Scalar Curvature Behavior for Finite Time Singularity of Kähler-Ricci Flow

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January 11, 2009

Abstract

In this short paper, we show that Kähler-Ricci flows over closed manifolds would have scalar curvature blown-up for finite time singularity. Certain control of the blowing-up is achieved with some mild assumption.

1 Introduction

In this short note, we consider the following Kähler-Ricci flow

\[
\frac{\partial \tilde{\omega}_t}{\partial t} = -\text{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t, \quad \tilde{\omega}_0 = \omega_0.
\]  

(1.1)

over a closed Kähler manifold \(X\) where \(\omega_0\) is any Kähler metric on \(X\). The short time existence of the solution is known from either R. Hamilton’s general existence result on Ricci flow in [3] or the fact that Kähler-Ricci flow is indeed parabolic.

By the optimal existence result on Kähler-Ricci flow as in [1] or [8], we know the classic solution of (1.1) exists exactly as long as the cohomology class \([\tilde{\omega}_t]\) from formal computation remains to be Kähler. The actually meaning will be explained later.

It then comes down to the study of the behavior of the metric solution when approaching the flow singularity. In this work, we focus on the study of the case when the flow singularity happens at some finite time. Let’s state the main theorems below.

**Theorem 1.1.** Kähler-Ricci flow (1.1) either exists for all time, or the scalar curvature blows up (from above) at some finite time, i.e.

\[
\sup_{X \times [0,T]} |R(\tilde{\omega}_t)| = +\infty
\]

where \(T\) is the finite singular time.
The blow-up would be from above in sight of the classic result on the lower bound of scalar curvature. Let’s point out that the statement of this theorem also holds for other usual versions Kähler-Ricci flows by simple rescaling consideration.

**Remark 1.2.** There have been some fundamental results regarding the finite time blowing-up of Ricci flow. In fact, it’s known that curvature operator blows up by R. Hamilton’s work [4] and Ricci curvature blows up by N. Sesum’s work [5].

In the case of finite time singularity, suppose we have a holomorphic map
\[ F : X \rightarrow Y \]
where \( Y \) is an analytic variety smooth near the image \( F(X) \) and there is a Kähler metric, \( \omega_M \), in a neighborhood of \( F(X) \) such that \( [\tilde{\omega}_t] = [F^*\omega_M] \). Then we have the following control of the blowing-up of scalar curvature.

**Theorem 1.3.** In the above setting, for the flow \( \{\tilde{\omega}_t\} \),
\[ R(\tilde{\omega}_t) \leq \frac{C}{(T-t)^2} \]
where \( C \) is a positive constant.

The motivation of the setting of this theorem is the semi-ampleness of the cohomology limit at singular time. It is of quite some interest in algebraic geometry as explained in [8], for example.

**Acknowledgment 1.4.** The author would like to thank R. Lazarsfeld for pointing out the result by J. Demailly and M. Paun in [2] which is absolutely crucial to conclude Theorem 1.1 for general closed (Kähler) manifolds. The discussion with J. Song is also valuable for this result. The comparison with the result by G. Perelman as mentioned in the last section is suggested by J. Lott. The author cannot thank his advisor, G. Tian, enough for introducing him to this interesting topic and constant encouragement along the way.

\section{2 Proof of Theorem 1.1}

The proof is by contradiction. We assume the scalar curvature is uniformly bounded along the flow with finite time singularity. One then makes use of J. Song and G. Tian’s parabolic Schwarz Lemma (as in [7]) and some basic computations on this Kähler-Ricci flow to get some uniform control of the flow metric. The contradiction then comes from the general result on the existence of Kähler-Ricci flow and the numerical characterization of Kähler cone on closed Kähler manifolds by J. Demailly and M. Paun. The rest of this section contains the detailed argument.
As usual when dealing with Kähler-Ricci flow, we need the scalar version of (1.1) described below as in [8]. Let
\[ ω_t = ω_∞ + e^{-t}(ω_0 - ω_∞) \]
where \[ ω_∞ = K_X = −\text{Ric}(Ω) \] for some smooth volume form \( Ω \) over \( X \). Then set \( \tilde{ω}_t = ω_t + \sqrt{-1}\partial\bar{∂}u \) and one has the following parabolic evolution equation for \( u \),
\[ \frac{∂u}{∂t} = \log \left( \frac{(ω_t + \sqrt{-1}\partial\bar{∂}u)^n}{Ω} \right) - u, \quad u(\cdot, 0) = 0. \] (2.1)

Now we state the following optimal existence result of Kähler-Ricci flow (as in [1] and [8]) mentioned in Introduction.

**Proposition 2.1.** (1.1) (or (2.1) equivalently) exists as long as \( [ω_t] \) remains Kähler, i.e. the solution is for the time interval \([0, T]\) where \( T = \sup\{t | [ω_t] \text{ is Kähler}\} \).

The finite time singularity means \( [ω_T] \) is on the boundary of the (open) Kähler cone, and thus no longer Kähler. Clearly it’s ”numerically effective” using the natural generalization of the notion from algebraic geometry.

From now on, we consider the flow existing only for some finite interval \([0, T]\). As usual, the \( C’s \) below might stand for different positive constants. In case that the situation is more subtle, lower indices are used to tell them apart. The argument is organized into three steps.

- **Volume Form (Lower) Bound**

With the bounded scalar curvature assumption, we can easily derive the uniform control on the volume form along the flow, using the following evolution of volume form,
\[ \frac{∂ω^n_t}{∂t} = n \frac{∂ω_t}{∂t} ∧ ω^{n-1}_t \]
\[ = n(−\text{Ric}(ω_t) − ω_t) ∧ ω^{n-1}_t \]
\[ = (−R − n)ω^n_t. \]

This gives \( |\frac{∂u}{∂t} + u| ≤ C \) as \( ω^n_t = e^{\frac{∂u}{∂t} + u}Ω \).

**Remark 2.2.** Instead of the assumption on scalar curvature, one can also directly assume positive lower bound for the volume form or equivalently, \( \frac{∂u}{∂t} ≥ −C \) since we are considering the finite time singularity case. This simple observation actually brings up a very intuitive understanding of Theorem 1.1: the flow (2.1) is stopped at some finite time because the term in log is tending to 0, i.e. no uniform lower bound.

- **Metric Estimate**

We begin with the inequality from parabolic Schwarz Lemma. In this note, the Laplacian \( Δ \) without lower index, is always with respect to the changing metric along the flow, \( ω_t \).
Let \( \phi = \langle \tilde{\omega}_t, \omega_0 \rangle > 0 \). Using computation for (1.1) in [7], one has
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \log \phi \leq C_1 \phi + 1, \tag{2.2}
\]
where \( C_1 \) is a positive constant depending on the bisectional curvature of \( \omega_0 \).

It’s quite irrelevant that \( \omega_0 \) is the initial metric for the Kähler-Ricci flow. In fact, it doesn’t have to be a metric over \( X \) which is an interesting part of this lemma as indicated in [7]. This is useful for the proof of Theorem 1.3.

Applying Maximum Principle to (2.1) gives \( u \leq C \). Take \( t \)-derivative to get
\[
\frac{\partial}{\partial t} \left( e^t \frac{\partial u}{\partial t} \right) = \Delta \left( e^t \frac{\partial u}{\partial t} \right) - e^{-t} \langle \tilde{\omega}_t, \omega_0 - \omega_\infty \rangle - \frac{\partial u}{\partial t}.
\]

It can be reformulated into the following two equations,
\[
\frac{\partial}{\partial t} \left( e^t \frac{\partial u}{\partial t} \right) = \Delta \left( e^t \frac{\partial u}{\partial t} \right) - \langle \tilde{\omega}_t, \omega_0 - \omega_\infty \rangle,
\]
\[
\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} + u \right) = \Delta \left( \frac{\partial u}{\partial t} + u \right) - n + \langle \tilde{\omega}_t, \omega_\infty \rangle.
\]

Their difference gives
\[
\frac{\partial}{\partial t} \left( (e^t - 1) \frac{\partial u}{\partial t} - u \right) = \Delta \left( (e^t - 1) \frac{\partial u}{\partial t} - u \right) + n - \langle \tilde{\omega}_t, \omega_0 \rangle. \tag{2.3}
\]

By Maximum Principle, this gives
\[
(e^t - 1) \frac{\partial u}{\partial t} - u - nt \leq C,
\]
which together with the upper bound of \( u \) and local bound for \( \frac{\partial u}{\partial t} \) near \( t = 0 \) would provide
\[
\frac{\partial u}{\partial t} \leq C.
\]

The upper bounds on \( \frac{\partial u}{\partial t} \) and \( u \) together with \( |\frac{\partial u}{\partial t} + u| \leq C \) from volume control give the uniform (lower) bounds on \( \frac{\partial u}{\partial t} \) and \( u \).

Multiply (2.3) by a large enough constant \( C_2 > C_1 + 1 \) and combining it with (2.2), one arrives at
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left( \log \phi + (e^t - 1) \frac{\partial u}{\partial t} - u \right) \leq nC_2 + 1 - (C_2 - C_1) \phi \leq C_3 - \phi. \tag{2.4}
\]

Apply Maximum Principle for \( \log \phi + (e^t - 1) \frac{\partial u}{\partial t} - u \). Considering the place where it achieves maximum value, one has
\[
\phi \leq C,
\]
and so
\[ \log \phi + (e^t - 1) \frac{\partial u}{\partial t} - u \leq C. \]

Hence we conclude \( \langle \tilde{\omega}_t, \omega_0 \rangle \leq C \) using the bound on \( \frac{\partial u}{\partial t} \) and \( u \). This trace bound, together with \( \tilde{\omega}_n^t \leq C \omega_0^n \), provide the uniform bound of \( \tilde{\omega}_t \) as metric, i.e. \( C^{-1} \omega_0 \leq \tilde{\omega}_t \leq C \omega_0 \).

- **Contradiction**

The metric (lower) bound makes sure that for any fixed analytic variety in \( X \), the integral of the proper power of \( \tilde{\omega}_t \) is bounded away from 0, and so the limiting class \( [\omega_T] \) would have positive intersection with any analytic variety by taking the cohomology limit. Then by Theorem 4.1 in [2], we conclude that \( [\omega_T] \) is actually Kähler which contradicts with the assumption of finite time singularity at \( T \) in sight of Proposition 2.1.

Hence we have finished the proof of Theorem 1.1.

**Remark 2.3.** In sight of this numerical characterization of Kähler cone for any general closed Kähler manifold by J. Demailly and M. Paun, the blowing-up of curvature operator or Ricci curvature in closed Kähler case is fairly obvious. The situation of scalar curvature is the first non-trivial statement.

# 3 Proof of Theorem 1.3

Now we derive certain control of the blowing-up of scalar curvature by mainly following the argument in [10]. The argument is also organized in three steps.

- **0-th Order Estimates**

  \( u \leq C \) is directly from (2.1). \( t \)-derivative of (2.1) is

  \[ \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} \right) = \Delta \left( \frac{\partial u}{\partial t} \right) - e^{-t} \langle \tilde{\omega}_t, \omega_0 - \omega_\infty \rangle - \frac{\partial u}{\partial t}, \]

  which has the following variations,

  \[ \frac{\partial}{\partial t} (e^t \frac{\partial u}{\partial t}) = \Delta (e^t \frac{\partial u}{\partial t}) - \langle \tilde{\omega}_t, \omega_0 - \omega_\infty \rangle, \]

  \[ \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} + u \right) = \Delta \left( \frac{\partial u}{\partial t} + u \right) - \langle \tilde{\omega}_t, \omega_\infty \rangle. \]

  A proper linear combination of these equations provides the following "finite time version" of the second equation,

  \[ \frac{\partial}{\partial t} \left( (1 - e^{t-T}) \frac{\partial u}{\partial t} + u \right) = \Delta \left( (1 - e^{t-T}) \frac{\partial u}{\partial t} + u \right) - \langle \tilde{\omega}_t, \omega_T \rangle. \]
The difference of the original two equations gives
\[ \partial_t \left( (1 - e^t) \frac{\partial u}{\partial t} + u \right) = \Delta \left( (1 - e^t) \frac{\partial u}{\partial t} + u \right) - n + \langle \tilde{\omega}_t, \omega_0 \rangle, \]
which implies the "essential decreasing" of metric potential along the flow, i.e.
\[ \frac{\partial u}{\partial t} \leq nt + C e^t - 1. \]

Notice that this estimate only depends on the initial value of \( u \) and its upper bound along the flow. It is uniform away from the initial time.

Another \( t \)-derivative gives
\[ \frac{\partial}{\partial t} \left( \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \right) = \Delta \left( \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \right) - \frac{\partial^2 u}{\partial t^2} - \frac{\partial \tilde{\omega}_t}{\partial t}^2 \]
Take summation with the first \( t \)-derivative to arrive at
\[ \frac{\partial}{\partial t} \left( \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \right) = \Delta \left( \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \right) - \frac{\partial^2 u}{\partial t^2} - \frac{\partial \tilde{\omega}_t}{\partial t}^2, \]
which gives
\[ \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \leq Ce^{-t}. \]
This implies the "essential decreasing" of volume form along the flow, i.e.
\[ \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} + u \right) \leq Ce^{-t}, \]
which also induces
\[ \frac{\partial u}{\partial t} \leq Ce^{-t}. \]

Let’s rewrite the metric flow equation as follows,
\[ \text{Ric}(\tilde{\omega}_t) = -\sqrt{-\text{det}(\tilde{\omega}_t + \frac{\partial u}{\partial t}))} - \omega_\infty. \]

Taking trace with respect to \( \tilde{\omega}_t \) for the original metric flow equation and the one above, we have
\[ R = e^{-t} \langle \tilde{\omega}_t, \omega_0 - \omega_\infty \rangle - \Delta \left( \frac{\partial u}{\partial t} \right) - n = -\Delta \left( u + \frac{\partial u}{\partial t} \right) - \langle \tilde{\omega}_t, \omega_\infty \rangle, \]
where \( R \) denotes the scalar curvature of \( \tilde{\omega}_t \). Using the equations above, we also have
\[ R = -n - \partial \left( \frac{\partial u}{\partial t} + u \right), \]
and so the estimate got for \( \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} + u \right) \) before is equivalent to the well known fact for scalar curvature.
We only consider smooth solution of Kähler-Ricci flow in \([0, T) \times X\) with finite time singularity at \(T\). At this moment, we only need that the smooth limiting background form \(\omega_T \geq 0\). It is essentially equivalent to assume \([\omega_T]\) has a smooth non-negative representative and presumably weaker than the class being "semi-ample", i.e. the existence of map \(F\) before Theorem 1.3.

Recall the following equation used before
\[
\frac{\partial}{\partial t} \left( (1 - e^{t-T}) \frac{\partial u}{\partial t} + u \right) = \Delta \left( (1 - e^{t-T}) \frac{\partial u}{\partial t} + u \right) - n + \langle \tilde{\omega}_t, \omega_T \rangle
\]
with the "\(T\)" in the equation chosen to the "\(T\)" above. With \(\omega_T \geq 0\), by Maximum Principle, one has
\[
(1 - e^{t-T}) \frac{\partial u}{\partial t} + u \geq -C.
\]
Together with the upper bounds, we conclude
\[
|(1 - e^{t-T}) \frac{\partial u}{\partial t} + u| \leq C.
\]

• Parabolic Schwarz Estimate

Use the following setup as in [7] for the map \(F\) before the statement of Theorem 1.3. Let \(\varphi = \langle \tilde{\omega}_1, F^* \omega_M \rangle\), then one has, over \([0, T) \times X\),
\[
(\frac{\partial}{\partial t} - \Delta) \varphi \leq \varphi + C \varphi^2 - H,
\]
where \(C\) is related to the bisectional curvature bound of \(\omega_M\) near \(F(X)\) and \(H \geq 0\) is described as follows. Using normal coordinates locally over \(X\) and \(Y\), with indices \(i, j\) and \(\alpha, \beta\), \(\varphi = |F_i|^2\) and \(H = |F_{ij}|^2\) with all the summations. Notice that the normal coordinate over \(X\) is changing along the flow with the metric. Using this inequality, one has
\[
(\frac{\partial}{\partial t} - \Delta) \log \varphi \leq C \varphi + 1.
\]

**Remark 3.1.** For application, the map \(F\) is coming from the class \([\omega_T]\) with \(Y\) being some projective space \(\mathbb{CP}^N\), and so \(\omega_T\) is \(F^* \omega\) where \(\omega\) is (some multiple of) Fubini-Study metric over \(Y\).

Define
\[
v := (1 - e^{t-T}) \frac{\partial u}{\partial t} + u
\]
and we know \(|v| \leq C\) for the previous step. We also have
\[
(\frac{\partial}{\partial t} - \Delta)v = -n + \langle \tilde{\omega}_t, \omega_T \rangle = -n + \varphi.
\]
After taking a large enough positive constant $A$, the following inequality is true,

\[
\left(\frac{\partial}{\partial t} - \Delta\right)(\log \varphi - Av) \leq -C\varphi + C.
\]

Since $v$ is bounded, Maximum Principle can be used to deduce $\varphi \leq C$, i.e. \( \langle \bar{\omega}_t, \omega_T \rangle \leq C \).

- **Gradient and Laplacian Estimates**

  In this part, we derive gradient and Laplacian estimates for $v$. Recall that

  \[
  (\partial_t - \Delta)v = -n + \varphi, \quad \varphi = \langle \bar{\omega}_t, \omega_T \rangle.
  \]

  Standard computation gives:

  \[
  \left(\frac{\partial}{\partial t} - \Delta\right)(|\nabla v|^2) = |\nabla v|^2 - |\nabla \nabla v|^2 - |\nabla \bar{\nabla} v|^2 + 2\text{Re}(\nabla \varphi, \nabla v),
  \]

  \[
  \left(\frac{\partial}{\partial t} - \Delta\right)(\Delta v) = \Delta v + (\text{Ric}(\bar{\omega}_t), \sqrt{-1} \partial \bar{\partial} v) + \Delta \varphi.
  \]

  Again, all the $\nabla$, $\Delta$ and $\langle \cdot, \cdot \rangle$ are with respect to $\bar{\omega}_t$ and $\nabla \nabla v$ is just $\partial \bar{\partial} v$.

  Consider $\Psi := \frac{|\nabla v|^2}{C - v}$. Since $v$ is bounded, one can easily make sure the denominator is positive, bounded and also away from 0. We have the following computation,

  \[
  \left(\frac{\partial}{\partial t} - \Delta\right)\Psi = \left(\frac{\partial}{\partial t} - \Delta\right)\frac{|\nabla v|^2}{C - v}
  \]

  \[
  = \frac{1}{C - v} \cdot \frac{\partial}{\partial t} (|\nabla v|^2) + \frac{|\nabla v|^2}{(C - v)^2} \cdot \frac{\partial v}{\partial t} - \frac{\left(\frac{|\nabla v|^2}{C - v}\right)_i}{(C - v)^2} \frac{\partial v}{\partial t} - \frac{v_i \frac{|\nabla v|^2}{C - v}}{(C - v)^2} - \frac{2\text{Re}(\nabla v, \nabla |\nabla v|^2)}{(C - v)^2} - \frac{2|\nabla v|^4}{(C - v)^3}.
  \]

  Plug in the results from before and rewrite the differential equality for $\Psi$ below,

  \[
  \left(\frac{\partial}{\partial t} - \Delta\right)\Psi = \frac{(-n + \varphi)|\nabla v|^2}{(C - v)^2} + \frac{|\nabla v|^2 - |\nabla \nabla v|^2 - |\nabla \bar{\nabla} v|^2}{C - v} + \frac{2\text{Re}(\nabla \varphi, \nabla v)}{C - v} - \frac{2\text{Re}(\nabla v, \nabla |\nabla v|^2)}{(C - v)^2} - \frac{2|\nabla v|^4}{(C - v)^3},
  \]

  (3.1)
The computations below are useful next.

\[
|\langle \nabla v, \nabla |\nabla v|^2 \rangle| = |v_i (v_j v_{ji})| \\
= |v_i v_j v_{ji} + v_i v_j v_{ji}| \\
\leq |\nabla v|^2 (|\nabla v| + |\nabla v|) \\
\leq \sqrt{2} |\nabla v|^2 (|\nabla v|^2 + |\nabla v|^2)^{\frac{1}{2}}.
\]

\[
\nabla \Psi = \nabla \left( \frac{|\nabla v|^2}{C-v} \right) = \frac{\nabla(|\nabla v|^2)}{C-v} + \frac{|\nabla v|^2 \nabla v}{(C-v)^2}.
\]

Together with the bounds for \(\varphi\) and \(C - v\), we can have the following computation with \(\epsilon\) representing small positive constant (different from place to place),

\[
\frac{\partial}{\partial t} - \Delta \Psi \\
\leq C|\nabla v|^2 + \epsilon \cdot |\nabla \varphi|^2 - C(|\nabla \nabla v|^2 + |\nabla \nabla v|^2) + \\
- (2 - \epsilon) \text{Re} \left( \nabla \Psi, \frac{\nabla v}{C-v} \right) - \epsilon \cdot \frac{\text{Re}(\nabla v, |\nabla v|^2)}{(C-v)^2} - \epsilon \cdot \frac{|\nabla v|^4}{(C-v)^3} \\
\leq C|\nabla v|^2 + \epsilon \cdot |\nabla \varphi|^2 - C(|\nabla \nabla v|^2 + |\nabla \nabla v|^2) + \\
- (2 - \epsilon) \text{Re} \left( \nabla \Psi, \frac{\nabla v}{C-v} \right) + \epsilon \cdot (|\nabla \nabla v|^2 + |\nabla \nabla v|^2) - \epsilon \cdot |\nabla v|^4 \\
\leq C|\nabla v|^2 + \epsilon \cdot |\nabla \varphi|^2 - (2 - \epsilon) \text{Re} \left( \nabla \Psi, \frac{\nabla v}{C-v} \right) - \epsilon \cdot |\nabla v|^4.
\]

We need a few more calculations to set up Maximum Principle argument. Recall that \(\varphi = \langle \tilde{\omega}, \omega_T \rangle\) and,

\[
\frac{\partial}{\partial t} - \Delta \varphi \leq \varphi + C \varphi^2 - H.
\]

With the description of \(H\) before and the estimate for \(\varphi\), i.e. \(\varphi \leq C\) from Schwarz estimate, we can conclude that

\[H \geq C|\nabla \varphi|^2.\]

Now one arrives at

\[
\frac{\partial}{\partial t} - \Delta \varphi \leq C - C|\nabla \varphi|^2. \quad (3.2)
\]

We also have

\[
|\left( \nabla \varphi, \frac{\nabla v}{C-v} \right)| \leq \epsilon \cdot |\nabla \varphi|^2 + C \cdot |\nabla v|^2. \quad (3.3)
\]

Now consider the function \(\Psi + \varphi\). By choosing \(\epsilon > 0\) small enough above (which also affects the choices of \(C\)'s), we have

\[
\frac{\partial}{\partial t} - \Delta (\Psi + \varphi) \leq C + C|\nabla v|^2 - \epsilon \cdot |\nabla v|^4 - (2 - \epsilon) \text{Re} \left( \nabla (\Psi + \varphi), \frac{\nabla v}{C-v} \right).
\]
At the maximum value point of $\Psi + \varphi$, we know $|\nabla v|^2$ can not be too large. It’s then easy to conclude the upper bound for this term, and so for $\Psi$. Hence we have bounded the gradient, i.e. 

$$|\nabla v| \leq C.$$ 

Now we want to do similar thing for the Laplacian, $\Delta v$. Define the function $\Phi := \frac{C - \Delta v}{C - v}$. Similar computation as before gives the following 

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Phi = \left(\frac{\partial}{\partial t} - \Delta\right)(\frac{C - \Delta v}{C - v})$$ 

$$= -\frac{1}{C - v} \cdot \left(\frac{\partial}{\partial t} - \Delta\right)\Delta v + \frac{C - \Delta v}{(C - v)^2} \cdot \left(\frac{\partial}{\partial t} - \Delta\right)v + \frac{2\text{Re}(\nabla v, \nabla \Delta v)}{(C - v)^2} +$$ 

$$- \frac{2|\nabla v|^2(C - \Delta v)}{(C - v)^3}$$ 

$$= -\frac{1}{C - v} \cdot (\Delta v + (\text{Ric}(\bar{\omega}_t), \sqrt{-1}\partial\bar{\partial}v) + \Delta \varphi) + \frac{C - \Delta v}{C - v} \cdot (-n + \varphi)$$ 

$$+ \frac{2\text{Re}(\nabla v, \nabla \Delta v)}{(C - v)^2} - \frac{2|\nabla v|^2(C - \Delta v)}{(C - v)^3}. \quad (3.4)$$ 

We also have $\nabla (\frac{C - \Delta v}{C - v}) = \frac{(C - \Delta v)(\nabla v) - \nabla \Delta v}{C - v}$. Recall that it is already known $(0 \leq \varphi \leq C$. The following inequality follows from standard computation as in [7] and has actually been used for parabolic Schwarz estimate, 

$$\Delta \varphi \geq (\text{Ric}(\bar{\omega}_t), \omega_T) + H - C\varphi^2,$$

where $H \geq C|\nabla \varphi|^2 \geq 0$ from the bound of $\varphi$ as mentioned before. Now we have 

$$\text{Ric}(\bar{\omega}_t), \sqrt{-1}\partial\bar{\partial}v) + \Delta \varphi \geq (\text{Ric}(\bar{\omega}_t), \sqrt{-1}\partial\bar{\partial}v + \omega_T) - C. \quad (3.5)$$ 

We are considering the case $T < \infty$. Recall that $v = (1 - e^{t-T})\frac{\partial u}{\partial t} + u$ and $\omega_T = \omega_\infty + e^{-T}(\omega_0 - \omega_\infty)$. We have 

$$\text{Ric}(\bar{\omega}_t) = -\sqrt{-1}\partial\bar{\partial} \left( \frac{\partial u}{\partial t} + u \right) - \omega_\infty$$ 

$$= -\sqrt{-1}\partial\bar{\partial}v - \omega_T - e^{t-T} \sqrt{-1}\partial\bar{\partial} \left( \frac{\partial u}{\partial t} + u \right) - e^{-T}(\omega_0 - \omega_\infty)$$ 

$$= -\sqrt{-1}\partial\bar{\partial}v - \omega_T - e^{t-T} \left( \sqrt{-1}\partial\bar{\partial} \frac{\partial u}{\partial t} - e^{-t}(\omega_0 - \omega_\infty) \right)$$ 

$$= -\sqrt{-1}\partial\bar{\partial}v - \omega_T - e^{t-T} \frac{\partial \tilde{w}_t}{\partial t}$$ 

$$= -\sqrt{-1}\partial\bar{\partial}v - \omega_T - e^{t-T} (-\text{Ric}(\bar{\omega}_t) - \tilde{w}_t),$$ 

which gives 

$$(1 - e^{t-T})\text{Ric}(\bar{\omega}_t) = -\sqrt{-1}\partial\bar{\partial}v - \omega_T + e^{t-T}\tilde{w}_t,$$ 

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and so
\[
\text{Ric}(\bar{\omega}_t) = -\frac{\sqrt{1}\partial\bar{\omega}}{1 - e^{-T}} + \frac{e^{t-T}}{1 - e^{-T}}\bar{\omega}_t,
\]
\[
(1 - e^{-T})R = -\Delta v + \langle \bar{\omega}_t, \omega_T \rangle + ne^{t-T}.
\]

As \( R \geq -C \) and \( 0 \leq \langle \bar{\omega}_t, \omega_T \rangle \), we have \( \Delta v \leq C \).

Now we can continue the estimation (3.5) as follows.
\[
(\text{Ric}(\bar{\omega}_t), \sqrt{1}\partial\bar{\omega}) + \Delta \varphi \geq \text{Ric}(\bar{\omega}_t), \sqrt{1}\partial\bar{\omega}_t + \omega_T) - C
\]
\[
\geq \left( -\frac{\sqrt{1}\partial\bar{\omega}_t + \omega_T}{1 - e^{-T}} + \frac{e^{t-T}}{1 - e^{-T}}\bar{\omega}_t, \sqrt{1}\partial\bar{\omega}_t + \omega_T \right) - C
\]
\[
= \frac{|\sqrt{1}\partial\bar{\omega}_t + \omega_T|^2}{1 - e^{-T}} + \frac{e^{t-T}(\Delta v + \langle \bar{\omega}_t, \omega_T \rangle)}{1 - e^{-T}} - C.
\]

As \( C^{-1}(T - t) \leq 1 - e^{-T} \leq C(T - t) \) for \( t \in [0, T) \), using \( \Delta v = \Delta v - C + C \) and \( 0 \leq \langle \bar{\omega}_t, \omega_T \rangle \leq C \), we have
\[
(\text{Ric}(\bar{\omega}_t), \sqrt{1}\partial\bar{\omega}) + \Delta \varphi
\]
\[
= -\frac{|\sqrt{1}\partial\bar{\omega}_t + \omega_T|^2}{1 - e^{-T}} + \frac{e^{t-T}(\Delta v + \langle \bar{\omega}_t, \omega_T \rangle)}{1 - e^{-T}} - C
\]
\[
\geq -\frac{1}{T - t} \left( (1 + \epsilon)|\sqrt{1}\partial\bar{\omega}_t|^2 + C|\omega_T|^2 \right) - \frac{C}{T - t} \left( C - \Delta v \right) - \frac{C}{T - t}
\]
\[
\geq -\frac{1 + \epsilon}{T - t} |\sqrt{1}\partial\bar{\omega}_t|^2 - \frac{C}{T - t} (C - \Delta v) - \frac{C}{T - t}.
\]

Now we can continue the computation for \( \Phi \), (3.4) as follows.
\[
\frac{\partial}{\partial t} - \Delta \Phi \leq \frac{C}{T - t} + \frac{C}{T - t} \cdot (C - \Delta v) + \frac{(1 + \epsilon)|\nabla v|^2}{(T - t)(C - v)} - 2\text{Re} \left( \nabla \Phi, \frac{\nabla v}{C - v} \right).
\]

Using \( \Phi = \frac{C - C}{C - v} \geq C(C - \Delta v) \), one arrives at
\[
\frac{\partial}{\partial t} - \Delta ((T - t)\Phi) \leq C + C(C - \Delta v) + \frac{(1 + \epsilon)|\nabla v|^2}{C - v} - 2\text{Re} \left( \nabla ((T - t)\Phi), \frac{\nabla v}{C - v} \right)
\]
In sight of (3.2) and (3.3), we have,
\[
\frac{\partial}{\partial t} - \Delta \varphi \leq C - 4\text{Re} \left( \nabla \varphi, \frac{\nabla v}{C - v} \right) + C|\nabla v|^2.
\]

Also, (3.1) can be rewritten as
\[
\frac{\partial}{\partial t} - \Delta \Psi \leq \frac{(-n + \varphi)|\nabla v|^2}{(C - v)^2} + \frac{|\nabla v|^2 - |\nabla v|^2 - |\nabla v|^2}{C - v} + 2\text{Re} \left( \nabla \varphi, \frac{\nabla v}{C - v} \right) - 2\text{Re} \left( \Psi, \frac{\nabla v}{C - v} \right).
\]
Using the bound for $|\nabla v|$ and choosing $\epsilon < 1$, we have

$$
\frac{\partial}{\partial t} - \Delta \left((T-t)\Phi + 2\Psi + 2\varphi\right)
\leq C + C \cdot (C - \Delta v) - 2\text{Re} \left(\nabla ((T-t)\Phi + 2\Psi + 2\varphi), \frac{\nabla v}{C-v}\right) - C|\nabla \overline{\nabla v}|^2
\leq C + C \cdot (C - \Delta v) - 2\text{Re} \left(\nabla ((T-t)\Phi + 2\Psi + 2\varphi), \frac{\nabla v}{C-v}\right) - C(C - \Delta v)^2
$$

where $|\nabla \overline{\nabla v}|^2 \geq C(\Delta v)^2 \geq C(C - \Delta v)^2 - C$ is used for the second $\leq$.

Now we apply Maximum Principle. At maximum value point of the function $(T-t)\Phi + 2\Psi + 2\varphi$, we have $C - \Delta v \leq C$. Using the bounds on $\Psi$ and $\varphi$, $(T-t)\Phi + 2\Psi + 2\varphi \leq C$ and so

$$
\Phi \leq \frac{C}{T-t}, \text{ i.e. } \Delta v \geq -\frac{C}{T-t}.
$$

Finally since $(1 - e^{t-T})R = -\Delta v - \langle \bar{\omega}_t, \omega_T \rangle + ne^{t-T}$, we conclude that

$$
R \leq \frac{C}{(T-t)^2}.
$$

Theorem 1.3 is proved.

4 Further Remarks

There are several closely related results worth mentioning. The last two remarks below should give people the idea about the essential difference between finite time and infinite time singular case for Kähler-Ricci flow.

- In [6], following Perelman’s idea, Sesum and Tian proved that for $X$ with $c_1(X) > 0$, for any initial Kähler metric $\omega$ such that $[\omega] = c_1(X)$, the Kähler-Ricci flow

$$
\frac{\partial \bar{\omega}_t}{\partial t} = -\text{Ric} (\bar{\omega}_t) + \bar{\omega}_t
$$

has uniformly bounded scalar curvature and diameter for $\bar{\omega}_t$ where $t \in [0, \infty)$. Using simply rescaling of time and metric, one can see for our flow (1.1) with $[\omega_0] = c_1(X)$,

$$
R(\bar{\omega}_t) \leq \frac{C}{T-t}
$$

for $t \in [0, T)$ where the finite singular time $T = \log 2$, which is a better control than Theorem 1.3 for this special case.

- For the infinite time limiting case, the scalar curvature would be bounded if the infinite time limiting class provides a smooth holomorphic fibration for $X$, i.e. the map $F$ as in our setting is a smooth fibration. This is actually proved in [7] if one only considers smooth collapsing case.
• For the infinite time limiting case, the scalar curvature would also be bounded if the limiting class is ”semi-ample and big”, i.e. the (possibly singular) image of the map $F$ is of the same dimension as $X$. This result is proved in [10]. The more recent work of Yuguang Zhang, [9], has given a nice application of it.

References

[1] Cascini, Paolo; La Nave, Gabriele: Kähler-Ricci flow and the minimal model program for projective varieties. arXiv:math/0603064 (math.AG).

[2] Demailly, Jean-Pierre; Paun, Mihai: Numerical characterization of the Kähler cone of a compact Kähler manifold. Ann. of Math. 159 (2004), 1247–1274.

[3] Hamilton, Richard S.: Three-manifolds with positive Ricci curvature. J. Differential Geom. 17 (1982), no. 2, 255-306.

[4] Hamilton, Richard S.: The formation of singularities in the Ricci flow. Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.

[5] Sesum, Natasa: Curvature tensor under the Ricci flow. Amer. J. Math. 127 (2005), no. 6, 1315–1324.

[6] Sesum, Natasa; Tian, Gang: Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman). J. Inst. Math. Jussieu 7 (2008), no. 3, 575–587. 53C44.

[7] Song, Jian; Tian, Gang: The Kähler-Ricci flow on surfaces of positive Kodaira dimension. Invent. Math. 170 (2007), no. 3, 609–653.

[8] Tian, Gang; Zhang, Zhou: On the Kähler-Ricci flow on projective manifolds of general type. Chinese Annals of Mathematics - Series B, Volume 27, Number 2, 179–192.

[9] Zhang, Yuguang: Miyaoka-Yau inequality for minimal projective manifolds of general type. arXiv:0812.0462 (math.DG) (math.AG).

[10] Zhang, Zhou: Scalar Curvature Bound for Kähler-Ricci Flows over Minimal Manifolds of General Type. arXiv:0801.3248 (math.DG).

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