CONVERGENCE OF STATIONARY RADIAL BASIS FUNCTION-SCHEMES FOR EVOLUTION EQUATIONS

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Abstract. We establish precise convergence rates for semi-discrete schemes based on Radial Basis Function interpolation, as well as approximate approximation results for such schemes. Our schemes use stationary interpolation on regular grids, with basis functions from a general class of functions generalizing one introduced earlier by M. Buhmann. Our results apply to parabolic equations such as the heat equation or Kolmogorov-Fokker-Planck equations associated to Lévy processes, but also to certain hyperbolic equations.

1. Introduction

We establish convergence rates for semi-discrete numerical schemes based on stationary Radial Basis Function (RBF) interpolation. We do this for the classical heat equation but also for more general constant coefficient (translation invariant) pseudo-differential evolution equations. Our analysis applies to (suitably regularized versions of) the fractional heat equation and other Kolmogorov-Fokker-Planck equations associated to Lévy processes, but also to certain non-parabolic equations such as the free Schrödinger equation or hyperbolic equations such as the half-wave and transport equations.

The scheme we consider is an RBF-version of the method of lines, implemented on regular square grids, and we examine the convergence of the scheme when the grid size tends to 0, using basis functions which scale with the grid. We refer to Buhmann [4], Wendland [17] and Fasshauer [7] for general introductions to Radial Basis Function interpolation, and to the beginning of sections 5 and 6 below for a precise description of the scheme; roughly speaking, RBF-interpolation seeks to interpolate a function by a linear combination of translates of a given basis function, centered in the interpolation points. The coefficients can be computed by solving a linear system whose coefficient matrix, under appropriate conditions on the basis function, is always non-singular. To turn this into a scheme for solving a linear evolution equation on $\mathbb{R}^n$, one takes the coefficients to be time-dependent and asks that the equation be satisfied exactly in the interpolation points, leading to a system of ordinary differential equations. An advantage of using a regular grid for the interpolation points is that the interpolation problem can be reduced to the construction of a single Lagrange function (also called cardinal function), which greatly simplifies the theoretical analysis. The advantage of using basis functions which scale with the grid, known as stationary interpolation, is that the condition number of the coefficient matrix of the linear system which, in a practical implementation of the scheme, has to be solved numerically, becomes independent of the grid-size.

The numerical performance of such a scheme was examined in [2], [9], [14], [5], [6], motivated by applications in mathematical finance. Our main results will relate the order of convergence of the scheme to the degree of the operator and to the approximation order of the underlying RBF interpolation, as function of the grid-size. The latter will only be algebraic, since we use stationary interpolation, but can be arbitrarily large, depending on the basis function and the degree of smoothness of the initial condition. We will furthermore show that under certain circumstances we can have approximate approximation, in the sense that, in case of non-convergence of the scheme to the true solution, one can nevertheless get arbitrarily close to the real solution by an appropriate choice of basis function. More generally, if the scheme does converge, one can, for initial values which are sufficiently smooth and for grid-sizes which are small but not too small, observe an apparent order of convergence which is bigger
than the actual one. This was observed numerically in [2]. The notion of approximate approximation was introduced by Maz’ya [11], and analyzed in detail for approximation using Gaussian kernels in Maz’ya and Schmidt [12], [13]. Amongst other things, we generalize their work to more general basis functions.

We will in fact not restrict ourselves to particular examples of Radial Basis Functions, such as the generalized multiquadrics or the polyharmonic basis functions, but carry out our analysis for a general class of basis functions which we introduce in section 2. Since this class is a generalization of one introduced by Martin Buhmann in [3] and by which it was inspired, we have called it the Buhmann class. We will analyze the properties of our scheme in Fourier space and, for that reason, first re-examine in section 3 the convergence of RBF-interpolation on regular grids from the Fourier point of view by deriving precise estimates for the Wiener norm of the difference between a function and its RBF-interpolant. Our convergence theorems have a non-zero intersection with classical results of Buhman and Powell (see [4] and its references), generalizing these in some respects. Despite the use of the Wiener norm we can allow certain classes of polynomially increasing functions. The Fourier transform of such a function will have a non-integrable algebraic singularity at 0, and the allowed order of the singularity (and therefore the allowed growth of the function) will depend on the basis function which is used for the interpolation. In section 4 we show that the convergence rates which we found in section 3 are best possible, and discuss approximate approximation.

The next two sections examine the convergence of the RBF-variant of the method of lines, first, in section 5, for the in many respects typical case of classical heat equation before indicating, in section 6, how these results extend to more general pseudo-differential evolution equations. We show that the scheme converges at a rate of $h^{8-q}$, where $q$ is the order of the operator ($q = 2$ for the heat equation) and $\kappa$ the order of convergence of the underlying RBF-interpolation scheme, which is also the order of the singularity in 0 of the Fourier transform of the basis function which is used. We show that this rate is in general optimal, but that we can also have approximate approximation, in the sense that for appropriate basis functions which are sufficiently ”flat” and with sufficiently smooth initial data there can be an apparent higher order of convergence when $h$ is ”small but not too small”, which is determined by the degree of smoothness of the initial data. This provides an explanation for the empirical convergence rates which were observed in [2].

One obvious limitation of the present paper is that we have restricted ourselves to interpolation on regular grids of scaled integer points, whereas one of the strengths of the RBF method is that one can use arbitrarily scattered interpolation points, opening up the way to adaptive methods. This flexibility may become important when dealing with variable coefficient linear differential operators or with non-linear ones. We note however, in our defense, that the often-used Finite Difference methods are usually restricted to regular grids also, and that even on regular grids RBF methods can have definite advantages over FD methods when treating non-local operators, as they do not discretize the operator, and can therefore be better suited when the latter has a singular kernel, such as for the Kolmogorov backward equation of certain Lévy processes: see [2]. Another limitation is that we only have treated translation invariant evolution equations. These do however already include large classes of operators which are of interest for applications, such as the fractional heat equation or other Kolmogorov-Fokker-Planck equations associated to Lévy processes. It would obviously be of interest to generalize our results to variable coefficient PDEs, but this will require other methods. We also have (primarily) examined convergence in Wiener norm: convergence of the scheme in the $L^2$ norm or more general Sobolev norms will be treated elsewhere. We finally want to note that although we have used stationary RBF interpolation, our analysis can be extended to the non-stationary case, when one uses the same basis function for all grid-sizes. In this case one can have exponential rates of convergence, as first discovered by Madych and Nelson [10] for interpolation of functions in the so-called native space of the basis function. This also will be examined elsewhere.

Notations: $C$ denotes the usual ”variable constant”, whose exact numerical value is allowed to change from one occurrence to the other. We use the following convention for the Fourier transform $\hat{f} = \mathcal{F}(f)$
of an integrable function $f$ on $\mathbb{R}^n$:
$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i(x,\xi)}dx,$$
$(x,\xi)$ being the Euclidean inner product on $\mathbb{R}^n$. We will routinely use the extension of the Fourier transform $\mathcal{F}$ to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the usual Schwarz-space of rapidly decreasing functions.

For $s \in \mathbb{R}$, let $L^s_{\ast}(\mathbb{R}^n)$ be the space of measurable functions on $\mathbb{R}^n$ for which

$$||f||^s_{0,s} := \int_{\mathbb{R}^n} |\xi|^s|f(\xi)|d\xi < \infty,$$
and $L^s_{\ast}(\mathbb{R}^n) := L^1(\mathbb{R}^n) \cap L^1_{\ast}(\mathbb{R}^n)$, its non-homogeneous version, with norm $\int |f|(1 + |\xi|)^s d\xi$. We will also need the "mixed" spaces

$$\hat{L}^s_{r,s} := \{ f \in \hat{L}^1(\mathbb{R}^n) : ||f||^s_{r,s} := \int_{\mathbb{R}^n} |f(\xi)|(|\xi|^r \wedge |\xi|^s)d\xi < \infty \},$$
for $r \leq s$, where $a \wedge b := \min(a,b)$. Note that this scale of spaces is increasing in $r$ for $r \leq s$, and that $L^1_{\ast}(\mathbb{R}^n) \subset \hat{L}^s_{r,s}(\mathbb{R}^n)$, while $\hat{L}^s_{r,s}(\mathbb{R}^n) = L^1_{\ast}(\mathbb{R}^n)$.

Finally, we define weighted sup-norm spaces $L^\infty_{\ast}(\mathbb{R}^n)$ of measurable functions such that

$$||f||^\infty_{\ast,s} := \sup_{x \in \mathbb{R}^n} (1 + |x|)^s|f(x)|,$$
where the sup is the essential supremum, as usual. If $s < 0$, an element $f$ of $L^\infty_{\ast}(\mathbb{R}^n)$ is of polynomial growth of order at most $|s|$: $|f(x)| \leq C (1 + |x|)^{|s|}$ on $\mathbb{R}^n$, with $C = ||f||^\infty_{\ast,s}$.

Derivatives of functions $f = f(x)$ on $\mathbb{R}^n$ will be denoted by $\partial_x f(x)$ or by $f^{(\alpha)}(x)$, $\alpha \in \mathbb{N}^n$ a multi-index. If $K \in \mathbb{N}$ and $\lambda \in (0,1]$, then $C^K_{\lambda}(\mathbb{R}^n)$ will denote the Hölder space of $K$-times differentiable functions on $\mathbb{R}^n$ with bounded derivatives of all orders, such that the derivatives of order $K$ satisfy a uniform Hölder condition on $\mathbb{R}^n$ with exponent $\lambda$, provided with the norm

$$\sum_{|\alpha| \leq K} ||f^{(\alpha)}||^\infty + \sum_{|\alpha| = K} ||f^{(\alpha)}||^{0,\lambda},$$
where $||g||^0_{0,\lambda} := \sup_{\xi \neq \eta} |g(\xi) - g(\eta)|/|\xi - \eta|^\lambda$.

Finally, $[x]$ and $\lfloor x \rfloor$ denote the usual floor and ceiling functions, defined as the greatest, respectively smallest integer which is less than, respectively greater than a real number $x$; note that $[x] = [x] + 1$ if $x \notin \mathbb{N}$, while $[x] = [x] = x$ otherwise.

## 2. A class of basis functions for interpolation on a regular grid

### 2.1. The Buhmann class

We introduce a flexible class of basis functions which is well-suited for stationary interpolation on regular grids. Since this class of functions is a generalisation of the one introduced earlier by Buhmann (called admissible in there) we will call it the Buhmann class. From the onset, we will allow non-radial basis functions, radiality not being essential for most of the theory (as is of course well known).

**Definition 2.1.** For $\kappa \geq 0$ and $N > n$ we define the Buhmann class $\mathcal{B}_{\kappa,N}(\mathbb{R}^n)$ as the set of functions $\varphi \in C(\mathbb{R}^n)$ such that

(i) $\varphi$ is of polynomial growth of order strictly less than $\kappa$, in the sense that $\varphi \in L^\infty_{-\kappa+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$.

(ii) (Regularity and strict positivity.) The restriction to $\mathbb{R}^n \setminus 0$ of the Fourier transform $\hat{\varphi} := \mathcal{F}(\varphi)$ (in the sense of tempered distributions) can be identified with a function in $C^{n+|\kappa|+1}(\mathbb{R}^n \setminus 0)$, which we will continue to denote by $\hat{\varphi}$, which is pointwise strictly positive: $\hat{\varphi}(\eta) > 0$ for all $\eta \in \mathbb{R}^n \setminus 0$.

(iii) (Elliptic singularity at 0.) There exist positive constants $c,C$ such that for all $|\alpha| \leq n + |\kappa| + 1$,

$$|\partial^\alpha \varphi| \leq C |\eta|^{n-|\alpha|}, \quad |\eta| \leq 1,$$
while also
\begin{equation}
\hat{\varphi}(\eta) \geq c|\eta|^{-\kappa}, \quad |\eta| \leq 1.
\end{equation}

(iv) (Decay at infinity.) There exist positive constants $C_\alpha$, $|\alpha| \leq n + |\kappa| + 1$, such that
\begin{equation}
|\partial^\alpha_\eta \hat{\varphi}(\eta)| \leq C_\alpha |\eta|^{-N}, \quad |\eta| \geq 1.
\end{equation}

We use the term "elliptic" for condition (iii) because of the resemblance of (4) and (5) with the ellipticity condition on symbols in pseudo-differential theory (where the singularity would be at infinity). The significance of $n + |\kappa| + 1$ is that this is the smallest integer which is strictly greater than $n + \kappa$. (Note that if $\kappa \notin \mathbb{N}$, then $n + |\kappa| + 1 = n + |\kappa|$.) Conditions (ii) and (iii) for derivatives up till this order will imply polynomial decay of order $n + \kappa$ of the associated Lagrange interpolation function which we will define below. Requiring higher order differentiability would not improve this rate of decay: $n + \kappa$ is best possible, under condition (iii).

In most of the results of this paper, strict positivity of $\hat{\varphi}$ on $\mathbb{R}^n \setminus 0$ could have been replaced by the weaker condition that the "periodisation" of $\hat{\varphi}$, $\sum_k \hat{\varphi}(\eta + 2\pi k)$, be pointwise strictly positive on all of $\mathbb{R}^n$, as in [3]: note that by (6) with $\kappa = 0$, this series converges absolutely on $\mathbb{R}^n \setminus \mathbb{Z}^n$, given that $N > n$, while it can be set equal to $\infty$ on $\mathbb{Z}^n$, in view of (5). Since for most of the radial basis functions used in practice, $\hat{\varphi}(\eta)$ itself is already strictly positive, we have opted to impose the stronger condition, which also simplifies the proofs.

Remarks 2.2. (i) Buhmann [3] studied stationary RBF interpolation on regular grids for a slightly more restricted class of radial basis functions. The main difference between his original class and the one of our definition 2.1 (besides, as already mentioned, Buhmann requiring strict positivity of the periodisation of $\hat{\varphi}$ instead of of $\hat{\varphi}$ itself) lies in condition (iii), where Buhmann asks that for small $|\eta|$, $\hat{\varphi}(\eta)$ be asymptotically equivalent to a positive multiple of $|\eta|^{-\kappa}$ modulo an relative error which has to be sufficiently small: $\hat{\varphi}(\eta) = A|\eta|^{-\kappa}(1 + h(\eta))$ with $A > 0$ and $|\partial^\alpha_\eta h(\eta)| = O(|\eta|^{\varepsilon - |\alpha|})$ as $\eta \to 0$ for $|\alpha| \leq n + |\kappa| + 1$, with an $\varepsilon > |\kappa| - \kappa$. Under these conditions Buhmann proved the existence of a unique Lagrange function for interpolation on $\mathbb{Z}^n$, constructed as an infinite linear combination of translates of $\varphi$, which moreover decays as $|x|^{-n-\kappa}$ at infinity. This fundamental result remains true for $\varphi$’s in $\mathcal{B}_{\kappa,N}(\mathbb{R}^n)$: see theorem 2.3 below and and its proof in Appendix A. The condition that $\varepsilon > |\kappa| - \kappa$ is in our treatment made unnecessary by lemma A.2.

(ii) All conditions in definition 2.1 except the first are on the Fourier transform of $\varphi$. One can show (cf. Appendix A) that if the Fourier transform of a polynomially increasing function $\varphi$ satisfies (ii), (iii) and (iv), then there exists a function $\tilde{\varphi}(x)$ which grows at most as $\max(|x|^{-\kappa-n}\log|x|, 1)$ at infinity (and, slightly better, as $\max(|x|^{-\kappa-n}, 1)$ if $\kappa \notin \mathbb{N}$) and a polynomial $P(x)$ such that
\begin{equation}
\varphi(x) = \tilde{\varphi}(x) + P(x).
\end{equation}

The function $\tilde{\varphi}$ is unique modulo polynomials of degree $|\kappa| - n$. If we moreover require $\varphi$ to have polynomial growth of order strictly less than $\kappa$, as in definition 2.1, then $P(x)$ will be a polynomial of degree at most $|\kappa| - 1$ (which is $|\kappa|$ if $\kappa \notin \mathbb{N}$, and $\kappa - 1$ if $\kappa \in \mathbb{N}$). Note that the Fourier transform of a polynomial is a linear combination of derivatives of the delta-distribution in $0$, and therefore equals $0$ on $\mathbb{R}^n \setminus 0$.

(iii) The condition that $N > n$ will suffice for convergence of the RBF interpolants on regular grids $h\mathbb{Z}^n$ as $h \to 0$, but will have to be strengthened to $n > N + k$ for convergence of the RBF schemes for solving parabolic PDEs and PIDEs which are of order $k$ (in the space variables).
The usual examples of radial basis functions, such as the generalised multi-quads, cubic and higher order splines, thin plate splines, inverse multi-quads and Gaussians, are Buhmann class.

One can show that if $\varphi \in L^\infty_p(\mathbb{R}^n), p \in \mathbb{N}$, satisfies conditions (ii) - (iv) of definition $\mathbb{N}$, then $\varphi$ conditionally positive definite of order $\mu$, where $\mu$ is the smallest integer such that $2\mu > \max([\kappa] - n, p, 0)$: for this it would in fact be sufficient that $\hat{\varphi}|_{\mathbb{R}^n \setminus 0}$ is locally integrable, satisfies $\mathbb{N}$ with $\alpha = 0$ and is integrable on $\{ |\eta| \geq 1 \}$. One can therefore, by standard RBF theory, interpolate an arbitrary function on a finite set $X$ of points by a linear combination of translates of $\varphi$ plus a polynomial of degree $\mu - 1$, provided the set $X$ is unisolvent for this class of polynomials: see for example $\mathbb{H}$. This involves solving a linear system of equations. The next theorem establishes the existence and main properties of a Lagrange function in terms of which the solution of the interpolation problem on $\mathbb{Z}^n$ can be simply expressed.

\textbf{Theorem 2.3.} Suppose that $\varphi \in \mathfrak{B}_{\kappa,N}(\mathbb{R}^n)$ with $\kappa \geq 0$ and $N > n$. Then there exist coefficients $c_k, k \in \mathbb{Z}^n$, such that the series

\[ L_1(x) := L_1(\varphi)(x) := \sum_{k \in \mathbb{Z}^n} c_k \varphi(x - k) \]

converges absolutely and uniformly on compacta and defines a Lagrange function for interpolation on $\mathbb{Z}^n$:

\[ L_1(j) = \delta_{0j}, \ j \in \mathbb{Z}^n. \]

The function $L_1$ satisfies the bound

\[ |L_1(x)| \leq C(1 + |x|)^{-\kappa-n}, \ x \in \mathbb{R}^n, \]

and its Fourier transform is given by

\[ \hat{L}_1(\eta) = \frac{\hat{\varphi}(\eta)}{\sum_k \hat{\varphi}(\eta + 2\pi k)}. \]

Moreover, at the points of $2\pi \mathbb{Z}^n$, $\hat{L}_1$ satisfies the Fix-Strang conditions:

\[ \hat{L}_1(2\pi k + \eta) = \delta_{0k} + O(|\eta|^\kappa), \ \eta \to 0. \]

See Appendix A for the proof. We make the trivial but for the sequel important observation that

\[ \sum_{k \in \mathbb{Z}^n} \hat{L}_1(\eta + 2\pi k) = 1. \]

\textbf{Remarks 2.4.} (i) We will write $L_1(\varphi)$ if we want to stress the dependence on the basis function $\varphi$, otherwise we will simply write $L_1$. The subindex 1 in $L_1$ is a notational reminder that $L_1$ is a Lagrange function for interpolation on the standard grid $\mathbb{Z}^n$ with width 1. For stationary RBF interpolation on the scaled grids $h\mathbb{Z}^n$ one uses the scaled basis functions $\varphi(x/h)$, whose associated Lagrange functions then simply are $L_h(x) := L_1(x/h)$. If $f \in L^\infty_p(\mathbb{R}^n)$ for some $p < \kappa$, then $s_h[f](x) := \sum_j f(hj)L_1(h^{-1}x - j)$ is an infinite linear combination of translates of $\varphi$ which will interpolate $f$ on $h\mathbb{Z}^n$, where the series converges absolutely and uniformly on compacta, in view of the growth restriction on $f$.

(ii) One important point of the theorem is that the basis function $\varphi$ need not decay at infinity, but is allowed to grow polynomially. A high order of growth will in fact lead to a high order convergence of the stationary RBF interpolants $s_h[f]$ to $f$ as $h \to 0$, since this will translate into a strong singularity in 0 of the Fourier transform of $\hat{\varphi}$ in the form of a large $\kappa$, which implies that $\hat{L}_1$ will satisfy the Fix-Strang conditions to a high order. The latter then implies a convergence rate of $O(h^\kappa)$ in sup-norm, as shown by Buhmann $\mathbb{K}$ (under suitable conditions on $f$); see also $\mathbb{H}$, Chapter 4. We will prove such convergence theorems for the Wiener norm instead of the sup-norm, using an entirely different approach: see theorems $\mathbb{S}$ and $\mathbb{S}$. Note that, contrary to $\varphi$, the Lagrange function $L_1$ will decay at infinity, as shown by $\mathbb{S}$, and this the more rapidly the higher $\kappa$ is. In particular, $L_1$ is integrable if
\( \kappa > 0 \) and its Fourier transform then exists in the classical sense, as an absolutely convergent integral. It is possible for \( L_1(x) \) to have faster decay: Buhmann [3] shows that if \( \hat{\varphi}(\eta) \sim |\eta|^{-\kappa} \) as \( \eta \to 0 \) with \( \kappa \in 2\mathbb{N} \), then
\[
L_1(x) \leq C(1 + |x|)^{-\kappa-n-\varepsilon},
\]
while there are examples of \( \varphi \) for which \( L_1(x) \) decays exponentially: see [4] for details and references.

(iii) The proof of theorem 2.3 shows that the coefficients \( c_{-k} \) are precisely the Fourier coefficients of \( (\sum_k \hat{\varphi}(\eta + 2\pi k))^{-1} \). They satisfy bounds analogous to the ones satisfied by \( L_1 \): \( |c_k| = O(|k|^{-\kappa}) \). This guarantees that the defining series for \( L_1(x) \) converges absolutely and uniformly on compacta including when \( \kappa = 0 \), in view of condition (i) of definition 2.1.

Since the denominator of (9) is \( 2\pi \)-periodic and positively bounded away from 0, \( \hat{L}_1(\eta) \) will have the same decay as \( \hat{\varphi}(\eta) \) as \( |\eta| \to \infty \). We state this as a lemma, for later reference:

**Lemma 2.5.** There exists a constant \( C_n > 0 \) such that for all \( \ell \in \mathbb{Z}^n \setminus 0 \),
\[
(12) \quad \max_{\eta \in [\pi, \pi]^n} |\eta|^{-\kappa} |\hat{L}_1(\eta + 2\pi \ell)| \leq C|\ell|^{-N}.
\]

**Proof.** The function \( \sum_k \hat{\varphi}(\eta + 2\pi k) \) is periodic and, by the positivity and ellipticity of \( \hat{\varphi} \) at 0, bounded from below by \( c|\eta|^{-\kappa} \) for some \( c > 0 \). Hence
\[
|\hat{L}_1(\eta + 2\pi \ell)| = \frac{|\varphi(\eta + 2\pi \ell)|}{\sum_k \hat{\varphi}(\eta + 2\pi k)} \leq C|\eta|^{\kappa}|\eta + 2\pi \ell|^{-N},
\]
which implies (12). \( \square \)

Another useful lemma clarifies the smoothness properties of \( \hat{L}_1 \):

**Lemma 2.6.** There exist constants \( C_\alpha \) such that for each multi-index \( \alpha \) with \( |\alpha| \leq n + |\kappa| + 1 \) and all \( k \in \mathbb{Z}^n \),
\[
(13) \quad \left| \partial_\eta^\alpha \left( \hat{L}_1(\eta + 2\pi k) - \delta_{0k} \right) \right| \leq \frac{C_\alpha}{(1 + |k|)^{\lambda}} |\eta|^{\kappa - |\alpha|},
\]
for \( \eta \neq 0 \) in a neighborhood of 0. In particular, if \( \kappa > 0 \) then \( \hat{L}_1 \) belongs to the Hölder space \( C_b^{\kappa-1, \lambda}(\mathbb{R}^n) \), with \( \lambda = \kappa - (|\kappa| - 1) \).

Note that \( |\kappa| - 1 = |\kappa| \) if \( \kappa \) is non-integer, but that it is equal to \( \kappa - 1 \) if \( \kappa \) is a positive integer, so that \( \lambda = 1 \) then.

**Proof of lemma 2.6.** This is elementary: if we let \( \hat{\varphi}_{\text{per}}(\eta) := \sum_k \hat{\varphi}(\eta + 2\pi k) \), then applying Leibnitz’s rule to the product \( \hat{L}_1 \hat{\varphi}_{\text{per}} = \hat{\varphi} \) yields that
\[
(\partial_\eta^\alpha \hat{L}_1) \hat{\varphi}_{\text{per}} = \partial_\eta^\alpha \hat{\varphi} - \sum_{\beta < \alpha} \binom{\alpha}{\beta} \partial_\eta^\beta \hat{L}_1 \partial_\eta^{\alpha-\beta} \hat{\varphi}_{\text{per}}.
\]

The estimate (13) for \( k \neq 0 \) now follows by induction on \( \alpha \), using that \( \partial_\eta^\alpha \hat{\varphi}_{\text{per}}(\eta + 2\pi k) = \partial_\eta^\alpha \hat{\varphi}_{\text{per}}(\eta) = O(|\eta|^{-\kappa-|\alpha|}) \), together with (6) of definition 2.1 and lemma 2.5 (to start the induction). If \( k = 0 \), we use the same argument, starting from
\[
\hat{\varphi}_{\text{per}} \left( \hat{L}_1 - 1 \right) = \hat{\varphi} - \hat{\varphi}_{\text{per}},
\]
on observing that the right hand side is \( C^{\kappa+1,n+1} \) near 0, since equal to \( \sum_{k \neq 0} \hat{\varphi}(\eta + 2\pi k) \). Finally, the fact that \( \hat{L}_1(\eta + 2\pi k) - \delta_{0k} \) is \( O(|\eta|^\kappa) \) implies that all derivatives of order up to \( |\kappa| \), if \( \kappa \notin \mathbb{N} \), or \( \kappa - 1 \), if \( \kappa \in \mathbb{N} \setminus 0 \), exist and are 0. Their continuity in 0 follows from (13). \( \square \)
Remark 2.7. We briefly pause to examine the differentiability of $\hat{L}_1$ if $\kappa \in \mathbb{N}$. Letting $g(\eta) := \sum_{k \neq 0} f(\eta + 2\pi k)$ and $\psi(\eta) := |\eta|^{\kappa} g(\eta)$, we have that

$$\hat{L}_1(\eta) = \frac{\psi(\eta)}{\psi(\eta) + |\eta|^{\kappa} g(\eta)}.$$

This shows that $\hat{L}_1$ cannot be $C^\infty$ in 0 if $\kappa \in \mathbb{N}$ is not even, even if $\psi$ is (note that then $\psi(0) \neq 0$ given that $\varphi$ is Buhmann class). If $\kappa \in 2\mathbb{N}$, then $\hat{L}_1$ will be as smooth as $\psi(\eta)$ is in 0, and therefore as $\hat{\varphi}$ is away from 0.

We finally note that to construct numerical PDE schemes using RBF interpolation one will obviously need sufficient differentiability of $L_1$. The proof of theorem 2.8 given in appendix A also yields existence and decay of derivatives of $L_1$, provided $N$ is chosen sufficiently large:

**Theorem 2.8.** Suppose that $k \in \mathbb{N}$ and let $\varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n)$ with $N > n + k$. Then $L_1 \in C^k(\mathbb{R}^n)$. Moreover, $|\partial_\alpha^p L_1(x)| = O(|x|^{-\kappa - n})$ as $|x| \to \infty$, for all $|\alpha| \leq k$.

See also theorem 3.1 below.

### 3. Convergence of RBF-interpolants

As a preparation for our analysis of numerical RBF-schemes for the heat equation and other evolution equations, we first revisit the convergence of stationary RBF interpolants, providing an alternative perspective on classical convergence theorems of Buhmann and Powell. As stated in the introduction, we will limit ourselves to stationary interpolation on regular grids $h\mathbb{Z}^n$, meaning that we let the basis function scale with the grid-size: $\varphi_h(x) := \varphi(x/h)$. The associated Lagrange function scales similarly, and the RBF interpolant $s_h[f]$ of a given function $f : \mathbb{R}^n \to \mathbb{R}$ can be conveniently written as

$$s_h[f](x) = \sum_j f(hj)L_1(\frac{x}{h} - j).$$

where $L_1$ is the Lagrange function of theorem 2.8. Here, and below, sums over $j$, $k$, $\ell$, etc. are understood to be over $\mathbb{Z}^n$. Note that the use of the Lagrange function eliminates the need for inverting the coefficient matrix $(\varphi_h(hj-hk))_{j,k} = (\varphi(j-k))_{j,k}$ in the standard formulation of RBF interpolation.

The decay at infinity of $L_1$ easily implies that the series (14) converges absolutely if $f$ is of polynomial growth of order strictly less than $\kappa$, in the sense that $f \in L^\infty_p(\mathbb{R}^n)$ for some $p < \kappa$.

Throughout this section, we fix a basis function $\varphi = \varphi_1 \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n)$ with $N > n$ and $\kappa > 0$: see subsection 3.2 below for the case of $\kappa = 0$; although there can be no convergence then, this case is nevertheless of interest since we can have approximate convergence in the sense of Maz’ja and Smith: see section 4 below. We will systematically work in Fourier-space, and examine convergence of $s_h[f]$ to $f$ in Wiener norm,

$$||f||_A := ||\hat{f}||_1,$$

except for the end of this section where we will briefly examine convergence in weighted sup-norms. Convergence in Wiener norm of course trivially implies convergence in Chebyshev or uniform norm, since $||f||_\infty \leq ||f||_A$.

#### 3.1. Convergence in Wiener norm

We begin by computing the Fourier transform of $s_h[f]$ for Schwarz-class functions $f$. For sufficiently rapidly decaying functions $g$, let us define the function $\Sigma_h(g)$

$$\Sigma_h(g)(\xi) := \left(\sum_{k} g(\xi + 2\pi h^{-1} k)\right)\hat{L}_1(h\xi).$$

\[\text{In the present, idealized, set-up of interpolation on } h\mathbb{Z}^n \text{ that coefficient matrix is infinite; in practice, one would have to truncate the matrix: } |j|, |k| \leq N \text{ (where, } |j| = |j|_{\infty} = \max_i |j_i| \text{ with } N \sim h^{-1}, \text{ taking larger and larger sections of the matrix as } h \to 0. \text{ One would also have to truncate the series for } L_1, \text{ leading to quasi-interpolation.}\]
The map $\Sigma_h : g \to \Sigma_h(g)$ will play an important rôle in what follows. We note that $\Sigma_h$ is a contraction with respect to the $L^1$-norm: indeed, by the positivity of $\hat{L}_1$ and monotone convergence,

\[
||\Sigma_h(g)||_1 \leq \sum_k \int_{\mathbb{R}^n} |g(\xi + 2\pi h^{-1}k)| \hat{L}_1(h\xi) \, d\xi
\]

\[
= \int_{\mathbb{R}^n} |g(\xi)| \left( \sum_k \hat{L}_1(h\xi + 2\pi k) \right) \, d\xi
\]

\[
= ||g||_1,
\]

in view of (11); $\Sigma_h$ therefore extends to a contraction on $L^1(\mathbb{R})$. We also note that if $g \in L^1(\mathbb{R}^n)$, then the defining series for $\Sigma_h(g)$ converges absolutely a.e., since

\[
\int_{[0,\pi]^n} \sum_k |g(\xi + 2\pi k)| \, d\xi = \int_{\mathbb{R}^n} |g(\xi)| \, d\xi < \infty.
\]

**Lemma 3.1.** If $f \in S(\mathbb{R}^n)$ then $s_h[f] \in L^1(\mathbb{R}^n)$ and then

\[
(16) \quad \hat{s_h[f]}(\xi) = \Sigma_h(\hat{f})(\xi).
\]

**Proof.** Since $\kappa > 0$, $L_1$ is integrable by theorem 2.3 and therefore $||s_h[f]||_1 \leq \left( h^n \sum_j |f(hj)| \right) ||\hat{f}||_1$. Applying Fubini’s theorem to the function $(j, x) \to f(hj)L_1(h^{-1}x - j)e^{-i(x, \xi)}$ on $\mathbb{Z}^n \times \mathbb{R}^n$ one finds

\[
\hat{s_h[f]}(\xi) = \left( \sum_j f(hj)e^{-ih(j, \xi)} \right) h^n \hat{L}_1(h\xi)
\]

\[
= \left( \sum_k \hat{f}(\xi + 2\pi h^{-1}k) \right) \hat{L}_1(h\xi),
\]

(17)

where for the second line we used the Poisson summation formula: $\sum_j g(j) = \sum_k \hat{g}(2\pi k)$, with $g(x) := f(hx)e^{-ih(x, \xi)}$. \hfill \Box

We can then already state a first convergence theorem:

**Theorem 3.2.** Let $\kappa > 0$. Then there exists a constant $C = C_\varphi > 0$ such that for all tempered functions $f$ for which $\hat{f} \in L^1_\kappa(\mathbb{R}^n)$ and for all positive $h \leq 1$,

\[
(18) \quad ||f - s_h[f]||_A \leq Ch^\kappa ||f||^\kappa_{\infty} = Ch^\kappa \int_{\mathbb{R}^n} |\hat{f}(\xi)| |\xi|^\kappa \, d\xi.
\]

The condition that $\hat{f} \in L^1_\kappa(\mathbb{R}^n)$ implies a certain smoothness: $f$ must have continuous and bounded derivatives of order $[\kappa]$.

**Proof.** We first note that if $\hat{f} \in L^1(\mathbb{R}^n)$, then $\hat{s_h[f]} = \Sigma_h(\hat{f})$: the hypothesis on $\hat{f}$ implies that $f$ is a bounded continuous function. It follows that $s_h[f]$ is well-defined, by (3), and that there exists a constant $C > 0$ such that for all $h \leq 1$,

\[
(19) \quad ||s_h[f]||_{\infty} \leq C||f||_{\infty}.
\]

Indeed, $|s_h[f](x)| \leq ||f||_{\infty} \sum_j |L_1(h^{-1}x - j)|$: the right hand side is $h$-periodic, and its sup on $\{|x| \leq h/2\}$ can be estimated by a constant times $\sum_j |L_1(j)|$, which converges since $\kappa > 0$.

The Fourier transform of $s_h[f]$ therefore exists as a tempered distribution. We show using a density argument that $\hat{s_h[f]} = \Sigma_h(\hat{f})$: since $\hat{f} \in L^1(\mathbb{R}^n)$, then there exists a sequence $f_\nu \in S(\mathbb{R}^n)$ such that $||\hat{f_\nu} - \hat{f}||_1 \to 0$. Consequently $\Sigma_h(\hat{f_\nu}) \to \Sigma_h(f)$ in $L^1$ and therefore also as tempered distributions. On the other hand, $||s_h[f_\nu] - s_h[f]||_{\infty} \leq C||f - f_\nu||_{\infty} \leq C||\hat{f} - \hat{f_\nu}||_1 \to 0$, so $s_h[f_\nu] \to s_h[f]$ as tempered
distributions also. Hence \( \Sigma_{h}(\hat{f}_{\nu}) = s_{h}[f_{\nu}] \rightarrow s_{h}[\hat{f}] \), and consequently \( s_{h}[\hat{f}] = \Sigma_{h}(\hat{f}) \). As a consequence, \( s_{h}[\hat{f}] \) has finite Wiener norm if \( f \) has.

We now observe that since \( 0 \leq \hat{L}_{1} \leq 1 \), and using the montone convergence,

\[
||f - s_{h}[\hat{f}]||_{1} \leq \int_{\mathbb{R}^{n}} |\hat{f}(\xi)|\left(1 - \hat{L}_{1}(h\xi)\right)d\xi + \sum_{k \neq 0} \int_{\mathbb{R}^{n}} |\hat{f}(\xi + 2\pi h^{-1}k)|\hat{L}_{1}(h\xi)d\xi,
\]

where we used (11) once more. The Fix-Strang condition (10) in 0 then implies (18) with (20)

\[
\text{Equation (22) means that the restriction to } \mathbb{R}^{n} \text{ the right hand side of (18) still makes sense for certain } \hat{f} \text{ having a non-integrable singularity at 0. Allowing such singularities means allowing } f \text{'s which grow at a certain polynomial rate, and we can prove the following extension of theorem 3.2}
\]

**Theorem 3.4.** Let \( f \) be a tempered function on \( \mathbb{R}^{n} \) such that \( |f(x)| \leq C(1 + |x|)^{p} \) for some \( p < \kappa \), and such that

\[
\hat{f}|_{\mathbb{R}^{n} \setminus 0} \in \hat{L}_{\kappa}^{1}(\mathbb{R}^{n}).
\]

Then \( s_{h}[\hat{f}] - \hat{f} \) is in \( L^{1}(\mathbb{R}^{n}) \), and

\[
||s_{h}[\hat{f}] - f||_{1} \leq C h^{\kappa}||\hat{f}||_{1,\kappa}^{p}.
\]

Equation (22) means that the restriction to \( \mathbb{R}^{n} \setminus 0 \) of the tempered distribution \( \hat{f} \) can be identified with a locally integrable function which is integrable with respect to the weight \( |\xi|^{n} \) and therefore is in \( \hat{L}_{\kappa}^{1}(\mathbb{R}^{n}) \), if we interpret it as an almost everywhere defined function on \( \mathbb{R}^{n} \), whose norm we then simply denote by \( ||\hat{f}||_{1,\kappa}^{p} \) instead of the more correct \( ||\hat{f}|_{\mathbb{R}^{n} \setminus 0}||_{1,\kappa}^{p} \).

**Proof.** We first check that \( s_{h}[\hat{f}] \) is a tempered distribution: this is a consequence of the estimate

\[
||s_{h}[\hat{f}]||_{\infty,-p} \leq C||f||_{\infty,-p}, \quad p \geq 0.
\]
which can be shown as follows: first note that \( f \to s_h[f] \) commutes with translations by elements of \( h\mathbb{Z}^n \): if \( k \in \mathbb{Z}^n \), then
\[
 s_h[f](x - kh) = s_h[f(\cdot - hk)](x).
\]
Let \(|\cdot| = |\cdot|_\infty \) be the \( \ell^\infty \)-norm on \( \mathbb{R}^n \). If \(|x| \leq h/2 \) and \( f \in L^\infty_{-p}(\mathbb{R}^n) \) with \( p < \kappa \), then
\[
 |s_h[f](x)| \leq ||f||_{\infty,-p} \left( 1 + \sum_{|j| \geq 1} \frac{(1 + h/j)^p}{(1 + |h^{-1}x - j|^{n+n})} \right) \leq C||f||_{\infty,-p},
\]
since \(|h^{-1}x - j| \geq |j|/2 \) if \(|j| \geq 1 \). Next, if \(|x - hk| \leq h/2 \) with \( k \in \mathbb{Z}^n \), then by translation invariance,
\[
 |s_h[f](x)| \leq C||f(\cdot + hk)||_{\infty,-p} \leq C(1 + h|k|)^p||f||_{\infty,-p},
\]
which implies (24). The next lemma identifies the Fourier transform if \( s_h[f] \).

**Lemma 3.5.** Suppose that \( |f(x)| \leq C(1+|x|)^p \) for some \( p < \kappa \) and that \( \hat{f}|_{\mathbb{R}^n \setminus 0} \in L^1(\mathbb{R}^n \setminus 0, \min(|\xi|^\kappa, 1)d\xi) \).
Then the tempered distribution \( s_h[f] - \hat{f} \) can be identified with the function
\[
 (\hat{L}_1(h\xi) - 1) \hat{f}(\xi) + \sum_{k \neq 0} \hat{f}(\xi + 2\pi h^{-1}k)\hat{L}_1(h\xi), \quad \xi \neq 0,
\]
which is in \( L^1(\mathbb{R}^n) \).

The proof of the lemma involves extending \( \hat{f} \) to a continuous linear functional on the Hölder spaces \( C_b^{\kappa-1,\lambda}(\mathbb{R}^n) \) with \( \lambda = \kappa - ((\kappa) - 1) \) (so that \( \lambda = \kappa - |\kappa| \) if \( \kappa \notin \mathbb{N} \), and \( \lambda = 1 \) otherwise), and using this to define \( \Sigma_h(\hat{f}) \) as a tempered distribution. In order not to interrupt the flow of the argument with distribution-theoretical technicalities, we postpone the proof to Appendix B. Note that the individual terms of (25) are integrable on account of the Fix-Strang conditions satisfied by \( \hat{L}_1 \), and that the \( L^1 \)-norm of (24) can be bounded by the \( L^1 \)-norm of \( 2(\hat{L}_1(h\xi) - 1)|\hat{f}(\xi)| \), using once more that the sum of translates of \( \hat{L}_1 \) by elements of \( (2\pi)\mathbb{Z}^n \) is identically equal to one.

The lemma implies the estimate (20), and the theorem follows as before.

**Example 3.6.** If \( p \geq 0 \) and if \( f \in C^{[p]+n+1}(\mathbb{R}^n) \) satisfies
\[
 |\partial_x^\alpha f(x)| \leq C_\alpha (1 + |x|)^{p-|\alpha|}, \quad |\alpha| \leq |p| + n + 1,
\]
with \( p \geq 0 \) then one can show that \( \hat{f}|_{\mathbb{R}^n \setminus 0} \in C^{\kappa-1,\lambda}(\mathbb{R}^n \setminus 0) \), and that \( |\hat{f}(\xi)| \leq C|\xi|^{-p-n} \) near 0 while \( \hat{f}(\xi) = O(|\xi|^{-[p]-n-1}) \) at infinity: if (20) holds for all \( \alpha \), this follows for example from Stein [13], proposition 1 of Chapter VI. Examination of the proof shows that we only need the number of derivatives indicated. It follows that \( |\xi|^\kappa \hat{f}(\xi) \) is integrable if \( p < \kappa \) and theorem 3.3 therefore applies to such functions.

We briefly compare our theorem 3.3 with convergence results of Buhmann and Powell, cf. [1] and its references. Theorem 4.6 of [1] states that if \( \kappa \notin \mathbb{N} \) and if \( f \in C^{[\kappa]}(\mathbb{R}^n) \) such that
\[
 \max_{|\alpha| = [\kappa], [\kappa]-1} ||\partial_x^\alpha f||_\infty < \infty,
\]
then \( ||s_h[f] - f||_\infty \leq C h^\kappa \). If \( \kappa \) is an odd integer, Buhmann loc cit., theorem 4.7, needs an additional degree of differentiability, with the derivatives of order \( \kappa+1 \) again bounded. Note that these conditions imply that \( f \) is of polynomial growth of order at most \([\kappa] - 1 \). On the other hand, theorem 3.3 covers cases when \( f \) can have stronger growth, and derivatives of order \([\kappa] - 1 \) do not need to be bounded, for example \( f(x) = (1 + |x|)^{p/2} \) with \([\kappa] < p < [\kappa] \) if \( \kappa \notin \mathbb{N} \) and \( \kappa - 1 < p < \kappa \) if \( \kappa \in \mathbb{N}^* \). We also note that condition (22) implies that \( f \) is \( C^{[\kappa]} \) (a consequence of the integrability of \( \hat{f}(\xi)|\xi|^\kappa \) on \( |\xi| \geq 1 \) and the smoothness of the inverse Fourier transform of any compactly supported distribution) without the derivatives of order \([\kappa] \) necessarily being bounded (because of the singularity at 0) and we believe this order of differentiability to be the maximal one necessary.
Remark 3.7. If $\kappa \notin \mathbb{N}$ then theorem 3.4 remains true if $\hat{f}|_{\mathbb{R}^n\setminus 0}$ can be identified with a Borel measure $\nu$ on $\mathbb{R}^n \setminus 0$ for which $|\xi|^{\kappa} \in L^1(\mathbb{R}^n, d|\nu|)$. The estimate (23) then generalises to an estimate for the variation norm of $\Sigma(\hat{f}) - \hat{f}$ (as measure on $\mathbb{R}^n$) which then implies a uniform estimate

$$\|s_h[f] - f\|_\infty \leq C h^{\kappa} \int_{\mathbb{R}^n} |\xi|^{\kappa} d|\nu|(\xi).$$

We next observe that if one is satisfied with a slower rate of convergence, the growth condition on $\hat{f}$ at infinity can be weakened. Recall the spaces $\hat{L}_{r,\kappa}(\mathbb{R}^n)$ defined in (2).

Corollary 3.8. (of the proof). Suppose that $0 < r \leq \kappa$ and that $f \in L^\infty_{\kappa+r}(\mathbb{R}^n)$ for some $\varepsilon > 0$ such that $\hat{f}|_{\mathbb{R}^n\setminus 0} \in \hat{L}^1_{r,\kappa}(\mathbb{R}^n)$. Then for all $h \leq 1$,

$$\|s_h[f] - \hat{f}\|_1 \leq C h^r \|\hat{f}\|_{r,\kappa}^\varepsilon.$$

Here, the constant $C$ can be taken the same as in theorems 3.4 and 3.2 (cf. the proof of the latter).

Proof. The hypotheses on $\hat{f}$ certainly imply that $\hat{f}|_{\mathbb{R}^n} \in L^1(\mathbb{R}^n, \min(|\xi|^{\kappa}, 1) d\xi)$, so we can apply lemma 3.5. In particular, the estimate (20) still holds. Now split this integral into three parts over the ranges $h|\xi| \leq h$, $h \leq h|\xi| \leq 1$ and $h|\xi| \geq 1$, and use the Fix - Strang condition in 0, together with the trivial bound $|\eta|^{\kappa} \leq |\eta|^r$ if $|\eta| \leq 1$ to estimate:

$$\int_{\mathbb{R}^n} \left(1 - \mathring{L}_1(h\xi)\right) |\hat{f}(\xi)| d\xi \leq C \left(\int_{h|\xi| \leq h} (h|\xi|)^{\kappa} |\hat{f}(\xi)| d\xi + \int_{h < h|\xi| \leq 1} (h|\xi|)^{\kappa} |\hat{f}(\xi)| d\xi + \int_{h|\xi| > 1} |\hat{f}(\xi)| d\xi\right)$$

$$\leq C \left(h^\kappa \int_{|\xi| \leq 1} |\xi|^\kappa |\hat{f}(\xi)| d\xi + h^r \int_{1 \leq |\xi| \leq h^{-1}} |\xi|^\kappa |\hat{f}(\xi)| d\xi + h^r \int_{|\xi| \geq h^{-1}} |\hat{f}(\xi)| d\xi\right)$$

$$\leq C h^r \int_{\mathbb{R}^n} |\hat{f}(\xi)| |\xi|^r d\xi,$$

□

The corollary shows that there is an interplay between the order of convergence of the RBF interpolants and the smoothness of the function which is interpolated, as quantified by the decay of $|\hat{f}(\xi)|$ at infinity. The singularity of $\hat{f}(\xi)$ at 0 can be of order $|\xi|^{-\kappa-n+\varepsilon}$, as before, which is compatible with $f(x)$ having polynomial growth of order less than $\kappa$.

3.2. The case of $\kappa = 0$. The results of this section, and in particular the basic estimate (20), remain true if $\kappa = 0$, which includes for example Gaussian basis functions. The reader may of course wonder why one would want to consider this case, since this estimate then does not imply that $s_h[f]$ will converge to $f$ and, as we will see in section 4, it won’t. The reason is that we can still have approximate approximation, in the sense that lim sup$_{h \to 0} \|s_h[f] - f\|_A$ can be made arbitrarily small for functions $\hat{f}$ whose Fourier transform decay sufficiently rapidly, by an appropriate choice of basis function: see theorem 4.4 and the discussion following it. There is a minor technical problem, in that theorems 2.3 no longer guarantees that $L_1(x)$ decays sufficiently rapidly to be integrable, though it may for particular examples such as the Gaussian, or under stronger conditions on $\varphi$, as per Buhmann’s result for integer pair mentioned before. The interpolating function $s_h[f]$ is therefore no longer be guaranteed to be in $L^1(\mathbb{R}^n)$, even if $f$ is rapidly decreasing. Its Fourier transform will still exist as a tempered distribution, and lemma 3.1 will still be true, as an easy approximation argument shows: $\sum_{|j| \leq N} f(hj) L_1(h^{-1}x - j) \to s_h[f]$ in $S'(\mathbb{R}^n)$ so $(\sum_{|j| \leq N} f(hj)e^{-ih(j,\xi)}) h^\kappa \mathring{L}_1(h\xi) \to s_h[f]$ as tempered distributions. But $(\sum_j f(jh)e^{-ih(j,\xi)}) h^\kappa \mathring{L}_1(h\xi)$ converges to $\Sigma_h(\hat{f})$.\]
Next, the series for $s_h[f]$ will not necessarily converge if $\hat{f} \in L^1(\mathbb{R}^n)$ and we need to impose an additional condition that $f \in L_\kappa^n(\mathbb{R}^n)$ for some $\varepsilon > 0$, as in theorem 3.4. Alternatively, we can observe that if $f$ is integrable, then the defining series for $s_h[f]$ is summable in the sense that if $\chi \in \mathcal{S}(\mathbb{R}^n)$ with $\chi(0) = 1$ and if we let

$$s_h^\varepsilon[f](x) := \sum_j \chi(\varepsilon j h)f(hj)L_1(h^{-1}x - j),$$

then, as $\varepsilon \to 0$, $s_h^\varepsilon[f]$ converges uniformly on $\mathbb{R}^n$ to a continuous function whose Fourier transform is $\Sigma_h(\hat{f})$, independently of the choice of $\chi$. To show this, observe that the Fourier transform of the left hand side is equal to $(2\pi)^{-n}\Sigma_h(\hat{\chi}_\varepsilon \ast \hat{f})$, where $\hat{\chi}_\varepsilon(\xi) = \varepsilon^{-n}\hat{\chi}(-\varepsilon \xi)$. Since (by a classical result on convolution with approximate identities),

$$(2\pi)^{-n}\hat{\chi}_\varepsilon \ast \hat{f} \to \hat{f}$$

in $L^1$ (observing that $(2\pi)^{-n}\int_{\mathbb{R}^n} \hat{\chi}_\varepsilon(-\xi)d\xi = \chi(0) = 1$), and since $\Sigma_h$ is a contraction, it follows that $s_h^\varepsilon[f]$ converges in Wiener norm, and therefore in sup-norm, to the inverse Fourier transform of the integrable function $\Sigma_h(\hat{f})$. If we now define $s_h[f]$ as the limit of the $s_h^\varepsilon[f]$, the estimate (20) follows as before. We in fact only need the Fourier transform of $\chi$ to be integrable, but this excludes taking for $\chi$ the characteristic function of a cube centered at 0, which would entail ordinary convergence of the series for $s_h[f](x)$.

### 3.3. Convergence in weighted sup-norms

Although our focus in this paper is on convergence in the Wiener norm, one can allow $f$’s for which $\hat{f}$ does not necessarily coincide with a function or a measure on $\mathbb{R} \setminus 0$, but is a more general distribution, if we replace the Wiener norm by one of the weighted sup-norms $|| \cdot ||_{\infty,-p}$. We give an example which can be deduced from theorem 3.2 by an approximation argument.

**Theorem 3.9.** Let $p \in \mathbb{N}$, $p < \kappa$ and let $f$ be a tempered function on $\mathbb{R}^n$ whose Fourier transform can be written as

$$\hat{f} = \sum_{|\alpha| \leq p} \partial^\alpha_{\xi} \nu_\alpha,$$

with $\nu_\alpha$ complex Borel measures on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} (1 + |\eta|)^p d|\nu_\alpha|(|\eta|) < \infty.$$

Then

$$||s_h[f] - f||_{\infty,-p} \leq C_f h^\kappa,$$

where we can take

$$C_f = C \cdot \sum_{|\alpha| \leq p} ||(1 + |\eta|)^p||_{L^1(|\nu_\alpha|)},$$

for some positive constant $C$ independent of $f$.

**Proof.** The hypothesis on $\hat{f}$ imply that $f$ is continuous and of polynomial growth of order at most $p$: $||f||_{\infty,-p} < \infty$. Let $\chi_R(x) := \chi(x/R)$, where $\chi = \chi_1 \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \chi \leq 1$, $\chi(x) = 1$ on $B(0,1)$. We will apply theorem 3.2 to $\chi_R f$ and for that purpose first bound $||\chi_R f||_{1,\kappa}$.

**Lemma 3.10.** For $f$ as in theorem 3.9 and $R \geq 1$,

$$||\chi_R f||_{1,\kappa} \leq C_f R^p,$$

with $C_f$ as in 3.2.
Proof. Since \( \hat{f} \) is a Fourier transform, we find that (writing \( \hat{\chi}^{(\alpha)} \) for \( \partial_{\xi}^{\alpha} \hat{\chi} \))

\[
(2\pi)^n |\hat{f} \chi_R|_{1, \infty} \leq \sum_{|\alpha| \leq p} |\hat{f} \chi_R|_{1, \infty} \leq \sum_{|\alpha| \leq p} |\hat{f} \chi_R|_{1, \infty} \leq \sum_{|\alpha| \leq p} |\hat{f} \chi_R|_{1, \infty}
\]

The lemma follows by observing that \( (1 + |\eta + R^{-1} \xi|) \leq (1 + |\eta|)(1 + R^{-1} |\xi|) \leq (1 + |\eta|)(1 + |\xi|) \) and using the rapid decay of \( \hat{\chi} \).

\[ \square \]

Proof of theorem 3.4 (continued). By theorem 3.2

\[ ||s_h[f \chi_R] - f \chi_R||_{\infty} \leq C_f R^p h^\kappa. \]

We next compare \( s_h[f] \) with \( s_h[\chi_R f] \): since \( \chi_R(x) = 1 \) for \( |x| \leq R \),

\[ |s_h[f](x) - s_h[\chi_R f](x)| = \sum_{h|j| \geq R} |f(hj) - \chi_R(hj)f(hj)|L_1(h^{-1}x - j)| \leq 2 \sum_{h|j| \geq R} |f(hj)| |L_1(h^{-1}x - j)|. \]

Now if \( |x| \leq R/2 \), then \( |hj| \geq R \) implies that \( |x - hj| \geq |hj|/2 \) so that \( |h^{-1}x - j| \geq |j|/2 \). Hence, by the decay at infinity of \( L_1 \),

\[ \sup_{|x| \leq R/2} \sum_{h|j| \geq R} |f(hj)| |L_1(h^{-1}x - j)| \leq ||f||_{\infty} R^p \sum_{h|j| \geq R} |j|^{p^\kappa - n} \leq C ||f||_{\infty} R^p h^\kappa \]

since we can for example bound the sum by a constant times \( \int_{|y| \geq R/h} |y|^{p^\kappa - n} dy \) (recall that \( p < \kappa \)).

Writing \( s_h[f] - f = s_h[f] - s_h[\chi_R f] + s_h[\chi_R f] - \chi_R f + \chi R f - f \), these estimates imply that

\[ R^{-p} \sup_{|x| \leq R/2} |s_h[f](x) - f(x)| \leq C_f h^\kappa, \]

for \( R \geq 1 \), which implies the theorem.

\[ \square \]

Examples of functions \( f \) which satisfy the hypothesis of theorem 3.9 are the inverse Fourier transforms of compactly supported distributions of order \( p < \kappa \) since, by a structure theorem going back to Laurent Schwartz, such a compactly supported distribution can be written in the form (29).

4. Approximate approximation

It is easy to show that the approximation error in the Wiener norm cannot in general go to 0 faster than \( h^\kappa \): if \( \hat{f} \in L^1(\mathbb{R}^n) \) has compact support, then the supports of \( \hat{f}(\cdot + 2\pi k/h) \) will be disjoint if \( h \) is sufficiently small. It follows that

\[
|s_h[f] - \hat{f}|_1 = \int_{\mathbb{R}^n} |\hat{f}(\xi)(\hat{L}_1(h\xi) - 1)|d\xi + \sum_{k \neq 0} \int_{\mathbb{R}^n} |\hat{f}(\xi + 2\pi k/h)|\hat{L}_1(h\xi)d\xi
\]

\[ = \int_{\mathbb{R}^n} |\hat{f}(\xi)| (1 - \hat{L}_1(h\xi))d\xi + \sum_{k \neq 0} \int_{\mathbb{R}^n} |\hat{f}(\xi)|\hat{L}_1(h\xi + 2\pi k)d\xi \]

\[ = 2 \int_{\mathbb{R}^n} |\hat{f}(\xi)| \left( 1 - \hat{L}_1(h\xi) \right) d\xi, \]

since \( \sum_k \hat{L}_1(\eta + 2\pi k) = 1 \). If we define

\[
\hat{L}_1 := \hat{L}_1(\varphi) := 2 \lim_{\eta \to 0} \inf_{\eta \to 0} \frac{1 - \hat{L}_1(\eta)}{|\eta|^\kappa}
\]

(34)
then Fatou’s lemma implies that
\begin{equation}
\liminf_{h \to 0} h^{-\kappa} \| s_h[f] - f \|_A \geq \mathcal{L}_\kappa \int_{\mathbb{R}^n} |\xi|^\kappa |\hat{f}(\xi)| \, d\xi.
\end{equation}

We will see below that $\mathcal{L}_\kappa > 0$. The inequality (35) remains valid if $\hat{f}$ is not compactly supported but decays sufficiently fast at infinity; see theorem 4.3 below. Here we first examine the corresponding upper bound.

As we just noted, one cannot in general do better that $O(h^\kappa)$ for the approximation error. However, for suitable basis functions $\varphi$ and for $\hat{f}(\xi)$ which decay sufficiently fast at infinity we may observe a higher apparent rate of convergence for $h$’s which are small but not too small. If $\varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n)$, we let
\begin{equation}
\mathcal{T}_\kappa := \mathcal{T}_\kappa(\varphi) := 2 \limsup_{|\eta| \to 0} \frac{1 - \mathcal{L}_1(\eta)}{|\eta|^\kappa}.
\end{equation}

A slight modification of the proof of theorems 4.2 and 4.4 then gives the following more precise estimate for the approximation error.

**Theorem 4.1.** Let $\varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n)$ with $\kappa \geq 0$ and suppose that $f \in L^\infty_p(\mathbb{R}^n)$ for some $p < \kappa$ such that for some $s > \kappa$,
\[ \hat{f}|_{\mathbb{R}^n \setminus 0} \in L^1(\mathbb{R}^n) \cap L^s(\mathbb{R}^n) = L^1(\mathbb{R}^n, \max (|\xi|^\kappa, |\xi|^s)) \].

Then there exists for each $\varepsilon > 0$, a constant $C_\varepsilon = C_{\varepsilon,\varphi}$ such that
\begin{equation}
\| s_h[f] - f \|_A \leq (1 + \varepsilon) \mathcal{T}_\kappa(\varphi) h^\kappa \| \hat{f} \|_1^0 + C_\varepsilon h^s \| \hat{f} \|_1^s.
\end{equation}

Concretely, the condition on $\hat{f}$ means that
\[ \int_{|\xi| \leq 1} |\hat{f}| |\xi|^\kappa d\xi + \int_{|\xi| \geq 1} |\hat{f}| |\xi|^s d\xi < \infty. \]

The theorem implies that if $\mathcal{T}_\kappa(\varphi)$ is very small, and $\hat{f} \in L^1(\mathbb{R}^n)$ with $s > \kappa$ then the rate of convergence for small, but not too small $h$’s will at first appear to be $h^s \ll h^\kappa$, up to the point that the first term dominates and the error saturates at a level comparable to $\mathcal{T}_\kappa(\varphi) h^\kappa$. This is the phenomenon of approximate approximation which was discovered by Maz’ya [11] in the context of quasi-interpolation; see also Maz’ya and Schmidt [13]. The quasi-interpolants these authors consider are, in our notation, $\sum_j f(jh) \varphi_h(x - jh)$ with $\varphi(x)$ of the form $\phi(x/c)$, where $\phi$ is rapidly decreasing (for example, a Gaussian) and $c > 0$ is called a shape parameter. Since we are in the case of $\kappa = 0$, their quasi-interpolants will not converge to $f(x)$, but it is shown in [13] that if $\phi$ is smooth, satisfies certain moment conditions and decays sufficiently rapidly at infinity, and if $f$ has bounded derivatives up till order $L$, then by choosing $c$ sufficiently large one can achieve an apparent order of convergence of $h^L$ up to a small saturation error which goes to 0 as $c$ tends to infinity. This should be compared with theorem 4.1 when $\kappa = 0$, in which case there will also be no actual convergence and where the required smoothness of $f$ is formulated in terms of its Fourier transform. Of course, this theorem concerns the exact interpolants instead of the quasi-interpolants. We will use shape parameters below to construct basis functions $\varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n)$ with small $\mathcal{T}_\kappa(\varphi)$.

We will encounter similar approximate approximation phenomena when studying convergence rates of RBF schemes in sections 5 and 6 below.

\footnote{The index $\kappa$ is a reminder of the degree of the singularity of $\varphi$ at 0, and therefore of the natural convergence rate of the RBF interpolants.}
Proof of theorem 4.1. It suffices, by (20), to bound \( \|(1 - \tilde{L}_1(h\xi))\hat{f}(\xi)\|_1 \). Let \( \tilde{T} := \tilde{T}_\kappa(\varphi) \). Then if \( \varepsilon > 0 \), there exists a \( \rho(\varepsilon) > 0 \) such that if \( h\xi < \rho(\varepsilon) \), then \( 0 \leq 1 - \tilde{L}_1(h\xi) \leq \frac{1}{2}(1 + \varepsilon)\tilde{T} \cdot h^\kappa|\xi|^\kappa \), and
\[
\|(1 - \tilde{L}_1(h\xi))\hat{f}(\xi)\|_1 \leq \int_{|\xi| \leq \rho(\varepsilon)} (1 - \tilde{L}_1(h\xi))|\hat{f}(\xi)|\,d\xi + 2\int_{|\xi| \geq \rho(\varepsilon)} |\hat{f}(\xi)|\,d\xi \leq \frac{1}{2}(1 + \varepsilon)\tilde{T}h^\kappa\int_{|\xi| \leq \rho(\varepsilon)/h} |\hat{f}(\xi)|\,|\xi|^\kappa\,d\xi + 2\rho(\varepsilon)^{-\kappa}h^\kappa\int_{|\xi| > \rho(\varepsilon)/h} |\hat{f}(\xi)|\,|\xi|^\kappa\,d\xi, \]
which implies the theorem. \( \square \)

Corollary 4.2. If \( \hat{f} \) satisfies the conditions of theorem 4.1, then
\[
\limsup_{h \to 0} h^{-\kappa}\|s_h[f] - f\|_A \leq \tilde{T}_\kappa(\varphi)\int_{\mathbb{R}^n} |\xi|^\kappa|\hat{f}(\xi)|\,d\xi. \tag{38} \]

The next theorem complements this upper bound by the lower bound (35) when \( \hat{f} \) is not necessarily compactly supported.

Theorem 4.3. Let \( f \) satisfy the hypothesis of theorem 4.1, \( f \in L^\infty_p(\mathbb{R}^n) \) for some \( p < \kappa \) and \( \hat{f} |_{\mathbb{R}^n \setminus 0} \in L^1(\mathbb{R}^n, \max(|\xi|^\kappa, |\xi|^s)) \) for some \( s > \kappa \). Then
\[
L_\kappa(\varphi)\int_{\mathbb{R}^n} |\xi|^\kappa|\hat{f}(\xi)|\,d\xi \leq \liminf_{h \to 0} h^{-\kappa}\|s_h[f] - f\|_A \leq \limsup_{h \to 0} h^{-\kappa}\|s_h[f] - f\|_A \leq \tilde{T}_\kappa(\varphi)\int_{\mathbb{R}^n} |\xi|^\kappa|\hat{f}(\xi)|\,d\xi. \]

Proof. We only need to establish the lower bound. If \( |\xi|_\infty = \max_j |\xi_j| \) is the \( \ell^\infty \)-norm on \( \mathbb{R}^n \), let \( Q_h = \{ \xi \in \mathbb{R}^n : |\xi|_\infty \leq \pi/h \} = [-\pi/h, \pi/h]^n \), the cube centered at 0 with sides \( 2\pi/h \), and let \( Q_h(\ell) = h^{-1}\ell + Q_h \). Then
\[
\|s_h[f] - \hat{f}\|_1 = \sum_\ell \int_{Q_h(\ell)} \left| \sum_k \left( \tilde{L}_1(h\xi) - \delta_{0,k} \right) \hat{f}(\xi + 2\pi k/h) \right|\,d\xi = \sum_\ell \int_{Q_h} \left| \sum_k \left( \tilde{L}_1(h\xi + 2\pi k\ell) - \delta_{0,k} \right) \hat{f}(\xi + 2\pi(k + \ell)/h) \right|\,d\xi \]
so that
\[
\|s_h[f] - \hat{f}\|_1 \geq \sum_\ell \int_{Q_h} \left| \left( \tilde{L}_1(h\xi + 2\pi \ell) - \delta_{0,-\ell} \right) \hat{f}(\xi) \right|\,d\xi - \sum_\ell \int_{Q_h} \sum_{k \neq -\ell} \left| \left( \tilde{L}_1(h\xi + 2\pi \ell) - \delta_{0,-k} \right) \hat{f}(\xi + 2\pi(k + \ell)/h) \right|\,d\xi. \tag{39} \]
The double sum in the second line can be bounded by
\[
\sum_\ell \sum_{k \neq -\ell} \int_{Q_h} \left| \tilde{L}_1(h\xi + 2\pi \ell) \hat{f}(\xi + 2\pi(k + \ell)/h) \right|\,d\xi + \sum_{\ell \neq 0} \int_{Q_h} \left| \left( \tilde{L}_1(h\xi + 2\pi \ell) - 1 \right) \hat{f}(\xi + 2\pi(k + \ell)/h) \right|\,d\xi \leq \left( \sum_\ell \frac{C}{1 + |\ell|^N} + 2 \right) \int_{|\xi|_\infty \geq \pi/h} |\hat{f}(\xi)|\,d\xi \leq Ch^\kappa\int_{\mathbb{R}^n} |\hat{f}(\xi)|\,|\xi|^\kappa\,d\xi, \]
where we used that
\[
\sup_{Q_h} |\hat{L}_1(h\xi + 2\pi\ell)| = \sup_{\eta \in \hat{Q}_1} \hat{L}_1(\eta + 2\pi\ell) \leq \frac{C}{(1 + |\ell|)^\nu}.
\]
The first line of (39), on account of \(\hat{L}_1\) taking values in \([0, 1]\), equals
\[
\int_{Q_h} \left( (1 - \hat{L}_1(h\xi) + \sum_{\ell \neq 0} \hat{L}_1(h\xi + 2\pi\ell) \right) |\hat{f}(\xi)| \, d\xi = 2 \int_{\mathbb{R}^n} (1 - \hat{L}_1(h\xi)) |\hat{f}| 1_{Q_h} \, d\xi,
\]
where \(1_{Q_h}\) is the indicator function of \(Q_h\). Since \(1_{Q_h} \rightarrow 1\) as \(h \rightarrow 0\), the lower bound now follows once more by Fatou’s lemma and the definition of \(L_\kappa(\varphi)\). \(\square\)

The next proposition gives a simple explicit formula for \(L_\kappa\) and \(\hat{L}_\kappa\):

**Proposition 4.4.** For \(\varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n)\) with \(\kappa \geq 0\) and \(N > n\), let
\[
A = A(\varphi) := \liminf_{\eta \rightarrow 0} |\eta|^{\kappa} \hat{\varphi}(\eta), \quad \overline{A} := \overline{A}(\varphi) := \limsup_{\eta \rightarrow 0} |\eta|^{\kappa} \hat{\varphi}(\eta).
\]
Then if \(\kappa > 0\),
\[
\hat{L}_\kappa(\varphi) = 2 A \sum_{k \neq 0} \hat{\varphi}(2\pi k), \quad L_\kappa(\varphi) = 2 A \sum_{k \neq 0} \varphi(2\pi k),
\]
while if \(\kappa = 0\),
\[
\hat{L}_0(\varphi) = \frac{2 \sum_{k \neq 0} \hat{\varphi}(2\pi k)}{\overline{A} + \sum_{k \neq 0} \hat{\varphi}(2\pi k)}, \quad L_0(\varphi) = \frac{2 \sum_{k \neq 0} \varphi(2\pi k)}{\overline{A} + \sum_{k \neq 0} \varphi(2\pi k)}.
\]
Note that \(A(\varphi) > 0\) by definition 2.1 (iii) and that the series in these formulas converges absolutely since \(N > n\).

**Proof.** If \(L_1 = L_1(\varphi)\) is the Lagrange function associated to \(\varphi\), then
\[
0 \leq 1 - \hat{L}_1(\eta) = \frac{\sum_{k \neq 0} \hat{\varphi}(\eta + 2\pi k)}{\sum_k \hat{\varphi}(\eta + 2\pi k)}.
\]
If we let \(R(\eta) := \sum_{k \neq 0} \hat{\varphi}(\eta + 2\pi k)\), then \(R\) is continuous (even \(C^{[\kappa]+n+1}\)) in a neighborhood of 0. Since
\[
1 - \hat{L}_1(\eta) = \frac{R(\eta)}{|\eta|^{\kappa} \hat{\varphi}(\eta) + |\eta|^{\kappa} R(\eta)},
\]
(41) and (42) follow upon letting \(\eta \rightarrow 0\). \(\square\)

**Corollary 4.5.** If \(\lim_{\eta \rightarrow 0} |\eta|^{-\kappa} \hat{\varphi}(\eta)\) exists, then \(L_\kappa(\varphi) = \hat{L}_\kappa(\varphi) = l_\kappa(\varphi)\), and
\[
\lim_{h \rightarrow 0} h^{-\kappa} \| s_h [f] - f \|_A = l_\kappa(\varphi) \| f \|_{1,\kappa},
\]
for \(f\) as in theorem 4.7 with \(s > \kappa\).

We can often construct basis functions with small \(\hat{L}_\kappa(\varphi)\) by introducing a so-called shape-parameter \(c\) and taking \(\varphi\) of the form \(\varphi(x) = \phi(x/c) := \phi_c(x)\) with \(c\) large, for suitable \(\phi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n)\):

**Proposition 4.6.** Suppose that \(\phi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n)\) with \(\kappa \geq 0\) and \(N > \max(\kappa, n)\). Then \(\lim_{c \rightarrow \infty} \hat{L}_\kappa(\phi_c) = 0\).

**Proof.** Since \(\hat{\phi}_c(\eta) = c^n \hat{\phi}(c\eta)\), it follows that \(A(\phi_c) = c^n A(\phi)\), and therefore, by (41), if \(\kappa > 0\),
\[
\hat{L}_\kappa(\phi_c) = 2 \frac{c^n}{A(\phi)} \sum_{k \neq 0} \hat{\phi}(2\pi ck) \leq \frac{C c^{\kappa-N} \sum_{k \neq 0} |k|^{-N}}{A(\phi)}
\]
for \(c \rightarrow \infty\).
which tends to 0 as $c \to \infty$ under the stated conditions on $\kappa$. The case of $\kappa = 0$ follows by observing that

$$T_0(\phi_c) \leq \frac{2}{A(\phi_c)} \sum_{k \neq 0} \hat{\phi}_c(2\pi k),$$

and proceeding as before.

Examples of basis functions $\phi$ which satisfy the conditions of the corollary are the Gaussians (for which $\kappa = 0$) and the generalized multiquadrics, whose Fourier transforms decay exponentially at infinity, but none of the homogeneous basis functions, since for these $\kappa = N$: see examples [5,12] below for more discussion. In fact, for a Gaussian or a multiquadric, $\hat{\phi}(\xi)$ decays exponentially at infinity, and $T_\kappa(c)$ will decay exponentially in $c$.

4.1. Dependence of the constants on the shape parameter. If we want to apply (47) with $\varphi = \phi_c$ as above (and some fixed $\varepsilon$), it becomes interesting to ask how the constant $C_\kappa = C_{\varepsilon,\varphi}$ depends on the shape-parameter $c$. On expects it to go to infinity with $c$, and the question then is at what rate exactly. We will see that under reasonable assumptions on $\hat{\phi}(\eta)$, this constant behaves like $c^\delta$.

We first analyse its dependence on $\varphi$ for a general $\varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n)$. From the proof, $C_{\varepsilon,\varphi}$ is proportional to $\rho(\varepsilon, \varphi)^{-\delta}$, where $\rho(\varepsilon, \varphi)$ is any positive number such that $2|\eta|^{-\kappa}(1 - \tilde{L}_1(\eta)) < (1 + \varepsilon)\tilde{T}_\kappa$ for $|\eta| \leq \rho(\varepsilon)$. If we put

$$R(\eta) := R_\varphi(\eta) = \sum_{k \neq 0} \hat{\varphi}(\eta + 2\pi k),$$

then

$$0 \leq \frac{1 - \tilde{L}_1(\eta)}{|\eta|^\kappa} = \frac{R(\eta)}{|\eta|^{\kappa}(\hat{\varphi}(\eta) + R(\eta))} \leq \frac{R(\eta)}{|\eta|^{\kappa}\hat{\varphi}(\eta)}$$

If we introduce the family of semi-norms

$$p_r(\psi) := \sum_{k \neq 0} \sup_{|\eta| \leq r} |\psi(\eta + 2\pi k)|, \quad r > 0,$$

then $|R(\eta) - R(0)| \leq p_r(|\nabla \hat{\varphi}|) |\eta|$ for $|\eta| \leq r$, where $|\nabla \varphi|$ the euclidean norm of the gradient, so that

$$0 \leq 2R(\eta) < 2(1 + \varepsilon)R(0) \text{ if } |\eta| \leq \rho_1,$$

where

$$\rho_1 := \rho_1(\varepsilon, \varphi) := \min(r, \varepsilon R(0)/p_r(|\nabla \hat{\varphi}|)).$$

On the other hand, there exists a $\rho_2 := \rho_2(\varepsilon, \varphi)$ such that $|\eta|^\kappa \hat{\varphi}(\eta) \geq (1 - \varepsilon)A(\varphi)$ if $|\eta| \leq \rho_2$, where $A = A(\varphi)$ was defined in proposition [4.1] above. Combining these two estimates and observing that $\tilde{T}_\kappa = 2R(0)/A$ we see that

$$0 \leq \frac{1 - \tilde{L}_1(\eta)}{|\eta|^\kappa} \leq \left(\frac{1 + \varepsilon}{1 - \varepsilon}\right) \tilde{T}_\kappa$$

if $|\eta| \leq \rho$ where

$$\rho := \rho(\varepsilon, \varphi) := \min(\rho_1, \rho_2),$$

and we get an approximate approximation estimate in the form

$$\|s_{h(\varphi)}[f] - f\|_A \leq \frac{1 + \varepsilon}{1 - \varepsilon} \tilde{T}_\kappa(\rho^\kappa) \|\hat{f}\|^2_{\kappa,\kappa} + \frac{h^\kappa}{\rho \kappa^\delta} \|\hat{f}\|^2_{\kappa,\kappa}.$$  

We now derive lower bounds for $\rho$ when $\varphi = \phi_c$ for a fixed $\varepsilon$, which we will sometimes drop from the notations. First of all, there exists a $r(\phi) = r(\phi; \varepsilon)$ such that $\hat{\phi}(\eta) \geq (1 - \varepsilon)A(\phi)$ if $|\eta| \leq r(\phi)$. This implies that $|\eta|^\kappa \hat{\phi}(\eta) = c^n|\eta|^\kappa \hat{\phi}(c\eta) \geq c^{n-\kappa} \frac{1}{2}A(\phi) = \frac{1}{2}A(\phi_c)$ if $c|\eta| \leq r(\phi)$, so we can take $\rho_2(\varepsilon, \phi_c) = r(\phi)/c$.

The behavior of $R_{\phi_c}(0)/p_{\varepsilon,N}(\nabla \hat{\phi}_c)$ for large $c$ depends on the asymptotic behavior at infinity of $\hat{\phi}$ and of $\nabla \hat{\phi}$. We consider the case of a polynomially decaying $\hat{\phi}$ and that of an exponentially decaying one.
Example 4.7. (Polynomially decaying $\hat{\phi}$) Suppose that
\begin{equation}
\inf_{|\eta| \geq 2\pi} |\eta|^N \hat{\phi}(\eta) =: a > 0.
\end{equation}
Then
\begin{equation}
R_{\hat{\phi}}(0) \geq ac^{n-N} \sum_{k \neq 0} (2\pi|\eta|)^{-N}.
\end{equation}
On the one hand, we have for any $\varphi \in \mathcal{B}_{k,N}(\mathbb{R}^n)$ that
\[
p_c(\nabla \varphi) \leq \left( \sum_{k \neq 0} (2\pi(|k| - \frac{1}{2})^{-N}) \right) \sup_{|\eta| \geq \pi} |\eta|^N |\nabla \hat{\varphi}(\eta)|,
\]
which implies that
\[
p_c(|\nabla \hat{\varphi}|) \leq C \cdot c^{n+1-N} \sup_{|\eta| \geq \pi} |\eta|^N |\nabla \hat{\varphi}(\eta)|
\]
which can certainly be bounded from above by a constant times $c^{n+1-N}$ if $c \geq 1$. Taking $r = \pi$ in (45) we therefore find that $\rho_1(\varepsilon, \phi_c) \geq Cc^{-1}$ for some constant $C$. It follows that $\rho(\varepsilon, \phi_c) \geq Cc^{-1}$ for some constant $C$ and (17) implies an estimate
\[
\|s_k[f] - f\|_A \leq C_1 c^{N-N} \|\hat{f}\|_{1,s}^0 + C_2 c^s \|\hat{f}\|_{1,s}^0,
\]
with constants $C_1$ and $C_2$ which depend on $\phi$ (and on $\varepsilon$), but not on the shape parameter $c$. Neglecting the numerical values of these constants, the second term will dominate the first as long as $h \gg c^{-1-N/(s-\kappa)}$ and the estimate will be relevant for $h$'s in the range $c^{-1-N/(s-\kappa)} \ll h \ll c^{-1}$ (since we want $(ch)^s \ll 1$).

Example 4.8. (Exponentially decaying $\hat{\phi}$) Suppose that $\phi \in \mathcal{B}_{k,N}(\mathbb{R}^n)$ is such that there exists constants $p \geq 0$ and $C > 0$ for which
\begin{equation}
C^{-1} |\eta|^{-p} e^{-|\eta|} \leq \hat{\phi}(\eta) \leq C |\eta|^{-p} e^{-|\eta|}, \quad |\eta| \geq 1,
\end{equation}
and
\[
|\nabla \hat{\phi}| \leq C |\eta|^{-p} e^{-|\eta|}, \quad |\eta| \geq 1.
\]
An example of such a $\phi$ is given by the multi-quadric on $\mathbb{R}^n$: see Example 5.12 below.

Since $\hat{\phi}_c(\eta) = c^p \hat{\phi}(c\eta)$, $p_c(|\nabla \hat{\phi}_c|)$ will then be bounded by a constant times
\[
c^{n+1} \sum_{k \neq 0} \sup_{|\eta| \leq r} \frac{e^{-c|\eta+2\pi k|}}{|\eta+2\pi ck|^p}
\]
Using that if $|\eta| \leq r$, $e^{-2|\eta+2\pi k|} \leq e^{cr} e^{-2\pi c |k|}$ and, assuming wlog that $r \leq \pi$, that $|\eta + 2\pi k| \geq 2\pi(|k| - \frac{1}{2}) \geq \pi |k|$ for $|k| \geq 1$, we find that this expression is bounded by
\[
c^{n+1} e^{cr} \sum_{k \neq 0} \frac{e^{-2\pi c |k|}}{(\pi c |k|)^p},
\]
and using the first inequality of (44), it follows that $p_c(|\nabla \hat{\phi}_c|)$ is bounded by a constant times $cR_{\phi_c}(0)$. Remembering (14), it follows that $\rho_1(\phi_c)$ is bounded from below by a constant times $\min(\pi, r, e^{-1})$. This still has $r$ as a free parameter, and taking $r = e^{-1}$ and remembering the estimate for $\rho_2(\phi_c)$ from the previous example, we find that we can take $\rho(\phi_c) \geq C \cdot c^{-1}$. Since (50) implies that $L_c \leq C \cdot c^{n-p} e^{-2\pi c}$, we now have the approximate approximation estimate
\[
\|s_k[f] - f\|_A \leq C_1 c^{n-p} e^{-2\pi c} \|\hat{f}\|_{1,\kappa}^0 + C_2 c^s \|\hat{f}\|_{1,s}^0,
\]
which is relevant for the range $c^{-1-p/(s-\kappa)} e^{-2\pi c/(s-\kappa)} \ll h \ll c^{-1}$, where $c \gg 1$.  

**Example 4.9.** (The Gaussian) A final interesting example is that of the Gaussian, with \( \hat{\phi}(\eta) = e^{-|\eta|^2} \).

In this case, \( \nabla \hat{\phi}_c(\eta) = c^n + 2e^{-|\eta|^2} \) and \( p_r(\nabla \hat{\phi}_c) \) for \( r \leq \pi \) is bounded by a constant times

\[
e^{c^n + 2|\eta + 2\pi k|^2} \leq e^{c^n + 2e^{(1-\epsilon)2\pi k^2}} \sum_{k \neq 0} e^{-c^2(1-\epsilon) |2\pi k|^2}.
\]

for any \( 0 < \epsilon < 1 \), where we used the inequality \( (a - b)^2 \geq (1 - \epsilon) a^2 - (1 - 1)b^2 \) with \( a = 2\pi |k| \) and \( b = |\eta| \). Asymptotically, for large \( c \), this behaves like \( c^n + 2e^{(1-\epsilon)2\pi^2(1-1)c^2} \), while \( R(\phi_c)(0) e^n + 2 \approx e^{-4\pi^2 c^2} \), and we can choose \( \rho_1(\phi_c) \) (with a fixed \( \epsilon \)) to be bounded from below by a constant times

\[
\min \left( \pi, r, e^{-4\pi^2 c^2} e^{(1-1)c^2} \right).
\]

If we now take the free parameters \( r \) and \( \epsilon \) equal to \( e^{-2} \) and remember that \( \rho_1(\phi_c) \approx c^{-1} \), we conclude that we can choose \( \rho_c(\phi_c) \approx c^{-2} \), and we have the following approximate approximation estimate for Gaussian RBF interpolation (recalling that \( \kappa = 0 \) in this case):

\[
||s_{h}[f] - f||_A \leq C_1 e^{-4\pi^2 c^2} ||f||_1 + C_2 e^{2\epsilon} h^s ||f||_1^0
\]

where the constant \( C_1 \) can be taken arbitrarily close to 1.

## 5. Convergence of stationary RBF schemes for the heat equation

### 5.1. An RBF scheme for the heat equation

We introduce an RBF scheme for the Cauchy problem for the classical heat equation,

\[
\begin{align*}
\partial_t u(x, t) &= \Delta u(x, t), \quad x \in \mathbb{R}^n, t > 0 \\
u(x, 0) &= f(x),
\end{align*}
\]

\( \Delta = \sum_{j=1}^{n} \partial_{x_j}^2 \) being the Laplace operator, and examine its convergence. The scheme is a variant of the classical method of lines, and looks for approximate solutions \( u_h \) of the form

\[
u_h(x, t) = \sum_{k \in \mathbb{Z}^n} c_k(t; h) L(\cdot - k),
\]

where \( c_k(\cdot; h) : [0, \infty) \to \mathbb{R} \) are differentiable functions. Here, \( L \) is the Lagrange interpolation function of theorem 3.3 associated to a given basis function \( \varphi \in \mathcal{B}_{k,N}(\mathbb{R}^n) \) which we fix. We assume throughout this section that \( N > n + 2 \) and \( \kappa > 0 \). We will see below that for the scheme to converge we will need that \( \kappa > 2 \) while for \( \kappa = 2 \) we can have approximate convergence. The coefficients \( c_k(t; h) \) of \( u_h \) are determined by requiring that \( u_h \) solve \( \hat{u}_t \) exactly in the points of \( h\mathbb{Z}^n \):

\[
\partial_t u_h(jh, t) = \Delta u_h(jh, t), \quad \forall j \in \mathbb{Z}^n,
\]

while \( u_h(x, 0) \) is taken to be equal to \( s_{h}[f](x) \), the RBF-interpolant of \( f \). This leads to the following initial value problem for the coefficients \( c_j(t; h) \):

\[
\begin{align*}
dc_j \left( t; h \right) &= \Delta \sum_{k} \left( \Delta L_1(j - k)c_k(t; h) \right) \\
c_j(0; h) &= f(jh).
\end{align*}
\]

Since this is an infinite system of ODEs we first discuss existence and uniqueness of solutions in suitable Banach spaces.

For \( s \in \mathbb{R} \), let

\[
\ell^\infty_s := \ell^\infty_s(\mathbb{Z}^n) := \{ (c_j)_{j \in \mathbb{Z}^n} : ||c||_{\infty, s} := \sup_j (1 + |j|)^s |c_j| < \infty \}.
\]

One easily verifies that the convolution operator

\[
A := A_L : (c_j)_{j} \mapsto \left( \sum_k \Delta L_1(j - k)c_k \right)_{j},
\]

for \( s \in \mathbb{R} \), let

\[
\ell^\infty_s := \ell^\infty_s(\mathbb{Z}^n) := \{ (c_j)_{j \in \mathbb{Z}^n} : ||c||_{\infty, s} := \sup_j (1 + |j|)^s |c_j| < \infty \}.
\]

One easily verifies that the convolution operator

\[
A := A_L : (c_j)_{j} \mapsto \left( \sum_k \Delta L_1(j - k)c_k \right)_{j},
\]
is a bounded operator on $\ell^\infty_p$ if $0 \leq p < \kappa$. Indeed,
\[
(1 + |j|)^{-p} \sum_k \Delta L_1(j - k)c_k \leq \sum_k \left( (1 + |j|)^{-p}((1 + |k|)^p|\Delta L_1(j - k)|) \right) ||c||_{\infty,-p}
\]
\[
\leq \left( \sum_k (1 + |j - k|)^p|\Delta L_1(j - k)| \right) ||c||_{\infty,-p},
\]
using that $(1 + |k|) \leq (1 + |j - k|)(1 + |j|)$. The sum of the series on the right is independent of $j$ and finite if $p < \kappa$, by theorem 2.8. It follows that if we let $c(t) := (c_j(t))_j$, and if the initial value $c(0) \in \ell^\infty_p$ for some $p \in [0, \kappa)$, the system (51) has a unique $\ell^\infty_p$-valued solution which is given by $c(t) = e^{h^{-2}tA_1}(c(0))$. Next, if for $c \in \ell^\infty_p$ we let (with some abuse of notation)
\[
s_h[c](x) := \sum_{j \in \mathbb{Z}^n} c_j L_1(h^{-1}x - j),
\]
then $s_h : c \to s_h[c]$ is a bounded linear operator from $\ell^\infty_p \to L^\infty_p(\mathbb{R})$ if $0 \leq p < \kappa$. Indeed, using the decay of $L_1$,
\[
\frac{||s_h[c]||_{\infty,-p}}{||c||_{\infty,-p}} \leq \sup_{x \in \mathbb{R}^n} (1 + |x|)^{-p} \sum_j \frac{(1 + |j|)^p}{(1 + h^{-1}x - j)^{\kappa + n}} \leq \sup_{y \in \mathbb{R}^n} (1 + |y|)^{-p} \sum_j \frac{(1 + |j|)^p}{(1 + |y - j|)^{\kappa + n}} \leq \sup_{y \in \mathbb{R}^n} \left( \frac{1 + |j|}{1 + h|y|} \right)^p \sum_j \frac{(1 + |j|)^p}{(1 + |y - j|)^{\kappa + n}}.
\]
The sum on the right converges and defines a 1-periodic continuous function on $\mathbb{R}^n$ which is therefore uniformly bounded, while the factor in front can be estimated by $\max(1, h^{-n})$.

If $f \in L^\infty_p(\mathbb{R}^n)$, we can in particular take $c(0) = f|_{h\mathbb{Z}^n}$, and
\[
u_h[f](x, t) := \sum_{j \in \mathbb{Z}^n} c_j L_1(h^{-1}x - j),
\]
is the unique function (52) whose coefficients satisfy (53). We summarize this discussion in the following lemma:

**Lemma 5.1.** If $0 \leq p < \kappa$ and if $f \in L^\infty_p(\mathbb{R}^n)$ then there is a unique function $u_h = u_h[f] \in C^1([0, \infty); L^\infty_p(\mathbb{R}^n))$ of the form (52) which satisfies (53) and $f \to u_h[f](\cdot, t)$ is a bounded linear map on $L^\infty_p(\mathbb{R})$, for each $t \geq 0$ and $h > 0$.

In particular, for each fixed $t$, $u_h(x, t)$ has tempered growth in $x$, and thus possesses a well-defined Fourier transform, which we will study next.

### 5.2. Convergence of the scheme in Wiener norm

We start by computing the Fourier transform of $u_h = u_h[f]$. Let us introduce the auxiliary function $G(\eta)$ on $\mathbb{R}^n$ by
\[
G(\eta) := G_\varphi(\eta) := \sum_k |\eta + 2\pi k|^{2}\hat{L}_1(\eta + 2\pi k) = \frac{\sum_k |\eta + 2\pi k|^2 \hat{\varphi}(\eta + 2\pi k)}{\sum_k \hat{\varphi}(\eta + 2\pi k)}
\]
where the series converges absolutely, since $N > n + 2$.

**Lemma 5.2.** Let $\varphi \in \mathcal{B}_{\kappa, N}(\mathbb{R})$ with $N > n + 2$ and $\kappa > 0$. If $\hat{f} \in L^1(\mathbb{R}^n)$, then the Fourier transform of $u_h(x, t)$ with respect to $x$ is given by
\[
\widehat{u}_h(\xi, t) = e^{-th^{-2}G(h\xi)} \overline{s_h[\hat{f}]}(\xi).
\]
Proof. Since \((\Delta L_1(j))_{j \in \mathbb{Z}^n}\) is in \(\ell^1 := \ell^1(\mathbb{Z}^n)\), it follows that \(A_L\) is a bounded operator on \(\ell^1\), and hence the system \((5.3)\) has an \(\ell^1\)-valued solution \(c(t) = (c_j(t))_j\) if the initial value \(c(0) \in \ell_1\). In particular, if \(c(0) = f|_{h\mathbb{Z}^n}\) with \(f \in \mathcal{S}(\mathbb{R}^n)\), then the function \((x, k) \mapsto c_k(t; h)L_1(h^{-1}x - k)\) is absolutely integrable on \(\mathbb{Z}^n \times \mathbb{R}^n\) for each fixed \(t \geq 0, h > 0\), and an application of Fubini’s theorem shows that the Fourier transform of \(u_h(\cdot, t)\) is given by \(\hat{u}_h(\xi, t) = h^n \hat{L}_1(h\xi) \gamma_h(\xi, t)\), where
\[
\gamma_h(\xi, t) := \sum_{j \in \mathbb{Z}} c_j(t; h)e^{-ih(j, \xi)},
\]
the series being absolutely convergent. By \((5.3)\),
\[
\partial_t \gamma_h(\xi, t) = h^{-2} \left( \sum_j \Delta L_1(j)e^{-ih(j, \xi)} \right) \gamma_h(\xi, t)
\]
\[
= h^{-2} \left( \sum_k \hat{\Delta L_1}(h\xi + 2\pi k) \right) \gamma_h(\xi, t)
\]
\[
= -h^{-2}G(h\xi)\gamma_h(\xi, t),
\]
where the second line follows from the Poisson summation formula, whose application is justified by the decay at infinity of \(\Delta L_1\) and its Fourier transform. Hence \(\partial_t \hat{u}_h(\xi, t) = -h^{-2}G(h\xi)\hat{u}_h(\xi, t)\) which, together with the initial condition \(u_h(x, 0) = s_h[f](x)\) implies \((5.7)\).

If \(f \in L^1(\mathbb{R}^n)\), \((5.7)\) follows by an approximation argument: if \(f_\nu \to f\) in \(L^1\) with \(f_\nu\) rapidly decreasing, then \(f_\nu \to f\) in \(L^\infty\), so by lemma \((5.3)\) \(u_h[f_\nu](\cdot, t) \to u_h[f](\cdot, t)\) in \(L^\infty\) also, since \(\kappa > 0\). Hence their Fourier transforms converge in \(\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)\). On the other hand, \(s_h[f_\nu] = \Sigma_h(f_\nu) \to \Sigma_h(f) = s_h[f]\) in \(L^1\) and since \(G\) is non-negative, \(e^{-h^{-2}G(h\xi)}s_h[f_\nu](\xi) \to e^{-h^{-2}G(h\xi)}s_h[f](\xi)\) in \(L^1\), and therefore in \(\mathcal{S}'\).

The following proposition lists some useful properties of \(G\).

**Proposition 5.3.** Suppose that \(\varphi \in \mathbb{B}_{\kappa, N}\) with \(N > n + 2\) and let \(G := G_\varphi\) be defined by \((5.7)\). Then

(i) \(G\) is a positive \(2\pi\)-periodic function, and \(G(\eta) = 0\) iff \(\eta \in 2\pi \mathbb{Z}^n\).

(ii) There exists a constant \(C > 0\) such that \(0 \leq G(\eta) - |\eta|^2 \leq C|\eta|^\kappa\) for \(|\eta| \leq \pi\).

(iii) \(G\) belongs to the Hölder space \(G_b^{[\kappa]-1, \lambda}(\mathbb{R}^n)\) with \(\lambda = \kappa - ([\kappa] - 1)\).

**Proof.** (i) The periodicity is obvious and the positivity of \(G\) is an immediate consequence of the positivity of \(\hat{\varphi}\). Next, \(G(\eta) = 0\) iff \(|\eta + 2\pi k|^2 \hat{L}_1(\eta + 2\pi k) = 0\) for all \(k\). Since \(\hat{L}_1\) is non-zero outside of \((2\pi)\mathbb{Z}^n \setminus 0\), this implies that \(\eta \in 2\pi \cdot \mathbb{Z}^n\). Conversely, any such \(\eta\) is a zero, given that \(\hat{L}_1(2\pi k) = \delta_{0k}\).

Assertion (ii) follows from
\[
G(\eta) - |\eta|^2 = \frac{\sum_{k \neq 0}(|\eta + 2\pi k|^2 - |\eta|^2)\hat{\varphi}(\eta + 2\pi k)}{\sum_k \hat{\varphi}(\eta + 2\pi k)}
= \frac{\sum_{k \neq 0}(4\pi^2|k|^2 + 4\pi(\eta, k))\hat{\varphi}(\eta + 2\pi k)}{\sum_k \hat{\varphi}(\eta + 2\pi k)}
\]
and the behaviour of \(\hat{\varphi}(\eta)\) for small \(\eta\).

Finally, (iii) follows from lemma \((5.3)\). \(\square\)

Property (iii) allows us to extend lemma \((5.3)\) to functions of polynomial growth: compare lemma \(3.5\).
Lemma 5.4. Suppose that $f \in L^\infty_p(\mathbb{R}^n)$ for some $p < \kappa$ such that $\hat{f}|_{\mathbb{R}^n \setminus 0} \in L^1(\mathbb{R}^n, (|\xi|^{\kappa} \wedge 1)\,d\xi)$. Then the identity (57) holds in the sense of distributions for each $t \geq 0$. Moreover, $u_h[\hat{f}](\cdot, t) - \hat{u}(\cdot, t)$ can be identified with the function

\begin{equation}
-\text{th}^{-2}G(h\xi) \left( s_h[\hat{f}] - \hat{f} \right)(\xi) + \left( e^{-\text{th}^{-2}G(h\xi)} - 1 \right) e^{-t|\xi|^2} \hat{f}(\xi),
\end{equation}

which moreover is integrable on $\mathbb{R}^n$.

Here, and below, $\hat{f}$ without argument will indicate the distribution, and $\hat{f}(\xi)$ the function with which it can be identified on $\mathbb{R}^n \setminus 0$; we recall that by lemma 3.3, $s_h[\hat{f}] - \hat{f}$ can be identified with an $L^1$-function. That the second term of (58) is $L^1$ follows from lemma 5.3 ii).

Proof. We just clarify the statement of the lemma, and refer to Appendix II for the proof, which uses elements of the proof of lemma 3.3. The proof of that lemma shows that $s_h[\hat{f}] = \Sigma_h(\hat{f})$ extends to a continuous linear functional on $C_b^{(\kappa)-1,\lambda}(\mathbb{R}^n)$. Since (i) and (iii) of proposition 5.3 imply that the function $e^{-\text{th}^{-2}G(h\xi)}$ is in $C_b^{(\kappa)-1,\lambda}(\mathbb{R}^n)$, its product with $\Sigma_h(\hat{f})$ is well-defined as an element of the dual of $C_b^{(\kappa)-1,\lambda}(\mathbb{R}^n)$.

We can now show convergence in Wiener norm of the $u_h$ to the solution of the Cauchy problem (57): recall again (2).

Theorem 5.5. Let $\varphi \in \mathfrak{B}_{\kappa, N}(\mathbb{R}^n)$ with $N > n + 2$ and $\kappa > 2$ and suppose that $f \in L^\infty_p(\mathbb{R}^n)$ for some $p < \kappa$ such that

$\hat{f}|_{\mathbb{R}^n \setminus 0} \in L^1_{\kappa-2,N}(\mathbb{R}^n)$

Let $u_h := u_h[\hat{f}]$ and let $u$ be the solution to the Cauchy problem (57) with initial value $f$. Then there exists a constant $C = C_{\varphi}$ independent of $h$ and $f$ such that for $0 < h \leq 1$, $\quad$

\begin{equation}
||u_h(\cdot, t) - u(\cdot, t)||_A \leq C \max(t, 1)||\hat{f}||_{L^\kappa-2,N} \quad h^{\kappa-2}.
\end{equation}

In particular, $u_h$ converges to $u$ in sup-norm at a rate of $h^{\kappa-2}$.

Proof. Note that the conditions on $f$ are weaker than those of theorem 3.4. Indeed, we will be applying corollary 3.8 with $r = \kappa - 2$. By lemma 5.4 we can estimate $||\hat{u}_h(\xi, t) - \hat{u}(\xi, t)||_1$ by

\begin{equation}
\left| e^{-\text{th}^{-2}G(h\xi)} \left( s_h[\hat{f}] - \hat{f} \right) \right|_1 + \left| \left( 1 - e^{-\text{th}^{-2}G(h\xi)} \right) e^{-t|\xi|^2} \hat{f} \right|_1.
\end{equation}

Since $G$ is positive, the first term can be bounded by $||s_h[\hat{f}] - \hat{f}||_1 \leq C||\hat{f}||_{L^\kappa-2,N} \quad h^{\kappa-2}$, by corollary 3.8. To estimate the second term, we first use proposition 5.3 ii) together with the inequality $(1 - e^{-x}) \leq x$ for $x \geq 0$ to estimate the integral over $|\xi| \leq \pi$ by

\begin{equation}
\int_{|\xi| \leq \pi} \left( 1 - e^{-\text{th}^{-2}G(h\xi)} \right) e^{-t|\xi|^2} |\hat{f}(\xi)| \,d\xi
\leq C \int_{|\xi| \leq h^{\kappa-2}} t|\xi|^\kappa |\hat{f}(\xi)| e^{-t|\xi|^2} \,d\xi
\leq Ch^{\kappa-2} \int_{|\xi| \leq 1} t|\xi|^\kappa |\hat{f}(\xi)| \,d\xi + \sup_{z} \left( |z|^2 e^{-\frac{1}{2}|z|^2} \right) \int_{|\xi| \leq \pi/h} |\xi|^{\kappa-2} |\hat{f}(\xi)| \,d\xi
\leq C \max(t, 2e^{-1}) h^{\kappa-2} ||\hat{f}||_{L^\kappa-2,N},
\end{equation}

assuming wlog that $h \leq \pi$. Since the integral over $|\xi| \geq h^{-1}$ can be bounded by

\begin{equation}
\int_{|\xi| \geq h^{-1}} \left| e^{-\text{th}^{-2}G(h\xi)} - e^{-t|\xi|^2} \right| |\hat{f}(\xi)| \,d\xi
\leq 2\pi^{-\kappa} h^{\kappa-2} \int_{|\xi| \geq h^{-1}} |\xi|^{\kappa-2} |\hat{f}(\xi)| \,d\xi
\leq 2\pi^{-\kappa} h^{\kappa-2} ||\hat{f}||_{L^\kappa-2,N},
\end{equation}

we have the desired estimate.
the theorem follows, where in fact we have established the more precise bound
\[
\left( C_{1,\varphi} \max(t, 2e^{-1}) + C_{2,\varphi} + 2\pi^{-\left(\kappa-2\right)} \right) \|\hat{f}\|_{\kappa-2}^{\kappa} h^{\kappa-2},
\]
with \( C_{1,\varphi} := \sup_{0<|\eta| \leq \pi} (G(\eta) - |\eta|^2)/|\eta|^\kappa \) and \( C_{2,\varphi} := \sup_{0<|\eta|} (1-\hat{L}_1(\eta))/|\eta|^\kappa \), the constant of corollary \ref{corollary:convergence-rate}.

**Remarks 5.6.** (i) If we strengthen the hypothesis on \( \hat{f} \) to \( \hat{f} \mid_{\Re^n\setminus0} \in \hat{L}_{1,\kappa-2}(\Re^n) \), we obtain an error bound of \( Ch^{\kappa-2}\|\hat{f}\|_{\kappa-2}^{\kappa} \) with a constant \( C \) which is independent of \( t \).

(ii) The estimate for the integral over \( |\xi| \geq \pi/h \) is quite rough, but note that since \( h^{-2}G(h\xi) \) is \( 2\pi/h \)-periodic and equal to 0 in points of \( 2\pi h^{-1}\Z^n \), \( e^{-h^{-2}G(h\xi)} - e^{-t|\xi|^2} \) can get arbitrarily close to 1 on this set. A similar remark applies to the first term of \eqref{eq:convergence-representation}. There may be room for improvement, by further analyzing the contribution of a small neighborhood of this set of points.

It is not difficult to verify that \( h^{\kappa-2} \) is the exact order of approximation if \( \kappa > 2 \), and that the scheme does not converge if \( \kappa = 2 \). Let
\[
\underline{g}_\kappa = g_{\kappa,\varphi} := \liminf_{\eta \to 0} \frac{G_\varphi(\eta) - |\eta|^2}{|\eta|^\kappa}.
\]
We will see in proposition \ref{prop:convergence-rate} below that \( \underline{g}_\kappa > 0 \).

**Theorem 5.7.** Let \( f \in L^\infty_{-p}(\Re^n) \) for some \( p < \kappa \) such that \( \hat{f} \mid_{\Re^n\setminus0} \in \hat{L}_{1,\kappa-2}(\Re^n) \) for some \( k \in (\kappa-2, \kappa] \). Then if \( \kappa > 2 \),
\[
\liminf_{h \to 0} h^{-\kappa+2}\|u_h(\cdot, t) - u(\cdot, t)\|_A \geq \underline{g}_\kappa \int_{\Re^n} |\xi|^\kappa |\hat{f}(\xi)| e^{-t|\xi|^2} d\xi,
\]
while if \( \kappa = 2 \),
\[
\liminf_{h \to 0} \|u_h - u\|_A \geq \int_{\Re^n} \left(1 - e^{-\underline{g}_\kappa |\xi|^2}\right) e^{-t|\xi|^2} |\hat{f}(\xi)| d\xi.
\]

**Proof.** Since \( ||\hat{s}_h[f] - \hat{f}||_1 = O(h^k) \) and \( k > \kappa - 2 \), lemma \ref{lemma:convergence-representation}, corollary \ref{corollary:convergence-rate} and Fatou’s lemma imply that
\[
\liminf_{h \to 0} h^{-\kappa+2}\|\hat{u}_h(\cdot, t) - \hat{u}(\cdot, t)\|_1 \geq \int_{\Re^n} \liminf_{h \to 0} h^{-\kappa+2} \left(1 - e^{-h^{-2}R(h\xi)}\right) e^{-t|\xi|^2} |\hat{f}(\xi)| d\xi
\]
where we have put \( R(\eta) = G(\eta) - |\eta|^2 \). By the mean value theorem applied to the exponential, there exist \( \zeta = \zeta(h, \xi, t) \in [0, 1] \) such that
\[
\left(1 - e^{-h^{-2}tR(h\xi)}\right) = h^{-2}tR(h\xi)e^{c_h^{-2}R(h\xi)}.
\]
If \( \kappa > 2 \), then \( R(h\xi) \to 0 \) as \( h \to 0 \) by proposition \ref{prop:convergence-rate}(ii), and \eqref{eq:convergence-representation} follows from
\[
\liminf_{h \to 0} h^{-\kappa}\|R(h\xi)| = \underline{g}_\kappa |\xi|^\kappa.
\]
If \( \kappa = 2 \), then
\[
\liminf_{h \to 0} \left(1 - e^{-h^{-2}tR(h\xi)}\right) = 1 - e^{-\underline{g}_\kappa |\xi|^2},
\]
which proves \eqref{eq:convergence-representation}. □
5.3. Approximate approximation properties of the scheme. As we have just seen, our RBF scheme for the heat equation does not converge if $\kappa = 2$, which is for example the case if $n = 1$ and the basis function is the Hardy multiquadric. It turns out that we can then still achieve an arbitrarily small absolute error by an appropriate choice of the basis function, e.g. by introducing a shape parameter. This is again an approximate approximation phenomenon of the type encountered in section 4, and which if $\kappa > 2$ will again take the form of an apparent rate of convergence better than $O(h^{\kappa-2})$ up to some threshold $h_0$, for initial conditions $f$ whose Fourier transform decay sufficiently rapidly at infinity. Suppose that $\varphi \in \mathcal{B}_{2,N}(\mathbb{R}^n)$ with $N > n + 2$ and $\kappa \geq 2$, and let the initial condition $f$ be as in lemma 5.3. Let

$$\overline{g}_\kappa := \overline{g}_\kappa(\varphi) := \limsup_{\eta \to 0} \frac{G(\eta) - |\eta|^2}{|\eta|^\kappa}.$$ (65)

**Theorem 5.8.** Let $\kappa \geq 2$ and $s > \kappa - 2$. If $s > \kappa$ then there exist for any $\varepsilon > 0$ a constant $C_\varepsilon$ which does not depend on $t \geq 0$ such that if $\int_{\mathbb{R}^n \setminus \{0\}} \in L^1_{\kappa} \cap L^1_s$,

$$|u_h(\cdot,t) - u(\cdot,t)|_A \leq (1 + \varepsilon)\overline{g}_\kappa h^{\kappa-2} \int_{\mathbb{R}^n} t|\xi|^\kappa |\hat{f}(\xi)| e^{-\varepsilon |\xi|^2} d\xi + (1 + \varepsilon)\overline{g}_\kappa h^\kappa \|\hat{f}\|^2_{s,\kappa}$$

while if $\kappa - 2 < s \leq \kappa$ and $\hat{f} \in L^1_{s,\kappa}$, then

$$|u_h(\cdot,t) - u(\cdot,t)|_A \leq (1 + \varepsilon)\overline{g}_\kappa h^{\kappa-2} \int_{\mathbb{R}^n} t|\xi|^\kappa |\hat{f}(\xi)| e^{-\varepsilon |\xi|^2} d\xi + C_\varepsilon h^s \|\hat{f}\|^2_{s,\kappa}$$

for sufficiently small $h$. If $\kappa = 2$, we can replace the first term on the right in these inequalities by

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)| \left(1 - e^{-\varepsilon |\xi|^2} \right) e^{-\varepsilon |\xi|^2} d\xi.$$ 

**Proof.** We adapt the proof of theorem 5.3. If $\varepsilon > 0$, there exists a $r(\varepsilon) > 0$ such that $|G(\eta) - |\eta|^2| > (1 + \varepsilon)\overline{g}_\kappa|\eta|^\kappa$ if $|\eta| < r(\varepsilon)$. Hence, assuming wlog that $r(\varepsilon) \leq \pi$,

$$\int_{|\xi| \leq r(\varepsilon)} \left(1 - e^{-\varepsilon |\xi|^2} \right) e^{-\varepsilon |\xi|^2} |\hat{f}(\xi)| d\xi \leq \int_{|\xi| \leq r(\varepsilon)} \left(1 - e^{-\varepsilon |\xi|^2} \right) e^{-\varepsilon |\xi|^2} |\hat{f}(\xi)| d\xi \leq (1 + \varepsilon)\overline{g}_\kappa h^{\kappa-2} \int_{\mathbb{R}^n} t|\xi|^\kappa |\hat{f}(\xi)| e^{-\varepsilon |\xi|^2} d\xi.$$ 

We estimate the integral over $|\xi| \geq r(\varepsilon)/h$ by $2r(\varepsilon)^{-s} h^s \int_{|\xi| \geq r(\varepsilon)/h} |\hat{f}(\xi)| |\xi|^s d\xi$, which we bound by $2r(\varepsilon)^{-s} h^s \|\hat{f}\|^2_{s,\kappa}$, and by $\|\hat{f}\|^2_{s,\kappa}$ if $s > \kappa$, assuming in the latter case that $h \leq r(\varepsilon)$. If we finally use theorem 4.1 to estimate first term of (69) when $s > \kappa$, and corollary 5.8 when $s \leq \kappa$, the theorem follows. \hfill \Box

**Corollary 5.9.** Let $\kappa \geq 2$ and suppose that $g_\kappa := \lim_{\eta \to 0} |G(\eta) - |\eta|^2|/|\eta|^\kappa$ exists, so that $g_\kappa = \overline{g}_\kappa =: g_\kappa$. Suppose that $\int_{\mathbb{R}^n \setminus \{0\}} \in L^1_{s,\kappa}(\mathbb{R}^n)$ for some $s > \kappa$. Then

$$\lim_{h \to 0} h^{-(\kappa-2)} |u_h(\cdot,t) - u(\cdot,t)|_A = \begin{cases} g_\kappa t \int_{\mathbb{R}^n} |\hat{f}(\xi)| |\xi|^\kappa e^{-t|\xi|^2} d\xi, & \kappa > 2, \\ g_\kappa \int_{\mathbb{R}^n} |\hat{f}(\xi)| \left(1 - e^{-2t|\xi|^2} \right) e^{-t|\xi|^2} d\xi, & \kappa = 2. \end{cases}$$

Compare with corollary 4.3. It follows from proposition 5.10 below that $g_\kappa$ exists iff $\lim_{|\eta| \to 0} |\eta|^\kappa \hat{f}(\eta)$ exists. One can also give a direct proof of this corollary using Lebesgue’s dominated convergence theorem: see the proof of theorem 6.8 below.
If \( \mathcal{J}_\kappa \) and \( \mathcal{T}_\kappa \) are small, then theorem 5.8 with \( \epsilon \) for example equal to \( \max(\mathcal{J}_\kappa, \mathcal{T}_\kappa) \) shows one may observe a higher apparent rate of convergence than the actual rate for small but not too small \( h \)'s if \( \hat{f}(\xi) \) decays sufficiently rapidly at infinity. As in section 4, we can construct basis functions \( \varphi \) with small \( g_\kappa(\varphi) \) by taking these of the form \( \varphi(x) = \phi(c^{-1}x) \) and letting \( c \to \infty \). We start by deriving explicit formulas for \( \mathcal{J}_\kappa(\varphi) \) and \( g_\kappa(\varphi) \). Recall the definition of \( A(\varphi) \) and \( \mathcal{A}(\varphi) \) in proposition 4.4.

**Proposition 5.10.** We have

\[
\mathcal{J}_\kappa(\varphi) = \frac{1}{A(\varphi)} \sum_{k \neq 0} |2\pi k|^2 \hat{\varphi}(2\pi k), \quad g_\kappa(\varphi) = \frac{1}{A(\varphi)} \sum_{k \neq 0} |2\pi k|^2 \hat{\varphi}(2\pi k)
\]

**Proof.**

\[
G(\eta) - |\eta|^2 = \sum_{k \neq 0} |\eta + 2\pi k|^2 \hat{\varphi}(\eta + 2\pi k) - |\eta|^2
= \sum_{k \neq 0} \left( 4\pi(\eta, k) + 4\pi^2|k|^2 \right) \hat{\varphi}(\eta + 2\pi k)
= \frac{g(\eta)}{\hat{\varphi}(\eta) + h(\eta)}
\]

where \( g(\eta) := \sum_{k \neq 0} \left( 4\pi(\eta, k) + 4\pi^2|k|^2 \right) \hat{\varphi}(\eta + 2\pi k) \) and \( h(\eta) := \sum_{k \neq 0} \hat{\varphi}(\eta + 2\pi k) \) are continuous in a neighborhood of 0. It follows that

\[
\limsup_{\eta \to 0} \left| \frac{G(\eta) - |\eta|^2}{|\eta|^\kappa} \right| = \limsup_{\eta \to 0} \left| \frac{g(\eta)}{|\eta|^\kappa \hat{\varphi} + |\eta|^\kappa h(\eta)} \right| = \frac{g(0)}{A} = \frac{\sum_k 4\pi^2|k|^2 \hat{\varphi}(2\pi k)}{A},
\]

with \( A = \mathcal{A}(\varphi) \). The formula for \( g_\kappa(\varphi) \) follows similarly. \( \square \)

Note that \( \mathcal{T}_\kappa(\varphi) \leq \mathcal{J}_\kappa(\varphi) \). Also note that \( g_\kappa > 0 \) since \( \mathcal{A} < \infty \) and \( \mathcal{J}_\kappa < \infty \) since \( A > 0 \), by the ellipticity condition on \( \hat{\varphi} \) at 0.

If we take \( \varphi(x) := \phi_c(x) = \phi(x/c) \), with \( \phi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n) \), then \( \hat{\varphi}(\eta) = c^n \hat{\phi}(c\eta) \), and \( \mathcal{A}(\phi_c) = c^{\kappa-N} A(\phi) \). It follows that

\[
\mathcal{J}_\kappa(\phi_c) = c^\kappa A(\phi) \sum_{k \neq 0} |k|^2 \hat{\phi}(2\pi ck)
\leq Cc^{\kappa-N} \sum_{k \neq 0} |k|^{2-N},
\]

where the series converges since \( N > n + 2 \).

**Corollary 5.11.** If \( \phi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n) \) with \( N > \max(n + 2, \kappa) \), then \( \mathcal{J}_\kappa(\phi_c) \to 0 \) as \( c \to \infty \).

**Examples 5.12.** (i) Hardy’s multiquadric with shape parameter \( c \) is defined by

\[
\varphi(x) := -\sqrt{|x|^2 + c^2}, \quad x \in \mathbb{R}^n.
\]

where the minus sign serves to make \( \hat{\varphi}(\eta) \) positive. Note that \( \varphi(x) = c\phi(x/c) \) with \( \phi(x) := -\sqrt{|x|^2 + 1} \), so that we are in the situation of corollary 5.11 except for an irrelevant multiplicative factor of \( c \). The Fourier transform of \( \varphi \) on \( \mathbb{R}^n \setminus 0 \) is given by

\[
\hat{\varphi}(\eta) = \pi^{-1}(2\pi c)^{(n+1)/2} |\eta|^{-(n+1)/2} K_{(n+1)/2}(c|\eta|),
\]

where \( K_{\nu} \) is the MacDonald function, or modified Bessel function of the second kind: see for example cf. Baxter [1]. The limiting form of \( K_{\nu} \) for small values of the argument implies that as \( \eta \to 0 \),

\[
\hat{\varphi}(\eta) \simeq A_n |\eta|^{-n-1} \quad \text{(with } A_n = 2^{n+1} \pi^{(n+1)/2} \Gamma \left( \frac{n+1}{2} \right) \text{)), so that } \kappa = n + 1, \text{ and our RBF-scheme for the heat equation will converge if } n \geq 2, \text{ at a rate of } h^{n-1}. \text{ The MacDonald function is known to decay}
exponentially at infinity, so that we can apply corollary 5.11 to conclude that $\hat{T}_c(\phi_c)$ and $\overline{g}_c(\phi_c) \to 0$ as $c \to \infty$. In fact, these will converge to 0 at an exponential rate since $\sum_k |k|^2 \hat{\phi}(2\pi ck)$ does.

If $n = 1$, then $\kappa = 2$, and the scheme will not converge. However, corollary 5.11 together with theorem 5.8 shows that we can make the error arbitrarily small by taking the shape parameter $c$ sufficiently large, with moreover an arbitrarily large apparent order of convergence for small but not-too-small $h$’s if the Fourier transform of the initial value decays sufficiently rapidly at infinity. At first sight, this may seem strange, because we are after all simply performing an additional scaling by $c$, and we are already using scaled basis functions $\varphi_h(x) = \varphi(h^{-1}x)$ for our interpolation. Note, though, that we are interpolating with $\phi_{ch}$ on $h\mathbb{Z}^n$, and not on $c h \mathbb{Z}^n$.

(ii) If we take a homogeneous basis function, $\phi(x) = |x|^p$ with $p > 0$, then $\hat{\phi}(\eta)$ is proportional to $|\eta|^{-p-n}$ on $\mathbb{R}^n \setminus 0$, so that $\kappa = n + p = N$ and corollary 5.11 does not apply, as indeed it shouldn’t: if $\phi$ is homogeneous, then $\hat{T}_1(\phi_c)$ and $G(\phi_c)$ are independent of $c$, and therefore $\overline{g}_c(\phi_c)$ and $\overline{T}_c(\phi_c)$ also.

**Remark 5.13.** One can perform a similar analysis of the $c$-dependence of the constant $C_c$ of theorem 6.8 as the one we did in subsection 4.1 with similar conclusions; we skip the details.

6. Convergence of stationary RBF schemes for pseudo-differential evolution equations

The results of the previous section remain valid for a large class of constant coefficient pseudo-differential evolution equations

$$\partial_t u + a(D)u = 0, \quad t > 0, \tag{68}$$

under suitable conditions on the symbol $a = a(\xi)$, notably $\text{Re} a(\xi) \geq 0$. The operator $a(D)$ is defined by $\widehat{a(D)f} = a\hat{f}$ initially with domain $\mathcal{S}(\mathbb{R}^n)$, for example. We are in fact restricting ourselves to a rather special class of pseudo-differential operators, the Fourier multiplier operators or convolution operators: if $a$ is a tempered distribution and if $f$ is a Schwarz class function, then $a(D)f$ is the convolution of $f$ with the inverse Fourier transform of $a$. These can also be considered as constant coefficient pseudo-differential operators, general pseudo-differential operators having symbols which also depend on $x$. The latter are outside of the scope of this paper, but the multiplier operators we consider here already contain many interesting examples, such as the fractional Laplacians or the generators of Lévy processes. Regarding the latter, the equation [68] occurs in mathematical finance, for derivative pricing in exponential Lévy models, and has been treated numerically in [2] using the RBF scheme we investigate here, with good results. We note that, from a theoretical point of view, convergence of this type of stationary scheme for convolution operators is not obvious, since these, as integral operators, are in general non-local (except if $a(\xi)$ is a polynomial), and we already know from section 3 that to obtain good convergence we will need to use basis functions with polynomial growth. To understand why such schemes nevertheless perform well numerically, as for example observed in [2], was a main motivation for this paper.

As regards the conditions on the symbols, we will work with the class $S^q_0 := S^q_0(\mathbb{R}^n)$ of $C^\infty$-functions $a : \mathbb{R}^n \to \mathbb{C}$ for which there exists for each multi-index $\alpha$ a constant $C_\alpha$ such that

$$|\partial_\alpha^q a(\xi)| \leq C_\alpha (1 + |\xi|)^q, \quad \xi \in \mathbb{R}^n. \tag{69}$$

We will not need the faster $(1 + |\xi|)^{-\alpha}$ decay for $\partial_\alpha^q a$ which is a standard requirement in much of pseudo-differential theory and which for example is satisfied by the symbols of partial differential operators. The requirement of having $C^\infty$-symbols is slightly restrictive, and a priori excludes symbols such as $|\xi|^q$ for non-integer but positive $q$, but the theory presented here will apply to regularized versions of such symbols, for example, replacing $|\xi|^q$ by $(1 - \chi)|\xi|^q$ with $\chi \in C^\infty_c(\mathbb{R}^n)$ equal to 1 on a neighborhood of 0. Modifying a symbol on a compact set will not affect the singularities of the (distributional) kernel of $a(D)$ but will change its decay properties at infinity. Care then has to be taken with the growth properties of the initial values $f$ which we allow for [68] (equivalently, the singularities at 0 of $\hat{f}$), in the various convergence theorems, which makes the statements of these theorems more complicated. On the other hand, if $a \in S^q_0$ has non-negative real part, then the
solution of (58) with initial value \( f \) makes sense for any tempered distribution \( f \), being the inverse Fourier transform of \( e^{-|\xi|^2} \). We therefore limit ourselves here to smooth symbols.

We first examine the action of \( a(D) \) on \( L_1 \):

**Theorem 6.1.** Let \( \kappa \geq 0 \). If \( a \in S_0^0(\mathbb{R}^n) \) and if \( \varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n) \) with \( N > n + q \), then \( a(D)L_1 \) is a bounded continuous function and there exists a constant \( C > 0 \) such that \( |a(D)L_1(x)| \leq C(1 + |x|)^{-(n+\kappa)} \).

The proof is similar to that of the bound (6) of theorem 2.3; see appendix A. In fact, here, and below, it would have sufficed to require (69) only for \( |\alpha| \leq \lceil \kappa \rceil + n + 1 \). We suppose from now on that \( N > n + q \).

The second condition we will need to put on the symbol is that it has a non-negative real part:

\[
\text{Re } a(\xi) \geq 0.
\]

Perhaps curiously, we do not need \( a(\xi) \) or \( \text{Re } a(\xi) \) to be elliptic. In particular, our results below will for example also apply to the free Schrödinger operator, for which \( a(\xi) = i|\xi|^2 \), or the regularized "half-wave equation", with \( a(\xi) = |\xi| \) outside of a neighborhood of 0. The heat equation obviously also falls within the class of allowed evolution equations, as do the Kolmogorov-Fokker-Planck equations associated to certain Lévy processes: see example 6.1 ii) below.

The proofs in this section will be similar to the ones for the classical heat equation in section 5, and we will only signal the differences.

We will extend the results of section 5 to the Cauchy problem for (68) for symbols satisfying (69), (70). The proofs will be similar to the ones for the classical heat equation in section 5, and we will mainly signal major differences. We are again interested in solving (68) with initial value \( f \) using a semi-discrete scheme which is the RBF-variant of the classical method of lines, looking for approximate solutions of the form (71), where \( L_1 \) is the Lagrange function on \( \mathbb{Z}^n \) associated to a basis function \( \varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n) \) with \( \kappa > 0 \) and \( N > n + q \): cf. theorem 6.1. The coefficients \( c_k(t;h) \) of \( u_h \) are again determined by requiring that \( u_h \) solve (68) exactly in the interpolation points:

\[
\partial_t u_h(hj,t) = -a(D)u_h(jh,t) \quad \text{for } j \in \mathbb{Z}^n.
\]

This now leads to the (infinite) system of ODEs

\[
\frac{dc_{j}(t;h)}{dt} = - \sum_k a(h^{-1}D_x)(L_1)(j - k)c_k(t;h)
\]

where \( a(h^{-1}D_x) \) has symbol \( a(h^{-1}\xi) \). We again have to solve this system with initial condition \( c_k(0) = f(hj) \). One shows as in lemma 5.1 that if \( p < \kappa \) then there exists a unique solution in \( C^\infty([0, \infty), L^\infty_p) \) and that, as a consequence, \( u_h[f](\cdot, t) \) is in \( L^\infty_p(\mathbb{R}^n) \) if \( f \in L^\infty_p(\mathbb{R}^n) \) with norm bounded by a constant times that of \( f \).

**Remark 6.2.** A noteworthy feature of the RBF-scheme is that we do not need to discretize the operator \( a(D) \), contrary to for example Finite Difference schemes, but only need to know its action on \( L_1 \) (or on \( \varphi \) when working with irregularly spaced interpolation points). This is an advantage when the operator is a singular integral operator: see [2] for concrete examples and further discussion.

To analyze the RBF scheme we introduce the auxiliary function \( G'_a \) on \( \mathbb{R}^n \times \mathbb{R}_{>0} \) defined by

\[
G'_a(\xi;h) := \sum_k a(\xi + 2\pi h^{-1}k)\hat{L}_1(h\xi + 2\pi k) = \sum_k a(\xi + 2\pi h^{-1}k)\hat{\varphi}(h\xi + 2\pi k),
\]

where the series converges absolutely since \( N > n + q \). To make the connection with the previous section note that if \( a(\xi) = |\xi|^2 \) then \( G'_a(\xi) = h^{-2}G(h\xi) \), with \( G \) given by (56).
We then can state the following convergence theorem.

**Theorem 6.4.** Let \( u \in L^1(\mathbb{R}^n) \), and if \( a \in S_0^\beta \) satisfies (70), then the Fourier transform with respect to \( x \) of \( u_h(x,t) \) is given by

\[
\hat{u}_h(\xi,t) = e^{-tG^*_a(\xi;h)}s_h[f](\xi).
\]

Since \( G^*_a(\xi;h) \) is in \( C^\beta_{\rho}(\mathbb{R}^n \setminus 0) \), we can extend this formula to initial values \( f \) of polynomial growth strictly less than \( \kappa \) whose Fourier transform coincides on \( \mathbb{R}^n \setminus 0 \) with an element of \( L^1(\mathbb{R}^n, (|\xi|^{\kappa} \wedge 1)d\xi) \). We also note that \( G^*_a(\xi;h) \) is \( 2\pi/h \)-periodic in \( \xi \) and has non-negative real part. Its zero-set contains \( 2\pi h^{-1}Z \setminus 0 \) but may be bigger. We have the following basic estimate which generalizes proposition 5.3(b).

**Proposition 6.3.** Suppose that \( a \in S_0^\beta(\mathbb{R}^n) \) for some \( q \in \mathbb{R} \), and let \( \varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n) \) with \( N > n + q \). Then there exists a constant \( C \) such that for all \( h < 1 \),

\[
|G^*_a(\xi;h) - a(\xi)| \leq C h^{\kappa - \max(q,0)}|\xi|^\kappa, \quad |\xi| \leq \pi/h.
\]

**Proof.** We have

\[
G^*_a(\xi;h) - a(\xi) = a(\xi) \left( \hat{L}_1(h(\xi) - 1) + \sum_{k \neq 0} a \left( \xi + \frac{2\pi}{h} \right) \hat{L}_1(h(\xi - 2\pi k)) \right).
\]

The first term is bounded by a constant times \( (1 + |\xi|^q)|h\xi|^{\kappa} \leq C h^{\kappa - \max(q,0)}|\xi|^{\kappa} \) if \(|h\xi| \leq \pi\). As for the other terms, \( |\xi + 2\pi k/h| \) is comparable to \(|k|/h|\xi| \leq \pi/h\), so \(|a(\xi + 2\pi k/h)| \leq C h^{-q}|k|^q \), and by (72),

\[
\hat{L}_1(h\xi + 2\pi k) \leq C|h\xi|^q|k|^{-N}, \quad |h\xi| \leq \pi,
\]

so that

\[
\sum_{k \neq 0} |a \left( \xi + \frac{2\pi}{h} \right) \hat{L}_1(h(\xi - 2\pi k))| \leq C h^{\kappa - q}|\xi|^\kappa \sum_{k \neq 0} |k|^{-N} \leq C h^{\kappa - q}|\xi|^\kappa.
\]



We will suppose from now on that \( f \in L^\infty_{\rho}([\mathbb{R}^n]) \) for some \( p < \kappa \) such that \( \hat{f}|_{\mathbb{R}^n \setminus 0} \in L^1(\mathbb{R}^n,(|\xi|^{\kappa} \wedge 1)d\xi) \). The unique solution of the initial value problem in the space of tempered distributions is given by \( \hat{u}(\xi,t) = e^{-ta(\xi)}\hat{f} \) which is well-defined as a product of a tempered distribution and a \( C^\infty \)-function of all whose derivatives have at most polynomial growth. One shows, using the arguments of Appendix B, that \( u_h(x,t) \) is a continuous function of polynomial growth of order at most \( \kappa \) for each \( t > 0 \), and that the Fourier transform of \( u_h(x,t) - u(x,t) \) is given by

\[
e^{-th^{-2}G^*_a(\xi;h)} \left( s_h[f](\xi) - \hat{f}(\xi) \right) + \left( e^{-t(G^*_a(\xi;h) - a(\xi))} - 1 \right) e^{-ta(\xi)}\hat{f}(\xi).
\]

We then can state the following convergence theorem.

**Theorem 6.4.** Let \( a \in S_0^\beta(\mathbb{R}^n) \), \( \text{Re} a(\xi) \geq 0 \), and suppose that \( \kappa \geq q \). Then there exists a constant \( C \) such that if \( f \) is a function of polynomial growth of order strictly less than \( \kappa \) such that \( \hat{f}|_{\mathbb{R}^n \setminus 0} \in L^1(\mathbb{R}^n) \), then

\[
||u_h(\cdot,t) - u(\cdot,t)||_A \leq C \max(t,1) \cdot h^{\kappa - \max(q,0)}||f||_0^e.
\]

**Proof.** The proof is similar to the proof of theorem 6.5 with a small twist. By proposition 6.3 and the elementary inequality \(|e^z - 1| \leq |z|e^{\max(Re z,0)}\) for \( z \in \mathbb{C} \), and since \(3\)

\[
\text{Re}(a(\xi) - G^*_a(\xi;h)) \leq \text{Re} a(\xi)(1 - \hat{L}_1(h(\xi))) \leq C h^{\kappa}|\xi|^\kappa \text{Re} a(\xi) \leq \frac{1}{2} \text{Re} a(\xi),
\]

\(3\)In the case of the heat equation \((a(\xi) = |\xi|^2)\), \( a - G^*_a \) was negative for small enough \( h|\xi| \), but this is not clear for \( \text{Re}(a - G^*_a) \) for general \( a \), in particular when \( \text{Re} a \) is not elliptic.
for sufficiently small $h|\xi|$, there exists an $\rho > 0$ such that if $h|\xi| \leq \rho$, then
\[
\left| e^{-tG_a^*(\xi h)} - e^{-t a(\xi)} \right| = e^{-t \Re a(\xi)} \left| e^{-t(G(\xi h) - a(\xi))} - 1 \right| \leq C t h^{\kappa - \max(q,0)} |\xi|^\kappa e^{-\frac{1}{2}t \Re a(\xi)},
\]
which in absence of further hypotheses on the symbol $a(\xi)$ we simply bound by $C h^{\kappa - p} |\xi|^\kappa$. The rest of the proof proceeds as before: since $\Re G_a^\tau \geq 0$,
\[
||u_\tau(t) - u(\cdot, t)||_A \leq \int |e^{-tG_a^*} - e^{-t a}| |\hat{f}| d\xi + ||s_h[\hat{f}] - \hat{f}||_1.
\]

The last term can be estimated using theorem 5.4 while the integral can be bounded by
\[
\int_{|\xi| \leq \rho/h} |e^{-tG_a^*} - e^{-t a}| |\hat{f}| d\xi + 2 \int_{|\xi| \geq \rho} |\hat{f}| d\xi
\]
Now use (76) for the first integral, and bound the second one by $(h/\rho)^\kappa \int_{|\xi| \geq \rho/h} |\xi|^\kappa |\hat{f}| d\xi \leq (h/\rho)^\kappa ||\hat{f}||_{1,\kappa,\kappa}^\circ.

We note that we do not get an improved convergence rate beyond $O(h^{\kappa})$ if $a$ is of negative order $q < 0$, even if $\hat{f}$ were rapidly decreasing and $a$ were smoothing. We therefore limit ourselves to operators of non-negative order: $q \geq 0$.

Also note that, on comparing theorem 6.4 for $a(\xi) = |\xi|^2$ with theorem 5.5 we require a stronger decay of $\hat{f}$ at infinity, which translates into two additional degrees of smoothness (two extra derivatives) of $f$. If we assume that $\Re a(\xi)$ is elliptic, then
\[
(77) \quad \sup_{\mathbb{R}^n} |t| |\xi|^2 e^{-\frac{1}{2}t \Re a(\xi)} < \infty,
\]
is independent of $t$ and theorem 6.4 remains valid if $\int_{\mathbb{R}^n \setminus 0} \in L^1(\mathbb{R}^n, |\xi|^{-q} \wedge |\xi|^\kappa)$, on replacing $||\hat{f}||_{1,\kappa,\kappa}^\circ$ by $||\hat{f}||_{1,\kappa-q,\kappa}^\circ$: compare the proof of theorem 5.5.

The argument can be refined to give a rough approximate approximation estimate: the proof above shows that if $r \leq \min(\rho, \pi)$ and if $s > \kappa - q \geq 0$,
\[
||u_\tau(t) - u(\cdot, t)||_A \leq C(r) t \cdot h^{\kappa - q} ||\hat{f}||_{1,\kappa,\kappa}^\circ + \frac{h^s}{r^{\kappa}} ||\hat{f}||_{1,\kappa,\kappa}^\circ + ||s_h[\hat{f}] - \hat{f}||_1,
\]
where
\[
(78) \quad C(r) := \sup_{h|\xi| \leq r} \frac{|a(\xi) - G_a^*(\xi; h)|}{h^{\kappa-q} |\xi|^\kappa},
\]
On closer examination, the proof of proposition 6.4 shows that
\[
C(r) \leq (4\pi)^q ||a||_{\infty,-q} \left( \sup_{|\eta| \leq r} \frac{1 - \hat{L}_1(\eta)}{|\eta|^\kappa} + \sum_{k \neq 0} |k|^q \sup_{|\eta| \leq r} \frac{\hat{L}_1(\eta + 2\pi k)}{|\eta|^\kappa} \right),
\]
where we recall that $||a||_{\infty,-q} = \sup\xi (1 + |\xi|)^{-q} |a(\xi)|$ (the constant in front is not optimal). From this it easily follows that
\[
(79) \quad \limsup_{r \to 0} C(r) \leq (4\pi)^q ||a||_{\infty,-q} \cdot T_{\kappa,q},
\]
where we’ve put
\[
T_{\kappa,q} := T_{\kappa,q}(\varphi) := A^{-1} \sum_{k \neq 0} (1 + |k|^q) \hat{\varphi}(2\pi k),
\]
with $A$ defined by (40); note that the series converges since $N > n + q$, by assumption.

Note that $2T_{\kappa,0}(\varphi) = T_{\kappa}(\varphi) = T_{\kappa}$. Equations (78), (80) and theorems 4.1 (when $s > \kappa$) and 5.4 (when $\kappa - q < s \leq \kappa$) now imply the following rough approximate approximation estimate:
Theorem 6.5. Suppose that $a \in S^0_\lambda(\mathbb{R}^n)$, $0 \leq q \leq \kappa$, with $\text{Re}a(\xi) \geq 0$. Then there exists, for each $\varepsilon > 0$ a constant $C_\varepsilon$ such that for $s > \kappa - q$ and $\hat{f}|_{\mathbb{R}^n \setminus L \in L^1_{S^\kappa}(\mathbb{R}^n)}$, the error $||u_h(\cdot, t) - u(\cdot, t)||_A$ can be bounded by

$$
(1 + \varepsilon)(C_a \mathcal{I}_{q,k,t} \cdot h^\kappa - q + \mathcal{I}_{\kappa} h^\kappa) \|\hat{f}\|_\kappa + C_\varepsilon h^s ||\hat{f}||_s
$$

with $C_a = (4\pi)^q ||a||_{S^0_\lambda}$, where the second term can obviously be absorbed in the final one if $s \leq \kappa$.

We call this a rough approximate approximation theorem since the bound does not correctly reflect the $a$-dependence. In particular, there is no corresponding lower bound for the approximation error. It does however imply that if $\mathcal{I}_{q,k}(\varphi)$ is small, then the error will appear to be smaller that $O(h^{\kappa-q})$ for not-too-small $h > 0$. It can in particular be applied with basis functions $\varphi = \phi_\kappa$ depending on a large shape parameter $c$, since one easily shows that $\mathcal{I}_{\kappa,q}(\phi_\kappa) \to 0$ as $c \to \infty$ if $\phi \in \mathfrak{B}_{\kappa,N}(\mathbb{R}^n)$ with $N > \max(n + q, \kappa)$; cf. corollary [5.11] and its proof. One also shows that the constant $C_\varepsilon$ in (81) (for any fixed $\varepsilon > 0$) again behaves like $c^\kappa$ for the $\phi$’s we considered in subsection 4.1.

To obtain a more precise result, and in particular an asymptotic lower bound for the approximation error, we put an additional hypothesis on $a$.

Definition 6.6. We will say that $a \in S^0_\lambda(\mathbb{R}^n)$ is asymptotically homogeneous at infinity if $\partial_\xi a \in S^\kappa(\mathbb{R}^n)$ for some $\varepsilon > 0$ and if

$$
\lim_{\lambda \to \infty} \frac{a(\lambda \xi)}{\lambda^q} =: a_\infty(\xi)
$$

exists for all $\xi \in \mathbb{R}^n \setminus 0$.

We will also assume for simplicity that $\lim_{\eta \to 0} |\eta|^{\kappa} \tilde{\varphi}(\eta) =: A(\varphi) =: A$ exists: if not, one has to replace $A$ by the corresponding liminf or limsup, and replace equalities by inequalities at the appropriate places in theorem [L8] below. Recall that $A > 0$ by the definition of the Buhmann class.

Lemma 6.7. If $a \in S^0_\lambda(\mathbb{R}^n)$, $q > 0$, is asymptotically homogeneous at infinity, then

$$
\lim_{h \to 0} \frac{G_a^\ast(\xi; h) - a(\xi)}{h^{\kappa-q} |\xi|^\kappa} =: g_{a,\kappa}
$$

exists and is equal to $A^{-1} \sum_{k \neq 0} a_\infty(2\pi k) \tilde{\varphi}(2\pi k)$.

Proof. The hypotheses on $a$ easily imply that

$$
\lim_{h \to 0} h^q a(\xi + 2\pi k/h) = a_\infty(2\pi k),
$$

for all $k \neq 0$ and all $\xi$. It follows that for each $\xi \in \mathbb{R}^n$,

$$
\lim_{h \to 0} h^q \sum_{k \neq 0} a(\xi + 2\pi k/h) \tilde{\varphi}(h\xi + 2\pi k) = \sum_{k \neq 0} a_\infty(2\pi k) \tilde{\varphi}(2\pi k),
$$

where the interchange of summation and limit can be justified by Lebesgue’s dominated convergence theorem, using that $N > n + q$. Since

$$
\frac{G_a^\ast(\xi; h) - a(\xi)}{h^{\kappa-q} |\xi|^\kappa} = \frac{h^q \sum_{k \neq 0} (a(\xi + 2\pi k/h) - a(\xi)) \tilde{\varphi}(h\xi + 2\pi k)}{h^{\kappa-q} |\xi|^\kappa \sum_{k \neq 0} \tilde{\varphi}(h\xi + 2\pi k)}
$$

the lemma follows. \hfill \Box

Theorem 6.8. Suppose that $a \in S^0_\lambda(\mathbb{R}^n)$, $q > 0$, is asymptotically homogeneous and let the initial value $f$ be as in theorem [4,4] in particular, $\hat{f}|_{\mathbb{R}^n \setminus L} \in L^1_{S^\kappa}(\mathbb{R}^n)$. Then if $\kappa > q$,

$$
\lim_{h \to 0} h^{\kappa-q} ||u_h(\cdot, t) - u(\cdot, t)||_A = |g_{a,\kappa}| \int_{\mathbb{R}^n} t |\xi|^\kappa e^{-t \text{Re}a(\xi)} |\hat{f}(\xi)| d\xi,
$$

while if $\kappa = q$ this limit equals

$$
\int_{\mathbb{R}^n} \left| 1 - e^{-t g_{a,\kappa} |\xi|^\kappa} \right| e^{-t \text{Re}a(\xi)} |\hat{f}(\xi)| d\xi,
$$
where we note that $\text{Re } g_{a,\kappa} \geq 0$.

**Proof.** First of all, since $\text{Re } G^*_a \geq 0$ and $\|sh[\hat{f}] - \hat{f}\|_1 = O(h^\kappa)$, the limit we have to compute is equal to

$$
\lim_{h \to 0} \int_{\mathbb{R}^n} h^{-(\kappa - q)} |e^{-t(G^*_a - a)} - 1| e^{-t\text{Re } a} |\hat{f}| d\xi.
$$

If $\kappa > q$ then for each $\xi$, $G^*_a(\xi, h) - a(\xi) \to 0$ as $h \to 0$, and consequently, using the Taylor expansion of the exponential function,

$$
\lim_{h \to 0} h^{-(\kappa - q)} |e^{-t(G^*_a - a)} - 1| = |g_{a,\kappa}| |\xi|^\kappa,
$$

while if $\kappa = q$, this limit equals $|e^{-tg_{a,\kappa}}|\xi|^q - 1|$. The dominated convergence theorem, using the estimate (\ref{6.7}), then shows that

$$
\lim_{h \to 0} h^{-(\kappa - q)} \int_{\mathbb{R}^n} e^{-t(G^*_a - a)} - 1 |\hat{f}| \cdot 1_{\{|\xi| \leq r/h\}} d\xi
$$

exists and is equal to the right hand side of (\ref{6.8}) if $r > 0$ is sufficiently small. The integral over $h|\xi| \geq r$ can as before be bounded by $Ch^\kappa \|\hat{f}\|_{1,\kappa}$ which goes to 0 when multiplied by $h^{-(\kappa - q)}$ if $q > 0$, which proves the theorem. \qed

The theorem shows that the approximation order $O(h^{\kappa-q})$ is exact if $g_{a,\kappa} \neq 0$, but the latter can be 0, for example if $g_{a,\kappa}(\eta) = 0$ and $\hat{\varphi}(\eta)$ is even, in which case $a_{\infty}$ would have to be purely imaginary to comply with (\ref{6.6}). A concrete example is given by the constant-coefficient transport equation

$$
\partial_t u + v \cdot \nabla u = 0, \quad v \in \mathbb{R}^n.
$$

**Remark 6.9.** We include some observations on the case of $q = 0$, which has some relevance for applications: for example, the transition probability densities of a compound Poisson process satisfy a pseudo-differential equation of the type considered here, with a symbol $a$ which is bounded together with its derivatives under suitable hypotheses on the probability distribution of the jumps (e.g. when the latter has moments of all, or sufficiently high, order).

If $a \in S_0^0$ with $q = 0$, we have to substract $\frac{1}{2} l_{\kappa}(\varphi)u(\xi)$ from the right hand side of (\ref{6.8}) (where we recall that $l_{\kappa}(\varphi) = 2A^{-1} \sum_{k \neq 0} \hat{\varphi}(2\pi k)$), making the limit depend on $\xi$. Also, the contributions of the $L^1$-norms of $e^{-tG^*_a}(sh[\hat{f}] - \hat{f})$ et de $(e^{-tG^*_a} - e^{-ta})f_1_{h|\xi| \leq r}$ are now of the same order $O(h^\kappa)$, as is our estimate for the norm of $(e^{-tG^*_a} - e^{-ta})f_1_{h|\xi| \geq r}$, and cannot be separated anymore as before. We can nevertheless impose a stronger decay at infinity to control the latter, and use corollary (\ref{6.10}) to show that if $\hat{f} \in L^1(\max\{|\xi|^{\kappa}, |\xi|^s\})$ for some $s > \kappa$, then

$$
\lim_{h \to 0} \sup_{h|\xi| \leq r} h^{-\kappa} \|u_h(\cdot, t) - u(\cdot, t)\|_A \leq l_{\kappa}(\varphi)\|\hat{f}\|_{1,\kappa} + \int_{\mathbb{R}^n} \left( g_{a,\kappa} - \frac{1}{2} l_{\kappa}(\varphi) a(\xi) \right) |\xi|^\kappa |\hat{f}(\xi)| e^{-t\text{Re } a(\xi)} d\xi.
$$

A further analysis shows that if, for example, $\hat{f}$ has compact support, then we have the more precise result

$$
\lim_{h \to 0} h^{-\kappa} \|u_h(\cdot, t) - u(\cdot, t)\|_A = \int_{\mathbb{R}^n} \left( \frac{1}{2} l_{\kappa}(\varphi) + |g_{a,\kappa} - \frac{1}{2} l_{\kappa}(\varphi)(1 + a(\xi))| \right) |\xi|^\kappa |\hat{f}| e^{-t\text{Re } a} d\xi,
$$

a result which remains true if the real part of $a \in S_0^0$ is not necessarily positive, and which implies (\ref{6.8}) if it is. Since the gain with respect to the latter seems relatively modest, we skip the details.

We cannot deduce from lemma (\ref{6.10}) a precise approximate approximation estimate similar to theorem (\ref{5.8}) since the limit isn’t \textit{uniform in } $h|\xi|^\kappa$. In fact, as we will show now, lower order parts of the symbol will also occur in such an estimate. We consider the case of a constant coefficient partial differential operator, $p = p(\xi)$ a polynomial of degree $q \in \mathbb{N}$ with non-negative real part: $\text{Re } p(\xi) \geq 0$. We write

$$
p = \sum_{j=0}^q p_{q-j},$$

where we note that $\text{Re } g_{a,\kappa} \geq 0$.
with $p_{q-j}(\xi)$ homogeneous of degree $q-j$. Then $G^*_p(\xi; h) = h^{-\nu}G_p(h\xi)$, where

$$G_p(\eta) := \sum_k p_p(\eta + 2\pi k)L_1(\eta + 2\pi k).$$

and, as is easily verified,

$$\lim_{\eta \to 0} \frac{G_p(\eta) - \rho(\eta)}{\eta^\kappa} = g_{p,\kappa},$$

for $\nu > 0$, while $G_{p_0} = p_0$ since $p_0$ is a constant. Since

$$G^*_p(\xi; h) - p(\xi) = \sum_{j=0}^{q-1} h^{-j} (G_{p_{q-j}}(h\xi) - p_{q-j}(h\xi)),$$

it follows that for each $\varepsilon > 0$ there exists a $\rho = \rho(\varepsilon) > 0$ such that

$$\sup_{|h| \leq \rho} \frac{|G^*_p(\xi; h) - p(\xi)|}{h^\kappa |\xi|^\kappa} \leq \sum_{j=0}^{q-1} h^j (|g_{p_{q-j},\kappa}| + \varepsilon).$$

Since $\text{Re}(p(\xi) - G^*_p(\xi; h)) \leq |h\xi|^\kappa \text{Re} p(\xi) < \varepsilon \text{Re} p(\xi)$ if $|h\xi| = \varepsilon$ is sufficiently small, the arguments above can be used to prove the following estimate:

**Theorem 6.10.** For all $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that if $\hat{f}(\xi) \neq 0 \in L_1^1(\mathbb{R}^n) \cap L_4^1(\mathbb{R}^n)$,

$$||u_h(\cdot, t) - u(\cdot, t)||_A \leq \sum_{j=0}^{q-1} h^{\kappa-q+j} (|g_{p_{q-j},\kappa}| + \varepsilon) \int_{\mathbb{R}^n} |\xi|^\kappa |\hat{f}(\xi)|e^{-(1-\varepsilon)\text{Re} \xi} \text{d}x$$

$$+ \sum_{j=0}^{q-1} h^j (|g_{p_{q-j},\kappa}| + \varepsilon) ||\hat{f}||_2^2 + C_\varepsilon h^\kappa ||\hat{f}||_2^2,$$

where $|g_{p_{q-j},\kappa}| + \varepsilon$ can be replaced by $(1+\varepsilon)|g_{p_{q-j},\kappa}|$ if $g_{p_{q-j},\kappa} \neq 0$.

**Examples 6.11.** We finally mention some examples of pseudo-differential evolution equations which are of interest for applications, and to which our results apply.

(i) The Kolmogorov - Fokker - Planck equation associated to a Lévy process $(X_t)_{t \geq 0}$ on $\mathbb{R}^n$. Recall that according to the Lévy - Khintchine theorem such a process is completely characterized by its characteristic function, $E(e^{i\xi X_t}) = e^{t\psi(\xi)}$ with

$$\psi(\xi) = i\mu, \xi - \frac{1}{2} (\Sigma \xi, \xi) + \int_{\mathbb{R}^n \setminus 0} \left( e^{i(\xi, x)} - 1 - i(\xi, x) \chi(x) \right) \text{d}\nu(x),$$

where $\Sigma$ is a positive semi-definite linear operator and where $\nu$ is a positive Borel measure on $\mathbb{R}^n \setminus 0$ such that

$$\int_{\mathbb{R}^n \setminus 0} (|x|^2 + 1) \text{d}\nu(x) < \infty,$$

called the Lévy measure; here $\chi$ is a compactly supported function which is equal to 1 on a neighborhood of 0, and can be taken smooth, if necessary.

If, for a given $f$, we let $u(t, t) = E(f(x + X_t))$, then $u$ satisfies (LS) with $a(\xi) := -\psi(\xi)$ and initial value $f$. Note that $a(\xi)$ satisfies (B) since

$$\text{Re} \psi(\xi) = -\frac{1}{2} (\Sigma \xi, \xi) + \int_{\mathbb{R}^n \setminus 0} (\cos(x \xi) - 1) \text{d}\nu(x) \leq 0.$$

Under appropriate hypotheses on the Lévy-measure $\nu$ one can derive symbol-type estimates for $a(\xi)$. For example, when $\text{d}\nu(x) = |x|^{-d}h(x) \text{d}x$ with $q < n + 2$, and $h(x)$ a rapidly decreasing continuous function, then $a \in S^q_0$ if $\Sigma \neq 0$, and in $S^q_0 - n$ if $V = 0$: cf. remark [A.3] in Appendix A below. Examples of such processes are the jump-diffusion processes and the CGMY-processes of mathematical finance, which were treated numerically in [2], [5] and [6] with different choices of basis functions (respectively the multi-quadric, inverse multi-quadric and the cubic spline). If $\Sigma = 0$ and if $\nu$ is a finite measure on $\mathbb{R}^n$ of total mass $\lambda$, we have the special case of a (multi-dimensional) compound Poisson process with
intensity $\lambda$ and drift $\mu + \int_{\mathbb{R}^n} x \chi d\nu(x)$, $\lambda^{-1}\nu$ then being the probability law of the jumps. The symbol will then be in $C^{|\alpha|+n+1}_{b}(\mathbb{R}^n)$ if $\nu$ has moments of this order.

(ii) The special case of symmetric stable Lévy-processes leads to the fractional heat equation:

$$\partial_t u + (-\Delta)^s u = 0,$$

with $s \in (0, 1)$. Here the symbol, $a(\xi) = |\xi|^{2s}$ is not smooth in 0, so our theory does not apply directly, but it applies to regularized versions with $|\xi|^{2s}$ replaced by $\chi(\xi)|\xi|^{2s}$, where $\chi \in C_\infty(\mathbb{R}^n)$ is equal to 1 outside of a small ball. The theory of this section can nevertheless be extended to certain families of symbols having singularities in $\xi = 0$. The presence of such singularities cause a number of technical problems, notably with regards to the initial values $f$ which are allowed in theorem 6.4 and the other theorems of this section: in the case of the fractional heat equation, $e^{-t|\xi|^{2s}\hat{f}}$ needs to be defined as a tempered distribution, so it becomes natural to require that $\hat{f}|_{\mathbb{R}^n\setminus 0} \in L^1(\mathbb{R}^n, (|\xi|^{2s} \wedge 1)d\xi)$. On the other hand one can use lemma [\ref{A.2} to show that the inverse Fourier transform of $e^{-t|\xi|^{2s}}$ decays as $(1 + |x|)^{n+2s}$ at infinity, so $f$ will have to be taken integrable with respect to this weight. The details of this case will be treated elsewhere. Note that once such a theory is in place, we can consider generalizations of theorem 6.10 to general polyhomogeneous symbols.

(iii) Certain non-parabolic operators, such as the transport equation $\partial_t u = v \cdot \nabla u$, the so-called "half wave equation" $\partial_t u = i\sqrt{-\Delta} u$, modified so as to make its symbol smooth in a neighborhood of 0, or the free Schrödinger equation, $i\partial_t u = -\Delta u$: for these, the symbol $a(\xi)$ is purely imaginary.

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Appendix A. Proof of theorem 2.3

We prove the existence and main properties of the cardinal function associated to a basis function \( \varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n) \) as stated in theorem 2.3. As already mentioned, this was done by Buhmann [3], [4] for a more restricted class of radial basis functions. The main difference in our treatment and that of [3] is the use of lemma A.2 below, relating the decay at infinity of the Fourier transform of a function with its behavior in 0, which allows one to go beyond the class of basis functions considered by Buhmann.

Before embarking upon the proof, it may be interesting to observe that the estimates (44) are similar to the symbol conditions of pseudodifferential calculus, except that the latter concern the behavior at infinity\(^4\) instead of at 0. From this point of view, (44) corresponds to having an elliptic symbol, whence our terminology.

We note that as a consequence of conditions (ii) and (iii) of definition 2.1

\[
|\partial_\eta^a(\hat{\varphi}^{-1})| \leq C|\eta|^{-|\alpha|}, \quad |\eta| \leq 1, |\alpha| \leq n + |\kappa| + 1.
\]

Turning to the proof of theorem 2.3 following Buhmann we start by defining \( L_1 \) as the inverse Fourier transform of the right hand side of (44), observing that since the latter is an integrable function, by definition 2.1(iv), \( L_1 \) is a well-defined continuous function. We first show that \( L_1(x) \) has the proper decay at infinity.

**Theorem A.1.** Let \( \varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n) \), and let

\[
L_1 := \mathcal{F}^{-1} \left( \frac{\hat{\varphi}(\cdot)}{\sum_{k \in \mathbb{Z}^n} \hat{\varphi}(\cdot + k)} \right).
\]

Then there exists a positive constant \( C \) such that

\[
|L_1(y)| \leq C(1 + |y|)^{-\kappa - n}, \quad y \in \mathbb{R}^n.
\]

The proof will use the following lemma, which basically is a special case of a classical estimate for kernels of convolution operators: see Stein [14], proposition 2 of Chapter VI, section 4.4.

**Lemma A.2.** Let \( p > -n \) and let \( a \in C^{[p]+1}(\mathbb{R}^n \setminus 0) \) be supported in some ball \( B(0, R) \) such that

\[
|\partial_\xi^\alpha a(\xi)| \leq C|\xi|^{p-|\alpha|}, \quad |\xi| \leq R, \quad |\alpha| \leq |p| + n + 1.
\]

Then the inverse Fourier transform \( k = \mathcal{F}^{-1}(a) \) satisfies

\[
|k(x)| \leq C_1(1 + |x|)^{-p - n}, \quad x \neq 0,
\]

with a constant \( C_1 \leq c_n C \), where \( c_n \) only depends on \( n \).

Stein in fact shows that if (92) is satisfied at all orders, without \( a \) necessarily being compactly supported, then \( k \) can be identified with a \( C^\infty \)-function away from 0, satisfying \( |\partial_\xi^\alpha k(x)| \leq C_\alpha |x|^{-p - n - |\alpha|} \) for all \( \alpha \) and all \( x \). This result is stated and proven there for \( p = 0 \), but the proof generalizes to any \( p > -n \). We only need this estimate for \( k(x) \) itself, in which case we only need (92) for the limited number of derivatives of \( a \) indicated, and we furthermore only need it for large \( |x| \) (note that if \( a \) has compact support, \( k \) is continuous, even \( C^\infty \), and Stein’s estimate for \( k(x) \) at 0 becomes trivial). The proof in [14] uses the Paley-Littlewood decomposition. An elementary proof of lemma A.2 can be given by writing

\[
k(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \chi(|x|\xi)a(\xi)e^{i(x,\xi)} \, d\xi + (2\pi)^{-n} \int_{\mathbb{R}^n} (1 - \chi(|x|\xi))a(\xi)e^{i(x,\xi)} \, d\xi.
\]

\(^4\)Indeed, if (44) were required for all orders \( \alpha \) (with constants which may then depend on \( \alpha \)), then \( \chi(\xi)\hat{\varphi}(\xi/|\xi|^2) \in S^1_{\kappa,0}(\mathbb{R}^n) \), where \( \chi \) is a \( C^\infty \)-function such that \( 1 - \chi \) is compactly supported, and \( S^1_{\kappa,0}(\mathbb{R}^n) \) is the standard symbol class of order \( p \) (cf. [14]).

\(^5\)Note that \( |p| + n + 1 \geq 1 \) since \( p > -n \).
where \( \chi \in C^\infty(\mathbb{R}^n) \) with bounded derivatives such that \( \chi(\xi) = 0 \) for \( |\xi| \leq 1 \), \( \chi(\xi) = 1 \) for \( |\xi| \geq 2 \), and integrating the first integral by parts \(|p| + n + 1 \).

**Proof of theorem A.7.** Let \( \chi_0 \in C^\infty(\mathbb{R}^n) \) such that \( \chi_0(\eta) = 1 \) in a neighbourhood of 0 and \( \text{supp}(\chi_0) \subset (-\pi, \pi)^k \). For \( k \in \mathbb{Z}^n \), define \( \chi_k \) by \( \chi_k(\eta) := \chi(\eta + 2\pi k) \) and note that the supports of the \( \chi_k \) are disjoint. Finally, let \( \chi_c := 1 - \sum_k \chi_k \) ("c" for "complement"), so that \( \chi_c \) together with the \( \chi_k \)'s form a partition of unit. Then

\[
L_1(x) = \ell_c(x) + \sum_{k \in \mathbb{Z}^n} \ell_k(x),
\]

where

\[
\ell_k = \mathcal{F}^{-1} \left( \chi_k(\eta) \frac{\hat{\varphi}(\eta)}{\sum_{\nu} \hat{\varphi}(\eta + 2\pi \nu)} \right), \quad k \in \mathbb{Z}^n \text{ or } k = c.
\]

We examine the decay in \( x \) of each term separately.

**Decay of \( \ell_c \).** The function \( \chi_c(\eta)/\sum_k \hat{\varphi}(\eta + 2\pi k) \) is in \( C^{[|\kappa|+n+1]}(\mathbb{R}^n) \), since the denominator is a strictly positive periodic function which is \( C^{[|\kappa|+n+1]} \) on the complement of \( (2\pi \mathbb{Z})^n \) and therefore on the support of \( \chi_c \). Multiplying with \( \hat{\varphi} \), we find that \( \hat{L}_c(\eta) \) is \( C^{[|\kappa|+n+1]} \) with integrable derivatives of all orders, which implies by the usual integration by parts argument that \( |\ell_c(x)| \leq C(1+|x|)^{-|\kappa|-n} \).

**Note** that \( \hat{L}_1(\eta) \) is at best \( C^{[|\kappa|} \) in the points of \( 2\pi \mathbb{Z}^n \), so integration by parts will not give the required decay for of \( \ell_k, k \in \mathbb{Z}^n \). We use lemma A.2 instead.

**Decay of \( \ell_0 \).** Since

\[
\hat{\ell}_0(\eta) - \chi_0(\eta) = \chi_0(\eta) \left( \hat{L}_1(\eta) - 1 \right) = -\chi_0(\eta) \left( \frac{\hat{\varphi}(\eta) - \sum_{k \neq 0} \hat{\varphi}(\eta + 2\pi k)}{1 + \sum_{\nu} \hat{\varphi}(\eta + 2\pi \nu)} \right)
\]

and since \( \sum_{k \neq 0} \hat{\varphi}(\eta + 2\pi k) \) is \( C^{[|\kappa|+n+1]} \) on the support of \( \chi_0 \), the estimates (89) easily imply that \( \hat{\ell}_0(\eta) - \chi_0(\eta) \) satisfies condition (92) of lemma A.2 with \( p = \kappa \) (for \( \psi := \hat{\varphi}^{-1} \sum_{k \neq 0} \hat{\varphi}(\cdot + 2\pi k) \) does, and then also \( \psi/(1+\psi) \)). It follows that \( |\ell_0(x)| \leq C(1+|x|)^{-\kappa-n} \), since \( \mathcal{F}^{-1}(\chi_0) \) is rapidly decreasing.

**Decay of \( \ell_k, k \neq 0, c \).** This is similar, except that we have to pay attention to the size of the constant in front of the \( (1+|x|)^{-\kappa-n} \). The Fourier transform \( \hat{\ell}_k(\eta) \) will now be supported near \( \eta = -2\pi k \). Shifting by \( 2\pi k \), we see that

\[
\hat{\ell}_k(\eta - 2\pi k) = \chi_0(\eta) \varphi(\eta - 2\pi k) \frac{\hat{\varphi}(\eta)}{1 + \sum_{\nu \neq 0} \hat{\varphi}(\eta + 2\pi \nu)}
\]

is supported in a small neighbourhood of 0, with derivatives of order \( |\alpha| \leq |\kappa| + n + 1 \) bounded by \( C(1 + |k|)^{-N} |\eta|^{\kappa+|\alpha|} \), with \( C \) independent of \( k \). Lemma A.2 then implies that

\[
|\ell_k(x)| = \left| \ell_k(x) 2^{2\pi k(x)} \right| \leq C(1 + |k|)^{-N}(1 + |x|)^{-\kappa-n}.
\]

Since \( N > n \) by assumption, summation over \( k \in \mathbb{Z}^n \) completes the proof.

\[\square\]

The same arguments prove theorem 6.1: \( a(L_1)(L_1) = a_c + \sum_k a_c \hat{\ell}_k \), and \( a_c \) has \( [\kappa]+n+1 \) derivatives which are integrable since \( N + q < n \), as has \( a\chi_0 \). Finally, \( a\chi_0(L - 1) \) and \( a\chi_1 \hat{L}_1, k \neq 0 \) satisfy the hypothesis (92) of lemma A.2, the latter with constants bounded by \( C|k|^{-(N-q)} \) which are summable since \( N > n + q \).

We continue with the proof of theorem 4.3: The expressions for \( \hat{\ell}_k \) above also show that \( \hat{L}_1 \) satisfies the Strang-Fix conditions (10). Once we have defined \( L_1 \) through its Fourier transform, it is immediate
to check that $L_1(k) = \delta_{0k}$ for $k \in \mathbb{Z}^n$: indeed, by the $2\pi$-periodicity of the denominator, writing the integral over $\mathbb{R}^n$ as a sum of integrals over translates of $(-\pi, \pi)^n$,

$$L_1(k) = \int_{\mathbb{R}^n} \frac{\hat{\varphi}(\eta)}{\sum_{\nu} \hat{\varphi}(\eta + 2\pi \nu)} e^{i k \eta} \, d\eta \bigg/ (2\pi)^n$$

$$= \int_{(-\pi, \pi)^n} \sum_{\nu} \hat{\varphi}(\eta + 2\pi \nu) e^{i k \eta} \, d\eta \bigg/ (2\pi)^n$$

$$= \int_{(-\pi, \pi)^n} e^{i k \eta} \, d\eta \bigg/ (2\pi)^n$$

$$= \delta_{0k}.$$ 

It remains to recognise $L_1$ as a sum of translates of $\varphi$. To show this we first write the denominator of (96) as a Fourier series:

$$\left( \sum_k \hat{\varphi}(\eta + 2\pi k) \right)^{-1} = \sum_k c_k e^{i k \eta}.$$ 

One verifies by the similar arguments as the ones of the proof of theorem A.1 that

$$|c_k| \leq C (1 + |k|)^{-\kappa-n},$$

write

$$c_k = (2\pi)^{-n} \int_{(-\pi, \pi)^n} \chi_0(\eta) e^{i (\eta, k)} \, d\eta + (2\pi)^{-n} \int_{(-\pi, \pi)^n} \sum_{\nu} \hat{\varphi}(\eta + 2\pi \nu) e^{i (\eta, k)} \, d\eta,$$

and estimate the first integral using lemma A.2 and the second by integrating by parts.

It follows that (96) converges absolutely. We then claim that

$$L_1(x) = \sum_k c_k \varphi(x - k),$$

where the series converges absolutely and uniformly on compacta, by (97), since $\varphi(x)$ grows at most as $(1 + |x|)^{\kappa-\epsilon}$, by assumption. Formally, (88) follows by writing

$$L_1(x) = \int_{\mathbb{R}^n} \left( \sum_k c_k e^{i k \eta} \right) \hat{\varphi}(\eta) e^{i nx} \, d\eta \bigg/ (2\pi)^n = \sum_k c_k \varphi(x + k),$$

except that the final step does not make sense for an arbitrary $\varphi \in \mathcal{B}_{\kappa,N}(\mathbb{R}^n)$ since $\hat{\varphi}(\eta)$ will not be integrable in 0 if $\kappa > n$ and even if it is, when $\kappa < n$, it might differ from integration against the function $\hat{\varphi}(\eta)$ by a distribution supported in 0.

We have to carefully distinguish between the tempered distribution $\hat{\varphi}$ and the locally integrable function $\eta \to \hat{\varphi}(\eta)$ with which it can be identified on $\mathbb{R}^n \setminus 0$. The relation between the two is given by the following identity: there exist constants $c_\alpha$, $|\alpha| \leq \kappa - 1$ such that for all $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \hat{\varphi}, \psi \rangle = \int_{|\eta| \leq 1} \hat{\varphi}(\eta) \left( \psi(\eta) - \sum_{|\alpha| \leq \kappa - 1} \frac{\psi^{(\alpha)}(0)}{\alpha!} \eta^\alpha \right) \, d\eta$$

$$+ \int_{|\eta| \geq 1} \hat{\varphi}(\eta) \psi(\eta) \, d\eta$$

$$+ \sum_{|\alpha| \leq \kappa - 1} (-1)^{|\alpha|} c_\alpha \psi^{(\alpha)}(0),$$

where the sum is interpreted as empty if $\kappa < n$. Indeed, the first integral converges since $|\eta|^{\kappa-n+1} \hat{\varphi}(\eta)$ has an integrable singularity at 0. The sum of the two integrals on the right defines a tempered distribution. If we denote this distribution by $\Lambda_\varphi$ then the restriction of $\Lambda_\varphi$ to $\mathbb{R}^n \setminus 0$ can be identified with the function $\hat{\varphi}(\eta)$. The difference $\hat{\varphi} - \Lambda_\varphi$ is then supported in 0, and therefore a linear combination
\[ \sum_{|\alpha| \leq p} c_\alpha \delta_0^{(\alpha)} \] of derivatives of the delta distribution in 0. To bound \( p \), we use the following lemma, whose proof we postpone till the end of his section:

**Lemma A.3.** If \( \kappa \geq n \), then the inverse Fourier transform \( \mathcal{F}^{-1}(\Lambda_\varepsilon) \) is a continuous function which is bounded by \( C(|x|^{\kappa-n} + 1) \) for non-integer \( \kappa \) and by \( C(|x|^{\kappa-n} \log |x| + 1) \) if \( \kappa \) is a positive integer.

Since the inverse Fourier transform of \( \sum_{|\alpha| \leq p} c_\alpha \delta_0^{(\alpha)} \) is a polynomial of order \( p \), and since, by assumption, \( \varphi(x) \), has polynomial growth of order strictly less than \( \kappa \), it follows that \( p < \kappa \), which is equivalent to \( p \leq [\kappa] - 1 \).

We now use (100) to prove that (98) holds as tempered distributions, that is, if \( \psi \in \mathcal{S}(\mathbb{R}^n) \), then

\[ \langle L_1, \hat{\psi} \rangle = \left\langle \sum_k c_k \varphi(\cdot + k), \hat{\psi} \right\rangle. \]  

If we let

\[ \Psi(\eta) := \frac{\psi(\eta)}{\sum_k \hat{\varphi}(\eta + 2\pi k)}. \]

then \( \Psi \) is \( C^{[\kappa]} \) if \( \kappa \notin \mathbb{N} \), and \( C^{\infty, 1} \) if \( \kappa \in \mathbb{N}^* \), with all its derivatives rapidly decreasing. To obtain a function in the Schwartz class \( \mathcal{S}(\mathbb{R}^n) \) we convolve with \( \chi_\varepsilon(x) := \varepsilon^{-n} \chi(x/\varepsilon) \), where \( \chi \in C_c^{\infty}(\mathbb{R}^n) \) with integral 1. Let \( \Psi_\varepsilon := \chi_\varepsilon * \Psi \). Then we first claim that

\[ \langle \hat{\varphi}, \Psi_\varepsilon \rangle \rightarrow \int_{\mathbb{R}^n} \hat{L}_1(\eta)\psi(\eta) \, d\eta, \]

To show this it suffices to consider the case that \( \kappa \geq n \), since \( \hat{\varphi} \) is integrable if \( \kappa < n \) and \( \Psi_\varepsilon \) converges uniformly. If we assume for example that \( \text{supp}(\hat{\chi}) \subset B(0, 1) \) then by the Taylor expansion with remainder there exists a constant \( C > 0 \) such that for all \( \varepsilon \leq 1 \),

\[ \left| \Psi_\varepsilon(\eta) - \sum_{|\alpha| \leq [\kappa] - n} \frac{\Psi_\varepsilon^{(\alpha)}(0)}{\alpha!} \eta^\alpha \right| \leq C \max_{|\beta| = [\kappa] - n + 1} \sup_{B(0, 1)} |\Psi_\varepsilon^{(\beta)}| : |\eta|^{[\kappa] - n + 1} \]

\[ \leq C \max_{|\beta| = [\kappa] - n + 1} \sup_{B(0, 2)} |\Psi_\varepsilon^{(\beta)}| : |\eta|^{[\kappa] - n + 1}, \]

where we note that if \( n \geq 2 \) or if \( \kappa \notin \mathbb{N}^* \), then derivatives of \( \Psi \) of order \( [\kappa] - n + 1 \) exist, while if \( n = 1 \) and \( \kappa \in \mathbb{N}^* \), these derivatives exist a.e. but are uniformly bounded, and the estimate remains true. Next, \( \Psi_\varepsilon^{(\alpha)}(x) \rightarrow \Psi^{(\alpha)}(x) \) for \( |\alpha| \leq [\kappa] - 1 \) since \( \Psi \) is \( C^{[\kappa]} \). Furthermore, since \( (\sum_k \hat{\varphi}(\eta + 2\pi k))^{-1} \) vanishes of order \( \kappa \) in 0, it follows that \( \Psi^{(\alpha)}(0) = 0 \) for \( |\alpha| \leq [\kappa] - 1 \) and (101) follows by dominated convergence.

Since \( \Psi_\varepsilon \) is Schwartz-class, we have \( \langle \hat{\varphi}, \Psi_\varepsilon \rangle = \langle \varphi, \hat{\Psi}_\varepsilon \rangle \). By (98) \( \Psi = (\sum_k c_k e^{-i(k, \eta)}) \psi(\eta) \), which can be interpreted as the product of a tempered distribution and a test function, whose Fourier transform equals

\[ \hat{\Psi}(x) = \sum_k c_k \hat{\varphi}(x - k). \]

One easily verifies using (97) that \( |\hat{\Psi}(x)| \leq C(1 + |x|)^{-\kappa-n} \).

Since \( \hat{\Psi}_\varepsilon(x) = \hat{\chi}_\varepsilon(x) \hat{\Psi}(x) \), and since \( |\varphi(x)| \leq C(1 + |x|)^{\kappa-\rho} \) for some \( \rho > 0 \), Lebesgue’s dominated convergence theorem then shows that

\[ \langle \varphi, \hat{\Psi}_\varepsilon \rangle = \int_{\mathbb{R}^n} \varphi(x) \hat{\Psi}_\varepsilon(x) \hat{\chi}_\varepsilon(x) \, dx \]

\[ \rightarrow \int_{\mathbb{R}^n} \varphi(x) \left( \sum_k c_k \hat{\varphi}(x - k) \right) \, dx. \]
Finally, one checks that the functions \( (x, k) \rightarrow c_k \varphi(x) \hat{\psi}(x - k) \) and \( (x, k) \rightarrow c_k \varphi(x + k) \hat{\psi}(x) \) are integrable on \( \mathbb{R}^n \times \mathbb{Z}^n \) with respect to the product of the Lebesgue measure and the counting measure. A double application of Fubini’s theorem then shows that the right hand side equals
\[
\int_{\mathbb{R}^n} \left( \sum_k c_k \varphi(x + k) \right) \hat{\psi}(x) \, dx,
\]
which proves (100). The pointwise identity (98) follows since both sides are continuous.

**Proof of lemma** (A.3) The lemma presumably is classical, but since we could not locate a suitable reference (apart from the well-known case of homogeneous \( \hat{\varphi} \)), we sketch a proof for convenience of the reader. If \( \kappa < n \), the inverse Fourier transform is a bounded function, so suppose that \( \kappa \geq n \). Since \( 1_{\{ |\eta| \geq 1 \}} \hat{\varphi}(\eta) \) is integrable, its inverse Fourier transform is a bounded continuous function, and it therefore suffices to examine the inverse Fourier transform of the tempered distribution defined by the first integral on the right hand side of (99). This distribution being of compact support, its inverse Fourier transform is the function \( k(x) \) obtained by taking \( \psi(\eta) = (2\pi)^{-n} e^{i(x, \eta)} \):

\[
k(x) := (2\pi)^{-n} \int_{|\eta| \leq 1} \hat{\varphi}(\eta) \left( e^{i(x, \eta)} - \sum_{j \leq \nu} \frac{i^j (x, \eta)^j}{j!} \right) \, d\eta,
\]

where we put \( \nu := |\kappa| - n \). This can be bounded by

\[
|k(x)| \leq C \int_{|\eta| \leq 1} |\eta|^{-\kappa} \left| e^{i(x, \eta)} - \sum_{j \leq \nu} \frac{i^j (x, \eta)^j}{j!} \right| \, d\eta
= C|x|^{\kappa - n} \int_{|\eta| \leq |x|} |\eta|^{-\kappa} \left| e^{i(x, \eta)} - \sum_{j \leq \nu} \frac{i^j (x, \eta)^j}{j!} \right| \, d\eta.
\]

Split the integral into an integral over \( |\eta| \leq c \) and one over the complement, where \( c > 0 \) is some fixed number and where we assume wlog that \( |x| > c \). The first integral converges absolutely, since

\[
\left| e^{i(x, \eta)} - \sum_{j \leq \nu} \frac{i^j (x, \eta)^j}{j!} \right| \leq \frac{1}{\nu!} |(x, \eta)|^{\nu + 1} \leq \frac{|\eta|^{\nu + 1}}{\nu!},
\]

and we can bound its contribution to \( k(x) \) by \( C|x|^{\kappa - n} \). As for the integral over \( |\eta| > c \), it can be bounded by a constant times

\[
|x|^{\kappa - n} \sum_{\nu=0}^{\nu} \int_c^{|x|} r^{-\kappa+j+n-1} \, dr = |x|^{\kappa - n} \sum_{j=0}^{\nu} \frac{1}{j - \kappa + n} \left( |x|^{j - \kappa + n} - c^{j - \kappa + n} \right),
\]

assuming that \( \kappa \notin \mathbb{N} \). Since \( j - \kappa + n \leq \nu - \kappa + n \leq 0 \) by the definition of \( \nu \), this will be bounded by \( C|x|^{\kappa - n} \). Finally, if \( \kappa \in \mathbb{N} \), \( \kappa \geq n \), then \( \nu = \kappa - n \) and

\[
|x|^{\kappa - n} \sum_{j=0}^{\kappa - n} \int_c^{|x|} r^{j - (\kappa - n) - 1} \, dr
= |x|^{\kappa - n} \sum_{j=0}^{\kappa - n - 1} \frac{1}{j - \kappa + n} \left( |x|^{j - (\kappa - n)} - c^{j - (\kappa - n)} \right) + \log(|x|/c)
\leq C|x|^{\kappa - n} \log |x| + 1.
\]
Remark A.4. The only hypotheses on \( \hat{\phi}(\eta) \) we needed for this lemma is that it be integrable on \( \{ |\eta| \geq 1 \} \) and that \( \hat{\phi}(\eta) = O(|\eta|^{-n}) \) near 0. If we strengthen the first assumption to
\[
|\eta|^r |\hat{\phi}(\eta)| 1_{\{ |\eta| \geq 1 \}} \in L^1(\mathbb{R}^n),
\]
where \( r \in \mathbb{N} \), then \( k \) will be \( r \)-times differentiable, and we will have that
\[
|\partial_k^r k(x)| \leq \begin{cases} C(|x|^{\max(k-n-|\alpha|,0)} + 1) & \kappa \notin \mathbb{N} \\ C(|x|^{\max(k-n-|\alpha|,0)} \log |x| + 1) & \kappa \in \mathbb{N}, \end{cases}
\]
for \( |\alpha| \leq r \): it suffices to observe that if \( k(x) \) is given by \( \Gamma(\eta) \) then its derivative of order \( \alpha \) is given by the same formula with \( \hat{\phi}(\eta) \) replaced by \( (i\eta)^\alpha \hat{\phi}(\eta) \).

These estimates can be used to obtain symbol estimates for the generator of pure-jump \( \text{Lévy} \) processes with \( \text{Lévy} \) measure \( \psi \)
\[ \psi \]
inverse Fourier transform of \( \Lambda \)
\[ \Lambda \]
proof of lemma F.

We next observe that \( \Lambda_{\eta} = -\eta \) then is, modulo a function in \( C_b^\infty \), equal to the symbol of the generator of the \( \text{Lévy} \) process, and the estimates show this symbol to be in \( \mathcal{S}_0^{\max(q-n,0)} \) if \( q \notin \mathbb{N} \), and in \( \mathcal{S}_0^{\max(q-n,0)+\varepsilon} \) for any \( \varepsilon > 0 \) otherwise (even a bit better, since the first few derivatives will decay relative to the symbol itself). Examples are given by the CGMY-processes which are used in financial modeling.

APPENDIX B. Some technical proofs

B.1. Proof of lemma 5.5. Let \( F \in L^1(\mathbb{R}^n \setminus 0, (|\xi|^{\kappa} \wedge 1) d\xi) \), where \( a \wedge b := \min(a,b) \) and \( \kappa \geq 0 \). Then \( F \) gives rise to a tempered distribution \( \Lambda_F \in \mathcal{S}'(\mathbb{R}^n) \) defined as follows: if \( g \in C_c^\infty(\mathbb{R}^n) \) be equal to 1 on a neighbourhood of 0, we put
\[
\langle \Lambda_F, \psi \rangle := \int_{\mathbb{R}^n} \left( \psi(\xi) - \sum_{|\alpha| \leq |\kappa|-1} \psi^{(\alpha)}(0) \xi^\alpha / \alpha! \right) g(\xi) F(\xi) d\xi + \int_{\mathbb{R}^n} (1 - g(\xi)) F(\xi) \psi(\xi) d\xi, \quad \psi \in \mathcal{S}(\mathbb{R}^n).
\]

The integral converges since \( \psi - \sum_{|\alpha| \leq |\kappa|-1} \psi^{(\alpha)}(0) \xi^\alpha / \alpha! = O(|\xi|^{[\kappa]}) = O(|\xi|^{\kappa}) \) in a neighbourhood of 0 and defines a distribution of order \( |\kappa| - 1 \). Note that \( \Lambda_F \) coincides on \( \mathbb{R}^n \setminus 0 \) with the locally integrable function \( F \).

We next observe that \( \Lambda_F \) extends to a continuous linear functional on the Hölder space \( C_b^{[\kappa]-1,\lambda} := C_b^{[\kappa]-1,\lambda}(\mathbb{R}^n) \) with \( \lambda = \kappa - ([\kappa] - 1) \). Indeed, if \( \psi \in C_b^{K,\lambda}(\mathbb{R}^n) \), then the Taylor expansion formula with integral remainder term easily implies that
\[
\left| \psi(\xi) - \sum_{|\alpha| \leq K} \psi^{(\alpha)}(0) \xi^\alpha / \alpha! \right| \leq C \left( \sum_{|\beta| = K} ||\psi^{(\beta)}||_{0,\lambda} \right) |\xi|^{K+\lambda},
\]
which shows, with \( K = [\kappa] - 1 \) and \( \lambda = \kappa - ([\kappa] - 1) \), that \( \langle \Lambda_F, \psi \rangle \) is well-defined and continuous.

We can, in particular, let \( \Lambda_F \) act on the imaginary exponentials \( \xi \to e^{i(x,\xi)} \). The function
\[
\tilde{F} : x \to (2\pi)^{-n} \left( \Lambda_F, e^{i(x,\xi)} \right).
\]
is then found to be bounded by \( C(1 + |x|^\kappa) \), since \( ||e^{i(x,\xi)}||_{K,\lambda} \leq C(1 + |x|^{K+\lambda}) \), and one easily verifies that the inverse Fourier transform of \( \Lambda_F \) coincides with \( \tilde{F} \). If \( \kappa \in \mathbb{N} \) one has the stronger estimate
\[
|\tilde{F}(x)| = o(|x|^{\kappa}), \quad |x| \to \infty,
\]
which can be seen as follows: write $F = \chi F + (1 - \chi)F$ with $\chi$ the characteristic function of a small ball around 0. Since $(1 - \chi)F$ is integrable, its inverse Fourier transform tends to 0 at infinity, by the Riemann-Lebesgue lemma. We can therefore wlog assume that $F$ is supported in $\{s = 1\}$. If we apply \[63\] with $\psi(x) = e^{i(x,\xi)}$ then

$$F(x) = \sum_{|\alpha| = \kappa} \int_{\mathbb{R}^n} F(\xi) \frac{(ix)^\alpha \xi^\alpha}{\alpha!} \left( \int_0^1 (1 - s)^{\kappa - 1} - \frac{1}{(\kappa - 1)!} ds \right) e^{is(x,\xi)} ds =: \sum_{|\alpha| = \kappa} (ix)^\alpha \int_0^1 \hat{F}_\alpha(sx) \frac{(1 - s)^{\kappa - 1}}{(\kappa - 1)!} ds,$$

where $\hat{F}_\alpha(x)$ is the inverse Fourier transform of the $L^1$-function $\xi \to \xi^\alpha F(\xi)$. By the Riemann-Lebesgue lemma, $\hat{F}_\alpha(sx) \to 0$ as $s \to \infty$, for all $s \in (0, 1)$, and the same is true for the integral over $s \in [0, 1]$, by the dominated convergence theorem (the $\hat{F}_\alpha$ are bounded). Hence $F(x)/|x|^\kappa \to 0$ for $x \to \infty$, as claimed.

Now let $f$ be a measurable function on $\mathbb{R}^n$ of polynomial growth of order strictly less than $\kappa$, such that its Fourier transform $\hat{f}$ (in the sense of tempered distributions) satisfies

$$\hat{f}|_{\mathbb{R}^n \setminus 0} \in L^1(\mathbb{R}^n, (|\xi|^\kappa \wedge 1)d\xi).$$

We write $\Lambda_{\hat{f}}$ for $\Lambda_{\hat{f}|_{\mathbb{R}^n \setminus 0}}$. Then $\hat{f} - \Lambda_{\hat{f}}$ is a distribution which is supported in 0, and therefore of the form $\sum_{|\alpha| \leq N} c_\alpha \delta_0^{(\alpha)}$ for certain $N \in \mathbb{N}$ and $c_\alpha \in \mathbb{C}$ with $\sum_{|\alpha| = N} |c_\alpha| \neq 0$. Since the inverse Fourier transform of $\hat{f} - \Lambda_{\hat{f}}$ is a polynomial of degree $N$, it follows that $N \leq [\kappa] - 1$, the largest integer which is strictly smaller than $\kappa$, since otherwise $|f(x)|$ would grow at a rate of at least $|x|^{|\kappa|}$ in certain directions. If $\kappa \notin \mathbb{N}$ this would contradict the bound $\hat{f}(x) = 0(|x|^\kappa)$, and if $\kappa \in \mathbb{N}$ this would contradict \[106\].

In follows that $\hat{f} = \Lambda_{\hat{f}} + \sum_{|\alpha| \leq N} c_\alpha \delta_0^{(\alpha)}$ also extends to a continuous linear functional on $C^{[\kappa] - 1, \kappa - [\kappa] - 1}$.

We exploit this to define $\Sigma_h(\hat{f})$ by duality.

If $\psi \in \mathcal{S}(\mathbb{R}^n)$, we let

$$\Sigma_h(\psi) := \sum_k \psi(\xi + 2\pi h^{-1} k) \hat{L}_1(h\xi + 2\pi k).$$

Note that $\Sigma_h(\psi)$ is the (formal) (real) adjoint of $\Sigma_h$. By lemma \[2.6\] $\Sigma_h(\psi)$ is $C^{[\kappa] - 1, \lambda}$ with $\lambda = \kappa - ([\kappa] - 1)$ and uniformly bounded together with all its derivatives, since $2\pi h^{-1}$-periodic. In fact, this is true even if $\psi \in C^{[\kappa] - 1, \lambda}$ with the same $\lambda$, on account of the decay at infinity of $\hat{L}_1$. We can then define $\Sigma_h(\hat{f})$, as a tempered distribution and, more generally, as a bounded linear functional on $C^{[\kappa] - 1, \lambda}(\mathbb{R}^n)$ by

$$\left\langle \Sigma_h(\hat{f}), \psi \right\rangle := \left\langle \hat{f}, \Sigma_h(\psi) \right\rangle.$$

We next check that $\Sigma_h(\hat{f})$ is the Fourier transform, in distribution sense, of $s_h[f]$. This is done by a standard approximation argument, with some care with the spaces in which the approximating sequence converges. We first note that we can assume without loss of generality that $\hat{f}$ is compactly supported: indeed, we can write $f = f_1 + f_2$ with $\hat{f}_1$ compactly supported and $\hat{f}_2 \in L^1(\mathbb{R}^n)$, and we know already that $s_h[\hat{f}_2] = \Sigma_h(\hat{f}_2)$.

\[\text{e.g. by using the Taylor formula with integral remainder in the form}\]

$$\psi(\xi) - \sum_{|\alpha| \leq \kappa - 1} \psi^{(\alpha)}(0) \frac{\xi^\alpha}{\alpha!} = \int_0^1 \frac{(1 - s)^{\kappa - 1}}{(\kappa - 1)!} ds \psi(s)\xi ds,$$

where $\psi(s) := \psi(s\xi)$
We first check that which is valid both for Schwarz-class functions \(\psi\). Next, the function \((\chi \ast \psi)\) is, and \(\psi \in C^{K,\lambda}\), then \(\psi^{(\alpha)}(x) \rightarrow \psi^{(\alpha)}(x)\) pointwise on \(\mathbb{R}^n\) for all \(|\alpha| \leq K\), while a trivial estimate shows that \(||\psi^{(\alpha)}||_{0,\lambda} \leq ||\psi^{(\alpha)}||_{0,\lambda}\), uniformly in \(\varepsilon > 0\), for \(|\alpha| = K\). This, together with the remainder estimate [105], the integrability of \(\tilde{f}(\xi)(|\xi|^\alpha \wedge 1)\) and Lebesgue’s dominated convergence theorem, implies that \(\langle \Lambda_{\tilde{f}}, \psi \rangle \rightarrow \langle \Lambda_{\tilde{f}}, \psi \rangle\). Since also \(\langle \delta^{(\alpha)}, \psi \rangle \rightarrow \langle \delta^{(\alpha)}, \psi \rangle\) for all \(|\alpha| \leq K\), the lemma follows.

The lemma immediately implies that if \(\psi \in S(\mathbb{R}^n)\), then \((\hat{f} \ast \chi_{\varepsilon}, \Sigma_h(\psi)) \rightarrow (\hat{f}, \Sigma_h'(\psi))\), so \(\Sigma_h(\hat{f} \ast \chi_{\varepsilon}) \rightarrow \Sigma_h(\hat{f})\) in \(S'(\mathbb{R}^n)\) and even in \((C^{K,\lambda})'\) with \(K\) and \(\lambda\) as above.

On the other hand, if we let \(f_\varepsilon\) be the inverse Fourier transform of \(\hat{f} \ast \chi_{\varepsilon}\), then \(f_\varepsilon \in S(\mathbb{R}^n)\) since \(\hat{f} \ast \chi_{\varepsilon}\) is, and \(s_h[f_\varepsilon] = \Sigma_h(\hat{f} \ast \chi_{\varepsilon})\). We have that \(f_\varepsilon(x) = (2\pi)^n f(x) \tilde{\chi}(\varepsilon x)\), with \(\tilde{\chi}\) the inverse Fourier transform of \(\chi\), so \(\tilde{\chi} \in \mathcal{S}(\mathbb{R}^n)\) and \((2\pi)^n \tilde{\chi}(0) = 1\). By hypotheses, \(f \in L^\infty_p\) for some \(p < \kappa\). If \(a > 0\) such that \(p + a < \kappa\), then by [24], writing \(\tilde{\chi} := (2\pi)^n \tilde{\chi}\),

\[
\|s_h[f_\varepsilon] - s_h[f]\|_{\infty,-(p+a)} \leq C \|f(\tilde{\chi}(\varepsilon) - 1)\|_{\infty,-(p+a)} \leq C \|f\|_{\infty,-p} \sup_{x \in \mathbb{R}^n} \frac{\tilde{\chi}(\varepsilon x) - 1}{(1 + |x|^a)} \rightarrow 0,
\]

as \(\varepsilon \rightarrow 0\), using for example the first order Taylor expansion for \(\tilde{\chi}\) for \(|x| \leq \varepsilon^{-1/2}\) plus a trivial estimate for \(|x| > \varepsilon^{-1/2}\). This certainly implies that \(s_h[f_\varepsilon] \rightarrow s_h[f]\) in \(S'(\mathbb{R}^n)\), so we conclude that \(s_h[f_\varepsilon] = \Sigma_h(\hat{f} \ast \chi_{\varepsilon}) \rightarrow \Sigma_h(\hat{f})\) as distributions.

We finally show that \(\Sigma_h(\hat{f}) = \tilde{f} + F\), where \(F\) is the (distribution obtained by integrating against the function

\[
F(\xi) = \hat{f}(\xi)(\hat{L}_1(h\xi) - 1) + \sum_{k \neq 0} \hat{f}(\xi + 2\pi h^{-1} k)\hat{L}_1(h\xi).
\] (109)

We first check that \(F\) is well-defined and in \(L^1\): first of all, each of the terms on the right hand side is in \(L^1\), on account of the Fix-Strang conditions satisfied by \(\hat{L}_1\) and the integrability of \((|\xi|^\alpha \wedge 1)\hat{f}(\xi)\). Next, the function \((\xi, k) \rightarrow \hat{f}(\xi + 2\pi h^{-1} k)(\hat{L}_1(h\xi) - \delta_{0k})\) is absolutely integrable on \(\mathbb{R}^n \times \mathbb{Z}^n\) with respect to the product of Lebesgue measure and the counting measure, since

\[
\sum_k \int_{\mathbb{R}^n} |\hat{f}(\xi + 2\pi h^{-1} k)| |(\hat{L}_1(h\xi) - \delta_{0k})| d\xi = \int_{\mathbb{R}^n} (1 - \hat{L}_1(h\xi))|\hat{f}(\xi)|d\xi + \sum_{k \neq 0} \hat{L}_1(h\xi + 2\pi k)|\hat{f}(\xi)|d\xi = 2 \int_{\mathbb{R}^n} (1 - \hat{L}_1(h\xi))|\hat{f}(\xi)|d\xi.
\]
Fubini’s theorem then implies that $F(\xi)$ is well-defined for almost all $\xi \in \mathbb{R}^n$ and that $F \in L^1(\mathbb{R}^n)$. If $\psi \in \mathcal{S}(\mathbb{R}^n)$, then a double application of Fubini will show that

$$
\int_{\mathbb{R}^n} F(\xi) \psi(\xi) d\xi
= \int_{\mathbb{R}^n} \left( \psi(\xi)(\hat{L}_1(h\xi) - 1) + \sum_{k \neq 0} \psi(\xi + 2\pi h^{-1}k)\hat{L}_1(h\xi + 2\pi k) \right) \hat{f}(\xi) d\xi
= \int_{\mathbb{R}^n} (\Sigma_h'(\psi) - \psi) \hat{f}(\xi) d\xi.
$$

Since, by the Fix-Strang conditions (10), all derivatives of order $\leq \lfloor \kappa \rfloor - 1$ of $\Sigma_h(\psi) - \psi$ in 0 are 0, the last integral is equal to $\langle \hat{f}, \Sigma_h' \psi - \psi \rangle = \langle \Sigma_h(\hat{f}) - \hat{f}, \psi \rangle$, and therefore $\Sigma_h(\hat{f}) - \hat{f} = F$, which finishes the proof of lemma 5.3.

**Remark B.2.** The lemma and its proof generalizes to $f$’s such that $|\xi|^n \wedge 1$ is integrable, provided that $\kappa \notin \mathbb{N}$ (the reason being that we then no longer have (10)).

**B.2. Proof of lemma 5.4.** It again suffices to consider the case of compactly supported $\hat{f}$’s. We use the notations of the proof of lemma 5.3 above: in particular, let $f_\varepsilon$ be the inverse Fourier transform of $\hat{f} \ast \chi_{\varepsilon}$ where $\chi_{\varepsilon} = \varepsilon^{-n} \chi(./\varepsilon)$ is an approximation of the identity. We have seen that $\Sigma_h(\hat{f}_\varepsilon) \to \Sigma_h(\hat{f})$ in $(C^{K,\lambda})'$, where $K = \lfloor \kappa \rfloor - 1$ and $\lambda = \kappa - K$. Since $e^{-h^{-2}tG(h\cdot)} \in C^{K,\lambda}$ this implies that

$$
e^{-h^{-2}tG(h\cdot)} \Sigma_h(\hat{f}_\varepsilon) \to e^{-h^{-2}tG(h\cdot)} \Sigma_h(\hat{f})$$

in $(C^{K,\lambda})'$ and hence in $\mathcal{S}'(\mathbb{R}^n)$.

On the other hand, we have seen in the proof of lemma 5.3 that $f_\varepsilon \to f$ in $L^\infty_{p-a}$ if $a > 0$. Hence by lemma 5.1, if $a < \kappa - p$ then $u_h[f_\varepsilon] \to u_h[f]$ in $L^\infty_{p-a}$ and therefore as tempered distributions. This implies that

$$
e^{-h^{-2}tG(h\cdot)} \Sigma_h(\hat{f}_\varepsilon) = u_h[f_\varepsilon] \to u_h[f],$$

where we used lemma 5.2. Hence $\widetilde{u_h[f]} = e^{-h^{-2}tG(h\cdot)} \Sigma_h(\hat{f}) = e^{-h^{-2}tG(h\cdot)} \Sigma_h(\hat{f}_\varepsilon)$ as tempered distributions, as claimed. We finally prove (68): if we let

$$g(\xi, t; h) := e^{-t(h^{-2}G(h\xi) - |\xi|^2)} - 1,$$

then $g$ is a $C^{K,\lambda}$-function of $\xi$ and

$$\widetilde{u_h[f]}(\xi, t) - \tilde{u}(\xi, t) = e^{-th^{-2}G(h\cdot)} \left( \Sigma_h(\hat{f}_\varepsilon) - \hat{f} \right) + (g(\cdot, t; h) - 1) e^{-t|\cdot|^2} \tilde{f}.$$

Since $g(\xi, t; h)$ vanishes to order $|\xi|^p$ in $\xi = 0$, by proposition 5.3(ii), the representation $\hat{f} = \Lambda_{\hat{f}} + \sum_{|\alpha| \leq \lfloor \kappa \rfloor - 1} c_\alpha \delta^{(\alpha)}$ from the proof of lemma 5.3 shows that the distribution $g(\cdot, t; h) \hat{f}$ can be identified with the locally integrable function $\xi \to g(\xi, t; h) \hat{f}(\xi)$. \qed