Quantum Cylindrical Waves and Sigma Models

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We analyze cylindrical gravitational waves in vacuo with general polarization and develop a viewpoint complementary to that presented recently by Niedermaier showing that the auxiliary sigma model associated with this family of waves is not renormalizable in the standard perturbative sense.

1. INTRODUCTION

Einstein-Rosen waves, i.e. linearly polarized cylindrical waves, have been extensively studied in the literature, both from the classical and quantum viewpoints [1, 2, 3, 4, 5]. They provide a non-trivial reduction of General Relativity (GR) that retains some important features of the full theory. In particular, they are described by a genuine field theory, with one local degree of freedom. However, one of the most distinctive features of GR, namely non-linearity, is not fully realized in this system inasmuch as the reduced phase space of the Einstein-Rosen waves is equivalent to that of an auxiliary linear theory, without appealing to perturbative techniques. This property implies the presence of two relevant Hamiltonians for the system: an auxiliary free Hamiltonian $H_0$, that generates the evolution in the auxiliary linear theory, and a physical Hamiltonian, $E(H_0) = 2(1 - e^{-H_0/2})$, which is a non-linear function of the former [3]. The coexistence of these two Hamiltonians has unexpected physical consequences in the classical and quantum realms [5, 7]. In our discussion, we adopt a system of units such that $c = \hbar = 8G_3 = 1$, where $G_3$ denotes the effective Newton constant per unit length in the direction of the symmetry axis.
A further step in increasing the complexity of the system, with the aim at investigating the kind of behavior expected in quantum gravity, is to consider midi-superspaces that incorporate the non-linear character of GR without leaving the field theoretic framework. In this sense, the cylindrically symmetric case with general polarization is a natural candidate to work with, since it provides a non-linear generalization of the Einstein-Rosen waves which consists of two interacting fields [8].

In principle, one might think of quantizing the family of cylindrical waves with general polarization by taking advantage of the two Hamiltonians structure that arises in the system. First, one would employ standard perturbative techniques to analyze and quantize the auxiliary Hamiltonian. Assuming that this perturbative quantization were feasible, one would then use the functional relation between the auxiliary and physical Hamiltonians in order to construct a quantum theory for the cylindrical model. Nonetheless, we will see that this quantization scheme cannot be fully accomplished owing to the non-renormalizability of the auxiliary theory.

The perturbative quantization of cylindrical gravitational waves with general polarization has been considered previously by Niedermaier [9]. However, there are some relevant differences between the approach followed in those references and ours. Namely, the cylindrical reduction of the gravitational theory discussed in Ref. [9] does not include the surface term necessary to render the action functionally differentiable. Actually, this term turns out to supply the physical Hamiltonian $E(H_0)$ after fixing the time gauge [10]. In this sense, the results obtained by Niedermaier refer in fact to the auxiliary model of the two Hamiltonians approach commented above. On the other hand, in the present paper we follow from the very beginning a path integral approach, incorporating both a gauge fixing and the corresponding Fadeev-Popov determinant. This simplifies drastically the structure of the sigma model attained in Ref. [9], therefore leading to much easier computations. From this perspective, our discussion can be regarded as a complementary analysis supporting the arguments of Niedermaier about the non-renormalizability, modulo a function, of the cylindrical gravitational waves.

The rest of paper is organized as follows. Section 2 contains the basic definitions concerning the geometry and dynamics of the cylindrical waves in vacuo. In that section, we also derive an expression for the partition function of the cylindrical gravitational theory, written in terms of the reduced phase space variables, and identify the auxiliary
and physical Hamiltonians of the system. In Sec. 3 we discuss the renormalizability of the auxiliary model. We will show that the model is non-renormalizable in the standard sense, but belongs to a wider class of models that are renormalizable modulo a function, as claimed in Ref. [9]. Finally, we summarize our results and conclude.

2. CYLINDRICAL WAVES

General cylindrical gravitational waves in vacuo are described by space-times whose metric $g$ admits a two-parametric, Abelian, and orthogonally transitive group of isometries. These space-times are equipped with two linearly independent, spacelike Killing fields, one of them axial, $X_{(\theta)}$, and the other translational, $X_{(z)}$. These Killing fields commute, $[X_{(\theta)}, X_{(z)}] = 0$, and generate a space-time foliation orthogonal to the isometry orbits (orthogonal transitivity). Note that, unlike what happens for the Einstein-Rosen waves, the Killing vectors are not assumed to be mutually orthogonal, $g(X_{(z)}, X_{(\theta)}) \neq 0$. This allows the presence of an additional degree of freedom with respect to the linearly polarized case.

One can always choose coordinates $(t, r, \theta, z) \in \mathbb{R} \times \mathbb{R}^+ \times S^1 \times \mathbb{R}$, adapted to the symmetry of the system, such that the line element takes the form

$$ds^2 = -(N^\perp)^2 dt^2 + e^{\gamma - \psi}(N^r dt + dr)^2 + R^2 e^{-\psi} d\theta^2 + e^\psi (dz + \phi d\theta)^2.$$ 

The smooth fields $\psi, \phi, \gamma, R, N^\perp$ (lapse), and $N^r$ (radial shift) depend only on the $(t, r)$ coordinates, satisfy suitable boundary conditions (at the symmetry axis $r = 0$ and at spatial infinity $r \to \infty$) [8, 11], and parameterize the different vacuum cylindrical metrics. The scalar fields $\psi$ and $\phi$ have a precise geometrical meaning, for they are related with the norm and the scalar product of the Killing vectors,

$$\psi = \log [g(X_{(z)}, X_{(z)})], \quad \phi = \frac{g(X_{(\theta)}, X_{(z)})}{g(X_{(z)}, X_{(z)})}.$$ 

The dynamics of these cylindrical space-times is inherited from GR via a symmetry reduction [8, 10]. In this way, one arrives at the action

$$S[N^\perp, N^r, \gamma, R, \psi, \phi, \pi_\gamma, \pi_R, \pi_\psi, \pi_\phi] =$$

$$\int_{t_1}^{t_2} \left\{ \int_0^\infty \left[ \pi_\gamma \dot{\gamma} + \pi_R \dot{R} + \pi_\psi \dot{\psi} + \pi_\phi \dot{\phi} - \left( N^\perp \mathcal{H}_\perp + N^r \mathcal{H}_r \right) \right] dr - 2 \left( 1 - e^{-\gamma/2} \right) \right\} dt,$$ 

where $\mathcal{H}_\perp = \frac{1}{2} g^{\perp\perp} \left( g_{\perp\perp} \partial_\perp \partial_\perp - \partial_\perp \right)$ and $\mathcal{H}_r = \frac{1}{2} g^{rr} \left( g_{rr} \partial_r \partial_r - \partial_r \right)$. These Hamiltonians are enforced as constraints on the space-time. The action (1) will be used to derive the equations of motion for the fields $N^\perp, N^r, \gamma, R, \psi, \phi$.
where the dot and the prime denote, respectively, the time and radial partial derivatives. The boundary term involving the value of $\gamma$ at $r = \infty$, $2(1 - e^{-\gamma\infty/2})$, must be included to render the action functionally differentiable \[11\]. We have also defined the constraints

\[
\mathcal{H}_\perp := e^{(\psi - \gamma)/2} \left[ -\pi_\gamma \pi_R + 2R'' - R'\gamma' + \frac{1}{2R} \left( R^2\psi'^2 + \pi_\psi^2 + R^2e^{-2\psi}\pi_\phi^2 + e^{2\psi}\phi'^2 \right) \right] \approx 0,
\]

\[
\mathcal{H}_r := -2\pi'_\gamma + \pi_\gamma \gamma' + \pi_R R' + \pi_\psi \psi' + \pi_\phi \phi' \approx 0,
\]

which generate the gauge symmetries of the theory, namely “bubble” time evolution and radial diffeomorphisms.

With the aim at developing a quantum theory for the cylindrical model described by the action \[11\], we will evaluate the partition function $Z$ by means of a path integral approach. In order to make sense of this path integral, we adopt a gauge fixing, and introduce the associated Fadeev-Popov determinant to ensure that the result is independent of the particular gauge chosen. Employing the usual cylindrical gauge \[12\]

\[
R(t, r) = r, \quad \pi_\gamma(t, r) = 0,
\]

we obtain the expression

\[
Z = \int [D\Psi][D\Pi] \times \prod_{\mathbf{r}} \delta \left[ \pi_\gamma(\mathbf{r}) \right] \delta \left[ R(\mathbf{r}) - \mathbf{r} \right] \det_{\mathbf{r}} \left( \begin{array}{cc} \{ \pi_\gamma(\mathbf{r}), \mathcal{H}_\perp \} & \{ \pi_\gamma(\mathbf{r}), \mathcal{H}_r \} \\ \{ R(\mathbf{r}) - \mathbf{r}, \mathcal{H}_\perp \} & \{ R(\mathbf{r}) - \mathbf{r}, \mathcal{H}_r \} \end{array} \right) \times \exp \left\{ i \int_{t_1}^{t_2} \int_0^\infty \left[ \pi_\gamma \dot{\gamma} + \pi_R \dot{R} + \pi_\psi \dot{\psi} + \pi_\phi \dot{\phi} - \left( N^+ \mathcal{H}_\perp + N^r \mathcal{H}_r \right) \right] dr dt \right\} \times \exp \left\{ -i \int_{t_1}^{t_2} 2 \left( 1 - e^{-\gamma\infty/2} \right) dt \right\},
\]

where

\[
[D\Psi] = [DN^\perp][DN^r][D\gamma][DR][D\psi][D\phi],
\]

\[
[D\Pi] = [D\pi_{N^\perp}][D\pi_{N^r}][D\pi_\gamma][D\pi_R][D\pi_\psi][D\pi_\phi],
\]

and

\[
\{ R(\mathbf{r}) - \mathbf{r}, \mathcal{H}_r(\mathbf{r}) \} = R'(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}) ,
\]

\[
\{ R(\mathbf{r}) - \mathbf{r}, \mathcal{H}_\perp(\mathbf{r}) \} = -e^{[\Psi(r) - \gamma(r)]/2} \pi_\gamma(r) \delta(\mathbf{r} - \mathbf{r}) ,
\]

\[
\{ \pi_\gamma(\mathbf{r}), \mathcal{H}_r(\mathbf{r}) \} = -\pi_\gamma(r) \partial_\mathbf{r} \delta(\mathbf{r} - \mathbf{r}) ,
\]

\[
\{ \pi_\gamma(\mathbf{r}), \mathcal{H}_\perp(\mathbf{r}) \} = e^{[\Psi(r) - \gamma(r)]/2} R'(\mathbf{r}) \partial_\mathbf{r} \delta(\mathbf{r} - \mathbf{r}) + \frac{1}{2} \mathcal{H}_\perp(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}) .
\]
The special form of the action and the simplicity of the gauge fixing term allow us to perform explicitly the integrals on the non-physical sector. Thus, at the end of the day, we can write down a formula for $Z$ exclusively in terms of the variables of the reduced phase space, avoiding the use of the standard Fadeev-Popovghosts. Note first that it is straightforward to integrate $\pi_\gamma$ and $R$ thanks to the gauge fixing. One can also easily evaluate the integrals in $N^\perp$ and $N^r$, which lead to Dirac deltas of the constraints. By making use of these deltas, it is then possible to integrate out $\gamma$ and $\pi_R$, attaining an expression for the partition function (modulo a multiplicative constant $N$) which involves only the fields $\psi$ and $\phi$:

$$
Z = N \int [D\psi][D\phi][D\pi_\psi][D\pi_\phi] \\
\times \exp \left\{ i \int_{t_1}^{t_2} \left[ \int_0^\infty \left( \pi_\psi \dot{\psi} + \pi_\phi \dot{\phi} \right) dr - 2 \left( 1 - e^{-H_0/2} \right) \right] dt \right\}.
$$

Here, we have defined

$$
H_0 := \frac{1}{2} \int_0^\infty \left[ r\dot{\psi}^2 + \frac{\pi_\psi^2}{r} + re^{-2\psi} \pi_\phi^2 + \frac{e^{2\psi} \phi'^2}{r} \right] dr = \gamma_\infty.
$$

Some comments are in order. The non-linear character of the Hamiltonian in the reduced phase space prevents us from performing the exact integration in the remaining variables. On the other hand notice that, just like in the linearly polarized case, two relevant Hamiltonians emerge in the system, as can be read out from expression (3). Namely, an auxiliary Hamiltonian $H_0$, that results from the integration of a local density, and the physical Hamiltonian

$$
E(H_0) := 2 \left( 1 - e^{-H_0/2} \right),
$$

which is a non-linear function of $H_0$ and, as a consequence, encodes a non-local dynamics. It is worth emphasizing that, in contrast to the situation found for linear polarization, $H_0$ is not the Hamiltonian of a free theory in terms of the scalar fields $\psi$ and $\phi$, although it reduces to the free Hamiltonian of the Einstein-Rosen model when the field $\phi$ and its momentum are switched off (i.e., set equal to zero).

Even though we have followed a path integral approach, equivalent results about the reduced phase space can be attained in other different manners; e.g., in a mathematically precise framework, by making use of standard Hamiltonian techniques [12], or, in a
rather formal way, by just plugging into the Dirac-Schrödinger action for gravity (equivalent to the Einstein-Hilbert action up to boundary terms) the gauge fixed line element
\[ ds_G^2 = e^{\gamma-\psi}(-dT^2 + dr^2) + r^2 e^{-\psi}d\theta^2 + e^\psi(dz + \phi d\theta)^2. \]

With this last procedure, one gets
\[ S[ds_G^2] = S_{\sigma}[\psi, \phi] := \int_{T_1}^{T_2} \int_0^\infty \left[ \left( \psi^2 - \psi'^2 \right) + \frac{e^{2\psi}}{r^2} \left( \phi^2 - \phi'^2 \right) \right] drdT, \tag{4} \]
an action that leads in fact to the \( H_0 \) Hamiltonian (owing to our choice of time \( T \)).

As we have commented, the non-linear character of \( H_0 \) poses a serious obstacle for completing the path integration explicitly. Nevertheless, one can try to use perturbative techniques in order to extract relevant physical information. In particular, a natural way to proceed is to study first the perturbative quantization of the auxiliary Hamiltonian \( H_0 \) and, if this task can be achieved, use the results in a subsequent step to deal with the physical Hamiltonian \( E(H_0) \). In the next section we will explore this possibility.

### 3. PERTURBATIVE QUANTIZATION OF THE AUXILIARY MODEL

The results of the previous section lead us to consider the perturbative quantization of the auxiliary theory described by the action \( S_{\sigma}[\psi, \phi] \), defined in Eq. (4). This theory belongs to the class of \( x \)-dependent sigma models studied by Osborn in the late 80’s [13].

The \( x \)-dependent sigma models are theories of maps of the type \( \Phi : X \to Y, x \mapsto \Phi(x) \) between pseudo-Riemannian manifolds \((X, \eta)\) and \( \{(Y, \gamma_x)\}_{x \in X} \), where the metric of the target manifold \( Y \) has an explicit dependence on the points of the base manifold. The simplest action for these sigma models is a direct generalization of the standard one, namely
\[ S_{\sigma}[\Phi] = \frac{1}{2} \int_X \gamma_{ab} \left[ \Phi(x), x \right] \frac{\partial \Phi^a}{\partial x^\mu} \frac{\partial \Phi^b}{\partial x^\nu} \eta^{\mu\nu}(x) \sqrt{|\eta(x)|} \, dx. \]

Particularizing this notation to the \( H_0 \) theory considered here, we identify the base manifold as \( X = \mathbb{R} \times \mathbb{R}^+ \), with coordinates \( x = (T, r) \), and metric [14]
\[ \eta_{TT} = -\eta_{rr} = 1. \]
The target manifold $Y = \mathbb{R}^2$ has coordinates $\Phi = (\psi, \phi)$, and the non-zero metric elements are

$$
\gamma_{\psi\psi}(r) = r, \quad \gamma_{\phi\phi}(\psi; r) = \frac{e^{2\psi}}{r}.
$$

Therefore, the target manifold $(Y, \gamma_r)$ is a symmetric space, with a curvature tensor that depends only on $\psi$, apart from an explicit dependence on the coordinate $r$ of the base manifold, namely

$$
\mathcal{R}_{abcd}(\psi; r) = \frac{1}{r} \left[ \gamma_{ad}(\psi; r) \gamma_{bc}(\psi; r) - \gamma_{ac}(\psi; r) \gamma_{bd}(\psi; r) \right].
$$

Without the explicit $r$-dependence, this would be the best possible theoretical scenario. The dependence on $r$ calls for a careful study of the perturbative properties of the model and, in particular, of its renormalizability.

In order to carry out a detailed analysis of the renormalizability of the action $S_c[\psi, \phi]$, one can employ the standard dimensional regularization and minimal subtraction scheme [13]. The bare target metric can be written by means of the standard pole-loop expansion as

$$
\gamma^B_{ab}(\psi; r) = \mu^{-\epsilon} \left[ \gamma_{ab}(\psi; r) + \sum_{\nu \geq 1} \sum_{l \geq \nu} \frac{\lambda^l}{(2\pi)^l} T^{(\nu, l)}_{ab}(\gamma(\psi; r); r) \right],
$$

where $\epsilon = 2 - d$ is the dimensional regularization parameter, $\mu$ is a scale associated with the dimensional regularization, and $\lambda$ is a loop counter ($\lambda \propto \hbar$). The first loop contributions to the bare metric can be computed by using the formulas in Osborn’s paper [13, 13]. Thus, the 1-loop and 2-loops contributions turn out to be, respectively,

$$
T^{(1,1)}_{ab}(\gamma(\psi; r); r) = R_{ab}(\psi; r) = -\frac{1}{r} \gamma_{ab}(\psi; r),
$$

$$
T^{(1,2)}_{ab}(\gamma(\psi; r); r) = \frac{1}{4} R_{acde}(\psi; r) R_{b}^{\ cde}(\psi; r) = \frac{1}{2r^2} \gamma_{ab}(\psi; r).
$$

Actually, owing to the fact that the target manifold is a space of ($r$-dependent) constant curvature, all the loop contributions take the simple form

$$
T^{(1, l)}_{ab}(\gamma(\psi; r); r) = \frac{C_l}{r^l} \gamma_{ab}(\psi; r) \quad (C_l \text{ constant}).
$$

As a consequence, the renormalized metric of the sigma model can be written as

$$
\gamma^B_{ab}(\psi; r) = \mu^{-\epsilon} \left[ 1 + \frac{1}{\epsilon} H_1(r) + \frac{1}{\epsilon^2} H_2(r) + \cdots \right] \gamma_{ab}(\psi; r) = H(r) \gamma_{ab}(\psi; r),
$$

where $H_i(r)$ are suitable functions.
where
\[ H_1(r) = -\frac{\lambda}{2\pi r} + \frac{\lambda^2}{8\pi^2 r^2} + O\left(\frac{\lambda^3}{r^3}\right). \]

Explicit formulas for the other functions \( H_i(r) \) can be calculated using Feynman diagrams techniques.

From the relation \( \gamma^B_{ab}(\psi;r) = H(r)\gamma_{ab}(\psi;r) \) between the bare metric and the original one, where \( H(r) \) is a non-constant function, we conclude that the auxiliary \( H_0 \) model defined in Eq. (4) is not renormalizable in the standard sense. This is just a reflection of the severe problems that arise in any perturbative approach to the quantization of GR. The usual perturbative techniques of quantum field theory cannot be successfully implemented even within the “simple” framework provided by the vacuum cylindrically symmetric waves. On the other hand, although it is possible to carry out a non-perturbative quantization of this family of waves [16], the physics behind such a quantization is really difficult to extract, at least as far as the space-time metric is concerned, and cannot be interpreted in a standard perturbative way because of the non-renormalizability of the model.

The kind of non-renormalizability that affects vacuum cylindrical gravity has been studied by Niedermaier [9]. The key idea is that action (4) belongs to a wider class of \( r \)-dependent sigma models parameterized by an arbitrary function \( h(r) \). In our case, one has to consider the \( h \)-family of actions
\[
\mathcal{S}^h_\sigma[\psi, \phi] = \int_{T_1}^{T_2} \int_0^\infty h(r) \left[ \left( \dot{\psi}^2 - \psi'^2 \right) + \frac{e^{2\psi}}{r^2} \left( \dot{\phi}^2 - \phi'^2 \right) \right] drdT. \tag{5}
\]

The studied case of cylindrical waves with general polarization and the gauge fixing (2) would correspond to \( h(r) = r \). The metric of the target space and its Riemann tensor become now \( h \)-dependent, namely
\[
\gamma^h_{ab}(\psi;r) := h(r)\gamma^0_{ab}(\psi;r) = h(r) \begin{pmatrix} 1 & 0 \\ 0 & e^{2\psi}/r^2 \end{pmatrix},
\]
\[
\mathcal{R}^h_{abcd} = \frac{1}{h} \left( \gamma^h_{ad}\gamma^h_{bc} - \gamma^h_{ac}\gamma^h_{bd} \right).
\]

Given the special dependence on \( h \), action (5) defines a maximally symmetric \( r \)-dependent sigma model. Moreover, the presence of an adjustable function \( h \) in the family
of actions (5) preserves renormalizability in this broader sense suggested by Niedermaier and supports the asymptotic safety scenario. If we compute the bare metric using the formulas of Ref. [13],

\[
\begin{align*}
(\gamma^h)^B_{ab} &= \mu \varepsilon \left[ 1 + \frac{1}{\varepsilon} \left( -\frac{\lambda}{2\pi h} + \frac{\lambda^2}{8\pi^2 h^2} + \cdots \right) + \cdots \right] \gamma^h_{ab} \\
&= \mu \varepsilon \left[ 1 + \frac{1}{\varepsilon} \left( -\frac{\lambda}{2\pi h} + \frac{\lambda^2}{8\pi^2 h^2} + \cdots \right) + \cdots \right] h \gamma^0_{ab} = h^B \gamma^0_{ab} = \gamma^h_{ab},
\end{align*}
\]

we realize that the renormalization of the metric \(\gamma^h_{ab}\) amounts to a renormalization of the function \(h\), so that we end up with an element of the same family of actions.

4. CONCLUSIONS

Parallel to the situation found for Einstein-Rosen waves, we have seen that two Hamiltonians emerge in the analysis of the general cylindrically symmetric reduction of GR. The physical Hamiltonian \(E(H_0)\) is a non-linear function of an auxiliary Hamiltonian \(H_0\), defined in terms of a local density. Nonetheless, in contrast with the relative simplicity of the linearly polarized case where \(H_0\) corresponds to a free scalar field, the auxiliary Hamiltonian \(H_0\) describes now a theory of two interacting fields. Moreover, this theory is not renormalizable in the usual sense. Owing to this non-renormalizability, the standard perturbative approach fails in the attempt to construct a satisfactory quantum field theory.

Our results agree with those of Niedermaier [9], and in this sense our discussion can be seen as lending further support to his arguments. In this respect, although the two types of approaches (Niedermaier’s and ours) lead to similar conclusions, there exist some relevant differences that make this analysis complementary to a certain extent. A specific feature of our approach is that we have carefully included the surface terms needed for the differentiability of the action from the very beginning. These boundary terms supply (after fixing the time gauge, with unit lapse at spatial infinity) a physical Hamiltonian \(E(H_0)\) that is a complicated function of the local, auxiliary Hamiltonian \(H_0\). It is only the latter of these Hamiltonians that can be treated with standard sigma model techniques. In addition, an advantage of our approach is that, using path integral methods and Fadeev-Popov determinants, we have been able to fix the gauge freedom of the system without introducing ambiguities in the action. The outcome is that the auxiliary Lagrangian
which determines the sigma model acquires a much simpler form, compared with that considered in Ref. [9]. This considerably simplifies all the calculations involved. Furthermore, the rationale presented here seems easy to generalize to other midi-superspace models of interest, like e.g. the family of Gowdy cosmological solutions with two polarizations and the spatial topology of a three-torus [8, 17]. For this family of solutions, one would expect a situation very similar to that encountered for cylindrical gravity, except for the interchange of the roles played in the $x$-dependent sigma model by the time and the spatial coordinate of the base manifold.

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These formulas have been widely used in the quantum field theory literature. Strictly speaking, they are deduced by computing the determinant of an elliptic operator defined on the base manifold, and their numerical coefficients may depend on the topology. Although our base manifold is not compact, we assume that the standard formulas for the loop-pole coefficients apply as well to our case.

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