FEYNMAN DIAGRAMS AND LAX PAIR EQUATIONS

GABRIEL BĂDIŢOIU AND STEVEN ROSENBERG

Abstract. We find a Lax pair equation corresponding to the Connes-Kreimer Birkhoff factorization of the character group of a Hopf algebra. This flow preserves the locality of counterterms. In particular, we obtain a flow for the character given by Feynman rules, and relate this flow to the Renormalization Group Flow.

1. Introduction

In the theory of integrable systems, many classical mechanical systems are described by a Lax pair equation associated to a coadjoint orbit of a semisimple Lie group, for example via the Adler-Kostant-Symes theorem [1]. Solutions are given by a Birkhoff factorization on the group, and in some cases, this technique extends to loop group formulations of physically interesting systems such as the Toda lattice [10, 16]. By the work of Connes-Kreimer [4], there is a Birkhoff factorization of characters on general Hopf algebras, in particular on the Kreimer Hopf algebra of 1PI Feynman diagrams. In this paper, we reverse the usual procedure in integrable systems: we construct a Lax pair equation \( \frac{dL}{dt} = [L, M] \) on the Lie algebra of infinitesimal characters of the Hopf algebra whose solution is given precisely by the Connes-Kreimer Birkhoff factorization (Theorem 5.9). The Lax pair equation is nontrivial in the sense that it is not an infinitesimal inner automorphism. The main technical issue, that the Lie algebra of infinitesimal characters is not semisimple, is overcome by passing to the double Lie algebra with the simplest possible Lie algebra structure. In particular, the Lax pair equation induces a flow for the character given by Feynman rules in dimensional regularization. This flow has the physical significance that it preserves locality, the independence of the character’s counterterm on the mass parameter.

In §§1-4, we introduce a method to produce a Lax pair on any Lie algebra from equations of motion on the double Lie algebra. In §5 we apply this method to the particular case of the Lie algebra of infinitesimal characters of a Hopf algebra, and prove Theorem 5.9.

The Renormalization Group Flow (RGF) usually considered in quantum field theory is a flow on the character group \( G_A \), while the Lax pair flow is on the corresponding Lie algebra \( g_A \) of infinitesimal characters. There are various bijections from \( g_A \) to \( G_A \), and via these bijections we can compare the Lax pair flow to the RGF. These flows are not the same, so we study how physically significant quantities behave under the Lax pair flow. In
§6, we derive an equation for the flow of the \( \beta \)-function of characters \( \varphi_t \in G_A \) associated to the Lax pair flow via the exponential map \( \exp : g_A \to G_A \) (Corollary 6.11). In §7, we first show that the Lax pair flow is trivial on primitives in the Hopf algebra. We then use Manchon’s bijection \( \tilde{R}^{-1} : g_A \to G_A \) to prove various locality results (Theorems 7.5, 7.16). The \( \beta \)-function flow defined via \( \tilde{R}^{-1} \) itself satisfies a Lax pair equation (Theorem 7.7). Thus \( \tilde{R}^{-1} \) is much better behaved than the exponential map. In §8, we work out several examples of this theory, and in particular keep track of the leading log terms.

An alternative algebraic geometric approach to Lax pair equations is to apply spectral curve techniques to linearize the flow on the Jacobian of the spectral curve. Unfortunately, in the worked example of §8, the spectral curve is reducible, and the only invariants we find are trivial. We hope to find examples with nontrivial invariants in the future.

We would like to thank Dirk Kreimer for suggesting we investigate the connection between the Connes-Kreimer factorization and integrable systems, and Dominique Manchon for helpful conversations.

2. The double Lie algebra and its associated Lie Group

There is a well known method to associate a Lax pair equation to a Casimir element on the dual \( g^* \) of a semisimple Lie algebra \( g \) \cite{10}. The semisimplicity is used to produce an \( \text{Ad} \)-invariant, symmetric, non-degenerate bilinear form on \( g \), allowing an identification of \( g \) with \( g^* \). For a general Lie algebra \( g \), there may be no such bilinear form. To produce a Lax pair, we need to extend \( g \) to a larger Lie algebra with the desired bilinear form. We do this by constructing a Lie bialgebra structure on \( g \), whose definition we now recall (see e.g. \cite{11}).

**Definition 2.1.** A Lie bialgebra is a Lie algebra \((g, [\cdot, \cdot])\) with a linear map \( \gamma : g \to g \otimes g \) such that

a) \( {}^1\gamma : g^* \otimes g^* \to g^* \) defines a Lie bracket on \( g^* \),

b) \( \gamma \) is a 1-cocycle of \( g \), i.e.

\[
\text{ad}^{(2)}_x(\gamma(y)) - \text{ad}^{(2)}_y(\gamma(x)) - \gamma([x, y]) = 0,
\]

where \( \text{ad}^{(2)}_x : g \otimes g \to g \otimes g \) is given by \( \text{ad}^{(2)}_x(y \otimes z) = \text{ad}_x(y) \otimes z + y \otimes \text{ad}_x(z) = [x, y] \otimes z + y \otimes [x, z] \).

A Lie bialgebra \((g, [\cdot, \cdot], \gamma)\) induces an Lie algebra structure on the **double Lie algebra** \( g \oplus g^* \) by

\[
[X, Y]_{g \oplus g^*} = [X, Y],
\]

\[
[X^*, Y^*]_{g \oplus g^*} = {}^1\gamma(X \otimes Y),
\]

\[
[X, Y^*] = \text{ad}^*_X(Y^*),
\]

for \( X, Y \in g \) and \( X^*, Y^* \in g^* \), where \( \text{ad}^* \) is the coadjoint representation given by \( \text{ad}^*_X(Y^*)(Z) = -Y^*(\text{ad}_X(Z)) \) for \( Z \in g \).
Since it is difficult to construct explicitly the Lie group associated to the Lie algebra \( g \oplus g^* \), we will choose the trivial Lie bialgebra given by the cocycle \( \gamma = 0 \) and denote by \( \delta = g \oplus g^* \) the associated Lie algebra. Let \( \{Y_i, i = 1, \ldots, l\} \) be a basis of \( g \), with dual basis \( \{Y^*_i\} \). The Lie bracket \([\cdot, \cdot]\) on \( \delta \) is given by

\[
[Y_i, Y_j]_\delta = [Y_i, Y_j], \quad [Y_i^*, Y_j^*]_\delta = 0, \quad [Y_i, Y_j^*]_\delta = -\sum_k c^j_{ik} Y_k^*,
\]

where the \( c^j_{ik} \) are the structure constants: \([Y_i, Y_j] = \sum_k c^j_{ik} Y_k\). The Lie group naturally associated to \( \delta \) is given by the following proposition.

**Proposition 2.2.** Let \( G \) be the simply connected Lie group with Lie algebra \( g \) and let \( \theta : G \times g^* \to g^* \) be the coadjoint representation \( \theta(g, X) = \text{Ad}^*_G(g)(X) \). Then the Lie algebra of the semi-direct product \( \tilde{G} = G \ltimes g^* \) is the double Lie algebra \( \delta \).

**Proof.** The Lie group law on the semi-direct product \( \tilde{G} \) is given by

\[(g, X) \cdot (g', X') = (gg', X + \theta(g, X')).\]

Let \( \tilde{g} \) be the Lie algebra of \( \tilde{G} \). Then the bracket on \( \tilde{g} \) is given by

\[ [X, Y^*]_{\tilde{g}} = d\theta(X, Y^*), \quad [X, Y]_{\tilde{g}} = [X, Y], \quad [X^*, Y^*]_{\tilde{g}} = 0, \]

for left-invariant vector fields \( X, Y \) of \( G \) and \( X^*, Y^* \in g^* \). We have \( d\theta(X, Y^*) = d\text{Ad}^*_G(X)(Y^*) = [X, Y^*]_\delta \) since \( d\text{Ad}_G = \text{ad}_g \). \( \square \)

The main point of this construction is existence of a good bilinear form on the double.

**Lemma 2.3.** The natural pairing \( \langle \cdot, \cdot \rangle : \delta \otimes \delta \to \mathbb{C} \) given by

\[ \langle (a, b^*), (c, d^*) \rangle = d^*(a) + b^*(c), \quad a, c \in g, \quad b^*, d^* \in g^*, \]

is an \( \text{Ad} \)-invariant symmetric non-degenerate bilinear form on the Lie algebra \( \delta \).

**Proof.** By [11], this bilinear form is ad-invariant. Since \( \tilde{G} \) is simply connected, the Ad-invariance follows. As an explicit example, we have

\[ \text{Ad}_{\tilde{G}}((g, 0))(Y_i, 0) = (\text{Ad}_G(g)(Y_i), 0), \quad \text{and} \quad \text{Ad}_{\tilde{G}}((g, 0))(0, Y^*_j) = (0, \text{Ad}_G^*(g)(Y^*_j)), \]

from which the invariance under \( \text{Ad}_{\tilde{G}}(g, 0) \) follows. \( \square \)

### 3. The Loop Algebra of a Lie Algebra

Following [11], we consider the loop algebra

\[ L\delta = \{L(\lambda) = \sum_{j=M}^N \lambda^j L_j \mid M, N \in \mathbb{Z}, L_j \in \delta\}. \]

The natural Lie bracket on \( L\delta \) is given by

\[
\left[ \sum \lambda^i L_i, \sum \lambda^j L'_j \right] = \sum_k \lambda^k \sum_{i+j=k} [L_i, L'_j].
\]
Set
\[
L_{\delta_+} = \{L(\lambda) = \sum_{j=0}^{N} \lambda^i L_j \mid N \in \mathbb{Z}^+ \cup \{0\}, L_j \in \delta\}
\]
\[
L_{\delta_-} = \{L(\lambda) = \sum_{j=-M}^{-1} \lambda^i L_j \mid M \in \mathbb{Z}^+, L_j \in \delta\}.
\]

Let \(P_+ : L\delta \rightarrow L\delta_+\) and \(P_- : L\delta \rightarrow L\delta_-\) be the natural projections and set \(R = P_+ - P_-\).

The natural pairing \(\langle \cdot, \cdot \rangle\) on \(\delta\) yields an \(\text{Ad}\)-invariant, symmetric, non-degenerate pairing on \(L\delta\) by setting
\[
\langle \sum_{i=0}^{N} \lambda^i L_i, \sum_{j=0}^{N'} \lambda^j L'_j \rangle = \sum_{i+j=-1} \langle L_i, L'_j \rangle.
\]

For our choice of basis \(\{Y_i\}\) of \(g\), we get an isomorphism
\[
I : L(\delta^*) \rightarrow L\delta
\]
with
\[
I \left( \sum_{i} L_i Y_i^i \right) = \sum_{i} L_i Y_i^{*i} \lambda^{1-i}.
\]

We will need the following lemmas.

**Lemma 3.1.** \([1]\) We have the following natural identifications:
\[
L_{\delta_+} = L(\delta^*)_+ \text{ and } L_{\delta_-} = L(\delta^*)_-.
\]

**Lemma 3.2.** \([16, \text{Lem. 4.1}]\) Let \(\varphi\) be an \(\text{Ad}\)-invariant polynomial on \(\delta\). Then
\[
\varphi_{m,n}[L(\lambda)] = \text{Res}_{\lambda=0}(\lambda^{-n}\varphi(\lambda^m L(\lambda)))
\]
is an \(\text{Ad}\)-invariant polynomial on \(L\delta\) for \(m, n \in \mathbb{Z}\).

As a double Lie algebra, \(\delta\) has an \(\text{Ad}\)-invariant polynomial, the quadratic polynomial
\[
\psi(Y) = \langle Y, Y \rangle
\]
associated to the natural pairing. Let \(Y_{i+1} = Y_i^*\) for \(i \in \{1, \ldots, l = \dim(g)\}\), so elements of \(L\delta\) can be written \(L(\lambda) = \sum_{j=1}^{2l} \sum_{i=-M}^{N} L_i^j Y_j \lambda^i\). Then the \(\text{Ad}\)-invariant polynomials\n\[
\psi_{m,n}(L(\lambda)) = \text{Res}_{\lambda=0}(\lambda^{-n}\psi(\lambda^m L(\lambda))),(3.2)
\]
defined as in Lemma 3.2 are given by
\[
\psi_{m,n}(L(\lambda)) = 2 \sum_{j=1}^{l} \sum_{i+k-n+2m=-1} \sum_{i} L_i^j L_k^{j+i} L_i Y_j Y_k^i . \]

Note that powers of \(\psi\) are also \(\text{Ad}\)-invariant polynomials on \(\delta\), so
\[
\psi_{m,n}(L(\lambda)) = \text{Res}_{\lambda=0}(\lambda^{-n}\psi^k(\lambda^m L(\lambda))) (3.4)
\]
are Ad-invariant polynomials on $L\delta$. It would be interesting to classify all Ad-invariant polynomials on $L\delta$ in general.

4. THE LAX PAIR EQUATION

Let $P_+, P_-$ be endomorphisms of a Lie algebra $\mathfrak{h}$ and set $R = P_+ - P_-$. Assume that

$$[X,Y]_R = [P_+X,P_+Y] - [P_-X,P_-Y]$$

is a Lie bracket on $\mathfrak{h}$. From [16, Theorem 2.1], the equations of motion induced by a Casimir (i.e. Ad-invariant) function $\varphi$ on $\mathfrak{h}^*$ are given by

$$\frac{dL}{dt} = -\text{ad}^*_{\mathfrak{h}}M \cdot L,$$

for $L \in \mathfrak{h}^*$, where $M = \frac{1}{2}R(d\varphi(L)) \in \mathfrak{h}$.

Now we take $\mathfrak{h} = (L\delta)^* = L(\delta^*)$, with $\delta$ a finite dimensional Lie algebra and with the understanding that $(L\delta)^*$ is the graded dual with respect to the standard $\mathbb{Z}$-grading on $L\delta$. Let $P_\pm$ be the projections of $L\delta^*$ onto $L\delta^*_\pm$. After identifying $L\delta^* = L\delta$ and $\text{ad}^* = -\text{ad}$ via the map $I$ in (3.1), the equations of motion (4.1) can be written in Lax pair form

$$\frac{dL}{dt} = [M,L],$$

where $M = \frac{1}{2}R(I(d\varphi(L(\lambda)))) \in L\delta$, and $\varphi$ is a Casimir function on $L\delta^* = L\delta$ [16, Theorem 2.1]. Finding a solution for (4.2) reduces to the Riemann-Hilbert (or Birkhoff) factorization problem. The following theorem is a corollary of [1, Theorem 4.37] [16, Theorem 2.2].

**Theorem 4.1.** Let $\varphi$ be a Casimir function on $L\delta$ and set $X = I(d\varphi(L(\lambda)))) \in L\delta$, for $L(\lambda) = L(0)(\lambda) \in L\delta$. Let $g_\pm(t)$ be the smooth curves in $L\tilde{G}$ which solve the factorization problem

$$\exp(-tX) = g_-(t)^{-1}g_+(t),$$

with $g_+(0) = e$, and with $g_+(t) = g_+(\lambda)(t)$ holomorphic in $\lambda \in \mathbb{C}$ and $g_-(t)$ a polynomial in $1/\lambda$ with no constant term. Let $M = \frac{1}{2}R(I(d\varphi(L(\lambda)))) \in L\delta$. Then the integral curve $L(t)$ of the Lax pair equation

$$\frac{dL}{dt} = [L,M]$$

is given by

$$L(t) = \text{Ad}_{L\tilde{G}}g_\pm(t) \cdot L(0).$$

This Lax pair equation projects to a Lax pair equation on the loop algebra of the original Lie algebra $\mathfrak{g}$. Let $\pi_1$ be either the projection of $\tilde{G}$ onto $G$ or its differential from $\delta$ onto $\mathfrak{g}$. This extends to a projection of $L\delta$ onto $L\mathfrak{g}$. The projection of (4.2) onto $L\mathfrak{g}$ is

$$\frac{d(\pi_1(L(t)))}{dt} = [\pi_1(L), \pi_1(M)],$$

(4.4)
since \( \pi_1 = d\pi_1 \) commutes with the bracket. Thus the equations of motion (4.2) induce a Lax pair equation on \( Lg \), although this is not the equations of motion for a Casimir on \( Lg \).

**Theorem 4.2.** The Lax pair equation of Theorem 4.1 projects to a Lax pair equation on \( Lg \).

**Remark 4.3.** The content of this theorem is that a Lax pair equation on the Lie algebra of a semi-direct product \( G \ltimes G' \) evolves on an adjoint orbit, and the projection onto \( g \) evolves on an adjoint orbit and is still in Lax pair form. Lax pair equations often appear as equations of motion for some Hamiltonian, but the projection may not be the equations of motion for any function on the smaller Lie algebra. We thank B. Khesin for this observation.

When \( \psi_{m,n} \) is the Casimir function on \( L\delta \) given by (3.2), \( X \) can be written nicely in terms of \( L(\lambda) \).

**Proposition 4.4.** Let \( X = I(d\psi_{m,n}(L(\lambda))) \). Then

\[
X = 2\lambda^{-n+2m}L(\lambda).
\]

**Proof.** Write \( L(\lambda) = \sum_{i,j} L^j_i \lambda^i Y_j \). By formula (3.3), we have

\[
\frac{\partial \psi_{m,n}}{\partial L^p_t} = \begin{cases} 2L_{n-1-2m-p}^{t+l}, & \text{if } t \leq l \\ 2L_{n-1-2m-p}^{t-1}, & \text{if } t > l. \end{cases}
\]

Therefore

\[
X = I(d\psi_{m,n}(L(\lambda))) = \sum_{p,t} \frac{\partial \psi_{m,n}}{\partial L^p_t} \lambda^{-1-p}Y^*_t
\]

\[
= 2\lambda^{-n+2m} \sum_p \left( \sum_{t=1}^{l} L_{n-1-2m-p}^{t+l} Y_{t+1}^n \lambda^{n-1-2m-p} + \sum_{t=l+1}^{2l} L_{n-1-2m-p}^{t-l} Y_{t-l}^n \lambda^{n-1-2m-p} \right)
\]

\[
= 2\lambda^{-n+2m}L(\lambda).
\]

\[\square\]

5. **The main theorem for Hopf algebras**

In this section we give formulas for the Birkhoff decomposition of a loop in the Lie group of characters of a Hopf algebra and produce the Lax pair equations associated to the Birkhoff decomposition. We present two approaches, both motivated by the Connes-Kreimer Hopf algebra of 1PI Feynman graphs. First, in analogy to truncating Feynman integral calculations at a certain loop level, we truncate a (possibly infinitely generated) Hopf algebra to a finitely generated Hopf algebra, and solve Lax pair equations on the
finite dimensional piece (Theorem 5.4). We also discuss the compatibility of solutions related to different truncations. Second, we solve a Lax pair equation associated to the full Hopf algebra, but for a restricted family of Casimirs (Theorem 5.9).

Let $\mathcal{H} = (\mathcal{H}, 1, \mu, \Delta, \varepsilon, S)$ be a graded connected Hopf algebra over $\mathbb{C}$. Let $\mathcal{A}$ be a unital commutative algebra with unit $1_\mathcal{A}$. Unless stated otherwise, $\mathcal{A}$ will be the algebra of Laurent series; the only other occurrence in this paper is $\mathcal{A} = \mathbb{C}$.

**Definition 5.1.** The character group $G_\mathcal{A}$ of the Hopf algebra $\mathcal{H}$ is the set of algebra morphisms $\phi: \mathcal{H} \to \mathcal{A}$ with $\phi(1) = 1_\mathcal{A}$. The group law is given by the convolution product

$$(\psi_1 \ast \psi_2)(h) = \langle \psi_1 \otimes \psi_2, \Delta h \rangle;$$

the unit element is $\varepsilon$.

**Definition 5.2.** An $\mathcal{A}$-valued infinitesimal character of a Hopf algebra $\mathcal{H}$ is a $\mathbb{C}$-linear map $Z: \mathcal{H} \to \mathcal{A}$ satisfying

$$\langle Z, hk \rangle = \langle Z, h \rangle \varepsilon(k) + \varepsilon(h) \langle Z, k \rangle.$$

The set of infinitesimal characters is denoted by $g_\mathcal{A}$ and is endowed with a Lie algebra bracket:

$$[Z, Z'] = Z \ast Z' - Z' \ast Z, \text{ for } Z, Z' \in g_\mathcal{A},$$

where $\langle Z \ast Z', h \rangle = \langle Z \otimes Z', \Delta(h) \rangle$. Notice that $Z(1) = 0$.

For a finitely generated Hopf algebra, $G_\mathbb{C}$ is a Lie group with Lie algebra $g_\mathbb{C}$, and for any Hopf algebra and any $\mathcal{A}$, the same is true at least formally.

We recall that $\delta = g_\mathbb{C} \oplus g_\mathbb{C}^*$ is the double of $g_\mathbb{C}$ and the $g_\mathbb{C}^*$ is the graded dual of $g_\mathbb{C}$. We consider the algebra $\Omega \delta = \delta \otimes \mathcal{A}$ of formal Laurent series with values in $\delta$

$$\Omega \delta = \{ L(\lambda) = \sum_{j=-N}^{\infty} \lambda^j L_j \mid L_j \in \delta, N \in \mathbb{Z} \}.$$

The natural Lie bracket on $\Omega \delta$ is

$$\left[ \sum \lambda^i L_i, \sum \lambda^j L'_j \right] = \sum_k \lambda^k \sum_{i+j=k} [L_i, L'_j].$$

Set

$$\Omega \delta_+ = \{ L(\lambda) = \sum_{j=0}^{\infty} \lambda^j L_j \mid L_j \in \delta \}$$

$$\Omega \delta_- = \{ L(\lambda) = \sum_{j=-N}^{-1} \lambda^j L_j \mid L_j \in \delta, N \in \mathbb{Z}^+ \}.$$

Recall that for any Lie group $K$, a loop $L(\lambda)$ with values in $K$ has a Birkhoff decomposition if $L(\lambda) = L(\lambda)^{-1} L(\lambda)_+$ with $L(\lambda)^{-1}$ holomorphic in $\lambda^{-1} \in \mathbb{P}^1 - \{0\}$ and $L(\lambda)_+$ holomorphic in $\lambda \in \mathbb{P}^1 - \{\infty\}$. In the next lemma, $\tilde{G}$ refers to $G \ltimes g^*$ as in Prop. 2.2.
Theorem 5.3. Every \((g, \alpha) \in \Omega \tilde{G} = G_A \times_{A \Delta A} \mathfrak{g}_A^*\) has a Birkhoff decomposition \((g, \alpha) = (g_-, \alpha_-)^{-1}(g_+, \alpha_+)^{-1}\) with \((g_+, \alpha_+)^{-1}\) holomorphic in \(\lambda\) and \((g_-, \alpha_-)^{-1}\) a polynomial in \(\lambda^{-1}\) without constant term.

Proof. We recall that \((g_1, \alpha_1)(g_2, \alpha_2) = (g_1g_2, \alpha_1 + \text{Ad}^*(g_1)(\alpha_2))\). Thus \((g, \alpha) = (g_-, \alpha_-)^{-1}(g_+, \alpha_+)\) if and only if \(g = g^{-1}_+\) and \(\alpha = \text{Ad}^*(g^{-1}_-)(\alpha_- + \alpha_+)\). Let \(g = g^{-1}_+\) be the Birkhoff decomposition of \(g\) in \(G_A\) given in \([11, 17, 15]\). Set \(\alpha_+ = P_+(\text{Ad}^*(g_-)(\alpha))\) and \(\alpha_- = -P_-(\text{Ad}^*(g_-)(\alpha))\), where \(P_+\) and \(P_-\) are the holomorphic and pole part, respectively. Then for this choice of \(\alpha_+\) and \(\alpha_-\), we have \((g, \alpha) = (g_-, \alpha_-)^{-1}(g_+, \alpha_+)\). Note that the Birkhoff decomposition is unique. \(\square\)

For a finitely generated Hopf algebra, we can apply Theorems 4.1, 4.2 to produce a Lax pair equation on \(L\delta\) and on the loop space of infinitesimal characters \(L\mathfrak{g}\). However, the common Hopf algebras of 1PI Feynman diagrams and rooted trees are not finitely generated.

As we now explain, we can truncate the Hopf algebra to a finitely generated Hopf algebra, and use the Birkhoff decomposition to solve a Lax pair equation on the infinitesimal character group of the truncation. A graded Hopf algebra \(\mathcal{H} = \oplus_{n \in \mathbb{N}} \mathcal{H}_n\) is said to be of finite type if each homogeneous component \(\mathcal{H}_n\) is a finite dimensional vector space. Let \(B = \{T_i\}_{i \in \mathbb{N}}\) be a minimal set of homogeneous generators of the Hopf algebra \(H\) such that \(\deg(T_i) \leq \deg(T_j)\) if \(i < j\) and such that \(T_0 = 1\). For \(i > 0\), we define the \(\mathbb{C}\)-valued infinitesimal character \(Z_i\) on generators by \(Z_i(T_j) = \delta_{ij}\). The Lie algebra of infinitesimal characters \(\mathfrak{g}\) is a graded Lie algebra generated by \(\{Z_i\}_{i > 0}\). Let \(\mathfrak{g}^{(k)}\) be the vector space generated by \(\{Z_i | \deg(T_i) \leq k\}\). We define \(\deg(Z_i) = \deg(T_i)\) and set

\[
[Z_i, Z_j]_{\mathfrak{g}^{(k)}} = \begin{cases} 
[Z_i, Z_j] & \text{if } \deg(Z_i) + \deg(Z_j) \leq k \\
0 & \text{if } \deg(Z_i) + \deg(Z_j) > k
\end{cases}
\]

We identify \(\varphi \in G_\mathbb{C}\) with \(\{\varphi(T_i)\} \in \mathbb{C}^N\) and on \(\mathbb{C}^N\) we set a group law given by \(\{\varphi_1(T_i)\} \oplus \{\varphi_2(T_i)\} = \{(\varphi_1 \ast \varphi_2)(T_i)\}\). \(G^{(k)} = \{\varphi(T_i)\}_{i | \deg(T_i) \leq k} | \varphi \in G_\mathbb{C}\) is a finite dimensional Lie subgroup of \(G_\mathbb{C} = (\mathbb{C}^N, \oplus)\) and the Lie algebra of \(G^{(k)}\) is \(\mathfrak{g}^{(k)}\). There is no loss of information under this identification, as \(\varphi(T_i) = \varphi(T_i)\varphi(T_j)\).

Let \(\delta^{(k)}\) be the double Lie algebra of \(\mathfrak{g}^{(k)}\) and let \(\tilde{G}^{(k)}\) be the simply connected Lie group with \(\text{Lie}(\tilde{G}^{(k)}) = \delta^{(k)}\) as in Proposition 2.2. The following theorem is a restatement of Theorem 4.4 in our new stage.

Theorem 5.4. Let \(\mathcal{H} = \oplus_n \mathcal{H}_n\) be a graded connected Hopf algebra of finite type, and let \(\psi : L\delta^{(k)} \rightarrow \mathbb{C}\) be a Casimir function (e.g. \(\psi(L) = \psi_{m,n}(L(\lambda)) = \text{Res}_{\lambda=0}(\lambda^m \psi(\lambda^n L(\lambda)))\)) with \(\psi : \delta^{(k)} \times \delta^{(k)} \rightarrow \mathbb{C}\) the natural paring of \(\delta^{(k)}\). Set \(X = I(d\psi(L_0))\) for \(L_0 \in L\delta^{(k)}\).
Then the solution in $L^{\delta(k)}$ of
\begin{equation}
\frac{dL}{dt} = [L, M]_{L^{\delta(k)}}, \quad M = \frac{1}{2} R(I(d\psi(L)))
\end{equation}
with initial condition $L(0) = L_0$ is given by
\begin{equation}
L(t) = \text{Ad}_{L^{G_0}} g_{\pm}(t) \cdot L_0,
\end{equation}
where $\exp(-tX)$ has the Connes-Kreimer Birkhoff factorization
\[\exp(-tX) = g_{\pm}(t)^{-1} g_{\pm}(t).\]

Remark 5.5. (i) If $L_0 \in L^{\delta}$, there exists $k \in \mathbb{N}$ such that $L_0 \in L^{\delta(k)}$. Indeed $L_0 \in L^{\delta}$ is generated over $\mathbb{C}[\lambda, \lambda^{-1}]$ by a finite number of $\{Z_i\}$, and we can choose $k \geq \max \{\deg(Z_i)\}$.

(ii) While the Hopf algebra of rooted trees and the Connes-Kreimer Hopf algebra of 1PI Feynman diagrams satisfy the hypothesis of Theorem 5.4, the Feynman rules character does not lie in $L^{G_0}$, as explained below.

In the next sections, we will investigate the relationship between the Lax pair flow $L(t)$ and the Renormalization Group Equation. In preparation, we project from $L^{\delta(k)}$ to $L^{G_0}$ via $\pi_1$ as in §4.

Corollary 5.6. Let $\psi$ be a Casimir function on $L^{\delta(k)}$. Set $L_0 \in L^{G_0(k)} \subset L^{\delta(k)}$, $X = \pi_1(I(d\psi(L_0)))$. Then the solution of the following equation in $L^{G_0(k)}$
\begin{equation}
\frac{dL}{dt} = [L, M_1]_{L^{G_0(k)}}, \quad M_1 = \pi_1(\frac{1}{2} R(I(d\psi(L))))
\end{equation}
with initial condition $L(0) = L_0$ is given by
\begin{equation}
L(t) = \text{Ad}_{L^{G_0}} g_{\pm}(t) \cdot L_0,
\end{equation}
where $\exp(-tX)$ has the Connes-Kreimer Birkhoff factorization in $L^{G_0(k)}$
\[\exp(-tX) = g_{\pm}(t)^{-1} g_{\pm}(t).\]

Remark 5.7. (i) For Feynman graphs, this truncation corresponds to halting calculations after a certain loop level. From our point of view, this truncation is somewhat crude. $g^{(k)}$ is not a subalgebra of $g$, and if $k < \ell$, $g^{(k)}$ is not a subalgebra of $g^{(\ell)}$. Although the Casimirs $\psi_{m,n}$ and the exponential map restrict well from $g$ to $g^{(k)}$, the Birkhoff decomposition $\exp(-tX)$ of $X \in L^{G_0(k)}$ is very different from the Birkhoff decompositions in $L^{G_0}, L^{G_0(\ell)}$. In fact, if $g \in G^{(k)}$ has Birkhoff decomposition $g = g_{\pm}^{-1} g_{\pm}$ in $G$, there does not seem to be $f(k) \in \mathbb{N}$ such that $g_{\pm} \in G^{(f(k))}$. Nevertheless, in the last section we will follow standard procedure and present calculations of truncated Hopf algebras.

(ii) It would interesting to know, especially for the Hopf algebras of Feynman graphs or rooted trees, whether there exists a larger connected graded Hopf algebra $H'$ containing $H$ such that the associated infinitesimal Lie algebra $\text{Lie}(G'_C)$ is the double $\delta$. This would provide a Lax pair equation associated to an equation of motion on the infinitesimal Lie
algebra of $\mathcal{H}'$. The most natural candidate, the Drinfeld double $\mathcal{D}(\mathcal{H})$ of $\mathcal{H}$, does not work since the dimension of the Lie algebra associated to $\mathcal{D}(\mathcal{H})$ is larger than the dimension of $\delta$.

In [1], Connes and Kreimer give a Birkhoff decomposition for the character group of the Hopf algebra of 1PI graphs, and in particular for the Feynman rules character $\varphi(\lambda)$ given by minimal subtraction and dimensional regularization. The truncation process treated above does not handle the Feynman rules character, as the Feynman rules character and the toy model character of the Hopf algebra of rooted trees considered in §8 are not polynomials in $\lambda, \lambda^{-1}$, but Laurent series in $\lambda$. Thus Corollary 5.6 does not apply, as in our notation $\log(\varphi(\lambda)) \in \Omega g \setminus Lg$. This and Remark 5.7(i) force us to consider a direct approach in $\Omega g$ as in next theorem. However, we cannot expect that the Lax pair equation is associated to any Hamiltonian equation, and we replace Casimirs with Ad-covariant functions.

**Definition 5.8.** [17] Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. A map $f : \mathfrak{g} \to \mathfrak{g}$ is Ad-covariant if $\text{Ad}(g)(f(L)) = f(\text{Ad}(g)(L))$ for all $g \in G$, $L \in \mathfrak{g}$.

**Theorem 5.9.** Let $\mathcal{H}$ be a connected graded commutative Hopf algebra with $\mathfrak{g}_A$ the associated Lie algebra of infinitesimal characters with values in Laurent series. Let $f : \mathfrak{g}_A \to \mathfrak{g}_A$ be an Ad-covariant map. Let $L_0 \in \mathfrak{g}_A$ satisfy $[f(L_0), L_0] = 0$. Set $X = f(L_0)$. Then the solution of

$$\frac{dL}{dt} = [L, M], \quad M = \frac{1}{2} R(f(L))$$

with initial condition $L(0) = L_0$ is given by

$$L(t) = \text{Ad}_G g_\pm(t) \cdot L_0,$$

where $\exp(-tX)$ has the Connes-Kreimer Birkhoff factorization

$$\exp(-tX) = g_-(t)^{-1} g_+(t).$$

**Proof.** The proof is similar to [16, Theorem 2.2]. First notice that

$$\frac{d}{dt} \left( \text{Ad}(g_-(t)^{-1} g_+(t)) \cdot L_0 \right) = \frac{d}{dt}(\exp(-tX)L_0 \exp(tX))$$

$$= -\exp(-tX)XL_0 \exp(tX) + \exp(-tX)L_0X \exp(tX)$$

$$= \exp(-tX)[X, L_0] \exp(tX) = 0,$$

which implies $\text{Ad}(g_-(t)^{-1} g_+(t)) \cdot L_0 = L_0$ and $\text{Ad}(g_-(t)) \cdot L_0 = \text{Ad}(g_+(t)) \cdot L_0$. Set $L(t) = \text{Ad}(g_\pm(t)) \cdot L_0 = g_\pm(t)L_0g_\pm(t)^{-1}$. As usual,

$$\frac{dL}{dt} = \left[ \frac{dg_\pm(t)}{dt} g_\pm(t)^{-1}, L(t) \right],$$

so

$$\frac{dL}{dt} = \frac{1}{2} \left[ \frac{dg_+(t)}{dt} g_+(t)^{-1} + \frac{dg_-(t)}{dt} g_-(t)^{-1}, L(t) \right].$$
The Birkhoff factorization \( g_+(t) = g_-(t) \exp(-tX) \) gives
\[
\frac{dg_+(t)}{dt} = \frac{dg_-(t)}{dt} \exp(-tX) + g_-(t)(-X) \exp(-tX),
\]
and so
\[
\frac{dg_+(t)}{dt} g_+(t)^{-1} = \frac{dg_-(t)}{dt} g_-(t)^{-1} + g_-(t)(-X)g_-(t)^{-1}.
\]
Thus
\[
2M = R(f(L(t))) = R(f(\text{Ad}(g_-(t)) \cdot L_0)) = R(\text{Ad}(g_-(t)) \cdot f(L_0)) = R(\text{Ad}(g_-(t)) \cdot X) = -R \left( \frac{dg_+(t)}{dt} g_+(t)^{-1} \right) + R \left( \frac{dg_-(t)}{dt} g_-(t)^{-1} \right) = -\frac{dg_+(t)}{dt} g_+(t)^{-1} - \frac{dg_-(t)}{dt} g_-(t)^{-1}.
\]
Here we use \( (\frac{dg_+(t)}{dt} g_+(t)^{-1})(x) \in A_\pm \) for \( x \in H \). Thus \( \frac{dL}{dt} = [L, M] \).

If \( f : g_A \to g_A \) is given by \( f(L) = 2\lambda^{-n+2m}L \), then \( f \) is Ad-covariant and \([f(L_0), L_0] = [2\lambda^{-n+2m}L_0, L_0] = 0\).

**Corollary 5.10.** Let \( H \) be a connected graded commutative Hopf algebra with \( g_A \) the Lie algebra of infinitesimal characters with values in Laurent series. Pick \( L_0 \in g_A \) and set \( X = 2\lambda^{-n+2m}L_0 \). Then the solution of
\[
\frac{dL}{dt} = [L, M], \quad M = R(\lambda^{-n+2m}L)
\]
with initial condition \( L(0) = L_0 \) is given by
\[
L(t) = \text{Ad}_{g_A} g_\pm(t) \cdot L_0,
\]
where \( \exp(-tX) \) has the Connes-Kreimer Birkhoff factorization
\[
\exp(-tX) = g_-(t)^{-1} g_+(t).
\]

**Remark 5.11.** Let \( \varphi \) be the Feynman rules character. We can find the Birkhoff factorization of \( \varphi \) itself within this framework by adjusting the initial condition. Namely, set \( L_0(\lambda) = \frac{1}{2} \lambda^{n-2m} \exp^{-1}(\varphi(\lambda)) \). Then \( \exp(X) = \varphi \) by Prop. 4.3 so the solution of \( 5.7 \) involves the Birkhoff factorization \( \varphi = g_-(1)^{-1} g_+(1) \). Namely, we have
\[
L(-1) = \frac{\lambda^{n-2m}}{2} \text{Ad}_{g_A} g_\pm(-1) \exp^{-1}(\varphi).
\]

**6. The Connes-Kreimer \( \beta \)-function**

The flow of characters usually considered in quantum field theory is the renormalization group flow (RGF). In contrast, the Lax pair flow lives on the Lie algebra of the character group. Since the \( \beta \)-function of the RGF is an element of the Lie algebra of the \( \mathbb{C} \)-valued characters, it is natural to examine the relationship between the Lax pair equations and the \( \beta \)-function. In this section, we continue to work in the general setup of Hopf algebras and character groups.
Here we consider two flows for the $\beta$-function. First, we extend the (scalar) beta function of a local character $\varphi$ (see (6.1)) to an infinitesimal character $\tilde{\beta}_\varphi$ (Lemma 6.6). This “beta character” has already appeared in the literature: $\tilde{\beta}_\varphi = \lambda \tilde{R}(\varphi)$, in the language of [15] explained below (Lemma 6.7), but it seems worth highlighting. For certain Casimirs, we show that the beta character is a fixed point of the Lax pair flow (Theorem 6.8).

It is more important and more difficult to consider the flow of the $\beta$-function itself. Namely, given a character $\varphi$, we can set $L_0 = \log(\varphi)$ and study the $\beta$-functions of the characters $\varphi(s) = \exp(L(s))$. In Theorem 6.10, we give a differential equation for $\beta_{\varphi(s)}$.

To define the beta character, we recall material from [5, 8, 15]. Throughout this section, $\mathcal{A}$ denotes the algebra of Laurent series.

Let $H = \bigoplus_n H_n$ be a connected graded Hopf algebra. Let $Y$ be the biderivation on $H$ given on homogeneous elements by

$$Y : H_n \rightarrow H_n, \quad Y(x) = nx \quad \text{for} \ x \in H_n.$$  

**Definition 6.1.** [15] We define the bijection $\tilde{R} : G_A \rightarrow \mathfrak{g}_A$ by

$$\tilde{R}(\varphi) = \varphi^{-1} * (\varphi \circ Y).$$

Consider the semidirect product Lie algebra $\tilde{g}_A = \mathfrak{g}_A \rtimes \mathbb{C} \cdot Z_0$, where $Z_0$ acts via $[Z_0, X] = X \circ Y$ for $X \in \mathfrak{g}_A$. Let $\{\theta_t\}_{t \in \mathbb{C}}$ be the one-parameter group of automorphisms of $H$ given by

$$\theta_t(x) = e^{nt}x, \quad \text{for} \ x \in H_n.$$  

Then $\varphi \mapsto \varphi \circ \theta_t$ is an automorphism of $G_A$. Let $\tilde{G}_A$ be the semidirect product

$$\tilde{G}_A = G_A \rtimes \mathbb{C},$$

with the action of $\mathbb{C}$ on $G_A$ given by $\varphi \cdot t = \varphi \circ \theta_t$. $G_A$ has Lie algebra $\mathfrak{g}_A$.

We now define a second action of $\mathbb{C}$ on $G_A$. For $t \in \mathbb{C}$ and $\varphi \in G_A$ we define $\varphi^t(x)$ on an homogeneous element $x \in H$ by

$$\varphi^t(x)(\lambda) = e^{t|\lambda|} \varphi(x)(\lambda),$$

for any $\lambda \in \mathbb{C}$, where $|x|$ is the degree of $x$.

**Definition 6.2.** Let

$$(6.1) \quad G^0_A = \{\varphi \in G_A \mid \frac{d}{dt}(\varphi^t)_- = 0\},$$

be the set of characters with the negative part of the Birkhoff decomposition independent of $t$. Elements of $G^0_A$ are called local characters.

The dimensional regularized Feynman rule character $\varphi$ is local. Referring to [5, 8], the physical meaning of locality is that the counterterm $\varphi_-$ does not depend on the mass parameter $\mu$: $\frac{\partial \varphi_-}{\partial \mu} = 0$. 

**Proposition 6.3 (5 14 8).** Let \( \varphi \in G^\Phi_A \). Then the limit
\[
F\varphi(t) = \lim_{\lambda \to 0} \varphi^{-1}(\lambda) \ast \varphi'(\lambda)
\]
exists and is a one-parameter subgroup in \( G_A \cap G_C \) of scalar valued characters of \( \mathcal{H} \).

Notice that \( (\varphi^{-1}(\lambda) \ast \varphi'(\lambda))(\Gamma) \in A_+ \) as
\[
\varphi^{-1}(\lambda) \ast \varphi'(\lambda) = \varphi_+^{-1} \ast \varphi_- \ast (\varphi'_-) \ast (\varphi'_+) = \varphi_+^{-1} \ast (\varphi'_+).
\]

**Definition 6.4.** For \( \varphi \in G^\Phi_A \), the \( \beta \)-function of \( \varphi \) is defined to be \( \beta_\varphi = -(\text{Res}(\varphi_-)) \circ Y \).

We have \( [5] \)
\[
\beta_\varphi = \frac{d}{dt} \bigg|_{t=0} F_{\varphi^{-1}}(t),
\]
where \( F_{\varphi^{-1}}, \) the one-parameter subgroup associated to \( \varphi^{-1} \), also belongs to \( G^\Phi_A \).

To relate the \( \beta \)-function \( \beta_\varphi \in g_C \) to our Lax pair equations, which live on \( g_A \), we can either consider \( g_C \) as a subset of \( g_A \), or we can extend \( \beta_\varphi \) to an element of \( g_A \). Since \( g_C \) is not preserved under the Lax pair flow, we take the second approach.

**Definition 6.5.** For \( \varphi \in G^\Phi_A \), \( x \in H \), set
\[
\tilde{\beta}_\varphi(x)(\lambda) = \frac{d}{dt} \bigg|_{t=0} (\varphi^{-1} \ast \varphi')(x)(\lambda).
\]

The following lemma establishes that \( \tilde{\beta} \) is an infinitesimal character.

**Lemma 6.6.** Let \( \varphi \in G^\Phi_A \).

i) \( \tilde{\beta}_\varphi \) is an infinitesimal character in \( g_A \).

ii) \( \tilde{\beta}_\varphi \) is holomorphic (i.e. \( \tilde{\beta}_\varphi(x) \in A_+ \) for any \( x \)).

**Proof.** i) For two homogeneous elements \( x, y \in \mathcal{H} \), we have:
\[
\varphi'(xy) = e^{t|x|\lambda} \varphi(xy) = e^{t|x|\lambda} \varphi(x)e^{t|y|\lambda} \varphi(y) = \varphi'(x) \varphi'(y).
\]
Therefore \( \varphi \ast \varphi' \in G_A \). Since \( \varphi^{-1} \ast \varphi^0 = e \) we get
\[
\frac{d}{dt} \bigg|_{t=0} \varphi^{-1} \ast \varphi' \in g_A.
\]

ii) Since \( \frac{d}{dt}(\varphi'(-)) = 0 \), we get
\[
\tilde{\beta}_\varphi = (\varphi_+)^{-1} \ast \varphi_- \ast ((\varphi'_-)^{-1} \ast (\varphi'_+)) = (\varphi_+)^{-1} \ast (\varphi'_+).
\]
Then
\[
\tilde{\beta}_\varphi(x) = (\varphi_+)^{-1}(x')(\varphi'_+)(x'') = (\varphi_+)(S(x'))(\varphi'_+)(x'')
\]
Therefore \( \tilde{\beta}_\varphi(x) \in A_+ \). \( \square \)
Lemma 6.7. If $\varphi \in G^\varphi_A$ then 

(i) $\bar{\beta}_\varphi = \lambda \bar{R}(\varphi)$,

(ii) $\beta_\varphi = \Ad(\varphi_+(0))(\bar{\beta}_\varphi|_{\lambda=0})$,

(iii) $\bar{\beta}_{\varphi-}(x)(\lambda = 0) = -\bar{\beta}_\varphi(x)$.

Proof. (i) For $\Delta(x) = x' \otimes x''$, we have 

$$\bar{\beta}_\varphi(x)(\lambda) = \frac{d}{dt}\big|_{t=0} (\varphi^{-1} \star \varphi')(x)(\lambda) = \varphi^{-1}(x') \frac{d}{dt}\big|_{t=0} (\varphi')(x'')$$

$$= \varphi^{-1}(x') \lambda \cdot \deg(x'') \varphi(x'') = \lambda \varphi^{-1}(x') \varphi \circ Y(x'') = \lambda(\varphi^{-1} \star (\varphi \circ Y))(x) = \lambda \bar{R}(\varphi)(x).$$

(ii) The cocycle property of $\bar{R}$, $\bar{R}(\phi_1 \star \phi_2) = \bar{R}(\phi_2) + \phi_2^{-1} \star \bar{R}(\phi_1) \star \phi_2$, implies that 

(6.2)  
$$\lambda \bar{R}(\varphi) = \lambda \bar{R}(\varphi_-^{-1} \star \varphi_+) = \lambda \bar{R}(\varphi_+) + \varphi_-^{-1} \star \lambda \bar{R}(\varphi_-^{-1}) \star \varphi_+.$$

Since $\bar{R}(\varphi_+) = \varphi_+^{-1} \star (\varphi_+ \circ Y)$ is always holomorphic and since $\lambda \bar{R}(\varphi_-^{-1}) = \text{Res}(\varphi_-^{-1}) \circ Y = -\text{Res}(\varphi_-) \circ Y = \beta$ by [13] Theorem IV.4.4, when we evaluate (6.2) at $\lambda = 0$ we get $\bar{\beta}(\varphi)|_{\lambda=0} = \Ad(\varphi_+^{-1}(0)) \beta$.

(iii) The Birkhoff decomposition of $\varphi_- = (\varphi_-)^{-1} \star (\varphi_-)_+$ is given by $(\varphi_-)_- = \varphi_-^{-1}$ and $(\varphi_-)_+ = \varepsilon$. By definition, $\beta_{\varphi_-} = -\text{Res}((\varphi_-)_-) \circ Y = -\text{Res}(\varphi_-^{-1}) \circ Y = \text{Res}(\varphi_-) \circ Y = -\beta_{\varphi}$.

Applying (ii) to $\varphi_-$, we get

$$-\beta_{\varphi} = \beta_{\varphi_-} = \Ad(\varepsilon|_{\lambda=0})(\bar{\beta}_{\varphi_-}|_{\lambda=0}) = \bar{\beta}_{\varphi_-}|_{\lambda=0}.$$

\[\square\]

If $\varphi \in G^\varphi_A$, the Lax pair equation in Corollary [5.10] for $L_0 = \bar{\beta}_\varphi$ is 

(6.3)  
$$\frac{d}{ds}\bar{\beta}_\varphi(s) = [\bar{\beta}_\varphi(s), M],$$

where $M = R(\lambda^{-n+2m} \bar{\beta}_\varphi(s))$ and the solution is given by 

(6.4)  
$$\bar{\beta}_\varphi(s) = \Ad(g_+(s)) \bar{\beta}_\varphi(0)$$

for $g_+(s)$ given by the Birkhoff decomposition $\exp(s \lambda^{-n+2m} \bar{\beta}_\varphi) = g^{-1}_+(s) \star g_+(s)$.

The next theorem shows that the $\beta$-function is a fixed point of the Lax pair flow for certain Casimirs. Of course this is not the same as having the $\beta$-function a fixed point of the RGF.

Theorem 6.8. $\bar{\beta}_\varphi(s)$ and therefore $\beta_\varphi(s) = \Ad(\varphi_+(s)|_{\lambda=0})(\bar{\beta}_\varphi(s)|_{\lambda=0})$ are constant under the Lax flow if $-n+2m \geq 0$.

Proof. We drop $s$ from the notation. If $-n+2m \geq 0$ then 

$$M = R(\lambda^{-n+2m} \bar{\beta}_\varphi) = \lambda^{-n+2m} \bar{\beta}_\varphi,$$
since $\tilde{\beta}_\phi$ is holomorphic by Lemma 6.6 and Theorem 7.1. (The proof of this Theorem is independent of this section.) So the Lax pair equation becomes

$$\frac{d}{ds} \tilde{\beta}_\phi = [\tilde{\beta}_\phi, \lambda^{-n+2m} \tilde{\beta}_\phi] = \lambda^{-n+2m}[\tilde{\beta}_\phi, \tilde{\beta}_\phi] = 0.$$  

□

Now we consider the more interesting case of the flow $\beta_{\psi(s)}$ of the $\beta$-function of exponentiated infinitesimal characters. We first establish some simple properties of $\phi^t$.

**Lemma 6.9.** Let $\phi \in G_\mathcal{A}$.

(i) $(\phi * \psi)^t = \phi^t * \psi^t$,

(ii) $(\phi^{-1})^t = (\phi^t)^{-1}$.

**Proof.** We have

$$(\phi * \psi)^t(x) = e^{t|z|\lambda}(\phi * \psi)(x) = \sum_{(x')} e^{t|z'|\lambda}(\phi(x'))\psi(x'').$$

$$= \sum_{(x')} e^{t(|x'| + |x''|)\lambda}(\phi(x'))\psi(x'') = e^{t|z'|\lambda}(\phi(x')) e^{t|z''|\lambda}(\phi(x'')) = \phi^t(x')\psi^t(x'').$$

Therefore

$$\phi^t * (\phi^{-1})^t = (\phi * \phi^{-1})^t = \epsilon^t = \epsilon = \phi^t * (\phi^t)^{-1},$$

so $(\phi^{-1})^t = (\phi^t)^{-1}$.

□

The exponential map $\exp : g_\mathcal{A} \to G_\mathcal{A}$ is a bijection. Therefore, we can transfer the Lax pair flow on $g_\mathcal{A}$ to a flow on $G_\mathcal{A}$, and study the associated flow of beta characters.

**Theorem 6.10.** Let $\phi \in G_\Phi^\mathcal{A}$. Let

$$\dot{\psi}(s) = [\psi(s), M]$$

be the Lax pair from Theorem 5.9 with $\psi(0) = \psi = \log(\phi)$. Let $\phi(s) = \exp(\psi(s))$. For

$$\dot{\bar{\beta}}_{\phi(s)} = \frac{d}{dt} \bigg|_{t=0} \phi(s)^{-1} * (\phi(s))^t,$$

we have

$$\frac{d}{ds} \bar{\beta}_{\phi(s)} = [\bar{\beta}_{\phi(s)}, \phi^{-1}(s) * d\exp[\log \phi(s), M]]$$

$$+ \lambda(\phi^{-1}(s) * d\exp[\log \phi(s), M]) \circ Y.$$
Omitting some stars, we have

$$\frac{d}{ds} \tilde{\beta}_{\varphi(s)}(x) = \frac{d}{ds} \frac{d}{dt} \bigg|_{t=0} (\varphi^{-1}(s) \ast \varphi^t(s))(x)$$

$$= \frac{d}{dt} \frac{d}{ds} \bigg|_{t=0} (\varphi^{-1}(s) \frac{d}{ds} \exp \psi(s) \varphi^{-1}(s) \varphi^t(s) + \varphi^{-1}(s)(\frac{d}{ds} \exp \psi(s))^t)(x)$$

$$= \frac{d}{dt} \frac{d}{dt} \bigg|_{t=0} (\varphi^{-1}(s)d \exp \psi(s) \varphi^{-1}(s) \varphi^t(s) + \varphi^{-1}(s)(d \exp \psi(s))^t)(x)$$

$$= (-\varphi^{-1}(s)d \exp[\psi(s), M] \tilde{\beta}_{\varphi(s)} + \frac{d}{dt} \bigg|_{t=0} \varphi^{-1}(s)(d \exp[\psi(s), M])^t)(x).$$

(6.6)

The last term in (6.6) is

$$\frac{d}{dt} \bigg|_{t=0} (\varphi^{-1}(s)(d \exp[\psi(s), M])^t)(x)$$

$$= \frac{d}{dt} \bigg|_{t=0} (\varphi^{-1}(s) \varphi(s)^t(\varphi(s)^t)^{-1}(d \exp[\psi(s), M])^t)(x)$$

$$= \frac{d}{dt} \bigg|_{t=0} (\varphi^{-1}(s) \varphi(s)^t) ((\varphi(s)^t)^{-1}(d \exp[\psi(s), M])^t) \bigg|_{t=0}(x)$$

$$+ (\varphi^{-1}(s) \varphi(s)^t) \bigg|_{t=0} \frac{d}{dt} \bigg|_{t=0} ((\varphi(s)^t)^{-1}(d \exp[\psi(s), M])^t)(x)$$

$$= (\tilde{\beta}_{\varphi(s)} \ast (\varphi(s)^{-1}d \exp[\psi(s), M]))(x) + \frac{d}{dt} \bigg|_{t=0} ((\varphi(s)^{-1}(d \exp[\psi(s), M])^t)(x)$$

$$= (\tilde{\beta}_{\varphi(s)} \ast (\varphi(s)^{-1}d \exp[\psi(s), M]))(x) + (\lambda(\varphi(s)^{-1}(d \exp[\psi(s), M]) \circ Y)(x)$$

Substituting back into (6.6) we get (6.5).

Corollary 6.11. Let $\varphi \in G^\phi_\mathcal{A}$. Let $\tilde{\psi}(s) = [M, \psi(s)]$ be the Lax pair from Theorem 5.9 with $\psi(0) = \psi = \log(\varphi)$. Let $\varphi(s) = \exp(\psi(s))$ and assume that $\varphi(s) \in G^\phi_\mathcal{A}$ for all s. Then

$$\frac{d}{ds} \beta_{\varphi(s)} = \text{Ad}(\varphi(s)_+(0)) \left( [\tilde{\beta}_{\varphi(s)}, \varphi^{-1}(s) \ast d \exp[\log(\varphi(s)), M] \bigg] \bigg|_{\lambda=0} \right.$$  

$+ \text{Res} \left( (\varphi^{-1}(s) \ast d \exp[\log(\varphi(s)), M] \circ Y) \right).$

Proof. This follows from the previous Theorem and Lemma 6.7.

Remark 6.12. In general, $\varphi \in G^\phi_\mathcal{A}$ does not imply $\varphi(s) = \exp(\psi(s)) \in G^\phi_\mathcal{A}$ for all s (see Theorem 7.14). A simple example with $\varphi(s) \in G^\phi_\mathcal{A}$ is given by a holomorphic $\varphi$ (i.e $\varphi(x) \in \mathcal{A}_+$) with $-n + 2m = 0$. Indeed $(\varphi^t)_- = \varepsilon$ as $\varphi^t$ is holomorphic, so $(\varphi^t)_-$ does not depend on $t$. From the Taylor series of the exponential, $\exp(-s \log(\varphi))$ has only a holomorphic part so $g_-(s) = \varepsilon$. Therefore the solutions $\psi(s)$ of the Lax pair equation are constant, so $\varphi(s) = \varphi(0) \in G^\phi_\mathcal{A}$.
7. The Lax pair flow and the renormalization group flow

The Lax pair flow lives on the Lie algebra $\mathfrak{g}A$ of infinitesimal characters, while the beta character flow is on the Lie group $G_A$ of characters. Theorem 6.10 and Corollary 6.11 show that under the exponential map $\exp : \mathfrak{g}A \to G_A$, the corresponding flow of beta characters and $\beta$-functions are not in Lax pair form. The main point of this section is that the bijection $\tilde{R}^{-1} : \mathfrak{g}A \to G_A$ of [13] is much better behaved: under $\tilde{R}^{-1}$, local characters remain local under the Lax pair flow (Theorem 7.5), and the beta characters and the $\beta$-functions satisfy Lax pair equations (Theorems 7.7). In contrast, we give a rooted trees example of the nonlocality of the Lax pair flow of characters using the exponential map.

7.1. The pole order under the Lax pair flow. To begin, we investigate the dependence of the pole order of the Lax pair flow $L(t)$ on the pole order of the initial condition $L_0$ and the Casimir function (e.g. the functions $\psi_{m,n}$). In the rooted trees case, the computations are considerably simplified using the normal coordinates of [3], which we refer to for details.

Let $H$ be the Hopf algebra of rooted trees and $T$ the set of trees. We choose an order on $T = \{ t_i \}_{i \in \mathbb{N}}$ such that $\deg(t_i) \leq \deg(t_j)$ for any $i < j$ and such that $h(t_i) \geq h(t_j)$ for any trees $t_i, t_j$ with $\deg(t_i) = \deg(t_j)$ and $i < j$. Here $\deg(t)$ is the number of vertices of $t$, and $h(t)$ is the height of the tree $t$, the length of the path from the root to the deepest node in the tree. For example, we can choose

$$ t_0 = 1_T, \quad t_1 = , \quad t_2 = 1, \quad t_3 = \lambda, \quad t_4 = \lambda, \quad t_5 = \lambda, \quad t_6 = \lambda, \quad t_7 = \lambda, \quad t_8 = \lambda. $$

We recall that the exp : $\mathfrak{g}A \to G_A$ is bijective with inverse log : $G_A \to \mathfrak{g}A$ given by

$$ \log(\varphi) = \sum_{k=1}^{\infty} (-1)^{k-1} (\varphi - \varepsilon)^k / k. $$

Set $f_0 = 1_T$ and let $\{ f_i \}_{i \in \mathbb{N}}$ be the normal coordinates, i.e. $f_i$ is the forest in $H$ satisfying

$$ \log(\varphi)(t_i) = (\varphi - \varepsilon)(f_i), $$

for every character $\varphi$. For example,

$$ f_1 = , \quad f_2 = 1 - \frac{1}{2}1, \quad f_3 = 1 - \lambda + \frac{1}{3}3, \quad f_4 = \lambda - \lambda + \frac{1}{6}6. $$

$$ f_5 = 1 - \frac{1}{2}1^2 + \frac{3}{4}3^2, \quad f_6 = \lambda - \frac{3}{2}\lambda + \frac{1}{2}21. $$

For a ladder tree $t$, the forest $f$ given by $\log(\varphi)(t) = (\varphi - \varepsilon)(f)$ for every character $\varphi$; is a primitive element of the Hopf algebra $H$.

We identify $\varphi \in G_A$ with $\{ \varphi(f_i) \}_{i \in \mathbb{N}} \in A^\mathbb{N}$ and call the $\varphi(f_i)$ the $i$-component of $\varphi$.

Since $\varphi(f_0) = 1$ for all $\varphi$, we drop the 0-component.
We use Sweedler’s notation for the reduced coproduct \( \tilde{\Delta}(x) = x' \otimes x'' \), where \( \tilde{\Delta}(x) = \Delta(x) - x \otimes 1_T - 1_T \otimes x \). Notice that \( \deg(x') + \deg(x'') = \deg(x) \) and \( 1 \leq \deg(x'), \deg(x'') < \deg(x) \). For \( x \neq 1_T \) and \( \tilde{\Delta}(x') = (x')' \otimes (x'')' \), we have

\[
((\varphi_1 \varphi_2) \varphi_1^{-1})(x) = ((\varphi_1 \varphi_2) \otimes \varphi_1^{-1}, x \otimes 1_T + 1_T \otimes x + x' \otimes x'')
\]

\[
= (\varphi_1 \varphi_2)(x) + \varphi_1^{-1}(x) + (\varphi_1 \varphi_2)(x') \varphi_1^{-1}(x'')
\]

\[
= \varphi(x + \varphi_2(x) + \varphi(x') \varphi_2(x'') + \varphi_1^{-1}(x) + (\varphi_1(x') + \varphi_2(x')
\]

\[
+ \varphi_1((x')') \varphi_2((x'')') \varphi_1^{-1}(x'')
\]

\[
= \varphi_2(x) + \varphi((x')' \varphi_2(x'') + (\varphi_1(x) + \varphi_1^{-1}(x) + \varphi_1((x') \varphi_2((x'')'))
\]

\[
+ \varphi_2((x') \varphi_1^{-1}(x'') + \varphi_1((x')' \varphi_2((x'')') \varphi_1^{-1}(x''))
\]

\[
= \varphi_2(x) + \varphi((x')' \varphi_2(x'') + \varphi_2(x') \varphi_1^{-1}(x'') + \varphi_1((x')' \varphi_2((x'')') \varphi_1^{-1}(x''))
\]

Differentiating with respect to \( \varphi_2 \) and setting \( L = \dot{\varphi}_2 \) gives the adjoint representation:

\[
(7.1) \quad \operatorname{Ad}(\varphi_1)(L)(x) = L(x) + \varphi_1(x') L(x'') + L(x') \varphi_1(Sx'')
\]

\[
+ \varphi_1((x')') L((x'')') \varphi_1(Sx''),
\]

where \( S \) is the antipode of the Hopf algebra.

**Theorem 7.1.** i) If the initial condition \( L_0 \in \mathfrak{g}_A \) is holomorphic in \( \lambda \), then the solution \( L(t) = \operatorname{Ad}(g_+(t))L_0 \) of the Lax pair equation is holomorphic in \( \lambda \).

ii) If \( L_0 \in \mathfrak{g}_A \) has a pole of order \( n \), then \( L(t) = \operatorname{Ad}(g_+(t))L_0 \) has a pole of order at most \( n \).

**Proof.** By (7.1), we have

\[
(7.2) \quad \operatorname{Ad}(g_+(t))(L_0)(x) = L_0(x) + g_+(t)(x') L_0(x'') + L_0(x') g_+(t)(Sx'')
\]

\[
+ g_+(t)((x')') L_0((x'')') g_+(t)(Sx'').
\]

Notice that \( g_+(t)(x) \) is holomorphic for \( x \in H \). If \( L_0 \) is holomorphic, then every term of the right hand side of (7.2) is holomorphic, so \( \operatorname{Ad}(g_+(t))(L_0) \) is holomorphic. Since multiplication with a holomorphic series cannot increase the pole order, \( L(t) \) cannot have a pole order greater than the pole order of \( L_0 \). \( \square \)

We can also use normal coordinates to measure the nontriviality of the Lax pair flow.

**Theorem 7.2.** If \( f_i \) is a primitive element of \( H \) (e.g. \( f_i \) corresponds to a ladder tree), then the i-component \( L(t)(f_i) \) of the Lax pair flow is does not depend on \( t \).

**Proof.** It is shown in [3] that \( \varphi^{-1}(f_i) = -\varphi(f_i) \), for every character \( \varphi \) and \( i \geq 1 \). Since \( f_i \) is a primitive element, the inner automorphism \( C_g : G_A \to G_A, C_g(h) = ghg^{-1} \), satisfies \( C_g(h)(f_i) = g(f_i) + h(f_i) - g(f_i) = h(f_i) \). Therefore, \( \operatorname{Ad}(g)L_0(f_i) = (L_0)_i \), where \( (L_0)_i \) is the i-component of \( L_0 \). \( \square \)
Thus everything of interest in the Lax pair flow occurs off the primitives, e.g. the normal coordinate $f_4$ corresponding to $\Lambda$ is the first component in the Hopf algebra on which the Lax pair is nonconstant.

### 7.2. The Lax pair flow, the RGE flow, and locality.

We now investigate whether the Lax pair flow can ever be identified with the RGF. Some identification is necessary: the RGF $(\varphi^t)_+(\lambda = 0)$ lives in the Lie group of characters $G_\mathbb{C}$, while the Lax pair flow $L(t)$ lives in a Lie algebra $\mathfrak{g}_\mathcal{A}$. To match these flows, we can transfer the Lax pair flow to the Lie group level using either of the maps $\tilde{R}^{-1}$ and exp, namely by defining

(7.3) \[ \varphi_t = \tilde{R}^{-1}(L(t)) \quad \text{and} \quad \chi_t = \exp(L(t)) \]

and then setting $\lambda = 0$.

The most naive hope would be that $\varphi_t$ or $\chi_t$ coincide with the RGF $\varphi^t$, perhaps after a rescaling of the parameter $t$. We shall see that this fails even in the trivial case. As usual, we take $\mathcal{A}$ to be the algebra of Laurent series.

**Proposition 7.3.** On a commutative, co-commutative, graded connected Hopf algebra $\mathcal{H}$, $\varphi_t \neq \varphi^t$ and $\chi_t \neq \varphi^t$.

**Proof.** If $\tilde{R}(\varphi^t) = L(t)$ then by Definition 6.1

\[ (\varphi^t \circ Y)(\Gamma) = (\varphi^t \star L(t))(\Gamma) \]

for every $\Gamma \in \mathcal{H}$. For a primitive homogenous element $\Gamma \in \mathcal{H}_n$ we get

(7.4) \[ |\Gamma| e^{\|\Gamma\|\lambda} \varphi(\Gamma) = e^{\|\Gamma\|\lambda} \varphi(\Gamma)L(t)(1) + e^{0\|\Gamma\|\lambda} \varphi(\Gamma)L(t)(\Gamma). \]

Therefore

\[ L(t)(\Gamma) = |\Gamma| e^{\|\Gamma\|\lambda} \varphi(\Gamma). \]

Since $\mathcal{H}$ is co-commutative, its Lie bracket is abelian. Thus the left hand side of (7.4) is constant in $t$, while the right hand side is not.

The same argument works for $\chi_t$ on $\mathcal{H}$. \qed

In a positive direction, we will show that locality of characters is preserved under the Lax pair flow, using the identification given by $\tilde{R}$. This indicates that $\tilde{R}$ is more useful than the exponential map.

Recall from [13, Theorem IV.4.1] that $\lambda \tilde{R} : G_\mathcal{A} \rightarrow \mathfrak{g}_\mathcal{A}$ restricts to a bijection from $G^\Phi_\mathcal{A}$ to $\mathfrak{g}_\mathcal{A}^+$, where $\mathfrak{g}_\mathcal{A}^+$ is the set infinitesimal characters on $\mathcal{H}$ with values in $\mathcal{A}^+$. In this sense, $\lambda \tilde{R}$ is better behaved than $\tilde{R}$, as the following locality result shows.

**Proposition 7.4.** For a local character $\varphi \in G^\Phi_\mathcal{A}$, let $L(t)$ be the solution of the Lax pair equation (5.5) with initial condition $L_0 = \lambda \tilde{R}(\varphi)$ and any Ad-covariant function $f$. Let $\tau_t$ be the flow of characters given by

\[ \tau_t = (\lambda \tilde{R})^{-1}(L(t)). \]
Then \( \tau_t \) is a local character for all \( t \).

**Proof.** By [8],

\[
\tilde{R}(\varphi \ast \xi) = \tilde{R}(\xi) + \xi^{-1} \ast \tilde{R}(\varphi) \ast \xi.
\]

Taking \( \xi = g_+(t)^{-1} \) and multiplying by \( \lambda \), we get

\[
\lambda \tilde{R}(\varphi \ast g_+(t)^{-1}) = \lambda \tilde{R}(g_+(t)^{-1}) + g_+(t) \ast \lambda \tilde{R}(\varphi) \ast g_+(t)^{-1}.
\]

Since \( \varphi \in G^\Phi_A \) and \( g(t)^+_1 \) is an element in \( G_A \) without polar part, by [15, Lemma IV.4.3.], \( \varphi \ast g(t)^+_1 \) is local. Thus \( \lambda \tilde{R}(\varphi \ast g_+(t)^{-1}) \in A_+ \). We have \( g(t)^+_1 \in G^\Phi_A \), simply because \( g(t)^+_1 \) does not have a polar part, so \( \lambda \tilde{R}(g(t)^+_1) \) is holomorphic. It follows that

\[
\tau_t = (\lambda \tilde{R})^{-1}(\text{Ad}(g_+(t))L_0) = (\lambda \tilde{R})^{-1}(g_+(t) \ast L_0 \ast g_+(t)^{-1}) \in G^\Phi_A
\]

\( \square \)

We can now show that locality of characters is preserved under the Lax pair flow via the \( \tilde{R} \) identification.

**Theorem 7.5.** For a local character \( \varphi \in G^\Phi_A \), let \( L(t) \) be the solution of the Lax pair equation (5.5) for any \( \text{Ad} \) -covariant function \( f \), with the initial condition \( L_0 = \tilde{R}(\varphi) \). Let \( \varphi_t \) be the flow given by

\[
\varphi_t = \tilde{R}^{-1}(L(t)).
\]

Then \( \varphi_t \) is a local character for all \( t \).

**Proof.** We show that the flow \( \tau_t \) constructed in the previous Proposition with the initial condition \( L_0 = \lambda \tilde{R}(\varphi) \), for the \( \text{Ad} \) -covariant function \( h : g_A \to g_A \) given by \( h(L) = f(\lambda^{-1}L) \), is equal to the flow \( \varphi_t \) constructed for the \( \text{Ad} \) -covariant function \( f \), with the initial condition \( L_0 = \tilde{R}(\varphi) \). We have

\[
\tau_t = (\lambda \tilde{R})^{-1} \left( g_+(t) \ast \lambda \tilde{R}(\varphi) \ast g_+^{-1}(t) \right) = \tilde{R}^{-1} \left( g_+(t) \ast \tilde{R}(\varphi) \ast g_+^{-1}(t) \right),
\]

where \( g_+(t) \) is given by the Birkhoff decomposition of

\[
\exp(-tf(\tilde{R}(\varphi))) = \exp(-th(\lambda \tilde{R}(\varphi))).
\]

Therefore the two \( g_+(t) \) involved in the definitions of \( \varphi_t \) and \( \tau_t \) coincide, so \( \varphi_t = \tau_t \). \( \square \)

In contrast to Theorem [6.10], it is immediate that the flow of beta characters associated to \( \tilde{R} \) is in Lax pair form.

**Lemma 7.6.** For a local character \( \varphi \in G^\Phi_A \), let \( \varphi_t \) be the flow from Theorem 7.5. Then

\[
\frac{d\tilde{\beta}_{\varphi_t}}{dt} = [\tilde{\beta}_{\varphi_t}, M],
\]

(7.5)
Proof. By Lemma 6.7, we get $\tilde{\beta}_{\varphi_t} = \lambda \tilde{R}(\varphi_t) = \lambda L(t)$. Then
\[
\frac{d\tilde{\beta}_{\varphi_t}}{dt} = \frac{d(\lambda \tilde{R}(\varphi_t))}{dt} = \frac{\lambda dL(t)}{dt} = \lambda [L(t), M] = [\tilde{\beta}_{\varphi_t}, M].
\]

The corresponding $\beta$-functions also satisfy a Lax pair equation.

**Theorem 7.7.** For a local character $\varphi \in C^\phi_{\mathfrak{A}}$, let $L(t)$ be the Lax pair flow of Corollary 5.10 with initial condition $L_0 = \tilde{R}(\varphi)$. Let $\varphi_t = \tilde{R}^{-1}(L(t))$. Then

(i) for $-n + 2m \geq 1$, $\varphi_t = \varphi$ and hence $\beta_{\varphi_t} = \beta_\varphi$ for all $t$.

(ii) For $-n + 2m \leq 0$, $\beta_{\varphi_t} \in \mathfrak{g}_\mathbb{C}$ satisfies
\[
\frac{d\beta_{\varphi_t}}{dt} = [\beta_{\varphi_t}, -\frac{d((\varphi_t) + (0))}{dt}((\varphi_t) + (0))^{-1} + 2\text{Ad}((\varphi_t) + (0))(\text{Res}(\lambda^{-n+2m-2}\beta_{\varphi_t}))].
\]

Proof. By Theorem 7.5, $\varphi_t$ are local characters, so by [15, Theorem IV.4.], $\tilde{\beta}_{\varphi_t} = \lambda L(t) = \lambda \tilde{R}(\varphi_t)$ is holomorphic.

(i) If $-n + 2m \geq 1$, then $\lambda^{-n+2m}L(t)$ is holomorphic, which implies
\[
M = R(\lambda^{-n+2m}L(t)) = \lambda^{-n+2m}L(t) = \lambda^{-n+2m-1}\tilde{\beta}_{\varphi_t}.
\]

$L(t)$ satisfies the Lax pair equation
\[
\frac{dL}{dt} = [L, M] = [L, \lambda^{-n+2m}L] = \lambda^{-n+2m}[L, L] = 0.
\]

Thus $L(t) = L_0$ for all $t$, which gives $\varphi_t = \tilde{R}^{-1}(L(t)) = \tilde{R}^{-1}(L_0) = \varphi$ for all $t$.

(ii) For $-n + 2m \leq 0$, we have
\[
M = R(\lambda^{-n+2m}L(t)) = \lambda^{-n+2m}L(t) - 2P_{-}(\lambda^{-n+2m}L(t)) = \lambda^{-n+2m-1}\tilde{\beta}_{\varphi_t} - 2P_{-}(\lambda^{-n+2m-1}\tilde{\beta}_{\varphi_t}).
\]

Let becomes
\[
\frac{d\tilde{\beta}_{\varphi_t}}{dt} = -2[\tilde{\beta}_{\varphi_t}, P_{-}(\lambda^{-n+2m-1}\tilde{\beta}_{\varphi_t})] = -2[P_{+}(\lambda^{-n+2m-1}\tilde{\beta}_{\varphi_t}), \lambda^{-n+2m+1}P_{-}(\lambda^{-n+2m-1}\tilde{\beta}_{\varphi_t})].
\]

Expand $\tilde{\beta}_{\varphi_t}$ as
\[
\tilde{\beta}_{\varphi_t} = \sum_{k=0}^{\infty} \tilde{\beta}_k(t)\lambda^k.
\]

Then
\[
\frac{d\tilde{\beta}_{\varphi_t}}{dt} = -2 \left[ \sum_{k=n-2m+1}^{\infty} \tilde{\beta}_k(t)\lambda^{k-n+2m-1}, \sum_{j=0}^{n-2m} \tilde{\beta}_j(t)\lambda^j \right],
\]

and evaluating at $\lambda = 0$ gives
\[
(7.6) \quad \frac{d\tilde{\beta}_0(t)}{dt} = -2[\tilde{\beta}_{n-2m+1}(t), \tilde{\beta}_0(t)] = 2[\tilde{\beta}_0(t), \tilde{\beta}_{n-2m+1}(t)].
\]
Using the facts that $\text{Ad}(g)$ is a Lie algebra homomorphism and $(d/dt)\text{Ad}(g(t))X = [(dg(t)/dt)g^{-1}(t), \text{Ad}(g(t))X]$ (see the proof of Theorem 7.4), we get

$$
\frac{d\beta_{\varphi_t}}{dt} = \frac{d}{dt} \left( \text{Ad}(\varphi_t)(\tilde{\beta}_0(t)) \right)
$$

(7.7) $= \left[ \frac{d((\varphi_t)+0)}{dt}((\varphi_t)+0)^{-1}, \text{Ad}(\varphi_t)(\tilde{\beta}_0(t)) \right] + \text{Ad}((\varphi_t)+0) \left( \frac{d\tilde{\beta}_0(t)}{dt} \right)

= \left[ \frac{d((\varphi_t)+0)}{dt}((\varphi_t)+0)^{-1}, \beta_{\varphi_t} \right] + 2 \left[ \text{Ad}((\varphi_t)+0)\tilde{\beta}_0(t), \text{Ad}((\varphi_t)+0)\tilde{\beta}_{n-2m+1}(t) \right]

= \left[ \beta_{\varphi_t}, -\frac{d((\varphi_t)+0)}{dt}((\varphi_t)+0)^{-1} + 2\text{Ad}((\varphi_t)+0) \left( \text{Res}(\lambda^{-n+2m-2}\tilde{\beta}_{\varphi_t}) \right) \right],

since $\tilde{\beta}_{n-2m+1}(t) = \text{Res}(\lambda^{-n+2m-2}\tilde{\beta}_{\varphi_t})$. \qed

Local characters satisfy the abstract Renormalized Group Equation [9], which we now recall. For a local character $\varphi \in G^\Phi_A$, the renormalized character is defined by $\varphi_{\text{ren}}(t) = (\varphi^t)_+(\lambda = 0)$.

**Theorem 7.8.** For $\varphi \in G^\Phi_A$, the renormalized character $\varphi_{\text{ren}}$ satisfies the abstract Renormalized Group Equation:

$$
\frac{\partial}{\partial t} \varphi_{\text{ren}}(t) = \beta_{\varphi} \ast \varphi_{\text{ren}}(t).
$$

Here our parameter $t$ corresponds to $e^t$ in [9].

In light of Theorem 7.5, we can ask for the relation between $(\varphi_t)_{\text{ren}}(s)$ and $\varphi_{\text{ren}}(s)$ corresponding to $\varphi_t$ and $\varphi$. In §9, we consider a toy model character on a Hopf algebra of rooted trees and show that these renormalized characters differ.

We can also show that for certain initial conditions, the flow $\tau_t$ is constant.

**Proposition 7.9.** If $\varphi \in G^\Phi_A$ and $\varphi_+ = \varepsilon$ (i.e. $\varphi$ has only a pole part), then the flow $\tau_t$ of Proposition 7.4 for the Ad-covariant function $f(L) = \lambda^{-n+2m}L$ has $\tau_t = \varphi$ for all $t$.

**Proof.** If we show that either $g_+(t) = \varepsilon$, then

$$
\tau_t = (\lambda\tilde{R})^{-1}(g_+(t) \ast \lambda\tilde{R}(\varphi) \ast g_+(t)^{-1}) = (\lambda\tilde{R})^{-1}(\varepsilon \ast \lambda\tilde{R}(\varphi) \ast \varepsilon^{-1}) = \varphi.
$$

$g_+(t)$ are given by the Birkhoff decomposition of

$$
g(t) = \exp(-2t\lambda^{-n+2m}L_0) = \sum_{k=0}^{\infty} \frac{(-2t\lambda^{-n+2m})^k L_0^k}{k!},
$$

where $L_0 = \lambda\tilde{R}(\varphi) \in \mathfrak{g}_C$ [15, Theorem IV.4.4]. We split the problem into two cases depending on the sign of $-m+2n$. If $-m+2n \geq 0$, then $g(t)(x) \in \mathcal{A}_+$ for any $x$, which implies $g_-(t) = \varepsilon$. Similarly, if $-m+2n < 0$, then $g(t)(x) \in \mathcal{A}_-$ for any $x$, which implies $g_+(t) = \varepsilon$. Notice that the right hand side of (7.8) is a finite sum, namely up to $k = \deg(x)$ when evaluated on $x \in \mathcal{H}$. \qed
We next study the locality of the flow \( \chi_t \) defined in (7.3) for the usual Lax pair flow \( L(t) \). Thus for an initial character \( \varphi \) and \( L_0 = \log(\varphi) \),

\[
\chi_t = \exp(g_+(t) * L_0 * g_+(t)^{-1}) = g_+(t) * \exp(L_0) * g_+(t)^{-1} = g_+(t) * \varphi * g_+(t)^{-1},
\]

with \( \chi_0 = \varphi \). As before, in normal coordinates \( \chi_t \) is trivial on primitives.

**Lemma 7.10.** If \( \varphi \in G^\Phi_A \) and \( f_i \) is a primitive element, then \( \chi_t(f_i) \) does not depend on \( t \).

**Proof.** \( \chi_t(f_i) = g_+(t)(f_i) + \varphi(f_i) - g_+(t)(f_i) = \varphi(f_i) \).

We now present some calculations showing the interplay between the Lax pair flow and locality.

For the first example, we construct a nontrivial Hopf subalgebra on which \( \chi_t \) is local. Let \( \mathcal{H}^2 \) be the Hopf subalgebra generated by the following trees

\[
t_0 = 1_T, \quad t_1 = *, \quad t_2 = 1, \quad t_4 = \Lambda,
\]

together with any set of ladder trees. The first normal coordinate of \( \chi_t \) to depend on \( t \) is \( f_4 \), corresponding to \( \Lambda \). Let \( G^2_A \) be the group of characters associated to the \( \mathcal{H}^2 \).

**Proposition 7.11.** For \( \varphi \in G^2_A \), let \( \chi_t \) be the flow of characters on \( \mathcal{H}^2 \) given by

\[
\chi_t = \exp(L(t)),
\]

where \( L(t) \) is the solution of the Lax pair equation (5.5) for any Ad-covariant function with the initial condition \( L_0 = \log(\varphi) \). Then \( \chi_t \) is local for all \( t \).

**Proof.** Let \( \pi \) denote the projection to the pole part of a Laurent series. By [3], \( \tilde{\Delta}(f_4) = f_1 \otimes f_2 - f_2 \otimes f_1 \), so

\[
(\chi_t^s)(f_4) = \pi(\chi_t^s(f_4) + (\chi_t^s)(f_1)\chi_t^s(f_2) - (\chi_t^s)(f_2)\chi_t^s(f_1)).
\]

Subtracting from this equation the corresponding equation for \( t = 0 \), and remembering that \( \chi_t(f_1) \) and \( \chi_t(f_2) \) do not depend on \( t \) by Lemma 7.10, we get

\[
(\chi_t^s)(f_4) = (\varphi^s)(f_4) - \pi(\chi_t^s(f_4) - \varphi^s(f_4)).
\]

We have

\[
\pi(\chi_t^s(f_4) - \chi^s(f_4)) = \pi(e^{3s\lambda}(-2g_+(t)(f_2)\varphi(f_1) + 2g_+(t)(f_1)\varphi(f_2))) \quad = \pi(-2g_+(t)(f_2)\pi(e^{3s\lambda}\varphi(f_1)) + 2g_+(t)(f_1)\pi(e^{3s\lambda}\varphi(f_2)) \quad = \pi(-2g_+(t)(f_2)\pi(e^{3s\lambda}\varphi(f_1)) + 2g_+(t)(f_1)\pi(e^{3s\lambda}\varphi(f_2))).
\]

Since \( \varphi \in G^2_A \), both \( \pi(e^{s\lambda}\varphi(f_1)) = -(\varphi^s)(f_1) \) and \( \pi(e^{2s\lambda}\varphi(f_2)) = -(\varphi^s)(f_2) \) are independent of \( s \). By rescaling \( s \), \( \pi(e^{3s\lambda}\varphi(f_1)) \) and \( \pi(e^{3s\lambda}\varphi(f_2)) \) are independent of \( s \). Therefore \( \pi(\chi_t^s(f_4) - \varphi^s(f_4)) \) is independent of \( s \), which finishes the proof. \( \square \)
Remark 7.12. We can apply the previous proposition to the Hopf subalgebra of Feynman diagram generated by the empty graph and the graphs

\[ A_1 = \begin{array}{c}
\end{array}, A_2 = \begin{array}{c}
\end{array}, A_3 = \begin{array}{c}
\end{array}, A_4 = \begin{array}{c}
\end{array}, A_5 = \begin{array}{c}
\end{array} \]

and with \( \varphi \) the Feynman rules character. The characters \( \chi_t \) restricted to this Hopf algebra are all local.

To investigate how \( \chi_t \) fails to be local on a non-ladder tree with a larger number of vertices, we consider the Hopf subalgebra \( \mathcal{H}^3 \) generated by

\[ t_0 = 1, \quad t_1 = \lambda, \quad t_2 = 1, \quad t_3 = \Lambda, \quad t_4 = \lambda, \quad t_5 = \Lambda, \quad t_6 = \lambda, \quad t_7 = \lambda, \quad t_8 = \Lambda \]

together with any set of ladder trees. Let \( f_i \) be the corresponding normal coordinates.

The next lemma gives the pole order of a local character \( \varphi \in C^\Phi_A \) on primitives.

Lemma 7.13. If \( \varphi \) is a local character and \( f_i \) is primitive, then both \( \varphi(f_i) \) and \( L_0(f_i) = \log(\varphi)(f_i) \) have a pole of order at most one.

Proof. If \( \deg(f_i) = d \) and \( \varphi = \sum_{k=-n}^{\infty} \varphi_k \lambda^k \) is the Laurent expansion of \( \varphi \), then

\[
(\varphi^\ast)(f_i) = -\pi(e^{\varphi(f_i)}) = -\pi(\varphi^{-n}(f_i) \lambda^{-n} + (\varphi^{-n}(f_i) s d + \varphi^{-n+1}(f_i)) \lambda^{-n+1} + o(\lambda^{-n+2}))
\]

If \( \varphi \) has a pole, then \( \varphi^\ast = \varphi^- \) implies \( -n + 1 = 0 \).

Proposition 7.14. Let \( \varphi \) be a local character on \( \mathcal{H}^3 \) and let \( \chi_t \) be the flow of characters given by

\[ \chi_t = \exp(L(t)), \]

where \( L(t) \) is the solution of the Lax pair equation (7.8) with the initial condition \( L_0 = \log(\varphi) \). Then \( \chi_t \) is local on \( \mathcal{H}^3 \) for all \( t \) if and only if either

\[ \varphi^-(f_1) = 0 \quad \text{or} \quad 3\varphi^-(f_1)(\varphi^+(f_2))_{\lambda=0} = \varphi^-(f_2)(\varphi^+(f_1))_{\lambda=0}. \]

The point is that (7.9) is unlikely to hold.

Proof. By [3],

\[
\Delta(f_8) = \frac{3}{2} f_1 \otimes f_4 - \frac{3}{2} f_4 \otimes f_1 - \frac{1}{2} f_1 \otimes f_1 f_2 - \frac{1}{2} f_2 f_2 \otimes f_1 + \frac{1}{2} f_1 f_1 f_2 \otimes f_2 + \frac{1}{2} f_2 f_1 f_1. \]

Since \( (\chi_t^\ast)(f_4) \) does not depend on \( s \) (Prop. 7.11) and since \( \chi_t(f_1) \) and \( \chi_t(f_2) \) do not depend on \( t \), after cancelations of terms involving only the primitives \( f_1 \) and \( f_2 \), we get

\[
(\chi_t^\ast)(f_8) = (\varphi^\ast)(f_8) \]

(7.10)

\[
= -\pi\left(\chi_t^\ast(f_8) - \varphi^\ast(f_8) + \frac{3}{2} \varphi^-(f_1)(\chi_t^\ast(f_4) - \varphi^\ast(f_4)) - \frac{3}{2} \varphi^\ast(f_1)((\chi_t^\ast)(f_4) - \varphi^-(f_4))\right)
\]

(7.11)
By Lemma 7.13, \( \varphi(f_1) \) and \( \varphi(f_2) \) have poles of order at most one. Set

\[
\varphi(f_1) = \sum_{k=-1}^{\infty} a_k \lambda^k \quad \text{and} \quad \varphi(f_2) = \sum_{k=-1}^{\infty} b_k \lambda^k.
\]

From the proof of Proposition 7.11,

\[
\chi^*_t(f_4) - \varphi^*(f_4) = e^{3\lambda}(2g_+(t)(f_2)\varphi(f_1) + 2g_+(t)(f_1)\varphi(f_2))
\]

We have \( g_-(t)(f_1) = \exp(-2t\lambda^{-n+2m}L_0)(f_1) = -\pi(-2t\varphi(f_1)) = 2ta_-\lambda^{-1} \), and \( g_+(t)(f_1) = -2t(a_0 + a_1\lambda + o(\lambda^2)) \). Similarly \( g_-(t)(f_2) = 2tb_-\lambda^{-1} \). \( (7.12) \) becomes

\[
\chi^*_t(f_4) - \varphi^*(f_4) = 2te^{3\lambda}((-b_-a_0 + a_-b_0)\lambda^{-1} + o(\lambda^0)),
\]

which implies

\[
\frac{\pi}{2}\varphi_-(f_1)(\chi^*_t(f_4) - \varphi^*(f_4)) - \frac{3}{2}\varphi^*(f_1)((\chi_t)_-(f_4) - \varphi_-(f_4))
\]

\[
= \frac{3}{2}\pi((-a_-\lambda^{-1}e^{3\lambda}(4t((-b_-a_0 + a_-b_0)\lambda^{-1} + o(\lambda^0))))
\]

\[
- \frac{3}{2}\pi(e^{s\lambda}(a_-\lambda^{-1} + o(\lambda^0)))(-1)(b_-a_0 + a_-b_0)\lambda^{-1})
\]

\[
= -12sta_1(-b_-a_0 + a_-b_0)\lambda^{-1} + P(\lambda^{-1}),
\]

where \( P(\lambda^{-1}) \) is some polynomial in \( \lambda^{-1} \) independent of \( s \). We get

\[
\pi\left(\chi^*_t(f_8) - \varphi^*(f_8)\right)
\]

\[
= \pi\left(e^{4\lambda}(4g_+(t)(f_1)g_+(t)(f_2) - 3g_+(t)(f_3))\varphi(f_1))\right)
\]

\[
+ 3\pi\left(e^{4\lambda}(g_+(t)(f_1)^2\varphi(f_2)) + \pi\left(3e^{4\lambda}g_+(t)(f_1)\varphi(f_1)\right)
\]

Since \( \varphi(f_1) \) and \( \varphi(f_2) \) have poles of order at most one, the first two terms of the right hand side of \( (7.13) \) do not depend on \( s \). Since \( \varphi \in G^{3uf}_A \),

\[
\varphi^*(f_4) = -\pi\left(e^{3\lambda}\varphi(f_4) + \varphi^*(f_1)e^{2\lambda}\varphi(f_2) - \varphi^*(f_2)e^{s\lambda}\varphi(f_1)\right)
\]

is independent of \( s \), and so \( \pi(e^{3\lambda}\varphi(f_4) - a_-b_-s\lambda^{-1}) \) is also independent of \( s \). By rescaling \( s \), \( \pi(3e^{3\lambda}\varphi(f_4) - 4a_-b_-s\lambda^{-1}) \) does not depend on \( s \). In conclusion, the terms independent of \( s \) in \(-\chi^*_t(f_8) + \varphi^*(f_8)\) are

\[
-12sta_1(-b_-a_0 + a_-b_0)\lambda^{-1} + 4a_-b_-s\lambda^{-1}(-2ta_0).
\]

Therefore \( \chi^*_t(f_8) \) is independent of \( s \) if and only if either \( a_- = 0 \) or \( 3a_-b_0 = a_0b_- \). Similar computations hold for the normal coordinates \( f_6 \) and \( f_7 \). \( \square \)

Remark 7.15. The choice \(-n+2m = 0\) in the Proposition is just for the sake of concreteness. A more detailed analysis reveals the following:

- If \(-n+2m \geq 1\) then \( \chi_t \) is local on \( \mathcal{H}^3 \) without any additional conditions. Indeed, in this case \( L_0(f_1) \) and \( L_0(f_2) \) are holomorphic and thus \( g_-(t)(f_1) = g_-(f_2) = 0 \), which implies that \( \chi^*_t(f_4) - L_0(f_4) = 0 \) for every \( t \). By \( (7.10) \) we get that \( (\chi^*_t)_-(f_8) \) does not depend on \( s \). Similar statements hold for \( f_6 \) and \( f_7 \).
- If \(-n + 2m = -1\), the situation is similar to Proposition 7.14, namely \(\chi_t\) is local on \(\mathcal{H}^3\) if and only if

\[
\varphi_-(f_1)(\varphi_+(f_2)|_{\lambda=0}) = \varphi_-(f_2)(\varphi_+(f_1)|_{\lambda=0})
\]

and either

\[
\varphi_-(f_1) = 0 \quad \text{or} \quad 3\varphi_-(f_1)(\frac{\partial \varphi_+(f_2)}{\partial \lambda}|_{\lambda=0}) = \varphi_-(f_2)(\frac{\partial \varphi_+(f_1)}{\partial \lambda}|_{\lambda=0}).
\]

For \(-n + 2m \in \mathbb{Z}^+\), the flows \(\chi_t\) and \(L(t)\) gain locality, in the sense that they become constant on larger Hopf subalgebras as \(-n + 2m\) increases. Indeed, \(\chi_t\) and \(L(t)\) are constant on the Hopf algebra generated by the primitives (e.g., the normal coordinates associated to ladder trees. In contrast, if we decrease \(-n + 2m < 0\), we preserve locality only when an increasing number of conditions are fulfilled.

7.3. The Lax pair flow of the \(\beta\)-function. Recall from §6 that the beta characters for the exponentiated Lax pair flow \(\exp(L(t))\) do not themselves satisfy a Lax pair equation. In the next theorem, we reverse this procedure by taking a Lax pair flow \(L(t)\) starting at the \(\beta\)-function of a character, and then producing characters \(\xi_t\) whose \(\beta\)-functions are \(L(t)|_{\lambda=0}\).

**Theorem 7.16.** Let \(\varphi \in G^0_A\) and let \(L(t)\) be the flow given by Theorem 5.9 with the initial condition \(L_0 = \beta_\varphi\). Let \(\xi_t = (\lambda \tilde{R})^{-1}(L(t)|_{\lambda=0})\). Then \(\xi_t\) is local for all \(t\). The \(\beta\)-function of \(\xi_t\) satisfies

\[
\beta_{\xi_t} = L(t)|_{\lambda=0}.
\]

Moreover, \(\xi(0) = \varphi_-\).

**Proof.** Since \(L_0 = \beta_\varphi\) is scalar valued, by Theorem 7.14 \(L(t) = \text{Ad}(g_+(t))(L_0)\) is holomorphic. Therefore, \(L(t)\) can be evaluated at \(\lambda = 0\). \(L(t)|_{\lambda=0}\) is also scalar valued, so by [15] Theorem IV.4.4], \(\xi_t \in G^0_{A-}\), the set of local characters taking values in \(\lambda^{-1}\mathbb{C}[\lambda^{-1}]\). In particular, \(\xi_t\) is local. By Lemma 6.7, \(\tilde{\beta}_{\xi_t} = (\lambda \tilde{R})(\xi_t) = L_t|_{\lambda=0}\), so \(\tilde{\beta}_{\xi_t}\) must be constant in \(\lambda\). This implies

\[
\beta_{\xi_t} = \tilde{\beta}_{\xi_t}|_{\lambda=0} = L(t)|_{\lambda=0}.
\]

It follows from the Connes-Kreimer scattering formula [15] that \(\xi(0) = \varphi_-\). \(\square\)

8. Worked examples

In this section we give some explicit computations on two Hopf algebras which illustrate results in previous sections. We first consider the Hopf algebra \(\mathcal{H}^1\) generated by the following trees:

\[
t_0 = 1_T, \quad t_1 = ., \quad t_2 = 1, \quad t_4 = \lambda, \quad t_8 = \lambda^2,
\]

and the regularized toy model character \(\varphi = \varphi(q, \mu, \lambda)\) (see [12] 6) given on trees by

\[
\varphi(T)(q, \mu, \lambda) = (q/\mu)^{-\lambda \text{deg}(T)} \prod_v B_{w(T_v)}(\lambda).
\]
Here the product is taken over all vertices $v$ of the tree $T$, $w(T_v)$ is the number of vertices of the subtree $T_v$ of $T$ which has $v$ as a root, and $B_j(\lambda) = B(j\lambda, 1-j\lambda)$ for $j \in \mathbb{N}^*$, with $B$ the Euler beta function. Referring to [8], $q$ is interpreted as a dimensional external parameter, and $\mu$ is the ‘t Hooft mass. $\varphi$ has enough similarity of realistic QFT calculations to be worth considering [2, 6, 12, 13]. Set $b = q/\mu$ and $a = \log(b)$. Thus terms in $a$ (or $\log(q^2/\mu^2) = 2a$ as in e.g. [13]) are the leading log terms in the various expansions. We have $\varphi(\ast) = b^{-\lambda}B_1(\lambda)$, $\varphi(1) = b^{-2\lambda}B_2(\lambda)B_1(\lambda)$, $\varphi(\lambda) = b^{-3\lambda}B_3(\lambda)B_1(\lambda)^2$, $\varphi(\lambda^2) = b^{-4\lambda}B_4(\lambda)B_1(\lambda)^3$, etc. In the normal coordinates $f_i$, the Laurent series of $\varphi(f_i)$ (cf. [3]) are given by

$$\varphi(f_1) = \frac{1}{\lambda} - a + o(\lambda), \quad \varphi(f_2) = \frac{\pi^2}{4} + o(\lambda),$$

$$\varphi(f_4) = \frac{7\pi^2}{36\lambda} - \frac{7\pi^2 a}{12} + o(\lambda), \quad \varphi(f_8) = \frac{\pi^2}{12\lambda^2} - \frac{\pi^2 a}{3\lambda} + o(\lambda^0).$$

The character $\varphi$ is local, and the Lax pair flow on $H^1$ $\varphi_t = \tilde{R}^{-1}(L(t))$ as in Corollary 5.10 with $L(t) = Ad(g_{\pm}(t))L_0$, $-m + 2n = 0$, and initial condition $L_0 = \tilde{R}(\varphi)$ is given by

$$\varphi_t(f_1) = \varphi(f_1), \quad \varphi_t(f_2) = \varphi(f_2), \quad \varphi_t(f_4) = \frac{\pi^2(7 + 24t)}{36\lambda} - \frac{\pi^2}{12}(7 + 16t)a + o(\lambda),$$

$$\varphi_t(f_8) = \frac{\pi^2(1 + 6t)}{12\lambda^2} - \frac{\pi^2}{6}(2 + 15t + 9t^2)a + \frac{\pi^2}{144}(\pi^2(83 + 288t + 126t^2) + 12(8 + 51t + 27t^2)a^2) + o(\lambda).$$

The flow $L(t)$ has poles of order at most one:

$$L(t)(f_1) = \frac{1}{\lambda} - a + o(\lambda), \quad L(t)(f_2) = \frac{\pi^2}{2} + o(\lambda),$$

$$L(t)(f_4) = \frac{\pi^2}{3} + \frac{2\pi^2 t}{\lambda} + (-\pi^2 - 4\pi^2 t)a + o(\lambda),$$

$$L(t)(f_8) = \frac{-2(\pi^2 t(2 + 3t)a)}{\lambda} + \frac{\pi^2}{6}(\pi^2(7 + 37t + 21t^2) + 6t(8 + 9t)a^2) + o(\lambda).$$

This confirms that $\lambda L(t) = (\lambda \tilde{R})(\varphi_t) = \tilde{\beta}_{\varphi_t}$ is holomorphic, which implies that $\varphi_t$ is local on $H^1$. It can be explicitly checked that $(\varphi_t^-)_-$ does not depend on $s$:

$$(\varphi_t^-)_-(f_1) = -\frac{1}{2}, \quad (\varphi_t^-)_-(f_2) = 0, \quad (\varphi_t^-)_-(f_4) = \frac{\pi^2(1 - 12t)}{18\lambda},$$

$$(\varphi_t^-)_-(f_8) = -\frac{\pi^2(1 - 12t)}{24\lambda^2} + \frac{3\pi^2 t(1 + t)a}{2\lambda}.$$ 

The Connes-Kreimer $\beta$-functions $\beta_{\varphi_t} = -\text{Res}(\varphi_t^-) \circ Y$ are:

$$\beta_{\varphi_t}(f_1) = 1, \quad \beta_{\varphi_t}(f_2) = 0, \quad \beta_{\varphi_t}(f_4) = -\frac{\pi^2}{6} + 2\pi^2 t, \quad \beta_{\varphi_t}(f_8) = -6\pi^2 t(1 + t)a.$$
The associated RGFs \((\varphi_t)_{\text{ren}}(s) = (\varphi^+_t)_{\text{ren}}|_{\lambda=0}\), which all satisfy the abstract RGE, are

\[
(\varphi_t)_{\text{ren}}(s)(f_1) = (s-a), \quad (\varphi_t)_{\text{ren}}(s)(f_2) = \frac{\pi^2}{4},
\]

\[
(\varphi_t)_{\text{ren}}(s)(f_4) = \frac{\pi^2}{12}(s + 24st - (1 + 16t)a),
\]

\[
(\varphi_t)_{\text{ren}}(s)(f_8) = \frac{\pi^2}{96}(12s^2 + \pi^2(11 + 136t + 84t^2) - 24s(1 + 20t + 24t^2)a + 12(1 + 22t + 18t^2)a^2).
\]

Thus in this model there is a polynomial dependence in \(t\) of the leading log terms in each of \(L(t), \beta_{\varphi_1}\) and \((\varphi_t)_{\text{ren}}\), although the first diagram with a nonzero leading log term differs.

By the recursion formula in \([14, (26)]\), the next to ... leading log terms in the Green’s functions will then also depend polynomially on \(t\). We conjecture that this polynomial dependence extends to the Feynman rules character on the full Hopf algebra.

We emphasize that the Renormalized Group Flows \((\varphi_t)_{\text{ren}}\) and \(\varphi_{\text{ren}}(s) = (\varphi_t)_{\text{ren}}(s)|_{t=0}\) of the characters \(\varphi_t\) and \(\varphi\) are different. While \(\varphi_t(f_1), \varphi_t(f_2)\) are independent of \(t\), we have

\[
(\varphi_t)_{\text{ren}}(s)(f_3) = (s - 4t)(\varphi_t(f_1) - \varphi_t(f_2)\varphi(f_1)).
\]

We have used \(\varphi_t = \bar{R}^{-1}(\text{Ad}(g_-(t))\bar{R}((\varphi))\) for this calculation, since it is easier to extract the pole part of a Laurent series than the holomorphic part, but we could also use \(\varphi_t = \bar{R}^{-1}(\text{Ad}(g_+(t))\bar{R}(\varphi))\). In this case, we get

\[
\varphi_t(f_3) = (s - 4t)(\varphi_t(f_1) - \varphi_t(f_2)\varphi(f_1)).
\]

As a check, we verify that \((\varphi_t)_{\text{ren}}(s)(f_3)\) and \((\varphi_t)_{\text{ren}}(s)(f_4)\) are equal. Let \(\pi\) denote the projection onto the pole part of a Laurent series. Then

\[
\varphi_t(f_1)\varphi(f_2) - \varphi_t(f_2)\varphi(f_1) = (\varphi(f_1) - \pi(\varphi(f_1))\varphi(f_2)(\varphi(f_2) - \pi(\varphi(f_2)))\varphi(f_1)
\]

\[
= -\pi(\varphi(f_1)\varphi(f_2) - (\pi(\varphi(f_2)))\varphi(f_1)
\]

\[
= \varphi(f_1)\varphi(f_2) - \varphi(f_2)\varphi(f_1).
\]

The computations for the character \(\chi_t\) in \((7.3)\) associated to the toy model character \(\varphi\) with \(-n + 2m = 0\) give

\[
(\chi_t^\varphi)(f_1) = \frac{1}{\lambda}, \quad (\chi_t^\varphi)(f_2) = 0, \quad (\chi_t^\varphi)(f_4) = \frac{-\frac{\pi^2}{18} + \pi^2t}{\lambda},
\]

\[
(\chi_t^\varphi)(f_8) = \frac{\pi^2}{24\lambda^2} + \frac{\pi^2t(18s + (5 + 18t)a)}{6\lambda}.
\]

In agreement with Theorem \([7,11]\), \((\chi_t^\varphi)\) is independent of \(s\) when evaluated on \(f_1, f_2\) and \(f_4\). However, \((\chi_t^\varphi)(f_8)\) is not local. We confirm that the necessary condition \([7,9]\) for locality in Theorem \([7,14]\) does not hold. Indeed, \(\varphi(1/z) = 1/z \neq 0\) and \(3(-1/z)(\pi^2) \neq 0 \cdot a\). The \(\beta\)-function on \(f_1, f_2, f_4\) is given by

\[
\beta_{\chi_t}(f_1) = 1, \quad \beta_{\chi_t}(f_2) = 0, \quad \beta_{\chi_t}(f_4) = -\frac{\pi^2}{6} + 3\pi^2t.
\]
The renormalized character \((\chi_t)_{\text{ren}}(s)\) is given by

\[
(\chi_t)_{\text{ren}}(s)(f_1) = s - a, \quad (\chi_t)_{\text{ren}}(s)(f_2) = \frac{\pi^2}{4}, \quad (\chi_t)_{\text{ren}}(s)(f_4) = \frac{\pi^2}{12}(s + 36st - (1 + 24t)a),
\]

and satisfies the abstract RGE.

Let \(\mathcal{H}^2\) be the Hopf subalgebra generated by the trees

\[
t_0 = 1_T, \quad t_1 = \ast, \quad t_2 = 1, \quad t_3 = \Lambda, \quad t_4 = \Lambda, \quad t_5 = \mathbb{1}.
\]

For \(T \in \{t_1, \ldots, t_5\}\), let \(Z_T\) be the corresponding infinitesimal character. The Lie algebra \(g_2\) of scalar valued infinitesimal characters of \(\mathcal{H}^2\) is generated by \(Z_{t_1}, \ldots, Z_{t_5}\). Let \(G_1\) be the scalar valued character group of \(\mathcal{H}^2\), and let \(G_0\) be the semi-direct product \(G_1 \rtimes \mathbb{C}\) given by

\[
(g, t) \cdot (g', t') = (g \cdot \theta_t(g'), t + t'),
\]

where \(\theta_t(g)(T) = e^{t \deg(T)}g(T)\) homogenous \(T\). Define a new variable \(Z_0\) with \([Z_0, Z_t] = \deg(t)Z_t\), so formally \(Z_0 = \frac{d}{dt}\). The Lie algebra \(g_0\) of \(G_0\) is generated by \(Z_0, Z_{t_1}, \ldots, Z_{t_5}\).

The conditions a) and b) in Definition 2.1 of a Lie bialgebra can be written in a basis as a system of quadratic equations. We can solve this system explicitly, e.g. via Mathematica. It turns out that there are 43 families of Lie bialgebra structures \(\gamma\) on \(g_0\). In more detail, the system of quadratic equations involves 90 variables. Mathematica gives 1 solution with 82 linear relations (and so 8 degrees of freedom), 7 solutions with 83 linear relations, 16 solutions with 84 linear relations, 13 solutions with 85 linear relations, 5 solutions with 86 linear relations, and 1 solution with 87 linear relations.

To any Lax equation with a spectral parameter, one can associate a spectral curve and study its algebro-geometric properties (see [16]). In our case, we consider the adjoint representation \(\text{ad} : \delta \rightarrow \mathfrak{gl}(\delta)\) and the induced adjoint representation of the loop algebra. The spectral curve is given by the characteristic equation of \(\text{ad}(L\lambda)\):

\[
\Gamma_0 = \{(\lambda, \nu) \in \mathbb{C} - \{0\} \times \mathbb{C} \mid \det(\text{ad}(L\lambda) - \nu\text{Id})) = 0\}.
\]

The theory of the spectral curve and its Jacobian usually assumes that the spectral curve is irreducible. For all 43 families of Lie bialgebra structures on \(\delta\), on the associated Lie algebra \(\text{ad}(\delta)\) all eigenvalues of the characteristic equation are zero, and the zero eigenspace is nine dimensional. The spectral curve itself is the union of degree one curves. Thus each irreducible component has a trivial Jacobian, and the spectral curve theory breaks down. The integrability of these Lax pair equation remains open for future investigations.

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Institute of Mathematics of the Romanian Academy, PO Box 1-764, 014700 Bucharest, Romania and Max-Planck-Institut für Mathematik, P.O. Box 7280, D-53072 Bonn, Germany. baditoiu@math.bu.edu

Department of Mathematics and Statistics, Boston University, Boston, MA 02215, USA. sr@math.bu.edu