Isotropic Subspaces of Schur Modules

Leesa B. Anzaldo

February 5, 2018

Abstract

It is a well-known fact that over the complex numbers and for a fixed $k$ and $n$, a generic $s$ in $\text{Sym}^2 V^*$ vanishes on some $k$-dimensional subspace of $V$ if and only if $n \geq 2k$. Tevelev found exact conditions for the extension of this statement for general symmetric and skew-symmetric multilinear forms, and we extend his work to all possible symmetric types, which corresponds to Schur modules for a general partition.

1 Introduction

Given a generic homogeneous quadratic polynomial over $\mathbb{C}$, when does its zero set contain a $k$-dimensional subspace? Geometrically, this is equivalent to asking when a generic degree 2 projective hypersurface contains a $(k-1)$-dimensional linear subspace. The answer comes from the well-known fact about symmetric bilinear forms: for a generic $s \in \text{Sym}^2 V^*$, there exists a $k$-dimensional subspace $W$ of $V$ such that $s|_W = 0$ if and only if $n \geq 2k$. One can generalize this question to symmetric multilinear forms: if $V = \mathbb{C}^n$, when does a generic $s \in (\text{Sym}^d V)^*$ vanish on some $k$-dimensional subspace of $V$? Tevelev answers this question for not only symmetric, but also for skew-symmetric multilinear forms in [4]. Putting aside some exceptions, Tevelev shows that this occurs exactly when

$$n \geq \frac{(d+k-1)}{d} + k \quad \text{or} \quad n \geq \frac{k}{d} + k$$

for symmetric or skew-symmetric multilinear forms, respectively.

Here, we consider forms whose symmetries are not considered by Tevelev; this is done by studying the vanishing of $s \in (S_\lambda V)^*$, where $\lambda$ is a nonempty partition which is neither a single row nor a single column partition and $S_\lambda$ is the Schur functor associated with $\lambda$. We show that for an $n$-dimensional vector space $V$, partition $\lambda$, and $k \geq 3$ such that $2 \leq \ell(\lambda) \leq k$ and $\lambda_1 \geq 2$, a generic $s \in (S_\lambda V)^*$ is $k$-isotropic if and only if

$$n \geq \frac{\dim (S_\lambda C^k)}{k} + k.$$

Using combinatorial tools, we prove several inequalities about Schur polynomials to obtain this result. Geometrically, we can interpret the main result as precise conditions for the existence of a $k$-dimensional subspace $W$ of $V$ such that $\text{Flag}_\lambda(W)$ is in the zero locus of $s$. 

1
2 Preliminaries

Let $V = \mathbb{C}^n$ and $\mathcal{R}$ be the tautological subbundle of $Gr(k, V)$. Recall that for a partition $\lambda$, the Schur module $S_\lambda M$ is a functor with respect to a module $M$, namely the image of the Schur map [1, 76]. For $s \in H^0(Gr(k, V), S_\lambda \mathcal{R}^*)$, a $k$-dimensional subspace $W$ of $V$ is isotropic with respect to $s$ if $s(W) = 0$. Moreover, we say $s$ is $k$-isotropic if there exists a subspace $W$ of $V$ that is isotropic with respect to $s$. Recall the following theorem (see [5, Corollary 4.1.9])

**Theorem 2.1** (Borel-Weil). If $\lambda$ is a partition, then as representations of $\text{GL}(V)$,

$$H^0(Gr(k, V), S_\lambda \mathcal{R}^*) = (S_\lambda V)^*.$$  

Tevelev used the Borel-Weil theorem to generalize the notion of isotropic subspace for symmetric bilinear forms: given $s \in \text{Sym}^d V^*$ or $s \in \Lambda^d V^*$, a subspace $W$ of $V$ is isotropic with respect to $s$ if $s|_W = 0$. We generalize the definition even further for Schur modules (and this is compatible with the definition for multilinear forms):

**Definition 2.2.** Let $\lambda$ be a partition. For $s \in (S_\lambda V)^*$, a subspace $W$ of $V$ is isotropic with respect to $s$ if $s|_{S_\lambda W} = 0$. □

We answer the following question: for generic $s \in (S_\lambda V)^*$, when does there exist an isotropic subspace $W$ of $V$ with respect to $s$? Tevelev gives necessary and sufficient conditions for the existence of isotropic subspaces with respect to symmetric or skew-symmetric multilinear forms in Theorem 2.4. One could answer this question using the fact that $s \in H^0(Gr(k, V), S_\lambda \mathcal{R}^*)$ is $k$-isotropic if and only if $c_{\text{top}}(S_\lambda \mathcal{R}^*) \neq 0$, but computing the top Chern class is hard in general:

**Example 2.3.** Let $V = \mathbb{C}^7$ and $k = 5$, and take $s \in \Lambda^3 V^*$. By the splitting principle, there exist line bundles $L_1, \ldots, L_5$ from the flag bundle associated with the tautological subbundle of $Gr(5, 7)$ such that

$$c(\Lambda^3 \mathcal{R}^*) = c \left( \sum_{1 \leq i < j < k \leq 5} L_i^{-1} L_j^{-1} L_k^{-1} \right).$$

If the Chern roots of $\mathcal{R}^*$ are denoted by $x_i$’s, then

$$c_{\text{top}}(\Lambda^3 \mathcal{R}^*) = \prod_{1 \leq i < j < k \leq 5} (x_i + x_j + x_k).$$

The Borel presentation of the cohomology ring of $Gr(5, 7)$ gives us

$$H^*(Gr(5, 7)) \otimes \mathbb{Q} = \mathbb{Q}[x_1, \ldots, x_7]^{S_5 \times S_2}/I$$

which is the ring of invariant polynomials where $S_5$ acts on $x_1, \ldots, x_5$ and $S_2$ acts on $x_6, x_7$, which we then mod out by the ideal $I$ of all positive degree symmetric functions [3, 138]. Therefore, it is enough to determine whether $c_{\text{top}}(\Lambda^3 \mathcal{R}^*)$ is in $(p_1, \ldots, p_7)$, the ideal generated by the power sum symmetric polynomials in $x_1, \ldots, x_7$. This is easily answered using Macaulay2, but difficult by hand:
The output is true, and hence the top Chern class is 0, so there does not exist a 5-dimensional isotropic subspace of V with respect to a generic s. We arrive at the same conclusion using Tevelev’s theorem:

\textbf{Theorem 2.4 (Tevelev).} Let \( s \in S^dV^* \) or \( s \in \Lambda^dV^* \) be a form in general position. The space \( V \) contains a \( k \)-dimensional isotropic subspace with respect to \( s \) if and only if

\[ n \geq \frac{(d+k-1)}{d} k + k \quad \text{or} \quad n \geq \frac{k}{\binom{n}{k}} k + k, \]

with the following exceptions:

1. if \( s \in S^2V^* \) or \( s \in \Lambda^2V^* \) is a form in general position, then \( V \) contains a \( k \)-dimensional isotropic subspace if and only if \( n \geq 2k \);

2. if \( s \in \Lambda^{n-2}V^* \) is in general position and \( n \) is even, then \( V \) contains a \( k \)-dimensional isotropic subspace if and only if \( k \leq n - 2 \);

3. if \( s \in \Lambda^3V^* \) is in general position and \( n = 7 \), then \( V \) contains a \( k \)-dimensional isotropic subspace if and only if \( k \leq 4 \).

\section{Main Theorem}

We give a similar criterion for all partitions \( \lambda \) not included in Tevelev’s theorem (except for \( \lambda = \emptyset \), which is not interesting).

\textbf{Theorem 3.1.} Let \( V \) be an \( n \)-dimensional vector space, \( \lambda \) be a partition, and take \( k \geq 3 \) such that \( 2 \leq \ell(\lambda) \leq k \) and \( \lambda_1 \geq 2 \). Then a generic \( s \in (S_\lambda V)^* \) is \( k \)-isotropic if and only if

\[ n \geq \frac{\dim(S_\lambda C^k)}{k} + k. \]

Notice that when rearranged, the inequality can be written as

\[ \dim(Gr(k,n)) \geq \dim(S_\lambda C^k). \]

\textbf{Example 3.2.} Let \( V = \mathbb{C}^6 \) and \( k = 3 \), and take \( s \in S_{(2,1)}V^* \). By the splitting principle, we can write

\[ c(S_\lambda \mathcal{R}^*) = c \left( \sum_T L_1^{-T(1)} L_2^{-T(2)} L_3^{-T(3)} \right) \]
where the $L_i$'s are line bundles, $T$ is a semistandard Young tableau of shape $\lambda$ with entries in $\{1, 2, 3\}$ (see 3.3 for the definition), and $T(i)$ is the number of boxes in $T$ labeled with $i$. If the Chern roots of $R^*$ are denoted by $x_i$'s, then we can find out if the top Chern class is 0 by determining whether the product

$$(2x_1 + x_2)(2x_1 + x_3)(x_1 + 2x_2)(x_1 + x_2 + x_3)^2(x_1 + 2x_3)(2x_2 + x_3)(x_2 + 2x_3)$$

is in $(p_1, \ldots, p_6)$, the ideal generated by the power sum symmetric polynomials in $x_1, \ldots, x_6$.

This can be answered using Macaulay2:

```
QQ[x_1..x_6];
p = k -> sum(apply(6,i->x_(i+1)^k));
f = (2*x_1+x_2)*(2*x_1+x_3)*(x_1+2*x_2)*(x_1+x_2+x_3)^2
*(x_1+2*x_3)*(2*x_2+x_3)*(x_2+2*x_3);
I = ideal(f);
J = ideal(apply(6, i-> p(i+1)));
isSubset(I,J)
```

Our final output is false and hence there exists a 3-dimensional isotropic subspace with respect to a generic $s$. This verifies the conclusion of Theorem 3.1.

For a geometric interpretation of Theorem 3.1, recall that the more general version of the Borel-Weil theorem [5, Theorem 4.1.8] says there exists a line bundle $L(\lambda)$ such that $H^0(Flag_{\lambda}(V), L(\lambda)) = (S_\lambda V)^*$ as representations of $GL(V)$. Then the zero locus of $s$ in $(S_\lambda V)^*$, denoted by $Z(s)$, is a subvariety of $Flag_{\lambda}(V)$. Therefore, we have the following consequence:

**Corollary 3.3.** Let $V$ be an $n$-dimensional vector space and $\lambda$ be a partition. For a generic $s \in (S_\lambda V)^*$, there exists a $k$-dimensional subspace $W$ of $V$ such that $Flag_{\lambda}(W)$ is in the zero locus of $s$ if and only if

$$n \geq \frac{\dim (S_\lambda C^k)}{k} + k.$$

The forward direction of Theorem 3.1 is implied by the following general fact, denoted here by Lemma 3.4 which Tevelev also uses in the analogous direction of his proof. In the case of our theorem, $X$ is our Grassmannian and $E = S_\lambda \mathcal{R}^*$ in the lemma below. Notice $S_\lambda \mathcal{R}^*$ is generated by global sections, i.e. for any $W \in Gr(k, n)$, the map $H^0(Gr(k, n), S_\lambda \mathcal{R}^*) \to (S_\lambda W)^*$, where $s \mapsto s|_{S_\lambda W}$, is surjective. This is true because $H^0(Gr(k, n), S_\lambda \mathcal{R}^*) = (S_\lambda C^n)^*$ and $S_\lambda W \hookrightarrow S_\lambda C^n$ is injective.

**Lemma 3.4.** Let $X$ be a connected variety of dimension $n$ and $\mathcal{E}$ be a rank $r$ vector bundle on $X$. Assume $\mathcal{E}$ is generated by global sections. If $r > n$, then $Z(s) = \emptyset$ for almost all $s \in H^0(X, \mathcal{E})$.

**Proof.** Define $Z = \{(s, x) \in H^0(X, \mathcal{E}) \times X : s(x) = 0\}$, and let $\pi_1 : Z \to H^0(X, \mathcal{E})$, $\pi_2 : Z \to X$ be projection maps. Let $ev_x : H^0(X, \mathcal{E}) \to \mathcal{E}_x$ take $s \mapsto s(x)$ for $x \in X$. By definition, for any $s \in H^0(X, \mathcal{E})$ and $x \in X$,

$$\pi_1^{-1}(s) \cong Z(s)$$
\[ \pi_2^{-1}(x) = \{ x \in H^0(X, \mathcal{E}) : s(x) = 0 \} = \ker ev_x. \]

Since \( \mathcal{E} \) is generated by global sections, \( ev_x \) is surjective and hence
\[ \dim \pi_2^{-1}(x) = (\dim H^0(X, \mathcal{E})) - r. \]

Since \( (0, x) \in Z \) for all \( x \in X \), \( \pi_2 \) is a surjective map between irreducible varieties,
\[ \dim Z = \dim X + \max_{x \in X} \{ \dim \pi_2^{-1}(x) \} = n + (\dim H^0(X, \mathcal{E})) - r \]
so \( \dim Z < \dim H^0(X, \mathcal{E}) \). This implies \( \pi_1 \) is not surjective, and hence \( \overline{\pi_1(Z)} \) is a closed proper subvariety of \( H^0(X, \mathcal{E}) \). Hence, if \( s \) is in the open subset \( H^0(X, \mathcal{E}) \setminus \overline{\pi_1(Z)} \), then \( Z(s) = \pi_1^{-1}(s) = \emptyset. \)

Notice that in the proof of Lemma 3.4, \( \pi_2 \) is a projective map because it can be factored as \( Z \to \mathbb{P}^n \times X \to X \) where the first map is an isomorphism of \( Z \) onto a closed subset of \( \mathbb{P}^n \times X \), and the second map is the projection of \( \mathbb{P}^n \times X \) onto \( X \). Therefore, \( \pi_2(Z) \) is closed, so we have the following corollary:

**Corollary 3.5.** Under the same assumptions as Theorem 3.1, if \( n \geq \frac{\dim(S_{\lambda} V^*)}{k} + k \), then every \( s \in (S_{\lambda} V^*) \) is \( k \)-isotropic.

To prove the reverse direction of Theorem 3.1, we use a general version of a lemma by Tevelev [4, p.849].

**Lemma 3.6** (Tevelev). Let \( V \) be an \( n \)-dimensional vector space and \( \lambda \) be a partition. If
\[ \dim (S_\lambda C^{k-i}) \leq (k-i)(n-k-i) \]  
for all \( i = 0, \ldots, \min \{k, n-k\} \), then for generic \( s \in (S_{\lambda} V^*) \), \( V \) contains a \( k \)-dimensional isotropic subspace with respect to \( s \).

In order to show that the inequalities above are satisfied, we compute \( \dim (S_\lambda C^{k-i}) \) by evaluating a Schur polynomial \( s_\lambda \) in \( (1, 1, \ldots, 1) \) and applying tools from combinatorics.

**Definition 3.8.** For a partition \( \lambda \), a **semistandard Young tableau** is a Young diagram of shape \( \lambda \) filled with some positive integers so that rows are weakly increasing from left to right and columns are strictly increasing from top to bottom. If \( \lambda \) is a partition, then \( s_\lambda(1^n) \) is the number of semistandard Young tableaux with the shape \( \lambda \) and filled with entries from \( \{1, \ldots, n\} \).

If \( b \) is any box in \( \lambda \), then the **content** of \( b \) is \( j - i \) if \( b \) is in the \( i \)th row from top to bottom and the \( j \)th column from left to right; this is denoted by \( c(b) \). The **hook length** at \( b \) is the number of squares below and to the right of \( b \), including \( b \) once, denoted by \( h(b) \).

**Example 3.9.** These are all possible semistandard Young tableaux of shape \( \lambda = (2, 1) \) filled with entries from \( \{1, 2, 3\} \):

\[
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 1 & 3 \\
\end{array} \quad \begin{array}{ccc}
1 & 2 & 2 \\
2 & 3 & 2 \\
\end{array} \quad \begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 3 \\
\end{array} \quad \begin{array}{ccc}
1 & 3 & 2 \\
3 & 3 & 3 \\
\end{array} \quad \begin{array}{ccc}
2 & 2 & 3 \\
3 & 3 & 3 \\
\end{array} \quad \begin{array}{ccc}
2 & 3 & 3 \\
3 & 3 & 3 \\
\end{array}
\]

Therefore, \( s_{(2,1)}(1, 1, 1) = 8 \). The values for hook length and content for boxes of \( \lambda \) are filled in below:

\[
h: \begin{array}{ccc}
3 & 1 \\
1 & & \\
\end{array} \quad c: \begin{array}{ccc}
0 & 1 \\
-1 & & \\
\end{array}
\]
Theorem 3.10 (Hook-Content Formula). Let $\lambda$ be a partition and $b$ be any box in $\lambda$. Then
\[
s_\lambda(1^n) = \prod_{b \in \lambda} \frac{n + c(b)}{h(b)}.
\]

We use the following well-known result [1, p.77]:

Theorem 3.11. Let $\lambda$ be a partition. Then
\[
\dim (S_\lambda(C^n)) = s_\lambda(1^n).
\]

We can prove that $n \geq \frac{\dim (S_\lambda(C^k))}{k} + k$ implies $\dim (S_\lambda(C^{k-i})) \leq (k-i)(n-k-i)$ for most values of $i \in \{0, \ldots, \min\{k, n-k\}\}$ by showing that
\[
\frac{\dim (S_\lambda(C^{k-i}))}{k-i} \geq \frac{\dim (S_\lambda(C^{k-i-1}))}{k-i-1} + 1. \tag{3.12}
\]

Tevelev uses induction to prove (3.12); for example, he gives the following lemma used for $Sym^dV^*$ where $d \geq 3$:

Lemma 3.13. If $d \geq 3$ and $\alpha \geq 2$, then
\[
\frac{\binom{d+\alpha-1}{d}}{\alpha} \geq \frac{\binom{d+\alpha-2}{d}}{\alpha - 1} + 1.
\]

However, this quickly becomes difficult for general partitions. This can be seen in the following examples of hooks and rectangular partitions because $\dim (S_\lambda(C^n))$ is no longer a single binomial coefficient. One can perform a painful induction in particular cases, but it is hard to generalize.

Example 3.14. If $\lambda = (d, 1)$ where $d \geq 2$, then by the Hook-Content Formula,
\[
s_\lambda(1^n) = \frac{n}{d+1} \cdot \frac{n+1}{d-1} \cdot \frac{n+2}{d-2} \cdots \frac{n+d-2}{1} \cdot \frac{n-1}{1} = \frac{n(n+d-1)(n+d-2)\cdots(n-1)}{(d+1)(d-1)\cdots1} = \frac{d(n-1)}{d+1} \binom{n+d-1}{d}.
\]

Example 3.15. If $\lambda = (d, d)$ where $d \geq 2$, then
\[
s_\lambda(1^n) = \frac{n+d-1}{(n-1)(d+1)} \binom{n+d-2}{d}^2.
\]

More generally, if $\lambda = (d, \ldots, d)$ have $l$ parts where $d, l \geq 2$, then
\[
s_\lambda(1^n) = \frac{(l-1)!}{(n-1)(d+1)} \left(\binom{n+d-l}{d}\right)^{l-1} \prod_{j=1}^{l-1} \left(\frac{n+d-j}{j(d+l-j)}\right)^j.
\]
4 Proof of Main Theorem

Now, we give inequalities that will assist in proving (3.12).

Lemma 4.1. Let $\lambda$ be a nonempty partition.

1. For $k \geq 2$,
   \[
   \frac{s_\lambda(1^k)}{k} \geq \frac{s_\lambda(1^{k-1})}{k-1}. \tag{4.2}
   \]

2. If, in addition, $2 \leq \ell(\lambda) \leq k - 1$, then
   \[
   \frac{s_\lambda(1^k)}{k} \geq \frac{s_\lambda(1^{k-1})}{k-1} + \frac{1}{k}. \tag{4.3}
   \]

Proof. To prove the second part of the lemma, notice (4.3) is equivalent to
   \[
   k(s_\lambda(1^k) - s_\lambda(1^{k-1})) \geq s_\lambda(1^k) + k - 1.
   \]

Let $g_\lambda(k)$ be the number of semistandard Young tableaux with shape $\lambda$ with entries in \{1, \ldots, k\} and labeled with at least one $k$. Since
   \[s_\lambda(1^k) = g_\lambda(k) + s_\lambda(1^{k-1}),\]
we can prove the equivalent statement
   \[(k - 1)g_\lambda(k) \geq s_\lambda(1^{k-1}) + k - 1.\]

Let $\mu$ be the subpartition of $\lambda$ obtained by removing the box in the last column of the last row of $\lambda$. Let $\nu$ be the partition obtained by adding a box to the end of the first row of $\mu$. Then
   \[(k - 1)g_\lambda(k) \geq (k - 1)s_\mu(1^{k-1})
   = s_1(1^{k-1})s_\mu(1^{k-1})
   \geq s_\lambda(1^{k-1}) + s_\nu(1^{k-1})
   \geq s_\lambda(1^{k-1}) + k - 1.\]

If we label a partition of shape $\mu$ with entries in \{1, \ldots, k - 1\}, reattach a box to $\mu$ in order to obtain $\lambda$, and label this new box with $k$, then we obtain a semistandard Young tableau of shape $\lambda$ with entries in \{1, \ldots, k\}; this proves the first line above. The second line is obvious. Since $\ell(\lambda) \geq 2$, $\lambda \neq \nu$, so the third line follows by Pieri’s rule. Since $\ell(\lambda) \leq k - 1$, this implies that $\ell(\nu) \leq k - 1$, so $s_\nu(1^{k-1}) \neq 0$. Moreover, we obtain a semistandard Young tableau if the last box in the first row of $\nu$ is filled with any integer in \{1, \ldots, k - 1\}; the remaining boxes in the first row are labeled with 1; and for the remaining rows, the boxes in the $i$th row are labeled with $i$. Therefore, we have at least $k - 1$ semistandard Young tableaux of shape $\nu$ with entries in \{1, \ldots, k - 1\}, proving the fourth line.
Now we prove the first part of the lemma. It is clearly true when \( \lambda = (1) \). Otherwise, we again choose \( \mu \) to be the subpartition of \( \lambda \) obtained by removing the box in the last column of the last row of \( \lambda \). Notice that (4.3) is equivalent to

\[
(k - 1)g_\lambda(k) \geq s_\lambda(1^{k-1}).
\]

Using a similar reasoning as above, we obtain

\[
(k - 1)g_\lambda(k) \geq (k - 1)s_\mu(1^{k-1}) = s_1(1^{k-1})s_\mu(1^{k-1}) \geq s_\lambda(1^{k-1}). \qedhere
\]

**Definition 4.4.** Let \( \lambda \) be a partition. If \( \mu \) is a subpartition of \( \lambda \) such that \( \lambda/\mu \) is a skew shape whose columns contain at most one box each, then \( \lambda/\mu \) is a **horizontal strip**. We denote the collection of all horizontal strips by \( HS \).

The following is a well-known fact:

**Proposition 4.5.** For any partition \( \lambda \),

\[
s_\lambda(1^k) = \sum_{\lambda/\mu \in HS} s_\mu(1^{k-1}).
\]

**Proof.** We can partition the collection of all semistandard Young tableau of shape \( \lambda \) with entries in \( \{1, \ldots, k\} \) into subsets based on the placement of \( k \)'s. Since \( k \) can appear at most once in each column of \( \lambda \), the size of such a subset is the same as the number of semistandard Young tableau of some unique \( \mu \subset \lambda \) such that \( \lambda/\mu \in HS \) and labeled with entries in \( \{1, \ldots, k-1\} \). \( \square \)

**Lemma 4.6.** Let \( \lambda \) be a partition and take \( k \geq 3 \). If \( 1 \leq \ell(\lambda) \leq k - 2 \) and \( \lambda \neq (1), (2), (1,1) \), then

\[
\frac{s_\lambda(1^k)}{k} \geq \frac{s_\lambda(1^{k-1})}{k-1} + 1.
\]

**Proof.** We perform induction on \( k \). The case when \( k = 3 \) corresponds to symmetric forms, which was proved by Tevelev’s lemma 3.13.

Now let \( k > 3 \) and \( \lambda \) be a partition satisfying \( 1 \leq \ell(\lambda) \leq k - 2 \) and not equal to \( (1), (2), \) or \( (1,1) \). By induction, we suppose that for any partition \( \mu \) not equal to \( (1), (2), \) or \( (1,1) \), and satisfying \( 1 \leq \ell(\mu) \leq k - 3 \), then

\[
\frac{s_\mu(1^{k-1})}{k-1} \geq \frac{s_\mu(1^{k-2})}{k-2} + 1.
\]

If \( \ell(\lambda) = 1 \), then we can use Lemma 3.13. Otherwise, we use Proposition 4.5 several times in the computation below.
\[
\frac{s_\lambda(1^k)}{k} = \frac{1}{k} \sum_{\lambda/\mu \in \text{HS}} s_\mu(1^{k-1})
\]

\[
= \frac{1}{k} \sum_{\lambda/\mu \in \text{HS}} s_\mu(1^{k-1}) + \frac{1}{k} \sum_{\lambda/\mu \in \text{HS}} s_\mu(1^{k-1}) + \frac{1}{k} \sum_{\lambda/\mu \in \text{HS}} s_\mu(1^{k-1})
\]

\[
= \frac{k - 1}{k} \sum_{\lambda/\mu \in \text{HS}} s_\mu(1^{k-1}) + \frac{k - 1}{k} \sum_{\lambda/\mu \in \text{HS}} s_\mu(1^{k-1}) + \frac{k - 1}{k} \sum_{\lambda/\mu \in \text{HS}} s_\mu(1^{k-1})
\]

\[
= \frac{k - 1}{k(k - 2)} \sum_{\lambda/\mu \in \text{HS}} s_\mu(1^{k-2}) + \frac{k - 1}{k} + \frac{1}{k}
\]

\[
= \frac{(k - 1)^2 s_\lambda(1^{k-1})}{k(k - 2)} + \frac{k - 1}{k} + \frac{1}{k}
\]

\[
\geq \frac{s_\lambda(1^{k-1})}{k - 1} + \frac{k - 1}{k} + \frac{1}{k}
\]

Line (4.8) follows by the induction hypothesis on the first sum and Lemma 4.1 applied to the remaining sums. After rearranging terms and noticing \((k - 1)^2 > k(k - 2)\), we have line (4.9). In Line (4.9), the first sum has at least one summand because we can let \(\mu\) be the subpartition of \(\lambda\) obtained by removing the last row of \(\lambda\); and the second sum has at least one summand because we can take \(\mu\) to be \(\lambda\). This proves line (4.10).
We can now prove the proof of the main theorem:

**Proof of Theorem 3.7** Since $s_{\lambda}(1^j) = 0$ for $j < \ell(\lambda)$, by Tevelev’s Lemma 3.6 it suffices to show

$$\dim(S_{\lambda}C^{k-i}) \leq (k-i)(n-k-i)$$

for all $i = 0, \ldots, k - \ell(\lambda)$.

Suppose $2 \leq \ell(\lambda) < k$. Since $i \neq k$, if we assume

$$n \geq \frac{\dim(S_{\lambda}C^k)}{k} + k,$$

then using the Lemma 4.6,

$$n \geq \frac{s_{\lambda}(1^k)}{k} + k \geq \frac{s_{\lambda}(1^{k-1})}{k-1} + k + 1 \geq \frac{s_{\lambda}(1^{k-2})}{k-2} + k + 2 \geq \frac{s_{\lambda}(1^{\ell(\lambda)+2})}{\ell(\lambda)+2} + k + k - \ell(\lambda) - 2 \geq \frac{s_{\lambda}(1^{\ell(\lambda)+1})}{\ell(\lambda)+1} + k + k - \ell(\lambda) - 1.$$

This proves the inequality (3.7) for $i = 0, \ldots, k - \ell(\lambda) - 1$.

If $\lambda$ is a rectangle, then

$$\dim(S_{\lambda}C^{\ell(\lambda)}) = s_{\lambda}(1^{\ell(\lambda)}) = 1 \leq \ell(\lambda)(n - \ell(\lambda))$$

because $\ell(\lambda) > 1$, so the inequality for $i = k - \ell(\lambda)$ is true.

If $\lambda$ is not a rectangle, then let $\mu$ be the partition obtained by removing all columns of height $\ell(\lambda)$ from $\lambda$; therefore, $s_{\lambda}(1^{\ell(\lambda)}) = s_{\mu}(1^{\ell(\lambda)})$. If $\mu = (1)$, then

$$\dim(S_{\lambda}C^{\ell(\lambda)}) = s_{\mu}(1^{\ell(\lambda)}) = \ell(\lambda) \leq \ell(\lambda)(n - \ell(\lambda))$$

because $\ell(\lambda) < k \leq n$. If $\mu = (2)$ or $(1,1)$, then our assumption, $n \geq \frac{\dim(S_{\lambda}C^k)}{k} + k$, says

$$n \geq \frac{3k+1}{2}, \quad n \geq \frac{3k-1}{2},$$

which imply the desired inequalities

$$n \geq \frac{3\ell(\lambda) + 1}{2} = \frac{s_{\lambda}(1^{\ell(\lambda)})}{\ell(\lambda)} + \ell(\lambda),$$
\[
n \geq \frac{3\ell(\lambda) - 1}{2} = \frac{s_\lambda(1^{\ell(\lambda)})}{\ell(\lambda)} + \ell(\lambda),
\]
respectively. Otherwise, we can prove (3.12) for our remaining case
\[
\frac{s_\lambda(1^{\ell(\lambda)+1})}{\ell(\lambda) + 1} \geq \frac{s_\mu(1^{\ell(\lambda)+1})}{\ell(\lambda) + 1}
\geq \frac{s_\mu(1^{\ell(\lambda)})}{\ell(\lambda)} + 1
= \frac{s_\lambda(1^{\ell(\lambda)})}{\ell(\lambda)} + 1.
\]

The first inequality is true because given a semistandard Young tableau of shape \(\mu\) filled with entries from \([1, \ldots, \ell(\lambda) + 1]\), one can obtain a semistandard Young tableau of shape \(\lambda\) filled with entries from \([1, \ldots, \ell(\lambda) + 1]\) in the following way: adjoin a rectangle to the left of \(\mu\) in order to obtain the shape \(\lambda\), and label the entire \(i\)th row of the rectangle with \(i\). Since \(\ell(\mu) \leq \ell(\lambda) - 1\), we can apply Lemma 4.6 to obtain the second inequality. The last line follows because the rectangle removed from \(\lambda\) in order to obtain \(\mu\) must be constant along rows when filled with integers \(1, \ldots, \ell(\lambda)\), and this is done in exactly one way. Therefore, we’ve proved the case when \(i = k - \ell(\lambda)\).

If \(\ell(\lambda) = k\), then we need only show (3.7) holds for \(i = 0\), but this is our assumption on \(n\).

References

[1] Fulton, William and Joe Harris. *Representation theory. A first course.* Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991.

[2] Grayson, Daniel R. and Michael E. Stillman, Macaulay2, a software system for research in algebraic geometry. Available at [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/).

[3] Manivel, Laurent. *Symmetric functions, Schubert polynomials and degeneracy loci.* Translated from the 1998 French original by John R. Swallow. SMF/AMS Texts and Monographs, 6. Cours Spécialis [Specialized Courses], 3. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001.

[4] Tevelev, E. A. *Isotropic subspaces of polylinear forms.* (Russian) Mat. Zametki 69 (2001), no. 6, 925–933; translation in Math. Notes 69 (2001), no. 5-6, 845–852.

[5] Weyman, Jerzy. *Cohomology of vector bundles and syzygies.* Cambridge Tracts in Mathematics, 149. Cambridge University Press, Cambridge, 2003.