The volume of stationary black holes and the meaning of the surface gravity

William Ballik and Kayll Lake

Department of Physics, Queen’s University, Kingston, Ontario, Canada, K7L 3N6
(Dated: June 3, 2010)

The purpose of this paper is to discuss the importance of the rate of growth of the invariant four-volume $V$ of stationary non-degenerate black holes and how this rate of growth is related to their surface gravity. Now whereas the three-volume of a black hole depends on the choice of slicing \[2\], and in general the associated three-volumes are finite, the full four-volume is usually never discussed since it is formally infinite. (Consider, for example, the four-volume of the region on and below the future horizons in the Kruskal-Szekeres plane.) Even if we consider the black hole as created by the gravitational collapse of an object, and so consider only part of the Kruskal-Szekeres plane on and below the future horizon and to the future of the boundary surface of the collapsing object, the four-volume still diverges as we integrate to the infinite future. However, we need not integrate to the infinite future and so in effect consider the evolution of $V$. In this paper, to start, we examine explicitly in regular coordinates the four-volume bounded by the horizon, the central singularity and two distinct ingoing null cones in the Schwarzschild spacetime. The situation is shown schematically in Figure 1. This introductory calculation points out a relation which we eventually show is a universal feature of stationary black holes: The invariant 4-volume $V$ of a black hole grows as $V \propto \ln(\lambda)$ where $\lambda$ is the affine generator of the horizon and the constant of proportionality is the Parikh volume $\nu$. The form is also of interest for regular black holes. The form is also of interest for regular black holes.

II. BACKGROUND

The static spacetimes under consideration here can be written in the form

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2_2$$

(1)

where $d\Omega^2_2$ is the metric of a unit two-sphere ($d\theta^2 + \sin^2 \theta d\phi^2$) and $f = f(r)$, a polynomial with simple root(s) which locate the horizon(s). Within the context of classical general relativity the form (1) is of course well known as it includes the Reissner-Nordström-de Sitter class of black holes. The form is also of interest for regular black holes (those without internal singularities), a subject that goes back many years \[3\]. The curious appearance of (1) (that is, $\partial_t g_{rr} = -1$) has been discussed recently by Jacobson \[4\] who showed, amongst other things, that the central feature of (1) is the vanishing radial null-null component of the Ricci tensor. Alternatively, consider
The metric
\[ ds^2 = -f dr^2 + \frac{dr^2}{j} + r^2 d\Omega_2^2 \]
where \( j = j(r) \). Since \( f \) is a polynomial with simple root(s) let us write
\[ f(r) = (r - a)h(r) \]
where \( h(a) \neq 0 \). Whereas for II all scalars constructed from the Riemann tensor are finite for \( r > 0 \) and \( h \in C^2 \), for II these scalars diverge at \( r = a \) unless \( j(a) = 0 \). This, in some measure, helps to explain the prevalence of II. However, the from II is defective at \( r = a \) and so for our considerations of \( V \) we need regular coordinates.

III. SCHWARZSCHILD

We begin with a calculation of the complete invariant 4-volume of a Schwarzschild black hole in regular coordinates. As shown elsewhere, the Kruskal-Szekeres form of the Schwarzschild metric can be given explicitly as
\[ ds^2 = (2M)^2 d\hat{s}^2 \]
where
\[ d\hat{s}^2 = \frac{-4}{(1 + L)e^{1 + L}} du dv + (1 + L)^2 d\Omega_2^2 \]
with \( L = L(-uv/e) \) where \( L \) is the Lambert W function. Trajectories with tangents \( K^\alpha = e^L(1 + L)^{\delta^\alpha_0} \) (constant \( u = u_0, \theta \) and \( \phi \)) are radial null geodesics given by
\[ v(\lambda) = \lambda e^{-u_0 \lambda/e} \]
where \( \lambda \) is an affine parameter (defined, of course, only up to a linear transformation) and we note the expansion
\[ \nabla_\alpha K^\alpha = \frac{-2v_0}{e(1 + L)}. \]
Trajectories with tangents \( M^\alpha = e^L(1 + L)^{\delta^\alpha_0} \) (constant \( v = v_0, \theta \) and \( \phi \)) are radial null geodesics given by
\[ u(\lambda) = \lambda e^{-v_0 \lambda/e} \]
and we now note the expansion
\[ \nabla_\alpha M^\alpha = \frac{-2v_0}{e(1 + L)}. \]
On the horizons \( u = 0 \) and \( v = 0 \) then \( v \) and \( u \) are affine parameters. The only singularity in I occurs for \( L = -1 \). That is, \( uv = 1 \).

As is well known, the invariant 4-volume over some region \( R \) of spacetime is given by
\[ V = \int \sqrt{|g|}^4 dx \]
where \( g \) is the determinant of the metric and the integration is over \( R \). For the present calculation, \( R \) is defined in Figure 2. From II and (II) and (II) we have
\[ V = (2M)^4 \int_0^{2\pi} \int_0^\pi \int_0^{1/v} \frac{1 + L}{e^{1+L}} du dv \sin(\theta) d\theta d\phi \]
from which we find
\[ V = \frac{8\pi}{3} (2M)^4 \ln(\frac{v}{\delta}) \]
so that with \( u_0 = 0 \) from II we arrive at the manifestly invariant statement
\[ \frac{dV}{d\lambda} = \frac{8\pi}{3} (2M)^4 \frac{1}{\kappa \lambda}. \]
irrespective of initial conditions. This relation can be written in the equivalent form
\[ \frac{dV}{d\ln(\lambda)} = \frac{3V}{\kappa \lambda} \]
where \( \kappa = \frac{1}{4\pi^2} \) is the surface gravity (see below) and \( 3V = \frac{1}{4\pi^2} (2M)^3 \) is the Euclidean 3-volume (also see below as the Euclidean form is due to the symmetry).

Defining
\[ V^* = \frac{dV}{d\ln(\lambda)} \]
we rewrite II as
\[ \kappa = \frac{3V}{V^*}. \]
we arrive back at (4) with (5). However, about a distinct root, say $r = b \neq a$, a new chart must be constructed about $r = b$. With $f = 1 - 2M/r$ we arrive back at (11) with (6).

B. Volume

It is convenient to start with the intermediate form (20) so that with (24) we have, prior to specifying boundary conditions,

$$\mathcal{V} = 4\pi\int \left( \int \frac{r^2}{2\kappa} du \right) \frac{dv}{\kappa v}.$$  (29)
We set the horizons as above and defining the form $u$ and $v$ and so from (29) and (30)

$$K = \frac{dv}{d\lambda}. \quad (33)$$

As explained below, a shell-type construction is needed for the consideration of cosmological horizons.

To interpret $v$ we now use the regular form (27) with (25). Trajectories with tangents $\mathcal{K}^\alpha = \frac{4}{\pi(r)} \delta^\alpha_u$ (constant $u = u_0, \theta$ and $\phi$) are radial null geodesics with expansion

$$\nabla_\alpha K^\alpha = -8u_0 \frac{dr}{K(r)r d(uv)}. \quad (32)$$

Trajectories with tangents $\mathcal{M}^\alpha = \frac{4}{\pi(r)} \delta^\alpha_u$ (constant $v = v_0, \theta$ and $\phi$) are radial null geodesics with expansion

$$\nabla_\alpha M^\alpha = -8v_0 \frac{dr}{K(r)r d(uv)}. \quad (33)$$

We set the horizons $r = a$ at $u = 0$ and $v = 0$ so that $v$ and $u$ are affine parameters. We can now rewrite (31) in the form

$$\frac{dV_s}{d\lambda} = \frac{3V_s}{\kappa \lambda}. \quad (34)$$

Note that section III above can be recovered form this section by setting $r_1 = 0$ and $f = 1 - 2M/r$. Proceeding as above and defining

$$V_s \equiv \frac{dV_s}{d\ln(\lambda)} \quad (35)$$

we rewrite (34) as

$$\kappa = \frac{3V_s}{V_s}. \quad (36)$$

We see that $\kappa$ is in effect independent of the thickness of the shell $s$ and so we view it as a property of the horizon alone.

V. GENERAL $f$ - COSMOLOGICAL HORIZONS

The foregoing argument applies directly to cosmological (and Cauchy) horizons with the insertion of an absolute value in the definition of the surface gravity in (25). We emphasize this point here by way of a direct calculation in de Sitter space in regular coordinates.

A complete covering of de Sitter space is given by

$$ds^2 = \frac{3}{\Lambda} ds^2 \quad (37)$$

where

$$ds^2 = \frac{-4}{(1 - uv)^2} du dv + \left(\frac{1 + uv}{1 - uv}\right)^2 d\Omega^2. \quad (38)$$

and we consider the region $0 < r \equiv \sqrt{\frac{4}{\Lambda}(1 + uv)/(1 - uv)} < \infty$. Trajectories with tangents $\mathcal{K}^\alpha = (1 - u_0 v)^2 \delta^\alpha_v$ (constant $u = u_0, \theta$ and $\phi$) are radial null geodesics given by

$$v(\lambda)u_0 = 1 - \frac{1}{u_0 \lambda} \quad (39)$$

where $\lambda$ is an affine parameter and $u_0 \neq 0$. If $u_0 = 0$ then the geodesic is affinely parameterized by $v$. We note the expansion

$$\nabla_\alpha K^\alpha = 4u_0 \left(\frac{1 - u_0 v}{1 + u_0 v}\right). \quad (40)$$

Trajectories with tangents $\mathcal{M}^\alpha = (1 - uv)^2 \delta^\alpha_v$ (constant $v = v_0, \theta$ and $\phi$) are radial null geodesics given by

$$u(\lambda)v_0 = 1 - \frac{1}{v_0 \lambda} \quad (41)$$

for $v_0 \neq 0$. If $v_0 = 0$ then the geodesic is affinely parameterized by $u$. We now note the expansion

$$\nabla_\alpha M^\alpha = 4v_0 \left(\frac{1 - u_0 v}{1 + u_0 v}\right). \quad (42)$$

On the cosmological horizons $r = \sqrt{\frac{3}{\Lambda}}$, $u = 0$ or $v = 0$ and so $v$ and $u$ are affine parameters. There are no singularities in (38). Note that $r = 0$ for $uv = -1$ and $r = \infty$ for $uv = 1$. To calculate a finite volume $\mathcal{V}$ for a cosmological horizon we integrate from $r = \sqrt{\frac{3}{\Lambda}}$ out to (say) $\epsilon \sqrt{\frac{3}{\Lambda}}$ where $\epsilon > 1$. The situation considered is qualitatively similar to Figure II but note that $r$ is now increasing along $N$. We now have

$$V_s = 4\pi \left(\frac{3}{\Lambda}\right)^2 \int_0^v \int_0^{\Delta/\epsilon} \left(1 + uv\right)^2 du dv \quad (43)$$

where $0 < \Delta \equiv (\epsilon - 1)/(\epsilon + 1) < 1$. We find

$$V_s = \frac{4\pi}{3} \left(\frac{3}{\Lambda}\right)^2 (\epsilon^3 - 1) \ln(\frac{v}{\delta}) \quad (44)$$

so that we arrive at the manifestly invariant statement

$$\frac{dV_s}{d\lambda} = \frac{4\pi}{3} \left(\frac{3}{\Lambda}\right)^2 (\epsilon^3 - 1) \frac{1}{\lambda}. \quad (45)$$

irrespective of initial conditions. It is a simple matter to show that (45) is equivalent to (34) and therefore (25).

All calculations up to this point can be considered merely motivational (but we think important) for the brief argument that now follows.
VI. THE SURFACE GRAVITY $\kappa$

Parikh [2] has considered the volume

$$3V^s = \int \sqrt{|g|}^3 dx = \frac{dV}{dT} \tag{46}$$

for spacetimes with non-degenerate Killing vectors $\eta^\alpha$ with Killing parameter $T$ where $\eta^\alpha \nabla_\alpha T = 1$. (Because the Parikh volume is a rate, we have introduced a superscript $s$ for consistency with [15].) It is important to note that the integrand in [46] refers to the full spacetime and not a slice of it. For static spherically symmetric spacetimes it is easy to show that the Parikh volume at a horizon $r = a$ is simply the Euclidean 3-volume $4\pi a^3/3$. However, away from (say) spherical symmetry, this will not be the case. Now let us write

$$\kappa \equiv \frac{3V^s}{V^s} = \frac{1}{\lambda} \frac{d\lambda}{dT} \equiv \kappa. \tag{47}$$

The right hand equivalence is a usual definition of the surface gravity $\kappa$ (see, for example, Wald [13]). The left hand equivalence is our interpretation of $\kappa$, as verified explicitly in the foregoing motivational calculations. Further explicit calculations seem essentially pointless in view of obvious equality in the center of [47], and so we relegate the explicit verification in the Kerr metric to the Appendix.

VII. DISCUSSION

The usual physical meaning given to the surface gravity is, as explained for example by Poisson [4], “the force required of an observer at infinity to hold a particle (of unit mass) stationary at the horizon” (think of the Schwarzschild case). The interpretation given here, that the surface gravity is the ratio of the Parikh volume to the rate of change of the invariant four-volume for a shell of arbitrary (but non-vanishing) thickness bounded by the horizon, is a local interpretation that would appear to be new. An important question is, can we use this to gain further insights into black hole mechanics? The most obvious conclusion regards the third law of black hole mechanics? The question of obvious equality in the center of (47), and so we relegate the explicit verification in the Kerr metric to the Appendix.

Appendix: Kerr

In standard Kerr coordinates $(r, \theta, \phi, \psi)$ (e.g. Poisson [4] equation (5.65)) the null generator of the outer horizon $(r_+ = m + \sqrt{a^2 - m^2})$ can be given as

$$p^\alpha = (0, 0, \frac{a}{\sqrt{m^2 - a^2}} \exp(-\kappa r), \frac{1}{\kappa} \exp(-\kappa r)) \tag{A.1}$$

where

$$\kappa = \frac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}. \tag{A.2}$$

We have

$$V = \int \frac{1}{\lambda} \frac{d\lambda}{r^2} \int_0^{2\pi} \int_0^\pi \int_0^{r_+} \sin(\theta)(r^2 + a^2 \cos^2(\theta))^2 dr d\theta d\phi d\psi \tag{A.3}$$

from which we obtain

$$\frac{dV}{d\ln \lambda} = \frac{4}{3} \pi r_+ (r_+^2 + a^2) \frac{1}{\kappa}. \tag{A.4}$$

We note that Parikh [3] has already shown that

$$3V^s = \frac{4}{3} \pi r_+ (r_+^2 + a^2) \tag{A.5}$$

and so with A.4 and A.5 we arrive back at [47] without s. To insert s we simply integrate from $r_0 < r_+$. 

[1] Electronic Address: lake@astro.queensu.ca

[2] For a recent discussion see B. S. DiNunno and R. A. Matzner, Gen. Rel. Grav. 42 63 (2010) arXiv:0801.1734

[3] M. K. Parikh, Phys. Rev. D 73 124021 (2006) arXiv:hep-th/0508108

[4] See, for example, E. Poisson, A Relativist’s Toolkit: The Mathematics of Black-hole Mechanics (Cambridge University Press, Cambridge, 2004), V. P. Frolov and I. D. Novikov, Black Hole Physics (Kluwer Academic Publishers, Dordrecht, 1998), and L. M. Burko and A. Ori, Internal Structure of Black Holes and Spacetime Singularities (Institute of Physics, Bristol, 1997).

[5] For an extensive list of references on these topics see S. Conboy and K. Lake, Phys. Rev. D 71 124017 (2005)
We use geometrical units and a signature of \(+2\) throughout.

All tangents, and associated expansions, are, of course, determined only up to a multiplicative constant. We include constants of interest.

This is defined by $L(x)e^{L(x)} = x$. See, for example, R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, Advances in Computational Mathematics 5, 329 (1996).

This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World-Wide-Web from the address http://grtensor.org