Raising and lowering operators and their factorization for generalized orthogonal polynomials of hypergeometric type on homogeneous and non-homogeneous lattice

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Abstract

We complete the construction of raising and lowering operators, given in a previous work, for the orthogonal polynomials of hypergeometric type on non-homogeneous lattice, and extend these operators to the generalized orthogonal polynomials, namely, those difference of orthogonal polynomials that satisfy a similar difference equation of hypergeometric type.

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1 Introduction

Recently we have presented a paper on the raising and lowering operators for the orthogonal polynomials (OP) of hypergeometric type [5] (in connection with the factorization method defined by Hull and Infeld). That paper covered only OP of continuous and discrete variable on an uniform lattice, as well as orthonormal functions of continuous and discrete variable.

In this work we continue the construction of raising and lowering operators for OP on non-homogeneous lattice. The starting point is also the Rodriguez formula and the fundamental properties of OP given by Nikiforov and collaborators [1], [18] that include the q-analog of classical OP of discrete variable.

Atakishiyev and collaborators [13] [14] extended the classification of Nikiforov to OP of discrete variable defined by Andrews and Askey and proved that they satisfy a difference equation only in the case of $x(s)$ linear, quadratic q-linear and q-quadratic.

The construction of raising and lowering operators on non-uniform lattice was also worked out by Alvarez-Nodarse and Costas-Santos [15][16] for the lattice $x(s) = c_1 q^n + c_2 q^m + c_3$.

Similar work was carried out by Smirnov for OP of hypergeometric type on homogeneous lattice [20] and non-homogeneous lattice [21]although the raising and lowering operators are defined with respect to two indices: $n$, the order of polynomials and $m$, the order of difference derivative of polynomials. In our work we define the raising and lowering operator with respect to one index only, $n$ or $m$. 
In order to complete the classification of the OP of hypergeometric type we include the generalized classical OP that satisfy also a difference/differential equation of hypergeometric type.

Since all classical OP of discrete variable lead in the limit to the corresponding OP of continuous variable, we start in section 2 with the raising and lowering operators for generalized OP of continuous variable with respect to the index \( n \), using Rodrigues formula. In section 3 we repeat the same construction for generalized classical OP of discrete variable on homogeneous lattice. In section 4 we extend the construction to classical OP of discrete variable on non-homogeneous lattice in the general case, when \( x(s) = c_1 s^2 + c_2 s + c_3 \) or \( x(s) = c_1 q^s + c_2 q^{-s} + c_3 \).

In section 5 we complete the picture with the construction of raising and lowering operators for generalized classical OP on non-homogeneous lattice, that include the \( q \)-analog of classical OP of discrete variable. In all these cases, the raising and lowering operators are given with respect to one index, say, \( n \), but the same operator can be considered, written in appropriate form, the raising and lowering operator with respect to index \( m \).

## 2 Raising and lowering operators for generalized classical orthogonal polynomials of continuous variable

Let \( y_n(x) \) be an orthogonal polynomials of continuous variable satisfying the differential equation \([1]\)

\[
\sigma(x)y''_n(x) + \tau(x)y'_n(x) + \lambda_n y_n(x) = 0,
\]

where \( \sigma(x) \) and \( \tau(x) \) are polynomials of at most second and first degree, respectively, and

\[
\lambda_n = -n \left( \tau' + \frac{1}{2} (n - 1) \sigma'' \right).
\]

It can be proved that the derivatives of \( y_n(x) \), namely, \( y^{(m)}_n(x) = v_{mn}(x), m = 0, 1, \ldots, n - 1 \), satisfy a similar equation:

\[
\sigma(x)v''_{mn}(x) + \tau_m v'_{mn}(x) + \mu_{mn} v_{mn}(x) = 0,
\]

with \( \tau_m = \tau(x) + m \sigma(x) \) and \( \mu_{mn} = -(n - m) \left( \tau' + \frac{n+m-1}{2} \sigma'' \right), m = 0, 1, \ldots, n - 1 \).

We call these polynomials generalized orthogonal polynomials of hypergeometric type, some particular examples of which are the Legendre and Laguerre, Hermite, Jacobi generalized orthogonal polynomials. [2]

The polynomials of hypergeometric type satisfy an orthogonality property with respect to the weight function \( \rho(x) \)

\[
\int_a^b y_n(x) y_{n'}(x) \rho(x) dx = \delta_{nn'} d_n^2.
\]
Similarly the generalized orthogonal polynomials satisfy

\[ \int_a^b v_{mn}(x)v_{mn}(x)\rho_m(x)dx = \delta_{ln}d_{mn}^2, \]  

(5)

where \( d_n^2 \) and \( d_{mn}^2 \) are normalization constants.

It can be proved \( [3] \)

\[ d_{mn}^2 = d_{nn}^2 \left( \prod_{k=0}^{n-1} \mu_{kn} \right)^{-1}, \]

\[ d_{nn}^2 = d_{nn}^2 \left( \prod_{k=0}^{n-1} \mu_{kn} \right)^{-1}, \]

from which \( d_{nn}^2 \) can be eliminated, therefore

\[ d_{mn}^2 = d_{nn}^2 \prod_{k=0}^{m-1} \mu_{kn}, \]  

(6)

where \( d_{nn}^2 \) and \( d_{mn}^2 \) are given in the tables \( [6] \).

The generalized orthogonal polynomials of hypergeometric type can be calculated from the weight function \( \rho_m(x) = \sigma(x)^m \rho(x) \), with the help of the Rodrigues formula:

\[ v_{mn}(x) = \frac{A_{mn}B_n}{\sigma^m(x)\rho(x)} \frac{d^{n-m}}{dx^{n-m}} \{ \rho_n(x) \}, \]  

(7)

where

\[ A_{mn} = (-1)^m \prod_{k=0}^{m-1} \mu_{kn} = \frac{n!}{(n-m)!} \prod_{k=0}^{m-1} \left( -\frac{\lambda_{n+k}}{n+k} \right). \]  

(8)

The leading coefficients of the orthogonal polynomial \( y_n(x) = a_n x^n + b_n x^{n-1} + \ldots \) can be calculated \( [4] \)

\[ a_n = B_n \prod_{k=0}^{n-1} \left( -\frac{\lambda_{n+k}}{n+k} \right), \]  

(9)

hence, it follows, \( A_{mn}B_n = n!a_n \).

We address ourselves to the construction of the raising and lowering operators for the generalized orthogonal polynomials using the Rodrigues formula as we did in a recent work \( [5] \).

We have from (7)

\[ v_{m,n+1}(x) = \frac{A_{m,n+1}B_{n+1}}{\sigma^m(x)\rho(x)} \frac{d^{n+1-m}}{dx^{n+1-m}} \{ \rho_n(x) \} = \frac{A_{m,n+1}B_{n+1}}{\sigma^m(x)\rho(x)} \frac{d^{n-m}}{dx^{n-m}} \{ \tau_n(x)\rho_n(x) \} = \]

\[ = \frac{A_{m,n+1}B_{n+1}}{\sigma^m(x)\rho(x)} \left\{ \tau_n(x) \frac{d^{n-m}}{dx^{n-m}} \{ \rho_n(x) \} + (n-m)\tau_n' \frac{d^{n-m-1}}{dx^{n-m-1}} \{ \rho_n(x) \} \right\} = \]

\[ = \frac{B_{n+1}}{B_n} \left\{ \frac{A_{m,n+1}}{A_{mn}} \tau_n(x)v_{mn}(x) + (n-m)\frac{A_{m,n+1}}{A_{m+1,n}} \tau_n' \sigma(x)v_n'(x) \right\} = \]

\[ = \frac{B_{n+1}}{B_n} \left\{ \frac{n+1}{n-m+1} \frac{\lambda_{n+m}}{\lambda_n} \tau_n(x)v_{mn}(x) - \frac{n+1}{n-m+1} \frac{\lambda_n}{\lambda_n} \tau_n' \sigma(x)v_n'(x) \right\}. \]  

(10)
The right hand can be considered the raising operator that, when applied to $v_{mn}(x)$ gives a new polynomial of higher order $v_{m,n+1}(x)$.

In order to evaluate the lowering operator we need a recurrence relation for the generalized polynomials. We write

$$xv_{mn}(x) = \sum_{k=0}^{n+1} c_{kn}v_{mk}(x),$$

$$c_{kn} = \frac{1}{d_{mk}^2} \int_a^b v_{mk}(x) x v_{mn} \rho_m(x) dx.$$  \hspace{1cm} (11)

From the orthogonality condition (5) we deduce

$$\int_a^b v_{mn}(x) x^r \rho_m(x) dx = 0 \quad \text{for} \quad r < n - m.$$

Since $x^p_k(x)$ is a polynomial of order $k + 1 - m$ it follows that $c_{kn} = 0$ if $k + 1 - m < n - m$, or $k + 1 < n$. Hence

$$xv_{mn} = \tilde{\alpha}_n v_{m,n+1}(x) + \tilde{\beta}_n v_{mn}(x) + \tilde{\gamma}_n v_{m,n+1}(x),$$  \hspace{1cm} (12)

where $\tilde{\alpha}_n = c_{n+1,n}$, $\tilde{\beta}_n = c_{nm}$, $\tilde{\gamma}_n = c_{n-1,n}$

The coefficients $\tilde{\alpha}_n$, $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ can be expressed in terms of the squared norm $d_{mn}^2$ and the leading coefficients $a_n$ and $b_n$ in $y_n(x)$.

From (11) it can be proved that $d_{mk}^2 c_{kn} = d_{mn}^2 c_{nk}$.

Since $\tilde{\alpha}_{n-1} = c_{n,n-1}$, $\tilde{\gamma}_n = c_{n-1,n}$, if we put $k = n - 1$ we obtain

$$c_{n-1,n} d_{m,n-1}^2 = c_{n,n-1} d_{mn}^2,$$

hence

$$\tilde{\gamma}_n = \tilde{\alpha}_{n-1} \frac{d_{mn}^2}{d_{m,n-1}^2}.$$  \hspace{1cm} (13)

Introducing the expansion $y_n(x) = a_n x^n + b_n x^{n-1} + \ldots$ in (12) and comparing the coefficients of the highest terms, we have

$$a_n (n - m + 1) = \tilde{\alpha}_n a_{n+1}(n + 1),$$

$$b_n (n - m) = \tilde{\beta}_n b_{n+1} n + \tilde{\gamma}_n a_n n.$$

Hence

$$\tilde{\alpha}_n = \frac{a_n}{a_{n+1}} \frac{n - m + 1}{n + 1},$$  \hspace{1cm} (13)

$$\tilde{\beta}_n = \frac{b_n}{a_n} \frac{(n - m)}{n} - \frac{b_{n+1}}{a_{n+1}} \frac{n + 1 - m}{n + 1},$$  \hspace{1cm} (14)

$$\tilde{\gamma}_n = \frac{a_{n-1}}{a_n} \frac{n - m}{n} \frac{d_{mn}^2}{d_{m,n-1}^2}.$$  \hspace{1cm} (15)
Substituting (9) in $\tilde{\alpha}_n$ we obtain

$$\tilde{\alpha}_n = \frac{-B_n}{B_{n+1}} \frac{n-m+1}{n+1} \frac{\lambda_n}{\lambda_{2n}} \frac{2n}{2n+1} \frac{2n+1}{\lambda_{2n+1}}.$$  \hspace{1cm} (16)

Hence (10) can be written

$$\tilde{\alpha}_n \frac{\lambda_{2n}}{2n} v_{m,n+1}(x) = \left\{ \frac{\lambda_{n+m}}{n+m} \frac{\tau_n(x)}{\tau'_n} v_{mn}(x) - \sigma(x) v'_m(x) \right\}.$$  \hspace{1cm} (17)

Inserting (12) in (17) we obtain

$$\tilde{\gamma}_n \frac{\lambda_{2n}}{2n} v_{m,n-1}(x) = \left\{ -\frac{\lambda_{n+m}}{n+m} \frac{\tau_n(x)}{\tau'_n} + \frac{\lambda_{2n}}{2n} (x - \tilde{\beta}_n) \right\} v_{mn}(x) + \sigma(x) v'_m(x)$$  \hspace{1cm} (18)

The right hand side of (17) and (18) can be considered the raising and lowering operators for the generalized classical orthogonal polynomials with respect to the index $n$. All the constants $\tilde{\alpha}_n$, $\tilde{\beta}_n$, $\tilde{\gamma}_n$, $\lambda_n$, $\tau'_n$ can be calculated from the tables [6].

Now we define the orthonormalized function

$$\psi_{mn}(x) = d_{mn}^{-1} \sqrt{\rho_m(x)} v_{mn}(x),$$  \hspace{1cm} (19)

hence

$$\psi'_m(x) = \frac{1}{2} \rho'_m(x) \psi_{mn}(x) + d_{mn}^{-1} \sqrt{\rho_m(x)} v'_m(x) = \frac{1}{2} \frac{\tau_{m-1}(x)}{\sigma(x)} \psi_{mn}(x) + d_{mn}^{-1} \sqrt{\rho_m(x)} v'_m(x)$$  \hspace{1cm} (20)

Multiplying (17) by $d_{mn}^{-1} \sqrt{\rho_m(x)}$ and substituting (20) in (17) we get

$$\tilde{\alpha}_n \frac{\lambda_{2n}}{2n} \frac{d_{m,n+1}}{d_{mn}} \psi_{m,n+1}(x) = \frac{\lambda_{n+m}}{n+m} \frac{\tau_n(x)}{\tau'_n} \psi_{mn}(x) + \frac{1}{2} \frac{\tau_{m-1}(x)}{\sigma(x)} \psi_{mn}(x) - \sigma(x) \psi'_m(x) =$$

$$= L^+(x,n) \psi_{m,n}(x)$$  \hspace{1cm} (21)

Similarly

$$\tilde{\gamma}_n \frac{\lambda_{2n}}{2n} \frac{d_{m,n-1}}{d_{mn}} \psi_{m,n-1}(x) = \left\{ -\frac{\lambda_{n+m}}{n+m} \frac{\tau_n(x)}{\tau'_n} + \frac{\lambda_{2n}}{2n} (x - \tilde{\beta}_n) - \frac{1}{2} \frac{\tau_{m-1}(x)}{\sigma(x)} \right\} \psi_{mn}(x) +$$

$$+ \sigma(x) \psi'_m(x) = L^-(x,n) \psi_{m,n}(x)$$  \hspace{1cm} (22)

that can be considered the raising and lowering operators for the generalized orthonormalized functions $\psi_{mn}(x)$. These operators are mutually adjoint with respect to the scalar product of unit weight.

Following the same procedure as in [5] we can factorize the raising and lowering operators as follows:

$$L^-(x,n+1)L^+(x,n) = \mu(n) - \sigma(x)H(x,n)$$

$$L^+(x,n)L^-(x,n+1) = \mu(n) - \sigma(x)H(x,n+1)$$

where

$$\mu(n) = \frac{\lambda_{2n}}{2n} \frac{\lambda_{2n+2}}{2n+2} \tilde{\alpha}_n \tilde{\gamma}_{n+1}$$

and $H(x,n)$ is the differential operator derived from the left hand side of (3) after substituting $\psi_{mn}(x)$ instead of $v_{mn}(x)$.

Notice that the factorization of the raising and lowering operators is defined in a basis independent manner, which is equivalent to the Infeld-Hull method.
3 Raising and lowering operators for generalized classical orthogonal polynomials of discrete variable on uniform lattice

Let $y_n(x)$ be an orthogonal polynomial of discrete variable satisfying the difference equation [7]

$$\sigma(x) \Delta \nabla y_n(x) + \tau(x) \Delta y_n(x) + \lambda_n y_n(x) = 0,$$  \hspace{1cm} \text{(23)}

where $\sigma(x)$ and $\tau(x)$ are polynomials of at most of second and first degree, respectively,

$$\lambda_n = -n \left( \tau' + \frac{1}{2}(n-1)\sigma'' \right), \hspace{1cm} \text{(24)}$$

and the forward and backward difference operators are, respectively,

$$\Delta f(x) = f(x + 1) - f(x), \quad \nabla f(x) = f(x) - f(x - 1).$$

It can be proved [8] that the differences of $y_n(x)$, namely $\Delta^m y_n(x) = v_{mn}(x)$ satisfy similar equation of hypergeometric type:

$$\sigma(x) \Delta \nabla v_{mn}(x) + \tau_m(x) \Delta v_{mn}(x) + \mu_{mn} v_{mn}(x) = 0$$ \hspace{1cm} \text{(25)}

with

$$\tau_m(x) = \tau(x + m) + \sigma(x + m) - \sigma(x),$$

$$\mu_{mn} = \lambda_n - \lambda_m = - (n - m) \left( \tau' + \frac{n + m - 1}{2} \sigma'' \right), \quad m = 0, 1, \ldots, n - 1.$$ 

We call the polynomials $v_{mn}(x)$ the generalized classical orthogonal polynomials of discrete variable, among them we find the Hahn, Chebyshev, Meixner, Kravchuk and Charlier polynomials.

The classical orthogonal polynomials of discrete variable satisfy an orthogonality property with respect to the weight function $\rho(x)$

$$\sum_{x=a}^{b-1} y_\ell(x)y_n(x)\rho(x) = \delta_{\ell n} d_n^2. \hspace{1cm} \text{(26)}$$

Similarly the generalized classical orthogonal polynomials of discrete variable satisfy the orthogonality property

$$\sum_{x=a}^{b-1} v_{m\ell}(x)v_{mn}(x)\rho_m(x) = \delta_{m\ell} d_{mn}^2,$$ \hspace{1cm} \text{(27)}

where $d_n^2$ and $d_{mn}^2$ are normalization constants. It can be proved [9]

$$d_{mn}^2 = d_{nn}^2 \left( \prod_{k=m}^{n-1} \mu_{kn} \right)^{-1}, \quad d_0^2 = d_{mn}^2 \left( \prod_{k=0}^{n-1} \mu_{kn} \right)^{-1}.$$
If we eliminate \( d_{nn} \) in the above equations we get

\[
d_{mn}^2 = d_{0n}^2 \prod_{k=0}^{m-1} \mu_{kn}. \tag{28}
\]

The generalized classical orthogonal polynomials of discrete variable can be calculated from the weight function \( \rho_m(x) \) with the formula [9]:

\[
v_{mn}(x) = \frac{A_{mn} B_n \nabla^{n-m} \{ \rho_n(x) \}}{\rho_m(x)}, \tag{29}
\]

where

\[
A_{mn} = \frac{n!}{(n-m)!} \prod_{k=0}^{m-1} \left( -\frac{\lambda_{n+k}}{n+k} \right), \tag{30}
\]

\[
B_n = \frac{\Delta^m y_n(x)}{A_{nn}}. \tag{31}
\]

The leading coefficients of the classical orthogonal polynomial of discrete variable \( y_n(x) = a_n x^n + b_n x^{n-1} + \ldots \), are given by [10]

\[
a_n = B_n \prod_{k=0}^{n-1} \left( -\frac{\lambda_{n+k}}{n+k} \right) \tag{32}
\]

from which it follows \( A_{nn} B_n = n! a_n \).

We have now all the necessary ingredients to construct the raising and lowering operators for the generalized orthogonal polynomials of discrete variable in analogy with those of continuous variable. From (29) we have

\[
v_{m,n+1}(x) = \frac{A_{m,n+1} B_{n+1}}{\rho_m(x)} \nabla^{n-m+1} \{ \rho_{n+1}(x) \} =
\]

\[
= \frac{A_{m,n+1} B_{n+1}}{\rho_m(x)} \nabla^{n-m} \{ \Delta \rho_{n+1}(x-1) \} =
\]

\[
= \frac{A_{m,n+1} B_{n+1}}{\rho_m(x)} \nabla^{n-m} \{ \tau_n(x) \rho_n(x) \} =
\]

\[
= \frac{A_{m,n+1} B_{n+1}}{\rho_m(x)} \{ \tau_n(x) \nabla^{n-m} \rho_n(x) + (n-m) \tau_n' \nabla^{n-m-1} \rho_n(x-1) \} \tag{33}
\]

From (29) we have

\[
\nabla^{n-m} \{ \rho_n(x) \} = \frac{\rho_m(x)}{A_{mn} B_n} v_{mn}(x)
\]

\[
\nabla^{n-m-1} \{ \rho_n(x-1) \} = \frac{\sigma(x) \rho_m(x)}{A_{m+1,n} B_n} \Delta^{n+1} y_n(x-1) = \frac{\sigma(x) \rho_m(x)}{A_{m+1,n} B_n} = \nabla v_{mn}(x)
\]

Substituting the last two expressions in (33) and using (30) we obtain

\[
v_{m,n+1}(x) = \frac{B_{n+1}}{B_n} \left\{ \frac{n+1}{n+1-m} \frac{\lambda_{n+1} - \lambda_n}{n+m} \tau_n(x) v_{mn}(x) - \frac{n+1}{n+1-m} \frac{n}{\lambda_n} \tau_n' \sigma(x) \nabla v_{mn}(x) \right\} \tag{34}
\]
that raises in one step the order of the generalized polynomials in terms of the polynomials $v_{mn}(x)$ and $\nabla v_{mn}(x)$.

In order to evaluate the lowering operator we calculate a recurrence relation for the generalized orthogonal polynomials of discrete variable. We write

$$x v_{mn}(x) = \sum_{k=0}^{n+1} c_{kn} v_{mk}(x),$$

$$c_{kn} = \frac{1}{d_{mk}^2} \sum_{x=a}^{b-1} v_{mk}(x) x v_{mn}(x) \rho_m(x). \quad (35)$$

As in the case of the continuous variable $c_{kn} = 0$, if $k + 1 < n$. Hence

$$x v_{mn}(x) = \tilde{\alpha}_n v_{m,n+1}(x) + \tilde{\beta}_n v_{mn}(x) + \tilde{\gamma}_n v_{m,n-1}(x) \quad (36)$$

where $\tilde{\alpha}_n = c_{n+1,n}$, $\tilde{\beta}_n = c_{nn}$, $\tilde{\gamma}_n = c_{n-1,n}$.

From (35) it follows that $d_{mk}^2 c_{kn} = d_{mn}^2 c_{mk}$.

Since $\tilde{\alpha}_{n-1} = c_{n,n-1}$, $\tilde{\gamma}_n = c_{n-1,n}$, if we put $k = n - 1$, we get $c_{n-1,n} d_{m,n-1}^2 = c_{n,n-1} d_{m,n}^2$, hence

$$\tilde{\gamma}_n = \frac{d_{m,n}^2}{d_{m,n-1}^2} \tilde{\alpha}_{n-1}.$$

Introducing the expansion $y_n(x) = a_n x^n + b_n x^{n-1} + \ldots$ in (36), comparing the coefficients of the highest terms, and using

$$\Delta^m x^n = n(n-1) \ldots (n-m+1) x^{n-m} + \frac{m}{2} n(n-1) \ldots (n-m) x^{n-m+1} + \ldots$$

we obtain

$$a_n (n-m+1) = \tilde{\alpha}_n a_{n+1} (n+1)$$

$$a_n n(n-m) \frac{m}{2} + b_n (n-m) = \tilde{\alpha}_n b_{n+1} + \tilde{\beta}_n a_n + \tilde{\gamma}_n a_{n+1} (n+1) \frac{m}{2}.$$

From these relations and (32) we obtain

$$\tilde{\alpha}_n = \frac{a_n}{a_{n+1}} \frac{n-m+1}{n+1} = - \frac{B_{2n}}{B_{n+1}} \frac{\lambda_{n} 2n + 1}{\lambda_{n} 2n + 1 - n - m + 1}, \quad (37)$$

$$\tilde{\beta}_n = \frac{b_n}{a_n} \frac{(n-m)}{n} = \frac{b_{n+1}}{a_{n+1}} \frac{n-m+1}{n+1} - \frac{m}{2}, \quad (38)$$

$$\tilde{\gamma}_n = \frac{a_{n-1}}{a_n} \frac{n-m}{n} \frac{d_{m,n}^2}{d_{m,n-1}^2}. \quad (39)$$

Hence (34) can be written

$$\tilde{\alpha}_n \frac{\lambda_{2n}}{2n} v_{m,n+1}(x) = \frac{\lambda_{n+m}}{n+m} \frac{\tau_n(x)}{\tau'_n} v_{mn}(x) - \sigma(x) \nabla v_{mn}(x) \quad (40)$$

Inserting (36) in (40) we get

$$\tilde{\gamma}_n \frac{\lambda_{2n}}{2n} v_{m,n-1}(x) = - \frac{\lambda_{n+m}}{n+m} \frac{\tau_n(x)}{\tau'_n} v_{mn}(x) + \frac{\lambda_{2n}}{2n} (x - \tilde{\beta}_n) v_{mn}(x) + \sigma(x) \nabla v_{mn}(x) \quad (41)$$
The right side of (40) and (41) can be considered the raising and lowering operators with respect to the index \( n \) for the generalized orthogonal polynomials of discrete variable on homogeneous lattice.

All the constants \( \tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_n, \lambda_n, \mu_n \) can be calculated from the tables [11]. Obviously, when \( m = 0 \), \( \tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_n \) become, respectively, \( \alpha_n, \beta_n, \gamma_n \).

Now we define the orthonormal function of discrete variable

\[
\phi_{mn}(x) = d_{mn}^{-1} \sqrt{\rho_m(x)} v_{mn}(x)
\]

Using the identity

\[
\frac{\nabla \rho_m(x)}{\rho_m(x)} = \frac{\tau_{m-1}(x)}{\tau_{m-1}(x) + \sigma(x)}
\]

and the properties of the backwards operator we get

\[
\nabla \phi_{mn}(x) = \sqrt{\frac{\sigma(x)}{\tau_{m-1}(x) + \sigma(x)}} d_{mn}^{-1} \sqrt{\rho_m(x)} \nabla v_{mn}(x) +
\]

\[
\frac{\tau_{m-1}(x)}{\sqrt{\sigma(x) + \sqrt{\tau_{m-1}(x) + \sigma(x)}}} \phi_{mn}(x)
\]

Multiplying both sides of (40) by \( d_{mn}^{-1} \sqrt{\rho_m(x)} \) \( \nabla v_{mn}(x) \) obtained in (42), we get

\[
\tilde{\alpha}_n \frac{\lambda_{2n}}{2n} \frac{d_{m,n+1}}{d_{m,n}} \phi_{m,n+1}(x) = L^+(x, n) \phi_{mn}(x) =
\]

\[
= \left\{ \frac{\lambda_n \tau_n(x)}{n + m} + \frac{\sqrt{\sigma(x)} \tau_{m-1}(x)}{\sqrt{\sigma(x)} + \sqrt{\tau_{m-1}(x) + \sigma(x)}} \right\} \phi_{mn}(x) +
\]

\[- \sqrt{\sigma(x) - (\tau_{m-1}(x) + \sigma(x))} \nabla \phi_{mn}(x) \]

Similarly

\[
\tilde{\gamma}_n \frac{\lambda_{2n}}{2n} \frac{d_{m,n-1}}{d_{m,n}} \phi_{m,n-1}(x) = L^-(x, n) \phi_{mn}(x) =
\]

\[
= \left\{ -\frac{\lambda_n \tau_n(x)}{n + m} + \frac{\lambda_{2n}}{2n} \left( x - \tilde{\beta}_n \right) - \frac{\sqrt{\sigma(x)} \tau_{m-1}(x)}{\sqrt{\sigma(x)} + \sqrt{\tau_{m-1}(x) + \sigma(x)}} \right\} \phi_{mn}(x) +
\]

\[- \sqrt{\sigma(x) \left( \tau_{m-1}(x) + \sigma(x) \right)} \nabla \phi_{mn}(x) \]

The expressions (43) and (44) can be considered the raising and lowering operators, respectively, for the generalized orthonormal functions on homogeneous lattice. These operators are mutually adjoint with respect to the scalar product of unit weight.

Notice that in (43) and (44) the last term is proportional to \( \nabla \phi_{mn}(s) \), which in the continuous limit becomes the derivative \( \psi'_{mn}(x) \).

As in [5] we can factorize the raising and lowering operators as follows:

\[
L^-(x, n + 1)L^+(x, n) = \mu(n) + \mu(x + 1, n) H(x, n)
\]

\[
L^+(x, n)L^-(x, n + 1) = \mu(n) + \mu(x, n - 1) H(x, n + 1)
\]

where

\[
\mu(n) = \frac{\lambda_{2n}}{2n} \frac{\lambda_{2n+2}}{2n+2} \tilde{\alpha}_n \tilde{\gamma}_{n+1},
\]
\[ \mu(x,n) = \frac{\lambda_n}{n} \frac{\tau_n(x)}{\tau'_n} - \sigma(x) \]

and \( H(x,n) \) is the difference operator derived from the left hand side of (25) after substituting \( \phi_{mn}(x) \) instead of \( v_{mn}(x) \).

4  Raising and lowering operators for classical orthogonal polynomials of a discrete variable on nonuniform lattice

Let \( y(s) \) a function of discrete variable satisfying the difference equation with respect to the lattice function \( x(s) \)

\[ \sigma(s) \frac{\Delta}{\Delta x(s-1/2)} \left\{ \nabla y(s) \frac{\Delta x(s)}{\Delta x(s)} \right\} + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) = 0 \]  

(45)

where \( \sigma(s) \equiv \sigma[x(s)] , \tau(s) \equiv \tau[x(s)] \) are functions of \( x(s) \) of at most of second and first degree, respectively.

It can be proved [12] that the functions \( v_k(s) \) connected with the solutions \( y(s) \) by the relations

\[ v_k(s) = \frac{\Delta v_{k-1}(s)}{\Delta x_{k-1}(s)} , \quad v_0(s) = y(s) \]  

(46)

\[ x_k(s) = x \left( s + \frac{k}{2} \right) , \quad k = 0, 1, 2, \ldots \]

satisfy the difference equation

\[ \sigma(s) \frac{\Delta}{\Delta x_{k}(s-1/2)} \left\{ \nabla v_k(s) \frac{\Delta x_k(s)}{\Delta x_k(s)} \right\} + \tau_k(s) \frac{\Delta v_k(s)}{\Delta x_k(s)} + \mu_k v_k(s) = 0 \]  

(47)

where

\[ \tau_k(s) = \frac{\sigma(s+k) - \sigma(s) + \tau(s+k) \Delta x(s+k-1/2)}{\Delta x(s+k-1/2)} \]  

(48)

\[ \mu_k = \lambda + \sum_{m=0}^{k-1} \frac{\Delta \tau_m(s)}{\Delta x_m(s)} = \lambda + \sum_{m=0}^{k-1} \tau'_m \]

(49)

provided the lattice functions \( x(s) \) have the form

\[ x(s) = c_1 s^2 + c_2 s + c_3 \quad \text{or} \]

\[ x(s) = c_1 q^s + c_2 q^{-s} + c_3 \]

(50)

(51)

with \( c_1, c_2, c_3, q, \) arbitrary constants.

When \( \mu = 0 \) for \( k = n \) in (47) \( v_n = \text{const} \). It can be proved that when \( k < n \), \( v_k(s) \) is a polynomial in \( x_k(s) \) and in particular for \( k = 0 \), \( v_0(s) = y(s) \) is a polynomial of degree \( n \) in \( x(s) \) satisfying (45).
An explicit expression for \( \lambda_n \), when \( \mu_n = 0 \), is given by

\[
\lambda_n = \frac{-\cosh \omega}{\sinh \omega} \left\{ \cosh (n-1) \omega \tau' + \frac{1}{2} \sinh (n-1) \omega \sigma'' \right\}
\]

(52)

where \( \omega = \frac{1}{2} \ln q \), or \( q = e^{2 \omega} \). For the square lattice (50) \( \omega = 0 \); and for the \( q \)-lattice (51) we have

\[
\frac{\sinh n \omega}{\sinh \omega} = q^{n/2} - q^{-n/2} \equiv [n]_q
\]

(52)

The polynomials solutions of (45) satisfy the following orthogonality condition with respect to the weight functions \( \rho(s) \), namely,

\[
\sum_{s=a}^{b-1} y_v(s) y_n(s) \rho(s) \Delta x \left( s - \frac{1}{2} \right) = \delta_{vn} d_n^2
\]

(53)

Similarly for the differences of the polynomials \( y_n(s) \), namely,

\[
v_{mn}(s) = \Delta^{(m)} [y_n(s)] = \Delta_{m-1} \Delta_{m-2} \ldots \Delta_0 [y_n(s)], \quad \Delta_k \equiv \frac{\Delta}{\Delta x_k(s)}
\]

it holds

\[
\sum_{s=a}^{b-k-1} v_{mt}(s) v_{mn}(s) \rho_m(s) \Delta x_m \left( s - \frac{1}{2} \right) = \delta_{km} d_{mn}^2
\]

(54)

where \( \rho_m(s) = \rho(s + m) \prod_{i=1}^{m} \sigma(s + i) \)

It can be proved that the normalization constants satisfy

\[
d_{mn}^2 = d_n^2 \left( \prod_{k=m}^{n-1} \mu_{kn} \right)^{-1}, \quad d_0^2 = d_n^2 \left( \prod_{k=0}^{n-1} \mu_{kn} \right)^{-1}
\]

from which \( d_{mn}^2 \) can be eliminated:

\[
d_{mn}^2 = d_0^2 \prod_{k=m}^{n-1} \mu_{kn}
\]

(55)

A particular solution of (45) when \( \lambda = \lambda_n \) is given by the Rodrigues type formula

\[
y_n(s) = \frac{B_n}{\rho(s)} \Delta^{(n)} \rho(n) = \frac{B_n}{\rho(s)} \nabla \left[ \Delta x_1(s) \nabla \left[ \Delta x_2(s) \nabla \ldots \nabla \right[ \rho(s) \right]
\]

(56)

A solution of (47) when \( \mu_k \) is restricted to \( \lambda_n \), namely, \( \mu_{mn} = \mu_m(\lambda_n) = \lambda_n - \lambda_m, 0, 1, \ldots n-1 \), is given by:

\[
v_{mn}(s) = \frac{A_{mn} B_n}{\rho_m(s)} \Delta^{(n-m)} \rho(n) = \frac{A_{mn} B_n}{\rho_m(s)} \nabla \left[ \Delta x_m(s) \nabla \left[ \Delta x_{m+1}(s) \nabla \ldots \nabla \right[ \rho(s) \right]
\]

(57)

where

\[
A_{mn} = (-1)^m \prod_{k=0}^{m-1} \mu_{kn} = \frac{[n]!}{[n-m]!} \prod_{k=0}^{m-1} \lambda_{n+k}
\]

(58)

\[
B_n = A_{nn}^{-1} \Delta^{(n)} y_n(s)
\]
Formulas (56) and (57) can be written in terms of the mean difference operator [13]
\[ \delta f(s) = f\left(s + \frac{1}{2}\right) - f\left(s - \frac{1}{2}\right) = \nabla f\left(s + \frac{1}{2}\right), \]
that is to say,
\[ y_n(s) = \frac{B_n}{\rho(s)} \left[ \frac{\delta}{\delta x(s)} \right]^n \rho_n\left(s - \frac{n}{2}\right) \] (59)
\[ v_{mn}(s) = \frac{A_{mn}B_n}{\rho_{m}(s)} \left[ \frac{\delta}{\delta x\left(s + \frac{m}{2}\right)} \right]^{n-m} \rho_n\left(s - \frac{n}{2} + \frac{m}{2}\right) \] (60)

In order to obtain the raising and lowering operators of the classical orthogonal polynomials on non-homogeneous lattice, we apply the Rodrigues formula (56)
\[ y_{n+1}(s) = \frac{B_{n+1}}{\rho(s)} \nabla_{n+1} \left\{ \rho_{n+1}(s) \right\} = \frac{B_{n+1}}{\rho(s)} \nabla \nabla_{x1(s)} \cdots \nabla \nabla_{x_{n+1}(s)} \left\{ \rho_{n+1}(s) \right\} \]

Since
\[ \nabla \rho_{n+1}(s) = \Delta \rho_{n+1}(s - 1) = \Delta s\{\sigma(s)\rho_n(s)\} = \Delta x_n(s - \frac{1}{2}) = \tau_n(s)\rho_n(s), \]
using (59) we have
\[ y_{n+1}(s) = \frac{B_{n+1}}{\rho(s)} \left[ \frac{\delta}{\delta x(s)} \right]^n \left\{ \tau_n(s)\rho_n(s) \right\} = \frac{B_{n+1}}{\rho(s)} \left[ \frac{\delta}{\delta x(s)} \right]^n \left\{ \tau_n\left(s - \frac{n}{2}\right)\rho_n\left(s - \frac{n}{2}\right) \right\} = \]
\[ = \frac{B_{n+1}}{\rho(s)} \left\{ \tau_n(s) \left[ \frac{\delta}{\delta x(s)} \right]^n \rho_n\left(s - \frac{n}{2}\right) + \frac{\text{sh} n \omega}{\text{sh} \omega} \tau_n' \left[ \frac{\delta}{\delta x\left(s - \frac{1}{2}\right)} \right]^{n-1} \rho_n\left(s - \frac{n}{2} - \frac{1}{2}\right) \right\} \] (61)

The last step can be proved by induction for both cases of \( x(s) \) on non-homogeneous lattice (50) and (51).

First of all, we transform the properties of these functions [14] given by
\[ x(s + n) - x(s) = \frac{\text{sh} n \omega}{\text{sh} \omega} \nabla x\left(s + \frac{n + 1}{2}\right) \]
\[ x(s + n) + x(s) = \text{ch} n \omega x(s) + \text{const.} \]

into the difference relations
\[ \delta x\left(s + \frac{n}{2}\right) - \delta x\left(s - \frac{n}{2}\right) = \frac{\text{sh} n \omega}{\text{sh} \omega} \left\{ \delta x\left(s + \frac{1}{2}\right) - \delta x\left(s - \frac{1}{2}\right) \right\} \] (62)
\[ \frac{1}{2} \left\{ \delta x\left(s + \frac{n}{2}\right) + \delta x\left(s - \frac{n}{2}\right) \right\} = \text{ch} n \omega \delta x(s) \] (63)

Suppose it is true that for any two functions of discrete variable it holds
\[ \left( \frac{\delta}{\delta x(s)} \right)^n \left\{ f(s)g(s) \right\} = f\left(s + \frac{n}{2}\right) \left( \frac{\delta}{\delta x(s)} \right)^n g(s) + \]
\[ + \frac{\text{sh} n \omega \delta f\left(s + \frac{n-1}{2}\right)}{\text{sh} \omega \delta x\left(s + \frac{n-1}{2}\right)} \left( \frac{\delta}{\delta x\left(s - \frac{1}{2}\right)} \right)^{n-1} g\left(s - \frac{1}{2}\right) + \ldots \]
Then using the properties of the mean operator we have:

\[
\left( \frac{\delta}{\delta x(s)} \right)^{n+1} \{ f(s)g(s) \} = f \left( s + \frac{n + 1}{2} \right) \left( \frac{\delta}{\delta x(s)} \right)^{n+1} g(s) + \\
+ \frac{\delta f \left( s + \frac{n}{2} \right)}{\delta x(s)} \left( \frac{\delta}{\delta x(s)} \right)^n g \left( s - \frac{1}{2} \right) + \\
+ \frac{\text{sh} n \omega}{\text{sh} \omega} \frac{\delta f \left( s + \frac{n}{2} \right)}{\delta x(s)} \left( \frac{\delta}{\delta x(s)} \right)^{n-1} g \left( s - \frac{1}{2} \right) + \ldots
\]

The second and third term on the right hand side can we written:

\[
\frac{\delta f \left( s + \frac{n}{2} \right)}{\delta x(s)} \left( \frac{\delta}{\delta x(s)} \right)^n g \left( s - \frac{1}{2} \right)
\]

Using (62) and (63) the expression between curly brackets is equal to \( \text{sh}(n+1)\omega/\text{sh}\omega \), therefore

\[
\left( \frac{\delta}{\delta x(s)} \right)^{n+1} \{ f(s)g(s) \} = f \left( s + \frac{n + 1}{2} \right) \left( \frac{\delta}{\delta x(s)} \right)^{n+1} g(s) + \\
+ \frac{\text{sh}(n+1)\omega}{\text{sh}\omega} \frac{\delta f \left( s + \frac{n}{2} \right)}{\delta x(s)} \left( \frac{\delta}{\delta x(s)} \right)^{n-1} g \left( s - \frac{1}{2} \right) + \ldots
\]

as required. Substituting \( f(x) = \tau_n \left( s - \frac{n}{2} \right) \) and \( g(s) = \rho_n \left( s - \frac{n}{2} \right) \), the terms of lower degree become zero, due to the properties of function \( \tau_n(s) \). Therefore (61) is proved.

Using (60) for \( m = 1 \) we have

\[
\frac{\nabla y_n(s)}{\nabla x(s)} = \Delta y_n(s-1) \Delta x(s-1) = v_1n(s-1) = \frac{A_1nB_n}{\rho_1(s-1)} \left( \frac{\delta}{\delta x(s)} \right)^{n-1} \rho_n \left( s - \frac{n}{2} - \frac{1}{2} \right)
\]

Therefore, (61) can be written

\[
y_{n+1}(s) = \frac{B_{n+1}}{B_n} \left\{ \tau_n(s)y_n(s) + \text{sh} n \omega \frac{\tau'_n}{\text{sh} \omega} A_1n \sigma(s) \frac{\nabla y_n(s)}{\nabla x(s)} \right\}
\]

Alvarez-Nodarse and Costas-Santos [15] [16] have given the same formula for the lattice (51). Here we have proved the similar expression for both cases (50) and (51).

From (64) we can calculate the raising and lowering operators. Instead, we proceed to the general case in section 5, and then take the value \( m = 0 \).
5 Raising and lowering operators for generalized classical orthogonal polynomials of discrete variable on non-uniform lattice

From (57) and (60) we obtain

\[
v_{m,n+1}(s) = \frac{A_{m,n+1} B_{n+1}}{\rho_m(s)} \nabla_{n+1}^{(n+1-m)} \{ \rho_{n+1}(s) \} = \]

\[
= \frac{A_{m,n+1} B_{n+1}}{\rho_m(s)} \nabla_n^{(n-m)} \{ \tau_n(s) \rho_n(s) \} = \]

\[
= \frac{A_{m,n+1} B_{n+1}}{\rho_m(s)} \left( \frac{\delta}{\delta x (s + \frac{m}{2})} \right)^{n-m} \left\{ \tau_n \left( s - \frac{n-m}{2} \right) \rho_n \left( s - \frac{n-m}{2} \right) \right\} = \]

\[
= \frac{A_{m,n+1} B_{n+1}}{\rho_m(s')} \left( \frac{\delta}{\delta x (s')} \right)^{n-m} \left\{ \tau_n \left( s' - \frac{n}{2} \right) \rho_n \left( s' - \frac{n}{2} \right) \right\} \]

With respect to the new variable \( s' = s + \frac{m}{2} \), this expression can be easily differentiated as in (61) giving

\[
v_{m,n+1}(s) = \frac{A_{m,n+1} B_{n+1}}{\rho_m(s)} \left\{ \tau_n(s) \left( \frac{\delta}{\delta x (s + \frac{m}{2})} \right)^{n-m} \rho_n \left( s - \frac{n-m}{2} \right) + \right. \]

\[
+ \frac{\text{sh}(n-m) \omega}{\text{sh} \omega} \tau_n \left( \frac{\delta}{\delta x (s + \frac{m}{2})} \right)^{n-m-1} \rho_n \left( s - \frac{n}{2} + \frac{m-l}{2} \right) \} \]

From (60) we get

\[
\nabla v_{m,n}(s) \over \nabla x(s) = \frac{\Delta v_{m,n}(s-1)}{\Delta x(s-1)} = v_{m,n+1}(s-1) = \]

\[
= \frac{A_{m+1,n} B_n}{\rho_{m+1}(s-1)} \left( \frac{\delta}{\delta x (s + \frac{m-1}{2})} \right)^{n-m-1} \rho_n \left( s - \frac{n}{2} + \frac{m-l}{2} \right) \}

Using this result and the values for \( A_{m,n} \) given in (58) we get the raising operator for \( v_{mn}(s) \), namely,

\[
v_{m,n+1}(s) = \frac{B_{n+1}}{B_n} \left\{ \left[ \frac{n+1}{n+1-m} \right] \left[ \frac{n}{n+m} \right] \frac{\lambda_{n+m}}{\lambda_n} \tau_n(x) v_{mn}(x) \right. \]

\[
- \left. \left[ \frac{n+1}{n+1-m} \right] \left[ \frac{n}{n+m} \right] \frac{\lambda_{n+1}}{\lambda_n} \tau_n(x) \right\} \frac{\nabla v_{m,n}(s)}{\nabla x(s)} \}

(65)

with \([n] \equiv \frac{\text{sh} \omega}{\text{sh} \omega} \) corresponding to all values of lattice functions \( x(s) \) given in (50) (51).

In order to construct the lowering operator we use the recurrence relation

\[
x_m(s) v_{mn}(s) = \tilde{\alpha}_n v_{m,n+1}(s) + \tilde{\beta}_n v_{mn}(s) + \tilde{\gamma}_n v_{m,n-1}(s) \]

(66)

where \( x_m(s) = x \left( s + \frac{m}{2} \right) \) and \( v_{mn}(s) \equiv \Delta^{(m)} y_n(s) \)

We introduce the expansion \( y_n(s) = a_n x^n(s) + b_n x^{n-1}(s) + \ldots \) in the recurrence relation (66). We have two cases [17]
a) Quadratic lattice: \( x(s) = s(s+1) \).

\[
\Delta^{(m)} x^n(s) = n(n-1) \ldots (n-m+1) x_m^{n-m}(s) + \frac{1}{12m} n(n-1) \ldots (n-m)(2n-2m+1) x_m^{n-m-1}(s)
\]

which after substitution in the recurrence relation (66) gives

\[
\tilde{\alpha}_n = \frac{a_n}{a_{n+1}} \frac{n-m+1}{n+1} \quad (67)
\]

\[
\tilde{\beta}_n = \frac{b_n}{a_n} \left[ \frac{n-m}{n} - \frac{b_{n+1}}{a_{n+1}} \frac{n-m+1}{n+1} - \frac{3}{12m} \right] \quad (68)
\]

\[
\tilde{\gamma}_n = \frac{a_{n-1}}{a_n} \frac{n-m}{n} \frac{d_n^2}{d_{m,n-1}^2} \quad (69)
\]

b) Exponential lattice \( x(s) = Aq^s + Bq^{-s} \)

\[
\Delta^{(m)} x^n(s) = [n] [n-1] \ldots [n-m+1] x_m^{n-m-1}(s) + C x_m^{n-m-3}(s) + \ldots
\]

which after substitution in the recurrence relation (66) gives

\[
\tilde{\alpha}_n = \frac{a_n}{a_{n+1}} \frac{[n-m+1]}{[n+1]} \quad (70)
\]

\[
\tilde{\beta}_n = \frac{b_n}{a_n} \left[ \frac{[n-m]}{[n]} - \frac{b_{n+1}}{a_{n+1}} \frac{[n-m+1]}{[n+1]} \right] \quad (71)
\]

\[
\tilde{\gamma}_n = \frac{a_{n-1}}{a_n} \frac{[n-m]}{[n]} \frac{d_n^2}{d_{m,n-1}^2} \quad (72)
\]

Since \( a_n = B_n \prod_{k=0}^{n-1} \left( -\frac{\lambda_{n+k}}{[n+k]} \right) \) we obtain

\[
\tilde{\alpha}_n = -\frac{B_n}{B_{n+1}} \frac{\lambda_n}{[n]} \frac{[2n]}{\lambda_{2n}} \frac{[2n+1]}{\lambda_{2n+1}} \frac{[n-m+1]}{[n+1]}, \quad (73)
\]

which after substituting in (65) gives

\[
\tilde{\alpha}_n \frac{\lambda_{2n}}{[2n]} v_{m,n+1}(s) = \frac{\lambda_{n+m}}{[n+m]} \frac{\tau_{n}(s)}{\tau_n} v_{mn}(s) - \sigma(s) \frac{\nabla v_{mn}(s)}{\nabla x(s)} \quad (74)
\]

Inserting the recurrence relation (66) in (74) we get

\[
\tilde{\gamma}_n \frac{\lambda_{2n}}{[2n]} v_{m,n-1}(s) = \left\{ -\frac{\lambda_{n+m}}{[n+m]} \frac{\tau_{n}(s)}{\tau_n} + \frac{\lambda_{2n}}{[2n]} \right\} v_{mn}(s) + \sigma(s) \frac{\nabla v_{mn}(s)}{\nabla x(s)} \quad (75)
\]

The last two equation can be considered the raising and lowering operators of generalized orthogonal polynomials on non-uniform latices for the functions (50) and (51). In the first case the parameter \([n]\) should be taken as \(n\).

In order to complete the picture, we define an orthonormal function

\[
\Omega_{mn}(s) = d_{mn}^{-1} \sqrt{\rho_m(s)} \ v_{mn}(s) \quad (76)
\]
Using the properties of the difference operator and the identity

\[
\frac{\nabla \rho_m(s)}{\rho_m(s)} = \frac{\tau_{m-1}(s) \Delta x_{m-1} \left( s - \frac{1}{2} \right)}{\sigma(s) + \tau_{m-1}(s) \Delta x_{m-1} \left( s - \frac{1}{2} \right)}
\]

(77)
we get

\[
\nabla \Omega_{mn}(s) = \sqrt{\frac{\sigma(s)}{\sigma(s) + \tau_{m-1}(s) \Delta x_{m-1} \left( s - \frac{1}{2} \right)}} d^{-1}_{mn} \sqrt{\rho_m(s)} \nabla v_{mn}(s) + \frac{1}{\sqrt{\sigma(s) + \tau_{m-1}(s) \Delta x_{m-1} \left( s - \frac{1}{2} \right)}} \times \frac{\tau_{m-1}(s) \Delta x_{m-1} \left( s - \frac{1}{2} \right)}{\sigma(s) + \tau_{m-1}(s) \Delta x_{m-1} \left( s - \frac{1}{2} \right)} \Omega_{mn}(s)
\]

(78)

Multiplying both sides of (74) by \(d^{-1}_{mn} \sqrt{\rho_m(s)}\) and substituting the value \(d^{-1}_{mn} \sqrt{\rho_m(s)}\) \(\nabla v_{mn}(s)\) obtained in (78) we get

\[
\tilde{\alpha}_n \frac{\lambda_{2n}}{2n} \frac{d_{m,n+1}}{d_{mn}} \Omega_{m,n+1}(s) = L^+(s, n) \Omega_{mn}(s) = \left\{ \frac{\lambda_{m+n}}{[n + m]} \frac{\tau_n(s)}{\tau_n'(s)} + \frac{\sqrt{\sigma(s) \tau_{m-1}(s)}}{\sqrt{\sigma(s) + \tau_{m-1}(s) \Delta x_m \left( s - \frac{1}{2} \right)}} \frac{\nabla x_m \left( s + \frac{1}{2} \right)}{\nabla x(s)} \right\} \Omega_{mn}(s) - \sqrt{\sigma(s) \sigma(s) + \tau_{m-1}(s) \Delta x_{m-1} \left( s - \frac{1}{2} \right)} \frac{\nabla \Omega_{mn}(s)}{\nabla x(s)}
\]

(79)

Similarly

\[
\tilde{\gamma}_n \frac{\lambda_{2n}}{2n} \frac{d_{m,n-1}}{d_{mn}} \Omega_{m,n-1}(s) = L^-(s, n) \Omega_{mn}(s) = \left\{ -\frac{\lambda_{m+n}}{[n + m]} \frac{\tau_n(s)}{\tau_n'(s)} + \frac{\lambda_{2n}}{2n} \left( s - \bar{\beta}_n \right) - \frac{\sqrt{\sigma(s) \tau_{m-1}(s)}}{\sqrt{\sigma(s) + \tau_{m-1}(s) \Delta x_m \left( s - \frac{1}{2} \right)}} \frac{\nabla x_m \left( s + \frac{1}{2} \right)}{\nabla x(s)} \right\} \times \Omega_{mn}(s) + \sqrt{\sigma(s) \sigma(s) + \tau_{m-1}(s) \Delta x_{m-1} \left( s - \frac{1}{2} \right)} \frac{\nabla \Omega_{mn}(s)}{\nabla x(s)}
\]

(80)

The last two expressions can be considered the raising and lowering operators for the generalized orthonormal functions on non-homogeneous lattices of the type (50) and (51). It can be proved that these operators are mutually adjoint with respect to the scalar product of unit weight.

As in the previous sections we can factorize the raising and lowering operators as follows:

\[
L^-(s, n + 1) L^+(s, n) = \mu(n) + u(s + 1, n) H(s, n)
\]

\[
L^+(s, n) L^-(s, n + 1) = \mu(n) + u(s, n - 1) H(s, n + 1)
\]
where
\[
\mu(n) = \frac{\lambda_{2n}}{[2n]} \frac{\lambda_{2n+2}}{[2n+2]} \tilde{\alpha}_n \tilde{\gamma}_{n+1},
\]
\[
u(s, n) = \frac{\lambda_n}{[n]} \frac{\tau_n(s)}{\tau'_n} - \frac{\sigma(s)}{\nabla x(s)}
\]
and \(H(s, n)\) is the difference operator derived from the left side of (47) after substituting \(\Omega_{mn}(s)\) instead of \(\omega_{mn}(s)\) given in (76). Notice that the expressions for the factorization of the raising and lowering operators become the same expressions (32) and (33) given in [15].

6 Conclusions

We have developed the construction of raising and lowering operators for classical OP of discrete variable on non-homogeneous lattice extended also to the generalized OP on homogeneous and non-homogeneous lattice.

In the last case (generalized OP) the raising and lowering operators can be defined with respect to the index \(n\), the order of the OP, or with respect to the index \(m\), the order of the difference derivative of the generalized OP, or both.

In our work we have taken into account only the index \(n\), although we have suggest how to complement the calculus with the index \(m\). We have also introduced the orthonormal functions of unit weight, more suitable to quantum mechanical applications.

Our presentation leads to an easier way for the continuous limit (compair with a different presentation in [5]).

We have already worked out some physical application of raising and lowering operators on homogeneous and non-homogeneous lattice. For instance, the quantum mechanical models for the harmonic oscillator in one dimension (Kravchuk OP), the hydrogen atom (generalized Meixner OP) [22], the Heisenberg equation of motion on the lattice (Hahn OP) [23] Dirac and Klein-Gordon equation on a homogeneous lattice (discrete exponential function). [24] [25]

Finally the connection between OP on non-homogeneous lattice and the 3nj-Wigner coefficients and its application to spin networks models in quantum gravity are now in progress.

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