Brick tiling

S I Ben-Abraham\textsuperscript{1} and D Flom\textsuperscript{2}

\textsuperscript{1} Department of Physics, Ben-Gurion University of the Negev, Beer Sheba, Israel
E-mail: shelomo.benabraham@gmail.com
\textsuperscript{2} Department of Electro-Optical Engineering, Ben-Gurion University of the Negev, Beer Sheba, Israel
E-mail: dvir.flom@gmail.com

Abstract. We propose a generalization of the well-known two-dimensional table tiling to three dimensions – the brick tiling. We develop a color/digit code for this tiling. We also discuss possible implications for higher dimensions.

1. Introduction

The work reported in this paper is part of a project to generalize standard one-dimensional sequences and two-dimensional (2D) structures to higher dimensions [1-4]. Here we consider the well-known two-dimensional table tiling and put forward its generalization to three dimensions which we call the brick tiling. The table tiling was thoroughly discussed by E Arthur Robinson Jr [5]. The interested reader will find wider and deeper mathematical background in the important paper by Boris Solomyak [6] and the seminal book by Jean-Paul Allouche and Jeffrey Shallit [7] and that by Michael Baake and Uwe Grimm [8].

2. Preliminaries

A brick tiling (including the table tiling) in any dimension (if it exists) is a lattice substitution system [9]. That means that it is supported by a lattice $\mathbb{Z}^d$. Its prototiles are lattice animals (alias $d$-polyominoes, i.e. $d$-dimensional polyominoes or $d$-poly-cubes) that is connected unions of $d$-dimensional cubic unit cells ($d$-cubes) [10]. A 2D brick, for instance, consists of two squares.

In $d$ dimensions, a standard brick $B_d$ (in what follows, simply brick) is a $d$-dimensional cuboid with edges of $2^0 = 1$, $2^1 = 2$, \ldots, $2^{d-1}$ units; hence it consists of $2^{d-1} \sum_{k=0}^{d-1} k$ unit cubes $q_d$. When there can be no confusion we omit the subscript $d$. We focus here on three dimensions (3D). There, the prototiles are $B_3$'s, i.e. $1 \times 2 \times 4$ cuboids (octominoes). In what follows we shall call them protobricks. A $g$ times inflated brick will be denoted $B_d(g)$. Thus, $B_d(0) \equiv B_d$ is a protobrick.

It is easy to see that in $d$ dimensions the protobricks come in $d!$ orientations; for convenience we shall call them colors.
In any dimension $d$, a brick is a rep-tile. Paraphrasing Robinson [5] we say: A $d$-dimensional rep-tile ($d$-rep-tile, for short) is a polytope that can be tiled by a finite number of smaller, congruent copies of itself. Obviously, the process can be repeated ad infinitum. By the same token, a rep-tile produces, by inflation, an infinite tiling.

### 3. Two dimensions – table tiling

To keep this paper self-contained we briefly recapitulate the essential features of the table tiling (alias domino tiling). In the present context we may call it 2D brick tiling. Fig. 1 (adapted from Robinson [3]) shows its inflation and labeling. Clearly, there are $2! = 2$ protobricks $B_2$ differing in orientation; each $B_2$ consists of $2^{d-1} = 2^1 = 2$ unit squares $q_2$.

![Figure 1. The table tiling inflation and the labeling of its protobricks.](image)

The substitution associated with the inflation is (cf. [5], [8])

$$
0 \rightarrow \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}, \quad 1 \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad 2 \rightarrow \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}, \quad 3 \rightarrow \begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix}.
$$

### 3. Three dimensions – brick tiling

A consistent generalization of the table tiling must comply with the following conditions:

(a) the inflated brick $B_d(1)$ consists of $2^d$ protobricks $B_d$; in other words, the inflation factor is $2^d$;

(b) all bricks are mirror symmetric w.r.t. to their median planes;

(c) there is maximum contact between the protobricks $B_d$ within the inflated brick $B_d(1)$;

(d) the inflated brick contains a maximum of colors subject to condition (c).

In 3D there are $3! = 6$ protobricks characterized by their orientation w.r.t. Cartesian coordinate axes $x, y, z$ and briefly labeled by digits and/or colors. They are explicitly listed in Eq. (2) and shown in Fig. 2 along with generation 1 of their inflation. Here the first entry in the parentheses refers to the direction of the long edge, the second to the direction of the medium edge and the third to the direction of the short edge. It can be seen that, indeed, the inflated bricks $B_3(1)$ (a) consist of $2^3 = 8$ protobricks, (b) are mirror symmetric w.r.t. the median planes, (c) satisfy maximum contact between the proto-bricks $B_3$, and (d) contain three colors. Four colors would violate the symmetry condition.
The six protobricks of the 3D brick tiling and their inflation are shown in Fig. 2. It is easy to see that the starting orientation is an invariant of the inflation. While the structure would eventually in the limit fill the entire lattice $\mathbb{Z}^3$ it will stay anisotropic and keep the original shape. Fig. 3 shows the hull of the second generation $B_3(2)$ starting with $B_3(0) = 0$ (yellow). The second generation contains $8^2 = 2^6 = 64$ protobricks and hence $8^3 = 2^9 = 512$ unit cubes. Likewise, the third generation contains $8^3 = 2^9 = 512$ protobricks and hence $8^4 = 2^{12} = 4096$ unit cubes. The numbers rapidly become staggering.

$$0 \text{ (yellow)} = (xyz), \quad 1 \text{ (green)} = (xzy),$$

$$2 \text{ (orange)} = (yzx), \quad 3 \text{ (blue)} = (yxz),$$

$$4 \text{ (red)} = (zxy), \quad 5 \text{ (violet)} = (zyx).$$

Figure 2. The six protobricks of the 3D brick tiling and their inflation.

Figure 3. Hull of the second generation of the three-dimensional brick tiling starting with orientation 0 (yellow).
4. Arbitrary dimensions

What can be said in the present context about arbitrary dimensions? Bricks exist in all dimensions. A brick tiling fulfilling conditions (a)–(d) certainly exists in one, two and three dimensions. The 1D case is ridiculously degenerate but perfectly valid. Trying to construct a 4D brick tiling we ran into contradictions. One may, though, extend the 2D solution to 3D and even higher dimensions by leaving the shorter edges idle. In other words, one may construct the 2D table tiling using real dominoes. In an analogous way one may extend the 3D brick tiling. This, of course, is not what we really intend. Perhaps, by somewhat relaxing the conditions (a)–(d) one might succeed to generalize the brick tiling to higher dimensions. That remains to be seen.

5. Concluding remarks

The 2D table tiling and its code are quite simple and they seem to be related to the 2D chair tiling and its code be some kind of duality. However, in contrast to the chair tiling whose Fourier spectrum is pure point that of the table tiling already in 2D contains a singular continuous part. It is reasonable to conjecture that that is true in 3D as well. It is altogether rather clear that the 3D brick tiling is much more complex and very different from the 3D chair tiling.

In analogy to 2D we also developed a lattice substitution tiling color coding for the 3D brick tiling. This is reported in our companion paper [11] in this volume.

References

[1] Ben-Abraham S I and Quandt D 2011 Aperiodic structures and notions of order and disorder Philos. Mag. 91 2718-2727
[2] Ben-Abraham S I, Quandt A and Shapira D 2013 Multidimensional paperfolding systems Acta Cryst. A69 123-130
[3] Ben-Abraham S I, Flom D, Richman R A and Shapira D 2014 Multidimensional "paperfolding" structures – three and four dimensions Acta Phys. Pol. A 126 435-437
[4] Lee J.-Y., Flom D. and Ben-Abraham S I, 2016 Multidimensional period doubling structures Acta Cryst. A72 391-394
[5] Robinson E A Jr 1999 On the table and the chair Indag. Mathem. N.S. 10 581-599
[6] Solomyak B 1996 Dynamics of self-similar tilings Ergodic Th. Dynam. Syst. 17 695-738
[7] Allouche J-P and Shallit J 2003 Automatic Sequences: Theory, Applications, Generalizations (Cambridge University Press)
[8] Baake M. and Grimm U 2013 Aperiodic Order Volume 1: A Mathematical Invitation (Cambridge University Press)
[9] Lee J.-Y. and Moody R V 2001 Lattice substitution systems and model sets Discrete Comput. Geom. 25 173-201
[10] Golomb S W 1996 Polyominoes: Puzzles, Patterns, Problems, and Packings 2nd ed (Princeton University Press)
[11] Flom D and Ben-Abraham SI 2016 Color coding for the brick tiling J. Phys. Conf. Series (this volume)