ON RADON MEASURES INVARIANT UNDER HOROSPHERICAL FLOWS ON GEOMETRICALLY INFINITE QUOTIENTS

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Abstract. We consider a locally finite (Radon) measure on $\text{SO}^+(d, 1)/\Gamma$ invariant under a horospherical subgroup of $\text{SO}^+(d, 1)$ where $\Gamma$ is a discrete, but not necessarily geometrically finite, subgroup. We show that whenever the measure does not observe any additional invariance properties then it must be supported on a set of points with geometrically degenerate trajectories under the corresponding contracting 1-parameter diagonalizable flow (geodesic flow).

1. Introduction and Statement of Results

Consider a geometrically infinite discrete subgroup $\Gamma$ of $G = \text{SO}^+(d, 1)$, the group of orientation preserving isometries of the hyperbolic $d$-space $\mathbb{H}^d$. We study the locally finite horospherical invariant measures on $G/\Gamma$.

Classification of invariant measures under horospherical flows on hyperbolic manifolds has a long history beginning with Furstenberg’s proof of unique ergodicity of the horocycle flow on compact hyperbolic surfaces [7]. More generally, finite measures on $G/\Gamma$ which are invariant and ergodic under the horospherical flow are either supported on periodic horospheres bounding a cusp or are equal to the volume measure (which therefore must be finite). This was shown by Dani [6] and Veech [23] for $G/\Gamma$ of finite volume; for general quotients this result is a special case of Ratner’s measure classification theorem [17]. However, when considering $G/\Gamma$ with infinite volume the natural measure classification question is not classifying finite measures but rather that of classifying locally finite, a.k.a. Radon, measures.

When considering Radon measures there is a clear distinction between geometrically finite and geometrically infinite discrete subgroups. The cusps of geometrically finite manifolds are somewhat more complicated than that of finite volume manifolds, as they may have non-maximal rank. We say an orbit of the horospherical flow on $G/\Gamma$ bounds a cusp if its projection to the corresponding hyperbolic orbifold $\mathbb{H}^d/\Gamma$ is the boundary of a horoball emanating from one of the cusps of $\mathbb{H}^d/\Gamma$. Equivalently, a horospherical orbit bounds a cusp if its orbit under the one-parameter $\mathbb{R}$-split group contracting the horosphere tends to a parabolic limit point of $\Gamma$ — see [5]. This is the natural generalization of periodic orbits of the a horosphere on finite volume quotients.

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of $G$, and somewhat loosely we shall say a measure on $G/\Gamma$ is non-periodic if it gives zero measure to the horospheres bounding a cusp.

Geometrically finite manifolds turn out to exhibit a unique recurrent and non-periodic horospherical invariant Radon measure. This result is due to Roblin \cite{19} over the unit tangent bundle, extending earlier work by Burger \cite{3}, and due to Winter \cite{23} over the full frame bundle.

It was discovered by Babillot and Ledrappier \cite{2} that there is no such uniqueness phenomenon in the geometrically infinite setting. In particular, Babillot and Ledrappier showed that Abelian covers of compact hyperbolic surfaces support an uncountable family of horocycle invariant ergodic and recurrent Radon measures. Sarig \cite{20} showed that the class of measures constructed by Babillot and Ledrappier are the only horocycle invariant Radon measures on Abelian covers of compact surfaces. Later Ledrappier and Sarig \cite{11} extended the measure classification result to all regular covers of finite volume surfaces (though in this case the classification one gets is somewhat less explicit, similar to the measure classification results we give here; see \S5.2). Ledrappier \cite{10} extended these results to the unit tangent bundle of normal covers of compact manifolds of variable negative curvature and arbitrary dimension. Oh and Pan \cite{15} recently strengthened Ledrappier’s result to address the horospherical flow over the full frame bundle of Abelian covers of compact hyperbolic manifolds of arbitrary dimension.

A key step in all the above is obtaining additional invariance properties of the given horospherically invariant measure, specifically proving that in the cases considered by these authors such measures (unless periodic) are quasi-invariant with respect to the geodesic flow.

Let $\{a_t\}_{t\in \mathbb{R}}$ denote the 1-parameter $\mathbb{R}$-diagonal group in $G$ and let $U < G$ be the associated unstable horospherical subgroup. The flow given by $a_t$ on $G/\Gamma$ projects to the geodesic flow on the unit tangent bundle of $\mathbb{H}^d/\Gamma$. Denote by $N_G(U)$ the normalizer of $U$ in $G$. One consequence of our main result is the following:

**Corollary 1.1.** Let $\Gamma_0$ be a geometrically finite Zariski dense discrete subgroup of $G$ and let $\{e\} \neq \Gamma \triangleleft \Gamma_0$, i.e. $G/\Gamma$ is a regular cover of $G/\Gamma_0$. Let $\mu$ be a $U$-ergodic and invariant Radon measure on $G/\Gamma$. Then one of the following three possibilities holds:

1. $\mu$ is supported on a wandering horosphere.
2. $\mu$ is supported on a lift to $G/\Gamma$ of a horosphere in $G/\Gamma_0$ bounding a cusp.
3. $\mu$ is $N_G(U)$-quasi-invariant.

Note that this result is new even in the case of $d = 2$.

More generally, we aim to understand what conditions on $\Gamma$ or $\mu$, a horospherically invariant Radon measure, imply $N_G(U)$-quasi-invariance. Inspired by Sarig’s results on “weakly tame” surfaces \cite{21}, we give a general condition for quasi-invariance depending on the geometric “scenery” encountered by geodesic rays $\{a_t^{-1}x\}_{t \geq 0}$ for $\mu$-a.e. $x \in G/\Gamma$. This relation between the geometric scenery encountered and the invariance properties of $\mu$ is exemplified in the
following statement, proven in \([5,4]\) which is a special case of Theorem \([1,3]\) below:

**Corollary 1.2.** Let \(\Gamma \subset \text{SO}^+(d,1)\) be a purely-hyperbolic discrete subgroup
(we use the term hyperbolic to refer to loxodromic elements as well) whose
length spectrum \(\ell(\Gamma)\) is bounded away from 0, i.e. \(\inf \ell(\Gamma) > 0\).
Let \(\mu\) be any \(U\)-ergodic and invariant Radon measure on \(G/\Gamma\).
Then at least one of the following holds:

1. \(\mu\) is quasi-invariant under some hyperbolic element of \(N_G(U)\).
2. \(\lim_{t \to \infty} \text{rad}_{\text{inj}}(a_{-t}x) = \infty\) for \(\mu\)-a.e. \(x\).

Where \(\text{rad}_{\text{inj}}(y)\) is the supremal injectivity radius at the point \(y \in G/\Gamma\).

We note that this result is somewhat reminiscent to a result in the
topological category by Maucourant and Schapira (\([13, \text{Cor. 3.2}]\)) classifying
points in \(G/\Gamma\) whose \(U\)-orbits are dense in the non-wandering set for \(U\).

In this paper we give a sufficient condition for \(N_G(U)\)-quasi-invariance applicable
to horospherical flows over the frame bundle of a wide variety of
hyperbolic manifolds of arbitrary dimension.

While heavily inspired by Ledrappier and Sarig, our proof uses a
significantly different approach, one which does not rely on any symbolic
representation of the dynamics at hand. In particular, we give a new proof of Sarig’s
quasi-invariance result for weakly tame surfaces \([21]\) as well as \([11]\). Our
starting point is an insight that we learned from Ratner’s proof of the horocy-
cle measure classification theorem in her expository paper \([18]\) which roughly
states that whenever \(a_{-t}x \approx a_{-t}y\) then the respective ergodic averages at \(x\)
and \(y\) of scale \(e^t\) satisfy

\[
e^{-t(d-1)} \int_{a_{-t}B^U_t a_{-t}} f(ux)du \approx e^{-t(d-1)} \int_{a_{-t}B^U_t a_{-t}} f(uy)du,
\]

where \(B^U_t\) denotes a unit ball around the identity \(e\) in \(U\). We adapt this argument
to suit the infinite measure setting and make crucial use of Hochman’s
multipart parameter ratio ergodic theorem \([3]\).

1.1. **Statement of the Main Results.** Let \(G = \text{SO}^+(d,1)\) be the group
of orientation preserving isometries of hyperbolic \(d\)-space, equipped with a
right-invariant metric. We shall make use of the following subgroups of \(G\):

- \(A = \{ a_t \}_{t \in \mathbb{R}}\) — the Cartan subgroup.
- \(U = \{ u : a_{-t}ua_t \to e \text{ as } t \to \infty \}\) — the unstable horospherical subgroup.
- \(N\) — the stable horospherical subgroup.
- \(K \cong \text{SO}(d)\) — the maximal compact subgroup.
- \(M = Z_K(A)\) — the centralizer of \(A\) in \(K\).

The normalizer of \(U\) in \(G\) is \(N_G(U) = MAU\). Given a discrete subgroup
\(\Gamma \subset G\), the space \(G/\Gamma\) is identified with the frame bundle \(\mathcal{F}\mathbb{H}^d/\Gamma\). The left
action of \(A\) on \(G/\Gamma\) corresponds to the geodesic frame flow on \(\mathcal{F}\mathbb{H}^d/\Gamma\) and the
action of \(M\) corresponds to a rotation of the frames around the direction of
the geodesic.
Given a $U$-ergodic and invariant Radon measure (e.i.r.m.) $\mu$ on $G/\Gamma$ denote by $H_\mu$ the stabilizer in $MA$ of the measure class of $\mu$, i.e.

$$H_\mu = \text{Stab}_{MA}([\mu]) = \{a' \in MA : a'.\mu \sim \mu\}.$$ 

Note that $H_\mu$ is a closed subgroup of $MA$.

**Definition 1.1.** Given any hyperbolic element $\gamma \in G$ define

$$\alpha(\gamma) = a' \in MA$$

whenever the following two conditions hold:

1. $u_0 \gamma = n_0 a' u_0$ for some $u_0 \in U$ and $n_0 \in N$.
2. $\|u'\|_U < \|u\|_U$ for all $u \in U$.

Where $\|\cdot\|_U$ is some fixed bi-invariant metric on the group $U$. Set $\alpha(\gamma) = e$ whenever such $a', n_0, u_0$ do not exist or whenever $\gamma$ is non-hyperbolic (i.e. elliptic or parabolic).

The map $\alpha$ is well defined and may be thought of as a map associating to a hyperbolic element $\gamma$ a specific representative to the action of $\gamma$ on its axis in the hyperbolic frame bundle $F\mathbb{H}^d$. See §3 for further discussion.

We can now state our main theorem:

**Theorem 1.3.** Let $\Gamma < G$ be any discrete subgroup. Let $\mu$ be a $U$-e.i.r.m. on $G/\Gamma$, then

$$\alpha \left( \bigcap_{n=1}^{\infty} \bigcup_{t \geq n} a_{-t} g \Gamma g^{-1} a_t \right) \subseteq H_\mu \text{ for } \mu\text{-a.e. } g \Gamma.$$ 

Given $x = g \Gamma \in G/\Gamma$ denote

$$S_x = \bigcap_{n=1}^{\infty} \bigcup_{t \geq n} a_{-t} g \Gamma g^{-1} a_t.$$ 

Recall that $a_{-t} g \Gamma g^{-1} a_t$ is the stabilizer of the point $a_{-t} x$ in $G/\Gamma$. Hence $S_x$ may be viewed as the set of accumulation points of elements of $\Gamma$ as “seen” from the viewpoint of the geodesic ray $\{a_{-t} x\}_{t \geq 0}$.

Note that the map $x \mapsto S_x$ is $U$-invariant, i.e. $S_{u \cdot x} = S_x$ for any $u \in U$. It is also measurable, as a map into the space of closed subsets of $G$. Hence given an ergodic measure $\mu$, there exists $S_\mu \subset G$ satisfying

$$S_x = S_\mu \text{ for } \mu\text{-a.e. } x \in G/\Gamma.$$ 

Thus an equivalent formulation of Theorem 1.3:

**Theorem (Alternative formulation of Theorem 1.3).** Let $\Gamma < G$ be any discrete subgroup. Let $\mu$ be a $U$-e.i.r.m. on $G/\Gamma$, then $\alpha(S_\mu) \subseteq H_\mu$.

We derive the following condition for $MA$-quasi-invariance:

**Corollary 1.4.** Let $\Gamma$ be any discrete subgroup of $G$ and let $\mu$ be any $U$-e.i.r.m. on $G/\Gamma$. If $S_\mu$ contains a Zariski dense subgroup of $G$ then $\mu$ is $MA$-quasi-invariant.
As in \cite{1, 3, 11, 21}, such quasi-invariance implies a representation of the above invariant measures using $\Gamma$-conformal measures on the boundary $S^{d-1}$ of $\mathbb{H}^d$. Any $U$-e.i.r.m. on $G/\Gamma$ which is $MA$-quasi-invariant lifts to a measure $\tilde{\mu}$ on $G$ presented in $S^{d-1} \times M \times A \times U$ coordinates as:

$$d\tilde{\mu} = e^{3t} dv dm dt du$$

where $\nu$ is a $\Gamma$-conformal measure with parameter $\beta$ on $S^{d-1}$; see §5.2.

1.2. Structure of the Paper. The paper is organized as follows — Section 2 presents some of the main ideas of the proof in an illustrative special case. Sections 3 and 4 contain the proof of Theorem 1.3. In Section 5 we deduce corollaries 1.1, 1.2 and 1.4 from our main result.

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2. Special case of Theorem 1.3

An important ingredient in the proof of Theorem 1.3 is the ratio ergodic theorem for non-singular actions of $\mathbb{R}^n$, proven by Hopf for $n = 1$ and by Hochman \cite{8} for all $n$:

**Theorem 2.1** (Ratio Ergodic Theorem - Hochman). Let $\| \cdot \|$ be any norm on $\mathbb{R}^n$ and let $(u_s)$ be a free, non-singular and ergodic $\mathbb{R}^n$-action on a standard $\sigma$-finite measure space $(X, \mu)$. Given any $f, h \in L^1(X)$ with $\int h d\mu \neq 0$:

$$\frac{\int_{\|s\| < S} f(u_s x) ds}{\int_{\|s\| < S} h(u_s x) ds} \underset{s \to \infty}{\longrightarrow} \frac{\int f d\mu}{\int h d\mu}$$

$\mu$-almost surely.

**Definition 2.1.** A point $x \in G/\Gamma$ is called $\mu$-generic if it satisfies the ratio ergodic theorem for $\mu$ w.r.t all functions in $C_c(G/\Gamma)$, i.e. if

$$\frac{\int_{\|u\| < S} f(u x) du}{\int_{\|u\| < S} g(u x) du} \underset{s \to \infty}{\longrightarrow} \frac{\int f d\mu}{\int g d\mu}$$

for all $f, g \in C_c(G/\Gamma)$ with $\int g d\mu \neq 0$. 

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Note that the separability of $C_c(G/\Gamma)$ implies $\mu$-a.e. $x \in G/\Gamma$ is $\mu$-generic.

A particularly illuminating special case of our main result is the case when for $\mu$-a.e. $x$ the geodesic ray $(a_{-t}x)_{t>0}$ enters arbitrarily small neighborhoods of closed geodesics of some approximate fixed type $a' \in MA \smallsetminus M$. More precisely, whenever for $\mu$-a.e. $x = g\Gamma$ there exist sequences $t_n \to \infty$ and $\gamma_n \in a_{-t_n}g\Gamma g^{-1}a_{t_n}$ satisfying $\gamma_n \to a'$.

At the heart of the proof is the following observation - If the points $a_{-t}x$ and $a_{-t}y$ are close then the ergodic $U$-averages of scale $Ce^t$ around $x$ and $y$ are also close (in an appropriate sense; see Figure 1). For any $f \in C_c(G/\Gamma)$, taking $y = a'x$, arbitrarily large $t$ and using the fact that $a'$ centralizes $a_t$ forces the ergodic averages of $f$ and $f \circ a'$ to be comparable, implying $a'$-quasi-invariance.

**Proposition 2.2** (Special case). Let $\mu$ be a U-e.i.r.m on $G/\Gamma$. Assume that $a' \in \mathcal{S}_\mu \cap (MA \smallsetminus M)$ then $a' \in H_\mu$.

**Lemma 2.3.** Let $a' \in MA$ and $0 < \varepsilon$ with $B^{MA}_\varepsilon(a') \cap H_\mu = \emptyset$. Then for any $0 < \eta$ there exists a function $0 \leq f \in C_c^1(G/\Gamma)$ with $\int f d\mu > 0$ and

$$\frac{\int f \perp d\mu}{\int f d\mu} < \eta$$

where

$$f \perp(x) = \max\{f(v\ell x) : v \in B_\delta^N(e), \; \ell \in B^{MA}_\varepsilon(a')\}$$

for some $\delta = \delta(\eta, f)$.

**Proof.** Given any $\mu$-generic point $x \in G/\Gamma$ and any element $\ell$ in the normalizer of $U$, the point $\ell x$ is $\ell, \mu$-generic. Let $K$ be any compact set of positive $\mu$-measure consisting of $\mu$-generic points, by the above observation we get that the set

$$K_\perp = \bigcup_{\ell \in B^{HA}_\delta(a')} \ell^{-1}K$$

**Figure 1**

For any $f \in C_c(G/\Gamma)$, taking $y = a'x$, arbitrarily large $t$ and using the fact that $a'$ centralizes $a_t$ forces the ergodic averages of $f$ and $f \circ a'$ to be comparable, implying $a'$-quasi-invariance.
satisfies $K_1 \cap K = \emptyset$. Taking $\chi_K \leq f \leq 1$ with $\int (f - \chi_K) d\mu < \frac{\eta}{2} \mu(K)$ and taking $0 < \delta$ sufficiently small ensures the function

$$f_\perp = \max\{v(\ell x) : \ell \in B_\varepsilon^M(a'), \ell \in B_\varepsilon^M(a')\}$$

is very close to $\chi_{K_1}$ and satisfies $\int f_\perp d\mu < \eta \mu(K)$, as required. \qed

Proof of Proposition 2.2. The group $H_\mu$ is closed in $MA$ since

$$H_\mu = \bigcap_{f,g \in C_c(X)} \{\ell \in MA : \ell \mu(f) \cdot \mu(g) = \mu(f) \cdot \ell \mu(g)\}.$$

Assume by contradiction that $a' \notin H_\mu$, then there exists an $0 < \varepsilon$ for which $B_\varepsilon^M(a') \cap H_\mu = \emptyset$. The groups $U$ and $\mathbb{R}^{d-1}$ are isomorphic as Lie groups via a map $s \mapsto u(s)$. Conjugation by $a'$ induces a similarity map $kO$ on $\mathbb{R}^{d-1}$, where $O \in SO(d-1)$ and $0 < k$. Both $S_\mu$ and $H_\mu$ are closed under inversion hence we may assume without loss of generality that

$$(a')^{-1} u(s) a' = u(kO s)$$

is contracting, i.e $k < 1$.

Let $f, f_\perp$ be the functions constructed in Lemma 2.3 for $\eta < \frac{\varepsilon}{3}$ and $\varepsilon$. Let $x = g\Gamma$ be a $\mu$-generic point with $a' \in S_\mu$. By the definition of $S_\mu$ there exist sequences $t_n \to \infty$ and $\gamma_n \to a'$ for which

$$\gamma_n(a_{-t_n}g) = (a_{-t_n}g)^{-1} \gamma_n(a_{-t_n}g) \in \Gamma.$$ (2.1)

Note that all $s \in \mathbb{R}^{d-1}$ satisfy

$$u(s)\gamma_n \to u(s) a' = a'u(kO s) \in NMAU$$

where the set $NMAU$ is open in $G$. Let $B$ be the open unit ball in $\mathbb{R}^{d-1}$ (the specific norm is irrelevant), one can define smooth functions

$$\varphi_n : B \to \mathbb{R}^{d-1}, v_n : B \to N \text{ and } \ell_n : B \to MA$$

satisfying for all large $n$ that

$$u(s)\gamma_n = v_n(s)\ell_n(s)u(\varphi_n(s))$$

and consequently

$$u(\ell_n s)g\Gamma = a_{t_n}u(s)a_{-t_n}g\Gamma \overset{2.1}{=} a_{t_n}(u(s)\gamma_n)a_{-t_n}g\Gamma =
\ell_n v_n(s)\ell_n(s)u(\varphi_n(s))\ell_n(s)u(\ell_n \varphi_n(s))g\Gamma.$$ (2.2)

The map $v_n$ uniformly tends to $e$ on $B$ while the conjugation $a_{t_n}v_n(s)a_{-t_n}$ further contracts this term. Hence

\begin{align*}
d_G(\ell_n s)x, a' u(\ell_n \varphi_n(s))x &< \min\{\delta, \varepsilon\}
\end{align*}

for all large $n$, where $\delta$ is the constant in the construction of $f_\perp$ in Lemma 2.3. This implies the inequality

$$f(u(\ell_n s)x) \leq f_\perp(u(\ell_n \varphi_n(s)))$$

for all $s \in B$. 7
Note that while contracting the errors in $NMA$, conjugation by $a_{tn}$ expands the $U$-direction. In particular, it is not necessarily true that $e^{tn}\varphi_n$ is pointwise close to $e^{tn}kO$. Despite of this, integration over all of $B$ under these two changes of variables is in fact comparable. More precisely, since $\varphi_n \to kO$ uniformly on $B$ we know

$$|\text{Jac}_s(\varphi_n)| > \frac{1}{2}k$$

and

$$\varphi_n(B) \subset B$$

for all large $n$, where $\text{Jac}_s(h)$ denotes the Jacobian of the function $h$ at the point $s$. This implies

$$\int_{e^{tn}B} f(u(s)x)ds = e^{tn} \int_B f(u(e^{tn}s)x)ds \leq e^{tn} \int_B f_\perp(u(e^{tn}\varphi_n(s))x)ds \leq 2k^{-1} \int_{e^{tn}\varphi_n(B)} f_\perp(u(s)x)ds \leq 2k^{-1} \int_{e^{tn}B} f_\perp(u(s)x)ds.$$

In conclusion, we have shown that for all large $n$

$$1 = \frac{\int_{|s|<e^{tn}} f(u(s)x)ds}{\int_{|s|<e^{tn}} f(u(s)x)ds} \leq 2k^{-1} \frac{\int_{|s|<e^{tn}} f_\perp(u(s)x)ds}{\int_{|s|<e^{tn}} f(u(s)x)ds}$$

which is in contradiction to the ratio ergodic theorem and the choice of $\eta$. □

3. The Horosphere Reparameterization Map

Let $P = NMA$ be a minimal parabolic subgroup of $G$. Let $\pi : G \to P\backslash G$ be the projection onto $P\backslash G \cong M\backslash K \cong S^{d-1}$. Identifying $G$ with $\mathcal{F}\mathbb{H}^d$, the frame bundle of hyperbolic $d$-space, this quotient map corresponds to the projection of frames onto the limit points in $\partial\mathbb{H}^d$ they tend to under the geodesic flow $a_t$ as $t \to +\infty$.

Let $\omega \in G$ be any representative of the non-trivial coset in the Weyl group of $G$, i.e $e \neq [\omega] \in W = C_G(A)\backslash N_G(A) \cong \mathbb{Z}_2$ and $P\omega \neq P$. Using the fact that $\omega N\omega = U$ one can reformulate the Bruhat decomposition of $G$ as

$$G = G\omega = (P\omega P \cup P)\omega = PU \cup P\omega$$

where $PU$ is a Zariski open set in $G$. Note that $U \cap P = \{e\}$ implying that the restricted function

$$\pi|_U : U \to (P\backslash G) \setminus \{P\omega\}$$
is a bijection. Any element $\gamma \in G$ acts by right multiplication on $P \setminus G \cong S^{d-1}$ as a Möbius transformation. This action commutes with $\pi$ therefore any horosphere $U \gamma$ projects injectively via $\pi$ onto $(P \setminus G) \setminus \{P \omega \gamma \}$. Let 

$$v_\gamma = (\pi|_U)^{-1}(P \omega \gamma^{-1})$$

be the point in $U$ satisfying $\pi(v_\gamma) = P \omega$, whenever such point exists.

Denote by $\iota : N \times MA \times U \rightarrow G$ the multiplication map $(n, ma, u) \mapsto nmau$. This map is an injective diffeomorphism onto its image $PU$.

Let $\Phi = (\Phi_N, \Phi_{MA}, \Phi_U)$ be the following smooth map 

$$\Phi : G \times U \setminus \{(g, v_g)\}_{g \in G} \rightarrow N \times MA \times U$$

defined by 

$$\Phi(\gamma, u) = \iota^{-1}(u \gamma).$$

Note that for any point $g \Gamma \in G/\Gamma$, any element $\gamma' \in \Gamma$ and corresponding $\gamma = g \gamma' g^{-1} \in \text{Stab}_G(g \Gamma)$, the following identity holds for all $u \in U$ in the domain of definition of $\Phi$:

$$u g \Gamma = u \gamma g \Gamma = \Phi_N(\gamma, u) \Phi_{MA}(\gamma, u) \Phi_U(\gamma, u) g \Gamma.$$ 

Therefore $\Phi$ may be viewed as a reparameterization map for the horospherical orbit $Ug \Gamma$ in $G/\Gamma$.

Corresponding to any hyperbolic (loxodromic) $\gamma \in G$ there exist two distinct fixed points for the action of $\gamma$ on $P \setminus G$ by $Pg \mapsto Pg \gamma$, one attracting fixed point and one repelling fixed point. Let $P \gamma^{-1}$ be the attracting fixed point of $\gamma$, then the map $\gamma \mapsto P \gamma^{-1}$ defined on the open subset of hyperbolic elements in $G$ is continuous. For any hyperbolic $\gamma$ with $P \gamma^{-1} \neq P \omega$ denote 

$$u_\gamma = (\pi|_U)^{-1}(P \gamma^{-1})$$

Since $\pi(u_\gamma) = \pi(u_\gamma)$ one has $\Phi_U(\gamma, u_\gamma) = u_\gamma$ and

$$u_\gamma \gamma = \Phi_N(\gamma, u_\gamma) \Phi_{MA}(\gamma, u_\gamma) u_\gamma$$

in which case 

$$\alpha(\gamma) = \Phi_{MA}(\gamma, u_\gamma)$$

as in Definition [1].

Whenever $P \gamma^{-1} = P \omega$ then $P \gamma^{-1} \neq P \omega$, therefore given any hyperbolic element $\gamma$, amongst $\alpha(\gamma)$ and $\alpha(\gamma^{-1})$ at least one is non-trivial.

The element $\alpha(\gamma) \in MA$ is a conjugate of $\gamma$ representing the action along its axis in $\mathbb{H}^d$. Indeed, [1] implies

$$3n \in N \quad \alpha(\gamma) = (nu_\gamma) \gamma(nu_\gamma)^{-1}.$$ 

Fix some bi-invariant metrics on $U, N$ and $MA$ and denote $B^U_r, B^N_r, B^MA_r$ the open balls of radius $r$ around the identity in these groups respectively. As before, let $\text{Jac}_u(F)$ denote the Jacobian of the function $F$ at the point $u$.

**Lemma 3.1.** Let $\gamma_0$ be any hyperbolic element of $G$. Let $P \gamma_0$ and $P \gamma_0^+$ be the attracting and repelling fixed points for $\gamma_0$ in $P \setminus G$ respectively. If $P \gamma_0^- \neq P \omega$ and $P \gamma_0^+ \neq P$ then for any $\varepsilon$ there exist:
• an open neighborhood $V_0 \subset G$ of $\gamma_0$
• constants $k \in \mathbb{N}$, $0 < c$, $0 < R$
• balls $D$ and $I$ around $e$ in $U$
such that for any $\gamma \in V_0$:

1. the map $\Phi(\gamma^k, \cdot)$ is defined and smooth on $D$
2. the map $\Phi(\cdot, \cdot)$ is defined and smooth on $E = I \cdot \Phi_U(\gamma^k, D)$
3. $B_r^U \subseteq \Phi_U(\gamma^k, B_r^U) \cdot (\Phi_U(\gamma^k, e))^{-1} \subseteq B_r^U$ for all $r > 0$ satisfying $B_r^U \subseteq D$
4. $\forall u \in E \forall v \in D \ c < |\text{Jac}_u(\Phi_U(\gamma, \cdot))| < 1$ and $c < |\text{Jac}_v(\Phi_U(\gamma^k, \cdot))| < 1$
5. $\Phi(\gamma^k, D) \subseteq B_R^N \times B_R^{MA} \times B_R^U$
6. $\Phi(\gamma, E) \subseteq B_R^N \times B_R^{MA} \times B_R^U$

and moreover:

7. $\Phi_{MA}(\gamma, E) \subseteq B_{\varepsilon}^{MA}(\alpha(\gamma_0))$, where $B_{\varepsilon}^{MA}(\alpha(\gamma_0))$ denotes the ball of radius $\varepsilon$ around $\alpha(\gamma_0)$ in $MA$
8. $\forall v \in D \ \Phi_U(\gamma, I \cdot \Phi_U(\gamma^k, v)) \subseteq I \cdot \Phi_U(\gamma^k, v)$

Proof. Use the map $\pi|_U$ to induce a metric $d_\pi$ onto $(P\backslash G) \backslash \{P\omega\}$ so that for any $Pg \neq P\omega$ and $0 < r$,

$$B_{d_\pi}^r(Pg) = \pi(B_r^U((\pi|_U)^{-1}(Pg))).$$

Figure 2. Lemma 3.1 in the hyperbolic ball model. Here we identify $P\backslash G$ with $\partial \mathbb{H}^3 = S^2$, and the subgroup $U$ with the image of $Ue$ under the projection $G \to T^1 \mathbb{H}^3$ to the unit tangent bundle of $\mathbb{H}^3$. 

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Let $R : G \subset P^1(G)$ be the smooth (contravariant) action on $P^1(G)$ by right multiplication $R_\gamma : Pg \mapsto Pg\gamma$. Since $R_\gamma$, contract points near $k_{\gamma_0}$, there exist $0 < r_1 < r_2$ and a neighborhood $V_1 \subset G$ of $\gamma_0$, satisfying for every $\gamma \in V_1$

$$R_\gamma(B_{r_2}^{d+}(Pk_{\gamma_0}^-)) \subseteq B_{r_1}^{d+}(Pk_{\gamma_0}^-).$$

Moreover, we can choose $r_2$ small enough so that

$$P\omega^{-1} \notin B_{r_2}^{d+}(Pk_{\gamma_0}^-).$$

Since $P\gamma_0^* Pk_{\gamma_0}^+ \neq P$,

$$P\gamma_0^* \rightarrow Pk_{\gamma_0}^-$$ and $P\omega^{-1} \rightarrow Pk_{\gamma_0}^+$ as $\ell \rightarrow \infty$.

Hence there exist $0 < \rho, k$ and a neighborhood $V_0 \subset V_1$ of $\gamma_0$ satisfying for every $\gamma \in V_0$ and every $Pg \in B_{\rho}^{d+}(P)$

$$B_{r_1}^{d+}(Pk_{\gamma_0}^-) \subseteq B_{(r_2+r_1)/2}^{d+}(R_\gamma k(Pg)) \subseteq B_{r_2}^{d+}(Pk_{\gamma_0}^-)$$

and

$$P\omega^{-k} \notin B_{r_2}^{d+}(P).$$

This implies in particular

$$R_\gamma \left( B_{(r_2+r_1)/2}^{d+}(R_\gamma k(Pg)) \right) \subseteq B_{(r_2+r_1)/2}^{d+}(R_\gamma k(Pg)).$$

Let $D = B_\rho^{d+}$ and $I = B_{(r_2+r_1)/2}$. For all $\gamma \in V_0$ the maps $\Phi(\gamma^k, \cdot)$ and $\Phi(\gamma, \cdot)$ are defined on $\overline{D}$ and

$$I \cdot (\pi|_U)^{-1} \left( R_\gamma k \left( B_{\rho}^{d+}(P) \right) \right)$$

respectively. Notice that the maps $\Phi_U(\gamma^k, \cdot)$ and $R_\gamma$ are conjugate since

$$\Phi_U(\gamma^k, \cdot) = (\pi|_U)^{-1} \circ R_\gamma \circ (\pi|_U)$$

for any $\gamma \in G$. Item (8) is directly implied from (3.3) using the conjugation relation above. Continuity of the map $\Phi$ implies item (7), after further restriction of $V_0$ if necessary. After possibly increasing $k$, items (3) and (4) follow from the fact that the maps $\Phi_U(\gamma^k, \cdot)$ and $\Phi_U(\gamma, \cdot)$ have non-vanishing Jacobians on the compact sets

$$\overline{D} \text{ and } I \cdot \overline{\Phi_U(\gamma^k, D)}$$

by (3.4), and are contracting.

\[ \square \]

4. Proof of Theorem 1.3

Given an element $g \in G$ or a set $F \subset G$ we denote $g^{au} := a_1 g a_{-1}$ and similarly $F^{au} := a_1 F a_{-1}$, for any $t \in \mathbb{R}$. Recall $B_r^u$ denotes the ball of radius $r$ around $e$ in $U$ w.r.t. some fixed bi-invariant norm. Similarly for $B^N_r$ and $B^MA_r$.

Given functions $h_1, h_2 \in L^1(G/\Gamma)$, a point $x \in G/\Gamma$ and a bounded measurable set $E \subset U$ denote:

$$\mathcal{R}(h_1, h_2, x, E) = \frac{\int_E h_1(ux)du}{\int_E h_2(ux)du}$$

where

$$\int_E h_1(ux)du = \int_{\mathbb{R}} h_1(ux)du.$$
Hence, since \( a_t \)-normalizes \( U \), and expands \( U \) for \( t > 0 \), the ratio ergodic theorem may be stated as follows: fixing \( B = B_r^U \),
\[
\lim_{t \to \infty} \mathcal{R}(h_1, h_2, x, B^{a_t}) = \frac{\int h_1 d\mu}{\int h_2 d\mu} \quad \text{for } \mu\text{-a.e. } x \in G/G,
\]
whenever \( \int h_2 d\mu \neq 0 \).

We now begin the proof of Theorem 1.3. Let \( \gamma_0 \in S_\mu \) be a hyperbolic element for which \( \alpha(\gamma_0) \in MA \) is well defined, i.e. \( Pk^-_{\gamma_0} \neq P_\omega \) where \( Pk^-_{\gamma_0} \) is the attracting fixed point of \( \gamma_0 \) in \( P \backslash G \). Without loss of generality we may assume the repelling fixed point admits \( Pk^+_{\gamma_0} \) in \( MA \). Indeed, whenever \( Pk^+_{\gamma_0} = P \) then \( \gamma_0 \in P \) and consequently \( \alpha(\gamma_0)^{-1} = (\alpha(\gamma_0))^{-1} \) in which case we may replace \( \gamma_0 \) by \( \gamma_0^{-1} \).

We need to show \( \alpha(\gamma_0) \in H_\mu \). Assume otherwise by contradiction. Then there exists an \( \varepsilon > 0 \) for which \( B_{\varepsilon}^{MA}(\alpha(\gamma_0)) \), the \( \varepsilon \)-neighborhood of \( \alpha(\gamma_0) \) in \( MA \), is disjoint from \( H_\mu \). Let \( V_0, D, I, k, c, R \) be the sets and constants chosen in Lemma 3.1 for \( \varepsilon \) and \( \gamma_0 \).

Fix some \( 0 < \eta_1 < \varepsilon/2 \). By Lemma 2.3 there exists a function witnessing the singularity \( \ell, \mu \perp \mu \) for all \( \ell \in B_{\varepsilon}^{MA}(\alpha(\gamma_0)) \), more precisely there exists \( f \in C_c(G/G) \) and a constant \( \delta > 0 \) for which the function
\[
f_\perp(x) = \max\{f(\ell x) : \ell \in B_{\varepsilon}^{MA}(\alpha(\gamma_0))\}
\]
satisfies
\[
\frac{\int f_\perp d\mu}{\int f d\mu} < \eta_1.
\]

Fix some open precompact set \( \Omega \subset G/G \) with \( \mu(\Omega) > 0 \) and let \( \tilde{\Omega} \) be any compact set containing a neighborhood of \( \bigcup_{\ell \in B_{\varepsilon}^{MA}} \ell^{-1}\Omega \).

Let \( \eta_2 > 0 \) be sufficiently small, to be made explicit later. By the ratio ergodic theorem we may choose a constant \( T_1 > 0 \) large enough so that
\[
\tilde{\Omega}_1 = \left\{ x \in \tilde{\Omega} \left| \mathcal{R}(f_\perp, f, x, I^{a_t}) < \eta_1 \right. \forall t \geq T_1 \right\}
\]
satisfies
\[
\frac{\mu(\tilde{\Omega} \setminus \tilde{\Omega}_1)}{\mu(\tilde{\Omega})} < \frac{1}{4}\eta_2.
\]

We additionally require that \( T_1 \) be large enough so that for any \( n \in B_{R_{\mu}}^N \) and \( \ell \in B_{R_{\mu}}^{MA} \)
\[
(n^{a_{T_1}} \cdot \ell)^{-1}\Omega \subseteq \tilde{\Omega}
\]
and
\[
n^{a_{T_1}} \in B_{\delta}^N
\]
with \( R \) as in Lemma 3.1.
Since $\mathcal{S}_\mu \cap V_0 \neq \emptyset$ there exists some $T_2 > T_1$ for which the set

$$\tilde{\Omega}_2 = \left\{ x = g\Gamma \in \tilde{\Omega}_1 \mid \bigcup_{t \in [T_1, T_2]} a_{-t}g\Gamma g^{-1}a_t \cap V_0 \neq \emptyset \right\}$$

satisfies

$$\frac{\mu(\tilde{\Omega} \setminus \tilde{\Omega}_2)}{\mu(\tilde{\Omega})} < \min \left\{ \frac{\eta_2}{2}, \frac{\mu(\Omega)}{4\mu(\Omega)} \right\}.$$

This condition together with the ratio ergodic theorem ensures the existence of a point $x_0 \in G/\Gamma$ and a constant $L > e^{\eta_2}$ satisfying

$$\mathcal{R}(\chi_{\tilde{\Omega}_2}, \chi_{\tilde{\Omega}}, x_0, B_L^U) > 1 - \eta_2$$

$$\mathcal{R}(\chi_{\Omega \cap \tilde{\Omega}_2}, \chi_{\Omega \cap \tilde{\Omega}}, x_0, B_L^U) > \frac{\mu(\Omega)}{2\mu(\Omega)}$$

A key component of Hochman’s proof of the ratio ergodic theorem is a multi-dimensional variant of the Chacon-Ornstein Lemma [8, Thm. 1.2] stating that for any fixed $r > 0$:

$$\int_{\tilde{\Omega}_r} h_1 (ux) du$$

$$\int_{B_R^U} h_2 (ux) du \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

where $\tilde{\Omega}_r = \tilde{\Omega} \setminus \tilde{\Omega}_2$ and $h_1, h_2 \in L^\infty(\mu) \cap L^1(\mu)$ with $\int h_2 d\mu \neq 0$. We may thus choose $L$ large enough so as to additionally satisfy

$$\int_{\tilde{\Omega}_r} \chi_{\Omega \cap \tilde{\Omega}_2} (ux_0) du < \frac{1}{2} \int_{B_L^U} \chi_{\Omega \cap \tilde{\Omega}_2} (ux_0) du$$

where $R' = R + \text{radius}(D)$.

In what follows, we shall focus our attention to the orbit $\{ux_0\}_{u \in B_L^U}$ of $x_0$. Let $m_U$ denote Haar measure on $U$. Given a set $E$ in $G$ we denote

$$O_E = \{ u \in B_L^U \mid ux_0 \in E \}.$$

Sets of particular interest are $O_{\Omega}$, $O_{\tilde{\Omega}}$ and $O_{\tilde{\Omega}_2}$.

Assumption (4.4) implies that out of the time in which $x_0$’s $U$-orbit spends inside $\tilde{\Omega}$, close to a full proportion of that time is spent inside the set $\tilde{\Omega}_2$, i.e. $m_U(O_{\tilde{\Omega}} \setminus O_{\tilde{\Omega}_2})/m_U(O_{\tilde{\Omega}})$ is small (less than $\eta_2$). The heart of the proof is an argument employing the maps $\Phi_U(\gamma^k_u, \cdot)$ to “push” an impossible proportion of $O_{\tilde{\Omega}}$ into $O_{\tilde{\Omega}_2} \setminus O_{\tilde{\Omega}}$.

We show there exists a constant $C_0 > 0$ independent of $\eta_2$ for which

$$m_U(O_{\tilde{\Omega}} \setminus O_{\tilde{\Omega}_2}) \geq C_0 \cdot m_U(O_{\Omega} \cap O_{\tilde{\Omega}_2})$$

implying

$$\eta_2 \geq \frac{\eta_2}{2} = \frac{m_U(O_{\tilde{\Omega}} \setminus O_{\tilde{\Omega}_2})}{m_U(O_{\tilde{\Omega}})} \geq C_0 \frac{m_U(O_{\Omega} \cap O_{\tilde{\Omega}_2})}{m_U(O_{\tilde{\Omega}})} \geq C_0 \frac{\mu(\Omega)}{2\mu(\Omega)}.$$

Taking $\eta_2$ small enough yields the desired contradiction.
In order to prove (4.8) we construct a family of open sets contained in $O_{\tilde{\Omega}} \setminus O_{\tilde{\Omega}_2}$ and use a covering argument to bound the total mass of these sets.

Fix $g_0 \in G$ for which $x_0 = g_0 \Gamma$. By definition, for any $u_0 \in O_{\tilde{\Omega}_2}$ there exists $t_{u_0} \in [T_1, T_2]$ and $\gamma_{u_0} \in G$ with

$$
\gamma_{u_0} \in \left( a^{-t_{u_0}} (u_0 g_0) \Gamma (u_0 g_0)^{-1} a_{t_{u_0}} \right) \cap V_0
$$

with $V_0 \subset G$ as in Lemma 3.1. To the element $u_0 \in O_{\tilde{\Omega}_2}$ we associate the following auxiliary function $\Psi_{u_0} : D \to U$ defined by

$$
\Psi_{u_0}(v) = (\Phi_U (\gamma_{u_0}^k, v))^{a_{t_{u_0}}} u_0.
$$

**Lemma 4.1.** For all $u_0 \in O_{\tilde{\Omega}_2}$

$$
\Psi_{u_0}(D) \cap O_{\tilde{\Omega}_2} = \emptyset.
$$

The proof of this lemma follows along the lines of the proof of Proposition 2.2 given in §2. Recall that points in $O_{\tilde{\Omega}_2}$ have a “good” ratio ergodic average, in the sense of (4.1). In Proposition 2.2 we have shown that given a $\mu$-generic point for which its geodesic past at time $t$ is close to a closed geodesic of some approximate type, we may reparameterize its horospherical orbit to deduce it has a “bad” ratio ergodic average over a ball in $U$ of size $e^t$. In the case where $t$ may be taken to be arbitrarily large we derive a direct contradiction to the ratio ergodic theorem.

The underlying reason that was implicitly behind the argument establishing that the ratio ergodic average on a ball of size $e^t$ is “bad” in this case is the fact that $\Phi_U(\gamma_{u_0}, \cdot)$ maps a ball around the identity into itself whenever $\gamma_{u_0}$ is sufficiently close to a contracting normalizing element of $U$. For a general $\gamma_{u_0}$ this is not necessarily true.

By the way we have set things up, balls centered at points in $\Psi_{u_0}(D)$ are mapped into a subset of themselves by $\Phi_U(\gamma_{u_0}, \cdot)$, in which case “bad” ratio ergodic averages are implied, that is to say these points are not contained in $O_{\tilde{\Omega}_2}$, as claimed in (4.10).

**Figure 3.** Elements in Lemma 4.1 associated to a point $u_0 \in O_{\tilde{\Omega}_2}$. 

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Proof of Lemma 4.1. Fix $u_0 \in O_{\tilde{\Omega}_2}$ and set $\gamma = \gamma_{u_0}$ and $t = t_{u_0}$. By item (1) of Lemma 3.1 the map $\Phi(\gamma^k, \cdot)$ is well defined on $D$. Fix $v \in D$ and denote

$$w = \Psi_{u_0}(v) = (\Phi_U(\gamma^k, v))^a_t u_0.$$  

We would like to prove $w \notin O_{\tilde{\Omega}_2}$. To that end, it would suffice to show

$$\mathcal{R}(f_\perp, f, wx_0, I^{a_t}) > \eta_1.$$  

By (4.9), the element

$$\gamma' = (u_0g_0)^{-1}a_t \gamma a_{-t}(u_0g_0)$$

is contained in $\Gamma$. For any $v$ in the domain of definition of $\Phi_U(\gamma, \cdot)$ we have

$$v^{a_t} u_0 g_0 \gamma' = a_t v a_{-t} u_0 g_0 \gamma' = a_t (v \gamma) a_{-t} u_0 g_0 =$$

$$= (\Phi_N(\gamma, v) \cdot \Phi_{MA}(\gamma, v) \cdot \Phi_U(\gamma, v)) a_{-t} u_0 g_0 =$$

$$= (\Phi_N(\gamma, v))^a_t \cdot \Phi_{MA}(\gamma, v) \cdot (\Phi_U(\gamma, v))^a_t \cdot u_0 g_0.$$

Denote

$$v' = \Phi_U(\gamma^k, v).$$

By item (2) of Lemma 3.1 the map $\Phi(\gamma, \cdot)$ is well defined on $I \cdot v'$. Hence for any $u \in I$ and $w$ as in (4.11), equation (4.12) implies

$$u^{a_t} w x_0 = (uv')^{a_t} u_0 x_0 =$$

$$= (\Phi_N(\gamma, uv'))^{a_t} \cdot \Phi_{MA}(\gamma, uv') \cdot (\Phi_U(\gamma, uv'))^{a_t} \cdot u_0 x_0.$$  

Lemma 3.1 ensures the different terms of (4.13) are well controlled. Item (6) states that $\Phi_N(\gamma, uv') \in B^N_R$ hence by the choice of $T_1$ in (4.3) we are ensured that conjugation by $a_t$ admits

$$(\Phi_N(\gamma, uv'))^{a_t} \in B^N_\delta.$$  

On the other hand item (7) of Lemma 3.1 gives that

$$\Phi_{MA}(\gamma, uv') \in B^MA_\varepsilon(\alpha(\gamma_0)).$$  

Recall the definition of the function $f_\perp$:

$$f_\perp(x) = \max\{f(v\ell x) : v \in B_\delta^N(e), \ell \in B^MA_\varepsilon(\alpha(\gamma_0))\}.$$  

We thus see that

$$f(u^{a_t} wx_0) \leq f_\perp ((\Phi_U(\gamma, uv'))^{a_t} \cdot u_0 x_0).$$

In order to estimate the ratio ergodic average along $I^{a_t} wx_0$ we need some control over $\Phi_U$. Item (8) of Lemma 3.1 applied to the point $v'$ states that

$$\Phi_U(\gamma, Iv') \subseteq Iv'$$

and item (4) states that

$$c < |\text{Jac}_{uv'}(\Phi_U(\gamma, \cdot))|.$$
All of the above support the following calculation:

\[
\int_I f(u^{a_t}w x_0)du \leq \int_I f_\perp \left( (\Phi_U(\gamma, uv'))^{a_t} \cdot u_0 x_0 \right) du \leq \\
\leq c^{-1} \int_{\Phi_U(\gamma, I \gamma')} f_\perp(u^{a_t} u_0 x_0)du \leq \\
\leq c^{-1} \int_{I \gamma'} f_\perp(u^{a_t} u_0 x_0)du = \\
\leq c^{-1} \int_{I} f_\perp(u^{a_t} w x_0)du
\]

hence

\[
1 = R(f, f, w x_0, I^{a_t}) \leq c^{-1} \cdot R(f_\perp, f, w x_0, I^{a_t}).
\]

Since \( \eta_1 < \nicefrac{\epsilon}{2} \) this implies \( w \notin \tilde{O}_2 \) as required. \( \square \)

As a corollary of the lemma above we obtain the following:

**Lemma 4.2.** For any \( u_0 \in \tilde{O}_2 \)

\[(4.17) \quad \Psi_{u_0} \left( D \cap (O_{\Omega} u_0^{-1})^{a_t} \right) \subseteq \tilde{O}_1 \setminus \tilde{O}_2 \]

where \( t = t_{u_0} \) as above.

**Proof.** Let \( v \in D \cap (O_{\Omega} u_0^{-1})^{a_t} \). By the definition of \( O_{\Omega} \) we have

\[v^{a_t} u_0 x_0 \in O_{\Omega} x_0 \subset \Omega.\]

On the other hand, as in (4.12), we know

\[(4.18) \quad v^{a_t} u_0 x_0 = (\Phi_N(\gamma^k, v))^{a_t} \cdot \Phi_{MA}(\gamma^k, v) \cdot (\Phi_U(\gamma^k, v))^{a_t} u_0 x_0\]

therefore

\[\left( \Phi_U(\gamma^k, v) \right)^{a_t} u_0 x_0 \in \left( (\Phi_N(\gamma^k, v))^{a_t} \cdot \Phi_{MA}(\gamma^k, v) \right)^{-1} \Omega.\]

Denote

\[n_v = \Phi_N(\gamma^k, v) \quad \text{and} \quad \ell_v = \Phi_{MA}(\gamma^k, v).\]

By item (5) in Lemma 3.1 we have \( n_v \in B^N_R, \ell_v \in B^{MA}_R \). The construction of \( \hat{\Omega} \) and the choice of \( T_1 \) in (4.2) ensure \( (n^\alpha_v \cdot \ell_v)^{-1} \Omega \subseteq \hat{\Omega} \). Hence we may conclude

\[\Psi_{u_0}(v) \in O_{\tilde{\Omega}}.\]

Since \( \Psi_{u_0} \left( D \cap (O_{\Omega} u_0^{-1})^{a_t} \right) \) is contained in \( \Psi_{u_0}(D) \) we deduce by Lemma 4.1 it is disjoint from \( O_{\tilde{O}_2} \), implying the claim. \( \square \)

We have thus shown that to every point in \( \tilde{O}_2 \) corresponds an open set contained in \( \tilde{O}_1 \setminus \tilde{O}_2 \). Our aim is to show these sets add up to a non-negligible \( m_U \) measure. This is the content of the following covering argument.

Given any \( u_0 \in \tilde{O}_2 \) we define

\[J_{u_0} = \Psi_{u_0} \left( D^{a_t}(\gamma^k) \cap (O_{\Omega} u_0^{-1})^{a_t} \right).\]
Note that $D^{a^{(t_2)}} \subset D$, since $c \in (0, 1)$, and therefore
\[ J_{u_0} \subseteq \Psi_{u_0} \left( D \cap (O_\Omega u_0^{-1})^{a-t_{u_0}} \right) \subseteq O_{\hat{\Omega}} \setminus O_{\hat{\Omega}_2}. \]

**Lemma 4.3.** There exists a constant $C_0 > 0$, depending only on the group $U$, the set $D$ and the constants $c$ and $R$ of Lemma 3.1, satisfying
\[ m_U \left( \bigcup_{u_0 \in O_{\hat{\Omega}_2}} J_{u_0} \cap B_{L}^{U} \right) > C_0 \cdot m_U(O_\Omega \cap O_{\hat{\Omega}_2}). \]

Note that the set on the left hand side is open in $U$ and hence measurable.

**Proof.** Recall the set $D$ is a ball in $U$ around the identity, for convenience we will use the metric notation $D = B^{U}_0$ throughout the proof. Also recall
\[ (B^U_r)^{a_s} = a_s B^U_r a_{-s} = B^U_{r e^s} \]
for any $r > 0$ and $s \in \mathbb{R}$.

Set $R' = R + \rho$. Consider the following open cover of $(O_\Omega \cap O_{\hat{\Omega}_2}) \cap B_{R'}^{U} :$
\[ \mathcal{F} = \left\{ B^U_{r e^{t_{u_0}}} : u_0 \in (O_\Omega \cap O_{\hat{\Omega}_2}) \cap B_{R'}^{U} \right\}. \]

By the Besicovitch covering theorem [12, Theorem 2.7] there exists a constant $P$, independent of $\mathcal{F}$ and $(O_\Omega \cap O_{\hat{\Omega}_2}) \cap B_{R'}^{U}$, and a countable sub-cover $\mathcal{F}' \subset \mathcal{F}$ of balls centered at $F = \{u_1, u_2, ...\}$ for which
\[ \sum_{B' \in \mathcal{F}'} \chi_{B'} \leq P. \]

We will focus our attention to the associated collection of sets $\{J_{u_i}\}_{u_i \in F}$. For any $u_i \in F$ choose $t_i \in [T_1, T_2]$ and $\gamma_i \in G$ satisfying
\[ \gamma_i \in (a-t_i u_0 g)\Gamma (u_0 g)^{-1} a_t_i) \cap V_0. \]

Item (3) in Lemma 3.1 gives that
\[ B_{c r'} \cdot \Phi_U (\gamma_i, e) \subseteq \Phi_U (\gamma_i, B_{c r'}^{U}) \subseteq B_{c r'} \cdot \Phi_U (\gamma_i, e) \]
for all $0 < r < \rho$.

Recall the explicit definitions of $\Psi_{u_i}$ and $J_{u_i}$:
\[ \Psi_{u_i} (\cdot) = \left( \Phi_U (\gamma_i, \cdot) \right)^{a t_i} u_i \]
\[ J_{u_i} = \left( \Phi_U (\gamma_i, B_{c r'}^{U} \cap (O_\Omega u_i^{-1})^{-a_{-t_i}}) \right)^{a t_i} u_i. \]

If we denote
\[ w_i = \Psi_{u_i} (e) = \left( \Phi_U (\gamma_i, e) \right)^{a t_i} u_i. \]

then the bounds given in (4.20) imply
\[ J_{u_i} \subseteq B_{c r'}^{U} w_i \]
and
\[ B_{c r'}^{U} w_i \subseteq \Psi_{u_i} (D) \subseteq B_{c r'}^{U} w_i. \]
Item (5) of Lemma 3.1 states, in particular, that $\Phi_U(\gamma_{u_i}, e) \in B^U_{R}$ implying $w_i \in B^U_{R_{e^t_i}} u_i$. Therefore

$$B^U_{cpe^t_i} w_i \subseteq \Psi_{u_i}(D) \subseteq B^U_{R_{e^t_i}} u_i$$

where $R' = R + \rho$ as defined earlier.

We would like to bound $\sum_{u_i \in F} \chi_{J_{u_i}}$. Set

$$\beta = \frac{c\rho}{2R'} < 1.$$ 

We claim that given any $u_i, u_j \in F$ if $J_{u_i} \cap J_{u_j} \neq \emptyset$ then

$$\beta \leq e^{t_j - t_i} \leq \beta^{-1}$$

and

$$d_U(u_i, u_j) < 2\beta^{-1}R' e^{t_i}.$$ 

Indeed, if $\beta \leq e^{t_j - t_i} \leq \beta^{-1}$ and $\|u_i - u_j\|_U \geq 2\beta^{-1}R' e^{t_i}$ then

$$d_U(u_i, u_j) \geq \max\{2R' e^{t_i}, 2R' e^{t_j}\}$$

and by (4.22) we are ensured that $\Psi_{u_i}(D) \cap \Psi_{u_j}(D) = \emptyset$ and consequently also $J_{u_i} \cap J_{u_j} = \emptyset$. On the other hand, if $e^{t_j - t_i} < \beta$ and $v \in J_{u_i} \cap J_{u_j}$ then by (4.21) and (4.22) we have

$$d_U(u_j, w_i) \leq d_U(u_j, v) + d_U(v, w_i) < R' e^{t_j} + \frac{1}{2}cpe^{t_i}$$

$$< \beta R' e^{t_j} + \frac{1}{2}cpe^{t_i} = cpe^{t_i}.$$ 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The different sets in Lemma 4.3 associated to a point $u_i \in F$.}
\end{figure}
implying \( u_j \in \Psi_{u_i}(D) \) and hence \( O_{\Omega_2} \cap \Psi_{u_i}(D) \neq \emptyset \), in contradiction to Lemma 4.1, establishing the claim.

To any \( u_j \in F \) let
\[
B_j = B_{\frac{U}{c\rho e^t}}^U u_j
\]
be the corresponding ball in \( F' \). We have thus shown that for any \( u_i \), if \( J_{u_j} \) intersects \( J_{u_i} \) then
\[
\frac{1}{2} c\rho e^t \leq \text{radius}(B_j) \leq \frac{1}{2} c\rho e^{-1} e^t
\]
and
\[
B_j \cap B_{2\beta^{-1} R e^t}^U u_i \neq \emptyset.
\]
Furthermore, Since \( c\rho < R' \), we have
\[
B_j \subset B_{2\beta^{-1} R e^t}^U u_i
\]
The number of such \( B_j \)-s contained in \( B_{2\beta^{-1} R e^t}^U u_i \) is bounded above by
\[
P \cdot \frac{m_U(B_{\frac{U}{c\rho e^t}}^U u_i)}{m_U(B_{2\beta^{-1} R e^t}^U u_i)} = P \left( \frac{6R'}{c\rho e^{2t}} \right)^{d-1} = 3^{d-1} P \left( \frac{2(R + \rho)}{c\rho} \right)^{3(d-1)} = C
\]
where \( d - 1 = \dim U \) and \( P \) is the Besicovitch covering constant from (4.19). Consequently if we denote the above constant by \( C \) then
\[
\sum_{u_i \in F} \chi_{J_{u_i}} \leq C.
\]
Note that since \( F \subseteq B_{L - R e^t}^U \), by (1.22) we know all \( J_{u_i} \) are contained in \( B_{L}^U \). We may therefore conclude
\[
m_U \left( \bigcup_{u_i \in O_{\Omega_2}} J_u \cap B_{L}^U \right) \geq m_U \left( \bigcup_{u_i \in F} J_{u_i} \right) \geq C^{-1} \sum_{u_i \in F} m_U(J_{u_i}).
\]
We shall now estimate the \( m_U \)-measure of each \( J_{u_i} \). Let \( \varphi_i : U \to U \) be defined by
\[
\varphi_i(u) = u^{a_i, u_i}
\]
for all \( u \in U \). We may rewrite the definition of \( J_{u_i} \) as
\[
J_{u_i} = \left( \varphi_i \circ \Phi_U(\gamma_{u_i}^k, \cdot) \right) \left( B_{\frac{U}{c\rho e^t}}^U u_i \cap O_{\Omega} \right)^{a_i, u_i} = \left( \varphi_i \circ \Phi_U(\gamma_{u_i}^k, \cdot) \circ \varphi_i^{-1} \right)(B_i \cap O_{\Omega})
\]
Note that the Jacobian \( \text{Jac}_u(\varphi_i) \) is constant \( \equiv e^{(d-1)t_i} \) over all of \( U \). By the uniform bound on the Jacobian of \( \Phi_U(\gamma_{u_i}^k, \cdot) \) on \( D \) given in item (4) of Lemma 3.1 we have
\[
|\text{Jac}_v(\varphi_i \circ \Phi_U(\gamma_{u_i}^k, \cdot) \circ \varphi_i^{-1})| > c
\]
for all \( v \in D \), and consequently
\[
m_U(J_{u_i}) > c \cdot m_U(B_i \cap O_{\Omega})
\]

Hence
\[
m_U \left( \bigcup_{u \in O_{\tilde{\Omega}_2}} J_u \cap B^U_L \right) \geq C^{-1} \sum_{u \in F} m_U(J_u) \geq
\]
\[
\geq \frac{c}{C} \sum_{u \in F} m_U(B_i \cap O_{\tilde{\Omega}}) \geq \frac{c}{C} m_U((O_{\tilde{\Omega}} \cap O_{\tilde{\Omega}_2}) \cap B^U_{L-Re_{\tilde{T}_2}})
\]
where the last inequality follows from the fact that \( F' \) is a cover of the set \((O_{\tilde{\Omega}} \cap O_{\tilde{\Omega}_2}) \cap B^U_{L-Re_{\tilde{T}_2}}\). Making use of assumption (4.7) which states
\[
m_U((O_{\tilde{\Omega}} \cap O_{\tilde{\Omega}_2}) \cap B^U_{L-Re_{\tilde{T}_2}}) \geq \frac{1}{2} m_U(O_{\tilde{\Omega}} \cap O_{\tilde{\Omega}_2})
\]
we conclude
\[
m_U \left( \bigcup_{u \in O_{\tilde{\Omega}_2}} J_u \cap B^U_L \right) \geq \frac{c}{2C} m_U(O_{\tilde{\Omega}} \cap O_{\tilde{\Omega}_2}),
\]
proving the claim with \( C_0 = \frac{c}{2C} \).

Joining the results of lemmata 4.1–4.3 we deduce
\[
m_U(O_{\tilde{\Omega}} \setminus O_{\tilde{\Omega}_2}) \geq m_U \left( \bigcup_{u \in O_{\tilde{\Omega}_2}} J_u \cap B^U_L \right) \geq C_0 \cdot m_U(O_{\tilde{\Omega}} \cap O_{\tilde{\Omega}_2})
\]
thus concluding the proof of Theorem 1.3.

5. Consequences of the Main Theorem

5.1. MA-quasi-invariance. We shall now prove our first corollary of the main theorem, giving a sufficient condition for MA-quasi-invariance in the context of a general discrete group \( \Gamma \):

**Corollary 1.4.** Let \( \Gamma \) be any discrete subgroup of \( G \) and let \( \mu \) be any U-e.i.r.m. on \( G/\Gamma \). If \( \mathcal{S}_\mu \) contains a Zariski dense subgroup of \( G \) then \( \mu \) is MA-quasi-invariant.

The corollary above for \( d = 2 \) includes the case of weakly-tame surfaces introduced by Sarig in [21]. Such hyperbolic surfaces satisfy a condition that all geodesic rays \( \{a_t \gamma \Gamma \}_{t \geq 0} \) intersect an infinite number of times pairs of pants with bounded “norm” (sum of the lengths of closed geodesics bounding the pair of pants). This condition implies the existence of a sequence \( t_n \to \infty \) for which \( a_{-t_n} g \gamma g^{-1} a_{t_n} \) contains a free group generated by two hyperbolic elements \( \gamma_1 \) and \( \gamma_2 \), with uniformly separated fixed points on \( \mathbb{H}^2 \). This in turn implies that \( \mathcal{S}_\mu \) contains a non-elementary and Zariski dense subgroup.

**Proof of Corollary 1.4.** Let \( H = \overline{\langle \alpha(S_\mu) \rangle} \) be the metric closure of the group generated by \( \alpha(S_\mu) \) in MA. By Theorem 1.3 \( H \subseteq H_\mu \). We want to show that \( MA \subseteq H \).
For a hyperbolic element $\gamma$ let

$$\ell(\gamma) = \inf_{z \in H^d} d_{H^d}(z, \gamma.z)$$

denote the translation length of $\gamma$. Recall that $\ell(ma_t) = |t|$ for any $ma_t \in MA$.

Kim [9], generalizing a theorem by Dal’bo [4] (see also [5, Ch.III Thm.3.6]), proved the length spectrum of any non-elementary subgroup of $G$ is non-arithmetic. Therefore there exist $\gamma_1, \gamma_2 \in S_\mu$ for which $\langle \ell(\gamma_1), \ell(\gamma_2) \rangle$ is a dense subgroup of $\mathbb{R}$. We show in §3 that given any hyperbolic element $\gamma$, amongst $\alpha_p \gamma_1 \alpha_p^{-1}$, $\alpha_p \gamma_2 \alpha_p^{-1}$ at least one is not equal to $e$. Hence without loss of generality we may assume $\alpha_p \gamma_i \alpha_p^{-1} \neq e$ for both $i = 1, 2$. Since $\alpha_p \gamma_i \alpha_p^{-1}$ and $\gamma_i$ are conjugate, they have equal translation lengths and therefore

$$\langle \ell(\alpha_p \gamma_1), \ell(\alpha_p \gamma_2) \rangle$$

is dense in $\mathbb{R}$. We thus conclude $M \backslash \alpha(S_\mu)$ generates a dense subgroup in $M \backslash MA$ implying $M \backslash H = A$. This completes the proof for $d = 2$.

We claim that for $d \geq 4$ the commutator subgroup of $H$ admits $[H, H] = M$. Recall equation (3.2) in §3 stating that for any hyperbolic $\gamma$ with $\alpha_p \gamma \neq e$ there exist $n \in \mathbb{N}$ and $u \in U$ satisfying

$$\gamma = (nu)^{-1}\alpha(\gamma)(nu).$$

Suppose $\alpha(S_\mu)$ is not Zariski dense in the group $MA$ (considered as a real algebraic group). Then there is a subvariety $V \subset MA$ of strictly lower dimension containing $\alpha(S_\mu)$.

The map $\Psi : N \times MA \times U \rightarrow G$ defined by $(n, a', u) \mapsto (nu)^{-1}a'(nu)$ is algebraic, and is a dominant morphism (the image contains a Zariski open dense set). Under the above non-density assumption $S_\mu \cap \text{Image}(\psi)$ is contained in $\Psi(N \times V \times U)$. Since $\Psi(N \times V \times U)$ is a constructible subset of $G$ of lower dimension, it follows that $S_\mu$ is contained in a proper subvariety of $G$, a contradiction.

Therefore $\alpha(S_\mu)$ is Zariski dense in the group $MA$. Consequently, $[H, H]$ is Zariski dense in $[MA, MA] = [M, M]$ (considered as an algebraic group). The group $M$ is isomorphic as a Lie group to the simple Lie group $\text{SO}(d-1)$, hence $[M, M] = M$ (also as Lie groups) and moreover any closed proper subgroup of $M$ is contained in a proper algebraic subgroup of $M$. It follows that $[H, H] = [M, M] = M$ for $d \geq 4$.

The case of $d = 3$ is handled separately in the following lemma.  

**Lemma 5.1.** Let $\Sigma$ be a Zariski dense subgroup of

$$G = \text{SO}^+(3, 1) \cong \text{PSL}_2(\mathbb{C}),$$

then $\langle \alpha(\Sigma) \rangle$ is dense in $MA$ w.r.t the metric topology.

It seems this fact was already known to the experts; in particular, as pointed out to us by Hee Oh this lemma follows from combining [15, Prop. 6.3] with [13, Thm. 1.1].
Proof of Lemma 5.1. Denote 
\[ H = \langle \alpha(\Sigma) \rangle. \]
We have seen in the proof of the general case of Corollary 1.4 that \( H \) is Zariski dense in \( MA \) and also \( M \setminus H = A \). In particular, \( H \) is a closed uncountable subgroup of \( MA \) and is hence non-discrete. We shall think of \( MA \subset SL_2(\mathbb{C}) \) as \( \mathbb{C}^\infty \).

Assume by contradiction that \( H \not\leq MA \). Fixing a branch of the logarithm on \( \mathbb{C} \) containing 1 and considering \( \log H \) on a small neighborhood of 1, we may deduce \( H^0 \), the identity component of \( H \), is a one-parameter subgroup of the form \( \{ te^{i\omega} \}_{t \in \mathbb{R}} \) for some \( \omega \in \mathbb{C} \).

We claim \( \omega \notin \mathbb{R} \cup i\mathbb{R} \). Indeed, the group \( H \) has at most a countable set of connected components and therefore the fact that \( M \setminus H = A \) implies \( \omega \notin i\mathbb{R} \). On the other hand, if \( \omega \) was contained in \( \mathbb{R} \) then \( H \) would have been equal to a finite union of rays emanating from the origin, contradicting the fact that \( H \) is Zariski dense in \( MA \) (over \( \mathbb{R} \)). We thus conclude \( H^0 \) is a winding spiral in \( \mathbb{C}^\infty \) and

\[ H = \bigcup_{0 \leq t \in m-1} \{ \theta e^{i\omega} \}_{t \in \mathbb{R}} \]

is a finite union of rotated copies of \( H^0 \), where \( \theta \in \mathbb{C} \) is an \( m \)-th root of unity.

In \( SL_2(\mathbb{C}) \) the map \( \alpha \) sends every hyperbolic element in \( G \) to its eigenvalue inside the unit disc. Therefore \( \langle \alpha(\Sigma) \rangle \) is the group generated by all eigenvalues of hyperbolic elements of \( \Sigma \).

Claim. There exists \( \gamma_0 \in \Sigma \) with eigenvalues \( \rho, \rho^{-1} \) contained in \( H^0 \) and satisfying \( \text{trace}(\gamma_0^2k) \in H^0 \) for all large \( k \).

Assuming the claim above, there exist \( s_k \in \mathbb{R} \) and \( t > 0 \) with

\[ e^{s_k \omega} = \text{trace}(\gamma_0^{2k}) \quad \text{and} \quad e^{i\omega} = \rho. \]

Since \( \text{trace}(\gamma_0^{2k}) = \rho^{2k} + \rho^{-2k} \) we know

\[ e^{s_k \omega} = e^{2kt\omega} + e^{-2kt\omega} \]

and

\[ e^{-4kt\omega} = e^{(s_k-2kt)\omega} - 1. \]

Recall that \( \text{Re}(\omega) \neq 0 \) hence Equation (5.1) implies

\[ |s_k - 2kt| \to 0. \]

Let \( \text{arg}(z) \in \mathbb{R}/2\pi\mathbb{Z} \) denote the complex argument of a number \( z \in \mathbb{C}^\infty \). Set

\[ \varepsilon = d_{\mathbb{R}/2\pi\mathbb{Z}}(\text{arg}(\omega), 2\text{arg}(\omega)), \]

the distance between \( \text{arg}(\omega) \) and \( 2\text{arg}(\omega) \) in \( \mathbb{R}/2\pi\mathbb{Z} \), and note that \( \varepsilon > 0 \) since \( \omega \notin \mathbb{R} \). Equation (5.1) implies the following relation

\[ k(4t\text{Im}(\omega)) \equiv -\text{arg}(e^{(s_k-2kt)\omega} - 1) \equiv -\text{arg} \left( \frac{e^{(s_k-2kt)\omega} - 1}{s_k - 2kt} \right) \quad \text{mod} \ 2\pi \]
for all large \( k \). Using the Taylor expansion \( e^{r \omega} = 1 + r \omega + O(r^2) \) we deduce
\[
k(4 \text{Im}(\omega)) \equiv -\arg(\omega) + O(s_k - 2kt) \mod 2\pi.
\]
Set \( \beta = 4t \text{Im}(\omega) \) and let \( k_0 \) be large enough so that for all \( k \geq k_0 \)
\[
d_{\mathbb{R}/2\pi\mathbb{Z}}(k\beta, -\arg(\omega)) < \varepsilon/4.
\]
On the one hand we are ensured that
\[
d_{\mathbb{R}/2\pi\mathbb{Z}}(2k_0\beta, -\arg(\omega)) < \varepsilon/4,
\]
but on the other hand, multiplication by 2 in \( \mathbb{R}/2\pi\mathbb{Z} \) implies
\[
d_{\mathbb{R}/2\pi\mathbb{Z}}(2k_0\beta, -2\arg(\omega)) < \varepsilon/2
\]
and consequently
\[
d_{\mathbb{R}/2\pi\mathbb{Z}}(\arg(\omega), 2\arg(\omega)) < 3\varepsilon/4
\]
in contradiction to the definition of \( \varepsilon \) in (5.2).

We are left with proving the existence of such a \( \gamma_0 \in \Sigma \) as claimed. Recall that the group \( \langle \alpha(\Sigma) \rangle \) is generated by the eigenvalues of all hyperbolic elements in \( \Sigma \), therefore conjugating \( \Sigma \) would leave this group unchanged, i.e. \( \langle \alpha(g\Sigma g^{-1}) \rangle = \langle \alpha(\Sigma) \rangle \) for any \( g \in G \).

Let \( \eta \) be any fixed hyperbolic element of \( \Sigma \). By conjugating \( \Sigma \) if necessary we may assume without loss of generality that \( \eta \) is a diagonal matrix in \( M\mathbb{A} \)
\[
\eta = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}
\]
with \( \lambda \in \mathbb{C}^\times \) and \( |\lambda| > 1 \). Note that \( \lambda, \lambda^{-1} \in H \).

Let
\[
\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Sigma.
\]
The element
\[
\eta^{-n}\gamma\eta^{-n} = \begin{bmatrix} \lambda^{-2na} & b \\ c & \lambda^{2nd} \end{bmatrix}
\]
of \( \Sigma \) has an eigenvalue \( e_n^+ = \lambda^{2na} + O(\lambda^{-2n}) \). This can be seen by considering the trace of the matrix, which is the sum of its two reciprocal eigenvalues. Therefore both \( e_n^+ \) and \( \lambda^{-2n} \) are contained in \( H \) implying \( e_n^+ \lambda^{-2n} \in H \) for all \( n \). Due to the fact that \( H \cup \{0\} \) is closed in \( \mathbb{C} \) we conclude \( d \in H \cup \{0\} \).

The 22-coefficient of the matrix \( \gamma\eta^n\gamma^{-1} \) in \( \Sigma \) is equal to
\[
[\gamma\eta^n\gamma^{-1}]_{22} = \lambda^{-n}ad - \lambda^nbc.
\]
By the argument above
\[
(\lambda^{-n}ad - \lambda^nbc)\lambda^{-n} = -bc + \lambda^{-2n}ad
\]
is contained in \( H \cup \{0\} \) for all \( n \) implying \( -bc \in H \cup \{0\} \). Denote \( \Delta(\gamma) = -bc \).
We have thus shown \( \Delta(\gamma) \in H \cup \{0\} \) for any \( \gamma \in \Sigma \).

Furthermore,
\[
\gamma^2 = \begin{bmatrix} * & b \cdot \text{trace}(\gamma) \\ c \cdot \text{trace}(\gamma) & * \end{bmatrix}
\]
implying

\[(5.3) \Delta(\gamma^2) = \Delta(\gamma) \cdot (\text{trace}(\gamma))^2 \in H \cup \{0\}.\]

Since \(\Sigma\) is Zariski dense in \(G\) there exists a hyperbolic element \(\gamma_1\) satisfying \(\Delta(\gamma_1^m) \neq 0\) where \(m = |H/H_0|\). Denote \(\gamma_0 = \gamma_1^m\). By \((5.3)\) and the fact that \(\Delta(\gamma_0) \in H\), we know \((\text{trace}(\gamma_0))^2 \in H\). Actually,

\[\Delta(\gamma) \neq 0 \implies \Delta(\gamma^2) = \Delta(\gamma) \cdot (\text{trace}(\gamma))^2 \neq 0\]

whenever \(\gamma\) is hyperbolic (i.e. has \(|\text{trace}| \geq 2\), ensuring

\[(\text{trace}(\gamma_0^{2k}))^2 \in H \quad \text{for all } k.\]

Denote \(\gamma_0\)'s eigenvalue outside the unit disc by \(\rho \in \mathbb{C}^\times\). Since \(\gamma_0 = \gamma_1^m\) we are ensured that \(\rho, \rho^{-1} \in H^0\). Recall

\[(\text{trace}(\gamma_0^{2k}))^2 = (\rho^{2k} + \rho^{-2k})^2 = \rho^{4k} + \rho^{-4k} + 2\]

implying

\[|(\text{trace}(\gamma_0^{2k}))^2 - \rho^{4k}| < 3.\]

Therefore whenever \(k\) is large enough \((\text{trace}(\gamma_0^{2k}))^2\) and \(\rho^{4k}\) are contained inside the same coset of \(H^0\), or in other words \((\text{trace}(\gamma_0^{2k}))^2 \in H^0\). The subgroup \(H^0\) is isomorphic to \((\mathbb{R}, +)\) and therefore contains a unique square root to any element, implying

\[\text{trace}(\gamma_0^{2k}) \in H^0\]

for all large \(k\), as claimed. \(\square\)

Assuming weaker conditions on \(S_\mu\) we receive:

**Corollary 5.2.** Let \(\Gamma\) be any discrete subgroup of \(G\) and let \(\mu\) be any \(U\)-e.i.r.m. on \(G/\Gamma\). If \(S_\mu\) contains a non-elementary subgroup of \(G\), then the projected measure \((\pi_\mu)_*\mu\) on \(M/G/\Gamma \cong T^1(\mathbb{H}^d/\Gamma)\) is \(A\)-quasi-invariant.

The claim follows directly from non-arithmeticity of the length spectrum [9] and Theorem 1.3, as described in the proof of Corollary 1.4.

**5.2. Measure Decomposition.** Let \(Q = UMA\) be a minimal parabolic. Denote by \(\pi\) the quotient map \(g \mapsto Qg\) onto \(Q\backslash G \cong S^{d-1}\). The group \(G\) is diffeomorphic to

\[U \times M \times A \times (Q\backslash G) \cong U \times M \times A \times S^{d-1}\]

The left action of \(UMA\) in these coordinates is easily computed as

\[(5.4) \quad \text{uma}(u', m', a', \xi) = (u(u')^{ma}, mm', aa', \xi)\]

where \(\xi \in Q\backslash G\) and conjugation is denoted by \(g^h = hgh^{-1}\).

Fix \(\xi \in Q\backslash G\) and two points \(g, h \in G\). Denote by \(g_\xi\) some element in \(Kg\) satisfying \(\pi_\mu(g_\xi) = \xi\), respectively for \(h_\xi \in Kh\). Using the \(KAU\) decomposition we write

\[h_\xi^{-1} = k_{h_\xi(g, h)}u.\]
The function \( b_\xi : G \times G \rightarrow \mathbb{R} \) above is called the Busemann function. This function is usually defined over \( \mathbb{H}^d \cong K \backslash G \) and satisfies
\[
|b_\xi(g, h)| = \lim_{t \to \infty} d_{\mathbb{H}^d}(Ka_{-t}g\xi, Ka_{-t}h\xi).
\]

Given the point \( p_\xi = (e, e, e, \xi) \) one can verify that the right action of an element \( \gamma \in \Gamma \) is given by
\[
(e, e, e, \xi).\gamma = (u(p_\xi\gamma), m(p_\xi\gamma), a_{b_\xi(\gamma^{-1}, e)}, \xi\gamma^{-1})
\]
for some \( u(p_\xi\gamma) \in U \) and \( m(p_\xi\gamma) \in M \) depending on \( \xi \) and \( \gamma \). Therefore
\[
(u, m, a_\xi, \xi).\gamma = \left( u(u(p_\xi\gamma))^{ma_\xi}, mm(p_\xi\gamma), a_{t+b_\xi(\gamma^{-1}, e)}, \xi\gamma^{-1} \right).
\]

A measure \( \nu \) on \( S^{d-1} \cong Q \backslash G \) is called \( \Gamma \)-conformal with parameter (or dimension) \( \beta \) if
\[
d_{\gamma_\xi}\nu(\xi) = e^{-\beta b_\xi(\gamma^{-1}, e)}
\]
for every \( \gamma \in \Gamma \) and \( \xi \in S^{d-1} \), where \( \gamma \) acts on \( Q \backslash G \) on the right.

**Lemma 5.3.** Let \( \mu \) be a \( U \)-ergodic and invariant and \( MA \)-quasi-invariant Radon measure on \( G/\Gamma \). Then \( \mu \) is induced by a measure \( \tilde{\mu} \) on \( G \) presented in \( U \hat{\times} M \hat{\times} A \hat{\times} S^{d-1} \) coordinates as
\[
d\tilde{\mu} = e^{\beta t} d\nu dmdtdu
\]
where \( \nu \) is a \( \Gamma \)-conformal measure with parameter \( \beta \) on \( S^{d-1} \).

For the case of \( d = 2 \) this correspondence was observed already in Burger’s paper [3, §2.1], and has been highlighted by Bézivin in her work, e.g. [11]; cf. also [11] by Ledrappier and Sarig. Note that this correspondence also works the opposite way: starting from a \( \Gamma \)-conformal probability measure on \( S^{d-1} \) one can use the above recipe to obtain a \( U \)-invariant Radon measure on \( G/\Gamma \).

**Proof.** Recall that whenever an ergodic measure \( \mu \) is quasi-invariant with respect to a group \( H \) then there exists some character \( \chi : H \rightarrow \mathbb{R} \) satisfying for all \( h \in H \)
\[
h.\mu = \chi(h)\mu.
\]
Since \( M \) is compact it has no non-trivial continuous characters into \( \mathbb{R} \) implying \( \mu \) is \( M \)-invariant. On the other hand, \( A \)-quasi-invariance implies there exists \( \alpha \in \mathbb{R} \) for which
\[
a_\alpha.\mu = e^{\alpha t} \mu.
\]
Let \( \tilde{\mu} \) be the lift of \( \mu \) to a \( \Gamma \)-invariant measure on \( G \). Therefore \( \tilde{\mu} \) may be presented in \( U \times M \times A \times S^{d-1} \) coordinates as
\[
d\tilde{\mu} = e^{\beta t} d\nu dmdtdu
\]
where \( \nu \) is some finite measure on \( S^{d-1} \). Note that \( \beta = \alpha - (d - 1) \) since (5.4) implies that applying \( a_\alpha \) on the left induces a change of variable \( u' = u^{a_\alpha} \) with \( \frac{du'}{du} = e^{(d-1)\mu} \).
We will show that $\Gamma$-invariance of $\tilde{\mu}$ implies that $\nu$ is $\Gamma$-conformal with parameter $\beta$. Fix $\gamma \in \Gamma$ and denote $B_\xi = b_\xi(\gamma^{-1}, e)$. Given $f \in C_c(G)$ we compute
\[
\int f d\tilde{\mu} = \int d\gamma \tilde{\mu} = \int (u, m, \xi, \gamma) d\tilde{\mu} = 
\]
\[
= \int f (u(p_{\xi\gamma}))^m a_t, mm(p_{\xi\gamma}), a_t + B_\xi, \xi, \gamma^{-1}) d\tilde{\mu}
\]
note that $u(p_{\xi\gamma})$ is independent of $u$ and $m(p_{\xi\gamma})$ independent of $m$, therefore by Fubini’s theorem we get
\[
... = \int f (u, m, a_t + B_\xi, \xi, \gamma^{-1}) d\tilde{\mu} = 
\]
\[
= \int (f \circ a_{B_\xi})(u^{-a_{-B_\xi}}, m, a_t, \xi, \gamma^{-1}) d\tilde{\mu} = 
\]
\[
= \int f (u^{-a_{-B_\xi}}, m, a_t, \xi, \gamma^{-1}) d(a_{B_\xi} \tilde{\mu}) = 
\]
\[
= \int f (u^{-a_{-B_\xi}}, m, a_t, \xi, \gamma^{-1}) e^{a_{B_\xi}} d\tilde{\mu} = 
\]
using the change of variable $u' = u^{-a_{-B_\xi}}$ with $du' = e^{(d-1)B_\xi} du$ gives
\[
... = \int f (u', m, a_t, \xi, \gamma^{-1}) e^{(a-(d-1))B_\xi + \beta} d\nu dm dt du' = 
\]
\[
= \int f (u', m, a_t, \xi) e^{\beta(B_\xi + t)} d\gamma \nu dm dt du'.
\]
Thus implying
\[
\frac{d\gamma \nu}{d\nu}(\xi) = e^{-\beta B_\xi (\gamma^{-1}, e)}
\]
as required. \qed

5.3. Regular covers of Geometrically Finite Manifolds. Recall the following definition of a geometrically finite discrete group:

**Definition 5.1.** [16, Chapter 12.4]

1. A fundamental polyhedron $P \subseteq \mathbb{H}^d$ of a subgroup $\Gamma_0$ is called exact if every face of $P$ is the intersection of $P$ and $P \gamma$ for some $\gamma \in \Gamma_0$.

2. A polyhedron $P$ is called geometrically finite if for each point $x \in \overline{P} \cap \partial \mathbb{H}^d$ there exists a neighborhood $V$ such that all faces of $\overline{P}$ meeting $V$ also pass through $x$.

3. A discrete group $\Gamma_0 < G$ is called geometrically finite if it has a convex, exact, geometrically finite fundamental polyhedron in $\mathbb{H}^d$.

This definition is equivalent to $\Gamma_0$ having only conical and parabolic limit points, see [16, Theorem 12.4.5]. For $d = 2, 3$ this definition is equivalent to $\Gamma_0$ having a finite-sided fundamental polyhedron in $\mathbb{H}^d$. Both lattices and convex cocompact subgroups are examples of geometrically finite discrete subgroups.
Whenever \(\{e\} \neq \Gamma < \Gamma_0\) the space \(G/\Gamma\) is a regular cover of the geometrically finite hyperbolic manifold \(G/\Gamma_0\).

**Corollary 1.1.** Let \(\Gamma_0\) be a geometrically finite Zariski dense discrete subgroup of \(G\) and let \(\{e\} \neq \Gamma < \Gamma_0\). Let \(\mu\) be a U-\(e.i.r.m.\) on \(G/\Gamma\). Then the following trichotomy holds, either:

1. \(\mu\) is supported on a wandering horosphere.
2. \(\mu\) is supported on a lift to \(G/\Gamma\) of a horosphere in \(G/\Gamma_0\) bounding a cusp.
3. \(\mu\) is MA-quasi-invariant.

We remark that (as pointed out to us by Hee Oh), this result gives a new proof that the Burger-Roblin measure on \(G/\Gamma\) for geometrically finite Zariski dense subgroups is the unique U-invariant non-periodic recurrent measure on \(G/\Gamma\), based on Sullivan’s theorem that there is only one atom-free \(\Gamma\)-conformal measure supported on the limit set [22], in particular, that the Burger-Roblin measure is ergodic on \(G/\Gamma\). This case was shown by Winter [24]; Cf. also [14, Thm. 5.7] for another proof of this measure classification in the geometrically finite context.

The group \(G\) may be identified with the frame bundle of hyperbolic \(d\)-space. Let \(\pi_\Gamma : G \to \partial \mathbb{H}^d\) denote the projection onto the boundary of hyperbolic space, where \(\pi_\Gamma(g)\) is defined as the limit of the geodesic trajectory \(\{a_{t}g\}_{t \geq 0}\).

Recall the limit set \(\Lambda_\Gamma\) associated to a discrete group \(\Gamma < G\) is defined as the set of accumulation points of \(g\Gamma\) in \(\partial \mathbb{H}^d\) for some (and hence any) \(g \in G\). An equivalent definition for the limit set of \(\Gamma\) is the minimal closed \(\Gamma\)-invariant subset of \(\partial \mathbb{H}^d\).

A limit point \(\xi \in \Lambda_\Gamma\) is called conical if for some (hence every) \(g \in \pi_\Gamma^{-1}(\xi)\) the geodesic ray \(\{a_{t}g\Gamma\}_{t \geq 0}\) in \(G/\Gamma\) returns infinitely often within a bounded distance of \(g\Gamma\). A limit point is called parabolic if it is fixed by a parabolic element of \(\Gamma\).

Denote by \(\Lambda_{\Gamma_0}^{par}\) and \(\Lambda_{\Gamma_0}^{con}\) the parabolic and conical limit points of \(\Lambda_{\Gamma_0}\). Note that \(\Lambda_{\Gamma_0}^{par}\) is countable. For a geometrically finite group \(\Gamma_0\), any limit point is either conical or parabolic [16, Theorem 12.4.5], i.e.

\[
\Lambda_{\Gamma_0} = \Lambda_{\Gamma_0}^{par} \cup \Lambda_{\Gamma_0}^{con}.
\]

Whenever \(\{e\} \neq \Gamma < \Gamma_0\) then \(\Lambda_\Gamma = \Lambda_{\Gamma_0}\). Clearly \(\Lambda_\Gamma \subseteq \Lambda_{\Gamma_0}\). Indeed, for \(\xi \in \Lambda_\Gamma\) there exists \(g \in G\) and \(\gamma_i \in \Gamma\) with

\[
\xi = \lim_{i \to \infty} g\gamma_i.
\]

Therefore, given any \(\gamma_0 \in \Gamma_0\)

\[
\xi\gamma_0 = \lim_{i \to \infty} g\gamma_i\gamma_0 = \lim_{i \to \infty} (g\gamma_0)\gamma_i
\]

for \(\gamma_i \in \Gamma\), where the last equality follows from the fact that \(\gamma_0\) normalizes \(\Gamma\). Since the limit set of a discrete group does not depend on the choice of “base point” \(g\) (or \(g\gamma_0\)) we conclude \(\xi\gamma_0 \in \Lambda_\Gamma\) implying \(\Lambda_\Gamma\) is \(\Gamma_0\)-invariant and hence by minimality \(\Lambda_{\Gamma_0} \subseteq \Lambda_\Gamma\), as claimed.

We may thus conclude

\[
(5.5) \quad \Lambda_\Gamma = \Lambda_{\Gamma_0}^{par} \cup \Lambda_{\Gamma_0}^{con}.
\]
Proof of Corollary 1.1. Let $\pi_- : G \to \partial H^d$ be the projection discussed above. Note that $\pi_-(ug) = \pi_-(g)$ for any $g \in G$ and $u \in U$. By (5,3) we know

$$\partial H^d = (\partial H^d \setminus \Lambda) \cup \Lambda_{\Gamma_0}^{\text{par}} \cup \Lambda_{\Gamma_0}^{\text{con}}$$

This partition of $\partial H^d$ induces via $\pi_-$ a partition of $G$ into $U$-invariant sets. Ergodicity implies either:

1. $\mu$ is supported inside $\pi_-(\partial H^d \setminus \Lambda)/\Gamma$
2. $\mu$ is supported inside $\pi_-(\Lambda_{\Gamma_0}^{\text{par}})/\Gamma$
3. $\mu$ is supported inside $\pi_-(\Lambda_{\Gamma_0}^{\text{con}})/\Gamma$

These three cases correspond respectively to the ones presented in the theorem. In case (1) the measure $\mu$ is supported on a set of horospheres wandering into a funnel. Ergodicity implies that $\mu$ is supported on a single wandering horosphere.

For case (2), by ergodicity (and the fact that $\Lambda_{\Gamma_0}^{\text{par}}$ is a countable set) there is a single $\Gamma$-orbit $\xi, \Gamma \in \Lambda_{\Gamma_0}^{\text{par}}/\Gamma$ so that $\mu$ is supported inside $\pi_-(\xi, \Gamma)/\Gamma$. Applying ergodicity once again we see that $p_\ast \mu$ must give full measure to a single horosphere bounding the cusp around $\xi, \Gamma_0$.

Assume case (3), then for $\mu$-a.e $g\Gamma$, the projected geodesic ray

$$p(\{a_{-t}g\Gamma\}_{t \geq 0}) = \{a_{-t}g\Gamma_0\}_{t \geq 0}$$

returns infinitely often to a compact set in $G/\Gamma_0$. Hence there exist sequences $\gamma_n \in \Gamma_0$ and $t_n \to \infty$ for which $a_{-t_n}g\gamma_n$ converges to some $g_0 \in G$. Using the fact that $\Gamma < \Gamma_0$ we deduce

$$a_{-t_n}g\gamma_n^{-1}a_{t_n} = a_{-t_n}g\left(\gamma_n \Gamma \gamma_n^{-1}\right) g^{-1}a_{t_n} \to g_0 \Gamma g_0^{-1}$$

and therefore

$$g_0 \Gamma g_0^{-1} \subseteq S_\mu.$$ 

Note that $Z\ast d(\Gamma)$, the Zariski closure of $\Gamma$, is invariant under conjugation by $G = Z\ast d(\Gamma_0)$. Therefore since $\Gamma$ is a simple Lie group we deduce $\Gamma$ and also $g_0 \Gamma g_0^{-1}$ are Zariski dense. Corollary 1.4 implies the measure $\mu$ is MA-quasi-invariant. \hfill \Box

5.4. Bounded Injectivity Radius. The injectivity radius at a point $x$ in $G/\Gamma$ is defined as

$$\text{rad}_{\text{inj}}(x) = \sup\{r : v_1x \neq v_2x \quad \forall v_1, v_2 \in B^G_r\}.$$ 

Corollary 1.2. Let $\Gamma < G$ be a purely-hyperbolic discrete subgroup with $\inf \ell(\Gamma) > 0$. Let $\mu$ be any U-e.i.r.m. Then one of the following holds:

1. $\lim_{t \to \infty} \text{rad}_{\text{inj}}(a_{-t}x) = \infty$ for $\mu$-a.e. $x$.
2. $H_\mu$ is cocompact in MA.

Whenever $H_\mu$ is cocompact in MA there exists a character $\chi : H_\mu \to \mathbb{R}$ satisfying

$$\ell_\mu = \chi(\ell)\mu \quad \text{for all } \ell \in H_\mu.$$
This character may be uniquely extended to all of $MA$. Indeed, $\chi$ extends trivially to $MH_\mu$ and extends linearly to $MA$ using the fact that $M \setminus MH_\mu$ is cocompact in $M \setminus MA \approx \mathbb{R}$. The measure
\[ \lambda = \int_{MA/H_\mu} \chi(\ell^{-1}) \ell, \mu \ dM_{A/H_\mu}(\ell) \]
on $G/\Gamma$ is Radon $U$-invariant and $MA$-quasi-invariant. Furthermore, by our construction $ma.\lambda = \chi(ma)\lambda$ for any $ma \in MA$. In other words, under these conditions the measure $\mu$ is an ergodic component of a measure of the form $e^{\beta t}d\nu dm dt du$ where $\nu$ is a $\Gamma$-conformal measure on $S^{d-1}$.

Proof. Whenever $\text{rad}_{\text{inj}}(g\Gamma) < R$ there exist $v_1, v_2 \in B^G_R$ with $v_1 \neq v_2$ satisfying $v_1 v_2^{-1} g\Gamma = g\Gamma$, implying
\[ v_1 v_2^{-1} \in g\Gamma g^{-1} \cap B^{G}_{2R}. \]
Note that the map
\[ x \mapsto \liminf_{t \to \infty} \text{rad}_{\text{inj}}(a_{-t} x) \]
from $G/\Gamma$ to $(0, \infty]$ is $U$-invariant and measurable and therefore constant $\mu$-a.s.
Assume that
\[ \liminf_{t \to \infty} \text{rad}_{\text{inj}}(a_{-t} x) < \infty \quad \mu\text{-a.s}. \]
Then for $\mu$-a.e. $g\Gamma \in G/\Gamma$ there exists $R > 0$ and sequences $t_n \to \infty$ and $\gamma_n \in G \setminus \{e\}$ satisfying
\[ \gamma_n \in a_{-t_n} g\Gamma g^{-1} a_{t_n} \cap B^{G}_{2R}. \]
Let $\gamma_0 \in B^{G}_{2R}$ be an accumulation point of the sequence $\gamma_n$. Since the length of a hyperbolic element is preserved by conjugation, and since $\inf \ell(\Gamma) = c > 0$ we are ensured that $\gamma_0$ is hyperbolic with $\ell(\gamma_0) \geq c$. Therefore $S_\mu$ contains a non-trivial hyperbolic element and Theorem 1.3 implies $M \setminus MH_\mu \neq \{e\}$, proving the claim. \hfill $\Box$

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