Angular momentum non-conserving decays in isotropic media

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Abstract

Various processes that are forbidden in the vacuum due to angular momentum conservation can occur in a medium that is isotropic and does not carry any angular momentum. We illustrate this by considering explicitly two examples. The first one is the decay of a spin-0 particle into a photon and another spin-0 particle, using a model involving the Yukawa interactions of the scalar particles with a charged fermion field. The second one involves the decay of a neutrino into another neutrino and a graviton, in the standard model of particle interactions augmented with the linearized gravitational couplings.

1 Introduction

Some physical processes that are not allowed to occur in the vacuum can occur in the presence of a background medium. We can classify such processes into two broad classes. The first one refers to those processes that are disallowed in the vacuum for kinematic reasons. One classic example of this type is the Cerenkov radiation of a charged particle, which can occur in a medium due to the fact that the photon dispersion relation is modified by the background effects. We are not concerned with processes of this type here.

The second class consists of processes that do not occur because the transition matrix element is zero in the vacuum. In general, whenever that is the case, it can be attributed to some conservation laws, which are in turn the consequences of the symmetries of the Lagrangian. It is common to refer to such process as being forbidden. Thus, when the medium is not invariant under the full symmetry group of the Lagrangian, the relevant transition matrix element can be non-zero when the background effects are taken into account.
In a recent paper [1] we considered a subset of processes in this class, namely those which are forbidden by helicity arguments, or angular momentum conservation. We specifically considered the radiative decay of a spin-0 particle into another spin-0 particle, the decay of a spin-1 particle into two photons, the gravitational decay of a spin-0 particle into another spin-0 particle and the gravitational decay of a spin-1/2 particle into another spin-1/2. By performing a form-factor analysis in each case, we reviewed the arguments that show that the amplitude for the process in the vacuum vanishes, and then demonstrated that the amplitude need not vanish if the process occurs in a medium, even if the medium is homogeneous and isotropic and therefore does not carry any net angular momentum.

The aim of the present work is to pursue this further to confirm that this is indeed the case, by computing the amplitude for such processes in viable models and verifying that it does not vanish due to some unexpected reason. Here we consider the radiative decay of a scalar particle and the gravitational decay of a neutrino as illustrative examples of that kind of process. In the scalar decay case, the model consists of two electrically neutral scalar fields coupled to a charged fermion field via Yukawa interactions, and the medium is assumed to consist of a thermal background of the charged fermions. In the neutrino case, the medium is a thermal background of electrons and, in order to include the graviton interactions, the Standard Model couplings are supplemented with the linearized gravitational couplings of the particles involved. In either case, the thermal background is parametrized by the Fermi-Dirac momentum distribution functions of the background particles in the usual way.

The fact that these processes are forbidden in the vacuum by angular momentum conservation arguments, implies that, in the medium, their angular distribution and differential decay rates have a distinctive form. This could lead to observable consequences in specific physical contexts despite the fact that there may exist other competing processes with comparable total rates.

In general, the presence of the medium modifies the dispersion relations and wave function normalization of the particles that participate in the process. Those corrections affect the kinematics, but their relative importance depends on the particular application and the physical context of the calculation. Here we assume that the situation is such that those corrections are negligible. However, the inclusion of those corrections in the calculation of the rates is straightforward [2], and it should be kept in mind that they may be important and need to be included in specific applications.

In Sec. 2 we consider in detail the scalar radiative decay. There we summarize the the form-factor analysis presented in Ref. [1], show that the on-shell amplitude depends only on one form factor, and the expressions for the total and the differential decay rates in terms of the form factor are given. We then carry out the calculation of the amplitude in a simple model involving a background of charged fermions, and the one-loop formula for the on-shell form factor is obtained in terms of integrals over the background fermion distribution functions. The integrals are evaluated explicitly for some particular cases of the distribution functions. The analogous calculations for the flavor-changing gravitational decay of a neutrino are presented in Sec. 3. Sec. 4 contains some general and concluding remarks.
2 Radiative decay of a spinless particle

2.1 Kinematical considerations

In this section, we consider the process

$$\phi(p) \rightarrow \phi'(p') + \gamma(q),$$

(2.1)

where, $\phi$ and $\phi'$ denote the scalar (spin-0) particles, $\gamma$ denotes the photon, and $p$, $p'$ and $q$ denote the corresponding momentum vectors. As mentioned in the Introduction, we neglect the effects of the medium on the dispersion relations and wave function normalizations in the calculation of the decay rate. While it should be kept in mind that in general those corrections must be taken into account in specific applications, they are not essential for our purposes here. Moreover, they can be included in the calculations that follow in a straightforward way if needed. Therefore for our purposes, we assume the vacuum on-shell relations

$$p^2 = m^2,$$

$$p'^2 = m'^2,$$

$$q^2 = 0.$$  

(2.2)

The medium is assumed to consist of a homogenous and isotropic thermal background of particles. Besides the thermodynamic variables such as temperature and chemical potentials, such a medium is characterized by its velocity four vector $v^\mu$.[3][4].

The amplitude can be written in the form

$$\mathcal{M} = \epsilon^{\mu}(q) j_\mu,$$

(2.3)

where $\epsilon^\mu(q)$ is the photon polarization vector which satisfies

$$q^\mu \epsilon_\mu(q) = 0,$$

(2.4)

and $j_\mu$ is the matrix element of the electromagnetic current, which satisfies the transversality condition

$$q^\mu j_\mu = 0.$$  

(2.5)

In general $j_\mu$ is a function of $p_\mu$ and $q_\mu$, which are the only independent momenta in the problem. When the process takes place in a medium, $j_\mu$ can also depend on $v_\mu$. In Ref. [1], it was pointed out that the most general form for the on-shell vertex function $j_\mu$ subject to the transversality condition of Eq. (2.5) is

$$j_\mu = a[p \cdot q \, v^\mu - q \cdot v \, p^\mu] + b \epsilon_{\mu\alpha\beta\gamma} p^\alpha q^\beta v^\gamma.$$  

(2.6)

This implies that the decay amplitude can be written as

$$\mathcal{M} = (a F^*_\mu\nu + b \tilde{F}^*_\mu\nu) v^\mu p^\nu,$$  

(2.7)
where
\[ F_{\mu \nu} = \epsilon_{\mu q \nu} - q_{\mu} \epsilon_{\nu}, \] (2.8)
and \( \tilde{F}_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \alpha \beta} F^{\alpha \beta} \) is the dual. The form factors \( a \) and \( b \) appearing in the amplitude are Lorentz invariant functions of \( p, q \) and \( v \).

The term associated with the form factor \( b \) is parity violating due to the presence of the Levi-Civita tensor. In a system in which the interactions are parity conserving and the background medium is parity symmetric, i.e., is not spin-polarized, the form factor \( b \) is zero. In the calculation that we carry out subsequently in this paper, we assume that this is the case, and therefore we set \( b = 0 \) and write
\[ \mathcal{M} = a F_{\mu \nu}^* v^\mu p^\nu. \] (2.9)

Without loss of generality, we will take the form factor \( a \) to be real.

Choosing the convention in which the \( z \)-axis points in the direction of the motion of the decaying particle, the components of the vectors \( v^\mu \) and \( p^\mu \) in the rest frame of the medium are
\[ v^\mu = (1, \vec{0}), \quad p^\mu = (E, \vec{P}), \] (2.10)
and we will denote by
\[ V = P/E \] (2.11)
the velocity of the decaying particle. The differential decay rate is then
\[ \frac{d\Gamma}{d(\cos \theta)} = \frac{\omega_0}{16\pi m^2} \frac{(1 - V^2)^{3/2}}{(1 - V \cos \theta)^2} |\mathcal{M}|^2, \] (2.12)
where \( \theta \) is the angle between \( \vec{q} \) and the \( z \)-axis,
\[ \omega_0 = \frac{m^2 - m'^2}{2m}, \] (2.13)
and
\[ |\mathcal{M}|^2 = \sum_{\text{pol}} |\mathcal{M}|^2, \] (2.14)
with the sum being over the two polarization states of the photon. It may be convenient to express the differential rate in terms of the photon energy, which is given by
\[ \omega = \frac{m^2 - m'^2}{2E(1 - V \cos \theta)}. \] (2.15)
Thus,
\[ \frac{d\Gamma}{d\omega} = \frac{1}{16\pi m^2} \frac{(1 - V^2)}{V} |\mathcal{M}|^2, \] (2.16)
and the total rate is
\[ \Gamma = \frac{1}{16\pi m^2} \frac{(1 - V^2)}{V} \int_{\omega_0 r}^{\omega_0/r} d\omega |\mathcal{M}|^2, \tag{2.17} \]
where
\[ r = \sqrt{\frac{1 - V}{1 + V}}, \tag{2.18} \]
and \( \omega_0 \) has been defined in Eq. (2.13).

For the present case, in which the amplitude is given by Eq. (2.9),
\[ |\mathcal{M}|^2 = a^2 P^2 \omega^2 \sum_{\text{pol}} |\vec{e} \cdot \hat{z}|^2 \]
\[ = a^2 P^2 \omega^2 \sin^2 \theta, \tag{2.19} \]
where we should remember that \( \omega \) and \( \theta \) are related by Eq. (2.15). Thus, using Eqs. (2.12), (2.13) and (2.15)
\[ \frac{d\Gamma}{d(\cos \theta)} = \frac{a^2}{16\pi} \left( \frac{m^2 - m_f^2}{2m} \right)^3 V^2 (1 - V^2)^{3/2} \sin^2 \theta \frac{\sin^2 \theta}{(1 - V \cos \theta)^4}. \tag{2.20} \]

The integration that remains to obtain the total rate, either from Eq. (2.20) or Eq. (2.17), cannot be performed until we have the explicit formulas for the form factor, since in general it depends on \( \theta \) or, equivalently, \( \omega \). Here we observe that, since the functional dependence of the form factor on \( \theta \) will be different depending on conditions of the fermion background, so will be the angular distribution.

### 2.2 The model and the diagrams

In order to perform calculations to evaluate the form factors we consider a model containing two neutral scalar fields \( \phi \) and \( \phi' \), and a charged fermion \( f \), with the interaction Lagrangian
\[ \mathcal{L}_Y = -\lambda \bar{f} f \phi - \lambda' \bar{f} f \phi', \tag{2.21} \]
in addition to the standard electromagnetic coupling of \( f \). The medium is assumed to be a thermal background of the fermions \( f \). The lowest order diagrams are shown in Fig. 1 where the internal line in the loops represent the fermions in the thermal background. We write the fermion thermal propagator in the form
\[ iS(l) = i(\not{l} + m_f)T(l), \tag{2.22} \]
where
\[ T(l) = \Delta(l) - 2\pi i \delta(l^2 - m_f^2)\eta(l), \tag{2.23} \]
with
\[ \Delta(l) = \frac{1}{l^2 - m_f^2 + i\epsilon}, \] (2.24)
and
\[ \eta(l) = \frac{\Theta(l \cdot v)}{e^{\beta(l \cdot v - \mu_f)}} + 1 + \frac{\Theta(-l \cdot v)}{e^{-\beta(l \cdot v - \mu_f)}} + 1, \] (2.25)
\( \beta \) and \( \mu_f \) being the inverse temperature and chemical potential of the fermion gas, respectively.

The contributions to \( j^\mu \) from the two diagrams are given by
\[
j^{(a)}_\mu = i4e_f l\lambda' \int \frac{d^4l}{(2\pi)^4} L_\mu(l+q,l+p,l)T(l+q)T(l+p)T(l), \] (2.26)
\[
j^{(b)}_\mu = i4e_f l\lambda' \int \frac{d^4l}{(2\pi)^4} L_\mu(l,l-p,l-q)T(l)T(l-p)T(l-q), \] (2.27)
where \( e_f \) is the electric charge of the fermion \( f \), and
\[
L_\mu(p_1,p_2,p_3) \equiv \frac{1}{4} \text{Tr} \left[ \gamma_\mu (p_1 + m_f)(p_2 + m_f)(p_3 + m_f) \right],
\]
\[
= p_1p_2 \cdot p_3 - p_2p_3 \cdot p_1 + p_3p_1 \cdot p_2 + (p_1p_2 + p_2p_3 + p_3p_1)m_f^2. \] (2.28)

This expression confirms the statement made in Sec. 2.1, that the form factor \( b \) that appears in Eq. (2.6) vanishes in this model. As stated there, this is a consequence of parity invariance of the interactions of Eq. (2.21) and the polarization independence of the fermion distribution functions in the medium.

Using the relationships
\[
L_\mu(p_1,p_2,p_3) = L_\mu(p_3,p_2,p_1),
L_\mu(-p_1,-p_2,-p_3) = -L_\mu(p_1,p_2,p_3), \] (2.29)
Eqs. (2.26) and (2.27) can be combined in the form

\[ j_\mu = 4e \lambda \lambda' \int \frac{d^4l}{(2\pi)^4} L_\mu(l + q, l + p, l) \times \left[ T(l + q)T(l + p)T(l) - T(-l - q)T(-l - p)T(-l) \right]. \]  

(2.30)

When Eq. (2.22) is substituted into Eq. (2.30) various terms are produced. The purely vacuum terms, which do not contain any factor involving \( \eta(l) \), give zero since \( \Delta(-l) = \Delta(l) \). The terms containing three factors of \( \eta \) also yield zero due to the on-shell conditions implied by the delta function in Eq. (2.25). The terms containing two factors of \( \eta \) will give an absorptive contribution to the amplitude. Here we will assume that the kinematic regime is such that those absorptive terms are zero, i.e., that the initial \( \phi \) state is below the \( f \bar{f} \) pair production threshold. Therefore the only the terms that survive are those that contain one factor of \( \eta \), and by making appropriate shifts of the integration variables in some of those terms, we obtain

\[ j_\mu = 4e \lambda \lambda' \int \frac{d^4l}{(2\pi)^3} \delta(l^2 - m_f^2)[\eta(l) - \eta(-l)] \left\{ L_\mu(l + q, l + p, l)\Delta(l + p)\Delta(l + q) \right. \\
+ L_\mu(l - p + q, l - p)\Delta(l - p)\Delta(l - p + q) \\
+ L_\mu(l, l + p - q, l - q)\Delta(l - q)\Delta(l - q) \left. \right\}. \]  

(2.31)

### 2.3 Transversality condition

The transversality condition \( q_\mu j^\mu = 0 \) is easily verified explicitly with the help of the identity

\[ (p_1 - p_3)^\mu L_\mu(p_1, p_2, p_3) = (p_2 \cdot p_3 + m_f^2)\Delta^{-1}(p_1) - (p_2 \cdot p_1 + m_f^2)\Delta^{-1}(p_3), \]  

(2.32)

which follows simply from Eq. (2.28). Using this identity to rewrite the various factors of \( q^\mu L_\mu \) that appear when we contract Eq. (2.31) with \( q^\mu \), we obtain

\[ q^\mu j_\mu = 4e \lambda \lambda' \int \frac{d^4l}{(2\pi)^3} \delta(l^2 - m_f^2)[\eta(l) - \eta(-l)] \left\{ \Delta(l + p)[l \cdot (l + p) + m_f^2] \right. \\
+ \Delta(l - p)[l \cdot (l - p) + m_f^2] - \Delta(l - p + q)[l \cdot (l - p + q) + m_f^2] \\
- \Delta(l + p - q)[l \cdot (l + p - q) + m_f^2] \left. \right\}, \]  

(2.33)

where we have also used the relation

\[ \delta(l^2 - m_f^2)\Delta^{-1}(l^2 - m_f^2) = 0. \]  

(2.34)

Since the factor \( \eta(l) - \eta(-l) \) in the integrand of Eq. (2.33) is odd under \( l \to -l \), while the other factor, within the curly brackets, is even, the integrand is odd and therefore Eq. (2.3) is verified.
2.4 Evaluation of the form factor

Using the trace formula of Eq. (2.28), we can write down the individual terms in \( j_\mu \) from Eq. (2.31). For this, it is convenient to define the integrals

\[
I_\alpha(p_1, p_2) \equiv \int \frac{l_\alpha}{(p_1^2 + 2l \cdot p_1)(p_2^2 + 2l \cdot p_2)},
\]

(2.35)

\[
I(p_1, p_2) \equiv \int \frac{1}{(p_1^2 + 2l \cdot p_1)(p_2^2 + 2l \cdot p_2)},
\]

(2.36)

where

\[
\int l_\equiv \int \frac{d^4l}{(2\pi)^3} \delta(l^2 - m_f^2)\eta(l) - \eta(-l).
\]

(2.37)

These integrals satisfy the relations

\[
I_\alpha(-p_1, -p_2) = I_\alpha(p_1, p_2),
\]

(2.38)

\[
I(-p_1, -p_2) = -I(p_1, p_2).
\]

(2.39)

In addition, since \( q^2 = 0 \), both integrals are odd in \( q \) if either \( p_1 = q \) or \( p_2 = q \). Using these properties, we can write the vertex function in the form

\[
j_\mu = 4eJ\lambda\eta' \left[(4m_f^2 - p \cdot p') \left[I_\mu(p, p') + I_\mu(p, q) - I_\mu(p', q)\right]
+ p_\mu \left[(p + p')_\alpha I^\alpha(p, p') - q_\alpha I^\alpha(p, q) - q_\alpha I^\alpha(p', q) + 4m_f^2 I(p, p')\right]\right],
\]

(2.40)

where we have omitted terms proportional to \( q_\mu \) that do not contribute to the amplitude due to Eq. (2.4). In order to determine the form factor \( a \) from this expression, we only need to look for the terms proportional to \( v_\mu \), according to Eq. (2.6). Such terms can arise only from the integrals \( I_\mu \), which can be combined in the form

\[
I_\mu(p, p') + I_\mu(p, q) - I_\mu(p', q) = -(m^2 - m'^2)\bar{I}_\mu(p, p'),
\]

(2.41)

where

\[
\bar{I}_\mu(p, p') = \int \frac{l_\mu}{2l \cdot q(m^2 + 2l \cdot p)(m'^2 + 2l \cdot p')}.
\]

(2.42)

The result of the integration can be expressed in the form

\[
\bar{I}_\mu(p, p') = Av_\mu + Bp_\mu + B'p'_\mu,
\]

(2.43)

where \( A, B \) and \( B' \) are Lorentz invariants, and the form factor \( a \) is then determined as

\[
a = 4eJ\lambda\eta' (m^2 + m'^2 - 8m_f^2)A.
\]

(2.44)

The integral defined in Eq. (2.42) cannot be calculated analytically in the general case, as is common for this type of integral that involve the thermal distribution functions. For
definiteness here we consider the case of a non-relativistic fermion gas for which a simple result can be obtained. Thus, assuming that 
\[ |l_0| \approx m_f, \quad |\vec{l}| \ll m_f, \] (2.45)
then as a first approximation we can neglect \( \vec{l} \) altogether and replace \( l_0 \) by \( m_f \) in Eq. (2.38). As a result, \( \tilde{I}_\mu \) is proportional to \( v_\mu \), so that the coefficients \( B \) and \( B' \) are zero, while
\[ A = \frac{1}{2\omega(m^2 + 2m_f E)(m'^2 + 2m_f E')} \int \frac{d^4l}{(2\pi)^3} \delta(l^2 - m_f^2)[\eta(l) - \eta(-l)], \] (2.46)
where \( E' = E - \omega \).
Therefore,
\[ a = e_f \lambda \lambda'(n_f - n_f') \frac{m^2 + m'^2 - 8m_f^2}{2m_f \omega (m^2 + 2m_f E)(m'^2 + 2m_f E')}, \] (2.48)
where \( n_f \) and \( n_f' \) are the number densities of the fermions \( f \) and their antiparticles, respectively.

If the background is charge-symmetric, i.e., contains an equal number of particles and antiparticles, the form factor, and thereby the amplitude, is zero as Eq. (2.48) clearly shows. This result actually follows more generally from the presence of the factor \( \eta(l) - \eta(-l) \) in Eq. (2.37), and it holds whether the fermion gas is non-relativistic or not. On the other hand, Eq. (2.48) confirms explicitly that the amplitude is non-zero in general, giving a non-zero rate for the radiative decay process.

Finally, the explicit expression for the differential and total decay rates in this limit (non-relativistic gas) are given by substituting Eq. (2.48) into Eqs. (2.17) and (2.20), and remembering the relation given in Eq. (2.15). The final integration that remains to obtain the total rate using either Eq. (2.20) or Eq. (2.17), is straightforward but cumbersome. We do not pursue this further here since the details are similar to next case that we consider, which we treat in full.

3 Gravitational decay of a neutrino

3.1 Kinematical considerations
We now consider the process
\[ \nu_1(p) \rightarrow \nu_2(p') + G(q) \] (3.1)
where \( G \) denotes the graviton. The amplitude can be written as
\[ i\mathcal{M} = -i\kappa \epsilon^{\lambda \rho \nu \sigma}(q)\bar{u}_2(p')\Gamma_{\lambda \rho}u_1(p), \] (3.2)
where $\epsilon^{\lambda\rho}$ is the polarization tensor of the graviton, $\Gamma_{\lambda\rho}^{(I)}$ is the vertex function, and $\kappa$ is related to the Newton gravitational constant $G$ by
\[
\kappa = \sqrt{8\pi G}.
\]
\[\text{(3.3)}\]

As we have already mentioned, in the calculation of the rate we neglect the effects of the medium in the dispersion relations and wave function renormalization of the external particles. Therefore, the spinors satisfy the vacuum Dirac equation while the on-shell conditions for the graviton polarization tensor are
\[
\begin{align*}
\epsilon^{\lambda\rho}(q) &= \epsilon_{\rho\lambda}(q), \\
\eta^{\lambda\rho} \epsilon^{\rho\lambda}(q) &= 0, \\
q^{\lambda} \epsilon^{\lambda\rho}(q) &= 0.
\end{align*}
\]
\[\text{(3.4)}\]

The vertex function is constrained by Lorentz invariance and must be transverse to the graviton momentum. Subject to these conditions, the general form for the vertex function has been obtained in Ref. [1]. It was noted there that the vertex function can be decomposed into a tensor and a pseudotensor,
\[
\Gamma_{\lambda\rho} = \Gamma_{\lambda\rho}^{(T)} + \Gamma_{\lambda\rho}^{(P)} \gamma_5,
\]
\[\text{(3.5)}\]
and that for each $\Gamma_{\lambda\rho}^{(I)}$ with $I = T, P$, the terms that contribute to the vertex function with an on-shell graviton are of the form
\[
\begin{align*}
\Gamma_{\lambda\rho}^{(I)} &= \left[ -\frac{q \cdot v}{p \cdot q} p_{\lambda} p_{\rho} - \frac{p \cdot q}{q \cdot v} v_{\lambda} v_{\rho} + \{p v\}_{\lambda\rho}\right] (a^{(I)}_S + b^{(I)}_S \gamma_5) \\
&\quad + \left[ \frac{q \cdot v}{(p \cdot q)^2} g_{\lambda\rho} p_{\lambda} p_{\rho} - \frac{q \cdot v}{q \cdot p} \{p \gamma\}_{\lambda\rho} + \{v \gamma\}_{\lambda\rho}\right] (a^{(I)}_g + b^{(I)}_g \gamma_5),
\end{align*}
\]
\[\text{(3.6)}\]
where the $a$’s and the $b$’s are form factors, and
\[
\{AB\}_{\lambda\rho} = A_\lambda B_\rho + A_\rho B_\lambda.
\]
\[\text{(3.7)}\]

For the present calculation, the on-shell amplitude takes a simpler form. First, the transversality of the vertex function implies that we can make the replacement
\[
\epsilon_{\lambda\rho} \to \epsilon_{\lambda\rho} + X_{\lambda} q_{\rho} + q_{\lambda} X_{\rho}
\]
\[\text{(3.8)}\]
with any $X_\lambda$, which allows us to choose the graviton polarization tensor such that
\[
v^\lambda \epsilon_{\lambda\rho} = 0.
\]
\[\text{(3.9)}\]
Second, since the neutrinos are much lighter compared to the charged leptons and the weak gauge bosons, we can neglect the masses in calculating the leading term of the amplitude. This also enables us to use the massless limit formulas
\[
\begin{align*}
\bar{u}_2(p') \gamma_\mu u_1(p) &= 0, \\
\bar{u}_2(p') \gamma_5 u_1(p) &= 0.
\end{align*}
\]
\[\text{(3.10)}\]
Finally, chirality arguments dictate that the terms with or without the $\gamma_5$ in the amplitude are not independent. In fact, in the massless limit, the chirality invariance of the standard model interactions imply that the terms $a_5^{(i)}$ and $b_5^{(i)}$ cannot be present, while the remaining ones actually appear combined in the form

$$iM = -i\kappa\varepsilon^{\mu\nu}\bar{u}_2(p')\left[cp_{\mu}p_{\nu}\gamma_5 + d(p_{\mu}\gamma_\nu + p_{\nu}\gamma_\mu)\right]Lu_1(p),$$

where $c$ and $d$ independent coefficients that must be computed. Eq. (3.11) is the most general form of the on-shell amplitude in the chiral limit.

The formulas already given in Eqs. (2.12) and (2.17) for the angular distribution and the decay rate apply in this case as well. In order to evaluate $|M|^2$ we use the relation

$$u_1\bar{u}_1 = \frac{L}{p},$$

which is satisfied by the neutrino spinor in the massless approximation that we have used, and the analogous relation for $u_2$. The graviton polarization tensor can be expressed in terms of the spin-1 polarization vector of definite helicity $\epsilon_\mu^s$ (with $s = \pm$), which are such that

$$\epsilon_\mu^s = (0, \vec{\epsilon}_s),$$

with

$$\vec{\epsilon}_s \cdot \vec{q} = 0.$$

A particularly convenient representation follows by defining the unit vectors

$$\hat{q} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),$$
$$\vec{e}_1 = (\sin \phi, -\cos \phi, 0),$$
$$\vec{e}_2 = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta).$$

The spin-1 polarization vectors of definite helicity are given by

$$\vec{\epsilon}_s = \frac{1}{\sqrt{2}}(\vec{e}_1 + is\vec{e}_2),$$

and the graviton polarization tensor for a definite helicity is

$$\epsilon^{\mu\nu}_s = \epsilon^\mu_s \epsilon^\nu_s.$$

In the most general case, the coefficients $c$ and $d$ introduced in Eq. (3.11) can have different phases, which is a signal of $CP$-nonconserving effects. However, as is well known, such effects do not show up at one-loop when their only source is the mixing matrix. Thus, for our present purposes, we can assume that $c$ and $d$ are relatively real. The generalization to the the case in which $\text{Im} \ c^{*}d \neq 0$ is straightforward. In this way we then obtain

$$|M|^2 = 2\kappa^2 P^4 \sin^4 \theta \left\{|c|^2 E(E - \omega) + 4|d|^2 + 2c^{*}d(2E - \omega)\right\}$$

where we have used the relations

$$\sum_s |\vec{\epsilon}_s \cdot \hat{z}|^2 = \sin^2 \theta,$$
$$\sum_s |\epsilon_s \cdot \hat{z}|^4 = \frac{1}{2} \sin^4 \theta.$$
The diagrams

We take the medium to be a thermal background of electrons. To one-loop order, $\Gamma_{\lambda\rho}$ is determined by the diagrams shown in Fig. 2. Those diagrams contribute only to the gravitational coupling of $\nu_e$ and not to the other flavors, which lead to non-diagonal couplings between the neutrino mass eigenstates through the neutrino mixing matrix. There is another set of one-loop diagrams involving the $Z$ boson, but they contribute equally to the gravitational vertex of all the neutrino flavors $\nu_{e,\mu,\tau}$, and therefore do not contribute to the off-diagonal couplings between the mass eigenstates of standard, weak SU(2)-doublet neutrinos. They are relevant if the decay process involves a so-called sterile neutrino, but we do not consider that possibility here.

There are additional diagrams in which one or both of the vector boson lines are replaced by their corresponding unphysical Higgs partners. As we will see, the leading contributions from the $W$ exchange diagrams to the decay amplitude is $O(1/M_W^2)$, and therefore the Higgs exchange diagrams are relevant in general. However, for simplicity, we will carry out the calculation in the limit $m_e \to 0$, in which case the diagrams involving the unphysical Higgs particles do not contribute. Thus, our results are strictly applicable for those physical environments in which the electrons in the background can be considered to be relativistic.

There are also diagrams in which the graviton line comes out from one of the external
neutrino legs are one-particle reducible and do not contribute to the leading order term of the off-diagonal vertex function \( \Gamma_{\lambda\rho} \). The proper way to take those diagrams into account in the calculation of the amplitude for any given process, is by choosing the external neutrino spinors \( u_i(p) \) to be the (properly normalized) solutions of the effective Dirac equation for the propagating neutrino mode in the medium, instead of the spinor representing the free-particle solution of the equation in the vacuum. Since the off-diagonal vertex function is zero at the tree-level, those corrections yield higher order contributions to the decay amplitude, which we neglect.

Therefore, we will compute the integral expressions defined by the diagrams shown in Fig. 2 in a manner that is consistent with the approximations and idealizations that we have outlined above.

3.3 One-loop expressions for the vertex

We consider situations in which the \( W \)-boson mass is much larger than all the other energy scales, so that it is valid to neglect the thermal effects in the \( W \)-boson propagator. In addition, we can expand the amplitude in inverse powers of the \( W \)-boson mass and keep only the leading terms. The one-loop expressions for the neutrino gravitational vertex function under the above conditions has been determined in previous works \([6, 7, 5]\), and the results that are relevant for our present purposes can be summarized as follows.

Up to terms of order \( 1/M_W^4 \), the vertex function can be written in the form

\[
\Gamma_{\lambda\rho} = U_{e1} U_{e2}^* \left[ \Lambda_{\lambda\rho} + G_{\lambda\rho} + H_{\lambda\rho} \right], \tag{3.20}
\]

where \( U \) is the neutrino mixing matrix. The terms contained in \( \Lambda_{\lambda\rho} \) are the leading order terms in the Fermi constant and they were derived in the first two references cited above. Those terms do not contribute to the on-shell amplitude, which is most easily seen by noticing that those terms are momentum-independent and therefore do not produce any contribution of the type shown in Eq. (3.11).

The remaining terms in Eq. (3.20) are the \( O(1/M_W^4) \) terms, which were determined in Ref. [5]. They are momentum-dependent and the relevant ones for the present work.

The part denoted \( G_{\lambda\rho} \) contains all the \( O(1/M_W^4) \) terms that are independent of the weak gauge parameter \( \xi \), while \( H_{\lambda\rho} \) contains the rest, which depend on \( \xi \). Borrowing the results from that reference, we quote below the relevant formulas for the contribution from each of the diagrams in Fig. 2 to these two sets of terms.

Let us consider \( G_{\lambda\rho} \) first, which contains all the \( O(1/M_W^4) \) terms that are \( \xi \)-independent, and let us start with diagram (a). The contribution from this diagram is given by

\[
G_{\lambda\rho}^{(a)} = \frac{-g^2}{2} \int \frac{d^4l}{(2\pi)^3} \delta(l^2) \eta(l) \gamma^\alpha L \left[ \frac{(\not{l} - \not{q}) V_{\lambda\rho}(l, l - q) \Delta_W^{(4)}(l - p)}{(l - q)^2} \right] \gamma^\beta L + \frac{\not{q} V_{\lambda\rho}(l + q, l) (\not{l} - \not{q}) \Delta_W^{(4)}(l - p')}{(l + q)^2} \gamma^\beta L, \tag{3.21}
\]
where
\[
\Delta^{(4)}_{W_{\mu\nu}}(k) = \frac{1}{M_W^4} \left( k^2 \eta_{\mu\nu} - k_\mu k_\nu \right),
\]
and
\[
V_{\lambda\rho}(k,k') = \frac{1}{4} \left[ \gamma_\lambda(k + k')_\rho + \gamma_\rho(k + k')_\lambda \right] - \frac{1}{2} \eta_{\lambda\rho} \left[ k + k' \right]
\]
is the tree-level electron gravitational coupling (in the massless limit), and \( \eta(l) \) has been defined in Eq. (2.2). The factor of \( \delta(l^2) \) appears in Eq. (3.21) because we have taken the limit \( m_e \to 0 \), as explained earlier. It should be noted that we have retained only the contribution due to the thermal part of the electron propagator since, as already mentioned, the on-shell amplitude vanishes in the vacuum. Similarly, for the other diagrams,

\[
G^{(b)}_{\lambda\rho} = \frac{g^2}{2} \int \frac{d^4l}{(2\pi)^4} \delta(l^2) \eta(l) \gamma^\alpha L \gamma^\beta L \\
\times \left[ -\frac{1}{M_W^4} C_{\lambda\rho\alpha\beta}(l - p', l - p) + \eta^{\mu\nu} a_{\lambda\rho\mu\nu}^{(4)}(l - p) + \eta^{\mu\nu} a'_{\lambda\rho\mu\nu}^{(4)}(l - p') + \Delta^{(4)}_{W_{\mu\nu}}(l - p') \right]
\]

\[
G^{(c)}_{\lambda\rho} = -\frac{g^2}{2} a_{\lambda\rho\beta\mu} \int \frac{d^4l}{(2\pi)^4} \delta(l^2) \eta(l) \gamma_\alpha L \gamma_\beta L \Delta^{(4)}_{W_{\alpha\beta}}(l - p'),
\]

\[
G^{(d)}_{\lambda\rho} = -\frac{g^2}{2} a_{\lambda\rho\mu\nu} \int \frac{d^4l}{(2\pi)^4} \delta(l^2) \eta(l) \gamma^\alpha L \gamma^\beta L \Delta^{(4)}_{W_{\alpha\beta}}(l - p),
\]

where the tensors \( a_{\lambda\mu\rho\nu}, a'_{\lambda\rho\mu\nu}, \) and \( C_{\lambda\rho\mu\nu} \) that appear in these expressions are related to the various gravitational vertices that appear in the diagrams, and are given by

\[
a_{\lambda\mu\rho\nu} = \eta_{\lambda\rho} \eta_{\mu\nu} - \frac{1}{2} \left( \eta_{\lambda\mu} \eta_{\rho\nu} + \eta_{\rho\mu} \eta_{\lambda\nu} \right),
\]

\[
a'_{\lambda\rho\mu\nu} = \eta_{\lambda\rho} \eta_{\mu\nu} - \left( \eta_{\lambda\mu} \eta_{\rho\nu} + \eta_{\rho\mu} \eta_{\lambda\nu} \right),
\]

\[
C_{\lambda\mu\rho\nu}(k,k') = \eta_{\lambda\rho}(\eta_{\mu\nu} k \cdot k' - k_\mu k'_\nu) - \eta_{\mu\nu}(k_\lambda k'_{\rho} + k_{\lambda} k'_{\rho}) + k_\nu(\eta_{\lambda\mu} k'_{\rho} + \eta_{\rho\mu} k'_{\nu}) + k_\mu(\eta_{\lambda\nu} k'_{\rho} + \eta_{\rho\nu} k'_{\lambda}) - k \cdot k'(\eta_{\lambda\mu} \eta_{\rho\nu} + \eta_{\rho\mu} \eta_{\lambda\nu}).
\]

The expression for the \( H^{(a,b,c,d)}_{\lambda\rho} \), which contain the \( \xi \)-dependent terms, are given by the same expressions as Eqs. (3.21) and (3.24), but with the replacements

\[
\Delta^{(4)}_{W_{\alpha\beta}} \rightarrow \Delta^{(\xi)}_{W_{\alpha\beta}} = \frac{\xi k_\mu k_\nu}{M_W^4},
\]

\[
C_{\lambda\rho\mu\nu}(k,k') \rightarrow \xi \left\{ \eta_{\lambda\rho} k_\mu k'_\nu - k'_\rho k_\mu \eta_{\rho\nu} - k'_\rho k_\mu \eta_{\lambda\nu} - k_\lambda k'_\nu \eta_{\rho\mu} - k_\lambda k'_\nu \eta_{\lambda\mu} \right\}.
\]

Before extracting the physical part from these expressions, we consider the conditions required by the gauge invariance of the weak and the gravitational interactions.
3.4 Weak gauge invariance

The gauge invariance of the weak interactions requires that the on-shell amplitude be independent of the parameter $\xi$. It turns out that the $H$ terms satisfy

$$\bar{u}_2(p')H^{(x)}_{\lambda\rho}u_1(p) = 0 \quad (x = a, b, c, d), \quad (3.27)$$

so that the requirement is indeed satisfied. In order to show this, let us consider specifically the first term $x = a$, where

$$H^{(a)}_{\lambda\rho} = \frac{g^2}{2} \int \frac{d^4l}{(2\pi)^3} \delta(l^2)\eta(l)\gamma^\alpha L \left[ \frac{(\not{l} - \not{q})V_{\lambda\rho}(l,l - q)\Delta^{(\xi)}_{\alpha\beta}(l - p)}{(l - q)^2} \right] \gamma^\beta L. \quad (3.28)$$

Using Eq. (3.26), the first term in the square brackets involves the factor $\not{l}(\not{l} - \not{p})u_1(p)$, which reduces to zero when the massless Dirac equation for $u_1$ and the delta function $\delta(l^2)$ are taken into account. Similarly, the second term in square brackets contains the factor $\bar{u}_2(p')(\not{l} - \not{p}')\not{l}$, which also reduces to zero for similar reasons.

Similar arguments hold for the remaining $H$-terms, which we do not consider explicitly any further. Therefore, the conclusion is that only the $G$ terms, which are independent of $\xi$, contribute to the physical amplitude.

3.5 Physical part of the amplitude

A by-product of Eq. (3.27) is that the $k_\mu k_\nu$ term in Eq. (3.22) does not contribute to the physical amplitude and therefore for the present purposes we can adopt

$$\Delta^{(4)}_{\alpha\beta\mu\nu}(k) = \frac{k^2}{M_W^4} \eta_{\mu\nu}, \quad (3.29)$$

in Eqs. (3.21) and (3.24). In the evaluation of the expressions the $G_{\lambda\rho}^{(x)}$, we can ignore the terms that do not contribute to the on-shell amplitude due to the on-shell relations given in Eqs. (3.4) and (3.10). Denoting the latter by $G''_{\lambda\rho}^{(x)}$ we then write

$$G_{\lambda\rho}^{(x)} = \left( \frac{g^2}{M_W^4} \right) G'_{\lambda\rho}^{(x)} + G''_{\lambda\rho}^{(x)}, \quad (3.30)$$

where the $G'_{\lambda\rho}^{(x)}$ contain the terms that survive. A straightforward evaluation of the integral expressions then yields

$$G'_{\lambda\rho}^{(a)} = \left[ 2J_{\lambda\rho\alpha} + p_\lambda J_{\alpha\kappa} + p_{\kappa} J_{\lambda\alpha} \right] \gamma^\alpha L,$$
\[ G'_{\lambda \rho}^{(b)} = -2 \left[ J_{\alpha \lambda} p_\lambda J_{\beta \rho} - p_\rho J_{\alpha \lambda} + p_\lambda p_\rho J_{\alpha} \right] \gamma^\alpha L + \left[ J_{\lambda \alpha} - p_\lambda J_\alpha \right] [(p + p')^\alpha \gamma_\rho - 2p_\rho \gamma^\alpha] L + \left[ J_{\rho \alpha} - p_\rho J_\alpha \right] [(p + p')^\alpha \gamma_\lambda - 2p \gamma^\alpha] L + \left[ J^\alpha (p + p')_\beta - p \cdot p' J_\alpha \right] (\eta_{\lambda \alpha} \gamma_\rho + \eta_{\rho \alpha} \gamma_\lambda) L , \]

\[ G'_{\lambda \rho}^{(c)} = -p'^\alpha \left( J_{\lambda \alpha} \gamma_\rho + J_{\rho \alpha} \gamma_\lambda \right) L , \]

\[ G'_{\lambda \rho}^{(d)} = -p^\alpha \left( J_{\lambda \alpha} \gamma_\rho + J_{\rho \alpha} \gamma_\lambda \right) L , \]

(3.31)

where we have introduced the definitions

\[ J_{\alpha_1 \alpha_2 \cdots \alpha_n} = \int \frac{d^4 l}{(2\pi)^3} \delta(l^2) \eta(l^\alpha l_\alpha l_{\alpha_2} \cdots l_{\alpha_n}) . \]

(3.32)

The integral \( J_{\alpha \lambda \rho} \) cancels when all the terms are added, while the Lorentz structure of \( J_\alpha \) and \( J_{\alpha \beta} \) imply that they are of the form

\[ J_\alpha = A_{11} v_\alpha , \]

\[ J_{\alpha \beta} = A_{20} \eta_{\alpha \beta} + A_{22} v_\alpha v_\beta . \]

(3.33)

Thus, substituting Eq. (3.33) into Eq. (3.31) we obtain

\[ G'_{\lambda \rho} = 2A_{11} p_\lambda p_\rho \not{L} + [3A_{20} - A_{11} (p + p') \cdot v] (p_\mu \gamma_\rho + p_\rho \gamma_\lambda) L . \]

(3.34)

which, together with Eqs. (3.20) and (3.30), establishes that the amplitude is of the form given in Eq. (3.11), with

\[ c = 2 U_{e_1} U_{e_2}^* \frac{g^2}{M_W^4} A_{11} \]

\[ d = U_{e_1} U_{e_2}^* \frac{g^2}{M_W^4} [3A_{20} - (2E - \omega) A_{11}] . \]

(3.35)

### 3.6 Evaluation of the form factors

The definitions in Eq. (3.33) imply the following expressions for the coefficients \( A_{11} \) and \( A_{20} \),

\[ A_{11} = v^\lambda J_\lambda , \]

\[ A_{20} = \frac{1}{3} \left( J^\lambda_\lambda - v^\lambda v^\rho J_{\lambda \rho} \right) , \]

(3.36)

which give the integral formulas

\[ A_{11} = \frac{1}{2} \int \frac{d^3 l}{(2\pi)^3} (f_e - f_{\bar{e}}) , \]

\[ A_{20} = \frac{1}{6} \int \frac{d^3 l}{(2\pi)^3} \left| \bar{f} \right| (f_e + f_{\bar{e}}) \]

(3.37)
where \( f_{e,\bar{e}} \) are the distribution functions of electrons and positrons respectively. In the massless limit that we have employed these integrals can be evaluated exactly in terms of the temperature \( T \) and chemical potential \( \mu \) of the electron gas, to yield

\[
A_{11} = \frac{\mu}{12} \left( T^2 + \frac{\mu^2}{\pi^2} \right),
\]
\[
A_{20} = -\frac{1}{12} \left[ \frac{7\pi^2}{60} T^4 + \frac{1}{2}\mu^2 T^2 + \frac{1}{4\pi^2} \mu^4 \right].
\]

(3.38)

The explicit formulas for the form factors \( c \) and \( d \) are obtained by substituting Eq. (3.38) in Eq. (3.35).

### 3.7 Decay rate

Substituting Eq. (3.35) in Eq. (3.18),

\[
|\mathcal{M}|^2 = 8 |U_{e1} U_{e2}^*|^2 \left( \frac{g^2 \kappa}{M_W} \right)^2 P^4 \sin^4 \theta \left( C_1 + C_2 \frac{\omega}{\omega_0} \right),
\]

(3.39)

where

\[
C_1 = (3A_{20} - EA_{11})^2,
\]
\[
C_2 = \omega_0 A_{11} (3A_{20} - EA_{11}),
\]

(3.40)

with \( \omega_0 \) given in Eq. (2.13). The decay rate is then obtained by substituting Eq. (3.39) into Eq. (2.17). Using Eq. (2.15) to express \( \theta \) in terms of \( \omega \) and making the change of variable \( \omega = \omega_0 y \), we get

\[
\Gamma = \frac{\omega_0}{2\pi V} \left( \frac{g^2 \kappa m}{M_W} |U_{e1} U_{e2}^*| \right)^2 I,
\]

(3.41)

where

\[
I = \int_r^{1/r} dy \left[ \sqrt{1 - V^2} \left( 1 + \frac{1}{y^2} \right) - \frac{2}{y} \right]^2 (C_1 + C_2 y),
\]

(3.42)

The integral \( I \) is of course trivial but the resulting formulas are cumbersome. Introducing

\[
R_n \equiv \int_r^{1/r} dy \frac{1}{y^n} = \begin{cases} 
\log \left( \frac{1}{r} \right) & n = 1 \\
\frac{1}{n-1} \left[ r^{n+1} - r^{n-1} \right] & n \neq 1,
\end{cases}
\]

(3.43)

and noting that \( R_{-n+2} = R_n \), the result can be written in the form

\[
I = \sum_{n=1}^{4} \alpha_n R_n,
\]

(3.44)
It should be noted that, for small $V$,  
\[ R_n = 2V + O(V^2) \]  
(3.46)
and
\[ \sum_n \alpha_n = O(V^2) , \]  
(3.47)
which imply that
\[ \sum_n \alpha_n R_n = O(V^2) . \]  
(3.48)
Thus, despite the overall factor of $1/V$ in Eq. (3.41), the rate vanishes for $V = 0$, which confirms the statements given in the Introduction based on general grounds.

As we have already mentioned, we have neglected the electron mass in the calculation of the amplitude. Therefore, the results that we have obtained are applicable in situations in which the electron gas is extremely relativistic. In particular, the following the hierarchical relationship
\[ T, \mu \gg m_e \gg m > m' \]  
(3.49)
must hold among the relevant parameters involved, where \( T, \mu \) on the left-hand side means “either \( T \) or \( \mu \), or both”. However, the final formulas still depend on some parameters including the initial neutrino velocity and the state of the electron gas. To give an idea of how the rate depends on them, we consider two extreme cases regarding the values of \( T \) and \( \mu \) in Eq. (3.38). For these purposes we write the formula for the decay rate in the form

\[
\Gamma = \frac{\omega_0}{2\pi} \left( g^2 \kappa m |U_{e_1}U_{e_2}^*| \right)^2 \left( \frac{C_1}{M_W^8} \right) F(V),
\]  
(3.50)

with

\[
F(V) \equiv \frac{1}{VC_1} \sum_{n=0}^{4} \alpha_n R_n.
\]  
(3.51)

### 3.7.1 Charge-symmetric medium

We consider first a medium with zero chemical potential. As Eq. (3.38) shows, in this case

\[
A_{11} = 0, \quad A_{20} = -\frac{7\pi^2}{720} T^4.
\]  
(3.52)

Looking at Eq. (3.40), we find that \( C_2 \) vanishes in this case, whereas \( C_1 \) is given by

\[
C_1 = \frac{49\pi^4}{57600} T^8.
\]  
(3.53)

The rate of the gravitational decay is then given by

\[
\Gamma = \frac{49\pi^3}{115200} \omega_0 \left( g^2 \kappa m |U_{e_1}U_{e_2}^*| \right)^2 \left( \frac{T}{M_W} \right)^8 F_1,
\]  
(3.54)

where \( F_1 \) is given by Eq. (3.51) but setting \( C_2 = 0 \). Furthermore, in this case the angular distribution of the gravitons has the specific form

\[
\frac{1}{\Gamma} \frac{d\Gamma}{d\cos \theta} \propto \frac{\sin^4 \theta}{(1-V \cos \theta)^2},
\]  
(3.55)

which depends on the velocity of the initial neutrino, but is independent of any other parameters.

Fig. 3 shows the plot of \( F_1 \) as a function of \( E/m \). For low energy neutrinos, \( F_1 \rightarrow 0 \) as \( E \rightarrow m \), as already commented. For high energy \( (E \gg m) \) neutrinos,

\[
\alpha_1 \simeq -4 \left( \frac{m}{E} \right) C_1,
\]
\[
\alpha_2 \simeq 4C_1,
\]
\[
\alpha_3 \simeq -4 \left( \frac{m}{E} \right) C_1,
\]
\[
\alpha_4 \simeq \left( \frac{m}{E} \right)^2 C_1,
\]  
(3.56)

while

\[ R_1 \simeq 2 \log \left( \frac{E}{m} \right), \]
\[ R_n \simeq \frac{2^{n-1}}{n-1} \left( \frac{E}{m} \right)^{n-1}, \quad (n \neq 1) \quad (3.57) \]

and therefore \( F_1 \) grows as

\[ F_1 \simeq F^{(ER)} \equiv \frac{8E}{3m}. \quad (3.58) \]

The result for the rate remains valid as long as the relationship \( E \ll M_W \) is maintained.

### 3.7.2 Completely degenerate medium

We now consider the limit \( T = 0 \). For this case Eq. (3.38) gives

\[ A_{11} = \frac{\mu^3}{12\pi^2}, \quad A_{20} = -\frac{\mu^4}{48\pi^2}, \quad (3.59) \]

which in turn yield

\[ C_1 = \frac{\mu^8}{2^8 \times 9\pi^4} (3 + 4E/\mu)^2, \]
\[ C_2 = -\frac{4\omega_0}{4E + 3\mu}. \quad (3.60) \]

However, notice that the relations in Eq. (3.49) imply in this case that

\[ \left| \frac{C_2}{C_1} \right| \ll 1, \quad (3.61) \]

for all the values of the parameters consistent with the assumptions that we have made. Therefore, the decay rate in this case is given by

\[ \Gamma = \frac{\omega_0}{2^9 \times 9\pi^5} \left( g^2 \kappa m |U_{e1}U_{\nu_e}^\ast| \right)^2 \left( \frac{\mu}{M_W} \right)^8 (3 + 4E/\mu)^2 F_1, \quad (3.62) \]

where \( F_1 \) is the same function that appears in Eq. (3.54). Therefore in this case the rate grows proportional to \( E \) for \( E \ll \mu \), or \( E^3 \) for \( E \gg \mu \).

### 4 Concluding remarks

These two example calculations explicitly confirm the suggestion that angular momentum violating processes can occur in a medium which is completely isotropic, even though the state of the medium does not change at all in the process. We can understand intuitively how this happens as follows. To elaborate this argument we use a radiative decay process, like the one described in Sec. 2 as an example.
Consider the decay of a particle $\psi$ to a particle $\psi'$ and a photon in the rest frame of the decaying particle. Denoting the total angular momentum of the initial and final states by $\vec{J}$ and $\vec{J}'$ respectively, in the vacuum we have

\[
\vec{J} = \vec{L}_\psi + \vec{S}_\psi, \\
\vec{J}' = \vec{L}_{\psi'} + \vec{L}_\gamma + \vec{S}_{\psi'} + \vec{S}_\gamma,
\]

where $L_x$ and $S_x$ stand for orbital angular momentum and spin of each particle. Angular momentum conservation implies in particular that

\[
\langle \vec{J} \cdot \hat{q} \rangle = \langle \vec{J}' \cdot \hat{q} \rangle,
\]

where the brackets in the left and right-hand sides denote the expectation value with respect to the initial and final states, respectively. However, since $\vec{p} = 0$, the orbital angular momentum of the decaying particle is zero and, since and $\vec{q} = -\vec{p}'$, neither $\vec{L}_{\psi'}$ nor $\vec{L}_\gamma$ have a component along $\hat{q}$. Therefore, conservation of angular momentum implies

\[
\langle \vec{S}_\psi \cdot \hat{q} \rangle = \langle \vec{S}_{\psi'} \cdot \hat{q} + \vec{S}_\gamma \cdot \hat{q} \rangle,
\]

so that the helicity along the direction of motion of the final particles is conserved in this frame. Since the helicity can only be zero for the scalars and $\pm 1$ for the photon, the process involving scalars is forbidden.

The diagrams of Fig. 1 in which one of the fermion lines denotes the thermal on-shell part of the fermion propagator, physically corresponds to the process

\[
f(l) + \psi(p) \rightarrow f(l) + \psi'(p') + \gamma(q),
\]

where $f$ is the fermion in the background medium. Then in place of Eq. (4.3), the relevant condition is now

\[
\langle \vec{L}_f \cdot \hat{q} + \vec{S}_f \cdot \hat{q} + \vec{S}_\psi \cdot \hat{q} \rangle = \langle \vec{L}_f \cdot \hat{q} + \vec{S}_f \cdot \hat{q} + \vec{S}_{\psi'} \cdot \hat{q} + \vec{S}_\gamma \cdot \hat{q} \rangle,
\]

where the various terms on either side of this equation should of course be added according to the angular momentum addition rules. Even assuming that we may drop the spin contributions of the background fermions upon averaging over an unpolarized background, their orbital angular momentum does not vanish because in the rest frame of the decaying particle the fermions appear to be moving in a preferred direction. Therefore, helicity is not a good quantum number anymore and the helicity argument does not apply. The argument can be extended with minor modifications to the gravitational decay as well.

It should be emphasized that, while the model of Sec. 2 is a toy model, in Sec. 3 we have worked with nothing else than the standard model augmented with the linearized gravitational couplings. Thus, the effects calculated in Sec. 3 constitute real predictions of the standard model, subject to the approximations we have made. The fact that these processes are forbidden in the vacuum by angular momentum conservation arguments, implies that, in the medium, their angular distribution and differential decay rates have a distinctive form. This could lead to observable consequences in specific physical contexts despite the fact that there may exist other competing processes with comparable total rates.
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