Hamiltonian Formalism of Bimetric Gravity In Vierbein Formulation

Josef Kluson

Department of Theoretical Physics and Astrophysics
Faculty of Science, Masaryk University
Kotlářská 2, 611 37, Brno
Czech Republic

Abstract

This paper is devoted to the Hamiltonian analysis of bimetric gravity in vierbein formulation. We identify all constraints and determine their nature. We also show an existence of additional constraint so that the scalar mode can be eliminated.

1E-mail: klu@physics.muni.cz
1 Introduction

Bimetric theories of gravity are based on the idea to join the two tensors $\hat{g}_{\mu\nu}$ and $\hat{f}_{\mu\nu}$ in a symmetric way when each tensor has its own Einstein-Hilbert action and then couple these actions through a non-derivative mass term. The presence of this term reduces the separate coordinate invariances to a single one \[1, 2\]. If we set one metric as the background metric without any dynamics we find that the bimetric theory is reduced to a single metric massive gravity theory with the mass term that at the linear limit leads to the Fierz-Pauli free theory \[3\]. However it was shown soon that this theory propagates ghost modes at non-linear level \[4, 5\]. On the other hand new form of the massive term was proposed recently in \[6, 7, 10, 11\] that was shown to be ghost free even at the non-linear level \[12, 9\], see also \[13, 40, 41\].

This form of the massive gravity was further generalized in \[14\] where the dynamical gravity was coupled to the general reference metric. Then it was the small step to the generalization of given construction to the bimetric gravity when the fixed reference metric becomes dynamical with its own Einstein-Hilbert action \[15\]. It was also argued there and in \[12\] that this theory is ghost free. However this analysis was discussed in \[34\] where it was argued that the analysis performed in \[12\] does not show an existence of the additional constraint in case of the bimetric gravity \[3\].

The non-linear massive gravity and bimetric gravity that are claimed that are ghost free are based on the specific form of the potential that contains the square root of $\hat{g}_{\mu\nu} \hat{f}_{\nu\rho}$. This is rather awkward structure which makes very difficult to find an extra constraint that could eliminate the Boulware-Deser ghost. However as was shown in beautiful paper \[8\] the square root structure suggests that the vierbein variables $E_{\mu}^{A}$ could be the appropriate one for the formulation of consistent bimetric theories. In more details, completely new multimetric interacting spin-2 theories were proposed in \[8\] using the powerful vierbein formulation of the general relativity and corresponding mass terms. It was argued there that due to the specific form of the interaction terms the action is linear in lapses and shifts which implies an existence of additional constraints that could eliminate non-physical modes. However we mean that the Hamiltonian analysis presented in given paper was not complete. In particular, the constraints corresponding to the diagonal diffeomorphism were not identified and it was not shown that they are the first class constraints.

The goal of this paper is to fill this gap and perform the Hamiltonian analysis of the bimetric gravity in vierbein formulation with the simplest form of the potential between two vierbeins $E_{\mu}^{A}$ and $F_{\mu}^{A}$. We explicitly show that it is crucial to analyze the time developments of the constraints corresponding broken spatial rotation. It is also important to stress that when we use the parametrization of the vierbein as in \[8\] we should interpret $p_{\mu}$-that will be defined below-as a dynamical variable with no time dependence in the action. As a result the conjugate momentum vanishes and is the primary constraint of the theory. Then the requirement of the preservation

\[2\]Bimetric theories of gravity were studied intensively in the past, see for example \[44, 45, 46\].

\[3\]The Hamiltonian analysis of bimetric gravity was also performed in \[35, 36, 57, 38\].
of given constraint leads to another secondary constraint that was not consider in 
the literature so far.

Very important point is to identify the constraints that are generators of diagonal 
diffeomorphism. To do this we follow [33] when we introduce new variables that are 
functions of $N, N^i$ and $M, M^i$ which are lapses and shifts in $\hat{g}_{\mu\nu}$ and $\hat{f}_{\mu\nu}$ respectively. 
Then we determine eight new secondary constraints where four of them should cor- 
respond to the generators of diagonal diffeomorphism on condition that the Poisson 
brackets between new Hamiltonian constraint $\mathcal{R}$ closes on the constraints surface. 
The similar analysis was performed previously in case of bimetric gravity formulated 
with metric variables in [35, 36]. It turns out that in case of bimetric theory in vierbein formulation the situation is more complicated and we have to take into 
account the presence of the new secondary constraints. Then we are able to show 
that the Poisson bracket between Hamiltonian constraints vanish on the constraint 
surface. On the other hand one can ask the question why we use the variables in- 
troduced in [33] in case of bimetric gravity formulated with metric variables in case 
of the bimetric gravity formulated using vierbein formalism. The answer is that we 
mean that they are the only variable where it is possible to identify generators of 
diagonal diffeomorphism that is difficult to identify with the help of another choose 
of variables. Further, using this formalism we can easily see an analogy between 
bimetric theory formulated using either metric of vierbein variables.

With the help of this result we proceed to the analysis of the consistency of all 
constraints during the time development of the system. Now due to the very re- 
markable structure of the vierbein formulation of bi-gravity we find an existence of 
additional constraint which leads to the elimination of the scalar mode in the same 
way as in case of non-linear massive gravity [9, 12]. This result confirms the results 
derived in [47]. More precisely, in [47] canonical analysis of bimetric gravity formulated 
in vierbein formalism where the spin connection is treated as an independent 
field was performed with elegant formulations of the secondary constraints that are 
responsible for the elimination of the ghosts. On the contrary our analysis is more 
closely related to the formulation introduced in [8] where the spin connection is 
not considered as an independent field however the constraints responsible for the 
elimination of ghost are much more complicated.

This paper is organized as follows. In the next section (2) we introduce the bi- 
metric gravity in vierbein formalism and find its Hamiltonian, identify all constraints 
and determine their constraint structure. In Conclusion (3) we outline our results. 
Finally in Appendix we review the Hamiltonian formulation of general relativity 
action formulated in the vierbein formalism.
2 Vierbein Formulation of Bimetric Gravity

General vierbein can be written in the upper triangular form and we denote this vierbein with hat

\[
\hat{E}_A^\mu = \begin{pmatrix} N & N^i e_i^a \\ 0 & e^a_i \end{pmatrix}, \quad \hat{E}_A^\mu = \begin{pmatrix} \frac{1}{N^i} & 0 \\ -\frac{N^i}{N} & e^i \end{pmatrix}
\] (1)

where \( N \) and \( N^i \) are the 4 time-like components. The spatial vielbeins \( e_i^a \) contain 9 components that are related to the spatial part of the metric by

\[
g_{ij} = e_i^a e_j^b \delta_{ab}.
\] (2)

Now by writing out the metric of this vierbein we find

\[
\hat{g}_{\mu\nu} = E_A^\mu \hat{E}_B^\nu \eta_{AB} = \begin{pmatrix} -N^2 + N^i N_i & N_i \\ N_i & g_{ij} \end{pmatrix}, \quad \eta_{AB} = \text{diag}(-1,1,1,1)
\] (3)

that means that \( N \) and \( N^i \) are the usual lapse and shifts that appear in the ADM decomposition of the metric \[25\,26\,27\]. Note that the inverse metric has the form

\[
\hat{g}^{\mu\nu} = \hat{E}_A^\mu \hat{E}_B^\nu \eta^{AB}.
\] (4)

Then by definition \[4\]

\[
\hat{E}_A^\mu \hat{E}_A^\nu = \delta_\mu^\nu, \quad \hat{E}_A^\mu \hat{E}_B^\mu = \delta_B^A, \\
e_i^a e_j^b = \delta_j^i, \quad e_i^a e_i^b = \delta^a_b.
\] (5)

The upper triangular form does not fix the local Lorentz invariance since it leaves a residual spatial rotation. There are 4 components in \( N, N^i \) and 9 in the spatial vielbein. The remaining 3 components of the vielbein have been fixed by using the upper triangular gauge choice. It is possible to formulate an arbitrary vierbein as the action of same boost on the upper triangular vierbein. Note that for 3–dimensional vector \( p_a \) we define a standard Lorentz boost as

\[
\Lambda(p)_B^A = \begin{pmatrix} \gamma & \frac{p_b}{\gamma + p^a}p_a \\ p^a \delta^a_b + \frac{1}{\gamma + p^a}p^a p_b \end{pmatrix}, \gamma = \sqrt{1 + p_a p^a},
\] (6)

where \( p^a = \delta_{ab} p_b \) and where by definition

\[
\eta_{AB} \Lambda_C^A \Lambda_D^B = \eta_{CD}
\] (7)

so that

\[
(\Lambda^{-1})_B^A = \begin{pmatrix} \gamma & -\frac{p_b}{\gamma + p^a}p_a \\ -p_a \delta_a^b + \frac{p_a p_b}{\gamma + p^a} \end{pmatrix}.
\] (8)

\[4\]For review of vierbein formalism, see \[32\].
The boost takes the 4-dimensional vector \((1, 0, 0, 0)\) into the unit normalized 4-vector
\[
\Lambda^A_B \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ p^a \end{pmatrix}.
\]

(9)

Then we write the general vierbein as the standard boost of the upper triangular vierbein
\[
E_\mu^A = \Lambda(p)^A_B \hat{E}_\mu^B = \begin{pmatrix} N + N^i e_i^a p_a & Np^b + N^i e_i^a (\delta_a^b + \frac{1}{\gamma+1} p_a p^b) \\ e_i^a p_a & e_i^a (\delta_a^b + \frac{1}{\gamma+1} p_a p^b) \end{pmatrix}.
\]

(10)

We see that 16 components of the general vierbein are now parameterized by the 4 components of \(N\) and \(N^i\) together with 9 components of the spatial vielbein \(e_i^a\) and 3 components of \(p_a\).

It is important that the Einstein-Hilbert action is invariant under local Lorentz transformation. As a result it is possible to partially fix the gauge and express the Einstein-Hilbert action using the upper triangular form. This fact greatly simplifies the Hamiltonian formalism of general relativity in vierbein formalism. The detailed analysis is performed in the Appendix A.

Now we are ready to proceed to the vierbein formulation of the bimetric gravity when we consider bigravity with two metrics
\[
\hat{g}_{\mu\nu} = E_{\mu}^A E_{\nu}^B \eta_{AB}, \quad \hat{f}_{\mu\nu} = F_{\mu}^A F_{\nu}^B \eta_{AB}
\]

(11)

with Einstein-Hilbert actions for both of these metrics. Then without the interaction term the action is invariant under two separate local Lorentz transformations
\[
E_{\mu}^A(x) = \Lambda_{(g)}^{A}B(x) E_{\mu}^B(x), \quad F_{\mu}^A(x) = \Lambda_{(f)}^{A}B(x) F_{\mu}^B(x).
\]

(12)

The action is also invariant under two diffeomorphisms
\[
E_{\mu}^A(x) df_{(1)}^\mu = E_{\nu}^A(x) dx^\nu, \quad F_{\mu}^A(x) df_{(2)}^\mu = F_{\nu}^A(x) dx^\nu.
\]

(13)

Then we consider the action in the form \([8]\]
\[
S = \frac{M_9^2}{2} \int d^4 x (\det E) R[E] + \frac{M_f^2}{2} \int d^4 x (\det F) R[F] - \\
- \mu^2 \int d^4 x \sum_{n=0}^{4} \beta_n (\det E) S_n (E^{-1} F),
\]

(14)

where \(\mu^2 = \frac{1}{8} m^2 M_f^2\) and where \(S_n\) are symmetric polynomials whose explicit definitions can be found in \([8]\). It was shown here that they can be written in terms of
traces of the matrix $M$ as

\[
S_0(M) = 1,
S_1(M) = [M],
S_2(M) = \frac{1}{2!}([M]^2 - [M^2]),
S_3(M) = \frac{1}{3!}([M]^3 - 3[M][M^2] + 2[M^3]),
S_4(M) = \frac{1}{4!}([M]^4 - 6[M^2][M]^2 + 8[M][M^3] + 3[M^2]^2 - 6[M^4]),
\]

(15)

where $[M]$ means the trace of the matrix $M$. In what follows we restrict ourselves to the simplest non-trivial case corresponding to $\beta_0 = \beta_2 = \beta_3 = \beta_4 = 0$, $\beta_1 = 1$ which however captures the main property of given theory.

Now we proceed to the Hamiltonian analysis of the bimetric theory in the vierbein formulation. We use the parametrization of the general vierbein introduced in (10). Explicitly

\[
E_A^\mu = \Lambda(p) A^A B_E^\mu, \quad F_A^\mu = \Lambda(l) A^A B_F^\mu,
\]

(16)

where

\[
\hat{E}_A^\mu = \begin{pmatrix} N & N_i e_i^a \\ 0 & e_i^a \end{pmatrix}, \quad \hat{F}_A^\mu = \begin{pmatrix} M & M_i f_i^a \\ 0 & f_i^a \end{pmatrix},
\]

\[
\hat{E}_A^\mu = \begin{pmatrix} \frac{1}{N} & e_i^a \\ -\frac{N_i}{N} & e_i^a \end{pmatrix}, \quad \hat{F}_A^\mu = \begin{pmatrix} \frac{1}{M} & 0 \\ -\frac{M_i}{M} & f_i^a \end{pmatrix},
\]

(17)

where $g_{ij} = e_i^a e_j^b \delta_{ab}$, $f_{ij} = f_i^a f_j^b \delta_{ab}$.

To proceed further we use the fact that bi-gravity is invariant under diagonal local Lorentz transformation which implies that we can partially gauge fix this gauge by imposing $l_a = 0$ [8]. Note also that since Einstein-Hilbert actions are invariant under local transformations the action depends on $p_a$ through the potential term only. Explicitly we find

\[
S_1(E^{-1} F) = \text{Tr}(E^{-1} \hat{F}) = \frac{M}{N} \gamma + \frac{1}{N} (M_i f_i^b p_b - N_i f_i^b p_b) + e_i^a f_i^a + \frac{1}{\gamma + 1} (e_i^a p^a)(f_j^b p_b).
\]

(18)

Using the Hamiltonian analysis performed in Appendix we find following Hamiltonian

\[
H = \int d^3 x (N R_0^{(g)} + M R_0^{(f)} + N^i R_i^{(g)} + M^i R_i^{(f)} + \mu^2 Ne \mathcal{V} + \\
\quad + \Lambda^{ab} E_{ab}^{(g)} + \Lambda_{(g)} E_{ab}^{(f)}),
\]

(19)
where
\[
\begin{align*}
\mathcal{R}_o^{(g)} & = \frac{1}{M_j^2\sqrt{g}} \pi_{ij} \mathcal{G}_{ijkl} \pi^{kl} - M_j^2 \sqrt{g} R^{(g)} , \\
\mathcal{R}_o^{(f)} & = \frac{1}{M_f^2\sqrt{f}} \rho_{ij} \mathcal{G}_{ijkl} \rho^{kl} - M_f^2 \sqrt{f} R^{(f)} , \\
\mathcal{R}_1^{(g)} & = -2g_{ij} \nabla_k \pi^{kj} , \\
\mathcal{R}_1^{(f)} & = -2f_{ij} \tilde{\nabla}_k \rho^{kj} , \\
\mathcal{V} & = \frac{M}{N} \gamma + \frac{1}{N} (M^i f^b p_b - N^i f^b p_b) + \epsilon_i f^a + \frac{1}{\gamma + 1} (\epsilon^j a f^a)(f^b p_b) \tag{20}
\end{align*}
\]
and where \(\pi^{ij}\) and \(\rho^{ij}\) are momenta conjugate to \(g_{ij}\) and \(f_{ij}\) respectively. Further \(\nabla_i\) and \(\tilde{\nabla}_i\) are covariant derivatives evaluated using the metric components \(g_{ij}\) and \(f_{ij}\) respectively. Finally note that \(\mathcal{G}^{ijkl}\) and \(\tilde{\mathcal{G}}^{ijkl}\) are de Witt metrics defined as
\[
\mathcal{G}^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - g^{ij}g^{kl} , \\
\tilde{\mathcal{G}}^{ijkl} = \frac{1}{2} (f^{ik}f^{jl} + f^{il}f^{jk}) - f^{ij}f^{kl} \tag{21}
\]
with inverse
\[
\mathcal{G}_{ijkl} = \frac{1}{2} (g_{ik}g_{jl} + g_{il}g_{jk}) - \frac{1}{2} g_{ij} g_{kl} , \\
\tilde{\mathcal{G}}_{ijkl} = \frac{1}{2} (f_{ik}f_{jl} + f_{il}f_{jk}) - \frac{1}{2} f_{ij} f_{kl} \tag{22}
\]
that obey the relation
\[
\mathcal{G}_{ijkl} \mathcal{G}^{klnm} = \frac{1}{2} (\delta^m_i \delta^n_j + \delta^m_j \delta^n_i) , \\
\tilde{\mathcal{G}}_{ijkl} \tilde{\mathcal{G}}^{klnm} = \frac{1}{2} (\delta^m_i \delta^n_j + \delta^m_j \delta^n_i) . \tag{23}
\]
Also note that \(e \equiv \det e\). We have also included the primary constraints \(L_{ab}^{(g)} \approx 0\), \(L_{ab}^{(f)} \approx 0\) into definition of the Hamiltonian.

An important point is to identify four constraints that are generators of the diagonal diffeomorphism\footnote{\emph{Note due to the specific form of the interaction term we have the action that is linear in \(N\) and \(M\) and hence the first guess would be that given constraints arise as the linear combinations of \(\mathcal{R}_o^{(g)}\), \(\mathcal{R}_o^{(f)}\) and \(\mathcal{R}_1^{(g)}\) and \(\mathcal{R}_1^{(f)}\). We checked this possibility however we found that it does not work due to the presence of the constraint \(k^n \approx 0\) defined below. The requirement of the preservation of the constraints \(k^n \approx 0\) led to the secondary constraints that were functions of \(N\) and \(M\). Then it was very difficult to identify four first class constraints that are generators of diagonal diffeomorphism. It turned out that these generators can be identified very easily using the ansatz introduced in \cite{33} even if it was proposed for the case of bimetric gravity formulated using metric variables.}}. In order to do this we proceed as in \cite{33} and introduce following variables
\[
\begin{align*}
\bar{N} & = \sqrt{NM} , \\
n & = \sqrt{N M} , \\
\bar{N}^i & = \frac{1}{2} (N^i + M^i) , \\
n^i & = \frac{N^i - M^i}{\sqrt{NM}} , \\
N & = \bar{N} n , \\
M & = \frac{\bar{N}}{n} , \\
M^i & = \bar{N}^i - \frac{1}{2} n^i \bar{N} , \\
N^i & = \bar{N}^i + \frac{1}{2} n^i \bar{N} . \tag{24}
\end{align*}
\]
Note that their conjugate momenta are the primary constraints of the theory
\[ \bar{P} \approx 0 \>, \> p \approx 0 \>, \> P_i \approx 0 \>, \> p_i \approx 0 \]  
with following non-zero Poisson brackets
\[ \{ \bar{N}(x), P(y) \} = \delta(x - y) \>, \> \{ n(x), p(y) \} = \delta(x - y) \>, \> \{ \bar{N}^i(x), P_j(y) \} = \delta^i_j \delta(x - y) \].

It is also important to stress that the absence of the time derivative of \( p_a \) in the action implies following primary constraint
\[ k^a \approx 0 \],
where \( k^a \) is momentum conjugate to \( p_a \) with non-zero Poisson bracket
\[ \{ p_a(x), k^b(y) \} = \delta^b_a \delta(x - y) \].

We also have to identify the constraints that are generators of the diagonal spatial rotations of vielbeins \( e_i^a, f_i^a \). These constraints are given as the linear combinations of \( L^{(g)}_{ab}, L^{(f)}_{ab} \) and \( k^a \). Explicitly, we introduce following set of the constraints
\[ L^{\text{diag}}_{ab} \approx 0 \>, \> L^{\text{br}}_{ab} \approx 0 \>, \> k^a \approx 0 \],
where
\[ L^{\text{diag}}_{ab} = e_{ia} \pi^i_b - e_{ib} \pi^i_a + f_{ia} \rho^i_b - f_{ib} \rho^i_a + p_a k_b - p_b k_a \],
\[ L^{\text{br}}_{ab} = e_{ia} \pi^i_b - e_{ib} \pi^i_a - f_{ia} \rho^i_b + f_{ib} \rho^i_a - p_a k_b + p_b k_a \].

where \( \pi^i_a, \rho^i_a \) are momenta conjugate to \( e_i^a, f_i^a \) respectively. Collecting all these terms together we find following form of the Hamiltonian
\[ H = \int d^3x (\bar{N} \bar{\mathcal{R}} + \bar{N}^i \bar{\mathcal{R}}_i + \Lambda^{ab}_{\text{diag}} L_{ab} + \Lambda^{ab}_{\text{br}} L^{br}_{ab} + v_a k^a + v_n p + v^i p_i + V \bar{N} \bar{P} + V^i P_i) \],

where
\[ \bar{\mathcal{R}} = n \bar{\mathcal{R}}_0^{(g)} + \frac{1}{n} \bar{\mathcal{R}}_0^{(f)} + \frac{1}{2} n \bar{\mathcal{R}}^i_0^{(g)} - \frac{1}{2} n \bar{\mathcal{R}}^i_0^{(f)} + \mu^2 e \bar{\mathcal{V}} \>, \> \bar{\mathcal{R}}_i = \bar{\mathcal{R}}^i_0^{(g)} + \bar{\mathcal{R}}^i_0^{(f)} \].
where

\[
\tilde{\mathcal{V}} = \frac{\gamma}{n} - n^i f^a_i p_a + n e^i_a f^a + \frac{n}{\gamma + 1} (e^i_a p^b) (f^b_i p_b).
\]

(33)

Now we proceed to the analysis of time development of the primary constraints (25) and (27)

\[
\begin{align*}
\partial_t P &= \{P, H\} = -\mathcal{R} \approx 0, \\
\partial_t P_i &= \{P_i, H\} = -\mathcal{R}_i \approx 0, \\
\partial_t p &= \{p, H\} = -\mathcal{R}_0^g + \frac{1}{n^2} \mathcal{R}_0^f - \mu^2 e \frac{\delta \tilde{\mathcal{V}}}{\delta n} \equiv \mathcal{C}_n \approx 0, \\
\partial_t p_i &= \{p_i, H\} = -\frac{1}{2} (\mathcal{R}_i^g - \mathcal{R}_i^f) - \mu^2 e \frac{\delta \tilde{\mathcal{V}}}{\delta n} \equiv \mathcal{S}_i \approx 0, \\
\partial_t k^a &= \{k^a, H\} = -\mu^2 e \frac{\delta \tilde{\mathcal{V}}}{\delta p_a} \equiv \mathcal{K}^a \approx 0.
\end{align*}
\]

(34)

Finally we have to the check the preservation of the constraints \(L_{ab}^{\text{diag}} \approx 0, L_{ab}^{\text{br}} \approx 0\). Firstly due to the fact that \(\mathcal{R}_0^g, \mathcal{R}_0^f, \mathcal{R}_i^g, \mathcal{R}_i^f\) have vanishing Poisson brackets with \(L_{ab}^{(g)}, L_{ab}^{(f)}\) according to (116) we find that they have also vanishing Poisson brackets with both \(L_{ab}^{\text{diag}}\) and \(L_{ab}^{\text{br}}\). Then the non-zero contribution could follow from the Poisson bracket between \(L_{ab}^{\text{diag}}, L_{ab}^{\text{br}}\) and \(\tilde{\mathcal{V}}\). Now with the help of the following Poisson brackets

\[
\begin{align*}
\{L_{ab}^{\text{diag}}(x), e^i_c(y)\} &= (e_{ib} \delta^c_a - e_{ia} \delta^c_b)(x) \delta(x - y), \\
\{L_{ab}^{\text{diag}}(x), e^i_n(y)\} &= (\delta_{ac} e^i_b - \delta_{bc} e^i_a)(x) \delta(x - y), \\
\{L_{ab}^{\text{diag}}(x), f^i_c(y)\} &= (f_{ib} \delta^c_a - f_{ia} \delta^c_b)(x) \delta(x - y), \\
\{L_{ab}^{\text{diag}}(x), f^i_n(y)\} &= (\delta_{ac} f^i_b - \delta_{bc} f^i_a)(x) \delta(x - y), \\
\{L_{ab}^{\text{diag}}(x), p_c(y)\} &= -(p_a \delta_{bc} - p_b \delta_{ac})(x) \delta(x - y), \\
\{L_{ab}^{\text{diag}}(x), p_c p^c(y)\} &= 0.
\end{align*}
\]

(35)
we find that the constraint $L_{ab}^{\text{diag}} \approx 0$ is preserved during the time evolution of the system while the requirement of the preservation of the constraint $L_{ab}^{\text{br}}$ implies

$$\partial_t L_{ab}^{\text{br}}(x) = \{ L_{ab}^{\text{br}}(x), H \} = 2\mu^2 N \epsilon_n[(e^j_b f_{ja} - e^j_a f_{jb}) + \frac{1}{\gamma + 1}[(p_a e^j_b - p_b e^j_a)f_j^d p_d ]] \equiv 2\mu^2 c \epsilon n \mathcal{T}_{ab} \approx 0 ,$$

(37)

where we introduced new secondary constraints $\mathcal{T}_{ab} = -\mathcal{T}_{ba}$

$$\mathcal{T}_{ab} = e^j_b f_{ja} - e^j_a f_{jb} + \frac{1}{\gamma + 1}[(p_a e^j_b - p_b e^j_a)f_j^d p_d].$$

(38)

As we will see below the existence of these constraints will be crucial for the consistency of the theory.

### 2.1 Calculation of the Poisson brackets between $\bar{R}, \bar{R}_i$

Before we proceed to the analysis of the stability of all constraints we would like to show that the Poisson brackets between the constraints $\bar{R}$ and $\bar{R}_i$ vanish on the constraint surface spanned by all constraints. To begin with we introduce the smeared form of the constraint $\bar{R}$

$$\mathbf{T}(N) = \int d^3x N \bar{R}.$$  

(39)

Then using the Poisson brackets given in (110) and following similar analysis as in case of metric formulations of bigravity we obtain [35, 36]

$$\{ \mathbf{T}_T(N), \mathbf{T}_T(M) \} = \frac{1}{2} \mathbf{T}_S((N \partial_i M - M \partial_i N)n^2 g^{ij}) + \frac{1}{2} \mathbf{T}_S((N \partial_i M - M \partial_i N) \frac{1}{n^2} f^{ij}) +$$

$$+ \frac{1}{4} \mathbf{T}_S((N \partial_i M - M \partial_i N)n^i n^j) -$$

$$- \mathbf{G}_S((N \partial_i M - M \partial_i N)n^2 g^{ij}) + \mathbf{G}_S((N \partial_i M - M \partial_i N) \frac{1}{n^2} f^{ij}) -$$

$$- \frac{1}{2} \mathbf{G}_T((N \partial_i M - M \partial_i N)n^i n) + \int d^3x (N \partial_i M - M \partial_i N)\Sigma^i ,$$

(40)
where
\[ G_T(N) = \int d^3x N \mathcal{G}_n, \quad G_S(N^i) = \int d^3x N^i \mathcal{S}_i, \]
and where
\[ \Sigma^i[\mathcal{V}] = \frac{n^i}{n} - \frac{n}{\gamma} e^a_i f^a_j n^j - \frac{n}{\gamma + 1} e^a_i b^b j^j f^a_j p_a + \frac{n^2}{\gamma + 1} e^a_i b^b j^j f^a_j p_a - \frac{1}{n} f^a_i p_a. \]

(42)

Note also that we used the extended version of the constraint \( \bar{\mathcal{R}}_i \) given in (47) and we omitted terms proportional to \( L^{(q)}_{ab} \), \( L^{(f)}_{ab} \) given in (116).

Our goal is to show that \( \Sigma^i[\mathcal{V}] \) vanishes on the constraint surface. To do this we use the fact that \( n^i \) can be expressed from the constraint \( K^a \)
\[ n^i = f^i_a \left( \frac{1}{\gamma} f^a p_a - n \left( e^a_i b^b j^j f^a_j p_a \right) \right) + \frac{1}{\gamma + 1} e^a_i b^b j^j f^a_j p_a + H_a K^a, \]

(43)

where \( H_a \) are functions that depend on the phase space variables whose explicit form is not important for us. To proceed further we use the fact that from the constraint \( T_{ab} \) we derive
\[ f^i_a f^b_j p_b e^j a = \frac{1}{\gamma} \left( e^b p_b + \frac{1}{\gamma + 1} f^i_a p_a (f^d_k p_d) (e^k_p b^b) \right) - \frac{f^i_a}{2\gamma \sqrt{e} \sqrt{f}} G^{ab} p_b. \]

(44)

Inserting this expression into (43) we find
\[ n^i = \frac{1}{\gamma} f^i_a p_a + \frac{n}{\gamma} e^i p_b \]

(45)

up to terms proportional to \( T_{ab} \) and \( K^a \). Finally inserting this result into (42) and after some calculations we find the desired result
\[ \Sigma^i[\mathcal{V}] = F_a K^a + G^{ab} T_{ab} \approx 0. \]

(46)

Then collecting (40) together with (46) we find that the Poisson bracket between \( \bar{\mathcal{R}} \) is proportional to the constraints \( \bar{\mathcal{R}}_i, \mathcal{G}_n, \mathcal{S}_i, K^a, T_{ab} \) which means that it vanishes on the constraint surface. This is very important result. Note also an importance of the constraints \( K^a, T_{ab} \) for the closure of the Poisson brackets between \( \bar{\mathcal{R}} \).

As the next step we calculate the Poisson brackets with the constraints \( \bar{\mathcal{R}}_i \). However it turns out that it is more convenient to consider its following extension
\[ \bar{\mathcal{R}}_i = \partial_i n p + \partial_i n^j p_j + \partial_j (n^i p_i) + \partial_k p_a k^a + + \bar{\mathcal{R}}_i^{(e)} + \frac{1}{2} \omega^a_i b^a (e) L^{(e)}_{ab} + \bar{\mathcal{R}}_i^{(f)} + \frac{1}{2} \omega^a_i b^a (f) L^{(f)}_{ab}. \]

(47)
Let us define its smeared form

\[ T_S(N^i) = \int d^3x N^i \mathcal{R}_i. \]  

(48)

Then we find following Poisson brackets

\[
\begin{align*}
\{ T_S(N^i), e^c \} &= -\partial_i N^j e^c_j - N^j \partial_j e^c_i, \\
\{ T_S(N^i), e^i \} &= \partial_j N^i e^j_c - N^j \partial_j e^i_c, \\
\{ T_S(N^i), f^i_c \} &= \partial_j N^i f^j_c - N^j \partial_j f^i_c, \\
\{ T_S(N^i), f^c_i \} &= -\partial_i N^j f^c_j - N^j \partial_j f^c_i, \\
\{ T_S(N^i), n^i \} &= -N^j \partial_j n^i + \partial_j N^i n^j, \\
\{ T_S(N^i), p_a \} &= -N^i \partial_a p_a, \\
\{ T_S(N^i), n \} &= -N^i \partial n \end{align*}
\]

(49)

which shows that \( T_S(N^i) \) is the generator of the diagonal spatial diffeomorphism.

Now we are ready to proceed to the calculation of the Poisson bracket between \( T_S(N^i) \) defined in (48) and \( T_T(N) \). In fact, using (49) we easily find

\[
\{ T_S(N^i), e\tilde{V} \} = -N^k \partial_k [e\tilde{V}] - \partial_k N^k e\tilde{V}. 
\]

(50)

Finally we should calculate the Poisson bracket between \( T_S(N^i) \) and \( \mathcal{R}^{(f)}_i, \mathcal{R}^{(g)}_i \) and \( \mathcal{R}_0^{(g)}, \mathcal{R}_0^{(f)} \). This is really easy task using the results given in (116) so that we find

\[
\{ T_S(N^i), T_T(N) \} = T_T(N^i \partial_i M) 
\]

(51)

up to the terms proportional to the primary constraints \( L_{ab}^{(g)} \approx 0, L_{ab}^{(f)} \approx 0 \). In the same way we can find that

\[
\{ T_S(N^i), T_S(M^j) \} = T_S((N^i \partial_i M^j - M^j \partial_i N^j)). 
\]

(52)

Using these results we are ready to proceed to the analysis of the stability of constraints.
2.2 Analysis of stability of constraints

In this section we perform the analysis of the stability of all constraints. Note that for the potential $\mathcal{V}$ given in (33) the constraints $G_n, S_i, K^a$ have the form

$$G_n = -R_0^{(g)} + \frac{1}{n^2} R_0^{(f)} + \mu^2 e \left( \frac{\gamma}{n^2} - e^a_i p^a_i - \frac{1}{1 + \gamma} (e^a_i p^a_i)(f^b_i p^b_i) \right),$$

$$S_i = -\frac{1}{2}(R_0^{(g)} - R_0^{(f)}) + \mu^2 e f^a_i p^a_i,$$

$$K^a = \mu^2 e \left( \frac{p^a}{\gamma n} - n^i f^a_i - n(e^d_j a f^d_j) f^b_i p^b_i \right) + \frac{n}{1 + \gamma} (f^a_j e^j_b p^b + f^b_j p^b e^ja),$$

$$T_{ab} = (e^j_b f^a_j - e^j_a f^b_j) + \frac{1}{1 + \gamma} (p_a e^j_b - p_b e^j_a) f^d_j p^d .$$

(53)

It turns out that these constraints could be simplified considerably. First of all we have following relation

$$\bar{R} + n^i S_i + n G_n = \frac{2}{n} (R_0^{(f)} + \mu^2 e \gamma)$$

(54)

so that we see that we can consider as an independent constraint following one

$$G'_n = R_0^{(f)} + \mu^2 e \gamma .$$

(55)

In previous section we also found the relation

$$n^i = \frac{1}{\gamma n} f^i_a p^a + \frac{n}{\gamma} e^{ib} p^b + H_a K^a + G^{ab} T_{ab}$$

(56)

so that it is possible to define new independent constraint $\tilde{K}^i$

$$\tilde{K}^i = n^i - \frac{1}{\gamma n} f^i_a p^a - \frac{n}{\gamma} e^{ib} p^b .$$

(57)

Then we have following set of independent secondary constraints $G'_n, \tilde{K}^i, S_i, T_{ab}$ so that the total Hamiltonian has the form

$$H_T = \int d^3x (\bar{N}\bar{R} + \bar{N}^i \bar{R_i} + \Lambda^{ab}_{\text{diag}} L_{ab} + V_{\bar{N}} \bar{P} + V^i \bar{P}_i +$$

$$+ \Gamma_n G'_n + \Omega_i \tilde{K}^i + \Gamma^i S_i + \Gamma^{ab} T_{ab} + v_n p + v^i p_i + v^a K^a + v^{ab} L^{br}_{ab} .$$

(58)

Now we are ready to proceed to the analysis of the stability of all constraints. We begin with the constraints $p_i \approx 0$

$$\partial_t p_i = \{p_i, H_T\} \approx -\mu^2 e \Omega_i = 0$$

(59)
that has the solution $\Omega_i = 0$, where $\Omega_i$ is Lagrange multiplier corresponding the constraint $\tilde{\mathcal{K}}^i$. In case of $p$ we find
\[
\partial_t p = \{p, H_T\} \approx 0 .
\]
Further, the requirement of the preservation of $L_{ab}^{br}$ takes the form
\[
\partial_t L_{ab}^{br} = \{L_{ab}^{br}, H_T\} = \Gamma^{cd} \Delta_{L_{ab}^{br}, G_{cd}} = 0 ,
\]
where $\{L_{ab}^{br}(x), G_{cd}(y)\} = \Delta_{L_{ab}^{br}, G_{cd}}(x)\delta(x - y)$ and where we used the fact that $\Omega_i = 0$ together with $\{L_{ab}^{br}(x), S_i(y)\} = 0$.

Then it can be explicitly checked that the matrix $\Delta_{L_{ab}^{br}, G_{cd}}$ is non-singular and hence the solution of (61) is $\Gamma^{cd} = 0$.

Using these results it is easy to analyze the requirement of the preservation of the constraints $k^a \approx 0$
\[
\partial_t k^a = \{k^a, H_T\} \approx -\mu^2 e\frac{p^a}{\gamma} - \Gamma^i \mu^2 e f_i^a = 0
\]
so that we obtain
\[
\Gamma^i = -\frac{p^a}{\gamma} f^i_a \Gamma_n .
\]
This result however suggests to consider as an independent constraint following one
\[
\tilde{\mathcal{G}}_n = \mathcal{R}_{(f)}^g + \mu^2 e \gamma \frac{1}{\gamma} S_i f^i_a P^a
\]
and following total Hamiltonian
\[
H_T = \int d^3x (\tilde{\mathcal{R}} + \tilde{\mathcal{R}} + \Lambda_{\text{diag}} L_{ab} + V_N \tilde{P} + V_i P_i +
\Gamma^i S_i + \Omega_i \tilde{\mathcal{K}}^i + \tilde{\mathcal{G}}_n + v^a p + v^j p_i + v_a k^a + v^{ab} L_{ab}^{br} + \Gamma^{ab} T_{ab} ) .
\]
Repeating the analysis as above we find that $p$ is trivially preserved and also $\Omega_i = \Gamma^{ab} = 0$. Further, the time evolution of the constraint $k^a \approx 0$ is given by equation
\[
\partial_t k^a = \{k^a, H_T\} \approx -\Gamma^i \mu^2 e f_i^a = 0
\]
that due to the fact that $f_i^a$ is non-singular implies that $\Gamma^i = 0$. Finally it is also clear that $\tilde{P}, P_i$ are trivially preserved.

Now we proceed to the analysis of the time evolution of the constraint $\tilde{\mathcal{G}}_n, S_i, \tilde{\mathcal{K}}^i$ together with $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{R}}_i$. First of all it is easy to see that that the secondary constraints $\tilde{\mathcal{G}}_n, S_i, \tilde{\mathcal{K}}^i, \tilde{\mathcal{G}}_{ab}$ are invariant under diagonal spatial diffeomorphism. Then
with the help of (51) and (52) we find that $\bar{R}_i$ are preserved during the time evolution of the system.

More interesting situation occurs in case of the time evolution of the constraints $\tilde{G}_n$ and $\bar{R}$ which is mainly determined by following Poisson bracket

$$\left\{ T_T(N), \int d^3x M \tilde{G}_n \right\} \approx \int d^3x N(x) M(x) \tilde{G}_n^{II}(x) ,$$

where

$$\tilde{G}_n^{II} = \tilde{\nabla}_i (f^{ij} \mathcal{R}_j^{(f)}) + \frac{1}{2} n^i \mu^2 \partial_i \gamma e + \ldots ,$$

and where $\ldots$ means other terms that depend on phase space variables. Note that the explicit form of $\tilde{G}_n^{II}$, which is very complicated, is not important for us. However it is crucial and non-trivial fact that the Poisson bracket (68) does not contain terms proportional to $M \partial_i N$ or $M \partial_i N$. Then the local form (68) has the form

$$\left\{ \bar{R}(x), \tilde{G}_n(y) \right\} \approx \tilde{G}_n^{II}(x) \delta(x - y)$$

so that there are no derivative of the delta function on the right side of the previous equation. This fact is very important for the consistency of given theory.

Now we are ready to proceed to the analysis of the consistency of the secondary constraints. In case of $\tilde{G}_n$ we obtain

$$\partial_t \tilde{G}_n(x) = \left\{ \tilde{G}_n(x), H_T \right\} \approx -\bar{N}(x) \tilde{G}_n^{II}(x)$$

using the fact that the Poisson bracket $\left\{ \tilde{G}_n(x), \tilde{G}_n(y) \right\}$ is weakly zero as follows from

$$\left\{ \int d^3x N \tilde{G}_n(x), \int d^3y M \tilde{G}_n(y) \right\} =$$

$$= \frac{1}{2} T_S((N \partial_i M - M \partial_i N) f^{ij}) + \frac{1}{4} T_S \left( \frac{1}{\gamma^2} f^i_a p^a f^j_b p^b (\partial_i N M - M \partial_i N) \right) +$$

$$+ \int d^3x (N \partial_i M - M \partial_i N) f^{ij} S_j -$$

$$- \frac{1}{2} \int d^3x (N \partial_i M - M \partial_i N) \frac{f^i_a p^a}{\gamma} \tilde{G}_n' \approx 0 .$$

(72)

On the other hand the time evolution of the constraint $\bar{R}$ is equal to

$$\partial_t \bar{R}(x) = \left\{ \bar{R}(x), H_T \right\} \approx \Gamma_n \tilde{G}_n^{II}(x) = 0$$

(73)

using (40),(46) and (51) together with the fact that $\Gamma^i = \Gamma^{ab} = \Omega_i = 0$. Now it is crucial to find non-trivial solution of (73). In case when $\tilde{G}_n^{II}$ were constant on the
whole phase space we would find that the only possible solution is $\Gamma_n = 0$. Then from (71) we would also find $N = 0$ and hence we should interpret $\mathcal{R}$ together with $\mathcal{G}_n$ as the second class constraints. However this is very unsatisfactory result since it would imply the lack of the Hamiltonian constraint while the theory is manifestly invariant under diagonal diffeomorphism. Fortunately $\mathcal{G}_n^{II}$ depends on the phase space variables so that it is more natural to obey (73) when we say that $\mathcal{G}_n^{II}$ is an additional constraint imposed on the system.

Now with this interpretation we find that (71) vanishes on the constraint surface when $\Gamma_n = 0$. As the next step we will analyze the requirement of the preservation of the constraints $S_i, T_{ab}, \mathcal{K}^i$ which however simplifies considerably due to the fact that $\Gamma_n = \Gamma^i = \Gamma^{ab} = \Omega_i = 0$. We start with the constraint $S_i$

$$ \partial_t S_i = \{S_i, H_T\} \approx \int d^3x N(x) \{S_i, \mathcal{R}(x)\} + f_i^a v_a = 0 , $$

(74)

using also the fact that $\{S_i(x), L_{ab}^{br}(y)\} = 0$. Now due to the fact that the matrix $f_i^a$ is non-singular we find that this equation can be solved for $v_a$.

In case of the constraints $T_{ab}$ we find

$$ \partial_t T_{ab} = \{T_{ab}, H_T\} \approx \int d^3x N(x) \{T_{ab}, \mathcal{R}(x)\} + \Delta_{T_{ab},k^e}v_c + \Delta_{T_{ab},L_{cd}^{br}}v^{cd} = 0 , $$

(75)

where the matrix $\Delta_{T_{ab},k^e}$ is defined as

$$ \{T_{ab}(x), k^e(y)\} \equiv \Delta_{T_{ab},k^e}(x)\delta(x - y) . $$

(76)

Now using the fact that the matrix $\Delta_{T_{ab},L_{cd}^{br}}$ is non-singular and that $v_a$ were determined by (74) we find that the equation (75) can be explicitly solved for $v^{ab}$.

Finally we proceed to the analysis of the equation of motion of the constraint $\mathcal{K}^i$

$$ \partial_t \mathcal{K}^i = \{\mathcal{K}^i, H_T\} \approx \int d^3x N(x) \{\mathcal{K}^i, \mathcal{R}(x)\} + v^n \Delta_{\mathcal{K}^i,p} + $$

$$ + v^{ab} \Delta_{\mathcal{K}^i,L_{ab}^{br}} + v^i + v_c \Delta_{\mathcal{K}^i,k^e} = 0 , $$

(77)

where we defined

$$ \{\mathcal{K}^i(x), p(y)\} = \left[ -\frac{1}{\gamma n^2} f^{i,a,p^a} + \frac{1}{\gamma} e^{ib} p_b \right] \delta(x - y) \equiv \Delta_{\mathcal{K}^i,p}(x)\delta(x - y) , $$

$$ \{\mathcal{K}^i(x), L_{ab}^{br}(y)\} = \frac{2n}{\gamma} (e^{i,a,p^{a}} - e^{i,b,p^b})(x)\delta(x - y) \equiv \Delta_{\mathcal{K}^i,L_{ab}^{br}}(x)\delta(x - y) , $$

$$ \{\mathcal{K}^i(x), k^e(y)\} = \frac{1}{\gamma^2} p^i(\mathcal{K}^i - n^i)\delta(x - y) - \frac{1}{\gamma} \left( \frac{1}{n} f^{i,e} + n e^{i,e} \right) \delta(x - y) \approx $$

$$ \approx - \left( \frac{1}{\gamma^2} p^i n^i + \frac{1}{\gamma} \frac{n}{\gamma} f^{i,e} + \frac{n}{\gamma} e^{i,e} \right) \delta(x - y) \equiv \Delta_{\mathcal{K}^i,k^e}(x)\delta(x - y) . $$

(78)
We see that (77) can be solved for \( v^i \) knowing the Lagrange multipliers \( v_{ab}, v_c, v_n \). Note that \( v_n \) is still undetermined which is the reflection of the fact that \( p \approx 0 \) is the first class constraint.

Finally we should check the stability of all constraints with the constraint \( \tilde{G}_{n}^{II} \approx 0 \) included. However it turns out that there is non-zero Poisson bracket between \( \tilde{G}_{n}^{II} \approx 0 \) and \( \tilde{G}_{n} \approx 0 \) and these are the second class constraints. Then the analysis of the stability of all constraints is the same as above and we will not repeat here. Let us outline our results and determine the physical degrees of freedom of given theory. We have \( N_{f.c.c.} = 12 \) first class constraints \( \tilde{R}, \tilde{R}_{i}, \tilde{P}_{i}, P_{i}, L_{ab}^{diag}, p \). Then we have \( N_{s.c.c.} = 20 \) second class constraints \( p_{i}, k^{a}, L_{ab}^{br}, \tilde{G}_{n}, \tilde{G}_{n}^{II}, S_{i}, K^{i}, T_{ab} \). We also have \( N_{ph.s.d.f.} = 58 \) phase space degrees of freedom \( \tilde{N}, \tilde{P}, \tilde{P}_{i}, n, p, n_{i}, p_{i}, p_{a}, k^{a}, e_{i}^{a}, \pi^{i}_{a}, f^{a}_{i}, \rho^{a}_{i} \). Then the number of physical degrees of freedom \( N_{p.d.f.} \) is

\[
N_{p.d.f.} = N_{ph.s.d.f.} - 2N_{f.c.c.} - N_{s.c.c.} = 14
\]

(79)

that could be interpreted as 4 physical degrees of freedom of the massless graviton, 10 physical degrees of freedom corresponding to the massive graviton. In other words we have shown that the bi-gravity in the vierbein formulation is ghost free.

### 3 Conclusion

This paper was devoted to the Hamiltonian analysis of the bimetric theory of gravity in the form introduced in [8]. We found corresponding Hamiltonian and determined the primary constraints of the theory. Then we analyzed the requirement of the preservation of these constraints and we determined corresponding secondary constraints. Finally we determined conditions when these constraints are preserved and we found that there is an additional constraint. As a result the constraint structure of given theory suggests that this theory is free of ghosts.

However it is still important to stress that even if the non-linear massive gravity is ghost free this does not mean that given theory is consistent. In fact, it was shown

---

\footnote{It is necessary to stress one important point. From the form of the constraint \( \tilde{G}_{n}^{II} \) we find that \( \{ \tilde{G}_{n}^{II}, \tilde{R} \} \neq 0 \) and hence we could say that \( \tilde{R} \) is second class constraint. Clearly this is rather unsatisfactory result since we would have three second class constraints while we should expect two second class constraints and one first class constraint. In order to see how to resolve this puzzle let us consider the case where the constraints \( \tilde{R}, \tilde{G}_{n}, \tilde{G}_{n}^{II} \) do not depend on spatial coordinates keeping in mind that extension of this analysis to the more general case is straightforward. Note that we can use this approximation since we know that \( \{ \tilde{G}_{n}(x), \tilde{G}_{n}(y) \} \approx 0 \) as follows from (72). Then the requirement of the stability of the constraints \( \tilde{R}, \tilde{G}_{n}, \tilde{G}_{n}^{II} \) implies following equations

\[
\partial_{i}\tilde{R} \approx \{ \tilde{R}, \tilde{G}_{n}^{II} \} \lambda_{n}^{II} = 0 \text{ that has solution } \lambda_{n}^{II} = 0.
\]

Then we also find that the constraint \( \tilde{G}_{n} \) is preserved. Finally the requirement of the stability of the constraint \( \tilde{G}_{n}^{II} \) implies following equation

\[
\partial_{i}\tilde{G}_{n}^{II} \approx \{ \tilde{G}_{n}^{II}, \tilde{R} \} N + \lambda_{n} \{ \tilde{G}_{n}^{II}, \tilde{G}_{n}^{II} \} = 0 \text{ that can be schematically solved as } \lambda_{n} = -\frac{\{ \tilde{G}_{n}^{II}, \tilde{R} \}}{\{ \tilde{G}_{n}^{II}, \tilde{G}_{n}^{II} \}} N.
\]

Then it is natural to define new Hamiltonian constraint \( \tilde{R}' = \tilde{R} - \{ \tilde{G}_{n}^{II}, \tilde{R} \} \tilde{G}_{n}^{II} \). Now it is easy to see that \( \{ \tilde{G}_{n}, \tilde{R}' \} = \{ \tilde{G}_{n}^{II}, \tilde{R}' \} = 0 \) and hence we find one first class constraint \( \tilde{R}' \) and two second class constraints \( \tilde{G}_{n}, \tilde{G}_{n}^{II} \) as expected.}
that non-linear massive gravity suffers from the superluminality in its decoupling limit \[16, 17, 18\]. It was also shown that generally contain the tachyonic modes \[19, 20\]. Further, the analysis of cosmological properties of non-linear massive gravity showed that it exhibits the ghost instabilities about its homogeneous solutions \[21, 22, 23\], see also \[42, 43\]. On the other hand it was shown very recently in \[48\] that non-linear bimetric theory of gravity could lead to viable cosmology under some conditions. In fact, the bimetric theory of gravity has one important advantage with respect to non-linear massive gravity when the second metric is not fixed by hand but it is dynamical as well. Clearly bimetric theory of gravity is very promising generalization of gravity that deserves to be studied further.

Acknowledgement:
I would like to thank S. Alexandrov for very useful discussions and for his finding of the crucial error in the first version of this paper. This work was supported by the Grant agency of the Czech republic under the grant P201/12/G028.

A Appendix: Hamiltonian Formalism of General Relativity in Vierbein Formulation

In this Appendix we perform the Hamiltonian formalism of the general relativity in vierbein formulation. We mostly follow \[28, 29, 30, 31\].

Let us consider the general relativity Lagrangian density written in the form

\[
\mathcal{L} = M_p^2 \det E(\Omega_A^B \Omega_C^B - \frac{1}{2} \Omega^{CAB} \Omega_{CBA} - \frac{1}{4} \Omega^{BAC} \Omega_{BAC}),
\]

where

\[
\Omega^{CAB} = E^\mu_C E^\nu_A \partial_\mu E^B_\nu, E^\mu_A = E^\mu_B \eta^{BA},
\]

and where \(\mu, \nu, \cdots = 0, 1, 2, 3\) and where \(A, B, \cdots = 0, 1, 2, 3\). Note that by definition we have two covariant derivatives \(\hat{\nabla}_\mu\) and \(\hat{\nabla}_\mu\). \(\hat{\nabla}_\mu\) is covariant with respect to both general coordinate transformations in spacetime as well as local Lorentz transformations on the flat index while \(\hat{\nabla}_\mu\) is covariant under general coordinate transformations. We have

\[
\hat{\nabla}_\mu \lambda^\nu_A = \partial_\mu \lambda^\nu_A + \hat{\Gamma}^\nu_\mu\gamma \lambda^{\gamma A} + \hat{\omega}^A_\mu B \lambda^\nu_B,
\]

\[
\hat{\nabla}_\mu \lambda^\nu_A = \partial_\mu \lambda^\nu_A + \hat{\Gamma}^\nu_\alpha\gamma \lambda^{\gamma A}.
\]

We require that these covariant derivatives are compatible with the vierbein and

\(^7\)However quiet recently the improved version of non-linear massive gravity was proposed in \[24\] that is claimed to be unitary with all degrees of freedom propagating on a homogeneous, isotropic and self-accelerating de Sitter background.
the metric
\[
\hat{D}_\mu E^\nu_A = 0 ,
\hat{\nabla}_\mu \hat{g}_{\nu\sigma} = \hat{D}_\mu \hat{g}_{\nu\sigma} = 0 .
\] (83)
where \(\hat{g}_{\mu\nu} = E^A_\mu E^B_\nu \eta_{AB}\). Note that from (83) we obtain
\[
\hat{\nabla}_\mu E^\nu_A = -\hat{\omega}^A_\mu B E^\nu_B .
\] (84)
From (83) and requiring \(\hat{\Gamma}^\rho_{[\mu\nu]} = 0\) it is possible to uniquely determine \(\hat{\Gamma}^\rho_{\mu\nu}\) and \(\hat{\omega}^A_\mu B\) as functions of vierbein \(E^A_\mu\). Explicitly, the first equation in (83) can be solved as
\[
\hat{\omega}^A_\mu B = \frac{1}{2} e_C^a (\Omega^{CAB} + \Omega^{BCA} - \Omega^{ABC}) .
\] (85)
Let us now consider following 3 + 1 decomposition of tetrad
\[
E_0^A = NN^A + N^a V^A_a , E_i^A = V_i^A , N^B \eta_{AB} V_i^B = 0 , N^A \eta_{AB} N^B = -1 ,
\] (86)
where \(i,j,k, \cdots = 1,2,3\) and \(a,b,c, \cdots = 1,2,3\). The inverse vielbein obeys
\[
E^\mu_B E^C_\mu = \delta^C_B , E^\mu_A E^\nu_A = \delta^\mu_\nu .
\] (87)
Using this decomposition it is rather straightforward perform the Legendre transform using this decomposition. However it is more convenient to partly break the manifest Lorentz invariance in such a way that the vierbein takes the upper triangular form (1). In this case we identify \(V_i^a\) with \(e_i^a\) where \(e_i^a\) defines the three dimensional metric \(g_{ij} = e_i^ae_j^b \delta_{ab}\).

Now using (86) and also the partial gauge fixing we obtain following decomposition of \(\Omega^{ABC}\)
\[
\Omega^0_0^b^a = \frac{1}{N} e^i_a (\partial_0 e_i^b - N^j \partial_j e_i^b - e^i_0 e_j^b \partial_0 N_j) ,
\]
\[
\Omega^0_0^a = \frac{1}{N} e^i_a \partial_i N ,
\]
\[
\Omega^a_0^b = 0 ,
\]
\[
\Omega^c_{ab} = e^i_a e^j_b (\partial_i e_j^c - \partial_j e_i^c) .
\] (88)
The general relativity Lagrangian now takes the form
\[
\mathcal{L} = M_P^2 N e \left( -\frac{1}{2} \Omega^{(ab)}_0 \Omega^{ab}_0 + \Omega^a_0 \Omega^a_0 b + 2 \Omega^a_0 \Omega^0_0 b - \frac{1}{4} \Omega^{abc} \Omega_{abc} - \frac{1}{2} \Omega^{abc} \Omega_{acb} + \Omega^c_0 \Omega^a_0 b \right) .
\] (89)
where $e = \det e_i^a$ and where we have following convention
\[
X^{(ab)} = X^{ab} + X^{ba}, \quad X^{(ab)} X_{(ab)} = 2X^{(ab)} X_{ba}.
\] (90)

Note that we can write
\[
Ne^{(3)} R = Ne(\Omega_b^a \Omega^c_{ca} - \frac{1}{4} \Omega^{abc} \Omega_{abc} - \frac{1}{2} \Omega^{abc} \Omega_{acb}) + \\
+ \nabla_i (N e^{ja} \nabla_j e^i a) - 2 e^{ja} \partial^i e^j a g^{jk} \partial_k N,
\] (91)

where $\nabla_i$ is covariant derivative compatible with $g_{ij}$ so that $\nabla_i g_{jk} = 0$. Then neglecting the surface term we find that (89) has the form that is suitable for the Hamiltonian formulation
\[
\mathcal{L} = M_g^2 Ne \left( -\frac{1}{2} \Omega^{0ab} \Omega^0_{(ab)} + \Omega^0_a \Omega^0_b + \Omega^{(3)} R \right),
\] (92)

where
\[
\Omega^{0ab} = \delta^{ac} \Omega^0_c b, \quad \Omega_{0ab} = -\Omega^0_b \delta_{dc}, \quad \Omega^0_{ab} = -\Omega^0 b^b.
\] (93)

Note that (91) implies
\[
\Omega^{0(ab)} \Omega_{0ab} = \frac{1}{2} \Omega^{0(ab)} \Omega_{0(ab)}.
\] (94)

Then from (92) we find the momenta $\pi_i^a$ conjugate to $e_i^a$
\[
\pi_i^a = \frac{\delta \mathcal{L}}{\delta \partial_t e_i^a} = M_g^2 e^i c \Omega^{0(cd)} \delta_{da} - 2 \Omega^0 b^i a
\] (95)

so that
\[
\Omega^0_a = -\frac{1}{4e} e_i^a \pi^i a.
\] (96)

Then it is easy to express $\Omega^{0(ab)}$ as function of $\pi_i^a$ and $e_i^a$
\[
\Omega^{0(ab)} = \frac{1}{e} (e_i^a \pi^i c \delta^{ab} - \frac{1}{2} \delta^{ab} e_i^a \pi^i c).
\] (97)

Using this result we easily find the Hamiltonian from (92) and hence corresponding Hamiltonian
\[
\mathcal{H} = N \mathcal{R} + N^i \mathcal{R}_i,
\] (98)

where
\[
\mathcal{R} = \frac{1}{4M_g^2} (e_i^a \pi^i c \delta^{ab} \delta_{ae} e_j^c \pi^j b - \frac{1}{2} (e_i^a \pi^i a)^2) - M_g^2 e^{(3)} R,
\]
\[
\mathcal{R}_i = -e_i^b \partial_j \pi^j b,
\] (99)
where $\mathcal{D}_i$ is covariant derivative compatible with $e^a_i$

$$\mathcal{D}_i e^a_j = 0 .$$  \hspace{1cm} (100)

Note also that we neglected the total derivative terms in the Hamiltonian \[98\]. It is also important to stress that \[95\] implies following primary constraints

$$L_{ab} = e_{ia} \pi^i_b - e_{ib} \pi^i_a \approx 0$$  \hspace{1cm} (101)

By definition $L_{ab}$ are antisymmetric so that there are three constraints $L_{ab}$ in three dimensions.

To proceed further we need an explicit form of $\mathcal{D}_i \pi^j_a$. Since $\pi^i_a$ is the density of weight one we have\footnote{Note that the spin connection has following prescription when it acts on object with upper and lower Lorentz indices 

$$\mathcal{D}_i X^a_b = \partial_i X^a_b + \omega^a_i c X^c_b - \omega^c_i b X^a_c$$  \hspace{1cm} (102) } \[30\].

$$\mathcal{D}_i \pi^j_a = \partial_i \pi^j_a + \Gamma^j_i k a \pi^k_a - \Gamma^m_j i m \pi^j a - \omega^b_i a \pi^j b$$  \hspace{1cm} (103)

so that

$$\mathcal{D}_i \pi^i_a = \partial_i \pi^i_a + \omega^b_i a \pi^i b$$  \hspace{1cm} (104)

It is also convenient to introduce the notation

$$\pi^{ij} = \frac{1}{4}(\pi^i_a e^j a + e^i a \pi^j_a)$$  \hspace{1cm} (105)

that it is similar as the notation used in \[30\]. Then it is easy to see that the Hamiltonian constraint $\mathcal{R}$ takes the familiar form

$$\mathcal{R} = \frac{1}{\sqrt{g M^2}} \mathcal{G}_{ijkl} \pi^{ij} \pi^{kl} - M^2 g^{(3)} R ,$$  \hspace{1cm} (106)

where

$$\mathcal{G}_{ijkl} = \frac{1}{2}(g_{ik} g_{jl} + g_{id} g_{jk} - g_{ij} g_{kl}) , \quad \mathcal{G}^{ijkl} = \frac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk} - g^{ij} g^{kl}) ,$$  \hspace{1cm} (107)

that obey the relation

$$\mathcal{G}_{ijkl} g^{kln} = \frac{1}{2}(\delta^{m}_i \delta^{n}_j + \delta^{m}_j \delta^{n}_i) .$$  \hspace{1cm} (108)

Note also that in the same way we can write

$$\mathcal{R}_i = -e^a_i \mathcal{D}_j \pi^j_b = -2 \nabla_i \pi^i_j$$  \hspace{1cm} (109)
using the fact that
\[-2 \nabla_i \pi^i_j \approx -\nabla_i \pi^i_a e_j^a - \pi_i^a \nabla_i e^a_j =
\]
\[-\nabla_i \pi^i_a a_j^a + \pi^a_i \omega^b_a e_j^b = -e_i^a D_j \pi^a_b\]
\] (110)

using also the fact that
\[\pi^i_j = \pi^i g_{kj} = \frac{1}{2} \pi^i a e^a_j + L_{ad} e^a_a e^d_j .\]
\] (111)

By definition the canonical variables are \(e^i_a\) and \(\pi^i_j\) with following canonical Poisson brackets
\[\{e^i_a(x), \pi^j_b(y)\} = \delta^i_j \delta^a_b \delta(x - y)\]
\] (112)

so that we obtain
\[\{g_{ij}(x), \pi^{kl}(y)\} = \frac{1}{2} (\delta^k_l \delta^i_j + \delta^i_l \delta^k_j) \delta(x - y)\]
\] (113)

On the other hand from (110) and from (112) we find
\[\{\pi^{ij}(x), \pi^{kl}(y)\} = \frac{1}{16} (g^{il} L_{kj} + g^{jl} L_{ki} + g^{jk} L_{li} + g^{ik} L_{lj}) \delta(x - y) = \]
\[\mu^{ijkl} \delta(x - y) .\]
\] (114)

This result implies that there are additional terms when we calculate the Poisson brackets between the constraints as was nicely shown in [30]. More precisely, let us introduce the smeared form of the constraints \(R, \mathcal{R}_i\) and \(L_{ab}\)
\[T_T(N) = \int d^3 x N R , \quad T_S(N^i) = \int d^3 x N^i R_i , \quad L(N_{ab}) = \int d^3 x N_{ab} L_{ab} .\]
\] (115)

Then, following [30] we find
\[\{T_T(N), T_T(M)\} = T_S((N \partial_t M - M \partial_t N) g^{ij}) , \]
\[\{T_S(N^i), T_T(M)\} = T_T(N^i \partial_t M) + \int d^3 x \nabla_j N^i \lambda_i^j M , \]
\[\{T_S(N^i), T_S(M^j)\} = T_S((N^i \partial_t M^j - M^j \partial_t N^i)) + \int d^3 x \nabla_k N^i \mu_i^k \lambda^l M^j , \]
\[\{T_T(N), L_{ab}(N^i)\} = 0 , \{T_S(N^i), L(N_{ab})\} = 0 , \]
\[\{L_{ab}(x), L_{cd}(y)\} = (\eta_{ad} L_{bc} + \eta_{bc} L_{ad} - \eta_{bd} L_{ac} - \eta_{ac} L_{bd}) \delta(x - y) .\]
\] (116)
where

\[ \lambda_{ij} = -4\mu^{ijkl}K_{kl} = -\frac{1}{2}(K^j_kL^{ik} - K^i_kL^{jk}), \]

\[ K_{ij} = \frac{1}{\sqrt{g}} \left( \frac{1}{2} \pi^{mn} g_{nm} g_{ij} - \pi_{ij} \right). \]

(117)

We see that there are additional terms on the right side of the Poisson brackets between constraints that are proportional to the primary constraints \( L_{ab} \). These terms also vanish on the constraints surface. For that reason we will not write the explicit form of these terms in the calculations performed in the main body of the paper.

References

[1] A. Salam and J. A. Strathdee, “Nonlinear realizations. 1: The Role of Goldstone bosons,” Phys. Rev. 184 (1969) 1750.

[2] C. J. Isham, A. Salam and J. A. Strathdee, “Spontaneous breakdown of conformal symmetry,” Phys. Lett. B 31 (1970) 300.

[3] M. Fierz, W. Pauli, “On relativistic wave equations for particles of arbitrary spin in an electromagnetic field,” Proc. Roy. Soc. Lond. A173 (1939) 211-232.

[4] D. G. Boulware and S. Deser, “Can gravitation have a finite range?,” Phys. Rev. D 6 (1972) 3368.

[5] D. G. Boulware and S. Deser, “Inconsistency of finite range gravitation,” Phys. Lett. B 40 (1972) 227.

[6] C. de Rham, G. Gabadadze and A. J. Tolley, “Resummation of Massive Gravity,” Phys. Rev. Lett. 106 (2011) 231101 [arXiv:1011.1232 [hep-th]].

[7] C. de Rham, G. Gabadadze and A. J. Tolley, “Ghost free Massive Gravity in the St"{u}ckelberg language,” Phys. Lett. B 711 (2012) 190 [arXiv:1107.3820 [hep-th]].

[8] K. Hinterbichler and R. A. Rosen, “Interacting Spin-2 Fields,” JHEP 1207 (2012) 047 [arXiv:1203.5783 [hep-th]].

[9] S. F. Hassan, A. Schmidt-May and M. von Strauss, “Proof of Consistency of Nonlinear Massive Gravity in the St"{u}ckelberg Formulation,” arXiv:1203.5283 [hep-th].

[10] S. F. Hassan and R. A. Rosen, “Resolving the Ghost Problem in non-Linear Massive Gravity,” Phys. Rev. Lett. 108 (2012) 041101 [arXiv:1106.3344 [hep-th]].
[11] S. F. Hassan and R. A. Rosen, “On Non-Linear Actions for Massive Gravity,” JHEP 1107 (2011) 009 [arXiv:1103.6055 [hep-th]].

[12] S. F. Hassan and R. A. Rosen, “Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity,” JHEP 1204 (2012) 123 [arXiv:1111.2070 [hep-th]].

[13] J. Kluson, “Non-Linear Massive Gravity with Additional Primary Constraint and Absence of Ghosts,” Phys. Rev. D 86 (2012) 044024 [arXiv:1204.2957 [hep-th]].

[14] S. F. Hassan, R. A. Rosen and A. Schmidt-May, “Ghost-free Massive Gravity with a General Reference Metric,” JHEP 1202 (2012) 026 [arXiv:1109.3230 [hep-th]].

[15] S. F. Hassan and R. A. Rosen, “Bimetric Gravity from Ghost-free Massive Gravity,” JHEP 1202 (2012) 126 [arXiv:1109.3515 [hep-th]].

[16] A. Gruzinov, “All Fierz-Paulian massive gravity theories have ghosts or superluminal modes,” arXiv:1106.3972 [hep-th].

[17] C. Burrage, C. de Rham, L. Heisenberg and A. J. Tolley, “Chronology Protection in Galileon Models and Massive Gravity,” JCAP 1207 (2012) 004 [arXiv:1111.5549 [hep-th]].

[18] P. de Fromont, C. de Rham, L. Heisenberg and A. Matas, “Superluminality in the Bi- and Multi- Galileon,” arXiv:1303.0274 [hep-th].

[19] S. Deser and A. Waldron, “Inconsistencies of massive charged gravitating higher spins,” Nucl. Phys. B 631 (2002) 369 [hep-th/0112182].

[20] S. Deser, M. Sandora and A. Waldron, “Nonlinear Partially Massless from Massive Gravity?,” Phys. Rev. D 87, 101501 (R) (2013) [arXiv:1301.5621 [hep-th]].

[21] A. De Felice, A. E. Gumrukcuoglu and S. Mukohyama, “Massive gravity: nonlinear instability of the homogeneous and isotropic universe,” Phys. Rev. Lett. 109 (2012) 171101 [arXiv:1206.2080 [hep-th]].

[22] A. De Felice, A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, “Nonlinear stability of cosmological solutions in massive gravity,” JCAP 1305 (2013) 035 [arXiv:1303.4154 [hep-th]].

[23] A. De Felice, A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, “On the cosmology of massive gravity,” arXiv:1304.0484 [hep-th].

[24] A. De Felice and S. Mukohyama, “Towards consistent extension of quasidilaton massive gravity,” arXiv:1306.5502 [hep-th].
[25] E. Gourgoulhon, “3+1 formalism and bases of numerical relativity,” [gr-qc/0703035].

[26] R. L. Arnowitt, S. Deser, C. W. Misner, “The Dynamics of general relativity,” [gr-qc/0405109].

[27] M. Chaichian, M. Oksanen and A. Tureanu, “Arnowitt-Deser-Misner representation and Hamiltonian analysis of covariant renormalizable gravity,” Eur. Phys. J. C 71 (2011) 1657 [Erratum-ibid. C 71 (2011) 1736] [arXiv:1101.2843].

[28] P. Peldan, “Actions for gravity, with generalizations: A Review,” Class. Quant. Grav. 11 (1994) 1087 [gr-qc/9305011].

[29] H. Nicolai and H. J. Matschull, “Aspects of canonical gravity and supergravity,” J. Geom. Phys. 11 (1993) 15.

[30] M. Henneaux, “Poisson Brackets Of The Constraints In The Hamiltonian Formulation Of Tetrad Gravity,” Phys. Rev. D 27 (1983) 986.

[31] J. M. Charap, M. Henneaux and J. E. Nelson, “Explicit Form Of The Constraint Algebra In Tetrad Gravity,” Class. Quant. Grav. 5 (1988) 1405.

[32] J. Yepez, “Einstein’s vierbein field theory of curved space,” [arXiv:1106.2037].

[33] T. Damour and I. I. Kogan, “Effective Lagrangians and universality classes of nonlinear bigravity,” Phys. Rev. D 66 (2002) 104024 [hep-th/0206042].

[34] J. Kluson, “Is Bimetric Gravity Really Ghost Free?,” [arXiv:1301.3296].

[35] J. Kluson, “Hamiltonian Formalism of General Bimetric Gravity,” [arXiv:1303.1652].

[36] J. Kluson, “Hamiltonian Formalism of Particular Bimetric Gravity Model,” [arXiv:1211.6267].

[37] V. O. Soloviev and M. V. Tchichikina, “Bigravity in Kuchar’s Hamiltonian formalism. 1. The General Case,” [arXiv:1211.6530].

[38] V. O. Soloviev and M. V. Tchichikina, “Bigravity in Kuchar’s Hamiltonian formalism. 2. The special case,” [arXiv:1302.5096].

[39] M. Henneaux and C. Teitelboim, “Quantization of gauge systems,” Princeton, USA: Univ. Pr. (1992) 520 p

[40] D. Comelli, F. Nesti and L. Pilo, “Massive gravity: a General Analysis,” [arXiv:1305.0236].
[41] D. Comelli, F. Nesti and L. Pilo, “Weak Massive Gravity,” arXiv:1302.4447 [hep-th].

[42] L. Berezhiani, G. Chkareuli and G. Gabadadze, “Restricted Galileons,” arXiv:1302.0549 [hep-th].

[43] L. Berezhiani, G. Chkareuli, C. de Rham, G. Gabadadze and A. J. Tolley, “Mixed Galileons and Spherically Symmetric Solutions,” arXiv:1305.0271 [hep-th].

[44] A. H. Chamseddine, A. Salam and J. A. Strathdee, “Strong Gravity and Supersymmetry,” Nucl. Phys. B 136 (1978) 248.

[45] A. H. Chamseddine, “Matrix gravity and massive colored gravitons,” Phys. Rev. D 70 (2004) 084006 hep-th/0406263.

[46] A. H. Chamseddine, “Spontaneous symmetry breaking for massive spin two interacting with gravity,” Phys. Lett. B 557 (2003) 247 hep-th/0301014.

[47] S. Alexandrov, “Canonical structure of Tetrad Bimetric Gravity,” Gen. Rel. Grav. 46 (2014) 1639 arXiv:1308.6586 [hep-th].

[48] A. De Felice, A. E. Gumrukcuoglu, S. Mukohyama, N. Tanahashi and T. Tanaka, “Viable cosmology in bimetric theory,” arXiv:1404.0008 [hep-th].