INEQUALITIES FOR SELECTED EIGENVALUES
OF THE PRODUCT OF MATRICES

BO-YAN XI AND FUZHEN ZHANG

Abstract. The product of a Hermitian matrix and a positive semidefinite matrix has only real eigenvalues. We present bounds for sums of eigenvalues of such a product.

1. Introduction

Let $A$ be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. If some, say $k$, of the eigenvalues of $A$ are selected, they may be indexed by a sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Hence, $\lambda_{i_1}(A) \geq \lambda_{i_2}(A) \geq \cdots \geq \lambda_{i_k}(A)$.

A classical result of Wielandt [15] states that if $A$ and $B$ are $n \times n$ Hermitian matrices and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, then

\begin{equation}
\sum_{t=1}^{k} \lambda_{i_t}(A + B) \leq \sum_{t=1}^{k} \lambda_{i_t}(A) + \sum_{t=1}^{k} \lambda_{i_t}(B).
\end{equation}

A reversed inequality follows from (1.1) by replacing $A$ and $B$ with $-A$ and $-B$, respectively:

\begin{equation}
\sum_{t=1}^{k} \lambda_{i_t}(A) + \sum_{t=1}^{k} \lambda_{n-t+1}(B) \leq \sum_{t=1}^{k} \lambda_{i_t}(A + B).
\end{equation}

There is a great amount of research on the partial sums of selected eigenvalues (see, e.g., [10, 12, 13] or [11, Chap. 9] and [2, Chap. III]) as well as on the characterization of the eigenvalues of the sum of Hermitian matrices (see, e.g., [3, 5, 7, 8]).

Inequalities analogous to (1.1) and (1.2) for the product of two matrices are presented in [14]: If $A$ and $B$ are $n \times n$ positive semidefinite matrices, then

\begin{equation}
\sum_{t=1}^{k} \lambda_{i_t}(A) \lambda_{n-t+1}(B) \leq \sum_{t=1}^{k} \lambda_{i_t}(AB) \leq \sum_{t=1}^{k} \lambda_{i_t}(A) \lambda_{i_t}(B).
\end{equation}

The eigenvalues of the product of two Hermitian matrices need not be real. For example, for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, the eigenvalues of $AB$ are $1 \pm i$. Thus, inequalities (1.3) do not extend to partial sums of eigenvalues of the product of two Hermitian matrices. However, requiring one matrix to be positive semidefinite (PSD) ensures that the eigenvalues of the product are all real; that is, if $A$ is Hermitian and $B$ is PSD, then $AB$ and $B^{1/2}AB^{1/2}$ have the same eigenvalues, so $AB$ has only real eigenvalues.

2010 Mathematics Subject Classification. Primary 15A42; Secondary 47A75.

Key words and phrases. Eigenvalue, Hermitian matrix, inequality, positive semidefinite matrix.

The first author was supported in part by the National Natural Science Foundation of China Grant No. 11361038. The second author was supported in part by NSU Research Scholar grant.
Eigenvalue problem is of central importance in matrix analysis and related areas. Usually, inequalities for selected eigenvalues involve two Hermitian matrices for sum and two positive semidefinite matrices for product. Nevertheless, the results on the partial sums of selected eigenvalues of the product of one PSD matrix and one Hermitian matrix are fragmentary. A result of this kind is, for example, a celebrated theorem of Ostrowski (see, e.g., [6, p. 283]) which states that, for Hermitian \( A \) and positive definite \( B \), \( \lambda_i(AB) = \theta_i \lambda_i(A) \), where \( \theta_i \in [\lambda_n(B), \lambda_1(B)] \).

The purpose of this paper is to present inequalities on the partial sums of selected eigenvalues of the product of a PSD matrix and a Hermitian matrix. Our results generalize some existing ones such as inequalities \([3]\).

2. Eigenvale inequalities for Hermitian and PSD matrices

In [17, Theorem 3], inequalities concerning \( \sum_{i=1}^k \lambda_i(AB) \) are shown, where \( A \) is Hermitian and \( B \) is positive semidefinite. These inequalities are about the sum of the \( k \) largest eigenvalues of \( AB \). In this section, we show inequalities for selected eigenvalues; that is, we present inequalities concerning \( \sum_{t=1}^k \lambda_{i_t}(AB) \).

We borrow Wielandt’s min-max representation (see, e.g., [2, p. 67]) for the eigenvalues of Hermitian matrices, which is used in the proof of our main result.

**Theorem 2.1** (Wielandt [15]). If \( A \in \mathbb{C}^{n \times n} \) (the set of \( n \times n \) complex matrices) is Hermitian and \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \), then

\[
\sum_{t=1}^k \lambda_{i_t}(A) = \max_{S_1 \subset \cdots \subset S_k \subset \mathbb{C}^n} \min_{x_1 \in S_1} \sum_{t=1}^k x_{i_t}^* A x_{i_t},
\]

where \( \delta_{rs} \) is the Kronecker delta and \( x^* \) is the conjugate transpose of \( x \in \mathbb{C}^n \).

Denote the inertia of an \( n \times n \) Hermitian \( A \) by \( (\pi_+, \nu_-, \delta_0) \), where \( \pi_+, \nu_-, \delta_0 \) are the numbers of positive, negative and zero eigenvalues of \( A \), respectively (see, e.g., [16, p. 255]). Let \( n_A \) be the number of nonnegative eigenvalues of \( A \), namely, \( n_A = \pi_+ + \delta_0 \). For any Hermitian matrix \( A \in \mathbb{C}^{n \times n} \), we have

\[
(2.1) \quad n_A + n_{-A} = \pi_+ + \nu_- + 2\delta_0 = n + \delta_0.
\]

Let \( k_A \) be the number of nonnegative eigenvalues in \( \{\lambda_{i_1}(A), \lambda_{i_2}(A), \ldots, \lambda_{i_k}(A)\} \) for the given index sequence \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \). So, if \( k_A = 0 \), then all \( \lambda_{i_1}(A), \lambda_{i_2}(A), \ldots, \lambda_{i_k}(A) \) are negative; if \( k_A = k \), then each of \( \lambda_{i_1}(A), \lambda_{i_2}(A), \ldots, \lambda_{i_k}(A) \) is positive or zero. It is always true that \( k_A \leq n_A \) and \( \lambda_{k_A} \geq \lambda_{n_A} \).

Now we are ready to present our main theorem. In what follows, our convention is that a summation in the form \( \sum_{t=p}^q \) over \( t \) vanishes if \( p > q \).

**Theorem 2.2.** Let \( A \in \mathbb{C}^{n \times n} \) be Hermitian, let \( B \in \mathbb{C}^{n \times n} \) be positive semidefinite, and let \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \). Then

\[
(2.2) \quad \sum_{t=1}^k \lambda_{i_t}(AB) \leq \sum_{t=1}^{k_A} \lambda_{i_t}(A) \lambda_{t}(B) + \sum_{t=k_A+1}^k \lambda_{i_t}(A) \lambda_{n-k+t}(B)
\]

and

\[
(2.3) \quad \sum_{t=1}^k \lambda_{i_t}(AB) \geq \sum_{t=1}^{k_A} \lambda_{i_t}(A) \lambda_{n-t+1}(B) + \sum_{t=k_A+1}^k \lambda_{i_t}(A) \lambda_{k-t+1}(B).
\]
Proof. Note that if \( k_A = 0 \), then the first terms (summations) on the right-hand
sides of (2.2) and (2.3) are absent. If \( k_A = k \), then the second terms disappear.
For \( 1 \leq k_A < k \), the first terms on the right-hand sides of (2.2) and (2.3) are
nonnegative, while the second terms are nonpositive.

We divide the proof of the theorem into five cases.

Case (1). If \( n_A = n \), then \( A \) is PSD and \( k_A = k \). (2.2) and (2.3) are
immediate.

Case (2). If \( n_A = 0 \), then \( k_A = 0 \) and \( -A \) is positive definite. We set \( j_t \) as follows:
\[
1 \leq j_t = n - i_k + 1 < \cdots < j_t = n - i_{k-t+1} + 1 < \cdots < j_k = n - i_1 + 1 \leq n.
\]

By inequalities (1.3), we have
\[
\sum_{t=1}^{k} \lambda_{j_t}(-A)\lambda_{n-t+1}(B) \leq \sum_{t=1}^{k} \lambda_{j_t}(-AB) \leq \sum_{t=1}^{k} \lambda_{j_t}(-A)\lambda_{t}(B).
\]

Because \( \lambda_{j_t}(-A) = -\lambda_{n-j_t+1}(A) \) and \( j_t = n - i_{k-t+1} + 1 \), we obtain

\[(2.4) \sum_{r=1}^{k} \lambda_{i_r}(A)\lambda_{k-r+1}(B) \leq \sum_{r=1}^{k} \lambda_{i_r}(AB) \leq \sum_{r=1}^{k} \lambda_{i_r}(A)\lambda_{n-k+r}(B).\]

Note that on the right hand side, \( \lambda_{i_{k-t+1}}(A) \) is paired (multiplied by) \( \lambda_{n-t+1}(B) \),
\( t = 1, 2, \ldots, k \), namely, \( \lambda_{i_r}(A) \) is paired with \( \lambda_{n-k+r}(B) \), \( r = 1, 2, \ldots, k \). Likewise,
on the left, \( \lambda_{i_r}(A) \) is paired with \( \lambda_{k-r+1}(B) \), \( r = 1, 2, \ldots, k \). It follows that
\[(2.5) \sum_{r=1}^{k} \lambda_{i_r}(A)\lambda_{k-r+1}(B) \leq \sum_{r=1}^{k} \lambda_{i_r}(AB) \leq \sum_{r=1}^{k} \lambda_{i_r}(A)\lambda_{n-k+r}(B).\]

Inequalities (2.2) and (2.3) follow immediately.

If \( A \) has no positive eigenvalues, then \( -A \) is PSD which can be dealt with as
above. We assume below that \( A \) has both positive and negative eigenvalues.

Case (3). Let \( 1 \leq \pi_+ \), \( 1 \leq \nu_- \), and \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n_A \). Then all
\( \lambda_{i_1}(A), \lambda_{i_2}(A), \ldots, \lambda_{i_k}(A) \) are nonnegative, namely, \( k_A = k \). We show that
\[(2.6) \sum_{t=1}^{k} \lambda_{i_t}(A)\lambda_{n-t+1}(B) \leq \sum_{t=1}^{k} \lambda_{i_t}(AB) \leq \sum_{t=1}^{k} \lambda_{i_t}(A)\lambda_{t}(B).\]

We may assume that \( A \) is a diagonal matrix as the inequalities are invariant
under unitary similarity. For the upper bound, using the approach of splitting (see
[9] p. 381 or [6] p. 250), we write \( A = A_+ + A_- \), where \( A_+ \) is the diagonal matrix
with all the positive (if any) eigenvalues of \( A \) on the main diagonal (plus some
zeros), and \( A_- \) is the diagonal matrix with all the negative (if any) eigenvalues of
\( A \) on the main diagonal (plus some zeros). Then \( A_+ \) is positive semidefinite and
\( \lambda_{i_1}(A), \lambda_{i_2}(A), \ldots, \lambda_{i_k}(A) \) are all contained on the main diagonal of \( A_+ \) as \( i_k \leq n_A \).

Since \( A \leq A_+ \) (i.e., \( A_+ - A \) is PSD), we have \( \lambda_t(AB) \leq \lambda_t(A_+B) \) for every \( t, \)
and moreover, \( \lambda_{i_t}(A_+) = \lambda_{i_t}(A) \) for \( t \leq k \). For the upper bound, by (2.3), we get
\[
\sum_{t=1}^{k} \lambda_{i_t}(AB) \leq \sum_{t=1}^{k} \lambda_{i_t}(A_+B) \leq \sum_{t=1}^{k} \lambda_{i_t}(A_+)\lambda_{t}(B) = \sum_{t=1}^{k} \lambda_{i_t}(A)\lambda_{t}(B).
\]

For the lower bound, we need to use the Wielandt’s min-max representation.
Bear in mind that $\lambda_t(A) \geq 0$, $t = 1, 2, \ldots, n_A$, and $\lambda_i\,(A) \geq 0$, $t = 1, 2, \ldots, k$. Let $A = \text{diag}(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$. The standard column unit vectors $e_i = (0, \ldots, 0, 1, 0 \cdot \cdot \cdot, 0)^T$ are eigenvectors corresponding to $\lambda_i(A)$, $i = 1, 2, \ldots, n$.

We assume $B$ is nonsingular (or use continuity with $B_\varepsilon = B + \varepsilon I$, $\varepsilon > 0$). Let

$$S_t = \text{Span}(B^{\frac{1}{2}}e_1, B^{\frac{1}{2}}e_2, \ldots, B^{\frac{1}{2}}e_t), \quad t = 1, 2, \ldots, k.$$  

Then $S_1 \subset S_2 \subset \cdots \subset S_k$ and $\dim(S_t) = i_t$. The min-max representation reveals

$$\sum_{t=1}^{k} \lambda_i\,(AB) \geq \min_{u_t \in S_t} \sum_{t=1}^{k} u_t^* B^{\frac{1}{2}}AB^{\frac{1}{2}}u_t = \min_{u_t \in S_t} \sum_{t=1}^{k} \frac{u_t^* B^{\frac{1}{2}}AB^{\frac{1}{2}}u_t}{(B^{\frac{1}{2}}u_t)^*(B^{\frac{1}{2}}u_t)} = \frac{1}{\sum_{t=1}^{k} |a_j|^2} \sum_{j=1}^{i_t} |a_j|^2 \lambda_j \geq \lambda_i\,(A), \quad t = 1, 2, \ldots, k.$$  

For $u_t \in S_t$, let $u_t = \sum_{j=1}^{i_t} a_j B^{\frac{1}{2}}e_j$, $a_1, a_2, \ldots, a_{i_t} \in \mathbb{C}$, $t = 1, 2, \ldots, k$. Then we have

$$u_t^* B^{\frac{1}{2}}AB^{\frac{1}{2}}u_t = \sum_{j=1}^{i_t} |a_j|^2 \lambda_j \geq \lambda_i\,(A), \quad t = 1, 2, \ldots, k.$$  

Let $C = \sum_{t=1}^{k} \lambda_i\,(A)u_tu_t^* \in \mathbb{C}^{n \times n}$, where $\{u_1, u_2, \ldots, u_k\}$ is an orthonormal set in $\mathbb{C}^n$. Then $C$ is PSD and $\lambda_i\,(C) = \lambda_i\,(A)$ for each $t$. We obtain

$$\sum_{t=1}^{k} \lambda_i\,(AB) \geq \min_{u_t \in S_t} \sum_{t=1}^{k} \lambda_i\,(A)u_t^*Bu_t = \min_{u_t \in S_t} \text{tr}(CB) \geq \min_{u_t \in S_t} \sum_{t=1}^{k} \lambda_i\,(C)\lambda_{n-t+1}(B) = \sum_{t=1}^{k} \lambda_i\,(A)\lambda_{n-t+1}(B).$$  

We note here that it would be nice if we could derive the lower bound from the upper bound with $A$ replaced by $-A$, without using the Wielandt’s min-max representation. However, this approach doesn’t work as $\lambda_i\,(-A) \leq 0$. Moreover, like the lower bound, the upper bound can also be obtained by using the min-max representation with suitably selected subspaces.

Case (4). Let $1 \leq \pi_+, \leq \nu_-$, and $n_A < i_1 < i_2 < \cdots < i_k \leq n$. Then none of $\lambda_{i_1}(A), \lambda_{i_2}(A), \ldots, \lambda_{i_k}(A)$ is positive. Consider $-A$ with $1 \leq j_1 = n - i_k + 1 < \cdots < j_k = n - i_1 + 1 \leq n$. Since $j_k = n - i_1 + 1 \leq n - n_A \leq n - A$ (see (2.1)), we apply case (3) to $-A$ and $B$ to get the desired inequalities (2.5) as in case (2).

Case (5). Let $1 \leq \pi_+, \leq \nu_-$, and $1 \leq i_1 < \cdots < i_{k_A} \leq n_A < i_{k_A+1} < \cdots < i_k \leq n$. Then

$$\sum_{t=1}^{k} \lambda_i\,(AB) = \sum_{t=1}^{k_A} \lambda_i\,(AB) + \sum_{t=k_A+1}^{k} \lambda_i\,(AB).$$  

For $1 \leq i_1 < \cdots < i_{k_A} \leq n_A$, using (2.6), we have

$$\sum_{t=1}^{k_A} \lambda_i\,(A)\lambda_{n-t+1}(B) \leq \sum_{t=1}^{k_A} \lambda_i\,(AB) \leq \sum_{t=1}^{k_A} \lambda_i\,(A)\lambda_t(B).$$  

(2.7)
For \( n_A < i_{k_A+1} < \cdots < i_k \leq n \), set \( j_t = i_{k_A+t}, \ t = 1, \ldots, k', \) where \( k' = k - k_A. \) Then \( n_A < j_1 < \cdots < j_{k'} \leq n. \) By case (4), we have (for the upper bound)

\[
\sum_{t=k_A+1}^{k} \lambda_i(AB) = \sum_{t=1}^{k'} \lambda_i(AB) \leq \sum_{t=1}^{k'} \lambda_{i_{k_A+t}}(A)\lambda_{n-k'+t}(B) = \sum_{r=k_A+1}^{k} \lambda_i(AB)\lambda_{n-k-r}(B)
\]

(2.8)

and (for the lower bound)

\[
\sum_{t=k_A+1}^{k} \lambda_i(AB) = \sum_{t=1}^{k'} \lambda_i(AB) \geq \sum_{t=1}^{k'} \lambda_{j_{k_A+t}}(A)\lambda_{k'-t+1}(B) = \sum_{r=k_A+1}^{k} \lambda_i(AB)\lambda_{k-r+1}(B)
\]

(2.9)

Combining inequalities (2.7), (2.8) and (2.9) results in (2.2) and (2.3).

\[\square\]

**Remark 2.3.** Li and Mathias [9, Theorem 2.3] showed some inequalities in partial products of the form \( \prod_{t=1}^{k} \lambda_i(AB) \), where \( A \) is Hermitian and \( B \) is PSD.

It is known that for positive numbers, inequalities of partial products imply those of partial sums. Simply put in the language of majorization, log-majorization implies weak-majorization (see, e.g., [1] p. 232 or [16] p. 345). Thus, it is tempting to obtain the results on partial sums from Li and Mathias’ result on partial products. However, this is only possible for \( |\lambda_i(AB)| \) and \( |\lambda_i(A)| \) by observing that \( \frac{\lambda_i(AB)}{\lambda_i(A)} > 0 \). Since \( \lambda_i(AB) \) and \( \lambda_i(A) \) are paired with the same sign (maybe both negative), we don’t see how the absolute values are dropped so that \( \lambda_i(AB) \) and \( \lambda_i(A) \) appear in a partial sum without being paired as a quotient.

**Remark 2.4.** Hoffman’s min-max representation (see, e.g., [1] 2.17) in the product form \( \prod_{t=1}^{k} \lambda_i(\cdot) \) for positive semidefinite matrices does not generalize to Hermitian matrices as Li and Mathias showed by example [9, pp. 411-412]; that is, there is no multiplicative analog of Wielandt’s min-max representation (Theorem 2.1) for Hermitian matrices. Therefore, it is impossible to derive sum inequalities \( \sum_{t=1}^{k} \lambda_i(\cdot) \) from product inequalities \( \prod_{t=1}^{k} \lambda_i(\cdot) \) for Hermitian matrices through majorization. In view of this, our Theorem 2.2 appears to be important.
Example 2.5. Let \( n = 3, k = 2 \), and let
\[
A = \begin{pmatrix}
1 & 2 & 0 \\
2 & 1 & 0 \\
0 & 0 & -4
\end{pmatrix}, \quad B = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]
Then
\[
\lambda_1(A) = 3, \quad \lambda_2(A) = -1, \quad \lambda_3(A) = -4; \\
\lambda_1(B) = 3, \quad \lambda_2(B) = 2, \quad \lambda_3(B) = 1; \\
\lambda_1(AB) = 3, \quad \lambda_2(AB) = -3, \quad \lambda_3(AB) = -8.
\]
Thus, \( n_A = 1 \). We consider the cases (I) \( i_1 = 1, i_2 = 2, k_A = 1 \); (II) \( i_1 = 1, i_2 = 3, k_A = 1 \); and (III) \( i_1 = 2, i_2 = 3, k_A = 0 \), to get, respectively,
\[
\sum_{t=1}^{k_A} \lambda_{i_t}(A) \lambda_t(B) + \sum_{t=k_A+1}^{k} \lambda_{i_t}(A) \lambda_{n-k+t}(B)
= \begin{cases}
\lambda_1(A) \lambda_1(B) + \lambda_2(A) \lambda_3(B) = 8, & \text{(I)} \\
\lambda_1(A) \lambda_1(B) + \lambda_3(A) \lambda_3(B) = 5, & \text{(II)} \\
\lambda_2(A) \lambda_2(B) + \lambda_3(A) \lambda_3(B) = -6, & \text{(III)}
\end{cases}
\]
and
\[
\sum_{t=1}^{k_A} \lambda_{i_t}(A) \lambda_{n-t+1}(B) + \sum_{t=k_A+1}^{k} \lambda_{i_t}(A) \lambda_{k-t+1}(B)
= \begin{cases}
\lambda_1(A) \lambda_3(B) + \lambda_2(A) \lambda_1(B) = 0, & \text{(I)} \\
\lambda_1(A) \lambda_3(B) + \lambda_3(A) \lambda_1(B) = -9, & \text{(II)} \\
\lambda_2(A) \lambda_2(B) + \lambda_3(A) \lambda_1(B) = -14. & \text{(III)}
\end{cases}
\]
Recall the trace inequality (see, e.g., [21 p. 78] or [61 p. 255]) for Hermitian \( A, B \):
\[
(2.10) \quad \sum_{t=1}^{n} \lambda_t(A) \lambda_{n-t+1}(B) \leq \text{tr}(AB) = \sum_{t=1}^{n} \lambda_t(AB) \leq \sum_{t=1}^{n} \lambda_t(A) \lambda_t(B).
\]
As we noted in Section 1, the product of two Hermitian matrices may have nonreal eigenvalues. So, the trace in (2.10) cannot be replaced by partial sums in general. Theorem 2.2 presents partial sum inequalities for the product of one Hermitian matrix and one PSD matrix.

We present a few results that are immediate from Theorem 2.2.

Corollary 2.6. Let \( A \in \mathbb{C}^{n \times n} \) be a stable Hermitian matrix (i.e., the eigenvalues of \( A \) are located on the left half-plane) and \( B \in \mathbb{C}^{n \times n} \) be positive semidefinite. Then
\[
\sum_{t=1}^{k} \lambda_{i_t}(A) \lambda_{k-t+1}(B) \leq \sum_{t=1}^{k} \lambda_{i_t}(AB) \leq \sum_{t=1}^{k} \lambda_{i_t}(A) \lambda_{n-k+t}(B).
\]
Proof. Note that \( k_A = 0 \) or \( \lambda_{i_t}(A) = 0 \) for \( t = 1, 2, \ldots, k_A \) in (2.2) and (2.3). \( \Box \)
Corollary 2.7. Let $A \in \mathbb{C}^{n \times n}$ be Hermitian and $B \in \mathbb{C}^{n \times n}$ be positive semidefinite. Then for any positive eigenvalue $\lambda_i(AB)$ and negative eigenvalue $\lambda_i(AB)$ (if any)

$$\lambda_s(A)\lambda_n(B) + \lambda_t(A)\lambda_1(B) \leq \lambda_s(AB) + \lambda_t(AB) \leq \lambda_s(A)\lambda_1(B) + \lambda_t(A)\lambda_n(B).$$

Proof. Take $k = 2$ and $1 \leq i_1 < i_2 < \cdots < i_t \leq n$. Then $\lambda_s(A) > 0 > \lambda_t(A)$. In (2.2), there is one term in each of the two summations; the same is true for (2.3). □

Corollary 2.8. Let $A \in \mathbb{C}^{n \times n}$ be Hermitian and $B \in \mathbb{C}^{n \times n}$ be positive semidefinite. If $\lambda_p(AB)$ is the smallest positive eigenvalue of $AB$ and $\lambda_q(AB)$ is the largest negative eigenvalue of $AB$ (if any), then their distance (gap near 0) is bounded as

$$\lambda_p(AB) - \lambda_q(AB) \leq [\lambda_p(A) - \lambda_q(A)]\lambda_1(B).$$

Proof. Since $\lambda_p(AB) > 0$ and $\lambda_q(AB) < 0$, we have $\lambda_p(A) > 0 > \lambda_q(A)$. Setting $k = 1$ and $i_1 = t$, for each $t = 1, 2, \ldots, n$ in Theorem 2.2, we obtain

$$\lambda_1(A)\lambda_n(B) \leq \lambda_1(AB) \leq \lambda_1(A)\lambda_1(B), \quad \text{if } \lambda_1(A) \geq 0$$

and

$$\lambda_t(A)\lambda_1(B) \leq \lambda_t(AB) \leq \lambda_t(A)\lambda_n(B), \quad \text{if } \lambda_t(A) \leq 0.$$ 

In particular,

$$\lambda_p(AB) \leq \lambda_p(A)\lambda_1(B) \quad \text{and} \quad \lambda_q(A)\lambda_1(B) \leq \lambda_q(AB).$$

The desired inequality follows immediately by subtraction. □

Corollary 2.8 provides an estimate for the gap between two eigenvalues of $AB$ near zero (on both sides) in terms of the eigenvalues of $A$ and $B$. The ratio $[\lambda_p(AB) - \lambda_q(AB)]/[\lambda_p(A) - \lambda_q(A)]$ is bounded above by the spectral norm of $B$. Thus, if neither $A$ nor $-A$ is stable Hermitian, then an application (multiplication) of a strictly contractive PSD matrix $B$ to $A$ narrows the gap between the positive and negative eigenvalues of $A$. (A matrix is said to be strictly contractive if its spectral norm is less than one.)

3. COMPARISON OF THE BOUNDS

Recall from the proof of Theorem 2.2 the splitting of the real diagonal matrix $A$: $A = A_+ + A_-$, where $A_+$ is the diagonal matrix with all positive (if any) eigenvalues of $A$ on the main diagonal (plus some zeros), and $A_-$ is the diagonal matrix with all negative (if any) eigenvalues of $A$ on the main diagonal (plus some zeros). It is natural to ask what inequalities would be derived if one applies (1.1), (1.2), and (1.3) to $AB = A_+B + A_-B$. We present the inequalities obtained by the splitting approach in the following proposition; we then show that these inequalities are in general weaker than the ones in Theorem 2.2. We show the case for upper bound.

Proposition 3.1. Let $A \in \mathbb{C}^{n \times n}$ be Hermitian and let $B \in \mathbb{C}^{n \times n}$ be positive semidefinite. For $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we have

$$\sum_{t=1}^{k} \lambda_{i_t}(AB) \leq \sum_{t=1}^{k} \lambda_{i_t}(A)\lambda_t(B) + \sum_{t=n_A+1}^{k} \lambda_t(A)\lambda_{n-k+t}(B).$$
Proof. Note that $A_+$ is positive semidefinite and all $\lambda_{i_1}(A), \lambda_{i_2}(A), \ldots, \lambda_{i_k}(A)$ are contained on the main diagonal of $A_+$. So, $\lambda_{i_t}(A_+) = \lambda_{i_t}(A)$ for $t = 1, 2, \ldots, k_A$, and $\lambda_{i_t}(A_+) = 0$ for $t > k_A$. Observe $\lambda_{i_t}(A - B) = 0$ for $t \leq n_A$. It follows that

$$\sum_{t=1}^{k} \lambda_{i_t}(AB) = \sum_{t=1}^{k} \lambda_{i_t}(A_+ B + A_- B) \leq \sum_{t=1}^{k} \lambda_{i_t}(A_+ B) + \sum_{t=1}^{k} \lambda_{i_t}(A_- B) \leq \sum_{t=1}^{k_A} \lambda_{i_t}(A_+) \lambda_{i_t}(B) + \sum_{t=1}^{k} \lambda_{i_t}(A_-) \lambda_{n-k+t}(B) = \sum_{t=1}^{k} \lambda_{i_t}(A) \lambda_{i_t}(B) + \sum_{t=n_A+1}^{k} \lambda_{i_t}(A) \lambda_{n-k+t}(B). \quad \square$$

In comparison, the first term on the right hand side of (3.1) is the same as that of (2.2), while the second terms are different. We denote the second term in (2.2) by $T_1$ and the second term in (3.1) by $T_2$. We show that $T_1 \leq T_2$ as follows.

$$T_1 = \sum_{t=k_A+1}^{k} \lambda_{i_t}(A) \lambda_{n-k+t}(B) = \sum_{t=k_A+1}^{n_A} \lambda_{i_t}(A) \lambda_{n-k+t}(B) + \sum_{t=n_A+1}^{k} \lambda_{i_t}(A) \lambda_{n-k+t}(B) \leq \sum_{t=n_A+1}^{k} \lambda_{i_t}(A) \lambda_{n-k+t}(B) \leq \sum_{t=n_A+1}^{k} \lambda_{i_t}(A) \lambda_{n-k+t}(B) = T_2.$$

Acknowledgement. The second author appreciates discussions with Roger Horn, Chi-Kwong Li, and Ren-cang Li during the preparation of the manuscript.

References

[1] A.R. Amir-Moez, Extreme Properties of Linear Transformations, Polygonal Publishing, New Jersey, 1990.
[2] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.
[3] R. Bhatia, Linear Algebra to Quantum Cohomology: The Story of Alfred Horn’s Inequalities, The American Mathematical Monthly Vol. 108, No. 4 (April, 2001) 289–318.
[4] F. Hiai and D. Petz, Introduction to Matriz Analysis and Applications, Springer, 2014.
[5] A. Horn, Eigenvalues of sums of Hermitian matrices, Pacific J. Math. 12 (1962) 225–241.
[6] R. Horn and C. Johnson, Matrix Analysis, Cambridge University Press, 2nd edition, 2013.
[7] A Knutson and T. Tao, The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture, J. Amer. Math. Soc. 12 (1999) 1055–1090.
[8] A. Knutson and T. Tao, Honeycombs and sums of Hermitian matrices, Notices Amer. Math. Soc., 48 (2001) 175–186.
[9] C.-K. Li and R. Mathias, The Lidskii-Mirsky-Wielandt theorem additive and multiplicative versions, Numer. Math. 81, No. 3 (1999) 377–413.
EIGENVALUE INEQUALITIES

[10] V.B. Lidski, The proper values of the sum and product of symmetric matrices (in Russian), Doklady Akad. Nauk SSSR 75 (1950) 769–772.

[11] A.W. Marshall, I. Olkin, and B.C. Arnold, Inequalities: Theory of Majorization and Its Application, 2nd edition, Springer, 2011.

[12] M.F. Smiley, Inequalities related to Lidskii, Proc. Amer. Math. Soc. 19 (1968) 1029–1034.

[13] R.C. Thompson and L.J. Freede, On the eigenvalues of sums of Hermitian matrices, Linear Algebra Appl. 4 (1971) 369–376.

[14] B.-Y. Wang and F. Zhang, Some inequalities for the Eigenvalues of the Product of Positive Semidefinite Hermitian Matrices, Linear Algebra Appl. 160 (1992) 113–118.

[15] H. Wielandt, An extremum property of sums of eigenvalues, Proc. Amer. Math. Soc. 6 (1955) 106–110.

[16] F. Zhang, Matrix Theory: Basic Results and Techniques, 2nd edition, Springer, 2011.

[17] F. Zhang and Q.-L. Zhang, Eigenvalue inequalities for matrix product, IEEE Trans. Automat. Contr., Vol. 51, No. 9 (August 2006) 1506–1509.

(B.-Y. Xi) College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China
E-mail address: baoyintu78@imun.edu.cn

(F. Zhang) Department of Mathematics, Nova Southeastern University, 33301 College Ave., Fort Lauderdale, FL 33314, USA
E-mail address: zhang@nova.edu