A note on optimal regularity and regularizing effects of point mass coupling for a heat-wave system

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Abstract

We consider a coupled 1D heat-wave system which serves as a simplified fluid-structure interaction problem. The system is coupled in two different ways: the first, when the interface does not have mass and the second, when the interface does have mass. We prove an optimal regularity result in Sobolev spaces for both cases. Furthermore, we show that point mass coupling regularizes the problem and quantify this regularization in the sense of Sobolev spaces.

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1 Introduction

In this note we analyze a coupled system consisting of the linear wave equation and the linear heat equation coupled through a common interface. This system can be viewed as a simplified fluid-structure interaction problem. Fluid-structure interaction (FSI) problems naturally arise in many applications and have been extensively studied from both analytical and numerical point of view (see e.g. [11, 19, 22] and references within). Despite recent progress, the development of a comprehensive well-posedness and regularity theory for FSI problems still remains a challenge. One of the main difficulties in analysis of FSI problems is hyperbolic-parabolic coupling and corresponding mismatch in regularity of solutions. The purpose of this note is to analyze this mismatch on the simplified problem and to answer the following two questions:

1. What is the optimal regularity for the considered system in the following sense: what is the minimal regularity for the wave component that allows the heat component to develop full parabolic regularity?

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2. What is the answer to the first question in the case when the interface has a mass, i.e. when coupling is realized through the point mass? Does the coupling through the point mass provide additional regularity to the problem?

We believe that answers to these questions for the simplified problem will give us better understanding of more complex and realistic FSI models.

Let us now briefly describe the main result of this paper (Theorem 2). Let \( u_0 \) be the initial data for the heat equation and let \((v_0, v_1)\) be the initial data for the wave equation. We prove that:

1. Optimal regularity is obtained for \((u_0, v_0, v_1)\) \( \in H^{2s+1/2} \times H^{s+1} \times H^s, s \geq 0 \). Then the solution of the heat component \( u \) satisfies \( u \in L^2(H^{2s+3/2}) \) and the solution of the wave component \( v \) satisfies \( v \in C(H^{s+1}) \), where \( L^2(H^s) \) is abbreviation for \( L^2(0, T; H^s(\Omega)) \). Notice that the obtained function spaces are non-symmetric and are not connected to the energy of the problem (on neither level).

2. If the coupling is done through the point mass, the system gains \( 1/2 \) of the derivative in a sense that initial data \((u_0, v_0, v_1)\) \( \in H^{2s+1/2} \times H^{s+1/2} \times H^{s-1/2} \) (i.e. with \( 1/2 \) derivative less in the wave component) produce the solution with the same regularity as in the case without point mass, i.e. \( u \in L^2(H^{2s+3/2}) \) and \( v \in C(H^{s+1}) \). This regularization effect of the interface with mass was noticed in [20] where the authors considered a more realistic moving boundary fluid-multi-layered structure problem motivated by blood flow applications (for a similar effect in a different context see [12]). However, in this work we quantify this regularization and give an explicit formula that explains the mechanism behind this regularization.

1.1 Brief literature review

The same simplified FSI model as in this note was analyzed in [24, 25, 26, 27] where the authors addressed boundary control, observability, stabilization and long time behavior of the solution. Rational decay rates for this model have also been studied in [2, 10].

In the context of strong regular solutions for FSI problems where both the fluid and the solid occupy a domain with the same spatial dimension (i.e. an elastic body is not described with some lower dimensional model), the following results have been obtained. A linear FSI problem on a fixed domain where 2D or 3D Stokes equations are coupled with the equations of 2D or 3D linear elasticity were studied in [8]. The existence and uniqueness of the strong solution was proved with initial data \((u_0, v_0, v_1)\) \( \in H^2 \times H^2 \times H^2 \). A similar problem was considered in [4], where the existence and uniqueness of solution \((u, v)\) \( \in L^2(H^2) \times L^\infty(H^2) \) was obtained with initial data \((u_0, v_0, v_1)\) \( \in H^2 \times H^2 \times H^1 \). The authors noted that additional regularity for the initial structure displacement is needed in order to take advantage of the parabolic regularity for the
fluid component (Remark 1.2 and Theorem 2.1). An analogous result for the nonlinear FSI problem defined on the fixed domain was proved in [3].

D. Coutand and S. Shkoller proved the existence, locally in time, of a unique, regular solution for a moving boundary FSI problem between a viscous, incompressible fluid in 3D and a 3D structure, immersed in the fluid, where the structure was modeled by the equations of linear [7], or quasi-linear [8] elasticity. In [7] initial data have the following regularity $(u_0, v_0, v_1) \in H^5 \times H^3 \times H^2$, while the solution satisfies $(u, v) \in L^2(H^3) \times C(H^3)$. Kukavica and Tuffaha [15, 16] considered a similar problem where the structure was modeled by a linear wave equation. In [15] they proved existence, locally in time, of solution $(u, v) \in L^\infty(H^3) \times C(H^3)$ was established with initial data $(u_0, v_0, v_1) \in H^4 \times H^3 \times H^2$. Recently, a similar problem was studied in [21] where the authors proved the existence of a unique solution $(u, v) \in L^2(H^{2+l}) \times C(H^{7/4+l/2})$ with initial data $(u_0, v_0, v_1) \in H^{1+l} \times H^{3/2+l+\beta} \times H^{1/2+l+\beta}$, where $l \in (1/2, 1)$, $\beta > 0$.

A nonlinear, unsteady, moving boundary, fluid-structure interaction (FSI) problem in which the structure is composed of two layers: a thick layer, and a thin layer which serves as a fluid-structure interface with mass was studied in [20, 23] where the existence of a weak solution was proved. The authors noted that the presence of a thin fluid-structure interface with mass regularizes solutions of the coupled problem. These observations were numerically confirmed in [5]. This is reminiscent of the result from [12] where two linear wave equations were coupled via elastic interface with mass and the well-posedness result was proved by taking advantage of the regularizing effects of the elastic interface.

We would like to emphasize that in most of the references cited in this short overview, the considered models are much more complicated and realistic than the model considered in this note. Therefore, it is not clear that the presented optimal regularity result can also be obtained in these cases. However, we believe that the presented analysis will provide better understanding of asymmetric regularity for the parabolic-hyperbolic systems and of regularization by point mass coupling (or coupling via elastic interface in a more realistic case).

2 Problem description

We consider the following coupled problems of parabolic-hyperbolic type which can be viewed as a simplified fluid structure problem and a fluid-composite structure problem, respectively.

**Problem 1.** (plain heat-wave coupling)

Find $(u, v)$ such that

$$
\begin{align*}
\partial_t u &= \partial^2_x u, & (0, T) \times (-1, 0), \\
\partial_t^2 v &= \partial^2_x v, & (0, T) \times (0, 1),
\end{align*}
$$

(1)
\[
\begin{align*}
\{ & \quad u(t, 0) = \partial_t v(t, 0), \quad t \in (0, T), \\
& \quad \partial_x u(t, 0) = \partial_x v(t, 0), \quad t \in (0, T), \\
& \quad u(t, -1) = v(t, 1) = 0, \quad t \in (0, T), \\
& \quad u(0, x) = u_0(x), \quad x \in (-1, 0), \\
& \quad v(0, x) = v_0(x), \quad \partial_t v(0, x) = v_1(x), \quad x \in (0, 1). 
\end{align*}
\]

Coupling conditions (2) can be viewed as continuity of velocity (2) (kinematic coupling condition) and continuity of normal stresses (dynamic coupling condition) for the simplified FSI model (see e.g. [27]).

Let \( h(t) \) denote the displacement of the point mass which serves as the heat-wave interface.

**Problem 2.** (coupling through point mass)

*Find \((u, v, h)\) such that*

\[
\begin{align*}
\{ & \quad \partial_t u = \partial_x^2 u, \quad \text{in} \quad (0, T) \times (-1, 0), \\
& \quad \partial_t^2 v = \partial_x^2 v, \quad \text{in} \quad (0, T) \times (0, 1), \\
& \quad u(t, 0) = h'(t) = \partial_t v(t, 0), \quad t \in (0, T), \\
& \quad h''(t) = \partial_x v(t, 0) - \partial_x u(t, 0), \quad t \in (0, T), \\
& \quad u(t, -1) = v(t, 1) = 0, \quad t \in (0, T), \\
& \quad u(0, x) = u_0(x), \quad x \in (-1, 0), \\
& \quad h(0) = h_0, \quad h'(0) = h_1, \\
& \quad v(0, x) = v_0(x), \quad \partial_t v(0, x) = v_1(x), \quad x \in (0, 1). 
\end{align*}
\]

Notice that dynamic coupling condition (5) is exactly the second Newton’s Law of motion which states that the point mass acceleration is balanced by difference of normal stresses from the wave and the heat equations.

The first step is to reformulate Problem 1 and Problem 2 in terms of trace function \( g \) defined on \((0, T)\). The main tool will be Neumann to Dirichlet operator for the heat equation. Since we also use D’Alambert formula, time \( T \) will depend on a slope of the characteristics, in particular in the considered case we assume \( T = 1 \). However, our argument can be iterated and the results can be extended to arbitrary \( T \) (including \( T = \infty \)), see Remark [5].

**Definition 1** (Neumann to Dirichlet operator). *Let \( g : [0, 1] \to \mathbb{R} \) and \( u_0 : [-1, 0] \to \mathbb{R} \). Furthermore, let \( Sg \) be a solution of the following initial boundary value problem:*

\[
\begin{align*}
\partial_t (Sg) &= \partial_x^2 (Sg), \quad \text{in} \quad (0, 1) \times (-1, 0), \\
(Sg)(., -1) &= 0, \quad (Sg)(0, .) = u_0, \quad \partial_x (Sg)(0, .) = g.
\end{align*}
\]

*We define Neumann to Dirichlet operator \( L \) with the following formula:*

\[
(L_{u_0}g)(t) := (Sg)(t, 0), \quad t \in (0, T),
\]

where equality is taken in the trace sense.
Remark 1. Poincaré-Steklov Neumann to Dirichlet operator was used in [6] in the analysis of the so-called added-mass effect and its connection to stability issues for the numerical schemes for the FSI problems involving the lower dimensional elastic models.

Before stating the main properties of \( L_{u_0} \) we need to define the function spaces appropriate for analysis of parabolic problems (see e.g. [18]).

\[
H^{s,2s}((0, T) \times (-1, 0)) = L^2(0, T; H^{2s}(-1, 0)) \cap H^s(0, T; L^2(-1, 0)), \ s \geq 0. \tag{9}
\]

Proposition 1. Let \( u_0 \in H^1_0(-1, 0) \) and \( g \in H^{1/4}(0, 1) \). Then \( L_{u_0} \) is well defined and

\[
L_{u_0}g = L_{u_0}0 + L_0g, \tag{10}
\]

where \( L_{u_0}0 \in H^{3/4}_0(0, 1) \) and \( L_0 : H^{1/4}(0, 1) \to H^{3/4}_0(0, 1) \) is an isomorphism.

Proof. The proof is a direct consequence of Theorem 4.4.3 from [18], which states that problem (7) has a unique solution for \( u_0 \in H^1_0(-1, 0) \) and \( g \in H^{1/4}(0, 1) \). Since problem (7) is linear and the trace operator is also linear, operator \( L_{u_0} \) can be written as a sum of \( L_{u_0}0 \) and \( L_0g \). Finally, \((L_0g)(0) = 0\) because functions from \( H^{1,2}((0, 1) \times (-1, 0)) \) satisfy the compatibility conditions (see [18], Theorem 4.2.3.). \( \square \)

We use D’Alambert formula in the wave subdomain to rewrite the full coupled problem as a problem on the interface. Let us denote with \( I \) the area where the solution of the wave equation is only influenced by the interface

\[
I = \{(t, x) \in (0, 1) \times (0, \frac{1}{2}) : |2t - 1| \leq 1 - 2x\},
\]

and by \( II \) we denote the area where the solution of the wave equation is only influenced by the initial data (Figure 1)

\[
II = \{(t, x) \in (0, \frac{1}{2}) \times (0, 1) : |2x - 1| \leq 1 - 2t\}.
\]

Solution \( v \) on \( II \) is given by d’Alembert’s formula:

\[
v(t, x) = \frac{1}{2}(v_0(x-t) + v_0(x+t)) + \frac{1}{2} \int_{x-t}^{x+t} v_1(s)ds, \quad (t, x) \in II. \tag{11}
\]

Similarly as in [12], we exploit the fact that 1D wave equation is symmetric in \( t \) and \( x \) variables and therefore \( v \) on \( I \) is also given by d’Alembert’s formula when the roles of \( t \) and \( x \) are switched, and we consider the region \( I \) as being influenced only by the boundary data on \((0, 1) \times \{0\}\). Therefore, we have the following formula for solution \( v \):

\[
v(t, x) = \frac{1}{2}(h(t-x) + h(t+x)) + \frac{1}{2} \int_{t-x}^{t+x} \partial_x v(s, 0)ds, \quad (t, x) \in I. \tag{12}
\]
where \( h(t) = v(t, 0) \). Now, assuming the continuity of \( v \) on the ray given by formula \( t = x \) and using formulas \((12)\) and \((11)\) at point \((\frac{t}{2}, \frac{t}{2})\) we get:

\[
\begin{align*}
    h(t) + \int_{0}^{t} \partial_x v(s, 0) ds &= v_0(t) + \int_{0}^{t} v_1(s) ds.
\end{align*}
\]

By differentiating this equality we get

\[
\begin{align*}
    c(s) + \partial_x v(s, 0) &= v'_0(s) + v_1(s),
\end{align*}
\]

where \( c(t) = h'(t) \) is the interface velocity. Now we are in position to reformulate Problems 1 and 2 in terms of the Neumann to Dirichlet operator \( L_{uo} \).

**Proposition 2.** Let \( L_{uo} \) be Neumann to Diriclet operator defined by \((8)\) and let \( g = \partial_x u(., 0) \). Then for the wave equation initial data \((v_0, v_1)\) the following statements hold:

1. **Problem 1** is formally equivalent to the following problem:
   
   Find \( g \) such that
   
   \[
   (I + L_{uo}) g = v'_0 + v_1. \tag{14}
   \]

2. **Problem 2** is formally equivalent to the following problem:
   
   Find \( g \) such that
   
   \[
   (I + L_{uo}) g + (L_{uo} g)' = v'_0 + v_1. \tag{15}
   \]

**Proof.** First we notice that from kinematic coupling condition \((2)_{1}\) (or \((5)_{1}\)) and the definition of operator \( L_{uo} \) we have

\[
    c = h' = \partial_v(., 0) = u(., 0) = L_{uo}(\partial_x u(., 0)) = L_{uo} g.
\]

Now \((14)\) follows directly from \((13)\) and the dynamic coupling condition \((2)_{2}\), i.e. \( g = \partial_x u(., 0) = \partial_x v(., 0) \). Similarly, \((15)\) follows directly from \((13)\) and the dynamic coupling condition \((5)_{2}\), i.e. \( \partial_x v(., 0) = \partial_x u(., 0) + h'' = g + c' \). \(\square\)
Remark 2. The extra term \((L_{u_0}g)'\) in [15] comes from the inertia of the point mass located at the interface between the heat and the wave equation domains.

Using the Dirichlet to Neumann formulation of problems 1 and 2 we show next that for problem 2, which contains the interface with point mass, less regularity of the initial data is required to recover the same interface regularity as in problem 1, which has no point mass at the interface. More precisely, we have the following.

**Proposition 3.** Let \(u_0 \in H^1_0(-1,0)\).

1. Let \((v_0, v_1) \in H^{5/4}(0,1) \times H^{1/4}(0,1)\). Then there exists a unique solution \(g \in H^{1/4}(0,1)\) to problem [14].
2. Let \((v_0, v_1) \in H^{3/4}(0,1) \times H^{-1/4}(0,1)\). Then there exists a unique solution \(g \in H^{1/4}(0,1)\) to problem [15].

**Proof.** Using Proposition 1, equation (10), we can rewrite (14) in the following equality:

\[(I + L_0)g = v_0 + v_1 - L_{u_0}0.\]

Notice that the right hand side of the above equation is a \(H^{1/4}(0,1)\) function. Furthermore, we can view \(L_0\) as an operator on \(H^{1/4}(0,1)\), i.e. \(L_0 : H^{1/4}(0,1) \to H^{1/4}(0,1)\). Now, \(L_0\) is a compact operator on \(H^{1/4}(0,1)\) because of compactness of embedding \(H^{3/4}(0,1) \hookrightarrow H^{1/4}(0,1)\). Therefore, we can use Fredholm alternative (see e.g. Theorem 6.6 in [4]) and to complete the proof of statement 1, it only remains to prove \(\text{ker}(I + L_0) = \{0\}\).

Let us take \(g \in \text{ker}(I + L_0)\), i.e. \(L_0g = -g\). By multiplying (7) by \(Sg\), integrating on \((0,1) \times (-1,0)\) and integrating by parts, we get the following equality:

\[\frac{1}{2} \frac{d}{dt} \|Sg\|_{L^2((-1,0))}^2 = -\| \partial_x (Sg) \|_{L^2((0,1) \times (-1,0))}^2 - \|g\|_{L^2((0,1))}^2.\]

Now from the initial condition \(Sg(0,.) = 0\), we conclude \(Sg = 0\) and therefore \(g = Sg(.,0) = 0\). This concludes the proof of the first statement of the Proposition.

Let us prove the second statement. We denote the right-hand side of (15) by \(f = v_0' + v_1\). We can formally solve ODE (15) for unknown \(L_{u_0}g\) and get

\[(L_{u_0}g)(t) = \int_0^t e^{s-t}(f(s) - g(s))ds.\]

By using Proposition 11 we get

\[L_0g + \int_0^t e^{s-t}g(s)ds = \int_0^t e^{s-t}f(s)ds - L_{u_0}0.\]

We apply \(L_0^{-1}\) to obtain the following formulation, which is formally equivalent to (15):

\[(I + W)g = L_0^{-1}\left( \int_0^t e^{s-t}f(s)ds - L_{u_0}0 \right),\]  

(16)
where \((Wg)(t) = L_0^{-1}\left(\int_0^t e^{s-t} g(s)ds\right)\). We again use Fredholm alternative to prove the existence result. Namely, \(W\) is a well defined operator on \(H^{1/4}(0,1)\). Furthermore, it is compact since \(\text{Im}(W) \subset H^{3/4-\varepsilon}(0,1), \varepsilon > 0\) (by integration we gain one derivative and by applying Dirichlet to Neumann operator \(L_0^{-1}\) we lose half of a derivative, see Propositions 1 and 4). Therefore it only remains to prove \(\ker(I + W) = \{0\}\).

Let \(g \in \ker(I + W)\). Then

\[
(L_0g)(t) = -\int_0^t e^{s-t} g(s)ds.
\]

Using a calculation analogous to the one in the first part of the proof we get the following equality:

\[
\frac{1}{2} \frac{d}{dt} \|Sg\|_{L^2(-1,0)}^2 = -\|\partial_x (Sg)\|_{L^2((0,1) \times (-1,0))}^2 - \int_0^1 g(t) \int_0^t e^{s-t} g(s)dsdt. \tag{17}
\]

To show that the right-hand side of \(17\) is negative, we define \(G(t) = \int_0^t e^{s-t} g(s)ds\). Straightforward calculation yields \(g = G' + G\). Therefore, we have

\[
-\int_0^1 g(t) \int_0^t e^{s-t} g(s)dsdt = -\int_0^1 (G' + G)G = -\frac{1}{2} G^2(1) - \int_0^1 G^2 \leq 0.
\]

Now, from \(17\) we deduce \(g = 0\), i.e. \(\ker(I + W) = \{0\}\).

Hence, we proved the existence of unique \(g \in H^{1/4}(0,1)\) satisfying \(16\), where integral \(\int_0^t e^{s-t} f(s)ds\) is understood in a dual \(H^{-1/4}(0,1)\) sense. By applying \(L_0\) on \(16\) and by differentiating the resulting equation, we show that \(g\) is a unique solution to the problem \(15\), where equality \(15\) is understood in the distributional sense.

Let us conclude this section by proving the existence theorem for original coupled Problems 1 and 2. Before stating the theorem, let us define the hyperbolic solution spaces as follows:

\[
V^s((0,T) \times (0,1)) = \{v \in C([0,T]; H^s(0,1)) : \partial_t^k v \in C([0,T]; H^{s-k}(0,1)), k = 1, \ldots, [s] \}, \ s \geq 0.
\]

**Theorem 1.** Let \(u_0 \in H^1(-1,0)\) and let \(g \in H^{1/4}(0,1)\) be given by Proposition 3. Furthermore, let \(c = L_{u_0}g, u = Sg, \) where \(L_{u_0}\) and \(S\) are defined by \(8\) and \(7\), respectively; and \(h(t) = \int_0^t c(t)dt\). Finally, let \(v\) be a solution to the following initial boundary value problem for the wave equation:

\[
\begin{align*}
\partial_t^2 v &= \partial_x^2 v, \quad \text{in} \ (0,1) \times (0,1), \\
v(t,0) &= h(t), \ v(t,1) = 0, \ t \in (0,1), \\
v(0,x) &= v_0(x), \ \partial_t v(0,x) = v_1(x), \ x \in (0,1).
\end{align*}
\]  

Then the following statements hold:
Proposition 4. Let prove an analogue of Proposition 1 for these initial data. Precisely, we have
\[ \frac{\partial}{\partial x} u, \frac{\partial}{\partial v} v \] where \( L \) is a hidden regularity theorem for the wave equation to justify trace conditions on the interface for Problem 2, are satisfied in the trace sense, where one has to use the so-called hidden regularity theorem for the wave equation to justify trace \( \partial_x v(\cdot, 0) \). More precisely, we have \( \partial_x v(\cdot, 0) \in L^2(0, 1) \) (see e.g. \[17\]).

Remark 3. Notice that the regularity assumptions for the initial data \((v_0, v_1)\) in the case of coupling through point mass are higher in Theorem \[7\] than in Proposition \[3\] \((H^1 \text{ vs } H^{3/4})\). This is due to the fact that one needs certain regularity for the solution of the wave equation to make sense of the traces needed in the coupling conditions. In formulation \[15\] the coupling conditions are “encoded” in operator \( L_0 \) and are implicit, so one can define lower regularity solutions.

Remark 4. The displacement of the interface is regularized because of the parabolic regularity. Namely, we have \((h, c) \in H^{7/4}(0, 1) \times H^{3/4}(0, 1)\), which is a gain of 1/2 derivative w.r.t. the wave displacement and velocity \((v, \partial_v v)\). However, the wave that is reflected from the interface is not regularized due to the low regularity of \( g = \partial_x u(\cdot, 0) \in H^{1/4}(0, 1) \) in the case of Problem \[7\] or due to the low regularity of \( h'' \in L^2(0, 1) \) in the case of Problem \[3\].

3 Optimal regularity

In order to prove regularity results for the considered problems, one has to assume that initial data satisfy certain compatibility conditions. However, to avoid technical complications and to make the text more accessible to the reader, we have chosen the simplest kind of compatibility conditions rather than pursuing full generality.

Therefore, we begin by assuming that the initial data are in \( H^s_0 \) spaces, and prove an analogue of Proposition \[7\] for these initial data.

**Proposition 4.** Let \( u_0 \in H^{s+1/2}_0(-1, 0) \) and \( g \in H^s_0(0, 1) \), \( s \geq 1 \), \( s \neq n + \frac{1}{2} \), \( s \neq n + \frac{3}{4} \), \( n \in \mathbb{N} \). Then \( L_{u_0} \) is well defined and \( L_{u_0}g = L_{u_0}0 + L_{0g} \), where \( L_{u_0}0 \in H^{s+1/2}_0(0, 1) \) and \( L_0 : H^s_0(0, 1) \to H^{s+1/2}_0(0, 1) \) is an isomorphism.

**Proof.** The proof is the same as the proof of Proposition \[7\] with a change that in this case we use the regularity result for parabolic problems (Theorem 4.6.2 in \[18\]). Restrictions on parameter \( s \) are necessary to satisfy the assumptions of the theorem we are applying. \(\square\)
A consequence of this proposition is the following regularity result for Problems 14 and 15.

**Corollary 1.** Let \( u_0 \in H^{2s+1/2}(0,1) \), then \( u \in C^s(0,1) \times C^s(0,1) \times C^s(0,1) \), \( u' \in H^{s-1/2}(0,1) \times H^{s-3/2}(0,1) \), \( v' \in H^{s-3/2}(0,1) \).

1. Let \( (u_0, v_0, v_1) \in H^{s+1}(0,1) \times H^{s}(0,1) \), \( v_1' \in H^{s}(0,1) \). Then there exists a unique solution \( g \in H^{s}(0,1) \) to problem 14.

2. Let \( (u_0, v_0, v_1) \in H^{s-1/2}(0,1) \times H^{s-3/2}(0,1) \), \( v_1' \in H^{s-3/2}(0,1) \). Then there exists a unique solution \( g \in H^{s}(0,1) \) to problem 15.

**Proof.** The proof is analogous to the proof of Proposition 3, where we use Proposition 3 instead of Proposition 1.

**Theorem 2.** Let \( u_0 \in H^{2s+1/2}(0,1) \), then \( u \in C^s(0,1) \times C^s(0,1) \times C^s(0,1) \), \( u' \in H^{s-1/2}(0,1) \times H^{s-3/2}(0,1) \), \( v' \in H^{s-3/2}(0,1) \).

1. Let \( (u_0, v_0, v_1) \in H^{s+1}(0,1) \times H^{s}(0,1) \), \( v_1' \in H^{s}(0,1) \), then

\[
(u, v) \in H^{s+3/4,2s+3/2}(0,1) \times V^{s+1}((0,1) \times (0,1))
\]

is a unique solution to Problem 2.

2. Let \( (u_0, v_1) \in H^{r+1/2}(0,1) \times H^{r-1/2}(0,1) \), \( v_1' \in H^{r-1/2}(0,1) \), then

\[
(u, r, v) \in H^{r+3/4,2r+3/2}((0,1) \times (-1,0)) \times V^{r+3/2}(0,1) \times V^{r+1/2}((0,1) \times (0,1))
\]

is a unique solution to Problem 3, where \( r = \max\{s, 1/2\} \).

**Proof.** The proof is a direct consequence of Theorem 1 and the regularity of \( g \), which is a consequence of Corollary 1. Notice that assumptions \( u_0 \in H^{2s+1/2}(-1,0) \), \( g \in H^{0}(0,1) \), \( v_0 \in H^{s+1}(0,1) \) (or \( H^{r+1/2}(0,1) \) in the case of Problem 2) and \( v_1' \in H^{s}(0,1) \) (or \( H^{r-1/2}(0,1) \) in the case of Problem 2) ensure that compatibility conditions for the wave and the heat equation are satisfied. Furthermore, \( r \geq 1/2 \) ensures that the trace \( \partial_x v, \partial_t v \) in coupling condition 3 is well-defined.

**Remark 5. On a global in time solution**

Using the same techniques as in Theorems 2 and 3, one could prove the existence of a global in time solution by restarting the proof from time \( t = 1 \) and reiterating the procedure. This would yield a global in time solution since the length of time interval on every step would be 1. However, in order to complete the proof, one would have to work with general compatibility conditions and a non-zero righthand side. Since in this note we are primarily interested in optimal regularity and regularizing effects of a point mass coupling, we skip the details for technical simplicity.
4 Conclusions

In this note we study the heat-wave systems of equations which are coupled at the interface between the two respective domains in two different ways: with and without point mass at the interface. These systems can be viewed as simplified FSI models. We prove an optimal regularity theorem for both systems. The regularity theorem is optimal in the sense that we have minimal regularity assumptions for the wave initial data for which we can use the full parabolic regularity for the heat component of the solution. A further increase in the regularity of the wave initial data would not yield an increase in the regularity of the heat component of the solution. We were also interested in the regularity of the interface and the regularizing effects of a point mass coupling. Our analysis revealed the following properties of the considered coupled systems:

1. Even in the case of the “plain” heat-wave coupling, namely the case without point mass, the interface displacement is regularized by a degree of $1/2$ of the derivative w.r.t. to the displacement of the wave component of the solution. This effect is a consequence of the parabolic regularity of the heat component. However, the wave which is reflected from the interface has the same regularity as the incoming wave, so there is no regularization in the wave component.

2. The point mass coupling regularizes the problem by $1/2$ of the derivative in a sense that the wave initial data needed for the optimal regularity result have $1/2$ derivative less (in the sense of Sobolev spaces) than in the “plain” heat wave coupling. Moreover, the interface displacement is now regularized by a degree of one derivative w.r.t. to the displacement of the wave component of the solution. However, there is still no regularization in the wave component of the solution.

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