QUANTUM STATISTICAL MECHANICS OF Q-LATTICES AND NONCOMMUTATIVE GEOMETRY

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ABSTRACT. After recalling some basic notions of quantum statistical mechanics, we explain the Bost-Connes system that relates the structure of the maximal abelian extension of \( \mathbb{Q} \) to the space of KMS\( \beta \) states of a \( C^* \)-dynamical system. Afterwards, we study briefly the Connes-Marcolli GL\(_2 \)-system as a generalization of the former system.

INTRODUCTION

In [BC], Bost and Connes established a new bridge between number theory and physics via operator algebras by introducing a quantum statistical model describing the Galois theory of maximal abelian extension of \( \mathbb{Q} \). Their work has motivated further developments in using noncommutative geometry to tackle the problem of the explicit class field theory of number fields, to name a few [Co, CM1, CMR1, HP, J, LLaN], as well as several new directions in the field of operator algebras, see for example [CuLi, KLnQ, L1, L2, LR1, LR2, T]. Here, we only study some of the developments in the application of noncommutative geometry to the explicit class field theory problem. We shall see that the notion of \( \mathbb{Q} \)-lattices plays an important role in new advances, so we focus on this notion. Our study here is far from being complete or detailed. Therefore, we refer the interested reader to [CM1, CM2] for more details and more complete lists of references.

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1. Basics of quantum statistical mechanics

Definition 1.1. (a) A \( C^* \)-dynamical system is a pair \((A, \sigma)\) such that \( A \) is a \( C^* \)-algebra, called the algebra of observable, and \( \sigma \) is a one parameter group of automorphisms of \( A \), called the time evolution of the system, such that for \( s, t \in \mathbb{R} \), we have \( \sigma_{s+t} = \sigma_s \sigma_t \), and \( \sigma \) is strongly continuous, that is for every \( a \in A \) the map \( t \mapsto \sigma_t(a) \) from \( \mathbb{R} \) into \( A \) is continuous in norm topology.

(b) A bounded linear functional \( \varphi : A \to \mathbb{C} \) is called a state on \( A \) , if

\[
\|\varphi\| = 1,
\]
\[ \varphi(aa^*) \geq 0, \quad \forall a \in A. \]

**Definition 1.2.** Let \((A, \sigma)\) be a \(C^*\)-dynamical system and \(\varphi\) be a state on \(A\). For \(0 < \beta < \infty\), we say \(\varphi\) is a **KMS state at inverse temperature \(\beta\)** on \((A, \sigma)\), or simply a **KMS \(\beta\) state** on \(A\), if it satisfies **KMS condition**: For for every \(x, y \in A\), there is a bounded holomorphic function \(F_{x,y}(z)\) on the open strip \(0 < \text{Im}z < \beta\), continuous on the closure of the strip, such that for all \(t \in \mathbb{R}\) we have

\[
F_{x,y}(t) = \varphi(x \sigma_t(y)), \\
F_{x,y}(t + i\beta) = \varphi(\sigma_t(y)x).
\]

A **KMS state at \(\infty\)** on \((A, \sigma)\) or simply a **ground state** on \(A\) is a weak limit of **KMS \(\beta\) states** \(\varphi_{\beta}\)'s as \(\beta \to \infty\),

\[
\varphi_{\infty} := \lim_{\beta \to \infty} \varphi_{\beta}(a), \quad \forall a \in A.
\]

In thermodynamics \(\beta = 1/kT\), where \(k\) is the Boltzmann constant, and \(T\) is the temperature of the system. For simplicity, we assume \(k = 1\).

**Example 1.3.** Let \(A = M_n(\mathbb{C})\) be the algebra of \(n \times n\) matrices with complex entries. Any one parameter group of automorphisms \((\sigma_t)_{t \in \mathbb{R}}\) of \(A\) has the form

\[
\sigma_t(x) = e^{itH}xe^{-itH}, \quad \forall x \in A, t \in \mathbb{R},
\]

for some self-adjoint operator \(H \in A\). Then for \(\beta > 0\) one has a unique KMS \(\beta\) state, called the **Gibbs equilibrium state**, given by

\[
\varphi_{\beta}(x) = \frac{\text{Tr}(e^{-\beta H}x)}{\text{Tr}(e^{-\beta H})}, \quad \forall x \in A.
\]

(1.1)

The normalizer of above state, i.e.

\[
\text{Tr}(e^{-\beta H})
\]

is called the **partition function** of the state. Due to the fact that

\[
\varphi_{\beta}(\sigma_t(x)) = \varphi_{\beta}(x), \quad \forall x \in A, t \in \mathbb{R},
\]

the Gibbs state is an equilibrium state with respect to every time evolution of the system, hence the name.

If one accepts the Gibbs formalism for thermodynamics, namely one describes a thermodynamics by a \(C^*\)-dynamical system \((A, \sigma)\), then KMS \(\beta\) states on \((A, \sigma)\) play the role of equilibrium states, that is for every KMS \(\beta\) state \(\varphi\) we have

\[
\varphi(\sigma_t(a)) = \varphi(a), \quad \forall a \in A, t \in \mathbb{R}.
\]

In contrary to the above example, KMS \(\beta\) states are not necessarily unique. To describe the space of KMS \(\beta\) states on a \(C^*\)-dynamical system \((A, \sigma)\), we need the following definition:

**Definition 1.4.**

(a) Any compact simplex in a locally convex topological vector space \(E\) is called a **Choquet simplex**.

(b) Let \(M\) be a von Neumann algebra. The center of \(M\) is defined as \(Z(M) := M \cap M'\) and \(M\) is called a **factor**, if \(Z(M) = 1\mathbb{C}\).

(c) Let \(\varphi\) be a state on a \(C^*\)-algebra \(A\). We say \(\varphi\) is a **factor state** if \(\pi_\varphi(A)''\), the enveloping von Neumann algebra of the cyclic representation associated to \(\varphi\), is a factor.
Proposition 1.5. ([BtR]) Let $(A, \sigma)$ be a $C^*$-dynamical system and $\beta > 0$. Then the space of KMS$_\beta$ states on $(A, \sigma)$ is a compact Choquet simplex, and the extreme points are factor states.

Notation. For $0 < \beta \leq \infty$, the set of extreme points of KMS$_\beta$ states, which is also called the set of extremal KMS$_\beta$ states, is denoted by $E$.  

2. The Bost-Connes system, 1-dimensional case

In [BC], Bost and Connes constructed a $C^*$-dynamical system $(A, \sigma)$, with an action of the idèles class group of $\mathbb{Q}$ modulo the connected component of the identity as symmetry. The Riemann zeta function appears as the partition function of the system. In this section, we describe the Bost-Connes (or briefly BC) system.

Definition 2.1. The underlying $C^*$-algebra $A$ of BC system is generated by two types of operators $\{e(r); r \in \mathbb{Q}/\mathbb{Z}\}$, and $\{\mu_n; n \in \mathbb{N}\}^\times$ subject to following conditions:

(a) $\mu_n \mu_n^* = 1$, $\forall n$,
(b) $\mu_m \mu_n = \mu_n \mu_m$, $\forall m, n$,
(c) $e(0) = 1, e(r + s) = e(r)e(s)$, and $e(r)^* = e(-r)$, $\forall r, s$,
(d) $\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{m=0}^{\infty} e(r)$, $\forall n, r$.

The time evolution of the system is given by

$$\sigma_t(\mu_n) := n^it \mu_n, \quad \sigma_t(e(r)) := e(r).$$

Remarks 2.2. (a) Bost and Connes defined this $C^*$-algebra also as the reduced Hecke $C^*$-algebra of the Hecke pair $(P^+_{\mathbb{Z}}, P^+_{\mathbb{Q}})$, where

$$P^+_{\mathbb{Z}} := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}; n \in \mathbb{Z} \right\},$$

and

$$P^+_{\mathbb{Q}} := \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}; a, b \in \mathbb{Q}, \text{ and } a > 0 \right\}.$$

(b) In [LR], Laca and Raeburn described the above $C^*$-algebra $A$ as a semigroup $C^*$-crossed product. In terms of the above definition, they used Relation (c) to construct a representation of $C^*(\mathbb{Q}/\mathbb{Z})$ in $A$. Then, Relations (a), (b) define an action of $\mathbb{N}^\times$ on $A$ by isometries. Afterwards, Relation (d) shows that $(e, \mu)$ is a covariant pair for the dynamical system $(C^*(\mathbb{Q}/\mathbb{Z}), \mathbb{N}^\times, \beta)$, where $\beta$ is the action of $\mathbb{N}^\times$ on $C^*(\mathbb{Q}/\mathbb{Z})$ by endomorphisms which is defined as

$$\beta_n(i(r)) := \frac{1}{n} \sum_{j=1}^{\infty} i\left(\frac{r + j}{n}\right), \quad \forall n \in \mathbb{N}^\times, \text{ and } r \in \mathbb{Q}/\mathbb{Z},$$

where $i: \mathbb{Q}/\mathbb{Z} \to C^*(\mathbb{Q}/\mathbb{Z})$ is the natural embedding of discrete group $\mathbb{Q}/\mathbb{Z}$ in its group $C^*$-algebra. The existence of a covariant pair implies the existence of the semigroup $C^*$-crossed product of the above semigroup $C^*$-dynamical system, and

$$A = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\beta \mathbb{N}^\times.$$
2.1. The Bost-Connes system in terms of 1-dimensional \( \mathbb{Q} \)-lattices.

In Remarks 2.2, we saw two different formulations of BC system. There are yet two more formulations for this system. Its underlying \( C^* \)-algebra can be obtained as a \( C^* \)-algebra associated to 1-dimensional \( \mathbb{Q} \)-lattices as well as a groupoid \( C^* \)-algebra. The notion of \( \mathbb{Q} \)-lattices was first initiated by Connes and Marcolli, [CM1], in order to generalize BC system to higher dimensions and find a way to tackle the problem of explicit class field theory of real quadratic fields. Thereby, Connes-Marcolli (or simply CM) GL2-system was invented. Afterwards, Connes, Marcolli, and Ramachandran introduced the notion of a \( K \)-lattice for \( K \) being an imaginary quadratic field in [CMR1], see also [CM2, CMR2]. In [CMR1], they constructed a quantum statistical model, similar to BC system, for explicit class field theory of imaginary quadratic fields. Finally, Ha and Paungam generalized Bost-Connes-Marcolli system for an arbitrary Shimura datum in [HP]. As one may notice, the key concept in recent developments of the subject is the notion of \( \mathbb{Q} \)-lattices. Therefore, we study it here.

**Definition 2.3.** (a) An (n-dimensional) \( \mathbb{Q} \)-lattice in \( \mathbb{R}^n \) is a pair \((\Lambda, \varphi)\), where \( \Lambda \) is an \( n \)-dimensional lattice, that is a discrete subgroup of \( \mathbb{R}^n \) of rank \( n \), or equivalently, a free subgroup generated by \( n \) linearly independent vectors, and

\[
\varphi: \mathbb{Q}^n / \mathbb{Z}^n \longrightarrow \mathbb{Q} \Lambda / \Lambda
\]

is a homomorphism of abelian groups. We denote the space of \( n \)-dimensional \( \mathbb{Q} \)-lattices by \( \mathcal{L}_n \).

(b) Two \( \mathbb{Q} \)-lattices \((\Lambda_1, \varphi_1)\), and \((\Lambda_2, \varphi_2)\) are called **commensurable**, and denote by \((\Lambda_1, \varphi_1) \sim (\Lambda_2, \varphi_2)\), if

\[
\mathbb{Q} \Lambda_1 = \mathbb{Q} \Lambda_2,
\]

and

\[
(\varphi_1 - \varphi_2)(x) \in \Lambda_1 + \Lambda_2, \quad \forall x \in \mathbb{Q}^n / \mathbb{Z}^n.
\]

The latter condition can also be read as \( \varphi_1 \) and \( \varphi_2 \) are equal modulo \( \Lambda_1 + \Lambda_2 \).

**Lemma 2.4.** The commensurability of \( \mathbb{Q} \)-lattices is an equivalence relation.

**Proof.** We only show that commensurability is transitive. Let \((\Lambda_1, \varphi_1) \sim (\Lambda_2, \varphi_2)\), and \((\Lambda_2, \varphi_2) \sim (\Lambda_3, \varphi_3)\). Obviously \( \mathbb{Q} \Lambda_1 = \mathbb{Q} \Lambda_3 \).

Let \( \Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3 \), then \( (\varphi_1 - \varphi_3)(x) \in \Lambda \) for all \( x \in \mathbb{Q} / \mathbb{Z} \). To see that it actually belongs to \( \Lambda_1 + \Lambda_3 \) we need to show that \( \Lambda_1 + \Lambda_3 \) is of finite index in \( \Lambda \).

\( \mathbb{Q} \Lambda_1 = \mathbb{Q} \Lambda_2 \) implies that \( m \Lambda_2 \subset \Lambda_1 \) for some \( m \in \mathbb{N} \). Thus \( m \Lambda_2 + \Lambda_1 + \Lambda_3 \subset \Lambda_1 + \Lambda_3 \).

Let \( e_1, \cdots, e_n \) generate \( \Lambda_2 \), then \( \{ \sum_{i=1}^n m_i e_i + \Lambda_1 + \Lambda_3 : 1 \leq m_i \leq m \} \) is a complete set of cosets of \( \Lambda_1 + \Lambda_3 \) in \( \Lambda \), so \( \Lambda_1 + \Lambda_3 \) is of finite index in \( \Lambda \).

Thus, there exists \( M \in \mathbb{N} \) such that for all \( x \in \mathbb{Q}^n / \mathbb{Z}^n \)

\[
M(\varphi_1 - \varphi_3)(x) \in \Lambda_1 + \Lambda_3.
\]

This implies that

\[
(\varphi_1 - \varphi_3)(x) = M(\varphi_1 - \varphi_3)(\frac{x}{M}) \in \Lambda_1 + \Lambda_3, \quad \forall x \in \mathbb{Q}^n / \mathbb{Z}^n.
\]

\( \square \)
1-dimensional $\mathbb{Q}$-lattices. One easily checks that every $\mathbb{Q}$-lattice $(\Lambda, \varphi)$ in $\mathbb{R}$ can be written as $(\lambda \mathbb{Z}, \lambda \rho)$ for some $\lambda > 0$ and some $\rho \in \text{Hom}_\mathbb{Z}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$.

Thus, the space $\mathcal{L}_1/\mathbb{R}^+$ of 1-dimensional $\mathbb{Q}$-lattices up to scaling can be identified by $\mathbb{R}$. On the other hand, one can formulate the equivalence classes of 1-dimensional $\mathbb{Q}$-lattices under the equivalence relation of commensurability as the orbits of an action of $\mathbb{N}^\times$ on the space of $\mathbb{Q}$-lattices.

**Lemma 2.5.** Two $\mathbb{Q}$-lattices $(\lambda_1 \mathbb{Z}, \lambda_1 \rho_1)$, $i = 1, 2$ are commensurable if and only if there exist $m, n \in \mathbb{N}$ such that

$$m\lambda_1 = n\lambda_2,$$

and

$$(n\rho_1 - m\rho_2)(x) \in \mathbb{Z}, \quad \forall x \in \mathbb{Q}/\mathbb{Z}.$$ 

Therefore, the orbit space of the action $n(\rho) := n\rho$ on $\mathcal{L}_1/\mathbb{R}^+$ is the space of commensurability classes of 1-dimensional $\mathbb{Q}$-lattices up to scaling. To study this space as a noncommutative quotient space, we consider the semigroup $C^*$-crossed product of the induced action on the algebra of continuous complex functions on $\mathbb{R}$.

**Definition 2.6.** Define $\alpha : \mathbb{N}^\times \to \text{Aut}(C(\mathbb{R}))$ by

$$\alpha_n f(\rho) := \begin{cases} f(n^{-1} \rho) & \text{if } \rho \in n\mathbb{R} \\ 0 & \text{otherwise.} \end{cases}$$

The semigroup $C^*$-crossed product $C(\mathbb{R}) \rtimes_{\alpha} \mathbb{N}^\times$ of the above action is the noncommutative quotient space of 1-dimensional $\mathbb{Q}$-lattices up to scaling by the equivalence relation of commensurability.

**Remarks 2.7.**

(a) Let $\mathbb{A}_f$ denote the ring of finite adèles on $\mathbb{Q}$, namely

$$\mathbb{A}_f = \prod_{\text{res}} \mathbb{Q}_p := \bigcup_F \prod_{p \in F} \mathbb{Q}_p \times \prod_{p \notin F} \mathbb{Z}_p,$$

where the union is taken over all finite families of rational prime numbers and $\mathbb{Q}_p$ (resp. $\mathbb{Z}_p$) is the field of $p$-adic numbers (resp. the ring of $p$-adic integers). The topology of $\mathbb{A}_f$ is defined such that each set in the above union is an open set. Then

$$\hat{\mathbb{Z}} := \prod_{p \text{ prime}} \mathbb{Z}_p$$

is the maximal compact subring of $\mathbb{A}_f$, and it is shown in [W] that $\mathbb{A}_f/\hat{\mathbb{Z}}$ and $\mathbb{Q}/\mathbb{Z}$ are isomorphic as abelian groups. This isomorphism gives rise to the following isomorphism

$$j : \hat{\mathbb{Z}} \to \mathbb{R}, \quad j(a)(x) := ax, \quad \forall x \in \mathbb{A}_f/\hat{\mathbb{Z}}, \forall a \in \hat{\mathbb{Z}}.$$ 

Therefore, we can consider $\mathbb{R}$ as a compact abelian group.

(b) The following map shows the Pontrjagin duality between $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{R}$.

$$e : \mathbb{Q}/\mathbb{Z} \to \mathbb{R}, \quad e(r)(\rho) := e^{2\pi i r(\rho)}, \quad \forall r \in \mathbb{Q}, \forall \rho \in \mathbb{R}.$$
(c) Under the above duality $\alpha$, the action of $\mathbb{N}^\times$ on $C(R)$, corresponds to $\beta$, the action of $\mathbb{N}^\times$ on $C^*(\mathbb{Q}/\mathbb{Z})$, that is the following diagram is commutative.

\[
\begin{array}{ccc}
C^*(\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\Gamma} & C(R) \\
\downarrow{\beta} & & \downarrow{\alpha} \\
C^*(\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\Gamma} & C(R)
\end{array}
\]

where $\Gamma$ is the Gelfand transform, i.e. $\Gamma(i(r))(\rho) = \rho(r)$, for $r \in \mathbb{Q}/\mathbb{Z}$.

Thus, we have the following isomorphism:

\[C(R) \times_\alpha \mathbb{N}^\times \simeq C^*(\mathbb{Q}/\mathbb{Z}) \times_\beta \mathbb{N}^\times.\]

We summarize the above remarks as the following theorem:

**Theorem 2.8.** The $C^*$-algebra of the BC system can be realized as the noncommutative quotient space of $\mathbb{Q}$-lattices up to scaling under the equivalence relation of commensurability.

### 2.2. Groupoid approach to the Bost-Connes system.

Now we construct the groupoid $G$ of the equivalence relation of commensurability on 1 dimensional $\mathbb{Q}$-lattices up to scaling. The groupoid $C^*$-algebra of this groupoid is another description of the Bost-Connes $C^*$-algebra.

The groupoid $G$ is defined by

\[G := \{(r, \rho) \in \mathbb{Q}^+ \times \mathbb{R}; r\rho \in \mathbb{R}\},\]

with the composition defined for two elements of $G$ by

\[(r_1, \rho_1) \circ (r_2, \rho_2) := (r_1 r_2, \rho_2), \quad \text{if } r_2 \rho_2 = \rho_1,\]

and the source and the target are given by

\[s : G \to \mathbb{R}, \quad (r, \rho) \mapsto \rho,\]
\[t : G \to \mathbb{R}, \quad (r, \rho) \mapsto r\rho.\]

The convolution product on $C(G)$, the algebra of complex continuous functions on $G$, is defined by

\[f_1 \ast f_2 (r, \rho) := \sum_{s\rho \in \mathbb{R}} f_1 (rs^{-1}, s\rho) f_2 (s, \rho).\]

The above sum is finite, because if $\frac{r}{s} \in \mathbb{Q}/\mathbb{Z}$, then $\rho(\frac{r}{s}) \neq 0$ implies $\frac{1}{s} \rho \notin \mathbb{R}$, and

\[Z = \left\{ \frac{r}{s} \in \mathbb{Q}/\mathbb{Z}; \rho(\frac{r}{s}) = 0 \right\}\]

is finite for every $\rho \neq 0$. The involution on $C(G)$ is defined by

\[f^*(r, \rho) := \overline{f(r^{-1}, r\rho)}.\]

**Proposition 2.9.** ([CM1]) Let $\mathcal{R}_1$ denote the groupoid of the equivalence relation of commensurability in 1-dimensional $\mathbb{Q}$-lattices. The map

\[\eta(r, \rho) = ((r^{-1}\mathbb{Z}, \rho); (Z, \rho)), \quad \forall (r, \rho) \in G,\]

defines an isomorphism of locally compact étale groupoids between $G$ and the quotient $\mathcal{R}_1/\mathcal{R}_1^*$ of the equivalence relation of commensurability on the space of 1-dimensional $\mathbb{Q}$-lattices by the natural scaling action of $\mathbb{R}_+^*$.

The above proposition allows us to consider BC $C^*$-algebra as $C^*(G)$, the groupoid $C^*$-algebra of $G$.
2.3. The Bost-Connes system and number theory.

The following theorem is the main result of Bost and Connes in [BC], which describes the space of KMS\(_\beta\) state for all \(\beta\)’s.

**Theorem 2.10.** (a) In the range \(0 < \beta \leq 1\) there is a unique KMS\(_\beta\) state. Its restriction to \(\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]\), the image of \(\mathbb{Q}/\mathbb{Z}\) in BC \(C^*\)-algebra, is of the form

\[
\varphi_\beta(e(a/b)) = b^{-\beta} \prod_{p \text{ prime, } p|b} \frac{1 - p^{-\beta}}{1 - p^{-1}}.
\]

(b) For \(1 < \beta \leq \infty\), the extreme KMS\(_\beta\) states, are parameterized by embeddings \(\iota : \mathbb{Q}^{ab} \hookrightarrow \mathbb{C}\), and on \(\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]\) we have

\[
\varphi_{\beta,\iota}(e(a/b)) = Z(\beta)^{-1} \sum_{n=1}^{\infty} n^{-\beta} \iota(\zeta_n^{a/b}),
\]

where the partition function \(Z(\beta) = \zeta(\beta)\) is the Riemann zeta function and \(\zeta_{a/b}\) is the root of unity associated to \(a/b\).

(c) For \(\beta = \infty\), every \(\varphi \in \mathcal{E}_\infty\) maps \(\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]\) into \(\mathbb{Q}^{ab} \subset \mathbb{C}\), and the Galois group \(\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})\) acts on the values of states in \(\mathcal{E}_\infty\) restricted to \(\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]\). Then, the class field theory isomorphism \(\theta : \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \to \mathbb{R}^*\) intertwines the actions of the Galois group with the action of \(\mathbb{R}^*\) by symmetries of \(\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]\), that is

\[
\gamma(\varphi(x)) = \varphi(\theta(\gamma)(x)),
\]

where \(\gamma \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}), \varphi \in \mathcal{E}_\infty, x \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]\), or equivalently, the following diagram commutes.

\[
\begin{array}{ccc}
\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] & \xrightarrow{\varphi} & \mathbb{Q}^{ab} \subset \mathbb{C} \\
\downarrow{\theta(\gamma)} & & \downarrow{\gamma} \\
\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] & \xrightarrow{\varphi} & \mathbb{Q}^{ab} \subset \mathbb{C}
\end{array}
\]

**The explicit class field theory of \(\mathbb{Q}\) and the BC system.**

The main theorem of the class field theory is the following isomorphism for any number field \(K\), a finite extension of \(\mathbb{Q}\),

\[
\theta : \text{Gal}(K^{ab}/K) \to \frac{C_K}{D_K},
\]

where \(K^{ab}, C_K,\) and \(D_K\) are respectively the maximal abelian extension of \(K\), the group of idèle classes of \(K\), and the connected component of the identity in the group of idèle classes.

A theorem of Kronecker and Weber states \(Q^{ab}\) is isomorphic to \(Q^{cycl}\) the cyclotomic extension of \(\mathbb{Q}\), the extension obtained by adding all roots of unity to \(\mathbb{Q}\), and \(C_K/D_K\) is isomorphic to \(R^*\), the group of of invertible elements of \(R\). Thus, the elements of \(Q/\mathbb{Z}\) are the generators of \(Q^{ab}\), and the above theorem illustrates the action of the Galois group \(\text{Gal}(Q^{ab}/\mathbb{Q})\) on the generators of \(Q^{ab}\) explicitly in terms of the action of \(\mathbb{R}^*\) on \(Q[Q/\mathbb{Z}]\) via the BC \(C^*\)-dynamical system and its ground states. Surprisingly, the explicit description of the generators of \(K^{ab}\) and the action of the Galois group \(\text{Gal}(K^{ab}/K)\) has been done only for \(\mathbb{Q}\) and imaginary quadratic fields \(Q(\sqrt{-d})\), while the similar description for more general number fields, in particular
real quadratic fields, is the subject of Hilbert’s 12th problem. As we mentioned before, the case of imaginary quadratic fields was treated in [CMR1]. The above theorem brought some hopes to find an answer to this problem at least for real quadratic fields \( \mathbb{Q}(\sqrt{d}) \) using noncommutative geometry. Indeed, the Connes-Marcolli GL2-system is an attempt towards this goal.

3. The Connes-Marcolli system, the 2-dimensional case

The observation summarized in Remarks 2.7 led Connes and Marcolli, [CM1], to consider the noncommutative quotient space of 2-dimensional \( \mathbb{Q} \)-lattices up to scaling and equivalence relation of commensurability as the generalization of BC system. Similar to the description of BC \( C^* \)-algebra as a groupoid \( C^* \)-algebra, Connes and Marcolli described 2-dimensional system as the completion of the convolution algebra over \( R_2/C^* \), where \( R_2 \) is the groupoid of the equivalence relation of commensurability on 2-dimensional \( \mathbb{Q} \)-lattices, and \( C^* \) represents the natural scaling in \( \mathbb{R}^2 \) considered by non-zero complex numbers. In this section, we review these results briefly.

An arbitrary 2-dimensional \( \mathbb{Q} \)-lattice can be written in the form 
\[
(\Lambda, \varphi) = (\lambda(\mathbb{Z} + \mathbb{Z} \tau), \lambda \rho),
\]
for some \( \lambda \in \mathbb{C}^* \), \( \tau \in \mathbb{H} \), \( \rho \in M_2(\mathbb{R}) = \text{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2) \), where \( \mathbb{H} = \{ x + iy \in \mathbb{C}; y > 0 \} \) is the upper half plane. In order to define the action of \( \text{GL}_2^+(\mathbb{R}) \), we choose a basis \( \{ e_1 = 1, e_2 = i \} \) of \( \mathbb{C} \) as a 2-dimensional real vector space. We set \( \Lambda_0 := Ze_1 + Ze_2 \). Then every element \( \rho \in M_2(\mathbb{R}) \) defines a homomorphism 
\[
\rho : \mathbb{Q}^2/\mathbb{Z}^2 \rightarrow \mathbb{Q}\Lambda_0/\Lambda,
\]
\[
\rho(a) = \rho_1(a)e_1 + \rho_2(a)e_2.
\]

Let \( \Gamma = \text{SL}(2, \mathbb{Z}) \). We define the action of \( \Gamma \times \Gamma \) on the space 
\[
\mathcal{U} := \{(g, \rho, \alpha) \in \text{GL}_2^+(\mathbb{Q}) \times M_2(\mathbb{R}) \times \text{GL}_2^+(\mathbb{R}) : g\rho \in M_2(\mathbb{R}) \}
\]
by 
\[
(\gamma_1, \gamma_2) (g, \rho, \alpha) = (\gamma_1 g^{-1}\gamma_2, \gamma_2 \rho, \gamma_2 \alpha).
\]

The following proposition is the analogue of Proposition 2.8 for the 2-dimensional case.

**Proposition 3.1.** ([CM1]) Let \( R_2 \) denote the groupoid of the equivalence relation of commensurability on 2-dimensional \( \mathbb{Q} \)-lattices. \( R_2 \) can be parameterized by the quotient of \( \mathcal{U} \) under the action of \( \Gamma \times \Gamma \) via the map
\[
\eta : \frac{\mathcal{U}}{\Gamma \times \Gamma} \rightarrow R_2
\]
\[
[(g, \rho, \alpha)] \mapsto ((\alpha^{-1} g^{-1} \Lambda_0, \alpha^{-1} \rho), (\alpha^{-1} \Lambda_0, \alpha \rho)).
\]

Since two \( \mathbb{Q} \)-lattices \( (\lambda_k, \varphi_k), k = 1, 2 \) are commensurable if and only if for any \( \lambda \in \mathbb{C}^* \), \( (\lambda \Lambda_k, \lambda \varphi_k) \) are commensurable, we should consider \( R_2/C^* \), where the action of \( C^* \) is defined by the embedding \( \mathbb{C}^* \) into \( \text{GL}_2^+(\mathbb{R}) \) by
\[
a + ib \in \mathbb{C}^* \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}).
\]

However, this action is not free, so \( R_2/C^* \) is not a groupoid anymore. But one still can define a convolution product on \( C_c(R_2/C^*) \), the algebra of continuous complex...
functions on $\mathcal{R}_2/\mathbb{C}^*$ with compact support. First, we observe that (3.2) gives rise to the following isomorphism:

\[
\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix} \mapsto \frac{ai + b}{ci + d}.
\]

Thus, we can identify $\mathcal{R}_2/\mathbb{C}^*$ with the quotient of the space $U := \{(g, \rho, \alpha) \in \text{GL}_2^+(\mathbb{Q}) \times M_2(\mathbb{R}) \times H; g \rho \in M_2(\mathbb{R})\}$ under the action of $\Gamma \times \Gamma$ defined in (3.1). Now, one may consider elements of $C_c(\mathcal{R}_2/\mathbb{C}^*)$ as those continuous complex functions on $U$ that are invariant under the action $\Gamma \times \Gamma$. Then, the convolution product is defined by

\[
f_1 \ast f_2(g, \rho, z) := \sum_{s \in \Gamma \setminus \text{GL}_2^+(\mathbb{Q}); \ s \rho \in M_2(\mathbb{R})} f_1(gs^{-1}, sp, s(z))f_2(s, \rho, z),
\]

and the involution is defined by

\[
f^*(g, \rho, z) := \frac{f(g^{-1}, g \rho, g(z))}{|\det(g)|}.
\]

Afterwards, the above algebra is represented as a subalgebra of operators on a Hilbert space $\mathcal{H}$. The completion of the latter algebra is the underlying $C^*$-algebra of the Connes-Marcolli $\text{GL}_2$-system $A_2$ and the time evolution is given by

\[
\sigma_t(f)(g, \rho, z) := \det(g)^{it}f(g, \rho, z)
\]

As before, let $\Gamma = \text{SL}(2, \mathbb{Z})$ and $\hat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$. Let $\Gamma$ act on $H$ via linear fractional transformations, that is, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in H$, we have $g(z) = \frac{az + b}{cz + d}$. Moreover, let $\Gamma$ act on $\hat{\mathbb{Z}}$ componentwise, i.e. for $(m_p)_p \in \hat{\mathbb{Z}}$ and $g \in \Gamma$, we have $g((m_p)_p) = (gm_p)_p$. Combining these two actions componentwise, we obtain an action of $\Gamma$ on $H \times \text{GL}_2(\hat{\mathbb{Z}})$. Then, the space of KMS$_\beta$ states of the Connes-Marcolli $\text{GL}_2$-system is determined as follows:

**Theorem 3.2.** The KMS$_\beta$ states of the $\text{GL}_2$-system are characterized as follows:

(a) For $\beta < 1$ there are no KMS$_\beta$ states.
(b) For $1 < \beta \leq 2$ there is a unique KMS$_\beta$ state.
(c) For $\beta > 2$ there is a one-to-one affine correspondence between KMS$_\beta$ states and probability measures on $\Gamma \setminus (H \times \text{GL}_2(\hat{\mathbb{Z}}))$. In particular, extremal KMS$_\beta$ states are in bijection with $\Gamma$-orbits in $H \times \text{GL}_2(\hat{\mathbb{Z}})$.

Some parts of the above theorem was proved in [CM1]. For the complete proof and more discussions on the above theorem and the case $\beta = 1$ see Theorems 3.7 and 4.1 as well as Remarks 3.8 and 4.8 of [LLaN].

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