KIRCHHOFF TYPE EQUATIONS WITH STRONG SINGULARITIES

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(Communicated by Changfeng Gui)

Abstract. An optimal condition is given for the existence of positive solutions of nonlinear Kirchhoff PDE with strong singularities. A byproduct is that 

\(-2\) is no longer the critical position for the existence of positive solutions of PDE's with singular potentials and negative powers of the form:

\[-|x|^{\alpha} \Delta u = u^{-\gamma} + g(x)u^q \text{ in } \Omega,\]

\(u > 0 \text{ in } \Omega,\]

\(u = 0 \text{ on } \partial \Omega,\]

where \(\alpha \in (0,N)\) and \(-\gamma \in (-\infty,-1)\). 

1. Introduction. Let \(\Omega\) be a smooth bounded domain of \(\mathbb{R}^N, N \geq 3\) and let \(H^1_0(\Omega)\) be the standard Sobolev space consisting of functions which vanish on the boundary of \(\Omega\) and whose gradient is in \(L^2(\Omega)\). We consider Kirchhoff type equations

\[
\begin{aligned}
-(a + b \int_{\Omega} |\nabla u|^2 \, dx) \Delta u &= f(x)u^{-\gamma} + g(x)u^q \quad \text{in } \Omega, \\
u > 0 &\quad \text{in } \Omega, \\
u = 0 &\quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(a, b > 0, q \in (0,1)\), and \(-\gamma \in (-\infty,-1)\). Such equations are related to the stationary version of Kirchhoff equations presented by Kirchhoff [27]. It should be noted here that the term \((\int_{\Omega} |\nabla u|^2 \, dx) \Delta u\) means that the above equation is nonlocal, namely, it is not a pointwise identity. Since the work by Lions [32], people have paid much attention to Kirchhoff type equations and a lot of classical results have been obtained (see, e.g., [35, 43] and references therein). The singular cases are investigated recently. In 3-dimension, Liu and the first author [37] proved that the nonlinear case \(f(x)u^{-\gamma} + \lambda g(x)\frac{x^s}{|x|^r}\), with \(0 \leq s < 1, 3 < p < 5 - 2s, -\gamma > -1\) has two positive solutions provided \(\lambda > 0\) small, and more recently by Liao et al. [31] for higher dimension. When \(-\gamma < -1\) then various difficulties arise in the analysis and it is the purpose of this paper to face them in the Kirchhoff case (1).

Singular elliptic equations have been intensively studied since last seventies because of its wide applications to physical models, for example, boundary layer phenomenon in fluid mechanics, chemical heterogenous catalysts, glacial advance etc. (see, e.g. [1-12,14-26,28-34,36-42] and references therein). For the singular boundary theory, we refer the reader to the books by Agarwal and O’Regan [1], and Hernández and Mancebo [25] for an excellent introduction to the subject.

2000 Mathematics Subject Classification. Primary: 35J70, 35J20; Secondary: 35J60. 
Key words and phrases. Kirchhoff type equations, strong singularity, singular potential. 
The authors are supported by NSFC grants 11571339 and 11771468.
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By solutions we mean here solutions in $H^1_0(\Omega)$, i.e. $u \in H^1_0(\Omega)$ satisfying $u(x) > 0$ in $\Omega$ and for all $\psi \in H^1_0(\Omega)$,

$$a \int_{\Omega} \nabla u \cdot \nabla \psi + b \left( \int_{\Omega} |\nabla u|^2 \right) \int_{\Omega} \nabla u \cdot \nabla \psi - \int_{\Omega} \frac{f(x)}{u^\gamma} \psi - \int_{\Omega} g(x) u^q \psi = 0.$$

We consider the following functional on $H^1_0(\Omega)$:

$$I(u) = \frac{1}{2} a \|u\|^2 + \frac{1}{4} b \|u\|^4 - \frac{1}{1-\gamma} \int_{\Omega} f(x)|u|^{1-\gamma} dx - \frac{1}{1+q} \int_{\Omega} g(x)|u|^{1+q} dx.$$

It should be noted that, since $-\gamma \in (-\infty, -1)$, the functional $I$ is not well defined on $H^1_0(\Omega)$. In [30], when $b = 0, g \equiv 0$, Lazer-McKenna announced that if $f(x) \in C^0(\overline{\Omega})$, $f(x) > 0$, $\forall x \in \overline{\Omega}$, then for each $-\gamma < 0$ (1) has a unique solution $u_{-\gamma} \in C^{2+\alpha}(\overline{\Omega}) \cap C(\Omega)$, $u_{-\gamma}$ is not in $C^1(\Omega)$ if $-\gamma < -1$, and $u_{-\gamma}$ is in $H^1_0(\Omega)$ if and only if $-\gamma > -3$ (the gradient of $u_{-\gamma}$ blows up at $\partial \Omega$ if $-\gamma < -1$ and out of $L^2$-space if $-\gamma \leq -3$). It seems that the unique solution $u_{-\gamma}$ become more and more steeper near $\partial \Omega$ as $-\gamma \to -\infty$. More recent development in this direction, appeared in Boccardo and Orsina [5] stating that when $f(x)$ is a nonnegative $L^1$-function (not identically zero), (1) with $b = 0, g \equiv 0$ has a $H^1_0(\Omega)$-solution $u$ for each $-\gamma < -1$ and $u_{-\gamma}^0 \in H^1_0(\Omega)$, in [38] by the first author stating that when $f(x)$ is a positive $L^1$-function satisfying a compatible condition, (1) with $b = 0, g \equiv 0$, has a $H^1_0(\Omega)$-solution for each $-\gamma < -1$, and especially in [39] by the first author and Zhang the relation between the standard Sobolev space $H^1_0(\Omega)$ and the critical position $-3$ is proved.

Monographs on the special case $f(x) \sim [\text{dist}(x, \partial \Omega)]^p$ (ie, there exist two positive constants $c_1, c_2$ such that $c_1 [\text{dist}(x, \partial \Omega)]^p \leq f(x) \leq c_2 [\text{dist}(x, \partial \Omega)]^p \forall x \in \Omega$) are in Gui-Lin [22], Diaz, Hernandez and Rakotoson [11], and Zhang and Cheng [42]. We also refer to Giacomoni and Saoudi [19] when $-\gamma > -3$, Hirano, Saccon and Shioji [26] for $L^1_{loc}(\Omega)$-solutions when $-\gamma < -1$.

The following theorem is the first result of this paper. It gives the necessary and sufficient conditions for the existence of a $H^1_0(\Omega)$-solution to the strong singular Kirchhoff equation.

**Theorem 1.1.** Assume that $f \in L^1(\Omega)$, $f > 0$ a.e. in $\Omega$, $0 \leq g \in L^\infty(\Omega)$ and that $a > 0, b > 0, q \in (0, 1)$ and $-\gamma \in (-\infty, -1)$. Then the problem

$$\begin{cases}
-(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \frac{f(x)}{u^\gamma} + g(x) u^q & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

admits a solution $u_{-\gamma} \in H^1_0(\Omega)$ if and only if there exists $u_0 \in H^1_0(\Omega)$ such that

$$\int_{\Omega} f(x)|u_0|^{1-\gamma} dx < +\infty. \quad (2)$$

The proof of Theorem 1.1 is motivated by our recent work [38] on strong singularity with the difficulty that, how to do the weak convergence with the nonlocal term $(\int_{\Omega} |\nabla u|^2 dx) \Delta u$.

**Remark 1.** Theorem 1.1 shows that, if the coefficient of the nonlinearity with negative power vanish at $\partial \Omega$ quickly enough, the singular equation can admit a $H^1_0(\Omega)$-solution even when $-\gamma \leq -3$. For example,

$$\begin{cases}
-(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \frac{(1-|x|)^p}{u^\gamma} + g(x) u^q, & \text{in } B_1 \\
u > 0, & \text{in } B_1 \\
u = 0, & \text{on } \partial B_1
\end{cases}$$
where \( B_1 \subset \mathbb{R}^N, N \geq 3 \) is the unit ball, and \( l \) is a positive number. It is easily checked that \( u_0(x) = (1 - |x|)^\alpha, \alpha > 1/2 \) belongs to \( H^1_0(B_1) \). We let \( -\gamma > -\frac{l+\alpha+1}{\alpha} \), so that
\[
\int_{B_1} (1 - |x|)^l |u_0|^{1-\gamma} dx = \int_{B_1} (1 - |x|)^{l+\alpha(1-\gamma)} dx \leq \frac{N\omega_N}{l+\alpha(1-\gamma)+1} < \infty
\]
where \( \omega_N \) is the volume of the unit \( N \)-sphere. We then obtain the existence of a \( H^1_0 \)-solution to (1) with \( f(x) = (1 - |x|)^l \) for \( -\gamma > -\frac{l+\alpha+1}{\alpha} \) and \( l > 0 \), thanks to (2). Clearly, \( -\gamma \to -\infty \) as \( l \to +\infty \). The condition \( -\gamma > -\frac{l+\alpha+1}{\alpha} \) is just the sufficient and necessary condition derived in [6] for the special case \( f(x) = [\text{dist}(x, \partial \Omega)]^l \).

It is well-known that if \( \Omega \) is a domain of \( \mathbb{R}^N \) containing \( 0, N \geq 2, p > 1 \) and \(-\alpha \leq -2\), then
\[
\begin{cases}
-\Delta u = |x|^{-\alpha}u^p & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
admits no positive solution (c.f.[6]). In contrast, it is shown that in presence of negative power nonlinearities \(-2 \) is no longer the critical position for the existence of positive solutions. As a remarkable application of Theorem 1.1, we derive our second result as following:

**Theorem 1.2.** Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^N \) containing \( 0, N \geq 3, 0 \leq g \in L^\infty(\Omega) \) and let \( a > 0, b \geq 0, 0 < q < 1 \). If \(-\alpha \in (-N, 0) \) and \(-\gamma \in (-3, -1)\), then the problem
\[
\begin{cases}
-(a + b \int_\Omega |\nabla u|^2 dx) \Delta u = |x|^{-\alpha}u^{-\gamma} + g(x)u^q & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(3)
admits a positive solution \( u \in H^1_0(\Omega) \).

We then establish the following property of the \( H^1_0 \)-solution in Theorem 1.2 which is based on test functions computations.

**Theorem 1.3.** Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^N \) containing \( 0, N \geq 3, 0 \leq g \in L^\infty(\Omega) \) and let \( a > 0, b \geq 0, 0 < q < 1 \). If \(-\alpha < -2 \) and \(-\gamma < 0\), then
\[
\begin{cases}
-(a + b \int_\Omega |\nabla u|^2 dx) \Delta u = |x|^{-\alpha}u^{-\gamma} + g(x)u^q & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(4)
admits no bounded positive solution.

**Notation.** In the rest of the paper we make use of the following notation:
\( C, c_i, i = 1, 2, \cdots \) denotes (possibly different) constants;
We denote by \( \| u \|^2 = \int_\Omega |\nabla u|^2 dx \) the Dirichlet norm in \( H^1_0(\Omega) \).

2. **Proof of Theorem 1.1.** We first establish two lemmas to establish the validity and connection of the two constrained sets:
\[
\mathcal{N}_1 := \{ u \in H^1_0(\Omega) : a\|u\|^2 + b\|u\|^4 - \int_\Omega f(x)|u|^{1-\gamma} dx - \int_\Omega g(x)|u|^{1+q} dx \geq 0 \},
\]
\[
\mathcal{N}_2 := \{ u \in H^1_0(\Omega) : a\|u\|^2 + b\|u\|^4 - \int_\Omega f(x)|u|^{1-\gamma} dx - \int_\Omega g(x)|u|^{1+q} dx = 0 \}.
\]

**Lemma 2.1.** The constrained sets \( \mathcal{N}_i, i = 1, 2 \) are nonempty.
Proof. Taking $u \in H^1_0(\Omega)$ with
$$\int_{\Omega} f(x)|u|^{1-\gamma} < \infty,$$
then define the function $U : (0, +\infty) \to \mathbb{R}$ by
$$U(t) := at^{1-q}||u||^2 + bt^{3-q}||u||^4 - t^{-\gamma} \int_{\Omega} f(x)|u|^{1-\gamma}, \forall t > 0.$$ Clearly, $U$ is increasing on $t > 0$, with \(\lim_{t \to +\infty} U(t) = +\infty\) and \(\lim_{t \to 0^+} U(t) = -\infty\). Since
$$\frac{dI(tu)}{dt} = at^q||u||^2 + bt^{3-q}||u||^4 - t^{-\gamma} \int_{\Omega} f(x)|u|^{1-\gamma} dx - t^q \int_{\Omega} g(x)|u|^{1+q},$$
then it follows that there exists the unique minimizer $t(u) > 0$ such that
$$I(tu) \geq I(t(u)u), \quad \forall t > 0 \quad (\text{so that } t(u)u \in \mathcal{N}_2)$$
that is, $I(t(u)u) = \min_{t \geq 0} I(tu)$. In particular, the assumption (2) of Theorem 1.1 implies the existence of $t(u_0) > 0$ such that $t(u_0)u_0 \in \mathcal{N}_2$, and hence $\mathcal{N}_1(\mathcal{N}_2)$ and $\mathcal{N}_2$ are not empty. \hfill $\square$

Also needed will be:

**Lemma 2.2.** $\mathcal{N}_1$ is an unbounded closed set in $H^1_0(\Omega)$.

**Proof.** Clearly, since $tu_0 \in \mathcal{N}_1$ for all $t \geq t(u_0)$, $\mathcal{N}_1$ is unbounded in $H^1_0(\Omega)$. The closeness of $\mathcal{N}_1$ follows easily from Fatou’s Lemma. However, it should be noted that $\mathcal{N}_2$ is not anymore a closed set in $H^1_0(\Omega)$ since $\int_{\Omega} f(x)|u|^{1-\gamma} dx$ is not continuous on $H^1_0(\Omega)$ as $-\gamma < -1$. On the other hand, since $-\gamma < -1$, the reverse form of Hölder’s inequality implies that, for every $u \in \mathcal{N}_1$ (note that $u \neq 0$ as $1 - \gamma < 0$)
$$a||u||^2 + b||u||^4 - \int_{\Omega} g(x)|u|^{1+q} \geq \int_{\Omega} f(x)|u|^{1-\gamma} \geq \left( \int_{\Omega} f(x)^{1/q} \right)^q \left( \int_{\Omega} |u|^{1-\gamma} \right)^{1-\gamma}. $$
Therefore, $||u|| \geq C > 0$ for all $u \in \mathcal{N}_1$ as $\int_{\Omega} f(x)^{1/q} dx > 0$. \hfill $\square$

We are now in a position to establish Theorem 1.1.

**Proof of Theorem 1.1.** Necessity is obvious. We shall justify the sufficiency. We exploit the best minimizing sequence for $\inf_{\mathcal{N}_1} I$ (13)), that is, $(u_n) \subset \mathcal{N}_1$ satisfying:
(i) $I(u_n) < \inf_{\mathcal{N}_1} I + \frac{1}{n}$, (ii) $I(u_n) \leq I(u) + \frac{1}{n}||u_n - u||$, \(\forall u \in \mathcal{N}_1\). As $\mathcal{N}_1$ is a closed set in $H^1_0(\Omega)$ by Lemma 2.2. Since $I(u) = I(|u|)$, we may assume $u_n \geq 0$, and thanks to $u_n \in \mathcal{N}_1$, we have that
$$\int_{\Omega} f(x)|u_n|^{1-\gamma} dx < +\infty,$$
which implies that $u_n(x) > 0 \ \text{a.e.in} \ \Omega$ since $f(x) > 0 \ \text{a.e.in} \ \Omega$ and $-\gamma < -1$. Moreover, $-\gamma = -1$ and $0 < q < 1$ yield that $I(u)$ is coercive on $\mathcal{N}_1$ and therefore $(u_n)$ is bounded in $H^1_0(\Omega)$. Hence there exists $u_{-\gamma} \in H^1_0(\Omega)$ such that, after passing to a subsequence, $u_n \rightharpoonup u_{-\gamma}$ weakly in $H^1_0(\Omega)$, strongly in $L^2(\Omega)$, pointwise a.e. in $\Omega$. This implies $u_{-\gamma} \geq 0$. Thanks to $(u_n) \subset \mathcal{N}_1$ and to Fatou’s Lemma, we have that
$$\int_{\Omega} f(x)(u_{-\gamma})^{1-\gamma} dx < \infty,$$
which in turn implies \( u_{-\gamma}(x) > 0 \) a.e. in \( \Omega \). We now claim that \( u_{-\gamma} \in \mathcal{N}_2 \) and \( \inf_{\mathcal{N}_1} I \) is achieved at such a \( u_{-\gamma} \). Indeed,

Case 1. Suppose that \( (u_n) \subset \mathcal{N}_1 \setminus \mathcal{N}_2 \) for all \( n \) large.

Fix \( \varphi \in H^1_0(\Omega) \), \( \varphi \geq 0 \) by now. Notice that, with \( (u_n) \subset \mathcal{N}_1 \setminus \mathcal{N}_2 \) and \( \gamma < -1 \), there holds that

\[
a\|u_n\|^2 + b\|u_n\|^4 - \int_{\Omega} g(x)u_n^{1+q} > \int_{\Omega} f(x)u_n^{1-\gamma}dx \geq \int_{\Omega} f(x)(u_n + t\varphi)^{1-\gamma}dx, \forall t \geq 0.
\]

Subsequently, choose \( t > 0 \) sufficiently small so that

\[
a\|u_n + t\varphi\|^2 + b\|u_n + t\varphi\|^4 - \int_{\Omega} g(x)(u_n + t\varphi)^{1+q} > \int_{\Omega} f(x)(u_n + t\varphi)^{1-\gamma}dx.
\]

Hence, \( u_n + t\varphi \in \mathcal{N}_1 \). In virtue of (i) and (ii), then we have

\[
\left( \frac{\|\varphi\|}{n} + \frac{1}{2}a(\|u_n + t\varphi\|^2 - \|u_n\|^2) + \frac{1}{4}b(\|u_n + t\varphi\|^4 - \|u_n\|^4) \right) - \frac{1}{1+q} \int_{\Omega} g(x)((u_n + t\varphi)^{1+q} - u_n^{1+q}) \geq \frac{1}{1-\gamma} \int_{\Omega} f(x)((u_n + t\varphi)^{1-\gamma} - u_n^{1-\gamma})
\]

and dividing by \( t > 0 \) and passing to the liminf as \( t \to 0 \), we obtain with Fatou’s Lemma that

\[
\frac{\|\varphi\|}{n} + (a + b\|u_n\|^2) \int_{\Omega} \nabla u_n \cdot \nabla \varphi - \int_{\Omega} g(x)u_n^{\gamma} \varphi \geq \int_{\Omega} \liminf_{t \to 0} f(x) (u_n + t\varphi)^{1-\gamma} - u_n^{1-\gamma} \int_{\Omega} f(x)(u_n + t\varphi)^{1-\gamma} - u_n^{1-\gamma}
\]

\[
= \int_{\Omega} f(x)u_n^{-\gamma} \varphi \quad \text{(since } u_n(x) > 0 \text{ a.e. in } \Omega), \quad (6)
\]

which leads after letting \( n \to +\infty \), thanks once again to Fatou’s Lemma, to

\[
\int_{\Omega} f(x)(u_{-\gamma})^{-\gamma} \varphi dx < +\infty,
\]

for every \( \varphi \in H^1_0(\Omega), \varphi \geq 0 \). Then \( \int_{\Omega} f(x)(u_{-\gamma})^{1-\gamma}dx < \infty \) (applied with \( \varphi = u_{-\gamma} \)) so that the above argument (5) in Lemma 2.1 gives the existence of \( t(u_{-\gamma}) > 0 \) such that \( I(t(u_{-\gamma})u_{-\gamma}) = \min_{t > 0} I(tu_{-\gamma}) \), thus

\[
\inf_{\mathcal{N}_1} I = \lim_{n \to \infty} I(u_n)
\]

\[
= \lim_{n \to \infty} \left[ \frac{1}{2}a\|u_n\|^2 + \frac{1}{4}b\|u_n\|^4 + \frac{1}{\gamma - 1} \int_{\Omega} f(x)u_n^{1-\gamma}dx - \frac{1}{1+q} \int_{\Omega} g(x)u_n^{1+q}dx \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{1}{2}a\|u_n\|^2 + \frac{1}{4}b\|u_n\|^4 + \frac{1}{\gamma - 1} \int_{\Omega} f(x)u_n^{1-\gamma}dx - \frac{1}{1+q} \int_{\Omega} g(x)(u_{-\gamma})^{1+q}dx \right]
\]

\[
\geq \lim_{n \to \infty} \frac{1}{2}a\|u_{-\gamma}\|^2 + \lim_{n \to \infty} \frac{1}{4}b\|u_{-\gamma}\|^4 + \lim_{n \to \infty} \left[ \frac{1}{\gamma - 1} \int_{\Omega} f(x)u_n^{1-\gamma}dx \right]
\]

\[
- \frac{1}{1+q} \int_{\Omega} g(x)(u_{-\gamma})^{1+q}dx
\]

\[
\geq \frac{1}{2}a\|u_{-\gamma}\|^2 + \frac{1}{4}b\|u_{-\gamma}\|^4 + \frac{1}{\gamma - 1} \int_{\Omega} f(x)(u_{-\gamma})^{1-\gamma}dx - \frac{1}{1+q} \int_{\Omega} g(x)(u_{-\gamma})^{1+q}dx
\]

\[
= I(u_{-\gamma}) \geq I(t(u_{-\gamma})u_{-\gamma}) \geq \inf_{\mathcal{N}_2} I \geq \inf_{\mathcal{N}_1} I.
\]
By comparing the previous argument (5), we obtain that \( t(u_{-\gamma}) = 1 \), which shows that
\[
u_{-\gamma} \in \mathcal{N}_2, \quad \inf_{\mathcal{N}_1} I = I(u_{-\gamma}). \tag{7}
\]
Moreover, \( u_{-\gamma} \) satisfies that for every \( \varphi \in H^1_0(\Omega), \varphi \geq 0, \)
\[
(a + b\|u_{-\gamma}\|^2) \int_\Omega \nabla u_{-\gamma} \cdot \nabla \varphi - \int_\Omega g(x)(u_{-\gamma})^q \varphi \geq \int_\Omega f(x)(u_{-\gamma})^{-\gamma} \varphi, \tag{8}
\]
which is not a direct consequence of (6) by the weakly lower semi-continuity of \( \| \cdot \| \) regarding the term \( (a + b\|u_n\|^2) \int_\Omega \nabla u_n \cdot \nabla \varphi \) (the sign of \( \int_\Omega \nabla u_{-\gamma} \cdot \nabla \varphi \) is indefinite). Indeed, as \( \|u_n\| \) is bounded, we may assume, up to subsequences, that \( \|u_n\| \to A \), so that from the weakly lower semi-continuity of \( \| \cdot \| \) we must have \( \|u_{-\gamma}\| \leq \liminf_{n \to \infty} \|u_n\| = \lim_{n \to \infty} \|u_n\| = A \). Actually, \( \|u_{-\gamma}\| = A \), which leads to \( u_n \to u_{-\gamma} \) strongly in \( H^1_0(\Omega) \). If not, then \( \|u_{-\gamma}\| < A \), we would have
\[
I(u_{-\gamma}) = \inf_{\mathcal{N}_1} I = \lim_{n \to \infty} I(u_n)
\]
\[
= \lim_{n \to \infty} \left[ \frac{1}{2} a \|u_n\|^2 + \frac{1}{4} b|\varphi|^4 + \frac{1}{4} \|\varphi\|^4 - \frac{1}{\gamma - 1} \int_\Omega f(x)u_n^{1-\gamma} dx - \frac{1}{1 + q} \int_\Omega g(x)u_n^{1+q} dx \right]
\]
\[
= \frac{1}{2} a A^2 + \frac{1}{4} b A^4 + \lim_{n \to \infty} \left[ \frac{1}{\gamma - 1} \int_\Omega f(x)u_n^{1-\gamma} dx - \frac{1}{1 + q} \int_\Omega g(x)(u_{-\gamma})^{1+q} dx \right]
\]
\[
\geq \frac{1}{2} a A^2 + \frac{1}{4} b A^4 + \frac{1}{\gamma - 1} \int_\Omega f(x)(u_{-\gamma})^{1-\gamma} dx - \frac{1}{1 + q} \int_\Omega g(x)(u_{-\gamma})^{1+q} dx
\]
\[
> \frac{1}{2} a \|u_{-\gamma}\|^2 + \frac{1}{4} b|\varphi|^4 + \frac{1}{\gamma - 1} \int_\Omega f(x)(u_{-\gamma})^{1-\gamma} dx - \frac{1}{1 + q} \int_\Omega g(x)(u_{-\gamma})^{1+q} dx
\]
\[
= I(u_{-\gamma})
\]

a contradiction. The strong convergence: \( u_n \to u_{-\gamma} \) strongly in \( H^1_0(\Omega) \) leads then to the estimate (8) of \( \int_\Omega f(x)(u_{-\gamma})^{-\gamma} \varphi^q dx \), thanks to Fatou’s Lemma and (6).

Case 2. There exists a subsequence of \( (u_n) \) (we still call \( u_n \)), which belongs to \( \mathcal{N}_2 \).

Fix \( \varphi \in H^1_0(\Omega), \varphi \geq 0 \). Since \( -\gamma < -1 \),
\[
\int_\Omega f(x)(u_n + t\varphi)^{1-\gamma} dx \leq \int_\Omega f(x)u_n^{1-\gamma} dx < \infty, \forall t \geq 0.
\]
By the previous argument (5), we get the existence of some function \( f_{n,\varphi} : [0, +\infty) \to (0, +\infty) \) such that
\[
f_{n,\varphi}(0) = 1, \quad f_{n,\varphi}(t)u_n + t\varphi) \in \mathcal{N}_2, \forall t \geq 0.
\]
It follows from the dominated convergence theorem and the facts \( -\gamma < -1 \), \( \int_\Omega f(x)u_n^{1-\gamma} dx < +\infty \), that \( f_{n,\varphi}(t) \) is continuous on \( t \geq 0 \). However, we have no idea whether or not \( f_{n,\varphi}(t) \) is differentiable. In order to prove \( u_{-\gamma} \in \mathcal{N}_2 \), we define \( f'_{n,\varphi}(0) \) by
\[
f'_{n,\varphi}(0) := \lim_{t \to 0^+} \frac{f_{n,\varphi}(t) - f_{n,\varphi}(0)}{t} \in [-\infty, +\infty].
\]
If the limit does not exist, we let \( t_k \to 0 \) (instead of \( t \to 0 \)) with \( t_k > 0 \) chosen in such a way that \( f'_{n,\varphi}(0) = \lim_{k \to \infty} \frac{f_{n,\varphi}(t_k) - f_{n,\varphi}(0)}{t_k} \in [-\infty, +\infty] \). We shall estimate \( f'_{n,\varphi}(0) \).
We recall that \( f_{n,\varphi}(t)(u_n + t\varphi) \in \mathcal{N}_2 \), \( u_n \in \mathcal{N}_2 \) with
\[
0 = a f^2_{n,\varphi}(t)||u_n + t\varphi||^2 + b f^4_{n,\varphi}(t)||u_n + t\varphi||^4
- f_{n,\varphi}^{-1}(t) \int_{\Omega} f(x)(u_n + t\varphi)^{-1} \gamma dx - f_{n,\varphi}^{1+q}(t) \int_{\Omega} g(x)(u_n + t\varphi)^{1+q} dx,
\]
\[
0 = a ||u_n||^2 + b ||u_n||^4 - \int_{\Omega} f(x)u_n^{-1} \gamma dx - \int_{\Omega} g(x)(u_n)^{1+q} dx.
\]

Therefore, by the continuity of \( f_{n,\varphi}(t) \), \( \forall t \geq 0 \), we write that
\[
0 = \{ a[f_{n,\varphi}(t) + 1]||u_n + t\varphi||^2 + 4b[f_{n,\varphi}(0) + o(1)]||u_n + t\varphi||^4
- (1 - \gamma)[f_{n,\varphi}(0) + o(1)]^{-\gamma} \int_{\Omega} f(x)(u_n + t\varphi)^{-1} \gamma dx
- (1 + q)[f_{n,\varphi}(0) + o(1)]^q \int_{\Omega} g(x)(u_n + t\varphi)^{1+q} \}
\frac{f_{n,\varphi}(t) - 1}{t}
+ a \frac{||u_n + t\varphi||^2 - ||u_n||^2}{t} + b \frac{||u_n + t\varphi||^4 - ||u_n||^4}{t}
- \frac{1}{t} \int_{\Omega} f(x) \left[ (u_n + t\varphi)^{-1-\gamma} - u_n^{-\gamma} \right]
- \frac{1}{t} \int_{\Omega} g(x) \left[ (u_n + t\varphi)^{1+q} - u_n^{1+q} \right],
\]
and letting \( t \to 0 \), we then obtain
\[
0 \geq f_{n,\varphi}'(0) \left\{ 2a||u_n||^2 + 4b||u_n||^4 + (\gamma - 1) \int_{\Omega} f(x)u_n^{-1} \gamma dx
- (1 + q) \int_{\Omega} g(x)u_n^{1+q} \right\}
+ (2a + 4b||u_n||^2) \int_{\Omega} \nabla u_n \cdot \nabla \varphi - (1 + q) \int_{\Omega} g(x)u_n^{q}\varphi
\]
\[
= f_{n,\varphi}'(0) \left\{ a(1 - q)||u_n||^2 + b(3 - q)||u_n||^4 + (\gamma + q) \int_{\Omega} f(x)u_n^{-1} \gamma dx \right\}
+ (2a + 4b||u_n||^2) \int_{\Omega} \nabla u_n \cdot \nabla \varphi - (1 + q) \int_{\Omega} g(x)u_n^{q}\varphi,
\] (since \( u_n \in \mathcal{N}_2 \))

which implies that \( f_{n,\varphi}'(0) \neq +\infty \). Furthermore, we get, thanks to the fact that \( (u_n) \subset \mathcal{N}_2(\subset \mathcal{N}_1) \) is bounded in \( H_0^1(\Omega) \), and to Lemma 2.2: \( ||u|| \geq C > 0 \) for all \( u \in \mathcal{N}_1 \), that
\[
f_{n,\varphi}'(0) \leq C_1 \text{ uniformly in } n
\] (9)
for suitable constant \( C_1 > 0 \). On the other hand, \( f_{n,\varphi}'(0) \equiv -\infty \) and bounded from below uniformly for all \( n \) large enough that
\[
\frac{||u_n||}{n} + \frac{(1 - q)aC^2}{1 - \gamma} < 0.
\]
We assume \( f_{n,\varphi}'(0) \in [-\infty, 0] \) for all \( n \) large now. Otherwise, the uniform boundedness from below is obvious. Indeed, from (ii) consider
\[
\frac{1}{n} \left( \frac{1 - f_{n,\varphi}(t)}{t} \right) ||u_n|| + \frac{1}{n} f_{n,\varphi}(t)||\varphi|| \geq \frac{1}{n} ||u_n - f_{n,\varphi}(t)(u_n + t\varphi)|| \frac{1}{t}
\]
\[
\geq |I(u_n) - I(f_{n,\varphi}(t)(u_n + t\varphi))| \frac{1}{t},
\]
Then, thanks to \((u_n) \subset N_2\), we can write
\[
\|\varphi\|_n \int_{\Omega} f_{n,\varphi}(t) \geq \left( \frac{\|u_n\|}{n} - a \left( \frac{1}{2} + \frac{1}{\gamma - 1} \right) \right) \left[ f_{n,\varphi}(t) + 1 \right] \|u_n + t\varphi\|^2 \\
+ \left( 1 + \frac{1+q}{\gamma - 1} \right) \left[ f_{n,\varphi}(0) + o(1) \right]^q \int_{\Omega} g(x)(u_n + t\varphi)^{1+q} \\
- 4b \left( \frac{1}{4} + \frac{1}{\gamma - 1} \right) \left[ f_{n,\varphi}(0) + o(1) \right]^q \|u_n + t\varphi\|^4 \\
- a \left( \frac{1}{2} + \frac{1}{\gamma - 1} \right) \|u_n + t\varphi\|^2 - \|u_n\|^2 \\
- b \left( \frac{1}{4} + \frac{1}{\gamma - 1} \right) \|u_n + t\varphi\|^4 - \|u_n\|^4 \\
+ \left( \frac{1}{\gamma - 1} + \frac{1}{1+q} \right) \int_{\Omega} g(x) \frac{(u_n + t\varphi)^{1+q} - u_n^{1+q}}{t} \, dx.
\]
Note here that \(0 < q < 1, -\gamma < -1, u_n \in N_2\) and \(\|u\| \geq C > 0\) for all \(u \in N_1\) and that, as a consequence,
\[
-2a \left( \frac{1}{2} + \frac{1}{\gamma - 1} \right) \|u_n\|^2 - b \left( 1 + \frac{4}{\gamma - 1} \right) \|u_n\|^4 + \left( 1 + \frac{1+q}{\gamma - 1} \right) \int_{\Omega} g(x) u_n^{1+q} \\
= -\frac{1}{\gamma - 1} \left[ a(\gamma + 1)\|u_n\|^2 + b(\gamma + 3)\|u_n\|^4 - (\gamma + q) \int_{\Omega} g(x) u_n^{1+q} \right] \\
= -\frac{1}{\gamma - 1} \left[ a(1 - q)\|u_n\|^2 + b(3 - q)\|u_n\|^4 + (\gamma + q) \int_{\Omega} f(x) u_n^{1-}\right] \\
\leq -\frac{(1 - q)aC^2}{\gamma - 1},
\]
so that, letting \(t \to 0^+\) direct calculation yields
\[
\|\varphi\|_n \geq f'_{n,\varphi} \left[ \frac{\|u_n\|}{n} - \frac{(1 - q)aC^2}{\gamma - 1} \right] - 2 \left( \frac{1}{2} + \frac{1}{\gamma - 1} \right) a \int_{\Omega} \nabla u_n \cdot \nabla \varphi \\
- \left( 1 + \frac{4}{\gamma - 1} \right) b\|u_n\|^2 \int_{\Omega} \nabla u_n \cdot \nabla \varphi + \left( 1 + \frac{1+q}{\gamma - 1} \right) \int_{\Omega} g(x) u_n^{3}\varphi.
\]
Thus the above inequality leads, thanks to the facts that \((u_n)\) is bounded in \(H_0^1(\Omega)\), \(-\gamma < -1\) and \(0 < q < 1\), to
\[
f'_{n,\varphi} \neq -\infty, \quad f'_{n,\varphi}(0) \geq c_1 \text{ uniformly in all } n \text{ large} \tag{10}
\]
for suitable constant \(c_1 \in \mathbb{R}\). The uniform boundedness of \(f'_{n,\varphi}(0)\) follows from (9) and (10). Now, we check (7)-(8) in Case 2.

Fix \(\varphi \in H_0^1(\Omega), \varphi \geq 0\). From (ii) we consider again
\[
\frac{1}{n} \left[ \frac{f_{n,\varphi}(t) - 1}{t} \|u_n\| + f_{n,\varphi}(t)\|\varphi\| \right] \geq \frac{1}{n} \|f_{n,\varphi}(t)(u_n + t\varphi) - u_n\| \frac{1}{t} \\
\geq [I(u_n) - I(f_{n,\varphi}(t)(u_n + t\varphi))] \frac{1}{t},
\]
By the continuity of $f_{n,\varphi}(t), \forall t \geq 0$, we then have that
\[
\frac{||u_n||}{n} \left| \frac{f_{n,\varphi}(t)}{f_{n,\varphi}(t)} - 1 \right| + \frac{||\varphi||}{n} f_{n,\varphi}(t) \\
\geq - \frac{1}{2} a[f_{n,\varphi}(t) + 1]|u_n + t\varphi|^2 - b[f_{n,\varphi}(0) + o(1)]^3 |u_n + t\varphi|^4 \\
+ [f_{n,\varphi}(0) + o(1)]^{-\gamma} \int_{\Omega} f(x)(u_n + t\varphi)^{1-\gamma} dx \\
+ [f_{n,\varphi}(0) + o(1)]^{q} \int_{\Omega} g(x)(u_n + t\varphi)^{1+q} \frac{f_{n,\varphi}(t) - 1}{t} \\
- \frac{a}{2} \frac{||u_n + t\varphi||^2 - ||u_n||^2}{t} - b \frac{||u_n + t\varphi||^4 - ||u_n||^4}{t} \\
+ \frac{1}{1-\gamma} \int_{\Omega} \frac{f(x)[(u_n + t\varphi)^{1-\gamma} - u_n^{1-\gamma}]}{t} + \frac{1}{1+q} \int_{\Omega} g(x) \frac{(u_n + t\varphi)^{1+q} - u_n^{1+q}}{t}.
\]

Since $u_n \in N_2$, $f_{n,\varphi}(0) \neq \pm \infty$, and $-\gamma < -1$, we obtain that
\[
\frac{||u_n||}{n} \left| \frac{f_{n,\varphi}(0)}{f_{n,\varphi}(0)} \right| + \frac{||\varphi||}{n} \\
\geq - (a||u_n||^2 + b||u_n||^4 - \int_{\Omega} f(x)u_n^{1-\gamma} - \int_{\Omega} g(x)u_n^{1+q}) f_{n,\varphi}'(0) \\
- (a + b||u_n||^2) \int_{\Omega} \nabla u_n \cdot \nabla \varphi + \int_{\Omega} g(x)u_n^q \varphi \\
+ \int_{\Omega} \liminf_{t \to 0^+} f(x)(u_n + t\varphi)^{1-\gamma} - f(x)u_n^{1-\gamma} \\
= - (a + b||u_n||^2) \int_{\Omega} \nabla u_n \cdot \nabla \varphi + \int_{\Omega} f(x)u_n^{-\gamma} \varphi + \int_{\Omega} g(x)u_n^q \varphi,
\]
which leads, thanks to Fatou’s lemma and the uniform boundedness of $(u_n)$ and $f_{n,\varphi}'(0)$ for all $n$ large, to
\[
\int_{\Omega} f(x)(u_n - \gamma) \varphi dx < +\infty,
\]
for every $\varphi \in H^1_0(\Omega), \varphi \geq 0$. By taking the same argument as in Case 1 we also have that
\[
\inf_{N_1} I = I(u_{-\gamma}), \quad u_{-\gamma} \in N_2,
\]
and, for every $\varphi \in H^1_0(\Omega), \varphi \geq 0$,
\[
(a + b||u_{-\gamma}||^2) \int_{\Omega} \nabla u_{-\gamma} \cdot \nabla \varphi - \int_{\Omega} g(x)u_{-\gamma}^q \varphi \geq \int_{\Omega} f(x)u_{-\gamma}^{-\gamma} \varphi,
\]
in Case 2.

Thanks to (7), (8), (11) and (12), we can conclude that, up to subsequences, $u_n \to u_{-\gamma}$ strongly in $H^1_0(\Omega), u_{-\gamma} \in N_2$ although $u_n \in N_1$, $\inf_{N_1} I = I(u_{-\gamma})$ and
\[
(a + b||u_{-\gamma}||^2) \int_{\Omega} \nabla u_{-\gamma} \cdot \nabla \varphi dx - \int_{\Omega} f(x)(u_{-\gamma})^{-\gamma} \varphi dx - \int_{\Omega} g(x)u_{-\gamma}^q \varphi dx \geq 0,
\]
for every $\varphi \in H^1_0(\Omega), \varphi \geq 0$.

We are ready to prove that $u_{-\gamma}$ is a weak solution of (1).
Let $\psi \in H^1_0(\Omega)$ be fixed. Since $u_{\gamma} \in N_2$, using (13), we write with $\varphi = (u_{\gamma} + t\psi)^+, t > 0$ that

$$0 \leq \frac{1}{t} (a + b\|u_{\gamma}\|^2) \int_{\Omega} \nabla u_{\gamma} \cdot \nabla (u_{\gamma} + t\psi)^+$$

$$- \frac{1}{t} \int_{\Omega} f(x)(u_{\gamma})^{-\gamma}(u_{\gamma} + t\psi)^+ - \frac{1}{t} \int_{\Omega} g(x)u^2_{\gamma}(u_{\gamma} + t\psi)^+$$

$$= \frac{1}{t} (a + b\|u_{\gamma}\|^2) \int_{\Omega} \nabla u_{\gamma} \cdot \nabla (u_{\gamma} + t\psi) - \frac{1}{t} \int_{\Omega} f(x)(u_{\gamma})^{-\gamma}(u_{\gamma} + t\psi)$$

$$- \frac{1}{t} \int_{\Omega} g(x)u^2_{\gamma}(u_{\gamma} + t\psi) - \frac{1}{t} (a + b\|u_{\gamma}\|^2) \int_{|u_{\gamma} + t\psi| < 0} \nabla u_{\gamma} \cdot \nabla (u_{\gamma} + t\psi)$$

$$+ \frac{1}{t} \int_{|u_{\gamma} + t\psi| < 0} f(x)(u_{\gamma})^{-\gamma}(u_{\gamma} + t\psi) + g(x)u^2_{\gamma}(u_{\gamma} + t\psi)$$

$$\leq \frac{1}{t} \left[ a\|u_{\gamma}\|^2 + b\|u_{\gamma}\|^4 - \int_{\Omega} f(x)(u_{\gamma})^{1-\gamma} - \int_{\Omega} g(x)u^{1+\gamma} \right]$$

$$+ (a + b\|u_{\gamma}\|^2) \int_{\Omega} \nabla u_{\gamma} \cdot \nabla \psi - \int_{\Omega} f(x)(u_{\gamma})^{-\gamma} \psi + g(x)u^2_{\gamma} \psi$$

$$- (a + b\|u_{\gamma}\|^2) \int_{|u_{\gamma} + t\psi| < 0} \nabla u_{\gamma} \cdot \nabla \psi$$

$$= (a + b\|u_{\gamma}\|^2) \int_{\Omega} \nabla u_{\gamma} \cdot \nabla \psi - \int_{\Omega} f(x)(u_{\gamma})^{-\gamma} \psi + g(x)u^2_{\gamma} \psi$$

(since $u_{\gamma} \in N_2$)

$$- (a + b\|u_{\gamma}\|^2) \int_{|u_{\gamma} + t\psi| < 0} \nabla u_{\gamma} \cdot \nabla \psi.$$ 

The proposition follows from $\text{meas}|u_{\gamma} + t\psi| < 0| \to 0$ as $t \to 0^+$ and the arbitrariness of $\psi \in H^1_0(\Omega)$, that is, for every $\psi \in H^1_0(\Omega)$,

$$(a + b\|u_{\gamma}\|^2) \int_{\Omega} \nabla u_{\gamma} \cdot \nabla \psi - \int_{\Omega} f(x)(u_{\gamma})^{-\gamma} \psi - \int_{\Omega} g(x)u^2_{\gamma} \psi = 0.$$

3. **Proof of Theorem 1.2.** *Proof of Theorem 1.2.* We let $\varphi_1$ be the first positive eigenfunction of $-\Delta$ in $\Omega$ with Dirichlet boundary condition, that is, $-\Delta \varphi_1 = \lambda_1 \varphi_1, \varphi_1|_{\partial \Omega} = 0$, with $\lambda_1$ the first Dirichlet eigenvalue of $-\Delta$. It is well known that

$$\int_{\Omega} \varphi_1^{-l}(x)dx < \infty \quad (14)$$

if and only if if $l > -1$ (cf.[30]). By the result of [30, Theorem 1 and 2], we also know that: for any $-\gamma \in (-3, -1)$, there exists at least one $H^1_0$-function $u_0 \in C^2(\Omega) \cap C(\overline{\Omega}) \cap H^1_0(\Omega)$ such that

$$d_0 \varphi_1^{\frac{1}{\gamma}}(x) \leq u_0(x) \leq d_1 \varphi_1^{\frac{1}{\gamma}}(x), \forall x \in \Omega. \quad (15)$$

Then we get the existence of some constants $0 < c_2 < C_2$ such that $0 < c_2 \leq u_{\gamma}(x) \leq C_2$ in a neighborhood of 0, say $B_{r_0}(0)$ the Euclidean ball of center 0 and radius $r_0$. We now check (2) in Theorem 1.1 applied to $f(x) := |x|^{-\alpha}$ with the above function $u_0(x)$: for any $-\alpha \in (-N, 0), -\gamma \in (-3, -1),$

$$\int_{\Omega} |x|^{-\alpha} u_0^{-\gamma} dx < \infty.$$
Indeed, thanks to (14), (15) and to the fact that $-\alpha > -N$, we have that

$$\int_{\Omega} |x|^{-\alpha} u_0^{1-\gamma} dx = \int_{B_{\rho_0}(0)} |x|^{-\alpha} u_0^{1-\gamma} dx + \int_{\Omega \setminus B_{\rho_0}(0)} |x|^{-\alpha} u_0^{1-\gamma} dx$$

$$\leq c_2^{1-\gamma} \int_{B_{\rho_0}(0)} |x|^{-\alpha} dx + r_0^{-\alpha} \int_{\Omega \setminus B_{\rho_0}(0)} u_0^{1-\gamma} dx$$

$$\leq c_2^{1-\gamma} \int_{B_{\rho_0}(0)} |x|^{-\alpha} dx + r_0^{-\alpha} d_0^{1-\gamma} \int_{\Omega \setminus B_{\rho_0}(0)} \varphi_1^{2(1-\gamma)}(x) dx$$

$$\leq c_2^{1-\gamma} \int_{B_{\rho_0}(0)} |x|^{-\alpha} dx + r_0^{-\alpha} d_0^{1-\gamma} \int_{\Omega \setminus B_{\rho_0}(0)} \varphi_1^{2(1-\gamma)}(x) dx < +\infty,$$

since $\frac{2(1-\gamma)}{1+\gamma} > -1$ if and only if $-\gamma > -3$. Then (2) holds. This also ends the proof of Theorem 1.2.

4. **Proof of Theorem 1.3.** Proof of Theorem 1.3. Assume by contradiction that, \( \sup_{\Omega} u(x) < +\infty \). Then it will always be possible to choose a sequence of test functions \( \psi_\delta \subset C_0^\infty(\Omega) \), which would violate the equation satisfied by \( u \):

$$a \int_{\Omega} \nabla u \cdot \nabla \psi + b \left( \int_{\Omega} |\nabla u|^2 dx \right) \int_{\Omega} \nabla u \cdot \nabla \psi - \int \Omega |x|^{-\alpha} u^{-\gamma} \psi - \int \Omega g(x) u^q \psi = 0. \ (16)$$

Just take, for instance, \( \psi_\delta \in C_0^\infty(\Omega) \) such that \( 0 \leq \psi_\delta \leq 1 \), \( \psi_\delta \equiv 0 \) in \( B_\delta(0) \), \( \psi_\delta \equiv 1 \) in \( B_{3\delta/3}(0) \setminus B_{2\delta/3}(0) \), \( \psi_\delta \equiv 0 \) in \( \Omega \setminus B_{2\delta}(0) \) and \( |\Delta \psi_\delta| \leq \frac{C(N,|\beta|)}{d^{N-\alpha}(\Omega,\partial \Omega)} \) in \( \Omega \). As is well known, for any \( \Omega' \subset \subset \Omega \), where \( \Omega \) is a bounded open set in \( \mathbb{R}^N \) with \( N \geq 1 \), we can construct some \( \psi(x) \in C_0^\infty(\Omega) \) be such that \( 0 \leq \psi(x) \leq 1 \) on \( \Omega \), \( \psi(x) \equiv 1 \) on \( \Omega' \) and \( |\Delta \psi(x)| \leq \frac{C(N,|\beta|)}{d^{N-\alpha}(\Omega,\partial \Omega)} \) on \( \Omega \) by taking advantage of mollifiers. Thus the above equation leads, thanks to (16), to the definition of \( \psi_\delta(x) \) and to the facts that \( g(x) \geq 0, -\gamma < 0, \) to

$$\int \Omega (|x|^{-\alpha} u^{-\gamma} + g(x) u^q) \psi_\delta dx = \int_{B_{2\delta}(0) \setminus B_{\delta}(0)} (|x|^{-\alpha} u^{-\gamma} + g(x) u^q) \psi_\delta dx$$

$$\geq \int_{B_{2\delta}(0) \setminus B_{\delta}(0)} |x|^{-\alpha} u^{-\gamma} \psi_\delta dx \geq \left( \sup_{\Omega} u \right)^{-\gamma} \int_{B_{2\delta}(0) \setminus B_{\delta}(0)} |x|^{-\alpha} \psi_\delta dx$$

$$\geq \left( \sup_{\Omega} u \right)^{-\gamma} \int_{B_{2\delta}(0) \setminus B_{\delta}(0)} |x|^{-\alpha} dx$$

$$= \left( \sup_{\Omega} u \right)^{-\gamma} \left\lbrack \left( \frac{5}{3} \right)^{N-\alpha} - \left( \frac{4}{3} \right)^{N-\alpha} \right\rbrack \frac{N\omega_N}{N-\alpha} \delta^{N-\alpha},$$

while

$$\left( a + b||u||^2 \right) \int \Omega \nabla u \cdot \nabla \psi_\delta dx = (a + b||u||^2) \int \Omega \nabla \cdot -\Delta \psi_\delta dx$$

$$\leq (a + b||u||^2) \int \Omega u \cdot |\Delta \psi_\delta| dx = (a + b||u||^2) \int_{B_{2\delta}(0) \setminus B_{\delta}(0)} u \cdot |\Delta \psi_\delta| dx$$

$$\leq \sup_{\Omega} u (a + b||u||^2) \frac{C(N)}{\delta^2} (2^N - 1) \omega_N \delta^N,$$
where $\omega_N$ is the volume of the standard unit sphere in $\mathbb{R}^N$. We thus obtain that
\[
(\sup_{\Omega} u)^{1+\gamma} \geq \frac{1}{(a+b||u||^2)} \left[ \left(\frac{5}{3}\right)^{N-\alpha} - \left(\frac{4}{3}\right)^{N-\alpha} \right] \frac{N}{C(N)(N-\alpha)(2^N-1)} \delta^{2-\alpha}
\]
a contradiction as $\delta \to 0^+$, thanks to $-\alpha < -2$. This ends the proof of Theorem 1.3. \qed

Acknowledgments. The authors thank the anonymous referee for carefully reading this paper and suggesting many useful comments. This work was supported by NSFC Grants 11571339 and 11771468.

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Received October 2017; revised January 2018.

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