Generalizing the Noether theorem for Hopf-algebra spacetime symmetries

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ABSTRACT

Over these past few years several quantum-gravity research groups have been exploring the possibility that in some Planck-scale nonclassical descriptions of spacetime one or another form of nonclassical spacetime symmetries might arise. One of the most studied scenarios is based on the use of Hopf algebras, but previous attempts were not successful in deriving constructively the properties of the conserved charges one would like to obtain from the Hopf structure, and this in turn did not allow a crisp physical characterization of the new concept of spacetime symmetry. Working within the example of \(\kappa\)-Minkowski noncommutative spacetime, known to be particularly troublesome from this perspective, we observe that these past failures in the search of the charges originated from not recognizing the crucial role that the noncommutative differential calculus plays in the symmetry analysis. We show that, if the properties of the \(\kappa\)-Minkowski differential calculus are correctly taken into account, one can easily perform all the steps of the Noether analysis and obtain an explicit formula relating fields and energy-momentum charges. Our derivation also exposes the fact that an apparent source of physical ambiguity in the description of the Hopf-algebra rules of action, which was much emphasized in the literature, actually only amounts to a choice of conventions and in particular does not affect the formulas for the charges.
1 Introduction

There has been quite some interest recently (see, e.g., Refs. [1, 2, 3]) in the hypothesis that the short-distance (Planck-scale) structure of spacetime, which according to a popular “quantum-gravity intuition” [4] should be highly nontrivial, might be such to require a new description of spacetime symmetries. In particular, the description of the “Minkowski limit” [5] of quantum gravity might require some deformation of the Poincaré symmetries. So far this idea has been mostly debated at a rather abstract conceptual level [5], without the support of a fully-worked-out theoretical picture that could at least illustrate the type of phenomena to be expected from a deformation of Poincaré symmetry. One candidate nonclassical spacetime which several authors have considered from this perspective is the $\kappa$-Minkowski noncommutative spacetime [6, 7], with the characteristic space/time noncommutativity given by

\begin{align}
[x_j, x_0] &= i\lambda x_j \\
[x_l, x_j] &= 0,
\end{align}

where the length scale $\lambda$ is usually expected to be of the order of the Planck length. Some arguments based on “mathematical analogies”[1] would suggest that the symmetries of $\kappa$-Minkowski should be described in terms of a $\kappa$-Poincaré Hopf algebra [6, 7, 10], but it was never established whether these formal observations are sufficient to ensure, in the sense needed for physics applications, the presence of nonclassical symmetries governed somehow by the $\kappa$-Poincaré Hopf algebra. In particular the conserved charges associated with the $\kappa$-Poincaré (would-be-)symmetry transformations have never been obtained.

In the absence of an actual result the charges have been characterized on the basis of various heuristic arguments, but these arguments lead to rather puzzling conclusions, including the fact that the energy-momentum charges appear to be affected by an ambiguity: a given field in $\kappa$-Minkowski would appear to carry different energy-momentum charges depending on the choice of ordering convention made in describing the field [11, 12]. Other authors (see, e.g., Ref. [13]) have argued that the Hopf-algebra structures might after all not reflect the presence of any symmetry: the Hopf-algebra structures could be just a fancy mathematical formalization of a rather trivial break down of symmetry.

We here attempt to bring the debate on Planck-scale-deformed spacetime symmetries, at least in the $\kappa$-Minkowski framework, beyond heuristics. We show that previous failures to derive energy-momentum conserved charges were due to the adoption of a rather naive description of translation transformations, which in particular did not take into account the properties of the noncommutative $\kappa$-Minkowski differential calculus. By taking properly into account the properties of the differential calculus one encounters no obstruction in following all the steps of the Noether analysis and obtain an explicit formula relating fields and energy-momentum charges. We find four energy-momentum conserved charges using the invariance of the theory under the four $\kappa$-Poincaré translation transformations, and this shows that Hopf algebras can be used to describe genuine spacetime symmetries. The result we here derive confirms that in $\kappa$-Minkowski there is a nonlinear Planck-scale modification of the energy-momentum relation, but the

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1 The space indices $j, l$ take values in $\{1, 2, 3\}$ while 0 is the time index. We shall later also use the spacetime indices $\mu, \nu, \alpha$, which take values in $\{0, 1, 2, 3\}$.

2 Rather than our length scale $\lambda$ a majority of authors use the energy scale $\kappa$, which is the inverse of $\lambda$ ($\lambda \rightarrow 1/\kappa$).

3 One notices that $\kappa$-Minkowski and the $\kappa$-Poincaré Hopf algebra form a “Heisenberg double” [8, 9], i.e. $\kappa$-Minkowski and $\kappa$-Poincaré are linked, as algebras, in a way that is rather similar to the relationship between classical Minkowski spacetime and the classical Poincaré Lie algebra.
nonlinearity intervenes in a way that differs significantly from what had been conjectured on the basis of some heuristic arguments. And it is noteworthy, in light of the mentioned debate on the possibility of a puzzling ordering-convention dependence of the symmetry analysis in $\kappa$-Minkowski, that our result expressing the energy-momentum charges as functions of the field configuration does not depend in any way on the choice of ordering convention for the description of the fields.

2 Fields and translation generators

Fields in $\kappa$-Minkowski (functions of the $\kappa$-Minkowski noncommutative spacetime coordinates \((1.2)\)) are conveniently introduced \[14\] in terms of a basis of "Fourier exponentials":

$$f(x) = \int d^4q \tilde{f}(\vec{q}, q_0)e^{i\vec{q}\cdot\vec{x}}e^{-iq_0x_0}, \quad (2.1)$$

where the $q_\mu$ are ordinary commuting variables and $\int d^4q$ is an ordinary integral. Therefore a (noncommutative) field $f(x)$ is identified in terms of a (commutative) field $\hat{f}(\vec{q}, q_0)$.

The type of description of fields given in (2.1) suggests straightforward generalizations to $\kappa$-Minkowski of some familiar commutative-spacetime formulas. For example, one is immediately led to a notion of integration\[5\] on the $\kappa$-Minkowski coordinates by simply posing that

$$\int d^4x e^{i\vec{q}\cdot\vec{x}}e^{-iq_0x_0} = \delta(\vec{q}, q_0), \quad (2.2)$$

which leads to

$$\int f(x)d^4x = \hat{f}(0,0). \quad (2.3)$$

It is much emphasized in the relevant literature \[6,11,12\] that translation generators, $P^\mu$, in $\kappa$-Minkowski can be introduced with a "classical action"

$$P^\mu(e^{i\vec{q}\cdot\vec{x}}e^{-iq_0x_0}) = q^\mu(e^{i\vec{q}\cdot\vec{x}}e^{-iq_0x_0}) \quad (2.4)$$

(the action on any field is of course fully characterized, in light of (2.1), once the action on the exponentials $e^{i\vec{q}\cdot\vec{x}}e^{-iq_0x_0}$ is given).

This type of classical-action description of translation generators finds encouragement from many arguments \[6,11,12\], but it leads to a puzzle, which is exposed upon observing that one may of course choose to describe our noncommutative fields equivalently using different ordering conventions for the basis of exponentials, such as

$$f(x) = \int d^4q \tilde{f}_{II}(\vec{q}, q_0)e^{-iq_0x_0/2}e^{i\vec{q}\cdot\vec{x}}e^{-iq_0x_0/2}, \quad (2.5)$$

\^{4}For the $d^4q$ in (2.1) one may introduce\[15\] an integration measure in order to attribute certain desired transformation properties to the fields. This will not play a role in our analysis.

\^{5}Besides the integration over the $\kappa$-Minkowski coordinates one can also describe the product rule for noncommutative fields (inherited from the commutation relations among spacetime coordinates) in terms of an equivalent deformed rule of product (a generalized “Moyal star product” \[14\]) for the associated commutative fields. Within our analysis it is not too cumbersome to work directly with the product of noncommutative fields, so we skip the step of introducing the star product.
which adopts a “time-symmetrized ordering convention” instead of the “time-to-the-right ordering convention” adopted in (2.1). Since the \( \kappa \)-Minkowski commutation relations, \(|\mathcal{L}|\), are such that \( \epsilon^{\hat{q}^2 \hat{x}} e^{-\hat{q}^2 x_0} = e^{-\hat{q}^2 x_0/2} e^{i\epsilon \lambda q_0/2} \hat{q}^2 \hat{x} e^{-i\hat{q}^2 x_0/2} \) the same field \( f(x) \) can indeed be equivalently described in terms of a corresponding \( \hat{f}(\hat{q}, q_0) \), according to (2.1), or in terms of \( \hat{f}_I(\hat{q}, q_0) \), according to (2.5), and the two descriptions are simply related by \( \hat{f}_I(\hat{q}, q_0) = e^{-3\lambda q_0/2} \hat{f}(e^{-\lambda q_0/2} \hat{q}, q_0) \). But these two equivalent descriptions of fields lead to genuinely different “classical actions” of the translation generators. In fact, according to the “time-symmetrized ordering convention” one would introduce translation generators through

\[
P_I^\mu(e^{-\hat{q}^2 x_0/2} e^{i\hat{q}^2 \hat{x}} e^{-i\hat{q}^2 x_0/2}) = q^\mu(e^{-i\hat{q}^2 x_0/2} e^{i\hat{q}^2 \hat{x}} e^{-i\hat{q}^2 x_0/2}) ,
\]

and the \( P_I^\mu \) are truly different from the \( P^\mu \), as one sees by verifying that

\[
P_I^j(e^{i\hat{q}^2 \hat{x}} e^{-i\hat{q}^2 x_0}) = e^{\lambda q_0/2} q^j(e^{i\hat{q}^2 \hat{x}} e^{-i\hat{q}^2 x_0}) = e^{\lambda P_0/2} P^j(e^{i\hat{q}^2 \hat{x}} e^{-i\hat{q}^2 x_0}) .
\]

This puzzling “ordering ambiguity” is confined to the description of translation generators. All other structures appear in fact to be independent of the choice of ordering convention, including the description of rotation and boost generators (as shown in Ref. [12] and the notion of integration (whose independence on the choice of ordering follows from \( \hat{f}_{II}(0, 0) = \hat{f}(0, 0) \)). But most of the interest in \( \kappa \)-Minkowski noncommutativity originates from some expectations concerning its translation sector, and therefore this ordering ambiguity has played an important role in the development of the relevant research area. The most discussed candidate physical effect in \( \kappa \)-Minkowski is a possible anomalous relation between energy and momentum (anomalous dispersion), which could be interesting since it is conjectured to arise at a level that is not far from the reach of forthcoming experimental studies [16][17][18][19][20][21]. But energy and momentum are the charges associated to translation invariance, and the presence of alternative descriptions of the translation generators is feared to introduce an ambiguity in the corresponding description of energy-momentum charges.

### 3 Translation transformations and differential calculus

Having described the much-debated “translations problem” for \( \kappa \)-Minkowski, we now start introducing the first tools needed for our proposed solution of the problem. Our first key observation takes as starting point another well-known characterization of translation transformations, which rather than focusing on the generators concerns the infinitesimal translation parameters. As customary in the commutative limit one views an infinitesimal translation as a map \( x_\mu \rightarrow x_\mu + \epsilon_\mu \). When this concept is enforced in \( \kappa \)-Minkowski one of course finds that the translation parameters must have nontrivial algebraic properties

\[
[\epsilon_j, x_0] = i\lambda \epsilon_j , \quad [\epsilon_j, x_k] = 0 , \quad [\epsilon_0, x_\mu] = 0
\]

in order to ensure that the “point” \( x + \epsilon \) still belongs to the \( \kappa \)-Minkowski spacetime:

\[
[x_j + \epsilon_j, x_0 + \epsilon_0] = i\lambda (x_j + \epsilon_j) , \quad [x_i + \epsilon_i, x_j + \epsilon_j] = 0 .
\]
Of course these algebraic relations reflect the known properties\(^6\) of the \(\kappa\)-Minkowski differential calculus \([24]\) (the \(\epsilon\)'s describe the differences between the coordinates of two spacetime points and are therefore related to the \(dx\)'s of the differential calculus).

In order to perform the Noether analysis we must describe the action of translation transformations on the fields \(f\), which will be of the type \(f \rightarrow f + df\). It is crucial for our analysis to observe that the two known facts about translations in \(\kappa\)-Minkowski, the form of the generators \(P_\mu\) and the properties of the infinitesimal translation parameters \(\epsilon_\mu\), must be combined in the description of the \(df\), as already clearly encoded in the classical-spacetime formula \(df = i [P_\mu f(x)] \epsilon_\mu\). And the fact that in the \(\kappa\)-Minkowski case the transformation parameters have nontrivial algebraic properties confronts us with another ordering issue: as \(df\) we could take \(i [P_\mu f(x)] \epsilon_\mu\) or, for example, \(\{i [P_\mu f(x)] \epsilon_\mu + i \epsilon_\mu [P_\mu f(x)]\}/2\). There is clearly an infinity of different formulations of the \(df\) which all reduce to \(df = i [P_\mu f(x)] \epsilon_\mu\) in the classical-spacetime (commutative) limit.

The definition of \(df\) is however not to be treated as a freedom allowed by the formalism: the exterior derivative operator \(d\) must of course satisfy the Leibnitz rule \(d(f \cdot g) = f \cdot dg + df \cdot g\). For the description of translations within the time-to-the-right ordering convention, \(i.e.\) based on the translation generators \(P_\mu\) of \([24]\), we considered the following ansatz for \(df\)

\[
\begin{align*}
df &= i \left( \sum_n A_n \epsilon_\alpha_n [P_\mu f]^{\frac{1}{2} \alpha_n} \right) \\
&= i \left( \sum_n A_n \epsilon_\alpha_n [P_\mu f]^{\frac{1}{2} \alpha_n} \right)
\end{align*}
\]  
\(3.3\)

where \(A_n\) and \(\alpha_n\) are real numbers, and \(\sum_n A_n = 1\) (meaning that we allowed \(df\) to be written as a sum of terms with a variety of possible ordering conventions for the position of the transformation parameters with respect to the generators). One then easily finds that the requirement (??) singles out the formula

\[
\begin{align*}
df &= i \epsilon_\mu P_\mu f(x) \\
&= i \epsilon_\mu P_\mu f(x)
\end{align*}
\]  
\(3.4\)

It is through this formula, involving both generators and transformation parameters, that one truly characterizes the translation transformations. The exclusive knowledge of the properties of the translation generators is clearly insufficient.

Now that we have a genuine description of translation transformations, obtained working with the time-to-the-right ordering convention, it is natural to ask whether a truly different description of translation transformations is obtained adopting the time-symmetrized ordering convention. In order to investigate this issue we considered this alternative ansatz for \(df\)

\[
\begin{align*}
df_{II} &= i \left( \sum_n B_n \epsilon_\beta_n [P_{II} f]^{\frac{1}{2} \beta_n} \right) \\
&= i \left( \sum_n B_n \epsilon_\beta_n [P_{II} f]^{\frac{1}{2} \beta_n} \right)
\end{align*}
\]  
\(3.5\)

formulated in terms of the generators \(P_{II}^\mu\) encountered working with the time-symmetrized ordering convention. It is easy to verify that once again the requirement (??) leads to a single possibility:

\[
\begin{align*}
df_{II} &= i \epsilon_\mu^{1/2} [P_{II}^\mu f(x)]^{1/2} \\
&= i \epsilon_\mu^{1/2} [P_{II}^\mu f(x)]^{1/2}
\end{align*}
\]  
\(3.6\)

\(^6\) As an alternative to the 4D differential calculus, whose properties are reflected in the commutation relations \([33]\), one may consider a 5D differential calculus \([22,23]\). The artifact of a 5D differential calculus for the 4D \(\kappa\)-Minkowski spacetime can be motivated \([22,24]\) on the basis of the desire to adapt the structure of the differential calculus to some features of the rotation/boost sector of the \(\kappa\)-Poincaré Hopf algebra. For our analysis, which concerns translation symmetry, the 4D differential calculus should suffice.
This turns out to be the way in which the formalism is telling us that there is a unique concept of translation transformation (a unique $df$) in spite of the availability of different choices of translation generators. In fact, one easily verifies \[25\] that $df_{II} = df$.

4 Noether analysis

In the previous Section we obtained the needed starting point for the Noether derivation of the conserved charges: the transformations for which we intend to find associated conserved charges are now properly characterized in terms of a map $f \rightarrow f + df$, rather than merely at the level of the generators. In doing this we accidentally solved one of the most debated problems for $\kappa$-Minkowski theories: while at the (insufficient) level of description based exclusively on the generators it appeared that there would be an ambiguity in the definition of translation transformations, we found that the algebraic properties of the $\kappa$-Minkowski translation parameters are such that different choices of formulation of the generators lead to the same actual transformation (same formula for the associated $df$). We now test our concept of translation invariance and our proposed generalization of the Noether theorem within the most studied \[7, 11, 12\] theory formulated in $\kappa$-Minkowski spacetime: a theory for a massless scalar field $\Phi(x)$ governed by the Klein-Gordon-like equation\[7\]

$$C_\lambda(P_\mu) \Phi \equiv \left[ \left( \frac{2}{\lambda} \right)^2 \sinh^2 \left( \frac{\lambda P_0}{2} \right) - e^{\lambda P_0} P_\mu P^\mu \right] \Phi = 0 \ , \quad (4.1)$$

whose most general solution can be written as

$$\Phi(x) = \int d^4 k \tilde f(k_0, \vec{k}) e^{i k \cdot x} e^{-ik_0 x_0} \delta(C_\lambda(k_\mu)) \ . \quad (4.2)$$

The equation of motion \[4.1\] can be derived from the following action

$$S[\Phi] = \int d^4 x \mathcal{L}[\Phi(x)] = \int d^4 x \frac{1}{2} \tilde{P}_\mu \Phi \tilde{P}^\mu \Phi$$

where we introduced the compact notation $\tilde{P}_\mu$,

$$\tilde{P}_0 = \left( \frac{2}{\lambda} \right) \sinh(\lambda P_0/2) \quad \tilde{P}_j = e^{\lambda P_0/2} P_j \ , \quad (4.4)$$

which also allows to rewrite $C_\lambda(P_\mu)$ as $\tilde{P}_\mu \tilde{P}^\mu$.

One easily finds that under a coordinate transformation $x \rightarrow x'$ the action varies according to

$$\delta S[\Phi] = \int d^4 x \left( \mathcal{L}[\Phi'(x')] - \mathcal{L}[\Phi(x)] \right) = - \frac{1}{2} \int d^4 x \left\{ e^{\lambda P_0/2} \left[ \left( [\tilde{P}_\mu \tilde{P}^\mu] \Phi \right) \delta \Phi \right] + e^{-\lambda P_0/2} \left[ \delta \Phi (\tilde{P}_\mu \tilde{P}^\mu) \Phi \right] \right\} +$$

$$+ \int d^4 x \left\{ \frac{1}{2} \tilde{P}^\mu \left[ e^{\lambda P_0/2} \tilde{P}_\mu \Phi \delta \Phi + \delta \Phi e^{-\lambda P_0/2} \tilde{P}_\mu \Phi \right] + \mathcal{L}[\Phi'(x')] - \mathcal{L}[\Phi(x)] \right\} \quad (4.5)$$

\[7\] This equation reduces to the Klein-Gordon equation in the $\lambda \rightarrow 0$ limit, and its form was proposed (see, e.g., Refs. \[7, 11, 12\]) using as guidance the idea that it should be an operator that commutes with all the generators in the $\kappa$-Poincaré Hopf algebra.
In deriving (4.5) it is useful to observe that for the action of the operator $\tilde{P}_\mu$ on a product of our noncommutative fields the following property holds:

$$\tilde{P}_\mu[f(x)g(x)] = [\tilde{P}_\mu f(x)][e^{i\xi_{P_0}g(x)}] + [e^{-\frac{i}{2}P_0 f(x)}][\tilde{P}_\mu g(x)] \quad (4.6)$$

for any fields $f(x)$ and $g(x)$.

Clearly the terms in the first pair of curly brackets in (4.5) simply reflect the fact that this is indeed an action that generates (4.1) as equation of motion. The terms in the second pair of curly brackets in (4.5) should be used to obtain the form of the conserved currents when $x \to x'$ is a symmetry transformation.

For our purposes it is necessary to analyze the variation of action specifically under a translation transformation $(x \to x + \epsilon$ and $\Phi \to \Phi + d\Phi)$:

$$\delta S[\Phi] = -\frac{1}{2} \int d^4x \left[ (\tilde{P}_\alpha \tilde{P}_\mu \Phi) + (\tilde{P}_\mu \tilde{P}_\alpha \Phi) \right] + i \int d^4x \epsilon^{\mu\nu} P_\mu \mathcal{L} =$$

$$= \frac{i}{2} \int d^4x \epsilon^{\mu\nu} \left[ (\tilde{P}_\mu \tilde{P}_\alpha \Phi) + (\tilde{P}_\alpha \tilde{P}_\mu \Phi) \right] + i \int d^4x \epsilon^{\mu\nu} P_\mu \mathcal{L} \quad (4.7)$$

where we used (3.10), the scalar-field transformation properties of $\Phi$, and the observation that from (3.10) one finds $\Phi e^\epsilon = e^{-\lambda \delta_{ij} P_0} \Phi$. Then using (4.6) one obtains

$$\delta S[\Phi] = -\frac{i}{2} \int d^4x \epsilon^{\mu\nu} \tilde{P}_\alpha \left[ (\tilde{P}_\mu \tilde{P}_\alpha \Phi) + (\tilde{P}_\alpha \tilde{P}_\mu \Phi) \right] +$$

$$+ \frac{i}{2} \int d^4x \epsilon^{\mu\nu} \left[ (\tilde{P}_\mu \tilde{P}_\alpha \Phi) + (\tilde{P}_\alpha \tilde{P}_\mu \Phi) \right] + i \int d^4x \epsilon^{\mu\nu} P_\mu \mathcal{L} \quad (4.8)$$

We are of course interested in evaluating $\delta S[\Phi]$ for fields which are solutions of the equation of motion (4.11), for which, since $\tilde{P}_\alpha \tilde{P}^\alpha \Phi = 0$, the term in curly bracket in (4.8) vanishes. It is easy to verify, also using the equation of motion and hence

$$\int d^4x \epsilon^{\mu\nu} P_\mu f(x) g(x) = \int d^4x f(x) g(x) \quad (4.9)$$

that $\delta S$ can be rewritten in the form

$$\delta S = i \int d^4x \left\{ \epsilon^\mu P^\nu J_{\mu\nu} \right\} \quad (4.10)$$

where

$$J_{\mu\nu} = \frac{1}{2} \left( \tilde{P}_\mu e^{(\delta_{ij} + \frac{1}{2} \lambda P_0) \Phi} (\tilde{P}_\nu \Phi) + \frac{1}{2} (\tilde{P}_\nu \Phi) \tilde{P}_\mu e^{-\lambda P_0/2 \Phi} - \delta_{ij} J_j \tilde{P}_j \mathcal{L} \right)$$

$$J_{0\mu} = \frac{1}{2} \left( \tilde{P}_\mu e^{(\delta_{ij} + \frac{1}{2} \lambda P_0) \Phi} (\tilde{P}_\nu \Phi) + \frac{1}{2} (\tilde{P}_\nu \Phi) \tilde{P}_\mu e^{-\lambda P_0/2 \Phi} - \delta_{i0} J_0 \tilde{P}_0 \mathcal{L} \right) \quad (4.11)$$

And by spatial integration of the $J_{\mu\nu}$ one obtains as hoped four time-independent quantities $Q_{\mu}$, the conserved charges. For example for the $Q_j$ charges,

$$Q_j = \int d^3x J_{0j} = \frac{1}{2} \int d^3x [\tilde{P}_0 e^{-\lambda P_0/2 \Phi} (\tilde{P}_j \Phi) + (\tilde{P}_j \Phi) \tilde{P}_0 e^{-\lambda P_0/2 \Phi}] \quad (4.12)$$
using a Fourier expansion of the field $\Phi$ solution of (4.6), one finds

\[
Q_j = \frac{1}{2} \int d^4 k dp_0 \phi(k) \phi(p_0, -k e^{\lambda k_0} e^{3\lambda k_0} \left[ \frac{1 - e^{-\lambda p_0}}{\lambda} - \frac{e^{\lambda k_0} - 1}{\lambda} \right] k_j e^{i(k_0 + p_0) x_0} \delta \left( \left( \frac{2}{\lambda} \right)^2 \sin^2 \left( \frac{\lambda k_0}{2} \right) - e^{\lambda k_0} k^2 \right)
\]

whose time independence is most easily seen by considering separately the two possibilities, $(p'_0, k'_0)$ and $(p''_0, k''_0)$, allowed by the last delta function in (4.13):

\[
p'_0 = -k'_0 \quad e^{-\lambda p'_0} = 2 - e^{\lambda k''_0} \quad (4.14)
\]

Since the only possible sources of time dependence in (4.13) are in factors of the form $e^{i(k_0 + p_0)x_0}$ the possibility $(p'_0, k'_0)$ does not give rise to any time dependence. And for the possibility $(p''_0, k''_0)$ one easily verified that the whole integrand vanishes.

The time independence of $Q_0$ can be verified analogously [25], and actually it is possible to rewrite [25] all the charges in an explicitly time-independent manner:

\[
Q_\mu = \int d^3 x J_{0\mu} = \int d^4 p \frac{e^{3\lambda p_0}}{2} p_\mu \tilde{\Phi}(p_0, \vec{p}) \tilde{\Phi}(-p_0, -e^{\lambda p_0} \vec{p}) P_0 \mathfrak{D}(C_\lambda(p_\mu)) . \quad (4.15)
\]

The fact that these energy-momentum charges $Q_\mu$ are indeed time independent confirms that the Noether analysis has been successful.

And it is rather clear from the form of (4.15) that the energy-momentum relation is Planck-scale-(\(\lambda\))-deformed with respect to the special-relativistic (Poincaré-Lie-algebra) limit. It is however also easy to see that for “realistic” field configurations, carrying energy much greater than the Planck energy scale but obtained combining Fourier exponentials with frequencies much lower than the Planck frequency, the Planck-scale correction is always negligibly small. A simple way to characterize this feature is found by considering a “regularized plane wave” solution [26]:

\[
\Psi(x) = \frac{1}{2} \frac{1}{\sqrt{|p| V}} e^{i\vec{p} \cdot \vec{x} - i\omega_\lambda x_0} + e^{-i\vec{p} \cdot \vec{x} - i\omega_\lambda x_0} , \quad (4.16)
\]

where $V$ represents a 3D normalisation volume in the space-time and behaves as a regulator and $\omega_\lambda(\vec{p})$ stands for one of the two real solutions of $\omega_\lambda(\vec{p}) = (2/\lambda)^2 \sin^2 \left( \lambda \omega_\lambda / 2 \right) - \vec{p}^2 e^{\lambda \omega_\lambda} = 0$. This would be a field with characteristic frequency scale $\omega_\lambda$ that carries energy $Q_0 = \hbar \omega_\lambda$. Substituting the field (4.16) in the formulas (4.15) one finds that

\[
\left( \frac{2}{\lambda} \right)^2 \sin^2 \left( \frac{\lambda Q_0}{2} \right) - e^{\lambda Q_0} Q_0^2 = 0 \quad (4.17)
\]

Of course, in the special-relativistic $\lambda \to 0$ limit one recovers the standard energy-momentum relation $Q_0^2 - Q_\lambda^2 = 0$ (for our massless fields). For $\lambda \neq 0$ some corrections are present, but these corrections quickly disappear if we increase the intensity of the field. Indeed for the field $\Psi_\alpha(x) = \alpha \Psi(x)$, obtained multiplying our “regularized plane wave” $\Psi(x)$ by a real number $\alpha$, one finds a $1/\alpha^2$ suppression of the correction, and in the $\alpha \to \infty$ limit the dispersion relation regains its special-relativistic form.
5 Closing remarks

The results here reported provide a safe point of anchorage for the debate on the properties of energy-momentum charges in $\kappa$-Minkowski. Some of the properties conjectured on the basis of previous heuristic arguments did emerge in our analysis, including the presence of some nonlinearity in the energy-momentum relation for massless fields. But the type of nonlinearity which emerged from our analysis differs from all the forms that had been previously conjectured. And we also showed that some expectations based on those heuristic arguments are incorrect. In particular, we found that the energy-momentum charges carried by a field do not depend in any way on the choice of ordering convention adopted in describing that field.

To our knowledge the characterization of translation symmetries in $\kappa$-Minkowski that emerges from our Noether analysis is the first explicit physical formulation of a non-classical (“quantum”) spacetime symmetry. The possibility that some sort of non-classical spacetime symmetry could be relevant for Planck-scale physics has been extensively discussed, but always merely at the level of the properties of some algebras of would-be symmetry generators, without establishing the properties of the associated charges, and without ever really establishing whether the new algebraic structures would result in something genuinely new for some physical observables. Indeed, as mentioned in the Introduction, some authors had argued that by introducing certain types of new properties for the generators one might only be providing fancy mathematics for structures which would not amount to any new symmetry. Our translations in $\kappa$-Minkowski are instead truly a new symmetry, and we have found that the fact that the generators close on a Hopf algebra leads to nontrivial properties for some key physical observables (energy-momentum charges).

While it is significant, from a conceptual perspective, that such a characterization of these Hopf-algebra spacetime symmetries is finally available, our result does not necessarily provide support for the idea that these symmetries should play a role in the short-distance structure of spacetime. The presence of a deformation of the energy-momentum dispersion relation is not in itself too worrisome since there are independent arguments \[1, 2, 3\] to motivate the possibility of this feature in Planck-scale physics, but some readers will understandably be puzzled by the emergence of a “nonuniversal” dispersion-relation formula: as shown in the previous section the energy-momentum charges of different fields in $\kappa$-Minkowski are related by different dispersion relations. And for realistically-large field configurations the correction terms are negligibly small. So this first actual (non-heuristic) encounter with a Hopf-algebra spacetime symmetry is rather challenging at the conceptual level (by requiring that we make sense of a nonuniversal dispersion relation, which the theory in principle accommodates), and not much valuable from a phenomenological perspective since for all practical purposes the associated new effects are quantitatively irrelevant.

Future studies may explore whether something of greater value for phenomenology is obtained if one manages to generalize our result (which concerns classical fields in our “quantum” spacetime) to the case of quantum fields. It seems plausible that, while classical fields are essentially unaffected by the symmetry deformation, quantum particles in $\kappa$-Minkowski spacetime be affected by a significant modification of the dispersion relation. We have not yet attempted this generalization since at present one finds in the literature several alternative proposals \[27, 28, 29\] of a quantum field theory in $\kappa$-Minkowski spacetime, all unsatisfactory on one or another ground \[30\].
Another possible direction for future studies is the one of considering other noncommutative spacetimes. It appears safe to assume that our line of analysis is applicable to other noncommutative spacetimes, but we were unable to propose a general recipe. The ingredients clearly should include a Noether analysis and a proper combination of symmetry generators and transformation parameters; however, the way in which we combined these ingredients made use of some peculiarities of $\kappa$-Minkowski. Probably the easiest generalization of our Noether analysis should apply to other spacetimes which, like $\kappa$-Minkowski, are of “Lie-algebra type” $^{[3]}$ $([x_\mu, x_\nu] = C^a_{\mu\nu} x_a)$. For spacetimes of “canonical type” $([x_\mu, x_\nu] = \theta_{\mu\nu})$ the key issues are not in the translation sector but in the boost/rotation sector, and this will perhaps require a bigger effort for the generalization of our procedure.

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