Effective Action and Conformal Phase Transition in Three–Dimensional QED

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Abstract

The effective action for local composite operators in QED$_3$ is considered. The effective potential is calculated in leading order in $1/N_f$ ($N_f$ is the number of fermion flavors) and used to describe the features of the phase transition at $N_f = N_{cr}$, $3 < N_{cr} < 5$. It is shown that this continuous phase transition satisfies the criteria of the conformal phase transition, considered recently in the literature. In particular, there is an abrupt change of the spectrum of light excitations at the critical point, although the phase transition is continuous, and the structure of the equation for the divergence of the dilatation current is essentially different in the symmetric and nonsymmetric phases. The connection of this dynamics with the dynamics in QCD$_4$ is briefly discussed.

11.30.Rd, 11.10.Kk, 11.10.Gh, 12.20.Ds
I. INTRODUCTION

Noncompact quantum electrodynamics in $2 + 1$ dimensions ($QED_3$) with $N_f$ flavors of four-component fermions [1,2] is a gauge theory with rich dynamics, reminiscent of four-dimensional quantum chromodynamics ($QCD_4$).

Studying the Schwinger-Dyson (SD) equation for the fermion self-energy in leading order in $1/N_f$ expansion [3] showed the existence of a critical number of fermion flavors $N_{cr}$, $3 < N_{cr} < 5$, below which there is the dynamical breakdown of the flavor $U(2N_f)$ symmetry and fermions acquire a dynamical mass. These conclusions were confirmed in the lattice computer simulations of noncompact lattice $QED_3$ [4].

The presence of a critical $N_{cr}$ in $QED_3$ is intriguing especially because of a possibility of the existence of an analogous critical $N_f = N_{cr}$ in $QCD_4$, suggested by both analytical studies [5–8] and lattice computer simulations of $QCD_4$ [9–11].

However there is still controversy concerning the existence of a finite $N_{cr}$ in $QED_3$: some authors argue that the generation of a fermion mass occurs at all values of $N_f$ in $QED_3$ [12,13]. Despite quite intensive studies of this issue [14,15], the problem is still open. The main difficulty is that at present there is no systematic approach to studying nonperturbative strong coupling dynamics in gauge theories. A possibility to get insight into the phase transition at $N_f = N_{cr}$ might be a low energy effective action approach, which has been very successful in studying the phase transition at finite temperatures in $QCD_4$ [16]. Such an approach is useful when there are few relevant (massless) degrees of freedom at the critical point. For example, at $T = T_{cr}$ in $QCD_4$, the relevant degrees of freedom are $N_f^2$ (or $N_f^2 - 1$) pseudoscalar and $N_f^2$ scalar mesons: because of a non-zero temperature, and therefore non-zero Matsubara frequencies for quarks, all other excitations are effectively massive at $T = T_{cr}$. As a result, the effective theory is given by a three dimensional renormalizable lagrangian with the complex matrix field $\phi_{ij}$, $i,j = 1,2,\ldots,N_f$, describing $2N_f^2$ scalar and pseudoscalar mesons [16]. This allows one to reduce the problem to a known universality class of three dimensional models.

Should such an approach (reducing the infrared dynamics to that of a known universality class) work in the case of the phase transition at $N_f = N_{cr}$ in $QED_3$? We believe that the answer is ”no”. The point is that there are too many relevant degrees of freedom at $N_f = N_{cr}$ in $QED_3$. Besides a composite field $\varphi$, describing $2N_f^2$ Nambu-Goldstone (NG) bosons (corresponding to the spontaneous breakdown $U(2N_f) \rightarrow U(N_f) \times U(N_f)$) and their $2N_f^2 - 1$ flavor partners (for details, see below), there are massless fermions and massless photon at $N_f = N_{cr}$. Therefore, unlike the temperature phase transition in $QCD_4$, the effective action includes even more degrees of freedom than the initial lagrangian. This in turn implies that the effective action for the field $\varphi$, that can be obtained when light fermion and photon fields are integrated out, is necessary non-local and non-renormalizable. In other words, the universality class of the phase transition in $QED_3$ is different from those described by renormalizable (or even local) lagrangians for the field $\varphi$.

Still, it would be interesting to find the effective action for $\varphi$ in $QED_3$, describing the phase diagram of the theory. In this paper, we make a step in realizing this program and calculate the effective potential in leading order in $1/N_f$ expansion. Our main goal is to figure out how the structure of the potential reflects some peculiar properties of the phase transition at $N_f = N_{cr}$ in $QED_3$ pointed out in Refs. [4,7]. It gives an example of the conformal phase
transition, the conception which generalizes the Berezinsky-Kosterlitz-Thouless (BKT) phase transition (taking place in two dimensions) to higher dimensions.

We stress that our aim is much more modest than to prove the existence of a finite $N_\text{cr}$ in QED$_3$. We will just derive the effective potential, accepting, following Ref. [3], that the leading order in $1/N_f$ is a reliable approximation in the theory. Then we will see that the structure of this potential indeed corresponds to the conformal phase transition. In particular, it will be shown that unlike the usual, $\sigma$-model-like, continuous phase transition, there is an abrupt change of the spectrum of light excitations at the critical point in QED$_3$. Also, the realization of the conformal symmetry is very different in the symmetric and non-symmetric phases.

II. THE MODEL. GENERAL PROPERTIES.

The lagrangian density for massless QED$_3$ is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi} i \dot{D} \psi,$$

(1)

where $D_\mu = \partial_\mu - i e A_\mu$, $\dot{D} = \gamma^\mu D_\mu$, and four-component fermion fields carry the flavor index $i = 1, 2, \ldots, N_f$. The three $4 \times 4$ $\gamma$-matrices can be taken to be

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}.$$  

(2)

Recall that $(\sigma_3, i\sigma_1, i\sigma_2)$ and $(-\sigma_3, -i\sigma_1, -i\sigma_2)$ make two inequivalent representations of the Clifford algebra in $2 + 1$ dimensions.

There are two matrices

$$\gamma^3 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(3)

that anticommute with $\gamma^0, \gamma^1$ and $\gamma^2$. Therefore for each four-component spinor, there is a global $U(2)$ symmetry with generators

$$I, \quad \frac{1}{i} \gamma^3, \quad \gamma^5, \quad \text{and} \quad \frac{1}{2}[\gamma^3, \gamma^5],$$

(4)

and the full symmetry is then $U(2N_f)$.

The lagrangian density with a mass term $m\bar{\psi}\psi$ is invariant under parity transformations defined as

$$P : \quad \psi(x^0, x^1, x^2) \rightarrow \frac{1}{i} \gamma^3 \gamma^1 \psi(x^0, -x^1, x^2).$$

(5)

The generation of a parity-invariant dynamical mass for fermions leads to the spontaneous breakdown of the $U(2N_f)$ down to $U(N_f) \times U(N_f)$ with the generators

$$\frac{\lambda^a}{2}, \quad \frac{1}{2} \gamma^3 \gamma^5,$$

(6)
\( \alpha = 0, 1, \ldots, N_f^2 - 1 \). The corresponding \( 2N_f^2 \) NG bosons are:

\[
S^\alpha \sim \bar{\psi} \frac{\lambda^\alpha}{2} \gamma^5 \psi
\]

\((N_f^2 \text{ scalars})\), and

\[
P^\alpha \sim \bar{\psi} \frac{\lambda^\alpha}{2} \gamma^3 \psi
\]

\((N_f^2 \text{ pseudoscalars})\). There are also their \( 2N_f^2 \) massive flavor partners:

\[
S^\alpha \sim \bar{\psi} \frac{\lambda^\alpha}{2} \psi
\]

\((N_f^2 \text{ scalars})\), and

\[
P^\alpha \sim \bar{\psi} \frac{\lambda^\alpha}{2} \frac{1}{2} [\gamma^3, \gamma^5] \psi
\]

\((N_f^2 \text{ pseudoscalars})\). Notice that these \( 4N_f^2 \) bosons can be described by the hermitian matrix field

\[
\phi^i_j = \psi^i \psi_j^\dagger
\]

which can be decomposed into a traceless part and its trace defined as:

\[
\varphi \equiv \phi - \frac{1}{8N_f} \gamma^0 [\gamma^5, \gamma^3] \chi, \quad \chi \equiv tr(\frac{1}{2} \gamma^0 [\gamma^5, \gamma^3] \phi) \sim P^0.
\]

The field \( \varphi \) is assigned to the adjoint representation of \( SU(2N_f) \), and \( \chi \) is a singlet. As to the vacuum group \( U(N_f) \times U(N_f) \), the \( 2N_f^2 \) NG bosons are assigned to the representation \( (N_f, N_f^2) \oplus (N_f^2, N_f) \), and the \( 2N_f^2 \) massive bosons are assigned to the representation \( (N_f \times N_f^*, 1) \oplus (1, N_f \times N_f^*) = (N_f^2 - 1, 1) \oplus (1, N_f^2 - 1) \oplus 2(1, 1) \) of its maximal semi-simple subgroup \( SU(N_f) \times SU(N_f) \). One of the two singlets \((1, 1)\) corresponds to the pseudoscalar \( \chi = \frac{1}{2} \bar{\psi} [\gamma^3, \gamma^5] \psi \sim P^0 \), and another to the scalar \( \sigma \equiv \bar{\psi} \psi \sim S^0 \). The vacuum expectation value \( \langle 0 | \sigma | 0 \rangle \) is an order parameter describing the spontaneous breakdown \( U(2N_f) \rightarrow U(N_f) \times U(N_f) \).

Recall that besides the generation of a parity-invariant mass, corresponding to \( \sigma_c \equiv \langle 0 | \sigma | 0 \rangle \neq 0 \), there might be the generation of a \( U(2N_f) \)-invariant mass corresponding to \( \chi_c \equiv \langle 0 | \chi | 0 \rangle \neq 0 \), thus violating parity. Here we accept arguments of Ref. [19] in the support of a solution with a parity-invariant vacuum.

In leading order in \( 1/N_f \), in Landau gauge, the SD equation for the fermion propagator \( G(p) = (A(p) \hat{p} - \Sigma(p))^{-1} \) is reduced to the equations [3,8]:

\[
A(p) = 1,
\]

\[
\Sigma(p) = m_0 + \frac{\alpha}{\pi^2 N_f \hat{p}} \left[ \int_0^p \frac{k \Sigma(k)}{k^2 + \Sigma^2(k)} \frac{k}{p} + \frac{\Lambda}{\hat{p}} \int_0^p \frac{k \Sigma(k)}{k^2 + \Sigma^2(k)} \frac{p}{k + \frac{\alpha}{8}} \right],
\]

\( \text{Eq. } 13 \)
where $\alpha \equiv e^2 N_f$, $\Lambda$ is an ultraviolet cutoff, and, for generality, the bare fermion mass $m_0$ is introduced.

Notice that the fermion mass function is neglected in the vacuum polarization in this approximation. The reliability of this approximation has been studied in Refs. [14,15]. In particular, in Ref. [15], to keep $A(p)=1$, a non-local gauge was used. It was shown then that including the mass function in the vacuum polarization does not lead to qualitative changes of the results of Ref. [3]: the scaling law for the dynamical mass function near the critical point has the same form (see Eq. (29) below), with a somewhat different value of $N_{cr}$: $N_{cr} = 4.3$ instead $N_{cr} = 3.2$. Moreover, this new value agrees with the result of the second paper of Ref. [3], incorporating the next-to-leading corrections in $1/N_f$ to the SD equation in Landau gauge.

That study suggests that neglecting a fermion mass function in the vacuum polarization is a reasonable approximation for a small ($\Sigma_0 << \alpha$) fermion mass, and this condition is fulfilled for $N_f$ close to the critical point $N_{cr}$. Therefore including the mass function in the vacuum polarization seems do not change the qualitative features of the phase transition at $N_f = N_{cr}$. Since the main aim of this paper is to give a qualitative description of this phase transition, the use of this approximation seems appropriate.

We will return to equation (13) in the next section.

III. EFFECTIVE POTENTIAL FOR LOCAL COMPOSITE FIELDS IN $QED_3$

We will consider the effective action (generating functional for proper vertices) of the local composite field $\varphi$ in $QED_3$. Actually we will calculate only the effective potential, i.e. the part of the action without derivatives of $\varphi$, though we will also describe properties of other terms in the effective action which are necessary for establishing the origin of the phase transition.

In the derivation of the effective potential, we will closely follow Ref. [20], where the effective potential for local composite fields in quenched $QED_4$ was derived.

The effective action is defined in the standard way. First, one introduces a generating functional for Green’s functions of the field $\varphi$:

$$ Z(J) = \exp(iW(J)) = \int d\eta \exp \left[ i \int d^3 x \left( L(x) + tr [J(x) \varphi(x)] \right) \right], \quad (14) $$

where the $\eta$ integration is functional, $J(x)$ is the source for $\varphi(x)$, and $L(x)$ is the lagrangian density (1) (the symbol $\eta$ represents all the fundamental fields (fermion and photon ones) of the model).

The effective action for the field $\varphi$ is a Legendre transform of the functional $W(J)$:

$$ \Gamma(\varphi_c) = W(J) - \int d^3 x tr [J(x) \varphi_c(x)], \quad (15) $$

where $\varphi_c(x) \equiv \langle 0 | \varphi(x) | 0 \rangle$. From Eqs. (14) and (15) one finds that the following relations are satisfied:

$$ \frac{\delta W}{\delta J(x)} = \varphi_c(x), \quad (16) $$

$$ \frac{\delta \Gamma}{\delta \varphi_c(x)} = -J(x). \quad (17) $$
The effective action $\Gamma$ can be expanded in powers of derivatives of the field $\phi_c$:

$$\Gamma(\phi_c) = \int d^3x \left[ -V(\phi_c) + \frac{1}{2}Z(\phi_c)tr(\partial_\mu \phi_c \partial^\mu \phi_c) + \ldots \right], \quad (18)$$

where $V(\phi_c)$ is the effective potential.

The calculation of the effective potential is reduced to finding the Legendre transform of the functional $W(J)$ with the source $J$ independent of coordinates $x$. The field $\phi_c$ is a matrix in the flavor space describing $4N_f^2 - 1$ boson fields.

Let us first consider the case when a constant source is introduced for the composite field $\sigma = \bar{\psi}\psi$: $L \rightarrow L + J\bar{\psi}\psi$. Then the source term in $Z(J)$ has the form of a bare mass term, with $m_0 \equiv -J$. Eq. (17) now becomes

$$\frac{dV}{d\sigma} = -m_0. \quad (19)$$

Therefore

$$V = -\int m_0(x)dx, \quad (20)$$

where $m_0(\sigma_c)$ defines the dependence of $m_0 = -J$ on the condensate $\sigma_c = \langle 0|\sigma|0 \rangle = \langle 0|\bar{\psi}\psi|0 \rangle$, which is an order parameter of the spontaneous breakdown $U(2N_f) \rightarrow U(N_f) \times U(N_f)$. The function $m_0(\sigma_c)$ can be defined from the SD equations (13).

The effective potential $V$ is calculated in the Appendix. It is:

$$V = \frac{A^2N_f^2\Sigma_0^3}{32} \left[ -\frac{7}{3} + \frac{3}{4\nu^2} - \frac{1}{\nu} \sin 2\theta - \left(1 + \frac{3}{4\nu^2}\right) \cos 2\theta \right] + \frac{A^2N_f\Lambda \Sigma_0^3(\frac{9}{4} + \nu^2)}{4\pi^2\nu^2 \Lambda_{np}} \sin^2(\theta + \nu\delta_1), \quad (21)$$

where $\nu = \sqrt{\lambda - 1/4}$, $\lambda = 8/\pi^2 N_f$, $\Lambda$ is an ultraviolet cutoff, $\Lambda_{np} \equiv \alpha/8$ is an ultraviolet cutoff for nonperturbative dynamics, like $\Lambda_\infty \equiv 4\pi F_\pi \sim 1$ Gev in QCD, $\Sigma_0 \equiv (p^2)|_{p=0}$ is an infrared fermion mass parameter,

$$\theta = \nu \log \frac{\Lambda_{np} e^{\delta_0}}{\Sigma_0}, \quad \delta_0 = 3 \log 2 + \frac{\pi}{2} - 2, \quad \delta_1 = \frac{1}{\nu} \arctan \frac{2\nu}{3},$$

and

$$A = \frac{\sqrt{\pi}}{2(1 + \frac{8}{\pi^2 N_f})} \left| \frac{\Gamma(1 + i\nu)}{\Gamma(\frac{1}{2} + \frac{i\nu}{2})\Gamma(\frac{3}{2} + \frac{i\nu}{2})} \right|. \quad (21)$$

Notice that expression (21) for $V$ is valid at $\Sigma_0 << \Lambda_{np} = \alpha/8$.

The dependence of $\Sigma_0$ on the condensate $\sigma_c$ is defined from the equation:

$$\sigma_c = \langle 0|\bar{\psi}\psi|0 \rangle = -\frac{2N_f \Lambda m_0}{\pi^2} + \sigma_{np}^c, \quad (22)$$

where the nonperturbative part of the condensate is
\[ \sigma_{np}^{\text{up}} = \frac{N_f^2}{4} \Lambda_{np}^{2} \frac{d^2}{dp} \left[ \frac{N_f^2}{4} \Lambda_{np}^2 \Sigma_0 \right]_{p = \Lambda_{np}} \simeq \frac{N_f^2}{4} \Lambda_{np}^2 \Sigma_0 \left( \frac{d}{dp} F\left( \frac{1}{4} - i \nu, \frac{1}{4} + i \nu, \frac{3}{2}; \frac{p^2}{\Sigma_0^2} \right) \right)_{p = \Lambda_{np}}, \tag{23} \]

\( F \) is a hypergeometric function (for details, see the Appendix). The parameter \( m_0 \) is expressed through \( \Sigma_0 \) as

\[ m_0 = \frac{A \Sigma_0^{3/2} \sqrt{9/4 + \nu^2}}{2 \nu \Lambda_{np}^{1/2}} \sin(\nu \log \frac{\Lambda}{\Sigma_0} + \nu (\delta_0 + \delta_1)). \tag{24} \]

Eqs. (21), (22), (23), and (24) define \( V \) as an implicit, and rather complicated, function of \( \sigma_c \). However, the phase diagram of the theory can be established by studying \( V \) as a function of the fermion mass \( \Sigma_0 \). It is convenient to rewrite Eq. (21) as

\[ V = \frac{A^2 N_f^2 \Sigma_0^2}{32} \left[ -7 + \frac{3}{4 \nu^2} + \left( 1 - \frac{3}{4 \nu^2} \right) \cos(2\theta + 2\nu \delta_1) - \frac{2}{\nu} \sin(2\theta + 2\nu \delta_1) \right] + \frac{A^2 N_f \Lambda \Sigma_0^2 (9/4 + \nu^2)}{4 \pi^2 \nu^2 \Lambda_{np}} \sin^2(\theta + \nu \delta_1). \tag{25} \]

Then the gap equation is:

\[ \frac{dV}{d\Sigma_0} = \frac{A^2 N_f^2 \Sigma_0^2}{16} \left\{ \left[ -7 + \frac{9}{4 \nu^2} \right] \sin(\theta + \nu \delta_1) + \left( 2 \nu + \frac{15}{2 \nu} \right) \cos(\theta + \nu \delta_1) \right\} + \frac{4 \Lambda (9/4 + \nu^2)}{N_f \pi^2 \nu^2 \Lambda_{np}} \left\{ 3 \sin(\theta + \nu \delta_1) - 2 \nu \cos(\theta + \nu \delta_1) \right\} \sin(\theta + \nu \delta_1) = 0. \tag{26} \]

This equation yields the following solutions:

\[ \Sigma_0^{(n)} = \Lambda_{np} \exp \left( -\frac{n \pi}{\nu} + \delta_0 + \delta_1 \right), \quad n = 1, 2, \ldots, \tag{27} \]

\[ \Sigma_0^{(n)} = \Lambda_{np} \exp \left( -\frac{n \pi}{\nu} + \delta_0 + \delta_1 - \frac{2}{3} \right), \quad n = 1, 2, \ldots. \tag{28} \]

One can check that while all the solutions (27) correspond to minima of \( V \), solutions (28) correspond to maxima of the potential. Actually, only the global minimum, corresponding to \( n = 1 \), defines the stable vacuum [22]. Therefore the dynamical mass is

\[ m_{\text{dyn}} \equiv \bar{\Sigma}_0 = \Lambda_{np} \exp \left( -\frac{\pi}{\nu} + \delta_0 + \delta_1 \right). \tag{29} \]

As it has to be, it coincides with \( \bar{\Sigma}_0 \) of Ref. [3], derived from the SD equation.

Few comments are in order:

1. Since expressions (21) and (25) for \( V \) are valid only if \( \Sigma_0 << \Lambda_{np} \), the solution (29) exists when \( 0 < \nu << 1 \), i.e. when \( 0 < \frac{8}{\pi^2 \sqrt{\nu} \left( \frac{N_f}{2} \right)^{3/2}} - \frac{1}{4} << 1 \). Therefore, in this approximation, there is a critical value \( N_f = N_{cr} = \frac{32}{\pi^2} \simeq 3.24 \), and expression (25) is valid in the near-critical, scaling, region.
2. The last term in expression (21) for $V$ is connected with a perturbative contribution, i.e. with ultraviolet dynamics at $p \gg \Lambda_{np} = \alpha/8$, where the dimensionless running coupling constant is weak (see Eq.(32) below). This term occurs because of the presence of a source $J = -m_0 \neq 0$ outside the extrema of $V$. On the other hand, since this term is proportional to $\sin^2(\theta + \nu \delta_1)$, the dynamical mass (29), defined at the minimum, is independent of the perturbative contribution. Notice also that while this contribution is proportional to $N_f$, the nonperturbative contribution in $V$ is proportional to $N^2_f$.

The perturbative term is divergent. It is connected with the point that we calculate the effective action for local composite operators, which is a generating functional for proper vertices (Green’s functions) of these operators. It is known that the standard renormalizations do not remove divergences from such Green’s functions (for example, see Sec.12.15 in the book [22]). These divergences are connected with perturbative short-range fluctuations, having nothing with infrared non-perturbative dynamics of bound states. The receipt of dealing with them in studying bound states (used, for example, in the case of QCD sum rules, dealing with Green’s functions of colorless composite operators) is known: one has just to subtract them. Then we are led to the non-perturbative effective potential:

$$V_{np} = \frac{A^2 N_f^2 \Sigma^3}{32} \left[ -\frac{7}{3} + \frac{3}{4\nu^2} - \frac{1}{\nu} \sin 2\theta - \left( 1 + \frac{3}{4\nu^2} \right) \cos 2\theta \right].$$

(30)

It contains only contributions from the nonperturbative dynamics and is independent of the cutoff $\Lambda$. Notice that in the symmetric phase, with $N_f > N_{cr}$, the parameters $\theta$ and $\nu$ in Eq. (30) are imaginary.

Henceforth the cutoff $\Lambda$ is put equal infinity and we will consider the region of the nonperturbative dynamics with momenta $p < \Lambda_{np}$.

3. We have derived the effective potential for the composite operator $\sigma = \bar{\psi}\psi$. In principle, one can derive the effective potential for other $4N^2_f - 2$ boson fields, described by the composite field $\varphi$, by introducing additional $4N^2_f - 2$ sources independent of $x$. The calculations of this potential at $N_f > 1$ are involved, and the expression for $V$ should be very cumbersome. We have not succeeded in getting it. Fortunately, the effective action for $\sigma_c$, with all other fields taken equal zero, is sufficient for our purposes. The reason is the following. On the one hand, since $\sigma_c$ is the only order parameter of the spontaneous breakdown $U(2N_f) \rightarrow U(N_f) \times U(N_f)$, the effective action for $\sigma_c$ defines the mass of $\sigma$ particle both in symmetric and nonsymmetric phases. On the other hand, as was already pointed out in the previous section, in the symmetric phase, all the $4N^2_f - 1$ bosons connected with the composite field $\varphi$ are assigned to the same (adjoint) irreducible representation of the $SU(2N_f)$. This point will be enough for us for proving that the phase transition in $QED_3$ is indeed conformal.

IV. CONFORMAL PHASE TRANSITION IN $QED_3$

The characteristic feature of the phase transition at $N_f = N_{cr}$ in $QED_3$ is that the scaling function
\[ f(N_f) = \exp \left( -\frac{\pi}{\nu} + \delta_0 + \delta_1 \right), \quad \nu = \frac{1}{2} \sqrt{\frac{N_{cr}}{N_f} - 1} \] (31)

\((N_{cr} = \frac{32}{\pi} \text{ in this approximation},\) has an essential singularity at \(N_f = N_{cr}.)\) In particular, considering the scaling function \(f(z)\) as an analytic function of the complex variable \(z = N_f,\) one finds that while \(\lim_{z \to z_{cr}} f(z) = 0\) as \(z\) goes to \(z_{cr}\) from the side of nonsymmetric phase \((N_f < N_{cr}),\) \(\lim_{z \to z_{cr}} f(z) \neq 0\) as \(z \to z_{cr}\) from the side of the symmetric phase with \(N_f > N_{cr}.\) This feature implies that the continuous phase transition in \(QED_3\) satisfies the criteria of the conformal phase transition (CPT) introduced in Ref. [7]. (Notice that since \(N_f\) appears analytically in the path integral of the theory, one can give a nonperturbative meaning to the theory with noninteger \(N_f).\)

As shown in Ref. [7], the CPT is characterized by the following two general features: a) unlike the usual (\(\sigma\)-model-like) continuous phase transition, there is an abrupt change of the spectrum of light excitations at the critical point, though the CPT is a continuous phase transition; b) the realization of the conformal symmetry is very different in the symmetric and nonsymmetric phases: in particular, the equation for the divergence of the dilatation current, \(\partial^\mu D_\mu,\) has essentially different structures in those two phases.

In this section, we will discuss how these two features are reflected in the effective action in \(QED_3.\)

\(QED_3\) is a super-renormalizable theory where the coupling constant is dimensional. However, as was pointed already in Ref. [2], in the infrared region, with \(p << \Lambda_{np} = \alpha/8,\) the conformal symmetry is a good symmetry in the symmetric phase with \(N_f > N_{cr}\) and \(m_{dyn} = 0\) (at least in leading order in \(1/N_f\)). The point is that in that region, the dimensional coupling constant \(\alpha\) drops out from the SD equations for Green’s functions: this, in particular, can be seen on the example of the SD equation for the mass function \(\Sigma(p)\) (see Eq.(A4) in the Appendix). This point is also connected with the fact that the dimensionless running coupling constant

\[ \bar{\alpha}(p) \equiv \frac{\alpha}{8p(1 + \Pi(p))} \] (32)

\((\Pi(p)\) is a polarization operator\) has an infrared stable fixed point \(\bar{\alpha} = 1\) in the symmetric phase [2].

Of course, in the nonsymmetric phase, because of the generation of the fermion dynamical mass, the conformal symmetry is broken.

As it was already pointed out in the previous section, the parameter \(\Lambda_{np}\) is an effective ultraviolet cutoff of the nonperturbative dynamics. Since, in leading order in \(1/N_f,\) the polarization operator \(\Pi(p) = \alpha/8p = \Lambda_{np}/p\) in the symmetric phase [2], the running coupling \((32)\) is \(\bar{\alpha} = 1\) at all values of \(p\) as \(\Lambda_{np} \to \infty.\) Therefore the dynamics is conformal invariant in that phase in this limit.

The physical meaning of this limit is the following. The near-critical dynamics around \(N_f = N_{cr}\) implies that, for such values of \(N_f,\) masses of light excitations have to be much less than the parameter \(\Lambda_{np}.\) We are interested in the low energy dynamics of these excitations, when their energies are much less than \(\Lambda_{np}.\) The question is whether there are such light excitations described by the composite field \(\sigma\) (and, in general, by the field \(\varphi).\) If they are, then the field \(\Sigma_0,\) describing their fluctuations around the minimum of the effective potential
\( V_{np} \) (see Eq. (30)), has also to be much less than \( \Lambda_{np} \). This in turn implies that the potential \( V_{np} \), describing the nonperturbative dynamics, has to be much less than \( \Lambda_{np}^3 \).

This leads us to studying the auxiliary "continuum" limit \( \Lambda_{np} \) goes to infinity with \( \Sigma_0 \) and \( V_{np} \) being finite (the critical dynamics). As usual for critical dynamics [22], such a limit can be interpreted as a renormalization of the parameter \( N_f \). In other words, one has to find such a dependence of \( N_f \) on \( \Lambda_{np} \) that as \( \Lambda_{np} \) goes to infinity, the effective potential \( V_{np} \) would be finite for finite \( \Sigma_0 \).

It is not difficult to show from Eqs. (29) and (30) that in the nonsymmetric phase the non-perturbative effective potential is in this limit:

\[
V_{np} = A^2 N_{cr}^2 \Sigma_0^3 \left( -\frac{1}{24} + \frac{1}{8} \log \frac{\Sigma_0}{\Sigma_0} + \frac{3}{64} \log^2 \frac{\Sigma_0}{\Sigma_0} \right).
\]

As it has to be, \( N_f \), defined from Eq. (29), is fixed as \( \Lambda_{np} \rightarrow \infty \): \( N_f = N_{cr} \).

But what is the form of the potential at \( N_f = N_{cr} \) from the side of the symmetric phase in this limit? [1] It is easy to show that in the symmetric phase, at finite \( \Sigma_0 \), the potential \( V_{np} \) (30) diverges as \( \Lambda_{np} \) goes to infinity (on technical side, it is connected with the point that \( \sin 2\theta \) and \( \cos 2\theta \) in the potential become hyperbolic functions for \( N_f > N_{cr} \)). In particular, for \( N_f \) close to \( N_{cr} \), the potential is:

\[
V_{np} \rightarrow \frac{A^2 N_{cr}^2 \Sigma_0^3}{32} \left[ -\frac{10}{3} - 2 \log \frac{\Lambda_{np}e^{\delta_0}}{\Sigma_0} + \frac{3}{2} \log^2 \frac{\Lambda_{np}e^{\delta_0}}{\Sigma_0} \right] \rightarrow \infty
\]

as \( \Lambda_{np} \rightarrow \infty \).

On the other hand, the kinetic term and terms with a larger number of derivatives (the structure of which is the same in the symmetric and nonsymmetric phases) have to be finite in this limit. Indeed, the most severe ultraviolet divergences always occur in an effective potential. Since the potential \( V_{np} \) (34) diverges only logarithmically, all other terms in the effective action are finite.

This situation implies that, in the symmetric phase, there are no \textit{light} particles described by the potential \( V_{np} \). If they exist, they are heavy (with \( M \sim \Lambda_{np} \)) and therefore decouple from infrared dynamics.

In the present case, \( \sigma \) boson is such a particle. However, since, as was already pointed out in Sec.2, in the symmetric phase, all the \( 4N_f^2 - 1 \) particles described by the field \( \varphi \) are in the same (adjoint) representation of the \( SU(2N_f) \), all of them decouple there. Therefore, in the limit \( \Lambda_{np} \rightarrow \infty \), the symmetric phase in massless \( QED_3 \) is a Coulomb-like, conformal-invariant phase, describing interactions between massless fermions and photons.

On the other hand, the non-perturbative effective potential, and the whole effective action, for this composite field is finite as \( \Lambda_{np} \rightarrow \infty \) in the nonsymmetric phase (see Eq.(33)).

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\(^1\)Since \( N_f \) is dimensionless, it is actually a function of the ratio \( \Lambda_{np}/\mu \), where \( \mu \) is a renormalization group parameter of the dimension of mass. It can be introduced in different ways. For example, one condition might be that the potential \( V_{np} \) is equal to \( (\mu)^2 \) at \( \Sigma_0 = \mu \). In the end of the calculations, if this limit exists, \( \mu \) can be expressed through such physical parameters as the mass of \( \sigma \) boson (in both symmetric and asymmetric phases) or the dynamical fermion mass in the asymmetric phase (see equation (33)).
This reflects the point that in that phase, besides photons and fermions, there are other light (with a mass \( M << \Lambda_{np} \)) particles: 2\( N_f \) massless composite NG bosons and 2\( N_f - 1 \) massive composite bosons with a mass \( M \sim m_{\text{dyn}} \equiv \Sigma_0 << \Lambda_{np} \) (in the scaling, near-critical, region). Moreover, the conformal symmetry is explicitly broken in the continuum limit in the nonsymmetric phase. Indeed, as follows from Eq. (33),

\[
\langle 0 | \partial^\mu D_\mu | 0 \rangle = \langle 0 | \theta^\mu \mu | 0 \rangle = 4V_{np}(\Sigma_0) \bigg|_{\Sigma_0 = \Sigma_0} = -\frac{A^2 N_f^2}{6} \Sigma_0^3 \neq 0,
\]

where \( \theta^\mu \mu \) is the energy-momentum tensor. Therefore there is a conformal anomaly in this phase, and the vacuum expectation value \( \langle 0 | \theta^\mu \mu | 0 \rangle \) plays here the same role as the gluon condensate in QCD.

Thus, in leading order in \( 1/N_f \), the phase transition in \( QED_3 \) possesses all the characteristic features of the conformal phase transition. In particular, in agreement with the general conclusion of Ref. [7], there is no Ginzburg-Landau like effective action describing the near-critical dynamics in both symmetric and broken phases.

It is instructive to compare the dynamics in \( QED_3 \) with the dynamics in \( QED_3 \) with a Chern-Simons (CS) term. As is known [23, 24], the CS term enforces the phase transition to be first order. Therefore an abrupt change of the spectrum of light excitations at the critical point is natural in that case [24]. In \( QED_3 \) without the CS term, there is still an abrupt change of the spectrum, though the phase transition is continuous.

It is amazing how closely the dynamics in \( QED_3 \) resembles the dynamics in quenched \( QED_4 \) [22, 23]. The scaling law for \( m_{\text{dyn}} \) in \( QED_4 \) has the form (29) with \( \Lambda_{np} \) replaced by cutoff \( \Lambda \) and \( N_{cr}/N_f \) replaced by \( \alpha^{(4)}/\alpha_{cr}^{(4)} \), where the dimensionless critical coupling \( \alpha_{cr}^{(4)} \sim 1 \). The effective potential (30) also closely resembles the effective potential calculated in Refs. [20, 4]. This implies that long-range dynamics, provided by strong Coulomb-like (\( \sim 1/r \)) forces, are essentially the same in these two models.

Recently, the existence of the conformal phase transition (CPT) in quenched \( QED_4 \) has been confirmed by calculating the spectrum of composites directly from Schwinger-Dyson equations for Green’s functions of local composite operators [20]. It would be worth considering such an approach in \( QED_3 \).

The CPT is a conception extending the BKT phase transition [18] (taking place in two dimensions) to higher dimensions. To see this in \( QED_3 \), it is convenient to consider the continuum limit \( \Lambda_{np} \to \infty \) with \( m_{\text{dyn}} = \Sigma_0 \) being fixed as a renormalization of the parameter \( N_f^{-1} \). Then, the scaling law (29) implies the following \( \beta \) function at \( N_f < N_{cr} \):

\[
\beta(N_f^{-1}) = \frac{\partial N_f^{-1}}{\partial \log \Lambda_{np}} = -\frac{\pi}{32} \left( N_{cr}/N_f - 1 \right)^{3/2}.
\]

Clearly, this function has an ultraviolet stable fixed point at \( N_f = N_{cr} \). On the other hand, since at \( N_f > N_{cr} \), where \( m_{\text{dyn}} = 0 \), the infrared dynamics with \( p << \Lambda_{np} = \alpha/8 \) is essentially independent of \( \Lambda_{np} \), the \( \beta \) function is identically zero. Therefore there is a line of fixed points at \( N_f > N_{cr} \).

Such a renormalization group (RG) structure coincides with that corresponding to the BKT phase transition.
Of course, real $QED_3$ has a fixed integer $N_f$ and a finite $\Lambda_{np} = \alpha/8$. Still, the $\beta$-function \[(36)\] is useful in the description of its dynamics. Let us consider a RG invariant quantity $X$. Then:

$$\frac{dX}{d\log \Lambda_{np}} = \frac{\partial X}{\partial \log \Lambda_{np}} + \beta(N_f^{-1}) \frac{\partial X}{\partial N_f^{-1}} = 0. \quad (37)$$

This relation demonstrates that the $\beta$ function $\beta(N_f^{-1})$ defines how $X$ depends on the coupling constant $\alpha = 8\Lambda_{np}$.

Thus the CPT yields extension of the BKT phase transition to higher dimensions: the crucial property of those phase transitions is the presence of an essential singularity in the mass (energy) gap at the critical point. However, there is an essential difference between the realization of the CPT in two and higher dimensions. While there cannot be spontaneous breakdown of any continuous symmetry in two dimensions \[27\], there is no such a restriction in higher dimensions. In particular, there is a genuine spontaneous flavor symmetry breaking in $QED_3$.

V. CONCLUSION

In this paper we have analyzed the structure of the effective action in $QED_3$ in leading order in $1/N_f$ and showed that it reflects the existence of the conformal phase transition at $N_f = N_{cr}$. How much of this picture survives beyond the $1/N_f$ expansion is still an open issue, though results of the lattice computer simulations \[3\] are encouraging.

The existence of such a phase transition, which is an extension of the BKT phase transition to higher dimensions (2+1, in this case), is interesting in itself. Also, the CPT in $QED_3$ may be connected with nonperturbative dynamics in condensed matter, in particular, with dynamics of a non-Fermi liquid \[28,29\].

The most interesting issue is the connection of the phase transition in $QED_3$ with a possibility of the existence of an analogous phase transition in a (3+1)-dimensional $SU(N_c)$ gauge theory \[4,5\]. In $QED_3$ the critical number $N_f = N_{cr}$ separates two phases with very different dynamics. At $N_f > N_{cr}$ there is a Coulomb phase describing interactions of massless photons and fermions; at $N_f < N_{cr}$ the rich dynamics with both spontaneous flavor symmetry breaking and confinement is realized. These two nonperturbative phenomena are intimately connected in this model: at $N_f < N_{cr}$ massive fermions decouple from infrared dynamics, thus leading to a potential growing with a distance, and therefore to confinement. As was already mentioned above, strong Coulomb-like forces play important role in providing the CPT in $QED_3$.

Is a similar picture realized in a (3+1)-dimensional $SU(N_c)$ gauge theory? What is the interplay between the dynamics provided by strong Coulomb-like forces and those connected with topologically nontrivial fluctuations, like instantons, monopoles, etc.? $QED_3$, with its rich dynamics, is a fruitful laboratory for studying those complicated issues.

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APPENDIX A: DERIVATION OF THE EFFECTIVE POTENTIAL

In this Appendix we derive expression (21) for the effective potential \( V \).

First, we need to recall some properties of the solution of the SD equation (13). This equation will be used to extract the dependance of the parameter \( m_0 \) on the condensate \( \sigma_c \).

Differentiating the second of Eqs.(13) with respect to \( p \), one finds that \( \Sigma(p) \) satisfies the differential equation

\[
\frac{d}{dp} \left[ p^2 (p + \alpha/8)^2 \frac{d\Sigma}{dp} \right] = -\frac{\alpha}{\pi^2 N_f} \frac{p^2 \Sigma(p)}{p^2 + \Sigma^2(p)}
\]

with infrared (IRBC) and ultraviolet (UVBC) boundary conditions:

\[
p^2 \frac{d\Sigma}{dp} \bigg|_{p=0} = 0 \quad \text{(IRBC)},
\]

\[
\left[ \frac{p(p + \alpha/8)}{2p + \alpha/8} \frac{d\Sigma}{dp} + \Sigma(p) \right] \bigg|_{p=\Lambda} \simeq \left[ \frac{\Lambda d\Sigma}{2} \frac{dp}{dp} + \Sigma(p) \right] \bigg|_{p=\Lambda} = m_0 \quad \text{(UVBC)}.
\]

Following Ref. [3], one can approximate the equation (A1) by writing it in two regions as follows:

\[
\frac{d}{dp} \left[ p^2 \frac{d\Sigma}{dp} \right] = -\frac{8}{\pi^2 N_f} \frac{p^2 \Sigma}{p^2 + \Sigma^2_0}, \quad p < \Lambda_{np} \equiv \frac{\alpha}{8},
\]

\[
\frac{d}{dp} \left[ p^3 \frac{d\Sigma}{dp} \right] = -\frac{2\alpha}{\pi^2 N_f} \Sigma, \quad p > \Lambda_{np},
\]

where, in Eq.(A4), \( \Sigma^2(p) \) in the denominator is replaced by \( \Sigma^2_0 \equiv \Sigma(p)|_{p=0} \); in Eq.(A3), at \( p > \Lambda_{np} \), \( \Sigma^2(p) \) in the denominator is neglected. This is justified if \( \Sigma^2(p) \ll \Lambda_{np} \); as we will see, it is indeed correct in the near-critical region.

The parameter \( \Lambda_{np} = \alpha/8 \) is an effective ultraviolet cutoff for nonperturbative dynamics in the model: at \( m_0 = 0 \), the dynamical mass function rapidly decreases at \( p \geq \Lambda_{np} \) (see below). This parameter plays here the same role as the parameter \( \Lambda_\chi = 4\pi F_\pi \sim 1\text{Gev} \) in QCD4 [21,22].

Taking into account IRBC (A3), the solution to Eq.(A4) is expressed through a hypergeometric function:

\[
\Sigma(p) = \Sigma_0 F \left( \frac{1}{4} - \frac{i\nu}{2}, \frac{1}{4} + \frac{i\nu}{2} \right; \frac{3}{2}, \frac{p^2}{\Sigma^2_0} \right).
\]
where $\nu = \sqrt{\frac{3}{\pi^2 N_f} - \frac{1}{4}}$. At $p \gg \Sigma_0$, its asymptotics is

$$\Sigma(p) = A_0 \frac{\Sigma_0^{3/2}}{\nu \sqrt{p}} \sin \left( \nu \log \frac{p}{\Sigma_0} + \nu \delta_0 \right),$$

where

$$A_0 = \frac{\sqrt{\pi}}{2} \left| \frac{\Gamma(1 + i\nu)}{\Gamma(\frac{1}{4} + \frac{\nu}{2}) \Gamma(\frac{5}{4} + \frac{\nu}{2})} \right|$$

and

$$\delta_0 = \frac{1}{\nu} \arg \left[ \frac{\Gamma(1 + i\nu)}{\Gamma(\frac{1}{4} + \frac{\nu}{2}) \Gamma(\frac{5}{4} + \frac{\nu}{2})} \right]$$

(notice that $\delta_0 = 3 \log 2 + \frac{\pi^2}{2} - 2$ as $\nu \to 0$).

Solutions of Eq.(A5) can in principle be expressed through Bessel functions, but, for our purposes, it is sufficient to display them as a series in $x = p/\alpha$:

$$\Sigma = x^s \left( C_0 + \frac{C_1}{x} + \cdots \right) \Rightarrow \Sigma = C_0 \left( 1 + \frac{2\alpha}{\pi^2 N_f p} \right) + C_2 \frac{\alpha^2}{p^2} + \cdots. \quad (A10)$$

Then we find from Eq.(A3):

$$m_0 = C_0 \left( 1 + \frac{1}{\pi^2 N_f} \frac{\alpha}{\Lambda} \right). \quad (A11)$$

Notice that Eqs.(A10) and (A11) imply that at $m_0 = 0$ the dynamical mass function rapidly (as $1/p^2$) decreases with increasing $p$ in the region with $p \geq \Lambda_{np}$. Therefore, as was already stated above, $\Lambda_{np} \sim \alpha$ is indeed an ultraviolet cutoff for nonperturbative dynamics.

Matching now solutions of equations (A4) and (A5) at $p = \Lambda_{np}$ (i.e. equating the values of the functions and their derivatives), one gets:

$$\Sigma(p) \bigg|_{p=\Lambda_{np}} = C_0 \left( 1 + \frac{2\alpha}{\pi^2 N_f \Lambda_{np}} \right) + C_2 \frac{\alpha^2}{\Lambda_{np}^2}, \quad (A12)$$

$$\frac{d\Sigma}{dp} \bigg|_{p=\Lambda_{np}} = \frac{2C_0}{\pi^2 N_f \Lambda_{np}^2} - 2C_2 \frac{\alpha^2}{\Lambda_{np}^3}. \quad (A13)$$

Eqs.(A11),(A12), and (A13) imply

$$m_0 = \frac{1}{1 + \frac{8}{\pi^2 N_f}} \left( \Sigma + \Lambda_{np} \frac{d\Sigma}{dp} \right) \bigg|_{p=\Lambda_{np}} \quad (A14)$$

(here we neglect a term of order $\frac{\alpha}{\Lambda}$). Then, using this equation and Eq.(A7), we arrive at the equation
\[ m_0 = \frac{A \Sigma_0^{3/2}}{2\nu \sqrt{\Lambda_{np}}} \sqrt{\frac{9}{4} + \nu^2 \sin \left( \nu \log \frac{\Lambda_{np}}{\Sigma_0} + \nu(\delta_0 + \delta_1) \right)}, \]  
(A15)

where

\[ A = \frac{A_0}{1 + \frac{8\pi^2 N_f}{3}}, \quad \delta_1 = \frac{1}{\nu} \arctan \frac{2\nu}{3}. \]

At \( m_0 = 0 \) we find solutions for the dynamical mass (compare with Eq. (27)):

\[ \Sigma_0^{(n)} = \Lambda_{np} \exp \left( -\frac{\pi n}{\nu} + \delta_0 + \delta_1 \right), \quad n = 1, 2, \ldots. \]  
(A16)

Eq. (20) implies that \( V \) as a function of \( \Sigma_0 \) is:

\[ V(\Sigma_0) = -\int m_0(x) \frac{d\sigma_c}{dx} dx. \]  
(A17)

The function \( m_0(\Sigma_0) \) is given by Eq. (A15). Let us determine the function \( \frac{d\sigma_c}{dx} \). The condensate \( \sigma_c \) is:

\[ \sigma_c = \langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle = -\text{tr}(0 | T \psi(x) \bar{\psi}(y) | 0) \bigg|_{y \to x} = -i \text{tr}G(x, x) \]

\[ = -\frac{i}{(2\pi)^3} \text{tr} \int d^3p \frac{1}{p - \Sigma(p)} = -\frac{2N_f}{\pi^2} \int dp \frac{p^2 \Sigma(p)}{p^2 + \Sigma^2(p)}. \]  
(A18)

The nonperturbative part of the condensate corresponds to the integration region with \( p < \Lambda_{np} \):

\[ \sigma_{np} = -\frac{2N_f}{\pi^2} \int_0^{\Lambda_{np}} dp \frac{p^2 \Sigma(p)}{p^2 + \Sigma^2(p)} = \frac{N_f^2}{4} \Lambda_{np}^2 \left. \frac{d\Sigma}{dp} \right|_{p=\Lambda_{np}}. \]  
(A19)

Here the last equality follows directly from equation (A4).

Because, as follows from Eqs. (A10) and (A11), \( \Sigma(p \to \infty) = m_0 \), the perturbative part of the condensate is \( \sigma_{pt} = -\frac{2N_f}{\pi^2} m_0 \Lambda_0 \). This and equation (A19) lead to expression (22) for the condensate \( \sigma_c = \sigma_{np} + \sigma_{pt} \).

Eqs. (22), (A15) and (A17) lead to expression (21) for the effective potential. Notice that the expression for the potential in the symmetric phase, with \( N_f > N_{cr} \), is an analytic continuation of the potential in the broken phase, with \( N_f < N_{cr} \).
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