RATIONAL CHEREDNIK ALGEBRAS AND DIAGONAL COINVARIANTS OF $G(m,p,n)$

RICHARD VALE

ABSTRACT. We construct a quotient ring of the ring of diagonal coinvariants of the complex reflection group $W = G(m,p,n)$ and determine its graded character. This generalises a result of Gordon for Coxeter groups. The proof uses a study of category $\mathcal{O}$ for the rational Cherednik algebra of $W$.

1. Introduction

1.1. Let $h$ be a finite–dimensional complex vector space. An element $s \in \text{End}(h)$ is called a complex reflection if $\text{rank}_h(1-s) = 1$ and $s$ has finite order. A finite group generated by complex reflections is called a complex reflection group. If $W$ is a complex reflection group then the ring of invariants $\mathbb{C}[h]^W$ is a polynomial ring by the Shepherd-Todd theorem [Ben93, Theorem 7.2.1] and if $\mathbb{C}[h]^+_W$ denotes the elements with zero constant term then it is well-known that the ring of coinvariants

\[
\frac{\mathbb{C}[h]}{(\mathbb{C}[h]^+_W)}
\]

is a finite–dimensional algebra isomorphic to $\mathbb{C}W$ as a $W$–module. There is interest in analogues of this construction with the representation $h \oplus h^*$ in place of $h$, see for example [Hai03]. The ring

\[
D_W := \frac{\mathbb{C}[h \oplus h^*]}{(\mathbb{C}[h \oplus h^*]^+_W)}
\]

is called the ring of diagonal coinvariants of $W$. The ring $D_W$ has a natural grading with $\text{deg}(h^*) = 1$ and $\text{deg}(h) = -1$. The following result was conjectured by Haiman and proved in Gordon [Gor03]:

Theorem. [Gor03] Let $W$ be a finite Coxeter group of rank $n$ with Coxeter number $h$ and sign representation $\varepsilon$. Then there exists a $W$–stable quotient ring $R_W$ of $D_W$ with the properties:

(1) $\text{dim}(R_W) = (h + 1)^n$.

(2) $R_W$ is graded with Hilbert series $t^{hn/2}(1 + t + \cdots + t^h)^n$.

(3) The image of $\mathbb{C}[h]$ in $R_W$ is $\mathbb{C}[h]/(\mathbb{C}[h]^+_W)$.

(4) The character $\chi$ of the $W$–module $R_W \otimes \varepsilon$ satisfies $\chi(w) = (h + 1)^{\text{dim}\ker(1-w)}$ $\forall w \in W$.

1.2. In [Val], this result was generalised to the complex reflection groups $G(m,1,n)$. The aim of this paper is to obtain a further generalisation to the groups $G(m,p,n)$, with some mild restrictions on $m,p,n$. The following result will be proved:
Theorem. Let \( W = G(m, p, n) \) where \( m \neq p \) and let \( \mathfrak{h} \) be the reflection representation of \( W \). Let \( d = m/p \). Then there exists a \( W \)-stable quotient ring \( S_W \) of \( D_W \) with the properties:

1. \( \dim(S_W) = (m(n-1)+d+1)^n \).
2. \( S_W \) is graded with Hilbert series \( t^{-n-m(1)}(1 + t + \cdots + t^{m(n-1)+d})^n \).
3. The image of \( \mathbb{C}[\mathfrak{h}] \) in \( S_W \) is \( \mathbb{C}[\mathfrak{h}]/\langle \mathbb{C}[\mathfrak{h}] \rangle \).
4. The character \( \chi \) of \( S_W \otimes \wedge^{n} \mathfrak{h}^* \) as a \( W \)-module satisfies \( \chi(w) = (m(n-1)+d+1)^{\dim \ker(1-w)} \) \( \forall w \in W \).

1.3. Theorem [1.2] is proved by obtaining \( R_W \) as the associated graded module of a finite-dimensional module over the rational Cherednik algebra of \( W \). The properties of this module are derived by studying the category \( \mathcal{O} \) for the rational Cherednik algebra. This is also the method that will be used to prove Theorem [1.2].

1.4. The structure of the paper is as follows. In Section 3, the rational Cherednik algebra \( H_\kappa \) is introduced for \( W = G(m, p, n) \). This is a certain deformation of the skew group algebra \( \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W \) which depends on parameters \( \kappa \in \mathbb{C}^{m/p} \). Next, in Section 4 we recall some important properties of category \( \mathcal{O} \) for \( H_\kappa \), including the Knizhnik-Zamolodchikov functor \( \text{KZ} \), which we use to relate category \( \mathcal{O} \) to the category of modules over a Hecke algebra \( \mathcal{H} \). A parametrisation of the simple \( \mathcal{H} \)-modules given by Genet and Jacon [GJ] enables us to prove that, for suitable choices of the parameters, there is only one finite-dimensional simple object \( L(\text{triv}) \) in category \( \mathcal{O} \) (Theorem 6.2). We then define, in Section 7, a one-dimensional \( H_\kappa \)-module \( \Lambda^\psi \) for “shifted” values of the parameters \( \kappa' \), and, using a shift isomorphism due to Berest and Chalykh [BC], we construct the shifted module

\[ L := H_\kappa e_\kappa \otimes e_\kappa H_\kappa e_\kappa \Lambda^\psi \]

where \( e_\kappa \) is a certain idempotent in \( \mathbb{C} W \). The associated graded module \( \text{gr} L \) is, up to tensoring by a one-dimensional \( W \)-module, naturally a quotient of the ring of diagonal coinvariants. But \( L \) is also a finite-dimensional object of category \( \mathcal{O} \), and we are able to use our results on category \( \mathcal{O} \) to show that it is isomorphic to \( L(\text{triv}) \). By results of Chmutova and Etingof [CE03], \( L(\text{triv}) \) is well-understood, and this enables us to compute the Hilbert series and character of \( L \) and hence of \( \text{gr} L \), proving Theorem [1.2]. These calculations are given in Section 8.

1.5. The main difference between our proof and the proof of Theorem [1.2] is that for some of the Coxeter groups considered in [Gor03], the rational Cherednik algebra depends on only one parameter, and there is only one choice of parameter for which the proof will work. In the \( G(m, p, n) \) case with \( m^{neq p} \), there is greater freedom to choose the parameters, and hence it is possible to have a lot of control over the simple modules in category \( \mathcal{O} \), by considering what happens when the parameters are chosen generically. We note here that in the \( m = p \) case the rational Cherednik algebra usually depends on only one parameter, and hence the proof of Theorem [1.2] will not work for the groups \( G(m, m, n) \). However, it is likely that an analogue of Theorem [1.2] can be proved for these groups, following the arguments of [Gor03]. We hope to return to this in future work.

1.6. It may appear that, in the case \( m = 1 \), the Hilbert series of \( S_W \) should be \( t^{-n-m(1)}(1 + t + \cdots + t^m)^n \), which does not generalise Theorem [1.2] in type \( A \). However, in order to make Theorem [1.2] and Theorem [1.1] agree in this case, we should write the Hilbert series of \( S_W \) as
\[ t^{-\dim \mathfrak{h} - m(\dim \mathfrak{h})/2}(1 + t + \cdots + t^{m(n-1)+d})^{\dim \mathfrak{h}}. \]

Note that the type A case of Theorem 1.1 does not follow from Theorem 1.2 because we will assume throughout that \( m > 1 \).

1.7. Acknowledgements. The research described here will form part of the author’s PhD thesis at the University of Glasgow. The author thanks K. A. Brown and I. Gordon for suggesting this problem and for their advice and encouragement. The author also wishes to thank Y. Berest and O. Chalykh for allowing us to look at a preliminary version of their paper [BC], and O. Chalykh for explaining in detail to us the proof of Theorem 7.3.

2. The group \( G(m, p, n) \)

2.1. Let \( \mathfrak{h} \) be an \( n \)-dimensional complex vector space equipped with a sesquilinear form \( \langle -, - \rangle \). Let \( m \geq 1 \) and let \( p \) be a natural number such that \( p \mid m \). We fix the notation \( d = m/p \) and \( \varepsilon = e^{2\pi i m} \) throughout. The complex reflection group \( G(m, p, n) \) is defined to be the subgroup of \( GL(\mathfrak{h}) \) consisting of those matrices with exactly one nonzero entry in each row and column, such that the nonzero entries are powers of \( \varepsilon \) and such that the \( d \)th power of the product of the nonzero entries is 1.

2.2. Complex reflections. We wish to identify the complex reflections in \( G(m, p, n) \). It turns out that the set of complex reflections depends on \( (m, p, n) \), so we will make the following assumption

**Assumption.** From now on, assume \( m > p \). Equivalently, \( d > 1 \).

Under this assumption, the complex reflections in \( G(m, p, n) \) are as follows. If \( \{y_1, \ldots, y_n\} \) is an orthonormal basis of \( \mathfrak{h} \) then the set \( S \) of complex reflections in \( W \) consists of the elements \( s_{q^j}^i \) for \( 1 \leq i \leq n \) and \( 1 \leq q \leq d - 1 \), and \( \sigma_{ij}^{(\ell)} \) for \( 1 \leq i < j \leq n \) and \( 0 \leq \ell \leq m - 1 \), defined by:

\[
\begin{align*}
    s_{q^j}^i(y_i) &= \varepsilon^{q^j}y_i, & s_{q^j}^i(y_j) &= y_j, & j \neq i, \\
    \sigma_{ij}^{(\ell)}(y_i) &= \varepsilon^{-\ell}y_j, & \sigma_{ij}^{(\ell)}(y_j) &= \varepsilon^{\ell}y_i, & \sigma_{ij}^{(\ell)}(y_k) &= y_k, & k \neq i, j.
\end{align*}
\]

2.3. We now list the \( W \)-conjugacy classes in \( S \). For each \( q \), \( \{s_{q^j}^i|1 \leq i \leq n\} \) form a single conjugacy class in \( S \). If \( n \geq 3 \) or \( n = 2 \) and \( p \) is odd, then \( \{\sigma_{ij}^{(\ell)}|i < j, 0 \leq \ell \leq m - 1\} \) also form a single conjugacy class in \( S \). For convenience, we now make the following assumption.

**Assumption.** Either \( n \geq 3 \), or \( n = 2 \) and \( p \) is odd.

Under assumptions 2.2 and 2.3 we see that there are exactly \( d \) \( W \)-conjugacy classes of complex reflections. Furthermore, it follows from [BMR98, Section 3] that the defining representation \( \mathfrak{h} \) is irreducible when these assumptions hold. We are now in a position to construct the rational Cherednik algebra of \( W \).

**Remark.** Theorem 1.2 still holds in the case where \( n = 2 \) and \( p \) is even. To avoid clutter, the modifications necessary to prove this case are explained in Section 9.
3. The rational Cherednik algebra

In [DO03] and [GGOR03] we have the following definition. Let $A$ be the set of reflection hyperplanes of $W$ and for $H \in A$ let $W_H = \text{stab}_W(H)$, a cyclic group of order $e_H$. For $1 \leq i \leq e_H - 1$, let $\varepsilon_{H,i} = \frac{1}{e_H} \sum_{w \in W_H} \det(w)^i w$. For $H \in A$, let $\{k_{H,i}\}_{i=0}^{e_H}$ be a family of scalars such that $k_{H,i} = k_{H,i}$ whenever $H, H'$ are in the same $W$–orbit, and $k_{H,0} = k_{H,e_H} = 0$ for all $H$. For each $H \in A$, pick a linear form $\alpha_H \in \mathfrak{h}^*$ with kernel $H$, and choose $\alpha_H \in \mathfrak{h}$ such that $\mathbb{C} \alpha_H$ is a $W_H$–stable complement to $H$ and $\alpha_H(\alpha_H^\vee) = 2$.

The rational Cherednik algebra is defined to be the quotient of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$ with these parameters by

$$[y, x] = \langle y, x \rangle + \sum_{H \in A} \langle \alpha_H, y \rangle \langle \alpha_H^\vee, x \rangle \frac{e_H - 1}{2} \sum_{j=0}^{e_H-1} (k_{H,j+1} - k_{H,j}) \varepsilon_{H,j}$$

for all $y \in \mathfrak{h}$ and all $x \in \mathfrak{h}^*$, where $\langle -, - \rangle$ here denotes the evaluation pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$.

**Remark.** In this paper, the signs in the commutation relation have been chosen so that our parameters $k_{H,i}$ are the same as those of the paper [GGOR03]. However, we will also use results from the paper [GC], in which the parameters denoted $k_{H,i}$ are the negatives of those given here. See Section 7.3.

3.1. In the case $W = G(m, p, n)$, we may write out the commutation relation more explicitly. Let $H_i$ be the reflection hyperplane of $s_i^{pr}$ and $H_{ij\ell}$ be the reflection hyperplane of $\sigma_{ij}^{(\ell)}$. Let $\{y_1, \ldots, y_n\}$ be the standard basis of $\mathfrak{h}$ and $\{x_1, \ldots, x_n\}$ the dual basis of $\mathfrak{h}^*$. Then we may choose $\alpha_{H_i} = x_i$, $\alpha_{H_i}^\vee = 2y_i$, $\alpha_{H_{ij\ell}} = x_i - \varepsilon^\ell x_j$ and $\alpha_{H_{ij\ell}}^\vee = y_i - \varepsilon^\ell y_j$. We have $e_H = d$ and $\varepsilon_{H_{ij\ell}} = \frac{1}{d} \sum_{r=0}^{d-1} \varepsilon^{pr} s_i^{pr}$, and $e_{H_{ij\ell}} = 2$ and $e_{H_{ij\ell}} = \frac{1}{d} (1 + (-1)^j \sigma_{ij}^{(\ell)})$. The commutation relation (1) becomes

$$[y_a, x_b] = \delta_{ab} + \sum_{i=1}^{n} \delta_{ia} \delta_{ib} \sum_{j=0}^{d-1} (k_{j+1} - k_j) \sum_{r=0}^{d-1} \varepsilon^{pr} s_i^{pr}$$

$$+ \sum_{1 \leq i < j \leq n} \sum_{\ell=0}^{m-1} \frac{1}{2} (\delta_{ia} - \varepsilon^\ell \delta_{ja})(\delta_{ib} - \varepsilon^{-\ell} \delta_{jb}) 2k_{00} \sigma_{ij}^{(\ell)}$$

where $k_{H_{ij\ell}} = \kappa_{00}$ for all $i, j, \ell$ and $k_{H_{ij}} = \kappa_{j}$. We will denote the rational Cherednik algebra with these parameters by $H_\kappa$.

3.2. It was proved in [EG02] Theorem 1.3 that $H_\kappa$ satisfies a PBW–property, that is, it is isomorphic as a vector space to $\mathbb{C}[\mathfrak{h}] \otimes CW \otimes \mathbb{C}[\mathfrak{h}^*]$ via the multiplication map.

3.3. Let $H_\kappa(G(m, p, n))$ denote the rational Cherednik algebra of $G(m, p, n)$ with parameters $\kappa$. We will make considerable use of the fact that there is an embedding

$$H_\kappa(G(m, p, n)) \hookrightarrow H_\mu(G(m, 1, n))$$

for an appropriate choice of $\mu$. This observation is essentially due to Dunkl and Opdam [DO03].
Theorem. Given $\kappa = (\kappa_0, \kappa_1, \ldots, \kappa_{d-1})$, define $\mu = (\mu_0, \mu_1, \ldots, \mu_{d-1}, \mu_d, \ldots, \mu_{m-1})$ by $\mu_0 = \kappa_0$, $\mu_0 = 0$, $\mu_i = \kappa_i/p$, $1 \leq i \leq d - 1$, and $\mu_{s+t} = \mu_t$ for $1 \leq s \leq p - 1$ and $1 \leq t \leq d - 1$. Then $H_{\kappa}(G(m,p,n))$ is the subalgebra of $H_{\mu}(G(m,1,n))$ generated by $\mathfrak{h}$, $\mathfrak{h}^*$, $\sigma^{(\ell)}_i$ for all $i, j, \ell$, and $s^p_i$, $1 \leq i \leq n$.

Proof. Write $H_{\kappa} := H_{\kappa}(G(m,p,n))$ and $H_{\mu} := H_{\mu}(G(m,1,n))$. We need to check that the copies of $\mathfrak{h}$, $\mathfrak{h}^*$ in $H_{\mu}$ obey the commutation relations for $H_{\kappa}$. It is simply a question of substituting the $\mu$ values into (2) above, with $p = 1$. We obtain, in $H_{\mu}$,

$$[y_a, x_b] = \delta_{ab} + \sum_{i=1}^{n} \delta_{ia} \delta_{ib} \left[ \sum_{j=0}^{m-1} (\mu_{j+1} - \mu_j) \sum_{r=0}^{m-1} \varepsilon^r s^r_i \right]$$

$$+ \sum_{1 \leq i < j \leq n} \sum_{\ell=0}^{m-1} (\delta_{ia} - \varepsilon^\ell \delta_{ja})(\delta_{ib} - \varepsilon^{-\ell} \delta_{jb}) \mu_{00} \sigma^j_i$$

where $\varepsilon := e^{2\pi i}$. This may be rewritten as

$$[y_a, x_b] = \delta_{ab} + \sum_{1 \leq i < j \leq n} \sum_{\ell=0}^{m-1} (\delta_{ia} - \varepsilon^\ell \delta_{ja})(\delta_{ib} - \varepsilon^{-\ell} \delta_{jb}) \kappa_{00} \sigma^j_i + \sum_{i=1}^{n} \delta_{ia} \delta_{ib} \frac{1}{p} \sum_{q=0}^{m-1} x_q s^q_i$$

where

$$x_q = \sum_{j=0}^{d-1} \varepsilon^{qj}(\kappa_{j+1} - \kappa_j) + \sum_{j=d}^{2d-1} \varepsilon^{qj}(\kappa_{[j+1]} - \kappa_{[j]}) + \cdots + \sum_{j=(p-1)d}^{pd-1} \varepsilon^{qj}(\kappa_{[j+1]} - \kappa_{[j]})$$

where $[j]$ denotes the remainder modulo $d$. Write $q = ap + b$, $0 \leq a \leq d - 1$, $0 \leq b \leq p - 1$. Then

$$x_q = \sum_{j=0}^{d-1} \varepsilon^{qj}(\kappa_{j+1} - \kappa_j)$$

$$= \sum_{j=0}^{d-1} \sum_{r=0}^{p-1} \varepsilon^{q(j+rd)}(\kappa_{j+1} - \kappa_j)$$

$$= \sum_{j=0}^{d-1} \alpha_j(\kappa_{j+1} - \kappa_j)$$

where $\alpha_j = \varepsilon^{qj} \sum_{r=0}^{p-1} \varepsilon^{qrd} = \varepsilon^{qj} \sum_{r=0}^{p-1} (e^{2\pi i} r)^b$. So

$$\alpha_j = \begin{cases} 
\varepsilon^{aqj} & \text{if } q = ap, \\
0 & \text{if not.}
\end{cases}$$

And so

$$x_q = \sum_{j=0}^{d-1} \alpha_j(\kappa_{j+1} - \kappa_j) = \begin{cases} 
0 & \text{if } q \neq ap, \\
p \sum_{j=0}^{d-1} \varepsilon^{aqj}(\kappa_{j+1} - \kappa_j) & \text{if } q = ap.
\end{cases}$$
So in $H_\mu$ we have
\[
[y_a, x_b] = \delta_{ab} + \sum_{i=1}^{n} \delta_{ia} \delta_{ib} \sum_{q=0}^{m-1} \sum_{j=0}^{d-1} \varepsilon^{apj} (\kappa_{j+1} - \kappa_j) s_i^q + \sum_{1 \leq i < j \leq n} \sum_{\ell=0}^{m-1} (\delta_{ia} - \varepsilon^\ell \delta_{ja})(\delta_{ib} - \varepsilon^{-\ell} \delta_{jb}) \kappa_0 \sigma_{ij}^\ell.
\]
The first term on the right hand side is $\sum_{i=1}^{n} \delta_{ia} \delta_{ib} \sum_{q=0}^{m-1} \sum_{j=0}^{d-1} \varepsilon^{apj} (\kappa_{j+1} - \kappa_j) s_i^q$. So $y_a, x_b$ obey the commutation relation for $H_\kappa$.

To finish the proof of the theorem, we may define a map
\[
T(\mathfrak{h} \oplus \mathfrak{h}^*) * G(m, p, n) \rightarrow H_\mu
\]
in the obvious way. We have checked above that the commutation relations for $H_\kappa$ are in the kernel, and it is easily seen that the other relations for $H_\kappa$ are in the kernel as well. Thus, there is a well-defined map
\[
H_\kappa \rightarrow H_\mu.
\]
To check that this is injective, consider an element of $H_\kappa$ which is mapped to zero. Write it in terms of a PBW-basis of $H_\kappa$, and observe that all the coefficients must therefore be 0, since a PBW-basis of $H_\kappa$ is mapped into a subset of a PBW-basis of $H_\mu$.

3.4. **The Dunkl representation.** It is well-known (see for instance, [DO03, EG02 Proposition 4.5]) that $H_\kappa$ acts on $\mathbb{C}[\mathfrak{h}] \otimes \text{triv}$ where \text{triv} denotes the trivial representation of $W$. Furthermore, this action is faithful, and if $\mathbb{C}[\mathfrak{h}] \otimes \text{triv}$ is identified with $\mathbb{C}[\mathfrak{h}]$, then the action of $y \in \mathfrak{h}$ is given by a differential–difference operator called a Dunkl operator:
\[
T_y = \partial_y + \sum_{H \in A} \frac{\langle \alpha_H, y \rangle}{\alpha_H} \sum_{i=1}^{e_H-1} e^{H \delta_{H,i}} e_i H, i
\]
If $\mathfrak{h}^{\text{reg}} = \mathfrak{h} \setminus \cup_{H \in A} H$ then the Dunkl representation defines an injective homomorphism
\[
H_\kappa \hookrightarrow \mathcal{D}(\mathfrak{h}^{\text{reg}}) * W
\]
called the Dunkl representation. If $\delta = \prod_{H \in A} \alpha_H \in \mathbb{C}[\mathfrak{h}]$, then $\mathbb{C}[\mathfrak{h}^{\text{reg}}] = \mathbb{C}[\mathfrak{h}]_\delta$ and the induced map
\[
H_\kappa |_{\mathfrak{h}^{\text{reg}}} := H_\kappa \otimes \mathbb{C}[\mathfrak{h}]^{\text{reg}} \rightarrow \mathcal{D}(\mathfrak{h}^{\text{reg}}) * W
\]
is an isomorphism ([GGOR03, Theorem 5.6]).

3.5. Let $\theta_\mu : H_\mu \rightarrow \mathcal{D}(\mathfrak{h}^{\text{reg}}) * G(m, 1, n)$ be the Dunkl representation of $H_\mu$ and let $\theta_\kappa : H_\kappa \rightarrow \mathcal{D}(\mathfrak{h}^{\text{reg}}) * G(m, p, n)$ be the Dunkl representation of $H_\kappa$. Regarding $H_\kappa$ as a subalgebra of $H_\mu$, we wish to show that $\theta_\mu |_{H_\kappa} = \theta_\kappa$. For this, it suffices to check that $\theta_\mu(y) = \theta_\kappa(y)$ for all $y \in \mathfrak{h}$. But $\theta_\mu(y)$ and $\theta_\kappa(y)$ may be regarded as differential-difference operators acting on the polynomial ring $\mathbb{C}[\mathfrak{h}]$, so it suffices to check that their values on polynomials are the same. If $p \in \mathbb{C}[\mathfrak{h}]$ then $\theta_\mu(y)(p) = y \cdot p \otimes 1$ where we identify $p \in \mathbb{C}[\mathfrak{h}]$ with $p \otimes 1 \in \mathbb{C}[\mathfrak{h}] \otimes \text{triv}$. But $y \cdot p \otimes 1 = [y, p]_\mu \otimes 1$ where $[y, p]_\mu$ denotes the commutator in $H_\mu$. This may be written in terms of commutators $[y, x]_\mu$ for $x \in \mathfrak{h}^*$. But $[y, x]_\mu = [y, x]_\kappa$ for all $y \in \mathfrak{h}$, $x \in \mathfrak{h}^*$, where $[y, x]_\kappa$ denotes the commutator in $H_\kappa$. So $[y, p]_\mu \otimes 1 = [y, p]_\kappa \otimes 1 = \theta_\kappa(y)(p)$. So $\theta_\kappa(y) = \theta_\mu(y)$ as required. We have proved the following lemma.
Lemma. If $H_\kappa$ is the rational Cherednik algebra of $G(m,p,n)$, and we consider $H_\kappa$ as a subalgebra of $H_\mu$ as in Theorem 9.3, then the Dunkl representation $\mathbb{C}[h] \otimes \text{triv}$ of $H_\mu$ restricts to the Dunkl representation of $H_\kappa$.

4. Category $\mathcal{O}$

4.1. In this section, we will review the theory for a general complex reflection group $W$ and its rational Cherednik algebra $H_\kappa$ depending on some collection of complex parameters $\kappa = (k_{H,i})_{H \in A, 0 \leq i \leq \epsilon_H - 1}$.

4.2. Following [BEG03], let $\mathcal{O}$ be the abelian category of finitely-generated $H_\kappa$–modules $M$ such that for $P \in \mathbb{C}[h^*]^W$, the action of $P - P(0)$ is locally nilpotent. Let $\text{Irrep}(W)$ denote the set of isoclasses of simple $W$–modules. Given $\tau \in \text{Irrep}(W)$, define the standard module $M(\tau)$ by:

$$M(\tau) = H_\kappa \otimes_{\mathbb{C}[h^*]^W} \tau$$

where for $p \in \mathbb{C}[h^*], w \in W$ and $v \in \tau$, $pw \cdot v := p(0)wv$.

4.3. In [DO03], it is proved that $M(\tau)$ has a unique simple quotient $L(\tau)$, and [GGOR03] prove that $\{L(\tau) | \tau \in \text{Irrep}(W)\}$ is a complete set of nonisomorphic simple objects of $\mathcal{O}$, and that every object of $\mathcal{O}$ has finite length.

4.4. By [GGOR03], if $z := \sum_{H \in A} \sum_{i=1}^{\epsilon_{H,i}} e_H k_{H,i} x_{H,i} \in Z(\mathbb{C}W)$, and $\mathfrak{d} := \sum_i x_i y_i \in H_\kappa$, then $\mathfrak{e}_\kappa = \mathfrak{e}_u := \mathfrak{d} - z$ has the property that $[\mathfrak{e}_u, x] = x$ for all $x \in h^*$ and $[\mathfrak{e}_u, y] = -y$ for all $y \in h$ and $[\mathfrak{e}_u, w] = 0$ for all $w \in W$. The action of $\mathfrak{e}_u$ on $M(\tau)$ is diagonalisable and the eigenspaces are $\mathbb{C}[h]_d \otimes \tau$, $d \geq 0$, where $\mathbb{C}[h]_d$ denotes the homogeneous polynomials in $\mathbb{C}[h]$ of degree $d$. The eigenvalue of $\mathfrak{e}_u$ on $\mathbb{C}[h]_d \otimes \tau$ is $d - \theta(z)$, where $\theta(z)$ is the eigenvalue of $z$ on $\tau$. In particular, the lowest eigenvalue of $\mathfrak{e}_u$ on $M(\tau)$ is $-\theta(z)$.

4.5. A useful alternative definition of category $\mathcal{O}$ is quoted in [CE03 Section 2.1]. Category $\mathcal{O}$ may be defined as the category of $H_\kappa$–modules $V$ such that $V$ is a direct sum of generalised $\mathfrak{e}_\kappa$ eigenspaces, and such that the real part of the spectrum of $\mathfrak{e}_\kappa$ is bounded below. It is clear from this definition that every finite-dimensional $H_\kappa$–module belongs to $\mathcal{O}$.

4.6. The group $B_W := \pi_1(\mathfrak{h}^\text{reg}/W)$ is called the braid group of $W$. In [GGOR03], a functor $KZ : \mathcal{O} \rightarrow \mathbb{C}B_W - \text{mod}$ is constructed as follows: If $M \in \mathcal{O}$ then $M|_{\mathfrak{h}^\text{reg}} := \mathbb{C}[\mathfrak{h}^\text{reg}] \otimes_{\mathbb{C}[h]} M$ is a finitely-generated module over $\mathbb{C}[\mathfrak{h}^\text{reg}] \otimes_{\mathbb{C}[h]} H_\kappa \cong \mathcal{D}(\mathfrak{h}^\text{reg}) \ast W$. In particular, $M$ is a $W$–equivariant $\mathcal{D}$–module on $\mathfrak{h}^\text{reg}$ and hence corresponds to a $W$–equivariant vector bundle on $\mathfrak{h}^\text{reg}$ with a flat connection $\nabla$. The horizontal sections of $\nabla$ define a system of differential equations on $\mathfrak{h}^\text{reg}$ which, by a process described in [BMR98] and [Rom], give a monodromy representation of $\pi_1(\mathfrak{h}^\text{reg}/W)$. By definition, $KZ(M)$ is the monodromy representation of $\pi_1(\mathfrak{h}^\text{reg}/W)$ associated to $M$. 

7
4.7. By [BMR98, 4.12] and [GGOR03, Section 5.25], the monodromy representation factors through the Hecke algebra \( \mathcal{H} \) of \( W \). This is the quotient of \( \mathbb{C}B_W \) by the relations:

\[
(T - 1) \prod_{j=1}^{e_H-1} (T - \det(s)^{-j}e^{-2\pi i k_{H,j}})
\]

for \( H \in \mathcal{A}, s \in W \) the reflection around \( H \) with nontrivial eigenvalue \( e^{2\pi i / e_H} \), and \( T \) an \( s \)-generator of the monodromy around \( H \). The parameters differ from those given in [GGOR03] because the idempotent \( \varepsilon_j(H) \) of \( \text{[BMR98]} \) is the \( \varepsilon_{-j,H} \) of \( \text{[GGOR03]} \).

4.8. Therefore, KZ gives a functor \( \text{KZ} : \mathcal{O} \to \mathcal{H} - \text{mod} \). By [GGOR03, Section 5.3], KZ is exact, and if \( \mathcal{O}_{\text{tor}} \) is the full subcategory of those \( M \) in \( \mathcal{O} \) such that \( M_{|\text{reg}} = 0 \) then KZ gives an equivalence \( \mathcal{O}/\mathcal{O}_{\text{tor}} \cong \mathcal{H} - \text{mod} \text{[GGOR03]} \text{Theorem 5.14].} \)

5. The Hecke Algebra of \( G(m,p,n) \)

We now identify the algebra \( \mathcal{H} = \mathcal{H}(\kappa_{00}, \kappa_1, \ldots, \kappa_{d-1}) \) through which KZ factors, in the case of \( W = G(m,p,n) \). By [BMR98, Prop 4.22], this algebra is generated by \( (T_s)_{s \in \mathcal{N}(D)} \) where \( \mathcal{N}(D) \) is the set of nodes of the braid diagram \( D \) of \( W \). The generators \( T_s \) are subject to the braid relations defined by \( D \), together with the relations of [L7]. From the braid diagram in [BMR98 Table 1] we see that if \( p > 2 \) then \( \mathcal{H} \) is generated by \( T_s, T_{t_2}, T_{t_2'}, T_{t_3}, \ldots, T_{t_n} \) subject to the following relations:

\[
T_s T_{t_2} T_{t_2} - T_{t_2} T_{t_2} T_s = 0
\]

\[
T_{t_2} T_{t_3} T_{t_2'} - T_{t_2} T_{t_2'} T_{t_3} = 0
\]

\[
T_{t_2} T_{t_3} T_{t_2'} - T_{t_3} T_{t_2} T_{t_3} = 0
\]

\[
T_{t_3} T_{t_2} T_{t_2'} T_{t_2'} - T_{t_2} T_{t_2'} T_{t_3} T_{t_2'} = 0
\]

\[
T_{t_2} T_s(T_{t_2} T_{t_2'})_{p-1} - T_s(T_{t_2'} T_{t_2})_p = 0
\]

\[
[T_{t_i}, T_{t_j}] = 0 \quad i \geq 3
\]

\[
[T_{t_2}, T_{t_i}] = 0 \quad i \geq 4
\]

\[
[T_{t_2'}, T_{t_i}] = 0 \quad i \geq 4
\]

\[
[T_{t_i}, T_{t_j}] = 0 \quad i, j \geq 3, |i - j| \geq 2
\]

\[
T_{t_i} T_{t_{i+1}} T_{t_i} - T_{t_{i+1}} T_{t_i} T_{t_i} = 0 \quad i \geq 3
\]

\[
(T_s - 1) \prod_{j=1}^{d-1} (T_s - \varepsilon^{-pj}e^{2\pi i k_{s,j}}) = 0
\]

\[
(T_{t_i} - 1)(T_{t_i} + e^{2\pi i k_{00}}) = 0 \forall i
\]

\[
(T_{t_2'} - 1)(T_{t_2'} + e^{2\pi i k_{00}}) = 0
\]

where if \( x, y \) are generators then \( (xy)^r \) denotes the word \( (xy)^{r/2} \) if \( r \) is even or \( (xy)^{(r-1)/2}x \) if \( r \) is odd. If \( p = 2 \) then we see from [BMR98, Table 2] that \( \mathcal{H} \) is the algebra described above, except that the relation \( T_{t_3} T_{t_2} T_{t_2} T_{t_3} T_{t_2} T_{t_2} T_{t_3} = 0 \) is omitted.
We wish to verify that the algebra presented by these generators and relations is the same as the Hecke algebra of $G(m, p, n)$ as defined in [GJ] 2.A. We set $a_0 = T_1$, $a_1 = -T_{i_1}$, $a_2 = -T_{i_2}$ and $a_i = -T_{i_i}$ for $i \geq 3$. In the $p = 2$ case, we see that we have exactly the relations of [GJ] 2.A., except that [GJ] have the additional relation $(a_1a_2a_3)^2 = (a_3a_1a_2)^2$. However, this additional relation follows from the other relations, since $(T_s - 1) \prod_{j=1}^{d-1}(T_s - \epsilon^{-nj}e^{2\pi ikj}) = 0$ implies that $a_0 = T_s$ is invertible, and we can then check that $(a_1a_2a_3)^2a_0 = (a_3a_1a_2)^2a_0$, using the relations listed above.

If $p > 2$, we again have the same relations as [GJ] 2.A., except that the relation

$$a_0a_1a_2 = (q^{-1}a_1a_2)^2-pa_2a_0a_1 + (q-1) \sum_{k=1}^{p-2}(q^{-1}a_1a_2)^{1-k}a_0a_1$$

(3)

of [GJ] has been replaced by the relation

$$a_2a_0(a_1a_2)_{p-1} = a_0(a_1a_2)_p.$$

(4)

An explicit calculation shows that in the presence of the other relations, (3) and (4) are equivalent, where $q = e^{2\pi ik00}$. We therefore obtain the following lemma.

**Lemma.** The Hecke algebra $\mathcal{H}$ through which the functor $KZ$ factors is isomorphic to the Hecke algebra denoted $\mathcal{S}_{m,p,n}(\mathbb{C})$ in [GJ], with parameters $q = e^{2\pi ik00}$ and $x_1 = 1$, $x_j = \epsilon^{-p(j-1)}e^{-2\pi ik_{j-1}}$ for $j > 1$.

We will need some facts about the representation theory of this algebra. Specifically, we will need to use the parametrisation of the simple modules that is the main result of [GJ] Section 3).

5.1. As in [GJ] 2.B], we make the following definitions. For $1 \leq i \leq m$, write $i = sp+t$, with $0 \leq s \leq d-1$ and $1 \leq t \leq p$. Let $\eta_p = e^{\frac{2\pi i}{p^d}} = \epsilon^d$ and let $Q_i = \eta_p^{-1}y_{s+1}$ where $y_{s+1}$ is chosen so that $y_p^p = x_{s+1}$. In this way, we get a new sequence of complex numbers $Q := (Q_1, \ldots, Q_m)$. Now we follow [GJ] 2.C]. Let $\Pi^m_n$ denote the set of multipartitions $\lambda = (\lambda^{(i)})_{1 \leq i \leq m}$ of $n$ with $m$ parts. Then there is a permutation $\varpi$ that acts on $\Pi^m_n$ as follows. The permutation $\varpi$ may be expressed in cycle notation as

$$\varpi = (1, 2, \ldots, p)(p+1, p+2, \ldots, 2p) \cdots ((d-1)p+1, \ldots, dp).$$

The action of $\varpi$ on a multipartition $\lambda = (\lambda^{(i)})$ is defined by $\varpi(\lambda)^{(i)} = \lambda^{\varpi^{-1}(i)}$. Let $\mathcal{L}$ be a set of representatives of the orbits of this action of $\varpi$ on $\Pi^m_n$ and for $\lambda \in \Pi^m_n$, let $\alpha_{\lambda} = \min\{k \in \mathbb{N}_{> 0} | \varpi^k(\lambda) = \lambda\}$.

5.2. In Theorem 5.2 below, the set of Kleshchev multipartitions in $\Pi^m_n$ is defined with respect to the parameters $q$ and $Q = (Q_1, \ldots, Q_m)$. To define the set of Kleshchev multipartitions, we first need the definition of the residue of a node in a multipartition $(\lambda^{(1)}, \ldots, \lambda^{(m)})$. If $x$ is a node in column $j(x)$ and row $i(x)$ of $\lambda^{(k)}$, we define the residue $\text{res}(x) = Q_kq^{j(x)-i(x)}$. We say $y \notin \lambda$ is an addable $a$–node if $\lambda \cup \{y\}$ is a multipartition and $\text{res}(y) = a$. We say $y \in \lambda$ is a removable $a$–node if $\lambda \setminus \{y\}$ is a multipartition and $\text{res}(y) = a$. A node $x \in \lambda^{(i)}$ is said to be below a node $y \in \lambda^{(j)}$ if either $i > j$ or else $i = j$ and $x$ is in a lower row than $y$. A removable node $x$ is called a normal $a$–node if whenever $y$ is an addable $a$–node which is below $x$ then there are more removable $a$–nodes between $x$ and $y$ than there are addable $a$–nodes. A removable $a$–node is called good if it is the highest normal $a$–node of
\( \lambda \). The set of Kleshchev multipartitions is defined inductively by declaring that the empty multipartition is Kleshchev, and that a multipartition \( \lambda \) is Kleshchev if and only if there is a node \( y \in \lambda \) which is a good \( a \)-node, for some \( a \), such that \( \lambda \setminus \{ y \} \) is Kleshchev. A more detailed exposition, including examples, may be found in the introduction to the paper [AM00].

By the remark following [GJ, Theorem 3.1], we have the following theorem.

**Theorem.** [GJ] Suppose \( q \) is not a root of unity. Then the set of simple \( \mathcal{H}_{m,p,n}^{x,y} \)-modules is in bijection with the set \( \{ (\lambda,i) | \lambda \in \Lambda^0 \cap \mathcal{L} \text{ and } i \in [0, \frac{p}{q} - 1] \} \) where \( \Lambda^0 \) denotes the set of Kleshchev multipartitions in \( \Pi_n^m \), and \( \mathcal{L} \) is defined as above.

5.3. In this paper, we will be primarily interested in values of the parameters such that the equation

\[
dk_1 + m(n - 1)k_{00} = -1 - m(n - 1) - d
\]

holds. If this is the case, then

\[
q^p(n-1) = \varepsilon^{-p}e^{-2\pi ik_1} = x_2.
\]

We will use the parametrisation given by Theorem 5.2 to work out how many simple modules \( \mathcal{H} \) has for generic choices of \( k_{00}, k_1, \ldots, k_{d-1} \) satisfying (5). This will be used in the next section to give an upper bound on the number of finite-dimensional simple objects in category \( \mathcal{O} \).

### 6. Simple modules for the Hecke algebra

6.1. In this section, we have the standing assumptions that the parameters are \( q = e^{2\pi ik_{00}} \) and \( Q = (Q_1, \ldots, Q_m) \) where the \( Q_i \) are defined as above. In particular, \( Q_{sp+t} = \eta_{p-1}^{t-1}x_{s+1}^{t-1} \). But \( x_{s+1} = \varepsilon^{-p}\varepsilon^{-2\pi ik} \) so \( Q_{sp+t} = \eta_{p-1}^{t-1}\varepsilon^{-s}\varepsilon^{-2\pi ik} = \varepsilon^{d(t-1)-s}\varepsilon^{-2\pi ik}/p \). In particular, \( Q_1 = 1, Q_2 = \eta_p, \ldots, Q_p = \eta_{p-1}^{p-1} \) and \( Q_{p+1} = y_2 = q^{-1}, Q_{p+2} = \eta_p q^{-1}, \ldots, Q_{2p} = \eta_{p-1}^{p-1}q^{-1} \). We claim that, for these values of \( Q_i \) and \( q \), and for a generic choice of the \( k_{00} \) and \( k_i \), there are exactly \( p \) multipartitions in \( \Pi_n^m \) which are not Kleshchev and they can be described as follows.

Let \( \rho = \square \cdots \square \) a partition of \( n \), and for \( 1 \leq i \leq m \), define \( \rho_i \in \Pi_n^m \) to be the multipartition with \( \rho \) in the \( i^{\text{th}} \) place and \( \emptyset \) everywhere else. Then we have the following lemma.

**Lemma.** With the above choices of \( Q_i \) and \( q \), the non-Kleshchev multipartitions in \( \Pi_n^m \) are precisely the \( \rho_i \) where \( 1 \leq i \leq p \).

**Proof.** First, note that for a generic choice of the parameters, \( k_{00} \notin \mathbb{Q} \) and so we may assume that \( q \) is not a root of unity.

Let \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)}) \in \Pi_n^m \). Suppose that \( \lambda \neq \rho_i \) for \( 1 \leq i \leq p \). We must show that \( \lambda \) is Kleshchev. We will show that we can repeatedly remove good nodes from \( \lambda \) until we reach the empty partition. First, let \( i > 2p \). We will show that we can reduce to the case \( \lambda^{(i)} = \emptyset \). Recall that \( Q_i \) is of the form \( y_{a} \eta_{p}^{b} \) for some \( a \) and \( b \). Thus, the residue of a node in \( \lambda^{(i)} \) is of the form \( q^{c}y_{a} \eta_{p}^{b} \) for some \( a, b, c \). If this is equal to the residue of a node in some \( \lambda^{(j)} \) where \( j \neq i \), then we must have \( q^{c}y_{a} \eta_{p}^{b} = q^{c^*}y_{a'} \eta_{p}^{b'} \) for some \( a', b', c' \). So \( q^{c-c^*} = y_{a}y_{a}^{-1} \eta_{p}^{b-b'} \).
\[
\exp(2\pi i x (c - c')) + \frac{2\pi i a' - 1}{m} + \frac{2\pi i \kappa_a}{p} - \frac{2\pi i a - 1}{m} - \frac{2\pi i \kappa_a}{p} - \frac{2\pi i (d - b) - b')}{m} = 1.
\]

For a generic choice of the \(\kappa_i\) and \(\kappa_0\), this can only happen if \(c = c'\) and \(a = a'\). But then this forces \(i = j\). Hence, the only nodes in \(\lambda\) which can have the same residue as a node in \(\lambda^{(i)}\) are the other nodes in \(\lambda^{(i)}\). Let \(x\) be the rightmost node in the bottom row of \(\lambda^{(i)}\). Any node with the same residue as \(x\) has residue \(q^{j(x) - i(x)} Q_i\) and so must lie on the same diagonal as \(x\). But, by the choice of \(x\), there can be no addable or removable nodes on the same diagonal as \(x\). So \(x\) is a good node, and we may remove \(x\). Continuing inductively, the nodes of \(\lambda^{(i)}\) may be removed one at a time, and we conclude that the multipartition with \(\lambda^{(i)}\) replaced by \(\emptyset\) is Kleshchev. Note that this is not necessarily a multipartition of \(n\); it is a multipartition of the integer \(\sum_{j \neq i} |\lambda^{(j)}| \leq n\).

We are now reduced to checking that all the multipartitions of the form
\[
\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(p)}, \ldots, \lambda^{(2p)}, \emptyset, \emptyset, \ldots, \emptyset)
\]
with \(\sum_i |\lambda^{(i)}| \leq n\) and \(\lambda \neq \rho_i, 1 \leq i \leq p\), are Kleshchev. We now show that we can reduce to the case \(\lambda^{(2p)} = \lambda^{(2p-1)} = \ldots = \lambda^{(p+1)} = \emptyset\). First, consider \(\lambda^{(2p)}\). Suppose \(\lambda^{(2p)}\) has \(b\) rows and that the lowest row has length \(\lambda^{(i)}\). Let \(x\) be the rightmost node of the bottom row of \(\lambda^{(2p)}\). Then the residue of \(x\) is \(q^{a-b} \cdot q^{n-1} \eta_p^{-1} = r\). Then \(x\) is a normal \(r\)-node because \(\lambda^{(2p)}\) has no addable \(r\)-nodes which are below \(x\), since such a node would have to lie on the same diagonal as \(x\), which is impossible by choice of \(x\). And if \(i > 2p\) then \(\lambda^{(i)} = \emptyset\) has no addable \(r\)-node, since the only node that can be added to \(\lambda^{(i)}\) has residue \(Q_i = Q_{sp+t}\) for some \(s \geq 2\) and some \(t\). This equals \(e^{d(t-1)} e^{-2\pi i \kappa/s\rho} \) which cannot equal \(r\) for a generic choice of \(\kappa_s\). Hence, there are no addable \(r\)-nodes of \(\lambda\) which are below \(x\), so \(x\) is a normal \(r\)-node. We must show that \(\lambda\) contains no higher normal \(r\)-nodes. Certainly \(\lambda^{(2p)}\) contains no higher normal \(r\)-node, since any \(r\)-node must lie on the same diagonal as \(x\) and so cannot be removable. Any normal \(r\)-node not in \(\lambda^{(2p)}\) must lie in \(\lambda^{(p)}\), because no power of \(q\) can be equal to a power of \(\eta_p\) since \(q\) is not a root of unity. Suppose then that the node \(y \in \lambda^{(p)}\) has residue \(r\). Say \(y\) lies in row \(i(y)\) and column \(j(y)\) of \(\lambda^{(p)}\). Then \(q^{j(y) - i(y)} \eta_p^{-1} = q^{a-b+n-1} \eta_p^{-1}\). Hence, \(j(y) - i(y) = a-b+n-1\). Suppose \(\lambda^{(p)}\) has \(d\) columns. Then \(a-b+n-1 = j(y) - i(y) \leq d-1\). So \(n+1 \leq a \leq b+d\). But \(b+d \leq |\lambda^{(p)}| + |\lambda^{(p)}| \leq n\), a contradiction. Hence, no such \(y\) exists, and \(x\) is the highest normal \(r\)-node, and so is good. Removing \(x\) and continuing inductively, we may take \(\lambda^{(2p)} = \emptyset\). The same argument shows that we may take \(\lambda^{(p+i)} = \emptyset\), \(1 \leq i \leq p\), as claimed.

We are now reduced to showing that those \(\lambda\) of the form \(\lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)}, \emptyset, \ldots, \emptyset)\) with \(\lambda \neq \rho_i, 1 \leq i \leq p\), are Kleshchev. First, consider \(\lambda^{(p)}\). Let \(b\) be the number of columns of \(\lambda^{(p)}\). Then \(b < n\) since \(\lambda \neq \rho_p\). If \(x \in \lambda^{(p)}\) then \(\text{res}(x) = q^{j(x) - i(x)} \eta_p^{-1}\) where \(j(x) - i(x) \leq b - 1\). Let \(x\) be the rightmost node in the bottom row of \(\lambda^{(p)}\) and let \(r = \text{res}(x)\). Then \(x\) is the only removable \(r\)-node in \(\lambda\) since there can be no \(r\)-nodes in \(\lambda^{(i)}\) with \(i \neq p\), again because \(q\) is not a root of unity. Furthermore, the only way there can be an addable \(r\)-node below \(x\) is if such a node can be added to the empty diagram \(\lambda^{(2p)}\). Such a node would have residue \(q^{n-1} \eta_p^{-1}\) and so we would have to have \(j(x) - i(x) = n - 1 \leq b - 1\). This contradicts \(b < n\). Hence, \(x\) is a good node and may be removed. Continuing inductively, we may remove all
the nodes in $\lambda^{(p)}$. The same argument works for all the $\lambda^{(i)}$, $1 \leq i \leq p$, and so we can get to the empty partition from $\lambda$ by successively removing good nodes. Hence, $\lambda$ is Kleshchev.

To complete the proof, we observe that $\rho_i$ is not Kleshchev for $1 \leq i \leq p$, since the only removable node in $\rho_i$ is the node at the end of the row $\rho_i^{(i)}$. This node has residue $q^{n-1}\eta_{p-1}^{i-1}$. It is not normal as there is an addable node in $\rho_i$ which is the unique node that may be added to the empty diagram $\rho_i^{(p+i)}$, and this node also has residue $q^{n-1}\eta_{p-1}^{i-1}$. □

**Corollary.** For generic values of the parameters satisfying $d\kappa_1 + m(n-1)\kappa_{00} = -1 - m(n-1) - d$, the algebra $\mathcal{H}$ has $|\text{Irrep}(W)| - 1$ nonisomorphic simple modules.

**Proof.** By Theorem 5.2, the simples are indexed by pairs $(\lambda, i)$ such that $\lambda \in \Lambda^0 \cap \mathcal{L}$ and $0 \leq i \leq \frac{\lambda}{\bar{\lambda}} - 1$. Let $T$ be the set of all such pairs. Note that $\{\rho_1, \rho_2, \ldots, \rho_p\}$ is an orbit of $\omega$ on $\Pi^n_q$. Let $S$ be the set of all pairs $(\lambda, i)$ with $\lambda \in \mathcal{L}$ and $0 \leq i \leq \frac{\lambda}{\bar{\lambda}} - 1$. Choose $\rho_1$ to be the representative of the orbit of $\rho_1$ in $\mathcal{L}$. Then $(\rho_1, 0) \in S$ and $(\rho_1, 0) \notin T$. Furthermore, if $\lambda \in \mathcal{L}$ and $\lambda \neq \rho_1$ then $\lambda$ is Kleshchev by 6.1 and so $(\lambda, i) \in T$ for all $0 \leq i \leq \frac{\lambda}{\bar{\lambda}} - 1$. Hence, $T = S \setminus \{(\rho_1, 0)\}$ and so $|T| = |S| - 1$. It remains to show that $|S| = |\text{Irrep}(W)|$.

By [Ari95, Theorem 2.6], $|S|$ is the number of simple modules for the generic version $KH$ of the Hecke algebra $\mathcal{H}$, defined over the field $K = \mathbb{C}(q, x_1, \ldots, x_d)$ with $q$ and the $x_i$ being indeterminates. But by the proof of [BMR98, Theorem 4.24], $KH$ is isomorphic to the group algebra $KW$ and so $|S| = |\text{Irrep}(W)|$. □

6.2. Application to category $\mathcal{O}$. We may use Corollary 6.1 together with the functor $\text{KZ}$, to get some information about the category $\mathcal{O}$ of $H_{\mu}$-modules.

**Theorem.** Let $\kappa_i, \kappa_{00}$ be chosen generically so that $d\kappa_1 + m(n-1)\kappa_{00} = -1 - m(n-1) - d$. Then there is exactly one finite-dimensional simple module in category $\mathcal{O}$, namely $L(\text{triv})$.

**Proof.** First, recall from Section 4.5 that $\text{KZ}$ is an exact functor, and that every $\mathcal{H}$-module is the image of some object of $\mathcal{O}$ under $\text{KZ}$. Suppose $X$ is a simple $\mathcal{H}$-module. Then, using exactness of $\text{KZ}$, we can find a simple object $L(\tau) \in \mathcal{O}$ such that $\text{KZ}(L(\tau)) = X \neq 0$. Since $\text{KZ}(L(\tau)) \neq 0$, we get $L(\tau)|_{\text{reg}} \neq 0$ and hence $\dim(L(\tau)) = \infty$. Since each of the $|\text{Irrep}(W)| - 1$ simple $\mathcal{H}$-modules is then the image of some finite-dimensional $L(\tau)$, we conclude that at least $|\text{Irrep}(W)| - 1$ of the $L(\tau)$ are infinite-dimensional. (This argument is based on [BEG03, Lemma 3.11]).

It remains to show that $L(\text{triv})$ is finite-dimensional. It was shown in Lemma 3.5 that the Dunkl representation $\mathbb{C}[h] = M(\text{triv})$ of $H_\kappa$ is the restriction to $H_\kappa$ of the Dunkl representation of the larger algebra $H_\mu$, where $H_\mu$ is the rational Cherednik algebra for $G(m, 1, n)$ with parameters $\mu_{00} = \kappa_{00}$ and $\mu_i = \kappa_i/p$. It follows from the hypothesis on $\kappa$ that

$$m\mu_1 + m(n-1)\mu_{00} = -1 - m(n-1) - d.$$ 

In the terminology of [CE03], this says that $(\mu_{00}, \mu_1, \ldots, \mu_{m-1})$ belongs to the set $E_r$ where $r = m(n-1) + d + 1 = m(n-1) + q$ where $q = d + 1$ and $1 \leq q \leq m - 1$. We are now in a position to apply [CE03, Proposition 4.1]. Let $\mathfrak{h}_q$ be the representation of $G(m, 1, n)$ with $\mathfrak{h}_q = \mathbb{C}^n$ as a vector space, and on which $S_\sigma$ acts by permuting the coordinates, and $s_i$ acts by multiplying the $i$th coordinate by $e^{-d-1}$. Then [CE03, Proposition 4.1] states that the polynomial representation $M(\text{triv})$ of $H_\mu$ contains a copy of $\mathfrak{h}_q$ in degree $m(n-1) + d + 1$ consisting of singular vectors (vectors which are killed by $\mathfrak{h} \subset H_\mu$). Furthermore, if we set $Y_c$ to be the quotient of $M(\text{triv})$ by the ideal generated by this copy of $\mathfrak{h}_q$, then [CE03...
Theorem 4.3(i) states that \( \tilde{Y}_c \) is finite-dimensional provided \( \kappa_{00} \notin \mathbb{Q} \). Hence, there is an exact sequence of \( H_\mu \)-modules

\[ M(\text{triv}) \to \tilde{Y}_c \to 0. \]

Since \( H_\kappa \subset H_\mu \), these are also \( H_\kappa \)-module maps, and so the Dunkl representation \( M(\text{triv}) \) of \( H_\kappa \) has a finite-dimensional quotient. It follows that the unique smallest quotient \( L(\text{triv}) \) of the \( H_\kappa \)-module \( M(\text{triv}) \) is finite-dimensional, as required.

6.3. This ends our study of the Hecke algebra and category \( \mathcal{O} \). We now wish to apply Theorem 6.2 to study a special object in \( \mathcal{O} \) whose associated graded module will yield the desired quotient of the ring of diagonal coinvariants.

7. Shifting

7.1. In this section we will construct a one-dimensional \( H_\kappa \)-module \( \Lambda \) for particular values of \( \kappa \), and then construct a shifted version of \( \Lambda \) which will be a finite-dimensional \( H_\kappa \)-module \( L \). In the next section we will show how \( L \) is related to the diagonal coinvariants of \( W \).

7.2. A one-dimensional module.

**Theorem.** Suppose we choose the parameters \( \kappa_{00}, \kappa_i \) such that \( d\kappa_1 + m(n-1)\kappa_{00} = -1 \). Then \( H_\kappa \) has a one-dimensional module \( \Lambda \) on which \( \mathfrak{h} \) and \( \mathfrak{h}^* \) act by zero and \( W = G(m, p, n) \) acts by the trivial representation.

**Proof.** Let \( \Lambda \) be the trivial \( W \)-module. We can make \( \Lambda \) into a \( T(\mathfrak{h} \oplus \mathfrak{h}^*) \times W \)-module by making \( \mathfrak{h}, \mathfrak{h}^* \) act by 0. So we are reduced to showing that the defining relations of \( H_\kappa \) act by 0 on \( \Lambda \). The only relation that may cause difficulty is the commutation relation of section 3.1. We need to check that the commutation relation between \( y_a \) and \( x_b \) acts by 0 on \( \Lambda \) for all \( a, b \). Recall that

\[
[y_a, x_b] = \delta_{ab} + \sum_{i=1}^{n} \delta_{ia} \delta_{ib} \left[ \sum_{j=0}^{d-1} (\kappa_{j+1} - \kappa_j) \sum_{r=0}^{d-1} \varepsilon^{pr} j^r \right]
\]

\[+ \sum_{1 \leq i < j \leq n} \sum_{\ell=0}^{m-1} \frac{1}{2} (\delta_{ia} - \varepsilon^{\ell} \delta_{ja}) (\delta_{ib} - \varepsilon^{-\ell} \delta_{jib}) 2\kappa_{00} \sigma_{ij}^{\ell} \] (6)

First, if \( a \neq b \) then the right hand side of (6) becomes

\[
\sum_{1 \leq i < j \leq n} \sum_{\ell=0}^{m-1} (\delta_{ia} - \varepsilon^{\ell} \delta_{ja}) (\delta_{ib} - \varepsilon^{-\ell} \delta_{jib}) \kappa_{00} \sigma_{ij}^{\ell}
\]

which acts on \( \Lambda \) by the scalar

\[
\sum_{1 \leq i < j \leq n} \sum_{\ell=0}^{m-1} (\delta_{ia} - \varepsilon^{\ell} \delta_{ja}) (\delta_{ib} - \varepsilon^{-\ell} \delta_{jib}) \kappa_{00} = \kappa_{00} \sum_{1 \leq i < j \leq n} \sum_{\ell=0}^{m-1} (-\varepsilon^{\ell} \delta_{ja} \delta_{ib} - \varepsilon^{-\ell} \delta_{ia} \delta_{jb})
\]

which vanishes since \( \sum_{\ell=0}^{m-1} \varepsilon^{\ell} = 0 \).
Second, if \( a = b \) then we must show that
\[
1 + \left[ \sum_{j=0}^{d-1} (\kappa_{j+1} - \kappa_j) \sum_{r=0}^{d-1} \varepsilon^{prj} \right] + \sum_{1 \leq i < j \leq m} (\delta_{ij} - \varepsilon^t \delta_{ja}) (\delta_{ia} - \varepsilon^{-t} \delta_{ja}) \kappa_{00}
\]
vanishes. So we must show that \( 1 + d\kappa_1 + \kappa_{00}(m(n - a) + (a - 1)m) = 0 \), which holds by the hypothesis. \( \square \)

7.3. A shift isomorphism. We require the following theorem from \([BC]\).

**Theorem** (Berest, Chalykh). Let \((\kappa_0, \kappa_1, \ldots, \kappa_{d-1}) \in \mathbb{C}^d \) and define \( \kappa'_0 = \kappa_0 + 1, \kappa'_1 = \kappa_1 + 1 \) and \( \kappa'_i = \kappa_i \) for all other \( i \). Then there is an isomorphism
\[
\psi : eH_{\kappa'}e \rightarrow ezeH_{\kappa}e
\]
where \( e \) is the symmetrising idempotent \( e = \frac{1}{|W|} \sum_{w \in W} w \in CW \) and \( e_z = \frac{1}{|W|} \sum_{w \in W} \det(w)w \).

**Proof.** This is a minor modification of \([BC\) Theorem 5.8(2)]. \( \square \)

7.4. Given \( \kappa \) with \( d\kappa_1 + m(n - 1)\kappa_{00} = -1 - m(n - 1) - d \), define \( \kappa' \) as in the statement of theorem 7.3. Then \( d\kappa'_1 + m(n - 1)\kappa'_{00} = -1 \) and hence by theorem 7.2 there is a one-dimensional module \( \Lambda \) for \( H_{\kappa'} \) on which \( W \) acts trivially. Hence, \( e\Lambda = \Lambda \) becomes an \( eH_{\kappa'}e \)-module. Via the isomorphism of 7.3, we may define a \( ezeH_{\kappa}e \)-module \( \Lambda_{\psi} \). Finally, we set
\[
L = H_{\kappa}e\Lambda_{\psi} \otimes_{ezeH_{\kappa}e} \Lambda_{\psi},
\]
an \( H_{\kappa} \)-module.

Then, because \( H_{\kappa}e\Lambda_{\psi} \) is a finite \( ezeH_{\kappa}e \)-module (this follows by considering the associated graded modules, for instance), \( L \) is finite-dimensional, and it then follows from Section 4.5 that \( L \) belongs to the category \( O \) of \( H_{\kappa} \)-modules.

Suppose we have chosen \( \kappa \) generically. Then we may apply the results of the previous section. In particular, Theorem 6.2 says that the only finite-dimensional simple object in category \( O \) is \( L(\text{triv}) \). Since \( L \) has a composition series with composition factors \( L(\tau), \tau \in \text{irrep}(W) \), we see that every composition factor of \( L \) must be \( L(\text{triv}) \). There is a functor \( F : H_{\kappa} \rightarrow ezeH_{\kappa}e \) defined by \( FM = e\Lambda M \). This is an exact functor and it takes a composition series of \( L \) to a composition series of \( FL \cong \Lambda_{\psi} \). Hence, \( ezeL(\text{triv}) \neq 0 \) and \( L \cong L(\text{triv}) \). This proves the following lemma.

**Lemma.** If \( \kappa_1, \kappa_{00} \) are chosen generically then the \( H_{\kappa} \)-module \( L \) is isomorphic to \( L(\text{triv}) \).

8. A quotient ring of the diagonal coinvariants

8.1. We follow the proof of \([Gor03\) Section 5] to obtain the desired ring \( S_W \) of Theorem \([1.2]\). Choose generic \( \kappa' \) with \( d\kappa'_1 + m(n - 1)\kappa'_{00} = -1 \), let \( \kappa \) be defined as above, and define
\[
L = H_{\kappa}e\Lambda_{\psi} \otimes_{ezeH_{\kappa}e} \Lambda_{\psi}
\]
as above. Consider the filtration on \( H_{\kappa} \) with \( \deg(h) = \deg(h^*) = 1 \) and \( \deg(W) = 0 \), and the associated graded module \( \text{gr}L \). As in \([Gor03\), one obtains a surjection of \( \mathbb{C}[h \oplus h^*] \ast W \)-modules:
\[
\mathbb{C}[h \oplus h^*] \ast W e_{\psi} \otimes_{\mathbb{C}[h \oplus h^*]w} \text{gr}\Lambda_{\psi} \twoheadrightarrow \text{gr}L.
\]
By definition, \( \mathbb{C}[h \oplus h^*]_W \) acts on \( \text{gr}\Lambda_{\psi} \) by 0, and hence \( S_W := \text{gr}L \otimes \wedge^n h \) is a quotient ring of \( D_W = \mathbb{C}[h \oplus h^*]/(\mathbb{C}[h \oplus h^*]_W) \). We wish to determine the graded character of \( S_W \). To
do so, we will determine the graded character of $L \cong L(\text{triv})$. This requires a more delicate study of the results of [CE03] that were used above.

8.2. Recall from the proof of [DO03] that we have the $H_\kappa$–module $\tilde{Y}_c$ which is a finite-dimensional quotient of the Dunkl representation $M(\text{triv})$ obtained by factoring out an ideal $J$ of the polynomial ring $\mathbb{C}[\mathfrak{h}] = M(\text{triv})$, where $J$ is generated by a copy of $\mathfrak{h}_q$ in degree $m(n-1)+d+1$ consisting of singular vectors. Call this copy $U$. Note that $U$ is an irreducible representation of $G(m,p,n)$ since it follows from the definition of $\mathfrak{h}_q$ (quoted in the proof of [DO03] that $U \cong \mathfrak{h}^*$ as $G(m,p,n)$–modules. Now we are in a position to apply [CE03] Theorem 2.3 with $W = G(m,p,n)$. This says that we have a BGG-resolution

$$0 \leftarrow \tilde{Y}_c \leftarrow M(\text{triv}) \leftarrow M(U) \leftarrow \cdots \leftarrow M(\wedge^n U) \leftarrow 0.$$

Hence in the Grothendieck group $K_0(\mathcal{O})$, we have an equality

$$[\tilde{Y}_c] = \sum_{i=0}^{n} (-1)^i [M(\wedge^i \mathfrak{h}^*)].$$

Now, by theorem [DO03] there is only one finite-dimensional simple module in $\mathcal{O}$, namely $L(\text{triv})$. Hence, $[\tilde{Y}_c] = a \cdot [L(\text{triv})]$ for some $a \geq 1$. By [DO03] Section 2.5 (32), there is an ordering on $\text{Irrep}(W)$ such that the matrix with entries $[M(\tau) : L(\sigma)]$ is unipotent upper triangular. Therefore the classes $[M(\tau)]$ give a $\mathbb{Z}$–basis for $K_0(\mathcal{O})$ and there is a unique expression

$$[L(\text{triv})] = \sum_{\tau} c_\tau [M(\tau)]$$

with $c_\tau \in \mathbb{Z}$. Hence, $a \cdot c_{\text{triv}} = 1$ and so $a = 1$. It follows that $[\tilde{Y}_c] = [L(\text{triv})]$ and hence $\tilde{Y}_c \cong L(\text{triv})$. Furthermore,

$$[L(\text{triv})] = \sum_{i=0}^{n} (-1)^i [M(\wedge^i \mathfrak{h}^*)]$$

in $K_0(\mathcal{O})$.

8.3. To prove Theorem 1.2, we will require the following lemma. Recall from Section 4.4 that the element $z \in \mathbb{C}W \subset H_\kappa$ is defined by $z = \sum_{H \in \mathcal{A}} \sum_{i=1}^{e_{H-1}} e_{H,k} \varepsilon_{H,i} \varepsilon_{H,i}$. When $W = G(m,p,n)$ we have

$$z = \sum_{i=1}^{n} \sum_{i=1}^{d-1} \kappa_i \sum_{j=0}^{d-1} \varepsilon_{H,j} s_{i}^{p,j} + \kappa_{00} \sum_{i < j} \sum_{0 \leq r \leq m-1} (1 - \sigma_{ij}^{(r)}).$$

Lemma. $z$ acts on $\wedge^i \mathfrak{h}^*$ by the scalar $i(d\kappa_1 + m(n-1)\kappa_{00})$.

Proof. Since $z$ is central, it acts on $\wedge^i \mathfrak{h}^*$ by the scalar $\chi_{\wedge^i \mathfrak{h}^*}(z)/\binom{n}{i}$. Choosing a basis for $\mathfrak{h}^*$ for which $s_i^j$ is diagonal, we see that $\chi_{\wedge^i \mathfrak{h}^*}(s_i^j) = \varepsilon_j^{(n-1)} + \binom{n-1}{i}$. Similarly, $\chi_{\wedge^i \mathfrak{h}^*}(\sigma_{ij}^{(r)}) = \binom{n-1}{i} - \binom{n-1}{i+1}$. Substituting these values into the expression for $z$ gives the result. \qed
8.4. All of the properties of $S_W$ listed in Theorem 1.2 are immediate consequences of the following theorem. Recall that $D_W$ is graded with deg($\mathfrak{h}$) = −1 and deg($\mathfrak{h}^*$) = 1 and that this grading is $W$–stable. In general, if $M$ is a $\mathbb{Z}$–graded module and $\chi_k$ is the character of the $k^{th}$ graded piece then the graded character of $M$ is defined to be the formal power series $\sum_i \chi_i t_i$.

Theorem. The graded character of $\text{gr} L = S_W \otimes \wedge^n \mathfrak{h}^*$ is

$$w \mapsto t^{-n-m(n_2)} \frac{\det |_{\mathfrak{h}^*} (1 - t^{m(n-1)+d+1}w)}{\det |_{\mathfrak{h}^*} (1 - tw)}.$$  

Proof. Recall the element $e_\mathfrak{h} \in H_\mathfrak{h}$ from Section 4.3. By Lemma 8.3, $z$ acts by 0 on the trivial representation $\text{triv} = \wedge^0 \mathfrak{h}^*$ of $W$ and hence $e_\mathfrak{h}$ also acts by 0 on the trivial representation $\text{triv}$. Hence, the eigenvalue of $e_\mathfrak{h}$ on the subspace $\mathbb{C}[\mathfrak{h}]_d \otimes \text{triv} \subset M(\text{triv})$ is $d$.

By [CE03] Theorem 4.2, the representation $\tilde{Y}_c = L(\text{triv})$ of $H_\mathfrak{h}$ is isomorphic as a graded $G(m,1,n)$–module to $U_{m(n-1)+d+1}^{(2)}$ where $U_{m(n-1)+d+1} = \mathbb{C}[u] / (u^{m(n-1)+d+1})$, regarded as a representation of $\mathbb{Z}_m = (s_1)$ via $s_1(u) = z^{-1}u$, and where $S_n$ acts by permuting the factors of the tensor product. Since $\{u^i | 0 \leq i \leq m(n-1) + d\}$ is a basis of $U_{m(n-1)+d+1}$, we may define distinct basis elements $a_i$ by $a_i := u^{m(i-1)+1}$, $1 \leq i \leq n$. Then the element $v := \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(n)}$ affords the representation $\wedge^n \mathfrak{h}^*$ of $G(m,1,n)$, and lies in degree $n + \sum_{i=1}^n m(i - 1) = n + m(n_2)$. Hence, $e_\mathfrak{h}v = (n + m(n_2))v$. Note that $v$ also affords the representation $\wedge^n \mathfrak{h}^*$ of $G(m,p,n)$ when we consider $\tilde{Y}_c$ as a $W = G(m,p,n)$–module.

We define $\mathfrak{h} = e_\mathfrak{h} - n - m(n_2) \in H_\mathfrak{h}$. We now calculate the graded character of $L = L(\text{triv})$ with respect to the $\mathfrak{h}$–eigenspaces. We have shown above that in the Grothendieck group of $\mathcal{O}$,

$$[L(\text{triv})] = \sum_{i=0}^n (-1)^i [M(\wedge^i \mathfrak{h}^*)].$$

Now, by Lemma 8.3 $z$ acts by $-i(m(n-1) + d + 1)$ on $\wedge^i \mathfrak{h}^*$, and hence, by Section 4.4, the lowest eigenvalue of $\mathfrak{h}$ on $M(\wedge^i \mathfrak{h}^*)$ is $i(m(n-1) + d + 1) - n - m(n_2)$. Therefore, the graded character of $M(\wedge^i \mathfrak{h}^*)$ is

$$t^{-n-m(n_2)} \frac{\chi_{\wedge^i \mathfrak{h}^*}(w)t^{i(m(n-1)+d+1)}}{\det |_{\mathfrak{h}^*} (1 - tw)}.$$  

But $\det |_{\mathfrak{h}^*} (1 - t^{m(n-1)+d+1}w) = \sum_i (-1)^i \chi_{\wedge^i \mathfrak{h}^*}(w)t^{i(m(n-1)+d+1)}$ (this follows readily from diagonalising $w$) which gives the graded character of $L(\text{triv})$ with respect to the $\mathfrak{h}$–eigenspaces as

$$w \mapsto t^{-n-m(n_2)} \frac{\det |_{\mathfrak{h}^*} (1 - t^{m(n-1)+d+1}w)}{\det |_{\mathfrak{h}^*} (1 - tw)}.$$  

By the definition of the diagonal coinvariant ring $D_W$, there is a unique copy of the trivial representation in $D_W$, which lies in degree 0, and hence a unique copy of $\wedge^n \mathfrak{h}^*$ in $S_W \otimes \wedge^n \mathfrak{h}^*$, and hence a unique copy of $\wedge^n \mathfrak{h}^*$ in $L(\text{triv})$ (since $S_W \otimes \wedge^n \mathfrak{h}^* = \text{gr} \tilde{L}$, which is isomorphic to $L$ as a $W$–module). The unique copy of $\wedge^n \mathfrak{h}^*$ in $L(\text{triv})$ must be spanned by $v$. But $\mathfrak{h}v = 0$ and hence, by Lemma 7.4, $\mathfrak{h}$ must act by 0 on the element $e_\mathfrak{c} \otimes 1 \in L$ which affords the unique copy of $\wedge^n \mathfrak{h}^*$ in $L$. But because the grading induced by $\mathfrak{h}$ on $L$ gives $e_\mathfrak{c} \otimes 1$ degree 0.
and \( x \in \mathfrak{h}^* \) degree 1, and \( y \in \mathfrak{h} \) degree \(-1\), we see that \( \text{gr} L \) has the same graded character as \( L(\text{triv}) \), which proves the theorem. \( \square \)

8.5. **Proof of Theorem 1.2 (3).** This is similar to a proof in [Gor03, Section 5]. It is well-known (see for example [Kan94]) that the ring of coinvariants \( A = \mathbb{C}[\mathfrak{h}]/\mathbb{C}[\mathfrak{h}]_W^W \) satisfies Poincaré duality. Therefore the highest degree graded component of \( A \), which lies in degree \( \sum_i (d_i - 1) \) where the \( d_i \) are the degrees of the fundamental invariants of \( W \), is an ideal of \( A \) which is contained in every nonzero ideal. This ideal is called the socle of \( A \). The socle lies in degree \( m \). The image of \( \mathbb{C}[\mathfrak{h}] \) in \( S_W \) corresponds to the subspace \( \mathbb{C}[\mathfrak{h}] e_x \otimes \Lambda^\psi \) of \( L \). If \( p \in \mathbb{C}[\mathfrak{h}]_W^W e_x \) then by the definition of the shift isomorphism \( \psi \) given in [BC], we see that \( \psi(epe) = e_epe_x \). It follows that in \( L \) we have:

\[
p \otimes \Lambda^\psi = e_epe_x \otimes \Lambda^\psi = e_x \otimes e_epe_x \Lambda^\psi = e_x \otimes epe \cdot \Lambda = 0.
\]

Thus the ideal generated by \( \mathbb{C}[\mathfrak{h}]_W^W \) annihilates \( e_x \otimes \Lambda^\psi \). On the other hand, the quotient \( \mathbb{C}[\mathfrak{h}]/\mathbb{C}[\mathfrak{h}]_W^W \) contains a unique (up to scalar) element of maximal degree \( m(\frac{n}{2}) + nd - n \), say \( q \). The space \( \mathbb{C} q \) is the socle of \( \mathbb{C}[\mathfrak{h}]/\mathbb{C}[\mathfrak{h}]_W^W \). We claim \( q e_x \otimes \Lambda^\psi \neq 0 \). By the PBW theorem, any element of \( H_\kappa \) can be written as a sum of terms of the form \( p_\pm \otimes wp_\pm \) where \( p_\pm \in \mathbb{C}[\mathfrak{h}]^\kappa \), \( p_+ \in \mathbb{C}[\mathfrak{h}] \) and \( w \in W \). Since \( p_\pm \) and \( w \) do not increase degree, it would follow if \( q e_x \otimes \Lambda^\psi \) were zero, then \( L \) could have no subspace in degree \( m(\frac{n}{2}) + nd - n \). But the Hilbert series of \( L \) has highest order term \( t^{-n-m(\frac{n}{2})+mn(n-1)+nd} = t^{m(\frac{n}{2})+nd-n} \). Thus \( q e_x \otimes \Lambda^\psi \) is non–zero and \( \mathbb{C}[\mathfrak{h}] e_x \otimes \Lambda^\psi \) is isomorphic to \( (\mathbb{C}[\mathfrak{h}]/\mathbb{C}[\mathfrak{h}]_W^W) e_x \otimes \Lambda^\psi \). This proves Theorem 1.2 (3). \( \square \)

9. **Appendix: the case \( W = G(m, p, 2) \) with \( p \) even.**

In this case, we have \( d + 1 \) conjugacy classes of complex reflections in \( W \). In the notation of Section 2.2 they are \( C_q := \{ s_{i}^{q} | 1 \leq i \leq q \} \), (for \( 1 \leq q \leq d - 1 \)), \( C_{odd} := \{ \sigma_{12}^{(\ell)} | \ell \ odd \} \) and \( C_{even} := \{ \sigma_{12}^{(\ell)} | \ell \ even \} \). Thus, the rational Cherednik algebra depends on \( d + 1 \) complex parameters: \( \kappa_1, \kappa_2, \ldots, \kappa_{d-1} \) corresponding to \( C_1 \), \( \kappa_{odd} \) corresponding to \( C_{odd} \) and \( \kappa_{even} \) corresponding to \( C_{even} \). We choose these parameters so that \( \kappa_{odd} = \kappa_{even} =: \kappa_{00} \), so our parameters become \( (\kappa_{00}, \kappa_1, \ldots, \kappa_{d-1}) \). We must now verify that the rest of the proof of Theorem 1.2 still works.

Using [BMR98, Table 1, Table 2], we see that in the definition of the Hecke algebra \( \mathcal{H} \) of Section 5, we get the same braid relations and the same relation for \( T_x \) as in Section 5 for the \( p = 2 \) case, but the relations for the \( T_i \) are:

\[
(T_{i_2} - 1)(T_{i_2} + e^{2\pi i \kappa_{00}^{even}}) = 0
\]

\[
(T_{i_2} - 1)(T_{i_2} + e^{2\pi i \kappa_{odd}^{00}}) = 0
\]

Thus, when \( \kappa_{odd} = \kappa_{even} \), we get the Hecke algebra of [GJ, 2.A]. Furthermore, in [CE03 Section 4.1], it is assumed that \( \kappa_{odd} = \kappa_{even}^{00} \), so the constructions of [CE03] are still valid. So the proofs of Sections 5 and 5.2 go through in the present case. Hence, Theorem 1.2 holds.

The only potential obstacle to completing the proof is the shift isomorphism of Theorem 7.3. But, under the shift isomorphism \( e \mathcal{H}_{\kappa} e \rightarrow e e \mathcal{H}_{\kappa} e \) from [BC], the parameters \( \kappa_{00}^{odd} \) and \( \kappa_{00}^{even} \) are both shifted by 1, so we can regard \( \kappa_{00} \) as also being shifted by 1. The rest of the proof of Theorem 1.2 now goes through.
References

[AM00] S. Ariki and A. Mathas. The number of simple modules of the Hecke algebras of type $G(r, 1, n)$. Math. Z., 233(3):601–623, 2000.

[Ari95] Susumu Ariki. Representation theory of a Hecke algebra of $G(r, p, n)$. J. Algebra, 177(1):164–185, 1995.

[BC] Y. Berest and O. Chalykh. Quasi-invariants of complex reflection groups. In preparation.

[BEG03a] Y. Berest, P. Etingof, and V. Ginzburg. Cherednik algebras and differential operators on quasi-invariants. Duke Math. J., 118(2):279–337, 2003.

[BEG03b] Y. Berest, P. Etingof, and V. Ginzburg. Finite-dimensional representations of rational Cherednik algebras. Int. Math. Res. Not., (19):1053–1088, 2003.

[Ben93] D. J. Benson. Polynomial invariants of finite groups, volume 190 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1993.

[BMR98] M. Broué, G. Malle, and R. Rouquier. Complex reflection groups, braid groups, Hecke algebras. J. Reine Angew. Math., 500:127–190, 1998.

[CE03] T. Chmutova and P. Etingof. On some representations of the rational Cherednik algebra. Represent. Theory, 7:641–650 (electronic), 2003.

[DO03] C. F. Dunkl and E. M. Opdam. Dunkl operators for complex reflection groups. Proc. London Math. Soc. (3), 86(1):70–108, 2003.

[EG02] P. Etingof and V. Ginzburg. Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. Invent. Math., 147(2):243–348, 2002.

[GGOR03] V. Ginzburg, N. Guay, E. Opdam, and R. Rouquier. On the category $\mathcal{O}$ for rational Cherednik algebras. Invent. Math., 154(3):617–651, 2003.

[GJ] G. Genet and J. Jacon. Modular representations of cyclotomic Hecke algebras of type $G(r, p, n)$. arXiv:math.RT/0409297.

[Gor03] I. Gordon. On the quotient ring by diagonal invariants. Invent. Math., 153(3):503–518, 2003.

[Hai03] M. Haiman. Combinatorics, symmetric functions, and Hilbert schemes. In Current developments in mathematics, 2002, pages 39–111. Int. Press, Somerville, MA, 2003.

[Kan94] R. Kane. Poincaré duality and the ring of coinvariants. Canad. Math. Bull., 37(1):82–88, 1994.

[Rou] R. Rouquier. Representations of rational Cherednik algebras. arXiv:math.RT/0504600.

[Val] R. Vale. Diagonal coinvariants of $\mathbb{Z}_m \wr S_n$. arXiv:math.RT/0505416.

Department of Mathematics, University of Glasgow, Glasgow, G12 8QW, U.K.

E-mail address: rv@maths.gla.ac.uk