Photon orbital angular momentum and mass in a plasma vortex

F. Tamburini\textsuperscript{1}, A. Sponselli\textsuperscript{1}, B. Thidé\textsuperscript{2}(a) and J. T. Mendonça\textsuperscript{3}

\textsuperscript{1} Department of Astronomy, University of Padova - vicolo dell’ Osservatorio 3, I-33122 Padova, Italy, EU
\textsuperscript{2} Swedish Institute of Space Physics, Physics in Space, Ångström Laboratory - P. O. Box 537, SE-751 21, Uppsala, Sweden, EU
\textsuperscript{3} IPFN and CFIF, Instituto Superior Técnico - Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal, EU

received on 12 May 2010; accepted by R. A. Treumann on 19 May 2010
published online 4 June 2010

PACS 52.35.We – Plasma vorticity
PACS 14.70.Bh – Photons
PACS 03.50.De – Classical electromagnetism, Maxwell equations

Abstract – We analyse the Anderson-Higgs mechanism of photon mass acquisition in a plasma and study the contribution to the mass from the orbital angular momentum acquired by a beam of photons when it crosses a spatially structured charge distribution. To this end we apply Proca-Maxwell equations in a static plasma with a particular spatial distribution of free charges, notably a plasma vortex, that is able to impose orbital angular momentum (OAM) onto light. In addition to the mass acquisition of the conventional Anderson-Higgs mechanism, we find that the photon acquires an additional mass from the OAM and that this mass reduces the Proca photon mass.

Introduction. – Influenced by results derived in 1962 by Schwinger [1], Anderson showed, in 1963, that a photon propagating in a plasma acquires a mass, \( \mu_\gamma = \frac{\hbar \omega_p}{c^2} \), where \( \omega_p \) is the plasma frequency [2]. In this Anderson-Higgs process the photon acquires an effective mass because of its interaction with plasmons, or better, a hidden gauge invariance in the plasma [3]. In order to study photons that have acquired an effective mass, it is convenient to replace Maxwell’s equations by Proca-Maxwell equations [4,5]. In this letter, we use this approach to analyse the contribution to the mass from the orbital angular momentum acquired by a beam of photons as it traverses a spatially structured charge distribution.

The orbital angular momentum (OAM) of light is a newly recognised observable of electromagnetic (EM) fields that is intimately related to optical vortices (OVs), phase defects embedded in certain particular light beams. OAM of light beams can be generated by the imprinting of vorticity onto the phase distribution of the original beam when it crosses inhomogeneous non-linear optical systems [6] or particular spatial structures such as fork holograms or spiral phase plates. Such beams can be mathematically described by a superposition of Laguerre-Gaussian (L-G) modes characterized by the two integer-valued indices \( l \) and \( p \), or by Kummer beams [7]. The azimuthal index \( l \) describes the number of twists of the helical wavefront and the radial index \( p \) gives the number of radial nodes of the mode.

The EM field amplitude of a generic L-G mode, in a plane perpendicular to the direction of propagation, is

\[
F_{pl}(r, \varphi) = \left( \frac{(l + p)!}{4\pi p!} \right)^\frac{1}{2} \left( \frac{r^2}{w^2} \right)^{\frac{|l|}{2}} L_p^{|l|} \left( \frac{r^2}{w^2} \right) \exp \left( -\frac{r^2}{2w^2} \right) e^{il\varphi},
\]

obeying the orthogonality condition

\[
\int_0^\infty r \, dr \int_0^{2\pi} F_{pl}^* F_{pl'} \, d\varphi = \delta_{pp'} \delta_{ll'},
\]

where \( w \) is the beam waist and \( L_n^m \) is the associated Laguerre polynomial. The phase factor \( \exp(-il\varphi) \) is associated with an OAM of \( \hbar l \) per photon, and a phase singularity is embedded in the wavefront, along the propagation axis, with a topological charge \( l \) [8,9]. The intensity distribution of an L-G mode with \( p = 0 \) has an annular shape with a central dark hole where the intensity vanishes because of total destructive interference.

As is well known, not only the linear momentum of light but also its angular momentum can propagate to infinity [10–12]. The OAM property of the field remains stable during the propagation in free space and has been experimentally verified down to the single-photon limit [13]. It has also been studied theoretically [14].

A photon in vacuum has its intrinsic angular momentum, the spin (SAM), that can assume only two values,
σ = ±1 corresponding to transverse left-hand and right-hand circular polarisation, respectively. If the photon were massive, then it would carry a third independent longitudinal polarisation component along the direction of propagation [15]. The exchange of angular momentum between a photon beam and a plasma vortex and the possible excitation of photon angular momentum states in a plasma was analysed in ref. [16]. There can also be an exchange of angular momentum between electromagnetic and electrostatic waves in a plasma due to stimulated Raman and Brillouin scattering processes [17]. The properties of plasmons carrying OAM were analysed in ref. [18].

In this letter we show that the OAM acquired by a photon in a spatially structured plasma can be interpreted as an additional mass-like term that appears in Proca equations. More specifically, we study the propagation of a photon with wavelength \( \lambda \) in a static helicoidally distributed plasma with step \( q_0 = \lambda / p \), where \( p \) is an integer [17]. This apparent mass term shows that structured spatial and temporal inhomogeneities of matter distribution can impose properties onto single quanta. This can be interpreted as a manifestation of a Mach principle in classical and quantum electrodynamics. Henceforth in this letter, we use natural units \( c = e = h = G = 1 \).

**Proca equations and OAM in a plasma vortex.** –

The Maxwell-Proca Lagrangian density \( \Lambda \) describes the kinetic potential of a massive EM field in which there appears a mass term \( \mu_0^2 A_\mu A^\mu / 2 \) because of the chosen gauge invariance:

\[
\Lambda = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu + \frac{1}{2} \mu_0^2 A_\mu A^\mu,
\]

(3)

where \( \mu_0^{-1} \) is the reduced Compton wavelength associated with the photon rest mass and \( j_\mu = (\rho, -j) \) is the 4-current. \( F^{\mu\nu} \) is the electromagnetic tensor and \( A_\mu \) the 4-vector potential. From the Lagrangian density, one obtains the covariant form of the Proca-Maxwell equations

\[
\frac{\partial F_{\mu\nu}}{\partial x_\nu} + \mu_0^2 A_\mu = 4\pi j_\mu
\]

(4)

that leads to the Proca wave equation for \( A_\mu \)

\[
(\Box - \mu_0^2) A_\mu = -4\pi j_\mu.
\]

(5)

When one considers a photon propagating in a plasma, the usual Maxwell equations can be replaced by the set of Proca-Maxwell equations in which there appears a mass-like term for the photon due to light-matter interaction [4]. The usual formulation of Proca-Maxwell equations is obtained by expanding eq. (4) in terms of the electric \( E \) and magnetic \( B \) fields, that, in the presence of charges and currents \( \rho \) and \( j \), are

\[
\nabla \cdot E = 4\pi \rho - \mu_0^2 \phi,
\]

\[
\nabla \times E = -\frac{\partial B}{\partial t},
\]

\[
\nabla \cdot B = 0,
\]

\[
\nabla \times B = 4\pi j + -\frac{\partial E}{\partial t} - \mu_0^2 A,
\]

(6)

and for \( \mu_0 \to 0 \), they smoothly reduce to Maxwell’s equations. The Poynting vector for massive photons depends directly on both the scalar and vector potentials

\[
S = \frac{1}{4\pi}(E \times B + \mu_0^2 \phi A)
\]

(7)

and also the energy density has an explicit dependency on the potentials

\[
\rho = \frac{1}{8\pi}(E^2 + B^2 + \mu_0^2 \phi^2 + \mu_0^2 A^2).
\]

(8)

While the Lorentz invariance remains valid, the gauge invariance is lost because the potentials become observable through the acquired energy densities \( \mu_0^2 \phi^2 / 8\pi \) and \( \mu_0^2 A^2 / 8\pi \), respectively, due to the interaction of the photon with the plasma or its confinement within the observable Universe.

Let us consider a transverse EM wave propagating through an isotropic plasma, cast to form a helicoidal static plasma vortex. The heavy ions constitute a neutralising background, and their motion can, in the first approximation, be neglected. The motion of free electrons forms a three-dimensional current \( j = -n v \), where \( n \) is the electron number density, and \( v \) the velocity of the electrons in the medium derived from the electron fluid equations

\[
\frac{\partial n}{\partial t} + \nabla \cdot n v = 0,
\]

(9)

\[
\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{1}{m}(E + v \times B).
\]

(10)

Thermal and relativistic mass effects are ignored. In the first approximation, the mean electron velocity in the static plasma vortex becomes \( v = v_0(r, t) + \delta v \), where \( v_0 \) is the background velocity and \( \delta v \) is the perturbation associated with the propagating EM wave. The electron number density is

\[
n = n_0 + \tilde{n}(r, z) \cos(l_0 \varphi + q_0 z),
\]

(11)

where \( n_0 \) is the background plasma density and the helicoidal density perturbation in the plasma is given by the latter term, expressed in cylindrical coordinates, \( r \equiv (r, \varphi, z) \). The coordinate \( z \) is the axis of symmetry around which the electron spiral is winding and along which the EM wave is propagating.

Similarly as in a spiral phase plate, the number of electrons affecting the EM wave depends on \( z \) and can
vary slowly on a scale much larger than the spatial period $z_0 = 2\pi/q_0$, where $q_0$ is the helix step. For a typical double vortex one obtains $l_0 = 1$. Neglecting for a moment the rotation of plasma and considering the case of a static helical perturbation, the current density perturbation in the plasma is $j = -n(r)\delta v$, and the propagation equation of the electric field becomes

$$\frac{\partial^2 E}{\partial t^2} - \nabla^2 E = 0,$$  \hspace{0.5cm} (12)

where $\omega_p^2 = \omega_{p0}^2[1 + \epsilon(r, \varphi, z)]$ and $\omega_{p0}$ is the unperturbed plasma frequency, given by $\omega_{p0}^2 = 4\pi n_0/m$. The quantity $\epsilon$ represents the perturbation. We consider solutions of the form

$$E(r, t) = a(r) \exp \left( -i\omega t + i \int^z k(z') \, dz' \right),$$  \hspace{0.5cm} (13)

where $\omega$ is the EM wave frequency and $a(r)$ is the amplitude, varying slowly along $z$ such that [17]

$$\left| \frac{\partial a}{\partial z} \right| \ll \frac{2\omega}{k} \frac{\partial a}{\partial z}. \hspace{0.5cm} (14)$$

We can then write the wave equation in a perturbed paraxial approximation form

$$\left( \nabla^2 + 2ik \frac{\partial}{\partial z} - \omega^2 \right) a = 0 \hspace{0.5cm} (15)$$

and the dispersion relation, connecting $k$ and $\omega$, is

$$k^2 = \omega^2 - \omega_p^2 \left[1 + \epsilon(r, \varphi, z)\right]. \hspace{0.5cm} (16)$$

In our case, a general solution to the wave equation in the paraxial approximation can be represented in a basis of orthogonal L-G modes, having the amplitude

$$a(r, \varphi, z) = \sum_{pl} b_{pl}(r, z) e^{il\varphi} \exp \left( -\frac{r^2}{2w^2} \right) \hat{e}_{pl}, \hspace{0.5cm} (17)$$

where $w \equiv w(z)$ is the beam waist, $\hat{e}_{pl}$ are unit polarisation vectors, $A_{pl}$ are the amplitudes of each mode and

$$b_{pl}(r, z) = c_{pl}(z) \sqrt{\frac{(l+p)!}{4\pi pl}} \left( \frac{r^2}{w^2} \right)^{|l|} L_p^{|l|} \left( \frac{r^2}{w^2} \right), \hspace{0.5cm} (18)$$

where the function $L_p^{|l|}$ represents the associated Laguerre polynomial. As usual, the integers $p$ and $l$ are the radial and the azimuthal quantum numbers, respectively. The electric field then becomes

$$E(r, t) = \sum_{pl} E_{pl}(r) \exp \left( -i\omega t + i \int^z k(z') \, dz' \right), \hspace{0.5cm} (19)$$

with

$$E_{pl}(r) = c_{pl}(z) F_{pl}(r, \varphi) \hat{e}_{pl}, \hspace{0.5cm} (20)$$

where $F_{pl}(r, \varphi)$ is given by formula (1).

When a vortex perturbation $\epsilon(r, \varphi, z)$ is present, the modes will couple according to

$$\frac{\partial}{\partial z} c_{pl}(z) = -\frac{i}{2k} \sum_{p' l'} K(p, p' l' l) c_{p' l'}, \hspace{0.5cm} (21)$$

where $K(p, p' l' l)$ are the coupling coefficients. If the EM wave does not carry OAM and the mode coupling is sufficiently weak that the zero OAM mode dominates over the entire interaction region, one obtains a coupling

$$K(p, p' l' l) = \pi \omega_{p0}^2 \frac{i}{n_0} \delta_{pp'} \frac{n l (z')}{k(z')} e^{i\gamma n_0 z'} \left(2\pi + i\gamma n_0 z'\right). \hspace{0.5cm} (22)$$

If we assume the same polarisation state for all the interacting modes, the field mode amplitudes describe the rate of transfer of OAM from the static plasma vortex to the EM field

$$c_{p, \pm l}(z) = \pi \frac{c(0)}{2\pi^2} \int_0^z \omega_{p0}^2 \frac{n l (z')}{k(z')} e^{i\gamma n_0 z'} \, dz'. \hspace{0.5cm} (23)$$

A more general solution, where the amplitude of the initially excited mode is allowed to change, is discussed in ref. [16]. The initial OAM state $l_0$ of the electromagnetic beam decays into other states $(l_i + u_l)$ on a length scale approximately determined by the inverse of the coupling constant, showing an effective exchange of OAM states between the photons and the plasma.

The photon mass in Proca equations is defined as $m_\gamma = \mu_\gamma$ and the effective photon mass is also related to the plasma frequency $m_{pl} = \omega_p$ [2]. By comparing these two definitions, one obtains the equivalence between the inverse of the characteristic length in a plasma, $\mu_\gamma$, and the plasma frequency, $\mu_\gamma = \omega_p$. When considering the EM wave equation of the Proca field one obtains a Klein-Gordon equation for the 4-vector potential

$$\Box - \mu_\gamma^2 A_\mu = -4\pi j_\mu \hspace{0.5cm} (24)$$

with the constraint derived from the massive photon in a medium $(\Box A_\mu = 0)$. By differentiating this expression with respect to time, and considering, in addition to the previous calculations, the simplest case where $\mu_\gamma$ is a constant in time, one obtains

$$\Box - \mu_\gamma^2 \frac{\partial}{\partial t} A_\mu = -4\pi \frac{\partial}{\partial t} j_\mu. \hspace{0.5cm} (25)$$

We now consider the spatial components applied to the case of photons moving in a plasma. In particular, we consider a plasma with a well-defined structure of a static plasma vortex, in which the density $n = n(r)$ is not a function of time. Being the set of equations independent of the temporal part, we pass from the four-dimensional notation to a three-dimensional vectorial notation without losing in generality. The current density has the form $j = -n(r)\nu$, and its derivative becomes

$$\frac{\partial}{\partial t} j_r = - \left( \delta \nu \frac{\partial}{\partial t} n(r) + n(r) \frac{\partial}{\partial t} \delta \nu \right) = -n(r) \frac{\partial}{\partial t} \nu \hspace{0.5cm} (26)$$
and the wave equation of the Proca EM field becomes
\[
\left(\square - \mu^2_\gamma\right)\frac{\partial}{\partial t} A = -4\pi n(r)\frac{\partial}{\partial t} v.
\]
(27)

Using \( E = -\nabla \phi - \frac{\partial}{\partial t} A \) one obtains the wave equation for the EM field:
\[
\left(\square - \mu^2_\gamma\right)(E + \nabla \phi) = 4\pi n(r)\frac{\partial}{\partial t} v.
\]
(28)

Assuming that \( v \) is parallel to \( E \), and \( v = v_0 + \delta v \), where \( \delta v \) is the perturbation of electrons velocity associated with the propagating EM wave, we obtain for \( E \neq 0 \) the wave equation for the EM field in the plasma,
\[
\left[\square - \mu^2_\gamma \left(1 + \frac{\hat{v} \cdot \nabla \phi}{|E|}\right) - 4\pi \frac{n(r)\delta v - \hat{v} \cdot \nabla \phi}{|E|}\right] E = 0,
\]
where \( \hat{v} = v/|v| \) is the direction vector of the velocity field, \( \hat{v} \) the unit vector of velocity and \( \delta v = \hat{v} \cdot \partial v = |\partial v| \). When comparing this equation with eq. (12), derived from the electric-field propagation equation in the case of a static plasma vortex, one obtains
\[
\mu^2_\gamma \left(1 + \frac{\hat{v} \cdot \nabla \phi}{E}\right) + 4\pi \frac{n(r)\delta v - \hat{v} \cdot \nabla \phi}{E} = \omega_p^2,
\]
(30)

that implies a direct relationship between the effective mass that a photon acquires in a plasma, the plasma frequency and the orbital angular momentum because of the peculiar spatial distribution:
\[
\mu^2_\gamma = \frac{E}{E + \hat{v} \cdot \nabla \phi} \frac{\omega_p^2|1 + \varepsilon(r, \varphi, z)| - 1}{E + \hat{v} \cdot \nabla \phi} \times \left(4\pi \delta v |n_0 + \hat{v} \cos(l_0\varphi + q_0z)| - 4\pi \hat{v} \cdot \nabla \phi\right).
\]
(31)

The new mass component is a fictitious term, generated by the interaction of photons with the plasma and cannot be ascribed to an intrinsic property of the photon.

When the electron number density exhibits certain spatial properties, such as vortices, any photon has an associated virtual mass term that is smaller than that expected from Proca equations in a homogeneous plasma, because of a negative term that corresponds to a precise orbital angular momentum component.

**Conclusions.** – We have investigated the problem of photon mass in a plasma and show that part of the acquired mass term is related to the orbital angular momentum of light imposed by certain spatial distributions of the plasma. We focused our attention on the simplest case of spatial distributions described by a static plasma vortex. This approach shows that also the spatial distribution of charges can impose OAM and an additional mass term that reduces the effective mass of the photon inside a non-structured plasma, degrading the variation in OAM. The spiral-like plasma structure induces OAM states in most of the scattered photons and the stochastic interaction value responsible for the Proca mass term is instead transformed into an organised state of light, with the result of reducing the averaged mass term [19].

***

One of the authors (BT), gratefully acknowledges the financial support from the Swedish Research Council (VR). FT acknowledges the financial support of CARIPARO in the 2006 program of excellence.

**REFERENCES**

[1] Schwinger J., *Phys. Rev.*, **125** (1962) 397.
[2] Anderson P. W., *Phys. Rev.*, **130** (1963) 439.
[3] Mendonça J. T., *Theory of Photon Acceleration* (IOP Publishing, Bristol) 2001, ISBN 0-7503-0711-0.
[4] Hora H., *Plasmas at High Temperature and Density* (Springer) 1991.
[5] Goldhaber A. S. and Nieto M. M., *Rev. Mod. Phys.*, **82** (2010) 939.
[6] Arecchi F. T., Giacomelli G., Ramazza P. L. and Residori S., *Phys. Rev. Lett.*, **67** (1991) 3749.
[7] Anzolin G., Tamburini F., Bianchini A. and Barbieri C., *Phys. Rev. A*, **79** (2009) 033845.
[8] Allen L., Beijersbergen M. W., Spreeuw R. J. C. and Woerdman J. P., *Phys. Rev. A*, **45** (1992) 8185.
[9] Vaziri A., Weih G. and Zeilinger A., *J. Opt. B: Quantum Semiclass. Opt.*, **4** (2002) 47.
[10] Jackson J. D., *Classical Electrodynamics*, 3rd edition (Wiley & Sons, New York, NY) 1999.
[11] Schwinger J., DeRaad L. L. jr., Milton K. A. and Tsai W., *Classical Electrodynamics* (Perseus Books Reading, Mass) 1998, ISBN 0-7382-0056-5.
[12] Thidé B., *Electromagnetic Field Theory*, 2nd edition (Dover Publications, Mineola, NY) 2010, in press, ISBN 978-0-486-4773-2, www.plasma.uu.se/CED/Book.
[13] O’Neil A. T., MacVicar I., Allen L. and Padgett M. J., *Phys. Rev. Lett.*, **88** (2002) 053601.
[14] Tamburini F. and Vicino D., *Phys. Rev. A*, **78** (2008) 052116.
[15] Heitler W., *The Quantum Theory of Radiation*, 3rd edition, *The International Series of Monographs on Physics* (Clarendon Press, Oxford) 1954.
[16] Mendonça J. T., Thidé B., Bergman J. E. S., Mohammadi S. M., Eliasson B., Baan W. and Then H., arXiv:0804.3221 [physics.plasm-ph] (2008).
[17] Mendonça J. T., Thidé B. and Then H., *Phys. Rev. Lett.*, **102** (2009) 185005(4).
[18] Mendonça J. T., Ali S. and Thidé B., *Phys. Plasmas*, **16** (2009) 112103.
[19] Andrews D. L. (Editor), *Structured Light and Its Applications* (Elsevier, Burlington, Mass.) 2008.