Random pullback attractor of a non-autonomous local modified stochastic Swift-Hohenberg with multiplicative noise

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Abstract: In this paper, we study the existence of the random pullback attractor of a non-autonomous local modified stochastic Swift-Hohenberg equation with multiplicative noise in Stratonovich sense. It is shown that a random pullback attractor exists in $\mathbb{H}^0(D)$ when its external force has exponential growth. Due to the stochastic term, the estimate are delicate, we overcome this difficulty by using the Ornstein-Uhlenbeck(O-U) transformation and its properties.

Keywords: Swift-Hohenberg equation; Random pullback attractor; Non-autonomous random dynamical system

1. Introduction

The Swift-Hohenberg(S-H) type equations arise in the study of convective hydrodynamical, plasma confinement in toroidal and viscous film flow, was introduced by authors in [1]. After that, Doelman and Standstede [2] proposed the following modified Swift-Hohenberg equation for a pattern formation system near the onset to instability

$$u_t + \Delta^2 u + 2\Delta u + au + b |\nabla u|^2 + u^3 = 0,$$  \hspace{1cm} (1.1)

where $a$ and $b$ are arbitrary constants.

We can see from the above equation that there exist two operators $\Delta$ and $\Delta^2$, these two operators have some symmetry, for any $u \in H_0^1(D)$, the inner product $(\Delta u, u) = - (\nabla u, \nabla u) = \|\nabla u\|^2$, for any $u \in H_0^2(D)$, $(\Delta^2 u, u) = (\Delta u, \Delta u) = \|\Delta u\|^2$, $\Delta$ is antisymmetry and $\Delta^2$ is symmetry. We will use the symmetry principle study S-H equation.

The dynamical properties of the S-H equation are important for the studies pattern formation system have been extensively investigated by many authors; see [3-8]. Polat [8] establish the existence of global attractor for the system (1.1), and then Song et al. [7] improved the result in $\mathbb{H}^k$.

Recently for non-autonomous modified S-H equation:

$$du + (\Delta^2 u + 2\Delta u + au + b |\nabla u|^2 + u^3 - g(x,t)) dt = lu \circ dW(t),$$  \hspace{1cm} (1.2)

it has also attracted the interest of many authors. If $l = 0$, equation (1.2) becomes a non-autonomous modified S-H equaiton. Park [9] proved the existence of $\mathcal{D}$-pullback attractor when the external force has exponential growth, Xu et al.[10] established the existence of uniform attractor when the external force $g(x,t)$ satisfies translation bounded, these results need the spatial variable in two dimensions. When $l = 0$, equation (1.2) becomes a non-autonomous stochastic S-H equation, if $|b| << 1$ is a constant, Guo et al.[11] investigated the equation when $g(x,t) = 0$ and proved the existence of random attractor which need the spatial variable in one dimension. For $g(x,t) = 0$, to the best of our knowledge, the existence of random $\mathcal{D}$-pullback attractor for equation (1.2) has not yet considered.
In this paper, we consider the following one dimensional non-autonomous local modified stochastic S-H equation with multiplicative noise:
\[
    du + (\Delta^2 u + 2u_{xx} + au + bu_{x}^2 + u^3 - g(x,t))dt = lu \circ dW(t), \text{ in } D \times [\tau, \infty),
\]
\[
    u = u_x = 0, x \in \partial D, \tag{1.3}
\]
\[
    u(x, \tau) = u_\tau, x \in D \tag{1.4}
\]
Where $D$ is a bounded open interval, $\Delta u$ means $u_{xx}$, and $\Delta^2 u$ means $u_{xxxx}$, $|b| < 4$, $a$ and $l$ are arbitrary constants. $W(t)$ is a two-sided real-valued Wiener process on a probability space which will be specified later. For the external force $g(x) \in L^2_{\text{loc}}(R, L^2(D))$, we assume that there exist $M > 0$ and $\beta > 0$ such that
\[
    a - \beta - 5 < 0, \quad \|g(x, t)\|^2 \leq M e^{\gamma |t|}, \text{ for any } t \in \mathbb{R}, \quad 0 \leq \gamma < \frac{3\beta}{11}. \tag{1.6}
\]
The assumption is same as [8, 12], through simple calculation, for all $t \in R$, we have
\[
    H(t) := \int_{-\infty}^{t} e^{\beta s} \|g(x, s)\|^2 ds < \infty, \quad \int_{-\infty}^{t} e^{-\frac{8\beta}{9} s} H^{\frac{11}{3}}(s)ds < \infty. \tag{1.7}
\]
An outline of this paper is as follows: In section 2, we recall some basic concepts about random $\mathcal{D}$-pullback attractors. In Section 3, we prove that the stochastic dynamical system generated by (1.3) exists a random $\mathcal{D}$-pullback attractor in $H^2_0(D)$.

2.2 Preliminaries

There are many research results on random attractors and related issues. The reader is referred to [13-19] for more details, we only list the definitions and abstract result.

Let $(X, || \cdot ||_X)$ be a separable Banach space with Borel $\sigma$-algebra $B(X)$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In this paper, the term $\mathbb{P}$-a.s.(the abbreviation for $\mathbb{P}$ almost surely) denotes that an event happens with probability one. In other words, the set of possible exception may be non-empty, but it has probability zero.

**Definition 2.1.** ([15,16,20,21]) $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical systems if $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$ is $(B(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable, and $\theta_0$ is the identity on $\Omega$, $\theta_{s+t} = \theta_s \theta_t$ for all $t, s \in \mathbb{R}$ and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

**Definition 2.2.** ([12,13,18]) A non-autonomous random dynamical system (NRDS) $(\varphi, \theta)$ on $X$ over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is a mapping
\[
    \varphi(t, \tau, \omega) : X \rightarrow X, \quad (t, \tau, \omega, x) \rightarrow \varphi(t, \tau, \omega)x, \tag{1.8}
\]
which represents the dynamics in the state space $X$ and satisfies the properties
   (i) $\varphi(t, \tau, \omega) = \text{the identity on } X$;
   (ii) $\varphi(t, \tau, \omega) = \varphi(t, s, \theta_{s-t}, \omega) \varphi(s, \tau, \omega)$ for all $\tau \leq s \leq t$;
   (iii) $\omega \rightarrow \varphi(t, \tau, \omega)x$ is $\mathcal{F}$-measurable for all $t \geq \tau$ and $x \in X$.

In the sequel, we use $\mathcal{D}$ to denote a collection of some families of nonempty bounded subsets of $X$:
\[
    \mathcal{D}' \subseteq \mathcal{D}, \mathcal{D}' = \{D(t, \omega) : t \in \mathbb{R}, \omega \in \Omega\}. \tag{1.9}
\]

**Definition 2.3.** ([12,13,18]) A set $B \subseteq \mathcal{D}$ is called a random $\mathcal{D}$-pullback bounded absorbing set for NRDS $(\varphi, \theta)$ if for any $t \in \mathbb{R}$ and any $D' \in \mathcal{D}$, there exists $\tau_0(t, D')$ such that
\[
    \varphi(t, \tau, \theta_{\tau-\tau_0}, \omega)D(t, \tau, \omega, \omega) \subseteq B(t, \omega) \quad \text{for any} \quad \tau \leq \tau_0. \tag{1.10}
\]

**Definition 2.4.** ([12,13,18]) A set $\mathcal{A} = \{A(t, \omega) : t \in \mathbb{R}, \omega \in \Omega\}$ is called a random $\mathcal{D}$-pullback attractor for $(\varphi, \theta)$ if the following hold:
   (i) $A(t, \omega)$ is a random compact set;
   (ii) $\mathcal{A}$ is invariant; that is, for $\mathbb{P}$-a.s. $\omega \in \Omega$, and $\tau \leq t$, $\varphi(t, \tau, \omega)A(\tau, \omega) = A(t, \theta_{t-\tau}, \omega)$;
(iii) $A$ attracts all set in $D$; that is, for all $B' \in D$ and $P$-a.s. $\omega \in \Omega$, 
$$\lim_{t \to \infty} d(\phi(t, \tau, \theta_{\tau}, \omega)) B(\tau, \theta_{\tau}, \omega), A(t, \omega)) = 0.$$  

Where $d$ is the Hausdorff semimetric given by $\text{dist}(B, A) = \sup_{b \in B} \inf_{a \in A} \| b - a \|_x$.

**Definition 2.5.** ([14, 17]) A NRDS $(\phi, \theta)$ on a Banach space $X$ is said to be pullback flattening if for every random bounded set $B' = \{ B(t, \omega) : t \in \mathbb{R}, \omega \in \Omega \} \subset D$, for any $\varepsilon > 0$ and $\omega \in \Omega$ there exists a $T(B', \varepsilon, \omega) < t$ and a finite dimensional subspace $X_\varepsilon$ such that 

(i) $P(\bigcup_{\tau \in T} \phi(t, \tau, \theta_{\tau}, \omega) B(\tau, \theta_{\tau}, \omega))$ is bounded, and 

(ii) $\| (I - P)(\bigcup_{\tau \in T} \phi(t, \tau, \theta_{\tau}, \omega) B(\tau, \theta_{\tau}, \omega)) \|_X < \varepsilon,$

where $P : X \to X_\varepsilon$ is a bounded projector.

**Theorem 2.1.** ([14, 17]) Suppose that $(\phi, \theta)$ is a continuous NRDS on a uniformly convex Banach space $X$. If $(\phi, \theta)$ possesses a random $D$-pullback bounded absorbing sets $B' = \{ B(t, \omega) : t \in \mathbb{R}, \omega \in \Omega \}$ and $(\phi, \theta)$ is pullback flattening, then there exists a random $D$-pullback attractor $A = \{ A(t, \omega) : t \in \mathbb{R}, \omega \in \Omega \}$.

### 3. Random pullback attractor for modified Swift-Hohenberg

In this section, we will use abstract theory in section 2 to obtain the random $D$-pullback attractor for equation (1.3)-(1.5). First we introduce an Ornstein-Uhlenbeck process, 

$$z(\theta_{\tau}(\omega)) = -\int_{-\infty}^{0} e^{t}(\theta_{\tau}(\omega)) d\tau, t \in \mathbb{R}.$$ 

We known from [6], it is the solution of Langevin equation 

$dz + zdt = dW(t).$ 

$W(t)$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, P)$, where 

$$\Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \},$$ 

$\mathcal{F}$ is the Borel algebra induced by the compact open topology of $\Omega$, and $P$ is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. We identify $\omega(t)$ with $W(t)$, i.e., 

$$W(t) = W(t, \omega) = \omega(t), t \in \mathbb{R}.$$ 

Define the Wiener time shift by 

$$\theta_{\tau}\omega(s) = \omega(s + \tau) - \omega(t), \omega \in \Omega, t, s \in \mathbb{R}.$$ 

Then $(\Omega, \mathcal{F}, P, \theta_{\tau})$ is an ergodic metric dynamical system.

From [15, 16, 20], it is known that the random variable $z(\omega)$ is tempered and there exists a $\theta_{\tau}$-invariant set of full measure $\hat{\Omega} \subset \Omega$ such that for all $\omega \in \hat{\Omega}$: 

$$\lim_{t \to \infty} \frac{|z(\theta_{\tau}(\omega))|}{|t|} = 0, \quad \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} |z(\theta_{\tau}(\omega))| ds = 0,$$ 

and for any $\varepsilon > 0$, there exists $\rho(\varepsilon) > 0$, such that 

$$|z(\theta_{\tau}(\omega))| \leq \rho(\omega) + \varepsilon |t|, \quad \int_{0}^{t} |z(\theta_{\tau}(\omega))| ds \leq \rho(\omega) + \varepsilon |t|.$$ 

Let $v(s) = e^{-iz(\theta_{\tau}(\omega))} u(s)$, then 

$$dv = -le^{-iz(\theta_{\tau}(\omega))} u(s) dz + e^{-iz(\theta_{\tau}(\omega))} du.$$ 

Using Langevin equation, combined with the original equation (1.3), we get 

$$\frac{dv}{ds} + \Delta^{2}v + 2v_{ss} + (a - \lambda z)v + be^{z(\theta_{\tau}(\omega))} |v_{s}|^{2} + e^{2z(\theta_{\tau}(\omega))} v_{s}^{3} = e^{-iz(\theta_{\tau}(\omega))} g(x, s), \text{in } D \times [\tau, \infty),$$ 

$$v = 0 \text{ on } \partial D \times [\tau, \infty),$$ 

$$v(x, \tau) = v_{\tau} = e^{-iz(\theta_{\tau}(\omega))} u_{\tau}(x) \text{ in } D.$$ 


Equation (1.3)-(1.5) are equivalent to equation (3.3)-(3.5), by a standard method, it can be proved that the problem (3.3)-(3.5) is well posed in $H^1_0(D)$, that is, for every $\tau \in \mathbb{R}$ and $v_0 \in H^1_0(D)$, there exits a unique solution $v \in C([\tau, \infty), H^1_0(D))$ (see e.g. [2, 22]). Furthermore, the solution is continuous with respect to the initial condition $v_0$ in $H^1_0(D)$. To construct a non-autonomous random dynamical system $\{V(t, \tau, \omega)\}$ for problem (3.3)-(3.5), we define $V(t, \tau, \omega): H^1_0(D) \rightarrow H^1_0(D)$ by $V(t, \tau, \omega)v_0$. Then the system $\{V(t, \tau, \omega)\}$ is a non-autonomous random dynamical system in $H^1_0(D)$.

We now apply abstract theory in Section 2 to obtain the random $\mathcal{D}$-pullback attractors for non-autonomous modified Swift-Hohenberg equation, by the equivalent, we only consider the random $\mathcal{D}$-pullback attractor of equation (3.3)-(3.5).

For convenience, the $U^p(D)$ norm of $u$ will be denoted by $\|u\|_{p,1}$, $H = L^2(D)$ with a scalar product and the norm of Sobolev spaces $W^1_0(D)$ by $\|u\|_{1,p}$, we regard the space $H^1_0(D)$ endowed with the norm $\|u\|_{2,2} = \|\Delta u\|_2$, $c$ or $c(\omega)$ denote the arbitrary positive constants, which only depend on $\omega$ and may be different from line to line and even in the same line.

For our purpose that the following Gagliardo-Nirenberg inequality will be used.

**Lemma 3.1.** (Gagliardo-Nirenberg Inequality). Let $D$ be an open, bounded domain of the Lipschitz class in $\mathbb{R}^n$. Assume that $1 \leq p \leq \infty, 1 \leq q \leq \infty, 1 \leq r, 0 < \theta \leq 1$ and let

$$k - \frac{n}{p} \leq \theta(m - \frac{n}{q}) + (1 - \theta)\frac{n}{r}.$$ 

Then the following inequality holds:

$$\|u\|_{k,p} \leq c(D)\|u\|_{r,q}^{\theta} \|u\|_{m,q}^{1 - \theta}.$$ 

**Lemma 3.2.** For all $t \geq \tau$, the following inequality holds:

$$\|v(t, \tau, \theta)\|^2 \leq c(\omega)(e^{-\beta(t-\tau)}\|v_0\|^2 + 1 + e^{-\beta H(t)}),$$

and

$$\int_{\tau}^{t} e^{2\beta(t-\tau)z(\theta, \omega)dr} \|v_{x}\|^2 ds \leq c(\omega)(e^{-\beta(t-\tau)}\|v_{x}\|^2 + 1 + e^{-\beta H(t)}).$$

**Proof.** Let $v(s)$ or $v$ denotes $v(s, \tau, \theta, \omega)$ be the solution of equation (3.3)-(3.5). Taking the inner product of equation (3.3) with $v$, we get

$$\frac{1}{2} \frac{d}{ds} \|v\|^2 + \|\Delta v\|^2 + 2(\Delta v, v) + (a - lz)\|v\|^2 + be^{2z(\theta, \omega)}(v_{x}^2, v)\|v\|_4^4$$

$$+ e^{2z(\theta, \omega)}\|v\|_4^4 = e^{-z(\theta, \omega)}(g(x, s), v).$$

Using Young inequality, we get

$$\|2\Delta v, v\| \leq \frac{1}{4} \|\Delta v\|^2 + 4\|v\|^2.$$ 

By integration by parts, we obtain

$$be^{2z(\theta, \omega)}(v_{x}^2, v) = be^{2z(\theta, \omega)} \int_{D} v_{x}^2 v dx = -be^{2z(\theta, \omega)} \int_{D} (v_{x}^2 v + vv_{x}^2) dx,$$

thus

$$be^{2z(\theta, \omega)}(v_{x}^2, v) = -\frac{b}{2} e^{2z(\theta, \omega)} \int_{D} v_{x}^2 v_{x} dx.$$

Applying the Hölder inequality and Young inequality, we get

$$be^{2z(\theta, \omega)} \|v_{x}^2 \| \leq \frac{b}{2} e^{2z(\theta, \omega)} \int_{D} v_{x}^2 v_{x} dx \leq \frac{b}{2} e^{2z(\theta, \omega)} \|v_{x}\| \|v\|_{4}^2 \leq \frac{b}{16\eta} \|v_{x}\| \|v\|_{4}^4 \leq \frac{b^2}{16\eta} e^{2z(\theta, \omega)} \|v\|_{4}^4.$$
and

\[ e^{-\varepsilon g(x, s)} \leq ||v||^2 + \frac{1}{4} e^{-2\varepsilon g(x, s)}||v||^2. \]

For convenience, we take \( \eta = \frac{1}{4} \), \( |b| < 2 \) \( (|b| < 4 \), the same conclusion hold), we obtain

\[ \frac{d}{ds}||v||^2 + ||\Delta v||^2 + 2(a - \varepsilon g(x, s))||v||^2 + 2(1 - b^2) e^{2\varepsilon g(x, s)}||v||^2 \leq 1 + e^{-2\varepsilon g(x, s)}||v||^2. \]

Taking \( \beta > 0 \) such that \( a - \beta - 5 < 0 \), we have

\[ \frac{d}{ds}||v||^2 + ||\Delta v||^2 + 2(\beta - \varepsilon g(x, s))||v||^2 + 2(1 - b^2) e^{2\varepsilon g(x, s)}||v||^2 \leq -2(a - \beta - 5)||v||^2 + \frac{1}{2} e^{-2\varepsilon g(x, s)}||v||^2. \]

By the Sobolev imbedding \( L^1(D) \subset L^2(D) \) and Young inequality, we get

\[ -2(a - \beta - 5)||v||^2 \leq c||v||^2 \leq 2(1 - \frac{b^2}{4}) e^{2\varepsilon g(x, s)}||v||^2 + ce^{-2\varepsilon g(x, s)}. \]

Thus we obtain

\[ \frac{d}{ds}||v||^2 + ||\Delta v||^2 + 2(\beta - \varepsilon g(x, s))||v||^2 \leq ce^{-2\varepsilon g(x, s)}(1+(g(x, s))||v||^2) \]

Multiply this by \( e^{2\beta(t - 1) t e^{2\beta(t - 1)} ||v||^2 ds \) and integrating from \( \tau \) to \( t \), we have

\[ ||v(t)||^2 + \int_{\tau}^{t} e^{2\beta(t - 1) t e^{2\beta(t - 1)} ||v(r)||^2 ds \]

\[ \leq e^{-2\beta(t - 1) t e^{2\beta(t - 1)} ||v||^2 + \int_{\tau}^{t} e^{-2\beta(t - 1) t e^{2\beta(t - 1)} (1+(g(x, s))||v||^2) ds \}

\[ \leq e^{-2\beta(t - 1) t e^{2\beta(t - 1)} ||v||^2 + \int_{\tau}^{t} e^{-2\beta(t - 1) t e^{2\beta(t - 1)} (1+(g(x, s))||v||^2) ds \}

From (3.2), we get

\[ 2\int_{\tau}^{t} \lambda(t, \omega) d\tau \leq \mu(\omega) + \beta(t - \tau), \quad 2\int_{\tau}^{t} \lambda(t, \omega) - 2\lambda(t, \omega) \leq \mu(\omega) + \beta(t - \tau). \]

Then we have

\[ ||v(t)||^2 + \int_{\tau}^{t} e^{2\beta(t - 1) t e^{2\beta(t - 1)} ||v||^2 ds \]

\[ \leq c(\omega)(e^{-2\beta(t - 1) t e^{2\beta(t - 1)} ||v||^2 + \int_{\tau}^{t} e^{-2\beta(t - 1) t e^{2\beta(t - 1)} (1+(g(x, s))||v||^2) ds \}

\[ \leq c(\omega)(e^{-2\beta(t - 1) t e^{2\beta(t - 1)} ||v||^2 + 1 + e^{-\beta(t - 1) t e^{2\beta(t - 1)} (g(x, s))||v||^2) ds \}

Thus we get the desired results. \( \square \)

**Lemma 3.3.** For all \( t \geq \tau \), the following inequality hold:

\[ ||\Delta v(t, \tau, \lambda_{\tau}, \omega)||^2 \leq c(\omega)[(1 + \frac{1}{t - \tau}) e^{-2\beta(t - 1) t e^{2\beta(t - 1)} ||v||^2 + \frac{11}{3} \beta(t - 1) t e^{2\beta(t - 1)} ||v||^2 \frac{22}{3} + e^{-2\beta(t - 1) t e^{2\beta(t - 1)} ([H(t) + \frac{1}{3} e^{2\beta(t - 1) t e^{2\beta(t - 1)} H^3(s) ds])] \]

**Proof.** Taking inner product of equation (3.3) with \( \Delta^2 v \), we have

\[ \frac{1}{2} \frac{d}{ds} ||\Delta v||^2 + ||\Delta^2 v||^2 + 2(\Delta v, \Delta^2 v) + (a - \varepsilon g(x, s)) ||\Delta v||^2 + \beta e^{2\varepsilon g(x, s)} (v_1^2, \Delta^2 v) + e^{2\varepsilon g(x, s)} (v_3, \Delta^2 v) = e^{-2\varepsilon g(x, s)} (g(x, s), \Delta^2 v). \]

\[ (3.8) \]

\[ (3.9) \]
By the Hölder inequality, Young inequality and Gagliardo-Nirenberg inequality, we get

\[-2(\Delta v, \Delta^2 v) \leq \frac{1}{8} \| \Delta^2 v \|^2 + 8 \| \Delta v \|^2,\]

\[|b| e^{2\varepsilon (\theta, \omega)} \left| (v_x^2, \Delta^2 v) \right| \leq \frac{1}{8} \| \Delta^2 v \|^2 + ce^{\| v \|^2} \| \Delta^2 v \|^2 \leq \frac{1}{8} \| \Delta^2 v \|^2 + ce^{\frac{11}{18} \| v \|^2} \| \Delta^2 v \|^2,\]

\[e^{2\varepsilon (\theta, \omega)} \left| (v^3, \Delta^2 v) \right| \leq ce^{2\varepsilon (\theta, \omega)} \| v \|^3 \| \Delta^2 v \|^2 \leq ce^{2\varepsilon (\theta, \omega)} \| v \|^3 \| \Delta^2 v \|^2 + ce^{\frac{1}{8} \| \Delta^2 v \|^2} \| v \|^3,\]

\[= ce^{\frac{5}{2} \varepsilon (\theta, \omega)} \| \Delta^2 v \|^2 \| v \|^3 \leq \frac{1}{8} \| \Delta^2 v \|^2 + ce^{\frac{11}{18} \| v \|^2} \| \Delta^2 v \|^2,\]

\[e^{-\varepsilon (\theta, \omega)} (g(x, s), \Delta^2 v) \leq \frac{1}{8} \| \Delta^2 v \|^2 + 2e^{-2\varepsilon (\theta, \omega)} \| g(x, s) \|^2.\]

Putting all these inequalities together, we deduce

\[\frac{d}{ds} \| \Delta v \|^2 + 2(\beta - 2\varepsilon) \| \Delta v \|^2 \leq 2(8 + \beta - \eta) \| \Delta v \|^2 + c(e^{\frac{16}{3} \varepsilon (\theta, \omega)} \| v \|^3 + e^{-2\varepsilon (\theta, \omega)} \| g(x, s) \|^2).\]

Multiplying this by \((s - \tau) e^{2\beta t - 2\varepsilon (\theta, \omega) dr}\) and integrating it over \((\tau, t)\), we get

\[(t - \tau) e^{2\beta t - 2\varepsilon (\theta, \omega) dr} \| \Delta v(t) \|^2 \leq \int_{\tau}^{t} e^{2\beta t - 2\varepsilon (\theta, \omega) dr} \| \Delta v \|^2 ds + \int_{\tau}^{t} \left( \frac{1}{t - \tau} e^{-2\beta (t - s)} + e^{2\varepsilon (\theta, \omega) dr} \right) \| g(x, s) \|^2 ds.\]

Then we have

\[\| \Delta v(t) \|^2 \leq c(1 + \frac{1}{t - \tau}) \int_{\tau}^{t} e^{-2\beta (t - s)} + e^{2\varepsilon (\theta, \omega) dr} \| \Delta v \|^2 ds + \left[ \frac{1}{t - \tau} e^{-2\beta (t - s)} + e^{2\varepsilon (\theta, \omega) dr} \right] \| g(x, s) \|^2 ds.\]

By (3.2), (3.6) and the inequality \((a + b)^r \leq c(a^r + b^r) (a, b > 0, r \geq 1)\), we get

\[\int_{\tau}^{t} e^{-2\beta (t - s)} + e^{2\varepsilon (\theta, \omega) dr} \| \Delta v \|^2 ds \leq c(\omega) \int_{\tau}^{t} e^{-2\beta (t - s)} \| \Delta v \|^2 ds \leq c(\omega) \int_{\tau}^{t} e^{-2\beta (t - s)} \| \Delta v \|^2 ds \leq c(\omega) e^{-\beta (t - r)} \left[ e^{11 r} \| v \|^2 + 1 + e^{-\beta s} H(s) \right],\]

\[\leq c(\omega) e^{-\beta (t - r)} \left[ e^{11 r} \| v \|^2 + 1 + e^{-\beta s} H(s) \right],\]

By (3.2), (3.6) and the inequality \((a + b)^r \leq c(a^r + b^r) (a, b > 0, r \geq 1)\), we get

\[\int_{\tau}^{t} e^{-2\beta (t - s)} + e^{2\varepsilon (\theta, \omega) dr} \| \Delta v \|^2 ds \leq c(\omega) \int_{\tau}^{t} e^{-2\beta (t - s)} \| \Delta v \|^2 ds \leq c(\omega) \int_{\tau}^{t} e^{-2\beta (t - s)} \| \Delta v \|^2 ds \leq c(\omega) \int_{\tau}^{t} e^{-2\beta (t - s)} \| \Delta v \|^2 ds \leq c(\omega) e^{-\beta (t - r)} \left[ e^{11 r} \| v \|^2 + 1 + e^{-\beta s} H(s) \right] ds.\]

Thus, we have

\[\| \Delta v(t, \tau) \|^2 \leq c(\omega) \int_{\tau}^{t} e^{-2\beta (t - s)} \| \Delta v \|^2 ds \leq c(\omega) \int_{\tau}^{t} e^{-2\beta (t - s)} \| \Delta v \|^2 ds \leq c(\omega) e^{-\beta (t - r)} \left[ e^{11 r} \| v \|^2 + 1 + e^{-\beta s} H(s) \right] ds.\]

We complete the proof of Lemma 3.3. □

Let \( \mathcal{R} \) be the set of all function \( r : \mathbb{R} \to (0, +\infty) \) such that \( \lim_{r \to +\infty} e^{\beta r^2(t)} = 0 \) and denote by \( \mathcal{D} \) the class of all families \( \mathcal{D} = \{ D(t) : t \in \mathbb{R} \} \) such that \( D(t) \subset \overline{B}(r(t)) \) for some \( r(t) \in \mathcal{R} \), \( \overline{B}(r(t)) \) denote the closed ball in \( H^2_{\Omega}(D) \) with radius \( r(t) \). Let
\[ r_1^2(t) = 2c(\omega)[1 + e^{-\beta t}(H(t)) + \int_{-\infty}^{t} e^{-\frac{8\beta s}{3}} H^2(s) ds] \]  
(3.10)

By lemma 3.3 for any \( \hat{D} \in \mathcal{D} \) and \( t \in \mathbb{R} \), there exists \( \tau_0(\hat{D}, t, \omega) < t \) such that
\[
\| \Delta v(t, \tau, \theta^{-\omega}, \omega) \| \leq r_1(t), \text{ for any } \tau < \tau_0.
\]

Since \( 0 \leq \gamma < \frac{3\beta}{11} \), simple calculation imply that \( r_1(t) \in \mathcal{R} \), which say that the \( \mathcal{B}(r_1(t)) \) be a family of random \( \mathcal{D} \)-pullback bounded absorbing sets in \( H_0^2(D) \) and \( \{ \mathcal{B}(r_1(t)) \} \in \mathcal{D} \).

**Theorem 3.1.** The non-autonomous random dynamical system to problem (1.1)-(1.3) possesses a unique random \( \mathcal{D} \)-pullback attractor in \( H_0^2(D) \).

**Proof.** We need only prove that the dynamical system (3.3)-(3.5) satisfies the pullback flattening condition. Since \( A^{-1} \) is a continuous compact operator in \( H_0^2(D) \), by the classical spectral theorem, there exists a sequence \( \{ \lambda_j \}_{j=1}^{\infty} \) satisfying
\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_j \rightarrow +\infty, \text{ as } j \rightarrow +\infty,
\]
and a family of elements \( \{ e_j \}_{j=1}^{\infty} \) of \( H_0^2(D) \) which are orthonormal in \( H \) such that
\[
A e_j = \lambda_j e_j, \forall j \in \mathbb{N}^+.
\]

Let \( H_m = \text{span}\{e_1, e_2, \cdots, e_m\} \) in \( H \) and \( P_m : H \rightarrow H_m \) be an orthogonal projector. For any \( v \in H \) we write
\[
v = P_m v + (I - P_m)v = v_1 + v_2.
\]

Taking inner product of (3.3) with \( \Delta^2 v_2 \) in \( H \), we get
\[
-2(\Delta v, \Delta^2 v_2) - \frac{1}{2} \int ds \| \Delta v_2 \|^2 + \frac{1}{2} \| \Delta^2 v_2 \|^2 + 2(\Delta v, \Delta^2 v_2) + (a - lz)\| \Delta v_2 \|^2 \\
+ be^{b(\theta^{-\omega})}(\| \nabla v_2 \|^2, \Delta^2 v_2) + e^{2b(\theta^{-\omega})}(v_1, \Delta^2 v_2) = e^{-b(\theta^{-\omega})}(g(x, s), \Delta^2 v_2).
\]

By the Hölder inequality, Young inequality and Gagliardo-Nirenberg inequality, we get
\[
-2(\Delta v, \Delta^2 v_2) - \frac{1}{8} \| \Delta^2 v_2 \|^2 + 8\| \Delta v \|^2,
\]
\[
- be^{b(\theta^{-\omega})}(v_1, \Delta^2 v_2) \leq \frac{1}{8} \| \Delta^2 v_2 \|^2 + 2b^2 e^{2b(\theta^{-\omega})}\| v_1 \|^4
\]
\[
\leq \frac{1}{8} \| \Delta^2 v_2 \|^2 + ce^{2b(\theta^{-\omega})}\| v \|^4 \| \Delta v \|^4
\]
\[
= \frac{1}{8} \| \Delta^2 v_2 \|^2 + ce^{2b(\theta^{-\omega})}\| v \|^4 \| \Delta v \| \quad (\theta = \frac{1}{4})
\]
\[
\leq \frac{1}{8} \| \Delta^2 v_2 \|^2 + \frac{1}{2} \| \Delta v \|^2 + ce^{4b(\theta^{-\omega})}\| v \|^6.
\]
Putting all these inequalities together, we have
\[
\frac{d}{ds} \left( \| \Delta v_s \|^2 + \| \Delta v_{3s} \|^2 + 2(a - lc) \| \Delta v_s \| \right) \leq 18 \| \Delta v_s \|^2 + e^{4z(\Theta - \omega_z)} \| v \|^6 + e^{8z(\Theta - \omega_z)} \| v \|^4 + e^{-2z(\Theta - \omega_z)} \| g(x,s) \|^2.
\]

\[\lambda_n \| \Delta v_s \|^2 \leq \| \Delta v_{3s} \|^2,\]
which imply that
\[
\frac{d}{ds} \left( \| \Delta v_s \|^2 + (\lambda_n - 2lz) \| \Delta v_s \|^2 \right) \leq c(\| \Delta v_s \|^2 + e^{4z(\Theta - \omega_z)} \| v \|^6 + e^{8z(\Theta - \omega_z)} \| v \|^4 + e^{-2z(\Theta - \omega_z)} \| g(x,s) \|^2).
\]

Multiply this by \((s - \tau) e^{\lambda_n - 2l \int_{\tau}^{s} z(\Theta - \omega_z) ds}\) and integrating from \(\tau\) to \(t\), we obtain
\[
(t - \tau) e^{\lambda_n - 2l \int_{\tau}^{s} z(\Theta - \omega_z) ds} \| \Delta v_s \|^2 \leq c\left[ \int_{\tau}^{t} (1 + s - \tau) e^{\lambda_n - 2l \int_{\tau}^{s} z(\Theta - \omega_z) ds} \| \Delta v_s \|^2 ds \right.
\]
\[
+ \left. \int_{\tau}^{t} (s - \tau) e^{\lambda_n - 2l \int_{\tau}^{s} z(\Theta - \omega_z) ds} \left( e^{4z(\Theta - \omega_z)} \| v \|^6 + e^{8z(\Theta - \omega_z)} \| v \|^4 + e^{-2z(\Theta - \omega_z)} \| g(x,s) \|^2 \right)ds \right]
\]

Thus we get
\[
\| \Delta v_s \|^2 \leq c\left[ (1 + \frac{1}{t - \tau}) \int_{\tau}^{t} e^{\lambda_n - 2l \int_{\tau}^{s} z(\Theta - \omega_z) ds} \| \Delta v_s \|^2 ds + \int_{\tau}^{t} e^{\lambda_n - 2l \int_{\tau}^{s} z(\Theta - \omega_z) ds + 8z(\Theta - \omega_z)} \| v \|^6 ds + \int_{\tau}^{t} e^{\lambda_n - 2l \int_{\tau}^{s} z(\Theta - \omega_z) ds - 2z(\Theta - \omega_z)} \| g(x,s) \|^2 ds \right]
\]
\[
\leq c(\omega)(1 + \frac{1}{t - \tau}) \int_{\tau}^{t} e^{(\lambda_n - \beta)x(s - \tau)} \| \Delta v_s \|^2 ds + \int_{\tau}^{t} e^{(\lambda_n - \beta)(s - \tau)} \| v \|^6 ds + \int_{\tau}^{t} e^{(\lambda_n - \beta)(s - \tau)} \| g(x,s) \|^2 ds
\]
\[
= c(\omega)(I_1 + I_2 + I_3 + I_4).
\]

By simple calculation, we find that there exists \(N \in \mathbb{N}\), \(\forall n > N\), \(\lambda_n - \beta > \beta\), and
\[
I_1 \leq (1 + \frac{1}{t - \tau}) \int_{\tau}^{t} e^{(\beta)(s - \tau)} \| \Delta v_s \|^2 ds < \infty, \quad e^{\lambda_n - 2l \int_{\tau}^{s} z(\Theta - \omega_z) ds} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]

According to the Lebesgue dominated convergent theorem, we obtain
\[
I_1 \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Using (3.6), we get
\[
I_2 \leq c\left[ \int_{\tau}^{t} e^{(\lambda_n - \beta)(s - \tau)} \| v_s \|^2 + 1 + e^{-\beta t} H(s) \right] ds
\]
\[
\leq c\left[ \int_{\tau}^{t} e^{(\lambda_n - \beta)(s - \tau)} \| v_s \|^6 + 1 + e^{-3\beta t} H^3(s) \right] ds
\]
\[
\leq c\left[ e^{-\beta t} \frac{e^{-3\beta t}}{\lambda_n - 4\beta} \| v_s \|^6 + \frac{1}{\lambda_n - \beta} + \frac{e^{-4\beta t}}{\lambda_n - 4\beta} H^3(t) \right] \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
\[
I_3 \leq c \int_\tau^t e^{(\lambda_n - \beta)(s-t)} (e^{-\beta(s-t)} \| v_s \|^2 + 1 + e^{-\beta t} H(s))^2 \, ds
\]

\[
\leq c \int_\tau^t e^{(\lambda_n - \beta)(s-t)} (e^{-\beta(s-t)} \| v_s \|^4 + 1 + e^{-2\beta t} H^2(s)) \, ds
\]

\[
\leq c [e^{\beta} - \frac{e^{-2\beta(t-t)}}{\lambda_n - 3\beta} \| v_t \|^4 \frac{1}{\lambda_n - \beta} + \frac{e^{-3\beta t}}{\lambda_n - 3\beta} H^2(t)] \to 0 \text{ as } n \to \infty,
\]

and

\[
I_4 \leq e^{-\beta t} \int_\tau^t e^{\beta s} \| g(x,s) \|^2 ds, \quad e^{(\lambda_n - \beta)(s-t)} \| g(x,s) \|^2 \to 0 \text{ as } n \to \infty,
\]

Again using Lebesgue dominated convergent theorem, we get

\[
e^{(\lambda_n - \beta)(t-t)} \| g(x,s) \|^2 ds \to 0 \text{ as } n \to \infty.
\]

In summary, we obtain that the terms on the right hand of inequality (3.13) tend to 0 as \( n \to \infty \), which say that \( \| v_\tau(t, \tau, \Theta_{\tau,s}, \omega) \| \to 0 \), i.e., the random dynamical system (3.3)-(3.5) satisfies pullback flattening.

\]

4. Conclusions

This paper extends the existence of pullback attractor of non-autonomous modified S-H equation to the case of non-autonomous stochastic modified S-H equation with multiplicative noise. In the concrete experiment, the random term in the equation is more consistent with the actual problem. For S-H equation with multiplicative noise, the external force has exponential growth, we have proved that the equation exists a random \( \mathcal{D} \)-pullback attractor in one dimension. In the future work, we will continue to investigate whether the same results can be obtained when the spatial dimension is two-dimensional or n-dimensional.

Author Contributions: All the authors have equal contribution to this study. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the National Natural Science Foundation of China (11761044, 11661048) and the key constructive discipline of Lanzhou City University (LZCU-ZDJSXK-201706).

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

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