Ergodic measures with multi-zero Lyapunov exponents inside homoclinic classes

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Abstract

We prove that for $C^1$ generic diffeomorphisms, if a homoclinic class $H(P)$ contains two hyperbolic periodic orbits of indices $i$ and $i+k$ respectively and $H(P)$ has no domination of index $j$ for any $j \in \{i+1, \cdots, i+k-1\}$, then there exists a non-hyperbolic ergodic measure whose $(i+l)$th Lyapunov exponent vanishes for any $l \in \{1, \cdots, k\}$, and whose support is the whole homoclinic class.

We also prove that for $C^1$ generic diffeomorphisms, if a homoclinic class $H(P)$ has a dominated splitting of the form $E \oplus F \oplus G$, such that the center bundle $F$ has no finer dominated splitting, and $H(p)$ contains a hyperbolic periodic orbit $Q_1$ of index $\dim(E)$ and a hyperbolic periodic orbit $Q_2$ whose absolute Jacobian along the bundle $F$ is strictly less than 1, then there exists a non-hyperbolic ergodic measure whose Lyapunov exponents along the center bundle all vanish and whose support is the whole homoclinic class.

1 Introduction

Since the middle of last century, the dynamics of hyperbolic systems are well understood by dynamicists. Hyperbolic systems have many good properties, for example $\Omega$-stability and existence of Markov partition. However, it was shown by R. Abraham and S. Smale [AS] that the hyperbolic systems are not dense among all the differential dynamical systems. Pesin’s theory [P] gives a new notion of hyperbolicity called non-uniform hyperbolicity, which also exhibits asymptotic expansion and contraction rate on the tangent space but may not have uniform bounds for the expansion and contraction time. The example by [CLR] shows that there exists a non-uniform hyperbolic system exhibiting homoclinic tangencies. Hence, non-uniformly hyperbolic system in general is not hyperbolic. Nevertheless, a series of works by Y. Pesin and A. Katok (for example [K] and [P]) show that many good properties of hyperbolic systems would survive in the non-uniformly hyperbolic setting, for example shadowing property and existence of stable and unstable manifolds. Then, it’s natural to ask if the non-uniformly hyperbolic systems are dense among all the differential systems. The first counterexample was given by [KN] in a global setting (some special partially hyperbolic diffeomorphisms), showing that the existence of non-hyperbolic ergodic measures is persistent. Recently, another example is given by [BBD2] in a local setting.

Let $M$ be a smooth compact Riemannian manifold of dimension $d$ without boundary. Denote by $\text{Diff}^1(M)$ the space of $C^1$ diffeomorphisms of $M$. Consider a diffeomorphism $f \in \text{Diff}^1(M)$. By Oseledets’s Theorem [O], for an $f$-invariant ergodic measure $\nu$, there exist $d$ numbers $\chi_1(\nu, f) \leq \chi_2(\nu, f) \leq \cdots \leq \chi_d(\nu, f)$ and a $\nu$-full measure set $\Lambda$ which is invariant under $f$, satisfying that for any $x \in \Lambda$ and any vector $v \in T_xM \setminus \{0\}$, there exists $i \in \{1, 2, \cdots, d\}$ such that

$$\lim_{k \to \infty} \frac{1}{k} \log \|Df^k v\| = \chi_i(\nu, f).$$

The number $\chi_i(\nu, f)$ is called the $i^{th}$ Lyapunov exponent of $\nu$. The measure $\nu$ is called hyperbolic, if all of its Lyapunov exponents are non-zero. In particular, if $\nu$ is an atomic measure distributed
averagely on a periodic orbit \( P = \text{Orb}(p) \), then its \( i^{th} \) Lyapunov exponent is also called the \( i^{th} \) Lyapunov exponent of \( P \) and is denoted by \( \chi_i(p, f) \) or \( \chi_i(P, f) \). Assume \( E \) is a \( Df \)-invariant subbundle of \( T_\Lambda M \), then the Lyapunov exponents corresponding to the vectors in \( E \) are called the Lyapunov exponents along \( E \). The number of negative Lyapunov exponents of a hyperbolic measure \( \nu \) (or a hyperbolic periodic orbit \( P \)) is called the index of \( \nu \) (or \( P \)), denoted by \( \text{Ind}(\nu) \) (or \( \text{Ind}(P) \)).

The dynamics of a system essentially concentrates on the set of points that have some recurrence properties, the chain recurrent set for instance, which splits into disjoint invariant compact sets called chain recurrence classes. By [BC], for \( C^1 \)-generic diffeomorphisms (i.e. diffeomorphisms in a dense \( G_\delta \) subset of \( \text{Diff}^1(M) \)), the chain recurrent set coincides with the closure of the set of periodic points and each chain recurrence class containing a periodic orbit \( P = \text{Orb}(p) \) coincides with its homoclinic class \( H(P, f) \): the closure of the transverse intersections of the stable and unstable manifolds of \( P \).

Given an invariant compact set \( \Lambda \). We say that \( \Lambda \) admits a \( T \)-dominated splitting for a positive integer \( T \), if the tangent bundle has a non-trivial \( Df \)-invariant splitting \( T_\Lambda M = E \oplus F \) such that

\[
\| Df^T|_{E(x)} \| \| Df^{-T}|_{F(f^T(x))} \| < \frac{1}{2} \quad \text{for any } x \in \Lambda.
\]

We say \( \Lambda \) admits a dominated splitting, if it admits a \( T \)-dominated splitting for some positive integer \( T \). The dimension of the bundle \( E \) is called the index of the dominated splitting.

Recall that a property is called a generic property if it is satisfied for a dense \( G_\delta \) subset of \( \text{Diff}^1(M) \). There are some previous works to characterize the non-hyperbolicity of homoclinic classes by the existence of non-hyperbolic ergodic measures supported on it, for example [DG, BDG, CCGWY]. Some method is introduced in [GIKN] to obtain the ergodicity of weak-\( \ast \) limit of atomic measures supported on periodic orbits, and it is developed in [DG, BDG].

**Theorem 1** ([DG, BDG]). For generic \( f \in \text{Diff}^1(M) \), consider a hyperbolic periodic orbit \( P \) of index \( i \). Assume that the homoclinic class \( H(P, f) \) contains a hyperbolic periodic orbit \( Q \) of index \( i - 1 \), then \( H(P, f) \) supports a non-hyperbolic ergodic measure, whose \( i^{th} \) Lyapunov exponent vanishes.

If moreover \( H(P, f) \) admits a dominated splitting \( T_{H(P, f)}M = E \oplus F \oplus G \) with \( \dim(E) = i - 1 \) and \( \dim(F) = 1 \), then \( H(P, f) \) supports a non-hyperbolic ergodic measure whose Lyapunov exponent along the bundle \( F \) vanishes and whose support equals \( H(P, f) \).

Based on the results of [DG, BDG] and combined to the results of [BCDG, Wa], a recent work of [CCGWY] shows that for \( C^1 \)-generic diffeomorphisms, if a homoclinic class is not hyperbolic, then it supports a non-hyperbolic ergodic measure. Moreover, if the homoclinic class contains periodic orbits of different indices, then one can obtain a non-hyperbolic ergodic measure whose support is the whole homoclinic class.

The non-hyperbolic ergodic measures in the discussions above can be only assured to have one vanishing Lyapunov exponent. The example in [BD1] shows that there exist iterated function systems (IFS) persistently exhibiting non-hyperbolic ergodic measures with all the Lyapunov exponents vanished. Here, we restate the question posed in [BD1].

**Question 1.** Does there exist an open set \( U \) of diffeomorphisms such that for any \( f \in U \), there exists an ergodic measure with more than one vanishing Lyapunov exponents ?

Also one can ask a similar question for homoclinic classes.

**Question 2.** Under what kind of assumption, does there exist a non-hyperbolic ergodic measure supported on a homoclinic class with more than one vanishing Lyapunov exponents ?
Inspired by Theorem 1 we would like to consider the question that:

If a homoclinic class contains periodic points of indices $i$ and $i + k$ respectively, $k > 0$, does there exist an ergodic measure supported on the homoclinic class such that all its $(i + 1)\text{th}$ to $(i + k)\text{th}$ Lyapunov exponents vanish?

Obviously, it is not true if $H(P, f)$ admits a dominated splitting of index $i + j$ for some $1 \leq j \leq k - 1$. What happens when there is no such dominated splitting over the class? We state our first result, which partially answers Question 1.

**Theorem A.** For generic diffeomorphism $f \in \text{Diff}^1(M)$, consider a hyperbolic periodic orbit $P$. Assume the homoclinic class $H(P, f)$ satisfies the following properties:

- $H(P, f)$ contains hyperbolic periodic orbits of indices $i$ and $i + k$ respectively, where $i, k > 0$;
- for any integer $1 \leq j \leq k - 1$, there is no dominated splitting of index $i + j$ over $H(P, f)$.

Then there exists an ergodic measure $\nu$ whose support is $H(P, f)$ such that the $(i + j)\text{th}$ Lyapunov exponent of $\mu$ vanishes for any $1 \leq j \leq k$.

**Remark 1.1.** Considering the support of the non-hyperbolic ergodic measure, in Theorem A the case when $k = 1$ can be obtained as a combination of Theorem 1 above and Theorem B of [CCGWY]: if there is a dominated splitting into three bundles, then one can apply Theorem 1 otherwise Theorem B of [CCGWY] concludes.

One has the following direct corollary of Theorem A which generalizes the “moreover” part of Theorem 1 in the sense that one can obtain non-hyperbolic ergodic measure with more than one vanishing Lyapunov exponents.

**Corollary 1.2.** For generic diffeomorphism $f \in \text{Diff}^1(M)$, consider a hyperbolic periodic orbit $P$. Assume that the homoclinic class $H(P, f)$ has a dominated splitting $T_{H(P, f)}M = E \oplus F \oplus G$. Assume, in addition, that the followings are satisfied:

- $H(P, f)$ contains hyperbolic periodic orbits of indices $\dim(E)$ and $\dim(E \oplus F)$ respectively,
- the center bundle $F$ has no finer dominated splitting.

Then there exists an ergodic measure $\nu$ whose Lyapunov exponents along the bundle $F$ vanish, and whose support is $H(P, f)$.

We point out that the assumption of existence of both periodic orbits of indices $\dim(E)$ and $\dim(E \oplus F)$ is important. We can give an example based on the results of [BB, BV], showing that if there is no periodic orbit of index $\dim(E \oplus F)$ inside the homoclinic class, the conclusion of Corollary 1.2 may not be valid. Actually, in the example, the center bundle $F$ has no finer domination but $F$ is uniformly volume expanding, which forbids to have non-hyperbolic ergodic measures with all zero center Lyapunov exponents (See the details in Section 3). One can also ask the following question, to consider the case when the center bundle $F$ is not volume expanding.

**Question 3.** In the assumption of Corollary 1.2, if we replace the existence of hyperbolic periodic orbit of index $\dim(E \oplus F)$ by the existence of hyperbolic periodic orbit whose absolute Jacobian along center bundle $F$ is strictly less than 1, does there exist an ergodic measure $\nu$ supported on $H(P, f)$ such that all the Lyapunov exponents of $\nu$ along $F$ vanish?

The following theorem gives an affirmative answer to Question 3. For a periodic orbit $Q = \text{Orb}(q)$, we denote by $\pi(Q)$ (or $\pi(q)$) its period.

**Theorem B.** For generic diffeomorphism $f \in \text{Diff}^1(M)$, consider a hyperbolic periodic orbit $P$. Assume that the homoclinic class $H(P, f)$ admits a dominated splitting $T_{H(P, f)}M = E \oplus F \oplus G$. Assume, in addition, that we have the following:
– $H(P, f)$ contains a hyperbolic periodic orbit of index $\dim(E)$ and a hyperbolic periodic point $q \in H(P, f)$ such that
\[ |\text{Jac}(Df^n|_{F(q)})| < 1; \]
– the center bundle $F$ has no finer dominated splitting.

Then there exists an ergodic measure $\nu$ whose support is $H(P, f)$, such that all the Lyapunov exponents of $\nu$ along $F$ vanish.

**Remark 1.3.** (1) It’s clear that Corollary 1.2 is also implied by Theorem B.

(2) We point out here that, under the assumption of Theorem B, by applying Theorem 1 of [BCDG] inductively, one can obtain that $H(P, f)$ contains periodic points with indices equal to $i + k - 1$ whose $(i + k)^{th}$ Lyapunov exponent (positive but) arbitrarily close to 0.

Let’s explain a little bit about the relation between Theorem B and Corollary 1.2. If the index of $q$ in Theorem B is no less than $i + k$, then we can conclude Theorem B directly from Corollary 1.2.

If the index of $q$ is smaller than $i + k$, indeed by the no-domination assumption along $F$ and the technics of [BB], we can do an arbitrarily small perturbation to get a new hyperbolic periodic orbit of index $i + k$. However, we do not know whether or not the new generated periodic orbits are still contained in the homoclinic class.

The proof of Theorem B is not by finding a hyperbolic periodic orbit of index $i + k$ in the homoclinic class. We use a little different strategy from the proof of Theorem A to give the proof.

A more general statement than Theorem B can be expected to be true. We state it as the following question.

**Question 4.** For generic $f \in \text{Diff}^1(M)$, consider the finest dominated splitting $E_1 \oplus \cdots \oplus E_k$ over a homoclinic class $H(P, f)$. Assume that there exist two saddles $q_1, q_2$ in the class such that $|\text{Jac}(Df^n|_{E_1(q_1)})| > 1$ and $|\text{Jac}(Df^n|_{E_1(q_2)})| < 1$ where $i \leq j$. Then for any $i \leq l \leq j$, does there exist an ergodic measure whose Lyapunov exponents along the bundle $E_l$ all vanish?

**Organization of the paper**

In Section 2, we give some definitions and some known results. Section 3 and Section 4 give the proof of Theorem A and Theorem B respectively. Section 5 gives an example which shows that the assumption of existence of both periodic orbits of index $\dim(E)$ and $\dim(E \oplus F)$ in Corollary 1.2 is important.

**2 Preliminary**

In this section, we collect the notations and known results that we need in this paper.

**2.1 Lyapunov exponents**

In this subsection, we state an expression of Lyapunov exponents for an ergodic measure.

Let $f \in \text{Diff}^1(M)$ and $\nu$ be an $f$-ergodic measure. We denote by
\[ \chi_1(\nu, f) \leq \cdots \leq \chi_d(\nu, f) \]
all the Lyapunov exponents of $\nu$ counted by multiplicity. We define a continuous function on $M$ as:
\[ L^n_i(x, f) = \frac{1}{n} \log \| \wedge^i Df^n(x) \|. \]

Then, for $\nu$-a.e. $x \in M$, we have that
\[ \chi_i(\nu, f) = \lim_{n \to \infty} \left( L^n_{d-i+1}(x, f) - L^n_{d-i}(x, f) \right). \]
2.2 Chain recurrence

Let \((X, d)\) be a compact metric space and \(f\) be a homeomorphism on \(X\). Given two points \(x, y \in X\), we define the relation \(x \dashv y\) if and only if for any \(\epsilon > 0\), there exist finite points \(x = z_0, z_1, \ldots, z_k = y\), where \(k \geq 1\), such that

\[
d(f(z_i), z_{i+1}) \leq \epsilon, \quad \text{for any } 0 \leq i \leq k-1.
\]

We define the relation \(x \vdash \dashv y\) if and only if \(x \dashv y\) and \(y \dashv x\).

The chain recurrent set of \(f\) is defined as

\[
R(f) = \{x \in X : x \vdash \dashv x\}.
\]

It’s well known that \(\vdash \dashv\) is an equivalent relation on \(R(f)\). Hence, \(R(f)\) can be decomposed into different equivalent classes, each of which is called a chain recurrence class.

Homoclinic classes can also be defined in the following way.

**Definition 2.1.** Assume that \(f\) is a diffeomorphism in \(\text{Diff}^1(M)\) and \(P, Q\) are two hyperbolic periodic orbits of \(f\). We say that \(P\) and \(Q\) are homoclinically related, if \(W^u(P)\) has non-empty transverse intersections with \(W^s(Q)\), and vice versa, denoted by \(W^u(P) \cap W^s(Q) \neq \emptyset\) and \(W^s(P) \cap W^u(Q) \neq \emptyset\). We call the closure of the set of periodic orbits homoclinically related to \(P\) the homoclinic class of \(P\) and denote it as \(H(P, f)\) or \(H(P)\) for simplicity.

The following lemma is from [BC].

**Lemma 2.2.** For \(C^1\)-generic diffeomorphisms, the chain recurrence class of a hyperbolic periodic orbit \(Q\) coincides with its homoclinic class \(H(Q)\).

2.3 A criterion to the ergodicity of convergence

The period of a periodic orbit \(P = \text{Orb}(p)\) is denoted by \(\pi(P)\). We define a relation between two periodic orbits called good approximation which is given in [DG, BDG].

**Definition 2.3.** Given a dynamical system \((K, f)\). Let \(X, Y\) be two periodic orbits of \(f\). We say that \(X\) is a \((\delta, \kappa)\)-good approximation of \(Y\), for some \(\delta > 0\) and \(\kappa \in (0, 1]\) if there exist a subset \(\tilde{X} \subset X\) and a map \(\Pi : \tilde{X} \to Y\) such that:

- \(#\tilde{X} / #X > \kappa;\)
- \(#(\Pi^{-1}(y))\) is independent of \(y\), where \(y\) belongs to \(Y;\)
- \(d(f^i(x), f^i(\Pi(x))) < \delta, \) for any \(i = 0, \cdots, \pi(Y) - 1\) and any \(x \in \tilde{X}\).

Here, we state a criterion which is first used in [GKN] and developed in [DG, BDG] showing that with some good approximation assumption, a sequence of periodic measures converges to an ergodic measure.

**Lemma 2.4 ([DG, BDG]).** Given a system \((K, f)\). Let \(\{X_n\}\) be a sequence of periodic orbits. Assume that \(X_{n+1}\) is a \((\delta_n, \kappa_n)\)-good approximation of \(X_n\) for each \(n \in \mathbb{N}\), where \(\{\delta_n\}\) and \(\{\kappa_n\}\) are two sequences of positive numbers no more than \(1\) satisfying:

\[
\sum_{n \geq 0} \delta_n < \infty \quad \text{and} \quad \prod_{n \geq 0} \kappa_n \in (0, 1].
\]

Then the dirac measure supported on \(X_n\) converges to an ergodic measure \(\nu\) and the support of \(\nu\) is given by

\[
\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} X_k.
\]
2.4 Perturbation technics

Let \( A_1, \ldots, A_l \in GL(d, \mathbb{R}) \) and denote by \( B = A_l \circ A_{l-1} \circ \cdots \circ A_1 \). Let \( \lambda_1(B), \ldots, \lambda_d(B) \) be the eigenvalues of \( B \), counted by multiplicity and satisfying

\[
|\lambda_1(B)| \leq \cdots \leq |\lambda_d(B)|.
\]

The \( i^{th} \) Lyapunov exponent of \( B \) is defined as

\[
\chi_i(B) = \frac{1}{l} \log |\lambda_i(B)|.
\]

We say that \( B \) has simple spectrum if all the Lyapunov exponents of \( B \) are mutually different.

We state a version of Theorem 4.11 in [BB] adapted to our situation. A similar result can be found in [G].

**Lemma 2.5.** For any \( d \geq 2, \epsilon > 0, \) and \( R > 1 \), there exist two positive integers \( T, l_0 \) such that:

Given \( l \) linear maps \( A_1, \ldots, A_l \in GL(d, \mathbb{R}) \) with \( l \geq l_0 \) such that \( \|A_i\|, \|A_i^{-1}\| < R \). Assume that \( B = A_l \circ A_{l-1} \circ \cdots \circ A_1 \) has no \( T \)-domination of index \( j \) for any \( j \in \{i_0 + 1, \cdots, i_0 + k_0 - 1\} \)

For any \( k_0 \) numbers \( \xi_1, \cdots, \xi_{k_0} \) satisfying:

- \( \xi_{k_0} \geq \cdots \geq \xi_1 \);
- \( \sum_{i=1}^j \xi_i \geq \sum_{i=1}^j \chi_{i_0+i}(B) \), for any \( j = 1, \cdots, k_0 \);
- \( \sum_{i=1}^{k_0} \xi_i = \sum_{i=1}^{k_0} \chi_{i_0+i}(B) \).

Then there exist \( l \) one-parameter families of linear maps \( \{(A_i, t)_{t \in [0, 1]}\}_{i=1}^l \) such that:

1. \( A_{i, 0} = A_i \) for each \( i \);
2. \( \|A_{i, t} - A_i\| < \epsilon \) and \( \|A_i^{-1, t} - A_i^{-1}\| < \epsilon \), for each \( i \) and any \( t \in [0, 1] \);
3. Consider the linear map \( B_t = A_{i, t} \circ A_{i-1, t} \circ \cdots \circ A_1, t \), then the Lyapunov exponents of \( B_t \) satisfy the following:

- \( \chi_j(B_t) = \chi_j(B) \), for any integer \( j \in [1, i_0] \cup [i_0 + k_0 + 1, d] \);
- \( \sum_{j=1}^{k_0} \chi_{i_0+j}(B_t) = \sum_{j=1}^{k_0} \chi_{i_0+j}(B) \), for any \( t \in [0, 1] \);
- For any \( j \in [1, k_0] \), the function \( \sum_{i=1}^j \chi_{i_0+i}(B_t) \) with respect to variable \( t \) is non-decreasing;
- \( \chi_{i_0+j}(B_1) = \xi_j \), for any \( j = 1, \cdots, k_0 \).

**Remark 2.6.** In particular, we can take \( \xi_1 = \cdots = \xi_{k_0} = \frac{1}{\epsilon_0} \sum_{i=1}^{k_0} \chi_{i_0+i}(B) \) in Lemma 2.5.

The following lemma shows that for periodic orbit of large period, we can do certain small perturbation to make it have simple spectrum.

**Lemma 2.7.** [BC Lemma 6.6] Given a positive number \( K \). For any \( \epsilon > 0 \), there exists an integer \( N \) such that for any \( n \geq N \) and any matrices \( A_1, \cdots, A_n \in GL(2, \mathbb{R}) \) satisfying that \( \|A_i\| < K \) and \( \|A_i^{-1}\| < K \) for any \( i = 1, \cdots, n \).

Then there exist matrices \( B_1, \cdots, B_n \in GL(2, \mathbb{R}) \) such that

- \( \|A_i - B_i\| < \epsilon \) and \( \|A_i^{-1} - B_i^{-1}\| < \epsilon \) for any \( i = 1, \cdots, n \);
- the matrix \( B_n \circ \cdots \circ B_1 \) has simple spectrum.
Remark 2.8. The original statement of Lemma 6.6 in [BC] is for the matrices in $\text{SL}(2,\mathbb{R})$, but with the assumption that the norm of the matrices and its inverse are uniformly bounded, the same conclusion is also true directly from [BC, Lemma 6.6].

We state a generalized Franks lemma by N. Gourmelon [Go], which allows us to do a Franks-type perturbation along a hyperbolic periodic orbit which keeps some homoclinic or heteroclinic intersections.

Lemma 2.9 (Franks-Gourmelon Lemma). Given $\epsilon > 0$, a diffeomorphism $f \in \text{Diff}^1(M)$ and a hyperbolic periodic orbit $Q = \text{Orb}(q)$ of period $n$. Consider $n$ one-parameter families of linear maps $\{(A_{i,t})_{i\in[0,1]}\}_{t=0}^{n-1}$ in $\text{GL}(d,\mathbb{R})$ satisfying the following properties:

- $A_{i,0} = Df(f^i(q))$ for any integer $i \in [0,n-1]$;
- $\| A_{i,t} - Df(f^i(q)) \| < \epsilon$ and $\| A_{i,t}^{-1} - Df^{-1}(f^{i+1}(q)) \| < \epsilon$, for any $t \in [0,1]$;
- $A_{n-1,t} \circ \cdots \circ A_{0,t}$ is hyperbolic for any $t \in [0,1]$.

Then for any neighborhood $U$ of $Q$, any number $\eta > 0$ and any pair of compact sets $K^s \subset W^s(Q,f)$ and $K^u \subset W^u(Q,f)$ which do not intersect $U$, there is a diffeomorphism $g \in \text{Diff}^1(M)$ which is $\epsilon$-$C^1$-close to $f$, such that

- $g$ coincides with $f$ on $Q \cup M \setminus U$,
- $Dg(g^i(q)) = A_{i,1}$, for any $i \in \{0,1,\ldots,n-1\}$,
- $K^s \subset W^s(Q,g)$ and $K^u \subset W^u(Q,g)$.

Definition 2.10. Consider a diffeomorphism $f \in \text{Diff}^1(M)$. An invariant compact set $\Lambda$ is said to admit a partially hyperbolic splitting, if there is a splitting $T\Lambda M = E^s \oplus E^c \oplus E^u$ such that, the splittings $(E^s \oplus E^c) \oplus E^u$ and $E^s \oplus (E^c \oplus E^u)$ are dominated splittings, and the bundle $E^s$ (resp. $E^u$) is uniformly contracting (resp. expanding). Moreover, at least one of the two extreme bundles $E^s$ and $E^u$ is non-degenerate.

Consider two hyperbolic periodic points $p$ and $q$ of indices $i$ and $i+k$ respectively. We say that $p$ and $q$ form a heterodimensional cycle if $W^s(P)$ has transverse intersections with $W^s(Q)$ along the orbit of some point $y$, and $W^s(P)$ has quasi-transverse intersections with $W^u(Q)$ along the orbit of some point $x$, i.e. $T_xW^s(P) + T_xW^u(Q)$ is a direct sum. We say $p$ and $q$ form a partially hyperbolic heterodimensional cycle, if the $f$-invariant compact set $\mathcal{C} = \text{Orb}(x) \cup \text{Orb}(y)$ admits a partially hyperbolic splitting of the form $T\mathcal{C} M = E^s \oplus E^c \oplus E^u$, where $\text{dim}(E^s) = i$ and $\text{dim}(E^c) = k$. Moreover, for any $x \in \mathcal{C}$, we denote by $W^{ss}(x)$ (resp. $W^{uu}(x)$) the strong stable manifold (resp. strong unstable manifold) of $x$ which is tangent to the bundle $E^s$ (resp. $E^u$) at $x$.

We have the following theorem from [BDPR] to obtain transition between two periodic orbits of different indices.

Theorem 2.11. [BDPR Theorem 3.1 and Lemma 3.5] Consider a diffeomorphism $f \in \text{Diff}^1(M)$. Let $p$ and $q$ be two hyperbolic periodic points of indices $i$ and $i+k$ respectively and denote by $P$ and $Q$ their orbits respectively. Assume that there exist dominated splitting $T_P M = E_1(P) \oplus E_2(P) \oplus E_3(P)$ and $T_Q M = E_1(Q) \oplus E_2(Q) \oplus E_3(Q)$ satisfying that $\text{dim}(E_1(P)) = \text{dim}(E_1(Q)) = i$ and $\text{dim}(E_2(P)) = \text{dim}(E_2(Q)) = k$. Assume, in addition, that $P$ and $Q$ form a heterodimensional cycle. Denote by $M_P$ and $M_Q$ the two linear maps:

\[ Df^{\pi(P)}(p) : T_P M \to T_P M \text{ and } Df^{\pi(Q)}(q) : T_Q M \to T_Q M. \]

Then for any $C^1$-neighborhood $\mathcal{U}$ of $f$, for any two neighborhoods $U_P$ and $U_Q$ of $P$ and $Q$ respectively, there are two matrices $T_0$ and $T_1$, and two integers $t_0$ and $t_1$, such that for any two positive integers $m$ and $n$, there is a diffeomorphism $g \in \mathcal{U}$ with a periodic point $p_1$, satisfying the following properties:

\[ Dg^{\pi(P)}(p_1) : T_{p_1} M \to T_{p_1} M \text{ and } Dg^{\pi(Q)}(q_1) : T_{q_1} M \to T_{q_1} M. \]
Theorem 2.13. \( g \) and \( Dg \) coincide with \( f \) and \( Df \) on \( P \cup Q \) respectively.

- For \( i = 1, 2, 3 \), we have that \( T_0(E_i(p)) = E_i(q) \) and \( T_1(E_i(q)) = E_i(p) \);

- The period of \( p_1 \) equals \( t_0 + t_1 + n\pi(P) + m\pi(Q) \);

- the matrix \( Dg^{\pi(p)}(p_1) : T_{p_1}M \to T_{p_1}M \) is conjugate to

\[
T_1 \circ M_Q^n \circ T_0 \circ M_P^n;
\]

- we denote by \( P_1 \) the orbit of \( p_1 \) under \( g \), then we have:

\[
\#(P_1 \cap U_P) \geq n\pi(P) \text{ and } \#(P_1 \cap U_Q) \geq m\pi(Q).
\]

Remark 2.12. By Lemma 4.13 in [BDPR], if the periodic orbits \( P \) and \( Q \) admit another dominated splitting of the same index, the two matrices \( T_0 \) and \( T_1 \) can be chosen to preserve the two dominated splitting at the same time.

2.5 Generic diffeomorphisms

Let \( f \in \text{Diff}^1(M) \), \( P \) and \( Q \) be two hyperbolic periodic orbits of \( f \). We say that \( P \) and \( Q \) are robustly in the same chain recurrence class, if there exists a \( C^1 \) small neighborhood \( U \) of \( f \) such that for any \( g \in U \), the continuation \( P_g \) of \( P \) and the continuation \( Q_g \) of \( Q \) are in the same chain recurrence class. A periodic orbit \( P \) is said to have simple spectrum, if the \( d \) Lyapunov exponents of \( P \) are mutually different. Denote by \( \text{Per}(f) \) the set of periodic points of \( f \).

The following theorem summarizes some generic properties for \( \text{Diff}^1(M) \), see for example [ABCDW, EC, BDPR, BDV, CCGWY, DG].

Theorem 2.13. There exists a residual subset \( \mathcal{R} \) of \( \text{Diff}^1(M) \) such that for any \( f \in \mathcal{R} \), we have the followings:

1. \( f \) is Kupka-Smale.

2. Any chain recurrence class containing a hyperbolic periodic orbit \( P \) coincides with the homoclinic class \( H(P, f) \). Hence two homoclinic classes either coincide or are disjoint.

3. Given a hyperbolic periodic orbit \( P \), there exists a neighborhood \( U \) of \( f \) such that the map \( g \mapsto H(P, g) \) is well defined and \( f \) is a continuous point of this map.

4. Given a homoclinic class \( H(P, f) \), for any hyperbolic periodic orbit \( Q \) contained in \( H(P, f) \), we have that \( P \) and \( Q \) are robustly in the same chain recurrence class.

5. Consider a non-trivial homoclinic class \( H(P, f) \), the set

\[
\{q \in \text{Per}(f) : \text{Orb}(q) \text{ has simple spectrum and is homoclinically related to } P\}
\]

is dense in \( H(P, f) \).

6. Consider a hyperbolic periodic orbit \( P \) of index \( i \), whose homoclinic class contains a hyperbolic periodic orbit \( Q \) of index \( i + k \) for some integer \( k > 0 \). If there exist dominated splitting \( T_PM = E_1(P) \oplus E_2(P) \oplus E_3(P) \) and \( T_QM = E_1(Q) \oplus E_2(Q) \oplus E_3(Q) \) satisfying that \( \dim(E_1(P)) = \dim(E_1(Q)) = i \) and \( \dim(E_2(P)) = \dim(E_2(Q)) = k \). Then arbitrarily \( C^1 \)-close to \( f \), there is a diffeomorphism \( g \), satisfying that:

- \( g \) and \( Dg \) coincide with \( f \) and \( Df \) on \( P \cup Q \) respectively,
Consider a hyperbolic periodic orbit $P$ with simple spectrum whose homoclinic class $H(P, f)$ is non-trivial. Then for any $\epsilon > 0, \delta > 0$ and $\kappa \in (0, 1)$, there is a hyperbolic periodic orbit $Q$ homoclinically related to $P$, such that the following properties are satisfied:

- $Q$ has simple spectrum and is $\epsilon$-dense in $H(P, f)$;
- $Q$ is a $(\delta, \kappa)$-good approximation of $P$;
- $|\chi_i(Q, f) - \chi_i(P, f)| < \epsilon$, for any $i \in \{1, 2, \cdots, d\}$.

### 3 Ergodic measure with multi-zero Lyapunov exponents for the case controlled by norm: Proof of Theorem A

#### 3.1 Proof of Theorem A

The following proposition is the main step for proving Theorem A.

**Proposition 3.1.** For generic $f \in \Diff^1(M)$. Consider a non-trivial homoclinic class $H(P, f)$ of a hyperbolic periodic orbit $P$ of index $i$. Assume that

- there is a hyperbolic periodic orbit $Q$ of index $i + k$ contained in $H(P, f)$, where $k \geq 1$;
- there is no dominated splitting of index $i + j$ over $H(P, f)$, for any $j = 1, 2, \cdots, k - 1$.

Then there is a constant $\chi > 0$ such that for any $\gamma > 0$ and any hyperbolic periodic orbit $P_0$ with simple spectrum which is homoclinically related to $P$, there is a hyperbolic periodic orbit $P_1$ homoclinically related to $P$, satisfying the following properties:

1. $P_1$ is $\gamma$-dense in $H(P, f)$ and has simple spectrum;
2. $\chi_{i+k}(P_1, f) < \frac{3}{4} \cdot \chi_{i+k}(P_0, f)$;
3. $P_1$ is a $(\gamma, 1 - \frac{\chi_{i+k}(P_0, f)}{\chi_{i+k}(P_0, f)})$-good approximation of $P_0$.

Using Proposition 3.1, we give the proof of Theorem A.

**Proof of Theorem A** By item 2 of Theorem 2.13, we can assume that $P$ is of index $i$. We take the positive constant $\chi$ from Proposition 3.1. We will inductively construct a sequence of periodic orbits $\{P_n\}$ with simple spectrum, a sequence of positive numbers $\{\gamma_n\}$ and a sequence of integers $\{N_n\}$ satisfying the following properties:

1. $\chi_{i+k}(P_{n+1}, f) < \frac{3}{4} \cdot \chi_{i+k}(P_n, f)$;
2. $P_n$ is homoclinically related to $P$ and is $\frac{1}{2\gamma_n}$-dense inside $H(P, f)$;
3. the constants $\gamma_n$ and $N_n$ satisfy that:
   - $\gamma_n < \frac{1}{2} \gamma_{n-1}$ and $N_n > N_{n-1}$;
   - for any point $x \in B_{2\gamma_n}(P_n) \cap H(P, f)$, we have that
     
     $0 < L_{d-1}^{N_n}(x, f) - L_{d-i-1}^{N_n}(x, f)$,
     
     $0 < L_{d-i-1}^{N_n}(x, f) - L_{d-i-k}^{N_n}(x, f) < 2k \cdot \chi_{i+k}(P_n, f)$;
4. $P_{n+1}$ is $(\gamma_n, 1 - \frac{\chi_{i+k}(P_n, f)}{\chi_{i+k}(P_n, f)})$-good approximation of $P_n$. 


Choice of $P_0$, $N_0$ and $\gamma_0$ First we construct for $n = 0$. By the item 3 and item 7 of Theorem 2.13 we can choose a hyperbolic periodic orbit $P_0$, with simple spectrum, which is homoclinically related to $P$ and is $\frac{1}{N}$-dense inside $H(P, f)$. Hence the item 2 is satisfied.

By the definition of the function $L^0_j(x, f)$, there exists an integer $N_0$ such that for any $y \in P_0$, we have that
\[
0 < L^0_{d-i}(y, f) - L^0_{d-i-1}(y, f),
\]
\[
0 < L^0_{d-i}(y, f) - L^0_{d-i-k}(y, f) < \frac{3k}{2} \cdot \chi_{i+k}(P_0, f).
\]
By the uniform continuity of the functions $L^0_{d-i}(x, f)$ and $L^0_{d-i-k}(x, f)$, there exists a number $\gamma_0 > 0$ such that for any point $x \in B_{2\gamma_0}(P_0) \cap H(P, f)$, we have that
\[
0 < L^0_{d-i}(x, f) - L^0_{d-i-1}(x, f),
\]
\[
0 < L^0_{d-i}(x, f) - L^0_{d-i-k}(x, f) < 2k \cdot \chi_{i+k}(P_0, f).
\]
Hence the item 3 is satisfied. Notice that we do not have to check the items 1, 4 for $n = 0$.

Construct $P_n$, $N_n$ and $\gamma_n$ inductively Assume that $P_j$, $N_j$ and $\gamma_j$ are already defined for any $j \leq n$. We apply $P_n$, $\gamma_n$, and $\frac{1}{2^{n+1}}$ to Proposition 3.1 then we get a periodic orbit $P_{n+1}$ with simple spectrum, satisfying that:

- $\chi_{i+k}(P_{n+1}, f) < \frac{3}{2} \cdot \chi_{i+k}(P_n, f)$;
- $P_{n+1}$ is homoclinically related to $P$ and is $\frac{1}{2^{n+1}}$-dense in $H(P, f)$;
- $P_{n+1}$ is $(\gamma_n, 1 - \frac{\chi_{i+k}(P_n, f)}{\chi_{i+k}(P_n, f)})$-good approximation of $P_n$.

Then the items 1, 2, 4 are satisfied.

By the definition of the function $L^n_j(x, f)$, there is an integer $N_{n+1} > N_n$ satisfying that: for any $y \in P_{n+1}$, we have that
\[
0 < L^{N_{n+1}}_{d-i}(y, f) - L^{N_{n+1}}_{d-i-1}(y, f),
\]
\[
0 < L^{N_{n+1}}_{d-i}(y, f) - L^{N_{n+1}}_{d-i-k}(y, f) < \frac{3k}{2} \cdot \chi_{i+k}(P_{n+1}, f).
\]
By the uniform continuity of the functions $L^{N_{n+1}}_{d-i}(x, f)$ and $L^{N_{n+1}}_{d-i-k}(x, f)$, there exists a number $\gamma_{n+1} \in (0, \frac{1}{4} \gamma_n)$ such that for any point $x \in B_{2\gamma_{n+1}}(P_{n+1}) \cap H(P, f)$, we have
\[
0 < L^{N_{n+1}}_{d-i}(x, f) - L^{N_{n+1}}_{d-i-1}(x, f),
\]
\[
0 < L^{N_{n+1}}_{d-i}(x, f) - L^{N_{n+1}}_{d-i-k}(x, f) < 2k \cdot \chi_{i+k}(P_{n+1}, f).
\]

End of proof of Theorem A By Lemma 2.4 the sequence of ergodic measures $\delta_{P_n}$ converges to an ergodic measure $\nu$ whose support is $H(P, f)$. We will show that $\nu$ has $k$ vanishing Lyapunov exponents.

Claim 3.2. The $(i + j)^{th}$ Lyapunov exponent of $\nu$ equals zero, for any $j = 1, 2, \cdots, k$.

Proof. By Definition 2.3 there exist a subset $P_n$ of $P_0$ and a map $\Pi_n : \bar{P}_n \mapsto P_{n-1}$ for each $n \geq 2$. Consider the set $K_n = \Pi_n^{-1} \circ \Pi_{n-1}^{-1} \circ \cdots \circ \Pi_1^{-1}(P_0)$, then we have that
\[
\delta_{P_n}(K_n) \geq \prod_{i=0}^{n-1} \left(1 - \frac{\chi_{i+k}(P_i, f)}{\chi_{i+k}(P_i, f)}\right).
\]
We denote by \( K = \bigcap_{n=1}^{\infty} \bigcup_{l=1}^{\infty} K_l \),
then we have that \( \nu(K) \geq \lim_{n \to \infty} \delta_{P_n}(K_n) > 0 \).

On the other hand, for any point \( x \in B_{2\gamma_n}(P_n) \cap H(P,f) \), we have that
\[
0 < L_{d-i}^N(x,f) - L_{d-i-1}^N(x,f),
0 < L_{d-i}^N(x,f) - L_{d-i-k}^N(x,f) < 2k \cdot \chi_{i+k}(P_n,f).
\]

Since \( P_{n+1} \) is a \((\gamma, 1 - \frac{\chi_{i+k}(P_n)}{\chi_i(P_n)})\)-good approximation of \( P_n \), we have that \( K \) is contained in the \( \sum_{i=n}^{\infty} \gamma_i \) neighborhood of \( P_n \), therefore is contained in \( 2\gamma_n \) neighborhood of \( P_n \). As a consequence, for any \( y \in K \), we have the following
\[
0 < L_{d-i}^N(y,f) - L_{d-i-1}^N(y,f),
\]
where
\[
0 < L_{d-i}^N(y,f) - L_{d-i-k}^N(y,f) < 2k \cdot \frac{3^n}{4^n} \cdot \chi_{i+k}(P,f).
\]

Since \( \nu \) is ergodic, for \( \nu \)-a.e. point \( y \), we have that
\[
\sum_{j=1}^{k} \chi_{i+j}(\nu,f) = \lim_{n \to +\infty} \left( L_{d-i}^n(y,f) - L_{d-i-1}^n(y,f) \right),
\]
\[
\chi_{i+1}(\nu,f) = \lim_{n \to +\infty} \left( L_{d-i}^n(y,f) - L_{d-i-1}^n(y,f) \right).
\]

By the fact that \( \nu(K) > 0 \) and the formulas (3.1) and (3.2), we can see that
\[
\sum_{j=1}^{k} \chi_{i+j}(\nu,f) = 0.
\]

By the formulas (5.1) and (5.4), we get that \( \chi_{i+1}(\nu,f) \geq 0 \). Then by the fact that \( \chi_{i+1}(\nu,f) \leq \chi_{i+2}(\nu,f) \leq \cdots \leq \chi_{i+k}(\nu,f) \), we have that
\[
\chi_{i+j}(\nu,f) = 0, \text{ for any } j = 1, 2, \cdots, k.
\]

This ends the proof of Theorem A. \( \square \)

Now it remains to prove Proposition 3.1

3.2 Good approximation with weaker center Lyapunov exponents: Proof of Proposition 3.1

The proof of Proposition 3.1 is based on the following perturbation Lemma:

**Lemma 3.3.** Consider a diffeomorphism \( f \in \text{Diff}^1(M) \). Let \( P \) and \( Q \) be two hyperbolic periodic orbits of indices \( i \) and \( i+k \) respectively. Assume that \( Q \) and \( P \) form a partially hyperbolic heterodimensional cycle \( K \) with the splitting \( T_KM = E^s \oplus E^c \oplus E^u \). Assume, in addition, that
\[
\chi_{i+j}(P,f) = \log \mu > 0 \text{ and } \chi_{i+j}(Q,f) = \log \lambda < 0, \text{ for any } j = 1, 2, \cdots, k. \quad (\star)
\]

Then for any \( \gamma > 0 \) and any \( C^1 \) neighborhood \( U \) of \( f \), there exist a diffeomorphism \( g \in U \) and a hyperbolic periodic orbit \( P_1 \) of \( g \) such that

1. \( P_1 \) has simple spectrum;
2. \( g \) and \( Dg \) coincide with \( f \) and \( Df \) on the set \( P \cup Q \) respectively;
3. \( \frac{1}{4} \cdot \chi_{i+k}(P, g) < \chi_{i+1}(P_1, g) < \chi_{i+k}(P_1, g) < \frac{1}{2} \cdot \chi_{i+k}(P, g) \);
4. \( P_1 \) is a \((\gamma, 1 + \frac{\log \mu}{2 \log \lambda - \log \mu})\)-good approximation of \( P \);
5. \( W^{ss}(P_1) \) has transverse intersections with \( W^s(P) \) and \( W^{uu}(P_1) \) has transverse intersections with \( W^u(Q) \).

**Remark 3.4.** If \( P \) and \( Q \) are robustly in the same chain recurrence class, the last item of Lemma 3.3 implies that \( P_1 \) is robustly in the same chain recurrence class with \( P \) and \( Q \).

The idea of the proof of Lemma 3.3 is that we mix two hyperbolic periodic orbits of different indices to get a new periodic orbit with weaker center Lyapunov exponents.

**Proof of Lemma 3.3.** We fix a small number \( \gamma > 0 \) and a neighborhood \( \mathcal{U} \) of \( f \). There exists \( \epsilon > 0 \) such that the \( \epsilon \) neighborhood of \( f \) is contained in \( \mathcal{U} \). There is a small number \( 0 < \theta < 1 \), such that for any \( k \in \mathcal{U} \) and any two points \( z_1, z_2 \) satisfying \( d(z_1, z_2) < \theta \cdot \gamma \), we have that

\[
d(h^i(z_1), h^i(z_2)) < \frac{\gamma}{2}, \quad \text{for any } i \in [-\pi(P), \pi(P)].
\]

We take two neighborhoods \( U_P \) and \( U_Q \) of \( P \) and \( Q \) respectively, such that \( U_P \) is contained in the \( \theta \cdot \gamma \)-neighborhood of \( P \) and is disjoint from \( U_Q \).

**Construction of the periodic orbit \( P_1 \)** Let \( P = \text{Orb}(p) \) and \( Q = \text{Orb}(q) \). We denote by \( M_P \) and \( M_Q \) the two linear maps:

\[
Df^{\pi(P)}(p) : T_p\mathcal{M} \to T_{p\mathcal{M}} \quad \text{and} \quad Df^{\pi(Q)}(q) : T_q\mathcal{M} \to T_{q\mathcal{M}}.
\]

Since \( P \) and \( Q \) form a partially hyperbolic heterodimensional cycle \( K \), by Theorem 2.11 there are two matrices \( T_0, T_1 \) and two integers \( t_0, t_1 \) such that for any two integers \( m \) and \( n \), there is a diffeomorphism \( g \), which is \( \frac{\lambda}{4} \)-C\(^1\)-close to \( f \) and has a periodic orbit \( P_1 = \text{Orb}(p_1, g) \), satisfying the following properties:

- \( g \) and \( Dg \) coincide with \( f \) and \( Df \) respectively on \( P \cup Q \),
- the matrix \( Dg^{\pi(p_1)}(p_1) : T_{p_1}\mathcal{M} \to T_{p_1}\mathcal{M} \) is conjugate to \( T_1 \circ M^n_{Q} \circ T_0 \circ M^m_{P} \),
- \( \pi(p_1) = t_0 + t_1 + n\pi(P) + m\pi(Q) \),
- \( \#(P_1 \cap U_P) \geq n\pi(P) \), and \( \#(P_1 \cap U_Q) \geq m\pi(Q) \).

Moreover, by the continuity of partial hyperbolicity and the local stable and unstable manifolds of hyperbolic periodic orbit, by taking \( U_P \) and \( U_Q \) small enough at first, we have that \( W^{ss}(P_1, g) \) intersects \( W^s_{loc}(P, g) \) transversely and \( W^{uu}(P_1, g) \) intersects \( W^u_{loc}(Q, g) \) transversely.

By the second item of Theorem 2.11 we can take proper coordinates at \( T_P\mathcal{M} \) and \( T_Q\mathcal{M} \), under which we have:

\[
M_P = \begin{pmatrix} A_s & 0 & 0 \\ 0 & A_c & 0 \\ 0 & 0 & A_u \end{pmatrix}, \quad M_Q = \begin{pmatrix} B_s & 0 & 0 \\ 0 & B_c & 0 \\ 0 & 0 & B_u \end{pmatrix}
\]

\[
T_1 = \begin{pmatrix} C_s & 0 & 0 \\ 0 & C_c & 0 \\ 0 & 0 & C_u \end{pmatrix}, \quad T_0 = \begin{pmatrix} D_s & 0 & 0 \\ 0 & D_c & 0 \\ 0 & 0 & D_u \end{pmatrix}
\]
Choice of the integers \(m\) and \(n\) We will adjust \(m, n\) to get the periodic orbit that satisfies the properties stated in Lemma 3.3 We take \(\eta > 0\) which will be decided later.

Claim 3.5. There exists an integer \(N_\eta\) such that for any \(m \geq N_\eta\) and \(n \geq N_\eta\), we have that all the center Lyapunov exponents of \(P_1\) belong to the interval:

\[
\left[ \frac{m \cdot \pi(P) \cdot \log \mu + n \cdot \pi(Q) \cdot \log \lambda}{m \cdot \pi(P) + n \cdot \pi(Q)} - 2\eta, \frac{m \cdot \pi(P) \cdot \log \mu + n \cdot \pi(Q) \cdot \log \lambda}{m \cdot \pi(P) + n \cdot \pi(Q)} + 2\eta \right].
\]

Proof. By the Equation (\(\star\)) in the assumption of Lemma 3.3 there exists an integer \(N_1(\eta)\) such that for any \(m, n \geq N_1(\eta)\), we have that

\[
\log \mu - \eta < \frac{1}{m \cdot \pi(P)} \log m(A_m^c) \leq \frac{1}{m \cdot \pi(P)} \log \| A_m^c \| < \log \mu + \eta;
\]

\[
\log \lambda - \eta < \frac{1}{n \cdot \pi(Q)} \log m(B_n^c) \leq \frac{1}{n \cdot \pi(Q)} \log \| B_n^c \| < \log \lambda + \eta.
\]

As a consequence, for any unit vector \(v \in E^c(P_1)\) and \(k \in \mathbb{N}\), we have that

\[
\| Dg^{k \cdot \pi(P_1)} v \| \leq \left( \| C_c \| \cdot \| D_c \| \right)^k \exp \left( k \cdot m \cdot \pi(P) \cdot (\log \mu + \eta) + k \cdot n \cdot \pi(Q) \cdot (\log \lambda + \eta) \right),
\]

\[
\| Dg^{k \cdot \pi(P_1)} v \| \geq \left( m(C_c) \cdot m(D_c) \right)^k \exp \left( k \cdot m \cdot \pi(P) \cdot (\log \mu - \eta) + k \cdot n \cdot \pi(Q) \cdot (\log \lambda - \eta) \right).
\]

Hence,

\[
\frac{1}{k \cdot \pi(P_1)} \log \| Dg^{k \cdot \pi(P_1)} v \| \leq \frac{\log (\| C_c \| \cdot \| D_c \|)}{\pi(P_1)} + \frac{m \cdot \pi(P) \cdot (\log \mu + \eta) + n \cdot \pi(Q) \cdot (\log \lambda + \eta)}{\pi(P_1)}.
\]

\[
\frac{1}{k \cdot \pi(P_1)} \log \| Dg^{k \cdot \pi(P_1)} v \| \geq \frac{\log (m(C_c) \cdot m(D_c))}{\pi(P_1)} + \frac{m \cdot \pi(P) \cdot (\log \mu - \eta) + n \cdot \pi(Q) \cdot (\log \lambda - \eta)}{\pi(P_1)}.
\]

By the fact that \(\pi(P_1) = m \pi(P) + n \pi(Q) + t_0 + t_1\) and the matrices \(C_c, D_c\) are independent of \(m\) and \(n\), there exists an integer \(N_2(\eta)\) such that for any \(m, n \geq N_2(\eta)\), we have that

\[
\cdot \frac{-\eta}{2} < \frac{\log (m(C_c) \cdot m(D_c))}{\pi(P_1)} \leq \frac{\log (\| C_c \| \cdot \| D_c \|)}{\pi(P_1)} < \frac{\eta}{2},
\]

\[
\cdot \frac{m \cdot \pi(P) \cdot \log \mu + n \cdot \pi(Q) \cdot \log \lambda}{\pi(P_1)} \leq \frac{m \cdot \pi(P) \cdot \log \mu + n \cdot \pi(Q) \cdot \log \lambda}{m \pi(P) + n \pi(Q)} < \frac{\eta}{2}.
\]

We take \(N_\eta = \max\{N_1(\eta), N_2(\eta)\}\). When \(m, n \geq N_\eta\), we have that all the center Lyapunov exponents of \(P_1\) would belong to the interval:

\[
\left[ \frac{m \cdot \pi(P) \cdot \log \mu + n \cdot \pi(Q) \cdot \log \lambda}{m \cdot \pi(P) + n \cdot \pi(Q)} - 2\eta, \frac{m \cdot \pi(P) \cdot \log \mu + n \cdot \pi(Q) \cdot \log \lambda}{m \cdot \pi(P) + n \cdot \pi(Q)} + 2\eta \right].
\]

This ends the proof of Claim 3.5 \(\square\)

To guarantee the item 3, we only need that

\[
\frac{m \pi(P) \cdot \log \mu + n \pi(Q) \cdot \log \lambda}{m \pi(P) + n \pi(Q)} + 2\eta < \frac{1}{2} \log \mu
\]

and

\[
\frac{m \pi(P) \cdot \log \mu + n \pi(Q) \cdot \log \lambda}{m \pi(P) + n \pi(Q)} - 2\eta > \frac{1}{4} \log \mu.
\]
By the choice of the numbers $\theta$ and $\gamma$, to guarantee the item 4, we only need that
\[
\frac{m\pi(P)}{m\pi(P) + n\pi(Q)} > 1 + \frac{\chi_{i+k}(P, g)}{2\chi_{i+k}(Q, g) - \chi_{i+k}(P, g)} = 1 + \frac{\log \mu}{2 \log \lambda - \log \mu}.
\] (3.7)

By calculation, to satisfy the inequalities (3.5), (3.6) and (3.7), we only have to show that there exist $m, n$ large enough such that the following is satisfied:
\[
\max \left\{ \frac{-2 \log \lambda}{\log \mu}, \frac{\log \mu - 4 \log \lambda + 8\eta}{3 \log \mu - 8\eta} \right\} < \frac{m\pi(P)}{n\pi(Q)} < \frac{\log \mu - 2 \log \lambda - 4\eta}{\log \mu + 4\eta}.
\] (3.8)

When $\eta$ is chosen small, we have the following inequality
\[
\max \left\{ \frac{-2 \log \lambda}{\log \mu}, \frac{\log \mu - 4 \log \lambda + 8\eta}{3 \log \mu - 8\eta} \right\} < \frac{\log \mu - 2 \log \lambda - 4\eta}{\log \mu + 4\eta}.
\] (3.9)

By Claim 3.5, the inequality (3.9) and the density of rational numbers on real line, there exist $m, n$ arbitrarily large satisfying the inequality (3.8).

By an arbitrarily $C^1$ small perturbation, the eigenvalues of the periodic orbit $P_1$ are of multiplicity one (might have complex eigenvalue). Since the period of $P_1$ can be chosen arbitrarily large, by Lemma 2.7, after another small Franks-type perturbation, we have that the periodic orbit $P_1$ has simple spectrum.

This ends the proof of Lemma 3.3.

**Remark 3.6.** One can see from the proof of Lemma 3.3 that the perturbation is done in very small neighborhood of the heterodimensional cycle $K$.

Now we are ready to give the proof of Proposition 3.1.

**Proof of Proposition 3.1.** We can see that the properties stated in Proposition 3.1 are persistent under $C^1$ small perturbation. Let $\mathcal{R}$ be the residual subset of $\text{Diff}^1(M)$ from Theorem 2.13. Notice that for any $f \in \mathcal{R}$, by the item 3 of Theorem 2.13, there is a periodic orbit $Q_0$ with simple spectrum which is homoclinically related to $Q$. We take $\chi = -\chi_{i+k}(Q_0, f) > 0$.

We only need to show that given $f \in \mathcal{R}$, for any $\zeta > 0$ and $\gamma > 0$, there are a diffeomorphism $g$ which is $\zeta$-$C^1$-close to $f$ and a hyperbolic periodic orbit $P_1$ of $g$, such that the following properties are satisfied:

1. $g$ coincides with $f$ on $P_0 \cup Q_0$;
2. $P_1$ is robustly in the chain recurrence class of $P_g$;
3. $P_1$ has simple spectrum and the Hausdorff distance $d_H(P_1, H(P_g, g)) < \gamma$;
4. $\chi_{i+k}(P_1, g) < \frac{1}{2} \cdot \chi_{i+k}(P_0, g)$;
5. $P_1$ is a $(\gamma, 1 - \chi_{i+k}(P_0, f), \chi_{i+k}(P_0, f))$-good approximation of $P_0$.

Then Proposition 3.1 can be proved by a standard Baire argument.

By item 4 of Theorem 2.13, we can require that $\zeta$ is chosen small enough such that after any $\zeta$-perturbation, the continuations of $P$, $Q$, $P_0$, and $Q_0$ are still robustly in the same chain recurrence class. We take $0 < \epsilon < \frac{1}{2}$, then there exist $T > 0$ and $l_0$ satisfying Lemma 2.5.
**Perturb to get a heterodimensional cycle** Since $H(P, f)$ admits no dominated splitting of index $j$ for any $j \in \{i+1, \cdots, i+k-1\}$, there is a number $\delta_0 \in (0, \frac{1}{2})$ such that for any compact invariant subset $\Lambda$ of $H(P, f)$, if $d_H(\Lambda, H(P, f)) < \delta_0$, then $\Lambda$ admits no $T$-dominated splitting of index $j$ for any $j \in \{i+1, \cdots, i+k-1\}$.

We fix a positive number $\delta < \min\{\delta_0, \frac{1}{3}\chi_{i+k}(P_0, f)\}$ small enough such that the following is satisfied:

$$\frac{\chi_{i+k}(P_0, f) + \delta}{-2\chi_{i+k}(Q_0, f) + \chi_{i+k}(P_0, f) - \delta} < \frac{\chi_{i+k}(P_0, f)}{-2\chi_{i+k}(Q_0, f) + \chi_{i+k}(P_0, f)}.$$  \hspace{1cm} \text{(3.10)}

We take a number $\kappa$ such that

$$\kappa \in \left(\frac{2\chi_{i+k}(P_0, f) - 3\chi_{i+k}(Q_0, f)}{3\chi_{i+k}(P_0, f) - 3\chi_{i+k}(Q_0, f)}, 1\right).$$

We apply the item 7 of Theorem 2.13 to the constants $\delta$ and $\kappa$, then there exist two hyperbolic periodic orbits $P' = \text{Orb}(p')$ and $Q' = \text{Orb}(q')$ such that:

- $P'$ and $Q'$ are homoclinically related to $P_0$ and $Q_0$ respectively;
- Both $P'$ and $Q'$ are $\delta/2$ dense in $H(P, f)$ and have simple spectrum.
- $P'$ is a $(\frac{\kappa}{2}, \kappa)$-good approximation of $P_0$ and $Q'$ is $(\frac{\kappa}{2}, \kappa)$-good approximation of $Q_0$.
- For each $j \in \{1, \cdots, d\}$, we have that

$$|\chi_j(P', f) - \chi_j(P_0, f)| < \delta \text{ and } |\chi_j(Q', f) - \chi_j(Q_0, f)| < \delta.$$  \hspace{1cm} \text{(3.11)}

- Both of the periods of $P'$ and $Q'$ are larger than $l_0$.

By item 4 of Theorem 2.13, we can do an arbitrarily $C^1$ small perturbation, keeping $P'$ and $Q'$ homoclinically related to $P$ and $Q$ respectively and without changing the Lyapunov exponents of $P'$ and $Q'$, such that $P'$ and $Q'$ form a partially hyperbolic heterodimensional cycle. For simplicity, we still denote this diffeomorphism as $f$.

Notice that the periodic orbits $P'$ and $Q'$ have no $T$-domination of index $j$ for any $j \in \{i+1, \cdots, i+k-1\}$.

**Equalize the center Lyapunov exponents of both $P'$ and $Q'$** By Lemma 2.9 and Remark 2.6 there exist $\pi(P')$ one-parameter families $\{(A_t, t)\}_{t \in [0,1]}^{\pi(P')-1}$ and $\pi(Q')$ one-parameter families $\{(B_m, t)\}_{t \in [0,1]}^{\pi(Q')-1}$ in $GL(d, \mathbb{R})$ such that:

- $A_{t, 0} = \text{Df}(f^l(p'))$ and $B_{m, 0} = \text{Df}(f^m(q'))$, for any $l, m$;
- $\|A_{t, t} - \text{Df}(f^l(p'))\| < \epsilon$ and $\|A_{t, t}^{-1} - \text{Df}^{-1}(f^{l+1}(p'))\| < \epsilon$, for any $t \in [0, 1]$;
- $\|B_{m, t} - \text{Df}(f^m(q'))\| < \epsilon$ and $\|B_{m, t}^{-1} - \text{Df}^{-1}(f^{m+1}(q'))\| < \epsilon$, for any $t \in [0, 1]$;
- $A_{\pi(P')-1, t} \cdots A_{0, t}$ and $B_{\pi(Q')-1, t} \cdots B_{0, t}$ are hyperbolic, for any $t \in [0, 1]$;
- For any integer $s \in [1, i] \cup [i + k + 1, d]$, we have that

$$\chi_s(A_{\pi(P')-1, t} \cdots A_{0, t}) = \chi_s(P', f) \text{ and } \chi_s(B_{\pi(Q')-1, t} \cdots B_{0, t}) = \chi_s(Q', f);$$

- $\chi_i(A_{\pi(P')-1, 1} \cdots A_{0, 1}) = \chi_i+k(A_{\pi(P')-1, 1} \cdots A_{0, 1}) = \frac{1}{k} \left\{ \sum_{j=i+1}^{i+k} \chi_j(P', f) \right\}$
- $\chi_i(B_{\pi(Q')-1, 1} \cdots B_{0, 1}) = \chi_i+k(B_{\pi(Q')-1, 1} \cdots B_{0, 1}) = \frac{1}{k} \left\{ \sum_{j=i+1}^{i+k} \chi_j(Q', f) \right\}$. 

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Fix a small number $\eta > 0$. Since $P'$ and $Q'$ form a heterodimensional cycle and $P'$ is homoclinically related to $P_0$, there exist four points $x, y, z, w \in M$ such that

- $x \in W^s(P') \cap W^u(Q')$ and $y \in W^u(P') \cap W^s(Q')$;
- $z \in W^s(P') \cap W^u(P_0)$ and $w \in W^u(P') \cap W^s(P_0)$.

We take $K^s = \{x, z\}$ and $K^u = \{y, w\}$, and we choose a small neighborhood $U$ of $P'$ such that $U$ is disjoint from $\text{Orb}^-(x) \cup \text{Orb}^-(z)$, $\text{Orb}^+(y) \cup \text{Orb}^+(w)$, $Q'$, $Q_0$ and two homoclinic orbits between $Q'$ and $Q_0$, whose $\omega$-limit sets are $Q'$ and $Q_0$ respectively. By Lemma 3.3, there exists an $\epsilon$ perturbation $g_1$ whose support is contained in $U$ such that

- $g_1$ keeps $P'$;
- $Dg_1(f^l(p')) = A_{l,1}$, for any $l = 0, \cdots, \pi(P') - 1$;
- $x \in W^s(P', g_1) \cap W^u(Q', g_1)$ and $y \in W^u(P', g_1) \cap W^s(Q', g_1)$.
- $W^s(P', g_1)$ intersects $W^u(P_0, g_1)$ transversely at the point $z$ and $W^u(P', g_1)$ intersects $W^s(P_0, g_1)$ transversely at the point $w$.

Following the same way above, we choose a small neighborhood $V$ of $Q'$ which is disjoint from certain homoclinic intersection and some orbit segments, and we apply Lemma 3.3. At the end, we get an $\epsilon$ perturbation $g_2$ of $g_1$ such that

- For diffeomorphism $g_2$, the periodic orbits $P'$, $P$, $Q$ and $Q'$ are robustly in the same chain recurrence class;
- $P'$ and $Q'$ form a partially hyperbolic heterodimensional cycle;
- $Dg_2(f^l(p')) = A_{l,1}$ and $Dg_2(f^m(q')) = B_{m,1}$, for any integer $l \in [0, \pi(P') - 1]$ and $m \in [0, \pi(Q') - 1]$.

To sum up, the diffeomorphism $g_2$ is $2\epsilon$-$C^1$-close to $f$ and satisfies that:

**S1.** $g_2$ coincides with $f$ on $P_0 \cup Q_0 \cup P' \cup Q'$;

**S2.** $\chi_j(P_0, g_2) = \chi_j(P_0, f)$ and $\chi_j(Q_0, g_2) = \chi_j(Q_0, f)$, for any $j = 1, 2, \cdots, d$;

**S3.** $\chi_j(P', g_2) = \chi_j(P', f)$ and $\chi_j(Q', g_2) = \chi_j(Q', f)$, for any $j \in [1, i] \cup [i + k + 1, d]$;

**S4.** $\chi_{i+1}(P', g_2) = \chi_{i+k}(P', g_2)$ and $\chi_{i+1}(Q', g_2) = \chi_{i+k}(Q', g_2)$;

**S5.** $\sum_{j=i+1}^{i+k} \chi_j(P', g_2) = \sum_{j=i+1}^{i+k} \chi_j(P', f)$ and $\sum_{j=i+1}^{i+k} \chi_j(Q', g_2) = \sum_{j=i+1}^{i+k} \chi_j(Q', f)$.

As a consequence, the diffeomorphism $g_2$ satisfies the assumptions of Lemma 3.3.

**Construction of the periodic orbit $P_1$.** By Lemma 3.3 there exist a diffeomorphism $g$ which is $\epsilon$-$C^1$-close to $g_2$, hence is $\zeta$-$C^1$-close to $f$, and a hyperbolic periodic orbit $P_1$ of index $i$ for the diffeomorphism $g$ such that

- $\chi_{i+k}(P_1, g) < \frac{1}{2} \cdot \chi_{i+k}(P', g)$;
- $P_1$ has simple spectrum;
- $g$ coincides with $g_2$ in a small neighborhood of $P_0 \cup Q_0$;
• $g$ and $Dg$ coincide with $g_2$ and $Dg_2$ on $P' \cup Q'$ respectively;

• $P_1$ is a $(\frac{\gamma}{10}, 1 + \frac{x_{i+k}(P',g_2)}{2x_{i+k}(Q',g_2) + x_{i+k}(P',g_2)})$-good approximation of $P'$.

Moreover, by Remark 3.4 we have that $P_1$ is robustly in the same chain class with $P_g$ and $Q_g$. By the choice of $\delta$ and $\gamma$, we have that $d_H(P_1, H(P, g)) < \gamma$. Then the items 1, 2, 3 are satisfied.

By the properties S4 and S5, we have that

\[
0 < \chi_{i+k}(P', g_2) < \chi_{i+k}(P', f) \quad \text{and} \quad \chi_{i+k}(Q', g_2) < \chi_{i+k}(Q', f) < 0,
\]

which implies that

\[
-2\chi_{i+k}(P', g_2) + \chi_{i+k}(P', f) < \chi_{i+k}(P', f) < -2\chi_{i+k}(Q', f) + \chi_{i+k}(P', f).
\]

(3.12)

By the inequalities (3.10), (3.11) and (3.12), we have that

\[
\frac{\chi_{i+k}(P', g_2)}{-2\chi_{i+k}(Q', g_2) + \chi_{i+k}(P', f)} < \frac{\chi_{i+k}(P_0, f)}{-2\chi_{i+k}(Q_0, f) + \chi_{i+k}(P_0, f)}.
\]

Recall that $P'$ is a $(\frac{\gamma}{10}, \kappa)$ good approximation of $P_0$ (for the diffeomorphisms $f$ and $g$), hence we have that $P_1$ is a $(\gamma, \kappa \cdot (1 + \frac{\chi_{i+k}(P_0, f)}{2\chi_{i+k}(Q_0, f) - \chi_{i+k}(P_0, f)}))$ good approximation of $P_0$.

By the choice of $\kappa$, we have that

\[
\kappa \cdot \left(1 + \frac{\chi_{i+k}(P_0, f)}{\frac{1}{2} \chi_{i+k}(Q_0, f) - \chi_{i+k}(P_0, f)}\right)
\]

\[
> \left(\frac{\chi_{i+k}(P_0, f) - \frac{3}{2} \chi_{i+k}(Q_0, f)}{\frac{1}{2} \chi_{i+k}(P_0, f) - \frac{1}{2} \chi_{i+k}(Q_0, f)}\right) \cdot \left(1 + \frac{\chi_{i+k}(P_0, f)}{\frac{1}{2} \chi_{i+k}(Q_0, f) - \chi_{i+k}(P_0, f)}\right)
\]

\[
= \frac{\chi_{i+k}(Q_0, f) - \chi_{i+k}(P_0, f)}{\chi_{i+k}(Q_0, f) - \chi_{i+k}(P_0, f)}
\]

\[
= 1 - \frac{\chi_{i+k}(P_0, f)}{\chi_{i+k}(P_0, f)}
\]

Hence, $P_1$ is a $(\gamma, 1 - \frac{\chi_{i+k}(P_0, f)}{\chi_{i+k}(P_0, f)})$ good approximation of $P_0$. This implies that the item 5 is satisfied.

Besides, by the choice of $\delta$, we have the following estimation for the maximal center Lyapunov exponent of $P_1$:

\[
\chi_{i+k}(P_1, g) < \frac{1}{2} \cdot \chi_{i+k}(P', g) = \frac{1}{2} \cdot \frac{1}{k} \sum_{j=1}^{k} \chi_{i+j}(P', f) + \frac{1}{2} \cdot \frac{1}{k} \sum_{j=1}^{k} \chi_{i+j}(P_0, g) + \delta \leq \frac{3}{4} \cdot \chi_{i+k}(P_0, g).
\]

Hence the item 4 is satisfied. This ends the proof of Proposition 3.1. \qed

4 Ergodic measure with multi-zero Lyapunov exponents for the case controlled by Jacobian: Proof of Theorem B

Consider a diffeomorphism $f \in \text{Diff}^1(M)$ and a homoclinic class $H(P, f)$ admitting a dominated splitting of the form $T_{H(P, f)} M = E \oplus F \oplus G$. We denote by $k = \dim(F)$. For any periodic orbit
$Q = \text{Orb}(q)$ contained in $H(P_0)$. The mean Lyapunov exponent along the bundle $F$ of $Q$ is defined as

$$L^F(Q, f) = \frac{1}{k \cdot \pi(Q)} \log |\text{Jac}(Df^{\pi(Q)}|_{F(q)})|.$$ 

Notice that $L^F(Q, f)$ is the average of the Lyapunov exponents of $Q$ along the bundle $F$.

### 4.1 Proof of Theorem B

The main ingredient for the proof of Theorem B is the following proposition.

**Proposition 4.1.** For generic diffeomorphism $f \in \text{Diff}_1^1(M)$, consider a hyperbolic periodic orbit $P$ of index $i$. Assume the homoclinic class $H(P, f)$ admits a dominated splitting $T_{H(P, f)}M = E \oplus F \oplus G$, such that $\dim(E) = i$. Assume, in addition, that we have the following:

- $H(P, f)$ contains a hyperbolic periodic orbit $Q = \text{Orb}(q)$, whose index is no larger than $\dim(E \oplus F)$, such that $|\text{Jac}(Df^{\pi(Q)}|_{F(q)})| < 1$;
- the center bundle $F$ has no finer dominated splitting.

Then there exists a constant $\rho \in (0, 1)$ which only depends on $Q$, such that for any hyperbolic periodic orbit $P_0$ with simple spectrum, which is homoclinically related to $P$, and any $\gamma > 0$, there exists a hyperbolic periodic point $P_1$ with simple spectrum such that:

1. $L^F(P_1, f) < \rho \cdot L^F(P_0, f)$;
2. $P_1$ is homoclinically related to $P$ and is $\gamma$ dense inside $H(P, f)$;
3. $P_1$ is $(\gamma, 1 - \frac{L^F(P_0, f)}{L^F(P_0, f) - L^F(Q, f)})$ good approximation of $P_0$.

The proof of Proposition 4.1 is left to the next subsection. Now, we follow the strategy of the Proof of Theorem A to give the proof of Theorem B.

**Proof of Theorem B.** We denote by $i = \dim(E)$. By item 3, item 5 and item 7 of Theorem 2.13, we can assume that $P$ is of index $i$ and has simple spectrum. Let $\rho \in (0, 1)$ be the number in Proposition 4.1 which only depends on $Q$.

We will inductively get a sequence of periodic orbits $\{P_n\}$, a sequence of positive numbers $\{\epsilon_n\}$ and a sequence of integers $\{N_n\}$ satisfying the following properties:

- $\epsilon_n < \frac{1}{2} \epsilon_{n-1}$;
- $L^F(P_{n+1}) < \rho \cdot L^F(P_n)$;
- $P_{n+1}$ is homoclinically related to $P$ and is $\epsilon_n$ dense inside $H(P, f)$;
- $P_{n+1}$ is $(\epsilon_n, 1 - \frac{2L^F(P_n, f)}{2L^F(P_n, f) - L^F(Q, f)})$ good approximation of $P_n$;
- For any point $x \in B_{2\epsilon_n}(P_n) \cap H(P, f)$, we have that

$$0 < \frac{1}{N_n} \log m(Df^{N_n}|_{F(x)}) \leq \frac{1}{N_n} \log \|Df^{N_n}|_{F(x)}\| < 2 \chi_{i+k}(P_n).$$
Choice of \( P_0, N_0 \) and \( \epsilon_0 \) Let \( P_0 = P \), then there exists an integer \( N_0 \) large enough such that for any \( y \in P_0 \), we have that

\[
0 < \frac{1}{N_0} \log m(Df^{N_0}|_{F(y)}) \leq \frac{1}{N_0} \log \| Df^{N_0}|_{F(y)} \| < \frac{3}{2} \chi_{i+k}(P_0).
\]

By the uniform continuity of the functions \( \log \| Df^{N_0}|_{F(x)} \| \) and \( \log m(Df^{N_0}|_{F(x)}) \), there exists a number \( \epsilon_0 > 0 \) such that for any point \( x \in B_{2\epsilon_0}(P_0) \cap H(P, f) \), we have that

\[
0 < \frac{1}{N_0} \log m(Df^{N_0}|_{F(x)}) \leq \frac{1}{N_0} \log \| Df^{N_0}|_{F(x)} \| < 2\chi_{i+k}(P_0).
\]

Construct \( P_n, N_n \) and \( \epsilon_n \) inductively Assume that \( P_i, N_i \) and \( \epsilon_i \) are already defined for any \( i \leq n \). We apply \( P_n \) and \( \epsilon_n \) to the Proposition 4.1, then we get a periodic orbit \( P_{n+1} \) which is homoclinically related to \( P_n \) such that

- \( L^F(P_{n+1}) < \rho \cdot L^F(P_n) \);
- \( P_{n+1} \) is \( \epsilon_n \) dense in \( H(p, f) \);
- \( P_n \) is \( (\epsilon_n, 1 - \frac{2L^F(P_n, f)}{2L^F(P_n, f) - L^F(Q, f)}) \) good approximation of \( P_n \).

Then there exists an integer \( N_{n+1} \) large enough such that for any \( y \in P_{n+1} \), we have that

\[
0 < \frac{1}{N_{n+1}} \log m(Df^{N_{n+1}}|_{F(y)}) \leq \frac{1}{N_{n+1}} \log \| Df^{N_{n+1}}|_{F(y)} \| < \frac{3}{2} \chi_{i+k}(P_{n+1}).
\]

By the uniform continuity of the functions \( \log \| Df^{N_{n+1}}|_{F(x)} \| \) and \( \log m(Df^{N_{n+1}}|_{F(x)}) \), there exists a number \( \epsilon_{n+1} \in (0, \frac{1}{2} \epsilon_n) \) such that for any point \( x \in B_{2\epsilon_{n+1}}(P_{n+1}) \cap H(P, f) \), we have

\[
0 < \frac{1}{N_{n+1}} \cdot \log m(Df^{N_{n+1}}|_{F(x)}) \leq \frac{1}{N_{n+1}} \log \| Df^{N_{n+1}}|_{F(x)} \| < 2\chi_{i+k}(P_{n+1}).
\]

End of proof of Theorem B Since \( 1 - \frac{2L^F(P_n, f)}{2L^F(P_n, f) - L^F(Q, f)} \) exponentially tends to 1 and \( \sum\epsilon_n \) converges, by Lemma 2.3, the sequence of ergodic measures \( \delta_{P_n} \) converges to an ergodic measure \( \nu \) whose support is \( H(p, f) \).

Claim 4.2. The Lyapunov exponents of \( \nu \) along the center bundle \( F \) are all zero.

Notice that \( \chi_{i+k}(P_n) \leq k \cdot L^F(P_n) \leq k \cdot \rho^n L^F(P_0) \). The proof of Claim 4.2 follows the proof of the Claim 3.2. The only difference is that we control the sum of the center Lyapunov exponents by the function \( \frac{1}{N_n} \log \| Df^{N_n}|_{F} \| \) instead of the function \( L_d^{d-i-k} - L_d^{d-i-k} \).

This ends the proof of Theorem B

\[ \square \]

Now it remains to prove Proposition 4.1

4.2 Good approximation with weaker center Jacobian: Proof of Proposition 4.1

The proof of Proposition 4.1 is based on the following perturbation lemma:

Lemma 4.3. Let \( P \) and \( Q \) be two hyperbolic periodic orbits of \( f \in \text{Diff}^1(M) \) with different indices. Assume that
- $Q$ and $P$ form a partially hyperbolic heterodimensional cycle $K$. In other words, $K$ admits a partially hyperbolic splitting of the form

\[ T_K M = E^s \oplus E^c \oplus E^u, \]

where $\dim(E^s) = \text{Ind}(P)$ and $\dim(E^s \oplus E^c) = \text{Ind}(Q)$;

- there exists another dominated splitting over $K$ of the form

\[ T_K M = E^s \oplus F \oplus G \]

such that $\dim(F) \geq \dim(E^c)$;

- all the Lyapunov exponents of $Q$ along $E^c$ are equal.

- all the Lyapunov exponents of $P$ along $E^c$ are equal and are larger than $L^F(P, f)/2$;

- $L^F(Q, f) < 0$.

Then there exists a number $\rho \in (0, 1)$ which only depends on $Q$, such that for any $\gamma > 0$ and any $C^1$ neighborhood $U$ of $f$, there exists $g \in U$ together with a hyperbolic periodic orbit $P'$ of index $\text{Ind}(P)$, with simple spectrum such that

1. $g = f$ and $Dg = Df$ on $P \cup Q$;

2. $L^F(P', g) < \rho \cdot L^F(P, g)$;

3. $P'$ is $(\gamma, 1 - \frac{L^F(P, g)}{L^F(P, g) - L^F(Q, g)})$ good approximation of $P$;

4. $W^{ss}(P', g)$ has transverse intersections with $W^u(P, g)$ and $W^{uu}(P', g)$ has transverse intersections with $W^s(Q, g)$, corresponding to the partially hyperbolic splitting $T_K M = E^s \oplus E^c \oplus E^u$.

**Remark 4.4.**

1. Once again, if $P$ and $Q$ are robustly in the same chain recurrence class, the fourth item above implies that $P'$, $P$ and $Q$ are robustly in the same chain recurrence class;

2. Actually, the constant $\rho$ is only and continuously depends on the mean Lyapunov exponent of $Q$ along the bundle $E^c$ and the mean Lyapunov exponent of $Q$ along $F$.

The idea of the proof of Lemma 4.3 is that we mix two hyperbolic periodic orbits with different sign of mean Lyapunov exponents to get a new hyperbolic periodic orbit with weaker mean Lyapunov exponent along the bundle $F$.

Similar to Section 3, we complete the proof of Proposition 4.1 by proving Lemma 4.3. To prove Lemma 4.3, we first follow the strategy of the proof of Lemma 3.3 to linearize the system in a small neighborhood of the cycle $K$ by an arbitrarily small perturbation, then by another arbitrarily small perturbation, we get a periodic orbit. At the end, we will adjust the time of periodic orbit staying close to $P$ and $Q$ respectively.

**Proof of Lemma 4.3.** By the assumption, we can denote by $\log \mu$ and $\log \lambda$ the Lyapunov exponents of $P$ and $Q$ along $E^c$ respectively. Then we have that

\[ \frac{L^F(P, f)}{2} < \log \mu < L^F(P, f) \quad \text{and} \quad \log \lambda < L^F(Q, f). \]

Denote by $P = \text{Orb}(p, f)$ and $Q = \text{Orb}(q, f)$.

We fix a small number $\gamma > 0$ and a neighborhood $U$ of $f$. There exists $\epsilon > 0$ such that the $\epsilon$ neighborhood of $f$ is contained in $U$. There is a small number $0 < \theta < 1$, such that for any $h \in U$, if $d(z_1, z_2) < \theta \cdot \gamma$, we have that

\[ d(h^i(z_1), h^i(z_2)) < \frac{\gamma}{2}, \quad \text{for any} \quad i \in \left[ -\pi(P), \pi(P) \right]. \]

We take two neighborhoods $U_P$ and $U_Q$ of $P$ and $Q$ respectively, such that $U_P$ is contained in the $\theta \cdot \gamma$-neighborhood of $P$ and is disjoint from $U_Q$. 
Construction of the periodic orbit $P_1$ Similar to the proof of Lemma 3.3 consider the splitting $T_K M = E^s \oplus E^c \oplus E^u = E^s \oplus F \oplus G$ and the two neighborhoods $U_P$ and $U_Q$ by Theorem 2.11 and Remark 2.12, there are two matrices $T_0, T_1$ and two positive integers $t_0, t_1$ such that for any two integers $m$ and $n$, there exist $g \in \mathcal{U}$ and a hyperbolic periodic orbit $P_1 = \text{Orb}(p_1, g)$ satisfying $s$ that:

- $g = f$ and $Dg = Df$ on $P \cup Q$,
- The linear maps $T_0 : T_p M \mapsto T_q M$ and $T_1 : T_q M \mapsto T_p M$ preserve the two dominated splittings,
- $\pi(P_i) = m\pi(P) + n\pi(Q) + t_0 + t_1$,
- $Dg^{\pi(P_i)}(p_1)$ is conjugate to $T_1 \circ Df^{m\pi(Q)}(q) \circ T_0 \circ Df^{n\pi(P)}(p)$,
- $\#(P_1 \cap U_P) \geq m\pi(P)$ and $\#(P_1 \cap U_Q) \geq n\pi(Q)$,
- $W^s(P_1, g) \cap W^u(P, g)$ and $W^{uu}(P_1, g) \cap W^s(Q, g)$ corresponding to the splitting $T_K M = E^s \oplus E^c \oplus E^u$.

As a consequence of the first item above, we have that

- $L^F(P, f) = L^F(P, g)$ and $L^F(Q, f) = L^F(Q, g)$;
- $\chi_j(P, f) = \chi_j(P, g)$ and $\chi_j(Q, f) = \chi_j(Q, g)$, for any $j = 1, \ldots, d$.

For simplicity, we denote them by $L^F(P), L^F(Q), \chi_j(P)$ and $\chi_j(Q)$.

Since $T_0$ and $T_1$ preserve the dominated splittings, by choosing the proper coordinates, we assume that, corresponding to the two splittings, the two matrices $T_1$ and $T_0$ have the following forms respectively:

$$
T_0 = \begin{pmatrix}
D_s & 0 & 0 \\
0 & D_c & 0 \\
0 & 0 & D_u
\end{pmatrix} = \begin{pmatrix}
D_s & 0 & 0 \\
0 & D_F & 0 \\
0 & 0 & D_G
\end{pmatrix},
$$

$$
T_1 = \begin{pmatrix}
C_s & 0 & 0 \\
0 & C_c & 0 \\
0 & 0 & C_u
\end{pmatrix} = \begin{pmatrix}
C_s & 0 & 0 \\
0 & C_F & 0 \\
0 & 0 & C_G
\end{pmatrix}.
$$

Then we have that $Dg^{\pi(P_i)}(p_1)|E^c$ is conjugate to

$$
C_c \circ Df^{n\pi(Q)}(q)|_{E^c} \circ D_c \circ Df^{m\pi(P)}(p)|_{E^c},
$$

and $Dg^{\pi(P_i)}(p_1)|F$ is conjugate to

$$
C_F \circ Df^{n\pi(Q)}(q)|_{F} \circ D_F \circ Df^{m\pi(P)}(p)|_{F}.
$$

Choice of $m$, $n$ and $\rho$ We will adjust $m$ and $n$ to get a periodic orbit satisfying the conclusion of Lemma 4.3. Let $\eta > 0$ be a small number which will be decided later.

Claim 4.5. There exists an integer $N_\eta$ such that for any $m, n \geq N_\eta$, we have that

- all the Lyapunov exponents of $P_1$ along the bundle $E^c$ would belong to the interval

$$
\left[ m\pi(P) \cdot \log \mu + n\pi(Q) \cdot \log \lambda \over m\pi(P) + n\pi(Q) \right] - \eta, \left[ m\pi(P) \cdot \log \mu + n\pi(Q) \cdot \log \lambda \over m\pi(P) + n\pi(Q) \right] + \eta.
$$


• the mean Lyapunov exponent of $P_1$ along the bundle $F$ would belong to the interval
\[
\left[ \frac{m\pi(P) \cdot L^F(P) + n\pi(Q) \cdot L^F(Q)}{m\pi(P) + n\pi(Q)} - \eta, \frac{m\pi(P) \cdot L^F(P) + n\pi(Q) \cdot L^F(Q)}{m\pi(P) + n\pi(Q)} + \eta \right].
\]

The proof of Claim 4.5 is just like the proof of Claim 3.5 and we omit the proof here.

To guarantee the item 2 and that $P_1$ has the same index as $P$, we only need to require that there exists a number $\rho \in (0, 1)$ which will be decided later, such that:
\[
\frac{m\pi(P) \cdot L^F(P) + n\pi(Q) \cdot L^F(Q)}{m\pi(P) + n\pi(Q)} + \eta < \rho \cdot L^F(P)
\]
and
\[
\frac{m\pi(P) \cdot \log \mu + n\pi(Q) \cdot \log \lambda}{m\pi(P) + n\pi(Q)} - \eta > 0,
\]
which are equivalent to
\[
\frac{\eta - \log \lambda}{\log \mu - \eta} < \frac{m\pi(P)}{n\pi(Q)} < \frac{\rho \cdot L^F(P) - L^F(Q) - \eta}{L^F(P) - \rho \cdot L^F(P) + \eta}.
\]

**Claim 4.6.** There exists $\rho \in (0, 1)$ such that the following inequality is satisfied:
\[
- \frac{\log \lambda}{\log \mu} < \frac{\rho \cdot L^F(P) - L^F(Q)}{L^F(P) - \rho \cdot L^F(P)}.
\]

**Proof.** The proof consists in solving the following inequality:
\[
\frac{\rho \cdot L^F(P) - L^F(Q)}{L^F(P) - \rho \cdot L^F(P)} > - \frac{\log \lambda}{\log \mu},
\]
which is equivalent to
\[
\rho > \frac{L^F(P) \cdot \log \mu - \log \lambda \cdot L^F(P)}{L^F(P) \cdot \log \mu - \log \lambda}.
\]
By assumption that $\log \mu \in \left( \frac{L^F(P)}{2}, L^F(P) \right)$, we have the estimation:
\[
\frac{L^F(Q) \cdot \log \mu - \log \lambda \cdot L^F(P)}{L^F(P) \cdot \log \mu - \log \lambda} = \frac{L^F(Q) \cdot \log \mu - \log \lambda}{L^F(P) \cdot \log \mu - \log \lambda} < \frac{L^F(Q) - \log \lambda}{-2 \log \lambda} = 1 + \frac{L^F(Q)}{-2 \log \lambda}.
\]
We only need to take
\[
\rho = 1 + \frac{L^F(Q)}{-2 \log \lambda} \in (0, 1).
\]
Notice that $\rho$ only depends on $L^F(Q)$ and $\log \lambda$. 

We fix the value of $\rho$ that we get from Claim 4.6, then when $\eta$ is chosen small enough, we have that
\[
\frac{\eta - \log \lambda}{\log \mu - \eta} < \frac{\rho \cdot L^F(P) - L^F(Q) - \eta}{L^F(P) - \rho \cdot L^F(P) + \eta}.
\]
By the density of rational numbers among $\mathbb{R}$ and Claim 4.5, there exist $m$ and $n$ arbitrarily large such that
\[
\frac{m\pi(P)}{n\pi(Q)} \in \left( \frac{\eta - \log \lambda}{\log \mu - \eta}, \frac{\rho \cdot L^F(P) - L^F(Q) - \eta}{L^F(P) - \rho \cdot L^F(P) + \eta} \right),
\]
which implies that $P_1$ satisfies the properties of the conclusion of Lemma 4.3 except the item 3.

Now, we only need to check that the choice of $m$ and $n$ guarantees the item 3. By the fact that
\[
\frac{m\pi(P)}{n\pi(Q)} > - \frac{\log \lambda}{\log \mu}, L^F(P) > \log \mu \text{ and } L^F(Q) > \log \lambda,
\]
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we have that
\[
\frac{m \pi(P)}{m \pi(P) + n \pi(Q)} > \frac{-\log \lambda}{\log \mu} + 1 = \frac{-\log \lambda}{\log \mu} - L^F(Q) \frac{-L^F(P)}{L^F(P) - L^F(Q)} = 1 - \frac{L^F(P)}{L^F(P) - L^F(Q)}.
\]

Hence by taking \(m\) and \(n\) much larger than \(t_0 + t_1\), we have that \(P'\) is \((\gamma, 1 - \frac{L^F(P)}{L^F(P) - L^F(Q)})\) good approximation of \(P\). Just as the part of the proof of Lemma 3.3 \(P\) can be chosen with simple spectrum. This ends the proof of Lemma 4.3.

**Remark 4.7.** From the proof above, one can see that \(\rho\) only depends on \(L^F(Q)\) and the average of the Lyapunov exponents of \(Q\) along \(E^c\).

Now, we can give the proof of Proposition 4.1 whose proof is quite similar to that of Proposition 3.1.

**Proof of Proposition 4.1.** We denote by \(k = \dim(F)\) and assume that \(\text{Ind}(Q) = i + k_0\), then we have \(0 < k_0 \leq k\).

We can see that the properties stated in Proposition 4.1 are persistent under \(C^1\) small perturbation. Let \(\mathcal{R}\) be the residual subset of \(\text{Diff}^1(M)\) from Theorem 2.13. We only need to show that given \(f \in \mathcal{R}\), there is \(\rho \in (0, 1)\) such that for any \(\zeta > 0\) and \(\gamma > 0\), there exist a diffeomorphism \(g\) which is \(\zeta\)-\(C^1\)-close to \(f\), and a \(g\) hyperbolic periodic orbit \(P_1\) of index \(i\) satisfying the followings:

\(H_1\). \(g\) coincides with \(f\) on \(P_0 \cup Q\);
\(H_2\). \(P_1\) is robustly in the chain recurrence class of \(P_g\);
\(H_3\). \(P_1\) has simple spectrum and \(d_H(P_1, H(P_g, g)) < \gamma\);
\(H_4\). \(L^F(P_1, g) < \rho \cdot L^F(P_0, g)\);
\(H_5\). \(P_1\) is a \((\gamma, 1 - \frac{L^F(P_0, g)}{L^F(P_0, g) - L^F(Q, g)})\)-good approximation of \(P_0\).

Then Proposition 4.1 can be proved by a standard Baire argument.

**The previous settings.** By item 4 of Theorem 2.13 we can require that \(\zeta\) is chosen small enough such that after any \(\zeta\)-perturbation, the continuations of \(P_0\), \(P\) and \(Q\) are still robustly in the same chain recurrence class.

We take \(0 < \epsilon < \frac{1}{2}\), then there exist \(T > 0\) and \(t_0\) satisfying Lemma 2.6.

We denote by
\[
\log \lambda = \frac{1}{k_0} \sum_{j=i+1}^{i+k_0} \chi_j(Q, f) \quad \text{and} \quad \rho_0 = 1 + \frac{L^F(Q, f)}{-2 \log \lambda}.
\]

Since \(H(P, f)\) admits no dominated splitting of index \(j\) for any \(j \in \{i+1, \ldots, i+k-1\}\), there is a number \(\delta_0 \in (0, \frac{1}{2})\) such that for any compact invariant subset \(\Lambda \subset H(P, f)\), if \(d_H(\Lambda, H(P, f)) < \delta_0\), then \(\Lambda\) admits no \(T\)-dominated splitting of index \(j\) for any \(j \in \{i+1, \ldots, i+k-1\}\).

Notice that \(\log \lambda < 0\), \(L^F(Q, f) < 0\), \(0 < L^F(P_0, f)\) and \(\rho_0 \in (0, 1)\), hence we have that

\[
\rho_0 = 1 + \frac{L^F(Q, f)}{-2 \log \lambda} < 1 + \frac{1}{2} \rho_0;
\]

\[
\frac{L^F(P_0, f)}{L^F(P_0, f) - L^F(Q, f)} \in \left(0, \frac{2L^F(P_0, f)}{2L^F(P_0, f) - L^F(Q, f)}\right).
\]

As a consequence, we can take a number \(\delta \in (0, \frac{1}{2} \rho_0)\) small enough such that:
We apply the item 7 of Theorem 2.13 to the constants \(\delta\) and \(\kappa\), then there exist two hyperbolic periodic orbits \(P' = \text{Orb}(p')\) and \(Q' = \text{Orb}(q')\), with simple spectrum such that:

- \(P'\) and \(Q'\) are homoclinically related to \(P_0\) and \(Q\) respectively;
- Both \(P'\) and \(Q'\) are \(\delta/2\) dense in \(H(P, f)\);
- \(P'\) is a \((\frac{\delta}{10}, \kappa)\)-good approximation of \(P_0\) and \(Q'\) is \((\frac{\delta}{10}, \kappa)\)-good approximation of \(Q\);
- \(|L^F(P', f) - L^F(P_0, f)| < \delta\) and \(|L^F(Q', f) - L^F(Q, f)| < \delta\);
- \(\sum_{j=i+1}^{i+k_0} \chi_j(Q') - \sum_{j=i+1}^{i+k_0} \chi_j(Q) < \delta\);

Both of the periods of \(P'\) and \(Q'\) are larger than \(l_0\).

By item 6 of Theorem 2.13, we can do an arbitrarily \(C^1\) small perturbation, keeping \(P'\) and \(Q'\) homoclinically related to \(P\) and \(Q\) respectively and without changing the Lyapunov exponents of \(P'\) and \(Q'\), such that \(P'\) and \(Q'\) form a partially hyperbolic heterodimensional cycle.

### Equalize the center Lyapunov exponents of both \(P'\) and \(Q'\)

By Lemma 2.14, there exist \(\pi(P')\) one-parameter families \(\{A_{l, t}\}_{t \in [0, 1]}\) with simple spectrum such that:

- \(A_{0,0} = Df(f^l(p'))\) and \(B_{m,0} = Df(f^m(q'))\), for any \(l, m\);
- \(\| A_{l, t} - Df(f^l(p')) \| < \epsilon\) and \(\| A_{l, t}^{-1} - Df^{-1}(f^{l+1}(p')) \| < \epsilon\), for any \(t \in [0, 1]\);
- \(\| B_{m, t} - Df(f^m(q')) \| < \epsilon\) and \(\| B_{m, t}^{-1} - Df^{-1}(f^{m+1}(q')) \| < \epsilon\), for any \(t \in [0, 1]\);
- \(A_{\pi(P') - 1, t} \circ \cdots \circ A_{0, t}\) and \(B_{\pi(Q') - 1, t} \circ \cdots \circ B_{0, t}\) are hyperbolic, for any \(t \in [0, 1]\);
- For any integer \(s \in [1, i] \cup [i + k + 1, d]\), we have that \(\chi_s(A_{\pi(P') - 1, t} \circ \cdots \circ A_{0, t}) = \chi_s(P', f)\).
- For any integer \(s \in [1, i] \cup [i + k_0 + 1, d]\), we have that \(\chi_s(B_{\pi(Q') - 1, t} \circ \cdots \circ B_{0, t}) = \chi_s(Q', f)\).
- \(\chi_{i+1}(A_{\pi(P') - 1, 1} \circ \cdots \circ A_{0, 1}) = \chi_{i+k_0}(A_{\pi(P') - 1, 1} \circ \cdots \circ A_{0, 1}) \in (\frac{1}{2}L^F(P'), L^F(P'))\);
- \(\chi_{i+k_0+1}(A_{\pi(P') - 1, 1} \circ \cdots \circ A_{0, 1}) = \chi_{i+k}(A_{\pi(P') - 1, 1} \circ \cdots \circ A_{0, 1}) \geq L^F(P')\). 

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• \( \chi_{i+1}(B_{\pi(Q')} - 1, 1 \circ \cdots \circ B_{0,1}) = \chi_{i+k_0}(B_{\pi(Q')} - 1, 1 \circ \cdots \circ B_{0,1}) = \frac{1}{k_0} \sum_{j=i+1}^{i+k_0} \chi_j(Q', f) \).

Similar to the proof of Proposition 3.1 by Franks-Gourmelon Lemma, there exists an \( \epsilon \) perturbation \( g_1 \) of \( f \), which preserves the partially hyperbolic heterodimensional cycle formed by \( P' \) and \( Q' \), such that

S. \( g_1 \) coincides with \( f \) on \( P_0 \cup Q \cup P' \cup Q' \);

S2. \( \chi_j(P_0, g_1) = \chi_j(P_0, f) \) and \( \chi_j(Q, g_1) = \chi_j(Q, f) \), for any \( j = 1, 2, \cdots, d \);

S3. \( \text{Ind}(P', g_1) = \text{Ind}(P', f) \) and \( \text{Ind}(Q', g_1) = \text{Ind}(Q', f) \);

S4. \( \chi_{i+1}(P', g_1) = \chi_{i+k_0}(P', g_1) \in \left( \frac{1}{2}L^F(P', f), L^F(P', f) \right) \);

S6. \( \chi_{i+1}(Q', g_1) = \chi_{i+k_0}(Q', g_1) \);

S7. \( L^F(P', g_1) = L^F(P', f) \) and \( L^F(Q', g_1) = L^F(Q', f) \).

**Construction of the periodic orbit \( P_1 \)** By Lemma 4.3 and Remark 4.3, we have that there exist \( g \in \text{Diff}^1(M) \), which is \( \epsilon \)-\( C^1 \)-close to \( g_1 \) and therefore \( \zeta - C^1 \)-close to \( f \), and a \( g \)-hyperbolic periodic orbit \( P_1 \) of index \( i \) such that

- \( L^F(P_1, g) < \rho_0 \cdot L^F(P', g_1) \);
- \( P_1 \) has simple spectrum;
- \( g \) coincides with \( g_1 \) on \( P_0 \cup Q \cup P' \cup Q' \);
- the Lyapunov exponents of \( P_0, Q, P' \) and \( Q' \) with respect to \( g_1 \) are equal to those with respect to \( g \), hence are equal to those with respect to \( f \);
- \( P_1 \) is a \( \left( \frac{\gamma_1}{10}, 1 - \frac{L^F(P', f)}{L^F(P', f) - L^F(Q', f)} \right) \)-good approximation of \( P' \);
- \( P_1 \) is robustly in the same chain class with \( P_g \) and \( Q_g \).

By the choice of \( \delta \) and \( \gamma \), we have that \( d_H(P_1, H(P, g)) < \gamma \). Then the items \( H_1, H_2, H_3 \) are satisfied.

By the choice of \( \delta \), we have the following estimation for the mean center Lyapunov exponent \( L^F(P_1) \) of \( P_1 \):

\[
L^F(P_1, g) < \rho_0 \cdot L^F(P', g_1) = \rho_0 \cdot L^F(P', f) < \rho_0 \cdot L^F(P_0, f) + \delta < \frac{1 + \rho_0}{2} \cdot L^F(P_0, g).
\]

We only need to take \( \rho = \frac{1 + \rho_0}{2} \), hence item \( H_4 \) is satisfied.

Besides, by the choice of \( \kappa \), we have that

\[
\kappa \cdot \left( 1 - \frac{L^F(P', f)}{L^F(P', f) - L^F(Q', f)} \right) > \frac{2L^F(P_0, f) - L^F(Q, f)}{3L^F(P_0, f) - L^F(Q, f)} \cdot \frac{-L^F(Q, f)}{2L^F(P_0, f) - L^F(Q, f)}
\]

\[
= 1 - \frac{3L^F(P_0, f)}{3L^F(P_0, f) - L^F(Q, f)}.
\]

Since \( P' \) is \( (\frac{\gamma_1}{10}, \kappa) \) good approximated of \( P_0 \), by the inequality above, \( P_1 \) is a \( \left( \gamma, 1 - \frac{2L^F(P_0, f)}{2L^F(P_0, f) - L^F(Q, f)} \right) \)-good approximation of \( P_0 \). Then item \( H_5 \) is satisfied.

This ends the proof of Proposition 4.1. \( \square \)
5 Partially hyperbolic homoclinic classes with volume expanding center bundle

In this section, we give an example showing that Corollary 1.2 may be not true if there is no periodic orbit of index $\dim(E \oplus F)$. We first give some known results about normally hyperbolic submanifolds in Section 5.1 and the example will be given in Section 5.2.

5.1 Stability of normally hyperbolic compact manifolds

Let $f \in \text{Diff}^1(M)$. A compact invariant submanifold without boundary $N$ of $M$ is called normally hyperbolic, if there exists a partially hyperbolic splitting of the form $T^*_N M = E^s \oplus TN \oplus E^u$.

We state a simple version of Theorem 4.1 in [HPS] which gives the stability theorem for normally hyperbolic compact submanifold.

**Theorem 5.1.** Let $f \in \text{Diff}^1(M)$ and $N$ be a compact normally hyperbolic submanifold. We denote by $i : N \to M$ the embedding map from $N$ to $M$.

There exists a $C^1$ small neighborhood $U$ of $f$ such that for any $g \in U$, there exists a $C^1$ embedding map $i_g : N \to M$, such that $N_g = i_g(N)$ is $g$-normally hyperbolic. Moreover, $i_g$ would tend to $i$ in the $C^1$ topology, if $g$ tends to $f$.

**Remark 5.2.** The map $i_g^{-1}|_{N_g} \circ g \circ i_g$ is $C^1$-conjugate to the restriction of the map $g$ to $N_g$ and is $C^1$ close to $f$ if $g$ is $C^1$ close to $f$.

5.2 An example

Ch. Bonatti [B] (see also Section 6.2 in [BV]) constructs an open set $U$ of $C^1$ diffeomorphism on $\mathbb{T}^3$ such that for any $f \in U$, we have the following:

- $f$ is robustly transitive;
- There exist a periodic orbit of index one having a complex eigenvalue and a periodic orbit of index two.

By Theorem 2 and Theorem 4 in [BDP], for the diffeomorphism $f \in U$, there exists a partially hyperbolic splitting of the form $T^*_f \mathbb{T}^3 = E^{ss} \oplus E^c \oplus E^{uu}$, where $\dim(E^{ss}) = 1$ and the center bundle $E^c$ is volume expanding without any finer dominated splitting.

Now, we consider a north-south diffeomorphism $h$ on $\mathbb{S}^1$ such that the expanding rate of $h$ at the source $Q$ is strictly larger than the norm of $f$. We denote by

$$\tilde{f} = f \times h : \mathbb{T}^3 \times \mathbb{S}^1 \to \mathbb{T}^3 \times \mathbb{S}^1.$$ 

By Theorem 5.1 and continuity of partial hyperbolicity, there exists a $C^1$ neighborhood $\mathcal{V}$ of $\tilde{f}$ such that any $\tilde{g} \in \mathcal{V}$ has a partially hyperbolic repelling set $\Lambda_{\tilde{g}}$ diffeomorphic to $\mathbb{T}^3 \times \{Q\}$ admitting a splitting of the form $T^*_\Lambda_{\tilde{g}} \mathbb{T}^4 = E^{ss} \oplus E^c \oplus E^{uu}$ where $E^{ss} \oplus E^c = T\Lambda_{\tilde{g}}$. Then by Remark 5.2, we have that $\tilde{g}|_{\Lambda_{\tilde{g}}}$ is transitive and the dynamics $\tilde{g} : \Lambda_{\tilde{g}} \to \Lambda_{\tilde{g}}$ is $C^1$-conjugated to a diffeomorphism $C^1$ close to the dynamics $f : \mathbb{T}^3 \to \mathbb{T}^3$. Then the bundle $E^c|_{\Lambda_{\tilde{g}}}$ is volume expanding and there is a periodic orbit of index one with complex eigenvalues along the bundle $E^c$ contained in $\Lambda_{\tilde{g}}$. Hence the bundle $E^c|_{\Lambda_{\tilde{g}}}$ also has no finer dominated splitting. As a consequence, we have the following conclusion.

**Lemma 5.3.** For generic diffeomorphism in $\mathcal{V}$, there is a partially hyperbolic homoclinic class, such that any ergodic measure supported on it has at least one positive Lyapunov exponent along $E^c$.
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