NON-TRIVIAL SHAFAREVICH-TATE GROUPS OF ELLIPTIC CURVES

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Abstract. We characterize quadratic twists of \( y^2 = x(x - a^2)(x + b^2) \) with Mordell-Weil groups and 2-primary part of Shafarevich-Tate groups being isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^2 \) under certain conditions. We also obtain the distribution result of these elliptic curves.

Keywords Shafarevich-Tate groups, Full 2-torsion, Cassels pairing, Gauss genus theory, independence property, residue symbol

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1. Introduction

In our previous paper [18], we use Cassels pairing to characterize congruent elliptic curves \( y^2 = x^3 - n^2x \) with Mordell-Weil ranks zero and 2-primary parts of Shafarevich-Tate groups being isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^2 \) provided that all prime divisors of \( n \) are congruent to 1 modulo 4. On the other hand side, we use the independence property of residue symbols to obtain corresponding distribution results in [19]. These tools play an important role in the proof of the breakthrough of Smith [2] on the distribution of 2-Selmer groups. The goal of this paper is to generalize these methods to quadratic twist family of elliptic curves with full rational 2-torsion points.

Let \( a \) and \( b \) be coprime integers. Denote by \( E = E_{a,b} \) the elliptic curve

\[ E : y^2 = x(x - a^2)(x + b^2). \]

Then the quadratic twist family of \( E \) consists of the elliptic curves

\[ E^{(n)} : y^2 = x(x - a^2n)(x + b^2n), \]

where \( n \) runs over all non-zero square-free integers. Note that if \( a = b = 1 \), these are the congruent elliptic curves. To state our main theorem, we introduce some notation. For a positive square-free integer \( m \) and a positive integer \( k \), the 2\(^k\)-rank \( h_{2^k}(m) \) of the ideal class group \( \mathcal{A} = \mathcal{A}_m \) of \( \mathbb{Q}(\sqrt{-m}) \) is defined to be

\[ \dim_{\mathbb{F}_2} 2^{k-1} \mathcal{A}/2^k \mathcal{A}. \]

Here the group operation of \( \mathcal{A} \) is written additively.

Theorem 1. Let \( (a, b, c) \) be any positive primitive integer solution to \( a^2 + b^2 = 2c^2 \) such that the dimension of the 2-Selmer group of \( E \) is two. Denote by \( n \) a positive square-free integer such that all prime factors of \( n \) are congruent to \( \pm 1 \) modulo 8 and quadratic residues modulo any prime divisor \( p \) of \( abc \). If \( n \equiv 1 \pmod{8} \), then the following are equivalent:

1. \( E^{(n)}(\mathbb{Q}) \) and \( \mathbb{I}(E^{(n)}(\mathbb{Q}))[2^\infty] \) are isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^2 \),
2. \( h_4(n) = 1 \) and \( h_8(n) = 0 \).

From elementary number theory, if \( (a, b, c) \) is any positive primitive integer solution to \( a^2 + b^2 = 2c^2 \), then \( a, b \) and \( c \) are the absolute values of \( 4k^2 - 4k - 1, 4k^2 + 4k - 1 \) and \( 4k^2 + 1 \) respectively, where \( k \) is an integer. For example, if \( k = 2, 3, 6, 7, 9, 10, 11, \cdots \).
then the dimensions of the 2-Selmer groups of the corresponding $E_{a,b}$ are two. Among all positive integers no larger than 50, there are 19 such $E_{a,b}$.

To state another theorem, we have to use Gauss genus theory (refer to §2.4 of this paper or §3 of [18] for more details). If $m$ is odd and $h_4(m) = 1$, then there are exactly two divisors $d_1$ and $d_2$ of $2m$ which correspond to the non-trivial element of $2\mathcal{A}_m \cap \mathcal{A}_m[2]$, where $\mathcal{A}_m[2]$ denotes the ideal classes with trivial squares. Furthermore, the product of the odd parts of $d_1$ and $d_2$ is $m$.

**Theorem 2.** Let $(a, b, c)$ be any positive primitive integer solution to $a^2 + b^2 = 2c^2$ such that the dimension of the 2-Selmer group of $E$ is two. Assume that $n$ is a positive square-free integer such that all prime factors of $n$ are quadratic residues modulo $4p$ with $p$ any prime divisor of $abc$. If $n$ is congruent to 1 modulo 8, then the following are equivalent:

1. $E(n)(\mathbb{Q})$ and $\text{III}(E(n)/\mathbb{Q})[2^\infty]$ are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$,
2. $h_4(n) = 1$ and $h_8(n) \equiv \frac{d-1}{4} \pmod{2}$.

Here $d$ denotes the odd part of $d_0$ which corresponds to the non-trivial element of $2\mathcal{A}_m \cap \mathcal{A}[2]$.

Now we explain the idea of the proof of Theorem 1 and 2. Via the exact sequence

\[0 \to E(n)(\mathbb{Q})/2E(n)(\mathbb{Q}) \to \text{Sel}_2(E(n)) \to \text{III}(E(n)/\mathbb{Q})[2] \to 0,\]

we derive that (1) implies $s_2(n) = 2$. Here $s_2(n)$ is the pure 2-Selmer rank

\[\dim_{\mathbb{F}_2} \text{Sel}_2(E(n))/E(n)(\mathbb{Q})[2].\]

By Gauss genus theory, $h_4(n)$ is closely related to the Rédei matrix $R_n$; we have parallel results between $s_2(n)$ and the generalized Monsky matrix $M_n$ by Proposition 3. Then we get that $s_2(n) = 2$ if and only if $h_4(n) = 1$. Cassels [11] introduced a skew-symmetric pairing on the pure 2-Selmer group $\text{Sel}_2(E(n))/E(n)(\mathbb{Q})[2]$. We can show that (1) is equivalent to the non-degeneracy of the Cassels pairing provided that $h_4(n) = 1$. According to Cassels pairing and Gauss genus theory, the non-degeneracy of the Cassels pairing under this condition is equivalent to (2).

To give the distribution result on the elliptic curves in Theorem 2, we first introduce some notation. Let $a, b, c$ be coprime positive integers such that $a^2 + b^2 = 2c^2$ and the dimension of the 2-Selmer group of $E$ is two. Let $k$ be a fixed positive integer. We denote by $\mathcal{D}_k(x)$ the set of positive square-free integers $n = p_1 \cdots p_k \leq x$ satisfying

- $n$ is congruent to 1 modulo 8, and
- all $p_i$ are quadratic residues modulo $4p$ with $p$ any prime divisor of $abc$.

We define $\mathcal{P}_k(x)$ to be all $n \in \mathcal{D}_k(x)$ such that

\[\text{rank}_{\mathbb{Z}} E(n)(\mathbb{Q}) = 0 \quad \text{and} \quad \text{III}(E(n)/\mathbb{Q})[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^2.\]

Denote by $C_k(x)$ the set of positive square-free integers $n \leq x$ with exactly $k$ prime factors. Then the independence property of Legendre symbols of Rhoades [14] implies

\[\#C_k(x) \sim \frac{1}{(k-1)!} \frac{x(\log x)^{k-1}}{\log x}.\]

Here the symbol "~" and many other symbols "≪, $O(\cdot), o(\cdot), \text{Li}(x)"" are standard notation in analytic number theory, it can be found in many references such as Iwaniec-Kowalski [13]. Let $[\frac{x}{2}]$ be the maximal integer no larger than $k/2$. We define $\{u_k : k \in \mathbb{N}\}$ to be the decreasing sequence $\left\{\prod_{i=1}^{[\frac{k}{2}]} (1 - 2^{1-2i}) : k \in \mathbb{N}\right\}$ with limit $u \approx 0.419$.

**Theorem 3.** Let $a, b$ and $c$ be coprime positive integers such that $a^2 + b^2 = 2c^2$ and the dimension of the 2-Selmer group of $E$ is two. Then for any positive integer $k$,

\[\#\mathcal{P}_k(x) \sim 2^{-kk^2-k-2} \left(u_k + (2^{1-k} - 2^{-k})u_{k-1}\right) \cdot \#C_k(x).\]
Here $k'$ is the number of different prime factors of $abc$.

The key ingredient of the proof of Theorem 3 is the independence property of residue symbols (Theorem 4), which reduces counting $\#E_k(x)$ to counting certain symmetric $k \times k$ matrices over $\mathbb{F}_2$.

In the end of this introduction, we introduce the arrangement of this paper. In Section 2, we introduce some preliminary results and several residue symbols. Section 3 is focused on the matrix representation of 2-Selmer group. We prove that $E^{(n)}(\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ is Section 4. We devote Section 5 to prove Theorem 1 and 2. We use the method of Cremona-Ododi [10] to prove the independence property of residue symbols (Theorem 4) in Section 6. In the last section, we prove the distribution result (Theorem 3).

2. Preliminary section

2.1. Identification of 2-Selmer Group.

Let $E/\mathbb{Q}$ be an elliptic curve with full rational 2-torsion points defined by

$$E : y^2 = (x - a_1)(x - a_2)(x - a_3). \quad (2.1)$$

Then we can identify (see Cassels [11]) the 2-Selmer group $\text{Sel}_2(E)$ of $E$ with

$$\{ \Lambda = (d_1, d_2, d_3) \in (\mathbb{Q}^\times / \mathbb{Q}^\times 2)^3 \mid d_1d_2d_3 \in \mathbb{Q}^\times 2, \ D_\Lambda(\mathbb{A}) \neq \emptyset \}.$$  

Here $\mathbb{A}$ is the adele ring of $\mathbb{Q}$ and $D_\Lambda$ is a genus one curve defined by

$$H_1 : (a_2 - a_3)t^2 + d_2u_2^2 - d_3u_3^2 = 0,$$

$$H_2 : (a_3 - a_1)t^2 + d_3u_3^2 - d_1u_1^2 = 0,$$

$$H_3 : (a_1 - a_2)t^2 + d_1u_1^2 - d_2u_2^2 = 0. \quad (2.2)$$

Moreover, $E(\mathbb{Q})/2E(\mathbb{Q})$ can be embedded into $\text{Sel}_2(E)$. If $(x, y) \notin E(\mathbb{Q})[2]$ is a rational point on $E$, the embedding is given by $(x, y) \mapsto (x - a_1, x - a_2, x - a_3)$. The 2-torsion point $(a_1, 0)$ corresponds to

$$(a_1 - a_2)(a_1 - a_3), a_1 - a_2, a_1 - a_3.$$ 

Similar correspondences are defined for the 2-torsion points $(a_2, 0)$ and $(a_3, 0)$.

2.2. Cassels Pairing.

For a general elliptic curve $E/\mathbb{Q}$, Cassels [11] defined a skew-symmetric bilinear pairing $\langle \cdot, \cdot \rangle$ on the pure 2-Selmer group $\text{Sel}_2(E)$, which is an $\mathbb{F}_2$-vector space defined by $\text{Sel}_2(E)/E(\mathbb{Q})[2]$. We assume that $E$ is defined by the equation $(2.1)$, and we use the identification of $\text{Sel}_2(E)$ in §2.1. Let $\Lambda = (d_1, d_2, d_3)$ be any element of $\text{Sel}_2(E)$ and $D_\Lambda$ the corresponding genus one curve associated to $\Lambda$. Since $H_i$ is locally solvable everywhere, there is a $Q_i \in H_i(\mathbb{Q})$ by Hasse-Minkowski principle. We define $L_i$ to be a linear form in three variables (all of $t, u_1, u_2, u_3$ except $u_i$) such that $L_i = 0$ defines the tangent plane of $H_i$ at $Q_i$. Then we call $L_i$ the tangent linear form of $H_i$ at $Q_i$. Moreover, we consider it as a linear form in $u_1, u_2, u_3$ and $t$ with the coefficient of $u_i$ being zero. As $D_\Lambda$ is locally solvable everywhere, there are enough points on $D_\Lambda(\mathbb{A})$ such that we may choose $P = (P_p) \in D_\Lambda(\mathbb{A})$ with all $\prod_{i=1}^3 L_i(P_p)$ non-vanishing. Given any $\Lambda' = (d'_1, d'_2, d'_3) \in \text{Sel}_2(E)$, the local Cassels pairing $\langle \Lambda, \Lambda' \rangle_p$ is defined to be

$$\prod_{i=1}^3 \langle L_i(P_p), d'_i \rangle_p.$$ 

Here $p$ is any rational prime or infinity, and $\langle \cdot, \cdot \rangle_p$ denotes the Hilbert symbol at $\mathbb{Q}_p$ ($\mathbb{Q}_\infty = \mathbb{R}$ if $p = \infty$). Then the Cassels pairing $\langle \Lambda, \Lambda' \rangle$ is given by

$$\prod_p \langle \Lambda, \Lambda' \rangle_p.$$
Here $p$ runs over all places of $\mathbb{Q}$.

Cassels [1] proved that this pairing is well-defined, namely it is independent of the choices of $P, Q$, and the representatives of the cosets of $\Lambda$ and $\Lambda'$. Since skew-symmetry over $\mathbb{F}_p$ is also symmetry, the left kernel and the right kernel of the Cassels pairing are the same. To show its kernel, we first introduce some notation. From the short exact sequence

$$0 \to E[2] \to E[4] \xrightarrow{2\lambda} E[2] \to 0,$$

we can derive the long exact sequence

$$0 \to E(\mathbb{Q})[2]/2E(\mathbb{Q})[4] \to \text{Sel}_2(E) \to \text{Sel}_4(E) \to \text{ImSel}_4(E) \to 0.$$

Cassels showed that the kernel of this pairing is $\text{ImSel}_4(E)/E(\mathbb{Q})[2]$. The following lemma shows that almost all the local Cassels pairings are trivial.

Lemma 1 (Cassels [1] Lemma 7.2). The local Cassels pairing $\langle \Lambda, \Lambda' \rangle_p = +1$ if $p$ satisfies

1. $p \neq 2, \infty$;
2. The coefficients of $H_i$ and $L_i$ are all integral at $p$ for $1 \leq i \leq 3$;
3. Modulo $D_\Lambda$ and $L_i$ by $p$, they define a curve of genus 1 over $\mathbb{F}_p$ together with tangents to it.

2.3. Residue Symbols.

In this subsection, we will introduce several residue symbols. The first residue symbol is the additive Jacobi symbol. Let $d$ be a positive odd integer and $m$ an integer coprime to $d$. Then we define the additive Jacobi symbol $\left[ \frac{m}{d} \right]$ if the Jacobi symbol $\left( \frac{m}{d} \right) = -1$ and 0 otherwise.

Since other residue symbols involve the Gaussian integer ring $\mathbb{Z}[i]$, we first recall some conceptions related to $\mathbb{Z}[i]$. A prime element $\lambda$ of $\mathbb{Z}[i]$ is called Gaussian if it is not a rational prime. An integer $\theta \in \mathbb{Z}[i]$ is called primary if $\theta \equiv 1 \pmod{2 + 2i}$. In particular, any primary integer can be written uniquely as the product of primary primes. We use $N$ to denote the norm of an element or an ideal of $\mathbb{Z}[i]$.

The second residue symbol is the general Legendre symbol over $\mathbb{Z}[i]$. Let $p$ be a prime ideal of $\mathbb{Z}[i]$ coprime to $(1 + i)$. The general Legendre symbol $\left( \frac{a}{p} \right)$ is the unique element of $\{0, \pm 1\}$ such that $\alpha^{N_\mathbb{F}_p - 1} \equiv \left( \frac{a}{p} \right) \pmod{p}$. We refer to Page 196 of Hecke [3]. In particular, if $\lambda$ is the unique primary prime in $p$, we put $\left( \frac{a}{\lambda} \right) = \left( \frac{a}{p} \right)$. If $\lambda$ has a factorization $\prod_{i=1}^k \lambda_i$ of primary primes, then we define $\left( \frac{a}{\lambda} \right)$ to be $\prod_{i=1}^k \left( \frac{a}{\lambda_i} \right)$.

The third residue symbol is the quartic residue symbol. We refer to Ireland-Rosen [13]. Assume that $\lambda$ is a prime element coprime to $(1 + i)$. For a Gaussian integer $\alpha$, the quartic residue symbol $\left( \frac{\alpha}{\lambda} \right)_4$ is defined to be the unique element of $\{0, \pm 1, \pm i\}$ such that $\alpha^{N_{\mathbb{F}_p} - 1} \equiv \left( \frac{\alpha}{\lambda} \right)_4 \pmod{\lambda}$. Let $\lambda_1$ and $\lambda_2$ be two coprime Gaussian primes. We have the quartic reciprocity law

$$\left( \frac{\lambda_1}{\lambda_2} \right)_4 = \left( \frac{\lambda_2}{\lambda_1} \right)_4 (-1)^{N_{\mathbb{F}_p} - 1} .$$

Assume that $\lambda$ has a factorization $\prod_{i=1}^k \lambda_i$ of primary primes. We define $\left( \frac{a}{\lambda} \right)_4$ to be $\prod_{i=1}^k \left( \frac{a}{\lambda_i} \right)_4$.

The last residue symbol is the rational quartic residue symbol. Let $p$ be a rational prime congruent to 1 modulo 4. So there are exactly two primitive primes $\lambda$ and $\bar{\lambda}$ lying above $p$. Here $\bar{\lambda}$ is the complex conjugate of $\lambda$ and $p = \lambda \bar{\lambda}$. If $q$ a rational integer such that $\left( \frac{a}{p} \right)_4 = 1$, then the two quartic residue symbols $\left( \frac{q}{\lambda} \right)_4$ and $\left( \frac{q}{\bar{\lambda}} \right)_4$ take the same value, and we use the symbol $\left( \frac{a}{p} \right)_4$ to denote any of them. Moreover, if $d$ is a positive integer such
that all prime factors of \( d \) are congruent to 1 modulo 4, then \( (\frac{a}{d})_4 \) is defined to be

\[
\prod_{p \mid d} \left( \frac{q}{p} \right)^{v_p(d)}
\]

provided that \( q \) is a rational integer satisfying \( \left( \frac{q}{p} \right) = 1 \) for any \( p \mid d \). Here \( v_p(d) \) is the \( p \)-adic valuation of \( d \).

### 2.4. Gauss Genus Theory.

In this subsection, we briefly summarize Gauss genus theory. One can refer to §3 of [18] for a detailed proof.

Let \( K \) be an imaginary quadratic number field with ideal class group \( \mathcal{A} \). We write the multiplication of ideal classes additively. Then the 2\(^{-}\)-rank \( h_2(A) \) of \( A \) is defined to be

\[ \dim_{\mathbb{Z}/2^{t}-1} A/2^{t}A \]

with \( t \) any positive integer. By classical Gauss genus theory, the 2-rank \( h_2(A) \) equals to \( t \) minus the number of different prime factors of the fundamental discriminant \( D \) of \( K \). In fact, \( h_2(A) \) equals to the dimension of \( A[2] \), which is the set consisting of ideal classes killed by 2. In addition, \( A[2] \) is an elementary abelian 2-group generated by \( (p, \alpha_0) \) with \( p \) any prime factor of \( D \) and \( 2\alpha_0 = D + \sqrt{D} \).

As to the 4-rank, we can easily deduce that \( h_4(A) = \dim_{\mathbb{Z}/2^{t}-1} A \cap A[2] \). Therefore, the study of the 4-rank is reduced to that of \( A \cap A[2] \). The key tool to study \( A \cap A[2] \) is the Rédei matrix. They are closely tied via \( D(K) \cap N_{K/Q}(K^\times) \). Here \( D(K) \) is the set of positive square-free divisors of \( D \) and \( N_{K/Q} \) is the norm map from \( K \) to \( \mathbb{Q} \). In fact, we have a two to one epimorphism

\[
\theta : D(K) \cap N_{K/Q}(K^\times) \longrightarrow 2A \cap A[2]
\]

with \( \theta(d) = [(d, \alpha_0)] \). Note that \( D(K) \) is a group under the group operation \( d_1 \oplus d_2 = \frac{d_1d_2}{(d_1,d_2)^2} \). Moreover, the kernel of \( \theta \) is \( \{1, D' \} \) with \( D' \) the square-free part of \( D \).

To connect \( D(K) \cap N_{K/Q}(K^\times) \) with the Rédei matrix, we first introduce some notation. Let \( p_1, \ldots, p_t \) be the different prime divisors of \( D \). We assume that \( p_t = 2 \) if \( 2 \mid D \). The Rédei matrix \( R = (r_{ij})(t-1) \times t \) of \( K \) is an \( \mathbb{F}_2 \) matrix defined by \( r_{ii} = \left[ \frac{D/p_t}{p_t} \right] \) and \( r_{ij} = \left[ \frac{p_i}{p_j} \right] \) if \( i \neq j \). Here \( p_i^* = (-1)^{\frac{n-1}{2}}p_i \).

**Lemma 2.** Assume that \( w \) is a positive odd integer satisfying \( (w, D) = 1 \) and \( \left( \frac{D}{w} \right) = 1 \) for any prime divisor \( p \) of \( w \). Let \( W = \left( \left[ \frac{w}{p_1} \right], \ldots, \left[ \frac{w}{p_{t-1}} \right] \right)^T \). Then we have an isomorphism

\[
\{ d \in D(K) \mid dw \in N_{K/Q}(K^\times) \} \longrightarrow \{ Y \in \mathbb{F}_2^t \mid RY = W \},
\]

where the map is given by \( d \mapsto Y_d := (v_{p_1}(d), \ldots, v_{p_t}(d))^T \) and its inverse is \( (y_1, \ldots, y_t)^T \mapsto \prod_{i=1}^{t} p_i^{y_i} \).

Choose \( w = 1 \) in this lemma, we obtain the following isomorphism

\[
D(K) \cap N_{K/Q}(K^\times) \longrightarrow \{ Y \in \mathbb{F}_2^t \mid RY = 0 \}.
\]

Consequently, \( h_4(A) = t - 1 - \text{rank}_{\mathbb{F}_2} R \).

Like the 4-rank, the study of the 8-rank \( h_8(A) \) is equivalent to that of \( 4A \cap A[2] \). This is equivalent to determine which \( [a] \in 2A \cap A[2] \) still lies in \( 4A \). We assume that \( K = \mathbb{Q}(\sqrt{-n}) \), where \( n = p_1 \cdots p_k \) is a positive square-free odd integer. Assume that \( 2^r d \) lies in \( D(K) \cap N_{K/Q}(K^\times) \) such that \( d \) is a non-trivial divisor of \( n \) and \( r \) equals to 0, 1. Then the following equation

\[
dx^2 + \frac{n}{d}y^2 = 2^rz^2
\]

has a non-trivial integer solution.
Lemma 3. Assume that $n, d, r, R$ are as above and $(u, v, w)$ is a positive primitive integer solution to \((2.3)\). Let $W = \left(\left[\frac{u}{p_1}\right], \ldots, \left[\frac{u}{p_k}\right]\right)^T$. Then $[a] \in 4A$ if and only if there is a $Y \in \mathbb{F}_2^{k+1}$ such that $RY = W$, where $a = (2'd, a_0)$.

2.5. Analytic Results.

Given a number field $F$, let $n$ and $O$ be its degree and ring of algebraic integers respectively. We define $\Delta$ and $N_F$ to be the discriminant of $F$ and the norm from $F$ to $\mathbb{Q}$ respectively. We call a non-zero element $\gamma \in F$ totally positive if it is positive under all real embeddings provided that $F$ has a real embedding. If $F$ has no real embedding, all non-zero elements of $F$ are totally positive.

Let $\mathfrak{p}$ be an integral ideal of $O$. We denote by $I(\mathfrak{p})$ the group of all the fractional ideals that are coprime to $\mathfrak{p}$. We use $P_1$ to denote the group consisting of the principal fractional ideals $(\gamma)$ such that $\gamma$ is totally positive and $\gamma \equiv 1 \pmod{\mathfrak{p}}$. Here the notation $\gamma \equiv 1 \pmod{\mathfrak{p}}$ denotes $\gamma \in O_\mathfrak{p}$ and $\gamma \equiv 1 \pmod{p^{\nu(\mathfrak{p})}}$ for every prime ideal $p | \mathfrak{p}$, where $O_\mathfrak{p}$ is the integer ring of $F_\mathfrak{p}$.

If $\chi$ is a character of $I(\mathfrak{p})/P_1$ with $\mathfrak{p}$ an integral ideal, then we view it as a character on $I(\mathfrak{p})$ call $\chi$ a character modulo $\mathfrak{p}$. In addition, if a fractional ideal $a$ is not coprime to $\mathfrak{p}$, we define $\chi(a) = 0$. Let $\Lambda(a)$ be the Mangoldt function defined by

$$
\Lambda(a) = \begin{cases} 
\log N_Fp & \text{if } a = p^n \text{ with } m \geq 1, \\
0 & \text{otherwise.}
\end{cases}
$$

Then $\psi(x, \chi)$ is defined to be

$$
\psi(x, \chi) = \sum_{N_Fp \leq x} \chi(a) \Lambda(a).
$$

The following explicit formula (Proposition 1) of $\psi(x, \chi)$ is proved in P114 of Iwaniec-Kowalski [7].

**Proposition 1.**

If $\chi$ is a non-principal character modulo an integral ideal $\mathfrak{p}$ and $1 \leq T \leq x$, then

$$
(2.4) \quad \psi(x, \chi) = - \sum_{|\text{Im} \rho| \leq T} \frac{x^\rho - 1}{\rho} + O_F\left(xT^{-1} \cdot \log x \cdot \log(x^n \cdot N_F)\right).
$$

Here $\rho$ runs over all the zeros of $L(s, \chi)$ with $0 \leq \text{Re} \rho \leq 1$ and $O_F$ means the implied constant only depends on $F$.

Note that the first term of the formula \((2.4)\) is not estimated. It can be estimated by the same way as the classical case, and we omit its proof. We derive the explicit formula

$$
(2.5) \quad \psi(x, \chi) = \frac{x^{\beta'}}{\beta'} + R(x, T)
$$

with

$$
R(x, T) \ll x \cdot \log^2(xN_F) \cdot \exp\left(- \frac{c_1 \log x}{\log(TN_F)}\right) + xT^{-1} \log x \cdot \log |x^nN_F| + x^{1/4} \log x.
$$

Here $c_1$ is a positive constant and the term $-x^{\beta'/\beta'}$ occurs only if $\chi$ is a real character such that $L(s, \chi)$ has a zero $\beta'$ satisfying

$$
\beta' > 1 - \frac{c_2}{\log N_F}
$$

with $c_2$ a positive constant.

For further application, we introduce Siegel Theorem and Page Theorem over $F$. The following Proposition 2 is Siegel Theorem over $F$, and the references are Fogels [4, 5, 6].

**Proposition 2.** Let $\chi$ be a character modulo an integral $\mathfrak{p}$ and $D = |\Delta|N_F > D_0 > 1$. 
(i) There is a positive constant $c_3 = c_3(n)$ such that in the region

$$\text{Re}(s) > 1 - \frac{c_3}{\log D(1 + |\text{Im}(s)|)} > \frac{3}{4}$$

there is no zero of $L(s, \chi)$ if $\chi$ is complex, and for at most one real character $\chi'$ there may be a simple zero $\beta'$ of $L(s, \chi')$.

(ii) Let $\beta'$ be the exceptional zero of the exceptional character $\chi'$ modulo $\dagger$. Then for any $\epsilon > 0$ there exists a positive constant $c_4 = c_4(n, \epsilon)$ such that

$$1 - \beta' > c_4(n, \epsilon)D^{-\epsilon}.$$ 

Proposition 3 is Page Theorem over $F$. One can refer to Hoffstein-Ramakrishnan [9].

**Proposition 3.** For any $Z \geq 2$ and $c_5$ a suitable constant, there is at most a real primitive character $\chi$ to a modulus $\dagger$ with $N_{F^\dagger} \leq Z$ such that $L(s, \chi)$ has a real zero $\beta$ satisfying

$$\beta > 1 - \frac{c_5}{\log Z}.$$ 

### 3. Representation of 2-Selmer Group

Now we apply the result of §2.1 to the elliptic curve

$$E_{a,b} : y^2 = x(x - a)(x + b)$$

with $a$ and $b$ positive odd integers. For positive square-free integer $n$, we consider the elliptic curve $E_{a,b}^{(n)} = E^{(n)}$

$$E^{(n)} : y^2 = x(x - an)(x + bn).$$

Choosing $a_1 = an, a_2 = -bn$ and $a_3 = 0$ in (2.2), we get the following identification

$$\text{Sel}_2(E^{(n)}) = \{ \Lambda = (d_1, d_2, d_3) \in (\mathbb{Q}/\mathbb{Q}^2)^3 \mid d_1d_2d_3 \in \mathbb{Q}^2, D_\Lambda(\Lambda) \neq \emptyset \}$$

with $D_\Lambda$ the genus one curve defined by

$$\begin{cases} 
H_1 : -bnt^2 + d_2u_2^2 - d_3u_3^2 = 0, \\
H_2 : -ant^2 + d_3u_3^2 - d_1u_1^2 = 0, \\
H_3 : 2cnt^2 + d_1u_1^2 - d_2u_2^2 = 0.
\end{cases}$$

Here $a + b = 2c$. If $(x, y) \in E^{(n)}(\mathbb{Q})$ is not a 2-torsion point, it corresponds to $(x - an, x + bn, x)$. Moreover, the four elements $(an, 0), (-bn, 0), (0, 0)$ and $O$ of $\mathcal{E}^{(n)}(\mathbb{Q})[2]$ correspond to $(2ac, 2cn, an), (-2cn, 2bc, -bn), (-an, bn, -ab)$ and $(1, 1, 1)$ respectively.

In this section, we always assume that $a, b$ and $c$ are coprime positive odd integers such that

$$a + b = 2c.$$ 

#### 3.1. Local Solvability Conditions on $D_\Lambda$

We denote by $a = a_1a_2^2, b = b_1b_2^2$ and $c = c_1c_2^2$ with $a_1, b_1$ and $c_1$ square-free integers.

**Lemma 4.** Assume that $n$ is a positive square-free integer coprime to $2abc$. Let $\Lambda = (d_1, d_2, d_3)$ with $d_i$ non-zero square-free integers such that $d_1d_2d_3$ is a square.

1. If $p \nmid 2abc$, then $D_\Lambda(\mathbb{Q}_p) \neq \emptyset$ if and only if $p \nmid d_1d_2d_3$.
2. $D_\Lambda(\mathbb{R})$ is non-empty if and only if $d_3 > 0$.
3. If $D_\Lambda(\mathbb{Q}_2)$ is non-empty, then $d_1$ and $d_2$ have the same parity.
4. If $d_1$ and $d_2$ are odd, then $D_\Lambda(\mathbb{Q}_2)$ is non-empty if and only if either $4 \mid d_1 - d_3, 8 \mid d_1 - d_2$ or $4 \mid d_1 + an, 8 \mid d_1 - d_2 + 2cn$. 


Proof. (1) Let \( p \) be a prime such that \( p \mid 2abc n \). If \( p \mid d_1 d_2 d_3 \), then \( p \) divides exactly two of \( d_1, d_2 \) and \( d_3 \). We assume that \( p \mid d_1 \) and \( p \mid d_2 \). Since we are dealing with homogeneous equations, we may assume that \( u_1, u_2, u_3 \) and \( t \) are \( p \)-adic integers and at least one of them is a \( p \)-adic unit. By comparing \( p \)-adic valuations of both sides of \( H_3 \), we get that \( p \mid t \). Similarly, from \( H_2 \) we derive that \( p \mid u_3 \). Then via \( H_1 \) and \( H_2 \) we infer that \( p \mid u_2 \) and \( p \mid u_1 \). So \( p \mid (t, u_1, u_2, u_3) \), which is impossible. Thus \( D_\Lambda(\mathbb{Q}_p) \) is empty provided that \( p \mid d_1 d_2 d_3 \).

Now we assume that \( p \nmid d_1 d_2 d_3 \). We will use Weil conjecture for curve (see Silverman [12] P134) to show that \( D_\Lambda(\mathbb{Q}_p) \) is non-empty. Note that \( D_\Lambda \) modulo \( p \) gives rise to a smooth projective curve over \( \mathbb{F}_p \) by \( p \nmid 2nabcd_1 d_2 d_3 \). Weil conjecture implies that

\[
Z(D_\Lambda, T) = \frac{P_1(T)}{(1 - T)(1 - pT)}
\]

with \( P_1(T) \in \mathbb{Z}[T] \) factoring as

\[
P_1(T) = (1 - \alpha T)(1 - \overline{\alpha} T)
\]

over \( \mathbb{C} \). Here \( \alpha \) has norm \( p^2 \) and \( Z(D_\Lambda, T) \) is the zeta function of \( D_\Lambda \) over \( \mathbb{F}_p \) given by

\[
Z(D_\Lambda, T) = \exp\left( \sum_{m=1}^{\infty} \frac{\#D_\Lambda(\mathbb{F}_p^m) T^m}{m} \right),
\]

where \( \#D_\Lambda(\mathbb{F}_p^m) \) denotes the number of points of \( D_\Lambda \) over \( \mathbb{F}_p^m \). Therefore,

\[
\sum_{m=1}^{\infty} \frac{\#D_\Lambda(\mathbb{F}_p^m) T^m}{m} = \log(1 - \alpha T) \log(1 - \overline{\alpha} T) - \log(1 - T) - \log(1 - p T).
\]

Comparing the coefficients of \( T \) in above equation, we have

\[
\#D_\Lambda(\mathbb{F}_p) = 1 + p - (\alpha + \overline{\alpha}) \geq 1 + p - 2\sqrt{p} > 0.
\]

Therefore, \( D_\Lambda(\mathbb{Q}_p) \) is non-empty by Hensel’s Lemma.

(2) If \( d_2 < 0 \), we see that \( H_1(\mathbb{R}) \) is not solvable for the coefficients on its left hand side are all negative. Conversely, if \( d_2 > 0 \), then \( D_\Lambda(\mathbb{R}) \) is obviously solvable.

(3) Now we assume that \( D_\Lambda(\mathbb{Q}_2) \) is non-empty. Assume that \( d_1 \) and \( d_2 \) have different parity. We first treat the case that \( d_1 \) is even and \( d_2 \) is odd. Then \( d_3 \) is even by \( d_1 d_2 d_3 \in \mathbb{Q}^{\times 2} \). Considering the 2-adic valuations of both sides of \( H_2 \) and \( H_3 \), we get that \( t \) and \( u_2 \) are even. From similar considerations on \( H_1 \) and \( H_3 \), we derive that \( u_3 \) and \( u_1 \) are also even. So \( u_1, u_2, u_3 \) and \( t \) are even, which is impossible. Similar arguments show that the case that \( d_1 \) is odd and \( d_2 \) is even is also impossible.

(4) Let \( d_1 \) and \( d_2 \) be odd integers. First, we assume that \( D_\Lambda(\mathbb{Q}_2) \) is non-empty. Since we are dealing with homogeneous equations, we may assume that \( u_1, u_2, u_3 \) and \( t \) are 2-adic integers and at least one of them is odd. Viewing \( H_3 \) as a congruence modulo 4, we see that \( u_1 \) and \( u_2 \) are odd. From \( H_2 \) we infer that exactly one of \( t \) and \( u_3 \) is even. Now we divide this into two subcases according to the parity of \( t \).

Case (i): \( t \) is odd. Then \( u_3 \) is even. Considering \( H_2 \) and \( H_3 \) as congruences modulo 4 and 8 respectively, we get that \( 4 \mid d_1 + an \) and \( 8 \mid d_1 - d_2 + 2cn \).

Case (ii): \( t \) is even. Then \( u_3 \) is odd. Viewing \( H_2 \) and \( H_3 \) as congruences modulo 4 and 8 respectively, we obtain that \( 4 \mid d_1 - d_3 \) and \( 8 \mid d_1 - d_2 \).

Finally, we assume that either \( 4 \mid d_1 - d_3 \) and \( 8 \mid d_1 - d_2 \) or \( 4 \mid d_1 + an \) and \( 8 \mid d_1 - d_2 + 2cn \). According to these conditions, we divide it into two subcases.

Case (iii): \( 4 \mid d_1 - d_3 \) and \( 8 \mid d_1 - d_2 \). We have \( d_3 \equiv d_1 d_2 \equiv 1 \mod 8 \). Since \( 4 \mid d_1 - d_3 \), we get \( d_1 \equiv 1 \mod 4 \). If \( d_1 \equiv 1 \mod 8 \), then \( d_1 \) has square root in \( \mathbb{Z}_2 \) and we choose \( t = 0 \) and \( u_i = d_i^{1/2} \) for \( 1 \leq i \leq 3 \). If \( d_1 \equiv 5 \mod 8 \), then \( d_1 + 4an \equiv 1 \mod 8 \) and
Assume that the remaining cases can be proved similarly. This completes the proof of the lemma.

Let \( d_1 + 8cn \equiv d_2 \pmod{8} \); so \( (d_1 + 8cn)d_2^{-1} \) and \( (d_1 + 4an)d_3^{-1} \) have square roots in \( \mathbb{Z}_2 \); in addition, we choose \( t = 2 \), \( u_1 = 1 \),

\[
    u_2 = \sqrt{(d_1 + 8cn)d_2^{-1}} \quad \text{and} \quad u_3 = \sqrt{(d_1 + 4an)d_3^{-1}}.
\]

Case (iv): \( 4 \mid d_1 + an \) and \( 8 \mid d_1 - d_2 + 2cn \). If \( 8 \mid d_1 + an \), then we choose \( t = 1 \) and \( u_3 = 0 \); so \( -and_1^{-1} \) and \( -(an + 2cn)d_2^{-1} \) are congruent to 1 modulo 8, and they have square roots in \( \mathbb{Z}_2 \); let \( u_1 \) and \( u_3 \) be any square roots of \( -and_1^{-1} \) and \( -(an + 2cn)d_2^{-1} \) respectively. If \( 4 \parallel d_1 + an \), we choose \( t = 1 \) and \( u_3 = 2 \); so \( (4 - an)d_1^{-1} \) and \( 2(cn + 4 - an)d_2^{-1} \) are congruent to 1 modulo 8, and they have square roots in \( \mathbb{Z}_2 \); we define \( u_1 \) and \( u_2 \) to be any square roots of \( (4 - an)d_1^{-1} \) and \( 2(cn + 4 - an)d_2^{-1} \) respectively. Therefore, in any case we have \( D_\Lambda(Q_2) \neq \emptyset \).

This completes the proof of the lemma. \( \square \)

Assume that \( n \) is a positive square-free integer coprime to \( 2abc \). By Lemma 4 any element of \( Sel_2(E^{(n)}/E^{(n)}(Q)[2] \) has a unique representative \( \Lambda = (d_1, d_2, d_3) \) with \( d_i \) positive square-free integers satisfying

\[
    d_1d_2d_3 \in \mathbb{Q}^{\times 2}, \quad d_2 > 0, \quad d_1|nabc \text{ and } d_2|nabc.
\]

**Lemma 5.** Assume that \( n \) is a positive square-free integer coprime to \( 2abc \) and \( \Lambda = (d_1, d_2, d_3) \) with \( d_i \) positive square-free integers such that (3.3) holds. Let \( p \) be a prime divisor of \( n \).

- If \( p \nmid d_1 \) and \( p \nmid d_2 \), then \( D_\Lambda(Q_p) \neq \emptyset \) if and only if \( \left( \frac{d_1}{p} \right) = \left( \frac{d_2}{p} \right) = 1 \).
- If \( p \nmid d_1 \) and \( p \mid d_2 \), then \( D_\Lambda(Q_p) \) is non-empty if and only if \( \left( \frac{d_1}{p} \right) = \left( \frac{2ac}{p} \right) = 1 \) and \( \left( \frac{N/d_2}{p} \right) = \left( \frac{2ab}{p} \right) \), where \( N = abcn \).
- If \( p \mid d_1 \) and \( p \nmid d_2 \), then \( D_\Lambda(Q_p) \) is non-empty if and only if \( \left( \frac{N/d_1}{p} \right) = \left( \frac{-2ab}{p} \right) \) and \( \left( \frac{d_2}{p} \right) = \left( \frac{2bc}{p} \right) \).
- If \( p \mid d_1 \) and \( p \mid d_2 \), then \( D_\Lambda(Q_p) \) is non-empty if and only if \( \left( \frac{N/d_1}{p} \right) = \left( \frac{-bc}{p} \right) \) and \( \left( \frac{N/d_2}{p} \right) = \left( \frac{ac}{p} \right) \).

**Proof.** Assume that \( p \nmid d_1d_2 \). So \( p \nmid d_3 \). If \( D_\Lambda(Q_p) \) is non-empty, then by \( H_2 \) and \( H_3 \) we get \( \left( \frac{d_2d_3}{p} \right) = \left( \frac{d_1d_2}{p} \right) = 1 \), namely \( \left( \frac{d_1}{p} \right) = \left( \frac{d_2}{p} \right) = 1 \). Conversely, if \( \left( \frac{d_1}{p} \right) = \left( \frac{d_2}{p} \right) = 1 \), then we set \( t = 0 \) and \( u_i = d_i^{-\frac{1}{2}} \) for \( 1 \leq i \leq 3 \). So \( (t, u_1, u_2, u_3) \) lies in \( D_\Lambda(Q_p) \). The remained cases can be proved similarly. This completes the proof of the lemma. \( \square \)

**Lemma 6.** Assume that \( n \) is a positive square-free integer coprime to \( 2abc \) and \( \Lambda = (d_1, d_2, d_3) \) with \( d_i \) positive square-free integers such that (3.2) holds.

- Let \( p \) be a prime divisor of \( a \). Then \( D_\Lambda(Q_p) \neq \emptyset \) implies that \( p \nmid d_2 \). Under the condition \( p \nmid d_2 \), we have
  
  - if \( p \nmid d_1 \), then \( D_\Lambda(Q_p) \) is non-empty if and only if \( \left( \frac{d_2}{p} \right) = 1 \);
  
  - if \( p \mid d_1 \), then \( D_\Lambda(Q_p) \) is non-empty if and only if \( \left( \frac{bnd_2}{p} \right) = 1 \) provided that \( p \mid d_1 \) with \( p \nmid a_2 \) and \( \left( \frac{d_2}{p} \right) = \left( \frac{bn}{p} \right) = 1 \) provided that \( p \mid a_2 \).

- Let \( p \) be a prime divisor of \( b \). Then \( D_\Lambda(Q_p) \neq \emptyset \) implies that \( p \nmid d_1 \). Under the condition \( p \nmid d_1 \), we have
  
  - if \( p \nmid d_2 \), then \( D_\Lambda(Q_p) \neq \emptyset \) if and only if \( \left( \frac{d_1}{p} \right) = 1 \);
we derive that

\[ p \mid b_1 \text{ with } p \nmid b_2 \text{ and } \left( \frac{-\alpha u}{p} \right) = \left( \frac{d_1}{p} \right) = 1 \text{ provided that } p \mid b_2. \]

- Let \( p \) be a prime divisor of \( c \). Then \( D_\Lambda(\mathbb{Q}_p) \neq \emptyset \) implies that \( p \nmid d_3 \). Under the condition \( p \nmid d_3 \), we have

- if \( p \nmid d_1 d_2 \), then \( D_\Lambda(\mathbb{Q}_p) \) is non-empty if and only if \( \left( \frac{d_3}{p} \right) = 1 \);

- if \( p \mid d_1 \) and \( p \mid d_2 \), then \( D_\Lambda(\mathbb{Q}_p) \) is non-empty if and only if \( \left( \frac{\alpha d_3}{p} \right) = 1 \)

provided that \( p \mid c_1 \) with \( p \nmid c_2 \) and \( \left( \frac{\alpha u}{p} \right) = \left( \frac{d_3}{p} \right) = 1 \text{ provided that } p \mid c_2. \)

Proof. Let \( p \) be a prime divisor of \( a \). Assume that \( D_\Lambda(\mathbb{Q}_p) \) is non-empty. If \( p \mid d_2 \), then \( p \) divides exactly one of \( d_1 \) and \( d_3 \). We may assume that \( p \mid d_1 \). So \( p \nmid d_3 \). Via \( H_2 \) and \( H_3 \), we obtain that \( p \mid u_3 \) and \( p \mid t \). Then \( H_1 \) and \( H_2 \) imply that \( p \mid u_2 \) and \( p \mid u_1 \). So \( p \mid (t, u_1, u_2, u_3) \), which is impossible. Therefore, \( D_\Lambda(\mathbb{Q}_p) \neq \emptyset \) implies that \( p \nmid d_2 \).

Now we assume that \( p \nmid d_1 d_2 d_3 \). If \( D_\Lambda(\mathbb{Q}_p) \neq \emptyset \), then \( H_2 \) implies that \( \left( \frac{d_1 d_2}{p} \right) = 1 \), namely \( \left( \frac{d_2}{p} \right) = 1 \). Consequently, if \( \left( \frac{d_2}{p} \right) = 1 \), then we choose \( u_1 = \frac{d_2}{(d_1 d_2)} \). We have

\[
\begin{align*}
u_3^2 &= d_2 + \alpha d_3^{-1} t^2 = d_2 + O(p), \\
u_2^2 &= d_3 + 2c d_3^{-1} t^2,
\end{align*}
\]

where \( O(p) \) is \( w p \) with \( w \) some \( p \)-adic integer. The first equation is solvable for any given \( t \) by \( \left( \frac{d_2}{p} \right) = 1 \). Denote by \( \mathcal{R} \) the quadratic residue classes modulo \( p \). Then \( \mathcal{R} \) and \( d_3 + 2c d_3^{-1} \mathcal{R} \) have non-empty intersections for their cardinalities are \( \frac{p+1}{2} \). Thus we can choose \( t \) such that \( d_3 + 2c d_3^{-1} t^2 \) is a quadratic residue. So \( D_\Lambda(\mathbb{Q}_p) \) is non-empty by Hensel's Lemma.

Now we consider the case \( p \mid d_1 \) and \( p \mid d_3 \). Then \( p \nmid d_2 \). First, we treat the subcase \( p \mid a_2 \). If \( D_\Lambda(\mathbb{Q}_p) \) is non-empty, then \( \left( \frac{d_1 b u}{p} \right) = 1 \) by \( H_1 \). In addition, dividing \( H_2 \) by \( p \) we derive that \( \left( \frac{d_1/d_2}{p} \right) = 1 \), namely \( \left( \frac{d_2}{p} \right) = 1 \). Conversely, if \( \left( \frac{d_2}{p} \right) = \left( \frac{b u}{p} \right) = 1 \), then we choose \( u_1 = \frac{d_2}{(d_1 d_2)} \). We get

\[
\begin{align*}
u_3^2 &= d_2 + a_2 d_3^{-1} a_1 t^2 = d_2 + O(p), \\
u_2^2 &= d_3 + 2c d_3^{-1} t^2 = b d_3^{-1} t^2 + O(p).
\end{align*}
\]

These equations are solvable for \( \left( \frac{d_2}{p} \right) = \left( \frac{b u}{p} \right) = 1 \). So \( D_\Lambda(\mathbb{Q}_p) \) is non-empty. Finally, we treat the subcase \( p \mid a_1 \) with \( p \nmid a_2 \). If \( D_\Lambda(\mathbb{Q}_p) \) is non-empty, we get \( \left( \frac{d_2 b u}{p} \right) = 1 \) by \( H_1 \). Conversely, if \( \left( \frac{d_2 b u}{p} \right) = 1 \), then we choose \( u_1 = \frac{d_2}{(d_1 d_2)} \). We obtain

\[
\begin{align*}
u_3^2 &= d_2 + a_1 d_3^{-1} a_2^2 t^2, \\
u_2^2 &= d_3 + 2c d_3^{-1} t^2 = b d_2^{-1} t^2 + O(p).
\end{align*}
\]

The second equation is solvable for any \( t \) by \( \left( \frac{d_2 b u}{p} \right) = 1 \). Like the case \( p \nmid d_1 d_2 d_3 \), we know that there is a \( t \) such that \( d_2 + a_1 d_3^{-1} a_2^2 t^2 \) is a quadratic residue modulo \( p \). So \( D_\Lambda(\mathbb{Q}_p) \) is non-empty. Hence, we finish the proof of the case \( p \mid a \).

For the case \( p \mid b c \), we can prove similarly. This completes the proof of the lemma. □

**Lemma 7.** Assume that \( n \) is a positive square-free integer coprime to \( 2abc \) and \( \Lambda = (d_1, d_2, d_3) \) with \( d_i \) positive square-free odd integers such that \( (3,2) \) holds. If \( D_\Lambda(\mathbb{Q}_p) \) is non-empty for all odd primes \( p \) and \( p = \infty \), then \( D_\Lambda(\mathbb{Q}_2) \) is also non-empty.
Proof. We only prove the case that \(a, b\) and \(c\) are squares, the general case can be proved similarly but the process is much more complicated. Now we assume that \(a, b\) and \(c\) are squares. Define \(A = (a, d_1)\). Then Lemma 6 implies that \((\frac{bn}{A}) = (\frac{d_1}{A}) = 1\). In addition, \((\frac{2cn}{A}) = 1\) for \(a + b = 2c\). Since \(b\) and \(c\) are squares, we have

\[
\left(\frac{n}{A}\right) = 1, \quad \left(\frac{d_1}{A}\right) = 1, \quad \left(\frac{2}{A}\right) = 1.
\]

Denote by \(B = (b, d_2)\) and \(C = (c, d_1) = (c, d_2)\). Similarly we have

\[
\left(\frac{-n}{B}\right) = 1, \quad \left(\frac{d_1}{B}\right) = 1, \quad \left(\frac{2}{B}\right) = 1
\]

and

\[
\left(\frac{n}{C}\right) = 1, \quad \left(\frac{d_3}{C}\right) = 1, \quad \left(\frac{-1}{C}\right) = 1.
\]

Put \(n = n_1n_2n_3n_4\) with \((d_1, n) = n_3n_4\), \((d_2, n) = n_2n_4\) and \(n_4 = (d_1, d_2, n)\). Then we get that \(d_1 = ACn_3n_4\), \(d_2 = BCn_2n_4\) and \(d_3 = ABn_2n_3\). By Lemma 5, we have

\[
\frac{AC}{n_1} = \frac{n_3n_4}{n_1}, \quad \frac{BC}{n_2} = \frac{n_2n_4}{n_2}, \quad (*)
\]

\[
\frac{AC}{n_2} = \frac{2n_3n_4}{n_2}, \quad \frac{BC}{n_3} = \frac{2n_1n_3}{n_3}, \quad \frac{AC}{n_3} = \frac{-2n_1n_2}{n_3}, \quad \frac{BC}{n_4} = \frac{2n_2n_4}{n_4}
\]

From the second identity of equation (3.3) we have

\[
\left[\frac{n_2n_4}{A}\right] = \left[\frac{BC}{A}\right] = \left[\frac{A}{B}\right] + \left[\frac{A}{C}\right] + \left[\frac{-1}{A}\right] \left[\frac{-1}{B}\right]
\]

by the quadratic reciprocity law and \((\frac{1}{A}) = 1\) of (3.5). Via the second identities of (3.4) and (3.5), we get

\[
\left[\frac{n_2n_4}{A}\right] = \left[\frac{n_3n_4}{B}\right] + \left[\frac{n_2n_3}{C}\right] + \left[\frac{-1}{A}\right] \left[\frac{-1}{B}\right],
\]

\[
= \left[\frac{B}{n_3n_4}\right] + \left[\frac{C}{n_2n_3}\right] + \left[\frac{-1}{A}\right] \left[\frac{-1}{B}\right] + \left[\frac{-1}{B}\right] \left[\frac{-1}{n_3n_4}\right].
\]

Here we have used \((\frac{-1}{B}) = 1\) and the quadratic reciprocity law. The third, sixth, seventh and the last identities of (*) imply that

\[
\left[\frac{n_2n_4}{A}\right] = \left[\frac{-1}{n_4}\right] + \left[\frac{2}{n_2n_3}\right] + \left[\frac{A}{n_2n_4}\right] + \left[\frac{n_2n_3}{n_4}\right] + \left[\frac{n_3n_4}{n_2}\right] + \left[\frac{n_2n_4}{n_3}\right] - \left[\frac{-1}{A}\right] \left[\frac{-1}{B}\right] + \left[\frac{-1}{B}\right] \left[\frac{-1}{n_3n_4}\right].
\]

By the quadratic reciprocity law we get

\[
\left[\frac{2}{n_2n_3}\right] + \left[\frac{-1}{n_4}\right] = \left[\frac{-1}{A}\right] \left[\frac{-1}{B}\right] + \left[\frac{-1}{A}\right] \left[\frac{-1}{n_2n_4}\right] + \left[\frac{-1}{B}\right] \left[\frac{-1}{n_3n_4}\right] + \left[\frac{-1}{n_2}\right] \left[\frac{-1}{n_3}\right] + \left[\frac{-1}{n_2}\right] \left[\frac{-1}{n_4}\right] + \left[\frac{-1}{n_3}\right] \left[\frac{-1}{n_4}\right].
\]

Expanding the residue symbols \((\frac{-1}{n_i}) = \frac{-1}{n_i}\) for \(i = 2, 3\), we have

\[
\left[\frac{2}{n_2n_3}\right] + \left[\frac{-1}{n_4}\right] = -\left[\frac{-1}{A}\right] \left(Bn_2\right) + \left[\frac{-1}{n_4}\right] \left(\frac{-1}{ABn_2n_3}\right).
\]
Similarly, starting from the first identity of (*) we derive

\[
(3.7) \quad \left[ \frac{-1}{n_3} \right] = \left[ \frac{-1}{An_4} \right] \left[ \frac{-1}{Bn_1} \right] + \left[ \frac{-1}{n_3} \right] \left[ \frac{-1}{ABn_1n_4} \right],
\]

and starting from the second identity of (*) we obtain

\[
(3.8) \quad \left[ \frac{-1}{Bn_4} \right] = \left[ \frac{-1}{An_4} \right] \left[ \frac{-1}{Bn_4} \right] + \left[ \frac{-1}{n_2} \right] \left[ \frac{-1}{ABn_1n_4} \right].
\]

Now we use the identities (3.6), (3.7) and (3.8) to prove the lemma. By Lemma 1 it suffices to show that either \(8 \mid d_1 - d_2, 4 \mid d_1 - d_3 \) or \(8 \mid d_1 - d_2 + 2cn, 4 \mid d_1 + an\). Note that \(d_1 - d_2 = Cn_4(An_3 - Bn_2)\) and \(d_1 - d_2 + 2cn \equiv Cn_4(An_3 - Bn_2 + 2n_1n_2n_3) \pmod{8}\). Moreover, \(d_1 - d_3 \equiv An_3(n_4 - Bn_2) \pmod{4}\) and \(d_1 + an \equiv n_3n_4(A + n_1n_2) \pmod{4}\). So we reduce to showing that either \(8 \mid An_3 - Bn_2, 4 \mid n_4 - Bn_2 \) or \(8 \mid An_3 - Bn_2 + 2n_1n_2n_3, 4 \mid A + n_1n_2\).

Since \(A, B, n_2\) and \(n_3\) are odd, we know that either \(4 \mid An_3 - Bn_2\) or \(4 \mid An_3 - Bn_2 + 2n_1n_2n_3\). First, we consider the case \(4 \mid An_3 - Bn_2\). Then

\[
(3.9) \quad \left[ \frac{-1}{An_3} \right] = \left[ \frac{-1}{Bn_2} \right].
\]

Substituting this into (3.6) we get

\[
\left[ \frac{2}{n_2n_3} \right] = \left[ \frac{-1}{An_3n_4} \right] = \left[ \frac{-1}{Bn_2n_4} \right].
\]

Moreover, substituting (3.9) into (3.7) and (3.8) we have

\[
0 = \left[ \frac{-1}{An_3n_4} \right] \left[ \frac{-1}{Bn_1n_3} \right] = \left[ \frac{-1}{Bn_2n_4} \right] \left[ \frac{-1}{Bn_1n_3} \right],
\]
\[
0 = \left[ \frac{-1}{ABn_1n_4} \right] \left[ \frac{-1}{Bn_2n_4} \right] = \left[ \frac{-1}{n} \right] \left[ \frac{-1}{Bn_2n_4} \right].
\]

Adding these we deduce that \(\left[ \frac{2}{n_2n_3} \right] = \left[ \frac{-1}{Bn_2n_4} \right] = 0\). From the third identities of (3.3) and (3.4) we see that \(\left[ \frac{2}{An_3} \right] = \left[ \frac{2}{Bn_2} \right]\). Thus \(8 \mid An_3 - Bn_2\). Since \(\left[ \frac{-1}{Bn_2n_4} \right] = 0\), we have

\[
\left[ \frac{-1}{n} \right] = \left[ \frac{-1}{Bn_2} \right].
\]

So \(4 \mid n_4 - Bn_2\).

Finally, we consider the case \(4 \mid An_3 - Bn_2 + 2n_1n_2n_3\). Then we have

\[
(3.10) \quad \left[ \frac{-1}{ABn_2n_3} \right] = 1.
\]

Substituting this into (3.6) we obtain \(\left[ \frac{2}{n_2n_3} \right] = 0\). From (3.7) we derive that

\[
\left[ \frac{-1}{ABn_1n_4} \right] \left[ \frac{-1}{Bn_2n_4} \right] = 0.
\]

Via (3.8) we obtain

\[
\left[ \frac{-1}{Bn_1n_3} \right] = \left[ \frac{-1}{ABn_1n_4} \right] \left[ \frac{-1}{Bn_1n_3} \right].
\]

Adding these two equations we get \(\left[ \frac{-1}{Bn_1n_3} \right] = 0\) by noting that \(\left[ \frac{-1}{ABn_1n_4} \right] = 1 + \left[ \frac{-1}{n} \right]\). To prove \(8 \mid An_3 - Bn_2 + 2n_1n_2n_3\), it suffices to show that \(\left[ \frac{2}{An_3 + 2n_1n_2n_3} \right] = \left[ \frac{2}{Bn_2} \right]\). By the supplementary law of the quadratic reciprocity law, we have

\[
\left[ \frac{2}{An_3 + 2n_1n_2n_3} \right] = \left[ \frac{2}{An_3} \right] + \left[ \frac{-1}{An_1n_2} \right] + 1 = \left[ \frac{2}{An_3} \right] + \left[ \frac{-1}{Bn_1n_3} \right] = \left[ \frac{2}{An_3} \right].
\]
Noting that $\begin{bmatrix} \frac{2}{2n_1} \end{bmatrix} = 0$ and $\begin{bmatrix} \frac{2}{2} \end{bmatrix} = \begin{bmatrix} \frac{2}{2} \end{bmatrix} = 0$, we have $8 \mid An_3 - Bn_2 + 2n_1n_2n_3$. Via $\begin{bmatrix} -1 \end{bmatrix} = 0$ and (3.10), we have $\begin{bmatrix} -1 \end{bmatrix} = 1$, namely $4 \mid A + n_1n_2$.

This completes the proof of the lemma. □

3.2. Matrix Representation of 2-Selmer Group.

From Lemma 4, 5, 6 and 7, there is a matrix representation of the pure 2-Selmer group $\text{Sel}_2'(\mathcal{E}(n)) = \text{Sel}_2(\mathcal{E}(n))/\mathcal{E}(n)(\mathbb{Q})[2]$. For our purpose, we only give this matrix representation under the condition that $a, b, c$ are squares and $n$ is a positive square-free odd integer such that all prime factors $p$ of $n$ satisfy

$$\begin{equation}
(3.11)
\begin{bmatrix} p \\ q \end{bmatrix} = 1
\end{equation}
$$

with $q$ any prime divisor of $abc$.

To give the matrix representation of $\text{Sel}_2'(\mathcal{E}(n))$, we first introduce some notation. Denote by $n = p_1 \cdots p_k$. Assume that $a, b$ and $c$ have the prime decompositions

$$\begin{equation}
(3.12)
q_1^{k_1} \cdots q_{k_1}^{k_1}, \quad q_{k_1+1}^{t_{k_1+1}} \cdots q_{k_2}^{t_{k_2}} \quad \text{and} \quad q_{k_2+1}^{t_{k_2+1}} \cdots q_{k_3}^{t_{k_3}}
\end{equation}
$$

respectively. Here $k_1 \leq k_2 \leq k_3$ are non-negative integers, and all the $t_i$ are positive even integers. Let $\mathcal{E}(n)$ have all prime factors $(d_1, d_2, d_3)$ with $d_1$ positive integers satisfying (3.2). Since $d_1 \mid nac$ and $d_2 \mid nbc$, we put $d_1 = p_1^{d_1} \cdots p_k^{d_1} q_1^{k_1} q_{k_1+1}^{t_{k_1+1}} \cdots q_{k_2}^{t_{k_2}}$ and $d_2 = p_1^{d_2} \cdots p_k^{d_2} q_{k_2+1}^{t_{k_2+1}} \cdots q_{k_3}^{t_{k_3}}$. Here $x, y, z$ and $w$ are row vectors over $\mathbb{F}_2$ given by $(x_1, \cdots, x_k)$, $(y_1, \cdots, y_k)$, $(z_1, \cdots, z_{k_2+1}, \cdots, z_{k_3})$ and $(w_{k_1+1}, \cdots, w_{k_3})$ respectively.

First, we define the matrix $M_1$ which gives rise to the matrix representation of $\text{Sel}_2'(\mathcal{E}(n))$. To this purpose, we first introduce some $\mathbb{F}_2$ matrices. Let $F = (f_{ij})_{k_3 \times k_3}$ be the matrix defined by $f_{ii} = 0$ and $f_{ij} = q_i / q_j$ if $i \neq j$. We write $F$ in the following block matrix form

$$F = \begin{pmatrix}
F_1 & F_2 & F_3 \\
F_4 & F_5 & F_6 \\
F_7 & F_8 & F_9
\end{pmatrix}.$$

Here $F_1$ and $F_5$ have sizes $k_1 \times k_1$ and $(k_2 - k_1) \times (k_2 - k_1)$ respectively. Let $\Delta$ and $\Delta'$ be the diagonal matrices given by

$$\Delta = \text{diag}(1, \cdots, 1),$$

$$\Delta' = \text{diag}\left(\begin{bmatrix} -1 \\ q_{k_1+1} \end{bmatrix}, \cdots, \begin{bmatrix} -1 \\ q_{k_2} \end{bmatrix}\right),$$

where the size of $\Delta$ is $(k_3 - k_2) \times (k_3 - k_2)$.

Now we define $M_1$ to be the $(2k_3 - k_1) \times (2k_3 - k_2)$ matrix

$$M_1 = \begin{pmatrix}
F_1 & F_2 & F_3 \\
F_4 & F_6 & F_8 \\
F_7 & F_8 & \Delta' \\
\Delta & \Delta
\end{pmatrix}.$$

By Lemma 4, 6 and 7, we know that the map $(d_1, d_2, d_3) \mapsto (z, w)$ induces an isomorphism

$$\text{Sel}_2'(\mathcal{E}(n)) \rightarrow \left\{ (z, w) : M_1 \begin{pmatrix} z^T \\ w^T \end{pmatrix} = 0, \ z \in \mathbb{F}_2^{k_1+k_3-k_2}, w \in \mathbb{F}_2^{k_3-k_1} \right\}.$$

In fact, this can be verified by block matrices. Taking the first block row of $M_1 \begin{pmatrix} z^T \\ w^T \end{pmatrix} = 0$, we get $\begin{pmatrix} F_2 & F_3 \end{pmatrix} w^T = 0$. This is $\sum_{i=1}^{k_3} w_j q_j / q_i = 1$ for all $i \leq k_1$. From this we obtain...
that \((d_2/\eta) = 1\) for all \(i \leq k_1\), which is compatible with the case of \(q | a\) in Lemma 6. The remaining block rows can be checked similarly. We have to remark on the last block row of 
\[ M_1 \begin{pmatrix} z^T \\ w^T \end{pmatrix} = 0. \]
From this we obtain that \(\Delta(z'^T + w'^T) = 0\) with \(z' = (z_{k_2+1}, \ldots, z_{k_3})\) and \(w' = (w_{k_2+1}, \ldots, w_{k_3})\). Then \(z_i = w_i\) for \(i > k_2\), which is equivalent to \((c, d_3) = 1\).

Now we use \(M_1\) to give the matrix representation of \(\text{Sel}'_2(E^{(n)})\). We first introduce some notation. Let \(A = A_n = (a_{ij})_{k \times k}\) be the \(F_2\) matrix given by \(a_{ii} = \sum_{l \neq i} a_{il} + a_{ij} = \left[ \frac{p_i}{p_j} \right]\) if \(i \neq j\). Denote by \(G = (g_{ij})_{k \times k_3}\) the \(F_2\) matrix defined by \(g_{ij} = \left[ \frac{q_i}{p_j} \right]\). We write \(G\) in the block matrix form
\[ G = \begin{pmatrix} G_1 & G_2 & G_3 \\ \end{pmatrix} \]
with the sizes of \(G_1\) and \(G_2\) being \(k \times k_1\) and \(k \times (k_2-k_1)\) respectively. Let \(M_n\) be the Momsky matrix (see Appendix of Heath-Brown [15])
\[ \left( \begin{array}{cc} A + D_{-2} & D_2 \\ D_2 & A + D_2 \end{array} \right), \]
where \(D_u = \text{diag}\left( \left[ \frac{p_i}{p_1} \right], \ldots, \left[ \frac{p_i}{p_k} \right] \right)\). We define \(M_n\) to be the \(F_2\) matrix
\[ \left( \begin{array}{cc} M_n & G \\ \end{array} \right) \]
with
\[ G = \begin{pmatrix} G_1 & G_3 \\ G_2 & G_3 \end{pmatrix}. \]

**Proposition 4.** Let \(a, b, c\) be odd squares with factorization \([3.12]\) and \(n\) a positive square-free integer satisfying \([3.17]\). Then the following is an isomorphism
\[ \text{Sel}'_2(E^{(n)}) \rightarrow \ker M_n, \quad (d_1, d_2, d_3) \mapsto (x, y, z, w)^T, \]
where \((d_1, d_2, d_3)\) is the representative of an element of \(\text{Sel}'_2(E^{(n)})\) such that \([3.3]\) holds. Here \(x = (v_{p_1}(d_1), \ldots, v_{p_k}(d_1))\), \(y = (v_{p_1}(d_2), \ldots, v_{p_k}(d_2))\), \(w = (v_{q_{k+1}}(d_2), \ldots, v_{q_{k_3}}(d_2))\) and \(z = (v_{q_1}(d_1), \ldots, v_{q_{k_1}}(d_1), v_{q_{k+1}}(d_1), \ldots, v_{q_{k_3}}(d_1))\).

For any \((d_1, d_2, d_3)\) above, we put \(d_1' = d_1 d_1'\) with \(d_1' = (d_1, n)\) and \(d_2'' = (d_1, abc)\). Similarly, we set \(d_2 = d_2 d_2''\) and \(d_3 = d_2 d_3''\). Since \(n\) satisfies \([3.11]\), those \(d_1', d_2', d_3'\) and \(n\) occurred in the local solvability conditions for \(p | abc\) (Lemma 6) vanish. Note that \(d_1''\) and \(d_2''\) correspond to \(z\) and \(w\) respectively. Combing these, we have \(M_1 \begin{pmatrix} z^T \\ w^T \end{pmatrix} = 0\).

From \(a, b\) and \(c\) being squares, those \(a, b\) and \(c\) occurred in the local solvability conditions for \(p | n\) (Lemma 5) also vanish. Note that \(d_1'\) and \(d_2'\) correspond to \(x\) and \(y\) respectively. Observe the identity
\[ x_i \sum_{l \neq i} \left[ \frac{p_l}{p_i} \right] + \sum_{j \neq i} x_j \left[ \frac{p_j}{p_i} \right] = x_i \left[ \frac{n/d_1'}{p_i} \right] + (1 - x_i) \left[ \frac{d_1'}{p_i} \right]. \]
From Lemma 5 we get
\[ x_i \left[ \frac{n/d_1'}{p_i} \right] + (1 - x_i) \left[ \frac{d_1'}{p_i} \right] = \left[ \frac{d_1'}{p_i} \right] + x_i \left[ \frac{-d_1'}{p_i} \right] + y_i \left[ \frac{2}{p_i} \right]. \]
Similar result also holds for \(y\). From these we can derive that
\[ M_n \begin{pmatrix} x^T \\ y^T \end{pmatrix} + G \begin{pmatrix} z^T \\ w^T \end{pmatrix} = 0. \]
So the map is well-defined. Its injectivity is obvious; by Lemma 4, 5, 6 and 7 it is surjective.
4. Torsion Subgroup

In this section, we will prove that $E^{(n)}(\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$, where $E^{(n)}$ is defined by (1.1) with $(a, b, c)$ any positive primitive integer solution to $a^2 + b^2 = 2c^2$. Let $E$ be an elliptic curve with full 2-torsion points. According to Mazur’s classification theorem on torsion subgroup of elliptic curves over $\mathbb{Q}$ (see [12]), $E^{(n)}(\mathbb{Q})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}$ for some $m = 1, 2, 3, 4$. Ono [17] has the following characterization of $E^{(n)}(\mathbb{Q})$.

Lemma 8 (Ono). Let $E : y^2 = x(x-a)(x+b)$ be an elliptic curve over $\mathbb{Q}$ with $a, b$ integers. Then $E[2](\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$.

1. $E^{(n)}(\mathbb{Q})$ contains a point of order 4 if and only if one of the three pairs $[-a, b], [a, a+b]$ and $[-b, -a-b]$ consists of squares of integers.
2. $E^{(n)}(\mathbb{Q})$ has a point of order 8 if and only if there exist a positive integer $d$ and pairwise coprime integers $u, v$ and $w$ such that $u^2 + v^2 = w^2$ and $[d^2u^4, d^2v^4]$ is one of the three pairs in (1).
3. $E^{(n)}(\mathbb{Q})$ has a point of order 3 if and only if there exist a positive integer $d$ and pairwise coprime integers $u$ and $v$ such that $a = -(u^4 + 2u^3v)d^2$, $b = (v^4 + 2v^3u)d^2$ and $\frac{a}{v} \notin \{ -2, -\frac{1}{2}, -1, 1, 0 \}$.

Proposition 5. For any square-free integer $n$, $E^{(n)}(\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$.

Proof. From the definition of $E^{(n)}$, we see that $E^{(n)}(\mathbb{Q})$ contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. The proposition will be proved by showing that $E^{(n)}(\mathbb{Q})$ contains no point of order 4 and 3. Here we have used Mazur’s classification theorem on torsion subgroup of elliptic curves over $\mathbb{Q}$.

Now we show that $E^{(n)}(\mathbb{Q})$ has no point of order 4. Note that none of the three pairs $[-a^2n, b^2n], [a^2n, a^2n + b^2n = 2c^2n]$ and $[-b^2n, -a^2n - b^2n]$ consists of squares of integers. So (1) of Lemma 8 implies that $E^{(n)}(\mathbb{Q})$ contains no point of order 4.

So we remain to prove that $E^{(n)}(\mathbb{Q})$ contains no point of order 3. By (3) of Lemma 8, it suffices to show that the equation

$$\begin{align*}
   d^2u^3(u + 2v) &= -a^2n, \\
   d^2v^3(v + 2u) &= b^2n
\end{align*}$$

with $d$ a positive integer and $u, v$ pairwise coprime integers is not solvable. We will divide the proof of this into several steps.

First, as $a$ and $b$ are coprime integers, we get that the greatest common divisor $(-a^2n, b^2n)$ of $-a^2n$ and $b^2n$ is $|n|$, which is square-free. From equation (*), we get that $d | (-a^2n, b^2n) = |n|$. So $d = 1$ by noting that $d$ is a positive integer.

Second, we claim that $|n| = (u^4 + 2u^3v, v^4 + 2v^3u) = (3, u - v)$. Since $(u, v) = 1$, we have $(u, v + 2u) = 1$. Thus $(u^3, v^3(v + 2u)) = 1$. Similarly, $(u + 2v, v^3) = 1$. Therefore, $(u^4 + 2u^3v, v^4 + 2v^3u)$ equals to

$$(u + 2v, v^3(v + 2u)) = (u + 2v, v + 2u) = (u + 2v, u - v) = (u - v, 3u) = (3, u - v).$$

So $n$ divides $u + 2v$ and $v + 2u$. Therefore, the equation (*) is reduced to

$$\begin{align*}
   v^3 \cdot \frac{u+2u}{n} &= b^2, \\
   -u^3 \cdot \frac{u+2v}{n} &= a^2.
\end{align*}$$

Next, we assume that $v > 0$ to solve the equations (*1) and (*2). Since $v$ is coprime to $v + 2u$, from (*1) we derive that there are integers $b_1$ and $b_2$ which divide $b$ and satisfy

$$v^3 = b_1^2, \quad v + 2u = nb_2^2.$$
The first equation implies that $b_1 = b_3^2$ for some $b_3 | b$. Since $b$ is odd, there are odd integers $b_2$ and $b_3$ such that

$$v = b_3^2, \quad v + 2u = nb_2^2.$$  

Now we claim that $u < 0$. Otherwise $u > 0$, from $v + 2u = nb_2^2$ we get $n > 0$ and

$$na^2 = -u^2 \cdot (u + 2v) < 0$$

by (*2). Then we have $u^2 < 0$, which is impossible. Hence, we get $u < 0$. Like (*1), from (*2) we know that there are odd integers $a_2$ and $a_3$ such that

$$u = -a_3^2, \quad u + 2v = na_2^2.$$  

Combing this with equation (4.1), we derive that there are odd integers $a_2, a_3, b_2$ and $b_3$ such that

$$b_3^2 - 2a_3^2 = nb_2^2, \quad 2b_3^2 - a_3^2 = na_2^2.$$  

Viewing these equations as congruences modulo 8, we see that $n \equiv 1 \pmod{8}$ and $n \equiv -1 \pmod{8}$ respectively. These can’t be true at the same time. So the equations (*1) and (*2) are not solvable if $v > 0$.

Finally, the equations (*1) and (*2) are not solvable for $v < 0$. This can be proved similarly as the case $v > 0$.

Therefore, $E_{4a(n)}(Q)$ contains no point of order 3. This finishes the proof of the proposition. 

\[\square\]

5. Non-trivial Shafarevich-Tate Group

In this section, we always assume that $(a, b, c)$ is a positive primitive integer solution to $a^2 + b^2 = 2c^2$ and $n = p_1 \cdots p_k \equiv \pm 1 \pmod{8}$ is a positive square-free integer satisfying

$$\left( \frac{n}{q} \right) = 1$$

for any $1 \leq i \leq k$ and prime divisor $q$ of $abc$. Note that $E = E_{a, b} = E_{a^2, b^2}$ and $E_{(n)}(Q)[2]$ consists of

$$(2, 2n, n), (-2n, 2, -n), (-n, n, -1)$$

and $(1, 1, 1)$.

We assume that Sel$_2(E)$ has dimension two.

5.1. Proof of Theorem

In this subsection, we always assume that all prime divisors $p_i$ of $n$ are congruent to $\pm 1$ modulo 8. Then the Monsky matrix $M_n$ is of the form

$$\text{diag}(A + D_{-1}, A).$$

Lemma 9. Assume that all prime divisors $p_i$ of $n$ are congruent to $\pm 1$ modulo 8. Let $x = (x_1, \cdots, x_k)^T$ and $x_0 = (1, \cdots, 1)^T$ be two column vectors in $\mathbb{F}_2^k$. Denote by $d = \prod_{i=1}^k p_i^{x_i}$.

1. $Ax = 0$ if and only if $x^T(A + D_{-1}) = 0$.
2. If $(A + D_{-1})x = 0$, then $x^T A = 0$ if $d \equiv 1 \pmod{8}$ and $(x_0 - x)^T A = 0$ if $d \equiv -1 \pmod{8}$.
3. The dimension of the pure $2$-Selmer group Sel$_2(E_{(n)})$ is two if and only if $h_4(n) = 1$. If this is satisfied, then Sel$_2(E_{(n)})$ is generated by

$$\Lambda = (2, 2, 1), \quad \Lambda' = (d, 1, d).$$

Proof. For notational simplicity, we assume that $p_1 \equiv \cdots \equiv p_l \equiv 1 \pmod{8}$ and $p_{l+1} \equiv \cdots \equiv p_k \equiv -1 \pmod{8}$. From $n \equiv 1 \pmod{8}$ we derive that $k - l$ is even. In addition, we can divide the matrices $A, D_{-1}, x$ and $x_0$ into block matrices

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad D_{-1} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} z \\ w \end{pmatrix}, \quad x_0 = \begin{pmatrix} z_0 \\ w_0 \end{pmatrix}.$$
Here the sizes of $A_4$ and $I = \text{diag}(1, \cdots, 1)$ are $(k - l) \times (k - l)$, and the sizes of $z$ and $z_0$ are $l \times 1$. We hope that the matrices $A_1, A_2, A_3$ and $A_4$ are not confused with $A_n$ when $n = 1, 2, 3$ and 4. From the definition of $A$ and $D_{-1}$, we obtain that
\begin{equation}
A_1^T = A_1, \quad A_2^T = A_3 \quad \text{and} \quad A_4^T + A_4 + I = E.
\end{equation}
Here $E$ is a $(k - l) \times (k - l)$ matrix with all components being 1, and we have used the quadratic reciprocity law. Since $k - l$ is even, we get $Ew_0 = 0$.

1. If $Ax = 0$, then by the block forms of $A$ and $x$ we get that $A_1z + A_2w = 0$ and $A_3z + A_4w = 0$. While
\[x^T(A + D_{-1}) = (z^T w^T) \begin{pmatrix}
A_1 & A_2 \\
A_3 & A_4 + I
\end{pmatrix} = (z^T A_1 + w^T A_3 \quad z^T A_2 + w^T (A_4 + I)).
\]
Note that $(z^T A_1 + w^T A_3)^T = A_1z + A_2w = 0$ and $[z^T A_2 + w^T (A_4 + I)]^T = A_3z + (A_4^T + I)w$. We claim that $w$ has even many non-zero components. From this claim we deduce that $Ew = 0$, namely $(A_4^T + I)w = A_4w$ by (5.1). Therefore,
\[\left[z^T A_2 + w^T (A_4 + I)\right]^T = A_3z + A_4w = 0.
\]
Thus $x^T(A + D_{-1}) = 0$. Moreover, the above process is invertible.

So the proof of (1) is reduced to proving the claim. From the definition of $A$, we know that $Ax_0 = 0$, namely
\begin{equation}
A_1z_0 + A_2w_0 = 0 \quad \text{and} \quad A_3z_0 + A_4w_0 = 0.
\end{equation}
Like the expansion of $x^T(A + D_{-1})$, we get $x_0^T A = (0 \quad w_0^T)$. Therefore,
\[0 = x_0^T Ax = (0 \quad w_0^T) x = w_0^T w,
\]
which is equivalent to the claim.

2. If $d \equiv 1 \pmod{8}$, then $w$ also has even many non-zero components and the proof is the same as that of (1). We assume that $d \equiv -1 \pmod{8}$, namely $w$ has odd many non-zero components. Then
\begin{equation}
Ew = w_0 = Iw_0.
\end{equation}
Denote by $\bar{z} = z_0 - z$ and $\bar{w} = w_0 - w$. So $(x_0 - x)^T = (\bar{z}^T \quad \bar{w}^T)$. From $(A + D_{-1})x = 0$ we obtain that $A_1\bar{z} + A_2\bar{w} = 0$ and $A_3\bar{z} + (A_4 + I)\bar{w} = 0$. Observe that
\[(x_0 - x)^T A = (\bar{z}^T A_1 + \bar{w}^T A_3 \quad \bar{z}^T A_2 + \bar{w}^T A_4).
\]
But we have $(\bar{z}^T A_1 + \bar{w}^T A_3)^T = A_1\bar{z} + A_2\bar{w} = A_1z_0 + A_2w_0 - (A_1z + A_2w) = 0$ by (5.2). Moreover,
\[(\bar{z}^T A_2 + \bar{w}^T A_4)^T = A_3\bar{z} + A_4^T \bar{w} = A_3z_0 + A_4^T w_0 + A_3\bar{z} + A_4^T w
\]
\[= A_3z_0 + (A_4 + I)w_0 + A_3\bar{z} + (A_4 + I)w + Ew
\]
\[= Iw_0 + Ew = 0.
\]
Here we have used (5.2) and (5.3). Consequently, $(x_0 - x)^T A = 0$.

3. By Proposition 4 to find all the elements of $\text{Sel}_2'(E'(n))$ is equivalent to compute the kernel of $M_n$. This is equivalent to find all $(X, Y, Z, W)$ such that
\[\mathcal{M}_n \begin{pmatrix} X \\ Y \end{pmatrix} + \mathcal{G} \begin{pmatrix} Z \\ W \end{pmatrix} = 0 \quad \text{and} \quad M_1 \begin{pmatrix} Z \\ W \end{pmatrix} = 0.
\]
Here $X, Y \in \mathbb{F}_2^k$, $Z \in \mathbb{F}_2^{k_1+k_3-k_2}$, and $W \in \mathbb{F}_2^{k_3-k_1}$. Since $\text{Sel}_2(E)$ has dimension two, we know that $\ker M_1 = \{0\}$ by Proposition 4. This implies that $Z$ and $W$ are zero vectors. Thus we reduce to finding $X$ and $Y$ in $\mathbb{F}_2^k$ such that $\mathcal{M}_n \begin{pmatrix} X \\ Y \end{pmatrix} = 0$. Then $\dim_{\mathbb{F}_2} \text{Sel}_2'(E'(n)) = 2k - \text{rank} M_n$. 
From (1) we derive that \( \text{rank}A = \text{rank}(A + D_{-1}) \). Hence, \( \text{rank}\mathcal{M}_n = 2\text{rank}A \). But the Rédei matrix \( R \) of \( \mathbb{Q} = \sqrt{-n} \) takes the form \( \begin{pmatrix} A & 0 \\ \end{pmatrix} \). Therefore, \( \text{rank}R = \text{rank}A = k - h_4(n) \). Thus \( h_4(n) = 1 \) if and only if \( \text{rank}\mathcal{M}_n = 2k - 2 \). Hence, the dimension of \( \text{Sel}_p'(E^n) \) is two if and only if \( h_4(n) = 1 \).

Now we assume that \( h_4(n) = 1 \) to find representatives of \( \text{Sel}_p'(E^n) \). From above argument, it suffices to find \( X, Y \in \mathbb{F}_2^n \) such that \( \mathcal{M}_n \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \), namely \( \langle A + D_{-1}, X \rangle = 0 \) and \( AY = 0 \). As the rank of \( A \) is \( k - 1 \) and \( Ax_0 = 0 \), we get \( Y = 0 \) or \( x_0 \). Similarly, \( X = 0 \) or \( x \). Hence, \( \text{Sel}_p'(E^n) \) is generated by \((1, n, n)(2, 2n, n)\), the proof of (3) is complete.

This completes the proof of the lemma. \( \square \)

Now we can prove Theorem 1.

**Proof of Theorem 1.** From the exact sequence (1.2), we see that the necessary condition for (1) is \( \dim \text{ker}_\mathbb{F}_2 \text{Sel}_p'(E^n) = 2 \). This is equivalent to \( h_4(n) = 1 \) by Lemma 9. Now we assume this and use the notation of Lemma 9. Since \( h_4(n) = 1 \), from (3) of Lemma 9 we know that \( \text{Sel}_p'(E^n) \) is generated by \( \Lambda = (2, 2, 1) \) and \( \Lambda' = (d, 1, d) \).

First, we compute the Cassels pairing \( \langle \Lambda, \Lambda' \rangle \). The genus one curve \( D_A \) is

\[
\begin{aligned}
H_1 : & \quad -b^2nt^2 + 2u_2^2 - u_3^2 = 0, \\
H_2 : & \quad -a^2nt^2 + u_3^2 - 2u_2^2 = 0, \\
H_3 : & \quad c^2nt^2 + u_1^2 - u_2^2 = 0.
\end{aligned}
\]

From the definition of the Cassels pairing, we have to choose points on \( H_1(\mathbb{Q}) \). For \( H_3 \) we choose \( Q_3 = (0, 1, 1) \). So the corresponding tangent linear form \( L_3 \) of \( H_3 \) at \( Q_3 \) is

\[
L_3 : u_1 - u_2.
\]

For \( H_1 \) we will use Gauss genus theory to select a point. As \( R \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \), by Gauss genus theory (Lemma 2) \( 2 \) is a norm element, namely there is a positive primitive integer solution \( (\alpha, \beta, \gamma) \) to

\[
\alpha^2 + n\beta^2 = 2\gamma^2.
\]

Let \( Q_1 = (\beta, \gamma b, \alpha b) \). Then \( Q_1 \) lies in \( H_1(\mathbb{Q}) \) and the corresponding tangent linear form is

\[
L_1 : \beta nt^2 - 2\gamma u_2 + \alpha u_3.
\]

So by Lemma 1 \( \langle \Lambda, \Lambda' \rangle \) equals to

\[
\prod_{p||2abc} \left( L_1L_3(P_p), \frac{d}{p} \right)_p.
\]

Here \( P_p \) is any point on \( D_A(\mathbb{Q}_p) \) such that \( L_3(P_p) \) is non-vanishing. For any prime divisor \( p \) of \( abc \), we have \( \left( \frac{p}{p} \right) = 1 \); so we have \( \left( \frac{d}{p} \right) = 1 \). The local Cassels pairing is trivial at all \( p | abc \). In addition, it is also trivial at \( p = \infty \) by \( d > 0 \). So we only need to compute it at \( p = 2 \) and \( p | n \).

For \( p | n \), we choose the local point \( (t, u_1, u_2, u_3) \) such that

\[
t = 0, u_1 = 1, u_2 = -1, u_3 = 2.
\]

As \( (2\gamma + \alpha u_3)(2\gamma - \alpha u_3) = 4\gamma^2 - 2\alpha^2 = 2n\beta^2 \), we choose \( u_3 \) such that \( p | 2\gamma - \alpha u_3 \). Then \( p | 2\gamma + \alpha u_3 \) by the primitivity of \( (\alpha, \beta, \gamma) \). So

\[
\left( L_1L_3(P_p), \frac{d}{p} \right)_p = \left( 2(2\gamma + \alpha u_3), \frac{d}{p} \right)_p = \left( \frac{\gamma}{p} \right)^\delta.
\]

Here \( \delta \) is 1 if \( p | d \) and 0 otherwise.
For $p = 2$, we note that the local Cassels pairing is trivial if $d \equiv 1 \pmod 8$. So we only need to consider the case $d \equiv -1 \pmod 8$. Let $(t, u_1, u_2, u_3)$ be the local point satisfying
\[ t = 1, u_1 = 0, u_2 = e^2 n, u_3 = a^2 n. \]
We observe that
\[ (\beta n + \alpha u_3)(\beta n - \alpha u_3) = \beta^2 b^2 n^2 - \alpha^2 a^2 n = b^2 n(2\gamma^2 - \alpha^2) - \alpha^2 a^2 n \]
\[ = 2\gamma^2 b^2 n - 2\alpha^2 c^2 n = 2b^2 n\left(\gamma^2 - \frac{\alpha^2 c^2}{b^2}\right) \equiv 0 \pmod{16}. \]
So we may choose $u_3$ such that $8 | \beta n + \alpha u_3$. Then
\[ \left(L_1L_3(P_p), d\right)_2 = \left(-u_2(\beta n - 2\gamma u_2 + \alpha u_3), -1\right)_2 = \left(2\gamma u_2^2, -1\right)_2 = \left(\frac{-1}{\gamma}\right). \]
So we get
\[ \langle \Lambda, \Lambda' \rangle = \left(\frac{2}{d}\right) \text{ or } \left(\frac{\gamma}{d}\right) \left(\frac{-1}{\gamma}\right) \]
according to $d \equiv 1 \pmod 8$ or not.

Next, we show that (1) is equivalent to $h_4(n) = 1$ and the non-degeneracy of the Cassels pairing on $\text{Sel}_2'(E(n))$. From the following short exact sequence
\[ 0 \longrightarrow E(n)[2] \longrightarrow E(n)[4] \xrightarrow{x^2} E(n)[2] \longrightarrow 0, \]
we obtain the derived long exact sequence
\[ 0 \longrightarrow E(n)(\mathbb{Q})[2]/E(n)(\mathbb{Q})[4] \longrightarrow \text{Sel}_2(E(n)) \longrightarrow \text{Sel}_4(E(n)) \longrightarrow \text{ImSel}_4(E(n)) \longrightarrow 0. \]
By Proposition 5, we see that (1) is equivalent to $\dim_{\mathbb{R}} \text{Sel}_2'(E(n)) = 2$ and $\#\text{Sel}_2(E(n)) = \#\text{Sel}_4(E(n))$, which are equivalent to $\dim_{\mathbb{R}} \text{Sel}_2'(E(n)) = 2$ and $\#\text{ImSel}_4(E(n)) = 4$ by the long exact sequence. Noting that the kernel of the Cassels pairing on $\text{Sel}_2'(E(n))$ is $\text{ImSel}_4(E(n))/E(n)(\mathbb{Q})[2]$, we infer that (1) is equivalent to $h_4(n) = 1$ and the non-degeneracy of the Cassels pairing on $\text{Sel}_2'(E(n))$ by Lemma 9.

Finally, we connect the Cassels pairing with $h_8(n)$. We first assume that $d \equiv 1 \pmod 8$. As $2 | 2n$ is a norm element satisfying (5.4), $h_8(n) = 1$ if and only if $W \in \text{Im}R$ with $W = \left(\left[\frac{n}{p_1}\right], \cdots, \left[\frac{n}{p_k}\right]\right)^T$ by Lemma 8. Via (2) of Lemma 9, we see that $x^TA = 0$. As $A$ has rank $k - 1$ and $R = \left(\begin{array}{ccc} A & 0 \end{array}\right)$, we obtain that $W \in \text{Im}R$ if and only if $x^TW = 0$, namely $\left(\frac{\gamma}{d}\right) = 1$. Hence, the Cassels pairing is non-degenerate if and only if $h_8(n) = 0$ provided that $d \equiv 1 \pmod 8$.

Now we assume that $d \equiv -1 \pmod 8$. Then $(x_0 - x)^TA = 0$ by (2) of Lemma 9. Like the above case, $h_8(n) = 1$ if and only if $(x_0 - x)^TW = 0$, namely $\left(\frac{\gamma}{n/\gamma}\right) = 1$. Viewing (5.4) as a congruence modulo $\gamma$, we have $\left(\frac{-n}{\gamma}\right) = 1$. Since $n \equiv 1 \pmod 8$, we have $\left(\frac{\gamma}{n}\right) = \left(\frac{\gamma}{a}\right)$ from the quadratic reciprocity law. So the Cassels pairing $\langle \Lambda, \Lambda' \rangle$ is $\left(\frac{\gamma}{n/\gamma}\right)$ in this case. Hence, the Cassels pairing is non-degenerate if and only if $h_8(n) = 0$.

In summary, we have proved that (1) is equivalent to (2). This completes the proof of the theorem.

\[ \square \]

5.2. Proof of Theorem 2

Lemma 10. Assume that all prime divisors of $n$ are congruent to 1 modulo 4. Then the dimension of the pure 2-Selmer group $\text{Sel}_2'(E(n))$ is two if and only if $h_4(n) = 1$.

Proof. Like (3) of Lemma 9, it suffices to show that $\text{rank} \mathcal{M}_n = 2k - 2$ if and only if $h_4(n) = 1$. Note that the Monsky matrix takes the form
\[ \mathcal{M}_n = \begin{pmatrix} A + D_2 & D_2 \\ D_2 & A + D_2 \end{pmatrix}. \]
Here $D_2 = \text{diag}\left( \frac{2}{p_1}, \cdots, \frac{2}{p_k} \right)$. To relate $\mathcal{M}_n$ to the Rédei matrix $R$ of $n$, we perform some elementary linear transforms on the block matrix $\mathcal{M}_n$. Adding the first block row to the second, we have

$$
\begin{pmatrix}
A + D_2 & D_2 \\
A & A
\end{pmatrix}.
$$

Then adding the second block column to the first, we get

$$
\begin{pmatrix}
A & D_2 \\
A & A
\end{pmatrix}.
$$

Summating all the last $(k - 1)$ columns to the $(k + 1)$-th column, we derive

$$
\begin{pmatrix}
R & D'_2 \\
A & A'
\end{pmatrix}.
$$

Here $D'_2$ and $A'$ denote the matrices obtained from $D_2$ and $A$ respectively by deleting their first columns. Adding all the first $(k - 1)$ rows to the $k$-th row and then moving the $k$-th row as the last row, we yield

$$
\begin{pmatrix}
R_k & * \\
R_1^T
\end{pmatrix}.
$$

Here $R_i$ is the matrix obtained from the Rédei matrix $R$ by deleting its $i$-th row. Since every $p_i$ is congruent to 1 modulo 4 and $n$ is congruent to 1 modulo 8, we see that the column sum of $R$ is zero, namely the sum of any of $R$’s given column is zero. Thus

$$\text{rank} R_i = \text{rank} R = k - h_4(n).$$

From this and (5.5) we get

$$2\text{rank} R \leq \text{rank} \mathcal{M}_n \leq k - 1 + \text{rank} R.$$

Therefore, $\text{rank} R = k - 1$ if and only if $\text{rank} \mathcal{M}_n = 2k - 2$. Then the lemma follows from $\text{rank} R = k - h_4(n)$.

**Lemma 11.** Assume that all prime divisors of $n$ are congruent to 1 modulo 4 and $h_4(n) = 1$.

1. If the rank of $A$ is $k - 2$, then $\text{Sel}_2^l(E^{(n)})$ is generated by $(n, n, 1)$ and $(d, d, 1)$, where $d = \prod_1^k p_i^{x_i}$ with $x = (x_1, \cdots, x_k)^T \neq 0, x_0 = (1, \cdots, 1)^T \in \mathbb{F}_2^k$ satisfying $Ax = 0$.

2. If the rank of $A$ is $k - 1$, then $\text{Sel}_2^l(E^{(n)})$ is generated by $(n, n, 1)$ and $(d, d/d, d)$, where $d = \prod_1^k p_i^{x_i}$ with $x = (x_1, \cdots, x_k)^T$ and $b = \left( \frac{2}{p_1}, \cdots, \frac{2}{p_k} \right)^T$ satisfying $Ax = b$.

**Proof.** Like (3) of Lemma 9, we reduce to finding $X$ and $Y$ in $\mathbb{F}_2^k$ such that $\mathcal{M}_n \left( \begin{array}{c} X \\ Y \end{array} \right) = 0$, namely

$$
\begin{align*}
AX + D_2(X + Y) &= 0, \\
AY + D_2(X + Y) &= 0.
\end{align*}
$$

Adding these two equations, we get $A(X + Y) = 0$. So $X + Y$ lies in $\ker A$. According to $\text{rank} A_n$, we can divide this into two cases.

First, we deal with the case $\text{rank} A = k - 1$. Then $\ker A = \{0, x_0\}$. If $X + Y = 0$, then $X = Y \in \ker A$ by (5.6). These give rise to two elements $X = Y = 0$ or $X = Y = x_0$. So $(1, 1, 1)$ and $(n, n, 1)$ lie in $\text{Sel}_2^l(E^{(n)})$. If $X + Y = x_0$, then (5.6) implies that

$$AX = D_2x_0 = b.$$ 

In fact, $b$ is indeed in the image of $A$. From Gauss genus theory, we know that

$$\text{rank} \left( \begin{array}{c} A \\ b \end{array} \right) = k - h_4(n) = k - 1 = \text{rank} A.$$
Let $x$ be such that $Ax = b$. Then $X = x$ or $x_0 - x$. So these give rise to the remaining two elements $(d, n/d, n)$ and $(n/d, d, n)$ of $\text{Sel}_2(E^n)$.

Finally, we consider the case $\text{rank} A = k - 2$. If $X + Y = 0$, then $X = Y \in \ker A$ by (5.6). Thus there are four elements in $\text{Sel}_2(E^n)$ given by

$$(n, n, 1), \ (d, d, 1), \ (n/d, n/d, 1) \text{ and } (1, 1, 1)$$

with $d$ defined in the lemma. This completes the proof of the lemma.

Now we can prove Theorem 2.

**Proof of Theorem 2.** Like the proof of Theorem 1, a necessary condition for (1) is $h_4(n) = 1$. So we may assume that $h_4(n) = 1$. Then there are two cases according to the rank of $A$.

First, we consider the case $\text{rank} A = k - 2$. We use the notation defined in (1) of Lemma 1. Then we have $R \left( \begin{array}{c} x \\ 0 \end{array} \right) = (A \ b) \left( \begin{array}{c} x \\ 0 \end{array} \right) = Ax = 0$. Here $b = \left( \begin{array}{c} \frac{a}{p_1} \\ \vdots \\ \frac{a}{p_k} \end{array} \right)^T$.

So by Gauss genus theory (Lemma 2) and the epimorphism $\theta$ in (2.4), $d_0 = \prod_i p_i^{r_i}$ corresponds to the non-trivial element of $2A \cap A[2]$. So $d = d_0 = \prod_i p_i^{r_i}$. We claim that $d \equiv 5 \pmod{8}$. Since $h_4(n) = 1$, we know that rank $\left( \begin{array}{c} A \\ b \end{array} \right) = k - 1$ by Gauss genus theory. While rank $A = k - 2$, we see that $b$ is not in the image of $A$. As $A$ is symmetric and ker $A$ is generated by $x_0$ and $x$, a column vector $y \in F_k^2$ is in the image of $A$ if and only if $x_0^Ty = x^Ty = 0$. For the vector $b$, we have $x_0^Tb = \left( \frac{2}{m} \right) = 0$ for $n \equiv 1 \pmod{8}$. Thus $x^Tb \neq 0$. Note that $x^Tb = \left( \frac{2}{m} \right)$. Therefore, $d$ is congruent to 5 modulo 8.

By Lemma 10, the pure 2-Selmer group $\text{Sel}_2(E^n)$ has dimension two. Moreover, via Lemma 11 it is generated by

$$\Lambda = (d, d, 1), \ \Lambda' = (-1, 1, -1).$$

Note that $\Lambda' = (n, n, 1)(-n, n, -1)$. Now we compute the Cassels pairing $\langle \Lambda, \Lambda' \rangle$. For $\Lambda = (d, d, 1)$, the genus one curve $D_\Lambda$ associated to $\Lambda$ is given by

$$\begin{cases}
H_1 : & -b^2nt^2 + du_2^2 - u_3^2 = 0, \\
H_2 : & -a^2nt^2 + u_3^2 - du_2^2 = 0, \\
H_3 : & 2c^2nt^2 + du_2^2 - u_3^2 = 0.
\end{cases}$$

By Cassels pairing, we have to choose global points on $H_1$. For $H_3$, we choose the global point $Q_3 = (0, 1, 1) \in H_3(\mathbb{Q})$. Then the corresponding tangent linear form $L_3$ of $H_3$ at $Q_3$ is

$$L_3 : u_1 - u_2.$$ 

Now we are ready to choose a point on $H_1(\mathbb{Q})$. Since $d$ corresponds to the non-trivial element of $2A \cap A[2]$, Gauss genus theory implies that $d$ is a norm element, namely there is a positive primitive integer solution $(\alpha, \beta, \gamma)$ to

$$(5.7)$$

$$d\alpha^2 + d'\beta^2 = \gamma^2$$

with $dd' = n$. We may assume that $\alpha$ is even. If not, $\alpha$ is odd and $\beta$ is even. We can get a new solution

$$(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) = \left( d'\alpha - 2d'\beta - d\alpha, d\beta - 2d\alpha - d'\beta, (d + d')\gamma \right)$$

to (5.7). Thus $4 \parallel \alpha$ and $2 \parallel \beta$. Dividing the solution $(|\tilde{\alpha}|, |\tilde{\beta}|, |\tilde{\gamma}|)$ by gcd$(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$, we derive a positive primitive integer solution to (5.7) with corresponding $\alpha$ being even. From (5.7), we know that $(\beta, \gamma, b, dob)$ lies in $H_1(\mathbb{Q})$ with the corresponding tangent linear form

$$L_1 : d'\beta t - \gamma u_2 + \alpha u_3.$$ 

By Lemma 1 the Cassels pairing $\langle \Lambda, \Lambda' \rangle$ equals to

$$\prod_{p|2nabc} \left( L_1 L_3(P_p), -1 \right)_p.$$
Here $P_p$ is any point of $D\lambda(\mathbb{Q}_p)$ such that $L_t(P_p)$ is non-vanishing. Since any prime divisor $p$ of $n$ is congruent to 1 modulo 4, we know that the local Cassels pairing is trivial at $p \mid n$. This pairing is also trivial at $p \mid c$ for $-a^2 = b^2 - 2c^2 \equiv b^2 \pmod{p}$. Thus we reduce to computing the local Cassels pairing at $p \mid 2ab\infty$.

For $p = \infty$, we choose the local point $P_\infty = (t, u_1, u_2, u_3) = (0, 1, -1, \sqrt{d})$. Then

$$\left( L_tL_3(P_\infty), -1 \right)_\infty = \left( 2(\gamma + \alpha\sqrt{d}), -1 \right)_\infty = 1.$$

For $p = 2$, we choose the local point $P_2 = (t, u_1, u_2, u_3)$ such that $t = 2$, $u_2 = 1$, $u_1^2 = 1 + 8c^2d'$, $u_3^2 = d - 4b^2n$.

Here we used the fact that $d \equiv 5 \pmod{8}$. As $u_1^2 = 1 + 8c^2d' \equiv 9 \pmod{16}$, we may assume that $u_1 \equiv 3 \pmod{8}$. Since $\alpha$ is even, we have $2 \mid \alpha \pmod{8}$ by (5.7). Let $u_3$ be any square root of $d - 4b^2n$. Then $2 \mid d'\beta b + \frac{\alpha}{2} u_3$. The local Cassels pairing at $p = 2$ is

$$\left( L_tL_3(P_2), -1 \right)_2 = \left( u_1 - 1)(2d'\beta b + \alpha u_3 - \gamma), -1 \right)_2 = (2, -1)_2 \left( 2(d'\beta b + \frac{\alpha}{2} u_3) - \gamma, -1 \right)_2 = (-\gamma, -1)_2 = -\left( -\frac{1}{\gamma} \right).$$

For $p \mid ab$, we choose the local solution

$$t = 0, u_1 = 1, u_2 = -1, u_3^2 = d.$$ 

By (5.7), $(\gamma + \alpha u_3)(\gamma - \alpha u_3) = \gamma^2 - \alpha^2 u_3^2 = d'\beta^2$. If $p \mid \beta$, then we choose $u_3$ such that $p \mid \gamma - \alpha u_3$; so $p \mid \gamma + \alpha u_3$. If $p \mid \beta$, we choose $u_3$ to be any square root of $d$ and have $p \mid \gamma - \alpha u_3$. Thus in any case $\gamma + \alpha u_3$ is a $p$-adic unit. So the local pairing is

$$\left( L_tL_3(P_p), -1 \right)_p = (2(\gamma + \alpha u_3), -1)_p = 1.$$

Therefore, the Cassels pairing $\langle \Lambda, \Lambda' \rangle$ is

$$\langle \Lambda, \Lambda' \rangle = -\left( -\frac{1}{\gamma} \right).$$

Like Theorem 1 (1) is equivalent to $h_4(n) = 1$ and the non-degeneracy of the Cassels pairing.

Now we claim that $\left( -\frac{1}{\gamma} \right) = 1$ if and only if $h_8(n) = 1$ provided that $h_4(n) = 1$. As $d = d_0 \mid 2n$ corresponds to the non-trivial element of $2A \cap A[2]$ and $(d\alpha)^2 + n\beta^2 = d\gamma^2$ by (5.7), we see that $h_8(n) = 1$ if and only if $W \in \text{Im} R$ by Lemma 8. Here $W = \left( \frac{2}{p_1}, \cdots, \frac{2}{p_k} \right)^T$ and $R$ is the Rédéi matrix $\left( \begin{array}{c} a \\ b \end{array} \right)$. Since the rank of $R$ is $k - 1$ and $x_0^T R = 0$, a vector $z$ is in the image of $R$ if and only if $x_0^T z = 0$. We have $x_0^T W = \left( \frac{2}{n} \right)$. Consequently, $h_8(n) = 1$ if and only if $\left( \frac{2}{n} \right) = 1$. Viewing (5.7) as a congruence modulo $\gamma$, we see that $\left( \frac{2}{\gamma} \right) = 1$.

By $n \equiv 1 \pmod{4}$ and the quadratic reciprocity law, we have $\left( \frac{-1}{\gamma} \right) = \left( \frac{2}{n} \right)$. Therefore, $h_8(n) = 1$ if and only if $\left( \frac{-1}{\gamma} \right) = 1$.

So in the case $\text{rank} A = k - 2$, (1) is equivalent to (2) by noting that $d \equiv 5 \pmod{8}$.

Now we consider the case that $\text{rank} A = k - 1$. We use the notation of (2) of Lemma 11. Note that $R \left( \begin{array}{c} x \\ 1 \end{array} \right) = (A \ b) \left( \begin{array}{c} x \\ 1 \end{array} \right) = Ax + b = 0$. By Gauss genus theory,
\[ d_0 = 2 \prod_{i=1}^{k} p_i^{x_i} \] corresponds to the non-trivial element of \( 2A \cap A[2] \). Then \( d = \prod_{i=1}^{k} p_i^{x_i} \). From Lemma 11 we derive that \( \text{Sel}^0(E(\alpha)) \) is generated by

\[ \Lambda = (2d, 2d, 1), \quad \Lambda' = (-1, 1, -1). \]

Here we used the fact that \( \Lambda = (d, n/d, n)(2, 2n, n) \) and \( \Lambda' = (-n, n, -1)(n, n, 1) \).

Now we begin to compute the Cassels pairing \( \langle \Lambda, \Lambda' \rangle \). The genus one curve \( D_\Lambda \) is defined by

\[
\begin{align*}
H_1 & : -b^2n^2 + 2d^2u^2 - u^2 = 0, \\
H_2 & : -a^2n^2 + u^2 - 2d^2u = 0, \\
H_3 & : c^2d^2t^2 + u^2 - u^2 = 0.
\end{align*}
\]

Here \( d' = n/d \). According to the definition of the Cassels pairing, we first choose global points \( Q_1 \) on \( H_1(\mathbb{Q}) \). Let \( Q_3 = (0, 1, 1) \). Then \( Q_3 \) lies in \( H_3(\mathbb{Q}) \) and the corresponding tangent linear form \( L_3 \) is

\[ L_3 : u_1 - u_2. \]

Since \( 2d \) corresponds to the non-trivial element of \( 2A \cap A[2] \), by Gauss genus theory there is a positive primitive integer solution \((\alpha, \beta, \gamma)\) to

\[ da^2 + d' \beta^2 = 2\gamma^2. \]

Then \( Q_1 = (\beta, \gamma b, abb) \) lies in \( H_1(\mathbb{Q}) \). In addition, the tangent linear form \( L_1 \) of \( H_1 \) at \( Q_1 \) is

\[ L_1 : d' \beta b t - 2\gamma u_2 + \alpha u_3. \]

Like the case \( \text{rank}A = k - 2 \), the Cassels pairing \( \langle \Lambda, \Lambda' \rangle \) equals to

\[ \prod_{p|2ab\infty} \left( L_1 L_3(P_p), -1 \right)_p \]

with \( P_p \) any point on \( D_\Lambda(\mathbb{Q}_p) \) such that \( L_3(P_p) \) is non-vanishing. For \( p = \infty \), we choose the local point \((t, u_1, u_2, u_3) = (0, 1, -1, \sqrt{2d}) \). Then

\[ \left( L_1 L_3(P_\infty), -1 \right)_\infty = \left( 2(\alpha \sqrt{2d} + 2\gamma), -1 \right)_\infty = 1. \]

For \( p = 2 \), we choose a point \( P_2 = (t, u_1, u_2, u_3) \) such that

\[ t = 1, u_1 = 2 \left[ \frac{2}{d} \right], u_2 = c^2 d' + u_1^2, u_3 = a^2 n + 2du_1^2 \]

with \( \gamma u_2 \equiv 1 \pmod{4} \). Note that

\[
(d' \beta b + \alpha u_3)(d' \beta b - \alpha u_3) = d'^2 \beta^2 b^2 - \alpha^2(a^2 n + 2du_1^2)
\]

\[ = d'^2 (2\gamma^2 - d^2 \alpha^2) - \alpha^2(a^2 n + 2du_1^2) = 2d'^2 \left[ 2\gamma^2 - \alpha^2 \left( \frac{1}{2d'^2} (2c^2 n + 2du_1^2) \right) \right] \]

\[ = 2d'^2 \left[ \gamma^2 - \alpha^2 \left( \frac{c^2}{b^2 d'^2} + \frac{d}{b^2 d'^2} u_1^2 \right) \right] \equiv 0 \pmod{16}. \]

So we may choose \( u_3 \) such that \( 8 \mid d' \beta b + \alpha u_3 \). We have

\[
\left( L_1 L_3(P_2), -1 \right)_2 = \left( (u_1 - u_2)(d' \beta b + \alpha u_3 - 2\gamma u_2), -1 \right)_2
\]

\[ = (2, -1)_2(-\gamma, -1)_2(u_1 \gamma - 1, -1)_2 \]

\[ = \left( \frac{-1}{\gamma} \right) \left( \frac{2}{d} \right). \]

For \( p \mid a \), we put \( P_p = (t, u_1, u_2, u_3) \) with

\[ t = 1, u_1 = 0, u_2 = c^2 d', u_3 = a \sqrt{n}. \]
Observe that
\[
(d'b - 2\gamma u_2)(d'b + 2\gamma u_2) = d'^2b^2 - 4\gamma^2c^2d'
\equiv 2d'^2(d'b^2 - 2\gamma^2) \equiv -2c^2n\alpha^2 \pmod{p}.
\]
If \(p \mid \alpha\), we choose \(u_2\) such that \(p \mid d'b + 2\gamma u_2\); in addition, \(p \nmid d'b - 2\gamma u_2\), otherwise \(p \mid \beta\) which contradicts that \((\alpha, \beta, \gamma)\) is a positive primitive integer solution to (5.8). If \(p \nmid \alpha\), we have \(p \mid d'b \pm 2\gamma u_2\). So we can always choose \(u_2\) such that \(p \nmid d'b - 2\gamma u_2\). Since \(p \nmid a\), we get
\[
(L_1L_3(P_p), -1)_p = \left(-u_2(d'b - 2\gamma u_2 + \alpha u_3), -1\right)_p = (d'b - 2\gamma u_2, -1)_p = 1.
\]
Similarly, for \(p \mid b\), we have
\[
(L_1L_3(P_p), -1)_p = 1.
\]
In summary, we have
\[
\langle \Lambda, \Lambda' \rangle = \left(\frac{-1}{\gamma}\right)\left(\frac{2}{d}\right).
\]
Like the case rank \(A = k - 2\), (1) is equivalent to \(h_4(n) = 1\) and \(\langle \Lambda, \Lambda' \rangle = -1\), and Gauss genus theory implies that \(\left(\frac{1}{\gamma}\right) = 1\) if and only if \(h_8(n) = 1\) provided that \(h_4(n) = 1\). Therefore, (1) is equivalent to \(h_4(n) = 1\) and \(h_8(n) \equiv \left[\frac{2}{7}\right] \equiv \frac{d-1}{4} \pmod{2}\). This completes the proof of the theorem.

\[\square\]

6. Independence Property of Residue Symbols

In this section, we assume that \(a, b\) and \(c\) are coprime positive integers satisfying \(a^2 + b^2 = 2c^2\). Let \(q_1, \ldots, q_k\) be all the prime divisors of \(abc\).

We first introduce some notation. Given \(k \geq 2\), let \(\alpha = (\alpha_1, \ldots, \alpha_k)\) with all \(\alpha_j \in \{1, 5, 9, 13\}\) and \(\prod_{j=1}^{k} \alpha_j \equiv 1 \pmod{8}\). Assume that \(B = (B_{ij})_{k \times k}\) is a symmetric \(\mathbb{F}_2\) matrix with rank \(k - 2\) and \(Bz_0 = 0\). Here \(z_0 = (1, \cdots, 1)^T \in \mathbb{F}_2^k\). So there is a unique \(z = (z_1, \cdots, z_k)^T \neq 0, z_0\) such that \(Bz = 0\) and \(z_1 = 1\). Our goal in this section is to estimate the number of \(C_k(x, \alpha, B)\). Here \(C_k(x, \alpha, B)\) consists of all \(n = p_1 \cdots p_k \leq x\) satisfying

- \(p_1 < \cdots < p_k\) and \(p_l \equiv \alpha_l \pmod{16}\) for all \(1 \leq l \leq k\),
- \[\left[\frac{p_l}{p_l}\right] = B_{lj}\] for all \(1 \leq l < j \leq k\),
- \(\left[\frac{p_l}{q_j}\right] = 1\) for all \(1 \leq l \leq k\) and \(1 \leq j \leq k'\), and
- \(\left(\frac{d'}{d}\right)_4 = -1\) with \(d = \prod_{i=1}^{k} p_i^{z_i}\) and \(d' = \prod_{i=1}^{k} p_i^{1-z_i}\).

Due to the existence of the quartic residue symbols, we can’t estimate \#\(C_k(x, \alpha, B)\) directly. This problem can be solved by identifying \(C_k(x, \alpha, B)\) with a set counting corresponding integers over \(\mathbb{Z}[i]\). To this purpose, we first introduce some notation. Denote by \(\mathcal{P}\) the set of primary primes of \(\mathbb{Z}[i]\) with positive imaginary part. We define \(N\) to be the norm map from \(\mathbb{Z}[i]\) to \(\mathbb{Z}\). Let \(C_k'(x, \alpha, B)\) be all \(\eta = \lambda_1 \cdots \lambda_k\) satisfying

- \(N\eta \leq x\) and \(N\lambda_1 < \cdots < N\lambda_k\),
- \(\lambda_l \in \mathcal{P}\) and \(N\lambda_l \equiv \alpha_l \pmod{16}\) for all \(1 \leq l \leq k\),
- \(\left[\frac{N\lambda_l}{N\lambda_1}\right] = B_{lj}\) for all \(1 \leq l < j \leq k\),
- \(\left[\frac{N\lambda_l}{q_j}\right] = 1\) for all \(1 \leq l \leq k\) and \(1 \leq j \leq k'\), and
- \(\left(\frac{\lambda'}{\lambda}\right) = -1\) with \(\lambda = \prod_{l=1}^{k} \lambda_l^{z_l}\) and \(\lambda' = \prod_{l=1}^{k} \lambda_l^{1-z_l}\).
Lemma 12. The following map is a bijection

\[ C_k'(x, \alpha, B) \rightarrow C_k(x, \alpha, B), \quad \eta \mapsto N\eta. \]

Proof. Let \( \eta = \lambda_1 \cdots \lambda_k \) be an element of \( C_k'(x, \alpha, B) \) satisfying \( N\lambda_1 < \cdots < N\lambda_k \) and \( \lambda_l \in \mathcal{P} \) for all \( 1 \leq l \leq k \). Denote by \( p_l = N\lambda_l \) for all \( 1 \leq l \leq k \). To show that \( N\eta \in C_k(x, \alpha, B) \), we only need to verify \( \left( \frac{d}{p_l} \right)_4 \left( \frac{d}{p_j} \right)_4 = -1 \) with \( d = N\lambda \) and \( d' = N\lambda' \). Note that

\[
\left( \frac{p_l}{p_j} \right)_4 \left( \frac{p_j}{p_l} \right)_4 = \left( \frac{\lambda_l \lambda_j}{\lambda_j} \right)_4 \left( \frac{\lambda_j \lambda_j}{\lambda_l} \right)_4 = \left( \frac{\lambda_j \lambda_j}{\lambda_l} \right)_4 \left( \frac{\lambda_j \lambda_j}{\lambda_l} \right)_4 = \left( \frac{\lambda_j \lambda_j}{\lambda_l} \right)_4 \left( \frac{\lambda_j \lambda_j}{\lambda_l} \right)_4.
\]

Here we have used the quadratic reciprocity law for \( \left( \frac{\lambda_j \lambda_j}{\lambda_l} \right)_4 \) and \( \left( \frac{\lambda_j \lambda_j}{\lambda_l} \right)_4 \). From the definition of the quartic residue symbol, we have \( \left( \frac{\lambda_j \lambda_j}{\lambda_l} \right)_4 \left( \frac{\lambda_j \lambda_j}{\lambda_l} \right)_4 = 1 \). Consequently,

\[
\left( \frac{p_l}{p_j} \right)_4 \left( \frac{p_j}{p_l} \right)_4 = \left( \frac{\lambda_j \lambda_j}{\lambda_l} \right)_4.
\]

Therefore,

\[
\left( \frac{d'}{d} \right)_4 \left( \frac{d}{d'} \right)_4 = \left( \frac{\lambda'}{\lambda} \right)_4 = -1.
\]

So \( N\eta \) lies in \( C_k(x, \alpha, B) \). The map is obviously injective. The map is surjective by observing that for every rational prime \( p \equiv 1 \pmod{4} \) there is exactly one primary prime \( \eta \) with \( \eta \). This completes the proof of the lemma. \( \square \)

To use the idea of Cremona-Odoni [10], we introduce another set \( T(x) \). Here \( T(x) \) is the set of positive integers \( n = p_1 \cdots p_{k-1} \leq x \) satisfying

- \( p_1 < \cdots < p_{k-1} \),
- \( p_l \equiv \alpha_l \pmod{16} \) for all \( 1 \leq l \leq k-1 \),
- \( \left[ \frac{p_l}{p_j} \right] = B_{ij} \) for all \( 1 \leq l < j \leq k-1 \), and
- \( \left[ \frac{p_l}{\eta_j} \right] = 1 \) for all \( 1 \leq l \leq k-1 \) and \( 1 \leq j \leq k' \).

The independence property of Legendre symbols of Rhoades [14] implies

\[
(6.1) \quad \#T(x) \sim 2^{-(k'+3)(k-1)-(k-1)} \cdot \#C_{k-1}(x),
\]

where \( \binom{k}{j} \) is the binomial coefficient and \( C_k(x) \) is the set of all positive square-free integers \( n \leq x \) with exactly \( k \) prime factors. Like \( C_k(x, \alpha, B) \), we have to identify \( T(x) \) with another set \( T'(x) \). Here \( T'(x) \) is the set of \( \eta = \lambda_1 \cdots \lambda_{k-1} \) satisfying

- \( N\eta \leq x \) and \( N\lambda_1 < \cdots < N\lambda_{k-1} \),
- \( \lambda_l \in \mathcal{P} \) and \( N\lambda_l \equiv \alpha_l \pmod{16} \) for all \( 1 \leq l \leq k-1 \),
- \( \left[ \frac{N\lambda_l}{N\lambda_i} \right] = B_{ij} \) for all \( 1 \leq l < j < k \), and
- \( \left[ \frac{N\lambda_l}{\eta_j} \right] = 1 \) for all \( 1 \leq l < k \) and \( 1 \leq j \leq k' \).

Similarly, we have the following lemma.

Lemma 13. The following map is a bijection

\[ T'(x) \rightarrow T(x), \quad \eta \mapsto N\eta. \]

Now we can use the idea of Cremona-Odoni [10] to prove the independence property of residue symbols.
Theorem 4. For \( k \) an integer greater than 1, let \( \alpha = (\alpha_1, \cdots, \alpha_k) \) with \( \prod_{j=1}^{k} \alpha_j \equiv 1 \pmod{8} \) and all \( \alpha_j \in \{1, 5, 9, 13\} \). Assume that \( B = (B_{ij})_{k \times k} \) is a symmetric \( \mathbb{F}_2 \) matrix with rank \( k - 2 \) and the sum of its any given row being zero. Then

\[
\#C_k(x, \alpha, B) = \frac{1 + o(1)}{2^{3k + k'k + 1+ \binom{k}{2}}} \cdot \#C_k(x).
\]

Proof. Like Cremona-Ododi [10], we consider the comparison map

\[
f : C_k'(x, \alpha, B) \longrightarrow T'(x), \quad \eta \mapsto \eta/\tilde{\eta},
\]

where \( \tilde{\eta} \) lies in \( \mathcal{P} \) such that its norm is the maximal prime divisor of \( N\eta \). According to \( z_k = 0 \) or not, the proof can be divided into two cases.

We first consider the case \( z_k = 0 \). The next step is to consider the fiber of the comparison map \( f \). Let \( \epsilon = \prod_{j=1}^{k-1} \lambda_j \in T'(x) \) with \( N\lambda_1 < \cdots < N\lambda_{k-1} \) and all \( \lambda_j \in \mathcal{P} \). Then \( \epsilon \) lies in \( \text{Im} f \) if and only if there exists a \( \lambda_0 \in \mathcal{P} \) with \( N\lambda_0 \) lying in \( (N\lambda_{k-1}, x/N\epsilon) \) such that

1. \( N\lambda_0 \equiv \alpha_k \pmod{16} \) and \( \left[ \frac{N\lambda_0}{N\alpha} \right] = B_{lk} \) for all \( 1 \leq l \leq k - 1 \),
2. \( \left( \frac{N\lambda_0}{\eta^j} \right) = 1 \) for all \( 1 \leq j \leq k' \) and
3. \( \left( \frac{\lambda}{\chi} \right) = -\left( \frac{\epsilon/\lambda}{\chi} \right) \) with \( \lambda = \prod_{i=1}^{k-1} \lambda_i^2 \).

Then there is a unique subset \( \mathcal{A} = \mathcal{A}_\epsilon \) of \( \left( \mathbb{Z}[i]/16\epsilon_1 \epsilon_2 \mathbb{Z}[i] \right)^\times \) such that for any prime \( \theta \) the integer \( \theta\epsilon \lambda \) lies in \( C_k'(x, \alpha, B) \) if and only if \( \theta \in \mathcal{P} \) and \( \theta \in \mathcal{A} \) satisfying \( N\theta \in (N\lambda_{k-1}, N\epsilon) \). Here \( \epsilon_1 \) is the product of all primary primes lying in \( \mathcal{P} \) and lying above \( p \) with \( p \mid abc \) and \( p \equiv 1 \pmod{4} \), and \( \epsilon_2 \) is the product of all prime factors \( p \) of \( abc \) with \( p \equiv 3 \pmod{4} \). The cardinality of \( \mathcal{A} \) is evaluated in the following lemma.

Lemma 14. Let \( \varphi(\epsilon) \) be the cardinality of \( G = \left( \mathbb{Z}[i]/\epsilon \right)^\times \) with \( \epsilon = \epsilon_1 \epsilon_2 \mathbb{Z}[i] \). Then

\[
\#\mathcal{A} = 2^{-k''-4} \varphi(\epsilon).
\]

Proof of Lemma 14. From the definition of \( \mathcal{A} \), we see that \( \mathcal{A} \) represents those primary classes \( \beta \) of \( G \) such that

(a) \( \beta \equiv \alpha_k \pmod{16} \) and \( \left[ \frac{N\beta}{N\alpha} \right] = B_{lk} \) for all \( 1 \leq l \leq k - 1 \),
(b) \( \left( \frac{N\beta}{\eta^j} \right) = 1 \) for all \( 1 \leq j \leq k' \), and
(c) \( \left( \frac{\beta}{\chi} \right) = \left( \frac{\epsilon/\lambda}{\chi} \right) \) with \( \lambda = \prod_{i=1}^{k-1} \lambda_i^2 \).

Via Chinese Remainder Theorem we obtain the following isomorphism

\[
G \cong \left( \mathbb{Z}[i]/16\mathbb{Z}[i] \right)^\times \times \left( \prod_{l=1}^{k-1} \left( \mathbb{Z}[i]/\lambda_l \mathbb{Z}[i] \right)^\times \right) \times \left( \mathbb{Z}[i]/\epsilon_1 \mathbb{Z}[i] \right)^\times \times \left( \mathbb{Z}[i]/\epsilon_2 \mathbb{Z}[i] \right)^\times,
\]

where the map is given by \( \beta \mapsto (\beta_0, \cdots, \beta_{k-1}, \beta_1', \beta_2') \). Here \( \beta_i \) denotes the class \( \beta \pmod{\lambda_i \mathbb{Z}[i]} \) if \( 1 \leq l \leq k - 1 \) and \( \beta \pmod{16 \mathbb{Z}[i]} \) if \( l = 0 \), and \( \beta_j' \) denotes the class \( \beta \pmod{\epsilon_j \mathbb{Z}[i]} \) for \( j = 1, 2 \). Note that the residue symbol \( \left( \frac{\beta}{\chi} \right) \) is only non-trivial on those \( \left( \mathbb{Z}[i]/\lambda_i \mathbb{Z}[i] \right)^\times \)-component with \( z_l = 1 \). For any \( 1 \leq l \leq k - 1 \), the condition \( \left[ \frac{N\beta}{N\alpha} \right] = B_{lk} \) takes up a half of the \( \left( \mathbb{Z}[i]/\lambda_l \mathbb{Z}[i] \right)^\times \)-component. Here we have used the isomorphism

\[
\left( \mathbb{Z}[i]/\lambda_l \mathbb{Z}[i] \right)^\times \cong \left( \mathbb{Z}/N\lambda_l \mathbb{Z} \right)^\times \text{ by } \lambda_l \in \mathcal{P}.
\]

The condition \( \left( \frac{\beta}{\chi} \right) = \left( \frac{\epsilon/\lambda}{\chi} \right) \) selects another
This completes the proof of the lemma. 

Like above, \( \left( \frac{N\beta}{q_j} \right) = 1 \) selects a half of the \( \left( \mathbb{Z}/\mathbb{Z}[i] \right)^\times \) component provided that \( \lambda_j \mid \epsilon_1 \). For \( q_j \mid \epsilon_2 \), the condition \( \left( \frac{N\beta}{q_j} \right) = 1 \) also selects a half of the \( \left( \mathbb{Z}/q_j\mathbb{Z}[i] \right)^\times \) component by the composition of the homomorphisms

\[
\left( \mathbb{Z}/q_j\mathbb{Z}[i] \right)^\times \rightarrow \left( \mathbb{Z}/q_j\mathbb{Z} \right)^\times \rightarrow \{ \pm 1 \}.
\]

Here the last homomorphism is given by the Legendre symbol.

Now we consider the \( G' \)-component, where \( G' = \left( \mathbb{Z}/16\mathbb{Z}[i] \right)^\times \). The primary condition selects \( G_0 = 1 + (2 + 2i)\mathbb{Z}[i] \) of \( G' \) and \( \#G' = 4\#G_0 \). Then \( N\beta \equiv \alpha_k \pmod{16} \) selects a fourth of \( G_0 \). So

\[
\#G = 2^{k-k'-4}\varphi(\epsilon).
\]

This completes the proof of the lemma. 

For any \( \epsilon \in T'(x) \), let \( h(\epsilon) \) be the number of primes \( \theta \in \mathcal{P} \) such that \( \theta + 16\epsilon\epsilon_1\epsilon_2\mathbb{Z}[i] \) lies in \( \mathcal{A} \) and \( N\theta \) lies in \( (N\lambda_{k-1}, N\epsilon) \). Then we have

\[
\text{(6.2)} \quad \#C_k(x, \alpha, B) = \sum_{\epsilon \in T'(x)} h(\epsilon).
\]

We will divide the sum in (6.2) into several parts according to the norm of \( \eta \). To this purpose, we introduce some notation. We define \( \mu \) and \( \nu \) to be \( (\log x)^{100} \) and \( \exp \left( \frac{\log x}{(\log \log x)^{1/3}} \right) \) respectively. For a set \( M \) consisting of positive integers and a function \( g \) on \( \mathbb{Z}[i] \), we define

\[
\sum_{N\delta \in \mathcal{M}} g(\delta) = \sum_{\delta \in T'(x) \atop N\delta \in \mathcal{M}} g(\delta).
\]

Like Lemma 3.1 of Cremona-Odoni, we have the following lemma (the proof is the same as that of Cremona-Odoni).

**Lemma 15.** If \( m = 20 \) and \( n = \mu \), then

\[
\sum_{m < N\delta \leq n} \text{Li}(x/N\delta) = \mathcal{O} \left( \frac{x(\log \log x)^{k-1}}{\log x} \right).
\]

Similar estimation is true for \( m = \nu \) and \( n = x^{k-1} \). Moreover, we have

\[
\sum_{\mu < N\delta \leq \nu} \text{Li}(x/N\delta) \sim \#T'(x) \log log x.
\]

Now we divide the interval \([1, x]\) into five parts by the points \( 20, \mu, \nu \) and \( x^{k-1} \). First, for those \( \epsilon \in T'(x) \) satisfying \( N\epsilon \leq 20 \), we get \( h(\epsilon) \leq \pi(x/N\epsilon) \) by noting that every prime ideal corresponds to at most one primary prime element. Here \( \pi(y) \) denotes the number of those prime ideals with norm no larger than \( y \), and the prime ideal theorem says that

\[
\pi(y) \sim \text{Li}(y).
\]

So we have

\[
\sum_{N\epsilon \leq 20} h(\epsilon) = \mathcal{O}(\text{Li}(x)).
\]
Next, if $\epsilon \in T^*(x)$ and $20 < N\epsilon \leq \mu$, then we also have $h(\epsilon) = O(Li(x/N\epsilon))$. Thus Lemma 14 implies that

$$\sum_{20 < N\epsilon \leq \mu} h(\epsilon) = o\left(\frac{x(\log \log x)^{k-1}}{\log x}\right).$$

By the same reason, we obtain

$$\sum_{\nu < N\epsilon \leq x} h(\epsilon) = o\left(\frac{x(\log \log x)^{k-1}}{\log x}\right).$$

Next, if $N\epsilon > x^{\frac{k+1}{k}}$, then $N\lambda_{k-1} > x^{\frac{k}{k-1}}$. So $x/N\epsilon < x^{\frac{k}{k-1}} < N\lambda_{k-1} < N\theta$. Thus from the definition of $h(\epsilon)$ we get $h(\epsilon) = 0$ in this case. Therefore,

$$\sum_{x^{\frac{k+1}{k}} < N\epsilon \leq x} h(\epsilon) = 0.$$

From these estimations, we see that

$$\#C_k(x, \alpha, B) \sim \frac{1}{2} \sum_{\mu < N\epsilon \leq \nu} \pi'(x/N\epsilon, \mathscr{A}, 16\epsilon_1\epsilon_2) - \frac{1}{2} \sum_{\mu < N\epsilon \leq \nu} \pi'(N\lambda_{k-1}, \mathscr{A}, 16\epsilon_1\epsilon_2).$$

Here the factor $\frac{1}{2}$ comes from $\theta \in \mathcal{P}$, and $\pi'(y, \mathscr{B}, \gamma)$ denotes the number of prime elements $\theta$ of $\mathbb{Z}[i]$ such that $N\theta \leq y$ and $\theta + \gamma \mathbb{Z}[i]$ lies in $\mathscr{B}$. Noting that

$$\sum_{\mu < N\epsilon \leq \nu} \pi'(N\lambda_{k-1}, \mathscr{A}, 16\epsilon_1\epsilon_2) \ll \nu Li(\nu) = o\left(\frac{x(\log \log x)^{k-1}}{\log x}\right),$$

we derive

$$(6.3) \quad \#C_k(x, \alpha, B) \sim \frac{1}{2} \sum_{\mu < N\epsilon \leq \nu} \pi'(x/N\epsilon, \mathscr{A}, 16\epsilon_1\epsilon_2).$$

To estimate the sum in the right hand side of (6.3), we have to use the Dirichlet prime ideal theorem. This theorem only gives the distribution of prime ideals, while (6.3) requires the estimation on the number of prime elements. We can solve this problem by the following transformation. Via Theorem 6.1 of Lang [16], we yield the exact sequence

$$(6.4) \quad 1 \xrightarrow{c} \mathbb{Z}[i]^\times \xrightarrow{\Phi} (\mathbb{Z}[i]/c)^\times \xrightarrow{\Phi} I(c)/P(c) \xrightarrow{1}$$

with $c = \zeta_0$ defined in Lemma 14. Here $\Phi$ is the map induced from $\Phi'$ which sends every $c$-invertible element $a$ of $\mathbb{Z}[i]$ into the principal ideal $(\beta)$. For an ideal $\mathfrak{a}$ and a subset $\mathcal{S}$ of $I(\mathfrak{a})/\mathfrak{P}$, we define $\pi(y, \mathcal{S}, \mathfrak{a})$ to be the number of prime ideals $\mathfrak{p}$ such that $N\mathfrak{p} \leq y$ and $\mathfrak{p}\mathfrak{P}$ lies in $\mathcal{S}$. Let $\mathcal{S} = \mathcal{S}_\mathfrak{c} = \Phi(\mathfrak{A})$. Then we get

$$(6.5) \quad \pi'(y, \mathcal{S}, 16\epsilon_1\epsilon_2) = \pi(y, \mathcal{S}, \mathfrak{c}), \quad \#\mathcal{A} = \#\mathcal{S}$$

by noting that every prime ideal in a class of $\mathcal{S}$ corresponds to exactly one primary prime element. From (6.3) and (6.5) we get

$$(6.6) \quad 2\#C_k(x, \alpha, B) \sim \sum_{\mu < N\epsilon \leq \nu} \pi(x/N\epsilon, \mathcal{S}_\mathfrak{c}, \mathfrak{c}).$$

By the relation of $\psi(y, \mathcal{S}, \mathfrak{c})$ and $\pi(y, \mathcal{S}, \mathfrak{c})$, it suffices to estimate

$$(6.7) \quad \sum_{\mu < N\epsilon \leq \nu} \psi(x/N\epsilon, \mathcal{S}_\mathfrak{c}, \mathfrak{c}).$$
Here $\psi(y, \mathcal{T}, c)$ is given by

$$
\sum_{\substack{N_a \leq y \\ aP \in \mathcal{T}}} \Lambda(a).
$$

Via the orthogonality of characters and the exact sequence (6.4), we get

$$
\psi(y, \mathcal{T}, c) = \frac{4}{\varphi(c)} \sum_{\chi \bmod c} \psi(y, \chi) \sum_{[a] \in \mathcal{T}} \chi(a).
$$

Here $\chi$ runs over all characters of $I(c_\epsilon)/P_\epsilon$ and

$$
\psi(y, \chi) = \sum_{N_a \leq y} \Lambda(a) \chi(a).
$$

Applying this formula to (6.7), we derive

$$
\sum_{\mu<N \epsilon \leq \nu}^* \psi(x/N \epsilon, \mathcal{T}, c_\epsilon) = \left( I \right) + \left( II \right)
$$

with

$$
\left( I \right) = \sum_{\mu<N \epsilon \leq \nu}^* \frac{4 \# \mathcal{T}}{\varphi(\epsilon)} \psi(x/N \epsilon, \chi_0),
$$

$$
\left( II \right) = \sum_{\mu<N \epsilon \leq \nu}^* \frac{4}{\varphi(\epsilon)} \sum_{\chi \bmod c_\epsilon} \psi(x/N \epsilon, \chi) \sum_{[a] \in \mathcal{T}} \chi(a).
$$

Here $\sum'$ is the sum over all non-principal characters of a fixed modulus. We first treat the main term (I). Via Lemma 14 we obtain that

$$
\left( I \right) \sim 2^{-k-k'-2} \sum_{\mu<N \epsilon \leq \nu}^* \psi(x/N \epsilon) \sim 2^{-k-k'-2} \sum_{\mu<N \epsilon \leq \nu}^* \log(x/N \epsilon) \text{Li}(x/N \epsilon) \sim 2^{-k-k'-2} \log x \sum_{\mu<N \epsilon \leq \nu}^* \text{Li}(x/N \epsilon) \sim \frac{1}{(k-1) \cdot 2^{k+k'+2} \#T'(x)} \cdot \log x \cdot \log \log x.
$$

Now we assume that (II) is error term to prove the theorem. By (6.6) and (6.7), we have

$$
\#C_k(x, \alpha, B) \sim \frac{1}{(k-1) \cdot 2^{k+k'+3} \#T'(x)} \cdot \log \log x \sim \frac{1}{2^{k+k+3(k+1)/(2)}} \#C_k(x).
$$

Here we have used (6.1) and Lemma 13. So by lemma 12 the proof of the theorem is reduced to showing that (II) is an error term.

Like Cremona-Odoni, we have to separate out all the modulus in the sum (II) with possible Siegel zeros. We denote by $\text{cond}^1$ the conductor of the exceptional primitive conductor.
with \( Z \leq 256\nu \) in Page Theorem (Proposition 3). According to the modulus \( \hat{1} \) being a multiple of \( \hat{1}_1 \) or not, we can divide the sum (II) into the subsums (III) and (IV), where

\[
(III) = \sum_{\mu < N \leq \nu}^{*} 4 \sum_{\chi \mod \nu} \psi(x/N\epsilon, \chi) \sum_{[a] \in \mathcal{F}} \overline{\chi(a)}, \\
(IV) = \sum_{\mu < N \leq \nu}^{*} 4 \sum_{\chi \mod \nu} \psi(x/N\epsilon, \chi) \sum_{[a] \in \mathcal{F}} \overline{\chi(a)}.
\]

We use the trivial estimation \( \psi(x/N\epsilon, \chi) \ll \psi(x/N\epsilon) \) to bound (III) and get

\[
(III) \ll \sum_{\mu < N \leq \nu}^{*} \psi(x/N\epsilon) \ll x \sum_{\mu < N \leq \nu}^{*} (N\epsilon)^{-1} \\
= \frac{x}{N_{\hat{1}_1}} \sum_{\mu < N \leq \nu}^{*} t^{-1} \sum_{t \in T(\infty)} \sum_{\nu \leq N \leq \nu T_{\hat{1}_1}} 1 \ll \frac{x \log \nu}{N_{\hat{1}_1}}.
\]

Again by Page Theorem (Proposition 3) with \( Z = 256\nu \), there is a positive constant \( c_6 \) such that the Siegel zero \( \beta \) of the primitive character with modulus \( \hat{1}_1 \) has the property

\[
\beta > 1 - \frac{c_6}{\log 256\nu}.
\]

Via Siegel Theorem (Proposition 2), for any \( \epsilon' > 0 \) (not confused with our \( \epsilon \in T'(x) \)), there is a constant \( c' = c'(\epsilon', 2) > 0 \) such that

\[
\beta \leq 1 - c' (4N_{\hat{1}_1})^{-\epsilon'}.
\]

Taking \( \epsilon' = 1/200 \), we obtain \( N_{\hat{1}_1} \gg c'(\log \nu)^{100} \). Consequently,

\[
(III) \ll x (\log \nu)^{-99}.
\]

Next, we bound the sum (IV). There is no Siegel zero in (IV). So we can apply the explicit formula (2.5) with \( T = (N\epsilon)^4 \) to all the \( \psi(x/N\epsilon, \chi) \) in (IV) and obtain

\[
\psi(x/N\epsilon, \chi) \ll x (N\epsilon)^{-1} (\log x)^2 \exp \left( - \frac{c_7 \log(x/N\epsilon)}{\log N\epsilon} \right) + x (N\epsilon)^{-5} (\log x)^2 + x^{\frac{1}{4}} (N\epsilon)^{-\frac{1}{4}} (\log x/N\epsilon). 
\]

Correspondingly, (IV) is bounded (up to a constant) by the sum of the three sums

\[
(V) = \sum_{\mu < N \leq \nu}^{*} x (N\epsilon)^{-1} (\log x)^2 \exp \left( - \frac{c_7 \log(x/N\epsilon)}{\log N\epsilon} \right), \\
(VI) = \sum_{\mu < N \leq \nu}^{*} x (N\epsilon)^{-5} (\log x)^2, \\
(VII) = \sum_{\mu < N \leq \nu}^{*} x^{\frac{1}{4}} (N\epsilon)^{-\frac{1}{4}} (\log x/N\epsilon).
\]

For the sum (V), we have

\[
(V) \ll x (\log x)^2 \exp \left( - c_8 (\log \log x)^{100} \right) \sum_{\mu < N \leq \nu}^{*} (N\epsilon)^{-1} \\
\ll x (\log x)^3 \exp \left( - c_8 (\log \log x)^{100} \right).
\]
Similarly, we get
\[
(VI) \quad \ll x(\log x)^{2}\mu^{-3} \ll x(\log x)^{-200},
\]
\[
(VII) \quad \ll x^{\frac{3}{4}} \log x \cdot \nu^{\frac{3}{2}} \ll x^{\frac{1}{2}}.
\]
Consequently, the sum \((IV)\) is an error term. This completes the proof of the theorem. \(\square\)

7. Distribution Result

In this section, we assume that \(a, b, c\) are coprime positive integers such that \(a^2 + b^2 = 2c^2\) and the dimension of the 2-Selmer group of \(E\) is two. Let \(q_1, \cdots, q_n\) be all the prime divisors of \(abc\). Denote by \(k\) a fixed positive integer.

Let \(n = p_1 \cdots p_k \in \mathcal{D}_k(x)\) with \(p_1 < \cdots < p_k\). By Theorem 2, \(n\) lies in \(\mathcal{P}_k(x)\) if and only if \(h_4(n) = 1\) and \(h_8(n) = \frac{d-1}{4} \pmod{2}\). Here \(d\) is a certain divisor of \(n\). From the proof of Theorem 2, the characterization of \(n \in \mathcal{P}_k(x)\) is divided into two cases (according to the rank of \(A = A_n\) being \(k - 1\) or \(k - 2\)).

We first assume that \(\text{rank } A = k - 2\). Then \(h_4(n) = 1\) is equivalent to \(b \notin \text{Im } A\), where \(b = \left(\left[\frac{2}{p_1}\right], \cdots, \left[\frac{2}{p_k}\right]\right)^T\). Since \(\text{rank } A = k - 2\) and \(A \sigma_0 = 0\) with \(\sigma_0 = (1, \cdots, 1)^T \in \mathbb{F}_2^k\), there is a unique column vector \(z = (z_1, \cdots, z_k)^T \neq 0\), \(z_0 = 1\) such that \(A \sigma = b\). Then \(d = \prod_{l=1}^k p_l^{z_l}\). Jung-Yue \(\mathcal{S}\) (Theorem 3.3 (ii)) showed that in this case \(h_8(n) = 1\) is equivalent to
\[
(7.1) \quad \left(\frac{d}{d'}\right)_4 \left(\frac{d'}{d}\right)_4 = -1,
\]
where \(d' = \prod_{l=1}^k p_l^{1-z_l}\).

Now we assume that \(\text{rank } A = k - 1\). Then \(h_4(n) = 1\) and \(b \in \text{Im } A\) with \(b = \left(\left[\frac{2}{p_1}\right], \cdots, \left[\frac{2}{p_k}\right]\right)^T\). Let \(z = (z_1, \cdots, z_k)^T\) be a column vector with \(A \sigma = b\). Then \(d = \prod_{l=1}^k p_l^{z_l}\). Jung-Yue \(\mathcal{S}\) (Theorem 3.3 (iii) and (iv)) proved that \(h_8(n) = 1\) if and only if \(\left(\frac{2d}{d'}\right)_4 \left(\frac{2d'}{d}\right)_4 = (-1)^{\frac{n-1}{8}}\). Here \(d' = \prod_{l=1}^k p_l^{1-z_l}\).

Now we begin to prove Theorem 3

Proof of Theorem 3 Like Theorem 2 we also divide the proof into two cases.

First, we count the number \(N_1(x)\) of those \(n \in \mathcal{P}_k(x)\) with \(\text{rank } A_n = k - 2\). Obviously, we have \(k \geq 2\) in this case. We first introduce some notation. Let \(\mathcal{B}\) be the set of \(k \times k\) symmetric matrices \(B\) over \(\mathbb{F}_2\) with \(\text{rank } B = k - 2\) and \(B \sigma_0 = 0\), where \(\sigma_0 = (1, \cdots, 1)^T \in \mathbb{F}_2^k\). We define \(\mathcal{I}\) to be all the \(\alpha = (\alpha_1, \cdots, \alpha_k)\) with \(\prod_{l=1}^k \alpha_l \equiv 1 \pmod{8}\) and \(\alpha_l \in \{1, 5, 9, 13\}\) for \(1 \leq l \leq k\). Given \(B \in \mathcal{B}\), we denote by \(\mathcal{I}_B\) the set of \(\alpha \in \mathcal{I}\) such that \(b_\alpha\) does not lie in the image of \(B\). Here \(b_\alpha = \left(\left[\frac{2}{\alpha_1}\right], \cdots, \left[\frac{2}{\alpha_k}\right]\right)^T\). Note that \(b_\alpha^T \sigma_0 = 0\) by \(\prod_{l=1}^k \alpha_l \equiv 1 \pmod{8}\). For any \(B \in \mathcal{B}\) and \(\alpha \in \mathcal{I}_B\), by (7.1) those \(n = p_1 \cdots p_k \in \mathcal{P}_k(x)\) satisfying
\[
\begin{align*}
&\bullet \quad p_1 < \cdots < p_k \quad \text{and} \quad A_n = B, \\
&\bullet \quad \left(\frac{p_l}{p_{l'}}\right) = 1 \quad \text{for all } 1 \leq l \leq k \quad \text{and} \quad 1 \leq j \leq k', \quad \text{and} \\
&\bullet \quad p_l \equiv \alpha_l \pmod{16} \quad \text{for all } 1 \leq l \leq k
\end{align*}
\]
consist the set \(C_k(x, \alpha, B)\). Moreover, given \(B \in \mathcal{B}\) and \(\alpha \in \mathcal{I} - \mathcal{I}_B\), the intersection of \(C_k(x, \alpha, B)\) and \(\mathcal{P}_k(x)\) is empty. Therefore, the number \(N_1(x)\) of those \(n \in \mathcal{P}_k(x)\) with \(\text{rank } A_n = k - 2\) is
\[
N_1(x) = \sum_{B \in \mathcal{B}} \sum_{\alpha \in \mathcal{I}_B} \#C_k(x, \alpha, B) \sim 2^{-3k-k'k'-1-\left(\frac{1}{4}\right)} \cdot \#C_k(x) \cdot \sum_{B \in \mathcal{B}} \#\mathcal{I}_B.
\]
Here we have used Theorem 4
Now we count the number of $\mathcal{F}_B$ with $B$ given. First, given $b = (b_1, \cdots, b_k)^T$ with $b \notin \text{Im} B$ and $b^T z_0 = 0$, we count the number of $\alpha$ such that $b = b_\alpha$. As $b = b_\alpha$, we get \[
exists \frac{2}{a_l} = b_l \text{ for all } 1 \leq l \leq k. \text{ So any } \alpha_l \text{ has exactly two choices. Thus the number of } \alpha \text{ such that } b = b_\alpha \text{ is } 2^k. \text{ Next, we count the number of column vectors } b \text{ such that } b^T z_0 = 0 \text{ and rank } (B b) = k - 1. \text{ Since } (B b) z_0 = 0 \text{ and } B \text{ is symmetric, we get rank } B' = k - 2 \text{ and rank } (B' b') = k - 1. \text{ Here } B' \text{ is the matrix obtained from } B \text{ by deleting its last row and column, and } b' \text{ is the vector obtained from } b \text{ by deleting its last component. Thus } b' \text{ does not lie in the image of } B'. \text{ So there are } 2^{k-2} \text{ many such } b' \text{ and } b. \text{ Consequently, } \#\mathcal{F}_B = 2^{2k-2}. \text{ Then (7.2) implies that }
$$N_1(x) \sim 2^{-k-k'k-3 - \binom{k}{2}} \cdot \#C_k(x) \cdot \#\mathcal{B}.$$
The number of $\mathcal{B}$ can be obtained from the following result of Brown et al [1].

**Proposition 6.** For positive integers $r \leq k$, we denote by $\mathcal{B}_{k,r}$ the set of $k \times k$ symmetric matrices over $\mathbb{F}_2$ with rank $r$. Then
$$\#\mathcal{B}_{k,r} = u_{r+1} \binom{r+1}{2} \cdot \prod_{l=0}^{k-r-1} \frac{2^k - 2^l}{2^{k-r} - 2^l},$$

with $u_r$ defined above Theorem 3.

Note that the map sending $B$ to $B'$ induces a bijection between $\mathcal{B}$ and $\mathcal{B}_{k-1,k-2}$. So
$$\#\mathcal{B} = u_{k-1} 2^{\binom{k-1}{2}} \cdot (2^{k-1} - 1).$$

We get
$$N_1(x) \sim 2^{-k-k'k-3} (1 - 2^{1-k}) u_{k-1} \cdot \#C_k(x).$$

Finally, we counts the number $N_2(x)$ of $n \in \mathcal{P}_k(x)$ with $\text{rank } A_n = k - 1$. Like above (we refer to our previous paper [19] for detailed proof), we get
$$N_2(x) \sim 2^{-k-k'k-2} u_k \cdot \#C_k(x).$$

This finishes the proof of the theorem. \qed

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