ON EXAMPLES OF RANK-TWO SYMBOLIC SHIFTS

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Abstract. We study rank-two symbolic systems (as topological dynamical systems) and prove that the Thue-Morse sequence and quadratic Sturmian sequences are rank-two and define rank-two symbolic systems.

1. Introduction

Rank-one measure-preserving transformations have played an important role in ergodic theory since the pioneering work of Chacón [4]. In this paper, rather than considering the notion of rank-one in measurable dynamics we are interested in studying rank-one and higher rank systems strictly from the point of view of topological dynamics, in particular as symbolic shifts. Of course, symbolic systems have been used extensively in ergodic theory. In [13], Kalikow discusses a symbolic model for rank-one measure-preserving transformations, and Ferenczi [8] in his survey on rank-one finite measure-preserving transformations mentions the symbolic definition for rank-one transformations. Later in [3], Bourgain used a class of symbolic rank-one transformations for which he proved the Moebius disjointness law, and rank-one symbolic shifts are also considered in [1, 5, 7]. It was in [11] that Gao and Hill started a systematic study of (non-degenerate) rank-one shifts as topological dynamical systems, and proved several properties for rank-one symbolic shifts. In this paper we study higher rank systems and prove that the system defined by the Thue-Morse sequence and systems defined by quadratic Sturmian sequences are rank-two.

The terminology “rank-one” comes from “rank-one” cutting and stacking systems [4]. As shown by Kalikow [13], one can encode cutting and stacking systems as a shift on a symbolic system and he shows that the two systems are measurably isomorphic when the symbolic sequence is non-periodic. Gao and Hill introduced the notion of (symbolic) rank-one shifts as an extension to earlier ideas and developed various results associated to symbolic rank-one systems. We start by first defining rank-one words using the definition provided by Gao and Hill in [11].

Definition 1.1. Let $F$ be the set consisting of all finite words in the alphabet $\{0, 1\}$ that begin and end with 0. Let $V \in \{0, 1\}^\mathbb{N}$. We say that $V$ is built from $v \in F$ if there exists a sequence $\{a_i\}_{i \geq 1}$ of natural numbers such that $V = v^{a_1}v^{a_2}v^{a_3} \cdots$. Let

$$A_V = \{v \in F : V \text{ is built from } v\}.$$ 

We say that the infinite word $V$ is rank-one if $A_V$ is infinite.

Example. The infinite word

$$U = 0101010101 \cdots$$

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is a rank-one word since
\[ A_U = \{0, 010, 01010, 0101010 \cdots \} \]
is infinite.

A simple observation is that as long as \( A_V \neq \emptyset, 0 \in A_V \). This is because \( A_V \subseteq F \), and for all \( v \in F, v \) is built from 0. The definition of built from used here is extended for finite words in Definition 1.3 below.

**Example** A well-known example of a rank-one word is the Chacón sequence, see e.g. [6]. We define the Chacón sequence as the limit \( \lim_{n \to \infty} B_n \) where
\[
B_0 = 0, \\
B_{i+1} = B_i B_i 1 B_i 
\]
for all non-negative integers \( i \geq 0 \).

One can verify that the Chacón sequence satisfies the definition of a rank-one sequence. Another description of the Chacón sequence can be obtained by taking the limit of digit-wise substitutions starting from 0 with the rule: \( 0 \to 0010, 1 \to 1 \).

Gao and Hill gave another definition of rank-one words in [10].

**Definition 1.2.** For a finite word \( \alpha \) over the alphabet \( \{0, 1\} \), let \( |\alpha| \) denote the length of \( \alpha \). Let \( F \) be as before the set of finite words over the alphabet \( \{0, 1\} \) that start and end with 0. A **generating sequence** is an infinite sequence \((v_n)\) of finite words in \( F \) defined inductively by
\[
v_0 = 0, \\
v_{n+1} = v_n 1^{a_{n,1}} v_n 1^{a_{n,2}} \cdots v_n 1^{a_{n,q_n-1}} v_n 
\]
where \( q_n > 2 \) and \( a_{n,i}, 0 < i < q_n \) are non-negative integers. An infinite word \( V \in \{0, 1\}^\mathbb{N} \) is rank-one if there exists a generating sequence \((v_n)\) such that \( V \upharpoonright |v_n| = v_n \) for all \( n \in \mathbb{N} \), i.e. the first \( |v_n| \) symbols of \( V \) is \( v_n \) for any \( n \in \mathbb{N} \). We then write \( V = \lim_{n \to \infty} v_n \).

We will use Definition 1.1 and Definition 1.2 interchangeably. We thus want to point out that Definition 1.1 and Definition 1.2 are equivalent. Before we prove the equivalency of the two definitions, we first extend the definition of built from defined in Definition 1.1 to finite words.

**Definition 1.3.** We say that a finite word \( w \) is built from \( v \in F \) if \( w = v 1^{a_1} v 1^{a_2} v \cdots v 1^{a_k} v \) such that \( k \geq 1 \) and \( a_i, 1 \leq i \leq k \) are non-negative integers.

**Lemma 1.4.** For any \( V \in \{0, 1\}^\mathbb{N} \), if there exist \( x, y \in A_V \) with \( |y| > |x| \), then either
1. \( y \) is built from \( x \), or
2. there exists a word \( z \in A_V \) that is built from both \( x \) and \( y \).

**Proof.** Since \( A_V \neq \emptyset \), we can write
\[ V = 01^{a_{V,1}} 01^{a_{V,2}} 0 \cdots 
\]
for some non-negative integers \( a_{V,i} \) for all \( i > 0 \). Since all words in \( A_V \) are built from 0, we can write
\[ x = 01^{a_{x,1}} 01^{a_{x,2}} 0 \cdots 01^{a_{x,q_x-1}} 0 
\]
for some \( q_x \geq 1 \) and non-negative integers \( a_{x,i} \) for all \( 0 < i < q_x \), and
\[ y = 01^{a_{y,1}} 01^{a_{y,2}} 0 \cdots 01^{a_{y,q_y-1}} 0 
\]
for some \( q_y \geq 2 \) and non-negative integers \( a_{y,i} \) for all \( 0 < i < q_y \). Note that \( x \) can be 0. Since \( x, y \in A_V \), \( V \) is built from both \( x \) and \( y \), so

\[
a_{V,i} = a_{x,j} \quad \text{for all } i = j \pmod{q_x}, j \neq 0, \text{ and}
\]

\[
a_{V,k} = a_{y,l} \quad \text{for all } k = l \pmod{q_y}, l \neq 0
\]

Note that by definition, if \( x, y \in A_V \), \( V \uparrow |x| = x \) and \( V \uparrow |y| = y \). Since \( |y| > |x|, q_y > q_x \). Suppose \( q_y \) is divisible by \( q_x \), then we have

\[
y = x^{a_y/q_x} x^{a_x} x \cdots x^{a_y/(q_y/q_x - 1) q_x} x
\]

so \( y \) is built from \( x \), and we are done. Now suppose \( q_y \) is not divisible by \( q_x \). Define the integer \( q_z = \text{lcm}(q_x, q_y) \), the least common multiple of \( q_x \) and \( q_y \). Consider the word

\[
z = 01^{a_z} 01^{a_z} 0 \cdots 01^{a_z, q_z - 1} 0
\]

that only contains the first \( q_z \) 0’s of \( V \) such that \( V \uparrow |z| = z \). Clearly, \( z \) is built from both \( x \) and \( y \). Since

\[
a_{V,i} = a_{x,j} \quad \text{for all } i = j \pmod{q_x}, j \neq 0, \text{ and}
\]

\[
a_{V,k} = a_{y,l} \quad \text{for all } k = l \pmod{q_y}, l \neq 0,
\]

\( a_{V,n} \neq a_{z,m} \) only at \( m = 0 \) when \( n = 0 \pmod{q_z} \). Therefore, \( z \) must build \( V \), so \( z \in A_V \).

We are now ready to prove the next theorem which shows that the two definitions of rank-one words provided by Gao and Hill are equivalent.

**Theorem 1.5.** For any \( V \in \{0,1\}^\mathbb{N} \), the following are equivalent:

1. \( A_V \) is infinite.
2. There exists a generating sequence \( (v_n) \) such that \( V \uparrow |v_n| = v_n \) for all \( n \in \mathbb{N} \).

**Proof.** (1) \( \rightarrow \) (2): Suppose \( A_V \) is infinite. Note that for any \( x \in A_V \), \( V \uparrow |x| = x \) by definition. Thus, we only need to show that there exists a sequence \( (v_n) \) in \( A_V \) such that \( v_{i+1} \) is built from \( v_i \) for all \( i \in \mathbb{N} \). We can inductively define the generating sequence \( (v_n) \):

As a base case, we may define \( v_0 = 0 \in A_V \).

Then given \( v_n \in A_V \), we choose any \( u \in A_V \) such that \( |u| > v_n \). Such \( u \) exists because \( A_V \) is infinite. By Lemma 1.4, we know that either

1. \( u \) is built from \( v_n \), and we define \( v_{n+1} = u \), or
2. there exists \( u' \in A_V \) that is built from \( v_n \), and we define \( v_{n+1} = u' \).

We have thus defined the generating sequence \( (v_n) \).

(2) \( \rightarrow \) (1): Suppose now that there exists a generating sequence \( (v_n) \) such that \( V \uparrow |v_n| = v_n \) for all \( n \in \mathbb{N} \). By definition, for any \( n \in \mathbb{N} \),

\[
v_{n+1} = v_n 1^{a_n,1} v_n 1^{a_n,2} \cdots v_n 1^{a_n, q_n - 1} v_n
\]

and

\[
\lim_{n \to \infty} v_{n+1} = V = v_n 1^{a_1} v_n 1^{a_2} v_n \cdots
\]

for some sequence of natural numbers \( \{a_i\}_{i \geq 1} \). Now notice that for any \( n, k \) such that \( n > k \), \( v_n \) is built from \( v_k \). It then follows that \( V \) is built from \( v_n \) for all \( n \in \mathbb{N} \), so \( A_V \) is infinite. \( \square \)
Corollary 1.6. Definition 1.1 of a rank-one word is equivalent to Definition 1.2 of a rank-one word.

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2. The Thue-Morse Sequence

A well-known example of a binary sequence is the Thue-Morse sequence. The Thue-Morse sequence was first introduced by Thue in 1906 and later discovered again by Morse in 1921. However, it was already implicit in an 1851 paper of Prouhet, which is why it is sometimes also referred to as the Prouhet-Thue-Morse sequence [2].

In this section, we first give a formal definition of the Thue-Morse sequence, then prove some properties of the sequence before concluding that the Thue-Morse sequence is not a rank-one sequence.

The Thue-Morse sequence is generated by taking the limit of continuous digit-wise substitutions starting from 0 with the rule:

\[
0 \rightarrow 01 \\
1 \rightarrow 10
\]

Before we define the Thue-Morse sequence formally, we first introduce some notation.

Definition 2.1. We define \( x(i) \) as the \( i \)th digit of the bi-infinite word \( x \) for \( i \in \mathbb{Z} \), \( V(i) \) as the \( i \)th digit of the infinite word \( V \) for \( i \in \mathbb{N} \), and \( v(i) \) as the \( i \)th digit of the finite word \( v \) for \( 0 \leq i \leq |v| \).

For example, with the above definition of \( V(i) \), we can express an infinite word \( V \) as \( V = V_{(1)}V_{(2)}V_{(3)} \cdots \).

Definition 2.2. Let \( \alpha \) be any bi-infinite, infinite or finite word. We define the complement of \( \alpha \), denoted \( \alpha^c \), digit by digit such that:

\[
\alpha^c(i) = \begin{cases} 
0 & \text{if } \alpha(i) = 1, \\
1 & \text{if } \alpha(i) = 0
\end{cases}
\]

Definition 2.3. Define the Thue-Morse sequence as the limit \( \lim_{n \to \infty} M_n \) where

\[
M_0 = 0 \\
M_{i+1} = M_i M_i^c \text{ for all } i > 0
\]

The above definition of the Thue-Morse sequence can be interpreted as the word obtained by starting with 0 and successively appending the complement of the existing word obtained at the end of the existing word.
The Thue-Morse sequence is invariant under the substitution
\[ \phi(a) = \begin{cases} 
01 & \text{if } a = 0, \\
10 & \text{if } a = 1
\end{cases} \]

To be precise, for any finite word \( v \), we define
\[ \phi(v) = \phi(v(1))\phi(v(2))\cdots \phi(v(|v|)) \]
and for any infinite word \( V \), we define
\[ \phi(V) = \phi(V(1))\phi(V(2))\cdots \]

In fact, we can prove that the Thue-Morse sequence can also be defined as the limit \( \lim_{n \to \infty} \phi^n(0) \).

**Proposition 2.4.** For any \( n \in \mathbb{N} \), \( M_n = \phi^n(0) \) and \( M_n^c = \phi^n(1) \).

**Proof.** We prove that \( M_n = \phi^n(0) \) and \( M_n^c = \phi^n(1) \) using induction. For the base case, we note that \( M_1 = 01 = \phi(0), M_1^c = 10 = \phi(1) \). Now assume that \( M_n = \phi^n(0) \) and \( M_n^c = \phi^n(1) \) for some \( n \leq k \). We want to show that \( M_{n+1} = \phi^{n+1}(0) \) and \( M_{n+1}^c = \phi^{n+1}(1) \). From the inductive hypothesis, we know that
\[ M_{n+1} = M_n M_n^c \]
\[ = \phi^n(0)\phi^n(1) \]
\[ = \phi^n(01) \]
\[ = \phi^{n+1}(0) \]

and
\[ M_{n+1}^c = M_n^c M_n \]
\[ = \phi^n(1)\phi^n(0) \]
\[ = \phi^n(10) \]
\[ = \phi^{n+1}(1) \]

We thus conclude by induction that \( M_n = \phi^n(0) \) and \( M_n^c = \phi^n(1) \) for all \( n \in \mathbb{N} \). \( \square \)

**Proposition 2.5.** The Thue-Morse sequence, denoted as \( M \), is also the limit \( M = \lim_{n \to \infty} \phi^n(0) \).

**Proof.** From Proposition 2.4, \( M_n = \phi^n(0) \) for any \( n \in \mathbb{N} \). Thus,
\[ M = \lim_{n \to \infty} M_n = \lim_{n \to \infty} \phi^n(0) \]
\( \square \)

Using the inductive definition in Definition 2.3, we can immediately observe that at each iteration \( n \), \( M_n \) always starts with 0110 and ends with either 1001 or 0110. This observation is crucial in proving that the Thue-Morse sequence is not rank-one.

**Lemma 2.6.** For all \( n > 2 \), \( M_n \) always starts with 0110 and ends with either 1001 or 0110.

**Proof.** We prove this by induction. For our base case, note that \( M_3 = 0110 \) and \( M_4 = 01101001 \). Assume now that \( M_n \) starts with 0110 and ends with either 1001 or 0110 for some \( n \leq k \). We want to show that \( M_{n+1} \) starts with 0110 and ends with either 1001 or 0110. From Definition 2.3, \( M_{n+1} = M_n M_n^c \). Since \( M_n \) starts with 0110 by the inductive hypothesis, \( M_{n+1} \) starts with...
0110. Since $M_n$ ends with either 1001 or 0110 by the inductive hypothesis, $M_n^c$ ends with either $(1001)^c = 0110$ or $(0110)^c = 1001$, so $M_{n+1}$ ends with either 1001 or 0110. We conclude by induction that for all $n > 2$, $M_n$ starts with 0110 and ends with either 1001 or 0110.

It immediately follows that at each iteration, $M_n^c$ always begins with $(0110)^c = 1001$.

In order to prove that the Thue-Morse sequence is not rank-one, we use the notion of sub-word for finite and infinite words, and prove a property of the Thue-Morse sequence.

**Definition 2.7.** For any infinite word $V \in \{0, 1\}^\mathbb{N}$ and finite word $w \in \{0, 1\}^n$ for some $n \in \mathbb{N}$, we say that $w$ is a sub-word of $V$ if there exists an integer $k \geq 0$ such that $w(i) = V(i+k)$ for all $0 \leq i \leq n$. Similarly, for any finite word $v \in \{0, 1\}^m$ for some $m \in \mathbb{N}$ and finite word $w \in \{0, 1\}^n$ for some $n \in \mathbb{N}$, we say that $w$ is a sub-word of $v$ if there exists an integer $k$ such that $0 \leq k \leq m - n$ and $w(i) = v(i+k)$ for all $0 \leq i \leq n$.

In order to prove that the Thue-Morse sequence is not rank-one we use the fact that the Thue-Morse sequence, $M$, contains $\{01010, 10101\}$. Since $M$ is not a sub-word of $01010$ or $10101$, by Lemma 2.8, we can conclude that the Thue-Morse sequence is not rank-one.

**Lemma 2.8.** $01010$ and $10101$ are not sub-words of the Thue-Morse sequence.

**Proof.** We prove that the Thue-Morse sequence, $M$, does not contain $01010$ or $10101$ by induction. We observe that for $n \leq 3$, the first $2^n$ digits of $M$ do not contain $01010$ or $10101$ - the first $2^3 = 8$ digits of the Thue-Morse sequence is 01101001. We now assume that for some $n = k \geq 3$, the first $2^n$ digits of $M$ do not contain $01010$ or $10101$. We show that the first $2^{n+1}$ digits of $M$ do not contain $01010$ or $10101$. Similar to Definition 2.3, we denote $M_{n+1} = M \upharpoonright 2^{n+1}$ (i.e. the first $2^{n+1}$ digits of $M$). Note that from Definition 2.3, we know that $M_{n+1} = M_nM_n^c$. Thus, in order to show that $M_{n+1}$ does not contain $01010$ or $10101$, we only need to show that:

- $M_n$ does not contain 01010 or 10101. This follows from the inductive hypothesis.
- $M_n^c$ does not contain 01010 or 10101. This follows from the inductive hypothesis. Since $(01010)^c = 10101$ and $(10101)^c = 0110$, if $M_n^c$ contains 01010 or 10101, then $M_n$ contains 10101 or 01010.
- The middle eight digits of $M_{n+1} = M_nM_n^c$, $M_n(2^n-3)M_n(2^n-2)\cdots M_n(2^n)M_n^c(1)M_n^c(2)\cdots M_n^c(4)$ does not contain 01010 or 10101. From Lemma 2.6, the middle eight digits of $M_{n+1}$ for any $n$ is either 10011001 or 01101001, both of which do not contain 01010 and 10101.

Therefore, we can conclude that the Thue-Morse sequence, $M$, does not contain 01010 or 10101.

One could also use Lemma 2.8 to prove the known property that the Thue-Morse sequence is in fact free of cubes and the details are left to the reader.

For clarity, we divide our proof into two parts. First, we prove that the Thue-Morse sequence cannot be built from a word of even length. Then we prove that the Thue-Morse sequence cannot be built from a word of odd length other than 0. These two proofs are very much similar. Finally, putting the two parts together, we conclude that the Thue-Morse sequence is not rank-one.

**Lemma 2.9.** The Thue-Morse sequence is not built from a word of even length.
Then, there exists a finite word $v$ shorter than we will denote as Morse sequence starts with $W$. We first point out that Proof. Lemma 2.10. Thus, From Proposition 2.5, so $1$ is odd, we can write $v = 0a_1 \cdots a_kb_1 \cdots b_k0$ for some integer $k$ and digits $a_i, b_i \in \{0, 1\}$ for all $1 \leq i \leq k$. Recall that $\phi$ is the digit by digit substitution of $0 \rightarrow 01$ and $1 \rightarrow 10$, so 

$$\phi(v) = 01a_1a_1^c \cdots a_ka_k^c b_1b_1^c \cdots b_kb_k^c01$$

From Proposition 2.5, $\phi(M) = M$. Specifically, 

$$v = 0a_1 \cdots a_kb_1 \cdots b_k0 = 01a_1a_1^c \cdots a_ka_k^c$$

Thus, $a_k^c = 0$ and $a_k = 1$. Now $b_1$ is either 0 or 1:

- Suppose $b_1 = 0$. Since $M$ is built from $v$, $b_1b_1^c \cdots b_kb_k^c01 = v$, a contradiction as $v \in F$ implies that $v$ starts and ends with 0.
- Suppose $b_1 = 1$, so $b_1^c = 0$. Since $M$ is built from $v$, $b_1^c \cdots b_kb_k^c010 = v$, and $b_2 = 1$. $\phi(v)$ is a sub-word of $M$, and $a_ka_k^c b_1 b_2 b_2$ is a sub-word of $\phi(v)$, so it follows that $a_k a_k^c b_1 b_2 b_2$ is a sub-word of $M$. However, from Lemma 2.8, $a_k a_k^c b_1 b_2 b_2 = 10101$ cannot be a sub-word of $M$, a contradiction.

Therefore, we can conclude that the Thue-Morse sequence is not built from a word of even length.

\[ \square \]

Lemma 2.10. The Thue-Morse sequence cannot be built from a word of odd length other than 0.

Proof. We first point out that 0 clearly builds the Thue-Morse sequence. We observe that the Thue-Morse sequence starts with $01101001 \cdots$, so it is not built from any finite word of length shorter than 6. Suppose for contradiction that the Thue-Morse sequence is built from a word of odd length. Then, there exists a finite word $v$ with odd length $|v| > 6$ such that the Thue-Morse sequence, which we will denote as $M$, is built from $v$. In other words, $M \uparrow |v| = v$. Since $v \in F$ and length of $v$ is even, we can write $v = 0a_1 \cdots a_kb_1 \cdots b_k0$ for some integer $k$ and digits $a_i, b_i \in \{0, 1\}$ for all $1 \leq i \leq k$. Recall that $\phi$ is the digit by digit substitution of $0 \rightarrow 01$ and $1 \rightarrow 10$, so 

$$\phi(v) = 01a_1a_1^c \cdots a_k a_{k+1}^c b_1b_1^c \cdots b_kb_k^c01$$

From Proposition 2.5, $\phi(M) = M$. Specifically, 

$$v = 0a_1 \cdots a_{k+1}b_1 \cdots b_k0 = 01a_1a_1^c \cdots a_{k+1}$$

Thus, $a_{k+1} = 0$ and $a_{k+1}^c = 1$. Now $b_1$ is either 0 or 1:

- Suppose $b_1 = 0$. Since $M$ is built from $v$, $b_1b_1^c \cdots b_kb_k^c010 = v$, so $a_k a_k^c a_{k+1} = 010$. $\phi(v)$ is a sub-word of $M$, and $a_k a_k^c a_{k+1} b_1 b_1^c$ is a sub-word of $\phi(v)$, so it follows that $a_k a_k^c a_{k+1} b_1 b_1^c$ is a sub-word of $M$. However, from Lemma 2.8, $a_k a_k^c a_{k+1} b_1 b_1^c = 01010$ cannot be a sub-word of $M$, a contradiction.
- Suppose $b_1 = 1$, so $b_2^c = 0$. Since $M$ is built from $v$, there is an occurrence of $v$ starting at $b_2^c$. Thus, $b_2 = 1$, $b_2^c = 0$ and $a_2^c = (b_2^c)^c = 1$. $\phi(v)$ is a sub-word of $M$, and $b_1 b_1^c b_2 b_2^c b_3$ is a sub-word of $\phi(v)$, so it follows that $b_1 b_1^c b_2 b_2^c b_3$ is a sub-word of $M$. However, from Lemma 2.8, $b_1 b_1^c b_2 b_2^c b_3 = 10101$ cannot be a sub-word of $M$, a contradiction.
Therefore, we can conclude that the Thue-Morse sequence is not built from a word of odd length other than 0.

\[\square\]

**Theorem 2.11.** The Thue-Morse sequence is not rank-one.

**Proof.** Using Lemma 2.9 and Lemma 2.10, we conclude that the Thue-Morse sequence is not built from \(v \in F\) for any \(v \neq 0\). Therefore, by Definition 1.1, the Thue-Morse sequence is not rank-one. \[\square\]

3. **Rank-one Systems**

In this section, we give a formal definition of symbolic rank-one systems using a definition similar to the definition in [11]. We use [11] to characterize symbolic rank-one systems into two types: degenerate and non-degenerate rank-one systems. We explore some properties of each type of rank-one systems, and also properties of rank-one systems in general.

**Definition 3.1.** Let \(V \in \{0, 1\}^{\mathbb{N}}\) be an infinite word. Consider a dynamical system \((X, \sigma)\) with

\[X = \{x \in \{0, 1\}^\mathbb{Z} : \text{every finite sub-word of } x \text{ is a sub-word of } V\}\]

and the shift map \(\sigma: X \to X\) defined digit-wise for all \(x \in X\) as

\[\sigma(x)(i) = x(i+1)\]

for all \(i \in \mathbb{Z}\). It is clear that \(X \subseteq \{0, 1\}^\mathbb{Z}\) is compact and one can verify that \(\sigma\) is a homeomorphism of \(X\). We say that \((X, \sigma)\) is the rank-one system associated to \(V\) if \(V\) is a rank-one word. We define the orbit of any \(x \in X\) to be the set \(\{x, \sigma(x), \sigma^2(x), \ldots\}\).

**Definition 3.2.** Let \(V \in \{0, 1\}^{\mathbb{N}}\) be any binary infinite word. We say that \(V\) is periodic if for all \(i \in \mathbb{N}, V(i) = V(i+p)\) for some \(p \in \mathbb{N}\). We call the minimum such \(p\) the period of \(V\).

Note that for all periodic word \(V\), there exists a finite word \(v\) such that \(V = vvv \ldots\).

**Definition 3.3.** Let \(V\) be a rank-one word and \((X, \sigma)\) the rank-one system associated to \(V\). We say that \((X, \sigma)\) is degenerate if \(V\) is periodic. We say that \((X, \sigma)\) is non-degenerate if it is not degenerate.

**Proposition 3.4.** Let \((X, \sigma)\) be a degenerate rank-one system associated to \(V = vvv \ldots\). For all \(x \in X, x = \cdots vvv \cdots\) up to shifts.

**Proof.** Suppose not, i.e. suppose there exists a bi-infinite word \(x \in X\) that contains a finite sub-word \(u\) such that \(u\) is not a sub-word of \(vvv \cdots v\) for any number of \(v\)’s. By definition, every finite sub-word of \(x\) is a sub-word of \(V\), but \(u\) is a sub-word of \(x\) and not a sub-word of \(V\), a contradiction. Therefore, for all \(x \in X, x\) does not contain such a sub-word \(u\), so \(x = \cdots vvv \cdots\) up to shifts. \(\square\)

We note that since any degenerate rank-one system is associated to a periodic word with a finite orbit, a degenerate rank-one system always has a finite \(X\). For example, the rank-one system associated to \(V = 010101 \cdots\) is \(\{\cdots 010101 \cdots , \cdots 101010 \cdots \}, \sigma\}).

Non-degenerate rank-one systems are usually more complicated. Take for example, the rank-one system associated to the Chacón sequence. Not only is \(X\) in the rank-one system associated to the Chacón sequence infinite, it is actually uncountable. Indeed, we can prove that for any non-degenerate rank-one system, \(X\) is uncountable by showing that it is a non-empty perfect set.

**Proposition 3.5.** If \((X, \sigma)\) is a non-degenerate rank-one system associated to a rank-one word \(V \in \{0, 1\}^{\mathbb{N}}, then X is nonempty.**
Proof. Define \( x \in \{0, 1\}^\mathbb{Z} \) digit-wise as

\[
x(i) = \begin{cases} 
V(i) & \text{if } i > 0, \\
1 & \text{if } i \leq 0 
\end{cases}
\]

Consider the sequence \( \{\sigma^n(x)\}_{n \in \mathbb{N}} \subseteq \{0, 1\}^\mathbb{Z} \). Since \( \{0, 1\}^\mathbb{Z} \) is compact, the sequence must have a convergent sub-sequence that converges to a point \( x' \in \{0, 1\}^\mathbb{Z} \). Any finite sub-word \( u \) of \( x' \) has to be a sub-word of \( V \) since there exists some \( n \) such that \( u \) appears in \( \sigma^n(x) \) after the \((-n)\)th digit (where \( V \) begins in \( x' \)). Therefore, \( x' \in X \), so \( X \) is non-empty.

More generally, since we do not require that \( (X, \sigma) \) is non-degenerate or even rank-one in our proof, any system associated to an infinite word is non-empty.

**Proposition 3.6.** If \( (X, \sigma) \) is a non-degenerate rank-one system associated to a rank-one word \( V \in \{0, 1\}^\mathbb{N} \), then \( X \) is perfect.

**Proof.** From the definition it follows that \( X \) is closed. Now we want to show that \( X \) contains no isolated points. Specifically, we want to show that for any finite word \( u \) that appears in some \( x \in X \) at the \( i \)th digit, we want to show that there exists \( y \in X \) and \( y \neq x \) such that \( u \) appears in \( y \) at the \( i \)th digit. Since \( u \) is a sub-word of \( x \), \( u \) is a sub-word of \( V \), and hence \( u \) is a sub-word of some finite word \( v \) that \( V \) is built from. Since \( (X, \sigma) \) is non-degenerate, \( V \) is not periodic, so there exist \( a, b \in \mathbb{N}, a \neq b \) such that \( v1^a v1^b v \) is a sub-word of \( V \). Define \( z \in \{0, 1\}^\mathbb{Z} \) digit-wise as

\[
z(i) = \begin{cases} 
V(i) & \text{if } i > 0, \\
1 & \text{if } i \leq 0 
\end{cases}
\]

Since \( V \) is rank-one, any finite sub-word of \( V \) appears infinitely often in \( V \), so there exists a sequence \( \{\sigma^k(z)\} \subseteq \{\sigma^n(z)\}_{n \in \mathbb{N}} \) such that \( u \) appears at the \( i \)th digit as a sub-word of the first \( v \) in \( v1^a v1^b v \) for all elements in the sub-sequence. Since \( \{0, 1\}^\mathbb{Z} \) is compact, \( \{\sigma^k(z)\} \) must have a convergent sub-sequence that converges to a point \( x' \in \{0, 1\}^\mathbb{Z} \). Shifting \( x' | v1^a | \) times we get \( y = \sigma^{v1^a}(x') \). Notice that \( x', y \in X \), \( x' \neq y \), and \( u \) appears in both \( x' \) and \( y \) at the \( i \)th digit. Therefore, \( X \) is perfect. \( \square \)

In our proof of Proposition 3.6, we construct two distinct bi-infinite sequences in \( X \) for any non-degenerate rank-one system that contains the same finite sub-word at the same position. More generally, one can extend that construction to prove that for any finite sub-word and any position, there has to be an element in \( X \) that contains the sub-word at that specific position.

**Proposition 3.7.** Let \( (X, \sigma) \) be a rank-one system associated to a rank-one word \( V \in \{0, 1\}^\mathbb{N} \). For any finite sub-word \( u \) of \( V \) and any integer \( i \in \mathbb{Z} \), there exists some \( x \in X \) such that \( u \) appears in \( x \) at the \( i \)th digit.

**Proof.** Since \( V \) is rank-one, any finite sub-word of \( V \) appears infinitely often in \( V \), so \( u \) appears infinitely often in \( V \). Thus, there exists a sequence \( \{\sigma^k(\cdots 111V)\} \subseteq \{\sigma^n(\cdots 111V)\}_{n \in \mathbb{N}} \) such that \( u \) appears at the \( i \)th digit for all elements in the sequence. Since \( \{0, 1\}^\mathbb{Z} \) is compact, \( \{\sigma^k(\cdots 111V)\} \) must have a convergent sub-sequence that converges to a point \( x' \in \{0, 1\}^\mathbb{Z} \), and \( x \in X \). We have thus constructed \( x \in X \) such that \( u \) appears in \( x \) at the \( i \)th digit. \( \square \)
4. Beyond Rank-one

In this section, we define systems beyond rank-one. In particular, we define at most rank-$n$ systems and hence rank-$n$ systems. We also look at symbolic shift systems generated from infinite word in general, exploring properties of these systems.

For simplicity in our definition of at most rank-$n$ systems, we extend the definition that a finite word is built from a set of finite words. As mentioned earlier, we thank Su Gao for suggesting a definition of rank-two sequences.

**Definition 4.1.** We say that a finite word $w$ is **built from** a set $S \subseteq \mathcal{F}$ if

$$w = v_11^{a_1}v_21^{a_2} \cdots v_k1^{a_k}v_{k+1}$$

such that $k \geq 1$, $a_i$ and $1 \leq i \leq k$ are non-negative integers, each $v_j \in S$.

**Definition 4.2.** An infinite word $V \in \{0, 1\}^\mathbb{N}$ is **at most rank-$n$** if there exists an infinite sequence $\{(v_{i,1}), (v_{i,2}), \cdots, (v_{i,n})\} \in \mathcal{N}$ of sets of finite words in $\mathcal{F}$ defined inductively by

$$v_{0,j} = 0 \text{ for all } 1 \leq j \leq n$$

$$v_{i+1,j} \text{ is built from } \{v_{i,1}, v_{i,2}, \cdots, v_{i,n}\} \text{ for all } 1 \leq j \leq n$$

such that $V \upharpoonright |v_{i,1}| = v_{i,1}$ for all $i \in \mathbb{N}$. We then write $V = \lim_{i \to \infty} v_{i,1}$.

We want to point out that in this definition, $V \upharpoonright |v_{i,1}| = v_{i,1}$ implies that $v_{i+1,1} \upharpoonright |v_{i,1}| = v_{i,1}$ for any $i \geq 0$.

**Definition 4.3.** For $n > 1$, an infinite word $V \in \{0, 1\}^\mathbb{N}$ is **rank-$n$** if $V$ is at most rank-$n$ and not at most rank-$n - 1$.

For example, we can see that the Thue-Morse sequence, denoted as $M$, is at most rank-2. Consider the following definition of $M_{i,j}$:

$$M_{0,1} = 0 \text{ and } M_{0,2} = 0$$

$$M_{n+1,1} = M_{n,1}1M_{n,2}$$

$$M_{n+1,2} = M_{n,2}1M_{n,1}$$

if $n$ is odd

$$M_{n+1,1} = M_{n,1}11M_{n,2}$$

$$M_{n+1,2} = M_{n,2}M_{n,1}$$

if $n$ is even

for all $n > 0$.

It is simple to check that $M \upharpoonright |M_{i,1}| = M_{i,1}$ for all $i \in \mathbb{N}$ and $M = \lim_{i \to \infty} M_{i,1}$. Thus, we have shown that the Thue-Morse sequence is at most rank-2. By Theorem 2.11, the Thue-Morse sequence is not rank-one, so we can conclude that the Thue-Morse sequence is indeed rank-two.

**Definition 4.4.** Let $V \in \{0, 1\}^\mathbb{N}$ be a one-sided infinite word. Consider a dynamical system $(X, \sigma)$ with

$$X = \{x \in \{0, 1\}^\mathbb{Z} : \text{every finite sub-word of } x \text{ is a sub-word of } V\}$$

and the shift map $\sigma : X \to X$ defined digit-wise for all $x \in X$ as

$$\sigma(x(i)) = x(i+1)$$

We say that $(X, \sigma)$ is **at most rank-$n$** if $V$ is a rank-$n$ word. We say that $(X, \sigma)$ is a **rank-$n$ system** if $(X, \sigma)$ is at most rank-$n$ but not rank-$n - 1$. 
It is non-trivial to qualify the definition of a rank-$n$ system in Definition 4.4 such that $(X, \sigma)$ is associated to a rank-$n$ word and there are no words that are at most rank-$n-1$ that $(X, \sigma)$ is associated to. There are many infinite words that are associated to the same system. Two different infinite words of different ranks can be associated to the same symbolic shift system. In fact, we will show below that any symbolic shift system $(X, \sigma)$ is associated to infinitely many words.

For example, we can obtain infinite words that are not rank-one by adding a 0 to the beginning of the Chacón sequence:

$0 \ 0010 \ 0010 \ 1 \ 0010 \ 0010 \ 0010 \ 1 \ 0010 \ 0010 \ 0010 \ 1 \ 0010 \ 0010 \ 0010 \ 1 \ 0010 \ \cdots$

or by removing the first 0 of the Chacón sequence:

$010 \ 0010 \ 1 \ 0010 \ 0010 \ 0010 \ 1 \ 0010 \ 1 \ 0010 \ 0010 \ 1 \ 0010 \ \cdots$

Both words are not rank-one as they are not built from any finite word other than 0. However, the system associated to both words is rank-one since it is the same system system that the Chacón sequence (which is rank-one) is associated to any finite sub-word of either words above is a sub-word of $B$ and vice versa.

The example above shows a rank-one system that is trivially associated to a word that is not rank-one. Even though given a rank-one word the system associated to it is always rank-one, it is not obvious that given a rank-$n$ word whether the system associated to the word is rank-$n$.

**Lemma 4.5.** For any infinite words $V$ and $W$, let $(X_V, \sigma)$ be the system associated to $V$ and $(X_W, \sigma)$ be the system associated to $W$. If every finite sub-word $V$ is a sub-word of $W$, then $X_V \subseteq X_W$.

**Proof.** For any bi-infinite word $x \in X_V$, any finite sub-word of $x$ is a finite sub-word of $V$. Since every finite sub-word $V$ is a sub-word of $W$, any finite sub-word of $x$ is a finite sub-word of $W$. Thus, $x \in X_W$, so $X_V \subseteq X_W$. □

**Theorem 4.6.** For any infinite words $V$ and $W$, let $(X_V, \sigma)$ be the system associated to $V$ and $(X_W, \sigma)$ be the system associated to $W$. $X_V \subseteq X_W$ if and only if every finite sub-word $v$ that appears infinitely often in $V$ appears infinitely often in $W$.

**Proof.** $\Rightarrow$ We prove the contrapositive (if there exists some finite sub-word $v$ that appears infinitely often in $V$ but does not appear infinitely often in $W$, then $X_V \not\subseteq X_W$) by defining a bi-infinite word $x \in X_V$ such that $x \notin X_W$. Define $z \in \{0, 1\}^\mathbb{Z}$ digit-wise as

$$z(i) = \begin{cases} V(i) & \text{if } i > 0, \\ 1 & \text{if } i \leq 0 \end{cases}$$

Since $v$ appears infinitely often in $V$, there exists a sequence of bi-infinite words $\{x_i = \sigma^k(z) | x_i \}$ has an occurrence of $v$ at the $0^{th}$ digit for all $i \in \mathbb{N}$. Since $\{0, 1\}^\mathbb{Z}$ is compact, the sequence $\{x_i\}$ has a convergent sub-sequence. Let $x$ be the limit of the convergent sub-sequence. $x \in X_V$ because every finite sub-word of $x$ is a sub-word of $V$. Note that $v$ appears at the $0^{th}$ digit in $x$. If $W$ does not contain $v$, we are done, so we assume that $v$ only has finite occurrences in $W$. Let $k$ be the position of the last occurrence of $V$ in $W$. $x \notin X_W$ since the sub-word $x_{(-k-1)}x_{(-k)} \cdots x_{(|v|-2)}x_{(|v|-1)}$ ($v$ preceded by $k + 1$ symbols in $x$) is not a sub-word of $W$. 
\[ X_V \subseteq X_W \text{ if every finite sub-word that appears infinitely often in } V \text{ appears infinitely often in } W \] follows directly from Lemma 4.5.

Theorem 4.6 is very interesting as it leads to many corollaries of which infinite words are associated to the same system. It is one of the new results of this paper.

**Corollary 4.7.** Let \((X_V, \sigma)\) be the system associated to an infinite word \(V\) and \((X_W, \sigma)\) be the system associated to an infinite word \(W\). \(X_V = X_W\) if \(V\) and \(W\) differ in finitely many digits.

For example, the Chacón sequence and the infinite word generated by changing the first 0 in the Chacón sequence to a 1 are infinite words that are associated to the same symbolic shift system but differ in finitely many digits. Consider

\[
0010 \ 0010 \ 1 \ 0010 \ 0010 \ 1 \ 0010 \ 1 \ldots
\]

which is the Chacón sequence, and

\[
1010 \ 0010 \ 1 \ 0010 \ 0010 \ 0010 \ 1 \ 0010 \ 1 \ldots
\]

which is identical to the Chacón sequence in every digit except the first. For simplicity, we will call the second word \(V'_{\text{Chacón}}\). These two words have to share the same dynamical system \((X, \sigma)\) associated to them: for any bi-infinite word \(x\) that is in \(X_{\text{Chacón}}\), \(x\) cannot contain any finite sub-word \(u\) that is not contained in \(V'_{\text{Chacón}}\). Notice that \(x\) is a bi-infinite word, so any finite sub-word \(u\) of \(x\) has to be either preceded by 1 or 0. Since the only difference between \(V'_{\text{Chacón}}\) and the Chacón sequence is at the first digit, if \(x\) contains any finite sub-word \(u\) of the Chacón sequence that is not a sub-word of \(V'_{\text{Chacón}}\), \(u\) has to be at the start of the Chacón sequence and contain the first digit. Then, \(x\) contains either \(1u\) or \(0u\), which is not a sub-word of the Chacón sequence. Similarly, any \(x \in X'_{\text{Chacón}}\) cannot contain a finite sub-word \(u\) that is not contained in the Chacón sequence.

Given an infinite word \(V\), we can generate an infinite set of infinite words \(\{V_i\}_{i \in \mathbb{N}}\) that are associated to the same system by defining \(V_i\) to differ from \(V\) only at the \(i^{\text{th}}\) digit.

**Corollary 4.8.** Any symbolic shift system \((X, \sigma)\) is associated to infinitely many words.

We can also have infinite words that differ in every digit but are still associated to the same symbolic shift system.

**Corollary 4.9.** Let \((X_V, \sigma)\) be the system associated to an infinite word \(V\) and \((X_W, \sigma)\) be the system associated to an infinite word \(W\). If there exist \(i, j \in \mathbb{N}\) such that \(V(i)V(i+1)V(i+2)\ldots = W(j)W(j+1)W(j+2)\ldots\), then \(X_V = X_W\).

For example, the system associated to both 010101\ldots and 101010\ldots is

\[
(\{\ldots01\bar{0}101\ldots, \ldots10\bar{1}010\ldots\}, \sigma),
\]

but 010101\ldots and 101010\ldots differ in every digit.

**Corollary 4.10.** For any infinite words \(V\) and \(W\), let \((X_V, \sigma)\) be the system associated to \(V\) and \((X_W, \sigma)\) be the system associated to \(W\). \(X_V = X_W\) if and only if \(V\) and \(W\) have the same set of finite words that appear infinitely often in them.

For example, consider the two infinite words

\[
0 \ 1 \ 00 \ 01 \ 10 \ 11 \ 000 \ 001 \ 010 \ 011 \ 100 \ 101 \ 110 \ 111 \ 0000 \ldots
\]

and

\[
1 \ 0 \ 11 \ 10 \ 01 \ 00 \ 111 \ 110 \ 101 \ 100 \ 011 \ 010 \ 001 \ 000 \ 1111 \ldots
\]
where both words contain every finite binary word, but in different orders. The same symbolic shift system \( (\{0, 1\}^\mathbb{Z}, \sigma) \) is associated to the two words, but these two words differ in infinitely many digits and are not the same up to any shifts \( i, j \) described in Corollary 4.9.

Thus far, we have only seen examples of words and systems that are at most rank-two. Before ending our discussion for this section, we provide an example of a system that is rank-three. Consider the measure theoretic cut-and-stack representation of three Rokhlin towers shown in Figure 1. Similar to the Chacón transformation, in the first iteration, starting with three identical copies of \( \frac{2}{3} \) of the unit interval, we cut each interval into three segments of the same length. We then add two spacers of the same length as each segment - one on the second segment of the second block, and the other on the first segment of the third block. We now stack the nine segments into three blocks as illustrated by the arrows in Figure 1. In the symbolic definition, we represent each block (represented as red lines in Figure 1) as a 0 and each spacer (represented as blue wiggly lines in Figure 1) as a 1.

The resulting sequence \( W \) can be defined inductively as

\[
\begin{align*}
W_{0,1} &= 0, \\
W_{n+1,1} &= W_{n,1}W_{n,2}1W_{n,3}, \\
W_{n+1,2} &= W_{n,2}W_{n,3}W_{n,1}, \\
W_{n+1,3} &= W_{n,3}1W_{n,1}W_{n,2}
\end{align*}
\]

for all \( n > 0 \) such that \( W = \lim_{n \to \infty} W_{i,1} \).

As another example, consider the Bernoulli sequence, a sequence containing all possible finite words in 0 and 1. We show that the Bernoulli sequence is not rank \( n \) for any \( n \).

**Proposition 4.11.** The Bernoulli sequence is not finite rank.

**Proof.** Suppose it is rank \( n \), built from the words \( (a_{k,1}, a_{k,2}, a_{k,3}, \ldots, a_{k,n}) \). Then there exists some \( \ell \) satisfying \( 1 < \ell \leq n+1 \) such that \( a_{\ell,i} \), \( \ldots, a_{n,i} \) are strings of the form \( 0^p \) for all \( i \). Suppose first that \( \ell \neq n+1 \). Fix a large \( N > \binom{n}{2} + 1 \) and choose \( k \) so that each of \( a_{\ell,k}, \ldots, a_{n,k} \) contains \( 0^N \). Write \( a_{k,j} \) as \( 0^{\ell_j,k+1}w_{k,j}0^{\ell_j,k+2} \) where \( w_{k,j} \) starts and ends with 1. Let \( w \) be a word built from \( (a_{k,1}, \ldots, a_{k,n}) \).

Consider \( S \) the set of all possible integers that can be written as finite sums of \( |a_{\ell,i}|, |a_{\ell+1,i}|, \ldots, |a_{n,i}| \). Notice that gaps between consecutive elements of \( S \) are greater than \( N > \binom{n}{2} + 1 \). Choose an element \( L \) in \( S \) such that \( L \) is larger than \( 2|a_{k,i}| \) for all \( i \). We claim that one of \( 10^{L+1}, 10^{L+2}, 10^{L+3}, \ldots, 10^{L+\binom{n}{2}+1} \) cannot appear in the sequence generated by \( (a_{k,1}, \ldots, a_{k,n}) \). Since \( N > \binom{n}{2} + 1 \) and gaps between consecutive elements of \( S \) are at least \( N, L+1, \ldots, K + \binom{n}{2} + 1 \) are not in \( S \). Furthermore, the only other way \( 10^{L+k} \) for \( k \leq \binom{n}{2} + 1 \) appears in a string built by \( (a_{k,1}, \ldots, a_{k,n}) \) is \( 10^{\ell_j,k+1}0^L0^{\ell_j,k+2}1 \). This is because \( L \) is larger than \( 2|a_{k,i}| \) so \( 10^{L+k} \) cannot be a substring of...
\( a_{k,i} \) so it must appear as a substring of \( a_{k,i_1} 0^j a_{k,i_2} \) for some \( i_1, i_2 < \ell \). The only substring of that string that can possibly be of the form \( 10^{\ell+k} 1 \) is the substring
\[
10^{\ell_{i_1,k,i_1}} 0^{\ell_{i_2,k,i_2}} 1
\]
since consecutive gaps in \( L \) are larger than \( N \). There are \( \binom{n}{2} \) possible strings of that form, and hence one of the strings \( 10^{L+1}, \ldots, 10^{L+\left(\frac{n}{2}\right)+1} \) cannot be a substring of any word generated by \( a_{k,1}, \ldots, a_{k,n} \). Hence, the sequence generated by \( (a_{k,1}, \ldots, a_{k,n}) \) cannot possibly be the Bernoulli sequence as the Bernoulli sequence contains all possible finite words.

If \( \ell = n + 1 \), then one may simply choose \( L \) so that \( L \geq 2|a_{k,i}| \) for all \( i \). There are at most
\[
2 \max_i |a_{k,i}|
\]
consecutive zeroes, so we may conclude that the Bernoulli sequence is not finite rank.

\[\square\]

5. The Thue-Morse Symbolic System is not Rank-One

We have thus far proved that the Thue Morse sequence, denoted as \( M \), is a rank-two sequence. However, as we have seen in Section 4, it is still possible for the symbolic shift system associated to the Thue-Morse sequence, denoted as \((X_M, \sigma)\), to be rank-one. In this section, we prove that \((X_M, \sigma)\) is not rank-one. Gao and Ziegler in [12] have shown that an infinite odometer is not a factor of a rank-one shift. Gao suggested to the authors that one way to show that the Thue-Morse system is not rank-one is to show that it has an infinite odometer factor. While it is clear that the finite measure-preserving Morse transformation has the binary odometer as a factor, it is not clear whether this remains true in the symbolic case, in fact we ask whether an infinite odometer can be a factor of a symbolic rank-two shift. In this context we note that del Junco showed in [6] that the finite measure-preserving Thue-Morse transformation is rank-two (as a measure-preserving system).

Recall that in Section 4, we have shown that \( M \) is rank-two by recursively defining \( M_{i,j} \) as
\[
M_{0,1} = 0 \text{ and } M_{0,2} = 0
\]
\[
M_{n+1,1} = M_{n,1} 1 M_{n,2} \text{ and } M_{n+1,2} = M_{n,2} 1 M_{n,1} \text{ if } n \text{ is odd}
\]
\[
M_{n+1,1} = M_{n,1} 11 M_{n,2} \text{ and } M_{n+1,2} = M_{n,2} M_{n,1} \text{ if } n \text{ is even}
\]
for all \( n > 0 \) such that \( M = \lim_{i \to \infty} M_{i,1} \). We now use the fact the \( M \) is rank-two, i.e. \( M \) is built from \( \{M_{i,1}, M_{i,2}\} \) for any \( i \in \mathbb{N} \) to prove that the system \((X_M, \sigma)\) is not rank-one.

**Lemma 5.1.** For any \( i \in \mathbb{N} \), the longest sub-word that both \( M_{i,1} \) and \( M_{i,2} \) start with is 0, and the longest sub-word that both \( M_{i,1} \) and \( M_{i,2} \) end with is also 0.

**Proof.** We prove this by induction. The longest word that both \( M_{i,1} \) and \( M_{i,2} \) start with is 0 and the longest word that both \( M_{i,1} \) and \( M_{i,2} \) end with is also 0. For \( i = 0, M_{0,1} = 0 \) and \( M_{0,2} = 0 \). For \( i = 1, M_{0,1} = 0110 \) and \( M_{0,2} = 00 \). The inductive hypothesis clearly holds for \( i < 2 \).

Now suppose the inductive hypothesis holds for some \( i = n \), we want to show that the inductive hypothesis must also hold for \( i = n + 1 \). If \( n \) is odd,
\[
M_{n+1,1} = M_{n,1} 1 M_{n,2} \text{ and } M_{n+1,2} = M_{n,2} 1 M_{n,1}
\]
Similarly, if \( n \) is even,
\[
M_{n+1,1} = M_{n,1}11M_{n,2} \quad \text{and} \quad M_{n+1,2} = M_{n,2}M_{n,1}
\]
Notice that \( M_{n+1,1} \) always starts with \( M_{n,1} \) and ends with \( M_{n,2} \), but \( M_{n+1,2} \) always starts with \( M_{n,2} \) and ends with \( M_{n,1} \) for any \( n \). Since the induction hypothesis holds for \( i = n \), it must also hold for \( i = n + 1 \).

Therefore, the longest word that both \( M_{i,1} \) and \( M_{i,2} \) start with is 0 for all \( i \in \mathbb{N} \), and the longest word that both \( M_{i,1} \) and \( M_{i,2} \) end with is also 0 for all \( i \in \mathbb{N} \). \( \square \)

It is worth noting that when dealing with an infinite or bi-infinite word that is built from a finite word \( v \), an occurrence of \( v \) in the word at position \( i \) does not necessarily imply that there is an occurrence of \( v1^a \) for some \( a \geq 0 \) ending at position \( i - 1 \). For example, consider a rank-one word \( V \) defined as
\[
V_0 = 0 \\
V_1 = 00100 \\
V_{n+1} = V_n1V_n11V_n \quad \text{for} \quad n > 0
\]
and the system \((X_V, \sigma)\) associated to \( V \). Suppose we see an occurrence of \( V_1 \) in a bi-infinite word \( x = \cdots 00100 \cdots \) at the 0th position for some \( x \in X_V \). Knowing that \( V_1 \) builds \( x \), we might jump to the conclusion that \( V_1 \) at the 0th position must be preceded by the word \( V_11^a \) for some \( a \geq 0 \), but it is possible that the occurrence of \( V_1 \) at the 0th position is in fact part of an occurrence of \( V_2 \):
\[
\cdots 00100 1 00100 11 00100 1 00100 11 00100 \cdots
\]
Thus, given an infinite or bi-infinite word that is built from a finite word \( v \), it is important to define which occurrences of \( v \) in the word actually contribute to building the word.

Using a similar definition to expected occurrences defined in [11], [12] and [13], given an infinite or bi-infinite word that is built from a finite word \( v \), if there is a unique way to decompose the word to a collection of disjoint occurrences of \( v \)’s separated by 1’s, then we can call an occurrence of \( v \) an **expected occurrence** if it is an element of such collection. Specifically, given any any infinite word \( V \), if there exists a unique decomposition of \( V \) to the form
\[
V = v1^{a_1}v1^{a_2}v1^{a_3}v1^{a_4}v \cdots
\]
such that \( a_i \geq 0 \) for all \( i \in \mathbb{Z}^+ \), then each occurrence of \( v \) shown above is an expected occurrence. Similarly, for any bi-infinite word \( x \), if there exists a unique decomposition of \( x \) to the form
\[
x = \cdots v1^{a_{-2}}v1^{a_{-1}}v1^{a_0}v1^{a_1}v1^{a_2}v \cdots
\]
such that \( a_i \geq 0 \) for all \( i \in \mathbb{Z} \), then each occurrence of \( v \) shown above is an expected occurrence.

Kalikow showed in [13] that whether an occurrence of \( v \) is expected in a bi-infinite word can be resolved uniquely for any aperiodic bi-infinite words.

**Lemma 5.2.** (Kalikow) Given a bi-infinite word that is built from a finite word \( v \), if the entire word is aperiodic, then there is a unique way to decompose the word into expected occurrences of \( v \)’s separated by 1’s.
Note that for any infinite word that is built from a finite word \( v \), the decomposition of the word into expected occurrences is unique because the first occurrence of \( v \) has to be an expected occurrence. We also point out that the case which the word is periodic and thus cannot be decomposed uniquely into expected occurrences of \( v \) is trivial, since the word has to be at most rank-one.

**Lemma 5.3.** Let \( V \) be an infinite word. If \( V \) starts with 0 and is periodic, then \( V \) is rank-one.

**Proof.** Let \( k \) be the period of \( V \). Since \( V \) starts with 0, the first digit of \( V \), \( V_1 = 0 \). Define

\[
v_0 = 0, \quad v_n = V_1 V_2 \cdots V_{a_n} \quad \text{for} \ n > 0
\]

where \( a_n \) is the position of the last 0 within the first \( nk \) digits of \( V \). Clearly, each \( v_i \) builds \( V \), so \( \{v_i\}_{i \in \mathbb{N}} \subseteq A_V \), where \( A_V = \{v \in \mathcal{F} : V \text{ is built from } v \} \). Since \( \{v_i\}_{i \in \mathbb{N}} \) is infinite, \( A_V \) is infinite. Then, by Definition 1.1, \( V \) is rank-one. \( \square \)

To illustrate the proof, we provide an example of why an infinite word that starts with 0 is rank-one. For example, consider

\[
V = 010111011010 \ldots
\]

We define \( v_0 = 0, v_1 = 010, v_2 = 01011010, v_3 = 0101101011010, \ldots \) Each \( v_i \) builds \( V \), and there are infinitely many such \( v_i \)'s. Therefore, \( V \) is rank-one.

**Corollary 5.4.** Let \( V \) be an infinite word. If \( V \) is periodic and \( V \neq 111 \cdots \), then \( X_V \) is rank-one.

**Proof.** The corollary follows from Corollary 4.9 which shows that infinite words up to shifts have the same rank-one system. Given \( V \) periodic and \( V \neq 111 \cdots \), we can remove the leading digits of \( V \) up to the first 0 to get an infinite periodic word that starts with 0. \( \square \)

In [11], Gao and Hill proved the following lemma.

**Lemma 5.5.** (Gao, Hill [11]) Suppose \( V \) is a rank-one word and \( (X, \sigma) \) the rank-one system associated to \( V \). If \( x \in X \) contains an occurrence of 0, then \( x \) contains an occurrence of every finite subword of \( V \).

We extend this lemma to systems of at most rank-\( n \) for any \( n \).

**Lemma 5.6.** Let \( W \) be an infinite word that is at most rank-\( n \) and \( (X_W, \sigma) \) be the system associated to \( W \). For any \( x \in X_W \), if \( x \neq \cdots 11111 \cdots \), then every finite sub-word of \( W \) is a sub-word of \( x \).

**Proof.** Since \( W \) is at most rank-\( n \), by definition, there exists an infinite sequence

\[
\{w_{i,1}, w_{i,2}, \ldots, w_{i,n}\}_{i \in \mathbb{N}}
\]

of sets of finite words such that \( W \) can be built from \( \{w_{i,1}, w_{i,2}, \ldots, w_{i,n}\} \) for any \( i \). For any finite sub-word \( u \) of \( W \), \( u \) is a sub-word of \( w_{k,1} \) for some \( k \geq 0 \). If \( x \neq \cdots 11111 \cdots \), then \( x \) contains an occurrence of 0. Every occurrence of 0 in \( x \) is a part of an occurrence of \( w_{l,1} \) for some \( l \geq 0 \). Since both \( w_{l,1} \) and \( w_{k,1} \) has to be an occurrence of \( w_{\text{max}(l,k),1} \), if \( x \) contains an occurrence of 0, then \( x \) contains an occurrence of \( w_{\text{max}(m+1,k),1} \). Therefore, \( x \) contains an occurrence of \( u \). \( \square \)

Using this result we prove that \( X_M \), the system associated to the Thue-Morse sequence, is not rank-one. In our proof that \( X_M \) is not rank-one, given any finite word \( v \), we decompose \( M_{k,m} \) for some \( k > 0 \) and \( 0 \leq m \leq 1 \) (the \( k \)th iteration of the \( m \)th tower of \( M \)) into three sub-words such that the first and third sub-words have length \( < |v| \), the second sub-word is built from \( v \). To make it easier for the reader to follow the proof, we first show that such decomposition, if it exists, is necessarily unique.
Lemma 5.7. Suppose there exists some finite word \( v \) that builds some \( x \in X_M \). If there exist two distinct ways to decompose \( M_{k,1} \) with every \( k > N \) for some integer \( N \) into sub-words \( a, b_k, c_k \) and \( a', b'_k, c'_k \) such that

1. \( M_{k,1} = ab_kc_k = a'b'_k c'_k \),
2. \( |a| < |v|, |a'| < |v|, |c_k| < |v| \) and \( |c'_k| < |v| \),
3. \( b_k \) and \( b'_k \) are built from \( v \),
4. there exist finite words \( d_k, e_k, d'_k, e'_k \) such that \( d_ka = v, d'_ka' = v, c_ke_k = v \) and \( c'_ke'_k = v \),

then there exists \( x \in X_M \) that is periodic.

Proof. Suppose there exist two distinct ways to decompose \( M_{k,1} \) for every \( k > N \) for some integer \( N \) satisfying the conditions stated in the lemma. Consider the following two sequences of bi-infinite words \( \{y_i\}_{i \in \mathbb{N}} \) and \( \{z_i\}_{i \in \mathbb{N}} \):

\[
y_i = \cdots vvv d_k M_{k,1} e_k vvv \cdots \text{ such that } M_{k,1} \text{ starts at position } -k \text{ for } k = i + N
\]

\[
z_i = \cdots vvv d'_k M_{k,1} e'_k vvv \cdots \text{ such that } M_{k,1} \text{ starts at position } -k \text{ for } k = i + N
\]

From the compactness of \( \{0, 1\}^\mathbb{Z} \), we know that there exists \( S \subseteq \mathbb{N} \) such that \( \{y_i\}_{i \in S} \) is a convergent sub-sequence of \( \{y_i\}_{i \in \mathbb{N}} \). Let \( y \) be the limit of \( \{y_i\}_{i \in S} \). Note that \( y \in X_M \) since every finite sub-word of \( y \) is a sub-word of \( M_{k,1} \) for any \( k \) and thus is a sub-word of \( M \). For any \( i \), both \( z_i \) and \( y_i \) have \( M_{k,1} \) at the same position around 0, with \( k \) increasing as \( i \) increases. Thus,

\[
\lim_{i \to \infty} d(z_i, y_i) = 0
\]

so \( \{z_i\}_{i \in S} \) also converges to the same limit \( y \). However, the positions of expected occurrences of \( v \) in \( z_i \) and \( y_i \) for every \( i \) are different, so there exist two distinct ways of decomposing \( y \) into expected occurrences of \( v \). Thus, by Lemma 5.2, \( y \) must be periodic.

For clarity in proving that \( X_M \) is not rank-one and to avoid excessive notations, for any finite words \( a \) and \( b \), we denote an occurrence of \( a \) separated by some number (which could be none) of 1’s as \( a1^*b \).

Theorem 5.8. The system associated to the Thue-Morse sequence, \( X_M \), is not rank-one.

Proof. Suppose for contradiction that \( X_M \) is rank-one. Then, there exists a rank-one word \( V \) that is associated to \( X_M \). Let \( v \) be any finite sub-word of \( V \) that builds \( V \). For any \( x \in X_M \), \( x \) is not periodic since \( M \) does not contain any cubes [14] (three consecutive sub-words that are the same), so \( x \) contains an occurrence of 0. Thus, by Lemma 5.6, every finite sub-word of \( V \) is a sub-word of \( x \) and every finite sub-word of \( M \) is a sub-word of \( x \), so there exist a unique decomposition of \( x \) into expected occurrences of \( v \) and a unique decomposition of \( x \) into expected occurrences of \( \{M_{k,1}, M_{k,2}\} \) for any \( k \in \mathbb{N} \). Since every finite sub-word of \( M \) is a sub-word of \( x \), there are infinitely many sub-words in \( x \) of the form \( M_{k,1}1^*M_{k,1} \) and \( M_{k,2}1^*M_{k,1} \) for any \( k \in \mathbb{N} \). By Lemma 2.9 and Lemma 2.10, any expected occurrence of \( M_{k,1} \) in \( x \) does not begin with an expected occurrence of \( v \). By Lemma 5.7, since \( x \) is aperiodic, there exists some \( k \) with a unique decomposition of \( M_{k,1} \) such that \( M_{k,1} = abc \) and \( M_{k,2} \) ends with \( f \), where \( v = c1^*a = f1^*a \). Since \( f \) ends with 0, it must be the case that \( f = c \), which we know from Lemma 5.1 that \( f = c = 1 \).

Now either every expected occurrence of \( M_{k,2}1^*M_{k,2} \) starts and ends with \( v \), or there exists some expected occurrences of \( M_{k,2}1^*M_{k,2} \) that does not.

In the first case, consider any expected occurrence of

\[
M_{k,1}1^*M_{k,2}1^*M_{k,2}1^*M_{k,1}
\]
The first $M_{k,2}$ starts with $v$ and the second $M_{k,2}$ ends with $v$. Since $M$ does not contain any cubes, there does not exist three expected occurrences of $M_{k,2}$ in a row. so $M_{k,2}^1 M_{k,2}$ has to be followed by a $M_{k,1}$. If the first $M_{k,2}$ starts with an expected occurrence of $v$, then the second $M_{k,2}$ has to end with an expected occurrence of $v$, so the second $M_{k,1}$ has to start with an expected occurrence of $v$, a contradiction. Thus, there exists an expected occurrence of $M_{k,1}^1 M_{k,2}$ such that $c^1 d = v$, so $a = d$, which we know from Lemma 5.1 that $a = d = 0$.

In the second case, there exists an expected occurrence of $M_{k,2}^1 M_{k,2}$ such that $f^1 d = v$, so $a = d$, which we know from Lemma 5.1 that $a = d = 0$.

Therefore, in both cases, $a$ and $c$ can only be 0. Since the Thue-Morse sequence does not contain any cubes, it contains at most two consecutive 1’s, so there are no words with length greater than 4 that builds $x$. Therefore, $X_M$ cannot be rank-one. □

6. Substitution and Sturmian Sequences

In this section we investigate the rank of substitution and Sturmian sequences. Given a function $\zeta : \{0,1\} \rightarrow \bigoplus_{i=0}^{\infty} \{0,1\}$ and a sequence $w = w_0 w_1 w_2 \cdots \in \{0,1\}^\infty$, we write $\zeta(w)$ as $\zeta(w_0) \zeta(w_1) \zeta(w_2) \cdots$.

**Definition 6.1.** Given a function $\zeta : \{0,1\} \rightarrow \bigoplus_{i=0}^{\infty} \{0,1\}$, A substitution sequence for $\zeta$ is a sequence $u$ such that $\zeta(u) = u$.

**Example.** The Thue-Morse sequence is a substitution sequence for the substitution $0 \mapsto 01$ and $1 \mapsto 10$.

**Theorem 6.2.** All substitution sequences are rank at most 2.

**Proof.** Let $q$ denote the substitution sequence built by the substitution $\sigma$. For $k$ a positive integer, let $\sigma^{2^k}(0) = v_k z_k$ where $v_k$ starts and ends with 0 and $\sigma^{2^k}(1) = 1^{y_k} w_k z_k$ where $x_k, y_k, z_k$ are nonnegative integers. Represent $\sigma$ as $0 \mapsto a_0 a_1 a_2 \cdots a_n$ with $a_0 = 0$ and $1 \mapsto b_0 b_1 b_2 \cdots b_m$. Suppose $m > 0$ and $n > 0$ and also that there is some $j$ such that $b_j = 0$. We claim the following:

\[(6.1) \quad |v_{r+1}| > |v_r| \text{ and } |w_{r+1}| > |w_r|.\]

We first prove (6.1). By replacing $\sigma^{2^r}$ with an arbitrary substitution $\tilde{\sigma}$ (or simply by an induction argument), we may assume that $r = 1$. Let $n'$ be the maximal integer such that $1 \leq n' \leq n$ and for $i > n'$, $a_i = 1$ and $a_{n' - 1} = 0$. Such an $n'$ must exist since the first digit $a_0 = 0$. Let $\sigma^2(0) = 0 c_1 c_2 \cdots c_k$ and $\sigma^2(1) = d_0 d_1 d_2 \cdots d_p$. Choose $k'$ to be the maximal integer such that $1 \leq k' \leq k$ and for $i > k'$, $c_i = 1$ and $c_{k'-1} = 0$. We show that $k' > n'$. First, observe that $k' \geq n'$ since the first digit of $\sigma(0)$ is 0 so $0 c_1 \cdots c_n = 0 a_1, \ldots, a_n$. Then $k' > n'$ because there is some $b_j$ such that $b_j = 0$ so even if $a_i = 1$ for all $i \geq 0$, $c_\ell = 0$ for some $\ell > n$. Therefore, as $k' \geq \ell$, we must have $k' > n'$ and thus $|v_2| > |v_1|$.

Suppose $b_0 = b_m = 1$. Let $m_1$ be the least integer greater than 0 such that $b_{m_1} = 0$ and $m_2$ be the greatest integer less than $m$ such that $b_{m_2} = 0$. Let $p_1$ be the least integer greater than 0 such that $d_{p_1} = 0$ and $p_2$ be the greatest integer less than $p$ such that $d_{p_2} = 0$. Observe that $p_2 - p_1 > m_2 - m_1$. Indeed, $p_2 - p_1 \geq m_2 - m_1$ since the first digit is a 1 so $p_1 = m_1$ and $p_2 \geq m_2$. Because the first and last digits of $b$ is a 1 and the middle digit is a 0, we know that $m \geq 2$. Consequently, $p_1 = m_1$ and $p_2 > m_2$ so $p_2 - p_1 > m_2 - m_1$ so $|w_2| > |w_1|$.
Next, suppose $b_0 = 0$. Let $m'$ be the greatest integer such that $b_{m'} = 0$ and $p'$ the greatest integer such that $d_{p'} = 0$. Note once again that $p' \geq m'$ since $\sigma(0)$ starts with a 0 and $p' > m'$ since $m > 0$: the digit $b_1$ must be either a 0 or a 1 and if it were a 0, then $d_0 = d_1 = 0$ and $d_{m+1} = d_{m+2} = 0$ so $p' > m'$. Since $|w_2| > |w_1|$, $m$.

If $b_m = 0$, let $m''$ be the least integer such that $b_{m''} = 0$ and $p''$ be the least integer such that $d_{p''} = 0$. A similar argument as above shows that $p'' > m''$ and thus $|w_2| > |w_1|$.

Notice that $v_{k+1}$ and $w_{k+1}$ may be built from $v_k$ and $w_k$. This is because $\sigma^{2k+1}(0)$ and $\sigma^{2k+1}(1)$ can be built from $\sigma^{2k}(0)$ and $\sigma^{2k}(1)$ and because $\sigma^{2k+1} = \sigma^{2k} \circ \sigma^1$, it follows that $x_{k+1}, y_{k+1}$, $z_{k+1} \in \{x_k, y_k, z_k, 0\}$. Hence, $v_{k+1}$ and $w_{k+1}$ may be built from $v_k$ and $w_k$. In addition, $q$ since the first digit of $q$ is 0. Hence $v_k$ and $w_k$ are a sequence of words of increasing length that build $q$, so $q$ is rank-two if there exists $j$ such that $b_j = 0$ and if $m, n > 0$.

If $n = 0$, then the sequence is trivial and thus rank 1. If $m = 0$, suppose $b_0 = 0$. Then the substitution $\sigma^2$ satisfies the condition that $|\sigma^2(0)| > 2$ and $|\sigma^2(1)| > 2$ and that there is some symbol in $\sigma^2(1)$ that is 0, and we are in the case of $m, n > 0$ and there is some $j$ with $b_j = 0$. This leaves us with the last remaining case that $b_i = 1$. In this case, we show that $q$ is rank 1. If $a_1 \ldots a_N$ are all 1’s, then $q$ is simply 0111111111. If there exists some $a_i = 0$ for $i \geq 1$, then we claim that $|v_2| > |v_1|$. Once again, $|v_2| \geq |v_1|$ since the first digit of $v_1$ is 0. The problem that the substitution of the second 0 contains a 0 and that 0 is in a further place than $|v_1|$. Since $w_k$ are all empty, $v_{k+1}$ are all built from $v_k$ and $q$ is rank one.

**Example.** Let $\sigma$ denote the Fibonacci substitution $0 \mapsto 01$ and $1 \mapsto 0$ and $w$ the word $\lim_{n \to \infty} \sigma^n(0)$. As $w$ is a substitution sequence, it is rank at most two. We will show below that the dynamical system associated to the Fibonacci sequence is not rank-one, thus giving another example of a rank-two system that is not rank-one.

**Example.** Let $\sigma$ denote the Cantor substitution $0 \mapsto 010$ and $1 \mapsto 111$ and the sequence $w = \lim_{n \to \infty} \sigma^n(0)$. As remarked in the proof of the above theorem, since $\sigma(1)$ has no zeroes in it, the Cantor sequence is rank-one.

The Fibonacci sequence is an example of what is known as a *Sturmian sequence*.

**Definition 6.3.** A **Sturmian sequence** is an element $w$ of $\{0, 1\}^\mathbb{N}$ such that the number of words of length $n$ that is a subword of $w$ is $n + 1$.

We refer the reader to [9] for properties of the Sturmian sequence. The main properties of Sturmian sequences we use are the following:

- Sturmian sequences are not eventually periodic.
- In any Sturmian sequence, either 11 appears as a subword or 00 appears as a subword. Not both. If 11 appears, we say that the sequence is *type I* and if 00 appears, we say that the sequence is *type 0*.
- Sturmian sequences are balanced, meaning for all $n$ and for any two subwords of length $n$, the nonnegative difference between the number of 1’s between those two subwords of length $n$ is at most 1.

**Proposition 6.4.** Let $(X, d)$ denote the dynamical system for a type 0 Sturmian sequence. Then $(X, d)$ is not rank 1.
Proof. Let \( w \) denote the Sturmian sequence. Since all Sturmian sequences are balanced, between any two 1’s there can be either \( n \) or \( n + 1 \)’s. Let \( v \in X \) be a word such that \( X_v = X \). Since any subword of \( v \) is a subword of \( w \), \( v \) must satisfy the same properties that the sequence is balanced. Let \( v \) start with \( 0^k1 \) with \( 0 < k \leq n + 1 \). Suppose \( v \) is a rank-one word built by some word \( p \) which contains \( 0^k1 \) and ends with a 0. If \( k = n + 1 \), then \( v \) cannot contain two \( p \)’s in a row since otherwise, there would be \( n + 2 \) zeroes between any two ones, contradicting the fact that \( v \) is balanced. Thus, \( v \) must be of the form \( p1p1p \ldots \) contradicting nonperiodicity of \( w \). If \( p \) ends with \( 10^{n+1} \), then the next symbol in \( w \) must be a 1 then followed by a 0. Thus, \( v \) is of the form \( p1p1p1p \ldots \) which obviously cannot be the case since \( w \) is not periodic. If \( p \) ends with \( 10^n \) and the next digit is a 1, then \( v \) contains the sequence \( 10^n10^{k+1} \) which cannot happen if \( k < n \), but if \( k = n \) only possibly occurs if \( 10^{n+1}10^{n+1}1 \) does not occur. If \( n = 1 \), then \( 10^n10^{n+1} \) can’t happen by the hypothesis of the theorem. If \( n > 1 \), then \( pp \) can’t occur in \( v \) since otherwise, there would be \( 2n > n + 1 \) zeroes in a row. Finally, if \( p \) ends with less than \( n \) zeroes then \( v \) must be of the form \( pppp \ldots \), contradicting nonperiodicity of Sturmian sequences.

In the case of \( n = 1 \), we show via induction that such a \( p \) must be of the form \( 01010101 \ldots 10 \). Note that both \( pp \) and \( plp \) must appear in \( v \) since otherwise there would be periodicity. In addition, by the argument above, \( p \) must begin with 01 and since \( pp \) appears, it must end with 10. Since \( plp \) appears, 10101 appears and therefore since \( v \) is a Sturmian sequence, 1001001 cannot appear. Thus, \( p \) must end with 01010 so 1010101 appears in the Sturmian sequence. Now suppose \( 10^n \) appears in the Sturmian sequence where the length of \( (10)^n \) is less than that of \( p \). Then \( 100(10)^n-21001 \) cannot appear in the sequence and since \( pp \) appears, \( p \) must end with \( (10)^n \) and since \( plp \) appears, the Sturmian sequence contains \( (10)^{n+1} \). By induction, \( p \) must be of the form \( 010101 \ldots 10 \). Since \( v \) is rank 1, this would mean that the first \( |p| \) letters of \( v \) are \( p \) for infinitely many \( p \) of the form \( 01010 \ldots 10 \) so \( v \) would be periodic. This is a contradiction. \( \Box \)

Example. The Fibonacci substitution \( 0 \mapsto 01 \) and \( 1 \mapsto 0 \) does not contain 10101 and is a type 0 Sturmian sequence so it is not rank 1 and since it is a substitution sequence, it is rank 2. In fact, any dynamical system for a Sturmian sequence associated to a quadratic irrational (see [9] chapter 6) is rank 2 but not rank 1.

Proposition 6.5. Let \((X, d)\) denote the dynamical system for a type 1 Sturmian sequence. Then \((X, d)\) is not rank 1.

Proof. Let \( w \) denote the Sturmian sequence and suppose there exists a rank 1 \( v \in X \) such that \( X_v = X \). As in the previous proposition, we argue that between two zeroes in \( v \) and \( w \), only \( 1^n \) and \( 1^{n+1} \) appear for some \( n \geq 1 \). In addition, as before, both \( v \) and \( w \) must be balanced sequences that cannot be eventually periodic (making \( v \) a Sturmian sequence as well). Suppose \( p \) builds \( v \). Note that \( p \) must begin and end in 0. Both \( plnp \) and \( plp1p \) must appear in \( v \) because otherwise, \( v \) would be periodic.

If \( p \) begins with \( 01^n \) and ends with \( 1^{n+1}0 \), then since \( plnp \) and \( plp1p \) appears in \( v \), \( 1^{n+1}01^{n+1} \) and \( 01^n01^n0 \) appears in \( v \), contradicting the fact that \( v \) is balanced. By a similar argument, if \( p \) begins with \( 01^{n+1} \) and ends with \( 1^{n}0 \), the word \( 01^n01^n0 \) and \( 1^{n+1}01^{n+1} \) must appear in \( v \), contradicting the fact that \( v \) is balanced.

If \( p \) starts with \( 01^n \) and ends with \( 1^n0 \), since \( plnp \) and \( plp1p \) both appear in \( v \), \( 01^n01^n01^n0 \) and \( 01^n01^n1^n0 \) both appear in \( v \). We show via induction that \( p \) must be of the form \( 01^n01^n \ldots 1^n0 \).
If \( p \) starts with \( 01^n01^{n+1} \), then \( 1^{n+1}01^n01^{n+1} \) and \( 01^n01^n01^n0 \) appear in \( v \), contradicting balancedness of \( v \). Thus, \( p \) must begin with \( 01^n01^n0 \). Now suppose \( p \) begins with \( (01^n)^k0 \). If \( p \) begins with \( (01^n)^k01^{n+1} \), then since \( p1^n0p \) and \( p1^{n+1}p \) both appear in \( v \), the string \( 1^{n+1}(01^n)^k01^{n+1} \) and \( (01^n)^{k+2}0 \) both appear in \( v \). They are both strings of length \((n + 1)(k + 2) + 1 \) but the latter has \( n(k + 2) \) 1’s while the former has \( n(k + 2) + 2 \) 1’s, a contradiction to \( v \) being balanced. Since \( v \) is rank 1, it must be built with arbitrarily many \( p \), so it must begin with \( (01^n)^k \) for \( k \) arbitrarily large: i.e. it must be periodic. This is a contradiction.

If \( p \) starts with \( 01^{n+1} \) and ends with \( 1^{n+1}0 \), we argue similarly that \( p \) must be of the form

\[ 01^{n+1}01^n1 \ldots 1^{n+1}0. \]

Suppose \( p \) begins with \((01^n+1)^k01^n0 \). Then since \( v \) contains \( p1^n0p \) and \( p1^{n+1}p \), \( v \) must contain the string \( 1^{n+1}01^n1(01^n+1)^k \) and must contain \( 01^n(01^n+1)^k01^n0 \), both of which are length \((n + 2)(k + 1) + n + 1 \) but the former has \((n + 1)(k + 3) \) 1’s while the latter has \((n + 1)(k + 3) − 2 \) 1’s, a contradiction to \( v \) being balanced. Hence, \( v \) begins with \((01^n+1)^k \) for \( k \) arbitrarily large, so it is periodic. This is a contradiction.

7. Questions

Question 1. Do finite rank sequences have zero (topological) entropy? What is the complexity of a rank \( n \) system? As shown below, rank-one systems have complexity bounded by a quadratic polynomial, and hence have zero entropy.

**Proposition 7.1.** If \( v \) is rank-one, then it has zero entropy.

**Proof.** Let \( w \) be a rank-one word built by \((v_i)\). We count the number of different words of length \( |v_i| = m \). To do this, we just need to count the number of different words of length \( m \) within the words \( v_i1^av_i \) for \( 0 \leq a \leq m \). If \( a = 0 \), then we obtain at most \( m + 1 \) words, if \( a = 1 \), we obtain at most \( m + 2 \) words, and for arbitrary \( a \), we obtain at most \( m + 1 + a \) words. This totals to \( (m + 1)^2 + \frac{m(m + 1)}{2} \) and is \( O(m^2) \). Taking the logarithm and dividing by \( m \), this quantity tends to 0 as \( m \) tends to infinity. Hence, \( v \) has zero entropy.

Question 2. What is an example of a rank-three system that is not rank-two? The Thue-Morse sequence and the quadratic Sturmian sequences give examples of a rank-two system that is not rank-one.

Question 3. What is the rank of a general Sturmian Sequence?

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