BCK-algebras arising from block codes

Cristina FLAUT

Abstract. In this paper, we will provide an algorithm which allows us to find a BCK-algebra starting from a given binary block code.

Keywords: BCK-algebras; Block codes.

AMS Classification. 06F35

0. Introduction

BCK-algebras were first introduced in mathematics in 1966 by Y. Imai and K. Iseki, through the paper [4], as a generalization of the concept of set-theoretic difference and propositional calculi. The class of BCK-algebras is a proper subclass of the class of BCI-algebras and there exist several generalizations of BCK-algebras as for example: generalized BCK-algebras [3], dual BCK-algebras [9], BE-algebras [1], [8]. These algebras form an important class of logical algebras and have many applications to various domains of mathematics, such as: group theory, functional analysis, fuzzy sets theory, probability theory, topology, etc.

For other details about BCK-algebras and about some new applications of them, the reader is referred to [2], [5], [6], [10], [11], [12], [13].

One of the recent applications of BCK-algebras was given in the Coding Theory. In Coding Theory, a block code is an error-correcting code which encode data in blocks. In the paper [7], the authors constructed a finite binary block-codes associated to a finite BCK-algebra. At the end of the paper, they put the question if the converse of this statement is also true.

In the present paper, we will prove that, in some circumstances, the converse of the above statement is also true.

1. Preliminaries

Definition 1.1. An algebra $(X, \ast, \theta)$ of type $(2, 0)$ is called a BCI-algebra if the following conditions are fulfilled:

1) $((x \ast y) \ast (x \ast z)) \ast (z \ast y) = \theta$, for all $x, y, z \in X$;
2) \((x \ast (x \ast y)) \ast y = \theta\), for all \(x, y \in X\);
3) \(x \ast x = \theta\), for all \(x \in X\);
4) For all \(x, y, z \in X\) such that \(x \ast y = \theta, y \ast x = \theta\), it results \(x = y\).

If a BCI-algebra \(X\) satisfies the following identity:
5) \(\theta \ast x = \theta\), for all \(x \in X\);

A BCK-algebra \(X\) is called \textit{commutative} if \(x \ast (x \ast y) = y \ast (y \ast x)\), for all \(x, y \in X\) and \textit{implicative} if \(x \ast (y \ast x) = x\), for all \(x, y \in X\).

The partial order relation on a BCK-algebra is defined such that \(x \leq y\) if and only if \(x \ast y = \theta\).

If \((X, \ast, \theta)\) and \((Y, \circ, \theta)\) are two BCK-algebras, a map \(f : X \to Y\) with the property \(f(x \ast y) = f(x) \circ f(y)\), for all \(x, y \in X\), is called a \textit{BCK-algebra morphism}. If \(f\) is a bijective map, then \(f\) is an \textit{isomorphism} of BCK-algebras.

In the following, we will use some notations and results given in the paper [7].

From now on, in whole this paper, all considered BCK-algebras are finite.

Let \(A\) be a nonempty set and let \(X\) be a BCK-algebra.

**Definition 1.2.** A mapping \(f : A \to X\) is called a \textit{BCK-function} on \(A\). A \textit{cut function} of \(f\) is a map \(f_r : A \to \{0, 1\}, r \in X\), such that \(f_r(x) = 1\), if and only if \(r \ast f(x) = \theta, \forall x \in A\).

A \textit{cut subset} of \(A\) is the following subset of \(A\)
\[A_r = \{x \in A : r \ast f(x) = \theta\}.\]

**Remark 1.3.** Let \(f : A \to X\) be a BCK-function on \(A\). We define on \(X\) the following binary relation
\[
\forall r, s \in X, r \sim s \text{ if and only if } A_r = A_s.
\]

This relation is an equivalence relation on \(X\) and we denote with \(\tilde{r}\) the equivalence class of the element \(r \in X\).

**Remark 1.4.** ([7] ) Let \(A\) be a set with \(n\) elements. We consider \(A = \{1, 2, ..., n\}\) and let \(X\) be a BCK-algebra. For each BCK-function \(f : A \to X\), we can define a binary block-code of length \(n\). For this purpose, to each equivalence class \(\tilde{x}, x \in X\), will correspond the codeword \(w_x = x_1x_2...x_n\) with \(x_i = j\), if and only if \(f_x(i) = j, i \in A, j \in \{0, 1\}\). We denote this code with \(V_X\).

Let \(V\) be a binary block-code and \(w_x = x_1x_2...x_n \in V, w_y = y_1y_2...y_n \in V\) be two codewords. On \(V\) we can define the following partial order relation:
\[w_x \preceq w_y \text{ if and only if } y_i \leq x_i, i \in \{1, 2, ..., n\}.\] (1.1)

In the paper [7], the authors constructed binary block-codes generated by BCK-functions. At the end of the paper they put the following question: \textit{for each binary block-code \(V\), there is a BCK-function which determines \(V\)?} The
answer of this question is partial affirmative, as we can see in Theorem 2.2 and Theorem 2.9.

2. Main results

Let \((X, \leq)\) be a finite partial ordered set with the minimum element \(\theta\). We define the following binary relation "\(*\)" on \(X\):

\[
\begin{align*}
\theta * x &= x \text{ and } x * x = \theta, \forall x \in X; \\
x * y &= \theta, \text{ if } x \leq y, x, y \in X; \\
x * y &= x, \text{ if } y < x, x, y \in X; \\
x * y &= y, \text{ if } x \in X \text{ and } y \in X \text{ can't be compared.}
\end{align*}
\]

(2.1.)

**Proposition 2.1.** With the above notations, the algebra \((X, *, \theta)\) is a non-commutative and non-implicative BCK-algebra. □

If the above BCK-algebra has \(n\) elements, we will denote it with \(C_n\).

Let \(V\) be a binary block-code with \(n\) codewords of length \(n\). We consider the matrix \(M_V = (m_{i,j})_{i,j \in \{1,2,...,n\}} \in M_n(\{0,1\})\) with the rows consisting of the codewords of \(V\). This matrix is called the matrix associated to the code \(V\).

**Theorem 2.2.** With the above notations, if the codeword \(11...1\) is in \(V\) and \(n\)-time the matrix \(M_V\) is upper triangular with \(m_{ii} = 1\), for all \(i \in \{1,2,...,n\}\), there are a set \(A\) with \(n\) elements, a BCK-algebra \(X\) and a BCK-function \(f : A \to X\) such that \(f\) determines \(V\).

**Proof.** We consider on \(V\) the lexicographic order, denoted by \(\leq_{\text{lex}}\). It results that \((V, \leq_{\text{lex}})\) is a totally ordered set. Let \(V = \{w_1, w_2, ..., w_n\}\), with \(w_1 \geq_{\text{lex}} w_2 \geq_{\text{lex}} ... \geq_{\text{lex}} w_n\). From here, we obtain that \(w_1 = 11...1\) and \(w_n = 00...01\) . On \(V\) we define a partial order \(\leq\) as in Remark 1.4. Now, \((V, \leq)\)

\[
\begin{align*}
\text{\text{(n-1)-time}}\quad w_1 &= w_i, i \in \{1,2,...,n\}. \text{ We remark that } w_1 = \theta \\
\text{\text{(n-1)-time}}\quad w_n &= 00...01. \text{ On } V \text{ we define a partial order } \leq \text{ as in Remark 1.4. Now, } (V, \leq)
\end{align*}
\]

is a partial ordered set with \(w_1 \leq w_i, i \in \{1,2,...,n\}\). We remark that \(w_1 = \theta\) is the "zero" in \((V, \leq)\) and \(w_n\) is a maximal element in \((V, \leq)\). We define on \((V, \leq)\) a binary relation "\(*\)" as in Proposition 2.1. It results that \(X = (V, *, w_1)\) becomes a BCK-algebra and \(V\) is isomorphic to \(C_n\) as BCK-algebras. We consider \(A = V\) and the identity map \(f : A \to V, f(w) = w\) as a BCK-function. The decomposition of \(f\) provides a family of maps \(V_{\leq} = \{f_r : A \to \{0,1\} / f_r(x) = 1\text{, if and only if } r * f(x) = \theta, \forall x \in A, r \in X\}\). This family is the binary block-code \(V\) relative to the order relation \(\leq\). Indeed, let \(w_k \in V, 1 < k < n, w_k = 00...0x_{i_k}...x_{i_k}, \ x_{i_k}...x_{i_k} \in \{0,1\}\). If \(x_{i_k} = 0\), it results that \(w_k \leq w_i, w_k \leq w_i, w_k \leq w_i\) and \(w_k \leq w_i, \theta\). If \(x_{i_k} = 1\), we obtain that \(w_i \leq w_k\) or \(w_i \leq w_k\) and \(w_k\) can't be compared, therefore \(w_k \leq w_i, \theta\). □
Remark 2.3. Using technique developed in [7], we remark that a BCK-algebra determines a unique binary block-code, but a binary block-code as in Theorem 2.2 can be determined by two or more algebras (see Example 3.1). If two BCK-algebras, \( A_1, A_2 \) determine the same binary block-code, we call them code-similar algebras, denoted by \( A_1 \sim A_2 \). We denote by \( \mathcal{C}_n \) the set of the binary block-codes of the form given in the Theorem 2.2.

Remark 2.4. If we consider \( \mathcal{B}_n \), the set of all finite BCK-algebras with \( n \) elements, then the relation code-similar is an equivalence relation on \( \mathcal{B}_n \). Let \( \mathcal{Q}_n \) be the quotient set. For \( V \in \mathcal{E}_n \), an equivalent class in \( \mathcal{Q}_n \) is \( \hat{V} = \{ B \in \mathcal{B}_n / B \) determines the binary block-code \( V \} \).

Proposition 2.5. The quotient set \( \mathcal{Q}_n \) has \( 2^{(n-1)(n-2)/2} \) elements, the same cardinal as the set \( \mathcal{C}_n \).

Proof. We will compute the cardinal of the set \( \mathcal{C}_n \). For \( V \in \mathcal{E}_n \), let \( M_V \) be its associated matrix. This matrix is upper triangular with \( m_{ii} = 1 \), for all \( i \in \{ 1, 2, ..., n \} \). We calculate in how many different ways the rows of such a matrix can be written. The second row of the matrix \( M_V \) has the form \((0, a_3, ..., a_n)\), where \( a_3, ..., a_n \in \{0, 1\} \). Therefore, the number of different rows of this type is \( 2^{n-2} \) and it is equal with the number of functions from a set with \( n-2 \) elements to the set \( \{0, 1\} \). The third row of the matrix \( M_V \) has the form \((0, 0, 1, a_4, ..., a_n)\), where \( a_4, ..., a_n \in \{0, 1\} \). In the same way, it results that the number of different rows of this type is \( 2^{n-3} \). Finally, we get that the cardinal of the set \( \mathcal{C}_n \) is \( 2^{n-2}2^{n-3}...2 = 2^{(n-1)(n-2)/2} \). □

Remark 2.6. If \( \mathcal{N}_n \) is the number of all finite non-isomorphic BCK-algebras with \( n \) elements, then \( \mathcal{N}_n \geq 2^{(n-1)(n-2)/2} \).

Remark 2.7. 1) Let \( V_1, V_2 \in \mathcal{E}_n \) and \( M_{V_1}, M_{V_2} \) be the associated matrices. We denote by \( r_j^{V_i} \) a row in the matrix \( M_{V_i}, i \in \{ 1, 2 \}, j \in \{ 1, 2, ..., n \} \). On \( \mathcal{E}_n \), we define the following totally ordered relation

\[ V_1 \succeq_{lex} V_2 \text{ if there is } i \in \{ 2, 3, ..., n \} \text{ such that } r_i^{V_1} = r_i^{V_2}, ..., r_{i-1}^{V_1} = r_{i-1}^{V_2} \text{ and } r_i^{V_1} \succeq_{lex} r_i^{V_2}, \]

where \( \succeq_{lex} \) is the lexicographic order.

2) Let \( V_1, V_2 \in \mathcal{E}_n \) and \( M_{V_1}, M_{V_2} \) be the associated matrices. We define a partially ordered on \( \mathcal{E}_n \)

\[ V_1 \preceq V_2 \text{ if there is } i \in \{ 2, 3, ..., n \} \text{ such that } r_i^{V_1} = r_i^{V_2}, ..., r_{i-1}^{V_1} = r_{i-1}^{V_2} \text{ and } r_i^{V_1} \preceq r_i^{V_2}, \]

where \( \preceq \) is the order relation given by the relation (1.1).

3) Let \( \Theta = (\theta_{ij})_{i,j \in \{ 1, 2, ..., n \}} \in \mathcal{M}(\{ 0, 1 \}) \) be a matrix such that \( \theta_{ij} = 1 \), \( i \leq j \), for all \( i,j \in \{ 1, 2, ..., n \} \) and \( \theta_{ij} = 0 \) in the rest. It results that the code \( \Omega \), such that \( M_0 = \Theta \), is the minimum element in the partial ordered set \( (\mathcal{E}_n, \preceq) \), where elements in \( \mathcal{E}_n \) are descending ordered relative to \( \succeq_{lex} \) defined in 1). Using the multiplication "*" given in relation (2.1) and Proposition 2.1, we obtain that \( (\mathcal{E}_n, \ast, \Omega) \) is a non-commutative and non-implicative BCK-algebra.
Due to the above remarks and relation (2.1), this BCK-algebra determines a binary block-code $V_{\mathcal{C}}$ of length $2^{\frac{n-1}{2}}$. Obviously, $\tilde{V}_{\mathcal{C}} \in \mathbb{C}_{2^{\frac{n-1}{2}(n-2)}}$.

Proposition 2.8. Let $A = (a_{i,j})_{i\in\{1,2,\ldots,n\}, j\in\{1,2,\ldots,m\}} \in M_{n,m}(\{0,1\})$ be a matrix with rows lexicographic ordered in the descending sense. Starting from this matrix, we can find a matrix $B = (b_{i,j})_{i,j\in\{1,2,\ldots,q\}} \in M_{q}(\{0,1\})$, $q = n + m$, such that $B$ is an upper triangular matrix, with $b_{ii} = 1$, $\forall i \in \{1,2,\ldots,q\}$ and $A$ becomes a submatrix of the matrix $B$.

Proof. We insert in the left side of the matrix $A$ (from the right to the left) the following $n$ new columns of the form $\underbrace{1111, 11110, \ldots, 1000}_{n \text{ rows}}$. It results a new matrix $D$ with $n$ rows and $n + m$ columns. Now, we insert in the bottom of the matrix $D$ the following $m$ rows: $\underbrace{0001, 1111, \ldots, 0001}_{n \text{ rows}}$. We obtained the asked matrix $B$. $\square$

Theorem 2.9. With the above notations, we consider $V$ a binary block-code with $n$ codewords of length $m, n \neq m$, or a block-code with $n$ codewords of length $n$ such that the codeword $\underbrace{11\ldots1}_{n \text{ time}}$ is not in $V$, or a block-code with $n$ codewords of length $n$ such that the matrix $M_V$ is not upper triangular. There are a natural number $q \geq \max\{m,n\}$, a set $A$ with $m$ elements and a BCK-function $f : A \to \mathcal{C}_q$ such that the obtained block-code $V_{\mathcal{C}}$ contains the block-code $V$ as a subset.

Proof. Let $V$ be a binary block-code, $V = \{w_1,w_2,\ldots,w_n\}$, with codewords of length $m$. We consider the codewords $w_1,w_2,\ldots,w_n$ lexicographic ordered, $w_1 \geq_{lex} w_2 \geq_{lex} \ldots \geq_{lex} w_n$. Let $M \in M_{n,m}(\{0,1\})$ be the associated matrix with rows the $w_1,\ldots,w_n$ in this order. Using Proposition 2.8, we can extend the matrix $M$ to a square matrix $M' \in M_q(\{0,1\}), q = m + n$, such that $M' = (m'_{i,j})_{i,j\in\{1,2,\ldots,q\}}$ is an upper triangular matrix with $m'_{ii} = 1$, for all $i \in \{1,2,\ldots,q\}$. If the first line of the matrix $M'$ is not $\underbrace{11\ldots1}_{n \text{ time}}$, then we insert the row $\underbrace{11\ldots1}_{q+1}$ as a first row and the column $\underbrace{00\ldots0}_{q}$ as a first column. Applying Theorem 2.2 for the matrix $M'$, we obtain a BCK-algebra $C_q = \{x_1,\ldots,x_q\}$, with $x_1 = \theta$ the zero of the algebra $C_q$ and a binary block-code $V_{\mathcal{C}}$. Assuming that the initial columns of the matrix $M$ have in the new matrix $M'$ positions $i_{j_1},i_{j_2},\ldots,i_{j_m} \in \{1,2,\ldots,q\}$, let $A = \{x_{j_1},x_{j_2},\ldots,x_{j_m}\} \subseteq C_q$. The BCK-function $f : A \to C_q, f(x_{j_i}) = x_{j_i}, i \in \{1,2,\ldots,m\}$, determines the binary block-code $V_{\mathcal{C}}$, such that $V \subseteq V_{\mathcal{C}}$. $\square$

3. Examples
Example 3.1. Let $V = \{0110, 0010, 1111, 0001\}$ be a binary block code. Using the lexicographic order, the code $V$ can be written $V = \{1111, 0110, 0010, 0001\} = \{w_1, w_2, w_3, w_4\}$. From Theorem 2.2, defining the partial order $\preceq$ on $V$, we remark that $w_1 \preceq w_i, i \in \{2, 3, 4\}$, $w_2 \preceq w_3, w_2$ can’t be compared with $w_4$ and $w_3$ can’t be compared with $w_4$. The operation "∗" on $V$ is given in the following table:

| ∗   | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|-----|-------|-------|-------|-------|
| $w_1$ | $w_1$ | $w_1$ | $w_1$ | $w_1$ |
| $w_2$ | $w_2$ | $w_1$ | $w_1$ | $w_2$ |
| $w_3$ | $w_3$ | $w_3$ | $w_1$ | $w_3$ |
| $w_4$ | $w_4$ | $w_4$ | $w_4$ | $w_1$ |

Obviously, $V$ with the operation "∗" is a BCK-algebra.

We remark that the same binary block code $V$ can be obtained from the BCK-algebra $(A, \circ, \theta)$ with BCK-function, $f: V \rightarrow V, f(x) = x$. From the associated Cayley multiplication tables, it is obvious that the algebras $(A, \circ, \theta)$ and $(V, *, w_1)$ are not isomorphic. From here, we obtain that BCK-algebra associated to a binary block-code as in Theorem 2.2 is not unique up to an isomorphism. We remark that the BCK-algebra $(A, \circ, \theta)$ is commutative and non implicative and BCK-algebra $(V, *, w_1)$ is non commutative and non implicative. Therefore, if we start from commutative BCK-algebra $(A, \circ, \theta)$ to obtain the code $V$, as in [7], and then we construct the BCK-algebra $(V, *, w_1)$, as in Theorem 2.2, the last obtained algebra lost the commutative property even that these two algebras are code-similar.

Example 3.2. Let $X$ be a non empty set and $\mathfrak{F} = \{f: X \rightarrow \{0, 1\} / f$ function\}. On $\mathfrak{F}$ is defined the following multiplication

$$(f \circ g)(x) = f(x) - \min\{f(x), g(x)\}, \forall x \in X.$$  

$(\mathfrak{F}, \circ, 0)$, where $0(x) = 0, \forall x \in X$, is an implicative BCK-algebra([12], Theorem 3.3 and Example 1).

If $X$ is a set with three elements, we can consider $\mathfrak{F} = \{000, 001, 010, 011, 100, 101, 110, 111\}$ the set of binary block-codes of length 3. We have the following multiplication table.
We obtain the following binary block-code
\[ V = \{11111111, 01010101, 00110011, 00010001, 00001111, 00000101, 00000011, 00000001\}, \]
with the elements lexicographic ordered in the descending sense. From Theorem 2.2, defining the partial order \( \preceq \) on \( V \) and the multiplication "\( \ast \)”, we have that \( (V, \ast, 11111111) \) is a non-implicative BCK-algebra and the algebras \( (V, \ast, 11111111) \) and \( (\mathcal{F}, \circ, 0) \) are code-similar.

**Example 3.3.** Let \( V = \{11110, 10010, 10011, 00000\} \) be a binary block code. Using the lexicographic order, the code \( V \) can be written
\[ V = \{11110, 10011, 10010, 00000\} = \{w_1, w_2, w_3, w_4\}. \]
Let \( M_V \in M_{4,5}(\{0,1\}) \) be the associated matrix,
\[
M_V = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
Using Proposition 2.8, we construct an upper triangular matrix, starting from the matrix \( M_V \). It results the following matrices:
\[
D = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
and
\[
B = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
Since the first row is not \( 11\ldots1 \), using Theorem 2.8, we insert a new row \( 11\ldots1 \) as a first row and a new column \( 10\ldots0 \) as a first column. We obtain the
The following matrix: $B' = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}$.

The binary block-code $W = \{w_1, \ldots, w_{10}\}$, whose codewords are the rows of the matrix $B'$, determines a BCK-algebra $(X, *, w_1)$. Let $A = \{w_6, w_7, w_8, w_9, w_{10}\}$ and $f : A \to X, f(w_i) = w_i, i \in \{6, 7, 8, 9, 10\}$ be a BCK-function which determines the binary block-code $U = \{11111, 11110, 10011, 10010, 00000, 01111, 00111, 00011, 00001\}$. The code $V$ is a subset of the code $U$.

**Conclusions.** In this paper, we proved that to each binary block-code $V$ we can associate a BCK-algebra $X$ such that the binary block-code generated by $X, V_X$, contains the code $V$ as a subset. In some particular case, we have $V_X = V$.

From Example 3.1 and 3.2, we remark that two code-similar BCK-algebras can’t have the same properties. For example, some algebras from the same equivalence class can be commutative and other non-commutative or some algebras from the same equivalence class can be implicative and other non-implicative. As a further research, will be very interesting to study what common properties can have two code-similar BCK-algebras.

Due to this connection of BCK-algebras with Coding Theory, we can consider the above results as a starting point in the study of new applications of these algebras in the Coding Theory.

**Acknowledgements**

The author thanks Professor Arsham Borumand Saeid for having brought [7] to my attention.

**References**

[1] S. Abdullah, A. F. Ali, JIFS, Applications of $N$-structures in implicative filters of BE-algebras, will appear in J. Intell. Fuzzy Syst., DOI 10.3233/IFS-141301.

[2] J. S. Han, H. S. Kim, J. Neggers, On linear fuzzifications of groupoids with special emphasis on BCK-algebras, J. Intell. Fuzzy Syst., 24(1)(2013), 105-110.

[3] S. M. Hong, Y. B. Jun, M. A. Öztürk, Generalizations of BCK-algebras, Sci. Math. Jpn. Online, 8(2003), 549–557
[4] Y. Imai, K. Iseki, *On axiom systems of propositional calculi*, Proc. Japan Academy, *42*(1966), 19-22.

[5] K. Iséki, S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Jpn. *23*(1978), 1-26.

[6] Young Bae Jun, *Soft BCK/BCI-algebras*, Comput. Math. Appl., *56*(2008), 1408–1413.

[7] Y. B. Jun, S. Z. Song, *Codes based on BCK-algebras*, Inform. Sciences., *181*(2011), 5102-5109.

[8] H. S. Kim and Y. H. Kim, *On BE-algebras*, Sci. Math. Jpn. Online e-2006(2006), 1199-1202.

[9] K. H. Kim, Y. H. Yon, *Dual BCK-algebra and MV -algebra*, Sci. Math. Jpn., *66*(2007), 247-253.

[10] A.B. Saeid, *Redefined fuzzy subalgebra (with thresholds) of BCK/BCI-algebras*, Iran. J. Math. Sci. Inform, *4*(2)(2009), 9-24.

[11] A. B. Saeid, M. K. Rafsanjani, D. R. Prince Williams, *Another Generalization of Fuzzy BCK/BCI-Algebras*, Int. J. Fuzzy Syst., 14(1)(2012), 175-184.

[12] Z. Samaei, M. A. Azadani, L. Ranjbar, *A Class of BCK-Algebras*, Int. J. Algebra, 5(28)(2011), 1379 - 1385.

[13] X. Xin, Y. Fu, *Some results of convex fuzzy sublattices*, J. Intell. Fuzzy Syst., 27(1)(2014), 287-298.

Cristina FLAUT
Faculty of Mathematics and Computer Science, Ovidius University,
Bd. Mamaia 124, 900527, CONSTANTA, ROMANIA

http://cristinaflaut.wikispaces.com/ http://www.univ-ovidius.ro/math/
e-mail: cflaut@univ-ovidius.ro; cristina.flaut@yahoo.com