D-GONALITY OF MODULAR CURVES AND BOUNDING TORSIONS

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Abstract. We study the problem of $d$-gonality of the modular curve $X_0(N)$. As a result, we can give an upperbound of the level $N$ by means of $d$. This generalizes Ogg’s result on hyperelliptic modular curves ($d = 2$) in ([O]). As a corollary of this result, we prove an analogue of the strong Uniform Boundedness Conjecture for elliptic curves defined over the function fields of curves. If a base curve is $d$-gonal, we can bound orders of torsions of Mordell-Weil groups in terms of $d$ uniformly.

§0 Introduction

Let $N$ be a positive integer, and let $\Gamma_0(N)$ be the subgroup of $\Gamma = SL_2(\mathbb{Z})/(\pm 1)$ defined by matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $N$ dividing $c$. Then $\Gamma_0(N)$ acts on the upper half plane $\mathbb{H}$ properly discontinuously, and let

$$Y_0(N)_C = \Gamma_0(N) \backslash \mathbb{H}.$$ 

The modular curve $X_0(N)_C$ is the compactification of $Y_0(N)_C$ obtained by adding the cusps. A smooth projective curve $C$ defined over an algebraically closed field $k$ is called $d$-gonal if there exists a finite morphism $f : X \to \mathbb{P}^1_k$ of degree $d$. For example, if a smooth curve $C$ is 1-gonal, $C$ is isomorphic to $\mathbb{P}^1_k$, and if $C$ is 2-gonal, then either $g(C) \leq 1$ or $C$ is a hyperelliptic curve of genus $g(C) \geq 2$.

In this paper, we shall consider the following problem.

Problem 0.1. If $X_0(N)_C$ is $d$-gonal, can one give a bound for $N$ by means of $d$?

By the genus formula of $X_0(N)_C$, one can determine the case when $g(N) := g(X_0(N)_C) = 0$, which gives the answer for $d = 1$. (One knows that $N \leq 25$ for $d = 1$.) For $d = 2$, Ogg [O] classified all hyperelliptic modular curves $X_0(N)_C$, and it gives us the sharp bound $N \leq 71$. In this paper, we deal with Problem 0.1 for general $d$.

Our main theorem can be stated as follows.

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Theorem 0.2. (cf. Corollary 3.5.) Let $X_0(N)$ be the modular curve of level $N$. If $X_0(N) \subset C$ is $d$-gonal, then we have

\begin{equation}
N \leq \begin{cases}
25 & \text{if } d = 1 \\
71 & \text{if } d = 2 \\
(48(d-1)^2 + 35)(36(d-1)^2 + 47) & \text{if } d \geq 3
\end{cases}.
\end{equation}

Note that we can obtain better bound for $N$ odd (see Corollary 3.3).

Before going into the sketch of the proof of Theorem 0.2, we explain the motivation of Problem 0.1. Our original motivation is a function field analogue of the so-called strong Uniform Boundedness Conjecture (abbreviated as sUBC). According to [Kam-Ma], we denote by $\Phi(d)$ the set of all isomorphism classes of finite abelian groups occurring as the full groups of torsion in the Mordell-Weil groups of elliptic curves over number fields $K$ with absolute degree $[K : \mathbb{Q}] \leq d$. Then sUBC states that $\Phi(d)$ is finite.

Let $d$ be a positive integer. A prime number $p$ is called a torsion prime for degree $d$ if there is a number field $K$ of degree $d$, an elliptic curve $E$ defined over $K$, and a $K$-rational point $P$ of $E$, of order $p$. Let $S(d)$ denote the set of torsion primes of degree $\leq d$. Then Kamienny and Mazur [Kam-Ma] showed that $\Phi(d)$ is finite if and only if $S(d)$ is finite (see also [E]). Recently, Merel [Me] showed that for $p \in S(d)$

$$p < d^{3d^2},$$

which completes the proof of sUBC. (Note that one could not give an explicit bound for $N$ occurring as an exponent of a group in $\Phi(d)$. The complete descriptions of $\Phi(d)$ for $d = 1, 2$ is known (see [Ma], [Kam-Ma], [E] and references therein).

In this paper, we prove the following theorem which can be viewed as an analogue of sUBC in the function field case. Let $C$ be a smooth projective curve defined over the field of complex numbers and set $K = \mathbb{C}(C)$, the function field of $C$. Let $E$ be a non-constant elliptic curve defined over $K$. If $C$ is $d$-gonal, then there exists an extension of fields $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(t) \hookrightarrow K$ of degree $d$. (Note that such a field extension may not be unique even if we assume that $d$ is minimal.)

Now we denote by $\Phi_{fun}(d)$ the set of all isomorphism classes of finite abelian groups occurring as the full groups of torsion in the Mordell-Weil groups of non-constant elliptic curves over a function field $K$ with an extension $\mathbb{C}(\mathbb{P}^1) \hookrightarrow K$ of degree $\leq d$. Then we propose an function field analogue of sUBC as: $\Phi_{fun}(d)$ is finite.

This seems to be a quite natural question, which originally arose in a seminar at Kochi (July, 1995) with H. Tokunaga and the second author. Furthermore they realized that the analogue can be reduced to the problem of $d$-gonality of modular curves.

From Theorem 0.2, we obtain the following theorem, which obviously implies that $\Phi_{fun}(d)$ is finite.
Theorem 0.3 (An analogue of sUBC for function field case). (cf. Theorem 4.3.) Let $E$ be a non-constant elliptic curve defined over the function field $K$ of a complex smooth projective curve $C$. Assume that $C$ is $d$-gonal and the Mordell-Weil group $E(K)$ has a torsion element of order $N$. Then

$$N \leq B(d)$$

where $B(d)$ is the right hand side of the inequality (0.1).

Let us give a short summary of this paper. In §1, we recall necessary results on curves. One important remark is that $d$-gonality of a curve descends to a curve below via a finite morphism, and another is the so-called inequality of Castelnuovo-Severi. In §2, we will prove “Tower Theorem 2.1”, which is one of our main contributions in this paper. In §3, we will generalize a beautiful argument of Ogg [O] and Harris-Silverman [Ha-Si] to obtain a bound of level $N$ by means of $d$. From Tower Theorem 2.1, we obtain a finite morphism $f : X_0(N)_{\mathbb{Q}} \to C'$ defined over $\mathbb{Q}$ of degree $d' \leq d$ such that $g(C') \leq (d/d' - 1)^2$. For a prime $p$ not dividing $N$, $X_0(N)_{\mathbb{Q}}$ has a good reduction at $p$. When $g(C') \geq 1$, applying Good Reduction Lemma 5.1, $C'$ has also good reduction at $p$ and we obtain a finite morphism $f_{\mathbb{F}_p} : X_0(N)_{\mathbb{F}_p} \to C'_{\mathbb{F}_p}$ of degree $d$ defined over $\mathbb{F}_p$.

By Weil’s theorem of the analogue of the Riemann hypothesis, we can bound the number of $\mathbb{F}_{p^2}$-rational points of $C'_{\mathbb{F}_p}$ from the above by $g(C')$, hence by $(d/d' - 1)^2$. Therefore the number of $\mathbb{F}_{p^2}$-rational points of $X_0(N)_{\mathbb{F}_p}$ is bounded from above only by $d$. On the other hand, we have enough $\mathbb{F}_{p^2}$-rational points of $X_0(N)_{\mathbb{F}_p}$ from points corresponding to supersingular elliptic curve cusps. For example, if $p = 2$, $\#(X_0(N)_{\mathbb{F}_2})$ is at least $\frac{1}{12}(N + 1) + 2$. This argument gives us bound of $N$ by means of $d$ and a proof of Theorem 0.2. In §4, we prove Theorem 0.3 by using Theorem 0.2 and the moduli property of $X_0(N)$. In §5, we shall prove the Good Reduction Lemma 5.1 which is our another contribution in this paper.

Let us mention some results related to ours. In the case of function field $K = k(C)$ with $\text{char } k = 0$, a bound of $|E(K)_{\text{tors}}|$ in terms of the genus $g = g(C)$ was given first by Levin [Le], and later (with sharper bound) by Hindry and Silverman [Hi-Si]. A rough estimate in [Hi-Si, Theorem 7.2, (c)] gives us

$$|E(K)_{\text{tors}}| \leq 144(g + 1)^{2/3}.$$  

(See [Hi-Si, Theorem 7.2, (a),(b)] for refined bounds.) The bound in [Hi-Si, Theorem 7.2, (a)] follows from the function field analogue of Szpiro’s conjecture ([Hi-Si, Theorem 5.1]), which was originally proved by Kodaira and Shioda (cf. [Shi, Proposition 2.8]). Note that since the genus $g$ can be arbitrarily large with $d$ fixed, so it is still far from the uniform bound by $d$.

As far as the set $\Phi_{\text{fun}}(d)$ is concerned, Cox-Parry [C-P] determined the set $\Phi_{\text{fun}}(1)$ completely. (See also [M-P]). We can also determine the set $\Phi_{\text{fun}}(2)$ completely. (See Nguyen’s report of our joint work [Ng]).

When $k$ is a finite field with $q$ elements of $\text{char } k = p > 0$, we remark that our method in this paper gives a bound of order of prime-to-$p$-part of $E(K)_{\text{tors}}$ by
means of $d$. Our bound inevitably depends on $d$ and the characteristic $p$ but not depend on $q$. Recently, Goldfeld and Szpiro gave a bound for $|E(K)_{tors}|$ when $k$ is a finite field with $q$ elements (cf. [G-S, Theorem 13]). Their bound depends on $q$, the genus $g$ and the inseparablility degree of the $j$-function $C \to \mathbb{P}^1$ associated to a minimal model of $E/K$.

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§1 Some results on curves

In this section, we recall some known results on curves which we use later. Let $k$ be an algebraically closed field of any characteristic $p \geq 0$.

Definition 1.1. A smooth projective curve $C$ defined over $k$ is called $d$-gonal if there exists a finite morphism $f : C \to \mathbb{P}^1_k$ of degree $d$, or equivalently, if $C$ has a $g^1_d$. We call such a map $f$ a $d$-gonal map.

If $C$ admits a $d$-gonal map $f : C \to \mathbb{P}^1_k$, then it induces a field extension $k(\mathbb{P}^1_k) = k(t) \hookrightarrow K = k(C)$ of degree $d$, and conversely such a field extension gives a $d$-gonal map $f : C \to \mathbb{P}^1_k$.

Remark 1.2. One may like to use the term “$d$-gonality” for the case with $d$ minimal. However the definition above allows us to state our results in simpler form and the reader can easily apply the results to the case with $d$ minimal. Moreover we remark that a $d$-gonal map $f : C \to \mathbb{P}^1_k$ may not be unique up to automorphism of $C$ and $\mathbb{P}^1_k$ even if $d$ is minimal.

Moreover, for $d \geq (1/2)g + 1$, any curve of genus $g$ is $d$-gonal, and for $d < (1/2)g + 1$, there exist curves of genus $g$ having no $d$-gonal map. On the other hand, there exists a hyperelliptic curve of genus $g$ (2-gonal curve) for any $g \geq 2$.

First of all, we recall the following lemma. Though it may be well-known to experts, for completeness, we will give a proof which is valid for any algebraically closed base field $k$. (cf. [Ne, Theorem VII.2]).

Lemma 1.3. Let $h : C_1 \to C_2$ be a finite morphism between two smooth projective curves. If $C_1$ is $d$-gonal, then $C_2$ is also $d$-gonal.

Proof. Let $f : C_1 \to \mathbb{P}^1$ be a rational function of degree $d$. Let $K_i, (i = 1, 2)$ denote the rational function fields of $C_i (i = 1, 2)$. Then we have a field extension

\[ k(\mathbb{P}^1_k) = k(t) \hookrightarrow K = k(C) \]
$K_2 \hookrightarrow K_1$ of degree $m$. Take an algebraic closure $\overline{K}_1$ of $K_1$, and let $l$ and $s$ be the separable and inseparable degrees of the extension $K_1/K_2$. (When char of $k$ is 0, $s = 1$ and when char $k = p > 0$, then $s = p^e$ for some $e \geq 0$.) For any element $h \in K_1$, we can define the norm $N_{K_1/K_2}(h)$ as

$$N_{K_1/K_2}(h) = \prod_{i=1}^{l} (\sigma_i(h^s))$$

where $\sigma_i : K_1 \hookrightarrow \overline{K}_1, i = 1, \cdots, l$ are different $K_2$-embeddings of $K_1$ into $\overline{K}_1$. Then the norm $N_{K_1/K_2}(f)$ gives a morphism $C_2 \rightarrow \mathbb{P}^1$ of degree $\leq d$. It is easy to see that for some constant $\lambda \in k$ the norm $N_{K_1/K_2}(f - \lambda)$ gives a finite morphism $C_2 \rightarrow \mathbb{P}^1$ exactly of degree $d$.

The next important result, which we use in the proof of Tower Theorem 2.1, is the so-called inequality of Castelnuovo-Severi.

**Lemma 1.4.** (The inequality of Castelnuovo-Severi). Let $C, C_1, C_2$ be smooth projective curves over $k$ of genera $g, g_1, g_2$ respectively and let $\pi_1 : C \rightarrow C_1$ and $\pi_2 : C \rightarrow C_2$ surjective morphisms of degrees $d_1, d_2$ respectively. Assume that the induced map

$$(1.1) \quad \pi_1 \times \pi_2 : C \rightarrow C_1 \times C_2$$

is birational onto its image. Then one has

$$(1.2) \quad g \leq d_1 g_1 + d_2 g_2 + (d_1 - 1)(d_2 - 1).$$

**Proof.** Though there are many proofs of this inequality (cf. [H, Ch. V], [Gr]), we will give a proof using genus formula and the Hodge index theorem. Set $S = C_1 \times C_2$ and $D = \pi_1 \times \pi_2(C) \subset S$. Then one can calculate the virtual genus $p_a(D)$ of a curve $D$ on a surface $S$ as $p_a(D) = (K_S + D) \cdot D/2 + 1$. Then by assumption that $C \rightarrow D$ is birational, we have $g(C) \leq p_a(D)$. On the other hand, from the assumption, we obtain $K_S \cdot D = (g_1 - 1)d_1 + (g_2 - 1)d_2$ and the Hodge index theorem implies that $D^2 \leq 2d_1 d_2$ (see [H, Ex. 1.9, Ch. V]). Therefore we obtain the inequality.

\section*{2 Tower theorem}

In this section, we prove the following theorem which we call “Tower theorem”. Let $k$ be a perfect field and fix an algebraic closure $\overline{k}$ of $k$ and let $G_k = Gal(\overline{k}/k)$ be the Galois group of the extension $\overline{k}/k$.

**Theorem 2.1.** Let $C$ be a projective smooth curve defined over a perfect field $k$ and $f : C \rightarrow \mathbb{P}^1_k$ a dominant morphism defined over $\overline{k}$ of degree $d$. Then there exists a smooth projective curve $C'$ defined over $k$ and a dominant morphism

$$f' : C \rightarrow C'$$
defined over $k$ of degree $d'$ dividing $d$ such that

\begin{equation}
(2.1) \quad g(C') \leq (d/d' - 1)^2.
\end{equation}

**Proof.** At first, we consider the following

**Claim:** There exists a tower of (projective smooth) curves over $\overline{k}$

\begin{equation}
(2.2) \quad C \rightarrow C_n \rightarrow \cdots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \simeq \mathbb{P}^1
\end{equation}

satisfying the following conditions:

(i) $h_i : C_i \rightarrow C_{i-1}$ and $f_n : C \rightarrow C_n$ are morphisms of degree $e_i \geq 2$ and of degree $d_n$ defined over $\overline{k}$.

(ii) Set $f_i : C \rightarrow C_i$ and $d_i = \deg f_i$. Then we have $d = d_0 > d_1 > \cdots > d_n \geq 1$ and for any $0 \leq i \leq n$

\[ g(C_i) \leq (d/d_i - 1)^2. \]

(iii) For any $\sigma \in G_k := \text{Gal}(\overline{k}/k)$, the morphism

\[ f_n \times f_n^\sigma : C \rightarrow C_n \times C_n^\sigma \]

has degree $d_n = \deg f_n$ onto its image.

Let us first explain how the theorem follows from this claim. Consider the morphism $f_n \times f_n^\sigma : C \rightarrow C_n \times C_n^\sigma$ for any $\sigma \in G_k$ and let $D = f_n \times f_n^\sigma(C)$ be the image. Let $p_1 : D \rightarrow C_n$ and $p_2 : D \rightarrow C_n^\sigma$ be the natural projections. Since $f_n$ and $f_n^\sigma$ factor through $C \rightarrow D$ and $p_1$ and $p_2$ respectively, from the assumption (iii), we know that $\deg p_i = (\deg f_n)/\deg(C \rightarrow D) = d_n / d_n = 1$, which implies that the projection maps $p_1$ and $p_2$ are birational. Hence we infer that $C_n \simeq C_n^\sigma$ (over $\overline{k}$) for any $\sigma \in G_k$. This shows that $C_n$ is isomorphic (over $\overline{k}$) to a curve $B$ defined over $k$. Replacing $C_n$ by this new curve $B$, we obtain a morphism $h : C \rightarrow B$ of degree $d_n$. Though $h$ may not be defined over $k$, for any $\sigma \in G_k$, the argument given above shows that $h : C \rightarrow B$ and $h^\sigma : C \rightarrow B$ differ by an automorphism of $B$. That is, we have $\alpha_\sigma \in \text{Aut}(B \otimes \overline{k})$ such that

\[ h^\sigma = \alpha_\sigma \circ h. \]

It is easy to see that $\alpha : G_k \rightarrow \text{Aut}(B \otimes \overline{k})$ defines a cocycle in $H^1(G_k, \text{Aut}(B \otimes \overline{k}))$. Then by general theory of twisting (cf. [Si., X, Theorem 2.2]), $\alpha$ corresponds to a twist, that is, there exists a smooth curve $C'$ defined over $k$ and $\overline{k}$-isomorphism $\lambda : C' \rightarrow B$ such that $\alpha_\sigma = \lambda^\sigma \lambda^{-1}$. Now the map $\lambda^{-1} \circ h : C \rightarrow C'$ gives a dominant morphism defined over $k$ of degree $d_n$ and $g(C') = g(B) = g(C_n) \leq (d/d_n - 1)^n$, which implies the theorem.

Now we show how one can obtain a tower in the claim.

Take $\sigma \in G_k$, and consider the map

\[ f \times f^\sigma : C \rightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k. \]

Let $D = f \times f^\sigma(C) \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the image. If for any $\sigma \in G_k$, $\deg f \times f^\sigma = \deg f = d$, then we can take $C_n = C_0(\simeq \mathbb{P}^1_k)$ as the normalization of $D$ and
\( f_n = f_0 : C \rightarrow C_0 \) as the natural morphism to obtain the desired tower. (Note that \( d_n = d \) and \( g(C_0) = 0 \).) Hence in this case we get a tower \((2.2)\) as desired.

Otherwise, for some \( \sigma \in G_k \), the degree of \( f \times f^\sigma \) onto its image is \( d_1 < d = d_0 \). Then setting \( C_1 = \) the normalization of the image \( f \times f^\sigma (C) \subset \mathbb{P}^1 \times \mathbb{P}^1 \), we obtain finite morphisms

\[
\begin{align*}
  h_1 : C_1 &\rightarrow C_0 \simeq \mathbb{P}^1_k, \\
  h_1^\sigma : C_1 &\rightarrow C_0 \simeq \mathbb{P}^1_k, \\
  f_1 : C &\rightarrow C_1
\end{align*}
\]

such that \( f = h_1 \circ f_1 \), \( f^\sigma = h_1^\sigma \circ f_1 \) with \( e_1 = \deg h_1 = \deg h_1^\sigma \geq 2 \) and \( d_1 = \deg f_1 = d/e_1 < d \). Then since \( h_1 \times h_1^\sigma : C_1 \rightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k \) is birational onto its image, the inequality of Castelnuovo-Severi implies that

\[
g(C_1) \leq (e_1 - 1)^2 = (d/d_1 - 1)^2.
\]

After continuing these procedures, we may assume that we have a tower of (smooth) curves

\[ C \rightarrow C_i \cdots \rightarrow C_1 \rightarrow C_0 \simeq \mathbb{P}^1 \]

satisfying only the conditions (i), (ii) up to a level \( i > 0 \). For any \( \sigma \in G_k \) and the given morphism \( f_i : C \rightarrow C_i \), consider the morphism

\[
f_i \times f_i^\sigma : C \rightarrow C_i \times C_i^\sigma.
\]

If for all \( \sigma \in G_k \) \( f_i \times f_i^\sigma \) is birational onto its image, then the tower also satisfies the condition (iii). Then we can stop the procedure.

Otherwise, for some \( \sigma \in G_k \), the degree of \( f_i \times f_i^\sigma \) is \( d_{i+1} < d_i \). Then again let \( C_{i+1} \) be the normalization of the image of \( f_i \times f_i^\sigma(C) \) and let \( f_{i+1} : C \rightarrow C_{i+1} \) be the induced morphism and \( C_{i+1} \rightarrow C_i \times C_i^\sigma \) the induced map birational onto its image. Since the degree of each projection \( C_{i+1} \rightarrow C_i \) and \( C_{i+1} \rightarrow C_i^\sigma \) is \( e_{i+1} = d_i/d_{i+1} \geq 2 \), from the inequality of Castelnuovo-Severi (Lemma 1.4) and the assumption \( g(C_i) \leq (d/d_i - 1)^2 \), we obtain:

\[
(2.3) \quad g(C_{i+1}) \leq 2e_{i+1} \cdot g(C_i) + (e_{i+1} - 1)^2 \leq 2e_{i+1} \cdot (d/d_i - 1)^2 + (e_{i+1} - 1)^2.
\]

Since \( d_i = d_{i+1} \cdot e_{i+1} \), we obtain \( d/d_{i+1} = (d/d_i) \cdot e_{i+1} \), and since \( e_{i+1} \geq 2 \), we can easily see that

\[
(2.4) \quad 2e_{i+1}(d/d_i - 1)^2 + (e_{i+1} - 1)^2 \leq ((d/d_i) \cdot e_{i+1} - 1)^2 = (d/d_{i+1} - 1)^2.
\]

Hence, together with \((2.3)\), we obtain \( g(C_{i+1}) \leq (d/d_{i+1} - 1)^2 \) as desired. This procedure stops after a finite number of steps and this completes the proof of the claim.

\section*{§3 \( d \)-gonality of the modular curve \( X_0(N) \)}

In this section, we show our main theorem on \( d \)-gonality of the modular curves \( X_0(N)_C \) of level \( N \).
The main idea of the proof obviously goes back to a beautiful argument of Ogg in [O], which determines the complete list of hyperelliptic modular curves.

For a positive integer \(N\), let \(\Gamma_0(N)\), \(Y_0(N)\), and \(X_0(N)\) be as in Introduction. Classically, it is well-known that \(Y_0(N)\) and \(X_0(N)\) admit structures of algebraic curves \(Y_0(N)\) and \(X_0(N)\) defined over \(\mathbb{Q}\), and moreover by Igusa ([I]), there exist smooth models of \(Y_0(N)\) and \(X_0(N)\) over \(\mathbb{Z}[1/N]\). The smooth model \(Y_0(N)/\mathbb{Z}[1/N]\) can be considered as the coarse moduli space classifying a pair \((E, C)\) of an elliptic curve \(E\) with a cyclic subgroup of order \(N\) and \(X_0(N)/\mathbb{Z}[1/N]\) is its natural compactification. (See also [D-R] and [Ka-Ma] for more modern treatment of these facts.) Therefore for a prime \(p\) with \(p \nmid N\), \(X_0(N)\) has a good reduction at \(p\).

Let \(N\) be a positive integer and \(p\) a prime such that \(p \nmid N\) and \(X_0(N)\) the smooth projective curve over \(\mathbb{F}_p\) obtained by a reduction of \(X_0(N)\). For any extension field \(k\) of \(\mathbb{F}_p\), an \(k\)-rational point of \(X_0(N)\) corresponds to an isomorphism class of a pair \((E, C)\) of an elliptic curve with a cyclic subgroup of order \(N\) defined over \(k\).

The following argument is essentially due to Ogg ([O, Theorem 3]), which gives the lower bounds of number of \(\mathbb{F}_p^2\)-rational points of \(X_0(N)\). (See also [Ha-Si, Lemma 6].) If there exists a supersingular elliptic curve \(E\) defined over \(\mathbb{F}_p\), its Frobenius endomorphism \(\pi_p\) satisfies \(\pi_p^2 = -p\). Therefore any cyclic subgroup \(C \subset E\) of order \(N\) is defined over \(\mathbb{F}_p^2\), which yields a point \((E, C) \in X_0(N)\).

For any prime \(p\), set

\[
(3.1) \quad s(p) = \sum_{E/\mathbb{F}_p, \text{supersingular}} \frac{1}{|\text{Aut}(E)|}.
\]

Moreover let \(\nu(N)\) be the number of distinct prime factors of \(N\), and set

\[
(3.2) \quad \phi(N) = [\text{SL}_2(\mathbb{Z})/ \pm 1 : \Gamma_0(N)] = N \prod_{p \mid N} (1 + 1/p).
\]

From the argument above, we can obtain the following lemma. For detailed proofs, see [O, Theorem 3] and [Ha-Si, Lemma 6].

**Lemma 3.1.** Let \(p \nmid N\) be a prime. Then we have

\[
(3.3) \quad \#X_0(N)_{\mathbb{F}_p^2}(\mathbb{F}_p^2) \geq 2^{\nu(N)} + 2s(p)\phi(N).
\]

Moreover for \(p = 2, 3\), \(s(2) = 1/24, s(3) = 1/12\), hence we have

\[
(3.4) \quad \#X_0(N)_{\mathbb{F}_4}(\mathbb{F}_4) \geq 2^{\nu(N)} + \frac{1}{12}\phi(N), \quad \text{if} \quad 2 \nmid N,
\]

and

\[
(3.5) \quad \#X_0(N)_{\mathbb{F}_9}(\mathbb{F}_9) \geq 2^{\nu(N)} + \frac{1}{6}\phi(N), \quad \text{if} \quad 3 \nmid N.
\]
Theorem 3.2. Let $X_0(N)$ be the modular curve of level $N$. If $X_0(N)_{\mathbb{C}}$ is $d$-gonal, that is, if it admits a finite map $f : X_0(N)_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ of degree $d$, then we have the following.

1) If $N$ is odd, then we have

\[ \frac{1}{12} \phi(N) + 2^\nu(N) \leq \begin{cases} 5d & \text{if } d = 1, 2 \\ 4(d-1)^2 + 5 & \text{if } d \geq 3 \end{cases}. \]

2) If $3 \nmid N$, then we have

\[ \frac{1}{6} \phi(N) + 2^\nu(N) \leq \begin{cases} 10d & \text{if } d = 1, 2 \\ 6(d-1)^2 + 10 & \text{if } d \geq 3 \end{cases}. \]

Proof. From Tower Theorem 2.1, we obtain a smooth projective curve $C_{\mathbb{Q}}$ defined over $\mathbb{Q}$ and a finite morphism $f : X_0(N)_{\mathbb{Q}} \longrightarrow C_{\mathbb{Q}}$ defined over $\mathbb{Q}$ of degree $d'$ such that $1 \leq d' \leq d$ and $g(C) \leq (d/d' - 1)^2$. First assume that $g(C) \geq 1$. For a prime $p \nmid N$, the curve $X_0(N)_{\mathbb{Q}}$ has good reduction at $p$, hence by Good Reduction Lemma 5.1 (see §5), $C_{\mathbb{Q}}$ has also good reduction at $p$ and we obtain a finite morphism

\[ f \times \mathbb{F}_p : X_0(N)_{\mathbb{F}_p} \longrightarrow C_{\mathbb{F}_p} \]

defined over $\mathbb{F}_p$. Let $\mathbb{F}_q$ be the finite extension of $\mathbb{F}_p$ with $q = p^2$. Then since $g(C) \leq (d/d' - 1)^2$, by Weil’s theorem of the analogue of the Riemann hypothesis, we can bound the number of $\mathbb{F}_q$-rational points of $C_{\mathbb{F}_q}$ as

\[ \#C_{\mathbb{F}_q}(\mathbb{F}_q) \leq 1 + 2g(C)\sqrt{q} + q = 1 + 2pg(C) + p^2. \]

Since $g(C) \leq (d/d' - 1)^2$, from (3.8) above, we obtain

\[ \#(X_0(N)_{\mathbb{F}_q}(\mathbb{F}_q)) \leq d' \cdot \#C_{\mathbb{F}_q}(\mathbb{F}_q) \leq d' \cdot (1 + 2(d/d' - 1)^2p + p^2). \]

Now fixing $d$, set $H(p, d') = d' \cdot (1 + 2(d/d' - 1)^2p + p^2)$. For $1 \leq d' \leq d$, we can easily see that

\[ H(p, d') \leq \max\{H(p, 1), H(p, d)\} = \max\{p^2 + 1 + 2p \cdot (d - 1)^2, (p^2 + 1)d\} \]

From this and Lemma 3.1, putting $p = 2, 3$, we obtain the assertions (1) and (2). If $g(C) = 0$, then we have a finite morphism $X_0(N)_{\mathbb{F}_p} \longrightarrow C''_{\mathbb{F}_p}$ defined over $\mathbb{F}_p$ but of degree $1 \leq d'' \leq d$ where $C''_{\mathbb{F}_p}$ is a rational curve defined over $\mathbb{F}_p$. In this case, the bound becomes better than the former case, which completes the proof of theorem.

By using obvious inequalities $\phi(N) \geq N + 1$ and $\nu(N) \geq 1$, we obtain the following corollary.
Corollary 3.3. Let \( X_0(N) \) be the modular curve of level \( N \). If \( X_0(N) \) is \( d \)-gonal, that is, if it admits a finite map \( f : X_0(N) \rightarrow \mathbb{P}^1_N \) of degree \( d \), then we have the following.

1) If \( N \) is odd, then we have

\[
(3.10) \quad N \leq \begin{cases} 60d - 25 & \text{if } d = 1, 2 \\ 48(d - 1)^2 + 35 & \text{if } d \geq 3 \end{cases}.
\]

2) If \( 3 \nmid N \), then we have

\[
(3.11) \quad N \leq \begin{cases} 60d - 11 & \text{if } d = 1, 2 \\ 36(d - 1)^2 + 47 & \text{if } d \geq 3 \end{cases}.
\]

Remark 3.4. Since we know the genus formula of \( X_0(N) \), we know all the cases with \( g(N) := g(X_0(N)) \leq 1 \). If \( g(N) = 0 \), we have \( N = 1, \ldots, 10, 12, 13, 16, 18, 25 \), and if \( g(N) = 1 \), we have \( N = 11, 14, 15, 17, 19, \ldots, 21, 24, 27, 32, 36, 49 \). Ogg [O] showed that \( X_0(N) \) is a hyperelliptic curve of \( g(N) \geq 2 \) if and only if \( N = 22, 23, 26, 28, 29, 30, 31, 33, 35, 37, 39, 41, 46, 47, 50, 59, 71 \), (i.e. 19 values). This implies that if \( X_0(N) \) is 2-gonal then \( N \leq 71 \). This is obviously better than our bounds in Corollaries 3.3 and 3.5, because we use only rough estimates of \( \phi(N) \) and \( \nu(N) \).

For a general positive integer \( N \), write \( N = 2^i \cdot M \) such that \( M \) is odd. Then we have natural finite morphisms

\[
\varphi_1 : X_0(N) \rightarrow X_0(2^i) \quad \varphi_2 : X_0(N) \rightarrow X_0(M).
\]

By Lemma 1.3, if \( X_0(N) \) is \( d \)-gonal, then both of \( X_0(2^i) \) and \( X_0(M) \) are \( d \)-gonal. Hence we have the following corollary which gives a bounds for \( N \) by a polynomial in \( d \) of degree \( \leq 4 \).

Corollary 3.5. Let \( X_0(N) \) be the modular curve of level \( N \). If \( X_0(N) \) is \( d \)-gonal, that is, if it admits a finite map \( f : X_0(N) \rightarrow \mathbb{P}^1_N \) of degree \( d \), then we have the following.

\[
(3.12) \quad N \leq \begin{cases} (60d - 25)(60d - 11) & \text{if } d = 1, 2 \\ (48(d - 1)^2 + 35)(36(d - 1)^2 + 47) & \text{if } d \geq 3 \end{cases}.
\]

Remark 3.6. Let \( k = \mathbb{F}_p \) and fix an algebraic closure \( \overline{k} \) of \( k \). For \( N \) such that \( p \nmid N \), assume that the smooth projective curve \( X_0(N) \) is \( d \)-gonal. From Tower Theorem 2.1, we obtain a smooth projective curve \( C' \) defined over \( k \) and a finite morphism \( f : X_0(N)_k \rightarrow C' \) of degree \( d' \), \( 1 \leq d' \leq d \) satisfying that \( g(C') \leq (d/d' - 1)^2 \). From the same arguments as in Theorem 1.2, we obtain an inequality

\[
\#(X_0(N)_{\mathbb{F}_p^2}(\mathbb{F}_p^2)) \leq \max\{p^2 + 1 + 2p \cdot (d - 1)^2, (p^2 + 1) \cdot d\}.
\]
Then from Lemma 3.1, (3.3), we obtain an inequality
\[ 2^{\nu(N)} + 2s(p)\phi(N) \leq \max\{p^2 + 1 + 2p \cdot (d - 1)^2, (p^2 + 1) \cdot d\}. \]

Hence if \( s(p) > 0 \), then we obtain a bound for \( N \) by a constant only depending on \( d \) and \( p \).

§4 Bounding orders of torsions of Mordell-Weil group.

Let \( C \) be a smooth projective curve defined over an algebraically closed field \( k \) with the rational function field \( K = k(C) \), and let \( E \) be an elliptic curve defined over \( K \). Then we obtain a relatively minimal elliptic surface \( \pi : E \to C \) associated to \( E/K \). We call \( E/K \) is “constant”, if its \( K/k \)-trace is non-trivial. If \( E/K \) is constant, then the associated family \( \pi : C \to C \) is birational equivalent to \( E_0 \times C \) with an elliptic curve \( E_0 \) over \( k \). It is known that the Mordell-Weil group \( E(K) \) of \( E \) is finitely generated, if \( E/K \) is not constant (cf. [La]).

**Lemma 4.1.** Let \( C \) and \( E/K \) be as above, and assume that the characteristic of the base field \( k \) is zero and \( E/K \) is not constant. Then if the Mordell-Weil group \( E(K) \) has a cyclic subgroup of order \( N > 1 \). Then one obtains a surjective morphism \( h : C \to X_0(N)_K \).

**Proof.** Let \( \pi^0 : E^0 \to C^0 \) denote the morphism obtained by restricting \( \pi \) to the maximal open set \( E^0 \) on which \( \pi \) is smooth. Now assume that the Mordell-Weil group \( E(K) \) has a cyclic subgroup of order \( N \), it defines a cyclic group of order \( N \) on each fiber \( E_t \) for \( t \in C^0 \). Then since \( Y_0(N)_K \) is the moduli space of pairs \( (E, D) \) (see [D-R] and [Kat-Ma]), we have a natural non-constant morphism \( h^0 : C^0 \to Y_0(N) \), which extends to a finite morphism \( h : C \to X_0(N) \).

From Lemma 4.1 together with Lemma 1.3, we have the following

**Proposition 4.2.** Let \( C \) be a smooth projective curve defined over \( k \) with the rational function field \( K \) and assume that \( C \) is \( d \)-gonal. If there exists a non-constant elliptic curve \( E \) over \( K \) whose Mordell-Weil group \( E(K) \) has a cyclic group of order \( N \), then the modular curve \( X_0(N) \) is also \( d \)-gonal.

Together with Corollaries 3.3 and 3.5, this proposition implies the following theorem, which may be considered as an analogue of strong uniformly boundedness conjecture in the function field case.

**Theorem 4.3.** Let \( k \) be an algebraically closed field of characteristic zero and \( C \) be a smooth projective curve defined over \( k \), and let \( K = k(C) \) be the function field of \( C \). Then if there exists a non-constant elliptic curve \( E \) defined over \( K \) such that its Mordell-Weil group \( E(K) \) admits a torsion element of order \( N \). Then we have a polynomial function \( B(d) \) in \( d \) such that

\[ N \leq B(d). \]
For example, we take \( B(d) \) as the right hand side of (3.12).

§5 A lemma of good reduction of morphisms

In this section, we shall use the following lemma, which we have used in the proof of Theorem 3.2.

Lemma 5.1. Let \( C_1 \) and \( C_2 \) be projective smooth curves defined over \( \mathbb{Q} \) both of which are geometrically irreducible, and let \( f : C_1 \to C_2 \) be a dominant morphism of degree \( d \) which is also defined over \( \mathbb{Q} \). Assume that \( C_1 \) has good reduction at a prime integer \( p > 0 \). Then we have the following.

1) Assume that \( g(C_1) \geq 1 \). Then \( C_2 \) has a good reduction at \( p \) and \( f \) induces a finite morphism of degree \( d \)

\[ f_s : C_{1,s} \to C_{2,s}, \]

defined over \( \mathbb{F}_p \) where \( C_{i,s} \), \( i = 1, 2 \) denote the smooth projective curves defined over \( \mathbb{F}_p \) obtained from \( C_i \) respectively.

2) If \( g(C_2) = 0 \), then we obtain a dominant morphism

\[ f'_s : C_{1,s} \to C' \]

of degree \( d' \leq d \) defined over \( \mathbb{F}_p \) such that \( C' \) is a smooth rational curve defined over \( \mathbb{F}_p \).

Proof. Let us set \( S = \text{Spec}(\mathbb{Z}_p) \) where \( \mathbb{Z}_p \) denotes the ring of \( p \)-adic integers, and denote by \( \eta \) and \( s \) the generic and the closed points of \( S \) respectively. Moreover, we set \( \hat{S} = \text{Spec}(\mathbb{Z}_p^{sh}) \), where \( \mathbb{Z}_p^{sh} \) is the strict henselization of \( \mathbb{Z}_p \) whose residue field \( k = \mathbb{Z}_p^{sh}/(p) \) is a fixed algebraic closure \( \mathbb{F}_p \) of \( \mathbb{F}_p = \mathbb{Z}_p/(p) \). For each curve \( C_i \) over \( \mathbb{Q} \), we obtain a regular \( \mathbb{Z}_p \)-model of \( C_i \)

\[ \pi_i : C_i \to S = \text{Spec}(\mathbb{Z}_p) \]

such that:

i) \( C_i \) is a regular scheme,

ii) \( \pi_i \) is a proper flat morphism and

iii) \( C_{i,\eta} \simeq C_i \times_{\mathbb{Q}} \mathbb{Q}_p \).

Such models can be obtained by the blowing up of any projective model of \( C_i \) over \( S \) by blowing up (cf. [Lip].) Moreover since \( C_1 \) has good reduction, we can assume that \( \pi_1 : C_1 \to S \) is smooth.

Now let us assume that \( g(C_2) \geq 1 \), Then, by Lichtenbaum [Lic] and Shafarevich [Sha], we may assume that \( \pi_2 : C_2 \to S \) is the minimal model. Let \( J_i, i = 1, 2 \) denote the Jacobian varieties of \( C_i \) respectively. Then since \( J_1 \) has good reduction at \( p \), it is easy to see that \( J_2 \) also has good reduction at \( p \). (For a proof, one may use Serre-Tate criterion [S-T, Cor. 2], or just use the Néron models of \( J_1 \) and \( J_2 \).)
Since there exists a dominant morphism $C_1 \longrightarrow C_2$ defined over $\mathbb{Q}$, we obtain a dominant $S$-rational map

$$\varphi : C_1 \cdots \longrightarrow C_2.$$ 

Now considering the base extension $\hat{S} \longrightarrow S$, we obtain the induced $\hat{S}$-rational map

$$\varphi : \overline{C}_1 \longrightarrow \overline{C}_2$$

$$\pi_1 \downarrow \quad \varphi \quad \downarrow \pi_2$$

$$\hat{S}$$

Note that $\overline{C}_2$ is also the minimal model over $\hat{S}$. Since $J_2$ has good reduction at $p$, it implies that $\overline{C}_2$ has at least semistable reduction at $p$ by [D-M, Theorem 2.4]. Consider the irreducible decomposition of $\overline{C}_{2,s}$:

$$\overline{C}_{2,s} = \sum_{i=1}^{l} T_i.$$ 

The dual graph of $\overline{C}_{2,s}$ has no cycle because the Néron model has no torus part, hence each irreducible component $T_i$ is smooth and because $J_2$ has good reduction at $p$, one has $\sum_{i=1}^{l} g(T_i) = g(C_2) \geq 1$. Now we claim that:

**Claim:** $\overline{C}_{2,s}$ has only one irrational component, say, $T_1$.

If we admit this claim, we can show that $\overline{C}_{2,s} = T_1$, which also implies that $\overline{C}_{2,s}$ is smooth. Let $l$ denote the the number of irreducible components of $\overline{C}_{2,s}$ and assume that $l > 1$. Then since the dual graph of $\overline{C}_{2,s}$ has no cycle, there exists a component $T_i$, $i \geq 2$ such that

$$(\overline{C}_{2,s} - T_i) \cdot T_i = 1.$$ 

Since $\overline{C}_{2,s} \cdot T_i = 0$, we have $(T_i)^2 = -1$. However since all components $T_j$, $j \geq 2$ are smooth rational curves, this implies that $T_i$ is an exceptional rational curve of the first kind, which contradicts to the minimality of the model $\overline{C}_2$.

Now it is easy to see that $\pi_2 : \overline{C}_2 \longrightarrow \hat{S}$ is smooth, and since $\hat{S} \longrightarrow S$ is faithfully flat, we conclude that $\pi_2$ is smooth.

Now we prove the claim. After a sequence of quadratic transformations with centers in the special fiber of $\pi_1$, we obtain a birational morphism $\tau : \Gamma \longrightarrow \overline{C}_1$ such that $\hat{S}$-rational map $\varphi$ induces a dominant $\hat{S}$-morphism $\phi : \Gamma \longrightarrow \overline{C}_2$. Note that since $\pi_1$ and $\pi_2$ are proper $\phi$ must be surjective. Consider the following commutative diagram (cf. [Sha]):

$$\begin{array}{c}
\Gamma \\
\tau \downarrow \varphi \downarrow \phi
\end{array}$$

$$\begin{array}{c}
\overline{C}_1 \\
\pi_1 \downarrow \quad \cdots \longrightarrow \quad \overline{C}_2 \\
\hat{S} \downarrow \pi_2
\end{array}$$
Set \( F = \overline{C_{1,s}} \), the special fiber of \( \pi_1 \) and let \( F' \) be the proper transform of \( F \) by \( \tau \), \( \overline{\Gamma_s} \) the special fiber of \( \tau \circ \pi_1 \), and write

\[
\overline{\Gamma_s} = F' + \sum_i m_i E_i.
\]

Here all of \( \{E_i\} \) are exceptional rational curves obtained by the blowing up \( \tau \). Now consider the morphism \( \phi_s : \overline{\Gamma_s} \rightarrow \overline{C_{2,s}} \). Since \( F' \) is the only one irrational component of \( \overline{\Gamma_s} \), \( \overline{C_{2,s}} \) has at most one irrational component, which proves the claim.

Next we prove assertion 2) of the lemma. Assume that \( g(C_2) = 0 \) and let

\[
f : C_1 \rightarrow C_2
\]

be the dominant morphism of degree \( d \).

Then we obtain a dominant \( \mathbb{Z}_p \)-rational map \( f : C_1 \cdots \rightarrow C_2 \). As in the former case, by Shafarevich [Sha], we obtain the commutative diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\phi} & C_2 \\
\tau & \xrightarrow{\phi} & \pi_2 \\
\pi_1 & \xrightarrow{\phi} & S \\
\end{array}
\]

Here \( \tau : \Gamma \rightarrow C_1 \) is obtained by a sequence of quadratic transformations with centers in the special fiber of \( \pi_1 \). Let \( F' \) be the proper transform of \( F = \overline{C_{1,s}} \). If the restriction of \( \tau \) to \( F' \simeq C_{1,s} \) is not constant, the image \( \phi(F') \) must be a rational curve \( \overline{C_{2,s}} \). The degree of the obtained map \( F' \rightarrow \overline{C_{2,s}} \) is equal to or less than \( d \). If the restriction of \( \phi \) to \( F' \) is constant, we will blow up \( C_2 \) at the closed point \( x = \tau(F') \) and its infinitely near points in order to make the morphism \( \tau \) flat. As mentioned in remark before the proof of [B-L-R, Prop. 6, 3.5] (see also [R-G]), there exists a blowing up \( V \rightarrow \overline{C_{2,s}} \) such that the induced morphism \( \phi' : \Gamma' \rightarrow V \) is flat. (Here \( \Gamma' \) is the schematic closure of \( \Gamma_1 \) in \( \Gamma \times_{C_2} V \).) Since \( \phi' \) is flat, it maps the proper transform of \( F' \simeq C_{1,s} \) by \( \Gamma' \rightarrow \Gamma \) onto a curve \( C' \) which is a rational curve over \( \mathbb{F}_p \) arising as an exceptional curve of the blowing up. Therefore, we obtain a surjective morphism \( C_{1,s} \rightarrow C' \), whose degree \( d' \) is less than or equal to \( d \), the degree of \( f \) at the generic fiber.

**Remark 5.2.** Without changing the above proof, we can extend the statement of Lemma 5.1 to the case of curves over a discrete valued field \( K \) with the integer ring \( R \) under the assumption that the residue field \( R/m \) is perfect.

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