KOLMOGOROV OPERATOR WITH THE VECTOR FIELD IN NASH CLASS

D. KINZEBULATOV AND YU. A. SEMÈNOV

Abstract. We establish sharp two-sided heat kernel bounds, Harnack inequality and Hölder continuity of bounded solutions for divergence-form parabolic equation with measurable uniformly elliptic matrix and the first-order term in a large class of locally unbounded vector fields containing $L^p$, $p > d$ as well as some vector fields $\not\in L^p_{\text{loc}}$, $p > 2$.

1. Introduction

The purpose of this paper is to extend the prominent result of E. De Giorgi and J. Nash on Hölder continuity of bounded solutions of divergence-form equation $(\partial_t - \nabla \cdot a \cdot \nabla)u = 0$ to the parabolic equation

$$(\partial_t + \Lambda)u = 0, \quad \Lambda = -\nabla \cdot a \cdot \nabla + b \cdot \nabla,$$

on $\mathbb{R}_+ \times \mathbb{R}^d$, $d \geq 3$, with $a$ a measurable uniformly elliptic matrix, i.e.

$$a = a^\top : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d,$$

$$\sigma I \leq a(x) \leq \xi I \quad \text{for a.e. } x \in \mathbb{R}^d \quad \text{for constants } 0 < \sigma < \xi < \infty \quad (H_{\sigma,\xi})$$

and $b : \mathbb{R}^d \to \mathbb{R}^d$ in a large class of locally unbounded measurable vector fields.

The existence and the precise form of the relationship between the integral characteristics of $a$ and $b$ and the regularity properties of solutions to (1) is one of the central problems in the theory of elliptic and parabolic PDEs. By the De Giorgi-Nash theory [DG, N], the bounded solutions to $(\partial_t + A)u = 0$, $A = -\nabla \cdot a \cdot \nabla$ are Hölder continuous and the heat kernel $e^{-tA}(x, y)$ satisfies two-sided Gaussian bounds, with the Hölder continuity exponent and the constants in the two-sided bounds depending only on $d$, $\sigma$, $\xi$. Further, the heat kernel $e^{-t\Lambda}(x, y)$ of (1) satisfies two-sided Gaussian bounds (Aronson [A]) and $t|\partial_t e^{-t\Lambda}(x, y)|$ satisfies the upper Gaussian bound (Eidelman-Porper [EP]) with constants that depend on $d$, $\sigma$, $\xi$, and $\|b_1\|_p + \|b_2\|_\infty$, $p > d$, where $b_1 + b_2 = b$.

Our first goal is to demonstrate, based on ideas of E. De Giorgi and J. Nash, that the constants in the two-sided bounds on $e^{-t\Lambda}(x, y)$, in the upper bound on $t|\partial_t e^{-t\Lambda}(x, y)|$, as well as Hölder continuity of bounded solutions to (1) (with smooth coefficients $a$, $b$) depend in fact on a much finer characteristic of $b$, that is, on the elliptic Nash norm of $b$:

$$n_e(b, h) := \sup_{x \in \mathbb{R}^d} \int_0^h \sqrt{e^{t\Delta} |b|^2(x)} \frac{dt}{\sqrt{t}} \quad (h > 0), \quad (N_e)$$

and only on the elliptic Nash norm.

As is well known, the existence of strong a priori estimates does not always mean that there is a satisfactory a posteriori regularity theory of the corresponding differential operator. Our second
goal is to develop an exhaustive a posteriori theory of \([1]\) assuming only that \(|b| \in L^2_{\text{loc}}\) and \(n_e(b,h)\) is sufficiently small for some \(h > 0\), using appropriate (i.e. consistent with the definition of the Nash norm) approximation of \(b\).

If \(b\) satisfies \(n_e(b,h) < \infty\), then we say that \(b\) belongs to the elliptic Nash class \(\mathbf{N}_e\) and write \(b \in \mathbf{N}_e\). The class \(\mathbf{N}_e\) contains the vector fields \(b\) with \(|b_1| \in L^p\), \(p > d\), and for such \(b\) one has \(\lim_{h \to 0} n_e(b,h) = 0\). Moreover, the class \(\mathbf{N}_e\) contains vector fields \(b\) with \(|b| \notin L^p_{\text{loc}}\) if \(p > 2\). In Section \(3\) we explain that \(\mathbf{N}_e\) is the analogue of the well known Kato class of vector fields \(K^{d+1}\) arising in the study of \([1]\) with a Hölder continuous matrix \(a\).

The elliptic Nash norm was introduced in \([S1]\) where the two-sided Gaussian bounds on the heat kernel \(e^{-t\Lambda}(x,y)\) were obtained under some additional to \(b \in \mathbf{N}_e\) assumptions.

The standard assumption on a locally unbounded \(b\) in \([1]\) found in the literature is the form-boundedness condition: \(b \cdot a^{-1} \cdot b \leq \delta (-\nabla \cdot a \cdot \nabla) + c\) (in the sense of quadratic forms) with \(\delta < 1\), for some constant \(c \geq 0\). Then the corresponding to \(\Lambda\) quadratic form on \(W^{1,2}\) is quasi \(m\)-accretive, and so it determines a unique operator \(\Lambda_2\) in \(L^2\) generating a holomorphic semigroup; the equation \([1]\) with \(\Lambda = \Lambda_2\) possesses a detailed regularity theory in \(L^2\), see Section \(3\).

For \(b \in \mathbf{N}_e\), the equation \([1]\) does not seem to admit any \(L^p\) theory with \(p > 1\) beyond the existence of the semigroup, but it admits a detailed \(L^1\) theory. Namely, in Theorem \([1]\) we construct an operator realization \(\Lambda_1\) of Kolmogorov operator \(\Lambda\) in \(L^1\) as the algebraic sum

\[
\Lambda_1 = A_1 + (b \cdot \nabla)_1, \quad D(\Lambda_1) = D(A_1),
\]

where \(A_1\) is the operator realization of \(-\nabla \cdot a \cdot \nabla\) in \(L^1\) and \((b \cdot \nabla)_1\) is the closure of \(b \cdot \nabla\) in the graph norm of \(A_1\), and show that

\[
e^{-t\Lambda_1} = s - L^1 - \lim_{\varepsilon \to 0} e^{-t\Lambda_1^\varepsilon} \quad \text{(loc. uniformly in } t \geq 0)\]

where \(\Lambda_1^\varepsilon = -\nabla \cdot a_\varepsilon \cdot \nabla + b_\varepsilon \cdot \nabla\) of domain \(D(\Lambda_1^\varepsilon)\) = \((1 - \Delta)^{-1} L^1\) with smooth \((a_\varepsilon, b_\varepsilon)\) approximating \((a,b)\) and essentially non-increasing the Nash norm: \(n_e(b_\varepsilon, h) \leq n_e(b, h) + \varepsilon\).

Armed with these results and a priori two-sided Gaussian bounds on \(e^{-t\Lambda^x}(x,y)\) of Theorem \(5\) we develop an exhaustive regularity theory of \([1]\), including a posteriori two-sided Gaussian bounds on the heat kernel \(e^{-t\Lambda^x}(x,y)\), the Harnack inequality, the Hölder continuity of bounded solutions of \([1]\), the strong Feller property, and the Gaussian upper bound on \(|t| e^{-t\Lambda^x}(x,y)|\) with the optimal (up to a strict inequality) exponent in the Gaussian factor, see Theorem \(1\). We also establish the bounds \(\|\nabla (\mu + \Lambda_1)^{-\alpha}\|_{1 \to 1} \leq C \mu^{\frac{2\alpha - d}{2\alpha - 1}}, \frac{1}{2} < \alpha \leq 1, \mu > \mu_0 > 0\) (\(\mu_0\) depends on \(d, \sigma, \xi, n_e(b,h)\)), and \(\|\nabla e^{-t\Lambda_1}\|_{1 \to 1} \leq c e^t, t > 0\), see Theorem \(2\).

**Notations and definitions.** We denote by \(\mathcal{B}(X,Y)\) the space of bounded linear operators between Banach spaces \(X \to Y\), endowed with the operator norm \(\|\cdot\|_{X \to Y}\). \(\mathcal{B}(X) := \mathcal{B}(X,X)\).

We write \(T = s - X - \lim_n T_n\) for \(T, T_n \in \mathcal{B}(X,Y)\) if \(\lim_n \|Tf - T_n f\|_Y = 0\) for every \(f \in X\).

Denote by \([L^p]^d\) and \([L^p]^{d \times d}\) the spaces of the \(d\)-vectors and the \(d\times d\)-matrices with entries in \(L^p \equiv L^p(\mathbb{R}^d, dx)\).

Put \(\langle f, g \rangle = \langle f \bar{g} \rangle := \int_{\mathbb{R}^d} f \bar{g} \, dx\) and \(\|\cdot\|_p\) := \(\|\cdot\|_{L^p_{\to L^p}}\).

\(C_\infty := \{f \in C(\mathbb{R}^d) \mid \lim_{|x| \to \infty} f(x) = 0\}\) endowed with the sup-norm.
Proposition 1. Let by continuity in the graph norm of $g$ with bound $\eta$ $B$ generator $-\partial_t$ is a semigroup $e^{-\Delta t}$. Then $C > 0$ is a constant generic if it depends only on the dimension $d$ and the constants $\sigma$ and $\xi$. It will be called generic if it also depends on the Nash norm $n_c(b, h)$. We write $c \neq c(\varepsilon)$ to emphasize that $c$ is independent of $\varepsilon$. Put

$$k(t, x, y) = k(t, x, y) := (4\pi t)^{-d/2} e^{-|x-y|^2/4t}, \quad \mu > 0.$$  

Recall that if $S$ and $T$ are linear operators in a Banach space $(Y, \| \cdot \|)$, then $S$ is said to be $T$-bounded if $D(S) \supset D(T)$ and there exist constants $\eta$ and $c$ such that $\|Sy\| \leq \eta(Ty\| + c\|y\|$ for all $y \in D(T)$. 

By $T | X$ we denote the restriction of $T$ to a subset $X \subset D(T)$. By $(T | X)_{Y \rightarrow Y}^{\text{clos}}$ we denote the closure of $T | X$ (when it exists). 

Let $T$ be closed. $D_T \subset D(T)$ is called a core of $T$ if $(T | D_T)_{Y \rightarrow Y}^{\text{clos}} = T$. 

Let $P$, $Q$ be linear operators in a Banach space $Y$. Assume that $Q$ is closed, $D(P)$ contains a core $D_Q$ of $Q$ and $\|Py\| \leq \eta(Qy\| + c\|y\|, y \in D_Q (\eta, c$ some constants). This inequality extends by continuity to $D(Q)$. An extension of $P$ obtained in this way, say $\tilde{P}$, is $Q$-bounded.

2. Main results

Let $A \equiv A_2$ be the operator in $L^2$ associated with the quadratic form $\langle \nabla u, a \cdot \nabla u \rangle$, $u \in W^{1,2}$. A standard application of the Beurling-Deny theory yields: $A$ generates a symmetric Markov semigroup $e^{-tA}$. Then

$$e^{-tA} := \left[ e^{-tA} | L^1 \cap L^2 \right]_{L^1 \rightarrow L^1}^{\text{clos}} \in \mathcal{B}(L^1), \quad t > 0.$$  

is a $C_0$ semigroup (this is a general fact from the theory of symmetric Markov semigroups). Its generator $-A_1$ is an appropriate operator realization of $-\nabla \cdot a \cdot \nabla$ in $L^1$.

In order to state our first result we need the following. Let $b \in [L^1]^{d'}$. In $L^1$ define operator $B_{\text{max}} \supset b \cdot \nabla$ of domain $D(B_{\text{max}}) := \{ f \in L^1 \mid f \in W^{1,1}_{\text{loc}}$ and $b \cdot \nabla f \in L^1 \}$. 

**Proposition 1.** Let $b \in \mathbb{N}_c$. Then $D(B_{\text{max}}) \supset D(A) \cap D(A_1)$ and $B_{\text{max}} | D(A_1) \cap D(A)$ extends by continuity in the graph norm of $A_1$ to $A_1$-bounded operator $(b \cdot \nabla)_1$:

$$\| (b \cdot \nabla)_1 f \|_1 \leq \eta \| A_1 f \|_1 + \eta \mu \| f \|_1, \quad f \in D(A_1),$$

with bound $\eta := \frac{1}{1 - e^{-c_0 \eta}} \eta \| A_1 f \|_1 + \eta \mu \| f \|_1, \quad f \in D(A_1),$

where $c_i (i = 3, 4, 5)$ are generic constants in the two-sided Gaussian bounds on the heat kernel $e^{-tA}(x, y)$ and its time derivative, see Theorem 4 below.

We need also the following result. Since $e^{-tA}$ and $e^{-tA}$ have the same integral kernel $e^{-tA}(x, y)$ which satisfies $|\partial_te^{-tA}(x, y)| \leq c_5 t^{-1} k_0(t, x - y)$, cf. Theorem 3 below, there exists a generic constant $C > 0$ such that $(Ct D_t e^{-tA})^n$ are uniformly (in $0 \leq t \leq 1$ and $n = 1, 2, \ldots$) bounded in
$B(L^1)$, and so, by a classical result [Y] Ch. IX, sect. 10,
\[
\|(\zeta + A_1)^{-1}\|_{1 \to 1} \leq \frac{M}{|\zeta|}, \quad \text{Re} \zeta > 0
\]
with generic constant $M$.

**Theorem 1.** Let $d \geq 3$, assume that the vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is in $N_e$ with the Nash norm
\[
n_e(b, h c) < \sqrt{\frac{\sigma c_4}{c_0}}
\]
for some $h > 0$ (the constants $c_0$, $c_4$ were introduced above).

The following is true:

(i) The algebraic sum $A_1 := A_1 + (b \cdot \nabla)_1$, $D(A_1) = D(A_1)$ generates a quasi bounded holomorphic semigroup $e^{-tA_1}$ in $L^1$ with the sector of holomorphy
\[
\{ z \in \mathbb{C} \mid | \arg z | < \frac{\pi}{2} - \theta \}, \quad \text{where} \quad \tan \theta = \sqrt{2 \left( \frac{M}{1 - \sqrt{\frac{c_0}{\sigma c_4}} n_e(b, h c)} - 1 \right)}.
\]
The operator $A_1$ is an operator realization of the formal Kolmogorov operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ in $L^1$.

(ii) 
\[
e^{-tA_1} = s \cdot L^1 \cdot \lim_{\epsilon \downarrow 0} e^{-tA_1^\epsilon} \quad (\text{loc. uniformly in } t \geq 0),
\]
where
\[
A_1^\epsilon := -\nabla \cdot a_{\epsilon} \cdot \nabla + b_{\epsilon} \cdot \nabla, \quad D(A_1^\epsilon) = W^{2,1}
\]
are the approximating operators, with the smooth matrices $a_{\epsilon} \in (H_{\sigma, \zeta})$ and the smooth vector fields $b_{\epsilon}$ constructed in such a way that
\[
a_{\epsilon} \to a \quad \text{strongly in } [L^2_{\text{loc}}]^{d \times d}, \quad b_{\epsilon} \to b \quad \text{strongly in } [L^2_{\text{loc}}]^d \quad \text{as } \epsilon \downarrow 0,
\]
and the Nash norms of $b_{\epsilon}$ for all small $\epsilon > 0$ are controlled by the Nash norm of $b$:
\[
n_e(b_{\epsilon}, h) \leq n_e(b, h) + \tilde{c} \epsilon \quad (\tilde{c} \text{ generic constant}).
\]
The semigroup $e^{-tA_1}$ conserves positivity and is a $L^\infty$ contraction (and so the convergence in (ii) holds for $e^{-tA_1}$ in $L^r$ for all $1 < r < \infty$).

Assuming that the Nash norm $n_e(b, h c_4)$ is sufficiently small, we further obtain:

(iii) For every $t > 0$, $e^{-tA_1}$ is an integral operator.

(iv) The heat kernel $e^{-tA_1}(x, y)$ (is the integral kernel of $e^{-tA_1}$) satisfies, possibly after redefinition on a measure zero set in $\mathbb{R}^d \times \mathbb{R}^d$, the lower and upper Gaussian bounds:

For every $\xi_1 > \xi$ there exist generic* constants $\sigma_1 \in ]0, \sigma]$ and $c_i > 0$, $\omega_i \geq 0$, $i = 1, 2$ such that
\[
c_1 e^{-\omega_1 k_{\sigma_1}}(t, x - y) \leq e^{-tA_1}(x, y) \leq c_2 e^{\omega_2 k_{\xi_1}}(t, x - y)
\]
for all $t > 0$, $x, y \in \mathbb{R}^d$.

(v) $e^{-tA_1}$ conserves probability:
\[
\langle e^{-tA_1}(x, \cdot) \rangle = 1 \quad \text{for every } x \in \mathbb{R}^d.
\]
(vi) For every $f \in L^1$, $u(t, \cdot) := e^{-t\Lambda_1} f(\cdot)$ is Hölder continuous (possibly after redefinition of a measure zero set in $\mathbb{R}^d \times \mathbb{R}^d$), i.e., for every $0 < \alpha < 1$ there exist generic $^*$ constants $C < \infty$ and $\beta \in [0, 1]$ such that for all $z \in \mathbb{R}^d$, $s > R^2$, $0 < R \leq 1$

$$|u(t, x) - u(t', x')| \leq C\|u\|_{L^\infty([s-R^2, s] \times B(z, R))} \left(\frac{|t-t'| + |x-x'|}{R}\right)^\beta$$

for all $(t, x)$, $(t', x') \in [s - (1 - \alpha^2)R^2, s] \times B(z, (1 - \alpha)R)$.

Furthermore, $u \geq 0$ satisfies the Harnack inequality: Let $0 < \alpha < \beta < 1$ and $\gamma \in [0, 1]$, then there exists a constant $K = K(d, \sigma, \xi, \alpha, \beta, \gamma) < \infty$ such that for all $(s, x) \in [R^2, \infty] \times \mathbb{R}^d$, $0 < R \leq 1$ one has

$$u(t, y) \leq K u(s, x)$$

for all $(t, y) \in [s - \beta R^2, s - \alpha^2 R^2] \times \bar{B}(x, \delta R)$.

(vii) $e^{-t\Lambda_C \infty} := \left[ e^{-t\Lambda_1} \upharpoonright C_\infty \cap L^1 \right]_{C_\infty \to C_\infty}$, $t > 0$ is a Feller semigroup in $C_\infty$ having the property $e^{-t\Lambda_C \infty} [L^\infty \cap L^1] \subset C_\infty$, $t > 0$. Moreover,

$$e^{-t\Lambda_C u} f(x) := \langle e^{-t\Lambda} (x, \cdot) f(\cdot) \rangle, \quad t > 0$$

is a Feller semigroup on $C_u$, the space of bounded uniformly continuous functions on $\mathbb{R}^d$.

(viii) For every $c_6 > \xi$ there exists a generic $^*$ constant $c_5$ such that

$$|\partial_t e^{-t\omega \Lambda_1} (x, y)| \leq c_5 t^{-1} k_{c_6}(t, x - y)$$

for all $t > 0$, $x, y \in \mathbb{R}^d$.

(ix) For every $1 < p < \infty$,

$$e^{-t\Lambda_p} := \left[ e^{-t\Lambda_1} \upharpoonright L^1 \cap L^p \right]_{L^p \to L^p}$$

is a quasi bounded holomorphic semigroup with the same sector of holomorphy as in (i).

Recall that a vector field $b$ is said to be form-bounded (with respect to $A \equiv A_2$) if there exist constants $0 < \delta < 1$ and $c(\delta) \geq 0$ such that the quadratic inequality

$$\|b_0 \varphi\|_2^2 \leq \delta \|A_{-2}^\frac{1}{2} \varphi\|_2^2 + c(\delta) \|\varphi\|_2^2 \quad (F_\delta(A))$$

is valid for all $\varphi \in D(A_{-2}^{\frac{1}{2}})$, where $b_0 := \sqrt{b \cdot a^{-1} \cdot b}$.

This is a large class of singular vector fields that contains, in particular, the vector fields $b = b_1 + b_2$ with bounded $b_2$ and $|b_1|$ in $L^d$, in the weak $L^d$ (e.g., $b_1(x) = c|x|^{-2}x$ by Hardy’s inequality), in the Campanato-Morrey class, see examples e.g., in [KGS] sect. 4.

**Theorem 2.** Let $d \geq 3$, assume that $b \in N_e$ with the same norm $n_e(b, h)$ as in Theorem (ii)-(ix) for some $h > 0$. Additionally, assume that $b \in F_\beta (-\Delta)$ for some $\beta < \infty$. Then

$$\|\nabla e^{-t\Lambda_1}\|_{1 \to 1} \leq \frac{C}{\sqrt{t}} e^{\omega_2 t}, \quad t > 0, \quad (3)$$

with constant $C$ depending on $d$, $\sigma$, $\xi$, $n_e(b, h)$, $\beta$ and $c(\beta)$. 
Remark 1. 1. The inclusion \(|b| \in L^p, p > d| \Rightarrow b \in N_e\) follows easily using \(\|e^{t\Delta}\|_{r \to \infty} \leq Ct^{-\frac{d}{2p}}\) upon taking \(r = \frac{p}{2}\):

\[
\sup_{x \in \mathbb{R}^d} \int_0^h \sqrt{e^{t\Delta}|b|^2(x)} \frac{dt}{\sqrt{t}} \leq \int_0^h \sqrt{\|e^{t\Delta}|b|^2\|_{\infty}} \frac{dt}{\sqrt{t}} \leq C\frac{2p}{p-d} \frac{p-d}{2} \|b\|_p < \infty.
\]

2. There exist \(b \in N_e\) such that, for any \(p > 2, |b| \not\in L^p\), e.g. consider

\[|b(x)| = 1_{B(0, e^{-1})}(x)|x_1|^{-\frac{1}{2}} \log |x_1|^{-\alpha}, \quad \alpha > \frac{1}{2}, \quad x = (x_1, \ldots, x_d).
\]

3. Comments

1. The following result was proved in [KIS] (the reader can compare it with Theorem 1). It establishes quantitative dependence of the regularity properties of solutions to \((\partial_t + \Lambda)u = 0\) with \(b \in F_r(A)\) on the value of \(\delta\).

Theorem 3. Let \(d \geq 3\). Assume that \(b \in F_r(A)\) for some \(0 < \delta < 4\). Set \(r_c := \frac{2}{2 - \sqrt{\delta}}\) and \(b_a := b \cdot a^{-1} \cdot b \in L_{loc}^2\). The following is true:

(i) Let \(1_n\) denote the indicator of \(\{x \in \mathbb{R}^d | b_n(x) \leq n\}\) and set \(b_n := 1_nb\). Then the limit

\[s\cdot L \cdot \lim_{n \to \infty} e^{-t\Lambda_r(b_n)}, \quad r \in I_c := [r_c, \infty[,
\]

where \(\Lambda_r(a, b_n) := A_r + b_n \cdot \nabla\), exists locally uniformly in \(t \geq 0\) and determines a positivity preserving, \(L^\infty\) contraction, quasi contraction \(C_0\) semigroup on \(L^r\), say, \(e^{-t\Lambda_r}(a, b)\).

(ii) One can define

\[e^{-t\Lambda_r(c, b)} := [e^{-t\Lambda_r(a, b)} | L^1 \cap L^r]_{L^r \to L^r}^{clos}, \quad r \in I_c^0.
\]

Then

\[\|e^{-t\Lambda_r(a, b)}\|_{r \to r} \leq e^{\omega_r}, \quad \omega_r = \frac{\lambda \delta}{2(r - 1)}, \quad r \in I_c := [r_c, \infty[.
\]

(iii) The interval \(I_c\) is the maximal interval of quasi contractive solvability.

(iv) For each \(r \in I_c^0\), \(e^{-t\Lambda_r(a, b)}\) is a holomorphic semigroup of quasi contractions in the sector

\[|\arg t| \leq \frac{\pi}{2} - \theta_r, \quad 0 < \theta_r < \frac{\pi}{2}, \quad \tan \theta_r \leq K(2 - r\sqrt{\delta})^{-1},
\]

where \(K = \frac{r-2}{\sqrt{r-1}} + r'\sqrt{\delta}\) if \(r \leq 2r_c\) and \(K = \frac{r-2+\sqrt{\delta}}{\sqrt{r-1}}\) if \(r > 2r_c\).

(v) \(e^{-t\Lambda_r(a, b)}, r \in I_c,\) extends to a positivity preserving, \(L^\infty\) contraction, quasi bounded holomorphic semigroup on \(L^r\) for every \(r \in I_m := \left]\frac{2}{2 - \sqrt{\delta}}\right],\infty[.

(vi) The interval \(I_m\) is the maximal interval of quasi bounded solvability.

(vii) For every \(r \in I_m\) and \(q > r\) there exist constants \(c_i = c_i(\delta, r, q), i = 1, 2\) such that the \((L^r, L^q)\) estimate

\[\|e^{-t\Lambda_r(a, b)}\|_{r \to q} \leq c_1 e^{c_2t} t^{-\frac{d}{2}(\frac{1}{2} - \frac{1}{q})}
\]

is valid for all \(t > 0\).

(viii) Let \(\delta < 1\), and let \(a_n \in (H_{\sigma,\xi}), b_n : \mathbb{R}^d \to \mathbb{R}^d, n = 1, 2, \ldots\) be smooth and such that

\[a_n \to a\text{ strongly in } [L^2_{loc}]^{d \times d}, \quad b_n \to b\text{ strongly in } [L^2_{loc}]^d
\]
and \( b_n \in \mathbf{F}_\delta(A^n) \) with \( c(\delta) \) independent of \( n \), where \( A^n \equiv -\nabla \cdot a_n \nabla \). Then
\[
e^{-t\Lambda_r(a,b)} = s\cdot L^r - \lim_{n\to \infty} e^{-t\Lambda_r(a_n,b_n)}
\]
whenever \( r \in I^d_c \), where \( \Lambda_r(a_n,b_n) = -\nabla \cdot a_n \nabla + b_n \cdot \nabla \) of domain \( W^{2,r} \).

**Remarks.**

(a) For \( \delta < 1 \), the corresponding to \( \Lambda \) quadratic form \( t[u] = \langle a \cdot \nabla u, \nabla u \rangle + \langle b \cdot \nabla u, u \rangle \), \( D(t) = W^{1,2} \) possesses the Sobolev embedding property \( t[u] \equiv c_\delta \| u \|^2_2 \), \( j = \frac{d}{d-2} \). This ceases to be true already for \( \delta = 1 \). The same occurs for \( 1 < \delta < 4 \) and \( r = r_c \).

(b) The intervals \( I_c, I_m \) are maximal already for \( a = I \) and \( b(x) = \sqrt{\frac{d-2}{2}}|x|-x \).

(c) Assertions (i)-(iv) are in fact valid for symmetric \( a \in [L^1_{loc}]^{d \times d} \) such that \( a \geq \sigma I, \sigma > 0 \), and \( b_a \in L^1 + L^\infty \), see [KIS, Theorem 4.2].

(d) While for \( b \in \mathbf{F}_\delta(A), \delta < 1 \) one first constructs the semigroup in \( L^2 \) (using the method of quadratic forms) and then proves the corresponding convergence results, in the case \( b \in \mathbf{F}_\delta(A) \), \( 1 \leq \delta < 4 \) the convergence result of Theorem 3(i) becomes the means of construction of the semigroup.

2. Recall that a vector field \( b : \mathbb{R}^d \to \mathbb{R}^d \) is said to belong to the Kato class \( \mathbf{K}^{d+1} \) if \( |b| \in L^1_{loc} \) and
\[
\kappa_{d+1}(b,h) := \sup_{x \in \mathbb{R}^d} \int_0^h e^{t\Delta} |b|(x) \frac{dt}{\sqrt{t}} < \infty
\]
for some \( h > 0 \).

If \( a = I \) or H"older continuous, then the condition “\( \kappa_{d+1}(b,h) \) is sufficiently small for some \( h > 0 \)” provides the upper Gaussian bound [S1], the Harnack inequality and the lower Gaussian bound on \( e^{-t\Lambda(x,y)} \) [Z1], see also [Z2]. The results in [Z1, Z2] were obtained, in fact, for the time-dependent case, i.e. for \( b = b(t,x) \) in the non-autonomous Kato class \( \mathbf{K}^{d+1} \) (introduced by Q. S. Zhang).

The Nash class \( \mathbf{N}_e \) is thus the analogue of the Kato class \( \mathbf{K}^{d+1} \) when \( a = a(x) \) is only measurable. Note that \( \mathbf{N}_e \subset \mathbf{K}^{d+1} \) as is immediate from \( e^{t\Delta} |b|(x) \leq \sqrt{e^{t\Delta}}|b|^2(x) \).

Note also that \( \mathbf{N}_e \cap \mathbf{F} \subset \mathbf{K}^d \subset \mathbf{F} \), where \( \mathbf{F} := \cup_{\beta > 0} \mathbf{F}_\beta(\Delta) \), and
\[
\mathbf{K}^d := \{|b| \in L^2_{loc} | \kappa_d(b,h) := \sup_{x \in \mathbb{R}^d} \int_0^h e^{t\Delta} |b|^2(x) \frac{dt}{\sqrt{t}} < \infty \text{ for some } h > 0 \}.
\]

Indeed, using \( b \in \mathbf{F} \), we have \( e^{t\Delta} b^2(x) \equiv \langle k(t,x,\cdot)b^2(\cdot) \rangle \leq \beta \| \nabla \sqrt{k(t,x,\cdot)} \|^2_2 + c(\beta) = \frac{\beta d}{2} + c(\beta) \) for some \( \beta > 0 \) and \( c(\beta) \). Therefore, for \( 0 < t \leq h \)
\[
e^{t\Delta} b^2(x) \leq \sqrt{\frac{\beta d}{8} + c(\beta)h} \sqrt{e^{t\Delta} b^2(x)} \frac{1}{\sqrt{t}}
\]
and so the condition \( b \in \mathbf{N}_e \) now yields the required. In turn, the inclusion \( \mathbf{K}^d \subset \mathbf{F} \) is well known (use the fact that \( b \in \mathbf{K}^d \) is equivalent to \( \| |b|(\lambda - \Delta)^{-1} \|_{1 \to 1} < \infty, \lambda > 0 \).

The principal difference between the cases covered by the Nash class \( \mathbf{N}_e \) (\( a \) is measurable) and the Kato class \( \mathbf{K}^{d+1} \) (\( a \) is H"older continuous) is as follows. For H"older continuous \( a \) one can appeal, in the proof of the two-sided bounds, to the estimate \( |\nabla_x e^{-t\Delta}(x,y)| \leq C t^{-\frac{1}{2}} e^{t\Delta}(x,y) \), which does not hold for merely measurable \( a \); for such \( a \) the role of the previous estimate is assumed by far-reaching inequalities \( \mathcal{N}(t) \leq \frac{2}{\delta }, \mathcal{N}(t) \leq \frac{2}{\delta } \), where \( \mathcal{N}(t), \dot{\mathcal{N}}(t) \) are the so-called Nash’s functions.
similar to $\langle \nabla x p \cdot \frac{a(x)}{p} \cdot \nabla x p \rangle$ employed by J. Nash \[N\], where $p \equiv p(t, x, y) = e^{-tA}(x, y)$, see Sections \[6\] and \[7\] for details.

3. Let us fix a continuous function $\phi : [0, \infty[ \to [0, \infty$ satisfying the following properties:

\[\begin{align*}
1) & \quad \phi(0) = 0, \\
2) & \quad \phi(t)/t \in L^1[0, 1].
\end{align*}\]

Put

$$ n_\phi(b, h) = \sup_{x \in \mathbb{R}^d} \int_0^h e^{t\Delta} b^2(x) \frac{dt}{\phi(t)}. $$

If $n_\phi(b, h) < \infty$ for some $h > 0$, then we write $b \in N_\phi$.

The class $N_\phi$ arises as the class providing the two-sided Gaussian on the heat kernel of $-\nabla \cdot a(t, x) \cdot \nabla + b(t, x) \cdot \nabla$, where $a(t, x)$ is a measurable uniformly elliptic matrix, see \[S2, LS\]. Since (for $b = b(x)$)

$$ \int_0^h \sqrt{\frac{e^{t\Delta} b^2(x)}{t}} dt \leq \left[ \int_0^h e^{t\Delta} b^2(x) \frac{dt}{\phi(t)} \right]^{\frac{1}{2}} \left[ \int_0^h \frac{\phi(t)}{t} dt \right]^{\frac{1}{2}}, $$

we have $N_\phi \subset N_e$ for every admissible $\phi$. Moreover, since $\phi$ is continuous and $\phi(0) = 0$, it is seen that $n_\phi(b, h) > k_d(b, h)$, and so $N_\phi \subset K^d$. Thus,

$$ N_\phi \subset N_e \cap K^d \subset K^{d+1} \cap K^d. $$

The need for more restrictive assumption “$b \in N_\phi$” when $a = a(t, x)$ is dictated by the subject matter: in the time-dependent case there are no estimates $N(t)$, $N'(t) \leq c(t)$ for any $c(t)$, cf. the previous comment.

4. Let us comment more on classes $K^{d+1}$ and $F$.

Note that $K^{d+1} \not\subset F$: There are $b \in K^{d+1}$ such that, for a given $p > 1$, $|b| \notin L^p_{loc}$, e.g. consider

$$ |b(x)| = 1_{B(0, 1)}(x)|x|^{-\alpha_2}, \quad 0 < \alpha_2 < 1. $$

On the other hand, already $[L^{d+1}] \not\subset K^{d+1}$, and so $F \not\subset K^{d+1}$. [Indeed, let

$$ |b(x)| = 1_{B(0, e^{-1})}(x)|x|^{-\alpha}, \quad \alpha > d^{-1}, \quad d \geq 3. $$

Then $|b|_d < \infty$ and $k_{d+1}(b, h) = \infty$.]

This dichotomy between the classes $K^{d+1}$ and $F$ was resolved in \[Ki, KS\] with development of the Sobolev regularity theory of $-\Delta + b \cdot \nabla$ for $b$ in the class

$$ F^{1/2} = \{ b \in L^1_{loc} \mid \lim_{\lambda \to \infty} \|b\|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}} \|_{2 \to 2} < \infty \} $$

(introduced in \[S1\] as the class responsible for the $(L^p, L^q)$ estimate on the semigroup) that contains $K^{d+1} + F := \{ b_1 + b_2 \mid b_1 \in K^{d+1}, b_2 \in F \}$.

By analogy, one can ask if it is possible to extend the convergence results in Theorem \[1\] and Theorem \[3\] or $(L^p, L^q)$ estimates, to $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ with a measurable $a \in (H^{\infty}_\sigma)$ and $b = b_1 + b_2$ with $b_1 \in N_e, b_2 \in F_\beta(A)$.

5. Theorem \[1\](iv), \[viii\] (the two-sided Gaussian bounds on the heat kernel and its time derivative) can be extended to more general operator

$$ \Lambda(a, b, \hat{b}) = -\nabla \cdot a \cdot \nabla + b \cdot \nabla + \nabla \cdot \hat{b} $$

with $a \in (H^{\infty}_\sigma)$, and $(b, \hat{b} \in N_e, \hat{b} \in F)$ or $(b, \hat{b} \in N_e, b \in F)$, provided that $n(b, h), n(\hat{b}, h)$ are sufficiently small. Note that the above assumptions on $b$ and $\hat{b}$ are non-symmetric, i.e. the presence...
of $b \in \mathbb{N}_e$ forces $\tilde{b}$ to be more regular: $\tilde{b} \in \mathbb{N}_e \cap \mathbb{F}$, and vice versa. We also note that here the form-boundedness assumption seems to be justified. The proof follows the argument in the present paper but with the Nash’s functions $\mathcal{N}, \tilde{\mathcal{N}}$ defined with respect to $u(t, x, y) := e^{-t\Lambda(a, b)}(x, y)$. We will address this matter in detail elsewhere.

6. Proposition 1 can be extended to non-local operators of the type $\Lambda = (\mu - \nabla \cdot a \cdot \nabla)^{\frac{\alpha}{2}} + b \cdot \nabla$, $1 < \alpha < 2$, with $b$ in an appropriate modification of the elliptic Nash class, see Remark 2 in Section 6.

7. In the course of the proof of Theorem 1(i) we obtain the resolvent representation as the K. Neumann series

$$(\zeta + \Lambda_1)^{-1} = (\zeta + A_1)^{-1}(1 + T_1)^{-1} \in \mathcal{B}(L^1), \quad \Re \zeta \geq \lambda_0,$$

where $\lambda_0 = \lambda_0(n_e(b, h)) > 0$, $T_1 := (b \cdot \nabla)_{1}(\zeta + A_1)^{-1} \in \mathcal{B}(L^1)$.

The latter yields $\|\nabla(\zeta + \Lambda_1)^{-1}\|_{1 \to 1} \leq c(\Re \zeta)^{-\frac{\alpha}{2}}$. [Indeed, $\|\nabla(\zeta + A_1)^{-1}\|_{1 \to 1} \leq c(\Re \zeta)^{-\frac{\alpha}{2}}$ (integrating $(\star)$ in $t \in [0, \infty]$ in the proof of Theorem 2), so the resolvent representation yields the required bound.] Also, for $1/2 < \alpha < 1$, $\|\nabla(\zeta + \Lambda_1)^{-\alpha}\|_{1 \to 1} \leq C(\Re \zeta)^{-\frac{\alpha}{2} + \frac{1}{2}}$.

8. In Theorem 2 we proved, although under the additional assumption $b \in \mathbb{F}$, that $\|\nabla e^{-tA_1}\|_{1 \to 1} \leq C_1 t^{-\frac{1}{2}} e^{\nu t} t^\nu$.

It is not clear how to extend the last bound and the bound in 7 to $\|\nabla e^{-t\Lambda_p}\|_{p \to p} \leq C_p t^{-\frac{1}{2}} e^{\nu t}$, $\|\nabla(\zeta + \Lambda_p)^{-1}\|_{p \to p} \leq c_p(\Re \zeta)^{-\frac{\alpha}{2}}$ (\star) for some $p > 1$. Of course, if also $b \in \mathbb{F}_\beta(A)$ with $\beta < 1$, then by standard theory $\|\nabla e^{-t\Lambda_2}\|_{2 \to 2} \leq C_2 t^{-\frac{1}{2}} e^{\nu t}$, $t > 0$ for constants $C_2, \nu_2$ depending on $d$, $\xi$, $\sigma$, $\beta$ and $c(\beta)$, and so (\star) follows by interpolation for all $p \in [1, 2]$ (similarly for $\nabla(\zeta + \Lambda_p)^{-1}$).

9. The authors do not know if there is a proof of the Harnack inequality for $\Lambda = -\nabla \cdot a \cdot \nabla + b \cdot \nabla$, $a \in (H_{\sigma, \xi}), b \in \mathbb{N}_e$ that does not use the lower bound on $e^{-tA}(x, y)$.

4. Heat kernel bounds on $e^{-tA}(x, y)$

Let $a \in (H_{\sigma, \xi}), 0 < \sigma < \xi < \infty$. Set $p(t, x, y) := e^{-tA}(x, y), A \equiv A(a)$.

Theorem 4. Fix $0 < c_2 < \sigma$ and $c_4 > \xi$. There exist constants $c_1, c_3 > 0$ that depend only on $d, c_2, c_4$ such that, for all $t > 0$, $x, y \in \mathbb{R}^d$,

$$p(t, x, y) \leq c_3 k_{c_4}(t, x - y)$$

and

$$c_1 k_{c_2}(t, x - y) \leq p(t, x, y).$$

Also, for a given $c_6 > \xi$ there is a generic constant $c_5$ depending on $c_6$ such that

$$t |\partial_t p(t, x, y)| \leq c_5 k_{c_6}(t, x - y)$$

for all $t > 0$, $x, y \in \mathbb{R}^d$.

The proof of (UBG$^p$) and (LGB$^p$) with some constants $c_2$ and $c_4$ is due to [A]. The proof of (UBG$^{2h}$) with some constant $c_6$ is due to [EP]. The proof of (UBG$^p$) and (UBG$^{2h}$) in the form as stated is due to [KS], and in a strengthened form, i.e. with polynomial factor, can be found in [Da]. The proof of (LGB$^p$) as stated is due to [Sl].
5. Nash's function $\mathcal{N}_\delta(t,x)$

Put $p(t,x,y) \equiv p_\varepsilon(t,x,y) := e^{-tA^\varepsilon(x,y)}$, where $A^\varepsilon := -\nabla \cdot a^\varepsilon \cdot \nabla$, $a^\varepsilon := E\varepsilon a$. Below we write for brevity $a \equiv a^\varepsilon$.

Define Nash's function

$$\mathcal{N}_\delta(t,x) := \langle \nabla p(t,\cdot,x) \cdot \frac{a(\cdot)}{k_\varepsilon(t,x-\cdot)} \cdot \nabla p(t,\cdot,x) \rangle, \quad \delta > 0.$$ 

In what follows, we apply $\mathcal{N}_\delta$ (and its counterpart $\hat{\mathcal{N}}_\delta$, see Section 8) with essentially the same purpose as J. Nash did himself [N].

**Proposition 2.** If $\delta = c_4$ then there exists a generic constant $c_0$ such that

$$\mathcal{N}_\delta(t,x) \leq \frac{c_0}{t}, \quad (t,x) \in [0, \infty) \times \mathbb{R}^d.$$

**Proof.** Write $\mathcal{N}_\delta = \langle \nabla p \cdot \frac{ap}{k_\varepsilon} \cdot \nabla p \rangle$. Integrating by parts and using the equation $(\partial_t + A^\varepsilon)p(t,\cdot,x) = 0$, we have

$$\mathcal{N}_\delta = \langle - \partial_t p, \frac{p}{k_\varepsilon} \rangle + \langle \nabla p \cdot \frac{ap}{k_\varepsilon} \cdot \nabla k_\varepsilon \rangle.$$

Let us show that the RHS is finite. By (UGB), (UGB$^{\theta,p}$) and by our choice of $\delta$,

$$|\langle - \partial_t p, \frac{p}{k_\varepsilon} \rangle| \leq c_3c_5t^{-1} \langle k_{\varepsilon}k_{\varepsilon} \rangle = \frac{c_3c_5}{t},$$

Due to (UGB$^p$) and a qualitative bound $|\nabla p(t,x,y)| \leq Ct^{-1/2}k_\varepsilon(t,x,y)$ (i.e. the constants $C$, $c$ depend on $\varepsilon$), we have $|\langle \nabla p \cdot \frac{ap}{k_\varepsilon} \cdot \nabla k_\varepsilon \rangle| < \infty$ and hence $\mathcal{N}_\delta < \infty$.

By quadratic inequalities and (UGB$^p$),

$$|\langle \nabla p \cdot \frac{ap}{k_\varepsilon} \cdot \nabla k_\varepsilon \rangle| \leq c_3\mathcal{N}_\delta^2 \langle \nabla k_\varepsilon \cdot \frac{a}{k_\varepsilon} \left( \frac{k_{\varepsilon}k_{\varepsilon}}{k_\varepsilon} \right) \cdot \nabla k_\varepsilon \rangle^{1/2},$$

and

$$\langle \nabla k_\varepsilon \cdot \frac{ak_{\varepsilon}k_{\varepsilon}}{k_\varepsilon} \cdot \nabla k_\varepsilon \rangle \leq \xi \langle \left( \nabla k_\varepsilon \right) \rangle \left( \frac{k_{\varepsilon}k_{\varepsilon}}{k_\varepsilon} \right) = \frac{\xi d}{2t} \leq \frac{1}{2t}.$$

and so

$$\mathcal{N}_\delta \leq 2\langle - \partial_t p, \frac{p}{k_\varepsilon} \rangle + \frac{c_3^2}{t} \langle \nabla k_\varepsilon \cdot \frac{a}{k_\varepsilon} \cdot \nabla k_\varepsilon \rangle \leq \frac{c_0}{t}, \quad \text{where } c_0 = 2c_3c_5 + \frac{d}{2}. \quad \square$$

6. Proof of Proposition 1

1. Let $1_\varepsilon, \varepsilon > 0$ be the indicator of $\{x \in \mathbb{R}^d \mid |x| \leq \varepsilon^{-1}, |b(x)| \leq \varepsilon^{-1}\}$. Define

$$b_\varepsilon := E_{\nu_\varepsilon}(1_\varepsilon b),$$

where, recall, $E_{\nu_\varepsilon} \equiv e^{\nu_\varepsilon \Delta}$, and $\nu_\varepsilon > 0$.

Define also $(b^2)_\varepsilon = E_{\nu_\varepsilon}(1_\varepsilon b^2)$ and set $g_{1,\varepsilon} := b_\varepsilon - 1_\varepsilon b$ and $g_{2,\varepsilon} := |(b^2)_\varepsilon - 1_\varepsilon b^2|$. In what follows, we select $\nu_\varepsilon$ so that $\nu_\varepsilon \downarrow 0$ sufficiently rapidly as $\varepsilon \downarrow 0$ so that $\|g_{1,\varepsilon}\|_2 \leq \varepsilon$ and $\|g_{2,\varepsilon}\|_q \leq \varepsilon^2$ for some $q \geq d$. Note that $(b^2)_\varepsilon \leq g_{2,\varepsilon} + b^2$. Since $\|1_{B(0,R)}(b_\varepsilon - b)\|_2 \leq \|g_{1,\varepsilon}\|_2 + \|1_{B(0,R)}(1_\varepsilon b - b)\|_2$, we have

$$b_\varepsilon \to b \quad \text{strongly in } [L^2_{loc}]^d.$$
The Nash norms of \( b_\varepsilon \) are controlled by the Nash norm of \( b_\varepsilon \):

**Lemma 1.** \( n_e(b_\varepsilon, h) \leq n_e(b, h) + c_2 h^{1/2}, \varepsilon > 0. \)

**Proof.** Clearly, \( (b_\varepsilon)^2 \leq (b^2)\varepsilon \), and so

\[
n_e(b_\varepsilon, h) \equiv \sup_{x \in \mathbb{R}^d} \int_0^h \sqrt{e^{t\Delta_b}(b_\varepsilon)^2(x)} \frac{dt}{\sqrt{t}} \leq n_e(b, h) + \sup_{x \in \mathbb{R}^d} \int_0^h \sqrt{e^{t\Delta}g_{2,\varepsilon}(x)} \frac{dt}{\sqrt{t}},
\]

where

\[
\sup_{x \in \mathbb{R}^d} \int_0^h \sqrt{e^{t\Delta}g_{2,\varepsilon}(x)} \frac{dt}{\sqrt{t}} \leq \int_0^h \sqrt{\|e^{t\Delta}g_{2,\varepsilon}\|_\infty} \frac{dt}{\sqrt{t}} \leq C_d \int_0^h t^{-\frac{d}{2q}} \|g_{2,\varepsilon}\|_q \frac{dt}{\sqrt{t}} \leq \sqrt{\|g_{2,\varepsilon}\|_q C_d} \frac{2}{1 - \frac{d}{2q}} h^{\frac{1}{2} - \frac{d}{4q}} \leq 4C_d h^{\frac{1}{2}}\varepsilon.
\]

\( \square \)

2. Now we can give

**Proof of Proposition \( \square \).** Set \( \delta := c_4 \). We will construct \( (b \cdot \nabla)g \) and prove

\[
\| (b \cdot \nabla)_1 g \|_1 \leq \eta \| (\zeta + A_1) g \|_1, \quad g \in D(A_1),
\]

with \( \eta := \frac{1}{1 - e^{-\Re \sqrt{\frac{\varepsilon}{h}}}} n_e(b, h\delta) \), for all \( \Re \varepsilon > 0 \), so taking \( \zeta := \mu > 0 \) we obtain the assertion of the proposition.

**Step 1.** Put \( B_1^\varepsilon := [b_\varepsilon \cdot \nabla \mid C^1_0]_{L^1 \rightarrow L^1} \) of domain \( W^{1,1} \), and

\[
T_1^\varepsilon := B_1^\varepsilon (\zeta + A_1^\varepsilon)^{-1} \in \mathcal{B}(L^1),
\]

where, recall, \( A_1^\varepsilon := -\nabla \cdot a_\varepsilon \cdot \nabla, a_\varepsilon \equiv E_\varepsilon a, D(A_1^\varepsilon) = W^{2,1} \). Since \( B_1^\varepsilon \) is closed, we can write

\[
T_1^\varepsilon f(x) = \int_0^\infty e^{-\zeta t} B_1^\varepsilon e^{-t A_1^\varepsilon} f(x) dt = \int_0^\infty e^{-\zeta t} \langle b_\varepsilon(x) \cdot \nabla x e(t, x, \cdot)f(\cdot) \rangle dt, \quad f \in W^{1,1}.
\]

Denote \( \mu := \Re \zeta \). We have

\[
\| T_1^\varepsilon f \|_1 \leq \sum_{j=0}^\infty e^{-j \mu h} \int_{j h}^{(j+1)h} \| B_1^\varepsilon e^{-t A_1^\varepsilon} f \|_1 dt = \sum_{j=0}^\infty e^{-j \mu h} \int_0^h \| B_1^\varepsilon e^{-t A_1^\varepsilon} e^{-j h A_1^\varepsilon} f \|_1 dt.
\]

By the Fubini Theorem and the Cauchy-Bunyakovsky inequality,

\[
\int_0^h \| B_1^\varepsilon e^{-t A_1^\varepsilon} e^{-j h A_1^\varepsilon} f \|_1 dt \leq \left\langle \int_0^h \langle \langle b_\varepsilon(x) \cdot \nabla x e(t, x, y) \|_x dt \right| e^{-j h A_1^\varepsilon} f(y) \rangle \right\rangle_y \\
\leq \sup_{y \in \mathbb{R}^d} \int_0^h \langle \langle b_\varepsilon(x) \cdot \nabla x e(t, x, y) \|_x dt \| f \rangle_1 \\
\leq \sup_{y \in \mathbb{R}^d} \int_0^h \sqrt{\langle k_\delta(t, x - y)(b_\varepsilon \cdot a_{\varepsilon^{-1}} b_\varepsilon)(x) \|_x \sqrt{N_\delta(t, y)} dt \| f \rangle_1,}
\]

(4)
where $N_\delta(t, y) \equiv \langle \nabla_x p_e(t, x, y) \cdot \frac{a_e(x)}{k_e(t, x-y)}, \nabla_x p_e(t, x, y) \rangle_x \leq \frac{\gamma}{\sigma}$ by Proposition 2. Therefore,

$$\int_0^h \|B_t^\epsilon e^{-tA_s^\epsilon} e^{-j A_s^\epsilon} f\|_1 dt \leq \sqrt{\frac{\gamma_0}{\sigma \delta}} n_\epsilon(b_\epsilon, h \delta)\|f\|_1$$

(we are applying lemma above)

$$\leq \sqrt{\frac{\gamma_0}{\sigma \delta}} (n_\epsilon(b, h \delta) + c_\delta h^{1/2} \delta^{1/2})\|f\|_1.$$ 

Thus,

$$\|T^\epsilon f\|_1 \leq \eta_\epsilon\|f\|_1, \quad \eta_\epsilon := \eta + \tilde{\epsilon}, \quad \Re \zeta > 0.$$ 

**Step 2.** Set $T f := b \cdot \nabla (\zeta + A)^{-1} f$, $f \in L^2$ and note that $\nabla (\zeta + A)^{-1} \to \nabla (\zeta + A)^{-1}$ strongly in $[L^2]^d$. The proof is standard: For $1 \leq i \leq d$, $f \in W^{-1,2}$, $\|\nabla (\zeta + A)^{-1} f - \nabla (\zeta + A)^{-1} f\|_2 =: M_\epsilon(f)$,

$$M_\epsilon(f) := \|\nabla (\zeta + A)^{-1} \nabla \cdot (a - a_\epsilon) \cdot \nabla (\zeta + A)^{-1} f\|_2 \leq \|\nabla (\zeta + A)^{-1} \nabla\|_{2 \to 2} \|a - a_\epsilon\| \cdot \nabla (\zeta + A)^{-1} f\|_2,$$

where $\|\nabla (\zeta + A)^{-1} \nabla\|_{2 \to 2} \leq C, C \neq C(\epsilon)$ and $\|a - a_\epsilon\| \cdot \nabla (\zeta + A)^{-1} f\|_2 \to 0$ (e.g. using the Dominated Convergence Theorem), so $M_\epsilon(f) \to 0$ as $\epsilon \downarrow 0$, in particular, for $f \in L^2$.

Therefore, since $b_\epsilon \to b$ strongly in $[L^2_{\text{loc}}]^d$,

$$T^\epsilon f \to Tf \quad \text{strongly in } L^1_{\text{loc}} \text{ as } \epsilon \downarrow 0. \quad (5)$$

Passing to a subsequence in $\epsilon$, if necessary, we have $T^\epsilon f \to Tf$ a.e. Applying Fatou’s Lemma, we have by Step 1, for all $f \in L^1 \cap L^2$,

$$\|Tf\|_1 \leq \liminf_{\epsilon} \|T^\epsilon f\|_1 \leq \eta\|f\|_1. \quad (6)$$

Let $T_1$ denote the extension of $T \upharpoonright L^1 \cap L^2$ by continuity to $L^1$.

**Step 3.** Since, by Step 2, $\|b \cdot \nabla (\zeta + A)^{-1} f\|_1 \leq \eta\|f\|_1$ for all $f \in L^1 \cap L^2$, $\Re \zeta > 0$, the operator $B := b \cdot \nabla \upharpoonright (D(A_1) \cap D(A)) : L^1 \to L^1$, and

$$\|b \cdot \nabla h\|_1 \leq \eta\|\zeta + A_1\| h\|_1, \quad h \in D(A_1) \cap D(A).$$

Since $D(A_1) \cap D(A) := (1 + A)^{-1}[L^1 \cap L^2]$ is a core of $A_1$, $B$ extends by continuity in the graph norm of $A_1$ to $A_1$-bounded operator $(b \cdot \nabla)_1$. The proof of Proposition 1 is completed. \hfill \Box

**Remark 2.** Fix $1 < \alpha < 2$ and assume that $b \in [L^2_{\text{loc}}]^d$ satisfies

$$\tilde{\eta}^\alpha(b, \mu) = \sup_{y \in \mathbb{R}^d} \int_0^\infty e^{-\mu t} \sqrt{e^t |b(y)\|} \frac{dt}{t^{\alpha/2}} < \infty, \quad \mu > 0.$$ 

Put $T^\epsilon_i := b_\epsilon \cdot \nabla (\mu + A)^{-1} f$. We note that a key bound $\|T^\epsilon_i f\|_1 \leq \tilde{\eta}\|f\|_1, f \in L^1$ remains valid with $\tilde{\eta} = \delta^{1/\alpha} \sqrt{\frac{\gamma_0}{\sigma} \tilde{\eta}_0(b, \mu \delta^{-1})}$ (cf. comment 6 in Section 3). Namely,

$$\|T^\epsilon_i f\|_1 \leq \left(\sup_{y} \int_0^\infty e^{-\mu t} t^{-\alpha/2-1} \langle k_\delta(t, t-y) b_\alpha(y) \rangle \sqrt{N_\delta(t, y)} dt\right)\|f\|_1 \quad (b_\alpha^2 = b_1 a_1 \cdot b)$$

$$\leq \delta^{1/\alpha} \sqrt{\frac{\gamma_0}{\sigma} \tilde{\eta}_0(b, \mu \delta^{-1})}\|f\|_1.$$ 

Above one can replace $\tilde{\eta}_0(b, \mu)$ by $\eta^\alpha(b, h) := \sup_{y \in \mathbb{R}^d} \int_0^h \sqrt{e^t |b(y)\|} \frac{dt}{t^{\alpha/2}}.$
7. Proof of Theorem 1

In the proof of Proposition 1 we established: \( T_1^\varepsilon := b_\varepsilon \cdot \nabla (\zeta + A_1^\varepsilon)^{-1} \), \( T_1 := (b \cdot \nabla)_1 (\zeta + A_1)^{-1} \), \( \text{Re} \zeta > 0 \) satisfy \( T_1 \in \mathcal{B}(L^1) \) and

\[
\|T_1^\varepsilon\|_{1 \rightarrow 1} \leq \eta + \tilde{c} \varepsilon, \quad \|T_1\|_{1 \rightarrow 1} \leq \eta.
\]

**Proposition 3.** \( T_1 = s \cdot L^1 \cdot \lim_{\varepsilon \to 0} T_1^\varepsilon \).

**Proof of Proposition 3.** Under the additional assumption \( b^2 \in L^1 + L^\infty \), the assertion of the proposition is evident (use (5) in the proof of Proposition 1). In general one has to employ the separation property of \( e^{-tA} \).

Since \( \sup_{\varepsilon > 0} \|T_1^\varepsilon\|_{1 \rightarrow 1}, \|T_1\|_{1 \rightarrow 1} < \infty \), it suffices to prove the claimed convergence on \( C_c^{\infty} \). Fix \( f \in C_c^{\infty} \) and then \( r > 0 \) by \( B(0, r) \supset \text{sprt} \, f \). Since by (5) \( T_1^\varepsilon f \rightarrow T_1 f \) strongly in \( L^1_{\text{loc}} \), the required convergence in (ii) would follow from (III) once we show that, for every \( \theta > 0 \), there exists \( R = R(r, \theta) > 0 \) such that

\[
\|1_{B^c(0, R)} T_1^\varepsilon f\|_1 \leq \theta \|f\|_1 \quad \text{for all } \varepsilon > 0 \text{ sufficiently small.}
\]

To prove the latter, we write

\[
1_{B^c(0, R)} T_1^\varepsilon f(x) = \int_0^\infty e^{-\xi t} (1_{B^c(0, R)}(x) b_\varepsilon(x) \cdot \nabla p_\varepsilon(t, x, \cdot) f(\cdot)) dt,
\]

where, recall, \( p_\varepsilon(t, x, y) = e^{-tA^\varepsilon_1}(x, y) \). Put \( \mu := \text{Re} \zeta \). Then

\[
\|1_{B^c(0, R)} T_1^\varepsilon f\|_1 \leq \sum_{j=0}^\infty e^{-j\mu h} \int_{j h}^{(j+1)h} \|1_{B^c(0, R)} B_1^\varepsilon e^{-tA^\varepsilon_1} f\|_1 dt
\]

\[
= \sum_{j=0}^\infty e^{-j\mu h} \int_0^h \|1_{B^c(0, R)} B_1^\varepsilon e^{-tA^\varepsilon_1} e^{-j \mu h} f\|_1 dt
\]

\[
= \sum_{j=0}^\infty e^{-j\mu h} \left[ \int_0^h \|1_{B^c(0, R)} B_1^\varepsilon e^{-tA^\varepsilon_1} 1_{B(0, m r)} e^{-j \mu h} f\|_1 dt \right]
\]

\[
+ \int_0^h \|1_{B^c(0, R)} B_1^\varepsilon e^{-tA^\varepsilon_1} 1_{B^c(0, mr)} e^{-j \mu h} f\|_1 dt \right] = \sum_{j=0}^\infty e^{-j\mu h} [I_j + J_j],
\]

where constant \( m \geq 1 \) is to be chosen. Arguing as in the proof of Step 1 of the proof of Proposition 1 and putting \( \delta := c_4 \), we obtain, for all \( j \geq 0 \),

\[
I_j \leq \sqrt{\frac{c_0}{\sigma \delta}} \sup_{y \in B(0, m r)} \int_0^h \sqrt{\langle k(t, y, \cdot) 1_{B^c(0, R)}(\cdot) b_\varepsilon(\cdot) \rangle^2 dt} \|e^{-khA^\varepsilon_1} f\|_1
\]

\[
\leq \left( \sqrt{\frac{c_0}{\sigma \delta}} M_R + 4C_d(h\delta)^{\frac{1}{2}} \varepsilon \right) \|f\|_1,
\]

where \( M_R := \sup_{y \in B(0, m r)} \int_0^h \sqrt{\langle k(t, y, \cdot) 1_{B^c(0, R)}(\cdot) b(\cdot) \rangle^2 dt} \sqrt{t}, \quad R > m r. \)
Clearly, \( J_0 = 0 \). For all \( j \geq 1 \) and \( \eta_0 = \sqrt{\frac{\alpha_0}{\sigma_0}} n_\varepsilon(b, h\delta) \),
\[
J_j \leq \eta_0 \|1_{B^c(0, m\varepsilon)} e^{-jhA_1} f\|_1
\]
(we are applying (UGB\( \bar{\varepsilon} \)) to \( e^{-jhA_1(x, y)} \))
\[
\leq \eta_0 c_3 (4\pi c_4 jh)^{-\frac{1}{2}} e^{-\frac{(m-1)^2 \varepsilon^2}{4\pi c_4 jh}} \|f\|_1.
\]
Thus, we have
\[
\|1_{B^c(0, R)} T^c_1 f\|_1 \leq \theta \|f\|_1,
\]
where
\[
\theta := \left( \frac{c_0}{\sigma \delta} M_R + 4C_d(h\delta)^\frac{1}{2} \frac{1}{1 - e^{-\mu h}} + C_\varepsilon \sum_{j=1}^{\infty} e^{-n_j h} (j \varepsilon + \frac{4}{3} e^{-\frac{(m-1)^2 \varepsilon^2}{4\pi c_4 jh}}) \right).
\]
It is clear that selecting \( m \) sufficiently large, we can make the second term in the RHS as small as needed.

We are left to prove the convergence \( M_R \to 0 \) as \( R \to \infty \).

\( (a_1) \) Fix \( n > 0 \) by \( k_\delta(t, z, y) \leq C_n k_\delta(t, z, 0) \) for all \( t > 0 \), \( z \in B^c(0, (m + n)r), y \in B(0, mr) \). Then
\[
M_R \leq C_n \int_0^h \sqrt{\langle k_\delta(t, 0, \cdot)1_{B^c(0, R)}(\cdot)|b(\cdot)|^2 \rangle} \frac{dt}{\sqrt{t}} \quad \forall R > (m + n)r.
\]
\( (a_2) \) Due to \( b \in N_\varepsilon \) the function
\[
w_R(t) := \sqrt{\langle k_\delta(t, 0, 0)1_{B^c(0, R)}(\cdot)|b(\cdot)|^2 \rangle} \frac{1}{\sqrt{t}}
\]
is in \( L^1([0, h]) \) for every \( R \geq 1 \). Moreover, it is seen from the definition of \( w_R \) that for every
\( 0 < t_1 < t_2 \leq h \), \( w_R(t_1) \leq C_{t_1, t_2 - t_1} w_R(t_2), C_{t_1, t_2 - t_1} \leq \infty \). Thus, \( w_R(t) \) is finite for all \( 0 < t \leq h \).

\( (a_3) \) \( w_R(t) \to 0 \) as \( R \to \infty \) for every \( 0 < t \leq h \).

Indeed, fix \( t \in [0, h] \). Set \( v_R(x) := k_\delta(t, x, 0)1_{B^c(0, R)}(x)|b(x)|^2 \). For a.e. \( x \in \mathbb{R}^d, v_R(x) \downarrow 0 \) as \( R \to \infty \), and \( v_R \leq v_1 \) a.e. on \( \mathbb{R}^d \) for all \( R \geq 1 \), where \( v_1 \) is summable. Hence by the Dominated Convergence Theorem, \( \langle v_R \rangle \to 0 \) as \( R \to \infty \), and so \( w_R(t) \to 0 \) as \( R \to \infty \).

\( (a_4) \) Due to \( (a_3) \) and \( w_R \leq w_1 \) for \( R \geq 1 \), the Dominated Convergence Theorem yields
\[
\int_0^h w_R(t) dt \to 0 \quad \text{as} \quad R \to \infty.
\]
Thus, \( M_R \to 0 \) as \( R \to \infty \). The proof of Proposition 3 is completed. \( \square \)

We are in position to complete the proof of Theorem 1. Recall \( \delta := c_4 \).

\( (i) \) By our assumption on \( n_\varepsilon(b, h\delta) \), there exists \( \lambda_0 > 0 \) such that
\[
\eta := \frac{1}{1 - e^{-\lambda_0 h}} \sqrt{\frac{c_0}{\sigma \delta} n_\varepsilon(b, h\delta)} < 1.
\]
By Proposition 3 \( \Lambda_1 \) is a closed densely defined operator. Using (4), we obtain that
\[
(\zeta + \Lambda_1)^{-1} = (\zeta + A_1)^{-1} (1 + T_1)^{-1} \in \mathcal{B}(L^1), \quad \Re \zeta > \lambda_0.
\]
Using (2), we obtain
\[
\| (\zeta + \Lambda_1)^{-1} \|_{1 \to 1} \leq \frac{M}{\zeta(1 - \eta)}, \quad \Re \zeta > \lambda_0
\]
completing the proof of the first part of assertion (i).
To prove the second part of (i), note that, in view of (7), the resolvent \( \zeta \mapsto (\zeta + \lambda_0 + \Lambda_1)^{-1} = \Theta(\zeta + \lambda_0) \) is holomorphic in the right-half plane \( \text{Re} \zeta > 0 \) and in \( |\zeta - \zeta_0| < \sqrt{2(M - 1)}|\zeta_0| \) for every \( \zeta_0 \) with \( \text{Re} \zeta_0 = 0 \) (see, if needed, the argument in [Y, Ch. IX, sect. 10]). Thus, \( e^{-z(\lambda_0+\Lambda_1)} \) is holomorphic in the sector

\[
\{ z \in \mathbb{C} \mid |\arg z| < \frac{\pi}{2} - \theta_{\lambda_0} \}, \quad \text{where} \quad \tan \theta_{\lambda_0} = \sqrt{2}\left(\frac{M}{1 - \eta} - 1\right).
\]

This completes the proof of assertion (i).

(ii) The claimed approximation \( \{ b_z \} \) was constructed in the proof of Proposition I. Let us show that

\[
(\lambda + \Lambda_t^\varepsilon)^{-1} \to (\lambda + \Lambda_t)^{-1} \quad \text{strongly in } L^1 \text{ as } \varepsilon \downarrow 0,
\]

which, by a standard result, implies the convergence of the semigroups.

Since \( (\lambda + \Lambda_t^\varepsilon)^{-1} = (\lambda + \Lambda_t^\varepsilon)^{-1}(1 + T_t^\varepsilon)^{-1}, (\lambda + \Lambda_t) = (\lambda + A_1)^{-1}(1 + T_t)^{-1} \), it suffices to show that 1) \( T_t^\varepsilon \to T_t \) and 2) \( (\lambda + A_t^\varepsilon)^{-1} \to (\lambda + A_1)^{-1} \) strongly in \( L^1 \) as \( \varepsilon \downarrow 0 \). 1) is Proposition 3 2) follows immediately from

\[
(\lambda + A^\varepsilon)^{-1} \to (\lambda + A)^{-1} \quad \text{strongly in } L^2
\]

and \( (\lambda + A^\varepsilon)^{-1}(x, y) \leq C(\lambda - c\Delta)^{-1}(x, y) \) for generic constants \( 0 < c, C < \infty \), an immediate consequence of (\text{UGB}^-).

To prove assertions (iii)-(ix) we need an a priori upper and lower Gaussian bounds on the heat kernel \( u(t, x, y) := e^{-t\Lambda_t^\varepsilon}(x, y) \):

**Theorem 5.** Fix \( \xi_1 > \xi \). Provided that \( n_\varepsilon(b_\varepsilon, h) \) is sufficiently small for some \( 0 < h \neq h(\varepsilon) \), there are constants \( 0 < \sigma_1 < \sigma \) and \( c_{\sigma_1}, c_{\xi_1} > 0, \omega_i \geq 0, i = 1, 2, \) such that, for all \( t > 0, x, y \in \mathbb{R}^d \),

\[
c_{\sigma_1}e^{-t\omega_1}k_{\sigma_1}(t, x - y) \leq u(t, x, y) \leq c_{\xi_1}e^{t\omega_1}k_{\xi_1}(t, x - y).
\]

The constants \( \sigma_1, c_{\sigma_1}, c_{\xi_1}, \omega_i \) depend on \( d, \xi_1 \) and \( n_\varepsilon(b_\varepsilon, h) \).

Theorem 5 is proved in Section 8.

(iii) The upper bound in (\text{UGB}^-) yields

\[
\| e^{-t\Lambda_t^\varepsilon} \|_{1 \to \infty} \leq c_2 e^{t\omega_2}t^{-\frac{d}{2}}, \quad t > 0, \quad \varepsilon > 0
\]

with generic* constants \( c_2, \omega_2 < \infty \). Using Theorem II(ii) and applying Fatou’s lemma, we obtain

\[
\| e^{-t\Lambda_1} \|_{1 \to \infty} \leq c_2 e^{t\omega_2}t^{-\frac{d}{2}}, \quad t > 0.\quad\text{Hence } e^{-t\Lambda_1} \text{ is an integral operator for every } t > 0.
\]

(iv) The a priori bounds (\text{UGB}^-) and Theorem II(ii) yield for every pair of bounded measurable subsets \( S_1, S_2 \subset \mathbb{R}^d \)

\[
c_{1}e^{t\omega_1}\langle 1_{S_1}, e^{t\sigma_1\Lambda_1}1_{S_2} \rangle \leq \langle 1_{S_1}, e^{-t\Lambda_1}1_{S_2} \rangle \leq c_2 e^{t\omega_2}\langle 1_{S_1}, e^{t\xi_1\Lambda_1}1_{S_2} \rangle.
\]

Since \( e^{-t\Lambda_1} \) is an integral operator for every \( t > 0 \), assertion (iv) follows by applying the Lebesgue Differentiation Theorem.

(v) For every \( \varepsilon > 0 \), \( e^{-t\Lambda_1}(x, \cdot) = 1, x \in \mathbb{R}^d \). Fix \( t > 0 \) and \( \Omega \subset \mathbb{R}^d \), a bounded open set. By the upper bound (\text{UGB}^-), for every \( \gamma > 0 \) there exists \( R = R(\gamma, t, \Omega) > 0 \) such that, for every \( x \in \Omega \), \( e^{-t\Lambda_1}(x, \cdot)1_{B_R(0)}(\cdot) < \gamma \), so \( e^{-t\Lambda_1}(x, \cdot)1_{B_R(0)}(\cdot) \geq 1 - \gamma \). Hence

\[
\langle 1_{\Omega}e^{-t\Lambda_1}1_{B_R(0)} \rangle \geq (1 - \gamma)|\Omega|.
\]
Applying Theorem \( \Pi(ii) \), we obtain
\[
\frac{1}{|\Omega|} \langle 1_{\Omega} e^{-t\Lambda} 1 \rangle \geq \frac{1}{|\Omega|} \langle 1_{\Omega} e^{-t\Lambda} 1_{B(0,R)} \rangle \geq 1 - \gamma.
\]

Applying the Lebesgue Differentiation Theorem, we obtain \( \langle e^{-t\Lambda}(x,\cdot) \rangle \geq 1 - \gamma \) for a.e. \( x \in \mathbb{R}^d \). In turn, the opposite inequality \( \langle e^{-t\Lambda}(x,\cdot) \rangle \leq 1 \) for a.e. \( x \in \mathbb{R}^d \) follows easily using Theorem \( \Pi(ii) \), and hence \( 1 \geq \langle e^{-t\Lambda}(x,\cdot) \rangle \geq 1 - \gamma \). The proof of (v) is completed.

(vi) Put \( u_\varepsilon(t,x) := e^{-t\Lambda_\varepsilon}f(x) \). Repeating the argument in [FS, sect. 3] which appeals to the ideas of E. De Giorgi, we obtain assertion (vi) for \( u_\varepsilon \). The result now follows upon applying Theorem \( \Pi(ii) \) and the Arzelà-Ascoli Theorem.

(vii) follows from (iv), (v) and (vi) using a standard argument for mollifiers.

(viii) is proved repeating the argument in [Da, sect. 2].

(ix) follows repeating the argument in [On].

8. Proof of Theorem 5

8.1. Auxiliary estimates. For a given \( \lambda > 0 \), denote
\[
k_\lambda := k_\lambda(\tau - s, y - \cdot) \quad \text{and} \quad \hat{k}_\lambda := k_\lambda(t - \tau, x - \cdot), \quad s < \tau < t
\]
and
\[
\langle \frac{(\nabla k_\lambda)^2}{k_\lambda} \rangle := \langle \frac{(\nabla k_\lambda(\tau - s, y - \cdot))^2}{k_\lambda(\tau - s, y - \cdot)} \rangle.
\]
The next three facts are evident:

(a1) \[
\langle \frac{(\nabla k_\lambda)^2}{k_\lambda} \rangle = \frac{d}{2\lambda} \frac{1}{\tau - s} = \langle \left( \frac{y - \cdot}{2\lambda(\tau - s)} \right)^2 k_\lambda(\tau - s, y - \cdot) \rangle,
\]
\[
\langle \frac{(\nabla \hat{k}_\lambda)^2}{k_\lambda} \rangle = \frac{d}{2\lambda} \frac{1}{t - \tau}.
\]

(a2) If \( \lambda < \lambda_1 \), then \( k_\lambda \leq \left( \frac{\lambda}{\lambda_1} \right)^{\frac{d}{2}} k_{\lambda_1} \).

(a3) If \( 2\delta > c_4 \), then
\[
k_{\delta c_4}^2 = \left( \frac{\delta^2}{2(\delta - c_4)c_4} \right)^{\frac{d}{2}} k_{\frac{\delta c_4}{\delta - c_4}}.
\]

(a4) \[
\begin{cases}
0 < 2\delta < \lambda \\
0 < \varepsilon < 1 \\
0 < \tau - s < (t-s)\varepsilon
\end{cases}
\Rightarrow
\begin{cases}
\hat{k}_\lambda k_\delta \leq c_2^2 k_\lambda \cdot k_\lambda^2(t - s, x - y), \\
where \quad c_- := (1 - \varepsilon)^{-d/2} \left( \frac{\lambda}{\lambda - \delta} \right)^{d/4}.
\end{cases}
\]

(a4) \[
\begin{cases}
0 < 2\delta < \lambda \\
\frac{2}{\lambda + \delta} < \varepsilon < 1 \\
(t-s)\varepsilon < \tau - s < t - s
\end{cases}
\Rightarrow
\begin{cases}
\hat{k}_\lambda k_\delta^2 \leq c_4^2 \hat{k}_\lambda \cdot k_\lambda^2(t - s, x - y), \\
where \quad c_+ := \varepsilon^{-d/2} \left( \frac{\lambda}{2\delta} \right)^{d/2} r^{-d/2}, \quad r = \frac{2(\lambda - \delta)\varepsilon - \lambda}{\lambda - 2\delta \varepsilon}.
\end{cases}
\]
Proof of (a4). Using \(ab \leq a^2 + 4^{-1}b^2\) and \(t - \tau \geq (1 - \varepsilon)(t - s)\) we have, for any \(\alpha \in \mathbb{R}^d\), \(\alpha \neq 0\),

\[
e^{\alpha(x-y)}k^2_{\delta}k_{\delta} = e^{\alpha(x-y)}k^2_{\delta}e^{\alpha(y)}k_{\delta}
\]

\[
\leq (1 - \varepsilon)^{-d}\langle 4\pi \lambda(t-s) \rangle^{-d/2}\cdot e^{\alpha(y)}k_{\delta}e^{\alpha(x)}k_{\delta}
\]

\[
= (1 - \varepsilon)^{-d}\langle \lambda(\lambda - 2\delta) \rangle^{-d/2}k^2_{\delta}e^{\alpha(y)}k_{\delta}
\]

Therefore,

\[
k^2_{\delta} \leq (1 - \varepsilon)^{-d}\langle \lambda(\lambda - 2\delta) \rangle^{-d/2}k^2_{\delta}e^{\alpha(y)}k_{\delta}
\]

Set \(\alpha = \frac{x-y}{\lambda(t-s)}\).

Proof of (a4). Using \(ab \leq a^2 + 4^{-1}b^2\) and \(\varepsilon(t-s) \leq \tau - s\) we have, for any \(\alpha \in \mathbb{R}^d\), \(\alpha \neq 0\) and \(r \in [0,1]\),

\[
e^{\alpha(x-y)}k^2_{\delta}k_{\delta} = e^{\alpha(x-y)}k^2_{\delta}e^{\alpha(x-y)}k_{\delta}
\]

\[
\leq \varepsilon^{-d}\langle \lambda/(2\delta) \rangle^{-d/2}e^{\alpha(y)}k_{\delta}e^{\alpha(x)}k_{\delta}
\]

\[
\leq \varepsilon^{-d}\langle \lambda/(2\delta) \rangle^{-d/2}k_{\delta}e^{\alpha(y)}k_{\delta}
\]

Using \(t - \tau \leq (1 - \varepsilon)(t - s)\) and taking into account our choice of \(r\) and \(\varepsilon\), we have

\[
\delta(t-s) + \frac{\lambda}{1-r}(t-\tau) = \delta(t-s) + \left(\frac{\lambda}{1-r} - \delta\right)(t-\tau)
\]

\[
\leq \delta(t-s) + \left(\frac{\lambda}{1-r} - \delta\right)(1-\varepsilon)(t-s) = \frac{\lambda}{2}(t-s).
\]

Therefore

\[
k^2_{\delta} \leq \varepsilon^{-d}\langle \lambda/(2\delta) \rangle^{-d/2}k_{\delta}e^{\alpha(y)}k_{\delta}
\]

Set \(\alpha = \frac{x-y}{\lambda(t-s)}\).

8.2. Nash’s function \(\hat{N}_\delta\). Let \(p(t,x,y)\) denote the fundamental solution of \(\partial_t + A^\varepsilon, A^\varepsilon \equiv -\nabla \cdot a_\varepsilon \cdot \nabla\). Put for brevity \(a \equiv a_\varepsilon\). Define

\[
\hat{N}_\delta(t-\tau, \tau-s, x, y) := \langle \nabla p(t-s, \cdot, x) \cdot a(\cdot)k_{\delta}(t-\tau, \cdot, x), \nabla p(t-s, \cdot, y) \rangle
\]

for all \(s < \tau < t, x, y \in \mathbb{R}^d\).

Proposition 4. Let \(c_4, c_6 < 2\delta < \lambda, c_0 \in \mathbb{R}^d\). There exists a generic constant \(\hat{c}_0\) such that

\[
\hat{N}_\delta(t-\tau, \tau-s, x, y) \leq \frac{\hat{c}_0}{t-\tau}
\]

for all \(t > 0, \tau - s < t - s, \tau \neq s, x, y \in \mathbb{R}^d\).

Proof. Write \(\hat{N}_\delta = \langle \nabla p \cdot a_k \cdot \nabla p \rangle\). Integrating by parts and using the equation \((\partial_t + A^\varepsilon)p(\tau-s, \cdot, y) = 0\), we obtain

\[
\hat{N}_\delta = \langle -\partial_t p \cdot \hat{k}_{\delta} \cdot \nabla p \rangle - \langle \nabla p \cdot \hat{k}_{\delta} \cdot \nabla k \rangle + 2\langle \nabla p \cdot \hat{k}_{\delta} \cdot \nabla k_{\delta} \rangle.
\]
By quadratic inequalities,
\[
|\langle \nabla p \cdot \frac{a p}{k_{2\delta}^2} \cdot \nabla \hat{k}_\lambda \rangle| \leq \frac{1}{4} \hat{N}_\delta + \langle \nabla \hat{k}_\lambda \cdot \frac{a p^2}{k_{2\delta}^4} \cdot \nabla \hat{k}_\lambda \rangle
\]
\[
\equiv \frac{1}{4} \hat{N}_\delta + M_1,
\]
\[
2|\langle \nabla p \cdot \frac{a p \hat{k}_\lambda}{k_{2\delta}^2} \cdot \nabla k_{2\delta} \rangle| \leq \frac{1}{4} \hat{N}_\delta + 4 \langle \nabla k_{2\delta} \cdot \frac{a p^2 \hat{k}_\lambda}{k_{2\delta}^4} \cdot \nabla k_{2\delta} \rangle
\]
\[
\equiv \frac{1}{4} \hat{N}_\delta + 4 M_2.
\]
Therefore,
\[
\hat{N}_\delta \leq 2 \langle - \partial_r p, \frac{\hat{k}_\lambda p}{k_{2\delta}^2} \rangle + 2 M_1 + 4 M_2.
\]

Let us estimate the terms in the RHS of (\ref{eq:main}).

By \textit{(UGB)}, \textit{(UGB$^q$)}, \textit{(UGB$^b$)}, and our choice of $\delta$,
\[
\langle - \partial_r p, \frac{\hat{k}_\lambda p}{k_{2\delta}^2} \rangle \leq c_3 c_5 (\tau - s)^{-1} \langle \frac{k_{c_4} k_{c_4} \hat{k}_\lambda}{k_{2\delta}^2} \rangle
\]
\[
\leq c_3 c_5 (\tau - s)^{-1} \left( \frac{(2\delta)^2}{c_4 c_6} \right)^{\frac{d}{2}} \langle \hat{k}_\lambda \rangle = c_3 c_5 (\tau - s)^{-1} \left( \frac{(2\delta)^2}{c_4 c_6} \right)^{\frac{d}{2}}.
\]

Taking into account that $\tau - s > \varepsilon (t - s) \Rightarrow \frac{1}{\tau - s} < \frac{1}{\varepsilon} \frac{1}{t - \tau}$, we thus obtain
\[
\langle - \partial_r p, \frac{\hat{k}_\lambda p}{k_{2\delta}^2} \rangle \leq c_3 c_5 \left( \frac{(2\delta)^2}{c_4 c_6} \right)^{\frac{d}{2}} \varepsilon \frac{1}{t - \tau}.
\]

Next, using $a_1$-$a_3$, we have:
\[
M_1 \leq \xi c_3^2 \left( \frac{k_{c_4}}{k_{2\delta}} \right)^2 \langle \frac{(\nabla \hat{k}_\lambda)^2}{k_\lambda} \rangle
\]
\[
\leq \xi c_3^2 \left( \frac{2\delta}{c_4} \right)^d \langle \frac{(\nabla \hat{k}_\lambda)^2}{k_\lambda} \rangle
\]
\[
= \xi c_3^2 \left( \frac{2\delta}{c_4} \right)^d \frac{1}{2\lambda t - \tau}.
\]
\[
M_2 \leq \xi c_3^2 \left( \frac{k_{c_4}}{k_{2\delta}} \right)^2 \langle \hat{k}_\lambda (\nabla \log k_{2\delta})^2 \rangle.
\]

where
\[
\left( \frac{k_{c_4}}{k_{2\delta}} \right)^2 = \left( \frac{2\delta}{c_4} \right)^d \exp \left[ - \frac{|y - \cdot|^2}{4(\tau - s)} \left( \frac{1}{c_4} - \frac{1}{2\delta} \right)^2 \right]
\]
\[
= \left( \frac{2\delta}{c_4} \right)^d \exp \left[ - \frac{|y - \cdot|^2}{4(\tau - s)} \right], \quad \gamma := \frac{\delta c_4}{2\delta - c_4},
\]
\[
(\nabla \log k_{2\delta})^2 = \left( \frac{y - \cdot}{2(\delta)(\tau - s)} \right)^2 = \frac{|y - \cdot|^2}{4\gamma(\tau - s)} \frac{1}{(2\delta)^2 \tau - s}.
\]
Since $0 < \eta < e^n$, we have therefore
\[
\left\langle \left( \frac{k_4}{k_2} \right)^2 \hat{k}_\lambda (\nabla \log k_2) \right\rangle \leq \left( \frac{2\delta}{c_4} \right)^d \frac{\gamma}{(2\delta)^2} \frac{1}{\tau - s} \langle \hat{k}_\lambda \rangle,
\]
and so
\[
M_2 \leq \xi c_3^2 \frac{2\delta}{c_4} \frac{c_4}{(2\delta - c_4)4\delta} \frac{1 - \epsilon}{\epsilon} \frac{1}{t - \tau}.
\]
Substituting the previous estimates into (□), we obtain
\[
\tilde{N}_\delta \leq 2c^5\left( \frac{2\delta}{c_4 c_6} \right)^{\frac{d}{2}} \frac{1 - \epsilon}{\epsilon} \frac{1}{t - \tau} + c_3 \left( \frac{2\delta}{c_4} \right)^d \left( 2 \cdot \frac{\xi d}{2\lambda} + 8 \cdot \frac{2\xi}{4\delta} \cdot \frac{c_4}{2\delta - c_4} \cdot \frac{1 - \epsilon}{\epsilon} \right) \frac{1}{t - \tau},
\]
as claimed. \hfill □

8.3. **Proof of the upper bound.** For brevity, $b \equiv b_\varepsilon$. We iterate the Duhamel formula
\[
u(t - s, x, y) = p(t - s, x, y) - \int_s^t \langle u(t - \tau, x, \cdot) b(\cdot) \cdot \nabla p(\tau - s, \cdot, y) \rangle d\tau.
\]
We obtain the series
\[
l(t - s, x, y) := \sum_{n=0}^{\infty} (-1)^n u_n(t - s, x, y),
\]
where $u_0(t - s, x, y) := p(t - s, x, y)$ and, for $n = 1, 2, \ldots,$
\[
u_n(t - s, x, y) := \int_s^t \langle u_{n-1}(t - \tau, x, \cdot) b(\cdot) \cdot \nabla p(\tau - s, \cdot, y) \rangle d\tau.
\]
In particular,
\[
u_1(t - s, x, y) = \int_s^t \langle p(t - \tau, x, \cdot) b(\cdot) \cdot \nabla p(\tau - s, \cdot, y) \rangle d\tau,
\]
and so
\[
u_1(t - s, x, y) \leq c_3 \int_s^t \langle k_{c_4} (t - \tau, x - \cdot) |b(\cdot) \cdot \nabla p(\tau - s, \cdot, y)| \rangle d\tau.
\]
Suppose that we are able to find generic* constants $h > 0$ and $C_h < 1$ such that the bound:
\[
\int_s^t \langle k_{c_4} (t - \tau, x - \cdot) |b(\cdot) \cdot \nabla p(\tau - s, \cdot, y)| \rangle d\tau \leq C_h k_{c_4} (t - s, x - y)
\]
is valid for all $x, y \in \mathbb{R}^d$ and $0 < t - s \leq h$.
Then $|\nu_1(t - s, x, y)| \leq c_3 C_h k_{c_4} (t - s, x - y)$, and by induction,
\[
|u_n(t - s, x, y)| \leq c_3 (C_h)^n k_{c_4} (t - s, x - y).
\]
Therefore, for all $0 < t - s \leq h$ and all $x, y \in \mathbb{R}^d$, the series $l(t - s, x, y)$ is well defined and
\[
|l(t - s, x, y)| \leq \frac{c_3}{1 - C_h} k_{c_4} (t - s, x - y).
\]
Repeating the standard argument we conclude that $l$ satisfies the Duhamel formula provided that $0 < t - s \leq h$. Then the uniqueness of $u(t - s, x, y)$ implies
\[
u = l \quad (0 < t - s \leq h),
\]
and the reproduction property of $u$ implies
\[
u(t - s, x, y) \leq \frac{c_3}{1 - C_h} e^{(t-s)\omega_h} k_{c_4} (t - s, x - y)
\]
for all $t - s > h$, where $\omega_h = \frac{1}{h} \log \frac{c_3}{1 - C h}$. Thus, we obtain the upper bound in (LUGB) of Theorem 5.

It remains to prove (4). Without loss of generality, $s = 0$. Set $b_a^2 := b \cdot a^{-1} \cdot b$ and denote

$$
\langle k_{\mu} b_a^2 \rangle := \langle k_{\mu}(\tau, y - \cdot) b_a^2(\cdot) \rangle, \quad \langle \hat{k}_{\mu} b_a^2 \rangle := \langle k_{\mu}(t - \tau, x - \cdot) b_a^2(\cdot) \rangle.
$$

Set

$$
I := \int_0^t \langle \hat{k}_{\lambda}(t - \tau, x - \cdot) | b(\cdot) \cdot \nabla p(\tau, \cdot, y) | \rangle d\tau.
$$

\textbf{Lemma 2.} Fix $\lambda > \xi$ and select constants $\delta, c_4$ such that

$$
\lambda > 2\delta > c_4 > \xi.
$$

Let $\frac{\lambda}{2(\lambda - \delta)} < \varepsilon < 1$, $r = \frac{2(\lambda - \delta) - \lambda}{\lambda - 2\varepsilon}$, and let $c_{\pm}$ be the constants defined in (4). Then, for all $t > 0$,

$$
I \leq (c_- M^- + c_+ M^+) k_{\lambda}(t, x, y),
$$

where

$$
M^-(t, x, y) := \int_0^{t\varepsilon} \sqrt{\langle \hat{k}_{\lambda} k_{\delta} b_a^2 \rangle} \sqrt{\langle \nabla p \cdot \frac{a}{k_{\delta}} \cdot \nabla p \rangle} d\tau,
$$

$$
M^+(t, x, y) := \int_{t\varepsilon}^t \sqrt{\langle \hat{k}_{\lambda} b_a^2 \rangle} \sqrt{\langle \nabla p \cdot \frac{a}{k_{\delta}} \cdot \nabla p \rangle} d\tau.
$$

\textbf{Proof.} Using quadratic inequality, we bound $\langle \hat{k}_{\lambda} | b \cdot \nabla p | \rangle^2$ in two ways:

$$
\langle \hat{k}_{\lambda} | b \cdot \nabla p | \rangle^2 \leq \langle \hat{k}_{\lambda} k_{\delta} b_a^2 \rangle \langle \nabla p \cdot \frac{a}{k_{\delta}} \cdot \nabla p \rangle
$$

and

$$
\langle \hat{k}_{\lambda} | b \cdot \nabla p | \rangle^2 \leq \langle \hat{k}_{\lambda} k_{\delta} b_a^2 \rangle \langle \nabla p \cdot \frac{a}{k_{\delta}} \cdot \nabla p \rangle,
$$

and hence

$$
I \equiv \int_0^t \langle \hat{k}_{\lambda} | b \cdot \nabla p | \rangle d\tau \leq I^- + I^+,
$$

where

$$
I^- := \int_0^{t\varepsilon} \sqrt{\langle \hat{k}_{\lambda} k_{\delta} b_a^2 \rangle} \sqrt{\langle \nabla p \cdot \frac{a}{k_{\delta}} \cdot \nabla p \rangle} d\tau,
$$

$$
I^+ := \int_{t\varepsilon}^t \sqrt{\langle \hat{k}_{\lambda} k_{\delta} b_a^2 \rangle} \sqrt{\langle \nabla p \cdot \frac{a}{k_{\delta}} \cdot \nabla p \rangle} d\tau.
$$

Now the assertion of Lemma 2 follows directly from (4) and Propositions 2 and 4. (Here we apply Propositions 2 with $\delta$ chosen as in Proposition 4, but it is not difficult to see, using (4), that its proof works for all $\delta > \xi$ although with different generic constant $c_0$.)

It remains to note that both $M_+, M_-$ in Lemma 2 are majorated by $c n_e(b, h)$ for appropriate multiple $c > 0$. Provided that $n_e(b, h)$ is sufficiently small, i.e. so that $C_h := (c_- + c_+) c n_e(b, h) < 1$, we obtain (4).
8.4. **Proof of the lower bound.** The analysis of the previous section and the (UGB) of Theorem 4 yield for $|x-y|^2 \leq t \leq h$

$$u(t, x, y) \geq p(t, x, y) - \sum_{n \geq 1} |u_n(t, x, y)|$$

$$\geq c_1 k_2 (t, x - y) - \frac{c_3 C_h}{1 - C_h} k_4 (t, x - y)$$

$$\geq \left( c_1 k_2 - \frac{e^{-\frac{1}{4c_2}}} {c_3 C_h - C_4} \right) (4\pi t)^{-\frac{d}{2}}$$

$$= rt^{-\frac{d}{2}}, \quad (**)$$

where $r > 0$ provided that $C_h$ is small enough, i.e. $\frac{C_h}{1 - C_h} < \frac{c_4}{c_3} \left( \frac{c_4}{c_2} \right) e^{-\frac{1}{4c_2}}$.

Now the standard argument ("small gains yield large gain", see e.g. [Da], Theorem 3.3.4) yields for all $x, y \in \mathbb{R}^d$, $t > 0$,

$$u(t, x, y) \geq re^{\nu_h t} t^{-\frac{d}{2}} \exp \left( -\frac{|x-y|^2}{4c_2 t} \right), \quad \nu_h = \frac{1}{h} \log r.$$  

The proof of Theorem 5 is completed.

9. **Proof of Theorem 2**

It suffices to carry out the proof on $C_c^\infty$ for smooth $a, b$, and then apply Theorem $(ii)$ using the closedness of the gradient.

First, let $0 < t \leq h$.

The Duhamel formula for $\nabla e^{-t\Lambda_1}$ yields:

$$\|\nabla e^{-t\Lambda_1} f\|_1 \leq \|\nabla e^{-t\Lambda_1} f\|_1 + \int_0^t \|\nabla e^{-(t-\tau)\Lambda_1} \|_{1 \to 1} \|b \cdot \nabla e^{-\tau\Lambda_1} f\|_1 d\tau, \quad f \in C_c^\infty. \quad (8)$$

We will need (proved below):

$$\|\nabla e^{-t\Lambda_1}\|_{1 \to 1} \leq C / \sqrt{t}, \quad (*)$$

$$\int_0^t \frac{C}{\sqrt{t-\tau}} \|b \cdot \nabla e^{-\tau\Lambda_1} f\|_1 d\tau \leq C \sup_{x \in \mathbb{R}^d} \int_0^t \frac{1}{\sqrt{t-\tau}} \sqrt{e^{\delta \tau} b_0^2(x)} \sqrt{N_{\delta}(\tau, x)} d\tau \|f\|_1, \quad (**)$$

$$N_{\delta}(\tau, x) \leq \frac{C_2}{\tau}, \quad (***)$$

where $N_{\delta}(\tau, x) := \langle \nabla u(\tau, x, \cdot) \cdot \frac{a(\cdot)}{s_1(\tau, x, \cdot)} \cdot \nabla u(\tau, x, \cdot) \rangle$, $u(\tau, x, y) = e^{-\tau\Lambda_1}(x, y)$, $\delta > \xi$, the constants $C_1, C_2, \omega$ are generic. We estimate the RHS of (**): write $\int_0^t = \int_0^{t/2} + \int_{t/2}^t$ and use (***). to obtain

$$\sup_{x \in \mathbb{R}^d} \int_0^{t/2} \frac{1}{\sqrt{t-\tau}} e^{\delta \tau} b_0^2(x) \sqrt{N_{\delta}(\tau, x)} d\tau \leq \sqrt{2C_2} \sup_{x \in \mathbb{R}^d} \int_0^{t/2} \sqrt{e^{\delta \tau} b_0^2(x)} d\tau \sqrt{\tau}$$

$$\leq \frac{\sqrt{2C_2}}{\sqrt{\delta t}} n_c(b, \frac{\delta h}{2}),$$
Clearly, \( t > h \). The latter yields the assertion of Theorem 2 for all
\[ 0 < t < h. \]

Also, for all \( t > h \), \( \| \nabla e^{-tA_1} \|_{1 \to 1} \leq \| \nabla e^{-hA_1} \|_{1 \to 1} \| e^{-(t-h)A_1} \|_{1 \to 1} \leq \frac{c}{\sqrt{h}} e^{(t-h)\omega_2} \) (cf. Theorem 11).

The latter yields the assertion of Theorem 2 for all \( t > 0 \).

It remains to prove \((\ast)-(\ast\ast)\).

\((\ast)\) was proved in Proposition 2

\[
\| \nabla e^{-tA_1} f \|_1^2 \leq \sigma^{-\frac{1}{2}} \sup_{x \in \mathbb{R}^d} N_\delta(t, x) \leq \frac{c_0}{t} \| f \|_1.
\]

The estimate \((\ast\ast)\) follows using quadratic inequality.

Thus, we are left to prove \((\ast\ast\ast)\). Integrating by parts, using the equation for \( u(t, x, y) \) and \((U\mathcal{B})\), \((U\mathcal{B}^\mathcal{H})\) (see Theorem 11(iv), (viii)), we obtain for \( 0 < t < h \)

\[
N_\delta^u(t, x) = \langle \nabla u \cdot \frac{a}{k_\delta} \cdot \nabla u \rangle = -\langle k_\delta^{-1} u \partial_t u \rangle - \langle k_\delta^{-1} u b \cdot \nabla u \rangle + \langle uk_\delta^{-2} \nabla k_\delta \cdot a \cdot \nabla u \rangle,
\]

\[
|\langle k_\delta^{-1} u \partial_t u \rangle| \leq \frac{c}{t}, \quad |\langle uk_\delta^{-2} \nabla k_\delta \cdot a \cdot \nabla u \rangle| \leq c|\langle \nabla k_\delta \cdot \frac{a}{k_\delta} \cdot \nabla u \rangle|.
\]

Clearly,

\[
|\langle \nabla k_\delta \cdot \frac{a}{k_\delta} \cdot \nabla u \rangle| \leq \frac{c}{\sqrt{t}} \sqrt{N_\delta^u(t, x)}.
\]

\[
|\langle k_\delta^{-1} u b \cdot \nabla u \rangle| \leq c \sqrt{e^{\delta t} \Delta b_0^2(x)} \sqrt{N_\delta^u(t, x)} \leq \frac{c}{\sqrt{t}} \sqrt{N_\delta^u(t, x)}.
\]

(due to \( e^{\delta t} \Delta b_0^2(x) \leq \frac{d\delta}{8\delta} \frac{1}{T} + c(\beta) \), see above). Now \((\ast\ast\ast)\) is evident.

The proof of Theorem 2 is completed.

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Université Laval, Département de mathématiques et de statistique, 1045 av. de la Médecine, Québec, QC, G1V 0A6, CANADA
Email address: damir.kinzebulatov@mat.ulaval.ca

University of Toronto, Department of Mathematics, 40 St. George Str, Toronto, ON, M5S 2E4, CANADA
Email address: semenov.yu.a@gmail.com