Fixed-Order $\mathcal{H}_2$-Conic Control

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Abstract— $\mathcal{H}_2$-conic controller design seeks to minimize the closed-loop $\mathcal{H}_2$ norm for a nominal linear system while satisfying the Conic Sector Theorem for nonlinear stability. This problem has only been posed with limited design freedom, as opposed to fixed-order design where all controller parameters except the number of state estimates are free variables. Here, the fixed-order $\mathcal{H}_2$-conic design problem is reformulated as a convergent series of convex approximations using iterative convex overbounding. A synthesis algorithm and various initializations are proposed. The synthesis is applied to a passivity-violated system with uncertain parameters and compared to benchmark controller designs.

I. INTRODUCTION

Recently, input-output (IO) analysis and stability theorems have emerged as useful tools in the fields of robust control, data-driven control, and robust neural networks [1]–[9]. For robust controller synthesis, conic-sector-based methods rely on more general stability results than the familiar Passivity and Small Gain Theorems, which often improves the performance-robustness tradeoff [10]–[18]. However, current design methods allow incomplete access to controller parameters. This erodes performance and impedes the addition of secondary objectives like sparsity promotion. Current designs also do not admit reduced order controllers. This paper proposes a fixed-order conic-sector-based design method that allows complete access to controller parameters for a controller of any specified dimension, improving design flexibility and admitting secondary objectives.

Performance versus robustness to model uncertainty is a fundamental tradeoff in automatic feedback control. This can be seen in design methods like $\mathcal{H}_2$/Small Gain control [19] and $\mathcal{H}_2$-SPR [20] control, where the $\mathcal{H}_2$ performance objective is optimized subject to the Small Gain and Passivity Theorems, respectively. These IO stability results guarantee closed-loop stability for nonlinear, uncertain plants.

Passivity and small gain, though widely used, are not always applicable. Several other results – large gain, γ-pasivity, and passivity indices – have been developed for cases that passivity and small gain cannot handle. It was recently shown that these are all special cases of the Extended Conic Sector Theorem [17]. Zames’ Conic Sector Theorem (CST) [21] predates many IO stability theorems, but it was relatively unused for several decades due to its analytical complexity. Even more general IO results – like QSR dissipativity [5], [22] – present similar analytical challenges. However, new tools have been developed to facilitate conic-sector-based analysis and design. Matrix inequality constraints that impose conic bounds on a linear system were developed using the KYP Lemma in the 90’s [18]. Since then, several design schemes attempting to minimize the $\mathcal{H}_2$-norm subject to the conic constraint have been proposed [10]–[15]. These “$\mathcal{H}_2$-conic” designs can be thought of as a stepping-stone to future QSR-dissipativity-based designs. However, simpler characterizations involved in conic properties facilitate interpretability and informed design choices.

The primary challenge in $\mathcal{H}_2$-conic design is the nonconvexity of the cubic objective and bilinear constraints. This makes the problem NP-hard, so no method can guarantee polynomial-time convergence to a global minimum [23]. The aforementioned design schemes instead restrict design freedom in various ways, resulting in a conservative convex problem or series thereof. The most straightforward method fixes the observer and designs only the state-estimate feedback matrix using a convex heuristic proxy for the $\mathcal{H}_2$ norm [10]. Other methods enforce a Luenberger structure on the controller and transform controller parameters into implicit variables to achieve convex constraints and a convex overbound on the $\mathcal{H}_2$-norm [13]–[15]. The restricted design space in these methods inevitably reduces performance, and forcing the controller order to match that of the plant is problematic for large systems. Further, incomplete access to controller parameters impedes addition of secondary objectives like sparsity promotion and structured communication, which are important to stability and security problems in distributed design [24]–[26]. The present work serves as an intermediate result towards structured $\mathcal{H}_2$-conic designs.

This paper addresses the above limitations by proposing a fixed-order $\mathcal{H}_2$-conic design scheme. In this context, “fixed-order” means the number of state estimates $n_e$ is fixed, and controller parameters $\hat{A}$, $\hat{B}$, and $\hat{C}$ (as defined in Section III) are accessible design variables. This is as opposed to a “Luenberger” controller, where the state-estimate dynamics matrix is assumed to have the structure $\hat{A} = A - B\hat{C} - BC_2$. This improved design is achieved by employing iterative convex overbounding (ICO) [27], [28] to pose the nonconvex problem as a convergent series of convex problems without obfuscating the controller parameters.

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II. Preliminaries

A. Notation

For a square matrix $P$, positive definiteness is denoted $P > 0$. Related properties (negative definiteness and positive/negative semi-definiteness) are denoted correspondingly. The identity matrix is $I$, the trace is $\text{tr}(\cdot)$, and $\lambda(\cdot)$ is the eigenvalues. For brevity, $\text{He}(M) = M + M^T$ and asterisks denote duplicate blocks in symmetric matrices. The $L_2$, Frobenius, and $H_2$ norms are denoted $\|\cdot\|_2$, $\|\cdot\|_F$, $\|\cdot\|_{H_2}$. Recall $y \in L_2$ if $\| y(x) \|_2 = \int_0^\infty \| y(t) \| dt < \infty$. Further, $y \in L_2$, if its truncation to $[0, T]$ is in $L_2 \forall T \geq 0$, where the truncation of $y$ is found by multiplying $y(t)$ by $0$ for $t > T$.

B. Review of Conic Sectors

Conic sectors, defined below, are an IO description of a set of operators. They can be viewed as a special case of dissipativity.

Definition 1: (Interior conic [10]) A square system, $G : L_{2e} \rightarrow L_{2e}$, is in the conic sector $\{a, b\}$, where $a < b$ and $0 < b$, denoted $G \in \operatorname{cone}(a, b)$, if $\forall u \in L_{2e}, T \in \mathbb{R}^+$,

$$\frac{1}{b} \|Gu\|_2^2 + \left(1 + \frac{a}{b}\right) \langle Gu, u \rangle_T - a\|u\|_2^2 \geq \beta,$$

where $\beta$ only depends on initial conditions. It is strictly in the conic sector, denoted $G \in \operatorname{cone}(a, b)$, if $G \in \operatorname{cone}(a+\delta, b-\delta)$, for some small $\delta > 0$.

The Conic Sector Lemma (CSL) below shows a matrix inequality expression of conic bounds for linear time invariant (LTI) systems without feedthrough.

Lemma 1: (Conic Sector Lemma [18]) A square LTI system, $G : L_{2e} \rightarrow L_{2e}$, with minimal space realization $(A, B, C, 0)$ is inside cone $(a, b)$, $a < b < b \ll a$ if and only if there exists $P = P^T > 0$, such that

$$\begin{bmatrix} PA + A^TP + C^TC & * \\ B^T P - \frac{ab}{2} C & abI \end{bmatrix} \leq 0 \quad (1)$$

or

$$\begin{bmatrix} PA + AP & PB \\ * & - \frac{(a-b)^2}{4b} I - \frac{(a+b)C^T}{2} \end{bmatrix} \leq 0 \quad (2)$$

or

$$P(A + \frac{a+b}{2ab}BC) + (A^T + \frac{a+b}{2ab}C^TB^T)P + \left(1 - \frac{(a-b)^2}{4ab}\right)C^TC - \frac{1}{ab}BB^TP \leq 0. \quad (3)$$

Equations 1–3 are equivalent. The conic sector with $a = 0$ and $b = \infty$ is identical to passivity. In fact, dividing Equation 1 by $b > 0$, setting $a = 0$, and taking the limit as $b \rightarrow \infty$ recovers the KYP Lemma for passivity [22, section 3].

In this paper, the CSL will be used in combination with the CST to ensure closed-loop IO stability. The CST statement below is more restricted than the original formulation in [21] but contains all cases needed in this paper.

Theorem 1: (CST [10], [21]) Consider the negative feedback interconnection of two square systems, $G_1 : L_{2e} \rightarrow L_{2e}$ and $G_2 : L_{2e} \rightarrow L_{2e}$, defined by $y_1 = G_1 u_1$, $y_2 = G_2 u_2$,

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C}^T & \tilde{D} \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{U} \end{bmatrix} + \begin{bmatrix} \tilde{F}^T & \tilde{G} \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{U} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \tilde{X} & \tilde{U} \end{bmatrix} = \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix},$$

which is an extension of the transformation used in [28], and contains all cases needed in this paper.

Fig. 1: Closed loops with (a) additive disturbances and (b) general disturbances and objectives.

$$u_1 = r_1 - y_2, \quad u_2 = r_2 - y_1, \quad y_1 = \tilde{A} \tilde{x} + \tilde{B} \tilde{u}, \quad y_2 = \tilde{C} \tilde{x} + \tilde{D} \tilde{u}.$$

III. Problem Statement

Consider the system in Figure 1b with plant and controller

$$\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} w \end{bmatrix} + \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} u \end{bmatrix}, \quad \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} w \end{bmatrix} + \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

Here, $x \in \mathbb{R}^m$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $w \in \mathbb{R}^r$, $z \in \mathbb{R}^q$, $\hat{x} \in \mathbb{R}^{n_1}$, $\tilde{y} \in \mathbb{R}^{n_2}$.

If a system is in any conic sector, it is in infinitely many conic sectors. An appropriate conic sector for controller synthesis is one that maximizes controller design space while robustly characterizing plant uncertainty. Such a sector can be identified analytically or with the CSL [10], frequency domain methods, [16], or data-driven methods [1], [2].

If $s(t)$ solves $\dot{s}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s(t)$, where the observability Gramian $Q = Q^T > 0$ solves $A^T Q + Q A = 0$ and $\tilde{A}_c = \begin{bmatrix} A & -B_2 C \\ B_1 & B_2 D_2 \end{bmatrix}$, where $\tilde{A}_c$ is the closed-loop state realization

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} B_c \\ C_c \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} -B_2 C \\ B_1 \end{bmatrix}.$$
beneficial to some secondary objectives, like sparsity promotion [24], because it collects all controller parameters into a single matrix, K. Now, the closed-loop state space realization and CSL inequality can be expressed as \( A_f = \tilde{A} + EKR \), \( B_{cl} = \tilde{B} + EKS \), \( C_{cl} = \tilde{C} + FKR \), and He(\( PKX \)) + \( K \leq 0 \), and Problem 4 can be equivalently expressed

\[
\min_{K, Q, P} J(K, Q) = \text{tr}((\tilde{B} + EKS)^T Q (\tilde{B} + EKS))
\]

\[
\text{s.t.} \quad \text{He}[Q(\tilde{A} + EKR)] + (\tilde{C} + FKR)^T (\tilde{C} + FKR) = 0
\]

\[
\text{He}[\tilde{P} \tilde{K}X] + \Gamma \leq 0
\]

\[
Q > 0, \quad P > 0.
\]

IV. MAIN RESULT

A. Design Algorithm

In this section, Theorem 2 establishes a convex optimization that minimizes an upper bound on the closed-loop \( H_2 \) norm subject to a conic controller constraint. The upper bound is nonconservative near the initialization point. Algorithm 1 uses Theorem 2 to iteratively reduce the \( H_2 \) norm from a feasible point to a local minimum. Corollary 1 shows that Algorithm 1 generates a sequence of monotonic decreasing costs bounded below, and it derives a finite bound on the number of iterations to convergence.

**Theorem 2**: Assume \( Q_0 \), \( K_0 \), and \( P_0 \) satisfy Equations 6–8. Then \( Q = Q_0 + \delta Q \), \( K = K_0 + \delta K \), and \( P = P_0 + \delta P \) also satisfy Equations 6–8, where \( \delta Q \), \( \delta K \), and \( \delta P \) solve

\[
\min_{\delta Q, \delta P, \delta K} J' = \text{tr}(\tilde{B}^T Q_0 + \delta Q \tilde{B}) + \text{tr}(Z) \quad (9)
\]

\[
\text{s.t.} \quad \begin{bmatrix}
Q_0 - Q_0^{-1} \delta QQ_0^{-1} & \\
S^T (K_0 + \delta K)^T E^T & Z
\end{bmatrix} \geq 0 \quad (10)
\]

\[
\begin{bmatrix}
\Pi_1 & \delta Q E K & \quad R^T \delta K^T \\
* & -W_1 & \quad 0 \\
* & * & -(W_1^{-1} + F^T F)^{-1}
\end{bmatrix} \leq 0 \quad (11)
\]

\[
\begin{bmatrix}
\Pi_2 & \delta P & \quad X^T \delta K^T \\
* & -W_2 & \quad 0 \\
* & * & -W_2
\end{bmatrix} \leq 0 \quad (12)
\]

and

\[
\Pi_1 = \text{He}[\tilde{A}^T \delta Q + \tilde{A}^T Q_0 + Q_0 EK_0 R + \delta QEKR] + Q_0 EKR + C^T FKR + C^T \delta KR
\]

\[
+ R^T K_0^T F\delta KR + C^T \tilde{C} + R^T \tilde{K}_0^T FKR, \\
\Pi_2 = \Gamma + \text{He}[P_0 K_0 X + P_0 \delta KX + \delta PK_0 X], \\
W_1 = W_1^T > 0 \quad \text{and} \quad W_2 = W_2^T > 0.
\]

Further, letting \( Q_{opt} \) and \( K_{opt} \) be the global minimizers of Problem 5, it is true that

\[
J'(K, Q) \geq J(K, Q),
\]

\[
J'(K_0, Q_0) = J(K_0, Q_0),
\]

\[
J'(K_{opt}, Q_{opt}) \leq J'(K, Q).
\]

**Proof**: The proof follows the overbounding technique from [27], begining by converting Objective 5 into a convex function with an added convex constraint, Equation 10. Then, ICO transforms Equations 6, 7 into Equations 11, 12, which are LMIs in \( \delta Q \), \( \delta K \), \( \delta P \). Last, it is shown that these transformations imply Equations 14–17.

To begin, the objective in Equation 5 is distributed as \( \text{tr}(\tilde{B}^T \tilde{Q} \tilde{B}) + \text{tr}(\text{He}[\tilde{B}^T \tilde{Q} EKR]) + \text{tr}(S^T K^T E^T Q EKR) \), the first term is convex in a single design variable. Using the cyclic property of the trace, the second term is equivalently \( \text{tr}(\text{He}[D_2 B_1^T Q_1^T B_1]) \), where \( Q_1^T \) is the top-right block of \( Q \) when partitioned appropriately. This term is zero due to the standard assumption that \( D_2 B_1^T = 0 \). The third term is cubic in \( Q \) and \( K \). Consider a new variable \( Z \) such that \( Z = S^T K^T E^T Q EKR \geq 0 \). Minimizing \( \text{tr}(Z) \) subject to this inequality is equivalent to minimizing the cubic term. Applying the Schur complement to the constraint yields an LMI in \( Z \), \( K \), \( Q^{-1} \). For any \( Q > 0 \), \( Q > 0 \), it is known that

\[
Q^{-1} \geq 2\tilde{Q}^{-1} - \tilde{Q}^{-1} \tilde{Q} \tilde{Q}^{-1} \geq 0.
\]

Applying this fact to the Schur complement yields

\[
\left[ \begin{array}{c}
2\tilde{Q}^{-1} - \tilde{Q}^{-1} \tilde{Q} \tilde{Q}^{-1} \\
(EKR)^T Z
\end{array} \right] \geq 0,
\]

which is linear in \( Q \), \( K \), and \( Z \). Now minimizing \( \text{tr}(Z) \) is conservative, which implies Equation 14.

Equation 6 is relaxed to a negative semi-definite inequality because the optimal solution converges to the equality boundary and negative semi-definiteness is sufficient to guarantee stability and a bounded solution domain. This bilinear inequality is treated with ICO [27]. Let \( K = K_0 + \delta K \) and \( Q = Q_0 + \delta Q \), where the new design variables \( \{\delta Q, \delta K\} \) represent change from a known feasible point, \( \{Q_0, K_0\} \). These substitutions yield \( \Pi_1 + \text{He}[\delta Q E \delta K R] + R^T \delta K^T E^T \delta K R \leq 0 \) where \( \Pi_1 \) is linear in the new variables.

\[
(\delta Q E L - R^T \delta K^T L^{-T})(\delta Q E L - R^T \delta K^T L^{-T}) =
\]

\[
\delta Q E L L^{-T} \tilde{E} E^T \delta K^T + R^T \delta K^T L^{-1} \delta K R + R^T \tilde{E} E^T \delta K^T + R^T \tilde{E} E^T \delta K^T
\]

for any invertible \( L \), which implies \( \text{He}[\delta Q E \delta K R] \leq \delta Q E W_1 \tilde{E} E^T \delta K^T + R^T \delta K^T W_1^{-1} \delta K R \) for any \( W_1 = W_1^T > 0 \). This allows Equation 6 to be rewritten conservatively as \( \Pi_1 + \delta Q E W_1 \tilde{E} E^T \delta K^T + R^T \delta K^T \leq 0 \) for any \( W_1 = W_1^T > 0 \). Now the quadratic terms are removed using the Schur complement, which yields directly Equation 11. An identical approach can be applied to Equation 7, which becomes Equation 12 with \( W_2 = W_2^T > 0, \quad P = P_0 + \delta P \), and

\[
\tilde{P} = P_0 + \delta P = \begin{bmatrix}
0 & P_0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

Applying the same change of variables in \( Q \) and \( K \) to Equation 19 and selecting \( Q = Q_0 \) yields Equation 10.

Thus any \( \delta K, \delta Q, \delta P, \delta Z \) satisfying Equations 10–13 implies \( K, Q, P \) satisfy Equations 6–8 and the costs are related by Equation 14. Further, if \( K = 0 \), hence \( Q = 0 \), \( P = 0 \), then Equations 11–13 reduce to Equations 6–8, and Equations 9–10 reduce to Equation 5. This implies Equation 15. Since \( K_0 \),
$\mathbf{P}_0$ and $\mathbf{Q}_0$ are feasible points, the cost given by Problem 9–13 cannot be worse than the initial cost, hence Equation 16. Equation 17 follows from the definition of the minimum and the conservatism introduced by Equation 18 and ICO.

Due to the properties of Problem 9–13 found in Theorem 2, Algorithm 1 gives a conic controller with a lower $\mathcal{H}_2$-norm than that of the initialization. This is formalized in Corollary 1. Initializations are considered in the sequel.

Algorithm 1:

Input: $\mathcal{G}$, $a_c$, $b_c$, $\mathbf{K}_0$, $\mathbf{Q}_0$, $\mathbf{P}_0$

Set $k = 0$, $\Delta J' = \infty$

while $\Delta J' > \varepsilon$ do

Solve Problem 9–13 with $\mathbf{K}_0 = \mathbf{K}_k$, $\mathbf{Q}_0 = \mathbf{Q}_k$, $\mathbf{P}_0 = \mathbf{P}_k$

$\mathbf{Q}_{k+1} = \mathbf{Q}_k + \delta \mathbf{Q}$

$\mathbf{P}_{k+1} = \mathbf{P}_k + \delta \mathbf{P}$

$\mathbf{K}_{k+1} = \mathbf{K}_k + \delta \mathbf{K}$

$\Delta J' = |J'_k(\mathbf{K}_{k+1}, \mathbf{Q}_{k+1}) - J'_k(\mathbf{K}_k, \mathbf{Q}_k)|$

$k = k + 1$

end while

Output: $\mathbf{K}_{k+1}$, $\mathbf{Q}_{k+1}$, $\mathbf{P}_{k+1}$

Corollary 1: At iteration $k$ of Algorithm 1, Problem 9–13 has a feasible solution whose cost, $J'_k(\mathbf{K}_k, \mathbf{Q}_k)$, is bounded above by the cost at the $(k - 1)^{st}$ iteration, $J'_k(\mathbf{K}_{k-1}, \mathbf{Q}_{k-1})$, and bounded below by the optimal cost of Problem 5, $J(\mathbf{K}_{opt}, \mathbf{Q}_{opt})$. Further, Algorithm 1 terminates in at most $(J'_k(\mathbf{K}_0, \mathbf{Q}_0) - J(\mathcal{H}_2))/\varepsilon$ iterations, where $J(\mathcal{H}_2)$ is the cost of the $\mathcal{H}_2$-optimal controller without the conic constraint.

B. Initialization

In this section, initialization schemes are proposed for the five parameters required to begin Algorithm 1. Two initialization schemes are proposed for $\mathbf{W}_1$ and $\mathbf{W}_2$ using various assumptions on $\delta \mathbf{K}$, $\delta \mathbf{Q}$, and $\delta \mathbf{P}$. Both attempt to minimize the conservatism introduced by $\mathbf{W}_1$ and $\mathbf{W}_2$. Meanwhile, $\mathbf{K}_0$, $\mathbf{Q}_0$, and $\mathbf{P}_0$ must be initialized together. These have the greatest impact on the solution as their initialization determines to which local minimum the problem will converge. An arbitrary initialization method is proposed for its simplicity, and its pitfalls are described. Then two heuristic methods are proposed which provide an initialization point that is “close” to the optimal controller without conic bounds.

1) $\mathbf{W}$ Initialization: The only necessary condition on $\mathbf{W}_1$ and $\mathbf{W}_2$ is positive definiteness, so the simplest option is to set both to identity. In fact, this choice works well in practice, as demonstrated in section V. This is because the conservatism introduced by $\mathbf{W}_1$ and $\mathbf{W}_2$ is dictated by the size of $\delta \mathbf{Q}\mathbf{E}\mathbf{W}_1\mathbf{E}^T + \delta \mathbf{P} - \mathbf{R}^T \delta \mathbf{K}^T \mathbf{W}_1^{-1} \delta \mathbf{K} \mathbf{R} + \delta \mathbf{P} \mathbf{W}_2 \delta \mathbf{P} + \mathbf{X}^T \delta \mathbf{K}^T \mathbf{W}_2^{-1} \delta \mathbf{K} \mathbf{X}$, respectively. Both terms are small when $\delta \mathbf{K}$, $\delta \mathbf{Q}$, and $\delta \mathbf{P}$ are small. An alternative heuristic seeks to minimize these terms directly, assuming that $\delta \mathbf{K}$, $\delta \mathbf{Q}$, and $\delta \mathbf{P}$ are of the same size and not arbitrarily small. Setting $\delta \mathbf{K}$, $\delta \mathbf{Q}$, and $\delta \mathbf{P}$ to identity, the “size” of the conservative terms may be defined as $\text{tr}(\mathbf{E}\mathbf{W}_1\mathbf{E}) + \text{tr}(\mathbf{R}^T \mathbf{W}_1^{-1} \mathbf{R})$ and $\text{tr}(\mathbf{W}_2) + \text{tr}(\mathbf{X}^T \mathbf{W}_2^{-1} \mathbf{X})$, respectively. These are minimized via

$$\min_{\mathbf{W}_1, \mathbf{V}_1} \text{tr}(\mathbf{E}\mathbf{W}_1\mathbf{E}) + \text{tr}(\mathbf{V}_1) \quad \text{s.t.} \quad \begin{bmatrix} \mathbf{W}_1 & \mathbf{R}^T \mathbf{V}_1 \end{bmatrix} \geq 0, \quad (20)$$

$$\min_{\mathbf{W}_2, \mathbf{V}_2} \text{tr}(\mathbf{W}_2) + \text{tr}(\mathbf{V}_2) \quad \text{s.t.} \quad \begin{bmatrix} \mathbf{W}_2^2 & \mathbf{X}^T \mathbf{V}_2 \end{bmatrix} \geq 0. \quad (21)$$

These optimization problems are derived by the same process that yields Equation 19 without applying Equation 18. Since $\mathbf{W}_1 = \mathbf{I}$ and $\mathbf{W}_2 = \mathbf{I}$ are feasible points for every permissible $\mathbf{E}$, $\mathbf{R}$, and $\mathbf{X}$, these problems are always solvable.

2) $\mathbf{KQP}$ Initialization: Arbitrary: The only requirement on $\mathbf{K}_0$, $\mathbf{Q}_0$, $\mathbf{P}_0$ is that they constitute a feasible point of Problem 5–8. Thus, any conic controller could form a basis for the initialization. Given a known conic controller, $\mathbf{K}_0$ is compiled directly from the parameters, the algebraic Riccati Equation 6 is solved for $\mathbf{Q} = \mathbf{Q}_0$, and a feasible $\mathbf{P} = \mathbf{P}_0$ is selected subject to Equations 7 and 8.

In practice, the choice of controller is further restricted. If $\mathbf{K}_0$ is small, then $\delta \mathbf{K}$ must be much smaller for Problem 5–8 to be near-nonconservative. In fact, if $\mathbf{K}_0 = 0$ (which is trivially interior conic for all $a_c, b_c$) then any $\delta \mathbf{K} \neq 0$ is infinitely conservative. In this case, Algorithm 1 immediately converges to the initialization point. Thus a sufficiently large $\mathbf{K}_0$ must be selected for good results, which complicates the already nontrivial task of constructing a conic controller.

3) $\mathbf{KQP}$ Initialization: ConicC: Even if an arbitrary controller can be directly constructed, it may not yield good results. While no initialization is guaranteed to achieve the global minimum due to the problem’s nonconvexity, heuristic methods aim to select an initialization point that is in some sense already “close” to the optimum. One such method is the ConicC algorithm [10]. This algorithm starts with the Luenberger optimal controller and changes $\mathbf{C}$ minimally to satisfy the conic bounds. The algorithm already calculates $\mathbf{P}_0$ and $\mathbf{K}_0$ is constructed from the resultant controller. Solving the algebraic Riccati Equation 6 then provides $\mathbf{Q}_0$. Any other conic design method can be similarly employed. The ConicC problem is always feasible given a stable target controller. This is because the problem is convex and Lemma 2 below gives a feasible point. The original ConicC formulation uses the $\mathcal{H}_2$-optimal Luenberger controller as the target. While the Luenberger controller is usually open-loop stable, it is not always [30]. Nonetheless, a different target controller can be selected arbitrarily or by a variety of stable compensator designs [31] to recover feasibility of the ConicC algorithm.

Lemma 2: All stable LTI systems without feedthrough are interior conic for some $a < b < c$. Further, $\forall \mathcal{G}: (\mathbf{A}, \mathbf{B}, \mathbf{C}, 0) \in \text{cone}(a, b, c) \subseteq \text{cone}(a, b, c, 0)$ where $a, b < 0$ and $b, c > 0$. $\forall \mathcal{G}': (\mathbf{A}, \mathbf{B}, \mathbf{C}', 0) = (\mathbf{A}, \mathbf{B}, \min(\{a, b, c\})\mathbf{C}, 0) \in \text{cone}(a, b, c, 0)$.

Proof: By the Lyapunov Stability Theorem [29], since $\mathbf{A}$ is Hurwitz, $\forall \mathbf{Q}, \exists \mathbf{P} = \mathbf{P}^T > 0$ such that $\mathbf{P}\mathbf{A} + \mathbf{A}^T \mathbf{P} < -\mathbf{Q}$. In particular, $\mathbf{Q} = \mathbf{C}_c^T + \mathbf{B}_c^T$ can be chosen. If $\gamma > \max(\lambda(\mathbf{P}))$, then $\mathbf{P} + \gamma \mathbf{B}_c^T < \mathbf{C}_c^T$. The Lyapunov equation then implies Equation 3 holds with $a = -\gamma, b = \gamma$. Thus by Lemma 1, $\mathcal{G} \in \text{cone}(-\gamma, \gamma)$. 

Further, if $\mathcal{G} \in \text{cone}(a,b)$ with $b > 0$ and $k \geq 0$, then $k\mathcal{G} \in \text{cone}(ka,kb)$ [21]. Scaling $\mathcal{G}$: $u \rightarrow y$ is achieved by scaling the output matrix, C. Suppose $a_0 < a$ and $b_0 > b$. In this case, $\frac{a_0}{a} < 1$ and $\frac{b}{b_0} < 1$, so $\min(\frac{a_0}{a}, \frac{b}{b_0})a_0 > a$ and $\min(\frac{a_0}{a}, \frac{b}{b_0})b_0 < b$. Thus, $\mathcal{G}' = \min(\frac{a_0}{a}, \frac{b}{b_0})\mathcal{G} \in \text{cone}(a,b)$. Further, if $a_0 > a$, then $\min(\frac{a_0}{a}, \frac{b}{b_0}) = \frac{b}{b_0}$, and $b_0 > a$, and vice versa for $b_0 < b$. Thus any interior conic system can be transformed to satisfy any interior conic bounds of opposing signs by scaling the output matrix.

4) KOP Initialization: ICO: An alternative method, based on a technique in [27], iteratively relaxes the optimal controller until it satisfies the constraints. First, the Luenberger optimal controller (or similar) is found, which provides $K_L$. Next, Equation 6 is solved with $K_L$ to get $Q_L$. Then $P_L$ is given by minimizing $\varepsilon$ over $P_L$ and $\varepsilon$ subject to $\text{He}(P_L, K, X) + \Gamma \leq \varepsilon I$ and $P_L > 0$. If $\varepsilon < 0$, then $(K_L, Q_L, P_L)$ is a feasible point, and the process is terminated. Otherwise, minimize $\varepsilon$ over $\delta K$, $\delta Q$, $\delta P$, $Z$, subject to Equations 10–13, where $K_0 = K_L$, $Q_0 = Q_L$, $P_0 = P_L$, and $0 < \varepsilon$ is replaced by $\leq \varepsilon I$ in Equation 12. Also add the constraint $\text{tr}(B^T(Q + \delta Q)B) + \text{tr}(Z) < (1 + \Delta)J_L$, where $\Delta$ is a small positive constant and $J_L$ is the cost of the initial controller. This extra constraint controls the rate at which the $H_2$ cost is relaxed. Iterate this new problem similarly to Algorithm 1, updating $J_L$ at each step, until $\varepsilon < 0$. If $\varepsilon < 0$ is achieved then the corresponding $K$, $Q$, $P$ is a feasible point. This algorithm is not guaranteed to converge, but it provides greater flexibility than ConicC when it does.

V. NUMERICAL EXAMPLE

Algorithm 1 is now applied to a vibration suppresion problem with parametric uncertainty and passivity violation. The plant is three masses connected by springs and dampers. The $i^{th}$ mass’ dynamics are given by $x_i = [p_i \ v_i]^T$, $\dot{x}_i = A_i x_i + \sum_{j \neq i} A_{ij} x_j + [0 \ 1]^T u_i + [0 \ 1]^T w_{1i}$, $v_i = C_{ii} x_i + w_{2i}$, $z_i = [p_i \ v_i u_i]^T$, $w_i = [w_{1i} \ w_{2i}]^T$, where $p_i$ and $v_i$ are the position and velocity of mass $i$, $A_{ii}$ and $A_{ij}$ are derived from Figure 2, and $C_{ij}$ depends on the chosen output measurement.

The idealized plant with velocity measured outputs and nominal parameters is $\mathcal{G}_1$. A more realistic version of the plant accounting for the parametric uncertainty and passivity violations is $\mathcal{G}_2$. Parametric uncertainty is introduced informally by considering the discrete set of parameters. Parameters $m_i$ and $k_i$ have nominal value 1 and off-nominal values 0.3 and 3, while $c_i$ has nominal value 0.05 and off-nominal values 0.01 and 0.1. This is not a full treatment of parametric uncertainty, merely a proof of concept. To introduce a passivity violation, the measured output is the position of each mass filtered by the approximate derivative filter $f(x) = \frac{2x}{s + 2}$. This filter causes a modest increase to the minimal $H_2$ norm but significant passivity violations.

All controllers in this example are designed for the state-space model of $\mathcal{G}_1$ and the conic sector of $\mathcal{G}_2$, then tested on $\mathcal{G}_2$. Conic sector $(-24.84, 62200)$ is sufficient for the discrete set of parameters considered here. More analysis would be required to show that the sector is sufficient to describe all parameter sets in the convex polytope.

Two fixed-order controllers, $C_{new}$ and $I_{new}$ are designed using Algorithm 1 with the proposed ConicC and ICO initializations, respectively. Both use the proposed $W$ initializations, which interestingly converge to the identity matrices in each case. The convergence criterion on Algorithm 1 is $\varepsilon = 5 \times 10^{-3}$. The ICO initialization uses increment $\Delta = 0.1$. Algorithm 1 and the ICO initialization experienced numerical issues, but these were remedied by regularizing the design variables by augmenting the objective with the squared Frobenius norms of $\delta K$, $\delta P$, and/or $\delta Q$ times a small constant $\gamma$. For Algorithm 1, $\gamma = 0.1$ was successful, and $\gamma = 10^{-3}$ was successful for the ICO initialization.

The novel $C_{new}$ and $I_{new}$ controllers’ performance is compared to several benchmark controllers: the $H_2$-Optimal controller as designed for $\mathcal{G}_1$ (which is passive), the ConicC controller [10], the Iterative Conic controller [13] with convergence criterion $\varepsilon = 10^{-6}$, and the open loop with no controller. Table I shows that $C_{new}$ and $I_{new}$ significantly improve over the ConicC and Iterative Conic controllers for the nominal parameters. Interestingly, the resulting controllers are substantially different even though their performance is similar. Figure 3 shows that the conic controllers are stable for every parameter set, whereas the (passive) $H_2$-Optimal controller is unstable for almost 40% of the simulations. Further, out of the five controllers that stabilize every simulation, $C_{new}$ performed the best for about 60% of simulations, while $I_{new}$, performed the best for about 40%. ConicC performed about 10% worse on all simulations, with Iterative Conic slightly worse still. The design curves in Figure 4 emphasize that $I_{new}$ required an order of magnitude more iterations and converges to a slightly worse performance than $C_{new}$.

VI. DISCUSSION

Motivated by the need to incorporate secondary objectives like structured communication into controller design, this paper develops a fixed-order $H_2$-conic synthesis that allows direct access to design variables. A convergent series of convex problems are posed that minimize the $H_2$ norm while maintaining desired conic bounds without restricting design freedom or transforming controller parameters into implicit variables. The method’s conservatism is concentrated mostly
in the choice of initialization, and initialization schemes are proposed based on successful heuristics.

Ultimately, the analytical results herein are intermediates to improved sparsity promoting and structured $\mathcal{H}_2$-conic designs. While those results are immanent, they remain nontrivial. Thus, complete development of those new designs are left as the subject of future work. Other future work includes extending these methods to more general IO stability parameters. These simulations also emphasize the value of $\mathcal{H}_2$-conic control in general, as all of the conic designs considered guaranteed stability for every parameter set, while the (passive) optimal controller was unstable for about 40%.

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