A commentary on the continued fraction by which the illustrious La Grange has expressed the binomial powers *

Leonhard Euler†

I. This illustrious man has converted the Binomial power \((1 + x)^n\), by his most singular method of logarithmic differentials, into this continued fraction:

\[
(1 + x)^n = 1 + \frac{nx}{1 + \frac{(1-n)x}{2 + \frac{(1+n)x}{3 + \frac{(2-n)x}{2 + \frac{(2+n)x}{5 + \frac{(3-n)x}{2 + \frac{(3+n)x}{7 + \ldots}}}}}}},
\]

which expression celebrates the marvelous property that whenever the exponent \(n\) is an integral number, either positive or negative, it is halted and is reduced to a finite form.

II. Seeing that this continued fraction does not proceed by a uniform law, but rather is interrupted, we may bring it to a uniform law, as it would be most desirable, if we should represent it in the following way by parts:

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\[
(1 + x)^n = 1 + \frac{n x}{A}, \quad A = 1 + \frac{(1-n)x}{2+B};
\]
\[
B = 3 + \frac{(2-n)x}{2+C}; \quad C = 5 + \frac{(3-n)x}{2+D}; \quad D = 7 + \frac{(4-n)x}{2+E};
\]

etc.

From here, we therefore will have by reducing:

\[
A = 1 + \frac{(1-n)x}{B+1} \quad = 1 + \frac{(1-n)x}{2} - \frac{(1-n)xx2}{2B+1+nx} = 1 + \frac{(1-n)x}{2} + \frac{(1-n)xx2}{B+1}x.
\]

In a similar way it will be:

\[
B = 3 + \frac{(2-n)x}{C+2} \quad = 3 + \frac{(2-n)x}{2} - \frac{(4-n)xx2}{2C+2+nx} = 3 + \frac{(2-n)x}{2} + \frac{(4-n)xx2}{C+2}x.
\]

In the very same way we shall have:

\[
C = 5 + \frac{(3-n)x}{D+3} \quad = 5 + \frac{(3-n)x}{2} - \frac{(9-n)xx2}{2D+3+nx} = 5 + \frac{(3-n)x}{2} + \frac{(9-n)xx2}{D+3}x.
\]

and so on.

III. Now if we were to substitute these values in succession in the place of \(A, B, C, \ldots\), the continued fraction will be induced to the following form:

\[
(1 + x)^n = 1 + \frac{n x}{1 + (1-n)x + \frac{nn-1xx4}{3(1+\frac{1}{2})x + \frac{nn-4xx4}{5(1+\frac{1}{2})x + \frac{nn-9xx4}{7(1+\frac{1}{2})x + \frac{nn-16xx4}{9(1+\frac{1}{2})x + \ldots}}}}}
\]
IV. So that we can remove these partial fractions, we set \( x = 2y \), so that this expression shall be obtained:

\[
(1 + 2y)^n = 1 + \frac{2ny}{1 + (1 - n)y} + \frac{(nn - 1)yy}{3(1 + y) + \frac{(nn - 4)yy}{5(1 + y) + \frac{(nn - 7)yy}{7(1 + y) + \text{etc.}}}}
\]

which form can easily be transformed into this:

\[
\frac{2ny}{(1 + 2y)^n - 1} = 1 + \frac{(nn - 1)yy}{3(1 + y) + \frac{(nn - 4)yy}{5(1 + y) + \text{etc.}}}
\]

Then \( ny \) is added to both sides, so that it will emerge

\[
\frac{ny(1 + (1 + 2y)^n)}{(1 + 2y)^n - 1} = 1 + y + \frac{(nn - 1)yy}{3(1 + y) + \frac{(nn - 4)yy}{5(1 + y) + \text{etc.}}}
\]

this expression is ordered enough that it may proceed to be regulated.

V. We will now divide both sides by \( 1 + y \), and the left member will come out as: \( \frac{ny}{1+y} \cdot \frac{(1+2y)^n+1}{(1+2y)^n-1} \). From the right side moreover, each of the fractions on the top and bottom should be divided by \( 1 + y \), and this form will extend:

\[
1 + \frac{(nn - 1)yy \cdot (1 + y)^2}{3 + \frac{(nn - 4)yy (1 + y)^2}{5 + \frac{(nn - 9)yy (1 + y)^2}{7 + \frac{(nn - 16)yy (1 + y)^2}{9 + \frac{(nn - 25)yy (1 + y)^2}{11 + \text{etc.}}}}}}
\]

VI. Then we may reduce this expression again for greater elegance, with it being set \( \frac{y}{1+y} = z \), so that it may thus be \( y = \frac{z}{1-z} \). Then presently the left member, on account of \( 1 + 2y = \frac{1+z}{1-z} \), will admit this form: \( \frac{ny[(1+z)^n+(1-z)^n]}{(1-z)^n-(1+z)^n} \), because of which it may therefore be equated to this continued fraction:

\[
1 + \frac{(nn - 1)zz}{3 + \frac{(nn - 4)zz}{5 + \frac{(nn - 9)zz}{7 + \frac{(nn - 16)zz}{9 + \text{etc.}}}}}
\]

which, by its elegance, merits the highest attention.

VII. Now, it is therefore manifest for this expression to always be halted whenever \( n \) is an integral number, either positive or negative. It is moreover
evident for the left member to retain the same value even if for \( n \) is written \(-n\). Namely, this fact comes forth from:

\[
\frac{-nz[(1 + z)^{-n} + (1 - z)^{-n}]}{(1 + z)^{-n} - (1 - z)^{-n}},
\]

which fraction, if multiplied above and below by \((1 - zz)^n\), induces this form:

\[
\frac{-nz[(1 - z)^n + (1 + z)^n]}{(1 - z)^n - (1 + z)^n} = \frac{nz[(1 + z)^n + (1 - z)^n]}{(1 + z)^n - (1 - z)^n},
\]

which is the same as the preceding expression. Thus it is the same whether the positive or negative of the letter \( n \) is taken.

VIII. Thus if we were to take \( n = \pm 1 \), the left member would be equal to 1, which is moreover the value of the right. Furthermore, by putting \( n = \pm 2 \), the left member will come forth as equal to \( 1 + zz \), and indeed the right member will also be equal to \( 1 + zz \). In a similar way, by taking \( n = \pm 3 \), the left part, and in turn the right, becomes \( 3(1 + 3z)^3 + 3z^3 \).

IX. Here one may deduce several conclusions of great importance, depending on whenever a vanishing or infinite value is taken for the exponent \( n \), but first of all for the case in which an imaginary value is taken for the letter \( z \), which leads to an outstanding conclusion, since this continued fraction shall nevertheless remain real, for which conclusion we will therefore take up first.

**Conclusion I.**

where \( z = t\sqrt{-1} \)

X. In this case therefore the continued fraction will have this form:

\[
1 - \frac{(nn - 1)tt}{3 - \frac{(nn - 4)tt}{5 - \frac{(nn - 9)tt}{7 - \frac{(nn - 16)tt}{9 - \ldots}}}},
\]

and to be sure the left part will now be:

\[
\frac{nt\sqrt{-1}[(1 + t\sqrt{-1})^n + (1 - t\sqrt{-1})^n]}{(1 + t\sqrt{-1})^n - (1 - t\sqrt{-1})^n},
\]
which not having been opposed by imaginary parts ought certainly to have a real value, which we shall now investigate. Then to this end we put \( t = \frac{\sin \phi}{\cos \phi} \), so that it will thus be \( t = \tan \phi \); then it will therefore be:

\[
(1 + t\sqrt{-1})^n = \frac{(\cos \phi + \sqrt{-1} \sin \phi)^n}{\cos \phi^n} = \frac{\cos n\phi + \sqrt{-1} \sin n\phi}{\cos \phi^n},
\]

and in a similar way:

\[
(1 - t\sqrt{-1})^n = \frac{(\cos \phi - \sqrt{-1} \sin \phi)^n}{\cos \phi^n} = \frac{\cos n\phi - \sqrt{-1} \sin n\phi}{\cos \phi^n}.
\]

Therefore by substituting these values our left member will come forth:

\[
\frac{2n\sqrt{-1} \cdot \tan \phi \cos n\phi}{2\sqrt{-1} \sin n\phi} = \frac{n \tan \phi \cos n\phi}{\sin n\phi} = \frac{n \tan \phi}{\tan n\phi}.
\]

XI. Therefore by putting \( \tan \phi = t \) we will have the following most remarkable continued fraction:

\[
\frac{nt}{\tan n\phi} = 1 - \frac{(nn - 1)tt}{3 - \frac{(nn - 4)tt}{5 - \frac{(nn - 9)tt}{7 - \text{etc.}}}},
\]

which then will be able to be represented in this way:

\[
\tan n\phi = \frac{nt}{1 - \frac{(nn-1)tt}{3 - \frac{(nn-4)tt}{5 - \frac{(nn-9)tt}{7 - \text{etc.}}}}},
\]

which expression therefore is able to be helpfully applied to the tangents of multiplied angles which are to be expressed by the tangent of the single angle \( t \). Thus if it were \( n = 2 \), we will have \( \tan 2\phi = \frac{2t}{1-tt} \). In the very same way if \( n = 3 \), it will be:

\[
\tan 3\phi = \frac{3t}{1 - \frac{8tt}{3 - tt}} = \frac{3t - t^3}{1 - 3tt}.
\]

In the most notable case presented whenever the exponent \( n \) is taken as less than infinite, it will then be \( \tan n\phi = n\phi \), therefore, by dividing both sides by \( n \), this form arises:

\[
\phi = \frac{t}{1 + \frac{t}{3 + \frac{tt}{5 + \frac{tt}{7 + \text{etc.}}}}},
\]
where the continued fraction is expressed by the tangent $t$ of the angle itself.

XII. We will now consider the case in which an infinite magnitude is taken for the exponent $n$, but when the angle $\phi$ less than infinite, and then too for the tangent $t$ of it less than infinite, so that it would thus be $n\phi = \theta$, and then also $nt = \theta$; then we will therefore have such a continued fraction:

$$\text{tg}\theta = \frac{\theta}{1 - \frac{\theta}{1 - \frac{\theta}{3 - \frac{\theta}{5 - \frac{\theta}{7 - \text{etc.}}}}}}$$

by which formula, from the given angle $\theta$, the tangent of it will be able to be determined, which expression will be able to be seen just as the reciprocal of the preceding.

Conclusion II.

where a vanishing exponent $n$ is taken:

XIII. Therefore in this case the continued fraction will be:

$$1 - \frac{zz}{3 - \frac{4zz}{5 - \frac{9zz}{7 - \frac{16zz}{9 - \text{etc.}}}}}$$

It is moreover to be noted for the left part to be $\frac{(1+z)^n-1}{n} = l(1+z)$, and for that reason $(1+z)^n = 1 + nl(1+z)$; in a similar way it will be: $(1-z)^n = 1 + nl(1-z)$, from which the left member comes forth as

$$nz\frac{2 + nl(1+z) + nl(1-z)}{nl(1+z) - nl(1-z)} = \frac{2z}{l \frac{1+z}{1-z}}$$

here therefore we will have such a form:

$$\frac{2z}{l \frac{1+z}{1-z}} = 1 - \frac{zz}{3 - \frac{4zz}{9zz}}$$

$$5 - \frac{16zz}{9 - \text{etc.}}$$
and then this logarithm may be expressed in the following way:

\[
\frac{1 + z}{1 - z} = \frac{2z}{1 - \frac{4z}{3 - \frac{4z}{5 - \text{etc.}}} - \text{etc.}}
\]

Conclusion III.

where an infinite magnitude is taken for the exponent \( n \)

XIV. Here therefore, so that a finite value may be obtained for this continued fraction, of which having been advanced a quantity less than infinity is to be taken for \( z \), it is put \( nz = v \), so that it would be \( z = \frac{v}{n} \), and thus our continued fraction will be:

\[
1 + \frac{v}{3 + \frac{v}{5 + \frac{v}{7 + \text{etc.}}}}
\]

Also, it is apparent for the left member to be \((1 + \frac{z}{n})^n = e^v\), and in a similar way \((1 - \frac{z}{n})^n = e^{-v}\); therefore the left member will have this form:

\[
\frac{v(e^v + e^{-v})}{e^v - e^{-v}} = \frac{v(e^{2v} + 1)}{e^{2v} - 1},
\]

and from this fact we will have this remarkable continued fraction:

\[
\frac{v(e^{2v} + 1)}{e^{2v} - 1} = 1 + \frac{v}{3 + \frac{v}{5 + \frac{v}{7 + \text{etc.}}}}
\]

whose transcendent value is actually able to be exhibited in this way by a series:

\[
1 + \frac{v}{1 + \frac{v}{1 \cdot 2} + \frac{v^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{v^6}{1 \cdot 2 \cdot 5} + \text{etc.}} = 1 + \frac{v}{1 + \frac{v}{1 \cdot 2 \cdot 3} + \frac{v^4}{1 \cdot 2 \cdot 3 \cdot 5} + \frac{v^6}{1 \cdot 2 \cdot 3 \cdot 7} + \text{etc.}}
\]