On Quantum Duality of Group Amenability

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Received: 3 November 2019; Accepted: 27 December 2019; Published: 2 January 2020

Abstract: In this paper, we investigate the co-amenability of compact quantum groups. Combining with some properties of regular C*-norms on algebraic compact quantum groups, we show that the quantum double of co-amenable compact quantum groups is unique. Based on this, this paper proves that co-amenability is preserved under formulation of the quantum double construction of compact quantum groups, which exhibits a type of nice symmetry between the co-amenability of quantum groups and the amenability of groups.

Keywords: compact quantum group; quantum duality; amenability; co-amenability; quantum double construction; Haar integral

MSC: 46L05; 46L65; 46L89

1. Introduction

Given a compact group $G$, denoted by $C(G)$ the C*-algebra of continuous functions on $G$, one can define a morphism

$$\Delta : C(G) \to C(G) \otimes C(G),$$

by $\Delta(f)(g_1, g_2) = f(g_1g_2)$, where $f \in C(G), g_1, g_2 \in G$, and $C(G) \otimes C(G)$ is naturally identified with $C(G \times G)$, which satisfies the co-associativity

$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta.$$

The morphism $\Delta$ is called a co-multiplication on $C(G)$, under which the pair $(C(G), \Delta)$ comes into being a compact quantum group defined in the sense of Woronowicz [1].

Definition 1 ([1]). Assume that $A$ is a C*-algebra with an identity and $\Delta : A \to A \otimes A$ is a unital *-homomorphism satisfying the following two relationships,

(i) $$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta,$$

(ii) the linear spans of $(1 \otimes A)\Delta(A)$ and $(A \otimes 1)\Delta(A)$ are each equal to $A \otimes A$.

Then, the pair $(A, \Delta)$ is called a compact quantum group (CQG).

For an arbitrary CQG $(A, \Delta)$, by [2], there exists a unique state $h_A$ on $A$ so that for all $a \in A$,

$$(id \otimes h_A)\Delta(a) = (h_A \otimes id)\Delta(a) = h_A(a)1,$$

which is called the Haar integral of $(A, \Delta)$. For the commutative CQG $(C(G), \Delta)$ associated to a classical compact group $G$ described as above, the Haar integral $h_{C(G)}$ is the integral with respect to the Haar measure on $G$, which has full support and, therefore, is faithful. However, the Haar integral on
an arbitrary CQG \((A, \Delta)\) needs not be always faithful. Each CQG \((A, \Delta)\) has a canonical dense Hopf *-subalgebra \((A_0, \Delta_0)\) linearly spanned by matrix entries of all finite dimensional co-representations of \((A, \Delta)\), where \(\Delta_0\) is given by restricting the co-multiplication \(\Delta\) from \(A\) to \(A_0\). In the article, we call \((A_0, \Delta_0)\) the associated algebraic CQG of \((A, \Delta)\) (algCQG).

Let \(\Gamma\) be a discrete group, and let \(C^c(\Gamma)\) and \(C^*(\Gamma)\) be its reduced and full group C*-algebras. \(\Gamma\) is called amenable if there exists an invariant mean on \(L^\infty(\Gamma)\). Endowed with co-multiplications \(\Delta_\Gamma\) and \(\Delta_r\), \((C^c(\Gamma), \Delta_\Gamma)\) and \((C^*(\Gamma), \Delta_r)\) come into being CQGs, which are called reduced and universal CQG, respectively. The Haar integral of \((C^c(\Gamma), \Delta_\Gamma)\) is faithful, but that of \((C^*(\Gamma), \Delta_r)\) may not be; the co-unit of \((C^*(\Gamma), \Delta_r)\) is norm-bounded, but that of \((C^c(\Gamma), \Delta_\Gamma)\) may not be. From [3], the Haar integral of \((C^*(\Gamma), \Delta_r)\) is faithful if and only if the co-unit of \((C^*(\Gamma), \Delta_r)\) is norm-bounded if and only if \(\Gamma\) is amenable. Under what conditions is the Haar integral on a CQG faithful and the co-unit norm-bounded? In [3], Bédos, Murphy, and Tuset defined the co-amenability of CQG, which can induce the faithfulness of its Haar integral and the norm-boundedness of its co-unit. As the quantum dual of group amenability, \((C^c(\Gamma), \Delta_\Gamma)\) is co-amenable if and only if \(\Gamma\) is amenable. Denote \(C[\Gamma]\) the group algebra of \(\Gamma\) equipped with its canonical Hopf *-algebra structure. By [3], \(C^c(\Gamma)\) and \(C^*(\Gamma)\) are the CQG completions of \(C[\Gamma]\). Under what conditions, for an arbitrary algCQG \((A_0, \Delta_0)\), is the CQG completion of \((A_0, \Delta_0)\) unique? Generally, it is not unique. However, in the co-amenable case, the answer is affirmative [3]. Moreover, in [4,5], Bédos, Murphy, and Tuset studied the amenability and co-amenability of algebraic quantum groups, a sufficient large quantum group class including CQGs and discrete quantum groups (DQGs), which admits a dual that is also an algebraic quantum group.

In the group case, a product of two discrete amenable groups is amenable; as a quantum counterpart, co-amenability is preserved under formulation of the tensor product of two CQGs [3]. In [6], we constructed the reduced and universal quantum double of two dually paired CQGs. Since the tensor product of two CQGs is a special case of quantum double of CQGs when the pairing is trivial, inspired by the underlying stability of co-amenability of CQGs and the symmetrical idea, in the article, we will focus on studying the stability of the co-amenability in the process of quantum double constructions. In Section 2, we first recall the definition of co-amenability of compact quantum groups, as well as some related properties, and then briefly present the quantum double construction procedure. By symmetric calculations, as used in the case of the group amenability, in Section 3, we show that the quantum double of CQGs is unique when the paired CQGs are both co-amenable and that co-amenability is preserved under formulation of the quantum double constructions of CQGs. Using this result, one can yield a co-amenable new CQG from a pair of co-amenable CQGs.

In the article, all algebras are considered over the complex field \(\mathbb{C}\). For the details on CQGs and C*-norms, we refer to [6–13]; and for the general conclusions for pairing and quantum double, we refer to [2,6,14–17]. In our proofs, we make use of a large quantity of calculations by the standard Sweedler notation.

2. Preliminaries

In this section, we first recall the definition of co-amenability of CQGs and some of its properties.

Let \((A, \Delta)\) be a CQG, \((A_0, \Delta_0)\) be the associated algCQG of \((A, \Delta)\), and \(h\) the Haar integral of \((A_0, \Delta_0)\). As is well known, \(h\) is faithful on \((A_0, \Delta_0)\) but need not be faithful on the C*-algebra \((A, \Delta)\). Set

\[
\Delta_r = A / N_h,
\]

where \(N_h\) is the left kernel of \(h\). Then, \(\Delta_r\) becomes a CQG, where its co-multiplication \(\Delta_r\) is defined as

\[
\Delta_r(\eta(a)) = (\eta \otimes \eta)\Delta(a),
\]
for all $a \in A$, where $\eta : A \rightarrow A_r$ is the canonical map. $(A_r, \Delta_r)$ is called the reduced quantum group of $(A, \Delta)$, where its co-unit $\varepsilon_r$, antipode $S_r$, and Haar state $\eta_r$ are determined by

$$ \varepsilon = \varepsilon_r \circ \eta, \quad \eta \circ S = S_r \circ \eta, \quad h = h_r \circ \eta, $$

respectively. What needs to be pointed out is that the co-unit $\varepsilon_r$ of $(A_r, \Delta_r)$ is faithful. However, generally, the co-unit $\varepsilon_r$ needs not be norm-bounded.

**Definition 2** ([3]). A CQG $(A, \Delta)$ is called co-amenable if the co-unit $\varepsilon_r$ of $(A_r, \Delta_r)$ is norm-bounded, where $(A_r, \Delta_r)$ is the reduced quantum group of $(A, \Delta)$.

With the following proposition, one can obtain the co-amenability of $(A, \Delta)$ without reference to the reduced quantum group $(A_r, \Delta_r)$.

**Proposition 1** ([3]). Let $(A, \Delta)$ be a CQG, and $h$ and $\varepsilon$ be its Haar integral and co-unit, respectively. Then, $(A, \Delta)$ is co-amenable if and only if $h$ is faithful and $\varepsilon$ is norm-bounded.

Assume that $(A, \Delta)$ and $(A_0, \Delta_0)$ are described as above. Let $\| \cdot \|_c$ be a C*-norm on $(A_0, \Delta_0)$, and let $(A_c, \Delta_c)$ be a compact quantum group completion of $(A_0, \Delta_0)$. $\| \cdot \|_c$ is called regular on $A_0$, if it is the restriction to $A_0$ of the C*-norm on $(A_c, \Delta_c)$. Define $\| \cdot \|_u$ on $A_0$ as

$$ \|a\|_u := \sup \|\pi(a)\|, $$

where the variable $\pi$ travels over all unital *-representations $\pi$ of $A_0$. It is not difficult to find that $\| \cdot \|_u$ is the greatest regular C*-norm on $A_0$. Denote $A_u$ as the C*-algebra completion of $A_0$ with respect to $\| \cdot \|_u$ and $\Delta_u$ the extension to $A_u$ of $\Delta$. Then, $(A_u, \Delta_u)$ is a CQG, which is called the universal quantum group of $(A, \Delta)$. Define $\| \cdot \|_r$ on $A_0$ as

$$ \|a\|_r := \|\eta(a)\|, $$

for all $a \in A_0$, which is the least regular C*-norm on $A_0$. Then, the underlying $A_r$ is the C*-algebraic completion of $A_0$ with respect to $\| \cdot \|_r$.

**Proposition 2** ([3]). Let $(A, \Delta)$ be a CQG, $(A_0, \Delta_0)$ be the associated algCQG of $(A, \Delta)$, and $\| \cdot \|_c$ a regular C*-norm on $A_0$. Then,

(i) For all $a \in A_0$,

$$ \|a\|_r \leq \|a\|_c \leq \|a\|_u. $$

(ii) $(A, \Delta)$ is co-amenable if and only if

$$ (A, \Delta) = (A_u, \Delta_u) = (A_r, \Delta_r). $$

Now, we recall the procedure of quantum double construction for CQGs simply exhibited in [11].

**Definition 3.** Let $(A, \Delta_A)$ and $(B, \Delta_B)$ be two dully paired CQGs, and let $(A_0, \Delta_{A_0})$ and $(B_0, \Delta_{B_0})$ be the associated algCQGs.

(i) Let $A_0$ and $B_0$ be two algCQGs, and $\langle \cdot, \cdot \rangle : A_0 \otimes B_0 \rightarrow \mathbb{C}$ be a bilinear form. Assume that they satisfy the relations

$$ \langle \Delta(a), b_1 \otimes b_2 \rangle = \langle a, b_1 b_2 \rangle, \quad \langle a_1 \otimes a_2, \Delta(b) \rangle = \langle a_1 a_2, b \rangle, \quad \langle a^*, b \rangle = \overline{\langle a, S_{B_0}(b)^* \rangle}, $$

$$ \langle a, 1_{B_0} \rangle = \varepsilon_{A_0}(a), \quad \langle 1_{A_0}, b \rangle = \varepsilon_{B_0}(b), \quad \langle S_{A_0}(a), b \rangle = \langle a, S_{B_0}(b) \rangle,$$
for all \(a_1, a_2, a \in A_0, b_1, b_2, b \in B_0\), where \(\varepsilon_{A_0}, \varepsilon_{B_0}\) denote the co-unit and antipode on \(A_0\) (resp. \(B_0\)), respectively. Then, \((A_0, B_0, \langle \cdot, \cdot \rangle)\) is called an algebraic compact quantum group pairing.

(2) Let \(\langle \cdot, \cdot \rangle : A \otimes B \rightarrow \mathbb{C}\) be a bilinear form. If \((A_0, B_0, \langle \cdot, \cdot \rangle)\) is an algebraic compact quantum group pairing, then the bilinear form is called a compact quantum group pairing, denoted by \((A, B, \langle \cdot, \cdot \rangle)\).

Let \((A, \Delta_A)\) and \((B, \Delta_B)\) be two dually paired CQGs, and let \((A_0, \Delta_{A_0})\) and \((B_0, \Delta_{B_0})\) be described as above. Denote by \(A_0 \otimes B_0\). It is well known that \(A_0 \otimes B_0\), the algebraic tensor product of \(A_0\) and \(B_0\), can be made into a linear space in a natural way. Under the multiplication map, \(m_D\) and involution \(*_D\) on \(A_0 \otimes B_0\) defined as the following:

\[
m_D((a, b)(a', b')) := \sum_{(a')(b)} (aa'_{(2)}, b_{(2)})(a'_{(1)}, S_{B_0}(b_{(3)}))(a'_{(3)}, b_{(1)}),
\]

\[
*_D(a, b) := \sum_{(a')(b)} (a^*_{(2)}, b^*_{(2)})(a^*_{(3)}, b_{(1)})(a^*_{(1)}, S^*_{B_0}b_{(3)}) \triangleq (a, b)^*,
\]

where \((a, b), (a', b') \in A_0 \otimes B_0\), \(A_0 \otimes B_0\) turn into a non-degenerate associative \(*\) algebra, which is similar to the classical Drinfeld’s quantum double \([18]\) in the pure algebra level, and then we denote it by \(D(A_0, B_0)\). To avoid using too many brackets, we will simplify \(m_D((a, b)(a', b'))\) as \((a, b)(a', b')\) and simplify \(S(a)\) as \(Sa\) in sequel.

Under the structure maps,

\[
\Delta_{D_0}(a, b) := \sum_{(a)(b)} (a_{(1)}, b_{(1)}) \otimes (a_{(2)}, b_{(2)}), \quad \varepsilon_{D_0}(a, b) := \varepsilon_{A_0}(a)\varepsilon_{B_0}(b),
\]

\[
S_{D_0}(a, b) := \sum_{(a)(b)} (S_{A_0}a_{(2)}, S_{B_0}b_{(2)}) < a_{(1)}, S_{B_0}b_{(3)} > < a_{(3)}, b_{(1)} > .
\]

\(D(A_0, B_0)\) forms a Hopf \(*\)-algebra. Furthermore, we have:

**Proposition 3.** (\(D(A_0, B_0), \Delta_{D_0}\)) is an \(\text{algCQG}\).

Define \(D_u(A, B) := \frac{D(A_0, B_0)}{\| \cdot \|_u}\), where for any \((a, b) \in D(A_0, B_0)\),

\[
\| (a, b) \|_u := \sup_{\pi} \| (a, b) \|.
\]

By Theorem 5.4.3 in \([19]\), \((D_u(A, B), \Delta_{D_u})\) is the universal compact quantum group of \(D(A_0, B_0)\), where \(\Delta_{D_u}\) is the extension to \(D_u(A, B)\) of \(\Delta_D\). Let \(h_{D_u}\) be the Haar state on \(D_u(A, B)\) and \((H, \Lambda, \pi)\) be the GNS- representation of \((D_u(A, B), \Delta_{D_u})\) for the Haar integral \(h_{D_u}\). Define

\[
D_r(A, B) := \pi(D_u(A, B)).
\]

Denote \(\Delta_{D_r}\) the extension to \(D_r(A, B)\) of \(\Delta_{D_u}\). Then, \((D_r(A, B), \Delta_{D_r})\) is the reduced quantum group of \(D(A_0, B_0)\), and its Haar integral \(h_{D_r}\) is faithful naturally.

**Proposition 4.** \((D_u(A, B), \Delta_{D_u})\) and \((D_r(A, B), \Delta_{D_r})\) are both CQGs.

**Definition 4.** \((D_u(A, B), \Delta_{D_u})\) and \((D_r(A, B), \Delta_{D_r})\) are called the universal and reduced quantum double of \(A\) and \(B\), respectively.
3. The Main Results

**Theorem 1.** Let \((A, B, \langle \cdot, \cdot \rangle)\) be a non-degenerate compact quantum group pairing. If \((A, \Delta_A)\) and \((B, \Delta_B)\) are two co-amenable CQGs, then \((D_u(A, B), \Delta_{Du}) = (D_r(A, B), \Delta_{Dr})\).

**Proof.** Suppose that \((A_0, \Delta_{A_0})\) and \((B_0, \Delta_{B_0})\) are the associated \(alg\)CQGs, respectively. Let \(\| \cdot \|_c\) be a regular C*-norm and \(A_c\) be the CQG completion of \((A_0, \Delta_{A_0})\). As described in Section 2, \(A_u\) and \(A_r\) are both CQG completions of \((A_0, \Delta_{A_0})\). Because \(A\) is co-amenable, by Proposition 2 (ii), there is a unique CQG completion for the associated \(alg\)CQGs \((A_0, \Delta_{A_0})\). Hence,

\[ A_r = A_c = A_u. \]

Analogously,

\[ B_r = B_c = B_u. \]

By Proposition 2 (i),

\[ \|a\|_r \leq \|a\|_c \leq \|a\|_u, \quad \|b\|_r \leq \|b\|_c \leq \|b\|_u, \]

for all \(a \in A_0\) and \(b \in B_0\). Combining with the equations \(A_r = A_u\) and \(B_r = B_u\), one can symmetrically obtain that

\[ \|a\|_r = \|a\|_u, \quad \|b\|_r = \|b\|_u. \]

So,

\[ \| \cdot \|_r = \| \cdot \|_c = \| \cdot \|_u \] \tag{1}

on \(A_0\) and \(B_0\). Moreover, Equation (1) also holds on \(A_0 \otimes B_0\). In fact, for any C*-norm \(\| \cdot \|\) on \(A_0 \otimes B_0\), we have

\[ \|a \otimes b\| = \|a\|\|b\|, \]

for all \(a \in A_0, b \in B_0\). Then,

\[ \|a \otimes b\|_u = \|a\|_u\|b\|_u = \|a\|_r\|b\|_r = \|a \otimes b\|_r. \]

From Proposition 2 (i),

\[ \|a \otimes b\|_r = \|a \otimes b\|_c = \|a \otimes b\|_u \]

for all \(a \otimes b \in A_0 \otimes B_0\).

Considering the multiplication rule on the quantum double \(D(A_0, B_0)\) (6), for any \((a, b) \in D(A_0, B_0)\),

\[ (a, b) = \sum_{(a)(b)} \langle a_{(1)}, S_{B_0}b_{(1)} \rangle \langle a_{(3)}, b_{(3)} \rangle a_{(2)} \otimes b_{(2)}. \] \tag{2}

From the above expression Equation (2), one can find that each element \((a, b)\) in \(D(A_0, B_0)\) is a linear combination of elements as \(c \otimes d \in A_0 \otimes B_0\). By the discussion in the underlying paragraph, we have

\[ \|a_{(2)} \otimes b_{(2)}\|_u = \|a_{(2)} \otimes b_{(2)}\|_r, \]

where \(a_{(2)}\) and \(b_{(2)}\) are as presented in Equation (2), which induces that

\[ \|(a, b)\|_u = \|(a, b)\|_r, \]

i.e., Equation (1) holds on \(D(A_0, B_0)\). Hence, \(D(A_0, B_0)\) has a unique CQG completion. Therefore, \((D_u(A, B), \Delta_{Du})\) coincides with \((D_r(A, B), \Delta_{Dr})\), i.e.,

\[ (D_u(A, B), \Delta_{Du}) = (D_r(A, B), \Delta_{Dr}). \]
In sequel, \((D_u(A, B), \Delta_{D_u})\) and \((D_r(A, B), \Delta_{D_r})\) will be denoted by \((D(A, B), \Delta_D)\).

**Theorem 2.** Let \((D(A, B), \Delta_D)\) be the quantum double of \((A, \Delta_A)\) and \((B, \Delta_B)\) based on a non-degenerate compact quantum group pairing \((A, B, \langle \cdot , \cdot \rangle )\). Assume that \((A, \Delta_A)\) and \((B, \Delta_B)\) are both co-amenable. Then, \((D(A, B), \Delta_D)\) is co-amenable.

**Proof.** By Proposition 1, we have to prove that the following two conditions hold.

(i) The Haar integral of \(D(A, B)\) is faithful.

Above all, we show that there exists a Haar integral \(h_{D_0}\) on it. For all \((a, b) \in D(A_0, B_0)\), we define

\[
h_{D_0}(a, b) := h_{A_0}(a) h_{B_0}(b).
\]

Denote \(bb^*\) by \(k\); then, we can obtain that

\[
h_{D_0}((a, b)(a, b)^*) = h_{D_0}((a, k)(a^*, 1_{B_0})) = \sum_{(a)(k)} h((aa^*_2, c_2))(a^*_1, S_{B_0}k_3)(a^*_2, k_{11}) = \sum_{(a)(k)} h_{A_0}(aa^*_2)h_{B_0}(c_2)(a^*_1, S_{B_0}k_3)(a^*_2, k_{11}) = \sum_{(a)(k)} h_{A_0}(aa^*_2)(a^*_1, S_{B_0}k_3)(a^*_3, h_{B_0}(k_{22})k_{11}) = \sum_{(a)(k)} h_{A_0}(aa^*_2)(a^*_1, S_{B_0}k_2)(a^*_3, 1_{B_0})h_{B_0}(k_{11}) = \sum_{(a)(k)} h_{A_0}(aa^*_2)(a^*_1, S_{B_0}k_2)(h_{B_0}(k_{11}).
\]

Considering \(h_{B_0} \circ S_{B_0} = h_{B_0}\), we have

\[
h_{D_0}((a, b)(a, b)^*) = \sum_{(a)(k)} h_{A_0}(aa^*_2)(a^*_1, h_{B_0} \circ S_{B_0}k_1)S_{B_0}k_2) = \sum_{(a)(k)} h_{A_0}(aa^*_2)(a^*_1, 1_{B_0})h_{B_0}(k) = \sum_{(a)(k)} h_{A_0}(aa^*_2)c_{A_0}(a^*_1)h_{B_0}(k) = h_{A_0}(aa^*)h_{B_0}(bb^*) \geq 0.
\]

Again, for all \((c, d) \in D(A_0, B_0)\), one can get

\[
h_{D_0}((a, b)(c, d)^*) = h_{D_0}((a, b)(a, b \otimes c, d^*)^*) = h_{D_0}((a, b)(a, b \otimes c, d^*)^*) = \sum_{(a)(b)(c)(d)} \langle a_1 \rangle S_{B_0}b_1 \langle a_3 \rangle b_3 \langle c_1 \rangle d_1 \langle c_2 \rangle d_2 \rangle h_{D_0}(a_2 \otimes b_2^*).h_{D_0}(c_2^*).d_2^* = \sum_{(a)(b)(c)(d)} \langle a_1 \rangle S_{B_0}b_1 \langle a_3 \rangle b_3 \langle c_1 \rangle d_1 \langle c_2 \rangle d_2 \rangle h_{D_0}(a_2 \otimes b_2^*).h_{D_0}(c_2^*).d_2^*.
\]
which implies that

\[ h_{D_0}((a, b)(c, d)^*) + (a, b)^*(c, d) \geq 0. \]

Therefore, \( h_{D_0} \) is positive on \( D(A_0, B_0) \). From the underlying formula, \( h_{D_0}((a, b)(a, b)^*) = 0 \) if and only if \( (a, b) = 0 \). Thus, \( h_{D_0} \) is a positive faithful linear functional on \( D(A_0, B_0) \). Considering the invariance of \( h_{A_0} \) and \( h_{B_0} \), we can get

\[ (t \otimes h_{D_0}) \Delta_{D_0}((a, b)) = (h_{D_0} \otimes t) \Delta_{D_0}((a, b)) = h_{D_0}((a, b))1, \]

for all \( (a, b) \in D(A_0, B_0) \).

Define \( h_D \) is the extension to \( (D(A, B), \Delta_D) \) of \( h_{D_0} \). It is easy to see that \( h_D \) is a Haar state on \( (D(A, B), \Delta_D) \) by the fact \( h_{D_0} \) is a Haar integral on \( D(A_0, B_0) \). Denote by \( h_A \) and \( h_B \) the Haar integrals on \( A \) and \( B \), respectively. Then, one can get that

\[ h_D = h_A \otimes h_B. \]

To prove \( h_D \) is faithful, it suffices to show that the Haar integral \( h_{D_0} \) of \( D_u(A, B) \) is faithful, since the Haar integral of \( D_1(A, B) \) is always faithful. Moreover, we just need to check the faithfulness of \( h_{D_u} \) on \( D_u(A, B) \setminus D(A_0, B_0) \).

Let \((a', b') \in D_u(A, B) \setminus D(A_0, B_0)\). From the definitions of \( D_u(A, B) \) and \( D(A_0, B_0) \), we have that

\[ (a', b') = \lim_{a \to a}(a, b)_a, \]

\[ (a, b) = \sum_{(a)(b)} (a_1, S_{B_0}b_{(1)})(a_3, b_{(3)})(a_2, b_{(2)}), \]

where \((a, b) \in D(A_0, B_0)\), "a"s are in some index set, and the limit is taken with respect to the universal C*-norm \( \| \cdot \|_u \) on \( D(A_0, B_0) \). Thus, \((a', b')\) can be rewritten as the following:

\[ (a', b') = \sum_{(a')(b')} (a'_{(1)}, S_{B_0}b'_{(1)})(a'_{(3)}, b'_{(3)})(a'_{(2)}, b'_{(2)}), \]

where \(a'_{(2)}\) is in \( A_u \setminus A_0 \) or \( b'_{(2)}\) is in \( B_u \setminus B_0 \). If \( h_{D_u}((a, b)(a, b)^*) = 0 \), then

\[ h_{D_u}((a', b')(a', b')^*) = h_{D_u}((\sum_{(a')(b')} (a'_{(1)}, S_{B_0}b'_{(1)})(a'_{(3)}, b'_{(3)})(a'_{(2)}, b'_{(2)})) (\sum_{(a')(b')} (a'_{(1)}, S_{B_0}b'_{(1)})(a'_{(3)}, b'_{(3)})(a'_{(2)}, b'_{(2)}))^*) \]

\[ = h_{D_u}((\sum_{(a')(b')} (a'_{(1)}, S_{B_0}b'_{(1)})(a'_{(3)}, b'_{(3)})(a'_{(2)}, b'_{(2)})) (\sum_{(a')(b')} (a'_{(1)}, S_{B_0}b'_{(1)})(a'_{(3)}, b'_{(3)})(a'_{(2)}, b'_{(2)}))^*) \]

\[ = \sum_{(a')(b')} (a'_{(1)}, S_{B_0}b'_{(1)})(a'_{(3)}, b'_{(3)})(a'_{(1)}, S_{B_0}b'_{(1)})(a'_{(3)}, b'_{(3)})(a'_{(2)}, b'_{(2)})(a'_{(2)}, b'_{(2)}) h_{D_u}(a'_{(2)}, b'_{(2)}) h_{D_u}(a'_{(2)}, b'_{(2)}) \]

\[ = 0. \]
Because \((A, \Delta_A)\) and \((B, \Delta_B)\) are both co-amenable, by Proposition 1, \(h_A\) and \(h_B\) are both faithful. Hence, \(h_A\) and \(h_B\) are also faithful. Combining with the underlying equation, we obtain that \(a'_{(2)} = 0\) and \(b'_{(2)} = 0\); thus, by (5), we get

\[
(a', b') = 0,
\]

which states that \(h_{D_B}\) is faithful on \(D_B(A, B)\).

(ii) The co-unit of \(D(A, B)\) is norm-bounded.

First, we show that \(\varepsilon_{D_0}\) defined as before Proposition 3 is a \(*\)-homomorphism. Using the definition of \(\varepsilon_{D_0}\), we have

\[
e_{D_0}(\langle a, b \rangle^*) = \varepsilon_{D_0}(\langle 1_{A_0}, b^* \rangle (a^*, 1_{B_0})) = \varepsilon_{D_0}(1_{A_0} b^*) \varepsilon_{D_0}(a^*, 1_{B_0}) = \varepsilon_{B_0}(b^*) \varepsilon_{A_0}(a^*) = (\varepsilon_{A_0}(a) \varepsilon_{B_0}(b))^* = (\varepsilon_{D_0}(a, b))^*.
\]

Let \(\varepsilon_A\) and \(\varepsilon_B\) be the co-units on \(A\) and \(B\), respectively. For all \((a, b) \in D(A, B)\), we define

\[
e_D(a, b) := \varepsilon_A(a) \varepsilon_B(b),
\]

i.e.,

\[
e_D = \varepsilon_A \otimes \varepsilon_B,
\]

which can be regarded as the extension to \((D(A, B), \Delta_D)\) of \(\varepsilon_{D_0}\).

Considering the continuity of extension of \(\varepsilon_{D_0}\) from \(D(A_0, B_0)\) to \(D(A, B)\), \(\varepsilon_D\) is a \(*\)-homomorphism and then the co-unit on \(D(A, B)\).

To prove that the co-unit \(\varepsilon_D\) on \(D(A, B)\) is norm-bounded, it suffices to show that the Haar integral \(\varepsilon_D\) of \(D_r(A, B)\) is norm-bounded with respect to the supremum norm, since the co-unit of \(D_B(A, B)\) is always norm-bounded. Moreover, we just need to check the norm-bounded-ness of \(\varepsilon_{D_0}\) on \(D_r(A, B) \setminus D(A_0, B_0)\). Let \((a', b') \in D_r(A, B) \setminus D(A_0, B_0)\). By a similar discussion, in Equations (3)–(5), we have

\[
(a', b') = \sum_{(a')(b')} \langle a'_{(1)}, S_{B_0} b'_{(1)}, a'_{(3)}, b'_{(3)} \rangle (a'_{(2)} \otimes b'_{(2)}),
\]

where \(a'_{(2)}\) is in \(A_r \setminus A_0\) or \(b'_{(2)}\) is in \(B_r \setminus B_0\). Since \(A\) and \(B\) are co-amenable, by Proposition 1, \(\varepsilon_A\) and \(\varepsilon_B\) are both norm-bounded. Hence, \(\varepsilon_A\) and \(\varepsilon_B\) are norm-bounded, i.e., there exist two positive number \(M_A\) and \(M_B\) such that

\[
\|\varepsilon_A\| = \sup_{\|a'\| = 1} |\varepsilon_A(a')| \leq M_A,
\]

and

\[
\|\varepsilon_B\| = \sup_{\|b'\| = 1} |\varepsilon_B(b')| \leq M_B.
\]

Thus,

\[
\|\varepsilon_{D_0}\| = \sup_{\|a', b'\| = 1} |\varepsilon_{D_0}(a', b')| = \sup_{\|a', b'\| = 1} \| \sum_{(a')(b')} \langle a'_{(1)}, S_{B_0} b'_{(1)}, a'_{(3)}, b'_{(3)} \rangle (a'_{(2)} \otimes b'_{(2)}) \| = \sup_{\|a', b'\| = 1} \| \sum_{(a')(b')} \langle a'_{(1)}, S_{B_0} b'_{(1)}, a'_{(3)}, b'_{(3)} \rangle \| \leq \sup_{\|a', b'\| = 1} \sum_{(a')(b')} \| \langle a'_{(1)}, S_{B_0} b'_{(1)}, a'_{(3)}, b'_{(3)} \rangle \| \leq K M_A M_B.
\]
where $K$ represents the supremum of $\{ \sum_{(a')} | \langle a'_1 S b'_1 \rangle \langle a'_3 b'_3 \rangle | \| (a', b') \|_r = 1 \}$ and is a finite positive real number, which states that $\varepsilon_D$ is norm-bounded. \qed

**Remark 1.** Consider the trivial case where $A = B = C(S^1)$, the $C^*$-algebra of continuous functions on the circle group $S^1$. Clearly, $D(A, B) = C(S^1 \times S^1) = C(T)$, where $T$ represents the 2-torus. It is easy to know that in this case $A$, $B$ and $D(A, B)$ are all co-amenable CQGs for their commutativity. In fact, we can also get the co-amenability of $D(A, B)$ by Theorem 2. The Haar integral $h_D$ on $C(T)$ is the integral with respect to the Haar measure $\mu$ on $T$. For all $f \in C(T)$, $t = (g_1, g_2) \in S^1 \times S^1 = T$, we have

$$h_D(f)(t) = \int_T f(t) d\mu(t) = \int_{S^1 \times S^1} f(g_1, g_2) d\mu_1(g_1) d\mu_2(g_2) = \int_{S^1} f(g_1) d\mu_1(g_1) \int_{S^1} f(g_2) d\mu_2(g_2) = h_A(f_1)(g_1) h_B(f_2)(g_2),$$

where $f_1, \mu_1$ and $f_2, \mu_2$ are the restrictions of $f$ and $\mu$ on $A$ and $B$, respectively. From the formula, since $h_A$ and $h_B$ are both faithful, $h_D$ is also faithful.

The co-unit $\varepsilon_D$ on $C(T)$ is the evaluation map on the unit of $T$, i.e., for all $f \in C(T)$,

$$\varepsilon_D(f) = f(e) = f(e_1, e_2),$$

where $e_1$ and $e$ are the units of $S^1$ and $T$, respectively. Thus, we have

$$\| \varepsilon_D \| = \sup_{\| f \| = 1} | \varepsilon_D(f) | = \sup_{\| f \| = 1} | f(e_1, e_2) | = \sup_{\| f \| = 1} | (f_1 \otimes f_2)(e_1, e_2) | = \sup_{\| (f_1, f_2) \| = 1} | f_1(e_1) | | f_2(e_2) | \leq 1.$$

By the formula, we have $\varepsilon_D$ is norm-bounded.

4. Conclusions

Based on the research for quantum double construction arising from co-amenable compact quantum groups and the $C^*$-norms on quantum groups, in the article, using the $C^*$-norm inequality and norm-bounded-ness of the co-unit on algebraic compact quantum groups, we prove that co-amenability is preserved under formulation of the quantum double construction of compact quantum groups.

The result not only presents the stability of the co-amenability of quantum groups in the quantum double construction process but also exhibits the nice quantum symmetry between the co-amenability of quantum groups and the amenability of group.

**Author Contributions:** All authors participated in the conceptualization, validation, formal analysis, and investigation, as well as the writing of the original draft preparation, reviewing, and editing. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work research was funded by the National Natural Science Foundation of China (Grant No. 11301380), the Natural Science Foundation of Tianjin (Grant No. 18JCYBJC18900) and the Higher School Science and Technology Development Fund Project in Tianjin (Grant No. 20131003).

**Acknowledgments:** The authors are grateful to the reviewers and to the editors for their valuable comments and suggestions which helped us improve the paper significantly.

**Conflicts of Interest:** The authors declare no conflict of interest.
Abbreviations

The following abbreviations are used in this manuscript:

- CQG: Compact Quantum Group
- CQGs: Compact Quantum Groups
- algCQG: algebraic Compact Quantum Group
- DQG: Discrete Quantum Group

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