On higher-order discriminants

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Abstract

For the family of polynomials in one variable $P := x^n + a_1 x^{n-1} + \cdots + a_n$, $n \geq 4$, we consider its higher-order discriminant sets $\{ \tilde{D}_m = 0 \}$, where $\tilde{D}_m := \text{Res}(P, P^{(m)})$, $m = 2, \ldots, n - 2$, and their projections in the spaces of the variables $a^k := (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$. Set $P^{(m)} := \sum_{j=0}^{n-m} c_j a_j x^{n-m-j}$, $P_{m,k} := c_k P - x^m P^{(m)}$. We show that $\text{Res}(\tilde{D}_m, \partial \tilde{D}_m/\partial a_k, a_k) = A_{m,k} B_{m,k} C_{m,k}$, where $A_{m,k} = a^{n-m-k}$, $B_{m,k} := \text{Res}(P_{m,k}, P_k')$ if $1 \leq k \leq n - m$ and $A_{m,k} = a^{n-m-k}$, $B_{m,k} := \text{Res}(P^{(m)}, P^{(m+1)})$ if $n - m + 1 \leq k \leq n$. The equation $C_{m,k} = 0$ defines the projection in the space of the variables $a^k$ of the closure of the set of values of $(a_1, \ldots, a_n)$ for which $P$ and $P^{(m)}$ have two distinct roots in common. The polynomials $B_{m,k}, C_{m,k} \in \mathbb{C}[a^{k^2}]$ are irreducible. The result is generalized to the case when $P^{(m)}$ is replaced by a polynomial $P_* := \sum_{j=0}^{n-m} b_j a_j x^{n-m-j}$, $0 \neq b_i \neq b_j$ for $i \neq j$.

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1 Introduction

In this paper we consider for $n \geq 4$ the general family of monic polynomials in one variable $P(x,a) := x^n + a_1 x^{n-1} + \cdots + a_n$, $x, a_j \in \mathbb{C}$. For its $m$th derivative w.r.t. $x$ we set $P^{(m)} := c_0 x^{n-m} + c_1 a_1 x^{n-m-1} + \cdots + c_{n-m} a_{n-m}$, where $c_j = (n-j)!/(n-m-j)!$. For $m = 1, \ldots, n - 1$ we define the $m$th order discriminant of $P$ as $\tilde{D}_m := \text{Res}(P, P^{(m)})$ which is the determinant of the Sylvester matrix $S(P, P^{(m)})$. We remind that $S(P, P^{(m)})$ is $(2n - m) \times (2n - m)$, its first (resp. $(n - m + 1)$st) row equals

$$(1, a_1, \ldots, a_n, 0, \ldots, 0) \quad \text{(resp.} \quad (c_0, c_1 a_1, \ldots, c_{n-m} a_{n-m}, 0, \ldots, 0) \quad \text{)} ,$$

the second (resp. $(n - m + 2)$nd) row is obtained from this one by shifting by one position to the right and by adding 0 to the left etc. We say that the variable $a_j$ is of quasi-homogeneous weight $j$ because up to a sign it equals the $j$th elementary symmetric polynomial in the roots of the polynomial $P$; the quasi-homogeneous weight of $x$ is 1.

There are at least two problems in which such discriminants are of interest. One of them is the Casas-Alvero conjecture that if a complex univariate polynomial has a root in common with each of its nonconstant derivatives, then it is a power of a linear polynomial, see [2], [16] and [17] and the claim in [15] that the answer to the conjecture is positive.

Another one is the study of the possible arrangements of the roots of a hyperbolic polynomial (i.e. real and with all roots real) and of all its nonconstant derivatives on the real line. This problem can be generalized to a class of polynomial-like functions characterized by the property their $n$th derivative to vanish nowhere. It turns out that for this class Rolle’s theorem gives only
necessary, but not sufficient conditions for realizability of a given arrangement by the zeros of a polynomial-like function, see [9], [10], [11] and [12]. Pictures of discriminants for the cases \( n = 4 \) and \( n = 5 \) can be found in [6]. Properties of the discriminant set \( \{ \tilde{D}_1 = 0 \} \) for real polynomials are proved in [14].

A closely related question to the one of the arrangement of the roots of a hyperbolic polynomial is the one to study \textit{overdetermined strata} in the space of the coefficients of the family of polynomials \( P \) (the definition is given by B. Z. Shapiro in [13]); these are sets of values of the coefficients for which there are more equalities between roots of the polynomial and its derivatives than expected. Example: the family of polynomials \( x^4 + ax^3 + bx^2 + cx + d \) depends on 4 parameters two of which can be eliminated by shifting and rescaling the variable \( x \) which gives (up to a nonzero constant factor) the family \( S := x^4 - x^2 + cx + d \). For \( c = 0, d = 1/2 \) the polynomial has two double roots \( \pm 1/\sqrt{2} \), and 0 is a common root for \( S' \) and \( S'' \). This makes three independent equalities, i.e. more than the number of parameters. For polynomials of small degree, overdetermined strata have been studied in [3] and [4]. The study of overdetermined strata is interesting both in the case of complex and in the case of real coefficients.

In what follows we enlarge the context by considering instead of the couple of polynomials \((P, P^m)\) the couple \((P, P_s)\), where \( P_s := \sum_{j=0}^{n-m} b_j a_j x^{n-m-j}, b_j \neq 0 \) and \( b_i \neq b_j \) for \( i \neq j \). By abuse of notation we set \( \tilde{D}_m := \text{Res}(P, P_s) \).

**Proposition 1.** The polynomial \( \tilde{D}_m \) is irreducible. It is a degree \( n \) polynomial in each of the variables \( a_j, j = 1, ..., n - m \), and a degree \( n - m \) polynomial in each of the variables \( a_j, j = n - m + 1, ..., n \). It contains monomials \( M_j := \pm b_j^2 a_j^n (1 - b_j/b_j')^2 a_j^{n - m - j}, j = 1, ..., n - m \), and \( N_s := \pm b_s^{m-s} a_j^{m-s} b_j^{m-s} a^{n-m+s}, s = 1, ..., m - 1 \). It is quasi-homogeneous, of quasi-homogeneous weight \( n(n - m) \). The monomial \( M_j \) (resp. \( N_s \)) is the only monomial containing \( a_j^n \) (resp. \( a_j^{n-m} \)).

**Proof.** We prove first the presence in \( \tilde{D}_m \) of the monomials \( M_j \) and \( N_s \). For each \( j \) fixed, \( 1 \leq j \leq n - m \), one can subtract the \((n - m + \nu)\)th row of \( S(P, P_s) \) multiplied by \( 1/b_j \) from its \( \nu \)th one, \( \nu = 1, ..., n - m \). We denote by \( T \) the new matrix. One has \( \det T = \det S(P, P_s) \) and the variable \( a_j \) is not present in the first \( n - m \) rows of \( T \). Thus there remains a single term of \( \det T \) containing \( n \) factors \( a_j \); it is obtained when the entries \( b_j a_j \) in positions \((n - m + \mu, j + \mu)\) of \( T \), \( \mu = 1, ..., n \), are multiplied by the entries \( a_n \) in positions \((\ell, n + \ell)\), \( \ell = j + 1, ..., n - m \), and by the entries \( 1 - b_j/b_j' \) in positions \((\ell, \ell), \ell = 1, ..., j \); this gives the monomial \( M_j \). (If when computing \( \det S(P, P_s) \) one chooses to multiply the \( n \) entries \( b_j a_j \), then they must be multiplied by entries of the matrix obtained from \( S(P, P_s) \) by deleting the rows and columns of the entries \( b_j a_j \). This matrix is block-diagonal, its upper left block is upper-triangular, with diagonal entries equal to \( 1 - b_j/b_j' \), its right lower block is lower-diagonal, with diagonal entries equal to \( a_n \). Hence \( M_j \) is the only monomial containing \( n \) factors \( a_j \).)

To obtain the monomial \( N_s \) one chooses in the definition of \( T \) above \( j = n - m \). Hence the first \( n - m \) rows of \( T \) do not contain the variable \( a_{n-m} \). The monomial \( N_s \) is obtained by multiplying the entries \( a_{n-m+s} \) in positions \((r, n - m + s + r), r = 1, ..., n - m \), by the entries \( b_{n-m} a_{n-m} \) in positions \((q, q), q = 2n - 2m + s + 1, ..., 2n - m \), and by the entries \( b_0 \) in positions \((n - m + p, p), p = 1, ..., n - m + s \). The monomial \( N_s \) is the only one containing \( n - m \) factors \( a_{n-m+s} \) (proved by analogy with the similar claim about the monomial \( M_j \)).

The matrix \( S(P, P_s) \) contains each of the variables \( a_j, j = 1, ..., n - m \) (resp. \( a_s, s = n - m + 1, ..., n \)) in exactly \( n \) (resp. \( n - m \)) of its columns. The presence of the monomials \( M_j \) (resp. \( N_s \)) in \( \tilde{D}_m \) shows that \( \tilde{D}_m \) is a degree \( n \) polynomial in the variables \( a_j \) and a degree \( n - m \) one in the variables \( a_s \).
Quasi-homogeneity of $\tilde{D}_m$ follows from the fact that its zero set and the zero sets of the polynomials $P$ and $P_\kappa$ remain invariant under the quasi-homogeneous dilatations $x \mapsto tx, \delta_\kappa \mapsto t^{\kappa}a_\kappa, \kappa = 1, \ldots, n$. Each of the monomials $M_j$ and $N_\kappa$ is of quasi-homogeneous weight $n(n - m)$.

Irreducibility of $\tilde{D}_m$ results from the impossibility to present simultaneously all monomials $M_j$ and $N_\kappa$ as products of two monomials, of quasi-homogeneous weights $u$ and $n(n - m) - u$, for any $1 \leq u \leq n(n - m) - 1$.

\[\boxempty{Notation 2.}\] For $Q, R \in \mathbb{C}[x]$ we denote by $\text{Res}(Q, R)$ the resultant of $Q$ and $R$ and we write $P^{(m)}$ for $d^m P/dx^m$. This refers also to the case when the coefficients of $Q$ and $R$ depend on parameters. We set $a := (a_1, \ldots, a_n)$ (resp. $a^j = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)$) and we denote by $A \simeq \mathbb{C}^n$ (resp. $A^j \simeq \mathbb{C}^{n-1}$) the space of the variables $a$ (resp. $a^j$). For $K, L \in \mathbb{C}[a]$ we write $S(K, L, a_k)$ and $\text{Res}(K, L, a_k)$ for the Sylvester matrix and the resultant of $K$ and $L$ when considered as polynomials in $a_k$. We set $\tilde{D}_{m,k} := \text{Res}(\tilde{D}_m, \partial \tilde{D}_m/\partial a_k, a_k)$. For a matrix $A$ we denote by $A_{k,\ell}$ its entry in position $(k, \ell)$ and by $[A]_{k,\ell}$ the matrix obtained from $A$ by deleting its $k$th row and $\ell$th column. By $\Omega$ (indexed, with accent or not) we denote throughout the paper nonspecified nonzero constants. By $P_{m,k}$ ($1 \leq k \leq n - m$) we denote the polynomial $b_k P - x^m P_\kappa$; its coefficients of $x^n$ and $x^k$ equal $b_k - b_0 \neq 0$ and 0.

\[\boxempty{Definition 3.}\] For $1 \leq m \leq n - 2$ we denote by $\Theta$ and $\tilde{M}$ the subsets of the hypersurface $\{\tilde{D}_m = 0\} \subset \mathcal{A}$ such that for $a \in \Theta$ (resp. for $a \in \tilde{M}$) the polynomial $P$ has a root which is a double root of $P_\kappa$ (resp. the polynomials $P$ and $P_\kappa$ have two simple roots in common). The remaining roots of $P$ and $P_\kappa$ are presumed simple and mutually distinct. We call the set $\tilde{M}$ the Maxwell stratum of $\{\tilde{D}_m = 0\}$.

In the present paper we prove the following theorem;

**Theorem 4.** Suppose that $2 \leq m \leq n - 2$. Then:

(1) The polynomial $\tilde{D}_{m,k}$ can be represented in the form

$$\tilde{D}_{m,k} = A_{m,k} B_{m,k} C_{m,k}^2,$$

where $A_{m,k} = a_{n-m-k}^n$ if $k = 1, \ldots, n - m$, and $A_{m,k} = a_{n-k}^n$ if $k = n - m + 1, \ldots, n$, $B_{m,k}$ and $C_{m,k}$ are irreducible polynomials in the variables $a^k$.

(2) One has $B_{m,k} = \text{Res}(P_{m,k}, P_{m,k}')$ if $k = 1, \ldots, n - m$, and $B_{m,k} = \text{Res}(P_\kappa, P_\kappa')$ if $k = n - m + 1, \ldots, n$.

(3) The equation $C_{m,k} = 0$ defines the projection in the space $A^k$ of the closure of the Maxwell stratum.

The paper is structured as follows. After some examples and remarks in Section 2, we justify in Section 3 the form of the factor $A_{m,k}$, see Proposition 9. Section 3 begins with Lemma 8 which gives the form of the determinant of certain matrices that appear in the proof of Theorem 4. Section 4 contains Lemma 12 and Statements 13, 14 and 15 (the latter claims that the factors $B_{m,k}$ and $C_{m,k}$ are irreducible). They imply that one has $\tilde{D}_{m,k} = A_{m,k} B_{m,k}^{s_{m,k}} C_{m,k}^{r_{m,k}}$, where $s_{m,k}$, $r_{m,k} \in \mathbb{N}$, see Remark 17. Thus after Section 4 there remains to show only that $s_{m,k} = 1$ and $r_{m,k} = 2$. In Section 5 we prove Theorem 3 in the case $m = n - 2$, see Proposition 18. In Section 6 we show that $s_{m,k} = 1$. We finish the proof of Theorem 4 in Section 7 by induction on $n$ and $m$, as follows. Statement 24 deduces formula (1) for $n = n_0 + 1, k = k_0 + 1$ from formula (1) for $n = n_0, k = k_0$. Statement 25 justifies formula (1) for $n = n_0, 2 \leq m < n_0 - 2, k = 1$ using formula (1) for $n = n_0, m = n_0 - 2, k = 1$ (recall that the latter is justified in Section 5).

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2 Examples and remarks

Although Theorem 4 speaks about the case $2 \leq m \leq n - 2$, our first example treats the case $m = 1$ in order to show its differences with the case $2 \leq m \leq n - 2$:

Example 5. For $n = 3$, $m = 1$ we set $P := x^3 + ax^2 + bx + c$, $P_* := x^2 + Aax + Bb$, $0 \neq A, B \neq 1$, $A \neq B$. Then

\[ \begin{align*}
\tilde{D}_1 &= (1 - A)B(B - A)a^2b^2 + (3AB - A - 2B)abc + c^2 + A^2(1 - A)a^3c + B(1 - B)2b^3 \\
\tilde{D}_{1,1} &= -A^2(A - 1)^2c(-27A^2(1 - A)c^2 + 4(A - B)3b^3)(-Ac^2 + (1 - A)B^2(1 - B)b^3)^2 \\
\tilde{D}_{1,2} &= -B^2(B - 1)^2c(-27B(1 - B)2c + 4(A - B)3a^3)(-1 - B)c + A(1 - A)2Ba^3)^2 \\
\tilde{D}_{1,3} &= -(4Bb + A^2a^2)((1 - B)b - A(1 - A)a^2)^2.
\end{align*} \]

The condition $P$ and $P_*$ to have two roots in common is tantamount to $P_*$ dividing $P$. One has $P = (x + a(1 - A))P_* + W_1x + W_0$, where

\[ W_1 := (1 - B)b - A(1 - A)a^2, \quad W_0 := c - B(1 - A)ab. \]

The quadratic factors in the above presentations of $\tilde{D}_{1,k}$, $k = 1$, 2 and 3, are obtained by eliminating respectively $a$, $b$ and $c$ from the system of equations $W_1 = W_0 = 0$ which is the necessary and sufficient condition $P_*$ to divide $P$.

In the particular case $A = 2/3, B = 1/3$ (i.e. $P_* = P'/3$) one obtains

\[ \begin{align*}
\tilde{D}_{1,1} &= (-2^6/3^{15})c(-27c^2 + b^3)^3, \quad \tilde{D}_{1,2} = (-2^6/3^{15})c(-27c + a^3)^3, \quad \tilde{D}_{1,3} = (2^4/3^6)(3b - a^2)^3.
\end{align*} \]

Remarks 6. (1) For $n \geq 4$, $m = 1$ and $P_* = P'$ a result similar to Theorem 4 holds true. Namely, if $n \geq 4$, then $\tilde{D}_{1,k}$ is of the form $A_{1,k}B_{1,k}^3C_{1,k}$, where for $m = 1$ the polynomials $B_{m,k}$ and $C_{m,k}$ are defined in the same way as for $2 \leq m \leq n - 2$ (with $P_* = P'$), but $A_{1,k} = \alpha_n {\min}(1, n - k) + \max(0, n - k - 2)$, see [2] and [3]. Hence for $m = 1$ and $P_* = P^{(m)}$ there are two differences w.r.t. the case $m \geq 2$ – the degree 3 (instead of 1) of $B_{1,k}$, and $A_{1,n-1} = a_n$ (instead of $A_{1,n-1} = 1$). This difference can be assumed to stem from the fact that for $m = 1$, if $P$ has a root of multiplicity $\geq 3$, then this is a root of multiplicity $\geq 2$ for $P'$. This explanation is detailed below and in Remark [10].

For $n = 4$ and for generic values of $b_j$ the polynomials $\tilde{D}_{1,k}$, up to a constant nonzero factor, are of the form

\[ \begin{align*}
\tilde{D}_{1,1} &= (b_1/b_0)^3(1 - b_1/b_0)^2a_3^2B_{1,1}C_{1,1}^2, \quad \tilde{D}_{1,2} = -(b_2/b_0)^2(1 - b_2/b_0)^2a_4B_{1,2}C_{1,2}^2, \\
\tilde{D}_{1,3} &= -(b_3/b_0)^2(1 - b_3/b_0)^3a_4B_{1,3}C_{1,3}^2, \quad \tilde{D}_{1,4} = B_{1,4}C_{1,4}^3,
\end{align*} \]

where the polynomials $B_{1,k}$ and $C_{1,k}$, when considered as polynomials in the variables $a_j$ and $b_j$, are irreducible. Set $b_1 = 3b_0/4$, $b_2 = b_0/2$, $b_3 = b_0/4$. This is the case $P_* = P'$; we write $B_{1,k}(b_1 = 3b_0/4, b_2 = b_0/2, b_3 = b_0/4) = B_{1,k}$ and $C_{1,k}(b_1 = 3b_0/4, b_2 = b_0/2, b_3 = b_0/4) = C_{1,k}$. In this case the polynomials $C_{1,k}$ become reducible; they equal $B_{1,k}C_{1,k}$ which explains the presence of the cubic factor $B_{1,k}^3$. 

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Thus for $m = 1$ the genericity condition $0 \neq b_j \neq b_i \neq 0$ (which we assume to hold true in the formulation of Theorem I) is not sufficient in order to have the presentation (I) for $\hat{D}_{m,k}$. At the same time imposing a more restrictive condition means leaving outside the most interesting case $P_* = P$.

(2) For $m = n - 1$ the analog of the factor $C_{m,k}$ does not exist because $P_*$ has a single root $-b_1/b_0$. For $P_* = P^{(n-1)} := n!(x + a_1/n)$ this is $x = -a_1/n$. In this case one finds that $\hat{D}_{n-1} = (-1)^n(n!)^nP(-a_1/n)$. To see this one subtracts for $j = 1, \ldots, n$ the $j$th column of the Sylvester matrix $S(P, x + a_1/n)$ multiplied by $-a_1/n$ from its $(j + 1)$st column. This yields an $(n+1) \times (n+1)$-matrix $W$ whose entry in position $(1, n+1)$ equals $P(-a_1/n)$ and which below the first row has units in positions $(\nu + 1, \nu)$, $\nu = 1, \ldots, n$, and zeros elsewhere. Hence $\det W = (-1)^nP(-a_1/n)$. There remains to remind that $\hat{D}_{n-1} = \det S(P, n!(x + a_1/n)) = (n!)^n \det W$.

One finds directly that $\hat{D}_{n-1,k} = \partial \hat{D}_{n-1}/\partial a_k = (-1)^n(n!)^n(-a_1/n)^{n-k}$, $2 \leq k \leq n$. To find also $\hat{D}_{n-1,1}$ one first observes that

$$P_{n-1,1}(x)/(n-1)! = -(n-1)x^n + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_n$$

and that $P(-a_1/n) = P_{n-1,1}(-a_1/n)/(n-1)!$. Hence up to a nonzero rational factor the determinants of the matrices $S(P_{n-1,1}, P_{n-1,1}')$ and $S(\hat{D}_{n-1}, \partial \hat{D}_{n-1}/\partial a_1, a_1)$ coincide, i.e. $\hat{D}_{n-1,1} = \hat{c}\text{Res}(P_{n-1,1}, P_{n-1,1}')$, $\hat{c} \in \mathbb{Q}$.

(3) The fact that the factor $C_{m,k}$ is squared (see formula (I)) is not astonishing. At a generic point of the Maxwell stratum the hypersurface $\{ \check{D}_{m} = 0 \} \subset \check{A}$ is locally the intersection of two analytic hypersurfaces, see Statement I[3]. Consider a point $\Psi \in \check{A}^k$ close to the projection $\Lambda_0$ in $\check{A}^k$ of a generic point $\Lambda \in \check{M}$. There exist two points $K_j \in \{ \check{D}_{m} = 0 \}$, $j = 1, 2$, which belong to these hypersurfaces and are close to $\Lambda$, and whose common projection in $\check{A}^k$ is $\Psi$. There exists a loop $\gamma \subset \check{A}^k$, $\Psi \in \gamma$, which circumvents the projection in $\check{A}^k$ of the set $\check{M} \cup \check{M}$ such that if one follows the two liftings on $\{ \check{D}_{m} = 0 \}$ of the points of $\gamma$ which at $\Psi$ are the points $K_j$, then upon one tour along $\gamma$ these liftings are exchanged. Hence in order to define the projection of $\check{M}$ in $\check{A}^k$ by the zeros of an analytic function one has to eliminate this monodromy of rank 2 by taking the square of $C_{m,k}$. For the case $m = 1$ a detailed construction of such a path $\gamma$ is given in [8].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{The sets $\{ \check{D}_{2} = 0 \}_{a=0,b=-1}$, $\{ \check{D}_{2} = 0 \}_{a=b=0}$ and $\{ \check{D}_{2} = 0 \}_{a=0,b=1}$ for $n = 4$.}
\end{figure}

**Example 7.** For $n = 4$ we consider the case of real polynomials. We write $P = x^4 + ax^3 + bx^2 + cx + d$ and we limit ourselves to the situation when $P_* := P^{(m)}$. On Fig. [1] we show the sets $\{ \check{D}_{1} = 0 \}_{a=0}$ and $\{ \check{D}_{2} = 0 \}_{a=0}$ when $b$, $c$ and $d$ are real. The sets $\{ \check{D}_{1} = 0 \}_{a=0}$ and
one can develop it w.r.t. that row or column in which there is a \( n \) entry. The matrix \( B \) contains 2 entries \( r \) and there is no entry \( j \). By doing so \( p - 1 \) times one finds that \( \det B = r_2 \cdots r_p \). The + sign of this product follows from the entries \( r_j \) being situated on the diagonal. When finding \( \det C \) one can develop it w.r.t. that row or column in which there is an entry \( q_v \) and there is no entry \( q_v \) and there is no entry \( r_j \). By doing so \( p - 1 \) times one finds that \( \det C = \pm q_2 \cdots q_p \) which proves the lemma.

In the present section we prove the following proposition:

**Proposition 9.** (1) For \( k = n - m + 1, \ldots, n \), the polynomial \( \tilde{D}_{m,k} \) is not divisible by any of the variables \( a_j \), \( j \neq n - m \).

(2) For \( k = 1, \ldots, n - m \), the polynomial \( \tilde{D}_{m,k} \) is not divisible by any of the variables \( a_j \), \( j \neq n \).

(3) For \( k = 1, \ldots, n - m \), the polynomial \( \tilde{D}_{m,k} \) is divisible by \( a_n^{n-m-k} \) and not divisible by \( a_n^{n-m-k+1} \).

(4) For \( k = n - m + 1, \ldots, n \), it is divisible by \( a_n^{n-k} \) and not divisible by \( a_n^{n-k+1} \).
Proof of part (1): We show first that for \( a_i = 0, \ n - m \neq i \neq k \), the polynomial \( \tilde{D}_m \) is of the form \( \Omega'^n a^n_{n-m} + \Omega''a^{n-k}_n a^{n-m}_k \). Indeed, in this case one can list the nonzero entries of the \((2n - m) \times (2n - m)\)-matrix \( S(P, P_e) \) and the positions in which they are situated:

\[
\begin{align*}
1 & \quad (j, j) \quad , \quad a_{n-m} \quad (j, j + n - m) \quad , \\
\mu & \quad (j, j + k) \quad , \quad j = 1, \ldots, n - m , \\
b_0 & \quad (\nu + n - m, \nu) \quad , \quad b_{n-m} a_{n-m} \quad (\nu + n - m, \nu + n - m) \quad , \quad \nu = 1, \ldots, n .
\end{align*}
\]

Subtract the \((\mu + n - m)\)th row multiplied by \(1/b_{n-m}\) from the \(\mu\)th one for \( \mu = 1, \ldots, n - m \). This makes disappear the terms \(a_{n-m}\) in positions \((j, j)\) while the terms 1 in positions \((j, j)\) become equal to \(\Omega_* := 1 - b_0/b_{n-m}\). The determinant of the matrix doesn't change. We denote the new matrix by \( T \). To compute \( \det T \) one can develop it \( n - k \) times w.r.t. the last column; each time one has a single nonzero entry in this column, this is \( b_{n-m} a_{n-m} \) in position \((2n - m - \ell, 2n - m - \ell), \ell = 0, \ldots, n - k - 1 \). The matrix \( T_1 \) which remains after deleting the last \( n - k \) rows and columns of \( T \) has the following nonzero entries, in the following positions:

\[
\begin{align*}
\Omega_* & \quad (j, j) \quad , \quad \mu \quad (j, j + k) \quad , \quad j = 1, \ldots, n - m , \\
b_0 & \quad (\nu + n - m, \nu) \quad , \quad b_{n-m} a_{n-m} \quad (\nu + n - m, \nu + n - m) \quad , \quad \nu = 1, \ldots, k .
\end{align*}
\]

Clearly \( \det T = (b_{n-m} a_{n-m})^{n-k} \det T_1 \). On the other hand the matrix \( T_1 \) satisfies the conditions of Lemma 8 with \( p = n - m + k \) and \( s = k \). Hence \( \det T_1 = \Omega'^n a^n_{n-m} + \Omega''a^{n-m}_n a^{n-m}_k \) and \( \tilde{D}_m = \Omega'^n a^n_{n-m} + \Omega''a^{n-k}_n a^{n-m}_k \).

But then the \((2n - 2m - 1) \times (2n - 2m - 1)\)-Sylvester matrix \( S^* := S(\tilde{D}_m, \partial \tilde{D}_m/\partial \mu, \mu) \) has only the following nonzero entries, in the following positions:

\[
\begin{align*}
\Omega'' a^{n-k}_{n-m} & \quad (j, j) \quad , \quad \Omega a^n_{n-m} \quad (j, j + n - m) \quad , \quad j = 1, \ldots, n - m - 1 , \\
(n - m) \Omega'' a^{n-k}_{n-m} & \quad (\nu + n - m - 1, \nu) \quad , \quad \nu = 1, \ldots, n - m .
\end{align*}
\]

Part (1) follows from \( \det S^* = \pm(\Omega a^n_{n-m})^{n-m-1}((n - m)\Omega'' a^{n-k}_{n-m})^{n-m} \neq 0 . \)

Proof of part (2): We prove that for \( a_i = 0, k \neq i \neq n \), the polynomial \( \tilde{D}_m \) is of the form \( \Omega'^n a^n_{n-m} + \Omega'' a^{n-m-k}a_k^n \). Indeed, we list below the nonzero entries of the matrix \( S(P, P_e) \) and their positions:

\[
\begin{align*}
1 & \quad (j, j) \quad , \quad a_k \quad (j, j + k) \quad , \\
a_n & \quad (j, j + n) \quad , \quad j = 1, \ldots, n - m , \\
b_0 & \quad (\nu + n - m, \nu) \quad , \quad b_k a_k \quad (\nu + n - m, \nu + k) \quad , \quad \nu = 1, \ldots, n .
\end{align*}
\]

One can develop \( n - m - k \) times \( \det S(P, P_e) \) w.r.t. its last column, where the only nonzero entries equal \( a_n \). Thus \( \det S(P, P_e) = \pm a^n a^{n-m-k} \det H \), where \( H \) is obtained from \( S(P, P_e) \) by
Its determinant equals \( \pm p^\Omega \). To make disappear the terms \( a_b \) nonzero entry list the nonzero entries of \( H \Omega \). The terms 1 in positions \((j, j)\) are replaced by 1 and \( k \text{ in positions } (j, j + k)\), \( j = 1, \ldots, k \),
\[
\begin{align*}
1 & \quad (j, j) \quad , \quad a_k \quad (j, j + k) \quad , \\
 a_n & \quad (j, j + n) \quad , \quad j = 1, \ldots, k \quad , \\
b_0 & \quad (\nu + k, \nu) \quad , \quad b_k a_k \quad (\nu + k, \nu + k) \quad , \quad \nu = 1, \ldots, n .
\end{align*}
\]
For \( \mu = 1, \ldots, k \) one can subtract the \((\mu + k)\)th row multiplied by \( 1/b_k \) from the \( \mu \)th one to make disappear the terms \( a_k \) in positions \((\mu, \mu + k)\); the entries 1 in positions \((\mu, \mu)\) change to \( \Omega^* := 1 - b_0/b_k \). We denote the newly obtained matrix by \( H_1 \). Obviously \( \det H_1 = \det H \); we list the nonzero entries of \( H_1 \) and their respective positions:
\[
\begin{align*}
\Omega^* & \quad (j, j) \quad , \quad a_n \quad (j, j + n) \quad , \quad j = 1, \ldots, k \quad , \\
b_0 & \quad (\nu + k, \nu) \quad , \quad b_k a_k \quad (\nu + k, \nu + k) \quad , \quad \nu = 1, \ldots, n .
\end{align*}
\]
One applies Lemma \( \mathcal{L} \) with \( p = n + k \), \( s = n \) to the matrix \( H_1 \) to conclude that \( \det H = \det H_1 = (\Omega^*)^k (b_k a_k)^n \pm b_0^k a_k^n \), so \( \tilde{D}_m = \det S(P, P_s) = \Omega^\dagger a_n^{n-m} + \Omega^\dagger a_n^{n-m-k} a_k^n \).

But then the \((2n - 1) \times (2n - 1)\)-Sylvester matrix \( S(\tilde{D}, \partial \tilde{D}/\partial a_k, a_k) \) has only the following nonzero entries, in the following positions:
\[
\begin{align*}
\Omega^\dagger a_n^{n-m-k} & \quad (j, j) \quad , \quad \Omega^\dagger a_n^{n-m} \quad (j, j + n) \quad , \quad j = 1, \ldots, n - 1 \quad , \\
n \Omega^\dagger a_n^{n-m-k} & \quad (\nu + n - 1, \nu) \quad , \quad \nu = 1, \ldots, n .
\end{align*}
\]
Its determinant equals \( \pm (\Omega^\dagger a_n^{n-m})^{n-1} (n \Omega^\dagger a_n^{n-m-k})^n \neq 0 \) which proves part (2).

Proof of part (3): For \( k = 1, \ldots, n - m \) the polynomial \( \tilde{D}_m \) contains the monomial \( M_k := \pm b_k^\Omega a_k^n \), and it does not contain any other monomial of the form \( \Omega^\dagger a_k^n \), where \( \tilde{E} \) is a product of powers of variables \( a_i \) with \( i \neq k \), see Proposition \( \mathcal{P} \).

Hence the first column of the \((2n - 1) \times (2n - 1)\)-matrix \( Y := S(\tilde{D}_m, \partial \tilde{D}_m/\partial a_k, a_k) \) contains only two nonzero entries, and these are \( Y_{1,1} = \pm b_k^\Omega (1 - b_0/b_k)^k a_n^{n-m-k} \) and \( Y_{n,1} = \pm nb_k^\Omega (1 - b_0/b_k)^k a_n^{n-m-k} \). Thus \( \Delta := \det Y \) is divisible by \( a_n^{n-m-k} \). We consider two cases:

Case 1: \( k = n - m \). We have to prove that \( \tilde{D}_{m,n-m}|_{a_n=0} \neq 0 \). Set \( a_j = 0 \) for \( n-m \neq j \neq n-1 \). Hence the nonzero entries of the matrix \( S(P, P_s) \) and their positions are
\[
\begin{align*}
1 & \quad (j, j) \quad , \quad a_{n-m} \quad (j, j + n - m) \quad , \\
a_{n-1} & \quad (j, j + n - 1) \quad , \quad j = 1, \ldots, n - m \quad , \\
b_0 & \quad (\nu + n - m, \nu) \quad , \quad b_{n-m} a_{n-m} \quad (\nu + n - m, \nu + n - m) \quad , \quad \nu = 1, \ldots, n .
\end{align*}
\]
One can subtract the \((j + n - m)\)th row multiplied by \( 1/b_{n-m} \) from the \( j \)th one, \( j = 1, \ldots, n - m \), to make disappear the terms \( a_{n-m} \) in the first \( n - m \) rows. This doesn’t change \( \det S(P, P_s) \). The terms 1 in positions \((j, j)\) are replaced by \( 1 - b_0/b_{n-m} \). Hence \( \tilde{D}_m \) is of the form \( \Omega_1 a_{n-m} + \Omega_2 a_{n-m}^{n-m} a_{n-m} \) (one first develops \( \det S(P, P_s) \) w.r.t. the last column, where there is a single nonzero entry \( b_{n-m} a_{n-m} \) in position \((2n - m, 2n - m)\), and then applies Lemma \( \mathcal{L} \) with \( p = 2n - m - 1 \) and \( s = n - 1 \).
Hence $R$ the rows and columns of the entries $b$

Lemma 10. \(\text{Lemma 10.}\)

Proof. \(\text{Proof.}\)

One can subtract the \((j + n - 1)\)st row from the \(j\)th one, \(j = 1, \ldots, n - 1\), to make disappear the terms \(\Omega_2 a_{n-1}^{n-m} (j, j + n - 1)\) in the first \(n - 1\) rows; the terms \(\Omega_1\) become \((1 - n)\Omega_1\). Hence det \(S^H = \Omega_3 a_{n-1}^{n(n-m)} \neq 0\).

Case 2: \(1 \leq k \leq n - m - 1\). To prove that \(\Delta\) is not divisible by \(a_n^{n-m-k+1}\) we develop it w.r.t. its first column:

\[
\Delta := (\pm b_k^k (1 - b_0 / b_k)^k a_n^{n-m-k}) (\text{det}([Y]_{1,1}) + (-1)^n \text{det}([Y]_{n,1})).
\]

Our aim is to show that for \(a_n = 0\) the sum \(Z := \text{det}([Y]_{1,1}) + (-1)^{n+1} \text{det}([Y]_{n,1})\) is nonzero; this implies \(a_n^{n-m-k+1} \neq 0\) not dividing \(\Delta\). Notice that for \(a_n = 0\) the only nonzero entries in the second column of \(Y\) (i.e. of \(Y|_{a_n = 0} = Y^0\)) are \(Y_{1,2}^0\) and \(Y_{n,2}^0 = (n - 1)Y_{1,2}^0\). Thus

\[
Z|_{a_n = 0} = (Y_{1,2}^0 + (-1)^{n+1} (-1)^n Y_{n,2}^0) \text{det}(Y^\dagger) = (1 - n(n-1)) Y_{1,2}^0 \text{det}(Y^\dagger),
\]

where the matrix \(Y^\dagger\) is obtained from \(Y^0\) by deleting its first two columns, its first and its \(n\)th rows.

Lemma 10. \(\text{Lemma 10.}\)

The entry \(Y_{1,2}^0\) is a not identically equal to 0 polynomial in the variables \(a_j, k \neq j \neq n\).

Proof. Indeed, this is the coefficient of \(a_k^{n-1}\) in \(R^0 := \text{Res}(P, P^*)|_{a_n = 0}\). The matrix \(S_* := S(P, P^*)|_{a_n = 0}\) has a single nonzero entry in its last column; this is \((S_*)_{2n-m, 2n-m} = b_{n-m} a_{n-m}\).

Then det \(M := [S_*]_{2n-m, 2n-m} (M = (2n - m - 1) \times (2n - m - 1))\).

For \(\nu = 1, \ldots, n - 1\) one can subtract the \((n - m + \nu)\)th row of \(M\) multiplied by \(1/b_k\) from its \(\nu\)th row to make disappear the terms \(a_k\) in its first \(n - m\) rows. The new matrix is denoted by \(M^1\); one has det \(M = \text{det} M^1\). The only terms of det \(M^1\) containing \(a_k^{n-1}\) are now obtained by multiplying the entries \(b_k a_k\) of the last \(n - 1\) rows of \(M^1\). To get these terms up to a sign one has to multiply \((b_k a_k)^{n-1}\) by det \(M^*\), where \(M^*\) is obtained from \(M^1\) by deleting the rows and columns of the entries \(b_k a_k\). The matrix \(M^*\) is block-diagonal, its left upper block is upper-triangular and its right lower block is lower-triangular. The diagonal entries of these blocks (of sizes \(k \times k\) and \((n - m - k) \times (n - m - k)\) equal \(1 - b_0 / b_k\) and \(a_{n-1}\). Hence

\[
Y_{1,2}^0 = \pm b_{n-m} a_{n-m} (1 - b_0 / b_k)^k b_k^{n-1} a_{n-1}^{n-m-k} \neq 0.
\]

There remains to prove that det \(Y^\dagger \neq 0\), see \((2)\). The matrix \(Y^\dagger\) is obtained as follows. Set \(D^\dagger := \tilde{D}^\dagger|_{a_n = 0} = \text{det} S_*\); recall that det \(S_* = b_{n-m} a_{n-m} \text{det} M^1\), see the proof of Lemma 10. Then det \(Y^\dagger = \text{det} D^\dagger\). Notice that \(D^\dagger\) is a degree \(n - 1\), not \(n\), polynomial in \(a_k\), therefore det \(Y^\dagger\) is \((2n - 3) \times (2n - 3)\). It suffices to show that for \(a_j = 0, j \neq k, n - m, n - 1\), one has det \(Y^\dagger \neq 0\). This results from det \(M^1|_{a_j = 0, k \neq j \neq n - 1}\) not having multiple roots (which we prove below).

One can develop \(n - m - k\) times det \(M^1\) w.r.t. its last column, where it has a single nonzero entry \(a_{n-1}\), to obtain det \(M^1 = \pm a_{n-1}^{n-m-k} \text{det} M^1\); \(M^1\) is \((n + k - 1) \times (n + k - 1)\), it is obtained from \(M^1\) by deleting the last \(n - m - k\) columns and the rows with indices \(k + 1, \ldots, n - m\). The matrix \(M^1\) satisfies the conditions of Lemma 8 with \(p = n + k - 1\) and \(s = k,\)

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the entries \( r_j \) from the lemma equal \( 1 - b_0/b_k \neq 0 \) (for \( j = 1, \ldots, k \)) or \( b_k a_k \) (for \( j = k + 1, \ldots, n + k - 1 \)); one has \( q_j = a_{n-1} \) (1 \( \leq j \leq k \)) or \( q_j = b_0 \) (\( k + 1 \leq j \leq n + k - 1 \)). Hence \( \det M_{1,1}^j \) (\( a_j, k \neq j \neq n+1 \)) = \( (1 - b_0/b_k)^k (b_k a_k)^{n-1} \pm b_0^{n-1} a_{n-1}^k \). For \( a_{n-1} \neq 0 \) it has \( n-1 \) distinct roots. Part (3) is proved.

\( \square \)

**Proof of part (4):** We use sometimes the same notation as in the proof of part (3), but with different values of the indices, therefore the proofs of the two parts of the proposition should be considered as independent ones. For \( k = n - m + 1, \ldots, n \), the polynomial \( \tilde{D}_m \) contains the monomial \( \Omega_{n+k+m} := \pm b_{n-m}^{n-k} a_{n-m}^{-k} b_k a_k^{n-m} \); it does not contain any other monomial of the form \( \Omega a_{n-m}^{k-1} D_m \), where \( D \) is a product of powers of variables \( a_i \) with \( i \neq k \), see Proposition [9].

The first column of the \((2n-2m-1) \times (2n-2m-1)\)-matrix \( Y := S(\tilde{D}_m, \partial \tilde{D}_m/\partial a_k, a_k) \) contains only two nonzero entries, namely \( Y_{1,1} = \pm b_{n-m}^{n-k} a_{n-m}^{-k} b_k \) and \( Y_{n,1} = \pm (n-m) b_{n-m}^{n-k} a_{n-m}^{k} b_k \). Thus \( \det Y \) is divisible by \( a_{n-m}^{n-k} \). We consider two cases:

**Case 1:** \( k = n \). We show that \( \det Y \neq 0 \) if \( a_{n-m} = 0 \). We prove this for \( a_j = 0, n - m - 1 \neq j \neq n \). In this case the nonzero entries of the matrix \( S(P, P_*) \) and their positions are

\[
\begin{align*}
1 & (j, j) , \quad a_{n-m-1} & (j, j + n - m - 1) , \\
\quad a_n & (j, j + n) , & j = 1, \ldots, n - m , \\
b_0 & \quad (\nu + n - m, \nu) , & b_{n-m-1} a_{n-m-1} & \quad (\nu + n - m, \nu + n - m - 1) , & \nu = 1, \ldots, n .
\end{align*}
\]

Subtracting the \( (j + n - m) \)th row multiplied by \( 1/b_{n-m-1} \) from the \( j \)th one for \( j = 1, \ldots, n - m \), one makes disappear the terms \( a_{n-m-1} \) in the first \( n - m \) rows. The only nonzero entry in the last column is now \( a_n \) in position \( (n - m, 2n - m) \), so

\[
\det S(P, P_*) = (-1)^n a_n \det \{ S(P, P_*) \}_{n-m, 2n-m} .
\]

The last matrix satisfies the conditions of Lemma [8] with \( p = 2n - m - 1 \), \( s = n \) and one finds that its determinant is of the form \( \Omega_5 a_{n-m-1}^n + \Omega_5 a_{n-m-1}^{n-1} \). Hence \( \det S(P, P_*) = (-1)^n a_n (\Omega_4 a_{n-m-1}^n + \Omega_5 a_{n-m-1}^{n-1}) \). This means that the matrix \( S(\tilde{D}_m, \partial \tilde{D}_m/\partial a_k, a_k) \) has only the following entries in the following positions:

\[
\begin{align*}
\Omega_5 & \quad (j, j) , & \Omega_4 a_{n-m-1}^n & \quad (j, j + n - m - 1) , & j = 1, \ldots, n - m - 1 , \\
(n-m) \Omega_5 & \quad (\nu + n - m - 1, \nu) , & \Omega_4 a_{n-m-1}^n & \quad (\nu + n - m - 1, \nu + n - m - 1) , & \nu = 1, \ldots, n .
\end{align*}
\]

One can subtract the \( (j + n - m - 1) \)st row from the \( j \)th one, \( j = 1, \ldots, n - m - 1 \), to make disappear the terms \( \Omega_4 a_{n-m-1}^n \) in the first \( n - m - 1 \) rows. The matrix becomes lower-triangular, with diagonal entries equal to \( (1 - n + m) \Omega_5 \) or to \( \Omega_4 a_{n-m-1}^n \), so its determinant is not identically equal to 0.

**Case 2:** \( n - m + 1 \leq k \leq n - 1 \). To prove that \( \det Y \) is not divisible by \( a_{n-m}^{n-k+1} \) we develop it w.r.t. its first column:

\[
\det Y = (\pm b_{n-m}^{n-k} a_{n-m}^{-k} b_k)(\det([Y]_{1,1}) + (-1)^{n-m}(n - m) \det([Y]_{n-m,1})) .
\]
Our aim is to show that for $a_{n-m} = 0$ the sum $U := \det([Y]_{1,1}) + (-1)^{n-m}(n-m) \det([Y]_{n-m,1})$ is nonzero; this implies $a_{n-k+1}$ not dividing $\det Y$. Notice that for $a_{n-m} = 0$ the only nonzero entries in the second column of $Y^0 := Y|_{a_{n-m}=0}$ are $Y^0_{1,2}$ and $Y^0_{n-m,2} = (n-m-1)Y^0_{1,2}$. Thus

$$U|_{a_{n-m}=0} = (Y^0_{1,2} + (-1)^{n+1}(-1)^{n-m}(n-m)Y^0_{n-m,2}) \det Y^\dagger$$

$$= (1 - (n-m)(n-m-1))Y^0_{1,2} \det Y^\dagger,$$

where the matrix $Y^\dagger$ is obtained from $Y^0$ by deleting its first two columns, its first and $(n-m)$th rows.

Lemma 11. The entry $Y^0_{1,2}$ is a not identically equal to 0 polynomial in the variables $a_j$, $k \neq j \neq n - m$.

Proof. Indeed, this is the coefficient of $a_k^{n-m-1}$ in $R^0 := \Res(P, P_*)|_{a_{n-m}=0}$. The matrix $S_* := S(P, P_*)|_{a_{n-m}=0}$ has a single nonzero entry in its last column; this is $(S_*)_{n-m,2n-m} = a_n$. Hence $R^0 = a_n \det M$, where $M := [S_*]_{n-m,2n-m}$ ($M$ is $(2n-m-1) \times (2n-m-1)$).

The only terms of $\det M$ containing $a_k^{n-m-1}$ are obtained by multiplying the entries $a_k$ of the first $n-m-1$ rows of $M$. To obtain these terms up to a sign one has to multiply $a_k^{n-m-1}$ by $\det M^*$, where $M^*$ is obtained from $M$ by deleting the rows and columns of the entries $a_k$. The matrix $M^*$ is block-diagonal, its left upper block is upper-triangular and its right lower block is lower-triangular. The diagonal entries of these blocks (of sizes $k \times k$ and $(n-m-k) \times (n-m-k)$) equal $b_0$ and $a_{n-m-1}$. Hence $Y^0_{1,2} = \pm a_n b_0 a_k^{n-m-1} \neq 0$.

There remains to prove that $\det Y^\dagger \neq 0$, see (3). The matrix $Y^\dagger$ is obtained as follows. Set $D^\dagger := \tilde{D}_m|_{a_{n-m}=0} = \det S_*$; recall that $\det S_* = a_n \det M$ (see the proof of Lemma 11). Then $Y^\dagger = S(D^\dagger, \partial D^\dagger/\partial a_k, a_k)$. Notice that $D^\dagger$ is a degree $n-m-1$, not $n-m$, polynomial in $a_k$, therefore $Y^\dagger$ is $(2n-2m-3) \times (2n-2m-3)$. It suffices to show that for $a_j = 0$, $k \neq j \neq n-m-1$, one has $\det Y^\dagger \neq 0$. This results from $\det M|_{a_j=0, k \neq j \neq n-m-1}$ not having multiple roots (which we prove below).

For $a_j = 0$, $k \neq j \neq n - m - 1$, one can develop $n-k$ times $\det M$ w.r.t. its last column in which there is a single nonzero entry $b_{n-m-1} a_{n-m-1}$ (on the diagonal). Hence $\det M = (b_{n-m-1} a_{n-m-1})^{n-k} \det M^\dagger$, where $M^\dagger$ is $(n-m+k-1) \times (n-m+k-1)$; it is obtained from $M$ by deleting the last $n-k$ rows and columns. The matrix $M^\dagger$ satisfies the conditions of Lemma 8 with $p = n-m+k-1$, $s = k$, $r_j = 1$, $q_j = a_k$ ($j = 1, \ldots, n-m-1$) or $r_j = a_{n-m-1}$, $q_j = b_0$ ($j = n-m, \ldots, n-m+k-1$). Hence $\det M^{\dagger} = a_k^{n-m-1} \pm b_0 a_k^{n-m-1}$. For $a_{n-m} = 0$ it has $n-m-1$ distinct roots.

\section{Some properties of the sets $\Theta$ and $\tilde{M}$}

Lemma 12. Suppose that all roots of $P_*, (a^0)$ $(a^0 \in A)$ are simple and nonzero and that $P_*, (a^0)$ and $P_*, (a^0)$ have exactly one root in common. Then for any $j = n-m+1, \ldots, n$, in a neighbourhood of $a^0 \in A$ the set $\{\tilde{D}_m = 0\}$ is locally the graph of a smooth analytic function in the variables $a^0$. If in addition all roots of $P_{m,k}, (a^0)$ are simple and nonzero ($1 \leq k \leq n-m$), then in a neighbourhood of $a^0 \in A$ the set $\{\tilde{D}_m = 0\}$ is locally the graph of a smooth analytic function in the variables $a^k$.

Proof. Denote by $[a]_{n-m}$ the first $n-m$ coordinates of $a \in A$. Any simple root of $P_*$ is locally (in a neighbourhood of $[a^0]_{n-m}$) the value of a smooth analytic function $\lambda$ in the variables $[a]_{n-m}$.
As \( \lambda([a^0]_{n-m}) \neq 0 \), the condition \( P(\lambda, a)/\lambda^j = 0, \ j < m \), allows to express \( a_{n-j} \) locally (for \( a_i \) close to \( a^{0}_i, \ i \neq j \)) as a smooth analytic function in the variables \( a^{n-j} \). Suppose that all roots of \( P_{m,k}(\cdot, a^{0}) \) are simple and nonzero. Then any of these roots is a smooth analytic function in the variables \( a^k \). This refers also to \( \mu \), the root in common of \( P \) and \( P_{\ast} \) which is also a root of \( P_{m,k} \). Hence one can express \( a_k \) as a function in \( a^k \) from the condition \( P(\mu, a)/\mu^{n-k} = 0 \). □

**Statement 13.** At a point of the Maxwell stratum the hypersurface \( \{ \tilde{D}_m = 0 \} \) is locally the transversal intersection of two smooth analytic hypersurfaces along a smooth analytic subvariety of codimension 2.

**Proof.** Suppose first that the roots in common of \( P \) and \( P_{\ast} \) are 0 and 1. The two conditions \( P_{\ast}(0) = P_{\ast}(1) = 0 \) define a codimension 2 linear subspace \( S \) in the space \( A \) of the variables \( a \). Adding to them the two conditions \( P(0) = P(1) = 0 \) means defining a codimension 2 linear subspace \( T \subset S \); hence \( T \) is a codimension 4 linear subspace of \( A \). The two linear subspaces \( \{ P(0) = 0 \} \) and \( \{ P(1) = 0 \} \) and their intersections with \( \{ P_{\ast}(0) = P_{\ast}(1) = 0 \} \) intersect transversally (along respectively \( \{ P(0) = P(1) = 0 \} \) and \( T \)).

By means of a linear change \( \tau : x \mapsto \alpha x + \beta, \ \alpha \in \mathbb{C}^*, \ \beta \in \mathbb{C} \), one can transform any pair of distinct complex numbers into the pair \((0,1)\). Hence at a point of \( T \) the Maxwell stratum is locally the direct product of \( T \) and the two-dimensional orbit of the group of linear diffeomorphisms induced in the space \( A \) by the group of linear changes \( \tau \). This proves the statement. □

**Statement 14.** (1) At a point of the set \( \Theta \) (see Definition 3) the set \( \{ \tilde{D}_m = 0 \} \) is not representable as the graph in the space \( A \) of a smooth analytic function in the variables \( a^j \), for any \( j = n - m + 1, \ldots, n \).

(2) At a point of the set \( \Theta \) this set is a smooth analytic variety of dimension \( n - 2 \) in the space of variables \( a \).

**Proof of part (1):** Suppose that for some \( a = a_0 \in A \) one has \( P_{\ast}(x_0, a_0) = P'_{\ast}(x_0, a_0) = 0 \). Suppose first that \( x_0 \neq 0 \). Consider the equation
\[
P_{\ast}(x, a_0) = \varepsilon, \quad \text{where} \quad \varepsilon \in (\mathbb{C}, 0).
\]
Its left-hand side equals \( P''_{\ast}(x_0, a_0)(x-x_0)^2/2 + o((x-x_0)^2) \) (with \( P''_{\ast}(a_0, x_0) \neq 0 \)). Thus locally (for \( x \) close to \( x_0 \)) one has
\[
x - x_0 = \left(2/P''_{\ast}(x_0, a_0)\right)^{1/2} \varepsilon^{1/2} + o(\varepsilon^{1/2}).
\]
In a neighbourhood of \( a_0 \in A \) one can introduce new coordinates two of which are \( x_0 \) and \( \varepsilon \). Indeed, one can write
\[
(n - m - 1)! P''_{\ast}/n! = (x-x_0)(x^{n-m-2} + g_1 x^{n-m-3} + \cdots + g_{n-m-2})
\]
\[
= x^{n-m-1} + b^*_1 a_1 x^{n-m-2} + \cdots + b^*_n a_n x^{n-m-1},
\]
where \( b^*_j = (n-j)!/(n-m-j-1)! n! \). Hence
\[
b^*_1 a_1 = g_1 - x_0, \quad b^*_2 a_2 = g_2 - x_0 g_1, \quad \cdots,
\]
\[
b^*_n a_n = g_{n-m-2} - x_0 g_{n-m-3}, \quad b^*_n a_n = -x_0 g_{n-m-2}.
\]
The Jacobian matrix $\partial(a_1, \ldots, a_{n-m-1})/\partial(x_0, g_1, \ldots, g_{n-m-2})$ is, up to multiplication of the columns by nonzero constants followed by transposition, the Sylvester matrix of the polynomials $x - x_0$ and $x^{n-m-2} + g_1 x^{n-m-3} + \cdots + g_{n-m-2}$. Its determinant is nonzero because $x_0$ is not a root of the second of these polynomials.

Thus in the space of the variables $(a_1, \ldots, a_{n-m-1})$ one can choose as coordinates $(x_0, g_1, \ldots, g_{n-m-2})$. The polynomial $P_*$ is a primitive of $P'$ and $(-\varepsilon)$ can be considered as the constant of integration, see (4), therefore $(g, \mu, a)$ allows to express values exist – the change $x \to x + \delta$, $\delta \in \mathbb{C}$, shifts simultaneously by $-\delta$ all roots of $P$ (hence of all its nonconstant derivatives as well).

Proof of part (2): Denote by $\xi$ the root of $P_*$ which is also a root of $P_*$ and of $P$. Then $\xi$ is a smooth analytic function in the variables $a^\dagger := (a_1, \ldots, a_{n-m-1})$. The condition $P_*(\xi, a) = 0$ allows to express $a_{n-m}$ as a smooth analytic function $\alpha$ in the variables $a^\dagger$. Set $a^\star := a_{n-m} = \alpha(a^\dagger)$. One can express $a_n$ as a smooth analytic function in the variables $a_j$, $n - m \neq j \neq n$, from the condition $P(\xi, a^\star) = 0$. Thus locally $\Theta$ is the graph of a smooth analytic vector-function in the variables $a_j$, $n - m \neq j \neq n$, with two components.

Statement 15. For $2 \leq m \leq n - 2$ the polynomials $B_{m,k}$ and $C_{m,k}$ defined in Theorem 4 are irreducible.

Proof. Irreducibility of the factor $B_{m,k}$ is proved by analogy with Proposition 9. (For $n-m+1 \leq k \leq n$ the analogy is complete because after the dilatations $a_j \mapsto a_j/b_j$, $j = 1, \ldots, n - m$, the polynomial $P_*$ becomes $b_0 P^*$, where $P^*$ is the polynomial $P$ defined for $n - m$ instead of $n$. For $1 \leq k \leq n - m$ the coefficients of the polynomial $P_{m,k}$ are not $a_j$ (we set $a_0 = 1$), but $(b_k - b_j)a_j$, and one can perform similar dilatations. Only the variable $a_k$ is absent; this, however, is not an obstacle to the proof of irreducibility. The details are left for the reader.)

Irreducibility of the factors $C_{m,k}$ can be proved like this. Denote by $\xi$ and $\eta$ two of the roots of $P_*$. They are multivalued functions of the coefficients $a_1, \ldots, a_{n-m}$. The system of two equations $P(\xi, a) = P(\eta, a) = 0$ allows to express for $\xi \neq \eta$ the coefficients $a_n$ and $a_{n-1}$ as functions of $a_1, \ldots, a_{n-2}$. These multivalued functions are defined over a Zariski dense open subset of the space of variables $(a_1, \ldots, a_{n-2})$ from which irreducibility of the set $\mathcal{M}$ follows. Hence its projections in the hyperplanes $A^k$ are also irreducible.

Remark 16. In the case $m = 1$ one cannot prove in the same way as above that the polynomials $C_{1,k}$ are irreducible because the coefficient $a_{n-1}$ is in fact $a_{n-m}$.

Remark 17. Proposition 9, Lemma 12, Statements 13 and 15 allow to conclude that $\tilde{D}_{m,k}$ is of the form $A_{m,k}B_{m,k}C_{m,k}$, where $s_{m,k}$, $r_{m,k} \in \mathbb{N}$. Indeed, the form of the factor $A_{m,k}$ is justified by Proposition 9. It follows from Lemma 12 and its proof that for $A_{m,k}B_{m,k}C_{m,k} \neq 0$ the polynomials $\tilde{D}_m$ and $\partial \tilde{D}_m/\partial a_k$, when considered as polynomials in $a_k$, have no root in common. Hence a priori $\tilde{D}_{m,k}$ is of the form $A_{m,k}B_{m,k}C_{m,k}$, with $s_{m,k}$, $r_{m,k} \in \mathbb{N} \cup \{0\}$ (implicitly
we use the irreducibility of $B_{m,k}$ and $C_{m,k}$ here). Statements [13] and [14] imply that one cannot have $s_{m,k} = 0$ or $r_{m,k} = 0$. To prove formula [11] now means to prove that $s_{m,k} = 1$, $r_{m,k} = 2$. This is performed in the next sections.

5 The case $m = n - 2$

Proposition 18. For $m = n - 2$, $n \geq 4$, one has $s_{m,k} = 1$ and $r_{m,k} = 2$.

Proof for $3 \leq k \leq n$. For $3 \leq k \leq n$ the polynomial $\tilde{D}_{n-2}$ is a degree 2 polynomial in $a_k$, see Proposition [11] so one can set $\tilde{\partial}D_{n-2}/\partial a_k = 2Ua_k + V$, where $U, V, W \in \mathbb{C}[a^k]$, $U \neq 0$. Hence $S(\tilde{D}_{n-2},\partial \tilde{D}_{n-2}/\partial a_k, a_k) = \left( \begin{array}{ccc} U & V & W \\ 2U & V & 0 \\ 0 & 2U & V \end{array} \right)$ and

$$\tilde{D}_{n-2,k} = U(4UW - V^2).$$

The second factor is up to a sign the discriminant of the quadratic polynomial (in the variable $a_k$) $Ua_k^2 + Va_k + W$. Up to a sign, $U$ is the determinant of the matrix $S^L$ obtained from $S(P, P_*)$ by deleting its first two rows and the columns, where its entries $a_k$ are situated. Hence $U = \omega a_2^{-k}$, $\omega \in \mathbb{C}^*$. Indeed, $S^L$ is block-diagonal, with diagonal blocks of sizes $k \times k$ (upper left) and $(n - k) \times (n - k)$ (lower right). They are respectively upper- and lower-triangular, with diagonal entries equal to $b_0$ and $b_1a_2$.

For $a_2 = 0$ the factor $4UW - V^2$ reduces to $-V^2 \in \mathbb{C}[a]$. From the following lemma we deduce (after its proof) that the factor $C_{n-2,k}$ must be squared.

Lemma 19. The polynomial $-V^2$ is a quadratic polynomial in the variables $a_i$, $i = 3, \ldots, n$, with the square of at least one of them present in $-V^2$. For $k < n$ (resp. $k = n$) it contains the monomial $a_k^2(b_0)^2(b_1a_1)^2(n-k)$ (resp. $a_{n-1}^2(b_0)^2(n-1)(b_1a_1)^2$).

Proof. Indeed, if $k < n$, then set $V^* := V|_{a_j=0,j\neq1,k,n}$ and $S^* := S(P, P_*)|_{a_j=0,j\neq1,k,n}$. There are two entries $a_k$ (resp. $a_1$ and $a_n$) in $S^*$, in positions $(1, k + 1)$ and $(2, k + 2)$ (resp. $(1, 2)$, $(2, 3)$, and $(1, n + 1)$, $(2, n + 2)$). The other nonzero entries of $S^*$ are $b_0$ (resp. $b_1a_1$) in positions $(\nu + 2, \nu)$ (resp. $(\nu + 2, \nu + 1)$), $\nu = 1, \ldots, n$. Thus

$$V^* = \det V^{**} + \det V^{***}, \quad \text{where} \quad V^{**} = [S^*]_{1,k+1}|_{a_k=0}, \quad V^{***} = [S^*]_{2,k+2}|_{a_k=0}.$$

The matrices $V^{**}$ and $V^{***}$ are $(n + 1) \times (n + 1)$. Hence

$$\det V^{**} = (-1)^n a_n \det [V^{**}]_{1,n+1}, \quad \det V^{***} = 0$$

(because all entries in the last column of $V^{***}$ equal 0). The matrix $[V^{**}]_{1,n+1}$ is block-diagonal, with diagonal blocks of sizes $k \times k$ (left upper, it is upper-triangular) and $(n - k) \times (n - k)$ (right lower, it is lower-triangular). Their diagonal entries equal respectively $b_0$ and $b_1a_1$. Thus

$$V^* = \det V^{**} = (-1)^n a_n (b_0)^k (b_1a_1)^{n-k}.$$

Hence for $k < n$ the term $-V^2$ contains the monomial $a_n^2(b_0)^2(b_1a_1)^2(n-k)$.

For $k = n$ we set $a_j = 0$, $j \neq 1$, $n - 1$, $n$, $S^\dagger := S(P, P_*)|_{a_j=0,j\neq1,n-1,n}$ and $V^\dagger := V|_{a_j=0,j\neq1,n-1,n}$. Hence

$$V^\dagger = \det V^{\dagger\dagger} + \det V^{\dagger\dagger\dagger}, \quad \text{where} \quad V^{\dagger\dagger} = [S^\dagger]_{1,n+1}|_{a_n=0}, \quad V^{\dagger\dagger\dagger} = [S^\dagger]_{2,n+2}|_{a_n=0}.$$
One has det $V^\dagger = 0$ (all entries in the last column are 0) and $V^{\dagger\dagger}$ has an entry $a_{n-1}$ in position $(1,n)$; no other entry of $V^{\dagger\dagger}$ depends on $a_{n-1}$. Hence det $V^{\dagger\dagger}$ contains the monomial $(-1)^{n+1} a_{n-1} \det [V^{\dagger\dagger}]_{1,n}$. The matrix $[V^{\dagger\dagger}]_{1,n}$ is block-diagonal, with diagonal blocks of sizes $(n - 1) \times (n - 1)$ (upper left, it is upper-triangular, with diagonal entries equal to $b_0$) and $1 \times 1$ (lower right, it equals $b_1 a_1$). Hence $-V^2$ contains the monomial $a_{n-1}^2 (b_0)^2 (b_1 a_1)^2$.

The factor $C_{n-2,k}$ is a linear function in the variables $a_3, \ldots, a_n$, with coefficients depending on $a_1$ and $a_2$. Indeed, set $P_\alpha := b_0 (x - \alpha) (x - \beta)$, $0 \neq \alpha \neq \beta \neq 0$. One can choose $(\alpha, \beta)$ as coordinates in the space $(a_1, a_2)$. The polynomial $P$ is obtained from $P_\alpha$ by rescaling of its coefficients followed by $(n - 2)$-fold integration with constants of integration of the form $\eta_s a_s$, $\eta_s \in \mathbb{Q}^*$, $s = 3, \ldots, n$. Consider the two conditions $P(\alpha, a)/\alpha^{n-k} = 0$ and $P(\beta, a)/\beta^{n-k} = 0$. Each of them is a linear form in the variables $a_3, \ldots, a_n$, with coefficients depending on $a_1$ and $a_2$; the one of $a_k$ equals 1. The projection of the Maxwell stratum in the space of the variables $a^k$ is given by the condition

$$\beta^{n-k} P(\alpha, a) - \alpha^{n-k} P(\beta, a) = 0 \quad (6)$$

Its left-hand side is a linear form in the variables $a_3, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n$, with coefficients depending on $\alpha$ and $\beta$. The presence of the monomial $a_{n-1}^2 (b_0)^2 (b_1 a_1)^2 (2(n-k))$ or $a_{n-1}^2 (b_0)^2 (b_1 a_1)^2 (2(n-k) - 1)$ in $\tilde{D}_{n-2,k}$ (see Lemma [19]) implies that the factor $C_{n-2,k}$ must be squared.

There remains to prove that $s_{n-2,k} = 1$, see Remark [17]. The left-hand side of equation (6) is divisible by $\alpha - \beta$. Represent this expression in the form $(\alpha - \beta) Q(\alpha, \beta, a)$. The polynomial $Q$ depends in fact on $\alpha + \beta = \beta b_1 a_1 / b_0$, $\beta = b_2 a_2 / b_0$ and $a$, hence this is a polynomial in $a$ (denoted by $K(a)$).

Clearly $K$ depends linearly on the variables $a_3, \ldots, a_n$. On the other hand $K$ is quasi-homogeneous. Hence $K$ is irreducible. Indeed, should $K$ be the product of two factors, then one of the two (denoted by $Z$) should not depend on any of the variables $a_3, \ldots, a_n$, i.e. $Z$ should be a polynomial in $a_1$ and $a_2$.

This polynomial should divide the coefficients of all variables $a_3, \ldots, a_n$ in $K$. But for $3 \leq s \leq n$ the coefficient of $a_s$ in $K$ equals $c_s := (\beta^{n-k} \alpha^{n-s} - \alpha^{n-k} \beta^{n-s}) / (\alpha - \beta)$. Hence $Z$ divides $c_s - \beta c_{s-1} = \alpha^{n-s} (\beta - \beta^{n-s})$ for all $s \neq k$, and by symmetry $Z$ divides $\alpha^{n-k} \beta^{n-s-1}$ for all $s \neq k$. Hence $Z = 1$ and the polynomial $C_{n-2,k}$ equals $(\beta^{n-k} P(\alpha, a) - \alpha^{n-k} P(\beta, a)) / (\alpha - \beta)$. Its quasi-homogeneous weight (QHW) is $2n - k - 1$ (notation: QHW($C_{n-2,k}$) = $2n - k - 1$).

Indeed, one has to consider QHW($\alpha$) and QHW($\beta$) to be equal to 1 because $\alpha$ and $\beta$ are roots of $P_\alpha$ and their QHW is the same as the one of the variable $x$.

Obviously QHW($\tilde{D}_{n-2,k}$) = $2$QHW($U$) + QHW($W$) because $\tilde{D}_{n-2,k} = U(4W - V^2)$ and $\tilde{D}_{n-2,k}$ is quasi-homogeneous. As $U = \omega a_2^{n-k}$, one has QHW($U$) = $2(n-k)$. The polynomial $\tilde{D}_{n-2,k}$ contains a monomial $\tilde{\omega} a_2^n$, $\tilde{\omega} \neq 0$ (see Proposition [1]). This monomial is contained also in $W = \tilde{D}_{n-2}|a_k=0$ hence QHW($\tilde{D}_{n-2}$) = QHW($W$) = $2n$. Thus

$$\text{QHW}(\tilde{D}_{n-2,k}) = 2 \text{QHW}(U) + \text{QHW}(\tilde{D}_{n-2}) = 6n - 4k$$

On the other hand one knows already that a priori $\tilde{D}_{n-2,k} = A_{n-2,k} B_{n-2,k}^{s_{n-2,k}} C_{n-2,k}^{2}$, $s_{n-2,k} \in \mathbb{N}$, $A_{n-2,k} = a_2^{n-k}$. Hence

$$s_{n-2,k} \text{QHW}(B_{n-2,k}) = \text{QHW}(\tilde{D}_{n-2,k}) - \text{QHW}(A_{n-2,k}) - 2 \text{QHW}(C_{n-2,k}) = 6n - 4k - 2(n-k) - 2(2n-k-1) = 2$$

and as $B_{n-2,k} = b_1^2 a_1^2 - 4b_0 b_2 a_2$, one has QHW($B_{n-2,k}$) = 2, so $s_{n-2,k} = 1$. \qed
Proof for $k = 1$ and $k = 2$. In order to deal with the cases $k = 1$ and $k = 2$ we need to know the degrees and quasi-homogeneous weights of certain polynomials in the variables $a$:

**Lemma 20.** (1) $\deg_{a_1} D_{n-2} = n$, $\deg_{a_2} D_{n-2} = n$;
(2) $QHW(D_{n-2}) = 2n$, $QHW(\partial D_{n-2}/\partial a_1) = 2n - 1$, $QHW(\partial D_{n-2}/\partial a_2) = 2n - 2$;
(3) $QHW(D_{n-2,1}) = n(3n - 2)$;
(4) $QHW(D_{n-2,2}) = 2n(n - 1)$;
(5) $QHW(B_{n-2,1}) = n(n - 1)$, $QHW(B_{n-2,2}) = n(n - 1)$;
(6) $QHW(C_{n-2,1}) = n(n - 1)$, $QHW(C_{n-2,2}) = (n - 1)/2$.

For $k = 1$ or $2$ one has to find positive integers $u$ and $v$ such that

$$QHW(\tilde{D}_{n-2,k}) = (2 - k)n + uQHW(B_{n-2,k}) + vQHW(C_{n-2,k}),$$

because $A_{n-2,k} = a_n^{2-k}$. For $k = 2$ parts (4), (5) and (6) of the lemma imply that $u = 1$, $v = 2$ is the only possible choice. For $k = 1$ there remain two possibilities - $(u,v) = (1,2)$ or $(u,v) = (2,1)$ - so we need another lemma as well:

**Lemma 21.** For $a_j = 0$, $j \neq 1$, $n - 1$, $n$, the polynomials $\tilde{D}_{n-2}$, $\tilde{D}_{n-2,1}$, $B_{n-2,1}$ and $C_{n-2,1}$ are of the form respectively (with $\Delta_i \neq 0$)

$$\tilde{D}_{n-2} = \Delta_1 a_n^2 + \Delta_2 a_n a_{n-1} + \Delta_3 a_n^2, \quad \tilde{D}_{n-2,1} = \Delta_4 a_n^{2n-1} a_{n-1} + \Delta_5 a_n^{3n-2},$$

$$B_{n-2,1} = \Delta_6 a_n^{n-1} + \Delta_7 a_n^2 \quad \text{and} \quad C_{n-2,1} = \Delta_8 a_n^{n-1}.$$ 

The lemma implies that it is possible to have $(u,v) = (1,2)$, but not $(u,v) = (2,1)$. Indeed, otherwise the product $\tilde{D}_{n-2,1} = A_{n-2,1} B_{n-2,1} C_{n-2,1}$, with $A_{n-2,1} = a_n$, should contain three different monomials whereas it contains only two.

**Proof of Lemma 20** Parts (1) and (2) follow directly from Proposition 1. To prove parts (3) and (4) one has to observe that as the polynomial $\tilde{D}_{n-2}$ contains a monomial $c^* a_n^2$, $c^* \neq 0$, the $(2n - 1) \times (2n - 1)$-Sylvester matrix $S_k^* := S(\tilde{D}_{n-2}, \partial \tilde{D}_{n-2}/\partial a_k, a_k)$, $k = 1$ or $2$, contains this monomial in positions $(j, j + n)$, $j = 1, \ldots, n - 1$ and only there. The matrix $S_k^*$ (resp. $S_k^0$) has entries $c^k a_n$, $c^k \neq 0$ (resp. $c^* \neq 0$) in positions $(\nu + n - 1, \nu)$, $\nu = 1, \ldots, n$. Hence $\tilde{D}_{n-2,k}$ contains a monomial $\pm (c^k a_n)^n (c^k a_n)^{n-1}$ for $k = 1$ and $\pm (c^* a_n^n) (c^* a_n^{n-2})$ for $k = 2$ whose quasi-homogeneous weight is respectively $n(3n - 2)$ and $2n(n - 1)$.

To prove part (5) recall that the $(2n - 1) \times (2n - 1)$-Sylvester matrix $S^0 := S(P_{n-2,k}, P'_{n-2,k})$, $k = 1$ or $2$, has entries of the form $c^* a_n$, $c^* \neq 0$, in positions $(j, j + n)$, $j = 1, \ldots, n - 1$ and only there, and constant nonzero terms in positions $(\nu + n - 1, \nu)$, $\nu = 1, \ldots, n$. Thus $B_{n-2,k}$ contains a monomial $\pm c^* \nu (a_n^n - 1, c^* \neq 0$ and $QHW(B_{n-2,k}) = n(n - 1)$.

For the proof of part (6) we need to recall that the factors $C_{n-2,k}$ are related to polynomials $P$ divisible by $P_k$. When one performs this Euclidean division one obtains a rest of the form $U^\top(a)x + V^\top(a)$, where $U^\top, V^\top \in \mathbb{C}[a]$, $QHW(U^\top) = n - 1, QHW(V^\top) = n$, $U^\top$ (resp. $V^\top$) contains monomials $\omega_1 a_1^{n-1}$ and $\omega_2 a_{n-1}$ (resp. $\omega_3 a_1^{n-2} a_2$ and $\omega_4 a_n$), $\omega_i \neq 0$. (To see that the monomials $\omega_1 a_1^{n-1}$ and $\omega_3 a_1^{n-2} a_2$ are present one has to recall that at each step of the Euclidean division one replaces a term $L x^k$, $L \in \mathbb{C}[a]$, by the sum $L(b_1/b_0)a_1 x^{k-1} - L(b_2/b_0)a_2 x^{k-2}$.)

To obtain the factor $C_{n-2,1}$ one has to eliminate $a_1$ from the system of equations $U^\top(a) = V^\top(a) = 0$, i.e. one has to find the subset in the space of variables $a_k^1$ for which $U^\top$ and $V^\top$ have a common zero when considered as polynomials in $a_1$. The $(2n - 3) \times (2n - 3)$-Sylvester matrix $S(U^\top, V^\top, a_1)$ contains terms $\omega_2 a_{n-1}$ in positions $(j, j + n - 1)$, $j = 1, \ldots, n - 2$, and
terms \( \omega_3 a_2 \) in positions \((\nu + n - 2, \nu)\), \(\nu = 1, \ldots, n - 1\). Hence \( C_{n-2,1} \) contains a monomial \( \pm(\omega_2 a_{n-1})^{n-2}(\omega_3 a_2)^{n-1} \), of quasi-homogeneous weight \( n(n-1) \).

The proof of the second statement of part (6) is performed separately for the cases of even and odd \( n \). If \( n \) is even, then \( U^\dagger \) (resp. \( V^\dagger \)) contains monomials \( \Omega_1 a_1 a_2^{(n-1)/2} \) and \( \Omega_{2a-1} \) (resp. \( \Omega_3 a_2^{(n-1)/2} \) and \( \Omega_{2a-1} \), \( \Omega_i \neq 0 \). The \((n-1) \times (n-1)\)-Sylvester matrix \( S(U^\dagger, V^\dagger, a_2) \) contains terms \( \Omega_{4a-1} \) in positions \((j, j+n/2)\), \(j = 1, \ldots, n/2-1\), and \( \Omega_1 a_1 \) in positions \((\nu+n/2-1, \nu)\), \(\nu = 1, \ldots, n/2\). Hence \( C_{n-2,2} \) contains a monomial \( \pm(\Omega_{4a-1})^{n/2-1}(\Omega_1 a_1)^{n/2} \), of quasi-homogeneous weight \( n(n-1)/2 \).

When \( n \) is odd, then \( U^\dagger \) (resp. \( V^\dagger \)) contains monomials \( \tilde{\Omega}_1 a_2^{(n-1)/2} \) and \( \tilde{\Omega}_{2a-1} \) (resp. \( \tilde{\Omega}_3 a_2^{(n-1)/2} \) and \( \tilde{\Omega}_{2a-1} \), \( \tilde{\Omega}_i \neq 0 \). The \((n-1) \times (n-1)\)-Sylvester matrix \( S(U^\dagger, V^\dagger, a_2) \) contains terms \( \tilde{\Omega}_{2a-1} \) in positions \((j, j+(n-1)/2)\), \(j = 1, \ldots, (n-1)/2\), and \( \tilde{\Omega}_3 a_1 \) in positions \((\nu+(n-1)/2, \nu)\), \(\nu = 1, \ldots, (n-1)/2\). Thus \( C_{n-2,2} \) contains a monomial \( \pm(\tilde{\Omega}_{2a-1} \tilde{\Omega}_3 a_1)^{(n-1)/2} \), of quasi-homogeneous weight \( n(n-1)/2 \).

\[\square\]

**Proof of Lemma 27** One can develop \( \det S(P, P_s) \) w.r.t. the last column in which there is a single nonzero entry \((a_n, \text{ in position } (2, n+2))\). Hence \( D_{n-2} = (-1)^n a_n \det S^2 \), where \( S^2 := [S(P, P_s)]_{2,n+2} \). The last column of \( S^2 \) contains only two nonzero entries \((a_n \text{ in position } (1, n+1) \text{ and } b_1 a_1 \text{ in position } (n, n+1)) \), therefore

\[
\det S^2 = (-1)^n a_n \det S^{21} + b_1 a_1 \det S^{22} , \quad \text{where } S^{21} := [S^2]_{1,n+1} \text{, } S^{22} := [S^2]_{n+1,n+1} .
\]

The matrix \( S^{21} \) is upper-triangular, with diagonal entries equal to \( b_0 \), so \( \det S^{21} = b_0^n \), while \( S^{22} \) contains only two nonzero entries in its last column \((a_{n-1} \text{ in position } (1, n) \text{ and } b_1 a_1 \text{ in position } (n, n)) \). Hence

\[
\det S^{22} = (-1)^{n+1} a_{n-1} \det S^{23} + b_1 a_1 \det S^{24} , \quad \text{where } S^{23} := [S^{22}]_{1,n} \text{, } S^{24} := [S^{22}]_{n,n} .
\]

The matrix \( S^{23} \) is upper-triangular, with diagonal entries equal to \( b_0 \), so \( \det S^{23} = b_0^{n-1} \). The matrix \( S^{24} \) becomes lower-triangular after subtracting its second row multiplied by \( 1/b_1 \) from the first one, with diagonal entries \( 1 - b_0/b_1, b_1 a_1, \ldots, b_1 a_1 \), from which the form of \( D_{n-2} \) follows.

Hence the \((2n-1) \times (2n-1)\)-Sylvester matrix \( S(D_{n-2}, \partial D_{n-2} / \partial a_1, a_1) \) has only the following nonzero entries, in the following positions:

\[
\begin{align*}
\Delta_1 a_n & \quad (j, j) , \\
\Delta_2 a_n a_{n-1} & \quad (j, j+n-1) , \\
\Delta_3 a_n^2 & \quad (j, j+n) , \\
\end{align*}
\]

\[
\begin{align*}
\frac{n\Delta_1 a_n}{j} & \quad (\nu + n - 1, \nu) , \\
\Delta_2 a_n a_{n-1} & \quad (\nu + n - 1, \nu + n - 1) ,
\end{align*}
\]

One can subtract the \((j+n-1)\)st row from the \( j \)th one \((j = 1, \ldots, n-1) \) to make disappear the terms \( \Delta_2 a_n a_{n-1} \) in positions \((j,j+n-1)\). This does not change the determinant; the entries \( \Delta_1 a_n \) in positions \((j, j)\) become \((1-n)\Delta_1 a_n\). The form of \( D_{n-2,1} \) follows now from Lemma 8.

For \( a_j = 0 \), \( j \neq 1, n-1, n \), the polynomial \( P_{n-2,1} \) is of the form \( \alpha_1 x^n + \alpha_2 a_{n-1} x + \alpha_3 a_n \), \( \alpha_i \neq 0 \), so the \((2n-1) \times (2n-1)\)-Sylvester matrix \( S(P_{n-2,1}, P'_{n-2,1}) \) has nonzero entries only
The new matrix satisfies the conditions of Lemma 8 with \( \alpha_1 \) at \((j,j)\), \( \alpha_2a_{n-1} \) at \((j,j+n-1)\), \( \alpha_3a_n \) at \((j,j+n)\), \( j = 1, \ldots, n-1 \),

\[ n\alpha_1 \text{ at } (\nu,\nu), \quad \alpha_2a_{n-1} \text{ at } (\nu,\nu+n-1), \quad \nu = 1, \ldots, n. \]

By analogy with the reasoning about \( \tilde{D}_{n-2,1} \) one finds that \( B_{n-2,1} = \Delta_0a_n^{n-1} + \Delta_7a_n^{n-1} \).

To justify the form of \( C_{n-2,1} \) it suffices to observe that for \( a_j = 0, \ j \neq 1, \ n-1, \ n, \) one has (see the definition of \( U^\dagger \) and \( V^\dagger \) in the proof of Lemma 20) \( U^\dagger = \alpha_4a_{n-1} + \alpha_5a_n^{n-1}, \ V^\dagger = \alpha_6a_n, \ \alpha_i \neq 0, \) so \( \deg a_i \), \( U^\dagger = n-1 \) and \( \deg a_i \), \( V^\dagger = 0. \) When eliminating \( a_1 \) from the system of equalities \( U^\dagger = V^\dagger = 0 \) one obtains \( \text{Res}(U^\dagger, V^\dagger, a_1) = 0, \) i.e. \( (\alpha_6a_n)^{n-1} = 0. \) \( \square \)

6 The proof of \( s_{m,1} = 1 \)

In the present section we prove the following

**Proposition 22.** With the notation of Remark 17 one has \( s_{m,1} = 1. \)

The proof of the proposition makes use of the following lemma:

**Lemma 23.** Set \( a_j = 0 \) for \( j \neq 1, \ell \) and \( n \), where \( n-m+1 \leq \ell \leq n-1 \). Then \( S(P, P_*) \) is of the form \( \Omega_1a_n^{n-m-1}a_1^n + \Omega_2a_n^{n-m-1}a_1\ell a_1^{n-\ell} + \Omega_3a_n^{n-m} \).

**Proof of Proposition 22.** Lemma 23 with \( \ell = n-1 \) implies that the matrix \( S(\tilde{D}_m, \partial \tilde{D}_m/\partial a_1, a_1) \) has only the following nonzero entries, in the following positions:

\[
\begin{align*}
\Omega_1a_n^{n-m-1} & \quad (j,j), \\
\Omega_3a_n^{n-m} & \quad (j,j+n), \\
n\Omega_1a_n^{n-m-1} & \quad (\nu+n-1, \nu),
\end{align*}
\]

\[
\begin{align*}
\Omega_2a_n^{n-m-1}a_n^{-1} & \quad (j,j+n-1), \\
\Omega_2a_n^{n-m-1}a_n^{-1} & \quad (\nu+n-1, \nu+n-1),
\end{align*}
\]

\[
\begin{align*}
\nu = 1, \ldots, n.
\end{align*}
\]

Subtract for \( j = 1, \ldots, n-1 \) its \((j+n-1)\)st row from the \( j \)th one. This preserves its determinant and leaves only the following nonzero entries, in the following positions:

\[
\begin{align*}
(1-n)\Omega_1a_n^{n-m-1} & \quad (j,j), \\
\Omega_3a_n^{n-m} & \quad (j,j+n), \\
n\Omega_1a_n^{n-m-1} & \quad (\nu+n-1, \nu),
\end{align*}
\]

\[
\begin{align*}
\Omega_2a_n^{n-m-1}a_n^{-1} & \quad (\nu+n-1, \nu+n-1),
\end{align*}
\]

\[
\begin{align*}
\nu = 1, \ldots, n.
\end{align*}
\]

The new matrix satisfies the conditions of Lemma 8 with \( p = 2n-1, \ s = n. \) Hence its determinant is of the form

\[
a_n^{(n-m-1)(2n-1)}(\Omega_4a_n^{n-m-1} + \Omega_5a_n^{n-m-1}), \tag{7}
\]

where \( \Omega_4 = ((1-n)\Omega_1)^{n-1}\Omega_2 \) and \( \Omega_5 = \pm\Omega_3^{n-1}\Omega_4^n. \) The polynomial \( \text{Res}(P_{m,1}, P'_{m,1}) \) contains monomials \( \alpha a_n^{n-m-1} \) and \( \beta a_n^{n-m-1}, \ \alpha \neq 0 \neq \beta; \) this can be proved by complete analogy with the
The matrix \( S \) so its determinant equals \( b \) so its determinant equals \( 1 \) multiplied by \( 1 \)

The matrix \( S \) shifts of this one by one position to the right. Developing of \( \det S \) there will be a single nonzero entry \( a \). One can develop the determinant \( n \times n \) of \( S \) by \( n \times n \) containing the entries \( 1, b \).

Recall that we have shown already (see Remark 17) that for each fixed the polynomials \( \tilde{D}_{m,k} \) for \( 0 \)

\[ \leq \]

If formula (1) is true for \( n \), Statement 25.

Statement 24. If formula (1) is true for \( n = n_0 \), \( k = k_0 \), then it is true for \( n = n_0 + 1 \), \( k = k_0 + 1 \).

Statement 25. If formula (1) is true for \( n = n_0 \), \( m = n_0 - 2 \), \( k = 1 \), then it is true for \( n = n_0 \), \( 2 \leq m < n_0 - 2 \), \( k = 1 \).

Proof of Statement 24. Recall that we have shown already (see Remark 17) that for each \( n \) fixed the polynomials \( \tilde{D}_{m,k} \) (2 \leq m \leq n - 2, 1 \leq k \leq n) are of the form \( A_{m,k}B_{m,k}C_{\tau_{m,k}}^\tau \), \( s_{m,k}, \tau_{m,k} \in \mathbb{N} \). Suppose that for \( 4 \leq n \leq n_0 \) one has \( s_{m,k} = 1 \), \( \tau_{m,k} = 2 \). (Using MAPLE one can obtain this result for \( n_0 = 4 \).) Set \( P(a,x) := x^{n_0} + a_1 x^{n_0-1} + \cdots + a_{n_0} a := (a_1, \ldots, a_{n_0}) \) and consider the polynomials \( F := ux^{n_0+1} + P \) and \( F_+ := b_{-1}ux^{n_0-m+1} + P_+ \), \( u \in (\mathbb{C},0) \), \( 0 \neq b_{-1} \neq b_j \) for \( 0 \leq j \leq n_0 - m \). They are deformations respectively of \( P \) and \( P_+ \). Our reasoning uses the following

### 7 Completion of the proof of Theorem 4

The matrix \( S(P, P_+) \) has only the following nonzero entries, in the following positions:

\[
\begin{array}{cccc}
1 & (j,j) & a_1 & (j,j+1) & a_\ell & (j,j+\ell) \\
\multicolumn{5}{c}{a_n} & (j,j+n) & a_\ell & (j,j+\ell) \\
b_0 & (\nu+n-m,\nu) & b_1a_1 & (\nu+n-m,\nu+1) & \multicolumn{3}{c}{n-m}, \nu = 1,\ldots,n.
\end{array}
\]

One can develop the determinant \( n - m - 1 \) times w.r.t. the last column in which each time there will be a single nonzero entry \( a_n \). Thus \( \det S(P, P_+) = \pm a_n^{n-m-1} \det S^\dagger \), where the first row of \( S^\dagger \) contains the entries \( 1, a_1, a_\ell \) and \( a_n \) in positions respectively \( (1,1), (1,2), (1,\ell+1) \) and \( (1,n+1) \); its second row is of the form \( (b_0, b_1a_1, 0, \ldots, 0) \) and the next rows are the consecutive shifts of this one by one position to the right. Developing of \( \det S^\dagger \) w.r.t. the last column yields

\[
\det S^\dagger = (-1)^n a_n \det[S^\dagger]_{1,n+1} + b_1a_1 \det[S^\dagger]_{n+1,n+1}.
\]

The matrix \( [S^\dagger]_{1,n+1} \) is upper-triangular, with diagonal entries equal to \( b_0 \) (hence \( \det[S^\dagger]_{1,n+1} = b_0^n \)). The determinant of the matrix \( S^\dagger_{\ell+1} := [S^\dagger]_{n+1,n+1} \) can be developed \( n - \ell - 1 \) times w.r.t. its last column, where each time it has a single nonzero entry \( b_1a_1 \) in its right lower corner:

\[
\det S^\dagger_{\ell+1} = (b_1a_1)^{n-\ell-1} \det S^{\dagger\ast},
\]

where \( S^{\dagger\ast} \) is \( (\ell+1) \times (\ell+1) \); it is obtained by deleting the last \( n - \ell - 1 \) rows and columns of \( S^\dagger \). The determinant \( \det S^{\dagger\ast} \) can be developed w.r.t. its last column:

\[
\det S^{\dagger\ast} = (-1)^\ell a_\ell \det[S^{\dagger\ast}]_{1,\ell+1} + b_1a_1 \det[S^{\dagger\ast}]_{\ell+1,\ell+1}.
\]

The matrix \( [S^{\dagger\ast}]_{1,\ell+1} \) (resp. \( [S^{\dagger\ast}]_{\ell+1,\ell+1} \)) is upper-triangular, with diagonal entries equal to \( b_0 \), so its determinant equals \( b_0^\ell \) (resp. becomes lower-triangular (after subtracting its second row multiplied by \( 1/b_1 \) from its first row), with diagonal entries equal to \( 1 - b_0/b_1, b_1a_1, \ldots, b_1a_1 \), so its determinant equals \( (1 - b_0/b_1)(b_1a_1)^{\ell-1} \). This implies the lemma.

\( \square \)
Observation 26. One has

\[ F = u(x^{n_0+1} + x^{n_0}/u + \sum_{j=0}^{n_0-1}(a_{n-j}/u)x^j) , \]

\[ F_* = u(b_{-1}x^{n_0-m+1} + b_0x^{n_0-m}/u + \sum_{j=0}^{n_0-1}(b_{-j}a_{n-j}/u)x^j) , \]

so after the change of parameters \( \tilde{a}_1 = 1/u, \tilde{a}_s = a_{s-1}/u, s = 2, \ldots, n_0 \) (which is well-defined for \( u \neq 0 \)) and the shifting by 1 of the indices of the constants \( b_j \), the polynomials \( F \) and \( F_* \) (up to multiplication by \( 1/u \)) become \( P \) and \( P_* \) defined for \( n_0 + 1 \) instead of \( n_0 \).

Lemma 27. The zero set of \( \text{Res}(F, F_*) \) for \( u \neq 0 \) is defined by an equation of the form \( \tilde{D}_m + uH/d = 0 \), where \( H \in \mathbb{C}[u, a] \) and \( d \neq 0 \).

Proof. Consider the \((2n_0 - m + 2) \times (2n_0 - m + 2)\)-Sylvester matrix \( \tilde{S} := S(F, F_*) \). Permute the rows of \( \tilde{S} \) as follows: place the \((n_0 - m + 2)\)nd row in second position while shifting the ones with indices \( 2, \ldots, n_0 - m + 1 \) by one position backward. This preserves up to a sign the determinant and yields a matrix \( T \) which we decompose in four blocks the diagonal ones being of size \( 2 \times 2 \) (upper left, denoted by \( T^* \)) and \((2n_0 - m) \times (2n_0 - m) \) (lower right, denoted by \( T^{**} \)); the left lower block is denoted by \( T^0 \) and the right upper by \( T^1 \). An easy check shows that

\[ T^* = \begin{pmatrix} u & 1 \\ b_{-1} & b_0 \end{pmatrix} , \quad T^{**}|_{u=0} = S(P, P_*) \]

and that the only nonzero entries of the left lower block \( T^0 \) are \( u \) and \( b_{-1}u \), in positions \((3, 2)\) and \((n_0 - m + 3, 2)\) respectively.

Divide the first column of \( T \) by \( u \) (we denote the thus obtained matrix by \( T^\dagger \)). This does not change the zero set of \( \det T \) for \( u \neq 0 \). For \( u = 0 \) the matrix \( T^\dagger \) is block-upper-triangular, with diagonal blocks equal to \( \begin{pmatrix} 1 & 1 \\ b_{-1} & b_0 \end{pmatrix} \) and \( S(P, P_*) \). Hence \( \det T^\dagger = d \det S(P, P_*) + uH(u, a) \),

\[ d := \det T^*|_{u=0} = b_0 - b_{-1} \neq 0, H \in \mathbb{C}[u, a] \] . Thus the zero set of \( \text{Res}(F, F_*) \) for \( u \neq 0 \) sufficiently small is defined by the equation \( \tilde{D}_m + uH/d = 0 \). \( \square \)

For \( u \neq 0 \) (resp. \( u = 0 \)) the quantity \( \det T^\dagger \) is a degree \( n_0 - m + 1 \) (resp. \( n_0 - m \)) polynomial in \( a_k \) for \( k = n_0 - m + 1, \ldots, n_0 \), and a degree \( n_0 + 1 \) (resp. \( n_0 \)) polynomial in \( a_k \) for \( k = 1, \ldots, n_0 - m \), see Proposition \( \square \) Hence for each \( k = 1, \ldots, n_0 \) there is one simple root \(-1/w_k(u, a)\) of \( \text{Res}(F, F_*) \) that tends to infinity as \( u \to 0 \). Thus one can set \( \text{Res}(F, F_*) = (1 + w_k(u, a)a_k)\tilde{D}_m^* \)

where \( \tilde{D}_m^*|_{u=0} = \tilde{D}_m \) and \( \deg \tilde{D}_m^* = n_0 - m \) (resp. \( n_0 \)) for \( k = n_0 - m + 1, \ldots, n_0 \) (resp. \( k = 1, \ldots, n_0 - m \)).

Lemma 28. Set \( E_m := \text{Res}(F, F_*) \) and \( \tilde{D}_m^* := \text{Res}(E_m, \partial E_m/\partial a_k, a_k) \). Then for \( u \neq 0 \) one has \( \tilde{D}_m^* = \Omega^{\phi} (a_{n_0}^{2(n_0-m-k)} \tilde{D}_m + uH_m(k(u, a))) \), where \( H_m \in \mathbb{C}[u, a] \).

Remark 29. One can set \( u := a_{n_0}^{2(n_0-m-k)}v \) to obtain the equality

\[ \tilde{D}_m^* = \Omega^{\phi} a_{n_0}^{2(n_0-m-k)}(\tilde{D}_m + vH_m(a_{n_0}^{2(n_0-m-k)}v, a)) \].

Now in a neighbourhood of each \( a_{n_0} \neq 0 \) fixed the zero set of \( \tilde{D}_m^* \) is defined by the equation \( \tilde{D}_m^* + vH_m(a_{n_0}^{2(n_0-m-k)}v, a) = 0 \), i.e. by deforming the equation \( \tilde{D}_m = 0 \).
Proof of Lemma 28. Indeed, Proposition 1 implies that \( \tilde{D}_m \) contains a monomial \( \Omega^b a^{n_0-m-k}_{a_n} a^{n_0-m-k}_{n_0} \), \( 1 \leq k \leq n_0-m \) (resp. \( \Omega^b a^{n_0-m-k}_{a_n} a^{n_0-m-k}_{n_0} \)), and this is the only monomial containing \( a^k_{a_n} \) (resp. \( a^k_{n_0-m} \)). Similarly, \( E_m \) contains a monomial \( I := u^{k+1} \Omega^a a^{n_0-m-1}_{a_n} a^{n_0-m-k}_{n_0} \), \( 1 \leq k \leq n_0-m \) (resp. \( J := u^{k+1} \Omega^a a^{n_0-m-1}_{a_n} a^{n_0-m-k}_{n_0} \)), and this is the only monomial containing \( a^k_{n_0+1} \) (resp. \( a^k_{n_0-m} \)). (The monomial \( I \) is obtained as follows: one subtracts for \( \nu = 1, \ldots, n_0-m+1 \), \( (\nu + n_0 - m + 1) \) row multiplied by \( 1/b_k \) from the \( \nu \)th one to make disappear the terms \( a_k \) in the first \( n_0-m+1 \) rows. The monomial \( J \) is the product of the terms \( b_k a_k \) in the last \( n_0+1 \) rows, the terms \( 1/(1-b_k) \) in the first \( k+1 \) rows and the terms \( a_{n_0} \) in the next \( n_0-m-k \) rows. The monomial \( J \) is obtained in a similar way. One has to assume that \( \text{QHW}(u) = -1 \).) Knowing that \( \text{deg}_{a_k} E_m = n_0+1 \) (resp. \( \text{deg}_{a_k} E_m = n_0-m+1 \)) for \( u \neq 0 \) and that \( \text{deg}_{a_k} \tilde{D}_m = n_0 \) (resp. \( \text{deg}_{a_k} \tilde{D}_m = n_0-m \)) one concludes that

\[
 E_m = u^{k+1} \Omega^a a^{n_0-m-1}_{a_n} a^{n_0-m-k}_{n_0} + \Omega^b a^{n_0-m-k}_{n_0} + E^*(u,a),
\]

(resp. \( E_m = u^{k+1} \Omega^a a^{n_0-m-1}_{a_n} a^{n_0-m-k}_{n_0} + \Omega^b a^{n_0-m-k}_{n_0} + E^**(u,a) \)),

where \( E^*, E^** \in \mathbb{C}[u,a] \), \( \text{deg}_{a_k} E^* \leq 0 \), \( \text{deg}_{a_k} E^** \leq 0-m \). The Sylvester matrix \( S(E_m, \partial E_m/\partial a_k, a_k) \) is \( (2n_0 + 1) \times (2n_0 + 1) \) (resp. \( (2n_0 - 2m + 1) \times (2n_0 - 2m + 1) \)). We permute its rows by placing the \( (n_0+1) \)th (resp. \( (n_0-m+1) \)th) row in second position while shifting by one position backward the second, third, \ldots, \( n_0 \)th (resp. \( (n_0-m) \)th) rows. The new matrix \( T^g \) can be block-decomposed, with diagonal blocks \( T^{ul} (2 \times 2, \text{upper left}) \) and \( T^{lr} \); the other two blocks are denoted by \( T^{ur} \) and \( T^{le} \). Hence

\[
 T^{ul} = \begin{pmatrix}
 u^{k+1} a^{n_0-m-k}_{n_0} & \Omega^b a^{n_0-m-k}_{n_0} + u X^1(u,a) \\
 (n_0+1) u^{k+1} a^{n_0-m-k}_{n_0} & (n_0+1) \Omega^b a^{n_0-m-k}_{n_0} + u X^2(u,a)
\end{pmatrix},
\]

(resp. \( T^{ul} = \begin{pmatrix}
 u^{k+1} a^{n_0-m-k}_{n_0} & \Omega^b a^{n_0-m-k}_{n_0} + u X^3(u,a) \\
 (n_0-m+1) u^{k+1} a^{n_0-m-k}_{n_0} & (n_0-m+1) \Omega^b a^{n_0-m-k}_{n_0} + u X^4(u,a)
\end{pmatrix} \)),

\( X^i \in \mathbb{C}[u,a] \). One has \( T^{lr}|_{u=0} = S(\tilde{D}_m, \partial \tilde{D}_m/\partial a_k, a_k) \). The block \( T^{le} \) has just two nonzero entries, in its second column, and \( T^{dl}|_{u=0} = 0 \). The first of these entries is in position \( (3,2) \) and equals \( u^{k+1} \Omega^b a^{n_0-m-k}_{n_0} \) (resp. \( u^{k+1} \Omega^b a^{n_0-m-k}_{n_0} \)). The second of them is in position \( (n_0+2,2) \) (resp. \( (n_0-m+2,2) \)) and equals \( n_0+1 u^{k+1} \Omega^b a^{n_0-m-k}_{n_0} \) (resp. \( (n_0-m+1) u^{k+1} \Omega^b a^{n_0-m-k}_{n_0} \)).

Thus for \( u \neq 0 \neq a_n \), the zero set of \( \tilde{D}^*_{m,k} \) is the one of \( \tilde{D}_{m,k} \). For \( u \neq 0 \) small enough this set does not change if one divides the first column of the matrix \( T^b \) by \( u^{k+1} \). We denote the new matrix by \( T^{bs} \). Obviously \( \det T^{bs} = -\Omega^b a^{n_0-m-k}_{n_0} (\tilde{D}_{m,k} + uH_{m,k}) \) (resp. \( \det T^{bs} = -\Omega^b a^{n_0-m-k}_{n_0} (\tilde{D}_{m,k} + uH_{m,k}) \)) for a suitably defined polynomial \( H_{m,k} \) which proves the lemma.

Further to distinguish between the sets \( \Theta \) and \( \tilde{M} \) (see Definition 3) defined for the polynomials \( P \) or \( F \) we write \( \Theta_P \) and \( M_P \) or \( \Theta_P \) and \( M_P \). Consider a point \( A \in \Theta_P \) and a germ \( G \) of an affine space of dimension 2 which intersects \( \Theta_P \) transversally at \( A \). Hence there exists a compact neighbourhood \( \mathcal{N} \) of \( A \) in the space \( \mathcal{S} \) such that the parallel translates of \( G \) which intersect \( \Theta_P \) at points of \( \mathcal{N} \), intersect \( \Theta_P \) transversally at these points. We assume that the value of \( a_{n_0} \) remains \( \geq \rho \) in \( \mathcal{N} \) for some \( \rho > 0 \). The restrictions of \( \tilde{D}_{m,k} \) to each of these translates are
smooth analytic functions each of which has one simple zero at its intersection point with $\Theta_F$; this follows from the factor $B_{m,k}$ participating in power 1 in formula (11) for $n = n_0$. Hence for all $u \in \mathbb{C}$ with $0 < |u| \ll \rho$ the restriction of $\tilde{D}_{m,k}^*$ to these translates are smooth analytic functions having simple zeros at the intersection points of the translates with $\Theta_F$.

But this means that the power of the factor $B_{m,k}$ in formula (11) applied to the polynomial $F$ is equal to 1 on the intersection of $\Theta_F$ with some open ball of dimension $n_0 + 1$ centered at $(0, A)$ in the space of the variables $(u, a)$. Hence this power equals 1 on some Zariski open dense subset of $\tilde{\Theta}$.

Thus the equality $s_{m,k} = 1$ is justified for $n = n_0 + 1$, $2 \leq k \leq n_0 + 1$ (because it is the coefficient of $x^{n_0-k}$, not of $x^{n_0+1-k}$ of $F$, that equals $a_k$).

Now we adapt the above reasoning to the situation, where instead of a point $A \in \Theta_F$ one considers a point $A \in \tilde{M}_P$. Each of the translates of $G$ intersects $\tilde{M}_P$ transversally, at just one point. The restriction of $\tilde{D}_{m,k}$ to the translate is a smooth analytic function having a double zero, so a priori the restriction of $\tilde{D}_{m,k}^*$ to it has either one double or two simple zeros. (Under an analytic deformation a double zero either remains such or splits into two simple zeros.) However two simple zeros is impossible because these zeros would be two points of $\tilde{M}_P$ whereas the translate contains just one point. Thus the power 2 of the factor $C_{m,k}$ is justified for some Zariski open dense subset of $\tilde{M}_F$. Once again, this is sufficient to claim that formula (11) is valid for $n = n_0 + 1$ and for $2 \leq k \leq n_0 + 1$.

Proof of Statement 25: Recall that by Remark 17 we have to show that for $n = n_0$ one has $s_{m,k} = 1$, $r_{m,k} = 2$. The first of these equalities was proved in Section 6 (see Proposition 22), so there remains to prove the second one.

As in the proof of Statement 24 we set $P(a, x) := x^{n_0} + a_1 x^{n_0-1} + \cdots + a_{n_0}$, $a := (a_1, \ldots, a_{n_0})$. We define the polynomial $P_\ast := x^2 + b_1 a_1 x + b_2 a_2$ to correspond to the case $m = n_0 - 2$ (i.e. $b_k \neq 0$, $1 \leq k \leq 2$). For $m = n_0 - 2$ Theorem 4 is proved in Section 5, so we assume that $m < n_0 - 2$ and we set $G := x^{n_0-m-2} P_\ast + u(b_3 a_3 x^{n_0-m-3} + \cdots + b_{n_0-m} a_{n_0-m})$, where $u \in (\mathbb{C}, 0)$ and for $i, j \geq 3$, $i \neq j$, one has $0 \neq b_i \neq b_j \neq 0$. Denote by $G^2$ the $(2n_0 - m) \times (2n_0 - m)$-matrix $S(P, G)$.

**Lemma 30.** One has $\det G^2|_{u=0} = a_{n_0}^{n_0-m-2} \det S(P, P_\ast) = a_{n_0}^{n_0-m-2} \tilde{D}_2$. Hence $G := \det G^2 = a_{n_0}^{n_0-m-2} \tilde{D}_2 + u H^2(u, a)$, $H^2 \in \mathbb{C}[u, a]$.

**Proof.** All nonzero entries of the matrix $G^2$ in the intersection of its last $n_0 - m - 2$ columns and rows are 0 for $u = 0$. One can develop $n_0 - m - 2$ times $\det G^2|_{u=0}$ w.r.t. its last column; each time there is a single nonzero entry in it which equals $a_{n_0}$. The matrix obtained from $G^2|_{u=0}$ by deleting its last $n_0 - m - 2$ columns and the rows with indices $m + 2$, ..., $n_0 - 1$ is precisely $S(P, P_\ast)$.

One can observe that $\det G^2$ and $\det G^2|_{u=0}$ are both degree $n_0$ polynomials in $a_1$. Assume that $a_{n_0}$ belongs to a closed disk on which one has $|a_{n_0}| \geq \rho^2 > 0$. Suppose that $|u| \ll \rho^2$, so one can consider the quantity $\tilde{D}_2 + (u/a_{n_0}^{n_0-m-2}) H^2(u, a)$ as a deformation of $\tilde{D}_2$. To this end we set $u := a_{n_0}^{n_0-m-2} v$, $v \in (\mathbb{C}, 0)$, see Remark 29. Now to prove Statement 25 one has just to repeat the reasoning from the last paragraph of the proof of Statement 24.

□
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