5D SYM on 3D Sphere and 2D YM

TERUHIKO KAWANO and NARIAKI MATSUMIYA

Department of Physics, University of Tokyo, Hongo, Tokyo 113-0033, Japan

It is shown by using localization that in five-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory on $S^3$, correlation functions in a sector are identical to correlation functions in two-dimensional bosonic Yang-Mills theory.

1 Introduction

It has been observed that a correlation function in a gauge theory gives the same result as the one in a matrix theory. The recent typical examples in our mind are four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on $S^4$ [1] and the ABJM model on $S^3$ [2], where the localization method was used to calculate the correlation functions exactly. A similar application of the localization method to five-dimensional supersymmetric Yang-Mills theory on $S^5$ has been attempted in [3].

Contrary to an application of the localization method to an $n$-dimensional supersymmetric gauge theory on an $n$-dimensional compact space, which has been seen to give rise to a matrix model, we will in this paper study five-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory put on Euclidean $\mathbb{R}^2 \times S^3$ by using the localization method.

To this end, one needs to pick up a supersymmetry as the BRST symmetry, and a correlation function of the BRST invariant operators can be calculated by using the localization method. In this paper, it will be seen that a correlation function of the BRST invariant operators in the supersymmetric Yang-Mills theory yields the one in two-dimensional bosonic Yang-Mills theory.

2 Five-Dimensional Super Yang-Mills Theory on $\mathbb{R}^2 \times S^3$

A vector supermultiplet in the five-dimensional $\mathcal{N} = 1$ Yang-Mills theory consists of a gauge field $v_M$, a real scalar field $\sigma$, an auxiliary field $D_{\dot{\alpha} \dot{\beta}}$, and a spinor field $\Psi_{\dot{\alpha}}$, where the indices $\dot{\alpha}, \dot{\beta}$ label the components of the fundamental representation $2$ of $SU(2)$ $R$-symmetry. The spinor field obeys the symplectic Majorana condition

$$(\Psi^{\dot{\beta}})^T C_5 \epsilon_{\dot{\beta} \dot{\alpha}} = (\Psi_{\dot{\alpha}}) \equiv \bar{\Psi}_{\dot{\alpha}},$$

where $T$ denotes the transpose, and $\epsilon_{\dot{\alpha} \dot{\beta}}$ is the invariant tensor of the $SU(2)$ $R$-symmetry. The auxiliary field $D_{\dot{\alpha} \dot{\beta}}$ is anti-Hermitian and in the adjoint representation of $SU(2)$.
R-symmetry;
\[ D_{\hat{\alpha} \hat{\beta}} = -(D^\hat{\beta} \hat{\alpha})^\dagger, \quad D_{\hat{\alpha} \gamma} \epsilon^{\hat{\gamma} \hat{\beta}} = D_{\hat{\beta} \gamma} \epsilon^{\hat{\gamma} \hat{\alpha}}, \quad D_{\hat{\alpha} \hat{\alpha}} = 0. \]

Our notations for the charge conjugation matrix \( C_5 \) and the gamma matrices \( \Gamma^M \) are explained in Appendix. We assume that the gauge group \( G \) is a simple Lie group. All the fields are in the adjoint representation of the gauge group \( G \), and are denoted in the matrix notation as

\[ \Phi = \Phi^A T^A \]

with the normalization \( \text{tr}[T^A T^B] = \delta^{AB} \).

On a flat Euclidean space \( \mathbb{R}^5 \), the Lagrangian \( \mathcal{L}_V \) is given by

\[ \text{tr}\left[ \frac{1}{4} v_{MN} v^{MN} - \frac{1}{2} D_M \sigma D^M - i \bar{\Psi} \Gamma^M D_M \Psi + g \bar{\Psi} [\sigma, \Psi] + \frac{1}{4} D_{\hat{\alpha} \hat{\beta}} D^\hat{\alpha} \hat{\beta} \right] \]

(1)

where \( v_{MN} \) is the field strength

\[ v_{MN} = \partial_M v_N - \partial_N v_M + ig [v_M, v_N], \]

of the gauge field \( v_M \), and the covariant derivatives \( D_M \Phi \) is given by

\[ D_M \Phi = \partial_M \Phi + ig [v_M, \Phi]. \]

The Lagrangian \( \mathcal{L}_V \) is left invariant under a supersymmetry transformation

\[ \delta^{(0)} v_M = -i \bar{\Sigma}_\hat{\alpha} \Gamma_M \Psi^\hat{\alpha}, \quad \delta^{(0)} \sigma = i \bar{\Sigma}_\hat{\alpha} \Psi^\hat{\alpha}, \quad \delta^{(0)} \Psi^\hat{\alpha} = -\frac{1}{2} \left( \frac{1}{2} v_{MN} \Gamma^{MN} \Sigma^\hat{\alpha} + \Gamma^M D_M \sigma \Sigma^\hat{\alpha} + D_{\hat{\alpha} \hat{\beta}} \Sigma^\hat{\beta} \right), \]

\[ \delta^{(0)} D_{\hat{\alpha} \hat{\beta}} = i \left[ D_M \bar{\Psi}^\hat{\beta} \Gamma^M \Sigma^\hat{\alpha} + \bar{\Sigma}_{\hat{\alpha}} \Gamma^M D_M \Psi^\hat{\alpha} + ig \left( [\sigma, \bar{\Psi}^\hat{\beta}] \Sigma^\hat{\alpha} + \bar{\Sigma}_{\hat{\alpha}} [\sigma, \Psi^\hat{\alpha}] \right) \right], \]

(2)

where the transformation parameter \( \Sigma^\hat{\alpha} \) is also a symplectic Majorana spinor;

\[ \bar{\Sigma}_{\hat{\alpha}} = (\Sigma^\hat{\beta})^T C_5 \epsilon_{\hat{\beta} \hat{\alpha}}. \]

When the system is put on \( \mathbb{R}^2 \times S^3 \), it is convenient to give the gauge field \( v^M \) and the spinor \( \Psi^\hat{\alpha} \) in terms of three-dimensional tensors and spinors as

\[ v^m (m = 1, 2, 3), \quad v_z = \frac{1}{2} (v_4 - iv_5), \quad v_{\bar{z}} = \frac{1}{2} (v_4 + iv_5), \]

\[ \Psi^\hat{\alpha}_1 = \lambda \otimes \chi_+ + \psi \otimes \chi_-, \quad \Psi^\hat{\alpha}_2 = C_3^{-1} \psi^* \otimes \chi_+ + C_3^{-1} \lambda^* \otimes \chi_-, \]

\[ D = D^1_1 + 2 v_{\bar{z}}z, \quad F = \frac{1}{2} D^1_2, \quad \bar{F} = \frac{1}{2} D^2_1, \]

where \( * \) denotes the complex conjugation, and the complex coordinates \( z, \bar{z} \) for \( \mathbb{R}^2 \) were introduced by \( z = x^4 + ix^5, \bar{z} = x^4 - ix^5 \). The two-dimensional spinors

\[ \chi_{\pm} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ \pm i \end{array} \right), \]
are the eigenvectors of $i \Gamma^4 \Gamma^5; i \Gamma^4 \Gamma^5 \chi_\pm = \pm \chi_\pm$.

For the supersymmetry transformation (2), in going onto the $\mathbb{R}^2 \times S^3$, we will pick up one of the Killing spinors $\epsilon$ on $S^3$ obeying that

$$\nabla_m \epsilon = \frac{i}{2} \gamma_m \epsilon,$$

and set

$$\Sigma^{\dot{a}=1} = \epsilon \otimes \chi_+ , \quad \Sigma^{\dot{a}=2} = C_3^{-1} \epsilon^* \otimes \chi_- .$$

On the $\mathbb{R}^2 \times S^3$, the supersymmetry transformation (2) no longer yields a closed algebra. Note here that the covariant derivative $D_m \lambda$ contains the spin connection $\omega_m$ of the unit round $S^3$ as

$$D_m \lambda = \partial_m \lambda + \frac{1}{4} \omega_m^{ab} \gamma^{ab} \lambda,$$

where $a, b = 1, 2, 3$ denote the tangent indices. In order to obtain a closed algebra, we will modify the transformation law of $D^{\dot{a}}{}_{\dot{b}}$ by adding

$$\delta^{\dot{a}}_\epsilon D = - \frac{1}{2} (\bar{\epsilon} \lambda - \bar{\lambda} \epsilon) , \quad \delta^{\dot{a}}_\epsilon F = \frac{1}{2} \epsilon^T C_3 \psi,$$

to them, respectively.

One then finds that the modified transformation $\delta_\epsilon = \delta^{\dot{a}}_\epsilon (0) + \delta^{\dot{a}}_\epsilon$,

$$\delta_\epsilon v_m = - i \left[ \bar{\epsilon} \gamma_m \lambda - \bar{\lambda} \gamma_m \epsilon \right] , \quad \delta_\epsilon v_z = - \bar{\epsilon} \psi , \quad \delta_\epsilon \sigma = i \left[ \bar{\epsilon} \lambda - \bar{\lambda} \epsilon \right] ,$$

$$\delta_\epsilon \lambda = - \frac{1}{2} \left[ \frac{1}{2} v_m \gamma^{mn} + \gamma^m D_m \sigma + D \right] \epsilon,$$

$$\delta_\epsilon \psi = - \left[ - i v_m \gamma^m \epsilon + i D_2 \sigma \epsilon + F C_3^{-1} \epsilon^* \right] ,$$

$$\delta_\epsilon D = i \left[ D_m \lambda \gamma^m \epsilon + \bar{\epsilon} \gamma^m D_m \lambda + i g \left( [\sigma, \lambda] \epsilon + \bar{\epsilon} [\sigma, \lambda] \right) + \frac{i}{2} (\bar{\epsilon} \lambda - \bar{\lambda} \epsilon) \right] ,$$

$$\delta_\epsilon F = i \left[ - \epsilon^T C_3 \gamma^m D_m \psi^a + 2 i \epsilon^T C_3 D_z \lambda^a + i g \epsilon^T C_3 [\sigma, \psi] - \frac{i}{2} \epsilon^T C_3 \psi \right] ,$$

yields the closed algebra

$$[\delta_\eta, \delta_\epsilon] v_m = - i \left( \xi^n \nabla_n v_m - D_m \omega \right) - (\bar{\epsilon} \gamma_m \eta - \bar{\eta} \gamma_m \epsilon) v^n ,$$

$$[\delta_\eta, \delta_\epsilon] v_z = - i \left( \xi^n \nabla_n v_z - D_2 \omega \right) , \quad [\delta_\eta, \delta_\epsilon] \sigma = - i \left( \xi^n \nabla_n \sigma + i g [\omega, \sigma] \right) ,$$

$$[\delta_\eta, \delta_\epsilon] \lambda = - i \left( \xi^n \nabla_n \lambda + i g [\omega, \lambda] \right) - (\bar{\epsilon} \eta - \bar{\eta} \epsilon) \lambda - \frac{1}{4} (\bar{\epsilon} \gamma_m \eta - \bar{\eta} \gamma_m \epsilon) \gamma^{mn} \lambda ,$$

$$[\delta_\eta, \delta_\epsilon] \psi = - i \left( \xi^n \nabla_n \psi + i g [\omega, \psi] \right) - (\bar{\epsilon} \eta - \bar{\eta} \epsilon) \psi - \frac{1}{4} (\bar{\epsilon} \gamma_m \eta - \bar{\eta} \gamma_m \epsilon) \gamma^{mn} \psi ,$$

$$[\delta_\eta, \delta_\epsilon] D = - i \left( \xi^n \nabla_n D + i g [\omega, D] \right) ,$$

$$[\delta_\eta, \delta_\epsilon] F = - i \left( \xi^n \nabla_n F + i g [\omega, F] \right) - 2 (\bar{\epsilon} \eta - \bar{\eta} \epsilon) F ,$$

with the transformation parameters

$$\xi^m = \bar{\epsilon} \gamma^m \eta - \bar{\eta} \gamma^m \epsilon , \quad \omega = \xi^n v_n + (\bar{\epsilon} \eta - \bar{\eta} \epsilon) \sigma,$$
where the covariant derivative $\nabla_m v_n$ includes the Levi-Civita connection $\Gamma^k_{mn}$ as
\[ \nabla_m v_n = \partial_m v_n - \Gamma^k_{mn} v_k. \]

On the $\mathbb{R}^2 \times S^3$, since the three-dimensional sphere $S^3$ is a curved space, one needs to replace all derivatives on spinors in the Lagrangian $L_V$ by the covariant derivative with the spin connection $\omega_m$. However, it isn’t enough to be invariant under the modified supersymmetry transformation (3). In fact, to the original Lagrangian $L_V$,
\[ \text{tr} \left[ \frac{1}{4} (v_{mm})^2 + 2 |v_{mz}|^2 - \frac{1}{2} (D_m \sigma)^2 - 2 D_z \sigma D_\sigma \sigma + \frac{1}{2} D^2 - 2 v_{zz} D + 2 \bar{F} F \right. \]
\[ \left. - 2 i \bar{\lambda} \gamma^m D_m \lambda + 2 i \bar{\psi} \gamma^m D_m \psi + 4 \bar{\psi} D_z \lambda + 4 D_z \bar{\lambda} \psi + 2 g \left( \bar{\lambda} [\sigma, \lambda] + \bar{\psi} [\sigma, \psi] \right) \right] \]
one needs to add the two terms
\[ L'_V = -\text{tr} \left[ \bar{\psi} \psi + \bar{\lambda} \lambda + \sigma \sigma + i \sigma (D - 4 v_{zz}) \right], \quad L_{CS} = -\epsilon^{mnl} \text{tr} \left[ v_m \partial_n v_l + i \frac{q}{3} v_m [v_n, v_l] \right]. \]
to obtain the supersymmetric total Lagrangian
\[ L = L_V + L'_V + \frac{1}{2} L_{CS}. \]

One can then verify that $\delta_\epsilon L = 0$.

## 3 Localization

In this section, we will calculate the partition function of the five-dimensional supersymmetric Yang-Mills theory on the $\mathbb{R}^2 \times S^3$ by using the localization method. In order to make the path integral well-defined, the bosonic fields $\sigma$, $D$, $F$, and $\bar{F}$ need to be analytically continued. Therefore, we will regard the scalar field $\sigma$ as taking pure imaginary values, and the auxiliary field $D$ as a real field. Further, $\bar{F} = F^\ast$.

To carry out the localization method, we will define the BRST transformation by setting $\bar{\epsilon}$ to zero in the supersymmetric transformation (3) and by replacing the Grassmann odd parameter $\epsilon$ by a Grassmann even one. It yields
\[ \delta_Q v_m = -i \bar{\lambda} \gamma_m \epsilon, \quad \delta_Q v_z = 0, \quad \delta_Q \bar{\psi} = \bar{\psi} \epsilon, \quad \delta_Q \sigma = i \bar{\lambda} \epsilon, \]
\[ \delta_Q \lambda = -\frac{1}{2} \left[ \frac{1}{2} v_{mm} \gamma^m + \gamma^m D_m \sigma + D \right] \epsilon, \quad \delta_Q \bar{\lambda} = 0, \]
\[ \delta_Q \psi = i \left[ v_{mz} \gamma^m - D_z \sigma \right] \epsilon, \quad \delta_Q \bar{\psi} = \bar{F} \epsilon^T C_3, \]
\[ \delta_Q D = -i \left[ D_m \bar{\lambda} \gamma^m \epsilon + i g [\sigma, \bar{\lambda}] \epsilon - \frac{i}{2} \bar{\lambda} \epsilon \right] \]
\[ \delta_Q F = i \epsilon^T C_3 \left[ -\gamma^m D_m \psi \epsilon + 2 i D_z \lambda \epsilon + i g [\sigma, \psi] - \frac{i}{2} \psi \right], \quad \delta_Q \bar{F} = 0, \]

which is in fact nilpotent; $\delta_Q^2 = 0$, as it should be. Using the BRST transformation (4), we will modify the Lagrangian $L$ into $L + t L_Q$ with a parameter $t$, where
\[ L_Q = \delta_Q \text{tr} \left[ (\delta_Q \lambda)^\dagger \lambda + (\delta_Q \psi)^\dagger \psi + \bar{\psi} (\delta_Q \bar{\psi})^\dagger \right]. \]
The bosonic part of the extra Lagrangian $L_Q$ gives
\[
L_Q^{(B)} = \frac{1}{2} \text{tr} \left[ \frac{1}{4} (v_{mn})^2 + 2 |v_{mz}|^2 - \frac{1}{2} D_m \sigma D_m \sigma - 2 D_z \sigma D_z \sigma + \frac{1}{2} D^2 
+ 2 |F|^2 + 2k_m (v_{mz} D_z \sigma - v_{mz} D_z \sigma + i \epsilon_{mnk} v_{nz} v_{kz}) \right]
\]
where the Killing vector $k_m$ was defined by
\[
k_m = \bar{\epsilon} \gamma_m \epsilon
\]
with the normalization $(\bar{\epsilon} \epsilon) = 1$. On the other hand, the fermionic part of $L_Q$ gives
\[
L_Q^{(F)} = i \text{tr} \left[ - \bar{\lambda} \gamma^m D_m \lambda - \frac{i}{2} \bar{\lambda} \lambda - i g \bar{\lambda} [\sigma, \lambda] + \bar{\psi} \gamma^m D_m \psi - \frac{i}{2} \bar{\psi} \psi - i g \bar{\psi} \gamma^m D_z \lambda 
- i k_m \bar{\psi} \gamma^m \psi + 2 i \bar{\lambda} D_z \psi - i \bar{\psi} D_z \lambda + i k_m \bar{\psi} \gamma^m D_z \lambda \right].
\]
In the large $t$ limit, $t \to \infty$, the fixed point, which is a solution to
\[
\left[ \frac{1}{2} v_{mn} \gamma^{mn} + \gamma^m D_m \sigma + D \right] \epsilon = 0, \quad [v_{mz} \gamma^m - D_z \sigma] \epsilon = 0, \quad F = 0,
\]
gives the dominant contribution to the partition function. In fact, the fixed point is given by
\[
v_m = 0, \quad D = 0, \quad F = 0, \quad v_z = v_z(z, \bar{z}), \quad \sigma = \sigma(z, \bar{z}), \quad D_z \sigma = 0. \quad (5)
\]
Substituting the background (5) into the original Lagrangian $L$, one finds that the additional Lagrangian $L'_V$ only contributes and yields
\[
L_{YM} = \text{tr}[-\sigma \sigma + 4i \sigma v_{zz}], \quad (6)
\]
which is the action of the two-dimensional Yang-Mills theory after eliminating the scalar field $\sigma$.

Around the fixed points, one needs to evaluate the path integral over the quantum fluctuations. Since the bosonic fields $\sigma$, $v_z$, and $v_{\bar{z}}$ have a non-trivial background as the fixed point, we will expand the fields as
\[
\sigma = \sigma(z, \bar{z}) + \frac{1}{\sqrt{t}} \bar{\sigma}(x^m, z, \bar{z}), \quad v_z = v_z(z, \bar{z}) + \frac{1}{\sqrt{t}} \bar{v}_z(x^m, z, \bar{z}),
\]
while the other fields are rescaled as $\Phi \to (1/\sqrt{t}) \bar{\Phi}$, as in [2].

One also needs the gauge-fixing procedure for the evaluation of the path integral. We will follow [2] and add to $L_Q$ the gauge-fixing term and the ghost term
\[
\text{tr}[c \nabla_m D^m c + B \nabla^m v_m].
\]
There remains the residual gauge symmetry, under which
\[
\sigma \to \sigma + ig \left[ \omega(z, \bar{z}), \sigma \right], \quad v_z \to v_z - D_z \omega(z, \bar{z}), \quad (7)
\]
where the gauge transformation parameter $\omega$ is constant on the $S^3$. Following [4, 5], one can make use of the residual symmetry (7) and $D_{\sigma}\sigma = 0$ in (5) to put the background $\sigma(z, \bar{z}), v_z(z, \bar{z})$ in the Cartan subalgebra of the Lie algebra of $G$ such that

$$
\sigma(z, \bar{z}) = \sum_{i=1}^{r} \sigma_i H_i, \quad v_z(z, \bar{z}) = \sum_{i=1}^{r} v^i_z(z, \bar{z}) H_i,
$$

(8)

where $H_i (i = 1, \cdots, r)$ are the generators of the Cartan subalgebra of rank $r$, and $\sigma_i (i = 1, \cdots, r)$ are constant with respect to $z, \bar{z}$.

Therefore, for the residual gauge symmetry (7), we will follow the same BRST quantization procedure as for the two-dimensional Yang-Mills theory in [4, 5]. The path-integral measure of the scalar field $\sigma(z, \bar{z})$ thus results in the finite-dimensional integral over $\sigma_i (i = 1, \cdots, r)$ and the determinant of the Fadeev-Popov ghosts.

One can thus see that the localization procedure has so far given the same Lagrangian (6), the same fixed points (8), and the same BRST gauge fixing procedure as for the Yang-Mills theory, - the exactly same results as in [4, 5], but, except for one point. In the two-dimensional Yang-Mills theory, for the two-dimensional gauge fields $v_z, v_{\bar{z}}$, the root part

$$
\sum_{\alpha \in \Lambda} v^\alpha_z(z, \bar{z}) E_{\alpha}, \quad \sum_{\alpha \in \Lambda} v^\alpha_{\bar{z}}(z, \bar{z}) E_{\alpha},
$$

(9)

where $\Lambda$ is the set of all the root of the Lie algebra of $G$, and the root generators $E_{\alpha}$ satisfy the algebra

$$
[H_i, E_{\alpha}] = \alpha_i E_{\alpha}, \quad [E_{\alpha}, E_{-\alpha}] = \sum_{i=1}^{r} \alpha_i H_i \equiv \alpha \cdot H,
$$

show up in the Lagrangian (6) as

$$
4g (\alpha \cdot \sigma) |v^\alpha_{\bar{z}}|^2,
$$

(10)

and yield the contributions to the partition function.

However, in our case, it no longer gives any contributions in the large $t$ limit, $t \to \infty$.

One then proceeds to the evaluation of the one-loop determinants from the Lagrangian $\mathcal{L}_Q$, which also contains the root part in (9) as the zero modes of $v_z, v_{\bar{z}}$ upon expanding them in terms of the harmonics on $S^3$. To this end, we will follow the same procedure as in [2, 6], - expanding all the fields in terms of the harmonics on $S^3$ and performing the Gaussian integration over them.

Up to an overall irrelevant normalization constant, the tedious calculation shows the exact cancellation between the bosonic degrees of freedom and fermionic ones, but, except for one pair. The zero modes of the scalar harmonics from $v_z, v_{\bar{z}}$ do not cancel out the contribution of one of the low-lying modes of the spinor harmonics from $\lambda, \psi$ to yield the same contribution as the discrepancy, which would come from (10) in the two-dimensional Yang-Mills theory;

$$
\int [d\Phi] e^{-\int d^5 x (\mathcal{L} + t\mathcal{L}_Q)} \rightarrow \int \prod_{\alpha \in \Lambda} [dv^\alpha_z dv^\alpha_{\bar{z}}] e^{-\int dz d\bar{z} 4g(\alpha \cdot \sigma)|v^\alpha_{\bar{z}}|^2}.
$$

The partition function in the supersymmetric Yang-Mills theory on $\mathbb{R}^2 \times S^3$ thus exactly reduces into the one in the bosonic Yang-Mills theory on $\mathbb{R}^2$, via the localization method.
4 Discussions

In the previous sections, we have seen that the five-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory can be put on the product space $\mathbb{R}^2 \times S^3$ with supersymmetry kept. Upon the localization, without losing any degrees of freedom, it reduces into the two-dimensional Yang-Mills theory; namely, a correlation function in the BRST invariant sector of the five-dimensional supersymmetric theory is identical to a correlation function in the two-dimensional bosonic theory.

A conceivable extension of this work would be inclusion of hypermultiplets into the five-dimensional theory [7]. It would be interesting to see what would happen in the two-dimensional theory and to understand the relation to the proposal in [8].

Acknowledgement

The authors would like to thank Yasutaka Fukuda and Satoshi Yamaguchi for collaborations at the early stage of this work. We are grateful to Yuji Tachikawa for helpful discussions and for a careful reading of the manuscript. We are also grateful to Kazuo Hosomichi for helpful discussions and for giving us crystal clear clear lectures about the papers [2, 6], where we have learnt all the techniques we needed for this work. The work of T. K. was supported in part by a Grant-in-Aid #23540286 from the MEXT of Japan.

Appendix

The five-dimensional gamma matrices $\Gamma^M$ ($M = 1, \cdots, 5$) satisfy

$$\{\Gamma^M, \Gamma^N\} = 2\delta^{MN},$$

and they are given in terms of the three-dimensional gamma matrices $\gamma^m = \sigma_m$ ($m = 1, 2, 3$) as

$$\Gamma^m = \gamma^m \otimes \sigma_2, \quad \Gamma^4 = 1 \otimes \sigma_1, \quad \Gamma^5 = 1 \otimes \sigma_3,$$

where $\sigma_{1,2,3}$ are the Pauli matrices.

The five-dimensional charge conjugation matrix $C_5$ satisfies

$$(\Gamma^M)^T = C_5 \Gamma^M C_5^{-1}, \quad (C_5)^T = -C_5,$$

where $T$ denotes the transpose of the matrix, and it may be given in terms of the three-dimensional charge conjugate matrix $C_3 = i\sigma_2$ as

$$C_5 = C_3 \otimes 1.$$
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