Right sign of spin rotation operator

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Abstract

For the fermion transformation in the space all books of quantum mechanics propose to use the unitary operator
\[ \hat{U}_n(\varphi) = \exp(-i\frac{\varphi}{2}(\hat{\sigma} \cdot \vec{n})), \]
where \( \varphi \) is angle of rotation around the axis \( \vec{n} \). But this operator turns the
spin in inverse direction presenting the rotation to the left. The error of
defining of \( \hat{U}_n(\varphi) \) action is caused because the spin supposed as simple vec-
tor which is independent from \( \hat{\sigma} \)-operator a priori. In this work it is shown
that each fermion marked by number \( i \) has own Pauli-vector \( \hat{\sigma}_i \) and both
of them change together. If we suppose the global \( \hat{\sigma} \)-operator and using
the Bloch Sphere approach define for all fermions the common quan-
tization axis \( z \) the spin transformation will be the same; the right hand rotation
around the \( \vec{n} \)-axis is performed by the operator
\[ \hat{U}^+_n(\varphi) = \exp(+i\frac{\varphi}{2}(\hat{\sigma} \cdot \vec{n})). \]
1 Introduction

The spin-vector is an analog of mechanical moment but its projection to any direction has a discrete values. In the case of particle with spin $\frac{1}{2}\hbar$ the projection will be equal to $s_z = +\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$. To account this duality the three Hermitian 2 × 2 matrices are used that forms the Pauli-operator $\hat{\sigma}$:

$$\hat{\sigma} = i\sigma_x + i\sigma_y + k\sigma_z,$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Since only $\sigma_z$ has a diagonal view its own vectors or spinors are pure states:

$$\chi_z(s_z = +\frac{1}{2}\hbar) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_z(s_z = -\frac{1}{2}\hbar) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

2 Eigenvectors of operator ($\hat{\sigma} \cdot \vec{r}$)

Let the unit vector $\vec{r}$ is defined in the spherical coordinate system by the values of zenith and azimutal angles $\theta$ and $\varphi$ (Fig. 1).

$$\chi_r(s_z = +\frac{1}{2}\hbar) = \begin{pmatrix} \cos\theta/2 \\ e^{i\varphi}\sin\theta/2 \end{pmatrix}, \quad \chi_r(s_z = -\frac{1}{2}\hbar) = \begin{pmatrix} \cos\theta e^{i\varphi}\sin\theta \\ e^{-i\varphi}\sin\theta - \cos\theta \end{pmatrix}.$$  

Figure 1: Absolute value of projection of the Pauli-vector $\hat{\sigma} = i\sigma_x + j\sigma_y + k\sigma_z$ to any direction $\vec{r}$ equals to unit: $|\sigma_r|^2 = 1$. That allows to present it like a $\Sigma$-sphere. Each point of its has two eigenvectors or spinors $\chi_r(s_z = +\frac{1}{2}\hbar)$ and $\chi_r(s_z = -\frac{1}{2}\hbar)$.  


In this case the scalar multiplying of $\hat{\sigma}$ and $\vec{r}$ has the next form:

$$\sigma \cdot \vec{r} = (\hat{\sigma} \cdot \vec{r}) = \sigma_x \sin \theta \cos \varphi + \sigma_y \sin \theta \sin \varphi + \sigma_z \cos \theta =$$

$$= \begin{pmatrix} \cos \theta & e^{-i\varphi} \sin \theta \\ e^{i\varphi} \sin \theta & -\cos \theta \end{pmatrix} = \begin{pmatrix} \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} & 2e^{-i\varphi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\ 2e^{i\varphi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} & -\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \end{pmatrix} =$$

$$= \chi(+) \otimes \chi(+^\ast) - \chi(-) \otimes \chi(-^\ast), \quad (4)$$

$$\chi(+) = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\varphi} \sin \frac{\theta}{2} \end{pmatrix}, \quad \chi(-) = \begin{pmatrix} -\sin \frac{\theta}{2} \\ e^{i\varphi} \cos \frac{\theta}{2} \end{pmatrix}. \quad (5)$$

Vectors $\chi(\pm)$ are unit and orthogonal among themselves:

$$|\chi(\pm)|^2 = 1, \quad \chi(-^\ast) \cdot \chi(+) = 0. \quad (6)$$

According to (4) it is obviously:

$$\sigma \cdot \vec{r} \chi(+) = \chi(+), \quad \sigma \cdot \vec{r} \chi(-) = -\chi(-). \quad (7)$$

Common phase of spinor’s elements is not informative therefore $\chi(-) \equiv -\chi(-)$. If the spin projection to the axis $z$ is positive $s_z = +\frac{1}{2} \hbar$ it can be presented as superposition of two states $s_\uparrow = +\frac{1}{2} \hbar$ and $-\frac{1}{2} \hbar$ and their amplitudes equal to $\cos \frac{\theta}{2}$ and $e^{i\varphi} \sin \frac{\theta}{2}$ respectively. If the spin projection is negative $s_z = -\frac{1}{2} \hbar$ the states $s_\downarrow = +\frac{1}{2} \hbar$ and $-\frac{1}{2} \hbar$ have amplitudes $-\sin \frac{\theta}{2}$ and $e^{i\varphi} \cos \frac{\theta}{2}$.

The Pauli-operator along the vector $\vec{r}$ can be associated with the Stern-Gerlach device as it is considered in the Feynman Lectures (V. III, Ch. 5, 6). The main of this device — quantization axis. When it coincides with the vector $\vec{r}$ the device rotation around $\vec{r}$ does not change the probability of spin states $s_\uparrow = +\frac{1}{2} \hbar$ and $-\frac{1}{2} \hbar$. The action of operator $\sigma \cdot \vec{r}$ is analogical — from all possible representations of spin it saves unchanged only own spinors $\chi(+) \text{ and } \chi(-)$ (5), i.e. only spin states along the vector $\vec{r}$.

1. The difference between them is concluded in fact that the Stern-Gerlach device measures not amplitudes but the probabilities of states $\uparrow$ and $\downarrow$ therefore the information about their relative phase $e^{i\varphi}$ is lost.

2. In the SU(2) algebra the definition of pure states is conditional because any representation of spin gives the full knowledge about its orientation in the space. If the $\chi(\uparrow) \text{ and } \chi(-)$ will be named as true then any others can be defined using their linear combinations. For example:

$$\cos \frac{\theta}{2} \chi(+) - \sin \frac{\theta}{2} \chi(-) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sin \frac{\theta}{2} \chi(+) + \cos \frac{\theta}{2} \chi(-) = e^{i\varphi} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (8)$$
3 Rotation of spin $\frac{1}{2}$ around an arbitrary axis

Consider a coordinate system rotation $(x, y, z) \rightarrow (x', y', z)$ around the axis $z$ by an angle $\varphi$ according to the right hand screw rule (Fig. 2).

![Figure 2](image-url)

Figure 2: The rotation of coordinate system around the axis $z$.

The matrix of transition has view:

$$ A = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} : \quad A \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} = \begin{pmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{pmatrix}. \quad (9) $$

Let the condition $\hat{\sigma} = \text{const}$, i.e. the spin Pauli-operator remains unchanging. In the system $(x', y', z)$ it can be presented as following:

$$ \hat{\sigma} = \vec{i}'(\sigma_x \cos \varphi + \sigma_y \sin \varphi) + \vec{j}'(-\sigma_x \sin \varphi + \sigma_y \cos \varphi) + \vec{k}\sigma_z. \quad (10) $$

On the other hand the new coordinates $\sigma_{x'}$ and $\sigma_{y'}$ can be defined by the unitary $2 \times 2$ matrices $u$ and $u^+$:

$$ \sigma_{x'} = u \sigma_x u^+ = \begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}, \quad \sigma_{y'} = u \sigma_y u^+ = \begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix}. \quad (11) $$

The solutions of equations (11) are:

$$ u \equiv u_z = \pm \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}, \quad u^+ \equiv u_z^+ = \pm \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}. \quad (12) $$

The marker $z$ says here about the identical transformation $\sigma_z = u \sigma_z u^+$ because this rotation carries out around the axis $z$. Without damage of our solution choosing in (12) the sign “+” the unitary matrix $u_z$ can be rewritten as:

$$ u_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \frac{\varphi}{2} - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sin \frac{\varphi}{2} = E \cos \frac{\varphi}{2} - i\sigma_z \sin \frac{\varphi}{2}. \quad (13) $$
The formula (13) propagates to the case of system rotation around an arbitrary axis $\vec{n}$ that leads to the unitary operator:

$$\hat{U}_{\vec{n}}(\varphi) = E \cos \frac{\varphi}{2} - i(\vec{\sigma} \cdot \vec{n}) \sin \frac{\varphi}{2} = \exp\left[-i\frac{\varphi}{2} (\vec{\sigma} \cdot \vec{n})\right],$$  \hspace{1cm} (14)

$$\hat{U}_{\vec{n}}(\varphi) \left( \sigma_x, \sigma_y, \sigma_z \right) \hat{U}_{\vec{n}}^+(\varphi) = \left( \sigma_{x'}, \sigma_{y'}, \sigma_{z'} \right).$$  \hspace{1cm} (15)

The transformation (15) allows to express the coordinates of vector $\vec{\sigma}$ in the system $(x', y', z')$ obtained from initial system $(x, y, z)$ by the rotation around the vector $\vec{n}$ by an angle $\varphi$ according to the right hand screw rule. For observer related with new system $(x', y', z')$ it will be look like the left hand rotation around $\vec{n}$ by an angle $-\varphi$. Since the fermion states with pure projections $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$ are connected with old axis $z$ by the matrix $\sigma_z$ (hold by its diagonal form) this rule will be true for the spin too:

**Lemma 1 (Spin rotation)** The rotation of the spin $\frac{1}{2}$ around an arbitrary axis $\vec{n}$ by an angle $\varphi$ corresponds to the operator $\hat{U}_{\vec{n}}^+(\varphi) = \exp\left[+i\frac{\varphi}{2} (\vec{\sigma} \cdot \vec{n})\right]$.

Let us to consider the rotation around the axis $z$ by the angle $90^\circ$:

$$\hat{U}_z^+ \equiv \hat{U}_z^+(\frac{\pi}{2}) = \frac{1}{\sqrt{2}} (E + i\sigma_z) = \frac{1 + i}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix},$$ \hspace{1cm} (16)

$$\sigma_y \rightarrow \hat{U}_z^+ \sigma_y \hat{U}_z = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x. \hspace{1cm} (17)$$

The projection $\sigma_y$ is replaced by the matrix $\sigma_x$. It means that the Pauli-vector turns around the axis $z$ to the right. Since the spin state $\chi_y = \frac{1}{\sqrt{2}}(\begin{pmatrix} 1 \\ i \end{pmatrix})$ is eigenvector of matrix $\sigma_y$ the same changing is performed with the fermion:

$$\chi_y \rightarrow \hat{U}_z^+ \chi_y = \frac{1 + i}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 \end{pmatrix} = e^{i\frac{\varphi}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \chi_x. \hspace{1cm} (18)$$

If this transformation $\hat{U}_z^+ \chi_y = \chi_x$ would be defined as the left rotation the connection between $\vec{\sigma}$-operator and our spin would be broken converting it to the element of external space such as ort-vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ as a minimum or we need to suppose that the spin and Pauli-operator are rotated in opposite directions.
Lemma 2 (Common moving)  The $\chi_r$ and $\sigma_r$ are changed identical.

Let us to take the $\sigma_r$ in the form \([1]\). Then:

$$u\sigma_r u^+ = u(\hat{\sigma} \cdot \vec{r}) u^+ = u\chi_{(+)} \otimes \chi_{(+)}^+ u^+ - u\chi_{(-)} \otimes \chi_{(-)}^+ u^+ =$$

$$= (u\chi_{(+)} \otimes (u\chi_{(+)}))^+ - (u\chi_{(-)} \otimes (u\chi_{(-)}))^+ = (\hat{\sigma}' \cdot \vec{r}) = \sigma_r'. \quad (19)$$

This transition $\sigma_r \rightarrow \sigma_r'$ is the transformation of eigenvectors $\chi_{(\pm)}$ of operator $\sigma_r$ to eigenvectors $\chi'_{(\pm)} = u\chi_{(\pm)}$ of operator $\sigma_r' = u\sigma_r u^+$,

quod erat demonstrandum.

The spin is related with the $\hat{\sigma}$-operator and both of them are changed in the space together. In particular the state with pure projection $s_z = +\frac{1}{2}\hbar$ is rotated to the same direction like the $\sigma_z$ matrix.

The transformation $\hat{\sigma} \rightarrow \hat{\sigma}'$ can be defined also by the $3 \times 3$ matrix $\hat{\sigma}' = A\hat{\sigma}$ and for scalar multiplying $(\hat{\sigma}' \cdot \vec{r})$ it gives the following:

$$(\hat{\sigma}' \cdot \vec{r}) = (A\hat{\sigma} \cdot \vec{r}) = (\hat{\sigma} \cdot A^T \vec{r}) = (\hat{\sigma} \cdot \vec{r}'') . \quad (20)$$

If the $A$-matrix in the system $(x, y, z)$ rotates the vector $\hat{\sigma}$ to the right it will be equal to the left rotation of vector $\vec{r} \rightarrow \vec{r}''$ in the own coordinates of Pauli-operator. As it was shown before, the direction of vector $\vec{r}$ on the $\Sigma$-sphere (Fig. \[\Pi\]) corresponds to the orientation of Stern-Gerlach device relative the fermion.
4 Common approaches to the determination of spin rotation operator

In all books of quantum mechanics the space rotation of spin has inverse definition of operator sign. For example in 8-th volume of Feynman’s Lectures \[1\] (V. III, Ch. 6-3, P. 6-15) we can find the analytical approach: if the xy-plane rotates around the axis \(z\) by the angle \(180^\circ\) the transformation of pure states \(s_z = +\frac{1}{2}\hbar\) and \(-\frac{1}{2}\hbar\) should add to them the phases \(e^{im\pi}\) and \(e^{-im\pi}\) respectively. Feynman chooses the \(m = +\frac{1}{2}\) and in the same time he talks about second solution \(m = -\frac{1}{2}\) that gives our definition of matrix \(u_z\) \[12\]. The main Feynman’s goal — to show the difference of two phases but it does not matter whether “+” or “−” since the relative shift between them will be equal to \(e^{i\pi} = -1\) anyway.

The standard solution can be find in the book of “Theoretical physics” \[2\] and here we consider the main points of this method. Supposing that new system \((x’, y’, z’)\) is produced from \((x, y, z)\) using rotation around the axis \(z\) by small angle \(\delta\varphi\) (Fig. 2, Form. 9). If the quantum object \(\psi(x, y, z)\) states unchanging then its representation in new system should be the same:

\[
\cos \delta\varphi \approx 1, \quad \sin \delta\varphi \approx \delta\varphi,
\]

\[
\psi(x, y, z) \equiv \psi’(x’, y’, z’) = \psi'(x + y \delta\varphi, -x \delta\varphi + y, z) = \]

\[
= \psi'(x, y, z) + y \delta\varphi \frac{\partial \psi'}{\partial x} - x \delta\varphi \frac{\partial \psi'}{\partial y} =
\]

\[
= \left[1 - \delta\varphi \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)\right] \psi'(x, y, z) =
\]

\[
= \left[1 - \frac{\delta\varphi}{\hbar} \hat{l}_z\right] \psi'(x, y, z) = \exp \left[-\frac{\delta\varphi}{\hbar} \hat{l}_z\right] \psi'(x, y, z). \tag{21}
\]

This rule \eqref{eq:21} is propagated to the case of rotation around an arbitrary axis \(\vec{n}\). Sum of rotations \(\sum \delta\varphi_i = \varphi\) gives the formula:

\[
\psi'(x, y, z) = \exp \left[+\frac{\delta\varphi}{\hbar} \hat{n}\right] \psi(x, y, z). \tag{22}
\]

The projection of angular momentum operator to the direction \(\vec{n}\) is replaced by the Pauli-operator and the wave function replaced by spinor:

\[
\hat{l}_n \rightarrow \frac{1}{2}\hbar \sigma_n, \quad \psi \rightarrow \chi \quad \Rightarrow \quad \chi' = \exp \left[+\frac{\varphi}{2} (\vec{\sigma} \cdot \vec{n})\right] \chi. \tag{23}
\]
Since the coordinate system is rotated around \( \vec{n} \) by the angle \( \varphi \) the observer related with new system \((x', y', z')\) will see it as a rotation to the angle \(-\varphi\), i.e. the formula (23) should produce the rotation of spin-vector to the left that contradicts with lemma [1]. Performed replacement in (23) can be justified in the sense that both spinor \( \chi \) and operator \( \sigma_n \) belong to the SU(2) algebra (form it). However the \( \chi \) and \( \sigma_n \) are entered to this formula independent among themselves, i.e. the spin behavior is not defined from the spin nature but are a kind of agreement. The problem is occurred also when in (21) we take the quantum mechanic definition of angular momentum operator \( \hat{l}_z \). Because in the Schrödinger and Klein-Gordon equations the quadratic form \( \nabla^2 \) is used the sign of particle momentum is free:

\[
\frac{\partial}{\partial x} \psi = \pm \frac{i}{\hbar} p_x \psi \quad \text{etc.} \quad (24)
\]

Choosing between “+” and “−” is a convention what wave function from \( \exp \left[ \frac{i}{\hbar} \vec{p} \vec{r} \right] \) or \( \exp \left[ -\frac{i}{\hbar} \vec{p} \vec{r} \right] \) has been taken. But in [2] this question is considered in a general view therefore the sign of operator \((\sigma \cdot \vec{n})\) in the (23) can be any and the direction of rotation around the vector \( \vec{n} \) is not defined at all.

In the «Course of Theoretical Physics» [3] (V. III, § 58) the rotation around the axis \( z \) is described by the matrix \( \hat{U}_z(\alpha) \) which coincides with the operator of lemma [1] However in the text it does not define exactly whether it is rotation of coordinate system or rotation of spin. Second rotation is performed around the axis \( ON \) (around \( knots axis \) according to Euler) but defined using the operator \( \hat{U}_y(\beta) \) that leads to an ambiguous definition for old and new coordinate systems. Therefore the \( \hat{U}_z(\alpha) \) should be understood in the sense of rotation of spin vector. The next transformation \( \hat{U}_z(\gamma) \) is supposed as a rotation around new axis \( z' \). This sequence of three rotations \( \hat{U}_z(\gamma)\hat{U}_y(\beta)\hat{U}_z(\alpha) \) which is shown on Fig. 20 in [3] can have only one interpretation — the spin is rotated together with its Pauli-vector as if repelled from the external space. In inverse case if the operator Pauli is fixed in the initial coordinate system \((x, y, z)\) these rotations should be performed around the axes \( z, y' \) and \( z'' \), i.e. total operator must be written as \( \hat{U}_z(\gamma)\hat{U}_y(\beta)\hat{U}_z(\alpha) \) regardless from the question what kind of rotation (right or left) is given by each matrix. If the transition into new system need to be done by the rotations around the axes of old basic then the matrices order should be inverse \( \hat{U}_z(\alpha)\hat{U}_y(\beta)\hat{U}_z(\gamma) \).
5 Own space of spin

Wrong interpretation of operator (14) is related with initial supposing that this operator acts to the spin as external but it is not true. According to (4) all three Pauli-matrices can be presented like a tensorial multiplying of orthogonal spin states among themselves:

\[ \sigma_z = \uparrow \otimes \uparrow - \downarrow \otimes \downarrow \], \hspace{1cm} (25a)
\[ \sigma_x = \uparrow \otimes \downarrow + \downarrow \otimes \uparrow \], \hspace{1cm} (25b)
\[ i\sigma_y = \uparrow \otimes \downarrow - \downarrow \otimes \uparrow \], \hspace{1cm} (25c)
\[ E = \uparrow \otimes \uparrow + \downarrow \otimes \downarrow \]. \hspace{1cm} (25d)

Unit matrix \( E \) is added here for complete set. By these definitions the matrices \( \sigma_x \), \( \sigma_y \) and \( \sigma_z \) become Hermitian and unitary automatically and satisfy to the commutation relations:

\[ \sigma_z = \sigma_z^+, \hspace{1cm} \sigma_z^2 = E, \hspace{1cm} \sigma_z \sigma_x = -\sigma_x \sigma_z = i\sigma_y \hspace{1cm} \text{etc.} \]

Spinors are called as eigenvectors of Pauli-operator which implies their subordinate position: the operator \( \sigma_r \) acts to the spin vector \( \vec{s} \) and represents its state as a spinor \( \chi_r \). But the formulas (25a, b, c) give inverse interpretation — the spin produces own Pauli-vector and is linked with it as a whole. We can say even that the Pauli-vector is a kind of form of the same spin. It does not matter how the spin is oriented in the laboratory coordinate system but anyway it has «own opinion» about the rules of changing in the external space. It is easy to understand using the sequence of rotations (Fig. 3). In the beginning the basic \((\vec{i}, \vec{j}, \vec{k})\) of system \((x, y, z)\) and basic \((\vec{m}, \vec{l}, \vec{n})\) of Pauli-operator are coincide. First rotation is performed around the \(y\)-axis by the operator \( \hat{U}_y = \hat{U}_l^+ \) that leads to the system \((x', y', z')\). Further the operator \( \hat{U}_n \) transforms the system to the next position \((x'', y'', z'')\). However the spin and Pauli-vector remained motionless and therefore the \( \hat{U}_n \) operator has the same form like a matrix \( u_z \) (12):

\[ \hat{U}_n = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix} = \exp \left[ -i\frac{\phi}{2} \sigma_z \right]. \]

This rotation is performed not around laboratory axis \(z\) but around the vector \(\vec{n}\), i.e. own \(z\)-axis of fermion where the spin produces the projection \(\sigma_z \) (25a).
Figure 3: Double transition to the new system. On the left the sequence of rotations of external space relative to the Pauli-operator drawn like a colored cube. First rotation around the vector $\vec{l}$ is left rotation and defined by the operator $\hat{U}^+_l$ that gives the system $(x', y', z')$. Further the operator $\hat{U}_n$ defines right hand rotation around $\vec{n}$ and moves the system to the position $(x'', y'', z'')$. On the right we change the point of view to return the laboratory system in initial state together with Pauli-operator.

This sequence of rotations (Fig. 3) can be presented by the other way following to the coordinate system $(x, y, z)$, i.e. considering it as fixed. In this case the state of Pauli-operator is changed (Fig. 4) but its transformations is defined using the same operators $\hat{U}^+_l$ and $\hat{U}_n$. It is obviously that both pictures 3.2 and 4.3 are identical, i.e. the final state in two different approaches is the same. In the books of quantum mechanic we read usually about coordinate system rotation but on the pictures the rotations of operator $\hat{\sigma}$ are shown.

Figure 4: The transformation of Pauli-operator in the laboratory system $(x, y, z)$. First rotation around the vector $\vec{l}$ to the right is given by the operator $\hat{U}^+_l$. On the second picture the left rotation is defined by the operator $\hat{U}_n$. Third picture shows the result of these two operations.
6 The Bloch Sphere

Let us introduce the concept of global operator $\hat{\sigma}$ which is fixed in the laboratory system $(x, y, z)$. Any other spin operator $\hat{\sigma}_1$ according to the Euler theorem can be obtained from one using a rotation around some axis $\vec{t}$ (Fig. 5).

![Figure 5: Symbolic presentation of two Pauli-operators $\hat{\sigma}$ and $\hat{\sigma}_1$ in the basics $(\vec{i}, \vec{j}, \vec{k})$ and $(\vec{m}, \vec{l}, \vec{n})$ respectively. They are related among themselves by the rotation around the $\vec{t}$-axis by the angle $\theta$ that equal to the unitary transformation: $\hat{\sigma}_1 = \hat{U}_{\vec{t}} \hat{\sigma} \hat{U}_{\vec{t}}^*$ where $\hat{U}_{\vec{t}} = E \cos \frac{\theta}{2} - i \sigma_t \sin \frac{\theta}{2}$ and $\sigma_t = (\hat{\sigma} \cdot \vec{t}) = (\hat{\sigma}_1 \cdot \vec{t})$ since the axis $\vec{t}$ is the same.

Let the operators $\hat{\sigma}$ and $\hat{\sigma}_1$ are defined by the standard way (2):

$$\begin{align*}
\sigma_x &= \sigma_{1m} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_y &= \sigma_{1l} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma_z &= \sigma_{1n} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}$$

The coordinates of $\hat{\sigma}_1$ in the system $(x, y, z)$ will respect to the formulas (15):

$$\begin{align*}
(\sigma_{1x}, \sigma_{1y}, \sigma_{1z}) &= \hat{U}_{\vec{t}}(\sigma_x, \sigma_y, \sigma_z)\hat{U}_{\vec{t}}^*, \\
\hat{U}_{\vec{t}} &= E \cos \frac{\theta}{2} - i(\hat{\sigma} \cdot \vec{t}) \sin \frac{\theta}{2}, \quad \hat{U}_{\vec{t}}^* - \text{left square}.
\end{align*}$$

For clarity let the projection of spin along the vector $\vec{n}$ equals to $+\frac{1}{2} \hbar$. Its state $\chi_{1n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is reflected to the system $(x, y, z)$ using inverse transformation: $\chi_{1z} = \hat{U}_{\vec{t}}^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. Consider the operator of right rotation around an arbitrary axis $\vec{q}$:

$$\begin{align*}
\hat{U}_{\vec{q}}^+ &= E \cos \frac{\theta}{2} + i(\hat{\sigma}_1 \cdot \vec{q}) \sin \frac{\theta}{2}, \\
(\hat{\sigma}_1 \cdot \vec{q}) &= \sigma_{1m} q_m + \sigma_{1l} q_l + \sigma_{1n} q_n = \sigma_{1x} q_x + \sigma_{1y} q_y + \sigma_{1z} q_z = \hat{U}_{\vec{t}}(\hat{\sigma} \cdot \vec{q})\hat{U}_{\vec{t}}^*.
\end{align*}$$
For scalar multiplying of spinors $\chi_{1n}$ and $\hat{U}_q^+ \chi_{1n}$ it gives the following:

\[
(\chi_{1n})^+ \hat{U}_q^+ \chi_{1n} = (\chi_{1n})^+ \hat{U}_t^+ \left( E \cos \frac{\theta}{2} + i(\hat{\sigma} \cdot \vec{q}) \sin \frac{\theta}{2} \right) \hat{U}_q^+ \chi_{1n} = \\
= \left( \hat{U}_t^+ \chi_{1n} \right)^+ \hat{R}^+_q \left( \hat{U}_q^+ \chi_{1n} \right), \quad \hat{R}^+_q = E \cos \frac{\theta}{2} + i(\hat{\sigma} \cdot \vec{q}) \sin \frac{\theta}{2}.
\]

It means that the operator $\hat{U}_q^+$ in the coordinate system of Pauli-operator $\hat{\sigma}_1$ changes the spinor $\chi_{1n} = (1\ 0)$ in the same way as the operator $\hat{R}^+_q$ changes its reflection $\chi_{1z} = (\alpha\ \beta)$ in the system $(x, y, z)$ of global operator $\hat{\sigma}$. Thus we can define the trivial rule:

**Lemma 3 (Arbitrary spin-space)** The rotation of spin vector around any axis $\vec{q}$ carries out identical at any choosing of quantization axis $z$.

Because the spin transformation depends only from the position of rotation axis and from the value of rotation angle it allows to use common for all fermions Pauli-operator $\hat{\sigma}$, i.e. we can work in the representation of Bloch Sphere (Fig. 6).

![Bloch sphere](image)

**Figure 6:** The Bloch sphere. The spin direction $\vec{s}$ is defined by the angles $\theta$ and $\varphi$. For each $\vec{s}$ the point on the Bloch sphere corresponds to the single spin state along the axis $z$: $\chi_z(\vec{s}) = \cos \frac{\theta}{2}(1\ 0) - e^{i\varphi} \sin \frac{\theta}{2}(0\ 1)$. The sign minus before low spinor element arises due to the inverse count of angle $\theta$ in comparison with the case on Fig. 1.

Global space of this *referee*-operator is a spin space in the sense that each spin is transformed here like in its own space, i.e. the right rotation around any axis $\vec{q}$ is given by the same form of unitary operator $\hat{U}_q^+(\varphi) = \exp \left( +i \frac{\varphi}{2} (\hat{\sigma} \cdot \vec{q}) \right)$. 

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7 Conclusion

1. Wrong interpretation of spin $\frac{1}{2}$ transformation is related with the casus that unitary operator $\hat{U}_n(\varphi) = \exp(-i\frac{\mathbf{n}}{2}(\hat{\sigma} \cdot \mathbf{n}))$ gives the right rotation not for spin vector but for the Stern-Gerlach device. Its axis of quantization is directed along the vector $\mathbf{r}$ and can moved on the $\Sigma$-sphere (Fig. 1) at the fixed spin projection $s_z = \pm \frac{1}{2}\hbar$. The Stern-Gerlach device can be associated with the operator $(\hat{\sigma} \cdot \mathbf{r})$ and its spinor $\chi_r(s_z = \pm \frac{1}{2}\hbar) = \cos \frac{\theta}{2}^{(1)} + e^{i\varphi} \sin \frac{\theta}{2}^{(1)}$ defines the amplitudes $\cos \frac{\theta}{2}$ and $e^{i\varphi} \sin \frac{\theta}{2}$ of projections $s_r = +\frac{1}{2}\hbar$ и $-\frac{1}{2}\hbar$.

2. Each fermion in own coordinate system $(x_i, y_i, z_i)$ produces own Pauli-vector $\hat{\sigma}_i$. The inverse unitary operator $\hat{U}_n^+(\varphi) = \exp(+i\frac{\mathbf{n}}{2}(\hat{\sigma}_i \cdot \mathbf{n}))$ performs the right rotation of spin vector in the external space from which this fermion seems to be repelled (Fig. 3).

3. If in the laboratory system $(x, y, z)$ we define the referee-operator $\hat{\sigma}$ which respects to the fixed position of Stern-Gerlach device the direction of spin is considered as free and defined by the vector $\mathbf{r}$ on the Bloch Sphere (Fig. 6). In this case the spin is quantized along the axis $z$ and its state is presented by the spinor $\chi_z(s_z = \pm \frac{1}{2}\hbar) = \cos \frac{\theta}{2}^{(1)} - e^{i\varphi} \sin \frac{\theta}{2}^{(1)}$ where the $\cos \frac{\theta}{2}$ and $-e^{i\varphi} \sin \frac{\theta}{2}$ are the amplitudes of projection $s_z = +\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$. The rotation of spin vector is given by the unitary operator $\hat{U}_n^+(\varphi) = \exp(+i\frac{\mathbf{n}}{2}(\hat{\sigma} \cdot \mathbf{n}))$.

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