Hillery-Type Squeezing in Fan-States

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Abstract

We study the Hillery-type, i.e. $N$-th power, amplitude squeezing in the fan-state $|\xi;2k,f\rangle_F$ characterized by $\xi \in \mathbb{C}$, $k = 1, 2, 3, \ldots$ and $f$ a nonlinear operator-valued function. We show that for a given $k$ there exists a critical $\xi_c$ such that for $0 < |\xi| \leq |\xi_c|$ squeezing occurs simultaneously in $2k$ directions for the powers $N$ which are a multiple of $2k$. This result does not depend on the concrete form of $f$, i.e. it holds true for both $f \equiv 1$ and $f \neq 1$. However, for $f \neq 1$, which is realized here in the ion trap context, the squeezing directions as well as the magnitude of $\xi$ can be controlled by adjusting the system driving parameters.

PACS numbers: 42.50.Dv

Keywords: Fan-state, squeezing directions

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I. INTRODUCTION

The squeezed state is a nonclassical state, which has been discovered for a long time (see, e.g., a recent review [1] and references therein). Besides recognized potential applications in detecting extremely weak forces, squeezed states have recently attracted much more attention in the newly emerging field of quantum information and quantum computation. By means of squeezed states schemes for teleportation of continuous quantum variables [2, 3, 4], quantum cryptography [5, 6], entanglement distribution [7], etc., have been proposed and some of them have been realized. The original squeezed state [8] has been generalized to different types of higher-order. One type of higher-order amplitude squeezing is suggested by Hong and Mandel [9]. Another qualitatively different type is introduced by Hillery [10] which is then developed further by several authors (see, e.g., [11, 12, 13]). The Hong-Mandel type higher-order amplitude squeezing has been investigated in the fan-state $|\xi; 2k, f\rangle_F$ which is introduced in [14] as a linear superposition of $2k$ $2k$-quantum nonlinear coherent states $|\xi_q; 2k, f\rangle$ ($q = 0, 1, ..., 2k - 1$) [15] in a phase-locked manner. Since there are states that exhibit Hong-Mandel type squeezing but not Hillery-type one and vice versa [10], we would like to examine in this paper whether the Hillery-type $N$-th power amplitude squeezing is possible and, if it is, how it behaves in the same fan-state. The normalized fan-state is defined as

$$
|\xi; 2k, f\rangle_F = \frac{1}{\sqrt{D_k(|\xi|^2)}} \sum_{q=0}^{2k-1} |\xi_q; 2k, f\rangle, \tag{1}
$$

where $k = 1, 2, 3, ...; \xi_q = \xi \exp(\frac{\imath \pi q}{2k})$, $\xi \in \mathbb{C}$, $f$ is in general an arbitrary real nonlinear operator-valued function (in particular, it may be that $f \equiv 1$) of $\hat{n} = a^\dagger a$ with $a$ ($a^\dagger$) the bosonic annihilation (creation) operator,

$$
D_k(|\xi|^2) = \sum_{m=0}^{\infty} \frac{|\xi|^{4km} |J_k(m)|^2}{(2km)! [f(2km)()]^{2k}|^2}, \tag{2}
$$

with

$$
J_k(m) = \sum_{n=0}^{2k-1} e^{\imath \pi nm} \tag{3}
$$

and the $2k$-quantum nonlinear coherent state is given by

$$
|\xi_q; 2k, f\rangle = \sum_{n=0}^{\infty} \frac{\xi_q^{2kn}}{\sqrt{(2kn)! f(2kn)()]^{2k}}} |2kn\rangle. \tag{4}
$$
with $|2kn\rangle$ a Fock state. The state $|ξ_{q}; 2k, f\rangle$ is a sub-state of the multi-quantum nonlinear coherent state which is by definition the eigenstate of the nonboson operator $a^{2k}f(\hat{n})$. The notation $(!)^{2k}$ appearing in Eqs. (2) and (4) is understood as follows

$$f(p)(!)^{2k} = \begin{cases} f(p)f(p - 2k)f(p - 4k)...f(q) & \text{if } p \geq 2k; 0 \leq q < 2k \\ 1 & \text{if } 0 \leq p < 2k \end{cases}$$

The notation $(!)^{2k}$ appearing in Eqs. (2) and (4) is understood as follows

The Hillery-type $N$-th power amplitude squeezing is associated with the operator $Q_N(\varphi)$ of the form

$$Q_N(\varphi) = \frac{1}{2}(a^N e^{-iN\varphi} + a^\dagger N e^{iN\varphi})$$

with $\varphi$ an angle determining the direction of $\langle Q_N(\varphi) \rangle$ in the complex plane. According to $[12, 13]$, a state $|...\rangle$ is said to be $N$-th power amplitude squeezed along the direction $\varphi$ if there exists a value of $\varphi$ such that the variance of $Q_N(\varphi)$ satisfies the following inequality

$$\langle (\Delta Q_N(\varphi))^2 \rangle < \frac{1}{4}\langle F_N \rangle$$

where

$$\langle F_N \rangle = \langle [a^N, a^\dagger N] \rangle.$$  

It is easy to express the variance as

$$\langle (\Delta Q_N(\varphi))^2 \rangle = \frac{1}{4}\langle F_N \rangle + \langle : (\Delta Q_N(\varphi))^2 : \rangle$$

where

$$\langle : (\Delta Q_N(\varphi))^2 : \rangle = \frac{1}{2}\{\langle a^\dagger N a^N \rangle + 3\Re[e^{-2iN\varphi}\langle a^{2N}\rangle] - 2(\Re[e^{-iN\varphi}\langle a^N\rangle]^2)\}$$

with $: ... :$ denotes a normal ordering of the operators. The $\langle F_N \rangle$ can also be normally ordered and its explicit form reads

$$\langle F_N \rangle = \sum_{q=1}^{N} \frac{N!N^{(q)}}{(N - q)!q!} \langle (a^\dagger)^{N-q}a^{N-q} \rangle$$

with $N^{(q)} = N(N - 1)...(N - q + 1)$. Combining the inequality and Eq. (8) we recognize that the state becomes squeezed whenever $\langle : (\Delta Q_N(\varphi))^2 : \rangle$ gets negative. For convenience, we define a measure of squeezing degree by a quantity $S$ scaled as

$$S = \frac{4\langle : (\Delta Q_N(\varphi))^2 : \rangle}{\langle F_N \rangle}.$$  

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Clearly that for a squeezed state \(-1 \leq S < 0\) and the ideal squeezing corresponds to \(S = -1\). Without any loss of generality we choose the real axis to be along the direction of \(\xi\) in the phase space. This allows us to treat \(\xi\) as a real number. In the fan-state, we have

\[
\langle a^{\dagger} a \rangle_k = \frac{\xi^{l-m}}{D_k(\xi^2)} I \left( \frac{l-m}{2k} \right) \sum_{n=0}^{\infty} \frac{\theta(2kn-m)\xi^{4k}n \mathcal{J}_k(n + \frac{l-m}{2k}) \mathcal{J}_k(n)}{(2kn-m)!f(2kn)(!)^2 f(2kn + l - m)(!)^2k} \tag{12}
\]

where \(\langle ... \rangle_k \equiv R \langle \xi; 2k, f; \ldots; \xi; 2k, f \rangle_F\). The function \(I(x)\) equals unity (zero) if \(x\) is an integer (non-integer) and \(\theta(x)\) is the step function: \(\theta(x) = 1\) (0) for \(x \geq 0\) \((x < 0)\). Putting Eqs. \((9)\) and \((10)\) into Eq. \((11)\) and making a simple trigonometric transformation, we arrive explicitly at

\[
S = \frac{2\{\langle a^{\dagger} a \rangle_k - \langle a \rangle_k^2 + \cos(2N\mathcal{J})(\langle a^{2N} \rangle_k - \langle a \rangle_k^2)\}}{\sum_{q=1}^{N} \frac{(N-Nq)q!}{(a^q)!a^{-q-k}}} \tag{13}
\]

where \(\langle a \rangle_k, \langle a^{2N} \rangle_k, \langle a^{1N} a \rangle_k\) and \(\langle a^q \rangle_{N-k} a^{N-k}\) are determined by \((12)\) if \(\{l, m\}\) is set to be \(\{0, N\}, \{0, 2N\}, \{N, N\}\) and \(\{N - q, N - q\}\).

In what follows we distinguish two cases: \(f \equiv 1\) which corresponds to light field and \(f \neq 1\) which may be associated with the vibrational motion of the center-of-mass of a trapped ion.

II. CASE \(f \equiv 1\)

A QED-based scheme to generate the fan-state with \(f \equiv 1\) which concerns radiation field in a cavity has been proposed in \([18]\). For \(f \equiv 1\) the expectation values \(\langle a \rangle_k\) and \(\langle a^{2N} \rangle_k\) simplify to

\[
\langle a \rangle_k = \frac{\xi^{-N}}{D_k(\xi^2)} I \left( \frac{-N}{2k} \right) \sum_{n=0}^{\infty} \frac{\theta(2kn-N)\xi^{4k}n \mathcal{J}_k(n - \frac{N}{2}) \mathcal{J}_k(n)}{(2kn-N)!}, \tag{14}
\]

\[
\langle a^{2N} \rangle_k = \frac{\xi^{-2N}}{D_k(\xi^2)} I \left( \frac{-N}{k} \right) \sum_{n=0}^{\infty} \frac{\theta(2kn-2N)\xi^{4k}n \mathcal{J}_k(n - \frac{N}{2}) \mathcal{J}_k(n)}{(2kn-2N)!}. \tag{15}
\]

Using the property of the function \(\mathcal{J}_k(n)\),

\[
\mathcal{J}_k(n) \mathcal{J}_k(n \pm n') = \begin{cases} 2k^2(1 + (-1)^n) & \text{if } n' \text{ is an even integer} \\ 0 & \text{if } n' \text{ is an odd integer} \end{cases}, \tag{16}
\]

it follows from Eqs. \((14)\) and \((15)\) that, for a given \(k\), both \(\langle a \rangle_k\) and \(\langle a^{2N} \rangle_k\) vanish if \(N \neq 2kp\) with \(p = 1, 2, 3, \ldots\). On the other hand,

\[
\langle a^{\dagger} a \rangle_k = \frac{4k^2}{D_k(\xi^2)} \sum_{n=0}^{\infty} \frac{\theta(4kn-N)\xi^{8k}n}{(4kn-N)!} \tag{17}
\]
is always positive. Hence, for \( N \neq 2kp \) the \( \varphi \)-dependence in \( S \) (see Eq. (13)) is killed and we always obtain \( S > 0 \), i.e. squeezing is impossible. As a consequence, squeezing turns out to be possible only for powers \( N \) that are a multiple of \( 2k \), a necessary condition for the Hillery-type squeezing in the fan-state. Yet, whether or not squeezing appears still depends on the value of \( \xi \). To see this let us analytically calculate \( S \) for a couple of concrete values of \( k \) and \( N \).

For \( k = 1 \) and \( N = 2 \) we have obtained explicitly

\[
S_{\varphi,N=2}^{(k=1)} = \frac{\xi^4 \{ \cosh(\xi^2) - \cos(\xi^2) + D_1(\xi^2) \cos(4\varphi) \}}{2\xi^2 (\sinh(\xi^2) - \sin(\xi^2)) + D_1(\xi^2)}
\]

with

\[
D_1(\xi^2) = 2[\cos(\xi^2) + \cosh(\xi^2)]
\]

and squeezing occurs whenever

\[
\cos(4\varphi) < h(\xi^2) = \frac{\cos(\xi^2) - \cosh(\xi^2)}{D_1(\xi^2)} \leq 0.
\]

The function \( h(\xi^2) \) equals zero at \( \xi = 0 \) and decreases for increasing \( |\xi| \). There is no squeezing for \( |\xi| > \xi_c = 1.2533 \) for which \( h(\xi^2) < -1 \) and no \( \varphi \) can be found to make \( S_{\varphi,N=2}^{(k=1)} \) negative. This defines the range of \( \xi \) within which squeezing takes place: \( 0 < |\xi| \leq |\xi_c| \).

Figure 1 is a 3D plot of \( S_{\varphi,N=2}^{(k=1)} \) as a function of \( |\xi| \) and \( \varphi \). Interestingly to notice that a maximal squeezing occurs simultaneously along two directions characterized by \( \varphi_1 = \pi/4 \) and \( \varphi_2 = 3\pi/4 \). The two coexistent directions of squeezing can alternatively be seen by a polar plot of \( S_{\varphi,N=2}^{(k=1)} \) in Fig. 2 with \( |\xi| = 0.8 \) which looks like an eight-winged bow. The shorter wings correspond to squeezing, while the longer ones to stretching. To better understand Fig. 2 let us also show in Fig. 3 the variance \( \langle (\Delta Q_N(\varphi))^2 \rangle \) itself as a function of \( \varphi \) for the same parameters as in Fig. 2, i.e. for \( \{k,N,|\xi|\} = \{1,2,0.8\} \). Taking into account Eqs. (8) and (11) yields

\[
\langle (\Delta Q_N(\varphi))^2 \rangle_k = \frac{(F_N)_k}{4} \left( 1 + S_{\varphi,N}^{(k)} \right)
\]

In Fig. 3 the variance \( \langle (\Delta Q_N(\varphi))^2 \rangle_k \) is represented by the solid curve, while the dashed circle of radius \( (F_N)_k / 4 \) determines the uncertainty region associated with the coherent state for which \( S_{\varphi,N}^{(k)} = 0 \) (Note, this corresponds to the center point in Fig. 2). It is clear that, along the directions \( \varphi = \pi/4 \) and \( \varphi = 3\pi/4 \), we have \( \langle (\Delta Q_N(\varphi))^2 \rangle_k < (F_N)_k / 4 \) signifying negativity of \( S_{\varphi,N}^{(k)} \), i.e. squeezing takes place. Along the directions \( \varphi = 0 \) and \( \varphi = \pi/2 \), however, the inequality \( \langle (\Delta Q_N(\varphi))^2 \rangle_k > (F_N)_k / 4 \) holds. That implies \( S_{\varphi,N}^{(k)} > 0 \), i.e. no
squeezing occurs. Since the difference between the solid curve and the dashed circle along \( \varphi = \pi/4 \) (and \( 3\pi/4 \)) is smaller than that along \( \varphi = 0 \) (and \( \pi/2 \)), the squeezing (stretching) gives rise to a shorter (longer) wing in Fig. 2. For \( N = 4, 6, 8, ... \) simultaneous squeezing along the same two directions may also result with appropriate values of \( \xi \) but the squeezing degree is much lower than that in the case with \( N = 2 \).

For \( k = 2 \) and \( N = 4 \) we have obtained explicitly
\[
S^{(k=2)}_{\varphi,N=4} = \frac{\xi^8}{4} \left[ \frac{\cosh(\xi^2) + \cos(\xi^2) - 2 \cosh(\frac{\xi^2}{\sqrt{2}}) \cos(\frac{\xi^2}{\sqrt{2}}) + D_2(\xi^2) \cos(8\varphi)}{3\xi^4[\cosh(\xi^2) - \cos(\xi^2) - 2 \sinh(\frac{\xi^2}{\sqrt{2}}) \sin(\frac{\xi^2}{\sqrt{2}})] + A(\xi^2) + 2B(\xi^2) - D_2(\xi^2)} \right],
\]
where
\[
D_2(\xi^2) = \cosh(\xi^2) + \cos(\xi^2) + 2 \cosh(\frac{\xi^2}{\sqrt{2}}) \cos(\frac{\xi^2}{\sqrt{2}}),
\]
\[
A(\xi^2) = 2\xi^6(\sinh(\xi^2) + \sin(\xi^2) - \sqrt{2}[\sinh(\frac{\xi^2}{\sqrt{2}}) \cos(\frac{\xi^2}{\sqrt{2}}) + \sin(\frac{\xi^2}{\sqrt{2}}) \cosh(\frac{\xi^2}{\sqrt{2}})]),
\]
\[
B(\xi^2) = 3\xi^4[\cosh(\xi^2) - \cos(\xi^2) - 2 \sinh(\frac{\xi^2}{\sqrt{2}}) \sin(\frac{\xi^2}{\sqrt{2}})] + 6C(\xi^2) + 2D_2(\xi^2)
\]
and
\[
C(\xi^2) = \xi^2[\sinh(\xi^2) - \sin(\xi^2) + \sqrt{2}(\sinh(\frac{\xi^2}{\sqrt{2}}) \cos(\frac{\xi^2}{\sqrt{2}}) - \sin(\frac{\xi^2}{\sqrt{2}}) \cosh(\frac{\xi^2}{\sqrt{2}}))].
\]

We display in Fig. 4 the polar plot of \( S^{(k=2)}_{\varphi,N=4} \) at \(|\xi| = 1.25\) as a function of \( \varphi \). In this case four directions of squeezing (the shorter wings) are clearly seen. These four directions along which squeezing is maximum are determined by \( \varphi_1 = \pi/8 \), \( \varphi_2 = 3\pi/8 \), \( \varphi_3 = 5\pi/8 \) and \( \varphi_4 = 7\pi/8 \).

For \( N = 8, 12, 16, ... \) simultaneous squeezing along the four above-mentioned directions may also result with appropriate values of \( \xi \) but the squeezing is much less pronounced compared to the case with \( N = 4 \).

III. CASE \( f \neq 1 \)

In general \( f \) is a nonlinear function of the boson number of a field. The concrete form of \( f \) depends on the physical system to be considered. For example, for the so-called “f-oscillator” [19, 20] whose frequency is not a constant but depends on the number of the quanta of a field, \( f \) can be given through \( q \)-deformed algebra [21, 22] as \( f = \sqrt{\sinh(n\lambda)/(n \sinh \lambda)} \) with \( \lambda \) a real parameter, or, for the so-called photon-added coherent states [23] one has \( f = 1 - m/(1+n) \) with the parameter \( m \) being a positive integer [24], etc. In this section, we deal with a
realistic situation associated with the phonon field of the center-of-mass vibrational motion of a trapped ion. A laser-driven scheme to produce the fan-state in this context has been presented in [25]. In this specific physical system the function $f$ and the quantity $\xi$ are determined by

$$f = \frac{(n-2k)! L_{n-2k}^{2k}(\eta^2)}{n! L_n^{0}(\eta^2)}$$

$$\xi^{2k} = -\frac{e^{i\phi_0} \Omega_0}{(i\eta)^{2k} \Omega_1}$$

where $L_i^m(x)$ is the $l$-th generalized polynomial in $x$ for parameter $m$, $\eta$ is the Lamb-Dicke parameter, $\phi = \phi_1 - \phi_0$ with $\phi_0(\phi_1)$ the phase of the driving laser which is resonant with (detuned to the $2k$-th red sideband of) the electronic transition of the trapped ion, and $\Omega_{0,1}$ are the Rabi frequencies. As followed from Eqs. (27) and (28), in this realistic system the function $f$ as well as the magnitude of $\xi$ can be controlled by adjusting the parameters of the driving lasers (through $\phi_{0,1}$, $\Omega_{0,1}$) and/or the trapping potential (through $\eta$). This provides an experimental means to tailor a fan-state on demand. As for the question “which power $N$ might give rise to squeezing for a given $k$”, it is not affected by the concrete form of the function $f$ since in fact this question depends merely on the property (16) of the $J_k(n)$ as argued in the preceding section. This means that independent of $f$ the $N$-th power amplitude squeezing is possible for $N = 2k$, $4k$, $6k$, .... However, squeezing actually arises only in certain ranges of $\xi^2$ and $\eta^2$. For each chosen value of $\eta$ there exists a critical $\xi_c(\eta)$ such that squeezing takes place when $0 < |\xi| \leq |\xi_c(\eta)|$. This is illustrated in Figure 5 which plots $S^{(k=1)}_{\varphi,N=2}$ for $\varphi = \pi/4$ and $\eta^2 = 0.05$ as a function of $\xi^2 : \xi_c(\eta = \sqrt{0.05}) = 1.0099$. The number of squeezing directions is determined in the same way as for $f \equiv 1$, i.e. squeezing if it exists is observed simultaneously and equally maximal at $2k$ directions. It is worthy to emphasize that the number of simultaneous squeezing directions is dictated only by $k$ but not by $N$, an interesting fact that can be justified from the symmetry property of the fan-state.

The figure of merit of the case $f \neq 1$ in comparison with $f \equiv 1$ is that we can manage the squeezing directions by controlling the physical parameters of the system under consideration. Namely, we could control $\eta$ and $\xi$ so that $\langle a^{2N} \rangle_k > \max\{\langle a^N \rangle_k^2, \langle a^{+N} a^N \rangle_k \}$ to have $2k$ squeezing directions determined by
\[ \varphi_j = \frac{(2j + 1)\pi}{4k} \quad \text{with} \quad j = 0, 1, \ldots, 2k - 1. \quad (29) \]

Or, if needed, we could chose \( \eta \) and \( \xi \) so that \( \langle a^{2N} \rangle_k < \min \{ \langle a^N \rangle_k^2, 2\langle a^N \rangle_k^2 - \langle a^N a^N \rangle_k \} \) to make squeezing along the other directions determined by

\[ \varphi_j = \frac{\pi j}{2k} \quad \text{with} \quad j = 0, 1, \ldots, 2k - 1. \quad (30) \]

Transparency, the squeezing directions determined by Eqs. (29) and (30) are “rotated” by \( \pi/(4k) \) relative to each other. In other words, by adjusting the system driving parameters we are able to interchange the direction of squeezing and stretching if we like.

**IV. CONCLUSION**

We have dealt with the Hillery-type \( N \)-power amplitude squeezing in the fan-state \( |\xi;2k,f\rangle_F \), Eq. (11), for both \( f \equiv 1 \) and \( f \neq 1 \). Independent of \( f \) we find out that for a given \( k \) squeezing may occur within a certain range of \( \xi \) for powers \( N \) that are a multiple of \( 2k \). The remarkable feature in the fan-state is that whenever squeezing exists it arises simultaneously and equally in \( 2k \) directions, as opposed to conventional states in which squeezing can be observed only along one direction. While the number of squeezing directions is fixed to \( 2k \) for a given \( k \), these directions themselves can be rotated by adjusting the physical system parameters when \( f \neq 1 \) as demonstrated in this work in the ion trap context. Though it can be hoped, it seems premature to see that the multidirectional character of squeezing might find an actual application in practice in general or in quantum information processing in particular. For example, it is known that for quantum continuous variables teleportation [2,3] the necessary resource, i.e. the two-mode squeezed entangled state, is produced by superimposing on a beamsplitter two single-mode squeezed states whose squeezing directions must be perpendicular to each other. Yet, as at present it remains unknown what happens and what are the benefits if one or both of the superimposed states would exhibit squeezing in more than one direction. Such kinds of work are exciting and do deserve further efforts.
Acknowledgments

M.D.T. thanks the Abdus Salam ICTP for the financial support and hospitality at Trieste. B.A.N. is founded by KIAS R&D Grant No. 03-0149-002.

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FIG. 1: $S = S^{k=1}_{\varphi,N=2}$ versus $|\xi|$ and $\varphi$ showing two coexistent directions of squeezing.

FIG. 2: Polar plot of $S^{(k=1)}_{\varphi,N=2}$ for $|\xi| = 0.8$. The shorter wings along the directions $\varphi = \pi/4$ and $\varphi = 3\pi/4$ are associated with squeezing, while stretching (longer wings) occurs along the directions $\varphi = 0$ and $\varphi = \pi/2$.

FIG. 3: The solid curve is the variance $\left< (\Delta Q_N(\varphi))^2 \right>_k$ as a function of $\varphi$ for the same parameters as in Fig. 2, i.e. $k = 1$, $N = 2$ and $|\xi| = 0.8$. The dashed circle (of radius $\langle F_N \rangle_k / 4$) represents the corresponding coherent state case.

FIG. 4: Polar plot of $S^{(k=2)}_{\varphi,N=4}$ for $|\xi| = 1.25$. Squeezing occurring simultaneously along the four directions (shorter wings) $\varphi = \pi/8$, $3\pi/8$, $5\pi/8$ and $7\pi/8$ are visual.

FIG. 5: $S = S^{k=2}_{\varphi,N=4}$ versus $\xi^2$ for $\varphi = \pi/4$ and $\eta^2 = 0.05$ in the ion trap context.
