A General Framework for a Class of Quarrels: The Quarrelling Paradox Revisited

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Abstract
If a measure of voting power assigns greater voting power to a player because it no longer effectively cooperates with another, then the measure displays the quarrelling paradox and violates the quarrel postulate. We provide formal criteria by which to judge whether a given conception of quarrelling is (a) reasonable and (b) fit to serve as the basis for a reasonable quarrel postulate. To achieve this, we formalize a general framework distinguishing between three degrees of quarrelling (weak, strong, cataclysmic), symmetric vs. asymmetrical quarrels, and reciprocal vs. non-reciprocal quarrels, and which thereby yields twelve conceptions of quarrelling, which encompasses the two conceptions proposed by Felsenthal and Machover and by Laruelle and Valenciano, respectively. We argue that the two existing formulations of the quarrel postulate based on these conceptions are unreasonable. In contrast, we prove that the symmetric, weak conception of quarrelling identified by our framework – whether reciprocal or not – is fit to serve as the basis for a reasonable quarrel postulate. Furthermore, the classic Shapley-Shubik index and Penrose-Banzhaf measure both satisfy the quarrel postulate based on a symmetric weak quarrel.

Keywords: voting power; quarrel paradox; Penrose-Banzhaf; Shapley-Shubik

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1 Introduction

It would be paradoxical if ruling out effective cooperation between two voters to achieve a jointly favoured outcome were somehow to increase their voting power. Following Kilgour (1974), the exclusion of effective cooperation between players has come to be known as a quarrel and, following Brams (1975), a would-be increase in voting power due to quarrelling has come to be known as the quarrelling paradox. A measure of voting power that displays the quarrelling paradox, in turn, can be said to violate a quarrel postulate.

Both the status and nature of the quarrelling paradox have been a matter of controversy and confusion. Brams (1975: 181-82) himself concluded that two classic indices of voting power, the Shapley-Shubik index (Shapley and Shubik 1954) and the normalized Banzhaf index (Banzhaf 1965, 1966), display the paradox, but this did not disturb his confidence in either. Indeed, he thought it revealed an important truth about voting power: “Although one might suspect that they [the two quarrelling players] could only succeed in hurting each other, it is a curious fact that the quarrel...may actually redound to their benefit,” such that “there is an incentive for them to quarrel and increase their share of the voting power.” Barry (1980: 193), by contrast, viewed Brams’ conclusion as “manifestly absurd”; he took the indices’ apparent susceptibility to the quarrelling paradox obviously to disqualify them as measures of voting power.1

Yet Barry’s criticisms miss the mark. First, the Banzhaf index is a normalization of the Penrose-Banzhaf measure of voting power (Penrose 1946; cf. Felsenthal and Machover 1998, 2004). As Felsenthal and Machover (1998: 40-41) note, by rescaling the Penrose-Banzhaf measure such that all players’ scores sum to one, the normalized Banzhaf index does not purport to measure – as Barry supposed – the amount of power a voter has; it measures, rather, the player’s “relative share of total power.” The Shapley-Shubik index is also a relative index for which all players’ scores sum to one. Even if an index of players’ relative share of voting power were to violate the quarrel postulate, this would not necessarily speak against the index in question. Because one’s relative share depends not only on one’s own power, but also on how much power others have, and because a change that is detrimental to one player’s voting power may be even more detrimental to others, there is no reason in general to require such indices to satisfy a quarrel postulate: two quarrelling players may hurt themselves, but hurt others (who depend on their cooperation, for example) even more (Felsenthal and Machover 1998: 240-41). Thus if players care about their power relative to others, they may indeed, as Brams thought, have reasons to instigate a quarrel. By contrast, if it is reasonable to expect measures of voting power to satisfy a quarrel postulate, it would be reasonable to expect it of absolute measures such as the Penrose-Banzhaf measure (from which the Banzhaf index is ultimately derivative). Second, as we shall argue, previous formulations of the quarrel postulate have been based on conceptions of quarrelling not fit to serve as the basis for a reasonable quarrel postulate.

Our task here is twofold: first, to articulate criteria by which to judge whether a given conception of quarrelling furnishes a reasonable interpretation of the concept of quarrelling2 and second, to

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1For a more sympathetic yet still ambivalent view of Brams’s conclusion, see Straffin (1982: 278-281).
2For the distinction between conception and concept, see, e.g., Rawls (1999: 5).
articulate criteria by which to judge whether it would be reasonable to expect a measure of voting power to satisfy a quarrel postulate grounded in a given conception. We must therefore keep track of two questions: Is a conception a reasonable conception of the concept of quarrelling? And is the conception fit to serve as the basis for a reasonable quarrel postulate? These questions are distinct: a conception of quarrelling may itself be reasonable, but it may nevertheless be unreasonable to expect measures of voting power to satisfy a quarrel postulate constructed on its basis.

We can model the imposition of a quarrel in two ways. First, we could model it by transforming the original voting structure itself, imposing a constraint on the voting rule mapping each possible complete vote configuration or division onto outcomes. Second, we could model it by keeping the original voting structure intact but transforming the voting situation by imposing constraints on the voting behaviour of the players acting within the structure (Laruelle and Valenciano 2005a). In the former case we model quarrelling by reducing or wholly neutralizing the two players’ ability effectively to cooperate in the divisions in which they vote together. In the latter, by contrast, we model quarrelling by imposing a constraint on the probability distribution of divisions; for example, we reduce or set to zero the probability of divisions in which the two quarrelling players vote together. The first approach yields a class of structural quarrel postulates, based on a structural model of quarrelling, while the second approach yields a class of behavioural quarrel postulates, based on a behavioural model of quarrelling. Although the second approach, insofar as it models a quarrel as a disinclination on the part of the quarrelling players to vote together, is perhaps more intuitive, the first approach, which models a quarrel as the inefficacy of co-action between the quarrelling players, is required for evaluating the reasonability of measures of voting power under the a priori assumption of equiprobable divisions. The class of behavioural quarrel postulates, because based on a conception of quarrelling modelled by constraining division probabilities, is suitable only for measures of voting power not restricted to equiprobable divisions.

Our focus here is on measures of a priori voting power. A priori voting power is one’s voting power solely in virtue of the formal voting structure itself, that is, the sets of actors, their action profiles, and alternative outcomes on the agenda, as well as the decision function that maps divisions (i.e., combinations of actions) onto outcomes (Felsenthal and Machover 1998, 2003, 2004). It therefore abstracts from the power one might have in virtue of the distribution of preferences (and consequent incentives for strategic interaction) within the voting structure – an abstraction modelled by assuming equiprobable divisions. The question we address is the correct formulation of the structural quarrel postulate that any reasonable measure of a priori voting power ought to satisfy.

Our thesis is that the two existing formulations of the structural quarrel postulate, by Felsenthal and Machover (1998) and Laruelle and Valenciano (2005a), respectively, face decisive shortcomings. We accordingly propose a structural quarrel postulate for binary voting games, based on a new conception of quarrelling, which is suitable for measures of a priori voting power and overcomes these defects. The quarrel postulate concerns the comparison of a voter’s voting power in two voting games: the initial voting game and a second game derived from it by inducing a quarrel. Since we shall define this derivation or transformation only from initial voting games that are binary and monotonic, we begin, in Section 2, by defining such games. Then, in Section 3, we identify
what we take to be the essence of the concept of a quarrel and, in its light, specify the necessary
and sufficient set of conditions that any transformation function should satisfy in order to count
as inducing a quarrel. We follow this, in Section 4, with the set of conditions that a conception of
quarrelling should satisfy to be fit to serve as the basis for a reasonable quarrel postulate, and then
show, in Sections 5 and 6, that the two existing conceptions violate these conditions. In particular,
we show why a reasonable quarrel postulate must be grounded in a monotonic and symmetric
conception of quarrelling. To identify our new conception and postulate, we formalize in Section 7
a general framework consisting of twelve classes of quarrel that enable us to distinguish between
degree of quarrelling (weak, strong, and cataclysmic) as well as between symmetric versus
asymmetrical quarrels (Section 8), and between reciprocal and non-reciprocal quarrels (Section 9).
This framework is broad enough to encompass the conceptions of quarrelling formulated by both
Felsenthal and Machover (1998) and Laruelle and Valenciano (2005a). We summarize our typology
and findings in Section 10, concluding that the most general reasonable quarrel postulate is based
on a symmetric weak quarrel (whether reciprocal or not). Finally, we conclude by showing that the
Shapley-Shubik index and Penrose-Banzhaf measure both satisfy this quarrel postulate (despite
failing to satisfy the Felsenthal-Machover and Laruelle-Valenciano postulates).

2 Voting Games

Let \([n] = \{1, 2, \ldots, n\}\) be a finite set of players. In a binary voting game, each player has two
strategies (voting YES or NO), and there are two possible outcomes (YES or NO). In such a game, a
division \(S = (S, [n] \setminus S)\) of the set \([n]\) is then an ordered partition of players where the first element
in the ordered pair is the set of YES-voters and the second element is the set of NO-voters in \(S\). Thus, for \(S = (S, \overline{S})\), the subset \(S \subseteq [n]\) comprises the set of YES-voters and the subset \(\overline{S} = [n] \setminus S\)
comprises the set of NO-voters. (Note the convention of representing a bipartitioned division by its
first element in blackboard bold.) We define \(\mathcal{D}\) as the set of all possible divisions \(S\) of \([n]\). The
voting game then corresponds to a function \(G(S)\) mapping the set of all possible divisions \(\mathcal{D}\) to the
set of alternative outcomes \(O = \{YES, NO\}\).

We say that any player whose vote corresponds to the division outcome is a successful player,
and that any division with a YES-outcome is a winning division. Let \(\mathcal{W}\) be the collection of all sets
of players \(S\) such that \(G(S) = YES\), that is, if each member of \(S\) were to vote YES, they would be
successful YES-voters. \(\mathcal{W}\), the collection of YES-successful subsets of \([n]\), commonly called winning coalitions, provides an alternative representation of the voting game \(G\).

Although we shall not restrict our definition of voting power to monotonic voting games, we
shall, for reasons that will become apparent, define quarrels only over voting games that are initially
monotonic (i.e., prior to the imposition of the quarrel):

Monotonicity: \(\forall T \subseteq S \ T \in \mathcal{W} \implies S \in \mathcal{W}\).

Monotonicity states that if a division outcome is YES, then the outcome of any division in which
at least the same players vote YES will also be YES. The literature on voting power also typically
assumes that voting games satisfy non-triviality:

Non-Triviality: \(\exists S \in \mathcal{W}\) and \(\exists S \notin \mathcal{W}\).
Non-triviality states that not all divisions yield the same outcome. Non-triviality is typically assumed because a voting game that fails to satisfy it is of little interest for measures of voting power: such a game would effectively be a structure in which voters never make a difference to the outcome and therefore automatically one in which voters have no voting power at all. (It is effectively not a voting game.) A monotonic binary voting game that satisfies non-triviality is called a simple voting game. Monotonicity and non-triviality together ensure that simple voting games also satisfy unanimity:

Unanimity: \( [n] \in W \) and \( \emptyset \notin W \).

3 What is a Reasonable Conception of a Quarrel?

We seek to define a class of quarrels and quarrel postulates for quarrels imposed on initially monotonic binary voting games. Our ultimate task is to define a conception of quarrelling that is itself reasonable, on the one hand, and fit to serve as the basis for a reasonable quarrel postulate, on the other. A conception of quarrelling is reasonable insofar as it captures core intuitions about what the concept of a quarrel is. A quarrel postulate is reasonable in the sense that it would be reasonable to expect a measure of voting power to satisfy it. In this section, we analyze the concept of a quarrel, and specify the core property we assume a reasonable conception should satisfy. We turn to the quarrel postulate in the next section.

We understand a quarrel to arise between two players when they no longer effectively cooperate. Since we assume that quarrels are imposed on an initially monotonotonic binary game – which renders sincere voting a dominant strategy and hence rules out strategic voting – we use the term cooperation to refer to a circumstance in which two players agree in their votes, that is, they vote on the same side, yes or no. Recall that the behavioural approach models a quarrel by reducing the likelihood the quarrelling players agree in their votes. The structural approach we pursue here, however, models a quarrel by reducing or neutralizing the effectiveness of their voting together on the same side. A quarrel is therefore operative under this approach only when their votes do indeed agree; the quarrel consists of the fact that, when the players vote on the same side, their votes do not work together effectively to contribute to the outcome for which they vote. Thus the difference between the two approaches consists in how each models the reduction or neutralization of effective cooperation that constitutes quarrelling: whereas the behavioural approach reduces or eliminates cooperation, the structural approach reduces or eliminates effectiveness.

Since the structural approach to modelling quarrelling is less intuitive than the behavioural one, we here motivate the former approach by presenting a model that furnishes an intuitive interpretation. We can model structural conceptions by disaggregating cooperation and effectiveness into two distinct moments or stages in the voting procedure. In stage 1, the players vote; to vote on the same side is to cooperate. In stage 2, the players on the same side must then jointly present their side’s votes to the tabulator. This second stage is when a quarrel may break out (between players

\[ \text{See the doubts Felsenthal and Machover (1998: 238) raise about the reasonability of a structural quarrel postulate for a priori measures of voting power on the grounds that “the concept of quarrelling oversteps the limits of the a prioristic terrain.”} \]
who, in stage 1, voted on the same side), with the result that those affected by the quarrel are unable or refuse to present their vote to the tabulator. It is only by presenting their votes together to the tabulator in the second stage that first-stage cooperation becomes effective. The model works by incorporating the quarrel at stage 2, thereby allowing us to maintain the a prioristic approach in stage 1, to which the quarrel postulate applies.

3.1 Three Quarrel Properties

To evaluate the reasonability of any (structural) conception of a quarrel, we begin by first identifying the core property that we assume any reasonable (structural) conception of a quarrel should possess, namely cooperative-success-reduction; an interpretation of that property, which we call internal motivation; and two further properties relevant for distinguishing the type of quarrel it is, namely, symmetry and reciprocality. We begin with an informal characterization of each property for intuition, and then formalize them.

We assume that the essence of a quarrel can be articulated by the property we label cooperative-success-reduction (CSR). We assume, in other words, that any conception must satisfy this property to be a reasonable conception of a quarrel, and that any conception satisfying it is ipso facto a reasonable conception. CSR holds that when the votes of two players $i$ and $j$ agree, the players never do better and sometimes do worse – that is, they are less successful overall in securing the outcome for which they both vote – if they quarrel than if they do not. Intuitively, when effective cooperation between two players diminishes, their effectiveness in helping each other to secure outcomes for which they both vote reduces. The “cooperative” in cooperative-success-reduction refers to the fact CSR requires a reduction in each player’s success in divisions in which their votes agree. (This contrasts with the players’ reduced success in divisions in which their votes disagree, which corresponds to the concept of an ambush, not quarrel. We shall return to the ambush concept in Section 7.)

We also assume that any conception of quarrelling must be compatible with interpreting the reduction or elimination of cooperative success as internally motivated, in the sense that the reason for reduction refers to a quarrelling player’s own agency. By contrast, consider a scenario in which the cooperative success of two unwavering friends is reduced because a third party interferes with the effectiveness of their cooperation: this scenario could also be formalized by CSR, but it does not correspond to the intuitive notion of a quarrel between the two players. It corresponds, rather, to the concept of what we might call an external attack.

For the second property, namely symmetry, consider the case in which, as a result of their quarrel, two players $i$ and $j$ no longer effectively cooperate to pass proposed legislation, but remain willing and able to cooperate effectively to block legislation. That is, consider the case in which the two players quarrel only on the yes side. This would be an asymmetric yes-quarrel. For intuition, imagine that the reason the players quarrel is they each suspect the other’s enmity, and so end up suspecting the advisability of the outcome for which they vote when their enemy also votes in favour. An asymmetric yes-quarrel might then arise because each player suspects the advisability only of yes-outcomes when the other also votes in favour. (Other interpretations of an asymmetric quarrel are also available: imagine the players suspect each other, but are extremely risk-averse...
when it comes to change and hence will accept anyone’s cooperation on the NO-side when they vote for the status quo, but not so on the YES-side, and are therefore quick to give in to their suspicions over any changes to the status quo. Thus we informally define a symmetric quarrel as one equally carried out on both the YES and NO sides.

Finally, consider the third property, namely reciprocity. Take the case in which one player, i, has a quarrel with j, but not vice versa. Here i does not effectively cooperate with j, even though j is willing and able effectively to cooperate with i were its cooperation effectively reciprocated. There are two ways to interpret this. First, it might be that i cannot effectively cooperate with j because someone else (whether j or not) has neutralized i’s capacity to do so. The trouble with this interpretation is that it does not correspond to the idea that i has a quarrel with j: recall that a quarrel intrinsically depends on the agency of the quarrelling player itself. This interpretation corresponds, rather, to the idea that someone other than i paralyzes i’s effective cooperation with j. Second, it might be that i does not effectively cooperate with j because i itself is unwilling (or undermines its own capacity) to do so. This corresponds to the intuitive concept of a quarrel as we construe it. But if j has no quarrel with i, and so is willing and able effectively to cooperate with j, then i’s quarrel with j is a non-reciprocal quarrel of i against j. For intuition, imagine that i suspects or cannot bear j and hence does not effectively cooperate with j, but that j is entirely immune to quarrelling of any kind: other players’ quarrels simply do not affect j’s willingness and capacity to function with others if they too are willing and able. Thus we informally define a reciprocal quarrel as one in which both players equally quarrel with each other.

We now turn to providing formal statements for each of these three properties.

### 3.2 Cooperative-Success-Reduction (CSR)

We say that, for any monotonic binary game, players i and j effectively cooperate with each other on the YES side if they are each YES-decisive in a division; and they effectively cooperate on the NO side if they are each NO-decisive. A player is decisive in a division if, holding all other votes constant, it could have unilaterally changed the outcome if it had voted differently. We say a player is YES-decisive if it votes YES and is decisive, and is NO-decisive if it votes NO and is decisive. For binary monotonic voting games, decisiveness can be formalized as follows: player i is YES-decisive in S if and only if i ∈ S ∈ W but S \ {i} /∈ W, and is NO-decisive if and only if i /∈ S /∈ W but S ∪ {i} ∈ W. Note, however, that for non-monotonic games, we must add that i is YES-decisive in S also if i ∈ S /∈ W but S \ {i} ∈ W, and that i is NO-decisive in S also if i /∈ S ∈ W but S ∪ {i} /∈ W. (In other words, in non-monotonic games a player’s vote may be decisive even when it disagrees with the outcome. Being decisive in this way is consistent with the notion of having and exercising voting power: it involves exercising voting power to satisfy one’s preferences by voting strategically. In non-monotonic – and, indeed, in non-binary – games, rational actors may have a reason to vote strategically against the outcome they prefer. By restricting our analysis to initially monotonic binary games, we initially abstract from this complication.)

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4Note that this formulation identifies a sufficient not necessary condition: it is does not imply that if either i or j is not decisive, the two do not effectively cooperate with each other.
Let \( G \) be a monotonic binary voting game, \( \mathcal{G} \) be the set of all logically possible monotonic binary voting games, \( \mathcal{CY} \subset \mathcal{G} \) be the set of all monotonic binary voting games that contain at least one division in which \( i \) and \( j \) effectively cooperate with each other on the yes-side, and \( \mathcal{CN} \subset \mathcal{G} \) be the set of all monotonic binary voting games that contain at least one division in which \( i \) and \( j \) effectively cooperate on the no-side. Finally, let \( \mathcal{G}' \) be a voting game derived from \( G \) by imposing a quarrel between players \( i \) and \( j \), and \( Q : \mathcal{G} \to \mathcal{G}' \) be the transformation rule or function that maps each division outcome in \( G \) to the corresponding division outcome in \( \mathcal{G}' \). The transformation rule \( Q \), by which the voting game \( \mathcal{G}' \) is derived from \( G \), can be thought of as formalizing a particular conception of what a quarrel is, that is, a formalization of the conception imposed on \( G \) and reflected in \( \mathcal{G}' \). We formulate CSR, which characterizes the transformation rule \( Q \), in terms of the yes-successful voter subsets \( W \) in \( G \) and \( \mathcal{W} \) in \( \mathcal{G}' \):

**Cooperative-Success-Reduction (CSR).** For any \( S \) containing neither \( i \) nor \( j \):

**On the Yes Side**

\[
\begin{align*}
\forall G \in \mathcal{G} : & \quad S \cup \{i,j\} \in \mathcal{W} \implies S \cup \{i,j\} \in \mathcal{W} \quad \text{(YQ-1)} \\
\forall G \in \mathcal{CY} : & \quad \exists S \mid S \cup \{i,j\} \notin \mathcal{W} \land S \cup \{i,j\} \in \mathcal{W} \quad \text{(YQ-2)}
\end{align*}
\]

**On the No Side**

\[
\begin{align*}
\forall G \in \mathcal{G} : & \quad S \in \mathcal{W} \iff S \in \mathcal{W} \quad \text{(NQ-1)} \\
\forall G \in \mathcal{CN} : & \quad \exists S \mid S \in \mathcal{W} \land S \notin \mathcal{W} \quad \text{(NQ-2)}
\end{align*}
\]

Here \( \land \) is the logical operator AND. Observe that CSR has two parts. The first, YQ, formalizes the idea that \( i \) and \( j \) have a quarrel on the yes-side. **[YQ-1]** states that if \( i \) and \( j \) vote yes in a division in which the outcome is no in the original game \( G \), then the outcome of that division will also be no once they quarrel in \( \mathcal{G}' \). This implies that \( i \) and \( j \) are never more yes-successful thanks to their quarrel when their votes agree on the yes side, which means their quarrel never renders cooperation between them more effective on the yes side. **[YQ-2]** states that, for any voting game \( G \) in which \( i \) and \( j \) effectively cooperate with each other on the yes side, there exists at least one division in which \( i \) and \( j \) vote yes, and whose outcome is yes in the original game \( G \) but no once they quarrel in \( \mathcal{G}' \). This implies that, for such games, in at least one division in which both players vote yes they are less yes-successful due to their quarrel, which means their quarrel renders cooperation between them less effective on the yes side on at least one occasion.

The second part, NQ, formalizes the idea that \( i \) and \( j \) have a no-quarrel. **[NQ-1]** states that if \( i \) and \( j \) vote no in a division in which the outcome is yes in the original game \( G \), then the outcome of that division will also be yes once they quarrel in \( \mathcal{G}' \). This implies that \( i \) and \( j \) are never more no-successful thanks to their quarrel, which means their quarrel never renders cooperation between them more effective on the no side. **[NQ-2]** states that, for any voting game \( G \) in which \( i \) and \( j \) effectively cooperate with each other on the no side, there exists at least one division in which \( i \) and \( j \) vote no, and whose outcome is no in the original game \( G \) but yes once they quarrel in \( \mathcal{G}' \). This implies that, for such games, in at least one division in which both players vote no they are less no-successful due to their quarrel, which means their quarrel renders cooperation between them less effective on the no side on at least one occasion.
Note that neither \textnormal{YQ-2} nor \textnormal{NQ-2} requires that the transformation reduce the quarrelling players’ success for all pairs $G$ and $\hat{G}$. This is because there may be no effective cooperation between them in the first place in the initial game $G$; if so, then there would be nothing for a quarrel to reduce. Consider, for example, a monotonic binary voting game $G$ with three players in which the first player is a dictator, such that $W = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$. In this structure, because player 1 unilaterally dictates the outcome, there can be no effective cooperation between it and other players in any case; thus imposing a quarrel between players 1 and 2 need not induce any change whatsoever in this voting game. But a transformation rule satisfying \textnormal{YQ-2} or \textnormal{NQ-2} will incur a change for other voting games.

We say that a conception of a quarrel satisfying YQ ($=\textnormal{YQ-1} \land \textnormal{YQ-2}$) satisfies CSR on the yes side; a conception satisfying NQ ($=\textnormal{NQ-1} \land \textnormal{NQ-2}$) satisfies CSR on the no side; and a conception satisfying both YQ and NQ satisfies CSR on both sides. On our account, any reasonable conception of a yes-quarrel must satisfy YQ; any reasonable conception of a no-quarrel must satisfy NQ; and any reasonable conception of a symmetric quarrel must satisfy CSR on both sides. This articulates what we assume is the core intuitive idea of quarrelling, namely, that a group containing a quarrelling couple will be less successful in cooperatively securing outcomes than it would have been had the couple not been quarrelling.

Note, finally, that neither \textnormal{YQ-2} nor \textnormal{NQ-2} requires that the transformation wholly neutralize or eliminate effective cooperation between the quarrelling players in $G$; it merely requires reduction. Quarrels that eliminate effective cooperation between the quarrelling players would satisfy a stronger variant of each condition, namely, that for any $S$ containing neither $i$ nor $j$:

$$\forall G \in \mathcal{G} : \quad S \cup \{i,j\} \notin \hat{W} \iff S \cup \{i\} \notin W \land S \cup \{j\} \notin W \quad (\textnormal{YQ’-2})$$

$$\forall G \in \mathcal{G} : \quad S \in \hat{W} \iff S \notin W \land S \cup \{i\} \in W \land S \cup \{j\} \in W \quad (\textnormal{NQ’-2})$$

\textnormal{YQ’-2} and \textnormal{NQ’-2} say that whenever, in a division in $G$, both $i$ and $j$ are yes-decisive or no-decisive, then in $\hat{G}$ they are both unsuccessful in that division. This means that their cooperation is wholly ineffective in $\hat{G}$ for any division. Satisfying \textnormal{YQ-2} and \textnormal{NQ-2} obviously implies satisfying \textnormal{YQ-2} and \textnormal{NQ-2} respectively. (All the conceptions of quarrelling we examine below satisfy \textnormal{YQ-2} and/or \textnormal{NQ-2} because they eliminate rather than merely reduce effective cooperation between the quarrelling players.)

### 3.3 Symmetry

To formalize our second property, symmetry, we define a game $G^C$ to be the complement of $G$ if, for any $S \subseteq [n]$,

$$S \in W \iff \bar{S} \notin W^C$$

Equivalently, $S$ is no-successful in $G^C$ if and only if it is yes-successful in $G$. Thus the complement $G^C$ would yield an identical game to $G$ if the yes/no vote labels were interchanged.

\footnote{We earlier left open the possibility of effective cooperation in divisions in which both players are not fully decisive, but in which one or both may be partially efficacious. But partial efficacy in one division (in which the players’ votes agree) is parasitic on being decisive in at least one division (in which the players’ votes agree) in the game.}
It follows that for a quarrel to be symmetric with respect to YES/NO, the games we obtain by imposing a quarrel on \( G \) and on \( G^C \) must themselves be complements! Any violation of this property implies that inducing the quarrel has a different effect when we swap the labels of the votes:

**Symmetry:** For any game \( G \in \mathbb{G} \), it holds that \( (\hat{G})^C = (\hat{G}^C) \).

An asymmetric quarrel, in turn, is any quarrel that is not symmetric. This includes purely asymmetric quarrels, in which the players quarrel on one side only (YES or NO), and quasi-symmetric quarrels, in which the players quarrel on both sides, but their quarrel on one side is not identical to the other. (We shall further formalize this once we introduce our framework). Since in this paper we treat only the pure case, we henceforth use “asymmetric” to refer to purely asymmetric quarrels.

### 3.4 Reciprocality

To formalize our third property, *reciprocality*, let \( \hat{G}^{i\rightarrow j} \) represent a purely non-reciprocal quarrel of \( i \) against \( j \), and \( \hat{G}^{i\leftrightarrow j} \) and \( \hat{G}^{j\leftrightarrow i} \) represent quarrels in which both \( i \) and \( j \) quarrel against each other. Reciprocality simply states that in the quarrel \( \hat{G} \) derived from \( G \), the player \( i \)’s quarrel against \( j \) is identical to \( j \)’s quarrel against \( i \):

**Reciprocality:** For any game \( G \in \mathbb{G} \), it holds that \( \hat{G} = \hat{G}^{i\leftrightarrow j} = \hat{G}^{j\leftrightarrow i} \).

A non-reciprocal quarrel, in turn, we define as any quarrel that is not reciprocal. This includes purely non-reciprocal quarrels, in which one player quarrels against another who has no quarrel with it, and quasi-reciprocal quarrels, in which both players quarrel with each other, but not equally. (Again, we shall further formalize this once we introduce our framework). Since in this paper we treat only the pure case, we henceforth use “non-reciprocal” to refer to purely non-reciprocal quarrels.

### 3.5 Reasonable Conceptions of Quarrelling

As far as *quarrelling* itself is concerned, any conception \( Q \) that satisfies CSR (on at least one side) counts as a reasonable formal conception – whether or not the conception satisfies symmetry or reciprocality. What symmetry and reciprocality do is help specify the quarrel’s type; we find nothing unreasonable per se about an asymmetric or non-reciprocal conception of quarrelling. One could even combine symmetry and reciprocality in various ways: a quarrel could be reciprocal on the YES side but not on the NO side, or \( i \)’s quarrel with \( j \) could be symmetric but \( j \)’s quarrel with \( i \) asymmetric. We leave aside these hybrid types for the sake of brevity. CSR leaves unspecified just how deleterious the quarrel is, and so how much less successful the group to which the quarrelling couple belongs would be. We specify this magnitude in Section 8 where we present a class of quarrels based on four degrees of quarrelling.
4 What is a Reasonable Quarrel Postulate?

To define the quarrel postulate and corresponding paradox, in addition to the concept of a quarrel we require that of voting power. Although we have defined quarrels only over initially monotonic binary voting games, we define a measure of voting power more broadly for binary voting games in general, as a function $\Psi$ that assigns to each player $i$ a nonnegative real number $\Psi_i \geq 0$ and that satisfies two sets of basic adequacy postulates concerning voting power under a prioristic assumptions (which we represent with the lower case $\psi$): the iso-invariance postulate, according to which the a priori voting power $\psi_i$ of any player $i$ remains the same between two isomorphic games; and the dummy postulates, according to which a player has zero a priori voting power if and only if it is a dummy (i.e., is not decisive in any division), and the addition of a dummy to a voting structure leaves other players’ a priori voting power unchanged (Felsenthal and Machover 1998: 222). We can now define the quarrel postulate in its standard form on the basis of a quarrel $\hat{G}$ between $i$ and $j$ derived, via $Q$, from a monotonic binary voting game $G$ (where $\hat{\psi}$ represents players’ a priori voting power in the derived game $\hat{G}$):

**Standard Quarrel Postulate:** A measure of voting power $\Psi$ satisfies the (standard) quarrel postulate based on $Q$ if and only if $\hat{\psi}_i \leq \psi_i$ and $\hat{\psi}_j \leq \psi_j$ for any $G \in \mathcal{G}$.

A measure of voting power displays the quarrelling paradox if and only if it violates the quarrel postulate. Thus, a violation arises if the a priori voting power of at least one of the quarrelling pair increases after the quarrel.

Just as it is possible to disaggregate voting power over binary games into its two components, namely, yes-voting power and no-voting power, it is also possible to disaggregate the quarrel postulate into two corresponding parts. We define yes-voting power as that part of a player’s voting power based solely on its potential yes-votes, and no-voting power as the part based solely on its potential no-votes. We therefore define a measure of yes-voting power for binary voting games as a function $\Psi^+$ that assigns to each player $i$ a nonnegative real number $\Psi^+_i \geq 0$, satisfies the basic adequacy postulates, and is a function of only those divisions in which $i$ votes yes. Symmetrically one can define a measure of no-voting power $\Psi^-$ such that $\Psi_i = \Psi^+_i + \Psi^-$. We can now define the corresponding components of the standard quarrel postulate:

**Yes-Voting-Power Quarrel Postulate:** A measure of voting power $\Psi$ satisfies the yes-voting-power quarrel postulate based on $Q$ if and only if $\hat{\psi}^+_i \leq \psi^+_i$ and $\hat{\psi}^+_j \leq \psi^+_j$ for any $G \in \mathcal{G}$.

**No-Voting-Power Quarrel Postulate:** A measure of voting power $\Psi$ satisfies the no-voting-power quarrel postulate based on $Q$ if and only if $\hat{\psi}^-_i \leq \psi^-_i$ and $\hat{\psi}^-_j \leq \psi^-_j$ for any $G \in \mathcal{G}$.

The satisfaction of both these non-standard postulates of course implies the satisfaction of the standard postulate.
4.1 Monotonicity

Whether or not the (standard) quarrel postulate is reasonable will depend on the conception of quarrelling on which it is based. The conception itself must of course be reasonable, that is, it must satisfy CSR. Although we define quarrels only over initially monotonic games, CSR permits the derived game (in which the two players quarrel) to be non-monotonic. We think this is as it should be: we find nothing unreasonable about a conception of quarrelling that induces non-monotonic voting games once the players quarrel. (Note that we have defined voting power over non-monotonic voting games as well.)

However, as we shall now argue, a conception of quarrelling that can result in non-monotonic voting games is not fit to serve as the basis of a reasonable quarrel postulate. That is, to be reasonable, a quarrel postulate must be based on a conception of quarrelling that satisfies:

**Monotonicity**: \[
\forall T \subseteq S \quad T \in \hat{W} \implies S \in \hat{W}.
\]

Recall our assumption that the initial game \(G\) is monotonic. The monotonicity property here states that the quarrel \(\hat{G}\) derived from \(G\) is also monotonic (that is, the transformation rule \(Q\) is monotonic in the sense that it preserves monotonicity in \(\hat{G}\)). This property is significant because if either \(G\) or \(\hat{G}\) departs from monotonicity, then it would not be reasonable to expect a measure of voting power to avoid succumbing to a quarrelling paradox, because the paradox, rather than reflecting any defect in the measure itself, may merely be an artifact of the voting structure’s departure from monotonicity. Intuitively, there is no reason to expect a measure of voting power to satisfy the quarrel postulate over non-monotonic games, because just as a set of players may, by definition, be more successful in a non-monotonic game if some of their members turn against them (that is to say, refuse to cooperate effectively with them), so too might some players become more powerful if others were to quarrel with them.

We can formalize this intuition as follows. Recall that monotonicity is violated in \(\hat{G}\) when there exists a violating pair of divisions \(\{S, T\}\) such that \(T \subset S \in W\) but \(S \notin \hat{W}\). Now if, for a division in \(G\) in which players \(i\) and \(j\) are both successful but \(i\) is not decisive, \(Q\) would render \(i\) and \(j\) unsuccessful when they quarrel in \(\hat{G}\) (but, being a pure, two-player quarrel, would leave divisions in which they disagree unchanged), then the transformation rule \(Q\) induces in \(\hat{G}\) non-monotonicity over quarrellers over that division and the one identical to it except for \(i\)’s vote:

**Non-Monotonicity Over Quarrellers (NMQ):** \(Q\) induces non-monotonicity over a quarrelling pair \(\{i, j\}\) in \(\hat{G}\) over division \(S\) if:

\[
\{(i, j) \in S \in W \land S \setminus \{i\} \in W \land S \notin \hat{W} \land S \setminus \{i\} \in \hat{W}\} \\
\lor \{(i, j) \notin S \notin W \land S \cup \{i\} \notin W \land S \in \hat{W} \land S \cup \{i\} \notin \hat{W}\}
\]

Here \(\lor\) is the logical operator OR. We say that a conception \(Q\) is disposed to induce non-monotonicity over quarrellers if, for any \(G\) containing divisions in which players \(i\) and \(j\) are both successful but \(i\) is not decisive, the transformation rule \(Q\) would render \(i\) and \(j\) unsuccessful in at least one such division when they quarrel in \(\hat{G}\):
Disposition to Induce Non-Monotonicity Over Quarrellers (DNQ): $Q$ is disposed to induce non-monotonicity over a quarrelling pair $\{i,j\}$ if:

$$\forall G \in \mathbb{G} : \exists S \mid Q \text{ satisfies NMQ for } \hat{G}$$

With this concept to hand, we can now present our first theorem, which provides the first step towards formalizing the intuition that a quarrel postulate based on a non-monotonic conception is unreasonable. (Proofs for this and subsequent theorems are presented in the Appendix.)

**Theorem 4.1.** If a reasonable conception of quarrelling $Q$ (which satisfies CSR) is disposed to induce non-monotonicity over quarrellers, then the standard quarrel postulate based on $Q$ will be violated by any measure of voting power $\Psi$.

A standard quarrel postulate based on a conception of quarrelling $Q$ that satisfies CSR but also DNQ is therefore unreasonable, i.e., the postulate would not be one that a reasonable measure of voting power should be expected to satisfy (because no measure could satisfy it!). Again, the point is not that non-monotonicity over quarrellers would disqualify $Q$ as a conception of what quarrelling is per se. A transformation rule may very well be non-monotonic over quarrellers and still adequately articulate the core intuition of what a quarrel is. The point is, rather, that a conception of quarrelling with property DVQ is not fit to serve as the basis for a reasonable quarrel postulate.

Now, it is true that inducing a quarrel between two players via a non-monotonic conception of quarrelling may result in a game that is non-motononic but not over those two quarrelling players. (This is because the non-monotonicity may result from pairs of divisions in which neither of these two players is decisive.) Nevertheless, as we shall now prove, any conception of a quarrel $Q$ that satisfies CSR, by which a non-monotonic game $\hat{G}$ is derived from an initially monotonic game $G$, will have property DNQ in general. This is because any such derived game will be non-monotonic over some pair(s) of players who would be quarrellers relative to some binary monotonic game from which the quarrelling game could be derived via $Q$.

**Theorem 4.2.** Let $\hat{G}$ be a non-monotonic game derived from a binary monotonic game $G' \in \mathbb{G}$ by imposing a quarrel $Q$ between two players $l$ and $m$, where $Q$ satisfies CSR. Then there exists a game $G \in \mathbb{G}$ such that $\hat{G}$ can be derived from $G$ by imposing the same conception of a quarrel $Q$ between players $i$ and $j$ and where $\hat{G}$ is non-monotonic over the quarrellers $i$ and $j$.

We therefore conclude that any conception of quarrelling that is non-motononic is not fit to serve as the basis for a reasonable quarrel postulate. This is because, by Theorem 4.2, any non-monotonic conception falls prey to Theorem 4.1.

Moreover, the greater the departures from monotonicity induced by a conception of quarrelling, the more unreasonable would be any quarrel postulate based on it. We accordingly provide a formal measure of how far a voting game departs from monotonicity. This measure is based upon both the magnitude and the multitude of the deviations from monotonicity.

Recall again that a deviation from monotonicity is given by a violating pair $\{T,S\}$, where $T \subset S$, and $T \in \mathcal{W}$ but $S \notin \mathcal{W}$. The key observation then is that the magnitude of the deviation
can be quantified by the cardinality of \( I = S \setminus T \). In particular, if a voting game is not monotonic then there is a violating pair \( \{T, S\} \) where \( I = S \setminus T \) consists of a single player \( i \), that is \( |I| = 1 \). Moreover, the greater the cardinality of the set \( I \) the greater the departure from monotonicity. This is because monotonicity implies that \( I \) itself is at least as successful a group as any individual member \( i \) (or, indeed, any strict subset \( J \) of \( I \)). Consequently, a violation in monotonicity caused by a pair \( \{T, T \cup I\} \) is more surprising and extreme than a violation caused by the pair \( \{T, T \cup J\} \) for some \( J \subset I \). The multitude of the violations is easy to quantify: it increases as the number of violating pairs increases.

The concept of \( k \)-monotonicity then quantifies the distance a voting game is from monotonicity by a parameter \( k \) that increases as \textbf{both} the magnitude \textbf{and} the multitude of the violations increase. Hence for a voting game to be far from monotonic it must have a large number of pairs whose induced violations are large. Formally, we define the measure by:

**\( k \)-monotonicity:** A voting game \( G \) is \( k \)-monotonic if for any group \( S \subseteq [n] \):

\[
S \notin \mathcal{W} \implies \forall T \subset S \exists K : |K| \leq k \text{ such that } T \setminus K \notin \mathcal{W}
\]

This definition says that if a group \( S \) is \textsc{yes}-unsuccessful then every subset \( T \) of \( S \) is either \textsc{yes}-unsuccessful as well, or is \textsc{yes}-successful but has \( k \) members without whom it would be \textsc{yes}-unsuccessful. This implies that \( 0 \)-monotonicity is exactly the property of monotonicity. To see this, observe that 0-monotonicity implies that we need not discard any members of \( T \) to ensure that it is \textsc{yes}-unsuccessful. Hence, if \( S \) is \textsc{yes}-unsuccessful, then so is \( T \) for any \( T \subset S \).

An alternative way to view \( k \)-monotonicity is to consider violations of the property: \( k \)-monotonicity is violated if \( S \notin \mathcal{W} \) and \( S \) contains a subset \( T \) such that not only \( T \in \mathcal{W} \), but also \( R \in \mathcal{W} \) for every \( R \subset T \) with up to \( k \) members fewer than \( T \). Thus, all such subsets \( R \) also exhibit a violation in monotonicity with \( S \). Indeed, if \( k \)-monotonicity does not hold then there are an exponential number in \( k \) of pairs \( (R, S) \) that violate monotonicity. It immediately follows that if \( k \)-monotonicity holds only for \( k \geq 1 \), then the voting rule is non-monotonic. Moreover, if \( G \) is \( k \)-monotonic then it is \((k+1)\)-monotonic, but not vice versa. In particular, for a \( k \)-monotonic voting rule, the smaller \( k \) can be, the less the voting rule departs from being monotonic. Since 1-monotonic voting rules are as near to being monotonic as a non-monotonic rule can be, we dub them quasi-monotonic. By contrast, if a voting rule is not \( k \)-monotonic for any fixed \( k \), then it violates monotonicity to the utmost extent possible; we dub such voting rules supremely non-monotonic.

4.2 Symmetry

To be fit to serve as the basis for a reasonable quarrel postulate in general, a conception of quarrelling must not only be monotonic, but also symmetric. This is because asymmetric conceptions, despite not being unreasonable per se, restrict the reasonability of a quarrel postulate based on them to a specific class of voting power measures. On the one hand, a violation of symmetry is innocuous for evaluating decisiveness measures of voting power, such as the Penrose-Banzhaf measure, which track (full) decisiveness only.\footnote{For decisiveness measures (and success measures), see Laruelle and Valenciano 2005b.} Decisiveness measures calculate a player’s voting power such that only
divisions in which the player is decisive count towards its voting power. Asymmetry is innocuous for this class of measures because every winning division in which a player is YES-decisive is mirrored by one and only one losing division in which the player is NO-decisive: these are the two divisions in which the decisive player’s vote varies but all other players’ votes are held constant. This implies that a player’s a priori YES-voting power is always equal to its a priori NO-voting power on such measures, $\psi_i^+ = \psi_i^-$. This is precisely why a player’s Penrose-Banzhaf a priori voting power, which by definition is equal to the proportion of all divisions in which the player is decisive, can be calculated, as is typically done, on the basis of only the divisions in which the voter is YES-decisive.

On the other hand, decisiveness measures are merely a subclass of the broader class of efficacy measures of voting power, which calculate a player’s voting power such that any division in which the player is casually efficacious may contribute to its voting power. Asymmetry is not innocuous for those efficacy measures of voting power, such as the Abizadeh-Vetta Recursive Measure, that track not just decisiveness or full efficacy, but also partial efficacy, and so base a player’s voting power on divisions in which it plays some causal role in producing the outcome – whether fully decisive or not. Asymmetry is not innocuous here because the partial efficacy of a successful YES-voter in winning divisions is not mirrored in the corresponding divisions in which the voter changes its vote but all other players’ votes are held constant: precisely because the voter is not (fully) decisive, its changed vote does not change the outcome; the voter therefore flips from being a partially efficacious, successful YES-voter to an inefficacious, unsuccessful NO-voter. Therefore, for such measures, a player’s a priori YES-voting power is not necessarily equal – as it is in decisiveness measures – to its a priori NO-voting power.

The upshot is that a conception of quarrelling that violates symmetry may be fit to serve as the basis of a reasonable quarrel postulate in the standard form for decisiveness measures, but not for measures of voting power in general. Alternatively, such a conception can be the basis for a reasonable quarrel postulate for efficacy measures in general, but only for a non-standard postulate specified for either YES- or NO-voting power only – rather than for voting power in general as in the standard postulate. In particular, if two voters quarrel only with respect to YES-outcomes, then their asymmetric YES-quarrel could be the basis for only a reasonable YES-voting-power quarrel postulate, but not for a reasonable quarrel postulate in standard form.

Monotonicity and symmetry exhaust our account of the properties a reasonable conception of quarrelling ought to have in order to be fit to serve as the basis of a reasonable quarrel postulate. As we shall see in Section 9, non-reciprocality does not pose any problems for the postulate’s reasonability.

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7 On the class of efficacy measures, partial efficacy, and the Abizadeh-Vetta Recursive Measure, see Abizadeh (preprint); Abizadeh and Vetta (2021). For partial efficacy and degrees of causation, see also Braham and van Hees 2009. More generally, for conditions that play a causal role but are not fully necessary for the effect, see McDermott 1995; Ramachandran 1997; Schaffer 2003.
5 Felsenthal and Machover’s Conception

We are now in a position to evaluate the reasonability of the two existing structural conceptions of quarrelling in the literature, and their fitness to serve as the basis for a reasonable quarrel postulate. We begin here with Felsenthal and Machover’s conception, and turn to Laruelle and Valciano’s conception in Section 6. We shall later show how each conception fits into the general framework for a class of quarrels we develop in Sections 7 to 9.

Felsenthal and Machover (1998: 237) base the (standard) quarrel postulate on the following conception of a quarrel. They say that
\[ i \text{ has a quarrel with } j \text{ if and only if } S \in W \implies i \notin S \text{ or } j \notin S. \]
This is equivalent to defining the yes-successful sets \( \hat{W} \) of \( \hat{G} \) as follows. For any \( S \subseteq [n] \), set
\[
S \notin \hat{W} \quad \text{if } \{i,j\} \subseteq S \\
S \in \hat{W} \iff S \in W \quad \text{otherwise}
\]
Thus any division in which both \( i \) and \( j \) vote yes loses: \( i \) and \( j \) cannot both be yes-successful. Call this conception of a quarrel an FM-quarrel or FM-rule.

To begin, observe that an FM-quarrel satisfies YQ. It is therefore a reasonable conception of a yes-quarrel, since it captures the concept’s core intuition. It is also a reciprocal conception.

The FM-quarrel, however, has two shortcomings. First, because it is restricted to a yes-quarrel – it does not satisfy NQ – it violates symmetry: an FM-quarrel models a quarrel between \( i \) and \( j \) over yes-outcomes, but not over no-outcomes. This implies, as argued earlier, that the FM-quarrel postulate (that is, the standard quarrel postulate based on an FM-quarrel) is not reasonable for measures of voting power in general: even if it were reasonable to expect decisiveness measures to satisfy it, it would not be so for efficacy measures in general.

Second, as Felsenthal and Machover themselves note, the FM-quarrel violates monotonicity. For example, suppose there exists a set \( S \), containing neither \( i \) nor \( j \), such that either \( S \cup \{i\} \in W \) or \( S \cup \{j\} \in W \) or both. Then monotonicity is violated in \( \hat{G} \) since \( S \cup \{i,j\} \notin \hat{W} \).

But perhaps this is a minor violation and the FM-conception of quarrelling produces voting games that are close to monotonic, say quasi-monotonic? No, an FM-rule not only violates monotonicity, it also violates \( k \)-monotonicity for any value of \( k \):

**Theorem 5.1.** The FM-rule is supremely non-monotonic.

We can also illustrate the FM-rule’s disposition to induce non-monotonicity over quarrellers with a simple example. Consider dictator-rule voting with three voters, one dictator and two dummies. Inducing an FM-quarrel between the dictator and a dummy in this voting game \( \hat{G} \) renders the dummy decisive in half of the eight divisions in \( \hat{G} \), thus causing any measure of voting power \( \Psi \) to violate the standard quarrel postulate. (The dummy becomes decisive in the two divisions in which they quarrel on the yes side, along with the two divisions that combine with each of these to compose a violating pair.)

An FM-quarrel is therefore not fit to be basis for a reasonable quarrel postulate. Felsenthal and Machover themselves, however, draw a more sweeping conclusion: not merely that it would

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\(^8\)As Felsenthal and Machover (1998: 238) note, the FM-rule also violates unanimity. In particular, for all \( \hat{G} \),
not be reasonable to expect measures of a priori voting power to satisfy a quarrel postulate based on a quarrel as they conceive it, but that it would not be reasonable to expect them to satisfy a quarrel postulate based on a structural conception of quarrelling in general. We think this more sweeping conclusion fails to appreciate the availability of other conceptions (to be presented below) that do satisfy monotonicity.

6 Laruelle and Valenciano’s Conception

In contrast to Felsenthal and Machover, who abandon the quarrel postulate as unreasonable, Laruelle and Valenciano (2005a) take the FM-quarrel’s failure to satisfy monotonicity as the occasion for proposing a revised conception of a quarrel on which to base the standard quarrel postulate. On their revised account, a game $G$ in which $i$ quarrels with $j$ is derived from $G$ by stipulating that the YES-successful sets $\hat{W}$ satisfy the properties:

$$ S \in \hat{W} \iff S \setminus \{i\} \in W \quad \forall S : j \in S $$

$$ S \in \hat{W} \iff S \cup \{i\} \in W \quad \forall S : j \notin S $$

To begin, note that Laruelle and Valenciano’s conception of a quarrel, which we call an $LV$-quarrel or $LV$-rule, satisfies both CSR and symmetry. It does, however, violate reciprocality: the formulation models a non-reciprocal quarrel that $i$ has against $j$. Player $i$’s LV-quarrel with player $j$ is different from $j$’s LV-quarrel with $i$.

**Theorem 6.1.** The $LV$-rule induces a non-reciprocal quarrel.

Although the violation of reciprocity does not fit Laruelle and Valenciano’s stated motivation of modelling Brams’ conception of a reciprocal quarrel, it is not a problem for the conception per se: as we formally demonstrate in Section 9, non-reciprocity does not disqualify a conception as a reasonable conception of a quarrel, nor as a reasonable basis for a quarrel postulate.

The LV conception does, however, suffer from a critical flaw: even though Laruelle and Valenciano propose their revision to Felsenthal and Machover’s conception on the grounds that the latter violates monotonicity, so too does their own proposal! For example, consider the same voting game $G$ with two players and $W = \{\{1\}, \{1, 2\}\}$. Now consider game $\hat{G}^{1 \rightarrow 2}$ that incorporates the quarrel of $i = 1$ against $j = 2$. As shown above, $\hat{W}^{1 \rightarrow 2} = \{\emptyset, \{1\}\}$. Thus $\emptyset \in \hat{W}$ and $\{1\} \in \hat{W}$ but $\{1, 2\} \notin \hat{W}$, which violates monotonicity. There is nothing special about this extremely small

[n] \notin \hat{W}$. This, however, has no significance for the other properties. In particular, if we modify the FM-rule such that $[n] \in \hat{W}$ and $\emptyset \notin \hat{W}$, thus satisfying unanimity, our arguments still apply. For example, the modified FM-rule is still supremely non-monotonic: we can see that Theorem 5.1 still holds by simply replacing $S = [n]$ in the proof with $S = [n] \setminus \ell$, for any player $\ell$. Similarly, all of our subsequent results apply when $G$ satisfies unanimity, regardless of whether $Q$ preserves unanimity in $\hat{G}$.

9As they put it, the paradox of quarrelling members is a merely “superficial” paradox (1998: 223, 241). They also suggest that “the concept of quarrelling oversteps the limits of the aptioristic terrain” and so “makes no sense” for efficacy measures of voting power such as $PB$ (1998: 238-239). However, as noted above, there exists a perfectly intuitive interpretation ready to hand for structural conceptions of quarrelling (appropriate for a postulate on a priori power).
example; similar violations arise in every voting game constructed under this formulation. An LV-quarrel therefore does not furnish the basis for a reasonable quarrel postulate.

Again, perhaps this is just a minor violation and the LV-rule produces voting games close to monotonic, say quasi-monotonic? No, like the FM-rule, the LV-rule produces voting games that are arbitrarily far from monotonic:

**Theorem 6.2.** The LV-rule is supremely non-monotonic.

7 A General Quarrel Framework

In light of the failure of existing specifications of the quarrel postulate, we seek a new conception of quarrelling fit to serve as the basis for a reasonable postulate. We pursue our goal by specifying a class of quarrels allowing for a variety of magnitudes of quarrel. This will give us a unified framework in which to compare alternate conceptions of quarrelling.

Recall that a (structural) quarrel consists in the reduction or neutralization of effective cooperation between the quarrelling players when they vote on the same side. Thus, given a voting game $G$, to incorporate a quarrel between $i$ and $j$ we must characterize the outcome for any $S \cup \{i,j\}$. This is precisely what cooperative-success-reduction does: CSR requires that, in the presence of a quarrel between players $i$ and $j$, any voting group containing both $i$ and $j$ not be more cooperatively successful than it would have been in the absence of a quarrel between players $i$ and $j$; and that, for those voting games in which $i$ and $j$ effectively cooperate, there be at least one group containing $i$ and $j$ that is less successful when they quarrel than it would have been absent the quarrel. Thus CSR, which comprises properties YQ on the yes side and NQ on the no side, furnishes the starting point for our framework.

To sharpen our analysis, however, we need to isolate the concept of a quarrel from other, distinct concepts, by imposing further constraints on the transformation rule $Q$. In particular, we should distinguish imposing a quarrel from three other ways in which a voting game could be transformed. Whereas imposing a quarrel reduces or neutralizes the effectiveness of cooperation, inducing enhanced cooperation produces the opposite effect: it transforms the original game by strengthening effective cooperation and therefore increasing the cooperative success of the players whose cooperation is enhanced. Quarrelling and enhanced cooperation both concern cooperation, or votes that agree: the transformation operates on those divisions in which the players concerned vote on the same side. Consider, by contrast, a transformation in which, in the derived game, player $i$ neutralizes the effectiveness of $j$’s vote in those divisions in which the two player’s votes disagree. This is what we earlier called an ambush of $j$ by $i$. Or consider the opposite of an ambush, which is induced when, in divisions in which $i$ and $j$ vote on opposite sides, $i$ enhances the effectiveness of $j$. We call this a betrayal (of the group with whom $i$ ostensibly cooperates) by $i$ in favour of $j$.

A transformation of the original game formally cannot combine, for a given pair of players, a quarrel and enhanced cooperation on the same side (yes or no), but a quarrel could be combined with an ambush. Formally, a quarrel between $i$ and $j$ could also be combined with a betrayal by $i$ in favour of $j$. (But it is difficult to give any substantive, intuitive interpretation to such a hybrid
transformation: if $i$ is motivated to quarrel with $j$ when their votes agree, it is difficult to see why $i$ would be motivated to betray its own side in favour of $j$ when their votes disagree.)

To rule out hybrid transformations, which combine a quarrel between two players with either an ambush or betrayal centred on their relationship, we therefore also impose, in addition to CSR, the following condition on the transformation rule $Q$:

**Non-Ambush/Betrayal ($\neg AB$).** For any $S$ containing neither $i$ nor $j$:

\[
\forall G \in \mathcal{G} : S \cup \{i\} \in \mathcal{W} \iff S \cup \{i\} \in \hat{\mathcal{W}} \quad (\neg AB-1)
\]

\[
\forall G \in \mathcal{G} : S \cup \{j\} \in \mathcal{W} \iff S \cup \{j\} \in \hat{\mathcal{W}} \quad (\neg AB-2)
\]

In $\neg AB-1$, the entailment from left to right says that for any division in which $i$’s and $j$’s votes disagree and $i$ votes YES, if the outcome is NO in $G$, then it is also NO in $\hat{G}$. This implies that $i$ is not more YES-successful and $j$ is not less NO-successful due to the transformation when their votes disagree. This means that $i$ does not ambush $j$ on the NO side, and (what is formally equivalent) $j$ does not betray its own NO side in favour of $i$. By contrast, the entailment from right to left says that for any division in which $i$’s and $j$’s votes disagree and $i$ votes YES, if the outcome is YES in $G$, then it is also YES in $\hat{G}$. This implies that $i$ is not less YES-successful and $j$ is not more NO-successful due to the transformation when their votes disagree. This means that $j$ does not ambush $i$ on the YES side, and (what is formally equivalent) $i$ does not betray its own YES side in favour of $j$.

In $\neg AB-2$, the entailment from left to right says that for any division in which $i$’s and $j$’s votes disagree and $i$ votes NO, if the outcome is NO in $G$, then it is also NO in $\hat{G}$. This implies that $i$ is not less NO-successful and $j$ is not more YES-successful due to the transformation when their votes disagree. This means that $j$ does not ambush $i$ on the NO side, and (what is formally equivalent) $i$ does not betray its own NO side in favour of $j$. By contrast, the entailment from right to left says that for any division in which $i$’s and $j$’s votes disagree and $i$ votes NO, if the outcome is YES in $G$, then it is also YES in $\hat{G}$. This implies that $i$ is not more NO-successful and $j$ is not less YES-successful due to the transformation when their votes disagree. This means that $i$ does not ambush $j$ on the NO side, and (what is formally equivalent) $j$ does not betray its own NO side in favour of $i$.

CSR and $\neg AB$ together provide six formulae that characterize our complete general framework for a pure quarrel (that is, a quarrel not combined with an ambush or betrayal):

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General Quarrel Framework. For any $S$ containing neither $i$ nor $j$:

**CSR: YQ**

- $\forall G \in G : S \cup \{i, j\} \in \hat{W} \Rightarrow S \cup \{i, j\} \in W$ (YQ-1)

- $\forall G \in CY : \exists S | S \cup \{i, j\} \notin \hat{W} \land S \cup \{i, j\} \in W$ (YQ-2)

**CSR: NQ**

- $\forall G \in G : S \in \hat{W} \Leftarrow S \in W$ (NQ-1)

- $\forall G \in CN : \exists S | S \in \hat{W} \land S \notin W$ (NQ-2)

**–AB**

- $\forall G \in G : S \cup \{i\} \in \hat{W} \iff S \cup \{i\} \in W$ (–AB-1)

- $\forall G \in G : S \cup \{j\} \in \hat{W} \iff S \cup \{j\} \in W$ (–AB-2)

CSR implies that the transformation imposes a quarrel, and –AB implies that it imposes only a quarrel.

Because CSR may be satisfied in numerous ways, this general quarrel framework leaves a large degree of flexibility for further specification. In particular, we can satisfy CSR by imposing more restrictive conditions that entail CSR but represent varying degrees or magnitude of quarrel, whether reciprocal or not. To simplify our analysis of these degrees of quarrelling, we begin by amending our general framework, in order to focus on quarrels on the yes side only. We introduce a general framework for asymmetric yes-quarrels by replacing NQ with the following condition:

**Zero No-Quarrel.** For any $S$ containing neither $i$ nor $j$:

- $\forall G \in G : S \in \hat{W} \iff S \in W$ (–NQ)

The entailment from left to right in –NQ states that for any division in which $i$ and $j$ vote no, if the outcome is no in $G$, then it is also no in $\hat{G}$. This implies that $i$ and $j$ are just as no-successful in $\hat{G}$ as in $G$, which means there is no no-quarrel between $i$ and $j$. (This obviously negates [NQ-2](#))

The entailment from right to left (which is obviously equal to [NQ-1](#)), states in turn that for any division in which $i$ and $j$ vote no, if the outcome is yes in $G$, then it is also yes in $\hat{G}$. This implies that neither is more no-successful in $\hat{G}$ than in $G$, which means there is no enhanced cooperation on the no side between $i$ and $j$ either.

With this simplified, asymmetric framework to hand, we now turn to fleshing out our framework by imposing varying degrees of quarrel. We shall do this by tightening YQ. For the sake of clarity we begin with reciprocal quarrels. We return to the symmetric framework in Section 8.4, and consider non-reciprocal quarrels in Section 9.

### 8 Three Degrees of Quarrel

A quarrel can adversely affect a group’s ability to cooperate effectively in different ways, depending on the strength of the quarrel. There are three intuitive ways to characterize the strength of a quarrel: as weak, strong, and cataclysmic.
To understand these concepts note that monotonicity has a simple interpretation in voting games and in coalition games more generally: a group or coalition $T$ is at least as effective as its most effective sub-group. That is, if the votes of $U \subset T$ are sufficient to secure a yes-outcome, then the votes of $T$ are also sufficient. Equivalently, if the sub-group $U$ is sufficient to secure a yes-outcome, then a group $T$ is also sufficient if the members of $U$ simply work together to the exclusion of, or in isolation from, the members of $T \setminus U$. The question is what impact a quarrel between two members $i$ and $j$ has on the group $T$’s ability to secure a yes-outcome. Assume that $i$ and $j$ have a reciprocal quarrel with each other in the group $T$. Then:

- A reciprocal weak quarrel renders $i$ and $j$ unable to function with each other.
- A reciprocal strong quarrel renders $i$ and $j$ unable to function with any member of $T$.
- A reciprocal cataclysmic quarrel renders every member of $T$ unable to function.

This gives us three degrees of quarrelling (or four degrees if we include the case in which there is no quarrel, where the quarrel is of zero magnitude). We have already formalized the notion of a zero or non-quarrel (on the no side) with $\neg NQ$. We now take up each of the three positive conceptions of quarrelling in turn.

### 8.1 Reciprocal Weak Quarrelling

Take any group $T = S \cup \{i, j\}$. In a weak quarrel, the sub-groups $S \cup \{i\}$ and $S \cup \{j\}$ still function as well as before. Thus the group $T$ may simply isolate or exclude one of $i$ or $j$ and obtain the best outcome achievable by just the votes of either $S \cup \{i\}$ or $S \cup \{j\}$. We can therefore define a reciprocal weak yes-quarrel by adopting the general asymmetric framework above, but tightening $YQ$ by replacing it with a further specification:

$$\forall G \in \mathcal{G} : S \cup \{i, j\} \in \hat{W} \iff S \cup \{i\} \in W \lor S \cup \{j\} \in W$$  

($YQ_w$)

$YQ_w$ is a specification of $YQ$ because, given that the original game $G$ is monotonic, satisfying $YQ_w$ entails satisfying both $YQ-1$ and $YQ-2$. This reciprocal weak conception is therefore a reasonable conception of a yes-quarrel: by construction, it satisfies $YQ$.

The entailment from left to right states that if, for any division in which $i$ votes yes in agreement with a set of voters excluding $j$, the outcome is no in $\hat{G}$, and if, for the division in which $j$ votes yes in agreement with that same set of voters, the outcome is also no in $\hat{G}$, then if $i$ and $j$ vote yes in agreement with that same set of voters, the outcome will also be no in the derived game $\hat{G}$. This implies that $i$ and $j$ cannot be more yes-successful in $\hat{G}$ than the most yes-successful of $i$ and $j$ without the other in $G$. This means that effective cooperation between $i$ and $j$ on the yes side is completely neutralized, that is, $\hat{G}$ comprises at least a weak yes-quarrel. By contrast, the entailment from right to left states that if, for any division in which $i$ votes yes in agreement with a set of voters excluding $j$, the outcome is yes in $G$, or if, for the division in which $j$ votes yes in agreement with that same set of voters, the outcome is also yes in $G$, then if $i$ and $j$ vote yes in agreement with that same set of voters, the outcome will also be yes in $\hat{G}$. This means that $\hat{G}$ does
not comprise a YES-quarrel any stronger than weak: in particular, it does not comprise a strong YES-quarrel.

Consider what a weak quarrel so-defined implies for a weighted voting game $G$, that is, a voting game in which each player’s vote has a fixed weight and a YES-outcome requires meeting a specified quota of total YES-vote-weights (Felsenthal and Machover 1998: 29-32). If $\hat{G}$ contains a reciprocal weak quarrel between $i$ and $j$, then both of them voting YES together in $S \cup \{i, j\}$ forces the weight of the weaker of the two players to 0.

Importantly, reciprocal weak quarrelling satisfies monotonicity.

**Theorem 8.1.** A game $\hat{G}$ with a reciprocal weak YES-quarrel is monotonic.

### 8.2 Reciprocal Strong Quarrelling

In a strong quarrel, not only are $i$ and $j$ unable to cooperate effectively with each other, $i$ and $j$ cannot individually do so with any other members of $T$ either. It is as if their quarrel so disturbs the quarrelling pair that they are no longer in a position to cooperate effectively with anyone. Hence the group $S \cup \{i, j\}$ can obtain the outcome achievable by just the votes of $S$. We can therefore define a reciprocal strong YES-quarrel by adopting the general asymmetric framework above, but further specifying condition $YQ$ as:

$$\forall G \in \mathcal{G} : S \cup \{i, j\} \in \hat{W} \iff S \in W$$

(YQ$_s$)

The entailment from left to right states that for any division in which $i$ and $j$ vote NO, if the outcome is NO in $G$, then for the division that is identical except both $i$ and $j$ vote YES, the outcome in $\hat{G}$ would still be NO. This implies that if there is a group excluding $i$ and $j$ that could not secure a YES-outcome on its own, but could do so with the cooperation of one of $i$ or $j$, that group could not do so in $\hat{G}$ if both quarrelling players join them to vote YES. This means that $i$ and $j$’s YES-quarrel is a strong quarrel. The entailment from right to left, by contrast, states that for any division in which $i$ and $j$ vote NO, if the outcome is YES in $G$, then for the division that is identical except both $i$ and $j$ vote YES, the outcome in $\hat{G}$ would still be YES. This means that if other players can secure a YES-outcome without $i$ or $j$’s vote, then they can do so even if $i$ and $j$ bring themselves and their quarrel to the YES side – by excluding or isolating the quarrelling pair. In other words, their YES-quarrel is not cataclysmic.

Consider what a reciprocal strong quarrel implies for a weighted voting game $G$. If $\hat{G}$ contains a strong quarrel between $i$ and $j$, then if both of them vote YES together in $S \cup \{i, j\}$, this forces the weight of **both** two players to 0.

Again, a reciprocal strong YES-quarrel clearly satisfies $YQ$: because the original game $G$ is monotonic, satisfying $YQ_s$ entails satisfying both $YQ$-1 and $YQ$-2. Reciprocal strong quarrelling does not, however, satisfy monotonicity. This is because a reciprocal strong quarrel affects not just the quarrellers themselves as a pair, but also implies that neither can effectively cooperate with any other members of $S \cup \{i, j\}$ either. Since by definition of a reciprocal strong quarrel the outcome of $S \cup \{i, j\}$ is equal to the outcome of $S$, monotonicity may be violated. This can be seen with a simple example. Consider the two-player game $G$ with YES-successful sets $W = \{\{1\}, \{2\}, \{1, 2\}\}$. When players 1 and 2 quarrel, we have a game $\hat{G}$ where $\{1\} \in \hat{W}$ and $\{2\} \in \hat{W}$, but $\{1, 2\} \notin \hat{W}$. 

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However, such local violations of monotonicity are the only possible violations. In particular, a reciprocal strong quarrel is quasi-monotonic:

**Theorem 8.2.** A game $\hat{G}$ with a reciprocal strong YES-quarrel is quasi-monotonic.

Nevertheless, as demonstrated by Theorem 4.2, even a quasi-monotonic conception is disposed to induce non-monotonicity over quarrellers, and hence falls prey to Theorem 4.1. This can be illustrated again, in the case of an asymmetric reciprocal strong YES-quarrel, by dictator-rule voting with three voters in $G$. If the dictator and a dummy quarrel in this way, this renders the dummy decisive in half of the eight divisions in $\hat{G}$, thus causing any measure of voting power $\Psi$ to violate the standard quarrel postulate. Such a quarrel is therefore not fit to be basis for a reasonable quarrel postulate.

### 8.3 Reciprocal Cataclysmic Quarrelling

Intuitively, a reciprocal cataclysmic quarrel between $i$ and $j$ is a quarrel so destructive that its effects spill beyond $i$ and $j$ and prevent any members of the group from effectively cooperating to secure an outcome whenever $i$ and $j$ join their ranks. A cataclysmic quarrel is so destructive we might dub it a war – a war between $i$ or $j$ that has, as it were, the collateral damage of ruining effective cooperation between (or the efficacy of) any players who vote on the side of the quarrelling couple. Since a reciprocal cataclysmic quarrel between $i$ and $j$ renders the group $S \cup \{i, j\}$ so ineffective that no members can function properly, the group must exclude all its members to function! Hence the group $S \cup \{i, j\}$ can obtain the outcome achievable by just the votes of $\emptyset$. We can therefore define a cataclysmic YES-quarrel by adopting the general asymmetric framework, but further specifying condition $YQ$ as:

$$\forall G \in \mathcal{G} : S \cup \{i, j\} \in \hat{W} \iff \emptyset \in \mathcal{W}$$

($YQ_c$)

This implies that a reciprocal cataclysmic YES-quarrel is precisely Felsenthal and Machover’s conception of a quarrel!

The entailment from left to right states that for the division in which everyone votes NO, if the outcome is NO in $G$, then for any division in which $i$ and $j$ vote YES, the outcome is NO in $\hat{G}$ – even if everyone else were to vote YES. This means that if $i$ and $j$ vote YES, their quarrel deluges any other YES-voters and renders them ineffective, i.e., this is a cataclysmic YES-quarrel. The entailment from right to left states that for the division in which everyone votes NO, if the outcome is YES in $G$ (which of course never holds if $G$ is non-trivial in addition to monotonic), then for any division in which $i$ and $j$ vote YES, the outcome is YES in $\hat{G}$. This implies that $i$ and $j$ are not less YES-successful in $\hat{G}$ than $\emptyset$, which means that if even $\emptyset$ is sufficient to secure a YES-outcome, then even a war between $i$ and $j$ cannot prevent YES.

The nature of this definition is illuminated by considering its effect on a weighted voting game $G$. If $\hat{G}$ contains a reciprocal cataclysmic quarrel between $i$ and $j$, then when they both vote YES together in $S \cup \{i, j\}$, this forces the weight of every player in $S \cup \{i, j\}$ to 0.

Although cataclysmic YES-quarrelling is a rather exaggerated type of quarrel, it is a perfectly reasonable conception: because satisfying $YQ_c$ entails satisfying $YQ-1$ and $YQ-2$, a reciprocal cataclysmic YES-quarrel satisfies $YQ$. But, as we have already proven in Theorem 5.1 when discussing
the FM-conception, a reciprocal cataclysmic quarrel not only violates monotonicity, it is supremely non-monotonic.

**Corollary 8.3.** A game $\hat{G}$ with a reciprocal cataclysmic YES-quarrel is supremely non-monotonic.

### 8.4 Symmetric Quarrelling

The degrees of quarrel we have analyzed for the YES side apply equally to the NO side. Therefore, with our classification by degree of quarrel in hand, we can now return to our General Quarrel Framework, and specify it further, by degree of quarrel, to yield three positive degrees of symmetric reciprocal quarrel. To do so, for each type of quarrel we must replace YQ and NQ with the corresponding weak, strong, or cataclysmic specification. Each type of symmetric reciprocal quarrel is therefore equal to the General Quarrel Framework, except we replace YQ and NQ in each case, respectively, with the following. For all $G \in G$ and for any $S$ containing neither $i$ nor $j$:

**Symmetric Reciprocal Weak Quarrel:**

\[
S \cup \{i,j\} \in \hat{W} \iff S \cup \{i\} \in W \lor S \cup \{j\} \in W \quad (YQ_w)
\]

\[
S \in \hat{W} \iff S \cup \{i\} \in W \land S \cup \{j\} \in W \quad (NQ_w)
\]

**Symmetric Reciprocal Strong Quarrel:**

\[
S \cup \{i,j\} \in \hat{W} \iff S \in W \quad (YQ_s)
\]

\[
S \in \hat{W} \iff S \cup \{i,j\} \in W \quad (NQ_s)
\]

**Symmetric Reciprocal Cataclysmic Quarrel:**

\[
S \cup \{i,j\} \in \hat{W} \iff \emptyset \in W \quad (YQ_c)
\]

\[
S \in \hat{W} \iff [n] \in W \quad (NQ_c)
\]

Since we have already furnished the verbal statement and intuitive meaning for the YES side, here we furnish the intuitive meaning only for the NO side. In $[NQ_w]$, the entailment from left to right means that $\hat{G}$ does not comprise a NO-quarrel any stronger than weak: in particular, it does not comprise a strong NO-quarrel. The entailment from left to right, in turn, means that effective cooperation between $i$ and $j$ on the YES side is completely neutralized, that is, that $\hat{G}$ comprises at least a weak NO-quarrel. In $[NQ_s]$ the entailment from left to right means that if other players can secure a NO-outcome without $i$ or $j$’s vote, then they can do so even if $i$ and $j$ bring themselves and their quarrel to the NO side, that is, their NO-quarrel is not cataclysmic. The entailment from left to right, in turn, means that $\hat{G}$ does comprise a strong NO-quarrel. Finally, in $[NQ_c]$ the entailment from left to right means that if even $[n]$ cannot prevent a NO-outcome, then even a war between $i$ and $j$ on the NO side will not stop a NO outcome. The entailment from right to left, in turn, means that if $i$ and $j$ bring their quarrel to the NO side, it deluges any other NO-voters and renders them ineffective, i.e., this is a cataclysmic NO-quarrel.

Now, just as in the asymmetric case, a reciprocal weak quarrel satisfies monotonicity:

**Theorem 8.4.** A game $\hat{G}$ with a symmetric reciprocal weak quarrel is monotonic.
Does a strong symmetric reciprocal quarrel mimic the asymmetric case in being quasi-monotonic? The surprising answer is no. Indeed, in the presence of a reciprocal strong quarrel, symmetry and monotonicity are completely incompatible:

**Theorem 8.5.** A game $\hat{G}$ with a symmetric reciprocal strong quarrel is supremely non-monotonic.

It is therefore unsurprising that symmetry and monotonicity are also completely incompatible in the presence of a reciprocal cataclysmic quarrel:

**Theorem 8.6.** A game $\hat{G}$ with a symmetric reciprocal cataclysmic quarrel is supremely non-monotonic.

### 9 Non-Reciprocal Quarrelling

Although we have only treated reciprocal quarrels up to now, our General Quarrel Framework encompasses, and can be easily used to formalize, non-reciprocal quarrels as well. We shall focus on a quarrel $i$ has with $j$ but not vice versa. Again, think of $j$ as entirely immune to quarrelling: no matter how much others around $j$ quarrel, with $j$ itself or with others, $j$ continues to be willing and able to cooperate effectively with whoever effectively reciprocates cooperation with it. But of course if $i$ quarrels with $j$, even non-reciprocally, then $i$ does not effectively cooperate with $j$.

We here provide the specification of YQ and NQ for a non-reciprocal weak, strong, and cataclysmic quarrel, respectively:

**Non-Reciprocal Weak Quarrel:**

$$S \cup \{i, j\} \in \hat{W} \iff S \cup \{j\} \in W \lor S \cup \{i\} \in W \quad (\text{YQ}_w)$$

$$S \in \hat{W} \iff S \cup \{j\} \in W \land S \cup \{i\} \in W \quad (\text{NQ}_w)$$

**Non-Reciprocal Strong Quarrel:**

$$S \cup \{i, j\} \in \hat{W} \iff S \cup \{j\} \in W \quad (\text{YQ}_s)$$

$$S \in \hat{W} \iff S \cup \{i\} \in W \quad (\text{NQ}_s)$$

**Non-Reciprocal Cataclysmic Quarrel:**

$$S \cup \{i, j\} \in \hat{W} \iff \{j\} \in W \quad (\text{YQ}_c)$$

$$S \in \hat{W} \iff [n] \setminus \{j\} \in W \quad (\text{NQ}_c)$$

The first thing to note is that a non-reciprocal weak quarrel is formally identical to a reciprocal weak quarrel. It immediately follows that it is monotonic, by Theorem 8.1 and Theorem 8.4.

**Corollary 9.1.** A game $\hat{G}$ with a non-reciprocal (symmetric or asymmetric) weak quarrel is monotonic.

The only thing that changes is the quarrel’s substantive meaning, i.e., the interpretation we give to the formal model: whereas in a reciprocal quarrel neither $i$ nor $j$ is willing or able to cooperate effectively with the other, in a non-reciprocal quarrel of $i$ against $j$, only $i$ is not willing or able to...
cooperate effectively with \( j \); \( j \) is in principle willing and able, but does not effectively cooperate with \( i \) because effective cooperation cannot be unilateral.

When we turn to a strong quarrel, by contrast, non-reciprocality does make a formal difference. This is because a strong quarrel reduces the cooperative efficacy of the quarrelling players with everyone else whose vote agrees with the quarrelling player’s vote. But in a non-reciprocal quarrel, the non-quarrelling player \( j \) is not quarrelling, and so, unlike in a reciprocal strong quarrel, can continue to cooperate effectively with the others despite \( i \)’s presence – just as the other players can. This is what \( \text{YQ}_s \) and \( \text{NQ}_s \) express.

The second thing to note is that this symmetric non-reciprocal strong quarrel is precisely the Laruelle and Valenciano conception of a quarrel! It is therefore non-monotonic; indeed, by Theorem 6.2, unlike a reciprocal strong quarrel, it is supremely non-monotonic!

**Corollary 9.2.** A game \( \hat{G} \) with a symmetric non-reciprocal strong quarrel is supremely non-monotonic.

Laruelle and Valenciano (2005a: 30-31) claim to have proven that no decisiveness measure of voting power displays the quarrelling paradox based on this conception, which, if true, would seem to call into question our claim that any standard postulate based on a non-monotonic conception of quarrelling is unreasonable. The problem with their proof, however, is that it relies on a formalization of the notion of decisiveness that is suited only for monotonic games, i.e., it counts a voter as decisive only if its vote agrees with the outcome (2005a: 22). They do this presumably because they did not realize that the LV-quarrel, like the FM-quarrel, also violates monotonicity. But as noted earlier, in non-monotonic games, players may have an incentive to vote strategically (i.e., vote against the outcome they prefer) and can be decisive even in divisions in which they have voted against the outcome. And once we adopt the formalization of decisiveness broad enough to cover non-monotonic games, their proof fails, as we can see with two simple counterexamples. Consider the three-player monotonic binary voting game for which \( \mathcal{W} = \{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}\} \). The Penrose-Banzhaf measure yields an a priori voting power for player 2 equal to \( \frac{1}{4} \), since it is decisive in two of the eight logically possible divisions. If we impose an LV-quarrel of player 1 against player 2, however, then player 2’s Penrose-Banzhaf measure rises to \( \frac{1}{2} \), in violation of the standard quarrel postulate based on the LV-quarrel. Even more starkly, in dictator-rule voting with three players, if the dictator engages in an LV-quarrel against one of the dummies, then the dummy is transformed into an (anti-)dictator.

The third thing to note is what happens when we combine asymmetry and non-reciprocality. Consider an asymmetric non-recipocpal strong yes-quarrel of \( i \) against \( j \). Again, it is easy to verify that such a quarrel does not satisfy monotonicity. Further, since such a quarrel is the asymmetric variant of an LV-quarrel, one might expect, as in Corollary 9.2, that it also be supremely non-monotonic. Yet in fact this is not the case: it is quasi-monotonic!

**Theorem 9.3.** A game \( \hat{G} \) with an asymmetric non-reciprocal strong yes-quarrel is quasi-monotonic.

Like the reciprocal version, an asymmetric non-recipocpal strong yes-quarrel is nevertheless disposed to induce non-monotonicity over quarrellers, as illustrated again by dictator-rule voting.
with three voters in $G$, which yields the same results as the reciprocal version. Such a quarrel is therefore not fit to be basis for a reasonable quarrel postulate.

Finally, consider non-reciprocal cataclysmic quarrels. Again here there is a difference between a non-reciprocal cataclysmic quarrel and a reciprocal one. Unlike all the other players whose votes agree with $i$’s, the non-quarrelling player $j$ is unaffected by $i$’s cataclysmic quarrel with it: as long as $j$ is a dictator in the original game $G$, i.e., $\{j\} \in W$ and $|n| \setminus \{j\} \notin W$, then it can continue to dictate the outcome no matter how cataclysmically $i$ wages war in $j$’s presence. This means the non-quarrelling player $j$ not only does not quarrel, but is immune to any quarrel – even a quarrel directed against itself.

The non-reciprocal version is, like the reciprocal one, non-monotonic. Indeed, as the reader might expect – and this time, unlike for strong non-reciprocal quarrels, there are no surprises – non-reciprocal cataclysmic quarrels are supremely non-monotonic, for both the asymmetric and symmetric cases.

**Theorem 9.4.** A game $\hat{G}$ with an asymmetric non-reciprocal cataclysmic YES-quarrel is supremely non-monotonic.

**Theorem 9.5.** A game $\hat{G}$ with a symmetric non-reciprocal cataclysmic quarrel is supremely non-monotonic.

## 10 A General Typology of Quarrels

We summarize in Table 1 our findings by presenting our typology of the twelve conceptions of (non-hybrid) quarrels and the extent to which they are monotonic. The only conception of a quarrel that is both monotonic and symmetric is a symmetric weak quarrel – whether reciprocal or non-reciprocal, which, recall, are formally identical.

| Quarrel       | Asymmetric Reciprocal | Asymmetric Non-Reciprocal | Symmetric Reciprocal | Symmetric Non-Reciprocal |
|---------------|------------------------|---------------------------|----------------------|----------------------------|
| Weak          | Monotonic              | Monotonic                 | Monotonic            | Monotonic                  |
|               | [Thm 8.1]              | [Thm 8.1+Cor 9.1]         | [Thm 8.4]            | [Thm 8.4+Cor 9.1]          |
| Strong        | Quasi-Monotonic        | Quasi-Monotonic           | Supremely Non-Monotonic | Supremely Non-Monotonic     |
|               | [Thm 8.2]              | [Thm 9.3]                 | [Thm 8.5]             | [Thm 9.2+Cor 9.2]          |
| Cataclysmic   | Supremely Non-Monotonic | Supremely Non-Monotonic   | Supremely            | Supremely                  |
|               | [Thm 5.1+Cor 8.3]      | [Thm 9.4]                 | [Thm 8.6]             | [Thm 9.5]                  |

| Table 1: Typology of a Class of Quarrels |

Although we do not treat hybrid quarrels here, the power of our framework is reflected in the fact that it permits us to classify, formally characterize, and investigate these as well. First, our framework would allow us to formalize and investigate a class of quasi-symmetric quarrels in which
voters $i$ and $j$ quarrel with each other on both the YES and NO sides, but in which their YES-querrel is different in degree from their NO-querrel. Similarly, it would allow us to formalize and investigate a class of quasi-reciprocal quarrels in which both $i$ and $j$ quarrel with each other, but in which $i$’s quarrel against $j$ is different in degree from $j$’s quarrel against $i$. Finally, it would allow us to formalize and investigate a class of quasi-symmetric, quasi-reciprocal quarrels in which the degree of quarrel differs not only on the YES and NO sides, but also between the two players.

11 Conclusion: A Reasonable Quarrel Postulate

We have presented a general framework for twelve conceptions of quarrelling, where each conception is distinguished according to its symmetry, reciprocality, and strength in quarrelling. We have also argued that a structural quarrel postulate, to be reasonable and suitable for measures of a priori voting power in general, must be based on a conception of a quarrel that is not only itself reasonable, but also monotonic and symmetric. Our framework has enabled us to identify a conception that fulfils these criteria: a symmetric, weak quarrel, whether reciprocal or non-reciprocal. This is the only conception in our framework that satisfies monotonicity and concerns both YES- and NO-voting power. It therefore furnishes the basis for a reasonable structural quarrel postulate – indeed, it is the only conception in our framework to do so.

We conclude by noting how the classic measures of voting power fare against the standard quarrel postulate based on a symmetric, weak quarrel.

Theorem 11.1. The Shapley-Shubik index satisfies the standard quarrel postulate based on a symmetric, weak quarrel.

Theorem 11.2. The Penrose-Banzhaf measure satisfies the standard quarrel postulate based on a symmetric, weak quarrel.

In contrast, the Shapley-Shubik index and the Penrose-Banzhaf measure – like any logically possible measure – fail to satisfy the standard quarrel postulate based on the non-monotonic conceptions of quarrel in the framework.

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Appendix of Proofs

11.1 Proofs for Section 4

Theorem 4.1. If a reasonable conception of quarrelling \( Q \) (which satisfies CSR) is disposed to induce non-monotonicity over quarrellers, then the standard quarrel postulate based on \( Q \) will be violated by any measure of voting power \( \Psi \).

Proof. For any measure \( \Psi \) (which, by definition, satisfies the dummy postulates), \( \psi_i=0 \) if and only if \( i \) is a dummy. We need to furnish an instance in which a dummy \( i \) in the initial voting game \( G \) is transformed into a non-dummy in the game derived by imposing a quarrel between \( i \) and \( j \) via a transformation rule \( Q \) that satisfies CSR but also DNQ.

Let \( G \) be a monotonic binary voting game in which player 1 is a dictator and player 2 a dummy. Now induce a quarrel between 1 and 2 that satisfies CSR, but which is disposed to induce non-monotonicity over the quarrellers 1 and 2. Then there must be at least one pair of divisions in the derived game \( \hat{G} \) such that: the quarrelling players’ votes agree in the first division and disagree in the second, where the second is identical to the first except for player 2’s vote; in the division in which they disagree, player 1 is successful and player 2 unsuccessful; and in the division in which players 1 and 2 agree, they are both unsuccessful. But since the two divisions are identical except for player 2’s vote, it follows that 2 is decisive in these two divisions in \( \hat{G} \). Therefore, \( \psi_2 = 0 \) but \( \psi_2 > 0 \), in violation of the standard quarrel postulate. \( \square \)

Theorem 4.2. Let \( \hat{G} \) be a non-monotonic game derived from a binary monotonic game \( \mathcal{G}' \in \mathcal{G} \) by imposing a quarrel \( Q \) between two players \( l \) and \( m \), where \( Q \) satisfies CSR. Then there exists a game \( G \in \mathcal{G} \) such that \( \hat{G} \) can be derived from \( G \) by imposing the same conception of a quarrel \( Q \) between players \( i \) and \( j \) and where \( \hat{G} \) is non-monotonic over the quarrellers \( i \) and \( j \).

Proof. So \( \hat{G} \) is non-monotonic. We have two cases. First, suppose the maximum cardinality subset \( S \) that exhibits non-monotonicity in \( \hat{G} \) has \( |S| = 1 \), i.e., \( S = \{i\} \), such that \( \emptyset \in \hat{W} \) but \( \{i\} \notin \hat{W} \). Let \( G \) agree with \( \hat{G} \) on every division except \((\emptyset, [n])\). In particular, \( \emptyset \notin W \). Thus \( G \) is monotonic. Now take any \( j \neq i \). Observe the \( i \) and \( j \) are both on the no-side of \((\emptyset, [n])\). It follows that the derivation of \( \hat{G} \) from \( G \) by imposing a quarrel between \( i \) and \( j \) does satisfy CSR. Furthermore, \( \hat{G} \) is non-monotonic over the quarrelling pair \( \{i, j\} \), as desired.

Otherwise, there exists a pair \( i, j \in [n] \) and a subset \( S \subseteq [n] \) such that \( S \cup \{i\} \in \hat{W} \) but \( S \cup \{i, j\} \notin \hat{W} \). We construct a monotonic \( G \) as follows. Take any set \( T \subseteq [n] \).

1. Suppose \( T \cap \{i, j\} \neq \emptyset \). If there exists an \( X \subseteq T \) such that \( X \in \hat{W} \), then let \( T \in W \).
2. Suppose \( T \cap \{i, j\} = \emptyset \). If there exists an \( X \supseteq T \) such that \( X \notin \hat{W} \), then let \( T \notin W \).
3. Otherwise let \( X \in \hat{W} \) if and only if \( X \in W \).

Observe that the derivation of \( \hat{G} \) from \( G \) satisfies CSR with respect to the quarrelling pair \( \{i, j\} \). Furthermore, by definition of \( S \), \( \hat{G} \) is non-monotonic over the quarrelling pair \( \{i, j\} \). It remains to show that \( G \) is monotonic. Suppose not. Then there is a set \( T \) such that \( T \in W \) and \( T \cup \{k\} \notin W \), for some \( k \notin T \). We have three cases:

\[ \text{The proof evidently relies on the fact that in non-monotonic games a player may be decisive even when its vote disagrees with the outcome.} \]

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(i) Suppose \( T \cap \{i,j\} \neq \emptyset \). Since \( T \cup \{k\} \notin \mathcal{W} \), by Rule (1) there is no \( X \subseteq T \cup \{k\} \) such that \( X \in \hat{\mathcal{W}} \). In particular, \( T \notin \hat{\mathcal{W}} \). But \( T \in \mathcal{W} \), so by Rule (1), there exists an \( X \subseteq T \) such that \( X \in \hat{\mathcal{W}} \), a contradiction.

(ii) Suppose \( T \cap \{i,j\} = \emptyset \) and \( k \in \{i,j\} \). Without loss of generality \( k = i \). Then \( (T \cup \{k\}) \cap \{i,j\} \neq \emptyset \). Since \( T \cup \{i\} \notin \mathcal{W} \), by Rule (1) there is no \( X \subseteq T \cup \{i\} \) such that \( X \in \hat{\mathcal{W}} \). In particular, \( T \notin \hat{\mathcal{W}} \). But \( T \in \mathcal{W} \), so by Rule (2) there does not exist an \( Y \supseteq T \) such that \( Y \notin \hat{\mathcal{W}} \). In particular, \( T \cup \{i\} \in \hat{\mathcal{W}} \). But then by Rule (1), \( T \cup \{i\} \in \mathcal{W} \), a contradiction.

(iii) Suppose \( (T \cup \{k\}) \cap \{i,j\} = \emptyset \). In particular \( k \notin \{i,j\} \). Since \( T \in \mathcal{W} \), by Rule (2) there does not exist a \( Y \supseteq T \) such that \( Y \notin \hat{\mathcal{W}} \). In particular, \( T \cup \{k\} \in \hat{\mathcal{W}} \). But \( T \cup \{k\} \notin \mathcal{W} \), which implies \( (T \cup \{k\}) \cap \{i,j\} \neq \emptyset \), a contradiction. \( \square \)

11.2 Proofs for Section 5

**Theorem 5.1** The FM-rule is supremely non-monotonic.

*Proof.* We must show the FM-rule is not \( k \)-monotonic, for any \( k \). Consider a game \( G \) whose *yes*-successful sets are given by \( \mathcal{W} = \{ S \subseteq [n] : S \neq \emptyset \} \). In particular, each player can ensure a *yes*-outcome simply by voting *yes*: each is a no-blocker.

Next take the derived game \( \hat{G} \) in which \( i \) quarrels with \( j \). Consider the group \( S = [n] \). Since \( i \) and \( j \) are both in \( [n] \), by definition of the FM-quarrel, \( [n] \notin \hat{\mathcal{W}} \). Now take \( T = [n] \setminus \{i\} \) and any \( U \subseteq T \). It follows that \( T \setminus U \subseteq [n] \setminus \{i\} \). In particular \( i \notin T \setminus U \). So by definition, \( T \setminus U \in \hat{\mathcal{W}} \) if and only if \( T \setminus U \in \mathcal{W} \). Since \( U \subseteq T \), \( T \setminus U \in \hat{\mathcal{W}} \). But \( U \) can take on any cardinality from 0 to \( n-2 \). Thus the FM-rule is not \( k \)-monotonic for any \( k \). \( \square \)

11.3 Proofs for Section 6

**Theorem 6.1** The LV-rule induces a non-reciprocal quarrel.

*Proof.* Take the two-player simple voting game \( G \) for which \( \mathcal{W} = \{\{1\}, \{1,2\}\} \). Now suppose (contrary to fact) that the LV-rule derives a reciprocal quarrel \( \hat{G} \) from \( G \), such that \( \hat{G} = \hat{G}^{i \leftrightarrow j} = \hat{G}^{j \leftrightarrow i} \).

On this supposition, first consider game \( \hat{G}^{1 \leftrightarrow 2} \), which incorporates the LV-quarrel of \( i = 1 \) with \( j = 2 \). Let’s calculate the *yes*-successful sets \( \hat{\mathcal{W}}^{1 \leftrightarrow 2} \). Observe that for any \( j \in S \), \( S \in \mathcal{W} \) if and only if \( S \setminus \{j\} \in \mathcal{W} \). Thus \( \{1,2\} \in \mathcal{W} \) and \( \{2\} \in \mathcal{W} \) if and only if \( \{2\} \in \mathcal{W} \). But \( \{2\} \notin \mathcal{W} \), so both \( \{1,2\} \notin \hat{\mathcal{W}} \) and \( \{2\} \notin \hat{\mathcal{W}} \). Next, for any \( j \notin S \), \( S \in \hat{\mathcal{W}} \) if and only if \( S \cup \{j\} \in \mathcal{W} \). Thus \( \emptyset \in \hat{\mathcal{W}} \) and \( \{1\} \in \hat{\mathcal{W}} \) if and only if \( \{1\} \in \mathcal{W} \). But \( \{1\} \in \mathcal{W} \), so both \( \emptyset \in \hat{\mathcal{W}} \) and \( \{1\} \in \hat{\mathcal{W}} \). Hence \( \hat{\mathcal{W}}^{1 \leftrightarrow 2} = \{\emptyset, \{1\}\} \).

Again supposing the LV-rule satisfies reciprocality, consider now \( \hat{G}^{2 \leftrightarrow 1} \), which incorporates the LV-quarrel of \( i = 2 \) with \( j = 1 \). Again, for any \( j \in S \), \( S \in \hat{\mathcal{W}} \) if and only if \( S \setminus \{j\} \in \mathcal{W} \). This implies \( \{1,2\} \notin \hat{\mathcal{W}} \) and \( \{1\} \in \hat{\mathcal{W}} \) if and only if \( \{1\} \in \mathcal{W} \). But \( \{1\} \in \mathcal{W} \), so both \( \{1,2\} \notin \hat{\mathcal{W}} \) and \( \{1\} \in \hat{\mathcal{W}} \). Next, for any \( j \notin S \), \( S \in \hat{\mathcal{W}} \) if and only if \( \{2\} \in \mathcal{W} \). But \( \{2\} \notin \mathcal{W} \), so both \( \emptyset \notin \hat{\mathcal{W}} \) and \( \{2\} \notin \hat{\mathcal{W}} \). It follows that \( \hat{\mathcal{W}}^{2 \leftrightarrow 1} = \{\{1\}, \{1,2\}\} \).

Recall, however, that \( \hat{\mathcal{W}}^{1 \leftrightarrow 2} = \{\emptyset, \{1\}\} \). This implies that \( \hat{G}^{2 \leftrightarrow 1} \neq \hat{G}^{1 \leftrightarrow 2} \), which contradicts our initial supposition that an LV-quarrel is reciprocal. \( \square \)
Theorem 6.2. The LV-rule is supremely non-monotonic.

Proof. We must show the LV-rule is not \( k \)-monotonic, for any \( k \). Consider a game \( G \) for which \( W = \{ S \subseteq [n] : i \in S \} \). Next take the LV-quarrel \( \hat{G} \) where \( i \) quarrels with \( j \). Take the group \( S = [n] \). Since \( j \in [n] \), the unanimous division \( S \) has, by definition, the same outcome in \( \hat{G} \) as \( ([n] \setminus \{i\}, \{i\}) \) does in \( G \), namely no. So \( S \notin W \).

Next take \( T = [n] \setminus \{j\} \) and any \( U \subseteq T \). It follows that \( T \setminus U \subseteq [n] \setminus \{j\} \). In particular \( j \notin T \setminus U \). So by definition, \( T \setminus U \) has the same outcome in \( \hat{G} \) as \( T \setminus U \cup \{i\} \) does in \( G \), namely yes. But \( U \) can take on any cardinality from 0 to \( n - 1 \). Thus the LV-rule is not \( k \)-monotonic for any \( k \).

11.4 Proofs for Section 8

Theorem 8.1. A game \( G \) with a reciprocal weak yes-quarrel is monotonic.

Proof. The original game \( G \) is monotonic. But the only subsets that have changed from yes-successful to yes-unsuccessful are of the form \( S \cup \{i, j\} \). So suppose we now have a violation in monotonicity involving \( S \cup \{i, j\} \). By definition, \( S \cup \{i\} \notin W \) and \( S \cup \{j\} \notin W \), and hence \( S \cup \{i\} \notin \hat{W} \) and \( S \cup \{j\} \notin \hat{W} \). So the violation must be due to a yes-successful set \( S \cup \{i, j\} \setminus k \).

Now consider \( S \cup \{j\} \setminus k \) and \( S \cup \{i\} \setminus k \). Since both \( S \cup \{i\} \notin W \) and \( S \cup \{j\} \notin W \), by monotonicity we have \( S \cup \{j\} \setminus k \notin W \) and \( S \cup \{i\} \setminus k \notin W \). But then, by definition, \( S \cup \{i, j\} \setminus k \notin \hat{W} \). This contradicts the supposition that \( S \cup \{i, j\} \) violates monotonicity in \( \hat{G} \).

Theorem 8.2. A game \( \hat{G} \) with a reciprocal strong yes-quarrel is quasi-monotonic.

Proof. Recall a voting rule is quasi-monotonic if it is 1-monotonic. So we must prove a game \( \hat{G} \) with a strong quarrel is 1-monotonic. Take any group \( S \). By definition, the divisions \( S \cup \{i\} \), \( S \cup \{j\} \), and all the other divisions whose yes-voters are subsets of \( S \) have the same outcome in \( \hat{G} \) as in \( G \). But the original game \( G \) is monotonic. It immediately follows that the 1-monotonic property (in fact, the 0-monotonic property) holds for any division \( T \) where \( T \subseteq S \cup \{i\} \) or \( T \subseteq S \cup \{j\} \).

Thus violations in 1-monotonicity can arise only for a division of the form \( S \cup \{i, j\} \setminus k \). So assume \( S \cup \{i, j\} \notin \hat{W} \). By definition of a strong quarrel this implies \( S \notin \hat{W} \). Now take any \( T \subseteq S \cup \{i, j\} \). We have four possibilities. First, assume that \( T \) contains both \( i \) and \( j \). Now \( T \setminus \{i, j\} \subseteq S \). So, by the monotonicity of \( G \) we have \( T \setminus \{i, j\} \notin \hat{W} \). Thus, by the definition of a strong quarrel, this implies \( T \notin \hat{W} \). This causes no violation of 1-monotonicity (nor, indeed, 0-monotonicity).

Second, assume that \( T \) contains \( i \) but not \( j \). Now set \( K = \{i\} \). Then \( T \setminus K = T \setminus \{i\} \subseteq S \). Then, by monotonicity, \( T \setminus K \notin \hat{W} \) because \( S \notin \hat{W} \). So, by definition of a strong quarrel, we have \( T \setminus K \notin \hat{W} \). Since \( |K| = |\{i\}| = 1 \), this is allowed by the definition of 1-monotonicity. The third case where \( T \) contains \( j \) but not \( i \) can be handled in a symmetric manner.

Fourth, assume that \( T \) contains neither \( i \) nor \( j \). Then \( T \subseteq S \). But \( S \notin \hat{W} \) so, by monotonicity, \( T \notin \hat{W} \). Now because \( T \) contains neither \( i \) nor \( j \), \( T \in \hat{W} \) if and only if \( T \in \hat{W} \). Thus \( T \notin \hat{W} \) and so causes no violation of 1-monotonicity (nor, indeed, 0-monotonicity). It follows that a game \( \hat{G} \) with a reciprocal strong yes-quarrel is quasi-monotonic.

Theorem 8.4. A game \( \hat{G} \) with a symmetric reciprocal weak quarrel is monotonic.
Proof. The original game $G$ is monotonic. We now have two cases. First, the only subsets that have changed from yes-successful to yes-unsuccessful are of the form $S \cup \{i, j\}$. So suppose we have a violation in monotonicity involving $S \cup \{i, j\}$. By definition, $S \cup \{i\} \not\in W$ and $S \cup \{j\} \not\in W$, and hence $S \cup \{i\} \not\in \hat{W}$ and $S \cup \{j\} \not\in \hat{W}$. So the violation must be due to a yes-successful set $S \cup \{i, j\} \setminus k$.

Now consider $S \cup \{j\} \setminus k$ and $S \cup \{i\} \setminus k$. Since $S \cup \{i\} \not\in W$ and $S \cup \{j\} \not\in W$, by monotonicity we have $S \cup \{j\} \setminus k \not\in W$ and $S \cup \{i\} \setminus k \not\in W$. But then by definition, $S \cup \{i, j\} \setminus k \not\in \hat{W}$. This contradicts the supposition that $S \cup \{i, j\}$ violates monotonicity in $\hat{G}$.

Second, the only subsets $S$ that have changed from yes-successful to yes-unsuccessful are of the form where $i, j \not\in S$. So suppose we have a violation in monotonicity involving $S$. By definition, $S \cup \{i\} \in W$ and $S \cup \{j\} \in W$ and hence $S \cup \{i\} \in \hat{W}$ and $S \cup \{j\} \in \hat{W}$. So the violation must be due to a yes-unsuccessful set $S \cup \{k\}$.

Now consider $S \cup \{j, k\}$ and $S \cup \{i, k\}$. Since $S \cup \{i\} \in W$ and $S \cup \{j\} \in W$, by monotonicity $S \cup \{j, k\} \in W$ and $S \cup \{i, k\} \in W$. But then by definition, $S \cup \{k\} \not\in \hat{W}$. This contradicts the supposition that $S$ violates monotonicity in $\hat{G}$. \hfill \Box

**Theorem 8.5.** A game $\hat{G}$ with a symmetric reciprocal strong quarrel is supremely non-monotonic.

Proof. We must show the symmetric reciprocal strong quarrel rule is not $k$-monotonic, for any $k$. Consider a game $G$ for which $W = \{S \subseteq [n] : S \cap \{i, j\} \neq \emptyset\}$. Thus $S$ is yes-successful if and only if it contains either $i$ or $j$ or both. Next consider the symmetric reciprocal strong quarrel $\hat{G}$ where $i$ quarrels with $j$. Take the group $S = [n]$. Since $i, j \in [n]$, the unanimous division $S$ has, by definition, the same outcome in $\hat{G}$ as $([n] \setminus \{i, j\}, \{i, j\})$ does in $G$, namely NO. So $S \not\in \hat{W}$.

Next take $T = [n] \setminus \{j\}$ and any $U \subseteq T$. It follows that $T \setminus U \subseteq [n] \setminus \{j\}$. In particular, $j \notin T \setminus U$. We have two possibilities. If $i \in T \setminus U$, then, by definition, $T \setminus U$ has the same outcome in $\hat{G}$ as $T \setminus U$ does in $G$, namely yes, since it contains $i$. By control, if $i \notin T \setminus U$, then by definition $T \setminus U$ has the same outcome in $\hat{G}$ as $T \setminus U \cup \{i, j\}$ does in $G$, namely yes. But $U$ can take on any cardinality from 0 to $n - 1$. Thus the symmetric reciprocal strong quarrel rule is not $k$-monotonic for any $k$. \hfill \Box

**Theorem 8.6.** A game $\hat{G}$ with a symmetric reciprocal cataclysmic quarrel is supremely non-monotonic.

Proof. We must show a symmetric reciprocal cataclysmic quarrel rule is not $k$-monotonic, for any $k$. Consider a game $G$ for which $W = \{S \subseteq [n] : S \neq \emptyset\}$. In particular, each player can ensure a yes-outcome simply by voting yes; that is, each is a NO-blocker.

Next take the symmetric reciprocal cataclysmic quarrel $\hat{G}$ where $i$ quarrels with $j$. Consider $S = [n]$. Since $i$ and $j$ are both in $[n]$, by definition of the quarrel, $S \not\in W$. Now take $T = [n] \setminus \{i\}$ and any $U \subseteq T$. We have two possibilities. If $j \in T \setminus U$, then, by definition, $T \setminus U$ has the same outcome in $\hat{G}$ as $T \setminus U$ does in $G$, namely yes. On the other hand, if $j \notin T \setminus U$, then $T \setminus U$ has the same outcome in $\hat{G}$ as $[n]$ does in $G$, namely yes. But $U$ can take on any cardinality from 0 to $n - 1$. Thus a symmetric reciprocal cataclysmic quarrel is not $k$-monotonic for any $k$. \hfill \Box
11.5 Proofs for Section 9

Theorem 9.3. A game $\hat{G}$ with an asymmetric non-reciprocal strong YES-quarrel is quasi-monotonic.

Proof. We must show $\hat{G}$ is 1-monotonic. Take any group $S$. By definition, the subsets $S$, $S \cup \{i\}$, $S \cup \{j\}$, and any other subset $T \subseteq S$ is a member of $W$ if and only if it is a member of $\hat{W}$. But the original game $G$ is monotonic. It immediately follows that the 1-monotonic property (in fact, the 0-monotonic property) holds for any $T \subseteq S \cup \{i\}$ or $T \subseteq S \cup \{j\}$.

Thus violations in 1-monotonicity can only arise for subsets of the form $S \cup \{i\}$. So assume $S \cup \{i\} \notin \hat{W}$. By definition of an asymmetric non-reciprocal strong quarrel, $S \cup \{j\} \notin W$. Now take any $T \subset S \cup \{i, j\}$. We have two possibilities. First, assume that $T$ contains $i$. Now $T \setminus \{i\} \subset S \cup \{j\}$. So, by the monotonicity of $G$, we have that $T \setminus \{i\} \notin W$. Thus, by the definition of an asymmetric non-reciprocal strong quarrel, $T \notin \hat{W}$. This is allowed by the definition of 1-monotonicity.

Second, assume that $T$ does not contain $i$. Then $T \subset S \cup \{j\}$. But $S \cup \{j\} \notin W$ so, by monotonicity, $T \notin W$. Now because $T$ does not contain $i$, $T \in W$ if and only if $T \in W$. Thus $T \notin \hat{W}$ and so causes no violation of 1-monotonicity (nor, indeed, 0-monotonicity). It follows that a game $\hat{G}$ with an asymmetric non-reciprocal strong quarrel is quasi-monotonic. $\square$

Theorem 9.4. A game $\hat{G}$ with an asymmetric non-reciprocal cataclysmic YES-quarrel is supremely non-monotonic.

Proof. We must show an asymmetric non-reciprocal cataclysmic quarrel rule is not $k$-monotonic, for any $k$. Consider a game $G$ for which $W = \{S \subseteq [n] : S \notin \{j\}\}$. In particular, each player except $j$ is a NO-blocker who can ensure a YES-outcome simply by voting YES.

Next take $\hat{G}$ where $i$ has an asymmetric non-reciprocal cataclysmic quarrel with $j$. Consider the group $S = [n]$. By $\hVQ$, since $i$ and $j$ are both in $[n]$, the unanimous division $S$ yields in $\hat{G}$ the same outcome as $j$ does in $G$, that is, a NO-outcome. Now take $T = [n] \setminus \{i, j\}$ and any $U \subset T$. It follows that $T \setminus U \subseteq [n] \setminus \{i, j\}$. In particular $i, j \notin T \setminus U$. So by definition, $T \setminus U \in W$ if and only if $T \setminus U \in \hat{W}$. Since $U \subset T$, $T \setminus U \in \hat{W}$. But $U$ can take on any cardinality from $0$ to $n - 3$. Thus the rule is not $k$-monotonic for any $k$. $\square$

Theorem 9.5. A game $\hat{G}$ with a symmetric non-reciprocal cataclysmic quarrel is supremely non-monotonic.

Proof. Now we must show a symmetric non-reciprocal cataclysmic quarrel rule is not $k$-monotonic, for any $k$. Consider a game $G$ for which $W = \{S \subseteq [n] : |S| \geq n - 1\}$. That is, any group that contains all players, or all players but one player, can secure a YES-outcome.

Next take the $\hat{G}$ where $i$ has a symmetric non-reciprocal cataclysmic quarrel with $j$. Consider the group $S = [n]$. By $\hVQ$, since $i$ and $j$ are both in $[n]$, the unanimous division $S$ yields in $\hat{G}$ the same outcome as $j$ does in $G$, that is, a NO-outcome. Now take $T = [n] \setminus \{i, j\}$ and any $U \subset T$. It follows that $T \setminus U \subseteq [n] \setminus \{i, j\}$. In particular $i, j \notin T \setminus U$. So by $\hNQ$, $T \setminus U \in \hat{W}$ if and only if $[n] \setminus \{j\} \in W$. Because $|[n] \setminus \{j\}| = n - 1$, it follows that $T \setminus U \in \hat{W}$. But $U$ can take on any cardinality from $0$ to $n - 3$. Thus the rule is not $k$-monotonic for any $k$. $\square$
11.6 Proofs for Section 11

**Theorem 11.1.** The Shapley-Shubik Index satisfies the standard quarrel postulate based on a symmetric, weak quarrel.

*Proof.* Let’s first recall the definition of voting power for the Shapley-Shubik Index. Let \( \sigma \) be an ordering (or permutation) of the \( n \) players, and let \( \Omega \) be the set of all orderings. We say that player \( i \) is *pivotal* for the ordering \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \) if \( \sigma_k = \{i\} \) and \( i \) is *yes*-decisive for \( S = \{\sigma_1, \sigma_2, \ldots, \sigma_k\} \). We remark that for a monotonic game, there is a unique pivotal player for each ordering. The a priori voting power of player \( i \) under the Shapley-Shubik Index is then supposed to be:

\[
\psi_{i}^{SS} = \sum_{\sigma \in \Omega} P(\sigma) \cdot \chi_i(\sigma) = \sum_{\sigma \in \Omega} \frac{1}{n!} \cdot \chi_i(\sigma)
\]

where

\[
\chi_i(\sigma) = \begin{cases} 
1 & \text{if } i \text{ is pivotal in } \sigma \\
0 & \text{otherwise}
\end{cases}
\]

Now consider the game \( \hat{G} \) that incorporates a symmetric, weak quarrel between \( i \) and \( j \) in the original game \( G \). We must prove \( \hat{\psi}_i^{SS} \leq \psi_i^{SS} \). Take any ordering \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \Omega \). Recall that \( \hat{G} \) is monotonic for a symmetric, weak quarrel. Therefore, it suffices to prove that if \( \sigma_k = \{i\} \) and \( i \) is *yes*-decisive for \( S = \{\sigma_1, \sigma_2, \ldots, \sigma_k\} \) in \( \hat{G} \) then it is also *yes*-decisive in \( G \). There are two cases:

(i) \( j \in S \). By assumption, \( S \setminus \{i\} \notin \hat{W} \) and \( S \in \hat{W} \). But \( j \in S \setminus \{i\} \) so, by definition of a general quarrel, the non-ambush condition implies that \( S \setminus \{i\} \notin W \). On the other hand, \( \{i, j\} \subseteq S \). Thus the cooperative-success-reduction condition (CSR:YQ-1) implies that \( S \setminus \{i\} \in W \). But this implies \( i \) is *yes*-decisive for \( S \) in the original game \( G \), as desired.

(ii) \( j \notin S \). By assumption, \( S \setminus \{i\} \notin \hat{W} \) and \( S \in \hat{W} \). Now, since \( i \in S \), the non-ambush condition implies that \( S \in W \). On the other hand, \( \{i, j\} \cap S \setminus \{i\} = \emptyset \). Thus the cooperative-success-reduction condition (namely, the contrapositive of CSR:NQ-1) implies that \( S \setminus \{i\} \notin W \). But this implies \( i \) is *yes*-decisive for \( S \) in the original game \( G \), as desired.

If follows that the Shapley-Shubik Index satisfies the standard quarrel postulate. \( \square \)

**Theorem 11.2.** The Penrose-Banzhaf Measure satisfies the standard quarrel postulate based on a symmetric, weak quarrel.

*Proof.* The a priori voting power of player \( i \) under the Penrose-Banzhaf Measure can be given by:

\[
\psi_{i}^{PB} = \sum_{S \subseteq [n] : i \in S} \frac{1}{2^{n-1}} \cdot \chi_i(S)
\]

where

\[
\chi_i(S) = \begin{cases} 
1 & \text{if } i \text{ is } \text{yes}-\text{decisive in } S \\
0 & \text{otherwise}
\end{cases}
\]

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Now consider the game $\hat{G}$ that incorporates a symmetric, weak quarrel between $i$ and $j$ in the original game $G$. Again, we must prove $\hat{\psi}^{PB} \leq \psi^{PB}$. This follows by a similar argument as in the proof of Theorem 11.4.