Global and Local: Synchronization and Emergence.

Mogens H. Jensen and Leo P. Kadanoff

Niels Bohr Institute,
University of Copenhagen,
Blegdamsvej 17, DK-2100,
Copenhagen, Denmark

and

The James Franck Institute,
The University of Chicago,
Chicago, Illinois 60637. USA

(Dated: February 6, 2022)

Abstract

When a dynamical system contains several different modes of oscillations it may behave in a variety of ways: If the modes oscillate at their own individual frequencies, it exhibits quasiperiodic behavior; when the modes lock to one another it becomes synchronized, or, as a third possibility, complex chaotic behavior may emerge. With two modes present, like an internal oscillation coupled to a periodic external signal, one obtains a highly structured phase diagram that exhibits these possibilities. In essence, the details are related to the difference between rational and irrational numbers. Natural system can display fragments of this phase diagram, thereby offering insights into their dynamical mechanisms.

*Electronic address: mhjensen@nbi.dk, lkadanoff@gmail.com
For centuries, physical scientists have studied the synchronization of oscillators. In 1665, Christiaan Huygens noticed that two clocks hanging on a wall (see Figure 1) tend to synchronize their pendula \[1\]. A similar scenario occurs with two metronomes placed on a piano: they interact through vibrations in the wood and will eventually coordinate their motion. In recent years, studies of oscillations and possible synchronizations \[2\] have become important research topics also in the biological sciences. Living species present us with a bewildering fauna of oscillators: cell cycles \[3\], circadian rhythms \[4\], calcium oscillations \[5\], pace maker cells \[6\], protein responses \[7-10\], hormone secretion \[11\], and so on. If such oscillators are in the neighborhood of each other - as they might well be in tissues, organs and cells- do they tend to synchronize?

Synchronization presents us with a surprising depth and complexity, not fully understood to this day. The phenomenon is at its basis related to the difference between rational and irrational numbers! Each of the two Huygens clocks or the two metronomes possesses a typical frequency (called an eigenfrequency) \(\omega_1\) and \(\omega_2\) respectively. For clocks and metronomes, the two frequencies are most likely quite similar in magnitude, \(\omega_1 \approx \omega_2\) - although maybe not exactly equal. When they interact through the wall or through the piano, they tend to approach each other via a phenomenon called frequency pulling or frequency locking: they try to synchronize by pulling their frequencies towards each other such that in the end the two effective frequencies (which we mark by primes) are exactly equal: \(\omega'_1 = \omega'_2\). This cor-
responds to a completely synchronized state and the more strongly the clocks/metronomes interact the more they tend to synchronize. Next, let us imagine that the original (bare) frequencies are such that $\omega_1 \approx 2\omega_2$, i.e. one frequency is roughly twice the size of the other. Again, the two oscillators will tend to pull and synchronize at a state where the effective frequencies are such that $\omega'_1 \approx 2\omega'_2$. In general, a synchronization will be likely as long as $Q\omega_1 \approx P\omega_2$ or $\omega_1/\omega_2$ can be approximately equal to $P/Q$, where $P$ and $Q$ are positive integer numbers. In other words, there will tend to be synchronization when the ratio of the bare frequencies of the two oscillator are close to a rational number. There exist many rational number, an infinity of them, but the ones formed from small integers "pull" best.

In a synchronized state, characterized by the rational number, $f = P/Q$, the dynamical motion of the clocks is periodic with period $Q$ so that it will return to the its initial state over and over again.

That is one possible behavior. But it is not the only one. It is certainly true that all possible ratios, $\omega_1/\omega_2$ are close to some $P/Q$ for some integer values of $P$ and $Q$. However, as the integers get larger, one needs an increasingly close approach between the two ratios before synchronization will occur. If there is no rational frequency ratio sufficiently close, the system will undergo a motion characterized by an irrational frequency ratio, $f$. The motion is just as orderly as the nearby periodic orbits, but it never repeats itself. The required closeness forms a pattern that depends in a complex manner upon the integers and upon the strength of the coupling between the frequencies.

Before describing the patterns formed by the different kinds of motion, we should spend a moment saying how they can be distinguished. The first and simplest technique is to plot the orbits formed by coordinates of the motion, as in Figure (2). As one can see from that figure, there are three kinds of orbits: locked, quasiperiodic, and chaotic. The orbits look different. They can be distinguished by their measured dimension [12–14] being respectively one, two, or non-integer in the three cases. A spectrum generator that measures the frequency content of each motion will sharply distinguish these patterns. The periodic motion will have one very sharp frequency line, plus overtones at multiples of that frequency. The quasiperiodic pattern will show two (or sometimes more) lines at fundamental frequencies not related by integer multiplication and then a host of other lines identifiable as integer multiples of the fundamental frequencies. In contrast, the chaotic motion will generate a smoothly varying frequency spectrum as in the third panel of Figure (3).
FIG. 2: Three kinds of orbits for two pendula. In leftmost plate we see the periodic motion of two locked pendula. The orbit simply repeats itself, and forms a one-dimensional curve. In contrast, the middle plate shows a quasiperiodic curve formed by two oscillations which retain the independent frequencies, and in this case, a frequency ratio which is an irrational number. Here the motion never repeats itself. It forms a smooth surface, that is to say it fills a two dimensional space. The rightmost plate presents the result of a strong coupling in which the motion becomes chaotic. The orbit is complex in that the motion is, for some periods, of one type, then switches to another type of motion with intervals that obey no simple law. The motion never repeats itself, but it does not fill any simple space. A measurement of the dimension of the orbit will show a fractal result, a dimension that is not equal to one, as in the locked case, nor two as in the quasiperiodic one, but is instead some number between two and three.

FIG. 3: Schematic plot of frequency spectra for the kinds of orbits shown in Figure 2. The three plates correspond to the cases of periodic/locked, quasiperiodic, and chaotic motion and show respectively line spectrum with a fundamental plus overtones, a line spectrum with mixtures of two incommensurate frequencies, and a smoothly varying spectrum.

It is not just the individual motion that are interesting: The different kinds of motions fit together into a richly interwoven phase diagram, see Figure 4. This plot shows how the kinds of motion can depend upon the parameter driving the behavior. A.A. Kolmogorov has
constructed a simplified mathematical model that can be used to describe the pattern of mingling of different kinds of orbits. We show that very rich and structured pattern in Figure (4). We shall argue below that elements of this figure may be found within the behavior of frequency locking in real systems. But for now, we ask the reader to bear with us and follow the description of the simplified model.

As mentioned, the tendency to synchronize depends on the interaction strength of the oscillators. Call this strength $K$ - for the clocks it is a measure of the vibration of the wall, for pace maker cells the intensity of the cell-to-cell interactions through a tissue. The results of a simplified model is shown in Figure (4), and it tells us how the frequencies can affect the mode locking. On the y-axis is the strength of the interaction $K$, on the x-axis the bare, original frequency $\omega_2$ of one of the oscillators. The other oscillator is assumed to have bare frequency equal to one. Without any interaction, that is when $K$ is equal to zero, the oscillators do not interact and the model makes the motion proceed with a frequency unchanged from the bare value, $\omega_2$. For irrational values of $\omega_2$ the motion is quasiperiodic, so that quasiperiodic behavior dominates the $K = 0$ line.

As soon as the two oscillators interact just the slightest amount, that is when $K$ assumes a value just above zero, the model displays frequency pulling so that it gives a region of synchronization whenever $\omega_2$ is in a small interval around each and every rational number, $P/Q$. Figure (4) shows these regions of frequency locking as blue regions. It has dark blue regions being very narrow at small $K$ and growing wider as $K$ gets larger. For each value of the rational number $P/Q$, there will be some region of locking. These regions are called called Arnold tongues. They are named after the mathematician Vladimir Arnold who did extensive studies of this model. As $K$ gets larger, the tongues widen and take up a larger and larger proportion of the frequency interval.

These frequency-locked regions do not exhaust the frequency interval between zero and one. For $K$ between zero and one, in addition to these locked regions, there are a host of quasiperiodic orbits, motions with frequency ratios, $f$, that behave like irrational values of $P/Q$. In these orbits, the motion remains orderly but, unlike in the periodic case, it never repeat itself. In Figure (4), the regions of quasiperiodic behavior are shown in white. These regions also include infinite numbers of narrow bands of synchronized orbits.

This part of the phase diagram is arranged in a very orderly fashion. As the frequency variable on the x-axis increases, each $\omega_2$ gives rise to an $f$, and these frequency-ratio values
FIG. 4: The pattern of frequency locking and unlocking in a simplified mathematical model. The $x$-axis on this graph is the original frequency of one of the two oscillators. The $y$-axis is the strength of the non-linear interaction between the oscillators. The blue regions are ones in which the oscillators are locked, the numbers attached to each region describes the frequency ratio for the locking. The white regions show intermixed quasi-periodic and periodic behavior, too finely intermingled to be separated by our plot. The green and red regions show similarly intermixed behavior, but now also including a chaotic element. The broken line at $K = 1$ shows the onset of chaotic behavior. Above this line, chaos is possible (indicated by a change in the blue color); below it there is only quasiperiodic and locked behavior.

all increase with increasing $\omega_2$. Here $f$ moves smoothly through irrational values but gets stuck for an interval, perhaps a very short interval, on each rational ratio.

These will be a quasi-periodic orbit for each irrational number. For increased interaction strength between the oscillators, the synchronization becomes stronger, and each rational interval tends to widen so that the irrational 'points' become more and more closely packed, maintaining however the property of having $f$ increase as one moves to the right in the figure.
For all $K$ between zero and one, both kinds of orbits, synchronized and quasiperiodic, occupy finite lengths of the frequency axis.

As $K$ increases further this area covered by the tongues continues to increase, until at $K = 1$, there is only an infinitesimal area left for the irrational orbits. At that point these quasiperiodic orbits occupy a "fractal" set with a dimension measured to be 0.870. That the dimension is less than one is an indication that at this value of $K$ these orbits cease to occupy a finite fraction of the frequency axis and instead are relegated to a set of zero length. The $K = 1$ line thus defines a complementary situation to that at $K = 0$: the rational and irrational numbers have exchanged their roles. Now the rational numbers fill up the line while the irrationals fill nothing. At $K = 1$ the irrationals are still all there, and they are infinitely more numerous than the rational numbers, but despite that they occupy zero length along the line.

At $K = 1$, there is a dramatic change in the behavior. The orderly progression of quasiperiodic orbits with continuously increasing frequency disappears. The immediate cause of this change is a region of the flow in which the derivative of the map defining our model passes through zero. Just above the $K = 1$ line the model begins to show a much richer behavior than heretofore. For some values of the model parameters, several different orbits, even orbits of different characters, are simultaneously possible. Which kind one sees depends upon the initial conditions of the motion. Chaotic orbits can be found (see plate c in Figure 2 and Figure 3) in which the long-term motion is quite unpredictable. However, regions of locked motion still exist. Figure 4 shows how the locked motion is intermixed with other kinds of orbits.

Most of the theoretical discussion up to here has been based upon attention to simplified models. This attention is justified because real systems, physical, biological or geological, show many of the same qualitative features as the models. The lunar month is incommensurate with the solar year, and this incommensuration is a real fact about earth. In a women’s dorm, women’s "monthly" periods do tend to lock, giving a surprising example of period-locking and indicating an unexpected mode of interaction. It is said that locusts pick their 11-, 13-, or 17- year cycle so that other species will find it hard to period-lock to them. We get an insight into the strength of non-linear couplings by noticing that, for example, predator-prey cycles tend to be quite chaotic. In each of these cases, we are analyzing real behavior by using pictures of simplified behavior, often pictures derived from the study of
simplified models.

Thus, the orbits observed in various sciences can be referred back to the orbits of simple dynamical systems. This referencing to simple models can also apply to entire regions of phase diagram like that in Figure 4. Each small region of this diagram is the result of varying parameters in the model over some small region of parameter space. Real systems can be expected to show the same sort of detailed features. One need only look for these features. Then by knowing the region in parameter space, one might hope to get some insights into biological function or geophysical history.

The point is that some features of models, particularly those involving how different kinds of motion arise and fit together, are "universal." That means that these features are to be found not only in simplified models, but in a wide variety of circumstances in which the same basic mechanics are at work. One such feature is the fractal dimension of the quasiperiodic orbits, 0.87..., at the onset of chaos \[17\]. Not only does this structure and this dimension appear in the model, the scenario has been experimentally verified in a number of physical systems, from onset of turbulence \[21\], Josephson junctions \[22, 23\], one-dimensional conductors \[24\], semiconductors \[25, 26\] and crystals \[27\]. The number close to 0.87... has even been observed in fluids \[21\], sliding charge-density-waves \[24\] and in Josephson simulators \[23\]. Similar studies might see the onset of chaos in biological systems. Such onset has been argued to be helpful to biological function \[28\]. The biological world contains an amazing number of coupled oscillators. What does total synchronization actually mean in a case of cell cycles? And will overlap of tongues lead to a chaotic state of the cell? Questions like these can be asked, and partially answered, by paying attention to model phase diagram like that of Figure 4.

---

[1] A copy of the letter on this topic to the Royal Society of London appears in C. Huygens, in *Oeuvres Completes de Christian Huygens*, edited by M. Nijhoff, Societe Hollandaise des Sciences, The Hague, The Netherlands (1893), Vol. 5, p. 246.

[2] A. Pikovsky, M. Rosenblum and J. Kurths, *Synchronization: a universal concept in nonlinear sciences* Cambridge University Press, Cambridge (2003).

[3] Tsai TY, Choi YS, Ma W, Pomerening JR, Tang C, Ferrell JE Jr. "Robust, tunable biological
oscillations from interlinked positive and negative feedback loops” Science Jul 4;321(5885):126-129 (2008)

[4] Q. Thommen et al, ”Robustness of circadian clocks to daylight fluctuations: hints from the picocellular Ostreococcus tauri”, PLoS Comput Biol 6(11) e1000990 (2010); B. Pfeuty, Q. Thommen, and M. Lefranc, ”Robust entrainment of circadian oscillators requires specific phase response curves”, Biophysical Journal 100, 2557 (2011).

[5] A. Goldbeter, ”Computational approaches to cellular rhythms” Nature 420, 238-245 (2002)

[6] B. O’Rourke, B.M. Ramza and E. Marban ”Oscillations of membrane current and excitability driven by metabolic oscillations in heart cells” Science 265 962 (1994).

[7] A. Hoffmann, A. Levchenko, M.L. Scott and D. Baltimore, The IκB-NF-κB signaling module: temporal control and selective gene activation. Science 298, 1241 (2002).

[8] D.E. Nelson et. al, Oscillations in nf-κb signaling control the dynamics of gene expression. Science 306, 704 (2004).

[9] S. Krishna, M.H. Jensen and K. Sneppen, ”Spiky oscillations in NF-kappaB signalling”, Proc.Nat.Acad.Sci. 103, 10840-10845 (2006).

[10] B. Mengel, A. Hunziker, L. Pedersen, A. Trusina, M.H. Jensen and S. Krishna, ”Modeling oscillatory control in NF-kB, p53 and Wnt signaling”, Current Opinion in Genetics and Development 20, 656-664 (2010).

[11] Waite, EJ, Kershaw, YM, Spiga, F and Lightman, SL. ’A glucocorticoid sensitive biphasic rhythm of testosterone secretion’, Journal of Neuroendocrinology, 21, (pp. 737-741), 2009.

[12] Benoit B. Mandelbrot, The Fractal Geometry of Nature W. H. Freeman (1982).

[13] P. Grassberger and I. Procaccia, Physica D 9, 198 (1983).

[14] J. Feder, Fractals, Plenum, New York (1988).

[15] The model constructs successive $x$-values, $x_0, x_1, \ldots$ from the equation $x_{k+1} = x_k + \omega_2 + (K/(2\pi))sin(2\pi x_k)$. The other frequency is given by the period of the sine function and is $\omega_1 = 1$. The modeler then asks whether the pattern of $x_k$ can be written as $x_k = x_0 + fk + G(x_k)$, where $f$ is a rational or irrational frequency ratio, with $G$ being a smooth periodic function obeying $G(x + 1) = G(x)$. If it can be written in this manner with a rational $f$, it is said to be periodic; if $f$ is irrational the result is said to be quasiperiodic.

[16] M. J. Feigenbaum, L. P. Kadanoff and S. J. Shenker, Quasiperiodicity in dissipative systems: A renormalization group analysis. Physica D5, 370-386 (1982).
[17] M. Jensen, P. Bak, and T. Bohr, Complete devil's staircase, fractal dimension and universality of mode-locking structure in the circle map, Phys. Rev. Lett. 50, 1637-1639 (1983); M. Jensen, P. Bak, and T. Bohr, Transition to chaos by interaction of resonances in dissipative systems. I. Circle maps, Phys. Rev. A 30, 1960-1969 (1984).

[18] See several fundamental books by V.I. Arnold, for instance V. I. Arnold, A. Avez, Ergodic Problems of Classical Mechanics, Addison-Wesley (1989).

[19] H. Hahn, "Infinity" in James R. Newman The World of Mathematics Simon and Schuster, New York, (1956), pp. 1593-1611.

[20] Martha McClintock ”Menstrual synchrony and suppression”, Nature 229, 244-245 (1971).

[21] J. Stavans, F. Heslot and A. Libchaber, Fixed winding number and the quasiperiodic route to chaos in a convective fluid Phys. Rev. Lett. 55, 596-599 (1985).

[22] P. Alstrom, M.H. Jensen and M.T. Levisen, Fractal Structure of Subharmonic steps in a Josephson junction: An Analog Computer Calculation, Physics Letters 103A 171-174 (1984).

[23] W. J. Yeh, Da-Ren He, and Y. H. Kao Fractal Dimension and Self-Similarity of the Devil’s Staircase in a Josephson-Junction Simulator Phys. Rev. Lett. 52, 480-480 (1984); Da-Ren He, W. J. Yeh, and Y. H. Kao Studies of return maps, chaos, and phase-locked states in a current-driven Josephson-junction simulator Phys. Rev. B 31 1359-1373 (1985).

[24] Stuart E. Brown, George Mozurkewich, and George Gruner Subharmonic Shapiro Steps and Devil’s-Staircase Behavior in Driven Charge-Density-Wave Systems Phys. Rev. Lett. 52, 2277-2380 (1984).

[25] A. Cumming and P. S. Linsay, Deviations from universality in the transition from quasiperiodicity to chaos, Phys. Rev. Lett. 59, 16331636 (1987).

[26] E. G. Gwinn and R. M. Westervelt, Frequency Locking, quasiperiodicity, and Chaos in Extrinsic Ge Phys. Rev. Lett. 57 1060-1063 (1986).

[27] S. Martin and W. Martienssen Circle Maps and Mode Locking in the Driven Electrical Conductivity of Barium Sodium Niobate Crystals Phys. Rev. Lett. 56, 1522-1525 (1986).

[28] S. Kauffman, ”The Origins of Order: Self-Organization and Selection in Evolution”, Oxford University Press, Oxford (1993).