Generalized Hirota bilinear identity  
and integrable q-difference  
and lattice hierarchies

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Abstract

Hirota bilinear identity for Cauchy-Baker-Akhieser (CBA) kernel is introduced as a  
basic tool to construct integrable hierarchies containing lattice and q-difference times.  
Determinant formula for the action of meromorphic function on CBA kernel is derived. This  
formula gives opportunity to construct generic solutions for integrable lattice equations.

1 Introduction.

This paper is a sequel of the work [1]. We introduce here generalized Hirota bilinear identity  
and develop a unified approach to continuous, lattice and q-difference variables in integrable  
hierarchies.

Generalized Hirota bilinear identity uses the function of two complex variables  
ψ(λ,μ),  
which is an analogue of Cauchy-Baker-Akhieser kernel on the Riemann surface [2] in Segal-  
Wilson type Grassmannian context [3]. In frame of ̄∂-dressing method this function is con-  
ected with the solution of nonlocal ̄∂-problem normalized by (λ − μ)−1 (see [4], [5]). The  
action of meromorphic function on the Grassmannian can be explicitly found in terms of the  
function ψ(λ,μ) in the form of elegant determinant formula, which has close ties with the  
Miwa’s formula for the τ-function in the model of free fermions [6] and Wronskian formulae  
for the composition of Darboux transformations.

Lattice versions and q-deformations of KP hierarchy and N-waves system are constructed.  
The determinant formula gives generic solution (i.e. solution corresponding to arbitrary Grass-  
mannian point) in explicit form. The same is valid for q-difference equations, but in this case,  
however, the determinants of infinite-dimensional matrices appear in the formula.

2 Generalized Hirota identity.

Due to the limited volume of this publication, we will take quite a formal start, introduc-  
ing from the beginning generalized Hirota bilinear identity as a basic tool for the following  
consideration

\[ \int_{\partial G}\chi(\lambda,\nu;g_1)g_1(\nu)g_2^{-1}(\nu)\chi(\nu,\mu;g_2)d\nu = 0, \quad (1) \]

Here χ(λ,μ;g) is a function of two complex variables λ,μ ∈ G and a functional of the group  
element g defining the dynamics (will be specified later), G is some set of domains of the

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complex plane, the integration goes over the boundary of $G$. By definition, the function $\chi(\lambda, \mu)$ possesses the following analytical properties: as $\lambda \to \mu$, $\chi \to (\lambda - \mu)^{-1}$ and $\chi(\lambda, \mu)$ is analytic function of two variables $\lambda, \mu$ for $\lambda \neq \mu$.

In another form, more similar to standard Hirota bilinear identity, the identity (1) can be written as

$$\int_{\partial G} \psi(\nu, \lambda, g_1) \psi(\lambda, \mu, g_2) d\lambda = 0,$$

(2)

where

$$\psi(\lambda, \mu, g) = g^{-1}(\lambda) \chi(\lambda, \mu, g) g(\mu)$$

The place of the function $\chi$ in the Segal-Wilson Grassmannian approach is outlined in [1]. In frame of $\bar{\partial}$-dressing method the analogue of this function is well-known (see [5]), in algebro-geometric technique the function $\psi(\lambda, \mu)$ corresponds to Cauchy-Baker-Akhieser kernel on the Riemann surface (see [2]). So the function $\psi(\lambda, \mu)$ introduced here could be called CBA kernel for Segal-Wilson type Grassmannian. Let us consider two linear spaces $W(g)$ and $W'(g)$ defined by the function $\chi(\lambda, \mu)$ via equations connected with the identity (1)

$$\int_{\partial G} \chi(\lambda, \nu; g) f(\nu; g) d\nu = 0,$$

(3)

$$\int_{\partial G} h(\nu; g) \chi(\nu, \mu; g) d\nu = 0,$$

(4)

here $f(\lambda) \in W$, $h(\lambda) \in W'$; $f(\lambda)$, $g(\lambda)$ are defined in $\bar{G}$. It is easy to check using the analytical properties of the function $\chi(\lambda, \mu)$ that these spaces possess the following properties

1. $W, W'$ contains a meromorphic function with arbitrary given divisor of poles in $G$ (completeness)
2. $W, W'$ is transversal to the space of holomorphic functions in $G$ (transversality).

So the function $\chi(\lambda, \mu)$ defines simultaneously a point and a dual point of ‘analytical Grassmannian’, where by ‘analytical Grassmannian’ we understand the set of linear spaces $W$ of functions in $\bar{G}$ possessing the properties 1,2. The function $\chi(\lambda, \mu)$ plays a role of the basic function in both of the spaces $W, W'$ (for $W$ with respect to the first variable, for $W'$ with respect to the second variable). It follows from the definition of linear spaces $W, W'$ that

$$f(\lambda) = 2\pi i \iint_{G} \left( \frac{\partial}{\partial \nu} f(\nu) \right) \chi(\lambda, \nu) d\nu \wedge d\bar{\nu},$$

$$g(\mu) = -2\pi i \iint_{G} \left( \frac{\partial}{\partial \nu} g(\nu) \right) \chi(\nu, \mu) d\nu \wedge d\bar{\nu},$$

(5)

these formulae in some sense provide an expansion of the functions $f, g$ in terms of the basic function $\chi(\lambda, \mu)$.

The dynamics of linear spaces $W, W'$ looks very simple

$$W(g) = gW_0; \quad W'(g) = g^{-1}W'_0,$$

(6)

here $W_0 = W(g = 1)$, $W'_0 = W'(g = 1)$ (the formulae (6) follow from identity (1) and the formulae (5)).

3 Determinant formula for action of rational $g(\lambda)$ on the CBA kernel

The action of arbitrary meromorphic function $g(\lambda)$, having in $G$ equal number of zeroes and poles (taking multiplicity into account), on the function $\chi(\lambda, \mu)$ can be found in explicit
form. This result was obtained in our work [1], using the properties of the space $W$. In the present work we would like to start from Hirota bilinear identity (1) and introduce a compact determinant formula for this action.

Let $g(\lambda)$ be some meromorphic function in $G$ having simple poles at the set of points $z_i$ and simple zeroes at the set of point $\hat{z}_i$, $0 < i < N$, and let the basic function $\chi_0(\lambda, \mu)$ be defined at the initial point (i.e. for $g(\lambda) = 1$). The problem is to find the solution of identity (12)

$$\int_{\partial G} \chi_0(\lambda, \nu) g^{-1}(\nu) \chi(\nu, \mu; g) d\nu = 0,$$

having analytical properties specified in the definition (see [1]). The answer is quite simple

$$\chi(g) = g(\lambda) g^{-1}(\mu) \frac{\Delta(\lambda, z_1, \ldots, z_N; \mu, \hat{z}_1, \ldots, \hat{z}_N)(\chi_0(\lambda, \mu))}{\Delta(z_1, \ldots, z_N; \hat{z}_1, \ldots, \hat{z}_N)(\chi_0(\lambda, \mu))} = \det(f_{ij}) := \det(f(z_i, \hat{z}_j)).$$

The function $\chi(\lambda, \mu, g)$ defined by the formula (3) satisfies the equation (7) and possesses necessary analytic properties. The most simple way to derive this formula is to use the consequence of the Hirota bilinear identity (1)

$$\int_{\partial G \times \partial G} \chi_0(\lambda, \nu) g^{-1}(\nu) \chi(\nu, \eta; g) g(\eta) \chi_0(\eta, \eta; g) d\eta d\nu = 0.$$  

(10)

For the arbitrary meromorphic function $g(\lambda)$, having poles with the multiplicity $n_i$ at the set of points $z_i$ and zeroes with the multiplicity $\hat{n}_i$ at the set of point $\hat{z}_i$

$$\chi(g) = g(\lambda) g^{-1}(\mu) \frac{\Delta((\lambda, 1), (z_1, n_1), \ldots, (z_N, n_N); (\mu, 1), (\hat{z}_1, \hat{n}_1), \ldots, (\hat{z}_N, \hat{n}_N))(\chi_0(\lambda, \mu))}{\Delta((z_1, 1), \ldots, (z_N, n_N); (\hat{z}_1, \hat{n}_1), \ldots, (\hat{z}_N, \hat{n}_N))(\chi_0(\lambda, \mu))},$$

(11) where

$$\Delta((z_1, n_1), \ldots, (z_N, n_N); (\hat{z}_1, \hat{n}_1), \ldots, (\hat{z}_N, \hat{n}_N))(\chi(\lambda, \mu)) = \det(\chi_{IJ}),$$

(12) the matrix $\chi_{IJ}$ consists of the elements

$$\left( \frac{\partial^{n_i-1} \partial^{\hat{n}_j-1}}{\partial \lambda^{n_i-1} \partial \mu^{\hat{n}_j-1}} \chi \right)(z_i, \hat{z}_j) = \chi_{IJ},$$

here $0 < n_i \leq \lambda$, $0 < \hat{n}_j \leq \hat{\lambda}$, $I = \sum_{k=0}^{i-1} n_i + \lambda_i$, $J = \sum_{k=0}^{i-1} \hat{n}_j + \hat{\lambda}_j$.

The function $\chi(\lambda, \mu)$ is connected with the $\tau$-function by the formula

$$\chi(\nu, \mu) = \frac{\tau(g \times \left( \frac{\lambda - \nu}{\lambda - \mu} \right))}{\tau(g)(\nu - \mu)}$$

(13)

We do not give the direct proof of the formula (13), though it seems more or less evident if $G$ is a unit circle. But for not 1-connected $G$ or $G$ consisting of several disconnected domains the interpretation of the right part is not so trivial, it goes beyond the standard Segal-Wilson approach. The formula (13) gives an opportunity to obtain Miwa’s formula [8] as a special case of the formula (11).
4 Introduction of lattice and q-difference variables to integrable hierarchies

A dependence of the function \(\chi(\lambda, \mu)\) on dynamical variables is hidden in the function \(g(\lambda)\). Usually these variables are continuous space and time variables, but it is possible also to introduce discrete (lattice) and q-difference variables into identity (1). We will consider the following functions

\[
g^{-1}_i = \exp(K_i x_i); \quad \frac{\partial}{\partial x_i} g^{-1} = K_i g^{-1}, \tag{14}
\]

\[
g^{-1}_i = (1 + l_i K_i)^{n_i}; \quad \Delta_i g^{-1} = \frac{g^{-1}(n_i + 1) - g^{-1}(n_i)}{l_i} = K_i g^{-1}, \tag{15}
\]

\[
g^{-1}_i = e_q(K_i y_i); \quad \delta^q_i g^{-1} = \frac{g^{-1}(q y_i) - g^{-1}(y_i)}{(q - 1) y_i} = K_i g^{-1}. \tag{16}
\]

Here \(K_i(\lambda)\) are rational functions. The function (14) introduces a dependence on continuous variable \(x_i\), the function (15) – on discrete variable \(n_i\) (the lattice parameter \(l_i\) may depend on \(n_i\), so it is possible to consider lattices with the changing step of the lattice) and the function (16) defines a dependence of \(\chi(\lambda)\) on the variable \(y_i\) (we will call it a q-difference variable).

The function \(e_q(y)\) has a representation

\[
e_q(y) = \left( \prod_{n=1}^{\infty} (1 + q^n y (q - 1)) \right)^{-1}, \tag{17}
\]

here we suggest that \(|q| < 1\).

To introduce a dependence on several variables (may be of different type), one should consider a product of corresponding functions \(g(\lambda)\) (all of them commute).

The structure of the functions (14-16) imposes some limitations on the choice of \(G\) in order to apply formula (11) to construct solutions of corresponding lattice and q-difference equations. Typically \(G\) should contain all zeroes and poles of the considered class of functions \(g\). In the case of function (15) it is just a finite set of neighborhoods of points, in the case (16), \(q \in \mathbb{R}\) \(G\) should contain neighborhoods of some curves in the complex plane, where group elements (16) have zeroes. Every function \(K_i(\lambda)\) may be defined on its own copy of the complex plane; then it needs a definition on another’s copy (we will use zero value on another’s copy). However, the choice of \(G\) is not so important for construction of equations.

5 Construction of equations

Equations in the right part of (14-16) and the boundary condition \(g(0) = 1\) characterize the corresponding functions (and give a definition of \(e_q(y)\)). These equations play a crucial role in the algebraic scheme of constructing integrable equations.

This scheme is based on the assumption of solvability of the problem (1) for some class of functions \(g(\lambda)\), on transversality property and on the existence of special operators, which transform \(W\) into itself.

Indeed, the relation (3) implies that if \(\chi(x, n, y, \lambda) \in W(x, n, y)\), then the functions

\[
D^c_i \chi = \frac{\partial}{\partial x_i} \chi + K_i(\lambda) \\
D^q_i \chi = \Delta_i \chi + T^q_i \chi K_i(\lambda) \\
D^q_i \chi = \delta^q_i \chi + T^q_i \chi K_i(\lambda) \tag{18}
\]
also belong to $W$, where $Tf(n) = f(n + 1)$, $Tqf(y) = f(qy)$. We can multiply the solution from the left by the arbitrary matrix function of additional variables, $u(x, n, y)\chi \in W$. So the operators $\delta_i^q$ are the generators of Zakharov-Manakov ring of operators, that transform $W$ into itself.

Combining this property with the transversality property, one obtains the differential relations between the coefficients of expansion of functions $\chi(x, n, y, \lambda)$ into powers of $(\lambda - \lambda_p)$ at the poles of $K_i(\lambda)$.

The derivation of equations in this case is completely analogous to the continuous case.

5.1 Lattice and q-difference KP hierarchy

The following derivation will be conducted for q-difference case, to get the difference case you should just change $\delta_i^q$ for $\Delta_i$ and $T_i$ for $T_i^q$.

The KP hierarchy corresponds to the special choice of the functions $K_i(\lambda)$: $K_i(\lambda) = (l_i)^{-1}\lambda^{-i}$. The transversality of the space $W$ implies linear equations (below we transformed operators $D_i^q$ to $\delta_i^q$ by the substitution $\psi(\lambda) = g(\lambda)\chi(\lambda, 0)$)

$$(\delta_i^q - \delta_i^{qj})\psi(\lambda) = \sum_{k=0}^{i-1} u_k(y)\delta_i^{qk}\psi(\lambda). \quad (19)$$

The q-difference KP hierarchy is the set of compatibility conditions for these linear equations.

5.2 q-difference N-waves equations

An arbitrary rational function $K_i(\lambda)$ may be treated as degenerate case of the function having simple poles. So generic equations in the hierarchy correspond to the operators $D_i$ with simple and distinct poles. These equations play a fundamental role, they can be constructed explicitly (see for the continuous case).

We use the functions $K_i(\lambda)$:

$$K_i(\lambda) = \sum_{\alpha=1}^{n_i} \frac{a_i^\alpha}{\lambda - \lambda_i^\alpha}, \quad (20)$$

where $a_i^\alpha, \lambda_i^\alpha \in \mathbb{C}, 1 \leq \alpha \leq n_i, \lambda_i^\alpha \neq \lambda_j^\beta$. Then due to transversality of the space $W$ the function $\chi(\lambda, \mu)$ satisfy the relations

$$D_i^q\chi(\lambda, \mu, y) - K_i(\mu)\chi(\lambda, \mu, y) - (T_i^q\chi(\lambda_i^\alpha, \mu, y))\chi(\lambda, \lambda_i^\alpha, y) = 0, \quad (21)$$

summation over $\alpha$ is understood. The other way to derive this equation is to use Hirota bilinear identity.

The substitution of the values $\lambda = \lambda_k^\gamma, \mu = \mu_j^\beta, i \neq j \neq k \neq i$ to (21) yields the equation

$$\delta_i^q\chi^{\beta\gamma}_{jk} + K_i(\lambda_i^\gamma)T_i^q\chi^{\beta\gamma}_{jk} - K_i(\lambda_j^\beta)\chi^{\beta\gamma}_{jk} - (T_i^q\chi^{\beta\gamma}_{ji})a_i^\alpha\chi^{\alpha\gamma}_{ik} = 0, \quad (22)$$

summation over $\alpha$ is understood. If different permutations $ijk$ and substitutions of the indices $\beta, \gamma$ are taken into account, (22) is a closed set of equations for the functions $\chi^{\beta\alpha}_{ji}(n)$ for chosen $ijk$.

Expression gives the solutions for the system of equations (22) and also for the lattice KP hierarchy at the arbitrary point of the lattice. The function $\chi_0(\lambda, \mu)$ defines the point of the Grassmannian and plays the role of spectral data.
The function $g(\lambda)$ (16), taking into account the representation (17), has infinite number of zeroes and poles. So application of the formula (11) to q-difference case requires the use of determinants of infinite-dimensional matrices.

We would like also to consider a simple example in which every function $K_i(\lambda)$ is defined on its own copy of the complex plane. Let’s consider the lattice case. We take a set of unit disks $D_i$ as $G$. The functions $K_i(\lambda)$ are chosen in the form

\[ K_i(\lambda) = \frac{1}{\lambda}, \quad \lambda \in D_i; \]
\[ K_i(\lambda) = 0, \quad \lambda \notin D_i. \]

We could use Zakharov-Manakov ring of operators to derive the equations, but it is easy in this case to obtain them directly from the identity (1). Performing the integration in the formula

\[ \int_{\partial G} \chi(\lambda, \nu, n)(1 + K_i(\nu)l_i)T_i\chi(\nu, \mu, n)d\mu = 0, \tag{23} \]

one obtains

\[ (1 + K_i(\lambda)l_i)T_i\chi(\lambda, \mu, n) - \chi(\lambda, \mu, n)(1 + K_i(\mu)l_i) = \chi(\lambda, 0_i, n)l_iT_i\chi(0_i, \mu, n) = 0 \tag{24} \]

where $0_i$ is a zero point of the corresponding region. Taking the equation (24) at $\lambda = 0_j, \mu = 0_k, i \neq j \neq k \neq i$ we get

\[ \Delta_i\chi_{jk} = (T_i\chi_{ji})\chi_{ik} \tag{25} \]

where $\chi_{ik}(n) = \chi(0_k, 0_j, n)$ If different permutations $ijk$ are taken into account, (25) is a closed set of equations for the functions $\chi_{ji}(n)$.

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References

[1] L.V. Bogdanov, hep-th 9401080, Teor. i Mat. Fiz., 99, 177 (1994)

[2] P.G. Grinevich, A.Yu. Orlov, in Modern problems of quantum field theory, Springer-Verlag 1989

[3] G.Segal and G.Wilson, Loop groups and equations of KdV type, Publ. Math. I.H.E.S. 61 (1985) 1.

[4] V.E. Zakharov and S.V.Manakov, Zap. Nauchn. Sem. LOMI 133 (1984) 77. V.E. Zakharov and S.V. Manakov, Funk. Anal. Ego Prilozh. 19 (1985) 11.

[5] L.V. Bogdanov and S.V. Manakov, J. Phys. A.:Math. Gen. 21 (1988) L537.

[6] T. Miwa, Proc. Jap. Acad. 58A (1982) 9.