Solutions of the functional tetrahedron equation connected with the local Yang – Baxter equation for the ferro-electric.

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Abstract: Local (or modified) Yang – Baxter equation (LYBE) gives the functional map from the parameters of the weights in the left hand side to the parameters of the correspondent weights in the right hand side of LYBE. Such maps solve the functional tetrahedron equation. In this paper all the maps associated with LYBE of the ferro-electric type with single parameter in each weight matrix are classified.

Key words: Tetrahedron equation, 2 + 1 integrability, local Yang – Baxter equation.

1 Introduction

As it is well known, local Yang – Baxter equations (LYBE-s), the proper 3D generalization of the zero – curvature condition [1], are tightly connected with solutions of the so – called functional tetrahedron equation. Usually, LYBE-s are used for the construction of three-dimensional discrete “classical” integrable models [2, 3]. However, in this paper the field of our interest is the functional tetrahedron equation itself and a class of its solutions, and the weights of LYBE we interpret as $L$ – operators in the three dimensional sense.

Recall first the general concepts. Let $L$ be the matrix of the weights, depending on some independent parameters, say, $\vec{x}$. The local YBE is

$$L_{12}(\vec{x}_a) L_{13}(\vec{x}_b) L_{23}(\vec{x}_c) = L_{23}(\vec{x}_c') L_{13}(\vec{x}_b') L_{12}(\vec{x}_a'),$$

(1)

where $\vec{x}_k'$ are some functions of $\vec{x}_k$:

$$\vec{x}_a' = f_1(\vec{x}_a, \vec{x}_b, \vec{x}_c), \quad \vec{x}_b' = f_2(\vec{x}_a, \vec{x}_b, \vec{x}_c), \quad \vec{x}_c' = f_3(\vec{x}_a, \vec{x}_b, \vec{x}_c).$$

(2)

It is supposed that:

- all $L$-s in (1) have the same functional structure, and differ only by their vector arguments,
being considered as the system of equations with respect to $\vec{x}_k$ (or with respect to $\vec{x}_k$), eq. (1) has the unique solution.

For given LYBE with the solution (2) associate an operator $R_{a,b,c}$, realizing the map

$$R_{a,b,c} : \vec{x}_a, \vec{x}_b, \vec{x}_c \rightarrow \vec{x}_a', \vec{x}_b', \vec{x}_c',$$

i.e. acting on the space of functions of $\vec{x}_a, \vec{x}_b, \vec{x}_c$ as follows:

$$R_{a,b,c} \cdot \phi[\vec{x}_a, \vec{x}_b, \vec{x}_c] = \phi[\vec{x}_a', \vec{x}_b', \vec{x}_c'] \cdot R_{a,b,c}.$$  (4)

Then the following formal equation, interpreted as the linear problem for three dimensional object $R$, holds:

$$L_{12}(\vec{x}_a) L_{13}(\vec{x}_b) L_{23}(\vec{x}_c) \cdot R_{a,b,c} = R_{a,b,c} \cdot L_{23}(\vec{x}_c) L_{13}(\vec{x}_b) L_{12}(\vec{x}_a).$$  (5)

Rather standard manipulations with the quadrilateral formed by six $L$-s with the arguments $\vec{x}_1, ..., \vec{x}_6$ allows one to prove from the uniqueness of LYBE the tetrahedron equation for $R$:

$$R_{1,2,3} \cdot R_{1,4,5} \cdot R_{2,4,6} \cdot R_{3,5,6} = R_{3,5,6} \cdot R_{2,4,6} \cdot R_{1,4,5} \cdot R_{1,2,3}.\quad (6)$$

This is the functional relation, the left and right sides of it are to be understood acting equivalently on the space of functions of six vector variables:

$$R_{123} \cdot \left( R_{145} \cdot \left( R_{246} \cdot \left( \phi[\vec{x}_1, ..., \vec{x}_6] \right) \right) \right) =

= R_{356} \cdot \left( R_{246} \cdot \left( R_{145} \cdot \left( \phi[\vec{x}_1, ..., \vec{x}_6] \right) \right) \right).$$  (7)

2 Local Yang–Baxter Equation

Now consider LYBE for simplest two–state ferro-electric weights:

$$L(a, b, c, d) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a & b & 0 \\
0 & c & d & 0 \\
0 & 0 & 0 & z
\end{pmatrix},$$  (8)

where the ferro-electric condition $z = bc - ad$. Being the special case of the free fermionic model, LYBE for this case can be rewritten in the equivalent but irreducible form (i.e. in the form where all equations are independent). This trick is well known, and there is no necessity do describe it here. Let

$$X_{12} = \begin{pmatrix}
a_1 & b_1 & 0 \\
c_1 & d_1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad X'_{12} = \begin{pmatrix}
a'_1 & b'_1 & 0 \\
c'_1 & d'_1 & 0 \\
0 & 0 & 1
\end{pmatrix},$$

2
Substituting equations, and the form of $X$ functional from rest of (10) and solving the appeared quite soon. Thus the subsequent step is to regard somebody of $a, b, c, d$ so one may conclude be a constant, etc.

Thus the irreducible part of LYBE is

$$X_{12} \cdot X_{13} \cdot X_{23} = X'_{13} \cdot X'_{12} \cdot X'_{23}.$$  

Thus there are nine independent relations for twelve “primed” variables. In general the entries of $X_{i,j}$, $a_k, b_k, c_k, d_k$, they are matrices. Equation (10) was investigated by I. G. Korepanov in this most general form, he proved the irreducibility of this equation and the existence of an unique (up to some “gauge” ambiguities) solution, and he pointed out the connection of eq. (10) with the functional tetrahedron equation first.

Now consider the case when $\vec{x} \equiv x$, this case we call as one – parameter functional space. $a, b, c, d$ are $C$ - number functions of $x$. In eq. (10) there are nine equations, and the form of $X(x)$ is to be chosen so that there are only three independent entries in (10). This problem is rather nontrivial, the first step of its solution is considering equations (11,12,13,14), this gives several permitted forms of $a(x), d(x)$ and $z(x)$. The second step is excluding primed variables from rest of (10) and solving the appeared functional equations for $b(x)$ and $c(x)$. These manipulations are simple but tedious.

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Give just as an example one of the possible reasonings. Suppose $a(x), d(x)$ and $z(x)$ are nontrivial. Then if the relations (13) are not equivalent, they would give an expression for $x'_1$ and $x'_2$ via $x_1$ and $x_2$. If so, (11,12,13,14) must be equivalent, and this gives either trivial solution $x'_k = x_k$, or contradiction. Thus both relations in (13) are equivalent, so one may conclude $z(x) = (1 - k)a(x)d(x)$, where $k \neq 1$ is a common constant. Substituting $c(x) = ka(x)d(x)/b(x)$ into (10), we obtain a functional contradiction quite soon. Thus the subsequent step is to regard somebody of $a(x), d(x)$ or $z(x)$ to be a constant, etc.

\[
X_{13} = \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix}, \quad X_{13}' = \begin{pmatrix} a'_2 & 0 & b'_2 \\ 0 & 1 & 0 \\ c'_2 & 0 & d'_2 \end{pmatrix}, \\
X_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix}, \quad X_{23}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a'_3 & b'_3 \\ 0 & c'_3 & d'_3 \end{pmatrix}. \tag{9}
\]
As the result it appears that there are only six (up to some equivalence) independent forms of \(X\) and so only six solutions of the functional TE of such kind. We'll numerate them by the Greece letters.

3 Solutions of the functional TE

3.1 Case \((\alpha)\)

\[
X(x) = \begin{pmatrix} 1 & x \\ 0 & k \end{pmatrix},
\]

\(k\) being a constant, this gives

\[
R_{123}: x_1, x_2, x_3 \rightarrow x_1, kx_2 + x_1x_3, x_3.
\]

Inverse map:

\[
R^{-1}_{123}: x_1, x_2, x_3 \rightarrow x_1, \frac{x_2 - x_1x_3}{k}, x_3.
\]

3.2 Case \((\beta)\)

\[
X(x) = \begin{pmatrix} 1 & x \\ k/x & 0 \end{pmatrix},
\]

this gives

\[
R_{123}: x_1, x_2, x_3 \rightarrow \frac{kx_2 + x_1x_3}{x_3}, x_1x_3, \frac{kx_2x_3}{kx_2 + x_1x_3}.
\]

Inverse map:

\[
R^{-1}_{123}: x_1, x_2, x_3 \rightarrow \frac{x_1x_2}{x_2 + x_1x_3}, \frac{x_1x_3}{k}, \frac{x_2 + x_1x_3}{x_1}.
\]

3.3 Case \((\gamma)\)

\[
X(x) = \begin{pmatrix} x & 0 \\ 1 - x & 1 \end{pmatrix},
\]

then

\[
R_{123}: x_1, x_2, x_3 \rightarrow \frac{x_3 - x_2 + x_1x_2}{x_3}, \frac{x_1x_2x_3}{x_3 - x_2 + x_1x_2}, x_3.
\]

Inverse map:

\[
R^{-1}_{123}: x_1, x_2, x_3 \rightarrow \frac{x_1x_2}{x_3 - x_1x_3 + x_1x_2}, x_3 - x_1x_3 + x_1x_2, x_3.
\]
3.4 **Case ($\delta$)**

\[ X(x) = \begin{pmatrix} x & 1 \\ 1 - x & 0 \end{pmatrix}. \]  

(24)

Then

\[ R_{123} : x_1, x_2, x_3 \rightarrow \frac{x_1x_2}{x_1 + x_3 - x_1x_3}, \frac{x_1 + x_3 - x_1x_3}{x_1 + x_3 - x_1x_2 - x_1x_3}. \]  

(25)

Here

\[ R^2 = 1. \]  

(26)

This transformation is connected with the Pentagon equation and is described in [3].

3.5 **Case ($\epsilon$)**

\[ X(x) = \begin{pmatrix} x & 1 + ix \\ 1 - ix & x \end{pmatrix}. \]  

(27)

then

\[ R_{123} : x_1, x_2, x_3 \rightarrow \frac{x_1x_2}{x_1 + x_3 + x_1x_2x_3}, \frac{x_1 + x_3 + x_1x_2x_3}{x_1 + x_3 + x_1x_2x_3}, \frac{x_2x_3}{x_1 + x_3 + x_1x_2x_3}. \]  

(28)

again with

\[ R^2 = 1. \]  

(29)

This is the electric network transformation, considering by R. M. Kashaev in [3]. Note that he realized this LYBE in terms of bosonic representation, while our case is the fermionic one.

3.6 **Case ($\zeta$)**

\[ X(x) = \begin{pmatrix} x & -s(x) \\ s(x) & x \end{pmatrix}, \]  

(30)

where \( s^2(x) = 1 - x^2 \). Thus the argument of \( X \) is not simply \( x \), but the pair \((x, s(x))\). This case is equivalent to Onsager’s star–triangle and also can be interpreted as the decomposition of holonomy group’s element with respect to the Euler bases, and \( X \)-s are the rotations:

\[ X(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}. \]  

(31)

This LYBE was also considered by R. M. Kashaev in [3].
Formally, in terms of single $x$, (10) gives

$$R_{123} : x_1, x_2, x_3 \rightarrow \frac{x_1 x_2}{F(x_1, x_2, x_3)}, \frac{x_2 x_3}{F(x_1, x_2, x_3)},$$

where $F$ can be found from

$$s(F) = s(x_2) x_1 x_3 - s(x_1) s(x_3),$$

and so this transformation is two-foiled one (a sign of $F$ is unessential, there may be two different signs of $s(x_1) s(x_2) s(x_3)$), and $R$-s from different foils are inverse.

### 3.7 Discussion

For given map $R_{123}$ there are a lot of equivalent maps and its descendants. Surely, if $R$ solves the functional TE, then $R^{-1}$ also solves it. Note that in all cases the inverse maps $R^{-1}$ are connected with $X^t$. Another automorphism is

$$R_{123} \rightarrow R'_{123} \equiv R_{321}.$$  

Also the functional tetrahedron equation admits the gauge freedom

$$R_{123} \rightarrow j_1 j_2 j_3 \circ R_{123} \circ j_1^{-1} j_2^{-1} j_3^{-1},$$

where $j^{-1}$ is the inverse function to $j$: $j^{-1}(j(x)) = j(j^{-1}(x)) = x$.

Consider the case when $j(x) = sx$, i.e. one can introduce common scaling factor for all variables, $x_k \rightarrow sx_k$, and one can choose it in different ways (putting it to zero or to something else). Thus from ($\delta$), as well as from ($\epsilon$), one can obtain

**Case ($\eta$):**

$$R_{123} : x_1, x_2, x_3 \rightarrow \frac{x_1 x_2}{x_1 + x_3}, x_1 + x_3, \frac{x_2 x_3}{x_1 + x_3}. $$

Also the structure of the tetrahedron equation allows us to impose an order relation

$$x_1 << x_2, x_3 << x_4, x_5, x_6,$$

so again ($\delta$) and ($\epsilon$) both give

**Case ($\theta$)**

$$R_{123} : x_1, x_2, x_3 \rightarrow x_1 \frac{x_2}{x_3}, x_3, x_2.$$
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