Exact solutions of the Harry Dym Equation using Lie group method
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Abstract

In this paper, we apply Lie symmetry group analysis to the Harry Dym equation. Then, the similarity reduction will be found, and the invariant solutions of Harry Dym equation is obtained from the solutions of reduced ordinary differential equations.

Keywords: Harry Dym equation, Lie group, similarity reduction, invariant solutions.

1. Introduction:

The linear and nonlinear fractional partial differential equations appear in many branches of science and engineering, that are in fluid mechanics, acoustic, electro-magnetism, analytical chemistry, biology, signal processing and other physical applications, see for instance [10, 20]. The Harry Dym equation has a strong links to the Korteweg-de Vries equation (Kdv) and the Sturm-Liouville operator [5, 19].

The nonlinear fractional Harry Dym equation has the following form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = u^3(x,t) \frac{\partial^3 u(x,t)}{\partial x^3}$$

(1)

Where $\alpha$ is a positive parameter describing the order of the fractional derivative and $u(x,t)$ is a function of the space $x$ and the time $t$.

In this paper, we consider the nonlinear fractional Harry Dym equation for $\alpha=1$, which takes the form

$$\frac{\partial u(x,t)}{\partial t} = u^3(x,t) \frac{\partial^3 u(x,t)}{\partial x^3}$$

(2)

Equation (2) is called classical nonlinear Harry Dym equation the time fractional Harry Dym equation.

There are many authors applied analytical and numerical methods to solve the classical nonlinear Harry Dym equation. Al-Khaled and Alquran [1], applied Adomian decomposition method (ADM) to obtain an approximate solution of time fractional generalized Harry Dym equation. Fonsca [4] solved the nonlinear Harry Dym equation using Lattice-Boltzmannn and a solitary wave methods. Ghiassi and Saleh [6] employed the homotopy analysis method (HAM) to obtain the approximate analytical solution of the nonlinear Harry-Dym (HD) equation.in[9], Iyiola and Gaba used q-homotopy analysis method (q-HAM) to obtain analytical solutions of the to time-fractional Harry Dym equation. Kumar et al. [11]used homotopy perturbation Sumudu transform method (HPSTM) and Adomian decomposition method (ADM) to solve nonlinear fractional Harry Dym equation. Kumar et al [12], applied the homotopy perturbation method (HPM) to obtain approximate solution of the time fractional Harry Dym equation. Maitama and Abdullahi [13] presented a new analytical method called the natural homotopy perturbation method (NHPM) for solving linear and nonlinear fractional partial differential equations, which is combined of natural transform (NTM) and homotopy perturbation methods (HPM). Mokhtari [14]applied Adomian decomposition method, He's variational iteration, direct integration, and power series methods to generate exact travelling wave solutions of the Harry Dym equation. Rawashdeh [16] employed a
new approach to solve fractional Harry Dym namely, fractional reduced differential transform method (FRDTM) and he found approximate solutions of the nonlinear Harry Dym equation. Shunmugarajan [17], the homotopy analysis method (HAM) is applied to get approximate solution of time fractional generalized Harry Dym equation. Soltani and Khorshidi[18] used the reconstruction of variational iteration (RVIM) and homotopy perturbation method (HPM) to obtain an analytical solution of Harry-Dym equation.

2. Method description

We apply Lie symmetry group analysis [2, 3, 7, 8, 15] for the nonlinear Harry-Dym (HD) equation (2) to derive symmetry generators of it. We consider one-parameter \( \varepsilon \)-Lie group point of transformations, which makes equation (2) invariant. These transformations are given by

\[
\begin{align*}
\xi &= \tilde{x}(x,t,u;\varepsilon) = x + \varepsilon \xi_1(x,t,u) + O(\varepsilon^2), \\
\tau &= \tilde{t}(x,t,u;\varepsilon) = t + \varepsilon \xi_2(x,t,u) + O(\varepsilon^2), \\
\theta &= \tilde{u}(x,t,u;\varepsilon) = u + \varepsilon \phi(x,t,u) + O(\varepsilon^2)
\end{align*}
\]

The infinitesimal generator associated with (3) is given by

\[
X = \xi_1(x,t,u) \frac{\partial}{\partial x} + \xi_2(x,t,u) \frac{\partial}{\partial t} + \phi(x,t,u) \frac{\partial}{\partial u}
\]

and the third prolongation of the infinitesimal generator (4) is given by

\[
Pr^{(3)} X = X + \phi' \frac{\partial}{\partial u} + \phi'' \frac{\partial}{\partial u^2} + \phi''' \frac{\partial}{\partial u^3}
\]

We consider

\[
\Delta = \frac{\partial u(x,t)}{\partial t} - u^3 \frac{\partial^3 u(x,t)}{\partial x^3}
\]

To calculate the infinitesimals \( \xi_1, \xi_2 \) and \( \phi \), applying (5) to (6), that is,

\[
Pr^{(3)} X(\Delta) = 0, \text{ when } \Delta = 0
\]

Eq. (7) is called condition of the invariant, and it leads to

\[
\phi' + \alpha \phi' - \varepsilon \phi''' = 0
\]

where

\[
\phi_{j_i}^{(i)} = D^{i-j} \phi_{i} - (D^{j} \xi_{i})u_{j} - (D_{i} \xi_{j}) u_{j} , i \neq j
\]

where \( x \) refers to the independent variables. Conditions on the infinitesimals \( \xi_1, \xi_2 \) and \( \phi \) are determined by equating coefficients of like derivatives of monomials in \( u_x \), \( u_t \) and higher derivatives by zero. This leads to a system of partial differential equations from which we can determine \( \xi_1, \xi_2 \) and \( \phi \). These equations are solved to get the infinitesimal solutions \( \xi_1, \xi_2 \) and \( \phi \) in the following forms:

\[
\begin{align*}
\xi_1(x,t,u) &= \frac{1}{2} c_3 x^2 + c_4 x + c_5 , \\
\xi_2(x,t,u) &= c_1 t + c_2 , \\
\phi(x,t,u) &= -\frac{1}{3} (c_1 - 3c_3 x - 3c_4 )u
\end{align*}
\]
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where \( c_i, i = 1, 2, ..., 5 \) are arbitrary constants, Eqs.\((10)\) shows that the nonlinear Harry-Dym (HD) equation \((2)\) has the following generators symmetry group:

\[
\begin{align*}
X_1 &= t \partial_t - \frac{1}{3} u \partial_u, \\
X_2 &= \partial_t, \\
X_3 &= \frac{1}{2} x^2 \partial_x + x u \partial_u, \\
X_4 &= x \partial_x + u \partial_u, \\
X_5 &= \partial_x
\end{align*}
\]

\((11)\)

2.1 Similarity Reduction and Similarity Solutions

We consider the generators \((X_2 - c X_3), (X_1 + X_4)\) and \((X_2 + X_4)\).

1. For the generator \(X_2 - c X_3\):

we get the similarity transformations

\[
\eta = x + c t, \quad u(x, t) = f(\eta)
\]

\((12)\)

Substituting Eq. \((12)\) in to Eq. \((2)\), we obtain the following ordinary differential equation:

\[
f^3(\eta) f''''(\eta) - c f'(\eta) = 0
\]

\((13)\)

Eq. \((13)\) can be written as

\[
f^{''''}(\eta) = c f^{-3}(\eta) f'(\eta)
\]

\((14)\)

Integrating Eq. \((14)\) with respect to \(\eta\), yields

\[
f''(\eta) + \frac{c}{2 f^2(\eta)} = C_1
\]

\((15)\)

Multiplying Eq. \((15)\) by \(f'(\eta)\) and integrating with respect to \(\eta\), yields

\[
(f'(\eta))^2 = \frac{c}{f(\eta)} + C_1 f(\eta) + C_2
\]

\((16)\)

Eq. \((16)\) gives us

\[
\eta = \pm \int \frac{f(\eta)}{C_1 f^2(\eta) + C_2 f(\eta) + c} \ df(\eta) + C_3
\]

\((17)\)

where \(C_1, C_2\) and \(C_3\) are integration constants.

Setting \(C_1 = C_2 = 0\), Eq. \((17)\) becomes

\[
\eta = \pm \int \frac{f(\eta)}{c} \ df(\eta) + C_3
\]

\((18)\)

Eq. \((18)\) leads to the following similarity solution

\[
f(\eta) = \left[ \pm \frac{3 \sqrt{c}}{2} (C_3 - \eta) \right]^{2/3}
\]

\((19)\)
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which is equivalent to

\[ f(\eta) = \left[ a \pm \frac{3\sqrt{c}}{2} \eta \right]^{2/3} \]  \hspace{1cm} (20)

Hence, the invariant solution of Eq. (2) is

\[ u(x,t) = \left[ a \pm \frac{3\sqrt{c}}{2} (x + ct) \right]^{2/3} \]  \hspace{1cm} (21)

where \( a = \pm \frac{3\sqrt{c}}{2} C_3 \).

2. For the generator \( X_1 + X_4 \):

we obtain the similarity transformations

\[ \eta = x/t, \quad u(x,t) = (x + t)^{2/3} f(\eta) \]  \hspace{1cm} (22)

Substituting Eq. (22) in to Eq. (2), we get

\[(\eta + 1)^3 f^3 f''(\eta) + 2(\eta + 1)^2 f - (\eta + 1)(\frac{2}{3} f^3 - \eta) f'(\eta) + \frac{2}{3} (\frac{4}{9} f^3 - 1) f(\eta) = 0 \]  \hspace{1cm} (23)

Solution of Eq. (23) is

\[ f(\eta) = e^{\int \left[ \frac{H(\chi) d\chi + c_1}{\chi} \right]^{2/3}} \]  \hspace{1cm} (24)

where

\[ \chi = \frac{(\eta + 1)^{2/3} f(\eta)}{\eta}, \quad H(\chi) = \frac{3\eta (\eta + 1)^{1/3}}{3\eta (\eta + 1) f'(\eta) - (\eta + 3) f(\eta)} \]  \hspace{1cm} (25)

3. For the generator \( X_2 + X_4 \):

Here we have the similarity reduction

\[ \eta = \ln x - t, \quad u(x,t) = x f(\eta) \]  \hspace{1cm} (26)

By using Eq. (26), Eq. (2) becomes

\[ f^3 (\eta) f''(\eta) - (f^3 (\eta) - 1) f'(\eta) = 0 \]  \hspace{1cm} (27)

Solution of Eq. (27) as the form

\[ \eta = \pm \int \frac{f(\eta)}{\sqrt{2 C_2 f(\eta) + f^3 (\eta) - 2 C_1 f^2 (\eta) - 1}} df(\eta) + C_3 \]  \hspace{1cm} (28)

Setting \( C_1 = C_2 = 0 \), Eq. (28) becomes

\[ \eta = \pm \int \frac{f(\eta)}{\sqrt{f^3 (\eta) - 1}} df(\eta) + C_3 \]  \hspace{1cm} (29)

The integrating in Eq. (29) leads to
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\[
\eta = \pm \frac{2}{3} \ln \left( \sqrt{f^3(\eta)} + \sqrt{f^3(\eta) - 1} \right) + C_3
\]

(30)

Conclusion
In this study, Lie symmetry group method for calculating the solution of the Harry Dym equation (2) is presented. Therefore, we have obtained the explicit solution (21) of the studied equation. The impliit solution of the Harry Dym equation (2) has been found in equations (24) and (30).

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الحلول المضبوطة لمعادلة هاري ديم باستخدام طريقة مجموعة لي

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الملخص

في هذا البحث طبقنا طريقة تناظر لي لمعادلة هاري ديم. باستخدام التحويلات المتماثلة تم تحويل هذه المعادلة إلى معادلة تفاضلية عادية ومن ثم حصلنا على الحل اللا تغييري لمعادلة هاري ديم من حلول المعادلة التفاضلية العادية.

الكلمات المفتاحية: معادلة هاري ديم، مجموعة لي، التحويلات المتماثلة، حلول لا تغيرية.