An analysis of dual-issue final-offer arbitration

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Abstract
We consider a final-offer arbitration problem between two players with two quantitative issues in dispute. Under reasonable assumptions we model the problem as a zero-sum two person game and show that a pair of pure strategies explicitly constructed are the unique minimax strategies for the two players.

Keywords Final-offer arbitration · Non-cooperative game theory · Multi-issue · Brams–Merrill

1 Introduction
Should negotiating parties fail to arrive at an agreeable solution, arbitration serves as a mechanism whereby a binding resolution may be reached. In conventional arbitration (CA), the disputing parties submit their cases to an agreed upon arbiter who has full power to craft whatever fair and just settlement he sees fit. It is widely accepted, however, that CA has a number of undesirable properties, in particular what has been called the “chilling effect”: since both parties know the arbiter will craft a compromise, they tend to take extreme positions. Since it is commonly held that a settlement reached through negotiation is preferable to a settlement reached through arbitration, one can view the purpose of a compulsory arbitration as motivating the parties to reach an agreement during negotiations. This is the paradox of arbitration: the best arbitration mechanism is that which is used least often.

It was Stevens (1966) who suggested a simple arbitration mechanism now known as Final-Offer Arbitration (FOA). In FOA, the arbiter must select one of the final offers submitted by the parties and has no prerogative to craft a compromise settlement. The theory was that such uncertainty in the final outcome would combat this chilling effect driving the two parties to make final offers that are “close” to one another, or better still motivate them to reach agreement during negotiations.
Since 1975 when FOA was adopted by Major League Baseball for salary disputes, variants of FOA have been used in various states in public sectors where labor does not have the right to strike (e.g. police, firefighters). A growing body of literature has been developed by legal scholars, economists and game theorists studying both the theoretical and empirical properties of FOA.

The first theoretical model of FOA was introduced by Crawford (1979). With the assumption that both parties know with certainty the arbiter’s opinion of a “fair” settlement, he showed that FOA would inevitably lead to the same outcome as conventional arbitration. Farber (1980), Chatterjee (1981) and Brams and Merrill (1983) independently developed game theoretic models of single-issue FOA for which players are uncertain of the arbiter’s behavior. Farber studied the effect of risk aversion by one of the parties, and derived the strategy pair which in many cases is a Nash equilibrium. Chatterjee and Brams and Merrill model the game as zero-sum and consequently assumed both parties are risk-neutral. Brams and Merrill provide sufficient conditions for the existence of a pure equilibrium. In all three models, the arbiter is assumed by the players to choose a “fair” settlement from a probability distribution commonly known to both players and select whichever player’s offer is closest in absolute value.

The basic model has been extended and analyzed in a number of ways in the literature. Samuelson (1991) developed a model of single-issue FOA where parties’ knowledge of a fair settlement is asymmetric. Kilgour (1994) studied the game theoretic properties of FOA and extended the Brams–Merrill model to allow for risk-aversion on the part of the players. Dickinson (2006) further showed that optimism on the part of the players, in the form of a biased prior distribution, drives the final-offers apart. Armstrong and Hurley (2002) generalized FOA and CA into a single model and showed that optimal offers under CA will always diverge more than those under FOA. Mazalov and Tokareva (2012) considered an extension where the decision is made by multiple arbitrators.

If multiple issues are in dispute, FOA has been primarily implemented in two ways (Stern 1975). Under Issue-by-Issue FOA (IBIFOA), the arbiter may craft a compromise of sorts from the two parties’ offers by choosing some components from one and some from the other. Alternatively, Whole Package FOA (WPFOA) requires that the arbiter select one offer in its entirety. A multi-issue model of FOA was first discussed by Crawford (1979) and further developed by Wittman (1986). Here the main concern was the existence of a Nash equilibrium under various assumptions. Wittman was also able to show in his model that increased risk-aversion leads a player to make a less extreme final-offer. Olson (1992) discussed how the single-issue model does not accurately reflect arbiter behavior when more than one issue is in dispute.

In his initial paper introducing FOA, Stevens cautions against the use of the “Whole Package” variant, stating that “such a system would run the danger of generating unworkable awards... the arbitration authority might be forced to choose between two extreme positions, each of which was unworkable” (Stevens 1966). Tulis (2013) elaborates: “One common criticism of package final-offer arbitration is that parties may be tempted to include outrageous offers”. He further claims that “issue-by-issue final offers... are more aligned with the objectives of final-offer arbitration”. We argue the opposite—that both players’ optimal strategy in a multiple-issue FOA is to make all final-offer components reasonable. Furthermore, the additional variance in the
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awards from WP, as opposed to IBI, acts as a greater motivator for the parties to reach agreement during negotiations. We show this by extending the model of Brams and Merrill to multiple-issues and proceed to explicitly construct a pure strategy pair, proving it is the unique optimal strategy pair in many cases.

The outline of the paper is as follows: in Sect. 2 we model the problem as a zero-sum two person game, with Players I and II as minimizer and maximizer respectively. In Sect. 3 we show that the game’s value is 0 and delineate some necessary properties of pure optimal\(^1\) strategies. Next in Sect. 4 we explicitly construct the only possible pure optimal strategies \(a^*\) and \(b^*\) by first assuming the existence of such strategies and then using necessary first-order conditions to solve for them. We then verify the second-order conditions hold which guarantee that the strategies are optimal within a small neighborhood. In Sect. 5 we show that these strategies are in fact globally optimal directly by fixing the strategy \(b^*\) of Player II and showing that for any arbitrary pure strategy \(a \neq a^*\) of Player I, the payoff is strictly positive. This is handled essentially by three cases (see Fig. 1). We divide \(\mathbb{R}^2\) into three regions: points on the circle centered at the origin passing through \(b^*\), points outside the circle and points within the circle. The critical part of the proof lies showing that a certain function is everywhere greater than the normal distribution function. This is done by using a suitable logistic function as an upper-bound for points outside the circle and a tangent line for points within the circle (see Fig. 2). Finally, in Sect. 6 we look at the variability of whole-package and issue-by-issue settlements and consider the effects of risk-aversion on equilibrium strategies. In the analysis which follows, the proofs of the simpler lemmas have been moved to the appendix to improve readability.

2 Dual-issue final-offer arbitration

As this is the first attempt (to the authors’ knowledge) to formulate a detailed model of the higher-dimensional FOA game, the intention here is not to create an exhaustively general model. Instead we wish to delineate a tractable model from which we can glean some insights, highlight the many interesting ways in which it can be extended and generalized, as well as discuss the challenges in doing so. Thus our model extends the model defined by Brams and Merrill (1983). Let Player I be the minimizer and Player II the maximizer in this zero-sum game. Let us consider the case where each player makes not a single valued offer, but an ordered pair \((x_i, y_i)\), \(i = 1, 2\). Let \(v(x, y)\) be the valuation of a settlement vector \((x, y)\) by Player I (consequently, Player II values this settlement as \(-v(x, y)\)).\(^2\) If the arbiter measures the “reasonable-ness” of a final-offer by a function \(R\), then the payoff of the game is given by

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\(^1\) By optimal we are referring to the minimax solution to the zero-sum game.
\(^2\) A non-zero-sum model in which players value the components differently is absolutely worthy of consideration. This analysis, however, requires a different method of analysis and is outside the scope of this paper.
\[ K((x_1, y_1), (x_2, y_2)) = \begin{cases} v(x_1, y_1) & R(x_1, y_1) > R(x_2, y_2) \\ v(x_2, y_2) & R(x_1, y_1) < R(x_2, y_2) \\ \frac{1}{2}v(x_1, y_1) + \frac{1}{2}v(x_2, y_2) & R(x_1, y_1) = R(x_2, y_2) \end{cases} \] (1)

where the arbiter chooses either offer with equal probability if they are equally reasonable (in his opinion). This raises three important questions: how do players value settlement bundles, how do we model the function \( R \) and how to model the uncertainty of the arbiter’s opinion of fairness.

We will assume that the two issues in dispute are quantitative in nature, with both players restricted to a strategy space \( S \) which is an arbitrarily large, but compact, subset of \( \mathbb{R}^2 \). Furthermore, we will assume the valuation is additive, namely \( v(x, y) = x + y \). An example of such a situation is one in which wage and workers compensation amounts are in dispute; workers’ compensation may be valued at the expected compensation amount (in the probabilistic sense). Even issues which are not monetary, such as number of sick days, may have a straightforward monetary valuation by the parties. While, it has been noted assumptions of additivity in valuation “suppress multidimensionality and, in fact, degenerate it into a univariate case” (Levhari and Paroush 1975), even with this simple assumption the model produces interesting results.

Both players are uncertain of the arbiter’s opinion of a fair settlement \((\xi, \eta)\), but assume that the arbiter (or a fact-finder) is sampling from relevant industry data to form an opinion. Thus, by the Central Limit Theorem, we may suppose that their common prior distribution for \((\xi, \eta)\) is a bivariate normal distribution, \( N(\mu, \Sigma) \) and it is common knowledge (Aumann). Let us assume without loss of generality that \( \mu = 0 \). We will assume that these issues are positively correlated across the industry, thus

\[ \Sigma = \begin{bmatrix} \sigma_x^2 & \rho \sigma_x \sigma_y \\ \rho \sigma_x \sigma_y & \sigma_y^2 \end{bmatrix}, \]

where \( \rho > 0 \).

In the multi-issue case, FOA is typically handled in one of two ways: Issue-by-Issue (IBI) or Whole-Package (WP). Under IBIFOA the arbiter rules independently on each issue presented. A compromise of sorts may be crafted in this way. If the arbiter uses the IBI mechanic, the players are engaged in two independently decided single-issue FOA games. By the Brams–Merrill Theorem (1983), we know that the unique optimal strategy pair of the players is given by

\[ (x_{1*}, y_{1*}) = \left(-\frac{\sigma_x \sqrt{2\pi}}{2}, -\frac{\sigma_y \sqrt{2\pi}}{2}\right) \quad (x_{2*}, y_{2*}) = \left(\frac{\sigma_x \sqrt{2\pi}}{2}, \frac{\sigma_y \sqrt{2\pi}}{2}\right). \] (2)

\(^3\) Otherwise, the game may not even posses a value.
4 Under WPFOA the arbiter must rule in favor of one final-offer vector in its entirety. It is in this variant that the choice of a distance criterion needs to be chosen by the arbiter. The “distance” from a final-offer point \((x_i, y_i)\) to \((\xi, \eta)\) may be determined in a number of ways. In perhaps the simplest case, the arbiter may take the net offer from each player and side with whichever is closer to \(\xi + \eta\). While this will reduce the game to the univariate case in a sense, the optimal strategy set for each player \(i = 1, 2\) in \(\mathbb{R}^2\) will be a line given by

\[
S^*_i = \left\{ \left( x^*_i, (-1)^i \sqrt{\frac{2\pi (\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}{2} - x^*_i} \right) : x^*_i \in \mathbb{R} \right\}.
\]

This raises the question as to how reasonable this decision criterion is for the arbiter to use in the first place. For our model, we will assume that the arbiter uses \(L_2\) (Euclidean) distance:

\[
D_{L_2}((x, y), (\xi, \eta)) = \sqrt{(\xi - x)^2 + (\eta - y)^2}.
\]

Of course, other distance metrics may be reasonable and are worthy of consideration. Suppose, for example, the arbiter wishes to measure deviation from fair component-wise; an \(L_1\) distance is appropriate in this case. If instead he finds a large component deviation particularly disagreeable, an \(L_\infty\) (Chebychev) distance may be used. These and other distance criteria, including a general \(L_p\) distance, Mahalanobis distance (1936) and standardized distance are worthy of analysis but beyond the scope of this paper. They are considered by the author in detail elsewhere.

3 Properties of dual-issue FOA under \(L_2\) distance

We now establish some properties of the game. Suppose Player I chooses pure strategy \(a = (x_1, x_2)\) and Player II chooses pure strategy \(b = (x_2, y_2)\), and the arbiter considers \((\xi, \eta)\) a fair settlement. We define \(C_1(a, b)\), as the set of points in \(\mathbb{R}^2\) which are strictly closer to Player \(i\)’s final-offer than to the other player’s, namely

\[
C_1(a, b) := \left\{ (x, y) : (x_1 - x)^2 + (y_1 - y)^2 < (x_2 - x)^2 + (y_2 - y)^2 \right\}, \quad (4)
\]

\[
C_2(a, b) := \left\{ (x, y) : (x_1 - x)^2 + (y_1 - y)^2 > (x_2 - x)^2 + (y_2 - y)^2 \right\}. \quad (5)
\]

It is immediately apparent that \(C_1(a, b) = C_2(b, a)\). The midset is

\[
Mid(a, b) := \left\{ (x, y) : (x_1 - x)^2 + (y_1 - y)^2 = (x_2 - x)^2 + (y_2 - y)^2 \right\}. \quad (6)
\]

4 In the single-issue game the player who is strictly closer in absolute difference to the arbiter’s opinion of a fair settlement wins. Because the first issue is settled with \(F(x)\) Gaussian with mean 0 and standard deviation \(\sigma_x\), the Brams–Merrill theorem has \(x^*_i = (-1)^i \frac{1}{2F(0)} = (-1)^i \frac{\sigma_x\sqrt{4\pi}}{2} \).
We observe that if \( a \neq b \) then \( \text{Mid}(a, b) \) is a line so, because \((\xi, \eta)\) follows a continuous distribution, \( P((\xi, \eta) \in \text{Mid}(a, b)) = 0 \). We can now define the expected payoff to Player II from \( I \)

\[
K(a, b) = \begin{cases} 
  x_1 + y_1 & a = b \\
  (x_1 + y_1)P((\xi, \eta) \in C_1(a, b)) + (x_2 + y_2)P((\xi, \eta) \in C_2(a, b)) & a \neq b 
\end{cases} \tag{7}
\]

The following are some consequential properties of the game.

**Lemma 1** \( K(a, b) = K(b, a) \).

The first property concerns the **anonymity of final-offers**; the arbiter essentially does not care which player submits which final-offer.

**Lemma 2** Let \( -a = (-x_1, -y_1) \) and \( -b = (-x_2, -y_2) \). Then \( K(-a, -b) = -K(a, b) \).

This is due to the symmetry of the bivariate normal distribution about \((0, 0)\). If the players negate their offers then they are effectively swapping roles.

**Lemma 3** Let \( -b = (-x_2, -y_2) \). Then \( K(-b, b) = 0 \).

In other words, because the probability distribution of \((\xi, \eta)\) is symmetric, the arbiter is indifferent with regards to negating offers.

**Lemma 4** The value of the game is zero.

It may seem deceptively obvious that a symmetric zero-sum game must have a value of zero, but we have no guarantee that a value exists at all. For example, consider a simple symmetric game where Player I and II choose \( x, y \in \mathbb{R} \), and I receives a payment from II of \( x - y \). The game has no value as \( \sup_y \inf_x (x - y) = -\infty \) while \( \inf_x \sup_y (x - y) = \infty \).

**Proof** Because the strategy space \( S \) of each player is compact and the payoff \( K \) is continuous, by the general minimax theorem of Ville the game has a value \( v \) in mixed strategies (see Parthasarathy and Raghavan 1971).

First suppose an optimal pure strategy pair \( a^*, b^* \) exists. Suppose \( v > 0 \). Then for any pure strategy \( a \) of Player I, \( K(a, b^*) \geq v > 0 \). But by Lemma 3 \( K(-b^*, b^*) = 0 \), contradicting that \( v > 0 \). Similarly it cannot be the case that \( v < 0 \). Therefore \( v = 0 \).

Now suppose that optimal mixed strategies \( F_1^*, F_2^* \) exist. Suppose \( v > 0 \). Then for any mixed strategy \( F_1 \),

\[
K(F_1, F_2^*) \geq v > 0. \tag{8}
\]

Player II may approximate the optimal strategy \( F_2^* \) by \( \hat{F}_2^* \) where probability mass is concentrated only on a finite symmetric subset \( T \subset S \) such that for \( \epsilon > 0 \) small enough and for any mixed strategy \( F_1 \),

\[\text{This claim follows from three arguments: first, by a well known theorem of Varadarajan the space of all probability measures } M(S) \text{ is a compact metric space in weak topology. Secondly, the set of all probability measures } M(S) \text{ is compact. Third, by the extreme value theorem of Carathéodory, } M(S) \text{ is compact.} \]
With a change of variables $c$

$$K(F_1, \hat{F}_2^*) \geq v - \epsilon > 0. \quad (9)$$

Define

$$g_1^*(x, y) = \hat{f}_2^*(-x, -y), \forall (x, y) \in T$$

and call the associated mixed strategy $G_1^*$.

$$K(G_1^*, \hat{F}_2^*) = \sum_{(a, b) \in T \times T} g_1^*(a) \hat{f}_2^*(b) K(a, b)$$

$$= \sum_{(a, b) \in T \times T} \hat{f}_2^*(-a) g_1^*(-b) K(a, b)$$

$$= \sum_{(a, b) \in T \times T} \hat{f}_2^*(-a) g_1^*(-b) K(b, a)$$

$$= -\sum_{(a, b) \in T \times T} g_1^*(-b) \hat{f}_2^*(-a) K(-b, -a)$$

With a change of variables $c = -b, d = -a$,

$$= -\sum_{(c, d) \in T \times T} g_1^*(c) \hat{f}_2^*(d) K(c, d)$$

$$= -K(G_1^*, \hat{F}_2^*)$$

Therefore $K(G_1^*, \hat{F}_2^*) = 0$, contradicting (8), so $v \leq 0$. In a similar manner we can show that $v \geq 0$. \hfill \Box

Having established that the game has a value, we know the players must have optimal mixed strategies. The key contribution of this paper is that in fact the players have optimal pure strategies.

**Lemma 5** $(x_2, y_2)$ is an optimal pure strategy for Player II if and only if $(-x_2, -y_2)$ is an optimal pure strategy for Player I.

This is to say, if optimal pure strategies do exist then they must be symmetric about the origin.

Recall from (7), that if Player I chooses $a = (x_1, y_1)$ and Player II chooses $b = (x_2, y_2)$, assuming $a \neq b$,

$$K(a, b) = (x_1 + y_1) P\left(\left(\xi, \eta\right) \in C_1(a, b)\right) + (x_2 + y_2) P\left(\left(\xi, \eta\right) \in C_2(a, b)\right)$$

$$= (x_1 + y_1) P\left(\left(\xi, \eta\right) \in C_1(a, b)\right) + (x_2 + y_2)[1 - \left(\left(\xi, \eta\right) \in C_1(a, b)\right)]$$

$$= (x_2 + y_2) + (x_1 + y_1 - x_2 - y_2) P\left(\left(\xi, \eta\right) \in C_1(a, b)\right).$$

measures under weak topology concentrated on finite subsets of a compact metric space $S$ are themselves dense in the space of all probability measures on $S$. Lastly, by a well known theorem of Prohorov, any compact subset $T$ of $M(S)$ is characterized by the property that given $\delta$ positive, there exists a compact subset of $C$ of $T$ such that $\mu(C) > 1 - \delta$ for all $\mu$ in the set $S$ (see Parthasarathy 2014).
Lemma 6 If a pure optimal strategy pair \( a^* = (x_1^*, y_1^*), b^* = (x_2^*, y_2^*) \) exists, then \( x_2^* \geq 0, \ y_2^* \geq 0 \) and \( x_1^* \leq 0, \ y_1^* \leq 0 \).

In other words, optimal pure strategies for Players I and II must be in quadrants III and I respectively.

**Proof** We know that if both players are playing optimally then the expected payoff is zero. Suppose only one of Player II’s offers is negative,\(^6\) WLOG let \( x_2^* < 0 \). By playing \((-x_2^*, -y_2^*)\), Player I is guaranteeing a zero expected payoff. Suppose Player I instead switches to \((x_2^*, -y_2^*)\). If \( y_2^* = 0 \) then the final offers are identical and the net award is \( x_2^* \). Therefore let us assume \( y_2^* > 0 \).

\[
K((x_2^*, -y_2^*), (x_2^*, y_2^*)) = (x_2^* + y_2^*) + (x_2^* - y_2^* - x_2^* - y_2^*)
\times P((\xi, \eta) \in C_1((x_2^*, -y_2^*), (x_2^*, y_2^*)))
\quad = x_2^* + y_2^* \left(1 - 2P((\xi, \eta) \in C_1((x_2^*, -y_2^*), (x_2^*, y_2^*)))\right)
\]

Since \( C_1 = \{(x, y) : y < 0\}, P((\xi, \eta) \in C_1) = P(\eta < 0) = \frac{1}{2} \). Therefore,

\[
K((x_2^*, -y_2^*), (x_2^*, y_2^*)) = x_2^* < 0.
\]

This contradicts that \((x_2^*, y_2^*)\) is an optimal pure strategy for Player II. Thus \( x_2^* \geq 0 \). Because the choice of component is arbitrary, \( y_2^* \geq 0 \) as well. The argument is the same to show that Player I’s component offers must be non-positive in order to play optimally. \(\square\)

### 4 Local optimality of pure strategies

Having established some of the properties of the game in question, we now derive a pure strategy pair for the players and show that it is locally optimal. Recall that

\[
K(a, b) = (x_2 + y_2) + (x_1 + y_1 - x_2 - y_2)P \text{ (Player I wins).} \tag{10}
\]

The event that “Player I wins” occurs precisely when the arbiter picks a random fair settlement \((\xi, \eta)\) and

\[
(x_1 - \xi)^2 + (y_1 - \eta)^2 < (x_2 - \xi)^2 + (y_2 - \eta)^2 \tag{11}
\]

which is equivalent to

\[
(x_2 - x_1)\xi + (y_2 - y_1)\eta < \frac{x_2^2 + y_2^2 - x_1^2 - y_1^2}{2} = w. \tag{12}
\]

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\(^6\) Player II cannot possibly be playing optimally if both \( x_2^* < 0 \) and \( y_2^* < 0 \), for in this case Player I may simply agree to the Player II’s final offer and happily accept a negative net settlement.
Letting \( \Omega = (x_2 - x_1)\xi + (y_2 - y_1)\eta \), we have \( \Omega \sim \mathcal{N}(0, \sigma_\Omega^2) \) where
\[
\sigma_\Omega^2 = (x_2 - x_1)^2\sigma_x^2 + 2(x_2 - x_1)(y_2 - y_1)\rho\sigma_x\sigma_y + (y_2 - y_1)^2\sigma_y^2.
\] (13)

Thus \( \Omega/\sigma_\Omega \) follows a standard normal distribution and we may express the expected payoff as
\[
K(a, b) = (x_2 + y_2) + (x_1 + y_1 - x_2 - y_2)\Phi(z)
\] (14)
where \( \Phi(z) \) is the distribution function of a standard normal random variable and
\[
z = \frac{w}{\sigma_\Omega} = \frac{x_2^2 + y_2^2 - x_1^2 - y_1^2}{2\sqrt{(x_2 - x_1)^2\sigma_x^2 + 2(x_2 - x_1)(y_2 - y_1)\rho\sigma_x\sigma_y + (y_2 - y_1)^2\sigma_y^2}}.
\] (15)

**Theorem 1** If \( \rho > \max\left\{ -\frac{\sigma_x^2 + \sigma_y^2}{4\sigma_x\sigma_y}, -\frac{3\sigma_x^2 + \sigma_y^2}{4\sigma_x\sigma_y}\right\} \) then the pure strategies given by
\[
(x_i^*, y_i^*) = \left( (-1)^i \sqrt{\frac{2\pi(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}{4}}, (-1)^i \sqrt{\frac{2\pi(\sigma_y^2 + 2\rho\sigma_x\sigma_y + \sigma_x^2)}{4}} \right)
\] (16)
for players \( i = 1, 2 \) constitute a locally optimal strategy pair.\(^7\)

**Proof** If the players have locally optimal pure strategies \( a^* \) and \( b^* \) then we must have all four first derivatives zero, namely
\[
\frac{\partial K}{\partial x_1} = \Phi(z) + (x_1 + y_1 - x_2 - y_2)\phi(z) \\
\quad \times \left( -\frac{x_1}{\sigma_\Omega} + \frac{(x_2 - x_1)\sigma_x^2 + (y_2 - y_1)\rho\sigma_x\sigma_y}{\sigma_\Omega^2} \right) = 0
\] (17)
\[
\frac{\partial K}{\partial y_1} = \Phi(z) + (x_1 + y_1 - x_2 - y_2)\phi(z) \\
\quad \times \left( -\frac{y_1}{\sigma_\Omega} + \frac{(x_2 - x_1)\rho\sigma_x\sigma_y + (y_2 - y_1)\sigma_y^2}{\sigma_\Omega^2} \right) = 0
\] (18)
\[
\frac{\partial K}{\partial x_2} = 1 - \Phi(z) + (x_1 + y_1 - x_2 - y_2)\phi(z) \\
\quad \times \left( \frac{x_2}{\sigma_\Omega} - \frac{(x_2 - x_1)\sigma_x^2 + (y_2 - y_1)\rho\sigma_x\sigma_y}{\sigma_\Omega^2} \right) = 0
\] (19)

\(^7\) Namely, \( \exists \varepsilon > 0 \) such that for pure strategies \( a^* \) and \( b^* \),
\[
\inf_{a \in N_\varepsilon(a^*)} K(a, b^*) = \sup_{b \in N_\varepsilon(b^*)} K(a^*, b) = K(a^*, b^*),
\]
where \( N_\varepsilon(\cdot) \) is an \( \varepsilon \)-neighborhood.
\[ \frac{\partial K}{\partial y_2} = 1 - \Phi(z) + (x_1 + y_1 - x_2 - y_2)\phi(z) \times \left( \frac{y_2}{\sigma_\Omega} - \frac{(x_2 - x_1)\rho\sigma_x\sigma_y + (y_2 - y_1)\sigma_y^2}{\sigma_\Omega^2} \right) = 0 \] \hspace{1cm} (20)

Since \( 0 < \Phi(z^*) < 1 \), \( x_1^* + y_1^* - x_2^* - y_2^* \neq 0 \). By adding (17) and (19) we have

\[ 0 = 1 + \frac{x_2^* - x_1^*}{\sigma_\Omega^*} (x_1^* + y_1^* - x_2^* - y_2^*)\phi(z^*) \] \hspace{1cm} (21)

and by adding (18) and (20) we have

\[ 0 = 1 + \frac{y_2^* - y_1^*}{\sigma_\Omega^*} (x_1^* + y_1^* - x_2^* - y_2^*)\phi(z^*). \] \hspace{1cm} (22)

From (21) and (22) we know that

\[ x_2^* - x_1^* = y_2^* - y_1^* = d^* \neq 0. \] \hspace{1cm} (23)

Note also that

\[ \sigma_\Omega^{*2} = d^*(\alpha + \beta), \] \hspace{1cm} (24)

where \( \alpha = \sigma_x^2 + \rho\sigma_x\sigma_y \) and \( \beta = \rho\sigma_x\sigma_y + \sigma_y^2 \). Furthermore, we now have that

\[ z^* = \frac{d^*((x_2^* + x_1^*) + (y_2^* + y_1^*))}{2\sigma_\Omega^*} = \frac{x_2^* + x_1^* + y_2^* + y_1^*}{2\sqrt{\alpha + \beta}} \text{sgn}(d^*). \] \hspace{1cm} (25)

We may now simplify the four equations derived from (17)–(20) as

\[ 0 = \Phi(z^*) + 2\phi(z^*) \left( \frac{\text{sgn}(d^*)x_1^*}{\sqrt{\alpha + \beta}} - \frac{\alpha}{\alpha + \beta}z^* \right) \] \hspace{1cm} (26)

\[ 0 = \Phi(z^*) + 2\phi(z^*) \left( \frac{\text{sgn}(d^*)y_1^*}{\sqrt{\alpha + \beta}} - \frac{\beta}{\alpha + \beta}z^* \right) \] \hspace{1cm} (27)

\[ 0 = 1 - \Phi(z^*) - 2\phi(z^*) \left( \frac{\text{sgn}(d^*)x_2^*}{\sqrt{\alpha + \beta}} - \frac{\alpha}{\alpha + \beta}z^* \right) \] \hspace{1cm} (28)

\[ 0 = 1 - \Phi(z^*) - 2\phi(z^*) \left( \frac{\text{sgn}(d^*)y_2^*}{\sqrt{\alpha + \beta}} - \frac{\beta}{\alpha + \beta}z^* \right) \] \hspace{1cm} (29)

By taking (26) + (27) - (28) - (29) and using (25) we get

\[ 0 = -2 + 4\Phi(z^*) + 2\phi(z^*) \left( \frac{x_1^* + y_1^* + x_2^* + y_2^*}{\sqrt{\alpha + \beta}} \text{sgn}(d^*) - 2\frac{\alpha + \beta}{\alpha + \beta}z^* \right) \]

\[ \frac{1}{2} = \Phi(z^*) + 2\phi(z^*)(2z^* - 2z^*) \]

\[ = \Phi(z^*) \]
Thus $z^* = 0$, and by simplifying the four equations (26)–(29) we get that $x_1^* = y_1^* = -x_2^* = -y_2^*$. We simplify $\sigma_{12}^2 = 4x_1^2(\alpha + \beta)$ and noting that $\phi(0) = \frac{1}{\sqrt{2\pi}}$, Eq. (26) becomes

$$0 = \frac{1}{2} + 2\phi(0) \left( \frac{\text{sgn}(d^*)x_1^*}{\sqrt{\alpha + \beta}} \right),$$

or equivalently

$$x_1^* = -\sqrt{\frac{2\pi(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}{4}} \frac{\text{sgn}(d^*)}{\sqrt[4]{\frac{\alpha + \beta}{\sqrt{\alpha + \beta}}}}.$$  

(30)

We have two solutions to the first order equations, namely $d^* > 0$ and $d^* < 0$. Let us choose the solution given by $d^* > 0$. To show that $b^*$ is a local maximum for Player II and $a^*$ is a local minimum for Player I we look at the second partial derivatives evaluated at $(a^*, b^*)$. It is straightforward to verify that

$$\left. \frac{\partial^2 K}{\partial x_1^2} \right|_{(a^*, b^*)} = \frac{\phi(0)}{\sqrt{\alpha + \beta}} \left( 3 - \frac{2\alpha}{\alpha + \beta} \right).$$

This is positive if and only if $\alpha + 3\beta > 0$, or equivalently, when

$$\rho > -\frac{3\sigma_x^2 + 3\sigma_y^2}{4\sigma_x\sigma_y}.$$  

(32)

Similarly,

$$\left. \frac{\partial^2 K}{\partial y_1^2} \right|_{(a^*, b^*)} = \frac{\phi(0)}{\sqrt{\alpha + \beta}} \left( 3 - \frac{2\beta}{\alpha + \beta} \right)$$

which is positive if and only if

$$\rho > -\frac{3\sigma_x^2 + \sigma_y^2}{4\sigma_x\sigma_y}.$$  

(33)

Note that it is impossible for both (32) and (33) to be unsatisfied, as this would imply that $\alpha < 0$ and $\beta < 0$ and thus $\alpha + \beta < 0$. It is likewise easily shown that

$$\left. \frac{\partial^2 K}{\partial y_1 \partial x_1} \right|_{(a^*, b^*)} = 0,$$

thus $K_{x_1} K_{y_1} - K_{x_1y_1}^2 > 0$ as long as

$$\rho > \max \left\{ -\frac{\sigma_x^2 + 3\sigma_y^2}{4\sigma_x\sigma_y}, -\frac{3\sigma_x^2 + \sigma_y^2}{4\sigma_x\sigma_y} \right\}.$$
It can be similarly verified that $b^*$ is a local maximum for Player II when Player I plays $a^*$, with the same condition on $\rho$. Thus we have a locally optimal pair of pure strategies.

It is important to highlight that if the issues are too negatively correlated, pure optimal strategies need not exist. For example, suppose $\sigma_x = 1, \sigma_y = \sqrt{3}$ and $\rho = -0.9$. If optimal pure strategies do exist, they must be

$$a^* = (-0.588627, -0.588627), b^* = (0.588627, 0.588627).$$

However,

$$K(a', b^*) = -0.003085 < 0 = K(a^*, b^*),$$

where $a' = (-0.572278, -1.029105)$. Because Player I has a strict incentive to deviate to a new pure strategy, $a^*$ is not optimal (and consequently, neither is $b^*$).

5 Global optimality of pure strategies

We now proceed to show that the pure strategies found in the preceding section are indeed globally optimal and thus represent the unique optimal strategy pair. We must first establish a few lemmas which will motivate the geometric interpretation of the model which follows.

**Lemma 7** Suppose Player II chooses strategy $b^* = (x_2^*, x_2^*)$. If Player I selects pure strategy $a = (x_1, y_1)$ then $z(a, b^*) = 0$ iff $(x_1, y_1)$ lies on the circle of radius $\sqrt{2}x_2^*$ centered at the origin. Furthermore, $x_1^2 + y_1^2 < 2x_2^*^2$ iff $z > 0$ and $x_1^2 + y_1^2 > 2x_2^*^2$ iff $z < 0$.

The proof is trivial.

**Lemma 8** If another pure strategy $a = (x_1, y_1) \neq -b^* = (-x_2^*, -x_2^*)$ exists such that $K(a, b^*) \leq 0$, then $x_1 + y_1 < 0$ and either

$$x_1^2 + y_1^2 < 2x_2^*^2 \quad \text{or} \quad x_1 + y_1 \leq -2x_2^*.$$

**Proof** Suppose $x_1 + y_1 \geq 0$. Because the net offer of Player II, $2x_2^* > 0$, and Player II has a positive probability $p$ of being chosen by the arbiter, the expected payoff

$$K(a, b^*) = p(2x_2^*) + (1-p)(x_1 + y_1) > 0.$$

This contradicts our assumption.

Suppose $x_1^2 + y_1^2 \geq 2x_2^*^2$. Then $z \leq 0$ by Lemma 7 and $\Phi(z) \leq \frac{1}{2}$. Suppose also that $x_1 + y_1 > -2x_2^*$. Then

$$x_1 + y_1 - 2x_2^* = -4x_2^* + \epsilon$$
If Player II fixes his strategy 

\[ b^* = (x_2^*, x_1^*) \]

we show that for all pure strategies 

\[ a \neq (-x_2^*, -x_1^*) \]

\( K(a, b^*) > 0 \). In Lemma 8 we show this is true in the gray shaded region. We consider cases of \( z : z = 0 \), the points on the blue circle, in Proposition 1; \( z > 0 \), points outside the circle, in Proposition 2; \( z < 0 \) (points within the circle) in Proposition 3.

for some \( 0 < \epsilon < 2x_2^* \). But then

\[
K(a, b^*) = 2x_2^* + (x_1 + y_1 - 2x_2^*)\Phi(z) \\
= 2x_2^* - (4x_2^* - \epsilon)\Phi(z) \\
\geq 2x_2^* - (4x_2^* - \epsilon) \frac{1}{2} \\
= \frac{\epsilon}{2} \\
> 0
\]

and this contradicts our assumption.

We now proceed to show that if Player II chooses pure strategy 

\[ b^* = (x_2^*, x_1^*) \]

and I deviates from 

\[ a^* = (-x_2^*, -x_1^*) \]

deviates from \( a^* = (-x_2^*, -x_1^*) \) to any other pure strategy \((x_1, y_1)\) then it will simply result in a positive expected payoff. For the remainder of the paper we will assume \( \rho \geq 0 \) and without loss of generality \( \sigma_x \leq \sigma_y \).

**Proposition 1** For all pure strategies 

\[ a = (x_1, x_2) \neq (-x_2^*, -x_1^*) \] such that 

\[ x_1^2 + y_1^2 = 2x_2^* \]

\( K(a, b^*) > 0 \).

---

8 It is the author’s conjecture that global optimality of pure strategies will hold with a weaker condition on \( \rho \), namely that of Theorem 1.
Proof If \( x_1^2 + y_1^2 = 2x_2^{*2} \), \( z(a, b^*) = 0 \), so

\[
K(a, b^*) = 2x_2^{*2} + (x_1 + y_1 - 2x_2^{*2}) \Phi(0)
\]

\[
= 2x_2^{*2} + (x_1 + y_1 - 2x_2^{*2}) \frac{1}{2}
\]

\[
= x_2^{*2} + \frac{x_1 + y_1}{2}.
\]

Geometrically we can see that \( x_1 + y_1 \) is minimized on the circle \( x_1^2 + y_1^2 = 2x_2^{*2} \) at \((-x_2^*, -x_2^*)\).

Against Player II’s strategy \( b^* = (x_2^*, x_2^*) \), any pure strategy \( a = (x_1, y_1) \) may be represented in terms of \( r \) and \( \theta \) as \((x_2^* + r \cos \theta, x_2^* + r \sin \theta)\). This will greatly facilitate the remaining proofs.\(^9\) In this representation, with \( t(\theta) = -(\cos \theta + \sin \theta) \), we can rewrite

\[
K(a, b^*) = 2x_2^* + r(\cos \theta + \sin \theta) \Phi(z) = 2x_2^* - rt(\theta) \Phi(z)
\]  

(34)

and

\[
z(r, \theta) = \frac{2x_2^{*2} - (x_2^* + r \cos \theta)^2 - (x_2^* + r \sin \theta)^2}{2r \sqrt{\sigma_\alpha^2 \cos^2 \theta + 2\rho \sigma_x \sigma_y \cos \theta \sin \theta + \sigma_\beta^2 \sin^2 \theta}} = \frac{2x_2^* t(\theta) - r}{2 \sqrt{\sigma_\theta^2}}
\]  

(35)

The following two lemmas are needed for Proposition 2.

Lemma 9 For \( z < 0 \), the scaled logistic function

\[
s(z) := \frac{1}{1 + \exp \left( -\sqrt{\frac{8}{\pi}} z \right)} > \Phi(z).
\]

Proof Consider \( s(z) - \Phi(z) \) and find the minimum:

\[
s'(z) - \Phi(z) = s(z)(1 - s(z)) \sqrt{\frac{8}{\pi}} - \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = 0
\]

\[
\Leftrightarrow e^{\sqrt{\frac{8}{\pi}} z} + 2 + e^{-\sqrt{\frac{8}{\pi}} z} = 4e^{z^2/2}
\]

\[
\Leftrightarrow 1 + \cosh \left( \sqrt{\frac{8}{\pi}} z \right) = 2e^{z^2/2}
\]

\[
\Leftrightarrow \cosh \left( \sqrt{\frac{8}{\pi}} z \right) = e^{z^2/4}
\]

\(^9\) For convenience we will define \( t(\theta) := -(\cos \theta + \sin \theta) \) and \( \sigma_\theta^2 := \sigma_\alpha^2 \cos^2 \theta + 2\rho \sigma_x \sigma_y \cos \theta \sin \theta + \sigma_\beta^2 \sin^2 \theta \). Note that \( t(\theta) = -\sqrt{2} \sin(\theta + \frac{\pi}{4}) \).
The three roots are at $z = 0$ and $z \approx \pm 1.7318300869742718735$. It is clear to see that $s(0) = \Phi(0)$. Because both $\Phi(z)$ and $s(z)$ are $\frac{1}{\sqrt{2\pi}}$ Lipschitz continuous, $s(z) - \Phi(z)$ is $\frac{\sqrt{2}}{\sqrt{2\pi}}$ Lipschitz continuous (and thus 1-Lipschitz continuous). Since $s(-1.7318300869742718735) - \Phi(-1.7318300869742718735) \approx .017671$, this point represents a local maximum, i.e. here the two curves are furthest apart.

Next observe that

$$\lim_{z \to -\infty} \frac{s(z)}{\Phi(z)} = \lim_{z \to -\infty} \frac{\cosh\left(\frac{\sqrt{2}}{\sqrt{2\pi}} z\right)}{e^{z^2/4}} = \lim_{z \to -\infty} \frac{e^{\sqrt{2}/\pi z} + e^{-\sqrt{2}/\pi z}}{e^{z^2/4}} = \lim_{z \to -\infty} e^{\sqrt{2}/\pi z - z^2/4} + \lim_{z \to -\infty} e^{-\sqrt{2}/\pi z - z^2/4} = \infty + 0.$$ 

It cannot be the case that $s(z) - \Phi(z) < 0$ at any point $z < 0$, as this would imply the existence of a local minimum of $s(z) - \Phi(z)$, of which we know there is none. Thus $s(z) > \Phi(z)$ for all $z < 0$. 

**Lemma 10** The general exponential curve $y = \beta e^{\alpha x}$ ($\alpha, \beta > 0$) and line $y = mx$ have:

\[
\begin{cases}
1 \text{ intersection at } x^* < 0 & \text{ if } m < 0 \\
\text{No intersection} & \text{ if } 0 \leq m < \alpha \beta e \\
1 \text{ intersection at } x^* = \frac{1}{\alpha} & \text{ if } m = \alpha \beta e \\
2 \text{ intersections} & \text{ if } m > \alpha \beta e
\end{cases}
\]

**Proof** First let $w = \alpha x$

$$y = \beta e^w \quad y = \frac{m}{\alpha} w$$

Now let $z = \frac{\gamma}{\beta}$

$$z = e^w \quad z = \frac{m}{\alpha \beta} w = \gamma w$$

If $\gamma = e$ then the two curves have a single intersection, and are tangent, at $w = 1$. If $0 \leq \gamma < e$ then they cannot intersect. If $\gamma > e$ they will intersect at two points.  

\begin{tikzpicture}
\end{tikzpicture}
Fig. 2 This sketch shows the curves $\Phi$ and $f$ defined below for a fixed $\theta$. To show that $a^*$ and $b^*$ are a globally optimal pure strategy pair, we show that $\Phi$ and $f$ intersect only at $r_0$ when $\theta = \frac{5\pi}{4}$ (Proposition 1) and everywhere else $f > \Phi$. To prove this, we show that for $r > r_0$, $f > s > \Phi$, where $s$ is a scaled logistic curve (Proposition 2), while for $r < r_0$, $f > y$, a line tangent to $\Phi$ (Proposition 3).

**Proposition 2** For all pure strategies $a = (x_1, x_2)$ such that $x_1^2 + y_2^2 > 2x_2^*\xi$, $K(a, b^*) > 0$.

**Proof** The claim may be equivalently expressed as

$$K(a, b^*) = 2x_2^* - rt(\theta)\Phi(z) > 0 \Leftrightarrow \Phi(z) < \frac{2x_2^*}{rt(\theta)} = f(r, \theta). \quad (36)$$

From Lemma 9, for $z < 0$, we note that a scaled logistic function

$$s(z) := \frac{1}{1 + \exp\left(-\frac{8}{\pi}z\right)} > \Phi(z).$$

We will indeed prove a stronger claim, namely that for all points $a$ in this region$^{10}$,

$$s(z) < f(r, \theta) \Leftrightarrow 2x_2^*\left(\frac{1}{s(z(r + \hat{r}, \theta))} - 1\right) > 2x_2^*\left(\frac{1}{f(r + \hat{r}, \theta)} - 1\right),$$

where $\hat{r} = \frac{2x_2^*}{rt(\theta)}$. Let us define the left and right side functions as $\hat{s}(r, \theta)$ and $\hat{f}(r, \theta)$ respectively. These may be explicitly written as

$^{10}$ Strictly speaking, for $\theta \in (\pi, \frac{5\pi}{4})$ (Case 4) the proof proceeds in a different way.
\[ s(r, \theta) = \sqrt{\pi (\alpha + \beta)} \exp \left( -\sqrt{\frac{8}{\pi}} \frac{2x_s^2 t(\theta)}{2\sigma_\theta} - r \right) \]

\[ = \sqrt{\pi (\alpha + \beta)} \exp \left( \frac{1}{t(\theta)} - t(\theta) \right) \frac{\sqrt{\alpha + \beta}}{\sigma_\theta} \exp \left( \frac{2}{\pi \sigma_\theta^2} r \right), \] (37)

and

\[ f(r, \theta) = 2x_s^2 \left( \frac{r + \frac{2x_s^2}{t(\theta)} t(\theta)}{2x_s^2} - 1 \right) = t(\theta) r. \] (38)

We have simply translated and scaled \( s \) and \( f \) so that, for fixed \( \theta \), \( \hat{s}(r, \theta) = be^{ar} \) and \( \hat{f}(r, \theta) = mr \). By Lemma 10, we need show that \( \forall \theta \in [\frac{3\pi}{4}, \frac{7\pi}{4}] \), \( t(\theta) \leq A(\theta) e \) and attains equality only at \( \theta = \frac{5\pi}{4} \), where

\[ A(\theta) := a(\theta) b(\theta) = \frac{\sqrt{\alpha + \beta}}{\sigma_\theta} \exp \left( \frac{1}{t(\theta)} - t(\theta) \right) \frac{\sqrt{\alpha + \beta}}{\sigma_\theta}. \] (40)

**Case 1:** \( \theta \in [\frac{3\pi}{4}, \pi] \cup [\frac{3\pi}{2}, \frac{7\pi}{4}] \)

In this case,

\[ A(\theta) e \geq e > 1 \geq t(\theta). \]

**Case 2:** \( \theta = \frac{5\pi}{4} \)

\[ A \left( \frac{5\pi}{4} \right) e = \frac{\sqrt{\alpha + \beta}}{\sqrt{\frac{1}{2}(\alpha + \beta)}} \exp \left( \frac{\sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}(\alpha + \beta)}} \sqrt{\alpha + \beta} \right) e = \sqrt{2} e^{-1+1} = \sqrt{2} = t \left( \frac{5\pi}{4} \right). \]

**Case 3:** \( \theta \in (\frac{5\pi}{4}, \frac{3\pi}{2}) \)

On this interval it is clear that \( t(\theta) \) is decreasing. We will show that on this interval \( A \) is bounded below by an increasing function \( A \) and that \( A \left( \frac{5\pi}{4} \right) = A \left( \frac{5\pi}{4} \right) \). First consider the partial derivatives with respect to \( \rho \) and \( \sigma_x \):

\[ \frac{dA}{d\sigma_x} = \frac{1}{2} \exp \left( \frac{1}{t(\theta)} - t(\theta) \right) \frac{\sqrt{\alpha + \beta}}{\sigma_\theta} \left( \frac{\sigma_\theta}{\sqrt{\alpha + \beta}} + \frac{1}{t(\theta)} - t(\theta) \right) \left( \frac{d}{d\sigma_x} \frac{\alpha + \beta}{\sigma_\theta^2} \right) \]

\[ \frac{dA}{d\rho} = \frac{1}{2} \exp \left( \frac{1}{t(\theta)} - t(\theta) \right) \frac{\sqrt{\alpha + \beta}}{\sigma_\theta} \left( \frac{\sigma_\theta}{\sqrt{\alpha + \beta}} + \frac{1}{t(\theta)} - t(\theta) \right) \left( \frac{d}{d\rho} \frac{\alpha + \beta}{\sigma_\theta^2} \right) \]

\[ \frac{d}{d\rho} \frac{\alpha + \beta}{\sigma_\theta^2} = \frac{2\sigma_x \sigma_y (\cos \theta - \sin \theta) (\rho \sigma_y^2 + \sigma_x \sigma_y) \cos \theta + (\rho \sigma_y^2 + \sigma_x \sigma_y) \sin \theta}{\sigma_y^4} \]

\[ \frac{d}{d\sigma_x} \frac{\alpha + \beta}{\sigma_\theta^2} = \frac{2\sigma_x \sigma_y (\cos \theta - \sin \theta) (\sigma_y^2 \cos \theta - \sigma_x^2 \sin \theta)}{\sigma_y^4} \]
For $\theta \in (\frac{5\pi}{4}, \frac{3\pi}{2})$:

$$\frac{d}{d\rho} \left( \frac{\alpha + \beta}{\sigma_0^2} \right) > 0, \quad \frac{d}{d\sigma_x} \left( \frac{\alpha + \beta}{\sigma_0^2} \right) > 0.$$  \hspace{1cm} (41)

and

$$G(\theta) = \frac{\sigma_0}{\sqrt{\alpha + \beta}} + \frac{1}{t(\theta)} t(\theta) > 0.$$  \hspace{1cm} (42)

Inequalities of (41) are self-evident. To justify (42), note that (41) implies that

$$\frac{d}{d\rho} \left( \frac{\sigma_0^2}{\alpha + \beta} \right) < 0, \quad \frac{d}{d\sigma_x} \left( \frac{\sigma_0^2}{\alpha + \beta} \right) < 0.$$  \hspace{1cm} (43)

Thus, for any fixed $\theta \in (\frac{5\pi}{4}, \frac{3\pi}{2})$,

$$\frac{\sigma_0^2}{\alpha + \beta} \geq \frac{\sigma_0^2}{\alpha + \beta} \bigg|_{\rho=1,\sigma_x=\sigma_y} = \frac{t(\theta)^2}{4}.$$  

Because $1 < t(\theta) < \sqrt{2}$,

$$G(\theta) = \frac{t(\theta)}{2} - \frac{1}{t(\theta)} t(\theta) = \frac{2 - t(\theta)^2}{2t(\theta)} > 0.$$ 

Since $A$ is an increasing function of $\rho$ and $\sigma_x$,

$$A(\theta) \geq A(\theta) := A(\theta)_{\rho=0,\sigma_x=0} = -\frac{1}{\sin \theta} \exp \left( \frac{-2 \cos \theta}{\sin \theta + \cos \theta} \right).$$ 

The derivative

$$\frac{dA(\theta)}{d\theta} = \exp \left( \frac{-2 \cos \theta}{\sin \theta + \cos \theta} \right) \frac{\cos \theta - 2 \sin^3 \theta}{\sin^2 \theta t(\theta)^2}$$ 

is positive for any $\theta \in (\frac{5\pi}{4}, \frac{3\pi}{2})$. Since $A(\frac{5\pi}{4}) = A(\frac{5\pi}{4})$ and $A(\theta)$ is increasing, we are done.

**Case 4:** $\theta \in (\pi, \frac{5\pi}{4})$

Recall from (36) that $K > 0$ is equivalent to

$$\Phi(z) < f(r, \theta) = \frac{2x^*_z}{rt(\theta)}.$$ 

Let us fix $r^* > 2\sqrt{2}x^*_z$. The proof proceeds via three claims:

**Claim 1:** $f(r^*, \theta)$ is a decreasing function on this interval.

This claim follows immediately after noting that $t(\theta)$ is an increasing function on $(\pi, \frac{5\pi}{4})$. 

$\square$ Springer
Claim 2: $\Phi(z(r^*, \theta))$ is an increasing function for $\theta \in (\pi, \frac{5\pi}{4})$.

Recall that $z = \frac{2x^2 t(\theta) - r}{2\sqrt{\sigma^2}}$. For $\theta \in (\pi, \frac{5\pi}{4})$, $r_0 = 2x^2 r(\theta)$ is increasing and attains its maximum value of $2\sqrt{2}x^* \theta^2$ when $\theta = \frac{5\pi}{4}$. Since $r^* > 2\sqrt{2}x^* \geq r_0$, $z(r^*, \theta) < 0$.

Because $d\sigma^2 = 2\rho \sigma_x \sigma_y (\cos^2 \theta - \sin^2 \theta) + 2(\sigma_y^2 - \sigma_x^2) \cos \theta \sin \theta > 0$, $\sigma^2$ is increasing on $(\pi, \frac{5\pi}{4})$.

Consider $|z| = \frac{r^* - 2x^2 t(\theta)}{2\sqrt{\sigma^2}}$. For $\theta \in (\pi, \frac{5\pi}{4})$, $t(\theta)$ increases so the numerator is decreasing. Meanwhile the denominator is increasing. Thus $\Phi(z(r^*, \theta))$ is an increasing function in $(\pi, \frac{5\pi}{4})$.

Claim 3: $\Phi(z(r^*, \frac{5\pi}{4})) < f(r^*, \frac{5\pi}{4})$.

If we fix $\theta = \frac{5\pi}{4}$, then the players are in the one-dimensional FOA game, and we already know that $a^* = -b^*$ (i.e. $r = 2\sqrt{2}x^*$) is the globally optimal strategy for Player I to play against $b^*$. Since we have fixed $r^* > 2\sqrt{2}x^*$, Player I is not playing optimally, so $K > 0$ which is equivalent to the claim.

From these three claims it follows that $K > 0$ for $r > 2\sqrt{2}x^*$ and $\theta \in (\pi, \frac{5\pi}{4})$. □

Now that we have shown that against $(x^*_2, x^*_2)$ all pure strategies $(x, y)$ for Player I outside the circle $x^2 + y^2 = 2x^2$ will give a positive expected payoff, we consider strategies within the circle.

Proposition 3 For all pure strategies $a = (x_1, x_2)$ such that $x_1^2 + y_1^2 < 2x^2$, $K(a, b^*) > 0$.

The proof relies on the concavity of the normal distribution function for $z > 0$.

Proof From Lemma 8, we need only show that $K(a, b^*) > 0$ for all $a$ in the semi-circle described by

$$\begin{cases} x + y < 0, \\ x^2 + y^2 < 2x^2. \end{cases}$$

In terms of $\theta$, we are restricting our attention to $\theta \in (\pi, \frac{3\pi}{2})$. For the angles $\theta$ in question,

$$t(\theta) > 1. \quad (44)$$

Recall from (34) that $K(a, b^*) > 0$ is equivalent to

$$\Phi(z) < f(r, \theta) = \frac{2x^*}{rt(\theta)}.$$
First we fix \( \tilde{\theta} \in (\pi, \frac{3\pi}{2}) \). Let \( r_0 = 2x^* t(\tilde{\theta}) \). Note by definition that \( z(r_0, \tilde{\theta}) = 0 \).

Since \( z = \frac{r - r_0}{2\sqrt{\sigma^2}} \), it is straightforward to show that

\[
\left. \frac{d}{dr} \Phi(z) \right|_{z=0} = \phi(z) \left. \frac{dz}{dr} \right|_{z=0} = \frac{1}{\sqrt{2\pi}} \frac{-1}{2\sqrt{\sigma^2}} \frac{-1}{2\sqrt{2\pi\sigma^2}}.
\]

Define \( y \) as the line tangent to \( \Phi \) at \((r_0, \frac{1}{2})\), specifically,

\[
y(r, \tilde{\theta}) = -\frac{r - r_0}{2\sqrt{2\pi\sigma^2}} + \frac{1}{2}.
\]

Note \( \Phi \) is a concave function for \( r < r_0 \). Therefore, \( \Phi(z(r, \tilde{\theta})) \leq y(r, \tilde{\theta}) \). To demonstrate that \( f > \Phi \) for all \( r < r_0 \), it suffices to show that \( f > y \) for all \( r \). Since \( f \) and \( y \) are both continuous functions and \( \lim_{r \to 0^+} f(r, \tilde{\theta}) = \infty \gg y(0, \tilde{\theta}) \), it suffices to show that \( f \neq y \) for any \( r \). If the two curves do intersect, then there is at least one solution to the equation

\[
\sqrt{2\pi(\alpha + \beta)} + t(\tilde{\theta}) \sqrt{\alpha + \beta} - \frac{1}{2} r \sqrt{2\pi\sigma^2} \tilde{\theta} + \frac{1}{2} t(\tilde{\theta}) \sqrt{2\pi(\alpha + \beta)} + \frac{1}{2} r \left( \frac{t(\tilde{\theta})}{2} \right) \sqrt{2\pi\sigma^2} = 0.
\]

We have a quadratic in \( r \). Let the vertex be

\[
\hat{r} = \frac{t(\tilde{\theta})}{4} + \frac{\sqrt{2\pi\sigma^2}}{2},
\]

and the discriminant, \( \Delta \), for the quadratic is

\[
\Delta = \left( \frac{t(\tilde{\theta})}{2} \right)^2 + \frac{8\pi(\alpha + \beta)\sigma^2}{t(\tilde{\theta})}.
\]

If \( \Delta < 0 \) then we are done. Let us assume that \( \Delta \geq 0 \). If \( f(r^*, \tilde{\theta}) = y(r^*, \tilde{\theta}) \), it must be that \( r^* < r_0 \); for \( r \geq r_0 \), \( f(r, \tilde{\theta}) > \Phi(z(r^*, \tilde{\theta})) > y(r^*, \tilde{\theta}) \). This gives us a condition, namely \( \hat{r} + \frac{\sqrt{\Delta}}{2} < r_0 \).
\[
\sqrt{\sigma^2_{\theta}} + \sqrt{\left( \frac{t(\tilde{\theta})}{2} \right)^2 + \sqrt{\sigma^2_{\theta}}} \leq \frac{\hat{r} + \sqrt{\Delta}}{2} < r_0
\]

in other words, \( t(\tilde{\theta}) < 1 \). But recall from (44) that \( t(\tilde{\theta}) > 1 \), which is a contradiction. \( \Box \)

The following is the main result.

**Theorem 2** If \( \rho \geq 0 \), then \( a^* = (-x^*_2, -x^*_2) \), \( b^* = (x^*_2, x^*_2) \) is the unique globally optimal pure strategy pair.

**Proof** This follows from Propositions 1, 2 and 3. Without any loss of generality, assume \( \sigma_x \leq \sigma_y \). If Player II plays pure strategy \( b^* \), then for any pure strategy \( a = (x_1, y_1) \), \( K(a, b^*) \geq 0 \), and equality is only achieved when \( a = a^* \). Similarly, if Player I plays \( a^* \), \( K(a^*, b) \leq 0 \) and equality is only achieved when \( b = b^* \). \( \Box \)

6 A comparison of issue-by-issue and whole package outcomes

Having shown that under an \( L_2 \) distance criterion there is a unique pure optimal strategy pair, we consider the question of whether the issue-by-issue or whole-package variant is more in line with the aims of FOA. First we look at the optimal net demands of the parties under the two scenarios.

**Proposition 4** Under the assumptions of this model, the net demand under WPFOA is no more extreme than under IBI.

**Proof** Under IBI, the net demand solution of Player II is \( \sqrt{\frac{2\pi}{2}} (\sigma_x + \sigma_y) \). Under WP, however, the net demand is

\[
\sqrt{\frac{2\pi}{2}} \sqrt{\sigma_x^2 + 2\rho \sigma_x \sigma_y + \sigma_y^2} \leq \sqrt{\frac{2\pi}{2}} (\sigma_x + \sigma_y) = \sqrt{\frac{2\pi}{2}} (\sigma_x + \sigma_y).
\]

The inequality is strict if \( \rho < 1 \). \( \Box \)
WPFOA seems to encourage more convergence to the middle ground in the sense that parties are less inclined to take extreme stances than if the issues were settled independently. Next we compare the uncertainty of the arbitrated outcomes. As Stevens says, “Generally speaking, this criterion generates just the kind of uncertainty about the location of the arbitration award that is well calculated to recommend maximin notions of prudence to the parties and, hence, compel them to seek security in agreement” (Stevens 1966). If FOA makes arbitration a costly alternative by its inherent uncertainty, we may compare the uncertainty (i.e. variance) between optimal strategies under the two mechanisms. It may come as no surprise that the arbitrated outcome in WPFOA has a higher variance.

**Proposition 5** The expected payoff is zero under both Issue-by-Issue and Whole-Package variants. If both players play optimally then the variances of the awards are \( \frac{\pi}{2} (\sigma_x^2 + \sigma_y^2) \) and \( \frac{\pi}{2} (\sigma_x^2 + 2 \rho \sigma_x \sigma_y + \sigma_y^2) \) respectively.

**Proof** Under IBIFOA, since the components are awarded independently. Let \( K_x \) and \( K_y \) be the awards for the first and second issue in dispute respectively. The variance is

\[
Var(K) = Var(K_x + K_y) = E(K_x^2) + E(K_y^2) = \frac{1}{2} \left( \frac{2\pi \sigma_x^2}{4} \right) + \frac{1}{2} \left( \frac{2\pi \sigma_y^2}{4} \right) = \frac{\pi}{2} (\sigma_x^2 + \sigma_y^2).
\]

Under WPFOA the variance is

\[
Var(K) = E(K^2) = \frac{1}{2} (2x^*_x)^2 + \frac{1}{2} (2x^*_y)^2 = 4x^*_2 \sigma^2 = \frac{\pi}{2} (\sigma_x^2 + 2 \rho \sigma_x \sigma_y + \sigma_y^2).
\]

The analysis thus far has assumed both parties to be risk neutral, however in practice at least one of the parties tends to be risk averse; indeed, if both parties were truly risk-neutral then the prospect of FOA would do little to motivate agreement among parties beyond the time and actual cost of the arbitration process. Maintaining the assumption of risk-neutrality, Çelen and Özgür (2016) introduce the concept of uncertainty-aversion to the theory of FOA to explain how FOA incentivizes negotiated settlements. From this perspective we see that WPFOA serves as a stronger motivator towards settlement so long as the issues are positively correlated.

Both theory and empirical accounts suggest that the more risk averse a player is, the closer his final offer moves to the middle ground. Furthermore, the more risk
averse party tends to win more often (Kilgour 1994; Wittman 1986). If we were to extend this model to accommodate a risk-averse player we would expect that his equilibrium strategy would move further towards the middle ground and consequently his probability of winning would increase.

Rather than fully develop the theory of the risk-sensitive variant of this scenario, let us consider an example to observe the affects of risk aversion. Let Player 1 be the employer and Player 2 be the employee demanding higher wages and benefits. Suppose \( \mu = (2, 3), \sigma_x = 1, \sigma_y = 2, \rho = 0.5 \). If both players are risk-neutral, the Nash equilibrium strategies are

\[
a^* = (0.342021, 1.342021), \quad b^* = (3.657979, 4.657979).
\]

If, however, Player 2 is risk-averse with utility function \( u_2(x, y) = \sqrt{x + y} \), the game becomes non zero-sum with expected payoff functions

\[
K_1(a, b) = -(x_1 + y_1)P_1 - (x_2 + y_2)(1 - P_1),
\]

\[
K_2(a, b) = \sqrt{x_1 + y_1}P_1 + \sqrt{x_2 + y_2}(1 - P_1).
\]

The Nash equilibrium can be found numerically to be

\[
a^{*'} = (0.423283, 1.20516), \quad b^{*'} = (3.285887, 4.067764).
\]

As expected, the risk-averse employee makes a more moderate demand, but the employer takes advantage of these concessions and takes a more extreme stance. The arbitrator more likely sides with the employee, as expected, noting that \( P_1 \) decreases from 0.5 to 0.423728. Furthermore, as found in the one-dimensional case (Hanany et al. 2007), with the introduction of risk-aversion a contract zone emerges, with both players preferring a negotiated packages with \( 4.424435 < x + y < 4.92772 \) to the expected arbitrated settlement.\(^{11}\) See Fig. 3. Treating the arbitrated outcome as a (random) disagreement point, the Nash bargaining solution is found to be \( x + y = 4.67435 \), quite near the midpoint of the contract zone, \( x + y = 4.67608 \).

To further develop our intuition about the effect of risk-aversion on player behavior, the parameters \( \rho \) and \( \sigma_Y \) were varied, along with the \( u_2(x, y) = (x + y)^r \) for \( r \in (0, 1] \), and equilibrium strategies were computed using R code. Visual summaries of the simulations are given in Fig. 4. In all cases we observe these common phenomena:

1. The equilibrium strategy of the risk-averse player shifts apparently linearly, becoming more moderate as \( r \) decreases. Meanwhile, the equilibrium strategy of the risk-neutral employer moves in a curve, the net offer decreasing.
2. As \( r \to 0 \), the equilibrium strategy of the risk-averse employee decreases more in the \( y \) than the \( x \) coordinate, and the difference is more extreme as \( \sigma_Y \) and/or \( \rho \) increase. This is somewhat counter-intuitive; while the second issue has greater

\[^{11}\text{Their expected payoffs are } -4.92772 \text{ and } 2.103434 \text{ respectively. Any settlement such that } \sqrt{x + y} > 2.103434 \leftrightarrow x + y > 4.244435 \text{ is more attractive to Player 2 and any settlement such that } -(x + y) > -4.92772 \leftrightarrow x + y < 4.92772 \text{ is more attractive to Player 1 than the expected arbitration outcome.}\]
Fig. 3 An illustration of the change in equilibrium strategies when Player 2 is risk-averse and the new contract zone (in blue). The density of the arbitrator’s opinion probability distribution is indicated in pink.

Fig. 4 Changes to Nash equilibrium strategies when Player I has utility function $u(x, y) = (x + y)^r$ for various powers $r \in (0, 1]$, $\mu = (3, 2)$, $\sigma_x = 1$. Player 1 strategy indicated by triangle, Player 2 strategy by circle. The expected arbitrated settlement is indicated by diamond, a line indicating the width of the contract zone and asterisk indicating where the Nash bargaining solution set intersects. For readability, the bounds of the contract zone region are shown in red only for $r \approx 0$. 
variability under $F$, the employee’s equilibrium demand of the first issue lies more standard deviations from the mean.

3. In equilibrium, the arbitrator probabilistically favors the risk-averse player. As risk-aversion increases, $P_1$ decreases from 0.5 to as low as 0.2823976 in the scenario where $\sigma_y = 2.5$

4. As expected, the contract zone widens as risk-aversion increases, and moves towards the preference of the risk-neutral player. This makes sense, as the upper bound of the contract zone is the midpoint of the two equilibrium strategies while the lower bound is the employee’s expected utility of the arbitration outcome.

5. The Nash bargaining solution appears to remain near the midpoint of the contract zone regardless of the level of risk-aversion, though slightly favoring the employee along this interval.

While these are only a few initial observations, they serve to motivate further investigation and development of the theory.

7 Conclusion

We have developed a model of multi-issue final-offer arbitration as a zero-sum game where both players are risk-neutral, issues under dispute are quantitative and the values are additive. In the bivariate case where the judge’s opinion is drawn from a normal distribution, if the two components are not too negatively correlated, we derived locally optimal pure strategies. If we further assume that the issues are positively correlated, these represent the unique optimal strategy pair. Furthermore, it was observed that in this case whole-package FOA leads to an outcome with greater variance than IBI, and would act as a greater motivator to reach agreement in negotiations. We have also observed the emergence of a multidimensional contract zone when one player exhibits risk-aversion, and using one risk-averse utility function we see how the equilibrium demands react to varying levels of risk-aversion.

This represents only an initial model of the multi-issue FOA game. As this is the first attempt (to the authors’ knowledge) to formulate a detailed model of the higher-dimensional FOA game, the intention here is not to create an exhaustively general model. Instead we wish to delineate a tractable model from which we can glean some insights, highlight the many interesting ways in which it can be extended and generalized, as well as discuss the challenges in doing so. Many variants are worthy of consideration. Firstly, it is worth studying the effect of a different choice of distance criterion on the part of the judge. It may be the case that the final-offer vectors must be standardized before making the ruling so that components of differing units may be compared. Furthermore, it is worthwhile to extend these results to higher dimension, to a general $d$-issue FOA game.

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8 Appendix

Proof of Lemma 1 If \( a = b \) the proof is trivial. Assume \( a \neq b \).

\[
K(a, b) = (x_1 + y_1)P((\xi, \eta) \in C_1(a, b)) + (x_2 + y_2)P((\xi, \eta) \in C_2(a, b))
\]
\[
= (x_1 + y_1)P((\xi, \eta) \in C_2(b, a)) + (x_2 + y_2)P((\xi, \eta) \in C_1(b, a))
\]
\[
= (x_2 + y_2)P((\xi, \eta) \in C_1(b, a)) + (x_1 + y_1)P((\xi, \eta) \in C_2(b, a))
\]
\[
= K(b, a)
\]

\[\square\]

Proof of Lemma 2 This proof makes use of two facts: first, \((\xi, \eta) \in C_i(a, b) \Leftrightarrow (\xi, \eta) \in C_i(-a, -b), i = 1, 2\). Secondly, \((\xi, \eta) \text{ and } (-\xi, -\eta)\) follow the same probability distribution.

\[
K(-a, -b) = (-x_1 - y_1)P((\xi, \eta) \in C_1(-a, -b))
\]
\[
+ (-x_2 - y_2)P((\xi, \eta) \in C_2(-a, -b))
\]
\[
= -\left((x_1 + y_1)P((\xi, \eta) \in C_1(-a, -b))
\right)
\]
\[
+ (x_2 + y_2)P((\xi, \eta) \in C_2(-a, -b))
\]
\[
= -\left((x_1 + y_1)P((\xi, \eta) \in C_1(a, b))
\right)
\]
\[
+ (x_2 + y_2)P((\xi, \eta) \in C_2(a, b))
\]
\[
= -K(a, b)
\]

\[\square\]

Proof of Lemma 3 This proof also relies on the fact that \((\xi, \eta) \text{ and } (-\xi, -\eta)\) follow the same probability distribution.

\[
K(-b, b) = (-x_2, -y_2)P((\xi, \eta) \in C_1(-b, b)) + (x_2 + y_2)P((\xi, \eta) \in C_2(-b, b))
\]
\[
= (x_2 + y_2)\left(P((\xi, \eta) \in C_2(-b, b)) - P((\xi, \eta) \in C_1(-b, b))\right)
\]
\[
= (x_2 + y_2)\left(P((\xi, \eta) \in C_2(-b, b)) - P((-\xi, -\eta) \in C_1(b, -b))\right)
\]
\[
= (x_2 + y_2)\left(P((\xi, \eta) \in C_2(-b, b)) - P((-\xi, -\eta) \in C_2(-b, b))\right)
\]
\[
= (x_2 + y_2)\left(P((\xi, \eta) \in C_2(-b, b)) - P((\xi, \eta) \in C_2(-b, b))\right)
\]
\[
= 0
\]

\[\square\]
Proof of Lemma 5 Suppose \( b^* = (x^*_2, y^*_2) \) is an optimal pure strategy for Player II. Because the value of the game is zero,

\[
K(a, b^*) \geq 0, \forall a.
\]  

(45)

If \(-b^*\) is not an optimal pure strategy for Player I then there exists \( b^\circ \) such that

\[
K(-b^*, b^\circ) > 0.
\]

By Lemmas 1 and 2,

\[
K(-b^\circ, b^*) = K(b^*, -b^\circ)
\]

\[
= -K(-b^*, b^\circ)
\]

\[
< 0
\]

but this contradicts (45), so it must be the case that \(-b^*\) is an optimal pure strategy for Player I. The converse of the lemma is shown in an analogous way. \(\square\)

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