HOLOMORPHIC ANOMALY EQUATIONS FOR $[\mathbb{C}^5/\mathbb{Z}_5]$  
DENIZ GENLIK AND HSIAN-HUA TSENG

ABSTRACT. We prove holomorphic anomaly equations for $[\mathbb{C}^5/\mathbb{Z}_5]$ based on the work of Lho [8].

To the memory of Bumsig Kim

CONTENTS

0. Introduction 1
0.1. Acknowledgement 3
1. On mirror theorem 3
2. Frobenius Structures 5
3. Lift of modified R-matrix 7
3.1. Canonical Lift 7
3.2. Preparations 8
4. Holomorphic anomaly equations 9
4.1. Formula for potentials 9
4.2. Proof of holomorphic anomaly equations 11
References 15

Date: November 30, 2022.

0. INTRODUCTION

The cyclic group $\mathbb{Z}_5$ acts naturally on $\mathbb{C}^5$ by letting its generator $1 \in \mathbb{Z}_5$ act by multiplication by the fifth root of unity

$$e^{2\pi\sqrt{-1}/5}.$$  

This action commutes with the diagonal action of the torus $T = (\mathbb{C}^*)^5$ on $\mathbb{C}^5$ and induces a $T$-action on $[\mathbb{C}^5/\mathbb{Z}_5]$. Consequently $[\mathbb{C}^5/\mathbb{Z}_5]$ is a toric Deligne-Mumford stack.

This paper is concerned with $T$-equivariant Gromov-Witten invariants of $[\mathbb{C}^5/\mathbb{Z}_5]$. By definition, these are the following integrals

$$(0.1) \quad \int_{\overline{M}_{g,n}^{\text{orb}}([\mathbb{C}^5/\mathbb{Z}_5],0)^{\text{vir}}} \prod_{k=1}^n \text{ev}_i^* (\gamma_k) \psi_i^{k_i}.$$  

Here, $[\overline{M}_{g,n}^{\text{orb}}([\mathbb{C}^5/\mathbb{Z}_5],0)^{\text{vir}}]$ is the (T-equivariant) virtual fundamental class of the moduli space $\overline{M}_{g,n}^{\text{orb}}([\mathbb{C}^5/\mathbb{Z}_5],0)$ of stable maps to $[\mathbb{C}^5/\mathbb{Z}_5]$. The classes $\psi_i \in H^2(\overline{M}_{g,n}^{\text{orb}}([\mathbb{C}^5/\mathbb{Z}_5],0), \mathbb{Q})$ are descendant classes. The evaluation maps

$$\text{ev}_i : \overline{M}_{g,n}^{\text{orb}}([\mathbb{C}^5/\mathbb{Z}_5],0) \to I[\mathbb{C}^5/\mathbb{Z}_5]$$

are given by

$$\text{ev}_i(M) = \left\{ \begin{array}{ll} 
1 & \text{if } M \text{ maps } i \text{ times to a point, and the hours are compatible with Lho’s setup,} \\
0 & \text{otherwise.} 
\end{array} \right.$$
take values in the inertia stack $I[\mathbb{C}^5/\mathbb{Z}_5]$ of $[\mathbb{C}^5/\mathbb{Z}_5]$. The classes $\gamma_i \in H^*_{\text{orb}}([\mathbb{C}^5/\mathbb{Z}_5]) := H^*(I[\mathbb{C}^5/\mathbb{Z}_5])$ are classes in the Chen-Ruan cohomology of $[\mathbb{C}^5/\mathbb{Z}_5]$.

Let
\[
\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in H^*_\text{pt}(BT) = H^*(BT)
\]
be the first Chern classes of the tautological line bundles of $BT = (BC^*)^5$. Then (0.1) takes value in $\mathbb{Q}(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Foundational treatments of orbifold Gromov-Witten theory can be found in many references. For compact target stacks, the original reference is [1]. For non-compact target stacks admitting torus actions, such as $[\mathbb{C}^5/\mathbb{Z}_5]$, one can define Gromov-Witten theory for them using virtual localization formula [5]. In this case, their Gromov-Witten theory should be understood as certain twisted Gromov-Witten theory of stacks. Generalities on twisted Gromov-Witten theory of stacks can be found in [2] and [15].

The main results of this paper concern structures of Gromov-Witten invariants (0.1), formulated in terms of generating functions. The definition of inertia stacks implies that
\[
I[\mathbb{C}^5/\mathbb{Z}_5] = \mathbb{C}^5/\mathbb{Z}_5 \cup \bigcup_{k=1}^4 B\mathbb{Z}_5.
\]

Let
\[
\phi_0 = 1 \in H^0_T([\mathbb{C}^5/\mathbb{Z}_5]), \phi_k = 1 \in H^0_T(B\mathbb{Z}_5), 1 \leq k \leq 4.
\]
Then $\{\phi_0, \phi_1, \phi_2, \phi_3, \phi_4\}$ is an additive basis of $H^*_T, \text{orb}([\mathbb{C}^5/\mathbb{Z}_5])$. The orbifold Poincaré dual $\{\phi^0, \phi^1, \phi^2, \phi^3, \phi^4\}$ of this basis is given by
\[
\phi^0 = 5\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\phi_0, \quad \phi^1 = 5\phi_4, \quad \phi^2 = 5\phi_3, \quad \phi^3 = 5\phi_2, \quad \phi^4 = 5\phi_1.
\]

To simplify notation, in what follows we set
\[
\phi_i := \phi_{j} \quad \text{if } j \equiv i \mod 5 \quad \text{and} \quad \phi^i := \phi^{j} \quad \text{if } j \equiv i \mod 5,
\]
for all $i \geq 0$ and $0 \leq j \leq 4$.

For $\phi_{c_1}, \ldots, \phi_{c_n} \in H^*_\text{orb}([\mathbb{C}^5/\mathbb{Z}_5])$, define the Gromov-Witten potential by
\[
\mathcal{F}_{g,n}^{[\mathbb{C}^5/\mathbb{Z}_5]}(\phi_{c_1}, \ldots, \phi_{c_n}) = \sum_{d=0}^{\infty} \frac{\Theta^d}{d!} \int_{\mathcal{M}_{g,n+d}([\mathbb{C}^5/\mathbb{Z}_5],0)} \prod_{k=1}^{n} \text{ev}^*_k(\phi_{c_k}) \prod_{i=n+1}^{n+d} \text{ev}^*_i(\phi_1).
\]

In the standard double bracket notation, this is
\[
\langle \langle \phi_{c_1}, \ldots, \phi_{c_n} \rangle \rangle_{g,n}^{[\mathbb{C}^5/\mathbb{Z}_5]} = \mathcal{F}_{g,n}^{[\mathbb{C}^5/\mathbb{Z}_5]}(\phi_{c_1}, \ldots, \phi_{c_n}).
\]

The main results of this paper are differential equations for these generating functions $\mathcal{F}_{g}^{[\mathbb{C}^5/\mathbb{Z}_5]}$ for $g \geq 2$ after the following specializations of equivariant parameters:
\[
\lambda_i = e^{\frac{2\pi \sqrt{-1} i}{5}}, \quad 0 \leq i \leq 4.
\]

There are two differential equations, given precisely in Theorems 4.6 and 4.7 below. Theorem 4.6 is an analogue of a main result of [10]. To the best of our knowledge, Theorem 4.7 does not have analogue in previous studies. Borrowing terminology from String Theory, we call these two differential equations holomorphic anomaly equations for $[\mathbb{C}^5/\mathbb{Z}_5]$.

Our approach to proving holomorphic anomaly equations is the same as that of [10] and is based on the fact that genus 0 Gromov-Witten theory of $[\mathbb{C}^5/\mathbb{Z}_5]$ yields a semisimple Frobenius structure on $H^*_\text{orb}([\mathbb{C}^5/\mathbb{Z}_5])$. Consequently, the cohomological field theory (in the sense of [6]) associated to the Gromov-Witten theory of $[\mathbb{C}^5/\mathbb{Z}_5]$ is semisimple. The Givental-Teleman classification [4],
of semisimple cohomological field theories can then be applied to yield an explicit formula for $\mathcal{F}_9^{[\mathbb{C}^5/\mathbb{Z}_5]}$, which can be used to prove holomorphic anomaly equations.

The rest of the paper is organized as follows. In Section 1, we state the mirror theorem for $[\mathbb{C}^5/\mathbb{Z}_5]$ and study certain power series arising from the $I$-function. In Section 2, we describe necessary ingredients of the Frobenius structure from Gromov-Witten theory of $[\mathbb{C}^5/\mathbb{Z}_5]$. In Section 3, we study lifting to certain ring of functions of an important ingredient called the $R$-matrix. Section 4 contains the main results of this paper. In Section 4.1, we give the formula for Gromov-Witten potentials of $[\mathbb{C}^5/\mathbb{Z}_5]$ arising from Givental-Teleman classification of semisimple CohFTs. In Section 4.2, we state the two holomorphic anomaly equations and use the formula in Section 4.1 to prove them.

0.1. Acknowledgement. D. G. is supported in part by a Special Graduate Assignment fellowship by Ohio State University’s Department of Mathematics and H.-H. T. is supported in part by Simons foundation collaboration grant.

1. On Mirror Theorem

In this Section we discuss mirror theorem for Gromov-Witten theory of $[\mathbb{C}^5/\mathbb{Z}_5]$. The $I$-function of $[\mathbb{C}^5/\mathbb{Z}_5]$ is defined to be

$$I(x, z) = \sum_{k=0}^{\infty} \frac{x^k}{z^k!} \prod_{b \leq k < b+\frac{5}{2}, (b)= \left(\frac{5}{2}\right)} (1 - (bz)^5) \phi_k.$$  \hspace{1cm} (1.1)

It is easy to see that $I$-function (1.1) of $[\mathbb{C}^5/\mathbb{Z}_5]$ is of the form

$$I(x, z) = \sum_{k=0}^{\infty} \frac{I_k(x)}{z^k} \phi_k.$$  \hspace{1cm} (1.2)

The small $J$-function for $[\mathbb{C}^5/\mathbb{Z}_5]$ is defined by

$$J(\Theta, z) = \phi_0 + \frac{\Theta \phi_1}{z} + \sum_{i=0}^{n-1} \phi_i \left( \frac{\phi_i}{z(z - \psi)} \right)_{[\mathbb{C}^n/\mathbb{Z}_n]} [0, 1].$$

The following is a consequence of the main result of [2].

**Proposition 1.1.** We have the following mirror identity,

$$J(T(x), z) = I(x, z),$$

with the mirror transformation

$$T(x) = I_1(x) = \sum_{k \geq 0} \frac{(-1)^5x^{5k+1}}{(5k+1)!} \left( \frac{\Gamma\left(k + \frac{1}{5}\right)}{\Gamma\left(k + \frac{1}{2}\right)} \right)^5.$$  \hspace{1cm} (1.3)

Define the operator

$$D : \mathbb{C}[[x]] \to x\mathbb{C}[[x]]$$

by

$$D f(x) = x \frac{df(x)}{dx}.$$  \hspace{1cm} (1.4)

Here, $(\cdot)$ is the fractional part of $\cdot$. 

Next, we consider the following series in $\mathbb{C}[\lbrack x \rbrack]$ arising from the $I$-function, which will be useful later:

$$
L = x \left(1 + \left(\frac{x}{5}\right)^{5}\right)^{-\frac{1}{5}},
$$

$$
C_1 = DI_1,
$$

(1.5)

$$
C_2 = D \left(\frac{DI_2}{C_1}\right),
$$

$$
C_3 = D \left(\frac{D(DI_3)}{C_2}\right).
$$

It is easy to verify that

$$
\frac{DL}{L} = 1 - \frac{L^5}{5^5}.
$$

(1.6)

In [8, Proposition 4], the following identity is given:

$$
C_1^2 C_2^2 C_3 = L^5.
$$

(1.7)

The following lemma is a direct result of the definition (6.2) of Gromov-Witten potential and the mirror map $\Theta = T(x)$.

**Lemma 1.2.** For $k \geq 1$, we have

$$
\frac{\partial^k \mathcal{F}_{g,n}^{[\mathbb{C}^5/\mathbb{Z}_5]}(\phi_{c_1}, \ldots, \phi_{c_n})}{\partial T^k} = \mathcal{F}_{g,n+k}^{[\mathbb{C}^5/\mathbb{Z}_5]}(\phi_{c_1}, \ldots, \phi_{c_n}, \phi_{1}, \ldots, \phi_{1}).
$$

We further define the following series:

$$
X_1 = \frac{DC_1}{C_1},
$$

$$
X_2 = \frac{DC_2}{C_2},
$$

(1.8)

$$
A_1 = \frac{1}{L} \left(\frac{DL}{L} - X_1\right),
$$

$$
A_2 = \frac{1}{L} \left(2 \frac{DL}{L} - X_1 - X_2\right),
$$

$$
B_i = \frac{1}{5^i} (D + X_1)^{i-1} X_1 \quad \text{for} \quad 1 \leq i \leq 4.
$$

In [8, Section 3], the following equations are given:

$$
B_4 = \left(1 - \frac{L^5}{5^5}\right) \left(2B_3 - \frac{7}{5} B_2 + \frac{2}{5} B_1 - \frac{24}{625}\right),
$$

(1.9)

$$
DX_2 = -10 \left(1 - \frac{L^5}{5^5}\right) + 10 \left(1 - \frac{L^5}{5^5}\right) X_1 + 5 \left(1 - \frac{L^5}{5^5}\right) X_2 - 2X_1^2 - 4DX_1 - 2X_1 X_2 - X_2^2.
$$

(1.10)

---

2Here, our $L$ differs from $L$ defined in [8] by a sign. Although the definitions of $C_1$ and $C_2$ look different from those in [8], it is easy to check that these definitions match with those in [8].
Since there is a linear relation between \( \{ A_1, A_2 \} \) and \( \{ X_1, X_2 \} \) with coefficients from the ring \( \mathbb{C}[L^0] \), we can rewrite these two equations in terms of \( A_i \)'s and their \( D \) differentials. For example, equation (1.10) can be rewritten as

\[
DA_2 = LA_1^2 + LA_2 - DA_1 - 15 \left( 1 - \frac{L_5}{5^5} \right) \frac{L_5}{5^5}.
\]

Moreover, these linear relations show that the differential ring

\[
\mathbb{C}[L^0][A_1, A_2, DA_1, DA_2, D^2 A_1, D^2 A_2, \ldots]
\]

is a quotient of the free polynomial ring

\[
\mathbb{P} := \mathbb{C}[L^0][A_1, DA_1, D^2 A_1, A_2].
\]

2. Frobenius Structures

In this Section, we spell out details of the Frobenius structure on \( H^*_{T,\text{Orb}} ([\mathbb{C}^5/\mathbb{Z}_5]) \) defined using genus 0 Gromov-Witten theory of \([\mathbb{C}^5/\mathbb{Z}_5] \). We refer to [7] for generalities of Frobenius structures.

Let \( \gamma = \sum t_i \phi_i \in H^*_{T,\text{Orb}} ([\mathbb{C}^5/\mathbb{Z}_5]) \). The full genus 0 Gromov-Witten potential is defined to be

\[
\mathcal{F}_0^{[\mathbb{C}^5/\mathbb{Z}_5]}(t, \Theta) = \sum_{n=0}^{\infty} \sum_{d=0}^{\infty} \frac{1}{n! d!} \int_{\overline{\mathcal{M}}_{6, n+d} ([\mathbb{C}^5/\mathbb{Z}_5], 0)} v_r \prod_{i=1}^{n+d} \text{ev}_i^*(\Theta \phi_i).
\]

In the basis \( \{ \phi_0, \phi_1, \phi_2, \phi_3, \phi_4 \} \) and under the specialization (2.3), the orbifold Poincaré pairing

\[
g(-, -) : H^*_{T,\text{Orb}} ([\mathbb{C}^5/\mathbb{Z}_5]) \times H^*_{T,\text{Orb}} ([\mathbb{C}^5/\mathbb{Z}_5]) \to \mathbb{Q}(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)
\]

has the matrix representation

\[
G = \frac{1}{5} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}.
\]

The quantum product \( \cdot \), at \( \gamma \in H^*_{T,\text{Orb}} ([\mathbb{C}^5/\mathbb{Z}_5]) \) is a product structure on \( H^*_{T,\text{Orb}} ([\mathbb{C}^5/\mathbb{Z}_5]) \). It can be defined as follows:

\[
g(\phi_i \cdot \gamma, \phi_j) := \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \mathcal{F}_0^{[\mathbb{C}^5/\mathbb{Z}_5]}(t, \Theta).
\]

In what follows, we focus on the quantum product \( \cdot_{\gamma=0} \) at \( \gamma = 0 \in H^*_{T,\text{Orb}} ([\mathbb{C}^5/\mathbb{Z}_5]) \), which we denote by \( \cdot \). Note that \( \cdot \) still depends on \( \Theta \).

It is proved in [8, Section 5] that the quantum product at \( 0 \in H^*_{T,\text{Orb}} ([\mathbb{C}^5/\mathbb{Z}_5]) \) is semisimple with the idempotent basis \( \{ e_0, e_1, e_2, e_3, e_4 \} \), that is,

\[
e_i \cdot e_j = \delta_{i,j} e_j.
\]

The corresponding normalized idempotent basis \( \{ \bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4 \} \) is given by

\[
\bar{e}_i = 5e_i \quad \text{for} \quad 0 \leq i \leq 4.
\]
In [8] Section 5, the transition matrix given by \( \Psi_{ij} = g(\bar{c}_i, \phi_j) \) is calculated to be

\[
\Psi = \frac{1}{5} \begin{bmatrix}
1 & \frac{1}{C_1} & \frac{1}{C_1^2} & \frac{C_2 C_3}{L} & \frac{C_4}{L} \\
1 & \zeta C_1 & \zeta^2 C_1 L & \zeta^3 C_2 C_3 & \zeta^4 C_3 L \\
1 & \zeta^2 C_1 & \zeta^3 C_1 C_2 & \zeta^4 C_2 C_3 & \zeta^5 C_3 L \\
1 & \zeta^3 C_1 & \zeta^4 C_1 C_2 & \zeta^5 C_2 C_3 & \zeta^6 C_3 L \\
1 & \zeta^4 C_1 & \zeta^5 C_1 C_2 & \zeta^6 C_2 C_3 & \zeta^7 C_3 L
\end{bmatrix}.
\]

Let \( \{u^0, u^1, u^2, u^3, u^4\} \) be canonical coordinates associated to the idempotent basis \( \{e_0, e_1, e_2, e_3, e_4\} \) which satisfy

\[
u^\alpha(t_i = 0, \Theta = 0) = 0.
\]

By [8] Lemma 6, we have

\[
d u^\alpha \frac{1}{d} = \zeta^\alpha L \frac{1}{x}
\]
at \( t = 0 \), for \( 0 \leq \alpha \leq 4 \).

The full genus 0 Gromov-Witten potential (2.1) satisfies the following property

\[
\mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_5]}(t, \Theta) = \mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_5]}(t|_{t_1 = 0}, \Theta + t_1).
\]

that is, \( \mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_5]}(t, \Theta) \) depends on \( t_1 \) and \( \Theta \) through \( \Theta + t_1 \). In particular, the operator

\[
\frac{\partial}{\partial t_1} - \frac{\partial}{\partial \Theta}
\]

annihilates \( \mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_5]}(t, \Theta) \).

Denote by

\[
R(z) = \text{Id} + \sum_{k \geq 1} R_{k} z^k \in \text{End}(H_{T,\text{Orb}}^*([\mathbb{C}^3/\mathbb{Z}_5]))[[z]]
\]

the \( R \)-matrix of the Frobenius structure associated to the (T-equivariant) Gromov-Witten theory of \( [\mathbb{C}^3/\mathbb{Z}_5] \) near the semisimple point 0. The \( R \)-matrix plays a central role in the Givental-Teleman classification of semisimple cohomological field theories. By definition of \( R \), the symplectic condition

\[
R(z) \cdot R(-z)^* = \text{Id}
\]

holds. The following flatness equation

\[
(z(d\Psi^{-1})R + z\Psi^{-1}(dR) + \Psi^{-1}R(dU) - \Psi^{-1}(dU)R = 0
\]

also holds, see [7] Section 4.6 and [3] Proposition 1.1. Here \( d = \frac{d}{dt} \).

Since \( \mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_5]}(t, \Theta) \) depends on \( t_1 \) and \( \Theta \) through \( \Theta + t_1 \), it follows that \( \Psi \) and \( R(z) \) also depend on \( t_1 \) and \( \Theta \) through \( \Theta + t_1 \). So we have

\[
\frac{\partial}{\partial t_1} \Psi = \frac{\partial}{\partial \Theta} \Psi, \quad \frac{\partial}{\partial t_1} R(z) = \frac{\partial}{\partial \Theta} R(z).
\]

In equation (2.5), we set \( t = 0 \) and only consider \( \frac{d}{dt_1} \). It follows that (2.5) can be written as

\[
z(\frac{d}{dt} \Psi^{-1})R + z\Psi^{-1}(\frac{d}{dt} R) + \Psi^{-1}R(\frac{d}{dt} U) - \Psi^{-1}(\frac{d}{dt} U)R = 0.
\]

\[\text{An argument for this (written for a different target space) from the CohFT viewpoint can be found in [13] Section 3.3}.\]
Since \( \frac{d}{d\theta} = \frac{dx}{d\theta} \frac{d}{dx} \)
after cancelling \( \frac{dx}{d\theta} \) and multiplying by \( x \), we rewrite the above equation as
\[
z(x \frac{d}{dx} \psi^{-1}(x)R + z\psi^{-1}(x \frac{d}{dx} R) + \psi^{-1}R(x \frac{d}{dx} U) - \psi^{-1}(x \frac{d}{dx} U)R = 0.
\]
By equating coefficients of \( z^k \), we see that
\[
\psi \left( D\psi^{-1} \right) R_{k-1} + DR_{k-1} + R_k (DU) - (DU) R_k = 0
\]
or equivalently,
\[
D \left( \psi^{-1}R_{k-1} \right) + \left( \psi^{-1}R_k \right) DU - \psi^{-1} (DU) \psi \left( \psi^{-1}R_k \right) = 0
\]
where \( D = x \frac{d}{dx} \) as before.

Now set \( t_1 = 0 \). By equation \( \ref{eq:2.3} \), we have
\[
DU = \text{diag}(L, L\zeta, \ldots, L\zeta^{n-1}).
\]
Let \( P_{i,j}^k \) denote the \((i, j)\)-entry of the matrix defined by \( P_k = \psi^{-1}R_k \) after being restricted to the semisimple point \( 0 \in H_{\text{T,Orb}}^*([\mathbb{C}^n/\mathbb{Z}_5]) \). Set
\[
\tilde{P}_{i,j}^k = \frac{L^i}{K^i} P_{i,j}^k \zeta^{(k+i)j}
\]
where \( 0 \leq i, j \leq 4 \) and \( k \geq 0 \). Then, the flatness equation \( \ref{eq:2.7} \) reads as
\[
\begin{align*}
\tilde{P}_{4,j}^k &= \tilde{P}_{0,j}^k + \frac{1}{L} D \tilde{P}_{0,j}^{k-1}, \\
\tilde{P}_{3,j}^k &= \tilde{P}_{4,j}^k + \frac{1}{L} D \tilde{P}_{4,j}^{k-1} + A_1 \tilde{P}_{4,j}^{k-1}, \\
\tilde{P}_{2,j}^k &= \tilde{P}_{3,j}^k + \frac{1}{L} D \tilde{P}_{3,j}^{k-1} + A_2 \tilde{P}_{3,j}^{k-1}, \\
\tilde{P}_{1,j}^k &= \tilde{P}_{2,j}^k + \frac{1}{L} D \tilde{P}_{2,j}^{k-1} - A_2 \tilde{P}_{2,j}^{k-1}, \\
\tilde{P}_{0,j}^k &= \tilde{P}_{1,j}^k + \frac{1}{L} D \tilde{P}_{1,j}^{k-1} - A_1 \tilde{P}_{1,j}^{k-1}.
\end{align*}
\]
We call equation \( \ref{eq:2.9} \) the \textit{modified flatness equations}.

By the methodology of \cite{16}, we obtain the following result.

\textbf{Lemma 2.1.} \textit{We have} \( \tilde{P}_{0,j}^k \in \mathbb{C}[L^{\pm 1}] \) \textit{for all} \( 0 \leq j \leq 4 \) \textit{and} \( k \geq 0 \).

3. \textbf{Lift of Modified R-matrix}

3.1. \textbf{Canonical Lift.} The functions \( \tilde{P}_{i,j}^k \in \mathbb{C}[[x]] \) in modified flatness equations have canonical lifts to the ring
\[
\mathbb{F} = \mathbb{C}[L^{\pm 1}][A_1, DA_1, D^2 A_1, A_2]
\]
via equation \( \ref{eq:1.6} \), equation \( \ref{eq:1.11} \) and the first four rows of the flatness equations \( \ref{eq:2.9} \) in the descending order. More precisely, we start with Lemma 2.1, that is
\[
\tilde{P}_{0,j}^k \in \mathbb{C}[L^{\pm 1}] \subset \mathbb{F}.
\]
Then, by equation (1.6), we obtain
\[ \tilde{P}_{4,j}^k = \tilde{P}_{0,j}^k + \frac{1}{L} D \tilde{P}_{0,j}^{k-1} \in \mathbb{C}[L^{\pm 1}] \subset \mathbb{F}. \]

By proceeding in a similar manner, we see that
\[ \tilde{P}_{3,j}^k = \tilde{P}_{4,j}^k + \frac{1}{L} D \tilde{P}_{4,j}^{k-1} + A_1 \tilde{P}_{4,j}^{k-1} \in \mathbb{C}[L^{\pm 1}][A_1] \subset \mathbb{F}, \]
\[ \tilde{P}_{2,j}^k = \tilde{P}_{3,j}^k + \frac{1}{L} D \tilde{P}_{3,j}^{k-1} + A_2 \tilde{P}_{3,j}^{k-1} \in \mathbb{C}[L^{\pm 1}][A_1, D A_1, A_2] \subset \mathbb{F}. \]

Lastly, using equation (1.11) we get
\[ \tilde{P}_{1,j}^k = \tilde{P}_{2,j}^k + \frac{1}{L} D \tilde{P}_{2,j}^{k-1} - A_2 \tilde{P}_{2,j}^{k-1} \in \mathbb{C}[L^{\pm 1}][A_1, D A_1, D^2 A_1, A_2] = \mathbb{F}. \]

This procedure gives us a canonical lift of \( \tilde{P}_{i,j}^k \in \mathbb{C}[[x]] \) to the free polynomial ring \( \mathbb{F} \), which we denote again as \( \tilde{P}_{i,j}^k \). We state this result in the following way.

**Lemma 3.1.** We have \( \tilde{P}_{i,j}^k \in \mathbb{F} \) for all \( 0 \leq i, j \leq 4 \) and \( k \geq 0 \).

### 3.2. Preparations

In this subsection, we use the lift \( \tilde{P}_{i,j}^k \in \mathbb{F} \) and prove two lemmas which will be used for the proof of holomorphic anomaly equations.

**Lemma 3.2.** The following identity holds
\[ \frac{\partial \tilde{P}_{i,j}^k}{\partial A_2} = \delta_{i,2} \tilde{P}_{3,j}^{k-1}. \]

**Proof.** It is clear that the degrees of \( \tilde{P}_{0,j}^k, \tilde{P}_{4,j}^k, \) and \( \tilde{P}_{3,j}^k \) in \( A_2 \) are all zero. Hence, we get
\[ \frac{\partial \tilde{P}_{0,j}^k}{\partial A_2} = \frac{\partial \tilde{P}_{4,j}^k}{\partial A_2} = \frac{\partial \tilde{P}_{3,j}^k}{\partial A_2} = 0. \]

The place where we see \( A_2 \) for the first time are the next two equations in (2.9),
\[ \tilde{P}_{2,j}^k = \tilde{P}_{3,j}^k + \frac{1}{L} D \tilde{P}_{3,j}^{k-1} + A_2 \tilde{P}_{3,j}^{k-1}, \]
\[ \tilde{P}_{1,j}^k = \tilde{P}_{2,j}^k + \frac{1}{L} D \tilde{P}_{2,j}^{k-1} - A_2 \tilde{P}_{2,j}^{k-1}. \]

From the first equation (3.2), we see that
\[ \frac{\partial \tilde{P}_{2,j}^k}{\partial A_2} = \tilde{P}_{3,j}^{k-1}. \]

Note that equation (1.11) gives
\[ \frac{\partial (D A_2)}{\partial A_2} = 2 L A_2. \]

Now, we compute the last derivative. By flatness equations (2.9) and equation (3.2), we obtain
\[ \frac{\partial \tilde{P}_{1,j}^k}{\partial A_2} = \frac{\partial \tilde{P}_{2,j}^k}{\partial A_2} + \frac{1}{L} \left( D \tilde{P}_{2,j}^{k-1} \right) = \tilde{P}_{3,j}^{k-1} - \frac{A_2}{\partial A_2} \tilde{P}_{2,j}^{k-1} \]
\[ = \frac{\partial \tilde{P}_{2,j}^k}{\partial A_2} + \frac{1}{L} \left( 2 L A_2 \tilde{P}_{3,j}^{k-2} + D \tilde{P}_{3,j}^{k-2} \right) - \tilde{P}_{2,j}^{k-1} - A_2 \frac{\partial \tilde{P}_{2,j}^{k-1}}{\partial A_2}. \]
Then, by equation (3.2) and again by flatness equations (2.9), we get
\[
\frac{\partial \tilde{P}^{k}_{i,j}}{\partial A_{2}} = \tilde{P}^{k-1}_{3,j} + 2A_{2}\tilde{P}^{k-2}_{3,j} + \frac{1}{L}D\tilde{P}^{k-2}_{3,j} - \tilde{P}^{k-1}_{2,j} - A_{2}\tilde{P}^{k-2}_{3,j}
\]
\[
= \tilde{P}^{k-1}_{3,j} + A_{2}\tilde{P}^{k-2}_{3,j} + \frac{1}{L}D\tilde{P}^{k-2}_{3,j} - \tilde{P}^{k-1}_{2,j} = 0.
\]
This completes the proof. □

**Lemma 3.3.** The following identity holds
\[
\frac{\partial \tilde{P}^{k}_{i,j}}{\partial (D^{2}A_{1})} = \delta_{i,1} \frac{1}{L^{2}} \tilde{P}^{k-3}_{4,j}.
\]

**Proof.** It is clear that the only \( \tilde{P}^{k}_{i,j} \in \mathbb{P} \) that has non-zero degree in \( D^{2}A_{1} \) is \( \tilde{P}^{k}_{1,j} \), and the degree of \( \tilde{P}^{k}_{1,j} \) in \( D^{2}A_{1} \) is 1. So, we obtain
\[
\frac{\partial \tilde{P}^{k}_{0,j}}{\partial (D^{2}A_{1})} = \frac{\partial \tilde{P}^{k}_{4,j}}{\partial (D^{2}A_{1})} = \frac{\partial \tilde{P}^{k}_{3,j}}{\partial (D^{2}A_{1})} = \frac{\partial \tilde{P}^{k}_{2,j}}{\partial (D^{2}A_{1})} = 0.
\]
The coefficient of \( D^{2}A_{1} \) in \( \tilde{P}^{k}_{1,j} \) descends from the coefficient of \( A_{1} \) in \( \tilde{P}^{k-2}_{3,j} \), which is \( \tilde{P}^{k-3}_{4,j} \). Keeping track of this term in the procedure of canonical lifting, we see that the coefficient of \( D^{2}A_{1} \) in \( \tilde{P}^{k}_{1,j} \) is
\[
\frac{1}{L^{2}} \tilde{P}^{k-3}_{4,j}.
\]
This completes the proof. □

### 4. Holomorphic Anomaly Equations

#### 4.1. Formula for potentials.

By general considerations, Gromov-Witten theory of \([\mathbb{C}^{5}/\mathbb{Z}_{5}]\) has the structure of a cohomological field theory (CohFT). We refer to [6] and [11] for discussions on CohFTs.

**Graphs.** We describe the graphs needed in the formula for Gromov-Witten potentials.

Recall that a **stable graph** \( \Gamma \) is a tuple
\[
\Gamma = (V_{\Gamma}, H_{\Gamma}, L_{\Gamma}, g : V_{\Gamma} \to \mathbb{Z}_{\geq 0}, \nu : H_{\Gamma} \cup L_{\Gamma} \to V_{\Gamma}, \iota : H_{\Gamma} \to H_{\Gamma}, \ell : L_{\Gamma} \to \{1, \ldots, m\})
\]
satisfying:

1. \( V_{\Gamma} \) is the vertex set with a genus assignment \( g : V_{\Gamma} \to \mathbb{Z}_{\geq 0} \),
2. \( H_{\Gamma} \) is the half-edge set equipped with an involution \( \iota : H_{\Gamma} \to H_{\Gamma} \),
3. \( E_{\Gamma} \) is the edge set defined by the orbits of \( \iota : H_{\Gamma} \to H_{\Gamma} \) in \( H_{\Gamma} \) (self-edges are allowed at the vertices) and the tuple \((V_{\Gamma}, E_{\Gamma})\) defines a connected graph,
4. \( L_{\Gamma} \) is the set of legs equipped with an isomorphism \( \ell : L_{\Gamma} \to \{1, \ldots, m\} \),
5. The map \( \nu : H_{\Gamma} \cup L_{\Gamma} \to V_{\Gamma} \) is a vertex assignment,
6. For each vertex \( v \), let \( n(v) = l(v) + h(v) \) be the valence of the vertex (where \( l(v) \) and \( h(v) \) are the number of legs and the number of edges attached to the vertex \( v \) respectively). Then, the following stability condition holds:
\[
2g(v) - 2 + n(v) > 0.
\]
The genus \( g(\Gamma) \) of a stable graph \( \Gamma \) is defined by
\[
g(\Gamma) = h^1(\Gamma) + \sum_{v \in V} g(v).\]

A decorated stable graph
\[ \Gamma \in G^\text{Dec}_{g,n}(5) \]
of order 5 is a stable graph \( \Gamma \in G_{g,n} \) with an extra assignment \( p : V_\Gamma \to \{0, 1, 2, 3, 4\} \) to each vertex \( v \in V_\Gamma \).

### 4.1.2. Formula for \( F_g \)

By the results stated in Section 2 the CohFT of Gromov-Witten theory of \([\mathbb{C}^5/\mathbb{Z}_5]\) is semisimple. By Givental-Teleman classification of semisimple CohFTs (see e.g. [11] for a survey), we can write Gromov-Witten potential as a sum over decorated stable graphs,
\[
F^{[\mathbb{C}^5/\mathbb{Z}_5]}_{g,n}(\phi_1, \ldots, \phi_{e_n}) = \sum_{\Gamma \in G^\text{Dec}_{g,n}(5)} \text{Cont}_\Gamma (\phi_1, \ldots, \phi_{e_n}).
\]

Details about how this formula works in general can be found in e.g. [12] and [10].

In order to state the contributions of graphs to \( F^{[\mathbb{C}^5/\mathbb{Z}_5]}_{g,n}(\phi_1, \ldots, \phi_{e_n}) \), we need to introduce the following series in \( \mathbb{C}[[x]] \):
\[
K_0, \quad K_1 = C_1, \quad K_2 = C_1C_2, \quad K_3 = C_1C_2C_3, \quad \text{and} \quad K_4 = C_1C_2^2C_3,
\]
and the following involution
\[
\text{Inv} : \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\},
\]
with \( \text{Inv}(0) = 0 \) and \( \text{Inv}(i) = 5 - i \) for \( 1 \leq i \leq 4 \).

**Proposition 4.1.** The contribution associated to a decorated stable graph \( \Gamma \in G^\text{Dec}_{g,n}(5) \) is
\[
\text{Cont}_\Gamma (\phi_1, \ldots, \phi_{e_n}) = \frac{1}{|\text{Aut}(\Gamma)|} \prod_{v \in V_\Gamma} \text{Cont}_\Gamma^A(v) \prod_{e \in E_\Gamma} \text{Cont}_\Gamma^A(e) \prod_{l \in L_\Gamma} \text{Cont}_\Gamma^A(l),
\]
where \( F(\Gamma) = |H_\Gamma \cup L_\Gamma| = n + |H_\Gamma| \) and \( \text{Cont}_\Gamma^A(v), \text{Cont}_\Gamma^A(e), \text{and} \text{Cont}_\Gamma^A(l) \), the vertex, edge and leg contributions with flag A-values \( (a_1, \ldots, a_n, b_1, \ldots, b_{|H_\Gamma|}) \) respectively, are given by

\[
\text{Cont}_\Gamma^A(v) = \sum_{k \geq 0} \frac{g(e_p(v), e_p(v))^{-2p - 2n(v) + k}}{k!} \times \int_{M_{g,v,n}(\psi_1, \ldots, \psi_{n(v) + k})} \psi^{a_1 \cdots a_{n(v)}}_1 \cdots \psi^{a_1 \cdots a_{n(v)}}_{n(v) + 1} \cdots \psi^{b_1 \cdots b_{n(v) + k}}_{n(v) + k},
\]
\[
\text{Cont}_\Gamma^A(e) = \frac{(-1)^{b_1 + b_2}}{5} \sum_{m = 0}^{b_2} (-1)^m \sum_{r = 0}^{4} \frac{\tilde{P}_{b_1 + m + 1}^{(r)(p(v_1))}}{\zeta^{(b_1 + m + 1 + \text{Inv}(r))p(v_1)}} \zeta^{(b_2 - m + r)p(v_2)},
\]
\[
\text{Cont}_\Gamma^A(l) = \frac{(-1)^{a(l)}}{5} \frac{\tilde{P}_{l}^{(e(l)), p(v(l))}}{L^{(a(l) + \text{Inv}(e(l))), p(v(l))}} \zeta^{(e(l) + \text{Inv}(e(l)))p(v(l))},
\]
where
\[
t_p(v) (z) = \sum_{k \geq 2} T_p(v) k^k z^k \quad \text{with} \quad T_p(v) k = \frac{(-1)^k}{n} \tilde{P}_{0, p(v)} k^{\text{Inv}(v)}.\]

\(4\)Notation: The values \( b_{v_1}, \ldots, b_{v_{n(v)}} \) and \( b_{e_1}, b_{e_2} \) are the entries of \( (a_1, \ldots, a_n, b_1, \ldots, b_{|H_\Gamma|}) \) corresponding to \( \text{Cont}_\Gamma^A(v) \) and \( \text{Cont}_\Gamma^A(e) \) respectively.
We should emphasize that Proposition 4.1 holds in $\mathbb{C}[[x]]$. Using Proposition 4.1 and lifting procedure in Section 3, we obtain the following lift of Gromov-Witten potential to certain polynomial rings.

**Theorem 4.2 (Finite generation property).** Let $\text{Cont}_1^A(v)$, $\text{Cont}_1^A(e)$, and $\text{Cont}_1^A(l)$ be as in Proposition 4.1. We have $\text{Cont}_1^A(v) \in \mathbb{C}[L^\pm]$, $\text{Cont}_1^A(e) \in \mathbb{F}$, and $\text{Cont}_1^A(l) \in \mathbb{F}[C_1, C_2, C_3]$. Hence, we have

$$\mathcal{F}_{g,n}[C^5/Z_5]^{C_1, \ldots, C_n} \in \mathbb{F}$$

and when there is no insertions, we have

$$\mathcal{F}_g^{C^5/Z_5} \in \mathbb{F}$$

where $\mathbb{F} = \mathbb{C}[L^\pm][A_1, DA_1, D^2A_1, A_2]$ as before.

**Proof.** By Lemma 2.1, we have $\text{Cont}_1^A(v) \in \mathbb{C}[L^\pm]$ since its expression involves only $\overline{P}_{0,j}$ s. By Lemma 3.1 and definitions of $K_i$ s, we see that $\text{Cont}_1^A(e) \in \mathbb{F}$ and $\text{Cont}_1^A(l) \in \mathbb{F}[C_1, C_2, C_3]$. Hence, results for Gromov-Witten potentials follow.

Depending on the insertions, we can give a better description of the polynomial ring that contains Gromov-Witten potentials. For example, by Proposition 4.1 we have

\begin{equation}
\mathcal{F}_{g,n}[C^5/Z_5]^{C_1, \ldots, C_n} \in \mathbb{F}[C_1^{-1}] = \mathbb{C}[L^\pm][A_1, DA_1, D^2A_1, A_2, C_1^{-1}]
\end{equation}

and the degree of $C_1^{-1}$ in $\mathcal{F}_{g,n}[C^5/Z_5]^{C_1, \ldots, C_n}$ is $n$. Then, we obtain the following result by Lemma 4.2.

**Corollary 4.3.** For all $k \geq 1$, we have

$$\frac{\partial^k \mathcal{F}_{g,n}[C^5/Z_5]}{\partial T^k} \in \mathbb{F}[C_1^{-1}] = \mathbb{C}[L^\pm][A_1, DA_1, D^2A_1, A_2, C_1^{-1}]$$

and the degree of $C_1^{-1}$ in $\frac{\partial^k \mathcal{F}_{g,n}[C^5/Z_5]}{\partial T^k}$ is $k$.

**4.2. Proof of holomorphic anomaly equations.**

**Lemma 4.4.** We have

$$\frac{\partial}{\partial A_2} \text{Cont}_1^A(e) = \frac{(-1)^{b_1+b_2}}{5} \frac{\overline{P}_{b_{v_1}}^{b_{v_2}}}{\zeta(b_{v_1+3})p(v_1) \zeta(b_{v_2+3})p(v_2)} \cdot$$

**Proof.** By Proposition 4.1 and Lemma 4.2 we obtain

\begin{align*}
\frac{\partial}{\partial A_2} \text{Cont}_1^A(e) &= \frac{(-1)^{b_1+b_2}}{5} \sum_{m=0}^{b_2} (-1)^m \sum_{r=0}^{4} \frac{\partial}{\partial A_2} \left( \frac{\overline{P}_{b_{v_1}+m}^{b_{v_2}}}{{\text{Inv}(r,p(v_1))} \zeta(b_{v_1+m+r})p(v_2)} \right) \\
&= \frac{(-1)^{b_1+b_2}}{5} \sum_{m=0}^{b_2} (-1)^m \frac{\overline{P}_{b_{v_1}+m}^{b_{v_2}}}{{\text{Inv}(r,p(v_1))} \zeta(b_{v_1+m+r})p(v_2)} \\
&= \frac{(-1)^{b_1+b_2}}{5} \sum_{m=0}^{b_2} (-1)^m \frac{\overline{P}_{b_{v_1}+m}^{b_{v_2}}}{{\text{Inv}(r,p(v_1))} \zeta(b_{v_1+m+r})p(v_2)} \cdot
\end{align*}
Lemma 3.3 instead of Lemma 3.2 and shift one of the summations by 1.

Proof. The strategy of proof is similar to that of Lemma 3.2. The only difference is that we use \( \bar{\lambda} \) instead of \( \lambda \). The only difference is that we use Lemma 3.3 instead of Lemma 3.2 and shift one of the summations by 1.

\[
\frac{\partial}{\partial A_2} \text{Cont}_A^A(\epsilon) = \frac{\partial}{\partial A_2} \text{Cont}_A^\lambda(\epsilon) = -\frac{1}{5} (b_{11} + b_{12}) \sum_{m=0}^{b_{12}} (-1)^m \frac{\bar{p}^{b_{11}+m+3} \bar{p}^{b_{12}-m}}{3.3.3} \zeta(3p(v_1)) \zeta(3p(v_2)) \]

Lemma 4.5. We have

\[
\frac{\partial}{\partial (D^2 A_1)} \text{Cont}_A^\lambda(\epsilon) = \frac{-1}{5L^2} (b_{11} + b_{12}) \sum_{m=0}^{b_{12}} (-1)^m \frac{\bar{p}^{b_{11}+m-2} \bar{p}^{b_{12}-m}}{4.4.4} \zeta(3p(v_1)) \zeta(3p(v_2))
\]

Proof. The strategy of proof is similar to that of Lemma 3.2. The only difference is that we use Lemma 3.3 instead of Lemma 3.2 and shift one of the summations by 3 rather than by 1.

Theorem 4.6. (The first holomorphic anomaly equation). For \( g \geq 2 \), we have

\[
\frac{C_3}{5L} \frac{\partial}{\partial A_2} F_{g-1,2}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\phi_2, \phi_2) + \frac{1}{2} \sum_{i=1}^{g-1} F_{g-i,1}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\phi_2, \phi_2) F_{i,1}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\phi_2)
\]

in \( \mathbb{F}[C_1, C_2, C_3] \).

Proof. Let \( \Gamma \in \mathcal{G}_{g,0}^\text{Dec}(5) \) be a decorated graph and \( \epsilon \in \mathcal{E}_\Gamma \) be an edge of \( \Gamma \) connecting two vertices \( v_1 \) and \( v_2 \). After deleting the edge \( \epsilon \), we obtain a new graph. (By deleting, we mean breaking the edge \( \epsilon \) into two legs \( \ell \) and \( \ell' \).) There are two possibilities for the resulting graph after deletion of edge \( \epsilon \):

(i) If it is connected, then we obtain an element of \( \mathcal{G}_{g-1,2}^\text{Dec}(5) \), which we denote as \( \Gamma_\epsilon^0 \). In this case, note that \( |\text{Aut}(\Gamma)| = |\text{Aut}(\Gamma_\epsilon^0)| \). Note that we have two possibilities to label two legs and there is no canonical choice for this labeling.

(ii) If it is disconnected, then the resulting graph has two connected components, which we denote as \( \Gamma_\epsilon^1 \in \mathcal{G}_{g-1,2}^\text{Dec}(5) \) and \( \Gamma_\epsilon^2 \in \mathcal{G}_{g-1,2}^\text{Dec}(5) \) where we have \( g = g_1 + g_2 \). In this case, for the cardinality of the automorphism group of the decorated stable graphs, we have \( |\text{Aut}(\Gamma)| = |\text{Aut}(\Gamma_\epsilon^1)\text{Aut}(\Gamma_\epsilon^2)| \) if \( \Gamma_\epsilon^1 \neq \Gamma_\epsilon^2 \). For the special case when \( \Gamma_\epsilon^1 = \Gamma_\epsilon^2 \), the graph \( \Gamma \) has a \( \mathbb{Z}_2 \)-symmetry given by interchanging \( \Gamma_\epsilon^1 \) and \( \Gamma_\epsilon^2 \). Hence \( |\text{Aut}(\Gamma)| = 2|\text{Aut}(\Gamma_\epsilon^1)|^2 \). Putting these together into a single formula, we get \( |\text{Aut}(\Gamma)| = (1 + \delta\Gamma_\epsilon^1 \Gamma_\epsilon^2)|\text{Aut}(\Gamma_\epsilon^1)\text{Aut}(\Gamma_\epsilon^2)| \).

By Proposition 4.1 and Lemma 4.4, we observe that

\[
\frac{\partial}{\partial A_2} \text{Cont}_A^\lambda(\epsilon) = \frac{-1}{5} (b_{11} + b_{12}) \sum_{m=0}^{b_{12}} (-1)^m \frac{\bar{p}^{b_{11}+m+3} \bar{p}^{b_{12}-m}}{3.3.3} \zeta(3p(v_1)) \zeta(3p(v_2))
\]

By Proposition 4.1 and Lemma 4.4, we observe that

\[
\frac{\partial}{\partial A_2} \text{Cont}_A^\lambda(\epsilon) = \frac{-1}{5} (b_{11} + b_{12}) \sum_{m=0}^{b_{12}} (-1)^m \frac{\bar{p}^{b_{11}+m+3} \bar{p}^{b_{12}-m}}{3.3.3} \zeta(3p(v_1)) \zeta(3p(v_2))
\]

\[
= 5 \left( \frac{L^3}{K^3} \right)^2 \begin{cases} \text{Cont}_A^\lambda_l(l) \text{Cont}_A^\lambda_l'(l') & \text{for the case (i)} \\ \text{Cont}_A^\lambda_l(l) \text{Cont}_A^\lambda_l'(l') & \text{for the case (ii)} \end{cases}
\]

with \( \ell(l) = \ell(l') = 2 \), i.e., with insertions \( \phi_2 \).
By definition of $K_3$ and equation (1.7), we also note that

$$\left( \frac{L^3}{K_3} \right)^2 = \frac{L}{C_3}.$$

Then, for case (i), we easily see that we have

$$\text{Cont}_{\Gamma^1_\xi}(\phi_1, \phi_2) = \frac{1}{|\text{Aut}(\Gamma^1_\xi)|} \sum_{\text{Aut}(\Gamma^1_\xi)} \prod_{A \in \mathbb{Z}_{\text{odd}}} \text{Cont}^A_{\Gamma^1_\xi}(v) \prod_{e \in E_{\Gamma^1_\xi}} \text{Cont}^A_{\Gamma^1_\xi}(e) \prod_{\text{lab}L^1_\xi} \text{Cont}^A_{\Gamma^1_\xi}(l)$$

(4.3)

$$= \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\text{odd}}} C_3 \frac{\partial \text{Cont}^A_{\Gamma}(\xi)}{\partial A_2} \prod_{v \in V_{\Gamma}} \text{Cont}^A_{\Gamma}(v) \prod_{e \in E_{\Gamma}} \text{Cont}^A_{\Gamma}(e).$$

Similarly, for case (ii), we observe the following

$$\text{Cont}_{\Gamma^2_\xi}(\phi_s) \text{Cont}_{\Gamma^2_\xi}(\phi_s)$$

(4.4)

$$= \frac{1}{|\text{Aut}(\Gamma^2_\xi)|} \sum_{A \in \mathbb{Z}_{\text{odd}}} \text{Cont}^A_{\Gamma^2_\xi}(t) \prod_{v \in V_{\Gamma^2_\xi}} \text{Cont}^A_{\Gamma^2_\xi}(v) \prod_{e \in E_{\Gamma^2_\xi}} \text{Cont}^A_{\Gamma^2_\xi}(e)$$

$$\times \frac{1}{|\text{Aut}(\Gamma^2_\xi)|} \sum_{A \in \mathbb{Z}_{\text{odd}}} \text{Cont}^A_{\Gamma^2_\xi}(t') \prod_{v \in V_{\Gamma^2_\xi}} \text{Cont}^A_{\Gamma^2_\xi}(v) \prod_{e \in E_{\Gamma^2_\xi}} \text{Cont}^A_{\Gamma^2_\xi}(e)$$

$$= \frac{(1 + \delta_{\Gamma^1_\xi, \Gamma^2_\xi})}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\text{odd}}} C_3 \frac{\partial \text{Cont}^A_{\Gamma}(\xi)}{\partial A_2} \prod_{v \in V_{\Gamma}} \text{Cont}^A_{\Gamma}(v) \prod_{e \in E_{\Gamma}} \text{Cont}^A_{\Gamma}(e).$$

By Lemma 2.1 and Theorem 4.2, we have the following vanishing result:

$$\frac{\partial \text{Cont}^A_{\Gamma}(v)}{\partial A_2} = 0$$

for any vertex $v \in V_{\Gamma}$.

Then, this vanishing result gives us the following

$$\frac{\partial \text{Cont}_{\Gamma}}{\partial A_2} = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\text{odd}}} \prod_{v \in V_{\Gamma}} \text{Cont}^A_{\Gamma}(v) \frac{\partial}{\partial A_2} \left( \prod_{e \in E_{\Gamma}} \text{Cont}^A_{\Gamma}(e) \right)$$

$$= \sum_{e \in E_{\Gamma}} \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\text{odd}}} \frac{\partial \text{Cont}^A_{\Gamma}(\xi)}{\partial A_2} \prod_{v \in V_{\Gamma}} \text{Cont}^A_{\Gamma}(v) \prod_{e \in E_{\Gamma}} \text{Cont}^A_{\Gamma}(e).$$

So, we have

$$\frac{C_3}{5L} \frac{\partial \text{Cont}_{\Gamma}}{\partial A_2} = \sum_{e \in E_{\Gamma}} \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\text{odd}}} \frac{C_3}{5L} \frac{\partial \text{Cont}^A_{\Gamma}(\xi)}{\partial A_2} \prod_{v \in V_{\Gamma}} \text{Cont}^A_{\Gamma}(v) \prod_{e \in E_{\Gamma}} \text{Cont}^A_{\Gamma}(e).$$

(4.5)

Then, summing equation (4.3) and equation (4.4) over all decorated stable graphs $\Gamma^0_\xi$ and $(\Gamma^1_\xi, \Gamma^2_\xi)$ we obtain

$$\langle \phi_2 \rangle_{g-1,2}^{[C^5/\mathbb{Z}_5]}$$

and

$$\sum_{i=1}^{g-1} \langle \phi_2 \rangle_{g-i,1}^{[C^5/\mathbb{Z}_5]} \langle \phi_2 \rangle_{i,1}^{[C^5/\mathbb{Z}_5]}$$
respectively. Then, by equation (4.5) we conclude that we have
\[2 C_3 \frac{\partial}{\partial A_2} \langle \langle \phi \rangle \rangle_{g/Z_5} = \langle \langle \phi_2, \phi_2 \rangle \rangle_{g-1,2} + \sum_{i=1}^{g-1} \langle \langle \phi \rangle \rangle_{g-i,1} \langle \langle \phi_2 \rangle \rangle_{i,1}\]
after summing over all decorated stable graphs \( \Gamma \). The reason we have 2 in front of the left hand side is due to not having a canonical order of labelings of each of the legs \( l_k \) and \( l'_k \) for case (i) and double counting of different genera of connected components\(^5\) for case (ii). This completes the proof.

\[\square\]

**Theorem 4.7** (The second holomorphic anomaly equation). For \( g \geq 2 \), we have
\[
\frac{C_2^2 C_3}{5L^3} \frac{\partial \mathcal{F}_{g}^{[C]} / \partial (D^2 A_1)}{\partial} = \frac{1}{2} \mathcal{F}_{g-1,2}^{[C]} (\phi_1, \phi_1) + \sum_{i=1}^{g-1} \mathcal{F}_{g-i,1}^{[C]} (\phi_1) \mathcal{F}_{i,1}^{[C]} (\phi_1)
\]
in \( \mathbb{F} [C_1, C_2, C_3] \).

**Proof.** The proof is similar to that of Theorem 4.6 with some technical difference. Instead of giving full details, this time we point out these different technicalities. Throughout the proof, let \( \Gamma, \tilde{\Gamma}, v_1, v_2, l_k, l'_k, \Gamma^0, \Gamma^1, \Gamma^2 \), “case (i)” and “case (ii)” be as in the proof of Theorem 4.6.

By Proposition 4.1 and Lemma 4.5 we have
\[
\frac{\partial}{\partial (D^2 A_1)} \text{Cont}^A_{i} (\varepsilon) = \frac{(-1)^b v_1 + b v_2}{5L^2} \left\{ \frac{\tilde{P}^{b_1-2}}{4_p} \frac{\tilde{P}^{b_2}}{4_p} \frac{1}{\zeta(b_1+2)p(\varepsilon)} + \frac{(-1)^b v_1 + b v_2}{5L^2} \left\{ \frac{\tilde{P}^{b_1-1}}{4_p} \frac{\tilde{P}^{b_2-1}}{4_p} \frac{1}{\zeta(b_1+3)p(\varepsilon)} + \frac{(-1)^b v_1 + b v_2}{5L^2} \left\{ \frac{\tilde{P}^{b_1}}{4_p} \frac{\tilde{P}^{b_2-2}}{4_p} \frac{1}{\zeta(b_1+4)p(\varepsilon)} \right\} \right\} \right\} \text{Cont}^A_{i} (\varepsilon)
\]
where right hand side of this equation is equal to
\[
5 \left( \frac{L^4}{K_4} \right)^2 \left( \text{Cont}^A_{i} (\varepsilon) \right) \text{Cont}^A_{i} (l_k) = \left( \text{Cont}^A_{i} (l_k) \right) \text{Cont}^A_{i} (l'_k) + \text{Cont}^A_{i} (l_k) \text{Cont}^A_{i} (l'_k) + \text{Cont}^A_{i} (l_k) \text{Cont}^A_{i} (l'_k)
\]
for the case (i), and it is equal to
\[
5 \left( \frac{L^4}{K_4} \right)^2 \left( \text{Cont}^A_{i} (l_k) \text{Cont}^A_{i} (l'_k) \right) = \left( \text{Cont}^A_{i} (l_k) \text{Cont}^A_{i} (l'_k) \right) \text{Cont}^A_{i} (l_k) + \text{Cont}^A_{i} (l_k) \text{Cont}^A_{i} (l'_k) + \text{Cont}^A_{i} (l_k) \text{Cont}^A_{i} (l'_k) + \text{Cont}^A_{i} (l_k) \text{Cont}^A_{i} (l'_k)
\]
for the case (ii), with \( \ell (l_k) = \ell (l'_k) = 1 \), i.e., with insertions \( \phi_1 \) for both cases. Here, by \( A_{b_{\alpha+r}-r} \) we mean the flag value \( b_{\alpha} \) is shifted by \( r \) in \( A \in \mathbb{Z}^{\mathbb{F} (\varepsilon)} \) where \( \varepsilon \) is \( \Gamma^0, \Gamma^1 \) or \( \Gamma^2 \). Since \( \tilde{P}^{k} = 0 \) for \( k < 0 \) and shifting \( b_{\alpha+1} + b_{\alpha+2} \) by 2 does not change signs in equation (4.6), we can view equation (4.6) as
\[
\frac{\partial \text{Cont}^A_{i} (\varepsilon)}{\partial (D^2 A_1)} = 5 \left( \frac{L^4}{K_4} \right)^2 \left\{ \text{Cont}^A_{i} (l_k) \text{Cont}^A_{i} (l'_k) \right\} \text{Cont}^A_{i} (l_k) \text{Cont}^A_{i} (l'_k) \text{Cont}^A_{i} (l'_k) \text{Cont}^A_{i} (l'_k) \text{Cont}^A_{i} (l'_k) \text{Cont}^A_{i} (l'_k) \text{Cont}^A_{i} (l'_k) \text{Cont}^A_{i} (l'_k)
\]
while following proof strategy of Theorem 4.6.

By definition of \( K_4 \) and equation (1.7), we also note that
\[
\left( \frac{L^4}{K_4} \right)^2 = \frac{L^3}{C_2^2 C_3}.
\]
The rest of the proof is just adaptation of proof of Theorem 4.6.

REFERENCES

[1] D. Abramovich, T. Graber, A. Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, Amer. J. Math. 130 (2008), no. 5, 1337–1398.
[2] T. Coates, A. Corti, H. Iritani, H.-H. Tseng, *Computing Genus-Zero Twisted Gromov-Witten Invariants*, Duke Math. J. 147 (2009), no.3, 377–438.
[3] A. Givental, *Elliptic Gromov-Witten invariants and the generalized mirror conjecture*, in: “Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)”, 107–155, World Sci. Publ., River Edge, NJ, 1998.
[4] A. Givental, *Symplectic geometry of Frobenius structures*, in: “Frobenius manifolds”, Aspects Math., E36, 91–112, Friedr. Vieweg, Wiesbaden, 2004.
[5] T. Graber, R. Pandharipande, *Localization of virtual classes*, Invent. Math. 135 (1999), 487–518.
[6] M. Kontsevich, Y. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. 164 (1994), 525–562.
[7] Y.-P. Lee, R. Pandharipande, *Frobenius manifolds, Gromov-Witten theory, and Virasoro constraints*, manuscript available from the authors’ websites.
[8] H. Lho, *Crepeant resolution conjecture for $\mathbb{C}^5/\mathbb{Z}_5$*, arXiv:1707.02910.
[9] H. Lho, R. Pandharipande, *Stable quotients and the holomorphic anomaly equation*, Adv. Math. 332 (2018), 349–402.
[10] H. Lho, R. Pandharipande, *Crepeant resolution and the holomorphic anomaly equation for $[\mathbb{C}^3/\mathbb{Z}_3]$*, Proc. London Math. Soc. (3) 119 (2019), 781–813.
[11] R. Pandharipande, *Cohomological field theory calculations*, Proceedings of the ICM (Rio de Janeiro 2018), Vol 1, 869–898, World Sci. Publications: Hackensack, NJ, 2018.
[12] R. Pandharipande, A. Pixton, D. Zvonkine, *Relations on $\overline{M}_{g,n}$ via 3-spin structures*, J. Amer. Math. Soc. 28 (2015), 279–309.
[13] R. Pandharipande, H.-H. Tseng, *Higher genus Gromov-Witten theory of $\text{Hilb}^n(\mathbb{C}^2)$ and CohFTs associated to local curves*, Forum of Mathematics, Pi (2019), Vol. 7, e4, 63 pages, arXiv:1707.01406.
[14] C. Teleman, *The structure of 2D semi-simple field theories*, Invent. Math. 188 (2012), 525–588.
[15] H.-H. Tseng, *Orbifold quantum Riemann-Roch, Lefschetz and Serre*, Geom. Topol. 14 (2010), 1–81.
[16] D. Zagier, A. Zinger, *Some properties of hypergeometric series associated with mirror symmetry*, In: “Modular forms and string duality”, 163–177, Fields Inst. Commun. 54, AMS 2008.

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, 100 MATHEMATIC, 231 WEST 18TH AVE., COLUMBUS, OH 43210, USA

Email address: genlik.1@osu.edu

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, 100 MATHEMATIC, 231 WEST 18TH AVE., COLUMBUS, OH 43210, USA

Email address: hhtseng@math.ohio-state.edu