Interpolated Policy Gradient: Merging On-Policy and Off-Policy Gradient Estimation for Deep Reinforcement Learning

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Abstract

Off-policy model-free deep reinforcement learning methods using previously collected data can improve sample efficiency over on-policy policy gradient techniques. On the other hand, on-policy algorithms are often more stable and easier to use. This paper examines, both theoretically and empirically, approaches to merging on- and off-policy updates for deep reinforcement learning. Theoretical results show that off-policy updates with a value function estimator can be interpolated with on-policy policy gradient updates whilst still satisfying performance bounds. Our analysis uses control variate methods to produce a family of policy gradient algorithms, with several recently proposed algorithms being special cases of this family. We then provide an empirical comparison of these techniques with the remaining algorithmic details fixed, and show how different mixing of off-policy gradient estimates with on-policy samples contribute to improvements in empirical performance. The final algorithm provides a generalization and unification of existing deep policy gradient techniques, has theoretical guarantees on the bias introduced by off-policy updates, and improves on the state-of-the-art model-free deep RL methods on a number of OpenAI Gym continuous control benchmarks.

1 Introduction

Reinforcement learning (RL) studies how an agent that interacts sequentially with an environment can learn from rewards to improve its behavior and optimize long-term returns. Recent research has demonstrated that deep networks can be successfully combined with RL techniques to solve difficult control problems. Some of these include robotic control (Schulman et al., 2016; Lillicrap et al., 2016; Levine et al., 2016), computer games (Mnih et al., 2015), and board games (Silver et al., 2016). One of the simplest ways to learn a neural network policy is to collect a batch of behavior wherein the policy is used to act in the world, and then compute and apply a policy gradient update from this data. This is referred to as on-policy learning because all of the updates are made using data that was collected from the trajectory distribution induced by the current policy of the agent. It is straightforward to compute unbiased on-policy gradients, and practical on-policy gradient algorithms tend to be stable and relatively easy to use. A major drawback of such methods is that they tend to be data inefficient, because they only look at each data point once. Off-policy algorithms based on Q-learning and actor-critic learning (Sutton et al., 1999) have also proven to be an effective approach to deep reinforcement learning such as in (Mnih et al., 2015) and (Lillicrap et al., 2016). Such methods
reuse samples by storing them in a memory replay buffer and train a value function or Q-function with off-policy updates. This improves data efficiency, but often at a cost in stability and ease of use.

Both on- and off-policy learning techniques have been shown to have advantages. Most recent research has worked with on-policy algorithms or off-policy algorithms, a few recent methods have sought to make use of both on- and off-policy data for learning (Gu et al., 2017; Wang et al., 2017; O’Donoghue et al., 2017). Such algorithms hope to gain advantages from both modes of learning, whilst avoiding their limitations. Broadly speaking, there have been two basic approaches in recently proposed algorithms that make use of both on- and off-policy data and updates. The first approach is to mix some ratio of on- and off-policy gradients or update steps in order to update a policy, as in the ACER and PGQ algorithms (Wang et al., 2017; O’Donoghue et al., 2017). In this case, there are no theoretical bounds on the error induced by incorporating off-policy updates. In the second approach, an off-policy Q critic is trained but is used as a control variate to reduce on-policy gradient variance, as in the Q-prop algorithm (Gu et al., 2017). This case does not introduce additional bias to the gradient estimator, but the policy updates do not use off-policy data.

We seek to unify these two approaches using the method of control variates. We introduce a parameterized family of policy gradient methods that interpolate between on-policy and off-policy learning. Such methods are in general biased, but we show that the bias can be bounded. We show that a number of recent methods (Gu et al., 2017; Wang et al., 2017; O’Donoghue et al., 2017) can be viewed as special cases of this more general family. Furthermore, our empirical results show that in most cases, a mix of policy gradient and actor-critic updates achieves the best results, demonstrating the value of considering interpolated policy gradients.

2 Preliminaries

A key component of our interpolated policy gradient method is the use of control variates to mix likelihood ratio gradients with deterministic gradient estimates obtained explicitly from a state-action critic. In this section, we summarize both likelihood ratio and deterministic gradient methods, as well as how control variates can be used to combine these two approaches.

2.1 On-Policy Likelihood Ratio Policy Gradient

At time $t$, the RL agent in state $s_t$ takes action $a_t$ according to its policy $\pi(a_t|s_t)$, the state transitions to $s_{t+1}$, and the agent receives a reward $r(s_t,a_t)$. For a parametrized policy $\pi_\theta$, the objective is to maximize the $\gamma$-discounted cumulative future return $J(\theta) = J(\pi) = E_{s_0 \sim \pi, a_0 \sim \pi} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t,a_t) \right]$. Monte Carlo policy gradient methods, such as REINFORCE (Williams, 1992) and TRPO (Schulman et al., 2015), use the likelihood ratio policy gradient of the RL objective,

$$\nabla_{\theta} J(\theta) = E_{\rho^\pi,s} \left[ \nabla_{\theta} \log \pi_\theta(a_t|s_t) \left( \hat{Q}(s_t,a_t) - b(s_t) \right) \right] = E_{\rho^\pi,s} \left[ \nabla_{\theta} \log \pi_\theta(a_t|s_t) \hat{A}(s_t,a_t) \right],$$

(1)

where $\hat{Q}(s_t,a_t) = \sum_{t'=t}^{\infty} \gamma^{t-t'} r(s_{t'},a_{t'})$ is the Monte Carlo estimate of the “critic” $Q^\pi(s_t,a_t) = E_{s_{t+1},a_{t+1},\ldots\sim\rho^\pi} [\hat{Q}(s_t,a_t)]$, and $\rho^\pi = \sum_{t=0}^{\infty} \gamma^t p(s_t = s)$ are the unnormalized state visitation frequencies, while $b(s_t)$ is known as the baseline, and serves to reduce the variance of the gradient estimate (Williams, 1992). If the baseline estimates the value function, $V^\pi(s_t) = E_{a_t \sim \pi(\cdot|s_t)} [Q^\pi(s_t,a_t)]$, then $\hat{A}(s_t)$ is an estimate of the advantage function $A^\pi(s_t,a_t) = Q^\pi(s_t,a_t) - V^\pi(s_t)$. Likelihood ratio policy gradient methods use unbiased gradient estimates (except for the technicality detailed by Thomas (2014)), but they often suffer from high variance and are sample-intensive.

2.2 Off-Policy Deterministic Policy Gradient

Policy gradient methods with function approximation (Sutton et al., 1999), or actor-critic methods, are a family of policy gradient methods which first estimate the critic, or the value, of the policy by $Q_\omega \approx Q^\pi$, and then greedily optimize the policy $\pi_\theta$ with respect to $Q_\omega$. While it is not necessary for such algorithms to be off-policy, we primarily analyze the off-policy variants, such as (Riedmiller, 2005; Degris et al., 2012; Heess et al., 2015; Lillicrap et al., 2016). For example, DDPG Lillicrap et al. (2016), which optimizes a continuous deterministic policy $\pi_\theta(a_t|s_t) = \delta(a_t = \mu_\theta(s_t))$, can be summarized by the following update equations, where $Q'_\omega$ denotes the target Q network (Lillicrap
Table 1: Prior policy gradient method objectives as special cases of IPG.

| β | μ | CV | Examples |
|---|---|---|---------|
| π | 0 | No | REINFORCE (Williams, 1992), TRPO (Schulman et al., 2015) |
| π | 0 | Yes | Q-Prop (Gu et al., 2017) |
| - | 1 | - | DDPG (Silver et al., 2014; Lillicrap et al., 2016), SVG(0) (Heess et al., 2015) |
| - | - | - | No |

This provides the following determined policy gradient through the critic:

$$\nabla_\theta J(\theta) \approx E_{\nu/\pi}[\nabla_\theta \log \pi_\theta(a_t|s_t)(Q(s_t, a_t) - \hat{Q}_w(s_t, a_t))] + E_{\nu}[\nabla_\theta Q_w(s_t, \mu_\theta(s_t))].$$

(3)

The gradient estimator combines both likelihood ratio and deterministic policy gradients in Eq. 1 and 3. It has lower variance and stable gradient estimates and enables more sample-efficient learning. However, one limitation of Q-Prop is that it uses only on-policy samples for estimating the policy gradient. This ensures that the Q-Prop estimator remains unbiased, but limits the use of off-policy samples for further variance reduction.

### 3 Interpolated Policy Gradient

Our proposed approach, interpolated policy gradient (IPG), mixes likelihood ratio gradient with $\hat{Q}$, which provides unbiased but high-variance gradient estimation, and deterministic gradient through an off-policy fitted critic $Q_w$, which provides low-variance but biased gradients. IPG directly interpolates the two terms from Eq. 1 and 3:

$$\nabla_\theta J(\theta) \approx (1 - \nu)E_{\rho^\pi}[\nabla_\theta \log \pi_\theta(a_t|s_t)\hat{A}(s_t, a_t)] + \nu E_{\rho}[\nabla_\theta \hat{Q}_w(s_t)].$$

(5)

where we generalized the deterministic policy gradient through the critic as $\nabla_\theta \hat{Q}_w(s_t) = \nabla_\theta E_{\pi}[Q_w^\pi(s_t)]$. This generalization is to make our analysis applicable with more general forms of the critic-based control variates, as discussed in the Appendix. This gradient estimator is biased from two sources: off-policy state sampling $\rho^\pi$, and inaccuracies in the critic $Q_w$. However, as we show in Section 4, we can bound the biases for all the cases, and in some cases, the algorithm still guarantees monotonic convergence as in Kakade & Langford (2002); Schulman et al. (2015).

#### 3.1 Control Variates for Interpolated Policy Gradient

While IPG includes $\nu$ to trade off bias and variance directly, it contains a likelihood ratio gradient term, for which we can introduce a control variate (CV) Ross (2006) to further reduce the estimator variance.
The expression for the IPG with control variates is below, where $A^w(s_t, a_t) = Q^w(s_t, a_t) - \bar{Q}^w(s_t)$.

$$\nabla_\theta J(\theta) \approx (1 - \nu) \nabla_\theta \log \pi_\theta(A(s_t, a_t) | \bar{A}(s_t, a_t)) + \nu \nabla_\theta \bar{Q}^w(s_t)$$

$$= (1 - \nu) \nabla_\theta \log \pi_\theta(A(s_t, a_t)) + \nu \nabla_\theta \bar{Q}^w(s_t)$$

$$\approx (1 - \nu) \nabla_\theta \log \pi_\theta(A(s_t, a_t)) + \nu \nabla_\theta \bar{Q}^w(s_t) + \mathbb{E}_{\rho^T}[\nabla_\theta \bar{Q}^w(s_t)].$$

The first approximation indicates the biased approximation from IPG, while the second approximation indicates replacing the $\rho^T$ in the control variate correction term with $\rho^\beta$ and merging with the last term. The second approximation is a design decision and introduces additional bias when $\beta \neq \pi$ but it helps simplify the expression to be analyzed more easily, and the additional benefit from the variance reduction from the control variate could still outweigh this extra bias. The biases are analyzed in Section 4. The likelihood ratio gradient term is now proportional to the residual in on- and off-policy advantage estimates $\hat{A}(s_t, a_t) - A^w(s_t, a_t)$, and therefore, we call this term residual likelihood ratio gradient. Intuitively, if the off-policy critic estimate is accurate, this term has a low magnitude and the overall variance of the estimator is reduced.

### 3.2 Relationship to Prior Policy Gradient and Actor-Critic Methods

Crucially, IPG allows interpolating a rich list of prior deep policy gradient methods using only three parameters: $\beta$, $\nu$, and the use of the control variate (CV). The connection is summarized in Table 1 and the algorithm is presented in Algorithm 1. Importantly, a wide range of prior work has only explored limited cases of the spectrum, e.g. $\nu = 0$, 1, with or without the control variate. Our work provides a thorough theoretical analysis of the biases, and in some cases performance guarantees, for each of the method in this spectrum and empirically demonstrates often the best performing algorithms are in the midst of the spectrum.

#### Algorithm 1 Interpolated Policy Gradient

**input** $\beta, \nu$, useCV

1. Initialize $w$ for critic $Q_w$, $\theta$ for stochastic policy $\pi_\theta$, and replay buffer $\mathcal{R} \leftarrow \emptyset$.
2. **repeat**
3. Roll-out $\pi_\theta$ for $E$ episodes, $T$ time steps each, to collect a batch of data $\mathcal{B} = \{s, a, r\}_{t=1:T,E}$ to $\mathcal{R}$
4. Fit $Q_w$ using $\mathcal{R}$ and $\pi_\theta$, and fit baseline $V_\phi(s_t)$ using $\mathcal{B}$
5. Compute Monte Carlo advantage estimate $\hat{A}_{t,e}$ using $\mathcal{B}$ and $V_\phi$
6. **if** useCV **then**
7. Compute critic-based advantage estimate $\tilde{A}_{t,e}$ using $\mathcal{B}$, $Q_w$, and $\pi_\theta$
8. Compute and center the learning signals $l_{t,e} = A_{t,e} - \hat{A}_{t,e}$ and set $b = 1$
9. **else**
10. Center the learning signals $l_{t,e} = \hat{A}_{t,e}$ and set $b = \nu$
11. **end if**
12. Multiply $l_{t,e}$ by $(1 - \nu)$
13. Sample $D = s_{1:M}$ from $\mathcal{R}$ and/or $\mathcal{B}$ based on $\beta$
14. Compute $\nabla_\theta J(\theta) \approx \sum_{m} \sum_{i} \nabla_\theta \log \pi_\theta(a_{t,e} | s_{t,e}) l_{t,e} + \sum_{m} \nabla_\theta \bar{Q}^w(s_{m})$
15. Update policy $\pi_\theta$ using $\nabla_\theta J(\theta)$
16. **until** $\pi_\theta$ converges.

### 3.3 $\nu = 1$: Actor-Critic methods

Before presenting our theoretical analysis, an important special case to discuss is $\nu = 1$, which corresponds to a deterministic actor-critic method. Several advantages of this special case include that the policy can be deterministic and the learning can be done completely off-policy, as it does not have to estimate the on-policy Monte Carlo critic $\hat{Q}$. Prior work such as DDPG Lillicrap et al. (2016) and related Q-learning methods have proposed aggressive off-policy exploration strategy to exploit these properties of the algorithm. In this work, we compare alternatives such as using on-policy exploration and stochastic policy with classical DDPG algorithm designs, and show that in some domains the off-policy exploration can significantly deteriorate the performance. Theoretically, we confirm this empirical observation by showing that the bias from off-policy sampling in $\beta$ increases
monotonically with the total variation or KL divergence between $\beta$ and $\pi$. Both the empirical and theoretical results indicate that well-designed actor-critic methods with an on-policy exploration strategy could be a more reliable alternative than with an on-policy exploration.

4 Theoretical Analysis

In this section, we present a theoretical analysis of the bias in the interpolated policy gradient. This is crucial, since understanding the biases of the methods can improve our intuition about its performance and make it easier to design new algorithms in the future. Because IPG includes many prior methods as special cases, our analysis also applies to those methods and other intermediate cases. We first analyze a special case and derive results for general IPG. All proofs are in the Appendix.

4.1 $\beta \neq \pi, \nu = 0$: Policy Gradient with Control Variate and Off-Policy Sampling

This section provides an analysis of the special case of IPG with $\beta \neq \pi, \nu = 1$, and the control variate. Plugging in to Eq. 6, we get an expression similar to Q-Prop in Eq. 4,

$$\nabla_\theta J(\theta) \approx \mathbb{E}_{\rho} \left[ \log \pi(a_t | s_t) \right] \left( \hat{A}(s_t, a_t) - A^\pi_w(s_t, a_t) \right) + \mathbb{E}_{\rho} \left[ \nabla_\theta \hat{Q}^\pi_w(s_t, a_t) \right],$$

(7)

except that it also supports utilizing off-policy data for updating the policy. To analyze the bias for this gradient expression, we first introduce $\hat{J}(\pi, \tilde{\pi})$, a local approximation to $J(\pi)$, which has been used in prior theoretical work (Kakade & Langford, 2002; Schulman et al., 2015). The derivation and the bias from this approximation are discussed in the proof for Theorem 1 in the Appendix.

$$J(\pi) = \hat{J}(\tilde{\pi}) + \mathbb{E}_{\rho} \left[ A^\hat{\pi}(s_t, a_t) \right] \approx \hat{J}(\tilde{\pi}) + \mathbb{E}_{\rho} \left[ A^\pi(s_t, a_t) \right] = J(\pi, \tilde{\pi}).$$

(8)

Note that $J(\pi) = \hat{J}(\pi, \tilde{\pi} = \pi)$ and $\nabla_\pi J(\pi) = \nabla_\pi \hat{J}(\pi, \tilde{\pi} = \pi)$. In practice, $\tilde{\pi}$ corresponds to policy $\pi_k$ at iteration $k$ and $\pi$ corresponds next policy $\pi_{k+1}$ after parameter update. Thus, this approximation is often sufficiently good. Next, we write the approximate objective for Eq. 7,

$$\tilde{A}^\pi_w(s_t) = \mathbb{E}_\pi \left[ A^\pi_w(s_t, a_t) \right] = \mathbb{E}_\pi \left[ Q_w(s_t, a_t) \right] - \mathbb{E}_\pi \left[ w(s_t, a_t) \right].$$

(9)

Note that $\tilde{J}^\beta,\nu=0(\pi, \tilde{\pi} = \pi) = \hat{J}(\pi, \tilde{\pi} = \pi) = J(\pi)$, and $\nabla_\pi \tilde{J}^\beta,\nu=0(\pi, \tilde{\pi} = \pi)$ equals Eq. 7. We can bound the absolute error between $\tilde{J}^\beta,\nu=0,\text{CV}(\pi, \tilde{\pi})$ and $J(\pi)$ by the following theorem, where $D^\text{KL}(\pi, \pi_\gamma) = \max_\pi \text{KL}(\pi(s), \pi_\gamma(s))$ is the maximum KL divergence between $\pi$ and $\gamma$.

**Theorem 1.** If $\epsilon = \max_s A^\pi_w(s), \zeta = \max_s A_\gamma(s)$, then

$$\left\| J(\pi) - \tilde{J}^\beta,\nu=0,\text{CV}(\pi, \tilde{\pi}) \right\|_1 \leq 2 \frac{\gamma}{(1 - \gamma)^2} \left( \epsilon \sqrt{D^\text{KL}(\tilde{\pi}, \beta)} + \zeta \sqrt{D^\text{KL}(\pi, \tilde{\pi})} \right).$$

Theorem 1 contains two terms: the second term confirms $\tilde{J}^\beta,\nu=0,\text{CV}$ is a local approximation around $\pi$ and deviates from $J(\pi)$ as $\pi$ deviates, and the first term bounds the bias from off-policy sampling using the KL divergence between the policies $\tilde{\pi}$ and $\beta$. This means that the algorithm fits well with policy gradient methods which constrain the KL divergence per policy update, such as covariant policy gradient (Bagnell & Schneider, 2003), natural policy gradient (Kakade & Langford, 2002), REPS (Peters et al., 2010), and trust-region policy optimization (TRPO) (Schulman et al., 2015).

4.1.1 Monotonic Policy Improvement Guarantee

Some forms of on-policy policy gradient methods have theoretical guarantees on monotonic convergence (Kakade & Langford, 2002; Schulman et al., 2015). Such guarantees often correspond to stable empirical performance on challenging problems, even when some of the constraints are relaxed in practice (Schulman et al., 2015; Duan et al., 2016; Gu et al., 2017). We can show that Algorithm 2, which is a variant of IPG, guarantees monotonic convergence. The proof is provided in the appendix.

Algorithm 2 is often impractical to implement; however, IPG with trust-region updates when $\beta \neq \pi, \nu = 1, \text{CV} = \text{true}$ approximates this monotonic algorithm, similar to how TRPO is an approximation to the theoretically monotonic algorithm proposed by Schulman et al. (2015).
Algorithm 2: Policy iteration with non-decreasing returns $J(\pi)$ and bounded off-policy sampling

1: Initialize policy $\pi_0$, and critic $Q_w$
2: repeat
3: Compute all advantage values $A^\pi(s,a)$, and choose any off-policy distribution $\beta_i$
4: Update critic $Q_w$ using any method (no requirement for performance)
5: Solve the constrained optimization problem:
6: $\pi_{i+1} \leftarrow \arg\max_{\pi} J^{\beta_i,\nu=0,CV}(\pi,\pi_i) - C \left( \sqrt{D_{KL}^{\pi,\beta_i}(\pi,\pi_i)} + \epsilon \sqrt{D_{KL}^{\pi,\beta_i}(\pi,\beta_i)} \right)$
7: subject to $\sum_\pi \pi(a|s) = 1 \quad \forall s$
8: where $C = \frac{2}{(2-\gamma^2\nu)}$, $\zeta = \max_s |A^\pi(s)|$, $\epsilon = \max_s |A_w^\pi(s)|$
9: until $\pi_i$ converges.

4.2 General Bounds on the Interpolated Policy Gradient

We can establish bias bounds for the general IPG algorithm, with and without the control variate, we might expect gains from mixing on- and off-policy gradients. An overarching aim of this paper is to help unify on-policy and off-policy policy gradient algorithms into a single conceptual framework. Our analysis examines how Q-Prop (Gu et al., 2017), PGQ (O’Donoghue et al., 2017), and ACER (Wang et al., 2017), which are all recent works that combine on-policy with off-policy learning, are connected to each other (see Table 1). IPG with PGQ (O’Donoghue et al., 2017), and ACER (Wang et al., 2017), which are all recent works that combine on-policy with off-policy learning, are connected to each other (see Table 1). IPG with

Theorem 2. If $\delta = \max_a |A^\pi(s,a) - A_w^\pi(s,a)|$, $\epsilon = \max_a |A_w^\pi(s)|$, $\zeta = \max_a |A^\pi(s)|$,

$J^{\beta_i,\nu}(\pi, \tilde{\pi}) \triangleq J(\tilde{\pi}) + (1 - \nu)E_{\mu^*,\pi}[A^\pi] + \nu \int E_{\mu^*,\pi}[A_w^\pi - A^\pi]$

$J^{\beta_i,\nu,CV}(\pi, \tilde{\pi}) \triangleq J(\tilde{\pi}) + (1 - \nu)E_{\mu^*,\pi}[A^\pi - A_w^\pi] + \int E_{\mu^*,\pi}[A_w^\pi]$

then,$\|J(\pi) - J^{\beta_i,\nu}(\pi, \tilde{\pi})\|_1 \leq \frac{\nu \delta}{1 - \gamma} + 2 \frac{\gamma}{(1 - \gamma)^2} \left( \nu \epsilon \sqrt{D_{KL}^{\max}(\tilde{\pi}, \beta_i)} + \epsilon \sqrt{D_{KL}^{\max}(\pi, \beta_i)} \right)$

$\|J(\pi) - J^{\beta_i,\nu,CV}(\pi, \tilde{\pi})\|_1 \leq \frac{\nu \delta}{1 - \gamma} + 2 \frac{\gamma}{(1 - \gamma)^2} \left( \nu \epsilon \sqrt{D_{KL}^{\max}(\tilde{\pi}, \beta_i)} + \epsilon \sqrt{D_{KL}^{\max}(\pi, \beta_i)} \right)$

This bound shows that the bias from directly mixing the deterministic policy gradient through $\nu$ comes from two terms: how well the critic $Q_w$ is approximating $Q^\pi$, and how close the off-policy sampling policy is to the actor policy. We also show that the bias introduced is proportional to $\nu$ while the variance of the high variance likelihood ratio gradient term is proportional to $(1 - \nu)^2$, so $\nu$ allows directly trading off bias and variance. Theorem 2 fully bounds bias in the full spectrum of IPG methods; this enables us to analyze how biases arise and interact and help us design better algorithms.

5 Related Work

An overarching aim of this paper is to help unify on-policy and off-policy policy gradient algorithms into a single conceptual framework. Our analysis examines how Q-Prop (Gu et al., 2017), PGQ (O’Donoghue et al., 2017), and ACER (Wang et al., 2017), which are all recent works that combine on-policy with off-policy learning, are connected to each other (see Table 1). IPG with $0 < \nu < 1$ and without the control variate relates closely to PGQ and ACER, but differ in the details. PGQ mixes in the Q-learning Bellman error objective, and ACER mixes parameter update steps rather than directly mixing gradients. And both PGQ and ACER come with numerous additional design details that make fair comparisons with methods like TRPO and Q-Prop difficult. We instead focus on the three minimal variables of IPG and explore their settings in relation to the closely related TRPO and Q-Prop methods, in order to theoretically and empirically understand in which situations we might expect gains from mixing on- and off-policy gradients.

Asides from these more recent works, the use of off-policy samples with policy gradients has been a popular direction of research (Peshkin & Shelton, 2002; Jie & Abbeel, 2010; Degris et al., 2012; Levine & Koltun, 2013). Most of these methods rely on variants of importance sampling (IS) to correct for bias. The use of importance sampling ensures unbiased estimates, but at the cost of considerable variance, as quantified by the ESS measure used by Jie & Abbeel (2010). Ignoring importance weights produces bias but, as shown in our analysis, this bias can be bounded. Therefore, our IPG estimators have higher bias than the sampling distribution deviates from the policy, while IS methods have higher variance. Among these importance sampling methods, Levine & Koltun (2013) evaluates on tasks that are the most similar to our paper, but the focus is on using importance sampling to include demonstrations, rather than to speed up learning from scratch.
6 Experiments

In this section, we empirically show that the three parameters of IPG can interpolate different behaviors and often achieve superior performance versus prior methods that are limiting cases of this approach. Crucially, all methods share the same algorithmic structure as Algorithm 1, and we hold the rest of the experimental details fixed. All experiments were performed on MuJoCo domains in OpenAI Gym (Todorov et al., 2012; Brockman et al., 2016), with results presented for the average over three seeds. Additional experimental details are provided in the Appendix.

6.1 $\beta \neq \pi, \nu = 0$, with the control variate

We evaluate the performance of the special case of IPG discussed in Section 4.1. This case is of particular interest, since we can derive monotonic convergence results for a variant of this method under certain conditions, despite the presence of off-policy updates. Figure 1a shows the performance on the HalfCheetah-v1 domain, when the policy update batch size is 5000 transitions (i.e. 5 episodes). “last” and “rand” indicate if $\beta$ samples from the most recent transitions or uniformly from the experience replay. “last05000” would be equivalent to Q-Prop given $\nu = 0$. Comparing “IPG-$\beta$-rand05000” and “Q-Prop” curves, we observe that by drawing the same number of samples randomly from the replay buffer for estimating the critic gradient, instead of using the on-policy samples, we get faster convergence. If we sample batches of size 30000 from the replay buffer, the performance further improves. However, as seen in the “IPG-$\beta$-last030000” curve, if we instead use the 30000 most recent samples, the performance degrades. One possible explanation for this is that, while using random samples from the replay increases the bound on the bias according to Theorem 1, it also decorrelates the samples within the batch, providing more stable gradients. This is the original motivation for experience replay in the DQN method (Mnih et al., 2015), and we have shown that such decorrelated off-policy samples can similarly produce gains for policy gradient algorithms. See Table 2 for results on other domains.

The results for this variant of IPG demonstrate that random sampling from the replay provides further improvement on top of Q-Prop, a strong baseline for sample-efficiency and stability. Note that these replay buffer samples are different from standard off-policy samples in DDPG or DQN algorithms, which often use aggressive heuristic exploration strategies. The samples used by IPG are sampled...
from prior policies that follow a conservative trust-region update, resulting in greater regularity but less exploration. In the next section, we show that in some cases, ensuring that the off-policy samples are not too off-policy is essential for good performance.

6.2 $\beta = \pi, \nu = 1$

In this section, we empirically evaluate another special case of IPG, where $\beta = \pi$, indicating on-policy sampling, and $\nu = 1$, which reduces to a trust-region, on-policy variant of a deterministic actor-critic method. Although this algorithm performs actor-critic updates, the use of a trust region makes it more similar to TRPO or Q-Prop than DDPG.

Results for all domains are shown in Table 2. Figure 1b shows the learning curves on Ant-v1. Although IPG-$\nu=1$ methods can be off-policy, the policy is updated every 5000 samples to keep it consistent with other IPG methods, while DDPG updates the policy on every step in the environment and makes other design choices Lillicrap et al. (2016). We see that, in this domain, standard DDPG becomes stuck with a mean reward of 1000, while IPG-$\nu=1$ improves monotonically, achieving a significantly better result. To investigate why this large discrepancy arises, we also ran IPG-$\nu=1$ with the same OU process exploration noise as DDPG, and observed large degradation in performance. This provides empirical support for Theorem 2. It is illuminating to contrast this result with the previous experiment, where the off-policy samples did not adversely alter the results. In the previous experiments, the samples came from Gaussian policies updated with trust-regions. The difference between $\pi$ and $\beta$ was therefore approximately bounded by the trust-regions. In the experiment with Brownian noise, the behaving policy uses temporally correlated noise, with potentially unbounded KL-divergence from the learned Gaussian policy. In this case, the off-policy samples result in excessive bias, wiping out the variance reduction benefits of off-policy sampling. In general, we observed that for the harder Ant-v1 and Walker-v1 domains, on-policy exploration is more effective, even when doing off-policy state sampling from a replay buffer. This results suggests the following lesson for designing off-policy actor-critic methods: for domains where exploration is difficult, it may be more effective to use on-policy exploration with bounded policy updates than to design heuristic exploration rules such as the OU process noise, due to the resulting reduction in bias.

6.3 General Cases of Interpolated Policy Gradient

Table 2 shows the results for experiments where we compare IPG methods with varying values of $\nu$; additional results are provided in the Appendix. $\beta \neq \pi$ indicates that the method uses off-policy samples from the replay buffer, with the same batch size as the on-policy batch for fair comparison. We ran sweeps over $\nu = \{0.2, 0.4, 0.6, 0.8\}$ and found that $\nu = 0.2$ consistently produce better performance than Q-Prop, TRPO or prior actor-critic methods. This is consistent with the results in PGQ (O’Donoghue et al., 2017) and ACER (Wang et al., 2017), which found that their equivalent of $\nu = 0.1$ performed best on their benchmarks. Importantly, we compared all methods with the same algorithm designs (exploration, policy, etc.), since Q-Prop and TRPO are IPG-$\nu=0$ with and without the control variate. IPG-$\nu=1$ is a novel variant of the actor-critic method that differs from DDPG (Lillicrap et al., 2016) and SVG(0) (Heess et al., 2015) due to the use of a trust region. The results in Table 2 suggest that, in most cases, the best performing algorithm is one that interpolates between the policy-gradient and actor-critic variants, with intermediate values of $\nu$. 

| Algorithm   | HalfCheetah-v1 | Ant-v1 | Walker-v1 | Humanoid-v1 |
|-------------|----------------|--------|-----------|-------------|
| $\beta = \pi$ | $\beta \neq \pi$ | $\beta = \pi$ | $\beta \neq \pi$ | $\beta = \pi$ |
| IPG-$\nu=0.2$ | 3356 | 3458 | 4237 | 4415 | 2047 | 1932 | 1231 | 920 |
| IPG-$\nu=0.2$ | 4216 | 4023 | 3943 | 3421 | 1896 | 1411 | 1651 | 1613 |
| IPG-$\nu=1$ | 2962 | 4767 | 3469 | 3780 | 2704 | 805 | 1571 | 1530 |
| Q-Prop | 4178 | 4182 | 3374 | 3479 | 2832 | 1692 | 1423 | 1519 |
| TRPO | 2889 | N.A. | 1520 | N.A. | 1487 | N.A. | 615 | N.A. |

Table 2: Comparisons on all domains with mini-batch size 10000 for Humanoid and 5000 otherwise. We compare the maximum of average test rewards in the first 10000 episodes (Humanoid requires more steps to fully converge; see the Appendix for learning curves). Results outperforming Q-Prop (or IPG-$\nu=0$ with $\beta = \pi$) are boldface. The two columns show results with on-policy and off-policy samples for estimating the deterministic policy gradient.
7 Discussion

In this paper, we introduced interpolated policy gradient methods, a family of policy gradient algorithms that allow mixing off-policy learning with on-policy learning while satisfying performance bounds. This family of algorithms unifies and interpolates on-policy likelihood ratio policy gradient and off-policy deterministic policy gradient, and includes a number of prior works as approximate limiting cases. Empirical results confirm that, in many cases, interpolated gradients have improved sample-efficiency and stability over the prior state-of-the-art methods, and the theoretical results provide intuition for analyzing the cases in which the different methods perform well or poorly. Our hope is that this detailed analysis of interpolated gradient methods can not only provide for more effective algorithms in practice, but also give useful insight for future algorithm design.

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8 Proof for Theorem 1

8.1 Local approximation objective with bounded bias

In the main paper, we introduced the approximate objective $\hat{J}(\pi, \tilde{\pi})$ to $J(\pi)$ during our theoretical analysis. In this section, we discuss the motivations behind this choice, referencing prior work (Kakade & Langford, 2002; Schulman et al., 2015).

First, the expected return $J(\pi)$ of a policy $\pi$ can be written as the sum of the expected return $J(\tilde{\pi})$ of another policy $\tilde{\pi}$ and the expected advantage term between the two policies in the equation, where $A^{\tilde{\pi}}(s_t, a_t)$ is the advantage of policy $\tilde{\pi}$, $J(\pi) = J(\tilde{\pi}) + \mathbb{E}_{\rho^{\pi}, \pi}[A^{\tilde{\pi}}(s_t, a_t)]$.

For the proof, see Lemma 1 in (Schulman et al., 2015). This expression is still not tractable to analyze because of the dependency of unnormalized state sampling distribution $\rho^{\pi}$ on $\pi$. Kakade & Langford (2002); Schulman et al. (2015) thus introduce a local approximation by replacing $\rho^{\pi}$ with $\rho^{\tilde{\pi}}$, $J(\pi) \approx J(\tilde{\pi}) + \mathbb{E}_{\rho^{\tilde{\pi}}, \tilde{\pi}}[A^{\tilde{\pi}}(s_t, a_t)] \triangleq \tilde{J}(\pi, \tilde{\pi})$.

We can show that $J(\pi) = \tilde{J}(\pi, \tilde{\pi} = \pi)$ and $\nabla_{\pi} J(\pi) = \nabla_{\tilde{\pi}} \tilde{J}(\pi, \tilde{\pi} = \pi)$, meaning that the $J(\pi)$ and $\tilde{J}(\pi, \tilde{\pi})$ match up to the first order. Schulman et al. (2015) then uses this property, in combination with minorization-maximization Hunter & Lange (2004), to derive a monotonic convergence proof for a variant of policy iteration algorithm. We also use this property to prove the monotonic convergence property in Algorithm 2, but we further derive the following lemma,

Lemma 3. If $\zeta = \max_{s} |\tilde{A}^{\pi, \tilde{\pi}}(s)|$, then

$$\left\| J(\pi) - \tilde{J}(\pi, \tilde{\pi}) \right\|_1 \leq 2\zeta \frac{\gamma}{(1-\gamma)^2} D_{TV}^{\max}(\tilde{\pi}, \pi) \leq 2\zeta \frac{\gamma}{(1-\gamma)^2} \sqrt{D_{KL}^{\max}(\tilde{\pi}, \pi)}$$

Proof. We define $\rho^{\pi}(s_t)$ as the marginal state distribution at time $t$ assuming that the agent follows policy $\pi$ from initial state $\rho^{\pi}_0(s_t)$ at time $t = 0$. Note that from the definition of $\rho^{\pi}$, $\rho^{\pi}(s) = \sum_{t=0}^{\infty} \gamma^t \rho^{\pi}_t(s_t = s)$. We can use the following lemma from Kahn et al. (2016), which is adapted from Ross et al. (2011) and Schulman et al. (2015).

Lemma 4. (Kahn et al., 2016)

$$\left\| \rho^\pi_t - \rho^\beta_t \right\|_1 \leq 2t D_{TV}^{\max}(\pi, \beta) \leq 2t \frac{D_{TV}^{\max}(\pi, \beta)}{\sqrt{D_{KL}^{\max}(\pi, \beta)}}$$ (10)
The full proof is below, where \( \bar{A}^\pi,\tilde{\pi}(s) = \mathbb{E}_\pi[A^\pi,\tilde{\pi}(s_t, a_t)] \) and \( A^\pi(s_t, a_t) \) is the advantage function of \( \tilde{\pi} \),

\[
\left\| J(\pi) - \bar{J}(\pi, \tilde{\pi}) \right\|_1 \\
= \left\| \mathbb{E}_{\rho^\pi}[\bar{A}^\pi,\tilde{\pi}(s)] - \mathbb{E}_{\rho^\pi}[\bar{A}^\pi,\tilde{\pi}(s)] \right\|_1 \\
\leq \sum_{t=0}^{\infty} \gamma^t \left\| \mathbb{E}_{\rho^\pi}[\bar{A}^\pi,\tilde{\pi}(s)] - \mathbb{E}_{\rho^\pi}[\bar{A}^\pi,\tilde{\pi}(s)] \right\|_1 \\
\leq \zeta \sum_{t=0}^{\infty} \gamma^t \left\| \rho^\pi - \rho^\pi_t \right\|_1
\]

(11)

This lemma is crucial in our theoretical analysis, as it allows us to tractably bound the biases of the full spectrum of IPG objectives \( \bar{J}^{\beta,\nu,TV}(\tilde{\pi}, \pi) \) against \( J(\pi) \).

8.2 Main proof for Theorem 1

Proof. We first prove the bound for \( \left\| \bar{J}(\pi, \tilde{\pi}) - \bar{J}^{\beta,\nu=0,TV}(\tilde{\pi}, \pi) \right\|_1 \). Using Lemma 4, the bound is given below, with a similar derivation process as in Lemma 3.

\[
\left\| \bar{J}(\pi, \tilde{\pi}) - \bar{J}^{\beta,\nu=0,TV}(\tilde{\pi}, \pi) \right\|_1 \\
= \left\| \bar{J}(\tilde{\pi}) + \mathbb{E}_{\rho^\tilde{\pi}}[\bar{A}^\tilde{\pi}(s_t, a_t)] - \bar{J}(\pi) - \mathbb{E}_{\rho^\pi}[\bar{A}^\pi(s_t, a_t)] \right\|_1 \\
= \left\| \mathbb{E}_{\rho^\tilde{\pi}}[\bar{A}^\tilde{\pi}(s)] - \mathbb{E}_{\rho^\pi}[\bar{A}^\pi(s)] \right\|_1 \\
\leq \sum_{t=0}^{\infty} \gamma^t \left\| \mathbb{E}_{\rho^\tilde{\pi}}[\bar{A}^\tilde{\pi}(s)] - \mathbb{E}_{\rho^\pi}[\bar{A}^\pi(s)] \right\|_1 \\
\leq \epsilon \sum_{t=0}^{\infty} \gamma^t \left\| \rho^\pi_t - \rho^\pi \right\|_1 \\
\leq 2\epsilon \left( \sum_{t=0}^{\infty} \gamma^t \right) D_{TV}^{\max}(\tilde{\pi}, \beta) \\
= 2\epsilon \gamma \left( \frac{1}{1-\gamma} \right)^2 D_{TV}^{\max}(\tilde{\pi}, \beta) \\
\leq 2\epsilon \gamma \left( \frac{1}{1-\gamma} \right)^2 \sqrt{D_{KL}^{\max}(\tilde{\pi}, \beta)}.
\]

(12)

Given this bound, we can directly derive the bound for \( \left\| \bar{J}(\pi, \tilde{\pi}) - \bar{J}^{\beta,\nu=0,TV}(\tilde{\pi}, \pi) \right\|_1 \) by combining with Lemma 3,

\[
\left\| J(\pi) - \bar{J}^{\beta,\nu=0,TV}(\tilde{\pi}, \pi) \right\|_1 \\
\leq \left\| J(\pi) - \bar{J}(\pi, \tilde{\pi}) \right\|_1 + \left\| \bar{J}(\pi, \tilde{\pi}) - \bar{J}^{\beta,\nu=0,TV}(\tilde{\pi}, \pi) \right\|_1 \\
\leq 2 \gamma \left( \epsilon \sqrt{D_{KL}^{\max}(\tilde{\pi}, \beta)} + \zeta \sqrt{D_{KL}^{\max}(\tilde{\pi}, \pi)} \right).
\]
9 Proof for Monotonic Convergence in Algorithm 2

We can prove that the algorithm guaranteeing monotonic improvement by first introducing the following corollary,

**Corollary 1.**

\[
J(\pi) \geq M(\pi, \tilde{\pi}) \geq M^{\beta, \nu=0, CV}(\pi, \tilde{\pi}), J(\tilde{\pi}) = M(\tilde{\pi}, \pi) = M^{\beta, \nu=0, CV}(\tilde{\pi}, \pi) \tag{14}
\]

where

\[
M(\pi, \tilde{\pi}) = \tilde{J}(\pi, \tilde{\pi}) - C\zeta \sqrt{D_{KL}^{\text{max}}(\pi, \tilde{\pi})}
\]

\[
M^{\beta, \nu=0, CV}(\pi, \tilde{\pi}) = \tilde{J}^{\beta, \nu=0}(\pi, \tilde{\pi}) - \zeta \sqrt{D_{KL}^{\text{max}}(\pi, \tilde{\pi}) + \epsilon \sqrt{D_{KL}^{\text{max}}(\tilde{\pi}, \beta)}}
\]

\[
C = \frac{2\gamma}{(1-\gamma)^2}, \zeta = \max_s |\bar{A}_{\pi, \tilde{\pi}}(s)|, \epsilon = \max_s |\bar{A}_{\pi, \tilde{\pi}}(s)|
\]

**Proof.** It follows from Theorem 1 in the main text and Theorem 1 in Schulman et al. (2015). $J(\tilde{\pi}) = M^{\beta, \nu=0, CV}(\tilde{\pi}, \tilde{\pi})$ since $\zeta = \epsilon = 0$ when $\pi = \tilde{\pi}$.

Given Corollary 1, we use minorization-maximization (MM) (Hunter & Lange, 2004) to derive Algorithm 2 in the main text, a policy iteration algorithm that allows using off-policy samples while guaranteeing monotonic improvement on $J(\pi)$. MM suggests that at each iteration, by maximizing the lower bound, or the minorizer, of the objective, the algorithm can guarantee monotonic improvement: $J(\pi_{i+1}) \geq M^{\beta, \nu=0, CV}(\pi_{i+1}, \pi_i) \geq M^{\beta, \nu=0, CV}(\pi_{i}, \pi_i) = J(\pi_i)$, where $\pi_{i+1} \leftarrow \arg\max_{\pi} M^{\beta, \nu=0, CV}(\pi, \pi_i)$. Importantly, the algorithm guarantees monotonic improvement regardless of the off-policy distribution $\beta$, or the performance of the critic $Q_\omega$. This result is a step toward achieving off-policy policy gradient with convergence guarantee of on-policy algorithms.\(^1\)

We compare our theoretical algorithm with Algorithm 1 in Schulman et al. (2015), which guarantees monotonic improvement in a general on-policy policy gradient algorithm. The main difference is the additional term, $C\zeta \sqrt{D_{KL}^{\text{max}}(\pi, \tilde{\pi})}$ to the lower bound. $D_{KL}^{\text{max}}(\pi, \tilde{\pi})$ is constant with respect to $\pi$, while $\epsilon = 0$ if $\pi = \tilde{\pi}$ and $\epsilon \geq 0$ if otherwise. This suggests that as $\beta$ becomes more off-policy, the gap between the lower bound and the true objective widens, proportionally to $\sqrt{D_{KL}^{\text{max}}(\pi, \tilde{\pi})}$. This may make each majorization step end in a place very close to where it started, i.e. $\pi_{i+1}$ very close to $\pi_i$, and slow down learning. This again suggests a trade-off that comes in as off-policy samples are used.

10 Proof for Theorem 2

We follow the same procedure as the proof for Theorem 1, where we first derive bounds between $\tilde{J}(\pi, \tilde{\pi})$ and the other local objectives, and then combine the results with Lemma 3.

To begin the proof, we first derive the bound for the special case where $\nu = 1$. Having $\nu = 1$, we remove the likelihood ratio policy gradient term, and get the following gradient expression,

\[
\nabla_\theta J(\theta) \approx E_\rho^{\pi}[\nabla_\theta \tilde{Q}_\omega^\pi(s_t)]. \tag{15}
\]

This is an off-policy actor-critic algorithm, and is closely related to DDPG (Lillicrap et al., 2016), except that it does not use target policy network and its use of a stochastic policy enables on-policy exploration, trust-region policy updates, and no heuristic additive exploration noise.

We can introduce the following bound on the local objective $\tilde{J}^{\beta, \nu=1}(\pi, \tilde{\pi})$, whose policy gradient equals 15 at $\pi = \tilde{\pi}$, similarly to the proof for Theorem 1 in the main text.

**Corollary 2.** If $\delta = \max_s |A^\pi(s, a) - A^{\tilde{\pi}}_\omega(s, a)|$, $\epsilon = \max_s |\tilde{A}^{\pi, \tilde{\pi}}_\omega(s)|$, and

\[
\tilde{J}^{\beta, \nu=1}(\pi, \tilde{\pi}) = J(\tilde{\pi}) + \mathbb{E}_{\rho^{\pi}}[\tilde{A}_{\pi, \tilde{\pi}}^\omega(s_t)], \tag{16}
\]

\(^1\)Schulman et al. (2015) applies additional bound, $\epsilon \geq 2\epsilon' \sqrt{D_{KL}^{\text{max}}(\pi, \tilde{\pi})}$ where $\epsilon' = \max_s |A^{\pi}_\omega(s, a)|$ to remove dependency on $\pi$. In our case, we cannot apply such bound on $\zeta$, since then the inequality in Theorem 1 is still satisfied but the equality is violated, and thus the algorithm no longer guarantees monotonic improvement.
then,
\[
\left\| \tilde{J}(\pi, \tilde{\pi}) - \hat{J}^{\beta, \nu = 1}(\pi, \tilde{\pi}) \right\|_1 \leq \frac{\delta}{1 - \gamma} + 2\epsilon \frac{\gamma}{(1 - \gamma)^2} \sqrt{D_{\text{max}}^\text{KL}(\tilde{\pi}, \beta)} \tag{17}
\]

Proof. We note that
\[
\left\| \tilde{J}(\pi, \tilde{\pi}) - \hat{J}^{\beta, \nu = 1}(\pi, \tilde{\pi}) \right\|_1
\]
\[
= \left\| \mathbb{E}_{\rho^\nu, \pi}[A^\pi(s_t, a_t)] - \hat{J}(\pi, \tilde{\pi}) - \tilde{J}^{\beta, \nu = 0}(\pi, \tilde{\pi}) \right\|_1
\]
\[
\leq \left\| \mathbb{E}_{\rho^\nu, \pi}[A^\pi(s_t, a_t)] - A^\pi_w(s_t, a_t) \right\|_1 + \left\| \tilde{J}(\pi, \tilde{\pi}) - \tilde{J}^{\beta, \nu = 0}(\pi, \tilde{\pi}) \right\|_1
\]
\[
\leq \sum_{t=0}^{\infty} \gamma^t \left\| \mathbb{E}_{\rho^\nu, \pi}[A^\pi(s_t, a_t)] - A^\pi_w(s_t, a_t) \right\|_1 + \left\| \tilde{J}(\pi, \tilde{\pi}) - \tilde{J}^{\beta, \nu = 0}(\pi, \tilde{\pi}) \right\|_1
\]
\[
\leq \frac{\delta}{1 - \gamma} + \frac{\delta}{1 - \gamma} + 2\epsilon \frac{\gamma}{(1 - \gamma)^2} \sqrt{D_{\text{max}}^\text{KL}(\tilde{\pi}, \beta)}.
\]

where the proof uses Theorem 1 at the last step.

Given Corollary 2 and Theorem 1, we are ready to prove the two bounds in Theorem 2.

Proof.
\[
\left\| \tilde{J}(\pi, \tilde{\pi}) - \hat{J}^{\beta, \nu}(\pi, \tilde{\pi}) \right\|_1
\]
\[
= \left\| \tilde{J}(\pi) + \mathbb{E}_{\rho^\nu, \pi}[A^\pi(s_t, a_t)] - J(\pi) - (1 - \nu)\mathbb{E}_{\rho^\nu, \pi}[A^\pi(s_t, a_t)] - \nu\mathbb{E}_{\rho^\nu}[\tilde{A}^\pi_w(s_t)] \right\|_1
\]
\[
= \nu \left\| \mathbb{E}_{\rho^\nu, \pi}[A^\pi(s_t, a_t)] - \mathbb{E}_{\rho^\nu}[\tilde{A}^\pi_w(s_t)] \right\|_1
\]
\[
= \nu \left\| \mathbb{E}_{\rho^\nu, \pi}[A^\pi(s_t, a_t)] - \mathbb{E}_{\rho^\nu}[\tilde{A}^\pi_w(s_t)] \right\|_1 + \nu \left\| \mathbb{E}_{\rho^\nu}[\tilde{A}^\pi_w(s_t)] - \mathbb{E}_{\rho^\nu}[\tilde{A}^\pi_w(s_t)] \right\|_1
\]
\[
= \nu \left\| \mathbb{E}_{\rho^\nu, \pi}[A^\pi(s_t, a_t)] - \tilde{A}^\pi_w(s_t, a_t) \right\|_1
\]
\[
\leq \frac{\delta}{1 - \gamma} + 2\epsilon \frac{\gamma}{(1 - \gamma)^2} \sqrt{D_{\text{max}}^\text{KL}(\tilde{\pi}, \beta)}.
\]

We combine these bounds with Lemma 3 to conclude the proof.

11 Control Variates for Policy Gradient

In this Section, we describe control variate choices for policy gradient methods other than the first-order Taylor expansion presented in Q-Prop (Gu et al., 2017).
11.1 Reparameterized Critic Control Variate

If action is continuous and the policy is a simple distribution such as a Gaussian, one option is to use the full $Q_w$ as the control variate and use Monte Carlo to estimate its expectation with respect to the policy. To reduce the variance of the correction term, we estimate the policy gradient, the reparameterization trick (Kingma & Welling, 2014) can be employed to reduce the variance. For a Gaussian policy $\pi_\theta(a_t|s_t) = \mathcal{N}(\mu_\theta(s_t), \sigma_\theta(s_t))$,

$$Q_w^*(s_t) = \mathbb{E}_\pi [Q_w(s_t, a_t)] = \mathbb{E}_{\epsilon \sim \mathcal{N}(0, 1)}[Q_w(s_t, \mu_\theta(s_t) + \epsilon \sigma(s_t))].$$

(20)

11.2 Discrete Critic Control Variate

Let $\pi_\theta(s_t) \in \mathbb{R}^k$ denote a probability vector over $k$ discrete actions, and $Q_w(s_t) \in \mathbb{R}^k$ denote the action-value function for the $k$ actions, as in DQN (Mnih et al., 2015).

$$Q_w^*(s_t) = \pi_\theta(s_t)^T \cdot Q_w(s_t).$$

(21)

11.3 NAF Critic Control Variate

For continuous control, it is also possible to use a more general critic. If the policy is locally Gaussian, i.e. $\pi_\theta(a_t|s_t) = \mathcal{N}(\mu_\theta(s_t), \Sigma_\theta(s_t))$, then the quadratic $Q_w$ from Normalized Advantage Function (NAF) (Gu et al., 2016) can be directly used,

$$Q_w(s_t, a_t) = A_w(s_t, a_t) + V_w(s_t)$$
$$A_w(s_t, a_t) = -\frac{1}{2}(a_t - \mu_w(s_t))^T P_w(s_t)(a_t - \mu_w(s_t)).$$

(22)

The deterministic policy gradient expression leads to,

$$Q_w^*(s_t) = V_w(s_t) - \frac{1}{2}\text{Tr}(P_w(s_t)\Sigma_\theta(s_t))$$
$$- \frac{1}{2}(\mu_\theta(s_t) - \mu_w(s_t))^T P_w(s_t)(\mu_\theta(s_t) - \mu_w(s_t)).$$

(23)

12 Supplementary Experimental Details

12.1 Hyperparameters

GAE($\lambda = 0.97$) (Schulman et al., 2016) is used for $\hat{A}$ estimation. Trust-region update in TRPO is used as the policy optimizer (Schulman et al., 2015). The standard $Q$-fitting routine from DDPG (Lillicrap et al., 2016) is used for fitting $Q_w$, where $Q_w$ is trained with batch size 64, using experience replay of size 1e6, and target network with $\tau = 0.001$. ADAM (Kingma & Ba, 2014) is used as the optimizer for $Q_w$. Policy network parametrizes a Gaussian policy with $\pi_\theta(a_t|s_t) = \mathcal{N}(\mu_\theta(s_t), \Sigma_\theta)$, where $\mu_\theta$ is a two-hidden-layer neural network of size 100 - 50 and tanh hidden nonlinearity and linear output, and $\Sigma_\theta$ is a diagonal, state-independent variance. For DDPG, the policy network is deterministic and additionally has tanh activation at the output layer. The critic function $Q_w$ is a two-hidden-layer neural network of size 100 - 100 with ReLU activation. We use the first-order Taylor expansion of $Q_w$ as the control variate, as in Q-Prop (Gu et al., 2017), except for $\nu = 1$ cases we use the reparameterized control variate described below. For IPG methods with the control variates, we further explored the standard and conservative variants, the technique proposed in Q-Prop (Gu et al., 2017), and the Taylor expansion variant with the reparameterized variant discussed in Section 11.1.

The trust-region step size for policy update is fixed to 0.1 for HalfCheetah-v1 and Humanoid-v1, and 0.01 for Ant-v1 and Walker2d-v1, while the learning rate for ADAM in critic update is fixed to 1e-4 for HalfCheetah-v1, Ant-v1, Humanoid-v1, and 1e-3 for Walker2d-v1. Those two hyperparameters are found by first running TRPO and DDPG on each domain, and picking the ones that give best performance for each domain. These parameters are fixed throughout the experiment to ensure fair comparisons.
Figure 2: IPG-$\nu = 0.2$-$\pi$-CV vs Q-Prop and TRPO on Humanoid-v1 with batch size 10000 in the first 10000 episodes. IPG-$\nu = 0.2$-$\pi$-CV, with a small difference of $\nu = 0.2$ multiplier, out-performs Q-Prop. All these methods have stable, monotonic policy improvement. The experiment is cut at 10000 episodes due to heavy compute requirement of Q-Prop and IPG methods, mostly from fitting the off-policy critic.

For all IPG algorithms, we use the first-order Taylor expansion control variate, as in Q-Prop, while for $\nu = 1$, we use the reparameterized control variate in Section 11.1 with Monte Carlo sample size $m = 1$. The first-order Taylor expansion control variate cannot be used with $\nu = 1$ directly, since then it does not provide the gradient for training the variance term of the policy.

The plots in the main text present the mean returns as solid lines, scatter plots of all runs in the background to visualize variability.

12.2 Additional Plot

Figure 2 shows additional plot on Humanoid-v1.