Latin trades and simplicial complexes

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July 9, 2009

Abstract

In this note we introduce the concept of the trade space of a latin square. Computations using Sage and the GAP package Simplicial Homology are presented.

1 Introduction

We first introduce the concept of a latin square and a latin bitrade. For a more detailed exposition and literature survey, see [8].

Definition 1.1. Let $N$ be a fixed set of size $n > 0$. A latin square $L$ of order $n$ is an $n \times n$ array with rows and columns indexed by $N$, and entries from the set $N$. Further, each $e \in N$ appears exactly once in each row and exactly once in each column. A partial latin square of order $n$ is an $n \times n$ array where each $e \in N$ occurs at most once in each row and at most once in each column.

Usually for index set $N$ we will use $[n] = \{1, 2, \ldots, n\}$ or sometimes $[n]_0 = \{0, 1, \ldots, n - 1\}$ when working with modulo arithmetic. Note that a latin square is a partial latin square with no empty cells. A latin square $L$ may also be represented as a set of ordered triples, where $(r, c, e) \in L$ denotes the fact that symbol $e$ appears in the cell at row $r$, column $c$, of $L$. Alternatively we write $L^\circ$ with binary operation $\circ$ such that $r \circ c = e$ if and only if $(r, c, e) \in L$. Similarly, a partial latin square $P$ may be written as $P^\circ$. We use setwise and binary operator notation interchangeably.

The size of a partial latin square $P$ is the number of filled cells, denoted by $|P| = |\{(r, c, e) \mid (r, c, e) \in P\}|$. A partial latin square $P'$ is contained in, or is a (partial) subsquare of $P$ if and only if $P' \subseteq P$.

*Supported by Eduard Cech center, grant LCS05.
1.1 Latin bitrades and trades

Definition 1.2. A latin bitrade \((T^\circ, T^\otimes)\) is a pair of partial latin squares such that:

1. \(\{(i, j) \mid (i, j, k) \in T^\circ\text{ for some symbol }k\} = \{(i, j) \mid (i, j, k') \in T^\otimes\text{ for some symbol }k'\}\);
2. for each \((i, j, k) \in T^\circ\) and \((i, j, k') \in T^\otimes\), \(k \neq k'\);
3. the symbols appearing in row \(i\) of \(T^\circ\) are the same as those of row \(i\) of \(T^\otimes\);
   the symbols appearing in column \(j\) of \(T^\circ\) are the same as those of column \(j\) of \(T^\otimes\).

The following definition is equivalent to Definition 1.2:

Definition 1.3. A latin bitrade \((T^\circ, T^\otimes)\) is a pair of partial latin squares \(T^\circ, T^\otimes \subseteq A_1 \times A_2 \times A_3\) such that:

(R1) \(T^\circ \cap T^\otimes = \emptyset\);
(R2) for all \((a_1, a_2, a_3) \in T^\circ\) and all \(r, s \in \{1, 2, 3\}, r \neq s\), there exists a unique \((b_1, b_2, b_3) \in T^\otimes\) such that \(a_r = b_r\) and \(a_s = b_s\);
(R3) for all \((a_1, a_2, a_3) \in T^\otimes\) and all \(r, s \in \{1, 2, 3\}, r \neq s\), there exists a unique \((b_1, b_2, b_3) \in T^\circ\) such that \(a_r = b_r\) and \(a_s = b_s\).

Note that (R2) and (R3) imply that each row (column) of \(T^\circ\) contains the same subset of \(A_3\) as the corresponding row (column) of \(T^\otimes\). Since all of the bitrades in this dissertation are latin bitrades, we usually shorten ‘latin bitrade’ to just ‘bitrade.’

For a bitrade \((T^\circ, T^\otimes)\) we refer to \(T^\circ\) as the trade, and \(T^\otimes\) as the disjoint mate. A particular trade may have more than one disjoint mate.

Example 1.4. Consider the following latin squares:

\[
\begin{array}{ccccc}
\diamond & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 3 & 2 & 4 \\
1 & 5 & 2 & 4 & 3 & 1 \\
L^\circ = 2 & 2 & 3 & 5 & 4 & 0 \\
3 & 3 & 4 & 1 & 0 & 5 \\
4 & 4 & 5 & 0 & 1 & 2 \\
5 & 1 & 0 & 2 & 5 & 3 \\
\end{array}
\quad \begin{array}{ccccc}
\otimes & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 3 & 1 & 2 & 4 \\
1 & 5 & 2 & 3 & 4 & 1 \\
L^\otimes = 2 & 2 & 4 & 5 & 3 & 0 \\
3 & 3 & 1 & 4 & 0 & 5 \\
4 & 4 & 5 & 0 & 1 & 2 \\
5 & 1 & 0 & 2 & 5 & 3 \\
\end{array}
\]

One possible bitrade \((T^\circ, T^\otimes)\) where \(T^\circ \subseteq L^\circ\) and \(T^\otimes \subseteq L^\otimes\) is shown below:

\[
\begin{array}{ccccc}
\diamond & 0 & 1 & 2 & 3 & 4 \\
0 & 1 & 3 & 4 \\
1 & 4 & 3 \\
T^\circ = 2 & 3 & 4 \\
3 & 4 & 1 \\
4 & 4 \\
5 & 5 \\
\end{array}
\quad \begin{array}{ccccc}
\otimes & 0 & 1 & 2 & 3 & 4 \\
0 & 3 & 1 \\
1 & 3 & 4 \\
T^\otimes = 2 & 4 & 3 \\
3 & 1 & 4 \\
4 & 4 \\
5 & 5 \\
\end{array}
\]

(1)
So \( L^\circ = (L^\circ \setminus T^\circ) \cup T^\circ \) and \( L^\circ = (L^\circ \setminus T^\circ) \cup T^\circ \).

**Example 1.5.** The following bitrade is simply a cyclic row-shift of \( T^\circ = \mathbb{Z}_3 \), the integers under addition modulo 3:

\[
\begin{array}{ccc}
\otimes & 0 & 1 & 2 \\
1 & 0 & 1 & 2 \\
1 & 2 & 0 & 1 \\
2 & 0 & 1 & 2 \\
\end{array}
\quad \begin{array}{ccc}
\otimes & 0 & 1 & 2 \\
0 & 1 & 2 & 0 \\
1 & 2 & 0 & 1 \\
2 & 0 & 1 & 2 \\
\end{array}
\]

**Example 1.6.** Here is a larger bitrade:

\[
\begin{array}{cccc}
\otimes & 1 & 2 & 3 & 4 \\
1 & 1 & 4 & 3 & 2 \\
2 & 4 & 3 & 2 & 1 \\
3 & 2 & 1 & 3 & 1 \\
4 & 3 & 1 & 4 & 1 & 3 \\
\end{array}
\quad \begin{array}{cccc}
\otimes & 1 & 2 & 3 & 4 \\
1 & 4 & 3 & 2 & 1 \\
2 & 3 & 4 & 2 & 1 \\
3 & 1 & 2 & 3 & 4 \\
4 & 1 & 3 & 2 & 1 & 3 \\
\end{array}
\]

### 1.2 Latin critical sets

**Definition 1.7.** A partial latin square \( C \subseteq L \) is a critical set if

1. \( C \) has unique completion to \( L \); and
2. no proper subset of \( C \) satisfies \( \square \)

**Example 1.8.** Latin square \( \mathbb{Z}_2 \) and a critical set:

\[
\begin{array}{cc}
0 & 1 \\
1 & 0 \\
\end{array}
\]

**Example 1.9.** Latin square \( L_3 \) and critical set \( P_3 \subset L_3 \):

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\
2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\
3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\
4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\
5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\
6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
\end{array}
\]

Fix a latin square \( L \) and define the trade space

\[ T_L = \{ T^\circ \mid T^\circ \subset L \text{ and } (T^\circ, T^\circ) \text{ is a bitrade} \} \]

**Lemma 1.10.** Let \( C \) be a critical set of the latin square \( L \). Then for any \( T \in T_L \), there exists \( c \in C \) such that \( c \in T \).

An long-standing open conjecture is that the size of the smallest critical set in any latin square of order \( n \) is \( \lfloor n^2/4 \rfloor \). See [2], [4] and [7] for recent work on this problem.
1.3 Simplicial complexes

We follow the presentation of Faridi [6]. A facet of a simplicial complex is a maximal face under inclusion. A vertex cover $A$ of a simplicial complex $\Delta$ is a subset of vertices of $\Delta$ such that any facet is adjacent to some $a \in A$. A minimal vertex cover $A$ is a vertex cover of $\Delta$ such that no proper subset of $A$ is a vertex cover.

Let $L$ be a latin square of order $n$. Form the simplicial complex $\Delta(L)$ with vertices corresponding to trades $T \subset L$, and edges $(T, T')$ for trades $T, T' \subset L$ such that $T \cap T' \neq \emptyset$, and so on for faces of higher dimensions.

**Lemma 1.11.** For each critical set $C$ of $L$ there exists a minimal vertex cover $A_C$ of $\Delta(L)$ and $|C| = |A_C|$.

**Proof.** Let $C = \{c_1, \ldots, c_m\}$ be a critical set in $L$. Create a family of $m$ sets

$$S_i = \{T \subseteq L \mid c_i \in T\}$$

where $T$ is a trade in $L$. We will show that the $S_i$ have a system of distinct representatives. Choose any $\ell$ of the sets, $S_{i_1}, \ldots, S_{i_\ell}$ and suppose that their union $S$ contains $l < \ell$ trades. By the pigeonhole principle there is an entry $c_{i_u}$ such that each trade that intersects $c_{i_u}$ also intersects some $c_{i_v}, v \neq u$. So the set $C \setminus \{c_{i_u}\}$ is a critical set, contradicting the minimality of $C$. Call the system of distinct representatives $A_C$.

To see that $A_C$ is a vertex cover of $\Delta(L)$, let $F$ be some facet of $\Delta(L)$. Pick any $T \in F$. Then there exists some $c_i \in C$ such that $c_i \in T$. We chose some $T$ as the representative of $S_i$. There is an edge $(T, \overline{T})$ due to the common element $c_i$, so $F$ is adjacent to $\overline{T} \in A_C$.

For minimality, suppose that $A_C \setminus \{T\}$ is a vertex cover for some trade $T$ in $L$. Suppose that $T$ was chosen as the representative of $S_i$. Let $U$ be any trade in $L$ and consider $C' = C \setminus \{c_i\}$. If $c_i \notin U$ then some other entry of $C$ covers $U$ since $C$ is a critical set. Otherwise, suppose that $c_i \in U$ then consider the facet $F$ containing $T$ and $U$. By assumption there is some $T' \in A_C \setminus \{T\}$ such that $T' \in F$. Then $T'$ must be the representative for $S_j$ (where $j \neq i$ since $A_C$ is a system of distinct representatives) so $c_j \in U$. In this way any trade $U$ is covered by some element of $C' = C \setminus \{c_i\}$, contradicting the minimality of $C$. $\square$

**Lemma 1.12.** For each minimal vertex cover $A$ of $\Delta(L)$ there exists a critical set $C_A$ of $L$ such that $|C_A| = |A|$.

**Proof.** Let $A$ be a minimal vertex cover of $\Delta(L)$. For each $T \in A$, let $x_T$ be an entry of $T$ contained in the intersection of the trades that make up the facet containing $T$. We show that $X$ is a critical set. Suppose that $U$ is some trade in $L$ and no $x_T$ is in $U$. The vertex $U$ is adjacent to some facet $F$ and by definition some $T' \in A$ is adjacent to this $F$. Then by definition $x_{T'} \in U$, a contradiction. Thus $X$ is a uniquely completable partial latin square.

For minimality, suppose that the set $X' = X \setminus \{x_T\}$ is uniquely completable for some $x_T \in X$. By the previous lemma, $X'$ is equivalent to a vertex cover
A' of the same size, contradicting the minimality of A. So X is minimal and |X| = |A|.

\[ \text{Remark 1.13.} \text{ A critical set of } L \text{ is equivalent to a vertex cover of } \Delta(L). \text{ The simplicial complex } \Delta(L) \text{ captures information about the intersections of all trades. The critical sets correspond to minimal vertex covers.} \]

Faridi’s paper ([6], Proposition 1) shows that a minimum vertex cover is the same as a minimal prime ideal in the facet ideal \( \mathcal{F}(\Delta) \) in the polynomial ring \( R = k[x_1, \ldots, x_n] \). In future research we would like to calculate more detailed information about the facet ideal and its minimal prime ideals.

\section{Computations}

In this section we present computations of the reduced homology groups \( H_k \) of the facet ideal \( \Delta(L) \) for various latin squares \( L \) of small order. Informally, the group \( H_k \) indicates the number of \( k \)-dimension ‘holes’ in the simplicial complex at hand. See [9] for more information about homology theory. We use a Sage script [1] and the GAP package Simplicial Homology to calculate the reduced homology groups. For more information about Sage see [10]. The Simplicial Homology package is available at [http://www.cis.udel.edu/~dumas/Homology/](http://www.cis.udel.edu/~dumas/Homology/).

In this section, \( B_n \) denotes the addition table for integers modulo \( n \) (also known as the back-circulant square). Also, let \( L_s \) denote the latin square corresponding to the group table of the elementary abelian 2-group of order \( 2^s \). Formally, these latin squares are defined by

\[
L_1 = \begin{array}{cc}
0 & 1 \\
1 & 0 \\
\end{array}
\]

and for \( s \geq 2 \),

\[
L_s = L_1 \times L_{s-1} = \{(x, y; z), (x, y + n/2; z + n/2), (x + n/2, y; z + n/2), (x + n/2, y + n/2; z) \mid (x, y; z) \in L_{s-1}\}
\]

For example,

\[
L_3 = \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\
2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\
3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\
4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\
5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\
6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0
\end{array}
\]

We now present computations for small orders:
| Latin square | Homology groups |
|--------------|-----------------|
| $B_3$        | [0, 0, 0, 10, 0, 0] |
| $B_4$        | [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] |
| $L_2$        | [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 8, 9, 0] |

So for $B_3$ we have $\tilde{H}_0 = 0, \tilde{H}_1 = 0, \tilde{H}_2 = 0, \tilde{H}_3 = 10, \tilde{H}_4 = 0, \tilde{H}_5 = 0$.

### 2.1 Intercalate homology

An intercalate is a latin trade of size four. Intercalates are interesting because they are the simplest (smallest) type of latin trade and in fact any trade can be written as a sum of intercalates. Another interesting question is whether there are latin squares with as many intercalates as possible, or as few as is possible (see for example [3]).

For a latin square $L$ we define the intercalate trade space as

$$\mathcal{I}_L = \{T^\circ \mid T^\circ \subset L, |T^\circ| = 4 \text{ and } (T^\circ, T^\circ) \text{ is a bitrade}\}.$$  

We can then create a simplicial complex where vertices are intercalates, and higher dimension simplices are collections intercalates that intersect in some common point. Here is the homology group information for the elementary abelian 2–group of various orders:

| Latin square | Intercalate homology groups |
|--------------|-----------------------------|
| $L_1$        | [0, 0, 0, 0]                |
| $L_2$        | [0, 21, 0, 0]               |
| $L_3$        | [0, 273, 0, 0]              |
| $L_4$        | [0, 2625, 0, 0]             |
| $L_5$        | [0, 22785, 0, 0]            |
| $L_6$        | [0, 189441, 0, 0]           |

For the back-circulant $B_n$ with $n \geq 4$ and $n$ even we have $\tilde{H}_1 = 0, \tilde{H}_2 = 0, \tilde{H}_3 = 0$, and only $\tilde{H}_0 \neq 0$. The values for $\tilde{H}_0$ are given below:

3, 8, 15, 24, 35, 48, 63, 80, 99, 120, 143, 168, 195, 224, 255, 288, 323, 360, 399, 440, 483, 528, 575, 624, 675, 728, 783, 840, 899, 960, 1023, 1088, 1155, 1224, 1295, 1368, 1443, 1520, 1599, 1680, 1763, 1848, 1935, 2024, 2115, 2208, 2303, 2400, 2499, 2600, 2703, 2808, 2915, 3024, 3135, 3248, 3363, 3480, 3599, 3720, 3843, 3968, 4095, 4224, 4355, 4488, 4623, 4760, 4899, 5040, 5183, 5328, 5475, 5624, 5775, 5928, 6083, 6240, 6399, 6560, 6723, 6888, 7055, 7224, 7395, 7568, 7743, 7920, 8099, 8280, 8463, 8648, 8835, 9024, 9215, 9408, 9603, 9800.

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