Linearized Controllability Analysis of Semilinear Partial Differential Equations

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Abstract: It is well-known that the controllability of finite-dimensional nonlinear systems can be established by showing the controllability of the linearized system. However, this classical result does not generalize to infinite-dimensional nonlinear systems. In this paper, we restrict ourselves to semilinear infinite-dimensional systems, and show that the exact controllability of the linearized system implies exact controllability of the nonlinear system. The restrictions concerning the nonlinear operator are similar to those that can be found in the literature about the linearized stability analysis of semilinear systems.

Keywords: nonlinear systems, infinite-dimensional systems, linearization, exact controllability

1. INTRODUCTION

Studying system-theoretic properties like controllability or observability for nonlinear infinite-dimensional systems is in general a very difficult task. Roughly speaking, systems theory for partial differential equations can be divided into formal (algebraic or geometric) methods like e.g. in Pommaret (1994) and Schöberl (2014) that are based on the structure of the equations, and functional-analytic methods that are rather based on the solutions. Whereas formal methods have been proven to be very successful for finite-dimensional nonlinear systems, in the infinite-dimensional case they suffer from the drawback that the function spaces for e.g. the state and the input cannot be properly specified. Thus, formal methods seem to be rather suited for proving negative results like non-controllability or non-observability, that can possibly be shown directly from the structure of the equations, see e.g. Kolar et al. (2018) or Kolar and Schöberl (2019). For proving positive results, in contrast, a functional-analytic approach seems to be indispensable.

For finite-dimensional nonlinear systems, it is well-known that stability, controllability, or observability can be established in a straightforward way by proving the corresponding property for the linearized system, see e.g. Nijmeijer and van der Schaft (1990) or Khalil (2002). Unfortunately, these classical results do not generalize to infinite-dimensional nonlinear systems. So far, the existing literature deals mainly with the linearized stability analysis of infinite-dimensional nonlinear systems, see e.g. Desch and Schappacher (1986), Smoller (1994), Kato (1995), Al Jamal et al. (2014), or Al Jamal and Morris (2018). Particularly, it is shown in Smoller (1994) that for semilinear systems with a nonlinear operator that is subject to certain restrictions the stability of the linearized system implies – like in the finite-dimensional case – the (local) stability of the original system. In the present paper, we pursue a similar approach with respect to the exact controllability problem for infinite-dimensional semilinear systems with distributed input. Exact controllability means that the controllability map of the system is surjective, and the basic idea of our approach is to apply the local surjectivity theorem to this controllability map in order to establish a connection between the exact controllability of the linearized system and the (local) exact controllability of the nonlinear system. Even though the controllability problem is quite different from the stability problem, we need conditions on the nonlinear operator of the semilinear system that are very similar to those in Smoller (1994). It should also be noted that in contrast to the stability analysis we are dealing here with non-autonomous systems, and, as mentioned in Schmid et al. (2019), there exist only very few papers on non-autonomous semilinear systems in the context of control theory.

2. PRELIMINARIES

Throughout the paper, we need the concept of a Fréchet derivative of maps between infinite-dimensional spaces.

Definition 1. (Fréchet Derivative) A map \( f : X \to Y \) from a Banach space \( X \) to a Banach space \( Y \) is Fréchet differentiable at \( x \in X \), if there exists a bounded linear operator \( Df(x) : X \to Y \) such that

\[
\lim_{\|h\|_X \to 0} \frac{\| f(x + h) - f(x) - Df(x)h \|_Y}{\| h \|_X} = 0. \tag{1}
\]

The map is Fréchet differentiable if it is Fréchet differentiable at every \( x \in X \), and it is continuously Fréchet differentiable if the Fréchet derivative \( Df(x) \) depends continuously on \( x \).

For bounded linear operators between Banach spaces, we use the usual operator norm

\[
\| T \| = \sup_{x \in D(T), x \neq 0} \frac{\| Tx \|_Y}{\| x \|_X}.
\]

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and denote it by $||\cdot||$ without a subscript.

Our approach to the linearized controllability analysis is based on the local surjectivity theorem, that can be found e.g. in Abraham et al. (1988).

**Theorem 2.** (Local Surjectivity Theorem) Let $X$ and $Y$ be Banach spaces and $V \subset X$ be open. If the map $f : V \subset X \to Y$ is continuously Fréchet differentiable and $Df(x_0)$ is surjective for some $x_0 \in V$, then $f$ is locally surjective. That is, there exist open neighborhoods $V_1$ of $x_0$ and $W_1$ of $f(x_0)$ such that $f|_{V_1} : V_1 \to W_1$ is surjective.

We also make frequent use of Gronwall’s lemma in the form presented in Zeidler (1986).

**Lemma 3.** (Gronwall) Let $f,g : [t_0, \tau] \to \mathbb{R}$ be continuous functions, with $g$ nondecreasing, and which, for fixed $K > 0$, satisfy the inequality

$$f(t) \leq g(t) + K \int_{t_0}^{t} f(s) \, ds, \quad \forall t \in [t_0, \tau].$$

Then

$$f(t) \leq g(t)e^{K(t-t_0)}, \quad \forall t \in [t_0, \tau].$$

3. SEMILINEAR SYSTEMS

We consider semilinear systems of the form

$$\dot{x}(t) = Ax(t) + f(x(t)) + Bu(t), \quad x(t_0) = x_0, \quad (2)$$

where $A$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on a Hilbert space $X$, the linear operator $B : U \to X$ is bounded, and the nonlinear map $f : X \to X$ is continuously Fréchet differentiable. Furthermore, we assume that $f(0) = 0$ and $Df(0) = 0$.

With these assumptions,

$$(x_s, u_s) = (0, 0)$$

is an equilibrium of the system (2), and the linearization about this equilibrium is given by

$$\Delta \dot{x}(t) = A\Delta x(t) + B\Delta u(t), \quad \Delta x(t_0) = \Delta x_0, \quad (3)$$

which is just the linear part of the system.

**Remark 4.** It should be noted that the linearization of the system is based on the Gâteaux derivative, see e.g. Al Jamal et al. (2014) or Al Jamal and Morris (2018). Since the operator $A$ is typically unbounded, the right-hand side of (2) does not possess a Fréchet derivative with respect to $x$.

As discussed in Pazy (1983), classical solutions of the semilinear system (2) satisfy the integral equation

$$x(t) = T(t)x(t_0) + \int_{t_0}^{t} T(t-s)(f(x(s)) + Bu(s)) \, ds. \quad (4)$$

Therefore, continuous solutions of the integral equation (4) are called mild solutions of (2), see also Curtain and Zwart (1995) for the linear case. For the controllability problem with $t_0 = 0$ and initial condition $x(0) = 0$, we have

$$x(t) = \int_{0}^{t} T(t-s)(f(x(s)) + Bu(s)) \, ds. \quad (5)$$

Throughout the paper, we assume that for some $\tau > 0$ and all inputs $u(t)$ in an open neighborhood

$$U \subset L_2([0, \tau]; U)$$

of $u(t) = 0$ the semilinear system (2) with $x(0) = 0$ has a unique mild solution on the interval $[0, \tau]$, i.e., a unique continuous solution of (5).

**Remark 5.** Proving the existence and uniqueness of solutions would require a (repeated) application of Banach’s fixed-point theorem.

In the following, we denote the solution for an input $u(t) \in U$ by

$$x(t) = S_t(u), \quad t \in [0, \tau].$$

The map

$$S_t(u) : U \to X \quad (6)$$

with $t = \tau$ is the controllability map of the semilinear system on $[0, \tau]$, and we call the system locally exactly controllable on $[0, \tau]$ if this map is locally surjective. In other words, for every final state $x(\tau) \in X$ in an open neighborhood of the origin there must exist an input trajectory $u(t)$ on the time interval $[0, \tau]$ such that

$$x(\tau) = S_t(u).$$

The basic idea is now to use the local surjectivity theorem in order to establish a connection between the local exact controllability of the semilinear system (2) and the exact controllability of the linearized system (3).

**Theorem 6.** Assume that the controllability map (6) of the semilinear system (2) satisfies the following conditions:

(A1) $S_t(u)$ is continuously Fréchet differentiable with respect to $u$ in an open neighborhood of $u = 0$.

(A2) The Fréchet derivative $DS_t(0)$ at $u = 0$ coincides with the controllability map

$$\int_{0}^{\tau} T(\tau-s)B\Delta u(s) \, ds : L_2([0, \tau]; U) \to X$$

of the linearized system (3) on $[0, \tau]$.

Then exact controllability of the linearized system (3) on $[0, \tau]$ implies local exact controllability of the semilinear system (2) on $[0, \tau]$.

**Proof.** The condition (A1) is a prerequisite for the application of the local surjectivity theorem. With condition (A2) and the local surjectivity theorem, surjectivity of the controllability map of the linearized system implies local surjectivity of the controllability map of the semilinear system. Consequently, exact controllability of the linearized system implies local exact controllability of the semilinear system.

The practical use of this theorem is of course limited, since it would require knowledge of the controllability map (6) of the semilinear system. Thus, in the remainder of the paper, we translate the conditions (A1) and (A2) of Theorem 6 into (sufficient) conditions on the nonlinear term $f$ of the system (2). The main difficulty consists in proving the continuous Fréchet differentiability of $S_t(u)$.

4. FRÉCHET DIFFERENTIABILITY OF THE CONTROLLABILITY MAP

In this section, we show that the conditions (A1) and (A2) of Theorem 6 are satisfied for all semilinear systems (2) where the nonlinear operator $f$ satisfies the following additional assumptions.
(B1) For every bounded set \( B \subset X \), there exist positive constants \( \alpha, \gamma \in \mathbb{R} \) such that
\[
\| f(x_1) - f(x_2) - Df(x_2)(x_1 - x_2) \|_X \leq \alpha \| x_1 - x_2 \|_X^{1+\gamma}
\]
for all \( x_1, x_2 \in B \).

(B2) The Fréchet derivative \( Df \) is locally Lipschitz continuous.

The assumption (B1) with \( \gamma = 1 \) is also used in Smoller (1994) for the linearized stability analysis of semilinear autonomous systems.

We proceed in several steps. First, we show that for all \( t \in [0, \tau] \) the map \( S_t(u) : U \to X \) is locally Lipschitz continuous (Lemma 7). Based on this result, we prove that \( S_t(u) \) is also Fréchet differentiable with respect to \( u \), and that the Fréchet derivative coincides with the solution operator (controllability map) of the linearized system (Theorem 8). Finally, we prove that \( S_t(u) \) is even continuously Fréchet differentiable (Theorem 9).

**Lemma 7.** There exists a constant \( c \) such that
\[
\| S_t(u_1) - S_t(u_2) \|_X \leq c \| u_1 - u_2 \|_{L_2}
\]
for all \( t \in [0, \tau] \) and \( u_1, u_2 \in U \).

**Proof.** Let \( x_1(t) = S_t(u_1) \) and \( x_2(t) = S_t(u_2) \) denote the solutions for two input trajectories \( u_1, u_2 \in U \). These solutions satisfy the integral equations
\[
x_1(t) = \int_0^t T(t-s)(f(x_1(s)) + Bu_1(s)) \, ds
\]
and
\[
x_2(t) = \int_0^t T(t-s)(f(x_2(s)) + Bu_2(s)) \, ds.
\]

For the norm of the difference \( x_1(t) - x_2(t) \), we get the estimate
\[
\| x_1(t) - x_2(t) \|_X \leq \int_0^t \| T(t-s)(f(x_1(s)) - f(x_2(s))) \|_X \, ds
\]
\[
+ \int_0^t \| T(t-s)(Bu_1(s) - Bu_2(s)) \|_X \, ds
\]
\[
\leq t \int_0^t \| T(t-s) \| \| f(x_1(s)) - f(x_2(s)) \|_X \, ds
\]
\[
+ k \| u_1 - u_2 \|_{L_2}
\]
\[
\leq ML \int_0^t \| x_1(s) - x_2(s) \|_X \, ds
\]
\[
+ k \| u_1 - u_2 \|_{L_2}
\]  
(7)

with \( M = \sup_{t \in [0, \tau]} \| T(t) \| \) and some \( k > 0 \). Since the continuous Fréchet differentiability of \( f \) guarantees only local Lipschitz continuity, the Lipschitz constant \( L \) depends of course on
\[
\sup_{u \in U, t \in [0, \tau]} \| S_t(u) \|_X
\]
i.e., on the maximal “size” of the solutions with input trajectories \( u \in U \). We have also used the fact that the term
\[
\int_0^t T(t-s)Bu(s) \, ds
\]
is just the controllability map of the linear part of the system and therefore bounded with some constant \( k \), see e.g. Curtain and Zwart (1995). Applying Gronwall’s lemma to (7) yields
\[
\| x_1(t) - x_2(t) \|_X \leq k \| u_1 - u_2 \|_{L_2} e^{MLt}
\]
and finally
\[
\| x_1(t) - x_2(t) \|_X \leq ke^{MLt} \| u_1 - u_2 \|_{L_2}, \quad \forall t \in [0, \tau],
\]
which completes the proof. \( \square \)

With help of Lemma 7, we can now show that the solutions \( x(t) = S_t(u) \) are Fréchet differentiable with respect to \( u \). The proof is divided in fact into two parts. First, we show that if the Fréchet derivative exists it coincides with the solution operator of the linearized system. Subsequently, we prove that the solution operator of the linearized system satisfies the condition (1) for a Fréchet derivative.

**Theorem 8.** For all \( t \in [0, \tau] \), the map
\[
S_t(u) : L_2([0, \tau]; U) \to X
\]
is Fréchet differentiable with respect to \( u \) at every \( \bar{u} \in U \). The Fréchet derivative \( DS_t(\bar{u}) \) coincides with the solution operator (controllability map)
\[
L_t \Delta u : L_2([0, \tau]; U) \to X
\]
of the linearized, time-variant system
\[
\Delta x(t) = (A + Df(S_t(\bar{u}))) \Delta x(t) + B \Delta u(t) \quad (8)
\]
with \( \Delta x(0) = 0 \).

**Proof.** If \( S_t(u) \) is Fréchet differentiable at \( \bar{u} \in U \) for all \( t \in [0, \tau] \), then we can differentiate the integral equation (5) and observe that the Fréchet derivative \( DS_t(\bar{u}) \) must satisfy the integral equation
\[
DS_t(\bar{u}) = \int_0^t T(t-s)Df(S_s(\bar{u}))DS_s(\bar{u}) \, ds
\]
\[
+ \int_0^t T(t-s)Bu(s) \, ds.
\]  
(9)

The solution of the linearized system (8) meets
\[
L_t \Delta u = \int_0^t T(t-s)Df(S_s(\bar{u}))L_s \Delta u \, ds
\]
\[
+ \int_0^t T(t-s)B \Delta u(s) \, ds,
\]  
(9)

and because of \( D(L_t \Delta u) = L_t \) the solution operator \( L_t \) satisfies exactly the same integral equation

1. The system (8) is the linearization of (2) along the trajectory \( x(t) = S_t(\bar{u}) \). For \( \bar{u} = 0 \) we have \( S_t(\bar{u}) = 0 \), and (8) becomes the linear, time-invariant system (3). Thus, the condition (A2) of Theorem 6 is contained as a special case.

2. It should be noted that the integrals in this equation are in general not Lebesgue integrals but Pettis integrals, see also Curtain and Zwart (1995).
For the norm of $\sigma$ into (12) yields with some rest $r$ according to the integral equations (5) and (9) yields for a Fréchet derivative of $S_t(u)$ at $\bar{u}$, cf. (1). For this purpose, we introduce the abbreviation

$$\sigma(t) = S_t(\bar{u} + \Delta u) - S_t(\bar{u}) - L_t\Delta u$$

for the sum in the numerator of (11). Substituting

$$S_t(\bar{u} + \Delta u) = \int_0^t T(t-s)f(S_s(\bar{u} + \Delta u))ds$$

$$S_t(\bar{u}) = \int_0^t T(t-s)f(S_s(\bar{u}))ds$$

$$L_t\Delta u = \int_0^t T(t-s)Df(S_s(\bar{u}))L_s\Delta u ds$$

according to the integral equations (5) and (9) yields

$$\sigma(t) = \int_0^t T(t-s)(f(S_s(\bar{u} + \Delta u)) - f(S_s(\bar{u}))) - Df(S_s(\bar{u}))L_s\Delta u) ds.$$  

(12)

Now we write $f(S_s(\bar{u} + \Delta u))$ as

$$f(S_s(\bar{u} + \Delta u)) = f(S_s(\bar{u})) + Df(S_s(\bar{u}))(S_s(\bar{u} + \Delta u) - S_s(\bar{u})) + r(s)$$

with some rest $r(s)$, and substituting

$$f(S_s(\bar{u} + \Delta u)) - f(S_s(\bar{u})) = Df(S_s(\bar{u}))(S_s(\bar{u} + \Delta u) - S_s(\bar{u})) + r(s)$$

into (12) yields

$$\sigma(t) = \int_0^t T(t-s)Df(S_s(\bar{u}))\sigma(s)ds + \int_0^t T(t-s)r(s)ds.$$  

For the norm of $\sigma(t)$ we get the estimate

$$\|\sigma(t)\|_X \leq \int_0^t \|T(t-s)\| \|Df(S_s(\bar{u}))\| \|\sigma(s)\|_X ds +$$

$$\int_0^t \|T(t-s)\| \|r(s)\|_X ds$$

$\leq M\bar{c} \int_0^t \|\sigma(s)\|_X ds + M \int_0^t \|r(s)\|_X ds$

with

$$\bar{c} = \sup_{t \in [0,\tau]} \|Df(S_t(\bar{u}))\|,$$

and applying Gronwall's lemma yields

$$\|\sigma(t)\|_X \leq \left(M \int_0^t \|r(s)\|_X ds\right) e^{M\bar{c}t}.$$  

(14)

Now we make use of the additional assumption (B1) to obtain an inequality for $\|r(s)\|_X$. Applying (B1) to the right-hand side of

$$r(s) = f(S_s(\bar{u} + \Delta u)) - f(S_s(\bar{u})) - Df(S_s(\bar{u}))(S_s(\bar{u} + \Delta u) - S_s(\bar{u}))$$

yields

$$\|\sigma(s)\|_X \leq \alpha \|S_s(\bar{u} + \Delta u) - S_s(\bar{u})\|_{1+\gamma}^X,$$

and with Lemma 7 we get

$$\|\sigma(s)\|_X \leq \alpha \|\Delta u\|_{L^2}^{1+\gamma}, \quad \forall s \in [0,\tau].$$

Substituting this inequality into (14) results in

$$\|\sigma(t)\|_X \leq M\alpha(1+\gamma)e^{M\bar{c}t} \|\Delta u\|_{L^2}^{1+\gamma}, \quad t \in [0,\tau],$$

and because of

$$\lim_{\|\Delta u\|_{L^2} \to 0} \frac{\|\sigma(t)\|_X}{\|\Delta u\|_{L^2}} = \lim_{\|\Delta u\|_{L^2} \to 0} \frac{M\alpha(1+\gamma)e^{M\bar{c}t} \|\Delta u\|_{L^2}^{1+\gamma}}{\|\Delta u\|_{L^2}} = 0$$

the condition (11) for a Fréchet derivative is indeed satisfied.

However, for the application of the local surjectivity theorem, it is not enough to prove the Fréchet differentiability. We have to show that $S_t(u)$ is continuously Fréchet differentiable, i.e., that the Fréchet derivative $DS_t(u)$ depends continuously on the point $u \in U$.

Theorem 9. For all $t \in [0,\tau]$, the map

$$S_t(u) : L_2([0,\tau]; U) \to X$$

is continuously Fréchet differentiable on $U$.

Proof. We have to show that the map $u \to DS_t(u)$ is continuous. If $DS_t(u_1)$ and $DS_t(u_2)$ are the Fréchet derivatives of $S_t(u)$ at $u_1, u_2 \in U$, then they satisfy the integral equations

$$DS_t(u_1) = \int_0^t T(t-s)(Df(S_s(u_1)))DS_s(u_1) + B) ds$$

and

$$DS_t(u_2) = \int_0^t T(t-s)(Df(S_s(u_2)))DS_s(u_2) + B) ds.$$  

The difference of these equations can be written as

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3 Provided that the integral equations have a unique solution, but it is well-known that the Fréchet derivative is unique if it exists.
\[ DS_t(u_1) - DS_t(u_2) = \int_0^t T(t-s) (Df(S_s(u_1))DS_s(u_1) - Df(S_s(u_2))DS_s(u_2)) ds \]
\[ = \int_0^t T(t-s) (Df(S_s(u_1))) (DS_s(u_1) - DS_s(u_2)) + (Df(S_s(u_1)) - Df(S_s(u_2))) DS_s(u_2) ds. \]

For the norm of the difference, we get the estimate
\[ \| DS_t(u_1) - DS_t(u_2) \| \leq \int_0^t \| T(t-s) \| \| Df(S_s(u_1)) \| \| DS_s(u_1) - DS_s(u_2) \| ds \]
\[ + \int_0^t \| T(t-s) \| \| Df(S_s(u_1)) - Df(S_s(u_2)) \| \| DS_s(u_2) \| ds \]
\[ \leq M_{c_1} \int_0^t \| DS_s(u_1) - DS_s(u_2) \| ds \]
\[ + M_{c_2} \int_0^t \| Df(S_s(u_1)) - Df(S_s(u_2)) \| ds \]

with
\[ c_1 = \sup_{u \in U, t \in [0, \tau]} \| Df(S_t(u)) \| \]
and
\[ c_2 = \sup_{u \in U, t \in [0, \tau]} \| DS_t(u) \|. \]

With the Lipschitz continuity
\[ \| Df(S_s(u_1)) - Df(S_s(u_2)) \| \leq c_3 \| S_s(u_1) - S_s(u_2) \|_{X} \]
of \( Df \) according to assumption (B2) and Lemma 7, we also get
\[ \| DS_s(u_1) - DS_s(u_2) \| \leq c_4 \| u_1 - u_2 \|_{L_2} \]
\[ \forall s \in [0, t]. \] Thus, (15) can be simplified to
\[ DS_t(u_1) - DS_t(u_2) \leq c_{c_3} c_4 \| u_1 - u_2 \|_{L_2}. \]

Applying Gronwall’s lemma yields
\[ \| DS_t(u_1) - DS_t(u_2) \| \leq M c_{c_4} t e^{M c_{c_1} t} \| u_1 - u_2 \|_{L_2}, \]
which shows that the map \( u \to DS_t(u) \) is continuous for all \( t \in [0, \tau] \).

With Theorem 8 and Theorem 9, we can finally state our main result.

**Theorem 10.** Consider a semilinear system (2) with a nonlinear term \( f \) that meets the conditions (B1) and (B2). If the linearized system (3) is exactly controllable on \([0, \tau]\), then the original system is locally exactly controllable on \([0, \tau]\).

**Proof.** We only have to show that the conditions (A1) and (A2) of Theorem 6 are satisfied. Condition (A1) follows from Theorem 9, and condition (A2) from Theorem 8 with \( \bar{u} = 0 \).

5. CONCLUSION

We have shown for a class of semilinear infinite-dimensional systems that exact controllability of the linearized system implies local exact controllability of the original system. The assumptions on the nonlinear operator are similar to those used in Smoller (1994) for the linearized stability analysis, i.e., Lyapunov’s indirect method. Future research will deal with the question whether these assumptions can be relaxed. A further interesting question is whether the approach with the local surjectivity theorem can also be applied to the approximate controllability problem. Such an extension is of course not at all straightforward. Since the infinite-dimensional spaces in the local surjectivity theorem are Banach spaces, i.e., complete, the intuitive idea of simply using the reachable subspace of the linearized system as target space is not directly applicable.

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