THE LOCAL THETA CORRESPONDENCE AND THE LOCAL GAN–GROSS–PRASAD CONJECTURE FOR THE SYMPLECTIC-METAPLECTIC CASE

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ABSTRACT. We prove the local Gan–Gross–Prasad conjecture for the symplectic-metaplectic case under some assumptions. This is the last case of the local Gan–Gross–Prasad conjectures. We also prove two of Prasad’s conjectures on the local theta correspondence in the almost equal rank case.

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1. Introduction

In [20] and [21], Gross and Prasad studied a restriction problem for special orthogonal groups over a local field and gave a precise conjecture. They and Gan ([10] and [11]) extended this conjecture to classical groups, which are called the local Gan–Gross–Prasad conjectures (GGP). These conjectures consist of four cases; the orthogonal, hermitian, the symplectic-metaplectic and skew-hermitian cases. The orthogonal, hermitian and skew-hermitian cases were proven by Waldspurger [55], [56], [57], [58] and Mœglin–Waldspurger [42], Bezuard-Plessis [6], [7], [8] and Gan–Ichino [13], respectively.

In this paper, we consider the orthogonal and the symplectic-metaplectic cases. Let $F$ be a non-archimedean local field of characteristic zero. We denote by $(V_{m+1}, \langle \cdot, \cdot \rangle_{V_{m+1}})$ (resp. $(W_{2n}, \langle \cdot, \cdot \rangle_{W_{2n}})$) an orthogonal space of dimension $m + 1$ (resp. a symplectic space of dimension $2n$). Let $V_m \subset V_{m+1}$ be a non-degenerate subspace of codimension 1, so that we have a natural inclusion $SO(V_m) \hookrightarrow SO(V_{m+1})$. We let $\widetilde{Sp}(W_{2n})$ be the metaplectic group, i.e., the unique non-split central extension of $Sp(W_{2n})$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{Sp}(W_{2n}) \longrightarrow Sp(W_{2n}) \longrightarrow 1.$$ 

In particular, we have a diagonal embedding

$$\Delta: SO(V_m) \hookrightarrow SO(V_m) \times SO(V_{m+1})$$

and a natural map

$$\Delta: \widetilde{Sp}(W_{2n}) \rightarrow \widetilde{Sp}(W_{2n}) \times Sp(W_{2n}),$$

where the first factor is the identity map and the second factor is the projection map.
In the orthogonal case, for an irreducible smooth representation $\sigma$ of $\text{SO}(V_m) \times \text{SO}(V_{m+1})$, one is interested in determining
\[
\text{dim}_C(\text{Hom}_{\Delta \text{SO}(V_m)}(\sigma, C)).
\]
We shall call this the Bessel case (B) of the GGP conjecture. On the other hand, we fix a non-trivial additive character $\psi$ of $F$, and let $\omega_\psi$ be a Weil representation of $\widehat{\text{Sp}(W_{2n})}$, which is given by the Heisenberg group associated to the symplectic space $(W_{2n}, 2\langle \cdot, \cdot \rangle_{W_{2n}})$. In the symplectic-metaplectic case, for an irreducible genuine smooth representation $\pi$ of $\widehat{\text{Sp}(W_{2n})} \times \text{Sp}(W_{2n})$, one is interested in determining
\[
\text{dim}_C(\text{Hom}_{\Delta \text{Sp}(W_{2n})}(\pi \otimes \overline{\omega_\psi}, C)).
\]
We shall call this the Fourier–Jacobi case (FJ) of the GGP conjecture.

By results of Aizenbud–Gourevitch–Rallis–Schiffmann [1] and Sun [51], it is known that the above Hom spaces have dimension at most 1. Hence, the main problems are to determine when the Hom spaces are nonzero. In [10], an answer for these problems is formulated in the framework of the local Langlands correspondence in a form proposed by Vogan [53], which treats the irreducible representations of all pure inner forms simultaneously.

More precisely, a pure inner form of $\text{SO}(V_m)$ is simply a group of the form $\text{SO}(V'_m)$, where $V'_m$ is an orthogonal space with the same dimension and discriminant as $V_m$. Hence, a pure inner form of $\text{SO}(V_m) \times \text{SO}(V_{m+1})$ is a group of the form
\[
\text{SO}(V'_m) \times \text{SO}(V'_{m+1}).
\]
We say that this pure inner form is relevant if $V'_m \subseteq V'_{m+1}$ and $V'_{m+1}/V'_m \cong V_{m+1}/V_m$ as orthogonal spaces. In this case, we have a diagonal embedding
\[
\Delta : \text{SO}(V'_m) \hookrightarrow \text{SO}(V'_m) \times \text{SO}(V'_{m+1}).
\]
On the other hand, $\text{Sp}(W_{2n}) \times \text{Sp}(W_{2n})$ has no non-trivial pure inner forms.

Let $G = \text{SO}(V_m) \times \text{SO}(V_{m+1})$ or $G = \widehat{\text{Sp}(W_{2n})} \times \text{Sp}(W_{2n})$. For an $L$-parameter $\phi$ of $G$, we should obtain a Vogan $L$-packet $\Pi_\phi$ consisting of certain irreducible smooth representations of $G$ and its (not necessarily relevant) pure inner forms $G'$. Here, to parametrize irreducible genuine representations of $\widehat{\text{Sp}(W_{2n})}$, we use the theta correspondence for $(\widehat{\text{Sp}(W_{2n})}, \text{O}(V_{2n+1}))$. Hence, by an $L$-parameter of $\widehat{\text{Sp}(W_{2n})}$, we mean one of $\text{SO}(V_{2n+1})$, i.e., a symplectic representation
\[
\phi_M : \text{WD}_F \to \text{Sp}(M)
\]
of the Weil–Deligne group $\text{WD}_F$ of $F$ with $\text{dim}_C(M) = 2n$. Moreover, for a fixed Whittaker datum $w = (B, \mu)$ of $G$, which is a conjugacy class of a pair of an $F$-rational Borel subgroup $B = TU$ of $G$ and a generic character $\mu$ of the unipotent radical $U(F)$ of $B(F)$, there exists a natural bijection
\[
\iota_w : \Pi_\phi \to \text{Irr}(\pi_0(S_\phi)),
\]
where we put $S_\phi = \text{Cent}(\text{Im}(\phi), G)$. For $\eta \in \text{Irr}(\pi_0(S_\phi))$, we write $\pi(\eta) = \iota_w^{-1}(\eta)$.

The local Langlands conjecture has been established for quasi-split symplectic and special orthogonal groups by Arthur [2] under an assumption on the stabilization of the twisted trace formula. For this assumption, see also the series of papers [59, 60, 61, 62, 63, 64, 65, 66] and [44]. In [15], Gan–Savin gave a parametrization (depending on the choice of a non-trivial additive character $\psi$ of $F$) of irreducible genuine representations of $\widehat{\text{Sp}(W_{2n})}$ by using the theta correspondence and the local Langlands correspondence for $\text{SO}(V_{2n+1})$.

The GGP conjectures can be roughly stated as follows:

**Conjecture 1.1 (B).** Let $G = \text{SO}(V_m) \times \text{SO}(V_{m+1})$.

1. Given a generic $L$-parameter $\phi$ of $G$, there exists a unique representation $\pi(\eta) \in \Pi_\phi$ such that $\pi(\eta)$ is a representation of a relevant pure inner form $G' = \text{SO}(V'_m) \times \text{SO}(V'_{m+1})$ and such that $\text{Hom}_{\Delta \text{SO}(V_m)}(\pi(\eta), C) \neq 0$;
2. There is a precise recipe for the unique character $\eta$.

**Conjecture 1.2 (FJ).** Let $G = \text{Sp}(W_{2n}) \times \text{Sp}(W_{2n})$ and fix a non-trivial additive character $\psi$ of $F$. 
Moreover, we assume that the uniqueness of generic representations in an L-
Hence we consider only the basic case in this paper.

In fact, there are GGP conjectures in general codimension cases (Conjecture 17.1 and 17.3 in [10]). However, [10] Theorem 19.1] says that the general codimension cases follow from the basic case (B) and (FJ). Hence we consider only the basic case in this paper.

The purpose of this paper is to establish the symplectic-metaplectic case (FJ), which is the last case of the local conjectures, as well as two conjectures of D. Prasad concerning local theta correspondence in almost equal rank cases, under the following assumptions:

(1) Given a generic L-parameter φ of G, there exists a unique representation π(η) ∈ Πφ such that \( \text{Hom}_{\Delta \overline{\text{Sp}(2n)}}(π(η)) \otimes \overline{\omega(\eta)}, \mathbb{C}) \neq 0; \)
(2) There is a precise recipe for the unique character η.

In §4 we will recall the recipes for the unique characters. Waldspurger [55, 56, 57, 58] shows the orthogonal case for tempered L-parameters, and Mœglin–Waldspurger [42] extends this result for generic L-parameters.

In this paper, we show the following:

Corollary 1.4. Namely, we can deduce the following corollary from Theorem 1.3.

In this paper, we show the following:

Theorem 1.3. Assume (LLC), (AM), (UGP), (GPR) and (B). Let φ and \( \tilde{\phi} \) be generic L-parameters of \( \text{Sp}(W_{2n}) \) and \( \tilde{\text{Sp}}(W_{2n}) \), respectively.

1. If \( \phi \) and \( \tilde{\phi} \) are tempered, then (FJ) is true for \( \phi \times \tilde{\phi} \).
2. If we further assume (NQ), (IS) and that there exists a quadratic character \( \chi \in F^\times \) such that the local \( L \)-function \( L(s, \phi \otimes \chi) \) is regular at \( s = 1 \), then (FJ) is true for \( \phi \times \tilde{\phi} \).

The argument in the proof of [10] Theorem 19.1] works when we restrict to the above cases respectively. Namely, we can deduce the following corollary from Theorem 1.3.

Corollary 1.4. Under the same assumptions as Theorem 1.3 Conjecture 17.1 and 17.3 in [10] are true.

We have some remarks on these assumptions. For quasi-split symplectic and special orthogonal groups, (LLC) and (AM) have been established by Arthur [2] under some assumption on the stabilization of the twisted trace formula. For this assumption, see also the series of papers [60, 61, 62, 63, 64, 65, 66] and [44]. One expects (UGP) in general, and it was established for split \( \text{SO}(V_{2n+1}) \) by [25]. However, (UGP) has not been proven for \( \text{Sp}(W_{2n}) \) or \( \text{SO}(V_{2n}) \). For (GPR) for split symplectic and special orthogonal groups, there are results in [57, 26, 4] and [24]. The assumption (NQ) is serious. For quasi-split orthogonal groups, the properties analogous to (NQ), which contain Arthur’s multiplicity formula, were proven by Arthur [2]. For \( \tilde{\text{Sp}}(W_{2n}) \) and \( \text{SO}(V_m) \), the property analogous to (IS) was established by Mœglin–Waldspurger [42].

We describe the main idea of the proof of Theorem 1.3. The method is analogous to the work of Gan–Ichino [13]. As in this paper, by the local theta correspondence, the Bessel and the Fourier–Jacobi cases of GGP conjectures are related. More precisely, there exists a see-saw diagram
with \( \text{disc}(V_1) = -1 \), and the associated see-saw identity holds:

\[
\text{Hom}_{\text{Sp}(W_{2n})}((\Theta_\psi,V_{2n+1},W_{2n})_\tau \otimes \omega_{-\psi}^\ast,\pi) \cong \text{Hom}_{O(V_{2n+1})}(\Theta_\psi,V_{2n+2},W_{2n})_\tau \otimes \omega_{-\psi}^\ast,\pi)
\]

for irreducible smooth representations \( \pi \) of \( \text{Sp}(W_{2n}) \) and \( \tau \) of \( O(V_{2n+1}) \). The left-hand side of the see-saw identity is related to the Fourier–Jacobi case (FJ), whereas the right-hand side is related to the Bessel case (B). Therefore, if we knew the local theta correspondence for \( (\text{Sp}(W_{2n}),O(V_{2n+2})) \) and \( (O(V_{2n+1}),\tilde{\text{Sp}}(W_{2n})) \) explicitly, then the see-saw identity would give the precise relation of (FJ) and (B).

More precisely, one considers the following statements:

(Θ) If \( \phi_\pi \) (resp. \( \phi_\tau \)) is a generic \( L \)-parameter of \( \text{Sp}(W_{2n}) \) (resp. \( O(V_{2n+1}) \)) and \( \pi \in \Pi_{\phi_\pi} \) (resp. \( \tau \in \Pi_{\phi_\tau} \)), then the big theta lift \( \Theta_\psi,V_{2n+2},W_{2n} \) \( (\tau) \) (resp. \( \Theta_\psi,V_{2n+1},W_{2n} \) \( (\tau) \)) is irreducible (if it is nonzero).

(Mp) If \( \tau \) \( SO(V_{2n+1}) \) has parameter \( (\phi_\pi,\eta_\pi) \) and \( \theta_\psi,V_{2n+1},W_{2n} \) \( (\tau) \) has parameter \( (\phi,\eta) \), then \( (\phi,\eta) \) can be described in terms of \( (\phi_\pi,\eta_\pi) \) explicitly.

(P1) Likewise, if \( \pi \) has parameter \( (\phi_\pi,\eta_\pi) \) and \( \sigma := \theta_\psi,V_{2n+2},W_{2n} \) \( (\pi) \) \( SO(V_{2n+2}) \) has parameter \( (\phi_\sigma,\eta_\sigma) \), then \( (\phi_\sigma,\eta_\sigma) \) can be described in terms of \( (\phi_\pi,\eta_\pi) \) explicitly.

(G) If \( \phi_\pi \) and \( \phi_\tau \) are generic parameters, then so are \( \phi_\sigma \) and \( \phi_\pi \).

Note that these statements are not true in general. See Proposition 1.5 and 1.6 below.

The statement (G) is needed to use the Bessel case (B). For \( \phi, \) by definition of generic parameter for \( \tilde{\text{Sp}}(W_{2n}) \), the statement (G) is true. (See 3.7) However, (G) is not always true for \( \phi_\pi \). The last assumption in Theorem 1.3 (2) implies (G) for \( \phi_\pi \). More precisely, we have the following:

**Proposition 1.5 (Lemma 5.1).** Suppose \( (GPR) \) for \( \text{Sp}(W_{2n}) \) and \( SO(V_{2n+2}) \). Let \( \phi_\pi \) be a generic \( L \)-parameter of \( \text{Sp}(W_{2n}) \) and \( \chi \) be the discriminant character of \( V_{2n+2} \). Then the \( L \)-parameter \( \phi_\sigma \) of \( SO(V_{2n+2}) \) given by \( \phi_\pi \) is generic if and only if the local \( L \)-function \( L(s,\phi_\pi \otimes \chi) \) is regular at \( s = 1 \).

In particular, if \( \phi_\pi \) is tempered, then \( \phi_\sigma \) is generic. Indeed, \( \phi_\sigma \) is also tempered.

In 10, D. Prasad has formulated a precise conjecture regarding (P1). He also has formulated one regarding (P2) as follows:

(P2) Let \( \tilde{\sigma} \) be an irreducible representation of \( O(V_{2n}) \). If an irreducible constituent of \( \sigma := \tilde{\sigma} \) \( SO(V_{2n}) \) has parameter \( (\phi_\sigma,\eta_\sigma) \) and \( \theta_\psi,V_{2n+1},W_{2n} \) \( (\tilde{\sigma}) \) has parameter \( (\phi_\pi,\eta_\pi) \), then \( (\phi_\pi,\eta_\pi) \) can be described in terms of \( (\phi_\sigma,\eta_\sigma) \) explicitly.

We shall also denote by (weak P1) (resp. (weak P2)) the part of Conjecture (P1) (resp. (P2)) concerning only the correspondence of \( L \)-parameters.

**Proposition 1.6.** We have the following:

1. (13 Theorem 8.1 (ii)] and [12 Proposition C.4 (i)] The statement (Θ) holds for tempered parameters.

2. (12 Proposition C.4 (ii)] + [12 p.40 Théorème (i)] If \( \phi_\sigma \) is generic, then (Θ) holds for \( \phi_\pi \).

3. ([13 Theorem 8.1 (iii)] The assumption (IS) implies (Θ) for \( \tilde{\sigma} \).

4. ([15 Corollary 1.2] and [12 Theorem C.5]) The statements (Mp), (weak P1) and (weak P2) hold.

Using the see-saw identity above and Proposition 1.6 we reduce Theorem 1.3 and (P2) to Conjecture (P1) and (B). In view of results of Waldspurger 55, 56, 57, 58 and Meglin–Waldspurger 72, it is enough to show the statement (P1), and our main result is the following theorem:

**Theorem 1.7.** Conjecture (P1) holds. Therefore we obtain Theorem 1.3 and (P2).

By [12 Proposition C.4], Conjecture (P1) for general parameters follows from that for tempered parameters. The proof of (P1) for tempered parameters consists four steps as follows:

1. We show that the cases when \( SO(V_{2n+2}) \) is not quasi-split can be reduced to those when \( SO(V_{2n+2}) \) is quasi-split.

2. We show that, in all but one case, the tempered but not discrete cases can be reduced to the discrete cases on smaller symplectic groups by using intertwining operators.

3. We show all but two discrete cases by a global argument.
(4) We show the three exceptional cases directly.

The assumption (NQ) is only used in the first step. To prove (P1), (P2) for quasi-split orthogonal groups and Theorem 1.3 (1), we do not need (NQ). In Theorem 1.3 (2), we need (NQ) only when \( L(s, \phi \otimes \chi) \) has a pole at \( s = 1 \) for any non-trivial quadratic character \( \chi \).

The exceptional cases include all cases of (P1) for \((\text{Sp}(W_2), \text{SO}(V))\). In the skew-hermitian case \([13]\), there are no exceptional cases. We explain why the exceptional cases occur. In the second step, the reason is that the group \( \text{SO}(V_2) \), which is associated to an orthogonal space \( V_2 \) with dimension 2, is abelian. This fact may make the notation and the situation more complicated. In the third step, the situation is more serious. For an irreducible (unitary) discrete series representation \( \pi \) of \( \text{Sp}(W_{2n}) \), we want to globalize \( \pi \) to an irreducible cuspidal automorphic representation \( \Pi = \otimes_v \Pi_v \) such that

- \( \Pi_v = \pi \) for some place \( v_0 \);
- \( \Pi_v \) is not discrete series for any \( v \neq v_0 \), so that (P1) is known for \( v \neq v_0 \);
- \( \Pi \) has a tempered \( \Lambda \)-parameter whose component group is equal to the local component group of the \( \lambda \)-parameter associated to \( \pi \);
- \( \Pi \) has a nonzero global theta lift to an orthogonal group which globalizes \( \text{O}(V_{2n+2}) \).

Then the desired result follows from Arthur’s multiplicity formula. To globalize \( \pi \), we use Shin’s result ([49, Theorem 5.13], which was proven using the trace formula). However, using this theorem, we must choose several finite places for bad places. The issue is whether the second condition is satisfied at the bad places, and in some two cases, it can not be achieved.

In the skew-hermitian case \([13]\), Gan and Ichino chose split places for such bad places. At split places, the local factor of a global unitary group becomes a general linear group, and (P1) for general linear groups is known. Hence, in the skew-hermitian case, there are no exceptional cases.

When \( \phi \) is a tempered \( L \)-parameter of \( \text{Sp}(W_{2n}) \) in the exceptional cases, we observe that the component group \( \pi_0(S_{\phi}) \) is “small”, and that almost all the elements in \( \Pi_{\phi} \) are \( \text{w} \)-generic for some Whittaker datum \( \text{w} \) of \( \text{Sp}(W_{2n}) \). When \( \text{SO}(V_{2n+2}) \) is quasi-split, we can show that for a Whittaker datum \( \text{w} \) of \( \text{Sp}(W_{2n}) \), there is a datum \( \text{w}' \) of \( \text{SO}(V_{2n+2}) \) such that if \( \pi \) is \( \text{w} \)-generic, then \( \Theta_{\psi, V, V_2, W_{2n}}(\pi)|_{\text{SO}(V_{2n+2})} \) is \( \text{w}' \)-generic. In the local Langlands correspondence, one expects that if \( \pi \) is \( \text{w} \)-generic, then \( t_{\text{w}}(\pi) \) is the trivial representation of the component group. Hence it is enough to study that \( t_{\text{w}} \circ t_{\text{w}}^{-1} \) for two Whittaker data \( \text{w} \) and \( \text{w}_0 \) of \( \text{Sp}(W_{2n}) \) or \( \text{SO}(V_{2n+2}) \). By Kaletha \([27]\), it is known for the local Langlands correspondence for quasi-split symplectic and special orthogonal groups. However, in the local Langlands correspondence proposed by Vogan, certain issues arise.

We explain these issues in more general setting. Let \( G \) be a quasi-split connected reductive group over a non-archimedean local field \( F \) of characteristic zero. A pure inner twist of \( G \) is a pair \((G', \psi, z)\), where \( G' \) is an algebraic group over \( F \), \( z \in Z^1(F, G) \) and

\[
\psi: G \rightarrow G'
\]

is an isomorphism over \( \overline{F} \) such that \( \psi^{-1} \circ \sigma(\psi) = \text{Ad}(z_\sigma) \) for any \( \sigma \in \text{Gal}(\overline{F}/F) \). We call such a \( G' \) a pure inner form of \( G \). In contrast to inner forms, \( G' \) may be also quasi-split over \( F \) even if the pure inner twist is non-trivial. For example, if \( V_{2n} \) is an orthogonal space with non-trivial discriminant, then \( \text{SO}(V_{2n}) \) has two pure inner forms both of which are quasi-split. Note that in this situation, \( G \) and \( G' \) are isomorphic over \( F \). Hence for an \( L \)-parameter \( \phi \) of \( G \), we may regard \( \phi \) as an \( L \)-parameter \( \phi' \) of \( G' \), and so that we should obtain two Vogan \( L \)-packets \( \Pi_{\phi} \) of \( G \) and \( \Pi_{\phi'} \) of \( G' \). These are subsets of the same set \( \coprod_{G_1} \text{Irr}(G_1(F)) \), where \( G_1 \) runs over pure inner forms of \( G \) (hence of \( G' \)). Do these two \( L \)-packets coincide? If so, for Whittaker data \( \text{w} \) and \( \text{w}' \) for \( G \) and \( G' \) respectively, what is the map

\[
\text{Irr}(\pi_0(S_{\phi})) \xrightarrow{\iota_{\phi}} \Pi_{\phi} \xrightarrow{\iota_{\phi}^{-1}} \text{Irr}(\pi_0(S_{\phi})).
\]

Note that Kaletha’s result is the case when \( G' = G \), but the problems we need to know are the case when \( G' \) is not necessarily the trivial pure inner form of \( G \). In Appendix A we will give an answer in a special case. As an application, we show (P1) in the three exceptional cases.
Acknowledgments. The author is grateful to Professor Atsushi Ichino for his helpful comments and suggestions. This work was supported by the Foundation for Research Fellowships of Japan Society for the Promotion of Science for Young Scientists (DC1) Grant 26-1322.

Notations. Let \( F \) be a non-archimedean local field with characteristic zero, \( \mathfrak{o}_F \) be the ring of integers of \( F \), \( \varpi \) be a uniformizer, \( q \) be the number of elements in the residue class field \( \mathfrak{o}_F/\varpi \mathfrak{o}_F \) and \( |.|_F \) be the normalized absolute value on \( F \) so that \( |\varpi|_F = q^{-1} \). We denote by \( \Gamma, W_F \) and \( WD_F = W_F \times SL_2(\mathbb{C}) \) the absolute Galois, Weil and Weil–Deligne groups of \( F \), respectively. Fix a non-trivial additive character \( \psi \) of \( F \). For \( c \in F^\times \), we define an additive character \( \psi_c \) or \( c\psi \) of \( F \) by
\[
\psi_c(x) = c\psi(x) = \psi(cx).
\]
Moreover we set \( \chi_c = (\cdot, c) \) to be the quadratic character of \( F^\times \) associated to \( c \in F^\times/F^\times_2 \). Here, \((\cdot, \cdot)\) is the quadratic Hilbert symbol of \( F \). For a totally disconnected locally compact group \( G \), we denote the set of equivalence classes of irreducible smooth representations of \( G \) by \( \text{Irr}(G) \). If \( G \) is the group of \( F \)-points of a linear algebraic group over \( F \), we denote by \( \text{Irr}_{\text{temp}}(G) \) the subset of \( \text{Irr}(G) \) of classes of irreducible tempered representations. For a topological group \( H \), we define the component group of \( H \) by \( \pi_0(H) = H/H^0 \), where \( H^0 \) is the identity component of \( H \). The Pontryagin dual (i.e., the character group) of a finite abelian group \( A \) is denoted by \( A^\vee \) or \( \hat{A} \).

2. Preliminaries

In this section, we recall some notions.

2.1. Whittaker data. Let \( G \) be a quasi-split connected reductive group over \( F \) and \( B = TU \) be an \( F \)-rational Borel subgroup of \( G \), where \( T \) is a maximal \( F \)-torus and \( U \) is the unipotent radical of \( B \). We denote the center of \( G \) by \( Z \). We call a character \( \mu \) of \( U(F) \) generic if its stabilizer in \( T(F) \) is equal to \( Z(F) \). A Whittaker datum of \( G \) is a conjugacy class of a pair \( \mathfrak{w} = (B, \mu) \), where \( B = TU \) is an \( F \)-rational Borel subgroup of \( G \) and \( \mu \) is a generic character of \( U(F) \). We say that \( \pi \in \text{Irr}(G(F)) \) is \( \mathfrak{w} \)-generic if
\[
\text{Hom}_{U(F)}(\pi, \mu) \neq 0.
\]
If \( \pi \) is \( \mathfrak{w} \)-generic for some Whittaker datum \( \mathfrak{w} \) of \( G \), then we say that \( \pi \) is generic.

2.2. Orthogonal spaces. Let \( V = V_m \) be a vector space of dimension \( m \) over \( F \) and
\[
(\cdot, \cdot)_V : V \times V \to F
\]
be a non-degenerate symmetric bilinear form. We take a basis \( \{e_1, \ldots, e_m\} \) of \( V \), and define the discriminant of an orthogonal space \( V \) by
\[
\text{disc}(V) = \text{disc}(V, (\cdot, \cdot)_V) = 2^{-m}(-1)^{\frac{(m-1)(m-3)}{2}} \det(((e_i, e_j)_V)_{i,j}) \mod F^\times_2 \subset F^\times/F^\times_2.
\]
Note that this differs from disc\((V, q)\) as in [10, p. 41], which is not used in this paper. Let \( \chi_V = (\cdot, \text{disc}(V)) \) be the character of \( F^\times \) associated with \( F(\sqrt{\text{disc}(V)})/F \).

We denote the anisotropic kernel of \( V \) by \( V_{\text{an}} \). The special orthogonal group \( SO(V) \) is quasi-split if and only if \( \dim(V_{\text{an}}) \leq 2 \). In this case, we choose a subset \( \{v_1, v_2^*\} \) of \( V \) such that
\[
(v_i, v_j)_V = (v_i^*, v_j^*)_V = 0, \quad (v_i, v_j)_V = \delta_{i,j},
\]
where \( n \) is the integer such that \( m = 2n + 1 \) or \( m = 2n + 2 \). We set
\[
X_k = Fv_1 + \cdots + Fv_k \quad \text{and} \quad X_k^* = Fv_1^* + \cdots + Fv_k^*
\]
for \( 1 \leq k \leq n \). Let \( V_1 \) be the orthogonal complement of \( X_n \oplus X_n^* \) in \( V \), so that \( V_1 \) is an orthogonal space of dimension \( 1 \) or \( 2 \). We denote by \( B = TU \) the \( F \)-rational Borel subgroup of \( SO(V) \) stabilizing the complete flag
\[
0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \ldots , v_n \rangle = X_n,
\]
where \( T \) is the \( F \)-rational torus stabilizing the lines \( Fv_i \) for \( i = 1, \ldots , n \).
If \( m = 2n + 1 \), then there is a unique \( T \)-orbit of generic characters of \( U \). Let \( V_1 = F e \). We choose a generic character \( \mu \) of \( U \) such that
\[
\mu(u) = \psi(\langle uw_2, v_i^* \rangle_V + \cdots + \langle uw_n, v_{n-1}^* \rangle_V + \langle uc, v_n^* \rangle_V)
\]
for \( u \in U \). We define a Whittaker datum \( w \) of \( \text{SO}(V) \) by \( w = (B, \mu) \).

Next, we suppose that \( m = 2n + 2 \) and \( \text{SO}(V) \) is quasi-split. Then there exist \( c, d \in F^\times \) such that
\[
V_1 \cong F[X]/(X^2 - d)
\]
as vector spaces, and the pairing of \( V_1 \) is given by
\[
(\alpha, \beta) \mapsto (\alpha, \beta)_{V_1} := c \cdot \text{tr}(\alpha \overline{\beta}),
\]
where \( \beta \mapsto \overline{\beta} \) is the involution on \( F[X]/(X^2 - d) \) induced by \( a + bX \mapsto a - bX \). In this case, we say that \( V \) is type \((d, c)\). Note that \( \text{disc}(V) = d \mod F^\times \). If \( V \) is type \((d, c)\), then we can take \( e, e' \in V_1 \) such that
\[
\langle e, e \rangle_V = 2c, \quad \langle e', e' \rangle_V = -2cd \quad \text{and} \quad \langle e, e' \rangle_V = 0.
\]
Then we define a generic character \( \mu_c \) of \( U \) by
\[
\mu_c(u) = \psi(\langle uw_2, v_i^* \rangle_V + \cdots + \langle uw_n, v_{n-1}^* \rangle_V + \langle uc, v_n^* \rangle_V).
\]

By [10 §12], the map \( c' \mapsto \mu_{c'} \) gives a bijection (not depending on \( \psi \))
\[
cN_{E/F}(E^\times)/F^\times 2 \to \{T\text{-orbits of generic characters of } U \},
\]
where \( E = F(\sqrt{d}) \). Note that \( V \) is both type \((d, c)\) and \((d, c')\) if and only if \( c' \in cN_{E/F}(E^\times) \). We define a Whittaker datum of \( \text{SO}(V) \) by \( w_c = (B, \mu_c) \). Note that \( w_c \) does not depend on the choice of \( \psi \).

### 2.3. Symplectic spaces

Let \( W = W_{2n} \) be a vector space of dimension \( 2n \) over \( F \) and
\[
\langle \cdot, \cdot \rangle_W : W \times W \to F
\]
be a non-degenerate symplectic form. The symplectic group \( \text{Sp}(W) \) is a split algebraic group over \( F \). We choose a basis \( \{w_i, w_i^* \}_{i = 1, \ldots, n} \) of \( W \) such that
\[
\langle w_i, w_j \rangle_W = \langle w_i^*, w_j^* \rangle_W = 0, \quad \langle w_i, w_j^* \rangle_W = \delta_{i,j}.
\]

We set
\[
Y_k = Fw_1 + \cdots + Fw_k \quad \text{and} \quad Y_k^* = Fw_1^* + \cdots + Fw_k^*.
\]

Let \( B' = T'U' \) be the \( F \)-rational Borel subgroup of \( \text{Sp}(W) \) stabilizing the complete flag
\[
0 \subset \langle w_1 \rangle \subset \langle w_1, w_2 \rangle \subset \cdots \subset \langle w_1, \ldots, w_n \rangle = Y_n,
\]
where \( T' \) is the \( F \)-split torus stabilizing the lines \( Fw_i \) for \( i = 1, \ldots, n \). For \( c \in F^\times \), we define a generic character \( \mu_{c}' \) of \( U' \) by
\[
\mu_{c}'(u') = \psi(\langle u'w_2, w_i^* \rangle_W + \cdots + \langle u'w_n, w_{n-1}^* \rangle_W + c\langle u'w_n^*, w_n^* \rangle_W).
\]

By [10 §12], the map \( c \mapsto \mu_{c}' \) gives a bijection (depending on \( \psi \))
\[
F^\times / F^\times 2 \to \{T'\text{-orbits of generic characters of } U' \}.
\]

We define a Whittaker datum of \( \text{Sp}(W) \) by \( w_{c}' = (B', \mu_{c}') \). We emphasize that \( w_{c}' \) depends on \( \psi \).
2.4. **Parabolic subgroups.** Fix a positive integer $k \leq m - 1$. Let $V = V_{2m}$ be an orthogonal space with $\dim(V) = 2m$ and type $(d, c)$. We set $X = X_d$ and $X^* = X_c^*$. Let $V_0$ be the orthogonal complement of $X \oplus X^*$ in $V$, so that $V_0$ is an orthogonal space of dimension $2m_0 = 2(m - k)$ over $F$. Let $P = P_k = M_P U_P$ be the maximal parabolic subgroup of $O(V)$ stabilizing $X$, where $M_P$ is the Levi component of $P$ stabilizing $X^*$. We have

$$M_P = \{ m_P(a) \cdot h_0 \mid a \in \GL(X), h_0 \in O(V_0) \},$$

$$U_P = \{ u_P(b) \cdot u_P(c) \mid b \in \Hom(V_0, X), c \in \Sym(X^*, X) \},$$

where

$$m_P(a) = \begin{pmatrix} a & 1_{V_0} \\ (a^*)^{-1} & 1_{V_0} \end{pmatrix},$$

$$u_P(b) = \begin{pmatrix} 1_X & b & -\frac{1}{2} bb^* \\ 1_{V_0} & -b^* & 1_{X} \end{pmatrix},$$

$$u_P(c) = \begin{pmatrix} 1_X & c \\ 1_{V_0} & 1_{X} \end{pmatrix}$$

and

$$\Sym(X^*, X) = \{ c \in \Hom(X^*, X) \mid c^* = -c \}.$$  

Here, the elements $a^* \in \GL(X^*)$, $b^* \in \Hom(X^*, V_0)$, and $c^* \in \Hom(X^*, X)$ are defined by requiring that

$$\langle ax, x' \rangle_V = \langle x, a^* x' \rangle_V, \quad \langle bv, x' \rangle_V = \langle v, b^* x' \rangle_V, \quad \langle cx, x'' \rangle_V = \langle x, c^* x'' \rangle_V$$

for $x \in X$, $x', x'' \in X^*$ and $v \in V_0$. Let $P^0 = P \cap SO(V)$ and $M^0_P = M_P \cap SO(V)$, so that $M^0_P \cong \GL(X) \times SO(V_0)$. We put

$$\rho_P = \frac{2m_0 + k - 1}{2}, \quad w_P = \begin{pmatrix} 1_{V_0} \\ -I_X \end{pmatrix},$$

where $I_X \in \Hom(X^*, X)$ is defined by $I_X v_i^* = v_i$ for $1 \leq i \leq k$. Then the modulus character $\delta_P$ of $P$ is given by

$$\delta_P(m_P(a) h_0 u_P) = | \det(a) |^{2\rho_P}$$

for $a \in \GL(X)$, $h_0 \in O(V_0)$ and $u_P \in U_P$.

Similarly, for a fixed positive integer $k \leq n$, we let $W = W_{2n}$ be a symplectic space with $\dim(W) = 2n$ and we set $Y = Y_k$ and $Y^* = Y_k^*$. We define $W_0$, $2n_0 = 2(n - k) = \dim(W_0)$, $Q = M_Q U_Q \subset \Sp(W)$, $m_Q(a), U_Q(b), U_Q(c)$, $\Sym(Y^*, Y) = \{ c \in \Hom(Y^*, Y) \mid c^* = -c \}$ and $I_Y$ as above. We put

$$\rho_Q = \frac{2n_0 + k + 1}{2}, \quad w_Q = \begin{pmatrix} 1_{W_0} \\ -I_Y \end{pmatrix},$$

Then the modulus character $\delta_Q$ of $Q$ is given by

$$\delta_Q(m_Q(a') g_0 u_Q) = | \det(a') |^{2\rho_Q}$$

for $a' \in \GL(Y')$, $g_0 \in \Sp(W_0)$ and $u_Q \in U_Q$.

2.5. **Representations of $SO(V)$ and $O(V)$.** Let $V = V_m$ be an orthogonal space over $F$ of dimension $m$. In this subsection, we recall some results about representations of $SO(V)$ and $O(V)$. Note that any irreducible representation of $O(V)$ is self-dual by a result in [40, Chapter 4. II. 1]. We fix $\varepsilon \in O(V) \setminus SO(V)$.

First we assume that $m = 2n + 1$ is odd. Then we can take $\varepsilon$ in the center of $O(V)$ and we have

$$O(V) = SO(V) \times \langle \varepsilon \rangle.$$  

Hence, if $\tilde{\sigma} \in \Irr(O(V))$, then $\tilde{\sigma}|SO(V)$ is also irreducible. Moreover for $\sigma \in \Irr(SO(V))$, there are exactly two extensions of $\sigma$ to $O(V)$. In particular, $\sigma$ is self-dual.

Next we assume that $m = 2n$ is even. For $\sigma \in \Irr(SO(V))$, we denote by $\sigma^\varepsilon$ the representation given by conjugating $\sigma$ by $\varepsilon$. The next proposition follows from the Clifford theory. See also [5, Lemma 4.1].

**Proposition 2.1.** (1) For $\sigma \in \Irr(SO(V))$, the following are equivalent:

- $\sigma^\varepsilon \cong \sigma$;

- there exists $\tilde{\sigma} \in \Irr(O(V))$ such that $\tilde{\sigma}|SO(V) \cong \sigma$;
Proposition 2.2. Let $\sigma \in \text{Irr}(\text{SO}(V))$. Then we have
\[
\sigma^\vee \cong \sigma \quad \text{or} \quad \sigma^\vee \cong \sigma^\varepsilon.
\]

Proof. See [21, Proposition 5.3]. \hfill \Box

The group $O(V)$ acts on $\text{Irr}(\text{SO}(V))$ by conjugation.

Lemma 2.3. Let $V = V_{2n}$.

(1) For $(\sigma, V_\sigma) \in \text{Irr}(\text{SO}(V))$ and $\varepsilon \in O(V) \setminus \text{SO}(V)$, we have
\[
\text{Ind}_{\text{SO}(V)}^{O(V)}(\sigma) \cong \text{Ind}_{\text{SO}(V)}^{O(V)}(\sigma^\varepsilon).
\]

(2) We decompose $V = X_1 + V_0 + X_1^*$ with $\dim(V_0) = 2n - 2$. We put $P = P_1 = M_P U_P \subset O(V)$ with $M_P \cong \text{GL}_1(F) \times O(V_0)$. Assume that $V_0 \neq 0$, i.e., $n > 1$. Then for $\sigma_0 \in \text{Irr}_{\text{temp}}(\text{SO}(V_0))$, $\varepsilon \in O(V_0) \setminus \text{SO}(V_0)$ and a unitary character $\chi$ of $F^\times = \text{GL}_1(F)$, we have
\[
\text{Ind}_{P_0}^{O(V)}(\sigma_0 \boxtimes \chi) \cong \text{Ind}_{P_0}^{SO(V)}(\sigma_0^\varepsilon \boxtimes \chi^{-1}).
\]

(3) For $k < n$, we decompose $V = X_k + V_0 + X_k^*$ with $\dim(V_0) = 2(n-k) > 0$. We put $P = P_k = M_P U_P \subset O(V)$ with $M_P \cong \text{GL}(X_k) \times O(V_0)$. Let $\tau \in \text{Irr}(\text{GL}(X_k))$ and $\sigma_0 \in \text{Irr}(O(V_0))$. Assume that $\sigma_0 := \sigma_0|\text{SO}(V_0)$ is irreducible. Then there is a canonical isomorphism
\[
\text{Ind}_{P_0}^{O(V)}(\tau \otimes \sigma_0)|\text{SO}(V) \cong \text{Ind}_{P_0}^{SO(V)}(\tau \otimes \sigma_0).
\]

The induced representation $\text{Ind}_{\text{SO}(V)}^{O(V)}(\sigma)$ can be realized on
\[
X_\sigma := \{ f : O(V) \to V_\sigma \text{ smooth} \mid f(hg) = \sigma(h)f(g) \text{ for any } h \in \text{SO}(V), g \in O(V) \}
\]
by $(g \cdot f)(x) = f(xg)$. Then the map
\[
f \mapsto [x \mapsto f(\varepsilon^{-1}x)]
\]
gives an isomorphism $X_\sigma \to X_{\sigma^\vee}$.
Note that \( \text{Ind}_{C}^{O(V)}(\sigma_{0} \boxtimes \chi) \) is a direct sum of irreducible representations of \( O(V) \). The assertion follows from Proposition 2.2 and the fact that

\[
\text{Ind}_{C}^{O(V)}(\sigma_{0} \boxtimes \chi)^{\vee} \cong \text{Ind}_{C}^{O(V)}(\sigma_{0}^{\vee} \boxtimes \chi^{-1}).
\]

The representation \( \text{Ind}_{C}^{O(V)}(\tau \otimes \sigma_{0}) \) can be realized on a space of smooth functions \( f \) on \( O(V) \). The map \( f \mapsto f|_{SO(V)} \) gives a desired isomorphism. \( \square \)

We define equivalent relations \( \sim_{\text{det}} \) on \( \text{Irr}(O(V)) \) and \( \sim_{\varepsilon} \) on \( \text{Irr}(SO(V)) \) by

\[
\bar{\sigma} \sim_{\text{det}} \sigma \otimes \text{det} \quad \text{and} \quad \sigma \sim_{\varepsilon} \sigma^{\varepsilon}
\]

for \( \bar{\sigma} \in \text{Irr}(O(V)) \) and \( \sigma \in \text{Irr}(SO(V)) \). Note that \( \bar{\sigma}|_{SO(V)} \cong (\bar{\sigma} \otimes \text{det})|_{SO(V)} \) and \( \text{Ind}_{SO(V)}^{O(V)}(\sigma) \cong \text{Ind}_{SO(V)}^{O(V)}(\sigma^{\varepsilon}) \). Hence, by Proposition 2.1, the restriction and the induction give a canonical bijection

\[
\text{Irr}(O(V))/\sim_{\text{det}} \leftrightarrow \text{Irr}(SO(V))/\sim_{\varepsilon}.
\]

For \( \sigma \in \text{Irr}(SO(V)) \), we denote the equivalence class of \( \sigma \) by

\[
[\sigma] = \{ \sigma, \sigma^{\varepsilon} \} \in \text{Irr}(SO(V))/\sim_{\varepsilon}.
\]

2.6. **Theta lifts.** We introduce the local theta correspondence induced by a Weil representation \( \omega_{\psi,V,W} \) of \( \text{Sp}(W) \times O(V) \) when \( \dim(W) = 2n \) and \( \dim(V) = 2m \), and recall some basic general results.

We fix a non-trivial additive character \( \psi \) of \( F \). Let \( W = W_{2n} \) and \( V = V_{2m} \). We denote a Weil representation of \( \text{Sp}(W) \times O(V) \) by \( \omega = \omega_{\psi,V,W} \). Let \( \pi \in \text{Irr}(\text{Sp}(W)) \). Then the maximal \( \pi \)-isotypic quotient of \( \omega \) is of the form

\[
\pi \boxtimes \Theta(\pi),
\]

where \( \Theta(\pi) = \Theta_{\psi,V,W}(\pi) \) is a smooth finite length representation of \( O(V) \). Hence, there exists an exact sequence of \( \text{Sp}(W) \times O(V) \)-modules:

\[
1 \longrightarrow S[\pi] \longrightarrow \omega \longrightarrow \pi \boxtimes \Theta(\pi) \longrightarrow 1,
\]

where the kernel \( S[\pi] \) is given by

\[
S[\pi] = \bigcap_{f \in \text{Hom}_{\text{Sp}(W)}(\omega, \pi)} \ker(f).
\]

Similarly, for \( \bar{\sigma} \in \text{Irr}(O(V)) \), we obtain a smooth finite length representation \( \Theta(\bar{\sigma}) = \Theta_{\psi,V,W}(\bar{\sigma}) \) of \( \text{Sp}(W) \). The maximal semi-simple quotient of \( \Theta(\pi) \) (resp. \( \Theta(\bar{\sigma}) \)) is denoted by \( \theta(\pi) = \theta_{\psi,V,W}(\pi) \) (resp. \( \theta(\bar{\sigma}) = \theta_{\psi,V,W}(\bar{\sigma}) \)). The Howe duality conjecture, which was proven by Waldspurger \[54\] if the residue characteristic is not 2 and by Gan–Takeda \[17, 18\] in general, says that \( \theta(\pi) \) and \( \theta(\bar{\sigma}) \) are irreducible (if they are nonzero). Moreover, we have the following:

**Proposition 2.4** (\[12\] Proposition C.4 (ii)). We set \( V = V_{2m} \) and \( W = W_{2n} \). Let \( \pi \in \text{Irr}_{\text{temp}}(\text{Sp}(W)) \) (resp. \( \bar{\sigma} \in \text{Irr}_{\text{temp}}(O(V)) \)) be a tempered representation.

1. Suppose that \( m = n + 1 \). Then \( \Theta(\pi) \) is either zero or an irreducible tempered representation of \( O(V) \) so that \( \theta(\pi) = \theta(\bar{\sigma}) \).

2. Suppose that \( m = n \). Then \( \Theta(\bar{\sigma}) \) is either zero or an irreducible tempered representation of \( \text{Sp}(W) \) so that \( \Theta(\bar{\sigma}) = \theta(\bar{\sigma}) \).

2.7. **Theta correspondence for unramified representations.** In this subsection, we assume the following conditions:

- the residue characteristic of \( F \) is \( p \neq 2 \);
- \( \psi \) is an unramified additive character of \( F \), i.e., \( \psi|_{\mathfrak{o}_{F}} = 1 \) and \( \psi|_{\mathfrak{w}^{-1}\mathfrak{o}_{F}} \neq 1 \);
- \( \dim(W) = 2n \) and \( \dim(V) = 2n + 2 \);
- \( V \) is type \( (d, 1) \) with \( d \in \mathfrak{o}_{F}^{\times} \).
In this case, the Hasse invariant of $V$ is 1. Recall that $V$ and $W$ have decompositions

$$V = X_n + V_1 + X_n^*,$$
$$W = Y_n + Y_n^*$$

with $V_1 = Fe + Fe'$, $X_n = Fv_1 + \cdots + Fv_n$, $X_n^* = Fv_1^* + \cdots + Fv_n^*$, $Y_n = Fw_1 + \cdots + Fw_n$ and $Y_n^* = Fw_1^* + \cdots + Fw_n^*$. If $d \in \mathfrak{o}_F^\times$, then put $v_0 = (e + u^{-1}e')/2$ and $v_0^* = (e - u^{-1}e')/2$, where we choose $u \in \mathfrak{o}_F^\times$ such that $d = u^2$. Then we have $(v_0, v_0)_V = (v_0^*, v_0^*)_V = 0$ and $(v_0, v_0^*)_V = 1$. If $d \notin \mathfrak{o}_F^\times$, then $V_0 \cong (E, N_{E/F})$, where $E/F$ is the quadratic unramified extension. We put

$$L_V = \begin{cases} \sigma_F v_1 + \cdots + \sigma_F v_n + \sigma_F v_0 + \sigma_F v_1^* + \cdots + \sigma_F v_n^*, & \text{if } d \in \mathfrak{o}_F^\times, \\ \sigma_F v_1 + \cdots + \sigma_F v_n + \sigma_E + \sigma_F v_n^* + \cdots + \sigma_F v_1^*, & \text{if } d \notin \mathfrak{o}_F^\times, \end{cases}$$
$$L_W = \sigma_F w_1 + \cdots + \sigma_F w_n + \sigma_F w_1^* + \cdots + \sigma_F w_n^*.$$ 

Here, $\sigma_E$ is the ring of integer of $E$. Let $K_V$ (resp. $K_W$) be the maximal compact subgroup of $SO(V)$ (resp. $Sp(W)$) which preserves the lattice $L_V$ (resp. $L_W$). Note that there are Iwasawa decompositions $SO(V) = BK_V$ and $Sp(W) = B'K_W$, where $B$ and $B'$ are the Borel subgroups of $SO(V)$ and $Sp(W)$ defined in \[2.4\] and \[2.3\] respectively. We have the following proposition.

**Proposition 2.5.** Assume the above conditions. If $\pi \in \operatorname{Irr}(Sp(W))$ has a nonzero $K_W$-fixed vector, then $\theta_V(\pi)$ has a nonzero $K_V$-fixed vector. In particular, $\theta_V(\pi)$ is nonzero, and so that it is irreducible.

**Proof.** It is well-known that if $\theta_V(\pi)$ is nonzero, then it has a nonzero $K_V$-fixed vector (see, e.g., [40, Chapter 5. I. 10 Théorème]). By [67] Lemma 8.6, [15] Proposition 6.2 and [16] Proposition 3, the theta lift $\Theta(\pi)$ is nonzero. 

**2.8. Theta correspondence and parabolic inductions.** We put $V = V_{2m}$ and $W = W_{2n}$. We set $m = n + 1$ and $\chi = \chi_V$. As in \[2.4\] for each $j \leq n$, we decompose

$$V = X_j + V_j + X_j^*$$
$$W = Y_j + Y_j^* + Y_j^*$$

where $V_j$ (resp. $W_j$) is the orthogonal complement of $X_j \oplus X_j^*$ in $V$ (resp. $Y_j \oplus Y_j^*$ in $W$). Let $P_j = M_{P_j}U_{P_j}$ (resp. $Q_j = M_{Q_j}U_{Q_j}$) be the parabolic subgroup of $O(V)$ (resp. $Sp(W)$) defined in \[2.4\]. Hence $M_{P_j} \cong GL(X_j) \times O(V_j)$ and $M_{Q_j} \cong GL(Y_j) \times Sp(W_j)$. For fixed $k \leq n$, we put $V_0 := V_k$, $X := X_k$, $P := P_k$, $W_0 := W_k$, $Y := Y_k$ and $Q := Q_k$.

The following result is well-known.

**Proposition 2.6.** Let $\pi \in \operatorname{Irr}_{temp}(Sp(W))$ and $\pi_0 \in \operatorname{Irr}_{temp}(Sp(W_0))$. Suppose that $\tilde{\pi} := \Theta(\pi) \neq 0$ and $\pi$ is an irreducible constituent of $\operatorname{Ind}_{\chi \otimes \pi_0}^{\chi}(\tau \otimes \pi_0)$ for some (unitary) discrete series representation $\tau$ of $GL_k(F) \cong GL(Y)$. Then we have $\tilde{\pi}_0 := \Theta(\pi_0) \neq 0$. Moreover, $\tilde{\pi}$ is an irreducible constituent of $\operatorname{Ind}_{\pi}^{\chi}(\tau \otimes \pi_0)$.

There are several similar results. For example, see Proposition C.1 and C.4 in [12] or Theorem 8.1 in [15]. For readers, we prove Proposition 2.6 in the rest of this section.

Let $\pi$, $\pi_0$, $\tilde{\pi}$ and $\tau$ be as in Proposition 2.6. We denote the full linear dual of $\tilde{\pi}$ by $\tilde{\pi}^\ast$. Then we have

$$0 \neq \tilde{\pi}^\ast = \Theta(\pi) = \operatorname{Hom}_{Sp(W)}(\omega_{\psi,W}, \chi_V)$$
$$\rightarrow \operatorname{Hom}_{Sp(W)}(\omega_{\psi,W}, \operatorname{Ind}_{\chi \otimes \pi_0}^{\chi}(\tau \otimes \pi_0) = \operatorname{Hom}_{Sp(W_0) \times GL(Y)}(R_Q(\omega_{\psi,W}, \tau \otimes \pi_0),$$

where $R_Q$ denotes the normalized Jacquet functor with respect to $Q$.

The normalized Jacquet module $R_Q(\omega_{\psi,W})$ has been computed by Kudla [33] Theorem 2.8 (ii)].

**Proposition 2.7.** The normalized Jacquet module $R_Q(\omega)$ of $\omega = \omega_{\psi,W}$ with respect to $Q$ has an $Sp(W_0) \times O(V) \times GL(Y)$-invariant filtration

$$R_Q(\omega) = R^0 \supset R^1 \supset \cdots \supset R^k \supset R^{k+1} = 0$$

with successive quotients given by

$$J^j = R^j/R^{j+1} \cong \operatorname{Ind}_{Sp(W_0) \times O(V) \times GL(Y)}^{Sp(W_0) \times P_j \times R_j}(S(\operatorname{Isom}(X_j, Y_j)) \otimes S(V_j \otimes Y_j^*))$$

Here,
\item Isom\((X_j, Y_j)\) is the set of invertible linear maps from \(X_j\) to \(Y_j\) and \(S(\text{Isom}(X_j, Y_j))\) is the space of locally constant compactly supported functions on \(\text{Isom}(X_j, Y_j)\);

\item \(R_j = M_{R_j}U_{R_j}\) is the parabolic subgroup of \(\text{GL}(Y_k)\) stabilizing 
\[Y_{k-j} = Fw_1 + \cdots + Fw_{k-j},\]
and \(M_{R_j}\) is the Levi subgroup of \(R_j\) stabilizing 
\[Y'_j = Fw_{k-j+1} + \cdots + Fw_k,\]
so that \(M_{R_j} \cong \text{GL}(Y_{k-j}) \times \text{GL}(Y'_j)\);

\item \(\text{Sp}(W_0) \times O(V_j)\) acts on \(S(V_j \otimes Y^*_0)\) by the Weil representation \(\omega_{0,j} = \omega_{W_j, W_0}\);

\item \(\text{GL}(Y_{k-j})\) acts by the scalar multiplication of the character \(\chi(\det) |J|_F^{m-n+\frac{1}{2}(k-j-1)}\);

\item \((aX, aY) \in \text{GL}(X_j) \times \text{GL}(Y'_j)\) acts on \(\varphi \in S(\text{Isom}(X_j, Y_j)) \otimes S(V_j \otimes Y^*_0)\) by 
\[[(aX, aY) \cdot \varphi](\alpha, x) = \chi(\det(aY)) |\det(aY)|_F^{\frac{1}{2}m-\frac{1}{2}(j+1)} |\det(aX)|_F^{m+\frac{1}{2}(j+1)} \varphi(aY^{-1}aX, x)\]
for \(\alpha \in \text{Isom}(X_j, Y_j)\) and \(x \in V_j \otimes Y^*_0\).

Using the notation in the above proposition, we have the following lemma:

**Lemma 2.8.** For \(j < k\), we have 
\[\text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y_k)}(J^j, \tau \chi \otimes \pi_0) = 0.\]

**Proof.** We have 
\[
\text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y_k)}(J^j, \tau \chi \otimes \pi_0)
= \text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y_{k-j}) \times \text{GL}(Y'_j)}(S(\text{Isom}(X_j, Y_j)) \otimes S(V_j \otimes Y^*_0), R_{R_j}(\tau \chi \otimes \pi_0)),
\]
where \(R_j\) is the parabolic subgroup \(\text{GL}(Y_k)\) opposite to \(R_j\). Since \(\tau \chi\) is a unitary discrete series representation, by [8, Proposition 9.5], the normalized Jacquet module \(R_{\text{GL}(Y_k)}(\tau \chi)\) is an irreducible discrete series representation 
\[|\det|^{\frac{1}{2}m-\frac{1}{2}(j+1)}|\det|^{m+\frac{1}{2}(j+1)}\varphi(aY^{-1}aX, x)\]
for \(\alpha \in \text{Isom}(X_j, Y_j)\) and \(x \in V_j \otimes Y^*_0\).

In particular, we have 
\[t_1 < t_2 \quad \text{and} \quad t_1 \cdot (k-j) + t_2 \cdot j = 0.\]

Hence, on \(R_{\text{GL}(Y_k)}(\tau \chi)\), the center of \(\text{GL}(Y_{k-j})\) acts by the character \(|\det|^{\frac{1}{2}m-\frac{1}{2}(k-j-1)}\) (up to a unitary character), whereas on \(S(\text{Isom}(X_j, Y_j)) \otimes S(V_j \otimes Y^*_0)\), it acts by the scalar multiplication of 
\[\chi(\det)|J|_F^{m-n+\frac{1}{2}(k-j-1)}\]
by Proposition 2.8. Since 
\[m - n + \frac{1}{2}(k-j-1) = \frac{1}{2}(k-j+1) > 0 \geq t_1,\]
we deduce that the above Hom space must be zero. \(\square\)

For a representation \(\mathcal{V}\) of a totally disconnected locally compact group \(G\), we denote by \(\mathcal{V}_\infty\) the smooth part of \(\mathcal{V}\), i.e., the \(G\)-submodule of smooth vectors of \(\mathcal{V}\).

Note that \((\sigma^\vee)_{\infty} \cong \sigma^\vee\). By this lemma, we see that \(\sigma^\vee\) is an \(O(V)\)-submodule of 
\[\text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y)}(\text{Ind}_{\text{Sp}(W_0) \times P \times \text{GL}(Y)}(S(\text{Isom}(X, Y)) \otimes \omega_{00}), \tau \chi \otimes \pi_0)_{\infty}.\]

**Lemma 2.9.** There is an isomorphism of \(O(V)\)-representations 
\[
\text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y)}(\text{Ind}_{\text{Sp}(W_0) \times P \times \text{GL}(Y)}(S(\text{Isom}(X, Y)) \otimes \omega_{00}), \tau \chi \otimes \pi_0)_{\infty}
\cong \text{Ind}_{P}^{O(V)}(\text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y)}(S(\text{Isom}(X, Y)) \otimes \omega_{00}, \tau \chi \otimes \pi_0)_{\infty}).
\]
We fix a special maximal compact subgroup of \( O(V) \) such that for \( \Phi \in \text{Isom}(X,Y) \) we have

\[
\Phi(p)(f(1,ph,1)) = \delta_P(p)\Phi(h)(f(1,h,1))
\]

for \( h \in O(V) \) and \( p \in P \). Here, \( \delta_P \) is the modulus character of \( P \) so that

\[
\delta_P(m_P(a)h_0u) = |\det(a)|^{2\rho_p}
\]

for \( a \in GL(X) \), \( h_0 \in O(V_0) \) and \( u \in U_P \). Hence we can consider the integral

\[
\langle \Phi'(f), \hat{v} \rangle := \int_{P \setminus O(V)} \langle \Phi(h)(f(1,h,1)), \hat{v} \rangle dh
\]

for \( \hat{v} \in V' \) and \( \hat{v}_0 \in V_{\pi_0} \). Clearly, this integral converges absolutely.

Let \( g_0 \in \text{Sp}(W_0) \) and \( a_Y \in GL(Y) \). Then we have

\[
\langle \Phi'([g_0,a_Y]f), \hat{v} \rangle = \int_{P \setminus O(V)} \langle \Phi(h)([g_0,a_Y]f)(1,h,1), \hat{v} \rangle dh
\]

\[
= \int_{P \setminus O(V)} \langle \Phi(h)(f(g_0,h,a_Y)), \hat{v} \rangle dh
\]

\[
= \int_{P \setminus O(V)} \langle [\tau \chi \otimes \pi_0](g_0,a_Y)\Phi(h)(f(1,h,1)), \hat{v} \rangle dh
\]

\[
= \langle [g_0,a_Y]\Phi'(f), \hat{v} \rangle.
\]

Hence we have

\[
\Phi' \in \text{Hom}_{\text{Sp}(W_0) \times GL(Y)}(\text{Ind}_{\text{Sp}(W_0) \times P \times GL(Y)}(\text{Isom}(X,Y) \otimes \omega_0), \tau \chi \otimes \pi_0).
\]

Moreover, for \( h_1 \in O(V) \), we have

\[
\langle [h_1] \Phi'(f), \hat{v} \rangle = \int_{P \setminus O(V)} \langle [h_1] \Phi(h)(f(1,h,1)), \hat{v} \rangle dh
\]

\[
= \int_{P \setminus O(V)} \langle \Phi(hh_1)(f(1,h,1)), \hat{v} \rangle dh
\]

\[
= \int_{P \setminus O(V)} \langle \Phi(h)(f(1,hh_1^{-1},1)), \hat{v} \rangle dh
\]

\[
= \int_{P \setminus O(V)} \langle \Phi(h)([h_1^{-1}f](1,h,1)), \hat{v} \rangle dh
\]

\[
= \langle \Phi'(h_1^{-1}f), \hat{v} \rangle = \langle [h_1] \Phi'(f), \hat{v} \rangle.
\]

In particular, we see that \( \Phi' \) is a smooth vector, i.e., \( \Phi' \in A \). We conclude that the map \( \Phi \mapsto \Phi' \) is an \( O(V) \)-homomorphism \( B \to A \).

We show that the map \( \Phi \mapsto \Phi' \) is injective. Assume that \( \Phi \) is nonzero. Then there are \( h \in O(V) \) and \( \varphi \in \text{S}(\text{Isom}(X,Y)) \otimes \omega_0 \) such that

\[
\Phi(h)(\varphi) \neq 0.
\]

We fix a special maximal compact subgroup \( K \) of \( O(V) \) so that \( O(V) = PK \). We may assume that \( h \in K \).

We take compact open subgroups \( K_1, K_2 \) of \( K \subset O(V) \) such that \( K_1 \) fixes \( \Phi \) and \( P \cap hK_2h^{-1} \) fixes \( \varphi \). We may assume that \( K_2 \subset K_1 \). There exists

\[
f \in \text{Ind}_{\text{Sp}(W_0) \times O(V) \times GL(Y)}(\text{Isom}(X,Y) \otimes \omega_0)
\]

such that \( \text{Supp}(f(1,1,1)) = PhK_2 \) and \( f(1,hk_2,1) = \varphi \) for \( k_2 \in K_2 \). Then we have

\[
\langle \Phi'(f), \hat{v} \rangle = \int_K \langle \Phi(k)(f(1,k,1)), \hat{v} \rangle dk
\]
This is nonzero for some \(\tilde{v} \in V\), and \(\tilde{v}_0 \in V\). This implies the injectivity.

Next, we show that the map \(\Phi \mapsto \Phi'\) is surjective. Let \(\Psi \in A\). Choose a compact open subgroup \(K'\) of \(O(V)\) which fixes \(\Psi\). We may assume that \(K' \subset K\). For \(\varphi \in S(\text{Isom}(X,Y)) \otimes \omega_{\partial 0}, h \in O(V)\) and a compact open subgroup \(K_1\) of \(K'\) such that \(P \cap hK_1h^{-1}\) fixes \(\varphi\), we define \(f_{\varphi,h,K_1}\) by

\[
f_{\varphi,h,K_1}(h',1) = \int_P \bigl[ p^{-1}\varphi \bigr] \cdot 1_{hK_1}(ph') \cdot \delta^{1/2}(p) \, dp,
\]

where we denote the characteristic function of a subset \(\Omega\) of \(O(V)\) by \(1_{\Omega}\), and \(dp\) is a left-invariant Haar measure on \(P\). We claim that \(f_{\varphi,h,K_1}\) can be extended to an element in

\[
\text{Ind}_{\text{Sp}(W_0) \times O(V) \times GL(Y)}^{\text{Sp}(W_0) \times P \times GL(Y)}(S(\text{Isom}(X,Y)) \otimes \omega_{\partial 0}).
\]

Indeed, for \(p_0 \in P\), we have

\[
f_{\varphi,h,K_1}(p,ph',1) = \int_P \bigl[ p^{-1}\varphi \bigr] \cdot 1_{hK_1}(ppho) \cdot \delta^{1/2}(p) \, dp
\]

\[
= \int_P \bigl[ p_0(p^{-1}\varphi) \bigr] \cdot 1_{hK_1}(ph') \cdot \delta^{1/2}(pp_0^{-1}) \times \delta_P(p_0) \, dp
\]

\[
= \delta^{1/2}(p_0) \times p_0 \cdot f_{\varphi,h,K_1}(h',1).
\]

This implies the claim. Moreover we see that \(\text{Supp}(f_{\varphi,h,K_1}(1,\cdot,1)) \subset PhK_1\) and

\[
f_{\varphi,h,K_1}(1,1,1) = \int_{PhK_1} \bigl[ p^{-1}\varphi \bigr] \cdot \delta^{1/2}(p) \, dp = \text{vol}(P \cap hK_1h^{-1}; P) \cdot \varphi.
\]

Now, we put

\[
\Psi'(h)(\varphi) = [K : K_1] \cdot \Psi(f_{\varphi,h,K_1}).
\]

It suffices to show the following:

1. \(\Psi'(h)(\varphi)\) is independent of the choice of \(K_1\);
2. \(\Psi'(h) \in \text{Hom}_{\text{Sp}(W_0) \times GL(Y)}(S(\text{Isom}(X,Y)) \otimes \omega_{\partial 0}, \tau X \otimes \pi_0)\) for any \(h \in O(V)\);
3. \(\Psi' \in B\);
4. if we put \(\Phi = \Psi'\), then \(\Phi' = c \cdot \Psi\) for some nonzero constant \(c\).

First, we show (1). Suppose that both \(K_1, K_2\) satisfy that \(K_1 \subset K'\) and that \(P \cap hK_1h^{-1}\) fixes \(\varphi\). We may assume that \(K_1 \subset K_2\). Since

\[
hK_2 = \bigcup_{K' \in K_1 \cap K_2} hK_1K'
\]

we have

\[
1_{hK_2}(h') = \sum_{K' \in K_1 \cap K_2} 1_{hK_1K'}(h') = \sum_{K' \in K_1 \cap K_2} 1_{hK_1}(h'k'^{-1})
\]

and so that

\[
f_{\varphi,h,K_2} = \sum_{K' \in K_1 \cap K_2} k'^{-1} \cdot f_{\varphi,h,K_1}.
\]

Since \(K' \in K_2 \subset K'\) fixes \(\Psi\), we have

\[
[K : K_2] \cdot \Psi(f_{\varphi,h,K_2}) = [K : K_2] \cdot \sum_{K' \in K_1 \cap K_2} \Psi(k'^{-1} \cdot f_{\varphi,h,K_1})
\]

\[
= [K : K_2] \cdot \sum_{K' \in K_1 \cap K_2} [k' \cdot \Psi](f_{\varphi,h,K_1})
\]

\[
= [K : K_2][K_2 : K_1] \cdot \Psi(f_{\varphi,h,K_1})
\]

\[
= [K : K_1] \cdot \Psi(f_{\varphi,h,K_1}).
\]
This shows that (1).

Next, we show (2). Let $\varphi_1, \varphi_2 \in \mathcal{S}(\text{Isom}(X,Y)) \otimes \omega_{00}$. If we fix $h \in \text{O}(V)$, then we can take a compact open subgroup $K_1'$ of $K'$ such that $P \cap hK_1'h^{-1}$ fixes $a_1 \varphi_1 + a_2 \varphi_2$ for any $a_1, a_2 \in \mathbb{C}$. Then we see that $f_{a_1 \varphi_1 + a_2 \varphi_2, h, K_1} = a_1 \cdot f_{\varphi_1, h, K_1} + a_2 \cdot f_{\varphi_2, h, K_1}$. Hence the map $\Psi'(h)$ is linear. Let $g_0 \in \text{Sp}(W_0)$ and $a_Y \in \text{GL}(Y)$. Put $\varphi' = (g_0, a_Y) \cdot \varphi$. Note that if $\varphi$ is fixed by $P \cap hK_1'h^{-1}$, then so is $\varphi'$. Moreover we have

$$f_{\varphi', h, K_1} = (g_0, a_Y) \cdot f_{\varphi, h, K_1}.$$  

Hence we have

$$\Psi'(h)(\varphi') = [K : K_1] \Psi(f_{\varphi', h, K_1}) = [K : K_1] \Psi((g_0, a_Y) \cdot f_{\varphi, h, K_1})$$  

$$= [K : K_1](\tau \otimes \pi_0)(g_0, a_Y) \Psi(f_{\varphi, h, K_1})$$  

$$= (\tau \otimes \pi_0)(g_0, a_Y) \Psi(h)(\varphi).$$

This shows that $\Psi'(h)$ is an $\text{Sp}(W_0) \times \text{GL}(Y)$-homomorphism. Moreover, if $p_0 \in P \cap hK_1'h^{-1}$, then we may take a compact open subgroup $K_1'$ of $K'$ such that $P \cap hK_1'h^{-1}$ fixes both $\varphi$ and $p_0^{-1} \cdot \varphi$, and $K_1$ is a normal subgroup of $K'$. We put $K_1' = h^{-1}p_0hK_1(h^{-1}p_0h)^{-1}$. Then $K_1' = K_1$ and we see that

$$ph'(h^{-1}p_0h)^{-1} \in hK_1' \iff p_0^{-1}ph' \in hK_1$$

for any $h' \in \text{O}(V)$ and $p \in P$. Hence we have

$$f_{p_0^{-1} \varphi, h, K_1}(1, h', 1) = \int_P \left[ p^{-1}(p_0^{-1} \varphi) \right] \cdot 1_{hK_1}(ph') \cdot \delta^{1/2}_P(p) dp$$  

$$= \int_P \left[ p^{-1} \varphi \right] \cdot 1_{hK_1}(p_0^{-1} ph') \cdot \delta^{1/2}_P(p_0^{-1} p) dp$$  

$$= \delta^{-1/2}_P(p_0) \int_P \left[ p^{-1} \varphi \right] \cdot 1_{hK_1'}(ph'(h^{-1}p_0h)^{-1}) \cdot \delta^{1/2}_P(p) dp$$  

$$= f_{\varphi, h, K_1}(1, h'(h^{-1}p_0h)^{-1}, 1) = f_{\varphi, h, K_1}(1, 1', 1).$$

Since $h^{-1}p_0h \in K'$ fixes $\Psi$, we have

$$\Psi'(h)(p_0^{-1} \varphi) = [K : K_1] \Psi(f_{p_0^{-1} \varphi, h, K_1}) = [K : K_1] \Psi((h^{-1}p_0h)^{-1} \cdot f_{\varphi, h, K_1})$$  

$$= [K : K_1][(h^{-1}p_0h)^{-1} \Psi](f_{\varphi, h, K_1}) = [K : K_1] \Psi(f_{\varphi, h, K_1}) = \Psi'(h)(\varphi).$$

This shows that $\Psi'(h)$ is fixed by $P \cap hK_1'h^{-1}$. Therefore, we deduce that

$$\Psi'(h) \in \text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y)}(\mathcal{S}(\text{Isom}(X,Y)) \otimes \omega_{00} \tau \chi \otimes \pi_0) \otimes \infty.$$

Next, we show (3). For $p_0 \in P$, we see that if $P \cap hK_1'h^{-1}$ fixes $p_0^{-1} \varphi$, then $P \cap p_0hK_1'h^{-1}p_0^{-1}$ fixes $\varphi$. Moreover we have

$$f_{p_0^{-1} \varphi, h, K_1}(1, h', 1) = \int_P \left[ p^{-1}(p_0^{-1} \varphi) \right] \cdot 1_{hK_1}(ph') \cdot \delta^{1/2}_P(p) dp$$  

$$= \int_P \left[ p^{-1} \varphi \right] \cdot 1_{hK_1}(p_0^{-1} ph') \cdot \delta^{1/2}_P(p_0^{-1} p) dp$$  

$$= \delta^{-1/2}_P(p_0) \cdot f_{\varphi, p_0h, K_1}(1, h', 1).$$

Hence

$$\Psi'(h)(p_0^{-1} \varphi) = [K : K_1] \Psi(f_{p_0^{-1} \varphi, h, K_1})$$  

$$= \delta^{-1/2}_P(p_0)[K : K_1] \Psi(f_{\varphi, p_0h, K_1}) = \delta^{-1/2}_P(p_0) \Psi'(p_0h)(\varphi).$$

This shows that

$$\Psi'(p_0h) = \delta^{1/2}_P(p_0)[p_0 \cdot \Psi'(h)].$$

For $k' \in K'$, we put $K'_1 = k'^{-1}K_1k'$. We see that if $P \cap hK_1'h^{-1}$ fixes $\varphi$, then so does $P \cap hK_1h^{-1}$. Moreover we have

$$f_{\varphi, hK_1'h, K'_1}(1, h', 1) = \int_P \left[ p^{-1} \varphi \right] \cdot 1_{hK_1'hK'_1}(ph') \cdot \delta^{1/2}_P(p) dp$$
and τ

Lemma 2.10. Let τY ∈ Irr(GL(Y)). We define I0 ∈ Hom(Y, X) by

\[ I_0 : w_i \mapsto v_i, \]

and τX ∈ Irr(GL(X)) by τX(aX) := τY(I_0^{-1}aXI_0) for aX ∈ GL(X). Then we have

\[ \text{Hom}_{GL(Y)}(\mathcal{S}(\text{Isom}(X,Y)), \tau_Y \chi) \cong (\tau_X)^* \]

as GL(X)-modules.
Proof. The map
\[ GL(Y) \ni \alpha \mapsto \alpha^{-1} \circ I_0^{-1} \in \text{Isom}(X,Y) \]
gives an isomorphism
\[ S(\text{Isom}(X,Y)) \cong \text{ind}_{I}^{\text{GL}(Y)}(C) \otimes \chi(\text{det}_Y) \mid \text{det}_Y |_{F} \]
as \text{GL}(Y)-modules, where \( l = m - (k+1)/2 \). Hence, as vector spaces, we have
\[ \text{Hom}_{\text{GL}(Y)}(S(\text{Isom}(X,Y)), \tau_Y \chi) \cong \text{Hom}_{\text{GL}(Y)}((\tau_Y \chi)^{\vee}, (\text{ind}_{I}^{\text{GL}(Y)}(C) \otimes \chi(\text{det}_Y) \mid \text{det}_Y |_{F})^{\vee}) \]
\[ \cong \text{Hom}_{\text{GL}(Y)}(\tau_Y^{\vee} \otimes |\text{det}_Y |_{F}, \text{Ind}_{I}^{\text{GL}(Y)}(C)) \]
\[ \cong \text{Hom}(\tau_Y^{\vee} \otimes |\text{det}_Y |_{F}, C) = (\tau_Y^{\vee})^* \]

We check the action of \( \text{GL}(X) \) on \( (\tau_Y^{\vee})^* \) via this isomorphism. Let \( \Phi \in \text{Hom}_{\text{GL}(Y)}(S(\text{Isom}(X,Y)), \tau_Y \chi) \). We define \( \Phi^{\vee} \in \text{Hom}_{\text{GL}(Y)}(\tau_Y^{\vee} \otimes |\text{det}_Y |_{F}, \text{Ind}_{I}^{\text{GL}(Y)}(C)) \) by
\[ \langle \Phi(\varphi), \tilde{v} \rangle = \int_{\text{GL}(Y)} \varphi(\alpha^{-1}I_0^{-1})\Phi^{\vee}(\tilde{v})(\alpha)d\alpha \]
for \( \varphi \in S(\text{Isom}(X,Y)) \) and \( \tilde{v} \in \mathcal{V}_{\tau_Y} \). Here, \( \langle \cdot, \cdot \rangle : \tau_Y \times \tau_Y^{\vee} \to C \) is a fixed nonzero \( \text{GL}(Y) \)-invariant pairing. Then the map \( \text{Hom}_{\text{GL}(Y)}(S(\text{Isom}(X,Y)), \tau_Y \chi) \to (\tau_Y^{\vee})^* \) is given by
\[ \Phi \mapsto [\tau_Y^{\vee} \ni \tilde{v} \mapsto \Phi^{\vee}(\tilde{v})(1_Y) \in C] \]

Let \( a_X \in \text{GL}(X) \) and \( a_Y \in \text{GL}(Y) \). Then we have
\[ \int_{\text{GL}(Y)} \varphi(\alpha^{-1}I_0^{-1})[a_X \Phi]^{\vee}(\tilde{v})(\alpha)d\alpha = ([a_X \Phi](\varphi), \tilde{v}) = \langle \Phi(a_X^{-1}\varphi), \tilde{v} \rangle \]
\[ = \langle (\tau_Y \chi)(a_Y)\Phi(a_X^{-1} \varphi), (\tau_Y \chi)^{\vee}(a_Y \tilde{v}) \rangle = \langle \Phi((a_X^{-1}, a_Y)\varphi), (\tau_Y \chi)^{\vee}(a_Y \tilde{v}) \rangle \]
\[ = \int_{\text{GL}(Y)} |\text{det}(a_Y)|_{F}^{-1} |\text{det}(a_X^{-1})|_{F}^{-1} \varphi(a_Y^{-1} \alpha^{-1}I_0^{-1}a_X^{-1})\Phi^{\vee}(\tau_Y \chi)(a_Y \tilde{v})(\alpha)d\alpha \]
\[ = \int_{\text{GL}(Y)} |\text{det}(a_Y)|_{F}^{-1} \varphi(\alpha^{-1}I_0^{-1})\Phi^{\vee}(\tau_Y \chi)(a_Y \tilde{v})(I_0^{-1}a_X^{-1}I_0 \alpha a_X^{-1})d\alpha \]

In particular, putting \( a_Y = I_0^{-1}a_X^{-1}I_0 \), we have
\[ [a_X \Phi]^{\vee} = [\tilde{v} \mapsto \Phi^{\vee}(\tau_Y \chi)(a_Y)] \circ \text{Ad}(a_Y) \]
Hence, via the map \( \text{Hom}_{\text{GL}(Y)}(S(\text{Isom}(X,Y)), \tau_Y \chi) \to (\tau_Y^{\vee})^* \), we have
\[ [a_X \Phi] \mapsto [\tilde{v} \mapsto \Phi^{\vee}(\tau_Y \chi)(I_0^{-1}a_X^{-1}I_0 \tilde{v})(1_Y)] \]
i.e., \( \text{GL}(X) \) acts on \( (\tau_Y^{\vee})^* \) by
\[ (\tau_Y^{\vee})^*(I_0^{-1}a_XI_0) = (\tau_X)^*(a_X) \]
This completes the proof. \( \square \)

Lemma 2.11. We have
\[ \text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y)}(S(\text{Isom}(X,Y)) \otimes \omega_{00}, \tau_Y \chi \otimes \pi_0) = \tau_X \otimes \Theta_{\varphi, V_0, W_0}(\pi_0)^{\vee} \]
as \( \text{GL}(X) \times \text{O}(V_0) \)-modules.

Proof. Since \( \pi_0 \) is an admissible representation of \( \text{Sp}(W_0) \), we have
\[ \text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y)}(S(\text{Isom}(X,Y)) \otimes \omega_{00}, \tau_Y \chi \otimes \pi_0) = \text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y)}(\omega_{00} \otimes \pi_0^\vee, S(\text{Isom}(X,Y))^\vee \otimes \tau_Y \chi) \]
Let \( \Phi \in \text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y)}(\omega_{00} \otimes \pi_0^\vee, S(\text{Isom}(X,Y))^\vee \otimes \tau_Y \chi) \). Then we have
\[ \Phi(g_0^{-1}x) = \Phi(x) = a_Y \cdot \Phi(x) \]
for any \( x \in \omega_{00} \otimes \pi_0^\vee, g_0 \in \text{Sp}(W_0) \) and \( a_Y \in \text{GL}(Y) \). In particular, \( \Phi(x) \) lies in
\[ (S(\text{Isom}(X,Y))^\vee \otimes \tau_Y \chi)^{\text{GL}(Y)} = \text{Hom}_{\text{GL}(Y)}(C, S(\text{Isom}(X,Y))^\vee \otimes \tau_Y \chi) \]
for any \( x \in \omega_0 \otimes \pi_0^\vee \). By Lemma \[2.10\] this space is isomorphic to \((\tau_X^{\psi})^\vee\). Moreover, there exists a compact open subgroup \( K \) of \( \text{GL}(X) \times \text{O}(V_0) \) such that
\[
\Phi(h_0^{-1} x) = \Phi(x) = a_X \cdot \Phi(x)
\]
for any \( x \in \omega_0 \otimes \pi_0^\vee \) and \((a_X, h_0) \in K\). In particular, some compact open subgroup of \( \text{GL}(X) \) fixes \( \Phi(x) \) for any \( x \in \omega_0 \otimes \pi_0^\vee \). Since \(((\tau_X^{\psi})^\vee)^\infty = (\tau_X^{\psi})^\vee \cong \tau_X \) is admissible, we deduce that \( \text{Im}(\Phi) \) is a finite dimensional subspace of \( \tau_X \). Let \( y_1, \ldots, y_r \) be a basis of \( \text{Im}(\Phi) \). Then we can write
\[
\Phi(x) = \sum_{i=1}^r y_i \otimes f_i(x)
\]
for some \( f_i: \omega_0 \otimes \pi_0^\vee \to \mathbb{C} \). We easily see that
\[
f_i \in \text{Hom}_{\text{Sp}(W_0)}(\omega_0 \otimes \pi_0^\vee, \mathbb{C})_\infty = (\Theta_{\psi, V_0, W_0}(\pi_0)^*)_\infty = \Theta_{\psi, V_0, W_0}(\pi_0)^\vee.
\]
We obtain a map
\[
\text{Hom}_{\text{Sp}(W_0) \times \text{GL}(Y)}(\omega_0 \otimes \pi_0^\vee, S(\text{Isom}(X, Y))^\vee \otimes \tau_X^\vee) \to \tau_X \otimes \Theta_{\psi, V_0, W_0}(\pi_0)^\vee.
\]
Clearly, this map is an \( \text{O}(V_0) \times \text{GL}(X) \)-isomorphism.

We conclude that
\[
\sigma^\vee \hookrightarrow \text{Ind}_P^{O(V)}(\tau \otimes \Theta_{\psi, V_0, W_0}(\pi_0)^\vee).
\]
Since \( \sigma = \Theta_{\psi, V, W}(\pi) \) and \( \Theta_{\psi, V_0, W_0}(\pi_0) \) are self-dual, we have
\[
\sigma \hookrightarrow \text{Ind}_P^{O(V)}(\tau \otimes \Theta_{\psi, V_0, W_0}(\pi_0)).
\]
This completes the proof of Proposition \[2.6\].

3. Local Langlands Correspondence

Through this paper, we assume the local Langlands correspondence for symplectic and special orthogonal groups. See \[2\]. In this section, we summarize some of its properties which are used in this paper. See also Appendix A.3 and \[10\].

3.1. Representations of \( WD_F \). Let \( M \) be a finite dimensional vector space over \( \mathbb{C} \). We say that a homomorphism \( \phi: WD_F \to \text{GL}(M) \) is a representation of \( WD_F \) if
\begin{itemize}
  \item \( \phi(\text{Frob}) \) is semi-simple, where \( \text{Frob} \in WD_F \) is a geometric Frobenius;
  \item the restriction of \( \phi \) to \( WD_F \) is smooth;
  \item the restriction of \( \phi \) to \( \text{SL}_2(\mathbb{C}) \) is algebraic.
\end{itemize}
We call \( \phi \) tempered if the image of \( WD_F \) is bounded. We say that \( \phi \) is orthogonal or self-dual with sign +1 (resp. symplectic or self-dual with sign −1) if there exits a non-degenerate bilinear form \( B: M \times M \to \mathbb{C} \) such that
\[
\begin{aligned}
B(\phi(w)x, \phi(w)y) &= B(x, y), \\
B(y, x) &= B(x, y) \quad (\text{resp. } B(y, x) = -B(x, y))
\end{aligned}
\]
for \( x, y \in M \) and \( w \in WD_F \). In this case, \( \phi \) is self-dual, i.e., \( \phi \) is equivalent to its contragredient \( \phi^\vee \).

For any positive integer \( n \), there exists a unique irreducible algebraic representation \( \nu_n \) of \( \text{SL}_2(\mathbb{C}) \) (or irreducible continuous representation \( \nu_n \) of \( \text{SU}(2) \)) with dimension \( n \). It is easy to see that
\[
\nu_n \text{ is } \begin{cases}
  \text{orthogonal} & \text{if } n \text{ is odd,} \\
  \text{symplectic} & \text{if } n \text{ is even.}
\end{cases}
\]

For a representation \( M \) of \( WD_F \), we define the \( L \)-factor \( L(s, M) \) and the \( \varepsilon \)-factor \( \varepsilon(s, M, \psi) \) as in \[52\]. It is well-known that if \( M \) is a symplectic representation of \( WD_F \), then \( \varepsilon(M) = \varepsilon(1/2, M, \psi) \) does not depend on the choice of \( \psi \), and \( \varepsilon(M) \in \{ \pm 1 \} \) (see, e.g., \[10\] Proposition 5.1).
3.2. Properties. Let $G$ be a quasi-split connected reductive algebraic group over $F$. We denote the Langlands dual group and the $L$-group of $G$ by $\hat{G}$ and $L^G = \hat{G} \rtimes W_F$, respectively. We call an $L$-homomorphism

$$\phi: WD_F \to \hat{G}$$

an $L$-parameter of $G$. We say that two $L$-parameters are equivalent if they are conjugate by an element in $\hat{G}$. We call an $L$-parameter $\phi$ tempered if $\phi(W_F)$ projects onto a relatively compact subset of $\hat{G}$. We denote the set of equivalence classes of $L$-parameters of $G$ by $\Phi(G)$. Let $\Phi_{\text{temp}}(G)$ be the subset of $\Phi(G)$ of classes of tempered $L$-parameters. For $\phi \in \Phi(G)$, we put

$$S_\phi = \text{Cent}(\text{Im}(\phi), \hat{G}).$$

A pure inner twist of $G$ is a tuple $(G', f, z)$, where $G'$ is an algebraic group over $F$, $f: G \to G'$ is an isomorphism over $\overline{F}$ and $z \in Z'(F, G)$ such that $f^{-1} \circ \gamma(f) = \text{Ad}(z_\gamma)$ for any $\gamma \in \Gamma$. Then we call $G'$ a pure inner form of $G$. There exists a canonical bijection

$$H^1(F, G) \leftrightarrow \{\text{The isomorphism classes of pure inner twists of } G\}.$$ 

More precisely, see Appendix A.2. Moreover, since $F$ is $p$-adic, there exists a canonical bijection (Kottwitz’s isomorphism)

$$H^1(F, G) \to \pi_0((\hat{G}^\Gamma)^D).$$

See [31, Proposition 6.4]. Note that $\pi_0((\hat{G}^\Gamma)^D)$ is a central subgroup of $\pi_0(S_\phi)$.

We are now ready to describe the desiderata for the Langlands correspondence.

1. There exists a canonical surjection

$$\bigcup_{G'} \text{Irr}(G'(F)) \to \Phi(G),$$

where $G'$ runs over all pure inner forms of $G$. For $\phi \in \Phi(G)$, we denote by $\Pi_\phi$ the inverse image of $\phi$ under this map, and call $\Pi_\phi$ the $L$-packet of $\phi$. The $L$-packet $\Pi_\phi$ is a finite set whose order is equal to $\#\text{Irr}(\pi_0(S_\phi))$. If $\phi \in \Phi_{\text{temp}}(G)$, then $\Pi_\phi \subset \bigcup_{G'} \text{Irr}_{\text{temp}}(G')$.

2. For a Whittaker datum $w$ of $G$, there exists a bijection

$$\iota_w: \Pi_\phi \to \text{Irr}(\pi_0(S_\phi))$$

which satisfies some character identities. More precisely, see Appendix A.3.

3. The cardinality of

$$\{\pi \in \Pi_\phi \mid \pi \text{ is } w\text{-generic}\}$$

is at most one, and is independent of $w$. If this number is one, we say that the $L$-parameter $\phi$ is generic. If $\phi$ is generic, then the unique $w$-generic representation $\pi$ in $\Pi_\phi$ corresponds to the trivial representation of $\pi_0(S_\phi)$ via the bijection $\iota_w$.

4. The $L$-parameter $\phi$ is generic if and only if the local adjoint $L$-function $L(s, \phi, \text{Ad})$ is regular at the point $s = 1$.

5. An irreducible representation $\eta$ of $\pi_0(S_\phi)$ has a central character of $\pi_0((\hat{G}^\Gamma)^D)$, which gives a class of $H^1(F, G)$, i.e., an isomorphism class of pure inner twists $G'$. Then $\iota_w(\eta) \in \text{Irr}(G'(F)).$

Once we fix a Whittaker datum $w$ of $G$, we denote by $\pi(\eta)$ the representation in $\Pi_\phi$ corresponding to $\eta \in \text{Irr}(\pi_0(S_\phi))$ via $\iota_w$.

The property (5) is the uniqueness of generic representations in an $L$-packet (UGP). The property (4) is the conjecture of Gross–Prasad and Rallis (GPR). Note that for $\phi \in \Phi_{\text{temp}}(G)$, the $L$-function $L(s, \phi, \text{Ad})$ is regular in the half-plane $\text{Re}(s) > 0$, so that $\phi$ is generic by (4).

In the rest of this section, we explain the desiderata of $L$-packets more precisely for symplectic and special orthogonal groups, which will be assumed in the rest of this paper.
3.3. Type $B_n$. Let $n \geq 1$ and $G = \text{SO}(V_{2n+1})$ be a quasi-split special orthogonal group. Then $\widehat{G} = \text{Sp}_{2n}(\mathbb{C})$ and $\mathbb{L}G = \text{Sp}_{2n}(\mathbb{C}) \times W_F$. An $L$-parameter $\phi$ of $\text{SO}(V_{2n+1})$ gives a self-dual representation $\phi : WD_F \rightarrow \text{GL}(M)$ with sign $-1$, $\dim_{\mathbb{C}}(M) = 2n$ and $\det(M) = 1$. The map $\overline{\phi} \mapsto \phi$ gives a bijection

$$\Phi(\text{SO}(V_{2n+1})) \rightarrow \overline{\Phi}(\text{SO}(V_{2n+1})) := \{ \phi : WD_F \rightarrow \text{Sp}(M) \mid \dim_{\mathbb{C}}(M) = 2n \} / \cong.$$ 

We identify $\phi$ with $\overline{\phi}$ via this bijection. Let $\overline{\Phi}_{\text{temp}}(\text{SO}(V_{2n+1}))$ be the subset of $\overline{\Phi}(\text{SO}(V_{2n+1}))$ of tempered representations.

Let $\phi \in \overline{\Phi}(\text{SO}(V_{2n+1}))$. We denote the centralizer of $\text{Im}(\phi)$ in $\text{Sp}(M)$ by $C_\phi$ and its component group by $A_\phi = C_\phi/C_\phi^\circ$. Then $\pi_0(S_\phi) \cong A_\phi$ if $\phi \mapsto \overline{\phi}$. If

$$M \cong \bigoplus_{i \in I^+} (V_i \otimes M_i) \oplus \bigoplus_{i \in I^-} (W_i \otimes N_i) \oplus \bigoplus_{j \in J} (U_j \otimes (P_j + P_j^\vee)),$$

where

- $M_i$ (resp. $N_i$) is an irreducible self-dual representation of $WD_F$ with sign $-1$ (resp. $+1$),
- $P_j$ is an irreducible representation of $WD_F$ which is not self-dual,

which are pairwise inequivalent, and $V_i$, $W_i$ and $U_j$ are multiplicity spaces, then the bilinear form on $M$ gives a non-degenerate bilinear form on $V_i$ (resp. $W_i$) of sign $1$ (resp. $-1$). Moreover, we have

$$C_\phi \cong \prod_{i \in I^+} O(V_i) \times \prod_{i \in I^-} \text{Sp}(W_i) \times \prod_{j \in J} \text{GL}(U_j).$$

For a semi-simple element $a \in C_\phi$, we denote by $M^a$ the $(-1)$-eigenspace of $a$ on $M$. Then $M^a$ gives a symplectic representation of $WD_F$. If $a_i \in O(V_i) \setminus \text{SO}(V_i)$, then $M^{a_i} = V_i^{a_i} \otimes M_i$. We have

$$A_\phi = \bigoplus_{i \in I^+} (\mathbb{Z}/2\mathbb{Z}) a_i \cong (\mathbb{Z}/2\mathbb{Z})^m,$$

where $m = \#I^+$. By Kottwitz’s isomorphism, we have $\#H^1(F, \text{SO}(V_{2n+1})) = 2$, so that there are exactly two pure inner forms of $G = \text{SO}(V_{2n+1})$. They are $\text{SO}(V_{2n+1})$ and $\text{SO}(V_{2n+1}^\vee)$, where $\dim(V_{2n+1}) = \dim(V_{2n+1}^\vee) = 2n + 1$ and $\text{disc}(V_{2n+1}) = \text{disc}(V_{2n+1}^\vee)$. We see that $\text{SO}(V_{2n+1})$ is split, but $\text{SO}(V_{2n+1}^\vee)$ is not quasi-split.

By [2] Theorem 1.5.1 (b), for $\phi \in \overline{\Phi}_{\text{temp}}(\text{SO}(V_{2n+1}))$, we obtain an $L$-packet $\Pi_\phi^A$, which is a finite subset of $\text{Irr}_{\text{temp}}(\text{SO}(V_{2n+1}))$, and a canonical bijection $\iota_{mn}^A : \Pi_\phi^A \rightarrow \pi_0(C_\phi/\{\pm 1\})$. We assume that there are a Vogan $L$-packet $\Pi_\phi \subset \text{Irr}_{\text{temp}}(\text{SO}(V_{2n+1})) \cup \text{Irr}_{\text{temp}}(\text{SO}(V_{2n+1}^\vee))$ such that $\Pi_\phi \cap \text{Irr}_{\text{temp}}(\text{SO}(V_{2n+1})) = \Pi_\phi^A$, and an extension $\iota_{mn} : \Pi_\phi \rightarrow \Pi_\phi^A$. We extend $L$-packets for general $\phi \in \overline{\Phi}(\text{SO}(V_{2n+1}))$ as follows. In general, we can write

$$\phi = \phi_1 \oplus \cdots \oplus \phi_r \oplus \phi_0 \oplus \phi_r^\vee \oplus \cdots \oplus \phi_1^\vee,$$

where

- $\phi_i$ is a $k_i$-dimensional representation of $WD_F$ for $i = 1, \ldots, r$, which is of the form $\phi_i = \phi_i^j \otimes | \cdot |_{\mathbb{P}_i}^s$ for some tempered representation $\phi_i^j$ of $WD_F$ and some real number $s_i$ such that $s_1 > \cdots > s_r > 0$, $n = n_0 + (k_1 + \cdots + k_r)$;
- $\phi_0 \in \overline{\Phi}_{\text{temp}}(\text{SO}(V_{2n+1}))$ if $n_0 \geq 1$, and $\phi_0 = 0$ if $n_0 = 0$.

We write

$$V_{2n+1}^\bullet = X_1 \oplus \cdots \oplus X_r \oplus V_{2n_0+1}^\bullet \oplus X_r^\vee \oplus \cdots \oplus X_1^\vee,$$

where $X_i$ and $X_i^\vee$ are $k_i$-dimensional totally isotropic subspaces of $V_{2n+1}$ such that $X_i \oplus X_i^\vee$ is non-degenerate, mutually orthogonal, and orthogonal to $V_{2n+1}^\bullet$. Let $P^o$ be the parabolic subgroup of $\text{SO}(V_{2n+1}^\bullet)$ stabilizing the flag

$$X_1 \subset X_1 \oplus X_2 \subset \cdots \subset X_1 \oplus \cdots \oplus X_r$$

and $M^o_\phi$ be the Levi component of $P^o$ stabilizing the flag

$$X_1^\vee \subset X_1^\vee \oplus X_2^\vee \subset \cdots \subset X_1^\vee \oplus \cdots \oplus X_r^\vee,$$
so that
\[ M_φ^N = GL(X_1) \times \cdots \times GL(X_r) \times SO(V_{2n+1}^*). \]
Then the L-packet \( Π_φ \) consists of the unique irreducible quotients \( π \) of the standard modules
\[ \text{Ind}_{F^0}^{SO(V_{2n+1}^*)} (τ_1 \otimes \cdots \otimes τ_r \otimes π_0), \]
where \( τ_i \) is the irreducible essentially tempered representation of \( GL(X_i) \) corresponding to \( φ_i \) for \( i = 1, \ldots, r \), and \( π_0 \) runs over elements in \( Π_{φ_0} \) (or \( π_0 = 1 \) if \( n_0 = 0 \) so that \( SO(V_{2n+1}) = \{1\} \)). In particular, if \( n_0 = 0 \), then \( Π_φ \) is singleton. If \( n_0 \geq 1 \), then the natural map \( A_{φ_0} \rightarrow A_φ \) is an isomorphism, and we define \( τ_m(π) \in A_φ \) by
\[ τ_m(π_0) = τ_m(π)|A_{φ_0}. \]

The desiderata (UGP) and (GPR) were proven in [26, Theorem C]. We say that \( φ \in \tilde{Φ}(SO(V_{2n+1})) \) is generic if \( Π_φ \) has a generic representation. Let \( \tilde{Φ}_{gen}(SO(V_{2n+1})) \) be the subset of \( \tilde{Φ}(SO(V_{2n+1})) \) of generic representations. Then we have a sequence
\[ \tilde{Φ}_{temp}(SO(V_{2n+1})) \subset \tilde{Φ}_{gen}(SO(V_{2n+1})) \subset \tilde{Φ}(SO(V_{2n+1})). \]

### 3.4. Type \( C_n \)

Let \( n \geq 1 \) and \( G = Sp(W_{2n}) \) be a (split) symplectic group. Then \( \hat{G} = SO_{2n+1}(C) \) and \( L^G = SO_{2n+1}(C) \times W_F \). An L-parameter \( φ \) of \( G \) gives a self-dual representation \( φ: WD_F \rightarrow GL(N) \) with sign +1, \( \dim C(N) = 2n + 1 \) and \( \det(N) = 1 \). The map \( φ \leftrightarrow \tilde{φ} \) gives a bijection
\[ \tilde{Φ}(Sp(W_{2n})) \rightarrow \tilde{Φ}(Sp(W_{2n})) := \{ φ: WD_F \rightarrow SO(N) \mid \dim C(N) = 2n + 1 \}/ \cong. \]
We identify \( \tilde{φ} \) with \( φ \) via this bijection. Let \( \tilde{Φ}_{temp}(Sp(W_{2n})) \) be the subset of \( \tilde{Φ}(Sp(W_{2n})) \) of tempered representations.

Let \( φ \in \tilde{Φ}(Sp(W_{2n})) \). We denote the centralizer of \( \text{Im}(φ) \) in \( O(N) \) by \( C_φ \) and its component group by \( A_φ = C_φ/C_φ^o \). The centralizer of \( \text{Im}(φ) \) in \( SO(N) \) is denoted by \( C_φ^+ \). We put \( A_φ^+ = \text{Im}(C_φ^+ \rightarrow A_φ) \). Then \( π_0(S_φ) \cong A_φ^+ \) if \( φ \rightarrow \tilde{φ} \). If
\[ N \cong \bigoplus_{i \in I^+} (V_i \otimes N_i) \oplus \bigoplus_{i \in I^-} (W_i \otimes M_i) \oplus \bigoplus_{j \in J} (U_j \otimes (P_j + P_j^\vee)) \]
where
- \( M_i \) (resp. \( N_i \)) is an irreducible self-dual representation of \( WD_F \) with sign −1 (resp. +1),
- \( P_j \) is an irreducible representation of \( WD_F \) which is not self-dual,
which are pairwise inequivalent, and \( V_i, W_i \) and \( U_j \) are multiplicity spaces, then the bilinear form on \( N \) gives a non-degenerate bilinear form on \( V_i \) (resp. \( W_i \)) of sign +1 (resp. −1). Moreover, we have
\[ C_φ \cong \prod_{i \in I^+} O(V_i) \times \prod_{i \in I^-} Sp(W_i) \times \prod_{j \in J} GL(U_j). \]
For a semi-simple element \( a \in C_φ \), we denote by \( N^a \) the \((-1)\)-eigenspace of \( a \) on \( N \). Then \( N^a \) gives an orthogonal representation of \( WD_F \). If \( a_i \in O(V_i) \setminus SO(V_i) \), then \( N^{a_i} = V_i^{a_i} \otimes N_i \). We have
\[ A_φ = \bigoplus_{i \in I^+} (\mathbb{Z}/2\mathbb{Z}) a_i \cong (\mathbb{Z}/2\mathbb{Z})^m, \]
where \( m = \# I^+ \). We see that \( A_φ^+ \) is the kernel of the character
\[ A_φ \ni a_i \mapsto (-1)^{\dim C(N_i)} \in \{ \pm 1 \}. \]
Hence we have \( A_φ^+ \cong (\mathbb{Z}/2\mathbb{Z})^{m-1} \) since some \( N_i \) has odd dimension.

By Kottwitz’s isomorphism, we have \#H^1(F,Sp(W_{2n})) = 1, so that there are no non-trivial pure inner forms of \( G = Sp(W_{2n}) \).
By [2] Theorem 1.5.1 (b), for \( \phi \in \tilde{\Phi}_{\text{temp}}(\text{Sp}(W_{2n})) \), we obtain an \( L \)-packet \( \Pi_\phi \), which is a finite subset of \( \text{Irr}_{\text{temp}}(\text{Sp}(W_{2n})) \), and a canonical bijection \( \iota_{w'}: \Pi_\phi \to \tilde{A}^+_\phi \) for each Whittaker datum \( w_c \) of \( \text{Sp}(W_{2n}) \). We extend \( L \)-packets for general \( \phi \in \tilde{\Phi}(\text{Sp}(W_{2n})) \) as follows. In general, we can write
\[
\phi = \phi_1 \oplus \cdots \oplus \phi_r \oplus \phi_0 \oplus \phi'_r \oplus \cdots \oplus \phi'_1,
\]
where
- \( \phi_i \) is a \( k_i \)-dimensional representation of \( \text{WD}_F \) for \( i = 1, \ldots, r \), which is of the form \( \phi_i = \phi'_i \otimes | \cdot |_F^{s_i} \) for some tempered representation \( \phi'_i \) of \( \text{WD}_F \) and some real number \( s_i \) such that \( s_1 > \cdots > s_r > 0 \), \( n = n_0 + (k_1 + \cdots + k_r) \);
- \( \phi_0 \in \tilde{\Phi}_{\text{temp}}(\text{Sp}(W_{2n})) \) if \( n_0 \geq 1 \), and \( \phi_0 = 1 \) if \( n_0 = 0 \).

We write
\[
W_{2n} = Y_1 \oplus \cdots \oplus Y_r \oplus W_{2n_0} \oplus Y^*_1 \oplus \cdots \oplus Y^*_r,
\]
where \( Y_i \) and \( Y^*_i \) are \( k_i \)-dimensional totally isotropic subspaces of \( W_{2n} \) such that \( Y_i \oplus Y^*_i \) is non-degenerate, mutually orthogonal, and orthogonal to \( W_{2n_0} \). Let \( Q \) be the parabolic subgroup of \( \text{Sp}(W_{2n}) \) stabilizing the flag
\[
Y_1 \subset Y_1 + Y_2 \subset \cdots \subset Y_1 + \cdots + Y_r
\]
and \( M_Q \) be the Levi component of \( Q \) stabilizing the flag
\[
Y^*_1 \subset Y^*_1 + Y^*_2 \subset \cdots \subset Y^*_1 + \cdots + Y^*_r,
\]
so that
\[
M_Q \cong \text{GL}(Y_1) \times \cdots \times \text{GL}(Y_r) \times \text{Sp}(W_{2n_0}).
\]
Then the \( L \)-packet \( \Pi_\phi \) consists of the unique irreducible quotients \( \pi \) of the standard modules
\[
\text{Ind}_Q^{\text{Sp}(W_{2n})}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi_0),
\]
where \( \tau_i \) is the irreducible essentially tempered representation of \( \text{GL}(Y_i) \) corresponding to \( \phi_i \) for \( i = 1, \ldots, r \), and \( \pi_0 \) runs over elements in \( \Pi_{\phi_0} \) if \( n_0 > 0 \). If \( n_0 = 0 \), then we ignore \( \pi_0 \). In particular, if \( n_0 = 0 \), then \( \Pi_\phi \) is singleton. If \( n_0 \geq 1 \), then the natural map \( A_{\phi_0} \to A_\phi \) is an isomorphism, and we define \( \iota(\pi) \in \tilde{A}_\phi \) by
\[
\iota(\pi) = \iota(\pi)|_{A_{\phi_0}}.
\]

The desideratum (UGP) has not been proven. For (GPR), there is a result in [37], which says that \( \phi \) is generic, i.e., \( \Pi_\phi \) has a generic element if and only if \( L(s, \phi, \text{Ad}) \) is regular at \( s = 1 \). Let \( \tilde{\Phi}_{\text{gen}}(\text{Sp}(W_{2n})) \) be the subset of \( \tilde{\Phi}(\text{Sp}(W_{2n})) \) of generic representations.

Recall that the set of Whittaker data of \( \text{Sp}(W_{2n}) \) is parametrized by \( F^\times / F^{x, 2} \) (once we fix a non-trivial additive character \( \psi \) of \( F \)). For \( c \in F^\times \) and a semi-simple element \( a \) in \( C^+_F \), we put \( Q_c(a) := \text{det}(N^c)(c) \). By [10] §4, this gives a character \( \eta_c \) of \( A^+_\phi \). The following proposition is a special case of [27] Theorem 3.3.

**Proposition 3.1.** For \( c_1, c_2 \in F^\times \), the diagram
\[
\begin{array}{ccc}
\Pi_\phi & \xrightarrow{\iota_{w_1}} & \tilde{A}^+_\phi \\
\end{array}
\]
is commutative.

We decompose \( W_{2n} = Y_n \oplus Y_n^* \) as in [27]. We define \( \delta \in \text{GL}(W_{2n}) \) by \( \delta|_{Y_n} = 1_{Y_n} \) and \( \delta|_{Y_n^*} = -1_{Y_n^*} \). Then \( \delta \in \text{GSp}(W_{2n}) \) with similitude factor \(-1\), and satisfies
\[
(\delta w, \delta w')|_{W_{2n}} = (w', w)|_{W_{2n}}
\]
for \( w, w' \in W_{2n} \). Hence for \( \pi \in \text{Irr}(\text{Sp}(W_{2n})) \), we have \( \pi^\vee \cong \pi^\delta \) by [10] Chapter 4. II. 1]. This implies that
\[
\iota_{w_1}(\pi^\vee) = \iota_{w_1}(\pi^\delta) = \iota_{w_1}(\pi) = \iota_{w_1}(\pi) \otimes \eta_{-1}.
\]
We say that \( \phi \in \widetilde{\Phi}(\text{Sp}(W_{2n})) \) is discrete if \( I^- = J = \emptyset \) and \( \dim(V_i) = 1 \) for each \( i \in I^+ \). In this case, \( C^+_\phi \) is a finite group and \( \Pi_\phi \) consists of irreducible (unitary) discrete series representations (i.e., square-integrable representations) of \( \text{Sp}(W_{2n}) \). If \( \phi \) is a tempered but not discrete parameter of \( \text{Sp}(W_{2n}) \), then we can write
\[
\phi = \phi_1 + \phi_0 + \phi'_Y,
\]
where

- \( \phi_1 \) is a \( k \)-dimensional irreducible representation of \( WD_F \) for some positive integer \( k \),
- \( \phi_0 \in \Phi_{\text{temp}}(\text{Sp}(W_{2n_0})) \) if \( n_0 = n - k \geq 1 \), and \( \phi_0 = 1 \) if \( n_0 = n - k = 0 \).

In this case, there is a natural embedding \( A^+_\phi \hookrightarrow A^+_\phi \), where we put \( A^+_\phi = 1 \) if \( n_0 = 0 \). We can decompose
\[
W_{2n} = Y_k + W_{2n_0} + Y_k^*
\]
and define a parabolic subgroup \( Q = Q_k = MQU_Q \) of \( \text{Sp}(W_{2n}) \) as in [23]. Hence,
\[
MQ \cong \text{GL}(Y_k) \times \text{Sp}(W_{2n_0}).
\]
Let \( \pi \) be the irreducible (unitary) discrete series representation of \( \text{GL}(Y_k) \) associated to \( \phi_1 \). For \( \pi_0 \in \Pi_{\phi_0} \), the induced representation \( \text{Ind}_{M}^{\text{Sp}(W_{2n})}(\pi \otimes \pi_0) \) decomposes to a direct sum of irreducible representations, and
\[
\{ \pi \in \text{Irr}(\text{Sp}(W_{2n})) \mid \pi \subset \text{Ind}_{M}^{\text{Sp}(W_{2n})}(\pi_0) \} = \{ \pi \in \Pi_{\phi} \mid \ell_{\pi}(\pi_0)|A^+_\phi = \ell_{\pi_0}(\pi_0) \}
\]
for any \( c \in F^\times \). Here, if \( n_0 = 0 \), then we ignore \( \pi_0 \) and interpret the right hand side to be \( \Pi_\phi \). By using an intertwining operator, we can determine \( \ell_{\pi}(\pi_0) \) completely. More precisely, see [33] below.

Let \( \Phi_{\text{disc}}(\text{Sp}(W_{2n})) \) be the subset of \( \Phi(\text{Sp}(W_{2n})) \) of discrete representations. Then we have a sequence
\[
\Phi_{\text{disc}}(\text{Sp}(W_{2n})) \subset \Phi_{\text{temp}}(\text{Sp}(W_{2n})) \subset \Phi_{\text{gen}}(\text{Sp}(W_{2n})) \subset \Phi(\text{Sp}(W_{2n})).
\]

3.5. Type \( D_n \). Let \( n \geq 1 \) and \( G = \text{SO}(V_{2n}) \) be a quasi-split special orthogonal group. We put \( E = F(\sqrt{\text{disc}(V_{2n})}) \). Then \( \widetilde{G} = \text{SO}_2(\mathbb{C}) \) and \( \mathbb{L} \text{SO}_2(\mathbb{C}) \times W_F \). The action of \( W_F \) on \( \text{SO}_2(\mathbb{C}) \) factors through \( W_F \rightarrow \text{Gal}(E/F) \). For \( \gamma \in W_F \) and \( \phi \in \Phi(\text{SO}(V_{2n})) \), we define \( \gamma \phi(x) = \gamma \cdot \phi(x) \cdot \gamma^{-1} \).

An \( L \)-parameter \( \phi \) of \( G \) gives a self-dual representation \( \phi : WD_F \rightarrow GL(N) \) with sign \( +1 \), \( \dim(C(N)) = 2n \) and \( \text{det}(N) = \chi_{V_{2n}} \). The map \( _\phi \mapsto \phi \) gives a surjective map
\[
\Phi(\text{SO}(V_{2n})) \rightarrow \Phi(\text{SO}(V_{2n})) := \{ \phi : WD_F \rightarrow O(N) \mid \dim(C(N)) = 2n, \text{det}(N) = \chi_{V_{2n}} \} .
\]
Note that \( _\phi \) and \( \phi \) give the same self-dual representation. By [10] Theorem 8.1, for \( \phi \in \Phi(\text{SO}(V_{2n})) \), the inverse image of \( \phi \) under this map has one or two elements. We say that \( \phi \in \Phi(\text{SO}(V_{2n})) \) is \( \epsilon \)-invariant if the inverse image of \( \phi \) under this map is singleton. Let \( \Phi_{\text{temp}}(\text{SO}(V_{2n})) \) be the subset of \( \Phi(\text{SO}(V_{2n})) \) of tempered representations.

We denote the centralizer of \( \text{Im}(\phi) \) in \( O(N) \) by \( C_\phi \) and its component group by \( A_\phi = C_\phi/C^+_\phi \). The centralizer of \( \text{Im}(\phi) \) in \( \text{SO}(N) \) is denoted by \( C^+_\phi \). We put \( A^+_\phi = \text{Im}(C^+_\phi \rightarrow A_\phi) \). Then \( \pi_0(S_\phi) \cong A^+_\phi \) if \( _\phi \mapsto \phi \).

If
\[
N \cong \bigoplus_{i \in I^+} (V_i \otimes N_i) \oplus \bigoplus_{i \in I^-} (W_i \otimes M_i) \oplus \bigoplus_{j \in J} (U_j \otimes (P_j + P_j'))
\]
where

- \( M_i \) (resp. \( N_i \)) is an irreducible self-dual representation of \( WD_F \) with sign \(-1 \) (resp. \(+1 \)),
- \( P_j \) is an irreducible representation of \( WD_F \) which is not self-dual, which are pairwise inequivalent, and \( V_i, W_i, \) and \( U_j \) are multiplicity spaces, then the bilinear form on \( N \) gives a non-degenerate bilinear form on \( V_i \) (resp. \( W_i \)) of sign \(+1 \) (resp. \(-1 \)). Note that \( \phi \) is \( \epsilon \)-invariant if and only if \( \dim(C(N_i)) \) is odd for some \( i \in I^+ \) (see [10] Theorem 8.1]). Moreover, we have
\[
C_\phi \cong \prod_{i \in I^+} O(V_i) \times \prod_{i \in I^-} \text{Sp}(W_i) \times \prod_{j \in J} \text{GL}(U_j) .
\]
For a semi-simple element $a \in C_φ$, we denote by $N^a$ the $(-1)$-eigenspace of $a$ on $N$. Then $N^a$ gives an orthogonal representation of $WD_F$. If $a_i \in O(V_i) \setminus SO(V_i)$, then $N^{a_i} = V_i^{a_i} \otimes N_i$. We have
\[
A_φ = \bigoplus_{i \in \mathcal{I}^+} (\mathbb{Z}/2\mathbb{Z}) a_i \cong (\mathbb{Z}/2\mathbb{Z})^m,
\]
where $m = \# \mathcal{I}^+$. We see that $A_φ^+$ is the kernel of the character
\[
A_φ \ni a_i \mapsto (-1)^{\dim(\mathcal{N}_i)}.
\]
Hence we have
\[
A_φ^+ \cong \begin{cases} 
(\mathbb{Z}/2\mathbb{Z})^{m-1} & \text{if } φ \text{ is } ε\text{-invariant}, \\
(\mathbb{Z}/2\mathbb{Z})^m & \text{otherwise}.
\end{cases}
\]

By Kottwitz’s isomorphism, we have
\[
\# H^1(F, SO(2n)) = \begin{cases} 
1 & \text{if } \dim(V_{2n}) = 2n = 2 \text{ and } \text{disc}(V_{2n}) = 1, \\
2 & \text{otherwise}.
\end{cases}
\]
Hence if $n = 1$ and $\text{disc}(V_{2n}) = 1$, then there are no non-trivial pure inner forms of $G = SO(2n)$. Otherwise, there are exactly two pure inner forms of $G = SO(2n)$. They are $SO(2n)$ and $SO(2n')$, where $\dim(V_{2n}) = \dim(V_{2n'}) = 2n$ and $\text{disc}(V_{2n}) = \text{disc}(V_{2n'})$. We see that
\[
SO(2n) \begin{cases} \text{split} & \text{if } \text{disc}(V_{2n}) = 1, \\
\text{quasi-split but not split} & \text{if } \text{disc}(V_{2n}) \neq 1,
\end{cases}
SO(2n') \begin{cases} \text{not quasi-split} & \text{if } \text{disc}(V_{2n'}) = 1, \\
\text{quasi-split but not split} & \text{if } \text{disc}(V_{2n'}) \neq 1.
\end{cases}
\]

By [2 Theorem 1.5.1 (b)], for $φ \in \Phi_{\text{temp}}(SO(2n))$, we obtain an $L$-packet $Π_φ^A$, which is a finite subset of $\text{Irr}_{\text{temp}}(SO(2n)) / \sim_ε$, and a canonical bijection $ι_{m_ε}^A : Π_φ^A \to π_0(C_φ / \{±1\})$ for each Whittaker datum $w_ε$ of $SO(2n)$. Moreover, the following are equivalent:

- $φ$ is $ε$-invariant;
- $Π_φ$ has an orbit of an $ε$-invariant element, i.e., an orbit of order one;
- $Π_φ$ consists of orbits of an $ε$-invariant element.

This is the reason why we say that $φ$ is $ε$-invariant. We assume that there are a Vogan $L$-packet $Π_φ \subset \bigcup_{V^* \in Ξ} \text{Irr}_{\text{temp}}(SO(V_{2n})) / \sim_ε$ such that $Π_φ \cap \text{Irr}_{\text{temp}}(SO(V_{2n})) / \sim_ε = Π_φ^A$, and an extension $ι_{m_ε} : Π_φ \to Π_φ^A$ of $ι_{m_ε}^A$. Moreover we assume that $φ$ is $ε$-invariant if and only if $Π_φ$ consists of orbits of order one. We summarize these results and assumptions in Table 2 below. Here, $[σ] \in Π_φ$ with $σ \in \text{Irr}(SO(V_{2n}))$, and $\tilde{σ}$ is an irreducible constituent of $\text{Ind}_{SO(V_{2n})}^{SO(V_{2n})}(σ)$.

**Table 2.**

| $φ$ | $ε$-invariant | not $ε$-invariant |
|-----|----------------|-------------------|
| $\dim(\mathcal{N}_i)$: odd | contained | not contained |
| $A_φ$ | $A_φ^+ \neq A_φ$ | $A_φ^+ = A_φ$ |
| order of $[σ]$ | one | two |
| $σ^2$ | $σ \cong σ^2$ | $σ \not\cong σ^2$ |
| $\text{Ind}_{SO(V_{2n})}^{SO(V_{2n})}(σ)$ | $\tilde{σ} \oplus (\tilde{σ} \otimes \det)$ | irreducible |
| $\tilde{σ} \otimes \det$ | $\tilde{σ} \not\cong \tilde{σ} \otimes \det$ | $\tilde{σ} \cong \tilde{σ} \otimes \det$ |
| $σ \otimes \det$ | irreducible | $σ \not\cong σ^2$ |
We extend \( L \)-packets for general \( \phi \in \Phi(\text{SO}(V_{2n})) \) as follows. In general, we can write
\[
\phi = \phi_1 \oplus \cdots \oplus \phi_r \oplus \phi_0 \oplus \phi'_r \oplus \cdots \oplus \phi'_{1},
\]
where
- \( \phi_i \) is a \( k_i \)-dimensional representation of \( WD_F \) for \( i = 1, \ldots, r \), which is of the form \( \phi_i = \phi'_i \otimes |·|^s_i \) for some tempered representation \( \phi'_i \) of \( WD_F \) and some real number \( s_i \) such that \( s_1 > \cdots > s_r > 0 \), \( n = n_0 + (k_1 + \cdots + k_r) \);
- \( \phi_0 \in \Phi_{\text{temp}}(\text{SO}(V_{2n})) \) if \( n_0 \geq 0 \), and \( \phi_0 = 0 \) if \( n_0 = 0 \).

First, we assume that \( n_0 > 0 \). We write
\[
V_{2n}^\bullet = X_1 \oplus \cdots \oplus X_r \oplus V_{2n_0}^\bullet \oplus X_1^* \oplus \cdots \oplus X_1^*,
\]
where \( X_i \) and \( X_i^* \) are \( k_i \)-dimensional totally isotropic subspaces of \( V_{2n}^\bullet \), such that \( X_i \oplus X_i^* \) is non-degenerate, mutually orthogonal, and orthogonal to \( V_{2n_0}^\bullet \). Let \( P^\circ \) be the parabolic subgroup of \( \text{SO}(V_{2n}^\bullet) \) stabilizing the flag
\[
X_1 \subset X_1 \oplus X_2 \subset \cdots \subset X_1 \oplus \cdots \oplus X_r
\]
and \( M^\circ_P \) be the Levi component of \( P^\circ \) stabilizing the flag
\[
X_i^* \subset X_i^* \oplus X_2^* \subset \cdots \subset X_i^* \oplus \cdots \oplus X_r^*,
\]
so that
\[
M^\circ_P \cong \text{GL}(X_1) \times \cdots \times \text{GL}(X_r) \times \text{SO}(V_{2n_0}^\bullet).
\]
Then the \( L \)-packet \( \Pi_\phi \) consists of the \( \text{O}(V_{2n}^\bullet) \)-orbits of the unique irreducible quotients \( \sigma \) of the standard modules
\[
\text{Ind}^{\text{SO}(V_{2n}^\bullet)}_{P^\circ}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \sigma_0),
\]
where \( \tau_i \) is the irreducible essentially tempered representation of \( \text{GL}(X_i) \) corresponding to \( \phi_i \) for \( i = 1, \ldots, r \), and \( \sigma_0 \) runs over elements in \( \Pi_\phi_{\text{gen}} \). Since \( V_{2n_0} \neq 0 \) and
\[
\text{Ind}^{\text{SO}(V_{2n}^\bullet)}_{P^\circ}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \sigma_0)^{\varepsilon} \cong \text{Ind}^{\text{SO}(V_{2n}^\bullet)}_{P^\circ}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \sigma_0^{\varepsilon})
\]
for \( \varepsilon \in \text{O}(V_{2n_0}) \), the \( L \)-packet \( \Pi_\phi \) is well-defined. The natural map \( A_{\phi_0} \to A_\phi \) is an isomorphism, and we define \( \iota(\pi) \in A_\phi \) by
\[
\iota(\pi_0) = \iota(\pi)|A_{\phi_0}.
\]
If \( n_0 = 0 \), then \( \text{disc}(\phi) = 1 \), so that \( \phi \in \Phi(\text{SO}(V_{2n})) \) where \( \text{SO}(V_{2n}) \) is split. In this case, the \( L \)-packet \( \Pi_\phi \) is singleton and the unique element of \( \Pi_\phi \) is the \( \text{O}(V_{2n}) \)-orbit consisting of the unique irreducible quotients \( \sigma \) and \( \sigma^+ \) of the standard modules
\[
\text{Ind}^{\text{SO}(V_{2n})}_{P^\circ}(\tau_1 \otimes \cdots \otimes \tau_r) \quad \text{and} \quad \text{Ind}^{\text{SO}(V_{2n})}_{P^\circ}(\tau_1 \otimes \cdots \otimes \tau_r)^{\varepsilon}.
\]

The desideratum (UGP) has not been proven. For (GPR), there is a result for a split group \( \text{SO}(V_{2n}) \) in [24], which says that \( \phi \) is generic, i.e., \( \Pi_\phi \) has an orbit of generic representations if and only if \( L(s, \phi, \text{Ad}) \) is regular at \( s = 1 \). Let \( \Phi_{\text{gen}}(\text{SO}(V_{2n})) \) be the subset of \( \Phi(\text{SO}(V_{2n})) \) of generic representations.

If \( \text{disc}(V_{2n}) = 1 \), then the set of Whittaker data of \( \text{SO}(V_{2n}) \) is parametrized by \( F^x/F^{x+2} \). If \( \text{disc}(V_{2n}) \neq 1 \), then both \( \text{SO}(V_{2n}) \) and \( \text{SO}(V_{2n}^\prime) \) are quasi-split, and the set
\[
\bigcup_{V_{2n}^\bullet, \eta} \{ \text{Whittaker data of } \text{SO}(V_{2n}^\bullet) \}
\]
is parametrized by \( F^x/F^{x+2} \). Note that \( \Phi(\text{SO}(V_{2n})) = \Phi(\text{SO}(V_{2n}^\prime)) \). An element in this set gives a (Vogan) \( L \)-packet of \( \text{SO}(V_{2n}) \) and one of \( \text{SO}(V_{2n}^\prime) \). They are both subsets of \( \cup_{V_{2n}^\bullet} \text{Irr}(\text{SO}(V_{2n}^\bullet))/\sim_\varepsilon \). By Appendix A.4 these \( L \)-packets coincide. Hence, in each cases, there exists a bijection
\[
\iota_{\text{gen}} : \Pi_\phi \to \widehat{A}_{\phi}^+,
\]
for any \( \phi \in \Phi(\text{SO}(V_{2n})) \) and \( \varepsilon \in F^x/F^{x+2} \). For \( \varepsilon \in F^x \) and a semi-simple \( a \) in \( C^+_\phi \), we put \( \eta_{\varepsilon}(a) := \det(N^a)(c) \). By [10, §4], this gives a character \( \eta_{\varepsilon} \) of \( A_{\phi}^+ \). Then we have the following proposition.
Proposition 3.2. For $c_1, c_2 \in F^\times$, the diagram

\[
\begin{array}{ccc}
\Pi_\phi & \xrightarrow{\iota_{\varpi_1}} & \widehat{A}^+_\phi \\
\| & & \| \\
\Pi_\phi & \xrightarrow{\iota_{\varpi_2/c_1}} & \widehat{A}^+_\phi \\
\end{array}
\]

is commutative.

The case when both $\varpi_{c_1}$ and $\varpi_{c_2}$ are Whittaker data of $SO(V_{2n})$ is Kaletha’s result ([27, Theorem 3.3]). The proof of this proposition is given in Appendix A.8.

We say that $\phi \in \tilde{\Phi}(SO(V_{2n}))$ is discrete if $I^- = J = \emptyset$ and $\dim(V_i) = 1$ for each $i \in I^+$. In this case, $C^+_\phi$ is a finite group and $\Pi_\phi$ consists of orbits of irreducible (unitary) discrete series representations (i.e., square-integrable representations) of $SO(V_{2n})$. If $\phi \in \tilde{\Phi}(SO(V_{2n}))$ is tempered but not discrete, then we can write

$$\phi = \phi_1 + \phi_0 + \phi_1^\vee,$$

where

- $\phi_1$ is a $k$-dimensional irreducible representation of $WD_F$ for some positive integer $k$,
- $\phi_0 \in \tilde{\Phi}_{\text{temp}}(SO(V_{2n_0}))$ if $n_0 = n - k \geq 1$, and $\phi_0 = 0$ if $n_0 = n - k = 0$.

First, we assume that $n_0 > 0$. In this case, there is a natural embedding $A^+_\phi \hookrightarrow A^+_\phi$. Let $[\sigma_0] \in \Pi_{\phi_0}$ with $\sigma_0 \in \text{Irr}(SO(V^*_n))$. We can decompose

$$V^*_{2n} = X_k + V^*_{2n_0} + X_k^*$$

and define a parabolic subgroup $P^o = P^o_k = M^o_k U_P$ of $SO(V^*_{2n})$ as in ([24]). Hence,

$$M^o_k \cong GL(X_k) \times SO(V^*_n).$$

Let $\tau$ be the irreducible (unitary) discrete series representation of $GL(X_k)$ associated to $\phi_1$. Then the induced representation $\text{Ind}^{\text{SO}(V^*_{2n})}_{P^o}(\tau \otimes \sigma_0)$ decomposes to a direct sum of irreducible representations, and

$$\{[\sigma] \mid \sigma \in \text{Irr}(SO(V^*_{2n})), \sigma \subset \text{Ind}^{\text{SO}(V^*_{2n})}_{P^o}(\tau \otimes \sigma_0)\} = \{[\sigma] \in \Pi_\phi \mid \iota_{\varpi_1}(\sigma)\}.$$ 

Note that the left hand side is independent of the choice of a representative $\sigma_0$ of $[\sigma_0]$. By using an intertwining operator, we can determine $\iota_{\varpi_1}(\sigma)$ completely. More precisely, see ([33], below. If $n_0 = 0$, then $\text{det}(\phi) = 1$, so that $\phi \in \tilde{\Phi}(SO(V_{2n}))$ where $SO(V_{2n})$ is split. Moreover the induced representation $\text{Ind}^{\text{SO}(V_{2n})}_{P^o}(\tau)$ decomposes to a direct sum of irreducible representations, and

$$\Pi_\phi = \{[\sigma] \mid \sigma \in \text{Irr}(SO(V_{2n})), \sigma \subset \text{Ind}^{\text{SO}(V_{2n})}_{P^o}(\tau)\}.$$ 

Let $\tilde{\Phi}_{\text{disc}}(SO(V_{2n}))$ be the subset of $\tilde{\Phi}(SO(V_{2n}))$ of discrete representations. Then we have a sequence

$$\tilde{\Phi}_{\text{disc}}(SO(V_{2n})) \subset \tilde{\Phi}_{\text{temp}}(SO(V_{2n})) \subset \tilde{\Phi}_{\text{gen}}(SO(V_{2n})) \subset \tilde{\Phi}(SO(V_{2n})).$$

3.6. Metaplectic groups. Let $\tilde{\text{Sp}}(W_{2n})$ be the metaplectic group, i.e., the non-trivial central extension of $\text{Sp}(W_{2n})$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \tilde{\text{Sp}}(W_{2n}) \longrightarrow \text{Sp}(W_{2n}) \longrightarrow 1.$$ 

We denote the set of equivalence classes of irreducible genuine representations of $\tilde{\text{Sp}}(W_{2n})$ by $\text{Irr}(\tilde{\text{Sp}}(W_{2n}))$.

We recall some results of Gan–Savin (Theorem 1.1 and Corollary 1.2 in [15]).

Theorem 3.3. Corresponding to the choice of a non-trivial additive character $\psi$ of $F$, there is a natural bijection given by the theta correspondence:

$$\text{Irr}(\tilde{\text{Sp}}(W_{2n})) \rightarrow \prod \text{Irr}((SO(V_{2n+1}^{(1)})),$n$$

where the union is taken over all the isomorphism classes of orthogonal spaces $V_{2n+1}^{(1)}$ over $F$ with $\dim(V_{2n+1}^{(1)}) = 2n + 1$ and $\text{disc}(V_{2n+1}^{(1)}) = 1$. 

More precisely, for $\pi \in \text{Irr}(\tilde{\text{Sp}}(W_{2n}))$, there is a unique $V_{2n+1}^{(1)}$ as above such that $\theta_{\psi,V_{2n+1}^{(1)},W_{2n}}(\pi)$ is nonzero, in which case the image of $\pi$ is $\theta(\pi) = \theta_{\psi,V_{2n+1}^{(1)},W_{2n}}(\pi)|_{\text{SO}(V_{2n+1}^{(1)})} \in \text{Irr}(\text{SO}(V_{2n+1}^{(1)}))$.

**Corollary 3.4.** Suppose that the local Langlands conjecture holds for $\text{SO}(V_{2n+1})$ with $\dim(V_{2n+1}) = 2n + 1$ and $\text{disc}(V_{2n+1}) = 1$. Then one has a surjection (depending on $\psi$)

$$L_\psi: \text{Irr}(\tilde{\text{Sp}}(W_{2n})) \to \tilde{\Phi}(\text{SO}(V_{2n+1})).$$

For $\phi \in \tilde{\Phi}(\text{SO}(V_{2n+1}))$, we denote by $\Pi_\phi$ the inverse image of $\phi$ under this map, and call $\Pi_\phi$ the $L$-packet of $\phi$. Moreover, the composition of $\iota_\phi$ and theta lifts gives a bijection (depending on $\psi$) of

$$\iota_\psi: \Pi_\phi \to \tilde{A}_\phi.$$

We put $\Phi(\tilde{\text{Sp}}(W_{2n})) := \Phi(\text{SO}(V_{2n+1}))$ and we call $\phi \in \Phi(\tilde{\text{Sp}}(W_{2n}))$ an $L$-parameter of $\tilde{\text{Sp}}(W_{2n})$. This is also an $L$-parameter of $\text{SO}(V_{2n+1})$. However we use

$$\tilde{\Phi}(\tilde{\text{Sp}}(W_{2n})) = \tilde{\Phi}(\text{SO}(V_{2n+1})) = \{ WD_F \to \text{Sp}(M) \mid \dim_C(M) = 2n \}/ \cong$$

in this paper. Note that there is a bijection $\Phi(\tilde{\text{Sp}}(W_{2n})) \to \tilde{\Phi}(\text{Sp}(W_{2n}))$. We regard $L_\psi$ as a map

$$L_\psi: \text{Irr}(\tilde{\text{Sp}}(W_{2n})) \to \tilde{\Phi}(\text{Sp}(W_{2n})).$$

By [15 Theorem 12.1], we have the following theorem.

**Theorem 3.5.** For $\pi \in \text{Irr}(\tilde{\text{Sp}}(W_{2n}))$ and $c \in F^\times$, we put $(L_\psi(\pi), t_\psi(\pi)) = (\phi, \eta)$ and $(L_\psi(\pi), t_\psi(\pi)) = (\phi_c, \eta_c)$, where $\phi: WD_F \to \text{Sp}(M)$ is a symplectic representation of $WD_F$. Then the following hold.

1. $\phi_c = \phi \otimes \chi_c$, where $\chi_c$ is the quadratic character associated to $c \in F^\times/F^\times 2$. In particular, we have a canonical identification $A_\phi = A_{\phi_c}$.

2. The characters $\eta$ and $\eta_c$ are related by

$$\eta_c(a)/\eta(a) = \varepsilon(M^a)\varepsilon(M^a \otimes \chi_c) \cdot \chi_c(-1) \dim_C(M^a)/2 \in \{ \pm 1 \}$$

for $a \in A_\phi = A_{\phi_c}$.

By the above theorem, we have a parametrization of $\text{Irr}(\tilde{\text{Sp}}(W_{2n}))$ using $\text{Irr}(\text{SO}(V_{2n+1}))$ with $\text{disc}(V_{2n+1}) = c$ as follows.

**Corollary 3.6.** Suppose that the local Langlands conjecture holds for $\text{SO}(V_{2n+1})$ with $\dim(V_{2n+1}) = 2n + 1$ and $\text{disc}(V_{2n+1}) = 1$. We fix $c \in F^\times$.

1. Corresponding to the choice of a non-trivial additive character $\psi$ of $F$, there is a natural bijection

$$\theta_\psi: \text{Irr}(\tilde{\text{Sp}}(W_{2n})) \to \bigsqcup \text{Irr}(\text{SO}(V_{2n+1}^{(c)}))$$

where the union is taken over all the isomorphism classes of orthogonal spaces $V_{2n+1}^{(c)}$ over $F$ with $\dim(V_{2n+1}^{(c)}) = 2n + 1$ and $\text{disc}(V_{2n+1}^{(c)}) = c$.

More precisely, for $\pi \in \text{Irr}(\tilde{\text{Sp}}(W_{2n}))$, there is a unique $V_{2n+1}^{(c)}$ as above such that $\theta_{\psi,V_{2n+1}^{(c)},W_{2n}}(\pi)$ is nonzero, in which case the image of $\pi$ is $\theta_{\psi,V_{2n+1}^{(c)},W_{2n}}(\pi)|_{\text{SO}(V_{2n+1}^{(c)})} \in \text{Irr}(\text{SO}(V_{2n+1}^{(c)}))$.

2. The diagram

$$\begin{array}{ccc}
\text{Irr}(\tilde{\text{Sp}}(W)) & \xrightarrow{\theta_\psi} & \bigsqcup \text{Irr}(\text{SO}(V_{2n+1}^{(c)})) \\
\downarrow \tilde{\Phi}(\tilde{\text{Sp}}(W_{2n})) & & \downarrow \\
\tilde{\Phi}(\tilde{\text{Sp}}(W_{2n})) & \xrightarrow{\otimes \chi_c} & \tilde{\Phi}(\tilde{\text{Sp}}(W_{2n})) = \tilde{\Phi}(\text{SO}(V_{2n+1}))
\end{array}$$
is commutative, where the bottom arrow is given by \( \eta \mapsto \eta_c \) as in the above theorem.

Proof. Let \((V_{2n+1}^{(c)}, \langle \cdot, \cdot \rangle_c)\) be an orthogonal space with \( \text{disc}(V_{2n+1}^{(c)}) = c \). We consider a new orthogonal space \((V_{c}^{(1)}, \langle \cdot, \cdot \rangle_{1})\) defined by \( V_{c}^{(1)} = V_{2n+1}^{(1)} \) as vector spaces, and \( \langle \cdot, \cdot \rangle_{1} = c^{-1} \langle \cdot, \cdot \rangle_{c} \). Then we have \( \text{disc}(V_{c}^{(1)}) = c^{-1} \cdot \text{disc}(V_{2n+1}^{(c)}) = 1 \) in \( F^x / F^{x^2} \). Moreover by explicit formulas of the Weil representations, we have

\[
\omega_{\psi, V_{2n+1}^{(c)}, W_{2n}} = \omega_{\psi, V_{c}^{(1)}, W_{2n}}.
\]

So the first assertion follows from Theorem 3.3. In addition, the bottom arrows in the diagrams of the second assertion are \( L_{\psi_c} \circ L_{\psi}^{-1} \) and \( \psi_c \circ \psi^{-1} \), respectively. Hence the second assertion follows from Theorem 3.5. \( \square \)

To parametrize \( \text{Irr}(\widetilde{\text{Sp}}(W_{2n})) \), we use \( L_{\psi} \) and \( \psi \). Based on the desideratum \((\text{GPR})\), we say that \( \phi \in \widetilde{\text{Sp}}(W_{2n}) \) is generic (resp. tempered) if so is \( \psi \) as an element in \( \Phi(\text{SO}(V_{2n+1})) \). We put \( \Phi_{\text{gen}}(\text{Sp}(W_{2n})) = \Phi_{\text{gen}}(\text{SO}(V_{2n+1})) \) and \( \Phi_{\text{temp}}(\text{Sp}(W_{2n})) = \Phi_{\text{temp}}(\text{SO}(V_{2n+1})) \).

As in [2] we can define a parabolic subgroup \( Q \) of \( \text{Sp}(W_{2n}) \) with the Levi subgroup isomorphic to \( \text{GL}_{n_1}(F) \times \cdots \times \text{GL}_{n_r}(F) \times \text{Sp}(W_{2n_0}) \) with \( n_1 + \cdots + n_r + n_0 = n \), and a Borel subgroup \( B' = T'U' \) of \( \text{Sp}(W_{2n}) \). Here, \( \text{GL}(F) \) is the double cover of \( \text{GL}_n(F) \), \( B' = T'U' \) is the Borel subgroup of \( \text{Sp}(W_{2n}) \) defined in [2] and \( B' \) is the inverse image of \( B' \) in \( \text{Sp}(W_{2n}) \). More precisely, see [13] [2]. Let \( \mu_{c} \) be the generic character of \( U' \) defined in [2]. We say that \( \pi \in \text{Irr}(\text{Sp}(W_{2n})) \) is \( w_{c} \)-generic if \( \text{Hom}_{\text{Sp}(W_{2n})}(\pi, \mu_{c}) \neq 0 \). The following proposition is a part of Theorem 1.3 (and Theorem 8.1) in [13].

**Proposition 3.7.** Let \( V = V_{2n+1} \) be an orthogonal space over \( F \) with \( \dim(V) = 2n+1 \) and \( \text{disc}(V) = 1 \), and \( W = W_{2n} \) be a symplectic space over \( F \) with \( \dim(W) = 2n \). For \( \sigma \in \text{Irr}(\text{SO}(V_{2n+1})) \), by Theorem 3.5, we can take a unique extension \( \tilde{\sigma} \in \text{Irr}(\text{O}(V)) \) of \( \sigma \) such that \( \Theta_{\psi, V, W}(\tilde{\sigma}) \) is nonzero, so that \( \Theta_{\psi, V, W}(\tilde{\sigma}) \in \text{Irr}(\text{Sp}(W_{2n})) \).

1. The representation \( \sigma \) is tempered if and only if \( \Theta_{\psi, V, W}(\tilde{\sigma}) \) is irreducible and tempered. In particular, in this case we have \( \Theta_{\psi, V, W}(\tilde{\sigma}) = \Theta_{\psi, V, W}(\tilde{\sigma}) \).

2. If \( \tilde{\sigma} \) is the unique irreducible quotient of

\[
\text{Ind}_{P_{\mathcal{O}}(V)}(\tau_1 | \det |_{\mathcal{O}}^{s_1} \cdots \det |_{\mathcal{O}}^{s_r} | \det |_{\mathcal{O}}^{s_0} |_{\mathcal{O}}^{s_0}),
\]

where \( P \) is a parabolic subgroup of \( \mathcal{O}(V) \) with the Levi subgroup isomorphic to \( \text{GL}_{n_1}(F) \times \cdots \times \text{GL}_{n_r}(F) \times \text{O}(V_0) \), \( \tau_1 \in \text{Irr}\text{temp}(\text{GL}_{n_1}(F)) \), \( \tau_0 \in \text{Irr}\text{temp}(\text{O}(V_0)) \), and \( s_1 > \cdots > s_r > 0 \). Then we have

\[
\text{Ind}_{Q}(\tau_1 | \det |_{\mathcal{O}}^{s_1} \cdots \det |_{\mathcal{O}}^{s_r} | \det |_{\mathcal{O}}^{s_0} \otimes \Theta_{\psi, V, W}(\tau_0)) \rightarrow \Theta_{\psi, V, W}(\tilde{\sigma}),
\]

where \( Q \) is the parabolic subgroup of \( \text{Sp}(W_{2n}) \) with the Levi subgroup isomorphic to \( \text{GL}_{n_1}(F) \times \cdots \times \{\pm 1\} \text{GL}_{n_r}(F) \times \text{Sp}(W_{2n_0}) \) and \( \dim(V_0) = \dim(W) + 1 \). In particular, \( \Theta_{\psi, V, W}(\tilde{\sigma}) \) is the unique irreducible quotient of the standard module \( \text{Ind}_{Q}(\tau_1 | \det |_{\mathcal{O}}^{s_1} \cdots \det |_{\mathcal{O}}^{s_r} | \det |_{\mathcal{O}}^{s_0} \otimes \Theta_{\psi, V, W}(\tau_0)) \).

3. If \( \sigma \) is \( w \)-generic, then \( \Theta_{\psi, V, W}(\tilde{\sigma}) \) is \( w_{c} \)-generic.

Let \( \phi \in \Phi_{\text{gen}}(\text{Sp}(W_{2n})) \) and \( c \in F^x \). Then \( \phi \otimes \chi_c \in \Phi_{\text{gen}}(\text{SO}(V_{2n+1})) \) by \((\text{GPR})\) for \( \text{SO}(V_{2n+1}) \). For the generic representation \( \sigma \in \Pi_{\phi \otimes \chi_c} \), by Proposition 3.7, the theta lift \( \Theta_{\phi, V_{2n+1}, W_{2n}}(\sigma) \) is \( w_{c} \)-generic (if it is nonzero), where \( \sigma \) is an extension of \( \sigma \) to \( \text{O}(V_{2n+1}) \). By Theorem 3.5, we have \( \Theta_{\phi, V_{2n+1}, W_{2n}}(\sigma) \in \Pi_{\phi} \).

Therefore, by Proposition 3.7 for \( \phi \in \Phi(\text{Sp}(W_{2n})) \), we conclude that

- if \( \phi \) is generic, then \( \Pi_{\phi} \) has a \( w_{c} \)-generic representation of \( \text{Sp}(W_{2n}) \) for each \( c \in F^x \);
• if $\phi$ is tempered, then $\Pi_\phi$ consists of tempered representations of $\tilde{\text{Sp}}(W_{2n})$.

Note that even if $\phi$ is not generic, $\Pi_\phi$ may have a generic representation of $\tilde{\text{Sp}}(W_{2n})$. We will give an example in \[13\] below.

In this paper, we assume that if the unique irreducible quotient of a standard module of $\tilde{\text{Sp}}(W_{2n})$ belongs to a generic $L$-packet, then this standard module is irreducible. This is the assumption (IS) in \[11\]. Note that the analogous property for symplectic and special orthogonal groups was proven by \[12\] p.40 Théorème (i)]. This assumption (IS) and Proposition \[37\] imply that if $\sigma \in \text{Irr}(\text{SO}(V_{2n+1}))$ belongs to a generic $L$-packet and $\tilde{\sigma}$ is an extension of $\sigma$ to $O(V_{2n+1})$ such that $\Theta_{\psi,V_{2n+1},W_{2n+1}}(\tilde{\sigma})$ is nonzero, then the big theta lift $\Theta_{\psi,V_{2n+1},W_{2n+1}}(\tilde{\sigma})$ is irreducible.

Let $(V,\langle \cdot,\cdot \rangle)$ be an orthogonal space with $\dim(V) = 2n + 1$ and $\text{disc}(V) = c$. We define a new orthogonal space $-V$ by $(V, -\langle \cdot, \cdot \rangle)$. Then by \[15\] Lemma 3.2], for $\tau \in \text{Irr}(O(V))$ with $\theta_{\psi,V,W}(\tau) \neq 0$, we have

$$\theta_{\psi,V,W}(\tau)^\vee \cong \theta_{\psi,-V,W}(\tau^\vee) \cong \theta_{\psi,-1,V,W}(\tau).$$

We put $\tilde{\pi} = \theta_{\psi,V,W}(\tau)$. Let $\phi_\tau \in \tilde{\Phi}(\text{SO}(V))$ be a representation of $W\Phi_F$ associated to $\tau|\text{SO}(V)$. We put $\eta_\tau = \iota_m(\tau|\text{SO}(V))$, $\phi = \mathcal{L}_\psi(\tilde{\pi}^\vee)$ and $\eta = \iota_\psi(\tilde{\pi}^\vee)$. Then by Corollary \[3.6\] we have

$$\phi_\tau = \phi \otimes \chi_{-c} \quad \text{and} \quad \eta_\tau = \eta_{-c},$$

where $\eta_{-c}$ is the character defined in Theorem \[3.5\].

4. SOME CONJECTURES

In this section, we explicate the statements of the local (Gan–)Gross–Prasad conjecture for the orthogonal case and the symplectic-metaplectic case as well as Prasad’s conjectures on the local theta correspondence for $(\text{Sp}(W_{2n}), \text{O}(V_{2m}))$ with

$$|2m - (2n + 1)| = 1.$$

4.1. The local Gross–Prasad conjecture. In this subsection, we state the local Gross–Prasad conjecture (local Gan–Gross–Prasad conjecture for the orthogonal case) proven by Waldspurger \[25\], \[30\], \[37\], \[38\] and Mœglin–Waldspurger \[42\].

Let $V$ be an orthogonal space over $F$ and $V'$ be a non-degenerate subspace of $V$ with codimension one. Assume that both $\text{SO}(V)$ and $\text{SO}(V')$ are quasi-split. Let $V^{\text{even}}$ and $V^{\text{odd}}$ be in $\{V, V'\}$ such that $\dim(V^{\text{even}}) \in 2\mathbb{Z}$ and $\dim(V^{\text{odd}}) \notin 2\mathbb{Z}$.

Let $\phi_M : W \Phi_F \rightarrow \text{Sp}(M)$ and $\phi_N : W \Phi_F \rightarrow \text{O}(N)$ be elements in $\tilde{\Phi}(\text{SO}(V^{\text{odd}}))$ and $\tilde{\Phi}(\text{SO}(V^{\text{even}}))$, respectively. Following \[10\] 
\[6\], for semi-simple elements $a \in C_M$ and $b \in C_N^+$, we put

$$\chi_N(a) = \varepsilon(M^a \otimes N) \det(M^a)(-1)^{\dimc(N)/2} \det(N)(-1)^{\dimc(M^a)/2},$$

$$\chi_M(b) = \varepsilon(M \otimes N^b) \det(M)(-1)^{\dimc(N^b)/2} \det(N^b)(-1)^{\dimc(M)/2}.$$

By \[10\] Theorem 6.2], $\chi_N$ and $\chi_M$ define characters on $A_M$ and on $A_N^+$, respectively.

We say that a pure inner form $G_1 = \text{SO}(V_1) \times \text{SO}(V'_1)$ of $G$ is relevant if $V'_1$ is a non-degenerate subspace of $V_1$ and $V_1/V'_1 \cong V/V'$ as orthogonal spaces. Then there is a natural embedding $\text{SO}(V'_1) \hookrightarrow \text{SO}(V_1)$. Hence we have a diagonal embedding

$$\Delta : \text{SO}(V'_1) \hookrightarrow \text{SO}(V'_1) \times \text{SO}(V_1).$$

The Gan–Gross–Prasad conjecture for the orthogonal case is as follows.

Conjecture 4.1 (B). Let $\phi_M : W \Phi_F \rightarrow \text{Sp}(M)$ and $\phi_N : W \Phi_F \rightarrow \text{O}(N)$ be in $\tilde{\Phi}_{\text{gen}}(\text{SO}(V^{\text{odd}}))$ and in $\tilde{\Phi}_{\text{gen}}(\text{SO}(V^{\text{even}}))$, respectively. We take $\sigma_M \in \Pi_{\phi_M}$ and $[\sigma_N] \in \Pi_{\phi_N}$ such that $\sigma = \sigma_M \otimes \sigma_N$ is a representation of a relevant pure inner form $G_1 = \text{SO}(V^{\text{odd}}) \times \text{SO}(V^{\text{even}})$ of $G$. Then one has

$$\text{Hom}_{\text{SO}(V'_1)}(\sigma, \mathbb{C}) \neq 0 \iff \iota_m(\sigma_M) \times \iota_m([\sigma_N]) = \chi_N \times \chi_M,$$

where $c = -\text{disc}(V^{\text{odd}})/\text{disc}(V^{\text{even}})$. 
Let \( \varepsilon \in O(V_1^e) \setminus SO(V_1^e) \). We can extend \( \sigma_M \) to \( \tilde{\sigma}_M \in \text{Irr}(O(V_1^\text{odd})) \). If \( f \in \text{Hom}_{\text{SO}(V_1^e)}(\sigma_M \boxtimes \sigma_N, \mathbb{C}) \), then we have \( f \circ (\tilde{\sigma}_M(\varepsilon) \otimes \text{id}) \in \text{Hom}_{\text{SO}(V_1^e)}(\sigma_M \boxtimes \tilde{\sigma}_N^*, \mathbb{C}) \). Hence, \( \text{(B)} \) is independent of the choice of \( [\sigma_N] \).

4.2. The local Gan–Gross–Prasad conjecture for the symplectic-metaplectic case. We fix a non-trivial additive character \( \psi \) of \( F \). Let \( W \) be a symplectic space over \( F \), and \( V_1 \) be an orthogonal space over \( F \) with \( \dim(V_1) = 1 \) and \( \text{disc}(V_1) = 1 \). Hence there exists \( v \in V_1 \) such that \( \langle v, v \rangle_{V_1} = 2 \). Then the space \( V_1 \otimes W \) has a symplectic form

\[
\langle \cdot, \cdot \rangle_{V_1} \otimes \langle \cdot, \cdot \rangle_W.
\]

Let \( \omega_\psi \) be the Weil representation of \( \tilde{\text{Sp}}(V_1 \otimes W) = \text{Sp}(W) \) given by the unique irreducible representation of the Heisenberg group \( H(V_1 \otimes W) \) associated to the symplectic space \( (V_1 \otimes W, \langle \cdot, \cdot \rangle_{V_1} \otimes \langle \cdot, \cdot \rangle_W) = (W, 2\langle \cdot, \cdot \rangle_W) \) with the central character \( \psi \).

Let \( \phi_M: WD_F \to \text{Sp}(M) \) and \( \phi_N: WD_F \to \text{SO}(N) \) be elements in \( \tilde{\Phi}(\text{Sp}(W)) \) and in \( \tilde{\Phi}(\text{Sp}(W)) \), respectively. We put \( N_1 = N \oplus \mathbb{C} \), which is an orthogonal representation of \( WD_F \) with even dimension. We may regard \( A_N^+ \) as a subgroup of \( A_N^+ \). Note that \( (A_N^+: A_N^+) \leq 2 \). Following \([10, \S 6]\), for semi-simple elements \( a \in C_M \) and \( b \in C_N^+ \), we put

\[
\chi_{N_1}(a) = \varepsilon(M^a \otimes N_1) \det(M^a)(-1)^{\dim_c(N_1)/2} \det(N_1)(-1)^{\dim_c(M^a)/2} ,
\]

\[
\chi_M(b) = \varepsilon(M \otimes N_1^t) \det(M)(-1)^{\dim_c(N_1^t)/2} \det(N_1^t)(-1)^{\dim_c(M)/2} .
\]

By \([10]\) Theorem 6.2, \( \chi_{N_1} \) and \( \chi_M \) define characters on \( A_M \) and on \( A_N^+ \), respectively.

The identity map and the projection map give a diagonal map

\[
\Delta: \tilde{\text{Sp}}(W) \to \tilde{\text{Sp}}(W) \times \text{Sp}(W).
\]

The local Gan–Gross–Prasad conjecture for symplectic-metaplectic case is as follows.

**Conjecture 4.2** (FJ). Let \( \phi_M: WD_F \to \text{Sp}(M) \) and \( \phi_N: WD_F \to \text{SO}(N) \) be elements in \( \tilde{\Phi}_{\text{gen}}(\text{Sp}(W)) \) and in \( \tilde{\Phi}_{\text{gen}}(\text{Sp}(W)) \), respectively. We denote by \( \Pi_{\phi_M} \subset \text{Irr}(\tilde{\text{Sp}}(W)) \) the inverse image of \( \phi_M \) under the map \( \mathcal{L}_\psi \). Then for \( \pi \in \Pi_{\phi_M} \) and \( \tilde{\pi} \in \Pi_{\phi_N} \), one predicts

\[
\text{Hom}_{\tilde{\text{Sp}}(W)}((\tilde{\pi} \boxtimes \pi) \otimes \omega_\psi^\vee, \mathbb{C}) \neq 0 \iff \iota_\psi(\pi) = \chi_{N_1} \times \chi_M|_{A_n^+}.
\]

There is a conjecture for general codimension cases (Conjecture 17.1 and 17.3 in \([10]\)). However, by \([10]\) Theorem 19.1, the general codimension cases are reduced to the basic case (FJ).

4.3. Even Weil representations. In this subsection, we give a representation \( \phi \in \tilde{\Phi}(\text{Sp}(W_2)) \) which is not generic, but \( \Pi_{\phi} \) has a generic representation of \( \tilde{\text{Sp}}(W_2) \).

Recall that \( \tilde{\Phi}(\text{Sp}(W_{2n})) \) is the set of equivalence classes of symplectic representations

\[
\phi: WD_F \to \text{Sp}(M)
\]

with \( \dim_c(M) = 2n \). Note that \( \tilde{\Phi}(\tilde{\text{Sp}}(W_{2n})) = \tilde{\Phi}(\text{SO}(V_{2n+1})) \).

Now, we set \( n = 1 \). The Weil representation \( \omega_\psi \) of \( \tilde{\text{Sp}}(W_2) = \tilde{\text{SL}}_2(F) \) decomposes

\[
\omega_\psi = \omega_\psi^e \oplus \omega_\psi^o
\]

into a sum of two irreducible representations, called even and odd Weil representations. More precisely, see \([16]\). We consider the even Weil representation \( \omega_\psi^e \).

First, note that \( \theta_{\psi, V_3, W_2}(\omega_\psi^e)|_{\text{SO}(V_3)} \) is the trivial representation \( 1_{\text{SO}(V_3)} \) of the split group \( \text{SO}(V_3) \). Hence the element \( \phi^e = \phi^e \) is \( \phi^e = \big| \big| \big|^{-1/2} \oplus \big| \big|^{-1/2} \big) \). Note that \( A_{\phi^e} \) is singleton. In particular, as an \( L \)-packet of \( \text{SO}(V_3) \), we have \( \Pi_{\phi^e} = \{ 1_{\text{SO}(V_3)} \} \) and so that \( \phi^e \) is not generic. However, by calculating the (twisted) Jacquet modules of \( \omega_\psi^e \), we see that \( \omega_\psi^e \) is \( \omega_\psi \)-generic but not \( \omega_\psi \)-generic for any \( c \in F^\times \setminus F^\times 2 \). We conclude that

- \( \Pi_{\phi^e} \subset \text{Irr}(\text{SO}(V_3)) \) has no generic representations of \( \text{SO}(V_3) \);
• $\Pi_\phi \subset \operatorname{Irr}(\widetilde{\operatorname{Sp}}(W_2))$ has a $w'_1$-generic representation of $\widetilde{\operatorname{Sp}}(W_2)$ but does not have $w'_c$-generic ones for any $c \in F^\times \setminus F^{\times 2}$.

In addition, we can show that for $\phi^e \in \widetilde{\Phi}(\widetilde{\operatorname{SL}}_2(F))$, Conjecture (FJ) does not hold:

**Lemma 4.3.** Let $\phi \in \widetilde{\Phi}(\operatorname{Sp}(W_2)) = \widetilde{\Phi}(\operatorname{SL}_2(F))$. Assume that $\phi$ is of the form $\phi = \chi \oplus 1 \oplus \chi^{-1}$ with $\chi^2 = 1$ and $\chi \neq 1$. Then there are no $\pi \in \Pi_\phi$ such that

$$\text{Hom}_{\Delta \widetilde{\operatorname{Sp}}(W_2)}((\pi \boxtimes \omega'_\phi) \otimes \overline{\omega'_\psi}, \mathbb{C}) \neq 0.$$  

**Proof.** We consider the following see-saw diagram:

$$\begin{array}{ccc}
\text{Sp}(W_2) \times \widetilde{\text{Sp}}(W_2) & \rightarrow & \text{O}(-V_3) \\
\downarrow & & \downarrow \\
\widetilde{\text{Sp}}(W_2) & \rightarrow & \text{O}(V_2) \times \text{O}(V_1)
\end{array}$$

with $\text{disc}(V_1) = -1$ and $\text{disc}(V_2) = \text{disc}(V_3) = 1$ so that $\text{disc}(-V_3) = -1$.

By Proposition 4.3 below, any $\pi \in \Pi_\phi$ is isomorphic to a theta lift $\Theta_{\psi, V_2, W_2}(\overline{\sigma})$ with $\overline{\sigma}|_{\text{SO}(V_2)} = \chi$. By the see-saw identity, we have

$$\text{Hom}_{\Delta \widetilde{\text{Sp}}(W_2)}((\pi \boxtimes \omega'_\phi) \otimes \overline{\omega'_\psi}, \mathbb{C}) \neq 0 \iff \text{Hom}_{\widetilde{\text{Sp}}(W_2)}(\pi \otimes \omega^- \psi, \omega'_\psi, \overline{\psi}) \neq 0 \iff \text{Hom}_{\text{O}(V_2)}(\Theta_{\psi, -V_3, W_2}(\omega^- \psi, \overline{\psi}, \overline{\tau}) \neq 0 \iff \text{Hom}_{\text{O}(V_2)}(\Theta_{-\psi, V_3, W_2}(\omega^- \psi, \overline{\psi}, \overline{\sigma}) \neq 0 \iff \text{Hom}_{\text{SO}(V_2)}(\Theta_{-\psi, V_3, W_2}(\omega^- \psi, \overline{\psi}, \overline{\sigma}) \neq 0.$$  

Note that $\text{SO}(V_3) \cong \text{PGL}_2(F)$ and via this isomorphism, $\text{SO}(V_2)$ corresponds to the subgroup of diagonal matrices of $\text{PGL}_2(F)$. Via this isomorphism, $\Theta_{-\psi, V_3, W_2}(\omega^- \psi, \overline{\psi})$ is the representation of $\text{PGL}_2(F)$ given by the induced representation $\text{Ind}_{B}^{\text{GL}_2(F)}(|\cdot|_{F}^{1/2} \otimes |\cdot|_{F}^{-1/2})$ of $\text{GL}_2(F)$ (with the trivial central character), where $B$ is the Borel subgroup of $\text{GL}_2(F)$ consisting of upper triangular matrices. Let $T$ be the subgroup of $\text{GL}_2(F)$ consisting of diagonal matrices, and $\overline{T}$ be the image of $T$ in $\text{PGL}_2(F)$. Then we have

$$\text{Hom}_{\Delta \widetilde{\text{Sp}}(W_2)}((\pi \boxtimes \omega'_\phi) \otimes \overline{\omega'_\psi}, \mathbb{C}) \neq 0 \iff \text{Hom}_{\overline{T}}(\text{Ind}_{B}^{\text{GL}_2(F)}(|\cdot|_{F}^{1/2} \otimes |\cdot|_{F}^{-1/2}), \chi \otimes \chi^{-1}) \neq 0 \iff \text{Hom}_{\text{GL}_2(F)}(\text{Ind}_{B}^{\text{GL}_2(F)}(|\cdot|_{F}^{1/2} \otimes |\cdot|_{F}^{-1/2}), \text{Ind}_{B}^{\text{GL}_2(F)}(\chi|\cdot|_{F}^{1/2} \otimes \chi^{-1}|\cdot|_{F}^{-1/2})) \neq 0.$$  

However, since $\chi \neq 1$, the last condition is not true.

Since $\chi$ is quadratic, the representation $\phi = \chi \oplus 1 \oplus \chi^{-1}$ is generic. In fact, $\phi$ is tempered. Lemma 4.3 says that Conjecture (FJ) is not true for $\phi \times \phi^e$.

### 4.4. Prasad’s conjectures

We consider the theta correspondences for $(\text{Sp}(W_{2n}), \text{O}(V_{2n+2}))$ and $(\text{O}(V_2), \text{Sp}(W_{2n}))$. Let $V^+ = V_{2m}^+$ be an orthogonal space with $\dim(V_{2m}^+) = 2m$ and type $(d, c)$. We denote by $V^- = V_{2m}^-$ an orthogonal space such that $\dim(V_{2m}^-) = \dim(V_{2m}^+) = 2m$ and $\text{disc}(V_{2m}^-) = \text{disc}(V_{2m}^+)$ but $V_{2m}^- \neq V_{2m}^+$. Note that $\text{SO}(V_{2m}^+)$ is quasi-split, and $\text{SO}(V_{2m}^-)$ is the non-trivial pure inner form of $\text{SO}(V_{2m}^+)$ (if $V_{2m}^- \text{ exists}$). We put $\chi_V = \chi_{V_{2m}^+}$.

First, we let $m = n + 1$. The following proposition is a special case of [12] Theorem C.5.

**Proposition 4.4.** Let $\phi \in \widetilde{\Phi}(\operatorname{Sp}(W_{2n}))$ and put

$$\phi' = (\phi \otimes \chi_V) \oplus 1 \in \widetilde{\Phi}(\text{SO}(V_{2n+2})).$$

1. If $\phi$ does not contain $\chi_V$, then

   • $\theta_{\psi, V^{\phi}_{2n+2}, W_{2n}}(\pi)$ is nonzero for any $\pi \in \Pi_\phi$ and $\bullet = \pm$;

   • $\sigma := \theta_{\psi, V^{\phi}_{2n+2}, W_{2n}}(\pi)|_{\text{SO}(V^{\phi}_{2n+2})}$ is irreducible and $[\sigma] \in \Pi_{\phi'}$;
the theta correspondence gives a bijection
\[ \theta_{\psi, \varphi^*} : \Pi_{\phi} \to \Pi_{\phi'} \cap \text{Irr}(SO(V_{2n+2}^*)) \].

(2) If \( \phi \) contains \( \chi_V \), then
- exactly one of \( \theta_{\psi, \varphi^*} \) is nonzero for any \( \pi \in \Pi_{\phi} \);
- if \( \theta_{\psi, \varphi^*} \) is nonzero, then \( \sigma := \theta_{\psi, \varphi^*} \) is irreducible and \( \sigma \in \Pi_{\phi'} \);
- the theta correspondence gives a bijection

\[ \theta_{\psi, \varphi^*} : \Pi_{\phi} \to \Pi_{\phi'} \].

There is a canonical injective map
\[ A_{\phi}^+ \to A_{\phi'}^+ \].

It is bijective if and only if \( \phi \) contains \( \chi_V \). This map induces a surjection of the character groups
\[ \tilde{A}_{\phi}^+ \to \tilde{A}_{\phi'}^+ \].

The first conjecture of Prasad (P1) predicts the behavior of characters associated to \( \pi \) and \( \theta_{\psi, \varphi^*} \).

**Conjecture 4.5** (P1). Let \( \phi \in \tilde{\Phi}(Sp(W_{2n})) \) and put \( \phi' = (\phi \otimes \chi_V) \oplus 1 \in \tilde{\Phi}(SO(V_{2n+2}^*)) \). For \( \pi \in \Pi_{\phi} \), we take \( \sigma := \theta_{\psi, \varphi^*} \) is irreducible and \( \sigma \in \Pi_{\phi'} \). Then one predicts

\[ \iota_{m_{c_0}}(\sigma) | A_{\phi}^+ = \iota_{m_{c_0}}(\pi) \]

for \( c_0 \in F^\times \). Namely, the diagram
\[
\begin{array}{ccc}
\Pi_{\phi} & \xrightarrow{\theta_{\psi, \varphi^*}} & \Pi_{\phi'} \\
\iota_{m_{c_0}} \downarrow & & \iota_{m_{c_0}} \\
\tilde{A}_{\phi}^+ & \xleftarrow{\tilde{A}_{\phi'}^+} & \tilde{A}_{\phi'}^+ \\
\end{array}
\]
is commutative.

Note that the maps \( \theta_{\psi, \varphi^*} \) and \( \iota_{m_{c_0}} \) depend on the choice of \( \psi \), but the maps \( \iota_{m_{c_0}} \) and \( \tilde{A}_{\phi}^+ \) do not. If \( A_{\phi}^+ = \{1\} \), then this conjecture has nothing to prove. In particular, if \( \phi \) is irreducible or \( \phi = \phi_1 \oplus 1 \oplus \phi_1' \), where \( \phi_1 \) is a sum of non-tempered representations, then this conjecture for \( \phi \) is true.

Next, we let \( m = n \). The following proposition is a special case of [12, Theorem C.5].

**Proposition 4.6.** Let \( \phi' \in \tilde{\Phi}(SO(V_{2n}^*)) \) and put
\[ \phi = (\phi' \otimes \chi_V) \oplus \chi_V \in \tilde{\Phi}(Sp(W_{2n})). \]

(1) If \( \phi' \) does not contain \( 1 \), then
- \( \theta_{\psi, \varphi^*} \) is nonzero for any \( \sigma \in \Pi_{\phi'} \) and any irreducible constituent \( \sigma \) of \( \text{Ind}_{SO(V_{2n}^*)}(\sigma) \);
- \( \theta_{\psi, \varphi^*} \) is irreducible and belongs to \( \Pi_{\phi} \);
- the theta correspondence gives a bijection

\[ \theta_{\psi, \varphi^*} : \bigcup_{[\sigma] \in \Pi_{\phi'}} \{ \sigma \in \text{Ind}_{SO(V_{2n}^*)}(\sigma) \} \leftrightarrow \Pi_{\phi} \].

(2) If \( \phi' \) contains \( 1 \), then
- for any \( \sigma \in \Pi_{\phi'} \), there exists a unique extension \( \sigma \) to \( O(V_{2n}^*) \) such that \( \theta_{\psi, \varphi^*} \) is nonzero;
- if \( \theta_{\psi, \varphi^*} \) is nonzero, then \( \theta_{\psi, \varphi^*} \) is irreducible and belongs to \( \Pi_{\phi} \);
- the theta correspondence gives a bijection

\[ \theta_{\psi, \varphi^*} : \Pi_{\phi'} \leftrightarrow \Pi_{\phi} \].
We have two remarks. First, for $[\sigma] \in \Pi_{\phi'}$, the induction $\text{Ind}_{\text{SO}(V_{2n}^*)}^{\text{O}(V_{2n}^*)}(\sigma)$ does not depend on the choice of a representation $\sigma$ by Lemma 2.3 [1]. Second, if $\phi'$ contains $1$, then $\phi'$ is $\varepsilon$-invariant. Hence any $[\sigma] \in \Pi_{\phi'}$ is an orbit of order one, and there are exactly two extensions to $\text{O}(V_{2n}^*)$ of $\sigma$.

There is a canonical injective map

$$A^+_\phi \to A^+_\phi.$$ 

It is not bijective if and only if $\tilde{\phi}'$ is \(\varepsilon\)-invariant and does not contain $1$. This map induces a surjection of the character groups

$$\widehat{A}^+_\phi \leftarrow \widehat{A}^+_\phi.$$

We put

$$\Pi_{\phi'} = \bigcup_{[\sigma] \in \Pi_{\phi'}} \left\{ \tilde{\sigma} \subset \text{Ind}_{\text{SO}(V_{2n}^*)}^{\text{O}(V_{2n}^*)}(\sigma) \mid \theta_{\psi, V_{2n}^*}, W_{2n}(\tilde{\sigma}) \neq 0 \right\} \subset \bigcup_{V_{2n}^*} \text{Irr}(\text{O}(V_{2n}^*)�).$$

If $\tilde{\sigma}$ is an irreducible constituent of $\text{Ind}_{\text{SO}(V_{2n}^*)}^{\text{O}(V_{2n}^*)}(\sigma)$, then the image of $\tilde{\sigma}$ under the map

$$\text{Irr}(\text{O}(V)) \to \text{Irr}(\text{O}(V))/\sim_{\text{det}} \to \text{Irr}(\text{O}(V))/\sim_\varepsilon$$

is $[\sigma]$. Here, the last map is given by the restriction. See [2,3] We define a map $\imath_{\text{m}_{\phi_0}}: \Pi_{\phi'} \to \widehat{A}^+_\phi$ by $\imath_{\text{m}_{\phi_0}}([\sigma]) := \imath_{\text{m}_{\phi_0}}([\sigma])$ for $c_0 \in F^\times$.

The second conjecture of Prasad (P2) predicts the behavior of characters associated to $\tilde{\sigma}$ and $\theta_{\psi, V_{2n}^*, W_{2n}}(\tilde{\sigma})$.

**Conjecture 4.7 (P2).** Let $\phi' \in \Phi(\text{SO}(V_{2n}^*))$ and put $\phi = (\phi \otimes \chi_V) \oplus \chi_V$. For $[\sigma] \in \Pi_{\phi'}$, we take an irreducible constituent $\tilde{\sigma} \subset \text{Ind}_{\text{SO}(V_{2n}^*)}^{\text{O}(V_{2n}^*)}(\sigma)$ such that $\pi = \theta_{\psi, V_{2n}^*}, W_{2n}(\tilde{\sigma})$ is nonzero, so that $\pi \in \Pi_{\phi}$. Then one predicts

$$\imath_{\text{m}_{\phi_0}}(\pi)|A^+_\phi = \imath_{\text{m}_{\phi_0}}(\tilde{\sigma}) = \imath_{\text{m}_{\phi_0}}([\sigma])$$

for $c_0 \in F^\times$ Namely, the diagram

$$\begin{array}{ccc}
\Pi_{\phi'} & \xrightarrow{\theta_{\psi, V_{2n}^*}, W_{2n}} & \Pi_{\phi} \\
| \imath_{\text{m}_{\phi_0}} & & | \imath_{\text{m}_{\phi_0}} \\
\widehat{A}^+_\phi & \leftarrow & \widehat{A}^+_\phi \\
\end{array}$$

is commutative.

Note that the maps $\theta_{\psi, V_{2n}^*}, W_{2n}$ and $\imath_{\text{m}_{\phi_0}}$ depend on the choice of $\psi$, but the maps $\imath_{\text{m}_{\phi_0}}$ and $\widehat{A}^+_\phi \leftarrow \widehat{A}^+_\phi$ do not. If $\phi'$ is irreducible or $\phi' = \phi'_1 \oplus (\phi'_1)^\vee$, where $\phi'_1$ is a sum of non-tempered representations, then this conjecture for $\phi'$ is true.

In this paper, we show (FJ), (P1) and (P2) under some assumptions.

5. Reductions to (P1) for Tempered L-parameters

In this section, we reduce (P1), (P2) and Theorem L.3 to (P1) for tempered representations. Through this section, we assume (P1) for tempered representations.

5.1. Some lemmas. Let $\text{Sp}(W_{2n})$ and $\text{SO}(V_{2n+2})$ be quasi-split symplectic and special orthogonal groups, respectively. We put $\chi_V = \chi_{V_{2n+2}}$. To prove Theorem L.3 we need the following two lemmas:

**Lemma 5.1.** Suppose (GPR) for $\text{Sp}(W_{2n})$ and $\text{SO}(V_{2n+2})$. Let $\phi \in \Phi_{\text{gen}}(\text{Sp}(W_{2n}))$ and put $\phi' = (\phi \otimes \chi_V) \oplus 1 \in \Phi(\text{SO}(V_{2n+2}))$. Then $\phi'$ is generic if and only if the local twisted $L$-function $L(s, \phi \otimes \chi_V)$ is regular at $s = 1$.

**Proof.** We compare the local adjoint $L$-functions attached to $\phi$ and $\phi'$. In general, we may write

$$\phi = \phi_1 \oplus \cdots \oplus \phi_r \oplus \phi_0 \oplus \phi_0^\vee \oplus \cdots \oplus \phi_1^\vee,$$

where

- $\phi_0 \in \Phi_{\text{temp}}(\text{Sp}(W_{2n_0}))$ or $\phi_0 = 1$;
Lemma 5.2. Let $A$ if $\Theta$ with $s_1 > s_2 > \cdots > s_r > 0$; $\phi'$ is a tempered representation of $WD_F$ with $\dim(\phi') = k_i$; $n = n_0 + (k_1 + \cdots + k_r)$.

Then we have

$$\frac{L(s, \phi', \text{Ad})}{L(s, \phi, \text{Ad})} = \frac{L(s, \phi_0', \text{Ad})}{L(s, \phi_0, \text{Ad})} \prod_{i=1}^{r} L(s, \phi_i \otimes \chi_V) L(s, \phi_i' \otimes \chi_V) = \frac{L(s, \phi_0', \text{Ad})L(s, \phi \otimes \chi_V)}{L(s, \phi_0, \text{Ad})L(s, \phi_0 \otimes \chi_V)},$$

where we put $\phi_0' = (\phi_0 \otimes \chi_V) \otimes 1$. Here, if $\phi_0 = 1$, then we interpret $L(s, \phi_0, \text{Ad})$ to be 1. Since $\phi_0$ and $\phi_0'$ are tempered, the $L$-functions $L(s, \phi_0', \text{Ad})$, $L(s, \phi_0, \text{Ad})$ and $L(s, \phi_0 \otimes \chi_V)$ are regular and nonzero at $s = 1$. So is $L(s, \phi, \text{Ad})$ since $\phi$ is generic. Hence, $\phi'$ is generic, i.e., $L(s, \phi', \text{Ad})$ is regular at $s = 1$ if and only if so is $L(s, \phi \otimes \chi_V)$.

In the proof of Lemma 5.1, we used (GPR) for $\text{Sp}(W_{2n})$ and $\text{SO}(V_{2n+2})$. By a similar argument, using (GPR) for $\text{Sp}(W_{2n})$ and a genericity of theta lifts of $(O(V_{2n+2}), \text{Sp}(W_{2n+2}))$ (c.f., Proposition 5.7 (2)), we can show that

$$L(s, \phi \otimes \chi_V)$$

is regular at $s = 1$ if $\phi'$ is generic.

**Lemma 5.2.** Let $\phi \in \hat{\Phi}(\text{Sp}(W_{2n}))$ and $\pi \in \Pi_{\phi}$. Assume that $\phi' = (\phi \otimes \chi_V) \oplus 1 \in \hat{\Phi}(\text{SO}(V_{2n+2}))$ is generic. Then the big theta lift $\Theta_{\psi, \pi}^{1, W_{2n+2}, \phi, \phi'}$ is irreducible (if it is nonzero). Hence,

$$\Theta_{\psi, \phi}^{1, W_{2n+2}, \pi} \circ \text{SO}(V_{2n+2}) = \Theta_{\psi, \phi}^{1, W_{2n+2}, \pi} \circ \text{SO}(V_{2n+2}).$$

**Proof.** If $\phi$ is tempered, the assertion is in Proposition 2.4. In general, we may write

$$\phi = \phi_0 + \cdots + \phi_r \oplus \phi_0 \oplus \phi_0' + \cdots \oplus \phi_0'',$$

as in the proof of the above lemma. Note that the canonical map $A_{\phi_0}^+ \to A_{\phi_0}^+$ is an isomorphism. Let $\pi \in \Pi_{\phi_0}$. Then $\pi$ is the unique irreducible quotient of the standard module

$$\text{Ind}_{Q}^{\text{Sp}(W_{2n})}(\tau_1 \chi_V | \text{det} | \tau_i \otimes \chi_V | \text{det} | \tau_i' \otimes \pi_0),$$

where $\tau_1$ is the tempered representation of $\text{GL}_k(F)$ associated to $\phi'$, and $\pi_0 \in \Pi_{\phi_0}$ with $\nu_{\phi_0} | \pi_0 \circ A_{\phi_0}^+ = \nu_{\phi_0} | \tau_0$. Here, if $n_0 = 0$, then we ignore $\pi_0$. By [12] Proposition C.4 (ii), we have a surjection

$$\text{Ind}_{P}^{\text{SO}(V_{2n+2})}(\tau_1 | \text{det} | \tau_i | \text{det} | \tau_i' \otimes \pi_0) \to \Theta_{\psi, \phi}^{1, W_{2n+2}, \pi_0}.$$

Here, if $n_0 = 0$, then we interpret $\Theta_{\psi, \phi_0, \phi_0, \phi_0}(\pi_0)$ to be the trivial representation of $O(V_2)$. In particular, if $\Theta_{\psi, \phi_0, \phi_0, \phi_0}(\pi_0)$ is nonzero, then so is $\Theta_{\psi, \phi_0, \phi_0, \phi_0}(\pi_0)$. By a similar argument to Lemma 2.3 (3), we have

$$\text{Ind}_{P}^{\text{SO}(V_{2n+2})}(\tau_1 | \text{det} | \tau_i | \text{det} | \tau_i' \otimes \pi_0) \text{SO}(V_{2n+2}) \equiv \text{Ind}_{P}^{\text{SO}(V_{2n+2})}(\tau_1 | \text{det} | \tau_i | \text{det} | \tau_i' \otimes \pi_0) \text{SO}(V_{2n+2}).$$

Since $\phi'$ is generic, by [12], p. 40 Théorème (i)), this standard module is irreducible. Therefore the quotient $\Theta_{\psi, \phi_0, \phi_0, \phi_0}(\pi_0) \text{SO}(V_{2n+2})$ is also irreducible.

**5.2. Existence of (FJ).** In this subsection, we prove the following proposition.

**Proposition 5.3.** Let

$$\phi_M : WD_F \to \text{Sp}(M) \quad \text{and} \quad \phi_N : WD_F \to \text{SO}(N)$$

be in $\hat{\Phi}_{\text{gen}}(\hat{\Phi}(W_{2n}))$ and $\hat{\Phi}_{\text{gen}}(\hat{\Phi}(W_{2n}))$, respectively. Assume (P1) and that

- any standard module of $\text{Sp}(W_{2n})$ whose unique irreducible quotient is in $\Pi_{\phi_M}$ is irreducible;
- $L(s, \phi_N \otimes \chi_d)$ is regular at $s = 1$ for some $d \in F^\times.$
Then there exist \( \bar{\pi} \in \Pi_{\phi_M} = L_{\psi}^{-1}(\phi_M) \) and \( \pi \in \Pi_{\phi_N} \) such that

\[
\text{Hom}_{\Delta \text{Sp}(W_{2n})}((\bar{\pi} \boxtimes \pi) \otimes \overline{\omega_{\psi}}, \mathbb{C}) \neq 0.
\]

**Proof.** We consider the following see-saw diagram:

\[
\begin{array}{ccc}
\text{Sp}(W_{2n}) & \text{Sp}(W_{2n}) & \text{O}(V_{2n+2}) \\
\text{Sp}(W_{2n}) & \text{O}(V_{2n+1}) \times \text{O}(V_1)
\end{array}
\]

with \( \text{disc}(V_1) = -1 \).

Note that \( \text{dim}_C(M) = 2n \) and \( \text{dim}_C(N) = 2n + 1 \). We put

\[
(\phi_\sigma, N_\sigma) = ((\phi_N \otimes \chi_d) \oplus 1, (N \otimes \chi_d) \oplus \mathbb{C}) \quad \text{and} \quad (\phi_\tau, M_\tau) = (\phi_M \otimes \chi_d, M \otimes \chi_d).
\]

Then \( \phi_\sigma \) is an \( \varepsilon \)-invariant element in \( \tilde{\Phi}(\text{SO}(V_{2n+2})) \) with \( \text{disc}(V_{2n+2}) = d \), and \( \phi_\tau \in \tilde{\Phi}(\text{SO}(V_{2n+1})) \) with \( \text{disc}(V_{2n+1}) = d \). Moreover, by Lemma 5.1 and (GPR) for \( \text{SO}(V_{2n+1}) \), we see that \( \phi_\sigma \) and \( \phi_\tau \) are generic.

By (B), there are a pair of orthogonal spaces \( V_{2n+1} \subset V_{2n+2} \), and \([\sigma] \in \Pi_{\phi_\sigma} \cap \text{Irr}(\text{SO}(V_{2n+2}))/_{\sim_\varepsilon}, \tau \in \Pi_{\phi_\tau} \cap \text{Irr}(\text{SO}(V_{2n+1}))\) such that

\[
\text{Hom}_{\Delta \text{SO}(V_{2n+1})}((\sigma \otimes \tau), \mathbb{C}) \neq 0.
\]

If we denote the orthogonal complement of \( V_{2n+1} \) in \( V_{2n+2} \) by \( V_1 \), then we have

\[
\text{disc}(V_1) = -\text{disc}(V_{2n+2})/\text{disc}(V_{2n+1}) = -1.
\]

By Proposition 3.4 and (the proof of) Lemma 5.2, we can find \( \pi \in \Pi_{\phi_N} \) such that

\[
\Theta_{\psi, V_{2n+2}, W_{2n}}(\pi) \otimes \text{SO}(V_{2n+2}) = \sigma.
\]

We claim that there exists an extension \( \tau_\bullet \) of \( \tau \) to \( \text{O}(V_{2n+1}) \) such that

\[
\text{Hom}_{\Delta \text{O}(V_{2n+1})}(\Theta_{\psi, V_{2n+2}, W_{2n}}(\pi) \otimes \tau_\bullet, \mathbb{C}) \neq 0.
\]

Let \( \varepsilon = -1 \) be the non-trivial element in the center of \( \text{O}(V_{2n+1}) \), so that \( \varepsilon \not\in \text{SO}(V_{2n+1}) \). Then there exist two extensions \( \tau_+ \) and \( \tau_- \) of \( \tau \) to \( \text{O}(V_{2n+1}) \) such that \( \tau_{\pm}(\varepsilon) = \pm 1 \). For \( f \in \text{Hom}_{\Delta \text{SO}(V_{2n+1})}((\sigma \otimes \tau, \mathbb{C}) \) with \( f \neq 0 \), we put

\[
f_{\pm} = f \circ (\Theta_{\psi, V_{2n+2}, W_{2n}}(\pi)(\varepsilon) \otimes \tau_{\pm}(\varepsilon)).
\]

We see that

\[
f_{\pm} \in \text{Hom}_{\Delta \text{SO}(V_{2n+1})}((\sigma \otimes \tau, \mathbb{C}).
\]

Moreover we have \( (f_{\pm})_{\pm} = f \) and \( f_- = -f_+ \). Since \( \text{dim}_C \text{Hom}_{\Delta \text{SO}(V_{2n+1})}((\sigma \otimes \tau, \mathbb{C}) = 1 \) by [1], we have \( f_{\pm} = a_{\pm} \cdot f \) for some \( a_{\pm} \in \{\pm 1\} \). If \( f_\ast = f \), then \( f \in \text{Hom}_{\Delta \text{O}(V_{2n+1})}(\Theta_{\psi, V_{2n+2}, W_{2n}}(\pi) \otimes \tau_\bullet, \mathbb{C}) \). Hence this Hom space is nonzero.

By the see-saw identity, we see that

\[
\text{Hom}_{\text{Sp}(W_{2n})}(\Theta_{\psi, V_{2n+1}, W_{2n}}(\tau_\bullet) \otimes \omega_{\psi}, \pi) \cong \text{Hom}_{\text{O}(V_{2n+1})}(\Theta_{\psi, V_{2n+2}, W_{2n}}(\pi), \tau_\bullet) \\
\cong \text{Hom}_{\Delta \text{O}(V_{2n+1})}(\Theta_{\psi, V_{2n+2}, W_{2n}}(\pi) \otimes \tau_\bullet, \mathbb{C}) \neq 0,
\]

since \( \tau_\bullet \cong \tau_\bullet \).

Putting \( \bar{\pi} := \Theta_{\psi, V_{2n+1}, W_{2n}}(\tau_\bullet) \), we see that

\[
\text{Hom}_{\text{Sp}(W_{2n})}(\pi_{\psi} \otimes \bar{\pi} \otimes \overline{\omega_{\psi}}, \mathbb{C}) \neq 0.
\]

Note that \( \pi \in \Pi_{\phi_N} \) implies that \( \pi_{\psi} \in \Pi_{\phi_N} \). By the assumption of standard modules and a remark in 3.3.6 we see that \( \bar{\pi} \) is irreducible. Moreover, by Corollary 3.6 we have \( \bar{\pi} \in \Pi_{\phi_M} \). This completes the proof. \( \square \)
5.3. Uniqueness of (FJ). In this subsection, we prove the following proposition.

**Proposition 5.4.** Let \( \phi_M \) and \( \phi_N \) be as in Proposition 5.3. Assume (P1) and the second condition of Proposition 5.3. Let \( \bar{\pi} \in \Pi_{\phi_M} = \mathcal{L}_{\psi}^{-1}(\phi_M) \) and \( \pi \in \Pi_{\phi_N} \). If they satisfy
\[
\text{Hom}_{\DeltaSp(W_{2n})}((\bar{\pi} \boxtimes \pi) \otimes \bar{\psi}, \mathbb{C}) \neq 0,
\]
then we have
\[
\eta_{\bar{\pi}} \times \eta_{\pi} = \chi N_1 \times \chi_M | A^+_\phi,
\]
where we put \( \eta_{\bar{\pi}} = \iota_{\psi}(\bar{\pi}) \) and \( \eta_{\pi} = \iota_{\psi}(\pi) \).

**Proof.** Note that
\[
\text{Hom}_{\DeltaSp(W_{2n})}((\bar{\pi} \boxtimes \pi) \otimes \bar{\psi}, \mathbb{C}) \neq 0 \iff \text{Hom}_{Sp(W_{2n})}((\bar{\pi} \otimes \omega - \psi, \pi^\vee) \neq 0.
\]

There exists a unique orthogonal space \( V_{2n+1} \) such that \( \dim(V_{2n+1}) = 2n + 1 \), \( \text{disc}(V_{2n+1}) = d \) and
\[
\theta_{\psi, V_{2n+1}, W_{2n}}(\tau) = \bar{\pi}
\]
for some \( \tau \in \text{Irr}(O(V_{2n+1})) \). Let \( \phi_\tau : WD_F \to \text{Sp}(M_F) \) be the element in \( \Phi(\text{SO}(V_{2n+1})) \) associated to \( \tau \mid SO(V_{2n+1}) \) and \( \eta_\tau = \iota_{w_1}(\tau \mid SO(V_{2n+1}) \in A^+_{\phi_\tau} \). Then by Corollary 3.6, we have
\[
M_\tau = M \otimes \chi_d,
\]
\[
\eta_\tau(\alpha) = \eta_{\bar{\pi}}(\alpha) \varepsilon(M^a) \varepsilon(M^a \otimes \chi_d) \chi_d(-1)^{\dim_C(M^a)/2}.
\]
Let \( V_{2n+2} = V_{2n+1} \oplus V_1 \) with \( \dim(V_1) = 1 \) and \( \text{disc}(V_1) = -1 \). Then we have \( \dim(V_{2n+2}) = 2n + 2 \) and \( \text{disc}(V_{2n+2}) = -\text{disc}(V_{2n+1}) \cdot \text{disc}(V_1) = d \). By the see-saw argument, we have
\[
0 \neq \text{Hom}_{Sp(W_{2n})}(\bar{\pi} \otimes \omega - \psi, \pi^\vee)
\]
\[
\cong \text{Hom}_{O(V_{2n+1})}((\Theta_{\psi, V_{2n+2}, W_{2n}}(\pi^\vee) \otimes \tau), \mathbb{C})
\]
\[
\cong \text{Hom}_{\DeltaSO(V_{2n+1})}(\Theta_{\psi, V_{2n+2}, W_{2n}}(\pi^\vee) \otimes \tau, C)
\]
\[
\cong \text{Hom}_{\DeltaSO(V_{2n+1})}(\Theta_{\psi, V_{2n+2}, W_{2n}}(\pi^\vee) \otimes \tau, C)
\]
since \( \pi^\vee \simeq \tau \). We write \( \bar{\sigma} = \Theta_{\psi, V_{2n+2}, W_{2n}}(\pi^\vee) \). Since \( \text{Hom}_{\DeltaSO(V_{2n+1})}(\bar{\sigma} \otimes \tau, \mathbb{C}) \neq 0 \), we have \( \bar{\sigma} \neq 0 \), and so that \( \bar{\sigma} \in \text{Irr}(O(V_{2n+2})) \) by Lemma 5.1 and 5.2. Let \( \phi_\sigma : WD_F \to O(N_{2n}) \) be the element in \( \Phi(\text{SO}(V_{2n+2})) \) associated to \( [\sigma] \) and put \( \eta_\sigma = \iota_{w_1}(\sigma) \in A^+_{\phi_\sigma} \), where \( \sigma := \bar{\sigma} \mid SO(V_{2n+2}) \in \text{Irr}(SO(V_{2n+2})) \). Then by (P1), we have
\[
N_\sigma = (N \otimes \chi_d) \oplus \mathbb{C}
\]
and
\[
\eta_\sigma | A^+_{\phi_{\sigma N}} = \eta_{\pi^\vee},
\]
where we put \( \eta_{\pi^\vee} = \iota_{w_{\pi^\vee}}(\pi^\vee) \).

By (GPR) for \( \text{SO}(V_{2n+1}) \) and Lemma 5.1 we see that \( \phi_\tau \in \Phi_{\text{gen}}(SO(V_{2n+1})) \) and \( \phi_\sigma \in \Phi_{\text{gen}}(SO(V_{2n+2})) \). Since \( \text{Hom}_{\DeltaSO(V_{2n+1})}(\sigma \otimes \tau, C) \neq 0 \) and \( 1 - \text{disc}(V_{2n+1})/\text{disc}(V_{2n+2}) ) = -1 \), by (B), we see that
\[
\eta_\sigma(\alpha) \cdot \iota_{w_{\bar{\sigma}}}(\alpha)(b) = \varepsilon(M^a \otimes N_\sigma) \text{det}(M^a(-1)^{\dim_C(N_\sigma)/2} \text{det}(N_\sigma)(-1)^{\dim_C(M^a)/2}
\]
\[
\times \varepsilon(M \otimes N_\sigma^b) \text{det}(M_\tau(-1)^{\dim_C(N_\sigma^b)/2} \text{det}(N_\sigma^b)(-1)^{\dim_C(M_\tau)/2}
\]
for \( a \in A_{\phi_M} = A_{\phi_\tau} \).

Now, for \( a \in A_{\phi_M} = A_{\phi_\tau} \), we have
\[
\varepsilon(M^a \otimes N_\sigma) = \varepsilon((M^a \otimes \chi_d) \otimes (N \otimes \chi_d) \otimes \mathbb{C}) = \varepsilon(M^a \otimes N) \varepsilon(M^a \otimes \chi_d)
\]
\[
= \varepsilon(M^a \otimes N_1) \varepsilon(M^a \otimes \chi_d),
\]
\[
\text{det}(M^a_\tau) = \text{det}(M^a) = 1, \quad \dim_C(M^a_\tau) = \dim_C(M^a).
\]
For \( b \in A^+_{\phi_{\sigma N}} \subset A^+_{\phi_\sigma} \), we have \( N_\sigma^b = N^b \otimes \chi_d \) and \( N_\tau^b = N^b \). Hence we have
\[
\varepsilon(M_\tau \otimes N_\sigma^b) = \varepsilon((M \otimes \chi_d) \otimes (N^b \otimes \chi_d)) = \varepsilon(M \otimes N_\sigma^b),
\]
\[
\text{det}(N_\sigma^b) = \text{det}(N^b) = \text{det}(N_\sigma^b), \quad \dim_C(N_\sigma^b) = \dim_C(N^b).
\]
Moreover we have
\[
\text{det}(N_\sigma) = \text{det}(N) \chi_d = \text{det}(N_1) \chi_{-d}, \quad \dim_C(N_\sigma) = \dim_C(N) + 1 = \dim_C(N_1).\]
Finally, by Proposition $3.2$ and a remark after Proposition $3.1$ we have
\[ \iota_{m-1}([\sigma]) (b) = \eta_\sigma (b) \det (N^b_\sigma)(-1) = \iota_{m_1'} (\pi')(b) \det (N^b)(-1) = \eta_\pi (b). \]

Therefore we have
\[
(\eta_\pi(a) \varepsilon (M^a) \varepsilon (M^b \otimes \chi_d) \chi_d (-1)^{\dim_c (M^a)/2} \eta_\pi (b) \\
= (\varepsilon (M^a \otimes N_1) \varepsilon (M^a \otimes \chi_d)) \det (M^a)(-1)^{\dim_c (N_1)/2} (\det (N_1) \chi_d)(-1)^{\dim_c (M^a)/2} \\
\times \varepsilon (M \otimes N_1^b) \det (M)(-1)^{\dim_c (N_1)/2} \det (N_1^b)(-1)^{\dim_c (M)/2}.
\]

This gives the desired equation for $\eta_\pi(a) \eta_\pi (b)$.

$\square$

5.4. (P1) and (P2) for tempered cases $\Rightarrow$ those for general cases. As we have seen in Proposition $3.1$ (resp. (P2)) is true for $\phi \in \tilde{\Phi}(\text{Sp}(W_{2n}))$ (resp. $\phi' \in \tilde{\Phi}(\text{SO}(V_{2n}))$) such that $A^+ = 1$ (resp. $A_{\phi'}^+ = 1$). Therefore, (P1) (resp. (P2)) for $\phi \in \tilde{\Phi}(\text{Sp}(W_{2n})) \setminus \tilde{\Phi}_{\text{temp}}(\text{Sp}(W_{2n}))$ (resp. $\phi' \in \tilde{\Phi}(\text{SO}(V_{2n})) \setminus \tilde{\Phi}_{\text{temp}}(\text{SO}(V_{2n}))$) follows from the tempered cases and the compatibility of the local Langlands correspondences, the Langlands quotients and the local theta correspondences. See [334, 335] and [12, Proposition C.4 (ii)].

5.5. Proof of (P2) for tempered cases. In this subsection, we show that (P1) implies (P2).

Let $V = V_{2n}$ and $W = W_{2n}$. We put $d = \dim (V)$ so that $\chi_V = \chi_d$. Let $\phi' \in \tilde{\Phi}_{\text{temp}}(\text{SO}(V))$ and put
\[
\phi = (\phi' \otimes \chi_d) \oplus \chi_d \in \tilde{\Phi}_{\text{temp}}(\text{Sp}(W)).
\]

Let $[\sigma] \in \Pi_{\phi'}$. Take an irreducible constituent $\tilde{\sigma}$ of $\text{Ind}_{\text{SO}(V^*)}^{\text{Sp}(V)} (\sigma)$ such that the theta lift $\pi = \Theta_{\psi, V^*, W} (\tilde{\sigma})$ is nonzero, so that $\pi \in \Pi_{\phi'}$. Fix $c_0 \in F^\times$. We have to show that
\[
\iota_{m_{c_0}} (\pi) | A^+_\phi' = \iota_{m_{c_0}} ([\sigma]).
\]

We define an orthogonal space $V^*_1$ by $V_1^* = Fe \oplus V^* \oplus Fe^*$ equipped with the pairing $\langle \cdot, \cdot \rangle_{V^*}$ which is an extension of $\langle \cdot, \cdot \rangle_{V^*}$ and satisfies $\langle e, e^* \rangle_{V^*} = (e^*, e^*)_{V^*} = 0$ and $\langle e, e^* \rangle_{V^*} = 1$. Consider the theta correspondence for $(\text{Sp}(W), O(V^*_1))$. Let $\omega = \omega_{\psi, V^*, W}$ and $\omega_1 = \omega_{\psi, V^*_1, W}$ be Weil representations of $\text{Sp}(W) \otimes O(V^*)$ and $\text{Sp}(W) \otimes O(V^*_1)$, respectively. Let $P = M_P U_P$ be the maximal parabolic subgroup of $O(V^*_1)$ stabilizing $F_e$, where $M_P$ is the Levi component of $P$ stabilizing $F_e$. Note that $M_P \cong O(V^*) \times \text{GL}(1, F)$. By a formula in [882], below, we see that there exists a surjective $\text{Sp}(W) \times O(V^*) \times \text{GL}(1, F)$-homomorphism
\[
\omega_1 \rightarrow \omega \boxtimes | \cdot |_{p}^\flat.
\]

This implies that if $\pi \in \text{Irr} (\text{Sp}(W))$ participates in the theta correspondence with $O(V^*)$, then so does in the theta correspondence with $O(V^*_1)$.

We put $\tilde{\sigma}_1 = \Theta_{\psi, V^*_1, W} (\pi)$ to be the theta lift to $O(V^*_1)$ and $\sigma_1 = \tilde{\sigma}_1 |_{\text{SO}(V^*_1)}$. As we have seen, this is nonzero, so that $\sigma_1$ is irreducible by Proposition $2.4$ and $3.1$. There exists an exact sequence of $\text{Sp}(W) \oplus O(V^*_1)$-modules:
\[
1 \rightarrow \mathcal{S}_1[\pi] \rightarrow \omega_1 \rightarrow \pi \boxtimes \tilde{\sigma}_1 \rightarrow 1,
\]

where the kernel $\mathcal{S}_1[\pi]$ is given by
\[
\mathcal{S}_1[\pi] = \bigcap_{f \in \text{Hom}_{\text{Sp}(W)} (\omega_1, \pi)} \ker (f).
\]

On the other hand, there exists a surjective $\text{Sp}(W) \times O(V^*) \times \text{GL}(1, F)$-homomorphism
\[
\omega_1 \rightarrow \omega \boxtimes | \cdot |_{p}^\flat \rightarrow \pi \boxtimes \tilde{\sigma} \boxtimes | \cdot |_{p}^\flat.
\]
Since this map kills $S_1[\pi]$, the diagram

$$
\begin{array}{c}
1 \\
\downarrow \\
S_1[\pi] \\
\downarrow \\
\omega_1 \\
\downarrow \\
\omega \otimes |/F \rightarrow \pi \otimes \overline{\sigma} \otimes |/F \\
\downarrow \\
\pi \otimes \overline{\sigma}_1 \\
\downarrow \\
1
\end{array}
$$

gives a surjective $\text{Sp}(W) \times O(V^*) \times \text{GL}(1,F)$-homomorphism

$$
\pi \otimes \overline{\sigma}_1 \rightarrow \pi \otimes \overline{\sigma} \otimes |/F,
$$

and so that we get a nonzero $O(V^*) \times \text{GL}(1,F)$-homomorphism

$$
\overline{\sigma}_1 \rightarrow \overline{\sigma} \otimes |/F.
$$

This implies that

$$
\text{Hom}_{SO(V^*)}((\sigma_1, \text{Ind}_{P_0}^{SO(V^*)}((\sigma \otimes 1)))
$$

is nonzero. Note that the induction $\text{Ind}_{P_0}^{SO(V^*)}((\sigma \otimes 1))$ does not depend on the choice of a representative of $[\sigma]$ by Lemma 2.3 (2). Hence we have

$$
\iota_{W_0}([\sigma_1])|A^+_\phi = \iota_{W_0}([\sigma]).
$$

Since $\iota_{W_0}([\sigma_1])|A^+_\phi = \iota_{W_0}(\pi)$ by (P1) and $A^+_\phi \subset A^+_0$, we have

$$
\iota_{W_0}(\pi)|A^+_\phi = \iota_{W_0}(\sigma)|A^+_{\phi} = \iota_{W_0}(\sigma)|A^+_{\phi'} = \iota_{W_0}(\sigma),
$$

as desired.

## 6. Globalization

In this section, we prove the following theorem:

**Theorem 6.1.** Suppose that $\text{SO}(V_{2n+2})$ is quasi-split. If (P1) for $\text{Sp}(W_{2n}), \text{O}(V_{2n+2})$ holds for $\phi \in \Phi_{\text{temp}}(\text{Sp}(W_{2n})) \setminus \Phi_{\text{disc}}(\text{Sp}(W_{2n}))$, then (P1) for $\text{Sp}(W_{2n}), \text{O}(V_{2n+2})$ holds for all $\phi \in \Phi_{\text{disc}}(\text{Sp}(W_{2n}))$, but the following two exceptional cases:

- $n = 1$, i.e., $\dim(\phi) = 3$;
- $\phi = \phi_1 + \chi$, where $\phi_1$ is an irreducible orthogonal representation of $\text{WD}_F$ with $\dim(\phi_1) = 2n$ and $\chi$ is a quadratic character of $F^\times$.

To prove this theorem, we use a global argument and Arthur’s multiplicity formula.

### 6.1. A-parameters and Arthur’s multiplicity formula

In this subsection, we review the theory of $A$-parameters and state Arthur’s multiplicity formula for symplectic and special orthogonal groups. More precisely, see [2 §1].

Let $\mathbb{F}$ be a number field and $\mathbb{A}_F$ be the adele ring of $\mathbb{F}$. For a connected reductive group $G$ over $\mathbb{F}$, we put

$$
G(\mathbb{A}_F)^1 = \{ x \in G(\mathbb{A}_F) \mid \| \chi(x) \| = 1 \text{ for } \chi \in X(G)_{\mathbb{F}} \},
$$

$$
\text{Sp}(W) \times O(V^*) \times \text{GL}(1,F)
$$

is a surjective $\text{Sp}(W) \times O(V^*) \times \text{GL}(1,F)$-homomorphism.
where $X(G)_{\mathbb{F}}$ is the additive group of characters of $G$ defined over $\mathbb{F}$. Note that $G(\mathbb{F}) \backslash G(\mathbb{A}_{\mathbb{F}})$ has finite volume, and there is a sequence

$$L^2_{\text{cusp}}(G(\mathbb{F}) \backslash G(\mathbb{A}_{\mathbb{F}})) \subset L^2_{\text{disc}}(G(\mathbb{F}) \backslash G(\mathbb{A}_{\mathbb{F}})) \subset L^2(G(\mathbb{F}) \backslash G(\mathbb{A}_{\mathbb{F}}))$$

de of right $G(\mathbb{A}_{\mathbb{F}})$-invariant Hilbert spaces, where $L^2_{\text{cusp}}(G(\mathbb{F}) \backslash G(\mathbb{A}_{\mathbb{F}}))$ is the subspace of $L^2(G(\mathbb{F}) \backslash G(\mathbb{A}_{\mathbb{F}}))$ consisting of cuspidal functions, and $L^2_{\text{disc}}(G(\mathbb{F}) \backslash G(\mathbb{A}_{\mathbb{F}}))$ is the maximal subspace of $L^2(G(\mathbb{F}) \backslash G(\mathbb{A}_{\mathbb{F}}))$ which decomposes under the action of $G(\mathbb{A}_{\mathbb{F}})$ into a direct sum of irreducible representations. We denote by $\mathcal{A}_{\text{cusp}}(G)$, $\mathcal{A}_2(G)$ and $\mathcal{A}(G)$ be the sets of irreducible automorphic unitary representations $\pi$ of $G(\mathbb{A}_{\mathbb{F}})$ whose restrictions to $G(\mathbb{A}_{\mathbb{F}})$ are irreducible constituents of $L^2_{\text{cusp}}(G(\mathbb{F}) \backslash G(\mathbb{A}_{\mathbb{F}}))$, $L^2_{\text{disc}}(G(\mathbb{F}) \backslash G(\mathbb{A}_{\mathbb{F}}))$ and $L^2(G(\mathbb{F}) \backslash G(\mathbb{A}_{\mathbb{F}}))$, respectively.

First, let $G = \text{GL}(N)$. Then it is known that the representations $\Sigma_2 \in \mathcal{A}_2(\text{GL}(N))$ are parametrized by pairs $(m, \Sigma_{\text{cusp}})$, where $m | N$ and $\Sigma_{\text{cusp}} \in \mathcal{A}_{\text{cusp}}(\text{GL}(m))$. More precisely, see [11]. Moreover, $\Sigma \in \mathcal{A}(\text{GL}(N))$ is the irreducible induced representation

$$\Sigma = \Sigma_1 \boxplus \cdots \boxplus \Sigma_r := \text{Ind}_{\mathbb{F}}^{\text{GL}(N)_{\mathbb{A}_{\mathbb{F}}}}(\Sigma_1 \otimes \cdots \otimes \Sigma_r)$$

for some $\Sigma_i \in \mathcal{A}_2(N_i)$ with $N_1 + \cdots + N_r = N$, where $\mathbb{F}$ is the standard parabolic subgroup of block upper triangular matrices in $\text{GL}(N)$ corresponding to the partition $(N_1, \ldots, N_r)$ of $N$. More precisely, see [2, §1.3].

We call an element in $\mathcal{A}(\text{GL}(N))$ a global $A$-parameter of $\text{GL}(N)$. In this paper, we do not assume the existence of the global Langlands group $\mathbb{L}_{\mathbb{F}}$. If $\mathbb{L}_{\mathbb{F}}$ were available to us, we would expect that there are canonical bijections

$$\mathcal{A}_{\text{cusp}}(\text{GL}(N)) \leftrightarrow \{N\text{-dimensional irreducible unitary “representations” of } \mathbb{L}_{\mathbb{F}}\},$$

$$\mathcal{A}_2(\text{GL}(N)) \leftrightarrow \{N\text{-dimensional irreducible unitary “representations” of } \mathbb{L}_{\mathbb{F}} \times \text{SU}(2)\},$$

$$\mathcal{A}(\text{GL}(N)) \leftrightarrow \{N\text{-dimensional semi-simple unitary “representations” of } \mathbb{L}_{\mathbb{F}} \times \text{SU}(2)\}.$$
By [2 Proposition 1.4.2], for $\Sigma = \Sigma_1 \boxplus \cdots \boxplus \Sigma_r \in \tilde{\Psi}_2(G_N)$ with $\Sigma_i \in \mathcal{A}_2(N_i)$, the representation $\psi_v = \psi_{1,v} \oplus \cdots \oplus \psi_{r,v}: WD_{F_v} \times SL_2(\mathbb{C}) \to GL_N(\mathbb{C})$ associated to the local factor $\Sigma_v \in \text{Irr}(GL_N(F_v))$ factors though $O_N(\mathbb{C}) \hookrightarrow GL_N(\mathbb{C})$. Here, $\psi_{i,v}$ is the $A$-parameter of $\Sigma_{i,v} \in \text{Irr}(GL_{N_i}(F_v))$. In particular, for $\Sigma \in \tilde{\Psi}_{2,\text{temp}}(G_N)$, we have $\psi_v = \phi_v: WD_{F_v} \to O_N(\mathbb{C})$, which is an element in $\tilde{\Phi}(G_N(F_v))$. Note that it has not been proven that the image of $\tilde{\Psi}_{2,\text{temp}}(G_N) \ni \Sigma \mapsto \phi_v \in \tilde{\Phi}(G_N,F_v)$ is contained in $\Phi_{\text{temp}}(G_N(F_v))$ since the generalized Ramanujan conjecture has not been established.

Let $\Sigma = \Sigma_1 \boxplus \cdots \boxplus \Sigma_r \in \tilde{\Psi}_{2,\text{temp}}(G_N)$ with $\Sigma_i \in \mathcal{A}_{\text{cusp}}(N_i)$, and $\phi_v = \phi_{1,v} \oplus \cdots \oplus \phi_{r,v} \in \tilde{\Phi}(G_N(F_v))$ be the associated representation of $WD_{F_v}$. We put

$$A_{\Sigma} = \bigoplus_i (\mathbb{Z} / 2\mathbb{Z}) a_{\Sigma_i} \cong (\mathbb{Z} / 2\mathbb{Z})^r,$$

and define $A_{\Sigma}^+$ by the kernel of the map $A_{\Sigma} \ni a_{\Sigma_i} \mapsto (-1)^{N_i} \in \{\pm 1\}$. Then we have a map

$$A_{\Sigma} \to C_{\phi_v} \to A_{\phi_v}, \quad a_{\Sigma_i} \mapsto -1_{\phi_{i,v}}.$$ 

This implies $A_{\Sigma} \to A_{\phi_v}$ for each place $v$, and we obtain the diagonal map

$$\Delta: A_{\Sigma} \to \prod_v A_{\phi_v}^+.$$ 

Let $\Sigma \in \tilde{\Psi}_2(G_N)$ and $\psi_v: WD_{F_v} \times SL_2(\mathbb{C}) \to O_N(\mathbb{C})$ be the local $A$-parameter associated to $\Sigma_v$. By [2 Theorem 1.5.1], there exists a local $A$-packet $\Pi_{\psi_v}$, which is a finite subset of

$$\begin{cases} 
\text{Irr}(\text{Sp}(W_{2n}(F_v))) & \text{if } G_N = \text{Sp}(W_{2n}), \\
\text{Irr}(\text{SO}(V_{2n}(F_v))/\sim_v) & \text{if } G_N = \text{SO}(V_{2n}).
\end{cases}$$

If $\Sigma \in \tilde{\Psi}_{2,\text{temp}}(G_N)$, then $\psi_v = \phi_v \in \tilde{\Phi}(G_N(F_v))$ and $\Pi_{\psi_v} = \Pi_{\phi_v}$ is the $L$-packet described in [3]. Note that for two local $A$-parameters $\psi_{1,v}$ and $\psi_{2,v}$ of $G_N$, the local $A$-packets $\Pi_{\psi_{1,v}}$ and $\Pi_{\psi_{2,v}}$ may intersect even if $\psi_{1,v} \neq \psi_{2,v}$. However, by [39 p.17 Corollaire], we have the following.

**Proposition 6.2.** Let $\psi_{1,v}, \psi_{2,v}: WD_{F_v} \times SL_2(\mathbb{C}) = W_{F_v} \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to GL_N(\mathbb{C})$ be two local $A$-parameters of $G_N,F_v$. Assume that $\Pi_{\psi_{1,v}} \cap \Pi_{\psi_{2,v}} \neq \emptyset$. Then $\psi_{1,v}|W_{F_v} \times \Delta SL_2(\mathbb{C}) \cong \psi_{2,v}|W_{F_v} \times \Delta SL_2(\mathbb{C})$. Here, we put $W_{F_v} \times \Delta SL_2(\mathbb{C}) = \{(w,a,a) \in W_{F_v} \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) | w \in W_{F_v}, a \in SL_2(\mathbb{C})\}$.

**Corollary 6.3.** Let $\Sigma \in \tilde{\Psi}_2(G_N)$ and $\psi_v$ be the local $A$-parameter associated to $\Sigma_v$ for a finite place $v$. Assume that $\Pi_{\psi_v} \cap \Pi_{\phi_v} \neq \emptyset$ for some $\phi_v \in \tilde{\Phi}_{\text{temp}}(G(N,F_v))$ with $\phi_v|SL_2(\mathbb{C}) = 1$, i.e. $\phi_v$ is a tempered representation of $W_{F_v}$. Then we have $\Sigma \in \tilde{\Psi}_{2,\text{temp}}(G_N)$.

**Proof.** Write $\Sigma = \Sigma_1 \boxplus \cdots \boxplus \Sigma_r \in \mathcal{A}(N)$, where $\Sigma_i \in \mathcal{A}_2(N_i)$ corresponds to $(m_i, \Sigma'_i)$ with $N = m_i n_i$ and $\Sigma'_i \in \mathcal{A}_{\text{cusp}}(m_i)$. We have to show that $n_i = 1$ for any $i$. Let $\phi_{i,v}$ be the $m_i$-dimensional representation of $WD_{F_v}$ corresponding to $\Sigma'_i$ in $\text{Irr}(GL_{m_i}(F_v))$. Then we can write

$$\phi_{i,v} = \bigoplus_{j=1}^{s_i} \phi'_{i,j,v} \boxtimes \nu_{n'_j}$$

as a representation of $WD_{F_v} = W_{F_v} \times SL_2(\mathbb{C})$, where $\phi'_{i,j,v}$ is a representation of $W_{F_v}$. Note that

$$\psi_v = \bigoplus_{i=1}^{r} \phi_{i,v} \boxtimes \nu_{n_i} = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s_i} \phi'_{i,j,v} \boxtimes \nu_{n'_j} \boxtimes \nu_{n_i}.$$ 

Then we have

$$\psi_v|W_{F_v} \times \Delta SL_2(\mathbb{C}) = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s_i} \phi'_{i,j,v} \boxtimes (\nu_{n'_j} \boxtimes \nu_{n_i}) = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s_i} \phi'_{i,j,v} \boxtimes \nu_{n'_j+n_i-2k+1}.$$ 

By the above proposition and the assumption, we have $n'_j + n_i - 2k + 1 = 1$ for $1 \leq k \leq \max\{n'_j, n_i\}$. This implies that $n_i = 1$ for any $i$, as desired. $\square$
For $\Sigma \in \tilde{\Psi}_{2,\text{temp}}(G_N)$ and a place $v$ of $F$, we have the local $L$-packet $\Pi_{\phi_v}$ of $G_N(F_v)$. We assume that if $G_N = \text{SO}(V_{2n})$, then $\phi_v$ is $\varepsilon$-invariant for any $v$. Then $\Pi_{\phi_v}$ consists of orbits of representations whose order is one, so that we may regard $\Pi_{\phi_v} \subset \text{Irr}(G_N(F_v))$. We fix a global Whittaker datum $w$ of $G_N$, which gives a local Whittaker datum $w_v$ of $G_N(F_v)$ for each places $v$ of $F$. Hence we can attach a global $A$-packet

$$
\Pi_{\Sigma} = \{ \pi = \bigotimes'_v \pi_v \mid \pi_v \in \Pi_{\phi_v} \cap \text{Irr}(G_N(F_v)), \eta_v = 1 \text{ for almost all } v \}
$$

of representations of $G_N(\mathbb{A}_F)$, where we put $\eta_v = \iota_{w_v}(\pi_v)$. Note that if both $G_{N,F_v}$ and $\pi_v$ are unramified, then $\eta_v = 1$ by [2, Theorem 1.5.1 (a)]. In general, for any $\Sigma \in \tilde{\Psi}_2(G_N)$, there exists a similar global $A$-packet $\Pi_{\Sigma}$, which consists (orbits of) irreducible representations of $G_N(\mathbb{A}_F)$.

The following theorem is a part of Theorem 1.5.2 in [2].

**Theorem 6.4** (Arthur’s multiplicity formula). Let $\Sigma \in \tilde{\Psi}_{2,\text{temp}}(G_N)$ and $\pi \in \Pi_{\Sigma}$.

1. Assume that $G_N = \text{Sp}(V_{2n})$. Then the following are equivalent:
   - $\pi$ occurs in $L^2_{\text{disc}}(G_N(F) \backslash G_N(\mathbb{A}_F))$;
   - the character $\left( \prod_v \eta_v \right) \circ \Delta$ of $A^+_F$ is trivial.

2. Assume that $G_N = \text{SO}(V_{2n})$ and the local $A$-parameter $\phi_v \in \tilde{\Phi}(\text{SO}(V_{2n}(F_v)))$ associated to $\Sigma_v$ is $\varepsilon$-invariant for any $v$. Then the following are equivalent:
   - $\pi$ occurs in $L^2_{\text{disc}}(G_N(F) \backslash G_N(\mathbb{A}_F))$;
   - the character $\left( \prod_v \eta_v \right) \circ \Delta$ of $A^+_F$ is trivial.

Moreover, any (orbit of) irreducible cuspidal automorphic representation of $G_N(\mathbb{A}_F)$ belongs in $\Pi_{\Sigma}$ for some $\Sigma \in \tilde{\Psi}_2(G_N)$.

6.2. **Irreducible orthogonal representations of $WD_F$.** Let $F$ be a non-archimedean local field with characteristic 0 and residue characteristic $p$. In this subsection, we make some remarks on irreducible orthogonal representations of $WD_F$.

Note that a character of $WD_F$ is a character of $F^\times$. In particular, an orthogonal character of $WD_F$ is a quadratic character of $F^\times$. Since $F^\times / F^\times \cong (\mathbb{Z}/2\mathbb{Z})^d$ with

$$
d = \begin{cases} 
2 & \text{if } p \neq 2, \\
2 + [F : \mathbb{Q}_2] & \text{if } p = 2,
\end{cases}
$$

there are exactly $2^d$ quadratic characters of $F^\times$ or $WD_F$. Note that if $\phi$ is an orthogonal representation of $WD_F$, then $\det(\phi)$ is a quadratic character of $F^\times$.

By Theorem 4 and the proof of Proposition 4 in [17], there are no irreducible orthogonal representations of $W_F$ with trivial determinant and dimension $n$ when $p \neq 2$ and

$$
\begin{cases} 
n \text{is odd and } n > 1, & \text{or} \\
n \equiv 2 \text{ mod } 4.
\end{cases}
$$

On the other hand, we have the following proposition:

**Proposition 6.5.** Assume that $p \neq 2$. Let $n$ be a positive integer and $\omega$ be a non-trivial quadratic character of $F^\times$. Then there are infinitely many inequivalent irreducible orthogonal representations of $W_F$ with dimension $2n$ and determinant $\omega^n$.

**Proof.** Let $E/F$ be the quadratic extension corresponding to $\omega$ by the local class field theory. Fix $s \in W_F \setminus W_E$. We say that a representation $\rho: WD_E \to \text{GL}(M)$ is conjugate-orthogonal if there exists a non-degenerate bilinear form $B: M \times M \to \mathbb{C}$ such that

$$
\begin{align*}
B(\rho(w)m, \rho(sws^{-1})n) &= B(m, n), \\
B(n, m) &= B(m, \rho(s^2)n)
\end{align*}
$$

for $m, n \in M$ and $w \in WD_E$. In particular, a conjugate-orthogonal character of $E^\times$ is a character of $E^\times$ whose restriction to $F^\times$ is trivial.
Note that there are infinitely many conjugate-orthogonal characters $\chi$, since $E^\times/F^\times$ is compact and $\#(E^\times/F^\times) = \infty$. By [12 Lemma 20.7], there exists an irreducible conjugate-orthogonal representation $\rho$ of $W_E$ with $\dim(\rho) = n$. Consider the induced representation $\phi_\chi := \text{Ind}_{W_E}^{W_F}(\rho \otimes \chi)$. Since $\rho \otimes \chi$ is a conjugate-orthogonal representation, we see that $\phi_\chi$ is an orthogonal representation of $W_F$ by [10 Lemma 3.5]. We have $\dim(\phi_\chi) = 2n$ and

$$\det(\phi_\chi) = \omega^n \otimes \det(\rho \otimes \chi)|F^\times = \omega^n.$$

We show that there exist at most finitely many conjugate-orthogonal characters $\chi$ such that $\phi_\chi$ is reducible. Note that

$$\phi_\chi| W_E \cong (\rho \otimes \chi) \oplus (\rho \otimes \chi)^* = (\rho \otimes \chi) \oplus (\rho^s \otimes \chi^{-1}).$$

So we see that:

$$\phi_\chi \text{ is reducible } \Rightarrow \mathbb{C} \not\cong \text{End}_{W_F}(\phi_\chi) \cong \text{Hom}_{W_E}(\phi_\chi| W_E, \rho \otimes \chi)$$

$$\Rightarrow \rho \otimes \chi \cong (\rho \otimes \chi)^* = \rho^s \otimes \chi^{-1}.$$

In particular, if $(\rho \otimes \chi) \not\cong (\rho \otimes \chi)^*$ for any $\chi$, then $\phi_\chi$ is irreducible for each $\chi$. Assume that $(\rho \otimes \chi) \cong (\rho \otimes \chi)^*$. Then for a conjugate-orthogonal character $\chi'$, we see that

$$\phi_{\chi\chi'} \text{ is reducible } \Rightarrow (\rho \otimes \chi) \otimes \chi'^2 \cong (\rho \otimes \chi).$$

In this case, by taking determinant, we have

$$\chi'^{2n} = 1.$$

There are only finitely many such $\chi'$. Hence we see that $\phi_\chi$ is irreducible for all but finitely many $\chi$.

Now, let $\chi$ and $\chi'$ be conjugate-orthogonal characters of $E^\times$. Then we see that

$$\phi_\chi \cong \phi_{\chi'} \Rightarrow \rho \otimes \chi' \cong \rho \otimes \chi \quad \text{or} \quad \rho \otimes \chi' \cong \rho^s \otimes \chi^{-1}.$$

This implies that the set $\{\chi' \mid \phi_{\chi'} \cong \phi_\chi\}$ is a finite set for each $\chi$. Hence there exist infinitely many conjugate-orthogonal characters $\chi$ such that $\phi_\chi$ are irreducible and mutually inequivalent, as desired. \qed

6.3. Globalization of quadratic characters. First, we globalize quadratic characters of $F^\times$.

**Lemma 6.6.** Let $F$ be a number field, $v$ be a finite place of $F$ and $m$ be a positive integer. Then there exists an extension $E/F$ such that

1. $[E:F] = m$;
2. $v$ splits completely over $E/F$.

**Proof.** Choose a finite place $v'$ of $F$ such that $v' \neq v$. By the weak approximation theorem, there exists $\alpha \in F^\times$ such that $\alpha \in F_{v'}^\times$ and $\text{ord}_v(\alpha) = 1$. We put $E = F(\sqrt{\alpha})$. By Eisenstein’s irreducibility criterion for $v'$, we have $[E:F] = m$. Since $E_v = F_v$, we obtain the second condition. \qed

**Lemma 6.7.** Let $F$ be a number field, $S$ be a finite set of finite places of $F$ and $\chi_v$ be a quadratic character of $F_v^\times$ for $v \in S$. Then there exists a quadratic character $\omega$ of $\mathbb{A}_F^\times/F^\times$ such that

$$\omega_v = \chi_v$$

for any $v \in S$.

**Proof.** This is a special case of the Grunwald–Wang theorem (see, e.g., Chapter X, Theorem 5 in [3]). \qed
6.4. Globalization of parameters. We prove Theorem 6.1. So we take \( \phi \in \Phi_{\text{disc}}(\text{Sp}(W_{2n})) \). We decompose
\[
\phi = \phi_1 + \cdots + \phi_r + \chi_1 + \cdots + \chi_s,
\]
where \( \phi_i \) is an irreducible orthogonal representation of \( WD_F \) with \( n_i := \dim(\phi_i) \geq 2 \), and \( \chi_j \) is a quadratic character of \( F^\times \). We may assume that \( \phi \) is reducible, i.e., \( r + s \geq 2 \), and \( (r, s) \neq (1, 1) \).

First, we suppose that \( r \geq 1 \). We may assume that
\[
\max_i \{ \dim(\phi_i) \} = \dim(\phi_1).
\]
By Lemma 6.6, we can find a totally complex field \( F \) and finite places \( v_0, v'_0 \) of \( F \) such that \( \mathbb{F}_{v_0} \cong \mathbb{F}_{v'_0} \cong F \). We put \( \chi_{j,v_0} = \chi_{j,v'_0} = \chi_j \). Moreover, we choose a finite place \( v_{c,1} \neq v_0, v'_0 \) of \( F \), which is not 2-adic. If \( n_1 \equiv 2 \mod 4 \) and \( s \geq 1 \), then we set \( \chi_{1,v_{c,1}} \) to be the non-trivial quadratic unramified character of \( F_{v_{c,1}}^\times \) and \( \chi_{j,v_{c,1}} = 1 \) for \( 2 \leq j \leq s \). Otherwise we put \( \chi_{j,v_{c,1}} = 1 \) for \( 1 \leq j \leq s \). By Lemma 6.7, we can find a quadratic character \( \chi_j \) of \( A_F^+ / F^\times \) such that
\[
\chi_{j,v} = \chi_{j,v_0}
\]
for \( 1 \leq j \leq s \) and \( v \in \{ v_0, v'_0, v_{c,1} \} \).

**Proposition 6.8.** Assume that \( r + s \geq 2 \), \( (r, s) \neq (1, 1) \) and \( r \geq 1 \). Let \( v = v_0' \). Then for \( 1 \leq i \leq r \), there exist orthogonal representations \( \phi_{1,v}, \ldots, \phi_{r,v} \) of \( WD_{F_v} \) such that

1. \( \phi_{i,v} \) is a multiplicity-free sum of irreducible orthogonal representations of \( W_{F_v} \) (i.e., \( \phi_{i,v} |_{\text{SL}_2(\mathbb{C})} = 1 \));
2. \( \dim(\phi_{i,v}) = n_i (= \dim(\phi_i)) \);
3. the representation
\[
\phi_v = \phi_{1,v} + \cdots + \phi_{r,v} + \chi_{1,v} + \cdots + \chi_{s,v}
\]
is not multiplicity-free;
4. \( \det(\phi_v) = 1 \);
5. the natural map \( A_\phi \to A_{\phi_v} \) given by
\[
A_\phi \ni -1_{\phi} \mapsto -1_{\phi_v} \in A_{\phi_v} \quad \text{and} \quad A_{\chi_j} \ni -1_{\chi_j} \mapsto -1_{\chi_{j,v}} \in A_{\chi_{j,v}}
\]
induces a injective map
\[
A_\phi^+ \to A_{\phi_v}^+.
\]

To prove this proposition, we prepare the following two lemmas.

**Lemma 6.9.** Let \( \chi, \omega \) be quadratic characters of \( F_0^\times \), \( n \geq 3 \) be an integer, and \( R \) be a finite set of irreducible 2-dimensional orthogonal representations of \( W_{F_v} \). Then unless \( n = 3 \) and \( \omega = \chi \), there exist infinitely many inequivalent orthogonal representations \( \phi \) of \( WD_{F_v} \) such that

1. \( \dim(\phi) = n \) and \( \det(\phi) = \omega \);
2. \( \phi \) is a multiplicity-free sum of irreducible orthogonal representations of \( W_{F_v} \);
3. \( \phi \) contains \( \chi \);
4. \( \phi \) contains an irreducible 2-dimensional orthogonal representation, which does not belong to \( R \).

**Proof.** Note that the restriction of an irreducible orthogonal representation \( \rho \) of \( WD_{F_v} \) with \( \dim(\rho) = 1 \) or 2 to \( \text{SL}_2(\mathbb{C}) \) is trivial. We consider several cases separately.

First, we assume that \( n = 4m \) with \( m \geq 1 \). Then we may take
\[
\phi = \left\{ \begin{array}{ll}
\rho_1 + \cdots + \rho_{2m-1} + \chi + \omega & \text{if } \chi \neq \omega, 1, \\
\rho'_1 + \cdots + \rho'_{2m-1} + \chi' + \omega & \text{if } \chi = \omega, \\
\rho'_1 + \cdots + \rho'_{2m-1} + \chi' + 1 & \text{if } \chi = 1, \omega \neq 1.
\end{array} \right.
\]

Here, \( \chi' \) is a non-trivial quadratic character of \( F_0^\times \) such that \( \chi' \neq \omega \), and \( \rho_1 \) and \( \rho'_1 \) are inequivalent irreducible 2-dimensional orthogonal representations of \( WD_{F_v} \) with \( \det(\rho_1) = \chi \) and \( \det(\rho'_1) = \chi' \).
Next, we assume that \( n = 4m + 1 \) with \( m \geq 1 \). Then we may take
\[
\phi = \begin{cases} 
\rho'_1 + \cdots + \rho'_{2m} + \omega & \text{if } \chi = \omega, \\
\rho_1 + \cdots + \rho_{2m-1} + \chi + 1 + \omega & \text{if } \chi \neq \omega, 1, \text{ and } \omega \neq 1, \\
\rho'_1 + \cdots + \rho'_{2m-1} + \chi' + 1 + \omega & \text{if } \chi = 1, \omega \neq 1, \\
\rho_1 + \cdots + \rho_{2m-2} + \chi + \rho'_1 + \rho''_1 & \text{if } \chi \neq 1, \omega = 1.
\end{cases}
\]

Here, \( \chi' \) is a non-trivial quadratic character of \( \mathbb{F}_v^\times \) such that \( \chi' \neq 1, \chi, \omega \), and \( \rho_i, \rho'_i \) and \( \rho''_i \) are inequivalent irreducible 2-dimensional orthogonal representations of \( WD_{\mathbb{F}_v} \) with \( \det(\rho_i) = \chi, \det(\rho'_i) = \chi' \) and \( \det(\rho''_i) = \chi' \chi'' \).

Next, we assume that \( n = 4m + 2 \) with \( m \geq 1 \). Then we may take
\[
\phi = \begin{cases} 
\rho'_1 + \cdots + \rho'_{2m} + \chi + \chi\omega & \text{if } \omega \neq 1, \\
\rho'_1 + \cdots + \rho'_{2m-1} + \rho''_1 + 1 + \chi & \text{if } \omega = 1, \chi \neq 1, \\
\rho'_1 + \cdots + \rho'_{2m-1} + \rho''_1 + \chi' \chi'' + 1 & \text{if } \omega = \chi = 1.
\end{cases}
\]

Here, \( \chi' \) and \( \chi'' \) are non-trivial quadratic characters of \( \mathbb{F}_v^\times \) with \( \chi' \neq \chi'', \) and \( \rho_i, \rho'_i \) and \( \rho''_i \) are inequivalent irreducible 2-dimensional orthogonal representations of \( WD_{\mathbb{F}_v} \) with \( \det(\rho_i) = \chi, \det(\rho'_i) = \chi' \) and \( \det(\rho''_i) = \chi'' \). In addition, we assume that \( \chi' \chi'' = 1 \) in the second case.

Finally, we assume that \( n = 4m + 3 \) with \( m \geq 0 \). Then we may take
\[
\phi = \begin{cases} 
\rho'_1 + \cdots + \rho'_{2m+1} + \chi & \text{if } \omega \neq 1, \\
\rho'_1 + \cdots + \rho'_{2m-1} + \rho''_1 + \rho''_1 + \chi & \text{if } \omega = \chi \text{ and } m > 0.
\end{cases}
\]

Here, \( \chi', \chi'' \) and \( \chi''' \) are non-trivial quadratic characters of \( \mathbb{F}_v^\times \), and \( \rho'_i, \rho''_i \) and \( \rho'''_i \) are inequivalent irreducible 2-dimensional orthogonal representations of \( WD_{\mathbb{F}_v} \) with \( \det(\rho'_i) = \chi', \det(\rho''_i) = \chi'' \) and \( \det(\rho'''_i) = \chi''' \). In addition, we assume that \( \chi' = \chi'' \) in the first case, and \( \chi' = \chi'' = 1 \) in the second case.

By the remark after Proposition 6.3.15 for a non-trivial quadratic character \( \chi' \) of \( \mathbb{F}_v^\times \), there are infinitely many inequivalent irreducible 2-dimensional orthogonal representations with determinant \( \chi' \). Hence, there are infinitely many \( \phi \) which satisfy the desired conditions. \( \square \)

**Lemma 6.10.** Let \( s \) be an odd positive integer and \( \omega \) be a non-trivial quadratic character of \( \mathbb{F}_v^\times \). Then for quadratic characters \( \chi_1, \ldots, \chi_s \) which are pairwise distinct, we have
\[\{\chi_1, \ldots, \chi_s\} \neq \{\chi_1\omega, \ldots, \chi_s\omega\} \).

**Proof.** Suppose that \( \{\chi_1, \ldots, \chi_s\} = \{\chi_1\omega, \ldots, \chi_s\omega\} \). Then the map \( \chi_i \mapsto \chi_i\omega \) induces a permutation of indices
\[\sigma: \{1, \ldots, s\} \rightarrow \{1, \ldots, s\}. \]

Since \( \omega \) is a non-trivial quadratic character, we have \( \sigma^2 = 1 \) and \( \sigma(i) \neq i \) for any \( 1 \leq i \leq s \). Such a permutation exists only when \( s \) is even. \( \square \)

**Proof of Proposition 6.8.** We consider several cases separately.

First, we assume that \( r \geq 2 \) and \( \dim(\phi_1), \dim(\phi_2) \geq 3 \). Then for \( 3 \leq i \leq r \), we take \( \phi_{i,v} \) such that
- \( \phi_{i,v} \) is a sum of irreducible orthogonal 2-dimensional representations and quadratic characters;
- \( \phi_{i,v} \) contains a 2-dimensional irreducible representation \( \rho_{i,v} ' \) which is not contained in \( \phi_{i',v} \) for any \( i' \neq i \).

Then we take \( \phi_{1,v} \) and \( \phi_{2,v} \) such that
- both \( \phi_{1,v} \) and \( \phi_{2,v} \) contain the trivial character 1;
- \( \phi_{i,v} \) contains a 2-dimensional irreducible representation \( \rho_{i,v} ' \) which is not contained in \( \phi_{i',v} \) for any \( i' \neq i \);
- \( \det(\phi_{i,v}) = 1 \).

This can be done by Lemma 6.9.

Next, we assume that \( r \geq 2 \), \( \dim(\phi_1) \geq 3 \) and \( \dim(\phi_i) = 2 \) for each \( 2 \leq i \leq r \). Then we let \( \phi_{i,v} = \phi_i \) for \( 2 \leq i \leq r \). We take \( \phi_{1,v} \) such that
Using Lemma 6.7, we globalize these characters to \( \omega \). This is a quadratic character of \( A \). We choose an orthogonal representation \( \phi \). This can be done. Indeed, if \( \dim(\phi_1) \geq 4 \) or \( s = 0 \), then it is clear by Lemma 6.9. So we assume that \( \dim(\phi_1) = 3 \). If \( s \geq 1 \), then \( s \geq 2 \) since \( \dim(\phi) \) is odd. Hence there exists \( j \) such that

\[
\chi_{j,v} \neq \det(\phi_{2,v} + \cdots + \phi_{r,v} + \chi_{1,v} + \cdots + \chi_{s,v}).
\]

By Lemma 6.9, we can take \( \phi_{1,v} \) which contains \( \chi_{j,v} \) and such that \( \det(\phi_v) = 1 \).

Next, we assume that \( r = 1 \) and \( \dim(\phi_1) \geq 3 \). Then \( s \geq 2 \) by assumption. Hence we can take \( \phi_{1,v} \) which contains \( \chi_{j,v} \) for some \( j \) and a 2-dimensional irreducible orthogonal representation.

Finally, we assume that \( \det(\phi_i) = 2 \) for each \( 1 \leq i \leq r \). Then we let \( \phi_{i,v} = \phi_i \) for \( 2 \leq i \leq r \). Put \( \omega_v = \det(\phi_1) \). Note that \( \omega_v \neq 1 \), and \( \chi_{1,v}, \ldots, \chi_{s,v} \) are pairwise distinct. Since \( s \) is odd, by Lemma 6.10 we can find \( \chi_j \) such that \( \chi_{j,v}\omega_v \notin \{\chi_{1,v}, \ldots, \chi_{s,v}\} \).

We take

\[
\phi_{1,v} = \chi_{j,v} + \chi_{j,v}\omega_v.
\]

Then in any cases, the conditions (1) to (4) hold. Moreover, we see that for each \( 1 \leq i \leq r \), the representation \( \phi_{1,v} \) contains an irreducible orthogonal representation which are not contained in \( \phi_{i,v} \) and \( \chi_{j,v} \) for any \( i' \neq i \) and any \( j \). In addition, \( \chi_{1}, \ldots, \chi_{s} \) are pairwise distinct. Therefore, the natural map

\[
A_{\phi}^+ \rightarrow A_{\phi_v}^+
\]

is injective. This completes the proof. \( \square \)

For \( 2 \leq i \leq r \), we choose finite places \( v_{c,i} \) of \( F \) such that

- \( v_{c,i} \notin \{v_0, v'_0, v_{c,i}\} \);
- \( v_{c,i} \) is not 2-adic;
- \( v_{c,2}, \ldots, v_{c,r} \) are pairwise distinct;
- \( \chi_{j,v_{c,i}} = 1 \) for any \( i,j \).

We choose an orthogonal representation \( \phi_{i,v_{c,i}} \) of \( WD_{F_{v_{c,i}}} \) for \( 1 \leq i \leq r \) such that

- if \( n_i = \dim(\phi_i) \) is odd, then \( \phi_{i,v_{c,i}}|SL_2(C) = v_{n_i} \) and \( \phi_{v_{c,i}}|W_{F_{v_{c,i}}} = 1 \);
- if \( n_i = \dim(\phi_i) \equiv 2 \mod 4 \), then \( \phi_{i,v_{c,i}}|SL_2(C) = 1 \) and \( \phi_{v_{c,i}}|W_{F_{v_{c,i}}} \) is an irreducible orthogonal representation such that \( \det(\phi_{v_{c,i}}) \) is the non-trivial unramified quadratic character of \( F_{v_{c,i}}^\times \);
- if \( n_i = \dim(\phi_i) \equiv 0 \mod 4 \), then \( \phi_{i,v_{c,i}}|SL_2(C) = 1 \) and \( \phi_{v_{c,i}}|W_{F_{v_{c,i}}} \) is an irreducible orthogonal representation with the trivial determinant.

For \( 2 \leq i \leq r \), we put

\[
\omega_{i,v} := \begin{cases} 
\det(\phi_{i,v}) & \text{if } v \in \{v_0, v'_0, v_{c,i}\}, \\
1 & \text{if } v \in \{v_{c,2}, \ldots, v_{c,r}\} \setminus \{v_{c,i}\}.
\end{cases}
\]

Moreover, we put

\[
\omega_{1,v_{c,i}} := \begin{cases} 
\det(\phi_{1,v_{c,i}})^{-1} & \text{if } s = 0 \text{ and } i = 2, \\
1 & \text{if } s \geq 1 \text{ or } i \geq 3.
\end{cases}
\]

Using Lemma 6.7, we globalize these characters to \( \omega_1 \). Then we obtain global characters \( \omega_2, \ldots, \omega_r \) and \( \overline{\chi}_{1}, \ldots, \overline{\chi}_{s} \). We put

\[
\omega_1 := (\omega_2 \cdot \omega_3 \cdot \overline{\chi}_1 \cdot \ldots \cdot \overline{\chi}_s)^{-1}.
\]

This is a quadratic character of \( \mathbb{A}^\times_F/\mathbb{F}^\times \) which satisfies

\[
\omega_{1,v} = \det(\phi_{1,v})
\]

for \( v \in \{v_0, v'_0, v_{1,c}\} \). Note that if \( s = 0 \), then \( r \geq 2 \).
Let $S_{\text{ram},i}$ (resp. $S'_{\text{ram},j}$) be the finite set of finite places $v \neq v_0, v'_0$ of $F$ such that $\omega_{i,v}$ (resp. $\chi_{j,v}$) is ramified. We put

$$S_{\text{ram}} = \bigcup_{i=1}^{r} S_{\text{ram},i} \cup \bigcup_{j=1}^{s} S'_{\text{ram},j}$$

and

$$T = \{ v \in S_{\text{ram}} \mid \omega_{j,v} \neq 1 \text{ whenever } \dim(\phi_j) = 2 \}.$$  

Note that $v_{c,i} \notin S_{\text{ram}}$ for any $1 \leq i \leq r$.

**Lemma 6.11.** For $v \in T$, there exists an orthogonal representation $\phi_{i,v}$ of $WD_{F_v}$ such that $\det(\phi_{i,v}) = \omega_{i,v}$ and $\phi_{i,v}$ satisfies the conditions $\mathbf{(1)}$ to $\mathbf{(4)}$ of Proposition 6.8.

**Proof.** The proof is similar to that of Proposition 6.8. \[\square\]

We put $S_0 = T \cup \{ v_0 \}$. Note that $v_0 \notin S$ and the natural map

$$A^+_\phi \rightarrow \prod_{v \in S_0} A^+_\phi_v$$

is injective.

If $v \in S_{\text{ram},i} \setminus T$, then by Proposition 6.3 and Lemma 6.9 we can choose a discrete parameter $\phi_{i,v}$ with determinant $\omega_{i,v}$. We set

$$S_i := \{ v_0, v_{c,i} \} \cup T \cup S_{\text{ram},i}$$

for $1 \leq i \leq r$.

Finally, we choose finite places $w_i, w_{d,i}, w_{s,i}$ of $F$ for $1 \leq i \leq r$ such that

- $w_i, w_{d,i}, w_{s,i} \not\in \bigcup_j S_j$;
- $\omega_{i,w} = \chi_{j,w} = 1$ for $1 \leq i \leq r$, $1 \leq j \leq s$ and $w \in \{ w_1, \ldots, w_r, w_{d,1}, \ldots, w_{d,r}, w_{s,1}, \ldots, w_{s,r} \}$;
- $w_1, \ldots, w_r, w_{d,1}, \ldots, w_{d,r}, w_{s,1}, \ldots, w_{s,r}$ are pairwise distinct.

Fix $1 \leq i \leq r$. If $\omega_i \neq 1$, then $\omega_i$ gives a quadratic extension $E_i/F$. We may write $E_i = F(\sqrt{\alpha_i})$. If $\omega_i = 1$, then we put $\alpha_i = 1$. Let $n_i = \dim(\phi_i)$. Note that $n_i \geq 2$.

If $n_i$ is even, then $\phi_{i,v} \in \Phi_{\text{disc}}(SO(V_{n_i,v}))$ with a quasi-split group $SO(V_{n_i,v})$ for $v \in S_i \cup \{ v_0 \}$, where $V_{n_i,v}$ is an orthogonal space over $F_v$ with $\text{disc}(V_{n_i,v}) = \alpha_i \pmod{F_v^{\times 2}}$. Take an irreducible discrete series representation $\pi_{i,v} \in \Pi_{\phi_{i,v}}$. We may assume that $\pi_{i,v} \in \text{Irr}(SO(V_{n_i,v}))$. There is an orthogonal space $V_{n_i}$ over $F$ such that $V_{n_i,v} = V_{n_i,w}$ for any $v \in S_i \cup \{ v_0 \}$, $\text{disc}(V_{n_i}) = \alpha_i \pmod{F_v^{\times 2}}$ and $\text{SO}(V_{n_i})$ is quasi-split over $F$. Then by [49, Theorem 5.13], one can find an irreducible cuspidal automorphic representation $\Pi_i$ of $\text{SO}(V_{n_i})(\mathbb{A}_F)$ such that

- $\Pi_{i,v} = \pi_{i,v}$ for $v \in S_i \cup \{ v_0 \}$;
- $\Pi_{i,v}$ is unramified for any $v \notin S_i \cup \{ v_0, w_i, w_{d,i}, w_{s,i} \}$;
- $\Pi_{i,w_{d,i}}$ is discrete series;
- $\Pi_{i,w_{s,i}}$ is supercuspidal.

By Theorem 6.1 we the representation $\Pi_i$ belongs to $\Pi_{\Sigma_i}$ for some $\Sigma_i \in \tilde{\Psi}_2(\text{SO}(V_{n_i}))$. Since $\phi_{i,v_0} \mid_{\text{SL}_2(\mathbb{C})} = 1$, by Corollary 6.3 we see that $\Sigma_i \in \tilde{\Psi}_{2,\text{temp}}(\text{SO}(V_{n_i}))$. In particular, $\Sigma_{i,v_{c,i}} \in \text{Irr}(\text{GL}_{n_i}(F_{v_{c,i}}))$ corresponds to $\phi_{i,v_{c,i}} : WD_{F_{v_{c,i}}} \rightarrow \text{GL}_{n_i}(\mathbb{C})$. Since this is an irreducible representation of $W_{F_{v_{c,i}}}$, we see that $\Sigma_{i,v_{c,i}}$ is supercuspidal, so that $\Sigma_i \in \text{A}_{\text{cusp}}(\text{GL}(n_i))$.

If $n_i$ is odd, then $\phi_{i,v} := \phi_{i,v} \otimes \omega_{i,v} \in \Phi_{\text{disc}}(\text{Sp}(W_{n_i-1,v}))$ for $v \in S_i \cup \{ v_0 \}$, where $W_{n_i-1,v}$ is a symplectic space over $F_v$. Take an irreducible discrete series representation $\pi'_{i,v} \in \Pi_{\phi_{i,v}}$. There is a symplectic space $W_{n_i-1}$ over $F$ such that $W_{n_i-1,v} = W_{n_i-1,w}$ for any $v \in S_i \cup \{ v_0 \}$. Then by [49, Theorem 5.13], one can find an irreducible cuspidal automorphic representation $\Pi'_i$ of $\text{Sp}(W_{n_i-1})(\mathbb{A}_F)$ such that

- $\Pi'_{i,v} = \pi'_{i,v}$ for $v \in S_i \cup \{ v_0 \}$;
- $\Pi'_{i,v}$ is unramified for any $v \notin S_i \cup \{ v_0, w_i, w_{d,i}, w_{s,i} \}$;
- $\Pi'_{i,w_{d,i}}$ is discrete series;
- $\Pi'_{i,w_{s,i}}$ is supercuspidal.
By Theorem [44] the representation \( \Pi'_i \) belongs to \( \Pi^s \) for some \( \Sigma'_i \in \tilde{\Phi}_2(\text{Sp}(\mathcal{W}_{n_i-1})) \). Since \( \phi'_{i,v_0} |_{\text{SL}_2(\mathbb{C})} = 1 \), by Corollary [45] we see that \( \Sigma'_i \in \tilde{\Phi}_{2,\text{temp}}(\text{Sp}(\mathcal{W}_{n_i-1})) \). In particular, \( \Sigma'_{i,v_{c,i}} \in \text{Irr} (\text{GL}_{n_i-1}(\mathbb{F}_{v_{c,i}})) \) corresponds to \( \phi'_{i,v_{c,i}} : WD_{v_{c,i}} \rightarrow \text{GL}_{n_i-1}(\mathbb{C}) \). Since this is an irreducible representation of \( \text{SL}_2(\mathbb{C}) \), we see that \( \Sigma'_{i,v_{c,i}} \) is a special representation. Since this special representation is tempered, by [9, Proposition 4.10], we see that \( \Sigma'_i \) is cuspidal. We put \( \Sigma_i = \Sigma'_i \otimes \omega_i \).

We set

\[
S = \{ v_0 \} \cup S_{\text{ram}} \cup \bigcup_{i=1}^{r} \{ v_{c,i}, w_i, w_{d,i}, w_{s,i} \}
\]

and

\[
\Sigma = (\Pi'_{i=1} \Sigma_i) \boxplus (\Pi^s_{j=1} \chi_j).
\]

Then \( \Sigma \in \tilde{\Phi}_{2,\text{temp}}(\text{Sp}(\mathcal{W}_{2n})) \), where \( \mathcal{W}_{2n} \) is a symplectic space over \( \mathbb{F} \) with dimension \( 2n = \sum_{i=1}^{r} n_i + s - 1 \).

Next, we consider the case

\[
\phi = \chi_1 + \cdots + \chi_s
\]

with \( s = 2n + 1 \geq 5 \). This case may occur only if \( p = 2 \).

**Proposition 6.12.** Suppose that \( s \geq 5 \). There exist a global field \( \mathbb{F} \), a finite place \( v_0 \) of \( \mathbb{F} \), quadratic characters \( \omega_1, \ldots, \omega_s \) of \( \mathbb{A}_F^\times /\mathbb{F}_v^\times \) and a finite set \( S \) of finite places of \( \mathbb{F} \) such that

- \( \mathbb{F}_{v_0} = \mathbb{F} \);
- \( \mathbb{F} \) is totally complex;
- \( \omega_i,v_0 = \chi_i \) for \( 1 \leq i \leq s \);
- \( S_0 = \{ v_1, \ldots, v_{s-1} \} \neq v_0 \);
- \( \omega_i,v_0 \) is ramified for \( 1 \leq i \leq s - 1 \);
- for any finite place \( v \neq v_0 \) of \( \mathbb{F} \),

\[
\# \{ i \mid 1 \leq i \leq s - 1 \text{ and } \omega_i,v_0 \text{ is ramified} \} \leq 1.
\]

Moreover we see that

- for any \( v \neq v_0 \), the local component

\[
\phi_v = \omega_1,v + \cdots + \omega_s,v
\]

is not multiplicity-free;
- the natural map

\[
A^+_\phi \rightarrow \prod_{v \in S_0} A^+_{\phi_v}
\]

is injective.

**Proof.** We choose \( \mathbb{F} \) and \( v_0 \). We construct \( v_1 \) and \( \omega_1 \) by induction on \( 1 \leq i \leq s - 1 \).

We take \( v_1 \neq v_0 \). We set \( \chi_{1,v_0} = \chi_1 \) and choose a ramified quadratic character \( \chi_{1,v_1} \) of \( \mathbb{F}_{v_1}^\times \). We globalize \( \{ \chi_{1,v} \} \) to \( \omega_1 \).

Suppose that we can construct \( v_1, \ldots, v_{i-1} \) and \( \omega_1, \ldots, \omega_{i-1} \) for \( i \geq 2 \). We choose \( v_i \neq v_0 \) such that \( \omega_j,v_i \) is unramified for any \( 1 \leq j \leq i - 1 \). We set \( \chi_{i,v_0} = \chi_i \) and choose a ramified quadratic character \( \chi_{i,v_i} \) of \( \mathbb{F}_{v_i}^\times \).

In addition, for any finite place \( v \neq v_0 \) such that \( \omega_j,v \) is ramified for some \( 1 \leq j \leq i - 1 \), we put \( \chi_{i,v} = 1 \). We globalize \( \{ \chi_{i,v} \} \) to \( \omega_i \).

By induction, we can construct \( S_0 = \{ v_1, \ldots, v_{s-1} \} \) and \( \omega_1, \ldots, \omega_{s-1} \). We put

\[
\omega_s = (\omega_1 \cdots \omega_{s-1})^{-1}.
\]

Since \( \det(\phi) = 1 \), we see that \( \omega_{s,v_0} = \chi_s \). By the construction, the conditions of ramification are satisfied.

Let \( v \neq v_0 \) be a finite place of \( \mathbb{F} \). Since \( s - 2 \geq 3 \) and there are only two quadratic unramified characters of \( \mathbb{F}_v^\times \), we see that there are \( 1 \leq i < j \leq s - 1 \) such that \( \omega_i,v \) and \( \omega_j,v \) are the same (unramified) character. Hence

\[
\phi_v = \omega_1,v + \cdots + \omega_s,v
\]

is not multiplicity-free.
Finally, we show that the natural map $A^+_{\phi} \to \prod_{v \in S_0} A^+_{\phi_v}$ is injective. Let $a_i$ be the element in $A^+_{\phi}$ corresponding to $\chi_i$. Then any element in $A^+_{\phi}$ can be written as

$$a = \varepsilon_1 a_1 + \cdots + \varepsilon_s a_s$$

for $\varepsilon_i \in \{0, 1\}$. Now we assume that

$$a \in \ker[A_{\phi} \to \prod_{v \in S_0} A_{\phi_v}].$$

Let $1 \leq i \leq s - 1$. Note that $\omega_{j,v_i}$ is ramified if and only if $j = i$ or $j = s$. Hence we see that if $\varepsilon_i = 1$, then $\varepsilon_j = 1$ for all $j$. Therefore we have

$$\ker[A_{\phi} \to \prod_{v \in S_0} A_{\phi_v}] \subseteq \{0, a_1 + \cdots + a_s\}.$$

However, $a_1 + \cdots + a_s \not\in A^+_\phi$ since $s$ is odd. Hence we see that

$$A^+_\phi \to \prod_{v \in S_0} A^+_\phi_v$$

is injective, as desired. \hfill $\Box$

We set $S = S_0$ and

$$\Sigma = \omega_1 \boxplus \cdots \boxplus \omega_s.$$

Then $\Sigma \in \mathcal{P}_{2,\text{temp}}(\Sp(\mathcal{W}_{2n}))$, where $\mathcal{W}_{2n}$ is a symplectic space over $\mathbb{F}$ with dimension $2n = s - 1$. We also put $\chi_j := \omega_j$.

6.5. Properties of $\Sigma$. We have completed the construction of a global tempered $A$-parameter $\Sigma$ unless $(r, s) = (1, 1)$ or $(r, s) = (0, 3)$. Let us examine some crucial properties of $\Sigma$.

**Proposition 6.13.** Suppose that $(r, s) \neq (1, 1), (0, 3)$. The local components of the $A$-parameter $\Sigma$ are given as follows:

1. at the place $v_0$, the local $A$-parameter associated to $\Sigma_{v_0}$ is $\phi$;
2. at the place $v \in S_0$, the local $A$-parameter associated to $\Sigma_v$ is $\phi_v$;
3. at the place $v \neq v_0$, the local $A$-parameter associated to $\Sigma_v$ is not discrete.

**Proof.** If $r = 0$, then the assertions are in Proposition 6.12. Hence we may assume that $r \geq 1$. The first two assertions are clear. We write $\phi'_{1,v}$ for the complement of $\phi_{1,v}$ in $\phi_v$, i.e.,

$$\phi_v = \phi_{1,v} + \phi'_{1,v}.$$
We choose local spaces $V_{\phi_v}$ and consider the local theta correspondence for $(\text{Sp}(n), \text{Sp}(n))$ for each $v$. Namely, we use $v_0$. As we have seen in [6.1], the global component group $A^+_n$ admits a natural map $A^+_n \rightarrow A^+_n$ for each place $v$. For $v = v_0$, this natural map is an isomorphism. On the other hand, by Proposition [6.8 and 6.12], we see that the diagonal map

$$A^+_n \rightarrow \prod_{v \in S_0} A^+_n$$

is injective. Hence, for any $\eta \in \hat{A^+_n}$, there exists $\eta_v \in \hat{A^+_n}$ for $v \in S_0$ such that

$$\left(\eta \otimes \bigotimes_{v \in S_0} \eta_v\right) \circ \Delta = 1_{A^+_n},$$

where

$$\Delta : A^+_n \rightarrow \prod_{v} A^+_n$$

is the diagonal map.

For $v \notin S_0 \cup \{v_0\}$, we put $\eta_v = 1_{A^+_n}$. Then for each $v$, we obtain the representation $\pi(\eta_v)$ of the local symplectic group $\text{Sp}(\mathbb{W}_{2n,v})$. By Arthur’s multiplicity formula (Theorem 6.4), we see that

$$\Pi = \bigotimes_v \pi(\eta_v)$$

is a representation of $\text{Sp}(\mathbb{W}_{2n})(\mathbb{A}_\mathbb{F})$ and occurs in the automorphic discrete spectrum of $\text{Sp}(\mathbb{W}_{2n})(\mathbb{A}_\mathbb{F})$. By [9 Proposition 4.10], we see that $\Pi$ is an irreducible cuspidal automorphic representation of $\text{Sp}(\mathbb{W}_{2n})(\mathbb{A}_\mathbb{F})$.

6.6. Global theta correspondence. Now we shall construct an orthogonal space $\mathbb{V}_{2n+2}$ of dimension $2n+2$ over $\mathbb{F}$, and consider the global theta correspondence for $(\text{Sp}(\mathbb{W}_{2n}), O(\mathbb{V}_{2n+2}))$.

Lemma 6.14. There is an orthogonal space $\mathbb{V}_{2n+2}$ of dimension $2n+2$ over $\mathbb{F}$ such that:

- at the place $v_0$, the local space $\mathbb{V}_{2n+2,v_0}$ is equal to the given orthogonal space $\mathbb{V}_{2n+2}$;
- for any place $v$, the local theta lift of $\Pi_v$ to $O(\mathbb{V}_{2n+2,v})$ with respect to $\Psi_v$ is nonzero;
- the group $SO(\mathbb{V}_{2n+2,v})$ is ramified over $F_v$ for some finite place $v \notin S \cup \{v_0\}$;
- for any place $v$, the group $SO(\mathbb{V}_{2n+2,v})$ is quasi-split.

Proof. We choose a finite place $v_0$ of $\mathbb{F}$ which is not contained in $S \cup \{v_0\}$. Note that the local $A$-parameter $\phi_v$ associated to $\Sigma_v$ is a sum of unramified characters. In particular, it does not contain a ramified character $\chi_v$. Then, by the weak approximation theorem, we can find $\alpha \in \mathbb{F}^\times$ such that

- $\text{disc}(\mathbb{V}_{2n+2}) = \alpha \text{ mod } F^\times$;
- $\alpha \notin F_v^{\times 2}$ for $v \in S \cup \{v_0\}$;
- $\chi_v = (\cdot, \alpha)_v$.

We choose local spaces $V_{2n+2,v}$ for $v \neq v_0$ such that

- $V_{2n+2,v_0} = \mathbb{V}_{2n+2}$;
- for any place $v \neq v_0$, the local theta lift of $\Pi_v$ to $O(\mathbb{V}_{2n+2,v})$ with respect to $\Psi_v$ is nonzero;
- $\text{disc}(\mathbb{V}_{2n+2,v}) = \alpha \text{ mod } F_v^{\times 2}$ for each $v \neq v_0$;
- if $\alpha \in F_v^{\times 2}$, then $SO(\mathbb{V}_{2n+2,v})$ is split;
- the Hasse invariant of $\mathbb{V}_{2n+2,v}$ is $1$ for almost all $v$.  

\[\square\]
We can choose such spaces by Proposition 2.5 since for almost all \( v \neq v_1 \), the conditions in [27, 28] are satisfied.

We can find an orthogonal space \( V_{2n+1} \) over \( F \) such that \( V_{2n+2,v} = V_{2n+2,v} \) for any \( v \neq v_1 \) and \( \text{disc}(\mathbb{V}) = \alpha \). Since \( \phi_{v_1} \) does not contain \( \chi_{v_1} \), the local theta lift of \( \Pi_{v_1} \) to \( O(V_{2n+2,v_1}) \) is nonzero. Moreover, for \( v \in S \cup \{ v_1 \} \), the group \( SO(V_{2n+2,v}) \) is quasi-split since \( \alpha \notin F_v \times 2 \).

Since the group \( SO(V_{2n+2,v}) \) is quasi-split for any place \( v \), by the Hasse–Minkowski theorem, we see that \( SO(V_{2n+2}) \) is quasi-split over \( F \). Consider the global theta lift \( \Pi' = \Theta_{\Psi,V_{2n+2},W_{2n},(\pi)} \) to \( O(V_{2n+2}) \).

**Proposition 6.15.** The global theta lift \( \Pi' = \Theta_{\Psi,V_{2n+2},W_{2n},(\pi)} \) is an irreducible cuspidal automorphic representation of \( O(V_{2n+2})(\mathbb{A}_F) \) such that \( \Pi'_{v_0} = \Theta_{\Psi,V_{2n+2},W_{2n},(\pi)} \).

**Proof.** Since the local \( A \)-parameter associated to \( \Sigma_{v_0} \) does not contain the discriminant character of \( V_{2n+2,v_1} \), by Proposition 2.6 we ensure that \( \Pi' \) is cuspidal.

To show that \( \Pi' \neq 0 \), we consider the standard \( L \)-function \( L(s,\Pi) \) of \( \Pi \) defined by [14, 36]. Observe that the partial \( L \)-function \( L_{\Sigma}^{S_{\cup\{v_0\}}}(s,\Sigma) \) of \( \Sigma \), so that

\[
L_{\Sigma}^{S_{\cup\{v_0\}}}(1,\Pi) = L_{\Sigma}^{S_{\cup\{v_0\}}}(1,\Sigma) = \left( \prod_{i=1}^{r} L_{\Sigma}^{S_{\cup\{v_0\}}}(1,\Sigma_i) \right) \left( \prod_{j=1}^{s} L_{\Sigma}^{S_{\cup\{v_0\}}}(1,\Sigma_j) \right) .
\]

By [23], we have

\[
L_{GJ}(1,\Sigma_i) \neq 0 \quad \text{and} \quad L_{GJ}(1,\Sigma_j) \neq 0 .
\]

Note that for an irreducible admissible representation \( \sigma_v \) of \( GL_n(F_v) \), by [36, 10], we have

\[
L(s,\sigma_v) = L_{GJ}(s,\sigma_v) L_{GJ}(s,\sigma_v^\vee) .
\]

By [36, Proposition 5], if \( \sigma_v \) is tempered and unitary, then the local standard \( L \)-function \( L(s,\sigma_v) \) is holomorphic at \( s = 1 \), and so is \( L_{GJ}(s,\sigma_v) \). Therefore we have

\[
L_{GJ}(1,\Sigma_i) \neq 0 \quad \text{and} \quad L_{GJ}(1,\Sigma_j) \neq 0 .
\]

Hence we see that \( L_{\Sigma}^{S_{\cup\{v_0\}}}(1,\Pi) \neq 0 \). On the other hand, since \( \Pi_v \) is tempered and unitary for \( v \in S \cup \{ v_0 \} \), by [36, Proposition 5], the local standard \( L \)-factor \( L(s,\Pi_v) \) at \( v \in S \cup \{ v_0 \} \) is holomorphic and nonzero at \( s = 1 \). Hence, we have

\[
L(1,\Pi) \neq 0 ,
\]

and it follows by [14, Theorem 1.4] that \( \Pi' \neq 0 \).

Finally, since the local theta lift \( \Theta_{\Psi_v,V_{2n+2,v},W_{2n,v},(\pi)} \) is irreducible, it follows by [34, Corollary 7.1.3] that \( \Pi' \) is irreducible. Therefore \( \Pi' \) is an irreducible cuspidal automorphic representation of \( O(V_{2n+2})(\mathbb{A}_F) \) such that

\[
\Pi'_{v_0} = \Theta_{\Psi,V_{2n+2},W_{2n},(\pi)} .
\]

6.7. **Completion of proof.** We have fixed \( a_0 \in F^\times \) such that \( a_0 \equiv c_0 \mod F_v v_0 = F^\times 2 \). This gives the local Langlands–Vogan parametrization of \( SO(V_{2n+2,v}) \) for each \( v \). Namely, we use \( i_{v_0,a_0} \). We may write

\[
\Pi = \bigotimes_v \pi(\eta_v) \quad \text{and} \quad \Pi'|SO(V_{2n+2}) = \bigotimes_v \pi'(\eta'_v)
\]

with \( \eta_v \in A^+_\phi \) and \( \eta'_v \in A^+_\phi' \), where \( \phi' = (\phi_v \otimes \chi_{V_{2n+2,v}}) \otimes 1 \) and \( \eta'_v = \eta' \). Note that \( \Pi'|SO(V_{2n+2}) \) is also irreducible as a representation of \( SO(V_{2n+2})(\mathbb{A}_F) \) since \( \pi'(\eta'_v)|SO(V_{2n+2,v}) \) is irreducible for any \( v \). Applying Arthur’s multiplicity formula (Theorem 6.3) to \( \Pi' \) and \( \Pi'|SO(V_{2n+2}) \), we see that

\[
\prod_v \eta_v(a_v) = 1 \quad \text{and} \quad \prod_v \eta'_v(a_v) = 1
\]

for any \( a \in A^+_\Sigma \), where \( a_v \) is the image of \( a \) in \( A^+_\phi \), which is the component group of the local \( A \)-parameter \( \phi_v \) associated to \( \Sigma_v \). However, for any places \( v \neq v_0 \), the element \( \phi_v \) in \( \Phi(Sp(W_{2n,v})) \) associated to \( \Pi_v = \pi(\eta_v) \) is not discrete. Hence, one knows that (P1) holds for \( \phi_v \). So we have

\[
\eta'_v(a_v) = \eta_v(a_v)
\]
for any $v \neq v_0$. Therefore, we conclude that at the place $v_0$, we have
\[ \eta'(a_{v_0}) = \eta(a_{v_0}) \]
for $a \in A^+_\phi$. Since $A^+_\phi \ni a \mapsto a_{v_0} \in A^+_\phi$ is an isomorphism, we have $\eta'|A^+_\phi = \eta$, as desired.

This completes the proof of Theorem 6.1.

6.8. Reduction to quasi-split cases. In this subsection, we prove that $(P1)$ when $SO(V_{2n+2})$ is quasi-split implies $(P1)$ when $SO(V_{2n+2}^\prime)$ is not quasi-split under some assumptions.

We denote by $V_{2n+2}^\pm$ the orthogonal spaces over $F$ such that $\dim(V_{2n+2}^\pm) = 2n+2$, $SO(V_{2n+2}^\pm)$ is split and $SO(V_{2n+2}^\pm)$ is not quasi-split. Note that $\dim(V_{2n+2}^\pm) = \dim(V_{2n+2}^-) = 1$ and $SO(V_{2n+2}^-)$ is a pure inner form of $SO(V_{2n+2}^\pm)$. Let $\phi' \in \tilde{\Phi}(SO(V_{2n+2}^\pm))$. Then we can write
\[ \phi' = \phi_1 \oplus \phi_0 \oplus \phi_2', \]
where $\phi_0$ (resp. $\phi_1'$) is a sum of irreducible representations of $WD_F$ which are orthogonal (resp. are not orthogonal). In this case, we see that $A^+_{\phi'} = A^+_{\phi_0}$. Note that $\det(\phi') = 1$. If $\dim(\phi_0) \leq 2$, then $A^+_{\phi_0}$ is singleton and $\Pi_{\phi_0} \subset \text{Irr}(SO(V_{2n+2}^\pm))/\sim_{\varepsilon}$. So we may assume that $\dim(\phi_0) \geq 4$. We put $2n_0 + 2 = \dim(\phi_0)$ and $k = \dim(\phi_1')$ for $n_0 \geq 1$. Let $P_n^\phi = P_n^\phi$ be the parabolic subgroup of $SO(V_{2n+2}^-)$ defined in (2.3) Hence the Levi component of $P_n^\phi$ is isomorphic to $GL_k(F) \times SO(V_{2n_0+2})$.

We make the following additional assumptions:
- Similar results in [6.1] for $SO(V_{2n+2}^-)$;
- If $\phi_0$ is $\varepsilon$-invariant, then for $\sigma_0 \in \Pi_{\phi_0} \cap \text{Irr}(SO(V_{2n+2}^-))$, the parabolic induction
\[ \sigma := \text{Ind}_{P_n^\phi}^{SO(V_{2n+2}^-)}(\tau \otimes \sigma_0) \]
is irreducible. Here, $\tau$ is the representation of $GL_k(F)$ corresponding to $\phi_1'$;
- Moreover, $[\sigma] \in \Pi_{\phi'}$ and $\iota_{w_{\phi_0}}([\sigma]) = \iota_{w_{\phi_0}}([\sigma_0])$ for $\sigma_0 \in F^\times$.

The collection of these assumptions is (NQ) in (1).

Theorem 6.16. Assume that $(P1)$ holds for the pair $(Sp(W_{2n}), O(V_{2n+2}))$ when $SO(V_{2n+2})$ is quasi-split. Then $(P1)$ holds for the pair $(Sp(W_{2n}), O(V_{2n+2}^-))$ under the above assumptions.

First, we show that:

Lemma 6.17. Suppose that $SO(V_{2n+2}^-)$ is not quasi-split. If $(P1)$ for $(Sp(W_{2n}), O(V_{2n+2}^-))$ holds for $\phi \in \tilde{\Phi}_{\text{temp}}(Sp(W_{2n}))$ which is a sum of irreducible orthogonal representations, then $(P1)$ for $(Sp(W_{2n}), O(V_{2n+2}^-))$ holds generally under the above assumptions.

Proof. Let $\phi \in \tilde{\Phi}_{\text{temp}}(Sp(W_{2n}))$. We can write
\[ \phi = \phi_1 \oplus \phi_0 \oplus \phi_2', \]
where $\phi_0$ (resp. $\phi_1'$) is a sum of irreducible representations of $WD_F$ which are orthogonal (resp. are not orthogonal). In this case, $A^+_{\phi} = A^+_{\phi_0}$. Put $2n_0 + 1 = \dim(\phi_0)$ and $k = \dim(\phi_1')$. If $\phi = \phi_1 \oplus \phi_0 \oplus \phi_2'$, then $\Pi_{\phi} = \{\pi\}$ is singleton and $\Theta_{\psi,V_{2n+2}^+,W_{2n}}(\pi) \neq 0$, so that $\Theta_{\psi,V_{2n+2}^+,W_{2n}}(\pi) = 0$. See also (7.2) below. Hence we may assume that $n_0 \geq 1$, i.e., $\dim(\phi_0) \geq 3$. In particular, $\phi_0 \in \tilde{\Phi}_{\text{temp}}(Sp(W_{2n_0}))$. Put $\phi_0' = \phi_0 \oplus 1$ and $\phi' = \phi \oplus 1$. If $\pi_0 \in \Pi_{\phi_0}$ and $\tau$ is the irreducible representation of $GL_k(F)$ corresponding to $\phi_1'$, then $\tau := \text{Ind}_{P_n^\phi}^{SO(V_{2n+2}^-)}(\tau \otimes \pi_0)$ is irreducible. Here, $Q = Q_k$ is the parabolic subgroup of $Sp(2n)$ defined in (2.4) so that the Levi component of $Q$ is isomorphic to $GL_k(F) \times Sp(W_{2n_0})$. Moreover we have $\iota_{w_{\phi_0}}(\pi) = \iota_{w_{\phi_0}}(\pi_0)$.

We put $\tilde{\sigma} = \Theta_{\psi,V_{2n+2}^-W_{2n}}(\pi), \tilde{\sigma}_0 = \Theta_{\psi,V_{2n+2}^-W_{2n}}(\pi), \sigma = \tilde{\sigma}|SO(V_{2n+2}^-)$ and $\sigma_0 = \tilde{\sigma}_0|SO(V_{2n+2}^-)$. If $\tilde{\sigma}$ is nonzero, then by Proposition 2.6 and Lemma 2.3, we see that $\tilde{\sigma}_0$ is also nonzero and $\sigma$ is an irreducible constituent of $\text{Ind}_{P_n^\phi}^{SO(V_{2n+2}^-)}(\tau \otimes \sigma_0)$. However, by assumptions (NQ), we see that $\text{Ind}_{P_n^\phi}^{SO(V_{2n+2}^-)}(\tau \otimes \sigma_0)$ is irreducible, so that this is isomorphic to $\sigma$. Moreover $\iota_{w_{\phi_0}}([\sigma]) = \iota_{w_{\phi_0}}([\sigma_0])$. Hence we have
\[ \iota_{w_{\phi_0}}([\sigma])|A^+_{\phi} = \iota_{w_{\phi_0}}([\sigma_0])|A^+_{\phi_0} = \iota_{w_{\phi_0}}(\pi_0) = \iota_{w_{\phi_0}}(\pi). \]
This completes the proof of Lemma 6.17.

By Lemma 6.17, it suffices to consider $\phi \in \tilde{\Phi}_{\text{temp}}(\text{Sp}(W_{2n}))$ such that $\phi$ is reducible and is a sum of irreducible orthogonal representations. We write

$$
\phi = \phi_1^1 + \cdots + \phi_r^r,
$$

where $\phi_i$ is an irreducible orthogonal representation of $W_D$, and $\phi_i \not\equiv \phi_j$ for $i \neq j$. Let $n_i = \dim(\phi_i)$. We may assume that both $n_1$ and $l_1$ are odd.

There exist a totally complex field $F$ and a finite place $v_0$ such that $F_{v_0} = F$. We will choose an orthogonal representation $\phi_{i,j,v}$ of $W_D$, for $1 \leq i \leq r$, $1 \leq j \leq l_1$, and for several finite places $v$ of $F$. By Lemma 6.6 we may assume that there exists a finite place $v_0^0 \neq v_0$ of $F$ such that $F_{v_0^0} = F$. We put $\phi_{i,j,v_0} = \phi_{i,j,v_0^0} = \phi_i$. We take a finite place $v_0''$ of $F$ such that $v_0'' \neq v_0, v_0'$. We put $\phi_{i,j,v_0''} = 1^{|m_i|}$.

Next, we choose finite places $v_{i,j}$ of $F$ for $1 \leq i \leq r$ and $1 \leq j \leq l_i$, which are distinct, are not 2-adic, and are not in $\{v_0, v_0', v_0''\}$. We take $\phi_{i,j,v_{i,j}}$ such that

1. If $n_i$ is odd, then $\phi_{i,j,v_{i,j}}|_{\text{SL}_2(C)}$ is irreducible and $\phi_{i,j,v_{i,j}}|_{W_{\Sigma_{i,j}}} = 1$;
2. If $n_i$ is even, then $\phi_{i,j,v_{i,j}}|_{\text{SL}_2(C)} = 1$ and $\phi_{i,j,v_{i,j}}|_{W_{\Sigma_{i,j}}} = 1$.

Since $\phi$ is reducible, we can choose a quadratic character $\omega_{i,j,v}$ for $v \in \{v_0, v_0', v_0''\} \cup (\bigcup_{i,j} \{v_{i,j}\})$ such that

1. $\omega_{i,j,v} = \text{det}(\phi_{i,j,v})$ for $v \in \{v_0, v_0', v_0''\} \cup (\bigcup_{i,j} \{v_{i,j}\})$;
2. $\prod_{i,j} \omega_{i,j,v} = 1$ for $v \in \{v_0, v_0', v_0''\} \cup (\bigcup_{i,j} \{v_{i,j}\})$.

Using Lemma 6.7, we globalize these characters to $\omega_{i,j}$. We may assume that there exist finite places $v_{i,j,j'}$ of $F$ for $1 \leq i \leq r$ and $1 \leq j < j' \leq l_i$ such that

1. they are distinct;
2. $v_{i,j,j'} \notin \{v_0, v_0', v_0''\} \cup (\bigcup_{i,j} \{v_{i,j}\})$;
3. $\omega_{i,j,v_{i,j,j'}}$ and $\omega_{i,j',v_{i,j,j'}}$ are distinct non-trivial quadratic characters of $F_{v_{i,j,j'}}^\times$.

Then by Proposition 6.6 and Lemma 6.9, there exist orthogonal representations $\phi_{i,j,v_{i,j,j'}}$ and $\phi_{i,j',v_{i,j,j'}}$ for $1 \leq i \leq r$ and $1 \leq j < j' \leq l_i$ such that

1. $\phi_{i,j,v_{i,j,j'}}$ and $\phi_{i,j',v_{i,j,j'}}$ are multiplicity-free sums of irreducible orthogonal representations;
2. $\dim(\phi_{i,j,v_{i,j,j'}}) = \dim(\phi_{i,j',v_{i,j,j'}}) = n_i$;
3. $\text{det}(\phi_{i,j,v_{i,j,j'}}) = \omega_{i,j,v_{i,j,j'}}$ and $\text{det}(\phi_{i,j',v_{i,j,j'}}) = \omega_{i,j',v_{i,j,j'}}$.

In particular, we have $\phi_{i,j,v_{i,j,j'}} \not\equiv \phi_{i,j',v_{i,j,j'}}$.

By a similar argument to [6.4], we globalize $\phi_{i,j,v}$ for $v \in \{v_0, v_0', v_0''\} \cup (\bigcup_{j' < j} \{v_{i,j',j}\}) \cup (\bigcup_{j' > j} \{v_{i,j',j}\})$ to $\Sigma_{i,j}$, which is an irreducible orthogonal cuspidal automorphic representation of $\text{GL}_{n_i}(A_F)$ whose central character is $\omega_{i,j}$. They are distinct. We set $\Sigma = \prod_{i,j} \Sigma_{i,j}$. Then $\Sigma \in \tilde{\Phi}_{\text{temp}}(\text{Sp}(W_{2n}))$, where $W_{2n}$ is a symplectic space over $F$ with dimension $2n$. Let $S_{\text{ram}}$ be the set of finite places of $F$ at which $\Sigma$ is ramified. We put

$$
S = \{(v_0', v_0'') \cup (\bigcup_{i,j} \{v_{i,j}\}) \cup (\bigcup_{i,j,j'} \{v_{i,j,j'}\}) \cup S_{\text{ram}} \} \setminus \{v_0\}.
$$

We fix a non-trivial additive character $\Psi$ of $\mathbb{A}_F$ and $a_0 \in F^\times$ such that $a_0 \equiv c_0 \text{ mod } F^{\times 2} = F^{\times 2}$. We use $\Psi_v$ and $a_0 \in F_v^\times$ for the local Langlands–Vogan parametrization of $\text{Sp}(W_{2n,v})$ for each $v$. Namely, we use $\kappa_{v_0'}$.

Let $\eta = \eta_{v_0} \in \hat{A}_F^\times$. We put $\eta_{v_0'} = \eta^{-1}$ and $\eta_0 = 1$ for $v \neq v_0, v_0'$. Then by Arthur’s multiplicity formula (Theorem 6.4 and [9] Proposition 4.10), we see that

$$
\Pi := \bigotimes_v \pi(\eta_v)
$$

is an irreducible cuspidal automorphic representation of $\text{Sp}(W_{2n})(\mathbb{A}_F)$.

By a similar argument to [6.6] there exists an orthogonal space $V_{2n+2}$ of dimension $2n + 2$ over $F$ such that:

1. at the place $v_0$, the local space $V_{2n+2,v_0}$ is equal to the given orthogonal space $V_{2n+2}^-$;
2. for any place $v \neq v_0$, the group $\text{SO}(V_{2n+2,v})$ is quasi-split;
• for any place $v$, the local theta lift of $H_v$ to $O(V_{2n+2, v})$ with respect to $\Psi_v$ is nonzero.

Then by a similar argument to [6, 0.6] we see that
\[ H' := \Theta_{\Psi, V_{2n+2}, W_{2n}}(H)SO(V_{2n+2}) = \bigotimes_v \pi'(\eta'_v) \]
is an irreducible cuspidal automorphic representation of $SO(V_{2n+2})$, where $\eta'_{v_0} = \eta'$. Applying Arthur’s multiplicity formula (Theorem 6.4) for $H$ and $H'$, we see that $\eta'|A^+_\phi = \eta$. This completes the proof of Theorem 6.16.

7. Exceptional cases

Let $V = V_{2m}$ and $W = W_{2n}$. In the rest of the paper, we assume that $SO(V)$ is quasi-split. Hence $V$ is type $(d, c)$ for some $d, c \in F^\times$. We put $E = F(\sqrt{d})$. Then we have the following theorem whose proof will be given in Remark below:

**Theorem 7.1.** Let $n > 1$. If (P1) holds for all $\phi_0 \in \Phi_{\text{temp}}(Sp(W_{2n_0}))$ for any $n_0 < n$, then (P1) holds for $\phi \in \Phi_{\text{temp}}(Sp(W_{2n})) \setminus \Phi_{\text{disc}}(Sp(W_{2n}))$, except for $\phi$ of the form
\[ \phi = \phi_1 + 1 + \phi'_1, \]
where $\phi_1$ is an irreducible representation of $WD_F$.

To finish the proof of (P1), it now remains to prove Theorem 7.1 and the following exceptional cases:

- $n = 1$;
- $\phi = \phi_1 + \chi$, where $\phi_1$ is an irreducible orthogonal representation of $WD_F$ with $\dim(\phi_1) = 2n$ and $\chi$ is a quadratic character of $F^\times$;
- $\phi = \phi_1 + 1 + \phi'_1$, where $\phi_1$ is an irreducible representation of $WD_F$.

In this section, we prove (P1) for these exceptional cases. In these cases, the component group $A^+_\phi$ is small, so that we can show (P1) directly by considering the genericity of theta lifts.

7.1. Genericity of theta lifts. Let $G$ be a group, $(\pi, \mathcal{V})$ be a representation of $G$ and $\chi$ be a character of a subgroup $H$ of $G$. We denote by $\pi(H, \chi)$ or $\mathcal{V}(H, \chi)$ the subspace of $\mathcal{V}$ spanned by
\[ \{ \pi(h)v - \chi(h)v \mid h \in H, v \in \mathcal{V} \}. \]

We define the Jacquet module $\pi[H, \chi]$ or $\mathcal{V}[H, \chi]$ by
\[ \mathcal{V}[H, \chi] = \mathcal{V}/\mathcal{V}(H, \chi). \]

Let $B = TU$ (resp. $B' = T'U'$) be the Borel subgroup of $O(V)$ (resp. $Sp(W)$) and $\mu_{c_0}$ (resp. $\mu'_{c_0}$) be the generic character of $U$ (resp. $U'$) for $c_0 \in c \cdot N_{E/F}(E^\times)$ defined in [2]. In this subsection, we consider the Jacquet module $\omega[U, \mu_{c_0}]$ of the Weil representation $\omega = \omega_{\phi, V, W}$. Note that $Sp(W)$ acts on $\omega[U, \mu_{c_0}]$ canonically.

**Theorem 7.2.** Let $m = n + 1$, so that $W = W_{2n}$ and $V = V_{2n+2}$. For $c_0 \in c \cdot N_{E/F}(E^\times)$, there exists an isomorphism
\[ \omega[U, \mu_{c_0}] \cong \text{ind}_{U'}^{Sp(W)}(\mu'_{c_0}) \]
as representations of $Sp(W)$. Here $\text{ind}_{U'}^{Sp(W)}$ is the compact induction.

**Proof.** The proof is similar to that of [23] Proposition 2.1. \hfill \square

Using this theorem, we have the genericity of theta lifts.

**Corollary 7.3.** Let $c_0 \in c \cdot N_{E/F}(E^\times)$ and $\pi$ be an irreducible tempered $\omega'_{c_0}$-generic representation of $Sp(W)$. Then $\sigma := \Theta_{\phi, V, W}(\pi)|SO(V)$ is irreducible and $\omega_{c_0}$-generic.

**Proof.** By a similar argument to the proof of [22] Proposition 4.1, we see that precisely one irreducible subquotient of $\sigma = \Theta_{\phi, V, W}(\pi)$ is $\omega_{c_0}$-generic. In particular, $\sigma \neq 0$. Moreover, by Proposition 2.4 and 4.4, we see that $\sigma = \tilde{\sigma}|SO(V)$ is irreducible. \hfill \square
7.2. Proof of (P1) for exceptional cases. We show (P1) for the exceptional cases.

**Proposition 7.4.** Assume that \( n = 1 \) and \( \text{SO}(V) \) is quasi-split. Then Conjecture (P1) is true.

**Proof.** Let \( \phi: WD_{\mathcal{F}} \to \text{SO}(N) \) be in \( \check{\Phi}(\text{Sp}(W)) \). Note that \( \text{dim}(\phi) = 3 \). We may assume that \( \phi \) is tempered and reducible. We put \( \phi' = (\phi \otimes \chi_{V}) \otimes 1 \in \check{\Phi}_{\text{temp}}(\text{SO}(V)) \). We consider several cases separately.

(1) \( \phi = \rho + \chi \), where \( \rho \) and \( \chi \) are an irreducible orthogonal representations with \( \text{dim}(\rho) = 2 \) and \( \text{dim}(\chi) = 1 \). Then we have \#\( A_{\phi}^{+} = 2 \) and so that \( \Pi_{\phi} = \{ \pi, \pi' \} \), where

- \( \iota_{\mathfrak{w}_{c_{0}}}(\pi) \) is the trivial character of \( A_{\phi}^{+} \) and;
- \( \iota_{\mathfrak{w}_{c_{0}}}(\pi') \) is the non-trivial character of \( A_{\phi}^{+} \).

In particular, \( \pi \) is \( \mathfrak{w}_{c_{0}}' \)-generic. Let \( s \) be an element of \( C_{\phi}^{+} \) such that \( \phi^{s} = \rho \). We see that \( A_{\phi}^{+} \) is generated by (the image of) \( s \), and \( \eta_{c}(s) = \det(\rho)(c') = \chi(c') \) for \( c' \in F^{x} \). Note that \( \chi = \det(\rho) \neq 1 \) since there are no irreducible orthogonal representations with dimension 2 and the trivial determinant. We choose \( c' \in F^{x} \) which satisfies \( \chi(c') \neq 1 \). Then \( A_{\phi}^{+} = \{ 1, \eta_{c'} \} \), so that \( \iota_{\mathfrak{w}_{c_{0}}}(\pi') = \eta_{c'} \). By Proposition 3.1, we have

\[
\iota_{\mathfrak{w}_{c_{0}}}(\pi') = \iota_{\mathfrak{w}_{c_{0}}}(\pi') \otimes \eta_{c'} = 1,
\]

so that \( \pi' \) is \( \mathfrak{w}_{c_{0}c'} \)-generic.

First, we assume that \( \chi_{V} = \chi \). Then \( A_{\phi}^{+} = A_{\phi}^{+} \) and we see that:

- \( \Theta_{\psi, V^{+}, W}(\pi) \) is \( \mathfrak{w}_{c_{0}}' \)-generic and;
- \( \Theta_{\psi, V^{-}, W}(\pi') \) is \( \mathfrak{w}_{c_{0}c'} \)-generic,

by Corollary 7.3 where \( V^{+} \) (resp. \( V^{-} \)) is type \( (d, c_{0}) \) (resp. \( (d, c_{0}c') \)). This implies that

- \( \iota_{\mathfrak{w}_{c_{0}}} (\Theta_{\psi, V^{+}, W}(\pi)) \) is the trivial character of \( A_{\phi}^{+} \) and;
- \( \iota_{\mathfrak{w}_{c_{0}}} (\Theta_{\psi, V^{-}, W}(\pi')) \) is the non-trivial character of \( A_{\phi}^{+} \),

as desired.

Next, we assume that \( \chi_{V} \neq \chi \) and \( \chi_{V} \neq 1 \). We may take \( c_{1}, c_{2} \in F^{x} \) such that

\[
\chi(c_{1}) = -1, \quad \chi_{V}(c_{1}) = 1, \quad \chi(c_{2}) = 1 \quad \text{and} \quad \chi_{V}(c_{2}) = -1.
\]

Then by Proposition 3.1 we see that:

- \( \pi \) is \( \mathfrak{w}_{c_{1}}' \)-generic and \( \mathfrak{w}_{c_{1}c_{2}} \)-generic and;
- \( \pi' \) is \( \mathfrak{w}_{c_{0}c_{1}}' \)-generic and \( \mathfrak{w}_{c_{0}c_{1}c_{2}} \)-generic.

Hence by Corollary 7.3 and Proposition 3.2, we see that:

- \( \Theta_{\psi, V^{+}, W}(\pi) \) is \( \mathfrak{w}_{c_{0}}' \)-generic so that \( \iota_{\mathfrak{w}_{c_{0}}} (\Theta_{\psi, V^{+}, W}(\pi)) = 1; \)
- \( \Theta_{\psi, V^{+}, W}(\pi') \) is \( \mathfrak{w}_{c_{0}c_{1}}' \)-generic so that \( \iota_{\mathfrak{w}_{c_{0}}} (\Theta_{\psi, V^{+}, W}(\pi')) = \eta_{c_{1}}; \)
- \( \Theta_{\psi, V^{-}, W}(\pi) \) is \( \mathfrak{w}_{c_{1}c_{2}}' \)-generic so that \( \iota_{\mathfrak{w}_{c_{0}}} (\Theta_{\psi, V^{-}, W}(\pi)) = \eta_{c_{2}} \) and;
- \( \Theta_{\psi, V^{-}, W}(\pi') \) is \( \mathfrak{w}_{c_{0}c_{1}c_{2}}' \)-generic so that \( \iota_{\mathfrak{w}_{c_{0}}} (\Theta_{\psi, V^{-}, W}(\pi')) = \eta_{c_{1}c_{2}} \),

where \( V^{+} \) (resp. \( V^{-} \)) is type \( (d, c_{0}) \) and \( (d, c_{0}c_{1}) \) (resp. type \( (d, c_{0}c_{2}) \) and \( (d, c_{0}c_{1}c_{2}) \)). This implies that Conjecture 4.3 is true in this case.

If \( \chi_{V} = 1 \), then the map \( \iota_{\mathfrak{w}_{c_{0}}}: \Pi_{\phi'} \to \overset{\sim}{A}_{\phi}^{+} \) gives a bijection

\[
\iota_{\mathfrak{w}_{c_{0}}}: \Pi_{\phi'} \cap \text{Irr}(\text{SO}(V)) \to \pi_{0}(C_{\phi'}^{+}/\{ \pm 1 \})\overset{\sim}{\rightarrow}.
\]

Here, we let \( V \) be type \( (1, 1) \) and denote by \( \{ \pm 1 \} \) the center of \( \text{SO}(4, \mathbb{C}) = (\text{SO}(V))\overset{\sim}{\rightarrow}. \) Corollary 7.3 and Proposition 3.1, 3.2 imply that the diagram

\[
\begin{array}{ccc}
\Pi_{\phi} & \overset{\Theta_{\psi, V,W}}{\longrightarrow} & \Pi_{\phi'} \cap \text{Irr}(\text{SO}(V)) \\
\iota_{\mathfrak{w}_{c_{0}}}: 1:1 & \downarrow & \downarrow \iota_{\mathfrak{w}_{c_{0}}}: 1:1 \\
A_{\phi}^{+} & \overset{\pi_{0}(C_{\phi'}^{+}/\{ \pm 1 \})}{\rightarrow} & \pi_{0}(C_{\phi'}^{+}/\{ \pm 1 \})
\end{array}
\]

is commutative, as desired.
(2) \( \phi = \chi + 1 + \chi^{-1} \) with \( \chi^2 \neq 1 \), or \( \phi = 1 + 1 + 1 \). Then \( \Pi_{\phi} = \{ \pi \} \) is singleton. Hence we have nothing to prove.

(3) \( \phi = \chi + 1 + \chi^{-1} \) with \( \chi^2 = 1 \) and \( \chi \neq 1 \). Then \( \Pi_{\phi} = \{ \pi, \pi' \} \). Note that \( s \in C^+_\phi \) which generates \( A^+_{\phi} \) satisfies that \( \eta_c(s) = \chi(c) \) for \( c \in F^\times \). Hence, by Proposition [3.1] we see that both \( \pi \) and \( \pi' \) are generic. The proof is similar to the case when \( \phi = \rho + \chi \).

(4) \( \phi = \chi_1 + \chi_2 + \chi_3 \), where \( \chi_1, \chi_2 \) and \( \chi_3 \) are distinct non-trivial quadratic characters. Let \( s_i \) be the element of \( C^+_{\phi} \) such that

\[
\eta_i|\chi_i = \chi_i, \quad s_i|\chi_j = -\chi_j \quad \text{for} \quad j \neq i
\]

for \( i = 1, 2, 3 \). Then we have

\[
A^+_{\phi} = \{1, s_1, s_2, s_3\} \cong (\mathbb{Z}/2\mathbb{Z})^2.
\]

In particular, we have \( \# \Pi_{\phi} = 4 \). For \( c \in F^\times \), we have

\[
\eta_c(s_j) = \chi_j(c)
\]

since \( \chi_1\chi_2\chi_3 = \det(\phi) = 1 \). We may take \( c_i \in F^\times \) such that

\[
\chi_i(c_i) = 1 \quad \text{and} \quad \chi_j(c_i) = -1 \quad \text{for} \quad j \neq i.
\]

Then we have

\[
\eta_c(s_j) = \chi_j(c_i) = \begin{cases} 1 & \text{if} \ j = i, \\ -1 & \text{if} \ j \neq i. \end{cases}
\]

Hence we see that the characters \( \eta_c \) are non-trivial and distinct, and so that

\[
\hat{A}^+_{\phi} = \{1, \eta_{c_1}, \eta_{c_2}, \eta_{c_3}\}.
\]

This implies that all \( \pi \in \Pi_{\phi} \) are generic. The proof is similar to the case when \( \phi = \rho + \chi \). This completes the proof.

**Proposition 7.5.** Assume that \( \text{SO}(V) \) is quasi-split. Then Conjecture (P1) is true for \( \phi \in \hat{\Phi}_{\text{temp}}(\text{Sp}(W)) \) of the form \( \phi = \rho + \chi \): \( WD_F \rightarrow \text{SO}(N) \), where \( \rho \) is an irreducible orthogonal representation of \( WD_F \) with dimensional \( 2n \) and \( \chi \) is a quadratic character of \( F^\times \).

**Proof.** If \( \det(\rho) \neq 1 \), the proof is similar to that of the above proposition. Hence we may assume that \( \det(\rho) = \chi = 1 \).

Let \( s \) be the element of \( A_{\phi} \) corresponding to \( \rho \). We see that \( A^+_{\phi} \) is generated by \( s \), and \( \Pi_{\phi} = \{ \pi, \pi' \} \). In this case, \( \eta_c(s) \) is the trivial character for any \( c \in F^\times \). In particular, if \( \pi \) is generic, then \( \pi' \) is not generic.

If \( \chi_V = 1 \), then \( A^+_{\phi} \) is bijective. We have \( \Pi_{\phi'} = \{[\sigma], [\sigma']\} \), where \( \sigma \) is generic and \( \sigma' \) is not so. Then we have \( \Theta_{\psi, V^\bullet, W}(\pi) = \sigma \) by Corollary [7.3] Since the map \( \Theta_{\psi, V^\bullet, W}: \Pi_{\phi} \rightarrow \Pi_{\phi'} \) is bijective, we have \( \Theta_{\psi, V^\bullet, W}(\pi') = \sigma' \). This is the desired correspondence.

If \( \chi_V \neq 1 \), then \( \# \Pi_{\phi'} = 4 \) and the map \( \Theta_{\psi, V^\bullet, W}: \Pi_{\phi} \rightarrow \Pi_{\phi'} \) is one-to-two. By Proposition [3.2] we see that exactly two of \( \Pi_{\phi'} \) are generic and they correspond to \( \eta \in \hat{A}^+_{\phi} \) such that \( \eta(a) = 1 \). This implies the desired correspondence. \( \square \)

**Proposition 7.6.** Assume that \( \text{SO}(V) \) is quasi-split. Then Conjecture (P1) is true for \( \phi \in \hat{\Phi}_{\text{temp}}(\text{Sp}(W)) \) of the form \( \phi = \phi_1 + 1 + \phi_1' \), where \( \phi_1 \) is an irreducible representation of \( WD_F \) with dimensional \( n \).

**Proof.** If \( \phi_1 \) is not orthogonal, then \( \Pi_{\phi} \) is singleton, so that there is nothing to prove. If \( \phi_1 \) is orthogonal, then the proof is similar to those of the above propositions. \( \square \)

8. Preparation for the proof of Theorem [7.1]

Let \( V = V_{2m} \) and \( W = W_{2n} \). We assume that \( V \) is type \((d, c)\). To prove Theorem [7.1] we need to introduce more notation.
8.1. Haar measures. Let \( k \leq \min\{m - 1, n\} \) be a positive integer. As in [24] we decompose

\[
V = X + V_0 + X^*, \quad W = Y + W_0 + Y^*
\]

with \( X = X_k, X^* = X_k^* \subset V \) and \( Y = Y_k, Y^* = Y_k^* \subset W \). Hence \( \dim(V_0) = 2m_0 := 2(m - k) \) and \( \dim(W_0) = 2n_0 := 2(n - k) \). Let \( P = P_k = M_P U_P \subset O(V) \) and \( Q = Q_k = M_Q U_Q \subset Sp(W) \) be the parabolic subgroups defined in [24]. Hence \( M_P \cong GL(X) \times O(V_0) \) and \( M_Q \cong GL(Y) \otimes Sp(W_0) \). We need to choose Haar measures on various groups. In particular, we shall define Haar measures on

\[
\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_V \otimes \langle \cdot, \cdot \rangle_W \quad \text{on} \quad V \otimes W \quad \text{over} \quad F, \quad \text{and the maps} \quad I_X : X^* \rightarrow X \quad \text{and} \quad I_Y : Y^* \rightarrow Y.
\]

We consider the following spaces and pairings:

- \((x, y) \mapsto \psi((x, I_Y^{-1} y))\) for \(x, y \in V \otimes Y\);
- \((x, y) \mapsto \psi((x, I_Y y))\) for \(x, y \in V \otimes Y^*\);
- \((x, y) \mapsto \psi((x, I_Y y))\) for \(x, y \in V_0 \otimes Y^*\);
- \((x, y) \mapsto \psi((I_X^{-1} x, y))\) for \(x, y \in X \otimes W_0\);
- \((x, y) \mapsto \psi((I_X x, y))\) for \(x, y \in X^* \otimes W_0\);
- \((x, y) \mapsto \psi((I_X x, I_Y y))\) for \(x, y \in X \otimes Y^*\);
- \((x, y) \mapsto \psi((I_X x, I_Y^{-1} y))\) for \(x, y \in X^* \otimes Y\);
- \((x, y) \mapsto \psi((I_X x, I_Y y))\) for \(x, y \in X \otimes Y^*\).

On these spaces, we take the self-dual Haar measures with respect to these pairings. Put

\[
e^{**} = v_1^* \otimes w_1^* + \cdots + v_k^* \otimes w_k^* \in X^* \otimes Y^*.
\]

- We transfer the Haar measure on \( V_0 \otimes Y^* \) to \( \Hom(V_0, X) \) via the isomorphism \( b \mapsto b^* e^{**} \) for \( b \in \Hom(V_0, X) \).
- We transfer the Haar measure on \( X^* \otimes W_0 \) to \( \Hom(W_0, Y) \) via the isomorphism \( b \mapsto b^* e^{**} \) for \( b \in \Hom(W_0, Y) \).

Furthermore:

- We transfer the Haar measure on \( X \otimes Y^* \) to \( \Hom(X^*, X) \) via the isomorphism \( c \mapsto c e^{**} \) for \( c \in \Hom(X^*, X) \). This Haar measure on \( \Hom(X^*, X) \) is self-dual with respect to the pairing \((c_1, c_2) \mapsto \psi((I_X^{-1} c_1 e^{**}, I_Y c_2 e^{**}))\).
- We take the Haar measure

\[
|2|_F^{-\frac{k(k-1)}{4}} dc
\]

on \( \Sym(X^*, X) \), where \( dc \) is the self-dual Haar measure with respect to the pairing \((c_1, c_2) \mapsto \psi((I_X^{-1} c_1 e^{**}, I_Y c_2 e^{**}))\).

- We transfer the Haar measure on \( X^* \otimes Y \) to \( \Hom(Y^*, Y) \) via the isomorphism \( c \mapsto c e^{**} \) for \( c \in \Hom(Y^*, Y) \). This Haar measure on \( \Hom(Y^*, Y) \) is self-dual with respect to the pairing \((c_1, c_2) \mapsto \psi((I_X c_1 e^{**}, I_Y^{-1} c_2 e^{**}))\).
- We take the Haar measure

\[
|2|_F^{-\frac{k(k-1)}{4}} dc
\]

on \( \Sym(Y^*, Y) \), where \( dc \) is the self-dual Haar measure with respect to the pairing \((c_1, c_2) \mapsto \psi((I_X c_1 e^{**}, I_Y^{-1} c_2 e^{**}))\).

Then:

- We take the Haar measure \( du = db dc \) on \( U_P \) for \( u = u_P(b) u_P(c) = u_P(c) u_P(b) \) with \( b \in \Hom(V_0, X) \) and \( c \in \Sym(X^*, X) \).
- Similarly, we define the Haar measure on \( U_Q \).

We note the following Fourier inversion formula:

**Lemma 8.1.** For \( \varphi \in \mathcal{S}(X \otimes Y^*) \), we have

\[
\int_{\Sym(Y^*, Y)} \left( \int_{\Hom(X^*, X)} \varphi(x e^{**}) \psi((x e^{**}, c' e^{**})) dx \right) dc' = \int_{\Sym(X^*, X)} \varphi(c e^{**}) dc.
\]
Proof. For $a \in \text{Hom}(X^*, X)$, we put

$$a_\pm = \frac{a \pm a^*}{2} \in \text{Hom}(X^*, X).$$

Then we have

$$\langle a_\pm x', x'' \rangle_V = \pm \langle x', a_\pm x'' \rangle_V$$

for $x', x'' \in X^*$. This implies that $a_- \in \text{Sym}(X^*, X)$. Moreover, the map

$$e^{a^*} \mapsto I_X^{-1} Y a_+ e^{a^*}$$

gives an element $a'$ in $\text{Hom}(Y^*, Y)$. We claim that $a' \in \text{Sym}(Y^*, Y)$. Indeed, for $y_1', y_2' \in Y^*$, we can write

$$y_i' = \langle e^{a^*}, x_i \rangle_V$$

for some $x_i \in X = \text{Hom}(X^*, F)$ and $i = 1, 2$. Then we have

$$\langle a' y_1', y_2' \rangle_W = \langle I_X^{-1} Y a_+ e^{a^*}, x_1 \otimes y_2' \rangle = \langle I_Y e^{a^*}, a_+ I_X^{-1} x_1 \otimes y_2' \rangle$$

$$= \sum_{j=1}^{k} \langle v_j', a_+ I_X^{-1} x_1 \rangle_V \langle w_j, y_2' \rangle_W$$

$$= \sum_{j=1}^{k} \langle v_j', a_+ I_X^{-1} x_1 \rangle_V \langle x_2 \otimes w_j, e^{a^*} \rangle$$

$$= \sum_{j=1}^{k} \langle v_j', a_+ I_X^{-1} x_1 \rangle_V \langle x_2, v_j' \rangle_V = \langle I_X^{-1} x_2, a_+ I_X^{-1} x_1 \rangle_V.$$

Therefore we have

$$\langle y_1', a' y_2' \rangle_W = -\langle I_X^{-1} x_1, a_+ I_X^{-1} x_2 \rangle_V = -\langle I_X^{-1} x_2, a_+ I_X^{-1} x_1 \rangle_V = -\langle a' y_1', y_2' \rangle_W,$$

so that $a' \in \text{Sym}(Y^*, Y)$ as desired. This implies that

$$\text{Hom}(X^*, X) e^{a^*} = \text{Sym}(X^*, X) e^{a^*} + I_X I_Y^{-1} \text{Sym}(Y^*, Y) e^{a^*}.$$

Comparing dimensions, we see that this decomposition is a direct sum. Note that the Haar measures we have fixed are compatible with this decomposition.

Now we consider the non-degenerate symmetric bilinear form $(x, y) \mapsto \langle I_X^{-1} x, I_Y y \rangle$ on $X \otimes Y^* = \text{Hom}(X^*, X) e^{a^*}$ over $F$. For $x \in \text{Sym}(X^*, X)$ and $y \in \text{Sym}(Y^*, Y)$, we have $\langle I_X^{-1} (x e^{a^*}), I_Y (I_X I_Y^{-1} ye^{a^*}) \rangle = \langle xe^{a^*}, ye^{a^*} \rangle$ and

$$\langle xe^{a^*}, ye^{a^*} \rangle = \langle y^* e^{a^*}, x^* e^{a^*} \rangle = \langle ye^{a^*}, xe^{a^*} \rangle = -\langle xe^{a^*}, ye^{a^*} \rangle,$$

since $x^* = -x$ and $y^* = -y$. Hence $(I_X^{-1} (xe^{a^*}), I_Y (I_X I_Y^{-1} ye^{a^*})) = 0$. This implies that the decomposition

$$\text{Hom}(X^*, X) e^{a^*} = \text{Sym}(X^*, X) e^{a^*} \oplus I_X I_Y^{-1} \text{Sym}(Y^*, Y) e^{a^*}$$

is an orthogonal direct sum. This yields the desired Fourier inversion formula. 

\[\Box\]

8.2. Weil representations. We recall some explicit formulas for the Weil representations. Recall that $V$ and $W$ have decompositions

$$V = X + V_0 + X^*, \quad W = Y + W_0 + Y^*$$

with $2m = \text{dim}(V)$, $2n = \text{dim}(W)$, $2m_0 = \text{dim}(V_0)$, $2n_0 = \text{dim}(W_0)$ and $k = \text{dim}(X) = \text{dim}(Y)$. Assume that $m_0 > 0$ and $n_0 > 0$. We denote $W_0 = Y_0 + Y_0^*$, where $Y_0 = W_0 \cap Y_0$, and $Y_0^* = W_0 \cap Y_0^*$. We have fixed a non-trivial additive character $\psi$ of $F$.

Let $H(W) = W \oplus F$ be the associated Heisenberg group, i.e., the multiplication law is given by

$$\langle w, t \rangle \cdot \langle w', t' \rangle = \left( w + w', t + t' + \frac{1}{2} \langle w, w' \rangle_W \right)$$
for \(w, w' \in W\) and \(t, t' \in F\). Let \(\rho\) be the Heisenberg representation of \(\mathcal{H}(W)\) on \(\mathcal{S}(Y^*_n)\) with the central character \(\psi\). Namely,
\[
\rho(y + y', t)\varphi(y'_1) = \psi(t + (y'_1, y)w + \frac{1}{2}(y', y)w)\varphi(y'_1 + y')
\]
for \(\varphi \in \mathcal{S}(Y^*_n), y \in Y_n, y', y'_1 \in Y^*_n\) and \(t \in F\).

For simplicity, we write:
- \(\omega\) for the Weil representation \(\omega_{\psi,V,W}\) of \(\text{Sp}(W) \times \text{O}(V)\) on a space \(S\);
- \(\omega_0\) for the Weil representation \(\omega_{\psi,V,W_0}\) of \(\text{Sp}(W_0) \times \text{O}(V)\) on a space \(S_0\);
- \(\omega_{00}\) for the Weil representation \(\omega_{\psi,V_0,W_0}\) of \(\text{Sp}(W_0) \times \text{O}(V_0)\) on a space \(S_{00}\).

We take the Schrödinger model
\[
S_{00} = \mathcal{S}(V_0 \otimes Y^*_0)
\]
of \(\omega_{00}\). We take a mixed model
\[
S_0 = S_{00} \otimes \mathcal{S}(X^* \otimes W_0) = \mathcal{S}(V_0 \otimes Y^*_0) \otimes \mathcal{S}(X^* \otimes W_0)
\]
of \(\omega_0\), where we regard \(S_0\) as a space of functions on \(X^* \otimes W_0\) with values in \(S_{00}\). Similarly, we take a mixed model
\[
S = \mathcal{S}(V \otimes Y^*) \otimes S_0 = \mathcal{S}(V \otimes Y^*) \otimes \mathcal{S}(V_0 \otimes Y^*_0) \otimes \mathcal{S}(X^* \otimes W_0)
\]
of \(\omega\), where we regard \(S\) as a space of functions on \(V \otimes Y^*\) with values in \(S_0\). Also, we write:
- \(\rho_0\) for the Heisenberg representation of \(\mathcal{H}(V \otimes W_0)\) on \(S_0\) with the central character \(\psi\);
- \(\rho_{00}\) for the Heisenberg representation of \(\mathcal{H}(V_0 \otimes W_0)\) on \(S_{00}\) with the central character \(\psi\).

**Lemma 8.2.** Let \(Q_0\) be the Siegel parabolic subgroup of \(\text{Sp}(W_0)\) stabilizing \(Y_0\), and define \(m_0(a') = m_{Q_0}(a') \in Q_0\) and \(u_0(c') = u_{Q_0}(c')\) for \(a' \in \text{GL}(Y_0)\) and \(c' \in \text{Sym}(Y^*_0, Y_0)\) as in [2.4]. We put \(I_0 = I_{Q_0} \in \text{Hom}(Y^*_0, Y_0)\) and \(w_0 = w_{Q_0} \in \text{Sp}(W_0)\) as in [2.4]. Then for \(\varphi \in S_0 = \mathcal{S}(V_0 \otimes Y^*_0)\) and \(x \in V_0 \otimes Y^*_0\), we have
\[
[\omega_0(1, h_0)\varphi](x) = \varphi(h_0^{-1}x), \quad h_0 \in \text{O}(V_0),
\]
\[
[\omega_0(m_0(a'), 1)\varphi](x) = \chi_{Y_0}(\det(a'))|\det(a')|^{m_0} \varphi(a'^*x), \quad a' \in \text{GL}(Y_0),
\]
\[
[\omega_0(u_0(c'), 1)\varphi](x) = \psi(\frac{1}{2}(c'x^t, x))\varphi(x), \quad c' \in \text{Sym}(Y^*_0, Y_0),
\]
\[
[\omega_0(0, 1)\varphi](x) = \gamma_{Y_0}^{-n_0} \int_{Y_0 \otimes Y_0} \varphi(I_0^{-1}z)\psi(-\langle z, x \rangle)dz.
\]

Here, \(dz\) is the self-dual measure on \(V_0 \otimes Y_0\) with respect to the pairing \(\langle x, y \rangle \mapsto \psi(\langle y, I_0^{-1}x \rangle)\), and \(\gamma_{Y_0}\) is a 4-th root of unity satisfying \(\gamma_{Y_0}^2 = \chi_{Y_0}(-1)\).

**Proof.** This formula is the Schrödinger model. \(\square\)

**Lemma 8.3.** The mixed model \(S' = \mathcal{S}(V \otimes Y^*) \otimes \mathcal{S}(V \otimes Y^*_0)\) of \(\omega\) is given as follows: For \(\varphi = \varphi_1 \otimes \varphi' \in \mathcal{S}(V \otimes Y^*) \otimes \mathcal{S}(V \otimes Y^*_0)\) and \((x_1, x') \in (V \otimes Y^*) \times (V \otimes Y^*_0)\), we have
\[
[\omega(g_0, h)\varphi](x_1, x') = \varphi_1(h^{-1}x_1) \cdot [\omega_0(g_0, h)\varphi'](x'), \quad (g_0, h) \in \text{Sp}(W_0) \times \text{O}(V),
\]
\[
[\omega(m(a'), 1)\varphi](x_1, x') = \chi_V(\det(a'))|\det(a')|^{m} \varphi_1(a'^*x_1) \cdot \varphi'(x'), \quad a' \in \text{GL}(Y),
\]
\[
[\omega(u_0(b'_0), 1)\varphi](x_1, x') = \varphi_1(x_1) \cdot [\rho_0(b'_0x_1, 0)\varphi'](x'), \quad b' \in \text{Hom}(W_0, Y),
\]
\[
[\omega(u_0(c'), 1)\varphi](x_1, x') = \psi(\frac{1}{2}(c'x_1, x_1))\varphi_1(x_1) \cdot \varphi'(x'), \quad c' \in \text{Sym}(Y^*, Y),
\]
\[
[\omega(0, 1)\varphi](x_1, x') = \gamma_Y^{-k} \int_{V_0 \otimes Y_0} \varphi(I_0^{-1}z)\psi(-\langle z, x_1 \rangle)dz \cdot \varphi'(x').
\]

**Proof.** This formula is obtained by the canonical isomorphism
\[
\mathcal{S}(V \otimes Y^*_n) \to \mathcal{S}(V \otimes Y^*) \otimes \mathcal{S}(V \otimes Y^*_0)
\]
given by the decomposition \(Y^*_n = Y^* \otimes Y^*_0\). \(\square\)
Lemma 8.4. For \( \varphi' = \varphi_2 \otimes \varphi_3 \in \mathcal{S}_0 = \mathcal{S}(V_0 \otimes Y_0^*) \otimes \mathcal{S}(X^* \otimes W_0) \), we have

\[
[w_0(g_0, h_0) \varphi'](x_2, x_3) = [w_0(0, g_0) \varphi_2](x_2) \cdot \varphi_3(g_0^{-1} x_3),
\]

\( (g_0, h_0) \in \text{Sp}(W_0) \times \text{O}(V_0) \),

\[
[w_0(1, m_P(a)) \varphi'](x_2, x_3) = |\det(a)|_{P_0}^{\varphi_2(x_2)} \cdot \varphi_3(a \cdot x_3),
\]

\( a \in \text{GL}(X) \),

\[
[w_0(1, u_P(b)) \varphi'](x_2, x_3) = [\rho_0(b^* x_3, 0) \varphi_2](x_2) \cdot \varphi_3(x_3),
\]

\( b \in \text{Hom}(V_0, X) \),

\[
[w_0(1, w_P(c)) \varphi'](x_2, x_3) = \psi(\frac{1}{2} \langle cx_3, x_3 \rangle) \varphi_2(x_2) \cdot \varphi_3(x_3),
\]

\( c \in \text{Sym}(X^*, X) \),

\[
[w_0(1, w_P) \varphi'](x_2, x_3) = \varphi_2(x_2) \cdot \varphi_3(I_X^{-1} z) \psi(-\langle z, x_3 \rangle) dz.
\]

Moreover we have

\[
[\rho_0(v + v_0 + v^*, 0) \varphi'](x_2, x_3) = \psi(\langle x_3, v \rangle + \frac{1}{2} \langle v^*, v \rangle) [\rho_0(0, 0) \varphi_2](x_2) \cdot \varphi_3(x_3 + v^*)
\]

for \( v \in X \otimes W_0, v_0 \in V_0 \otimes W_0 \) and \( v^* \in X^* \otimes W_0 \).

Proof. These formulas are given by the partial Fourier transform

\[
\mathcal{S}(V \otimes Y_0^*) \rightarrow \mathcal{S}(V_0 \otimes Y_0^*) \otimes \mathcal{S}(X^* \otimes W_0), \quad \varphi \mapsto \tilde{\varphi}
\]

defined by

\[
\tilde{\varphi}(x, y) = \int_{X \otimes Y_0^*} \varphi \left( \begin{array}{c} z \\ x \\ y_2 \end{array} \right) \psi(-\langle z, y_1 \rangle) dz
\]

for \( x \in V_0 \otimes Y_0^* \) and \( y = y_1 + y_2 \) with \( y_1 \in X^* \otimes Y_0 \) and \( y_2 \in X^* \otimes Y_0^* \). Here, \( dz \) is the self-dual measure on \( X \otimes Y_0^* \) with respect to the pairing \( \langle x, y \rangle \rightarrow \psi((I_X^{-1} x, I_0 y)) \).

In particular, the map \( \mathcal{S}_0 \ni \varphi \mapsto \varphi(\cdot, 0) \in \mathcal{S}_{\omega_0} \) gives a surjective \( \text{Sp}(W_0) \times \text{O}(V_0) \times \text{GL}(X) \)-homomorphism

\( \omega_0 \rightarrow \omega_0 \otimes |\cdot|_{P_0}^{\varphi} \).

This fact was used in [24].

By the above lemmas, we get a formula for the mixed model \( \mathcal{S} = \mathcal{S}(V \otimes Y^*) \otimes \mathcal{S}(V_0 \otimes Y_0^*) \otimes \mathcal{S}(X^* \otimes W_0) \) of \( \omega \).

8.3. Normalized intertwining operators. In this subsection, we define normalized intertwining operators which are used to describe the local Langlands correspondence.

In [24], for \( k \leq \min \{ m - 1, n \} \), we have defined parabolic subgroups \( P = P_k = M_P U_P \) of \( \text{O}(V) \) and \( Q = Q_k = M_Q U_Q \) of \( \text{Sp}(W) \) such that

\[
M_P \cong \text{GL}(X) \times \text{O}(V_0), \quad M_Q \cong \text{GL}(Y) \times \text{Sp}(W_0).
\]

We put \( P^\circ = P \cap \text{SO}(V) \) and \( M_P^\circ = M_P \cap \text{SO}(V) \cong \text{GL}(X) \times \text{SO}(V_0) \). Assume that \( k \) is even and \( \dim(V_0) = 2m_0 \geq 4 \). We identify \( \text{GL}(X) \) (resp. \( \text{GL}(Y) \)) with \( \text{GL}(F) \) using the basis \( \{ v_1, \ldots, v_k \} \) (resp. \( \{ w_1, \ldots, w_k \} \)). Hence we can define an isomorphism \( i : \text{GL}(Y) \rightarrow \text{GL}(X) \) via these identifications.

Let \( \tau \) be an irreducible tempered representation of \( \text{GL}_k(F) \) on a space \( \mathcal{V}_\tau \) with a central character \( \omega_\tau \). We may regard \( \tau \) as a representation of \( \text{GL}(X) \) or \( \text{GL}(Y) \) via the above identifications. For any \( s \in \mathbb{C} \), we realize the representation \( \tau_s := \tau \otimes |s|^{\varphi} \) on \( \mathcal{V}_\tau \) by setting \( \tau_s(a)v := |\det(a)|_F^{\varphi(a)} \tau(a)v \) for \( v \in \mathcal{V}_\tau \) and \( a \in \text{GL}_k(F) \). Let \( \sigma_0 \) (resp. \( \sigma_0 \)) be an irreducible tempered representation of \( \text{SO}(V_0) \) (resp. \( \text{Sp}(W_0) \)) on a space \( \mathcal{V}_{\sigma_0} \) (resp. \( \mathcal{V}_{\sigma_0} \)). Assume that \( \sigma_0 \) is \( \varepsilon \)-invariant, i.e., there exists \( \tilde{\sigma}_0 \in \text{Irr}(\text{O}(V_0)) \) such that \( \tilde{\sigma}_0|_{\text{SO}(V_0)} = \sigma_0 \). We may assume that \( \tilde{\sigma}_0 \) is realized on \( \mathcal{V}_{\sigma_0} \). We consider the induced representation

\[
\text{Ind}_{P \cap \text{SO}(V)}^{\text{O}(V)}(\tau_s \otimes \tilde{\sigma}_0) \quad (\text{resp. Ind}_{Q \cap \text{Sp}(W)}^{\text{Sp}(W)}(\tau_s \otimes \tilde{\sigma}_0))
\]

of \( \text{O}(V) \) (resp. \( \text{Sp}(W) \)), which is realized on the space of smooth functions \( \Phi_\tau : \text{O}(V) \rightarrow \mathcal{V}_\tau \otimes \mathcal{V}_{\sigma_0} \) (resp. \( \Phi_\tau : \text{Sp}(W) \\rightarrow \mathcal{V}_\tau \otimes \mathcal{V}_{\sigma_0} \)) such that

\[
\Phi_\tau(u_p m_P(a) h_0 h) = |\det(a)|_F^{\varphi(a)} \tau_0(h_0) \Phi_\tau(h)
\]

(resp. \( \Phi_\tau(u_Q m_Q(a^*) g) = |\det(a^*)|_F^{\varphi(a^*)} \tau_0(g_0) \Phi_\tau(g) \).
for any $u_P \in U_P$, $a \in \text{GL}(X)$, $h_0 \in \text{O}(V_0)$ and $h \in \text{O}(V)$ (resp. $u_Q \in U_Q$, $a' \in \text{GL}(Y)$, $g_0 \in \text{Sp}(W_0)$ and $g \in \text{Sp}(W)$). By Lemma 2.3, we have

$$\text{Ind}^{\text{Q}(V)}_{\text{P}}(\tau_\alpha \otimes \sigma_0)\text{SO}(V) \cong \text{Ind}^{\text{SO}(V)}_{\text{P}_0^+(\text{g})}(\tau_\alpha \otimes \sigma_0).$$

Let $A_P$ (resp. $A_Q$) be the split component of the center of $M^\ast_P$ (resp. $M^\ast_Q$) and $W(M^\ast_P) = \text{Norm}(A_P, \text{SO}(V))/M^\ast_P$ (resp. $W(M^\ast_Q) = \text{Norm}(A_Q, \text{Sp}(W))/M^\ast_Q$) be the relative Weyl group for $M^\ast_P$ (resp. $M^\ast_Q$). Note that $W(M^\ast_P) \cong W(M^\ast_Q) \cong \mathbb{Z}/2\mathbb{Z}$. We denote by $w$ (resp. $w'$) the non-trivial element in $W(M^\ast_P)$ (resp. $W(M^\ast_Q)$).

The definition of the normalized intertwining operators is very subtle because one has to choose the following data appropriately:

- a representative $\bar{w}$ of $w$ (resp. $\bar{w}'$ of $w'$);
- a Haar measure on $U_P$ (resp. $U_Q$) to define the unnormalized intertwining operator;
- a normalizing factor $\gamma(w, \tau_\alpha \otimes \sigma_0)$ (resp. $\gamma(w', \tau_\alpha \otimes \sigma_0)$);
- an intertwining isomorphism $\mathcal{A}_w$ (resp. $\mathcal{A}_{w'}$).

To do these, we need an $F$-splitting of $\text{SO}(V)$ (resp. $\text{Sp}(W)$).

Let $(B, T)$ (resp. $(B', T')$) be the pair of the Borel subgroup and the maximal torus of $\text{SO}(V)$ (resp. $\text{Sp}(W)$), which are defined in 2.2 (resp. 2.3). They are $F$-rational. The Lie algebra of $\text{SO}(V)$ (resp. $\text{Sp}(W)$) is given by

$$\text{Lie}(\text{SO}(V)) = \{x \in \text{End}(V) \mid \langle xv, v' \rangle_V + \langle v, xv' \rangle_V = 0, \text{ for any } v, v' \in V\}$$

(resp. $\text{Lie}(\text{Sp}(W)) = \{y \in \text{End}(W) \mid \langle yw, w' \rangle_W + \langle w, yw' \rangle_W = 0, \text{ for any } w, w' \in W\}$).

Let $v_i, v_i^\ast, e$ and $e'$ for $1 \leq i \leq m - 1$ (resp. $w_j, w_j^\ast$ for $1 \leq j \leq n$) be the elements in $V$ (resp. in $W$) as in 2.2. Hence

$$\langle e, e \rangle_V = 2c, \quad \langle e', e' \rangle_V = -2cd, \quad \langle e, e' \rangle_V = 0.$$

For $t \in T$ and $t' \in T'$, we define $t_i, t'_j \in F^\times$ and $a(t), b(t) \in F$ by

$$t_i = (tv_i, v_i^\ast)_V, \quad t'_j = (t'w_j, w_j^\ast)_W$$

and

$$\left\{\begin{array}{l}
t e = a(t) \cdot e + b(t) \cdot e', \\
t e' = b(t) d \cdot e + a(t) \cdot e'.
\end{array}\right.$$

Note that $a(t)^2 - b(t)^2 d = 1$. Then the simple roots of $T$ in $B \subset \text{SO}(V)$ (resp. $T'$ in $B' \subset \text{Sp}(W)$) are given by

$$\alpha_i(t) = t_i/t_{i+1}, \quad \alpha_{\pm}(t) = t_{m-1}(a(t) \pm b(t)\sqrt{d})$$

(resp. $\alpha'_j(t') = t'_j/t'_{j+1}, \quad \alpha_n(t') = t'_n^2$)

for $1 \leq i \leq m - 2$ (resp. $1 \leq j \leq n - 1$). We choose root vectors $x_{\alpha_i} \in \text{Lie}(\text{SO}(V))$ satisfying

$$x_{\alpha_i} v_i = \begin{cases} v_i & \text{if } i' = i + 1, \\
0 & \text{otherwise}, \end{cases} \quad x_{\alpha_i} v_i^\ast = \begin{cases} -v_{i+1}^\ast & \text{if } i' = i, \\
0 & \text{otherwise}, \end{cases} \quad x_{\alpha_i} e = x_{\alpha_i} e' = 0$$

for $1 \leq i \leq m - 2$ and

$$x_{\alpha_{\pm}} v_i = 0, \quad x_{\alpha_{\pm}} v_i^\ast = \begin{cases} -e \pm \sqrt{d}^{-1}e' & \text{if } i' = m - 1, \\
0 & \text{otherwise}, \end{cases}$$

$$x_{\alpha_{\pm}} e = v_{m-1}, \quad x_{\alpha_{\pm}} e' = \mp \sqrt{d}v_{m-1}.$$  

Also, we choose root vectors $y_{\alpha_j} \in \text{Lie}(\text{Sp}(W))$ satisfying

$$y_{\alpha_j} w_j = \begin{cases} w_j & \text{if } j' = j + 1, \\
0 & \text{otherwise}, \end{cases} \quad y_{\alpha_j} w_j^\ast = \begin{cases} -w_{j+1}^\ast & \text{if } j' = j, \\
0 & \text{otherwise}, \end{cases}$$

for $1 \leq j \leq n - 1$ and

$$y_{\alpha_n} w_n = 0, \quad y_{\alpha_n} w_n^\ast = \begin{cases} w_n & \text{if } j' = n, \\
0 & \text{otherwise}. \end{cases}$$
We define a splitting \( \mathfrak{spl}_{SO(V)} \) of \( SO(V) \) (resp. \( \mathfrak{spl}_{Sp(W)} \) of \( Sp(W) \)) by
\[
\mathfrak{spl}_{SO(V)} = (B, T, \{x_{\alpha\cdot}\}_{\alpha\cdot}) \quad (\text{resp. } \mathfrak{spl}_{Sp(W)} = (B', T', \{y_{\alpha\cdot}\}_{\alpha\cdot})).
\]
These are \( F \)-splittings.

Let \( \alpha^\vee \) be the simple coroot corresponding to \( \alpha \). This is a homomorphism \( \alpha^\vee : GL(1) \to T \) of algebraic groups, and gives a map
\[
d_{\alpha^\vee} : \text{Lie}(GL(1)) = \mathbb{G}_a \to \text{Lie}(T)
\]
of Lie algebras. We put \( H_{\alpha\cdot} = d_{\alpha^\vee}(1) \in \text{Lie}(T) \). We take a root vector \( x_{-\alpha\cdot} \) of the negative root \( -\alpha \cdot \) such that
\[
[x_{\alpha\cdot}, x_{-\alpha\cdot}] = H_{\alpha\cdot}.
\]
We denote by \( W(T, SO(V)) \) the Weyl group of \( T \) in \( SO(V) \). Let \( w_{\alpha\cdot} \in W(T, SO(V)) \) be the reflection with respect to \( \alpha \). Following the procedure of [35 §2.1], we take the representative \( \tilde{w}_{\alpha\cdot} \in SO(V) \) of \( w_{\alpha\cdot} \), defined by
\[
\tilde{w}_{\alpha\cdot} = \exp(x_{\alpha\cdot}) \exp(-x_{-\alpha\cdot}) \exp(x_{\alpha\cdot}).
\]
We put \( w_{\alpha\cdot} := w_{\alpha\cdot} \cdot w_{-\alpha\cdot} = w_{\alpha\cdot} \cdot \tilde{w}_{\alpha\cdot} \) and \( \tilde{w}_{\alpha\cdot} := \tilde{w}_{\alpha\cdot} \cdot \tilde{w}_{\alpha\cdot} \cdot \tilde{w}_{\alpha\cdot} \). Similarly, we take the representative \( \tilde{w}'_{\alpha\cdot} \) of the reflection \( w'_\alpha \in W(T', Sp(W)) \) with respect to \( \alpha \). Moreover, we define \( w_{p,q} \) for \( 1 \leq p, q \leq n \) by
\[
w_{p,q} = w_{\alpha_{p+1}} \cdots w_{\alpha_{n-1}} w_{\alpha_n} w_{\alpha_{n-1}} \cdots w_{\alpha_{p+1}} w_{\alpha_p}.
\]
Similarly, we define \( w'_{p,q}, \tilde{w}_{p,q} \) and \( \tilde{w}'_{p,q} \).

Let \( w_T \in W(T, SO(V)) \) (resp. \( w'_T \in W(T', Sp(W)) \)) be the representative of \( w \in W(M'_{\mathbb{P}}) \) (resp. \( w' \in W(M'_Q) \)) which stabilizes the simple roots of \( (B \cap M'_{\mathbb{P}}, T) \) (resp. \( (B' \cap M'_Q, T') \)). It is easily seen that
\[
w_T = w_{k,k} \cdot w_{k-1,k} \cdots w_{1,k} \quad (\text{resp. } w'_T = w'_{k,k} \cdot w'_{k-1,k} \cdots w'_{1,k})
\]
and this is a reduced decomposition of \( w_T \) (resp. \( w'_T \)). We take a representative \( \tilde{w} \in SO(V) \) of \( w \) (resp. \( \tilde{w}' \in Sp(W) \) of \( w' \)) defined by
\[
\tilde{w} = \tilde{w}_{k,k} \cdot \tilde{w}_{k-1,k} \cdots \tilde{w}_{1,k} \quad (\text{resp. } \tilde{w}' = \tilde{w}'_{k,k} \cdot \tilde{w}'_{k-1,k} \cdots \tilde{w}'_{1,k}).
\]
Then we see that
\[
\tilde{w} = \tilde{w} \cdot m_p(-c \cdot a) \quad (\text{resp. } \tilde{w}' = \tilde{w}' \cdot m'_Q(a)),
\]
where \( a \in GL_k(F) \cong GL(X) \cong GL(Y) \) is given by
\[
a = \begin{pmatrix}
-(-1)^{n-k+1} \\
-1^n \\
\end{pmatrix}
\]
We have defined an \( F \)-splitting \( \mathfrak{spl}_{SO(V)} \) of \( SO(V) \) (resp. \( \mathfrak{spl}_{Sp(W)} \) of \( Sp(W) \)). This splitting determines a Chevalley basis of \( \text{Lie}(SO(V)) \) (resp. \( \text{Lie}(Sp(W)) \)), and hence an invariant \( F \)-valued differential form of highest degree on \( U_P \) (resp. \( U_Q \)). Its absolute value, together with the self-dual Haar measure on \( F \) with respect to \( \psi \), gives a Haar measure \( du'_P \) on \( U_P \) (resp. \( du'_Q \) on \( U_Q \)). These are the measures that we take in the definition of the unnormalized intertwining operators.

**Lemma 8.5.** These measures satisfy
\[
du'_P = |c_F|^k du_P, \quad du'_Q = du_Q,
\]
where \( du_P \) and \( du_Q \) are the measures defined in [8.1].

**Proof.** We only show this lemma for \( du'_P \). We can prove this for \( du'_Q \) in a similar way.

Let \( B = TU \) be the Borel subgroup of \( SO(V) \) defined in [2.2]. For \( t \in T \), we defined \( t_i \in F^\times \) and \( a(t), b(t) \in F \) as above. We put
\[
\alpha_{i,j}(t) = t_i/t_j, \quad \alpha_{i,\pm} = t_i(a(t) \pm b(t)\sqrt{d}), \quad \beta_{i,j} = t_it_j.
\]
Then the set of roots of \( T \) in \( \text{Lie}(U) \) given by
\[
\{\alpha_{i,j}, \beta_{i,j} \mid 1 \leq i < j \leq m-1\} \cup \{\alpha_{i,\pm} \mid 1 \leq i \leq m-1\}.
\]
Note that $\alpha_{i,i+1} = \alpha_i$ and $\alpha_{m-1,\pm} = \alpha_\pm$ are the simple roots, so that we have defined the root vectors $x_{\alpha_{i,i+1}} = x_{\alpha_i}$ and $x_{\alpha_{m-1,\pm}} = x_{\alpha_\pm}$. Moreover we have
\begin{align*}
\alpha_{i,j} &= \alpha_i + \alpha_{i+1,j}, & \alpha_{i,\pm} &= \alpha_i, m-1 + \alpha_\pm, & \beta_{i,j} &= \alpha_{i,\pm} + \alpha_{j,\pm}.
\end{align*}
We define root vectors $x_{\alpha_i}, x_{\beta_i} \in \text{Lie}(\text{SO}(V))$ by
\begin{align*}
x_{\alpha_{i,j}} &= \{x_{\alpha_i}, x_{\alpha_{i+1,j}}\}, & x_{\alpha_{i,\pm}} &= \{x_{\alpha_i}, x_{\alpha_{m-1,\pm}}\}, & x_{\beta_{i,j}} &= \{x_{\alpha_{i,\pm}}, x_{\alpha_{j,\pm}}\}.
\end{align*}
Then the basis
\begin{align*}
\{x_{\alpha_{i,j}} \mid 1 \leq i \leq k < j \leq m-1\} \cup \{x_{\alpha_{i,\pm}} \mid 1 \leq i \leq k\} \cup \{x_{\beta_{i,j}} \mid 1 \leq i < j \leq m-1\}
\end{align*}
of $\text{Lie}(U_P)$ is a part of a Chevalley basis of $\text{Lie}(\text{SO}(V))$. Let $\{dx_{\alpha_i}, dx_{\beta_i}\}$ be the dual basis of the linear dual of $\text{Lie}(U_P)$, and put
\begin{align*}
\omega &= (\bigwedge dx_{\alpha_i}) \wedge (\bigwedge dx_{\beta_i}).
\end{align*}
This is an $F$-valued differential form of highest degree on $U_P$ (defined up to a multiplication of $\pm 1$).

On the other hand, we can identify $U_P$ with $F^l$ as a topological spaces by the map
\begin{align*}
u \rightarrow (\langle uw_j, u_i \rangle V)_{1 \leq i < j \leq m-1}, \quad (\langle uw_j, u_i \rangle V)_{1 \leq i < j \leq m-1}, \quad (\langle (uw_j, u_i \rangle V)_{1 \leq i < j \leq m-1},
\end{align*}
where
\begin{align*}
l = \dim(U_P) = 2k - \frac{3}{2}k - \frac{1}{2}k.
\end{align*}
This gives an $F$-valued differential form $\omega_0$ of highest degree on $U_P$. Then we have
\begin{align*}
\omega &= \pm (2\sqrt{d})^{-k}e^{\sum_{i=1}^{k}(m-1-i)} \omega_0.
\end{align*}

Let $dx$ be the self-dual Haar measure on $F$ with respect to $\psi$. We put $U_P(\mathfrak{o}_F)$ to be the subset of $U_P$ corresponding to $\mathfrak{o}_F$ via the above identification $U_P \cong F^l$. Then the measure $du_P$ on $U_P$ is defined by
\begin{align*}
\text{vol}(U_P(\mathfrak{o}_F), du_P) &= |(2\sqrt{d})^{-k}e^{\sum_{i=1}^{k}(m-1-i)} \text{vol}(\mathfrak{o}_F, dx)|^l.
\end{align*}
On the other hand, the measure $du_P$ defined in [B, 3] satisfies
\begin{align*}
\text{vol}(U_P(\mathfrak{o}_F), du_P) &= |(2c\sqrt{d})^{-k}e^{\sum_{i=1}^{k}(m-1-i)} \text{vol}(\mathfrak{o}_F, dx)|^l.
\end{align*}
Since
\begin{align*}
k + \sum_{i=1}^{k} (m-1-i) &= k \left(m - \frac{k + 1}{2}\right) = kp_P,
\end{align*}
we have
\begin{align*}
du_P' = |c|^{kp} \langle c \rangle^{\sum_{i=1}^{k}(m-1-i)} du_P = |c|^{kp} du_P,
\end{align*}
as desired. \hfill \square

We define unnormalized intertwining operators
\begin{align*}
\mathcal{M}(\bar{\omega}, \tau_s \otimes \bar{\sigma}_0) &\colon \text{Ind}_{P_L}^{O(V)}(\tau_s \otimes \bar{\sigma}_0) \rightarrow \text{Ind}_{P_L}^{O(V)}(\bar{w}(\tau_s \otimes \bar{\sigma}_0)), \\
\mathcal{M}(\bar{\omega}', \tau_s \otimes \pi_0) &\colon \text{Ind}_{P_L}^{\text{Sp}(W)}(\tau_s \otimes \pi_0) \rightarrow \text{Ind}_{P_L}^{\text{Sp}(W)}(\bar{w}'(\tau_s \otimes \pi_0))
\end{align*}
by (the meromorphic continuations of) the integrals
\begin{align*}
\mathcal{M}(\bar{w}, \tau_s \otimes \bar{\sigma}_0) \Phi_s(h) &= |c|^{kp} \int_{U_P} \Phi_s(\bar{w}^{-1}u_P h) du_P, \\
\mathcal{M}(\bar{\omega}', \tau_s \otimes \pi_0) \Phi_s'(g) &= \int_{U_Q} \Phi_s'(\bar{w}'^{-1}u_Q g) du_Q,
\end{align*}
where $w(\tau_s \otimes \bar{\sigma}_0)$ (resp. $w'(\tau_s \otimes \pi_0)$) is the representation of $M_P$ on $V_\tau \otimes V_{\bar{\sigma}_0}$ (resp. $M_Q$ on $V_\tau \otimes V_{\pi_0}$) given by $w(\tau_s \otimes \bar{\sigma}_0)(m_P) = (\tau_s \otimes \bar{\sigma}_0)(\bar{w}^{-1}m_P \bar{w})$ for $m_P \in M_P$ (resp. $w'(\tau_s \otimes \pi_0)(m_Q) = (\tau_s \otimes \pi_0)(\bar{w}'^{-1}m_Q \bar{w}')$ for $m_Q \in M_Q$). Since $\sigma_0$ is $\varepsilon$-invariant, by Lemma 2.33 [3], the operator $\mathcal{M}(\bar{w}, \tau_s \otimes \bar{\sigma}_0)$ gives an operator
\begin{align*}
\mathcal{M}(\bar{w}, \tau_s \otimes \sigma_0) &\colon \text{Ind}_{P_L}^{SO(V)}(\tau_s \otimes \sigma_0) \rightarrow \text{Ind}_{P_L}^{SO(V)}(\bar{w}(\tau_s \otimes \sigma_0)).
\end{align*}
Following [2, §2.3], we use the normalizing factors \( r(w, \tau \otimes \sigma_0) \) and \( r(w', \tau \otimes \pi_0) \) defined as follows. Let \( \phi_\tau, \phi_{\sigma_0} \) and \( \phi_{\pi_0} \) be the representations of \( WD_F \) corresponding to \( \tau, [\sigma_0] \) and \( \pi_0 \), respectively. Then we define \( r(w, \tau \otimes \sigma_0) \) and \( r(w', \tau \otimes \pi_0) \) by

\[
\begin{align*}
    r(w, \tau \otimes \sigma_0) &= \lambda(E/F, \psi)^k \frac{L(s, \phi_\tau \otimes \phi_{\sigma_0}^\vee)}{L(1 + s, \phi_\tau \otimes \phi_{\sigma_0}^\vee)} \frac{L(-2s, (\Lambda_2)^\vee \circ \phi_\tau)}{L(1 - 2s, (\Lambda_2)^\vee \circ \phi_\tau)} \varepsilon(s, \phi_\tau \otimes \phi_{\sigma_0}^\vee, \psi) \varepsilon(-2s, (\Lambda_2)^\vee \circ \phi_\tau, \psi), \\
    r(w', \tau \otimes \pi_0) &= \frac{L(s, \phi_\tau \otimes \phi_{\pi_0}^\vee)}{L(1 + s, \phi_\tau \otimes \phi_{\pi_0}^\vee)} \frac{L(-2s, (\Lambda_2)^\vee \circ \phi_\tau)}{L(1 - 2s, (\Lambda_2)^\vee \circ \phi_\tau)} \varepsilon(s, \phi_\tau \otimes \phi_{\pi_0}^\vee, \psi) \varepsilon(-2s, (\Lambda_2)^\vee \circ \phi_\tau, \psi),
\end{align*}
\]

where \( \Lambda_2 \) is the representation of \( GL_k(\mathbb{C}) \) on the space of skew-symmetric \((k, k)\)-matrices, and \( \lambda(E/F, \psi) \) is the Langlands \( \lambda \)-factor associated to \( E = F(\sqrt{d}) \). Note that \( \lambda(E/F, \psi)^2 = \omega_{E/F}(-1) = \chi_V(-1) \). Then the normalized intertwining operators

\[
R(w, \tau \otimes \sigma_0) := r(w, \tau \otimes \sigma_0)^{-1} \mathcal{M}(\overline{w}, \tau \otimes \sigma_0), \\
R(w', \tau \otimes \pi_0) := r(w', \tau \otimes \pi_0)^{-1} \mathcal{M}(\overline{w'}, \tau \otimes \pi_0)
\]

are holomorphic at \( s = 0 \) by [2, Proposition 2.3.1].

Now assume that \( w(\tau \otimes \sigma_0) \cong \tau \otimes \sigma_0 \), which is equivalent to \( \tau^\vee \cong \tau \). We take the unique isomorphism

\[
\mathcal{A}_w : V_\tau \otimes V_{\sigma_0} \to V_\tau \otimes V_{\sigma_0}
\]

such that:

- \( \mathcal{A}_w \circ w(\tau \otimes \sigma_0)(m) = (\tau \otimes \sigma_0)(m) \circ \mathcal{A}_w \) for any \( m \in M_P \);
- \( \mathcal{A}_w = \mathcal{A}_w' \otimes 1_{V_{\sigma_0}} \) with an isomorphism \( \mathcal{A}_w' : V_\tau \to V_\tau \) such that \( \Lambda \circ \mathcal{A}_w' = \Lambda \). Here \( \Lambda : V_\tau \to \mathbb{C} \) is the unique (up to a scalar) Whittaker functional with respect to the Whittaker datum \((B_k, \psi_{U_k})\), where \( B_k \) is the Borel subgroup consisting of upper triangular matrices in \( GL_k(F) \) and \( \psi_{U_k} \) is the generic character of the unipotent radical \( U_k \) of \( B_k \) given by \( \psi_{U_k}(x) = \psi(x_{1,2} + \cdots + x_{k-1,k}) \).

Similarly, we take the unique isomorphism

\[
\mathcal{A}_{w'} : V_\tau \otimes V_{\pi_0} \to V_\tau \otimes V_{\pi_0}
\]

We define self-intertwining operators

\[
R(w, \tau \otimes \sigma_0) : \text{Ind}_{P_0}^{\text{SO}(V)}(\tau \otimes \sigma_0) \to \text{Ind}_{P_0}^{\text{SO}(V)}(\tau \otimes \sigma_0), \\
R(w', \tau \otimes \pi_0) : \text{Ind}_Q^{\text{Sp}(W)}(\tau \otimes \pi_0) \to \text{Ind}_Q^{\text{Sp}(W)}(\tau \otimes \pi_0)
\]

by

\[
R(w, \tau \otimes \sigma_0) \Phi_s(h) = \mathcal{A}_w(R(w, \tau \otimes \sigma_0) \Phi_s)(h), \\
R(w', \tau \otimes \pi_0) \Phi'_s(g) = \mathcal{A}_{w'}(R(w', \tau \otimes \pi_0) \Phi'_s)(g)
\]

for \( \Phi_s \in \text{Ind}_{P_0}^{\text{SO}(V)}(\tau \otimes \sigma_0) \), \( \Phi'_s \in \text{Ind}_Q^{\text{Sp}(W)}(\tau \otimes \pi_0) \), \( h \in \text{SO}(V) \) and \( g \in \text{Sp}(W) \).

Put \( \phi_{\sigma} = \phi_\tau + \phi_{\sigma_0} + \phi_{\pi_0}^\vee \in \tilde{\Phi}_{temp}(\text{SO}(V)) \) (resp. \( \phi_{\sigma} = \phi_\tau + \phi_{\pi_0} + \phi_{\sigma_0}^\vee \in \tilde{\Phi}_{temp}(\text{Sp}(W)) \)). We take \([\sigma] \in \Pi_{\phi_{\sigma}}\) (resp. \( \pi \in \Pi_{\phi_{\pi}}\)) such that \( \sigma \subset \text{Ind}_{P_0}^{\text{SO}(V)}(\tau \otimes \sigma_0) \) (resp. \( \pi \subset \text{Ind}_Q^{\text{Sp}(W)}(\tau \otimes \pi_0) \)). Let \( w_\sigma \) (resp. \( w'_\pi \)) be the Whittaker datum of \( \text{SO}(V) \) (resp. \( \text{Sp}(W) \)) defined in [22]. Note that \( w_\sigma \) (resp. \( w'_\pi \)) is given by the splitting \( \mathfrak{sp}_{\text{SO}(V)} \) (resp. \( \mathfrak{sp}_{\text{Sp}(W)} \)) and the non-trivial additive character \( \psi \) we have fixed. The intertwining operators and the local Langlands correspondence are related as follows:

**Proposition 8.6.** Assume that \( \dim(V_0) = 2m_0 \geq 4 \) and \( \phi_{\sigma_0} \in \tilde{\Phi}_{temp}(\text{SO}(V_0)) \) is \( \varepsilon \)-invariant.

1. If \( \phi_\tau \) is an irreducible orthogonal representation of \( WD_F \) with \( \dim(\phi_\tau) \) is even, then we have

\[
R(w, \tau \otimes \sigma_0)|\sigma = \iota_m([\sigma])(a_\tau), \\
R(w', \tau \otimes \pi_0)|\pi = \iota_{w'_\pi}(\pi)(a'_\tau).
\]

Here, \( a_\tau \) (resp. \( a'_\tau \)) is the element in \( A^+_{\phi_{\sigma_0}} \) (resp. \( A^+_{\phi_{\pi_0}} \)) corresponding to \( \phi_\tau \).
(2) If $\phi_\tau = \phi_1 + \phi_2$ where $\phi_1$ and $\phi_2$ are distinct irreducible orthogonal representations of $\text{WD}_F$ with $\dim(\phi_1 + \phi_2)$ is even, then we have
\[ R(w, \tau \otimes \sigma_0)(\sigma = \iota_{\text{w}}, ([\sigma])(a_1 + a_2)), \]
\[ R(w', \tau \otimes \pi_0)(\pi = \iota_{\text{w'}}(\pi)(a'_1 + a'_2)). \]

Here, $a_i$ (resp. $a'_i$) is the element in $A_{\phi_i}$ (resp. $A_{\phi_i}$) corresponding to $\phi_i$ for $i = 1, 2$.

**Proof.** This follows from Theorem 2.2.1 and Theorem 2.4.1 in [2]. \[\square\]

Note that $a_\tau, a_1 + a_2 \in A_{\phi_\tau}$ and $a'_1, a'_2 \in A_{\phi_\tau}$ since $k = \dim(\phi_\tau)$ is even.

### 8.4. Zeta integrals of Godement–Jacquet

In this subsection, we review the theory of local factors for $\text{GL}(k)$ developed by Godement–Jacquet [19].

Let $\tau$ be an irreducible smooth representation of $\text{GL}_k(F)$ on a space $\mathcal{V}_\tau$ with a central character $\omega_\tau$. We write
\[ L(s, \tau) = L(s, \phi_\tau) \quad \text{and} \quad \varepsilon(s, \tau, \psi) = \varepsilon(s, \phi_\tau, \psi) \]
for the standard $L$-factor and $\varepsilon$-factor of $\tau$, where $\phi_\tau$ is the $k$-dimensional representation of $\text{WD}_F$ associated to $\tau$. Then the standard $\gamma$-factor of $\tau$ is defined by
\[ \gamma(s, \tau, \psi) = \varepsilon(s, \tau, \psi) \cdot \frac{L(1 - s, \tau^\vee)}{L(s, \tau)}. \]

For $s \in \mathbb{C}$, $\Phi \in \mathcal{S}(M_k(F))$ and a matrix coefficient $f$ of $\tau$, we put
\[ Z(s, \Phi, f) = \int_{\text{GL}_k(F)} \Phi(a)f(a)|\det(a)|_F^s da. \]

This integral is absolutely convergent for $\Re(s) > k - 1/2$ and admits a meromorphic continuation to $\mathbb{C}$. Moreover, the quotient
\[ \frac{Z(s + k - 1/2, \Phi, f)}{L(s, \tau)} \]
is an entire function of $s$.

**Lemma 8.7.** If $\tau$ is tempered, then $Z(s, \Phi, f)$ is absolutely convergent for $\Re(s) > (k - 1)/2$.

**Proof.** Put $t = \Re(s) > (k - 1)/2$. Fix a uniformizer $\varpi$ of $F$ and put
\[ t(a) = \begin{pmatrix} \varpi^{a_1} \\ & \ddots \\ & & \varpi^{a_k} \end{pmatrix} \in \text{GL}_k(F) \]
for $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k$. Let $B$ be the Borel subgroup of $G = \text{GL}_k(F)$ consisting of upper triangular matrices, and put $K = \text{GL}_k(0_F)$. We denote the modulus character of $B$ by $\delta_B$. Then we have
\[ \delta_B(t(a)) = \prod_{i < j} \left| \frac{\varpi^{a_i}}{\varpi^{a_j}} \right|_F = q^{-(k-1)a_1}q^{-(k-3)a_2} \cdots q^{(k-1)a_k}. \]

Since $F$ is non-archimedean, the integral formula
\[ \int_{\text{GL}_k(F)} |\Phi(g)f(g)| \det(g)|_F^s da \]
\[ = \sum_{a_1 \leq \cdots \leq a_k} \mu(t(a)) \int_{K \times K} |\Phi(k_1t(a)k_2)| \cdot |f(k_1t(a)k_2)| \cdot |\det(k_1t(a)k_2)|_F^s dk_1dk_2 \]
holds for some $\mu(t(a)) \geq 0$. Moreover, there exists a positive constant $A_0$ such that $\mu(t(a)) \leq A_0 \cdot \delta_B(t(a))$ for $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ with $a_1 \leq \cdots \leq a_k$ (see, e.g. [59] p.149).
We define the height function $\sigma(g)$ of $G$ by
\[
\sigma(g) = \max_{1 \leq i \leq k} \{ \log |g_{i,j}| r, \log |(g^{-1})_{i,j}| r \}.
\]

Harish-Chandra’s spherical function $\Xi(g)$ of $G$ is given by
\[
\Xi(g) = \int_K h_0(kg)dk,
\]
where $h_0 \in \text{Ind}_{B}^{G}(1)$ is the function whose restriction to $K$ is identically equal to 1. Note that $\Xi$ is a matrix coefficient of the tempered representation $\text{Ind}_{B}^{G}(1)$, and is $K$-invariant. It is known that there exist positive constants $A_1$, $A_2$ such that
\[
\Xi(t(a)) \leq A_1 \cdot \delta_B^{-1/2}(t(a)) \cdot (1 + \sigma(t(a)))^{A_2}
\]
for $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ with $a_1 \leq \cdots \leq a_k$ (see, e.g. [50, p.154, Theorem 4.2.1]). Since $\tau$ is tempered, any matrix coefficient $f$ of $\tau$ satisfies
\[
|f(g)| \leq B_1 \cdot \Xi(g) \cdot (1 + \sigma(g))^{B_2}
\]
for some positive constants $B_1$ and $B_2$.

Since $\sigma(k_1 g k_2) = \sigma(g)$ for $k_1, k_2 \in K$ and $g \in G$, we conclude that there are positive constants $C_1$ and $C_2$ such that
\[
\int_{GL_k(F)} |\Phi(g)f(g)| |\det(g)|^{r} | \sigma(g) | d\sigma(a) = C_1 \cdot \sum_{a_1 \leq \cdots \leq a_k} \delta_B^{1/2}(t(a)) (1 + \sigma(t(a)))^{C_2} | \det(t(a)) |^{r} \int_{K \times K} |\Phi(k_1 t(a) k_2)| dk_1 dk_2.
\]

We choose $\varepsilon > 0$ such that $(k-1)/2 - t + C_2 \varepsilon < 0$. Note that $\Phi$ has a compact support in $M_k(F)$, and so does the function
\[
M_k(F) \ni x \mapsto \int_{K \times K} |\Phi(k_1 x k_2)| dk_1 dk_2.
\]
This implies that there are constants $M > 0$ and $r > 0$ such that
\[
\int_{GL_k(F)} |\Phi(g)f(g)| |\det(g)|^{r} | \sigma(g) | d\sigma(a) = M \cdot \sum_{-r \leq a_1 \leq \cdots \leq a_k} \delta_B^{1/2}(t(a)) (1 + \sigma(t(a)))^{C_2} | \det(t(a)) |^{r} \]
\[
= M \cdot \sum_{-r \leq a_1 \leq \cdots \leq a_k} q^{a_1(-k-1)/2-t} q^{a_2(-k-3)/2-t} \cdots q^{a_k((k-1)/2-t)} (1 + \sigma(t(a)))^{C_2}.
\]

To see the convergence of this sum, we only consider the sum over $a_k \geq r \geq -a_1$. Then we have
\[
\sigma(t(a)) = \log(q^{a_k}).
\]
Moreover, we may assume that if $a_k \geq r$, then $1 + \log(q^{a_k}) \leq q^{-a_k}$. In this case, the sum over $a_k \geq r \geq -a_1$ is bounded by
\[
M \cdot \sum_{a_k \geq r} (a_k + r + 1)^{k-1} q^{-r(-k-1)/2-t} \cdots q^{-r((k-3)/2-t)} q^{a_k((k-1)/2-t+C_2 \varepsilon)}.
\]
This sum converges since $q^{(k-1)/2-t+C_2 \varepsilon} < 1$. \qed

Let $\hat{\Phi} \in \mathcal{S}(M_k(F))$ be the Fourier transform of $\Phi$ defined by
\[
\hat{\Phi}(x) = \int_{M_k(F)} \Phi(y) \psi(\text{tr}(xy)) dy,
\]
Proof. This follows from Lemma 8.2, 8.3 and 8.4.

9. Proof of Theorem 7.1

Now we can begin the proof of Theorem 7.1. This will be proven by an explicit construction of an equivariant map which realizes the theta correspondence. Let \( V = V_{2m} \) and \( W = W_{2n} \). In this section, we put \( m = n + 1 \).

9.1. Construction of equivariant maps. Recall that \( V \) and \( W \) have the decompositions
\[
V = X + V_0 + X^*, \quad W = Y + W_0 + Y^*
\]
with \( \dim(X) = \dim(Y) = k \). Assume that \( k \) is even and \( \dim(W_0) = 2n_0 \geq 2 \), so that \( \dim(V_0) = 2m_0 = 2(n_0 + 1) \geq 4 \). Using the basis \( \{v_1, \ldots, v_k\} \) of \( X \) (resp. \( \{w_1, \ldots, w_k\} \) of \( Y \), we identify \( \text{GL}(X) \) (resp. \( \text{GL}(Y) \)) with \( \text{GL}_k(F) \). Hence we can define an isomorphism \( i: \text{GL}(Y) \rightarrow \text{GL}(X) \) via these identifications. Put
\[
e = v_1 \otimes w_k^* + \cdots + v_k \otimes w_k^* \in X \otimes Y^*, \quad e^* = v_1^* \otimes w_1 + \cdots + v_k^* \otimes w_k \in X^* \otimes Y.
\]
Then we have \( i(a) e = a^* e \) and \( i(a)^* e^* = ae^* \) for \( a \in \text{GL}(Y) \).

Recall that \( W_0 = Y_0 + Y_0^* \). For \( \varphi \in S = \mathcal{S}(V \otimes Y^*) \otimes \mathcal{S}(V_0 \otimes Y_0^*) \otimes \mathcal{S}(X^* \otimes W_0) \), we define maps
\[
f(\varphi), \hat{f}(\varphi): \text{Sp}(W) \times \text{O}(V) \rightarrow \mathcal{S}(V_0 \otimes Y_0^*)
\]
by
\[
[f(\varphi)(g, h)](x_0) = [\omega(g, h)\varphi](\begin{pmatrix} e \\ 0 \\ 0 \end{pmatrix}, x_0, 0),
\]
\[
[\hat{f}(\varphi)(g, h)](x_0) = \int_{X \otimes Y^*} [\omega(g, h)\varphi](\begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}, x_0, 0)\psi(\langle z, e^* \rangle)dz.
\]
Here, we write an element in \( V \otimes Y^* \) as a block matrix
\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]
with \( x_1 \in X \otimes Y^* \), \( x_2 \in V_0 \otimes Y^* \) and \( x_3 \in X^* \otimes Y^* \).

Lemma 9.1. For \( f = f(\varphi) \) or \( f = \hat{f}(\varphi) \), we have
\[
f(u_Qg, u_ph) = f(g, h), \quad u_p \in U_p, u_Q \in U_Q, 
\]
\[
f(g_0g, h_0h) = \omega_0(g_0, h_0)f(g, h), \quad h_0 \in \text{O}(V_0), g_0 \in \text{Sp}(W_0),
\]
\[
f(m_Q(a)g, m_P(i(a))h) = \chi_N(\det(a))|\det(a)|^{\rho_1 + \rho_2}f(g, h), \quad a \in \text{GL}(Y).
\]

Proof. This follows from Lemma 8.2, 8.3 and 8.4.

Let \( \tau \) be an irreducible (unitary) tempered representation of \( \text{GL}_k(F) \) on a space \( \mathcal{V}_\tau \). We may regard \( \tau \) as a representation of \( \text{GL}(X) \) or \( \text{GL}(Y) \) via the above identifications. Let \( \pi_0 \) and \( \sigma_0 \) be irreducible tempered representations of \( \text{Sp}(W_0) \) and \( \text{O}(V_0) \) on spaces \( \mathcal{V}_{\pi_0} \) and \( \mathcal{V}_{\sigma_0} \), respectively. Fix nonzero invariant non-degenerate bilinear forms \( \langle \cdot, \cdot \rangle \) on \( \mathcal{V}_\tau \times \mathcal{V}_\tau, \mathcal{V}_{\pi_0} \times \mathcal{V}_{\pi_0} \) and \( \mathcal{V}_{\sigma_0} \times \mathcal{V}_{\sigma_0} \). Let
\[
\langle \cdot, \cdot \rangle: (\mathcal{V}_\tau \otimes \mathcal{V}_{\sigma_0}^*) \times \mathcal{V}_\tau \rightarrow \mathcal{V}_{\sigma_0}^*
\]
be the induced map.

Now we assume that
\[
\tilde{\sigma}_0 = \Theta_{\psi, V_0, W_0}(\pi_0).
\]
We fix a nonzero \( \text{Sp}(W_0) \times \text{O}(V_0) \)-equivariant map
\[
\mathcal{T}_{00}: \omega_0 \otimes \widetilde{\sigma}_0^\vee \to \pi_0.
\]
For \( \varphi \in \mathcal{S} = \mathcal{S}(V \otimes Y^*) \otimes \mathcal{S}(V_0 \otimes Y_0^*) \otimes \mathcal{S}(X^* \otimes W_0) \), \( \Phi_s \in \text{Ind}_P^{O(V)}(\tau_s \otimes \widetilde{\sigma}_0^\vee) \), \( g \in \text{Sp}(W) \), \( \tilde{v} \in \mathcal{V}_\tau \), and \( \tilde{v}_0 \in \mathcal{V}_{\sigma_0^\vee} \), we put
\[
\langle \mathcal{T}_s(\varphi, \Phi_s)(g), \tilde{v} \otimes \tilde{v}_0 \rangle = L(s, \tau)^{-1} \cdot \int_{U_P \mathcal{O}(V_0) \mathcal{O}(V)} \langle \mathcal{T}_{00}(f(\varphi)(g, h), \langle \Phi_s(h), \tilde{v} \rangle), \tilde{v}_0 \rangle dh.
\]
Note that \( \langle \Phi_s(h), \tilde{v} \rangle \in \mathcal{V}_{\sigma_0^\vee} \).

**Proposition 9.2.** We have the following:

1. The integral \( \langle \mathcal{T}_s(\varphi, \Phi_s)(g), \tilde{v} \otimes \tilde{v}_0 \rangle \) is absolutely convergent for \( \text{Re}(s) > 0 \) and admits a holomorphic continuation to \( \mathbb{C} \).
2. For \( \text{Re}(s) < 1 \), we have
\[
\langle \mathcal{T}_s(\varphi, \Phi_s)(g), \tilde{v} \otimes \tilde{v}_0 \rangle = L(s, \tau)^{-1} \cdot \int_{U_P \mathcal{O}(V_0) \mathcal{O}(V)} \langle \mathcal{T}_{00}(f(\varphi)(g, h), \langle \Phi_s(h), \tilde{v} \rangle), \tilde{v}_0 \rangle dh.
\]
3. The map
\[
\mathcal{T}_s: \omega \otimes \text{Ind}_P^{O(V)}(\tau_s \otimes \widetilde{\sigma}_0^\vee) \to \text{Ind}_Q^{\text{Sp}(W)}(\tau_s \chi \otimes \pi_0)
\]
is \( \text{Sp}(W) \times \text{O}(V) \)-equivariant.
4. For \( \Phi \in \text{Ind}_P^{O(V)}(\tau \otimes \widetilde{\sigma}_0^\vee) \) with \( \Phi \neq 0 \), there exists \( \varphi \in \omega \) such that
\[
\mathcal{T}_0(\varphi, \Phi) \neq 0.
\]

**Proof.** The proof is similar to those of Lemma 8.1, 8.2 and 8.3 in [13]. \( \square \)

**Corollary 9.3.** Let \( \pi_0 \) be an irreducible tempered representation of \( \text{Sp}(W_0) \). Assume that \( \tilde{\sigma}_0 = \Theta_{\psi, V_0, W_0}(\pi_0) \) is nonzero so that this is irreducible. Let \( \tau \) be an irreducible (unitary) tempered representation of \( \text{GL}_k(F) \cong \text{GL}(X) \cong \text{GL}(Y) \). Assume that both \( \tilde{\sigma} = \text{Ind}_P^{O(V)}(\tau \otimes \tilde{\sigma}_0) \) and \( \pi = \text{Ind}_Q^{\text{Sp}(W)}(\tau \chi \otimes \pi_0) \) are irreducible. Then we have
\[
\tilde{\sigma} = \Theta_{\psi, V, W}(\pi).
\]

**Proof.** The map \( \mathcal{T}_0 \) induces a nonzero \( \text{Sp}(W) \times \text{O}(V) \)-equivariant map
\[
\omega \to \pi \boxtimes \tilde{\sigma}.
\]
This shows that \( \tilde{\sigma} = \Theta_{\psi, V, W}(\pi) \). \( \square \)

### 9.2. **Compatibilities with intertwining operators.**

Now we shall explain a key property of the equivariant map we have constructed.

We have assumed that \( k = \dim(X) = \dim(Y) \) is even and \( \dim(V_0) = 2(n_0 + 1) \geq 4 \). Let \( w \in W(\mathcal{M}_P) \) and \( w' \in W(\mathcal{M}_Q) \) be the non-trivial elements in the relative Weyl groups. As in [8, 9] we take the representatives \( \tilde{w} \in \text{SO}(V) \) of \( w \) and \( \tilde{w}' \in \text{Sp}(W) \) of \( w' \) defined by
\[
\tilde{w} = w_P \cdot m_P(-c \cdot a) \quad \text{and} \quad \tilde{w}' = w_Q \cdot m_Q(a),
\]
where \( a \in \text{GL}_k(F) \cong \text{GL}(X) \cong \text{GL}(Y) \) is defined by
\[
a = \begin{pmatrix}
(-1)^{n-k+1} & \\
& \ddots
\end{pmatrix}
\]
We fix \( \tau, \pi_0 \) and \( \tilde{\sigma}_0 = \Theta_{\psi, V_0, W_0}(\pi_0) \). We shall write
\[
\mathcal{M}(\tilde{w}, s) = \mathcal{M}(\tilde{w}, \tau_s \otimes \tilde{\sigma}_0^\vee) \quad \text{and} \quad \mathcal{M}(\tilde{w}', s) = \mathcal{M}(\tilde{w}', \tau_s \chi V \otimes \pi_0)
\]
for the unnormalized intertwining operators, which are defined by the integrals
\[
\mathcal{M}(\tilde{w}, s) \Phi_s(h) = |c|_F^{kpp} \int_{U_P} \Phi_s(\tilde{w}^{-1} u_P h) du_P,
\]
Proof.
We have
\[ \mathcal{M}((\tilde{w}', s)|\psi_s(g)) = \int_{U_Q} \psi_s((\tilde{w}'^{-1}u_Q)g)du_Q \]
for \( \Phi_s \in \text{Ind}_{P}^{Q}(\tau_s \otimes \sigma_0^\lor) \) and \( \psi_s \in \text{Ind}_{Q}^{\text{Sp}(W)}(\tau_s \chi_V \otimes \pi_0) \). By the Howe duality, the diagram
\[
\begin{array}{ccc}
\omega \otimes \text{Ind}_{P}^{Q}(\tau_s \otimes \sigma_0^\lor) & \xrightarrow{\tau_s} & \text{Ind}_{Q}^{\text{Sp}(W)}(\tau_s \chi_V \otimes \pi_0) \\
1 \otimes \mathcal{M}(\tilde{w}, s) & & \mathcal{M}(\tilde{w}', s) \\
\omega \otimes \text{Ind}_{P}^{Q}(w(\tau_s \otimes \sigma_0^\lor)) & \xrightarrow{\tau_s} & \text{Ind}_{Q}^{\text{Sp}(W)}(w'(\tau_s \chi_V \otimes \pi_0))
\end{array}
\]
commutes up to a scalar. The following proposition determines this constant of proportionality explicitly.

**Proposition 9.4.** For \( \varphi \in \omega \) and \( \Phi_s \in \text{Ind}_{P}^{Q}(\tau_s \otimes \sigma_0^\lor) \), we have
\[ \mathcal{M}(\tilde{w}', s)\mathcal{T}_s(\varphi, \Phi_s) = \omega(\tau) \cdot |c|^s \cdot \gamma^{-k} \cdot L(s, \tau)^{-1} L(-s, \tau^\lor) \gamma(-s, \tau^\lor, \psi) \cdot T_{-s}(\varphi, \mathcal{M}(\tilde{w}, s)\Phi_s). \]

**Proof.** The proof is similar to that of [13, Proposition 8.4]. \( \square \)

Let \( \phi_{\tau}, \phi_0 \) and \( \phi_0^\lor \) be the representations of \( WD_F \) corresponding to \( \tau, \pi_0 \) and \( [\sigma_0] \), respectively, where we put \( \sigma_0 := \sigma_0|\text{SO}(V) \). Note that \( \sigma_0 \) is irreducible.

**Corollary 9.5.** For \( \varphi \in \omega \) and \( \Phi_s \in \text{Ind}_{P}^{Q}(\tau_s \otimes \sigma_0^\lor) \), we have
\[ \mathcal{R}(w', \tau_s \chi_V \otimes \pi_0)\mathcal{T}_0(\varphi, \Phi_s) = \omega(\tau) \cdot \alpha(s) \cdot \mathcal{T}_{-s}(\varphi, \mathcal{R}(w, \tau_s \otimes \sigma_0^\lor)\Phi_s), \]
where
\[ \alpha(s) = |c|^s \frac{\varepsilon(-s, \tau^\lor, \psi)}{\varepsilon(s, \tau, \psi)}. \]

In particular, if \( \tau \equiv \tau^\lor \), then \( \alpha(0) = 1 \) and
\[ \mathcal{R}(w', \tau_s \chi_V \otimes \pi_0)\mathcal{T}_0(\varphi, \Phi) = \omega(\tau) \cdot \mathcal{T}_0(\varphi, \mathcal{R}(w, \tau \otimes \sigma_0^\lor)\Phi) \]
for \( \Phi \in \text{Ind}_{P}^{Q}(\tau \otimes \sigma_0^\lor) \).

**Proof.** We have
\[ \alpha(s) = |c|^s \frac{r(w, \tau_s \otimes \sigma_0^\lor)}{r(w', \tau_s \chi_V \otimes \pi_0)} \cdot \gamma^{-k} \cdot \frac{L(-s, \tau^\lor)}{L(s, \tau)} \cdot \gamma(-s, \tau^\lor, \psi). \]
Since
\[ \phi_0^\lor = (\phi_0 \otimes \chi_V) \oplus 1, \]
we have
\[ \phi_{\tau} \otimes \phi_0^\lor = (\phi_{\tau} \otimes \chi_V) \otimes \phi_0^\lor \oplus \phi_{\tau}. \]
Moreover, we have
\[ \Lambda_2^\lor \circ (\phi_{\tau} \otimes \chi_V) = \Lambda_2^\lor \circ \phi_{\tau}. \]
Hence we have
\[ \frac{r(w, \tau_s \otimes \sigma_0^\lor)}{r(w', \tau_s \otimes \sigma_0^\lor)} = \lambda(E/F, \psi)^k \cdot L(s, \phi_{\tau}) \frac{L(1 + s, \phi_{\tau})}{\varepsilon(s, \phi_{\tau}, \psi)L(1 + s, \phi_{\tau})}. \]
Note that \( \lambda(E/F, \psi)^k \gamma^{-k} = 1 \) since \( k \) is even and \( \lambda(E/F, \psi)^2 = \gamma_1^2 = \chi_V(-1) \). Recall that
\[ \gamma(-s, \tau^\lor, \psi) = \frac{\varepsilon(-s, \tau^\lor, \psi)L(1 + s, \tau)}{L(-s, \tau^\lor)}. \]
Therefore, we have
\[ \alpha(s) = |c|^s \frac{\varepsilon(-s, \tau^\lor, \psi)}{\varepsilon(s, \tau, \psi)}. \]
Assume that \( \tau \equiv \tau^\lor \). By definition of \( \tilde{w} \) and \( \tilde{w}' \), the action of \( \tilde{w} \) on \( GL(X) \subset M_P \) coincides with that of \( \tilde{w}' \) on \( GL(Y) \subset M_Q \) via the identification \( i : GL(Y) \to GL(X) \). This implies that \( A_{w} = A_{w'} \) as \( \mathbb{C} \)-isomorphisms on \( Y_{\tau} \). Hence the last equation holds. \( \square \)
9.3. Completion of the proof. We set $W = W_{2n}$ and $V = V_{2n+2}$. We may assume that $V$ is type $(d, c)$. Fix $c_0 \in F^\times$. Let $\phi \in \hat{\Phi}_{\text{temp}}(\text{Sp}(W))$ such that
\[
\phi = (\phi_1 \otimes \chi_V) \oplus c_0 \oplus (\phi_1 \otimes \chi_V)^\vee,
\]
where $\phi_1$ is an irreducible tempered representation of $WD_F$ and $\phi_0 \in \hat{\Phi}_{\text{temp}}(\text{Sp}(W_0))$ with dim($W_0$) = $2n_0$.

Then we have $\phi \in \hat{\Phi}_{\text{temp}}(\text{Sp}(W_0))$ for the irreducible representation $\pi_0 \in \Pi_{\phi_0}$ of $\text{Sp}(W_0)$ with the associated character $\eta_0 = \iota_{\text{w}_{\phi_0}}(\pi_0) \in \hat{A}_{\phi_0}^+$ such that
\[
\eta|A_{\phi_0}^+ = \eta_0.
\]

Now, we assume that
\[
\bar{\sigma} = \Theta_{\psi', V, W}(\pi) \neq 0.
\]

Then by Proposition 2.6 we see that $\bar{\sigma}_0 = \Theta_{\psi', V_0, W_0}(\pi_0) \neq 0$ and $\bar{\sigma}$ is an irreducible constituent of $\text{Ind}_{\text{G2}}^{\text{O}(V)}(\tau \otimes \bar{\sigma}_0)$, where $V = X + V_0 + X^*$ with dim($V_0$) = $2n_0 + 2 \geq 4$. Let $\sigma := \bar{\sigma}|\text{SO}(V)$ and $\sigma_0 := \bar{\sigma}_0|\text{SO}(V_0)$. Note that these are irreducible and $\varepsilon$-invariant. We put
\[
\phi' = (\phi \otimes \chi_V) \oplus 1 \in \hat{\Phi}_{\text{temp}}(\text{SO}(V)) \quad \text{and} \quad \phi'_0 = (\phi_0 \otimes \chi_V) \oplus 1 \in \hat{\Phi}_{\text{temp}}(\text{SO}(V_0)).
\]

Then we have $[\sigma] \in \Pi_{\phi'}$ and $[\sigma_0] \in \Pi_{\phi'_0}$. The associated characters $\eta' = \iota_{\text{w}_{\phi}'}([\sigma]) \in \hat{A}_{\phi'}^+$ and $\eta'_0 = \iota_{\text{w}_{\phi'}_0}([\sigma_0]) \in \hat{A}_{\phi'_0}^+$ satisfy that
\[
\eta'|A_{\phi'_0}^+ = \eta_0.
\]

We need to show that $\eta'|A_{\phi}^+ = \eta$.

Consider the commutative diagram
\[
\begin{array}{c}
A_{\phi}^+ \xrightarrow{1/2} A_{\phi'}^+ \\
\uparrow \quad \uparrow \\
A_{\phi_0}^+ \xrightarrow{1/2} A_{\phi'_0}^+
\end{array}
\]
of natural embeddings. The map $A_{\phi}^+ \to A_{\phi'}^+$ (resp. $A_{\phi_0}^+ \to A_{\phi'_0}^+$) is bijective unless $\chi_V \subset \phi$ (resp. $\chi_V \subset \phi_0$), in this case, $(A_{\phi}^+ : A_{\phi'}^+) = 2$ (resp. $(A_{\phi_0}^+ : A_{\phi'_0}^+) = 2$). Since $n_0 < n$, we know that (P1) for $(\text{Sp}(W_0), \text{O}(V_0))$ holds by assumption, so that
\[
\eta'_0|A_{\phi'_0}^+ = \eta_0.
\]

Hence, we conclude that
\[
\eta'|A_{\phi_0}^+ = (\eta'|A_{\phi'_0}^+)|A_{\phi_0}^+ = \eta'_0|A_{\phi_0}^+ = \eta_0 = \eta|A_{\phi_0}^+.
\]

In particular, if $A_{\phi_0}^+ = A_{\phi}^+$, then $\eta'|A_{\phi}^+ = \eta$ as desired.

Finally, we assume that $A_{\phi_0}^+ \neq A_{\phi}^+$, which is equivalent that $\phi_1$ is orthogonal and $\phi_1 \otimes \chi_V$ is not contained in $\phi_0$. Then we have
\[
A_{\phi} = A_{\phi_0} \times (\mathbb{Z}/2\mathbb{Z})a_1,
\]
where $a_1$ is the element in $A_{\phi}$ corresponding to $\phi_1 \otimes \chi_V$. Note that $a_1 \in A_{\phi}^+$ if and only if $k_1$ is even.

First, we assume that $k = k_1$ is even. Then we have
\[
A_{\phi}^+ = A_{\phi_0}^+ \times (\mathbb{Z}/2\mathbb{Z})a_1.
\]

Since we already know that $\eta'|A_{\phi_0}^+ = \eta|A_{\phi_0}^+$, it suffices to show that $\eta'(a'_1) = \eta(a_1)$, where $a'_1$ is the element in $A_{\phi'}^+$ corresponding to $\phi_1$.

**Lemma 9.6.** We have
\[
\iota_{\text{w}_c}(\pi)(a_1) = \omega_r(c) \cdot \iota_{\text{w}_c}([\sigma])(a'_1).
\]
Proof. Note that \( \bar{\sigma} \) is a direct summand of \( \text{Ind}^O(V)(\tau \otimes \bar{\sigma}_0) \), so that \( \bar{\sigma}' \) is a submodule of \( \text{Ind}^O_P(V)(\tau \otimes \bar{\sigma}_0') \). Choose \( \Phi_0 \in \bar{\sigma}' \subset \text{Ind}^O_P(V)(\tau \otimes \bar{\sigma}_0') \) with \( \Phi_0 \neq 0 \). By Proposition 9.2, there is \( \varphi_0 \in \omega \) such that \( T_0(\varphi_0, \Phi_0) \neq 0 \). Then the map \( \varphi \mapsto T_0(\varphi, \cdot)\bar{\sigma}' \) gives a nonzero \( \text{Sp}(W) \times O(V) \)-equivariant map

\[ \omega \rightarrow \text{Ind}^\text{Sp}(W)(\tau \chi V \otimes \pi_0) \otimes (\bar{\sigma}')^\vee \cong \text{Ind}^\text{Sp}(W)(\tau \chi V \otimes \pi_0) \otimes \bar{\sigma}. \]

Since \( \bar{\sigma} = \Theta_{\varphi, V, W}(\pi) \), the image must be contained in \( \pi \otimes \bar{\sigma} \), so that \( T_0(\varphi_0, \Phi_0) \in \pi \). By Corollary 9.5 we have

\[ R(w', \tau_x \chi V \otimes \pi_0)T_0(\varphi_0, \Phi_0) = \omega_x(c) \cdot T_0(\varphi_0, R(w, \tau_x \otimes \bar{\sigma}_0))\Phi_0. \]

Note that \( \bar{\sigma}' \cong \bar{\sigma} \) and \( \bar{\sigma}_0' \cong \bar{\sigma}_0 \), and we regard \( \Phi_0 \) as an element in \( \sigma = \bar{\sigma}|\text{SO}(V) \). Since \( R(w', \tau_x \chi V \otimes \pi_0)|\pi = \iota_{w'}(\pi)(a_1') \) and \( R(w, \tau_x \otimes \bar{\sigma}_0')|\sigma = \iota_{w}(\sigma)(a_1) \) by Proposition 8.6, we have

\[ \iota_{w'}(\pi)(a_1')T_0(\varphi_0, \Phi_0) = \omega_x(c) \cdot T_0(\varphi_0, \iota_w([\sigma])(a_1))\Phi_0. \]

This shows the desired equation. \( \square \)

However, by Proposition 3.1 and 3.2 we know that

\[ \frac{\iota_{w'}(\pi)(a_1)}{\iota_{w}(\sigma)(a_1)} = n_{\bar{\sigma}_0}(a_1) = \det(\phi_1 \otimes \chi V)(e_0^{-1}) = \omega_x(c_0^{-1}) \]

and

\[ \frac{\iota_{w}(\sigma)(a_1')}{\iota_{w}(\sigma)(a_1')} = n_{\bar{\sigma}_0}(a_1') = \det(\phi_1(c/e_0) = \omega_x(c/e_0). \]

Since \( \omega_x^2 = 1 \), we have

\[ \eta(a_1) = \omega_x(c_0) \cdot \iota_{w'}(\pi)(a_1) = \omega_x(c_0) \cdot \iota_{w}(\sigma)(a_1') = \eta'(a_1') \]

as desired.

Next, we assume that \( k_1 \) is odd. Since \( \varphi_0 \) is an orthogonal representation of \( WD_F \) with odd dimensional, we can find an irreducible orthogonal representation \( \phi_2 \) of \( WD_F \) such that \( \phi_2 \otimes \chi \) is contained in \( \varphi_0 \) and \( k_2 = \dim(\phi_2) \) is odd. Let \( a_2 \) be the element in \( A_\phi \) corresponding to \( \phi_2 \otimes \chi V \). Since \( k = k_1 + k_2 \) is even, we have \( a_1 + a_2 \in A_\phi^+ \) and

\[ A_\phi^+ = A_\phi^+ \times (\mathbb{Z}/2\mathbb{Z})(a_1 + a_2). \]

Since we already know that \( \eta|A_\phi^+ = \eta|A_\phi^+ \), it suffices to show that \( \eta'(a_1' + a_2') = \eta(a_1 + a_2) \), where \( a_2' \) is the element in \( A_{\phi_2}^+ \) corresponding to \( \phi_2 \), which is contained in \( \phi_0 \).

We consider

\[ \phi_* = (\phi_2 \otimes \chi V) \oplus \phi \oplus (\phi_2 \otimes \chi V) \in \bar{\Phi}_{\text{temp}}(\text{Sp}(W_*)) \quad \text{and} \quad \phi_*' = \phi_2 \oplus \phi' \oplus \phi_2 \in \bar{\Phi}_{\text{temp}}(\text{SO}(V_*)), \]

where \( \dim(W_*) = 2n_* = 2(n + k_2) \) and \( \dim(V_*) = 2m_* = 2(m + k_2) \). Note that \( A_{\phi_*}^+ = A_{\phi_2}^+ \) and \( A_{\phi_*}' = A_{\phi_2}' \).

Consider the commutative diagram

\[
\begin{array}{ccc}
A_{\phi_*}^+ & \xrightarrow{1/2} & A_{\phi_*}' \\
\| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
A_{\phi}^+ & \xrightarrow{1/2} & A_{\phi}' \\
\| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
A_{\phi_0}^+ & \xrightarrow{1/2} & A_{\phi_0}' \\
\| & & \|
\end{array}
\]

of natural embeddings.

Let \( \bar{\sigma}_* = \text{Ind}^{O(V_*)}(\tau' \otimes \bar{\sigma}) \) and \( \pi_* = \text{Ind}^{\text{Sp}(W_*)}(\tau' \chi V \otimes \pi) \) be parabolic inductions, where \( \tau' \) is the irreducible representation of \( \text{GL}_{k_2}(F) \) corresponding to \( \phi_2 \), and \( P' = P_{k_2} \) (resp. \( Q' = Q_{k_2} \)) is the parabolic subgroup of \( \text{O}(V_*) \) (resp. \( \text{Sp}(W_*) \)) which have a Levi subgroup of the form \( \text{GL}_{k_2}(F) \times \text{O}(V) \) (resp. \( \text{GL}_{k_2}(F) \times \text{Sp}(W) \)) defined in 2.3. Since \( A_{\phi_*}^+ = A_{\phi}^+ \) and \( A_{\phi_*}' = A_{\phi}' \), we see that \( \pi_* \) (resp. \( \sigma_* \) := \( \bar{\sigma}_*|\text{SO}(V_*) \)) is an irreducible
representation of $\text{Sp}(W_s)$ (resp. $\text{SO}(V_s)$) corresponding to the same character $\eta_s = \eta$ (resp. $\eta'_s = \eta'$) to $\pi$ (resp. $[\sigma]$). By corollary 9.3 we have

$$\tilde{\sigma}_s = \Theta_{\psi,V_s,W_s}(\pi_s).$$

Moreover, $\pi_s$ (resp. $\tilde{\sigma}_s$) is an irreducible constituent of $\text{Ind}_{P_s}^{\text{Sp}(W_s)}(\tau_s \chi_V \otimes \pi_0)$ (resp. $\text{Ind}_{P_s}^{\text{O}(V_s)}(\tau_s \otimes \tilde{\sigma}_0)$).

Here, $\tau_s$ is the irreducible representation of $\text{GL}_{k_1+k_2}(F)$ corresponding to $\phi_1 \oplus \phi_2$, and $Q_s = Q_{k_1+k_2}$ (resp. $P_s = P_{k_1+k_2}$) is the parabolic subgroup of $\text{Sp}(W_s)$ (resp. $\text{O}(V_s)$) which has a Levi subgroup of the form $\text{GL}_{k_1+k_2}(F) \times \text{Sp}(W_0)$ (resp. $\text{GL}_{k_1+k_2}(F) \times \text{O}(V_0)$) defined in [28, 32].

By a similar argument to Lemma 9.3 we have

$$\tau_s(a_1 + a_2) = \omega_{\tau_s}(c) \cdot \tau_{m_s}([\sigma_s])(a_1 + a_2).$$

However, by Proposition 3.1 and 3.2 we know that

$$\frac{\tau_m([\pi_s])(a_1 + a_2)}{\tau_{m_0}([\pi_s])(a_1 + a_2)} = \eta_{c_0}(a_1 + a_2) = \det((\phi_1 \oplus \phi_2) \otimes \chi_V)(c_0^{-1}) = \omega_{\tau_s}(c_0^{-1})$$

and

$$\frac{\tau_m([\sigma_s])(a'_1 + a'_2)}{\tau_{m_0}([\sigma_s])(a'_1 + a'_2)} = \eta_{c/c_0}(a'_1 + a'_2) = \det(\phi_1 \oplus \phi_2)(c/c_0) = \omega_{\tau_s}(c/c_0).$$

Since $\omega_{\tau_s}^2 = 1$, we have

$$\eta(a_1 + a_2) = \eta_s(a_1 + a_2) = \omega_{\tau_s}(c_0) \cdot \eta_{m_s}([\sigma_s])(a_1 + a_2) = \omega_{\tau_s}(c_0) \cdot \eta_{m_0}([\sigma_s])(a'_1 + a'_2) = \eta_s'(a'_1 + a'_2) = \eta'(a'_1 + a'_2)$$

as desired. This completes the proof of Theorem 7.1.

**Appendix A. Transfer factors**

In this appendix, we recall the transfer factors and explain the expectations in the local Langlands correspondence. More precisely, see [28, 32, 33]. Comparing the transfer factors of pure inner twists, we prove Proposition 3.1 and 3.2.

For simplicity, we let $F$ be a non-archimedean local field of characteristic zero.

A.1. **Endoscopic data.** Let $G$ be a quasi-split connected reductive group over $F$. An endoscopic datum for $G$ is a tuple $(H, \mathcal{H}, s, \eta)$, where

- $H$ is a quasi-split connected reductive group defined over $F$;
- $\mathcal{H}$ is a split extension of $W_F$ by $\tilde{H}$;
- $s \in G$ is a semi-simple element;
- $\eta: \mathcal{H} \rightarrow L^1 G$ is an $L$-embedding

such that

1. the homomorphism $W_F \rightarrow \text{Out}(\tilde{H})$ given by $\mathcal{H}$ is identified with the homomorphism $W_F \rightarrow \text{Out}(H)$ provided by the rational structure of $H$ via the natural isomorphism $\text{Out}(\tilde{H}) \cong \text{Out}(H)$;
2. $\eta(\bar{H}) = \hat{G}_s^\circ = \text{Cent}(s, \hat{G})^\circ$;
3. a certain condition of $s$ and $\xi$.

An isomorphism from $(H, \mathcal{H}, s, \eta)$ to another such tuple $(H', \mathcal{H}', s', \eta')$ is an element $g \in \hat{G}$ such that

1. $g\eta(\mathcal{H})g^{-1} = \eta'(\mathcal{H}')$;
2. $gsg^{-1} \equiv s' \pmod{Z(\hat{G})}$.

Given an endoscopic datum $(H, \mathcal{H}, s, \eta)$ of $G$, we may replace it by an isomorphic one and assume that $s \in \eta(Z(H)^F)$.

Let $\phi: \text{WD}_F \rightarrow L^1 G$ be an $L$-parameter of $G$. We put $S_\phi = \text{Cent}(\text{Im}(\phi), \hat{G})$. For a semi-simple element $s \in S_\phi$, there exists an endoscopic datum $(H, \mathcal{H}, s, \eta)$ for $G$ such that

- $\tilde{H} = \hat{G}_s^\circ$;
- $\mathcal{H} = H \cdot \phi(W_F)$;
- $\eta: \mathcal{H} \rightarrow L^1 G$ is the natural embedding,
Then we call \((H, \mathcal{H}, s, \eta)\) an endoscopic datum associated to \(s \in S_\phi\). Note that \(s \in Z(\hat{H})^{\Gamma}\) and \(\text{Im}(\phi) \subset \mathcal{H}\).

It is not always true that \(\mathcal{H} = \mathcal{H}_s\). So we take a z-pair \(\mathfrak{z} = (H_s, \eta_\mathfrak{z})\) for \(\mathfrak{c} = (H, \mathcal{H}, s, \eta)\). We recall that \(H_\mathfrak{z}\) is an extension of \(H\) by an induced torus and \(\eta_\mathfrak{z}: \mathcal{H} \rightarrow \mathcal{H}_\mathfrak{z}\) is an \(L\)-embedding such that the diagram
\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\eta_\mathfrak{z}} & \mathcal{H}_\mathfrak{z} \\
\uparrow & & \uparrow \\
\hat{H} & \longrightarrow & \hat{H}_\mathfrak{z}
\end{array}
\]
is commutative, where the bottom arrow is the embedding \(\hat{H} \rightarrow \hat{H}_\mathfrak{z}\), which is the dual to the surjection \(H_\mathfrak{z} \rightarrow H\). If \(\mathfrak{c}\) is an endoscopic datum associated to \(s \in S_\phi\) for some tempered \(L\)-parameter of \(G\), then we obtain a tempered \(L\)-parameter \(\phi_\mathfrak{c} = \eta_\mathfrak{z} \circ \phi\) of \(H_\mathfrak{c}\).

A.2. **Transfer factors of pure inner twists.** Recall that a pure inner twist of \(G\) is a triple \((G', \psi, z)\), where
- \(G'\) is a connected reductive algebraic group over \(F\);
- \(\psi: G \rightarrow G'\) is an isomorphism over \(\overline{F}\);
- \(z \in \mathcal{Z}(F, G)\)

such that \(\psi^{-1} \circ \sigma(\psi) = \text{Ad}(z_\sigma)\) for \(\sigma \in \Gamma = \text{Gal}(\overline{F}/F)\). Then we call \(G'\) a pure inner form of \(G\).

Let \((G_1, \psi_1, z_1)\) and \((G_2, \psi_2, z_2)\) be two pure inner twists of \(G\). An isomorphism from \((G_1, \psi_1, z_1)\) to \((G_2, \psi_2, z_2)\) is a pair \((f, g)\), where \(f: G_1 \rightarrow G_2\) is an isomorphism over \(F\) and \(g \in G(\overline{F})\) such that \(z_2\sigma = g_{z_1, \sigma}\sigma(g^{-1})\) and the diagram
\[
\begin{array}{ccc}
G & \xrightarrow{\psi_1} & G_1 \\
\downarrow{\text{Ad}(g)} & & \downarrow{f} \\
G & \xrightarrow{\psi_2} & G_2
\end{array}
\]
is commutative. It is known that there exists a canonical bijection
\[
\{\text{isomorphism classes of pure inner twists of } G\} \rightarrow H^1(F, G),
\]
\[(G', \psi, z) \mapsto [z].\]

Let \(\mathfrak{c} = (H, \mathcal{H}, s, \eta)\) be an endoscopic datum for \(G\) and \(\mathfrak{z} = (H_\mathfrak{z}, \eta_\mathfrak{z})\) be a z-pair for \(\mathfrak{c}\). We may assume that \(\eta^{-1}(s) \in Z(\hat{H})^\Gamma\), and identify \(s\) with \(\eta^{-1}(s)\). Then for an inner twist \((G', \psi)\) of \(G\), these data give a relative transfer factor of \(G'\), which is a function
\[
\Delta[\mathfrak{c}, \mathfrak{z}, \psi]: H_{3,G,\text{sr}}(F) \times G'_{\text{sr}}(F) \times H_{3,G,\text{sr}}(F) \times G'_{\text{sr}}(F) \rightarrow \mathbb{C}.
\]
See [39 §3.7]. We also write \(\Delta[\mathfrak{c}, \mathfrak{z}] = \Delta[\mathfrak{c}, \mathfrak{z}, \text{id}]\). Here, \(G'_{\text{sr}}\) is the set of strongly regular semi-simple elements of \(G'\). We explain \(H_{3,G,\text{sr}}\). Fix pairs \((B, T)\) of \(G\), \((B, T)\) of \(\tilde{G}\), \((B_H, T_H)\) of \(H\) and \((B_H, T_H)\) of \(\hat{H}\), where by a pair we mean a tuple of a Borel subgroup and a maximal torus contained in it. Assume that \(\eta(B_H) \subset B\) and \(\eta(T_H) = T\).

Then we have an isomorphism
\[
\tilde{T}_H \cong T_H \xrightarrow{\eta} T \cong \hat{T},
\]
and so that we get an isomorphism \(\eta^*: T_H \rightarrow T\) over \(\overline{F}\). Now we assume that \(T_H\) is \(F\)-rational. Since \(\sigma(\eta^* \circ (\eta^*)^{-1})\) is given by an element in the Weyl group of \(T\) in \(G\), by Steinberg’s theorem, there exists \(g \in G_{\text{sr}}(\overline{F})\) such that \(\sigma(g)g^{-1}\) normalizes \(T\) and induces \(\sigma(\eta^*) \circ (\eta^*)^{-1}\). Then
\[
\xi := \text{Ad}(g^{-1}) \circ \eta^*: T_H \rightarrow g^{-1}Tg
\]
is an isomorphism over \(F\). Such an \(F\)-isomorphism \(\xi: T_H \rightarrow T' := g^{-1}Tg\) is called an admissible embedding of \(T_H\) in \(G\). For \(h \in T_H\), the element \(\xi(h)\) is called an image of \(h\), and the elements \(h\) and \(\xi(h)\) are said to be related. The set \(H_{3,G,\text{sr}}\) consists of the preimages in \(H_3\) of those elements of \(H\) that are related to elements of \(G_{\text{sr}}\).
An absolute transfer factor of $G$ is a function $\Delta[e, \delta]_{\text{abs}}: H_{\delta} \times G_{\text{sr}} \to \mathbb{C}$ such that this is nonzero for any pair $(\gamma, \delta)$ of related elements, and satisfies

$$\Delta[e, \delta](\gamma; \delta, \delta') = \frac{\Delta[e, \delta]_{\text{abs}}(\gamma; \delta)}{\Delta[e, \delta]_{\text{abs}}(\gamma'; \delta')},$$

for any two pairs $(\gamma, \delta)$ and $(\gamma', \delta')$ of related elements. This is not unique. By choosing a Whittaker datum $w$ for $G$, we obtain a normalization $\Delta[e, \delta, w]$ of the absolute transfer factor of $G$. See [28] [5.3].

Using $\Delta[e, \delta, w]$, we define a transfer factor $\Delta[e, \delta, \psi, z, w]$ of a pure inner twist $(G', \psi, z)$ of $G$. Let $\delta' \in G_{\text{sr}}'(F)$ and $\gamma_3 \in H_3(F)$ be related elements. We denote by $\gamma \in H(F)$ the image of $\gamma_3$ under the map $H_3 \to H$. By [28] Corollary 2.2, there exists $\delta \in G_{\text{sr}}(F)$ such that $\psi(\delta')$ and $\delta$ are $G(F)$-conjugate. We put

$$T = G_{\delta} = \text{Cent}(\delta, G) \quad \text{and} \quad S = H_{\gamma} = \text{Cent}(\gamma, H).$$

We take $g \in G(F)$ such that

$$\psi(\delta') = g\delta g^{-1}.$$

It is easily seen that

$$[\sigma \mapsto g^{-1} z_\sigma \sigma(g)] \in Z^1(F, T)$$

and the class of this element in $H^1(F, T)$ is independent of the choice of $g$. We denote this class by $\text{inv}(\delta, \delta')$. Since $\text{Ad}(g^{-1} z_\sigma \sigma(g))$ is identity on $T$, we see that $\psi \circ \text{Ad}(g)$ gives an $F$-isomorphism

$$T = \text{Cent}(\delta, G) \xrightarrow{\psi} T' = \text{Cent}(\delta', G').$$

This implies that $\gamma_3$ and $\delta$ are related. Hence, there exists a unique admissible embedding

$$\phi_{\gamma, \delta}: S \xrightarrow{\psi} T \subset G$$

such that $\phi_{\gamma, \delta}(\gamma) = \delta$. We denote by $s_{\gamma, \delta} \in T^\Gamma$ the image of $s$ under the map

$$Z(H)^\Gamma \hookrightarrow \tilde{S}^\Gamma \to T^\Gamma,$$

where the last map is induced by $\phi_{\gamma, \delta}^{-1}$. Let

$$\langle \cdot, \cdot \rangle: H^1(F, T) \times \pi_0(T^\Gamma) \to \mathbb{C}^\times$$

be the Tate–Nakayama pairing. Then we define $\Delta[e, \delta, \psi, z, w]$ by

$$\Delta[e, \delta, \psi, z, w](\gamma_3, \delta') = \Delta[e, \delta, w](\gamma_3, \delta) \cdot \langle \text{inv}(\delta, \delta'), s_{\gamma, \delta} \rangle^{-1}.$$
Irr(π₀(Sφ)) such that the diagram
\[
\begin{array}{ccc}
\Pi_\phi & \xrightarrow{1_w} & \text{Irr}(\pi_0(S_\phi)) \\
\downarrow & & \downarrow \\
H^1(F,G) & \longrightarrow & \pi_0(Z(G)^\Gamma)^D
\end{array}
\]
is commutative, where the bottom arrow is the Kottwitz map, the right arrow sends each irreducible representation to (the restriction of) its central character, and the left arrow is given by (G', ψ, z, π') → [z]. Also, we expect that \(\Pi_\phi\) contains a unique element ˙\(\pi\) = (G, id, 1, π) such that π is \(\mathfrak{w}\)-generic and that \(\iota_\mathfrak{w}(\hat{\pi})\) is the trivial character of \(\pi_0(S_\phi)\).

Given ˙\(\pi\) = (G', ψ, z, π') ∈ \(\Pi_\phi\), we write \(\langle s, \hat{\pi}\rangle_\mathfrak{w} = \text{tr}(\iota_\mathfrak{w}(\hat{\pi})(s))\) for \(s \in \pi_0(S_\phi)\). We expect that for a fixed pure inner twist \((\psi, z)\): G → G', the virtual character
\[
S\Theta_{\phi, m, \psi, z} = e(G') \sum_{\hat{\pi} \in \Pi_{\phi}} \langle 1, \hat{\pi}\rangle_\mathfrak{w} \cdot \Theta_{\hat{\pi}}
\]
is a stable function on \(G'(F)\) and is independent of \(\mathfrak{w}\), where \(e(G') \in \{±1\}\) is the sign defined in [30]. It is known that \(e(G') = 1\) if \(G'\) is quasi-split. For any semi-simple element \(s \in S_\phi\), we put
\[
\Theta_{\phi, m, \psi, z} = e(G') \sum_{\hat{\pi} \in \Pi_{\phi}} \langle s, \hat{\pi}\rangle_\mathfrak{w} \cdot \Theta_{\hat{\pi}}
\]

Let \(c = (H, \mathcal{H}, s, \eta)\) be an endoscopic datum associated to \(s \in S_\phi\). For simplicity, we assume that \(\mathcal{H} = \mathcal{L}H\). Then we may take the z-pair \(z = (H_1, \eta_1) = (H, \text{id})\) for \(c\). Let \(\phi_1\) be the \(L\)-parameter for \(H_1 = H\) given by \(\phi\). Let \(f'\) and \(f\) be smoothly compactly supported functions on \(H(F)\) and \(G'(F)\), respectively. For \(\gamma \in H_{sr}(F)\) and \(\delta' \in G'_{sr}(F)\), we put
\[
O_{\gamma}(f') = \int_{H_1(F) \backslash H(F)} f'(h^{-1}\gamma h)dh, \\
O_{\delta'}(f) = \int_{G'_1(F) \backslash G'(F)} f(g^{-1}\delta' g)dg,
\]
where we put \(H_1 = \text{Cent}(\gamma, H)\) and \(G'_1 = \text{Cent}(\delta', G')\). We also define
\[
SO_{\gamma}(f') = \sum_{\gamma'} O_{\gamma'}(f'),
\]
where \(\gamma'\) runs over a set of representatives for the \(H(F)\)-conjugacy classes in the \(H(\mathcal{F})\)-conjugacy class of \(\gamma\). More precisely, see [32 §5.5]. We say that \(f\) and \(f'\) have \(\Delta[c, z, \psi, m]\)-matching orbital integrals if
\[
SO_{\gamma}(f') = \sum_{\delta'} \Delta[c, z, \psi, m](\gamma, \delta') \cdot O_{\delta'}(f)
\]
for any \(\gamma \in H_{G, sr}\). Here, the sum is taken over a set of representatives for the \(G'(F)\)-conjugacy classes of \(\delta' \in G'(F)\) such that \(\delta'\) and \(\gamma\) are related. Finally, if \(f'\) and \(f\) have \(\Delta[c, z, \psi, m]\)-matching orbital integrals, then we expect that
\[
S\Theta_{\phi_1, \text{id}, 1}(f') = \Theta_{\phi_1, m, \psi, z}(f).
\]

A.4. Changing base points. Let \((\psi, z): G → G'\) be a pure inner twist. In contrast to inner twists, it may occur that \(G'\) is also quasi-split over \(F\) even if \((\psi, z)\) is non-trivial. Then we have the following:

**Lemma A.1.** There exists a bijection
\[
J_{\psi, z}: \Pi_{\text{temp}}(G) → \Pi_{\text{temp}}(G'),
\]
\((G_1, \psi_1, z_1, \pi_1) → (G_1, \psi_1 ° \psi^{-1}, \psi(z_1 z^{-1}), \pi_1)\).

**Proof.** Easy. \(\square\)
We denote the center of $G$ by $Z$. Let $(\psi', z') : G \to G'$ be a pure inner twist such that both $G$ and $G'$ are quasi-split. Then the cohomology class $[z'] \in H^1(F, G)$ is killed under the map
$$H^1(F, G) \to H^1(F, G_{ad}).$$

Hence there exists $z \in Z^1(F, Z)$ such that $[z] = [z']$ in $H^1(F, G)$. Then $(\psi', z') : G \to G'$ is isomorphic to the pure inner twist
$$(id, z) : G \to G.$$

In particular, we see that $G$ and $G'$ are isomorphic over $F$, and so that we may identify $LG$ with $L G'$. Let $\phi : WD_F \to LG$ be a tempered $L$-parameter of $G$. We may regard $\phi$ as an $L$-parameter of $G'$. We denote this $L$-parameter of $G'$ by $\phi'$, i.e.,
$$\phi : WD_F \overset{\phi}{\longrightarrow} LG = L G'.$$

Then we should obtain two finite subsets $\Pi_\phi \subset \Pi_{\text{temp}}(G)$ and $\Pi_{\phi'} \subset \Pi_{\text{temp}}(G')$.

**Question A.2.**

(1) Does $J_{\psi, z}(\Pi_\phi)$ coincide with $\Pi_{\phi'}$?

(2) If so, what is the map
$$\text{Irr}(\pi_0(S_\phi)) \overset{\iota^{-1}}{\longrightarrow} \Pi_\phi \overset{J_{\psi, z}}{\longrightarrow} \Pi_{\phi'} \overset{\iota_{\psi, z}}{\longrightarrow} \text{Irr}(\pi_0(S_{\phi'})) = \text{Irr}(\pi_0(S_\phi))$$

for Whittaker data $w$ and $w'$ of $G$ and $G'$, respectively?

In this subsection, we give an answer of this question when $z \in Z^1(F, Z)$ and $\psi = \text{id}$.

**Proposition A.3.** Fix $z \in Z^1(F, Z)$. Let $\iota = (\mathcal{H}, \mathcal{H}, s, \eta)$ be an endoscopic datum, $\zeta = (H_1, \eta_1)$ be a $z$-pair for $\mathcal{H}$ and $w$ be a Whittaker datum of $G$. For a pure inner twist $(\psi_1, z_1) : G \to G_1$, we put $\Delta_1 = \Delta[\iota, \zeta, \psi_1, z_1, w]$ and $\Delta_1' = \Delta[\iota, \zeta, \psi_1, z_1, z^{-1}, w]$. Then there exists $\alpha = \alpha[\iota, \zeta, \psi_1, z_1, z, w] \in \mathbb{C}^\times$ such that
$$\Delta_1'(\gamma_\zeta, \delta_1) = \alpha \cdot \Delta_1(\gamma_\zeta, \delta_1).$$

for $\gamma_\zeta \in H_{\zeta, G_{ad}}$ and $\delta_1 \in G_{1, \text{sr}}$.

**Proof.** Since the relative transfer factor $\Delta[\iota, \zeta, \psi_1]$ of $G_1$ is independent of both $z_1$ and $z_1 z^{-1}$, we have
$$\frac{\Delta_1'(\gamma_\zeta, \delta_1)}{\Delta_1'(\gamma_\zeta', \delta_1')} = \Delta[\iota, \zeta, \psi_1](\gamma_\zeta, \delta_1, \gamma_\zeta', \delta_1') = \frac{\Delta_1(\gamma_\zeta, \delta_1)}{\Delta_1(\gamma_\zeta', \delta_1')}$$

for any two pairs $(\gamma_\zeta, \delta_1)$ and $(\gamma_\zeta', \delta_1')$ of related elements. Hence the quotient
$$\alpha = \frac{\Delta_1'(\gamma_\zeta, \delta_1)}{\Delta_1(\gamma_\zeta, \delta_1)}$$

does not depend on the choice of a pair $(\gamma_\zeta, \delta_1)$ of related elements. This satisfies the desired equations. $\square$

We also denote $J_\zeta = J_{\psi, z} : \Pi_{\text{temp}}(G) \to \Pi_{\text{temp}}(G)$.

**Corollary A.4.** Let $z \in Z^1(F, Z)$. Then for a tempered $L$-parameter $\phi$ of $G$, we have
$$J_{\zeta}(\Pi_\phi) = \Pi_{\phi}.$$

**Proof.** We take $\iota = (G, LG, 1, \text{id})$ and $\zeta = (G, \text{id})$. Let $(G_1, \psi_1, z_1)$ be a pure inner twist of $G$, and $\Delta_1$, $\Delta_1'$ and $\alpha$ as in (the proof of) the above lemma. Then we see that $f^\iota$ and $f$ have $\Delta_1$-matching orbital integrals if and only if $f^\iota$ and $\alpha \cdot f$ have $\Delta_1$-matching orbital integrals. Hence we should have
$$S \Theta_{\phi, \psi_1, z_1, z^{-1}}(f) = S \Theta_{\phi, \psi_1, z_1}(\alpha \cdot f).$$

Since $\{ \Theta_{\pi_1} \mid \pi_1 \in \text{Irr}(G_1(F)) \}$ is linearly independent, we see that $(G_1, \psi_1, z_1, \pi_1) \in \Pi_\phi$ if and only if $(G_1, \psi_1, z_1 z^{-1}, \pi_1) \in \Pi_\phi$. $\square$
Let $s$ be a semi-simple element in $S_\phi$. We take an endoscopic datum $\epsilon = (H, \mathcal{H}, s, \eta)$ associated to $s$. Note that $s \in Z(\hat{H})^\Gamma$. Let $j = (H_\epsilon, \eta_3)$ be a $\epsilon$-pair for $\epsilon$. For a pair $(\gamma, \delta) \in H_{j,G_{sr}} \times G_{sr}$ of related elements, we denote the image of $\gamma_j$ under the map $H_j \rightarrow H$ by $\gamma$, and we put $S = \operatorname{Cent}(\gamma, H)$ and $T = \operatorname{Cent}(\delta, G)$. Then there exists a unique admissible embedding $\xi : S \rightarrow T \subset G$ such that $\xi(\gamma) = \delta$. The isomorphism $\xi^{-1}$ gives a map

$$Z(\hat{H})^\Gamma \rightarrow \widehat{S^\Gamma} \rightarrow \widehat{T^\Gamma},$$

We denote the image of $s$ under this map by $s_{\gamma, \delta}$. For $z \in Z^1(F, Z)$, we consider

$$\langle [z], s_{\gamma, \delta} \rangle,$$

where $(\cdot, \cdot) : H^1(F, T) \times \pi_0(\hat{T}^\Gamma) \rightarrow \mathbb{C}^\times$ is the Tate-Nakayama pairing, and $[z]$ is the image of $z$ under the map $Z^1(F, Z) \rightarrow H^1(F, Z) \rightarrow H^1(F, T)$.

**Theorem A.5.** The value $\langle [z], s_{\gamma, \delta} \rangle$ does not depend on the choice of a pair $(\gamma, \delta) \in H_{j,G_{sr}} \times G_{sr}$ of related elements. The map $s \mapsto \langle [z], s_{\gamma, \delta} \rangle$ gives a character $\chi_z$ of $\pi_0(S_\phi)$, and the diagram

$$\begin{array}{ccc}
\Pi_\phi & \overset{i}{\longrightarrow} & \operatorname{Irr}(\pi_0(S_\phi)) \\
\downarrow J_s & & \downarrow \otimes \chi_z \\
\Pi_\phi & \overset{i}{\longrightarrow} & \operatorname{Irr}(\pi_0(S_\phi))
\end{array}$$

is commutative.

**Proof.** As in Proposition A.3 for a pure inner twist $(\psi_1, z_1) : G \rightarrow G_1$, we put $\Delta_1 = \Delta(\epsilon, j, \psi_1, z_1, w)$ and $\Delta'_1 = \Delta(\epsilon, j, \psi_1, z_1, w)$. Let $(\gamma, \delta, \eta_1) \in H_{j,G_{sr}} \times G_{sr}$ be a pair of related elements. We take $\delta \in G(F)$ and $g \in G(F)$ such that $\psi_1^{-1}(\delta_1) = g^\delta g^{-1}$. We define $\operatorname{inv}(\delta, \delta_1)$ and $\operatorname{inv}'(\delta, \delta_1)$ in $H^1(F, T)$ by

$$\begin{align*}
\operatorname{inv}(\delta, \delta_1) &= [\sigma \mapsto g^{-1} z_1^1 \sigma (g)], \\
\operatorname{inv}'(\delta, \delta_1) &= [\sigma \mapsto g^{-1} z_1^1 \sigma^{-1} \sigma (g)].
\end{align*}$$

Then we have

$$\begin{align*}
\Delta_1(\gamma, \delta_1) &= \Delta(\gamma, \delta) \cdot \operatorname{inv}(\delta, \delta_1, s_{\gamma, \delta})^{-1}, \\
\Delta'_1(\gamma, \delta_1) &= \Delta(\gamma, \delta) \cdot \operatorname{inv}'(\delta, \delta_1, s_{\gamma, \delta})^{-1}.
\end{align*}$$

Hence we have

$$\alpha = \frac{\Delta'_1(\gamma, \delta_1)}{\Delta_1(\gamma, \delta_1)} = \frac{\operatorname{inv}(\delta, \delta_1, s_{\gamma, \delta})}{\operatorname{inv}'(\delta, \delta_1, s_{\gamma, \delta})} = \langle [z], s_{\gamma, \delta} \rangle.$$ 

Since $\alpha$ does not depend on the choice of $(\gamma, \delta_1)$, we see that $\langle [z], s_{\gamma, \delta} \rangle$ is independent of the choice of $(\gamma, \delta)$. We put $\chi_z(s) = \langle [z], s_{\gamma, \delta} \rangle$.

Since we should have

$$\Theta_{\phi, m, \phi, \psi, z, \pi_1}(f) = \Theta_{\phi, m, \psi, z_1}(\chi_z(s) \cdot f)$$

if $f^\phi$ and $f$ have $\Delta'_1$-matching orbital integrals, we see that

$$\langle s, J_s(\pi_1) \rangle_m = \chi_z(s) \cdot \langle s, \pi_1 \rangle_m$$

for $\pi_1 = (G_1, \psi_1, z_1, \pi_1) \in \Pi_\phi$. In particular, taking the unique element $\pi_1 = (G, \text{id}, 1, \pi)$ such that $\pi$ is $w$-generic, we see that $\chi_z$ is the character of an irreducible representation of $\pi_0(S_\phi)$. Since $\chi_z(1) = 1$, this representation has dimension 1, i.e., $\chi_z : \pi_0(S_\phi) \rightarrow \mathbb{C}^\times$ is a (1-dimensional) character. \qed

The character $\chi_z$ is also described as follows. Let $s \in S_\phi$ be a semi-simple element. We take an endoscopic datum $(H, \mathcal{H}, s, \eta)$ associated to $s$. Let $S \subset H$ and $T \subset G$ be maximal $F$-tori and $\eta^* : S \rightarrow T$ be the isomorphism given by fixed pairs of $G, \hat{G}, H$ and $\hat{H}$. Note that $s \in Z(\hat{H}^\Gamma) \subset \hat{S}^\Gamma$. We take $g \in G_{\text{sr}}(F)$ such that $\xi = \operatorname{Ad}(g^{-1}) \circ \eta^* : S \rightarrow g^{-1} T g$ is an admissible embedding. Then $Z \subset T$, and $\operatorname{Ad}(g^{-1})$ fixes all elements in $Z$. For $z \in Z^1(F, Z)$, we see that $\xi^{-1}(z) = (\eta^*)^{-1}(z) \in Z^1(F, \xi^{-1}(Z)) \subset Z^1(F, S)$. Then we have $\chi_z(s) = \langle (\eta^*)^{-1}(z), s \rangle$, where $\langle \cdot, \cdot \rangle : H^1(F, S) \times \pi_0(\hat{S}^\Gamma) \rightarrow \mathbb{C}^\times$ is the Tate-Nakayama pairing.
In particular, we have
\[
\iota_w \circ z = \iota_{w'}.
\]

In particular, we have \( \iota_w(\pi) = \iota_w(\tilde{\pi}) \otimes \chi_z \) for \( \tilde{\pi} \in \Pi_\phi \).

Proof. There is a similar result in [27]. According to Theorem 3.3 (or Lemma 3.2) in [27], there exists a Whittaker data of \( G \) then we have

\[
Hence we have \( \chi_z = (w,w')^{-1} \) and so that this is an isomorphism

\[
\begin{align*}
1 = \iota_w(J_z(\pi)) = \iota_{w'}(\pi) \otimes \chi_z = \iota_{w}(\pi) \otimes (w,w') \otimes \chi_z = (w,w') \otimes \chi_z.
\end{align*}
\]

Hence we have \( \chi_z = (w,w')^{-1} \), and so that

\[
\iota_w \circ J_z(\pi) = \iota_{w'}(\pi) \otimes \chi_z = \iota_{w}(\pi) \otimes (w,w')^{-1} = \iota_{w}(\pi)
\]

for any \( \pi \in \Pi_\phi \), as desired. \(\square\)

A.5. Examples. We calculate \( \chi_z \) for \( G = \text{SO}(V) \) with \( \dim(V) \in 2\mathbb{Z} \) and for \( G' = \text{Sp}(W) \). As an application, we prove Proposition 3.1 and 3.2. We fix a non-trivial additive character \( \psi_F : F \to \mathbb{C}^\times \).

First, we prepare a certain property of the Galois cohomology. For \( d \in F^\times \), we put \( E_d = F(\sqrt{d}) \). Let

\[
T = E_d^1 := \{(a,b) \in \mathbb{G}_m^2 \mid a^2 - b^2d = 1\}
\]

be a torus over \( F \). Note that \( T \cong \text{GL}_1 \) over \( E_d \). For \( \sigma \in \Gamma \), we denote the usual action on \( \text{GL}_1(T) \) by \( x \mapsto \sigma(x) \), and the action on \( T(F) \) by \( x \mapsto \sigma_d(x) \). If \( d \not\in F^{\times 2} \), then we have \( H^1(F,T) \cong \{\pm 1\} \) by Kottwitz’s isomorphism.

Lemma A.7. Let \( c \in F^\times \). We define \( z_{c,\sigma} \in E_d^\times = T(F) \) by

\[
z_{c,\sigma} = \frac{\sigma(\sqrt{c})}{\sqrt{c}} = \begin{cases} 1 & \text{if } \sigma|_{E_d} = \text{id}_{E_d}, \\ -1 & \text{otherwise}. \end{cases}
\]

Then the map \( z_c : \Gamma \ni \sigma \mapsto z_{c,\sigma} \in T(F) \) belongs to \( Z^1(F,T) \). Moreover, the map \( H^1(F,T) \ni \{\pm 1\} \) is given by

\[
[z_c] \mapsto (c,d),
\]

where \( \langle \cdot, \cdot \rangle \) is the quadratic Hilbert symbol of \( F \).

Proof. The first assertion is clear. To prove the last assertion, we may assume that \( d \not\in F^{\times 2} \). We define \( z'_{c,\sigma} \in E_d^\times = T(F) \) by

\[
z'_{c,\sigma} = \begin{cases} 1 & \text{if } \sigma|_{E_d} = \text{id}_{E_d}, \\ c & \text{otherwise}. \end{cases}
\]

It is easily seen that \( z'_c \in Z^1(F,T) \). Moreover we have

\[
z_{c,\sigma} z_{c,\sigma}^{-1} = \sigma_d(\sqrt{c}) \cdot \sqrt{c}^{-1}.
\]

Hence we have \( [z_c] = [z'_c] \) in \( H^1(F,T) \).

There exists an exact sequence

\[
1 \longrightarrow H^1(\text{Gal}(E_d/F), T(E_d)) \longrightarrow H^1(F,T) \longrightarrow H^1(E_d,T)_{\text{Gal}(E_d/F)}.
\]
Since $T$ is isomorphic to $\text{GL}_1$ over $E_d$, by Hilbert 90, we have $H^1(E_d, T) = 1$. Therefore we have

$$H^1(\text{Gal}(E_d/F), T(E_d)) \cong H^1(F, T).$$

Let $\sigma$ be the generator of $\text{Gal}(E_d/F)$. Then we have

$$\sigma_d(x) \cdot x^{-1} = \sigma(x)^{-1}x^{-1} = N_{E_d/F}(x)^{-1}$$

for $x \in T(E_d) \cong E_d^\times$. Hence we have $B^1(\text{Gal}(E_d/F), T(E_d)) \cong N_{E_d/F}(E_d^\times)$. Put $z_1 = 1$ and $z_\sigma = x$ for $x \in T(E_d) = E_d^\times$. Then $z \in Z^1(\text{Gal}(E_d/F), T(E_d))$ if and only if

$$1 = z_1 = z_{\sigma_\sigma} = z_\sigma \cdot \sigma_d(z_\sigma) = x \cdot \sigma_d(x) = x \cdot \sigma(x)^{-1},$$

i.e., $x \in F^\times$. Therefore we have

$$H^1(\text{Gal}(E_d/F), T(E_d)) \cong F^\times/N_{E_d/F}(E_d^\times).$$

Moreover we see that

$$[z_\sigma'] \neq 1 \text{ in } H^1(\text{Gal}(E_d/F), T(E_d)) \iff c \not\in N_{E_d/F}(E_d^\times) \iff (c, d) = -1$$

as desired. □

For a positive integer $n$ and $d, c \in F^\times$, we let $V = V_{2n, d, c}$ be an orthogonal space with dimension $2n$ and type $(d, c)$, and $W = W_{2n}$ be a symplectic space with dimension $2n$. We put $G_{n, d, c} = \text{SO}(V_{2n, d, c})$ and $G'_n = \text{Sp}(W_{2n})$. As in [2.2] we decompose

$$V_{2n, d, c} = X_{n-1, d, c} \oplus V_{2, d, c} \oplus X_{n-1, d, c}^*$$

with

$$X_{n-1, d, c} = Fv_{1, d, c} \oplus \cdots \oplus Fv_{n-1, d, c}, \quad X_{n-1, d, c}^* = Fv_{1, d, c}^* \oplus \cdots \oplus Fv_{n-1, d, c}^*, \quad V_{2, d, c} = Fe_{d, c} \oplus F'e_{d, c}^{'},$$

where

$$\langle e_{d, c}, e_{d, c} \rangle_{V_{2n, d, c}} = 2c, \quad \langle e_{d, c}^{'}, e_{d, c}^{'}, \rangle_{V_{2n, d, c}} = -2cd, \quad \langle e_{d, c}, e_{d, c}^{'}, \rangle_{V_{2n, d, c}} = 0.$$

For $c, c' \in F^\times$, we define an isomorphism of vector spaces

$$f : V_{2n, d, c} \rightarrow V_{2n, d, c'}$$

by

$$V_{2n, d, c} \ni X_{n-1, d, c} \ni v_{i, d, c} \mapsto v_{i, d, c'} \in X_{n-1, d, c'} \subset V_{2n, d, c'}, \quad V_{2n, d, c} \ni v_{i, d, c'} \mapsto e \in V_{2n, d, c} \subset V_{2n, d, c'}, \quad V_{2n, d, c} \ni v_{i, d, c'} \mapsto e' \in V_{2n, d, c'} \subset V_{2n, d, c'}, \quad V_{2n, d, c} \ni X_{n-1, d, c}^* \ni v_{i, d, c}^* \mapsto c' \cdot v_{i, d, c'}^* \in X_{n-1, d, c'}^* \subset V_{2n, d, c'}.$$

This satisfies

$$\langle f(x), f(y) \rangle_{V_{2n, d, c'}} = \frac{c'}{c}(x, y)_{V_{2n, d, c}}$$

for any $x, y \in V_{2n, d, c}$. Hence $f$ gives an isomorphism $\psi_{c'/c} : G_{2n, d, c} \rightarrow G_{2n, d, c'}$ over $F$. Let $Z_{2n, d, c}$ be the center of $G_{2n, d, c}$ and define $z_{c'/c} \in Z^1(F, Z_{2n, d, c})$ by

$$z_{c'/c} = \frac{\sigma(\sqrt{c'/c})}{\sqrt{c'/c}} \cdot 1.$$

Then $(\psi, z) = (\psi_{c'/c}, z_{c'/c}) : G_{2n, d, c} \rightarrow G_{2n, d, c'}$ be a pure inner twist of $G_{2n, d, c}$. This is trivial if and only if $c'/c \in N_{E_d/F}(E_d^\times)$, via this pure inner twist, we regard $G_{2n, d, c'}$ as a pure inner form of $G_{2n, d, c}$. If $c'/c \not\in N_{E_d/F}(E_d^\times)$, then we identify $\Pi_{\text{temp}}(G_{2n, d, c})$ with $\Pi_{\text{temp}}(G_{2n, d, c'})$ via $J_{\psi, z}$ in Lemma A.1.

Let $B = TU$ (resp. $B' = T'U'$) be the Borel subgroup of $G_{2n, d, c}$ (resp. $G'_{2n, d, c}$) and $\mu_c = (B, \mu_c)$ (resp. $\mu'_c = (B', \mu'_c)$) be the Whittaker datum of $G_{2n, d, c}$ (resp. $G'_{2n, d, c}$) defined in [2]. Via the isomorphism $\psi_{c'/c} : G_{2n, d, c} \rightarrow G_{2n, d, c'}$, the datum $\mu_c$ is transferred to $\mu'_{c'}$. 
Recall that an $L$-parameter $\phi$ of $G_{2n,d,c}$ (resp. $G'_{2n}$) gives an orthogonal representation

$$\phi: WD_F \to O(2n+2, \mathbb{C}) \quad (\text{resp. } \phi: WD_F \to SO(2n+1, \mathbb{C}))$$

with $\det(\phi) = \chi_{V_{2n,d,c}}$ (resp. $\det(\phi) = 1$). For a semi-simple element $s \in S_\phi$, we denote the $(-1)$-eigenspace of $s$ in $\phi$ by $\phi^s$. This is a subrepresentation of $\phi$. We put $\eta_{c_0}(s) = \det(\phi^s)(c_0)$ for $c_0 \in F^\times$. This gives a character $\eta_{c_0}$ of $\pi_0(S_\phi)$. Now we prove Proposition 3.1 and 3.2. More precisely, we prove the following proposition.

**Proposition A.8.** Let $G = G_{2n,d,c}$ or $G = G'_{2n}$, and $Z \cong \{\pm 1\}$ be the center of $G$. For $c_0 \in F^\times$, we define $z_{c_0} \in Z^1(F, Z)$ by $z_{c_0, \sigma} = \sigma(\sqrt{c_0})/\sqrt{c_0}$. Then we have $\chi_{z_{c_0}} = \eta_{c_0}$. Hence for a tempered $L$-parameter $\phi$ of $G$, the diagram

$$\begin{array}{ccc}
\Pi_\phi \overset{\iota_{\text{m.c.}}}{\longrightarrow} & Irr(\pi_0(S_\phi)) \\
\downarrow & \downarrow \circ \eta_{c'/c} \\
\Pi_\phi \overset{\iota_{\text{m.c.}}}{\longrightarrow} & Irr(\pi_0(S_\phi))
\end{array}$$

is commutative. Here, by the pure inner twist

$$G_{2n,d,c} \overset{(\text{id}, z_{c'/c})}{\longrightarrow} G_{2n,d,c} \overset{(\psi_{c'/c}, \iota_{\text{c'/c}})}{\longrightarrow} G_{2n,d,c'},$$

we regard $G_{2n,d,c'}$ as a pure inner form of $G_{2n,d,c}$.

**Proof.** We may assume that $\phi = \phi_1 \oplus \phi_2$, where $\phi_1$ and $\phi_2$ are orthogonal representations with $\dim(\phi_1) = 2k$ and $\dim(\phi_2) = \dim(\phi) - 2k$, and $s \in S_\phi$ satisfies $s|\phi_1 = -1$ and $s|\phi_2 = 1$. Then $\phi^s = \phi_1$. Let $(H, \mathcal{H}, s, \eta)$ be an endoscopic datum associated to $s$ such that

$$H = \begin{cases}
G_{2k,d_1,1} \times G_{2(n-k),d_1,1} & \text{if } G = G_{2n,d,c}, \\
G_{2k,d_1,1} \times G'_{2(n-k)} & \text{if } G = G'_{2n},
\end{cases}$$

where $d_1 \in F^\times$ satisfies that $\det(\phi_1) = (-1, d_1)$. Then we can take a maximal $F$-torus $S$ of $H$ which is isomorphic to

$$\begin{cases}
(GL^{k-1}_1 \times E_{d_1}^1) \times (GL^{n-k-1}_1 \times E_{d/d_1}^1) & \text{if } G = G_{2n,d,c}, \\
(GL^{k-1}_1 \times E_{d_1}^1) \times GL^{n-k}_1 & \text{if } G = G'_{2n},
\end{cases}$$

over $F$. This implies that

$$H^1(F, S) \cong \begin{cases}
H^1(F, E_{d_1}^1) \times H^1(F, E_{d/d_1}^1) \hookrightarrow \{\pm 1\} \times \{\pm 1\} & \text{if } G = G_{2n,d,c}, \\
H^1(F, E_{d_1}^1) \hookrightarrow \{\pm 1\} & \text{if } G = G'_{2n}.
\end{cases}$$

Via this isomorphism, (the image of) $[z_{c_0}]$ corresponds to $((c_0, d_1), (c_0, d/d_1))$ if $G = G_{2n,d,c}$, and to $(c_0, d_1)$ if $G = G'_{2n}$.

On the other hand, via the above isomorphism of $S$, we have

$$\pi_0(S^T) \cong \begin{cases}
\pi_0(E_{d_1}^1)^\Gamma \times \pi_0(E_{d/d_1}^1)^\Gamma & \text{if } G = G_{2n,d,c}, \\
\pi_0(E_{d_1}^1)^\Gamma & \text{if } G = G'_{2n}.
\end{cases}$$

Via this isomorphism, $s$ corresponds to the generator of $\pi_0(E_{d_1}^1)^\Gamma \subset \{\pm 1\}$. Therefore we have

$$\chi_{z_{c_0}}(s) = ([z_{c_0}], s) = (c_0, d_1) = \det(\phi_1)(c_0) = \eta_{c_0}(s).$$

First, we assume that $z_{c'/c} \neq 1$ in $H^1(F, G)$. In particular, $G = G_{2n,d,c}$ and $c'/c \notin N_{E_{d/F}}(E_d^1)$. By Theorem A.5 the diagram

$$\begin{array}{ccc}
\Pi_\phi \overset{\iota_{\text{m.c.}}}{\longrightarrow} & Irr(\pi_0(S_\phi)) \\
J_{0, s} \downarrow & \downarrow \circ \eta_{c'/c} \\
\Pi_\phi \overset{\iota_{\text{m.c.}}}{\longrightarrow} & Irr(\pi_0(S_\phi))
\end{array}$$


is commutative, where we put $(ψ, z) = (ψ_{e'/c}, z_{e'/c})$. Note that we have identified $Π_ϕ \subset Π_{\text{temp}}(G_{2n,d,c})$ with $Π_ϕ \subset Π_{\text{temp}}(G_{2n,d,c})'$ via $J_{ϕ,c}$.

Next, we assume that $[z_{c0}] = 1$ in $H^1(F, G)$, where we put $c_0 = e'/c$. Let $T$ be the maximal $F$-torus of $G$ stabilizing the lines $Fv_i$ or $Fw_i$ for any $i$. If $G = G_{2n}'$, then $T \cong \text{GL}_n^1$. Putting $t_{c_0} = (\sqrt{c_0}^{-1}, \ldots, \sqrt{c_0}^{-1}) \in T(F) < G(F)$ and let $g_{c_0} \in G_{\text{ad}}(F)$ be the image of $t_{c_0}$. Then we have $z_{c_0} = t_{c_0}^{-1} \cdot (g_{c_0})$, i.e., $[z_{c_0}] \in H^1(F, Z)$ is the image of $g_{c_0}$. We have $g_{c_0}B g_{c_0}^{-1} = B'$ and $µ_c \circ \text{Ad}(g_{c_0}^{-1}) = µ_{c_0}$. Hence we have $t_{w,c} \circ J_{z_{c_0}} = t_{w,c_{0}}$ by Proposition A.6.

If $G = G_{2n,d,c}$, then $T \cong \text{GL}_n^1 \times E_d^1$. We have $[z_{c_0}] = 1$ in $H^1(F, G)$ if and only if $c_0 \in N_{E_d/F}(E_d^1)$. Take $α_0, β_0 \in F$ such that $N_{E_d/F}(α_0 + β_0\sqrt{d}) = α_0^2 - β_0^2d = c_0$. We define $t_{c_0} \in T(F)$ by $t_{c_0}v_i = \sqrt{c_0}^{-1}v_i$ and

$$t_{c_0}e = \frac{α_0}{\sqrt{c_0}}e + \frac{β_0}{\sqrt{c_0}}e',$$

$$t_{c_0}e' = \frac{β_0}{\sqrt{c_0}}de + \frac{α_0}{\sqrt{c_0}}e'.$$

Note that $t_0 = \sqrt{c_0}^{-1}f$, where $f : V_{2n,d,c} \to V_{2n,d,c}' = V_{2n,d,c}$ is the isomorphism defined as above. Let $g_{c_0} \in G_{\text{ad}}(F)$ be the image of $t_{c_0}$. Then we have $z_{c_0} = t_{c_0}^{-1} \cdot (g_{c_0})$, i.e., $[z_{c_0}] \in H^1(F, Z)$ is the image of $g_{c_0}$. We have $g_{c_0}B g_{c_0}^{-1} = B$ and $µ_c \circ \text{Ad}(g_{c_0}^{-1}) = µ_{c_0}$. Hence we have $t_{w,c} \circ J_{z_{c_0}} = t_{w,c_{0}}$ by Proposition A.6.

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