QUANTIZATION OF SLODOWY SLICES

WEE LIANG GAN AND VICTOR GINZBURG

Abstract. We give a direct proof of (a slight generalization of) the recent result of Premet related to generalized Gelfand-Graev representations and of an equivalence due to Skryabin.

1. Introduction

1.1. Let \( \mathfrak{g} \) be a complex semisimple Lie algebra, let \( G \) be the adjoint group of \( \mathfrak{g} \), let \( \mathcal{O} \) be a nonzero nilpotent \( \text{Ad} G \)-orbit in \( \mathfrak{g} \), and let \( e \in \mathcal{O} \). By the Jacobson-Morozov Theorem, there is an \( \mathfrak{sl}_2 \)-triple \( (e, h, f) \) associated to \( e \), i.e. \( [h, e] = 2e \), \( [h, f] = -2f \), \( [e, f] = h \). Fix such an \( \mathfrak{sl}_2 \)-triple. The Slodowy slice to \( \mathcal{O} \) at \( e \) is defined to be the affine space \( e + \ker \text{ad} f \), see e.g. [Sl] §7.4. The same slice has been used already by Harish-Chandra [HC] §12-14, see also [Br].

Since the Killing form on \( \mathfrak{g} \) is nondegenerate, there is an isomorphism \( \Phi : \mathfrak{g} \to \mathfrak{g}^* \) such that \( \langle \Phi(e), f \rangle = 1 \). Let \( \chi = \Phi(e) \) and \( \mathcal{S} = \Phi(e + \ker \text{ad} f) \). We will show in [G] that \( \mathcal{S} \) has a natural Poisson structure. The aim of this paper is to construct a quantization of \( \mathcal{S} \).

1.2. Under the action of \( \text{ad} h \), we have a decomposition \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i) \), where

\[ \mathfrak{g}(i) = \{ x \in \mathfrak{g} \mid [h, x] = ix \}. \]

Note that there is a nondegenerate skew-symmetric bilinear form \( \omega \) on \( \mathfrak{g}(-1) \) defined by \( \omega(x, y) = \chi([x, y]) \) for all \( x, y \in \mathfrak{g}(-1) \). Fix an isotropic subspace \( \ell \) of \( \mathfrak{g}(-1) \), and denote by \( \ell^\perp \subset \mathfrak{g}(-1) \) the annihilator of \( \ell \) with respect to \( \omega \). Let

\[ \mathfrak{m}_\ell = \ell \bigoplus (\bigoplus_{i \leq -2} \mathfrak{g}(i)) \quad \text{and} \quad \mathfrak{n}_\ell = \ell^\perp \bigoplus (\bigoplus_{i \leq -2} \mathfrak{g}(i)). \]

Note that \( \mathfrak{m}_\ell \subset \mathfrak{n}_\ell \) and they are both nilpotent Lie subalgebras of \( \mathfrak{g} \). Also, \( \chi \) restricts to a character on \( \mathfrak{m}_\ell \).

Let \( U_\mathfrak{g} \) and \( U\mathfrak{m}_\ell \) be the universal enveloping algebras of \( \mathfrak{g} \) and \( \mathfrak{m}_\ell \) respectively. Denote by \( \mathbb{C}_\chi \) the 1-dimensional left \( U\mathfrak{m}_\ell \)-module obtained from the character \( \chi \) of \( \mathfrak{m}_\ell \), and let \( Q_\ell = U_\mathfrak{g} \otimes_{U\mathfrak{m}_\ell} \mathbb{C}_\chi \) be the induced left \( U\mathfrak{g} \)-module, equivalently, the quotient of \( U_\mathfrak{g} \) by the left ideal \( I_\ell \) generated by \( x - \chi(x) \), for all \( x \in \mathfrak{m}_\ell \). Now consider the unique extension of the adjoint action of \( \mathfrak{n}_\ell \) on \( \mathfrak{g} \) to derivations on \( U_\mathfrak{g} \). Note that \( I_\ell \) is stable under this action of \( \mathfrak{n}_\ell \) because if \( x \in \mathfrak{m}_\ell \) and \( n \in \mathfrak{n}_\ell \), then \( (x - \chi(x))m = n(x - \chi(x)) + [x, n] \), and \( \chi([x, n]) = 0 \). Thus, there is an induced \( \text{ad} \mathfrak{n}_\ell \)-action on \( Q_\ell \).

Let \( H_\ell = Q_\ell^{ad \mathfrak{n}_\ell} \) be the subspace of all \( x + I_\ell \subset Q_\ell \) such that \( nx - xn \in I_\ell \) for all \( n \in \mathfrak{n}_\ell \). Take any \( x + I_\ell, y + I_\ell \subset H_\ell \). We define an algebra structure on \( H_\ell \) by \( (x + I_\ell)(y + I_\ell) = xy + I_\ell \). We claim that this multiplication is well defined. To see this, we note that for any \( m \in \mathfrak{m}_\ell \) and \( y + I_\ell \in H_\ell \), by the definition of \( H_\ell \) we have: \( [m, y] \in [\mathfrak{m}_\ell, y] \subset \mathfrak{m}_\ell, y] \subset I_\ell \). Hence, we find:

\[ (m - \chi(m))y = ym - y\chi(m) + [m, y] = y(m - \chi(m)) \in I_\ell \subset I_\ell \].

It follows that \( I_\ell \cdot y \subset I_\ell \), and our claim is proved. Further, it is clear that \( H_\ell \) is closed under the multiplication, because if \( n \in \mathfrak{n}_\ell \), then \( nxy - yxn = (nx - xn)y + x(ny - yn) \in I_\ell \).
We will define in §4 the Kazhdan grading on $\mathbb{C}[S]$ and the Kazhdan filtration on $H_\ell$. We will then prove in §5 that $gr H_\ell$ is canonically isomorphic to $\mathbb{C}[S]$ as graded Poisson algebras, and $H_\ell$ is independent of the choice of $\ell$. Our proof is based on generalizations of some of Kostant’s results in [Ko], in which he considered the case when $e$ is a principal nilpotent element. Kostant’s results were generalized by Lynch [Ly] to the setup of ‘admissible’ parabolic subalgebras, which include our results in the special case of an even nilpotent element $e$. In particular, our proof of Proposition 4.2 below is very similar to an argument in [Ly], see also [Ko]. However, we believe that an $\mathfrak{sl}_2$-triple setting considered in the present paper is more natural than that of admissible parabolic subalgebras considered in [Ly], making all results much ‘cleaner’.

1.3. This work was inspired by Premet [Pr]. He proved the isomorphism between $gr H_\ell$ and $\mathbb{C}[S]$ in the case when $\ell$ is Lagrangian. His proof is based on results over algebraically closed fields of positive characteristics. Similar results were also obtained in [BT] using BRST cohomology. Recall that in the Lagrangian case, $Q_\ell$ is called a generalized Gelfand-Graev representation associated to $e$ (c.f. e.g. [Ka], [Ma], [Mœ], or [Ya]), we have an algebra isomorphism $\text{End}_{\mathfrak{g}}(Q_\ell)^{\text{op}} \to H_\ell : h \mapsto h(1 \otimes 1)$, and $H_\ell$ may be identified with the space of Whittaker vectors $\text{Wh}(Q_\ell) = \{v \in Q_\ell \mid xv = \chi(x)v, \forall x \in \mathfrak{m}_\ell\}$.

2. A decomposition lemma

2.1. It will be useful to define a linear action of $\mathbb{C}^*$ on $\mathfrak{g}$ which stabilizes $e + \text{Ker ad } f$. First, consider the Lie algebra homomorphism $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ defined by

$$
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \mapsto e, \quad \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \mapsto h, \quad \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} \mapsto f.
$$

This Lie algebra homomorphism exponentiates to a rational homomorphism $\tilde{\gamma} : \text{SL}_2(\mathbb{C}) \to G$. We put

$$
\gamma : \mathbb{C}^* \to G, \quad \gamma(t) = \tilde{\gamma} \begin{pmatrix}
t & 0 \\
0 & t^{-1}
\end{pmatrix}, \quad \forall t \in \mathbb{C}^*.
$$

Note that $(\text{Ad } \gamma(t))(e) = t^2 \cdot e$. The desired action of $\mathbb{C}^*$ on $\mathfrak{g}$, to be denoted by $\rho$, is defined by $\rho(t)(x) = t^2 \cdot (\text{Ad } \gamma(t^{-1}))(x)$, for all $x \in \mathfrak{g}$. Note that $\rho(t)(e + x) = e + \rho(t)(x)$. Thus, since $\rho(t)$ stabilizes $\text{Ker ad } f$, it also stabilizes $e + \text{Ker ad } f$. Note that $\lim_{t \to 0} \rho(t)(x) = e$, for all $x \in e + \text{Ker ad } f$, i.e. the $\mathbb{C}^*$-action on $e + \text{Ker ad } f$ is contracting.

2.2. Recall that the intersections of $e + \text{Ker ad } f$ with $\text{Ad } G$-orbits in $\mathfrak{g}$ are transversal. This is clear at $e$ since $\mathfrak{g} = [\mathfrak{g}, e] \oplus (\text{Ker ad } f)$. Thus, the adjoint action map $G \times (e + \text{Ker ad } f) \to \mathfrak{g}$ has a surjective differential at each point in some open neighborhood of $(1, e)$. It follows that at each point in some open neighborhood of $e$ in $e + \text{Ker ad } f$, the intersection of $e + \text{Ker ad } f$ with $\text{Ad } G$-orbits is transversal. By the contracting $\mathbb{C}^*$-action on $e + \text{Ker ad } f$, it follows that the same is true at all points of $e + \text{Ker ad } f$.

2.3. Let $N_\ell$ be the unipotent subgroup of $G$ with Lie algebra $\mathfrak{n}_\ell$, and let $\mathfrak{m}_\ell^\perp \subset \mathfrak{g}^*$ be the annihilator of $\mathfrak{m}_\ell$. The following key lemma is a generalization of [Ko] Theorem 1.2.

**Lemma 2.1.** The coadjoint action map $\alpha : N_\ell \times \mathcal{S} \to \chi + \mathfrak{m}_\ell^\perp$ is an isomorphism of affine varieties.
Proof. Given a subspace $V \subset \mathfrak{g}$, we will write $V^\perp$ for the annihilator of $V$ in $\mathfrak{g}$ (as opposed to $V^\perp$, the annihilator in $\mathfrak{g}^*$) with respect to the Killing form on $\mathfrak{g}$. The statement of the Lemma is equivalent to saying that the adjoint action map

$$\alpha : N_\ell \times (e + \text{Ker } f) \to e + \mathfrak{m}_\ell^\perp$$

is an isomorphism. First, note that $\mathfrak{sl}_2$ representation theory implies: $[\mathfrak{n}_\ell, e] \cap \text{Ker } f = 0$, and that the map $\mathfrak{n}_\ell \to [\mathfrak{n}_\ell, e]$, $x \mapsto [x, e]$, is a bijection. It follows easily that $\dim \mathfrak{m}_\ell^\perp = \dim \mathfrak{n}_\ell + \dim \mathfrak{g}(0) + \dim \mathfrak{g}(-1) = \dim [\mathfrak{n}_\ell, e] + \dim \text{Ker } f$. Thus, there is a direct sum decomposition

$$(2.2) \quad \mathfrak{m}_\ell^\perp = [\mathfrak{n}_\ell, e] \oplus \text{Ker } f$$

Next, define a $\mathbb{C}^*$-action on $N_\ell \times (e + \text{Ker } f)$ by

$$t \cdot (g, x) = (\gamma(t^{-1})g\gamma(t), \rho(t)(x)).$$

Note that for any $(g, x) \in N_\ell \times (e + \text{Ker } f)$, we have $\lim_{t \to 0} t \cdot (g, x) = (1, e)$. Also, the action of $\mathbb{C}^*$ on $e + \mathfrak{m}_\ell^\perp$ satisfies $\lim_{t \to 0} \rho(t)(x) = e$, for any $x \in e + \mathfrak{m}_\ell^\perp$. The action map $\alpha$ is $\mathbb{C}^*$-equivariant. Moreover, by $(2.2)$, $\alpha$ induces an isomorphism between the tangent spaces of the $\mathbb{C}^*$-fixed points $(1, e)$ and $e$. Thus, Lemma 2.1 follows from the following general result: An equivariant morphism $\alpha : X_1 \to X_2$ of smooth affine $\mathbb{C}^*$-varieties with contracting $\mathbb{C}^*$-actions which induces an isomorphism between the tangent spaces of the $\mathbb{C}^*$-fixed points must be an isomorphism.

To prove this, let $x_i$ be the $\mathbb{C}^*$-fixed point of $X_i$, $i = 1, 2$, let $T_i$ be the tangent space of $X_i$ at $x_i$, and write $\chi_{X_i} \in \mathbb{C}[\![t]\!]$ for the formal character of the coordinate ring of a $\mathbb{C}^*$-variety $X$. Since $\alpha$ induces an isomorphism $T_1 \cong T_2$, the pullback on coordinate rings $\alpha^* : \mathbb{C}[X_2] \to \mathbb{C}[X_1]$ is injective. The surjectivity of $\alpha^*$ follows from the equation: $\chi_{X_1} = \chi_{T_1} = \chi_{T_2} = \chi_{X_2}$, see [Gi] (7.7).

Remark. Even in the case of $e$ being the principal nilpotent, the proof above is much simpler than that of [Ko] Theorem 1.2. There is also an alternative proof of Lemma 2.1 based on an inductive argument similar to one used by Lynch [Ly].

3. Poisson structure on $S$

3.1. The space $\mathfrak{g}^*$ has a natural Poisson structure defined by

$$\{F_1, F_2\}(\xi) = \xi([dF_1(\xi), dF_2(\xi)]),$$

where $F_1, F_2 \in \mathbb{C}[\mathfrak{g}^*]$ and $\xi \in \mathfrak{g}^*$. The symplectic leaves of $\mathfrak{g}^*$ are the $\text{Ad}^* G$-orbits. From [22], we know that $S$ intersects the symplectic leaves transversally. Thus, to show that $S$ inherits a Poisson structure from that of $\mathfrak{g}^*$, it suffices (by [Va] Proposition 3.10) to verify that for any $\text{Ad}^* G$-orbit $O$ and $\xi \in O \cap S$, the restriction of the symplectic form on $T_\xi O$ to $T_\xi S \cap T_\xi O = T_\xi (S \cap O)$ is nondegenerate. Note that $T_\xi S = \Phi(\text{Ker } f)$, and the annihilator of $\Phi(\text{Ker } f)$ in $\mathfrak{g}$ is $[f, \mathfrak{g}]$. Thus, the null space of the restriction of the symplectic form to $T_\xi S \cap T_\xi O$ is

$$\Phi \left( \Phi^{-1}(\xi), [f, \mathfrak{g}] \cap \text{Ker } f \right).$$

This is 0 since $\Phi^{-1}(\xi) \in (e + \text{Ker } f)$. 


More details on the statements made above are provided in [7].

3.2. The Poisson structure on $\mathcal{S}$ can also be described via Hamiltonian reduction. For this, we will take $\ell$ to be a Lagrangian subspace of $\mathfrak{g}(-1)$ in $\S[3.2]$. Then $\mathfrak{m}_\ell = \mathfrak{n}_\ell$, and we denote both of them by $\mathfrak{m}$. Let $M$ be the unipotent subgroup of $G$ with Lie algebra $\mathfrak{m}$. The moment map $\mu : \mathfrak{g}^* \to \mathfrak{m}^*$ for the coadjoint action of $M$ on $\mathfrak{g}^*$ is just the restriction of functions from $\mathfrak{g}$ to $\mathfrak{m}$. Note that $\mu^{-1}(\chi|_\mathfrak{m}) = \chi + \mathfrak{m}^\perp$, where $\mathfrak{m}^\perp \subset \mathfrak{g}^*$ is the annihilator of $\mathfrak{m}$. Since $\chi|_\mathfrak{m}$ is a character on $\mathfrak{m}$, it is fixed under the coadjoint action of $M$. Moreover, $\mathfrak{e} + \mathfrak{m}^\perp$ is transversal to the $\text{Ad}G$-orbits in $\mathfrak{g}$. This is clearly true at $e$, hence locally around $e$, and hence it is true everywhere using the contracting $\mathbb{C}^*$-action $\rho$ (c.f. [2.2]). The transversality just proved implies in particular that, for any $\xi \in \mathfrak{g}$, we have $\mathfrak{g} = [\mathfrak{g}, \xi] + \mathfrak{m}^\perp$. It follows that $\chi|_\mathfrak{m}$ is a regular value for the restriction of $\mu$ to each symplectic leaf of $\mathfrak{g}^*$. Thus, by Lemma 2.1 we have a Hamiltonian reduction of the Poisson structure of $\mathfrak{g}^*$ to a Poisson structure on $\mathfrak{S}$ (c.f. [4a] Theorem 7.31). We remark that the symplectic form on each symplectic leaf of $\mathfrak{S}$ is obtained by symplectic reduction of the corresponding symplectic leaf of $\mathfrak{g}^*$. From this, and by the canonical embedding of $\mathcal{S}$ into $\chi + \mathfrak{m}^\perp$, it is easy to see that the Poisson structures on $\mathcal{S}$ defined in Sections 3.1 and 3.2 are the same. Moreover, the Poisson bracket $\{F_1, F_2\}$ for any $F_1, F_2 \in \mathbb{C}[[\mathcal{S}]]$ may be described as follows. Let $\pi : (\chi + \mathfrak{m}^\perp) \to (\chi + \mathfrak{m}^\perp)/\text{Ad}^*\mathfrak{m} \simeq \mathcal{S}$ be the projection map. Take an arbitrary extension $\tilde{F}_1$ of $F_1 \circ \pi$ to $\mathfrak{g}^*$, and an arbitrary extension $\tilde{F}_2$ of $F_2 \circ \pi$ to $\mathfrak{g}^*$. Then $\{F_1, F_2\} \circ \pi = \{\tilde{F}_1, \tilde{F}_2\} \circ \iota$, where $\iota : (\chi + \mathfrak{m}^\perp) \hookrightarrow \mathfrak{g}^*$.

4. Kazhdan grading and filtration

4.1. Now let us define a linear action $\rho^\sharp$ of $\mathbb{C}^*$ on $\mathfrak{g}^*$ such that $\rho^\sharp$ stabilizes $\mathcal{S}$. This action is defined by $\rho^\sharp(t)(\xi) = t^{-2}\text{Ad}^*(\gamma(t))(\xi)$ for all $\xi \in \mathfrak{g}^*$. If $\xi \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$, then $(\rho^\sharp(t)x, x) = t^{-4}(\xi, [\xi, x])$.

Note that we have an induced action on $\mathbb{C}[[\mathfrak{g}^*]]$ defined by $(\rho^\sharp(t)F)(\xi) = F(\rho^\sharp(t)^{-1}(\xi))$, where $F \in \mathbb{C}[[\mathfrak{g}^*]]$. The decomposition $\mathbb{C}[[\mathfrak{g}^*]] = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}[[\mathfrak{g}^*]](n)$, where

$$\mathbb{C}[[\mathfrak{g}^*]](n) = \{F \in \mathbb{C}[[\mathfrak{g}^*]] \mid \rho^\sharp(t)(F) = t^nF, \quad \forall t \in \mathbb{C}^*\},$$

gives $\mathbb{C}[[\mathfrak{g}^*]]$ the structure of a graded algebra. We call this the Kazhdan grading on $\mathbb{C}[[\mathfrak{g}^*]]$.

Similarly, since $\mathbb{C}^*$ acts on $\mathfrak{S}$ via $\rho^\sharp$, we may also speak of the Kazhdan grading on $\mathbb{C}[[\mathcal{S}]]$. Note however that the weights of $\rho^\sharp$ on $\Phi(\text{Ker ad} f)$ are negative integers. Thus, the Kazhdan grading on $\mathbb{C}[[\mathcal{S}]]$ has no negative graded components.

4.2. Let $\mathbb{S}\mathfrak{g}$ be the symmetric algebra of $\mathfrak{g}$. Identify $\mathbb{S}\mathfrak{g}$ with $\mathbb{C}[[\mathfrak{g}^*]]$. From this identification, $\mathbb{S}\mathfrak{g}$ acquires a graded algebra structure from the Kazhdan grading on $\mathbb{C}[[\mathfrak{g}^*]]$. This grading on $\mathbb{S}\mathfrak{g}$ may be described explicitly as follows. Let $\mathbb{S}\mathfrak{g} = \bigoplus_{n \geq 0} \mathbb{S}^n\mathfrak{g}$ be the standard grading of $\mathbb{S}\mathfrak{g}$. The action of ad $h$ on $\mathfrak{g}$ extends uniquely to a derivation on $\mathbb{S}\mathfrak{g}$. For any $i \in \mathbb{Z}$, let

$$(\mathbb{S}^n\mathfrak{g})(i) = \{x \in \mathbb{S}^n\mathfrak{g} \mid (\text{ad} h)(x) = ix\}.$$ 

The Kazhdan grading $\mathbb{S}\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} (\mathbb{S}\mathfrak{g})[n]$ is defined by letting $(\mathbb{S}\mathfrak{g})[n]$ be the subspace of $\mathbb{S}\mathfrak{g}$ spanned by all $(\mathbb{S}^j\mathfrak{g})(i)$ with $i + 2j = n$.

We will define a filtration on $U\mathfrak{g}$ such that the associated graded algebra is $\mathbb{S}\mathfrak{g}$ with the Kazhdan grading. Consider the standard filtration $U_0\mathfrak{g} \subset U_1\mathfrak{g} \subset \ldots \subset U_n\mathfrak{g} \subset \ldots$ of $U\mathfrak{g}$. The
action of of \(\text{ad} \, h\) on \(\mathfrak{g}\) extends uniquely to a derivation on \(U\mathfrak{g}\) which we will also denote by \(\text{ad} \, h\). For any \(i \in \mathbb{Z}\), let
\[
(U_n \mathfrak{g})(i) = \{ x \in U_n \mathfrak{g} \mid (\text{ad} \, h)(x) = ix \}.
\]

The **Kazhdan filtration**, \(\ldots \subset F_n U \mathfrak{g} \subset F_{n+1} U \mathfrak{g} \subset \ldots\), of \(U \mathfrak{g}\) is a \(\mathbb{Z}\)-filtration defined by letting \(F_n U \mathfrak{g}\) be the subspace of \(U \mathfrak{g}\) spanned by all \((U_j \mathfrak{g})(i)\) with \(i + 2j \leq n\). The Kazhdan filtration gives \(U \mathfrak{g}\) the structure of a filtered algebra. Observe that if \(x \in F_n U \mathfrak{g}\) and \(y \in F_m U \mathfrak{g}\), then \(xy - yx \in F_{n+m-2} U \mathfrak{g}\). Further, for any \(x \in \mathfrak{g}(n)\), \(y \in \mathfrak{g}(m)\), we have \(x \in F_{n+2} U \mathfrak{g}, y \in F_{m+2} U \mathfrak{g}\) and, moreover, the classes of \(x\) and \(y\) in \(\text{gr} \, U \mathfrak{g}\) commute. It follows that the natural map \(\mathfrak{g} \to \text{gr} \, U \mathfrak{g}\) extends uniquely to a well-defined graded algebra homomorphism: \(S \mathfrak{g} \to \text{gr} \, U \mathfrak{g}\). Moreover, the Poincaré-Birkhoff-Witt Theorem implies easily that this homomorphism is a bijection.

Note further that the canonical Poisson bracket on \(\text{gr} \, U \mathfrak{g}\) defined by \(\{ \mathfrak{g}_n \, x, \mathfrak{g}_m \, y \}\) corresponds, under the isomorphisms \(\text{gr} \, U \mathfrak{g} = S \mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]\), to the standard Poisson structure on \(\mathbb{C}^*\) (it suffices to check this on linear functions).

4.3. Let \(p : U \mathfrak{g} \to Q_\ell\) be the quotient map. The Kazhdan filtration on \(Q_\ell\) is defined by \(F_n Q_\ell = p(F_n U \mathfrak{g})\). This gives \(Q_\ell\) the structure of a filtered module over \(U \mathfrak{g}\). Note that \(\text{gr} \, Q_\ell\) has a commutative algebra structure since \(\text{gr} \, Q_\ell = \text{gr} \, U \mathfrak{g}/I_\ell\), where \(I_\ell\) is the left ideal generated by \(x - \chi(x)\), for all \(x \in \mathfrak{m}_\ell\). Further, \(F_n Q_\ell = 0\) for all \(n < 0\). The associated graded map \(\text{gr} \, p : S \mathfrak{g} \to \text{gr} \, Q_\ell\) is a surjective homomorphism of graded algebras whose kernel is the ideal generated by \(x - \chi(x)\) for all \(x \in \mathfrak{m}_\ell\). Under the identification of \(S \mathfrak{g}\) with \(\mathbb{C}[\mathfrak{g}^*]\), we see that \(\ker \, \text{gr} \, p\) is the ideal of all polynomial functions on \(\mathfrak{g}^*\) vanishing on \(\chi + \mathfrak{m}_\ell^{\perp}\). Since \(\mathfrak{m}_\ell^{\perp}\) is stable under the action of \(\mathbb{C}^*\) via \(\rho^\ell\), there is a Kazhdan grading on \(\mathbb{C}[\chi + \mathfrak{m}_\ell^{\perp}]\), and \(\text{gr} \, Q_\ell\) may be identified with \(\mathbb{C}[\chi + \mathfrak{m}_\ell^{\perp}]\) as graded algebras. Note that the weights of \(\rho^\ell\) on \(\mathfrak{m}_\ell^{\perp}\) are negative integers, which agrees with the fact that the Kazhdan grading on \(\text{gr} \, Q_\ell\) has no negative component.

4.4. The Kazhdan filtration on \(H_\ell\) is induced from the Kazhdan filtration on \(Q_\ell\) via the inclusion \(H_\ell \hookrightarrow Q_\ell\). Note that \(\text{gr} \, H_\ell \hookrightarrow \text{gr} \, Q_\ell\) is an injective homomorphism of graded algebras. Since \(\Phi(\ker \, \text{ad} \, f) \subset \mathfrak{m}_\ell^{\perp}\), we have a restriction homomorphism \(\nu : \mathbb{C}[\chi + \mathfrak{m}_\ell^{\perp}] \to \mathbb{C}[S]\). Recall that \(\text{gr} \, Q_\ell\) is identified with \(\mathbb{C}[\chi + \mathfrak{m}_\ell^{\perp}]\). Thus, there is a canonical homomorphism of graded algebras \(\nu : \text{gr} \, H_\ell \to \mathbb{C}[S]\).

Our goal is to give a simple direct proof of the following theorem.

**Theorem 4.1.** The canonical homomorphism \(\nu : \text{gr} \, H_\ell \to \mathbb{C}[S]\) is an isomorphism of graded Poisson algebras. Moreover, \(H_\ell\) is independent of the choice of an isotropic subspace \(\ell \subset \mathfrak{g}(-1)\).

5. \(n_\ell\)-cohomology of \(\text{gr} \, Q_\ell\) and \(Q_\ell\)

5.1. From now on, we will regard \(U \mathfrak{g}\) and \(Q_\ell\) as \(n_\ell\)-modules via the adjoint \(n_\ell\)-action. Thus, \(H_\ell = H^0(n_\ell, \mathfrak{g})\). Note that the map \(p : U \mathfrak{g} \to Q_\ell\) is \(n_\ell\)-equivariant.

Next, note that \(n_\ell\) is a graded subalgebra of \(\mathfrak{g}\), and hence it is also filtered. Clearly, \(U \mathfrak{g}\) and \(Q_\ell\) are Kazhdan filtered \(n_\ell\)-modules. Thus, \(\text{gr} \, U \mathfrak{g}\) and \(\text{gr} \, Q_\ell\) acquire the structure of Kazhdan graded \(n_\ell\)-modules, and \(\text{gr} \, p : \text{gr} \, U \mathfrak{g} \to \text{gr} \, Q_\ell\) is \(n_\ell\)-equivariant. The claim that \(\nu : \text{gr} \, H_\ell \to \mathbb{C}[S]\) is an isomorphism follows immediately from the following two propositions.
Proposition 5.1. $\nu : H^0(\mathfrak{n}_\ell, \text{gr } Q_\ell) \to \mathbb{C}[S]$ is an isomorphism, and $H^i(\mathfrak{n}_\ell, \text{gr } Q_\ell) = 0$ for all $i > 0$.

Proposition 5.2. $\text{gr } H^0(\mathfrak{n}_\ell, Q_\ell) = H^0(\mathfrak{n}_\ell, \text{gr } Q_\ell)$, and $H^i(\mathfrak{n}_\ell, Q_\ell) = 0$ for all $i > 0$.

5.2. To prove the above propositions, let us describe an action of $\mathfrak{n}_\ell$ on $\mathbb{C}[\chi + m_\ell^+]$ so that the identification of $\text{gr } Q_\ell$ with $\mathbb{C}[\chi + m_\ell^+]$ becomes also an identification of Kazhdan graded $\mathfrak{n}_\ell$-modules.

First, note that the adjoint action of $\mathfrak{n}_\ell$ on $\mathfrak{g}$ extends uniquely to derivations on $S\mathfrak{g}$. We have $\text{gr } U\mathfrak{g} = S\mathfrak{g}$ as Kazhdan graded $\text{ad } \mathfrak{n}_\ell$-modules. Note also that the Ad$^*$ $N_\ell$-action on $\mathfrak{g}^*$ induces an $N_\ell$-action on $\mathbb{C}[\mathfrak{g}^*]$. This $N_\ell$-action on $\mathbb{C}[\mathfrak{g}^*]$ is locally finite, so there is an infinitesimally induced action of $\mathfrak{n}_\ell$ on $\mathbb{C}[\mathfrak{g}^*]$. We have $S\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]$ as Kazhdan graded $\text{ad } \mathfrak{n}_\ell$-modules. Since $\chi + m_\ell^+$ is stable under the Ad$^*$ $N_\ell$-action, we also have an induced $N_\ell$-action on $\mathbb{C}[\chi + m_\ell^+]$, and hence also an infinitesimal action of $\mathfrak{n}_\ell$ on $\mathbb{C}[\chi + m_\ell^+]$. Clearly, the restriction homomorphism $\mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[\chi + m_\ell^+]$ is $\mathfrak{n}_\ell$-equivariant. Hence, $\text{gr } Q_\ell$ and $\mathbb{C}[\chi + m_\ell^+]$ are identified as $\mathfrak{n}_\ell$-modules.

5.3. We now prove Proposition 5.1. Note that if we let $N_\ell$ act on $N_\ell \times S$ by left translations on $N_\ell$, then $\alpha$ is $N_\ell$-equivariant. We have an $N_\ell$-module structure on $\mathbb{C}[N_\ell]$ defined by $(g \cdot F)(x) = F(g^{-1}x)$, where $g, x \in N_\ell$ and $F \in \mathbb{C}[N_\ell]$. Lemma 2.1 implies that $\mathbb{C}[N_\ell] \otimes \mathbb{C}[S] \cong \mathbb{C}[\chi + m_\ell^+]$ as algebras and $N_\ell$-modules. Since $H^0(\mathfrak{n}_\ell, \mathbb{C}[\chi + m_\ell^+])$ is precisely the $N_\ell$-invariants $\mathbb{C}[\chi + m_\ell^+]^{N_\ell}$, it follows that

$$H^0(\mathfrak{n}_\ell, \text{gr } Q_\ell) = H^0(\mathfrak{n}_\ell, \mathbb{C}[\chi + m_\ell^+]) = \mathbb{C}[\chi + m_\ell^+]^{N_\ell} \xrightarrow{\psi} \mathbb{C}[S].$$

Also, for $i > 0$,

$$H^i(\mathfrak{n}_\ell, \text{gr } Q_\ell) = H^i(\mathfrak{n}_\ell, \mathbb{C}[\chi + m_\ell^+]) = H^i(\mathfrak{n}_\ell, \mathbb{C}[N_\ell]) \otimes \mathbb{C}[S] = 0,$$

where the last equality follows from the fact that the standard cochain complex for Lie algebra cohomology with coefficients in $\mathbb{C}[N_\ell]$ is just the algebraic de Rham complex for $N_\ell$, and $N_\ell$ is isomorphic to an affine space, hence has trivial de Rham cohomology. This completes the proof of Proposition 5.1.

5.4. We now deduce Proposition 5.2 from Proposition 5.1 using spectral sequence. Note that $\mathfrak{n}_\ell$ is a negatively graded subalgebra of $\mathfrak{g}$, so its dual $\mathfrak{n}_\ell^*$ is positively graded; we write its decomposition as $\mathfrak{n}_\ell^* = \bigoplus_{i \geq 1} \mathfrak{n}_\ell^*(i)$. Consider the standard cochain complex for computing the $\mathfrak{n}_\ell$-cohomology of $Q_\ell$:

$$0 \to Q_\ell \to \mathfrak{n}_\ell^* \otimes Q_\ell \to \ldots \to \wedge^n \mathfrak{n}_\ell^* \otimes Q_\ell \to \ldots.$$

A filtration on $\wedge^n \mathfrak{n}_\ell^* \otimes Q_\ell$ is defined by letting $F_p(\wedge^n \mathfrak{n}_\ell^* \otimes Q_\ell)$ be the subspace of $\wedge^n \mathfrak{n}_\ell^* \otimes Q_\ell$ spanned by $(x_1 \wedge \ldots \wedge x_n) \otimes v$, for all $x_1 \in \mathfrak{n}_\ell^*(i_1), \ldots, x_n \in \mathfrak{n}_\ell^*(i_n)$ and $v \in F_Q Q_\ell$ such that $i_1 + \ldots + i_n + j \leq p$. This defines the structure of a filtered complex on (5.3). Taking the associated graded gives the standard cochain complex for computing the $\mathfrak{n}_\ell$-cohomology of $\text{gr } Q_\ell$.

Now consider the spectral sequence with

$$E_{0}^{p,q} = F_{p}(\wedge^{p+q} \mathfrak{n}_\ell^* \otimes Q_\ell)/F_{p-1}(\wedge^{p+q} \mathfrak{n}_\ell^* \otimes Q_\ell).$$
5. Note that if $\ell_1 \subset \ell_2$ are both isotropic subspaces of $g(-1)$, then we have a natural map $Q_{\ell_1} \to Q_{\ell_2}$ which gives a map $H_{\ell_1} \to H_{\ell_2}$. By above, the associated graded map $gr H_{\ell_1} \to gr H_{\ell_2}$ is an isomorphism of Kazhdan graded algebras, hence $H_{\ell_1} \to H_{\ell_2}$ is an isomorphism of Kazhdan filtered algebras. Taking $\ell_1 = 0$, we see that $H_{\ell_2}$ is independent of the choice of $\ell_2$.

To check that the Poisson structure of $H_{\ell_1}$ is that of $\mathbb{C}[S]$, we take $\ell_2$ to be a Lagrangian subspace. It suffices to prove that $gr H_{\ell_2}$ and $\mathbb{C}[S]$ have the same Poisson structures. This follows from the observation that the Poisson structure of the Kazhdan graded algebra $gr U g$ is same as the usual Poisson structure of $\mathbb{C}[g^*]$ and the discussion in §3.2.

This concludes the proof of Theorem 4.1.

### 6. Skryabin’s equivalence

6.1. We would like to make some remarks on results by Skryabin [Sk]. Throughout this section we take $\ell$ to be a Lagrangian subspace of $g(-1)$, and write $m = m_\ell$, $Q = Q_\ell$, and $H = H_\ell$. Let $C$ be the abelian category of finitely generated $U g$-modules on which $m - \chi(m)$ acts locally nilpotently for each $m \in m$. For $E \in C$, put $Wh(E) = \{x \in E \mid mx = \chi(m)x, \forall m \in m\}$. Observe that for $E \in C$ we have: $Wh(E) = 0 \iff E = 0$.

The following beautiful theorem of Skryabin and its proof in [Sk] are similar in spirit to the well-known result of Kashiwara on the equivalence of the category of $D$-modules on a submanifold with the category of $\mathcal{D}$-modules on the ambient manifold which are supported on the submanifold.

**Theorem 6.1.** The functor $V \mapsto Q \otimes_H V$ sets up an equivalence of the category of finitely generated left $H$-modules and category $C$. The inverse equivalence is given by the functor $E \mapsto Wh(E)$, in particular, the latter is exact.

6.2. We present an alternative proof of Theorem 6.1 along the lines of the preceeding section (cf. also [Ko] §4, and [Ly] Theorem 4.1).

Fix an $H$-module $V$ generated by a finite dimensional subspace $V_0$. View $H$ as a filtered algebra with respect to the Kazhdan filtration $F_i H$, and define an increasing filtration on $V$ by $F_i V = F_i H \cdot V_0$, for all $i$. This makes $V$ a filtered $H$-module, with associated graded $H$-module $gr V$. Lemma 2.1 and Theorem 4.1 yield $gr Q = \mathbb{C}[N] \otimes gr H$. We deduce

$$H^0(m, gr Q \otimes gr H gr V) = gr V \quad \text{and} \quad H^i(m, gr Q \otimes gr H gr V) = 0, \forall i > 0.$$ 

Further, the Kazhdan filtration on $Q$ and the filtration on $V$ give rise to a filtration on $Q \otimes_H V$, and since $gr Q$ is free over $gr H$, we have a canonical isomorphism $gr (Q \otimes_H V) \simeq gr Q \otimes_{gr H} gr V$. Now a spectral sequence argument very similar to that in the proof of Proposition 5.2 yields

$$H^0(m, Q \otimes_H V) = V \quad \text{and} \quad H^i(m, Q \otimes_H V) = 0, \forall i > 0,$$

where the cohomology is taken with respect to the $\chi$-twisted action of $m$.

Note that the equation on the left of (6.2) says that $Wh(Q \otimes_H V) = V$. Thus, to complete the proof of the Theorem it suffices to show that, for any $E \in C$, the canonical map $f : Q \otimes_H Wh(E) \to E$ is an isomorphism. Let $E'$ be the kernel, and $E''$ the cokernel of $f$. We
observe that $\text{Wh}(E') = E' \cap \text{Wh}(Q \otimes_H \text{Wh}(E))$, which is equal to $E' \cap \text{Wh}(E)$, by (6.2). But $\text{Wh}(E)$ does not intersect the kernel of $f$, hence $\text{Wh}(E') = E' \cap \text{Wh}(E) = 0$. Since $E'$ is clearly an object of $C$, this yields $E' = 0$. Hence $f$ is injective.

To prove surjectivity we write the long exact sequence of cohomology associated to the short exact sequence $0 \to Q \otimes_H \text{Wh}(E) \to E \to E'' \to 0$. We obtain:

$$0 \to H^0(\mathfrak{m}, Q \otimes_H \text{Wh}(E)) \xrightarrow{H^0(f)} H^0(\mathfrak{m}, E) \to H^0(\mathfrak{m}, E'') \to H^1(\mathfrak{m}, Q \otimes_H \text{Wh}(E)) \to \ldots .$$

In this formula, the $H^1$-term vanishes, and the map $H^0(f)$ is a bijection, due to (6.2). Hence, the long exact sequence yields $\text{Wh}(E'') = H^0(\mathfrak{m}, E'') = 0$. This forces $E'' = 0$, and the Theorem is proved.

6.3. Let $M$ be the unipotent algebraic subgroup of $G$ corresponding to the Lie algebra $\mathfrak{m}$, and let $\mathcal{B}$ denote the flag manifold for $G$. Let $\mathcal{V}$ be an $M$-equivariant $\mathcal{D}$-module on $\mathcal{B}$. Given an element $x \in \mathfrak{m}$, we write $x_\mathcal{D}$ for the action on $\mathcal{V}$ of the vector field corresponding to $x$ via the $\mathcal{D}$-module structure, and $x_M$ for the action on $\mathcal{V}$ obtained by differentiating the $M$-action arising from the equivariant structure. We say that $\mathcal{V}$ is an $\mathfrak{m}$-Whittaker $\mathcal{D}$-module with respect to the character $\chi : \mathfrak{m} \to \mathbb{C}$ if, for any $x \in \mathfrak{m}$ and $v \in \mathcal{V}$, we have $(x_\mathcal{D} - x_M)v = \chi(x) \cdot v$.

Note that the natural map $U\mathfrak{g} \to Q$ maps the center of $U\mathfrak{g}$ injectively into $H$ (c.f. [Pr] §6.2). Denote by $Z_+$ the augmentation ideal of the center of $U\mathfrak{g}$. Skryabin’s result combined with the Beilinson-Bernstein localization theorem implies the following.

**Proposition 6.3.** The category of finitely generated $H/Z_+H$-modules is equivalent to the category of $\mathfrak{m}$-Whittaker coherent $\mathcal{D}_\mathcal{B}$-modules (with respect to the character $\chi$).

7. Appendix: Intersections of a Slodowy slice with coadjoint orbits

In this appendix, we provide more details on the arguments in §3.1. These details were omitted in the original version of the paper published more than ten years ago; we had thought that those were routine arguments but since then we have not always been able to recall those arguments ourselves.

Recall the setup and notation of §3.2. Restricting the map $\mu$ to a coadjoint orbit $\mathcal{O}$ gives a moment map associated with the $M$-action on $\mathcal{O}$. According to §3.2, the element $\chi|_\mathfrak{m} \in \mathfrak{m}^*$ is a regular value of the map $\mu|_\mathcal{O}$. Thus, by standard results concerning fibers of moment maps over regular values, cf. eg. [GS] Theorem 2.5, we obtain the following

**Lemma 7.1.** Assume that the set $\Sigma := \mathcal{O} \cap (\chi + \mathfrak{m}^\perp)$ is non-empty.

Then, $\Sigma$ is a smooth coisotropic submanifold of the symplectic manifold $\mathcal{O}$. Moreover, the canonical nil-foliation on the coisotropic manifold $\Sigma$ is the foliation by the $M$-orbits in $\Sigma$.

Now, fix $\xi \in \mathcal{O} \cap \mathcal{S}$ and let $T_\xi(M \cdot \xi)$ be the tangent space to the $M$-orbit of $\xi$. By Lemma 7.1, $T_\xi(M \cdot \xi)$ is the tangent space to the nil-leaf through $\xi$. Further, the isomorphism of Lemma 2.1 implies a direct sum decomposition

$$\mathfrak{m}^\perp = T_\xi(M \cdot \xi) \oplus T_\xi\mathcal{S} = \text{ad} \mathfrak{m}(\xi) \oplus \Phi(\ker \text{ad} f).$$

Now, the claim from §3.1 is an immediate consequence of the above results, that is, we get
Corollary 7.2. The manifold \( \mathcal{O} \cap S \) is a symplectic submanifold of \( \mathcal{O} \), for any coadjoint orbit \( \mathcal{O} \); that is, the Kirillov-Kostant 2-form on \( \mathcal{O} \) gives, by restriction, a nondegenerate 2-form on \( \mathcal{O} \cap S \).

Here is an alternative, more direct proof of Corollary 7.2. We use the notation of §3.1 and let \( x = \Phi^{-1}(\xi) \). We identify \([x, g]\) with \( T_\xi \mathcal{O} \), write \( \langle -, - \rangle \) for the Killing form on \( g \) and \((-)^{\perp} \subset g \) for the annihilator with respect to the Killing form. The Kirillov-Kostant symplectic form on \([x, g]\) is then given by the formula

\[
[x, g] \times [x, g] \rightarrow \mathbb{C}, \quad [x, u] \times [x, v] \mapsto \langle x, [u, v] \rangle.
\]

Now let \( y = [x, v] \in T_\xi \mathcal{O} \) be such that \( \langle x, [u, v] \rangle = 0 \) holds for all \( u \in g \) such that \([x, u] \in \text{Ker}(\text{ad}(f))\). Our formula (3.2) claims that

\[
(7.3) \quad y \in [x, [f, g]].
\]

To see this, note first that \( \langle x, [u, v] \rangle = \langle [x, u], v \rangle \), so

\[
\langle a, v \rangle = 0 \quad \text{for all } a \in \text{Im}(\text{ad}(x)) \cap \text{Ker}(\text{ad}(f)).
\]

Thus, our assumption on \( y \) reads

\[
v \in \left( [x, g] \cap \text{Ker}(\text{ad}(f)) \right)^{\perp}.
\]

For any \( a \in g \), the linear map \( \text{ad} \ a : g \rightarrow g \) is skew-adjoint relative to the Killing form. Therefore, we obtain

\[
\left( \text{Im}(\text{ad}(x)) \cap \text{Ker}(\text{ad}(f)) \right)^{\perp} = \text{Im}(\text{ad}(x))^{\perp} + \text{Ker}(\text{ad}(f))^{\perp} = \text{Ker}(\text{ad}(x)) + \text{Im}(\text{ad}(f)).
\]

This proves (7.3) since we have

\[
y = [x, v] \in [x, \text{Ker}(\text{ad}(x))] + [x, [f, g]] = [x, [f, g]].
\]

To complete the proof we must show that, for any \( x \in e + \text{Ker}(\text{ad}(f)) \), one has

\[
(7.4) \quad [x, [f, g]] \cap \text{Ker}(\text{ad}(f)) = 0.
\]

To this end, we use direct sum decompositions

\[
(7.5) \quad [e, g] \oplus \text{Ker}(\text{ad}(f)) = g = [f, g] \oplus \text{Ker}(\text{ad}(e)).
\]

From the decomposition on the right we see that the map \( \text{ad} \ e : [f, g] \rightarrow [e, [f, g]] \) is a bijection. We deduce by continuity that, for any \( x \in g \) sufficiently close to \( e \), the map

\[
\text{ad} \ x : [f, g] \rightarrow [x, [f, g]]
\]

is also a bijection. It follows, that assigning to \( x \) the vector space \([x, [f, g]]\) gives a continuous map of a neighborhood of \( e \) in \( g \) to an appropriate Grassmannian.

Next, we observe that the first decomposition in (7.5) implies that \([e, [f, g]] \cap \text{Ker}(\text{ad}(f)) = 0\). Therefore, (7.4) holds for all \( x \) in a neighborhood of \( e \), by continuity. This implies (7.4) for any \( x \in e + \text{Ker}(\text{ad}(f)) \), using the \( \mathbb{C}^* \)-action.

There is also a completely algebraic proof of (7.4) as follows. Let \( y \in [x, [f, g]] \cap \text{Ker}(\text{ad}(f)) \), where \( x = e + k \) for some \( k \in \text{Ker}(\text{ad}(f)) \). Then, for some \( z \in g \),

\[
y = [e + k, [f, z]] = [e, [f, z]] + [k, [f, z]].
\]
Suppose that \([f, z] \neq 0\). If \(z \in g(i)\), then \([e, [f, z]]\) is a nonzero vector contained in \(g(i)\), but \([k, [f, z]]\) is contained in \(\bigoplus_{j<i} g(j)\). More generally, we can write
\[
z = z_{i_1} + \cdots + z_{i_r}, \quad z_i \in g(i), \quad i_1 < \cdots < i_r.
\]
Suppose \(i_s\) is the biggest \(i\) such that \([f, z_i] \neq 0\). Then \([e, [f, z_i]]\) is a nonzero vector in \(g(i_s)\), but \([e, [f, z_j]]\) for \(j \neq s\) and \([k, [f, z]]\) are all contained in \(\bigoplus_{j<i_s} g(j)\). Therefore, if we decompose \(y\) according to the direct sum \(g = \text{Ker}(\text{ad} f) \oplus [e, g]\), then \(y\) will have a nonzero component in \([e, g]\), which is a contradiction to \(y \in \text{Ker}(\text{ad} f)\). Hence, \([f, z] = 0\).

References

[Ba] D. Barbasch, *Fourier transforms of some invariant distributions on semisimple Lie groups and Lie algebras*, in Noncommutative harmonic analysis, 1–7, Lecture Notes in Math., 728, Springer, Berlin, 1979.

[BT] J. de Boer, T. Tjin, *Quantization and representation theory of finite \(W\)-algebras*, Comm. Math. Phys. 158 (1993), 485–516.

[Gi] V. Ginzburg, *Principal nilpotent pairs in a semisimple Lie algebra 1*, Invent. Math. 140 (2000), 511-561.

[GS] V. Guillemin, S. Sternberg, *Geometric quantization and multiplicities of group representations*. Invent. Math. 67 (1982), 515-538.

[HC] Harish-Chandra, *Invariant distributions on Lie algebras*, Amer. J. Math. 86 (1964), 271-309.

[Ka] N. Kawanaka, *Generalized Gelfand-Graev representations and Ennola duality*, in Algebraic groups and related topics, 175–206, Adv. Stud. Pure Math., 6, North-Holland, 1985.

[Ko] B. Kostant, *On Whittaker vectors and representation theory*, Invent. Math. 48 (1978), 101-184.

[Ly] T.E. Lynch, *Generalized Whittaker vectors and representation theory*, Thesis, M.I.T., 1979.

[Ma] H. Matumoto, *Whittaker modules associated with highest weight modules*, Duke Math. J. 60 (1990), 59-113.

[Me] C. Mœglin, *Modules de Whittaker et ideaux primitifs complètement premiers dans les algèbres enveloppantes*, C. R. Acad. Sci. Paris 303 (1986), 845-848.

[Pr] A. Premet, *Special transverse slices and their enveloping algebras*, preprint.

[Sk] P. Slodowy, *Simple singularities and simple algebraic groups*, Lecture Notes in Mathem., 815, Springer Verlag, 1980.

[Va] I. Vaisman, *Lectures on the geometry of Poisson manifolds*, Progress in Mathematics, 118 Birkhäuser Verlag, Basel, 1994.

[Va] H. Yamashita, *On Whittaker vectors for generalized Gelfand-Graev representations of semisimple Lie groups*, Proc. Japan Acad. Ser. A Math. Sci. 61 (1985), 213–216.

Department of Mathematics, University of Chicago, Chicago, IL 60637, U.S.A.

E-mail address: wlgan@math.uchicago.edu

Department of Mathematics, University of Chicago, Chicago, IL 60637, U.S.A.

E-mail address: ginzburg@math.uchicago.edu