HETEROTIC T-DUALITY AND THE RENORMALIZATION GROUP

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Abstract

We consider target space duality transformations for heterotic sigma models and strings away from renormalization group fixed points. By imposing certain consistency requirements between the $T$-duality symmetry and renormalization group flows, the one loop gauge beta function is uniquely determined, without any diagram calculations. Classical $T$-duality symmetry is a valid quantum symmetry of the heterotic sigma model, severely constraining its renormalization flows at this one loop order. The issue of heterotic anomalies and their cancelation is addressed from this duality constraining viewpoint.

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1 Introduction

Symmetry is a central concept in quantum field theory. Usually, one thinks of symmetries as transformations acting on the fields of a theory, leaving its partition function invariant. More fashionable these days is a different concept. This is the idea of duality symmetries, transformations on the parameter space of a theory which leave its partition function invariant. One such example is the well known target space duality (T-duality henceforth), see [1] for a complete set of references. Another important action on the parameter space of a quantum field theory is that of the renormalization group (RG henceforth), as RG transformations also leave the partition function invariant. The study of the interplay between the RG and duality symmetries then seems quite natural [2].

The idea of T-duality symmetry first came about in the context of string theory [1], but it was soon realized that a proof of its existence could be given directly from sigma model path integral considerations [3, 4]. On the other hand, sigma models are well defined two dimensional quantum field theories away from the conformal backgrounds that are of interest for string theory [3, 5]. A study was then initiated concerning the possibility of having T-duality as a symmetry of the quantum sigma model away from the (conformal) RG fixed points, when the target manifold admits an Abelian isometry. Central to this study was the aforementioned interplay between the duality symmetry and the RG. It was observed that this interplay translates to consistency conditions to be verified by the RG flows of the model; and that indeed they were verified by, and only by, the correct RG flows of the bosonic sigma model. Such a study was carried out up to two loops, order $O(\alpha'^2)$, in references [7, 8, 9, 10].

Such symmetry being verified in the bosonic sigma model – where the target fields are a metric, an antisymmetric field and a dilaton – one then wonders what happens for the supersymmetric extensions of such models. With relation to the $\mathcal{N} = 2$ supersymmetric sigma model [8], where the target fields are similar, the bosonic results do have something to say. This is due to the fact that this model asks for target Kähler geometry [11, 12], if it is to be supersymmetric. Including this extra constraint in the analysis of [8, 9] one then sees that when restricted to background Kähler tensor structures, the results obtained in there translate to the well known results for this supersymmetric sigma model. Corresponding results for the $\mathcal{N} = 1$ supersymmetric sigma model can also be obtained.

Another interesting supersymmetric extension of the bosonic sigma model is the heterotic sigma model [13]. One extra feature is that one now has a target gauge field. It is this new
coupling that we shall study in here, following the point of view in [7, 8, 9, 10]. We shall work
to one loop, order $O(\alpha')$, and we will see that $T$-duality is again a good quantum symmetry of
this sigma model. This shall be done by deriving consistency conditions for the RG flows of the
model under $T$-duality and observing that they are satisfied by, and only by, the correct RG
flows of the heterotic sigma model. However, yet another extra feature arises. In these models
the measure of integration over the quantum fields involves chiral fermions. Such fermions
produce potential anomalies, and we therefore have a first example where we can analyze the
interplay of $T$-duality and the RG flow in the presence of anomalies. It is then reasonable to
expect that the consistency conditions may have something to say about these anomalies, as
they need to cancel in order to define an RG flow.

One should finally remark that it is indeed interesting that duality, a symmetry which is
apparently entirely unaware of the renormalization structure of the model, should yield such
strong constraints as to uniquely determine the sigma model beta functions. Work similar in
spirit to the one we perform here has also been carried out in condensed matter systems [14, 15],
and more recently in systems that have a strong-weak coupling duality [16, 17].

Following [7, 8, 9, 10], let us begin with a theory with an arbitrary number of couplings, $g^i$,
$i = 1, \ldots, n$, and consider a duality symmetry, $T$, acting as a map between equivalent points in
the parameter space, such that,

$$ T g^i \equiv \tilde{g}^i = \tilde{g}^i(g). \quad (1.1) $$

Let us also assume that our system has a renormalization group flow, $R$, encoded by a set of
beta functions, and acting on the parameter space by,

$$ R g^i \equiv \beta^i(g) = \mu \frac{dg^i}{d\mu}, \quad (1.2) $$

where $\mu$ is some appropriate subtraction scale. Given any function on the parameter space of
the theory, $F(g)$, the previous operations act as follows:

$$ TF(g) = F(\tilde{g}(g)) \quad , \quad RF(g) = \delta F \frac{\delta}{\delta g^i}(g) \cdot \beta^i(g). \quad (1.3) $$

For a finite number of couplings the derivatives above should be understood as ordinary deriva-
tives, whereas in the case of the sigma model these will be functional derivatives, and the dot
will imply an integration over the target manifold.

The consistency requirements governing the interplay of the duality symmetry and the RG
can now be stated simply as,

$$ [T, R] = 0, \quad (1.4) $$
or in words: duality transformations and RG flows commute as motions in the parameter space of the theory. This amounts to a set of consistency conditions on the beta functions of our system:

$$\beta^i(\tilde{g}) = \frac{\delta \tilde{g}^i}{\delta g^j} \cdot \beta^j(\tilde{g}).$$  \hspace{1cm} (1.5)

As we shall see, this is a very strict set of requirements in our model.

The organization of this paper is as follows. In section 2 we will give a brief review of the heterotic sigma model, and on how to construct the T-duality transformation acting on the target space. Then, in section 3, we shall see how these transformations translate to a set of consistency conditions to be satisfied by the beta functions of the model. In section 4, we study such conditions in the heterotic sigma model; for the simpler case of torsionless backgrounds and paying special attention to the cancelation of anomalies. The results obtained in this section are then extended to torsionfull backgrounds in section 5, where the calculations are more involved. Finally, in section 6, we present a concluding outline.

2 Duality in the Heterotic Sigma Model

We shall start by reviewing the construction of the heterotic sigma model in $(1,0)$ superspace, and the standard procedure of dualizing such model. We will closely follow the main references on the subject, [13, 18, 19, 20], and refer to them for further details.

Superspace will have two bosonic coordinates, $z_0$ and $z_1$, and a single fermionic coordinate of positive chirality, $\theta$. The supersymmetry is $\mathcal{N} = \frac{1}{2}$ Majorana-Weyl, as only the left moving bosons have fermionic partners. We will consider two types of superfields, one scalar coordinate superfield and one spinor gauge superfield,

$$\Phi^\mu(z, \theta) = X^\mu(z) + \theta \lambda^\mu(z), \hspace{1cm} \Psi^I(z, \theta) = \psi^I(z) + \theta F^I(z).$$  \hspace{1cm} (2.1)

In here the $\Phi^\mu$ are coordinates in a $(d + 1)$-dimensional target manifold $\mathcal{M}$, so that $\mu = 0, 1, ..., d$, while the $\Psi^I$ are sections of a $G$-bundle over $\mathcal{M}$ with $n$-dimensional fibers, so that $I = 1, ..., n$. These spinor superfields transform under a representation $R$ of the gauge group $G$, with $n = \text{dim } R$. We will consider arbitrary $n, d$, even though for the heterotic superstring $d + 1 = 10$, $n = 32$ and $G = \text{Spin}(32)/\mathbb{Z}_2$ or $G = E_8 \otimes E_8$ [21]. Using light-cone coordinates, $z^\pm = \frac{1}{\sqrt{2}}(z^0 \pm z^1)$ and $\partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$, the superspace $(1,0)$ covariant derivative is written as:

$$D = \partial_\theta + i \theta \partial_+ \hspace{1cm} , \hspace{1cm} D^2 = i \partial_+.$$  \hspace{1cm} (2.2)
We consider the target manifold endowed with a metric \( g_{\mu\nu} \), antisymmetric tensor field \( b_{\mu\nu} \) and a gauge connection \( A_{\mu IJ} \) associated to the gauge group \( G \). The Lagrangian density of such model is given by [13, 19]:

\[
\mathcal{L} = -i \int d\theta \left\{ (g_{\mu\nu}(\Phi) + b_{\mu\nu}(\Phi))D\Phi^\mu \partial_- \Phi^\nu - i\delta_{I J} \Psi^I (D\Psi^J + A_{\mu J}^K(\Phi) D\Phi^\mu \Psi^K) \right\}. 
\] (2.3)

One should keep in mind that the action has an overall coefficient of \( \frac{1}{4\pi\alpha'} \), as usual. A good exercise is to do the \( \theta \) integration and eliminate the auxiliary fields. One should find:

\[
\mathcal{L} = (g_{\mu\nu} + b_{\mu\nu}) \partial_+ X^\mu \partial_- X^\nu + ig_{\mu\nu} \lambda^\nu(\partial_- \lambda^\nu + (\Gamma^\nu_{\rho\sigma} + \frac{1}{2} H^\nu_{\rho\sigma}) \partial_- X^\rho \lambda^\sigma) + 
\]

\[
+iv^I (\partial_+ \psi^I + A_{\mu J}^I \partial_+ X^\mu \psi^J) + \frac{1}{2} F_{\mu\nu IJ} \lambda^\mu \lambda^\nu \psi^I \psi^J, 
\] (2.4)

where,

\[
H_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu} \quad \text{and} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. 
\] (2.5)

We need to assume that the sigma model has an Abelian isometry in the target manifold, which will enable duality transformations [18, 19, 20]. Let \( \xi \) be the Killing vector that generates the Abelian isometry. The diffeomorphism generated by \( \xi \) transforms the scalar superfields, and the total action is invariant under the isometry only if we can compensate this transformation in the scalar superfields by a gauge transformation in the spinor superfields [18, 19]. This introduces a target gauge transformation parameter \( \kappa \), such that \( \delta_\xi A_\mu = \mathcal{L}_\xi A_\mu = \mathcal{D}_\mu \kappa \).

Choose adapted coordinates to the Killing vector, \( \xi^\mu \partial_\mu \equiv \partial_0 \), and split the coordinates as \( \mu, \nu = 0, 1, ..., d = 0, i \), so that the \( \mu = 0 \) component is singled out. In these adapted coordinates the isometry is manifest through independence of the background fields on the coordinate \( X^0 \). Moreover, in these coordinates the target gauge transformation parameter will satisfy [18, 19],

\[
\mathcal{D}_\mu \kappa \equiv \partial_\mu \kappa + [A_\mu, \kappa] = 0. 
\] (2.6)

The duality transformations are then [19, 20]:

\[
\tilde{g}_{00} = \frac{1}{g_{00}}, \quad \tilde{g}_{0i} = \frac{b_{0i}}{g_{00}}, \quad \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}}, 
\]

\[
\tilde{g}_{ij} = g_{ij} - \frac{g_{0i}g_{0j} - b_{0i}b_{0j}}{g_{00}}, \quad \tilde{b}_{ij} = b_{ij} - \frac{g_{0i}b_{0j} - g_{0j}b_{0i}}{g_{00}}, 
\] (2.7)

\[
\tilde{A}_{0IJ} = -\frac{1}{g_{00}} A_{0IJ}, 
\] (2.8)

\[
\tilde{A}_{iIJ} = A_{iIJ} - \frac{g_{0i}}{g_{00}} A_{0IJ}. 
\] (2.9)
where we have used \( \mu_{IJ} \equiv (\kappa - \xi^\alpha A_\alpha)_{IJ} \) following [18, 19], and which in adapted coordinates becomes \( \mu_{IJ} \equiv (\kappa - A_0)_{IJ} \). Observe that the gauge transformation properties of \( \kappa \) are such that \( \mu_{IJ} \) will transform covariantly under gauge transformations [18, 19]. Equations (2.7) are well known since [3, 4], and their interplay with the RG has been studied in [7, 8, 9, 10]. They shall not be dealt with in here, as to our one loop order there is nothing new to be found relative to the work in [7]. We shall rather concentrate on the new additions (2.8) and (2.9) yielding the duality transformations for the gauge connection.

There is one more duality transformation one needs to pay attention to, the one for the dilaton field. As is well known, in a curved world-sheet we have to include one further coupling in our action,

\[
\frac{1}{4\pi} \int d^2z \sqrt{h}R^{(2)}(X),
\]

where \( h = \det h_{ab} \), \( h_{ab} \) being the two dimensional world-sheet metric, and \( R^{(2)} \) its scalar curvature. \( \phi(X) \) is the background dilaton field in \( \mathcal{M} \). This term is required in order to construct the Weyl anomaly coefficients (see section 3). We should however point out that the addition of such coupling to the heterotic string is not entirely trivial as it is not invariant under the so-called kappa-symmetry [22, 23]. Taking into account the one loop Jacobian from integrating out auxiliary fields in the dualization procedure, one finds as usual the dilaton shift [4, 24]:

\[
\tilde{\phi} = \phi - \frac{1}{2} \ln g_{00}.
\]

Formulas (2.7-9) were obtained using classical manipulations alone. Only (2.11) involves quantum considerations. So, for this heterotic sigma model, we need to be careful in the following as there will be anomalies generated by the chiral fermion rotations in the quantum measure, and if so the original and dual action will not be equivalent. If we want these two theories to be equivalent one must find certain conditions on the background fields in order to cancel the anomalies. We shall see in the following that the consistency conditions (1.5) do have something to say on this matter.

3 Renormalization and Consistency Conditions

The renormalization of the heterotic sigma model has been studied in many references. Of particular interest to our investigations are the one loop beta functions [23, 26, 27, 28]. However, there are some subtleties we should point out before proceeding, as the one loop effective action is not gauge or Lorentz invariant. It happens that this non-invariance is of a very special kind,
organizing itself into the well known gauge and Lorentz Chern-Simons (order $O(\alpha')$) completion of the torsion \cite{13}. Then, starting at two loops, there are non-trivial anomalous contributions to the primitive divergences of the theory, and things get more complicated \cite{27}. None of these problems will be of concern to us to the order $O(\alpha')$ we shall be working to, appearing only at order $O(\alpha'^2)$. The one loop, order $O(\alpha')$, beta functions can be computed to be \cite{27, 28}:

$$
\beta^g_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} H_{\mu}^{\lambda\rho} H_{\lambda\rho\nu} + \mathcal{O}(\alpha'),
$$

(3.1)

$$
\beta^b_{\mu\nu} = -\frac{1}{2} \nabla^\lambda H_{\lambda\mu\nu} + \mathcal{O}(\alpha'),
$$

(3.2)

$$
\beta^A_{\mu} = \frac{1}{2} (D^\lambda F_{\lambda\mu} + \frac{1}{2} H_{\mu}^{\lambda\rho} F_{\lambda\rho}) + \mathcal{O}(\alpha'),
$$

(3.3)

where $R_{\mu\nu}$ is the Ricci tensor of the target manifold, $\nabla_\mu$ is the metric covariant derivative, and $D_\mu$ is the covariant derivative involving both the gauge and the metric connections.

Of special interest to us are the Weyl anomaly coefficients \cite{29, 30, 31, 22}, which are in general different from the RG beta functions. Their importance comes from the fact that while the definition of the sigma model beta functions ($\beta$) is ambiguous due to the freedom of target reparameterization, there is no such ambiguity for the Weyl anomaly coefficients ($\bar{\beta}$) which are invariant under such transformations. This, of course, is related to the fact that the $\bar{\beta}$-functions are used to compute the Weyl anomaly, while the $\beta$-functions are used to compute the scale anomaly \cite{29, 30}.

The advantage of using Weyl anomaly coefficients in our studies is then due to the fact that while both $\bar{\beta}$ and $\beta$ satisfy the consistency conditions (1.5), the $\bar{\beta}$-functions satisfy them exactly, while the $\beta$-functions satisfy them up to a target reparameterization \cite{7, 8}. Since both encode essentially the same RG information, in the following we shall simply consider RG motions as generated by the $\bar{\beta}$-functions. For the heterotic sigma model \cite{31, 22}, and for the loop orders considered in this work:

$$
\bar{\beta}^g_{\mu\nu} = \beta^g_{\mu\nu} + 2 \nabla_\mu \partial_\nu \phi + \mathcal{O}(\alpha'),
$$

(3.4)

$$
\bar{\beta}^b_{\mu\nu} = \beta^b_{\mu\nu} + H_{\mu}^{\lambda\rho} \partial_\lambda \phi + \mathcal{O}(\alpha'),
$$

(3.5)

$$
\bar{\beta}^A_{\mu} = \beta^A_{\mu} + F_{\mu}^{\lambda\rho} \partial_\lambda \phi + \mathcal{O}(\alpha').
$$

(3.6)

The consistency conditions (1.5) can now be derived. The couplings are $g^i \equiv \{g_{\mu\nu}, b_{\mu\nu}, A_{\mu}, \phi\}$, and the duality operation (1.1) is defined through (2.7-9) and (2.11). The RG flow operation is defined in (1.2), for our couplings, with the only difference that we shall consider $\bar{\beta}$-generated RG motions as previously explained. It is then a straightforward exercise to write down the
consistency conditions (1.5) for the heterotic sigma model. The consistency conditions associated to (2.7) and (2.11) have in fact been studied before [7, 8, 9] and are known to be satisfied by, and only by, (3.1-2) or (3.4-5). So, we shall not deal with them in here. The consistency conditions associated to the gauge field coupling are:

\[
\bar{\beta}^\Lambda_0 = \frac{1}{g_{00}} \bar{\beta}^A_0 + \frac{1}{g_{00}} (\kappa - A_0) \bar{\beta}^g_{00},
\]

(3.7)

\[
\bar{\beta}^\Lambda_i = \bar{\beta}^A_i - \frac{1}{g_{00}} ((\kappa - A_0)(\bar{\beta}^g_{0i} + \bar{\beta}^A_{0i}) - (g_{0i} + b_{0i})(\bar{\beta}^A_0)) + \frac{1}{g_{00}} (g_{0i} + b_{0i})(\kappa - A_0) \bar{\beta}^g_{00},
\]

(3.8)

where we have used the notation \(\bar{\beta}^\Lambda_\mu \equiv \bar{\beta}^A_\mu [\bar{g}, \bar{b}, \bar{A}, \phi]\). These are the main equations to be studied in this paper. The task now at hand is to see if these two conditions on the gauge field \(\bar{\beta}\)-functions are satisfied by – and only by – expressions (3.3), (3.6); and if so under what conditions are they satisfied. For that we need to perform a standard Kaluza-Klein decomposition of the target tensors. This procedure is familiar from previous work [7, 8, 9], and in particular we will use the formulas in the Appendix of [8], supplemented with the ones in the Appendix of this paper.

A final ingredient to such an investigation is the following [6, 26]. At loop order \(\ell\), the possible (target) tensor structures \(T^{\mu\nu...}\) appearing in the sigma model beta functions must scale as \(T^{\mu\nu...}(\Lambda g) = \Lambda^{1-\ell} T^{\mu\nu...}(g)\) under global scalings of the background fields. In our case at one loop, order \(O(\alpha')\), we have \(\ell = 1\). These tensor structures must obviously also share the tensor properties of the beta functions. In our case the gauge beta function is gauge covariant Lie algebra valued, with one lower tensor index.

4 Duality, the Gauge Beta Function and Heterotic Anomalies

Let us now start analyzing our main equations, (3.7) and (3.8), for the case of the heterotic sigma model, as described in section 2. As previously mentioned this model has chiral fermions that, when rotated, introduce potential anomalies into the theory. These anomalies need to be canceled if the dualization is to be consistent at the quantum level. However, our strategy in here is to see if we can get any information on this anomaly cancelation from our consistency conditions (3.7-8). So, we will set this question aside for a moment and directly ask: are the consistency conditions (3.7-8) verified by (3.3), (3.6)?

We choose to start with torsionless backgrounds. Such choice can be seen to extremely
simplify equation (3.8), as the metric is parameterized by:
\[ g_{\mu\nu} = \begin{pmatrix} a & 0 \\
0 & \bar{g}_{ij} \end{pmatrix}, \]  
(4.1)
and we take \( b_{\mu\nu} = 0 \). Therefore, there is also no torsion in the dual background \( \mathbb{B} \). In this simpler set up it shall be clearer how to deal with anomalies before addressing the case of torsionfull backgrounds (see section 5). All this said, equations (3.7-8) become,
\[ \bar{\beta}^{\hat{A}}_0 = \frac{1}{a} \beta^A_0 + \frac{1}{a^2} (\kappa - A_0) \bar{\beta}^g_{00}, \]  
(4.2)
\[ \bar{\beta}^{\hat{A}}_i = \bar{\beta}^A_i. \]  
(4.3)

Now, use the Kaluza-Klein tensor decomposition of (3.3), (3.6), under (4.1), and compute \( \bar{\beta}^{\hat{A}}_0 \) and \( \bar{\beta}^{\hat{A}}_i \) (see the Appendix). By this we mean the following. One should start with (3.3), (3.6), and decompose it according to the parameterization (4.1). We will obtain expressions for \( \bar{\beta}^{\hat{A}}_0 \) and \( \bar{\beta}^{\hat{A}}_i \). Then, dualize these two expressions by dualizing the fields according to the rules (2.7-9) and (2.11). This yields expressions for \( \bar{\beta}^{\hat{A}}_0 \) and \( \bar{\beta}^{\hat{A}}_i \). Finally, one should manipulate the obtained expressions so that the result looks as much as possible as a “covariant vector transformation” (1.5). Hopefully one would obtain (4.2-3), if the gauge beta functions are to satisfy the consistency conditions. However, the result obtained is:
\[ \bar{\beta}^{\hat{A}}_0 = \frac{1}{a} \beta^A_0 + \frac{1}{a^2} (\kappa - A_0) (-\bar{\beta}^g_{00}), \]  
(4.4)
\[ \bar{\beta}^{\hat{A}}_i = \bar{\beta}^A_i. \]  
(4.5)

The first thing we observe is that even though (4.5) is correct as we compare it to (4.3), (4.4) is not as we compare it to (4.2). There is an extra minus sign that should not be there. Could anything be wrong? A possibility that comes to mind is that nothing is wrong, and indeed (4.4) and (4.5) are the correct consistency conditions, implying that the duality transformations were incorrect to start with. In that case the duality transformation (2.8) would need to be modified in order to yield the correct consistency condition upon differentiation. Let us regard this consistency condition (4.4) as a differential relation: a one-form \( \bar{\beta}^{\hat{A}}_0 \) which is expressed in the one-form coordinate basis of a “2-manifold” with local coordinates \( \{ A_0, a \} \). But then, as,
\[ \frac{\partial}{\partial a} [\frac{1}{a}] = -\frac{1}{a^2} \neq \frac{1}{a^2} = \frac{\partial}{\partial A_0} [-\frac{1}{a^2} (\kappa - A_0)], \]  
(4.6)
we see that the consistency condition (4.4) is not integrable. Therefore we cannot modify the duality transformation rules.
Let us look at this situation from another perspective. We can make the consistency conditions (4.4-5) match (4.2-3) if we realize that what (4.4) is saying is that, in order for duality to survive as a quantum symmetry of the heterotic sigma model, we need to have,

\[(\kappa - A_0) \bar{\beta}_{00}^g = 0.\] (4.7)

We shall see that this is just the requirement of anomaly cancelation, in a somewhat disguised form – it is the way duality finds to say that these anomalies must be canceled, if the dualization is to be consistent at the quantum level.

As was mentioned before, equations (2.8-9) were obtained using classical manipulations alone. In general, however, there will be anomalies and in this case the original theory and its dual will not be equivalent. If we want the two theories to be equivalent one must find the required conditions on the target fields that make these anomalies cancel. The simplest way to do so is to assume that the spin and gauge connections match in the original theory, i.e., \(\omega = A\). Under this assumption, the duality transformation then guarantees that in the dual theory spin and gauge connections also match, \(\bar{\omega} = \bar{A}\). In the following we choose to cancel the anomalies according to such prescription.

There are two outcomes of such choice. The first one is that if the original theory is conformally invariant to \(O(\alpha')\), so is the dual theory. For the sigma model this means that flowing to a fixed point will be equivalent to dual flowing to the dual fixed point (observe that the duality operation (1.1) does map fixed points to fixed points). The second is that we are now required to have \(\mu = \Omega\), where we define:

\[\Omega_{\mu\nu} \equiv \frac{1}{2}(\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu),\] (4.8)

with \(\xi\) the Killing vector generating the Abelian isometry and \(\nabla_\mu\) the metric covariant derivative. In particular for our adapted coordinates \(\xi_\mu = g_{\mu0}\), and as the affine connection is metric compatible, \(\Omega = 0\). But then,

\[\mu_{IJ} = (\kappa - A_0)_{IJ} = 0,\] (4.9)

and we are back to (4.7). Then, the consistency conditions are satisfied as long as the anomalies are canceled.

Putting together the information in (4.7) and (4.9), let us address a few questions. The first thing we notice is that \(\bar{\beta}_0^A = 0\) as \(\kappa = A_0\) (recall that in adapted coordinates \(\kappa\) satisfies (2.6), and so \(F_{0i} = 0\)), which is consistent with the fact that the target gauge transformation...
parameter is not renormalized. Then, the consistency conditions become,

\[ \bar{\beta}_0^A = 0 , \quad \bar{\beta}_i^A = \beta_i^A, \]  

(4.10)

stating that the gauge beta function is self-dual under (2.8-9). But so, by (4.4-5) with (4.7) satisfied, this proves that (3.6) explicitly satisfies the consistency conditions (4.10) – to the one loop, order \( O(\alpha') \), we are working to.

Given that the gauge field \( \bar{\beta} \)-function satisfies the consistency conditions, the question that follows is whether the scaling arguments mentioned in section 3 joined with the consistency conditions (4.10) are enough information to uniquely determine (3.3). This would mean that (4.10) is verified by, and only by, the correct gauge RG flows of the heterotic sigma model. Replacing (3.6) in (4.10) and using the duality transformations, we obtain the beta function constraint:

\[ \beta_i^A = \beta_i^A + \frac{1}{2} F_{ik}^i \partial_k \ln a. \]  

(4.11)

On the other hand, according to scaling arguments the possible tensor structures appearing in the one loop, order \( O(\alpha') \), gauge beta function are:

\[ \beta^A = c_1 D_\lambda F_{\lambda \mu} + c_2 H_{\mu}^{\lambda \rho} F_{\lambda \rho}, \]  

(4.12)

where the notation is as in (3.3). Dealing with torsionless backgrounds (4.1) we set \( c_2 = 0 \), and are left with \( c_1 \) alone. Inserting (4.12) in (4.11) then yields,

\[ (c_1 - \frac{1}{2}) F_{ik}^i \partial_k \ln a = 0, \]  

(4.13)

and as the background is general (though torsionless), we obtain \( c_1 = \frac{1}{2} \) which is the correct result (3.3). Therefore, our consistency conditions were able to uniquely determine the one loop gauge field beta function, in this particular case of vanishing torsion. We shall later see that the same situation happens when one deals with torsionfull backgrounds.

A final point to observe is that the proof of \( \mu_{IJ} = 0 \) through (4.7) (and so, also the proof of validity of the consistency conditions) is telling us that only if the sigma model is consistent at the quantum level (no anomalies) can the duality symmetry be consistent at the quantum level (by having the consistency conditions verified). Still, one could argue that strictly speaking (4.7) requires either \( \mu_{IJ} = 0 \) or \( \bar{\beta}_0^g = 0 \). But we also need to cancel all anomalies in order to have an RG flow. So, if one wants to flow away from the fixed point along all directions in the parameter space, one needs to cancel the anomalies in such a way that \( \mu_{IJ} = 0 \) in the adapted coordinates to the Abelian isometry. Otherwise, if we were to choose an anomaly cancelation
procedure yielding non-vanishing $\mu_{IJ}$, it would seem that in order to preserve $T$-duality at the quantum level away from criticality, expression (4.7) would require that one could only flow away from the fixed point along specific regions of the parameter space (i.e., regions with $\beta_{00}^g = 0$). As we shall see next when we deal with torsionfull backgrounds, this is actually not a good option: the only reasonable choice one can make is $\mu_{IJ} = 0$.

5 Torsionfull Backgrounds

To complete our analysis, we are left with the inclusion of torsion to the previous results. We shall see that even though the calculations are rather involved, the results are basically the same. Let us consider the same situation as in the last section, with the added flavor of torsion. As in [4, 8], we decompose the generic metric $g_{\mu \nu}$ as:

$$g_{\mu \nu} = \left( \begin{array}{cc} a & a v_i \\ a v_i & \bar{g}_{ij} + a v_i v_j \end{array} \right),$$

so that $g_{00} = a$, $g_{0i} = a v_i$, and $g_{ij} = \bar{g}_{ij} + a v_i v_j$. The components of the antisymmetric tensor are written as $b_{0i} \equiv w_i$ and $b_{ij}$. We will also find convenient to define the following quantities, $a_i \equiv \partial_i \ln a$, $f_{ij} \equiv \partial_i v_j - \partial_j v_i$, and $G_{ij} \equiv \partial_i w_j - \partial_j w_i$. From (2.7) one finds that in terms of the mentioned decomposition, the dual metric and antisymmetric tensor are given by the substitutions $a \rightarrow 1/a$, $v_i \leftrightarrow w_i$, and $\tilde{b}_{ij} = b_{ij} + w_i v_j - w_j v_i$.

With all these definitions at hand, we proceed with the Kaluza-Klein decomposition of (3.3), (3.6), and compute $\beta_A^0$ (see the Appendix). From the discussion in section 4 it should be clear what we mean by this, and which are the several steps required to carry out such calculation. Again, one hopes to find (3.7) if the gauge beta function is to satisfy the consistency conditions in this torsionfull case. Yet again, this does not happen. Instead we obtain,

$$\tilde{\beta}_0^A = \frac{1}{a} \tilde{\beta}_0^A + \frac{1}{a^2} (\kappa - A_0) (-\beta_{00}^g) +$$

$$+ \frac{1}{2} (\kappa - A_0) (f_{ij} + \frac{1}{a} G^{ij}) f_{ij} + (f_{ij} + \frac{1}{a} G^{ij}) v_i F_{j0} + \frac{1}{a} v^i [A_0, F_{i0}].$$

(5.2)

At first this looks like a complicated result. However, we already have the experience from the torsionless case, and that should be enough information to guide our way. Indeed, recall the discussion on anomaly cancelation from section 4, and proceed to cancel the anomalies according to $\mu_{IJ} = 0$. Then one has $\kappa = A_0$, and as $\kappa$ satisfies (2.6) in these adapted coordinates we are working in, we also have $F_{i0} = 0$. Looking again at (5.2), one sees that the anomaly
cancelation condition – just like in the torsionless case – makes (5.2) match the consistency condition (3.7). Moreover, we also see from (5.2) that, unless we are to severely restrict the background fields, the only choice one can make in order to have the consistency conditions verified is to cancel the anomalies through $\mu_{IJ} = (\kappa - A_0)_{IJ} = 0$. Finally, observe that as the target gauge transformation parameter does not get renormalized, and $\kappa = A_0$, we will have in adapted coordinates $\bar{\beta}_0^A = 0$.

One is now left with the analysis of $\bar{\beta}_1^A$. Making use of all that has been said in the last paragraph this turns out to be a reasonable calculation as the consistency conditions (3.7-8) have once again become,

$$\bar{\beta}_0^A = 0, \quad \bar{\beta}_1^A = \bar{\beta}_i^A,$$

the same as (4.10). Computing $\bar{\beta}_1^A$ by the usual procedure (also see the Appendix), one then finds that it indeed satisfies the consistency conditions, modulo gauge transformations. This is reminiscent of the fact that the $\beta$-functions only satisfy the consistency conditions modulo target reparameterizations, as they are not invariant under such transformations. In here, the $\bar{\beta}$-functions themselves are not gauge invariant but gauge covariant. In particular, we can choose a gauge where the consistency conditions are explicitly verified, the gauge $A_0 = 0$.

So the gauge field $\bar{\beta}$-function satisfies the consistency conditions in the torsionfull case as well as it does in the torsionless case. One final question remains: are these consistency conditions enough information to compute the coefficients $c_1$ and $c_2$ in (4.12)? The constraint these conditions impose on the beta function is obviously the same as (4.11). So, when we insert (4.12) in (4.11) we obtain on one hand, (4.13). This is to be expected and allows us to determine $c_1 = \frac{1}{2}$. On the other hand we get the new relation,

$$\frac{1}{2}(c_1 - 2c_2)(av_i + w_i)f^{jk}F_{jk} = 0,$$

and as the background is general, we obtain $c_2 = \frac{1}{4}$ which is the correct result (3.3). Therefore, our consistency conditions were able to uniquely determine the one loop gauge field beta function. Thus, the consistency conditions (5.3) are verified by, and only by, the correct RG flows of the heterotic sigma model. In other words, classical target space duality symmetry survives as a valid quantum symmetry of the heterotic sigma model.
6 Conclusions

We have studied in this paper the consistency between RG flows and $T$-duality in the $d = 2$ heterotic sigma model. The basic statement $[T, R] = 0$ that had been previously studied in bosonic sigma models was shown to keep its full validity in this new situation, with the added bonus of giving us extra information on how one should cancel the anomalies (arising from chiral fermion rotations) of the heterotic sigma model. Moreover, contrary to previously considered cases \cite{7, 8, 9}, the requirement $[T, R] = 0$ enabled us to uniquely determine the (gauge field) beta function at one loop order, without any overall global constant left to be determined.

Having considered the cases of closed bosonic, heterotic and (to a certain extent) Type II strings/sigma models, a question that comes to mind is the following. What happens in the open string case? In the open string case, the duality transformations are \cite{32, 33},

\begin{align}
\tilde{A}_0 &= 0, & \tilde{A}_i &= A_i.
\end{align}

The consistency conditions associated to (6.1) are,

\begin{align}
\bar{\beta}^A_0 &= 0, & \bar{\beta}^A_i &= \bar{\beta}^A_i,
\end{align}

the same as (4.10). Again, using scaling arguments the only possible form of the gauge field beta function is (4.12). If actually the Weyl anomaly for this situation is the same as in (3.6), we conclude that also in here $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{4}$. Then, by the same line of arguments as in section 5, we also conclude that for the open string the statement $[T, R] = 0$ is true and determines the beta function exactly, ensuring that duality is a quantum symmetry of the sigma model.

One last sigma model to mention is a truncated version of the heterotic sigma model \cite{24}, where one gets rid of the $\lambda^\mu$ fermions in the Lagrangian (2.4). The consequences of such truncation are the loss of fermionic partners for the left moving bosons (thus destroying the (1,0) supersymmetry), and the fact that one no longer needs to rotate the $\psi^I$ fermions in the dualization procedure (thus removing the $\kappa$ parameter from expressions (2.8-9) and (3.7-8)). Considering the simpler case of torsionless backgrounds, one finds that the known $\bar{\beta}$-functions (3.6) satisfy the consistency conditions, modulo gauge transformations. Choosing a gauge where these conditions are explicitly verified ($A_0 = 0$), and following the standard dualization procedure \cite{24} one obtains that the gauge fixed duality transformations are the same as (6.1), and so the consistency conditions are the same as (4.10) or (6.2). Then, by the familiar line of arguments, $[T, R] = 0$ is true and determines the beta function exactly ensuring that duality is a quantum symmetry of this sigma model.
Such a basic statement \([T, R] = 0\) has now been shown to be alive and well in a wide variety of situations, possibly validating the claim in [8, 10] that it should be a more fundamental feature of the models in question than the invariance of the string background effective action.

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A Kaluza-Klein Tensor Decompositions

We list below all quantities relevant for our computations, as they were cited upon during the text. We shall consider in here the general torsionfull metric parameterization, as was done in section 5 (see expression (5.1) and the definitions that follow it in the text). To use these decompositions in section 4, all one needs to do is to set \(v_i = w_i = b_{ij} = f_{ij} = G_{ij} = 0\) in the following. The tensor decompositions are as follows, for both the gauge beta function, (3.3), (3.6), and the (00)-component of the metric beta function, (3.1), (3.4) (where barred quantities will refer to the metric \(\bar{g}_{ij}\)). Observe that expressions (A.2), (A.4), (A.6) and (A.8) below have \((\kappa - A_0)_{I,J} = 0\).

1. \(\nabla^\lambda F_{\lambda\mu}\):

\[
\nabla^\lambda F_{\lambda 0} = \bar{g}^{ij} (\partial_i F_{j0} - \bar{\Gamma}^k_{ij} F_{k0}) - \frac{1}{2} a_i F^i_0 - \frac{1}{2} a f^{ij} F_{ij} - av_i f^{ij} F_{j0}, \tag{A.1}
\]

\[
\nabla^\lambda F_{\lambda i} = \bar{g}^{jk} (\partial_j F_{ki} - \bar{\Gamma}^\ell_{jk} F_{\ell i} - \bar{\Gamma}^\ell_{ji} F_{k\ell}) + \frac{1}{2} a_k F^k_i - \frac{1}{2} a v_i f^{jk} F_{jk}, \tag{A.2}
\]

2. \([A^\lambda, F_{\lambda\mu}]\):

\[
[A^\lambda, F_{\lambda 0}] = -v^j [A_0, F_{j0}] + [A^i, F_{i0}], \tag{A.3}
\]

\[
[A^\lambda, F_{\lambda i}] = -v^j [A_0, F_{ji}] + [A^j, F_{ji}], \tag{A.4}
\]

3. \(H_{\mu} ^{\lambda\rho} F_{\lambda\rho}\):

\[
H_0^{\lambda\rho} F_{\lambda\rho} = 2 v_i G^{ij} F_{0j} - G^{ij} F_{ij}, \tag{A.5}
\]

\[
H_i ^{\lambda\rho} F_{\lambda\rho} = 2 v_j G_{ik} F^{kj} + H_{ijk} F^{jk}, \tag{A.6}
\]

4. \(F_{\mu} ^{\lambda} \partial_\lambda \phi\):

\[
F_0 ^{\lambda} \partial_\lambda \phi = F_0 ^{i} \partial_i \phi, \tag{A.7}
\]
\[ F^i_\lambda \partial_\lambda \phi = F^j_\lambda \partial_\lambda \phi, \quad (A.8) \]

5. \( \tilde{\beta}^0_{\mu \nu} \):

\[
\tilde{\beta}^0_{00} = -\frac{a}{2} [\nabla^i a_i + \frac{1}{2} a_i a^i - \frac{1}{2} f_{ij} f^{ij}] - \frac{1}{4} G_{ij} G^{ij} + a a^i \partial_i \phi. \quad (A.9)
\]

Finally one should also include, for the sake of completeness, the duality transformations acting on the gauge field strength tensor. These are derived directly from expressions (2.8-9), with the following results:

6. \((0i)\)-component:

\[
\tilde{F}_{0i} = \frac{1}{a} (F_{0i} - (\kappa - A_0) a_i), \quad (A.10)
\]

7. \((ij)\)-component:

\[
\tilde{F}_{ij} = F_{ij} - (v_j + \frac{1}{a} w_j) F_{0i} + (v_i + \frac{1}{a} w_i) F_{0j} - (\kappa - A_0) (f_{ij} + \frac{1}{a} G_{ij} + \frac{1}{a} (w_i a_j - w_j a_i)). \quad (A.11)
\]

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