Moebius Schrödinger

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(1). Let \( \mu(n) \) be the Moebius function and consider the Schrödinger operator on \( \mathbb{Z}_+ \)
\[
H = \Delta + \lambda \mu \quad (\lambda \neq 0 \text{ arbitrary}).
\] (1.0)

We prove the following

**Theorem 1.** For \( E \in \mathbb{R} \) outside a set of 0-measure, any solution \( \psi = (\psi_n)_{n \geq 0}, \psi_0 = 0, \psi \neq 0 \) of
\[
H\psi = E\psi
\]
satisfies
\[
\lim \frac{\log^+ |\psi_n|}{n} > 0.
\] (1.1)

Recalling the spectral theory of 1D Schrödinger operators with a random potential, Theorem 1 fits the general heuristic, known as the ‘Moebius randomness law’ (cf. [Sa]). The question whether (1.0) satisfies Anderson localization remains open and is probably difficult.

The fact that \( H \) has no ac-spectrum is actually immediate from the following result of Remling.

**Proposition 1.** ([R], Theorem 1.1): Suppose that the (half line) potential \( V(n) \) takes only finitely many values and \( \sigma_{ac} \neq \phi \). Then \( V \) is eventually periodic.
We will use again Proposition 1 later on, in the proof of the Theorem.

Let $X \subset \{0, 1, -1\}^\mathbb{Z}$ be the point-wise closure of the set $\{T^j\overline{w}; j \in \mathbb{Z}\}$, where $T$ is the left shift and $\overline{w}$ defined by

\[ \overline{w}_n = \begin{cases} \mu(n) & \text{for } n \in \mathbb{Z}_+ \\ 0 & \text{for } n \in \mathbb{Z}_- \end{cases} \]  

(2.1)

Let

\[ \nu_N = \frac{1}{N} \sum_{j=0}^{N-1} \delta_{T^j\overline{w}} \quad (\delta_x = \text{Dirac measure at } x) \]

and $\nu \in \mathcal{P}(X)$ a weak*-limit point of $\{\nu_N\}$.

Then $\nu$ is a $T$-invariant probability measure on $X$.

The only property of the Moebius function exploited in the proof of Theorem II is the following fact.

**Lemma 1.** For no element $\omega \in X, (\omega_n)_{n \geq 0}$ is eventually periodic, unless $\omega_n = 0$ for $n$ large enough. Similarly for $(\omega_n)_{n \leq 0}$.

**Proof.** Suppose $\omega$ eventually periodic. Hence there is $n_0 \in \mathbb{Z}_+$ and $d \in \mathbb{Z}_+$ such that

\[ \omega(n + d) = \omega(n) \quad \text{for } n \geq n_0. \]  

(2.2)

Take $N = 10^3(n_1 + d^3)$ and choose $n_1 \geq n_0$ and $k \in \mathbb{Z}_+$ such that

\[ \omega(n) = \mu(k + n) \quad \text{for } n \in [n_1, n_1 + N]. \]  

(2.3)

Let $d < p < 10d$ be a prime. Taking $n \in [n_1, n_1 + \frac{N}{2}]$, there is $0 \leq j < p^2$ such that $k + n + jd \equiv 0 \pmod{p^2}$ and thus $\mu(k + n + jd) = 0$. Since $n + jd \in [n_1, n_1 + N]$, (2.3), (2.2) imply that $\mu(k + n + jd) = \omega(n + jd) = \omega(n)$ and therefore $\omega = 0$ on $[n_1, n_1 + \frac{N}{2}]$, hence on $[n_1, \infty[. \square$

Denote for $\omega \in X$

\[ H_\omega = \Delta + \lambda \omega. \]  

(2.4)

Combined with Proposition \[ \square \] Lemma \[ \square \] implies

**Lemma 2.**

\[ \sigma_{ac}(H_\omega) = \phi \quad (\nu - a.e.) \]
Proof. Denoting $H^\pm_\omega$ the corresponding halfline $SO$’s, we have

$$\sigma_{ac}(H_\omega) = \sigma_{ac}(H^+_\omega) \cup \sigma_{ac}(H^-_\omega)$$

and these sets are empty, unless

$$\omega \in \bigcup_{k=1}^\infty \{ \omega \in X; \omega_n = 0 \text{ for all } n \geq k \text{ or all } n \leq -k \}.$$  \hspace{1cm} (2.5)

Clearly $\nu(2.5) = 0$.

The measure $\nu$ need not be $T$-ergodic, so we consider its ergodic decomposition

$$\nu = \int \nu_\alpha d\alpha.$$ \hspace{1cm} (2.6)

For each $\alpha$, let $\gamma_\alpha(E)$ be the Lyapounov exponent of $H_\omega$, i.e.

$$\gamma_\alpha(E) = \lim_{N \to \infty} \frac{1}{N} \log \left\| \prod_{0}^{N} \begin{pmatrix} E - \lambda \omega_n & -1 \\ 1 & 0 \end{pmatrix} \right\| \quad (\nu_\alpha \text{ a.e.}).$$ \hspace{1cm} (2.7)

Next, we apply Kotani’s theorem (for stochastic Jacobi matrices, as proven in [Si], Theorem 2).

**Proposition 2. (assuming $(\Omega, \mu, T)$ ergodic).**

If $\gamma(E) = 0$ on a subset $A$ of $\mathbb{R}$ with positive Lebesgue measure, then $E^{ac}_\omega(A) \neq 0$ for a.e. $\omega$.

($E^{ac}$ denote the projection on the ac-spectrum).

Apply Proposition 2 to $H_\omega$ on $(X, \nu_\alpha)$. By Lemma 2 $E^{ac}_\omega = 0$, $\nu_\alpha$ a.e., hence $\{ E \in \mathbb{R}; \gamma_\alpha(E) = 0 \}$ is a set of zero Lebesgue measure. For $E$ outside a subset $E_{\ast} \subset \mathbb{R}$ of zero Lebesgue measure, we have that $\gamma_\alpha(E) > 0$ for almost all $\alpha$ in (2.6), therefore

$$\liminf_{N \to \infty} \frac{1}{N} \log \left\| \prod_{0}^{N} \begin{pmatrix} E - \lambda \omega_n & -1 \\ 1 & 0 \end{pmatrix} \right\| \nu(d\omega) \geq$$

$$\int \left\{ \liminf_{N \to \infty} \frac{1}{N} \log \left\| \prod_{0}^{N} \begin{pmatrix} E - \lambda \omega_n & -1 \\ 1 & 0 \end{pmatrix} \right\| \nu_\alpha(d\omega) \right\} d\alpha \geq$$

3
\[
\int \gamma_\alpha(E) d\alpha > 0. \quad (2.8)
\]

Denoting \( R_N \) the restriction operator to \([1, N]\), let
\[
H^{(N)}_\omega = R_N H_\omega R_N
\]
\[
G^{(N)}_\omega(E) = (H^{(N)}_\omega - E + i0)^{-1} \quad (= \text{restricted Green's function}).
\]

Recall that by Cramer’s rule, for \(1 \leq k_1 \leq k_2 \leq N\)
\[
|G^{(N)}_\omega(E)(k_1, k_2)| = \frac{\det[H^{(k_1-1)}_\omega - E] \cdot \det[H^{(N-k_2)}_\omega - E]}{|\det[H^{(N)}_\omega - E]|} \quad (2.9)
\]
and also the formula
\[
M_N(E, \omega) = \prod_{N} \begin{pmatrix} E - \lambda \omega_n & -1 \\ 1 & 0 \end{pmatrix}
\]
\[
= \begin{bmatrix} \det[E - H^{(N)}_\omega] & \det[E - H^{(N-1)}_{T^2\omega}] \\ \det[E - H^{(N-1)}_\omega] & \det[E - H^{(N-2)}_{T^2\omega}] \end{bmatrix}. \quad (2.10)
\]

Using the above formalism, it is well-known how to derive from positivity of the Lyapounov exponent, bounds and decay estimates on the restricted Green’s functions. Since ergodicity of the measure is used, application to the preceding requires to start from the \(\nu_\alpha\).

For \(E \in \mathbb{R}, \delta, c > 0, M \in \mathbb{Z}_+\), define
\[
\Omega_{E, \delta, c, M} = \{ \omega \in X; \|G^{(M)}_\omega(E)\| < e^{\delta M} \text{ and } |G^{(M)}_\omega(E)(k, k')| < e^{-c|k-k'|} \}
\]
if \(1 \leq k, k' \leq M\) and \(|k - k'| > \delta M\). \quad (2.11)

Fix \(\alpha\) and \(\delta > 0\). Then \(E\) a.e
\[
\lim_{M \to \infty} \nu_\alpha(\Omega_{E, \delta, \frac{1}{2} \gamma_\alpha(E), M}) = 1. \quad (2.12)
\]

Using Fubini arguments and (2.6), we derive the following
Lemma 3. Given \( \varepsilon > 0 \), there is \( b > 0 \), such that for all \( \delta > 0 \), there is a subset \( E_\varepsilon \subset \mathbb{R} \), \( \text{mes} E_\varepsilon < \varepsilon \) and some scale \( M \) satisfying

\[
\nu(\Omega_{E_\varepsilon, \delta, b, N}) > 1 - \varepsilon \text{ for } E \notin E_\varepsilon \text{ and } N > M.
\] (2.13)

(3). Using the definition of \( \nu \), we re-express (2.13) in terms of the Moebius function.

Let \( H \) be as in (1.0). For \( I \subset \mathbb{Z}_+ \) an interval, denote

\[
H_I = R_I HR_I
\]
and

\[
G_I(E) = (H_I - E + io)^{-1}.
\] (3.1)

Let \( S = S_{E, \delta, M} \) be defined by

\[
S = \{ k \in \mathbb{Z}; \| G_{[k, k+N]}(E) \| < e^{\delta N} \text{ and } |G_{[k, k+N]}(E)(k', k'')| < e^{\delta N} \text{ if } k \leq k', k'' \leq k + N, |k' - k''| > \delta N \}.
\] (3.2)

Property (2.13) then translates as follows

\[
\lim_{\ell \to \infty} \frac{1}{\ell} |S \cap [1, \ell]| > \frac{1}{2}
\] (3.4)

for \( E \notin E_\varepsilon \) and \( N > M \). Here ‘lim’ refers to the Banach limit in the definition of \( \nu \).

Fix \( \varepsilon > 0 \) a small number, take \( 0 < b < \frac{1}{10} \) as in Lemma 3 and let \( \delta = b^{10} \).

Let \( E_\varepsilon \subset \mathbb{R} \), \( M > \delta^{-2} + \frac{1}{\varepsilon} \), satisfy the lemma. Hence, from (3.4)

\[
\lim_{\ell \to \infty} \frac{1}{\ell} |S_{E, \delta, M} \cap [1, \ell]| > \frac{1}{2} \text{ for } E \notin E_\varepsilon.
\] (3.5)

Choose \( \ell \gg M \) such that

\[
\frac{1}{\ell} |S_{E, \delta, M} \cap [1, \ell]| > \frac{1}{2} \text{ for } E \notin E_\varepsilon'
\]

where \( E_\varepsilon \subset E_\varepsilon' \subset \mathbb{R} \) satisfies

\[
\text{mes} E_\varepsilon' < 2\varepsilon.
\] (3.7)

Next we rely on a construction from [B], Lemma 6.1 and Corollary 6.54. We recall the statement

\[ \text{[Unsure of recall statement]} \]
Lemma 4. Let \(0 < c_0 < 1, \ 0 < c_1 < \frac{1}{10}\) be constants, \(0 < \delta < c_1^{10}\) and \(\ell \gg M > \delta^{-2}\).

Let
\[
A = v_n \delta_{nn'} + \Delta \quad (1 \leq n, n' \leq \ell)
\]
(hence \(A\) is an \(\ell \times \ell\) matrix) with diagonal \(v_n\) arbitrary, bounded, \(|v_n| = 0(1)\).

Let \(U \subset \mathbb{R}\) be a set of energies \(E\) such that for each \(E \in U\), the following holds:

There is a collection \(\{I_\alpha\}\) of disjoint intervals in \([1, \ell]\), \(|I_\alpha| = M\) such that for each \(\alpha\)
\[
\|(R_{I_\alpha}(A - E)R_{I_\alpha})^{-1}\| < \epsilon M \quad (3.9)
\]
and
\[
|(R_{I_\alpha}(A - E)R_{I_\alpha})^{-1}(k, k')| < e^{-c_1|k - k'|} \text{ for } k, k' \in I_\alpha, |k - k'| > \delta M \quad (3.10)
\]
holds, and
\[
\sum_\alpha |I_\alpha| > c_0 \ell. \quad (3.11)
\]

Then there is a set \(\mathcal{E}'' \subset \mathbb{R}\) so that
\[
\text{mes}(\mathcal{E}'') < \frac{1}{M} \quad (3.12)
\]
and for \(E \in U \setminus \mathcal{E}''\),
\[
\max_{1 \leq x \leq \frac{\ell}{10}} \max_{\ell y \geq \ell - \frac{c_0 \ell}{10}} |(A - E)^{-1}(x, y)| < e^{-\frac{1}{4} c_0 c_1 \ell}. \quad (3.13)
\]

The proof of Lemma 4 is a bit technical, but uses nothing more than the resolvent identity and energy perturbation.

Let \(v_n = \lambda \mu(n)\).

Take \(c_0 = \frac{1}{2}, c_1 = b, U = \mathbb{R} \setminus \mathcal{E}'_\epsilon\) with \(\mathcal{E}'_\epsilon\) as above:

Let \(\ell_0 \gg M\) satisfy (3.6). From the definition (3.3) of \(S_{E, \delta, M}\) and (3.6), we clearly obtain a collection \(\{I_\alpha\}\) of \(M\)-intervals in \([1, \ell]\) such that (3.9)-(3.11) hold.
It follows that for $E$ outside of the set $\mathcal{E}' = \mathcal{E}' \cup \mathcal{E}''$ of measure at most $2\varepsilon + \frac{1}{M} < 3\varepsilon$, one has for $b' \sim b$ that

$$\max_{1 \leq x \leq \frac{2\varepsilon + 1}{M}} |G_{[1,\ell]}(E)(x, y)| < e^{-b'\ell}. \quad (3.14)$$

Note that $b' > 0$ depends on $\varepsilon$ and $\nu$ and $\mathcal{E}''$ depends on $\ell$, which can be taken arbitrarily large in the subsequence of $\mathbb{Z}_+$ used to define $\nu$. Since this subsequence is arbitrary, it follows that there is some $b' = b_\varepsilon$ and $\ell_\varepsilon \in \mathbb{Z}_+$ such that for $\ell > \ell_\varepsilon$

$$\text{mes} \left\{ E \in \mathbb{R}; \max_{1 \leq x \leq \frac{2\varepsilon + 1}{M}} |G_{[1,\ell]}(E)(x, y)| > e^{-b'\ell} \right\} = \text{mes} \tilde{E}_\varepsilon < \varepsilon. \quad (3.15)$$

Assume $\psi = (\psi_n)_{n \geq 0}, \psi_0 = 0$ a solution of

$$H\psi = E\psi.$$

Taking $\ell$ large, one has by projection

$$H_{[1,\ell]}\psi^{(\ell)} + \psi_{\ell + 1} e_\ell = E\psi^{(\ell)} \quad (3.16)$$

where $\psi^{(\ell)} = \sum_{1 \leq x \leq \ell} \psi_x e_x, \{e_x\}$ the unit vector basis.

Hence

$$\psi^{(\ell)} = -\psi_{\ell + 1} G_{[1,\ell]}(E) e_\ell$$

and fixing some coordinate $x \geq 1$, for $\ell$ large enough

$$|\psi_x| \leq |\psi_{\ell + 1}| |G_{[1,\ell]}(E)(x, \ell)|. \quad (3.17)$$

Take $x$ with $\psi_x \neq 0$. Assuming

$$\lim_{n} \frac{\log^+ |\psi_n|}{n} = 0$$

it follows from (3.17) that

$$\lim_{\ell} \frac{1}{\ell} \log^+ |G_{[1,\ell]}(E)(x, \ell)|^{-1} = 0. \quad (3.18)$$
From the definition of $\tilde{E}_\ell$ in (3.15), this means that

$$E \in \bigcup_{\ell_0} \bigcap_{\ell \geq \ell_0} \tilde{E}_\ell$$

which is a set of measure $\leq \varepsilon$.

Letting $\varepsilon \to 0$, Theorem 1 follows.

(4). Taking into account the comment made prior to Lemma 1, our argument gives the following more general result, that can be viewed as a refinement of [R].

**Theorem 2.** Suppose that the (half line) potential $(V_n)_{n \geq 0}$ takes only finitely many values and satisfies the following property

$$\lim_{r \to \infty} \lim_{N \to \infty} \frac{1}{N} |\{1 \leq k \leq N; V_k = \omega_0, V_{k+1} = \omega_1, \ldots, V_{k+r} = \omega_r\}| = 0 \quad (4.1)$$

whenever $\omega = (\omega_r)_{r \geq 0}$ is a periodic sequence in the pointwise closure of the sequences $(V_{n+j})_{n \in \mathbb{Z}^+}$ ($j \in \mathbb{Z}^+$).

Then the Schrödinger operator $H = \Delta + V$ satisfies the conclusion of Theorem 1.

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**References**

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