THE CYCLIC HOMOLOGY AND K-THEORY OF CERTAIN ADELCIC CROSSED PRODUCTS

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Abstract. The multiplicative group of a global field acts on its adele ring by multiplication. We consider the crossed product algebra of the resulting action on the space of Schwartz functions on the adele ring and compute its Hochschild, cyclic and periodic cyclic homology. We also compute the topological K-theory of the $C^\ast$-algebra crossed product.

1. Introduction

Let $K$ be a global field. That is, $K$ is a finite extension of the rational numbers or the field $\mathbb{F}_p(t)$, where $\mathbb{F}_p$ is the field with $p$ elements for some prime $p$. The adele ring $\mathcal{A}_K$ over $K$ is a certain locally compact ring that contains $K$ as a discrete cocompact subfield. Its multiplicative group $\mathcal{A}_K^\times$ is a locally compact group as well for an appropriate topology. It contains $K^\times$ as a discrete subgroup. The quotient group $C_K := \mathcal{A}_K^\times/K^\times$ is the idele class group of $K$.

Let $S(\mathcal{A}_K)$ be the space of Bruhat-Schwartz functions on $\mathcal{A}_K$ as defined by François Bruhat in [2]. We equip it with a canonical bornology, so that $S(\mathcal{A}_K)$ is a nuclear, complete, convex bornological vector space. Both the pointwise product and the convolution define algebra structures on $S(\mathcal{A}_K)$. Hence the Fourier transform can be viewed as an operator on $S(\mathcal{A}_K)$. Since it intertwines the convolution and the pointwise product, the two multiplications on $S(\mathcal{A}_K)$ give rise to isomorphic algebras.

The group $\mathcal{A}_K^\times$ acts on $\mathcal{A}_K$ by multiplication. Thus $\mathcal{A}_K^\times$ and its discrete subgroup $K^\times$ act on $S(\mathcal{A}_K)$. We let $K^\times \ltimes S(\mathcal{A}_K)$ be the algebraic crossed product. As a bornological vector space, this is $\bigoplus_{g \in K^\times} S(\mathcal{A}_K)$ with the usual convolution product. We shall compute the Hochschild homology, the cyclic homology and the periodic cyclic homology of this crossed product. In addition, we also partially compute the topological K-theory of the enveloping $C^\ast$-algebra $K^\times \ltimes C_0(\mathcal{A}_K)$. We use $\ltimes$ for both the algebraic and the $C^\ast$-algebra crossed products. It should be clear from the context which crossed product is meant.

An important additional structure on these homology spaces is the action of the idele class group $C_K$. Since $\mathcal{A}_K^\times$ is commutative, its action on $S(\mathcal{A}_K)$ extends to an action on $K^\times \ltimes S(\mathcal{A}_K)$ by automorphisms. Clearly, the subgroup $K^\times$ acts by inner automorphisms on the crossed product. Therefore, it acts trivially on the various homology spaces of $K^\times \ltimes S(\mathcal{A}_K)$. Thus the action of $\mathcal{A}_K^\times$ descends to an action of $C_K$. We also describe the representation of $C_K$ on the homology spaces. We state our results and briefly summarize the way we obtain them in Section 2.

Our computation of Hochschild and cyclic homology is a byproduct of the spectral interpretation for poles and zeros of $L$-functions constructed by the author in [8]. This spectral interpretation is inspired by a similar construction by Alain...
Connes in [4]. A basic ingredient is the coinvariant space $S(A_K)/K^\times$. However, to see zeros of $L$-functions, we have to compare this space to another natural space $A(C_K)$ of functions on $C_K$. If $K$ is a global function field, then $A(C_K) = D(C_K)$ is the space of locally constant functions with compact support. If $K$ is an algebraic number field, then $A(C_K)$ is the space of smooth functions on $C_K$ with superexponentially decaying derivatives for $\ln |x| \to \pm \infty$. Roughly speaking, we compare the noncommutative quotient space $\mathbb{A}_K/K^\times$ to $\mathbb{A}_K^0/K^\times$. Since the action of $K^\times$ on $\mathbb{A}_K^0$ is free and proper, there is nothing noncommutative about the latter quotient space.

The results of [8] indicate that the spaces $\mathbb{A}_K/K^\times$ and $\mathbb{A}_K^0/K^\times$ are closely related. This also comes out in our homology and K-theory computations. The Hochschild, cyclic and periodic cyclic homology spaces of crossed products decompose naturally as a direct sum of a homogeneous and an inhomogeneous part. We recall this decomposition later. The inhomogeneous part vanishes for $\mathbb{A}_K/K^\times$, but it is not very interesting either: the inhomogeneous part can be detected by evaluation at the fixed point $0 \in \mathbb{A}_K$. The homogeneous parts of the various cyclic homology spaces for $K^\times \times S(\mathbb{A}_K)$ are very similar to the corresponding spaces for $A(C_K)$. For the K-theory computation, the relationship is not that strong but still perceptible.

Éric Leichtnam and Victor Nistor have already attempted to compute the cyclic theory of $Q^\times \times S(\mathbb{A}_Q)$ in [7]. They compute the inhomogeneous part correctly, but they make a fatal mistake for the homogeneous part, which leads to a wrong final result. In particular, their results for the Hochschild and cyclic homology of $Q^\times \times S(\mathbb{A}_Q)$ have not much in common with the corresponding spaces for $A(C_Q)$. The general theory of how to compute Hochschild and cyclic homology for group algebras and crossed products can be found in [1, 3, 5, 10, 11]. However, we only use some elementary facts from this theory. The result of our Hochschild homology is so simple that we can pass to the cyclic theories immediately.

Our K-theory computation uses the Baum-Connes conjecture for the Abelian group $K^\times$ and a spectral sequence for the topological side of the Baum-Connes assembly map for torsion free discrete groups. Again, we use special properties of our situation to simplify the computation.

## 2. Summary and statement of the results

For any discrete group crossed product $G \times A$, the Hochschild, cyclic and periodic cyclic homology spaces split naturally over conjugacy classes of $G$. The contribution of the trivial conjugacy class of $G$ is called the homogeneous part $(\cdots)_h$, the contribution of the other conjugacy classes is called the inhomogeneous part $(\cdots)_{inh}$.

The homogeneous part of the Hochschild homology of $K^\times \times S(\mathbb{A}_K)$ is easy to compute using results of [8]. To describe it explicitly, we split $\mathbb{A}_K = \mathbb{A}_f \times \mathbb{A}_\infty$, where $\mathbb{A}_f$ and $\mathbb{A}_\infty$ are the finite and infinite adele rings, respectively. Thus $\mathbb{A}_f$ is totally disconnected and $\mathbb{A}_\infty \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$. Let $\Lambda^n(\mathbb{A}_\infty)$ be the finite dimensional vector space of $n$-forms over the real vector space $\mathbb{A}_\infty$ and let $\Omega^n(\mathbb{A}_K)$ be the space of Bruhat-Schwartz functions on $\mathbb{A}_K$ taking values in $\Lambda^n(\mathbb{A}_\infty)$. The space $\Omega^n(\mathbb{A}_K)$ is naturally isomorphic to $HH_n(S(\mathbb{A}_K))$. Naturality includes the assertion that this isomorphism is $\mathbb{A}_K^0$-equivariant. Extending slightly the results of [8], we show that $H_j(K^\times \times \Omega^n(\mathbb{A}_K)) = 0$ for $j \geq 1$ and describe the coinvariant space as a space of functions on $C_K$. It follows easily that

$$HH_n(K^\times \times S(\mathbb{A}_K))_h \cong \Omega^n(\mathbb{A}_K)/K^\times.$$

This isomorphism on Hochschild homology can be realized by an explicit chain map from the Hochschild complex $(\Omega(K^\times \times S(\mathbb{A}_K)), b)$ to $\Omega^*(\mathbb{A}_K)/K^\times$ with vanishing boundary map. Up to a constant factor, this chain map also intertwines the
de Rham boundary map \( d_{dR}: \Omega^n(\mathbb{A}_K) \to \Omega^{n+1}(\mathbb{A}_K) \) and the boundary \( B \) on \( \Omega(K^\times \ltimes S(\mathbb{A}_K)) \). Hence it induces a quasi-isomorphism of the cyclic bicomplexes. Since the vertical boundary map in \( \Omega^*(\mathbb{A}_K) \) vanishes, this allows us to compute the cyclic homology and the periodic cyclic homology.

We need more notation in order to describe the result. Let \( \sigma: \mathbb{A}_K^\times \to \{ \pm 1 \} \) be the character that multiplies the signs of an idele at the real places and let \( \mathbb{C}(\sigma) \) be the corresponding 1-dimensional representation. There is a canonical integration map \( j: \Omega^*(\mathbb{A}_K) \to S(\mathbb{A}_\mathbb{Q}) \) that only depends on a choice of orientation on \( \mathbb{A}_\mathbb{Q} \). Hence it is an equivariant map to the representation \( S(\mathbb{A}_\mathbb{Q}) \otimes \mathbb{C}(\sigma) \) of \( \mathbb{A}_K^\times \cong \mathbb{A}_\mathbb{Q}^\times \rtimes \mathbb{Z}_K^\times \). It is easy to see that the kernel of \( j \) is a contractible complex. Hence \( (\Omega^*(\mathbb{A}_K), d_{dR}, j) \) is a resolution of \( S(\mathbb{A}_\mathbb{Q}) \otimes \mathbb{C}(\sigma) \). Our previous results show that this resolution is acyclic for group homology. Hence the complex \( (\Omega^*(\mathbb{A}_K)/K^\times, d_{dR}) \) computes the group homology \( H_j(K^\times, S(\mathbb{A}_\mathbb{Q}) \otimes \mathbb{C}(\sigma)) \). We obtain

\[
\text{HP}_j(K^\times \ltimes S(\mathbb{A}_\mathbb{Q}))_n \cong \bigoplus_{k \in \mathbb{Z}} H_{j+2k}(K^\times, S(\mathbb{A}_\mathbb{Q}) \otimes \mathbb{C}(\sigma))
\]

for \( j \in \mathbb{Z}/2 \) and a similar result for cyclic homology. It remains to describe \( H_j(K^\times, S(\mathbb{A}_\mathbb{Q}) \otimes \mathbb{C}(\sigma)) \). The higher group homology spaces vanish for \( K = \mathbb{Q} \) and more generally if \( K \) has at most one infinite place. Let \( K_{\infty} \subseteq K \) be the subring of algebraic integers and let \( K^\times_{\infty} \) be the multiplicative group of invertible algebraic integers \( K^\times_{\infty} \). It is known that \( K^\times_{\infty} \) is a product of the finite cyclic group of roots of unity in \( K^\times \) and a free Abelian group of rank \( u - 1 \), where \( u \) is the number of infinite places of \( K \). The action of \( K^\times_{\infty} \) on \( S(\mathbb{A}_\mathbb{Q}) \otimes \mathbb{C}(\sigma) \) factors through a compact group and hence is close to a trivial action. Hence it is no surprise that the group homology of \( \mathbb{Z}^{u-1} \) comes up. Indeed, there is an isomorphism

\[
H_k(K^\times, S(\mathbb{A}_\mathbb{Q}) \otimes \mathbb{C}(\sigma)) \cong D(\mathbb{A}_\mathbb{Q}^\times/K^\times) \otimes \Lambda^k \mathbb{C}^{u-1}
\]

where \( \mathbb{A}_\mathbb{Q}^\times \) denotes the closure of \( K^\times \) in \( \mathbb{A}_\mathbb{Q}^\times \).

The inhomogeneous part of the Hochschild homology can be described as follows. Evaluation at the fixed point \( 0 \in \mathbb{A}_K \) of the action of \( \mathbb{A}_K^\times \) induces an algebra homomorphism \( K^\times \ltimes S(\mathbb{A}_K) \to C(K^\times) \). This homomorphism induces an isomorphism on the inhomogeneous part of the Hochschild homology. Hence it also induces an isomorphism on the inhomogeneous parts of cyclic and periodic cyclic homology.

It is straightforward to compute the various homology spaces for \( C[K^\times] \), using that \( K^\times \) is isomorphic to a product of the finite cyclic group \( \mu_K \) of roots of unity in \( K^\times \) and a free Abelian group of infinite rank. Hence the Pontrjagin dual \( \hat{K}^\times \) is a product of the finite cyclic group \( \hat{\mu}_K \) and an infinite dimensional torus. The Fourier transform identifies \( C[K^\times] \) with the algebra of Laurent series on \( \hat{K}^\times \). Hence the \( n \)th Hochschild homology is the space \( \Omega^n(\hat{K}^\times) \) of differential \( n \)-forms on \( \hat{K}^\times \) with Laurent series as coefficients. The periodic cyclic homology is the de Rham cohomology \( H_{dR}^n(\hat{K}^\times) \) of \( \hat{K}^\times \). The latter is a direct sum, indexed by \( \hat{\mu}_K \), of copies of the exterior algebra in infinitely many generators.

Next we explain the K-theory computation. First, we have to compute the equivariant K-theory group \( K^{\mu_K}(C_0(\mathbb{A}_K)) \), where \( \mu_K \) is the group of roots of unity in \( K^\times \) as above. We obtain a natural splitting

\[
K^{\mu_K}(C_0(\mathbb{A}_K)) \cong K_0 \oplus K_{\text{inh}}
\]

that corresponds to the homogeneous and inhomogeneous parts of the homology computation. The summand \( K_{\text{inh}} \) is again detected by evaluation at \( 0 \in \mathbb{A}_K \). The summand \( K_0 \) is isomorphic to the space of compactly supported locally constant functions \( f: \mathbb{A}_\mathbb{Q} \to \mathbb{Z} \) that satisfy \( f(ax) = \sigma(a)f(x) \) for all \( a \in \mu_K \), where \( \sigma(a) = \pm 1 \). Thus we obtain an integral lattice in the representation \( S(\mathbb{A}_\mathbb{Q}) \otimes \mathbb{C}(\sigma) \) that occured in the computation of the periodic cyclic homology.
We can write
\[ K^\times / \mu_K K^\times C_0(\mathbb{A}_K) \cong K^\times / \mu_K \times (\mu_K K^\times C_0(\mathbb{A}_K)). \]
The group \( K^\times / \mu_K \) is torsion free and satisfies the Baum-Connes conjecture with coefficients. Hence \( K(K^\times C_0(\mathbb{A}_K)) \) is the limit of a spectral sequence with
\[ E^2_{pq} = H_p(K^\times / \mu_K, K^\times C_0(\mathbb{A}_K)) = H_p(K^\times / \mu_K, K^\times / \mu_K, K_{inh}). \]
We check that the spectral sequence already collapses at \( E^2 \). Since the group \( C_K \) acts differently on \( K_h \) and \( K_{inh} \), there is no interaction between the corresponding parts of the spectral sequence. Hence we only have to consider the homogeneous part of \( E^2_{pq} \). It describes the subquotients of a certain filtration on the “homogeneous part” of the K-theory. If \( K = \mathbb{Q} \) or more generally if \( K \) has just one infinite place, then \( E^2_{pq} = 0 \) for \( p \geq 1 \), so that we obtain the K-theory itself. In general, we should also describe the extensions by which we get the K-theory from the subquotients \( E^2_{pq} \). We do not do that.

Our computation of \( H_p(K^\times / \mu_K, K_h) \) proceeds as follows. We first restrict the action to \( K_{inh}^\times / \mu_K \subseteq K^\times / \mu_K \). This yields a space of functions from \( K_{inh} \) to an exterior algebra as for the periodic cyclic homology. However, these functions are allowed to take certain rational values, with a rather complicated rule for the denominators that are allowed at a point \( x \in K_{inh} \). Nevertheless, we can show that
\[ H_j(K^\times / K_{inh}^\times, H_h(K_{inh}^\times / \mu_K, K_h)) \cong 0 \]
for \( j > 0 \), so that
\[ E^2_{pq} \cong H_0(K^\times / K_{inh}^\times, H_p(K_{inh}^\times / \mu_K, K_h)) \]
The proof uses the filtrations on \( S(\mathbb{A}_K, \mathbb{Z}) \) and \( D(\mathbb{A}_K^\times, \mathbb{Z}) \) by the subspaces of functions that are ramified at at most \( n \) places. The subquotients of these filtrations on \( S(\mathbb{A}_K, \mathbb{Z}) \) and \( D(\mathbb{A}_K^\times, \mathbb{Z}) \) turn out to be isomorphic and can be described explicitly.

3. Preparations

We begin by recalling our notation regarding adeles and ideles and Bruhat-Schwartz spaces. We are going to use some general results about homological algebra for certain smooth convolution algebras on locally compact groups that are proved already in [8]. We identify \( K^\times \times S(\mathbb{A}_K) \) with a smooth convolution algebra on the crossed product group \( K^\times \times \mathbb{A}_K \) and show that this convolution algebra is isocohomological. This allows us to apply the usual recipe for computing the Hochschild homology of group algebras.

3.1. Adeles and Bruhat-Schwartz functions. We begin by recalling some basic notation and results concerning local and global fields and the space \( S(\mathbb{A}_K) \). More details can be found in [2,8,12].

Local fields can be described abstractly as non-discrete, locally compact fields. More concretely, the local fields can be defined as finite extensions of the fields \( \mathbb{R}, \mathbb{Q}_p \) and \( \mathbb{F}_p((t)) \) for prime numbers \( p \). Here \( \mathbb{F}_p((t)) \) is obtained by adjoining \( t^{-1} \) to the ring \( \mathbb{F}_p[[t]] \) of formal power series over \( \mathbb{F}_p \) and \( \mathbb{Q}_p \) denotes as usual the \( p \)-adic integers. Of course, the only finite extensions of \( \mathbb{R} \) are \( \mathbb{R} \) and \( \mathbb{C} \), these are the Archimedean or infinite local fields. The other local fields are called finite or non-Archimedean. The only finite extensions of \( \mathbb{F}_p((t)) \) are \( \mathbb{F}_q((t)) \), where \( q \) is a power of \( p \). We cannot list the finite extensions of \( \mathbb{Q}_p \), so easily.

Each local field is equipped with a canonical norm homomorphism, \( x \mapsto |x| \). This is the usual \( p \)-adic norm \( |x|_p \) for \( \mathbb{Q}_p \); the usual absolute value for \( \mathbb{R} \); and the square of the usual absolute value for \( \mathbb{C} \).

A global field is, by definition, a finite extension of \( \mathbb{Q} \) or \( \mathbb{F}_p((t)) \) for some prime \( p \). Global fields of the first kind are called algebraic number fields, those of the second
kind are called global function fields. A place of a global field $K$ is a dense embedding of $K$ into a local field. For instance, the places of $\mathbb{Q}$ are the embeddings $\mathbb{Q} \to \mathbb{Q}_p$ for the prime numbers $p$ and the single infinite place $\mathbb{Q} \to \mathbb{R}$. A place is called finite, infinite, real or complex depending on the local field $K_v$.

We denote the set of places of $K$ by $\mathcal{P}(K)$. Let $S \subseteq \mathcal{P}(K)$ be a subset that contains all infinite places of $K$ and let $\mathcal{C}S := \mathcal{P}(K) \setminus S$. We let

$$K_S := \{a \in K \mid |a|_v \leq 1 \text{ for all } v \in \mathcal{C}S\},$$
$$K^\times_S := \{a \in K \mid |a|_v = 1 \text{ for all } v \in \mathcal{C}S\}.$$  

Then $K_S \subseteq K$ is a subring and $K^\times_S$ is its multiplicative group of units. That is, $a \in K^\times_S$ if and only if both $a$ and $a^{-1}$ belong to $K_S$. If $K$ is an algebraic number field and $S$ is the set of all infinite places, then $K_S \subseteq K$ is equal to the ring of algebraic integers in $K$. We also denote this subring by $K_\infty$. If $K$ is a global function field, then there are no infinite places, so that $S = \emptyset$ is admissible. We let $K_\infty := K_\emptyset$ in this case.

The adele ring $\mathbb{A}_K$ of $K$ is the restricted direct product of the local fields $K_v$, where $v$ runs through the places of $K$. This is a locally compact ring, that is, its topology is locally compact and addition and multiplication are continuous. The embeddings $K \to K_v$ combine to a diagonal embedding $K \to \mathbb{A}_K$. Its range is a discrete and cocompact subfield of $\mathbb{A}_K$. The idele group $\mathbb{A}^\times_K$ is the multiplicative group of $\mathbb{A}_K$. It can also be described as the restricted direct product of the multiplicative groups $K^\times_v$ for $v \in \mathcal{P}(K)$ and is a locally compact Abelian group as well. The idele class group is the quotient group $\mathcal{C}_K := \mathbb{A}^\times_K/K^\times$. Since $K^\times \subseteq \mathbb{A}^\times_K$ is discrete, this is again a locally compact group.

The norm homomorphism on the groups $K^\times_v$ combine to a norm homomorphism

$$|\omega| : \mathbb{A}^\times_K \to \mathbb{R}^+_\times, \quad x \mapsto |x|,$$

which is trivial on $K^\times$ and hence descends to the quotient group $\mathcal{C}_K$.

Given a set $S$ of places of $K$, we can also define the $S$-adele ring $\mathbb{A}_S$ and the $S$-idele group $\mathbb{A}^\times_S$ by taking the restricted direct products of $K_v$ and $K^\times_v$ only over $v \in S$. We are particularly interested in the cases where $S$ consists of all finite or of all infinite places of $K$. These give rise to the rings $\mathbb{A}_f$ and $\mathbb{A}_\infty$ of finite and infinite adeles and the groups $\mathbb{A}^\times_f$ and $\mathbb{A}^\times_\infty$ of finite and infinite ideles. Let $n_K = [K : \mathbb{Q}]$ for algebraic number fields and $n_K = 0$ for global function fields. Then $\mathbb{A}_\infty \cong \mathbb{R}^{n_K}$ as a locally compact Abelian group.

The space $\mathcal{S}(\mathbb{A}_K)$ is defined by François Bruhat in [2]. We view it as a bornological vector space as in [8]. The product decomposition $\mathbb{A}_K = \mathbb{A}_f \times \mathbb{A}_\infty$ gives rise to a tensor product decomposition

$$\mathcal{S}(\mathbb{A}_K) \cong \mathcal{S}(\mathbb{A}_f) \otimes \mathcal{S}(\mathbb{A}_\infty).$$

Since $\mathbb{A}_f$ is totally disconnected and a direct union of compact subgroups, $\mathcal{S}(\mathbb{A}_f)$ is equal to the space of locally constant, compactly supported functions on $\mathbb{A}_f$. This is a direct union of finite dimensional subspaces and hence carries the fine bornology. Since $\mathbb{A}_\infty \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$, $\mathcal{S}(\mathbb{A}_\infty)$ is the usual Schwartz space on $\mathbb{R}^n$. A subset is bounded if and only if it is von Neumann bounded in the usual Fréchet topology on $\mathcal{S}(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{A}_f)$ carries the fine bornology, we have $\mathcal{S}(\mathbb{A}_K) \cong \mathcal{S}(\mathbb{A}_f) \otimes \mathcal{S}(\mathbb{A}_\infty)$, that is, the algebraic tensor product is already complete.

Both the pointwise multiplication $\cdot$ and the convolution $\ast$ (with respect to the additive structure) turn $\mathcal{S}(\mathbb{A}_K)$ into bornological algebras. The Fourier transform gives rise to an isomorphism between these two algebras, as follows. Since $K \subseteq \mathbb{A}_K$ is discrete, there is a non-trivial character $\psi : \mathbb{A}_K \to S^1$ with $\psi(a) = 1$ for all $a \in K$.  


Then the bilinear pairing
\[ \mathbb{A}_K \times \mathbb{A}_K \to S^1, \quad (x, y) \mapsto \psi(x \cdot y) \]
induces an isomorphism of locally compact Abelian groups \( \mathbb{A}_K \cong \hat{\mathbb{A}_K} \). We use this isomorphism to define the Fourier transform
\[ \mathfrak{f}: \mathcal{S}(\mathbb{A}_K) \to \mathcal{S}(\hat{\mathbb{A}_K}), \quad \mathfrak{f}(\xi) := \int_{\mathbb{A}_K} f(x) \psi(x \xi) \, dx. \]
We normalize the Haar measure \( dx \) on \( \mathbb{A}_K \) so that \( \mathfrak{f}^* f(x) = \mathfrak{f}(-x) \) is inverse to \( \mathfrak{f} \).

The Fourier transform is an isomorphism of bornological algebras
\[ (\mathcal{S}(\mathbb{A}_K), \ast) \cong (\mathcal{S}(\hat{\mathbb{A}_K}), \cdot). \]

The group \( \hat{\mathbb{A}_K} \) acts on \( \mathbb{A}_K \) by multiplication. Let
\[ \lambda_g f(x) := f(g^{-1} x), \quad \rho_g f(x) := |g| \cdot f(xg), \]
for \( g \in \hat{\mathbb{A}_K}, f \in \mathcal{S}(\mathbb{A}_K), x \in \mathbb{A}_K \). Then \( \lambda \) and \( \rho \) are smooth representations of \( \hat{\mathbb{A}_K} \) on \( \mathcal{S}(\mathbb{A}_K) \). The maps \( \lambda_g \) and \( \rho_g \) are algebra automorphisms with respect to pointwise multiplication and convolution, respectively. Since \( \mathfrak{f} \circ \lambda_g = \mathfrak{f} \circ \rho_g \), we have an isomorphism of dynamical systems
\[ (\mathcal{S}(\hat{\mathbb{A}_K}), \cdot, \lambda) \cong (\mathcal{S}(\mathbb{A}_K), \ast, \rho), \]
so that we need not distinguish between them.

We define \( K^\times \ltimes \mathcal{S}(\mathbb{A}_K) \) to be \( \bigoplus_{g \in K^\times} \mathcal{S}(\mathbb{A}_K) \) equipped with the direct sum bornology and the usual multiplication of a crossed product.

### 3.2. Isoconhomological smooth convolution algebras

There is a semidirect product group \( G := K^\times \ltimes \mathbb{A}_K \) with multiplication
\[ G \times G \to G, \quad (g_1, x_1) \cdot (g_2, x_2) = (g_1 g_2, g_2^{-1} x_1 + x_2). \]

The product in the crossed product
\[ A(G) := K^\times \ltimes (\mathcal{S}(\mathbb{A}_K), \ast) \]
is exactly the convolution for this group structure on \( G \). It is clear that \( A(G) \) contains \( D(G) \) as a dense subalgebra. The left and right regular representations of \( G \) on \( A(G) \) are smooth because the regular representation of \( \mathbb{A}_K \) on \( \mathcal{S}(\mathbb{A}_K) \) is smooth. Thus \( A(G) \) is a smooth convolution algebra on \( G \) in the sense of [8, Section 4.3].

We define the category \( \mathbf{M}_{A(G)} \) of essential (left) modules over \( A(G) \) as in [8]. We use the notation \( \mathbf{M}_{A(G)} \) and \( \mathbf{M}_{A(G)} \mathbf{M} \) for the categories of essential right and left \( A(G) \)-modules if we need both kinds of modules. We usually avoid the ugly notation \( A(G)\mathbf{M} \). Recall that \( \mathbf{M}_{A(G)} \) is isomorphic to a full subcategory of the category of smooth representations of \( G \). Let \( A(G)^{\text{op}} \) be the opposite algebra of \( A(G) \). Then the category of essential \( A(G)\)-bimodules is isomorphic to the category of essential modules over \( \mathcal{A}(G) \otimes A(G)^{\text{op}} \). The latter is a smooth convolution algebra on \( G \times G^{\text{op}} \cong G \times G \). The quickest way to see this is to observe that both categories of essential modules are isomorphic to the same category of smooth representations.

We require extensions in \( \mathbf{M}_{A(G)} \) to split as extensions of bornological vector spaces. This turns \( \mathbf{M}_{A(G)} \) into an exact category, so that we can form its derived category \( \mathbf{D}_{A(G)} \), whose objects are chain complexes over \( \mathbf{M}_{A(G)} \). Since modules over the zero algebra are nothing but (complete convex) bornological vector spaces, the category \( \mathbf{D}_0 \) is the homotopy category of chain complexes of bornological vector spaces. In the following, all bornological vector spaces are tacitly assumed complete.
and convex. We denote the total left derived functor of the balanced tensor product functor $\hat{\otimes}_{A(G)}$ by $L\hat{\otimes}_{A(G)}$. This is a functor

$$L\hat{\otimes}_{A(G)} : D_{A(G)} \times_{A(G)} D \to D_0.$$  

We obtain the more classical satellite functors by embedding $M_{A(G)} \to D_{A(G)}$ in the usual way and taking the homology of the chain complex $V L\hat{\otimes}_{A(G)} W$. It is convenient for statements to replace the usual Hochschild homology by a chain complex as well. We let $\text{H}^\bullet(A(G))$ be the image of the chain complex $(\Omega^n A(G), b)$ in $D_0$. Thus the Hochschild homology of $A(G)$ is the homology of $\text{H}^\bullet(A(G))$.

It is observed in [8] that the algebra $A(G)$ contains an approximate identity and that $A(G)$ is projective both as a left and right $A(G)$-module. This is true for any smooth convolution algebra and follows easily from the corresponding results for $D(G)$ proved in [9]. It follows that any essential module has a (linearly split) resolution by essential modules that are projective among not necessarily essential $A(G)$-modules. Thus it makes no difference whether we compute derived functor in the category of all modules or the category of essential modules. It also follows that $A(G)$ is $H$-unital. Hence we can rewrite Hochschild homology as a balanced tensor product functor

$$\text{H}^\bullet(A(G)) \cong A(G) L\hat{\otimes}_{A(G)} \hat{\otimes}_{A(G)^{op}} A(G).$$

Let $C(1)$ be the trivial representation of $G$, viewed as an essential right module over $D(G)$, and let $D(G)_{A\text{d}}$ and $A(G)_{A\text{d}}$ denote the representation of $G$ on $D(G)$ and $A(G)$ by conjugation, $g \cdot f(x) := f(g^{-1}xg)$. If $W$ is a smooth representation of $G$, we let

$$\text{H}(G, W) := C(1) L\hat{\otimes}_{D(G)} W \in D_0.$$  

By definition, the homology of this chain complex is the group homology of $G$. We would like to have an isomorphism $\text{H}^\bullet(A(G)) \cong \text{H}^\bullet(C(1), A(G)_{A\text{d}})$. However, this is not true for arbitrary smooth convolution algebras. It fails, for instance, for $\ell^1(\mathbb{Z})$, which is a smooth convolution algebra on $\mathbb{Z}$. We only get such an isomorphism if $A(G)$ is isocohomological in the sense of [8].

A smooth convolution algebra $A(G)$ on a locally compact group is called isocohomological if the fully faithful embedding $M_{A(G)} \to M_{D(G)}$ gives rise to a fully faithful functor $D_{A(G)} \to D_{D(G)}$. An equivalent condition is that $V L\hat{\otimes}_{A(G)} W \cong V L\hat{\otimes}_{D(G)} W$ for all $V \in D_{A(G)}$, $W \in D_{A(G)}$. Even more, it suffices to check

$$A(G) L\hat{\otimes}_{D(G)} A(G) \cong A(G) L\hat{\otimes}_{A(G)} A(G) \cong A(G)$$

because this implies the corresponding assertion for chain complexes of free essential $A(G)$-modules.

**Lemma 3.1.** The crossed product $K^\times \ltimes S(\mathbb{A}_K)$ is an isocohomological smooth convolution algebra on $K^\times \ltimes \mathbb{A}_K$.

**Proof.** Let $H := \mathbb{A}_K$ and $G = K^\times \ltimes \mathbb{A}_K$ and let $A(G) := K^\times \ltimes S(\mathbb{A}_K)$. Then $H$ is an open normal subgroup of $G$ and there is a natural isomorphism

$$A(G) \cong c\text{-Ind}^G_H S(H)$$

as left or right modules over $D(G)$. We compute

$$A(G) L\hat{\otimes}_{D(G)} A(G) \cong c\text{-Ind}^G_H S(H) L\hat{\otimes}_{D(G)} A(G) \cong S(H) L\hat{\otimes}_{D(H)} \text{Res}_G^H A(G) \cong S(H) L\hat{\otimes}_{S(H)} \text{Res}_G^H A(G) \cong \text{Res}_G^H A(G),$$

using that compact induction and restriction are adjoint ([9]) and that $S(\mathbb{A}_K)$ is isocohomological ([8]).
Proposition 3.2. Let \( A(G) \) be an isocohomological smooth convolution algebra on a locally compact group \( G \). Then

\[
\text{HH}(A(G)) \cong \text{H}(G, A(G)_{\text{Ad}}).
\]

Proof. The condition \( A(G) \mathbb{L}\hat{\otimes}_{D(G)} A(G) \cong A(G) \) is symmetric under passage to opposite algebras and hereditary for tensor products. Hence \( A(G) \hat{\otimes} A(G)^{\text{op}} \) is an isocohomological smooth convolution algebra on \( G \times G \). Therefore,

\[
\text{HH}(A(G)) \cong A(G) \mathbb{L}\hat{\otimes}_{A(G)\hat{\otimes}A(G)^{\text{op}}} A(G) \cong A(G) \mathbb{L}\hat{\otimes}_{D(G \times G)} A(G).
\]

It is shown in [9] that the bivariant functor \( \mathbb{L}\hat{\otimes}_{D(G \times G)} \) can be reduced to group homology. In our case, this yields the following. Let \( G \times G \) act on \( A(G) \hat{\otimes} A(G) \) by

\[
(g_1, g_2) \cdot f(h_1, h_2) = f(g_1^{-1} h_1 g_2, g_2^{-1} h_2 g_1).
\]

Then

\[
A(G) \mathbb{L}\hat{\otimes}_{D(G \times G)} A(G) \cong \mathbb{C}(1) \mathbb{L}\hat{\otimes}_{D(G \times G)} A(G) \hat{\otimes} A(G) \cong \mathbb{C}(1) \mathbb{L}\hat{\otimes}_{D(G_1 \times G_2)} (\mathbb{C}(1) \mathbb{L}\hat{\otimes}_{D(G_2)} A(G) \hat{\otimes} A(G)).
\]

Here \( G_1 \) and \( G_2 \) denote the first and second factor \( G \) in \( G \times G \). Reversing the above argument, we can rewrite

\[
\mathbb{C}(1) \mathbb{L}\hat{\otimes}_{D(G_2)} A(G) \hat{\otimes} A(G) \cong A(G) \mathbb{L}\hat{\otimes}_{D(G_2)} A(G) \cong A(G),
\]

where we used once again that \( A(G) \) is isocohomological. The resulting action of \( G = G_1 \) on \( A(G) \) is by conjugation, so that we obtain \( \mathbb{C}(1) \mathbb{L}\hat{\otimes}_{D(G)} A(G)_{\text{Ad}} \) as desired.

This proposition applies to \( G := K^\times \ltimes \check{A}_K \) and \( A(G) := K^\times \ltimes S(\check{A}_K) \) by Lemma 3.1. Hence we obtain

\[
\text{HH}(K^\times \ltimes S(\check{A}_K)) = \text{HH}(A(G)) \cong \text{H}(G, A(G)_{\text{Ad}}).
\]

3.3. The coaction on the homology spaces. Next we explain how the coaction of \( K^\times \) on the various cyclic homology groups comes about. Since \( K^\times \) is discrete, a coaction on a bornological vector space \( V \) is nothing but a direct sum decomposition \( V = \bigoplus_{g \in K^\times} V_g \). For coactions on algebras, we require \( A_g A_h \subseteq A_{gh} \) for all \( g, h \in G \). If \( A \) is any algebra on which \( K^\times \) acts by automorphisms, then \( K^\times \ltimes A := \sum_{g \in G} A_g \) with \( A_g := A \delta_g \subseteq K^\times \ltimes A \) carries such a coaction. We describe it by the partially defined function \( \text{deg}(a) := g \) for \( a \in A_g \). The complex \( \Omega(K^\times \ltimes A) \) of non-commutative differential forms inherits a “total” grading

\[
\text{deg}(a_0 da_1 \ldots da_n) := \text{deg}(a_0) \cdots \text{deg}(a_n).
\]

The boundary maps \( b \) and \( d \) preserves this grading because \( K^\times \) is commutative. Hence we obtain a direct sum decomposition

\[
\text{HH}(K^\times \ltimes A) \cong \bigoplus_{g \in K^\times} \text{HH}(K^\times \ltimes A)_g
\]

and a similar decomposition of Hochschild homology. Since \( B \) also preserves the coaction, we also have such decompositions for cyclic and periodic cyclic homology. It turns out that the 1-homogeneous parts play a special role. Hence we only distinguish the homogeneous part \( (\cdots)_1 \) and the inhomogeneous part \( \bigoplus_{g \neq 1} (\cdots)_g \).
4. Computation of the Hochschild homology

As in Section 3.2, we let $G := K^\times \ltimes \mathbb{A}_K$ and view

$$A(G) := K^\times \ltimes (S(\mathbb{A}_K),\ast)$$

as a smooth convolution algebra on $G$. We have seen there that the complex $\mathbb{H}(A(G))$ that computes the Hochschild homology of $A(G)$ is homotopy equivalent to the complex $\mathbb{H}(G, A(G)_{Ad})$ that computes the group homology of $G$ with coefficients in $A(G)$ with $G$ acting by conjugation. Since $G$ is defined as a semidirect product, we can compute this group homology in two stages as

$$\mathbb{H}(G, A(G)_{Ad}) \cong \mathbb{H}(K^\times, \mathbb{H}(\mathbb{A}_K, A(G)_{Ad})).$$

where we view $\mathbb{H}(\mathbb{A}_K, A(G)_{Ad})$ as an object of $D_{D(K^\times)}$ in the canonical way. If we worked with satellite functors, the above isomorphism would become the spectral sequence computing group homology for a group extension.

The subsets $\{g\} \times \mathbb{A}_K \subseteq G$ are invariant under conjugation because $K^\times$ is commutative. Hence

$$A(G)_{Ad} \cong \bigoplus_{g \in K^\times} S(\mathbb{A}_K)_g,$$

where $S(\mathbb{A}_K)_g$ is $S(\mathbb{A}_K)$ as a bornological vector space. The action of $G$ on $S(\mathbb{A}_K)$ comes from the action on the subset $\{g\} \times \mathbb{A}_K$ by conjugation. Since

$$(h, y)^{-1}(g, x)(h, y) = (h^{-1}, -hy)(g, x)(h, y) = (g, (1 - g^{-1})y + h^{-1}x),$$

the representation of $G$ on $S(\mathbb{A}_K)_g$ is given by

$$(h, y) \cdot_g f(x) = f(h^{-1}x + (1 - g^{-1})y)$$

for all $g, h \in K^\times$, $x, y \in \mathbb{A}_K$.

We have to distinguish the cases $g \neq 1$ and $g = 1$. We treat the case $g \neq 1$ first.

Then $1 - g^{-1}$ is invertible, so that the map $\Phi f(x) = f((1 - g^{-1})x)$ is a bornological isomorphism on $S(\mathbb{A}_K)_g$. It intertwines the representation $\cdot_g$ on $S(\mathbb{A}_K)_g$ described above and the standard representation

$$(h, y) \cdot f(x) = f(h^{-1}x + y).$$

Its restriction to $\mathbb{A}_K \subseteq G$ is the regular representation. Since $\mathbb{C}(1)$ is an essential module over $S(\mathbb{A}_K)$ and $S(\mathbb{A}_K)$ is isocohomological by [8], we obtain

$$\mathbb{H}(\mathbb{A}_K, S(\mathbb{A}_K)_g) = \mathbb{C}(1) \mathbb{L} \mathbb{S}_{D(\mathbb{A}_K)} S(\mathbb{A}_K)_{g} \cong \mathbb{C}(1) \mathbb{L} \mathbb{S}_{S(\mathbb{A}_K)} S(\mathbb{A}_K) \cong \mathbb{C}(1).$$

This homotopy equivalence is implemented by the $G$-invariant linear map

$$S(\mathbb{A}_K)_g \to \mathbb{C}(1), \quad f \mapsto \int_{\mathbb{A}_K} f(x) \, dx$$

and hence is an isomorphism in the derived category $D_{D(K^\times)}$. Therefore,

$$\mathbb{H}(K^\times \ltimes S(\mathbb{A}_K))_g \cong \mathbb{H}(K^\times, \mathbb{H}(\mathbb{A}_K, S(\mathbb{A}_K)_g)) \cong \mathbb{H}(K^\times, \mathbb{C}(1)).$$

The homology of $\mathbb{H}(K^\times, \mathbb{C}(1))$ is the group homology of the discrete group $K^\times$ in the usual sense. We will describe it more concretely below.

Now we consider the homogeneous part, that is, we let $g = 1$. The subgroup $\mathbb{A}_K$ acts trivially on $S(\mathbb{A}_K)$. Therefore,

$$\mathbb{H}(\mathbb{A}_K, S(\mathbb{A}_K)_1) \cong \mathbb{H}(\mathbb{A}_K, \mathbb{C}(1)) \mathbb{S} S(\mathbb{A}_K),$$

where $\mathbb{H}(\mathbb{A}_K, \mathbb{C}(1))$ is a chain complex of bornological vector spaces that computes the group homology of $\mathbb{A}_K$ with trivial coefficients. The direct product decomposition $\mathbb{A}_K \cong \mathbb{A}_1 \times \mathbb{A}_\infty$ gives rise to a tensor product decomposition for $D(\mathbb{A}_K)$ and hence to an isomorphism

$$\mathbb{H}(\mathbb{A}_K, \mathbb{C}(1)) \cong \mathbb{H}(\mathbb{A}_1, \mathbb{C}(1)) \mathbb{S} \mathbb{H}(\mathbb{A}_\infty, \mathbb{C}(1)).$$
We have already observed that $D(\mathbb{A}_f) \cong S(\mathbb{A}_f)$. Hence the Fourier transform identifies $(D(\mathbb{A}_f), \ast)$ and $(D(\mathbb{A}_f), \cdot)$. The space $\mathbb{A}_f$ is totally disconnected. Therefore, any essential module over $D(\mathbb{A}_f)$ is flat, that is, $V L \hat{\otimes}_{D(\mathbb{A}_f)} W = V \hat{\otimes}_{D(\mathbb{A}_f)} W$ for all $V, W$. In particular, $\mathbb{C}(1) L \hat{\otimes}_{D(\mathbb{A}_f)} \mathbb{C}(1) \cong \mathbb{C}(1)$. Since $\mathbb{A}_\infty \cong \mathbb{R}^{n_K}$, we have

$$H(\mathbb{A}_\infty, \mathbb{C}(1)) \cong H(\mathbb{R}, \mathbb{C}(1)) \hat{\otimes}_{\mathbb{S}^{n_K}} (\mathbb{A} \mathbb{R}) \hat{\otimes}_{\mathbb{S}^{n_K}} \mathbb{A}(\mathbb{R}^{n_K}).$$

Here $\Lambda(V)$ denotes the exterior algebra over a real vector space $V$. We have to describe the action of $K^\times$ on this space. Therefore, we write down the above computation carefully and naturally.

The space $\mathbb{A}_\infty$ carries a canonical real vector space structure. The action of $\mathbb{A}_\infty^\times$ on $\mathbb{A}_\infty$ is linear. Let $\Lambda \mathbb{A}_\infty$ be the exterior algebra over the real vector space $\mathbb{A}_\infty$. This carries an induced representation of $\mathbb{A}_\infty^\times$. We let $\mathbb{A}_f^\times$ act trivially on $\mathbb{A}_\infty$ and $\Lambda \mathbb{A}_\infty$ and denote the resulting representation of $\mathbb{A}_K^\times$ on $\Lambda \mathbb{A}_\infty$ by $\pi$. Consider the graded vector space

$$C(\mathbb{A}_\infty) := D(\mathbb{A}_\infty) \hat{\otimes} \Lambda \mathbb{A}_\infty$$

with the boundary map

$$b(f \otimes \xi_1 \wedge \cdots \wedge \xi_j) := \sum_{k=1}^j (-1)^{k-1} \frac{\partial f}{\partial \xi_k} \otimes \xi_1 \wedge \cdots \wedge \hat{\xi}_k \wedge \cdots \wedge \xi_j$$

of degree $-1$ and the augmentation map

$$f : C_0(\mathbb{A}_\infty) \cong D(\mathbb{A}_K) \rightarrow \mathbb{C}, \quad f \mapsto \int_{\hat{\mathbb{A}_K}} f(x) dx.$$

We have $f \circ b = 0$ and $b^2 = 0$ because the differential operators $\partial / \partial \xi$ for $\xi \in \mathbb{A}_\infty$ commute. Thus $(C(\mathbb{A}_\infty), b, f)$ is a bornological chain complex over $\mathbb{C}$. Writing $\mathbb{A}_\infty \cong \mathbb{R}^n$, we get $(C(\mathbb{A}_\infty), b) \cong (C(\mathbb{R}), b) \hat{\otimes} n$. It is easy to check that $(C(\mathbb{R}), b, f)$ is a linearly split resolution of $\mathbb{C}$. Hence so is $(C(\mathbb{A}_\infty), b, f)$.

We let $\mathbb{A}_K^\times$ and $\mathbb{A}_K$ act on $C(\mathbb{A}_\infty)$ by $\rho \hat{\otimes} \pi$ and by translation on the first tensor factor. Thus $\mathbb{A}_f^\times \ltimes \mathbb{A}_f \subseteq \mathbb{A}_K$ acts trivially. These representation piece together to a smooth representation of $\mathbb{A}_K^\times \ltimes \mathbb{A}_K$. The boundary map $b$ is equivariant for this representation and $f$ is an invariant linear functional. The space $C(\mathbb{A}_\infty)$ is a free essential module over $\mathbb{A}_\infty$. Since any smooth representation of $\mathbb{A}_f$ is flat, $C(\mathbb{A}_f)$ is flat as a smooth representation of $\mathbb{A}_K$. Hence we have constructed a resolution of the trivial representation $\mathbb{C}(1)$ in the category of smooth representations of $\mathbb{A}_K^\times \ltimes \mathbb{A}_K$ that is flat as a representation of $\mathbb{A}_K$. Hence we get an isomorphism

$$H(\mathbb{A}_K, \mathbb{C}(1)) \cong \mathbb{C}(1) L \hat{\otimes}_{D(\mathbb{A}_K)} \mathbb{C}(1) \cong \mathbb{C}(1) \hat{\otimes}_{D(\mathbb{A}_K)} (C(\mathbb{A}_\infty), b) \cong (\Lambda(\mathbb{A}_\infty), 0)$$

in the category $D_{\mathbb{A}_K^\times}$. Notice that the boundary map in this complex vanishes.

Putting everything together, we obtain a canonical isomorphism

$$H(K^\times \ltimes S(\mathbb{A}_K))_1 \cong H(K^\times, H(\mathbb{A}_K, S(\mathbb{A}_K)_1)) \cong H(K^\times, S(\mathbb{A}_K) \hat{\otimes} \Lambda \mathbb{A}_\infty).$$

We now recall some results of [8] that we are going to need. In [8], the representation $\lambda$ is used instead of $\rho$. This has no effect because the Fourier transform intertwines the two representations. Let $S$ be a sufficiently large finite set of places. Let $\mathcal{C}_S := \mathbb{A}_S^\times / K_S^\times$ be the $S$-idele class group. Let $F$ be the Fourier transform on $L^2(\mathbb{A}_S, dx)$. It descends to a unitary operator on the Hilbert space $L^2(\mathcal{C}_S, |x| d^nx)$. Let

$$\mathcal{T}(\mathcal{C}_S) := \{ f \in L^2(\mathcal{C}_S, |x| d^nx) \mid f|_{\alpha} \in S(\mathcal{C}_S) \text{ and } F(f)|_{\alpha} \in S(\mathcal{C}_S) \text{ for all } \alpha > 0 \}.$$

It is shown in [8] that

$$H(K^\times, S(\mathbb{A}_K)) \cong S(\mathbb{A}_K)/K^\times \cong \lim_{\rightarrow} \mathcal{T}(\mathcal{C}_S)$$
with suitable bornological embeddings $\mathcal{T}(C_S) \to \mathcal{T}(C_{S'})$ for $S \subseteq S'$. Moreover, the right hand side can be identified with a space of functions on $C_K$. The following lemma allows us to apply this to differential forms as well.

**Lemma 4.1.** There exists a bornological isomorphism on $S(A_K) \otimes \Lambda A_\infty$ that intertwines the representations $\rho \otimes \pi$ and $\rho \otimes \text{id}$.

**Proof.** Let $S$ be the set of infinite places of $K$. We write

$$S(A_K) \otimes \Lambda A_\infty = S(A_\ell) \otimes \bigotimes_{v \in S}(S(K_v) \otimes \Lambda K_v).$$

The representation of $A_\infty^S = \prod_{v \in S} K_v^\times$ is the external tensor product representation. Hence it suffices to construct such isomorphisms for Archimedean local fields, that is, for $\mathbb{R}$ and $\mathbb{C}$.

Consider first the case $K_v = \mathbb{R}$. The spaces $\Lambda^0 \mathbb{R}$ and $\Lambda^1 \mathbb{R}$ are both 1-dimensional. The action of $\mathbb{R}^\times$ on $\Lambda^0 \mathbb{R}$ is trivial, on $\Lambda^1 \mathbb{R}$ it is given by the identical quasi-character $x \mapsto x$. Hence $S(\mathbb{R}) \otimes \Lambda^1 \mathbb{R} \cong S(\mathbb{R}) \cdot x \subseteq S(\mathbb{R})$. Decompose functions in $S(\mathbb{R})$ into an even and odd component $f = f_+ + f_-$ and let $\Phi(f) := x df_+/dx + f_-$. It is easy to see that this map is an isomorphism $S(\mathbb{R}) \cong S(\mathbb{R}) \cdot x$.

Next we consider the case $K_v = \mathbb{C}$. Again there is nothing to do for $\Lambda^0 \mathbb{C}$. The space $\Lambda^1 \mathbb{C}$ has two generators corresponding to the holomorphic and antiholomorphic differentials $\partial/\partial z$ and $\partial/\partial \overline{z}$. Since multiplication preserves the complex structure, they span invariant subspaces of $\Lambda^1 \mathbb{C}$. On the first subspace, $\mathbb{C}^\times$ acts by the identical quasi-character $z \mapsto z$, on the second by $z \mapsto \overline{z}$. The space $\Lambda^2 \mathbb{C}$ is again 1-dimensional and acted upon by the quasi-character $z \mapsto z\overline{z}$. Hence we obtain the desired isomorphism for $\Lambda^2 \mathbb{C}$ by composing the isomorphisms for the two summands of $\Lambda^1 \mathbb{C}$. For symmetry reasons, it suffices to treat $S(\mathbb{C})\partial/\partial y$. We have to construct a $\mathbb{C}^\times$-equivariant isomorphism $S(\mathbb{C}) \cong S(\mathbb{C}) \cdot z$. We split $S(\mathbb{C})$ into homogeneous components with respect to the representation of the compact group $S^1 \subseteq \mathbb{C}^\times$ and group together the representations $\mathbb{C}^n$ for $n \geq 0$ into the positive and $\mathbb{C}^n$ for $n > 1$ into the negative part. Thus we write any $f \in S(\mathbb{C})$ uniquely as $f = f_+ + f_-$. The operator $\Phi f := z \partial f_+/\partial z + f_-$ is the desired isomorphism. \hfill \Box

It follows that

$$\text{HH}(K^\times \ltimes S(A_K))^1 \cong \text{HH}(K^\times, S(A_K) \otimes \Lambda A_\infty)$$

$$\cong (S(A_K) \otimes \Lambda A_\infty)/K^\times \cong S(A_K)/K^\times \otimes \Lambda A_\infty.$$

The expressions in the second line are graded bornological vector spaces viewed as chain complexes with vanishing boundary map. Hence their graded components are the homogeneous parts of the Hochschild homology. We also get the additional fact that the homology computation can be done using only homotopy equivalences. Therefore, it immediately implies results about Hochschild cohomology and also about Hochschild homology and cohomology for tensor products with $K^\times \ltimes S(A_K)$. The last isomorphism is not quite natural but can be chosen $A_K^\times$-equivariant. We can identify $S(A_K)/K^\times$ further with $\lim_{\to} \mathcal{T}(C_S)$ as in [8].

### 4.1. The inhomogeneous part of the cyclic homology theories

In order to pass to cyclic and periodic cyclic homology, we need explicit chain maps that implement the above homology computation. First we consider the inhomogeneous part. The integration map $(S(A_K), \ast) \to \mathbb{C}$ is an algebra homomorphism. Its Fourier transform is the map $(S(A_K), \cdot) \to \mathbb{C}$ of evaluation at 0. We get an $A_K^\times$-equivariant algebra homomorphism

$$\Phi: K^\times \ltimes S(A_K) \to K^\times \ltimes \mathbb{C}(1) = \mathbb{C}[K^\times].$$
Proposition 4.2. The map $\Phi$ induces a homotopy equivalence
\[ \text{HH}(K^\times \ltimes S(\mathcal{A}_K))_{\text{inh}} \to \text{HH}(C[K^\times])_{\text{inh}} \]
and hence isomorphisms between the inhomogeneous parts of the Hochschild homology, the cyclic homology and the periodic cyclic homology of $K^\times \ltimes S(\mathcal{A}_K)$ and $C[K^\times]$. The group $C_K$ acts trivially on these homology spaces.

This is essentially proved by our above computations. We make the assertion more explicit by describing the Hochschild homology of $C[K^\times]$. Let $\mu_K \subseteq K^\times$ be the subgroup of roots of unity. The group $K^\times/\mu_K$ is a free Abelian group of infinite rank. Hence we have an isomorphism $K^\times \cong \mu_K \times \mathbb{Z}^\infty$. Therefore,
\[ C[K^\times] \cong C[\mu_K] \otimes \mathbb{Z}^{\otimes \infty}, \]
where we define the infinite tensor product as an inductive limit of finite tensor products. Each factor $C[\mathbb{Z}]$ is a Laurent series ring. We view $C[K^\times]$ as the algebra of Laurent series on the Pontrjagin dual $\hat{K}^\times$. We let $\Omega(\hat{K}^\times)$ be the algebra of differential forms on $\hat{K}^\times$ with Laurent series as coefficients. This is a differential graded algebra over $C[K^\times]$, so that we obtain a canonical homomorphism of differential graded algebras $\Omega(C[K^\times]) \to \Omega(\hat{K}^\times)$. It is well-known that the map
\[ \text{HH}(\Omega(C[K^\times])) = (\Omega(C[K^\times]), b) \to (\Omega(\hat{K}^\times), 0) \]
is a homotopy equivalence. Hence the Hochschild homology of $C[K^\times]$ is $\Omega(\hat{K}^\times)$.

For any $g \in K^\times$, the $g$-homogeneous subspace of $\Omega^n(\hat{K}^\times)$ is canonically isomorphic to the group homology of $K^\times$. When we computed the inhomogeneous part of $\text{HH}(K^\times \ltimes S(\mathcal{A}_K))$, the only important ingredient was exactly the integration map $S(\mathcal{A}_K) \to C$. This shows that the isomorphism from $\text{HH}(K^\times \ltimes S(\mathcal{A}_K))_g$ to $\text{HH}(K^\times, C(1))$ that we constructed above agrees with the map induced by $\Phi$. Since $\Phi$ is an algebra homomorphism, it induces maps on cyclic and periodic cyclic homology. These maps are compatible with the decomposition into homogeneous and inhomogeneous parts. Since $\Phi$ induces an isomorphism on the inhomogeneous part of the Hochschild homology, it also induces isomorphisms on the inhomogeneous parts of the cyclic and periodic cyclic homologies. Since $\Phi$ is invariant under $\mathcal{A}_K$, the action of $C_K$ on the $g$-homogeneous part of the Hochschild homology is trivial. This finishes the proof of Proposition 4.2.

Let $N$ and $d_{\text{dR}}$ be the number operator (that is, $N = n$ on $n$-forms) and the de Rham differential on $\Omega(\hat{K}^\times)$. Then the homotopy equivalence in (2) intertwines $B$ and $Nd_{\text{dR}}$. Hence we obtain a chain map from the $(B, b)$-bicomplex to the bicomplex $(\Omega(\hat{K}^\times), Nd_{\text{dR}}, 0)$ that is a homotopy equivalence with respect to the vertical boundary maps $b$ and 0. Therefore, it is a homotopy equivalence between the total complexes as well. Since the total complex of the $(B, b)$-bicomplex computes the cyclic homology of $C[K^\times]$, we get
\[ \text{HC}_j(C[K^\times]) \cong \Omega(\hat{K}^\times)/d_{\text{dR}}\Omega^{j-1}(\hat{K}^\times) \oplus H^3_{\text{dR}}(\hat{K}^\times) \oplus H^4_{\text{dR}}(\hat{K}^\times) \oplus \cdots \]
for all $j \in \mathbb{N}$, where $H^j_{\text{dR}}$ denotes the de Rham cohomology of $\hat{K}^\times$. The periodicity operator $S$ can also be described easily, so that we get
\[ \text{HP}_j(C[K^\times]) = \prod_{k \in \mathbb{Z}} H^{j+2k}_{\text{dR}}(\hat{K}^\times). \]

We leave it to the reader to describe the coaction and decomposition into homogeneous and inhomogeneous parts of the Hochschild and cyclic homology. The coaction of $\mathbb{Z}^N$ on $\text{HP}_j(C[K^\times])$ is trivial. However, we still have a non-trivial coaction of $\mu_K$. An element of $H^j_{\text{dR}}(\hat{K}^\times)$ is homogeneous if and only if it is invariant under the action of $\mu_K$ that permutes the connected components of $\hat{K}^\times$. 


The homogeneous part of the cyclic homology theories. We can also describe an explicit chain map implementing our computation of the homogeneous part of the Hochschild homology. It is more convenient to equip \( \mathcal{S}(\mathbb{A}_K) \) with the pointwise product in order to bring into play the de Rham boundary map.

Let \( \mathcal{S}(\mathbb{A}_\infty) \) be the differential graded algebra of differential forms on \( \mathbb{A}_\infty \cong \mathbb{R}^{n_K} \) with Schwartz functions as coefficients. Let \( \mathcal{S}(\mathbb{A}_K) := \mathcal{S}(\mathbb{A}_\infty) \otimes \mathcal{S}(\mathbb{A}_f) \). We let \( N \) and \( d_{\text{DR}} \) be the number operator and the de Rham boundary maps on these spaces of differential forms. We let \( \mathbb{A}_K \) act on \( \mathcal{S}(\mathbb{A}_K) \) in the canonical way. Thus we can form the coinvariant space \( \mathcal{S}(\mathbb{A}_K)/K^\times \). The Fourier transform gives rise to a natural isomorphism

\[
\mathcal{F}: \mathcal{S}(\mathbb{A}_K) \xrightarrow{\cong} \mathcal{S}(\mathbb{A}_K) \otimes \Lambda\mathbb{A}_\infty.
\]

Hence our computation of the homogeneous part of the Hochschild homology yields

\[
\text{HH}_n(K^\times \ltimes \mathcal{S}(\mathbb{A}_K))_1 \cong (\mathcal{S}(\mathbb{A}_K) \otimes \Lambda\mathbb{A}_\infty)/K^\times \cong \mathcal{S}(\mathbb{A}_K)/K^\times.
\]

We can realize this by the following explicit map. The homogeneous part of \( \mathcal{S}(\mathbb{A}_K) \) is the closed linear span of monomials

\[
f_0(\delta_{g_0}, f_1) \cdots f_n(\delta_{g_n}), \quad d(f_1) \cdots d(f_n),
\]

with \( g_0 \cdots g_n = 1 \) and \( g_1 \cdots g_n = 1 \), respectively. We map them to the classes of the differential forms

\[
f_0(\lambda_{g_0}, f_1) \wedge \cdots \wedge d(\lambda_{g_0 \cdots g_{n-1}} f_n), \quad d f_1 \wedge d(\lambda_{g_1} f_2) \cdots \wedge d(\lambda_{g_{n-1}} f_n)
\]

in \( \mathcal{S}(\mathbb{A}_K)/K^\times \), respectively. The resulting map

\[
\Phi_1: \Omega(K^\times \ltimes \mathcal{S}(\mathbb{A}_K))_1 \to \mathcal{S}(\mathbb{A}_K)/K^\times
\]

evidently satisfies \( \Phi_1 \circ b = 0 \) and \( \Phi_1 \circ B = N d_{\text{DR}} \circ \Phi_1 \). A closer inspection of our above computations shows that \( \Phi_1 \) is a homotopy equivalence

\[
(\Omega(K^\times \ltimes \mathcal{S}(\mathbb{A}_K)), b)_1 \to (\mathcal{S}(\mathbb{A}_K)/K^\times, 0).
\]

Hence we can use \( \Phi_1 \) also to compute the cyclic and periodic cyclic homology spaces. As in the computation for the inhomogeneous part, we get

\[
\text{HP}_j(K^\times \ltimes \mathcal{S}(\mathbb{A}_K))_1 \cong \bigoplus_{k \in \mathbb{Z}} H_{j-2k}(\mathcal{S}(\mathbb{A}_K)/K^\times, (d_{\text{DR}})_*)
\]

and a similar result for the cyclic theories.

It remains to compute this homology more explicitly. We use once again that \( \mathcal{H}(K^\times, \mathcal{S}(\mathbb{A}_K)) \cong 0 \). This means that the complex \( (\mathcal{S}(\mathbb{A}_K), d_{\text{DR}}) \) is acyclic with respect to the group homology functor. We want to describe its homology. Fix an orientation on \( \mathbb{A}_\infty \) and let

\[
f: \mathcal{S}(\mathbb{A}_K) \cong \mathcal{S}(\mathbb{A}_\infty) \otimes \mathcal{S}(\mathbb{A}_f) \to \mathcal{S}(\mathbb{A}_f)
\]

be the usual integration map for forms of top degree. Let \( \sigma: \mathbb{A}_K \to \{\pm 1\} \) be the character that multiplies the signs of an idele at the real places of \( K \). Thus \( \sigma(a) = +1 \) if multiplication by \( a \) preserves orientation and \( \sigma(a) = -1 \) if multiplication by \( a \) reverses orientation. Hence \( f \) is an equivariant map to \( \mathcal{S}(\mathbb{A}_f) \otimes \mathbb{C}(\sigma) \). We claim that \( (\mathcal{S}(\mathbb{A}_K), d_{\text{DR}}, f) \) is a linearly split resolution of \( \mathcal{S}(\mathbb{A}_f) \otimes \mathbb{C}(\sigma) \). To prove this we write

\[
(\mathcal{S}(\mathbb{A}_K), d_{\text{DR}}) \cong (\mathcal{S}(\mathbb{A}_f), 0) \otimes (\mathcal{S}(\mathbb{R}), d_{\text{DR}}) \otimes n_K.
\]

It is elementary to verify that \( \mathcal{S}(\mathbb{R}) \xrightarrow{d_{\text{DR}}} \mathcal{S}(\mathbb{R}) ) \xrightarrow{\sigma} \mathbb{C} \) is a linearly split extension. This implies the claim. Since the complex \( (\mathcal{S}(\mathbb{A}_K), d_{\text{DR}}) \) is also acyclic for the group homology functor, we obtain

\[
H_j(\mathcal{S}(\mathbb{A}_K)/K^\times, (d_{\text{DR}})_*) \cong H_{n_k-j}(K^\times, \mathcal{S}(\mathbb{A}_f) \otimes \mathbb{C}(\sigma)).
\]
Thus our first step is to compute $K(A)$ then we have $E_{top}$ amenable. Therefore, the full and reduced crossed products agree and in the situation.

In order to get more precise results it is often better to use special structure present in the latter group can always be computed by a spectral sequence. However, in the case of rank $u$ where $u$ is the rank of $K^\infty$.

The space $\mathcal{D}(\mathbb{A}_k^\infty)$ is the direct union of the $\mathbb{A}_k^\infty$-invariant subspaces $\mathcal{D}(\mathbb{A}_k^\infty / U_k)$. Hence

$$H_*(K^\infty, \mathcal{D}(\mathbb{A}_k^\infty) \otimes \mathbb{C}(\sigma)) \cong \lim_{\mathcal{L}} H_*(K^\infty, \mathcal{D}(\mathbb{A}_k^\infty / U_k) \otimes \mathbb{C}(\sigma)).$$

For any $k \in \mathbb{N}$, the quotient space $\mathbb{A}_k^\infty / U_k$ is discrete and $L_k$ acts trivially on it. The induced action of $K^\infty / L_k$ on $\mathbb{A}_k^\infty / U_k$ is free. Hence the Shapiro Lemma yields

$$H_*(K^\infty, \mathcal{D}(\mathbb{A}_k^\infty / U_k) \otimes \mathbb{C}(\sigma)) \cong H_*(L_k) \otimes (\mathcal{D}(\mathbb{A}_k^\infty / U_k) \otimes \mathbb{C}(\sigma))/K^\infty.$$  

The latter tensor factor is naturally isomorphic to the space $\mathcal{D}(\mathbb{A}_k^\infty / U_k K^\infty)_{\sigma}$ of smooth functions $f: \mathbb{A}_k^\infty / U_k \to \mathbb{C}$ that satisfy $f(ax) = \sigma(a)f(x)$ for all $a \in K^\infty$, $x \in \mathbb{A}_k^\infty / U_k$. We can identify it naturally with $\mathcal{D}(\mathbb{A}_k^\infty / U_k K^\infty)$ by choosing a fundamental domain for the action of $K^\infty$. The group homology of the free Abelian group $L_k$ is an exterior algebra on $u$ generators.

Inspection shows that the passage from $U_k$ to $U_{k+1}$ is the tensor product of the obvious map on $\mathcal{D}(\mathbb{A}_k^\infty / K^\infty U_k)_{\sigma}$ and multiplication with certain constants that depend on the group $L_k/L_{k+1}$ on $H_*(L_k)$. Thus

$$H_*(K^\infty, \mathcal{D}(\mathbb{A}_k^\infty) \otimes \mathbb{C}(\sigma)) \cong \lim_{\mathcal{L}} \Lambda^* \mathbb{C}^u \otimes \mathcal{D}(\mathbb{A}_k^\infty / K^\infty U_k)_{\sigma} \cong \Lambda^* \mathbb{C}^u \otimes \mathcal{D}(\mathbb{A}_k^\infty \overline{K^\infty} )_{\sigma}.$$ 

Here $\overline{K^\infty}$ denotes the closure of $K^\infty$ in $\mathbb{A}_k^\infty$. That is, we obtain the space of functions $f: \mathbb{A}_k^\infty \to \Lambda^* \mathbb{C}^u$ that satisfy $f(ax) = \sigma(a)f(x)$ for all $a \in K^\infty$, $x \in \mathbb{A}_k^\infty$. The action of $\mathbb{A}_k^\infty$ on the right hand side is trivial on the first and the obvious action on the second tensor factor. The group $\mathbb{A}_k^\infty$ acts by $\sigma$.

This finishes our computation of the cyclic and periodic cyclic homology of the crossed product.

5. The Topological $K$-Theory of the Crossed Product

Let $A$ be the (complex) $C^*$-algebraic crossed product $K^\infty \rtimes C_0(A_K)$. We want to compute the topological $K$-theory $K(A)$. The group $K^\infty$ is Abelian and hence amenable. Therefore, the full and reduced crossed products agree and $K^\infty$ satisfies the Baum-Connes conjecture with arbitrary coefficients by a result of Nigel Higson and Gennadi Kasparov ([6]). Thus we can replace $K(A)$ by $K^{top}(K^\infty, C_0(A_K))$. The latter group can always be computed by a spectral sequence. However, in order to get more precise results it is often better to use special structure present in the situation.

Let $\mu_K \subseteq K^\infty$ be the subgroup of roots of unity and choose an isomorphism $K^\infty \cong \mu_K \times \mathbb{Z}^N$ as above. Let

$$B := \mu_K \rtimes C_0(A_K),$$

then we have $A \cong \mathbb{Z}^N \rtimes B$ and hence $K(A) \cong K^{top}(\mathbb{Z}^N, B)$. Since $\mathbb{Z}^N$ is torsion free, the spectral sequence for $K^{top}(\mathbb{Z}^N, B)$ is much simpler than for $K^{top}(K^\infty, A)$. Its $E^2$-term is

$$E^2_{pq} := H_p(\mathbb{Z}^N, K_q(B)).$$

Thus our first step is to compute $K(\mu_K \rtimes C_0(A_K)) \cong K^{\mu_K}(C_0(A_K))$ together with the representation of $K^\infty / \mu_K$ on it. To express the result in a uniform way for all
global fields, we need some ingredients. They all are $\mathbb{Z}/2$-graded Abelian groups together with a representation of $A_\infty^\times$.

Let $\text{Rep}(\mu_K)$ be the representation ring of $\mu_K$ in even degree and 0 in odd degree. Thus $\text{Rep}(\mu_K) \cong K_{\mu K}(\mathbb{C})$. The forgetful map $K_{\mu K}(\mathbb{C}) \to K(\mathbb{C})$ associates to each representation its dimension. We let $J\text{Rep}(\mu_K) \subseteq \text{Rep}(\mu_K)$ be its kernel. We let $A_\infty^\times$ act trivially on $\text{Rep}(\mu_K)$ and $J\text{Rep}(\mu_K)$. Let $n = n_K$ be the degree of $K$ over $\mathbb{Q}$, so that $A_\infty \cong \mathbb{R}^n$. Let $\sigma$ be the orientation character that already occurred in the periodic cyclic homology and let $\mathbb{Z}(\sigma)$ be the resulting representation of $A_\infty^\times$ on $\mathbb{Z}$. We view $\mathbb{Z}(\sigma)$ as a $\mathbb{Z}/2$-graded Abelian group in degree $n \mod (2)$.

For a totally disconnected locally compact space $F$ and a graded Abelian group $V$, we let $D(F, V)$ be the space of locally constant, compactly supported functions $F \to V$ with the obvious grading. If a group acts on $F$ and $V$, we equip $D(F, V)$ with the induced action. In particular, on $D(A_\infty, \mathbb{Z}(\sigma))$, the finite adeles act by translation and the infinite adeles act by $\sigma$. Finally, we write $V_{\mu K}$ for the subspace of $\mu_K$-invariants in a representation $V$ of $\mu_K$.

**Proposition 5.1.** There is an $A_\infty^\times$-equivariant grading preserving isomorphism

$$K_{\mu K}(C_0(A_\infty)) \cong J\text{Rep}(\mu_K) \oplus D(A_\infty, \mathbb{Z}(\sigma))_{\mu K}. $$

Evaluation at $0 \in A_\infty$ induces a map $K_{\mu K}(C_0(A_\infty)) \to K_{\mu K}(\mathbb{C})$, which sends the first summand isomorphically onto $J\text{Rep}(\mu_K) \subseteq K_{\mu K}(\mathbb{C})$.

**Proof.** Let $A_\infty^* := A_\infty \setminus \{0\}$. Evaluation at $0 \in A_\infty^*$ gives rise to an extension of $C^*$-algebras

$$C_0(A_\infty^* \times A_\infty^\times) \twoheadrightarrow C_0(A_\infty^\times) \to C_0(A_\infty).$$

This extension splits by the following $\mu_K$-equivariant $*$-homomorphism. Let $O \subseteq A_\infty$ be the maximal compact subring. Thus $x \in O$ if and only if $|x|_v \leq 1$ for all finite places $v$. Let $1_O$ be the characteristic function of $O$. The map $f \mapsto 1_O \otimes f$ is the desired section. Applying the K-theory functor, we get a split exact sequence

$$K_{\mu K}(C_0(A_\infty^* \times A_\infty)) \twoheadrightarrow K_{\mu K}(C_0(A_\infty^*)) \to K_{\mu K}(C_0(A_\infty)).$$

Since $A_\infty \cong \mathbb{R}^n$, we would like to use equivariant Bott periodicity to simplify the above expressions. For that we need to know whether $\mathbb{R}^n$ has a $\mu_K$-invariant spin structure. We distinguish three cases.

Suppose first that $K$ has no real places. Thus $\sigma = 1$ and $n$ is even. The space $A_\infty^\times$ is a complex manifold in a natural and in particular $A_\infty^\times$-invariant way. Since a complex structure is more than a spin structure, equivariant Bott periodicity applies and shows that $C_0(A_\infty^\times)$ is $KK_{\mu K}$-equivalent to $\mathbb{C}$. Therefore,

$$K_{\mu K}(C_0(A_\infty^\times)) \cong K_{\mu K}(\mathbb{C}) \cong \text{Rep}(\mu_K).$$

$$K_{\mu K}(C_0(A_\infty^* \times A_\infty^\times)) \cong K_{\mu K}(C_0(A_\infty^*)) \cong K(C_0(A_\infty^*/\mu_K))$$

$$\cong D(A_\infty^*/\mu_K, \mathbb{Z}) \cong D(A_\infty^*, \mathbb{Z})_{\mu K}.$$
same as the action of $-1$ on $\mathbb{C}$ and hence again preserves a complex structure. Hence we can apply equivariant Bott periodicity as in the purely complex case and get essentially the same results. The only difference is that $\mathbb{A}^\infty_\infty$ need not preserve the spin structure on $\mathbb{A}_\infty$ and therefore acts by the orientation character $\sigma$. This yields the assertions if $n$ is even.

Finally, suppose that $n$ is odd. Hence there is an odd number of real places. Again this forces $\mu_K = \{\pm 1\}$. Equivariant Bott periodicity allows us, as above, to remove all complex places and an even number of real places from $\mathbb{A}_\infty$. Thus $C_0(\mathbb{A}_\infty)$ is KK$\mu_K$-equivalent to $C_0(\mathbb{R})$ with the action of $\mu_K$ by reflection at the origin. A straightforward computation shows that evaluation at 0 induces an isomorphism $K^\mu_K(C_0(\mathbb{R})) \cong J\text{Rep}(\mu_K) \subseteq \text{Rep}(\mu_K)$. There exists a fundamental domain $F$ for the free action of $\mu_K$ on the totally disconnected space $\mathbb{A}^*_\infty$. This allows us to identify

$$K^\mu_K(C_0(\mathbb{A}^*_\infty \times \mathbb{A}_\infty)) \cong K^\mu_K(C_0(\mu_K \times F \times \mathbb{A}_\infty)) \cong K(F \times \mathbb{A}_\infty) \cong D(F, \mathbb{Z}(\sigma)).$$

By Lemma 5.2.

Define $\mathbb{Z}/2$-graded groups by

$$K_h := (\mathcal{D}(\mathbb{A}_f, \mathbb{Z}) \otimes \mathbb{Z}(\sigma))^\mu_K, \quad K_{inh} := J\text{Rep}(\mu_K).$$

They correspond to the homogeneous and inhomogeneous parts in the cyclic homology computations. We obtain

$$E^2_{pq} \cong H_p(K^\times/\mu_K, K_h) \oplus H_p(K^\times/\mu_K, K_{inh}).$$

**Lemma 5.2.** The boundary maps $d^2_{pq}: E^2_{pq} \rightarrow E^2_{p-2, q+1}$ vanish for all $p, q$.

**Proof.** If $n$ is even, this is clear because either the source or target of $d^2_{pq}$ vanish. Let $n$ be odd and let $g$ be the adele $(1_f, -1_\infty)$. That is, the value of $g$ at a place $v$ is $\pm 1$ depending on whether $v$ is infinite or not. Then $g$ acts as $-1$ on $J\text{Rep}(\mu_K)$ and as $(-1)^n = -1$ on the other summmand. Since the boundary map $d^2_{pq}$ is natural with respect to equivariant $*$-homomorphisms, it commutes with the action of $g$. Hence it cannot mix the two direct summands. Since each summand is concentrated in one parity, we get $d^2_{pq} = 0$. □

Lemma 5.2 implies that there is a filtration on $K_m(K^\times \ltimes C_0(\mathbb{A}_K))$ whose subquotients are isomorphic to $E^2_{pq}$ for $p + q = m$. Since $K^\times$ acts trivially on $K_{inh}$, we get natural isomorphisms

$$H_p(K^\times/\mu_K, K_{inh}) \cong K_{inh} \otimes H_p(\mathbb{Z}^N) \cong J\text{Rep}(\mu_K) \otimes \Lambda^p\mathbb{Z}^N,$$

where $\Lambda^pV$ is the $p$th exterior power of $V$. The evaluation map at 0 maps this group isomorphically onto the corresponding subspace of $K(K^\times \ltimes \mathbb{C}) \cong K(C_0(\mathbb{K}^\times))$.

The contribution of $K_h$ requires considerably more work. We proceed in two steps. We first deal with the subgroup $K^\times_{inh}/\mu_K$ and then use the spectral sequence

$$H_p(K^\times/\mu_K, H_q(K^\times_{inh}/\mu_K, K_h)) = H_{p+q}(K^\times/\mu_K, K_h).$$

Let $\mathcal{O} \subseteq \mathbb{A}_f$ be the maximal compact subring, that is, the set of all $x \in \mathbb{A}_f$ with $|x|_v \leq 1$ for all finite places $v$. There is a basis for the neighborhoods of zero in $\mathbb{A}_f$ consisting of compact, open subgroups of the form $x_m \cdot \mathcal{O}$ for a suitable sequence of ideles $(x_m)$. Thus

$$K_h = \mathcal{D}(\mathbb{A}_f, \mathbb{Z}(\sigma))^\mu_K \cong \varinjlim \mathcal{D}(\mathbb{A}_f/x_m\mathcal{O}, \mathbb{Z}(\sigma))^\mu_K.$$
The action of $K^\infty_\infty/\mu_K$ respects this inductive limit decomposition, so that

$$H_*(K^\infty_\infty/\mu_K, K_h) \cong \lim_{\to} H_*(K^\infty_\infty/\mu_K, \mathcal{D}(\mathbb{A}_t/x_m\mathcal{O}, \mathbb{Z}(\sigma))^{\mu_K}).$$

The space $\mathbb{A}_t/x_m\mathcal{O}$ is discrete. We let

$$L_m(y) := \{g \in K^\infty_\infty \mid gy - y \in x_m\mathcal{O}\}$$

be the stabilizer of the point $y + x_m\mathcal{O} \in \mathbb{A}_t/x_m\mathcal{O}$. This is a cocompact subgroup of $K^\infty_\infty$ because the action of $K^\infty_\infty$ on $\mathbb{A}_t$ factors through the compact group $\tilde{K}_\infty \subseteq O^\infty \subseteq \mathbb{A}_t$. Let $\hat{L}_m(y) := L_m(y)/(\mu_K \cap L_m(y))$. This is a cocompact subgroup of $K^\infty_\infty/\mu_K$ and hence a free Abelian group of rank $u$.

The space $\mathbb{A}_t/x_m\mathcal{O}$ is a disjoint union of orbits $K^\infty_\infty/L_m(y)$. Each orbit yields a contribution

$$H_*(K^\infty_\infty/\mu_K, \mathcal{D}(K^\infty_\infty/L_m(y), \mathbb{Z}(\sigma))^{\mu_K}).$$

To compute the latter, we have to distinguish three cases.

If $-1 \in L_m(y)$ and $\sigma(-1) = -1$, then there simply are no $\mu_K$-invariant functions $K^\infty_\infty/L_m(y) \to \mathbb{Z}(\sigma)$, so that we get 0. In the other cases, $\sigma$ vanishes on $\mu_K \cap L_m(y)$. Hence $\mathcal{D}(K^\infty_\infty/L_m(y), \mathbb{Z}(\sigma))^{\mu_K}$ is isomorphic to $\mathcal{D}(K^\infty_\infty/L_m(y)\mu_K, \mathbb{Z}(\sigma))$. This representation is induced from the representation $\mathbb{Z}(\sigma)$ of $L_m(y)$. Hence we obtain

$$H_*(K^\infty_\infty/\mu_K, \mathcal{D}(K^\infty_\infty/L_m(y)\mu_K, \mathbb{Z}(\sigma))) \cong H_*(\hat{L}_m(y), \mathbb{Z}(\sigma)).$$

If $\sigma$ does not vanish on $L_m(y)$, we obtain again 0. Otherwise, we obtain the group homology of $L_m(y)$, which is $\Lambda\hat{L}_m(y)$. The final result is therefore

$$H_*(K^\infty_\infty/\mu_K, \mathcal{D}(\mathbb{A}_t/x_m\mathcal{O}, \mathbb{Z}(\sigma))^{\mu_K}) \cong \bigoplus_{y \in \mathbb{A}_t/x_m\mathcal{O}\mid L_m(y) \subseteq \ker\sigma} \Lambda\hat{L}_m(y).$$

To understand how these spaces fit together in the inductive limit over $m$, we have to rewrite this result in a more functorial way. Let $L$ be a free Abelian group of rank $u$. Let $\omega \in \text{Hom}(\Lambda^n L, \mathbb{Z}) \cong \mathbb{Z}$ be one of the two generators. We define a biadditive pairing $\Lambda^p L \otimes \Lambda^{u-p} L \to \mathbb{Z}$ by

$$(\xi, \eta) := \omega(\xi \wedge \eta),$$

This pairing defines an isomorphism $\Lambda^p L \cong \text{Hom}(\Lambda^{u-p} L, \mathbb{Z})$.

Let $L' \subseteq L$ be a cocompact subgroup of a free Abelian group. Then we can consider the subspace $\Lambda(L' : L)$ of all $f \in \Lambda^p L \otimes \mathbb{Q}$ with $(f, x) \in \mathbb{Z}$ for all $x \in \Lambda^{u-p} L'$. This subspace does not depend on the choice of $\omega$. Explicitly, it looks as follows. Choose a basis $(g_1, \ldots, g_n)$ for $L$ such that $(n_1 g_1, \ldots, n_u g_u)$ is a basis for $L'$ with certain $n_j \in \mathbb{N}^\times$. Express all exterior forms in these bases. Then $\Lambda(L' : L)$ is equal to the subalgebra generated by the 1-forms $n_j^{-1} dg_j$. This subspace is non-canonically isomorphic to $\Lambda^p L'$.

For each $y \in \mathbb{A}_t$, we have a decreasing sequence of subgroups $\hat{L}_m(y) \subseteq K^\infty_\infty/K^\infty$. Let

$$\Lambda_y := \bigcup_{m \in \mathbb{N}} \Lambda(\hat{L}_m(y) : K^\infty_\infty/\mu_K) \subseteq \Lambda(K^\infty_\infty/\mu_K) \otimes \mathbb{Q}.$$

We can now reformulate the above computation as follows:

**Proposition 5.3.** The homology $H_*(K^\infty_\infty/\mu_K, K_h)$ is naturally isomorphic to the space of all functions in $\mathcal{D}(\mathbb{A}_t, \Lambda(K^\infty_\infty/\mu_K) \otimes \mathbb{Q}(\sigma))^{K^\infty}$ that satisfy the integrality condition $f(y) \in \Lambda_y$ for all $y \in \mathbb{A}_t$.

**Proof.** The space of functions that satisfies the integrality condition is the inductive limit of the spaces $\mathcal{D}(\mathbb{A}_t/x_m\mathcal{O}, \Lambda(\hat{L}_m(y) : K^\infty_\infty \otimes \mathbb{Z}(\sigma))^{K^\infty}$ for $m \to \infty$. For a fixed $m$ this space is isomorphic to the result of our computation for the group homology. The point is that these isomorphisms can be chosen natural and compatible with...
the inductive limit. Let \( L := K_x^\infty / \mu K \). The group homology of the free Abelian group \( L \) with coefficients \( K_h \) is computed by the Koszul complex \( (K_h \otimes \Lambda(L), \delta) \) with

\[
\delta(f \ dg_1 \wedge \cdots \wedge dg_k) := \sum_{j=1}^{k} (-1)^j (f - g_j \cdot f) \ dg_1 \wedge \cdots \wedge \widehat{dg}_j \wedge \cdots \wedge dg_k.
\]

Consider the graded \( \mathbb{Q} \)-vector space

\[
W := D(\mathcal{A}_f, \Lambda(L) \otimes \mathbb{Q}(\sigma))^{\hat{K}_x^\infty}
\]
as a complex with trivial boundary map.

The closure \( \overline{K_x^\infty} \subseteq \mathbb{A}_K^\infty \) of \( K_x^\infty \) is a compact group. Let \( \mu \) be the normalized Haar measure on \( \overline{K_x^\infty} \). We extend \( \sigma \) to a continuous character \( \mathcal{P} \) on \( \overline{K_x^\infty} \) and consider the integration map \( \alpha : K_h \otimes \Lambda(L) \to W \) defined by

\[
\alpha(f \ dg_i \wedge \cdots \wedge dg_k)(t) := \int_{\overline{K_x^\infty} / \mu K} f(at) \mathcal{P}(a) \ d\mu(a) \ dg_i \wedge \cdots \wedge dg_k.
\]

It is a chain map because the integration annihilates all coinvariants \( f - g \cdot f \) for \( g \in K_x^\infty \). The map on homology induced by \( \alpha \) is injective. This follows easily from our elementary computations with \( \mathcal{A}_f / x_m \mathcal{O} \). The assertion follows by explicitly identifying the image of \( H_*(K_h \otimes \Lambda(L)) \) in \( W \).

Let \( W \) be the graded Abelian group that we identified with the homology in Proposition 5.3. We still have to describe the homology of \( K_x^\infty / K_x^\infty \) with coefficients in \( W \). We use a filtration of \( W \) for this purpose. Let \( S \) be a finite set of finite places. Let \( T \) be the set of finite places not in \( S \), let \( \mathcal{O}_T \) be the set of all \( x \in \mathcal{A}_T \) that satisfy \(|x|_v \leq 1 \) for all \( v \in T \) and let \( 1_{\mathcal{O}_T} \in \mathcal{S}(\mathcal{A}_T) \) be its characteristic function.

A smooth function on \( \mathcal{A}_T \) is called \textit{unramified} outside \( S \) if it is a finite linear combination of functions of the form \( g \cdot (1_{\mathcal{O}_T} \otimes f) \) for some smooth function \( f \) on \( \mathcal{A}_S \) and some \( g \in \mathcal{A}_x^\infty \). We let \( W_S \subseteq W \) be the subspace of functions that are unramified outside \( S \). We clearly have \( W_S \subseteq W_S' \) for \( S \subseteq S' \). If we work rationally, then a smooth function is unramified outside \( S \) if and only if it is \( \mathcal{O}_T^\infty \)-invariant. However, functions in \( W_S \) must not only be \( \mathcal{O}_T^\infty \)-invariant, they also must satisfy a stronger integrality condition. Let

\[
L_m(y, S) := \{ g \in K_x^\infty \mid gy - y \in x_m \mathcal{O}_S \}.
\]

Then \( L_m(y) \subseteq L_m(y, S) \) for all \( S \). It is straightforward to see that \( W_S \) consists exactly of the \( \mathcal{O}_T^\infty \)-invariant elements of \( W \) that satisfy \( f(y) \in \bigcup_{m \in \mathbb{N}} \Lambda(L_m(y, S) : K_x^\infty / \mu K) \) for all \( y \in \mathcal{O}_T \).

For \( a \in \mathbb{N} \), let \( W_a \subseteq W \) be the subspace generated by all \( W_S \) with \(#S \leq a\). This is an increasing filtration on \( W \). We use this filtration to compute the homology with coefficients in \( W \). Let \( W_a^{(0)} \subseteq W_a \) be the subspace that is spanned by elements of the form \( 1_{\mathcal{O}_S} \otimes f_S \) for finite sets of places \( S \) with \(#S = a \) and suppf \( f_S \subseteq \mathcal{O}_T \). Let \( W_a^{(1)} := W_a^{(0)} \cap W_{a-1} \). Both \( W_a^{(0)} \) and \( W_a^{(1)} \) are representations of the compact open subgroup \( \mathcal{O}_T^\infty \subseteq \mathbb{A}_K^\infty \). Since this subgroup is open, compact induction from \( \mathcal{O}_T^\infty \) to \( \mathbb{A}_K^\infty \) makes sense for representations on Abelian groups. Hence we can define a representation \( \text{c-Ind}_{\mathcal{O}_T^\infty}^{\mathbb{A}_K^\infty} W_a^{(0)} / W_a^{(1)} \) of \( \mathbb{A}_K^\infty \) and a canonical equivariant map

\[
\Phi_a : \text{c-Ind}_{\mathcal{O}_T^\infty}^{\mathbb{A}_K^\infty} W_a^{(0)} / W_a^{(1)} \to W_a / W_{a-1}.
\]

\textbf{Lemma 5.4.} The map \( \Phi_a \) is an isomorphism for all \( a \in \mathbb{N} \).
Proof. First we check surjectivity. Let \( f \in W_S \) for some set of finite places \( S \) with \#\( S \leq a \). If \#\( S < a \), then \( f \in W_{a-1} \), so that we may assume \#\( S = a \). Write \( f = g \cdot 1_{O_T} \otimes f_S \). We will show that \( f \equiv f' \mod W_{a-1} \) for a function \( f' = g' \cdot 1_{O_T} \otimes f'_S \) with \( \text{supp} f'_S \subseteq A_S^\times \). Since \( A_i^\times \) acts transitively on \( A_S^\times \), the function \( f' \) belongs to the range of \( \Phi_a \). Hence \( \Phi_a \) must be surjective if we can modify \( f \) in this fashion.

If \( f \) is not yet supported in \( A_S^\times \), there is \( v \in S \) such that \( f|_{x_v=0} \) does not vanish. Write \( A_S = K_v \times A_{S \setminus \{v\}} \) and define

\[
    f_v(x_v,y) := \begin{cases} 
    f(0,y) & \text{if } |x_v|_v \leq 1; \\
    0 & \text{if } |x_v|_v > 1.
    \end{cases}
\]

This is again an element of \( W \). The reason for this is that the integrality condition at \((0,y)\) is stronger than the integrality condition at \((x_v,y)\) for \( x_v \neq 0 \) because \( L_m((0,y)) \supseteq L_m((x_v,y)) \). Hence \( f_v \in W_{S \setminus \{v\}} \). Subtracting \( f_v \) from \( f \), we obtain a function that vanishes whenever \( x_v = 0 \). Proceeding in this fashion for all \( v \in S \), we obtain a function supported in \( A_S^\times \). This establishes the surjectivity of \( \Phi_a \).

Next we verify injectivity. Let \( A \subseteq A_i^\times \) be a closed and open subset. Then multiplication by \( 1_A \) is an operator on \( W \) because the integrality condition is local. If \( A \) is invariant under \( O_i^\times \), then this operator also preserves the subspaces \( W_{S \setminus \{v\}} \) and hence the filtration. It also acts on the source of \( \Phi_a \) in an evident fashion.

Both the source and target of \( \Phi_a \) carry a filtration defined by the condition that \( |x|_f \leq n \) for all \( x \in \text{supp} f \). Using multiplication operators, we obtain easily that \( \Phi_a \) is injective.

The source of \( \Phi_a \) is a free module over \( K^\times /K^\times_\infty \) because \( K^\times /K^\times_\infty \) embeds in \( A_i^\times /O_i^\times \). Hence it is relatively projective, that is, projective for extensions that split as extensions of Abelian groups. The group \( W^{(0)}/W^{(1)} \) contains torsion elements, so that the source of \( \Phi_a \) is not free as an Abelian group. Nevertheless, we can prove by induction on \( a - b \) that \( W_{a}/W_b \) is relatively projective for all \( b \leq a \) because extensions of relatively projective modules are again relatively projective. Therefore, \( W = \lim_W W_a \) is relatively flat, so that \( H_m(K^\times /K^\times_\infty,W) = 0 \) for \( m \neq 0 \). This implies

\[
    H_p(K^\times /\mu_K,K_b) \cong H_p(K^\times /\mu_K,K_h)/K^\times \cong W/K^\times.
\]

Our filtration \( (W_a) \) also yields a filtration on \( W/K^\times \), whose subquotients are

\[
    H_0(K^\times /K^\times_\infty,W_a/W_{a-1}) \cong D(A_i^\times /K^\times O_i^\times) \otimes W^{(0)}_a/W^{(1)}_a.
\]

However, the extensions that come with this filtration are quite complicated because of the torsion in the groups \( W_a^{(0)}/W_a^{(1)} \).

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