Phenomenological signatures of two-body decays in deformed relativity

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Deformed relativistic kinematics are a framework which captures effects that are expected from particles and fields propagating on a quantum spacetime effectively. They are formulated in terms of a modified dispersion relation and a modified momentum conservation equation. In this work we use Finsler geometry to formulate deformed relativistic kinematics in terms of particle velocities. The relation between the Finsler geometric velocity dependent formulation and the original momentum dependent formulation allows us to construct deformed Lorentz transformations between arbitrary frames and the corresponding compatible momentum conservation equation, to first order in the Planck scale deformation of special relativity based on the $\kappa$-Poincaré algebra in the bicrossproduct basis. The deformed Lorentz transformations, as well as the deformed time dilation factor, contain terms that scale with the energy of the particle under consideration to the fourth power. This feature is an amplifier which opens access to observables that are close to the Planck scale with present technological capabilities. We derive how the distribution of particles in decays is affected and find, in the example of a pion decaying into a neutrino and a muon, how the ratio of expected neutrinos to muons with a certain energy, changes compared to the predictions based on special relativity. We also discuss the phenomenological consequences of this framework for cosmic-ray showers in the atmosphere.

I. INTRODUCTION

The search for the nature of spacetime when gravitational degrees of freedom are quantized has proven to be one of the most challenging quests of physical sciences. Many theoretical attempts have been pursued in the last decades [1–3], which has allowed the emergence of phenomenological proposals that gather some in common and observationally compelling properties of these approaches in a unified way that can be confronted with present or near future observations [4], specially in the multi-messenger astronomy era [5].

Among these proposals, a prominent role is played by the those that predict a violation or deformation of the special relativistic principles (referred as LIV and DSR scenarios, respectively) in the regime in which $\hbar \to 0$ and $G \to 0$, but that preserves the Planck energy scale $E_P = \sqrt{c^5\hbar/G} \approx 1.22 \times 10^{19}$ GeV. The phenomenological opportunities of this approach range among several areas, for instance threshold effects in particle interactions [6–8], time delays in the arrivals of relativistic particles with different energies due to modified dispersion relations (MDRs) [9–15]), gravitational lensing observations [16, 17], violation of CPT symmetry [18, 19], etc (we refer the reader to Refs. [4, 5, 20] for reviews in modern searches of these effects).

High-energy astrophysical messengers such as cosmic rays, gamma rays, and neutrinos have been widely used to search for signatures of quantum gravity (QG). Most investigations focused on searches for LIV signatures, both in the propagation of these cosmic messengers and in the atmospheric shower they induce (see, e.g., [5, 21–24] for reviews). Nevertheless, their potential for probing DSR theories have not yet been fully exploited, despite this being a clearly promising avenue for investigating QG phenomenology. One remarkable example of a problem whose solution might potentially hint at QG is the so-called muon puzzle, which consists in an $8\sigma$ excess in the measured number of muons in air showers initiated by ultra-high-energy cosmic rays (UHECRs) compared with theoretical predictions [25–29]. In this case, the muon puzzle could be a consequence of the changes in the kinematics associated with the decay of the particles that compose the shower.

Recently, in approaches that deform relativistic symmetries, the lifetime of fundamental particles has been shown to be an experimentally compelling observable with the potential of reaching the Planck scale in the near future.

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In [31] we used Finsler geometry, see for example [32] for an overview on Finsler geometry, to construct the geometric clock (the length measure for worldliness) which is compatible with DSR symmetries, and derived the dilated lifetime of particles. The corrections to the special relativistic time dilation are interoressed as arising from the effective spacetime probed by particles which are subject to kinematic corrections due to propagation on a quantum spacetime.

One of the key results that emerged in [31] consists in the identification of a deformed Lorentz transformation that relates the (4-)momenta of a particle in its rest frame to the lab frame simply from the relation between the momenta and the (4-)velocity derived by the Finsler function. Most interestingly, the first order Planck-scale correction is governed by the fourth power of the Lorentz factor. This dominant contribution is of high importance since it works as an amplifier to the Planck scale that can turn the lifetime of fundamental particles in accelerators as an observable that could be measured with Planck scale sensitivity.

We wish to diversify the analysis of observables which are derived from the transformations between different frames which move with a certain velocity relatively to each other, and/or seek to have a more complete description that is compatible with the relativistic principle. Therefor, it becomes necessary to find the deformed Lorentz transformations that connect arbitrary frames. In addition, since we work in a DSR approach, it becomes fundamental to investigate the composition law of momenta in a Finsler-DSR-compatible way [33], in order to preserve the nature of relativistic interactions for every inertial frame. Previous investigations have focused on an infinitesimal version of the symmetry transformations, i.e., without integrating the symmetry transformation parameters [34]. Here we construct the finite version of the deformed Lorentz transformation and its related composition law, in the sense that the trasnformation parameter which connects different frames does not need to be small, this is what we call a finite transformation. However, this should not be confused with the fact that we investigate first order in Planck-scale corrections, i.e. first order Finsler deviations from Minkowski spacetime which emerge from first order in Planck-scale deformed symmetries. The reason for this choice consists in the positive outcome due to the amplifying factor that grows with the fourth-power in the Lorentz factor that might bring us close to the Planck scale with accelerators or astrophysical observations. Investigating these issues is the main purpose of this paper.

The paper is organized as follows: In Section II, we derive the finite deformed Lorentz transformation between arbitrary frames in the bicrossproduct basis ς-Poincaré-inspired Finsler geometry. In Section III, we derive the Finsler-DSR-compatible composition law of energy and momentum and discuss its ambiguities. In Section IV, we discuss implications of these results for some equations that involve particles’ decays. In Section V, we draw our conclusions and prospects.

II. DEFORMED LORENTZ TRANSFORMATIONS FROM FINSLER GEOMETRY

Modified relativistic kinematics lead to a momentum dependent spacetime geometry which can be captured by a non-trivial geometry phase space (the cotangent bundle) of spacetime [35–37]. For applications, it is useful to reformulate the momentum dependent spacetime geometry in a dual way in terms of a velocity dependent geometry on the tangent bundle of spacetime, which leads to the mathematical framework of Finsler geometry [32].

Here, we assume as our working model the so called bicrossproduct basis-inspired Finsler geometry discussed in [31, 34]. In this case, the Finsler function in the Cartesian coordinate system, calculated from a Helmholtz action on the tangent bundle of spacetime, which leads to the mathematical framework of Finsler geometry [32].

From the Finsler function (1) it can be verified that the particle’s momenta satisfy the following MDR that is functionally equivalent to the Cartesian coordinates realization of the Casimir operator of the ς-Poincaré algebra in the bicrossproduct basis [34]

\[ m^2 = H(x, p) = g^{\mu\nu}(\dot{x}(p))p_\mu p_\nu = p_0^2 - \delta^{ij}p_ip_j - \ell p_0 \delta^{ij}p_ip_j. \]
In the above equation, $g^{\mu\nu}$ are the contravariant components of the Finsler metric $g_{\mu\nu} = \partial^2 (F^2/2)/\partial \dot{x}^\mu \partial \dot{x}^\nu$, and the momenta physical $p_\mu$, which satisfy the dispersion relation, are defined directly from $F$ as follows
\[ p_\mu = m \frac{\partial F}{\partial \dot{x}^\mu}. \] (4)

### A. Lorentz transformations between rest frame and lab frame

Calculating the momenta from (1) and parametrizing the particle’s worldline with the coordinate time $x^0$, we observe that the energy $E = p_0$ and spatial momenta $p_i$ are linked to the mass of the particle by the following relation
\[
\begin{align*}
    p_0(v) &= \gamma m - \frac{\ell}{2} m^2 (\gamma^2 - 1)(2\gamma^2 - 1), \\
    p_i(v) &= -v_i \gamma m + \ell m^2 v_i \gamma^4,
\end{align*}
\] (5) (6)
where $v^i = dx^i/dx^0$, $v^2 = \delta_{ij} v^i v^j$ and $\gamma^{-1} = \sqrt{1 - v^2}$ is what we call the velocity Lorentz factor given in terms of the particle’s velocity as measured by the laboratory (lab) frame. Notice that Eqs. (5) and (6) are, in fact, displaying the energy/momentum of a particle in its rest frame to the lab frame in which it moves with velocity $v$ and the rest frame, which is, in fact, a symmetry of the Minkowski spacetime.

As we verified in [31], Eqs. (5) and (6) are not only a transformation between quantities in the rest and the lab frames, but also constitute a symmetry of the Finsler spacetime, in the sense that the dispersion relation which is, in fact a symmetry of the Minkowski spacetime.

We verify in [31], Eqs. (5) and (6) are not only a transformation between quantities in the rest and the lab frames, but also constitute a symmetry of the Finsler spacetime, in the sense that the dispersion relation $H(x,p) = H(x,\tilde{p}) = m^2$ is preserved. In other words, for a particle at rest ($v = 0, \gamma = 1$), the dispersion relation implies for the momenta $p_0 = m$ and $p_i = 0$. If we consider the $p_\mu$ as function of $\gamma$ and apply the transformation
\[
\begin{align*}
    p_0 = p_0(1) \to \tilde{p}_0 = p_0(\gamma) = \frac{\partial}{\partial \dot{x}^0} F \quad \text{and} \quad p_i = p_i(1) \to \tilde{p}_i = p_i(\gamma) = \frac{\partial}{\partial \dot{x}^i} F,
\end{align*}
\] (7)
then $\tilde{p}_\mu$ also satisfies the dispersion relation. This implies that the just highlighted transformations are, in fact, deformed Lorentz transformations between the rest and the lab frames.

Eq. (5) also connects what we call the velocity Lorentz factor $\gamma$ and the momentum Lorentz factor $\bar{\gamma} = p_0/m$ as $\gamma = \bar{\gamma} + \frac{\ell m}{2}(1 - 3\bar{\gamma}^2 + 2\bar{\gamma}^4)$, which, along with the formulation of the clock postulate in Finsler geometry [31], gives the following prediction for the dilated lifetime of fundamental particles as measured in the lab frame
\[
\Delta t = \frac{p_0}{m} \Delta \tau \left[ 1 + \frac{\ell}{2} \left( \frac{m^2}{p_0} - 2p_0 + \frac{p_0^3}{m^2} \right) \right] = \gamma_{DSR} \Delta \tau,
\] (8)
where $\Delta \tau$ corresponds to the lifetime a fundamental particle in its rest frame. We define a modified Lorentz factor $\gamma_{DSR}$, which is derived from the geometric clock defined by the Finsler function (1) and which obeys the principle of relativity through the deformation of the Lorentz transformation. This verification is particularly important when one is analyzing the effects of the Finsler deformed relativistic approach on the phenomenology of the time dilation of particle’s lifetime in accelerators. In Fig. 1, we displayed how the lifetime of a pion (as a function of the energy in the lab frame), with mass around 140 MeV, would be dilated in the lab frame, when the particle propagates through the effective Finsler spacetime, if we assume that the lifetimes are connected by the clock postulate and the energy-dependence is determined by the deformed Lorentz symmetry.

Notice that one would have found a result that shares some similarities with this approach, if one had considered a different strategy through the identification of an “energy-momentum-dependent mass” $m_{LIV}(p)$, that is read from the MDR, as $p_0^2 - \delta^{ij} p_ip_j = m^2 + \ell p_0 \delta^{ij} p_ip_j = m_{LIV}^2$. From this one could define a modified Lorentz factor of the form $\gamma_{LIV} = p_0/m_{LIV} \approx p_0/m(1 - \frac{\ell}{2} \frac{p_0^3}{m^2})$. This second kind of deformed Lorentz factor has been considered in [38] to constrain quantum gravity effects from cosmic rays, by directly inserting it as a factor that dilates the lifetime of particles at rest $\gamma_{LIV} \Delta \tau$, furnishing constraints on the quantum gravity length scale $\ell \in (-5.95 \times 10^{-6}, 10^{-1})\ell_{Planck}$, from an MDR.

We point out that this constraint does not translate directly to our case, since the full shape of the deformed Lorentz factor derived from the energy dependent mass, does not share exactly the same factors as the one we derive from the geometric clock (2) defined by the Finsler function. We believe that further investigations would need to be carried out in order to properly compare the two proposals. Moreover, the derivation of the deformed time dilation
factor from the clock postulate defining Finsler function stems from first principles, namely the time measurement. The derivation via the introduction of an “energy-momentum-dependent mass” $m_{LIV}(p)$ is more ad hoc and, in our opinion, it is not immediately clear how it is related to an observers measurement of time and the geometric clock, from which we started.

In order to extend the analysis on particle lifetimes to particle decays in different frames, as we will do in Section IV, it is necessary to find the expression of the deformed Lorentz transformations in terms of the momenta of the particles and their relative velocity $v$. In this way, we are analysing these processes in a scenario that does not break Lorentz invariance and preserves the clock postulate.

### B. Lorentz transformations between arbitrary momenta

In order to proceed with this investigation, we assume, for simplicity, a spacetime in $1 + 1$ dimensions. We consider two inertial frames $S$ and $\tilde{S}$, that move with relative velocity $v$. We also assume that each observer assigns momenta $p_{\mu}$ and $\tilde{p}_{\mu}$ to a given particle. From this ansatz, we see that, to first order in $\ell$, the most general deformed transformation that connects these momenta $p_{\mu} \rightarrow \tilde{p}_{\mu}$ (and that reduces to (5) and (6) when $p_{\mu} = (m, 0)$) have to be of the following form:

\[
\tilde{p}_0 = \gamma(p_0 - vp_1) + \ell \left[ A p_0 p_1 + B p_1^2 - \frac{1}{2} p_0^2 (\gamma^2 - 1)(2\gamma^2 - 1) \right], \
\tilde{p}_1 = \gamma(p_1 - vp_0) + \ell \left( p_0^2 v\gamma^4 + F p_0 p_1 + G p_1^2 \right),
\]

where $A, B, F, G$ are general functions of $v$. The terms multiplying the perturbation parameter $\ell$ must be quadratic in the momenta for dimensional reasons. Enforcing the invariance of the MDR (3), in order to indeed derive a deformed symmetry transformation, we find the following necessary conditions for these functions:

\[
B = -\frac{A}{v} - \frac{(1 - v^2)\gamma^2 - v^2 - 1}{2(1 - v^2)}, \
F = -\frac{A}{v}, \
G = A - \frac{v[2v^2 - (1 - v^2)\gamma^2]}{2(1 - v^2)^2}.
\]

where $A$ is an arbitrary function of the relative velocity $v$ which parametrizes all possible consistent deformed Lorentz transformations. Setting $A = 0$, for simplicity, we find the following set of DSR transformations.

\[
[A(v, p)]_{\mu} = \tilde{p}_{\mu} = \begin{cases} 
\tilde{p}_0 = \gamma(p_0 - vp_1) + \ell \left[ \frac{p_0^2 \gamma^2 (2\gamma^3 - \gamma - 1) - p_0^2 (\gamma^2 - 1)(2\gamma^2 - 1)}{2} \right], \\
\tilde{p}_1 = \gamma(p_1 - vp_0) + \ell v p_0^2 \gamma^4 - \frac{p_0^2}{2} \gamma^2 (2\gamma^3 - 2\gamma - 1) .
\end{cases}
\]
In this notation, we are referring to \([\Lambda(v, p)]_{\mu}\) as the \(\mu\)-component of the transformed momenta \("p"\) using boost parameter \("v"\). As can be seen, in this case, the Planck-scale correction in the energy transformation is isotropic, while that of the spatial momentum is anisotropic. We also notice that this equation is dominated by a \(\gamma^4\) term when \(\gamma \gg 1\). The appearance of this \(\gamma^4\) term is where the possibility of detection of effects emerges, it works as an amplifier of Planck scale effects. On the other hand, in order to find the associated deformed boost generator, one must rely on the infinitesimal (in \(v\)) version of this transformation as discussed in [34, 39, 40]. They are given by (we also replace \(v \rightarrow -v\)):

\[
\begin{align*}
\tilde{p}_0 & \approx p_0 + vp_1, \\
\tilde{p}_1 & \approx p_1 + vp_0 - \ell v \left( p_0^2 + \frac{p_1^2}{2} \right).
\end{align*}
\]

Notice that this transformation on \(p_\mu\) is equivalent to the infinitesimal version of the \(\kappa\)-Poincaré boost in the bicrossproduct basis. This can be verified by writing the above transformation as \(\tilde{p}_\mu = p_\mu + \xi_\mu\), where

\[
\xi_\mu = \left( p_0 - \ell (p_0^2 + p_1^2/2) \right),
\]

from which it is possible to identify the boost generator of \(\kappa\)-Poincaré algebra in this basis (see [34, Eq. (5)]):

\[
N = \xi_\mu x^\mu = x^0 p_1 + x^1 p_0 - \ell x^1 \left( p_0^2 + \frac{p_1^2}{2} \right).
\]

Also in [34, section V] it is demonstrated that this generator can be found from the Killing vectors of the related Finsler geometry, when one writes the 4-velocities in terms of the momenta.

### III. COMPOSITION LAW

Another important piece for fulfilling the requirements of a relativistic deformed kinematics is the formulation of a compatible modified energy/momentum conservation law. This condition guarantees that all inertial observers will agree about the existence or prohibition of interactions between elementary particles.

For dimensional reasons, the most general composition law at first order perturbation is \(^1\)

\[
\begin{align*}
(p \oplus q)_0 &= p_0 + q_0 + \ell(\alpha p_0 q_0 + \beta p_1 q_1 + \omega p_0 q_1 + \eta p_1 q_0), \\
(p \oplus q)_1 &= p_1 + q_1 + \ell(\delta p_1 q_0 + \epsilon p_0 q_1 + \lambda p_1 q_1 + \mu p_0 q_0),
\end{align*}
\]

where \((\alpha, \beta, \omega, \eta, \delta, \epsilon, \lambda, \mu)\) are dimensionless parameters yet to be determined. Deformed relativistic compatibility is guaranteed if the action of the Lorentz transformation (14) on composed momenta fulfills a relation of the form:

\[
\Lambda(v, p \oplus q) = \Lambda(v_q, p) \oplus \Lambda(v_p, q),
\]

where, in general, it is possible that the boost parameters \(v_p\) and \(v_q\), appearing on the right hand side of this relation, can dependent on the momenta \(p\) and \(q\) respectively. A feature that is called “back-reaction”, which was originally proposed in [41] as a necessity to assure the relativistic nature of a composition law from the \(\kappa\)-Poincaré algebra in the bicrossproduct basis.

Here, the most general “back-reacting” parameters for first order deformations that we can use in the boosted composition law (20) are

\[
\begin{align*}
v_q &= v + \ell(Hq_0 + Jq_1), \\
v_p &= v + \ell(Mp_0 + Rp_1).
\end{align*}
\]

This form of back-reaction acting on both entries of the composition law has also been considered in [34]. We fully realize it in this paper and analyze its phenomenological consequences. The imposition of the relativistic condition

\[p \oplus 0 = p \quad \text{and} \quad 0 \oplus p = 0.\]

---

\(^1\) This law satisfies \(p \oplus 0 = p\) and \(0 \oplus p = 0\).
for each component of (20) from Eqs. (14) and (19) give the following set of conditions between the composition law parameters and those of the back-reaction:

\[ \alpha = 0 = \lambda, \quad (22a) \]
\[ \beta = \frac{2 + 2 \gamma^3 (J + R - 2) + 4 \gamma^4 - \gamma^2[4 + J + R - v \gamma (H + M)]}{2(\gamma - 1)}, \quad (22b) \]
\[ \delta = -\gamma \frac{1 + \gamma^2 (R - 1) + 2 \gamma^3 - \gamma (2 + R - v \gamma M)}{2(\gamma - 1)}, \quad (22c) \]
\[ \epsilon = \delta(R \to J; M \to H), \quad (22d) \]
\[ \omega = -\gamma \frac{M \gamma (\gamma - 1) + v \gamma [2 \gamma^2 + \gamma (R - 1) - 1]}{2(\gamma - 1)}, \quad (22e) \]
\[ \eta = \omega(R \to J; M \to H), \quad (22f) \]
\[ \mu = \frac{-2 + 2 \gamma^2 (2 + J + R) + 4 \gamma^3 + \gamma [J + R - 4 + v \gamma (H + M)]}{2 \nu}. \quad (22g) \]

Until this point, the only restriction that we imposed was the choice \( A = 0 \) in Eqs. (11), (12), (13), which was responsible for removing mixed terms between energy and momentum in the Lorentz transformation.

### A. Parity-invariant composition law

As can be seen from (3), the 3 + 1-dimensional case is invariant under parity transformations \((k_0 \to k_0, \vec{k} \to -\vec{k})\), where \(k\) describes momenta \(p, q\) and \(p \oplus q\). For this reason, we implement this symmetry also in the 1+1-dimensional case under consideration, so that the results could be translated to the general one. In order to have this property, we require \( \omega = \eta = \mu = 0 \) (the term \( \lambda \) is null due to (22a)). This gives the following set of conditions on the back-reaction parameters found from Eqs. (22a)–(22g)

\[ J = \frac{-H v \gamma}{1 + \gamma} + \frac{1 + \gamma - 2 \gamma^2}{\gamma}, \quad M = \frac{-R v \gamma}{\gamma - 1} - v (1 + 2 \gamma). \quad (23) \]

This restriction, allows us to two analyze two cases that are particularly interesting due to their simplicity, namely the cases of either undeformed momentum or undeformed energy compositions.

#### 1. Undeformed spatial momentum conservation

This case is particularly important due to its possible applications in the study of the decay of a particle in two others, when the analysis is done in the rest frame of the parent particle. It is realized when we add the choice \( \delta = 0 = \epsilon \) in (19) to \( \omega = \eta = \mu = 0 \).

It implies for the back-reaction parameters (23)

\[ R = J = \frac{1}{\gamma} - \gamma, \quad H = M = -v \gamma, \quad (24) \]

which also leads to \( \beta = 1 \). This way, the composition law reads

\[
\begin{align*}
(p \oplus q)_0 &= p_0 + q_0 + \ell p_1 q_1, \\
(p \oplus q)_1 &= p_1 + q_1,
\end{align*}
\]

which is compatible with deformed Lorentz transformations \( \Lambda(v, p) \) given by Eq. (14) for the back-reacting parameter

\[ v_k = v + \ell \left[ \left( \frac{1}{\gamma} - \gamma \right) k_1 - v \gamma k_0 \right], \quad (26) \]

where \( k \) refers to momenta \( p \) or \( q \). In this case, the back-reaction acts equally on the first and second argument of the composition law. One can also check this result by a straightforward calculation considering the infinitesimal transformations (15) and (16) and infinitesimal back-reacting parameter \( v_k = v(1 - \ell k_0) \).
2. Undeformed energy conservation

Another important case consists in not deforming the energy conservation, since, as we will see a posteriori will allow us to recover the known addition of momenta that is defined from the coproduct structure of the bicrossproduct basis of \( \kappa \)-Poincaré algebra. In fact, this condition is realized by requiring \( \beta = 0 \) in (19) (obviously, besides the conditions for the parity invariant case that spreads through this section). This gives the following condition:

\[
H = \frac{v[\gamma^2(1 + R) - 1]}{\gamma(\gamma - 1)},
\]

which implies in the following composition law

\[
\begin{aligned}
(p \oplus q)_0 &= p_0 + q_0, \\
(p \oplus q)_1 &= p_1 + q_1 + \ell(\delta p_0 q_0 - \epsilon p_0 q_1),
\end{aligned}
\]

where

\[
\delta = \frac{\gamma(\gamma^2 + R\gamma - 1)}{\gamma - 1}, \quad \epsilon = \frac{\gamma^3 + R\gamma^2 - 1}{\gamma - 1}.
\]

This is the general case. Choosing \( R = \gamma^{-1} - \gamma \), we obtain further simplifications which turn the composition law into the one from the bicrossproduct \( \kappa \)-Poincaré coproduct structure, [42],

\[
\begin{aligned}
(p \oplus q)_0 &= p_0 + q_0, \\
(p \oplus q)_1 &= p_1 + q_1 - \ell p_0 q_1.
\end{aligned}
\]

This law is compatible with deformed Lorentz transformations \( \Lambda(v, p) \) given by Eq. (14) for the back-reaction parameters

\[
\begin{aligned}
v_q &= v - \ell \left[ v \left( \frac{\gamma^2 - 1}{\gamma} \right) q_0 + \left( \gamma - \frac{1}{\gamma^2} \right) q_1 \right], \\
v_p &= v - \ell \left[ v\gamma p_0 + \left( \gamma - \frac{1}{\gamma} \right) p_1 \right],
\end{aligned}
\]

where we can see different kinds of back-reaction parameters. One can also check this result by a straightforward calculation considering the infinitesimal transformations (15) and (16) and infinitesimal back-reacting parameters \( v_q = v \) and \( v_p = v(1 - \ell p_0) \), which coincides with the back-reaction of the infinitesimal deformed Lorentz transformation extensively studied in the literature [41, 42].

From the literature on the bicrossproduct basis of the \( \kappa \)-Poincaré algebra, the finite back-reaction acts just on the second entry of the composition law (which in our case would be momenta \( q \)). In this case, we start with a very different finite form of Lorentz transformation, that coincides with the standard ones of [43] only at first order in \( v \). This kind of ambiguity between the Finsler and certain basis of \( \kappa \)-Poincaré is a subject that has been discussed in Ref. [34], in which this possible different back-reaction parameters are highlighted and demonstrated in the present paper.

IV. DECAY OF MASSIVE PARTICLE IN ITS REST FRAME

The discussion carried out in the previous sections find an useful application in the analysis of modified equations that govern cosmic rays showers due to particles decays. This is due to the fact that some capital expressions are calculated when comparing the rest frame of the decayed particle and the laboratory. Besides that, the composition of momenta is also an essential information needed to describe the spectrum of produced particles.

In this section we study the case of a parent particle of mass \( M \) decaying in two descendant particles with masses \( m_p \) and \( m_q \), with momenta \( p \) and \( q \), respectively. Concretely, let us consider the cases described in the previous section: undeformed momentum and undeformed energy conservation. Other cases should follow from similar procedures. Consider the deformed conservation law \( P_\mu = (p \oplus q)_\mu \). In the following subsection, we refer to frame \( "s" \) as the one in which the parent particle (the one with momenta \( P_\mu \)) is at rest.
A. Decayed particle’s momentum for undeformed spatial momentum conservation

From the rest frame condition and the composition law we find \( 0 = P_1^* = (p_1^* + q_1^*), \) and thus \( p_1^* = -q_1^*. \) Then, the modified dispersion relation (3) and the composition law (25) imply

\[
M^2 = (P_0^*)^2 = (p_0^* + q_0^* - \ell(p_1^*)^2)^2 = (p_0^*)^2 + (q_0^*)^2 + 2(p_0^*)(q_0^*) - 2\ell(p_1^*)^2[p_0^* + q_0^*].
\] (33)

Further, \( p \) and \( q \) themselves satisfy the MDR (3), and thus one can replace \( p_0 \) and \( q_0 \) as functions of \( p_1^*, \) \( q_1^*, \) \( m_p \) and \( m_q. \) After this substitution, (33) can be solved for \( p^* = |p_1^*| = |q_1^*| \) and we find

\[
p^* = \frac{\sqrt{M^4 - 2M^2(m_p^2 + m_q^2) + (m_p^2 - m_q^2)^2}}{2M}.
\] (34)

We notice that the composition law exactly compensates the effect of the MDR, furnishing the same expression that one would have found in the framework of Special Relativity (SR).

An interesting possibility to be analyzed consists in the decay of a particle into a massless and a massive one. In this case, we can set \( m_q = 0 \) (since this composition law is commutative, we would have found the same result have we set \( m_p = 0 \)). This way, the previous equation reduces to the following momentum of the produced massive and massless particles in the rest frame of the parent particle

\[
p^* = \frac{M}{2} \left( 1 - \frac{m_p^2}{M^2} \right).
\] (35)

Although this result coincides with the one derived from SR, the effect of the deformed boost transformation on each decay product will furnish important corrections in the analysis to be done in the next sections.

B. Decayed particle’s momentum for undeformed energy conservation

Now, we analyze the case that coincides with the composition law that follows from the coproduct structure of the bicrossproduct basis of \( \kappa \)-Poincaré algebra, given by Eq. (30). Again, we define \( P_\mu = (p \oplus q)_\mu, \) and consider the rest frame “\( \ell \)’” of the parent particle with momenta \( P_\mu. \) Now, from \( P_1^* = 0, \) we find a non-trivial relation between the momenta of the descendant particles 1 and 2. In fact, assuming on-shell particles dispersion relation (3) and the first order in \( \ell \) approach, we deduce

\[
p_1^* = -q_1^* + \ell(q_1^*)(p_0^*) \approx -q_1^* + \ell(q_1^*) \sqrt{m_p^2 + (q_1^*)^2} \approx -q_1^* + \ell(q_1^*) \sqrt{m_p^2 + (q_1^*)^2}.
\] (36)

Using the MDR (3) for the momenta \( p \) and \( q \) and \( P_\mu^*, \) i.e. the equation \( (P_\mu^*)^2 = M^2 = (p_0^*)^2 + (q_0^*)^2 + 2(p_0^*)(q_0^*) \) we can write the energies \( p_0^*, \) \( q_0^* \) as a function of the spatial momenta \( p_1^*, \) \( q_1^* \) and the masses \( m_p \) and \( m_q. \) In addition, with Eq. (36) we can finally express \( p_1^*, \) \( q_1^* \) as functions of the masses alone

\[
|p_1|^* = \frac{\sqrt{M^4 - 2M^2(p_0^* + q_0^*) + (p_0^2 - q_0^2)^2}}{2M} \left[ 1 - \ell(M^2 - m_p^2 - m_q^2) \right],
\] (37a)

\[
|q_1|^* = \frac{\sqrt{M^4 - 2M^2(p_0^* + q_0^*) + (m_p^2 - m_q^2)^2}}{2M}.
\] (37b)

We find corrections only for the first momentum in this composition law. In fact, since the composition law is non-commutative, we can have two distinct cases depending on the order in which the particles’ momenta “\( \ell \)” the deformed sum. To illustrate this issue, let us once again consider the case of the decay into a massive and a massless particle. If the massless particle is the first one in the composition law \( p \oplus q, \) i.e., if \( m_p = 0, \) we find the following relations:

\[
m_p = 0 \Rightarrow \begin{cases} 
|p_1|^* = \frac{M}{2} \left( 1 - \frac{m_q^2}{M^2} \right) \left[ 1 - \frac{\ell}{2M}(M^2 - m_q^2) \right], \\
|q_1|^* = \frac{M}{2} \left( 1 - \frac{m_p^2}{M^2} \right).
\end{cases}
\] (38)
On the other hand, if the massless one is the second particle, \( m_q = 0 \), we find

\[
m_q = 0 \Rightarrow \begin{cases} 
|p_1|^* = \frac{M}{2} \left(1 - \frac{m^2_q}{M^2} \right) [1 - \frac{\ell}{2M}(M^2 + m_q^2)] , \\
|q_1|^* = \frac{M}{2} \left(1 - \frac{m^2_q}{M^2} \right).
\end{cases}
\] (39)

So, for instance, consider a pion decaying into a muon and a neutrino \( \pi^\pm \to \mu^\pm + \nu_\mu(\bar{\nu}_\mu) \), (for simplicity we shall refer these processes simply as \( \pi \to \mu + \nu \)). If we have the composition law of the form \( p_\pi = p_\mu \oplus p_\nu \), where \( p_\pi, p_\mu \) and \( p_\nu \) refer to the energy/momentum of each of these particles, then the spatial momenta of the muon and the neutrino are given by the first and second expressions of (39), respectively. This means that only the relation between the momentum of the muon, its mass and the pion mass gets corrected in this frame. On the other hand, if we express the conservation law of this decay by \( p_\pi = p_\nu \oplus p_\mu \), then the spatial momenta of the neutrino and the muon are given by the first and second expressions of (38), respectively. This means that only the relation between the momentum of the neutrino, the mass of the muon and the pion mass gets corrected in this frame. Such situation does not happen in the previous subsection (undeformed momentum conservation) due to the commutativity of that composition law.

C. Deformed distributions of muon and neutrino from pion decay

From the discussion of the previous subsection, we can derive some important consequences for the decay distribution of particles. The starting point is the deformed Lorentz transformation (14) in which \( \gamma \) now is considered as velocity Lorentz factor and energy of the parent particle in the lab.

Following [44, Sec. 6.1], we can calculate the limits in the laboratory frame of the energies \( p_0 \) and \( q_0 \) of secondary particles in a two-body decay. In order to do that, we shall connect the rest frame of the parent particle to the lab and realize that, since this particle momenta is boosted with parameter \( \ell \), we shall boost the produced particles momenta \( p_\mu \) and \( q_\mu \) with the back-reacted parameters \( v_\text{p} \) and \( v_\text{q} \) of Eqs. (21b) and (21a). Then, the maximum and minimum energies are given by the following expressions (where \( v_{q^\ast,p^\ast} \) are the back-reaction parameters that apply to each momenta \( p_\mu \) and \( q_\mu \) and \( \gamma_{q^\ast,p^\ast} = 1/\sqrt{1 - v_{q^\ast,p^\ast}^2} \)):

\[
p_0^\pm = \gamma_{q^\ast} \left[(p_0^* \pm v_{q^\ast}p_0^*) + \frac{\ell}{2} \left[(p_0^*)^2\gamma_{q^\ast}(2\gamma_{q^\ast}^2 - \gamma_{q^\ast} - 1) - (p_0^*)^2\gamma_{q^\ast}^2 - 1 \right) \right], \quad (40a)
\]

\[
q_0^\pm = \gamma_{p^\ast} \left[(q_0^* \pm v_{p^\ast}q_0^*) + \frac{\ell}{2} \left[(q_0^*)^2\gamma_{p^\ast}(2\gamma_{p^\ast}^2 - \gamma_{p^\ast} - 1) - (q_0^*)^2\gamma_{p^\ast}^2 - 1 \right) \right], \quad (40b)
\]

which follow from the modified Lorentz transformation (14). Assuming SR in 3 + 1D, the values for the maximum and minimum energies are determined by the presence of the cosine of the angle that connects the trajectory of the parent particle and the lab, which multiplies the speed parameter \( v \) (see section 6.1 of [44]). Since the extrema of the cosine function are \( \pm 1 \), one derives the same above results when \( \ell \to 0 \) (back-reaction disappears). In our case of a DSR approach in 1 + 1D, we automatically find the \( \pm \) result, because in this case the parent particle can only move toward or backward the laboratory with speed \( v \) (as measured in the lab frame). Therefore, we do not lose generality in this derivation despite we are considering lower dimensions kinematics. From these expressions, we can calculate the energy differences \( \Delta p_0 = p_0^+ - p_0^- = 2\gamma_{q^\ast}v_{q^\ast}p_1^\ast \) and \( \Delta q_0 = q_0^+ - q_0^- = 2\gamma_{p^\ast}v_{p^\ast}q_1^\ast \), which apart from the presence of the back-reaction, their forms coincide with SR results. This shall furnish the following expressions for the normalized distributions of particles’ energies for a product of two-body decay of an unpolarized parent

\[
\frac{dn_p}{dp_0} = \frac{1}{\Delta p_0} = \frac{1}{2\gamma_{q^\ast}v_{q^\ast}p_1^\ast}, \quad (41)
\]

\[
\frac{dn_q}{dq_0} = \frac{1}{\Delta q_0} = \frac{1}{2\gamma_{p^\ast}v_{p^\ast}q_1^\ast}. \quad (42)
\]

Notice that the back-reaction found in the previous section plays an important role in this relativistic analysis, which shall give extra Planck-scale corrections. Other contributions emerge from the non-trivial relation between the velocity Lorentz factor and energy of the parent particle in the lab frame. As a first step, let us transform the momentum of the parent particle, with mass \( M \), from the lab frame (for which we write its momenta as \( P_\mu = (E_L, P_L) \)) to the its rest frame \( (P_\mu^* = (E^*, 0)) \) using Eq. (14):

\[
0 = P^* = \gamma (P_L - vE_L) + \ell v [E_L^2 \gamma^4 - \frac{P_L^2}{2} - v(2\gamma^3 - 2\gamma - 1)], \quad (43)
\]
from this expression, we can find an implicit relation for \( v \) involving the energy and momentum of the parent particle and \( \gamma \) (since it depends on \( v \) as well) to find

\[
v = \frac{P_L}{E_L} + \ell \left[ P_L \gamma^3 - \frac{P_L^3}{2E_L^2} (2\gamma^3 - 2\gamma - 1) \right].
\] (44)

Now, we transform the energy of the parent particle from its rest frame to the lab using Eq. (5) and isolate the velocity Lorentz factor \( \gamma \) to first order in \( \ell \)

\[
E_L = \gamma M - \frac{\ell}{2} M^2 (\gamma^2 - 1)(2\gamma^2 - 1)
\] (45)

\[
\Rightarrow \gamma = \frac{E_L}{M} + \frac{\ell}{2M^3} (2E_L^4 - 3E_L^2 M^2 + M^4).
\] (46)

From these expressions and the MDR (3), we can immediately find what would be one of the terms in the distribution equations if we were not considering back-reaction:

\[
\frac{1}{\gamma v} = M \left[ \frac{1}{P_L} - \ell \left( M \frac{P_L}{M} + \frac{P_L}{E_L} + \frac{P_L^3}{M^2 E_L} \right) \right].
\] (47)

Now, if we use Eqs. (21b) and (21a) in order to consider DSR corrections, we can find exactly the terms that will appear in Eqs. (41) and (42) in terms of the previous results:

\[
\frac{1}{\gamma_{p', q'} v_{p', q'}} = \frac{1}{\gamma v} + \left( v - v_{p', q'} \right) \frac{E_L^3}{MP_L^2}.
\] (48)

This equation depends on the deformed Lorentz transformation, the MDR (the first term) and the possibility of back-reaction (the second term), where its exact form will depend on the kind of composition law that one is considering. The other piece of information shall be furnished by the momentum of each produced particle \( p^*_1, q^*_1 \). In what follows, we shall assume the composition laws of the previous subsections in order to derive the modified distribution of particles.

Our case of study will consist in the decay of a massive particle into another massive particle and a massless one, so, as a prototype that simplifies the notation, we shall refer to the parent particle as a pion \( \pi \), the produced massive particle as the muon \( \mu \) and the massless one as a neutrino \( \nu \). Although neutrinos have a non-vanishing mass, the sum of the masses of the three neutrino flavours are strongly constrained from the CMB as \( \sum_m m_\nu < 0.12 \text{ eV} \) [45, 46], which is more stringent than the bounds from laboratory experiments [47]. From Eq. (34), this leads us to expect that assuming a non-null neutrino mass in the analysis of this paper would produce fixed corrections of order \( (m_\nu/m_\pi)^2 < 7.3 \times 10^{-19} \). Since this correction is fixed and does not present any amplifying factor, it should not interfere in the analysis to be carried out. For this reason, we perform our calculations assuming neutrinos as massless particles. Also, in order to present these expression in a form that is more familiar to the cosmic rays community, we continue referring to the parent particle momentum and energy in the lab frame as \( P_L \) and \( E_L \), and we refer to the energy of the produced particles as \( E_{\mu, \nu} \), mass of the muon as \( m_\mu \) and mass of the pion as \( M \).

1. Undeformed momentum conservation

As a first case, we consider the undeformed momentum conservation, that does not present ambiguities regarding the order in which the momenta are inserted in the composition law. We use Eq. (35) and the MDR (3) to find the energy of each produced particle in the frame *, which gives

\[
E^*_\mu, \nu = \frac{M^2 \pm m^2_\mu}{2M} + \ell \frac{(M^2 - m^2_\mu)^2}{8M^2},
\] (49)

where the plus and minus signs refer to muon and neutrino energies, respectively. Then, we use these expressions in Eqs. (41) and (42), using the back-reaction (26), along with Eq. (47) and (48). By labeling \( m_p = m_\mu \), this gives the

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2 This result could also be found by explicitly solving Eq. (44) for \( v \) (considering that \( \gamma = 1/\sqrt{1-v^2} \)) just as a function of \( P_L, E_L \), and by using Eq. (46).
following distribution of produced muons and neutrinos:

\[
\frac{dn_\mu}{dE_\mu} = \frac{1}{(1 - m_\mu^2/M^2)P_L} \left[ 1 + \ell \left( \frac{2MP_L^4 + 2E_LP_L^4 + E_L^3(P_L - 2M) + E_L^3(E_L^2 - 3P_L^2) - E_L^3P_L - m_\mu^2E_L^2(E_L + P_L)}{2M^4} \right) \right],
\]

(50)

\[
\frac{dn_\nu}{dE_\nu} = \frac{dn_\mu}{dE_\mu} + \frac{\ell}{(1 - m_\mu^2/M^2)P_L} \frac{m_\mu^2E_L^3}{M^4}.
\]

(51)

This is an expression that considers all contributions arising from the previous sections. However, it would also be interesting to set, here, the liming case, expected in the relativistic limit (\(E_L \gg M\)), in which we just keep the dominant term that shall govern this kind of effect at very high energies:

\[
\pi \rightarrow \nu + \mu \Rightarrow \begin{cases} \frac{dn_\mu}{dE_\mu} & \approx \frac{1}{(1 - m_\mu^2/M^2)E_L} \left( 1 - \ell m_\mu^2E_L^3/M^4 \right), \\ \frac{dn_\nu}{dE_\nu} & \approx \frac{1}{(1 - m_\mu^2/M^2)E_L} \left( 1 - \ell m_\mu^2E_L^3/M^4 \right), \end{cases}
\]

(52)

For the kind of decays that we are considering, \(m_\mu\) is of the same order of magnitude of \(M\), which is corrected roughly by an amplifying factor of the order \(\ell E_L^3/M^2\), which is typical of this Finsler approach and has great potential of connecting this phenomenological approach and observations even when \(\ell^{-1}\) is of the order of the Planck energy \(10^{19}\) GeV. The flat probability distribution for detecting muons in a certain energy range from the decay of pions is roughly by an amplifying factor of the order \(10^{19}\) GeV. The flat probability distribution for detecting muons in a certain energy range from the decay of pions is

\[
\frac{dn}{dE} \approx \frac{1}{(1 - m_\mu^2/M^2)E_L} \left( 1 - \ell m_\mu^2E_L^3/M^4 \right),
\]

(53)

we stress that for this composition law, the order in which the order in which particles are considered in the conservation law, in principle, matters. Nevertheless, notice that this limiting result coincides with the one found in the previous subsection (52), meaning that in the ultraviolet (UV) regime, the contributions due to the presence of a modified boost are dominant.

The second case consists in setting \(m_\mu = 0\), which means that the massless particle is the second one in the composition law (30). In this case, it would be necessary to set the momentum of the massive and massless particles as \([p_\nu]\) and \([q_\nu]\) from Eq. (39), respectively. Also using the back-reaction parameters to their respective particles in Eqs. (41) and (42), one can find the corresponding distributions. As before, we here write just the dominant term that appear from these calculations:

\[
\pi \rightarrow \mu + \nu \Rightarrow \begin{cases} \frac{dn_\mu}{dE_\mu} & \approx \frac{1}{(1 - m_\mu^2/M^2)E_L} \left[ 1 - \ell m_\mu^2E_L^3/M^4 \right], \\ \frac{dn_\nu}{dE_\nu} & \approx \frac{1}{(1 - m_\mu^2/M^2)E_L} \left[ 1 - \ell m_\mu^2E_L^3/M^4 \right]. \end{cases}
\]

(54)

If we considered all the terms that appeared in these distributions, we would notice that, indeed, Eqs. (53) and (54) differ due to the non-commutativity of the composition law. However, we notice that in the UV regime, the deformed boost transformation suppresses corrections to possible non-commutativities.
D. Some phenomenological consequences

From this analysis, we can draw some remarks that could guide future researches in this field, when considering the detection of particles, for instance, from cosmic rays. In this case, there are three main points that suffer corrections and are important for calculations of spectra of produced particles from decays [44]:

1. Deformed lifetimes of the particles, that furnishes the distance they travel;

The deformed lifetimes have been discussed in this Finsler approach in Ref. [31] and has also been recapped in this paper in Eq. (8).

2. Deformed distributions of particles;

The deformed distributions of particles have been analyzed for the case of a two-body decay (in which one of the particles is massive) in the previous sections. And from the discussion carried out by the end of Section (IV C 2), we can verify that in the UV regime (which is the one that we are interested), we notice only negligible differences between the different kinds of composition laws analyzed. Therefore, without loss of generality, one could pick the one that comes from the undeformed momentum conservation, given by Eq. (52).

3. Deformed minimum and maximum attained energies of each produced particle and, consequently, of the parent particle, for integration purposes.

The maximum and minimum energies for each particle are determined by the “+” and “−” signs in Eqs. (40a) and (40b). For simplicity, we shall proceed to the calculation assuming the undeformed momentum conservation case. To do this, we assume that \( p^r = q^r \) is given by (35), which, from the MDR (3) gives Eq. (49) as the energy of each particle. We also assume the back-reaction parameter (26) that uses momenta \( (p^r, q^r) \) and its respective velocity Lorentz factor \( \gamma_{p^r, q^r} \). From these steps, we derive the following results for the energy ranges (where we only keep dimensionless corrections that grow with the third power in the energy, since they should be ones that give a significant contribution in this regime)

\[
E_L = \frac{E_\mu^3}{M^2 (1 - r_M)} \times r_M E_L \lesssim E_\mu \lesssim E_L,
\]

\[
0 \lesssim E_\nu \lesssim (1 - r_M) E_L,
\]

where \( E_L \) is the energy of the parent particle (pion) in the lab frame and \( r_M = m_\mu^2/M^2 \). We see a shift in the minimum energy of the muon, although the upper threshold do no get significantly altered, which is compatible with the SR conservation of energy. Besides that, the limits for the neutrino energy do not modify in the UV limit, i.e., the corrections cancel out terms that grow with \( E_\mu^3 \). This same approximate limits are found when we analyze the cases of undeformed energy conservation.

In Fig. (2), we depict the different distributions for fixed \( E_L \) considering the energy ratio \( z = E_\mu/E_L \) as the variable (this can be found just by dividing the above limits by \( E_L \) and multiplying the distributions (52) by \( E_L \)). On the left, we see the neutrino distribution from the pion decay, i.e., \( d\nu/dz \times z \), where \( z = E_\nu/E_L \) (labeled as \( \pi \rightarrow \nu \)), and on the right, we depict the muon distribution due to the pion decay, i.e., \( d\mu/dz \times z \), where \( z = E_\mu/E_L \) (which we label as \( \pi \rightarrow \mu \)). We have set a high energetic case of \( E_L \approx 4.5 \times 10^{14} \) eV for graphical purposes. We see an enlargement of the energy range of muons when \( \ell > 0 \) (massless superluminal propagation), which is expected, since in a flat distribution, this energy range reduces its height, since \( dn/dE = (\Delta E)^{-1} \). The opposite behavior happens when \( \ell < 0 \) (massless subluminal propagation). We see no corrections in the neutrino case. For these plots, we used the exact form of each distributions (50) and (51), however, we would verify basically the same behavior had we considered just the cases described in the UV limit (52).

Inequalities (55) and (56) also modify the limits of the the parent particles’ energy \( E_L \) for the occurrence of the decay. This is an important observation if one aims to calculate the production spectra of these secondary particles, since these quantities are required for integration. For instance, considering this decay \( \pi \rightarrow \mu + \nu \), we would have the following pion energy limits (in the lab frame) that can give rise to muons and neutrinos, respectively:

\[
E_\mu \lesssim E_L \lesssim \left( 1 + \ell \frac{E_\mu^3}{M^2 r_M^3} (1 - r_M) \right) \frac{E_\mu}{r_M},
\]

\[
\frac{E_\nu}{1 - r_M} \lesssim E_L \lesssim \infty.
\]
As can be seen, the energy range for the estimation of the muon spectra, one needs to be careful because the upper integration limits should be modified in this DSR scenario. Besides these integration limits, it is important to consider the modified distribution and lifetimes of the involved particles (all of these quantities suffer modifications that grow with the amplifying factors stressed in this paper). Obviously, when \( \ell \to 0 \), we recover known results presented, for instance, in chapter 6 of [44]. We considered the case of the pion decay for simplicity reasons, but similar procedures would need to be carried out if one aims to analyze different channels, while still preserving the relativistic principle at the Planck scale. We can see, immediately that the decay \( \pi^0 \to \gamma \gamma \) (which could be found by just setting \( m_\mu \to 0 \), \( E_L \) as the neutral pion energy in the lab frame and multiplying the distribution by 2, since there are two identical decay products), the distribution and the energy limits do not carry \( E^3_L \) terms. So, the dominant contributions would emerge just from the dilated lifetime of \( \pi^0 \).

Another phenomenological consequence of the dilated lifetime of decaying particles in DSR are the different fluxes of particles resulting from the decay of, for instance, pions. The lifetimes of \( \pi^0 \) and \( \pi^\pm \) would be altered according to Eq. (8), and so would the relative fluxes of neutrinos, photons, and electrons. In cosmic-ray showers, this would directly translate into a difference in the hadronic component of the shower with respect to the electromagnetic one. Naturally, other unstable particles commonly found in atmospheric showers such as kaons and eta mesons would be affected in a similar fashion.

Interestingly, the MDR presented in Eq. (3) also has important implications for interpreting signatures of cosmic messengers related to two-body decays. For instance, the interpretation of electromagnetic observations across the whole spectrum together with a neutrino signal for the neutrino-emitting objects [48, 49] could have to be revisited, considering a possible enhancement or suppression of the neutrino flux with respect to the gamma-ray one, depending on whether we are dealing with a superluminal scenario (\( \ell > 0 \)) or subluminal (\( \ell < 0 \)). At ultra-high energies, this would also affect the expected fluxes of the long-sought cosmogenic neutrinos and photons, stemming from UHECR interactions with pervasive photon fields such as the CMB [50]. Furthermore, depending on how much the lifetime of a particle changes compared to the special-relativistic kinematics, effects such as synchrotron emission by charged particles could become a relevant energy-loss mechanism, depending on the magnetic magnetic field.

V. FINAL REMARKS

In a previous paper [31], a finite deformed Lorentz symmetry connecting the rest and laboratory frames emerged from Finsler geometry, containing an amplifying factor that can lead to observations close to Planck scale sensitivity. In this paper, we continued this analysis on two stages:

First, in order to have a more complete DSR framework, we generalized the \( \kappa \)-Poincaré-inspired results of [31] by
constructing finite deformed Lorentz transformations which connect the momenta of particles in two different frames (in $1 + 1D$) that move relatively to each other. This construction is necessary for the analysis of phenomenological possibilities of Finsler-DSR in a rich variety of contexts, like the analysis of cosmic rays.

In order to achieve the complete relativistic realization, we also built the general momentum composition law that is compatible with this finite transformation at first order in the deformation scale. To achieve this, we needed to introduce a back-reaction acting on both “entries” of the composition law, which means that the each boost parameter acting on momenta of a given particle depends on the momenta of the other particle. This condition is fundamental to guarantee that all inertial observers agree about the nature of the vertices of interactions between fundamental particles. In particular, we focused on the cases of undeformed spatial momentum conservation, and undeformed energy conservation that is compatible with the coproduct structure of the $\kappa$-Poincaré algebra in the bicrossproduct basis.

On the second stage, we applied these constructions (modified dispersion relation, Finsler-compatible deformed Lorentz transformation and composition law with back-reaction) to consider the decay of a massive parent particle into two descendant ones. We derived kinematical equations that allowed us to deduce the correction to the distributions of produced particles when one the products is massless. We verified that at the UV level, these quantities carry dimensionless corrections of the order $\ell E_L^3/M^2$ (where $\ell$ is expected to be of the order of the inverse of Planck energy and $M$, $E_L$ are the mass and energy of the parent particle as measured in the lab) which have a characteristic amplifying power that can bring us close to the Planck scale using particles so energetic as those presented in cosmic rays. This articles aims to set a pathway for a novel procedure for constraining the Planck scale from a connection with the particle and astroparticle physics community in a way that respects relativistic principles. The steps followed in this paper can be generalized for the sake of allowing the exploration of higher orders of perturbation in the Planck scale from the analysis of very-high energy particles, which complement and extend the studies started in [51].

The phenomenological implications of the DSR framework are immediate. For instance, it is fair to hypothesize that the muon puzzle alluded to in Section I could be a consequence of the changes in the lifetimes of particles. If this is indeed the case, the number of muons would not be the only observable affected – the flux of other particles would also change. Only a complete reinterpretation of the development of atmospheric showers within the DSR framework would allow for robust theoretical predictions that could be confronted with data from facilities such as the Pierre Auger Observatory [52] and its forthcoming enhancement dedicated to muon measurements, AMIGA [53], as well as GRANDProto300 [54], the prototype of the Giant Radio Array for Neutrino Detection (GRAND) [55]. Furthermore, astrophysical strategies combining multiple messengers could also be of use to test this idea using the fluxes of secondary particles stemming from the decay of, e.g., muons or pions.

This articles aims to set a pathway for a novel procedure for constraining the Planck scale from a connection with the particle and astroparticle physics community in a way that respects relativistic principles. The steps followed in this paper can be generalized for the sake of allowing the exploration of higher orders of perturbation in the Planck scale from the analysis of very-high energy particles, which complement and extend the studies started in [51].

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[1] C. Rovelli, “Loop quantum gravity,” Living Rev. Rel. 1 (1998) 1, arXiv:gr-qc/9710008.
[2] S. Mukhi, “String theory: a perspective over the last 25 years,” Class. Quant. Grav. 28 (2011) 153001, arXiv:1110.2569 [physics.pop-ph].
[3] R. Loll, “Discrete approaches to quantum gravity in four-dimensions,” Living Rev. Rel. 1 (1998) 13, arXiv:gr-qc/9805049.
[4] G. Amelino-Camelia, “Quantum-Spacetime Phenomenology,” Living Rev. Rel. 16 (2013) 5, arXiv:0806.0339 [gr-qc].
[36] J. M. Carmona, J. L. Cortés, and J. J. Relancio, “Relativistic deformed kinematics from momentum space geometry,” Phys. Rev. D100 (2019) no. 10, 104031, arXiv:1907.12298 [hep-th].

[37] C. Pfeifer and J. J. Relancio, “Deformed relativistic kinematics on curved spacetime – a geometric approach,” arXiv:2103.16626 [gr-qc].

[38] Pierre Auger Collaboration, P. Abreu et al., “Constraining Lorentz Invariance Violation using the muon content of extensive air showers measured at the Pierre Auger Observatory,” PoS ICRC2021 (2021) 340.

[39] G. Amelino-Camelia, “Doubly-Special Relativity: Facts, Myths and Some Key Open Issues,” Symmetry 2 (2010) 230-271, arXiv:1003.3942 [gr-qc].

[40] N. Jafari and M. R. R. Good, “Dispersion relations in finite-boost DSR,” Phys. Lett. B (2020) 135735, arXiv:2009.06096 [gr-qc].

[41] S. Majid, “Algebraic approach to quantum gravity. II. Noncommutative spacetime,” arXiv:hep-th/0604130.

[42] G. Gubitosi and F. Mercati, “Relative Locality in $\kappa$-Poincaré,” Class. Quant. Grav. 30 (2013) 145002, arXiv:1106.5710 [gr-qc].

[43] S. Majid and H. Ruegg, “Bicrossproduct structure of kappa Poincare group and noncommutative geometry,” Phys. Lett. B 334 (1994) 348–354, arXiv:hep-th/9405107.

[44] T. K. Gaisser, R. Engel, and E. Resconi, Cosmic Rays and Particle Physics. Cambridge University Press, 2016.

[45] Planck Collaboration, N. Aghanim et al., “Planck 2018 results. VI. Cosmological parameters,” Astron. Astrophys. 641 (2020) A6, arXiv:1807.06209 [astro-ph.CO]. [Erratum: Astron.Astrophys. 652, C4 (2021)].

[46] Particle Data Group Collaboration, P. Zyla et al., “Review of Particle Physics,” PTEP 2020 (2020) no. 8, 083C01.

[47] K. Assamagan et al., “Upper limit of the muon-neutrino mass and charged pion mass from momentum analysis of a surface muon beam,” Phys. Rev. D 53 (1996) 6065–6077.

[48] IceCube, Fermi-LAT, MAGIC, AGILE, ASAS-SN, HAWC, H.E.S.S., INTEGRAL, Kanata, Kiso, Kapteyn, Liverpool Telescope, Subaru, Swift NuSTAR, VERITAS, VLA/17B-403 Collaboration, M. G. Aartsen et al., “Multimessenger observations of a flaring blazar coincident with high-energy neutrino IceCube-170922A,” Science 361 (2018) no. 6398, eaat1378, arXiv:1807.08816 [astro-ph.HE].

[49] IceCube Collaboration, M. G. Aartsen et al., “Neutrino emission from the direction of the blazar TXS 0506+056 prior to the IceCube-170922A alert,” Science 361 (2018) no. 6398, 147–151, arXiv:1807.08794 [astro-ph.HE].

[50] R. Alves Batista, R. M. de Almeida, B. Lago, and K. Kotera, “Cosmogenic photon and neutrino fluxes in the Auger era,” JCAP 01 (2019) 002, arXiv:1806.10879 [astro-ph.HE].

[51] J. Carmona, J. Cortes, and J. Relancio, “Beyond Special Relativity at second order,” Phys. Rev. D 94 (2016) no. 8, 084008, arXiv:1609.01347 [hep-th].

[52] Pierre Auger Collaboration, A. Aub et al., “The Pierre Auger Cosmic Ray Observatory,” Nucl. Instrum. Meth. A 798 (2015) 172–213, arXiv:1502.01323 [astro-ph.IM].

[53] Pierre Auger Collaboration, A. Aub et al., “Calibration of the underground muon detector of the Pierre Auger Observatory,” JINST 16 (2021) no. 04, P04003, arXiv:2012.08016 [astro-ph.IM].

[54] GRAND Collaboration, V. Decoene, “GRANDProto300 experiment: a pathfinder with rich astroparticle and radio-astronomy science case,” PoS ICRC2019 (2020) 233, arXiv:1909.04893 [astro-ph.IM].

[55] GRAND Collaboration, J. Álvarez-Muñiz et al., “The Giant Radio Array for Neutrino Detection (GRAND): Science and Design,” Sci. China Phys. Mech. Astron. 63 (2020) no. 1, 219501, arXiv:1810.09994 [astro-ph.HE].