \([18]\)). Neuberger’s Dirac operator is determined through a configuration-wise operatorial (numerical) projection \([7]\) of the Wilson operator. In this case an explicit parametrization of the Dirac operator is not yet available; also the precise locality properties of this operator have not been established (see, however \([19]\)).

Based on universality argument, suggested by Leutwyler and Smilga’s \([20]\) sum rules for the eigenvalues of the chiral Dirac operator in a finite volume, Shuryak and Verbaarschot \([21]\) hypothesized that statistical properties of the spectrum of this operator at its lower edge are described by Random Matrix Theory (RMT, for a recent review cf. \([22]\)). This ansatz has been explicitly verified \([23]\). When chiral symmetry holds in the lattice theory, the chiral version of RMT (chRMT) should apply \([21]\) \([24]\) \([25]\). So, another way to check the effective restoration of the chiral symmetry by GW actions, is to compare \([26]\) the statistical properties of their spectrum with the prediction of chRMT. In this paper we also address this point.
The sequel of the paper is organized as follows. First we summarize the main properties of GW lattice fermions in relation to chiral symmetry; in Section 3 we report our numerical results about the spectrum of the Dirac operator, in particular the chirality of (quasi) zero modes, their dispersion on the real axis, and agreement of the index of the Dirac operator with the geometric definition for the topological charge. We also show the results of the calculation of the chiral condensate in a finite volume according to the prescription given in [9]. In Section 4 we concentrate on the study of the statistical properties of the eigenvalues of the Dirac operator, investigating the universality class at the lower edge of the spectrum. Comparison with the predictions of chRMT allows us to extract the infinite volume fermion condensate from lattice data. The last section is devoted to discussion and conclusions.

2 Ginsparg-Wilson fermions

The GWC for the fermion matrix $D_{x,x'}$ describing massless fermions reads [3, 4]

$$\frac{1}{2} \left\{ D_{x,x'}, \gamma^5 \right\} = \left( D \gamma^5 R D \right)_{x,x'}, \quad (1)$$

where $R_{x,x'}$ is a local matrix in coordinate space, i.e. whose matrix elements vanish exponentially with the distance. Usual lattice actions obey the general $\gamma^5$-hermiticity property $D^\dagger = \gamma^5 D \gamma^5$. In the particular case $R_{x,x'} = \frac{1}{2} \delta_{x,x'}$ the GWC then assumes the elegant form

$$D^\dagger D = \frac{1}{2} \left( D + D^\dagger \right). \quad (2)$$

Fixed point action. Any FP action of a block spin transformation (BST) satisfies the GWC [4]. Such an action is the FP of a recursion relation of the form,

$$D_{FP,x'y'}(V) = \kappa_{bs} \delta_{x'y'}$$

$$-\kappa_{bs}^2 \sum_{xy} \omega_{x'x}(U) \left( \frac{1}{D_{FP}(U)} + \kappa_{bs} \omega^\dagger \omega \right)_{xy} \omega^\dagger_{yy}(U), \quad (3)$$

where $U(V)$ is fixed by the solution of the pure-gauge part of the FP problem, $\omega(U)$ is the matrix in space-time and color indices defining the (gauge-covariant) block-spin average, and $\kappa_{bs}$ is a free parameter. The operator $R_{x,x'}$
has an expression [4] in terms of $\omega(U)$, and it can in general be assumed to be trivial in Dirac space. In the case of the non-overlapping BST considered here and in [13], the particular form [2] is fulfilled.

**Neuberger’s Dirac operator.** In this approach [6] the starting point is the Wilson Dirac operator at some value of $m \in (-1,0)$ corresponding to $\frac{1}{2D} < \kappa < \frac{1}{2D-2}$ ($\kappa$ is the well-known hopping parameter, not to be confused with the parameter $\kappa_{bs}$ of the block-spin transformation) and then one constructs

$$\mathcal{D}_{Ne} = 1 + \gamma_5 \epsilon(\gamma_5 \mathcal{D}_{Wi})$$

(4)

From the definition (4) it is evident that $\mathcal{D}_{Ne}$ satisfies (2) [7]. Here we consider the case $\kappa = \frac{1}{2}$, $m = -1$ (for a discussion in the context of this study cf. [16]).

### 2.1 Chiral properties

Dirac operators $\mathcal{D}$ obeying the GWC do have non-trivial spectral properties. For simplicity we consider here operators, like $\mathcal{D}_{Fp}$ and $\mathcal{D}_{Ne}$, satisfying the special version (2) of the GWC.

i. $[\mathcal{D}, \mathcal{D}^\dagger] = 0$, i.e. $\mathcal{D}$ is a normal operator; as a consequence, its eigenvectors form a complete orthonormal set.

ii. The spectrum lies on a unit circle in the complex eigenvalue-plane centered at $\lambda = 1$.

iii. The property (i), together with the hermiticity property of the fermion matrix, implies:

$$\gamma^5 v_\lambda \propto v_{\bar{\lambda}} \quad \text{if} \quad \lambda \neq \bar{\lambda},$$

$$\gamma^5 v_\lambda = \pm v_\lambda \quad \text{if} \quad \lambda = \bar{\lambda} \in \mathbb{R}.$$  

(5)

Here $v_\lambda$ denotes an eigenvector of $\mathcal{D}$ with eigenvalue $\lambda$. Just as in the continuum, the eigenvectors of complex-conjugate eigenvalues form doublets related through $\gamma^5$; moreover all real-modes have definite chirality. For the general form (1) of the GWC this property holds only for the zero modes.
2.2 The Atiyah-Singer theorem on the lattice.

The Atiyah-Singer index theorem (ASIT) in the continuum relates the topological charge \( Q(A) \) of a differentiable gauge field configuration \( A \) to the difference between the numbers of positive and negative chirality zero modes of the Dirac operator,

\[
Q(A) = \text{index}(A) \equiv n_+ - n_- .
\]  

(6)

On the lattice, an index for \( D \) may be defined in a way analogous to the continuum, explicitly expressed by the relation \[8\]

\[
\text{index}(U) = -\text{tr}(\gamma^5 R \mathcal{D}(U)) .
\]  

(7)

In the case of GW Dirac operators satisfying the special GW condition \[2\], the above relation comes out trivially considering that (due to their properties listed in the previous section) only the modes with real non-vanishing eigenvalues contribute to the trace in the r.h.s, where \( R = 1/2 \); since the overall chirality must be zero, it reproduces up to a sign \( \text{index}(U) \).

This \( \text{index}(U) \) can be used to define a fermionic lattice topological charge

\[
Q_{\text{ferm}}(U) \equiv \text{index}(U)
\]  

(8)

for which the ASIT is satisfied by definition.

In the case of the FP action the fermionic definition \[8\] coincides \[8\] with the pure-gauge quantity \( Q_{\text{FP}}(U) \), the FP topological charge \[28\] of the configuration \( U \):

\[
Q_{\text{ferm}}(U) \equiv \text{index}(U) = Q_{\text{FP}}(U) .
\]  

(9)

The non-obviousness of this relation relies on the fact that \( Q_{\text{FP}}(U) \) can be defined \textit{a priori} in the pure gauge theory, without any regard to the fermion part. We stress that this result is particular for a FP action, having no counterpart for a general (non-FP) GW action. Of course, in practical implementations one relies on approximate parametrizations of the FP Dirac operator and the strictness of the relation is consequently lost.

Not all fermionic formulations of the topological charge, associated to different lattice Dirac operators, are equivalent, since they can be more or less reminiscent of the continuum \[29\]. The FP definition is in this sense the most reliable, since it is essentially based on a renormalization group guided
procedure of interpolation (but maybe difficult to implement in realistic situations like QCD). Here, as a general criterion, we compare the results for $D_{Fp}$ and $D_{Ne}$ with the geometric definition

$$Q_{\text{geo}}(U) = \frac{1}{2\pi} \sum_x \text{Im} \ln(U_{12}(x)),$$

which is in this context the most natural one and the closest to the continuum definition. In $d = 2$ the geometric definition is the FP for a particular BST [27].

In two dimensions, a theorem of the continuum – the so-called Vanishing Theorem [30] – ensures that only either positive or negative chirality zero modes occur. It seems plausible that this theorem applies for FP actions [27]; we know of no proof that it should apply to GW fermions in general.

2.3 The fermion condensate

In [4] Hasenfratz proposed a subtraction procedure for the fermion condensate inspired by (1) (an analogous prescription for a physical lattice fermion condensate was proposed by Neuberger in an earlier paper [31] in the specific framework of the overlap formalism and for the particular case $R = 1/2$). The definition of the subtracted lattice condensate reads

$$\langle \bar{\psi} \psi \rangle_{\text{sub}} = -\frac{1}{V} \langle \text{tr} \left( \frac{1}{D} - \frac{1}{D} \right) \rangle_{\text{gauge}},$$

where $V$ is the (finite) space-time volume. The expectation value on the right-hand side denotes the gauge averaging including the determinant weight, corresponding to the full consideration of dynamical fermions. With (11) we may rewrite the right-hand side as

$$-\frac{1}{2V} \langle \text{tr} \left( \frac{1}{D} - \frac{1}{D} \right) \rangle_{\text{gauge}}.$$

Because of $\gamma_5$-hermiticity the trace in the above expression vanishes, except when a zero mode of $D$ occurs, in which case a regulator-mass $\mu$ must be introduced,

$$D \rightarrow D(\mu) = D + \mu \mathbf{1}.$$

In $d = 4$ for more than one flavor, $N_f > 1$, $\langle \bar{\psi} \psi \rangle_{\text{sub}}$ is an order parameter for the spontaneous breaking of the chiral symmetry: the contribution of the
zero modes to the gluon average vanishes when $\mu \to 0$ because of the damping effect of the fermion determinant. We conclude that for finite volume $\langle \bar{\psi} \psi \rangle_{\text{sub}}(\mu)$ vanishes in the chiral limit [4].

Here we consider $d = 2$ and $N_f = 1$; the chiral symmetry is broken by the $U(1)$ anomaly. Still we want to use the above expressions to study the condensate. In this situation $\langle \bar{\psi} \psi \rangle_{\text{sub}} \neq 0$ even in a finite volume. The configurations responsible for the non-zero fermion condensate in a finite volume are those from the $|Q| = 1$ sector; indeed, if $\mathcal{D}$ has just one zero mode – which is possible due to the ASIT and Vanishing Theorem only for $|Q| = 1$ – the quantity $(\det \mathcal{D} \text{ tr} \mathcal{D}^{-1})$ has a non-zero limit when $\mu \to 0$. The effect of the subtraction is in general (for any $N_f$) just to remove the contribution of the $Q = 0$ sector.

The situation in the infinite volume, when the fermion condensate is obtained through the sequence of limits $\lim_{\mu \to 0} \lim_{V \to \infty}$, is different. In this case the role of the exact zero modes is irrelevant, the quasi zero modes (the modes with small but non-vanishing eigenvalues) are responsible for the non-zero fermion condensate as the Banks-Casher formula [32] shows.

We observe (cf. [11, 14] as well as the discussion in [33]) that (11) may be rewritten in the form

$$\langle \bar{\psi} \psi \rangle_{\text{sub}} = -\frac{1}{V} \left\langle \text{tr} \left( \frac{1}{\mathcal{D}} \right) \right\rangle_{\text{gauge}} \quad \text{with} \quad \tilde{\mathcal{D}} = \mathcal{D}(1 - R \mathcal{D})^{-1}. \tag{14}$$

The redefined fermion matrix $\tilde{\mathcal{D}}$ is anti-hermitian and anti-commutes with $\gamma^5$:

$$\tilde{\mathcal{D}}^\dagger = -\tilde{\mathcal{D}}, \quad \{ \tilde{\mathcal{D}}, \gamma^5 \} = 0, \tag{15}$$

from which it follows that $\tilde{\mathcal{D}}$ has a purely imaginary spectrum,

$$\tilde{\mathcal{D}} v = -i \lambda v, \quad \lambda \in \mathbb{R}. \tag{16}$$

The zero modes of $\mathcal{D}$ coincide of course with those of $\tilde{\mathcal{D}}$. The replacement $\mathcal{D} \to \tilde{\mathcal{D}}$ makes manifest the chiral invariance implicit in the original operator (observe however that according to Hasenfratz’s subtraction prescription the dynamics of fermions is still given by the chirality-breaking matrix $\mathcal{D}$).

The spectral density of $\tilde{\mathcal{D}}$,

$$\rho(\lambda) = \frac{1}{V} \frac{dN}{d\lambda} \tag{17}$$
is obtained from the eigenvalue spectrum sampled with the weight of \((\det D)^{N_f}\). It complies with the Banks-Casher formula for the (infinite-volume) subtracted fermion condensate \([14]\),

\[
\lim_{\mu \to 0} \lim_{V \to \infty} \langle \bar{\psi} \psi \rangle_{\text{sub}}(\mu) = -\pi \lim_{\lambda \to 0} \lim_{V \to \infty} \rho(\lambda); \quad (18)
\]
the sequence of the limits is essential.

In the case of the simpler GWC \([2]\), the spectrum of \(\tilde{D}\) is obtained by mapping the spectrum of \(D\) (which in this case, we recall, lies on a unit circle in the complex plane) onto the imaginary axis by the stereographic projection:

\[
\lambda \to \lambda \left(1 - \frac{\lambda^2}{2}\right)^{-1}. \quad (19)
\]

The spectral density agrees with that of \(D\) in \(O(\lambda)\).

### 2.4 Spectral microscopic fluctuations

In the framework of QCD with arbitrary \(N_f\), Leutwyler and Smilga \([20]\) derived, through the calculation of low–energy effective lagrangians, sum rules which are obtained in a limit where QCD reduces to a simple matrix model. These arguments suggest \([21]\), that the symmetry of the theory (in the simplest case just the chiral one) rather than its detailed dynamical structure dictates the microscopic fluctuations of the spectrum of the massless Dirac operator; they should have a universal behavior in a scaling variable \(z = V \Sigma \lambda\) in the limit \(V \to \infty\). Here \(\Sigma\) denotes a free parameter related to the dynamics of the system. For fixed \(z\), \(\lambda \to 0\) in the thermodynamic limit and the part of the spectrum near zero – which is also the most relevant for the continuum limit – is under study.

The simplest model encompassing chiral symmetry is \([21]\) a chiral Random Matrix Theory. Since GW actions are expected to describe massless quarks and implicitly realize chiral symmetry on the lattice (explicitly for the matrix \(\tilde{D}\) which anti-commutes with \(\gamma_5\)), it is natural to investigate whether the predictions of chRMT for the microscopic fluctuations of the spectrum apply; for GW fermions cf. \([26,35]\). Previous studies for staggered lattice fermions, which partially restore chirality, where already accomplished for the SU(2) \([34,36,37,38]\) and the SU(3) \([39]\) lattice gauge theory in four dimensions.

Three classes of universality are predicted within chRMT \([25]\), corresponding to orthogonal, unitary or symplectic ensembles (chOE, chUE or...
universality of related correlation functions has been proved in [40]. Starting from Leutwyler and Smilga’s effective lagrangians, it is possible to show [23] that the microscopic fluctuations of QCD (with three colors) are described by chUE. In our context, QED$_2$, no additional symmetry for the Dirac operator beyond the chiral one is present; in this situation one also expects chUE universality [23].

Spectral fluctuations may be studied through a variety of probes (for an incomplete list of such studies within lattice gauge theory, cf. [34, 36, 37, 38, 39]). Here we consider just three of them:

- the probability distribution of the smallest eigenvalue $P(\lambda_{\text{min}})$;
- the microscopic spectral density $\rho_s(z)$

$$\rho_s(z) = \lim_{V \to \infty} \frac{1}{\Sigma} \rho(\frac{z}{V\Sigma}) \ ,$$

(20)

where $\rho(\lambda)$ is the spectral density [17];
- the number variance

$$\Sigma^2(S_0, S) = \langle (N(S_0, S) - \bar{N}(S_0, S))^2 \rangle_{\text{gauge}} \ ,$$

(21)

i.e. the variance of the number of eigenvalues in the interval $[S_0, S]$.

Different numbers of flavors $N_f$ correspond to different predictions from chRMT. In the terminology of lattice calculations, $N_f = 0$ corresponds to the quenched situation, while $N_f = 1$ (or more) to the dynamical one. Therefore comparison with chRMT allows us to check whether the dynamics of the fermions has been effectively included in the simulations [37]. In the dynamical setup, the Banks-Casher relation [18] relates the parameter $\Sigma$ to the (subtracted) fermion condensate in the infinite volume:

$$\Sigma = - \lim_{\mu \to 0} \lim_{V \to \infty} \langle \bar{\psi}\psi \rangle_{\text{sub}} \ .$$

(22)

In chRMT there are specific predictions concerning the spectral distribution in different topological sectors. This theory knows nothing about the dynamical content of gauge fields; the parameter $\nu$, which defines the topological sector, is the difference between the numbers of left- and right-handed

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1We thank J.-Z. Ma for drawing our attention on this particular test.
modes in the matrix representation. In the framework of gauge theories \(\nu\) corresponds to the index of the Dirac operator: for GW fermions it can be defined as in the continuum and produces a fermionic definition of the topological charge on the lattice (see Section 2.1, eq. (7) and (8)). This is not the case for other (non-GW) lattice actions, where the index – and as a consequence the topological charge – is not well defined causing ambiguities in the comparison with chRMT \[38\].

In our discussion of the numerical results we rely on dimensionless quantities. The lattice spacing \(a\) and the (physical) gauge coupling constant \(e\) are dimensionful and we assume the usual asymptotic relation to the dimensionless coupling \(\beta\),

\[
\sqrt{\beta} = \frac{1}{ea} .
\] (23)

In order to compare with the (theoretical) continuum values we therefore have to include corresponding factors of \(e\). The physical lattice size \((La)\) is then in dimensionless units

\[
Lae = \frac{L}{\sqrt{\beta}} .
\] (24)

For \(N_f = 1\) the Schwinger mass is

\[
a m_{Schwinger} = \frac{1}{\sqrt{\beta\pi}} .
\] (25)

The continuum, infinite volume value for the condensate is

\[- \langle \bar{\psi} \psi \rangle_{cont} = ce \text{ where } c = \frac{\exp(\gamma)}{2\pi\sqrt{\pi}} \approx 0.15989 ,\] (26)

(\(\gamma\) denotes the Euler constant) and its dimensionless lattice partner (corresponding to \(\langle \bar{\psi} \psi \rangle_{cont}/e\)) is

\[
\frac{\langle \bar{\psi} \psi \rangle_{lat}}{ea} = \langle \bar{\psi} \psi \rangle_{lat} \sqrt{\beta} .
\] (27)

3 The numerical analysis

The lattice gauge fields \(U_{x,\mu}\) are in the compact representation with the usual Wilson plaquette action.
In [13] the FP Dirac operator was parametrized as

$$D_{FP}(x, y) = \sum_{i=0}^{3} \sum_{x, f} \rho_i(f) \sigma_i U(x, f), \quad \text{with } y \equiv x + \delta f.$$  

(28)

Here \( f \) denotes a closed loop through \( x \) or a path from the lattice site \( x \) to \( y = x + \delta f \) (distance vector \( \delta f \)) and \( U(x, f) \) is the parallel transporter along this path. The \( \sigma_i \)-matrices denote the Pauli matrices for \( i = 1, 2, 3 \) and the unit matrix for \( i = 0 \). Geometrically the couplings have been restricted to lie within a \( 7 \times 7 \) lattice. The action obeys the usual symmetries; altogether it has 429 terms per site. The action was determined as the (numerically approximate) FP of the Dirac operator for gauge fields distributed according to the non-compact formulation with the Gaussian measure. Excellent scaling properties, rotational invariance and continuum-like dispersion relations were observed at various values of the gauge coupling \( \beta \) [13]. Here we study the action (28) only for the compact gauge field distributions (see, however, the results in [14]). In this case the action is not expected to exactly reproduce the FP of the corresponding BST, but nevertheless it is still a solution of the GWC; violations of the GWC are instead introduced by the parametrization procedure, which cuts off the less local couplings.

The situation is different for Neuberger’s operator, which is determined for each configuration on the basis of (4): in our simple context computer time was not really an obstacle and therefore we computed \( \epsilon(\gamma_5 D_Wi) \) through diagonalization with machine accuracy (cf. [16] for more details, as well as [11, 12, 19, 43] for more efficient approaches in \( d = 4 \)).

Essentially uncorrelated gauge configurations have been generated in the quenched setup. As a measure we have used the autocorrelation length for the geometric topological charge, which – for Metropolis updating – behaves like \( \exp(1.67\beta - 3) \) for the range of gauge couplings studied. Unquenching is obtained by including the fermionic determinant in the observables; better ways to include the dynamics of fermions with the present actions – in particular Neuberger’s operator – are still being developed [44]. From earlier experience [13, 14] we learned that our procedure is justifiable at least for the discussed model and the presented statistics. We perform our investigation on sets of 5000 configurations (for \( \beta = 2, 4 \)) and 10000 configurations (for \( \beta = 6 \)) for lattice size \( L = 16 \), and 5000 configurations (for \( \beta = 4, 6 \)) and size \( L = 24 \).
3.1 The spectrum

Fig. 1 shows the spectrum of the studied FP Dirac operator collectively for a subset of 25 uncorrelated gauge configurations in thermal equilibrium. While Neuberger's Dirac operator has (within computational precision, in our case 14 digits) exactly circular spectrum, in the case of the FP action the effect of the truncation in coupling space due to the ultra-local parametrization is displayed by an increased fuzziness for lower $\beta$.

The spectrum of the Dirac operator is gauge invariant and so it can be expanded in series of loop operators. At lowest order in the lattice spacing only few of them are independent, but when considering higher order corrections operators of larger (geometric) extent have to be introduced in order to have a complete set. So it may be argued that the truncation of the couplings to a $7 \times 7$ lattice for the present parametrization of the FP action implies an error in the form of an operator of some (high) dimension:

$$\lambda_{\text{FP}}(U) - \lambda_{\text{par}}(U) = O^{(k)}(U), \quad k = \dim[O^{(k)}].$$  \hspace{1cm} (29)

In order to estimate quantitatively the scaling behavior of the deviations of the spectrum from the ideal circular shape, we defined a mean deviation $|\lambda - 1|$ from the unit circle in an angular window of $|\arg(1 - \lambda)| < \pi/4$. The displayed behavior with $\beta$ of the average width (standard deviation) $\sigma$ of that distribution is $\sigma \propto 1/\beta^{2.41} \simeq a^5$, thus suggesting a dimension-5 operator $[14]$.

Another effect observed for the FP action and due to the truncation, is
Figure 2: Distribution of the (real) eigenvalues for the quasi zero modes for the FP action on a $16^2$ lattice; from left to right: $\beta = 2, 4, 6$ (observe the different scales of the abscissa). For convenience the entries have been normalized to the maximum entry.

Figure 3: Distribution of the chirality ($\bar{\psi}_\lambda, \gamma_5 \psi_\lambda$) of the quasi zero modes of the Dirac operator for the FP action on a $16^2$ lattice; from left to right: $\beta = 2, 4, 6$ (observe the different scales of the abscissa). We show only the positive chirality contribution. For convenience the entries have been normalized to the maximum entry.

The scattering of the zero modes on the real axis. The relative spread is tiny at large values of $\beta$ but becomes relevant when decreasing $\beta$. It damages the nice theoretical properties of the operator causing difficulties in the numerical approach (see the following). From Fig. 2 we see that the spread is $\sim 0.002$ for $\beta = 6$, $\sim 0.01$ for $\beta = 4$, while for $\beta = 2$ it becomes substantial.

A similar effect can be observed in the distribution of the chirality of the (almost) zero modes, displayed in Fig. 3. In this case, the accuracy of
Table 1: Upper part: percentage \( p(\beta) \) of configurations where the index of the Dirac operator index(\( U \)) agrees with the geometric charge. Lower part: percentage \( p(\beta) \) of configurations where the Vanishing Theorem is fulfilled.

| Op. | \( p(\beta = 2) \) | \( p(\beta = 4) \) | \( p(\beta = 6) \) |
|-----|-----------------|-----------------|-----------------|
| \( \mathcal{D}_{Ne} \) | 74.22 | 99.70 | 100.00 |
| \( \mathcal{D}_{Fp} \) | 96.58 | 100.00 | 100.00 |
| \( \mathcal{D}_{Ne} \) | 100.00 | 100.00 | 100.00 |
| \( \mathcal{D}_{Fp} \) | 91.00 | 99.84 | 99.96 |

realization of chiral symmetry is directly probed. For Neuberger’s operator the chirality of the zero modes is ±1 within numerical precision.

Table 1 gives the results of the comparison between the index of the Dirac operator index(\( U \)) (the modes are counted according to the sign of their chiralities) and the geometric charge of the gauge configuration. Observe that in the case of the FP action the agreement is excellent already at \( \beta = 2 \), thus indicating a close correspondence between the geometric definition and the FP topological charge. For this value of \( \beta \) the deviation is comparatively large for Neuberger’s operator, reflecting in this respect features of the Wilson operator entering (4).

The lower part of the table demonstrates that for \( \mathcal{D}_{Ne} \) zero modes have just one chirality here. For \( \mathcal{D}_{Fp} \) this situation is also rapidly approached towards the continuum limit, as predicted by the Vanishing Theorem.

### 3.2 The fermion condensate in a finite volume

The finite-volume fermion condensate is determined following the prescriptions of Section 2.3. In order to check the correctness of our determination we verify the Ward identity (see also (11)), valid in the limit \( \mu \to 0 \)

\[
- \lim_{\mu \to 0} \langle \bar{\psi} \psi \rangle_{\text{sub}}(\mu) = \lim_{\mu \to 0} \chi'(\mu) \tag{30}
\]

with \( \chi'(\mu) \equiv \frac{1}{V\mu} \langle (\text{index}(U))^2 \rangle_{\text{gauge}}(\mu) \). \( \tag{31} \)

In Table 2 we compare the values for both sides of (30) (extrapolated at \( \mu = 0 \)); excellent agreement between the two quantities is displayed by data for Neuberger’s action. In the case of the FP action, \( \chi'(\mu) \) is unstable
Figure 4: The subtracted chiral condensate as a function of $\mu$ for Neuberger’s (left) and FP (right) operator on a 16$^2$ lattice: $\beta = 2$ (circles), $\beta = 4$ (diamonds), $\beta = 6$ (triangles); the full symbols denote the zero-mass estimate obtained through a linear extrapolation. For $\beta \geq 4$ they agree within the error bars with the finite volume numbers (indicated by the pointers on the left-hand side) from theory [45].

Table 2: Subtracted condensate $\langle \bar{\psi} \psi \rangle_{sub}$ and $\chi'$ for the FP and Neuberger’s action (all obtained in the limit $\mu \to 0$). The last column indicates the continuum value in the corresponding physical volume [45].

| Op. | $L$ | $\beta$ | $\langle \bar{\psi} \psi \rangle_{sub}$ | $\chi'$ | Cont. |
|-----|-----|---------|-----------------|--------|-------|
| $D_{Ne}$ | 16  | 2       | 0.135(12)       | 0.135(18) | 0.1127 |
| $D_{Fp}$ | 16  | 4       | 0.0725(73)      | 0.0777  |       |
| $D_{Ne}$ | 16  | 4       | 0.0747(41)      | 0.0747(41) | 0.0777 |
| $D_{Fp}$ | 24  | 4       | 0.104(38)       | 0.0798  |       |
| $D_{Fp}$ | 16  | 6       | 0.0622(19)      | 0.0604  |       |
| $D_{Ne}$ | 16  | 6       | 0.0617(18)      | 0.0617(18) | 0.0604 |
| $D_{Fp}$ | 24  | 6       | 0.0621(36)      | 0.0650  |       |

for $\mu \to 0$: this is simply explained by the spread of the zero modes which prevents the cancellation between the denominator factor $\mu$ with the lowest eigenvalue in the fermion determinant (which is $\mu$ for an exact zero mode).
The subtracted condensate for the (approximate) FP action is plagued by similar problems. For $\beta = 4$ and $L = 24$ the determination of the condensate has large error bars and for $\beta = 2$ and $L = 16$ it is not possible at all due to the instability of the $\mu \to 0$ limit. Except for these two cases the relative accuracy of our results is between 3 and 6%. In Table 2 we also compare our data with the continuum value [45] in the corresponding physical volume, assuming the asymptotic relation (23); the discrepancy is always smaller than one standard deviation.

In Fig. 4 we report $\langle \bar{\psi}\psi \rangle_{\text{sub}}(\mu)$ for a range of $\mu$ values in the case of Neuberger (left) and FP (right) action; the expected linear asymptotic behavior allows for the extrapolation at $\mu = 0$.

4 Statistical properties of the spectrum

GW actions realize chiral symmetry on the lattice. This realization is implicit, in the sense that the Dirac operator breaks the chiral symmetry, as usually defined, by a local term $O(a)$. We make the symmetry manifest and construct a fermion matrix $\tilde{D}$ which anti-commutes with $\gamma^5$. This is achieved through the transformation proposed in (14). In this section we study the microscopic fluctuations of the spectrum of this operator.

4.1 The trivial sector: $N_f = 0$ and $\nu = 0$

Since chRMT gives specific predictions for given topological number $\nu$ and number of flavors $N_f$, it allows to disentangle the statistical properties of the spectrum of the lattice Dirac operators from the more subtle questions of the correct reproduction of the dynamics of fermions and of the identification of the topological charge with the index (see the discussion on this latter point in Section 2.4). Consequently, we start our investigation in the simplest case $N_f = 0$ and $\nu = 0$.

We expect that a gauge theory of the type studied here – without further symmetries – should lead to a spectral distribution in the universality class of chUE [25]. This is to be checked with the data.

Comparison of the statistics of the Dirac spectrum distribution with chRMT requires the knowledge of the inherent scale parameter $\Sigma$ that is related to the dynamical properties of the system. Technically $\Sigma$ may be determined by comparing spectral data with chRMT predictions for a par-
Figure 5: The ratio between the $\chi^2_u$ ($\chi^2$ with chUE) and $\chi^2_s$ ($\chi^2$ with chSE) in the case of the FP action as a function of the physical lattice size $L/\sqrt{\beta}$.

A simple way to determine $\Sigma$ is to look at the probability distribution of the smallest eigenvalue $P(\lambda_{\text{min}})$. This statistics contains spectral information on a region of small enough values of $\lambda$ not to be affected by the macroscopic, non-universal (i.e. not described by chRMT) component of the spectrum. Of course the macroscopic component can be easily eliminated from lattice data through an unfolding procedure, which – at this level – would introduce unnecessary arbitrariness, however.

As discussed, we expect results lying in the universality class of chUE. However, in order to decide on the applicability of chRMT in an unbiased approach, we tested the predictions [46] for the distribution of the smallest eigenvalue from all three variants of chRMT through a standard best-fit procedure on our lattice data. This test clearly ruled out the chOE, which has a completely different distribution shape. The two remaining ensembles chUE and chSE seemed to fit data in disconnected – complementary – regions of the lattice parameter space $(L, \beta)$.

In Fig. 5 we plot the ratio of $\chi^2$-values obtained for the fits to the two
distributions as a function of the physical size. We find that in the region of large physical volumes chUE is preferable (the ratio is smaller than unity), while in the region of small physical volumes chSE provides better fits to the data. The ‘transition’ seems to take place at $L/\sqrt{\beta} \simeq 9$. In Fig. 6 we compare our lattice results for $P(\lambda_{\text{min}})$ with the best fit prediction from chUE and chSE for two cases in the large-volume region (upper part) and in the small-volume one (lower part). In Table 3 we give the resulting values of $\Sigma$. Within chRMT this value is intrinsically volume-independent. The obtained value of $\Sigma$ is taken as an input parameter for the subsequent investigation.

For small physical volumes we find chSE distribution shapes; however, this result is in conflict with other properties of such ensembles. For chSE the eigenvalues should come in degenerate pairs, which is not true in our case.

Such a volume dependence of our results does not come unexpected and is related to the limitations of applicability of RMT. Actually, in QCD one expects RMT to be applicable \cite{20, 21} for

$$\frac{1}{m_{\text{QCD}}} \ll aL \ll \frac{1}{m_{\pi}}$$

which, in our case with $N_f = 1$ reduces to

$$\frac{1}{a m_{\text{Schwinger}}} \ll L.$$  \hspace{1cm} (33)

From the continuum results we expect on the lattice \cite{20}, and then \cite{33} reads $\sqrt{\pi} \ll L/\sqrt{\beta}$, to be compared with the discussed observations. We suspect that the observed behavior at small volume is outside the scope of RMT and that one may be misled by the superficial agreement with chSE. We return to a discussion in the last section.

**Microscopic spectral density.** Using $\Sigma$ as input parameter, as discussed, we checked the universal behavior of the microscopic spectral density $\rho_s(z)$. In this case the macroscopic unfolding of the lattice data was necessary (in this we followed essentially the suggestions in \cite{36}). We report the results in Fig. 7, again comparing with the chUE \cite{24} and chSE \cite{37, 36} predictions. We see a (universal) behavior compatible – within the accuracy of our data – with that found in the case of the distribution of the smallest eigenvalue, namely chUE in the large-volume region and chSE in the small-volume region.

\footnote{We thank T. Wettig for pointing this out to us.}
Figure 6: Distribution of the smallest eigenvalue $P(\lambda_{\text{min}})$ in the $\nu = 0$ topological sector for different lattices and values of $\beta$, for the FP and Neuberger’s Dirac operator. The full and dashed curves represent the theoretical predictions of chUE and chSE, respectively.

**Number variance.** A third test was accomplished by checking the number variance $\Sigma^2(S_0, S)$. We report the results (for the unfolded variable $z$) in Fig. 8 for the same collection of cases considered in Fig. 6 and 7. In the large volume case we see agreement with the theoretical prediction of chUE up to a value of $z \simeq 2$. This outcome is also consistent with the results for $\rho_s(z)$, where the lattice data agree with chUE up to the same value of $z$, corresponding to the support of the distribution of the smallest eigenvalue.
Figure 7: Microscopic spectral density $\rho_s(z)$ in the $\nu = 0$ topological sector for different lattices and values of $\beta$, for the FP and Neuberger’s Dirac operator. The full and dashed curves represent the theoretical predictions of chUE and chSE, respectively.

(see Fig. 6 and 7, upper part). The existence of an upper bound of $z$, $z_{\text{max}}$, for the applicability of chRMT is well-known in the literature (see for example [48]) and an estimate for $z_{\text{max}}$ is available [49] in $d = 4$: $z_{\text{max}} \sim f_\pi^2 (L a)^2$; such an estimate (related to the massless modes in $d = 4$) is absent in $d = 2$.

In the small volume case we observe instead disagreement with the analytical prediction of chSE [36] already for very small values of $z$, even if the overall shape seems to resemble the symplectic behavior.

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Figure 8: Number variance as a function of the unfolded variable \( z \): (a) large volume region, \( L = 16, \beta = 2 \) and Neuberger’s action (circles), \( L = 24, \beta = 4 \) and FP action (boxes); (b) small volume region, \( L = 16, \beta = 4 \) for Neuberger’s (triangles) and the FP action (diamonds). The full and dashed curves represent the theoretical predictions of chUE and chSE, respectively.

4.2 Topological sector: \( N_f = 0 \) and \( \nu = 1 \)

As discussed earlier (Section 2.4), in the case of GW actions a definition of the topological charge corresponding to the parameter \( \nu \) of chRMT is available: this is the index of the lattice Dirac operator.

In Fig. 9 and 10 we present our results for the distribution of the smallest eigenvalue and microscopic spectral density respectively, for configurations with \( \text{index}(U)=1 \) and compare with the corresponding predictions of chUE [50, 24] and chSE [37, 47, 36]. For these configurations we use the identical unfolding as in the previous section for \( \nu = 0 \) configurations (for corresponding \( L \) and \( \beta \) and, in particular, the same value of \( \Sigma \)). This procedure is consistent with the expectations from chRMT that \( \Sigma \) should not depend on the topological sector, since all such dependence is incorporated in the functional behavior of the distributions. We observe for \( P(\lambda_{\text{min}}) \) and \( \rho_s(z) \) a universal behavior, again consistent with the observations of the previous section.
Figure 9: Distribution $P(z_{\text{min}})$ of the smallest eigenvalue in the unfolded variable $z$ in the $\nu=1$ topological sector for different lattices and values of $\beta$, for the FP and Neuberger’s Dirac operator. The dotted and dashed curves represent the theoretical predictions of the chRMT ensemble for the values $\nu=0$ and $\nu=1$, respectively, with chUE for (a, b) and chSE for (c, d), cf. the discussion in the text.
Figure 10: Microscopic spectral density $\rho_s(z)$ in the $\nu = 1$ topological sector for different lattices and values of $\beta$, for the FP and Neuberger’s Dirac operator. The dashed curves represent the theoretical predictions of the chRMT ensemble for $\nu = 1$, with chUE for (a, b) and chSE for (c, d).

### 4.3 Dynamical fermions

Consistent with our unquenching procedure, we define the distributions for the setup with dynamical fermions by weighting the entries of the quenched distributions with the fermion determinant. For example, in the case of the
microscopic spectral density we take:
\[ \rho_s(z) = \frac{\sum_C \rho_s(C, z) \det D(C)}{\sum_C \det D(C)}, \quad (34) \]
where \( \rho_s(C, z) \) is the spectral distribution for an individual configuration \( C \); the quenched setup is recovered replacing \( \det D(C) \) with unity. This “unquenching” is formally correct; it is not optimal from the point of view of statistical reliability and one may expect large fluctuations of intermediate results. However, in our \( d = 2 \) studies (cf. [13, 16]) we found stable results for the given statistics.

In Fig. 11 and 12 we show the results for \( P(\lambda_{\text{min}}) \) and \( \rho_s(z) \) respectively in the case of configurations with \( \text{index}(U)=0 \). We compare these with the predictions of chUE [50, 24] and chSE [30, 47, 36] in the case \( N_f = 1, \nu = 0 \). In Fig. 12 we see that higher statistics for dynamical fermions would be desirable in particular for \( \rho_s(z) \).

The value of \( \Sigma \) used for the definition of the unfolded variable \( z \) was found by a best-fit procedure in the distribution of the smallest eigenvalues like in Section 4.1. In the large-volume case we find, consistent with the results of the quenched setup, agreement with chUE up to \( z \simeq 2 \).

The values of \( \Sigma \) found here provide an estimate of the fermion condensate in the infinite volume limit. In Table 3 we summarize these results for this quantity. These should also be compared with the direct determination in the finite volume of Section 3.2 in Table 2. Typically, the statistical error are smaller for the values obtained by the RMT techniques.

In Fig. 13 we display these estimates for the infinite volume condensate. The results obtained in the large volume region (using the chUE distributions) seem to approach the continuum with a linear dependence in the lattice spacing; those obtained in the small volume region appear more scattered.

Another check [34] of the chRMT predictions related to the microscopic spectral distribution involves the quantity [20]
\[ S_2 \equiv \frac{1}{\Sigma^2 V^2} \left\langle \sum_{\lambda>0} \frac{1}{\lambda^2} \right\rangle_{\text{gauge}}^{V \to \infty} \int dz \frac{\rho_s(z)}{z^2}. \quad (35) \]
Inserting the predictions from chRMT for \( \rho_s(z) \), one obtains [25]:
\[ S_2 = \frac{b}{8 \left( \frac{b \nu}{2} + \frac{b}{2} + N_f - 1 \right)}, \quad (36) \]
Figure 11: Distribution $P(z_{\text{min}})$ of the smallest eigenvalue in the unfolded variable $z$ for the unquenched ($N_f = 1$) data in the $\nu = 0$ topological sector, for the FP and Neuberger’s Dirac operator. The dotted and dashed curves represent the theoretical prediction of the best fitting chRMT ensemble (cf. Fig. 9) in the trivial sector for $N_f = 0$ and $N_f = 1$, respectively.

with $b = 2$ and 4 for chUE and chSE, respectively.

In Tables 4 and 5 we compare our values for $S_2$ with (36) in the large and the small (physical) volume region. We find better agreement for large volumes and for higher topological sectors.
Figure 12: Microscopic spectral density $\rho_s(z)$ for the unquenched ($N_f = 1$) data in the trivial topological sector for the FP and Neuberger’s Dirac operator. The dashed curves represent the theoretical predictions of the best fitting chRMT ensemble (cf. Fig. 9) for $N_f = 0$ and $N_f = 1$, respectively.

5 Discussion and conclusions

Both, the FP action and Neuberger’s Dirac operator, guarantee the restoration of the main features of chiral symmetry in the continuum limit. We checked this through the analysis of their spectrum, in particular the zero modes and their chirality. The GW relation allows one to define an index for the Dirac operator which is useful to define a topological charge on the lat-
Figure 13: The ratio of $\Sigma \sqrt{\beta}$ (in units of the continuum value $-\langle \bar{\psi} \psi \rangle_{\text{cont}} / e = c$, cf. (26)), obtained from the distribution of the smallest eigenvalue for the quenched (left) and unquenched (right) setup as a function of $ea = 1/\sqrt{\beta}$. Full (open) symbols denote results of the fit in the large (small) physical volume region, with unitary (symplectic) prediction; circles denote the FP action, boxes Neuberger’s operator.

| Op. | $L$ | $\beta$ | $N_f = 0$     | $N_f = 1$     | Ensemble | Cont.   |
|-----|-----|---------|---------------|---------------|----------|---------|
| $D_{\text{Ne}}$ | 16  | 2       | 0.1437(39)    | 0.1394(26)    | chUE     | 0.1131  |
| $D_{\text{Fp}}$ | 16  | 4       | 0.0946(14)    | 0.0983(15)    | chSE     | 0.0804  |
| $D_{\text{Ne}}$ | 16  | 4       | 0.0927(14)    | 0.0975(14)    | chSE     | 0.0804  |
| $D_{\text{Fp}}$ | 24  | 4       | 0.0933(11)    | 0.0927(19)    | chUE     | 0.0804  |
| $D_{\text{Fp}}$ | 16  | 6       | 0.0791(12)    | 0.0868(12)    | chSE     | 0.0653  |
| $D_{\text{Ne}}$ | 16  | 6       | 0.0783(12)    | 0.0865(12)    | chSE     | 0.0653  |
| $D_{\text{Fp}}$ | 24  | 6       | 0.0696(15)    | 0.0736(18)    | chUE     | 0.0653  |

Table 3: Values of the fermion condensate in the infinite volume obtained from comparison of lattice data with chRMT.

The additional property of the FP action to be perfect at the classical level (which is not the case for Neuberger’s operator) makes this definition particularly reliable, since it is related to a renormalization group guided procedure of interpolation of the gauge configuration. This results, as we observe, in an excellent agreement – good even at low $\beta$ – with the geometric
\[
N_f = 0 \quad N_f = 1 \\
\begin{array}{lcccc}
N_f + \nu & L=24 \, \beta=4 & L=16 \, \beta=2 & L=24 \, \beta=4 & L=16 \, \beta=2 \\
1 & 0.369(47) & 0.30(12) & 0.288(29) & 0.326(17) \\
2 & 0.157(6) & 0.159(7) & 0.1401(24) & 0.1567(20) \\
3 & 0.0938(24) & 0.0922(36) & 0.0941(10) & 0.1042(12) \\
4 & 0.0666(15) & 0.0607(13) & 0.0692(8) & 0.0758(9) \\
5 & 0.0508(16) & 0.0525(36) & 0.0554(8) & 0.0563(8) \\
6 & 0.0407(16) & 0.0402(20) & 0.0452(8) & 0.0455(11) \\
7 & & & 0.0392(12) & 0.0380(16) \\
\end{array}
\]

Table 4: Our values for \( S_2 \) in the large-volume region in the quenched and unquenched setup: \( L = 24 \) and \( \beta = 4 \) (FP action), \( L = 16 \) and \( \beta = 2 \) (Neuberger’s Dirac operator); chUE denotes the prediction (36) for \( b = 2 \).

\[
\begin{array}{cccccc}
\nu & \multicolumn{3}{c}{N_f = 0} & \multicolumn{3}{c}{N_f = 1} \\
& FP & Neub. & chSE & FP & Neub. & chSE \\
0 & 0.994(84) & 1.16(18) & 0.5 & 0.318(12) & 0.300(10) & 0.25 \\
1 & 0.2020(25) & 0.2003(22) & 0.1666 & 0.1562(11) & 0.1520(10) & 0.125 \\
2 & 0.1150(13) & 0.1143(13) & 0.1 & 0.1034(8) & 0.0997(7) & 0.0833 \\
3 & 0.0773(14) & 0.0768(14) & 0.0714 & 0.0739(9) & 0.0725(8) & 0.0625 \\
4 & 0.0552(19) & 0.0551(18) & 0.0555 & 0.0543(12) & 0.0536(12) & 0.05 \\
5 & 0.0468(48) & 0.0472(49) & 0.0454 & 0.0385(25) & 0.0457(43) & 0.0417 \\
\end{array}
\]

Table 5: The values \( S_2 \) in the small-volume region in the quenched and unquenched setup: \( L = 16 \) and \( \beta = 4 \); chSE denotes the prediction (36) for \( b = 4 \).

At the practical level, Neuberger’s approach seems to be most suitable when strictness of the chiral properties is required, as in the case of the calculation of the fermion condensate by subtraction. Parametrization uncertainties of the FP action seem to be a problem in this case, for small or even vanishing fermion mass. On the other hand, some experience [16] indicates that e.g. for dispersion relations, even the approximate perfectness of the FP action is a great advantage, while Neuberger’s operator introduces discretization effects as large as for the Wilson action. Of course, the present parametrization of the FP action could be improved (e.g. by sampling ac-
cording to the compact gauge action in the determination of the FP fermion action); there may be even the chance to reduce the effective number of parameters \[51\]. However, in view of the four-dimensional environment, where only few operators can be included in the action for practical reasons, the parametrization effects are likely to be a serious problem, at least for small fermion mass.

We also studied the statistical fluctuations of the spectra of the Dirac operators \(D_{\text{FP}}\) and \(D_{\text{Ne}}\). In all cases considered here (different values of \(\beta\) and lattice size) \(D_{\text{FP}}\) and \(D_{\text{Ne}}\) display very similar statistical properties. Concerning comparison with chRMT, the results show an unexpected complementarity between the symplectic and the unitary universal behavior, the latter coming into play for larger physical volumes. However, several aspects lead us to conclude that the symplectic behavior is an artifact of the small volume where RMT breaks down. The spectrum is not doubly degenerate as would be expected in the symplectic framework; it just looks like one half of a chSE spectrum. The number variance shows deviation from chSE already from very small values of \(z\). The distribution shape depends on the physical size of the volume. This is in agreement with the expected range of applicability of RMT, \(1/(am_{\text{Schwinger}}) \ll L\) as discussed in Section 4.1.

We conclude that RMT necessitates randomness at a lattice size \(L\) much larger than the correlation length of the system. In the (in physical units) large volume region chUE describes our data up to \(z_{\text{max}} \simeq 2\) (for the trivial sector). This distribution then should govern the continuum limit.

RMT provides a powerful tool to extract the infinite volume chiral condensate from finite (and hopefully small) lattices in the case of dynamical fermions. Our findings indicate, that some caution is in order. We argue that the physical volume should be not too small, since chRMT interpretation may produce misleading results in this situation.

Concerning the condensate as determined from the chUE fits in the large volume domain, we observe linear cut-off effects for both actions (Fig. 13). If this is not an artifact of this approach, the natural explanation would be, that one still has to improve the field operators. This is also true for the FP action, since perfectness applies to spectral quantities and its extension to other observables passes through the implementation of the RG improvement to the operator (in this case \(\bar{\psi}\psi\)) as well. Also, we did not attempt to introduce a renormalized coupling constant.

We find that studying the spectrum of GW operators is extremely helpful to identify chirality properties. With chRMT one learns how to disentangle
universal properties from the dynamical parameters like the fermion condensate. Our FP action and Neuberger’s operator lead to similar results in this context.

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References

[1] H. Nielsen and M. Ninomiya, Nucl. Phys. B 193 (1981) 173.

[2] M. Atiyah and I. M. Singer, Ann. Math. 93 (1971) 139.

[3] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D 25 (1982) 2649.

[4] P. Hasenfratz, Nucl. Phys. B (Proc. Suppl.) 63A-C (1998) 53.

[5] R. Narayanan and H. Neuberger, Phys. Rev. Lett. 71 (1993) 3251.

[6] H. Neuberger, Phys. Lett. B 417 (1998) 141.

[7] H. Neuberger, Phys. Lett. B 427 (1998) 353.

[8] P. Hasenfratz, V. Laliena, and F. Niedermayer, Phys. Lett. B 427 (1998) 125.

[9] P. Hasenfratz, Nucl. Phys. B 525 (1998) 401.

[10] M. Lüscher, Phys. Lett. B 428 (1998) 342.

[11] S. Chandrasekharan, hep-lat/9805015.

[12] I. Horvath, Phys. Rev. Lett. 81 (1998) 4063.

[13] C. B. Lang and T. K. Pany, Nucl. Phys. B (Proc. Suppl.) 63A-C (1998) 898; Nucl. Phys. B 513 (1998) 645.

[14] F. Farchioni, C. B. Lang, and M. Wohlgenannt, Phys. Lett. B 433 (1998) 377.

[15] F. Farchioni, I. Hip, C. B. Lang, and M. Wohlgenannt, hep-lat/9809049.

[16] F. Farchioni, I. Hip, and C. B. Lang, Phys. Lett. B 443 (1998) 214.

[17] S. Chandrasekharan, hep-lat/9810007.

[18] R. Narayanan, H. Neuberger, and P. Vranas, Phys. Lett. B 353 (1995) 507.

[19] P. Hernández, K. Jansen, and M. Lüscher, hep-lat/9808010.
[20] H. Leutwyler and A. Smilga, Phys. Rev. D 46 (1992) 5607.

[21] E. V. Shuryak and J. J. M. Verbaarschot, Nucl. Phys. A 560 (1993) 306.

[22] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Phys. Rep. 299 (1998) 189.

[23] P. H. Damgaard, Phys. Lett. B 424 (1998) 322; G. Akemann and P. H. Damgaard, Nucl. Phys. B 528 (1998) 411; J. C. Osborn, D. Toublan and J. J. M. Verbaarschot, [hep-th/9806110]; P. H. Damgaard, J. C. Osborn, D. Toublan and J. J. M. Verbaarschot, [hep-th/9811212].

[24] J. J. M. Verbaarschot and I. Zahed, Phys. Rev. Lett. 70 (1993) 3852.

[25] J. J. M. Verbaarschot, Phys. Rev. Lett. 72 (1994) 2531.

[26] K. Splittorff and A. D. Jackson, [hep-lat/9805018].

[27] E. Farchioni and V. Laliena, Phys. Rev. D 58 (1998) 054501.

[28] M. Blatter, R. Burkhalter, P. Hasenfratz, and F. Niedermayer, Phys. Rev. D 53 (1996) 923.

[29] F. Niedermayer, [hep-lat/9810026].

[30] J. Kiskis, Phys. Rev. D 15 (1977) 2329; N. K. Nielsen and B. Schroer, Nucl. Phys. B 127 (1977) 493; M. M. Ansourian, Phys. Lett. 70B (1977) 301.

[31] H. Neuberger, Phys. Rev. D 57 (1998) 5417.

[32] T. Banks and A. Casher, Nucl. Phys. 169 (1980) 103.

[33] T. W. Chiu and S. V. Zenkin, [hep-lat/9806019].

[34] M. E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J. J. M. Verbaarschot, and T. Wettig, Phys. Rev. Lett. 80 (1998) 1146.

[35] K. Splittorff, [hep-th/9810248].

[36] J.-Z. Ma, T. Guhr, and T. Wettig, Eur. Phys. J. A 2 (1998) 87, 425.

[37] M. E. Berbenni-Bitsch, S. Meyer, and T. Wettig, Phys. Rev. D 58 (1998) 071502.
[38] M. E. Berbenni-Bitsch, M. Göckeler, S. Meyer, A. Schäfer, and T. Wettig, hep-lat/9809058.

[39] P. H. Damgaard, U. M. Heller, and A. Krasnitz, hep-lat/9810060.
M. Göckeler, H. Hehl, P. E. L. Rakow, A. Schäfer, and T. Wettig, hep-lat/9811018.

[40] G. Akeman, P. H. Damgaard, U. Magnea and S. M. Nishigaki, Nucl. Phys. B 487 (1997) 721; P. H. Damgaard and S. M. Nishigaki, Nucl. Phys. B 518 (1998) 495; M. K. Sener and J. J. M. Verbaarschot, Phys. Rev. Lett. 81 (1998) 248.

[41] H. Neuberger, Phys. Rev. Lett. 81 (1998) 4060.

[42] R. G. Edwards, U. M. Heller, and R. Narayanan, Nucl. Phys. B 535 (1998) 403.

[43] A. Borici, hep-lat/9810064.

[44] C. Liu, hep-lat/9811008.

[45] I. Sachs and A. Wipf, Helv. Phys. Acta 65 (1992) 653.

[46] P. J. Forrester, Nucl. Phys. B 402 (1993) 709.

[47] T. Nagao and P. J. Forrester, Nucl. Phys. B 435 (1995) 401.

[48] M. E. Berbenni-Bitsch, M. Göckeler, T. Guhr et al., Phys. Lett. B 438 (1998) 14.

[49] J. Stern, hep-ph/9812082.
R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, Phys. Rev. Lett. 81 (1998) 264. J. C. Osborn and J. J. M. Verbaarschot, Nucl. Phys. B 525 (1998) 738; Phys. Rev. Lett. 81 (1998) 268.

[50] T. Nagao and P. J. Forrester, Nucl. Phys. B 509 (1998) 561.

[51] W. Bietenholz, Eur. Phys. J. C 6 (1999) 537.