Hybrid finite element formulation for geometrically nonlinear buckling analysis of truss with initial length imperfection

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Abstract. This paper presents a novel hybrid FEM-based approach for nonlinear buckling analysis of truss with initial length imperfection. The contribution deals with establishing two types of truss finite element (perfection and imperfection element) considering large displacement based on displacement formulation and mixed formulation. Therefore, the hybrid global equation system is developed by assembling perfection and imperfection truss elements. The incremental-iterative algorithm based on the arc-length method is used to establish calculation programs for solving geometrically nonlinear buckling analysis of truss with initial length imperfection. Using a written calculation program, the numerical test is presented to investigate the equilibrium path for plan truss with initial member length imperfection.

1 Introduction

Many truss members have initial geometric imperfections as a result of manufacturing, transporting, and handling processes. This initial member imperfection significantly influences the buckling behaviour of the truss structure. In recent years, many research works addressed the influence of geometrical imperfection on the behaviour of truss structures [1-4]. For solving the buckling problem of truss structure, the finite element method is considered the most popular and efficient method. In geometrical linear finite element analysis, the length imperfection usually is calculated by adding equivalent loads to the nodal external force vector. However, in geometrical nonlinear analysis, it cannot be used. Generally, the solution of nonlinear buckling problem of truss based on displacement finite element formulation requires the implementation of length imperfection to the material stiffness matrix. The operation of incorporating length imperfection considerably increases the difficulty in constructing and solving nonlinear incremental balanced equations of the system. For escaping difficulties of the mathematical treatment of imperfection, in [5] the author proposed an approach to formulate the nonlinear buckling problem of truss with imperfection based on mixed finite element formulation. The mixed model has significant advantage over displacement-based formulation model but increases the solving system dimension. Nowadays, the hybrid finite element approach is widely used to solve the

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nonlinear contact mechanic problem such as displacement-based finite elements are
difficult to solve [6-9]. In this work, the author proposes a novel hybrid finite element
approach for constructing the solving system of equation. The main idea is establishing two
types of truss finite element considering large displacement based on displacement
formulation and mixed formulation. The global equation system is developed by
assembling two types of proposed truss elements. The solving algorithm of geometrically
nonlinear buckling analysis of truss system is built by employing arc length method due to
its efficiency to predict the proper response and follow the nonlinear equilibrium path
through limit. Therefore, a new incremental-iterative algorithm for solving constructed
system of equation and calculation program is established. The numerical results are
presented to verify the efficiency of the proposed method.

2 Equilibrium equations for the truss elements considering large
displacements

For hybrid finite element formulation, the research proposed to discretize the truss system
into two types of the truss elements: first type element $e_I$ – perfection truss element; $e_{II}$ -
imperfection truss element with initial length imperfection $\Delta_l$ (shown in Fig.1).

![Fig. 1. Truss elements’ types](image)

Let us consider two-node truss elements $e_I$ and $e_{II}$ in the global coordinate system (X0Y) as
shown in Fig.2.

![Fig. 2. Truss elements $e_I$ and $e_{II}$ considering large displacements](image)

The following is designated

$\{X_i, X_j\}, \{X_i, Y_j\} : i^{th}$ and $j^{th}$ nodal coordinates in global coordinate system before and
after deformation;

$L_0$ và $L$ : distance between $i^{th}$ and $j^{th}$ node before and after deformation;
\( u_1, u_2, u_3, u_4 \) and \( P_1, P_2, P_3, P_4 \) : nodal displacements and forces in global coordinates; 
\( P_e \) : resultant external force at the \( i \)th cross section after deformation; 
\( u_i \equiv P_e = N \) : resultant external force at the \( i \)th cross section after deformation; 
\( A \) : cross sectional area of truss element; \( E \) : elastic modulus of material; \( N \) : axial load of truss element.

The length of the truss element after deformation is defined as

\[
L = \sqrt{(X_2 - X_1 + u_3 - u_4)^2 + (Y_2 - Y_1 + u_4 - u_2)^2} \quad (1)
\]

The axial deformation of perfection truss element and imperfection truss element are obtained

\[
\begin{align*}
(e_I) : & \quad \Delta L^{(e_I)} = L - L_0 = L - \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} \\
(e_{II}) : & \quad \Delta L^{(e_{II})} = L - L_e = L_0 + \Delta_e = L - \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} + \Delta_e
\end{align*}
\]  

(2.1)  
(2.2)

Work of internal axial force can be computed for each truss element as following

\[
\begin{align*}
(e_I) : & \quad \delta V^{(e_I)} = -\int_0^{L_0} \sigma_x \delta \varepsilon_x \, dV = -\int_A \sigma_x \, dA \delta \varepsilon_x, \, dx = -N \int_0^{L_0} \frac{\delta (\Delta dx)}{dx} \, dx = \\
& = -N \int_0^{L_0} \delta \left( \frac{\Delta dx}{dx} \right) = -N \delta \Delta L = N \sum_{i=1}^4 \frac{\partial \Delta L}{\partial u_i} \delta u_i
\end{align*}
\]

(3.1)

\[
\begin{align*}
(e_{II}) : & \quad \delta V^{(e_{II})} = -\int_0^{L_e} \sigma_x \delta \varepsilon_x \, dV = -\int_A \sigma_x \, dA \delta \varepsilon_x, \, dx = -N \int_0^{L_e} \frac{\delta (\Delta dx)}{dx} \, dx = \\
& = -N \int_0^{L_e} \delta \left( \frac{\Delta dx}{dx} \right) = -N \delta \Delta L = N \sum_{i=1}^4 \frac{\partial \Delta L}{\partial u_i} \delta u_i + \frac{\partial \Delta L}{\partial \Delta_e} \delta \Delta_e
\end{align*}
\]

(3.2)

For each truss element, the virtual external work can be defined as

\[
\begin{align*}
(e_I) : & \quad \delta W^{(e_I)} = P_1 \delta u_1 + P_2 \delta u_2 + P_3 \delta u_3 + P_4 \delta u_4 = \sum_{i=1}^4 P_i \delta u_i \\
(e_{II}) : & \quad \delta W^{(e_{II})} = P_1 \delta u_1 + P_2 \delta u_2 + P_3 \delta u_3 + P_4 \delta u_4 + P_e \delta \Delta_e = \sum_{i=1}^4 P_i \delta u_i + P_e \delta \Delta_e
\end{align*}
\]

(4.1)  
(4.2)

Combining equations (3) and (4), getting total work done by the applied forces and the inertial forces of a mechanical system

\[
\begin{align*}
(e_I) : & \quad \delta V^{(e_I)} + \delta W^{(e_I)} = -N \sum_{i=1}^4 \frac{\partial \Delta L}{\partial u_i} \delta u_i + \sum_{i=1}^4 P_i \delta u_i = \sum_{i=1}^4 \left\{ -N \frac{\partial \Delta L}{\partial u_i} + P_i \right\} \delta u_i = 0; \quad (5.1)
\end{align*}
\]

\[
\begin{align*}
(e_{II}) : & \quad \delta V^{(e_{II})} + \delta W^{(e_{II})} = -N \sum_{i=1}^4 \frac{\partial \Delta L}{\partial u_i} \delta u_i + \frac{\partial \Delta L}{\partial \Delta_e} \delta \Delta_e + \sum_{i=1}^4 P_i \delta u_i + P_e \delta \Delta_e = \sum_{i=1}^4 \left\{ -N \frac{\partial \Delta L}{\partial u_i} + P_i \right\} \delta u_i + \left\{ -N \frac{\partial \Delta L}{\partial \Delta_e} + P_e \right\} \delta \Delta_e = 0 \quad (5.2)
\end{align*}
\]
Based on the principle of virtual work, in equilibrium the virtual work of the forces applied to a system is zero, from equation (5) getting

\[
(e_f) : \begin{align*}
-N \frac{\partial AL}{\partial u_i} + P &= 0 \quad (i = 1, 2, 3, 4) \\
-N \frac{\partial AL}{\partial u_i} + P &= 0 \quad (i = 1, 2, 3, 4)
\end{align*}
\]

\[
(e_h) : \begin{align*}
N \frac{\partial AL}{\partial A_k} - P_e &= 0
\end{align*}
\]

Expressing axial force through deformation and adding deformation from the equation (2) to equation (6), having the system (7)

\[
(e_f) : \begin{align*}
q_i^{(ef)}(u) &= \frac{EA}{L_0} (L - L_0) \frac{\partial L}{\partial u_i} = P_i, \quad (i = 1, 2, 3, 4) \\
\text{or } q_i^{(ef)}(u^{(ef)}) &= p_i^{(ef)} \\
&i = 1, 2, 3, 4
\end{align*}
\]

\[
(e_h) : \begin{align*}
q_k^{(eh)}(u, A_j) &= \frac{EA}{L_e} (L - L_0 + A_j) \frac{\partial (L - L_0 + A_j)}{\partial u_i} = P_j \\
q_k^{(eh)}(u^{(eh)}), A_j) &= p_k^{(eh)} \\
&k = 1, 2, \ldots, 5
\end{align*}
\]

\[
u^{(ef)} = \{u_1, u_2, u_3, u_4\}^T; \quad u^{(eh)} = \{u_1, u_2, u_3, u_4, u_5 \equiv P_e\}^T
\]

Input incremental loading into the equation (7) and express in matrix format

\[
(e_f) : k^{(ef)}(u) \delta u = (P^{(ef)} + \Delta P^{(ef)}) - q^{(ef)}(u)
\]

\[
(e_h) : k^{(eh)}(u, A_j) \delta u = (P^{(eh)} + \Delta P^{(eh)}) - q^{(eh)}(u, A_j)
\]

Where the tangent stiffness matrices are written

\[
(e_f) : k^{(ef)}(u^{(ef)}) = \frac{\partial q^{(ef)}(u^{(ef)})}{\partial u^{(ef)}}, \quad (e_h) : k^{(eh)}(u^{(eh)}, A_j) = \frac{\partial q^{(eh)}(u^{(eh)}, A_j)}{\partial u^{(eh)}}
\]

The length imperfection \(A_j\) is considered in the tangent stiffness matrix \(k^{(eh)}(u^{(eh)}, A_j)\) of element \(e_{II}\).
3 Hybrid equation of truss system

The hybrid global equation of truss system (9) can be established by assembling all perfection and imperfection truss elements which were established above

\[ K(u)\delta u = (P + \Delta P) - q(u) \]  \hspace{1cm} (9)

Where

\[ u = \{u_1, u_2, \ldots, u_n\}^T ; q(u) = \{q_1(u), q_2(u), \ldots, q_n(u)\}^T ; P = \{P_1, P_2, \ldots, P_m\}^T \]

\[ q_i(u) = \sum_{e_i, e_H} \{q_i^{(e)}(u); q_i^{(e_H)}(u, \Delta_e)\} ; P_i = \sum_{e_i, e_H} \{P_i^{(e)}; P_i^{(e_H)}\} ; (i = 1, 2, \ldots, n) \]

\[ K_{ij}(u) = \sum_{e_i, e_H} \{k_{ij}^{(e)}(u); k_{ij}^{(e_H)}(u, \Delta_e)\} ; (i, j = 1, 2, \ldots, n) \]

“m” is a number of truss elements and “n” is number of unknowns;

Using arc length technique [10-11] the incremental-iterative algorithm is established for solving nonlinear system.

4 Numerical investigations

Based on proposed incremental-iterative algorithm, the calculation program to solve the example is written using Matlab software. The system is composed of bars made of the same material and had the same geometrical properties (system is shown in Fig. 3), having length imperfection \( \Delta_{e_5} = 2cm \). The geometric parameters, material parameters and loading parameters are given \( E = 2.10^4 kN / cm^2, A = 4cm^2, \Delta_{e_5} = 2cm \)

The unknowns of truss system are designated as shown in Fig. 3 for solving the nonlinear equation based on hybrid formulation n, including \( (u_1, u_2, u_3, u_4) \) - nodal displacement unknowns and \( (u_5 = N_5) \) - axial force unknown.

![Fig. 3. Examined system, designating the system unknowns for hybrid formulation](image-url)
Fig. 4. Load – displacement equilibrium path $P-u_2$, $P-u_4$

The calculating results are load-displacement and equilibrium path shown in Fig. 4.

5 Conclusion

The presented hybrid model does not require implementing the length imperfection to the mater stiffness matrix in geometrically nonlinear buckling analysis of truss system. This formulation provides an effective remedy to overcome the mathematical difficulty associated with the displacement-based formulation in establishing solving algorithm. Comparison to the mixed-based formulation, the current formulation has advantage of decreasing the dimension of the solving system.

References

1. M. Gordini, M. R. Habibi, M. H. Tavana, M. Amiri, M. T. Roudsari, The Open Civil Engineering Journal 12 (2018)
2. A. El-Sheikh, Engineering Structures 19, 7 (1997)
3. Z. Zhou, J. Wu and S. Meng, International Journal of Structural Stability and Dynamics 14, 3 (2014)
4. Zhong-Wei Zhao, Hai-Qing Liu, Bing Liang, and Ren-Zhang Yan, Advanced Steel Construction 15, 1 (2019)
5. Vu Thi Bich Quyen, Dao Ngoc Tien, Nguyen Thi Lan Huong, IOP Conf. Series: Materials Science and Engineering 960 (2020)
6. Manish Agrawal, Arup Nandy, C.S. Jog, Computer Methods in Applied Mechanics and Engineering 356, 1 (2019)
7. H. A. F. A. Santos, P. M. Pimenta & J. P. M. Almeida, Computational Mechanics 48 (2011)
8. Vida Niki and R. Emre Erkme, Canadian Journal of Civil Engineering 45, 4 (2018)
9. R. Emre Erkmen, Finite Elements in Analysis and Design 82, 32-45 (2014).
10. Crisfield, M. A., Comput. Struct. 13 (1981)
11. M. A. Crisfield, John Wiley & Sons Ltd. 1&2, (1997)