Abstract: Let $B_H(t), t \geq [0,T], T \in (0,\infty)$ be the standard Multifractional Brownian Motion (mBm), in this contribution we are concerned with the exact asymptotics of

$$P \left\{ \sup_{t \in [0,T]} B_H(t) > u \right\}$$

as $u \to \infty$. Mainly depended on the structures of $H(t)$, the results under several important cases are investigated.

Key Words: Multifractional Brownian motion; Supremum; Exact asymptotics; Pickands constants.

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

Multifractional Brownian motion generalize the fractional Brownian motion (fBm) of exponent $H$ to the case where $H$ is no longer a constant but a function of the time index of the process. It was introduced to overcome certain limitations of the classical fBm. Contrarily to fBm, the almost sure Hölder exponent of mBm is allowed to vary along the trajectory, a useful feature when one needs to model processes whose regularity evolves in time, such as internet traffic or images. The tail behaviors of the supremum of the fBm over certain time interval have been investigated in the literature, such as [1–4]. But there are rare results about the extreme of mBm. In this paper we try to extend the exact tail asymptotics to mBm case, i.e. we consider for

$$P \left\{ \sup_{t \in [T_1,T_2]} B_H(t) > u \right\}, u \to \infty.$$  (1)

Through this paper, $H(t) : [0,\infty) \to [H,\overline{H}] \subset (0,1)$, the index function of mBm, is a positive Hölder function of exponent $\lambda > 0$, i.e.

$$|H(t) - H(s)| \leq C |t - s|^{\lambda}, t, s \in [0,T],$$  (2)

where $C$ is a positive constant and satisfies

$$0 < \forall t \geq 0 H(t) \leq \min(1, \lambda).$$  (3)

First we give the definition of standard mBm $B_H(t), t \geq 0$ as in [5].

Definition 1.1. Let $Y_H(t), t \geq 0$ be a mBm and $H(t)$ satisfy (2) and (3). Then there exists a unique continuous positive function $C(H(t)), t \geq 0$ such that the process $B_H(t), t \geq 0$ defined by $B_H(t) = \frac{Y(t)}{C(H(t))}$ is continuous and verifies the following property

$$\text{Var} \left( \frac{B_H(t+h) - B_H(t)}{h^{H(t)}} \right) \to 1, \quad h \to 0.$$  

The process $B_H(t), t \geq 0$ is called Standard Multifractional Brownian Motion.

Then by [6] and [7], we have the following representation of the standard mBm.
Lemma 2.1. We introduce a lemma which is an important property of the mBm and some notation before we state our main results.

\[ B_{H}(t) = \frac{1}{C(H(t))} \int_{\mathbb{R}} e^{it\xi} - 1 \frac{dB(\xi)}{\xi^{H(t)+\frac{1}{2}}} \]

where \( B \) denotes standard Brownian motion and

\[ C(x) = \left( \frac{\pi}{x\Gamma(2x)\sin(\pi x)} \right)^{1/2} \]

2. MAIN RESULTS

We introduce a lemma which is an important property of the mBm and some notation before we state our main results.

Lemma 2.1. By [7], the explicit expressions for the autocovariance of \( B_{H}(t), t \geq 0 \) is

\[ \text{Cov}(B_{H}(t), B_{H}(s)) = \mathbb{E} \{ B_{H}(t)B_{H}(s) \} = D(H(s), H(t)) \left( s^{H(s)+H(t)} + t^{H(s)+H(t)} - |t-s|^{H(s)+H(t)} \right), \]

where

\[ D(x, y) = \frac{\sqrt{\Gamma(2x+1)\Gamma(2y+1)\sin(\pi x)\sin(\pi y)}}{2\Gamma(x+y+1)\sin(\frac{\pi(x+y)}{2})}. \]

Further by \( D(H(t), H(t)) = \frac{1}{2} \)

\[ \sigma_{H}^{2}(t) := \mathbb{E} \{ B_{H}^{2}(t) \} = t^{2H(t)}, \; t \geq 0. \]

By [8], the correlation function of \( B_{H}(t) \) satisfies for any \( t > 0 \)

\[ r_{H}(t, t+h) = 1 - \frac{1}{2} t^{-2H(t)}|h|^{2H(t)} + o(|h|^{2H(t)}), \; h \to 0. \]

Next we need to introduce some notation, starting with the well-known Pickands constant \( \mathcal{H}_{a} \) defined by

\[ \mathcal{H}_{a} = \lim_{S \to \infty} \frac{1}{S} \mathcal{H}_{a}[0, S], \quad \text{with} \quad \mathcal{H}_{a}[0, S] = \mathbb{E} \left\{ \sup_{t \in [0, S]} e^{\sqrt{2}B_{a}(t) - |t|^a} \right\} \in (0, \infty), \]

where \( S > 0 \) is a constant and \( B_{a}(t), t \in \mathbb{R} \) is a standard fractional Brownian motion (fBm) with Hurst index \( a/2 \in (0, 1] \). Further, define for \( a > 0 \)

\[ \mathcal{P}_{a} = \lim_{S \to \infty} \mathcal{P}_{a}[0, S], \quad \tilde{\mathcal{P}}_{a} = \lim_{S \to \infty} \mathcal{P}_{a}[-S, S], \]

with

\[ \mathcal{P}_{a}[0, S] = \mathbb{E} \left\{ \sup_{t \in [0, S]} e^{\sqrt{2}B_{a}(t) - |t|^a} \right\}. \]

These constants defined above play a significant role in the following theorems, see [9] for various properties of these constants and compare with, e.g., [10–23].

Now we return to our principal problem deriving below the exact asymptotic behaviour of (1).

Theorem 2.2. If \( \sigma_{H}(t) \), the standard deviation of \( B_{H}(t) \), attains its maximum over \([0, T]\) at \( t^{*} \in [0, T] \), and \( H(t) \) satisfies that

\[ H(t^{*} + h) = H(t^{*}) - c|h|^{\gamma} + o(|h|^{\gamma}), \; h \to 0, \; c > 0, \; \gamma \in (0, 1). \]

Then we have as \( u \to \infty \)

\[ \mathbb{P} \left\{ \sup_{t \in [0, T]} B_{H}(t) > u \right\} \sim \Psi (\mu) \begin{cases} (1 + \mathbb{1}_{[t^{*}\in [0, T)]}) \mathcal{H}_{a} a^{1/\alpha} b^{-1/\gamma} \Gamma \left( \frac{1}{\gamma} + 1 \right) \mu^{\frac{\alpha}{\gamma} - \frac{1}{2}}, & \text{if} \; \alpha < \gamma, \\ \tilde{\mathcal{P}}_{a}^{b/a}, & \text{if} \; \alpha = \gamma, \\ 1, & \text{if} \; \alpha > \gamma, \end{cases} \]
where
\[
\overline{P}_{\alpha}^{b/a} = \begin{cases} 
\mathcal{P}_{\alpha}^{b/a} , & \text{if } t^* \in \{0,T\}, \\
\mathcal{P}_{\alpha}^{b/a} , & \text{if } t^* \in (0,T),
\end{cases}
\]
\[
\mu = u^{*-H(t^*)} , \quad a = \frac{1}{2} t^{*-2H(t^*)} , \quad b = c \ln t^* , \quad \text{and } \alpha = 2H(t^*).
\]

Following are several special cases which can not be included in the last theorem.

**Theorem 2.3.** i) If
\[
H(t) = \frac{1}{\ln t} , \quad t \in [T_1, T_2], \quad e < T_1 < T_2 < \infty,
\]
we have as \( u \to \infty \)
\[
\mathbb{P} \left\{ \sup_{t \in [T_1, T_2]} B_H(t) > u \right\} \sim \mathcal{H}_a a^{1/\alpha} b^{-1} \mu^{2/\alpha} \Psi(\mu),
\]
where \( \mu = \frac{u}{\tau} , \quad a = \frac{1}{2} T_2^{-2H(T_2)} , \quad b = \frac{2}{\alpha^2 T_2 (\ln T_2)^2} , \quad \text{and } \alpha = 2H(T_2).
\]

ii) If
\[
H(t) = ct^\gamma , t \in [T_1, T_2], 0 < T_1 < T_2 < \infty, \quad c, \gamma > 0,
\]
we have as \( u \to \infty \)
\[
\mathbb{P} \left\{ \sup_{t \in [T_1, T_2]} B_H(t) > u \right\} \sim \Psi(\mu) \begin{cases} 
QH_a a^{1/\alpha} b^{-1} \mu^{2/\alpha} , & \text{if } \alpha < 1, \\
Q \overline{P}_{\alpha}^{b/a} , & \text{if } \alpha = 1, \\
Q , & \text{if } \alpha > 1,
\end{cases}
\]
where
\[
Q = \begin{cases} 
2 , & \text{if } \sigma_H(T_1) = \sigma_H(T_2), \\
1 , & \text{other,}
\end{cases} \quad \overline{T} = \begin{cases} 
T_1 , & \text{if } \sigma_H(T_1) > \sigma_H(T_2), \\
T_2 , & \text{if } \sigma_H(T_1) \leq \sigma_H(T_2),
\end{cases}
\]
\[
\mu = u^{\overline{T}^\gamma} , \quad a = \frac{1}{2} \overline{T}^{-2H(\overline{T})} , \quad b = c \overline{T}^{\gamma-1} \left( 1 + \gamma \ln \overline{T} \right) , \quad \text{and } \alpha = 2H(\overline{T}).
\]

iii) If \( H(t) \) is a differentiable function and decreases over \([T_1, T_2]\) with \( T_2 \leq 1 \) or increases over \([T_1, T_2]\) with \( T_2 \geq 1 \), we have as \( u \to \infty \)
\[
\mathbb{P} \left\{ \sup_{t \in [T_1, T_2]} B_H(t) > u \right\} \sim \Psi(\mu) \begin{cases} 
\mathcal{H}_a a^{1/\alpha} b^{-1} \mu^{2/\alpha} , & \text{if } \alpha < 1, \\
\overline{P}_{\alpha}^{b/a} , & \text{if } \alpha = 1, \\
1 , & \text{if } \alpha > 1,
\end{cases}
\]
where \( \mu = u^{T_2^{-H(T_2)}}, \quad a = \frac{1}{2} T_2^{-2H(T_2)}, \quad b = \frac{H(T_2)}{T_2} + H'(T_2) \ln T_2 \) and \( \alpha = 2H(T_2) \).

3. **Proofs**

**Proof of Theorem 2.2:** By (5), we have
\[
\sigma_H(t^* + h) = e^{H(t^* + h) \ln(t^* + h)} = e^{(H(t^*) - c||h||^\gamma + o(||h||^\gamma)) (\ln t^* + \ln(1 + \frac{h}{t^*})})
\]
\[
= e^{H(t^*) \ln t^* + \left( 1 + \frac{H(t^*)}{t^*} \ln \left( 1 + \frac{h}{t^*} \right) \right)^\gamma} - c(\ln t^*) ||h||^\gamma + o(||h||^\gamma)
\]
\[
= t^* H(t^*) \left( 1 - c(\ln t^*) ||h||^\gamma + o(||h||^\gamma) \right), \quad h \to 0.
\]

Thus by (4), \( B_H(t) \) is a centered Gaussian case with the standard deviation function and correlation function given as
\[
\sigma_H(t^* + h) = t^* H(t^*) \left( 1 - c \ln t^* ||h||^\gamma + o(||h||^\gamma) \right), \quad h \to 0,
\]
and
\[
\tau_H(t, s) = 1 - \frac{1}{2} t^{2H(t)} |s - t|^{2H(t)} + o(|s - t|^{2H(t)}), \quad s, t \to t^*.
\]

Further, since \( \sigma_H(t) \) attains its maximum over \([0, T]\) at a unique point \( t^* \in [0, T] \), by [24] [Proposition 3.9] the result follows.

\[\square\]

**Proof of Theorem 2.3:**

i) We have
\[
\sigma_H(t) = e^{H(t) \ln t} = e, \quad t \in [T_1, T_2].
\]

\( H(t) \) attains its minimum over \([T_1, T_2]\) at \( T_2 \) with
\[
H(T_2 - h) = \frac{1}{\ln T_2} + \frac{1}{2} (\ln T_2)^2 h + o(h^2), \quad h \downarrow 0,
\]

which combined with (4), we have that \( B_H(t) \) is a \( \alpha(t)\)-locally stationary Gaussian processes as in [8]. Thus the results follows from [8] [Theorem 2.1].

ii) We have \( H'(t) = c\gamma t^{\gamma - 1} \) and
\[
\sigma_H'(t) = e^{H(t) \ln t} \left( H'(t) \ln t + \frac{H(t)}{t} \right)
\]
\[
= t^{\gamma - 1} \left( c\gamma t^{\gamma - 1} \ln t + c\gamma - 1 \right), \quad t \in [T_1, T_2] \subset (0, \infty).
\]

Setting \( \sigma_H'(t) = 0 \), we get \( \tilde{t} = e^{-\frac{1}{\gamma}} \) and \( \sigma_H(t) \) is decreasing on \((0, \tilde{t})\) and increasing on \((\tilde{t}, \infty)\). Thus \( \sigma_H(t) \) attains its maximum over \([T_1, T_2]\) at \( T_1 \) or \( T_2 \).

Case 1: When \( \sigma_H(T_1) > \sigma_H(T_2) \), we have \( T_1 < e^{-\frac{1}{\gamma}} \) and
\[
H(T_1 + h) = cT_1^{\gamma} + c\gamma T_1^{\gamma - 1} h + o(h), \quad h \downarrow 0.
\]

Further,
\[
\sigma_H(T_1 + h) = e^{H(T_1 + h) \ln (T_1 + h)} = e^{cT_1^{\gamma} + c\gamma T_1^{\gamma - 1} h + o(h)} \left( \ln (T_1 + h) \right)
\]
\[
= e^{cT_1^{\gamma} \ln T_1 \left( 1 + \frac{h}{T_1} \right) + c\gamma T_1^{\gamma - 1} \ln T_1 + o(h)}
\]
\[
= T_1^{cT_1^{\gamma} \left( 1 + \frac{1}{\gamma} \ln T_1 \right) + o(h)}, \quad h \downarrow 0,
\]

which combined with (4) and [24] [Proposition 3.9] derives the result.

Case 2: When \( \sigma_H(T_1) < \sigma_H(T_2) \), we have \( \sigma_H(t) \) attains its maximum over \([T_1, T_2]\) at \( T_2 \),
\[
H(T_2 - h) = cT_2^{\gamma} - c\gamma T_2^{\gamma - 1} h + o(h), \quad h \downarrow 0,
\]

and
\[
\sigma_H(T_2 - h) = e^{H(T_2 - h) \ln (T_2 - h)} = e^{cT_2^{\gamma} - c\gamma T_2^{\gamma - 1} h + o(h)} \left( \ln (T_2 - h) \right)
\]
\[
= T_2^{cT_2^{\gamma} \left( 1 + \frac{1}{\gamma} \ln T_2 \right) + o(h)}, \quad h \downarrow 0,
\]
which combined with (4) and [24] [Proposition 3.9] derives the result.

Case 3: If $\sigma_H(T_1) = \sigma_H(T_2)$, i.e., $\sigma_H(t)$ attains its maximum over $[T_1, T_2]$ at two point $T_1$ and $T_2$, by [25] [Corollary 8.2], we have for $\delta \in (0, \frac{T_1-T_2}{2})$

$$
\mathbb{P}\left\{ \sup_{t \in [T_1, T_2]} B_H(t) > u \right\} \sim \mathbb{P}\left\{ \sup_{t \in [T_1, T_1+\delta]} B_H(t) > u \right\} + \mathbb{P}\left\{ \sup_{t \in [T_2-\delta, T_2]} B_H(t) > u \right\}, \ u \to \infty.
$$

Then for $\mathbb{P}\left\{ \sup_{t \in [T_1, T_1+\delta]} B_H(t) > u \right\}$ and $\mathbb{P}\left\{ \sup_{t \in [T_2-\delta, T_2]} B_H(t) > u \right\}$, using the similar argument as in Case 1 and Case 2, we get the asymptotics.

iii) When $H(t)$ is a decrease function over $[T_1, T_2]$, with $T_2 \leq 1$, we have

$$
\sigma'_H(t) = e^{H(t)\ln t} \left( H'(t) \ln t + \frac{H(t)}{t} \right) > 0, \ t \in [T_1, T_2] \subset (0, 1].
$$

When $H(t)$ is an increase function over $[T_1, T_2]$, with $T_2 \geq 1$, we have $\sigma_H(t) < 1 = \sigma_H(1)$ for $t \in (0, 1)$ and

$$
\sigma'_H(t) = e^{H(t)\ln t} \left( H'(t) \ln t + \frac{H(t)}{t} \right) > 0, \ t \in [1, \infty).
$$

Thus we have in both cases $\sigma_H(t)$ attains its maximum over $[T_1, T_2]$ at $T_2$ and

$$
H(T_2 - h) = H(T_2) - H'(T_2)h + o(h), \ h \downarrow 0.
$$

We have

$$
\sigma_H(T_2 - h) = e^{H(T_2-h)\ln(T_2-h)}
= e^{(H(T_2)-H'(T_2)h+o(h))(\ln(T_2+\ln(1-\frac{h}{T_2}))}
= e^{H(T_2)\ln(T_2)} \left( 1 - \frac{H(T_2)h}{T_2} \ln \left( 1 - \frac{h}{T_2} \right) - \frac{T_2}{h} - H'(T_2)h \ln T_2 + o(h) \right)
= T_2^{H(T_2)} \left( 1 - \frac{H(T_2)}{T_2} + H'(T_2) \ln T_2 \right) h + o(h), \ h \downarrow 0,
$$

which combined with (4) and [24] [Proposition 3.9] derives the result.

\[\Box\]

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