ON COLORED SET PARTITIONS OF TYPE $B_n$

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Abstract. Generalizing Reiner’s notion of set partitions of type $B_n$, we define colored $B_n$-partitions by coloring the elements in and not in the zero-block respectively. Considering the generating function of colored $B_n$-partitions, we get the exact formulas for the expectation and variance of the number of non-zero-blocks in a random colored $B_n$-partition. We find an asymptotic expression of the total number of colored $B_n$-partitions up to an error of $O(n^{-1/2} \log^{7/2} n)$, and prove that the centralized and normalized number of non-zero-blocks is asymptotic normal over colored $B_n$-partitions.

1. Introduction

It is well known that the centralized and normalized number of blocks over ordinary set partitions is asymptotic normal, which is due to Harper [13]. There are various studies on the asymptotic behavior of set partitions. Basic notions and results of limiting distributions can be found in [10,19,23]. In this paper, we focus on the limiting distribution of another set partition structure.

A partition of the set $[n] = \{1, 2, \ldots, n\}$ is a collection of pairwise disjoint sets $b_1, b_2, \ldots, b_m$ such that $\bigcup_{i=1}^m b_i = [n]$. The sets $b_i$ are called blocks. In 1997, Reiner [22] introduced the notion of set partition of type $B_n$ ($B_n$-partition for short), which was defined to be a partition $\pi$ of the set $[\pm n] = \{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$ such that for any block $b$ of $\pi$, $-b$ is also a block of $\pi$, and that $\pi$ contains at most one block $b$ satisfying $b = -b$. The block $b = -b$, if it exists, is called the zero-block. We call the pair $\pm b$ of blocks a block pair of $\pi$ if $b$ is not the zero-block.

Most attention that has been paid to $B_n$-partitions are devoted to its lattice structure. One of the main reasons seems to be the development in subspace arrangements. A main question regarding subspace arrangements is to study the structure of the complement subspace $C_A = V - \bigcup_{X \in A} X$, where $V$ is a finite-dimensional vector space, and $A$ a subspace arrangement of $V$. The Goresky-MacPherson formula [12] allows one to translate the problem of determining the homology groups of $C_A$ into the problem of studying certain groups related to the intersection lattice. In fact, the lattice of ordinary set partitions can be considered as the intersection lattice for the hyperplane arrangement corresponding to the root system of type $A$, see [3,16]; and the intersection lattice of the Coxeter arrangement of type $B_n$ corresponds to $B_n$-partitions, see [4].

Set partitions of type $B_n$ are also referred to as signed partitions. In some papers, for example [28], a $B_n$-partition is defined to be a partition of the set $[n] \cup \{0\}$ allowing barring elements in $[n]$ such that every element in the block containing the element 0 is unbarred, and that the minimum element in each block is unbarred. The block containing 0 is called the zero-block. In fact, this definition for

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$B_n$-partitions is equivalent to the one given by Reiner. This can be seen by identifying the block
\[ \{0, a_1, a_2, \ldots, a_r\} \]
containing 0 with Reiner’s zero-block
\[ \{\pm a_1, \pm a_2, \ldots, \pm a_r\}, \]
and identifying the block
\[ \{x_1, x_2, \ldots, x_s, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_t\} \]
with Reiner’s block pair
\[ \pm\{x_1, x_2, \ldots, x_s, -y_1, -y_2, \ldots, -y_t\}. \]
However, the wording “the number of non-zero-blocks” is therefore confusing because one non-zero-block in the latter definition corresponds to two non-zero-blocks in Reiner’s definition. To avoid this, we will use the latter definition throughout.

Similar to unbarring the minimal elements, we generalize $B_n$-partitions by coloring its elements. Let $n$, $m$ and $c$ be positive integers. Let $\pi$ be a partition of the set $[n] \cup \{0\}$. Let $C_1$ be a list of $c$ colors, and $C_2$ a list of $m$ colors. For any $x \in [n] \cup \{0\}$, let $b_x$ be the block of $\pi$ containing $x$. Then $b_0$ is the zero-block. For any $x \in [n]$, we color $x$ in
\[
\begin{cases}
\text{the first color in } C_2, & \text{if } x = \min b_x; \\
\text{some color in } C_1, & \text{if } x \neq \min b_x \text{ and } x \in b_0; \\
\text{some color in } C_2, & \text{if } x \neq \min b_x \text{ and } x \notin b_0.
\end{cases}
\]
In this way, every element $x \in [n]$ can be regarded as a pair $(x, cl)$, where $cl$ denotes the color of $x$. We call the resulting partition a $(c, m)$-colored $B_n$-partition, and denote the set of $(c, m)$-colored $B_n$-partitions by $\Pi_{n,m,c}$. For notation convenience, we can ignore the element 0 if $\pi$ contains the block $\{0\}$. For example, when $c = m = n = 2$, if we denote $C_1 = [c_1, c_2]$ and $C_2 = [m_1, m_2]$, then the set $\Pi_{2,2,2}$ consists of the following 11 partitions:
\[
\begin{align*}
(1, m_1)/& (2, m_1) & (1, m_1),& (2, m_1) & (1, m_1),& (2, m_2) \\
0, (1, c_1)/& (2, m_1) & 0, (1, c_2)/& (2, m_1) & 0, (2, c_1)/& (1, m_1) & 0, (2, c_2)/& (1, m_1) \\
0, (1, c_1),& (2, c_1) & 0, (1, c_1),& (2, c_2) & 0, (1, c_2),& (2, c_1) & 0, (1, c_2),& (2, c_2),
\end{align*}
\]
where we use the slash symbol “/” to partition the blocks.

In particular, the set of $(1,1)$-colored $B_n$-partitions has a transparent one to one correspondence with the set of ordinary partitions of $[n] \cup \{0\}$. The $(1,2)$-colored $B_n$-partitions is another representation of Reiner’s $B_n$-partitions. More references can be found from Sloane’s Online Encyclopedia of Integers Sequences (A039755). In fact, as will be seen, the number of $(1, m)$-colored $B_n$-partitions with $k$ non-zero-blocks is the Whitney numbers $W_m(n,k)$ of the second kind, that is, the number of elements of rank $k$ in the Dowling lattice $Q_n(G)$, where $G$ is a finite group of order $m$; see [2],[8].

In this paper, we will show that the centralized and normalized number of non-zero-blocks over colored $B_n$-partitions is asymptotic normal. In particular, we get the asymptotic normality over set partitions of type $B_n$. A classical method to prove the normality of some limiting distribution is the following criterion; see [21], page 286 and [1].

Let $A = (a_0, a_1, \ldots, a_n)$ be a sequence of nonnegative numbers. The sequence $A$ is said to be a Polya frequency sequence (PF sequence for short) of order $r$, if the infinite matrix $(a_{i-j}, i,j \geq 0$ where $a_k = 0$ for $k \notin \{0,1,\ldots,n\}$, is totally positive of order $r$. For background on total positivity, see Karlin [17]. It is well known that $A$ is a PF sequence if the polynomial $A(z) = \sum_{k=0}^n a_k z^k$ is either
a positive constant or real-rooted; see [25]. Let $P$ denote the probability distribution on $\{0,1,\ldots,n\}$ defined by normalization of $A$. We call $P$ a PF distribution if $A$ is a PF sequence.

**Theorem 1.1.** Let $(S_n)$ be a sequence of random variables such that $S_n$ has a PF distribution with mean $\mu_n$ and variance $\sigma_n^2$. The asymptotic distribution of $(S_n - \mu_n)/\sigma_n$ is standard normal if and only if $\sigma_n \to \infty$ as $n \to \infty$.

There are some standard methods to find the exact formulas for the expectation and variance in terms of $A(1)$, as well as many methods to prove the real-rootedness of certain polynomials; see [5,18] for example. For the problem we are going to handle, we have $A(1) = |\Pi_{n,m,c}|$. We will use the notation $T_n = |\Pi_{n,m,c}|$, and determine an asymptotic formula for it up to an error of $O(n^{-1/2} \log^{3/2} n)$. This enables us to prove that the variance tends to infinity. More analytic tools for asymptotics can be found in [7,10,20]. We should mention that to include full asymptotic expansions, Harris and Schoenfeld [15] extended the theory of Hayman-admissibility [14], and this extended theory is applicable to functions of at least double “exponential growth”, which is the case we treat in this paper. See also [9,11,24].

In [23] we give a recurrence relation of the cardinality $T_n$ and thus obtain the exact formulas of the expectation and variance of the number of non-zero-blocks of a random partition in $\Pi_{n,m,c}$. In [33] we derive an asymptotic expression of $T_n$, and prove that the centralized and normalized number of zero-blocks in a random $(c, m)$-colored $B_n$-partition is asymptotic normal by Theorem [34].

2. The recurrence relation

Let $T_{n,k}$ be the number of $(c, m)$-colored $B_n$-partitions with $k$ non-zero-blocks. It is clear that $T_{n,0} = c^n$ and $T_{n,n} = 1$. For convenience, let $T_{n,k} = 0$ if $k < 0$ or $k > n$. We claim that

$$T_{n,k} = T_{n-1,k-1} + (mk + c)T_{n-1,k}.$$  \(2.1\)

In fact, all $(c, m)$-colored $B_n$-partitions can be generated by inserting the element $n$ in some color into a $(c, m)$-colored $B_{n-1}$-partition. Let $\pi$ be a $(c, m)$-colored $B_{n-1}$-partition. One may insert the element $n$ into $\pi$ as a singleton non-zero-block. In this case, $n$ has to be colored in the given first color in $C_2$. This way generates all $(c, m)$-colored $B_n$-partitions such that $n$ is a singleton non-zero-block. So it contributes the summand $T_{n-1,k-1}$ to the recurrence \(2.1\). One may also insert $n$ into the zero-block of $\pi$, in any color in $C_1$. This way generates all $(c, m)$-colored $B_n$-partitions such that $n$ is in the zero-block. Therefore, it contributes the summand $cT_{n-1,k}$ to \(2.1\). The last possibility is that we insert $n$ into a non-zero-block of $\pi$. In this case, we have $k$ non-zero-blocks and $m$ colors in $C_2$ to choose. So this way generates all $(c, m)$-colored $B_n$-partitions such that $n$ is in a non-zero-block, and contributes the summand $mkT_{n-1,k}$ to \(2.1\). This proves the claim.

We mention that $T_{n,k}$ reduces to the Whitney number $W_m(n,k)$ of the second kind when $c = 1$. This can be seen straightforwardly from the recurrence

$$W_m(n,k) = W_m(n-1,k-1) + (mk + c)W_m(n-1,k),$$

with $W_m(n,0) = W_m(n,1) = 1$; see Benoumhani [2, Corollary 3].

Consider the polynomial $T_n(x) = \sum_{k=0}^{n} T_{n,k} x^k$. For example,

$$T_1(x) = x + c,$$

$$T_2(x) = x^2 + (m + 2c)x + c^2,$$

$$T_3(x) = x^3 + (3m + 3c)x^2 + (m^2 + 3cm + 3c^2)x + c^3.$$

Multiplying \(2.1\) by $x^k$ and summing it over $k = 1, 2, \ldots, n - 1$, we get

$$T_n(x) = (x + c)T_{n-1}(x) + mxT_{n-1}'(x), \quad n \geq 1.$$
Then we have the following real-rootedness result.

**Theorem 2.1.** For any \( n \geq 1 \), the polynomial \( T_n(x) \) has \( n \) distinct negative roots.

**Theorem 2.1** can be shown by using a criterion for generalized Sturm sequences given by Liu and Wang [18, Corollary 2.4]. In fact, they proved a more general result that any polynomial \( P_n(x) \) of degree \( n \) with nonnegative coefficients satisfying the recurrence relation

\[
P_n(x) = a_n(x)P_{n-1}(x) + b_n(x)P'_{n-1}(x),
\]

where \( a_n(x) \) and \( b_n(x) \) are real polynomials such that \( b_n(x) \leq 0 \) whenever \( x \leq 0 \), forms a generalized Sturm sequence, and is therefore real-rooted.

Let \( \xi_n \) be the random variable denoting the number of non-zero-blocks of a partition in \( \Pi_{n,m,c} \). It follows that

\[
\mathbb{P}(\xi_n = k) = T_{n,k}T_n
\]

Therefore, \( \xi_n \) is the normalization of \( (T_{n,0}, T_{n,1}, \ldots, T_{n,n}) \) and thus has a PF distribution by Theorem 2.1. Denote the expectation of \( \xi_n \) by \( E_n \), and the variance by \( V_n \). It is a standard technique [23] to find the expectation and variance by using the formulas

\[
E_n = \frac{T_n'(1)}{T_n} \quad \text{and} \quad V_n = E_n - E_n^2 + \frac{T_n''(1)}{T_n}.
\]

Note that both

\[
T_n'(1) = \sum_{k=0}^{n} kT_{n,k} \quad \text{and} \quad T_n''(1) = \sum_{k=0}^{n} k(k-1)T_{n,k}
\]

can be expressed in terms of \( T_n \) with the aid of the recurrence (2.1). Routine computations give the exact formulas for the expectation and variance of \( \xi_n \).

**Theorem 2.2.** We have

\[
E_n = \frac{T_{n+1}}{mT_n} - \frac{1 + c}{m} \quad \text{and} \quad V_n = \frac{T_{n+2}}{m^2T_n} - \frac{T_{n+1}^2}{m^2T_n^2} - \frac{1}{m}.
\]

In view of Theorem 1.1 and the above expression of \( V_n \), we are led to find an asymptotic expression for \( T_n \), which will be dealt with in the next section.

3. **An Asymptotic Expression of the Total Number \( T_n \)**

In this section, we will find an asymptotic expression for the total cardinality \( T_n \) by using the classical saddle point method, which is due to Schrödinger [26].

First, we seek the generating function

\[
F(z) = \sum_{n \geq 0} T_n \frac{z^n}{n!}
\]

In fact, for any \( 0 \leq k \leq n \), one may choose \( k \) elements from \( [n] \) to form the non-zero-blocks of a colored \( B_n \)-partition, and let the other \( n-k \) elements constitute the zero-block. Then, for any non-zero-block of size \( s \), there are \( m^{s-1} \) distinct colorings for that block. And for the zero-block of size \( n-k \), there are \( c^{n-k} \) ways of coloring its elements. Therefore,

\[
T_n = \sum_{k=0}^{n} \binom{n}{k} c^{n-k} \sum_{j=0}^{k} m^{k-j} S(k,j),
\]
where $S(k, j)$ is the Stirling numbers of the second kind. By using the generating function

$$\sum_n S(n, k) \frac{x^n}{n!} = \frac{1}{k!} (e^x - 1)^k$$

of Stirling numbers (see [27, page 34]), we deduce

(3.1) \hspace{1cm} F(z) = \exp\left(\frac{e^{mz} - 1}{m} + cz \right).

The saddle point of $F(z)$ is defined to be the value $z$ that minimizes $z^{-n}F(z)$, i.e., the unique positive solution $r$ of the equation

$$r(e^{mr} + c) = n.$$

Similar to the Lambert $W$ function $r = W(n)$ defined by $re^r = n$ (see [6]), it is easy to see that

(3.2) \hspace{1cm} r = \frac{\log n}{m} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right),
\hspace{1cm} e^{mr} = \frac{mn}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right).

Note that

$$\left(\frac{n}{r}\right)^n = \exp(n \log(e^{mr} + c)) = \exp(mrn + n \log(1 + ce^{-mr})).$$

Since $\log(1 + x) = x + O(x^2)$ as $x \to 0$, we infer that

(3.3) \hspace{1cm} \left(\frac{n}{r}\right)^n = e^{mrn + cnr} \left(1 + O\left(\frac{\log^2 n}{n}\right)\right).

Now we are in a position to give an estimate of $T_n$.

**Theorem 3.1.** We have

$$T_n = \frac{1}{\sqrt{mr + 1}} \exp\left(mrn - n + \frac{n}{mr} + 2cr - \frac{1 + c}{m}\right) \cdot \left(1 + O\left(\frac{\log^{7/2} n}{\sqrt{n}}\right)\right),$$

where $r$ is the unique positive solution of the equation $r(e^{mr} + c) = n$.

The proof of Theorem 3.1 follows directly from the theory of Hayman-admissibility, see [15]. In fact, it can be also directly proved by the saddle point method with aid of (3.1) and (3.3).

By Theorem 2.2 and Theorem 3.1, it is easy to deduce that

$$E_n \sim \frac{n}{\log n}.$$
Lemma 3.2. Let $c$ be a nonnegative integer, and $f(x) = x(e^{mx} + c)$. For any $i = 0, 1, 2$, let $t_i$ be the unique positive number such that $f(t_i) = n + i$. Then we have

\begin{align*}
(3.4) & \quad t_1 - t_0 = \frac{1}{mn} - \frac{1}{m^2nt_0} + O\left(\frac{1}{n \log^2 n}\right), \\
(3.5) & \quad t_2 - t_1 = \frac{1}{mn} - \frac{1}{m^2nt_0} + O\left(\frac{1}{n \log^2 n}\right), \\
(3.6) & \quad 2t_1 - t_0 - t_2 = \frac{1}{mn^2} + O\left(\frac{1}{n^2 \log n}\right), \\
(3.7) & \quad \frac{1}{t_0} + \frac{1}{t_2} - \frac{2}{t_1} = O\left(\frac{1}{n^2 \log^2 n}\right).
\end{align*}

Proof. First we prove (3.4). By Cauchy’s mean value theorem, there exists $t \in (t_0, t_1)$ such that

\[ f(t_1) - f(t_0) = (t_1 - t_0)f'(t). \]

Since $f(t_1) - f(t_0) = 1$ and $f'(t) = (mt + 1)e^{mt} + c$, we have

\[ \frac{1}{(mt_1 + 1)e^{mt_1} + c} \leq t_1 - t_0 \leq \frac{1}{(mt_0 + 1)e^{mt_0} + c}. \]

We can compute that

\[ \frac{1}{(mt_0 + 1)e^{mt_0} + c} - \frac{1}{mn} + \frac{1}{m^2nt_0} = \frac{n + cm^2t_0^3 - cm^2t_0^2}{m^2nt_0(mx + n - cm^2t_0^2)}. \]

Since $t_0 = O(\log n)$, we have

\[ \frac{1}{(mt_0 + 1)e^{mt_0} + c} = \frac{1}{mn} - \frac{1}{m^2nt_0} + O\left(\frac{1}{n \log^2 n}\right). \]

On the other hand, we can compute

\[ \frac{1}{(mt_1 + 1)e^{mt_1} + c} - \frac{1}{mn} + \frac{1}{m^2nt_0} = \frac{(n + 1)(1 + mt_1 - mt_0) + m^2t_0t_1^2c - m^2t_0c - m^2t_0t_1}{m^2nt_0(mx + n + 1 + mt_1 - cm^2t_1^2)}. \]

By (3.8) and (3.9), we have $t_1 - t_0 = O(1/n)$. Since $t_1 = O(\log n)$, we deduce that

\[ \frac{1}{(mt_1 + 1)e^{mt_1} + c} = \frac{1}{mn} - \frac{1}{m^2nt_0} + O\left(\frac{1}{n \log^2 n}\right). \]

In view of (3.8), (3.9) and (3.10), we infer that

\[ t_1 - t_0 = \frac{1}{mn} - \frac{1}{m^2nt_0} + O\left(\frac{1}{n \log^2 n}\right). \]

Along the same line, we can prove that

\[ t_2 - t_1 = \frac{1}{mn} - \frac{1}{m^2nt_1} + O\left(\frac{1}{n \log^2 n}\right). \]

Since

\[ \frac{1}{m^2nt_1} - \frac{1}{m^2nt_0} = \frac{t_0 - t_1}{m^2nt_0t_1} = O\left(\frac{1}{n^2 \log^2 n}\right), \]

we get the estimate (3.5). We pause here for the following proposition, which will imply the last two approximations.
Proposition 3.3. Let \( h(x) \) be a continuous function defined on the closed interval \([a, b]\). Suppose that \( h''(x) \) exists in the open interval \((a, b)\). Then for any \( c \in (a, b) \), there exists \( s \in (a, b) \) such that

\[
(3.11) \quad \frac{h(a)}{(a-b)(a-c)} + \frac{h(b)}{(b-a)(b-c)} + \frac{h(c)}{(c-a)(c-b)} = \frac{h''(s)}{2}.
\]

Proof. Let

\[
\begin{align*}
f_1(x) &= (a-b)h(x) + (b-x)h(a) + (x-a)h(b), \\
g_1(x) &= (a-b)(x-a)(x-b).
\end{align*}
\]

Then the left hand side of (3.11) becomes \( f_1(c)/g_1(c) \). Note that \( f_1(a) = g_1(a) = 0 \). By Cauchy’s mean value theorem, there exists \( s_1 \in (a, c) \) such that

\[
\frac{f_1(c)}{g_1(c)} = \frac{f_1(c) - f_1(a)}{g_1(c) - g_1(a)} = \frac{f_1(s_1)}{g_1(s_1)} = \frac{f_2(b) - f_2(a)}{g_2(b) - g_2(a)},
\]

where \( f_2(x) = h(x) - h'(s_1)x \) and \( g_2(x) = x^2 - 2s_1x \). Again, by Cauchy’s mean value theorem, there exist \( s_2 \in (a, b) \) and \( s \in (a, b) \) such that

\[
\frac{f_2(a) - f_2(b)}{g_2(a) - g_2(b)} = \frac{f_2'(s_2)}{g_2'(s_2)} = \frac{h'(s_2) - h'(s_1)}{2s_2 - 2s_1} = \frac{h''(s)}{2}.
\]

This completes the proof of Proposition 3.3.

Now we continue the proof of Lemma 3.2. Set \( h(x) = f(x), a = t_0, b = t_2, \) and \( c = t_1 \) in Proposition 3.3. Multiplying the formula (3.11) by \((t_1 - t_0)(t_2 - t_1)(t_2 - t_0)\) gives

\[
2t_1 - t_0 - t_2 = \frac{1}{2} f''(s)(t_1 - t_0)(t_2 - t_1)(t_2 - t_0).
\]

In view of (3.4) and (3.5), we deduce (3.6) immediately from the above formula. The estimate (3.7) can be derived along the same line, by taking

\[
h(x) = \frac{1}{x} \exp \left( \frac{m}{x} + c \right), \quad a = \frac{1}{t_2}, \quad b = \frac{1}{t_0}, \quad c = \frac{1}{t_1}
\]

in Proposition 3.4. This completes the proof of Lemma 3.2.

Here is the main result of this paper.

Theorem 3.4. The random variable \((\xi_n - E_n)/\sqrt{n}\) has an asymptotically standard normal distribution as \( n \) tends to infinity.

Proof. By Theorem 1.1 and Theorem 2.1 it suffices to prove \( \lim_{n \to \infty} V_n = \infty \), that is,

\[
\lim_{n \to \infty} \frac{1}{m^2} \left( T_{n+2} - \frac{T_{n+1}^2}{T_n} - m \right) = \infty.
\]

For \( i = 0, 1, 2, \) suppose that

\[
t_i(e^{mt_i} + c) = n + i.
\]

By Theorem 3.1, we have

\[
T_{n+i} = \frac{1}{\sqrt{mt_i + 1}} \exp \left( m(n+i)t_i - (n+i) + \frac{n+i}{mt_i} + 2ct_i - \frac{1+c}{m} \right) \cdot \left( 1 + O \left( \frac{\log^{7/2} n}{\sqrt{n}} \right) \right).
\]
Then we can compute
\[
\frac{T_{n+2}}{T_n} = \sqrt{\frac{m t_0 + 1}{m t_2 + 1}} e^{A} \left( 1 + O\left( \frac{\log^{7/2} n}{\sqrt{n}} \right) \right),
\]
where
\[
A = 2 m t_2 + \frac{2}{m t_2} + (m n + 2 c)(t_2 - t_0) - \frac{n}{m} \left( \frac{1}{t_0} - \frac{1}{t_2} \right) - 2.
\]
Similarly, we have
\[
\frac{T_{n+1}^2}{T_n^2} = \frac{m t_0 + 1}{m t_1 + 1} e^{B} \left( 1 + O\left( \frac{\log^{7/2} n}{\sqrt{n}} \right) \right),
\]
where
\[
B = 2 m t_1 + \frac{2}{m t_1} + (2 m n + 4 c)(t_1 - t_0) - \frac{2 n}{m} \left( \frac{1}{t_0} - \frac{1}{t_1} \right) - 2.
\]
Therefore,
\[
\frac{T_{n+2}}{T_n} - \frac{T_{n+1}^2}{T_n^2} = \left( \sqrt{\frac{m t_0 + 1}{m t_2 + 1}} e^{A} - \frac{m t_0 + 1}{m t_1 + 1} e^{B} \right) \left( 1 + O\left( \frac{\log^{7/2} n}{\sqrt{n}} \right) \right).
\]
By the estimates (3.14) and (3.15), we have
\[
\frac{m t_0 + 1}{m t_1 + 1} = 1 + O\left( \frac{1}{n \log n} \right),
\]
\[
\sqrt{\frac{m t_0 + 1}{m t_2 + 1}} = 1 + O\left( \frac{m t_0 + 1}{m t_2 + 1} - 1 \right) = 1 + O\left( \frac{1}{n \log n} \right).
\]
Substituting the above estimates into (3.15), we get
\[
\frac{T_{n+2}}{T_n} - \frac{T_{n+1}^2}{T_n^2} = (e^A - e^B) \left( 1 + O\left( \frac{\log^{7/2} n}{\sqrt{n}} \right) \right).
\]
By Cauchy’s mean value theorem, there exists a constant $C$ such that $B < C < A$ and
\[
e^A - e^B = (A - B)e^C.
\]
We can compute directly from (3.12) and (3.13) that
\[
A - B = 2m(t_2 - t_1) - (mn + 2c)(2t_1 - t_0 - t_2) + \frac{2}{m} \left( \frac{1}{t_2} - \frac{1}{t_1} \right) + n \left( \frac{1}{t_0} + \frac{1}{t_2} - \frac{2}{t_1} \right).
\]
By Lemma 3.2, we deduce that
\[
2m(t_2 - t_1) = \frac{2}{n} + O\left( \frac{1}{n \log n} \right),
\]
\[
-(mn + 2c)(2t_1 - t_0 - t_2) = -\frac{1}{n} + O\left( \frac{1}{n \log n} \right),
\]
\[
\frac{2}{m} \left( \frac{1}{t_2} - \frac{1}{t_1} \right) = O\left( \frac{1}{n \log^2 n} \right),
\]
\[
n \left( \frac{1}{t_0} + \frac{1}{t_2} - \frac{2}{t_1} \right) = O\left( \frac{1}{n \log^2 n} \right).
Substituting them into (3.17), we get

\[(3.18) \quad A - B = \frac{1}{n} \left( 1 + O \left( \frac{1}{\log n} \right) \right).\]

Similarly, we can derive that

\[A = 2mt_2 + O \left( \frac{1}{\log n} \right),\]
\[B = 2mt_1 + O \left( \frac{1}{\log n} \right).\]

By the estimates (3.2), we find that

\[e^A = e^{2mt_2} \left( 1 + O \left( \frac{1}{\log n} \right) \right) = \frac{m^2 n^2}{\log^2 n} \left( 1 + O \left( \frac{\log \log n}{\log n} \right) \right),\]
\[e^B = e^{2mt_1} \left( 1 + O \left( \frac{1}{\log n} \right) \right) = \frac{m^2 n^2}{\log^2 n} \left( 1 + O \left( \frac{\log \log n}{\log n} \right) \right).\]

Since \( B < C < A \), we have

\[(3.19) \quad e^C = \frac{m^2 n^2}{\log^2 n} \left( 1 + O \left( \frac{\log \log n}{\log n} \right) \right).\]

Substituting (3.19) and (3.18) into (3.16), we deduce that

\[(3.20) \quad e^A - e^B = \frac{m^2 n}{\log^2 n} \left( 1 + O \left( \frac{\log \log n}{\log n} \right) \right).\]

Substituting (3.20) into (3.15), and in view of Theorem 2.2, we obtain the approximation

\[V_n = \frac{1}{m^2} \left( \frac{T_{n+2}}{T_n} - \frac{T_{n+1}^2}{T_n^2} \right) - \frac{1}{m} \sim \frac{n}{\log^2 n}.\]

This completes the proof.

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