BELLMAN FUNCTION APPROACH TO THE SHARP CONSTANTS IN UNIFORM CONVEXITY

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ABSTRACT. We present a short proof of finding the sharp constants in uniform convexity by Bellman function technique.

1. Introduction

1.1. Bellman function in uniform convexity. Let $I$ be an interval of the real line. For an integrable function $f$ over $I$, we set $\langle f \rangle_I \defeq \frac{1}{|I|} \int_I f(s) \, ds$, and $\|f\|_p \defeq \langle |f|^p \rangle_I^{1/p}$. We recall the definition of uniform convexity of a normed space $(X, \|\cdot\|)$ (see [2]).

Definition 1. (Clarkson ’36) $X$ is uniformly convex if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, then $\frac{\|x + y\|}{2} \leq 1 - \delta$.

Constant of uniform convexity of $L^p(I)$ space is defined as follows:

$$\delta(\epsilon) = \inf \left\{ \left(1 - \frac{\|f + g\|_p}{2}\right) : \|f\|_p = 1, \|g\|_p = 1, \|f - g\|_p \geq \epsilon \right\}.$$

Remark 1. $L^p(I)$ space is uniformly convex if $\delta(\epsilon) > 0$.

O. Hanner (see [1]) gave an elegant proof of finding the constant $\delta(\epsilon)$ in $L^p(0,1)$ space for $p \in (1, \infty)$ in 1955. He proved two necessary inequalities (further called Hanner’s inequalities) in order to obtain constant $\delta(\epsilon)$. Hanner mentions in his note [1] that his proof is a reconstruction of some Beurling’s ideas given at a seminar in Upsala in 1945. Further we assume $p \in (1, \infty)$. In this note we present Bellman function approach in finding the constant $\delta(\epsilon)$. We also show that the Bellman function (2), which arise naturally, is a minimal concave function with the given boundary condition (3).

Let $\varphi(x_1, x_2) : \mathbb{R}^2 \to \partial \Omega$ be a continuous map, image of which is a boundary of a convex set $\Omega \subset \mathbb{R}^3$. We require that convex hull of $\partial \Omega$ is $\Omega$. Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. We introduce a Bellman function

$$B(x) = \sup_{f,g} \{ \langle H(f,g) \rangle_1, \langle \varphi(f,g) \rangle_1 = x \},$$

where $x = (x_1, x_2, x_3)$, and the supremum is taken over the measurable functions $f, g$ such that $\varphi(f,g)$ is integrable over $I$.

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Theorem 1. $B$ is minimal among concave functions $B$ on $\Omega$ with a boundary data $B(\varphi(x_1, x_2)) \geq H(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$.

Proof. It is easy to see that $B$ is defined on the set $\Omega$ and $B(\varphi(x_1, x_2)) \geq H(x_1, x_2)$. If $B$ is concave in $\Omega$ with a boundary data $B(\varphi(x_1, x_2)) \geq H(x_1, x_2)$ then Jensen’s inequality implies

$$\langle H(f, g) \rangle_{1} \leq \langle B(\varphi(f, g)) \rangle_{1} \leq B(\langle \varphi(f, g) \rangle_{1}).$$

This implies that $B(x) \leq B(x)$. We are going to show that $B$ is concave itself which will imply that $B$ is minimal concave function in $\Omega$ with the boundary data $B(\varphi(x_1, x_2)) \geq H(x_1, x_2)$. The reader can easily see that $B$ does not depend on the interval $I$. Without loss of generality we assume that $I = [0, 1]$. We want to show that for all $\alpha \in (0, 1)$

(1) $B(\alpha x + (1 - \alpha)y) \geq \alpha B(x) + (1 - \alpha)B(y)$ for all $x, y \in \Omega$.

Take any $\varepsilon > 0$. Given the point $x \in \Omega$ there exists a pair $(f_1, g_1)$ such that $\langle \varphi(f_1, g_1) \rangle_{[0, 1]} = x$ and $\langle H(f_1, g_1) \rangle_{[0, 1]} \geq B(x) - \varepsilon$. Similarly for the point $y \in \Omega$ we find $(f_2, g_2)$ such that $\langle \varphi(f_2, g_2) \rangle_{[0, 1]} = y$ and $\langle H(f_2, g_2) \rangle_{[0, 1]} \geq B(y) - \varepsilon$. We consider concatenation of the pairs $(f_1, g_1), (f_2, g_2)$, namely

$$\langle f(t), g(t) \rangle = \begin{cases} 
(f_1(\alpha t), g_1(\alpha t)), & t \in [0, \alpha], \\
(f_2 \left(\frac{t}{\alpha - 1}\right), g_2 \left(\frac{t}{\alpha - 1}\right)), & t \in [\alpha, 1].
\end{cases}$$

Note that $\langle \varphi(f, g) \rangle_{[0, 1]} = \alpha x + (1 - \alpha)y$ and $\langle H(f, g) \rangle_{[0, 1]} = \alpha B(x) + (1 - \alpha)B(y) - 2\varepsilon$. We recall that $\varepsilon$ was arbitrary, therefore we get 1. \hfill \Box

Further we restrict ourselves to the case when $\varphi(x_1, x_2) = (|x_1|^p, |x_2|^p, |x_1 - x_2|^p)$, $H(x_1, x_2) = |\theta x_1 + (1 - \theta)x_2|^p$ for some $\theta \in (0, 1)$. Hence we are interested to find a function

(2) $B(x) = \sup_{f, g} \{\langle |\theta f + (1 - \theta)g|^p \rangle_{1}, \langle |f|^p, |g|^p, |f - g|^p \rangle_{1} \} = x$.

We make several observations. Note that all variables $x_1, x_2, x_3$ are non-negative. Note also that the function $B$ is given on the convex cone

$$\Omega = \{x_1, x_2, x_3 \geq 0, \quad x_1^{1/p} + x_2^{1/p} \geq x_3^{1/p}, x_2^{1/p} + x_3^{1/p} \geq x_1^{1/p}, x_3^{1/p} + x_1^{1/p} \geq x_2^{1/p}\}.$$

The function $B$ is concave (it follows from the Theorem 1). Minkowski’s inequality implies that whenever $x \in \partial \Omega$ we must have $f = \lambda g$ for an appropriate $\lambda$. This allows us to find boundary data for the function $B$.

(3) $B(x_1, x_2, x_3) = \begin{cases} 
|\theta x_1^{1/p} - (1 - \theta)x_2^{1/p}|^{1/p}, & x_1^{1/p} + x_2^{1/p} = x_3^{1/p}, \\
|\theta x_2^{1/p} + x_1^{1/p}|^{1/p}, & x_2^{1/p} + x_3^{1/p} = x_1^{1/p}, \\
|\theta x_3^{1/p} + x_2^{1/p}|^{1/p}, & x_1^{1/p} + x_3^{1/p} = x_2^{1/p},
\end{cases}$

where $x_1, x_2$ and $x_3$ are nonnegative numbers. Theorem 1 implies that $B$ is minimal concave function on $\Omega$ with the given boundary condition (3). Further we assume $\theta = 1/2$. 
1.2. **Sharp constants in uniform convexity.** Note that

\[ \delta(\varepsilon) = 1 - \sup_{2p \geq x_3 \geq \varepsilon^p} (B(1,1,x_3))^{1/p}. \]

The reader can try to find the function \( B(x_1, x_2, x_3) \). However, one can avoid finding the exact value of \( B(x_1, x_2, x_3) \), and by Theorem 1 one can present an appropriate concave function \( B \) which majorize \( B \), however gives the exact value of \( \delta(\varepsilon) \).

We consider the case \( p \geq 2 \). Let’s consider the following function

\[ B(x_1, x_2, x_3) = \frac{x_1 + x_2}{2} - \frac{x_3}{2^p}. \]

Surely \( B \) is concave in \( \Omega \) and \( (B - B)|_{\partial \Omega} \geq 0 \) (see Appendixes). Therefore, Theorem 1 implies that \( B \geq B \) in \( \Omega \). Thus,

\[ \delta(\varepsilon) = 1 - \sup_{2p \geq x_3 \geq \varepsilon^p} (B(1,1,x_3))^{1/p} \geq 1 - \sup_{2p \geq x_3 \geq \varepsilon^p} (B(1,1,x_3))^{1/p} = \]

\[ 1 - (B(1,1,\varepsilon^p))^{1/p} = 1 - \left(1 - \frac{\varepsilon^p}{2^p}\right)^{1/p}. \]

If we show that \( B(1,1,\varepsilon^p) = B(1,1,\varepsilon^p) \) then this would imply that the estimate obtained above for \( \delta(\varepsilon) \) is sharp. Homogeneity of the functions \( B \) and \( B \) (i.e. \( B(\lambda x) = \lambda B(x) \) for all \( \lambda \geq 0 \)) implies that it is enough to prove the equality \( B(\varepsilon^{-p}, \varepsilon^{-p}, 1) = B(\varepsilon^{-p}, \varepsilon^{-p}, 1) \). We show that \( B(s, s, 1) = B(s, s, 1) \) for all \( s \geq 2^{-p} \). Take an arbitrary \( s \in (2^{-p}, \infty) \), Consider the points \( A = (2^{-p}, 2^{-p}, 1) \) and \( D(s) = (s, 1 - s^1/p^1, 1) \). Clearly \( A, D(s) \in \partial \Omega \). Let \( L_s(x) \) be a linear function such that \( L_s(A) = B(A) = 0 \), \( L_s(D(s)) = B(D(s)) = (1/2 + s^1/p)^p \). Concavity of \( B \) implies that \( B \geq L \) on the chord \( [A, D(s)] \) joining the points \( A \) and \( D(s) \). On the other hand one can easily see that for all \( a > 0 \), there exists sufficiently large \( s \) such that we have \( L_s(x) - B(x) < a \) for the points \( x \) belonging to the chord \( [A, D(s)] \).

Continuity of \( B \) and \( B \) finishes the story.

Now we consider the case \( 1 < p < 2 \). We set

\[ (g(s), f(s)) = (|1 - s^1/p^p|, (s^1/p - 1/2)^p) \text{ for } s \geq 2^{-p}. \]

Let \( s^* \in [2^{-p}, \infty) \) be the solution of the equation \( 2\varepsilon^{-p} = s^* + g(s^*) \). Consider the following function

\[ B(x_1, x_2, x_3) = x_3 f(s^*) + \frac{f'(s^*)}{1 + g'(s^*)} \left[ x_1 + x_2 - 2\varepsilon^{-p}x_3 \right]. \]

Surely \( B \) is concave in \( \Omega \) and \( (B - B)|_{\partial \Omega} \geq 0 \) (see Appendixes). Therefore, Theorem 1 implies that \( B \geq B \) in \( \Omega \). The inequality \( B_{x_3}''(1,1,x_3) \leq 0 \) follows from the inequality \( f(s)(1 + g'(s)) - f'(s)(s + g(s)) \leq 0 \) for all \( s \geq 2^{-p} \), which can be seen by direct computation. Thus,

\[ \delta(\varepsilon) = 1 - \sup_{2p \geq x_3 \geq \varepsilon^p} (B(1,1,x_3))^{1/p} \geq 1 - \sup_{2p \geq x_3 \geq \varepsilon^p} (B(1,1,x_3))^{1/p} = \]

\[ 1 - (B(1,1,\varepsilon^p))^{1/p} = 1 - \varepsilon (f(s^*))^{1/p}. \]
If we show that $B(1, 1, \varepsilon^p) = B(1, 1, \varepsilon^p)$ then this would imply that the estimate obtained above for $\delta(\varepsilon)$ is sharp. Homogeneity of the functions $B$ and $B$ implies that it is enough to prove the equality $B(\varepsilon^{-p}, \varepsilon^{-p}, 1) = B(1, 1, \varepsilon^p)$.

Consider the points $A = (s^*, g(s^*), 1)$ and $D = (g(s^*), s^*, 1)$. Clearly $A, D \in \partial \Omega$. Let $L(x)$ be a linear function such that $L(A) = B(A) = f(s^*)$, $L(D) = B(D) = f(s^*)$. Concavity of $B$ implies that $B(x) \geq L(x)$ on the chord $[A, D]$ joining the points $A$ and $D$. On the other hand one can easily see that $B(x) = L(x)$ for the points $x$ belonging to the chord $[A, D]$. Thus the fact $(\varepsilon^{-p}, \varepsilon^{-p}, 1) \in [A, D]$, which follows from the equality $2 \varepsilon^{-p} = s^* + g(s^*)$ finishes the proof. It is clear that the constant $\delta(\varepsilon) \in [0, 1]$ is the solution of the equation

$$
(1 - \delta(\varepsilon) + \frac{\varepsilon}{2})^p + |1 - \delta(\varepsilon) - \frac{\varepsilon}{2}|^p = 2.
$$

Thus we finish the current note. One can ask how did we find the functions $B$. These functions are tangent planes to the graph of actual Bellman function $B$ at point $(1, 1, \varepsilon^p)$. Unlike the actual Bellman function $B$, which has the implicit expression, the tangent planes $B$ to the graph $B$ have simple expression, so it was easy to work with tangent planes $B$, rather than with actual Bellman function $B$.

At the end of the note I would like to mention that the Bellman function (2) was born during the listening seminar given by N. K. Nikolskij at Chebyshev Laboratory in Saint Petersburg. Idea of investigating such type of Bellman functions comes from A. Volberg’s seminars given at the same laboratory. I would like to thank P. Zatitskiy and D. Stolyarov for long discussion and construction of the actual Bellman function (2). They should be considered as co-authors of this note.

2. Appendixes

We show that $(B - B)|_{\partial \Omega} \geq 0$. Homogeneity of $B$ and $B$ implies that without loss of generality we can assume $x_3 = 1$. In this case boundary condition (3) can be rewritten as follows $B(s, g(s), 1) = B(g(s), s, 1) = f(s)$ for all $s \geq 2^{-p}$. It is enough to show that $U(s) = B(s, g(s), 1) - f(s) \geq 0$ for $s \geq 2^{-p}$, because the other cases can be covered by symmetry i.e. $B(s, g(s), 1) = B(g(s), s, 1)$ and $B(s, g(s), 1) = B(g(s), s, 1)$.

**Case** $p \geq 2$. Let $V(s) = U(s^p)$. Note that $V(2^{-1}) = 0$, and $V'(s) \geq 0$ for $s \geq 1$. Note that $W(s) = V'(s^p) = (2^{-1}2s^{p-1} = 1 - (s - 1)^{p-1} - 2(1 - s/2)^{p-1}$ is a concave function for $s \in (1, 2)$, and it is nonnegative at the endpoints of this interval, hence $V'(s) \geq 0$ for $s \geq 2^{-p}$.

**Case** $1 < p < 2$. Note that $U(s^*) = 0$, the function $\frac{f(s)}{1 + g'(s)}$ is decreasing and $1 + g'(s) > 0$ for $s \geq 2^{-p}$. It is clear that $U'(s)(1 + g'(s)) = \frac{f'(s^*)}{1 + g'(s^*)} - \frac{f'(s)}{1 + g'(s)}$. Thus we see that $U'(s) < 0$ for $s \in [2^{-p}, s^*)$, $U'(s^*) = 0$, and $U'(s) > 0$ for $s \in (s^*, \infty)$. Hence $U(s) \geq 0$. 


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