NORMALITY AND COHEN-MACAULAYNESS OF LOCAL MODELS OF SHIMURA VARIETIES

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Abstract. We prove that in the unramified case, local models of Shimura variety with parahoric level structure are normal and Cohen-Macaulay.

INTRODUCTION

Local models of Shimura varieties are projective schemes over the spectrum of a discrete valuation ring. Their singularities are expected to model the singularities that arise in the reduction modulo p of Shimura varieties, with parahoric level structure. Local models also appear in the study of singularities of other moduli schemes (see Faltings [5] and Kisin [14]). We refer to the survey article by Pappas, Rapoport and Smithling [20] for more details.

The simplest case of local models is for modular curve with \( \Gamma_0(p) \)-level structure. In this case, the local model is obtained by blowing up the projective line \( \mathbb{P}^1_{\mathbb{Z}_p} \) over \( \text{Spec} (\mathbb{Z}_p) \) at origin of the special fiber \( \mathbb{P}^1_{\overline{F}_p} = \mathbb{P}^1_{\mathbb{Z}_p} \times_{\text{Spec} \mathbb{Z}_p} \text{Spec} \overline{F}_p \).

More generally, local models of Shimura varieties of PEL type with parahoric level structure were given by Rapoport and Zink in [22] and in the ramified PEL case, by Pappas and Rapoport [16], [17] and [19]. The constructions there are representation-theoretic and mostly done case-by-case.

Very recently, Zhu [27] (for equal characteristic analogy), Pappas and Zhu [21] made some new progress in the study of local models. They provide a group theoretic definition of local models that is not tied to a particular representation. The local model is constructed based on the “local Shimura data” \( (G, K, \{ \mu \}) \), where \( G \) is a connected reductive over \( \mathbb{Q}_p \), \( K \subset G(\mathbb{Q}_p) \) is a parahoric subgroup and \( \{ \mu \} \) is a geometric conjugacy class of one-parameter subgroup of \( G \). Assume furthermore that \( G \) splits over a tamely ramified extension of \( \mathbb{Q}_p \) and \( \mu \) is minuscule. In [21, Definition 7.1], Pappas and Zhu defined the local model \( M^{\text{loc}} \), which is a flat, projective scheme over \( \text{Spec} (\mathcal{O}_E) \). Here \( E \) is the reflex field of \( \mu \), \( \mathcal{O}_E \) is the ring of integers of \( E \) and \( k_E \) its residue field.

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It is conjectured in [21] that $M^{\text{loc}}$ is normal and Cohen-Macaulay. This question is also asked by Pappas, Rapoport and Smithling in [20]. In this paper, we’ll show that $M^{\text{loc}}$ is normal and Cohen-Macaulay in the unramified case. The precise statement will be found in Theorem 1.2.

Now we discuss the outline of the proof. The generic fiber of $M^{\text{loc}}$ is easy to understand. It is the Grassmannian variety associated to $\mu$. The special fiber $M^{\text{loc}} \otimes_{\mathcal{O}_E} k_E$, on the other hand, is much more difficult to understand. A basic technique, introduced by Görtz [8], is to embed the special fiber into an appropriate affine flag variety.

One of the main results in [21] is that the special fiber is the reduced union of affine Schubert varieties in the affine flag variety, indexed by the $\mu$-admissible set $\text{Adm}^K(\mu)$ of Kottwitz and Rapoport, and each irreducible component of the special fiber is normal and Cohen-Macaulay. This extends result of Görtz [8], [9], Pappas and Rapoport [16], [17], [19] on some Shimura varieties of PEL type. It is a deep result, based on the geometry of Schubert variety in affine flag varieties [7] and [18] and the coherence conjecture of [18] recently proved by Zhu in [27].

Now based on [20, Remark 2.1.3], it remains to prove that special fiber, as a whole, is Cohen-Macaulay. This is what we are going to do in this paper.

The statement is obvious when the special fiber is irreducible (see e.g. [21, Corollary 8.6]). The main difficulty appears when the special fiber has more than one irreducible components. In [8], Görtz proposes a combinatorial approach to this question and verify the statement for unramified unitary group of rank $\leq 6$ in this way (with the aid of computer). Later, He checked a few more cases for $GL_n$ and $GSp_{2n}$ with small $n$ (unpublished).

Our method here is quite different and more geometric. We now explain our strategy in more details. For simplicity, we only discuss the case where $G$ is split, of adjoint type and $K$ is an Iwahori subgroup of $G$. Let $\bar{k}_E$ be an algebraic closure of $k_E$. Let $LG = G_{\bar{k}_E[[u]]}$ be the loop group of $G_{\bar{k}_E}$ and $I$ be an Iwahori subgroup of $LG$. The map $G(\bar{k}_E[[u]]) \rightarrow G_{\bar{k}_E}$ sends $I$ to a Borel subgroup $B$ of $G_{\bar{k}_E}$. Let $\mathcal{F}l = LG/I$ be the affine flag variety.

The main idea is to relate the affine flag variety with the wonderful compactification $X$ [4] of $G_{\bar{k}_E}$ and the geometric special fiber $M^{\text{loc}} \otimes_{\mathcal{O}_E} \bar{k}_E$ with the boundary in $X$ of parabolic subgroup of $G$ associated to $\mu$.

The idea of relating affine flag variety with the wonderful compactification was motivated by Springer. In [25], Springer introduced a map from the loop group $LG$ to $X$, which factors through $G_{k_E[[u]]}/\mathcal{K}_1 \rightarrow X$. Here $\mathcal{K}_1$ is the kernel of the projection map $G(\bar{k}_E[[u]]) \rightarrow G_{\bar{k}_E}$. Notice that the natural map $LG/\mathcal{K}_1 \rightarrow \mathcal{F}l$ is a $B$-torsor. This map was used later in [10] in the study of affine Deligne-Lusztig varieties in affine flag.
Springer’s map is not continuous since it sends $G(\bar{k}_E[[u]])/\mathcal{K}_1$, which is closed in $LG/\mathcal{K}_1$, to $G_{\bar{k}_E}$, which is open in $X$. However, as we’ll see in Proposition 2.1, its restriction $G(\bar{k}_E[[u]])s_\lambda G(\bar{k}_E[[u]])/\mathcal{K}_1 \rightarrow X$ is a morphism. Here $\lambda$ is a coweight and $s_\lambda$ is the associated point in $G_{\bar{k}_E}(u)$.

In particular, for the minuscule coweight $\mu$, we have the following diagram

\[
\begin{array}{ccc}
\tilde{X}_\mu & \xrightarrow{\mu} & X_{\mu} \\
\downarrow & & \downarrow \\
X_{\mu} & \xrightarrow{Z'_{\mu}} & Z'_{\mu},
\end{array}
\]

where $X_{\mu} = G(\bar{k}_E[[u]])s_\mu G(\bar{k}_E[[u]])/I$ is a closed subscheme of $\mathcal{Y}_l$, $\tilde{X}_\mu = G(\bar{k}_E[[u]])s_\mu G(\bar{k}_E[[u]])/\mathcal{K}_1$, and $Z'_{\mu}$ is the codimension-one $G_{\bar{k}_E} \times G_{\bar{k}_E}$-orbit in $X$ corresponding to $\mu$. The two maps in the diagram are smooth morphisms with isomorphic smooth fibers.

The geometric special fiber $M_{loc} \otimes_{O_E} \bar{k}_E$ is a closed reduced subscheme of $X_{\mu}$. And there exists a closed reduced subscheme $A'$ of $Z'_{\mu}$ such that the inverse image of $M_{loc} \otimes_{O_E} \bar{k}_E$ in $Z_{\mu}$ equals to the inverse image of $A'$ in $Z_{\mu}$. Hence $M_{loc} \otimes_{O_E} \bar{k}_E$ is Cohen-Macaulay if and only if $A'$ is Cohen-Macaulay.

By the explicit description of $\tilde{B}$ in $X$ obtained by Brion [1] and Springer [23] and the description of $\mu$-admissible set obtained in a joint work with Lam [11], we’ll show that $A' = \tilde{B} \cap Z'_{\mu}$ and is an open subscheme of the boundary of $B$ in $X$. By a result of Brion and Polo [2], the boundary $\partial \tilde{B}$ is Cohen-Macaulay. Hence $A'$ is Cohen-Macaulay.

We finally obtain the Cohen-Macaulayness of the special fiber of local model.

In [27], Zhu introduced global Schubert varieties, which are the (generalized) equal characteristic counterpart of the local models. It is also worth mentioning that by a similar argument, in the unramified case, global Schubert varieties associated to minuscule coweights are normal and Cohen-Macaulay. It would be interesting to see if it is still the case for arbitrary coweights.

There is a different connection between local models and complete symmetric varieties, by Faltings in [6] and by Pappas (unpublished notes, see also [20, Chapter 8]). This approach doesn’t use loop groups and works when the level subgroup is close to maximal parahoric. It is interesting to see if the construction in this paper is related to this approach.

1. Local models

1.1. In this section, we recall the definition and some results on $M_{loc}$ in [21].
Let $\mathcal{O}$ be a discrete valuation ring with fraction field $F$ and perfect residue field $k$ of characteristic $p > 0$. Fix a uniformizer $\varpi$ of $\mathcal{O}$. Let $G$ be a connected reductive group over $\mathcal{O}$ scheme over $G$. Let $K$ be a parahoric subgroup of $G(F)$. Then $K$ is the stabilizer of a point $x$ in the Bruhat-Tits building of $G(F)$. Let $\mathcal{G}$ be the group scheme associated to $x$ in the sense of [21, Theorem 3.2]. It is a smooth affine group scheme over $\mathcal{O}$ represented by an ind-proper ind-scheme over $\mathcal{O}$ scheme over $\mathcal{O}$ scheme over $\mathcal{O}$. In particular, it is also the group scheme associated to $x$ in the sense of Pappas and Zhu. See [21, Proposition 6.3].

1.2. Let $\{\mu\}$ be a geometric conjugacy class of one parameter subgroups of $G$. Let $E$ be the reflex field of $\{\mu\}$, i.e. the field of definition of $\{\mu\}$. Let $E' = E F'$, where $F'$ is the maximal unramified extension $F'$ of $F$ in $\bar{F}$. As explained in [21, 7.a], there exists a representative of $\{\mu\}$ defined over $E'$ and this representative gives rise to an element $s_\mu$ in $L\mathcal{G}(E')$. Moreover, the $L^+\mathcal{G}$-orbit $(L^+\mathcal{G})_{E'} \cdot [s_\mu]$ in the affine Grassmannian $L\mathcal{G}/L^+\mathcal{G} \times_F E'$ is actually defined over $E$. In other words, there is an $E$-subvariety $X_\mu$ of $L\mathcal{G}/L^+\mathcal{G} \times_F E$ such that $X_\mu \times_E E' = (L^+\mathcal{G})_{E'} \cdot [s_\mu]$.

The generalized local model $M_{\mathcal{G},\mu}$ (in the sense of Pappas and Zhu) is the reduced scheme over $\text{Spec}(\mathcal{O}_E)$ which underlies the Zariski closure of $X_\mu$ in the ind-scheme $G_{\mathcal{G},\mathcal{O}_E}$.

1.3. Let $\bar{k}$ be an algebraic closure of $k$ and $F' = \bar{k}((u))$. Let $G' = \mathcal{G} \times_{\text{Spec}(\mathcal{O}[u])} \text{Spec}(F')$ be the base changing of $\mathcal{G}$ to $F'$. Then $G'$ splits over a tamely ramified extension $\bar{F}'$ of $F'$. Let $T'$ be the centralizer of a maximal split torus of $G'$. Let $I = \text{Gal}(\bar{F}'/F')$ and $X_*(T')_I$ be the coinvariants of the coweight lattice $X_*(T')$. Let $\tilde{W}$ be the Iwahori-Weyl group of $G'$ and $W_0 = N'(F')/T'(F')$ be the relative Weyl group of $G'$ over $F'$. There is a short exact sequence

$$1 \rightarrow X_*(T')_I \rightarrow \tilde{W} \rightarrow W_0 \rightarrow 1.$$  

For $\lambda \in X_*(T')_I$, we denote by $t_\lambda$ the corresponding translation element in $\tilde{W}_1$.

\[^1\text{Here we adapt the sign convention in [21]. In fact } t_\lambda \text{ equals to } t^\lambda \text{ in [11].} \]
1.4. Recall that $x$ is a point in the Bruhat-Tits building of $G(F)$. Let $P'_x \subset G'$ be the corresponding parahoric group scheme over $\tilde{k}[[u]]$. We choose a rational Borel $B'$ of $G'$ containing $T'$ in such a way that $P'_x$ is a standard parahoric group. As in [21, 8.d.1], this gives $\tilde{\rho}$ Coxeter group structure and hence a Bruhat order $\leq$. Choose a rational Borel set by $M$. Theorem 1.2. Suppose that $p \nmid |\pi_1(G_{der})|$. Then the special fiber $M_{\tilde{\rho},\mu} \otimes_{\mathcal{O}_F} k_E$ is reduced and each geometric irreducible component is normal and Cohen-Macaulay. Moreover, the geometric special fiber $M_{\tilde{\rho},\mu} \otimes_{\mathcal{O}_F} \tilde{k} = \mathcal{A}^{P'_x}(\mu)$ as closed subschemes of $Gr_{P'_x}$.

Here the condition that $p \nmid |\pi_1(G_{der})|$ is necessary to ensure that the corresponding loop group and affine Grassmannian variety are reduced. See [18, Remark 6.4].

The main theorem in this paper is as follows.

Theorem 1.2. Suppose that $p \nmid |\pi_1(G_{der})|$ and $\tilde{F}$ is an unramified extension of $F$ and $G$ is the Weil restriction $\text{Res}_{\tilde{F}/F}\tilde{G}$ for a split group $\tilde{G}$ over $\tilde{F}$. Then $M_{\tilde{\rho},\mu}$ is normal and Cohen-Macaulay.

2. Loop group and wonderful compactification

2.1. In this section, we assume that $G$ is split over $F$. Hence $G'$ is also split over $F'$, i.e., $G' = LH$ is the loop group for some connected reductive algebraic group $H$ over $\tilde{k}$. Let $T$ be a maximal torus of $H$ and $B \supset T$ be a Borel subgroup of $H$ such that $T' = T(F')$ and $B' = B(F')$. The pair $(B,T)$ determines the set of simple roots, which we denote by $S$. For any $J \subset S$, let $P_J \supset B$ be the standard parabolic subgroup of type $J$ and $P^+_J$ the opposite parabolic subgroup. Then $L_J = P_J \cap P^+_J$ is a standard Levi subgroup of $H$. For any parabolic subgroup $P$ of $H$, we denote by $U_P$ its unipotent radical.

2.2. Now we recall the variety $Z_J$ introduced by Lusztig in [15].

Let $J \subset I$. We define the action of $P_J^- \times P_J^-$ on $H \times H \times L_J$ by $(q,p) \cdot z = \pi_{P_J^-}(q)z\pi_{P_J^+}(p)^{-1}$. Here $\pi_{P_J^-} : P_J^- \to P_J^-/U_{P_J^-} \cong L_J$ and
We denote by $Z_{\pi} = \pi \circ \pi^{-1} \times_{P}\times L J$ its quotient space.

For any $h, h' \in H$ and $l \in L J$, we denote by $[h, h', l]$ the image of $(h, h', l)$ in $Z_{J}$. The $H \times H$-action on $Z_{J}$ is defined by $(h, h') \cdot [a, b, c] = [ha, h'b, c]$. We write $h_{J} = [1, 1, 1]$. This is the base point of $Z_{J}$.

2.3. Let $\pi : L^+ H \to H$ be the reduction modulo $\varpi$ map and $K_{1}$ be the kernel of $\pi$. Then $K_{1}$ is a normal subgroup of $L^+ H$. We define an action of $L^+ H \times L^+ H$ on $LH/k H_{1}$ by $(h, h') \cdot z_{K_{1}} = h z(h')^{-1} k_{1}$.

For any dominant coweight $\lambda \in X_{s}(T)$, set

$$I(\lambda) = \{ i \in S; \langle \lambda, \alpha_{i} \rangle = 0 \}$$

and

$$\tilde{X}_{\lambda} = L^+ H s_{\lambda} L^+ H / K_{1}.$$  

Then $\tilde{X}_{\lambda}$ is a single $L^+ H \times L^+ H$-orbit and is a locally closed subscheme of $LH/k_{1}$. Moreover $LH/k_{1} = \sqcup_{\gamma} \tilde{X}_{\lambda}$, where $\gamma$ runs over all the dominant coweights.

The following result provides a relation between $\tilde{X}_{\lambda}$ and $Z_{J}$.

**Proposition 2.1.** Let $\lambda$ be a dominant coweight and $J = I(\lambda)$. Then the map $L^+ H \times L^+ H \to Z_{J}$, $(h, h') \mapsto [\pi(h), \pi(h'), 1] = (\pi(h), \pi(h')) \cdot h_{J}$ induces a surjective $L^+ H \times L^+ H$-equivariant smooth morphism

$$s : \tilde{X}_{\lambda} \to Z_{J}$$

and each fiber is isomorphic to an affine space over $\bar{k}$ of dimension $\langle \lambda, 2\rho \rangle - \ell(w_{S})$. Here the action of $L^+ H \times L^+ H$ on $Z_{J}$ factors through the action of $H \times H$ on $Z_{J}$ defined in $\S$2.2, $\rho$ is the sum of all fundamental weights of $H$ and $w_{S}$ is the maximal element of $W_{0}$.

**Remark.** An analogy in mixed characteristic case is proved in a joint work with Wedhorn [13].

**Proof.** We first prove that $s$ is well-defined. We regard $H$ as a subgroup of $L^+ H$. Then $L^+ H = H K_{1} = K_{1} H$. Since the map $L^+ H \times L^+ H \to Z_{J}$ is $H \times H$-equivariant, it suffices to show that

(a) For $h, h' \in H$ with $h s_{\lambda}(h')^{-1} \in K_{1} s_{\lambda} K_{1}$, $(h, h') \cdot h_{J} = h_{J}$.

By assumption, $\emptyset \neq K_{1} h \cap s_{\lambda} K_{1} h's_{-\lambda} \subset L^+ H \cap s_{\lambda} L^+ H s_{-\lambda}$.

By [3, Theorem 2.8.7],

$$L^+ H \cap s_{\lambda} L^+ H s_{-\lambda} = (K_{1} \cap s_{\lambda} K_{1} s_{-\lambda})(K_{1} \cap s_{\lambda} H s_{-\lambda})(H \cap s_{\lambda} K_{1} s_{-\lambda})(H \cap s_{\lambda} H s_{-\lambda}).$$

We have that $K_{1} \cap s_{\lambda} H s_{-\lambda} = s_{\lambda} U_{P_{J}} s_{-\lambda}$, $H \cap s_{\lambda} K_{1} s_{-\lambda} = U_{P_{J}}$ and $H \cap s_{\lambda} K_{1} s_{-\lambda} = L_{J}$. Then there exists $z \in K_{1} \cap s_{\lambda} K_{1} s_{-\lambda}$, $l \in L_{J}$, $u \in U_{P_{J}}$ and $u' \in U_{P_{J}}$ such that

$$z(s_{\lambda} u's_{-\lambda}) ul \in K_{1} h \cap s_{\lambda} K_{1} h's_{-\lambda}.$$
Notice that \( s_\lambda u's_{-\lambda} \in \mathcal{K}_1 \). Hence \( z(s_\lambda u's_{-\lambda})ul \in \mathcal{K}_1 ul \) and \( h = ul \). Similarly, \( s_{-\lambda}us_\lambda \in \mathcal{K}_1 \) and \( s_{-\lambda}(z(s_\lambda u's_{-\lambda})ul)s_\lambda \in \mathcal{K}_1 u'l \). Hence \( h' = u'l \). Therefore
\[
(h, h') \cdot h_J = (ul, u'l) \cdot h_J = (u, u') \cdot h_J = h_J.
\]

(a) is proved.

Since \( L^+H \times L^+H \) acts transitively on \( \bar{X}_\lambda \) and on \( Z_J \) and the map is \( L^+H \times L^+H \)-equivariant, each fiber is isomorphic. Now we consider the fiber over \( h_J \). It is
\[
\{ \mathcal{K}_1 U_{P_J} l s_\lambda l^{-1} U_{P_J} \mathcal{K}_1; l \in L_J \}/\mathcal{K}_1 = \mathcal{K}_1 U_{P_J} l s_\lambda l^{-1} U_{P_J} \mathcal{K}_1/\mathcal{K}_1.
\]
Since \( s_{-\lambda}U_{P_J} \subset \mathcal{K}_1 \) and \( s_\lambda U_{P_J} s_{-\lambda} \subset \mathcal{K}_1 \), we have that
\[
\mathcal{K}_1 U_{P_J} l s_\lambda l^{-1} U_{P_J} \mathcal{K}_1/\mathcal{K}_1 = \mathcal{K}_1 s_\lambda \mathcal{K}_1/\mathcal{K}_1 \cong \mathcal{K}_1/(\mathcal{K}_1 \cap s_\lambda \mathcal{K}_1 s_{-\lambda}).
\]
This is an affine space of dimension \( \dim(\bar{X}_\lambda) - \dim(Z_J) = \dim(X_\lambda) + \dim(B) - \dim(G) = \langle \lambda, 2\rho \rangle - \ell(w_S). \)

2.4. Now we recall the definition and some elementary facts on the wonderful compactification. More details can be found in the survey article of Springer [24].

Let \( \mathcal{H} \) be the adjoint group of \( H \). The set of simple roots of \( \mathcal{H} \) is again denoted by \( S \). For any subgroup \( H' \) of \( H \), we denote by \( \mathcal{H}' \) the image of \( H' \) via the map \( H \to \mathcal{H} \).

Let \( X \) be the wonderful compactification of \( \mathcal{H} \) ([4], [26]). Roughly speaking, one start with a suitable finite-dimensional projective representation \( \rho : \mathcal{H} \to PGL(V) \) of \( H \), then \( X \) is defined to be the closure in \( PGL(V) \) of the image \( \rho(\mathcal{H}) \). The closure is independent of the choice of \( \rho \).

It is known that \( X \) an irreducible, smooth projective \((H \times H)\)-variety with finitely many \( H \times H \)-orbits indexed by the subsets \( J \) of \( S \). They are described as follows.

Let \( J \subset S \). Let \( H_J \) be the adjoint group of \( L_J \) (and hence of \( L_J \)). We define an action of \( P_J \times P_J \) on \( H \times H \times H_J \) in the same way as in §2.2 and denote by \( Z'_J = (H \times H) \times L_J \times H_J \) the quotient space. The group \( H \times H \) (and hence \( H \times H \)) acts on \( Z'_J \) in the same way as in §2.2. We denote by \( h'_J \) the image in \( Z'_J \) of \((1, 1, 1) \in H \times H \times H_J \). This is the base point of \( Z'_J \). It is known that \( X = \bigsqcup_{J \subset S} Z'_J \) is the union of \( H \times H \)-orbits.

For any locally closed subscheme \( Z \) of \( X \), we denote by \( \tilde{Z} \) the closure of \( Z \) in \( X \). The closure relation between \( H \times H \)-orbits on \( X \) is described as follows. For any \( J \subset S \),
\[
\tilde{Z}'_J = \bigsqcup_{J' \subset J} Z'_J
\]
In particular, \( Z_S = \mathcal{H} \) is the open orbit in \( X \) and for any maximal proper subset \( J \) of \( S \), \( Z'_J \) is a codimension-one orbit of \( X \) and hence
is open in the boundary $\partial X = X \setminus H$ of $H$. The closed orbit $Z'_\emptyset$ is isomorphic to $H/B \times H/B$.

2.5. Now we discuss the situation we may apply to the study of local models. By the choice of $B$ and $T$, $L^+\mathcal{P}'_x$ is the inverse image of $P_Y$ under $\pi$ for some $Y \subset S$ and $L^+\mathcal{P}'_x/\mathcal{K}_1 \cong P_Y$. Therefore the projection map $f_1 : LH/\mathcal{K}_1 \to LH/L^+\mathcal{P}'_x$ is a $P_Y$-torsor. Hence the map $f_1 : \tilde{X}_\mu \to X_\mu$ is a $P_Y$-torsor.

Now we have the following diagram

$$
\begin{array}{ccc}
X_\mu & \xrightarrow{f_1} & \tilde{X}_\mu \\
& \xleftarrow{s} & \downarrow{f_2} \\
& & Z_{I(\mu)} \\
\end{array}
$$

Here $f_2 : Z_{I(\mu)} \to Z'_{I(\mu)}$ is induced from the map $H \times H \times L_J \to H \times H \times H_J$ and hence is a smooth morphism with fibers isomorphic to the center of $L_J$.

Notice that $s, f_1, f_2$ are smooth morphisms with isomorphic smooth fibers for each map. Hence if $A$ is a closed reduced subscheme of $X_\mu$ and $A'$ is a closed reduced subscheme of $Z'_{I(\mu)}$ such that $f_2^{-1}(A) = (f_2 \circ s)^{-1}(A')$, then $A$ is Cohen-Macaulay if and only if $A'$ is Cohen-Macaulay.

3. Boundary of Parabolic Subgroup

In this section, we study the boundary in $X$ of the parabolic subgroup $P_Y$ of $H$. As we’ll see in the next section, this boundary is closely related to the geometric special fiber of local model in the sense of §2.5.

3.1. We first recall some results on the $B \times B$-orbits of $X$ obtained by Brion [1] and Springer [23].

For any $J \subset S$, we denote by $W_J$ the subgroup of $W_0$ generated by the simple reflections in $J$ and $W^J$ the set of minimal length representatives in $W_0/W_J$. Let $w_J$ be the maximal element in $W_J$.

For any $(x, y) \in W^J \times W_0$, we set

$$[J, x, y] = (Bx, By) \cdot h'_J \subset Z'_J.$$

By [23, Lemma 1.3], $X = \sqcup_{J \subset S} \sqcup_{x \in W^J} W_0 \cdot [J, x, y]$.

The closure relations between $B \times B$-orbits of $X$ is obtained in [23, Proposition 2.4]. The following simplified version is found in [12, Proposition 6.3].

**Proposition 3.1.** Let $J, J' \subset S$, $x \in W^J$, $x' \in W^{J'}$ and $y, y' \in W_0$. Then $[J', x', y'] \subset [J, x, y]$ if and only if $J' \subset J$ and there exists $u \in W_J$ such that $xu \leq x'$, $y' \leq yu$. 

3.2. Now we discuss some special cases that will be used in this paper.

For \( J \subset S \), we define a partial order \( \leq_J \) on \( W^J \times W_0 \) as follows. Let \((x, y), (x', y') \in W^J \times W_0\), we write \((x', y') \leq_J (x, y)\) if there exists \( u \in W_J \) such that \( xu \leq x' \), \( y' \leq yu \). Then \([J, x', y'] \subset [J, x, y]\) if and only if \((x', y') \leq_J (x, y)\).

The following joint result with Lam [11, Theorem 2.2] relates this partial order with the Bruhat order on the Iwahori-Weyl group.

**Proposition 3.2.** Let \( \mu \) be a minuscule coweight. Then

1. The map
   \[
   (W^{I(\mu)} \times W_0, \leq_{I(\mu)}) \to (W_0 t_\mu W_0, \leq), \quad (x, y) \mapsto xt_\mu y^{-1}
   \]
   is a bijection between posets. Here \( \leq \) is the restriction to \( W_0 t_\mu W_0 \) of the Bruhat order on \( W \).

2. Set \( Q_\mu = \{(x, y) \in W^{I(\mu)} \times W_0; y \leq x\} \). Then the restriction of the map in (1) gives a bijection from \( Q_J \) to the admissible set \( \text{Adm}(\mu) = \{x \in \dot{W}; \exists w \in W_0, x \leq t_w(\mu)\} \).

3.3. As a special case of Proposition 3.1, the closure of \( P_Y \) in \( X \) is described as follows

\[
\overline{P_Y} = [S, 1, w_Y] = \bigcup_{J \subset S} \bigcup_{x \in W^J, y \in W_{\min(W_Y y)\leq x}} [J, x, y].
\]

For our purpose, we need a different description of \( \overline{P_Y} \).

**Corollary 3.3.** For any \( J \subset S \),

\[
\overline{P_Y} \cap Z'_J = \bigcup_{w \in W^J} (P_Y w, P_Y w) \cdot h'_J \cap Z'_J.
\]

Proof. Define an action of \( B \times B \) on \( P_Y \times P_Y \) by \((b, b') \cdot (p, p', z) = (bp^{-1}, bp'(b')^{-1}, (b, b') \cdot z)\). Let \((P_Y \times P_Y) \times B \times B \) be the quotient space. The map \( P_Y \times P_Y \times \overline{B} \to X \), \((p, p', z) \mapsto (p, p') \cdot z \) induces a proper morphism \( (P_Y \times P_Y) \times B \times B \overline{B} \to X \). Hence the image equals to the closure of the image of \((P_Y \times P_Y) \times B \overline{B} \) in \( X \). So \((P_Y, P_Y) \cdot \overline{B} = \overline{P_Y} \) and

\[
\overline{P_Y} \cap Z'_J = (P_Y, P_Y) \cdot \overline{B} \cap Z'_J = (P_Y, P_Y) \cdot (\overline{B} \cap Z'_J).
\]

By Proposition 3.1 and §3.2,

\[
\overline{B} \cap Z'_J = \bigcup_{(x, y) \in W^J \times W_0, y \leq x} [J, x, y] \subset \bigcup_{w \in W^J} [J, w, w] \cap Z'_J.
\]

On the other hand, \( \overline{B} \cap Z'_J \) is closed in \( Z'_J \) and \([J, w, w] \subset \overline{B} \) for all \( w \in W^J \). Therefore \( \overline{B} \cap Z'_J = \bigcup_{w \in W^J} [J, w, w] \cap Z'_J \).

Hence

\[
\overline{P_Y} \cap Z'_J = (P_Y, P_Y) \cdot (\overline{B} \cap Z'_J) = \bigcup_{w \in W^J} (P_Y, P_Y) \cdot ([J, w, w] \cap Z'_J)
\]

\[
= \bigcup_{w \in W^J} (P_Y w, P_Y w) \cdot h'_J \cap Z'_J.
\]

\( \square \)
3.4. Define an action of $B$ on $P_Y \times \overline{B}$ by $b \cdot (p, z) = (pb^{-1}, (b, 1) \cdot z)$. Let $P = P_Y \times \overline{B}$ be the quotient space. The morphism

$$P_Y \times \overline{B} \to X, \quad (p, z) \mapsto (p, 1) \cdot z$$

induces a proper morphism $P \to X$. The image is the closure of the image of $P_Y \times \overline{B} \subset P$ and equals to $P_Y \cdot z$. We denote by $f : P \to P_Y$ this birational proper morphism.

By [2, Theorem 20 (iii)], $B$ and $P_Y$ are Cohen-Macaulay. Hence $P$ is also Cohen-Macaulay. We denote by $\omega_P$ the dualizing sheaf of $P$ and $\omega_{P_Y} = \omega_P$. We now recall a result of Brion and Polo [2].

Lemma 3.4. We keep the notations as above. Then

$$f^*\omega_P = \omega_{P_Y}.$$ 

Proof. By [2, Theorem 20 (ii)], $f_*\mathcal{O}_P = \mathcal{O}_{P_Y}$ and $R^i f_*\mathcal{O}_P = R^i f_*\omega_P = 0$ for $i \geq 1$. Also $P$ is Cohen-Macaulay. The lemma then follows from [2, Lemma 15].

3.5. We follow the notation in [2, Section 1].

Let $\hat{H}$ be the simply connected covering of $H$. Let $\bar{B}$ (resp. $\bar{T}$) be the preimage of $B$ (resp. $T$) in $\hat{H}$. For any $\lambda \in X_*(\bar{T})$, we denote by $\mathcal{L}_{\hat{H}/\bar{B}}(\lambda)$ the $\hat{H}$-linearized line bundle on $\hat{H}/\bar{B}$ whose geometric fiber at the point $\bar{B}/\bar{B}$ is the 1-dimensional representation of $\bar{B}$ corresponding to the character $-\lambda$ and denote by $\mathcal{L}_X(\mathcal{L})$ the line bundle on $X$ such that the restriction $Z_\lambda$ is $\mathcal{L}_{\hat{H}/\bar{B}}(\lambda) \boxtimes \mathcal{L}_{\hat{H}/\bar{B}}(-w_\lambda)\mathcal{L}$.

Let $g : P \to P_Y/\overline{B}$ be the projection map and $\omega_y$ be the relative dualizing sheaf. The dualizing sheaf $\omega_P$ is calculated in the proof of [2, Theorem 20]. It is

$$\omega_P = g^*\omega_{P_Y/\overline{B}} \otimes \omega_y = g^*\omega_{P_Y/\overline{B}} \otimes g^*\mathcal{L}_{P_Y/\overline{B}}(\rho) \otimes f^*\mathcal{L}_{P_Y/\overline{B}}(-\beta - \rho),$$

where $\beta$ is the sum of simple roots and $\rho$ is the sum of all fundamental weights of $\hat{H}$.

Similar to [2, Theorem 20], we have the following result.

Proposition 3.5. Let $\partial P_Y = P_Y \setminus P_Y$ be the boundary of $P_Y$ in $X$. Then the dualizing sheaf of $P_Y$ is locally isomorphic to the ideal sheaf of $\partial P_Y$ and $\partial P_Y$ is Cohen-Macaulay.

Proof. By the proof of [2, Theorem 20], the line bundle associated to $\partial P_Y = P_Y \cap \partial X$ is $\mathcal{L}_{P_Y/\overline{B}}(-\beta)$. By Lemma 3.4 and §3.5,

$$\omega_{P_Y} = f_*\omega_P = f_*(g^*(\omega_{P_Y/\overline{B}} \otimes \mathcal{L}_{P_Y/\overline{B}}(\rho)) \otimes f^*\mathcal{L}_{P_Y/\overline{B}}(-\beta - \rho))$$

$$= f_*g^*(\omega_{P_Y/\overline{B}} \otimes \mathcal{L}_{P_Y/\overline{B}}(\rho)) \otimes \mathcal{L}_{P_Y/\overline{B}}(-\rho) \otimes \mathcal{L}_{P_Y/\overline{B}}(-\beta).$$
where $J_{\partial \overline{P_Y}}$ is the ideal sheaf of $\partial \overline{P_Y}$.

In particular, $J_{\partial \overline{P_Y}}$ is locally isomorphic to $\omega_{\overline{P_Y}}$ and hence is Cohen-Macaulay of depth $\dim(\overline{P_Y})$. Now the exact sequence

$$0 \to J_{\partial \overline{P_Y}} \to O_{\overline{P_Y}} \to O_{\partial \overline{P_Y}} \to 0$$

yields that the sheaf $O_{\partial \overline{P_Y}}$ is Cohen-Macaulay of depth $\dim(\overline{P_Y}) - 1 = \dim(\partial \overline{P_Y})$.  
\[\square\]

4. Proof of the main theorem

4.1. In this section, we prove Theorem 1.2.

Let $n = [\tilde{F} : F]$ be the degree of $\tilde{F}$ over $F$. By definition, $G \otimes_F \tilde{F} = \tilde{G} \times \cdots \times \tilde{G}$ is a product of $n$-copies of $\tilde{G}$ and

$$M_{\tilde{G}, \mu} \otimes_{O_{\tilde{F}}} O_{\tilde{F}} = M_{\tilde{G}, \mu} \times \cdots \times M_{\tilde{G}, \mu}$$

is a product of $n$-copies of local model $M_{\tilde{G}, \mu}$ for $\tilde{G}$ over $\tilde{F}$.

Thus it suffices to consider the case where $G$ is split over $F$. We keep this assumption in the rest of this section.

4.2. The geometric special fiber $M_{\tilde{G}, \mu} \otimes_{O_{\tilde{F}}} \bar{k} = A_{P_Y}^{\nu}(\mu)$ is closed subschemes of $X_{\mu}$. Set $A_{P_Y}^{\nu}(\mu) = \cup_{w \in \text{Adm}(\mu)} L^{+}P_x^w L^{+}P_x^w \mathcal{K}_{1}$. Then $A_{P_Y}^{\nu}(\mu)$ is a reduced closed subscheme of $\tilde{X}_{\mu}$ and is the inverse image of $A_{P_Y}^{\nu}(\mu)$ under the map $f_1$.

Let $A_Y(\mu)$ be the reduced subscheme of $Z_{I(\mu)}'$, which equals to

$$\cup_{(x,y) \in Q_{\mu}} (P_Y x, P_Y y) \cdot h_{I(\mu)}'$$

as a set. Notice that $P_x^w = P_Y \mathcal{K}_1 = \mathcal{K}_1 P_Y$. By Proposition 3.2 (2), we have that $A_{P_Y}^{\nu}(\mu) = (f_2 \circ s)^{-1} A_Y(\mu)$.

The reduced schemes $A_{P_Y}^{\nu}(\mu)$ and $A_Y(\mu)$ are related in the sense of §2.5.

4.3. We'll then prove that $A_Y(\mu)$ is the scheme-theoretic intersection of $P_Y$ with $Z_{I(\mu)}'$.

We first show that $A_Y(\mu) = P_Y \cap Z_{I(\mu)}'$ set-theoretically.

By definition, $A_{P_Y}^{\nu}(\mu)$ is the union of the closures of $L^{+}P_x^w \mathcal{K}_1 P_Y$ in $L/H \cup L^+P_x^w$, where $w$ runs over elements in $W'_{(\mu)}$. By §2.5, $A_Y(\mu)$ is the union of the closures of $P_Y w \cdot h_{I(\mu)}'$ in $Z_{I(\mu)}'$. Hence by Corollary 3.3, $A_Y(\mu) = P_Y \cap Z_{I(\mu)}'$ as sets.

It remains to show that $P_Y \cap Z_{I(\mu)}'$ is reduced. We recall a result in [12, Proposition 6.2], which strengthened [2, Theorem 2].

Proposition 4.1. There exists a Frobenius splitting on $X$ that compatibly splits all the $B \times B$-orbit closures.
In particular, there exists a Frobenius splitting on $X$ that compatibly splits $\overline{P_Y}$ and $Z'_{I(\mu)}$. Therefore the scheme-theoretic intersection $\overline{P_Y} \cap Z'_{I(\mu)}$ is a split scheme and hence is reduced. Therefore the scheme-theoretic intersection $\overline{P_Y} \cap Z'_{I(\mu)}$ is also reduced.

4.4. Now we prove Theorem 1.2 for split case.

Since $\mu$ is minuscule, $I(\mu)$ is a maximal proper subset of $S$. Hence $Z'_{I(\mu)}$ is a open subscheme of $\partial X$. So $A^Y(\mu) = \overline{P_Y} \cap Z'_{I(\mu)}$ is a open subscheme of $\partial \overline{P_Y}$. Since $\partial \overline{P_Y}$ is Cohen-Macaulay, $A^Y(\mu)$ is also Cohen-Macaulay.

By §2.5, the geometric special fiber $A^{\mathfrak{p}^s}_Y(\mu)$ is Cohen-Macaulay and so is the special fiber $M_{G,\mu} \otimes_{O_E} k$. By Theorem 1.1, $M_{G,\mu} \otimes_{O_E} k$ is reduced and each irreducible component is normal. By [20, Remark 2.1.3], $M_{G,\mu}$ is normal and Cohen-Macaulay. This finishes the proof.

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