Projective isomorphisms between rational surfaces

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Abstract

We present a method for computing projective isomorphisms between rational surfaces
that are given in terms of their parametrizations. The main idea is to reduce the computation
of such projective isomorphisms to five base cases by modifying the parametric maps such
that the components of the resulting maps have lower degree. Our method can be used to
calculate affine, Euclidean and Möbius isomorphisms between surfaces.

Keywords: projective isomorphisms, surface automorphisms, rational surfaces, del Pezzo
surfaces, adjunction

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1 Introduction

Suppose we are given two rational surfaces in terms of their parametrizations. We reduce the
computation of the projective isomorphisms between these surfaces, to the more tractable problem
of finding projective isomorphisms between surfaces that are covered by lines or conics and that
belong to one of five different types. We shall discuss two of the types in more detail and leave
the remaining three types as future work. Our reduction method translates into Algorithm 3 and
its correctness follows from Theorem 1 and Theorem 2. We refer to [8, github] for a (partial)
implementation. We explain in Section 6 how to recover from projective isomorphisms between
surfaces, the affine, Euclidean and Möbius isomorphisms.

If \( f \) is a rational map, then we denote its domain by \( \text{dom} f \) and the Zariski closure of its image by
\( \text{img} f \). Let \( \mathcal{M} \) be defined as the set of all rational maps \( f: \text{dom} f \to \text{img} f \subseteq \mathbb{P}^{\dim f} \) defined over
some algebraically closed field \( F \) such that \( \text{dom} f \in \{ \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1 \} \). Here \( \dim f \in \mathbb{Z}_{\geq 0} \) denotes the
embedding dimension and we assume that \( \text{img} f \) is not contained in a hyperplane section of \( \mathbb{P}^{\dim f} \).

The group of biregular automorphisms of a variety \( Z \) is denoted by \( \text{Aut}(Z) \) and it is known that
The set of projective isomorphisms is defined as
\[ P(f, g) := \{ p \in \text{Aut}(P^{\dim f}) \mid p(\text{img } f) = \text{img } g \}. \]

In this article we address the following problem:

For given birational maps \( f, g \in M \) determine \( P(f, g) \).

A projective isomorphism \( p: \text{img } f \to \text{img } g \) induces a birational map
\[ g^{-1} \circ p \circ f: \text{dom } f \to \text{dom } g. \]

Our strategy is to compute the set of birational maps between the domains that are compatible with projective isomorphisms in this way. In [7] it was assumed that \( \text{dom } f = \text{dom } g = P^2 \) with both \( f \) and \( g \) base point free so that \( g^{-1} \circ p \circ f \) is a linear projective automorphism of \( P^2 \). In this article, we admit arbitrary base points and as a consequence we get compatible birational maps that blow up base points and contract curves. Fortunately, these blowups and contractions can be controlled using adjunction theory, and allow us to get a finite-dimensional set of candidates for the induced birational maps between the domains. Once we have this set of candidates, we write down the conditions expressing the statement that we really have a projective isomorphism. This will be a system of algebraic equations and its solutions correspond to the projective isomorphisms in \( P(f, g) \).

Isomorphisms between surfaces are of interest in geometric modeling. If \( \text{img } f \) and \( \text{img } g \) are surfaces in \( P^3 \) that are covered by lines, then [2, Section 3.1] and [3, Algorithm 5] provide methods for computing affine and Euclidean isomorphisms between \( \text{img } f \) and \( \text{img } g \). We refer to [1, Introduction] and [7, Introduction] for further references.

If \( \text{img } f \) is smooth, then \( \text{Aut}(\text{img } f) \) and its action on the Néron-Severi lattice \( N(\text{img } f) \) is of interest to algebraic geometers [4, 9, 15]. The relation of \( \text{Aut}(\text{img } f) \) to our paper is as follows: projective isomorphisms in \( P(f, f) \) correspond to automorphisms of \( \text{img } f \) whose induced action on \( N(\text{img } f) \) leaves the class of hyperplane sections invariant. We will clarify these notions in Section 2 in order to make our results also accessible to the geometric modeling community. We refer to [4, Introduction] for further references from the viewpoint of algebraic geometry.

The set of compatible reparametrizations is defined as
\[ R(f, g) := \{ r \in \text{bir}(\text{dom } f, \text{dom } g) \mid p \circ f = g \circ r \text{ for some } p \in P(f, g) \}, \]
where \( \text{bir}(\text{dom } f, \text{dom } g) \) is the set of birational maps between the domains. The idea of Algorithm 3 is to first compute a set \( S \) that contains the compatible reparametrizations \( R(f, g) \). We explain in Section 3 how to recover the projective isomorphisms \( P(f, g) \) from this super set \( S \).

In Section 4 we state Theorem 1 and Theorem 2 which reduce the computation of \( S \) to five base cases B1—B5. The base cases B1 and B2 are considered in Section 5, and the remaining three base cases are left as future work. In Section 6 we discuss some applications of our algorithm. See Example 10 for a full run of the algorithm in a concrete instance. Finally, we present the proof for Theorem 1 and Theorem 2 in Section 7.

2 Basic concepts and notation

In order to make this article accessible to a wide audience we recall some basic concepts from algebraic geometry and provide references. We will also introduce non-standard notation that will be used in the remaining sections of this article.

We define a sequential blowup as a birational morphism \( \pi: Z_{r+1} \to Z_1 \) between smooth surfaces together with blowups \( \pi_i: Z_{i+1} \to Z_i \) of points \( p_i \in Z_i \) for \( 1 \leq i \leq r \) such that \( \pi = \pi_1 \circ \ldots \circ \pi_r \). We
refer to $p_i$ as the center of the blowup and $E_{i+1} := \pi_i^{-1}(p_i)$ as a $(-1)$-curve (a curve isomorphic to $\mathbb{P}^1$ and with self-intersection $-1$). See [6, Example I.4.9.1 and Section V.3] for more information. If $p_i \in (\pi_{i-1} \circ \cdots \circ \pi_1)^{-1}(p_j)$ for some $r \leq i < j \leq 1$, then we say that $p_i$ is infinitely near to $p_j$. We call a point infinitely near if it is infinitely near to some point and simple otherwise.

Suppose $C_1 \subset Z_1$ is a curve and that $C_2 \subset Z_2$ is the Zariski closure of the preimage $\pi_1^{-1}(C_1 \setminus \{p_1\})$. We refer to [6, Remark V.3.5.2] for the definition of multiplicity of $C_1$ at the simple point $p_1$. The multiplicity of $C_1$ at an infinitely near point $p_2$ is defined as the usual multiplicity of $C_2$ at $p_2$.

Let $V$ be a vector space of forms on $Z_1$. The linear series of $V$ is defined as $|V| := \{\text{ZeroSet}(v) \mid v \in V\}$. The moving part of $V$ is defined as the vector space that is generated by the polynomial quotients $g_1/q, \ldots, g_n/q$, where $(g_1, \ldots, g_n)_V$ is a basis for $V$ and $q := \gcd(g_1, \ldots, g_n)$ is the greatest common polynomial divisor.

Suppose that $f : Z_1 \to \mathbb{P}^n$ is the rational map whose components generate $V$. The associated vector space $V_f$ of $f$ is defined as $V$. We say that $q$ is a base point of multiplicity $m$ of both $V$ and $f$, if there exists a sequential blowup $\pi$ such that $q = p_i$ for some $1 \leq i \leq r$ and if a general curve in $|V|$ has multiplicity $m > 0$ at $p_i$.

**Remark 1** (Algorithm 1 and Algorithm 2). We refer to [11, Algorithms 1 and 2] for the method and implementation of Algorithm 1 and Algorithm 2 (see alternatively [13] for a possibly faster implementation). We remark that in [11] a sequential blowup $\pi$ and its centers are represented in terms of a data structure that extracts only the part of $\pi$ that is needed for this article.

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**Algorithm 1**

- **input.** A vector space $V$ of forms on $Z_1 \in \{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1\}$.
- **output.** A sequential blowup $\pi : Z_{r+1} \to Z_1$ and $m_i \in \mathbb{Z}_{>0}$ such that $q$ is a base point of $V$ of multiplicity $m_i$ if and only if there exists a unique $1 \leq i \leq r$ such that $q = p_i \in Z_i$.

---

**Algorithm 2**

- **input.** A sequential blowup $\pi : Z_{r+1} \to Z_1$ with centers $p_i \in Z_i$ such that $Z_1 \in \{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1\}$. The vector space $W$ of all (bi-) degree $d$ of forms on $Z_1$. A set of multiplicities $m_i \in \mathbb{Z}_{>0}$ for $1 \leq i \leq r$.
- **output.** The subspace $V \subset W$ of forms whose zero set are curves that have multiplicity $\geq m_i$ at the base point $p_i$ for all $1 \leq i \leq r$.

---

**Example 1.** Suppose that $f : \mathbb{P}^2 \to \mathbb{P}^1$ maps $(x_0 : x_1 : x_2)$ to $(x_1^2 + x_2^2 : x_2 + x_1 x_0)$. We find that $p_1 := (1 : 1 : 0), p_2 := (1 : 1 : -i)$ and $p_3 := (1 : 0 : 0)$ are simple base points for $f$ with multiplicities $(m_1, m_2, m_3) = (1, 1, 1)$. The map $f$ has also a base point $p_4$ of multiplicity $m_4 = 1$ that is infinitely near to $p_3$.

**Definition 1.** Suppose that $V$ is a vector space of forms on a smooth projective surface $S$. Let $\mathbb{P}(V^\ast)$ denote the projectivization of the dual space of $V$ so that each point in $\mathbb{P}(V^\ast)$ corresponds to a codimension one subspace of $V$. Let $\varphi_V : Z \to \mathbb{P}(V^\ast)$ be defined as $\varphi_V(p) := \{v \in V \mid \nu(p) = 0\}$ for all $p \in Z$. A choice of basis is defined as an isomorphism $\beta : \mathbb{P}(V^\ast) \to \mathbb{P}^{\dim V - 1}$. The associated map $\varphi_V : Z \to \mathbb{P}^{\dim V - 1}$ is defined as $\beta \circ \varphi_V$ where $\beta$ is a choice of basis. We need to be careful that the definitions and assertions in this article that involve the notion of the associated map are independent of the choice of such a basis. Recall that we denote by $\text{img} \varphi_V$ the Zariski closure of the image of $\varphi_V$.
Definition 2. In this article we will assume that the components of maps in $\mathcal{M}$ as defined in Section 1 have a constant greatest common divisor. The component degree $\deg(f)$ of $f \in \mathcal{M}$ is defined as the (bi-)degree of the components of $f$. Let the sequential blowup $\pi: S \to \text{dom } f$ be the output of Algorithm 1 when it is applied to the associated vector space $V_f$. In this case, we call $\text{bmd } f := S$ the base model for $f$. \hfill \triangledown

Definition 3. Suppose that $f \in \mathcal{M}$ has base points $p_1, \ldots, p_r$. In this article the Néron-Severi lattice $N(\text{bmd } f)$ is an additive group together with an intersection product $\cdot : N(\text{bmd } f) \otimes N(\text{bmd } f) \to \mathbb{Z}$ that satisfies the following axioms:

- If $\text{dom } f = \mathbb{P}^2$, then $N(\text{bmd } f) \cong \langle e_0, e_1, \ldots, e_r \rangle_{\mathbb{Z}}$, where the only non-zero intersections between the generators are $e_0^2 = 1$ and $e_i^2 = -1$ for $1 \leq i \leq r$.
- If $\text{dom } f = \mathbb{P}^1 \times \mathbb{P}^1$, then $N(\text{bmd } f) \cong \langle \ell_0, \ell_1, \ldots, \ell_r \rangle_{\mathbb{Z}}$, where the only non-zero intersections between the generators are $\ell_0 \cdot \ell_1 = 1$ and $\ell_i^2 = -1$ for $1 \leq i \leq r$.

See forward Remark 2 for more information. \hfill \triangledown

Definition 4. Suppose that $f \in \mathcal{M}$ has base points $p_1, \ldots, p_r$ with multiplicities $m_1, \ldots, m_r$, respectively. First suppose that $\text{dom } f = \mathbb{P}^2$ and that $d := \deg(f)$. The class of $f$ is defined as

$$[f] = d e_0 - m_1 e_1 - \ldots - m_r e_r.$$  

The greatest common divisor of $f$ is defined as

$$\gcd([f]) := \gcd(d, m_1, \ldots, m_r).$$  

The canonical class associated to $f$ is defined as

$$\kappa_f := -3 e_0 + e_1 + \ldots + e_r.$$  

Conversely, suppose that $c := d e_0 - m_1 e_1 - \ldots - m_r e_r$ is a class in $N(\text{bmd } f)$ such that $d, m_1, \ldots, m_r > 0$. The associated vector space $V_c$ is defined as the output of Algorithm 2 with input $\pi$: $\text{bmd } f \to \text{dom } f$ and $d, m_1, \ldots, m_r$. We denote

$$h^0(c) := \dim V_c.$$  

If $M$ is the moving part of $V_c$, then the parametric map of $c$ is defined as

$$\Psi_c := \varphi: \text{dom } f \to \mathbb{P}^{h^0(c)-1}.$$  

The moving part of the class $c$ is defined as the following class:

$$[c] := [\Psi_c].$$  

If $\text{dom } f = \mathbb{P}^1 \times \mathbb{P}^1$ and $\deg(f) = (d_1, d_2)$, then the terminology is analogous except:

$$[f] = d_1 \ell_0 + d_2 \ell_1 - m_1 e_1 - \ldots - m_r e_r,$$

$$\gcd([f]) := \gcd(d_1, d_2, m_1, \ldots, m_r)$$  

and

$$\kappa_f := -2 \ell_0 - 2 \ell_1 + e_1 + \ldots + e_r.$$  

The reader is warned that the non-standard notation introduced in this definition will be used throughout this article. \hfill \triangledown

Example 2. The birational map $f: \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by

$$x \mapsto (x_3^4 - x_2^3 x_0 : x_1^2 x_0 : x_1 x_2^2 : x_0^3 - x_2^2 x_0),$$  

has simple base points $p_1 := (1 : 0 : 0), p_2 := (1 : 1 : 0)$ and $p_3 := (1 : 0 : 1)$ with multiplicities $m_1 := 2, m_2 := 1$ and $m_3 := 1$, respectively. The class of $f$ is $[f] = 3 e_0 - 2 e_1 - e_2 - e_3$, and for a particular choice of a basis, the parametric map $\Psi_f: \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ is defined as $x \mapsto (x_1^2 - x_2^2 x_0 : x_1 x_2^2 : x_1^2 x_2 x_0 : x_1 x_2^2 : x_1 x_2 x_0 : x_2^3 - x_2^2 x_0)$. Notice that $\dim f = 3 < h^0([f]) - 1 = 4$ and that $\text{bmd } f$ is $\mathbb{P}^2$ blown up in $p_1, p_2$ and $p_3$. Since $f$ is birational, $\deg(\text{im } f)$ is equal to the number of intersections outside the base points of the pullback along $f$ of two hyperplane sections of $\text{im } f$ to $\mathbb{P}^2$, and therefore we can deduce that $\deg(\text{im } f) = [f]^2 = 3$. In fact, for all $f \in \mathcal{M}$, either $\dim(\text{im } f) < 2$ or $\deg(\text{im } f) = \deg(f) \cdot [f]^2$, where $\deg(f)$ equals the number of points in a general fiber. \hfill \triangledown
Example 3. Suppose that \( f: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \) is defined by
\[
(y_0 : y_1 : y_2 : y_3) \mapsto (y_0^2 y_2^3 - 3y_0^2 y_1 y_2 y_3 + 3y_0^2 y_2 y_3 : y_0^2 y_1^2 y_2 + 3y_0^2 y_1 y_2^2 + 3y_0 y_1 y_2^3 : y_0 y_1 y_2^2 + y_0 y_1^2 y_2^2 + y_1^2 y_2^3).
\]
The components of this map form the vector space \( V_{[f]} \) of forms of bi-degree \((2,2)\) that have four simple base points of multiplicity one (\(i^2 = -1\) and \(j^2 = -\frac{1}{2}\)).

Thus \( [f] = 2\ell_0 + 2\ell_1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 \) so that \( \deg(\text{img } f) = \deg(f) \cdot [f]^2 = 4 \) with \( \deg(f) = 1 \) and \( h^0([f]) = 5 \). Let \( \tau_1 \) and \( \tau_2 \) denote the projection \( \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \) to the first and second component, respectively. Notice that the base points do not lie in general position, since \( \tau_1(p_1) = \tau_1(p_3) \), \( \tau_1(p_2) = \tau_1(p_4) \), \( \tau_2(p_1) = \tau_2(p_2) \) and \( \tau_2(p_2) = \tau_2(p_4) \). Therefore, it follows that
\[
\begin{align*}
h^0(\ell_0 - \varepsilon_1 - \varepsilon_3) &= 1, & h^0(\ell_1 - \varepsilon_1 - \varepsilon_4) &= 1, \\
h^0(\ell_0 - \varepsilon_2 - \varepsilon_4) &= 1, & h^0(\ell_1 - \varepsilon_2 - \varepsilon_3) &= 1.
\end{align*}
\]
The fibers of \( \tau_1 \) and \( \tau_2 \) that contain two base points are contracted via \( f \) to four complex conjugate isolated singularities in \( \text{img } f \). We remark that \( \text{img } f \) can be linearly projected to a quartic surface in \( \mathbb{P}^3 \) whose real points form a torus of revolution in \( \mathbb{R}^3 \).

Remark 2. We shall consider maps \( f \in \mathcal{M} \) whose base points are contained in some fix set of base points \( \{p_1, \ldots, p_r\} \) and the classes of such maps keep track of the multiplicities at these base points. We may think of the class of a map as a “generalized component degree” and the intersection product between these classes allows us to access power tools from algebraic geometry. The reason is that the Néron Severi lattice \( N(bmd f) \) is actually the set of divisor classes on \( bmd f \) modulo numerical equivalence. In this article we consider only rational surfaces and thus numerical equivalence and rational equivalence define the same equivalence relation on divisor classes. The element \( \kappa_f \) is the “canonical class” of \( bmd f \) and is defined as the divisor class that corresponds to the line bundle that is the determinant of the cotangent bundle of \( bmd f \).

We refer to [10, Section 1.1] for more information.

3 Isomorphisms from reparametrizations

Suppose that \( f, g \in \mathcal{M} \) are birational and suppose that \( \mathcal{S} \supseteq \mathcal{R}(f, g) \) is a family \((r_c)_{c \in \mathcal{I}}\) of reparametrizations indexed by \( \mathcal{I} \subseteq \mathbb{F}^t \) for some \( t > 0 \). In order to recover the projective isomorphisms \( P(f, g) \) from \( \mathcal{S} \) we show how to recover the index-set \( \mathcal{J} \subseteq \mathcal{I} \) so that \( \mathcal{R}(f, g) = (r_c)_{c \in \mathcal{J}} \).

Definition 5. Let \( \vec{v} \) be a column vector with \( m \) rows that consists of a basis for all forms of the same (bi-) degree. If \( f \) has \( n + 1 \) components, then the \textit{coefficient matrix} of \( f \) is defined as the \((n + 1) \times m \) matrix \( M_f \) such that \( M_f \cdot \vec{v} \) defines \( f \) as a column vector. Let \( M_f \) be a matrix whose columns form a basis for the kernel of \( M_f \). We denote the identity matrix and zero matrix by \( I \) and \( O \), respectively.

Example 4. If \( f \in \mathcal{M} \) such that \( \text{dom } f = \mathbb{P}^1 \times \mathbb{P}^1 \), \( \text{img } f \subseteq \mathbb{P}^3 \) and \( \text{cdeg}(f) = (2, 2) \), then we may choose
\[
\vec{v} = (s^2 u^2, s^2 u v, s^2 v^2, s t u^2, s t u v, s t v^2, t^2 u^2, t^2 u v, t^2 v^2)^	op,
\]
and we find that \( M_f \) is a \( 4 \times 9 \) matrix.

The index set \( \mathcal{J} \) such that \( \mathcal{R}(f, g) = (r_c)_{c \in \mathcal{J}} \) is recovered as follows:
\[
\mathcal{J} := \{ c \in \mathcal{I} \mid g \circ r_c \text{ has the same base points as } f, \quad \text{cdeg}(g \circ r_c) = \text{cdeg}(f) \quad \text{and} \quad M_{g \circ r_c} \cdot \ker M_f = O \}.
\]

(1)
The composition \( g \circ r_c \) is computed using first substitution after which we factor out the greatest common divisor of the resulting components. For computing the base points of \( f \) and enforcing these base points on \( g \circ r_c \), we use Algorithm 1 and Algorithm 2.

In order to recover \( \mathcal{P}(f, g) \) from \( \mathcal{R}(f, g) = (r_c)_{c \in \mathcal{J}} \), we consider the following two \( m \times m \) matrices that are constructed using matrix augmentation (\(|\cdot|\)):

\[
E_f := (M_f^\top | \ker M_f)^\top \quad \text{and} \quad E_{gor_c} := (M_{gor_c}^\top | \ker M_f)^\top.
\]

If \( U \in \mathbb{F}^{n+1 \times m+1} \) is a matrix, then we denote by \( \chi_U : \mathbb{P}^n \to \mathbb{P}^n \) the corresponding projective linear map.

**Proposition 1.** \( \mathcal{P}(f, g) = \{ \chi_U \in \text{Aut}(\mathbb{P}^n) \mid U \oplus I = E_{gor_c} \cdot E_f^{-1} \text{ and } c \in \mathcal{J}\} \).

**Proof.** Let the components of \( v \): \( \text{dom} \ f \to Z \subset \mathbb{P}^n \) form a basis of all forms of degree \( \text{cd}(f) \).

Notice that \( \text{img} f, \text{img} g \subset \mathbb{P}^n \) and \( m \geq n \). Let \( s : Z \to f \) and \( t : Z \to g \) be the linear projections defined by the coefficient matrices \( M_f \) and \( M_{gor_c} \), respectively. We require that \( M_{gor_c} \) and \( M_f \) have the same kernel so that the projection centers of \( s \) and \( t \) coincide. Indeed, the elements in \( \mathcal{P}(f, g) \) are via \( s \) and \( t \) compatible with the projective isomorphisms of \( Z \) that preserve this center of projection. In other words, if \( \mathcal{J}' = \{ c \in \mathcal{I} \mid r_c \in \mathcal{R}(f, g) \} \), then

\[
\mathcal{P}(f, g) = \{ \chi_U \in \text{Aut}(\mathbb{P}^n) \mid U \cdot M_f = M_{gor_c} \text{ for some } c \in \mathcal{J}'\}
\]

It remains to show that \( \mathcal{J}' \subseteq \mathcal{J} \). If \( c \in \mathcal{J}' \), then there exists a projective isomorphism \( p \in \mathcal{P}(f, g) \) such that \( p \circ f = g \circ r_c \) and thus \( c \in \mathcal{J} \) by definition, which concludes the proof.

We define the following index sets, where a matrix is *normalized* if it has non-zero determinant and if the first non-zero entry of the matrix has value one:

\[
\mathcal{I}_{\mathbb{P}^2} := \left\{ c \in \mathbb{F}^9 \mid \begin{pmatrix} c_0 & c_1 & c_2 \\ c_3 & c_4 & c_5 \end{pmatrix} \text{ is normalized} \right\},
\]

\[
\mathcal{I}_{\mathbb{P}^1 \times \mathbb{P}^1} := \left\{ c \in \mathbb{F}^8 \mid \begin{pmatrix} c_0 & c_1 \\ c_2 & c_3 \\ c_4 & c_5 \end{pmatrix} \text{ and } \begin{pmatrix} c_6 & c_7 \end{pmatrix} \text{ are both normalized} \right\}.
\]

We assume coordinates \( x = (x_0 : x_1 : x_2) \) and \( y = (y_0 : y_1 : y_2 : y_3) \) for \( \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \), respectively. If \( c \in \mathcal{I}_{\mathbb{P}^2} \), then we implicitly assume that the corresponding reparametrization \( r_c : \mathbb{P}^2 \to \mathbb{P}^2 \) is defined as

\[
r_c : x \mapsto (c_0 x_0 + c_1 x_1 + c_2 x_2 : c_3 x_0 + c_4 x_1 + c_5 x_2 : c_6 x_0 + c_7 x_1 + c_8 x_2).
\]

Similarly, if \( c \in \mathcal{I}_{\mathbb{P}^1 \times \mathbb{P}^1} \), then \( r_c : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \) is defined as

\[
r_c : y \mapsto (c_0 y_0 + c_1 y_1 : c_2 y_0 + c_3 y_1 : c_4 y_2 + c_5 y_3 : c_6 y_2 + c_7 y_3).
\]

We denote by \( \text{Aut}_c(\mathbb{P}^1 \times \mathbb{P}^1) \) the identity component of \( \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \).

**Lemma 1.** If the set of reparametrizations \( \mathcal{S} \) is defined by either \( \text{Aut}(\mathbb{P}^2) \) or \( \text{Aut}_c(\mathbb{P}^1 \times \mathbb{P}^1) \), then \( \mathcal{S} = (r_c)_{c \in \mathcal{I}_{\mathbb{P}^2}} \) and \( \mathcal{S} = (r_c)_{c \in \mathcal{I}_{\mathbb{P}^1 \times \mathbb{P}^1}} \), respectively.

**Proof.** It follows from [6, Example 7.1.1] that the biregular automorphisms of projective space are linear.

**Example 5.** Let \( f : \mathbb{P}^2 \to \mathbb{P}^3, x \mapsto (x_0^3 + x_1^3 + x_2^3 : x_0 x_1 : x_0 x_2 : x_1 x_2) \) be the parametrization of a Roman surface \( \text{img} f \) and suppose that \( g = f \). We will see in Example 9 that \( \mathcal{S} := (r_c)_{c \in \mathcal{I}_{\mathbb{P}^2}} \) contains the compatible reparametrizations \( \mathcal{R}(f, g) \). Since \( f \) is base point free, we find that

\[
\mathcal{J} = \{ c \in \mathcal{I}_{\mathbb{P}^2} \mid M_{gor_c} \cdot \ker M_f = \mathbf{0} \}.
\]
We choose the monomial basis $(x_0^2, x_0 x_1, x_0 x_2, x_1^2, x_1 x_2, x_2^2)$ so that
\[
M_f = \begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (\ker f)\top = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad M_{g_{9c}} =
\begin{pmatrix}
\begin{array}{cccccc}
c_1^2+c_3^2+c_6^2 & 2c_0c_1+2c_0c_3+2c_0c_6 & 2c_0c_2+2c_2c_3+2c_2c_6 & c_1^2+c_4^2+c_5^2 & c_1^2+c_4^2+c_5^2 & c_1^2+c_4^2+c_5^2 \\
c_0c_3 & c_1c_3+c_1c_4 & c_2c_4+c_2c_5 & c_1c_4 & c_2c_4+c_2c_5 & c_1c_4 \\
c_0c_6 & c_1c_6+c_1c_7 & c_2c_7+c_2c_8 & c_1c_7 & c_2c_7+c_2c_8 & c_1c_7 \\
c_0c_3 & c_1c_3+c_1c_4 & c_2c_4+c_2c_5 & c_1c_4 & c_2c_4+c_2c_5 & c_1c_4 \\
c_0c_6 & c_1c_6+c_1c_7 & c_2c_7+c_2c_8 & c_1c_7 & c_2c_7+c_2c_8 & c_1c_7 \\
\end{array}
\end{pmatrix}.
\]

We find that $|\mathcal{J}| = 24$ and 8 elements of $\mathcal{J}$ are listed below:

| $c_0$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ | $c_7$ | $c_8$ |
|------|------|------|------|------|------|------|------|------|
| 0    | ±1   | 0    | 1    | 0    | 0    | 0    | 0    | 1    |
| 0    | ±1   | 0    | 1    | 0    | 0    | 0    | 0    | 1    |
| 0    | ±1   | 0    | 0    | 1    | 1    | 0    | 0    | 0    |
| 0    | ±1   | 0    | 0    | 0    | 0    | -1   | 1    | 0    |

We obtain 8 more elements by interchanging columns $c_0 \leftrightarrow c_1$, $c_3 \leftrightarrow c_4$ and $c_6 \leftrightarrow c_7$. The remaining 8 elements of $\mathcal{J}$ are obtained by instead interchanging columns $c_1 \leftrightarrow c_2$, $c_4 \leftrightarrow c_5$ and $c_7 \leftrightarrow c_8$. If, for example, $c = (0, 1, 0, 1, 0, 0, 0, 1)$ and $U$ is the matrix such that $U \oplus 1 = E_{g_{9c}} \cdot E_f^{-1}$, then
\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

Indeed, we verify that $\text{img} \chi_c \circ f \subseteq \text{img} g$, where
\[
\text{img} g = \{ z \in \mathbb{P}^3 \mid z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2 - z_0 z_1 z_2 z_3 = 0 \}.
\]

See [8] for an implementation of this example. \hfill $\blacksquare$

## 4 Reduction to five base cases

This section will be concluded with the statements of Theorem 1 and Theorem 2. These main results imply a method that reduces the computation of a super set of compatible reparametrizations to five base cases. We shall delay the proof of the theorems until Section 7.

A condition is a map $\mathbf{c} : \mathcal{M} \to \{0, 1\}$, where 0 and 1 are identified with False and True, respectively. Let $\mathcal{M}_c := \{ f \in \mathcal{M} \mid \mathbf{c}(f) = 1 \}$ be the set of rational maps that satisfy the condition $\mathbf{c}$. A reducer consists of a condition $\mathbf{c}$ and a map $\mathbf{r} : \mathcal{M}_c \to \mathcal{M}$. We refer to $\mathbf{r}(f)$ as the reduction of $f$.

We call $\mathbf{q} : \mathcal{M} \to \mathbb{Z}^m$ for some $m \in \mathbb{Z}_{>0}$ a projective invariant if $\mathbf{q}(f)$ is a projective invariant of $\text{img} f \subset \mathbb{P}^\dim f$ for all $f \in \mathcal{M}$. A reducer $\mathbf{r}$ is compatible if its condition $\mathbf{c} : \mathcal{M} \to \{0, 1\} \subset \mathbb{Z}$ is a projective invariant and if
\[
\mathcal{R}(f, g) \subseteq \mathcal{R}(\mathbf{r}(f), \mathbf{r}(g))
\]

for all $f, g \in \mathcal{M}_c$.

**Remark 3.** Recall from Proposition 1 that for given birational maps $f, g \in \mathcal{M}$ we can recover the projective isomorphisms $\mathcal{P}(f, g)$ from a super set of the compatible reparametrizations $\mathcal{R}(f, g)$. Instead of finding a super set of $\mathcal{R}(f, g)$ it is sufficient to find a super set of $\mathcal{R}(\mathbf{r}(f), \mathbf{r}(g))$, where $\mathbf{r}$ is a compatible reducer. If $\mathbf{c}(f) \neq \mathbf{c}(g)$, then $\mathcal{P}(f, g) = \emptyset$ as the condition $\mathbf{c}$ of a compatible reducer $\mathbf{r}$ is a projective invariant. If $\mathbf{c}(f) = \mathbf{c}(g) = 0$, then we cannot further reduce the problem so that we are in a base case. In this section we will define three reducers $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2$ and we show in Theorem 1 that these reducers are compatible. The component degree of $\mathbf{r}_1(f)$ is equal to the component degree of $f$ minus three (see Proposition 3). Hence, it is easier to determine the super set of $\mathcal{R}(\mathbf{r}_1(f), \mathbf{r}_1(g))$ instead of $\mathcal{R}(f, g)$. In Theorem 2 we classify the base cases, namely the cases where $\mathbf{c}_0(f) = \mathbf{c}_1(f) = \mathbf{c}_2(f) = 0$. If both $f$ and $g$ are in a base case, then the computation of a super set of the compatible reparametrizations is better tractible as $\text{img} f$ and $\text{img} g$ are surfaces that are covered by lines or conics and thus theoretically well understood. Example 10
and Algorithm 3 in the next section shows that our theory can be converted into an algorithm for computing projective isomorphisms between rational surfaces in \( \mathbb{P}^n \) for all \( n > 1 \).

**Proposition 2.** If \( r : \mathcal{M}_e \to \mathcal{M} \) and \( s : \mathcal{M}_d \to \mathcal{M} \) are compatible reducers, then the reducer \( s \circ r \) with condition \( c(f) \cdot d(r(f)) \) for all \( f \in \mathcal{M} \) is compatible as well.

**Proof.** Straightforward consequence of the definitions. \( \square \)

Please recall Definition 4. We define \( p : \mathcal{M} \to \mathbb{Z}^3 \) as
\[
p(f) := (h^0([f]), |f|, \gcd[f]).
\]
The reducers \( r_0, r_1, r_2 \) with conditions \( c_0, c_1, c_2 \), respectively, are defined as follows where \( f \in \mathcal{M} \):

- \( r_0(f) := \Psi_0[f] \) and \( c_0(f) \) is defined as
  \[
  \dim f < h^0([f]) - 1.
  \]
- \( r_1(f) := \Psi_e \) with \( c := [f] + \kappa_f \) and \( c_1(f) \) is defined as
  \[
  h^0([f] + \kappa_f) > 1 \quad \land \quad \neg(|f|^2 > |c|^2 = c \cdot [c] = 0).
  \]
- \( r_2(f) := \Psi_b \) with \( b := \frac{\gcd[f]}{\gcd[f]} \) and \( c_2(f) \) is defined as
  \[
  \gcd[f] > 1.
  \]

**Example 6.** Let us consider \( f : \mathbb{P}^2 \to \mathbb{P}^3 \) in Example 5 which is defined as
\[
x \mapsto (x_0^2 + x_1^2 + x_2^2 : x_0 x_1 : x_0 x_2 : x_1 x_2).
\]
We have \([f] = 2e_0 \) and \( c_0(f) = 1 \). The reduction \( r_0(f) : \mathbb{P}^2 \to \mathbb{P}^5 \) is up to a choice of basis defined as
\[
x \mapsto (x_0^2 : x_1^2 : x_0 x_1 : x_0 x_2 : x_1 x_2).
\]
Since \([r_0(f)] = 2e_0 \) and \( \kappa_{r_0}(f) = -3e_0 \) it follows that \( h^0(r_0(f) + \kappa_{r_0}(f)) = 0 \) and thus \( c_1(r_0(f)) = 0 \). We verify that \( c_2(r_0(f)) = 1 \) as \( \gcd[r_0(f)] = 2 \) and thus the reduction \( (r_2 \circ r_0)(f) : \mathbb{P}^2 \to \mathbb{P}^2 \) is up to a choice of basis defined as \( x \mapsto (x : x : x) \).

**Proposition 3.** \( \operatorname{cdeg} r_i(f) < \operatorname{cdeg} f \) for all \( f \in \mathcal{M}_e \) and \( i \in \{1, 2\} \).

**Proof.** Straightforward consequence of the definitions. \( \square \)

**Theorem 1.** The function \( p \) is a projective invariant and the reducers \( r_0, r_1 \) and \( r_2 \) are compatible.

**Definition 6.** We say that \( f \in \mathcal{M} \) is characterized by a base case if
\[
c_0(f) = c_1(f) = c_2(f) = 0
\]
and either one of the following five cases holds:

- **B1.** \( h^0([f]) = 3, \ |f|^2 = 1 \) and \( \operatorname{img} f \approx \mathbb{P}^2 \).
- **B2.** \( h^0([f]) = 4, \ |f|^2 = 2 \) and \( \operatorname{img} f \) is a quadric surface.
- **B3.** \( h^0([f]) = |f|^2 + 1, \ 1 \leq |f|^2 \leq 8 \) and \( \operatorname{img} \Psi_0[f] \) is a del Pezzo surface, where \( \alpha := \max(4 - |f|^2, 1) \). Moreover, the map \( \Psi_0[f] \) is birational and if \( |f|^2 \geq 3 \), then \( \operatorname{img} \Psi_0[f] \) is covered by conics.
- **B4.** \( h^0(2[f] + \kappa_f) \geq 2, \ [2[f] + \kappa_f]^2 = 0 \) and \( \operatorname{img} f \) is a surface covered by lines.
- **B5.** \( h^0([f] + \kappa_f) \geq 2, \ [(f] + \kappa_f)^2 = 0 \) and \( \operatorname{img} f \) is a surface covered by conics or lines. \( \square \)
Example 7. Suppose that $f \in M$ is a rational map such that $\text{dom } f = \mathbb{P}^2$ and $|f| = 8e_0 - 5e_1 - 3e_2 - 3e_3$. We set $u := [f] + \kappa_f = 5e_0 - 4e_1 - 2e_2 - 2e_3$. Since $u$ is negative against the classes of lines defined by $e_0 - e_1 - e_2$ and $e_0 - e_1 - e_4$ it follows that $u \neq [u]$. We subtract these classes from $u$ and find that $[r_1(f)] = [u] = 3e_0 - 2e_1 - e_2 - e_3$ (see Example 2). We notice that $c_0(r_1(f)) = c_1(r_1(f)) = c_2(r_1(f)) = 0$, since $|r_1(f)| + \kappa_{r_1}(f) = -e_1$ and $\gcd[r_1(f)] = 1$. The class $v := 2[r_1(f)] + \kappa_{r_1}(f) = 3e_0 - 3e_1 - e_2 - e_3$ is negative against the classes $e_0 - e_1 - e_2$ and $e_0 - e_1 - e_4$. Therefore, $[v] = e_0 - e_1$ and thus $r_1(f)$ is characterized by base case B4. 

Recall Remark 3 for the purpose of the following theorem.

Theorem 2. Suppose that $f \in M$ such that $c_0(f) = c_1(f) = c_2(f) = 0$.

a) If $|f|^2 > 0$, then $f$ is characterized by base case B1, B2, B3, B4 or B5.

b) If $|f|^2 = 0$, then there exists no $g \in M_{e_1}$ such that $|g|^2 > 0$ and $f \in \{r_1(g), (r_2 \circ r_1)(g)\}$.

Example 8. If $f, g \in M$ such that $|f| = 6e_0 - e_1 - \cdots - e_9$ and $|g| = 3e_0 - 2e_1 - \cdots - 2e_9$, then $h^0(f) = 2$, $|f|^2 = 0$ and $f = r_1(g)$. However, $|g|^2 = 0$ and thus this is not a counter example to Theorem 2b.

5 Reparametrizations for B1 and B2

In this section we characterize a super set for $\mathcal{R}(f, g)$ in case $f, g \in M$ are characterized by base case B1 or B2. The main results of this article are translated into Algorithm 3.

Proposition 4 (B1). If $f, g \in M$ are both characterized by B1, then

$$\mathcal{R}(f, g) \subseteq S := \{g^{-1} \circ r_c \circ f \mid c \in \mathbb{I}_{\mathbb{P}^2}\}.$$

Proof. By Theorem 2 we have $\text{img } f \equiv \mathbb{P}^2 \equiv \text{img } g$ and $\text{Aut}(\mathbb{P}^2)$ is characterized in Lemma 1. 

Example 9 (B1). Let $f : \mathbb{P}^2 \to \mathbb{P}^3$ be defined as in Example 6 and recall that $(r_2 \circ r_0)(f) : \mathbb{P}^2 \to \mathbb{P}^2$ is a projective isomorphism. Thus if $f = g$, then it follows from Proposition 4 that $(r_c)_{c \in \mathbb{I}_{\mathbb{P}^2}}$ is a super set of $\mathcal{R}((r_2 \circ r_0)(f), (r_2 \circ r_0)(g))$ and thus also a super set of $\mathcal{R}(f, g)$.

Proposition 5 (B2). Suppose that $f, g \in M$ are characterized by base case B2. We consider the following set of classes of lines that are contained in $\text{img } f$:

$$\mathbf{F}(f) := \{c \in \mathcal{N}(|\text{img } f|) \mid c^2 = 0, |f| : c = 1, c = |c|\}.$$

a) If $|\mathbf{F}(f)| \geq 2$, then $\mathbf{F}(f) = \{a, b\}$, $\mathbf{F}(g) = \{u, v\}$ and

$$\mathcal{R}(f, g) \subseteq S := \{(\Psi_u \times \Psi_v)^{-1} \circ r_c \circ (\Psi_u \times \Psi_v) \mid c \in \mathbb{I}_{\mathbb{P}^1 \times \mathbb{P}^1}\} \cup \{(\Psi_u \times \Psi_v)^{-1} \circ r_c \circ (\Psi_b \times \Psi_a) \mid c \in \mathbb{I}_{\mathbb{P}^1 \times \mathbb{P}^1}\}.$$

b) If $|\mathbf{F}(f)| \leq 1$, then $\text{img } f$ and $\text{img } g$ are quadric cones and

$$\mathcal{R}(f, g) \subseteq S := \{g^{-1} \circ t^{-1} \circ r_c \circ s \circ f \mid c \in \mathcal{I}\}$$

where $\mathcal{I} := \{c \in \mathbb{F}^{16} \mid r_c \in \text{Aut}(\mathbb{P}^3) \text{ and } r_c(Q) = Q\}$ such that $Q \subset \mathbb{P}^3$ is some fixed quadric cone in diagonal form and $s, t \in \text{Aut}(\mathbb{P}^3)$ are projective isomorphisms such that $s(\text{img } f) = t(\text{img } g) = Q$.

Proof. a) We observe that $\mathbf{F}(f) = \{a, b\}$ where $a$ and $b$ are classes of intersecting lines on $\text{img } f$ so that $h^0(a) = h^0(b) = 2$ and $a \cdot b = 1$. Since $a^2 = 0$, we obtain after resolving its base points
a morphism $\xi_0$: $\text{bmd } f \to \mathbb{P}^1$ and thus the composition $\Psi_0 \circ f^{-1}$: $\text{img } f \to \mathbb{P}^1$ is a morphism as well. Moreover, the following maps are birational morphisms

$$\mu := (\Psi_u \times \Psi_v) \circ f^{-1} : \text{img } f \to \mathbb{P}^1 \times \mathbb{P}^1,$$
$$\nu := (\Psi_u \times \Psi_v) \circ g^{-1} : \text{img } g \to \mathbb{P}^1 \times \mathbb{P}^1.$$

Indeed, $\alpha \cdot b = 1$ and thus the preimages $\mu^{-1}(\{t\} \times \mathbb{P}^1)$ and $\mu^{-1}(\mathbb{P}^1 \times \{t'\})$ are lines that intersect at the unique point in $\text{img } f$ whose image is $(t, t') \in \mathbb{P}^1 \times \mathbb{P}^1$ for all $t, t' \in \mathbb{P}^1$. Now let the line $L \subset \text{img } f$ be defined by $f(\Psi_u^{-1}(t))$ for some $t \in \mathbb{P}^1$. For all $p \in \mathcal{P}(f, g)$ there exists $t' \in \mathbb{P}^1$ and $u \in \mathbb{F}(g)$ such that $p(L) \subset \text{img } g$ is defined by $g(\Psi_u^{-1}(t'))$. It follows that for all $p \in \mathcal{P}(f, g)$ there exists a birational map $r : \mathbb{P}^1 \to \mathbb{P}^1$ such that $r \circ \Psi_u \circ f^{-1} = \Psi_u \circ g^{-1} \circ p$. Therefore, for all $p \in \mathcal{P}(f, g)$ there exists $r \in \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ such that $r \circ \mu = \nu \circ p$. By interchanging $a$ and $b$ we may assume without loss of generality that $r \in \text{Aut}_0(\mathbb{P}^1 \times \mathbb{P}^1)$ so that the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ are not flipped by $r$. The proof is now concluded with Lemma 1.

For assertion b) we consider the symmetric matrices defined by the quadratic forms associated to $\text{img } f, \text{img } g \subset \mathbb{P}^3$. We use matrix diagonalization on these matrices to compute $Q \subset \mathbb{P}^3$ and $s, t \in \text{Aut}(\mathbb{P}^3)$.

The following example for case B2 explains how to compute $\mathcal{S}$ in case the quadrics are doubly ruled.

**Example 10** (B2). Suppose that we are given the following birational maps:

$$f : \mathbb{P}^2 \dasharrow X \subset \mathbb{P}^3, \quad (x_0 : x_1 : x_2) \mapsto (x_0^2 x_1^2 : x_0 x_1 x_2 : x_1^2 x_2^2 : x_0 x_1 x_2^2 + 2x_0^2 x_2),$$

$$g : \mathbb{P}^1 \times \mathbb{P}^1 \dasharrow Y \subset \mathbb{P}^3, \quad (y_0 : y_1 ; z_0 : z_1) \mapsto (y_0^3 y_1^3 y_2^3 : y_0^3 y_1^2 y_2^3 + y_0^3 y_1^3 + y_0^2 y_1^3 + y_0^3 y_1^3 + y_0^3 y_1^3 + y_0^3 y_1^3 + y_0^3 y_1^3 + y_0^3 y_1^3 + y_0^3 y_1^3 + y_0^3 y_1^3).$$

Our goal is to compute the projective isomorphisms $\mathcal{P}(f, g)$. We use Algorithm 1 to compute the base points of the linear series associated to $f$ and $g$. We find that $f$ has simple base points at $p_1 := (0 : 0 : 1), p_2 := (0 : 1 : 0)$ and $p_3 := (1 : 0 : 0)$ with multiplicities 3, 2 and 2, respectively. The infinitely near relations between the remaining 10 base points $p_4, \ldots , p_{13}$ of $f$ are as follows:

$$p_7 \leadsto p_6 \leadsto p_4 \leadsto p_1, \quad p_0 \leadsto p_8 \leadsto p_5 \leadsto p_2,$$

$$p_{10} \leadsto p_{11} \leadsto p_2, \quad p_{13} \leadsto p_{12} \leadsto p_3.$$

The simple base points of $g$ are $q_1 := (0 : 1 ; 0 : 1), q_2 := (0 : 1 ; 1 : 0), q_3 := (1 : 0 ; 0 : 1), q_4 := (1 : 0 ; 1 : 0)$ and have each multiplicity 2. The infinitely near relations between the remaining 8 base points $q_5, \ldots , q_{12}$ of $g$ are as follows:

$$q_6 \leadsto q_5 \leadsto q_1, \quad q_8 \leadsto q_7 \leadsto q_2, \quad q_{10} \leadsto q_9 \leadsto q_3, \quad q_{12} \leadsto q_{11} \leadsto q_4.$$

The multiplicities of the base points are encoded by the classes of $f$ and $g$:

$$[f] = 8e_0 - 3e_1 - 3e_2 - 2e_3 - 2e_4 - 2e_5 - e_6 - \ldots - e_{13} \quad \text{and}$$

$$[g] = 5\ell_0 + 5\ell_1 - 2e_1 - 2e_2 - 2e_3 - 2e_4 - e_5 - \ldots - e_{12}.$$

We observe that $[f]^2 = [g]^2 = \deg X = \deg Y = 26$. We apply Algorithm 2 and find that $h^0([f]) = h^0([g]) = 16$ and thus $(c_0(f), c_0(g)) = (1, 1)$ since $\dim f = \dim g = 3 < 16 - 1$.

We set $(\hat{f}, \hat{g}) := (r_0(f), r_0(g))$ so that $p(\hat{f}) = p(\hat{g}) = (16, 26, 1)$. Notice that $[\hat{f}] = [f]$ and $[\hat{g}] = [g]$ as a direct consequence of the definitions. We find that $(c_1(\hat{f}), c_1(\hat{g})) = (1, 1)$ and $h^0(\hat{f} + \kappa_{\hat{f}}) = h^0(\hat{g} + \kappa_{\hat{g}}) = 12$.

We set $(\tilde{f}, \tilde{g}) := (r_1(\hat{f}), r_1(\hat{g}))$ so that $p(\tilde{f}) = p(\tilde{g}) = (12, 14, 1)$ and

$$[\tilde{f}] = 5e_0 - 2e_1 - 2e_2 - e_3 - e_4 - e_5 \quad \text{and}$$

$$[\tilde{g}] = 3\ell_0 + 3\ell_1 - e_1 - e_2 - 2e_3 - e_4.$$
We remark that \([f] + \kappa_f = ([f] + \kappa_f)\) and \([g] + \kappa_g = ([g] + \kappa_g)\), unlike as we have seen in Example 7.

We find that \((c_1(f), c_1(g)) = (1, 1)\) and \(h^0(\hat{f} + \kappa_f) = h^0(\hat{g} + \kappa_g) = 4\).

We set \((\hat{f}, \hat{g}) := (r_1(\hat{f}), r_1(\hat{g}))\) so that \(p(\hat{f}) = p(\hat{g}) = (4, 2, 1)\) and

\[ [f] = 2e_0 - e_1 - e_2 \quad \text{and} \quad [g] = \ell_0 + \ell_1. \]

We verify that \(h^0(\hat{f} + \kappa_f) = h^0(\hat{g} + \kappa_g) = 0\) so that \((c_1(f), c_1(g)) = (0, 0)\). It follows from Theorem 2 that \(f\) and \(g\) are characterized by base case B2.

We find that \(F(\hat{f}) = \{a, b\}\) and \(F(\hat{g}) = \{u, v\}\) as defined at Proposition 5, where \(a := e_0 - e_1\), \(b := e_0 - e_2\), \(u := \ell_0\) and \(v := \ell_1\) so that

\[
\Psi_a \times \Psi_b : \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1, \quad x \mapsto (x_0 : x_1 : x_0 : x_2),
\]

\[
\Psi_b \times \Psi_a : \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1, \quad x \mapsto (x_0 : x_2 : x_0 : x_1), \quad \text{and}
\]

\[
\Psi_u \times \Psi_v : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, \quad y \mapsto y.
\]

We consider the following reparametrizations \(\mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1:\)

\[
s_c : x \mapsto (c_0 x_0 + c_1 x_1 : c_2 x_0 + c_3 x_1; c_4 x_0 + c_5 x_2 : c_6 x_0 + c_7 x_2),
\]

\[
t_c : x \mapsto (c_0 x_0 + c_1 x_2 : c_2 x_0 + c_3 x_2; c_4 x_0 + c_5 x_1 : c_6 x_0 + c_7 x_1),
\]

and we set \(S := \{s_c\}_{c \in I_{\mathbb{P}^1 \times \mathbb{P}^1}} \cup \{t_c\}_{c \in I_{\mathbb{P}^1 \times \mathbb{P}^1}}\). It follows from Proposition 5 that \(S \supseteq R(f, g)\).

Let us first consider the reparametrizations \(\{s_c\}_{c \in I_{\mathbb{P}^1 \times \mathbb{P}^1}}\). For general \(c \in \mathbb{P}^1 \times \mathbb{P}^1\) we observe that \(c \text{deg}(g \circ s_c) = 10\), although \(c \text{deg}(f) = 8\). We use [11, Algorithm 2] to compute

\[ \mathcal{J}' := \{c \in I \mid g \circ s_c \text{ has the same base points as } f\}, \]

and find that \(\mathcal{J}' = \mathcal{J}_0 \cup \mathcal{J}_1\), where

\[
\mathcal{J}_0 = \{c \in I_{\mathbb{P}^1 \times \mathbb{P}^1} \mid c_0 = c_4 = 1, c_1 = c_2 = c_5 = 0\},
\]

\[
\mathcal{J}_1 = \{c \in I_{\mathbb{P}^1 \times \mathbb{P}^1} \mid c_1 = c_5 = 1, c_0 = c_3 = c_4 = c_7 = 0\}.
\]

If we substitute \(s_c\) into \(g\) for \(c \in \mathcal{J}'\), then the greatest common divisor of the coefficients is \(x_0^2\). Therefore \(c \text{deg}(g \circ s_c) = 8 = c \text{deg}(f)\) as required. Next we enforce that the \(4 \times 4\) coefficient matrix \(M_{g \circ s_c}\) has the same kernel as the coefficient matrix \(M_f\) and compute the corresponding index sets:

\[
\mathcal{J}_0' := \{c \in \mathcal{J}_0 \mid M_{g \circ s_c} \cdot \ker M_f = 0\} = \{c \in \mathcal{J}' \mid c_7 = 2 c_8\} \quad \text{and}
\]

\[
\mathcal{J}_1' := \{c \in \mathcal{J}_1 \mid M_{g \circ s_c} \cdot \ker M_f = 0\} = \emptyset.
\]

Next, we perform the same procedure for \(\{t_c\}_{c \in I_{\mathbb{P}^1 \times \mathbb{P}^1}}\), but in this case we arrive at only empty-sets. Therefore, it follows that \(\mathcal{J}\) as defined at (1) is equal to \(\mathcal{J}_0'\). We apply Proposition 1 and recover \(P(f, g)\) in terms of a matrix parametrized in terms of \(c_5 \neq 0:\)

\[
U := \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & c_3 & 0 & 0 \\
0 & 0 & 32 c_8 & 0 \\
0 & 0 & 32 c_8 & 4 c_8
\end{bmatrix}.
\]

Indeed, if we substitute \(\chi_{U} \circ f\) with indeterminate \(c_3\) into the equation of \(Y \subseteq \mathbb{P}^3\), then we obtain 0. See [8] for an implementation of this example.

\[ \leq \]

Remark 4. The current state is summarized in Algorithm 3 whose correctness follows from Theorem 1 and Theorem 2. Recall that in case B3, B4 and B5, the surface \(\mathfrak{g} F\) is covered by lines or conics. Such surfaces are theoretically well understood, but a complete algorithmic description is left as future work.

\[ \leq \]

6 Applications of the algorithm

In this section we outline how to find the projective isomorphism that correspond to (non-) Euclidean isomorphisms between rational surfaces.

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Algorithm 3

- **Input.** Birational maps \( f, g \in M \).
- **Output.** The set of projective isomorphisms \( P(f, g) \).
- **Method.** We use the \# symbol for comments.

Compute the base points of \( f \) and \( g \) with Algorithm 1.

\[
\begin{align*}
\text{if } & p(f) \neq p(g) \quad \text{then return } \emptyset; \\
(\hat{f}, \hat{g}) & := (f, g); \\
\text{if } & (c_0(\hat{f}), c_0(\hat{g})) = (1, 1) \quad \text{then } (\hat{f}, \hat{g}) := (r_0(\hat{f}), r_0(\hat{g})); \\
\text{while } & (c_1(\hat{f}), c_1(\hat{g})) = (1, 1) \quad \text{do:} \\
(\hat{f}, \hat{g}) & := (r_1(\hat{f}), r_1(\hat{g})); \\
\text{if } & p(\hat{f}) \neq p(\hat{g}) \quad \text{then return } \emptyset; \\
\text{if } & h^0([\hat{f}]) = 3 \quad \text{and } [\hat{f}]^2 = 1 \quad \text{then } \# \text{ case B1} \\
& \quad \text{Set } S \text{ as defined in Proposition 4.} \\
\text{else if } & h^0([\hat{f}]) = 4 \quad \text{and } [\hat{f}]^2 = 2 \quad \text{then } \# \text{ case B2} \\
& \quad \text{Set } S \text{ as defined in Proposition 5.} \\
\text{else if } & h^0([\hat{f}]) = [\hat{f}]^2 + 1 \quad \text{and } 1 \leq [\hat{f}]^2 \leq 8 \quad \text{then } \# \text{ case B3} \\
& \quad \# \text{ Not considered in this article.} \\
\text{else if } & h^0([\hat{f}]) = 2[\hat{f}] + \kappa_j \geq 2 \quad \text{and } [2[\hat{f}] + \kappa_j]^2 = 0 \quad \text{then } \# \text{ case B4} \\
& \quad \# \text{ Not considered in this article.} \\
\text{else if } & h^0([\hat{f}] + \kappa_j) \geq 2 \quad \text{and } [[\hat{f}] + \kappa_j]^2 = 0 \quad \text{then } \# \text{ case B5} \\
& \quad \# \text{ Not considered in this article.} \\
\text{else: } & \text{return } \emptyset; \\
\text{Compute } P(f, g) \text{ from } S \supseteq R(f, g) \text{ using Proposition 1.} \\
\text{return } & P(f, g); \\
\end{align*}
\]
The Möbius quadric is defined as
\[ S^n := \{ x \in \mathbb{P}^{n+1} \mid -x_0^2 + x_1^2 + \ldots + x_{n+1}^2 = 0 \}. \]

Let \( \pi: S^n \rightarrow \mathbb{P}^n \) be the stereographic projection defined as
\[ \pi: S^n \rightarrow \mathbb{P}^n, \quad (x_0 : \ldots : x_{n+1}) \mapsto (x_0 - x_{n+1} : x_1 : \ldots : x_n), \]
with inverse \( \pi^{-1}: \mathbb{P}^n \rightarrow S^n \),
\[ z \mapsto (z_0^2 + z_1^2 + \ldots + z_n^2 : 2z_0z_1 : \ldots : 2z_0z_n : -z_0^2 + z_1^2 + \ldots + z_n^2). \]

Suppose that \( f, g \in \mathcal{M} \) are birational.

The affine isomorphisms between \( \text{img } f \) and \( \text{img } g \) are defined as
\[ \{ \rho \in \mathcal{P}(f, g) \mid \rho(\mathbb{H}_n) = \mathbb{H}_n \}, \]
where \( \mathbb{H}_n := \{ z \in \mathbb{P}^n \mid z_0 = 0 \} \).

The Euclidean isomorphisms between \( \text{img } f \) and \( \text{img } g \) are defined as
\[ \{ \rho \in \mathcal{P}(f, g) \mid \rho(\mathbb{E}_n) = \mathbb{E}_n \}, \]
where \( \mathbb{E}_n := \{ z \in \mathbb{H}_n \mid z_0^2 + \ldots + z_n^2 = 0 \} \). We remark that Euclidean isomorphisms are perhaps better known as “Euclidean similarities”.

The Möbius isomorphisms between \( \text{img } f \) and \( \text{img } g \) are defined as
\[ \{ \pi \circ \rho \circ \pi^{-1} \mid \rho \in \mathcal{P}(\pi^{-1} \circ f, \pi^{-1} \circ g) \text{ and } \rho(S^n) = S^n \}. \]

Notice that if \( \alpha: \mathbb{P}^n \rightarrow \mathbb{P}^n \) is a Möbius isomorphism such that \( \alpha(\text{img } f) = \text{img } g \), then \( \alpha \) is a birational quadratic map such that \( \alpha(\mathbb{E}_n) = \mathbb{E}_n \).

It is straightforward to recover the affine, Euclidean, or Möbius isomorphisms from the output \( \mathcal{P}(f, g) \) or \( \mathcal{P}(\pi^{-1} \circ f, \pi^{-1} \circ g) \) of Algorithm 3.

**Example 11.** We continue with Example 5, where \( f \) parametrizes a Roman surface and where the projective isomorphism \( \chi \in \mathcal{P}(f, f) \) corresponding to \( c = (0, 1, 0, 1, 0, 0, 0, 0, 1) \) is defined as \( \chi(x) = (x_0 : x_1 : x_3 : x_2) \). We check that \( \chi(\mathbb{E}_3) = \mathbb{E}_3 \) and thus \( \chi \) is a Euclidean isomorphism. In fact, we verify that \( |\mathcal{P}(f, f)| = |\{ \rho \in \mathcal{P}(f, f) \mid \rho(\mathbb{E}_3) = \mathbb{E}_3 \}| = 24 \) and thus the Roman surface admits 24 Euclidean symmetries, namely the symmetries of a tetrahedron. \( \triangleright \)

## 7 The proofs of the theorems

In this section we prove Theorem 1 and Theorem 2. We assume that the reader is familiar with the material of [6, Sections II.7 and V.3] and [12, Chapter 1].

**Definition 7.** Suppose that \( Z \) is a rational surface and recall that on rational surfaces the numerical- and rational-equivalence relations for divisor classes are the same. Let \( c \in N(Z) \) be a class such that \( h^0(c) > 0 \) and let \( V := H^0(Z, c) \) denote the vector space of global sections over the ground field \( \mathbb{F} \). We define \( \varphi_c: Z \rightarrow \mathbb{P}^{h^0(c)-1} \) as \( \varphi_c \) as defined in Definition 1. \( \triangleright \)

**Proof of Theorem 1.** We consider the birational morphism \( \pi: \text{bmd } f \rightarrow \text{dom } f \) that resolves the base locus of \( f \) so that the composition \( \Psi_{[f]} \circ \pi: \text{bmd } f \rightarrow \text{img } f \) is a morphism. Notice that \([f] \in N(\text{bmd } f)\) is the divisor class of the pullback of a hyperplane section along this morphism and that \( \kappa_f \) is the canonical class of \( \text{bmd } f \). We recall Definition 7 and notice that for all \( c \in N(\text{bmd } f) \) the map \( \varphi_c \) is up to a choice of basis equivalent to the morphism \( \Psi_c \circ \pi \). In particular, we will notice that the diagrams in this proof remain commutative when an arrow for \( \pi: \text{bmd } f \rightarrow \text{dom } f \) is included.

The case \( \mathcal{R}(f, g) = \emptyset \) is trivial and therefore we will assume that \( \mathcal{R}(f, g) \neq \emptyset \). Suppose that
\( \gamma \in \mathcal{R}(f, g) \) is an arbitrary but fixed compatible reparametrization and let \( \beta \in \mathcal{P}(f, g) \) be a projective isomorphism such that \( \beta \circ f = g \circ \gamma \).

First we observe that \( \mathcal{R}(r_i(f), r_i(g)) \) does not depend on the choice of basis of the associated maps \( r_i(f) \) and \( r_i(g) \) for all \( i \in \{0, 1, 2\} \) (see Definition 1).

In order to show that \( r_0 \) is compatible we need to show that the condition \( c_0 \) is a projective invariant and that \( \gamma \in \mathcal{R}(r_0(f), r_0(g)) \) if \( (c_0(f), c_0(g)) = (1, 1) \).

Let \( \hat{X} := \text{img} r_0(f) \), \( \hat{Y} := \text{img} r_0(g) \) and recall that \( r_0(f) = \Psi_1 \). As a consequence of the definitions, there exists a birational and degree preserving linear projection \( \rho_f : \hat{X} \rightarrow \text{img} f \) so that \( \rho_f \circ \Psi_1 = f \). Let \( \alpha : \text{bmd} \ f \rightarrow \text{bmd} \ g \) be the birational map that makes the diagram of Figure 1 commutative.

![Diagram](image)

**Figure 1:** See proof of Theorem 1.

It follows from the factorization theorem for birational maps [12, Corollary 1-8-4] that there exists a smooth surface \( S \) and birational morphisms \( s : S \rightarrow \text{bmd} \ f \) and \( t : S \rightarrow \text{bmd} \ g \) so that the diagram in Figure 2 commutes.

![Diagram](image)

**Figure 2:** See proof of Theorem 1.

Since \( s^*[f] = t^*[g] \) we find that \( \hat{X} = \text{img} \varphi_{s^*[f]} \) is projectively isomorphic to \( \hat{Y} = \text{img} \varphi_{t^*[g]} \). Hence, \( c_0 \) is a projective invariant and there exists a projective isomorphism \( \beta : \hat{X} \rightarrow \hat{Y} \) that makes the diagram in Figure 3 commutative. It follows that \( \gamma \in \mathcal{R}(r_0(f), r_0(g)) \) and thus \( r_0 \) is a compatible reducer as asserted.

In the remainder of the proof we assume that \( f = r_0(f) = \Psi_1 \). Since \( r_0 \) is compatible, this assumption is without loss of generality.

In order to show that \( r_2 \) is compatible we need to show that \( c_2 \) is a projective invariant and that \( \gamma \in \mathcal{R}(r_2(f), r_2(g)) \) if \( (c_2(f), c_2(g)) = (1, 1) \).

As before, let \( \alpha : \text{bmd} \ f \rightarrow \text{bmd} \ g \) be a birational map such that \( \beta \circ \varphi_{[f]} = \varphi_{[g]} \circ \alpha \). It follows from the factorization theorem for birational maps that there exists a smooth surface \( S \) and birational morphisms \( s : S \rightarrow \text{bmd} \ f \) and \( t : S \rightarrow \text{bmd} \ g \) so that the diagram in Figure 4 commutes.
Let $\hat{X} := \text{img } r_2(f)$, $\hat{Y} := \text{img } r_2(g)$, $a := \frac{1}{\gcd[f]}$ and $b := \frac{1}{\gcd[g]}$. We consider the diagram of Figure 5 that is commutative as a direct consequence of the definitions.

We observe that $s^*[f] = t^*[g]$ as these are the classes of the pullback of a hyperplane section of projectively isomorphic surfaces. Since $[f]$ and $[g]$ are orthogonal to the classes of $(-1)$-curves contracted by $s$ and $t$, respectively, we observe that $\gcd[f] = \gcd[s^*[f]]$ and $\gcd[g] = \gcd[s^*[g]]$ so that

$$\gcd[f] = \gcd[g] \quad \text{and} \quad s^*[a] = \frac{1}{\gcd[f]} s^*[f] = \frac{1}{\gcd[g]} t^*[g] = t^*[b].$$

It follows that $\hat{X} = \text{img } \varphi_{\ast a}$ is projectively isomorphic to $\hat{Y} = \text{img } \varphi_{\ast b}$. Hence, the condition $c_2$ is a projective invariant and there exists a projective isomorphism $\beta: \hat{X} \to \hat{Y}$ that makes the diagram in Figure 6 commutative. We conclude that $\gamma \in R(r_2(f), r_2(g))$ and thus $r_2$ is a compatible reducer as asserted.

We can now see that $p$ is a projective invariant as asserted. Indeed, $[f]^2 = \deg(\text{img } f)$ and $h^0([f])$ is the embedding dimension of the image of $r_0(f)$. Since $r_2$ is compatible it follows that $\gcd[f]$ is
a projective invariant as well.

It is only left to show that the reducer $r_1$ is compatible. Thus we need to show that $c_1$ is a projective invariant and $\gamma \in R(r_1(f), r_1(g))$ if $(c_1(f), c_1(g)) = (1, 1)$. We assume that without loss of generality that $c_1(f) = 1$.

As before, let $\alpha: \text{bmd } f \dashrightarrow \text{bmd } g$ be a birational map such that $\beta \circ \varphi[f] = \varphi[g] \circ \alpha$ and recall from the factorization theorem for birational maps that there exists a smooth surface $S$ and birational morphisms $s: S \to \text{bmd } f$ and $t: S \to \text{bmd } g$ so that the diagram in Figure 4 commutes. We set $\hat{X} := \text{img } r_1(f)$ and $\hat{Y} := \text{img } r_2(g)$. Let $\kappa$ denote the canonical class of $S$ and let $a$ denote the sum of the pullbacks of the classes of $(-1)$-curves that are contracted by $s$. We have $s^*\kappa_f = \kappa - a$ and $s^*\kappa = \kappa_f$ by [6, Proposition V.3.3]. We observe that

$$\varphi_{s^*[f]+\kappa_f} = \varphi_{[f]+\kappa} \circ s \quad \text{and} \quad s_*(s^*[f] + \kappa) = [f] + \kappa_f,$$

and thus $\varphi_{s^*[f]+\kappa}$ makes the diagram of Figure 7 commutative. We remark that if $\kappa \neq 0$, then the linear series $|s^*[f] + \kappa|$ has a fixed part, since $(s^*[f] + \kappa) \cdot a < 0$.

Since $s^*[f] = t^*[g]$ by construction, there exists a projective isomorphism $\hat{\beta} \in \mathcal{P}(r_1(f), r_1(g))$ making the diagram of Figure 8 commutative.

It follows that $c_1$ is a projective invariant and that $\gamma \in R(r_1(f), r_1(g))$. We conclude that the reducer $r_1$ is compatible as asserted.

We will now proceed with the proof of Theorem 2.

In the remainder of this section we suppose that $S$ is a smooth rational surface with canonical class $\kappa$.

The nef threshold of a class $\mathfrak{h} \in N(S)$ is defined as

$$\tau(\mathfrak{h}) := \sup \{ t \in \mathbb{R} \mid \mathfrak{h} + t\kappa \text{ is nef} \}.$$
Notice that $\kappa$ is not nef as $S$ is a rational surface. Hence $h$ is nef if and only if $\tau(h) \geq 0$. Recall from Definition 2 that the components of $f \in \mathcal{M}$ do not have a non-constant greatest common divisor, and thus $\tau([f]) \geq 0$.

**Lemma 2.** If $h \in N(S)$ such that $\tau(h) \geq 0$ and $h^2 > 0$, then $\tau(h) \in \mathbb{Q}_{\geq 0}$.

**Proof.** If $h$ is ample, then the assertion follows from the rationality theorem at [12, Theorem 1-2-11]. The proof of [12, Theorem 1-2-11] also works with $h$ nef and big instead of ample by using the Kawamata-Viehweg vanishing theorem [10, Theorem 4.3.1] instead of the Kodaira Vanishing theorem. We know from [10, Theorem 2.2.16 (bigness of nef divisors)] that $h$ is nef and big. □

**Lemma 3.** If $b, c \in N(S)$ such that $\tau(b), \tau(c) \geq 0$, then $b \cdot c \geq 0$.

**Proof.** See [10, Example 1.4.16]. □

**Lemma 4.** If $c \in N(S)$ such that $\tau(c) \geq 0$ and $c^2 > 0$, then

$$h^0(c + \kappa) = \frac{1}{2} c \cdot (c + \kappa) + 1.$$  

**Proof.** We know from [10, Theorem 2.2.16 (bigness of nef divisors)] that $c$ is nef and big. The assertion now follows from the Riemann-Roch theorem and the Kawamata-Viehweg vanishing theorem as stated at [10, Theorem 4.3.1]. □

**Lemma 5.** If $h \in N(S)$, $\tau(h) \geq 0$ and $(h + \tau(h)) \kappa)^2 \neq 0$, then $\tau(h) \in \mathbb{Z}_{\geq 0}$ and there exists a $(-1)$-curve $E \subset S$ such that $(h + \tau(h)) \cdot [E] = 0$.

**Proof.** There exists a curve $E \subset S$ such that $(h + \tau(h)) \cdot [E] = 0$ and $\kappa \cdot [E] < 0$. Indeed $E$ is a curve that determines the nef threshold. Recall from Lemma 3 that $(h + \tau(h)) \kappa)^2 > 0$. It follows from Hodge index theorem and the genus formula that $[E]^2 = \kappa \cdot [E] = -1$. This concludes the proof as $\tau(h) = h \cdot [E]$. □

**Lemma 6.** If $h \in N(S)$ and $c := \alpha h + \beta \kappa$ for some co-prime $\alpha, \beta \in \mathbb{Z}_{>0}$ such that

$$\tau(c - \kappa) \geq 0, \quad (c - \kappa)^2 > 0 \quad \text{and} \quad h^0(c) \leq 1,$$

then either $\tau(h) < \beta/\alpha$ or $c = 0$.

**Proof.** It follows from Lemma 4 that $c \cdot (c - k) \leq 0$. If $c \cdot (c - k) = 0$, then either $c^2 < 0$ or $c = 0$ by the Hodge index theorem. If $c^2 < 0$ or $c \cdot (c - k) < 0$, then $c$ is not nef by Lemma 3 and thus $\tau(h) < \beta/\alpha$. □
Lemma 7. If \( h \in N(S) \) such that
\[
\tau(h) > 0, \quad h^2 > 0 \quad \text{and} \quad h^0(h + \kappa) \leq 1,
\]
then \((h + \tau(h) \kappa)^2 = 0 \) and \( \tau(h) \leq 1 \).

Proof. It follows from Lemma 6 with \( \alpha = \beta = 1 \) that either \( \tau(h) < 1 \) or \( h + \kappa = 0 \) so that \( \tau(h) = 1 \). Recall from Lemma 3 that \((h + \tau(h) \kappa)^2 \geq 0 \). If \((h + \tau(h) \kappa)^2 > 0 \), then \( \tau(h) \in \mathbb{Z} > 0 \) by Lemma 5 so that we arrive at a contradiction.

Lemma 8. Suppose that \( M, F \in N(S) \) are the classes of the moving and fixed part of the linear series \([M + F]\) such that
\[
h^0(M + F) > 1 \quad \text{and} \quad M^2 = 0.
\]
a) If \( M \cdot [A] = [A]^2 = 0 \) for some curve \( A \subset S \), then \([A] = \beta M \) for some \( \beta \in \mathbb{Q} > 0 \).

b) \( M = \gamma \cdot C \) for some \( \gamma \in \mathbb{Z} > 0 \) and irreducible curve \( C \subset S \).

c) If \( \tau(M + F) \geq 0 \), then \( F = 0 \).

Proof. a) See [12, Lemma 1-2-10 and Proposition 1-2-16].

b) Let \( M = \sum_{i \in I} M_i \), where \( M_i \) are classes of irreducible curves. Since \( M^2 = 0 \) and \( \tau(M) \geq 0 \) we find that \( M_i \cdot M_j = 0 \) for all \( i, j \in I \) and thus this assertion is a consequence of a).

c) Suppose by contradiction that \( F \neq 0 \). We know from Lemma 3 that \((M + F)^2 \geq 0 \). Since \( M^2 = 0 \), \( \text{img} \varphi_{\alpha M} \) is a curve for all \( \alpha \in \mathbb{Z} > 0 \). The class of the fixed part of \( \alpha(M + F) = \alpha M + \alpha F \) is \( \alpha F \) and thus \( \text{img} \varphi_{\alpha M} \cong \text{img} \varphi_{\alpha(M + F)} \). Hence \( \varphi_{\alpha(M + F)} \) is not birational so that \((M + F)^2 = 0 \) by [10, Theorem 2.2.16 (bigness of nef divisors)]. Since \((M + F)^2 = (M + F) \cdot F + M \cdot F + M^2 = 0 \) and \( (M + F) \cdot F, M \cdot F, M^2 \geq 0 \) it follows that \( M \cdot F = F^2 = 0 \). We arrived at a contradiction as a) states that \( F \) must be a multiple of \( M \) and thus cannot be a fixed part.

Lemma 9. If \( h \in N(S) \) such that
\[
\tau(h) > 0, \quad h^2 > 0, \quad (h + \tau(h) \kappa)^2 = 0 \quad \text{and} \quad h + \tau(h) \kappa \neq 0,
\]
then \( h + \tau(h) \kappa = \gamma \cdot C \) for some \( \gamma \in \mathbb{Z} > 0 \) and a rational curve \( C \subset S \) such that
\[
\tau(h) \in \frac{1}{2} \mathbb{Z} > 0, \quad [C]^2 = 0, \quad \kappa \cdot [C] = -2 \quad \text{and} \quad h^0([C]) > 1.
\]

Proof. Recall from Lemma 2 that \( \tau(h) = \beta / \alpha \) for some co-prime \( \alpha, \beta \in \mathbb{Z} > 0 \). Thus \( \tau(\alpha h + \beta \kappa) \geq 0 \) and \( \tau(\alpha h + (\beta - 1) \kappa) \geq 0 \) so that \( h^0(\alpha h + \beta \kappa) \geq 1 \) by the Riemann-Roch theorem and Lemma 3. It follows from Lemma 8b that \( \alpha h + \beta \kappa = \gamma' \cdot [C] \) for some irreducible curve \( C \subset S \) and \( \gamma' \in \mathbb{Z} > 0 \). If \( h \cdot [C] = 0 \), then we arrive at a contradiction with Hodge index theorem and thus \( h \cdot [C] > 0 \). Notice that \( h \cdot [C] + \tau(h) \kappa \cdot [C] = 0 \) and thus \( \kappa \cdot [C] < 0 \) so that \( \kappa \cdot [C] = -2 \) by the genus formula. Therefore \( \tau(h) = \frac{1}{2} h \cdot [C] \) which concludes the proof.

Lemma 10. Suppose that \( h \in N(S) \) such that \( \tau(h) \geq 0 \) and \( h^2 > 0 \).

a) If \( \tau(h) = 0 \), then there exists a birational morphism \( \nu : S \to S' \) to a smooth surface \( S' \) such that
\[
\tau(\nu_* h) > 0, \quad \text{img} \varphi_{\nu_* h} = \text{img} \varphi_h, \quad h^0(\nu_* h) = h^0(h), \quad \gcd \nu_* h = \gcd h, \quad (\nu_* h)^2 = h^2 \quad \text{and} \quad \nu_* h \cdot \nu_* \kappa = h \cdot \kappa.
\]

b) If \( \epsilon := h + \kappa \) and \( h^0(\epsilon) > 1 \), then there exists a birational morphism \( \mu : S \to S' \) such that \( \text{dom} \mu = S \), \( \text{img} \mu \) is a smooth surface and
\[
\tau(\mu_* \epsilon) \geq 0 \quad \text{and} \quad \text{img} \varphi_{\mu_* \epsilon} = \text{img} \varphi_\epsilon.
\]
Proof. a) It follows from Lemma 5 that there exists a \((-1)\)-curve \(E \subset S\) such that \(h \cdot [E] = 0\). By Castelnuovo’s contraction theorem there exists a birational morphism \(ν_1 : S \to S_1\) that contracts \(E\) to a smooth point. If \(τ(ν_1, h) > 0\), then \(ν := ν_1\) and the remaining assertions are a straightforward consequence of basic intersection theory (see for example [6, Section V.3]). If \(τ(ν_1, h) = 0\), then we repeat the same argument for \(ν_1\cdot h\) and contract the resulting \((-1)\)-curve. Since \(\text{rank } N(S_1) < \text{rank } N(S) < ∞\) there will be a finite number of contractions and we set \(ν\) equal to the composition of these contractions.

b) Suppose that \(E \subset S\) is an irreducible curve such that \(c \cdot [E] < 0\). This implies that \(h^0([E]) = 1\). We have \(h \cdot [E] ≥ 0\) and thus \(κ \cdot [E] < 0\). By the Riemann-Roch formula and Serre duality we have \(|E|^2 - [E] \cdot κ ≤ 0\) and thus \(|E|^2 < 0\). Therefore \(E\) is a \((-1)\)-curve by the genus formula.

We now apply Castelnuovo’s contraction theorem as in a) and define \(μ\) as the compositions of contractions of \((-1)\)-curves that are negative against pushforwards of \(c\).

We call \((S, h)\) a reduction pair for \(f \in \mathcal{M}\) if the following four properties are satisfied:

1. \(τ(h) > 0\) and \(h^2 > 0\),
2. \(\text{img } ϕ_h = \text{img } Ψ_f\) (see Definition 7),
3. \((h^0(h), h^2, \gcd h) = (h^0([f]), [f]^2, \gcd [f])\),
4. \(h^0(h + κ) = h^0([f] + κ)\).

Lemma 11. There exists a reduction pair \((S, h)\) for all \(f \in \mathcal{M}\) such that \([f]^2 > 0\).

Proof. Let \((S, h) := (\text{bnd}\ f, [f])\). If \(τ(h) > 0\), then \((S, h)\) is a reduction pair for \(f\) as a direct consequence of the definitions. Now suppose that \(τ(h) = 0\) and let \(ν : S → S'\) be defined as in Lemma 10a. Since \(ν_2⋅κ\) is the canonical class of \(S'\), it follows from Lemma 4 that \(h^0(ν_2⋅h + ν_2⋅κ) = h^0(h + κ)\) so that \((S', ν_2⋅h)\) is a reduction pair for \(f\).

Lemma 12. If \((S, h)\) is a reduction pair for \(f \in \mathcal{M}\) such that

\[(h + τ(h))κ^2 = 0, \quad τ(h) ≤ 1 \quad \text{and} \quad h + τ(h)κ ≠ 0,\]

then \(f\) is characterized by either base case B4 or B5.

Proof. We know from Lemma 9 that \(τ(h) ∈ \{\frac{1}{2}, 1\}\) and \(h + τ(h)κ = γ [C]\) for some rational curve \(C ⊂ S\) such that \(h^0([C]) > 1\) and \(κ \cdot [C] = -2\). It follows that \(1 ≤ h \cdot [C] ≤ 2\). If \(h \cdot [C] = 1\), then \(ϕ_h(S)\) is covered by lines so that \(f\) is characterized by base case B4. If \(h \cdot [C] = 2\), then \(S\) is mapped by \(ϕ_h\) either 2:1 to a line or 1:1 to a conic. Hence \(f\) is in this case characterized by base case B5.

Lemma 13. If \((S, h)\) is a reduction pair for \(f \in \mathcal{M}\) such that

\[h + τ(h)κ = 0 \quad \text{and} \quad \gcd h = 1,\]

then \(f\) is characterized by base case either B1, B2 or B3.

Proof. By assumption \(h = -τ(h)κ\) and thus \(-κ\) is nef so that \(S\) is a weak del Pezzo surface [5, Definition 8.1.18]. It follows from [5, Theorem 8.1.15, Lemma 8.3.1, Theorem 8.3.2] that \(S\) is a weak del Pezzo surface of degree \(1 ≤ h^2 ≤ 8\) such that \(h^0(-κ) = κ^2 + 1\). Since \(\gcd h = 1\) we find that \(τ(h) ∈ \{\frac{1}{7}, \frac{1}{2}, 1\}\) by \([12, \text{Corollary 1-2-15 (boundedness of denominator)}]\). The assertions about birationality are a consequence of Reider’s theorem. Therefore \(f\) is characterized by base case B1, B2 or B3 if \(τ(h)\) is equal to \(\frac{1}{7}, \frac{1}{2}, 1\) respectively.
Lemma 14. If \( f \in \mathcal{M} \) and \( c := [f] + \kappa_f \) such that
\[
h^0(c) > 1 \quad \text{and} \quad [f]^2 > [c]^2 = c \cdot [c] = 0,
\]
then \( f \) is characterized by base case B5.

Proof. We know from Lemma 8 that \([c] = \gamma[C]\) for some irreducible curve \( C \subseteq S \) and \( \gamma \in \mathbb{Z}_{>0} \). We have \([f] \cdot [C] > 0\), otherwise we arrive at a contradiction with the Hodge index theorem. Since \([(f) + \kappa_f] \cdot [C] = 0\) it follows that \(\kappa_f \cdot [C] < [C]^2 = 0\). Hence \(\kappa_f \cdot [C] = -2\) by the genus formula so that \(C\) is a rational curve. The image of \(f\) is a surface, since \([f]^2 > 0\) and \([f] \cdot [C] = 2\). Therefore \(C\) is mapped by \(\varphi[f]\) to either a line or a conic. We conclude that \(f\) is characterized by base case B5.

Lemma 15. If \( f \in \mathcal{M} \) such that \([f]^2 = c_0(f) = c_1(f) = c_2(f) = 0\), then there exists no \( g \in \mathcal{M}_{c_1} \) such that \([g]^2 > 0\) and \( f \in \{ r_1(g), (r_2 \circ r_1)(g) \}\).

Proof. Suppose by contradiction that there exists \( g \in \mathcal{M}_{c_1} \) such that \([g]^2 > 0\) and \( f = r_1(g) \). Let \( c := [g] + \kappa_g \) and notice that \([c] = [r_1(g)] = [f]\) is the class of the moving part of the linear series \([c]\) associated to \(c\). Since \(c_1(g) = 1\), \([g]^2 > 0\) and \([c]^2 = 0\), we deduce from the definition of \(c_1\) that \(h^0(c) > 1\) and \([c] \cdot c \neq 0\). Thus \(c - [c] \neq 0\) and there exists an irreducible curve \(A \subseteq S\) such that \([A]\) is the class of a prime divisor of the fixed part with class \(c - [c]\). By Lemma 10b there exists a birational morphism \(\mu: bmd g \rightarrow \text{img } \mu\) such that \(\tau(\mu_*c) \geq 0\) and \(\text{img } \varphi_{\mu_+c} = \text{img } \varphi_c\). It follows from Lemma 8c that \((\mu_*A) = 0\) and \(\mu_*[A] = 0\). Hence \(A\) is contained the fiber of \(\varphi_c\). This implies that \([c] \cdot A = 0\) and thus we deduce that \([c] \cdot (c - [c]) = 0\). We arrived at a contradiction as \([c] \cdot (c - [c]) = [c] \cdot c = 0\).

Finally, suppose by contradiction that there exists \( g \in \mathcal{M}_{c_1} \) such that \([g]^2 > 0\) and \( f = (r_2 \circ r_1)(g) \). In this case \([h]^2 = c_0(h) = c_1(h) = 0\), where \( h := r_1(g) \). We now arrive at a contradiction by applying the above arguments with \(h\) instead of \(f\).

Proof of Theorem 2. a) If \(h^0([f] + \kappa_f) > 1\), then it follows from Lemma 14 that \(f\) is characterized by base case B5. Now suppose that \(h^0([f] + \kappa_f) \leq 1\). By Lemma 11 there exists a reduction pair \((S, h)\) for \(f\). It follows from Lemma 7 that \((h + \tau(h)) \kappa) = 2 = 0\) and \(\tau(h) \leq 1\). If \(h + \tau(h) \kappa \neq 0\), then it follows from Lemma 12 that \(f\) is characterized by base case B4 or B5. If \(h + \tau(h) \kappa = 0\), then we know from Lemma 13 that \(f\) is characterized by base case B1, B2 or B3.

b) This assertion follows from Lemma 15.

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