COHERENCE FOR INDEXED SYMMETRIC MONOIDAL CATEGORIES

CARY MALKIEWICH AND KATE PONTO

ABSTRACT. Indexed symmetric monoidal categories are an important refinement of bicategories — this structure underlies several familiar bicategories, including the homotopy bicategory of parametrized spectra, and its equivariant and fiberwise generalizations.

In this paper, we extend existing coherence theorems to the setting of indexed symmetric monoidal categories. The most central theorem states that a large family of operations on a bicategory defined from an indexed symmetric monoidal category are all canonically isomorphic. As a part of this theorem, we introduce a rigorous graphical calculus that specifies when two such operations admit a canonical isomorphism.

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1. Introduction

Wherever monoidal categories appear, Mac Lane’s coherence theorem [ML98, VII.] provides very convenient simplifications. This is certainly true in stable homotopy theory, since the monoidal categories in question are often complicated and opaque.

Mac Lane’s theorem admits many generalizations: braided categories [JS93], bicategories [Pow89], symmetric monoidal bicategories [GO13], and tricategories [GPS95, Gur06]. In this paper we prove a generalization for indexed symmetric monoidal categories with coproducts, also known as symmetric monoidal bifibrations. It can be stated roughly as follows.

Theorem 1.1. In an indexed symmetric monoidal category with coproducts $\mathcal{C}^-$, any two composites of the eight operations
\[ \boxtimes, I, f^*, f_!, \circ, \langle \langle \rangle \rangle, U_B, [A \xrightarrow{f} B] \]
that differ only in the order of application are canonically isomorphic.

A more precise statement is that we associate each composite of these operations to a map of graphs, and if two different composites result in the same map of graphs, the resulting functors are canonically isomorphic. We call this a coherence theorem because the canonical isomorphism it provides can be expressed in terms of more familiar isomorphisms, such as the associator for $\circ$, but in more than one way. We give an overview of these graphs and their associated operations in Theorem 2.2 and Table 2.3, and complete the formal statement of this theorem in Theorem 6.9.

Our interest in Theorem 1.1 comes from an ongoing program, motivated by [LM16, CP18], to refine classical fixed point invariants and connect those invariants to trace methods in algebraic $K$-theory. The following result is a first step in this program. It is a consequence of the coherence theorems in this paper and the results in [MP].

Theorem 1.2. If $X$ is a compact ENR and $f: X \to X$ is a continuous map then the Lefschetz number of $f^n$ agrees with the Lefschetz number of
\[ X \times \cdots \times X \xrightarrow{\Psi^n(f)} X \times \cdots \times X \]
\[ (x_1, x_2, \ldots, x_n) \mapsto (f(x_2), \ldots, f(x_n), f(x_1)). \]
Furthermore the Reidemeister traces of $f^n$ and $\Psi^n(f)$ also agree.

Our approach to this result is sufficiently formal that it also applies to the equivariant and fiberwise generalizations of the Lefschetz number and Reidemeister trace.

The core of the proof of Theorem 1.1 and the heart of this paper is the development of a string diagram calculus that describes in pictures when two composites of the basic operations are canonically isomorphic. In the setting of a monoidal category, this calculus can be described as follows. The objects are represented by vertices of a graph, and the monoidal product is represented by edges connecting these vertices. See Figure 1.3a. The coherence theorem for monoidal categories implies that the graph in Figure 1.3a specifies a well defined operation in the monoidal category, up to canonical isomorphism.

To generalize this to bicategories, recall that in many examples, the 1-cells in a bicategory play the same role as objects in a monoidal category. So we will represent each 1-cell by a (dark) labeled vertex. Since each 1-cell has a source and target 0-cell, we extend this vertex to a graph as in Figure 1.3b, with the incident edges labeled by 0-cells. The new vertices are colored white and left unlabeled. Connecting $n$ of these graphs gives the graph in Figure 1.3c. We think of this as a composable tuple $(M_1, \ldots, M_n)$ of 1-cells. Then we darken the unlabeled white vertices to indicate bicategorical composition. So Figure 1.3d represents the composite $M_1 \oplus \cdots \oplus M_n$. Of course, different orders
of darkening will give different parenthesizations of this product, but they all result in the same graph, which corresponds to the fact that the different parenthesizations are all canonically isomorphic.

Bicategorical composition is only allowed along graphs as in Figure 1.3c, but the examples we are interested in permit far more general and interesting compositions. The first of these is a shadow on a bicategory [Pon10] which adds a circular product. This product is represented graphically using a single cycle as in Figure 1.4b and it is defined for any tuple of 1-cells as in Figure 1.4a.

When we generalize to indexed symmetric monoidal categories, there is no longer any need to insist that the 1-cells are multiplied along a circle – it could be any finite graph. One such “multi-⊙” operation is pictured in Figure 1.5. We will see that we can move between these graphs by applying covering maps, in addition to darkening vertices.

We emphasize that this calculus, described in full detail in Part 2, is rigorous. So long as a graph adheres to certain combinatorial rules, it defines family of operations and coherent natural isomorphisms between them. This allows us to prove statements about indexed symmetric monoidal categories and their associated bicategories using pictorial arguments. We give examples of such arguments in Part 1.

This calculus is reminiscent of, but distinct from, other calculi that arise when studying two-dimensional topological field theories, e.g. [PS12, Bar14]. We conjecture that it
also applies to any compact closed bicategory [Sta16]. Using field-theoretic language as in [SP09], it should apply when one has a partial field theory valued in a symmetric monoidal bicategory where each 0-cell is 1-dualizable. The relevant bordism category would have 1-morphisms that are cobordisms with singularities (i.e. graphs with half-edges), while the 2-morphisms are trivial cobordisms (i.e. graphs times intervals), although the defect curves are allowed to be embedded in a nontrivial way.

The techniques we develop in the course of the proof of Theorem 1.1 also lead to other coherence results which we use in [MP] to prove Theorem 1.2. The most notable of these is the coherence theorem for bicategories with shadow. This result is so important to our treatment in [MP] that some of our definitions cannot even be stated without it.

**Theorem 1.6** (Theorem 9.12). In a bicategory with shadow $\mathcal{B}$, every composition of $\odot$, $\ominus$, and units that defines the circular product of $1.4b$ is canonically isomorphic. In other words, between any two such compositions, any two isomorphisms formed using the structure isomorphisms for $\mathcal{B}$ must coincide.

This result comes with a string diagram calculus of its own, in the sense that the graph in Figure 1.4b gives a single well-defined operation up to canonical isomorphism.

The proof of Theorem 1.1 also allows us to upgrade the result [PS12, 5.2] to the following, which we rely on in [MP].

**Theorem 1.7** (Theorem 14.1). Every map of indexed symmetric monoidal categories (with coproducts) induces a map of shadowed bicategories (i.e. a strong shadow functor).

Finally, as a consequence of Theorem 1.1 we show that, in the language of [MP],

**Theorem 1.8** (Theorems 3.4 and 3.6). If a bicategory arises from an indexed symmetric monoidal category (with coproducts), then it has an $n$-Fuller structure and a system of base-change objects for every $n$.

This is precisely the additional structure on the bicategory of parametrized spectra that we use in [MP] to prove Theorem 1.2.

These theorems are all special cases of a general coherence problem for natural isomorphisms. Namely, given a “2-dimensional” diagram of categories, functors, and natural isomorphisms, are the natural isomorphisms coherent with each other? In general the answer is “no,” so in each case where the answer is “yes,” we use combinations of universal properties and combinatorial arguments to prove coherence.

Our proof of Theorem 1.1 builds a set of tools for solving this general problem, adapted specifically for the examples we consider above. For instance, at several points we have to consider whether a collection of Beck-Chevalley isomorphisms is coherent. They do not fit together into a planar diagram, which would make such a statement reduce completely to standard pasting lemmas, the calculus of mates (e.g. [Shu11, Wer17]), or a formal black-box such as the statement that a higher category of spans acts on the categories in question (e.g. [Bar17, Hau18]). We therefore develop combinatorial coherence
results that hold when the diagram has a cubical or higher-dimensional staircase shape to it, and introduce the idea of a “rotation” of a diagram to link these combinatorial results back to diagrams that we can construct by hand. It is likely that these techniques have additional applications beyond the theorems listed above.

**Organization.** In Part 1 we introduce the string diagram calculus in an informal way and use it to prove Theorem 1.8. In Part 2 we give a completely rigorous statement of the string diagram calculus, as a pseudofunctor from a certain category of graphs into $\text{Cat}$. In Part 3 we introduce the remaining necessary machinery and prove that the calculus provides coherent isomorphisms. Along the way we also apply that machinery to prove Theorems 1.6 and 1.7.

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**Part 1. Applications of the calculus**

In this introductory part, we show how to use our string diagram calculus to recover various other results. This part is informal and its goal is to help the reader start to work with this calculus. (The arguments are rigorous when interpreted in the precise language from Part 2.) A reader less interested in the formal category theory may find this section sufficient for their needs. For a reader who is more interested in the precise formal statements, this part is an extended introduction to Part 2.

2. First examples

Informally, an **indexed symmetric monoidal category (with coproducts)** $\mathcal{C}^-$ over a cartesian monoidal category $S$ is

- a symmetric monoidal category $(\mathcal{C}^A, \otimes, I_A)$ for each object $A$ of $S$ and

- an adjunction $(f_! \dashv f^*) : \mathcal{C}^B \to \mathcal{C}^A$ for each morphism $f : A \to B$ in $S$ so that $f^*$ is strong symmetric monoidal.

We refer to $f^*$ as a pullback and $f_!$ as a pushforward, and frequently drop the term “coproducts” to keep the terminology compact. See Section 7.1 for the formal definition, which includes additional compatibility axioms for the above structure, and also Section 5 for the equivalent concept of a symmetric monoidal bifibration. Table 2.1 gives a list of common examples.

| $S$ | groups $A$ | topological spaces $A$ | topological spaces $A$
|---|---|---|---|
| $\mathcal{C}^A$ | $\mathbb{Z}[A]$-modules | spaces with a map to $A$ | parametrized spectra over $A$ |
| $\otimes$ | tensor over $\mathbb{Z}$ | fiberwise product | fiberwise smash product |
| $f^*$ | restriction of scalars | pullback | pullback |
| $f_!$ | extension of scalars | compose with $f$ | pushout |

**Table 2.1.** Examples of symmetric monoidal bifibrations
At its core our string diagram calculus is a coherence result that implies significant compatibility between a broad range of functors that arise in an indexed symmetric monoidal category. This perspective will be sufficient for the applications in this section and so we will be satisfied with the following intentionally vague formulation of Theorem 6.9.

**Theorem 2.2.** In an indexed symmetric monoidal category $\mathcal{C}$, composites of the functors in Table 2.3 that only differ in order of application are canonically isomorphic.

| Graph morphism | Functor on $\mathcal{C}/S$ |
|----------------|-----------------------------|
| §2.1 Darkening a white vertex whose adjacent vertices are black | Horizontal composition (2.6) or shadow (2.10) |
| Darkening a white vertex whose adjacent vertices are white | Add a unit 1-cell (2.5) |
| Darkening a white vertex adjacent to one white and one black vertex | Identity functor |
| §2.2 Trivial cover (Figure 2.14a) | External product (2.12) |
| §2.3 Darkening a labeled white vertex (Figure 2.17a) whose adjacent vertices are white | Add a base change object (Section 2.3) |
| Darkening a labeled white vertex adjacent to one white and one black vertex | Apply the pullback or pushforward functor |
| Collapse adjacent labeled white vertices, composing their functions | Identity functor |
| Collapse adjacent labeled dark vertices | Identity functor |
| §2.5 Nontrivial cover (Figure 2.23a) | External product (locally) (2.22) |

**Table 2.3.** Dictionary of string diagram operations

2.1. **Bicategories.** As a first example of the string diagram calculus we recover Shulman’s result [Shu08] that an indexed symmetric monoidal category defines a bicategory.

**Theorem 2.4.** [Shu08, 14.4, 14.11] If $\mathcal{C}$ is an $S$-indexed symmetric monoidal category, there is a bicategory $\mathcal{C}/S$ whose objects are the objects of $S$ and $\mathcal{C}/S(A,B) := \mathcal{C}^{A \times B}$. The unit object is the composition

\[(2.5) \quad \ast \rightarrow \mathcal{C}^{\ast} \xrightarrow{\pi_B} \mathcal{C}^B \xrightarrow{(\Delta_B)^*} \mathcal{C}^{B \times B}\]

where the first map picks out the unit $1_\ast$ in $\mathcal{C}^\ast$. The bicategorical composition is the composite

\[(2.6) \quad \mathcal{C}^{A \times B} \times \mathcal{C}^{B \times C} \xrightarrow{(\text{id} \times \text{id} \times \pi_{B \times C})^* \times (\pi_{B \times C} \times \text{id} \times \text{id})^*} \mathcal{C}^{A \times B \times B \times C} \times \mathcal{C}^{A \times B \times B \times C} \xrightarrow{(\text{id} \times \Delta_B \times \text{id})^* \times (\text{id} \times \Delta_B \times \text{id})^*} \mathcal{C}^{A \times C}\]

The operations in this theorem give the first entries in the dictionary (Table 2.3) that translates between maps of graphs and functors of $\mathcal{C}/S$.

**Proof.** Recall from the introduction that string diagrams for bicategories are built from paths as in Figure 2.7a. A tuple $(M_1, M_2, \ldots, M_n)$ of $n$ composable 1-cells can be represented by the labeled graph in Figure 1.3c and the bicategorical composition is represented in Figure 1.3d.

The associativity isomorphism can be deduced from the graph in Figure 2.7b. First darkening the vertex between the copies of $B$ and then the copies of $C$ is the composite
of functors
\[ \mathcal{B}(A, B) \times \mathcal{B}(B, C) \times \mathcal{B}(C, D) \xrightarrow{\otimes \times \text{id}} \mathcal{B}(A, C) \times \mathcal{B}(C, D) \xrightarrow{\otimes} \mathcal{B}(A, D) \]

Darkening in the opposite order gives the composite
\[ \mathcal{B}(A, B) \times \mathcal{B}(B, C) \times \mathcal{B}(C, D) \xrightarrow{\text{id} \times \otimes} \mathcal{B}(A, B) \times \mathcal{B}(B, D) \xrightarrow{\otimes} \mathcal{B}(A, D) \]

For the unitality isomorphism, we can darken the degree 2 white vertices in the graph in Figure 2.7c from left to right or right to left. In the first case we have the identity functor and in the second case we have the composite
\[ \mathcal{B}(A, B) \to \mathcal{B}(A, B) \times \mathcal{B}(B, B) \xrightarrow{\otimes} \mathcal{B}(A, B) \]

where the first map introduces the unit object.

Finally, the two coherences follow from Theorem 2.2 applied to Figures 2.7d and 2.7e.

\[ \square \]

A **shadow** for a bicategory \( \mathcal{B} \) is a 1-category \( \mathcal{T} \), a functor
\[ \langle \cdot \rangle : \mathcal{B}(A,A) \to \mathcal{T} \]
for each object \( A \) of \( \mathcal{B} \), and natural twist isomorphisms
\[ (2.8) \quad \theta : \langle M \otimes N \rangle \sim \langle N \otimes M \rangle \]
that are appropriately compatible with unit and associativity isomorphisms [Pon10].

**Theorem 2.9.** [PS12, 5.2] If \( \mathcal{C} \) is an \( \mathbf{S} \)-indexed symmetric monoidal category the functors
\[ (2.10) \quad \mathcal{C} B \times B \xrightarrow{(\Lambda_B)^*} \mathcal{C} B \xrightarrow{\pi_B} \mathcal{C}^{*} \]
define a shadow on the bicategory \( \mathcal{C}/\mathbf{S} \).

**Proof.** The shadow functor is represented by darkening the white vertex in Figure 2.11a. The two orders of darkening of the white vertices in the graph Figure 2.11b define the twist isomorphism. The compatibility of associativity and twist isomorphisms is given by the orders of darkening the white vertices in the graph Figure 2.11c. The unit conditions are given by the orderings of darkening the white vertices in the graph Figure 2.11d. \( \square \)
2.2. External Products. A trivial cover of a graph $G$ with labels on the edges is a map of graphs

$$IIG_i \rightarrow G$$

where, for each $i$, $G_i \rightarrow G$ is a homeomorphism of graphs and the map is the product of labels on edges. See Figure 2.14a. These maps along with the following construction of the external product allow us to add to our dictionary in Table 2.3.

In an indexed symmetric monoidal category the external product $\boxtimes$ is the composite

$$\mathcal{C}A \times \mathcal{C}B \xrightarrow{(\pi B)^\ast \times (\pi A)^\ast} \mathcal{C}A \times B \times \mathcal{C}A \times B \xrightarrow{\gamma} \mathcal{C}A \times B$$

This in turn defines an external product on the associated bicategory

$$\mathcal{C}_{\mathcal{S}}(A_1, B_1) \times \mathcal{C}_{\mathcal{S}}(A_2, B_2) \rightarrow \mathcal{C}_{\mathcal{S}}(A_1 \times A_2, B_1 \times B_2)$$

by the formula

$$\mathcal{C}_{\mathcal{S}}(A_1 \times B_1) \times \mathcal{C}_{\mathcal{S}}(A_2 \times B_2) \xrightarrow{\boxtimes} \mathcal{C}_{\mathcal{S}}(A_1 \times B_1 \times A_2 \times B_2) \xrightarrow{\gamma} \mathcal{C}_{\mathcal{S}}(A_1 \times A_2 \times B_1 \times B_2)$$

This admits generalizations for any number of inputs.

**Proposition 2.13.** If $\mathcal{C}$ is a $\mathcal{S}$-indexed symmetric monoidal category the external product defines a pseudo functor

$$\boxtimes: \underbrace{\mathcal{C}_{\mathcal{S}} \times \ldots \times \mathcal{C}_{\mathcal{S}}}_n \rightarrow \mathcal{C}_{\mathcal{S}}$$

for each natural number $n$.

**Proof.** On 0-cells, $\boxtimes$ is the Cartesian product $(A_1, \ldots, A_n) \sim \prod_i A_i$. On the 1-cells and 2-cells, it is represented by the covering map of graphs in Figure 2.14a.

Figures 2.14b and 2.14c each depict a map of graphs in which one vertex is colored. Interpreting that colored vertex as a map from the graph where the vertex is white to the graph where it is black, each figure depicts a commuting square of graphs. The associated natural isomorphisms finish the definition of the pseudofunctor $\boxtimes$. Their coherence follows from the diagrams in Figures 2.14d to 2.14f.

Let $m_{\boxtimes}: (\boxtimes M_i) \odot (\boxtimes N_i) \rightarrow \boxtimes (M_i \otimes N_i)$ denote the natural map defined by Figure 2.14b.

2.3. Base change. In an $\mathcal{S}$-indexed symmetric monoidal category $\mathcal{C}$, the base change objects for a morphism $f: A \rightarrow B$ in $\mathcal{S}$ are

$$\left( \begin{array}{c} B \xrightarrow{\ell} A \end{array} \right) := (\text{id}_A, f)_* \pi_A^* I \in \mathcal{C}^{A \times B}$$

To capture this structure in string diagrams, we allow any degree 2 vertex, either white or black, to be labeled by a morphism $f$ in $\mathcal{S}$. See Figure 2.17a for an example. Darkening a labeled white vertex will either
• add a base change object, or
• apply a pushforward or pullback functor.

The functor associated to the darkening of a particular vertex depends only on the colors of the adjacent vertices. Theorem 2.2 implies that there are isomorphisms

\[(f \times \text{id})^*(\text{id} \times g)^* M \cong (\text{id} \times g)^*(f \times \text{id})^* M.\]

If adjacent vertices of the same color are labeled by maps that point in the same direction, we allow those vertices to collapse and the associated functor is the identity functor.
We note that once a vertex is black, its label no longer matters, but we sometimes keep it because it helps us remember which map of graphs we performed.

**Lemma 2.16.** For a string of composable maps $B_{i-1} \overset{f_i}{\rightarrow} B_i$, there is a canonical isomorphism

$$m_{[]} : \left[ B_n \overset{f_n \circ \ldots \circ f_1}{\rightarrow} B_0 \right] \cong \left[ B_n \overset{f_n}{\rightarrow} B_{n-1} \right] \circ \ldots \circ \left[ B_2 \overset{f_2}{\rightarrow} B_1 \right] \circ \left[ B_1 \overset{f_1}{\rightarrow} B_0 \right].$$

**Proof.** Both sides arise from the map of graphs in Figure 2.17b. The product

$$\left[ B_n \overset{f_n \circ \ldots \circ f_1}{\rightarrow} B_0 \right]$$

is obtained by first darkening the white vertices that are labeled by $f_i$ as in Figure 2.17c. The term $\left[ B_n \overset{f_n \circ \ldots \circ f_1}{\rightarrow} B_0 \right]$ is obtained by first collapsing all the degree-two white vertices together, then darkening resulting white vertex, as in Figure 2.17d. To check that these isomorphisms are compatible with composition we pass through the graph depicted in Figure 2.17e.

![Diagram](image)

**Figure 2.17.** String diagrams for base change (Lemma 2.16)

### 2.4. Twisted external products

The **twisted external product** of $Q_i \in \mathcal{C}_S(A_{i-1}, B_i)$ is denoted by

$$\oplus Q_i \in \mathcal{C}_S(\prod A_i, \prod B_i),$$

and is defined by the map of graphs in Figure 2.18. It is isomorphic to the pullback of the external product $\boxtimes Q_i$ along the twist map $\gamma : \prod A_i \rightarrow \prod A_{i-1}$. 
Lemma 2.19. There is a natural isomorphism
\[ m_\Phi : (\oplus -) \circ (\sqcap -) \to (\oplus -) \]
so that the following diagram commutes for \( Q_i \in \mathcal{C}_S(A_{i-1}, B_i) \), \( M_i \in \mathcal{C}_S(B_i, C_i) \), and \( N_i \in \mathcal{C}_S(C_i, D_i) \).

\[
\begin{array}{c}
(\oplus Q_i) \circ ((\sqcap M_i) \circ (\sqcap N_i)) \\
\downarrow m_\Phi \circ \text{id} \\
(\oplus Q_i) \circ ((\sqcap M_i) \circ (\sqcap N_i))
\end{array}
\]

\[
\begin{array}{c}
(\oplus Q_i) \circ ((\sqcap M_i) \circ (\sqcap N_i)) \\
\downarrow m_\Phi \circ \text{id} \\
(\oplus Q_i) \circ ((\sqcap M_i) \circ (\sqcap N_i))
\end{array}
\]

Similarly, there is a natural isomorphism
\[ m_\Phi : (\sqcap -) \circ (\oplus -) \to (\sqcap (\_ -)) \]
so that the following diagram commutes for \( P_i \in \mathcal{C}_S(C_{i-1}, D_i) \), \( M_i \in \mathcal{C}_S(B_i, C_i) \), and \( N_i \in \mathcal{C}_S(A_i, B_i) \).

\[
\begin{array}{c}
(\sqcap N_i) \circ ((\sqcap M_i) \circ (\sqcap P_i)) \\
\downarrow m_\Phi \circ \text{id} \\
(\sqcap N_i) \circ ((\sqcap M_i) \circ (\sqcap P_i))
\end{array}
\]

\[
\begin{array}{c}
(\sqcap N_i) \circ ((\sqcap M_i) \circ (\sqcap P_i)) \\
\downarrow m_\Phi \circ \text{id} \\
(\sqcap N_i) \circ ((\sqcap M_i) \circ (\sqcap P_i))
\end{array}
\]

Proof. The proof follows that of Proposition 2.13. For the first half, replace the \( A_i \) in the source graph Figures 2.14b and 2.14f with \( A_{i-1} \). For the second half, in Figure 2.14b replace \( A_i, B_i \) with \( A_{i-1} \) and \( B_{i-1} \), and in Figure 2.14f do the same for \( A_i, B_i \), and \( C_i \).

In particular, for \( Q_i, P_i \in \mathcal{C}_S(A_{i-1}, B_i) \), \( M_i, N_i \in \mathcal{C}_S(B_i, C_i) \), and maps \( g_i : Q_i \to P_i \) and \( f_i : M_i \to N_i \) the following diagram commutes.

\[
(\oplus Q_i) \circ (\sqcap M_i) \overset{\text{id} \circ m_\Phi}{\longrightarrow} (\oplus Q_i) \circ (\sqcap M_i)
\]

\[
(\oplus P_i) \circ (\sqcap N_i) \overset{\text{id} \circ m_\Phi}{\longrightarrow} (\oplus P_i) \circ (\sqcap N_i)
\]

The corresponding diagram with the external product and twisted external product exchanged also commutes.

2.5. Untwisting maps. So far we have have only considered interval-shaped graphs and trivial covers of such. Now we will consider circle-shaped graphs and nontrivial covers of such, see Figure 2.23a.

Lemma 2.21. For \( P_i \in \mathcal{C}_S(B_{i-1}, B_i) \), there is a natural “untwisting” isomorphism
\[ \tau : \langle \oplus P_i \rangle \to \langle P_1 \circ \cdots \circ P_n \rangle \]
so the following three diagrams commute.

\[
\begin{array}{c}
\langle (\oplus P_i) \circ (\sqcap N_i) \rangle \\
\downarrow \phi \\
\langle (\oplus P_i) \circ (\sqcap N_i) \rangle
\end{array}
\]

\[
\begin{array}{c}
\langle (\oplus P_i) \circ (\sqcap N_i) \rangle \\
\downarrow \phi \\
\langle (\oplus P_i) \circ (\sqcap N_i) \rangle
\end{array}
\]

\[
\begin{array}{c}
\langle (\oplus P_i) \circ (\sqcap N_i) \rangle \\
\downarrow \phi \\
\langle (\oplus P_i) \circ (\sqcap N_i) \rangle
\end{array}
\]
\[ \langle \bigoplus Q_i \circ U_{\prod A_i} \rangle \xrightarrow{\sim} \langle \bigoplus Q_i \circ (\overline{U}_{A_i}) \rangle \xrightarrow{m_{Q_i}} \langle \bigoplus (Q_i \circ U_{A_i}) \rangle \xrightarrow{\tau} \langle Q_1 \circ U_{A_1} \circ \cdots \circ Q_n \circ U_{A_n} \rangle \]

\[ \langle \bigoplus Q_i \rangle \xrightarrow{\tau} \langle Q_1 \circ \cdots \circ Q_n \rangle \]

\[ \langle U_{\prod B_i} \circ (\bigoplus P_i) \rangle \xrightarrow{\sim} \langle \overline{U}_{B_i} \circ (\bigoplus P_i) \rangle \xrightarrow{m_{P_i}} \langle \bigoplus (U_{B_{i-1}} \circ P_i) \rangle \xrightarrow{\tau} \langle U_{B_n} \circ P_1 \circ \cdots \circ U_{B_{n-1}} \circ P_n \rangle \]

\[ \langle \bigoplus P_i \rangle \xrightarrow{\tau} \langle P_1 \circ \cdots \circ P_n \rangle \]

**Proof.** The untwisting isomorphism is induced by the covering map in Figure 2.23a. Covering first and then darkening gives \( \langle \bigoplus P_i \rangle \) since cutting the graph at the white vertices gives the cover used to define \( \bigoplus \). Darkening first gives \( \langle P_1 \circ \cdots \circ P_n \rangle \).

To check the first of the three diagrams we modify Figure 2.23a by having the black vertices alternate between \( P_i \) and \( N_i \). To check the remaining two we use the modification depicted in Figure 2.23b.

\[ \text{Figure 2.23. String diagram for Lemma 2.21} \]
Since the isomorphism $\tau$ is natural, for $Q_i, P_i \in \mathcal{C}[S](A_{i-1}, A_i)$ and maps $f_i : Q_i \to P_i$, the following square commutes.

\[
\begin{array}{ccc}
\langle \oplus Q_i \rangle & \xrightarrow{\tau} & \langle Q_1 \circ \cdots \circ Q_n \rangle \\
\downarrow \langle \oplus f_i \rangle & & \downarrow \langle f_1 \circ \cdots \circ f_n \rangle \\
\langle \oplus P_i \rangle & \xrightarrow{\tau} & \langle P_1 \circ \cdots \circ P_n \rangle
\end{array}
\]

3. Bicategories with additional structure

In this section we prove that $\mathcal{C}[S]$ has an $n$-Fuller structure and a system of base-change objects indexed by $S$. This supplies the axioms necessary for the proofs of [MP, Theorem 5.7 and Proposition 6.2]. A **shadowed $n$-Fuller structure** on a bicategory with shadow $\mathcal{B}$ consists of the following.

- A strong functor (pseudofunctor) of bicategories
  \[
  \Xi : \mathcal{B} \times \ldots \times \mathcal{B} \to \mathcal{B},
  \]

- A pseudonatural transformation
  \[
  \theta : \Xi \circ \gamma \to \Xi
  \]
  where $\gamma$ is the strong functor $\mathcal{B} \times \ldots \times \mathcal{B} \to \mathcal{B} \times \ldots \times \mathcal{B}$ that permutes the leftmost $\mathcal{B}$ to the right. For each $n$ tuple of objects $(A_1, \ldots, A_n)$ in $\mathcal{B}$ we denote the associated object of $\mathcal{B}(A_2 \times \ldots \times A_n \times A_1, A_1 \times \ldots \times A_n)$ by $T_{A_i}$ and the natural isomorphisms
  \[
  \theta : T_{A_i} \circ (\Xi M_i) \cong (\Xi M_{i+1}) \circ T_{B_i}
  \]
  for all $M_i \in \mathcal{B}(A_i, B_i)$.

- A natural isomorphism
  \[
  \tau : \langle T_{A_{i-1}} \circ \Xi Q_i \rangle \cong \langle Q_1 \circ \cdots \circ Q_n \rangle
  \]
  so that
  \[
  \langle T_{A_{i-1}} \circ \Xi R_i \circ \Xi S_i \rangle \cong \langle T_{A_{i-1}} \circ \Xi (R_i \circ S_i) \rangle \xrightarrow{\tau} \langle R_1 \circ S_1 \circ R_2 \circ \cdots \circ R_n \circ S_n \rangle
  \]
  \[
  \xrightarrow{\theta} \langle \Xi R_{i+1} \circ T_{B_i} \circ \Xi S_i \rangle \xrightarrow{\tau} \langle T_{B_i} \circ \Xi (S_1 \circ R_{i+1}) \rangle \xrightarrow{\theta} \langle S_1 \circ R_2 \circ \cdots \circ R_n \circ S_n \circ R_1 \rangle
  \]
  commutes for all $R_i \in \mathcal{B}(A_{i-1}, B_i)$ and $S_i \in \mathcal{B}(B_i, A_i)$.

Embedded in this definition are the three diagrams in Figure 3.1.

**Proposition 3.2.** If $\mathcal{C}$ is an $S$-indexed symmetric monoidal category there is a pseudonatural transformation

\[
\theta : \Xi \circ \gamma \to \Xi
\]

where $\gamma$ is the strong functor $\mathcal{C}[S] \times \ldots \times \mathcal{C}[S] \to \mathcal{C}[S] \times \ldots \times \mathcal{C}[S]$ that permutes the leftmost $\mathcal{C}[S]$ to the right.

**Proof.** We define the natural transformation $\theta$ using the trivial $n$-fold cover in Figure 3.3b. The definition of $\theta$ requires the darkening of many internal vertices, so in this proof we will label internal vertices by lowercase letters. (Note that these are not maps and we indicate the difference from the convention in Figure 2.17a by the presence of arrows.) The graphs in Figure 3.3 are examples of this convention. A sequence $abc$ denotes first darkening the $a$ vertex, then the $b$ vertex and finally the $c$ vertex.

We let $\diamond_b$ denote the map of graphs from the labeled trivial $n$-fold cover of a graph $G$ to $G$, whose labels to the right of vertex $b$ are the $n$-fold product $\prod_i B_i$, and to the left of
3.3a

\[ \begin{array}{c}
U_{\Pi A_{i+1}} \circ T_{A_i} \xrightarrow{\ell} T_{A_i} \leftarrow T_{A_i} \circ U_{\Pi A_i} \\
(\bigoplus U_{A_{i+1}}) \circ T_{A_i} \xrightarrow{\theta} T_{A_i} \circ (\bigoplus U_{A_i})
\end{array} \]

\text{(A) Unit compatibility}

\[ \begin{array}{c}
(T_{A_i} \circ \bigotimes M_i) \circ \bigotimes N_i \xrightarrow{a} T_{A_i} \circ (\bigotimes M_i \circ \bigotimes N_i)
\end{array} \]

\[ \begin{array}{c}
\bigotimes M_{i+1} \circ (T_{B_i} \circ \bigotimes N_i) \xrightarrow{\theta} \bigotimes (M_{i+1} \circ N_{i+1}) \circ T_{C_i}
\end{array} \]

\[ \begin{array}{c}
\bigotimes M_{i+1} \circ (\bigotimes N_{i+1} \circ T_{C_i}) \xrightarrow{a} (\bigotimes M_{i+1} \circ \bigotimes N_{i+1}) \circ T_{C_i}
\end{array} \]

\text{(B) Compatibility with \circ}

\[ \begin{array}{c}
\{T_{A_{i-1}} \circ (\bigotimes R_i \circ \bigotimes S_i)\} \xrightarrow{\{T_{A_{i-1}} \circ \bigotimes R_i \circ \bigotimes S_i\}} \{R_1 \circ S_1 \circ R_2 \circ \ldots \circ R_n \circ S_n\}
\end{array} \]

\[ \begin{array}{c}
\{\bigotimes R_{i+1} \circ T_{B_i} \circ \bigotimes S_i\} \xrightarrow{\bigotimes R_{i+1} \circ (T_{B_i} \circ \bigotimes S_i)} \{T_{B_i} \circ \bigotimes S_i \circ \bigotimes R_{i+1}\}
\end{array} \]

\[ \begin{array}{c}
\{T_{B_i} \circ (\bigotimes S_i \circ \bigotimes R_{i+1})\} \xrightarrow{T_{B_i} \circ (\bigotimes S_i \circ \bigotimes R_{i+1})} \{S_1 \circ R_2 \circ \ldots \circ R_n \circ S_n \circ R_1\}
\end{array} \]

\text{(C) Compatibility with the twist map}

\text{Figure 3.1. Commutative diagrams for shadowed } n \text{-Fuller structure}

\( b \) are labeled by \( \prod_i A_{i+1} \). See Figure 3.3a. The map \( \theta \) is the isomorphism between the two factorizations of the map in Figure 3.3b given by

\[ \bigotimes_d dcba \equiv \bigotimes_a abcd. \]

Note that we read this string of letters left to right, so the first step in \( \bigotimes_d dcba \) is to apply the twisted external product at \( d \). Then darken \( d \) followed by \( c, b \) and \( a \).

The compatibility of \( \theta \) and the unit isomorphism of \( \bigotimes \) (Figure 3.1a) is given by comparing the following eight orders for darkening and covering the graphs in Figure 3.3c.

\[ \bigotimes_e ecdba \xrightarrow{\ell} \bigotimes_e ecdba \xrightarrow{\bigotimes_a abcde} \bigotimes_a acbde \]

\[ \bigotimes_e edba \bigotimes_c edba \bigotimes_a cabde \]

\[ \bigotimes_e edba \xrightarrow{\theta} \bigotimes_c edba \xrightarrow{\bigotimes_a abde} \]
The compatibility for $\vartheta$ and the bicategorical composition (Figure 3.1b) follows by comparing the following eight orders of darkening and covering the graphs in Figure 3.3d.

\[
\begin{align*}
\Diamond_{ggefcedba} & \xrightarrow{\alpha} \Diamond_{gcedgfba} \\
\vartheta & \downarrow \\
\Diamond_{ddefgcb} & \xrightarrow{a} \Diamond_{ggefba} \\
\vartheta & \downarrow \\
\Diamond_{dcbaefg} & \xrightarrow{\vartheta} \Diamond_{abfg} \\
\vartheta & \downarrow \\
\Diamond_{abcdefg} & \xrightarrow{a} \Diamond_{acedabfg}
\end{align*}
\]

\[\square\]

**Figure 3.3.** Constructing and verifying that $\vartheta$ is a pseudonatural transformation

**Theorem 3.4.** If $\mathcal{C}$ is an $\mathbf{S}$-indexed symmetric monoidal category then $\mathcal{C}/_{\mathbf{S}}$ with the structure from [PS12, 5.2][Shu08, 14.4, 14.11] has a shadowed $n$-Fuller structure.

**Proof.** The required pseudo functor is defined in Proposition 2.13 and the pseudo natural transformation is in Proposition 3.2.

The isomorphism $\langle T_{A_{i+1}} \otimes Q_i \rangle \to \langle Q_1 \otimes \cdots \otimes Q_n \rangle$ is defined in Figure 3.5a by comparing the routes $\Diamond_{b\cdot c\cdot a}$ and $a_1 b_1 c_1 a_2 \cdots b_n c_n$. The compatibility condition (Figure 3.1c) is
given by comparing the following routes through Figure 3.5b.

\[ e \text{cbafed} \rightarrow a_1 b_1 c_1 \ldots a_n b_n c_n \nabla e f d \rightarrow a_1 b_1 c_1 \ldots a_n b_n c_n d_1 e_1 f_1 \ldots d_n e_n f_n \]

\[ e \text{efabcd} \]
\[ e \text{bafecd} \]
\[ e \text{bcafed} \]
\[ e \text{fedcba} \rightarrow d_1 e_1 f_1 \ldots d_n e_n f_n \nabla b \text{ba} f e c d \]
\[ e \text{eedcfa} \]
\[ e \text{fede} \]

\[ e \text{bca} \rightarrow d_1 e_1 f_1 \ldots d_n e_n f_n a_1 b_1 c_1 \ldots a_n b_n c_n \]

\[ \text{A} \]
\[ \text{B} \]

\[ \text{A}_{i-1} \quad \text{A}_i \]
\[ A_{i-1} \quad c_{i-1} \quad a_i \quad A_i \]
\[ A_{i-1} \quad b_{i-1} \quad b_i \quad A_i \]
\[ A_{i-1} \quad a_{i-1} \quad c_i \quad A_i \]

\[ \text{B}_{i-1} \quad \text{B}_i \]
\[ B_{i-1} \quad f_{i-1} \quad a_i \quad B_i \]
\[ B_{i-1} \quad e_{i-1} \quad b_i \quad c_i \quad B_i \]
\[ B_{i-1} \quad d_{i-1} \quad A_i \]
\[ B_{i-1} \quad b_{i-1} \quad a_{i-1} \quad f_i \quad A_i \]

\[ \text{(A) Isomorphism} \]
\[ \text{(B) Coherence} \]

**Figure 3.5. The shadowed Fuller structure**

If \( S \) is a cartesian monoidal 1-category, a **system of base-change objects for \( B \) indexed by \( S \)** is the following data and conditions.

- A pseudofunctor \([\cdot] : S \rightarrow B\).
- A vertical natural isomorphism \( \pi \) filling the square of pseudofunctors

\[
\begin{array}{ccc}
S^{\times n} & \xrightarrow{\Pi} & S \\
\downarrow & & \downarrow \\
B^{\times n} & \xrightarrow{\otimes} & B
\end{array}
\]

where \( \Pi \) denotes a fixed model for the \( n \)-fold product in \( S \).
• An equality $T_B = \prod B_{i+1} \cong \prod B_i$ so that the following diagram relating $\vartheta$, $\pi$ and the pseudofunctor structure commutes.

Theorem 3.6. If $\mathcal{C}$ is an $\mathbf{S}$-indexed symmetric monoidal category, the shadowed $n$-Fuller bicategory $\mathcal{C}/\mathbf{S}$ has base change objects.

Proof. The pseudofunctor's composition isomorphism is defined in Lemma 2.16. Its unit isomorphism is an identity map $\langle B , B \rangle = U_B$. Of the two needed coherences, one is covered by the more general statement in Lemma 2.16. The other is the observation that the unit and composition maps give the same isomorphism $\langle B , B \rangle \circ \langle B , A \rangle \cong \langle B , A \rangle$ because they both arise from the same string diagram. Figure 3.7a defines a canonical isomorphism $\boxtimes \langle B , f_i A_i \rangle \cong \prod B_i \prod f_i \prod A_i$ and this is the vertical natural isomorphism filling the square of pseudofunctors. For the coherence statement, all five terms come from Figure 3.7b.

\[
\begin{array}{c}
\prod_i \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B_i \xleftarrow{f_i} A_i
\end{array}
\end{array}
\end{array}
\right) \longrightarrow \prod B_i \prod A_i
\end{array}
\]
(A) The isomorphism relating $\boxtimes$ and base change

\[
\begin{array}{c}
\prod_i \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C_i \xleftarrow{g_i} B_i \xrightarrow{f_i} A_i
\end{array}
\end{array}
\end{array}
\right) \longrightarrow \prod C_i \prod A_i
\end{array}
\]
(B) Coherence for $\boxtimes$ and base change

FIGURE 3.7. Compatibility of base change and $\boxtimes$ (Theorem 3.6)

4. COMPARING THE FULLER TRACE AND MULTITRACE

In this section we give a second, shorter proof of [MP, Theorem 5.7 and Proposition 6.2], which are the most technical steps in the proof of Theorem 1.2. In this second proof, we bypass the $n$-Fuller structure and the arguments in [MP, Sections 5 and 6] and
instead use the string diagram calculus in a more direct way to compare the Fuller trace to the multitrace.

Recall (e.g. [MS06, 18] or [MP, Section 4]) that a 1-cell $M \in \mathcal{C}_S(A, B)$ is right dualizable if there is a 1-cell $M^*$ and 2-cells

$$\eta : U_A \to M \otimes M^*, \quad \epsilon : M^* \otimes M \to U_B$$

satisfying two triangle identities. If $M$ is dualizable, $Q \in \mathcal{C}_S(A, A)$, $P \in \mathcal{C}_S(B, B)$, and $\phi : Q \otimes M \to M \otimes P$, the trace of $\phi$ is the composite:

$$\langle Q \rangle \xrightarrow{id \otimes \eta} \langle Q \otimes M \otimes M^* \rangle \xrightarrow{\phi} \langle M \otimes P \otimes M^* \rangle \cong \langle P \otimes M^* \otimes M \rangle \xrightarrow{id \otimes \epsilon} \langle P \rangle.$$

More generally if $M_i \in \mathcal{B}(A_i, B_i)$ is right dualizable, $Q_i \in \mathcal{B}(A_{i-1}, A_i)$, $P_i \in \mathcal{B}(B_{i-1}, B_i)$ (subscripts taken mod $n$), the multitrace of maps

$$\phi_i : Q_i \otimes M_i \to M_{i-1} \otimes P_i,$$

denoted $\text{tr}(\phi_1, \ldots, \phi_n)$, is the composite:

$$\langle Q_1 \ldots Q_n \rangle \xrightarrow{id \otimes \eta_1 \ldots \eta_n} \langle Q_1 \otimes M_1 \otimes M_1^* \otimes Q_2 \otimes M_2^* \otimes \ldots \otimes M_n \otimes M_n^* \rangle \xrightarrow{\phi_1 \otimes \ldots \otimes \phi_n \otimes \eta} \langle M_n \otimes P_1 \otimes M_1^* \otimes M_1 \otimes P_2 \otimes M_2^* \otimes \ldots \otimes P_n \otimes M_n^* \rangle \xrightarrow{\theta} \langle P_1 \otimes M_1^* \otimes M_1 \otimes P_2 \otimes M_2^* \otimes \ldots \otimes P_n \otimes M_n^* \otimes M_n \rangle$$

The abstract Fuller construction $\Psi(\phi_1, \ldots, \phi_n)$ is the map

$$\bigoplus Q_i \otimes \Box M_i \xrightarrow{m^+} \bigoplus (Q_i \otimes M_i) \xrightarrow{\bigoplus \phi_i} \bigoplus (M_{i-1} \otimes P_i) \xrightarrow{m^+} \Box M_i \otimes \bigoplus P_i.$$

**Theorem 4.1.** [MP, Theorem 5.7] The following diagram commutes.

$$\begin{array}{ccc}
\langle \bigoplus Q_i \rangle & \xrightarrow{\tau} & \langle Q_1 \ldots Q_n \rangle \\
| & \text{tr}(\Psi(\phi_1 \ldots \phi_n)) & \text{tr}(\phi_1 \ldots \phi_n) \\
\langle \bigoplus P_i \rangle & \xrightarrow{\tau} & \langle P_1 \ldots P_n \rangle
\end{array}$$

**Proof.** The regions with one dotted edge in Figure 4.2 commute by definition of the dotted map. Each of the remaining regions commutes by the result noted in that region (or by the coherence theorem for bicategories with shadow, Theorem 9.12). \qed

For [MP, Proposition 6.2], $n$ commuting squares in $\mathbf{S}$

$$\begin{array}{c}
E_i \xrightarrow{f_i} E_{i-1} \\
p_i \downarrow & \\
B_i \xrightarrow{f_{i-1}} B_{i-1}
\end{array}$$

define a commuting square

$$\begin{array}{ccc}
\prod E_i & \xrightarrow{\Psi(f_1, \ldots, f_n)} & \prod E_i \\
\prod p_i & \downarrow & \prod p_i \\
\prod B_i & \xrightarrow{\Psi(f_1, \ldots, f_n)} & \prod B_i
\end{array}$$
where $\Psi(f_1, \ldots, f_n)$ is the composition

$$\prod E_i \overset{\prod f_i}{\longrightarrow} \prod E_{i-1} \overset{\gamma_i}{\longrightarrow} \prod E_i$$

and $\Psi(f'_1, \ldots, f'_n)$ is similar. The first squares define maps

$$\phi_i: \left[ B_{i-1} \overset{\bar{f}_i}{\longrightarrow} B_i \right] \otimes \left[ B_i \overset{p_i}{\longrightarrow} E_i \right] \rightarrow \left[ B_{i-1} \overset{p_{i-1}}{\longrightarrow} E_{i-1} \right] \otimes \left[ E_{i-1} \overset{f_i}{\longrightarrow} E_i \right]$$

for each $i$, and the second square defines a map

$$\phi: \left[ \prod B_i \overset{\Psi(f_1', \ldots, f_n')}{\longrightarrow} \prod B_i \right] \otimes \left[ \prod B_i \overset{p_i}{\longrightarrow} \prod E_i \right] \rightarrow \left[ \prod B_i \overset{p_{i-1}}{\longrightarrow} \prod E_{i-1} \right] \otimes \left[ \prod E_i \overset{\Psi(f_1, \ldots, f_n)}{\longrightarrow} \prod E_i \right].$$

The maps of graphs depicted in Figure 4.3 define isomorphisms

$$\pi: \bigoplus \left[ B_{i-1} \overset{\bar{f}_i}{\longrightarrow} B_i \right] \equiv \left[ \prod B_i \overset{\Psi(f_1', \ldots, f_n')}{\longrightarrow} \prod B_i \right]$$

$$\pi: \bigoplus \left[ E_{i-1} \overset{f_i}{\longrightarrow} E_i \right] \equiv \left[ \prod E_i \overset{\Psi(f_1, \ldots, f_n)}{\longrightarrow} \prod E_i \right]$$

$$\pi: \bigotimes \left[ B_i \overset{p_i}{\longrightarrow} E_i \right] \equiv \left[ \prod B_i \overset{p_i}{\longrightarrow} \prod E_i \right]$$
Proposition 4.4. [MP, Proposition 6.2] Let \( \mathcal{C} \) be an \( \mathbf{S} \)-indexed symmetric monoidal category. For any \( n \)-tuple of commuting squares in \( \mathbf{S} \) as above, where \( [B_i \xrightarrow{p_i} E_i] \) and \( [\prod B_i \xrightarrow{\prod p_i} \prod E_i] \) are dualizable, the following diagram commutes.

\[
\begin{align*}
&\left\langle \left( B_{i-1} \xrightarrow{f_i} B_i \right) \right\rangle \xrightarrow{\langle \pi \rangle} \left\langle \left[ \prod B_i \xrightarrow{\prod f_i} \prod B_i \right] \right\rangle \\
&\begin{array}{c}
\downarrow \tr(\Psi(\phi_1, \ldots, \phi_n)) \\
\end{array} \\
&\left\langle \left( E_{i-1} \xrightarrow{f_i} E_i \right) \right\rangle \xrightarrow{\langle \pi \rangle} \left\langle \left[ \prod E_i \xrightarrow{\prod f_i} \prod E_i \right] \right\rangle \\
&\begin{array}{c}
\downarrow \tr(\phi) \\
\end{array}
\end{align*}
\]

Proof. It is enough to verify Figure 4.5 commutes. The middle region of Figure 4.5 commutes by the definition of \( \phi \). The top region commutes using the string diagrams depicted in Figure 4.6. The bottom region commutes by the same argument. \( \square \)
Part 2. Precise statement of the calculus

The goal of this part is to give a formal statement of the calculus introduced in Part 1. Most of the work required to make the statement rigorous is focused in defining

- indexed categories and
- labeled graphs and their morphisms.

There are several equivalent structures that satisfy the (sketched) structure of an indexed category at the beginning of Part 1:

- Grothendieck fibrations,
- indexed categories, and
- coherent diagrams of functors.
Each has distinct advantages and disadvantages. We will use all of these structures in this paper. In this part we will work with fibrations since they have the simplest description and much of their structure is encoded by universal properties.

5. Grothendieck fibrations and constellations

A (Grothendieck) fibration over a category $\mathcal{S}$ is a category $\mathcal{C}$ and a functor $\Phi: \mathcal{C} \to \mathcal{S}$ such that for every pair $\Phi(X)$ consisting of an object in $\mathcal{C}$ and morphism in $\mathcal{S}$ there is a pullback $f^*X$, satisfying the universal property given in shorthand in Figure 5.1a. Arrows $f^*X \to X$ of this form are called cartesian arrows over $f$, and they are unique up to canonical isomorphism.

![Figure 5.1. Cartesian and Cocartesian arrows](image)

**Remark** 5.2. For an object $A$ in $\mathcal{S}$ let $\mathcal{C}^A$ be the subcategory $\Phi^{-1}(A)$. If $f: A \to B$ is a morphism in $\mathcal{S}$, we can define a function, abusively denoted $f^*$,

$$f^*: \text{ob} \mathcal{C}^B \to \text{ob} \mathcal{C}^A$$

where $f^*(X)$ is any object with a cartesian arrow over $f$ terminating at $X$. The universal property makes this into a functor $f^*: \mathcal{C}^B \to \mathcal{C}^A$ which is unique up to canonical isomorphism.

A symmetric monoidal fibration is

- a fibration $\Phi: \mathcal{C} \to \mathcal{S}$,
- a Cartesian monoidal structure on $\mathcal{S}$, and
- a symmetric monoidal structure $(\mathcal{C}, \otimes, I)$

so that

- $\Phi$ is a strict symmetric monoidal functor, and
- the tensor product of any two cartesian arrows in $\mathcal{C}$ is a cartesian arrow.

Note that for each pair of maps $f: A \to B$, $g: A' \to B'$ there is a canonical isomorphism

$$f^*X \otimes g^*Y \cong (f \times g)^*(X \otimes Y)$$

of functors $\mathcal{C}^B \times \mathcal{C}^{B'} \to \mathcal{C}^{A \times A'}$.

A collection of Beck-Chevalley squares in a Cartesian monoidal category $\mathcal{S}$ is a choice of commuting squares in $\mathcal{S}$ that includes the squares in the following list:
• for any pair of composable maps \( A \xrightarrow{f} B \xrightarrow{g} C \) and \( A' \xrightarrow{f'} B' \xrightarrow{g'} C' \), cf. [PS12, Fig 1(a-c)], the squares

\[
\begin{array}{c}
A \times A' \xrightarrow{1 \times f'} A \times B' \\
\downarrow f \times 1 \\
B \times A' \xrightarrow{1 \times f} B \times B'
\end{array}
\quad
\begin{array}{c}
A \xrightarrow{(1, g \circ f)} A \times C \\
\downarrow f \times 1 \\
B \xrightarrow{(1, g)} B \times C
\end{array}
\quad
\begin{array}{c}
A \times C \\
\downarrow (1, g \circ f) \\
A \times C
\end{array}
\]

• any square isomorphic to a Beck-Chevalley square (this includes commuting squares with two parallel isomorphisms), and

• any product of a Beck-Chevalley square and an object of \( S \).

We will refer to the first square as the **diagrammatic external product** of \( f \) and \( f' \).

**A symmetric monoidal bifibration (smbf)** is

• a symmetric monoidal fibration \( \Phi : \mathcal{C} \to S \) and

• a choice of Beck-Chevalley squares in \( S \) so that
  
  i. each pushout diagram can be filled in with a **cocartesian arrow** \( X \to f_! X \) with the universal property illustrated in Figure 5.1b,
  
  ii. the external tensor \( \boxtimes \) preserves the cocartesian arrows \( X \to f_! X \), so there are canonical isomorphisms

\[
f_! X \boxtimes g_! Y \cong (f \times g)_! (X \boxtimes Y)
\]

and

iii. for any Beck-Chevalley square

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow h \\
C \xrightarrow{k} D
\end{array}
\]

in \( S \), the natural transformation of functors \( \mathcal{C}^C \to \mathcal{C}^B \)

\[
f_! h^* \sim f_! h^* k_! k_1 \sim f_! f^* g^* k_1 \to g^* k_1
\]

is an isomorphism (the **Beck-Chevalley condition**).

If we drop all of the conditions involving tensor products \( \boxtimes \) then \( \mathcal{C} \) is merely a **bifibration**.

**Example 5.5** (Examples of fibrations and smbfs).

- If \( C \) is a category with finite products and pullbacks (e.g. topological spaces), the category \( C^\leftarrow \) of arrows in \( C \) is an smbfs with base category \( C \). The projection functor sends an arrow to its codomain, and the symmetric monoidal structure on \( C^\leftarrow \) is by the categorical product. The Beck-Chevalley squares consist of all the strict pullback squares in \( C \).

- There is an smbfs \( S \) of parametrized spectra over all base spaces. The base category is (CGWH) topological spaces, and the fiber category \( S^A \) is the category of parametrized orthogonal spectra over \( A \). The tensor product is the external smash product and the Beck-Chevalley squares are the strict pullback squares. Inverting the stable equivalences gives a smbfs \( h_! S \), whose tensor product is the left-derived external smash product, and whose Beck-Chevalley squares are the homotopy pullback squares. See [MP, Theorem 8.9] and [Mal].
• The previous example generalizes by allowing genuinely $G$-equivariant spectra over $G$-spaces, and/or by restricting to base spaces equipped with a reference map to a fixed space $B$.

• There is an smbf $\mathcal{M}_Z$ whose base category is groups and whose fiber category $\mathcal{M}_Z^A$ is the category of $\mathbb{Z}[A]$-modules. The tensor product is $\otimes = \otimes_Z$ and the Beck-Chevalley squares are those squares for which the map $C \times_A B \to D$ is an isomorphism of $C - B$ bisets. There is a similar smbf $\mathcal{M}_S$ whose base category is topological groups, and whose fiber category is the category of $\Sigma^\infty A$-module spectra, see [Mal].

• If $\mathcal{C}$ is a symmetric monoidal category with all colimits, and $\otimes_C$ preserves all colimits in each variable, then the indexed category $\mathcal{D}$ of all diagrams in $\mathcal{C}$ is a symmetric monoidal bifibration. The **diagrammatic external product** of $X$ over $I$ and $Y$ over $J$ is a diagram $X \boxtimes Y$ over $I \times J$ whose value at $(i, j)$ is $X(i) \otimes_C Y(j)$. The pullbacks are restrictions and the pushforwards are left Kan extensions. The Beck-Chevalley condition is similar to the one for $\mathcal{M}_Z$.

• Given a map of base categories $H: \mathcal{T} \to \mathcal{S}$ that strictly preserves products and Beck-Chevalley squares, and an smbf $\mathcal{C} \to \mathcal{S}$, the pullback $H^* \mathcal{C} \to \mathcal{T}$ has a canonical structure as an smbf. This can be generalized further, see [MP, Lemma 10.1].

Our focus in this paper is on the coherence between the four basic operations $I, \boxtimes, f^*$, and $f!$, along with the following operations built out of them.

\[
- \circ_B - : \mathcal{C}^B \times \mathcal{C}^B \to \mathcal{C}^B
\]

\[
M \circ_B N = (id_A \times \pi_B \times id_C)(id_A \times \Delta_B \times id_C)^*(M \boxtimes N)
\]

\[
U_B: * \to \mathcal{C}^B
\]

\[
M := (\Delta_B); \pi_B
\]

\[
(f, id_A): \pi_A^* I \to \pi_A^* I
\]

\[
(f, id_A): \pi_A^* I
\]

Note that $B \overline{\times} B = U_B$. Figures 5.6 to 5.9 depict four natural isomorphisms relating compositions of these operations. In these figures the smaller squares are filled by one of the natural transformations that describes the compatibility of $\boxtimes$, $(-)^*$, and $(-)_!$. (For readability, we omit the $\times$ symbol when taking products of categories and of maps.)

The definition of $\circ$ and $\otimes$ can be generalized to allow for different numbers of factors in the fiber categories, for instance:

\[
- \circ_B - : \mathcal{C}^{A_1 \times A_2 \times B} \to \mathcal{C}^{A_1 \times A_2}
\]

\[
\otimes_B : \mathcal{C}^{A_1 \times B} \to \mathcal{C}^A
\]

We capture these operations graphically using star graphs. A **star** is a tree with one central black vertex and $k$ white vertices, each joined to the black vertex by a single edge. A **constellation** is a disjoint union of finitely many stars.

If we label each edge of a star by an object $A_i$ of $\mathcal{S}$, then we associate to this star the category $\mathcal{C}^{[1]A}_i$. We extend this to labeled constellation by associating the product of the categories $\mathcal{C}^{[1]A}_i$, one for each star. See Figure 5.10. Then Figures 5.12 and 5.13 and Table 5.11 give operations on constellations and a dictionary that translates between operations on constellations and functors in $\mathcal{C}$.
This dictionary of ten operations is the first half of the string diagram calculus. The second half is a framework that accounts for when two different compositions of these operations define canonically isomorphic functors. This bookkeeping seems to be simpler if we place each constellation into a larger labeled graph and use that graph to dictate which operations can be applied.
Figure 5.7. Right unit isomorphism (and a left unit isomorphism defined similarly)

Figure 5.8. Shadow isomorphism

Figure 5.9. Base change composition
Figure 5.12. The first eight operations on constellations

(A) \(\emptyset \rightarrow 1\)  
(B) \(U_B\)  
(C) \([A \overset{f}{\leftarrow} B]\)

(D) \(f^*\) for \(f : A \rightarrow B\)  
(E) \(f_!\) for \(f : A \rightarrow B\)

(F) \(\odot_B\)  
(G) \(\square_B\)

(H) \(\boxtimes\)

Figure 5.13. Additional operations on constellations

(A) Pullback along \(A \times B \times \ast \rightarrow A \times B\)  
(B) Pullback along \(D \times C \rightarrow A \times B \times C\)
### Operation on constellations

| Operation | Description | Operation on \( \mathcal{C} \) | Page |
|-----------|-------------|---------------------------------|------|
| add a star with no leaves |  | the unit \( I \) | 5.12a |
| add a star with two leaves |  | \( U_B \) and \( \frac{A}{f} \rightarrow B \) | 5.12b, 5.12c |
| change the label on one of the leaves on one of the stars |  | \( f^* \) and \( f! \) | 5.12d, 5.12e |
| If two stars each have a leaf labeled by \( B \), join those two stars together and consume the two leaves labeled by \( B \). |  | \( \odot_B \) | 5.12f |
| If a star has two leaves labeled by \( B \), join a star to itself consuming two leaves labeled by \( B \). |  | \( \odot_B \) | 5.12g |
| join two stars together without consuming any leaves. |  | \( \boxtimes \) | 5.12h |
| create leaves labeled by the terminal object \( \ast \) of \( \mathcal{S} \) |  |  | 5.13a |
| join leaves labeled by \( A \) and \( B \) on the same star to make a leaf labeled by any space \( \sim \) \( = A \times B \) |  |  | 5.13b |

**Table 5.11.** Operations on constellations. Compare to Table 2.3.

### 6. Graphs

The objects of the category of **colored graphs**, denoted \( \mathcal{G} \), are pairs consisting of

- a finite graph \( G \) with no isolated vertices and
- a function \( V(G) \rightarrow \{ \text{black}, \text{white} \} \)

so that the degree of each white vertex is at most 2. Degree one white vertices are **external** white vertices and degree two white vertices are **internal** white vertices. The graph in Figure 6.2a has two external white vertices and one internal white vertex.

A graph morphism (with contractions) \( v : G \rightarrow H \) consists of functions

\[
v_{V} : V(G) \rightarrow V(H) \quad \text{and} \quad v_{E} : E(G) \rightarrow V(H) \cup E(H)
\]

so that for an edge \( e \) of \( G \) from \( v \) to \( v' \), \( v_{E}(e) \) is either an edge from \( v_{V}(v) \) to \( v_{V}(v') \), or the vertex \( v_{E}(e) = v_{V}(v) = v_{V}(v') \). If \( BV(G) \) is the set of black vertices, \( EV(G) \) is the set of external white vertices and \( IV(G) \) is set of internal white vertices, a **morphism** in \( \mathcal{G} \) is a graph morphism \( v : G \rightarrow H \) so that

\[
v_{V}(BV(G)) \subseteq BV(H), \quad v_{V}(EV(G)) \subseteq EV(G), \quad \text{and} \quad v_{V}(IV(G)) \subseteq IV(H) \cup BV(G)
\]

There are several important classes of morphisms in \( \mathcal{G} \):

- A **darkening morphism** is an isomorphism of graphs that changes some white vertices to black.

A **color-preserving morphism** sends every vertex to one of the same color.

- A **collapsing morphism** is a color-preserving morphism in which each edge and vertex in \( H \) has connected preimage. These are generated by maps that collapse edges between vertices of the same color.

- A **covering morphism** is a color-preserving morphism that does not collapse any edges to vertices. Equivalently, each vertex \( v \) in \( H \) has preimage consisting only of vertices with the same color as \( v \).

**Lemma 6.1.** Every morphism in \( \mathcal{G} \) factors, in a canonical way, into a darkening morphism, a collapsing morphism, and a covering morphism.
**Proof.** First darken a vertex if and only if its image is black. Then collapse an edge if and only if its image is a vertex. There is a unique way to factor the original morphism through the resulting graph, and the result is a covering morphism. □

We will represent some of the darken-then-collapse morphisms by coloring a subset of the internal white vertices. The morphism thus represented is the one that darkens the colored vertices, then collapses all edges between black vertices. For example, the morphism depicted in Figure 6.2c that sends the center three vertices to the black vertex can be represented by the graph in Figure 6.2b. Note that this shorthand cannot be used to describe morphisms that collapse an edge between white vertices to a point.

![Diagram](image)

**Figure 6.2. Bicategorical composition graphs**

We will use two colors on the same graph to represent a commuting square in \( \mathcal{G} \). For instance Figure 6.3a corresponds to the commuting square in Figure 6.3b where the blue dot represents the horizontal maps and the green dot represents the vertical maps.

### 6.1. Decorated graphs

Recall that \( S \) is a category. Let \( E\mathcal{G} \) be the category whose objects are tuples consisting of

- A colored graph \( G \) in \( \mathcal{G} \).
- A map \( s \times t : IV(G) \to E(G) \times E(G) \) that assigns a **source** and **target** to each internal white vertex. We require that
  - \( s(v) \neq t(v) \) for all \( v \in IV(G) \) and
  - if the edge \( e \) is incident to \( v_1 \) and \( v_2 \) and \( s(v_1) = e \) then \( t(v_2) = e \).
- Functions \( A_E : E(G) \to \text{ob}(S) \) and \( A_V : IV(G) \to \text{mor}(S) \) respecting the source and target maps.

![Diagram](image)

**Figure 6.3. Representing a commuting square using two colors**
We think of $G$ as a category whose objects are edges and morphisms are generated by internal white vertices. Then $A$ is a functor from this category to $\mathcal{S}$. In particular, we can extend the definition of $A_V$ to any string of (composable) internal white vertices by composing the maps. Similarly the maps $s$ and $t$ extend to strings of composable internal white vertices.

A morphism $(P, \iota): (G, A) \to (H, B)$ in $E\mathcal{G}$ is

- a morphism $P: G \to H$ in $\mathcal{G}$ so that

$$
\begin{array}{ccc}
IV(G) & \xrightarrow{P} & IV(H) \\
\downarrow{s \times t} & & \downarrow{s \times t} \\
E(G) \times E(G) & \xrightarrow{P_\times P_\times} & E(H) \times E(H)
\end{array}
$$

commutes and

- an edge identification map

$$
t_{P, e}: B(e) \xrightarrow{\cong} \prod_{\tilde{e} \in P^{-1}(e)} A(\tilde{e})
$$

for each edge $e \in E(H)$,

so that for each $v \in IV(H)$ and component $\tilde{v}$ of $P^{-1}(v)$ the square below commutes. 

(6.4) 
$$
\begin{array}{ccc}
A(s(\tilde{v})) & \xrightarrow{A(\tilde{v})} & A(t(\tilde{v})) \\
\downarrow{t_{P, s(\tilde{v})}} & & \downarrow{t_{P, t(\tilde{v})}} \\
B(s(v)) & \xrightarrow{B(v)} & B(t(v))
\end{array}
$$

More succinctly, a morphism $P$ is a map of graphs that induces a functor $G \to H$, together with an isomorphism between the functor $B$ and the right Kan extension of $A$ from $G$ to $H$.

We form a composition $P \circ Q$ by composing the maps of graphs, and post-composing each identification map $t_P$ with the product of the identification maps $t_Q, \tilde{e}$ for each $\tilde{e} \in P^{-1}(e)$.

**Example 6.5.** If $A$ and $B$ assign $IV(G)$ and $IV(H)$ to identity morphisms in $\mathcal{S}$ then we can visualize the morphism $P$ by ignoring the internal white vertices entirely. Each edge of the resulting graph $\tilde{H}$ is assigned to an object of $\mathcal{S}$, which is the product of the objects assigned to its preimage edges in $\tilde{G}$. If an edge of $\tilde{H}$ lies outside the image, it must be labeled by an empty product $\ast$.

A morphism $P: (G, A) \to (H, B)$ in $E\mathcal{G}$ is **darkening or collapsing** if

- the map of graphs $P: G \to H$ is such in $\mathcal{G}$, and so each edge $e$ has a unique preimage $\tilde{e}$, and
- the identifications $t_{P, e}: B(e) \cong A(\tilde{e})$ are identity maps in $\mathcal{S}$.

As before, we can give examples of darkening-then-collapsing morphisms in $E\mathcal{G}$ by starting with a labeled graph $(G, A) \in \text{ob}E\mathcal{G}$ and coloring some subset of the white vertices. For instance the graph in Figure 6.6a represents the morphism in Figure 6.6b. Figure 6.6c is a collapse that cannot be represented this way.

The morphism $P: (G, A) \to (H, B)$ is **covering** if $P: G \to H$ is covering. The maps in Figure 6.7 are two important examples of covering morphisms in $E\mathcal{G}$. As before, every morphism in $E\mathcal{G}$ factors into a darkening map, a collapsing map, and a covering map. The covering map is the only part of this three-fold factorization where the edge identification maps may be nontrivial.
A B = B C

(A) Collapsing map represented with color

A B B C → A C

(B) Collapsing map that can be represented using color

f = g → g ∘ f

(c) Collapsing map that can't be represented using color

Figure 6.6. Darkening and collapsing maps

\[ \coprod_i \left( A_{i-1} \quad A_i \quad A_i \quad B_i \right) \rightarrow \times A_i \times A_i \times A_i \times B_i \]

Figure 6.7. Covering maps

In the first map of Figure 6.7, \( \coprod \) is the disjoint union of graphs. It is the coproduct in \( \mathcal{G} \), but not in \( E^G \). (It becomes the coproduct if we drop the condition that the maps \( \iota_p, e \) are isomorphisms.) This operation makes \( E^G \) into a symmetric monoidal category.

6.2. Cutting and constellations. Every graph \( G \) in \( \mathcal{G} \) has a constellation hidden inside. The maximal cutting of \( G \), denoted \( \Psi(G) \), is the constellation obtained by

- collapsing all edges between black vertices,
- deleting all internal white vertices and edges between them, and
- capping off each remaining edge that no longer ends in a vertex with a new external white vertex.

See Figure 6.8. We define a maximal cutting \( \Psi(G, A) \) for labeled graphs in the same way.
An inert morphism in $E\mathcal{G}$ is any morphism $P : (G, A) \to (H, B)$ that is a composite of
- collapsing morphisms,
- darkening a single white vertex $v$, adjacent to precisely one white and one black vertex, such that $A(v)$ is an identity map, and/or
- a map that includes a new component consisting entirely of white vertices.

Intuitively, these are the morphisms that do not change the constellation $\Psi(G, A)$. Formally, each inert morphism induces an isomorphism of constellations $\Psi(G, A) \cong \Psi(H, B)$ along which the labels in $S$ coincide on the nose.

Any morphism $P : (G, A) \to (H, B)$ of $E\mathcal{G}$ induces a map of sets $\pi_0 \Psi(G, A) \to \pi_0 \Psi(H, B)$. This defines the cut components functor

$$\pi_0 \Psi : E\mathcal{G} \to \text{fin}.$$ 

For any collection of internal white vertices $T \subseteq IV(G)$ the cutting along $T$, or $\Psi(G, A; T)$, is the result of following the cutting algorithm but only deleting those vertices in $T$. For a diagram of graphs $F : I \to E\mathcal{G}$ or $\mathcal{G}$ a diagram of cut sets is a subset $T(i) \subseteq IV(F(i))$ for each object $i \in \text{ob} I$, such that for each morphism $i \to i'$,

$$F(p)^{-1}(T(i')) = T(i).$$

In other words, each map of graphs in the diagram sends each cut point to a cut point, and each non-cut point to a non-cut point. Then there is a diagram $\Psi(F; T) : I \to E\mathcal{G}$ or $\mathcal{G}$ whose objects are the resulting cut graphs $\Psi(F(i); T(i))$. We refer to this operation as cutting the diagram of graphs along $T(i)_{i \in \text{ob} I}$.

6.3. Statement of the calculus. Now we can give a precise statement of our string diagram calculus.

Theorem 6.9. Suppose $\mathcal{C}$ is an smbf over $S$. Then there exists a Grothendieck fibration $\mathcal{C}_{\mathcal{G}}$ over $E\mathcal{G}^{\text{op}}$, with the following properties.

i. (Value on objects) For each labeled graph $(G, A)$ of $E\mathcal{G}$, the fiber category $\mathcal{C}_{\mathcal{G}}^{(G, A)}$ contains, as an equivalent subcategory, the category associated to the constellation $\Psi(G, A)$:

$$\prod_{u \in \pi_0 \Psi(G, A)} \mathcal{C}^{\prod_{v \in \text{IV}(G)} A(v)}.$$
The remaining properties will describe the pullback functors on this subcategory.

**ii. (Inert morphisms)** Each inert morphism in \( E \) is sent to an identity functor, and the isomorphisms between these identity functors are identity natural transformations. Formally, for each inert morphism there is a canonical choice of pullback functor which is an isomorphism of categories, these canonical pullbacks are preserved by composition, and the isomorphisms between them are identity natural transformations.

As a consequence, if two diagrams in \( E \) differ by an inert transformation, then along a collection of canonical isomorphisms of categories, each one is assigned to the same functors and natural isomorphisms.

**iii. (Locality)** For any diagram \( F : I \to E \) and any cutting \( \Psi(F; T) \) of this diagram, both \( F \) and \( \Psi(F; T) \) are sent to the same pullback functors. Furthermore, a disjoint union of diagrams in \( E \) is sent to the product of the corresponding categories, functors, and natural isomorphisms.

**iv. (Ten operations)** In Figure 6.10, the morphism in \( E \) that darkens the vertex marked in red induces the listed functor. In Figure 6.11, the covering morphism pictured gives the desired functor. In all cases, the black vertices may have any number of edges in addition to the edge that meets the red vertex (so the case pictured is one of many possible cases). The four covering maps could alternatively be described by the more general statement that any covering morphism induces external products \( \boxtimes \) followed by pullback along the edge identification map.

**v. (Four isomorphisms)** The isomorphisms of functors assigned to the commuting squares in \( E \) illustrated by Figure 6.12 recover the associator, unitor, shadow, and base change isomorphisms defined in Figures 5.6 to 5.9.

![Diagram](image_url)

**Figure 6.10.** Darkening maps of graphs inducing six of the ten operations
(A) $\emptyset \overset{I}{\rightarrow}$
(B) $\emptyset \overset{\Box}{\rightarrow}$
(C) Pullback along $A \times B \times \overset{\sim}{\longrightarrow} A \times B$
(D) Pullback along $D \times C \overset{\sim}{\longrightarrow} A \times B \times C$

Figure 6.11. Covering maps of graphs inducing four of the ten operations

(A) left unit isomorphism
(B) right unit isomorphism
(C) associator
(D) shadow isomorphism
(E) base change composition isomorphism

Figure 6.12. Graphs for bicategories
Part 3. Proof of the calculus

In this part we construct the fibration \( C_g \) over \( \mathcal{E} \). This challenge is less about defining the fiber categories and pullback functors, and more about verifying the required compatibilities. We do this by reducing the problem to a larger collection of coherences that are more straightforward to check. We check an important collection of simpler coherences in Section 8 and give tools to assemble them into larger coherences in Section 9. We apply this to the case of interest in Section 11.

7. Equivalent formulations of Grothendieck fibrations

The first step of the proof is to explain how to pass back and forth between fibrations and indexed categories. This is necessary because we will build the fibration \( C_g \) using the combinatorial structure of an indexed category.

This section is an elaboration of [Shu08, 3.8], drawing out the parts that are most relevant for the comparisons we will use later.

7.1. Indexed categories. For a category \( S \), an \( S \)-indexed category \( C^- \) is a pseudofunctor \( S^{op} \to \text{Cat} \). In more detail, it assigns

- a category \( C^A \) to each object \( A \in S \),
- a pullback functor \( f^* : C^B \to C^A \) to each morphism \( A \xleftarrow{f} B \) in \( S \),
- a natural composition isomorphism
  \[ f^* \circ g^* \cong (g \circ f)^* \]
  of functors \( C^C \to C^A \) to each pair of morphisms \( A \xleftarrow{f} B \xrightarrow{g} C \), and
- a natural unit isomorphism \( \text{id}_{C^A} \cong (\text{id}_A)^* \) of functors \( C^A \to C^A \) to each object \( A \in S \).

These natural isomorphisms satisfy the same coherence conditions as those for a monoidal functor, the role of tensoring being played by composition of morphisms in \( S \) and composition of functors in \( C \).

A Grothendieck fibration \( C \) over \( S \) defines an \( S \)-indexed category using the fiber categories \( C^A \) and, for each morphism \( f \) in \( S \), any choice of pullback functor \( f^* \) as in Remark 5.2. Conversely, for an \( S \)-indexed category \( C^- \), the Grothendieck construction produces a Grothendieck fibration \( C \) whose objects are pairs \( (A, M \in \text{ob} C^A) \) and whose maps are pairs \( (f : A \to B, M \to f^* N) \) of a map in \( S \) and a map in \( C^A \). This defines equivalence of categories between \( S \)-indexed categories and Grothendieck fibrations over \( S \) (Proposition 7.8).

Along this correspondence, symmetric monoidal fibrations \( C \) correspond to \( S \)-indexed symmetric monoidal categories. These are indexed categories \( C^- \) in which the fibers \( C^A \) are symmetric monoidal, the pullbacks \( f^* \) are strong symmetric monoidal, and the composition and unit isomorphisms are monoidal as well. Note the product on \( C^A \) is the “internal” tensor product \( M \otimes N := \Delta_A^*(M \boxtimes N) \), not the “external” tensor product \( \boxtimes \).

In this case, \( C \) is an smbf if and only if its indexed category \( C^- \) has the following additional properties. This makes \( C^- \) into an \textit{indexed symmetric monoidal category with coproducts}.

i. Each pullback functor \( f^* \) has a left adjoint \( f_! \).

ii. for any \( f : A \to B \) in \( S \), and any \( M \in C^B \), \( N \in C^A \), the canonical map

\[ f_!(f^* M \otimes N) \to f_!(f^* M \otimes f^* f_! N) \cong f_! f^*(M \otimes f_! N) \to M \otimes f_! N \]

is an isomorphism (the projection formula), and

iii. the same Beck-Chevalley condition as before.
7.2. **Thick indexed categories.** If we return to the equivalence between fibrations and indexed categories above, the translation from a fibration to an indexed category requires a choice of pullback functor for each map in $S$. In this section we describe a factorization of this equivalence where we first pass from fibrations to a less restrictive version of an indexed category in which each map $f$ is assigned to several different, canonically isomorphic, pullback functors $f^*$. This is illustrated in Figure 7.1.

![Figure 7.1. Translations between equivalent formulations of Grothendieck fibrations](image)

A **thick object** in a category $C$ (also known as a clique or a contractible category) is a cofree category $\mathcal{A}$ (i.e. every morphism set is a singleton, so any two objects are connected by a unique isomorphism) and a functor $i: \mathcal{A} \rightarrow C$. We call the unique isomorphism between a pair of representatives in $\mathcal{A}$ the **canonical isomorphism**. (We will abuse notation and suppress the functor.)

A **thick map** $\mathcal{A} \rightarrow \mathcal{B}$ of thick objects is a collection of maps $i(a) \rightarrow i(b)$ in $C$ for every pair of objects $(a, b) \in \mathcal{A} \times \mathcal{B}$, commuting with the canonical isomorphisms in $\mathcal{A}$ and $\mathcal{B}$. Thick maps can be composed, and their compositions are equal if and only if they are equal on a single representative.

**Remark 7.2.** We could choose to only define the maps $i(a) \rightarrow i(b)$ for some nonempty collection of pairs $\mathcal{I} \subseteq \mathcal{A} \times \mathcal{B}$, commuting only with canonical isomorphisms between pairs in $\mathcal{I}$. This extends uniquely to a thick map $\mathcal{A} \rightarrow \mathcal{B}$. In particular, if we define the map on a single pair $(a, b)$, it extends in a unique way to all pairs.

Now we apply these definitions to a category of functors:

- A thick object of $\text{Fun}(C, D)$ is called a **thick functor**\(^1\), and a thick map between such is called a **thick natural transformation**.
- Thick functors $\mathcal{A} \rightarrow \text{Fun}(C, D)$ and $\mathcal{B} \rightarrow \text{Fun}(D, E)$ can be composed by taking their product as functors and composing with the composition functor:

$$\mathcal{A} \times \mathcal{B} \rightarrow \text{Fun}(C, D) \times \text{Fun}(D, E) \rightarrow \text{Fun}(C, E)$$

We will use the notation $\mathcal{F} \circ \mathcal{G}$ for this composition.

The following is a straightforward check of the definitions, as soon as we fix some strictification for the product in $\text{Cat}$.

**Lemma 7.3.** The above conventions define a 2-category $\mathbf{Cat}_{\text{thick}}$ of categories, thick functors and thick natural transformations. More generally any 2-category can be replaced by an equivalent one with the same 0-cells, the thick 1-cells, and the thick maps between such.

\(^1\)These should not be confused with anafunctors, a different way of thickening the notion of a functor.
Definition 7.4. A thick $S$-indexed category is a pseudofunctor $S^{\text{op}} \to \tilde{\text{Cat}}$. In more detail, it assigns

- a category $\mathcal{C}^A$ to each object $A \in S$,
- a thick functor $\mathcal{F}_f: \mathcal{C}^B \to \mathcal{C}^A$ to each morphism $A \xleftarrow{f} B$ in $S$,
- a thick natural isomorphism $\mathcal{F}_f \circ \mathcal{F}_g \cong \mathcal{F}_{g \circ f}$ of thick functors $\mathcal{C}^C \to \mathcal{C}^A$ to each pair of morphisms $A \xleftarrow{f} B \xrightarrow{g} C$, and
- a thick natural isomorphism $\ast \cdot \text{id}_{\mathcal{C}^A} \cong \mathcal{F}_{\text{id}_{\mathcal{A}}}$, i.e. a coherent isomorphism between the functors in $\mathcal{F}_{\text{id}_{\mathcal{A}}}$ and the identity functor of $\mathcal{C}^A$.

These natural isomorphisms have the same coherence conditions as before, which can be checked on any one representative $f^*$ for each thick functor $\mathcal{F}_f$.

Example 7.5. Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. An unbiased tensor product (cf. [Lei04, App. A]) is

- a functor $\bigotimes_T : \prod_T C \to C$,
- coherent isomorphisms filling the diagram

$$
\begin{array}{c}
\prod_U \mathcal{C} \\
\downarrow \otimes_U \\
\bigotimes_T \\
\end{array}
\xrightarrow{\Pi_U \otimes_{p^{-1}(U)} \Pi_T \mathcal{C}}
\begin{array}{c}
\prod_T \mathcal{C} \\
\end{array}
$$

for each map $p: U \to T$ of finite sets.

We also require $\bigotimes_T$ is $\otimes$ on the two point set.

This is much of the data required to define a thick $\text{fin}$-indexed category, but we haven’t verified the coherence conditions. Rather than doing this directly in this example, in the next section we develop useful tools for verifying these conditions and use them to complete this example.

7.3. Constructing indexed categories. As we observed in Remark 7.2, we can define a thick map $\mathcal{A} \to \mathcal{B}$ by defining a map on some of the pairs $(a, b)$ and then extending using the canonical isomorphisms. In particular, a thick map $\mathcal{A}' \to \mathcal{B}$ and a strictly commuting diagram

$$
\mathcal{A} \quad \longrightarrow \quad \mathcal{A}'' \quad \leftarrow \quad \mathcal{A}'
$$

define a thick map $\mathcal{A} \to \mathcal{B}$. We think of the functors $\mathcal{A} \to \mathcal{A}''$ and $\mathcal{A}' \to \mathcal{A}''$ as enlarging the categories $\mathcal{A}$ and $\mathcal{A}'$ to give more convenient choices of objects. We formalize this by defining an expansion of thick objects be a functor $\mathcal{A} \to \mathcal{B}$ so that

$$
\begin{array}{c}
\mathcal{A} \\
\downarrow \iota_{\mathcal{A}} \\
\mathcal{C}
\end{array}
\xrightarrow{\iota_{\mathcal{B}}}
\begin{array}{c}
\mathcal{B} \\
\end{array}
$$

strictly commutes. Every expansion gives a thick isomorphism $\mathcal{A} \to \mathcal{B}$. 
Lemma 7.6. The data of

- a category $S$,
- a category $C^A$ for each object $A$ of $S$,
- a thick functor $\mathcal{F}_f$ for each morphism $f$ in $S$,
- an expansion $\ast \to \mathcal{F}_{\text{id}_A}$ for each $A$ in $S$, i.e. a functor so that
  \[
  \begin{array}{ccc}
  \ast & \to & \mathcal{F}_{\text{id}_A} \\
  \downarrow & & \downarrow \\
  \text{Fun}(C^A, C^A)
  \end{array}
  \]
strictly commutes, and
- a composition expansion $\mathcal{F}_f \circ \mathcal{F}_g \to \mathcal{F}_{g \circ f}$, i.e. a functor such that
  \[
  \begin{array}{ccc}
  \mathcal{F}_f \times \mathcal{F}_g & \to & \mathcal{F}_{g \circ f} \\
  \downarrow & & \downarrow \\
  \text{Fun}(C^B, C^A) \times \text{Fun}(C^C, C^B) & \to & \text{Fun}(C^C, C^A)
  \end{array}
  \]
strictly commutes

so that the squares of functors

\[
\begin{array}{ccc}
\mathcal{F}_f \times \mathcal{F}_g \times \mathcal{F}_h & \to & \mathcal{F}_f \times \mathcal{F}_{h \circ g} \\
\downarrow & & \downarrow \\
\mathcal{F}_{g \circ f \circ h} & \to & \mathcal{F}_{h \circ g \circ f}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{F}_f \times \ast & \to & \mathcal{F}_f \times \mathcal{F}_{\text{id}_A} \\
\downarrow & & \downarrow \\
\mathcal{F}_{\text{id}_A \circ f} & \to & \mathcal{F}_{f \circ \text{id}_A}
\end{array}
\]

\[
\begin{array}{ccc}
\ast \times \mathcal{F}_f & \to & \mathcal{F}_{\text{id}_B \times f} \\
\downarrow & & \downarrow \\
\mathcal{F}_{\text{id} \times f} & \to & \mathcal{F}_{f \circ \text{id}_B}
\end{array}
\]

commute strictly, defines a thick $S$-indexed category.

Informally, to define a thick indexed category it is enough to give

- a contractible choice of functors that contains the identity $C^A \to C^A$ for each $A$ in $S$ and
- a contractible choice of pullback functors $f^*$ for each $f$ in $S$, containing the functors $f_1^* \ldots f_n^*$ for every factorization $f = f_n \circ \ldots \circ f_1$.

Example 7.7. Returning to Example 7.5, let $\text{Tree}$ be the set of planar directed trees such that

- each tree has a single outgoing leaf and
- each (internal) vertex has two incoming edges and one outgoing edge.

For a finite set $S$ let $\text{Tree}(S)$ be pairs $(\tau, \sigma)$ where $\tau \in \text{Tree}$ and $\sigma$ is an injective function from $S$ to the incoming leaves of $\tau$. $(\sigma$ does not have to be surjective.) For a morphism $p : S \to T$, let $\text{Tree}(p)$ be the sets

\[
((\tau_t, p^{-1}(t)))_{t \in T}
\]

where we think of the single outgoing leaf of $\tau_t$ as labeled by $t$.

To specify the thick functor associated to a morphism $p : S \to T$ of finite sets we define a functor on each tree in $\text{Tree}(p)$. A labeled directed tree $((\tau_t, p^{-1}(t)))_{t \in T}$ in $\text{Tree}(p)$ defines a functor where we apply $\otimes$ at each vertex of the tree and a unit $I$ at each unlabeled incoming edge.

If there is an isomorphism of planar directed graphs preserving the labeling by $S$ then the trees define the same functor. In particular, when forming $(A \otimes B) \otimes (C \otimes D)$, we don’t record the difference between taking $(A \otimes B)$ first or $(C \otimes D)$ first.

We can get from any one tree to another using associativity, unit, and symmetry isomorphisms, and the coherence theorem for symmetric monoidal categories guarantees
this gives a unique isomorphism between any two functors assigned to \( p : U \to T \). Past-
ing trees together defines the expansions in Lemma 7.6 so we have a thick \( \mathcal{Fin}^{op} \)-indexed category.

This approach to unbiased products should not be confused with the one in [HHM16, §6.3], even though they both use trees. Rather than starting with all of the unbiased products and using trees with vertices of any degree to encode the relations between them, we are starting only with a binary product and unit, and using binary trees to build the remaining unbiased products out of the binary product.

**Proposition 7.8.** There are equivalences of categories between Grothendieck fibrations over \( S \), thick indexed categories over \( S \), and indexed categories over \( S \).

**Proof.** The Grothendieck construction defines a fibration from an indexed category. Given a fibration we define an indexed category using the fiber categories and a choice of pullback. Alternatively, we could associate to \( f \) the category of all pullback functors \( f^* : \mathcal{C}^B \to \mathcal{C}^A \) from Remark 5.2. The remaining data for Lemma 7.6 is filled out by the fact that identity arrows are always cartesian, and compositions of two cartesian arrows are cartesian.

Given a thick \( S \)-indexed category \( \mathcal{C}^- \), we can define a thin one by selecting one object \( f^* \) from each category \( \mathcal{F}_f \). The composition and unit isomorphisms for these thin functors, and their coherence, descends directly from the analogous statements for the thick functors they came from.

Finally, a map of fibrations over \( S \) is a functor \( \mathcal{C} \to \mathcal{D} \) over the identity of \( S \) that pre-
serves cartesian arrows, and a map of indexed or thick indexed categories is a pseudonatural transformation of pseudofunctors \( S \to \text{Cat} \), respectively \( S \to \tilde{\text{Cat}} \). The fact these con-
structions give an equivalence between fibrations and indexed categories is well-known, e.g. [Shu08, 3.8]. By construction the passage from thick indexed to indexed is also full, faithful, and essentially surjective, so we have an equivalence. \( \square \)

We recall a standard corollary for future reference.

**Corollary 7.9** (Coherence theorem for indexed categories). In an \( S \)-indexed category \( \mathcal{C}^- \) for any two factorizations of the same morphism

\[
\phi = f_n \circ \ldots \circ f_1 = g_m \circ \ldots \circ g_1.
\]

any two isomorphisms \( f_1^* \ldots f_n^* \cong g_1^* \ldots g_m^* \) obtained by composing the composition and unit isomorphisms, are equal.

**8. Coherence in Symmetric Monoidal Bifibrations**

In this section we prove some fundamental or atomic compatibilities between \( \boxtimes, f^* \), and \( f_! \) in a symmetric monoidal bifibration \( \mathcal{C} \). These are the building blocks for the more complicated coherences we will need in later sections.

As an example of the atomic coherences, consider the commuting cube in Figure 8.1. This cube defines the four diagrams in Figure 8.2.
Each face in Figure 8.2 is filled by either by

- a composition isomorphism for pullbacks or pushforwards (shorthand ** and !!)
- or
- a Beck-Chevalley transformation (shorthand *!).

Each of these cubes defines a diagram of isomorphisms between the functors in that cube. For example the cube labeled by (** *) becomes the following diagram.

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 f^* g^* \delta^* \\
 \downarrow \\
 f^* \beta^* n^*
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 h^* k^* \delta^* \\
 \downarrow \\
 h^* \gamma^* q^*
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 \alpha^* m^* n^* \\
 \downarrow \\
 \alpha^* p^* q^*
\end{array}
\end{array}
\end{array}
\]

The remaining cubes are similar.

The next four atomic coherences give compatibilities between the above isomorphisms, and the unit and counit of the \((f, f^*)\). They are encoded by the triangular prisms in Figure 8.3. The square faces are filled as above, and triangular faces are filled by either
- a unit map for an adjunction \((f!, f^*)\) (shorthand \(u_f\)), or
- a counit map for an adjunction \((f!, f^*)\) (shorthand \(c_f\)).

![Figure 8.3](image)

**Figure 8.3.** Coherences \((u!\), \((u^*\), \((c!\), and \((c^*\)

The triangular prism labeled \((u!)\) denotes the diagram of isomorphisms

\[
\begin{array}{c}
  h! \\
  \downarrow \\
  k^*k!h!
\end{array}
\rightarrow
\begin{array}{c}
  h^*f^*f! \\
  \downarrow \\
  k^*k^!g^*f!
\end{array}
\]

(8.4)

The remaining triangular prisms are similar.

These eight coherences are recorded in abbreviated form in Table 8.5. In Section 8.1 we give a quick proof of them, before proceeding onwards and extending the table to include compatibility with \(\boxtimes\).

| **\(\boxtimes\)** | **!!\)** | **!*\)** | **u** | **c** |
|-------------------|---------|--------|------|------|
| **!*\)** | **!*\)** | **!*\)** | **u** | **c** |
| **!*\)** | **!*\)** | **!*\)** | **u** | **c** |

**Table 8.5.** The first eight atomic coherences

**Remark 8.6.** These coherences are well-known, and are usually phrased by saying that the bicategory of correspondences \(\text{Corr}(S)\) acts on the fibers of \(\Phi: \mathcal{C} \rightarrow S\) (e.g. [Bar17, Hau18]). For our goals, it turns out to be a little difficult to work directly with \(\text{Corr}(S)\), because we will have a large commuting diagram of spaces and will be traversing that diagram by zig-zags of pushforward and pullback maps.

**8.1. The first eight coherences.**

**Proposition 8.7.** The diagram of natural isomorphisms associated to each polyhedron in Figures 8.2 and 8.3, and recorded succinctly in Table 8.5, commutes.
Before proving Proposition 8.7 we recall two standard lemmas about Beck-Chevalley maps.

Lemma 8.8 (Pasting lemma). For any diagram of two pasted squares

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
C & \xrightarrow{k} & D \\
\downarrow{l} & & \downarrow{m} \\
E & \xrightarrow{n} & F
\end{array}
\]

the Beck-Chevalley map

\[f_1(l \circ h)^* \to (m \circ g)^* n_1\]

is the composite of coherence isomorphisms and Beck-Chevalley maps for the smaller squares

\[f_1(l \circ h)^* \equiv f_1 h^* l^* \to g^* k_1 l^* \to g^* m^* n_1 \equiv (m \circ g)^* n_1.\]

The same statement holds with the roles of the maps reversed.

Lemma 8.9 (Rearrangement lemma). For a single pullback square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
C & \xrightarrow{k} & D
\end{array}
\]

the Beck-Chevalley map

\[f_1 h^* \to f_1 h^* k^* k_1 \sim f_1 f^* g^* k_1 \to g^* k_1\]

is also given by

\[f_1 h^* \to g^* g_1 h^* \sim g^* k_1 h^* \to g^* k_1.\]

The proof of the pasting lemma uses the assumption that our isomorphisms between compositions of pullbacks are coherent, and the rearrangement lemma uses further that the adjunctions \((f_1, f^*)\) respect composition along the composition isomorphisms.

Proof of Proposition 8.7. (***) and (!!!)) These follow from coherence for an indexed category, Corollary 7.9.

(**!) and (!!!)) Cut the cube diagonally across the face with two *s (or two !s) to get two triangular prisms. Each of these is coherent by Lemma 8.8.

((u!), (u*), (c!), and (c*)) We prove \(u!\), the others are similar. Using Lemma 8.9, (8.4) expands to the following commutative diagram.

\[
\begin{array}{ccc}
h_1 & \xrightarrow{u_k h_1} & k^* k_1 h_1 \\
h_1 f^* f_1 & \xrightarrow{u_k h_1 f^* f_1} & k^* k_1 f^* f_1 \\
k^* g_1 f_1 & \xrightarrow{k^* g_1 f^* f_1} & k^* g_1 f^* f_1
\end{array}
\]

The two squares commute by whiskering and the triangle commutes by the triangle identity for \((f_1, f^*)\).
Adding external products. In this section we add another row to Table 8.5 giving us the table in Table 8.10.

The five new entries are associated to the diagrams of functors in Figures 8.12 and 8.13, coming from a \( U \)-tuple of commuting squares of spaces as pictured in Figure 8.11.

\[
\begin{array}{cccccc}
& \ast & \star & \ast & u & c \\
\ast & \star & \ast & \star & u & c \\
! & \star & \ast & \star & u & c \\
\otimes & \star & \ast & \star & u & c \\
\end{array}
\]

Table 8.10. The first thirteen atomic coherences

8.2. Adding external products. In this section we add another row to Table 8.5 giving us the table in Table 8.10.

The five new entries are associated to the diagrams of functors in Figures 8.12 and 8.13, coming from a \( U \)-tuple of commuting squares of spaces as pictured in Figure 8.11.

\[
\begin{array}{ccc}
& \mathcal{B}_u & D_u \\
B_u & g_u & D_u \\
A_u & h_u & C_u \\
\end{array}
\]

Figure 8.11. A product of commuting squares

\[
\begin{array}{c}
\prod_u \mathcal{B}_u \xrightarrow{\Pi(f_u)} \prod_u \mathcal{A}_u \xrightarrow{\text{id}} \prod_u \mathcal{B}_u \\
\prod_u \mathcal{A}_u \xrightarrow{\Pi(f_u)} \prod_u \mathcal{A}_u \xrightarrow{\text{id}} \prod_u \mathcal{A}_u \\
\prod_{u \in U} \mathcal{B}_u \xrightarrow{(\prod f_u)^*} \prod_{u \in U} \mathcal{B}_u \\
\prod_{u \in U} \mathcal{A}_u \xrightarrow{(\prod f_u)^*} \prod_{u \in U} \mathcal{A}_u \\
\end{array}
\]

(A) (u\(\otimes\))

\[
\begin{array}{c}
\prod_{u \in U} \mathcal{B}_u \xrightarrow{\text{id}} \prod_{u \in U} \mathcal{B}_u \\
\prod_{u \in U} \mathcal{A}_u \xrightarrow{\text{id}} \prod_{u \in U} \mathcal{A}_u \\
\prod_{u \in U} \mathcal{B}_u \xrightarrow{(\prod f_u)^*} \prod_{u \in U} \mathcal{B}_u \\
\prod_{u \in U} \mathcal{A}_u \xrightarrow{(\prod f_u)^*} \prod_{u \in U} \mathcal{A}_u \\
\end{array}
\]

(B) (c\(\otimes\))

Figure 8.12. Coherences (u\(\otimes\)) and (c\(\otimes\))

The square and triangular faces can be filled as above, but the square faces can also be filled by

- a canonical map commuting \(\otimes\) with \(f^*\) or \(f!\) (shorthand \(*\otimes\) and \(!\otimes\)).

In Figures 8.12 and 8.13 the set \( U \) always has either two or zero elements in it, in which case the functor labeled \(\otimes\) is either \(\otimes\) or \(I\), respectively. Of course, by composing several of these coherences together, we get a similar coherence for an arbitrary finite set \( U \).

**Proposition 8.14.** The diagram of natural isomorphisms associated to each polyhedron in Figures 8.12 and 8.13, and recorded succinctly in Table 8.10, commutes.

The following lemma helps clarify the statement of both Propositions 8.7 and 8.14, and we will use it for the proof of Proposition 8.14. Its proof is an easy application of universal properties of co/cartesian arrows.

**Lemma 8.15.** Each of the natural isomorphisms occurring in Table 8.10 is a thick natural isomorphism of thick functors.

As a result, Figures 8.2, 8.3, 8.12 and 8.13 can be interpreted as diagrams of thick isomorphisms, or as diagrams of thin isomorphisms by selecting one particular model.
for each pullback and pushforward. It does not matter for the proof – the diagram will commute in one sense if and only if it commutes in the other.

Proof. If \( U \) is empty, the coherences become trivial by picking the pullback or pushforward functors \( C^* \to C^* \) to be identity functors. So we may focus on the case where \( U \) has two points.

((u\,\otimes\,) and (c\,\otimes\,)) We prove (u\,\otimes\,); the proof of (c\,\otimes\,) is essentially dual. We refer to the two \( A_u \) spaces as \( A \) and \( A' \), and similarly for \( B \). Take any \((X, Y) \in C^A \times C^{A'} \), and tensor together the diagrams of cartesian and cocartesian arrows in Figure 8.16a to get the larger diagram in Figure 8.16b. A careful reasoning through the universal properties here shows that the parallelogram on the left commutes, and that this is the desired commuting diagram.

((\ast \otimes \,) and (\!,\otimes \,)) For (\ast \otimes \,), it suffices to cut the ** square along a diagonal to make two triangular prisms and argue that the interchanges \( \ast \otimes \) commute with composition isomorphisms. We check that all five functors take each pair \((X, Y) \) to a cartesian arrow over \( X \otimes Y \) over the same map in \( S \), and therefore the isomorphisms between then are coherent by the universal property. The proof of (\!,\otimes \,) is identical except that everything is a cocartesian arrow under \( X \otimes Y \).

((\ast \,\otimes\,)) Expand the Beck-Chevalley maps into the far-left and far-right columns of the diagram below. Subdivide as indicated. The hexagon in the center commutes by the (\ast \otimes \,) coherence. The trapezoid at the far left commutes by (c\,\otimes\,) and the one at the far right by (u\,\otimes\,). Of the remaining six regions, three commute by naturality of the isomorphism (\!,\otimes \,) and three commute by naturality of (\ast \,\otimes\,).
8.3. **Thick products.** There is a conceptual issue we have to fix before we can handle coherences with two external products. For an smbf $C$ and objects $A_i \in \text{ob} \mathcal{S}$, we might expect the external tensor product $\boxtimes$ to give a functor

$$\boxtimes_T: \prod_i C^{A_i} \to C^{(\prod_i A_i)}$$

that is well-defined up to canonical isomorphism. However, on closer inspection, that statement can’t be correct. The space $\prod_i A_i$ is only well-defined up to isomorphism, giving different values for the category $C^{(\prod_i A_i)}$, and the different models for the product $\boxtimes_T$.
land in these different categories. We will use the notion of thick object from Section 7.2 to handle this issue.

If \( T \) is a finite set, the **thick product** of a \( T \)-tuple \( \langle A_t \rangle_{t \in T} \) of objects of \( S \), denoted \( \prod_{t \in T} A_t \), is the thick object in \( S \) with one object for every model of the product of the \( A_t \) in \( S \). In other words, objects are pairs \((B, \langle B \to A_t \rangle_{t \in T})\) so that \( B \cong \prod_{t \in T} A_t \) along these maps. The canonical isomorphisms between different pairs are the ones arising by the universal property of the product. As a special case, \( \prod_{\emptyset} \) is the subcategory of \( S \) consisting of all terminal objects.

Let \( S_{\prod} \) be the category whose objects are all thick products and morphisms are thick maps between them. If \( S_{\prod} \) is the category of tuples in \( S \) and choice of product of the tuple, there is a zig-zag of functors

\[
S_{\prod} \rightleftarrows S_{\prod} \rightleftarrows S
\]

The functor \( S \to S_{\prod} \) regards an object as a product of a single object. The isomorphisms between different choices of products show \( S \to S_{\prod} \) is an equivalence. The functor \( S_{\prod} \to S_{\prod} \) is a choice of representative for a thick object. The canonical isomorphisms in the thick object provide the structure needed to show this functor is an equivalence.

**Lemma 8.17.** The functors \( S_{\prod} \rightleftarrows S_{\prod} \rightleftarrows S \) are equivalences of categories.

**Proof.** The functor \( S \to S_{\prod} \) regards an object as a product of a single object. The isomorphisms between different choices of products show \( S \to S_{\prod} \) is an equivalence. The functor \( S_{\prod} \to S_{\prod} \) is a choice of representative for a thick object. The canonical isomorphisms in the thick object provide the structure needed to show this functor is an equivalence. \( \square \)

For each tuple \( \langle A_t \rangle \), the **thick fiber** \( C_{\prod, A_t} \) is the pullback of the following diagram of categories.

\[
\begin{array}{ccc}
C & \to & S \\
\downarrow & & \downarrow \\
\prod_{t \in T} A_t & \to & S
\end{array}
\]

Different models for \( \bigotimes_T \) take values in the same thick fiber, addressing the problem described above. The trade off is that we have to generalize the coherences of Proposition 8.7 to use thick fibers. To set this up, we define a smbf \( C_{\prod} \) whose fibers are the thick fibers of \( C \).

**Proposition 8.18.** There is a bifibration \( C_{\prod} \to S_{\prod} \) whose fibers are the thick fibers \( C_{\prod, A_t} \).

**Proof.** The objects of \( C_{\prod} \) are tuples

\[
\langle \langle A_t \rangle_{t \in T}, X, \Phi(X) \cong \prod_{t} A_t \rangle
\]

for \( X \in C \). Forgetting \( \langle A_t \rangle_{t \in T} \) gives a map of sets \( \text{ob}(C_{\prod}) \to \text{ob}C \), and we define the morphisms of \( C_{\prod} \) by pulling back the morphism sets of \( C \) along this map. This defines the category \( C_{\prod} \) and an equivalence of categories \( C_{\prod} \rightleftarrows C \).

We directly check that a morphism in \( C_{\prod} \) is cartesian if and only if it is so in \( C \), hence \( C_{\prod} \to S_{\prod} \) is a fibration. The proof that \( C_{\prod} \) is an opfibration is identical. Because \( S \rightleftarrows S_{\prod} \) is an equivalence of categories, the Beck-Chevalley condition for \( C_{\prod} \) automatically holds on those squares that are isomorphic in \( S_{\prod} \) to a Beck-Chevalley square in \( S \). \( \square \)

**Remark 8.19.** The bifibration \( C_{\prod} \) is the pullback of \( C \) to \( S_{\prod} \), composed with the bifibration \( S_{\prod} \to S_{\prod} \).
Any choice of model for the product $\prod_{s \in S} A_s$ gives an inclusion of a thin fiber into the thick fiber

$$\mathcal{C}[\prod_s A_s] \subset \mathcal{C}[\prod_s A_s]$$

which is an equivalence of categories. More generally, we have the following result, which tells us that $\mathcal{C}$ is “contained inside” $\mathcal{C}[\prod]$. This gives us a direct way to compare the composition and Beck-Chevalley isomorphisms in $\mathcal{C}$ and $\mathcal{C}[\prod]$.

**Proposition 8.20.** For any diagram of thick products $D : I \to S[\prod]$, and any diagram $D' : I \to S$ obtained from it by choosing a representative for each thick product, there is a canonical inclusion $D' \star \mathcal{C} \to D \star \mathcal{C}[\prod \prod]$ of bifibrations over $I$.

**Proof.** Consider the full subcategory of $D \star \mathcal{C}[\prod \prod]$ consisting of those pairs $(i, Y)$ for which $\Phi(Y) = D'(i)$. There is a bijection between these and the objects of $D' \star \mathcal{C}$. By the construction of $\mathcal{C}[\prod \prod]$, along this bijection the morphism sets are identical, i.e. they correspond to the same pairs of morphisms in $I$ and $\mathcal{C}$. $\square$

**Proposition 8.21.** $\mathcal{C}[\prod] \supset S[\prod]$ is a symmetric monoidal bifibration. In addition, for each set $T$ and objects $\{A_t\}_{t \in T}$ of $S$, the unbiased tensor products of Example 7.5 define a thick $(T \downarrow \text{Fin})^{op}$-indexed category where the category associated to $T \to U$ is the product

$$\prod_{u \in U} \mathcal{C}[\prod_{t \to u} A_t].$$

**Proof.** The category $S[\prod]$ is cartesian monoidal. The unit is the empty tuple and the product concatenates tuples. For each pair of tuples there is a tensor product map $\boxtimes : \mathcal{C}[\prod_{t \in T} A_t] \times \mathcal{C}[\prod_{t \in T'} A_t] \to \mathcal{C}[\prod_{t \in T \cup T'} A_t]$ that applies $\boxtimes$ to the $\mathcal{C}$ coordinate, $\times$ to the $S$ coordinate, and takes the pair of products $B \equiv \prod_{t \in T} A_t$, $B' \equiv \prod_{t \in T'} A_t$ inside the thick product to $B \times B' \equiv \prod_{t \in T \cup T'} A_t$. There is also a unit map

$$I : \star \to \mathcal{C}[\prod]$$

that picks out $I$, $\star$, and $\star$.

The associator, unitor, and symmetry maps are defined by applying the relevant map in the $\mathcal{C}$ coordinate and observing that its image in $S$ lies underneath the only possible isomorphism we can pick in the $\prod_{t \in T} A_t$-coordinate (recall this latter category is contractible). This observation amounts to the fact that the map of products we get in $S$ respects the projection maps to the factors. The fact that these isomorphisms are coherent reduces to the same statement on the $\mathcal{C}$ coordinate.

Hence $\mathcal{C}[\prod]$ is a symmetric monoidal category and the projection to $S[\prod]$ is strict symmetric monoidal by construction. Since an arrow in $\mathcal{C}[\prod]$ is (co)cartesian if and only if its image in $\mathcal{C}$ is, it follows that $\boxtimes$ preserves (co)cartesian arrows in $\mathcal{C}[\prod]$. Hence it is an smbf.

Now we may build the desired indexed category. We repeat the above operation $\boxtimes$ but only for disjoint subsets of $T$. For each map $T \to U \xrightarrow{L} U'$, take the collection of all planar trees with inputs labeled by $U$, outputs labeled by $U'$, and all the conditions from Example 7.5. To each such tree we assign the functor

$$\prod_{u \in U} \mathcal{C}[\prod_{t \to u} A_t] \to \prod_{u' \in U'} \mathcal{C}[\prod_{t \to u} A_t]$$
that associates to each vertex \( \otimes \), and \( I \) to each incoming leaf to \( I \). Any two trees are connected by finitely many associator, unitor, and symmetry maps, giving the canonical isomorphisms between the functors for different trees. We define the necessary expansions by gluing trees together, as before, and the verification that the expansions commute proceeds in the same way as it did in Example 7.5.

### 8.4 Thick atomic coherences

Now we can prove more general versions of the thirteen coherences from Propositions 8.7 and 8.14 that use the thick fibers of \( C \), and add to this list two new coherences that involve composing tensor products together, giving the right-hand column in Table 8.22.

| * | ** | u | c |
| --- | --- | --- | --- |
| ** | *! | c* | *! |
| *! | u* | u* | c* |
| u | c | c | u |

**Table 8.22. The full list of fifteen atomic coherences**

The first eight coherences are just the diagrams in Figures 8.2 and 8.3 for the bifibration \( C \), so the proof in Proposition 8.7 still applies and there is nothing new to say. The next five coherences are depicted in Figures 8.24 and 8.25. The proof of these is essentially by reduction to Proposition 8.14. The final two coherences are depicted in Figure 8.26.

In each of these polyhedra, \( U \to V \to W \) are maps of finite sets, \( P, Q, R, S, \) and \( T \) are finite sets with maps to \( U \), \( \{ A_p \}_{p \in P} \) is an \( P \)-tuple of objects of \( S \) and similarly for \( \{ B_t \}_{t \in T}, \{ C_q \}_{q \in Q}, \{ D_r \}_{r \in R}, \) and for each \( u \in U \) we have a commuting square of maps of products depicted in Figure 8.23a (or a line as in Figure 8.23b).

![Diagram shapes for Proposition 8.27](image)

**Figure 8.23. Diagram shapes for Proposition 8.27**

![Thick coherences \( u \otimes \) and \( c \otimes \)](image)

**Figure 8.24. Thick coherences \( u \otimes \) and \( c \otimes \)**
In these figures, most of the square and triangular regions are filled by the same isomorphisms **, *, !, $u$, and $c$ as before, though in the bifibration $\mathcal{C}^P$. The map $\nabla$ refers to an unbiased tensor product as in Proposition 8.21, and we refer to its composition isomorphism as $\nabla$. The remaining squares are filled by isomorphisms $\nabla$ and $\land$, arising from the fact that the unbiased $\land$ also preserves tuples of (co)cartesian morphisms in $\mathcal{C}^P$. By Proposition 8.20, these new isomorphisms are the same as the isomorphisms above any time we restrict to a thin fiber.

**Proposition 8.27.** Each polyhedron in Figures 8.24 to 8.26 is coherent.

Again, for the proof it is helpful to have the following analog of Lemma 8.15. It is also proven using universal properties and the definition of an indexed category (for $\nabla$).

**Lemma 8.28.** Each of the natural isomorphisms **, *, !, $u$, $c$, $\nabla$, $\land$, and $\nabla$ is a thick natural isomorphism of thick functors.
Corollary 8.29. The coherences between the unit isomorphism \((\text{id}_A)^* \equiv \text{id}_{\mathcal{E}^A}\) and \(*, !\) or \(\otimes\) all hold. The same applies to the unit isomorphism \((\text{id}_A)_! \equiv \text{id}_{\mathcal{E}^A}\) and the unit isomorphism \(\otimes \text{id}_U \equiv \text{id}_{\left(\prod_{u \in U} \mathcal{E}^{(\prod_s A_s)}\right)}\).

Proof. By the previous lemma, these coherences concern thick isomorphisms of thick functors, so we are free to choose any model we wish to for each of the thick functors. In each of these three cases, the unit isomorphism can be modeled by a strict identity map, and with that model the proof of the coherence is trivial. □

Proof of Proposition 8.27. \(((u \otimes), (c \otimes), (\ast \ast \otimes), (!! \otimes), \text{and } (\ast ! \otimes))\) In each of these cases, the diagram of functors and natural isomorphisms is a product of diagrams indexed by \(V\), so we may assume without loss of generality that \(V = \ast\).

Once this is done, each vertical arrow is of the same form as the right-hand vertical arrow in Figure 8.30a. We fix a choice of planar tree as in Proposition 8.21 to model the \(U\)-fold product \(\prod_{u \in U}\). In particular, this model of the product brings us from the products \(\prod_{s \rightarrow u}\) to one model for \(\prod_{s \in S}\), and similarly for \(T, Q,\) and \(R\). This gives the arrow on the left in Figure 8.30a.

Making these replacements for each vertical arrow in the polyhedron, we get a new polyhedron that is coherent by repeated application of Proposition 8.14, one for each product or unit vertex in our planar tree. To prove the original polyhedron is coherent, it now suffices to check that each of its faces gives a coherent polyhedron between the old and new maps. One way to do this is to use Proposition 8.20 to regard the left-hand side of Figure 8.30b as subcategories of the right-hand side. Along this inclusion, the universal properties for the cartesian arrows coincide. Another way to do this is to focus on one object in the source category, and carefully select models for the pullback functors so that all of the natural isomorphisms in the diagram are identity maps on the images of that one object.

\(((\ast \otimes \otimes) \text{ and } (!! \otimes \otimes))\) Start with \((\ast \otimes \otimes)\). As above, we can assume \(W = \ast\), fix a model for each of the three \(\otimes\) operations along the triangle, then restrict attention to the equivalent subcategories illustrated in Figure 8.31. The two routes across the bottom triangle...
land in two different thin fibers inside \( C_{\mathcal{V}_S} \), which we will refer to as

\[
\prod_{u \in \mathcal{U}} \prod_{s \rightarrow u} A_s, \quad \prod_{v \in \mathcal{V}} \prod_{s \rightarrow v} A_s.
\]

Of the five functors we need to compare, two land in the thin fiber over \( \prod_{u \in \mathcal{U}} \prod_{s \rightarrow u} A_s \), and the other three land in the thin fiber over \( \prod_{v \in \mathcal{V}} \prod_{s \rightarrow v} A_s \). The first two take a \( \mathcal{U} \)-tuple

\[
\{X_u \in C_{\mathcal{V}_s}(\prod_{v \in \mathcal{V}} B_t)\}_{u \in \mathcal{U}}
\]

to a cartesian arrow that is a “pullback” of the first diagram below, the other three take the same \( \mathcal{U} \)-tuple to a cartesian arrow that is a “pullback of the second diagram.

\[
\prod_{u \in \mathcal{U}} \prod_{s \rightarrow u} A_s \rightarrow \prod_{u \in \mathcal{U}} \prod_{t \rightarrow u} B_t \quad \prod_{v \in \mathcal{V}} \prod_{s \rightarrow v} A_s \rightarrow \prod_{v \in \mathcal{V}} \prod_{t \rightarrow v} B_t.
\]

There is a canonical isomorphism between these pullback diagrams arising from the symmetric monoidal structure on \( C_{\prod} \). Each of the five isomorphisms in the triangular prism is a canonical isomorphism of pullbacks, either along the identity of one of these two diagrams, or along this canonical isomorphism between them. This guarantees the coherence. The proof of \( (\! \otimes \! \otimes) \) is the same except with “pushout” diagrams, and the roles of \( A_s \) and \( B_t \) switched. \( \Box \)

9. Coherence Theorems for Diagrams of Functors

In this section we describe a method for gluing together smaller coherence results into larger ones. We consider any \( \mathcal{P} \)-indexed category \( \mathcal{Q} : \mathcal{P}^{\text{op}} \rightarrow \mathcal{Cat} \), where the composition and unit isomorphisms are identity transformations. Let \( \mathcal{P} \times \mathcal{Q} \) denote the Grothendieck construction of \( \mathcal{Q} \). Then the following is an intuitive formulation of the main result (Theorem 9.18) of this section.

**Theorem 9.1.** To define a \( \mathcal{P} \times \mathcal{Q} \)-indexed category, it suffices to define an \( \mathcal{Q}_a \)-indexed category and a \( (\mathcal{P} \downarrow a) \)-indexed category for each object \( a \in \mathcal{P} \) so that
• the indexed categories agree on overlaps,
• there are swap relations (9.16) and
• the swap relations and indexed categories are compatible (Definition 9.17).

In particular, if $Q_-$ is a constant indexed category, then $P \times Q$ is just a product category, and this result tells us how to verify a coherence condition for functors and natural isomorphisms on $P \times Q$. In essence, it reduces to a coherence condition along $P$ and $Q$ separately, along some smaller coherence conditions between generators of one category and relations in the other category.

9.1. Coherent diagrams of functors. First we will formalize what it means to check coherence between a collection of functors and natural isomorphisms. The cubes and triangular prisms of Section 8 will serve as inspiration.

Let $S$ be a category presented as a set of objects, generators $G$ for the morphisms, and relations $R$ between the generators. A word $w = f_n \circ \ldots \circ f_1$ is any sequence of composable generators – this is not the same thing as a morphism in $S$, because it has more data.

If each $f_i$ is associated to a pullback functor $f_i^*$, let $w^* := f_n^* \circ \ldots \circ f_1^*$.

If each $f_i$ is assigned to a thick functor $F_{f_i}$, let $\mathcal{F}_w := \mathcal{F}_{f_1} \circ \ldots \circ \mathcal{F}_{f_n}$ using the composition notation from Section 7.1. In other words, $\mathcal{F}_w$ consists of all functors $w^* = f_n^* \circ \ldots \circ f_1^*$ where each $f_i^*$ is a functor in $\mathcal{F}_{f_i}$.

By convention, an empty word is a choice of object $a$ in $S$, and the morphism in $S$ associated to this word is the identity map $id_a$. The associated functor is always an identity functor, and the associated thick functor is always a singleton category picking out the identity functor.

Definition 9.2. A diagram of functors on $(S, G, R)$ is

- a category $\mathcal{C}^a$ for each object $a \in S$,
- a functor $f^*: \mathcal{C}^b \to \mathcal{C}^a$ for each generator $f: a \to b$, and
- a natural isomorphism of functors $w_1^* \equiv w_2^*: \mathcal{C}^b \to \mathcal{C}^a$ for each relation $w_1 = w_2$ in $S$. This isomorphism will be called a relation isomorphism.

A diagram of functors is coherent if any two natural isomorphisms $w_1^* \to w_2^*$ obtained by composing (horizontal compositions of) the relation isomorphisms agree as isomorphisms in $\text{Fun}(\mathcal{C}^b, \mathcal{C}^a)$. We also say that such a diagram of functors satisfies the coherence condition.

The coherence condition amounts to saying that, up to canonical isomorphism, each morphism $\phi: a \to b$ in $S$ is assigned to a well-defined pullback functor. So coherent diagrams of functors are another way of describing thick indexed categories. We formalize this in Proposition 9.4.

Definition 9.3. A thick diagram of functors on $(S, G, R)$ is

- a category $\mathcal{C}^a$ for each object $a \in S$,
- a thick functor $\mathcal{F}_f$ for each generator $f$, and
- a thick natural isomorphism $\mathcal{F}_{w_1} \equiv \mathcal{F}_{w_2}$ for each relation.

A thick diagram of functors is coherent if any one (equivalently, all) choices of one representative $f^* \in \mathcal{F}_f$ for each $f$ results in a coherent diagram of functors.

Proposition 9.4.

- Each thick $S$-indexed category can be combinatorialized into a coherent thick diagram of functors on $S$, using any set of generators and relations for $S$.
• Each coherent thick diagram of functors on $S$ can be de-combinatorialized into a thick $S$-indexed category.
• Performing the first operation and then the second recovers the original thick $S$-indexed category up to isomorphism.

Proof. In both directions, there is no need to change the categories $\mathcal{C}^0$, so we focus on the functors.

Given a thick indexed category $\mathcal{C}^-$ and a presentation of $S$, we combinatorialize it by assigning each generator $f$ to the given contractible category of pullback functors $\mathcal{F}_f$, and each relation $w_1 = w_2$ to the thick isomorphism $\mathcal{F}_{w_1} \simeq \mathcal{F}_{w_2}$ that on any pair of representatives is the canonical isomorphism $w_1^* \equiv w_2^*$ between the composites of pullbacks (using Corollary 7.9). These give thick maps because they are composed of composition and unit isomorphisms of $\mathcal{C}^-$, which are thick maps by definition. To check coherence, we pick one $f^*$ in each $\mathcal{F}_f$ and observe that any two isomorphisms $w_1^* \equiv w_2^*$ composed from the above relation isomorphisms, are themselves composed of composition and unit isomorphisms of $\mathcal{C}^-$. So by Corollary 7.9, any two such isomorphisms are identical.

Given a coherent thick diagram of functors on $\mathcal{C}^-$, we de-combinatorialize it as follows. For each morphism $\phi : a \to b$ in $S$, define $\mathcal{F}_\phi$ to be a contractible category with one object for each word $w = f_n \circ \ldots \circ f_1$ composing to $\phi$, and choice of $w^*$ from $\mathcal{F}_w$ (in other words, choice of $f_i^*$ from each $\mathcal{F}_{f_i}$). For any two words $v, w$ both composing to $\phi$, the coherence condition guarantees there is a unique thick isomorphism $\mathcal{F}_v \simeq \mathcal{F}_w$ obtained from the relation isomorphisms. This gives us coherent isomorphisms between all the resulting functors $w^*$, in other words a functor $\mathcal{F}_\phi \to \text{Fun}(\mathcal{C}^0, \mathcal{C}^0)$.

We use Lemma 7.6 to build the rest of the indexed category. The expansion $* \to \mathcal{F}_{\text{id}_a}$ picks out the empty word and the expansion $\mathcal{F}_\phi \circ \mathcal{F}_\gamma \to \mathcal{F}_{\gamma \circ \phi}$ concatenates the words. This respects the maps into the functor categories – on objects this is immediate and on morphisms this is because, by definition, relation isomorphisms are preserved by horizontal compositions. The three needed squares commute because concatenation of words is strictly associative and unital.

Finally, if we combinatorialize and then de-combinatorialize, for each $\phi$, the category of pullback functors $\phi^*$ is replaced by the category of all expressions $f_1^* \circ \ldots \circ f_n^*$ for factorizations $\phi = f_n \circ \ldots \circ f_1$ into generators. Corollary 7.9 gives a canonical isomorphism between any one of these composite pullback functors and $\phi^*$. This gives the data of an isomorphism of thick indexed categories. To check the coherence condition, we restrict to a thin indexed category inside each one. Then it is enough to check for each pair of composable morphisms $\phi, \gamma$, respectively each object $a$, that the following diagrams commute:

\[
\begin{array}{ccc}
(\gamma \circ \phi)^* & \cong & \phi^* \circ \gamma^* \\
\downarrow & & \downarrow \\
h_1^* \circ \ldots \circ h_k^* & \cong & (f_1^* \circ \ldots \circ f_n^*) \circ (g_1^* \circ \ldots \circ g_m^*)
\end{array}
\]

\[
\begin{array}{ccc}
\text{id}_{\phi^*} & \cong & \text{id}_{\phi^*} \\
\downarrow & & \downarrow \\
\text{id}_{\mathcal{F}_\phi} & \cong & \text{id}_{\mathcal{F}_\phi}
\end{array}
\]

Tracing through the definitions, in each case the bottom map is composed of composition and unit isomorphisms, therefore all the maps in the diagram are, and the diagram commutes by Corollary 7.9.

\[
\square
\]

9.2. Coherence along a cubical grid. Now we can build new indexed categories out of smaller ones, by thinking of both as coherent diagrams of functors. In practice, it is fairly easy to build a new diagram of functors, but not so easy to verify the coherence.
In this section we discuss the solution to this problem in a few simple cases, including when the diagram of functors is a cubical grid.

**Example 9.5.** Suppose $S$ is a category presented with no relations, for instance the category with two generators

\[
\begin{array}{c}
\circlearrowright \\
\circlearrowright \\
\end{array}
\]

Then the coherence condition for diagrams of functors on $S$ is always satisfied.

**Example 9.6.** Suppose $S$ is a planar poset, meaning it has finitely many generators and relations and they are represented as line segments and polygonal regions embedded in the plane:

When $S$ is not planar, we have to work harder. For a product $P \times Q$ it is almost enough to check coherence on $P$ and $Q$ separately. The only additional ingredient is coherence along the cubes formed by relations in $P$ and generators in $Q$, and by generators in $P$ and relations in $Q$:

**Proposition 9.7 (Cubical coherence).** Suppose categories $P$ and $Q$ have generators $G_P, G_Q$ and relations $R_P, R_Q$ respectively. To define a diagram of functors on the product $P \times Q$, it suffices to assign

- a category $\mathcal{C}^{a,b}$ to each pair of objects $(a, b)$ in $P \times Q$,
- a functor $f^\#: \mathcal{C}^{a,b} \to \mathcal{C}^{a,b'}$ to each object $a \in P$ and generator $f : b \to b'$ in $Q$,
- a functor $g^\#: \mathcal{C}^{a',b} \to \mathcal{C}^{a,b}$ to each generator $g : a \to a'$ in $P$ and object $b \in Q$,
- a relation isomorphism to each object $a \in P$ and relation in $Q$,
- a relation isomorphism to each relation in $P$ and object $b \in Q$, and
- a swap isomorphism $g^\# f^\# \cong f^\# g^\#$ of functors $\mathcal{C}^{a',b'} \to \mathcal{C}^{a,b}$ to each generator $g : a \to a'$ in $P$ and generator $f : b \to b'$ in $Q$.

To verify the coherence condition, it suffices to check

- for each object $a \in P$, the given functors and isomorphisms form a coherent diagram on $\{a\} \times Q$,
- for each object $b \in Q$, the given functors and isomorphisms form a coherent diagram on $P \times \{b\}$. 
• For each relation \( w_1 = w_2 \) in \( Q \) and generator \( g \) in \( P \) the square of natural isomorphisms

\[
\begin{array}{ccc}
(w_1)^\# g^\# & \cong & g^\# (w_1)^\# \\
\cong & & \cong \\
(w_2)^\# g^\# & \cong & g^\# (w_2)^\#
\end{array}
\]

commutes, where the horizontal isomorphisms are compositions of swaps \( g^\# f^\# \cong f^\# g^\# \) for each generator \( f \) in each word \( w_1 \) or \( w_2 \), and the vertical isomorphisms are relation isomorphisms, and

• the same for generators \( f \in Q \) and relations \( w_1 = w_2 \) in \( P \).

Note that we are using the notation \( f^\# \) since there are now two different notions of pullback.

**Proof.** This is a special case of Theorem 9.18 below, but for convenience we summarize the proof here.

First consider all functors that traverse a specified word \( g_1 \ldots g_n \) in the generators of \( P \), interspersed with any generators in \( Q \). If \( n = 0 \), the isomorphisms between these are all contained in a slice \( \{a\} \times Q \) and so they are coherent by assumption. For arbitrary \( n \), we argue that any isomorphism between two of them can be rewritten as an isomorphism that swaps \( g_n \) to the end of the word, leaves it fixed during all the other relations, then swaps \( g_n \) back in at the very end – this uses the assumed coherence for relations in \( Q \) and generators in \( P \). By induction on \( n \), any two isomorphisms between two words of this form must therefore be equal.

When we change the given word in \( P \) by a relation, we use the assumed coherence on relations in \( P \) and generators in \( Q \) to argue that we get the same map by first moving all the letters of \( P \) to the front of the word, applying the relation, and then shuffling them all back. The remaining coherence to check is between these relations, but this follows from the assumed coherence along the slices of the form \( P \times \{b\} \).

**Example 9.8.** A useful example of this proposition is given by taking \( P \) to be a square grid and \( Q \) to be a subdivided line segment, so that \( P \times Q \) is a grid of cubes. To show all the compositions of functors along the edges of this grid are isomorphic, one only needs to specify isomorphisms for each small two-dimensional square (i.e. the swap isomorphisms and the squares in \( P \)). Furthermore, to show that these isomorphisms are coherent, one only needs to check coherence on each small three-dimensional cube. (The commuting squares in the proposition become hexagons because each word has two letters, and these hexagons represent the six faces of a cube.)

**Remark 9.9.** The variant of Proposition 9.7 for thick diagrams of functors also holds, with the same proof. More generally, the proof applies with \( \text{Cat} \) replaced by any 2-category (such as \( \tilde{\text{Cat}} \)).

Since this theorem returns the same kind of data that it takes in, the generalization of this result for products of \( n \) different categories follows immediately by induction.

**Corollary 9.10.** Given categories \( P_1, \ldots, P_n \), each with generators and relations, to specify a coherent diagram on \( P_1 \times \ldots \times P_n \) it suffices to give a category for each tuple of objects, coherent diagrams on each “strand” category

\[
\{s_1\} \times \ldots \times \{s_{i-1}\} \times P_i \times \{s_{i+1}\} \times \ldots \{s_n\},
\]
and swap isomorphisms for each pair of generators from different categories $P_i$ and $P_j$. Then one has to verify for each generator from $P_i$ and relation from $P_j$ a coherence between swap and relation isomorphisms, and for each triple of generators from $P_i$, $P_j$, and $P_k$ a coherence between the swap isomorphisms.

9.3. **Application: coherence for bicategories with shadow.** In this section we prove a coherence theorem for bicategories with shadows generalizing the coherence theorem for bicategories [Pow89]. We prove this theorem by first constructing a coherent diagram of functors using Proposition 9.7.

Let $C$ be a finite graph consisting of a single cycle and $n + k$ vertices. Color $n$ of the vertices black and consider the remaining vertices white. Let $C$ be the category whose objects are colorings $c$ of the vertices of $C$ by white or black, such that black vertices of the original graph must be black in the coloring. There is a morphism $c ightarrow d$ if $d$ is darker than $c$. This category is isomorphic to the $k$-dimensional cube $(\bigcirc \rightarrow \bullet)^k$. It has a presentation with one generator for each darkening of a single vertex and one relation for each pair of darkenings of two distinct vertices.

Working clockwise on the graph $C$, label the black vertices with $1, \ldots, n$. Label each of the white vertices between the black vertices labeled by $i$ and $i + 1$ with $i$. For a coloring $c$, let $B(c)$ be the set of strings of consecutive black vertices in the coloring $c$. Then each $b \in B(c)$ determines a pair of integers $(i_b, j_b)$ where $i_b$ is the label of the white vertex before this string of black vertices and $j_b$ is the label of the white vertex after this string of black vertices. Each coloring determines a set of tuples $\{(i_b, j_b) \mid b \in B(c)\}$.

**Lemma 9.11.** Let $\mathcal{B}$ be a bicategory with a shadow valued in $T$. For a cycle graph $C$ with $n$ black vertices and 0-cells $A_1, \ldots, A_n$ there is a coherent diagram of functors on $C$ where the value on a darkening $c$ is

$$\prod_{b \in B(c)} \mathcal{B}(A_{i_b}, A_{j_b})$$

if $c$ has at least one white vertex, and $T$ if all vertices of $c$ are black.

**Proof.** Each generator is assigned a functor based on the colors of the vertices adjacent to the vertex changing color. We notate this using strings of white, gray and black dots as in the following table. The gray vertex is the vertex that is changing from white to black in this darkening.

| darkening | functor       |
|-----------|--------------|
| \(\bullet\bullet\bullet\) | unit         |
| \(\bigcirc\bullet\bullet\) | identity map |
| \(\bullet\bullet\bigcirc\bullet\bullet\) | identity map |
| \(\bullet\bullet\bullet\bullet\bullet\) | composition or shadow |

If we darken a pair of vertices and, after darkening, those vertices are in distinct strings of black vertices, we take the associated relation isomorphism to be the identity. Alternatively we are in one of the cases in the following table. To simplify notation, we represent a continuous string of black vertices by a single black vertex.
In the last case the transformation is the associator if there is a white vertex remaining after darkening these two, and a shadow isomorphism if there is no white vertex remaining.

To check the coherence condition using Corollary 9.10, it is sufficient to prove a coherence for every triple of darkened vertices. If at least one white vertex remains in $C$ after all three vertices are darkened, then the coherence follows from coherence for a bicategory [Pow89]. It remains to consider the case where we are darkening the last three vertices. There are either 1, 2, or 3 strings of black vertices between them. When there are two strings, four faces of the resulting cube are identity transformations and two faces are shadow isomorphisms $\theta$, and so it is trivially coherent. When there is one string, the resulting cube is the shadow unit coherence axiom [Pon10, 4.4.1]. When there are three strings, the resulting cube is the shadow associator coherence axiom [Pon10, 4.4.1]. Therefore this diagram of functors is coherent. □

There are several routes through $C$ that define the same functor, with the identity natural transformations between them. To recognize this, we say that a generator that darkens a vertex between a black and a white vertex is inert. A word in the generators that fills in a nonempty string of contiguous white vertices

\[
\bullet \circ \circ \ldots \circ \circ \circ \rightarrow \bullet \bullet \bullet \ldots \bullet \bullet \bullet,
\]

is a minimal filling if every generator but the last is inert. Similarly any word of the form

\[
\circ \circ \circ \ldots \circ \circ \circ \rightarrow \circ \bullet \bullet \ldots \bullet \bullet \bullet
\]

is a minimal filling if every generator but the first is inert. Any word of the form

\[
\bullet \circ \circ \ldots \circ \circ \circ \rightarrow \bullet \bullet \bullet \ldots \bullet \bullet \bullet
\]

is a minimal filling if every generator is inert.

When the string of darkened vertices has length one, Lemma 9.11 assigns the minimal filling of that string to a multiplication, shadow, unit, or identity functor. It is straightforward to check that:

- the same is true if the string has length greater than one,
- for any two different minimal fillings of the same string, the isomorphism from Lemma 9.11 relating the resulting functors is the identity, and
- for any two different strings, the isomorphism from Lemma 9.11 that swaps a minimal filling of the first string past a minimal filling of the second is the same as the isomorphism one would get if the two strings had length one.

In short, when using minimal fillings to fill long strings of white vertices, one can act as though the string has length one and all minimal fillings are the same.
Continuing to let $\mathcal{B}$ be a bicategory with shadow taking values in $\mathbf{T}$, for any $n$-tuple of 0-cells $A_1, \ldots, A_n$, a **circular product** is any functor

$$\mathcal{B}(A_n, A_1) \times \mathcal{B}(A_1, A_2) \times \cdots \times \mathcal{B}(A_{n-1}, A_n) \to \mathbf{T}$$

obtained by inserting units, multiplying in any order (respecting the cyclic ordering of the terms), and finally taking a shadow.\(^2\)

**Theorem 9.12.** Any two circular products are uniquely isomorphic by a sequence of the isomorphisms $\alpha$, $\ell$, $r$, and $\theta$ (and their inverses). Of course, this isomorphism may be obtained by several different formulas.

**Proof.** At least one such isomorphism exists since we may remove all the units and then use associators and the shadow isomorphism $\theta$ to rearrange into the standard form

$$(M_1, \ldots, M_n) \leadsto \langle \langle \ldots (M_1 \otimes M_2) \otimes M_3 \ldots \otimes M_n \rangle \ldots \rangle.$$  

Therefore it suffices to consider a self-isomorphism of this particular circular product made from the structure isomorphisms of $\mathcal{B}$, and prove it is the identity.

Fix a sequence of structure isomorphisms

$$F_0 \xrightarrow{\phi_1} F_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_N} F_N = F_0$$

where $F_0$ is the model for the circular product given in (9.13). This sequence of isomorphisms determines a (disconnected) planar, stratified graph consisting of $n$ trees. There is a vertex at level $i$ for each 1-cell in the expression $F_i$. Vertices in levels $i$ and $i + 1$ are adjacent if

* the vertices correspond to the same 1-cell or
* $\phi_i$ is a unit isomorphism $U_A \otimes (-) \to (-) \otimes U_A \to (-)$ and the vertices correspond to 1-cells involved in this isomorphism. (If however $\phi_i$ is the inverse of a unit isomorphism, these 1-cells are not adjacent.)

We regard the vertices in level $N$ as the roots of the trees. See Figure 9.14.

\(^2\)Strictly speaking, it is the pattern of units and multiplications that defines a circular product, not the resulting functor. If two different patterns happen by chance to produce the same functor, we regard them as different.
Define a path through the $M_1$ tree starting at the root by taking the left edge at each vertex of degree 3. At each leaf turn around and continue to choose the left edge. The choice of left and right is determined by the embedding and so the embedding defines a total order on the leaves (omitting the roots). Repeat on the trees from $M_2, \ldots, M_n$ (in order) to define a total order on all the leaves.

Consider a vertex $v$ at level $i$ of this graph. The vertex $v$ becomes a root if we restrict the graph to levels 0 to $i$. Let $l(v)$ be the set of leaves that are in the same tree $v$ in this truncated tree.

If there are $m$ total leaves, form a cycle graph $C$ with $m$ white vertices labeled consecutively by the numbers $1, \ldots, m$. Darken the vertices associated to the 1-cells $M_i$. Subdivide each edge with 1 additional white vertex, so that there are $2m$ vertices in total, $n$ of which are black.

We can now show that each of the functors $F_i$ is given by a path of darkenings through this graph, i.e. a word in the generators of $C \cong (\bigcirc \rightarrow \bullet)^{2m-n}$ from the initial object to the terminal object. To build $F_i$, we look at the vertices $v$ that are the roots of the above graph when it is truncated to levels 0 through $i$. For each such $v$, darken (using any minimal filling) all the vertices between the smallest value of $l(v)$ and the largest value. (Note that this is with respect to the cyclic order.) For vertices $v$ labeled by $M_j$, this operation gives the identity functor, while for each of the other vertices $v$ this operation inserts a unit. After this is done, the remaining strings of white vertices correspond to the tensor product and shadow operations in $F_i$. So, carry out a minimal filling of each of these, in any total ordering compatible with the tensor pattern that defines $F_i$.

In the above algorithm, there is a choice of order to insert the units, and the order to carry out the tensor products. So it describes not one but several words in $C$ that are all assigned to the functor $F_i$. Using the conventions in Lemma 9.11 and our discussion of minimal fillings, between any two such words, our diagram of functors assigns the identity transformation $F_i = F_i$.

It now remains to show that between $F_i$ and $F_{i+1}$, the isomorphism given by the diagram of functors in Lemma 9.11 is $\phi_i$. If $\phi_i$ is an associator, shadow, or inverse associator or shadow, then $F_i$ and $F_{i+1}$ have the same units, and the only thing that changes in the algorithm is the ordering of the tensorings. We move from one order to the other by swapping two minimal fillings. By the discussion of minimal fillings, this gives the desired isomorphism (associator, shadow, or inverse of such).

If $\phi_i$ is a left unit isomorphism, then $F_i$ has an unit that $F_{i+1}$ does not. After the first phase of the graph darkening, this means that $F_i$ and $F_j$ give identical pictures except in one segment where $F_i$ is the picture on the left while $F_{i+1}$ is the one on the right.

\[
\bigcirc \circ \circ \circ \ldots \circ \quad \text{and} \quad \bigcirc \bullet \bullet \bullet \ldots \bullet \bullet \circ
\]

For $F_j$, fix some word in $C$ arising from the above algorithm in which the extra unit is inserted after all the others. Modify this word by moving the last unit past every tensor product, until the one where it gets multiplied in. Then swap the order of the darkenings for this unit and the subsequent multiplication, so that both moves become inert and the unit disappears. Finally, swap the resulting two inert darkenings back past all the tensor products, until they happen just after the unit to the right is created. This gives us a valid word for $F_{i+1}$. The three modifications we made induce the identity, the left unit isomorphism, and the identity, so together we get the left unit isomorphism, as desired.

When $\phi_i$ is an inverse of a left unit isomorphism, the proof is the same except that the two functors give

\[
\bigcirc \bigcirc \bigcirc \bullet \bullet \ldots \bullet \circ \quad \text{and} \quad \bigcirc \bigcirc \bigcirc \bullet \bullet \ldots \bullet \circ.
\]
We start with a word for \(F_{i+1}\) where the extra unit is inserted last, move its insertion past the tensor products so that it is inserted before it is used, swap the order of the unit and multiplication to make both inert, then swap the resulting two inert darkenings further forward until they become a part of the next multiplication on the left of the segment depicted just above. This gives a valid route for \(F_i\), so again the isomorphism from \(F_{i+1}\) to \(F_i\) is the left unit isomorphism. Of course the right unit isomorphism and its inverse are handled by the same proof. \(\square\)

9.4. Coherence along a staircase grid. We now generalize Proposition 9.7. Suppose \(P\) is a category and \(Q_a : \mathbf{P}^{op} \to \mathbf{Cat}\) is a \(P\)-indexed category where the composition and unit isomorphisms are identity transformations. Let \(g^* : Q_{a'} \to Q_a\) denote the pullback functor associated to the morphism \(a \to a'\) of \(P\).

The Grothendieck construction of \(Q_a\), denoted \(P \times Q_a\), has an object for each pair \((a,b)\) with \(a \in \text{ob}P\) and \(b \in \text{ob}Q_a\), and a morphism \((a,b) \to (a',b')\) for each morphism \(g : a \to a'\) in \(P\) and \(f : b \to g^*(b')\). Let \(g_b : (a,g^*(b)) \to (a',b)\) denote the canonical cartesian arrow in \(P \times Q\) over \(g\) ending at \((a',b)\).

**Lemma 9.15.** A presentation for \(P\) and for each fiber \(Q_a\) define a presentation of \(P \times Q_a\).

**Proof.** Let \(G_P\) and \(R_P\) be generators and relations for \(P\). We think of these has horizontal generators and relations. Let \(G_{Q_a}\) and \(R_{Q_a}\) be generators and relations for each fiber \(Q_a\). We think of these as vertical generators and relations. For each horizontal generator \(g : a \to a'\) and vertical generator in \(f \in G_{Q_{a'}}\), fix a choice of decomposition of \(g^*(f)\) into generators in \(G_{Q_a}\). Note that this choice can be made canonically if \(g^*(f)\) is a generator in \(G_{Q_a}\).

Let \(G_{P \times Q}\) be the union of

- the vertical generators \(G_{Q_a}\) in each fiber and
- the horizontal morphism \(g_b\) for each generator \(g\) in \(P\) and object \(b\) in \(Q_{a'}\).

Let \(R_{P \times Q}\) be the union of

- the vertical relations \(v_1 = v_2\)
- in \(R_{Q_a}\) for each object \(a\) in \(P\),
- the horizontal relation \((w_1)_b = (w_2)_b\)
- for each relation \(w_1 = w_2\) in \(R_P\) (with common target \(a'\)) and each object \(b\) in \(Q_{a'}\),
- a swap relation \(f \circ g_b = g_{b'} \circ g^*(f)\) \((\ref{equation:swap})\)

for each horizontal generator \(g\) with target \(a'\) and vertical generator \(f : (a',b) \to (a',b')\). (The term \(g^*(f)\) represents the chosen decomposition of the morphism \(g^*(f)\) into vertical generators in \(Q_a\).)

Every morphism \((a,b) \to (a',b')\) in \(P \times Q\) factors in a canonical way as \(\gamma_{b'} \circ \phi\) with \(\gamma : a \to a'\) in \(P\) and \(\phi : (a,b) \to (a,\gamma^*(b'))\) in \(Q_a\). Both \(\phi\) and \(\gamma_{b'}\) can be decomposed into composites of maps in \(G_{P \times Q}\). If two words in these generators specify the same morphism in \(P \times Q\), we apply swap relations to put our two words into this standard form, with all the vertical letters on the right (applied first) and all horizontal letters on the left. Since they specify the same morphism in \(P \times Q\), the horizontal \(P\) parts and the vertical \(Q\) parts must separately agree. Therefore we may apply additional horizontal and vertical relations that bring one word to the other. \(\square\)
Any coherent diagram of functors we build on $\mathbf{P} \ltimes \mathbf{Q}$ using this presentation will in particular have to satisfy coherence for the swap relations. Since this condition is both important and laborious to state, we separate it out as a separate definition.

**Definition 9.17.** A diagram of functors on $\mathbf{P} \ltimes \mathbf{Q}$ with respect to the presentation in Lemma 9.15 is **coherent for swaps** if the following two conditions hold.

- (Coherence of swaps and vertical relations) For each horizontal generator $g \in \mathcal{G}_\mathbf{P}$ and vertical relation $v_1 = v_2$ of words $(a', b) \rightarrow (a', b')$ as below

$$
\begin{align*}
(a, g^*(b')) & \xrightarrow{\bar{g}_{b'}} (a', b') \\
g^*(v_1) & \xrightarrow{v_1} v_2 \\
(a, g^*(b)) & \xrightarrow{\bar{g}_b} (a', b)
\end{align*}
$$

the square of natural isomorphisms

$$
\begin{array}{ccc}
(g^*(v_1))^\# & \xrightarrow{\bar{g}_{b'}} & g^{(v_1)}_b^\# \\
\equiv & & \equiv \\
& \bar{g}_b & (g^*(v_2))^\# \xrightarrow{\bar{g}_{b'}} g^{(v_2)}_b^\#
\end{array}
$$

commutes, where the horizontal isomorphisms are compositions of swap isomorphisms

$$(g^*(f))^\# \bar{g}_{b'} = \bar{g}_b f^\#$$

and the vertical isomorphisms come from the vertical relations.

- (Coherence of swaps and horizontal relations) For each horizontal relation $w_1 = w_2$ and vertical generator $f$ as below (where $w = w_1$ but as a morphism in $\mathbf{P}$, not a word)

$$
\begin{align*}
(a, w^*(b')) & \xrightarrow{(w_1)_{b'}} (a', b') \\
& \xrightarrow{(w_2)_{b'}} f \\
(a, w^*(b)) & \xrightarrow{(w_2)_b} (a', b)
\end{align*}
$$

the square of natural isomorphisms

$$
\begin{array}{ccc}
(w^*(f))^\#(w_1)_b^\# & \xrightarrow{(w_1)_{b'}} (w_1)_b^\#f^\# \\
\equiv & & \equiv \\
& \bar{g}_b & (w^*(f))^\#(w_2)_b^\# \xrightarrow{(w_2)_{b'}} f^\#
\end{array}
$$

commutes, where as before the vertical isomorphisms come from horizontal relations and the horizontal isomorphisms are compositions of swaps. (For instance if $w_1 = g_1 g_2$, so $w_1^\# = \bar{g}_2 \bar{g}_1^\#$, we get one swap for $f$ and one for every letter of $g_1^*(f)$.)

Before we state the generalization of Proposition 9.7, we observe that $\mathbf{P}$ does not in general embed into $\mathbf{P} \ltimes \mathbf{Q}$, but for each object $a$ of $\mathbf{P}$ and $b$ of $\mathbf{Q}_a$, the rule $g \leadsto \bar{g}_b$ embeds the comma category $(\mathbf{P} \downarrow a)$ into the comma category $(\mathbf{P} \ltimes \mathbf{Q} \downarrow (a, b))$.

**Theorem 9.18** (Staircase coherence). A diagram of functors on the presentation of $\mathbf{P} \ltimes \mathbf{Q}$ in Lemma 9.15 is coherent if the following conditions are satisfied.
• (Vertical coherence) For each object \( a \in P \), the coherence condition along the fiber \( Q_a \).

• (Horizontal coherence) For each object \( a \in P \) and \( b \in Q_a \), the coherence condition along the comma category \( (P \downarrow a) \).

• The diagram is coherent for swaps (Definition 9.17).

Proof. Throughout the proof we fix a morphism \( \gamma_{b'} \circ \phi \) from \((a, b)\) to \((a', b')\) in \( P \times Q \), and consider only isomorphisms between words in the generators that compose to this particular morphism.

We begin by fixing a decomposition of \( \gamma \) into generators:

\[
a = a_0 \xrightarrow{g_1} a_1 \rightarrow \ldots \rightarrow a_{n-1} \xrightarrow{g_n} a_n = a'\]

We examine the category of functors \( \mathcal{C} \) composed from words in our generators for \( P \times Q \), whose \( P \)-letters are \( g_1, g_2, \ldots, g_n \). (The \( b_i \) subscripts on the terms \( g_i \) are allowed to vary. We suppress these subscripts but they are understood to still be there.) Between these functors, we take all compositions of the vertical relation isomorphisms in each fiber category \( Q_a \), and the swap isomorphisms. Our first step is to prove that these form a contractible category, in other words any two compositions of such give the same isomorphism. This is true when \( n = 0 \) by the assumption of vertical coherence.

By inductive hypothesis we can assume that it is contractible when it is defined using only \( n - 1 \) generators from \( G_P \). Returning to our original list of \( n \) generators, given an isomorphism between two functors in this category, we represent it by a sequence of relation and swap isomorphisms. Without loss of generality we may assume that the first steps are swaps that move \( g_n \) past everything to its right so that it is at the end of the word (by order conventions this means \( g_n \) is the final arrow in our decomposition of \( \gamma_{b'} \circ \phi \)), then swaps that switch \( g_n \) back to its original position, before applying the remaining steps.

Now examine the segment of our sequence of steps that starts with swapping \( g_n \) from the end of the word back to its original position, and ends the very first time \( g_n \) moves again. By commutativity of whiskering, we may re-order the steps in this segment so that, after \( g_n \) is moved to its original position, all subsequent steps that are relations \( v_1 = v_2 \) in \( Q_{a_{n-1}} \) are applied first, and then all the remaining steps (the ones taking place “to the left of \( g_n \)”) are applied second. By the assumed coherence between vertical relations and swaps, this gives the same isomorphism as the sequence in which \( g_n \) stays fixed at the beginning of the word while the corresponding sequence of relations \((g_n)^{(v_1)}(v_2)\) in \( Q_{a_{n-1}} \) is instead applied first on the right-hand side of \( g_n \), then the remaining relations from this segment, and then \( g_n \) is swapped to its original position, before getting swapped one additional time at the end of the segment of steps we have been examining (so that if \( g_n \) was originally followed by \( k \) letters in the word, it is now followed by \( k - 1 \) or \( k + 1 \) letters).

Apply this procedure again to the segment of steps that now begins with \( g_n \) being swapped up \( k - 1 \) or \( k + 1 \) times, and ends the next time it is swapped. Since our original sequence of steps swaps \( g_n \) finitely many times in total, we eventually finish and conclude that our original isomorphism is equal to one obtained by the sequence that first swaps \( g_n \) to the end, leaves it inert while all the other steps are carried out, and then at the very end, swaps \( g_n \) back to a new position. By inductive hypothesis, all such sequences of steps (in particular the ones in the middle where \( g_n \) is not moving) give the same isomorphism, and therefore all sequences of steps on our original word must also give the same isomorphism.
In summary, for each fixed word \( w^# = g_1^# g_2^# \ldots g_n^# \) of functors from generators of \( P \), the functors that traverse the letters of that word, plus any generators in the fibers \( Q^- \), form a thick object \( C_{w,\phi} \rightarrow \text{Fun}(g \cdot a \cdot b, i \cdot b \cdot c) \). The morphisms in \( C_{w,\phi} \) are generated by arbitrary swaps and vertical relations.

The next step is to compare these contractible categories together for two different choices of word in the generators of \( P \) that give the same morphism \( \gamma \). For each horizontal relation \( w_1 = w_2 \) in \( \text{RP} \), applied to a fixed sub-word of \( w^# = g_1^# \ldots g_n^# \) to obtain a new word \( \tilde{w}^# = h_1^# \ldots h_m^# \), we define a thick isomorphism \( C_{w,\phi} \rightarrow C_{\tilde{w},\phi} \) by applying a horizontal relation isomorphism \( (w_1)^#_{i_{b_i}} \equiv (w_2)^#_{i_{b_i}} \) to any word where the relevant letters for \( w_1 \) occur consecutively (so no letters from the fibers \( Q^- \) in between).

To check this is a thick isomorphism, we have to check that any two words with the same \( P \) -letters that contain this sub-word in different places give the same map. Any two such words admit an unique isomorphism in \( C_{w,\phi} \), and we choose to present this isomorphism by swapping all the \( P \) -letters to the end of the word one at a time, except we swap our chosen sub-word past each \( Q^- \) -letter all at once, then applying relations to the \( Q^- \) -letters on the left, then finally swapping the \( P \) -letters back to new positions, again going one at a time, except that when we swap our sub-word we swap the whole thing past each letter. By whiskering, almost all of these moves commute with the natural isomorphism associated to the relation in \( RP \) automatically. The only ones that don’t are the ones where we swap our sub-word in \( P \) past a letter in \( Q^- \), but these commute by the assumed coherence of swaps and horizontal relations.

Now the categories \( C_{w,\phi} \) for words \( w \) specifying \( \gamma \) are connected together along thick isomorphisms, and it remains to argue that these new isomorphisms together generate a larger contractible category \( C_{\gamma,\phi} \). It suffices to pick representative in each \( C_{w,\phi} \) and to check that our isomorphisms between these representatives are preserved by composition. In each case we pick the word where all the letters \( g_i^# \) are at the end the word, and some fixed word forming \( \phi \) is at the front of the word. The conclusion then follows immediately from the assumption of horizontal coherence.

Remark 9.19. A similar theorem applies to the forwards Grothendieck construction \( P \rightarrow \text{op} \) of a functor \( Q^- \rightarrow \text{Cat} \). In the theorem and lemma above replace pairs of the form \( (g^*(f), f) \) with pairs of the form \( (f, g_*(f)) \). The canonical factorization of each morphism into vertical-then-horizontal becomes horizontal-then-vertical. In the proof where we move a \( g_i^# \) to the end of the word, we instead move \( g_i^# \) to the beginning of the word.

10. Rotation of a Category

Suppose that \( \mathcal{C} \) is a bifibration over \( S \). In this section we establish a convention for forming a diagram of pullback functors, pushforward functors, and Beck-Chevalley isomorphism from a diagram in \( S \). We use the results of the previous two sections to give conditions under which this diagram of functors is coherent. We also give a counterexample demonstrating that coherence does not hold in general.

Let \( P \) be a polygon with a choice of orientation for each edge. A source for \( P \) is a vertex \( \text{Source}(P) \) in \( P \) so that for every vertex \( w \) in \( P \) there is a directed path

\[
\text{Source}(P) \rightarrow w.
\]

If \( P \) has a source, then either the edges of \( P \) form a pair of directed paths from \( \text{Source}(P) \) to a different vertex \( \text{Sink}(P) \), or else the edges of \( P \) are oriented compatibly. In the first case, \( \text{Sink}(P) \) is a sink for \( P \). In the second case, we take \( \text{Sink}(P) := \text{Source}(P) \). See Figure 10.1.
Suppose $P$ is a polygon with orientations on the edges. If we pick a function $\rho$ from the edges of $P$ to the set
\[
\{\text{preserve}, \text{flip}\},
\]
then it defines new orientations of the edges of $P$, where the orientations of the edges whose image is $\{\text{flip}\}$ have been reversed. We abuse notation at let $\rho(P)$ denote the polygon $P$ with this set of orientations on the edges. Note that this choice of orientations may or may not have a source.

We say the function $\rho$ is a **rotation** of $P$ if
- both $P$ and $\rho(P)$ have chosen sources and
- of the two directed paths $\text{Source}(P) \to \text{Sink}(P)$, $\text{Source}(\rho(P))$ lies on one of these paths and $\text{Sink}(\rho(P))$ lies on the other. See Figure 10.2.

If sources and sinks coincide, they can be regarded as lying on either of the directed paths. For example
\[
\text{Source}(P) = \text{Source}(\rho(P)) = \text{Sink}(P)
\]
is permissible as is
\[
\text{Source}(\rho(P)) = \text{Source}(P) = \text{Sink}(\rho(P)).
\]

**Remark 10.3.** For this discussion, it is helpful to imagine that we embed $P \to \mathbb{R}^2$ as a convex polygon so that all the arrows point downwards. Then we want to rotate the embedding, giving new choices of orientations for all of the edges.

If $\mathbf{I}$ is a category with presentation $(G,R)$, represent generators $f \in G$ as oriented edges, and the relations $(w_1 = w_2) \in R$ as polygons $P_{w_1 = w_2}$ built out of the labeled, oriented edges. Each relation polygon $P_{w_1 = w_2}$ has a canonical choice of source vertex. A **rotation** of $\mathbf{I}$ is a function $\rho: G \to \{\text{preserve, flip}\}$, together with a choice of vertex $\text{Source}(\rho(P_{w_1 = w_2}))$ for each relation, so that the condition discussed above (Figure 10.2) is satisfied on each relation polygon $P_{w_1 = w_2}$. The objects of **rotated category** $\rho(\mathbf{I})$ are the objects of $\mathbf{I}$. The generators of morphisms in $\rho(\mathbf{I})$ are in bijection with those of $\rho$ but if the image of a generator $a \to b$ is $\{\text{flip}\}$ it is regarded as a morphism $b \to a$ in $\rho(\mathbf{I})$. The relations are given by the polygons $\rho(P_{w_1 = w_2})$ with the orientations assigned by $\rho$.

**Example 10.4.**
\[ \text{Source}(P) \quad \text{Sink}(\rho(P)) \]

\[ \text{Source}(\rho(P)) \quad \text{Sink}(P) \]

**Figure 10.2.** Paths from source to sink

i. We can rotate the commuting square category

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\uparrow \\
\bullet
\end{array}
\]

by flipping the vertical edges. Alternatively, we could rotate the square by flipping all the edges.

ii. The cube-shaped category can be rotated by reversing the generators along one, two, or all three of the coordinate directions.

iii. If \( I \) is any category, the opposite category \( I^{\text{op}} \) is a rotation of \( I \).

iv. The \( \text{op} \) rotation can always be composed with a rotation \( \rho \), giving a new rotation \( \rho^{\text{op}} \), whose set of flipped generators is the complement of the flipped generators for \( \rho \). Abusively letting \( \rho \) denote the rotation of \( I^{\text{op}} \) that flips the same generators as \( \rho \), we get the following commuting diagram of rotations.

\[
\begin{array}{c}
I \\
\text{op} \downarrow \\
I^{\text{op}} \\
\rho^{\text{op}} \downarrow \\
\rho(I) \\
\rho(I^{\text{op}}) \downarrow \\
\rho(I) = \rho(I^{\text{op}}) = \rho(I)^{\text{op}}
\end{array}
\]

v. Let \( F : I \to J \) be a functor and \( \rho \) a rotation of \( I \). Then \( \rho \) also gives a rotation of the comma category \( (I \downarrow j) \) for each \( j \in \text{ob}J \). Each generator of \( (I \downarrow j) \) is a generator \( i \to i' \) of \( I \) together with a map \( i' \to j \). Each relation of \( (I \downarrow j) \) is similarly a relation in \( I \) with a map from the terminal object to \( j \). The rotation of \( (I \downarrow j) \) flips each generator \( i \to i' \to j \) if and only if the generator \( i \to i' \) in \( I \) was flipped. Note that the rotated comma category \( \rho(I \downarrow j) \) is **not** a comma category. It cannot be \( (\rho(I) \downarrow j) \), because this latter category does not even make sense – we do not have a functor \( \rho(I) \to J \).

vi. In the previous example we could also rotate the larger comma category \( (I \downarrow J) = J \times^{\text{op}} (I \downarrow j) \). We take as generators

- the maps \( F(i) \to F(i') \to j \) and
- the maps \( F(i) \to j \to j' \).

There is a relation for

- every relation in \( I \downarrow j \) and a map \( j \to j' \) and

\[ \text{Source}(P) \quad \text{Sink}(\rho(P)) \]

\[ \text{Source}(\rho(P)) \quad \text{Sink}(P) \]
for every $i \in I$, map $F(i) \to j$, and relation between two morphisms $j \Rightarrow j'$.

The rotated category is a (forwards) Grothendieck construction

$$\rho(I \downarrow J) = J \times^{\text{op}} \rho(I \downarrow J)$$

but (as before) it is not a comma category.

ii. If $I$ is a product of categories $P_1, \ldots, P_n$ and each category $P_i$ has a rotation $\rho^i$, there is a product rotation $\rho = \prod_i \rho^i$ on the product category $I$. Specifically, we go from the presentation on each of the categories $P_i$ to a presentation on $\prod P_i$ as follows. Each generator of $P_i$ and choice of object in $P_j$ for all $j \neq i$ defines a generator in $\prod P_i$. Each relation in $P_i$ and choice of object in $P_{j \neq i}$ defines a relation in $\prod P_i$. Finally, there is a swap relation for each pair of generators from $P_i, P_j, j \neq i$, and choice of object in the remaining categories.

Since each generator of $\prod P_i$ corresponds to a generator of one of the categories $P_i$, we define the product rotation $\rho$ by flipping such a generator if and only if $\rho^i$ flips it. The condition for each relation in a single category $P_i$ is satisfied by the fact that $\rho^i$ is a rotation, and the rotation condition for the swap relations is automatically satisfied. This finishes the construction of $\rho$ and the proof that it is a rotation.

In Items i and ii, the first is a two-fold product rotation, and the second is a three-fold product rotation.

Each functor $X : I \to S$ and rotation $\rho$ of $I$ defines a commutative diagram

$$
\begin{array}{ccc}
X(&\text{Source}(P))&\longrightarrow&X(&\text{Sink}(\rho(P)))
\end{array}
\begin{array}{ccc}
&\downarrow&
&\downarrow&
\end{array}
\begin{array}{ccc}
X(&\text{Source}(\rho(P)))&\longrightarrow&X(&\text{Sink}(P))
\end{array}
$$

for each relation. If $S$ is a category with a class of Beck-Chevalley squares, we say that $\rho$ respects Beck-Chevalley squares in $X$ if each of these squares is a Beck-Chevalley square in $S$.

**Example 10.5.**

i. If $X$ is the commuting square in $S$

$$
\begin{array}{ccc}
A&\longrightarrow&B
\end{array}
\begin{array}{ccc}
\downarrow f& & \downarrow g
\end{array}
\begin{array}{ccc}
C&\longrightarrow&D
\end{array}
\begin{array}{ccc}
\downarrow h& & \downarrow k
\end{array}
$$

and $\rho$ is the rotation that flips all of the edges, then $\rho$ respects Beck-Chevalley automatically, even if the above square is not Beck-Chevalley. However, if $\rho$ is the rotation that flips only the vertical edges, then $\rho$ respects Beck-Chevalley squares in $X$ if and only if the square above is Beck-Chevalley.

ii. The coherences ***, ***, ***, and !!! can be summarized: given any rotation of the cube category $I$ obtained by choosing an orientation for each coordinate direction, if the rotation respects Beck-Chevalley for a given cube of spaces, then the pullback and pushforward functors along the maps of that cube fit into a coherent diagram on the rotated category $\rho(I)$.

iii. For any rotation $\rho$, $\rho^{\text{op}}$ respects Beck-Chevalley squares if and only if $\rho$ respects Beck-Chevalley squares. In particular, the trivial rotation and its opposite respect Beck-Chevalley squares for every diagram $I \to S$. 
iv. If \( I = \prod P_i \), each \( P_i \) has a diagram \( A^i \) in \( S \), and \( \rho^i \) respects Beck-Chevalley in \( A^i \), then \( \rho \) respects Beck-Chevalley in the diagrammatic external product of the diagrams \( A^i \). Again the only non-trivial part of this claim is the swap relations, but the square in that case is an external product of two morphisms of \( S \), which must be a Beck-Chevalley square by the assumption (5.3).

The purpose of this language is to build diagrams of functors and natural isomorphisms out of diagrams in \( S \). As a motivating special case, given a Beck-Chevalley square in \( S \)

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
C & \xrightarrow{k} & D,
\end{array}
\]

we can rotate by flipping the vertical generators, and build a coherent diagram of functors on the rotated category using \( f^*, g^! \), \( h^! \), and \( k^* \) and the Beck-Chevalley map. If instead we flipped all the edges, we could build a diagram of functors on that category using the pullbacks \( f^*, g^*, h^* \), and \( k^* \); it isn’t necessary to even ask that the square is Beck-Chevalley. The following definition generalizes this idea to larger diagrams.

**Definition 10.6.** For a bifibration \( \mathcal{C} \to S \), a diagram \( A : I \to S \), and a rotation \( \rho \) respecting the Beck-Chevalley squares of \( A \), the **rotated diagram of functors** is the (thick) diagram of functors on \( \rho(I) \) that assigns

- the fiber \( \mathcal{C}^{A(i)} \) to \( i \in I \),
- the pullback \( A(f)^* \) to each non-flipped generator \( f \) of \( I \),
- the pushforward \( A(f)_! \) to each flipped generator \( f \), and
- for each relation square of the form

\[
\begin{array}{ccc}
\text{Source}(P) & \xrightarrow{f_1} & \cdots \xrightarrow{f_k} & \text{Source}(\rho(P)) \\
\downarrow{h_1} & & & \downarrow{g_1} \\
& \cdots & & \\
\downarrow{h_m} & & & \downarrow{g_l} \\
\text{Sink}(\rho(P)) & \xrightarrow{j_1} & \cdots & \xrightarrow{j_n} & \text{Sink}(P)
\end{array}
\]

the composite isomorphism

\[
(f_k)_! \cdots (f_1)_! (h_1)^* \cdots (h_m)^* \equiv (f_k \circ \cdots \circ f_1)_! (h_m \circ \cdots \circ h_1)^* \\
\equiv (g_l \circ \cdots \circ g_1)^* (j_n \circ \cdots \circ j_1)_! \\
\equiv (g_1)^* \cdots (g_1)^* (j_n)_! \cdots (j_1)_!
\]

where the first and third isomorphisms are from Corollary 7.9 and the second is Beck-Chevalley. Note that these are all thick isomorphisms by Lemma 8.15.

**Lemma 10.7.** When no edges are flipped, the above isomorphism is of the form \( * * \). Similarly, when every edge is flipped, the isomorphism is of the form \( !! \).

**Proof.** Suppose there are no maps \( f_i \) or \( j_i \), and we let \( g \) be the common composite of the \( g_i \) and the \( h_i \). Then the above composite becomes

\[
(h_1)^* \cdots (h_m)^* \equiv g^* \equiv \text{id}_! g^* \equiv \text{id}_! \text{id}_! g^* \equiv \text{id}_! \text{id}_! \rightarrow g^* \text{id}_! \equiv g^* \equiv (g_1)^* \cdots (g_1)^*
\]

Each cocartesian arrow defining \( \text{id}_! \) is an isomorphism whose inverse is cartesian. So \( \text{id}_! \) is actually a pullback functor and the unit and counit of the adjunction \( (\text{id}_!, \text{id}_!) \) arise
from the universal property of the pullback. Therefore each of the above isomorphisms is the canonical isomorphism ** between pullbacks. The case of !! has a dual proof. □

We finish this section with a general result that any sufficiently “cubical” rotated diagram of functors is coherent. This generalizes the coherences ***, ***, ***, and !!!. Then we give a counterexample showing that not every rotated diagram of functors is coherent.

**Proposition 10.8.** Suppose \( A : \prod P_i \to S \) is a diagram, and \( \rho = \prod \rho^i \) is a product rotation of \( \prod P_i \), respecting Beck-Chevalley squares in \( A \). Then the rotated diagram of functors associated to \( \rho \) is coherent if and only if the rotated diagram of functors associated to each strand

\[
\{s_1\} \times \ldots \times \{s_{i-1}\} \times P_i \times \{s_{i+1}\} \times \ldots \{s_n\}
\]

is coherent. (In particular, these are both true if each \( \rho^i(P_i) \) is a planar poset.)

**Proof.** By Corollary 9.10, it suffices to check a coherence condition for each triple of generators, each of which is of the form ***, ***, ***, and !!!! from Proposition 8.7, and a coherence condition for each generator from one strand and relation from another. By Definition 10.6, the relation isomorphism is composed of composition, unit, and Beck-Chevalley isomorphisms. The desired square of natural transformations therefore subdivides into several smaller squares, each of which commutes by one of the coherences ***, ***, ***, and !!! from Proposition 8.7, or the variants of these using the unit isomorphisms from Corollary 8.29. □

**Example 10.9.** Let \( I \) be obtained from two copies of the square-shaped poset (\( \bullet \leftarrow \bullet \rightarrow \bullet \)) by identifying the edges together. So it has 16 generators and 8 relations. See Figure 10.10a (the outside edges of both squares are identified). Let \( \rho \) be the rotation of \( I \) that flips all the maps \( \bullet \rightarrow \bullet \), so that \( \rho(I) \) has presentation in Figure 10.10b.

Let \( C \to S \) be the bifibration of parametrized retractive spaces, or parametrized spectra.\(^3\) Fix a choice of topological space \( B \) and non-trivial homeomorphism \( f : B \to B \). Form the diagram on \( I \) in Figure 10.10c. Then \( \rho \) respects Beck-Chevalley — this amounts to checking that 4 of the 8 squares in Figure 10.10c are Beck-Chevalley squares. The top and right route through either square gives the functor \( X \rightsquigarrow X \wedge B \). But each square gives a different automorphism of this functor: the first gives the automorphism that applies \( f \), and the second gives the identity. Therefore the associated diagram of functors is not coherent.

**11. EMBEDDING IN A CUBICAL CATEGORY**

In this section we give an embedding of \( E^{\mathcal{G}} \) in the category

\[
\text{Fin}^{\otimes} \left[ E^{\mathcal{G}} \otimes \rho \left[ (\bullet \leftarrow \bullet \rightarrow \bullet \right) \downarrow U \right] \right].
\]

This category is a sequence of Grothendieck constructions and so we build this category from right to left. In Section 11.1 we define a functor to finite sets. In Section 11.2 we use this functor to define the category above. In Section 11.3 we give more explicit descriptions of the objects and morphisms in (11.1).

\(^3\)It doesn’t matter much for this counterexample whether we take spaces or spectra on the nose or in the homotopy category.
11.1. The gray edges functor. Let $a \in \text{ob}E^{\mathcal{G}}$ be a labeled graph and $IV(a)$ denote the set of internal white vertices of $a$. A coloring of $a$ is an object of the category

$$(\bigcirc \rightarrow \bullet \rightarrow \bullet)^{IV(a)}.$$ 

This defines a coloring of the internal white vertices of $a$ by the three colors $\bigcirc, \bullet, \bullet$. Alternatively, it is a function

$$c : V(a) \rightarrow \{\bigcirc, \bullet, \bullet\}$$

so that black vertices must be sent to $\bullet$ and external white vertices to $\bigcirc$. If $c$ and $d$ are colorings, there is a morphism $c \rightarrow d$ if for every $v \in IV(a)$, $c(v) = d(v)$ or $d(v) = \bullet$. In this case we say the coloring $d$ is grayer than $c$.

The gray edges of a coloring $c : V(a) \rightarrow \{\bigcirc, \bullet, \bullet\}$ are

- single edges whose adjacent vertices are either $\bigcirc \bullet$ or $\bullet \bigcirc$, or
- strings of at least two edges, whose endpoints are either $\bigcirc \bullet, \bullet \bigcirc, \bigcirc \bullet$, or $\bullet \bullet$, and every vertex between them is $\bullet$.

If $s$ is a gray edge, the representative of $s$, $[s] \in E(a)$, is the first edge in $s$. We define “first” using the orientations of the internal white vertices in the string $s$.

The gray edges functor

$$(11.2) \quad \mathcal{G} : (\bigcirc \rightarrow \bullet \rightarrow \bullet)^{IV(a)} \rightarrow \text{fin}$$
takes the coloring \( c \) to the set of gray edges of \( c \), denoted \( \mathcal{G}(c) \). A morphism \( c \to d \) induces a map of sets \( \mathcal{G}(c) \to \mathcal{G}(d) \) that sends each gray edge in the coloring \( c \) to the gray edge containing it in the coloring \( d \).

**Example 11.3.** If \( \alpha \) is the graph

\[
\begin{array}{c}
B = B
\end{array}
\]

then \((\bigcirc \to \bullet \leftarrow \bullet)^{IV(\alpha)} \cong (\bigcirc \to \bullet \leftarrow \bullet)\) is a single zig-zag,

and the gray edges functor takes this to the zig-zag of finite sets.

\[
\{*, *\} \longrightarrow \{*\} \longleftarrow \emptyset
\]

If \( \alpha \) is the graph

\[
\begin{array}{c}
B = B = B
\end{array}
\]

then \((\bigcirc \to \bullet \leftarrow \bullet)^{IV(\alpha)} \cong (\bigcirc \to \bullet \leftarrow \bullet)^2\) is in Figure 11.4a.

\[
\begin{array}{c}
\{a\} \longrightarrow \{a, b, c\} \longleftarrow \{a, b, c\}
\end{array}
\]

\[
\begin{array}{c}
\{ab\} \longrightarrow \{abc\} \longleftarrow \{ab, c\}
\end{array}
\]

\[
\begin{array}{c}
\{b\} \longrightarrow \{bc\} \longleftarrow \{c\}
\end{array}
\]

(b) Image of the gray edges functor

\[
\begin{array}{c}
\{a\} \longrightarrow \{a, b\} \longleftarrow \{a, b, c\}
\end{array}
\]

\[
\begin{array}{c}
\{a\} \longrightarrow \{a\} \longleftarrow \{a, c\}
\end{array}
\]

\[
\begin{array}{c}
\{b\} \longrightarrow \{b\} \longleftarrow \{c\}
\end{array}
\]

(c) The gray edges functor with representatives

**Figure 11.4.** Example 11.3

If we label the three edges of the graph \( \alpha \) by \( a, b, c \), and label a gray edge by the word composed of all the edges it contains, the gray edges functor takes this category to the diagram of finite sets in Figure 11.4b. If we instead label each gray edge by its representative, and orient the white vertices from left to right, the diagram would be as in Figure 11.4c. In either case, the top-right and bottom-left squares are pushouts. The next lemma formalizes this observation.

For each graph \( \alpha \), coloring \( c \), and choice of an internal white vertex and an internal black vertex (in the coloring \( c \)), consider the square in \((\bigcirc \to \bullet \leftarrow \bullet)^{IV(\alpha)}\) given flipping
the two chosen vertices gray:

\[
\begin{array}{ccc}
\ldots \circ \ldots \circ & \longrightarrow & \ldots \circ \ldots \\
\downarrow & & \downarrow \\
\ldots \circ \ldots \circ & \longrightarrow & \ldots \circ \ldots 
\end{array}
\]

**Lemma 11.5.** The gray edges functor takes every such square to a pushout square of finite sets.

**Proof.** We first show the map from the pushout to $G(\ldots \circ \ldots \circ \ldots)$ is surjective. The only element of $G(\ldots \circ \ldots \circ \ldots)$ that may not be in the image of the left vertical map is the element corresponding to the first copy of $\circ$. This only happens when that gray vertex is surrounded by white vertices. In that case the bottom horizontal map hits this point, because flipping the second $\circ$ black does not affect this gray edge.

Now it suffices to take any gray edge in $G(\ldots \circ \ldots \circ \ldots)$ and show that the square consisting of its preimages is a pushout. If the chosen gray edge is has no internal vertices or is a string of gray vertices not containing the two flipped vertices, then we get a square of isomorphisms. If the chosen gray edge contains precisely one of the flipped vertices, then two of the parallel maps of the square are isomorphisms. Each of these cases gives a pushout square. In the final case, the chosen gray edge contains both of the flipped vertices. We check by hand in each of the nine cases in Figure 11.6 that the corresponding square is a pushout.

![Figure 11.6. The remaining cases of Lemma 11.5](image)

For the top row and right-hand column, the resulting square has two parallel maps that are isomorphisms. The remaining four all give the pushout square

\[
\begin{array}{c}
\{a, b, c\} \\
\downarrow \\
\{a, bc\}
\end{array} \quad \begin{array}{c}
\{a, b, c\} \\
\downarrow \\
\{abc\}
\end{array}
\]

These cases suffice because replacing the $\circ$ in the very center by any number of copies of $\circ$ (including zero copies) does not change the resulting square of sets. □
11.2. **Graph categories.** We can now build larger categories out of the categories of partial darkenings \((\bigcirc \to \bullet \leftarrow \bullet)^{IV(a)}\) for different graphs \(a\).

**Lemma 11.7.** The categories \((\bigcirc \to \bullet \leftarrow \bullet)^{IV(a)}\) form a strict \(E\mathcal{G}\)-indexed category.

**Proof.** Thinking of colorings as functions

\[ c : V(\alpha) \to \{\bigcirc, \bullet, \bullet\}, \]

for each map of graphs \(h : \alpha \to \beta\) and coloring \(d\) of the vertices of \(\beta\), we define the **pullback coloring** \(h^*d\) of \(\alpha\) by composing the functions

\[ V(\alpha) \to V(\beta) \to \{\bigcirc, \bullet, \bullet\}. \]

This pullback preserves the relation of being grayer, and therefore gives a functor

\[ h^* : (\bigcirc \to \bullet \leftarrow \bullet)^{IV(\beta)} \to (\bigcirc \to \bullet \leftarrow \bullet)^{IV(\alpha)}. \]

This defines a strictly associative and unital action of graphs on colorings. \(\square\)

The **twisted graph category** is the strict Grothendieck construction

\[ T\mathcal{G} := E\mathcal{G} \ltimes (\bigcirc \to \bullet \leftarrow \bullet)^{IV(-)}. \]

A morphism in \(T\mathcal{G}\) from \((\alpha, c)\) to \((\beta, d)\) is a map of graphs \(h : \alpha \to \beta\) along which \(h^*d\) is grayer than \(c\). The gray edges functor \((11.2)\) extends to a functor

\[ (\bigcirc \to \bullet \leftarrow \bullet)^{E\mathcal{G}} \to \mathcal{E}. \]

Each morphism gives a map of gray edges \(\mathcal{E}(c) \to \mathcal{E}(d)\) by sending each string of edges representing a point of \(\mathcal{E}(c)\) to the string that contains its image under \(h\).

For each finite set \(U\), let

\[ \left[(\bigcirc \to \bullet \leftarrow \bullet)^{IV(\alpha)} \downarrow U\right] \]

denote the comma category of the gray edges functor over \(U\).

**Lemma 11.9.** For each finite set \(U\), the categories \([(\bigcirc \to \bullet \leftarrow \bullet)^{IV(\alpha)} \downarrow U]\) form a strict \(E\mathcal{G}\)-indexed category.

**Proof.** The pullback operation is given by

\[ (\beta, d, \mathcal{E}(d) \to U) \leadsto (\alpha, h^*d, \mathcal{E}(h^*d) \to \mathcal{E}(d) \to U). \]

\(\square\)

The Grothendieck construction of this indexed category is canonically isomorphic to the comma category \([T\mathcal{G} \downarrow U]\) for the gray edges functor \((11.8)\) on \(T\mathcal{G}\):

\[ [T\mathcal{G} \downarrow U] \equiv \left[E\mathcal{G} \ltimes (\bigcirc \to \bullet \leftarrow \bullet)^{IV(-)} \downarrow U\right] \equiv \left[E\mathcal{G} \ltimes (\bigcirc \to \bullet \leftarrow \bullet)^{IV(-)} \downarrow U\right]. \]

Following Example 10.4\(v\), we define a rotation \(\rho\) on \([(\bigcirc \to \bullet \leftarrow \bullet)^{IV(\alpha)} \downarrow U]\) by defining a rotation on the cube \((\bigcirc \to \bullet \leftarrow \bullet)^{IV(\alpha)}\). We pick the rotation that flips every map \(\bullet \to \bullet\). As a result, there is a morphism

\[ (c, \mathcal{E}(c) \to U) \to (d, \mathcal{E}(d) \to U) \]

if and only if \(d\) is **darker** than \(c\). Next we rotate \(T\mathcal{G} \downarrow U\) by flipping the maps of the form \(\bullet \to \bullet\), and not flipping any of the maps in \(E\mathcal{G}\). This rotated category is a Grothendieck construction:

\[ \rho[T\mathcal{G} \downarrow U] = \rho \left[E\mathcal{G} \ltimes (\bigcirc \to \bullet \leftarrow \bullet)^{IV(-)} \downarrow U\right] \equiv E\mathcal{G} \ltimes \rho \left[(\bigcirc \to \bullet \leftarrow \bullet)^{IV(-)} \downarrow U\right]. \]

**Lemma 11.10.** The categories \(\rho[T\mathcal{G} \downarrow U]\) form a \(\mathcal{E}\mathcal{G}_{\text{fin}^{\text{op}}-\text{indexed}}\) category.
Proof. Each map of finite sets $U \rightarrow V$ defines a functor

$$\rho \left[ \left( \circ \rightarrow \bullet \rightarrow \bullet \right)^{IV(a)} \upharpoonright U \right] \rightarrow \rho \left[ \left( \circ \rightarrow \bullet \rightarrow \bullet \right)^{IV(a)} \upharpoonright V \right]$$

by composition with the map $\mathcal{G}(c) \rightarrow U$. This action is strictly associative and unital, giving a $\mathcal{F}in^{op}$-indexed category.

The **gigantic graph category** is the strict op-Grothendieck construction

$$\mathcal{F}in^{op}(\mathcal{T} \mathcal{G} \upharpoonright U) \cong \mathcal{F}in^{op} \left( \mathcal{E} \mathcal{F} \mathcal{G} \times \rho \left[ \left( \circ \rightarrow \bullet \rightarrow \bullet \right)^{IV(-)} \upharpoonright U \right] \right).$$

**Theorem 11.11.** There is an embedding

$$\mathcal{E} \mathcal{F} \mathcal{G} \rightarrow \mathcal{F}in^{op} \left( \mathcal{E} \mathcal{F} \mathcal{G} \times \rho \left[ \left( \circ \rightarrow \bullet \rightarrow \bullet \right)^{IV(-)} \upharpoonright U \right] \right).$$

To set up the proof, we need more explicit descriptions of the morphisms in some of the intermediate categories. These descriptions are in the next section.

### 11.3. Canonical forms

As in Example 10.4v,

$$\rho \left[ \left( \circ \rightarrow \bullet \rightarrow \bullet \right)^{IV(a)} \upharpoonright U \right]$$

is no longer a comma category. Every morphism is instead associated to a zig-zag of sets $\mathcal{G}(c)$, with a commuting map from the zig-zag to $U$.

The following lemma tells us how to simplify these zig-zags into a canonical form. The **common graying**, $\mathcal{G}_{c,d}$, of colorings $c$ and $d$ is their least upper bound in $\left( \circ \rightarrow \bullet \rightarrow \bullet \right)^{IV(a)}$, i.e., the coloring that is gray at every vertex where $c \neq d$, and equal to $c = d$ at the remaining vertices. Pullback preserves common grayings, meaning that for any two colorings $d,e$ of $\beta$, we have the equality $h^* \mathcal{G}_{d,e} = \mathcal{G}_{h^*d,h^*e}$.

**Lemma 11.12.** In the rotated category

$$\rho \left[ \left( \circ \rightarrow \bullet \rightarrow \bullet \right)^{IV(a)} \upharpoonright U \right],$$

if $d$ is darker than $c$ then the set of morphisms from $(c, \mathcal{G}(c) \rightarrow U)$ to $(d, \mathcal{G}(d) \rightarrow U)$ is in bijection with the set of maps $\mathcal{G}(\mathcal{G}_{c,d}) \rightarrow U$ making the diagram in Figure 11.13a commute. If $d$ is not darker than $c$, the morphism set is empty. For two such morphisms $c \rightarrow d \rightarrow e$, their composition is illustrated in Figure 11.13b. The square is a pushout of finite sets.

![Figure 11.13. The diagrams in Lemma 11.12](image)

**Proof.** Using the functor that forgets the map to $U$

$$\rho \left[ \left( \circ \rightarrow \bullet \rightarrow \bullet \right)^{IV(a)} \upharpoonright U \right] \rightarrow \rho \left[ \left( \circ \rightarrow \bullet \rightarrow \bullet \right)^{IV(a)} \equiv \left( \circ \rightarrow \bullet \rightarrow \bullet \right)^{IV(a)} \right],$$

it is clear that a morphism in $\rho \left[ \left( \circ \rightarrow \bullet \rightarrow \bullet \right)^{IV(a)} \upharpoonright U \right]$ from $c$ to $d$ can only exist if $d$ is darker than $c$. 
Next we prove that the morphisms can be put into canonical form in a unique way. Each morphism in the rotated comma category is presented by a sequence of maps, each of which turns one vertex one shade darker, together with a choice of map from the resulting zig-zag of gray edges to $U$:

\[
\begin{array}{cccc}
\mathcal{G}(c) & \cdots & \mathcal{G}(c_1) & \cdots & \mathcal{G}(c_2) & \cdots & \mathcal{G}(d)
\end{array}
\]

There are relations that swap the order of two consecutive maps. If the maps induced by $\mathcal{G}$ point in opposite directions, this creates a commuting square of the form in Lemma 11.5. The definition of a relation in a rotated comma category requires that this square be equipped with a map from the new vertex into $U$. The new vertex is either the initial or terminal one of a pushout square, so there is a unique map from the new vertex to $U$ that agrees with the others. Therefore it is possible to take every such zig-zag and modify it into one of the form

\[
\begin{array}{cccc}
\mathcal{G}(c) & \cdots & \mathcal{G}(c_1) & \cdots & \mathcal{G}(g_{c,d}) & \cdots & \mathcal{G}(d)
\end{array}
\]

To show that the resulting map $\mathcal{G}(g_{c,d}) \to U$ is determined uniquely by the map from the original zig-zag to $U$, it suffices to prove that the set $\mathcal{G}(g_{c,d})$ is the colimit of the diagram of sets formed by the original zig-zag. We do this by induction on the length of the zig-zag. Observe that $\mathcal{G}(g_{c,d})$ always receives a map because $g_{c,d} = c_i$ at every vertex where $g_{c,d}$ is not gray, hence $g_{c,d}$ is grayer than $c_i$.

For the inductive step we look at the last arrow of the zig-zag. Let $d'$ be the coloring just before this last step. Then we either have $d' \multimap d$ because $d'$ is gray in one spot where $d$ is black, or $d' \to d$ because $d'$ is white in one spot where $d$ is gray. In the first case, $g_{c,d'} = g_{c,d}$, and the colimit of the zig-zag from $c$ to $d'$ is also a colimit for the larger zig-zag, which proves $\mathcal{G}(g_{c,d})$ is the colimit of the larger zig-zag. In the second case, the claim follows once we show the square

\[
\begin{array}{ccc}
\mathcal{G}(d') & \to & \mathcal{G}(d) \\
\mathcal{G}(g_{c,d'}) & \to & \mathcal{G}(g_{c,d})
\end{array}
\]

is pushout. This is a sequence of pasted squares of the form from Lemma 11.5, one for each black vertex of $d$ that is turned gray in $g_{c,d}$, and is therefore also a pushout square.

Finally we prove the claim about compositions in $\rho \{ (\circ \to \bullet \to \bullet)^{V(a) \downarrow U} \}$. Since $\mathcal{G}(g_{c,d})$ is the colimit of the first half of the given zig-zag and $\mathcal{G}(g_{d,e})$ is the colimit of the second half, the colimit of the whole zig-zag must be the pushout of these two along $\mathcal{G}(d)$.

\[\square\]

**Corollary 11.14.** The category of factorizations of a fixed morphism $(c \to e, \mathcal{G}(g_{c,e}) \to U)$ in $\rho \{ (\circ \to \bullet \to \bullet)^{V(a) \downarrow U} \}$ is isomorphic to a product of categories of the form $(\circ \to \bullet)$, $(\bullet \to \bullet)$, and $(\circ \to \bullet \to \bullet)$, one for each vertex where $c$ and $e$ differ.

**Proof.** We define the functor by forgetting all the maps to $U$. By Lemma 11.12 this is an isomorphism. \[\square\]
Lemma 11.15. The gigantic graph category has objects

\[(U, \alpha, c, \mathcal{G}(c) \to U),\]

morphisms

\[(U, \alpha, c, \mathcal{G}(c) \to U) \to (V, \beta, d, \mathcal{G}(d) \to V)\]

given by a map of sets \(U \to V\), of graphs \(h : \alpha \to \beta\), and of sets \(\mathcal{G}(g_{c,h^*d}) \to V\) such that the diagram

\[
\begin{array}{ccc}
\mathcal{G}(c) & \to & U \\
\downarrow & & \downarrow \\
\mathcal{G}(g_{c,h^*d}) & \to & V \\
\uparrow & & \uparrow \\
\mathcal{G}(h^*d) & \to & \mathcal{G}(d)
\end{array}
\]

commutes, and composition

\[(U, \alpha, c, \mathcal{G}(c) \to U) \to (V, \beta, d, \mathcal{G}(d) \to V) \to (W, \gamma, e, \mathcal{G}(e) \to W)\]

given by composing the maps of finite sets, of graphs, and by taking the map \(\mathcal{G}(g_{c,h^*j^*e}) \to W\) given by the pushout

\[
\begin{array}{ccc}
\mathcal{G}(h^*d) & \to & \mathcal{G}(g_{c,h^*d}) \\
\downarrow & & \downarrow \\
\mathcal{G}(g_{h^*d,h^*j^*e}) & \to & \mathcal{G}(g_{c,h^*j^*e}) \\
\mathcal{G}(g_{d,j^*e}) & \to & W
\end{array}
\]

Proof. Lemma 11.12 and the definition of the Grothendieck construction imply in the rotated category

\[E\mathcal{G} \ltimes \rho \left[ (\bullet \to \bullet \to \bullet)^IV(-) \downarrow U \right],\]

the set of morphisms from \((\alpha, c, \mathcal{G}(c) \to U)\) to \((\beta, d, \mathcal{G}(d) \to U)\) is in bijection with the set of pairs \((h, \mathcal{G}(g_{c,h^*d}) \to U)\), where \(h : \alpha \to \beta\) is a graph morphism along which \(h^*d\) is darker than \(c\), and the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{G}(c) & \to & U \\
\downarrow & \nearrow & \\
\mathcal{G}(g_{c,h^*d}) & \to & \mathcal{G}(h^*d) \\
\downarrow & & \downarrow \\
\mathcal{G}(d) & \to & \mathcal{G}(d)
\end{array}
\]

Two such pairs \((h, \mathcal{G}(g_{c,h^*d}) \to U)\) and \((j, \mathcal{G}(g_{d,j^*e}) \to U)\) compose to \((j \circ h, \mathcal{G}(g_{c,j^*h^*e}) \to U)\) where the latter map is determined by the diagram in which the uppermost triangle is
a pushout square

Unwinding the definition of the op-Grothendieck construction gives the statement of the lemma.

Proof of Theorem 11.11. The image of a graph \( \alpha \) is \((\pi_0 \Psi \alpha, \alpha, \varphi, \mathcal{G}(\varphi) \to \pi_0 \Psi \alpha)\), where \( \Psi \alpha \) is the maximal cutting, \( \varphi \) is the lightest possible coloring, and the map \( \mathcal{G}(\varphi) \to \pi_0 \Psi \alpha \) sends each edge of the cut graph \( \Psi \alpha \) to its component.

The image of a map of graphs \( h: \alpha \to \beta \) is the map of sets \( \pi_0 \Psi \alpha \to \pi_0 \Psi \beta \) and the map of sets \( \mathcal{G}(\varphi_{\alpha, h, \cdot}) \to \pi_0 \Psi \beta \) that assigns each gray edge of \( \varphi_{\alpha, h, \cdot} \) in \( \alpha \) to the cut component containing its image in \( \beta \). Notice that since each vertex of \( \alpha \) and \( \beta \) is either black or white, the common graying \( \varphi_{\alpha, h, \cdot} \) is gray at precisely the white vertices of \( \alpha \) that are sent to black vertices of \( \beta \). This allows us to verify that the above map of sets is well-defined.

This assignment respects identity maps. To check it respects compositions \( \alpha \to \beta \to \gamma \), we take the image in the gigantic graph category then compose. Tracing through the definitions of the maps in the pushout in Lemma 11.15, each string of vertices in \( \alpha \) whose color is changed by the composite \( \alpha \to \gamma \) is ultimately assigned to the component of \( \pi_0 \Psi \gamma \) containing its image in \( \gamma \). By definition this is the morphism we get by composing the maps of graphs and then taking the image in the gigantic graph category.

12. Diagrams on Partially Darkened Graphs

This section is the technical climax of the paper. It is concerned with defining a coherent diagram of functors on the opposite of the gigantic graph category

\[ \mathcal{F}_{\mathrm{Gr}}^{\mathrm{op}} \times (\mathcal{E} \mathcal{G}^{\mathrm{op}} \times^{\mathrm{op}} \rho \delta (\circ \to \bullet \to \bullet)^{IV(a)} \uparrow U)^{\mathrm{op}}. \]

In other words, the functors point in the same direction as the gigantic graph category: forward along maps of sets, forward along maps of graphs, and from lighter to darker colorings. The construction of this diagram is a multi step process and those steps are outlined in Figure 12.1.

12.1. Diagrams in \( S \). We start by defining a diagram

\[ D_\alpha: (\circ \to \bullet \to \bullet)^{IV(a)} \to S \]

for each labeled graph \( \alpha \) in \( \mathcal{E} \mathcal{G} \). This diagram assigns the coloring \( c \), with set of gray edges \( \mathcal{G}(c) \), to the product of the objects of \( S \) that label the representatives of the gray edges. The morphisms are, roughly, induced by the gray edges functor and the morphisms that label the internal white vertices of \( \alpha \). We will first write out this definition precisely in a few simple examples, before making it precise in general. We will encounter familiar sequences of maps that encode the construction of bicategorical units
and multiplications from a symmetric monoidal bifibration; these formed the motivation for our definitions in this section and the previous one.

**Example 12.2.** Some very small examples are in Table 12.3.

**Table 12.3.** $D_\alpha$ for some small graphs

| $\alpha$ | domain of $D_\alpha$ | image of $D_\alpha$ |
|----------|----------------------|---------------------|
| $B = B$  | $\bullet \bullet \bullet \bullet \bullet \bullet$ | $B \times B \xrightarrow{\pi_B} *$ |
| $B = B$  | $\bullet \bullet \bullet \longrightarrow \bullet \bullet$ | $* \xrightarrow{\pi_B} B \xrightarrow{\Delta_B} B \times B$ |
| $B = B$  | $\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$ | $B \xrightarrow{id} B \xrightarrow{\Delta} B \times B \xrightarrow{\pi_1} B$ |

Generalizing these examples, if $\alpha$ is the graph

\[
\begin{array}{ccc}
  B & \longrightarrow & B \\
  \longrightarrow & \bullet & \longrightarrow
\end{array}
\]

with $n$ internal white vertices, the value of $D_\alpha$ on an object $c$ of $(\bullet \longrightarrow \bullet)^n$ is the space

\[
\prod_{\mathcal{E}(c)} B.
\]
A map of colorings \( c \to d \) induces a map of sets \( \mathcal{G}(d) \to \mathcal{G}(c) \) and a map on products \( \Pi_{\mathcal{G}(c)} B \to \Pi_{\mathcal{G}(d)} B \).

Since \((\Circle \leftarrow \bullet \to \bullet)^n\) is generated by maps of the form \( \bullet \to \Circle \) and \( \Circle \to \bullet \), Table 12.4 describes the action of \( D_a \) on all of the morphisms.

| \( \Circle \Circle \Circle \) | \( \Pi_{\mathcal{G}} \) | \( \Circle \bullet \bullet \) | \( B \) | \( \Circle \bullet \bullet \) | \( B \) |
|---|---|---|---|---|---|
| \( B \) | \( \uparrow \pi_B \) | \( B \times B \) | \( \downarrow \Delta_B \) | \( B \times B \) | \( B \) |

| \( \Circle \bullet \bullet \) | \( B \) | \( B \times B \) | \( \uparrow \Delta_B \) | \( B \times B \) | \( B \) |
|---|---|---|---|---|---|
| \( B \times B \) | \( \downarrow \Delta_B \) | \( B \times B \) | \( \downarrow \Delta_B \) | \( B \times B \) | \( B \) |

| \( \Circle \Circle \Circle \) | \( B \) | \( B \times B \) | \( \uparrow \Delta_B \) | \( B \times B \) | \( B \) |
|---|---|---|---|---|---|
| \( B \times B \) | \( \downarrow \Delta_B \) | \( B \times B \) | \( \downarrow \Delta_B \) | \( B \times B \) | \( B \) |

| \( \Circle \bullet \bullet \) | \( B \) | \( B \times B \) | \( \uparrow \Delta_B \) | \( B \times B \) | \( B \) |
|---|---|---|---|---|---|
| \( B \times B \) | \( \downarrow \Delta_B \) | \( B \times B \) | \( \downarrow \Delta_B \) | \( B \times B \) | \( B \) |

Table 12.4. Maps that generate the diagram on \((\Circle \leftarrow \bullet \to \bullet)^n\) from Example 12.2

The \( \mathbb{Z}/2 \times \mathbb{Z}/2 \)-symmetry of this table reflects the fact that the order of the word can be flipped, or the roles of \( \Circle \) and \( \bullet \) can be flipped, without affecting the above definitions.

The same definitions apply, without modification, to the following graphs as well.

Example 12.5. This generalizes the previous example. Consider the following labeled graph.

Let \( f_{i,j} \) be shorthand for the composite map \( B_i \to B_j \) when \( i < j \). The value of \( D_a \) on \( c \in (\Circle \leftarrow \bullet \to \bullet)^n \) is

\[
\prod_{s \in \mathcal{G}(c)} B_{[s]},
\]

where \( [s] \) is the representative of \( s \).

Then we follow Example 12.2 and define maps using Table 12.6. It is straightforward to check that each relation in the category \((\Circle \leftarrow \bullet \to \bullet)^n\) is sent to a commuting square, hence this defines a diagram in \( S \). Similarly, this can be generalized by allowing the black vertices on either of the two ends of the above graph to be external white vertices instead.
Now we can define $D_\alpha$ in general by bootstrapping these examples. For each $\alpha \in \mathcal{E}\mathcal{G}^{op}$, define

\begin{equation}
D_\alpha : (\bigcirc \to \bullet)^{IV(\alpha)} \to S
\end{equation}

by taking the diagrammatic external product of one copy of the diagram of Example 12.5 for each string of internal white vertices in $\alpha$ beginning and ending at a black or external white vertex.

**Lemma 12.8.** The product rotation of $(\bigcirc \to \bullet)^n$ that flips the generators $\bullet \to \bullet$ respects Beck-Chevalley squares in the diagram in Examples 12.2 and 12.5.

**Proof.** Each generator flips a single $\bullet$ to $\bigcirc$ or $\bullet$. We have to examine the square formed when we take two such generators acting on different copies of $\bullet$. If the two generators flip to the same color, this square is either preserved or completely flipped, so there is no Beck-Chevalley condition to check. When the colors are different but the two grey vertices belong to different gray edges, we get a diagrammatic external product of two maps, which is a Beck-Chevalley square in $S$ by assumption.

When the two $\bullet$s are in the same gray edge, we may without loss of generality assume that the one on the left is flipped to $\bigcirc$ and the one on the right is flipped to $\bullet$ (by inverting all colors if necessary). If the letter after the second $\bullet$ is a $\bigcirc$, the resulting square has two parallel isomorphisms and the condition is automatically satisfied. The remaining cases give the squares (with $i < j < k$)

\[
\begin{align*}
\begin{array}{c}
B_i \\
\downarrow f_i \\
\downarrow id
\end{array} & \quad \begin{array}{c}
B_i \times B_j \\
\downarrow (id, f_i) \\
\downarrow (id, f_i) \times id
\end{array} & \quad \begin{array}{c}
B_i \\
\downarrow id \\
\downarrow id
\end{array} & \quad \begin{array}{c}
B_i \times B_j \\
\downarrow (id, f_i) \\
\downarrow (id, f_i)
\end{array} & \quad \begin{array}{c}
\Pi_2 \\
\end{array}
\end{align*}
\]

which are Beck-Chevalley squares by the assumption (5.3).

12.2. **The labeled product category.** Before we extend this to a diagram on all of $T\mathcal{G}^{op}$, we observe that the above examples need a little more data attached to them. For
instance, one of the above diagrams contains the map id × Δ: B^2 → B^3. This gives us a pullback functor

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$$

for each fibration \(\mathcal{C}\) over \(S\). However we will also use the pullback functor

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}.$$ 

This requires us to remember not just the map \(B^2 \to B^3\), but also the data of how this map is a product of two simpler maps.

If \(S\) is a category, the objects of the labeled product category \(L(S)\) are pairs \((T, \{A_t\}_{t \in T})\) consisting of a finite set \(T\) and a collection of objects of \(S\) indexed by \(T\). A map

$$f: (T, \{A_t\}_{t \in T}) \to (U, \{B_u\}_{u \in U})$$

is a map of sets \(p_f: U \to T\) and, for each \(u \in p^{-1}(t)\), a map \(A_t \to B_u\). We abbreviate the object \((T, \{A_t\}_{t \in T})\) by \(\prod_t A_t\) and abbreviate \(\prod_{u \in p^{-1}(t)} B_u\) by \(\prod_{u \to t} B_u\).

The labeled product category \(L(S)\) has the same objects as the category of thick products \(S^\Pi\) from Section 8.3. This extends to a forgetful functor \(L(S) \to S^\Pi\), which is generally not an equivalence. For instance when \(S = \ast\) is the one-point category, \(S^\Pi\) is contractible but \(L(S) \simeq Fin^{op}\).

The category \(L(S)\) always has finite products, even if \(S\) does not, and they are given by concatenating the sets \(T\). If \(S\) has finite products, we can re-interpret the morphisms in \(L(S)\) as maps

$$A_t \to \prod_{u \to t} B_u,$$

where each \(B_u\) appears in exactly one such product.

If \(S\) has finite products and Beck-Chevalley squares, we say that a commuting square in \(L(S)\)

$$\begin{array}{ccc}
\prod_t A_t & \xrightarrow{\prod_U B_u} & \prod_U B_u \\
\downarrow & & \downarrow \\
\prod_V C_v & \xrightarrow{\prod_W D_w} & \prod_W D_w
\end{array}$$

is **Beck-Chevalley square** if for each \(t \in T\), the square of maps in \(S\)

$$\begin{array}{ccc}
A_t & \xrightarrow{\prod_{u \to t} B_u} & \prod_{u \to t} B_u \\
\downarrow & & \downarrow \\
\prod_{v \to t} C_v & \xrightarrow{\prod_{w \to t} D_w} & \prod_{w \to t} D_w
\end{array}$$

is a Beck-Chevalley square in \(S\). We emphasize that these squares do **not** have to be pullback squares in \(L(S)\). That would amount to the additional condition that \(T\) is the pushout of \(U\) and \(V\) along \(W\).\(^4\)

Note that a diagrammatic external product of two maps in \(L(S)\) is always a Beck-Chevalley square, as is a product of any Beck-Chevalley square with a single fixed object.

Using the gray edges functor \((\circ \leftarrow \bullet \to \bullet)^{IV(\alpha)} \to Fin^{op}\) as the functor to finite sets, the diagram \(D_\alpha\) in (12.7) lifts to a functor

$$\begin{array}{c}
(\circ \leftarrow \bullet \to \bullet)^{IV(\alpha)} \to L(S).
\end{array}$$

The proof of Lemma 12.8 shows the rotation above respects Beck-Chevalley in \(L(S)\), not just in \(S\). This will allow us to restrict to consistent choices of subsets of the products

\(^4\)This condition holds in the examples above, but it will fail when we pass to the entire category \(T^\Pi^{op}\).
that appear in the above examples, and conclude that the rotation still respects Beck-
Chevalley.

**Proposition 12.10.** The functor in (12.9) extends to a commuting diagram

\[ D : T'\mathcal{G}^{\text{op}} \rightarrow \mathcal{L}(S). \]

The rotation of \( T'\mathcal{G}^{\text{op}} \) that flips every \( \bullet \rightarrow \bullet \) respects Beck-Chevalley squares in \( D \).

**Proof.** The underlying functor to finite sets is the gray edges functor \( T'\mathcal{G} \rightarrow \text{fin} \). To extend this to a diagram

\[ T'\mathcal{G}^{\text{op}} = \mathcal{E}'\mathcal{G}^{\text{op}} \times^{\text{op}} (\circ \rightarrow \bullet)^{IV(a)} \rightarrow \mathcal{L}(S), \]

we first define a presentation of \( T'\mathcal{G}^{\text{op}} \). On \( (\circ \rightarrow \bullet)^{IV(a)} \) we use the product presentation. On \( \mathcal{E}'\mathcal{G}^{\text{op}} \) we use the presentation that has a generator for every morphism, and the relations \( \alpha \overset{h}{\rightarrow} \beta \overset{j}{\rightarrow} \gamma = \alpha \overset{h \circ j}{\rightarrow} \gamma \) and \( \alpha \overset{id_a}{\rightarrow} \alpha = \alpha \). Then the dual of Lemma 9.15 defines a presentation of the op-Grothendieck construction.

With this presentation, it is enough to assign each \((\alpha, c)\) to a labeled product indexed by \( \mathcal{G}(c) \), give the maps for the horizontal and vertical generators in a way that agrees with the gray edges functor, and then check the horizontal relations, vertical relations, and swap relations.

For the objects, vertical generators, and vertical relations, the diagram is \( D_a \) from (12.9). After composing with the functor to sets this functor agrees with the gray edges functor \( (\circ \rightarrow \bullet)^{IV(a)} \rightarrow \text{fin}^{\text{op}} \) from (11.8).

Each horizontal generator of \( T'\mathcal{G}^{\text{op}} \) is a backwards map \((\alpha, c) \leftarrow (\beta, h^*c)\) with \( h : \alpha \leftarrow \beta \) an arrow in \( E\mathcal{G} \) from \( \beta \) to \( \alpha \). We have to choose a morphism \( D_a(c) \rightarrow D_\beta(h^*c) \) in \( \mathcal{L}(S) \) whose map of sets \( \mathcal{G}(c) \leftarrow \mathcal{G}(h^*c) \) is induced by the gray edges functor. If \( a = (G, B_-) \) so that \( B_- \) denotes the edge labels, then for each point \( s \in \mathcal{G}(h^*c) \) with image \( t \in \mathcal{G}(c) \), we take the map in \( S \)

\[
B_{[t]} \xrightarrow{\cong} \prod_{t \in h^{-1}(s)} B_{[t]} \rightarrow B_{[s]}.
\]

That is, we take the composite of the product identification for \( h \) on the leading edge \([t]\) of \( t \), and the projection of the product onto one of its factors. We have \( h([s]) = [t] \) because the morphism of graphs is color-preserving.

We have a horizontal relation for every pair of composable horizontal generators. So, if we have three graphs \( \alpha \overset{h}{\rightarrow} \beta \overset{j}{\rightarrow} \gamma \) and a coloring \( c \) of \( \alpha \) that is pulled back to \( \beta \) and \( \gamma \), the gray edges functor gives maps of sets

\[
\mathcal{G}(c) \leftarrow \mathcal{G}(h^*c) \leftarrow \mathcal{G}(j^*h^*c).
\]

For each \( s \in \mathcal{G}(j^*h^*c) \), the two composites we wish to compare are equal by the following commuting diagram.
We also have a horizontal relation for every graph \( \alpha \). For each coloring \( c \) of \( \alpha \), the identity map \( \alpha \to \alpha \) induces the identity \( \mathcal{G}(c) \to \mathcal{G}(c) \), and the formula (12.11) induces the identity map \( B_{[s]} \to B_{[s]} \) for every \( s \in \mathcal{G}(c) \), so we get the identity map in \( \mathbf{L}(\mathcal{S}) \), as desired.

Finally, each swap relation in \( T\mathcal{G}^{\text{op}} \) is given by a backwards map of graphs \( h : \alpha \to \beta \) and pair of colorings \( c \to d \) of \( \alpha \), meaning that \( c \) is more gray than \( d \). Note that the resulting commuting square of finite sets

\[
\begin{array}{ccc}
\mathcal{G}(c) & \xrightarrow{\cong} & \mathcal{G}(h^*c) \\
\downarrow & & \downarrow \\
\mathcal{G}(d) & \xleftarrow{\cong} & \mathcal{G}(h^*d)
\end{array}
\]

is not a pushout square in general – this is why we did not impose this condition on a square in \( \mathbf{L}(\mathcal{S}) \) for it to be a Beck-Chevalley square.

For each \( s \in \mathcal{G}(h^*d) \), let \( [s]_{h^*c} \) denote its representative and let \( [s]_{h^*d} \) denote the representative of the larger gray edge in the coloring \( h^*c \) that contains \( s \). Similarly for \( h(s) \in \mathcal{G}(d) \), let \( [h(s)]_d \) and \([h(s)]_c \) denote its representatives in \( d \) and \( c \), respectively.

The two routes we wish to compare fit in the diagram

\[
\begin{array}{ccc}
B_{[h(s)]_c} & \xrightarrow{\cong} & \prod_\mathcal{G}(h(t)=[h(s)]_c) B_t \\
\downarrow & & \downarrow \\
B_{[h(s)]_d} & \xrightarrow{\cong} & \prod_\mathcal{G}(h(t)=[h(s)]_d) B_t \\
\end{array}
\]

This diagram can be subdivided into \( n \) smaller diagrams of the same shape, one for each vertex between \( [s]_{h^*c} \) and \( [s]_{h^*d} \). Each of the smaller diagrams commutes, the left-hand square by (6.4) and the right-hand square because the middle vertical is defined to be a product in which the right vertical is one of the factors. (When \( n = 0 \), commutativity is trivial because the vertical maps are identity maps.)

It remains to check the Beck-Chevalley squares. For the horizontal relations this is trivial, and for the vertical relations this follows from Lemma 12.8 and Example 10.5iv. For a swap relation, we have a square of sets as in (12.12), and for each \( \sigma \in \mathcal{G}(c) \) the relevant square of maps is

\[
\begin{array}{ccc}
B_{[\sigma]} & \xrightarrow{\cong} & \prod_\mathcal{G}(h(t)=[\sigma]) B_t \\
\downarrow & & \downarrow \\
\prod_{[s]_c=[\sigma]} B_{[s]_d} & \xrightarrow{\cong} & \prod_{[s]_c=[\sigma]} \left( \prod_{h(t)=[s]_d} B_t \right)
\end{array}
\]

Since both horizontal maps are isomorphisms, this is always a Beck-Chevalley square. In particular it is a Beck-Chevalley square any time \( c \to d \) is a generator of the form \( \circ \to \bullet \) in \( (\circ \leftarrow \bullet \to \circ)_{IV(\alpha)} \).

12.3. **Diagrams of functors on partially darkened graphs.** Now we will build a coherent diagram of functors on the opposite of the gigantic graph category that, roughly, assigns maps of finite sets to tensor products, maps \( \circ \to \bullet \) to pullbacks, and maps \( \bullet \to \bullet \) to pushforwards.
For the purpose of understanding what is going on, the reader may find it helpful to keep in mind the subcategory of the gigantic graph category where the graph \( \alpha \) is the associator isomorphism graph in Figure 2.7b. Our goal is to make a diagram of functors and isomorphisms that contains the associator isomorphism diagram from Figure 5.6. The relevant maps of spaces were captured in \( D_\alpha \) from the previous section. As we can see in Figure 5.6, we need to be able to pull back and push forward along sub-products of these maps. The first part of this subsection accomplishes this by defining a thick diagram of pullback and pushforward functors on \( \rho((I \downarrow U)^{op}) \) for any bifibration \( \mathcal{C} \) over \( S \), diagram \( D : I^{op} \to L(S) \), and finite set \( U \). This culminates in Proposition 12.15. The second part extends this by allowing \( U \) to vary and adding in tensor products – this is done in Theorem 12.20.

Now we begin. Given a diagram \( D : I^{op} \to L(S) \), composing with the forgetful functor \( L(S) \to fin^{op} \) gives a functor \( I \to fin \) that we suggestively call \( \mathcal{G} \). For each finite set \( U \), we therefore have an over category \( (I \downarrow U) \). For each \( u \in U \), we define a diagram on this over category \( (I \downarrow U)^{op} \to L(S) \) that sends \( i \) to the labeled product \( \prod_{\mathcal{G}(i) \to U} A_s \). Taking the product of these diagrams over all \( u \in U \) gives a functor \( D(U) : (I \downarrow U)^{op} \to \prod_U L(S) \).

In essence, we have “split up the labeled products in \( D \) along \( U \).” The following is immediate from the definitions.

**Lemma 12.13.** Given a commuting square in \( (I \downarrow U)^{op} \), if \( D \) of its image in \( I^{op} \) is a Beck-Chevalley square in \( L(S) \), then \( D(U) \) of the square is a \( U \)-tuple of Beck-Chevalley squares in \( L(S) \).

Composing with the forgetful map, we have a functor \( \overline{D(U)} : (I \downarrow U)^{op} \to \prod_U S \). Lemma 12.13 applies equally well to \( \overline{D(U)} \).

If \( \mathcal{C} \) is a symmetric monoidal bifibration over \( S \), recall that Proposition 8.21 defines a smbf \( \prod_U \mathcal{C} \prod \) on \( \prod_U S \). Therefore the pullback

\[
(\overline{D(U)})^* \left( \prod_U \mathcal{C} \prod \right)
\]

is a bifibration on \( (I \downarrow U)^{op} \). Applying any rotation of \( (I \downarrow U)^{op} \) that respects Beck-Chevalley squares therefore defines a thick diagram of pullback and pushforward functors.

**Proposition 12.15.** If

\[
D : \left( E \mathcal{G} \times (\circ \to \bullet \leftarrow \bullet)_{IV(a)} \right)^{op} \to L(S)
\]

is the diagram of Proposition 12.10 and \( \rho \) is the rotation that flips the \( \bullet \to \bullet \) generators, the thick diagram of functors built on

\[
\rho \left[ \left( E \mathcal{G} \times (\circ \to \bullet \leftarrow \bullet)_{IV(a)} \right)^{op} \downarrow U \right] \equiv E \mathcal{G}^{op} \times^{op} \rho \left[ (\circ \to \bullet \leftarrow \bullet)_{IV(a)} \downarrow U \right]^{op}
\]

as in (12.14) is coherent.

Before we prove this proposition, we summarize the construction:
• each triple \((\alpha = (G,A), c, \mathcal{G}(c) \to U)\) is sent to the product of thick fibers
\[ \prod_{u \in U} \mathcal{G}(A_{\mathcal{G}(c)}). \]

• Each generating morphism is sent to a pullback or pushforward functor along
the corresponding map of products defined by the diagram from (12.14).

• The functors go in the same direction as the morphisms of \(E^{\mathcal{G}}\), and in the direc-
tion \(\circ \to \bullet \to \circ\) in the fiber categories.

• The generators in \(E^{\mathcal{G}}\) and \(\circ \to \bullet \) go to pullbacks, while the generators \(\bullet \to \circ\) go
to pushforwards.

Proof. First we verify the above rotation \(\rho\) respects Beck-Chevalley, in other words
the square associated to each relation polygon in \((\mathcal{I} \downarrow U)^{op}\) satisfies the statement of
Lemma 12.13. Since \(\rho\) arises from a rotation on \(\mathcal{I}^{op}\), by Lemma 12.13 it is enough to
show that \(\rho\) respects Beck-Chevalley on \(\mathcal{I}^{op}\). This was proven in Lemma 12.8.

Next we describe the presentation associated to the rotation \(\rho\). The presentation of
\[ T^{\mathcal{G}}^{op} = \left( E^{\mathcal{G}} \ltimes (\circ \to \bullet \to \bullet)^{IV(\alpha)} \right)^{op} \]
constructed in Proposition 12.10 gives a canonical presentation of the comma category
and therefore a presentation of
\[ E^{\mathcal{G}}^{op} \ltimes^{op} \left( (\circ \to \bullet \to \bullet)^{IV(\alpha)} \downarrow U \right)^{op}. \]

Tracing through the definitions, we get a generator for

1. Each morphism \(\alpha \leftarrow \beta\) of \(E^{\mathcal{G}}^{op}\) and object in the fiber category

\[ \left( (\circ \to \bullet \to \bullet)^{IV(\alpha)} \downarrow U \right)^{op}, \]

2. Each object \(\alpha \in \text{ob} E^{\mathcal{G}}^{op}\) and generator of the fiber category.

• There are two kinds of generators in the fiber category, corresponding to the
two color changes, so we get three classes of relations in total.

We have one relation for

1. Each pair of composable morphisms in \(E^{\mathcal{G}}^{op}\) and object of the fiber category.

• Note the morphisms in \(E^{\mathcal{G}}^{op}\) are change of graphs.

2. Each object in \(E^{\mathcal{G}}^{op}\) and object of the fiber category.

3. Each object of \(E^{\mathcal{G}}^{op}\) and relation in the fiber category.

• There are three kinds of relations, corresponding to pairs of color changes.

4. Each morphism \(\alpha \leftarrow \beta\) in \(E^{\mathcal{G}}^{op}\) and generator in the fiber category over \(\alpha\).

• Again, there are two kinds of generators in the fiber category, so we list these
separately in Table 12.18.

See Table 12.16 for depictions of the generators and Tables 12.17 and 12.18 for the re-
lations. This presentation therefore coincides with the one we get by first passing to
a presentation of the comma categories, then taking the op-Grothendieck construction.
Therefore we can use the variant of Theorem 9.18 described in Remark 9.19 to prove
coherence.

For vertical coherence, we fix an \(\alpha\) and examine the fiber category

\[ \left( (\circ \to \bullet \to \bullet)^{IV(\alpha)} \downarrow U \right)^{op}. \]

It suffices to take an arbitrary morphism in this category and prove coherence on its fac-
torizations. By Corollary 11.14, this category is shaped like a product, and its rotation
Table 12.16. Functors on the opposite of the giant graph category.
(Morphisms are drawn left to right, so that morphisms in the giant graph category go right to left.)

is a product rotation. Therefore its rotated diagram of functors is coherent by Proposition 10.8. Note that by Lemma 10.7, each vertical relation is a square of the form **, *, or !.

For horizontal coherence, we fix α and a coloring c of α, and examine the resulting diagram of functors on the comma category of all graphs α ← β. Since none of these morphisms are flipped by ρ, every functor in the diagram is a pullback, and the isomorphisms between them are canonical isomorphisms of compositions of pullbacks by Lemma 10.7. So these are coherent by Corollary 7.9.

Using Lemma 10.7, if the vertical generator is ○ → • each swap relation goes to a square ** followed by several composition isomorphisms for *. If the vertical generator is • → • it goes to a Beck-Chevalley square *! followed by several composition isomorphisms for !. (The composition isomorphisms arise because a pushforward of a vertical generator is a composite of several vertical generators.)

For swaps and vertical relations, we observe that in a vertical *! relation, each generator pushes forward along α ← β to some composite of several generators, but each of the two pullbacks go to the same number of pullbacks, and each of the two pushforwards
| Relation | Natural Isomorphism |
|----------|-------------------|
| (R1) \((G, A) \xrightarrow{h} (H, B) \xrightarrow{j} (I, C)\) | canonical iso of pullbacks \((**)*\) |
| \(c \xrightarrow{h} h^* c \leftarrow j^* h^* c\) | \(\prod_{u \in U} C[I_{-u}]^{\bigotimes B[u]}\) |
| \(\mathcal{G}(c) \leftarrow \mathcal{G}(h^* c) \leftarrow \mathcal{G}(j^* h^* c)\) | \(\prod_{u \in U} C[I_{-u}]^{\bigotimes C[u]}\) |
| \(\downarrow U \quad \downarrow U \quad \downarrow U\) | \(s \in \mathcal{G}(c), \ t \in \mathcal{G}(h^* c), \ q \in \mathcal{G}(j^* h^* c)\) |
| (R2) | canonical iso of pullbacks |
| \((G, A) \xrightarrow{id} (G, A)\) | \(\prod_{u \in U} C[I_{-u}]^{\bigotimes A[u]} \leftarrow \prod_{u \in U} C[I_{-u}]^{\bigotimes A[u]}\) |
| \(c \leftarrow c\) | \(s \in \mathcal{G}(c)\) |
| \(\downarrow U \quad \downarrow U\) | | |
| (R3) | canonical iso of pullbacks \((**)\) |
| \((G, A)\) | \(\prod_{u \in U} C[I_{-u}]^{\bigotimes A[u]} \leftarrow \prod_{u \in U} C[I_{-u}]^{\bigotimes A[u]}\) |
| \(c_1 \leftarrow c_2\) | \(q \in \mathcal{G}(c_1), \ r \in \mathcal{G}(c), \ s \in \mathcal{G}(c_1), \ t \in \mathcal{G}(c_2)\) |
| \(\downarrow U \quad \downarrow U\) | | |
| (R3) | Beck-Chevalley \((*)\) |
| \((G, A)\) | \(\prod_{u \in U} C[I_{-u}]^{\bigotimes A[u]} \leftarrow \prod_{u \in U} C[I_{-u}]^{\bigotimes A[u]}\) |
| \(c_1 \leftarrow c_2\) | \(q \in \mathcal{G}(c_1), \ r \in \mathcal{G}(c_2), \ s \in \mathcal{G}(c_1), \ t \in \mathcal{G}(c_2)\) |
| \(\downarrow U \quad \downarrow U\) | | |
| (R3) | canonical iso of pushforwards \((!!)\) |
| \((G, A)\) | \(\prod_{u \in U} C[I_{-u}]^{\bigotimes A[u]} \leftarrow \prod_{u \in U} C[I_{-u}]^{\bigotimes A[u]}\) |
| \(c_2 \leftarrow c_1\) | \(q \in \mathcal{G}(c_2), \ r \in \mathcal{G}(c_1), \ s \in \mathcal{G}(c), \ t \in \mathcal{G}(c_1)\) |
| \(\downarrow U \quad \downarrow U\) | | |

Table 12.17. The non-swap relation isomorphisms used in Proposition 12.15. (Morphisms in the op category are drawn left to right and down to up.)
go to the same number of pushforwards. This allows us to express the relation between the composites in $\beta$ as a large square made up of several Beck-Chevalley squares glued together. By the Pasting Lemma 8.8 and induction, when this square is appended with composition isomorphisms, we get Beck-Chevalley for the resulting square of four maps. This observation, together with the $**!!$ cube from Proposition 8.14, is all we need to prove coherence of swaps with this vertical relation. The vertical relations $$ and $$ are similar but easier, using only $$ and coherence of pullbacks or $$ and coherence of pushforwards.

For swaps and horizontal relations, if the vertical generator is a pullback then all functors are pullbacks and coherence is by Corollary 7.9. If the horizontal relation is a unit isomorphism then the coherence becomes trivial by modeling $id^*$ by a strict identity, as in Corollary 8.29. The only remaining case is when the vertical generator is a pushforward and the horizontal relation is a composition isomorphism. In this case, we are checking coherence along a triangular prism in which the triangles are isomorphisms of pullbacks, two of the square faces are Beck-Chevalley maps, the third square face is several Beck-Chevalley maps pasted together, and two of the edges between the square faces are additionally fattened up with an isomorphism of pushforward functors. Using the pasting lemma twice, we replace the third square face with a single Beck-Chevalley map, then glue it to the previous square face. After this operation is performed the coherence is trivial.

\[\square\]

Remark 12.19. Suppose that $\alpha \rightarrow \beta$ is a map in $Eg^{op}$ that is the identity on the underlying graph, and every edge identification map is an identity map in $S$. (In other words, $\alpha$
is obtained from $\beta$ merely by darkening vertices.) Then for any coloring $c$ of $\alpha$, the pullback functor that Proposition 12.15 associates to $(\alpha, c) \to (\beta, h^*c)$ goes between identical categories, and contains the identity map.

**Theorem 12.20.** The diagram in Proposition 12.15 extends to a coherent thick diagram of functors on the opposite of the gigantic graph category

$$\mathcal{F}\mathcal{I}\mathcal{N} \times \mathcal{E}\mathcal{G} \times \rho \left[ (\bigcirc \to \bullet \leftarrow \bullet) \downarrow IV(\alpha) \downarrow U \right].$$

Again, this means the functors go in the same direction as the maps in $\mathcal{F}\mathcal{I}\mathcal{N}$ and $\mathcal{E}\mathcal{G}$, and in the direction $\bigcirc \to \bullet \leftarrow \bullet$.

**Proof.** Since the opposite of this category is a Grothendieck construction

$$\mathcal{F}\mathcal{I}\mathcal{N}^{\text{op}} \times \left( \mathcal{E}\mathcal{G}^{\text{op}} \times \rho \left[ (\bigcirc \to \bullet \leftarrow \bullet) \downarrow IV(\alpha) \downarrow U \right] \right)^{\text{op}},$$

we can use Theorem 9.18. We have a presentation of the fiber categories from Proposition 12.15. We present $\mathcal{F}\mathcal{I}\mathcal{N}$ with one generator for each morphism and one relation for each composable pair and for each object. For each finite set $U$, the categories, vertical generators and relations, and vertical coherence are as in Proposition 12.15.

We must specify a horizontal generator for

(G3) each map $V \leftarrow U$ and object $(\alpha, c, \mathcal{G}(c) \to U)$ of the fiber category over $U$, and a horizontal relation for

(R5) each pair of composable maps $W \leftarrow V \leftarrow U$ and each $(\alpha, c, \mathcal{G}(c) \to U)$,

(R6) each $U$ and each $(\alpha, c, \mathcal{G}(c) \to U)$, and

(R7) for every map of finite sets and generator in the fiber category.

- There are three generators in the fiber category. Two are color changes of vertices and the last is a change of graph.

See Table 12.16 for a depiction of the generator and Table 12.21 for the relations. In each of these cases, we regard every finite set, map of such, and relation of such as taking place in the category of finite sets under $T = \mathcal{G}(c)$, in other words $(\mathcal{G}(c) \downarrow \mathcal{F}\mathcal{I}\mathcal{N})^{\text{op}}$. The unbiased tensor products $\boxtimes$ of Proposition 8.21 define a thick indexed category on $(\mathcal{G}(c) \downarrow \mathcal{F}\mathcal{I}\mathcal{N})^{\text{op}}$, so along this identification they give a functor for each horizontal generator, and an isomorphism for each horizontal relation, satisfying the coherence condition.

Each vertical generator gives a forwards or backwards map in $L(S)$, which we write as

$$\prod_{s \in \mathcal{G}(c)} A_s \rightarrow \prod_{t \in \mathcal{G}(d)} B_t,$$

and maps of sets $\mathcal{G}(d) \to \mathcal{G}(c) \to U$. The pullback of this generator along $V \leftarrow U$ is vertical generator with the same map in $L(S)$, but with the maps of sets $\mathcal{G}(d) \to \mathcal{G}(c) \to U \rightarrow V$. So the square we must fill with the swap relation is

$$\prod_{v \in V} \mathcal{G}\left( \prod_{t \in \mathcal{G}(d)} B_t \right) \leftarrow \prod_{u \in U} \mathcal{G}\left( \prod_{t \in \mathcal{G}(d)} B_t \right)$$

We fill each of these with the canonical isomorphism arising from the fact that

$$\boxtimes : \prod_{V} \mathcal{G} \rightarrow \prod_{V} \mathcal{G}.$$
| Relation | Natural Isomorphism |
|----------|---------------------|
| (R5)     | $$(G,A) \in \mathfrak{S}(c)$$ $W \leftarrow V \leftarrow U$$ canonical iso of unbiased tensor products $$\bigotimes_{v \in V} \bigotimes_{u \in U} A_{u,v}$$ $s \in \mathfrak{S}(c)$ |
| (R6)     | $$(G,A) \in \mathfrak{S}(c)$$ $U \xrightarrow{id} U$ canonical iso of unbiased tensor products $$\prod_{u \in U} \bigotimes_{v \in V} A_{u,v} \cong \prod_{u \in U} \bigotimes_{v \in V} A_{u,v}$$ $s \in \mathfrak{S}(c)$ |
| (R7)     | $$(G,A) \xrightarrow{h} (H,B)$$ $c \xrightarrow{h^* c} \mathfrak{S}(h^* c)$$ $V \leftarrow U \leftarrow U$ tensor-pullback iso $$\bigotimes_{v \in V} \bigotimes_{u \in U} A_{u,v} \cong \bigotimes_{v \in V} \bigotimes_{u \in U} A_{u,v}$$ $s \in \mathfrak{S}(c), t \in \mathfrak{S}(h^* c)$ |
| (R7)     | $$(G,A) \equiv (G,A)$$ $g \xrightarrow{c} \mathfrak{S}(g) \xrightarrow{\mathfrak{S}(c)} U \leftarrow U$ tensor-pushback iso $$\bigotimes_{v \in V} \bigotimes_{u \in U} A_{u,v} \cong \bigotimes_{v \in V} \bigotimes_{u \in U} A_{u,v}$$ $s \in \mathfrak{S}(g), t \in \mathfrak{S}(c)$ |
| (R7)     | $$(G,A) = (G,A)$$ $d \xrightarrow{g} \mathfrak{S}(d) \xrightarrow{\mathfrak{S}(g)} U \leftarrow U$ tensor-pushforward iso $$\bigotimes_{v \in V} \bigotimes_{u \in U} A_{u,v} \cong \bigotimes_{v \in V} \bigotimes_{u \in U} A_{u,v}$$ $t \in \mathfrak{S}(d), s \in \mathfrak{S}(g)$ |

Table 12.21. The remaining relation isomorphisms introduced in Theorem 12.20
preserves tuples of (co)cartesian arrows. Of course, as before this is a thick isomorphism of thick functors.

Next we check coherence of swaps with horizontal relations. For the above vertical generator and the identity map of \( U \), we get a coherence that is trivial if we model the tensor product along id\( U \) by the strict identity. For the above vertical generator and a pair of maps \( W \to V \to U \), we get a triangular prism of the form \(* \boxtimes \) or \(! \boxtimes !\) from Figure 8.26.

Because the pullback of every vertical generator or relation is another vertical generator or relation of the same shape, the coherence of swaps with vertical relations is much more straightforward. The vertical relations here consist of all the relations (horizontal and vertical) appearing in the proof of Proposition 12.15. The relation arising from an object of \( E G \) and an object in \((\circ \to \bullet \to \bullet) IV(a)\) is trivial by the same trick of picking the identity. Each remaining vertical relation arises by a rotation of a \( U \)-tuple of diagrams in \( S^I \) of the same shape. This shape subdivides into triangles of pullback or pushforward maps, and Beck-Chevalley squares \(!!\). This reduces the coherence to \(*!!\) from Proposition 8.27, along with the variants of \(*!!\) and \(!!!\) where the pullback/pushforward maps form a triangle instead of a square. These in turn follow from the square variants by picking one of the maps of the square to be an identity map.

In summary, this coherent diagram specifies a thick functor for each generator, of which there are 4 kinds, and a natural isomorphism for each relation, of which there are 12 kinds. These are summarized in Tables 12.16 to 12.18 and 12.21.

13. Proof of the String Diagram Calculus

By combining Propositions 7.8 and 9.4, we can think of the coherent diagram of functors from Theorem 12.20 as a Grothendieck fibration over the opposite of the giant graph category. Pull this fibration back along the op of the map of Theorem 11.11

\[
E G^{op} \to \int_{G^{op}} \times \left( \left( \left( \left( E G^{op} \to \text{op} \rho \left[ \left( \circ \to \bullet \to \bullet \right) IV(a) \downarrow U \right] \right) \right) \right) \right)
\]

to get a fibration over \( E G^{op} \), which we call \( E G \). This is the fibration we will use for Theorem 6.9. Note that the pullback functors go in the same direction as the morphisms of \( E G \).

Proof of Theorem 6.9. The construction in the proof of Theorem 11.11 sends each graph \((G,A)\) to the object of the giant graph category given by \((a, \phi, \mathfrak{G}(\phi) \to \pi_0 \Psi(a))\), and each morphism \((G,A) \to (H,B)\) in \( E G^{op} \) to the composite of the four maps

\[
(G,A) \xleftarrow{h} (H,B) \xrightarrow{\phi} (H,B) \xrightarrow{\varphi} (H,B) \xrightarrow{h^* \phi} (H,B)
\]

\[
\mathfrak{G}(\phi) \xleftarrow{\mathfrak{G}(h^* \phi)} \mathfrak{G}(g^* \phi, h^* \phi) \xrightarrow{\mathfrak{G}(\phi)} \mathfrak{G}(\phi) \xleftarrow{\mathfrak{G}(\phi)} \mathfrak{G}(\phi)
\]

\[
\pi_0 \Psi(G,A) \Rightarrow \pi_0 \Psi(G,A) \Rightarrow \pi_0 \Psi(G,A) \Rightarrow \pi_0 \Psi(H,B).
\]

The first and last maps are generators. The second and third are in general composites of generators, one for each white vertex of \((H,B)\) that is sent to a black vertex in \((G,A)\).

i. (Value on objects) By the definitions in Proposition 12.15 and Theorem 12.20, each labeled graph \( a = (G,A) \) is sent to a product of thick fiber categories, one for each element of \( U = \pi_0 \Psi(a) \). The objects in the products are the representative
edge labels for the gray edges of \((\alpha, \emptyset)\), which are precisely the edges of the constellation \(\Psi(\alpha)\). The map \(\mathcal{G}(\emptyset) \to U\) sorts these edges by their components, so we get the product

\[
\prod_{u \in \pi_0 \Psi(G,A)} C(\prod_{e \in E(u)} A_e).
\]

The desired product of thin fiber categories lies inside.

**ii.** (Inert morphisms) Each of the three kinds of inert morphism \(\alpha \leftarrow \beta\) induces a bijection on all the horizontal maps of sets in (13.1). The associated maps of spaces are bijections on the underlying set, along which the tuples of spaces are identical. Pulling back or pushing forward along such an isomorphism, or tensoring along a bijection of finite sets, is modeled by the evident isomorphism of products of thick fibers that changes the sets \(E(-)\) and \(\pi_0 \Psi(G,A)\) by a bijection. It is straightforward to check that these bijections respect composition in \(E\mathcal{G}\). Furthermore in each of the 12 classes of relation, if every map of spaces is this kind of isomorphism of thick products, and we model each functor as this canonical isomorphism of categories, then the relation isomorphism is also the identity.

**iii.** (Locality) First we observe that the value of our diagram on any object of the gigantic graph category is determined by the sets \(G(c)\) and \(U\), and the \(U\)-tuple of thick products \(\prod_{s \to u} A_s\). Furthermore each relation, once we have decided which of the 12 classes of relation it is, depends only on the maps of these sets and the maps of thick products.

Suppose \(F : I \to E\mathcal{G}\) is a diagram and \(T = \{T(i)\}_{i \in \text{obj}}\) is a system of cut points as in Section 6.2. If we regard the edges of each cut graph as a subset of the edges of the original graph, and the black vertices as equivalence classes of black vertices from the original graph, which are joined together when we take the maximal cutting, then the two diagrams \(F\) and \(\Psi(F; T)\) send each object of \(I\) to two graphs that have identical values for \(U\), \(\mathcal{G}(\emptyset)\), and \(\prod_{s \to u} A_s\), hence they give the same category. Each morphism in \(I\) similarly gives the same maps of such sets and thick products, so (13.1) gives the same sequence of generators, which are sent to the same sequence of functors. For each relation, if we fix a composite of the 12 relations that bring us between the composites of the maps depicted in (13.1) for each generator, we see that both diagrams give the same natural isomorphism. This is enough to define an isomorphism of their associated indexed categories that is the identity at each object, hence an isomorphism between their associated fibrations that is the identity on each fiber category, hence they have the same pullback functors.

Similarly, if a diagram \(F\) is a disjoint union of diagrams \(F_k : I \to E\mathcal{G}\), then each occurrence of \(\mathcal{G}(c)\) or \(U\) is a disjoint union of the same sets for the diagrams \(F_k\). By the conventions in this section such disjoint unions are sent to products of categories, functors, and natural isomorphisms for each of the \(F_k\).

**iv.** (Ten operations) We start with the six operations in Figure 6.10, which are the maps \(\alpha \leftarrow \beta\) that are isomorphisms on the underlying graph. In this case the far-left morphism in (13.1) induces an identity functor on the associated categories and may be ignored. In the last four cases (Figures 6.10c to 6.10f) the three remaining generators in (13.1) go to a tensor product, pullback, and pushforward, that correspond precisely to the definition of \(\otimes_B\), \(U_B\), \(\begin{array}{c}A \\ \downarrow \end{array} \begin{array}{c}B \\ \downarrow \end{array}\), and \(\emptyset_B\). In the first two cases (Figures 6.10a and 6.10b) precisely one of the three remaining generators is non-trivial, and gives the desired pushforward or pullback.
The remaining four operations in Figure 6.11 are covered by the more general statement that any covering map $\alpha \twoheadrightarrow \beta$ induces a tensor product followed by pullback along the edge identification maps. In this case, the middle two generators in (13.1) are the ones that are trivial. The far-right generator induces the desired tensor product and the far-left generator induces the desired pullback.

(v) (Four isomorphisms) We start with the associator isomorphism. Our four morphisms in $E\mathcal{G}$ are sent under the rule in (13.1) to a composite of four generators each. As before, the far-left generator is an identity map and can safely be ignored for the purpose of computing the natural isomorphism between the two branches. This gives three generators each, corresponding to the twelve outside edges of the square in Figure 5.6. Next, notice that each of the nine squares in Figure 5.6 corresponds to one of the 12 relations in the gigantic graph category described in Tables 12.17, 12.18 and 12.21, except for the top-left square, which is a composite of two $\Diamond\Diamond$ relations. Therefore we can write a sequence of ten relations in the gigantic graph category that matches Figure 5.6. The four relations in the lower-right corner correspond to the pattern of partial darkenings in Figure 13.2a, where the morphisms are drawn in the same direction as the gigantic graph category. We verify in each region that the isomorphism provided by Theorem 12.20 matches the one used to define the associator, hence the relation isomorphism provided by the calculus is indeed the associator isomorphism.

For the shadow isomorphism the proof is almost the same, except that in Figure 5.8 we group together the categories separated by a $\simeq$ because they are in the same thick fiber. Then the top-left region is a canonical isomorphism between two models for $\otimes$ along the same map of sets. The remaining 4 regions correspond to 4 relations in the gigantic graph category, similar to the above. Therefore the relation isomorphism in our diagram is the desired shadow isomorphism.

For the right unit isomorphism, Figure 5.7 gives a triangle of functors but our morphisms in $E\mathcal{G}$ form a square, two sides of which are inert. We therefore extend the triangle to the square depicted in Figure 13.3. Now the proof proceeds as the earlier one for the associator isomorphism. (Note that even though the top-right region of Figure 13.3 consists only of identity functors, the map in the gigantic graph category we associate to each arrow is not an identity map. In particular, along the right-hand edge of the square, we are thinking of the first identity functor as a trivial tensor product, the second as a trivial pullback, and the third as a trivial pushforward.) The relevant diagram of partial darkenings is in Figure 13.2b.

For the base change composition isomorphism, we extend the triangle to the rectangle pictured in Figure 13.4. For the morphisms in $E\mathcal{G}$ drawn vertically,

**Figure 13.2. Partial darkenings**
the one on the right gives four generators in the gigantic graph category, the one on the left four morphisms, the middle two of which are composites of two generators. As before, the last one is trivial and may be ignored. In the horizontal direction, the last morphism corresponds to a collapsing map of graphs, not an isomorphism of graphs.

One might expect to use the pattern of partial darkenings in Figure 13.5a, however there is more than one square that requires multiple relations, because many of the arrows above are composites of generators. We therefore use the further subdivided pattern in Figure 13.5b. This gives a diagram of functors and isomorphisms arising from the giant graph category, which is like Figure 13.4 but the arrows are further subdivided. To prove the calculus provides the isomorphism we want, it suffices to connect each arrow of Figure 13.4 to a chain of arrows in this subdivided diagram by a composition isomorphism, and then prove that for each square of Figure 13.4 the resulting cube is coherent. (This is because along the outside edges of the rectangle, the “composition isomorphism” is the identity.)

In the first column of squares in Figure 13.4, we need two relations for the middle and bottom squares, stacked vertically. However the stacked composite of two \(*\otimes\) relations, together with \(*\) composition isomorphisms, is another \(*\otimes\) relation by the \(*\otimes\otimes\) coherence, and similarly for \(!\otimes\). This verifies the coherence condition in the first column.

In the second column in Figure 13.4, corresponding to the first column in Figure 13.5b, the coherence condition is easy to check on the middle square because all functors are pullbacks and all isomorphisms are canonical pullback isomorphisms. In the bottom square the coherence becomes the pasting lemma for Beck-Chevalley.
Figure 13.4. Extended base change composition isomorphism. Compare to Figure 5.9.

Figure 13.5. Pattern of partial darkenings for base change

In the third column of Figure 13.4, second column of Figure 13.5b, the middle square is trivial and the bottom square is populated by pushforwards and canonical isomorphisms of pushforwards, hence is coherent.

In the final column, middle square, the table of relations says that the calculus gives a \((**\) isomorphism followed by more canonical isomorphisms of pullbacks, so again our coherence is between canonical isomorphisms of pullbacks and therefore holds. In the bottom square the calculus gives a canonical isomorphism of pushforwards followed by Beck-Chevalley, but the isomorphism relating
this back to Figure 13.4 undoes the composition isomorphism, so we get the trivial coherence statement that the Beck-Chevalley map is equal to itself.

\[ \square \]

14. A LITTLE BIT OF Functoriality

A map of symmetric monoidal bifibrations \( H : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T) \) is a functor \( H_\circ : S \rightarrow T \) preserving products and Beck-Chevalley squares, and a strong symmetric monoidal functor \( H : \mathcal{C} \rightarrow \mathcal{D} \) that preserves cartesian and cocartesian arrows, such that the square

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{H} & \mathcal{D} \\
\pi_\mathcal{C} \downarrow & & \downarrow \pi_\mathcal{D} \\
S & \xrightarrow{H_\circ} & T
\end{array}
\]

strictly commutes and the isomorphism \( H(X \boxtimes Y) \rightarrow H(X \otimes Y) \) lies over the canonical map \( H_\circ(\mathcal{A}) \times H_\circ(\mathcal{B}) \equiv H_\circ(\mathcal{A} \times \mathcal{B}) \).

The purpose of this final section is to prove the following result. The ingredients we use form the building blocks for an extension of the string diagram calculus to smbf maps, but we will not pursue this further in the current paper.

**Theorem 14.1.** Each map of symmetric monoidal bifibrations \( H : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T) \) induces a strong shadow functor \( F : \mathcal{C}/S \rightarrow \mathcal{D}/T \), and a vertical isomorphism of pseudofunctors \( F \circ [\mathcal{C}] \approx [\mathcal{D}] \circ H_\circ \).

Recall that \([\mathcal{C}]\) is the base change pseudofunctor.

We need to define the terms in Theorem 14.1. A strong shadow functor \( F : \mathcal{C}/S \rightarrow \mathcal{D}/T \) is the appropriate notion of “homomorphism” for bicategories with a shadow. In detail, it consists of

- a function \( F_\circ : \text{ob} (\mathcal{C}/S) \rightarrow \text{ob} (\mathcal{D}/T) \),
- a functor \( F : \mathcal{C}/S (A, B) \rightarrow \mathcal{D}/T (F_\circ A, F_\circ B) \) for every pair of objects \( A, B \in \text{ob} \mathcal{C}/S \),
- a functor \( F : \mathcal{C}/S \rightarrow \mathcal{D}/T \),
- a natural isomorphism \( m_F : F(X) \otimes F(Y) \equiv F(X \otimes Y) \),
- a natural isomorphism \( s_F : \langle F(X) \rangle \equiv F\langle X \rangle \),
- a natural isomorphism \( i_F : U_{FA} \equiv F(U_A) \),

subject to the usual hexagon axiom for a pseudofunctor, a variant of this hexagon that uses the shadow instead of the associator, and the usual square axioms for the units. One can visualize these coherences by taking the two-dimensional diagrams in Figures 5.6 to 5.8 and adding an extra dimension that applies the functor \( F \) to each term.

A vertical natural isomorphism (or an isomorphism) between two pseudofunctors \( F, G : S \rightarrow \mathcal{D}/T \) consists of

- the condition that \( F = G \) on 0-cells, and
- a natural isomorphism \( F \equiv G \) on the morphism categories,

that satisfy a coherence for every pair of composable maps in \( S \) and a coherence for every object of \( S \). (This is the same as an invertible icon from [Lac10].) In this case \( S \) is a 1-category, so the isomorphism of base-change pseudofunctors \( F \circ [\mathcal{C}] \approx [\mathcal{D}] \circ H_\circ \) is an isomorphism \( \eta : F ([A, B]) \equiv [H_\circ A, H_\circ B] \) that commutes with the composition
and unit isomorphisms as illustrated below.

\[
\begin{align*}
F\left(\left[ A \xrightarrow{f} B \right]\right) \circ F\left(\left[ B \xrightarrow{g} C \right]\right) & \xrightarrow{\eta \circ \eta} \left[ \eta_{B}A \xrightarrow{\eta_{B}(f)} H_{B}B \right] \circ \left[ H_{B}B \xrightarrow{H_{B}(g)} H_{B}C \right] \\
& \xrightarrow{m_{f}} F\left(\left[ A \xrightarrow{f} B \right]\right) \circ \left[ B \xrightarrow{g} C \right] \\
& \xrightarrow{m_{i}l} \left[ H_{B}A \xrightarrow{H_{B}(g \circ f)} H_{B}C \right] \\
F\left(\left[ A \xrightarrow{g \circ f} C \right]\right) & \xrightarrow{\eta} \left[ H_{B}A \xrightarrow{H_{B}(g \circ f)} H_{B}C \right]
\end{align*}
\]

\[
\begin{align*}
U_{H_{B}A} & \xrightarrow{i_{F}} U_{H_{A}} \\
F(U_{A}) & \xrightarrow{i_{i}l} \left[ H_{B}A \xrightarrow{H_{B}(A)} H_{B}A \right] \\
F\left(\left[ A \xrightarrow{id} A \right]\right) & \xrightarrow{\eta} \left[ H_{B}A \xrightarrow{H_{B}(id)} H_{B}A \right]
\end{align*}
\]

The first of these can be visualized by crossing Figure 5.9 with an interval representing \( F \), as before, and proving that the resulting triangular prism is coherent. The second is the statement that on an identity map, \( \eta \) is essentially the same as \( i_{F} \).

The proof of Theorem 14.1 will occupy the remainder of this section. Note that the special case of \( H_{B} = id \), without the base-change objects, has already appeared as [PS12, 5.2].

Suppose that \((H_{A}, H_{B})\) is a map of symmetric monoidal bifibrations. We define

- \( F_{B} = H_{B} \) on objects,
- \( F : \mathcal{C}_{\mathcal{B}}(A, B) \to \mathcal{P}(F_{B}A, F_{B}B) \) is the composite

\[
\mathcal{C}A \times B \xrightarrow{H} \mathcal{P}(A \times B) \xrightarrow{*} \mathcal{P}(A) \times \mathcal{P}(B)
\]

and

- the shadow functor is

\[
\mathcal{C} \xrightarrow{H} \mathcal{P}(\mathcal{C}) \xrightarrow{*} \mathcal{P}.
\]

The four desired isomorphisms \( m_{F}, i_{F}, s_{F}, \) and \( \eta \) factor into six parts each. The case of \( m_{F} \) is illustrated below, the others are analogous.

\[
\begin{align*}
\mathcal{C}A \times B \times \mathcal{C}B \times \mathcal{C}C & \xrightarrow{H} \mathcal{C}A \times B \times \mathcal{C}B \times \mathcal{C}C \\
& \xrightarrow{*} \mathcal{C}A \times \mathcal{C}B \times \mathcal{C}C \\
& \xrightarrow{!} \mathcal{C}A \times \mathcal{C}B \times \mathcal{C}C
\end{align*}
\]

Three of these regions are familiar \(* \boxtimes, \boxtimes, \) and \(! \) isomorphisms. The other three interchange \( H \) with one of the three functors \( \boxtimes \) (or \( I \)), \( \boxtimes, \) or \( ! \). We call these isomorphisms \( H \boxtimes, H!, \) and \( H\boxtimes \). The first two arise by universal properties because \( H \) preserves \((co)\)cartesian arrows, the last is part of the symmetric monoidal structure on \( H \).
It remains to prove the coherence conditions for a strong shadow functor. The second condition for the vertical natural isomorphism is straightforward, because \( \eta \) has the same definition as \( i_F \) when the map \( f \) is the identity.

The remaining four coherences correspond to the four isomorphisms \( a, l, \theta, m \) depicted in Figures 5.6 to 5.9. Since these isomorphisms are defined using thick fibers, before we work with them we note the following.

**Lemma 14.2.** \( H \) induces an smbf map \( \mathcal{C} \prod \to \mathcal{D} \prod \).

**Proof.** The functor of base categories \( \mathcal{S} \prod \to \mathcal{T} \prod \) is obtained by keeping the set the same and applying \( H_b \) to each object of \( \mathcal{S} \) indexed by the set. For the total category, on objects we apply \( H \) and \( H_b \), and on morphisms we apply \( H \). This preserves (co)cartesian arrows because those are measured by their images in \( \mathcal{C} \) and \( \mathcal{D} \).

To give this functor a symmetric monoidal structure we use the given one for \( H \) in the \( \mathcal{D} \) coordinate and the canonical isomorphisms of coproducts of sets in the \( \mathcal{T} \prod \) coordinate. The coherence for these isomorphisms follows immediately from those for \( H \) and the universal property of the coproduct. \( \square \)

In particular, we can think of our definition of \( F \) as sitting inside the following larger operation on thick fiber categories.

\[
\mathcal{C}_{(\prod_{i=1}^2 A_i)} H \to \mathcal{D}_{(\prod_{i=1}^2 H_b(A_i))} \text{id}^{\ast} \to \mathcal{D}_{(\prod_{i=1}^2 H_b(A_i))}.
\]

Each of the desired coherences break apart into prisms, each of which is one each polygonal region in Figures 5.6 to 5.9 “times an interval” which applies \( H \). The following proposition therefore finishes the proof.

**Proposition 14.3.** Each polyhedron in Figures 14.4 to 14.8 is coherent.

**Proof.** ((\( H \ast \ast \) and \( H!! \))) These follow the same proof pattern as for \( \bullet \ast \ast \) and \( \bullet!! \) from Proposition 8.14 – all the isomorphisms are canonical isomorphisms of (co)cartesian arrows and are therefore coherent.

((\( Hc \) and \( Hu \))) These follow the same proof pattern as for \( \bigcirc c \) and \( \bigcirc u \) from Proposition 8.14. In essence, the earlier proof is a consequence of the fact that \( \bigcirc \) is a map of bifibrations.

((\( H\ast! \))) This follows from \( Hc, Hu, \) and \( H \ast \ast \).
((H ⊠ ⊠)) This follows from the coherence theorem for symmetric monoidal functors, see [ML98, XI.2].

((H ⊠ *) and ((H ⊠ !)) Without loss of generality V = * and U consists of either two points or zero points. If U has zero points, we can arrange so that only the front and back faces of the cube are not identity maps, and both give the isomorphism $H(I) \cong I$, hence the cube is coherent. So suppose U has two points. Of the six functors we need to compare, three define cartesian arrows that are “pullbacks” of the diagram on the left, the other
three give “pullbacks” of the diagram on the right.

\[
\begin{align*}
H(X) \boxtimes H(Y) &\quad
to
H_3(A) \times H_3(A') \quad
to
H_3(B) \times H_3(B') \\

H(X \boxtimes Y) &\quad
to
H_3(A \times A') \quad
to
H_3(B \times B')
\end{align*}
\]

These two pullback diagrams are isomorphic using the symmetric monoidal structure on \(H\). Coherence follows once we check that each of the six isomorphisms between our functors is a canonical isomorphism of pullbacks, either over the identity of one of the two diagrams or over the chosen isomorphism between them – this is where we use the assumption that \(H(X) \boxtimes H(Y) \to H(X \boxtimes Y)\) lies over \(H_3(A) \times H_3(B) \cong H_3(A \times B)\). The proof of \((H \boxtimes !)\) is dual. 

This finishes the proof of Theorem 14.1.

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E-mail address: malkiewich@math.binghamton.edu

BINGHAMTON UNIVERSITY, PO BOX 6000, BINGHAMTON, NY 13902

E-mail address: kate.ponto@uky.edu

UNIVERSITY OF KENTUCKY, 719 Patterson Office Tower, LEXINGTON, KY 40506