Tail approximations for sums of dependent regularly varying random variables under Archimedean copula models

Hélène COSSETTE\textsuperscript{*}, Etienne MARCEAU\textsuperscript{*}, Quang Huy NGUYEN\textsuperscript{†}
and Christian Y. ROBERT\textsuperscript{†}

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Abstract

In this paper, we compare two numerical methods for approximating the probability that the sum of dependent regularly varying random variables exceeds a high threshold under Archimedean copula models. The first method is based on conditional Monte Carlo. We present four estimators and show that most of them have bounded relative errors. The second method is based on analytical expressions of the multivariate survival or cumulative distribution functions of the regularly varying random variables and provides sharp and deterministic bounds of the probability of exceedance. We discuss implementation issues and illustrate the accuracy of both procedures through numerical studies.

Keywords: Tail approximation; Archimedean Copulas; Dependent regularly varying random variables; Conditional Monte Carlo simulation; Numerical Bounds

\textsuperscript{*}Université Laval, École d’Actuariat, 2425, rue de l’Agriculture, Pavillon Paul-Comtois, local 4177, Québec (Québec) G1V 0A6, Canada, Canada
\textsuperscript{†}Université de Lyon, Université Lyon 1, Institut de Science Financière et d’Assurances, 50 Avenue Tony Garnier, F-69007 Lyon, France
1 Introduction

A well-known problem in applied probability is the evaluation of the probability that the sum of \( n \) random variables (rvs) exceeds a certain level \( s \). This problem finds its way in many areas of application such as actuarial science, finance, quantitative risk management and, reliability. Different methods can be used to tackle this problem depending on the type of distributions of the random variables and their interaction as well as the values of \( n \) and \( s \).

In this paper, we consider the case of \( n \) positive heavy-tailed random variables that are linked through dependence structures based on Archimedean copulas. An \( n \)-dimensional copula \( C \) is a multivariate distribution on \([0,1]^n\) with uniform margins. Following [Ling, 1965], any Archimedean copula can be simply written as

\[
C(u_1, \ldots, u_n) = \Phi(\Phi^{-1}(u_1) + \cdots + \Phi^{-1}(u_n)), \quad (u_1, \ldots, u_n) \in [0,1]^n,
\]

where \( \Phi \) is a non-increasing function referred to as the generator of the copula \( C \) and \( \Phi^{-1} \) is the generalized inverse function of \( \Phi \) defined as \( \Phi^{-1}(x) = \inf\{t \in \mathbb{R}^+ : \Phi(t) \leq x\} \). The conditions under which a generator \( \Phi \) defines a proper \( n \)-dimensional copula are given in detail in [McNeil and Neslehová, 2009].

Let \( \mathbf{U} = (U_1, \ldots, U_n) \) be a random vector distributed as an Archimedean copula \( C \). For \( i = 1, \ldots, n \), we assume that the survival distribution function of the \( i \)-th rv of the sum is regularly varying, i.e. it satisfies \( \bar{F}_i(x) = 1 - F_i(x) = x^{-\alpha_i} l_i(x) \) where \( \alpha_i > 0 \) and \( l_i \) is a slowly varying function at infinity. We assume without loss of generality that \( \alpha_1 \leq \cdots \leq \alpha_n \). Throughout the paper we shall consider two sums that are linked through the Archimedean copula \( C \) in the following way

\[
S_n^X = \sum_{i=1}^{n} X_i \quad \text{and} \quad S_n^Y = \sum_{i=1}^{n} Y_i,
\]

where

\[
\mathbf{X} = (X_1, \ldots, X_n) \overset{d}{=} (F_1^{-1}(U_1), \ldots, F_n^{-1}(U_n)) \quad \text{and} \quad \mathbf{Y} = (Y_1, \ldots, Y_n) \overset{d}{=} (\bar{F}_1^{-1}(U_1), \ldots, \bar{F}_n^{-1}(U_n)),
\]

with \( F_i^{-1}(u_i) = \inf\{t \in \mathbb{R}^+ : F_i(t) \geq x\} \) since \( F_i \) is a non-decreasing function. Note that the multivariate cumulative distribution function of \( \mathbf{X} \) is given by

\[
\Pr(X_1 \leq x_1, \ldots, X_n \leq x_n) = C(F_1(x_1), \ldots, F_n(x_n)), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,
\]

while the multivariate survival distribution function of \( \mathbf{Y} \) is given by

\[
\Pr(Y_1 > y_1, \ldots, Y_n > y_n) = C(\bar{F}_1(y_1), \ldots, \bar{F}_n(y_n)), \quad (y_1, \ldots, y_n) \in \mathbb{R}^n.
\]

The Archimedean copula \( C \) is referred to as the copula of \( \mathbf{X} \) and as the survival copula of \( \mathbf{Y} \).

To approximate the probabilities \( z_X(s) = \Pr(S_n^X > s) \) and \( z_Y(s) = \Pr(S_n^Y > s) \) when \( s \) is large, one could use functions that are asymptotically equivalent to these probabilities. However there are few results concerning the asymptotic behaviours of these probabilities. Actually it strongly
depends on the tails of the Archimedean copulas (see [Charpentier and Segers, 2009] for a fine analysis of the several types of tails based on characteristics of the Archimedean generator) and strong assumptions have to hold to characterize these behaviours. For example, if the upper-tails of the Archimedean copula $C$ are independent, i.e. $\lim_{u \to 1} \Pr(U_i > u|U_j > u) = 0$ for all $i \neq j$ and if there exist $n - 1$ non-negative constants $c_2, \ldots, c_n$ such that $c_i = \lim_{s \to \infty} F_i(s)/F_1(s)$, then it can be shown that
\[
\lim_{s \to \infty} \frac{\Pr(S_n^X > s)}{F_1(s)} = 1 + \sum_{i=2}^{n} c_i.
\]
If the lower-tails of the Archimedean copula $C$ are rather independent, i.e. $\lim_{u \to 0} \Pr(U_i < u|U_j < u) = 0$ for all $i \neq j$, then
\[
\lim_{s \to \infty} \frac{\Pr(S_n^Y > s)}{F_1(s)} = 1 + \sum_{i=2}^{n} c_i
\]
(see e.g. [Jessen and H., 2006] or [Yuen and Yin, 2012]). [Sun and Li, 2010] studied the asymptotic behaviours of $z_X(s)$ and $z_Y(s)$ under the assumption of identically distributed marginals and specific upper or lower-tail dependence. Let $\beta > 0$ and $l_\Phi$ be a slowly varying function at infinity. If $1 - \Phi(x) = x^\beta l_\Phi(x^{-1})$, they proved that
\[
\lim_{s \to \infty} \frac{\Pr(S_n^X > s)}{\Pr(X_1 > s)} = q_n^C(\alpha, \beta),
\]
where $\alpha$ denotes the common tail index and
\[
q_n^C(\alpha, \beta) = \int_{\sum_{i=1}^{n} v_i^{-1/\alpha} > 1} \frac{\partial^n}{\partial v_1 \cdots \partial v_n} \sum_{1 \leq i_1, \ldots, i_j \leq n} (-1)^{j-1}(v_1^{1/\beta} + \ldots + v_j^{1/\beta})^{-\beta} dv_1 \cdots dv_n.
\]
If the generator rather satisfies $\Phi(x) = x^{-\beta} l_\Phi(x)$, they proved that
\[
\lim_{s \to \infty} \frac{\Pr(S_n^Y > s)}{\Pr(Y_1 > s)} = q_n^D(\alpha, \beta),
\]
where
\[
q_n^D(\alpha, \beta) = \int_{\sum_{i=1}^{n} v_i^{-\alpha/\beta} > 1} \frac{\partial^n}{\partial v_1 \cdots \partial v_n} \left(v_1^{-\alpha/\beta} + \ldots + v_n^{-\alpha/\beta}\right)^{-\beta} dv_1 \cdots dv_n,
\]
(see also [Wüthrich, 2003]). Although $q_n^C(\alpha, \beta)$ and $q_n^D(\alpha, \beta)$ are known, they do not have closed-form expressions and they can not be easily computed.

In this paper, we aim to provide two numerical methods for approximating $z_X(s)$ and $z_Y(s)$ for different values of $n$ and $s$ and choices of parameters and functions: $((\alpha_1, l_1), \ldots, (\alpha_n, l_n))$ and $\Phi$.

Our first method is based on conditional Monte Carlo and is ideally suited when $s$ and/or $n$ are large. The classical Monte Carlo method is easy to implement and can be applied in complex situations such as high dimensional calculations. However, it is well known that it is inadequate for small probability simulation since the relative errors (variation coefficients) are too large. [Asmussen and Glynn, 2007] introduced relative error as a measure of efficiency of an estimator and several definitions of efficient estimators. An unbiased estimator $Z(s)$ of the probability $z(s)$, with relative error $e(Z(s)) = \sqrt{\text{E}[Z^2(s)]/z(s)}$, is called (i) a logarithmically efficient estimator...
if \( \limsup_{s \to \infty} e(Z(s)) [z(s)]^\epsilon = 0 \) for all \( \epsilon > 0 \); (ii) an estimator with bounded relative error if \( \limsup_{s \to \infty} e(Z(s)) < \infty \); (iii) an estimator with vanishing relative error \( \limsup_{s \to \infty} e(Z(s)) = 0 \).

For sums of independent random variables, the most widely used alternatives to crude Monte Carlo computation of rare-event probabilities are conditional Monte Carlo and importance sampling. [Asmussen and Binswanger, 1997] propose a logarithmically efficient algorithm based on conditional Monte Carlo simulation using order statistics.

[Boots and Shahabuddin, 2001] use importance sampling to simulate ruin probabilities for subexponential claims and [Juneja and Shahabuddin, 2002] use importance sampling based on hazard rate twisting to simulate heavy-tailed processes. [Asmussen and Kroese, 2006] propose two algorithms which use importance sampling and conditional Monte Carlo and study their efficiency in the Pareto and Weibull case.

Estimating tail distribution of the sums of dependent random variables via simulation requires a specific expression for the dependence structure or a closed form expression for the conditional distribution functions, the case of elliptic distributions is an example. For an elliptic dependence structure, [Blanchet and Rojas-Nandayapa, 2011] proposed a conditional Monte Carlo estimator for the tail distribution of the sum of log-elliptic random variables and proved that it has a logarithmically efficient relative error. The sum of the log-elliptic random variables was also estimated by [Kortschak and Hashorva, 2013] using the simulation method introduced by [Asmussen and Kroese, 2006] and favorable results are presented especially in the multivariate log-normal case. [Asmussen et al., 2011] and [Blanchet et al., 2008] focus on the efficient estimation of sums of correlated lognormals using importance sampling and conditional Monte Carlo strategies. [Chan and Kroese, 2010], [Chan and Kroese, 2011] use conditional Monte Carlo notably in a credit risk setting under the t-copula model to estimate rare-event probabilities.

In this paper, we introduce four different estimators of the probabilities \( z_X(s) = \Pr(S_n^X > s) \) and \( z_Y(s) = \Pr(S_n^Y > s) \) using techniques of conditional Monte Carlo simulation. The main idea to build our estimators is to first isolate the known probabilities \( \Pr(M_n^X > s) \) or \( \Pr(M_n^Y > s) \) where \( M_n^A \) correspond to the maximum element of a given vector \( A \) (because \( \Pr(M_n^X > s) \) or \( \Pr(M_n^Y > s) \) have closed-form expressions in our framework), and then simulate conditionally on the values taken by these maxima. Two effective simulation techniques of vectors of Archimedean copula proposed in [Brechmann et al., 2013] and [McNeil and Nešlehová, 2009] will be used. We show that most of our estimators have bounded relative errors.

Our second method is based on analytical expressions of the survival multivariate distribution function and provides sharp, deterministic and numerical bounds of the probabilities using the same ideas as developed in [Cossette et al., 2014]. This method performs very well for cases when \( n \) is relatively small and effectively completes the conditional Monte Carlo method.

The outline of the paper is as follows. In Section 2, we present the two simulation techniques related to Archimedean copulas which are later used to develop the proposed estimators. These estimators are introduced, described and discussed in Section 3. Some results on their asymptotic efficiency are also given. Section 4 explains how to derive and compute the numerical bounds following the approach proposed in [Cossette et al., 2014]. Section 5 illustrates the accuracy of
both methods through a numerical study and discusses implementation issues.

2 Simulation and conditional simulation with Archimedean copulas

The classical simulation method for a dependent vector relies on conditional distributions. Consider a random vector \( \mathbf{U} \) with density function \( c(u_1, \ldots, u_n) \) which can be decomposed as the product of conditional densities

\[
c(u_1, \ldots, u_n) = c_{n|n-1,\ldots,1}(u_n|u_{n-1}, \ldots, u_1) \ldots c_{2|1}(u_2|u_1)c_1(u_1).
\]

The classical procedure of simulating vector such a vector \( \mathbf{U} \) is then: simulate \( u_1 \) based on \( c_1(u_1) \), simulate \( U_2 \) based on \( c_{2|1}(u_2|u_1) \), ..., simulate \( U_n \) based on \( c_{n|n-1,\ldots,1}(u_n|u_{n-1}, \ldots, u_1) \). Hence, the realization of vector \( \mathbf{U} \) is created by calculating \((n-1)\) times the inverses of conditional distribution functions. However it can be difficult and take quite an amount of time when the distribution function of \( \mathbf{U} \) is an Archimedean copula.

Another method for an Archimedean copula could be to consider the mixed exponential or frailty representation often used to model dependent lifetimes and discussed in, notably, [Marshall and Olkin, 1988], [McNeil, 2008] and [Hofert, 2008]. In this case, the Archimedean generator \( \Phi \) is the Laplace-Stieltjes transform of a non-negative random variable. Such a method thus requires to invert the Laplace-Stieltjes transform \( \Phi \) which can not always be evaluated explicitly.

To circumvent these problems, one can resort to two effective simulation techniques proposed in [Brechmann et al., 2013] and [McNeil and Nešlehová, 2009]. More precisely, [Brechmann et al., 2013] use the Kendall distribution function while [McNeil and Nešlehová, 2009] suggest a simulation method which relies on \( \ell_1 \)-norm symmetric distributions.

2.1 Brechmann, Hendrich and Czado’s approach

Arguing that the classical method does not work due to the problem of calculating the inverse functions of conditional distributions

\[
C_{j|j-1,\ldots,1}(u_j|u_{j-1}, \ldots, u_1) = \Pr(U_j \leq u_j|u_{j-1}, \ldots, u_1),
\]

[Brechmann, 2014] provides an algorithm to simulate Archimedean copulas using an intermediate variable \( Z \) whose distribution function is known as the Kendall distribution function (see [Barbe et al., 1996]). This conditional inverse simulation method eliminates the problems encountered with the numerical calculations of the inverse functions of \( C_{j|j-1,\ldots,1} \). We restate below two propositions which will prove useful for the understanding of the algorithm proposed by [Brechmann et al., 2013].
Proposition 1 (Barbe et al., 1996) Let \( \mathbf{U} \) be distributed as the Archimedean copula \( C \) with generator \( \Phi \) and let the random variable \( Z \) be defined as \( Z = C(\mathbf{U}) \). Then, the density function of \( Z \) is defined in terms of the generator \( \Phi \) as

\[
f_Z(z) = \frac{(-1)^{n-1}}{(n-1)!} (\Phi^\leftarrow(z))^{n-1} (\Phi^\leftarrow(1)^{(n)}(z)) \Phi^\leftarrow(z).
\]

Proposition 2 (Brechmann, 2014) Let \( \mathbf{U} \) be distributed as the Archimedean copula \( C \) with generator \( \Phi \) and let the random variable \( Z \) be defined as \( Z = C(\mathbf{U}) \). Then, the conditional distribution of \( U_j|Z,U_{j-1},...,U_1 \) for \( j = 1,...,n \) is

\[
F_{U_j|Z,U_{j-1},...,U_1}(u_j|z,u_{j-1},...,u_1) = \left( 1 - \frac{\Phi^\leftarrow(u_j)}{\Phi^\leftarrow(z) - \sum_{k=1}^{j-1} \Phi^\leftarrow(u_k)} \right)^{n-j}
\]

for \( 1 > u_j > \Phi \left( \Phi^\leftarrow(z) - \sum_{k=1}^{j-1} \Phi^\leftarrow(u_k) \right) \).

From these results, the inverse function of the conditional distribution function

\[
F_{U_j|Z,U_{j-1},...,U_1}(u_j|z,u_{j-1},...,u_1)
\]

can be calculated with an explicit formula. Indeed, if we have \( z,u_{j-1},...,u_1 \) as realizations of \( Z,U_{j-1},...,U_1 \) respectively and \( v \) as a realization of a uniform random variable in \((0,1)\), a realization of \( U_j \) is obtained with

\[
u_j = \Phi \left( \left( 1 - v^{1/(n-j)} \right) \left( \Phi^\leftarrow(z) - \sum_{k=1}^{j-1} \Phi^\leftarrow(u_k) \right) \right) \right).
\]

The conditional distribution of \( (Z|U_1) \) is given in the following proposition.

Proposition 3 (Brechmann et al., 2013) Let \( \mathbf{U} \) be distributed as the Archimedean copula \( C \) with generator \( \Phi \) and let the random variable \( Z \) be defined as \( Z = C(\mathbf{U}) \). Then, the conditional distribution \( F_{Z|U_1}(z|u_1) \) can be calculated by the Archimedean generator and its derivatives as

\[
F_{Z|U_1}(z|u_1) = (\Phi^\leftarrow(1)) (u_1) \sum_{j=0}^{n-2} \frac{(-1)^j}{j!} (\Phi^\leftarrow(z) - \Phi^\leftarrow(u_1))^j \Phi^\leftarrow(j+1)(z) \text{ for } z \in (0,u_1).
\]

Given the above propositions, the following algorithm is derived from [Brechmann et al., 2013] to generate a random vector \((X_1,...,X_n)\) from an \( n \)-dimensional Archimedean copula \( C \) with generator \( \Phi \) or a random vector \((Y_1,...,Y_n)\) from an \( n \)-dimensional survival Archimedean copula \( C \) with generator \( \Phi \).

**Algorithm 4** Brechmann et al. (2013)’s algorithm.

6
1. Generate a random variable $U_1$ uniformly distributed on $(0, 1)$.

2. Generate a random variable $(Z | U_1 = u_1)$ from (2).

3. For $j = 2, ..., n$ generate a random variable $(U_j | Z = z, U_1 = u_1, ..., U_{j-1} = u_{j-1})$ with

   $$u_j = \Phi \left( (1 - v^{1/(n-j)}) \left( \Phi_\leftarrow(z) - \sum_{k=1}^{j-1} \Phi_\leftarrow(u_k) \right) \right),$$

   where $V$ has been generated as a random variable uniformly distributed on $(0, 1)$.

4. Set $X_i = F_X^{-1} (U_i)$ or $Y_i = \bar{F}_Y^{-1} (U_i)$ for $i = 1, ..., n$.

As a consequence, it is easy to simulate a conditional Archimedean copula $U = (U_1, ..., U_n | U_1 \in [a, b])$ where $[a, b] \in (0, 1)$ by first simulating $U_1$ uniformly distributed on $[a, b]$ and then by following Steps 2-4 of the previous algorithm.

### 2.2 McNeil and Neslehová’s approach

[McNeil and Neslehová, 2009] give the conditions under which a generator $\Phi$ defines an $n$-dimensional copula by means of (1) and show that the close connection between Archimedean copulas and $\ell_1$-norm symmetric distributions, introduced by [Fang and Fang, 1988], and that allows a new perspective and understanding of Archimedean copulas. With such an insight, they are able to consider cases where an Archimedean generator is not completely monotone or equivalently is not equal to a Laplace transform of a non-negative random variable (see [Kimberling, 1974]).

**Theorem 5 (McNeil and Neslehová, 2009)** A real function $\Phi: [0, \infty) \to [0, 1]$ is the generator of an $n$-dimensional Archimedean copula if and only if it is an $n$-monotone function on $[0, \infty)$ i.e it is differentiable up to order $(n - 2)$ and the derivatives satisfy $(-1)^i \Phi^{(i)}(x) \geq 0, i = 0, 1, ..., n - 2$ for any $x$ in $[0, \infty)$ and further if $(-1)^{n-2} \Phi^{(n-2)}$ is non-increasing and convex in $[0, \infty)$.

**Definition 6 (Fang and Fang, 1988)** A random vector $X$ on $\mathbb{R}_+^n = [0, \infty)^n$ follows an $\ell_1$-norm symmetric distribution if and only if there exists a non-negative random variable $R$ independent of $W$, where $W = (W_1, ..., W_n)$ is a random vector distributed uniformly on the unit simplex $s_n$,

$$s_n = \{ x \in \mathbb{R}_+^n : \|x\|_1 = 1 \},$$

so that $X$ permits the stochastic representation

$$X \overset{d}{=} RW.$$ 

The random variable $R$ is referred to as the radial part of $X$ and its distribution as the radial distribution.
The following theorem establishes the connection between $\ell_1$-norm symmetric distributions and Archimedean copulas. More details and interesting results and comments in that regard can be found in [McNeil and Neslehová, 2009].

**Theorem 7** (McNeil and Neslehová, 2009) Let the random vector $U$ be distributed according to an $n$-dimensional Archimedean copula $C$ with generator $\Phi$. Then, $(\Phi^{-1}(U_1), \ldots, \Phi^{-1}(U_n))$ has an $\ell_1$-norm symmetric distribution with survival copula $C$ and radial distribution $F_R$ given by

$$F_R(x) = 1 - \sum_{j=0}^{n-2} (-1)^j \frac{x^j}{j!} \Phi(j)(x) - (-1)^{n-1} \frac{x^{n-1}}{(n-1)!} \Phi^{(n-1)+}(x), \quad x \in [0, \infty).$$

This last theorem implies

$$(U_1, \ldots, U_n) \overset{d}{=} (\Phi(RW_1), \ldots, \Phi(RW_n))$$

for $R$ a positive random variable with distribution function $F_R$ and $W$ a random vector uniformly distributed on the $n$-dimensional unit simplex $s_n$. Hence, since the vector $(X_1, \ldots, X_n)$ has marginal distribution functions $F_1, \ldots, F_n$ and $F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$ where $C$ is an Archimedean copula with generator $\Phi$, then

$$(X_1, \ldots, X_n) \overset{d}{=} (F_1^{-}(\Phi(RW_1)), \ldots, F_n^{-}(\Phi(RW_n))).$$

Since $(Y_1, \ldots, Y_n)$ has marginal distribution functions $F_1, \ldots, F_n$ with a dependence structure defined through an Archimedean survival copula $C$ with generator $\Phi$, then

$$(Y_1, \ldots, Y_n) \overset{d}{=} \left(\overline{F}_1^{-}(\Phi(RW_1)), \ldots, \overline{F}_n^{-}(\Phi(RW_n))\right).$$

This representation leads to the following sampling algorithm.

**Algorithm 8** McNeil and Neslehová’s algorithm.

1. Generate a vector $(E_1, \ldots, E_n)$ of $n$ iid exponential rvs with parameter 1. Calculate $W_i = E_i / \sum_{j=1}^n E_j$ such that $W$ is uniformly distributed on the $n$-dimensional unit simplex $s_n$.
2. Generate a random variable $R$ with distribution $F_R$ (see Theorem 7).
3. Return $U$ where $U_i = \Phi(RW_i)$ for $i = 1, \ldots, n$.
4. Set $X_i = F^{-}_{X_i}(U_i)$ or $Y_i = \overline{F}^{-}_{Y_i}(U_i)$ for $i = 1, \ldots, n$.

**3 Estimators of $z_X(s)$ and $z_Y(s)$**

In this section, we propose four different estimators of $z_X(s)$ and $z_Y(s)$. All estimators rely on a similar idea which is to decompose the probability of interest into different components. The known
components are exactly evaluated and the other ones, which are our main concern, are estimated by simulation.

To begin, we need to establish basic notations. Throughout, \( X_{-i} \) denotes the vector \( X = (X_1, ..., X_n) \) with the \( i \)-th component \( X_i \) removed and \( M_i^X \) (or \( M_i^X = \{X\} \)) corresponds to the \( i \)-th element of the vector \( X \) after rearranging the elements of \( X \) in a non-decreasing order. Obviously, \( M_1^X \) and \( M_n^X \) correspond to the minimum and maximum element of vector \( X \), respectively. The same convention holds for \( Y = (Y_1, ..., Y_n) \).

In what follows, we encounter frequently the evaluation of the probabilities \( \Pr(M_n^X > s) \) and \( \Pr(M_n^Y > s) \) which rely on the marginal distributions \( F_1, ..., F_n \) and the Archimedean generator \( \Phi \). They are obtained as follows

\[
\Pr(M_n^X > s) = 1 - C(F_1(s), ..., F_n(s))
\]

and

\[
\Pr(M_n^Y > s) = 1 - \overline{C}(\overline{F}_1(s), ..., \overline{F}_n(s)),
\]

with \( \overline{C}(u_1, ..., u_n) = \Pr(U_1 > u_1, ..., U_n > u_n) \).

### 3.1 First estimator

The first estimator of \( z_X(s) \) and \( z_Y(s) \) that we propose is based on the simulation technique used by [Brechmann et al., 2013] to generate sampled values of the conditional vector \( (U|U_1) \in [a, b] \) (see Section 2.1). The idea is to first isolate the known probability \( \Pr(M_n^X > s) \) and then condition on the value taken by the maximum \( M_n^X \).

Hence, we have

\[
z_X(s) = \Pr(M_n^X > s) + \Pr(S_{n}^{X_i} > s, s/n < M_n^X \leq s)
\]

\[
= \Pr(M_n^X > s) + \sum_{i=1}^{n} \Pr(s/n < X_i \leq s) \Pr(S_n^{X_i} > s, X_i = M_n^X | s/n < X_i \leq s).
\]

Then, it leads to the following estimator \( Z_{NR1}^X(s) \) for \( z_X(s) \):

\[
Z_{NR1}^X(s) = \Pr(M_n^X > s) + \sum_{i=1}^{n} (\overline{F}_i(s/n) - \overline{F}_i(s)) I_{\{S_n^{X_i} > s, X_i = M_n^{X_i}\}},
\]

where \( S_n^{X_i}, X_i \) and \( M_n^{X_i} \) correspond to the conditional random variables \( S_n^{X_i}, X_i \) and \( M_n^{X_i} \) given \( (s/n < X_i \leq s) \). The challenging problem here is the simulation of the vector

\[
X_i = (X_i^1, ..., X_i^n) = (X_1, ..., X_n | s/n < X_i \leq s)
\]

with an Archimedean copula as dependence structure. Given that \( X_i = F_i^+(U_i) \), this last vector
$X^i$ can be viewed as
\[
(X_1, ..., X_n | s/n < X_i \leq s) \overset{d}{=} (F_1^{-}(U_1), ..., F_n^{-}(U_n)) \mid s/n < F_i^{-}(U_i) \leq s)
\]
where $(U_1^{i+}, ..., U_n^{i+}) = (U_1, ..., U_n | F_i(s/n) < U_i \leq F_i(s))$.

Similarly for the random vector $Y$ with multivariate cumulative distribution function based on the Archimedean survival copula $C$, we have
\[
(Y_1, ..., Y_n | s/n < Y_i \leq s) \overset{d}{=} (\bar{F}_1^{-}(U_1), ..., \bar{F}_n^{-}(U_n)) \mid s/n < \bar{F}_i^{-}(U_i) \leq s)
\]
where $(U_1^{i-}, ..., U_n^{i-}) = (U_1, ..., U_n | \bar{F}_i(s/n) > U_i \geq \bar{F}_i(s))$. Hence, the first estimator for $y_Y(s)$ is given by
\[
Z_{NR1}^Y(s) = \Pr(M_n^Y > s) + \sum_{i=1}^n \left( \bar{F}_i(s/n) - \bar{F}_i(s) \right) I_{\{s_n^i > s, Y_i = M_n^Y \}}.
\]

We are now in a position to propose the following algorithms to generate realizations of both estimators $Z_{NR1}^X(s)$ and $Z_{NR1}^Y(s)$.

**Algorithm 9 (Estimator $Z_{NR1}^X(s)$) To generate a realization of $Z_{NR1}^X(s)$, proceed as follows:**

1. For $i = 1, ..., n$, independently simulate $U_i^{i+}$ uniformly distributed on $(F_i(s/n), F_i(s))$.
2. For each $U_i^{i+}$ in the first step, simulate $Z$ based on conditional distribution $F_{ZU_i^{i+}}$ and then simulate $(U_1^{i+}, ..., U_i^{i+}, U_{i+1}^{i+}, U_n^{i+})$.
3. For each $j = 1, ..., n$, compute $X_j^i = F_j^{-}(U_j^{i+})$ and return $I_{\{s_n^i > s, X_i^i = M_n^X \}}$ which takes value 0 or 1.
4. Return $Z_{NR1}^X(s) = \Pr(M_n^X > s) + \sum_{i=1}^n \left( \bar{F}_i(s/n) - \bar{F}_i(s) \right) I_{\{s_n^i > s, X_i^i = M_n^X \}}$.

**Algorithm 10 (Estimator $Z_{NR1}^Y(s)$) To generate a realization of $Z_{NR1}^Y(s)$, proceed as follows:**

1. For $i = 1, ..., n$, independently simulate $U_i^{i-}$ uniformly distributed on $(\bar{F}_i(s), \bar{F}_i(s/n))$.
2. For each $U_i^{i-}$ in the first step, simulate $Z$ based on conditional distribution $F_{ZU_i^{i-}}$ and then simulate $(U_1^{i-}, ..., U_i^{i-}, U_{i+1}^{i-}, U_n^{i-})$.
3. For each $j = 1, 2, ..., n$, compute $Y_j^i = \bar{F}_j^{i-}(U_j^{i-})$ and return $I_{\{s_n^i > s, Y_i^i = M_n^Y \}}$ which takes value 0 or 1.
4. Return $Z_{NR1}^Y(s) = \Pr(M_n^Y > s) + \sum_{i=1}^{n} (\overline{F}_i(s/n) - \overline{F}_i(s)) I\{S_{n_i}^{Y_i} > s, Y_i = M_n^Y\}$.

**Proposition 11** Estimators $Z_{NR1}^X(s)$ and $Z_{NR1}^Y(s)$ have bounded relative errors.

**Proof.** See Appendix. ■

It is important to note that the approach used here to estimate the sum of regularly varying random variables leads to estimators with a bounded relative error no matter the dependence structure between the random variables. The key element is the simulation of the conditional random vector $(X|s/n < X_i \leq s)$. Unfortunately, the numerical performance of $Z_{NR1}(s)$, as we will see in Section 5, is not as good as for the other estimators.

### 3.2 Second estimator

The construction of the second estimator is based on the stochastic representation of an Archimedean copula proposed by [McNeil and Neslehová, 2009]. As stated in Section 2.2, for a multivariate random vector $X$ with underlying Archimedean copula $C$ with generator $\Phi$, we have

$$(X_1, ..., X_n) \overset{d}{=} (F_1^{-1}(\Phi(RW_1)), ..., F_n^{-1}(\Phi(RW_n))),$$

where the distribution function of $R$ is given as in Theorem 7 and $W$ is a random vector uniformly distributed on $s_n$. This representation of the random vector $X$ permits to write the probability of interest $z_X(s)$ as follows:

$$z_X(s) = \Pr(M_n^X > s) + \Pr(S_n^X > s, M_n^X \leq s)$$

$$= \Pr(M_n^X > s) + \Pr\left(\sum_{i=1}^{n} F_i^{-1}(\Phi(RW_i)) > s, M_n \left\{ F_i^{-1}(\Phi(RW_i)) \right\} \leq s\right)$$

$$= \Pr(M_n^X > s) + \Pr\left(\sum_{i=1}^{n} F_i^{-1}(\Phi(RW_i)) > s, R \geq M_n \left\{ \frac{\Phi_i^{-1}(F_i(s))}{W_i} \right\} \right).$$

By conditioning on the random vector $W$, we have

$$\Pr\left(\sum_{i=1}^{n} F_i^{-1}(\Phi(RW_i)) > s, R \geq M_n \left\{ \frac{\Phi_i^{-1}(F_i(s))}{W_i} \right\} \right)$$

$$= E_{W} \left[ \Pr\left(\sum_{i=1}^{n} F_i^{-1}(\Phi(RW_i)) > s, R \geq M_n \left\{ \frac{\Phi_i^{-1}(F_i(s))}{W_i} \right\} \right| W \right].$$

Then, we obtain the following second estimator of $z_X(s)$ in terms of the known radial cumulative
distribution function $F_R$ given by

$$Z_{NR}^X(s) = \Pr(M_n^X > s) + \Pr \left( \sum_{i=1}^n F_i^{<} (\Phi(RW_i)) > s, R \geq M_n \left\{ \frac{\Phi^{<}(F_i(s))}{W_i} \right\} \right| W$$

$$= \Pr(M_n^X > s) + (F_R(U^X(W, s)) - F_R(L^X(W, s))) ,$$

where

$$U^X(W, s) = \sup \{ r \in \mathbb{R}^+ : \sum_{i=1}^n F_i^{<} (\Phi(rW_i)) \leq s \} \quad (3)$$

and

$$L^X(W, s) = M_n \left\{ \frac{\Phi^{<}(F_i(s))}{W_i} \right\} . \quad (4)$$

In a similar fashion for the random vector $Y$ with an underlying Archimedean survival copula, we obtain the estimator $Z_{NR}^Y(s)$ for $z_Y(s)$ which is given by

$$Z_{NR}^Y(s) = \Pr(M_n^Y > s) + (F_R(U^Y(W, s)) - F_R(L^Y(W, s))) ,$$

where

$$U^Y(W, s) = M_1 \left\{ \frac{\Phi^{<}(F_i(s))}{W_i} \right\} \quad (5)$$

and

$$L^Y(W, s) = \inf \{ r \in \mathbb{R}^+ : \sum_{i=1}^n F_i^{<} (\Phi(rW_i)) \geq s \}. \quad (6)$$

Note that if the marginal distributions are continuous and strictly increasing, then $U^X(W, s)$ and $L^Y(W, s)$ are the unique roots of equations $\sum_{i=1}^n F_i^{<} (\Phi(xW_i)) = s$ and $\sum_{i=1}^n F_i^{>} (\Phi(xW_i)) = s$ respectively.

The sampling algorithm to generate realizations of $Z_{NR}^X(s)$ and $Z_{NR}^Y(s)$ can be written down as outlined in the following.

**Algorithm 12** (Estimator $Z_{NR}^X(s)$) To generate a realization of $Z_{NR}^X(s)$, proceed as follows:

1. Let $(E_1, ..., E_n)$ be $n$ iid exponential rvs with parameter 1. Calculate $W_i = E_i / \sum_{j=1}^n E_j$.
2. Evaluate numerically $U^X(W, s)$ from (3) and $L^X(W, s)$ from (4).
3. Calculate derivatives $\Phi^{(j)}$ for $j = 1, ..., n - 1$ and then the radial distribution $F_R$.
4. Return $Z_{NR}^X(s) = \Pr(M_n^X > s) + F_R(U^X(W, s)) - F_R(L^X(W, s))$.

**Algorithm 13** (Estimator $Z_{NR}^Y(s)$) To generate a realization of $Z_{NR}^Y(s)$, proceed as follows:
1. Let \((E_1, ..., E_n)\) be \(n\) iid exponential rvs with parameter 1. Calculate \(W_i = E_i / \sum_{j=1}^{n} E_j\).

2. Evaluate numerically \(U^Y(W, s)\) from (3) and \(L^Y(W, s)\) from (4).

3. Calculate derivatives \(\Phi^{(j)}\) for \(j = 1, ..., n - 1\) and then the radial distribution \(F_R\).

4. Return \(Z_{NR2}^Y(s) = \Pr(M_n^Y > s) + F_R(U^Y(W, s)) - F_R(L^Y(W, s))\).

In the following proposition, the accuracy of our estimator \(Z_{NR2}^Y(s)\) is investigated under the assumption that the Archimedean generator is regularly varying.

**Proposition 14** For the random vector \(Y\) with an underlying Archimedean survival copula \(C\) with generator \(\Phi\) satisfying \(\Phi^{(n-2)}\) differentiable and \(\Phi(x) = x^{-\beta} l_\Phi (x)\) with \(\beta > 0\), then \(Z_{NR2}^Y(s)\) has a bounded relative error.

**Proof.** See Appendix. ■

### 3.3 Third estimator

This section presents the third estimator for \(z_X(s)\) which will show better numerical performances than the two previous estimators in the numerical study presented in a later section. The third estimator for \(z_Y(s)\) is based on the same idea and is not discussed. Let us separate the probability \(z_X(s)\) into the components

\[
\Pr(S_n^X > s) = \Pr(M_n^X > s) + z_1^X(s) + z_2^X(s),
\]

where \(z_1^X(s) = \Pr(S_n^X > s, M_{n-1}^X \leq \lambda s, M_n^X \leq s)\), \(z_2(s) = \Pr(S_n^X > s, M_{n-1}^X > \lambda s, M_n^X \leq s)\) and \(\lambda\) is a positive quantity less than \(1/n\). In \(z_1^X(s)\), the inequality \(M_n^X \leq \lambda s\) implies that there is only one variable taking a large value. Consequently, we estimate \(z_1^X(s)\) conditionally on \(X_{-i}\) when \(X_i = M_n^X\). In \(z_2^X(s)\), there are at least two variables taking large values, so it is coherent if we estimate \(z_2^X(s)\) conditionally on the uniform random vector \(W\) defined on the unit simplex \(s_n\).

#### 3.3.1 Estimators for \(z_1^X(s)\) and \(z_1^Y(s)\)

Let us develop the probability \(z_1^X(s)\) as

\[
\Pr(S_n^X > s, M_{n-1}^X \leq \lambda s, M_n^X \leq s) = \sum_{i=1}^{n} \Pr(S_n^X > s, M_{n-1}^X \leq \lambda s, M_n^X < s, X_i = M_n^X) = \sum_{i=1}^{n} \Pr(S_n^X > s, \max \{X_{-i}\} \leq \lambda s, X_i < s, X_i = M_n^X).
\]
By conditioning on $X_{-i}$, we obtain the following estimator $Z_{N^R3,1}^{[i]}(s)$ for $\Pr(S_n^X > s, \max\{X_{-i}\} \leq \lambda s, X_i < s, X_i = M_n^X)$:

$$Z_{N^R3,1}^{[i]}(s) = \Psi(X_{-i}, s),$$

where

$$\Psi(x_{-i}, s) = I_{\max\{x_{-i}\} \leq \lambda s} \Pr\left(s > X_i^* > s - \sum_{j=1, j \neq i}^n x_j \right)$$

with $X_i^* = (X_i | X_{-i} = x_{-i})$ and $i = 1, \ldots, n$. Note that, if $\max\{x_{-i}\} < \lambda s$, then

$$s - \sum_{j=1, j \neq i}^n x_j > (1 - (n - 1)\lambda)s > s/n > \lambda s \geq \max\{x_{-i}\}$$

which is coherent with $X_i = M_n^X > \max\{X_{-i}\}$. Estimator $Z_{N^R3,1}^X(s)$ for $z_1^X(s)$ is then defined by

$$Z_{N^R3,1}^X(s) = \sum_{i=1}^n Z_{N^R3,1}^{[i]}(s).$$

Under the assumption of identically distributed random variables $X_1, \ldots, X_n$, the estimator $Z_{N^R3,1}^X(s)$ coincides with Asmussen and Kroese’s estimator (see [Asmussen and Kroese, 2006]).

To perform the calculations, we need the conditional distribution of $X_i^* = (X_i | X_{-i} = x_{-i})$ for each $i = 1, \ldots, n$ which is given by

$$F_{X_i^*}(x_i) = \frac{\Phi^{(n-1)}\left(\sum_{j=1}^n \Phi^{-1}(F_j(x_j))\right)}{\Phi^{(n-1)}\left(\sum_{j=1, j \neq i}^n \Phi^{-1}(F_j(x_j))\right)}.$$

The method is similar to obtain $Z_{N^R3,1}^Y(s)$ except for the expression of the distribution of $Y_i^*$ which is slightly more difficult to derive.

**Proposition 15** Let $Y = (Y_1, \ldots, Y_n)$ with multivariate distribution defined with an Archimedean survival copula and marginals $F_1, \ldots, F_n$. The conditional cumulative distribution function of $Y_i^* = (Y_i | Y_{-i} = y_{-i})$ is

$$F_{Y_i^*}(y_i) = \Pr(Y_i^* \leq y_i) = 1 - \frac{\Phi^{(n-1)}\left(\sum_{j=1}^n \Phi^{-1}(F_j(y_j))\right)}{\Phi^{(n-1)}\left(\sum_{j=1, j \neq i}^n \Phi^{-1}(F_j(y_j))\right)}.$$

**Proof.** See Appendix. \[\square\]
3.3.2 Estimators for $z_2^X(s)$ and $z_2^Y(s)$

Given [McNeil and Nešlehová, 2009], we can write the probability $z_2^X(s) = \Pr(S_n^X > s, M_{n-1}^X \geq \lambda s, M_n^X \leq s)$ as

$$z_2^X(s) = \Pr \left( \sum_{i=1}^{n} F_i^+(\Phi(W_i R)) > s, M_{n-1} \{F_i^- (\Phi(W_i R))\} > \lambda s, M_n \{F_i^- (\Phi(W_i R))\} \leq s \right)$$

$$= \Pr \left( R < U^X(W, s), R < M_{n-1} \left\{ \frac{\Phi^- (F_i(\lambda s))}{W_i} \right\}, R \geq L^X(W, s) \right).$$

Conditioning on $W$, we have

$$z_2^X(s) = \mathbb{E}_W \left[ \Pr \left( R < U^X(W, s), R < M_{n-1} \left\{ \frac{\Phi^- (F_i(\lambda s))}{W_i} \right\}, R \geq L^X(W, s) \mid W \right) \right],$$

which leads to the estimator $Z_{NR3,2}^X(s)$ given by

$$Z_{NR3,2}^X(s) = F_R \left( U^X(W, s) \land M_{n-1} \left\{ \frac{\Phi^- (F_i(\lambda s))}{W_i} \right\} \right) - F_R \left( L^X(W, s) \right)$$

$$= F_R (U^X(W, s)) - F_R (L^X(W, s))$$

with $U^X(W^{(j)}, s)$, $L^X(W^{(j)}, s)$ as in (3) and (4) respectively, and

$$U^X(W^{(j)}, s) = U^X(W^{(j)}, s) \land M_{n-1} \left\{ \frac{\Phi^- (F_i(\lambda s))}{W_i^{(j)}} \right\}. \quad (7)$$

Similarly, under an Archimedean survival copula, we have

$$z_2^Y(s) = \Pr \left( \sum_{i=1}^{n} \bar{F}_i^-(\Phi(W_i R)) > s, M_{n-1} \{\bar{F}_i^- (\Phi(W_i R))\} \geq \lambda s, M_n \{\bar{F}_i^- (\Phi(W_i R))\} \leq s \right)$$

$$= \Pr(R > L^Y(W, s), R \geq M_2 \left\{ \frac{\Phi^- (\bar{F}_i(\lambda s))}{W_i} \right\}, R \leq U^Y(W, s)).$$

The estimator $Z_{NR3,2}^Y(s)$ is hence given by

$$Z_{NR3,2}^Y(s) = F_R (U^Y(W, s)) - F_R \left( L^Y(W, s) \lor M_2 \left\{ \frac{\Phi^- (\bar{F}_i(\lambda s))}{W_i} \right\} \right)$$

$$= F_R (U^Y(W, s)) - F_R (L^Y(W, s))$$

with $U^Y(W, s)$, $L^Y(W, s)$ as given in (5), (6) respectively, and

$$L^Y(W, s) = L^Y(W, s) \lor M_2 \left\{ \frac{\Phi^- (\bar{F}_i(\lambda s))}{W_i} \right\}. \quad (8)$$
3.3.3 Estimators for \( z_X(s) \) and \( z_Y(s) \)

The third estimators for \( z_X(s) \) and \( z_Y(s) \) are finally given by

\[
Z_{N^{R3}}^X(s) = \Pr(M_n^X > s) + Z_{N^{R3},1}^X(s) + Z_{N^{R3},2}^X(s) \quad \text{and} \quad Z_{N^{R3}}^Y(s) = \Pr(M_n^Y > s) + Z_{N^{R3},1}^Y(s) + Z_{N^{R3},2}^Y(s).
\]

**Algorithm 16** To generate a realization of \( Z_{N^{R3}}^X(s) \), proceed as follows:

1. Let \((E_1, \ldots, E_n)\) be \( n \) iid exponential rvs with parameter 1. Calculate \( W_i = E_i / \sum_{j=1}^n E_j \).
2. Compute \( U_i^X(W, s) \) from (7) and \( L_i^X(W, s) \) from (4).
3. For \( i = 1, \ldots, n \), simulate vector \( U_{-i} \) following \((n - 1)\) dimensional Archimedean copula of generator \( \Phi \).
   (a) Evaluate \( X_j = F_j^\Phi(U_j) \) for \( j \neq i \).
   (b) Evaluate \( Z_{N^{R3},1}^X(s) = I_{\{\max(x_{-i}) < \lambda s\}} \left(F_X^\star(s) - F_X^\star(s - \text{sum}(x_{-i}))\right) \).
   (c) Return \( Z_{N^{R3}}^X(s) = \Pr(M_n^X > s) + FR(U_X(W, s)) - FR(L_X(W, s)) + \sum_{i=1}^n Z_{N^{R3},1}^X(s) \).

**Algorithm 17** To generate a realization of \( Z_{N^{R3}}^Y(s) \), proceed as follows:

1. Let \((E_1, \ldots, E_n)\) be \( n \) iid exponential rvs of parameter 1, calculate \( W_i = E_i / \sum_{j=1}^n E_j \).
2. Calculate \( U_i^Y(W, s) \) from (5) and \( L_i^Y(W, s) \) from (8).
3. For \( i = 1, \ldots, n \), simulate vector \( U_{-i} \) following \((n - 1)\) dimensional Archimedean copula of generator \( \Phi \) and then calculate \( Y_j = F_j^\Phi(U_j) \) for \( j \neq i \). After that, calculate the value of \( Z_{N^{R3},1}^Y(s) = I_{\{\max(y_{-i}) < \lambda s\}} \left(F_Y^\star(s) - F_Y^\star(s - \text{sum}(y_{-i}))\right) \).
4. Return \( Z_{N^{R3}}^Y(s) = P(M_n^Y > s) + FR(U_Y(W, s)) - FR(L_Y(W, s)) + \sum_{i=1}^n Z_{N^{R3},1}^Y(s) \).

Unfortunately, the relative errors of \( Z_{N^{R3},1}^X(s) \) and \( Z_{N^{R3},1}^X(s) \) are not bounded if no assumption is made on the Archimedean generator. Consequently, the relative errors of \( Z_{N^{R3}}^X(s) \) and \( Z_{N^{R3}}^X(s) \) will not be bounded either in general. However, numerical performances of these estimators are better than \( Z_{N^{R2}}(s) \) in some situations when parameter \( \lambda \) takes appropriate values. Moreover, in almost all cases, \( Z_{N^{R3}}^X(s) \) and \( Z_{N^{R3}}^X(s) \) perform better than \( Z_{N^{R1}}^X(s) \) and \( Z_{N^{R1}}^Y(s) \) which we have proven to have a bounded relative error.

However we are able to prove the following result.

**Proposition 18** The estimator \( Z_{N^{R3},2}^Y(s) \) has bounded relative error.

**Proof.** See Appendix. ■
3.3.4 Fourth estimator

We propose in this section a fourth and final estimator of \( z_X(s) \) which is derived in a similar fashion as \( Z_{N R 3}^X(s) \), meaning that we split \( \Pr(S_n^X > s, M_n^X \leq s) \) into two parts. Note that the estimator under an Archimedean survival copula structure, denoted by \( Z_{Y 4}^Y(s) \) has bounded relative error without any assumption on \( \Phi \).

First, for a chosen \( \kappa \in (1/n, 1) \), we decompose \( z_X(s) \) into

\[
z_X(s) = \Pr(M_n^X > s) + \sum_{i=1}^{n} (\bar{F}_i(\kappa s) - \bar{F}_i(s)) \Pr(S_n^X > s, X_i = M_n^X | \kappa s < X_i \leq s)
+ \Pr \left( \sum_{i=1}^{n} \bar{F}_i^{-1}(\Phi(RW_i)) > s, R \geq M_n \left\{ \frac{\Phi^{-1}(F_i(\kappa s))}{W_i} \right\} \right).
\]

Following the same rationale as for the first and second estimator, we obtain the fourth estimator \( Z_{N R 4}^X(s) \) given by

\[
Z_{N R 4}^X(s) = \Pr(M_n^X > s) + \sum_{i=1}^{n} (\bar{F}_i(\kappa s) - \bar{F}_i(s)) \mathbb{I}_{\{S_n^X > s, X_i = M_n^X | \kappa s < X_i \leq s\}} + F_R(U_X(W, s)) - F_R(L_X^{\kappa}(W, s)),
\]

with \( X_j^{\kappa i} = (X_j | \kappa s < X_i \leq s; U_X(W, s)) \) as defined in (3) and

\[
L_X^{\kappa}(W, s) = M_n \left\{ \frac{\Phi^{-1}(F_i(\kappa s))}{W_i} \right\}.
\]

Similarly, for random vector \( Y \), we have

\[
z_Y(s) = \Pr(M_n^Y > s) + \sum_{i=1}^{n} (\bar{F}_i(\kappa s) - \bar{F}_i(s)) \Pr(S_n^Y > s, Y_i = M_n^Y | \kappa s < Y_i \leq s)
+ \Pr \left( \sum_{i=1}^{n} \bar{F}_i^{-1}(\Phi(RW_i)) > s, R \leq M_1 \left\{ \frac{\Phi^{-1}(F_i(\kappa s))}{W_i} \right\} \right)
\]

which leads to

\[
Z_{N R 4}^Y(s) = \Pr(M_n^Y > s) + \sum_{i=1}^{n} (\bar{F}_i(\kappa s) - \bar{F}_i(s)) \mathbb{I}_{\{S_n^Y > s, Y_i = M_n^Y | \kappa s \leq s\}} + F_R(U_Y(W, s)) - F_R(L_Y(W, s))
\]

as defined in (3) and

\[
L_Y^{\kappa}(W, s) = M_1 \left\{ \frac{\Phi^{-1}(F_i(\kappa s))}{W_i} \right\}.
\]
with $Y_{kj}^i = (Y_j|\kappa s < Y_i \leq s; L^Y(W, s))$ as defined in (6) and

$$U^Y_\kappa(W, s) = M_1 \left\{ \frac{\Phi_{\kappa}(F_i(\kappa s))}{W_i} \right\}. \quad (10)$$

We are able to prove that $Z^Y_{NR4}(s)$ is an estimator with bounded relative error.

**Algorithm 19** To generate a realization of $Z^X_{NR4}(s)$, proceed as follows:

1. For each $i = 1, \ldots, n$, simulate vector $(X_{\kappa i}^1, \ldots, X_{\kappa i}^n) = (X_1, \ldots, X_n|\kappa s < X_i \leq s)$, then calculate $Z^X_{NR4,1}(s) = \sum_{i=1}^n (\overline{F}_i(\kappa s) - F_i(s)) I_{\{S^X_{\kappa i} > s, X_{\kappa i}^1 = M^X_{\kappa i} = X_{\kappa i}^s\}}$.
2. Let $(E_1, \ldots, E_n)$ be $n$ iid exponential rvs of parameter 1, calculate $W_i = E_i / \sum_{j=1}^n E_j$.
3. Calculate $U^X(W, s)$ from (3) and $L^X_\kappa(W, s)$ from (9).
4. Return $Z^X_{NR4}(s) = \Pr(M^X_n > s) + Z^X_{NR4,1}(s) + \overline{F}_R(U^X(W, s)) - \overline{F}_R(L^X_\kappa(W, s))$.

**Algorithm 20** To generate a realization of $Z^Y_{NR4}(s)$, proceed as follows:

1. For each $i = 1, \ldots, n$, simulate vector $(Y_{\kappa i}^1, \ldots, Y_{\kappa i}^n) = (Y_1, \ldots, Y_n|\kappa s < Y_i \leq s)$, then calculate $Z^Y_{NR4,1}(s) = \sum_{i=1}^n (\overline{F}_i(\kappa s) - F_i(s)) I_{\{S^Y_{\kappa i} > s, Y_{\kappa i}^1 = M^Y_{\kappa i} = Y_{\kappa i}^s\}}$.
2. Let $(E_1, \ldots, E_n)$ be $n$ iid exponential rvs of parameter 1, calculate $W_i = E_i / \sum_{j=1}^n E_j$.
3. Calculate $U^Y_\kappa(W, s)$ from (10) and $L^Y(W, s)$ from (6).
4. Return $Z^Y_{NR4}(s) = \Pr(M^Y_n > s) + Z^Y_{NR4,1}(s) + \overline{F}_R(U^Y_\kappa(W, s)) - \overline{F}_R(L^Y(W, s))$.

**Proposition 21** $Z^Y_{NR4}(s)$ is an estimator with bounded relative error.

**Proof.** See Appendix. □

4 Numerical bounds for $z^X(s)$ and $z^Y(s)$

Inspired from the AEP algorithm in [Arbenz et al., 2011], [Cossette et al., 2014] have proposed sharp numerical bounds for $\Pr(S^X_n \leq s)$ when a closed-form expression is available for $\Pr(X_1 \leq x_1, \ldots, X_n \leq x_n)$. These bounds are recalled in a first subsection. In the next subsection, we propose an adaptation of this method for $\Pr(S^Y_n > s)$ assuming that a closed-form expression is available for $\Pr(Y_1 > y_1, \ldots, Y_n > y_n)$. 

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4.1 Numerical bounds for $z_X(s)$

Let us denote by $A_S^{(l,m)}(s)$ and $A_S^{(u,m)}(s)$ the bounds for $\Pr(S^n_X \leq s)$ with precision parameter $m \in \mathbb{N}^+$, such that

$$A_S^{(l,m)}(s) \leq \Pr(S^n_X \leq s) \leq A_S^{(u,m)}(s), \quad x \geq 0.$$

Briefly, for $n = 2$, $A_S^{(l,m)}(s)$ corresponds to the sum of the probabilities associated to $2^m - 1$ rectangles which lie strictly under the diagonal $x_1 + x_2 = s$, i.e.

$$A_S^{(l,m)}(s) = \sum_{i=1}^{2^m-1} \left( FX \left( \frac{i}{2^m s}, \frac{2^m - i}{2^m s} \right) - FX \left( \frac{(i-1)}{2^m s}, \frac{2^m - (i-1)}{2^m s} \right) \right). \quad (11)$$

Similarly, for $n = 2$, $A_S^{(u,m)}(s)$ is the sum of the probabilities associated to the $2^m$ rectangles strictly above the diagonal $x_1 + x_2 = s$, i.e.

$$A_S^{(u,m)}(s) = \sum_{i=1}^{2^m} \left( FX \left( \frac{i}{2^m s}, \frac{2^m + 1 - i}{2^m s} \right) - FX \left( \frac{(i-1)}{2^m s}, \frac{2^m + 1 - (i-1)}{2^m s} \right) \right). \quad (12)$$

For $n = 3$, the lower bound is given by

$$A_S^{(l,m)}(s) = \sum_{i_1=1}^{3^m - 2 \cdot 3^m - 1 - i_1} \sum_{i_2=1}^{3^m - i_1} \zeta_X^{(l,m)}(s; i_1, i_2), \quad (13)$$

where

$$\zeta_X^{(l,m)}(s; i_1, i_2) = \Pr \left( \frac{i_1 - 1}{3^m s} < X_1 \leq \frac{i_1}{3^m s}, \frac{i_2 - 1}{3^m s} < X_2 \leq \frac{i_2}{3^m s}, X_3 \leq \frac{3^m - i_1 - i_2}{3^m s} \right)$$

$$= FX \left( \frac{i_1}{3^m s}, \frac{i_2}{3^m s}, \frac{3^m - i_1 - i_2}{3^m s} \right) - FX \left( \frac{i_1 - 1}{3^m s}, \frac{i_2}{3^m s}, \frac{3^m - i_1 - i_2}{3^m s} \right)$$

$$- FX \left( \frac{i_1}{3^m s}, \frac{i_2 - 1}{3^m s}, \frac{3^m - i_1 - i_2}{3^m s} \right) + FX \left( \frac{i_1 - 1}{3^m s}, \frac{i_2 - 1}{3^m s}, \frac{3^m - i_1 - i_2}{3^m s} \right),$$

for $i_1 = 1, \ldots, 3^m - 2$ and $i_2 = 1, \ldots, 3^m - 1 - i_1$. The upper bound is given by

$$A_S^{(u,m)}(s) = \sum_{i_1=1}^{3^m} \sum_{i_2=1}^{3^m + 1 - i_1} \zeta_X^{(u,m)}(s; i_1, i_2), \quad (14)$$
with

\[
\zeta_{X}^{(u,m)}(s; i_1, i_2) = \Pr \left( \frac{i_1 - 1}{3m}s < X_1 \leq \frac{i_1}{3m}s, \frac{i_2 - 1}{3m}s < X_2 \leq \frac{i_2}{3m}s, X_3 \leq \frac{3m + 2 - i_1 - i_2}{3m}s \right) \\
= F_X \left( \frac{i_1}{3m}s, \frac{i_2}{3m}s, \frac{3m + 2 - i_1 - i_2}{3m}s \right) - F_X \left( \frac{i_1 - 1}{3m}s, \frac{i_2}{3m}s, \frac{3m + 2 - i_1 - i_2}{3m}s \right) \\
- F_X \left( \frac{i_1}{3m}s, \frac{i_2 - 1}{3m}s, \frac{3m + 2 - i_1 - i_2}{3m}s \right) + F_X \left( \frac{i_1 - 1}{3m}s, \frac{i_2 - 1}{3m}s, \frac{3m + 2 - i_1 - i_2}{3m}s \right)
\]

for \( i_1 = 1, ..., 3^m \) and \( i_2 = 1, ..., 3^m + 1 - i_1 \). Details for \( n > 3 \) are provided in [Cossette et al., 2014].

### 4.2 Numerical bounds for \( z_Y(s) \)

In this section, we propose an adaptation of this method assuming that a closed-form expression for the survival distribution function of the random vector \( Y \) is available. Our objective is to develop sharp numerical bounds, denoted \( B_{S}^{(l,m)}(s) \) and \( B_{S}^{(u,m)}(s) \), such that

\[
B_{S}^{(l,m)}(s) \leq z_Y(s) \leq B_{S}^{(u,m)}(s), \ s \geq 0.
\]

Clearly, we have

\[
B_{S}^{(l,m)}(s) = 1 - A_{S}^{(u,m)}(s) \quad \text{and} \quad B_{S}^{(u,m)}(s) = 1 - A_{S}^{(l,m)}(s), \ s \geq 0.
\]

However, to achieve our goal, the task is to rewrite expressions in (11) and (13) for \( A_{S}^{(u,m)}(s) \), and (12) and (14) for \( A_{S}^{(l,m)}(s) \) such that \( B_{S}^{(l,m)}(s) \) and \( B_{S}^{(u,m)}(s) \) can be defined in terms of \( F_Y \). We provide expressions of the lower and upper bounds for \( n = 2 \) and \( n = 3 \).

For \( n = 2 \), we have

\[
B_{S}^{(u,m)}(s) = \overline{A}_{S}^{(l,m)}(s) = 1 - A_{S}^{(l,m)}(s) \\
= \sum_{i=1}^{2^m - 1} \left( F_Y \left( \frac{i - 1}{2^m}s, \frac{2^m - i}{2^m}s \right) - F_Y \left( \frac{i}{2^m}s, \frac{2^m - i}{2^m}s \right) \right) \\
+ F_Y \left( \frac{2^m - 1}{2^m}s, 0 \right),
\]

and

\[
B_{S}^{(l,m)}(s) = \overline{A}_{S}^{(u,m)}(s) = 1 - A_{S}^{(u,m)}(s) \\
= \sum_{i=1}^{2^m} \left( F_Y \left( \frac{i - 1}{2^m}s, \frac{2^m + 1 - i}{2^m}s \right) - F_Y \left( \frac{i}{2^m}s, \frac{2^m + 1 - i}{2^m}s \right) \right) + F_Y \left( \frac{2^m}{2^m}s, 0 \right).
\]
For \( n = 3 \), we obtain

\[
B_S^{(u,m)}(s) = A_S^{(u,m)}(s) = 1 - A_S^{(l,m)}(s) \\
= \sum_{i_1=1}^{3^m} \sum_{i_2=1}^{3^m-1-i_1} \left( -F_Y \left( \frac{i_1-1}{3^m} s, \frac{i_1}{3^m} s, \frac{3^m-1-i_2}{3^m} s \right) + F_Y \left( \frac{i_1-1}{3^m} s, \frac{i_2}{3^m} s, \frac{3^m-i_1-i_2}{3^m} s \right) \\ + \sum_{i_1=1}^{3^m-2} \left( F_Y \left( \frac{i_1-1}{3^m} s, \frac{3^m-1-i_1}{3^m} s, 0 \right) - F_Y \left( \frac{i_1}{3^m} s, \frac{3^m-1-i_1}{3^m} s, 0 \right) \right) \\ + F_Y \left( \frac{3^m-2}{3^m} s, 0, 0 \right) \right).
\]

and

\[
B_S^{(l,m)}(s) = A_S^{(u,m)}(s) = 1 - A_S^{(u,m)}(s) \\
= \sum_{i_1=1}^{3^m} \sum_{i_2=1}^{3^m-1-i_1} \left( -F_Y \left( \frac{i_1-1}{3^m} s, \frac{i_2}{3^m} s, \frac{3^m+2-i_1-i_2}{3^m} s \right) + F_Y \left( \frac{i_1}{3^m} s, \frac{i_2}{3^m} s, \frac{3^m+2-i_1-i_2}{3^m} s \right) \\ + \sum_{i_1=1}^{3^m} \left( F_Y \left( \frac{i_1-1}{3^m} s, \frac{3^m+1-i_1}{3^m} s, 0 \right) - F_Y \left( \frac{i_1}{3^m} s, \frac{3^m+1-i_1}{3^m} s, 0 \right) \right) \\ + F_Y \left( \frac{3^m}{3^m} s, 0, 0 \right) \right).
\]

Expressions for lower and upper bounds for \( n > 3 \) can be derived in a similar way.

## 5 Numerical study

The numerical performances of the four estimators and the numerical bounds are discussed in this section. We shall first compare both approaches by considering small \( n = 2, 3 \) and from moderate to large \( s \). We then study the accuracy of the four estimators for the case \( n = 5 \) where the numerical bounds may not be computed in a reasonable time.

For both \( \mathbf{X} \) and \( \mathbf{Y} \), we assume that the marginal distributions are Pareto(\( \alpha_i, 1 \)), i.e. \( f_{X_i}(x) = \alpha_i/(1 + x)^{\alpha_i+1} \) for \( x > 0 \). For the dependence structure, we shall consider a Clayton or Gumbel copula.

The generator of the Clayton copula of parameter \( \theta \in (0, \infty) \) and its inverse function are given by

\[
\Phi(t) = \left( 1 + \frac{t}{\theta} \right)^{-\theta} \quad \text{and} \quad \Phi^{-1}(t) = \theta \left( t^{-1/\theta} - 1 \right).
\]

The derivatives of the generator are calculated as follows

\[
\Phi^{(k)}(t) = \left( 1 + \frac{1}{\theta} \right) \left( 1 + \frac{2}{\theta} \right) \ldots \left( 1 + \frac{k-1}{\theta} \right) \left( 1 + \frac{k}{\theta} \right)^{-\theta-k+1}.
\]
The formula for the $n$-dimensional copula is

$$C(u_1, \ldots, u_n) = \left( u_1^{-1/\theta} + \ldots + u_n^{-1/\theta} - (n - 1) \right)^{-\theta}.$$ 

Its Kendall’s tau is given by $\tau = \theta^{-1}/(2 + \theta^{-1})$. Note that the Clayton copula has a generator satisfying the assumptions of Proposition 14.

The generator of the Gumbel copula with parameter $b \in (0, 1)$ and its inverse function are given by

$$\Phi(t) = \exp(-x^b) \text{ and } \Phi^{-1}(t) = (-\log(t))^{1/b}.$$ 

The four derivatives of the generator are calculated as follows

$$\Phi^{(1)}(t) = \exp(-x^b) \left(-b t^{b-1} \right),$$

$$\Phi^{(2)}(t) = \exp(-t^b) \left(-b (b-1) t^{b-2} + b^2 t^{2b-2} \right),$$

$$\Phi^{(3)}(t) = \exp(-t^b) \left(-b (b-1)(b-2) t^{b-3} + 3b^2 (b-1) t^{2b-3} - b^3 t^{3b-3} \right),$$

$$\Phi^{(4)}(t) = \exp(-t^b) \left(-b (b-1)(b-2)(b-3) t^{b-4} + b^2 (b-1)(7b-11) t^{2b-4} - 6b^3 (b-1) t^{3b-4} + b^4 t^{4b-4} \right).$$

The formula for the $n$-dimensional Gumbel copula is

$$C(u_1, \ldots, u_n) = \exp \left( - \left[ (-\log(u_1))^{1/b} + \ldots + (-\log(u_n))^{1/b} \right]^{b} \right).$$

Its Kendall’s tau is given $\tau = 1 - b$.

### 5.1 Comparison of both approaches

#### 5.1.1 Numerical illustration for $z_X$

In the first example, Tables 1 and 2 provide the values of the four estimators and the numerical bounds of $z_X(s)$, for $n = 2, 3$, and $s = 1, 10^2, 10^4$, and $10^6$. The parameters of the Pareto distributions are given by $\alpha_1 = 0.9$, $\alpha_2 = 1.8$, $\alpha_3 = 2.6$, which come from Section 6 in [Arbenz et al., 2011] and Section 3.1 in [Cossette et al., 2014]. Note that, since the parameters of the Pareto distributions are different, the probability $z_X(s)$ is equivalent to $\Pr(X_1 > s)$ for large $s$ (e.g., $\Pr(X_1 > 10^6) = 3.9811E-06$).

#### 5.1.2 Numerical illustration for $z_Y$

For the second example, the values of the four estimators and the numerical bounds of $z_Y(s)$, for $n = 2, 3$, and $s = 1, 10^2, 10^3$, and $10^4$ are displayed in tables 3 and 4. The parameters of the
Table 1: Four estimators and numerical bounds of $z_X(s)$. Sum of two Pareto whose Clayton copula has Kendall’s $\tau$ equal to $\frac{3}{8}$.

| $s$  | $1 - A_S^{(u, 20)}(s)$ | $1 - A_S^{(l, 20)}(s)$ | $E(Z_{NR1})$ | $E(Z_{NR2})$ | $E(Z_{NR3})$ | $E(Z_{NR4})$ |
|------|------------------------|------------------------|-------------|-------------|-------------|-------------|
| 1    | 6.84165E-01           | 6.84165E-01           | 6.83859E-01 | 6.84258E-01 | 6.77340E-01 | 6.84340E-01 |
| 1E02 | 1.63096E-02           | 1.63096E-02           | 1.63378E-02 | 1.63088E-02 | 1.63853E-02 | 1.62861E-02 |
| 1E04 | 2.5128E-04            | 2.5128E-04            | 2.5125E-04  | 2.5129E-04  | 2.5125E-04  | 2.5125E-04  |
| 1E06 | 3.9811E-06            | 3.9811E-06            | 3.9811E-06  | 3.9811E-06  | 3.9811E-06  | 3.9811E-06  |

Table 2: Four estimators and numerical bounds of $z_X(s)$. Sum of three Pareto whose Clayton copula has Kendall’s $\tau$ equal to $\frac{1}{6}$.

| $s$  | $1 - A_S^{(u, 8)}(s)$ | $1 - A_S^{(l, 8)}(s)$ | $E(Z_{NR1})$ | $E(Z_{NR2})$ | $E(Z_{NR3})$ | $E(Z_{NR4})$ |
|------|------------------------|------------------------|-------------|-------------|-------------|-------------|
| 1    | 8.09108E-01           | 8.09173E-01           | 8.08747E-01 | 8.07925E-01 | 8.05646E-01 | 8.10322E-01 |
| 1E02 | 1.63381E-02           | 1.63428E-02           | 1.63361E-02 | 1.63411E-02 | 1.63800E-02 | 1.63198E-02 |
| 1E04 | 2.5128E-04            | 2.5128E-04            | 2.5127E-04  | 2.5129E-04  | 2.5129E-04  | 2.5127E-04  |
| 1E06 | 3.9811E-06            | 3.9811E-06            | 3.9811E-06  | 3.9811E-06  | 3.9811E-06  | 3.9811E-06  |

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Table 3: Four estimators and numerical bounds of $z_Y(s)$. Sum of two Pareto whose Clayton survival copula has Kendall’s $\tau$ equal to $\frac{1}{2}$.

| $s$  | $B_S^{(l,20)}(s)$ | $B_S^{(u,20)}(s)$ | $E(Z_{NR1})_{e(Z_{NR1})}$ | $E(Z_{NR2})_{e(Z_{NR2})}$ | $E(Z_{NR3})_{e(Z_{NR3})}$ | $E(Z_{NR4})_{e(Z_{NR4})}$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1    | 3.60712E-01    | 3.60712E-01    | 3.61435E-01    | 3.61003E-01    | 3.60885E-01    | 3.60812E-01    |
| 1E02 | 5.14701E-05    | 5.14702E-05    | 5.15248E-05    | 5.14659E-05    | 5.13841E-05    | 5.13841E-05    |
| 1E03 | 1.70171E-07    | 1.70172E-07    | 1.71695E-07    | 1.70997E-07    | 1.69923E-07    | 1.69923E-07    |
| 1E04 | 5.40553E-10    | 5.40554E-10    | 5.40001E-10    | 5.40002E-10    | 5.37363E-10    | 5.37363E-10    |

Table 4: Four estimators and numerical bounds of $z_Y(s)$. Sum of three Pareto whose Clayton survival copula has Kendall’s $\tau$ equal to $\frac{1}{2}$.

| $s$  | $B_S^{(l,20)}(s)$ | $B_S^{(u,20)}(s)$ | $E(Z_{NR1})_{e(Z_{NR1})}$ | $E(Z_{NR2})_{e(Z_{NR2})}$ | $E(Z_{NR3})_{e(Z_{NR3})}$ | $E(Z_{NR4})_{e(Z_{NR4})}$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1    | 4.99666E-01    | 5.00644E-01    | 5.00644E-01    | 5.00006E-01    | 5.00477E-01    | 5.00477E-01    |
| 1E02 | 1.35825E-04    | 1.36732E-04    | 1.34754E-04    | 1.36570E-04    | 1.35966E-04    | 1.35966E-04    |
| 1E03 | 4.58967E-07    | 4.62116E-07    | 4.61180E-07    | 4.60699E-07    | 4.60590E-07    | 4.60590E-07    |
| 1E04 | 1.46118E-09    | 1.47123E-09    | 1.47018E-09    | 1.46630E-09    | 1.46335E-09    | 1.46335E-09    |

Pareto distributions are equal, with $\alpha_1 = \alpha_2 = \alpha_3 = 2.5$. In this case, all components of the sum contribute to its large values.

### 5.1.3 Comments

For both random vectors $X$ and $Y$, the upper and lower bounds have been computed with the R Project for Statistical Computing. Computation time is rather fast and varies in function of the number of random variables $n$ and the precision parameter $m$. The evaluation of these bounds becomes time consuming starting at $n = 4$ contrarily to the conditional Monte Carlo estimators which can be rapidly obtained no matter the dimension $n$. For $n$ relatively small ($n = 2, 3$), the lower and upper bounds are close for any value of $s$. One can see with the results of both examples that the four conditional Monte Carlo estimators can produce values outside of the lower and upper bounds. Both methods are complementary in the sense that one would probably be more inclined to use the numerical bounds in small dimension and the conditional Monte Carlo method for $n \geq 5$. 

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5.2 Comparison of the four estimators when $n = 5$

We now compare the performance of our four estimators for $z_X$ and $z_Y$. We assume that the Pareto parameters are equal with $\alpha_1 = \ldots = \alpha_5 = 2.5$. The dependence is assumed to be defined by a Clayton copula, a Gumbel copula, a Clayton survival copula and a Gumbel survival copula. Using Kendall’s $\tau$, we study three levels of dependence. The weak level of dependence is when $\tau = 0.1$, the intermediate level of dependence is when $\tau = 0.5$ and the strong level of dependence is when $\tau = 0.9$.

For estimators $Z_{NR3}(s)$ and $Z_{NR4}(s)$, the choices of $\lambda$ and $\kappa$ are sensitive. In fact, we choose the values that minimize the numerical standard deviations of the estimators.
According to the numerical results, it is remarkable that $Z_{NR1}$ has bounded relative error. For example, under the assumption that the dependence is a Clayton survival copula with Kendall’s $\tau$ equal to 0.5, when $s$ increases from 20 (Table 5) to 200 (Table 6), the value of $z(s)$ decreases from 0.01639236 to 8.67011E-05, but the relative error of $Z_{NR1}$ does not change: 2.034 compared to 2.036.

Although $Z_{NR1}(s)$ is proved to have a bounded relative error under any dependence structure, the numerical performances of this estimator is not better than $Z_{NR2}(s)$. Note that $Z_{NR2}$ has bounded relative error only when the dependence structure is an Archimedean survival copula of generator $\Phi(x) = x^{-\beta} I_\Phi(x)$, that is the case of Clayton survival copula in this section. However, except the case of Clayton copula, $Z_{NR2}$ presents acceptable results in most cases. For example, in Table 5 and $\tau = 0.5$, under Gumbel survival copula, ratio $e(Z_{NR1})/e(Z_{NR2})$ equals to 37; or in Table 6 $\tau = 0.9$, this ratio under Gumbel copula is approximated to 15.

The construction of $Z_{NR3}$ is more complex than that of $Z_{NR2}$; however, the third estimator has no numerical improvement compared to the second one except for the case of Clayton copula. Indeed, in Table 6 and $\tau = 0.1$, the relative error of $Z_{NR3}$ is 0.480 while the relative error of $Z_{NR2}$ is 3.396 or in Table 6 and $\tau = 0.5$, the relative error of $Z_{NR3}$ is 0.182 while the relative error of $Z_{NR2}$ is 2.882. Under the other dependence structures, the relative errors of $Z_{NR3}$ and $Z_{NR2}$ are almost the same.

The fourth estimator has bounded relative error under Archimedean survival copula and it presents favorable numerical results even when the dependence structure is an Archimedean copula. For example, $Z_{NR4}$ has the smallest relative error under Clayton copula in all tables. Under Gumbel copula, except Table 5 where $s = 20$ and Kendall’s $\tau = 0.9$ or Table 6 where $s = 200$ and Kendall’s $\tau = 0.9$, $Z_{NR4}$ also has the smallest relative error. Under Archimedean survival copulas, there is not much difference between the relative errors of $Z_{NR2}$, $Z_{NR3}$ and $Z_{NR4}$.

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| Copulas     | E(Z_{NR1}) | E(Z_{NR2}) | E(Z_{NR3}) | E(Z_{NR4}) | e(Z_{NR1}) | e(Z_{NR2}) | e(Z_{NR3}) | e(Z_{NR4}) | s = 200, Kendall’s τ = 0.1 |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|----------------------------|
| Clayton    | 9.27623E-06 | 9.10775E-06 | 9.07197E-06 | 9.07119E-06 | 1.701      | 3.396      | 0.480      | 0.270      |
| Gumbel     | 3.19991E-05 | 3.16144E-05 | 3.16638E-05 | 3.13636E-05 | 3.216      | 0.752      | 0.626      | 0.584      |
| Survival Clayton | 2.1843E-05 | 2.18073E-05 | 2.17775E-05 | 2.17643E-05 | 3.582      | 0.130      | 0.165      | 0.145      |
| Survival Gumbel | 9.18493E-06 | 9.2342E-06 | 9.22524E-06 | 9.22872E-06 | 1.568      | 0.130      | 0.070      | 0.112      |
| s = 200, Kendall’s τ = 0.5 |
| Clayton    | 9.54965E-06 | 9.24395E-06 | 9.37001E-06 | 9.37181E-06 | 2.023      | 2.882      | 0.182      | 0.120      |
| Gumbel     | 7.84723E-05 | 7.9696E-05  | 7.91973E-05 | 7.96030E-05 | 2.148      | 0.273      | 0.240      | 0.230      |
| Survival Clayton | 8.67011E-05 | 8.63854E-05 | 8.60928E-05 | 8.61954E-05 | 2.036      | 0.111      | 0.113      | 0.133      |
| Survival Gumbel | 1.49943E-05 | 1.53712E-05 | 1.53317E-05 | 1.53699E-05 | 3.559      | 0.263      | 0.274      | 0.179      |
| s = 200, Kendall’s τ = 0.9 |
| Clayton    | 1.0871E-05  | 1.1563E-05  | 1.03202E-05 | 1.07798E-05 | 2.867      | 6.739      | 1.077      | 0.818      |
| Gumbel     | 9.18427E-05 | 1.09482E-04 | 1.07552E-04 | 1.09196E-04 | 1.973      | 0.134      | 0.096      | 0.127      |
| Survival Clayton | 1.14134E-04 | 9.27657E-05 | 9.27742E-05 | 9.27769E-05 | 1.721      | 0.017      | 0.017      | 0.015      |
| Survival Gumbel | 7.74801E-05 | 7.93332E-05 | 7.87621E-05 | 7.94657E-05 | 2.155      | 0.136      | 0.198      | 0.139      |

Table 6: Four estimators of $z_X(s)$ and $z_Y(s)$. Sum of five Pareto with $s = 200$. 

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7 Appendix

7.1 Proof of Proposition 3

From the conditional cumulative distribution function $F_{U_j|Z,U_{j-1},...,U_1}(u_j|z,u_{j-1},\ldots,u_1)$ with $j = 1$, we derive the conditional cumulative distribution function $F_{U_1|Z}$

$$F_{U_1|Z}(u_1|z) = \left(1 - \frac{\Phi^{-}(u_1)}{\Phi^{-}(z)}\right)^{n-1}$$

for $z < u_1 < 1$. Because the marginal density of $U_1$ is 1 on (0,1), with the density of $Z$ in Proposition 2, we have the conditional density of $Z|U_1$

$$f_{Z|U_1}(z|u_1) = \frac{(\Phi^{+})^{(1)}(u_1)}{(n-2)!} (\Phi^{-}(u_1) - \Phi^{-}(z))^{n-2} (\Phi^{+})^{(1)}(z) \Phi^{(n)}(\Phi^{-}(z))$$

for $0 < z < u_1$. The cumulative distribution function of $Z$ is obtained as follows:

$$F_{Z|U_1}(z|u_1) = (\Phi^{+})^{(1)}(u_1) \frac{(-1)^{n-2}}{(n-2)!} \int_{0}^{z} (\Phi^{-}(v) - \Phi^{-}(u_1))^{n-2} (\Phi^{+})^{(1)}(v) \Phi^{(n)}(\Phi^{-}(v)) \, dv$$

$$= - (\Phi^{+})^{(1)}(u_1) \frac{(-1)^{n-2}}{(n-2)!} \int_{\Phi^{-}(z)}^{\infty} (v - \Phi^{-}(u_1))^{n-2} \Phi^{(n)}(v) \, dv$$

$$= - (\Phi^{+})^{(1)}(u_1) \frac{(-1)^{n-2}}{(n-2)!} \int_{\Phi^{-}(z)}^{\infty} (v - \Phi^{-}(u_1))^{n-2} \Phi^{(n-1)}(v) \, dv$$

$$= - (\Phi^{+})^{(1)}(u_1) \frac{(-1)^{n-2}}{(n-2)!} \int_{\Phi^{-}(z)}^{\infty} (v - \Phi^{-}(u_1))^{n-2} \Phi^{(n-1)}(v) \, dv$$

$$+ (\Phi^{+})^{(1)}(u_1) \frac{(-1)^{n-2}}{(n-2)!} \int_{\Phi^{-}(z)}^{\infty} (n-2) (v - \Phi^{-1}(u_1))^{n-3} \Phi^{(n-1)}(v) \, dv.$$
7.2 Proof of Proposition 11

\[ \text{Var}(Z_{NR1}^X(s)) = \text{Var} \left( \sum_{i=1}^{n} (\mathcal{F}_i(s/n) - \mathcal{F}_i(s)) \mathbb{I}_{\{S_{n}^{X_i > s}, X_i^{i} = M_n^{X_i}\}} \right) \]
\[ = \sum_{i=1}^{n} (\mathcal{F}_i(s/n) - \mathcal{F}_i(s))^2 \text{Var} \left( \mathbb{I}_{\{S_{n}^{X_i > s}, X_i^{i} = M_n^{X_i}\}} \right) \]
\[ \leq \sum_{i=1}^{n} (\mathcal{F}_i(s/n))^2 \]
\[ \sim \sum_{i=1}^{n} n^{2\alpha_i} (\mathcal{F}_i(s))^2 \]
\[ \leq \left( \sum_{i=1}^{n} n^{2\alpha_i} \right) (z(s))^2 \]

The variance of $Z_{NR1}^Y(s)$ can be verified similarly.

7.3 Proof of Proposition 14

Because $\Phi^{(n-2)}$ is differentiable, the survival distribution function of the radius $R$ becomes

\[ \mathcal{F}_R(x) = \sum_{j=0}^{n-1} (-1)^j \frac{x^j}{j!} \Phi^{(j)}(x). \]

Following the property of the regularly varying function $\Phi(x) = x^{-\beta} l_{\Phi}(x)$, we have, for $j = 1, \ldots, (n-1)$,

\[ \lim_{x \to \infty} \frac{(-1)^j x^j \Phi^{(j)}(x)}{\Phi(x)} = \beta(\beta + 1) \ldots (\beta + j - 1) \]

and we can deduce

\[ \lim_{x \to \infty} \frac{\mathcal{F}_R(x)}{\Phi(x)} = \lim_{x \to \infty} \frac{\sum_{j=1}^{n-1} (-1)^j \frac{x^j}{j!} \Phi^{(j)}(x)}{\Phi(x)} = \sum_{j=1}^{n-1} \frac{\beta(\beta + 1) \ldots (\beta + j - 1)}{j!}. \]

We define $g(r) = \sum_{i=1}^{n} \mathcal{F}_i^{-} (\Phi(r))$ and $L^Y_0(s) = \inf\{r \in \mathcal{R}^+ : g(r) \geq s\}$. Because $\mathcal{F}_i^{-}$ and $\Phi$ are both non-increasing functions then for all $W \in s_n$ we have

\[ g(r) = \sum_{i=1}^{n} \mathcal{F}_i^{-} (\Phi(r \times 1)) \geq \sum_{i=1}^{n} \mathcal{F}_i^{-} (\Phi(rW_i)) \]
and then \( L^Y_0(s) \leq L^Y(W, s) \) for all \( W \in \mathfrak{s}_n \). Moreover, from the definition of \( L^Y_0(s) \), we have

\[
\max_{i=1,2,\ldots,n} F_i^{-1}( \Phi(L^Y_0(s))) \geq s/n
\]

and

\[
\Phi(L^Y_0(s)) \leq \max_{i=1,2,\ldots,n} F_i(s/n) \leq n^{\alpha_n} \max_{i=1,2,\ldots,n} F_i(s) \leq n^{\alpha_n} z(s).
\]

Thus, with \( \lim_{s \to \infty} L^Y_0(s) = \infty \), the second moment of \( Z^Y_{NR2}(s) \) is bounded in the following way

\[
\mathbb{E} \left[ (Z^Y_{NR2}(s))^2 \right] \leq 2 \left( \left( \Pr(M^Y_n > s) \right)^2 + \mathbb{E} \left[ (F_R(L^Y(W, s))^2) \right] \right)
\]

\[
\leq 2 \left( \left( \Pr(M^Y_n > s) \right)^2 + \left( F_R(L^Y_0(s))^2 \right) \right)
\]

\[
\sim 2 \left( \left( \Pr(M^Y_n > s) \right)^2 + \left( \sum_{j=1}^{n-1} \frac{\beta(\beta+1)\ldots(\beta+j-1)}{j!} \right)^2 \left( \Phi(L^Y_0(s)) \right)^2 \right)
\]

\[
\leq 2 \left( (z(s))^2 + \left( \sum_{j=1}^{n-1} \frac{\beta(\beta+1)\ldots(\beta+j-1)}{j!} \right)^2 \right) \times n^{2\alpha_n} (z(s))^2
\]

\[
\leq 2 \left( 1 + n^{2\alpha_n} \left( \sum_{j=1}^{n-1} \frac{\beta(\beta+1)\ldots(\beta+j-1)}{j!} \right)^2 (z(s))^2 \right)
\]

and the result follows.

### 7.4 Proof of Proposition 18

We start with an inequality between \( \Phi \) and \( F_R \). If \( \Phi \) is an \( n \)-monotone function, \( (n-1) \)-times differentiable and the random variable \( R \) has the distribution function satisfies (7) then we have

\[
\frac{\Phi(ax)}{(1-a)^{(n-1)}} \geq F_R(x), \ \forall x \in \mathbb{R}^+ \text{ and } a \in (0,1).
\]

Indeed, because \( \Phi \) is non-increasing function then there exists \( \mu \in (ax, x) \) such that

\[
\Phi(ax) = \sum_{k=0}^{n-2} (1-a)^k \frac{x^k}{k!} (-1)^k \Phi^{(k)}(x) + (1-a)^{(n-1)} \frac{x^{(n-1)}}{(n-1)!} (-1)^{(n-1)} \Phi^{(n-1)}(\mu).
\]

Following the property of \( n \)-monotone functions, \( (-1)^{(n-2)} \Phi^{(n-2)}(x) \) is a convex function, and then \( (-1)^{(n-1)} \Phi^{(n-1)}(x) \) is a non-increasing function, that means \( (-1)^{(n-1)} \Phi^{(n-1)}(\mu) \geq (-1)^{(n-1)} \Phi^{(n-1)}(x) \) because \( \mu \leq x \). Thus we have

\[
\Phi(ax) \geq \sum_{k=0}^{n-1} (1-a)^k (-1)^k \frac{x^k}{k!} \Phi^{(k)}(x)
\]
and
\[ \frac{\Phi(ax)}{(1-a)^{(n-1)}} \geq \sum_{k=0}^{n-1} (1-a)^{(k-n+1)}(-1)^k \frac{x^k}{k!} \Phi^{(k)}(x) \geq \tilde{F}_R(x). \quad (15) \]

To prove Proposition 18, first note that \( Z_{NR3,2}^Y(s) \) is bounded by \( \tilde{F}_R \left( L_{\lambda}^Y(W, s) \right) \) since
\[ Z_{NR3,2}^Y(s) = F_R \left( U^Y(W, s) \right) - F_R \left( L_{\lambda}^Y(W, s) \right) \leq F_R \left( L_{\lambda}^Y(W, s) \right). \]

Moreover, from the definition of \( \tilde{F}_R \left( L_{\lambda}^Y(W, s) \right) \) and \( M_2 \{ \Phi^{-1}(\tilde{F}_i(\lambda s))/W_i \} \), there exists two indexes \( i_1, i_2 \in 1, 2, \ldots, n \) such that
\[ L_{\lambda}^Y(W, s) \geq M_2 \left\{ \frac{\Phi^{-1}(\tilde{F}_{i_1}(\lambda s))}{W_{i_1}} \right\} = \frac{\Phi^{-1}(\tilde{F}_{i_1}(\lambda s))}{W_{i_1}} \lor \frac{\Phi^{-1}(\tilde{F}_{i_2}(\lambda s))}{W_{i_2}}. \]

Therefore,
\[
\begin{align*}
W_{i_1} L_{\lambda}^Y(W, s) &\geq \Phi^{-1}(\tilde{F}_{i_1}(\lambda s)) \\
W_{i_2} L_{\lambda}^Y(W, s) &\geq \Phi^{-1}(\tilde{F}_{i_2}(\lambda s))
\end{align*}
\]

implies
\[
\begin{align*}
\Phi \left( W_{i_1} L_{\lambda}^Y(W, s) \right) &\leq \tilde{F}_{i_1}(\lambda s) \\
\Phi \left( W_{i_2} L_{\lambda}^Y(W, s) \right) &\leq \tilde{F}_{i_2}(\lambda s).
\end{align*}
\]

Applying (15) with \( a = W_{ij}, j = 1, 2 \) and \( x = L_{\lambda}^Y(W, s) \), we have
\[
\begin{align*}
(1 - W_{i_1})^{n-1} \tilde{F}_R \left( L_{\lambda}^Y(W, s) \right) &\leq \Phi \left( W_{i_1} L_{\lambda}^Y(W, s) \right) \\
(1 - W_{i_2})^{n-1} \tilde{F}_R \left( L_{\lambda}^Y(W, s) \right) &\leq \Phi \left( W_{i_2} L_{\lambda}^Y(W, s) \right)
\end{align*}
\]

and for \( j = 1, 2 \) we have \( \tilde{F}_{i_j}(\lambda s) \sim \lambda^{-\alpha_{i_j}} \tilde{F}_{i_j}(s) \leq \lambda^{-\alpha_n} z(s) \). Finally,
\[
\mathbb{E} \left[ (Z_{NR3,2}^Y(s))^2 \right] \leq \mathbb{E} \left[ F_R \left( L_{\lambda}^Y(W, s) \right)^2 \right] \leq \mathbb{E} \left[ \left( (1 - W_{i_1})^{-(n-1)} \wedge (1 - W_{i_2})^{-(n-1)} \right)^2 \right] \lambda^{-2\alpha_n} (z(s))^2 \leq 2^{2n-2} \lambda^{-2\alpha_n} (z(s))^2.
\]

### 7.5 Proof of Proposition 15

The multivariate survival distribution function of \( Y = (Y_1, \ldots, Y_n) \) is given by
\[ \Pr(Y_1 > y_1, \ldots, Y_n > Y_n) = C \left( \tilde{F}_1(y_1), \ldots, \tilde{F}_n(y_n) \right) \]

and it follows that
\[ F_Y(y_1, \ldots, y_n) = \sum_{1 \leq i_1, \ldots, i_j \leq n} (-1)^j C(\tilde{F}_{i_1}(y_{i_1}), \ldots, \tilde{F}_{i_j}(y_{i_j})). \]
We can calculate the derivative of $F_Y(y_1, \ldots, y_n)$ following $y_{-i}$. Note that in the sum of $2^n$ elements, there are only two elements are different from 0 after taking the derivatives $(n - 1)$ times.

$$\frac{\partial^{(n-1)} F_Y(y_1, \ldots, y_n)}{\partial y_1 \cdots \partial y_{i-1} \partial y_{i+1} \cdots \partial y_n} = \Phi^{(n-1)} \left( \sum_{j \neq i} \Phi^{-1}(F_j(y_j)) \right) \prod_{j \neq i} \left[(\Phi^{-1})^{(1)}(F_j(y_j))f_j(y_j)\right]$$

$$- \Phi^{(n-1)} \left( \sum_{j=1}^n \Phi^{-1}(F_j(y_j)) \right) \prod_{j \neq i} \left[(\Phi^{-1})^{(1)}(F_j(y_j))f_j(y_j)\right]$$

and note that the density of $Y_{-i}$ is

$$f(y_{-i}) = \Phi^{(n-1)} \left( \sum_{j=1, j \neq i}^n \Phi^{-1}(F_j(y_j)) \right) \prod_{j=1, j \neq i} \left[(\Phi^{-1})^{(1)}(F_j(y_j))f_j(y_j)\right].$$

The conditional distribution of $Y_i^* = (Y_i|Y_{-i} = y_{-i})$ is then

$$\Pr(Y_i < y_i | Y_{-i} = y_{-i}) = 1 - \frac{(-1)^{n-1} \Phi^{(n-1)}(\sum_{j=1}^n \Phi^{-1}(F_j(y_j))) \prod_{j \neq i} \left[(\Phi^{-1})^{(1)}(F_j(y_j))f_j(y_j)\right]}{\Phi^{(n-1)}(\sum_{j=1, j \neq i}^n \Phi^{-1}(F_j(y_j))) \prod_{j \neq i} \left[(\Phi^{-1})^{(1)}(F_j(y_j))f_j(y_j)\right]}$$

$$= 1 - \frac{\Phi^{(n-1)}(\sum_{j=1}^n \Phi^{-1}(F_j(y_j)))}{\Phi^{(n-1)}(\sum_{j=1, j \neq i}^n \Phi^{-1}(F_j(y_j)))}.$$

### 7.6 Proof of Proposition 21

We have

$$\Pr(S_n^Y > s, M_n^Y \leq \kappa s) = \Pr(S_n^Y > s, M_n^Y \leq \kappa s, M_{n-1}^Y > \frac{1-\kappa}{n-1} s).$$

If we estimate this probability conditionally on $W \in s_n$ by the same method of estimating $Z_{N,R3,2}^Y(s)$, the value of $\lambda$ in this case is $\frac{1-\kappa}{n-1} \in (0, 1/n)$, the second moment of this estimator is upper bounded by $2^{2n-2} \left(\frac{1-\kappa}{n-1}\right)^{-2\alpha_n} \times [z^Y(s)]^2$. Thus, the variance of $Z_{N,R3,2}^Y(s)$ is bounded by

$$\forall \Pr(Z_{N,R4}^Y(s) \leq 2 \sum_{i=1}^n ([F_i(\kappa s) - F_i(s)])^2 \forall \Pr \left(1_{\{S_{n,i}^Y > s, Y_{n,i}^Y = M_{n,i}^Y\}} \right) + 2^{2n-1} \left(\frac{1-\kappa}{n-1}\right)^{-2\alpha_n} [z^Y(s)]^2$$

$$\leq 2 \sum_{i=1}^n (F_i(\kappa s))^2 + 2^{2n-1} \left(\frac{1-\kappa}{n-1}\right)^{-2\alpha_n} (z^Y(s))^2$$

$$\leq \left(2\kappa^{-2\alpha_n} + 2^{2n-1} \left(\frac{1-\kappa}{n-1}\right)^{-2\alpha_n}\right) (z^Y(s))^2.$$
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