LOGARITHMIC BUNDLES OF MULTI-DEGREE ARRANGEMENTS IN $\mathbb{P}^n$

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Abstract. Let $D = \{D_1, \ldots, D_\ell\}$ be a multi-degree arrangement with normal crossings on the complex projective space $\mathbb{P}^n$, with degrees $d_1, \ldots, d_\ell$; let $\Omega_{\mathbb{P}^n}(\log D)$ be the logarithmic bundle attached to it. First we prove a Torelli type theorem when $D$ has a sufficiently large number of components by recovering them as unstable smooth irreducible degree-$d_i$ hypersurfaces of $\Omega_{\mathbb{P}^n}(\log D)$. Then, when $n = 2$, by describing the moduli spaces containing $\Omega_{\mathbb{P}^2}(\log D)$, we show that arrangements of a line and a conic, or of two lines and a conic, are not Torelli. Moreover we prove that the logarithmic bundle of three lines and a conic is related with the one of a cubic. Finally we analyze the conic-case.

Key words Multi-degree arrangement, Hyperplane arrangement, Logarithmic bundle, Torelli theorem

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1. Introduction

In the complex projective space $\mathbb{P}^n$, let $D$ be a union of $\ell$ distinct smooth irreducible hypersurfaces with degrees $d_1, \ldots, d_\ell$, i.e. a multi-degree arrangement. We can map $D$ to $\Omega_{\mathbb{P}^n}(\log D)$, the sheaf of differential 1-forms with logarithmic poles on $D$. This sheaf was originally introduced by Deligne in [7] for arrangements with normal crossings. In this case, for all $x \in \mathbb{P}^n$, the space of sections of $\Omega_{\mathbb{P}^n}(\log D)$ near $x$ is defined as $<d \log z_1, \ldots, d \log z_k, dz_{k+1}, \ldots, dz_n>_{\mathcal{O}_{\mathbb{P}^n,x}}$, where $z_1, \ldots, z_n$ are local coordinates such that $D = \{z_1 \cdot \ldots \cdot z_k = 0\}$. In particular, $\Omega_{\mathbb{P}^n}(\log D)$ is a locally free sheaf over $\mathbb{P}^n$ and it is called logarithmic bundle.

A natural, interesting question is whether $\Omega_{\mathbb{P}^n}(\log D)$ contains information enough to recover $D$, which is the so-called Torelli problem for logarithmic bundles. In particular, if the isomorphism class of $\Omega_{\mathbb{P}^n}(\log D)$ determines $D$, then $D$ is called a Torelli arrangement.

In the mathematical literature, the first situation that has been analyzed is the case of hyperplanes. In [11] Dolgachev, Kapranov proved that, if $\ell \leq n + 2$, then two different arrangements give always the same logarithmic bundle and in [25] Vallès showed that, if $\ell \geq n + 3$, then we can reconstruct the hyperplanes from the logarithmic bundle (as its unstable hyperplanes, see Definition [2, 4]) unless they don't osculate a rational normal curve $C_n$ of degree $n$ in $\mathbb{P}^n$, in which case the logarithmic bundle is isomorphic to $E_{\ell-2}(C_n^\vee)$, the Schwarzenberger bundle ([21], [22]) of degree $\ell - 2$ associated
Recently, Dolgachev \cite{Dolgachev} and Faenzi, Matei, Vallès \cite{Vallès} answered to this problem in the case of hyperplanes that not necessarily satisfy the normal crossings property. Concerning the higher degree case, Ueda and Yoshinaga \cite{Ueda}, \cite{Yoshinaga} studied the case \( \ell = 1 \), characterizing generically the Torelli arrangements as the ones with \( d_1 \geq 3 \). In \cite{Angelini} we analyzed hypersurfaces of the same degree \( d \) and, by means of the unstable hypersurfaces of \( \Omega^1_{\mathbb{P}^n}(\log D) \) (see Definition \ref{def:unstable}), we proved a Torelli type theorem when \( \ell \geq \left( \frac{n+d}{d} \right) + 3 \). Pairs of quadrics are also investigated in \cite{Angelini}.

Very recently Ballico, Huh, Malaspina \cite{Ballico} and Dimca, Sernesi \cite{Dimca}, generalizing the techniques, respectively, of \cite{Angelini} and \cite{Yoshinaga}, answered to some Torelli type questions, respectively, in the case of logarithmic bundles over quadrics or product of projective spaces and for plane curves with nodes and cusps.

In this paper, after recalling some preliminary tools (§ 2, 3), we consider multi-degree arrangements with normal crossings on \( \mathbb{P}^n \) (§ 4), on \( \mathbb{P}^2 \) (§ 5, 6, 7) and conic-arrangements with normal crossings on \( \mathbb{P}^2 \) (§ 8, 9). In Theorem \ref{thm:main} by generalizing the arguments used in \cite{Angelini} for hypersurfaces of the same degree and by applying a reduction technique, we prove that if the number \( \ell_i \) of hypersurfaces of degree \( d_i \) in \( D \) satisfies \( \ell_i \geq \left( \frac{n+d}{d} \right) + 3 \), then we can generically recover the components of \( D \). In § 5, 6, 7 we focus on some line-conic cases on \( \mathbb{P}^2 \) and we prove that they are not of Torelli type (Corollaries \ref{cor:line-conic} \ref{cor:line-conic-2}). In particular, in Theorem \ref{thm:link} we show a link between arrangements of three lines and a conic and arrangements with a cubic in the projective plane. Finally, § 8 and § 9 are devoted to conics. The cases \( \ell \in \{1,2\} \) were studied in \cite{Angelini}; here we prove that for \( \ell \geq 4 \) a Torelli type result holds (Theorem \ref{thm:ell4}). \( \ell = 3 \) is still a bit mysterious.

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## 2. Preliminary definitions and notations

Let \( \mathbb{P}^n \) be the \( n \)-dimensional complex projective space with \( n \geq 2 \) and let \( D = \{D_1, \ldots, D_\ell\} \) be an arrangement on \( \mathbb{P}^n \), i.e. a family of smooth, irreducible, distinct hypersurfaces of \( \mathbb{P}^n \). Let assume that \( D \) has normal crossings, that is \( D \) is locally isomorphic (in the sense of holomorphic local coordinates changes) to a union of coordinate hyperplanes of \( \mathbb{C}^n \). For all \( i \in \{1, \ldots, \ell\} \), \( D_i = \{ f_i = 0 \} \) with \( f_i \in \mathbb{C}[x_0, \ldots, x_n]_{d_i} \) for certain \( d_i \); thus \( D = \{ f = 0 \} \), where \( f = f_1 \cdot \ldots \cdot f_\ell \) has degree \( d = d_1 + \ldots + d_\ell \). In particular, if all \( d_i \)'s are equal to 1 we speak about a hyperplane arrangement, if they are equal to 2 we deal with an arrangement of quadrics and so on.

In order to introduce the notion of sheaf of logarithmic forms on \( D \) we
refer to Deligne ([8], [7]). Let $U$ be the complement of $D$ in $\mathbb{P}^n$ and let $j$ be the embedding of $U$ in $\mathbb{P}^n$. We denote by $\Omega^1_U$ the sheaf of holomorphic differential 1-forms on $U$ and by $j_*\Omega^1_U$ its direct image sheaf on $\mathbb{P}^n$. Since $D$ has normal crossings, then for all $x \in \mathbb{P}^n$ there exists a neighbourhood $I_x \subset \mathbb{P}^n$ such that $I_x \cap D = \{z_1 \cdots z_k = 0\}$, where $\{z_1, \ldots, z_k\}$ is a part of a system of local coordinates. We have the following:

**Definition 2.1.** The sheaf of differential 1-forms on $\mathbb{P}^n$ with logarithmic poles on $D$ is the subsheaf $\Omega^1_{\mathbb{P}^n}(\log D)$ of $j_*\Omega^1_U$, such that, for all $x \in \mathbb{P}^n$,

$$\Gamma(I_x, \Omega^1_{\mathbb{P}^n}(\log D)) = \{s \in \Gamma(I_x, j_*\Omega^1_U) | s = \sum_{i=1}^k u_i d \log z_i + \sum_{i=k+1}^n v_i dz_i\}$$

where $u_i, v_i$ are locally holomorphic functions and $d \log z_i = \frac{dz_i}{z_i}$.

Another way to describe these sheaves, which is useful for more general divisors and is equivalent to the previous one in the normal crossings case, is the following, ([19], [20]):

**Definition 2.2.** The sheaf of diff. 1-forms on $\mathbb{P}^n$ with log. poles on $D$ is

$$\Omega^1_{\mathbb{P}^n}(\log D) = T(\log D)^\vee (-1),$$

where $T(\log D)$ is the kernel of the Gauss map $\mathcal{O}_{\mathbb{P}^n}^{n+1} (\partial_{0f}, \ldots, \partial_{nf}) \rightarrow \mathcal{O}_{\mathbb{P}^n}(d-1)$.

Since $D$ has normal crossings, $\Omega^1_{\mathbb{P}^n}(\log D)$ is a locally free sheaf of rank $n$, [7]. It is called the logarithmic bundle attached to $D$.

Definition 2.1 can be used, more in general, to introduce the logarithmic bundle of an arrangement with normal crossings $D$ on a smooth algebraic variety $X$ (see also [2]).

Our investigations are mainly based on the following:

**Theorem 2.3.** $\Omega^1_{\mathbb{P}^n}(\log D)$ admits the short exact sequences

1. $$0 \rightarrow \Omega^1_{\mathbb{P}^n} \rightarrow \Omega^1_{\mathbb{P}^n}(\log D) \xrightarrow{\text{res}} \bigoplus_{i=1}^\ell \mathcal{O}_{D_i} \rightarrow 0,$$

where res denotes the Poincaré residue morphism ([11]) and

2. $$0 \rightarrow \Omega^1_{\mathbb{P}^n}(\log D)^\vee \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \oplus \mathcal{O}_{\mathbb{P}^n}^{\ell-1} \xrightarrow{N} \bigoplus_{i=1}^\ell \mathcal{O}_{\mathbb{P}^n}(d_i) \rightarrow 0,$$

where $N$ is an $\ell \times (n + \ell)$ matrix depending on the $f_j$’s and their partial derivatives ([2]).

Our aim is to study the injectivity of the correspondence

3. $$D \mapsto \Omega^1_{\mathbb{P}^n}(\log D)$$
that is the Torelli problem for logarithmic bundles. In the case of 1 : 1 correspondence we call $\mathcal{D}$ an arrangement of Torelli type or, simply, a Torelli arrangement.

In the next section we recall the main results concerning this problem in the case of hyperplanes ([11], [25], [1]), of one smooth hypersurface ([24], [23], [2]), of many smooth hypersurfaces of degree $d \geq 2$ and of two smooth quadrics ([2]). In some of them, the components of $\mathcal{D}$ are recovered by looking at the set of unstable objects of $\Omega_{P^n}(\log \mathcal{D})$ of a given degree; in that sense we give the following:

**Definition 2.4.** Let $D \subset P^n$ be a hypersurface. We call $D$ unstable for $\Omega_{P^n}(\log \mathcal{D})$ if the following condition holds:

$$H^0(D, \Omega_{P^n}(\log \mathcal{D})) \neq \{0\}.$$  

**Remark 2.5.** Let us suppose that $D$ has $\ell = \ell_1 + \ldots + \ell_m$ components such that $\ell_i$ have degree $d_i$, $i \in \{1, \ldots, m\}$. Definition 2.4 is meaningful when $h^0(P^n, \Omega_{P^n}(\log \mathcal{D})) = \{0\}$, that is, by using the same arguments of Remark 5.3 of [2], when

$$m \sum_{i=1}^{m} (\ell_i \cdot d_i) > n + 1.$$  

**Remark 2.6.** In Lemma 5.4 of [2], by means of [4] we prove that each component $D_i$ of $\mathcal{D}$ is an unstable hypersurface of degree $d_i$ for $\Omega_{P^n}(\log \mathcal{D})$. As in Definition 2.4, we can introduce the notion of unstable hypersurface for $\Omega_{X}(\log \mathcal{D})$ when $X$ is a smooth algebraic variety and $\mathcal{D}$ is an arrangement with normal crossings on it. In a similar way we can prove that each element of $\mathcal{D}$ is unstable for $\Omega_{X}(\log \mathcal{D})$.

### 3. Some known Torelli type results

Let $\mathcal{H} = \{H_1, \ldots, H_\ell\}$ be a hyperplane arrangement with normal crossings on $P^n$. If $\ell \leq n + 2$, then $\mathcal{H}$ isn’t of Torelli type ([11]); otherwise we have the following result ([25], Theorem 3.1):

**Theorem 3.1.** If $\ell \geq n + 3$ then $\mathcal{H}$ is the set of unstable hyperplanes of $\Omega_{P^n}(\log \mathcal{H})$, unless $H_1, \ldots, H_\ell$ osculate a rational normal curve $C_n \subset P^n$ of degree $n$, in which case all the hyperplanes lying on $C_n \subset (P^n)^\vee$ are unstable and $\Omega_{P^n}(\log \mathcal{H}) \cong E_{\ell-2}(C_n^\vee)$, the Schwarzenberger bundle of degree $\ell - 2$ associated to $C_n^\vee$.

If $\mathcal{D} = \{D_1\}$, where $D_1 \subset P^n$ is a general hypersurface of degree $d_1$, then $\mathcal{D}$ is of Torelli type if and only if $d_1 \geq 3$ ([23], Theorem 1; [2], Proposition 6.1).

Now, let $\mathcal{D} = \{D_1, \ldots, D_\ell\}$ be an arrangement with normal crossings on $P^n$, with $\ell \geq 2$ and $d_i = d \geq 2$ for all $i \in \{1, \ldots, \ell\}$. By associating to $\mathcal{D}$ a hyperplane arrangement $\mathcal{H}$ in $P^{(n+d)-1}$ through the $d$-uple Veronese
embedding and by applying Theorem 3.1 we get the following result ([2], Theorem 5.5):

**Theorem 3.2.** If \( \ell \geq (n+d) + 3 \) and \( \mathcal{H} \) is a hyperplane arrangement with normal crossings whose components don’t osculate a rational normal curve of degree \((n+d) - 1\) in \( \mathbb{P}^{(n+d)-1} \), then \( \mathcal{D} \) is the set of smooth, irreducible, degree-\( d \) hypersurfaces of \( \mathbb{P}^n \) unstable for \( \Omega^1_{\mathbb{P}^n}(\log \mathcal{D}) \).

In ([2], Theorem 7.5) we prove also that if \( \ell = d = 2 \) then \( \mathcal{D} \) is not a Torelli arrangement. Indeed, by using the *simultaneous diagonalization* of the matrices of the smooth quadrics and a *duality* argument, we get that two such arrangements have isomorphic logarithmic bundles if and only if they have the same tangent hyperplanes.

In the next sections we present some recent results concerning multi-degree arrangements (§ 4, 5, 6, 7) and an almost complete description of the conic-case (§ 8, 9).

4. MANY MULTI-DEGREE HYPERSURFACES

Let \( \mathcal{D} = \{ D_1^{d_1}, \ldots, D_1^{d_{\ell_1}}, D_1^{d_2}, \ldots, D_2^{d_{\ell_2}}, \ldots, D_m^{d_m} \} \) be a multi-degree arrangement with normal crossings in \( \mathbb{P}^n \) such that the components \( D_i^{d_i} \) have degree \( d_i \), with \( i \in \{ 1, \ldots, m \} \) and \( d_m > d_{m-1} > \cdots > d_1 \); let denote by \( \Omega^1_{\mathbb{P}^n}(\log \mathcal{D}) \) the corresponding logarithmic bundle. When the number of components in \( \mathcal{D} \) is *sufficiently large*, the Torelli problem can be solved by generalizing the method performed in [2] and by applying a *reduction technique* inspired to the one adopted in [25]. So, let \( \mathcal{H}_{d_i} \) be the arrangement with \( \ell_i \) hyperplanes on \( \mathbb{P}^{N_i} \), with \( i \in \{ 1, \ldots, m \} \) and \( N_i = \binom{n+d_i}{d_i} - 1 \), associated to \( \{ D_1^{d_i}, \ldots, D_{\ell_i}^{d_i} \} \) by means of the \( d_i \)-uple Veronese embedding, i.e. \( \nu_{d_i} : \mathbb{P}^n \to \mathbb{P}^{N_i} \) and \( \nu_{d_i}([x_0, \ldots, x_n]) = [\ldots x^I \ldots] \), where \( x^I \) ranges over all monomials of degree \( d_i \) in \( x_0, \ldots, x_n \). Let assume that each \( \mathcal{H}_{d_i} \) has normal crossings on \( \mathbb{P}^{N_i} \) and let \( \Omega^1_{\mathbb{P}^{N_i}}(\log \mathcal{H}_{d_i}) \) be the associated logarithmic bundle. With the previous notations, let us consider the diagonal embedding:

\[ \nu : \mathbb{P}^n \to \mathbb{P} = \prod_{i=1}^m \mathbb{P}^{N_i} \]

\[ \nu([x_0, \ldots, x_n]) = [\nu_{d_1}([x_0, \ldots, x_n]), \ldots, \nu_{d_m}([x_0, \ldots, x_n])]. \]

Let \( p_i : \mathbb{P} \to \mathbb{P}^{N_i} \) be the \( i \)-th projection and let \( h_i = c_1(p_i^*(\mathcal{O}_{\mathbb{P}^{N_i}}(1))) \). By means of \( \nu \), we can associate to the multi-degree arrangement \( \mathcal{D} \) an arrangement \( \mathcal{A} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_m \) on \( \mathbb{P} \) such that \( \mathcal{A}_i \) is an irreducible divisor of class \( h_i \) which is the pull-back via \( p_i \) of \( \mathcal{H}_{d_i} \).

Let assume that \( \mathcal{A} \) has normal crossings and let \( \Omega^1_{\mathbb{P}}(\log \mathcal{A}) \) be its logarithmic bundle (see also [3] for some results concerning logarithmic bundles over product of projective spaces).
Remark 4.1. The following property holds:

\begin{equation}
\Omega^1_H(\log A) \cong \bigoplus_{i=1}^m p_i^*(\Omega^1_{P_{N_i}(\log \mathcal{H}_{d_i})}).
\end{equation}

Moreover, if \( \ell_i \geq N_i + 2 \), being \( \mathcal{H}_{d_i} \) with normal crossings, \( \Omega^1_{P_{N_i}(\log \mathcal{H}_{d_i})} \) is a Steiner bundle over \( P^{N_i} \), [11]. So, because of [6], \( \Omega^1_H(\log A) \) admits the short exact sequence

\begin{equation}
0 \rightarrow \bigoplus_{i=1}^m \mathcal{O}_P(-h_i)^{\ell_i - N_i - 1} \rightarrow \bigoplus_{i=1}^m \mathcal{O}_P^{\ell_i - 1} \rightarrow \Omega^1_H(\log A) \rightarrow 0.
\end{equation}

Now we can state and prove the main result concerning the Torelli problem for multi-degree arrangements with many components.

**Theorem 4.2.** Let \( \mathcal{D} \) be a multi-degree arrangement with normal crossings on \( P^n \) and let \( \mathcal{H}_{d_1}, \ldots, \mathcal{H}_{d_m} \), \( A \) be the corresponding arrangements, respectively, on \( P^{N_1}, \ldots, P^{N_m} \) and \( P \), in the sense of Veronese maps.

Assume that, for all \( i \in \{1, \ldots, m\} \):
1. \( \ell_i \geq N_i + 4 \)
2. \( A \) has normal crossings on \( P \)
3. \( \mathcal{H}_{d_i} \) has normal crossings on \( P^{N_i} \) and its hyperplanes don’t osculate a rational normal curve of degree \( N_i \) in \( P^{N_i} \).

Then \( \mathcal{D} = \{ D \subset P^n \text{ smooth irredu. hypers. of degree } d_i, \exists i \mid D \text{ satisfies } (4) \} \).

**Proof.** We perform a double inclusion argument. We observe that the inclusion \( \subset \) follows from Remark 2.6.

Thus, let assume that \( D \subset P^n \) is a smooth irreducible hypersurface of degree \( d_i \) which is unstable for \( \Omega^1_{P_{N_i}(\log \mathcal{D})} \), we want to prove that \( D \in \mathcal{D} \).

First let us suppose that the degree of \( D \) is the highest one, i.e. \( d_m \).

Our aim is to show that, denoted by \( H_{d_m} \subset P^{N_m} \) the hyperplane associated to \( D \) by means of \( \nu_{d_m} \), then \( H = p^*_m(H_{d_m}) \subset P \) satisfies

\begin{equation}
H^0(H, \Omega^1_H(\log A)|_{H}) \neq \{0\}.
\end{equation}

Indeed, if this is the case, \( H_{d_m} \) is an unstable hyperplane for \( \Omega^1_{P_{N_m}(\log \mathcal{H}_{d_m})} \) and so hypothesis 1. and 3. allow us to apply Theorem 3.1 which implies that \( H_{d_m} \in \mathcal{H}_{d_m} \). In particular, we get that \( D = D^m_j \in \mathcal{D} \) for certain \( j \in \{1, \ldots, \ell_m\} \).

Let denote by \( V \) the image of the map \( \nu \); since \( V \) is a non singular subvariety of \( P \) which intersects transversally \( A \), from Proposition 2.11 of [10] we get the following exact sequence

\begin{equation}
0 \rightarrow \mathcal{N}^\nu_{V,P} \rightarrow \Omega^1_H(\log A)|_V \rightarrow \Omega^1_V(\log A \cap V) \rightarrow 0
\end{equation}

where \( \mathcal{N}^\nu_{V,P} \) denotes the conormal sheaf of \( V \) in \( P \).

We remark that \( V \cong P^n \) and \( \mathcal{D} = A \cap V \), so if we restrict [9] to \( D \), we apply \( \mathcal{H}om(\cdot, \mathcal{O}_D) \) and then we pass to cohomology we get

\begin{align*}
0 \rightarrow H^0(D, \Omega^1_p(\log \mathcal{D})|_D) \rightarrow H^0(D, \Omega^1_p(\log A)|_D) & \rightarrow H^0(D, \Omega^1_p(\log A)|v_D) \rightarrow 0.
\end{align*}
Since $D$ is unstable for $\Omega^1_{\mathbb{P}^n}(\log D)$, necessarily it has to be
\begin{equation}
H^0(D, \Omega^1_{\mathbb{P}^n}(\log A)|^\vee_H) \neq \{0\}.
\end{equation}

Now, let do the tensor product of the ideal sheaf sequence of $V$ in $\mathbb{P}$ with $\Omega^1_{\mathbb{P}^n}(\log A)|^\vee_H$; we have the exact sequence
\[
0 \to \mathcal{I}_{V \cap H, H} \otimes \Omega^1_{\mathbb{P}^n}(\log A)|^\vee_H \to \Omega^1_{\mathbb{P}^n}(\log A)|^\vee_H \to \Omega^1_{\mathbb{P}^n}(\log A)|^\vee_D \to 0.
\]

Passing to cohomology we get
\[
0 \to H^0(H, \mathcal{I}_{V \cap H, H} \otimes \Omega^1_{\mathbb{P}^n}(\log A)|^\vee_H) \to H^0(H, \Omega^1_{\mathbb{P}^n}(\log A)|^\vee_H) \to H^0(D, \Omega^1_{\mathbb{P}^n}(\log A)|^\vee_D) \to H^1(H, \mathcal{I}_{V \cap H, H} \otimes \Omega^1_{\mathbb{P}^n}(\log A)|^\vee_H).
\]

We remark that to conclude the proof it suffices to show that
\begin{equation}
H^1(H, \mathcal{I}_{V \cap H, H} \otimes \Omega^1_{\mathbb{P}^n}(\log A)|^\vee_H) = \{0\}.
\end{equation}

Since hypothesis 1. and 3. hold, we are allowed to use (7), which, by applying $\mathcal{H}om(\cdot, \mathbb{P})$ it turns to be
\[
0 \to \Omega^1_{\mathbb{P}^n}(\log A)^\vee \to \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^n}(h_i-1) \to \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^n}(h_i)^{1-N_i} \to 0.
\]

If we do the tensor product with $\mathcal{I}_{V, \mathbb{P}}|_H$ and then we pass to cohomology, the previous sequence becomes
\begin{equation}
\cdots \to \bigoplus_{i=1}^m H^0(H, \mathcal{I}_{V \cap H, H} \otimes \mathcal{O}_{\mathbb{P}^n}(h_i)^{1-N_i}) \to \bigoplus_{i=1}^m H^1(H, \mathcal{I}_{V \cap H, H} \otimes \mathcal{O}_{\mathbb{P}^n}(h_i)^{1-N_i}).
\end{equation}

In order to prove (11) it suffices to show that
\begin{equation}
H^{1-k}(H, \mathcal{I}_{V \cap H, H}(kh_i)) = \{0\}
\end{equation}
for $k = \{0, 1\}$ and for all $i \in \{1, \ldots, m\}$. Being $V \cap H$ connected, from the induced cohomology sequence of the ideal sheaf sequence of $V$ in $\mathbb{P}$, restricted to $H$, we immediately get (13) for $k = 0$.

So, let consider the exact commutative diagram:
\[
\begin{array}{ccccccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \mathcal{I}_{V, \mathbb{P}}(h_i-h_m) & \mathcal{O}_{\mathbb{P}}(h_i-h_m) & \mathcal{O}_{\mathbb{P}^n}(d_i-d_m) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{I}_{V, \mathbb{P}}(h_i) & \mathcal{O}_{\mathbb{P}}(h_i) & \mathcal{O}_{\mathbb{P}^n}(d_i) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{I}_{V \cap H, H}(h_i) & \mathcal{O}_H(h_i) & \mathcal{O}_{V \cap H}(d_i) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
\]
Since $H^0(P, \mathcal{O}_P(h_i)) \to H^0(P^n, \mathcal{O}_{P_n}(d_i))$ is an isomorphism, we always get
\begin{equation}
H^0(P, \mathcal{I}_V P(h_i)) = H^1(P, \mathcal{I}_V P(h_i)) = \{0\}.
\end{equation}
Moreover, looking at the first row of the diagram, we obtain, for all $i$,
\begin{equation}
H^1(P, \mathcal{I}_V P(h_i - h_m)) = \{0\}
\end{equation}
By using (14) and (15), the first column of the diagram implies (13) for $k = 1$, as desired.

Now, let us suppose that $D$ has degree $d_i$ with $i \in \{m - 1, m - 2, \ldots, 1\}$. In order to prove that $D \in \mathcal{D}$, we apply a reduction technique to $\Omega^1_{P_n}(\log \{D_{d_i^n}, \ldots, D_{d_1^n}\})$ and to the hypersurfaces of $\mathcal{D}$ of highest degree $d_m$. Let’s start with $D_{d_m^n}$: since for this hypersurface (4) holds, there exists a non-zero surjective homomorphism $\Omega^1_{P_n}(\log D_{d_m^n}) \rightarrow \mathcal{O}_{D_{d_m^n}}$, which induces a surjective composed homomorphism $g_{d_m}
\begin{equation}
\Omega^1_{P_n}(\log \{D_{d_1^n}, \ldots, D_{d_{m-1}^n}\}) \rightarrow \Omega^1_{P_n}(\log \{D_{d_1^n}, \ldots, D_{d_{m-1}^n}\}) \rightarrow \mathcal{O}_{D_{d_m^n}}.
\end{equation}
Its kernel, denoted by $K_{d_m}$, turns to be a rank-$n$ vector bundle over $P^n$. If we apply the snake lemma to the commutative diagram
\begin{equation}
\begin{array}{cccc}
0 \rightarrow & \bigoplus_{i=1}^{m} \mathcal{O}_{P_n}(-d_i)^{\ell_i} & \rightarrow & \mathcal{O}_{P_n}(-1)^{n+1} \oplus \mathcal{O}_{P_n}^{(\sum_{i=1}^{m} \ell_i) - 1} \rightarrow \Omega^1_{P_n}(\log \mathcal{D}) \rightarrow 0 \\
\downarrow & & \downarrow & \downarrow g_{d_m} \\
0 \rightarrow & \mathcal{O}_{P_n}(-d_m) & \rightarrow & \mathcal{O}_{P_n}^{(\sum_{i=1}^{m-1} \ell_i) + (\ell_m - 1)} \rightarrow \mathcal{O}_{D_{d_m^n}} \rightarrow 0
\end{array}
\end{equation}
we get that $K_{d_m}$ admits the short exact sequence
\begin{equation}
0 \rightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_{P_n}(-d_i)^{\ell_i} \oplus \mathcal{O}_{P_n}(-d_m)^{\ell_m - 1} \rightarrow M_{d_m} \rightarrow K_{d_m} \rightarrow 0
\end{equation}
where $M_{d_m}$ is the $\left[ n + \left( \sum_{i=1}^{m-1} \ell_i \right) + (\ell_m - 1) \right] \times \left[ \left( \sum_{i=1}^{m-1} \ell_i \right) + (\ell_m - 1) \right]$ matrix obtained from the transpose of the matrix in (2) by removing the last column and row. So we have that
\begin{equation}
K_{d_m} \cong \Omega^1_{P_n}(\log \{D_{d_1^n}, \ldots, D_{d_1^{\ell_1}}, D_{d_2^n}, \ldots, D_{d_2^{\ell_2}}, \ldots, D_{d_m^n}, \ldots, D_{d_{m-1}^{\ell_{m-1}}}\})
\end{equation}
i.e. $K_{d_m}$ is the logarithmic bundle associated to $\mathcal{D} - \{D_{d_m^n}\}$. In particular, $D$ satisfies the condition
\begin{equation}
H^0(D, K_{d_m}^\vee |_D) \neq \{0\},
\end{equation}
that is $D$ is unstable for $K^{d_m}_{\ell_m}$. Indeed, if we apply $\mathcal{H}om(\cdot, \mathcal{O}_{P^n})$ to the short exact sequence

$$0 \to K^{d_m}_{\ell_m} \to \Omega^1_{P^n}(\log D) \xrightarrow{g_{\ell_m}} \mathcal{O}_{D^{d_m}_{\ell_m}} \to 0$$

we get

$$(17) \quad 0 \to \Omega^1_{P^n}(\log D)^\vee \to K^{d_m}_{\ell_m} \to \mathcal{O}_{D^{d_m}_{\ell_m}}(d_m) \to 0.$$ 

So, if we restrict $(17)$ to $D$ and then we consider the induced cohomology sequence, we obtain an injective map

$$H^0(D, \Omega^1_{P^n}(\log D)^\vee) \to H^0(D, K^{d_m}_{\ell_m} \vert_D),$$

which implies $(16)$. Now, starting from $K^{d_m}_{\ell_m}$, we iterate this technique for $D^{d_m}_{\ell_m-1}, D^{d_m}_{\ell_m-2}, \ldots, D^{d_m}_1$ and we get a sequence of rank-$n$ vector bundles $K^{d_m}_{\ell_m-1}, K^{d_m}_{\ell_m-2}, \ldots, K^d_1$ over $P^n$ such that, for all $s \in \{1, \ldots, \ell_m - 1\}$,

$$0 \to K^{d_m}_{\ell_m-s} \to K^{d_m}_{\ell_m-(s-1)} \xrightarrow{g_{\ell_m-s}} \mathcal{O}_{D^{d_m}_{\ell_m-s}} \to 0$$

is a short exact sequence and

$$K^{d_m}_{\ell_m-s} \cong \Omega^1_{P^n}(\log\{D^{d_1}_1, \ldots, D^{d_{\ell_1}}_1, \ldots, D^{d_m}_{1}, \ldots, D^{d_m}_{\ell_m-(s+1)}\}).$$

In particular

$$K^d_1 \cong \Omega^1_{P^n}(\log\{D^{d_1}_1, \ldots, D^{d_{\ell_1}}_1, \ldots, D^{d_{\ell_{m-1}}}_1\})$$

and the smooth irreducible hypersurface $D$ of degree $d_i$ is unstable for $K^d_1$. If $i = m - 1$, then $D$ is a hypersurface of highest degree in the arrangement $\mathcal{D} = \{D^{d_m}_1, \ldots, D^{d_m}_{\ell_m}\}$ and so, by repeating the computations of the first case of this proof, we get that there exists $j \in \{1, \ldots, \ell_m - 1\}$ such that $D = D^{d_m}_j$.

If $i = m - 2$, we apply the reduction technique to $K^{d_m}_1$ and to the hypersurfaces $\{D^{d_{m-1}}_{\ell_m-1}, \ldots, D^{d_{m-1}}_{1}\}$ and so on.

If $i = 1$, with this method $D$ turns to be unstable for the logarithmic bundle $\Omega^1_{P^n}(\log\{D^{d_1}_1, \ldots, D^{d_{\ell_1}}_1\})$ and so, from Theorem 3.2 it follows that there exists $j \in \{1, \ldots, \ell_1\}$ such that $D = D^{d_1}_j$, which concludes the proof. \hfill $\square$

We have the following:

**Corollary 4.3.** If $\ell_i \geq \binom{n+d_i}{d_i} + 3$, for all $i \in \{1, \ldots, m\}$, then the map

$$\mathcal{D} = \{D^{d_1}_1, \ldots, D^{d_{\ell_1}}_1, \ldots, D^{d_m}_{1}, \ldots, D^{d_m}_{\ell_m}\} \to \Omega^1_{P^n}(\log \mathcal{D})$$

is generically injective.

**Remark 4.4.** Hypothesis 1. of Theorem 4.2 implies 3.

**Remark 4.5.** We don’t know if we can state a Torelli type theorem like 4.2 without assuming 2. and 3.
In the case of arrangements with lines and conics in the projective plane, that is \( d_1 = 1 \) and \( d_2 = n = 2 \), hypothesis 1. of Theorem 1.2 translates in \( \ell_1 \geq 6 \) and \( \ell_2 \geq 9 \). In the next three sections we describe this kind of arrangements when \( \ell_1 \in \{1, 2, 3\} \) and \( \ell_2 = 1 \).

5. A CONIC AND A LINE

Let \( \mathcal{D} = \{L, C\} \) be an arrangement with normal crossings in \( \mathbb{P}^2 \) made of a line \( L \) and a conic \( C \). Without loss of generality, we can assume \( L = \{f_1 = 0\} \) and \( C = \{f_2 = 0\} \), with \( f_1 = x_0 \) and \( f_2 = \sum_{i,j=0}^{2} a_{ij} x_i x_j \), \( (a_{ij})_{0 \leq i,j \leq 2} \in GL(2, \mathbb{C}) \), so that, by applying Gaussian elimination to the matrix of \([2] \), we can get the minimal resolution for \( \Omega^1_{\mathbb{P}^2} \).

\[
(18) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^2} \rightarrow \Omega^1_{\mathbb{P}^2} \rightarrow 0
\]

with

\[
M = \begin{pmatrix}
2 \partial_1 f_2 \\
2 \partial_2 f_2 \\
-2 x_0 \partial_0 f_2
\end{pmatrix}.
\]

As a consequence we get that \( c_1(\Omega^1_{\mathbb{P}^2} ) = 0 \), \( c_2(\Omega^1_{\mathbb{P}^2} ) = 1 \) and, according to Bohnhorst-Spindler criterion \( [5] \), that \( \Omega^1_{\mathbb{P}^2} \) is a semistable vector bundle over \( \mathbb{P}^2 \). In this regard we give the following:

**Theorem 5.1.** Let \( \mathbf{M}^{ss}_{\mathbb{P}^2}(0, 1) \) be the family of semistable rank-2 vector bundles \( E \) over \( \mathbb{P}^2 \) with minimal resolution

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{t(\ell_1 \quad \ell_2 \quad q)} \mathcal{O}_{\mathbb{P}^2}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^2} \rightarrow E \rightarrow 0
\]

where \( \ell_1, \ell_2 \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \) and \( q \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \). Then the map

\[
\mathbf{M}^{ss}_{\mathbb{P}^2}(0, 1) \xrightarrow{\pi_2} \mathbb{P}^2
\]

\[
E \mapsto \{\ell_1 = 0\} \cap \{\ell_2 = 0\}
\]

is an isomorphism.

**Proof.** Let \( E \) and \( E' \) be two elements of \( \mathbf{M}^{ss}_{\mathbb{P}^2}(0, 1) \). We have to prove that the intersection point of \( \ell_1 \) and \( \ell_2 \) coincides with the one of \( \ell'_1 \) and \( \ell'_2 \) if and only if \( E \cong E' \). If the intersection point is the same, without loss of generality we can assume that \( \ell_1 = \ell'_1 = x_0 \) and \( \ell_2 = \ell'_2 = x_1 \). We remark that, for all \( x \in \mathbb{P}^2 \), \( E_x \) and \( E'_x \) are the cokernels of two rank-1 maps, in particular if \( x = [0, 0, 1] \) then \( q \) and \( q' \) have to contain the term \( x_2^2 \). Thus, \( E \cong E' \) if and only if there exists \( g_1, g_2 \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \) such that \( q' - q = g_1 x_0 + g_2 x_1 \). Direct computations allow to conclude the proof (for more details see [3]).

**Remark 5.2.** Each \( E \in \mathbf{M}^{ss}_{\mathbb{P}^2}(0, 1) \) is logarithmic for a line and a conic.
Remark 5.3. Theorem 5.1 asserts that $\Omega_{\mathbb{P}^2}^1(\log D)$ lives in 2-dimensional space, while the number of parameters associated to a line and a conic with normal crossings is 7. So we can immediately conclude that arrangements like these are not of Torelli type.

With the aid of the description given in Theorem 5.1, we get the following result, the proof of which is left to the reader:

**Proposition 5.4.** $\pi_2(\Omega_{\mathbb{P}^2}^1(\log D))$ is the pole of $L$ with respect to $C$.

We immediately get the following:

**Corollary 5.5.** Let $D = \{L, C\}$ and $D' = \{L', C'\}$ be arrangements with normal crossings in $\mathbb{P}^2$ given by a line and a conic. Then

$$\Omega_{\mathbb{P}^2}^1(\log D) \cong \Omega_{\mathbb{P}^2}^1(\log D')$$

if and only if the pole of $L$ with respect to $C$ coincides with the pole of $L'$ with respect to $C'$.

![Figure 1](image-url). $L$ is the polar line of $P$ with respect to $C$

**Remark 5.6.** These results can be extended in a natural way to the case of a multi-degree arrangement $D$ with normal crossings in $\mathbb{P}^n$, $n \geq 3$, made of a hyperplane $H$ and a smooth quadric $Q$. In this setting $\Omega_{\mathbb{P}^n}^1(\log D)$ is no more semistable over $\mathbb{P}^n$, but its isomorphism class is still described by the pole of $H$ with respect to $Q$. For more details, see [3].

6. A CONIC AND TWO LINES

Let $D = \{L_1, L_2, C\}$ be an arrangement with normal crossings in $\mathbb{P}^2$, where $L_i$ is a line and $C$ is a conic. We can assume that $L_1 = \{f_1 = 0\}$, $L_2 = \{f_2 = 0\}$ and $C = \{f_3 = 0\}$ where $f_1 = x_0$, $f_2 = x_1$ and $f_3 = \sum_{i,j=0}^2 a_{ij} x_i x_j$, $(a_{ij})_{0 \leq i, j \leq 2} \in GL(2, \mathbb{C})$, so that $\Omega_{\mathbb{P}^2}^1(\log D)$ fits in the minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^2 \to \Omega_{\mathbb{P}^2}^1(\log D) \to 0$$
where
\[
M = \begin{pmatrix}
2 \partial_2 f_3 \\
-2 x_0 \partial_0 f_3 \\
-2 x_1 \partial_1 f_3
\end{pmatrix}.
\]

In particular, (19) implies that the normalized bundle \( \Omega_{P^2}^1(\log D)(-1) \) belongs to \( M_{P^2}(-1, 2) \), the moduli space of rank-2 stable vector bundles over \( P^2 \) with \( c_1 = -1 \) and \( c_2 = 2 \). In the following result, which is likely to be known to experts, we give an interesting description of \( M_{P^2}(-1, 2) \); in order to state it, we denote by \( \sigma_2(\nu_2(P^2)) \) the 2-secant variety of the image \( \nu_2(P^2) \) of the quadratic Veronese map.

**Theorem 6.1.** \( M_{P^2}(-1, 2) \) is isomorphic to \( \sigma_2(\nu_2(P^2)) - \nu_2(P^2) \), the projective space of symmetric \( 3 \times 3 \) rank-2 matrices.

**Proof.** A vector bundle \( E \) lives in \( M_{P^2}(-1, 2) \) if and only if it is endowed with a short exact sequence like
\[
0 \longrightarrow \mathcal{O}_{P^2}(-3) \xrightarrow{\text{1}(\ell_1, q_1, q_2)} \mathcal{O}_{P^2}(-2) \oplus \mathcal{O}_{P^2}^2(-1) \longrightarrow E \longrightarrow 0
\]
where \( \ell_1 \in H^0(P^2, \mathcal{O}_{P^2}(1)) \) and \( q_1, q_2 \in H^0(P^2, \mathcal{O}_{P^2}(2)) \).

On the unique jumping line of \( E \), which is \( \{ \ell_1 = 0 \} \), the linear series given by \( q_1 \) and \( q_2 \) has two distinct double points, which we denote by \( P_1 \) and \( P_2 \). Then the map given by
\[
M_{P^2}(-1, 2) \longrightarrow \sigma_2(\nu_2(P^2)) - \nu_2(P^2)
\]
\[
E \longmapsto \{ P_1, P_2 \}
\]
is an isomorphism, which concludes the proof. \( \square \)

**Remark 6.2.** Theorem 6.1 implies that \( \Omega_{P^2}^1(\log D)(-1) \) is characterized by 4 parameters, while \( D \) needs 9 parameters to be described. So in this case \( D \) is not a Torelli arrangement.

**Remark 6.3.** The jumping line of \( \Omega_{P^2}^1(\log D) \) is \( \{ \partial_2 f_3 = 0 \} \) and it is the polar line with respect to \( C \) of \( L_1 \cap L_2 = [0, 0, 1] \). Moreover, the linear series on this line is given by \( L_1 \cup s_2 \) and \( L_2 \cup s_1 \), where \( s_2 \) is the polar line with respect to \( C \) of \( \{ \partial_2 f_3 = 0 \} \cap L_2 = [a_{22}, 0, -a_{02}] \) and \( s_1 \) is the polar line with respect to \( C \) of \( \{ \partial_2 f_3 = 0 \} \cap L_1 = \{ 0, a_{22}, -a_{12} \} \), that is \( s_2 = \{ a_{22} \partial_0 f_3 - a_{02} \partial_2 f_3 = 0 \} \) and \( s_1 = \{ a_{22} \partial_1 f_3 - a_{12} \partial_2 f_3 = 0 \} \). The logarithmic bundle \( \Omega_{P^2}^1(\log D) \) corresponds to the two intersection points \( \{ P_1, P_2 \} \) of \( C \) and \( \{ \partial_2 f_3 = 0 \} \).

**Corollary 6.4.** Let \( D = \{ L_1, L_2, C \} \) and \( D' = \{ L'_1, L'_2, C' \} \) be arrangements with normal crossings in \( P^2 \) made of two lines and a conic. Then
\[
\Omega_{P^2}^1(\log D) \cong \Omega_{P^2}^1(\log D') \iff \{ P_1, P_2 \} = \{ P'_1, P'_2 \}.
\]
7. A CONIC AND THREE LINES

Let $\mathcal{D} = \{L_1, L_2, L_3, C\}$ be an arrangement with normal crossings in $\mathbb{P}^2$ made of three lines and a conic, let say $L_1 = \{x_0 = 0\}$, $L_2 = \{x_1 = 0\}$, $L_3 = \{x_2 = 0\}$ and $C = \{f_4 = 0\}$ where

$$f_4 = \sum_{i,j=0}^{2} d_{ij} x_i x_j, \quad (d_{ij})_{0 \leq i,j \leq 2} \in GL(2, \mathbb{C}).$$

In this case (20) gives the minimal resolution for the logarithmic bundle

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^3 \rightarrow \Omega_{\mathbb{P}^2}^1(\log \mathcal{D}) \rightarrow 0$$

where

$$M = \begin{pmatrix} -x_0 \partial_0 f_4 \\ -x_1 \partial_1 f_4 \\ -x_2 \partial_2 f_4 \end{pmatrix}.$$ 

From (21) we get that $\Omega_{\mathbb{P}^2}^1(\log \mathcal{D})$ is stable and that its normalized bundle $\Omega_{\mathbb{P}^2}^1(\log \mathcal{D})(-1)$ lives in the moduli space $\mathcal{M}_{\mathbb{P}^2}(0, 3)$, which has dimension 9, as we can see in [17]. Since the number of parameters associated to three lines and a conic is 11, also in this case we can’t get a Torelli type theorem.

By using the second part of Theorem 2.3, we note that $\Omega_{\mathbb{P}^2}^1(\log \mathcal{D})(-1)$ admits an exact sequence like the one for the logarithmic bundle of a smooth plane cubic curve. The link between these two vector bundles is explained in the following result:

**Theorem 7.1.** Let $\mathcal{D}$ be the multi-degree arrangement with normal crossings on $\mathbb{P}^2$ given by $\{x_0 x_1 x_2 f_4 = 0\}$, where $f_4$ is as in [24]. Then there exists $\mathcal{D}' = \{D\}$, where $D \subset \mathbb{P}^2$ is a smooth cubic curve, such that

$$\Omega_{\mathbb{P}^2}^1(\log \mathcal{D}) \cong \Omega_{\mathbb{P}^2}^1(\log \mathcal{D}')(1).$$
Proof. We have to find $g \in H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(3))$ such that, for all $i \in \{0, 1, 2\}$,
\[
\partial_i g = e_0^i(-x_0 \partial_0 f_4) + e_1^i(-x_1 \partial_1 f_4) + e_2^i(-x_2 \partial_2 f_4)
\]
for certain $e_j^i \in \mathbb{C}$.

By using Schwarz’s theorem, from (22) we get, for all $i, h \in \{0, 1, 2\}, i \neq h$,
\[
\sum_{j=0}^{2} e_j^i \partial_i (x_j \partial_j f_4) = \sum_{j=0}^{2} e_j^h \partial_h (x_j \partial_j f_4).
\]

Let denote by $\{a_{uv}^j\}$ the coefficients of $x_j \partial_j f_4$ for $j \in \{0, 1, 2\}$; by using the identity principle for polynomials we get the following system of $9$ equations with variables $e_j^i$: for all $i, u, v \in \{0, 1, 2\}$, $i \neq u$,
\[
\sum_{j=0}^{2} a_{uv}^j e_j^i = \sum_{j=0}^{2} a_{uv}^j e_j^u.
\]

Since $a_{uv}^j$ depend on the coefficients of $f_4$, the matrix of (23) is:
\[
H = \begin{pmatrix}
  d_{01} & d_{01} & 0 & -2d_{00} & 0 & 0 & 0 & 0 & 0 \\
  0 & 2d_{11} & 0 & -d_{01} & -d_{01} & 0 & 0 & 0 & 0 \\
  0 & d_{12} & d_{12} & -d_{02} & 0 & -d_{02} & 0 & 0 & 0 \\
  d_{02} & 0 & d_{02} & 0 & 0 & 0 & -2d_{00} & 0 & 0 \\
  0 & d_{12} & d_{12} & 0 & 0 & 0 & -d_{01} & -d_{01} & 0 \\
  0 & 0 & 2d_{22} & 0 & 0 & 0 & -d_{02} & 0 & -d_{02} \\
  0 & 0 & 0 & d_{02} & 0 & d_{02} & -d_{01} & -d_{01} & 0 \\
  0 & 0 & 0 & 0 & d_{12} & d_{12} & 0 & -2d_{11} & 0 \\
  0 & 0 & 0 & 0 & 0 & 2d_{22} & 0 & -d_{12} & -d_{12}
\end{pmatrix}.
\]

It’s not hard to prove that $\text{rank } H = 8$, i.e the dimension of the space of solutions of (23) is $\infty^1$. So, let us assume that $e_j^i$ solve our system, we need a cubic polynomial $g$ such that conditions in (22) are satisfied. Let integrate with respect to $x_0$ the equation (22) with $i = 0$, we get
\[
g(x_0, x_1, x_2) = -2x_0 x_1^2 \left( x_0 \frac{x_0^0}{3} a_{00}^0 + \frac{x_1}{2} a_{01}^0 + \frac{x_2}{2} a_{02}^0 \right) + h(x_1, x_2) + 2x_0 \left[ \varepsilon_0^0 \left( \frac{x_0 x_1}{2} a_{01}^0 + x_1 x_2 a_{12}^0 \right) + \varepsilon_2^0 \left( \frac{x_0 x_2}{2} a_{02}^0 + x_1 x_2 a_{12}^0 \right) \right]
\]

where $h$ is a function to be determined. If we compute $\partial_1 g$ from (24), we substitute it in (22) with $i = 1$ and we integrate with respect to $x_1$ we get
\[
(25) \quad h(x_1, x_2) = x_0 x_1 \left[ \varepsilon_0^0 x_0 a_{00}^0 + 2 \varepsilon_1^0 \left( \frac{x_0}{2} a_{01}^0 + x_1 a_{11}^0 + x_2 a_{12}^0 \right) \right] + x_0 x_1 \left[ 2 \varepsilon_2^0 x_2 a_{12}^0 - \varepsilon_0^1 \left( 2x_0 a_{00}^0 + x_1 a_{01}^0 + 2x_2 a_{02}^0 \right) + i(x_2) \right] + \left[ \left( x_0 a_{00}^0 + 2 \frac{x_1}{3} a_{01}^0 + x_2 a_{12}^0 \right) - x_1 x_2 \varepsilon_1^2 \left( 2x_0 a_{02}^0 + x_1 a_{12}^0 + 2x_2 a_{22}^0 \right) \right]
\]
where we have to determine the function \( i \). Finally, if we compare \( \partial_2 g \) from [24] with [22] for \( i = 2 \), using also [25] and we integrate with respect to \( x_2 \), we can find explicitly, so that the required polynomial is

\[
g(x_0, x_1, x_2) = -\frac{2}{3} \varepsilon_0 a_0 x_0^3 - 2 \varepsilon_1 a_1 x_1 x_0^2 - 2 \varepsilon_2 a_2 x_2 x_0^2 - 2 \varepsilon_0 a_0 x_0^2 x_2 - 2 (\varepsilon_1 + \varepsilon_2) a_1 x_1 x_2 - 2 \varepsilon_2 a_2 x_2 x_2 - 2 \varepsilon_1 a_1 x_1 x_2 - \frac{2}{3} \varepsilon_2 a_2 x_2^3.
\]

**Remark 7.2.** The proof of Theorem 7.1 implies also Hermite’s Theorem (1868), which asserts that a net of conics can be regarded as the net of the polar conics with respect to a given cubic curve (see [12], book III, chapter III, section 29).

**Remark 7.3.** If we require that \( \partial_i g = x_i \partial_i f_4 \), for all \( i \in \{0, 1, 2\} \), then

\[
g(x_0, x_1, x_2) = \frac{2}{3} (a_0^0 x_0^3 + a_1^0 x_1^3 + a_2^0 x_2^3),
\]

provided that the conic is given by \( f_4(x_0, x_1, x_2) = d_0 x_0^2 + d_1 x_1^2 + d_2 x_2^2 \). So, let \( D = \{x_0 x_1 x_2 f_4 = 0\} \) and \( D' = \{x_0 x_1 x_2 f_4' = 0\} \) be two arrangements with normal crossings in \( \mathbb{P}^2 \) each of which with a conic given by a diagonalized quadratic form. \( D \) and \( D' \) correspond to a logarithmic bundle which is isomorphic to the logarithmic bundle of a smooth cubic like the one of [26]. Since in [24] it is proved that two smooth cubics which are both Fermat yield isomorphic logarithmic bundles, then \( \Omega_{\mathbb{P}^2}(\log D) \cong \Omega_{\mathbb{P}^2}(\log D') \).

**Remark 7.4.** Although we know that a multi-degree arrangement with three lines and a conic, because of parameters computations, isn’t Torelli and that Theorem 7.1 holds, in this case the problem of determining the fiber of \( \beta \) is still open.

### 8. Arrangements with Few Conics

Let \( D = \{C_1, \ldots, C_8\} \) be an arrangement of \( \ell \in \{4, \ldots, 8\} \) conics with normal crossings on \( \mathbb{P}^2 \). According to the results recalled in § 3, if \( \ell \in \{1, 2\} \) then we don’t get Torelli type arrangements. The case \( \ell = 3 \) will be discussed in § 9.

Let \( F^2_5 = \{(y, x) \in \mathbb{P}^2 \times \mathbb{P}_5 \mid x \in C_y\} \) be the incidence variety point-conic in \( \mathbb{P}^2 \times \mathbb{P}_5 \), where \( C_y \subset \mathbb{P}^2 \) denotes the conic defined by the point \( y \in \mathbb{P}_5 \) with the Veronese correspondence and let \( \alpha, \beta \) the restrictions to \( F^2_5 \) of the usual projections \( \alpha \) and \( \beta \):

\[
\begin{array}{ccc}
\mathbb{P}^2 & \leftarrow & \mathbb{P}_5 \\
\alpha & \leftarrow & \beta \\
\mathbb{P}^2 & \leftarrow & \mathbb{P}_5
\end{array}
\]
Remark 8.1. Let $UC(\Omega^1_{P_2}(\log D))$ be the set of unstable conics of $\Omega^1_{P_2}(\log D)$, in the sense of Definition 2.4. $UC(\Omega^1_{P_2}(\log D))$ coincides with the support of the first direct image sheaf $R^1(\beta_*\alpha^*\Omega^1_{P_2}(\log D)(-1))$: indeed, for all $y \in P_5$,

$$R^1(\beta_*\alpha^*\Omega^1_{P_2}(\log D)(-1))_y = H^1(\beta^{-1}(y), \alpha^*\Omega^1_{P_2}(\log D)(-1)|_{\beta^{-1}(y)}) = H^1(C_y, \Omega^1_{P_2}(\log D)(-1)|_{c_y}) = H^0(C_y, \Omega^1_{P_2}(\log D)^\vee_{|c_y})^\vee,$$

where the last inequality follows from Serre’s duality.

So, let do the tensor product with $O_{P^2}(-1)$ of (28) computed in the case of conics and let apply the functor $\beta_*\alpha^*$, we get:

$$(27) \quad 0 \rightarrow R^0\beta_*\alpha^*(O_{P^2}(-3))^\ell \rightarrow R^0\beta_*\alpha^*(O_{P^2}(-2)^3 \oplus O_{P^2}(-1)^{\ell-1}) \rightarrow$$

$$\rightarrow R^0\beta_*\alpha^*(\Omega^1_{P_2}(\log D)(-1)) \rightarrow R^1\beta_*\alpha^*(O_{P^2}(-3))^\ell \rightarrow$$

$$\rightarrow R^1\beta_*\alpha^*(O_{P^2}(-2)^{\ell-1} \oplus O_{P^2}(-1)^3) \rightarrow R^1\beta_*\alpha^*(\Omega^1_{P_2}(\log D)(-1)) \rightarrow 0.$$

In order to determine the terms in (27) we consider

$$0 \rightarrow O_{P^2 \times P_5}(-2, -1) \rightarrow O_{P^2 \times P_5} \rightarrow O_{F^2} \rightarrow 0,$$

we do the tensor product with $\alpha^*(O_{P^2}(t))$, where $t \in \{-1, -2, -3\}$ and we apply the functor $\beta_*$. In this way (27) becomes

$$(28) \quad 0 \rightarrow R^0\beta_*\alpha^*(\Omega^1_{P_2}(\log D)(-1)) \rightarrow (\Omega^1_{P_5})^\ell \rightarrow$$

$$\rightarrow (O_{P_5}(-1)^3 \oplus O_{P_5}(-1)^{\ell-1}) \rightarrow R^1\beta_*\alpha^*(\Omega^1_{P_2}(\log D)(-1)) \rightarrow 0.$$

Remark 8.2. In order to investigate $UC(\Omega^1_{P_2}(\log D))$, it suffices to study the cokernel of the map $F$ appearing in (28).

Remark 8.3. More in general, all the previous arguments can be applied to a vector bundle $E$ fitting in an exact sequence like the one of $\Omega^1_{P_2}(\log D)$.

Now, let us assume that $\ell = 4$. In what follows, by using Macaulay2 software system, we produce $D_0 = \{C_{0,1}, C_{0,2}, C_{0,3}, C_{0,4}\}$ such that

$$(29) \quad UC(\Omega^1_{P_2}(\log D_0)) = \{C_{0,1}, C_{0,2}, C_{0,3}, C_{0,4}\}.$$

Example

$D_0$ is made of four smooth random conics with normal crossings:

$C_{0,1} : 42x_0^2 - 50x_0x_1 + 9x_1^2 + 39x_0x_2 - 15x_1x_2 - 22x_2^2 = 0,$

$C_{0,2} : 50x_0^2 + 45x_0x_1 - 39x_1^2 - 29x_0x_2 + 30x_1x_2 + 19x_2^2 = 0,$

$C_{0,3} : -38x_0^2 + 2x_0x_1 - 36x_1^2 - 4x_0x_2 - 16x_1x_2 - 6x_2^2 = 0,$

$C_{0,4} : -32x_0^2 + 31x_0x_1 - 38x_1^2 - 32x_0x_2 + 31x_1x_2 + 24x_2^2 = 0.$

By multiplying the four polynomials defining the conics, we get the polynomial $f \in k[x_0, x_1, x_2]^8 = R_8$ associated to $D_0$, where $k$ is the field $Z_{101}$.

According to Definition 2.2, we consider the kernel $E$ of the Gauss map $M \in M_{0,4}(R)$ associated to the module defining $\Omega^1_{P_2}(\log D_0)$. Then we determine the elements of $UC(\Omega^1_{P_2}(\log D_0))$: as
we can see in Remark 8.1, $UC(\Omega^1_{\mathbb{P}^2}(\log D_0))$ is the zero locus of the order 4 minors of the matrix $Z$, whose cokernel is equal to the cokernel of $F$. In particular, posing $T = k[y_0, \ldots, y_5]$, $Z \in M_{4, 24}(T)$ is the product of $C \in M_{4, 12}(T)$ and $B \in M_{24, 12}(T)$, where $C$ is the matrix of variables needed to get $\Omega^1_{\mathbb{P}^2}$ and $B$ is the syzygy matrix of $A \in M_{12, 24}(T)$ whose entries are the coefficients of the polynomials in $M$. The ideal $J$ generated by the $4 \times 4$ minors of $Z$ has dimension 1 and degree 4, from which (29) follows.

This is the script of our algorithm.

```plaintext
k=ZZ/101
R=k[x_0..x_2]
ran=random(R^{1:0},R^{4:-2})
f=1_R; for t from 0 to rank source ran-1 do f=f*(ran_(0,t))
E=ker map(R^{1:-1+(degree f)_0},R^{3:0},diff(vars R,f))
M=(res dual E).dd_1
T=k[y_0..y_5]
coe=(M,k,i,j)->diff(symmetricPower(k,vars R),transpose(symmetricPower(k-2,vars R))*submatrix(M,{i},{j}))
coe2=(M,i,j)->diff(transpose(vars(R))*submatrix(M,{i},{j}),symmetricPower(2,vars R))
expa=(M,k)->matrix table(rank target M,rank source M,(i,j)->coe(M,k,i,j))
expa2=(M)->matrix table(rank target M,rank source M,(i,j)->coe2(M,i,j))
A=sub(matrix(expa(submatrix(M,{0..2},{0..3}),2), expa2(submatrix(M,{3..5},{0..3}))),T)
B=syz A
C=(id_(T^{4:0}))**(vars T)
Z=C*B
J=minors(4,Z)
dim J
degree J
```

Remark 8.4. The previous algorithm can be performed for all $\ell$. In particular, if $\ell = 5$ then we can get another example such that the unstable conics of the logarithmic bundle coincide with the conics of the arrangement.

Starting from the previous example, we can prove the following:

Theorem 8.5. If $\ell \geq 4$, then the map

$D = \{C_1, \ldots, C_\ell\} \longrightarrow \Omega^1_{\mathbb{P}^2}(\log D)$

is generically injective.

Proof. First, let us assume that $\ell = 4$. Let consider the incidence variety $W = \{ (D, C) \in (\mathbb{P}_5 \times \mathbb{P}_5 \times \mathbb{P}_5 \times \mathbb{P}_5) \times \mathbb{P}_5 | C \in UC(\Omega^1_{\mathbb{P}^2}(\log D)) \}$ and let $\overline{\alpha}, \overline{\beta}$ be the restrictions to $W$ of the projection morphisms, respectively, from $(\mathbb{P}_5 \times \mathbb{P}_5 \times \mathbb{P}_5 \times \mathbb{P}_5)$ and $\mathbb{P}_5$.

From the previous example we have that $\overline{\alpha}^{-1}(D_0) = D_0$. So, for all arrangements $D \in \mathbb{P}_5 \times \mathbb{P}_5 \times \mathbb{P}_5 \times \mathbb{P}_5$, $dim \overline{\alpha}^{-1}(D) \geq 0$ and $h^0(\overline{\alpha}^{-1}(D), \mathcal{O}_W) =$
lenght\(\pi^{-1}(D)\) \geq 4. To conclude the proof, it suffices to show that there exists \(V \subset P_5 \times P_5 \times P_5 \times P_5\) open such that, for all \(D \in V\)
\begin{equation}
(30) \quad \dim \pi^{-1}(D) = 0,
\end{equation}
\begin{equation}
(31) \quad \text{lenght} \pi^{-1}(D) = 4.
\end{equation}

We remark that the dimension \(d\) of the fiber given by the morphism \(\pi\) has the upper semicontinuity property ([16], chapter 1, section 8, corollary 3), which implies that \(\{w \in W \mid d(w) \geq 1\}\) is a closed subset in \(W\). So the set \(V_1 = \pi(\{w \in W \mid d(w) \leq 0\}) = \pi(\{w \in W \mid d(w) = 0\})\) is open in \(P_5 \times P_5 \times P_5 \times P_5\). By using the upper semicontinuity of the length of the fiber given by the morphism \(\pi\) (this fact is a consequence of theorem 12.8, chapter 3 of [14]; this theorem holds with the hypothesis of flatness, in our case we have the generic flatness) we get that the set \(\{D \in P_5 \times P_5 \times P_5 \times P_5 \mid \text{lenght} \pi^{-1}(D) \geq 5\}\) is closed in \(P_5 \times P_5 \times P_5 \times P_5\). As above, the set \(V_2 = \{D \in P_5 \times P_5 \times P_5 \times P_5 \mid \text{lenght} \pi^{-1}(D) \leq 4\} = \{D \in P_5 \times P_5 \times P_5 \times P_5 \mid \text{lenght} \pi^{-1}(D) = 4\}\) is open in \(P_5 \times P_5 \times P_5 \times P_5\). The points of the open set \(V = V_1 \cap V_2\) satisfy the required properties (30) and (31).

Now, if \(\ell \geq 5\), then we can apply the reduction technique, performed in the proof of Theorem 4.2, to \(\Omega^1_{P_5}(\log D)\) and to the conics of \(D\) at each step we get a logarithmic bundle of a conic-arrangement with one component less, till we reduce to the case of four conics, studied above. \(\square\)

9. THE CASE OF THREE CONICS

Let \(D = \{C_1, C_2, C_3\}\) be an arrangement of conics with normal crossings on \(P^2\). Let start by analyzing \(UC(\Omega^1_{P_5}(\log D))\). In order to do that, let consider the exact sequence (28) with \(\ell = 3\) \: \(UC(\Omega^1_{P_5}(\log D))\), the support of \(R^1\beta_3^*\alpha^*\Omega^1_{P_5}(\log D)(-1)\), is the maximal degeneration locus of the morphism \((\Omega^1_{P_5})^3 \to (\Omega^1_{P_5})^3 \oplus \Omega_{P_5}(-1)^2\), i.e. it coincides with the scheme \(D_{10}(F) = \{y \in P_5 \mid \text{rank}(F_y) \leq 10\}\), which, according to [18], has expected codimension 5 in \(P_5\) (we note that the computation of the expected codimension is meaningless when \(\ell \geq 4\)). If this is the case, the number of points in \(D_{10}(F)\) is determined by Porteous’ formula:
\begin{equation}
(32) \quad [D_{10}(F)] = \det[c_{1-i+j}(((\Omega_{P_5}(-1)^3) \oplus \Omega_{P_5}(-1)^2) - (\Omega^1_{P_5})^3)],
\end{equation}
where \(1 \leq i, j \leq 5\). The generic entry of the matrix (32) is the coefficient of the term of degree \((1 - i + j)\) in the formal series in one variable coming from the quotient of the Chern polynomials of \((\Omega_{P_5}(-1)^3) \oplus \Omega_{P_5}(-1)^2\) and \((\Omega^1_{P_5})^3\). Thus \([D_{10}(F)] = 21\). More in general, we get the following:

**Proposition 9.1.** Let \(E\) be a vector bundle over \(P^2\) such that
\[ 0 \to \mathcal{O}_{P_5}(-2)^3 \to \mathcal{O}_{P_5}(-1)^3 \oplus \mathcal{O}_{P_2}^2 \to E \to 0 \]
is exact and let \(UC(E)\) be the set of unstable conics of \(E\), in the sense of [4]. \(UC(E)\) is expected to be a 0-dimensional scheme of \(P_5\) with 21 points.
Remark 9.2. If we apply the algorithm performed in the Example of section 8 in the case of \( \ell = 3 \), we can find some arrangements \( \mathcal{D} \) such that \( UC(\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})) \) satisfies the expected properties of Proposition 9.1. More in detail, according to the notations introduced in such algorithm, the variety in \( \mathbb{P}_5 \) defined by the ideal \( J \) has 21 distinct points, which are smooth conics in \( \mathbb{P}^2 \): 3 of them belong to \( \mathcal{D} \) and the remaining 18 belong to a net quadrics in \( \mathbb{P}_5 \), which intersect in a K3-surface with 12 singular points, that don’t seem to be related to the 18 conics we are interested in. The problem of determining explicitly such 18 points or, equivalently, of finding a primary decomposition of \( J \) saturated with the ideals defining the points of \( \mathcal{D} \) in \( \mathbb{P}_5 \), seems to be hard, also with a computer.

According to Remark 9.2, instead of studying \( UC(\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})) \), we can focus on \( UL(\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})) \), the set of unstable lines of \( \Omega^1_{\mathbb{P}^2}(\log \mathcal{D}) \) in the sense of Definition 2.4. Let \( F_2^3 \) be the incidence variety point-line in \( \mathbb{P}^2 \times \mathbb{P}^2 \), i.e.

\[
F_2^3 = \{(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid x \in L_y\}
\]

where \( L_y \subset \mathbb{P}^2 \) is the line defined by \( y \in \mathbb{P}^2 \) and \( \overline{p}, \overline{q} \) be, respectively, the restrictions to \( F_2^3 \) of the projection maps \( p, q \) as in the following diagram:

\[
\begin{array}{ccc}
\mathbb{P}^2 & \overset{\overline{p}}{\leftarrow} & \mathbb{P}^2 \\
\downarrow & & \downarrow \\
F_2^3 & \overset{\overline{q}}{\leftarrow} & \mathbb{P}^2
\end{array}
\]

We remark that \( UL(\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})) \), as a subset of \( \mathbb{P}^2 \), is the support of \( R^1(\overline{q}_*\overline{p}^*\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})(-2)) \). Namely, if \( y \in \mathbb{P}^2 \) then we have that

\[
R^1(\overline{q}_*\overline{p}^*\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})(-2))_y = H^1(\overline{q}^{-1}(y), \overline{p}^*\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})(-2)|_{\overline{q}^{-1}(y)}) = \]

\[
= H^1(L_y, \Omega^1_{\mathbb{P}^2}(\log \mathcal{D})(-2)|_{L_y}) = H^0(L_y, \Omega^1_{\mathbb{P}^2}(\log \mathcal{D})^\vee|_{L_y})^\vee,
\]

where the last equality follows from Serre’s duality. In order to study this support, we apply the functor \( \overline{q}_*\overline{p}^* \) to the exact sequence (2) in the case of three conics twisted by \(-2\) and we get

\[
0 \longrightarrow R^0\overline{q}_*\overline{p}^*(\mathcal{O}_{\mathbb{P}^2}(-4)^3) \longrightarrow R^0\overline{q}_*\overline{p}^*(\mathcal{O}_{\mathbb{P}^2}(-3)^3 \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^2) \longrightarrow \\
\hspace{2cm} \longrightarrow R^0\overline{q}_*\overline{p}^*(\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})(-2)) \longrightarrow R^1\overline{q}_*\overline{p}^*(\mathcal{O}_{\mathbb{P}^2}(-4)^3) \longrightarrow \\
\hspace{2cm} \longrightarrow R^1\overline{q}_*\overline{p}^*(\mathcal{O}_{\mathbb{P}^2}(-3)^3 \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^2) \longrightarrow R^1\overline{q}_*\overline{p}^*(\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})(-2)) \longrightarrow 0.
\]

Our aim is to describe the terms of (34). So we do the tensor product of

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-1, -1) \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2} \longrightarrow \mathcal{O}_{F_2^3} \longrightarrow 0
\]

with \( p^*\mathcal{O}_\mathbb{P}(t) \), where \( t \in \{-4, -3, -2\} \) and then we apply \( q_* \). By using Serre’s duality and the Poincaré-Euler sequence, (34) turns to be

\[
0 \longrightarrow R^0\overline{q}_*\overline{p}^*(\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})(-2)) \longrightarrow R^1\overline{q}_*\overline{p}^*(\mathcal{O}_{\mathbb{P}^2}(-4)^3) \xrightarrow{G} \\
\hspace{2cm} \xrightarrow{G} (\Omega^1_{\mathbb{P}^2})^3 \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^2 \longrightarrow R^1\overline{q}_*\overline{p}^*(\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})(-2)) \longrightarrow 0
\]
where $R^1\mathcal{q}_*\mathcal{P}^*(\mathcal{O}_{\mathbf{P}^2}(−4))$ fits in the exact sequence

$$0 \to R^1\mathcal{q}_*\mathcal{P}^*(\mathcal{O}_{\mathbf{P}^2}(−4)) \to \mathcal{O}_{\mathbf{P}^2}(−1)^6 \to \mathcal{O}_{\mathbf{P}^2}^3 \to 0,$$

and it has rank $3$ over $\mathbf{P}^2$. The support of $R^1(\mathcal{q}_*\mathcal{P}^*(\log \mathcal{D}))(−2)$ is the maximal degeneration locus of the morphism $\mathcal{G}$ in $\{35\}$, i.e. it’s the scheme $D_7(\mathcal{G}) = \{y \in \mathbf{P}^2\colon \text{rank}(\mathcal{G}_y) \leq 7\}$. According to $\{18\}$, the expected codimension over $\mathbf{P}^2$ of $D_7(\mathcal{G})$ is $2$, that is we expect a finite number of unstable lines for $\Omega^1_{\mathbf{P}^2}(\log \mathcal{D})$. Assuming that $D_7(\mathcal{G})$ is $0$-dimensional, the number of its points is given by Porteous’ formula:

$$[D_7(\mathcal{G})] = det[c_{1−i+j}((\Omega^1_{\mathbf{P}^2})^3 \oplus \mathcal{O}_{\mathbf{P}^2}(−1)^3) − R^1\mathcal{q}_*\mathcal{P}^*(\mathcal{O}_{\mathbf{P}^2}(−4)^3))],$$

where $1 \leq i, j \leq 2$. The generic entry of $[D_7(\mathcal{G})]$ is the coefficient of the degree-$(1−i+j)$ term of the formal series in one variable defined as the quotient of the Chern polynomial of $(\Omega^1_{\mathbf{P}^2})^3 \oplus \mathcal{O}_{\mathbf{P}^2}(−1)^2$ with the one of $R^1\mathcal{q}_*\mathcal{P}^*(\mathcal{O}_{\mathbf{P}^2}(−4)^3)$. So $[D_7(\mathcal{G})] = 21$. These arguments imply the following:

**Proposition 9.3.** Let $\mathcal{D} = \{C_1, C_2, C_3\}$ be a normal crossing arrangement of conics in $\mathbf{P}^2$. $UL(\Omega^1_{\mathbf{P}^2}(\log \mathcal{D}))$ is expected to be a $0$-dimensional scheme of $\mathbf{P}^2$ with $21$ points.

**Remark 9.4.** The previous proposition holds, more in general, for all vector bundles $E$ over $\mathbf{P}^2$ admitting the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2}(−2)^3 \to \mathcal{O}_{\mathbf{P}^2}(−1)^3 \oplus \mathcal{O}_{\mathbf{P}^2}^2 \to E \to 0.$$
of $k[b, c]/J$ and let $\text{compb}$ (resp. $\text{compc}$) be the $21 \times 21$ matrix associated, with respect to $B$, to the linear map (companion)

$$k[b, c]/J \longrightarrow k[b, c]/J$$

defined by the multiplication by $b$ (resp. by $c$). By using eigenvectors techniques (6), the pair of parameters $(b_i, c_i)$ of a given unstable line is such that $b_i$ (resp. $c_i$) is an eigenvalue of $\text{compb}$ (resp. $\text{compc}$) and $b_i$ corresponds to the same eigenvector (up to a change of sign) of $c_i$.

If $\mathcal{D}_0$ is as above, then $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D}_0)$ has 21 unstable lines such that 11 are real.

**Figure 3.** $\mathcal{D}_0$ and the 11 real unstable lines plotted with [15]

**Remark 9.5.** As we can see in Figure 3, it seems to be hard but interesting to understand what these lines represent for the conic-arrangement and how it is possible to get the conics from them: we observe, for example, that they are not tangent lines and they don’t cross the conics in special points. So we can say that the three conics case represents still an open problem.

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