Guaraded Variable Automata over Infinite Alphabets

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Abstract. We define guaraded variable automata (GVAs), a simple ex-
tension of finite automata over infinite alphabets. In this model the tran-
sitions are labeled by letters or variables ranging over an infinite alpha-
bet and guarded by conjunction of equalities and disequalities. GVAs
are well-suited for modeling component-based applications such as web
services. They are closed under intersection, union, concatenation and
Kleene operator, and their nonemptiness problem is PSPACE-complete.
We show that the simulation preorder of GVAs is decidable. Our proof
relies on the characterization of the simulation by means of games and
strategies. This result can be applied to service composition synthesis.

1 Introduction

The simple and powerful formalism of finite automata is widely used for system
specification and verification. Considerable efforts have been devoted to extend
finite automata to infinite alphabets: finite memory automata [12], data au-
tomata [6], variable automata [10], usage automata [7], fresh-variable automata
[3], only to cite a few (see [15] for a survey). When developing formalisms over
infinite alphabets, the main challenge is to preserve as much as possible use-
ful properties such as compositionality (i.e. closure under basic operations) and
decidability of basic problems such as nonemptiness, membership, universality,
language containment, simulation, etc.

The language containment problem is a particularly important one in ap-
plications like formal verification. For instance, whether an implementation is
conform to a specification amounts to decide the containment $L(A) \subseteq L(B)$,
where $A$ (resp. $B$) is an automaton formalizing the behavior of the implemen-
tation (resp. specification), and $L(A)$ is the language of words recognized by
$A$.

The containment problem for finite automata (FAs) can be solved by using
determinization, in a complete but inefficient way. Moreover, for several classes
of automata over infinite alphabets, the containment problem turned out to
be undecidable. This is the case for finite memory automata [18] and variable
automata [10]. As a practical alternative approach, a simulation preorder can
be employed to underapproximate the containment relation (e.g. [8]). Indeed,
simulation-based techniques are sometimes more efficient. For instance a sim-
ulation between two finite automata can be computed in polynomial time. To
our knowledge, simulation has not been studied for the classes of automata over infinite alphabets from [12] and [10].

Our work is also motivated by the composition synthesis problem for web services in which the agents (i.e. client and the available services) exchange data ranging over an infinite domain. One of the most successful approaches to composition amounts to abstract services as finite-state automata (FA) and apply available tools from automata theory to synthesize a new service satisfying the given client requests from an existing community of services (e.g. [5,17]). In this setting synthesizing a new service amounts to compute a simulation relation of the client by the community of the available services, e.g. [5]. However it is not obvious whether the automata-based approach to service composition can still be applied with infinite alphabets since simulation often gets undecidable in extended models like Colombo (e.g. [1]). Following the approach initiated in [3] our objective is to define expressive classes of automata on infinite alphabets which are well-adapted to the specification and composition of services and enjoy nice closure properties and decidable simulation.

**Contributions.** In this paper we define _guarded variable automata_, or GVAs, a natural extension of finite automata over infinite alphabets. In this model the transitions are labeled by letters or variables ranging over an infinite alphabets and guarded by conjunction of equalities and disequalities. Besides, some variables are refreshed in some states, that is, these variables can be released so that new letters can be bound to them. The potential applicability of our model in verification (e.g. model checking [4]) and service composition [1] follows from the fact that GVAs are closed under intersection, union, concatenation and Kleene operator. The nonemptiness problem is shown to be PSPACE-complete for GVAs, and the membership is NP-Complete. However, their universality and containment problems are undecidable. We introduce a simulation preorder for GVAs and show its decidability. The proof relies on a game-theoretic characterization of simulation.

**Related work.** GVAs are closely related to the classes of automata in [12,10,3], but these classes are strictly included in GVAs. We show below that GVAs have the same expressivity as _finite-memory automata with non-deterministic reassignment_ (NFMA) [13]. Here we give the complexity status of nonemptyness for GVAs. This problem was not considered for NFMA in [13]. We also give a procedure to decide the simulation preorder for GVAs. Simulation has not been studied in [13,12,10] and considered only for the less expressive FVAs in [3].

**Paper organization.** Sec. 2 recalls standard notions. Sec. 3 introduces the new class of guarded variable automata. Subsec. 4 studies the expressiveness of GVAs with respect to NFMA. Sec. 5 studies closure properties and the complexity of Nonemptiness for GVAs. Sec. 6 introduces the simulation preorder of GVAs. Sec. 7 shows its decidability. Sec. 8 applies these results to service composition. Future work directions are given in Sec. 9. Missing proofs are provided in external appendices.
2 Preliminaries

Let $\mathcal{X}$ be a finite set of variables, $\Sigma$ an infinite alphabet of letters. A substitution is an idempotent mapping $\{x_1 \mapsto \alpha_1, \ldots, x_n \mapsto \alpha_n\} \cup \bigcup_{a \in \Sigma} \{a \mapsto a\}$ with variables $x_1, \ldots, x_n$ in $\mathcal{X}$ and $\alpha_1, \ldots, \alpha_n$ in $\mathcal{X} \cup \Sigma$. We call $\{x_1, \ldots, x_n\}$ its proper domain, and denote it by $dom(\sigma)$. We denote by $Dom(\sigma)$ the set $dom(\sigma) \cup \Sigma$. We define the function $V(\alpha)$ as the set $\{a \in \Sigma \mid \exists x \in dom(\sigma) \text{ s.t. } \sigma(x) = a\}$. If all the $\alpha_i, i = 1 \ldots n$ are letters then we say that $\sigma$ is ground. The empty substitution (i.e., with an empty proper domain) is denoted by $\emptyset$. The set of substitutions from $\mathcal{X} \cup \Sigma$ to a set $A$ is denoted by $\zeta_{\mathcal{X},A}$, or by $\zeta_{\mathcal{X}}$, or simply by $\zeta$ if there is no ambiguity. If $\sigma_1$ and $\sigma_2$ are substitutions that coincide on the domain $dom(\sigma_1) \cap dom(\sigma_2)$, then $\sigma_1 \cup \sigma_2$ denotes their union in the usual sense. If $dom(\sigma_1) \cap dom(\sigma_2) = \emptyset$ then we denote by $\sigma_1 \uplus \sigma_2$ their disjoint union. We define the function $V: \Sigma \cup \mathcal{X} \rightarrow P(\mathcal{X})$ by $V(\alpha) = \{\alpha\}$ if $\alpha \in \mathcal{X}$, and $V(\alpha) = \emptyset$, otherwise. For a function $F: A \rightarrow B$, and $A' \subseteq A$, the restriction of $F$ on $A'$ is denoted by $F|_{A'}$. If $k \in \mathbb{N}$ then we let $[k] = \{1, \ldots, k\}$.

A two-players game is a tuple $\langle Pos_E, Pos_A, M, p^* \rangle$, where $Pos_E, Pos_A$ are disjoint sets of positions: Eloise’s positions and Abelard’s positions. $M \subseteq (Pos_E \cup Pos_A) \times (Pos_E \cup Pos_A)$ is a set of moves, and $p^*$ is the starting position. A strategy for the player Eloise is a function $\rho: Pos_E \rightarrow Pos_E \cup Pos_A$, such that $\langle \varphi, \rho(\varphi) \rangle \in M$ for all $\varphi \in Pos_E$. A (possibly infinite) play $\pi = \langle \varphi_1, \varphi_2, \ldots \rangle$ follows a strategy $\rho$ for player Eloise iff $\varphi_{i+1} = \rho(\varphi_i)$ for all $i \in \mathbb{N}$ such that $\varphi_i \in Pos_e$. Let $W$ be a (possibly infinite) set of plays. A strategy $\rho$ is winning for Eloise from a set $S \subseteq Pos_E \cup Pos_A$ according to $W$ iff every play starting from a position in $S$ and following $\rho$ belongs to $W$.

3 Guarded variable automata

In this section we define formally the class of GVAs. It is an extension of FVAs [3] with logical constraints, called guards.

Let us first explain the main ideas behind GVAs. The transitions of a GVA are labeled with letters or variables ranging over an infinite set of letters. These transitions can also be labeled with guards consisting of equalities and disequalities. Its guard must be true for the transition to be fired. We emphasize that while reading a guarded transition some variables of the guard might be free and we need to guess their value. Finally, some variables are refreshed in some states, that is, variables can be freed in these states so that new letters can be assigned to them. Firstly, we introduce the syntax and semantics of guards.

**Definition 1.** The set $G$ of guards over $\Sigma \cup \mathcal{X}$ is inductively defined as follows: $G := true \mid \alpha = \beta \mid \alpha \neq \beta \mid G \land G$, where $\alpha, \beta \in \Sigma \cup \mathcal{X}$. We write $\sigma \models g$ if a substitution $\sigma$ satisfies a guard $g$.

We notice that adding the disjunction operator to the guards would not increase the expressivity of our model. A guard is atomic iff it is either true, an equality, or an inequality. Let $g_i, i = 1, \ldots, n$, be atomic guards. Then define the
free variables of a guard by $V(\bigwedge_{i=1}^n g_i) = \bigcup_{i=1}^n V(g_i)$ and $V(\alpha \sim \beta) = V(\alpha) \cup V(\beta)$, where $\sim \in \{=, \neq\}$ and $\alpha, \beta \in \Sigma \cup \mathcal{X}$. The application of a substitution $\gamma$ to a guard $g$, denoted by $\gamma(g)$, is defined in the usual way. The formal definition of GVAs follows.

**Definition 2.** A GVA $A = \langle \Sigma, \mathcal{X}, Q, Q_0, \delta, F, \kappa \rangle$ where $\Sigma$ is an infinite set of letters, $\mathcal{X}$ is a finite set of variables, $Q$ is a finite set of states, $Q_0 \subseteq Q$ is a set of initial states, $\delta : Q \times (\Sigma \cup \mathcal{X} \cup \{\varepsilon\}) \times \mathcal{G} \rightarrow 2^Q$ is a transition function where $\Sigma_A$ is a finite subset of $\Sigma$, $F \subseteq Q$ is a set of accepting states, and $\kappa : \mathcal{X} \rightarrow 2^Q$ is called the refreshing function.

The run of a GVA is defined over configurations. A configuration is a pair $(\gamma, q)$ where $\gamma$ is a substitution such that for all variables $x$ in $dom(\gamma)$, $\gamma(x)$ is the current value of $x$, and $q$ is a state of the GVA. Intuitively, when a GVA $A$ is in state $q$, and $(\gamma, q)$ is the current configuration, and there is a transition $q \xrightarrow{a} q'$ in $A$ then:

- if $a$ is a free variable (i.e. $a \in \mathcal{X} \setminus dom(\gamma)$) then $a$ stores the input letter and some values for all the other free variables of $\gamma(g)$ are guessed such that $\gamma(g)$ holds, and $A$ enters state $q' \in \delta(q, a, g)$,
- if $a$ is a bound variable or a letter (i.e. $a \in Dom(\gamma)$) and $\gamma(a)$ is equal to the input letter $l$ then some values for all the free variables of $\gamma(g)$ are guessed such that $\gamma(g)$ holds, and $A$ enters state $q' \in \delta(q, a, g)$.

In both cases when $A$ enters state $q'$ all the variables which are refreshed in $q'$ are freed. Thus the purpose of guards is to compare letters and to guess new letters that might be read afterward.

For a GVA $A$, we shall denote by $\Sigma_A$ the finite set of letters that appear in the transition function of $A$. We shall denote by $\kappa^{-1} : Q \rightarrow 2^\mathcal{X}$ the function that associates to each state of the GVA the set of variables being refreshed in this state. That is, $\kappa^{-1}(q) = \{x \in \mathcal{X} \mid q \in \kappa(x)\}$.

The formal definitions of configuration, run and recognized language follow.

**Definition 3.** Let $A = \langle \Sigma, \mathcal{X}, Q, Q_0, \delta, F, \kappa \rangle$ be a GVA. A configuration is a pair $(\gamma, q)$ where $\gamma$ is a substitution and $q \in Q$. We define a transition relation over the configurations as follows: $(\gamma_1, q_1) \xrightarrow{a} (\gamma_2, q_2)$, where $a \in \Sigma \cup \{\varepsilon\}$, iff there exists a substitution $\sigma$ such that $dom(\sigma) \cap dom(\gamma_1) = \emptyset$ and either:

i) $a \in \Sigma$ and in this case there exists a label $\alpha \in \Sigma \cup \mathcal{X}$ such that $q_2 \in \delta(q_1, \alpha, g)$, $(\gamma_1 \uplus \sigma)(\alpha) = a$, $(\gamma_1 \uplus \sigma) \models g$ and $\gamma_2 = (\gamma_1 \uplus \sigma)_D$, with $D = Dom(\gamma_1 \uplus \sigma) \setminus \kappa^{-1}(q_2)$. Or,

ii) $a = \varepsilon$ and in this case $(\gamma_1 \uplus \sigma) \models g$ and $\gamma_2 = (\gamma_1 \uplus \sigma)_D$, with $D = Dom(\gamma_1 \uplus \sigma) \setminus \kappa^{-1}(q_2)$.

We denote by $\Rightarrow^* \subset \Rightarrow$ the reflexive and transitive closure of $\Rightarrow$. For two configurations $c, c'$ and a letter $w \in \Sigma$, we write $c \xrightarrow{w} c'$ iff there exists two configurations $c_1$ and $c_2$ such that $c \xrightarrow{w} c_1 \xrightarrow{w} c_2 \xrightarrow{w} c'$. A finite word $w = w_1 w_2 \ldots w_n \in \Sigma^*$ is recognized by $A$ iff there exists a run $(\gamma_0, q_0) \xrightarrow{w_1} (\gamma_1, q_1) \xrightarrow{w_2} \ldots \xrightarrow{w_n} (\gamma_n, q_n)$, such that $q_0 \in Q_0$ and $q_n \in F$. The set of words recognized by $A$ is denoted by $L(A)$. 
Example 1. Let $A_1$ and $A_2$ be the GVAs depicted below where $x, y$ are variables and $\kappa_i$ the refreshing function of $A_i$, $i = 1, 2$, is defined by $\kappa_1(y) = \{p_0\}$ and $\kappa_2(x) = \kappa_2(y) = \{q_0\}$. We notice that while making the first loop over $p_0$, the variable $x$ of the guard ($y \neq x$) is free and its value is guessed. Then the variable $y$ is refreshed in $p_0$, and at each loop the input letter should be different than the value of the variable $x$ already guessed.

Hence, the language $L(A_1)$ consists of all the words in $\Sigma^*$ in which the last letter is different than all the other letters. This language can be recognized by a variable automaton [10] and by a NFMA [13] but not by a FMA [12]. On the other hand, the language $L(A_2) = \{w_1w'_1 \cdots w_nw'_n : w_i, w'_i \in \Sigma, n \geq 1, \text{ and } w_i \neq w'_i, \forall i \in [n]\}$ can be recognized by a FMA but not by a variable automaton.

4 Comparison between GVAs and NFMA

In this section we show that GVAs and NFMAs recognize the same languages. We recall that a NFMA [13] is a 8-tuple $F = \langle \Sigma, k, Q, q_0, u, \rho, \delta, F \rangle$ where $k \in \mathbb{N}^+$ is the number of registers, $Q$ is a finite set of states, $q_0 \in Q$ is the initial state, $u : [k] \rightarrow \Sigma$ is a partial function called the initial assignment of the $k$ registers, $\rho : \{(p, q) : (p, \varepsilon, q) \in \delta\} \rightarrow [k]$ is a function called the non-deterministic reassignment, $\delta : Q \times ([k] \cup \{\varepsilon\}) \times Q$ is the transition relation, and $F \subseteq Q$ is the set of final states. Intuitively, if $F$ is in state $p$, and there is an $\varepsilon$-transition from $p$ to $q$ and $\rho(p, q) = l$, then $F$ can non-deterministically replace the content of the $l^{th}$ register with an element of $\Sigma$ not occurring in any other register and enter state $q$. However, if $F$ is in state $p$, and the input symbol is equal to the content of the $l^{th}$ and $(p, l, q) \in \delta$ then $F$ may enter state $q$ and pass to the next input symbol. An $\varepsilon$-transition $(p, \varepsilon, q) \in \delta$ with $\rho(p, q) = l$, for a register $l \in [k]$, is denoted by $(p, \varepsilon/l, q)$.

![Fig. 1](image)

Fig. 1. A translation of NFMA to GVA. The registers of the NFMA $A$ are $\{1, \ldots, k\}$, they correspond to the variables $\{x_1, \ldots, x_k\}$ of the GVA $A'$. The variable $x_i$ is refreshed in the state $\tilde{q}$ of $A'$. 

\[\kappa(x_i) = \{q\}\]
On the one hand, we can show that there is a polynomial time translation of a NFMA (with $k$ registers) into a GVA (with $k$ variables) of linear size and recognizing the same language. More precisely, as shown in Figure 1:

- a transition $(p, m, p')$ of the NFMA is translated as such, i.e. to $(p, x_m, p')$
- a transition $(p, \varepsilon/l, p'')$ of the NFMA is translated to two transitions $(p, \varepsilon, \tilde{p})$ and $(\tilde{p}, (x_l, g), p''')$ where $g = \bigwedge_{i \in [k] \setminus \{l\}} (\varepsilon \neq x_i)$ and $x_l$ is refreshed in state $\tilde{p}$.

On the other hand, we show next that a GVA can be translated into a NFMA recognizing the same language. The idea is that the $\varepsilon$-transitions of the NFMA are used to encode the refreshing of the variables of the GVA. For this purpose we introduce an intermediary class of GVAs, called GVA#, in which the variables should have distinct values. Then we translate GVA# into NFMA.

**Definition 4.** Let GVA# be the subclass of GVAs such that every $A$ in GVA#, verifies i) $A$ has no constants, i.e. $\Sigma_A = \emptyset$, and ii) for every accessible configuration $(\sigma, q)$ of $A$ and for all $x, y \in \text{dom}(\sigma)$, $\sigma(x) \neq \sigma(y)$.

We show next that GVAs and GVA#'s recognize the same language, more precisely we have:

**Lemma 1.** For every GVA $A$ with $k$ variables and $n$ states there is a GVA# with $k + m$ variables and $O(n \cdot (k + m)!)$ states recognizing the same languages, where $m = |\Sigma_A|$. 

Every GVA# can be turned into a NFMA recognizing the same languages by encoding the refreshing of the variables of the GVA# with $\varepsilon$-transitions. Hence,

**Corollary 1.** For every GVA# with $k$ variables and $n$ states there exists a NFMA with $n \cdot k!$ states recognizing the same languages.

### 4.1 Succinctness of GVAs w.r.t. NFMAs

Though GVAs and NFMAs recognize the same languages, the latter are less adapted for service specification since their global requirement that distinct variables must have distinct instances is an obstacle to compositionality. Assume for instance that we want to compose a payment service $P(c)$ with a reservation service $R(c')$ where $c$ (resp. $c'$) is the payer (resp. traveller) name taking value in some infinite alphabet. If the services are specified as NFMAs a mediator (i.e. a service that delegates any client action to an appropriate service) would have to interact with two copies of the payment service, one in which the payer name is the same as the traveler, and one in which they differ. The choice by the mediator of one copy anticipates on the equalities that will be imposed by future messages (The mediator is non-deterministic in Milner’s weak determinacy sense [16, §11, Def 3]). Since GVAs do not impose this early (non-deterministic) choice, they are more suited to service specification and composition.
4.2 Weak Determinacy

In the rest of this section, a GVA or NFMA configuration is a couple \( s, m \) where \( s \) is an automaton state and \( m \) is an assignment to variables or registers. Given an automaton \( \mathcal{A} \), we denote \( s, m \xrightarrow{\varepsilon} s', m' \) if there is an \( \varepsilon \)-transition from the configuration \( s, m \) to the configuration \( s', m' \), and \( s, m \xrightarrow{x} s', m' \) if there exists a transition (whose guard is satisfied) from \( s, m \) labeled with the variable, register, or letter \( x \). We define the relation \( \Rightarrow A \) (reading of a register or letter \( x \)) as \( (\xrightarrow{\varepsilon} \circ \xrightarrow{x} A) \).

A run of \( \mathcal{A} \) is a sequence of configurations \( (s_i, m_i) \) such that for all \( 0 \leq i < n \) we have \( s_i, m_i \xrightarrow{\varepsilon} s_{i+1}, m_{i+1} \) such that \( s_0 \) is an initial state and \( m_0 \) is a possible initial assignment. It is accepting if \( s_n \) is a final state. Finally we say that a run \( s_0, m_0 \xrightarrow{x_1} \ldots \xrightarrow{x_n} s_n, m_n \) reads a word \( \omega = \omega_1 \ldots \omega_n \) if for all \( 1 \leq i \leq n \) we have \( \omega_i = m_i(x_i) \). In the rest of this section we omit the subscript in \( \Rightarrow A \) when the automaton \( \mathcal{A} \) is clear from the context.

First let us define active variables as the subset of variables whose values has not changed since the last time they were read.

**Definition 5.** (Active variables) In a run \( s_0, m_0 \xrightarrow{x_1} \ldots \xrightarrow{x_n} s_n, m_n \), the set \( A_i \) of variables active in state \( i \) is:

- \( \emptyset \) if \( i = 0 \)
- \( (A_{i-1} \setminus R_i) \cup \{x_i\} \) where \( R_i \) is the set of variables or registers \( x \) such that \( m_{i-1}(x) \neq m_i(x) \).

Since the automata we consider are fundamentally non-deterministic (e.g., in guessing a new value for a register) we introduce a notion of weak determinacy adapted from [16].

**Definition 6.** (Weakly deterministic automata) An automaton is weakly deterministic if for every word \( \omega \), if \( s_0, m_0 \xrightarrow{x_1} \ldots \xrightarrow{x_n} s_n, m_n \) and \( s'_0, m'_0 \xrightarrow{x_1} \ldots \xrightarrow{x_n} s'_n, m'_n \) are two runs reading \( \omega \) then:

- We have \( s_n = s'_n \)
- A variable is active in \( s_n, m_n \) iff it is active in \( s_n, m'_n \);
- For every active variable \( x \) in \( s_n, m_n \) we have \( m_n(x) = m'_n(x) \).

We remark that all the examples given in [13] are weakly deterministic NFMA.

4.3 Succinctness

We have seen that it is possible to translate in polynomial time an NFMA into a GVA that recognizes the same language. Conversely, we prove in this section that there exists a sequence of languages \( (\mathcal{L}_n)_{n \geq 0} \) that each can be recognized by a weakly deterministic GVA of size \( O(n^2) \) and a weakly deterministic NFMA of size \( \Omega(n^n) \). This justifies our assertion that GVA can be exponentially more succinct than NFMA in the class of weakly deterministic automata.
Proof. The proof proceeds by case analysis:

Lemma 2. If \( \text{transition (if it isn’t an } \varepsilon \text{)} \rangle \), the notation \( \varepsilon \) stands for being in a state \( s \) \( \omega \) beginning with \( \{ \forall a, |\omega_2|_a > 0 \Rightarrow |\omega_1|_a > 0 \} \)

In order to simplify the statements, we say that in a run reading a word of \( \mathcal{L}_n \) and ending in a final state:

- The initialization phase is the sub-run in which \( a_1 \cdot \ldots \cdot a_n \) is read (including the possible \( \varepsilon \)-transitions);
- The pivot is the state reached after reading \( b_n \) for the first time;

Proposition 1. Each \( \mathcal{L}_n \) is recognized by a GVA of \( 4n+2 \) states and \( 1+3n+n^2 \) transitions.

Proof. The constructed automaton has \( 2n \) variables, denoted \( x_{a_1}, \ldots, x_{a_n}, x_{b_1}, \ldots, x_{b_n} \) and is linear. The first \( n \) states read and instantiate the variables \( x_{a_1}, \ldots, x_{a_n} \), the \( n \) following states read and instantiate the variables \( x_{b_1}, \ldots, x_{b_n} \), with \( n \) available transitions from each state to its successor. For \( 1 \leq j \leq n \) there exists a transition reading \( b_i \) guarded with \( x_{b_i} = x_{a_j} \). The remaining \( 2n + 1 \) states and transitions check that the word belongs to the language by reading in sequence (and without refreshment) \( \# , x_{b_n}, \ldots, x_{b_1}, x_{a_n}, \ldots, x_{a_1} \). The last state is the final state. The last state is a final state.

Let us now prove that any NFMA recognizing \( \mathcal{L}_n \) must have at least \( n^n \) states. Given a word \( \omega \) of length \( n \) in which each letter occurs at most once (i.e., for all \( a \in \Sigma \) we have \( 0 \leq |\omega|_a \leq 1 \) ), we let \( \mathcal{L}_\omega \subseteq \mathcal{L}_n \) be the subset of \( \mathcal{L}_n \) of words beginning with \( \omega \). The following lemmatas hold for all NFMA. The notation \( s, m \) stands for being in a state \( s \) with the assignment to registers \( m \), and \( x \) is either an \( \varepsilon \)-transition (possibly with guessing) or the reading of a register or a constant. The notation \( \delta_{a,b} \) denotes the replacement of \( a \) by \( b \) in a configuration or in a transition (if it isn’t an \( \varepsilon \) one).

Lemma 2. If \( s, m \rightarrow^x s', m' \), \( a \in \text{codom}(m) \) and \( b \notin \text{codom}(m) \cup \text{codom}(m') \cup \{ x \} \) then \( s, m\delta_{a,b} \rightarrow^x s', m'\delta_{a,b} \).

Proof. The proof proceeds by case analysis:

- If \( x \) is an \( \varepsilon \)-transition:
  • If there is no refreshment or of a register that does not contain \( a \), then it is trivial;
  • If there is a refreshment of the register containing \( a \), then either we replace the new value by \( b \) in \( m' \) if \( a \) is guessed again, or we keep the new value. Since \( b \notin \text{codom}(m') \) this is a valid transition. Note that in this case \( a \notin \text{codom}(m') \), and thus \( m' = m'\delta_{a,b} \).
– Otherwise if the transition reads the register containing $a$, then the same transition, after the replacement of $a$ by $b$, will read $x\delta_{a,b} = b$. Otherwise it is trivial.

By recursion on the length of a run and by changing the possible occurrences of $b$ in the intermediate configurations (possible since the letter $b$ is never read in a transition) we obtain

**Lemma 3.** If $s, m \xrightarrow{\omega} s', m'$, $a \in \text{dom}(m)$ and $b \notin \text{dom}(m) \cup \text{dom}(m')$ and $|\omega|_b = 0$ then $s, m\delta_{a,b} \xrightarrow{\omega\delta_{a,b}} s', m'\delta_{a,b}$

The following lemmas hold for any weakly deterministic NFMA recognizing the language $L_n$.

**Lemma 4.** In any successful run recognizing a word in $L_\omega$:

1. Once a register $r$ is read during the initialization phase it is not refreshed before the pivot point;
2. If $r_1, \ldots, r_n$ are the registers read at the end of the initialization phase, then the rest of the run reads no other registers ($\varepsilon$-transitions and reading the letter # is still possible)

**Proof.** The first point is trivial, as Lemma 3 would otherwise permit us to construct another successful run in which the letter guessed is replaced by another one (not occurring in $\omega$), thereby contradicting the hypothesis that the NFMA recognizes exactly $L_n$.

The second point is a direct consequence of the first point, as other registers contain letters which do not occur in $\omega$, and thus which are not in any word belonging to $L_\omega$. But $L_\omega$ is exactly the subset of $L_n$ of words beginning with $\omega$, so a successful run reading first $\omega$ must recognize a word in $L_\omega$.

As a consequence, the possible configurations of the automaton are characterized by their values on the registers $r_1, \ldots, r_k$, and these values are fixed at the end of the initialization phase.

**Theorem 1.** A weakly deterministic NFMA recognizing $L_n$ has at least $n^n$ states.

**Proof.** Let $A$ be any weakly deterministic NFMA recognizing a language $L_n$. Let $\alpha \neq \beta$ be two words such that $\omega_\alpha = \omega \cdot \alpha \cdot \# \cdot \tilde{\alpha} \tilde{\omega}$ and $\omega_\beta = \omega \cdot \beta \cdot \# \cdot \tilde{\beta} \tilde{\omega}$ are in $L_\omega$.

Let $R_\alpha$ (resp. $R_\beta$) be a run recognizing $\omega_\alpha$ (resp. $\omega_\beta$), and $s_\alpha$ (resp. $s_\beta$) be the pivot point in $R_\alpha$ (resp. $R_\beta$). Let $s_\alpha^n, m_\alpha^n, s_\beta^n, m_\beta^n$ be a configuration reached after reading $\omega$ in each run. Since the automaton is weakly deterministic we have $s_\alpha^n = s_\beta^n$. By Lemma 4, point 1., the active variables in these states are the registers $r_1, \ldots, r_n$ storing the letters of $\omega$.

Let $p_\alpha, m_\alpha$ (resp. $p_\beta, m_\beta$) be the pivot configuration in the run recognizing $\omega_\alpha$ (resp. $\omega_\beta$). Again by Lemma 4, point 1., the active variables in these configurations are the registers $r_1, \ldots, r_n$ and they hold the same values in the two runs.
Assume that $p_\alpha = p_\beta$. Then Lemma 4, point 2, implies that the word $\omega \cdot \alpha \cdot \# \cdot \hat{\beta} \cdot \hat{\omega}$ is also recognized by $A$, even though it is not in $L_n$. Thus this contradicts the assumption the $A$ recognizes $L_n$.

As a consequence, there is at least as many pivot states as the number of possible words $\alpha$ such that $\omega \cdot \alpha \cdot \# \cdot \hat{\alpha} \cdot \hat{\omega} \in L_n$. Hence our lower bound $n^6$ on the number of pivot states, and thus on the size of any weakly deterministic NFMA recognizing $L_n$.

5 Properties of guarded variable automata

We study the closure properties of GVAs and some basic decision problems.

Since GVAs and NFMA recognize the same languages, GVAs inherit all the closure properties of NFMA. Hence,

**Theorem 2.** GVAs are closed under union, concatenation, Kleene operator and intersection. They are not closed under complementation.

Despite GVAs are not closed under complementation, FAs can be complemented within the class of GVAs. That is, given a FA $F$ there exists a GVA $A$ such that $L(A) = \Sigma^* \setminus L(F)$, see Proposition 1 in Appendix C.1. It is worth mentioning that FAs cannot be complemented within the subclass of FVAs.

We study the decidability and complexity of classical decision problems: Nonemptiness (given $A$, is $L(A) \neq \emptyset$?), Membership (given a word $w$ and $A$, is $w \in L(A)$?), Universality (given $A$, is $L(A) = \Sigma^*$?), and Containment (given $A_1$ and $A_2$, is $L(A_1) \subseteq L(A_2)$?).

**Theorem 3.** For GVAs, Membership is NP-complete, Universality and Containment are undecidable.

The undecidability of Containment and Universality is a consequence of the undecidability of these problems for NFMA [14]. However, the decidability of Containment if one of the GVAs is a finite automaton results from the fact that the intersection of the languages in this case is regular since the Cartesian product of a GVA and a FA yields a FA. The proof is the same as that of Lemma 17 of [3]. Hence,

**Proposition 2.** The containment problems between a GVA and a FA are decidable.

We show in the next subsection that nonemptiness for GVAs is PSPACE-Complete.

5.1 Nonemptiness is PSPACE-Complete

We recall that Nonemptiness is NL-Complete for both FVAs [3] and variable automata [10], and it is NP-Complete for FMAs [19]. Firstly we show that
Nonemptiness is in PSPACE. Given a GVA $\mathcal{A}$, we shall show that $\mathcal{A}$ recognizes a non-empty language over $\Sigma$ iff $\mathcal{A}$ recognizes a non-empty language over a finite set of letters. For this purpose, and in order to relate the two runs of $\mathcal{A}$ (the one over an infinite alphabet and the one over a finite alphabet) we introduce the relation of coherence between substitutions.

**Definition 7.** Let $C$ be a finite subset of $\Sigma$. The coherence relation $\forall_C \subseteq \zeta \times \zeta$ between substitutions is defined by $\bar{\sigma} \forall_C \sigma$ iff the three following conditions hold:

1. $\text{dom}(\bar{\sigma}) = \text{dom}(\sigma)$,
2. If $\bar{x}(x) \in C$ then $\bar{x}(x) = x(\bar{x})$, and if $x(\bar{x}) \in C$, then $\bar{x}(x) = x(\bar{x})$, for any variable $x \in \text{dom}(\bar{\sigma})$, and
3. for any variables $x, y \in \text{dom}(\sigma)$, $\bar{x}(x) = \bar{y}(y)$ iff $x(\bar{x}) = y(\bar{y})$.

We need to define a function $\Theta$ that will be used in the proof of Lemma 5 and other Lemmas. Given two sets $S_1$ and $S_2$ of letters such that $S_1 \cap S_2 = C \neq \emptyset$, $|S_1 \setminus S_2| \geq |X|$ and $|S_2 \setminus S_1| \geq |X|$, we can define a function:

$$\Theta^{S_1, S_2}_{C} : \xi_{x, S_1} \times \xi_{x, S_1} \times \xi_{x, S_2} \rightarrow \xi_{x, S_2}$$

(1)

such that given three substitutions $\sigma, \gamma, \sigma'$, such that $\text{dom}(\sigma) \cap \text{dom}(\gamma) = \emptyset$ and $\gamma \forall_C \gamma'$ construct a substitution $\gamma'' = \Theta^{S_1, S_2}_{C}(\sigma, \gamma, \sigma')$ such that $\sigma \gamma \forall_C \sigma' \gamma''$.

**Lemma 5.** Let $\mathcal{A}$ be a GVA over $\Sigma$ with $k$ variables and $m$ constants $\Sigma_\mathcal{A} = \{c_1, \ldots, c_m\}$. Let $\Sigma = \{a_1, \ldots, a_k, c_1, \ldots, c_m\}$. Then, $\mathcal{A}$ recognizes a non-empty language over $\Sigma^*$ if, and only if, it recognizes a non-empty language over $\Sigma^*$.

**Proof.** (Sketch) Let $C = \{c_1, \ldots, c_k\}$. We show that there is a run $(\sigma_0, q_0) \rightarrow \ldots \rightarrow (\sigma_n, q_n)$ over $\Sigma^*$ in $\mathcal{A}$ iff there is a run $(\sigma'_0, q_0) \rightarrow \ldots \rightarrow (\sigma'_n, q_n)$ over $\Sigma^*$ in $\mathcal{A}$ such that $\sigma_i \forall_C \sigma'_i$, for all $i = 0, \ldots, n$. The proof is by induction on $n$ in both directions. The base case $n = 0$ holds trivially since $\sigma_0 = \sigma'_0 = \emptyset$. Assume that the claim holds up to $n$ and let us prove it for $n + 1$.

$\Rightarrow$ Assume there is a transition $q_n \xrightarrow{\alpha_n, g_n} q_{n+1}$ in $\mathcal{A}$ where $\alpha_n \in \Sigma \cup \mathcal{X}$ and $g_n$ is a guard. From the induction hypothesis we have that $\sigma_n \forall_C \sigma'_n$. It follows that that $\sigma_n(g_n)$ holds iff $\sigma'_n(g_n)$ holds (Lemma 11 in Appendix A). Thus, the transition in $\mathcal{A}$ over $\Sigma$ is possible. We describe next this transition. From Definition 3 of the run of GVAs, there exists a substitution $\gamma_n : \forall (\sigma_n(\alpha_n)) \cup \forall (\sigma_n(g_n)) \rightarrow \Sigma$ such that $(\gamma_n \forall \sigma_n)(g_n)$ holds. Hence, we must find a substitution $\gamma'_n : \forall (\sigma'_n(\alpha_n)) \cup \forall (\sigma'_n(g_n)) \rightarrow \Sigma$ such that $(\gamma'_n \forall \sigma'_n)(g_n)$ holds. We define $\gamma'_n$ by $\gamma'_n = \Theta^{\Sigma, \Sigma}_{C}(\sigma_n, \gamma_n, \sigma'_n)$.

$\Leftarrow$ Same proof but we call the function $\Theta^{\Sigma, \Sigma}_{C}(\sigma'_n, \gamma'_n, \sigma_n)$.

**Definition 8.** (Restricted configuration) A restricted configuration of a GVA $\mathcal{A}$ with $k$ variables $x_1, \ldots, x_k$ and $m$ constants $c_1, \ldots, c_m$ is a tuple $(q, \alpha_1, \ldots, \alpha_k)$ where $\alpha_1, \ldots, \alpha_k \in \{a_1, \ldots, a_k, c_1, \ldots, c_m\}$.
Note that the number of different restricted configurations is exponentially bounded in the size of $A$.

**Lemma 6.** Assume $A$ is a GVA that recognizes a non-empty language. Then there is an accepting run $q_1 \to \ldots \to q_l$ such that $i \neq j$ implies $q_i \neq q_j$, and each $q_i$ is a restricted configuration.

**Proof.** By Lemma 5 and the assumption that $A$ recognizes a non-empty language, it recognizes a word in $\{a_1, \ldots, a_k, c_1, \ldots, c_m\}^*$. Let $\omega$ be such a word of minimal length, and let $q_1 \to \ldots \to q_l$ be an accepting run of $A$ recognizing $\omega$. Since all the variables are instantiated with constants in $\{a_1, \ldots, a_k, c_1, \ldots, c_m\}$, each $m_i$ is a restricted configuration of $A$. Furthermore, if a restricted configuration appears at steps $i$ and $j$ with $i \neq j$, it can be shown that $q_1 \to \ldots \to q_i \to q_{j+1} \to \ldots \to q_l$ is also a run for the automaton $A$ that contradicts the minimality of $\omega$. Hence for $i \neq j$ we have $q_i \neq q_j$. \qed

As a corollary, we obtain that if a GVA recognizes a non-empty language $L$ it has an accepting run consisting of restricted configurations that each appear at most once. Hence its length is less than the number of configurations, which can be encoded in binary in space linear in the size of $A$. We thus obtain:

**Theorem 4.** The nonemptiness problem for GVAs is in PSPACE.

As a direct consequence of Subsection 4, we also get:

**Corollary 2.** The nonemptiness problem for NFMA is in PSPACE.

To show that Nonemptiness of GVAs is PSPACE-hard, we reduce the reachability problem for bounded one-counter automata (known to be PSPACE-hard) to the nonemptiness problem of GVAs. In the rest of this section, we first present bounded one-counter automata, and then proceed to GVAs.

**Definition 9.** (Bounded one-counter automata [11]) A bounded one-counter automaton (Boca) is a tuple $(Q, b, \Delta, q_0)$ where $Q$ is a finite set of states, $b \in \mathbb{N}$ is a global counter bound, $q_0$ is the initial state, and $\Delta \subset Q \times [-b, b] \times Q$ is the transition relation.

A Boca configuration is a tuple $(q, p)$ with $q \in Q$ and $0 \leq p \leq b$. Given $\tau = (q, p, q') \in \Delta$ and two configurations $c_1 = (q_1, p_1)$ and $c_2 = (q_2, p_2)$ we denote $c_1 \to^\tau c_2$ if $q_1 = q$, $q_2 = q'$, $p_2 = p_1 + p$.

Note that mandating that $c_2$ is a configuration implies two implicit inequality testing $0 \leq p_1 + p \leq b$. The reachability problem for Boca consists in determining whether there exists a sequence of transitions starting from the configuration $(q_0, 0)$ and ending in a configuration $(q, p)$. This problem has been shown to be PSPACE-complete in [9]. We reduce it to GVA’s Nonemptiness as follows:

- Assuming $2^{k-1} \leq b \leq 2^k$, we build a GVA with $k+1$ variables $x_1, \ldots, x_k, r$, where each $x_i$ contains the $i^{th}$ bit in the binary representation of the counter $p$, and $r$ stands for a register that will hold the current carry of a bit-per-bit binary addition;
– Addition of a positive constant to the counter is encoded by a sequence of
$k + 1$ half-adders. Each half-adder reads $x_i$, and depending on the value of
$r$ and the $i$th bit of the number to add, refreshes $x_i$ and $r$, and sets (using
a guard) the correct new value for these two variables. A constant number
of states each necessary at each half-addition step, and thus the encoding is
linear in $k = \lceil \log_2 b \rceil$, and thus in the size of the input $\text{Boca}$. Note that the
result is greater than $2^{\lceil \log_2 b \rceil}$ if, and only if, the register $r$ contains 1 in the
end;
– Bitwise 2-complement of a register is similarly performed bit-per-bit, using
a number of states linear in $k$;
– Substraction of a positive integer is encoded by two bitwise 2-comple ment
computations on the counter around a positive integer addition. If the final
counter is negative the intermediate addition step will yield a number greater
than $2^k$;
– Finally, we test after each addition of a positive integer that the coun ter
is smaller or equal to $b$ by adding $2^k - b$ to it, and then substractiong this same
number;
– The final configuration $(q_f, p_f)$ is encoded by a transition from states of the
GVA encoding $q_f$ to its unique final state having as guard the equality test
of the variables $x_1, \ldots, x_k$ with the bits in the binary encoding of $p_f$.

The PSPACE-hardness of the reachability problem for $\text{Boca}$ and Theorem 4
imply:

**Theorem 5.** The nonemptiness problem for GVAs is PSPACE-complete.

Note that we need only two letters in $\Sigma_A$ in the encoding employed to prove the hardness.

6 Simulations for GVAs

We define and study the simulation preorder for GVAs, an extension of the
simulation preorder for FAs. To simplify the presentation, we shall only consider
in this section GVAs without $\varepsilon$-transitions and in which there is a unique initial
state and all the states are accepting. The definition of simulation preorder for
GVAs follows.

**Definition 10.** Let $A_1 = (\Sigma, X_1, Q_1, q_1^0, \delta_1, F_1, \kappa_1)$ and $A_2 = (\Sigma, X_2, Q_2, q_2^0, \delta_2, F_2, \kappa_2)$
be two GVAs where $X_1 \cap X_2 = \emptyset$. A simulation of $A_1$ by $A_2$ is a relation
$\preceq \subseteq (\zeta_{X_1, \Sigma} \times Q_1) \times (\zeta_{X_2, \Sigma} \times Q_2)$ such that:

– $(\emptyset, q_1^0) \preceq (\emptyset, q_2^0)$,
– if $(\sigma_1, q_1) \preceq (\sigma_2, q_2)$ and if $(\sigma_1, q_1) \xrightarrow{a} (\sigma_1', q_1')$ for $a \in \Sigma$ then there exists
a state $q_2' \in Q_2$ and a substitution $\sigma_2'$ such that $(\sigma_2, q_2) \xrightarrow{a} (\sigma_2', q_2')$ and
$(\sigma_1', q_1') \preceq (\sigma_2', q_2')$. 

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In order to study the decidability of the simulation, we provide next an equivalent game-theoretic definition in which we make explicit the evolution of the configurations.

**Definition 11.** Let \( A_1 = \langle \Sigma, X_1, Q_1, q_0^1, \delta_1, F_1, \kappa_1 \rangle \) and \( A_2 = \langle \Sigma, X_2, Q_2, q_0^2, \delta_2, F_2, \kappa_2 \rangle \) be two GVAs where \( X_1 \cap X_2 = \emptyset \). Let \( \text{Pos} \) be the set of positions reachable from \( p^* = ((\emptyset, q_1^0), (\emptyset, q_0^2)) \). By the set of moves \( M = M_A \cup M_E \), where:

\[
M_A = \{ ( (\sigma_1, q_1), \tau_2) \mid q_1^1 \in \delta_1(q_1, \alpha, g_1) \text{ and } \delta = \text{Dom}(\sigma_1 \cup \gamma) \setminus \kappa_1^{-1}(q_1^1) \text{ and } \sigma_1 \cup \gamma \models g_1 \}
\]

\[
M_E = \{ ( (\sigma_1, q_1), (\sigma_2, q_2), (\delta, q_2^2), (\sigma_3, q_3), \gamma(\sigma_3(\alpha))) \mid q_2^1 \in \delta_2(q_2, \beta, g_2) \text{ and } \delta_2 = \text{Dom}(\sigma_2 \cup \gamma) \setminus \kappa_2^{-1}(q_2^1) \text{ and } \gamma(\delta_2(\beta)) \text{ and } \gamma_2 \models \sigma_2(g_2) \}
\]

with \( \tau_2 \) a configuration in \( (X \times \Sigma \times Q_2) \), and \( \sigma_1, \sigma_2, \sigma_3 \) are substitutions. We let \( \text{Pos}_E = \text{Pos} \cap (X_1 \times Q_1) \times (X_2 \times Q_2) \) and \( \text{Pos}_A = \text{Pos} \cap (X_1 \times Q_1) \times (X_2 \times Q_2) \). The simulation game of \( A_1 \) by \( A_2 \), denoted by \( G(A_1, A_2) \), is the two-players game \( \langle \text{Pos}_E, \text{Pos}_A, M, p^* \rangle \). As usual, any infinite play is winning for Eloise, and any finite play is losing for the player who cannot move. And thus we write \( A_1 \preceq A_2 \).

The simulation problem for GVAs is the following: given two GVAs \( A_1 \) and \( A_2 \), is \( A_1 \preceq A_2 \)?

### 7 Decidability of the simulation problem

In this section we show that the simulation problem is decidable. The idea is that this problem can be reduced to a simulation problem over the same GVAs in which the two players instantiate the variables from a finite set of letters, as proven in Proposition 3.

**Definition 12.** Let \( A_1 = \langle \Sigma, X_1, Q_1, q_0^1, \delta_1, F_1, \kappa_1 \rangle \) and \( A_2 = \langle \Sigma, X_2, Q_2, q_0^2, \delta_2, F_2, \kappa_2 \rangle \) be two GVAs. Let \( k = |X_1| + |X_2| \). We define \( \overline{G}(A_1, A_2) \) to be the game obtained by restricting the codomain of \( \gamma \) to \( C_0 \) in the rules of Eloise \( M_E \) and Abelard \( M_A \) in Def. 11, where

\[
C_0 = (\Sigma_A \cup \Sigma_A) \cup \{ \overline{c_1}, \ldots, \overline{c_k} \}
\]
The following Lemma states an immediate property of the game $\mathcal{G}$.

**Lemma 7.** Let $A_1, A_2$ be two GVAs. Then, the game $\mathcal{G}(A_1, A_2)$ is finite.

In order to prove Proposition 3, we need to adapt the notion of coherence between substitutions given in Definition 7 to the coherence between game positions. The definition of the coherence between game positions, still denoted by $\mathcal{C}$, follows.

**Definition 13.** Let $C$ be a finite subset of $\Sigma$, and $A_1 = (\Sigma, X_1, Q_1^1, q_0^1, \delta_1, F_1, \kappa_1)$, and $A_2 = (\Sigma, X_2, Q_2, q_0^2, \delta_2, F_2, \kappa_2)$ be two GVAs s.t. $X_1 \cap X_2 = \emptyset$. Let $\text{Pos}_E$ (resp. $\text{Pos}_A$) be the set of Eloise’s (resp. Abelard’s) positions in the game $\mathcal{G}(A_1, A_2)$.

\begin{itemize}
  \item For any substitutions $\sigma, \tilde{\sigma}$ of proper domain included in $X_i$ ($i = 1, 2$) we have: 
    \[ ((\sigma_1, q_1), (\tilde{\sigma}_2, q_2))_A \mathcal{C} ((\sigma_1, q_1), (\sigma_2, q_2))_A \iff (\sigma_1 \cup \tilde{\sigma}_2) \mathcal{C} (\sigma_1 \cup \sigma_2). \]
  \item For any $\sigma, \tilde{\sigma}$ of proper domain included in $X_i$ ($i = 1, 2$), for any substitutions $\sigma, \tilde{\sigma}$ with proper domain included in $X_1$, we have: 
    \[ ((\sigma_1 \cup \tilde{\sigma}_2) \mathcal{C} ((\sigma_1 \cup \sigma_2, q_2), (\sigma_1, q_1))_A \mathcal{C} ((\sigma_1 \cup \sigma_2, q_2), (\sigma_1, q_1))_A. \]
\end{itemize}

In order to prove Proposition 3, we need to prove a technical Lemma:

**Lemma 8.** Let $A_1 = (\Sigma, X_1, Q_1^1, q_0^1, \delta_1, F_1, \kappa_1)$ and $A_2 = (\Sigma, X_2, Q_2, q_0^2, \delta_2, F_2, \kappa_2)$ be two GVAs, and let $\mathcal{X} = X_1 \cup X_2$. Let $\mathcal{G}(A_1, A_2)$ (resp. $\mathcal{G}(A_1, A_2)$) be the simulation game in which the two players instantiate the variables from $\Sigma$ (resp. $\Sigma_0$ defined in Eq. (2)). Let $\varphi^*$ and $\overline{\varphi}$ be their starting position respectively. Then, there is a function $f : \text{Pos}(\mathcal{G}(A_1, A_2)) \rightarrow \text{Pos}(\mathcal{G}(A_1, A_2))$ with $f(\varphi^*) = \overline{\varphi}$ and $\varphi \mathcal{C} f(\varphi)$ for all $\varphi \in \text{Pos}(\mathcal{G}(A_1, A_2))$, such that the following hold:

\begin{itemize}
  \item[i)] for all $\overline{\varphi} \in \text{Pos}_A(\mathcal{G}(A_1, A_2))$, if $\overline{\varphi} \rightarrow \overline{\varphi}'$ is a move of Abelard in $\mathcal{G}(A_1, A_2)$ and $f(\varphi) = \overline{\varphi}$ for some position $\varphi$ in $\mathcal{G}(A_1, A_2)$, then there exists a position $\varphi'$ in $\mathcal{G}(A_1, A_2)$ such that the move $\varphi \rightarrow \varphi'$ is possible in $\mathcal{G}(A_1, A_2)$ and $f(\varphi') = \overline{\varphi}'$. And,
  \item[ii)] for all $\varphi \in \text{Pos}_E(\mathcal{G}(A_1, A_2))$, if $\varphi \rightarrow \varphi'$ is a move of Eloise in $\mathcal{G}(A_1, A_2)$ then there exists a position $\overline{\varphi}'$ in $\mathcal{G}(A_1, A_2)$ such that the move $f(\varphi) \rightarrow \overline{\varphi}'$ is possible in $\mathcal{G}(A_1, A_2)$ and $f(\varphi') = \overline{\varphi}'$.
\end{itemize}

**Proof.** (Sketch) The main part of the proof consists in finding the right way to relate the instantiation of the variables in $\mathcal{G}(A_1, A_2)$ and $\mathcal{G}(A_1, A_2)$.

\begin{itemize}
  \item[i)] The function $\Theta^S_{C_1, C_2}$ of Eq (1) allows to construct the instantiation of the variables by Abelard in $\mathcal{G}$ out of the instantiation of variables by Abelard in $\mathcal{G}$ as follows. Assume $\overline{\varphi} = ((\sigma_1, q_1), (\tilde{\sigma}_2, q_2))_A$ is a position in $\mathcal{G}(A_1, A_2)$, and $\gamma$ is an instantiation made by Abelard from $\overline{\varphi}$ (i.e. $\gamma$ in the move $M_A$ of Def. 11), and that $\varphi = ((\sigma_1, q_1), (\sigma_2, q_2))_A$ is a position in $\mathcal{G}(A_1, A_2)$ such that $f(\varphi) = \overline{\varphi}$. Then, Abelard’s instantiation $\gamma$ from $\varphi$ is defined by $\gamma = \Theta^S_{C_1, C_2}(\sigma_1 \cup \tilde{\sigma}_2, \gamma, \sigma_1 \cup \sigma_2)$.
  \item[ii)] We define a function $\Xi^S_{C_1, C_2}$ which is similar to the function $\Theta^S_{C_1, C_2}$ but allows to construct the instantiation of the variables by Eloise in $\mathcal{G}$ out of the instantiation of variables by Eloise in $\mathcal{G}$. \hfill \square
Now we are ready to show that the games $G$ and $\mathcal{G}$ are equivalent. We recall that the variables in $G(A_1,A_2)$ are instantiated from the finite set of letters $C_0 = (\Sigma_{A_1} \cup \Sigma_{A_2}) \cup \{c_1,\ldots,c_k\}$, where $k = |X_1| + |X_2|$.

For the direction "⇒" we show that out of a winning strategy of Eloise in $G(A_1,A_2)$ we construct a winning strategy for her in $\mathcal{G}(A_1,A_2)$. For the direction "⇐" we show that each move of Abelard in $G(A_1,A_2)$ can be mapped to an Abelard move in $\mathcal{G}(A_1,A_2)$, and that Eloise response in $G(A_1,A_2)$ can be actually mapped to an Eloise move in $\mathcal{G}(A_1,A_2)$. Formally, we need to define a function $f : \text{Pos}(G(A_1,A_2)) \to \text{Pos}(\mathcal{G}(A_1,A_2))$ in order to make possible this mapping as shown in the Diagram on the right. It follows that this is sufficient to argue that if there is an infinite play in $G$ then we can construct an infinite play in $\mathcal{G}$ as well. We show in Lemma 8 that it is possible to construct the function $f$. The proof of the direction (⇐) is similar to the one of (⇒), we follow the same construction. Therefore,

**Proposition 3.** Let $A_1$ and $A_2$ be two GVAs. Then, Eloise has a winning strategy in $G(A_1,A_2)$ iff she has a winning strategy in $\mathcal{G}(A_1,A_2)$.

It follows from Lemma 7 and Proposition 3:

**Theorem 6.** The simulation problem for GVAs is decidable.

Note that the simulation problem for GVAs is in APSPACE: A position of $\mathcal{G}(A_1,A_2)$, has size linear in the size of $A_1$ and $A_2$, in order to encode the substitutions and states. Hence the number of different positions of $\mathcal{G}(A_1,A_2)$ is bounded exponentially in the size of $A_1$ and $A_2$. From this we can deduce that a polynomial space alternating Turing machine can solve the simulation game: universal states correspond to Abelard positions and existential states correspond to Eloise positions. Hence,

**Theorem 7.** The simulation problem for GVAs is in EXPTIME.

8 Application to service composition

We illustrate the practical use of GVAs through a service composition problem. In Fig 2 we have an e-commerce Web site allowing clients to open files, search for items in a large domain that can be abstracted as infinite and save them to an appropriate file depending on the type of the items (whether they are in promotion or not). The three agents: CLIENT, FILE and SEARCH communicate with messages ranging over a possibly infinite set of terms. The problem is to check whether FILE and SEARCH can be composed in order to satisfy the
CLIENT requests. Following [5,17] the problem reduces to find a simulation between CLIENT and the asynchronous product of FILE and SEARCH. The variables $x$ and $y$ are refreshed (i.e. freed to get a new value) when passing through the state $p_0$. In the same way variables $z$ and $w$ are refreshed at $p_2$. The variables $m$ and $n$ are refreshed at $q_0$; the variables $i$ and $j$ are refreshed at $r_0$. For saving space, a transition labeled by a term, say $\text{write}(m,n)$, abbreviates successive transitions labeled by the root symbol and its arguments, here $\text{write}$, $m$ and $n$, respectively.

![Diagram](image)

**Fig. 2.** PROM example.

In order to solve the PROM composition problem we define next the asynchronous product of GVAs which generalizes the asynchronous product of FAs as given in [17].

**Definition 14.** Given $n$ GVAs $A_i = \langle \Sigma_i, X_i, Q_i, Q_0^i, \delta_i, F_i, \kappa_i \rangle$, their asynchronous product $A_1 \otimes \cdots \otimes A_n$ is a GVA: $\langle \Sigma, X, Q, Q_0, F, \kappa \rangle$, where:

- $\Sigma = \bigcup_{i=1}^n \Sigma_i$, $X = \bigcup_{i=1}^n X_i$,
- $Q = Q_1 \times \cdots \times Q_n$, $Q_0 = Q_0^1 \times \cdots \times Q_0^n$, $F = F_1 \times \cdots \times F_n$,
- $\delta$ is defined by: $q \in \delta(p,t)$ iff for some $i$, $\pi_i(q) \in \delta_i(\pi_i(p),t)$, and for all $j \neq i$ we have that $\pi_j(q) = \pi_j(p)$, where $\pi_i$ denotes the projection along the $i$th-component, and
- $\kappa$ is defined by: $p \in \kappa(x)$ iff for some $i$, $\pi_i(p) \in \kappa_i(x)$.

Given a client specification and a community of available services, finding a simulation of the client by the community of services amounts to constructing a winning strategy for Eloise in the simulation game of the client by the asynchronous product of the available services. In the case of the PROM example,

\footnote{Up to variable renaming, we assume that $X_i \cap X_j = \emptyset$, for all $i \neq j.$}
a winning strategy for Eloise can be computed in the game $G(\text{CLIENT, FILE } \otimes \text{SEARCH})$, and thus the client requests can be satisfied in all cases.

Note that the asynchronous product of NFMAs cannot be defined easily due to the global constraint forcing the registers values to be distinct ones (as discussed in Subsec. 4). On the contrary, this construction is easy to specify with GVAs.

9 Conclusion

In future works we plan to investigate the decidability status of the containment problem left open for the subclass of GVAs without disequalities in the guards. Our result on GVAs simulation applies to the synthesis of web service composition. In this context, disequalities should be useful to express security policy enforcement on services in the spirit of [7,2].

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Appendices

A Proofs for Section 3

For two finite sets $A$ and $B$ we denote by $A^B$ the set of all total functions from $A$ to $B$. Let $\mathcal{X}$ and $\mathcal{X}'$ be two disjoint sets of variables, and let $\psi$ be a total function in $\mathcal{X}^{\mathcal{X}'}$, and let $g$ be a conjunction of equalities between variables in $\mathcal{X}$. Then define $g \sqsubset \psi$ iff there exists $x' \in \mathcal{X}'$ s.t. $\psi(x) = x'$ for all $x$ in $\mathcal{V}(g)$.

Lemma 9. (i.e. Lemma 1) For every GVA $A$ with $k$ variables and $n$ states there is a GVA* with $k + m$ variables and $O(n \cdot (k + m)!)$ states recognizing the same languages, where $m = |\Sigma_A|$.

Proof. Let $A = \langle \Sigma, \mathcal{X}, Q, Q_0, \delta, F, \kappa \rangle$ be a GVA with $\mathcal{X} = \{x_1, \ldots, x_k\}$.

Firstly, we transform the GVA $A$ into a GVA $\tilde{A}$ recognizing the same language and in which each state is labeled with the set of variables being free in this state. We define $\tilde{A} = \langle \Sigma, \mathcal{X}, Q, Q_0, F, \delta, \kappa \rangle$ by:

$$
\begin{align*}
Q &= \{(q, X) \mid q \in Q \text{ and } X \subseteq \mathcal{X}\}, \\
Q_0 &= \{(q, \mathcal{X}) \mid q \in Q_0\}, \\
F &= \{(q, X) \mid q \in F \text{ and } X \subseteq \mathcal{X}\}.
\end{align*}
$$

The transition function $\delta$ is defined by $(q', X') \in \delta((q, X), \alpha, g)$, where $\alpha \in \Sigma \cup \mathcal{X}$ and $g$ is a guard, if and only if, $q' \in \delta(q, \alpha, g)$ and $X' = (X \setminus \{(\alpha) \cup \mathcal{V}(g)\}) \cup \kappa^{-1}(q')$. Finally, the refreshing function $\kappa'$ is defined by $\kappa(x) = \{q, X \mid q \in \kappa(x)\}$.

Secondly, we can assume w.l.o. that $\tilde{A}$ has no constants and the variables are refreshed only in the states preceded by $\varepsilon$-transitions. The constants can be replaced by additional variables that have to be initialized with the related constants using an $\varepsilon$-transition outgoing from the initial state. And, if some variables, say $X \subseteq \mathcal{X}$, are refreshed in a state, say $q$, then we add an $\varepsilon$-transition $q \xrightarrow{\varepsilon} \tilde{q}$ where the variables $X$ are refreshed in $\tilde{q}$ instead of $q$ and the outgoing transitions of $q$ become the outgoing transitions of $\tilde{q}$. Thus, the guards of $\tilde{A}$ are of the form $\phi \land \phi'$ where $\phi$ (resp. $\phi'$) is a conjunction of equalities (resp. inequalities) between variables.

Thirdly, we let $\mathcal{A}'$ to be the GVA* $\mathcal{A}' = \langle \Sigma, \mathcal{X}', Q', Q'_0, \delta', F', \kappa' \rangle$ defined by

$$
\begin{align*}
\mathcal{X}' &= \{x'_1, \ldots, x'_k\} \\
Q' &= Q \times \mathcal{X}^{\mathcal{X}'} \\
Q'_0 &= Q_0 \times \mathcal{X}^{\mathcal{X}'} \\
\kappa' &= \kappa \times \mathcal{X}^{\mathcal{X}'}
\end{align*}
$$

and $\delta'$ is defined by

$$
\begin{align*}
((q_1, X_1, \psi_1), (\alpha, \psi_1(\gamma(x)), q_2, X_2, \psi_1)) \in \delta' \text{ iff } \begin{cases} 
(q_1, X_1), (\alpha, \psi_1(\gamma(x)), q_2, X_2) \in \delta \text{ and } \\
\alpha \neq \varepsilon \text{ and } \\
g \sqsubset \psi_1 \text{ and } \\
\mathcal{V}(\gamma(x)) = \text{codom}(\psi_1) \text{ and } \\
\mathcal{V}(\gamma(x)) \cap X_1 = \emptyset
\end{cases}
\end{align*}
$$

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B Proofs for Section 5

The claims in the following Lemma are not hard to prove. They will be used in the proofs of the main claims.

Lemma 10. Let $C \subseteq \Sigma$ be a finite set of letters, $\bar{\sigma}$ and $\sigma$ two substitutions, $x$, and $a$ a letter in $C$. The following hold.

1. If $\bar{\sigma} \triangleright_C \sigma$ then $|\text{dom}(\bar{\sigma})| = |\text{dom}(\sigma)|$.
2. If $\bar{\sigma} \triangleright_C \sigma$ and $D \subseteq \text{Dom}(\sigma)$ then $\bar{\sigma}|_D \triangleright_C \sigma|_D$.
3. If $(\bar{\sigma}_1 \uplus \bar{\sigma}_2) \triangleright (\sigma_1 \uplus \sigma_2)$ with $\text{dom}(\bar{\sigma}_i) = \text{dom}(\sigma_i)$, then $\bar{\sigma}_i \triangleright \sigma_i$, for $i = 1, 2$.
4. If $\bar{\sigma} \triangleright_C \sigma$ and $\gamma$ is a substitution with $\text{dom}(\gamma) \cap \text{dom}(\sigma) = \emptyset$ and $\text{dom}(\gamma) \subseteq C$, then $\bar{\sigma} \uplus \gamma \triangleright_C \sigma \uplus \gamma$.
5. If $\bar{\sigma} \triangleright_C \sigma$ with $\bar{\sigma}(y) = \bar{a}$ and $\sigma(y) = a$ for some variable $y$, and $x \notin \text{dom}(\sigma)$ then $\bar{\sigma} \uplus \{x \mapsto \bar{a}\} \triangleright_C \sigma \uplus \{x \mapsto a\}$.
6. If $\bar{\sigma} \triangleright_C \sigma$ and $\bar{a} \notin C \cup \text{dom}(\bar{\sigma})$ and $a \notin C \cup \text{dom}(\sigma)$ and $x \notin \text{dom}(\sigma)$ then $\bar{\sigma} \uplus \{x \mapsto \bar{a}\} \triangleright_C \sigma \uplus \{x \mapsto a\}$.

Notice that the opposite direction of the Item 3 of Lemma 10 does not hold in general.

Lemma 11. Let $\sigma$ and $\bar{\sigma}$ be two substitution, where $\sigma \triangleright_C \bar{\sigma}$, and let $g$ be a guard such that $\Sigma_g \subseteq C$. Then, $\sigma \models g$ iff $\bar{\sigma} \models g$.

Proof. By induction on the structure of $g$ in both directions.

$\Rightarrow$) If $g = (\alpha = x)$, where $\alpha \in \Sigma$ and $x \in \mathcal{X}$, then $\sigma(x) \in C$, hence $\bar{\sigma}(x) \in C$. Therefore $\sigma(x) = \bar{\sigma}(x) = \alpha$. If $g = (x = y)$, where $x, y \in \mathcal{X}$, then from the definition of $\triangleright_C$ we have that $\sigma(x) = \sigma(y)$ iff $\bar{\sigma}(x) = \bar{\sigma}(y)$. Thus the claim holds.

The case when $g = g_1 \wedge g_2$ follows from a direct application of the induction hypothesis.

$\Leftarrow$) This direction follows from the fact that $\sigma \triangleright_C \bar{\sigma}$ iff $\bar{\sigma} \triangleright_C \sigma$.

\[\square\]
For a substitution \( \sigma \) and a guard \( g \), we shall write \( \sigma \vdash g \) if there exists a substitution \( \sigma' \) such that \( \sigma \models \sigma'(g) \). Hence,

**Corollary 3.** Let \( \sigma \) and \( \bar{\sigma} \) be two substitution, where \( \sigma \not\preceq_C \bar{\sigma} \), and let \( g \) be a guard such that \( \Sigma_g \subseteq C \). Then, \( \sigma \vdash g \) iff \( \bar{\sigma} \not\vdash g \).

**Proof.**

\( \Rightarrow \) We show that if there exists a substitution \( \gamma \) such that \( \text{dom}(\gamma) = \mathbb{V}(g) \setminus \text{dom}(\sigma) \) and \( \sigma \models \gamma(g) \), then \( \bar{\sigma} \not\models \gamma(g) \). But this follows from Lemma 11.

\( \Leftarrow \) This direction follows from the fact that \( \preceq_C \) is symmetric relation.

\( \square \)

**Corollary 4.** Let \( \sigma, \bar{\sigma}, \gamma, \bar{\gamma} \) be substitutions, where \( \text{dom}(\gamma) \cap \text{dom}(\sigma) = \emptyset \) and \( \text{dom}(\bar{\gamma}) \cap \text{dom}(\bar{\sigma}) = \emptyset \). Let \( g \) be a guard such that \( \Sigma_g \subseteq C \). If \( \sigma \not\models_C \bar{\sigma} \not\models \bar{\gamma} \) then we have that \( \gamma \models (g) \iff \bar{\gamma} \not\models (g) \).

In what follows we let \( S_1, S_2 \) be two (possibly infinite) sets of letters with \( |S_1 \setminus S_2| > |X| \) and \( |S_2 \setminus S_1| > |X| \) and \( S_1 \cap S_2 \neq \emptyset \). Let \( C = S_1 \cap S_2 \).

**Definition 15.** We define the functions

\[
\Theta_C^{S_1,S_2}, \Theta_C^{S_1,S_2}: \xi_x.S_1 \times \xi_x.S_1 \times \xi_x.S_2 \rightarrow \xi_x.S_2
\]

as follows. Let \( M_1, \gamma_1 \in \xi_x.S_1, M_2 \in \xi_x.S_2 \). Then, \( \Theta_C^{S_1,S_2}(M_1, \gamma_1, M_2) \) is defined only when \( |\text{dom}(\gamma_1)| = 1 \) and \( \text{dom}(\gamma_1) \cap \text{dom}(M_1) = \emptyset \) and \( M_1 \not\preceq_C M_2 \) by:

\[
\Theta_C^{S_1,S_2}(M_1, \gamma_1, M_2) = \begin{cases} 
\gamma_1 & \text{if } \gamma_1(x) \in C \\
\{ x \mapsto M_2(y) \} & \text{if } \gamma_1(x) \in \text{codom}(M_1) \setminus C \text{ and } M_1(y) = \gamma_1(x), y \in X \\
\{ x \mapsto \text{get}(S_2 \setminus \text{codom}(M_2)) \} & \text{if } \gamma_1(x) \in S_1 \setminus (C \cup \text{dom}(M_1)) 
\end{cases}
\]

where \( \text{dom}(\gamma_1) = \{ x \} \).

And \( \Theta_C^{S_1,S_2}(M_1, \gamma_1, M_2) \) is defined only when \( \text{dom}(\gamma_1) \cap \text{dom}(M_1) = \emptyset \) by:

\[
\Theta_C^{S_1,S_2}(M_1, \gamma_1, M_2) = \begin{cases} 
\Theta_C^{S_1,S_2}(M_1, \gamma_1, M_2) & \text{if } \gamma_1 = 1 \\
\gamma_2 \triplus \Theta_C^{S_1,S_2}(M_1 \triplus \gamma'_1, \gamma''_1, M_2 \triplus \gamma'_2) & \text{if } \gamma_1 \geq 2, \gamma_1 = \gamma'_1 \triplus \gamma''_1 \text{ and } |\gamma'_1| = 1, \text{ and } \gamma'_2 = \Theta_C^{S_1,S_2}(M_1, \gamma'_1, M_2)
\end{cases}
\]

**Lemma 12.** Let \( M_1, \gamma_1 \in \xi_x.S_1 \) and \( M_2 \in \xi_x.S_2 \) be substitutions with \( \text{dom}(M_1) \cap \text{dom}(\gamma_1) = \emptyset \) and \( M_1 \not\preceq_C M_2 \). We have that

\[
(M_1 \triplus \gamma_1) \not\preceq_C (M_2 \triplus \Theta_C^{S_1,S_2}(M_1, \gamma_1, M_2))
\]

**Proof.** By induction on \( |\text{dom}(\gamma_1)| = 1 \).
Base Case. If $|\text{dom}(\gamma_1)| = 1$ then assume $\text{dom}(\gamma_1) = \{x\}$ and let $\gamma_2 = \Theta^S_1,S_2_{C}(M_1, \gamma_1, M_2)$. We distinguish three cases depending on $\gamma_1(x)$.

- If $\gamma_1(x) \in C$ then it follows from the definition of $\Theta^S_1,S_2_{C}$ that $\gamma_2 = \gamma_1$. From the Item 4 of Lemma 10 we get $M_1 \uplus \gamma_1 \not\preceq_C M_2 \uplus \gamma_1$.
- If $\gamma_1(x) \in \text{codom}(M_1) \setminus C$ then in this case we recall that $\gamma_2 = \{x \mapsto M_2(y)\}$ where $M_1(y) = \gamma_1(x)$ for some variable $y \in X$. The claim that $(M_1 \uplus \{x \mapsto \gamma_1(x)\}) \not\preceq_C M_2 \uplus \{x \mapsto M_2(y)\}$ follows from the Item 5 of Lemma 10.
- Otherwise, i.e. if $\gamma_1(x) \in S_1 \setminus (C \cup \text{codom}(M_1))$ then the claim that $(M_1 \uplus \{x \mapsto \gamma_1(x)\}) \not\preceq_C (M_2 \uplus \{x \mapsto \text{get}(S_2 \setminus \text{codom}(M_2))\})$ follows from the Item 6 of Lemma 10.

Induction Case. Assume $\gamma_1 = \gamma_1' \uplus \gamma_1''$ with $|\gamma_1'| = 1$. Let

$$
\begin{align*}
\gamma_2' &= \Theta^S_1,S_2_{C}(M_1, \gamma_1', M_2), \\
\gamma_2'' &= \Theta^S_1,S_2_{C}(M_1 \uplus \gamma_1', \gamma_1'', M_2 \uplus \gamma_2') \\
\gamma_2 &= \Theta^S_1,S_2_{C}(M_1, \gamma_1, M_2) = \gamma_2' \uplus \gamma_2''
\end{align*}
$$

From the induction hypothesis it follows that

$$
\begin{align*}
(M_1 \uplus \gamma_1' \uplus \gamma_1'') &\not\preceq_C M_2 \uplus \gamma_2' \\
(M_1 \uplus \gamma_1' \uplus \gamma_1'') &\not\preceq_C (M_2 \uplus \gamma_2') \uplus \gamma_2''
\end{align*}
$$

Therefore

$$
M_1 \uplus \gamma_1 \not\preceq_C M_2 \uplus \gamma_2
$$

\[\square\]

Lemma 13. (i.e. Lemma 5) Let $A$ be a GVA over $\Sigma$ with $n$ variables and $k$ constants $\Sigma_A = \{c_1, \ldots, c_k\}$. Let $\Sigma = \{a_1, \ldots, a_n, c_1, \ldots, c_k\}$. Then, $A$ recognizes a non-empty language over $\Sigma^*$ if, and only if, it recognizes a non-empty language over $\Sigma^*$. 

Proof. Let $C = \{c_1, \ldots, c_n\}$. We show that there is a run $(\sigma_0, q_0) \rightarrow \ldots \rightarrow (\sigma_n, q_n)$ over $\Sigma^*$ in $A$ if and only if there is a run $(\sigma_0, q_0) \rightarrow \ldots \rightarrow (\sigma'_n, q_n)$ over $\Sigma^*$ in $A$ such that $\sigma_i \not\preceq_C \sigma'_i$ for all $i = 0, \ldots, n$. The proof is by induction on $n$ in both directions. The base case $n = 0$ holds trivially since $\sigma_0 = \sigma'_0 = \emptyset$. Assume that the claim holds up to $n$ and let us prove it for $n + 1$.

$\Rightarrow$ Assume there is a transition $q_n \xrightarrow{\alpha_n \cdot q_n} q_{n+1}$ in $A$ where $\alpha_n \in \Sigma \cup X$ and $q_n$ is a guard. From the induction hypothesis we have that $\sigma_n \not\preceq_C \sigma'_n$. It follows from Lemma 11 that $\sigma_n(q_n)$ holds iff $\sigma'_n(q_n)$ holds. Thus, the transition is possible. We describe next this transition. From Definition 3 of the run of GVAs, there exists a substitution $\gamma_n : V(\sigma_n(\alpha_n)) \cup V(\sigma_n(q_n)) \rightarrow \Sigma$ such that $V(\gamma_n(\sigma_n(q_n)))$ holds. Hence, we must find a substitution $\gamma'_n : V(\sigma'_n(\alpha_n)) \cup V(\sigma'_n(q_n)) \rightarrow \Sigma$ such that $V(\gamma'_n(\sigma'_n(q_n)))$ holds. We define $\gamma'_n$ by
\[ \gamma'_n = \Theta_C^{\Sigma, \Sigma} (\gamma_n, \gamma'_n) \]. From Lemma 12 we have that \((\sigma_n \cup \gamma_n) \setminus \Delta_c, (\sigma'_n \cup \gamma'_n) \). Hence from Lemma 11 it follows that \((\gamma'_n \cup \gamma'_n)(g_n) \) holds. It remains to show that \(\sigma_{n+1} \setminus \Delta_c, \sigma'_{n+1} \). But \(\sigma_{n+1} = (\sigma_n \cup \gamma_n) \setminus \Delta_c \) and \(\sigma'_{n+1} = (\sigma'_n \cup \gamma'_n) \setminus \Delta_c \), for some set \(D \subseteq \Gamma \). From Item 9 of Lemma 10 it follows that \(\sigma_{n+1} \setminus \Delta_c, \sigma'_{n+1} \).

\(\Leftarrow\) Same proof but we call the function \(\Theta_C^{\Sigma, \Sigma} (\gamma'_n, \gamma_n) \).

\[ \Box \]

C Proofs for Section 5

C.1 Closure properties of GVAs

**Theorem 1.** GVAs are closed under concatenation, Kleene operator and intersection.

**Proof.** Let \(A_1 = (\Sigma_1, X_1, Q_1, q_0^1, \delta_1, F_1, \kappa_1) \) and \(A_2 = (\Sigma_2, X_2, Q_2, q_0^2, \delta_2, F_2, \kappa_2) \) be two GVAs. Up to variable renaming it is sufficient to consider the closure under the above operations for two GVAs that do not share variables.

The closure under Kleene operation and concatenation holds since GVAs have \(\varepsilon\)-transitions. More precisely, the Kleene closure \(A_1^* \) amounts to adding an (unguarded) \(\varepsilon\)-transition between the accepting states and initial states of \(A_1 \). And the concatenation \(A_1 \cdot A_2 \) amounts to adding an (unguarded) \(\varepsilon\)-transition between the accepting states of \(A_1 \) and the initial states of \(A_2 \).

The closure under intersection follows from the fact that the intersection of two GVAs \(A_1 \) and \(A_2 \) denoted by \(A_1 \cap A_2 \) can be defined as follows:

\[ A_1 \cap A_2 = (\Sigma_1 \cup \Sigma_2, X_1 \cup X_2, Q_1 \times Q_2, q_0^1 \times q_0^2, \delta, F_1 \times F_2, \kappa), \]

where \(\delta \) and \(\kappa \) are defined by:

\[
\begin{align*}
(q_1', q_2') \in \delta((q_1, q_2), (\alpha_1, (\alpha_1 = \alpha_2) \land g_1 \land g_2)) & \text{ iff } q_1' \in \delta_1(q_1, \alpha_1, g_1) \text{ and } q_2' \in \delta_2(q_2, \alpha_2, g_2). \\
(q_1, q_2) \in \kappa(x) & \text{ iff } q_1 \in \kappa_1(x) \text{ or } q_2 \in \kappa_2(x).
\end{align*}
\]

The proof that \(L(A_1) \cap L(A_2) = L(A_1 \cap A_2) \) is straightforward. \[ \Box \]

**Proposition 1.** The complement of a regular language is GVA-recognizable. That is, given a FA \(F \) there exists a GVA \(A \) such that \(L(A) = \Sigma^* \setminus L(F) \).

**Proof.** The construction of \(A \) is similar to the one for FAs (over a finite alphabet). We assume that \(F \) is deterministic. Firstly, we make the completion of \(F \), i.e. we construct an equivalent GVA so that for each state \(q \) of \(F \) and for each letter \(l \in \Sigma \) there is a unique transition outgoing from \(q \) that reads \(l \). Secondly, we swap the accepting and non-accepting states.

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Formally, assume \( F = \langle \Sigma, Q, p_0, \delta, F \rangle \), with \( Q = \{ q_1, \ldots, q_n \} \), and define \( A = \langle \Sigma, \mathcal{X}, Q', p_0, \delta', F', \kappa \rangle \) by

\[
\begin{align*}
\mathcal{X} &= \{ x_1, \ldots, x_n \} \\
Q' &= Q \cup \{ q_{1i}, q_{2i}, i = 1, \ldots, n \} \\
F' &= (Q \setminus F) \cup \{ q_{1i}, q_{2i}, i = 1, \ldots, n \} \\
\delta' &= \delta \cup \{ q_{i}, \Lambda_{\delta}(x_{a_j}) \} q_{1i} \quad \text{for all } a_j \text{ s.t. } q_{1i} a_j q_{1j} \in \delta \} \cup \{ q_{2i} \xrightarrow{a_j} q_{2j} \}
\end{align*}
\]

Notice that \( F \) rejects a word \( w \) if \( A \) accepts \( w \).

### C.2 Decision procedures for GVAs

**Theorem 2.** For GVAs, Membership is \( \text{NP-complete} \).

**Proof.** Let \( A \) be a GVA and \( w = w_1 \cdots w_n \) a word in \( \Sigma^* \).

For the upper bound of the membership, a non-deterministic polynomial algorithm guesses a path in \( A \) of length \( |w| \) such that the final state is accepting; and a series of substitutions \( \sigma_1, \ldots, \sigma_{|w|} \), where \( \sigma_i : \mathcal{X} \to \{ w_j, 1 \leq j \leq |w| \} \), then checks whether the corresponding run on \( w \) is possible. The lower bound, i.e. the \( \text{NP-hardness} \), follows from the fact that the membership problem for GVAs without guards, i.e. GVAs, is \( \text{NP-complete} \). The undecidability of the universality follows from [10] since this problem is undecidable for the class of variable automata which is a subclass of GVAs.

### D Proofs for Section 6

The definition of the function \( \Xi \) follows.

**Lemma 14.** Let \( \sigma_1, \sigma_2, \sigma_3, \gamma_2 \in \xi_{\mathcal{X}, S_1} \) and \( \alpha, \beta \in C \cup \mathcal{X} \) and \( \sigma'_1, \sigma'_2, \sigma'_3 \in \xi_{\mathcal{X}, S_2} \) and \( g_2 \in G \) be such that

\[
\begin{align*}
\gamma_2(\sigma_2(\beta)) &= \sigma_3(\alpha) \\
\sigma_3 \uplus \sigma_2 \uplus_C \sigma'_3 &\uplus \sigma'_2 \\
\sigma_1 \uplus_C \sigma'_1 &
\end{align*}
\]

(3)

There exists a function \( \Xi_{S_1, S_2}^C : \xi_{\mathcal{X}, S_1} \times (C \cup \mathcal{X})^2 \times G_{C} \times \xi_{\mathcal{X}, S_2} \to \xi_{\mathcal{X}, S_2} \)

such that \( (\sigma_1, \sigma_2, \sigma_3, \gamma_2, \alpha, \beta, g_2, \sigma'_1, \sigma'_2, \sigma'_3) \) \rightarrow \gamma'_2

which is defined only if Eq (3) holds and satisfies the following:

\[
\begin{align*}
\gamma'_2(\sigma'_2(\beta)) &= \sigma'_3(\alpha) \\
(\sigma_1 \uplus \gamma_1) \uplus (\sigma_2 \uplus \gamma_2) \uplus_C (\sigma'_1 \uplus \gamma'_1) &\uplus (\sigma'_2 \uplus \gamma'_2) \\
\gamma'_2 &\vdash \sigma'_3(g_2)
\end{align*}
\]

(A.1)  
(A.2)  
(A.3)
Proof. The construction of $\Xi_{C}^{S_{1},S_{2}}$ depends on $\sigma_{3}(\alpha)$

I.) If $\sigma_{3}(\alpha) \in C$, then in this case we have $\sigma_{3}(\alpha) = \sigma'_{3}(\alpha) \in C$. Hence $\sigma'_{2} = \sigma_{2}$.

Thus (A.1) holds. Furthermore we let:

$$\gamma'_{2} = \Theta_{C}^{S_{1},S_{2}}(\sigma_{3} \sqcup \sigma_{2}, \gamma_{2}, \sigma'_{3} \sqcup \sigma'_{2})$$

From Lemma 12 it follows that

$$\sigma_{3} \sqcup \sigma_{2} \sqcup \gamma_{1} \sqcup \gamma_{2} \sqcup \gamma'_{1} \sqcup \gamma'_{2}$$

From Eq (3) we have $\text{dom}(\sigma_{1}) \subseteq \text{dom}(\sigma_{3})$ and $\text{dom}(\sigma'_{1}) \subseteq \text{dom}(\sigma'_{3})$ and $\text{dom}(\sigma_{1}) = \text{dom}(\sigma'_{1})$, hence it follows from Item 2 of Lemma 10 that

$$\sigma_{1} \sqcup \sigma_{2} \sqcup \gamma_{2} \sqcup \sigma'_{1} \sqcup \sigma'_{2} \sqcup \gamma'_{2}$$

Therefore (A.2) holds. Finally (A.3) follows from Corollary 4.

II.) If $\sigma_{3}(\alpha) \in S_{1} \setminus C$, then $\alpha$ must be a variable, say $y_{1} \in X$. We distinguish two cases depending on $\sigma_{2}(\beta)$.

i.) If $\sigma_{2}(\beta)$ is a letter then in this case $\sigma_{2}(\beta) = \sigma_{3}(\alpha)$, and we let

$$\gamma'_{2} = \Theta_{C}^{S_{1},S_{2}}(\sigma_{3} \sqcup \sigma_{2} \sqcup \gamma_{1}, \gamma_{2}, \sigma'_{3} \sqcup \sigma'_{2} \sqcup \gamma'_{1})$$

And we must show that $\sigma'_{3}(\alpha) = \sigma'_{2}(\beta)$. Notice that $\beta$ must be a variable, say $y_{2} \in X$. On the other hand we have that $\{y_{1} \mapsto \sigma_{3}(\alpha), y_{2} \mapsto \sigma_{2}(\beta)\} \subseteq \sigma_{3} \sqcup \sigma_{2}$ and $\{y_{1} \mapsto \sigma'_{3}(\alpha), y_{2} \mapsto \sigma'_{2}(\beta)\} \subseteq \sigma'_{3} \sqcup \sigma'_{2}$. On the other hand, we have that $\sigma_{3}(\alpha) = \sigma_{2}(\beta)$ and $\sigma_{3} \sqcup \sigma_{2} \sqcup \sigma'_{1} \sqcup \sigma'_{2} \sqcup \gamma'_{2}$. Therefore $\sigma'_{3}(\alpha) = \sigma'_{2}(\beta)$, thus (A.1), (A.2) and (A.3) hold.

ii.) If $\sigma_{2}(\beta)$ is a variable, say $y_{2} \in X$, then $\sigma'_{3}(\beta) = \sigma_{2}(\beta) = \beta = y_{2}$, since $\sigma'_{2} \sqcup \sigma_{2}$. In this case we have $\{y_{2} \mapsto \sigma_{3}(\alpha)\} \subseteq \gamma_{2}$. Thus we let

$$\gamma'_{2} = \Theta_{C}^{S_{1},S_{2}}(\sigma_{3} \sqcup \sigma_{2} \sqcup \{y_{1} \mapsto \sigma_{3}(\alpha)\}, \gamma_{2}, \sigma'_{3} \sqcup \sigma'_{2} \sqcup \{y_{1} \mapsto \sigma'_{3}(\alpha)\})$$

And Eqs (A.1), . . . , (A.3) hold.

III.) If $\sigma_{3}(\alpha)$ is a variable, say $x_{1} \in X$, then $\sigma'_{3}(\alpha) = \sigma_{3}(\alpha) = \alpha = x_{1}$. We distinguish two cases depending on the nature of $\sigma_{2}(\beta)$.

I.) If $\sigma_{2}(\beta)$ is a letter then $\sigma'_{3}(\beta)$ is a letter as well since $\sigma'_{2} \sqcup \sigma_{2}$. This case is dual w.r.t. case II.ii).

II.) If $\sigma_{2}(\beta)$ is a variable, say $y_{2} \in X$, then $\sigma_{2}(\beta) = \sigma'_{3}(\beta) = \beta = y_{2}$ since $\sigma'_{2} \sqcup \sigma_{2}$. In this case we let $\gamma'_{2} = \Theta_{C}^{S_{1},S_{2}}(\sigma_{3} \sqcup \sigma_{2}, \gamma_{2}, \sigma'_{3} \sqcup \sigma'_{2})$.

\[\square\]

By using the functions $\Theta$ and $\Xi$ and their respective properties stated in Lemmas 12 and 14 we are ready to prove Lemma 8
Lemma 15. (i.e. Lemma 8) Let $A_1 = \langle \Sigma_1, X_1, Q_1, \delta_1, F_1, \kappa_1 \rangle$ and $A_2 = \langle \Sigma_2, X_2, Q_2, \delta_2, F_2, \kappa_2 \rangle$ be two GVAs, and let $X = X_1 \cup X_2$. Let $G(A_1, A_2)$ (resp. $\overline{G}(A_1, A_2)$) be the simulation game in which the two players instantiate the variables from $\Sigma$ (resp. $C_0$ defined in Eq. (2)). Let $p^*$ and $\overline{p}^*$ be their starting position respectively. Then, there is a function $f$:

$$f : \text{Pos}(G(A_1, A_2)) \rightarrow \text{Pos}(\overline{G}(A_1, A_2))$$

with $f(p^*) = \overline{p}^*$ and $\varphi \not\sim_C f(\varphi)$ for all $\varphi \in \text{Pos}(G(A_1, A_2))$, such that the following hold:

i) for all $\overline{\varphi} \in \text{Pos}_A(\overline{G}(A_1, A_2))$, if $\overline{\varphi} \rightarrow \overline{\varphi}'$ is a move of Abelard in $\overline{G}(A_1, A_2)$ and $f(\varphi) = \overline{\varphi}$ for some position $\varphi$ in $G(A_1, A_2)$, then there exists a position $\varphi'$ in $G(A_1, A_2)$ such that the move $\varphi \rightarrow \varphi'$ is possible in $G(A_1, A_2)$ and $f(\varphi') = \overline{\varphi}'$. And,

ii) for all $\varphi \in \text{Pos}_E(G(A_1, A_2))$, if $\varphi \rightarrow \varphi'$ is a move of Eloise in $G(A_1, A_2)$ then there exists a position $\overline{\varphi}$ in $\overline{G}(A_1, A_2)$ such that the move $f(\varphi) \rightarrow \overline{\varphi}'$ is possible in $\overline{G}(A_1, A_2)$ and $f(\varphi') = \overline{\varphi}'$.

Proof. The proof is by induction on $n$, the number of the moves made in $\overline{G}(A_1, A_2)$ plus the number of moves made in $G(A_1, A_2)$. The base case, i.e. when $n = 0$, trivially holds since the starting position of $\overline{G}(A_1, A_2)$ and of $G(A_1, A_2)$ is $((\emptyset, q_0^1), (\emptyset, q_0^2))$.

For the induction case let $\varphi_n \in \text{Pos}(G(A_1, A_2))$ and $\overline{\varphi}_n \in \text{Pos}(\overline{G}(A_1, A_2))$ where $f(\varphi_n) = \overline{\varphi}_n$. We discuss two cases depending whether $\overline{\varphi}_n$ and $\varphi_n$ are both Abelard positions or they are both Eloise positions.

i) If $\varphi_n \in \text{Pos}_A(G(A_1, A_2))$ and $\overline{\varphi}_n \in \text{Pos}_A(\overline{G}(A_1, A_2))$ then consider an Abelard move $\overline{m} = \overline{\varphi}_n \rightarrow \overline{\varphi}_{n+1}$ in $\overline{G}(A_1, A_2)$. In this case $\overline{m}$ is of the form:

$$\overline{m} = ((\overline{\varphi}_n), (\overline{q}_1, \overline{q}_2)) \rightarrow ((\overline{\varphi}_{n+1}), (\overline{\varphi}_1 \uplus \overline{\gamma}), (\overline{\varphi}_1 \uplus \overline{\gamma}, g_1)) \in \overline{G}(A_1, A_2)$$

where $\overline{q_1} \in D_1(\overline{\varphi}_1 \uplus \overline{\gamma}, g_1)$ and $D = \text{Dom}(\overline{\varphi}_1 \uplus \overline{\gamma}) \setminus \kappa_1^{-1}(\overline{q_1})$ and $\overline{\varphi}_1 \uplus \overline{\gamma} \vdash g_1$ and $\overline{\gamma} : V(\overline{\varphi}_1(g_1)) \setminus V(\overline{\varphi}_1(\alpha)) \rightarrow C_0$

From the induction hypothesis we have $\varphi_n \not\sim_C f(\varphi_n)$, that is, $\varphi_n \not\sim_C \overline{\varphi}_n$. Hence $\varphi_n = ((\sigma_1(q_1)), (\sigma_2(q_2)))$, for two substitutions $\sigma_1, \sigma_2$ where $(\overline{\sigma}_1 \uplus \overline{\sigma}_2) \not\sim_C (\sigma_1 \uplus \sigma_2)$. Thus Abelard move $m$ in $G(A_1, A_2)$ is defined by:

$$m = ((\sigma_1(q_1)), (\sigma_2(q_2))) \rightarrow ((\sigma_1 \uplus \gamma)(\sigma_1(q_1)), (\sigma_2(q_2)), (\sigma_1 \uplus \gamma))) \in G(A_1, A_2)$$

where $\gamma : V(\sigma_1(q_1)) \setminus V(\alpha) \rightarrow \Sigma$ is defined by

$$\gamma = \Theta^{C_0, \Sigma}_C(\sigma_1 \uplus \sigma_2, \gamma, \sigma_1 \uplus \sigma_2).$$

(4)
Notice that since $\bar{\sigma}_1 \not\models_C \sigma_1$ then $\text{dom}(\bar{\gamma}) = \text{dom}(\gamma)$. We let $f(\bar{\varphi}_{n+1}) = \bar{\varphi}_{n+1}$
Furthermore, we must show that
\[
\sigma_1 \not\models \gamma \vdash g_1
\] (5)

and that $\varphi_{n+1} \not\models_C f(\varphi_{n+1})$, i.e. to show that $\varphi_{n+1} \not\models_C \bar{\varphi}_{n+1}$. That is, we must show that:
\[
((\sigma_1 \not\models \gamma)\mid_D \cup (\sigma_1 \not\models \gamma)) \not\models \sigma_2. \ \not\models_C \ ((\sigma_1 \not\models \gamma)\mid_D \cup (\sigma_1 \not\models \gamma)) \not\models \sigma_2
\] (6)

From the definition of $\gamma$ in Eq (4) and by applying Lemma 12 we get: $(\sigma_1 \not\models \sigma_2) \not\models \gamma \not\models_C (\bar{\sigma}_1 \not\models \bar{\sigma}_2) \not\models \bar{\gamma}$. Therefore,
\[
(\sigma_1 \not\models \bar{\gamma}) \not\models \bar{\sigma}_2 \not\models_C (\bar{\sigma}_1 \not\models \bar{\sigma}_2)
\] (7)

On the one hand, it follows from the Item 3 of Lemma 10 that $(\sigma_1 \not\models \gamma) \not\models_C (\bar{\sigma}_1 \not\models \bar{\gamma})$. Since we already have $\bar{\sigma}_1 \not\models \bar{\gamma} \vdash g_1$, then it follows from Corollary 3 that $\sigma_1 \not\models \gamma \vdash g_1$. Thus Eq (5) is proved. On the other hand, since $M_D \subseteq M$ for any substitution $M$ and any $D \subseteq \text{dom}(M)$, then Eq (6) follows from Eq (7).

ii) If $\varphi_n \in \text{Pos}_E(\mathcal{G}(A_1, A_2))$ and $\bar{\varphi}_n \in \text{Pos}_E(\mathcal{G}(A_1, A_2))$ then consider an Eloise move $m = \varphi_n \rightarrow \varphi_{n+1}$ in $\mathcal{G}(A_1, A_2)$.

In this case $m$ of the form:
\[
m = (\sigma_1, q_1), (\sigma_2, q_2), (\sigma_3, \alpha))_{E} \rightarrow ((\sigma_1, q_1), ((\sigma_2 \not\models \gamma_2)\mid_D, q_2'), \bar{\varphi}_{n+1}
\] where $q_2' \in \delta_2(q_2, \beta, g_2)$ and $D_2 = \text{Dom}(\sigma_2 \not\models \gamma_2) \setminus \kappa^{-1}_2(q_2')$ and $\gamma_1(\sigma_3(\alpha)) = \gamma_2(\sigma_2(\beta))$ and $\gamma_2 \vdash \sigma_2(g_2)$ and $\gamma_2 : \mathcal{V}(\sigma_2(\beta)) \cup \mathcal{V}(\sigma_2(g_2)) \rightarrow \Sigma$

From the induction hypothesis we have that $\varphi_n \not\models f(\varphi_n)$, that is, $\varphi_n \not\models_C \bar{\varphi}_n$, therefore $\bar{\varphi}_n = ((\sigma_1, q_1), (\bar{\sigma}_2, q_2), (\bar{\sigma}_3, \alpha))_{E}$ for substitutions $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3$, such that $((\sigma_1 \cup \bar{\sigma}_1) \not\models (\bar{\sigma}_2 \not\models \bar{\sigma}_2)) \not\models_C (\bar{\sigma}_1 \cup \bar{\sigma}_3) \not\models \bar{\sigma}_2$.

The corresponding move $m$ in $\mathcal{G}(A_1, A_2)$ is defined by:
\[
\bar{m} = ((\bar{\sigma}_1, q_1), (\bar{\sigma}_2, q_2), (\bar{\sigma}_3, \alpha))_{E} \rightarrow ((\bar{\sigma}_1, q_1), ((\bar{\sigma}_2 \not\models \bar{\gamma}_2)\mid_D, q_2'), \bar{\varphi}_{n+1}
\] where $\bar{\gamma}_1(\bar{\sigma}_3(\alpha)) = \bar{\gamma}_2(\bar{\sigma}_2(\beta))$
and $\bar{\gamma}_2 \vdash \bar{\sigma}_2(g_2)$
and $\bar{\gamma}_2 \vdash \mathcal{V}(\bar{\sigma}_2(\beta)) \cup \mathcal{V}(\bar{\sigma}_2(g_2)) \rightarrow C_0$

where $\bar{\gamma}_2$ is defined by
\[
\bar{\gamma}_2 = \Xi^n_{\Sigma_C}C_0(\sigma_1, \sigma_2, \sigma_3, \gamma_2, \alpha, \beta, g_2, \bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3)
\]
From Eq (A.2) of Lemma 14 we get
\[ \sigma_1 \uplus \sigma_2 \uplus \gamma_2 \not\subseteq C \sigma_1 \uplus \bar{\sigma}_2 \uplus \bar{\gamma}_2 \]

From the Item 2 of Lemma 10 it follows that \( \varphi_{n+1} \not\subseteq C f(\varphi_{n+1}) \), i.e. \( \varphi_{n+1} \not\subseteq \bar{C} \varphi_{n+1} \), since
\[ \sigma_1 \uplus (\sigma_2 \uplus \gamma_2)|_{D_2} \not\subseteq C \sigma_1 \uplus (\bar{\sigma}_2 \uplus \bar{\gamma}_2)|_{D_2} \]

This ends the proof of the Lemma. \( \square \)