Directed polymers with tilted columnar disorder and Burgers-like turbulence

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The minimal energy variations of a directed polymer with tilted columnar disorder in two dimensions are shown numerically to obey a multiscaling at short distances which crosses over to global simple scaling at large distances. The scenario is analogous to that of structure functions in bifractal Burgers’ turbulence. Some scaling properties are predicted from extreme value statistics. The multiscaling disappears for zero tilt.

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The directed polymer in a random medium (DPRM) is a prototype model in the statistical physics of disordered systems. At zero temperature, $T$, for each realisation of a random potential landscape, the problem consists in the statistical characterisation of long directed paths (no overhangs) which minimise the total energy. Besides having ramifications and generalizations in different fields, the DPRM is directly related to flux lines in type II superconductors, domain walls in random ferromagnets and nonequilibrium fluctuations of growing interfaces in dimension $d = 2$. In the last case the DPRM energies can be mapped onto the interfacial profiles realised during the growth, and the quenched average over potential disorder amounts to a mean over growth histories.

Attention has been devoted in recent years to the effects of various forms of disorder on the asymptotic DPRM scaling properties. Particular interest focused on long-range disorder correlations which in the language of interfacial growth, can be either in "time" $t$ (longitudinal spatial variable) or in space $x$ (transversal). While the effects of spatial correlations are relatively well understood, long range temporal correlations (including those appropriate to $1/f$ noise) remain more problematic mainly due to a lack of Galilean invariance. Here, we study the case of a noise which is infinitely correlated along one space-time direction ($x = -t$), a situation we refer to as tilted columnar disorder. In the DPRM context, columnar disorder is of particular interest for describing flux lines in superconductors irradiated by heavy ions. The case in which the columns are tilted with respect to the (time) direction of the flux lines could be of relevance, e.g., for the behaviour of a flux line in the presence of a forest of splayed columnar defects. Such situations have been studied recently both theoretically and experimentally.

In this Letter we show that the DPRM with tilted columnar disorder in $d = 2$ and at $T = 0$ is characterised by two distinct critical regimes. The first is a global one, holding at large length scales and consistent with simple scaling. We fully elucidate this regime in terms of extreme value statistics. The second regime is local and holds at smaller length scales. It is characterised by intermittency and multifractal scaling. The two regimes match in a full scaling framework which displays close analogies and correspondences with the scaling regimes of a fully turbulent flow. The very existence of this unexpected local multiscaling is especially remarkable because the distributions generating it are quite standard and general, and do not necessarily involve long tails.

Consider a directed polymer that evolves on a $1 + 1$ dimensional lattice in a random energy landscape $\varepsilon(x, t)$. The energies $\varepsilon(x, t)$ associated to each site $(x, t)$ are taken from a probability distribution function (pdf) $P(\varepsilon)$. The partition function $Z(x, t)$ is then defined through the recursion relation

$$Z(x, t) = e^{-\varepsilon(x,t)/T} [Z(x-1, t-1) + Z(x+1, t-1)] \quad (1)$$

together with an initial condition (we take $Z(x, t = 0) = e^{-\varepsilon(x,0)/T}$ with $\varepsilon(x, 0)$ randomly distributed according to $P(\varepsilon)$). We will in particular study tilted columnar defects, $\varepsilon(x, t) = \varepsilon(x + t)$, and will compare them to the more standard columnar defects, $\varepsilon(x, t) = \varepsilon(x)$. Fig. 1 sketches the situation for the tilted case.

It is generally assumed that the recursion is a discrete version of the diffusion equation

$$\frac{\partial Z}{\partial t} = \frac{T}{2} \frac{\partial^2 Z}{\partial x^2} - \frac{1}{T} V(x, t)Z \quad (2)$$

FIG. 1. Tilted columnar defects are oriented along lines $x = t$ constant. At $T = 0$, the optimal paths (thin lines) follow columns with very low energies $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ (thick lines) for a very large fraction of time (see text). Periodic boundary conditions are applied at the horizontal sides of the strip.
where $V(x,t)$ is a random potential with zero mean. From (3) one obtains the KPZ (1) equation for the height $h$ of a growing interface by the usual Hopf-Cole transformation $h = T \log Z$

$$\frac{\partial h}{\partial t} = \frac{T}{2} \frac{\partial^2 h}{\partial x^2} + \frac{1}{2} \left( \frac{\partial h}{\partial x} \right)^2 - V(x,t)$$

Finally, (3) can in turn be mapped onto the Burgers equation [12]

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{T}{2} \frac{\partial^2 u}{\partial x^2} + \eta(x,t)$$

by letting $u = -\partial h/\partial x$, where $u$ is interpreted as the velocity of a fluid subject to a random force $\eta(x,t) = -\partial V/\partial x$, and with viscosity $T/2$.

Here, we will limit ourselves to the $T \to 0$ limit of (4). Denoting $\lim_{T \to 0} T \log Z = -E$, (4) reduces to

$$E(x,t) = \min[E(x-1,t-1), E(x+1,t-1)] + \varepsilon(x,t)$$

Taking $\varepsilon = 0$ in (3) one obtains, in the continuum limit, an equation with a nonlinearity $|\partial h/\partial x|$ [3]. Noise disorder generated when $\varepsilon \neq 0$ is however expected to transform such nonlinearity into the analytical one of (4). Using the numerical methods of ref. [14] we found evidence of such analytical nonlinearity [15].

In Ref. [3], some properties of the minimal energy path in columnar disorder ($\varepsilon = \varepsilon(x)$) were studied numerically. It was found that the disorder averaged transversal displacement $R(t) = \langle x_{\min}(t) \rangle$ of the minimal energy path grows as a power of time $R(t) \sim t^{1/2}$, where $z$ is highly non-universal, and can be calculated for various pdf’s $P$ using Flory arguments based on extreme value statistics. In particular for a Gaussian pdf, $z = 1$, but with logarithmic corrections. These arguments should remain valid for the tilted columnar disorder we study. Indeed, $z$ remains unchanged as we verified numerically for a few distributions $P(\varepsilon)$.

Here, we study the moments of the energy-energy structure function

$$G_q(r,t) = \langle |E(x + r,t) - E(x,t)|^q \rangle^{1/q}$$

and find that for the case of tilted defects, and for a very large class of pdf’s $P(\varepsilon)$, $G_q(r,t)$ shows two scaling regimes. The first one is global for $r > R(t) \sim t^{1/2}$, with $G_q(r,t)$ independent of $r$ and with $q$-independent simple scaling in time

$$G_q(r,t) \sim R(t)^{\alpha_q} \sim t^{\alpha_q/2} \quad r > R(t)$$

In contrast, for $r < R(t)$, there is strong evidence of multiscaling

$$G_q(r,t) \sim r^{\zeta_q} \quad 1 < r < R(t), \ t \ fixed$$

Figure 2 shows our numerical results for $P(\varepsilon) = 5|\varepsilon|^{-6}$ with $\varepsilon \leq -1$. These results were obtained from a numerical iteration of (4) on a strip with periodic boundary conditions and a typical width $2^{15}$ in both space and time directions. The results were averaged over $2 \times 10^3$ realisations of the random energy landscape. Qualitatively similar behaviour was found for a Gaussian, and for power law pdf’s like $P_+(\varepsilon) = 4e^{-\varepsilon^2}$ (for $\varepsilon \geq 1$) or even for pdf’s with a bounded support like $P_\nu(\varepsilon) = (\nu + 1)e^\nu, \varepsilon \in [0,1]$. We also investigated in detail the nearest neighbour energy difference which, when replaced by a space derivative, should correspond to the local velocity $u$ in (4). Also this quantity shows multiscaling, i.e.

$$\langle |u|^q \rangle^{1/q} \sim G_q(r = 1, t) \sim t^{\alpha_q/2}$$

where $\alpha_q$ is $q$-dependent. In Figure 3 we show our results for $\langle |u|^q \rangle^{1/q}(t)$ for a Gaussian distribution. Qualitatively similar behaviour was found for all $P$’s investigated.

The scaling behaviour of $G_q(r,t)$ as given in (6) and (7) can in fact be described by a form which was originally proposed for the structure functions of fully
developed turbulence \cite{1}, and later also for height correlators of certain models of epitaxial growth \cite{10}:

\[ G_q(r,t) \sim R(t)^{\alpha_q} r^{\epsilon_q} f_q \left( \frac{r}{R(t)} \right) \]  \hspace{1cm} (10)

where \( f_q(u) \to \text{constant for } u \to 0, \) and \( f_q(u) \to u^{-\epsilon_q} \) when \( u \to \infty. \) Consistency with (1) then implies the scaling identity

\[ \alpha_q + \epsilon_q = \alpha \]  \hspace{1cm} (11)

In the scalings (1)-(11), the DPRM minimal energy \( E(x,t) \) plays the role of a fluid velocity component in turbulence, while \( R(t) \) corresponds to the Reynolds number. Moreover, \([G_2(r = 1,t)]^2\) is an analog of the local average energy dissipation \cite{9,10}. In all cases investigated, (11) holds within the numerical accuracy. As an example, Table 1 shows our estimates for \( \alpha_q, \epsilon_q, \) and \( \alpha \) for some \( q \)-values for the case of \( P_2(\varepsilon) = 3\varepsilon^2 \) with \( \varepsilon \in [0,1]. \) Since in all cases \( q \epsilon_q \sim 1 \) for all \( q > 1, \) while roughly \( q \epsilon_q \sim q \) for \( q < 1, \) the multiscaling appears consistent with a bisingale, with \( \epsilon_q \) very universal with respect to the noise distribution. On the other hand, as discussed below, \( \alpha \) is less universal, and its classes appear to be at least as many as the possible asymptotic distributions in extremal value statistics. Bisingale is a form of multisingale already detected for velocities in Burgers turbulence \cite{10}. We will give full details of all exponent estimates for different \( P \)'s elsewhere. Eq. (11) reveals that, in presence of a tilt, the variations of minimal energy corresponding to even relatively small \( r \) are widely fluctuating in the same way as longitudinal velocity differences in a turbulent flow.

The geometry of minimal energy paths suggests an analytic determination of \( \alpha. \) Consider the minimal energy path starting somewhere and arriving in \( x_0 \) at time \( t \) (Fig. 1). Typically, this path will have followed a column with a favourable low energy \( \varepsilon_0 \) for a ‘macroscopic’ fraction of \( t. \) The particular path chosen will depend on a balance between the energy gained there and the energy it costs to get from the starting point to the column of energy \( \varepsilon_0 \) and then to \( x_0 \) in time \( t. \) As time goes on it pays to ‘travel’ to more distant but energetically more favourable columns. Now consider paths arriving at \( x_1 \neq x_0. \) For small \( |x_0 - x_1|, \) these paths will in general have followed for a large fraction of \( t \) the same low energy column. Thus, the difference in energy between paths ending at \( x_0 \) and at \( x_1 \) will arise in a short part of their histories, where they did not follow the same column. Considering sites \( x_2 \) that are further and further away from \( x_0, \) we will arrive at the situation where it is more favourable for the path ending at \( x_2 \) to follow another low energy column with energy \( \varepsilon_1, \) closer to \( x_2, \) for most of its history (Fig. 1). The typical distance between the different minimal energy columns must be of the order of \( R(t), \) the only relevant length scale in the problem. For the most part, each path follows the column with the lowest energy within a local region of size \( R(t). \) Subsequently, at scales above \( R(t) \) the energies of different paths are essentially independent, which accounts for the constant behaviour of \( G_q \) on these scales visible in Fig. 2. As explained above, the columns that attract most of the paths in a region of size \( R \) are such that their energy is minimal in a region of that size. The minimum \( \varepsilon \) in a set of \( R \) independent random variables drawn from \( P(\varepsilon) \) is distributed according to

\[ \hat{P}_R(\varepsilon) = R P(\varepsilon) \left[ 1 - \int_{-\infty}^{\varepsilon} P(\varepsilon') d\varepsilon' \right]^{R-1} \]  \hspace{1cm} (12)

Since for \( R > R(t) \) energy differences between two paths are, up to small corrections, given by the energy differences of the minimal energy columns they follow, and since these are expected to be independent, we may write

\[ [G_q(r,t)]^q \sim t^q \int \int |\varepsilon_1 - \varepsilon_2|^q \hat{P}_R(\varepsilon_1) \hat{P}_R(\varepsilon_2) d\varepsilon_1 d\varepsilon_2 \]  \hspace{1cm} (13)

It is not possible to evaluate (13) for arbitrary pdf’s analytically. However, from extreme values statistics \cite{13} it is known that for \( R \to \infty, \) \( \hat{P}_R \) converges to a limited number of possible asymptotic forms which only depend on the behaviour of \( P(\varepsilon) \) at \( -\infty. \) As an example, for a Gaussian \( P, \) or for other pdf’s that approach zero at \( -\infty \) faster then any power, \( \hat{P}_R(\varepsilon) \) converges to the Gumbel pdf \( P_G(\varepsilon) = \exp(\varepsilon - \exp y) \) where \( y = \sqrt{2\log R(\varepsilon + \sqrt{2\log R(\varepsilon)}). \) Inserting this into (13) one finds

\[ [G_q(r,t)]^q \sim t^q (\log R)^{-q/2} \]  \hspace{1cm} (14)

Since for the Gaussian distribution \( R(t) \sim t/\log t \), \( G_q(r,t) \) indeed satisfies (1), with \( \alpha = 1, \) up to logarithmic corrections. One can proceed similarly for other pdf’s. For distributions whose support is bounded from below such as \( P_+ \) or \( P_\mu(\varepsilon), \) \( \hat{P}_R \) converges for \( R \to \infty \) to the Weibull distribution, from which the universal value \( \alpha = 1 \) is again obtained, but without logarithmic corrections. Finally, for pdf’s approaching zero as a power law at \( -\infty \), i.e. \( P_-(\varepsilon) \sim |\varepsilon|^{-\mu-1}, \) \( \alpha \) is non universal and equals \( 1 + 1/\mu. \) These predictions are consistent with our estimates \( \alpha \sim 1.03 \) (\( P_+(\varepsilon) \) with \( \mu = 3), 1.05 \) (\( P_2(\varepsilon) \)) and \( 1.19 \) (\( P_- \) with \( \mu = 5). \) For the Gaussian case the data can be fitted well by the form (13).
We verified that when the tilt is removed and when $\varepsilon(x,t) = \varepsilon(x)$ is assumed, the multiscaling disappears for all $\varepsilon_1$'s studied. However, $\alpha$ remains unchanged. Indeed, our arguments for the its determination should be valid both in presence and in absence of tilt.

The difference between these two cases can be understood by referring again to Fig.1. Let us assume that $\varepsilon_1$ is particularly low. Only paths whose endpoint is above that column can take advantage of this low energy. As a consequence, there is a very big energy difference for optimal paths ending at sites $x$ that are close to, but on different sides of the column with energy $\varepsilon_1$. In contrast, if the tilt is absent, two optimal paths ending on different sides of a very low energy column can both follow that column for a long time, and hence energy differences tend to be smaller. The large energy differences in the tilted case lead to broad distributions and multiscaling.

It remains an interesting question to find out whether the multiscaling observed here is present for any value of the tilt angle, or whether it only appears above a certain non-zero critical angle.

In [3], the KPZ equation [8] with noise $V(x,t)$ that has long range correlations in time

$$ (V(x,t)V(x',t')) \sim |t - t'|^{2\theta - 1}\delta(x - x') $$

was studied. From a renormalisation group analysis it was expected that for $\theta$ sufficiently below $1/2$

$$ (1 + 2\theta)z = 2\alpha. $$

The case $\theta = 1/2$ marks the transition to a regime where infrared divergences become important and the renormalisation procedure breaks down. Dimensionally, the noise corresponding to columnar disorder in absence of tilt, shares the same correlation [13] with the $1/f$ noise ($\theta = 1/2$). Thus, once accepted that [2] at $T = 0$ is the appropriate continuum limit, our results suggest that for pdf's (e.g. Gaussian) whose extreme value statistics is given by the Gumbel distribution, [14] remains valid at $\theta = 1/2$, but with logarithmic corrections.

In summary, the differences in optimal energy of the DPRM with tilted columnar defects obey a form of multiscaling analogous to that valid for velocity structure functions in turbulence. An intermittent scaling regime matches a global simple scaling one. The global regime can be characterized on the basis of extreme value statistics arguments. The multiscaling at small scales is an unexpected feature of both deterministic and stochastic models of nonlinear diffusion. The multiscaling of the discrete counterpart of the velocity potential observed here is similar to the scaling behaviour of velocity increments of the Burgers equation solutions with other types of random forcing [13]. The case treated here is new also because, if [14] indeed provides the appropriate continuum limit, the bisingal is valid for increments of the velocity potential.

The research presented here can be extended in several directions. Besides obvious generalisations to finite temperature and higher dimension, the case of networks of splayed columnar defects deserves special attention [6]. It would be worth investigating whether the multiscaling observed here also shows up in these experimentally relevant situations.

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