COUNTING RATIONAL POINTS CLOSE TO $p$-ADIC INTEGERS AND APPLICATIONS IN DIOPHANTINE APPROXIMATION

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Abstract. We find upper and lower bounds on the number of rational points that are $\psi$-approximations of some $p$-adic integer. Lattice point counting techniques are used to find the upper bound result, and a Pigeon-hole principle style argument is used to find the lower bound result. We use these results to find the Hausdorff dimension for the set of $p$-adic weighted simultaneously approximable points intersected with $p$-adic coordinate hyperplanes. For the lower bound result we show that the set of rational points that $\tau$-approximate a $p$-adic integer form a set of resonant points that can be used to construct a local ubiquitous system of rectangles.

1. Introduction

The study of rational points on algebraic varieties, usually called Diophantine geometry, has a wide variety of applications in many areas of mathematics. A variation of this is the study of rational points that lie close to such algebraic varieties. In the setting of $\mathbb{R}^n$ there has been many results of this type, including counts on the number of rational points close to curves [7, 37, 34, 35, 24] and manifolds [4, 13, 22, 23]. In the $p$-adic setting less is known. In [2, 3] a bound on the number of rational points that lie on the curve $C_f = \{(x, x^2, \ldots, x^n) : x \in \mathbb{Z}_p\}$ were found, but as yet no other results are available. In this paper we provide an upper and lower bound on the number of rational points within a small neighbourhood of a $p$-adic integer. Such result allows us to find bounds on the number of rational points close to $p$-adic coordinate hyperplanes.

Fix a prime number $p \in \mathbb{N}$ and let $|.|_p$ denote the $p$-adic norm. Define the set of $p$-adic numbers $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ with respect to the $p$-adic norm. Denote by $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ the ring of $p$-adic integers. Let $x \in \mathbb{Z}_p$, $N \in \mathbb{N}$, and $\psi : \mathbb{N} \to \mathbb{R}_+$, with $\psi(q) \to 0$ as $q \to \infty$. We provide bounds on the cardinality of the set

$$Q(x, \psi, N) := \left\{ (q, q_1) \in \mathbb{N} \times \mathbb{Z} : 0 < q \leq N, |q_1| \leq N, \left| x - \frac{q}{q_1} \right|_p < \psi(N) \right\}.$$

If the approximation function $\psi$ is of the form $\psi(q) = q^{-\tau}$ we will use the notation $Q(x, \tau, N)$. Note that to get a result for general $x \in \mathbb{Z}_p$ we must apply some conditions on $x \in \mathbb{Z}_p$. For example, if $x \in \mathbb{Q}$ then for sufficiently large $N \in \mathbb{N}$ we have that $\#Q(x, \psi, N) \asymp N^2$, where by

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a \asymp b we mean there exists constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that $c_1 b \leq a \leq c_2 b$. Conversely, if $x$ is badly approximable and $\psi(q) < q^{-2-\epsilon}$ for some $\epsilon > 0$ then $\#Q(x, \psi, N) \ll 1$. In order to obtain good bounds on the cardinality of $Q(x, \psi, N)$ we use the Diophantine exponent, $\tau(x)$, where

$$\tau(x) := \sup\{\tau > 0 : |qx - q_1|_p < Q^{-\tau}, \text{ for i.m. } Q \in \mathbb{N}, \text{ with } 0 < q, |q_1| \leq Q, \}.$$ 

By a Theorem of Mahler [36] we have that for all $x \in \mathbb{Z}_p$, $\tau(x) \geq 2$. Further, by a result of Jarnik [27] we have that $\tau(x) \leq 2$ for almost all $x \in \mathbb{Z}_p$, with respect to the Haar measure $\mu_p$ on $\mathbb{Q}_p$, normalised by $\mu_p(\mathbb{Z}_p) = 1$.

We have the following results on the cardinality of $Q(x, \psi, N)$.

**Lemma 1.1.** Let $x \in \mathbb{Z}_p$ with Diophantine exponent $\tau(x)$ and let $\psi(q) = q^{-\tau}$ for some $\tau \in \mathbb{R}_+$ with $\max\{1, \tau(x) - 1\} < \tau < \tau(x)$. Then for any $\epsilon > 0$ there exists sufficiently large $Q \in \mathbb{N}$ such that

$$\#Q(x, \tau, Q) \leq N^{\tau(x) - \tau + \epsilon}.$$ 

Note by our previous remark on the Diophantine exponent that for almost all $x \in \mathbb{Z}_p$ if $\psi(q) = q^{-\tau}$ with $1 \leq \tau < 2$, then

$$\#Q(x, \psi, Q) \leq Q^{2-\tau+\epsilon}.$$ 

While Lemma 1.1 gives us an upper bound for all $x \in \mathbb{Z}_p$, provided the approximation function $\psi$ is 'close' to the function related to the Diophantine exponent, the bound given has an extra $Q^\epsilon$ term, which we believe is unnecessary. The following theorem offers an improvement in this respect.

**Theorem 1.2.** Let $x \in \mathbb{Z}_p$ and suppose that $\tau(x) = 2$. Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be an approximation function with $q^{-2} \leq \psi(q) < q^{-1}$. Then for sufficiently large $M \in \mathbb{N}$,

$$\#Q(x, \psi, M) \leq 6M^2\psi(M).$$

Again, as with Lemma 1.1 we can deduce that the above upper bound is true for almost all $x \in \mathbb{Z}_p$. This type of result has already been proven in the real case (see Lemma 6.1 of [8]). Lastly, we have the following lemma which provides a complimentary lower bound to the previous two results.

**Lemma 1.3.** Let $x \in \mathbb{Z}_p$ and $1 < \tau < 2$. Then for sufficiently large $Q \in \mathbb{N}$ we have that

$$\#Q(x, \tau, Q) \geq \frac{1}{p}Q^{2-\tau} - 1.$$
As with Theorem 1.2, the Euclidean version of this result has previously been proven, (see Lemma 3 of [33]). Further, as \( \tau < 2 \) we can choose \( Q \) large enough such that

\[
\#Q(x, \tau, Q) \geq \frac{1}{2p}Q^{2-\tau}.
\]

Thus combining this with Theorem 1.2 we have the expected result that \( Q(x, \tau, N) \asymp N^{2-\tau} \).

The proofs of Lemma 1.1 and Lemma 1.3 use elementary techniques. The proof of Theorem 1.2 is more substantial and uses \( p \)-adic approximation lattices and lattice counting techniques. Prior to the proofs of these results we give an example of their applications in Diophantine approximation.

2. \( p \)-adic Diophantine approximation

As an application of the main results in the previous section we consider the set of \( p \)-adic simultaneously approximable points over coordinate hyperplanes. Define the set of weighted simultaneously approximable points as follows. For an \( n \)-tuple of approximation functions \( \Psi = (\psi_1, \ldots, \psi_n) \), with \( \psi_i : \mathbb{N} \to \mathbb{R}^+ \) for \( 1 \leq i \leq n \), and \( q \in \mathbb{N} \) let

\[
\mathfrak{A}_q(\Psi) = \bigcup_{|q_i| \leq q} \{ x \in \mathbb{Z}_p^n : |qx_i - q_i|_p < \psi_i(q) \},
\]

where \( x = (x_1, \ldots, x_n) \). Define the set of weighted \( \Psi \)-approximable \( p \)-adic points as

\[
\mathfrak{W}_n(\Psi) := \limsup_{q \to \infty} \mathfrak{A}_q(\Psi).
\]

If the approximation functions have the form \( \psi(q) = q^{-\tau_i} \) for some exponents of approximation \( \tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_+^n \) we will use the notation \( \mathfrak{W}_n(\tau) = \mathfrak{W}_n(\Psi) \). By considering the Dirichlet style theorem for the set \( \mathfrak{W}_n(\tau) \) we have that \( \mathfrak{W}_n(\tau) = \mathbb{Z}_p^n \) provided that \( \sum_{i=1}^n \tau_i \leq n + 1 \). Let \( \mu_{p,n} \) denote the Haar measure of \( \mathbb{Q}_p^n \), normalized by \( \mu_{p,n}(\mathbb{Z}_p^n) = 1 \). Jarnik [27] showed that for \( \psi \) monotonic decreasing \( \mu_{p,n}(\mathfrak{W}_n(\psi)) \) has zero measure when

\[
\sum_{q=1}^{\infty} \psi(q)^n
\]

converges, and has full measure when \( \sum_{q=1}^{\infty} \psi(q)^n \) diverges. There are also developments of Jarnik’s Theorem to the weighted case [11] and linear forms [29].

For sets of zero Haar measure we use Hausdorff measure and Hausdorff dimension to provide more accurate notions of size. We briefly recap the definition and notation of Hausdorff measure and dimension. For a metric space \( (X, d) \), a set \( U \subset X \), and \( \rho > 0 \), define a \( \rho \)-cover of \( U \) as a sequence of balls \( \{B_i\} \) such that \( U \subset \bigcup_i B_i \) and for all balls \( r(B_i) \leq \rho \), where \( r(.) \) denotes the radius of
the ball. Define a dimension function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) as an increasing continuous function with \( f(r) \to 0 \) as \( r \to 0 \). Define the \( f \)-Hausdorff measure as

\[
\mathcal{H}^f(U) = \lim_{\rho \to 0^+} \inf \left\{ \sum_i f(r(B_i)) : \{B_i\} \text{ is a } \rho \text{-cover of } U \right\},
\]

where the infimum is taken over all \( \rho \)-covers of \( U \). When the dimension function \( f(x) = x^s \) we will use the notation \( \mathcal{H}^f = \mathcal{H}^s \). Define the Hausdorff dimension as

\[
\dim U = \inf \{ s \geq 0 : \mathcal{H}^s(U) = 0 \}.
\]

In \cite{11} it was proven that, for \( \sum_{i=1}^n \tau_i > n + 1 \),

\[
\dim \mathcal{W}_n(\tau) = \min_{1 \leq i \leq n} \left\{ \frac{n + 1 + \sum_{j=1}^n (\tau_i - \tau_j)}{\tau_i} \right\}.
\]

Correspondingly to results over \( \mathbb{Z}_p \) it would be desirable to obtain measure results for simultaneous \( p \)-adic approximable points over manifolds. In \cite{28} Kleinbock and Tomanov proved the extremality of \( p \)-adic manifolds provided some non-degeneracy conditions are satisfied. Generally a manifold \( \mathcal{M} \subset \mathbb{Z}_p^n \) is said to be extremal if almost all points, with respect to the induced Haar measure of the manifold, we have that \( \tau(x) = n + 1/n \) (see \cite{28} for more details). There are a variety of results for \( p \)-adic dual approximation, see for example \cite{31, 9, 15, 18, 30}, however results in the simultaneous case are lacking. Recently Oliveira \cite{31} produced a Khintchine-style Theorem for simultaneous \( p \)-adic approximation with denominators coming from \( p \)-adic balls. This result has a similar style to our result, with the difference being that our denominators come from a ball with radius tending to zero, rather than a fixed constant. Other than this result there are relatively few Khintchine-style results.

For the Hausdorff dimension there are recent results on simultaneously approximable points over \( \mathcal{C}_f = \{(x, x^2, \ldots, x^n : x \in \mathbb{Z}_p) \} \) for sufficiently large Diophantine exponents \cite{3, 16}. In the previous chapter a lower bound for the Hausdorff dimension was found for general \( n \)-dimensional normal curves. A key reason the upper bound could not be obtained was a lack in results on the behaviour of rational points close to \( p \)-adic curves. The results of the previous section give us a good understanding of the behaviour of rational points close to coordinate hyperplanes. The results of this chapter are closely related to a variety of results in the real case on Diophantine approximation over coordinate hyperplanes, see \cite{10, 32, 33}.

For a \( p \)-adic integer \( \alpha \in \mathbb{Z}_p \) and \( 1 \leq m \leq n \) define the coordinate hyperplane

\[
\Pi_{\alpha,m} := \{(x_1, \ldots, x_{m-1}, \alpha, x_{m+1}, \ldots, x_n) : (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n) \in \mathbb{Z}_p^{n-1} \} \subset \mathbb{Z}_p^n.
\]
For the set $W_n(\tau) \cap \Pi_{\alpha,m}$ we have the trivial result that

$$\dim W_n(\tau) \cap \Pi_{\alpha,m} \leq \dim \Pi_{\alpha,m} = n - 1,$$

with equality when $\sum_{i=1}^{n} \tau_i \leq n + 1$. In this paper we prove the following result on the Hausdorff dimension of $W_n(\Psi) \cap \Pi_{\alpha}$.

**Theorem 2.1.** Let $\Pi_{\alpha,m}$ be a coordinate hyperplane of $\mathbb{Z}_p^n$, let $\alpha \in \mathbb{Z}_p \setminus \mathbb{Q}$, and $1 \leq m \leq n$. Let $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_+^n$ be a weight vector with the properties that

$$1 < \tau_m < 2,$$

and

$$\sum_{i=1}^{n} \tau_i > n + 1.$$

Then for almost all $\alpha \in \mathbb{Z}_p \setminus \mathbb{Q}$ (with respect to the Haar measure),

$$\dim W_n(\tau) \cap \Pi_{\alpha,m} = \min_{1 \leq i \leq n \atop i \neq m} \left\{ \frac{n + 1 - \tau_m + \sum_{j \neq m} \tau_j \leq \tau_i (\tau_i - \tau_j)}{\tau_i} \right\}.$$

**Remark 2.2.** The constraints on $\tau_m$ ensure that we can apply Theorem 1.2. Note that we can use the same style of proof used to prove the upper bound of 2.1 in combination with Lemma 1.1 to prove that for $\max\{1, (\alpha) - 1\} < \tau_m < (\alpha)$ we have that

$$\dim W_n(\tau) \cap \Pi_{\alpha,m} \leq \min_{1 \leq i \leq n \atop i \neq m} \left\{ \frac{n + (\alpha) - 1 - \tau_m + \sum_{j \neq m} \tau_j \leq \tau_i (\tau_i - \tau_j)}{\tau_i} \right\}.$$

Proving the corresponding lower bound of this result is currently beyond our reach.

**Remark 2.3.** The lower bound over all $\tau_i$ ensures that we do not include the trivial case when $\sum_{i=1}^{n} \tau_i \leq n + 1$, in which case $W_n(\tau)$ has full dimension and so $\dim W_n(\tau) \cap \Pi_{\alpha,m} = n - 1$. In the special case where the approximation functions are the same i.e. $(\tau = (\tau, \ldots, \tau))$, then we have that, for $1 + \frac{1}{n} \leq \tau < 2$,

$$\dim W_n(\tau) \cap \Pi_{\alpha} = \frac{n + 1}{\tau} - 1.$$

Note that this gives us the expected dimension of the set of approximable points $W_n(\tau)$ less the codimension of the hyperplane $\Pi_{\alpha,m}$. 

5
For general approximation functions $\Psi = (\psi_1, \ldots, \psi_n)$, providing the limit

$$\psi^*_i = \lim_{q \to \infty} -\frac{\log(\psi(q))}{\log q}$$

exists and is positive finite for each $1 \leq i \leq n$, define $\Psi^* = (\psi^*_1, \ldots, \psi^*_n)$. Then we have the following corollary.

**Corollary 2.4.** Let $\Psi = (\psi_1, \ldots, \psi_n)$ with $\psi_i : \mathbb{N} \to \mathbb{R}_+$ for each $1 \leq i \leq n$ be an approximation vector with each $\psi_i$ having positive finite limit \([2]\). If $\Psi^*$ satisfy the same conditions as Theorem 2.1 then for almost all $\alpha \in \mathbb{Z}_p \setminus \mathbb{Q}$,

$$\dim W_n(\Psi) \cap \Pi_\alpha = \min_{1 \leq i \leq n} \left\{ \frac{n + 1 - \psi^*_m + \sum_{j=1, j \neq m}^n (\psi^*_i - \psi^*_j)}{\psi^*_i} \right\}.$$  

The corollary easily follows from the observation that

$$W_n(\Psi^* + \epsilon) \subseteq W_n(\Psi) \subseteq W_n(\Psi^* - \epsilon)$$

for any $\epsilon > 0$. As such limit \([2]\) exist for each $1 \leq i \leq n$ we can let $\epsilon \to 0$ to obtain the desired result. We remark that while Corollary 2.1 provides a result for general $\Psi$ with components satisfying \([2]\), there are many functions where such limits do not exist.

The following section provides auxiliary results needed to prove Theorem 2.1. In particular the framework for the Mass Transference Principle from rectangles to rectangles \([38]\) is provided. This result is crucial in finding the lower bound result of Theorem 2.1.

3. Auxiliary results

We provide a brief set of known results that we will use in the proof of Theorem 2.1. The first result we state can be considered as the $p$-adic version of Minkowski’s theorem for systems of linear forms. The proof is a straightforward application of the pigeon-hole principle and can be found in \([11]\).

**Lemma 3.1.** Let $L_i(x)$, with $i = 1, \ldots, n$, be linear forms with $p$-adic coefficients of $x = (x_0, x_1, \ldots, x_n)$. Let $\sum_{i=1}^n \tau_i = n + 1$ for $\tau_i \in \mathbb{R}_+$, and $H \geq 1$. Then there exists a non-zero rational integer vector $x = (x_0, x_1, \ldots, x_n)$ with

$$\max_{0 \leq i \leq n} |x_i| \leq H$$

satisfying the system of inequalities

$$|L_i(x)|_p < pH^{-\tau_i} \text{ for } i = 1, \ldots, n.$$
The following lemma generally states that the measure of a lim sup set of balls remains unaltered when the radius is multiplied by some constant. The Euclidean version of this result is well known and appears in a variety of texts, see [6]. The following version for ultrametric spaces was proven in [11].

**Lemma 3.2.** Let \((X, d)\) be a separable ultrametric space and \(\mu\) be a Borel regular measure on \(X\). Let \((B_i)_{i \in \mathbb{N}}\) be a sequence of balls in \(X\) with radii \(r_i \to 0\) as \(i \to \infty\). Let \((U_i)_{i \in \mathbb{N}}\) be a sequence of \(\mu\)-measurable sets such that \(U_i \subset B_i\) for all \(i\). Assume that for some \(c > 0\)

\[ |U_i| \geq c |B_i| \quad \text{for all } i. \]

Then the limsup sets

\[ \mathcal{U} = \limsup_{i \to \infty} U_i := \bigcap_{j=1}^{\infty} \bigcup_{i \geq j} U_i \quad \text{and} \quad \mathcal{B} = \limsup_{i \to \infty} B_i := \bigcap_{j=1}^{\infty} \bigcup_{i \geq j} B_i \]

have the same \(\mu\)-measure.

In particular, if we chose the approximation function \(\psi_i(q) = pq^{1+1/n}\) for each \(1 \leq i \leq n\) then by Lemma 3.1 we know \(\mathcal{W}_n(\Psi) = \mathbb{Z}_p\), and so by shrinking the lim sup set of balls by constant \(1/p\) Lemma 3.2 gives us that \(\mu_n(\mathcal{W}_n(\Psi/p)) = 1\).

Another key result in our proof of Theorem 2.1 is the following Mass Transference Principle type theorem. In order to state this theorem we need the notion of local ubiquity for rectangles, a variation of the notion of ubiquity introduced by Beresnevich, Dickinson, and Velani [6]. Fix an integer \(n \geq 1\), and for each \(1 \leq i \leq n\) let \((X_i, |.|_i, m_i)\) be a bounded locally compact metric space with \(m_i\) a \(\delta_i\)-Ahlfors probability measure. Consider the product space \((X, |.|, m)\), where

\[ X = \prod_{i=1}^{n} X_i, \quad m = \prod_{i=1}^{n} m_i, \quad |.| = \max_{1 \leq i \leq n} |.|_i. \]

For any \(x \in X\) and \(r \in \mathbb{R}_+\) define the open ball

\[ B(x, r) = \left\{ y \in X : \max_{1 \leq i \leq n} |x_i - y_i|_i < r \right\} = \prod_{i=1}^{n} B_i(x_i, r), \]

where \(B_i\) are the usual balls associated with the \(i\)th metric space. Let \(J\) be a infinite countable index set, and \(\beta : J \to \mathbb{R}_+\) a positive function. Let \(l_n, u_n\) be two sequences in \(\mathbb{R}_+\) such that \(u_n \geq l_n\) with \(l_n \to \infty\) as \(n \to \infty\). Define

\[ J_n = \{ \alpha \in J : l_n \leq \beta_\alpha \leq u_n \}. \]
Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function with $\rho(\beta_\alpha) \to 0$ as $\beta_\alpha \to \infty$. For each $1 \leq i \leq n$, let $\{R_{\alpha,i}\}_{\alpha \in J}$ be a sequence of subsets in $X_i$. As with the standard setting of ubiquitous systems define the resonant sets

$$ \left\{ R_\alpha = \prod_{i=1}^{n} R_{\alpha,i} \right\}_{\alpha \in J} .$$

For $\alpha = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ denote the set of hyperrectangles

$$ \Delta(R_\alpha, \rho(r)^{a}) = \prod_{i=1}^{n} \Delta(R_{\alpha,i}, \rho(r)^{a_i}), $$

where for some set $A$ and $b \in \mathbb{R}_+$

$$ \Delta(A, b) = \bigcup_{a \in A} B(a, b). $$

**Definition 3.3** (local ubiquitous system of rectangles). Call $\left\{ \{R_\alpha\}_{\alpha \in J}, \beta \right\}$ a local ubiquitous system of rectangles with respect to $(\rho, \alpha)$ if there exists a constant $c > 0$ such that for any ball $B \subset X$,

$$ \limsup_{n \to \infty} m \left( B \cap \bigcup_{\alpha \in J_n} \Delta(R_\alpha, \rho(U_n)^{a}) \right) \geq cm(B). $$

The second property needed to state the theorem is the following local scaling property, first introduced in [1]. While we will not need it in our use of the Theorem 3.5 we state it and include it in the final theorem for completeness.

**Definition 3.4** ($k$-scaling property). Let $0 \leq k < 1$ and $1 \leq i \leq n$. Then $\{R_{\alpha,i}\}_{\alpha \in J}$ has $k$-scaling property if for any $\alpha \in J$, any ball $B(x_i, r) \subset X_i$ with centre $x_i \in R_{\alpha,i}$, and $0 < \epsilon < r$ then

$$ c_2 r^{\delta_i k} \epsilon^{\delta_i (1-k)} \leq m_i \left( B(x_i, r) \cap \Delta(R_{\alpha,i}, \epsilon) \right) \leq c_3 r^{\delta_i k} \epsilon^{\delta_i (1-k)}, $$

for some constants $c_2, c_3 > 0$.

In our use the resonant sets will be sets of points, so $k = 0$. For $t = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$ define

$$ W(t) = \limsup_{\alpha \in J} \Delta \left( R_\alpha, \rho(\beta_\alpha)^{a+t} \right). $$

Given the above notations and definitions we can state the Mass Transference Principle from rectangles to rectangles (MTPRR) of [36].

**Theorem 3.5** (Mass Transference Principle from rectangles to rectangles). Under the settings above assume that $\left\{ \{R_\alpha\}_{\alpha \in J}, \beta \right\}$ satisfies the local ubiquity for systems of rectangles condition with
respect to \((\rho, a)\), and the \(k\)-scaling property. Then

\[
\dim W(t) \geq \min_{A_i \in A} \left\{ \sum_{j \in K_1} \delta_j + \sum_{j \in K_2} \delta_j + k \sum_{j \in K_3} \delta_j + (1 - k) \frac{\sum_{j \in K_3} a_j \delta_j - \sum_{j \in K_2} t_j \delta_j}{A_i} \right\},
\]

where \(A = \{a_i, a_i + t_i, 1 \leq i \leq n\}\) and \(K_1, K_2, K_3\) are a partition of \(\{1, \ldots, n\}\) defined as

\[
K_1 = \{j : a_j \geq A_i\}, \quad K_2 = \{j : a_j + t_j \leq A_i\} \setminus K_1, \quad K_3 = \{1, \ldots, n\} \setminus (K_1 \cup K_2).
\]

Hence, provided we can find a lim sup set of hyperrectangles that satisfy the local ubiquity property for rectangles then we have a lower bound for the corresponding lim sup set of shrunken hyperrectangles.

4. Proof of Theorem 2.1

We split the proof into the upper and lower bound, and solve each case separately. In both cases we will use the projection \(\pi : \mathbb{Z}_p^n \to \mathbb{Z}_p^{n-1}\), defined by

\[
(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n).
\]

By a well known theorem of Hausdorff theory (see Proposition 3.3 of [20]) as \(\pi\) is a bi-Lipschitz mapping over \(\mathcal{W}_{n}(\tau) \cap \Pi_{\alpha,m}\), we have that

\[
\dim \mathcal{W}_{n}(\tau) \cap \Pi_{\alpha,m} = \dim \pi(\mathcal{W}_{n}(\tau) \cap \Pi_{\alpha,m})).
\]

Consider the set of integers

\[
\mathcal{Q}(\alpha, \tau_m) := \{q \in \mathbb{N} : |q\alpha - q_m|_p < q^{-\tau_m}, \text{ for some } |q_m| \leq q\},
\]

and the union of sets

\[
\mathcal{A}^*_q(\tau) = \bigcup_{\substack{|q| \leq q \\ 1 \leq i \leq n}} \left\{ x \in \mathbb{Z}_p^{n-1} : |qx_i - q_i|_p < q^{-\tau_i} \right\}.
\]

Then,

\[
\pi(\mathcal{W}_{n}(\tau) \cap \Pi_{\alpha,m}) = \limsup_{q \in \mathcal{Q}(\alpha, \tau_m)} \mathcal{A}^*_q(\tau),
\]

hence we only need to find the upper and lower bounds for \(\dim \limsup_{q \in \mathcal{Q}(\alpha, \tau_m)} \mathcal{A}^*_q(\tau)\).
4.1. Upper bound. For the upper bound we take the standard cover of hyperrectangles used in the construction of \( \limsup_{q \in \mathcal{Q}(\alpha, \tau_m)} \mathcal{A}_q^*(\tau) \). By a standard geometrical argument we note that each hyperrectangle, centred at some \( \frac{4}{q} \in \mathcal{Q} \) in the construction of \( \mathcal{A}_q^*(\tau) \), can be covered by a finite collection of balls \( \mathcal{B}_q(\tau_i) \) of radius \( q^{-\tau_i} \), \( i \neq m \). Without loss of generality we can assume that 

\[
\tau_1 \geq \cdots \geq \tau_{m-1} \geq \tau_{m+1} \geq \cdots \geq \tau_n,
\]

since if not then we could take some bi-lipschitz mapping to reorder the coordinate axis such that this was the case. Hence for each \( j \leq i, \)

\[
\frac{q^{-\tau_j}}{q^{-\tau_i}} \leq 1.
\]

Hence in the product below we only consider the \( j \geq i \). By the above argument we have that the cardinality of \( \mathcal{B}_q(\tau_i) \) is

\[
\# \mathcal{B}_q(\tau_i) \ll \prod_{j=i}^{n} \frac{q^{-\tau_j}}{q^{-\tau_i}} = q^{\sum_{j=i, j \neq m}^{n} (\tau_i - \tau_j)}.
\]

As each \( \tau_i \)-approximation function is decreasing as \( q \) increases, for each interval \( 2^k \leq q < 2^{k+1} \) take \( q = 2^k \). Using that \( \mathcal{Q}(\alpha, \tau_m) \subseteq \bigcup_{k \in \mathbb{N}} \mathcal{Q}(\alpha, \tau_m, 2^k) \), then

\[
\mathcal{H}^s \left( \limsup_{q \in \mathcal{Q}(\alpha, \tau_m)} \mathcal{A}_q^*(\tau) \right) \leq \sum_{k=1}^{\infty} \sum_{q \in \mathcal{Q}(\alpha, \tau_m, 2^k)} (q)^{n-1} \# \mathcal{B}_q(\tau_i). (q^{-\tau_i})^s,
\]

\[
\overset{(a)}{\ll} \sum_{k=1}^{\infty} 2^{k(2-\tau_m)} (2^k)^{n-1} (2^k)^{\sum_{j=i, j \neq m}^{n} (\tau_i - \tau_j)} (2^k)^{-\tau_i} s,
\]

\[
\ll \sum_{k=1}^{\infty} 2^{k(n+1-\tau_m + \sum_{j=i, j \neq m}^{n} (\tau_i - \tau_j) - \tau_i s)},
\]

where (a) follows from Theorem\textsuperscript{1.2}. The above sum converges when

\[
s \geq \frac{n + 1 - \tau_m + \sum_{j=i, j \neq m}^{n} (\tau_i - \tau_j)}{\tau_i} + \epsilon,
\]

for any \( \epsilon > 0 \). This is true for each \( 1 \leq i \leq n, i \neq m \), and as \( \epsilon \) is arbitrary, we have that

\[
s \geq \min_{\begin{array}{c} 1 \leq i \leq n \\ i \neq m \end{array}} \left\{ \frac{n + 1 - \tau_m + \sum_{j=i, j \neq m}^{n} (\tau_i - \tau_j)}{\tau_i} \right\},
\]

completing the upper bound result. Note that the result of Remark 2.2 can similarly be obtained by replacing Theorem\textsuperscript{1.2} used at (a) by Lemma\textsuperscript{1.1}.
4.2. **Lower bound.** In order to use Theorem 3.5 to prove the lower bound of Theorem 2.1 we need to construct a ubiquitous system of rectangles. In following with the ubiquity setup for Theorem 3.5 let

\[ J = Q(\alpha, \tau_m), \quad R_{q,i} = \left\{ \frac{q}{q} \in \mathbb{Q} : |q| \leq q \right\}, \quad R_q = \prod_{i \neq m} R_{q,i}, \]

\[ \beta(q) = q, \quad \rho(q) = q^{-1}, \quad l_k = M^k, \quad u_k = M^{k+1}, \]

where \( M \in \mathbb{N} \) is a fixed integer to be determined later. Then we have that

\[ J_k = \left\{ q \in \mathbb{N} : M^k \leq q < M^{k+1} : \exists |q| \leq p \text{ s.t. } |q\alpha - q_m|_p < M^{-(k+1)\tau_m} \right\}. \]

Hence for some vector \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_+^n \),

\[ \Delta(R_{\alpha, \rho}(r)^{\mathbf{a}}) = \prod_{i=1}^{n} \bigcup_{|q_i| \leq q} B \left( \frac{q_i}{q} r^{-a_i} \right). \]

We prove the following.

**Proposition 4.1.** Let \( R_q, \rho, \) and \( J_k \) be as above, and let \( \bar{v} = (v_1, \ldots, v_{m-1}, v_{m+1}, \ldots, v_n) \in \mathbb{R}_+^n \) with each \( v_i > 1 \) and

\[ \sum_{i=1, i \neq m}^{n} v_i = n + 1 - \tau_m, \]

where \( 1 < \tau_m < 2 \). Then for any ball \( B = B(x, r) \subset \mathbb{Z}_p^{n-1} \), with centre \( x \in \mathbb{Z}_p^{n-1} \) and radius \( 0 < r < r_0 \) for some \( r_0 \in \mathbb{R}_+ \), there exists a constant \( c > 0 \) such that

\[ \mu_{n-1} \left( B \cap \bigcup_{q \in J_k} \Delta(R_q, \rho(u_k)^{\bar{v}}) \right) \geq c \mu_{n-1}(B), \]

provided \( M \geq 6^{\frac{1}{n-1}}3 \).

The proof of this result follows the same style of many similar results. For example see Theorem 1.3 of [12] for the one dimensional real case, or Proposition 5.1 of [11] for the \( n \)-dimensional \( p \)-adic case.

**Proof.** For any \( y = (y_1, \ldots, y_{m-1}, y_{m+1}, \ldots, y_n) \in (\mathbb{Z}_p \setminus \mathbb{Q})^{n-1} \), consider the system of inequalities

\[ \begin{cases} |q\alpha - q_m|_p < (M^{k+1})^{-\tau_m}, \\ |qy_i - q_i|_p < (M^{k+1})^{-y_i}, \quad 1 \leq i \leq n, \quad i \neq m, \\ \max_{1 \leq i \leq n} \{|q_i|, |q|\} \leq M^{k+1}. \end{cases} \]

(3)

By the condition on \( \bar{v} \) we have, by Lemma 3.1, that there exists a non-zero integer solution \( (q, q) \in \mathbb{Z}^n \times \mathbb{Z} \) to (3). Assume without loss of generality that \( q \geq 0 \) and consider the case where
$q = 0$. As $v_i > 1$ each of the inequalities of \((3)\) we are forced to conclude that each $q_i = 0$, which contradicts the fact that $(q, q)$ is a non-zero solution, thus $q > 0$. Further, if $(q, q)$ solves \((3)\) then $q \in \mathbb{Q}(\alpha, \tau_m, M^{k+1})$.

Thus, we have that

$$\mu_{n-1} \left( B \cap \bigcup_{q \in \mathbb{Q}(\alpha, \tau_m, M^{k+1})} \Delta(R_q, \rho(M^{k+1})^\circ) \right) = \mu_{n-1}(B).$$

Note that

$$\mu_{n-1} \left( B \cap \bigcup_{q \in \mathbb{Q}(\alpha, \tau_m, M^{k+1})} \Delta(R_q, \rho(M^{k+1})^\circ) \right) \leq \mu_{n-1} \left( B \cap \bigcup_{q \in \mathbb{Q}(\alpha, \tau_m, M^k)} \Delta(R_q, \rho(M^{k+1})^\circ) \right)$$

$$+ \mu_{n-1} \left( B \cap \bigcup_{q \in J_k} \Delta(R_q, \rho(M^{k+1})^\circ) \right),$$

and so

$$\mu_{n-1} \left( B \cap \bigcup_{q \in J_k} \Delta(R_q, \rho(M^{k+1})^\circ) \right) \geq \mu_{n-1}(B) - \mu_{n-1} \left( B \cap \bigcup_{q \in \mathbb{Q}(\alpha, \tau_m, M^k)} \Delta(R_q, \rho(M^{k+1})^\circ) \right).$$

At this point we only want the \(\frac{a}{q} \in R_q\) such that

$$B \cap B \left( \frac{a}{q}, \rho(M^{k+1})^\circ \right) \neq \emptyset.$$ 

For ball $B = B(x, r)$ with $x \in \mathbb{Z}_p^{n-1}$ and $r \in \{p^j : j \in \mathbb{Z}\}$, this is equivalent to the set of solutions to

\((4)\)

$$\left| \frac{x - q_i}{q} \right|_p < r, \quad 1 \leq i \leq n, \quad i \neq m.$$ 

For $q$ fixed and each $|q_i| \leq q$ by residue classes we have that there are at most

$$(2qr + 1)^{n-1}$$

suitable values of $q$. We can choose suitably large $k \in \mathbb{N}$ such that $M^k r > 1$, and so for each $q, |q_i| \leq M^k$ there are at most

\((5)\)

$$(3M^k r)^{n-1}$$
possible values of $q$ solving (4). Hence

$$
\mu_{n-1} \left( B \cap \bigcup_{q \in Q(\alpha, \tau_m, M^k)} \Delta(R_q, \rho(M^{k+1}v)) \right) \leq \sum_{q \in Q(\alpha, \tau_m, M^k)} \sum_{q \text{ solving } (4)} \mu_{n-1} \left( B \cap \Delta \left( \frac{q}{q}, \rho(M^{k+1}v) \right) \right),
$$

\[ \leq (a) \sum_{q \in Q(\alpha, \tau_m, M^k)} (3M^k)^{n-1}(M^{k+1})^{-1}\frac{n}{i \neq m} v_i, \]

\[ \leq (b) 6M^{k(2-\tau_m)}3^{n-1}M^{k(n-1)}M^{-(k+1)(n+1-\tau_m)}\mu_{n-1}(B), \]

\[ \leq 2.3^n M^{-n+1+\tau_m} \mu_{n-1}(B), \]

where (a) follows by (5) and (b) follows by Theorem 1.2 and our condition on $\tilde{v}$. As $M \geq 6^{\frac{1}{n-3}},$

\[ c = \left(1 - \frac{2.3^n}{M^{n+1-\tau_m}}\right) > 0. \]

Thus,

$$
\mu_{n-1} \left( B \cap \bigcup_{q \in Q_k} \Delta(R_q, \rho(M^{k+1}v)) \right) \geq c \mu_{n-1}(B).
$$

\[ \square \]

Given Proposition 4.1, we have that $(R_q, \beta)$ is a local ubiquitous system of rectangles with respect to $(\rho, \tilde{v})$, provided $\sum_{i=1}^n v_i = n + 1 - \tau_m$. Without loss of generality assume that $m = n$, and so $\tilde{v} = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}$. Given $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_+$ assume without loss of generality that $\tau_1 > \tau_2 > \cdots > \tau_{n-1}$ and define each $v_{n-i}$ recursively by

\[ v_{n-i} = \min \left\{ \tau_{n-i} - \varepsilon, \frac{n + 1 - \tau_n - \sum_{j=n-i}^{n-1} v_i}{n - i} \right\}. \]

By the condition on $\tau$ of Theorem 2.1, there exists a $k \in \{1, \ldots, n-1\}$ such that

\[ v_j = \frac{n + 1 - \tau_n - \sum_{j=n-k}^{n-1} v_i}{n - 1 - k}, \]

for all $1 \leq j \leq k$. Clearly each $v_i < \tau_i$ for $1 \leq i \leq n - 1$, and so the associated vector $t = (t_1, \ldots, t_n) \in \mathbb{R}^{n-1}$ is defined by

\[ t_i = \tau_i - v_i, \quad 1 \leq i \leq n - 1. \]

Consider the set

\[ A = \{ v_1, \ldots, v_{n-1}, \tau_1, \ldots, \tau_{n-1} \}. \]

For each $A_i \in A$ observe the following:
i) $A_i \in \{v_1, \ldots, v_{n-1}\}$: Then we have the sets

$$K_1 = \{1, \ldots, \max\{i, n-k\}\}, \quad K_2 = \{\max\{i+1, n-k+1\}, \ldots, n-1\}, \quad K_3 = \emptyset.$$ 

By Theorem 3.5, we have that

$$\dim \mathfrak{W}_n(\tau) \geq \min_{A_i} \left\{ \frac{\max\{i, n-k\} v_i + (n - \max\{i+1, n-k+1\}) v_i - \sum_{j=\max\{i+1, n-k+1\}}^{n-1} t_j}{v_i} \right\},$$

$$= \min_{A_i} \left\{ \frac{(n-1) v_i - \sum_{j=\max\{i+1, n-k+1\}}^{n-1} t_j}{v_i} \right\}. $$

Since $t_j = \varepsilon$ for $n-k < j \leq n-1$ the summation in the above equation can be made arbitrarily small to give that $\dim \mathfrak{W}_n(\tau) = n-1$.

ii) $A_i \in \{\tau_1, \ldots, \tau_{n-1}\}$: Since $\tau_i = v_i + \varepsilon$ for $n-k+1 \leq i \leq n-1$ the above argument covers such case, so we only need to consider $\tau_i$ for $1 \leq i \leq n-k$. For such $\tau_i$ we have the sets

$$K_1 = \emptyset, \quad K_2 = \{i, \ldots, n-1\}, \quad K_3 = \{1, \ldots, i-1\}.$$ 

Applying Theorem 3.5, we have

$$\dim \mathfrak{W}_n(\tau) \geq \min_{A_i} \left\{ \frac{(n-i) \tau_i + \sum_{j=1}^{i-1} v_j - \sum_{j=i}^{n-1} t_j}{\tau_i} \right\},$$

$$= \min_{A_i} \left\{ \frac{(n-i) \tau_i + (n-k)^{\left(\frac{n+1-\tau_n-\sum_{j=1}^{n-1} v_j}{n-k} - \sum_{j=1}^{n-k} t_j\right)}}{\tau_i} \right\},$$

$$= \min_{A_i} \left\{ \frac{n+1 - \tau_n + \sum_{j=i}^{n-1} (\tau_i - t_j)}{\tau_i} \right\}. $$

Combining i) and ii), and returning to the $\tau_m$ approximation over the $\alpha$ coefficient, we have that

$$\dim \limsup_{q \in Q(\alpha, \tau_m)} \mathfrak{W}_q^*(\tau) \geq \min_{1 \leq i \leq n} \left\{ \frac{n+1 - \tau_m + \sum_{j=i, j \neq m}^{n} (\tau_i - t_j)}{\tau_i} \right\},$$

completing the proof.

5. Proof of the counting results

Recall, we are aim to provide bounds on the set

$$\mathcal{Q}(x, \psi, N) := \left\{ (q, q_1) \in \mathbb{N} \times \mathbb{Z} : 0 < q \leq N, \ |q_1| \leq N, \ |x - \frac{q_1}{q}|_p < \psi(N) \right\}. $$
We begin with the proof of Lemma 1.3. This style of proof is not new and follows a similar idea to Euclidean version (see Lemma 3 of [33]).

**Proof of Lemma 1.3.** Fix \( x \in \mathbb{Z}_p \) and take \( t \in \mathbb{N} \) to be the integer such that
\[
p^{-t} \leq Q^{-\tau} < p^{-t+1}.
\]
Consider a set of open disjoint balls \( \{ B_i \}_{i=1}^{n} \), each with some centre point \( k_i \in \mathbb{Z} \) and radius \( p^{-t} \). Choose the set of points \( k_i \) such that \( \mathbb{Z}_p \subseteq \bigcup_{i=1}^{n} B_i \). Consider the \((Q+1)^2\) set of points of the form
\[
qx - q_1 \in \mathbb{Z}_p,
\]
for \((q, q_1) \in [0, Q]^2\). By the Pigeon-hole principle there exists at least one ball, say \( B_j \), such that there are at least
\[
\frac{(Q+1)^2}{p^t} > \frac{1}{p} Q^{2-\tau}
\]
points. As \( \tau < 2 \) we can choose \( Q \) sufficiently large enough such that \( p^{-1} Q^{2-\tau} > 2 \). Order the points \((q, q_1)\), that correspond to the set of points \( qx - q_1 \) contained in \( B_j \), in terms of the absolute value of the \( q \) component. Suppose that the pair \((m, m_1)\) is the smallest. Then for all other pairs, say \((q, q_1)\), with corresponding point contained in \( B_j \) we have that
\[
|(q - m)x - (q_1 - m_1) + k_j - k_j|_p < p^{-t}.
\]
Clearly \((q - m) \in (0, Q]\), and \((q_1 - m_1) \in [-Q, Q]\). Note that the above argument yields \( p^{-1} Q^{2-\tau} - 1 \) such points, completing the proof. \( \square \)

Lemma 1.1 is also a relatively simple proof. We are unable to find a proof that uses a similar argument, however we suspect such style of result has been used before.

**Proof of Lemma 1.1.** We use a proof by contradiction. Suppose that
\[
\# Q(x, \tau, Q) > 2 Q^{(\tau x) - \tau + \epsilon}.
\]
We use the following notations. Let \( X \in \mathbb{N} \) be an integer such that
\[
|x - X|_p < p^{-M},
\]
for some suitably large \( M \in \mathbb{N} \). Define \( V^+_Q \) and \( V^-_Q \) to be the sets
\[
V^+_Q := \{(q, q_1) \in \mathbb{N} \times \mathbb{Z} : 0 < q \leq Q, 0 \leq q_1 \leq Q, \},
\]
\[
V^-_Q := \{(q, q_1) \in \mathbb{N} \times \mathbb{Z} : 0 < q \leq Q, -Q \leq q_1 \leq 0, \}.
\]
Let \( t \in \mathbb{N} \) be the integer such that
\[
p^{-t} \leq Q^{-\tau} < p^{-t+1},
\]
and similarly $k \in \mathbb{N}$ be the integer such that
\[ p^{-k} \leq Q^{-(\tau(x)+\epsilon)} < p^{-k+1}. \]

Note that as $\tau(x) > \tau$, we have that $k \geq t$, and so $p^{k-t} \in \mathbb{N}$. Further, observe that
\[(7)\quad p^{k-t} < pQ^{\tau(x)-\tau+\epsilon}.\]

Lastly, by the definition of $\tau(x)$, we have that there exists finitely many $Q \in \mathbb{N}$ such that
\[
|qx - q_1|_p < Q^{-(\tau(x)+\epsilon)},
\]
for $0 < q, |q_1| \leq Q$. Hence our ‘sufficiently large $Q$’ is the value of $Q$ such that for any pair $0 < q, |q_1| \leq Q$,
\[(8)\quad |qx - q_1|_p \geq Q^{-(\tau(x)+\epsilon)},\]
for all $\epsilon > 0$. Consider the set of points in $Q(Q, \tau, x)$. Note that $(q, q_1) \in Q(Q, \tau, x)$ if and only if $(q, q_1) \in V_Q^+ \cup V_Q^-$, and
\[(9)\quad qX - q_1 \equiv 0 \mod p^t.\]

Thus, for all $(q, q_1) \in Q(Q, \tau, x)$ we have that
\[qX - q_1 = \lambda p^t,\]
for some $\lambda \in \mathbb{Z}$. Split the set of points in $Q(x, \tau, Q)$ into two disjoint sets, the set of pairs in $V_Q^+$, and the set of pairs in $V_Q^-$. As there are greater than $2Q^{\tau(x)-\tau+\epsilon}$ pairs, at least one of the sets has greater than $Q^{\tau(x)-\tau+\epsilon}$ pairs. Without loss of generality assume such set of points belong in $V_Q^+$. Considering the range of values of $\lambda p^t$ there are $p^{k-t}$ possible values of $\lambda p^t$ modulo $p^k$. By (6) and (7) we have, by the Pigeon-hole principle, that there exists at least two pairs, say $(a, a_1)$ and $(b, b_1)$, such that
\[(a - b)X - (a_1 - b_1) \equiv 0 \mod p^k.\]

This is equivalent to
\[|(a - b)x - (a_1 - b_1)|_p \leq p^{-k} \leq Q^{-(\tau(x)+\epsilon)},\]
with $(a - b, a_1 - b_1) \in V_Q^+ \cup V_Q^-$, as $0 < a - b \leq Q$ by our choice of ordering of $a, b$, and $|a_1 - b_1| \leq Q$ by the fact that the pairs $(a, a_1), (b, b_1) \in V_Q^+$. However, such result contradicts (8) which follows from the definition of $\tau(x)$, thus (6) must be false. \qed
5.1. **$p$-adic approximation lattices.** Prior to the proof of Theorem 1.2 we recall some basic definitions and results of Lattice theory that will be needed. Define a lattice $\Lambda$ as a discrete additive subgroup of $\mathbb{R}^n$. If $\Lambda \subseteq \mathbb{Z}^n$ the $\Lambda$ is an integer lattice. A set of linearly independent vectors $b_1, \ldots, b_n$ that generate $\Lambda$ is called a basis of $\Lambda$. Let $B$ be a $n \times n$ matrix with columns $b_i$, then call $B$ a basis matrix. Define the fundamental region as

$$F(B) := \left\{ \sum_{i=1}^{n} a_i b_i : a_i \in \mathbb{R}, \ 0 \leq a_i < 1 \right\}.$$  

A standard result of Lattice theory states that if $B$ is a basis matrix for $\Lambda$ then $F(B)$ contains no lattice points other than the origin (see Chapter 3, Lemma 6 of [17]).

The volume of the fundamental region can be found by taking the determinant of the basis matrix, that is $\text{vol}(F(B)) = |\det B|$. A basis matrix is not unique for each $\Lambda$, however for any lattice $\Lambda$ the volume of the fundamental region is the same regardless of choice of basis matrix. For this reason the notation $\text{vol}(F(B)) = |\det \Lambda|$ is used. If $U \in \mathbb{Z}^{n \times n}$ is a unimodular matrix and $B_1$ is a basis matrix for $\Lambda$ then $B_2 = B_1 U$ is also a basis matrix for $\Lambda$.

One property of lattices that are incredibly useful are the successive minima of a lattice. Let $B_n = B(0,1)$ denote the $n$-dimensional unit sphere. For $c \in \mathbb{R}_+$ we use the notation $cB_n = B(0,c)$. Define the successive minima of a lattice $\Lambda \subset \mathbb{R}^n$ of rank $n$ as the set of values

$$\lambda_i(\Lambda) := \min \{ \lambda > 0 : \dim(\Lambda \cap \lambda B) \geq i \},$$

for $i = 1, \ldots, n$. By Minkowski’s inequalities on the successive minima (see e.g. [21]) we have that

$$(10) \quad \frac{2^n}{n!} \det \Lambda \leq \text{vol}(B_n) \prod_{i=1}^{n} \lambda_i(\Lambda) \leq 2^n \det \Lambda.$$  

For a count on the number of lattice points within a convex body we have the follow theorem due to Blichfeldt [14].

**Theorem 5.1.** Let $\Lambda \subset \mathbb{R}^n$ be a lattice of rank $n$ and let $V \subset \mathbb{R}^n$ be a convex body such that $\dim(\Lambda \cap V) = n$. Then

$$\#(\Lambda \cap V) \leq n! \frac{\text{vol}(V)}{\det \Lambda} + n.$$  

The constant for such estimate can be excessively large, however in our use of the Theorem the size of such constant is irrelevant.

For the proof of Theorem 1.2 we use $p$-adic approximation lattices. First discovered by de Weger [19] who used them to prove a variety of results in classical $p$-adic Diophantine approximation,
including the $p$-adic analogue of Hurwitz Theorem. Recently $n$-dimensional forms of $p$-adic approximation lattices have been used to provide lattice based cryptosystems [25, 26]. In these papers both dual and simultaneous approximation lattices were discussed. In particular Dirichlet-style exponents were proven for simultaneous and dual approximation.

For a $n$-tuple of approximation functions $\Psi = (\psi_1, \ldots, \psi_n)$, a large integer $M \in \mathbb{N}$, and a fixed $x = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n$ define the $\Psi$-approximation lattice $\Lambda_M$ by

$$\Lambda_M = \{(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1} : |a_0 x_i - a_i|_p \leq \psi_i(M), \ 1 \leq i \leq n\}.$$ 

For any $x \in \mathbb{Z}_p^n$ we may write each $x_j$ as the $p$-adic expansion

$$x_j = \sum_{i=0}^{\infty} x_{j,i} p^i, \quad x_{j,i} \in \{0, 1, \ldots, p - 1\}.$$ 

Let $X_{j,M} \in \mathbb{Z}$ be the integer

$$X_{j,M} = \sum_{i=0}^{t_j} x_{j,i} p^i,$$

where each $t_j \in \mathbb{N}$ is the unique value associated with $M$ satisfying

$$p^{-t_j} \leq \psi_j(M) < p^{-t_j+1}.$$ 

Lastly, for each $1 \leq j \leq n$ let $\psi_{j,M} = p^{t_j}$. Then the set of vectors

$$(11) \quad B = \left\{\begin{pmatrix} 1 \\ X_{1,M} \\ \vdots \\ X_{n,M} \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_{1,M}^* \\ \vdots \\ \psi_{n,M}^* \end{pmatrix} \right\},$$

form a basis for $\Lambda_M$. The set $B$ can be proven to be a basis by considering the fundamental region $F(B)$ and showing the only lattice point contained is $0$. Given such basis we can deduce that

$$|\det \Lambda_M| = \prod_{i=1}^{n} \psi_{i,M}^* \asymp \left(\prod_{i=1}^{n} \psi_i(M)\right)^{-1}.$$ 

In the simultaneous case, $\Psi = (\psi, \ldots, \psi)$, it was proven in [26] that

$$\lambda_1(\Lambda_M) \ll \psi(M)^{-\frac{n}{n+1}}.$$ 

In order to prove Theorem 1.2 we find a lower bound on $\lambda_1(\Lambda_M)$ by only considering $x \in \mathbb{Z}_p$ satisfying certain Diophantine exponent properties.

Proof of Theorem 1.2.
For approximation function $\psi : \mathbb{N} \to \mathbb{R}_+$ with $\psi(q) \to 0$ as $q \to \infty$, fixed $x \in \mathbb{Z}_p$ and suitably large $M \in \mathbb{N}$ let $\psi^*_M$ and $X_M$ be defined as above. As $x \neq 0$ and $x \notin \mathbb{Q}$ we can choose suitably large $M$ such that $|X_M| > 2$. For example, suppose that $X_M = -2$ for all $M$. Then as $M \to \infty$, by our definition of $\psi^*_M$, $t \to \infty$. Thus

$$X_M = p - 2 + \sum_{i=1}^t (p - 1)p^i \to -2 \in \mathbb{Z}, \text{ as } t \to \infty.$$ 

As $\psi(q) < q^{-1}$ we can choose suitably large $M$ such that $\psi^*_M > 2M$. Further, $M$ is chosen suitably large enough such that for any $\epsilon > 0$

$$|q_0 x - q|_p \geq (M^{1/2-\epsilon})^{2+\epsilon}$$

for all $0 \leq q_0, |q| \leq M^{1/2-\epsilon}$. Note that such $M$ is possible since $\tau(x) \leq 2$.

By the above choices of $X_M$ and $\psi^*_M$ we have that

$$B = \left\{ \begin{pmatrix} 1 \\ X_M \end{pmatrix}, \begin{pmatrix} 0 \\ \psi^*_M \end{pmatrix} \right\}$$

form a basis for the lattice $\Lambda_M$ where

$$\Lambda_M = \{(a_0, a_1) \in \mathbb{Z}^2 : |a_0 x - a_1|_p < \psi(M)\}.$$ 

Further, we have that

$$\det \Lambda_M = \begin{vmatrix} 1 & 0 \\ X_M & \psi^*_M \end{vmatrix} = \psi^*_M.$$ 

Let $V$ be the convex body $V = [0, M] \times [-M, M]$. Consider the following two cases:

i) $\text{rank}(\Lambda_M \cap V) = 2$: By Theorem 5.1 we have that

$$\# \Lambda_M \cap V \leq 2 \frac{\nuol(V)}{\det \Lambda_M} + 2,$$

$$\leq 4M^2(\psi^*_M)^{-1} + 2,$$

$$\leq 6M^2 \psi(M),$$

where the last inequality holds due to the condition that $\psi(q) < q^{-2}$. This proves Theorem 1.2 for the rank 2 case.

ii) $\text{rank}(\Lambda_M \cap V) = 1$: When $\text{rank}(\Lambda_M \cap V) = 1$ all solution points lie on a line. Note that $(0, 0) \in \Lambda_M \cap V$, so the line is through the origin, so such set of solutions form a sublattice. Suppose $(a_0, a_1) \in \Lambda_M \cap V$ is the closest point to the origin, then all other points are a scalar
multiple of \((a_0, a_1)\). We show that since \(\tau(x) = 2\) we must have that \(|a_0|, |a_1| \geq \frac{\psi^*_M}{M}\). Suppose that \(|a_0|, |a_1| < \frac{\psi^*_M}{M} < (\psi^*_M)^{1/2-\epsilon}\) where \(\epsilon > 0\). Then it would imply that

\[
|a_0x - a_1|_p < (\psi^*_M)^{-1} = ((\psi^*_M)^{1/2-\epsilon})^{-\frac{2}{1-\epsilon}}.
\]

However, by our choice of \(M\) we have (12), a contradiction, hence \(|a_0|, |a_1| \geq \frac{\psi^*_M}{M}\). By properties of the Euclidean norm we have that

\[
\lambda_1(\Lambda_M) \geq \max\{|a_0|, |a_1|\} \geq \frac{\psi^*_M}{M}.
\]

Noting that any straight line contained in \(V\) is of length at most \(\sqrt{5}M\) we can deduce that in the rank 1 case

\[
\#(\Lambda_M \cap V) \leq \frac{\sqrt{5}M}{\lambda_1} + 1 \leq 2\sqrt{5}M^2\psi(M).
\]

Taking the maximum of the rank 1 and rank 2 case above we obtain our desired result. \(\square\)

6. Concluding remarks on Theorem 1.2

This article provides sharp bounds on the number of rational points close to almost all \(p\)-adic integers. While this result allows us to find simultaneous \(p\)-adic Diophantine approximation results on coordinate hyperplanes, it falls a long way short of providing results for Diophantine approximation sets on curves and manifolds. It is hoped the techniques used in this paper could be used to find rational points close to curves, we intend to follow this up with a subsequent paper.

With regards to the results of this paper, it would be desirable to produce bounds on the number of rational points close to \(n\)-dimensional \(p\)-adic integers. In particular for a fixed \(x = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n\) and \(n\)-tuple of approximation functions \(\Psi = (\psi_1, \ldots, \psi_n)\), we would like to find bounds on the set

\[
Q_n(x, \Psi, N) := \left\{(q, q_1, \ldots, q_n) \in \mathbb{N} \times \mathbb{Z}^n : 0 < q \leq N, |q_i| \leq N, \left|x_i - \frac{q_i}{q}\right|_p < \psi_i(N), 1 \leq i \leq n\right\}.
\]

Using the same ideas used in the proof of Theorem 1.2 it can be shown that the set \(Q_n(x, \Psi, M)\) can be described by a lattice \(\Lambda\) with basis \(\mathbf{11}\). Intersecting \(\Lambda\) with the the convex body \(V_n := [0, M] \times \prod_{i=1}^n[-M, M]\) we obtain the set \(Q_n(x, \Psi, M)\). In the cases where \(\text{rank } \Lambda \cap V_n\) is full it can be shown, via Blichfeldt’s Theorem, that

\[
Q_n(x, \Psi, M) \ll M^{n+1} \prod_{i=1}^n \psi_i(M),
\]

the expected upper bound for almost all \(x \in \mathbb{Z}_p^n\). However, for \(1 < \text{rank } (\Lambda \cap V_n) < n\) issues arise that cannot be solved by the methods of this chapter.
References

[1] D Allen and S Baker. A general mass transference principle. Selecta Math. (N.S.), 25(3):Paper No. 39, 38, 2019.
[2] D Badziahin and Y Bugeaud. On simultaneous rational approximation to a real number and its integral powers, II. New York J. Math., 26:362–377, 2020.
[3] D Badziahin, Y Bugeaud, and J Schleischitz. On simultaneous rational approximation to a $p$-adic number and its integral powers, ii. https://arxiv.org/abs/1511.06862
[4] V Beresnevich. Rational points near manifolds and metric Diophantine approximation. Ann. of Math. (2), 175(1):187–235, 2012.
[5] V Beresnevich, V Bernik, and E Kovalevskaya. On approximation of $p$-adic numbers by $p$-adic algebraic numbers. J. Number Theory, 111(1):33–56, 2005.
[6] V Beresnevich, D Dickinson, and S Velani. Measure theoretic laws for lim sup sets. Mem. Amer. Math. Soc., 179(846):x+91, 2006.
[7] V Beresnevich, D Dickinson, and S Velani. Diophantine approximation on planar curves and the distribution of rational points. Ann. of Math. (2), 166(2):367–426, 2007. With an Appendix II by R. C. Vaughan.
[8] V Beresnevich, A Haynes, and S Velani. Sums of reciprocals of fractional parts and multiplicative Diophantine approximation. Mem. Amer. Math. Soc., 263(1276):vii + 77, 2020.
[9] V Beresnevich and È Kovalevskaya. On Diophantine approximations of dependent quantities in the $p$-adic case. Mat. Zametki, 73(1):22–37, 2003.
[10] V Beresnevich, L Lee, R.C Vaughan, and S Velani. Diophantine approximation on manifolds and lower bounds for Hausdorff dimension. Mathematika, 63(3):762–779, 2017.
[11] V Beresnevich, J Levesley, and B Ward. Simultaneous $p$-adic Diophantine approximation. https://arxiv.org/abs/2101.05251
[12] V Beresnevich, F Ramírez, and S Velani. Metric Diophantine approximation: aspects of recent work. In Dynamics and analytic number theory, volume 437 of London Math. Soc. Lecture Note Ser., pages 1–95. Cambridge Univ. Press, Cambridge, 2016.
[13] V Beresnevich, R. C. Vaughan, S Velani, and E Zorin. Diophantine approximation on manifolds and the distribution of rational points: contributions to the convergence theory. Int. Math. Res. Not. IMRN, (10):2885–2908, 2017.
[14] H Blichfeldt. Report on the theory of the geometry of numbers. Bull. Amer. Math. Soc., 25(10):449–453, 1919.
[15] N Budarina. Simultaneous Diophantine approximation in the real and $p$-adic fields with nonmonotonic error function. Lith. Math. J., 51(4):461–471, 2011.
[16] Y Bugeaud, N Budarina, D Dickinson, and H O’Donnell. On simultaneous rational approximation to a $p$-adic number and its integral powers. Proc. Edinb. Math. Soc. (2), 54(3):599–612, 2011.
[17] JWS Cassels. An introduction to the geometry of numbers. Springer Science & Business Media, 2012.
[18] S Datta and A Ghosh. Diophantine inheritance for $p$-adic measures. https://arxiv.org/abs/1903.09362
[19] B. M. M. de Weger. Approximation lattices of $p$-adic numbers. J. Number Theory, 24(1):70–88, 1986.
[20] K Falconer. Fractal geometry. John Wiley & Sons, Ltd., Chichester, third edition, 2014. Mathematical foundations and applications.
[21] M Henk and F Xue. On successive minima-type inequalities for the polar of a convex body. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 113(3):2601–2616, 2019.

[22] J Huang. The density of rational points near hypersurfaces. Duke Math. J., 169(11):2045–2077, 2020.

[23] J Huang and J Liu. Simultaneous Approximation on Affine Subspaces. International Mathematics Research Notices, 11 2019. rnz190.

[24] MN Huxley. Area, lattice points, and exponential sums, volume 13 of London Mathematical Society Monographs. New Series. The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[25] H Inoue, S Kamada, and K Naito. Transference principle on simultaneous approximation problems of $p$-adic numbers and multidimensional $p$-adic approximation lattices. Linear Nonlinear Anal., 3(2):239–249, 2017.

[26] H Inoue and K Naito. The shortest vector problems in $p$-adic lattices and simultaneous approximation problems of $p$-adic numbers. Linear Nonlinear Anal., 3(2):213–224, 2017.

[27] V Jarník. Sur les approximations diophantiennes des nombres $p$-adiques. Rev. Ci. (Lima), 47:489–505, 1945.

[28] D Kleinbock and G Tomanov. Flows on $S$-arithmetic homogeneous spaces and applications to metric Diophantine approximation. Comment. Math. Helv., 82(3):519–581, 2007.

[29] E Lutz. Sur les approximations diophantiennes linéaires $P$-adiques. Actualités Sci. Ind., no. 1224. Hermann & Cie, Paris, 1955.

[30] A Mohammadi and A Salehi Golsefidy. Simultaneous Diophantine approximation in non-degenerate $p$-adic manifolds. Israel J. Math., 188:231–258, 2012.

[31] A Oliveira. Khintchine’s theorem with rationals coming from neighborhoods in different places. https://arxiv.org/abs/2006.14764.

[32] F Ramírez. Khintchine types of translated coordinate hyperplanes. Acta Arith., 170(3):243–273, 2015.

[33] F Ramírez, D Simmons, and F Süss. Rational approximation of affine coordinate subspaces of Euclidean space. Acta Arith., 177(1):91–100, 2017.

[34] J Schleischitz. On the spectrum of Diophantine approximation constants. Mathematika, 62(1):79–100, 2016.

[35] J Schleischitz. Diophantine approximation on polynomial curves. Math. Proc. Cambridge Philos. Soc., 163(3):533–546, 2017.

[36] V Sprindžuk. Mahler’s problem in metric number theory. Translated from the Russian by B. Volkmann. Translations of Mathematical Monographs, Vol. 25. American Mathematical Society, Providence, R.I., 1969.

[37] R. C. Vaughan and S. Velani. Diophantine approximation on planar curves: the convergence theory. Invent. Math., 166(1):103–124, 2006.

[38] B Wang and J Wu. Mass transference principle from rectangles to rectangles in Diophantine approximation. https://arxiv.org/abs/1909.00924.

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