On the runaway instability of self-gravitating torus around black holes

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Abstract. Black holes surrounded by self-gravitating tori are astrophysical systems which may naturally form following the core collapse of a massive star or the merger of two neutron stars. We present here results from fully general relativistic numerical simulations of such systems in order to assess the influence of the torus self-gravity on the onset of the so-called runaway instability. This instability, which might drive the rapid accretion of the disk on shorter timescales than those required to power a relativistic fireball, potentially challenges current models of gamma-ray bursts. Our simulations indicate that the self-gravity of the torus does not actually favour the onset of the instability.

1. Introduction

Relativistic self-gravitating tori orbiting black holes (BHs) are expected to form in a number of scenarios, such as in the merger of a binary system formed by either a BH and a neutron star (NS) or by two NSs (see e.g. [1] and references therein), and in the aftermath of the gravitational collapse of the rotating core of massive stars [2, 3]. Such tori have large average rest-mass densities, highly super-Eddington mass fluxes, and the angular momentum obeys sub-Keplerian distributions. State-of-the-art numerical simulations have recently started to provide quantitative estimates of the viability of such systems to form (see e.g. [4, 1]). As such systems are regarded as the central engine for gamma-ray bursts [5], understanding its formation, stability and dynamics is a stimulating task.

In this respect, the so-called runaway instability [6] may play a significant role as it could destroy the torus on dynamical timescales. In a marginally stable torus, the radial pressure gradient may drive the transfer of mass towards the BH through the cusp-like inner edge of the torus. Accretion increases the BH mass and spin and changes the gravitational field of the system. If the cusp moved deeper into the torus, mass accretion would increase, turning the accretion process runaway unstable. The runaway instability has so far been investigated using various approximations (see e.g. [6, 7, 8, 9]). Studies based on stationary models show that the self-gravity of the disk favours the instability, while there are additional parameters which have a stabilizing effect, namely the rotation of the BH and the radial distribution of specific angular momentum. Time-dependent, general relativistic hydrodynamical (GRHD) axisymmetric simulations of the runaway instability of tori around BHs have been performed.
by [10, 11], treating the dynamics of the gravitational field in an approximative way and neglecting the self-gravity of the torus. Overall, [10, 11] found that tori with constant distribution of specific angular momentum were unstable while non-constant (power law) angular momentum disks were stable.

In this work we present results which extend previous studies by incorporating an important missing ingredient in the modelling - the self-gravity of the torus. These results have also been discussed in [12], upon which the present article is based. In our work we use the initial data for a BH and a self-gravitating torus in the puncture framework derived by [13] and carry out fully relativistic simulations to investigate the onset of the runaway instability.

2. Basic equations

We follow the so-called BSSN formulation of the Einstein equations [14, 15]. Spacetime is hence foliated into a set of non-intersecting spacelike hypersurfaces, leading to the following line element $\text{ds}^2 = -(\alpha^2 - \beta_i\beta^i)\text{dt}^2 + 2\beta_i\text{dx}^i\text{dt} + \gamma_{ij}\text{dx}^i\text{dx}^j$, where $\alpha$, $\beta^i$ and $\gamma_{ij}$ are the lapse function, the shift vector, and the 3-metric, respectively. BSSN makes use of a conformal decomposition of the 3-metric, $\tilde{\gamma}_{ij} = e^{-\gamma}\gamma_{ij}$ and the conformal-traceless extrinsic curvature $\tilde{A}_{ij} = e^{-4\phi}(K_{ij} - \gamma_{ij}K/3)$, with the conformal factor $\phi$ satisfying $e^{4\phi} = \gamma^{1/3} \equiv \det(\gamma_{ij})^{1/3}$. In addition to evolution equations for $\tilde{\gamma}_{ij}$ and $\tilde{A}_{ij}$, there are evolution equations for the conformal factor $\phi$ (or $\chi \equiv e^{-4\phi}$), the trace of the extrinsic curvature $K$, and the “conformal connection functions” $\Gamma^i$ [15]. Moreover, there are two kinematical additional variables left undetermined, $\alpha$ and $\beta^i$. These are computed using the so-called “moving puncture gauge”, a combination of the “1+log” condition [16] for the lapse, and the “Gamma-freezing” condition for the shift vector [17].

The GRHD equations are the local conservation laws of momentum and energy, encoded in the stress-energy tensor $T^{\mu\nu}$, and of the matter density, $J^\mu$ (the continuity equation), i.e., $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu J^\mu = 0$, where $\nabla_\mu$ stands for the 4-dimensional covariant derivative. The density current is given by $J^\mu = \rho u^\mu$, where $u^\mu$ is the 4-velocity of the fluid and $\rho$ its rest-mass density. Following [18] the GRHD equations are written in flux-conservative form in cylindrical coordinates. Since the Einstein equations are solved only in the $y = 0$ plane with Cartesian coordinates, the hydrodynamic equations are rewritten in Cartesian coordinates for $y = 0$. By defining the vector of unknowns, $\mathbf{U}$, fluxes $\mathbf{F}^x$ and $\mathbf{F}^z$ along the $x$ and $z$ directions, and the vector of sources $\mathbf{S}$ the set of hydrodynamic equations can be cast in conservative form as

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}^x + \partial_z \mathbf{F}^z = \mathbf{S}, \quad (1)$$

We refer to [18] for further details on these equations, in particular for the expressions of the various vectors appearing in Eq. (1). To close the system of equations, we choose two possible equations of state, the so-called $\Gamma$-law equation of state (ideal fluid) given by $P = (\Gamma - 1)\rho\epsilon$, where $\epsilon$ is the specific internal energy, and a polytropic equation of state $P = \kappa\rho^\Gamma$. Here $\kappa$ is the polytropic constant, $\Gamma = 1 + 1/N$ and $N$ is the polytropic index.

3. Numerical approach and initial data

The numerical simulations are performed with the nada code [19] which solves the (BSSN) Einstein equations in conjunction with the “cartoon” method and the moving puncture approach. Correspondingly, the GRHD equations are solved with high-resolution shock-capturing schemes employing the HLLE approximate Riemann solver. The time update of the evolution equations is done using a 4th-order Runge-Kutta method of lines. The code has recently been validated against a number of stringent tests in vacuum and non-vacuum spacetimes [19]. For the simulations reported here we use an equally spaced ($x, z$) grid with a grid spacing $\Delta x = \Delta z = 0.05$ and $N_x \times N_z = 600 \times 600$ points to cover a computational domain, $0 \leq x \leq L$ and $0 \leq z \leq L$, with $L = 30$. 


Table 1. Main properties of the equilibrium models in units of \( c = G = M_\odot = 1 \) (unless shown otherwise). From left to right the columns report the name of the model, the type of specific angular momentum distribution, the torus-to-BH mass ratio, the position of the maximum density point \( r_{\text{max}} \), the position of the inner and outer radii of the torus \( r_{\text{in}} \) and \( r_{\text{out}} \), the maximum rest-mass density and the orbital period at the center of the torus \( t_{\text{orb}} \).

| Model | \( j - \text{law} \) | \( M_t/M_{\text{BH}} \) | \( r_{\text{max}} \) | \( r_{\text{in}} \) | \( r_{\text{out}} \) | \( \rho_{\text{max}} \) (g/cm\(^3\)) | \( t_{\text{orb}} \) |
|-------|-------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| E1    | const             | 0.1             | 7.10            | 9.81            | 14.17           | \( 1.233 \times 10^{14} \) | 223.81          |
| M1    | const             | 0.1             | 7.17            | 4.92            | 10.17           | \( 3.189 \times 10^{14} \) | 147.81          |
| M2    | const             | 1.0             | 8.87            | 4.02            | 19.97           | \( 2.202 \times 10^{14} \) | 199.54          |
| M3    | non-const         | 0.1             | 10.47           | 4.92            | 19.97           | \( 3.902 \times 10^{13} \) | 245.37          |
| M4    | non-const         | 0.5             | 10.02           | 4.07            | 19.97           | \( 1.538 \times 10^{14} \) | 229.91          |

The initial data for the numerical simulations of the system formed by a BH and a self-gravitating torus in axisymmetric equilibrium have been computed by [13], to which the reader is addressed for further details. The equilibrium configurations for the matter fields are obtained by assuming a perfect fluid stress-energy tensor, and adopting a polytropic equation of state. Initial configurations can be constructed with either constant or non-constant specific angular momentum distributions, defined as \( j \equiv h u_\phi \), where \( h \) is the enthalpy. For the simulations reported in this work we have considered tori around a Schwarzschild BH (of mass \( M_{\text{BH}} = 1 \)). The adiabatic index is \( \Gamma = 4/3 \) to mimic a degenerate relativistic electron gas, and the polytropic constant \( \kappa \) is chosen such that the torus-to-BH mass ratio, \( M_t/M_{\text{BH}} \), is 0.1, 0.5 or 1, depending on the model. A list of the models along with their main features is given in Table 1.

4. Results

We first test the ability of the code to keep the stationarity of a fluid configuration initially in equilibrium, model E1. This test is performed both in a fixed spacetime, which nevertheless has the contribution from the self-gravity of the torus, and in a fully dynamical spacetime. The accuracy check of the evolution is performed by comparing the stationarity and conservation of different fluid quantities over a timescale which is several times the dynamical one. Results for the fixed spacetime simulation show that at the end of the simulation (\( t = 1000M \)), the difference between the final total rest-mass with respect to the initial value is less than \( 10^{-3} \) while the central rest-mass density only differs by 1\% with the initial one. Correspondingly, for the dynamical spacetime evolution the difference between the final total rest mass with respect to the initial value is less than 0.1\%. Although it is due to accumulated numerical error during the evolution, this violation of the rest mass does not affect in any significant way the stability of the torus for the duration of our simulation. Similar results are obtained for the time evolution of the central rest-mass density and the total angular momentum. These two tests provide reassuring evidence of the ability of the code to keep the torus in equilibrium for evolutions longer than the characteristic dynamical timescales of these objects.

Next, we introduce a perturbation on the (marginally stable) equilibrium models models M1 to M4 to evaluate the appearance of the runaway instability. This is done by perturbing the \( v^r \) component of the 3-velocity of the torus, which drives a small mass transfer through the inner edge of the tori (as was also done in [20]). For each model, the numerical simulations are stopped at \( t \sim 2000 \), which corresponds to \( \sim 10 \) ms (between 8 to 10 orbital periods depending on the model), since after this value the growth of the Hamiltonian constraint violation would lead to a loss of accuracy for the spacetime evolution. Nevertheless, the considered timescale
Figure 1. Colour coded isodensity contours of the perturbed model M2 at four different stages of its evolution, from initial (top-left panel) to final time (bottom-right panel), $t \sim 10t_{\text{orb}}$. Despite the Kelvin-Helmholtz-like dynamics in the disk-external medium boundary the location of the torus remains firmly fixed in space.

would safely allow to identify the runaway instability, if present, since, according to [10, 11], it could even take place within one orbital period for the more massive models M2 and M4.

Figure 1 shows four different snapshots of the evolution of model M2, up to a a final time of $t \sim 10t_{\text{orb}}$ (bottom-right panel). The evolution of this model typifies the characteristic evolution of all models studied. The initial perturbation triggers the accretion of mass and angular momentum through the cusp and on to the BH, leading to a global readjustment of the effective (gravitational and centrifugal) field. This is sufficient to momentarily reduce the amplitude of the mass flux and stop the disk from further in-falling towards the BH. As a new equilibrium is reached the disk departs from the BH leading to a long-lasting cycle of oscillations. Our simulations show that such a quasi-periodic oscillatory behaviour, which had already been found in the test-fluid simulations of non-self-gravitating disks performed by [20], is also present when further physics is incorporated in the numerical modelling, namely self-gravity and fully dynamical spacetime and hydrodynamical evolutions. The colour coded isodensity contours
Figure 2. Left panel: Time evolution of the total rest-mass (top) and central rest-mass density, each of them normalized to its initial value, for the evolution of models M1 and M2. Right panel: Mass accretion rate evolution for models M1 and M2.

displayed in Fig. 1 also show the interesting dynamics at the boundary region separating the disk from the external medium. In particular, Kelvin-Helmholtz-driven eddies are seen being shed downwind from the edge of the disk during each oscillation (this is specially noticeable when these data are visualized in an animation).

To further analyze the system dynamics we plot in Figure 2 time evolutions of global quantities. The left panel of this figure displays the evolution of the normalized total rest-mass and central density for models M1 (solid and dashed lines) and M2 (dotted line). M1 is a $j$-constant model with an initial rest-mass of $M_{t,0} = 0.1M_{BH}$, thus representing a model with torus-to-BH mass ratio in broad agreement with the simulations of NS/NS or BH/NS mergers [1, 4, 21]. For this initial model, we consider two different perturbation amplitudes, $\eta = 0.01$ (solid line) and $\eta = 0.025$ (dashed line). As mentioned before, the perturbation triggers a phase of axisymmetric oscillations of the torus around its equilibrium which are present throughout the simulation. Such oscillations induce a small outflow of matter through the cusp towards the BH but the total rest-mass of the torus is not reduced in a significant way, as inferred from the upper panel of Figure 2. At the end of the simulation ($t \sim 10t_{orb}$) the rest-mass of the torus M1 is conserved up to about 1%. Therefore, the BH mass and spin do not increase notably, the torus showing no sign of the runaway instability. Further information is obtained from the right panel of Figure 2 which shows the time evolution of the mass accretion rate. This rate is larger the larger the initial perturbation is. For $\eta = 0.025$, the initial stage of very small mass transfer through the cusp of the torus is followed by a stage in which the oscillatory behavior of the mass flux, a signature of the induced oscillations, is obvious throughout the numerical evolution. Interestingly, during the oscillation phase, the mass flux does not increase in amplitude with time, as one would expect prior to the onset of the runaway instability (see [10, 11]). Instead, it reaches a maximum of about $\dot{M} \sim 0.1 M_{\odot}/s$ after the second orbital period to later decrease and oscillate around a lower value, never showing any sign of exponential growth. The effects of the self-gravity are of course more important for model M2 for which $M_{t} = M_{BH}$. Despite being a more massive disk the overall dynamics of M2 is very similar to that of the less massive torus, M1 (see dotted lines in Fig. 2). Again, the amplitude
in the evolution of the mass flux does not increase with time and does not lead to the runaway instability. Finally, to check the influence of the rotation law on the runaway instability [10] we also carry out two additional numerical evolutions for two \( j \)-non-constant models, M3 and M4, which have torus-to-BH mass ratios of 0.1 and 0.5 respectively. Despite the difference in the rotation law with respect to models M1 and M2, the dynamics is very similar to that reported in of Figure 2: disks are stable and show no exponential growth of the mass flux for timescales larger than the dynamical timescale.

5. Summary

We have presented results from fully general relativistic numerical simulations of a system formed by a BH surrounded by a self-gravitating torus in equilibrium (see also [19, 12]). Test simulations using fixed spacetime evolutions show that the torus remains in equilibrium around its initial configuration for more than 5 dynamical timescales, when the simulation was stopped. Correspondingly, by simulating the same model in a dynamical spacetime we found that our numerical code nada is able to keep the torus in equilibrium for the several hundred \( M \) that lasted the simulation. In addition, results from numerical simulations of marginally-stable self-gravitating tori, aimed at evaluating the influence of the torus self-gravity on the onset of the runaway instability, have also been presented. These simulations show that all models exhibit a persistent phase of axisymmetric oscillations around their equilibria for several dynamical timescales without the appearance of the runaway instability. As a result, the self-gravity of the torus does not play a critical role favoring the appearance of the runaway instability.

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