Travel Time and Heat Equation.
One space dimensional case

Masaru IKEHATA
Department of Mathematics, Faculty of Engineering
Gunma University, Kiryu 376-8515, JAPAN
10 Oct 2006

Abstract

The extraction problem of information about the location and shape of the cavity from a single set of the temperature and heat flux on the boundary of the conductor and finite time interval is a typical and important inverse problem. Its one space dimensional version is considered. It is shown that the enclosure method developed by the author for elliptic equations yields the extraction formula of a quantity which can be interpreted as the travel time of a virtual signal with an arbitrary fixed propagation speed that starts at the known boundary and the initial time, reflects at another unknown boundary and returns to the original boundary.

AMS: 35R30, 80A23
KEY WORDS: inverse initial boundary value problem, heat conduction, heat equation, indicator function, travel time, enclosure method

1 Introduction

In this paper we consider one space dimensional version of a typical and important inverse problem for the heat equation. Let $\Omega$ be a bounded domain of $\mathbb{R}^n (n = 1, 2, 3)$ with smooth boundary. Let $D$ be an open subset with Lipschitz boundary of $\Omega$ such that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected. Let $T$ be an arbitrary positive number. Let $u = u(x, t)$ be an arbitrary non constant solution of the heat conduction problem:

$$u_t = \Delta u \text{ in } (\Omega \setminus \overline{D}) \times ]0, T[,$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ in } \partial D \times ]0, T[,$$

$$u(x, 0) = 0 \text{ in } \Omega \setminus \overline{D}.$$

Here $\nu$ denotes the unit outward normal vector field on $\partial(\Omega \setminus \overline{D})$. The boundary condition for $\partial u/\partial \nu$ on $\partial D$ means that $D$ is perfectly insulated. Note that this is the simplest model of the heat conduction in a conductive body having a hole or cavity $D$.

The problem to be solved is
Inverse Problem. Extract information about the location and shape of $D$ from the single set of the observation data $u(x, t)$ and $\partial u/\partial \nu(x, t)$ on $\partial\Omega \times [0, T]$. Note that $T$ is fixed.

The problem comes from the thermal imaging and the solution method may have applications to, for example, the detection of corrosion. There are extensive studies of uniqueness and stability issue of Inverse Problem. In particular, it is known that the observation data uniquely determine general $D$ itself under a suitable condition on the heat flux on $\partial\Omega$. See Bryan-Caudill [2], Canuto-Rosset-Vessella [3] and references therein for more information about the issue.

In this paper we are especially interested in seeking an analytical formula for the purpose and start with one space dimensional version of the problem. In one space dimensional case there is a way of calculating the so-called response operator from a general single set of the observation data. Moreover in Avdonin-Belishev-Rozhkov [1] it is shown that from the response operator one can extract the spectral data. Thus in one space dimensional case the problem can be reduced to the inverse spectral problem which has been studied well. However, the reduction is based on the boundary controllability for the heat equation and not an easy way. To our best knowledge there is no attempt for finding the extraction formula of information about the location and shape of $D$ from the single set of the observation data without reducing to other inverse problems. In this paper, we present a direct approach using the idea of the enclosure method for elliptic equations introduced by the author [4].

So what is the enclosure method? Consider a non constant solution of the elliptic problem:

$$\triangle u = 0 \text{ in } \Omega \setminus \overline{D},$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D.$$

In [4] we considered the problem of extracting information about the location and shape of $D$ in two dimensions from the observation data that is single set of Cauchy data of $u$ on $\partial\Omega$. Assuming that $D$ is given by the inside of a polygon with an additional condition on the diameter, we established an extraction formula of the convex hull of $D$ from the data. In the method a special exponential solution of the Laplace equation played the central role. The solution takes the form $e^{-\tau s}e^{\tau x \cdot (\omega + i\omega^\perp)}$ where $\tau (> 0)$ and $s$ are parameters; both $\omega$ and $\omega^\perp$ are unit vectors and satisfy $\omega \cdot \omega^\perp = 0$. The solution divides the space into two half planes which have a line $\{x \mid x \cdot \omega = s\}$ as the common boundary. In one part $\{x \mid x \cdot \omega > s\}$ the solution is growing as $\tau \to \infty$ and in another part $\{x \mid x \cdot \omega < s\}$ decaying. We define a function $I_{\omega, \omega^\perp}(\tau, s)$ of the independent variable $\tau$ with parameter $s$ which is called the indicator function and can be calculated from the observation data:

$$I_{\omega, \omega^\perp}(\tau, s) = e^{-\tau s} \int_{\partial\Omega} \left\{-\frac{\partial}{\partial \nu} e^{\tau x \cdot (\omega + i\omega^\perp)} u + \frac{\partial u}{\partial \nu} e^{\tau x \cdot (\omega + i\omega^\perp)}\right\}ds.$$

In [4] it is clarified that the asymptotic behaviour of the indicator function as $\tau \to \infty$ depends on the position of half plane $x \cdot \omega > s$ relative to $D$. Recall the support function $h_D(\omega) = \sup_{x \in D} x \cdot \omega$. We say that $\omega$ is regular if the set $\{x \mid x \cdot \omega = h_D(\omega)\} \cap \partial D$ consists of only one point. Then we established: for regular $\omega$ $I_{\omega, \omega^\perp}(\tau, s)|_{s = h_D(\omega)}$ is truly algebraic
decaying as $\tau \to \infty$. This means that: there exist positive constants $A$ and $\mu$ such that

$$\lim_{\tau \to \infty} \tau^\mu |I_{\omega,\omega^\perp}(\tau, s)|_{s=h_D(\omega)} = A. \quad (1.1)$$

This fact is the core of the enclosure method. Since we have the trivial identity

$$I_{\omega,\omega^\perp}(\tau, t) = e^{-\tau(t-s)} I_{\omega,\omega^\perp}(\tau, s),$$

from (1.1) one could conclude that: if $s > h_D(\omega)$, then the indicator function is decaying exponentially; if $s = h_D(\omega)$, then the indicator function is decaying truly algebraically; if $s < h_D(\omega)$, then the indicator function is growing exponentially. Moreover by taking the logarithm of both sides of (1.1), we obtained also the one line formula

$$\lim_{\tau \to \infty} \frac{\log |I_{\omega,\omega^\perp}(\tau, 0)|}{\tau} = h_D(\omega). \quad (1.2)$$

By integration by parts (1.1) is equivalent to the statement: the integral

$$e^{-\tau h_D(\omega)} \int_{\partial D} u \frac{\partial}{\partial \nu} e^{\tau x \cdot (\omega+i\omega^\perp)} ds \quad (1.3)$$

is truly algebraic decaying as $\tau \to \infty$. The key point for this fact is, roughly speaking, there does not exist a harmonic extension in a neighbourhood of the point $x_0$ with $x_0 \cdot \omega = h_D(\omega)$.

This is the essence of the idea of the enclosure method. However this is the case when the governing equation is elliptic [5, 6, 7]. How can one apply the enclosure method to non elliptic case? This is another motivation of the study and here we give an answer to this question.

Now state our one space dimensional problem. Let $a > 0$. Let $u = u(x, t)$ be an arbitrary solution of the problem:

$$u_t = u_{xx} \text{ in }]0, a[ \times ]0, T[, \quad (1.4)$$

$$u_x(a, t) = 0 \text{ for } t \in ]0, T[, \quad u(x, 0) = 0 \text{ in }]0, a[.$$

Then the problem is: extract $a$ from $u(0, t)$ and $u_x(0, t)$ for $0 < t < T$.

Let $c$ be an arbitrary positive number. Let

$$v(x, t) = e^{-z^2 t} e^{xz} \quad (1.5)$$

where $\tau$ satisfies $\tau > c^{-2}$ and

$$z = -ct \left( 1 + i \sqrt{1 - \frac{1}{c^2 \tau}} \right).$$

The function $v$ is a complex valued function and satisfies the backward heat equation $v_t + \Delta v = 0$. Moreover $e^{z^2 t} v$ has the special character

- if $s < cx + t$, then $\lim_{\tau \to \infty} e^{z^2 t} |v(x, t)| = 0$
• if $s > cx + t$, then $\lim_{\tau \to \infty} e^{\tau s} |v(x, t)| = \infty$.

The half plane $\{(x, t) \mid s > cx + t\}$ in the space time plays the same role as the half plane $\{x \mid x \cdot \omega > s\}$ for elliptic case. Changing $c$ means changing normal vector of the line $cx + t = s$ and corresponds to changing $\omega$.

**Definition 1.1.** Given $c > 0$, $s \in \mathbb{R}$, define the *indicator function* $I_c(\tau; s)$ by the formula

$$I_c(\tau; s) = e^{\tau s} \int_0^T \left(-v_x(0, t)u(0, t) + u_x(0, t)v(0, t)\right) dt, \quad \tau > c^{-2}$$

where $u$ satisfies (1.4) and $v$ is the function given by (1.5).

Our main result is the following extraction formula.

**Theorem 1.1.** Assume that we know a positive number $M$ such that $M \geq 2a$. Let $T$ and $T'$ satisfy $0 < T' \leq T$. Let $c$ be an arbitrary positive number satisfying $Mc < \min \{T, 2T'\}$. Assume that $u_x(0, t)$ coincides with a polynomial of $t$ on the interval $[0, T']$ one of whose coefficients are not 0. The formula

$$\lim_{\tau \to \infty} \frac{\log |I_c(\tau; 0)|}{\tau} = -2ca$$

and the following statements are true:
- if $s \leq 2ca$, then $\lim_{\tau \to \infty} |I_c(\tau; s)| = 0$;
- if $s > 2ca$, then $\lim_{\tau \to \infty} |I_c(\tau; s)| = \infty$.

This is an unexpected result. Under the condition on $T$ integration by parts gives

$$I_c(\tau; s)|_{s=ca} \sim e^{\tau ca} \int_0^T \left(v_x(a, t)u(a, t) - u_x(a, t)v(a, t)\right) dt$$

modulo exponentially decaying as $\tau \to \infty$. Since the set $]a, \infty[ \times ]0, T]$ and $s = ca$ correspond to $D$ and $s = h_D(\omega)$, respectively and the right hand side of (1.7) corresponds to the integral (1.3) in the elliptic case as explained above, from the past experience we expected that this right hand side decays truly algebraically as $\tau \to \infty$. If it is true, then we automatically obtain the formula

$$\lim_{\tau \to \infty} \frac{\log |I_c(\tau; 0)|}{\tau} = -ca$$

which corresponds to the formula (1.2). However, the fact is different from the expected value. See Figure 1 below.
One heuristic, however, technical explanation for this phenomenon is the following. One knows that \( u(x, t) \) can be extended to the domain \([a, 2a] \times [0, T]\) by the reflection \( x \mapsto 2a - x \) at \( x = a \) as a solution of the heat equation. Then integration by parts gives

\[
e^{\tau ca} \int_0^T (v_x(a, t)u(a, t) - u_x(a, t)v(a, t)) \, dt \sim e^{\tau ca} \int_0^T (v_x(2a, t)u(0, t) + u_x(0, t)v(2a, t)) \, dt.
\]

However, the function \( e^{\tau ca} v(x, t) \) is exponentially decaying as \( \tau \to \infty \) in a neighbourhood of the set \( \{2a\} \times [0, T] \). Therefore one concludes the exponential decaying of the function \( I_c(\tau; s)|_{s=ca} \). This argument suggests the existence of the phase function of \( I_c(\tau; s)|_{s=ca} \) with a negative real part. Theorem 1.1 clarifies that it is essentially \(-\tau ca\).

So the next question about (1.6) is: what is the value \( 2ca \)? One interpretation for this is: it gives us the travel time of a signal with propagation speed \( 1/c \) that starts at the boundary \( x = 0 \) and the initial time \( t = 0 \), reflects by another boundary \( x = a \) and returns to \( x = 0 \). See Figure 2 below.

This is a quite attractive interpretation. From this point of view the restriction \( T > Mc \) is quite reasonable since \( Mc \) gives an upper bound of the travel time \( 2ca \). The idea behind this interpretation is the belief: solution of heat equation \( u_t = u_{xx} \) can be considered as a suitable superposition of solutions of the wave equations \( u_{tt} = \frac{1}{c^2} u_{xx} \) in an appropriate sense. This belief is coming from a well known fact for the solution of the initial value problem for the heat equation. Given initial temperature \( f = f(x), x \in \mathbb{R}^n \) with compact support, let \( u = u(x, t) \) be a solution of the wave equation \( u_{tt} = \Delta u \) for \( x \in \mathbb{R}^n, t > 0 \) with initial values \( u(x, 0) = f(x) \) and \( u_t(x, 0) = 0 \). Then it is well known (e.g., see [8])

Figure 1: (1.6) extracts the line \( cx + t = 2ca \) not \( cx + t = ca \).
Figure 2: (1.6) extracts the travel time $2ca$ of the signal with propagation speed $1/c$ started at the boundary $x = 0$.

that the function

$$v(x, t) = 2 \int_0^\infty e^{-\frac{s^2}{4t}} u(x, s) ds$$

yields a solution of the heat equation $v_t = \Delta v$ for $x \in \mathbb{R}^n$, $t > 0$ with initial values $v(x, 0) = f(x)$. Then the change of variable $s = \sqrt{t} \xi$ yields

$$v(x, t) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\xi^2/4} u(x, \xi \sqrt{t}) d\xi.$$

Replacing $t$ with $t^2$, we obtain the quite impressive formula

$$v(x, t^2) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\xi^2/4} u(x, \xi t) d\xi.$$

In one dimensional case, we have

$$v(x, t^2) = \frac{1}{2 \sqrt{\pi}} \int_0^\infty e^{-\xi^2/4} (f(x + \xi t) + f(x - \xi t)) d\xi.$$

The point is: the function $u(x, \xi t)$ of the independent variables $x$, $t$ satisfies the wave equation with propagation speed $\xi$, $u_{tt} = \xi^2 \Delta u$ with initial values $u(x, 0) = f(x)$ and $u_t(x, 0) = 0$. This means that $v(x, t^2)$ really contains a signal with an arbitrary propagation speed $\xi = 1/c$ and can be obtained as a superposition of signals with several propagation speeds. If this is true also for a solution of (1.4), then it is reasonable to
expect that the observation data coming from the heat equation should contain some information coming from the wave equation with arbitrary propagation speed \( \xi = 1/c \). This is another role of \( c \). In fact, in the final section we will see that a similar formula to (1.6) is valid for the wave equation with propagation speed \( 1/c \). This suggests that the indicator function for the heat equation is a mathematical instrument that picks up a signal coming from the corresponding wave equation with propagation speed \( 1/c \).

Note that, in Section 4 the reader will see that another, rather technical restriction \( T' > Mc/2 \) is redundant. However, following the discovery order, we keep the original statement since we think that the proof of Theorem 1.1 presented in Section 3 under the restriction and the homogeneous Neumann boundary condition is still interesting. The proof is a time dependent approach and is based on

* a representation formula of the solution of (1.4)
* the explicit form of the eigenvalues and eigenfunctions for the Laplacian with Neumann boundary condition
* the asymptotic expansion of the special function

\[
S_1(w) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k^2 + w^2}, \quad \text{Re } w > 0
\]  

(1.8)

which has been established by Olver [9]. Thus, one may think that the proof gives an application of the special function \( S_1(w) \) to an inverse problem.

In a forthcoming paper, we will study the corresponding problems for the heat equations with *variable coefficients*.

It should be pointed out that having Theorem 1.1 does not mean the end of the inverse problem. Needless to say, the real data always contain an error. The next problem is to consider the case when the data are given by \( u(0, t) + E_1(t) \) and \( u_x(0, t) + E_2(t) \) where the size of \( E_1 \) and \( E_2 \) are dominated by a positive number \( \delta \). Here we mean by the size of an arbitrary square integrable function \( E = E(t) \) on \( ]0, T[ \) the \( L^2 \) norm of \( E \) over \( ]0, T[ \).

We assume that

\[
\|(E_1, E_2)\| \equiv \|E_1\|_{L^2(0, T)} + \|E_2\|_{L^2(0, T)} \leq \delta.
\]

Define the corresponding indicator function (at \( s = 0 \)) by the formula

\[
I_c(\tau; E_1, E_2) = \int_0^T (-v_x(0, t)(u(0, t) + E_1(t)) + (u_x(0, t) + E_2(t))v(0, t)) \, dt, \quad \tau > c^{-2}
\]

where \( v \) is the function given by (1.5).

We assume that \( \delta \) is known and *sufficiently small*. The following asymptotic formula (1.10) says that if one chooses a suitable \( \tau \) depending on \( \delta \), then the ratio

\[
\frac{\log |I_c(\tau; E_1, E_2)|}{\tau}
\]

gives an approximate value of \(-2ca\).

**Corollary 1.2.** Given \( \delta \) and \( \sigma \in ]0, 1[ \) define

\[
\tau_\sigma(\delta) = \frac{\sigma}{T}|\log \delta|.
\]  

(1.9)
Then, as $\delta \to 0$ the formula
\[
\sup_{\| (E_1, E_2) \| \leq \delta} \left| \frac{\log |I_c(\tau; \delta; E_1, E_2)|}{\tau \delta} + 2ca \right| = O \left( \frac{\log \log \| \delta \|}{\log \| \delta \|} \right),
\] (1.10)
is valid.

Needless to say, (1.9) is one of possible choices of $\tau$. In practice this choice of $\tau$ will be a problem, however, it would be interested in doing a numerical testing of the formula (1.6) as done for (1.2) in [7]. This belongs to our future study.

A brief outline of this paper is as follows. Theorem 1.1 is proved in Section 3. The proof is based on an asymptotic behaviour of an integral involving $u$ on $\{ a \} \times [0, T']$ in the case when $u_x(0, t)$ is given by a constant on $[0, T']$, as derived in Section 2. From the proof one may think that it is quite difficult to cover other boundary conditions at $x = a$. However, in Section 4 we present a simpler approach that is based on the transform of $u$ into $w$ by the formula
\[
w(x, \tau) = \int_0^T u(x, t)e^{-z^2t}dt
\] where
\[
z^2 = \tau + i2c^2\tau^2\sqrt{1 - \frac{1}{c^2\tau}}.
\]
We see that the asymptotic behaviour of $w(a, \tau)$ as $\tau \to \infty$ yields a corresponding formula for the cavity with the Robin boundary condition. We believe that this idea will work also for the multidimensional case.

In the final section we give an application of this approach to the wave equation.

2 A key lemma. Time dependent approach

The integration by parts gives
\[
I_c(\tau; s) = -e^{ras} \int_0^T u(a, t)v_x(a, t)dt - e^{ras} \int_0^a u(x, T)v(x, T)dx.
\]
Since $v(x, t) = e^{-z^2t}e^{xz}$, we obtain
\[
I_c(\tau; s) = -ze^{ras}e^{az} \int_0^T u(a, t)e^{-z^2t}dt - e^{ras}e^{-z^2T} \int_0^a u(x, T)e^{xz}dx.
\] (2.1)
Rewrite this as
\[
I_c(\tau; s) = -ze^{ras}e^{2az} \times e^{-az} \int_0^T u(a, t)e^{-z^2t}dt
\]
\[
-ze^{ras}e^{2az} \times e^{-az} \int_0^T u(a, t)e^{-z^2t}dt - e^{ras}e^{-z^2T} \int_0^a u(x, T)e^{xz}dx.
\] (2.2)
Let $s = 2ca$. Since $T$ and $T'$ satisfy $T > 2ca$ and $T' > ca$, respectively, we see that the second and third terms of (2.2) have the estimates $O(e^{-\tau(T-ca)})$ and $O(e^{-\tau(T-2ca)})$, respectively. These yield
\[
I_c(\tau; s)|_{s=2ca} \sim -ze^{-2ia\tau} \sqrt{1 - \frac{1}{c^2\tau}} e^{-az} \int_0^T u(a, t)e^{-z^2t}dt
\] (2.3)
modulo exponentially decaying as $\tau \to \infty$.

Therefore it suffices to prove that
\[
e^{-az} \int_0^{T'} u(a, t) e^{-z^2 t} dt
\]
is truly algebraic decaying as $\tau \to \infty$.

For the purpose we recall a known representation formula of the solution.

Let $\Omega$ be an arbitrary bounded domain of $\mathbb{R}^n$. Let $u$ satisfy
\[
u_t = \Delta u \text{ in } \Omega \times ]0, T[, \quad u(x, 0) = 0 \text{ in } \Omega.
\]
The representation formula of $u$ which is given below involves an associated elliptic problem with the parameter $t$ and eigenfunctions for the Laplacian with the Neumann condition. This is taken from [2].

Let $v = v(x, t)$ be the unique solution of the elliptic problem depending on $t$:
\[
\Delta v = \frac{1}{|\Omega|} \int_{\partial \Omega} \frac{\partial u}{\partial \nu}(x, t) dS(x) \text{ in } \Omega, \\
\frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu}(\cdot, t) \text{ on } \partial \Omega, \\
\int_{\Omega} v dx = 0.
\]

Let $\{\lambda_k\}_{k=0}^\infty$ and $\{\Psi_k\}_{k=0}^\infty$ be the all of eigenvalues and the corresponding complete orthogonal system of the Laplacian in $\Omega$ with the Neumann boundary condition:
\[
\Delta \Psi_k + \lambda_k \Psi_k = 0 \text{ in } \Omega, \\
\frac{\partial \Psi_k}{\partial \nu} = 0 \text{ on } \partial \Omega, \\
\int_{\Omega} |\Psi_k(x)|^2 dx = 1
\]
and
\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.
\]

Note that $\Psi_k$ with $k \geq 1$ satisfies
\[
\int_{\Omega} \Psi_k(x) dx = 0
\]
since $\Psi_0(x) = |\Omega|^{-1/2}$.

**Lemma 2.1 ([2])**. The formula
\[
u(x, t) = v(x, t) + \frac{1}{|\Omega|} \int_{0}^{t} \left( \int_{\partial \Omega} \frac{\partial u}{\partial \nu} dS \right) dt
\]
\[
- \sum_{k=1}^\infty \left( e^{-\lambda_k t} \int_{\Omega} v(x, 0) \Psi_k(x) dx + \int_{0}^{t} e^{-\lambda_k(t-s)} \left( \int_{\Omega} v_t(x, s) \Psi_k(x) dx \right) ds \right) \Psi_k(x),
\]

9
is valid.

In the present case the domain $\Omega$ is given by $]0, a[$. $v$ takes the form

$$v(x, t) = -\frac{u_x(0, t)}{6a}(3x^2 - 6ax + 2a^2).$$

$\Psi_k$ and $\lambda_k$ for $k \geq 1$ are given by

$$\Psi_k(x) = \sqrt{\frac{2}{a}} \cos \frac{k\pi}{a} x, \quad \lambda_k = \left( \frac{k\pi}{a} \right)^2.$$

A direct computation yields

$$\int_0^a v(x, 0)\Psi_k(x)\,dx = -u_x(0, 0)\sqrt{\frac{2}{a}}\lambda_k^{-1}$$

and

$$\int_0^a v_t(x, s)\Psi_k(x)\,dx = -u_{xt}(0, s)\sqrt{\frac{2}{a}}\lambda_k^{-1}.$$

Therefore from Lemma 2.1 we obtain

$$u(x, t) = v(x, t) - \frac{1}{a} \int_0^t u_x(0, t)\,dt$$

$$+ \sqrt{\frac{2}{a}} \sum_{k=1}^{\infty} \lambda_k^{-1} \left( e^{-\lambda_k t}u_x(0, 0) + \int_0^t e^{-\lambda_k (t-s)}u_{xt}(0, s)\,ds \right)\Psi_k(x).$$

Letting $x \uparrow a$ in this formula we obtain the expression of the boundary value of $u$ at $x = a$

$$u(a, t) = \frac{a}{6}u_x(0, t) - \frac{1}{a} \int_0^t u_x(0, t)\,dt$$

$$+ \frac{2}{a} \sum_{k=1}^{\infty} (-1)^k \lambda_k^{-1} \left( e^{-\lambda_k t}u_x(0, 0) + \int_0^t e^{-\lambda_k (t-s)}u_{xt}(0, s)\,ds \right).$$

Now consider the simpler case

$$u_x(0, t) = 1, \quad 0 < t < T'.$$

In this case we have

$$u(a, t) = \frac{a}{6} - \frac{t}{a} + \frac{2}{a} \sum_{k=1}^{\infty} (-1)^k \lambda_k^{-1} e^{-\lambda_k t}, \quad 0 < t < T'. \quad (2.4)$$

The following result is the key for the enclosure method and not trivial.

**Lemma 2.2.**

$$e^{-az} \int_0^{T'} u(a, t)e^{-z^2t}\,dt$$

is algebraic decaying as $\tau \rightarrow \infty$. 

10
Proof. One sees that
\[ \int_0^{T'} e^{-\lambda_k t} e^{-z^2 t} dt = \frac{1}{\lambda_k + z^2} \left( 1 - e^{-(\lambda_k + z^2) T'} \right) . \]

This gives
\[ e^{-az} \int_0^{T'} \sum_{k=1}^\infty (-1)^k \lambda_k^{-1} e^{-\lambda_k t} e^{-z^2 t} dt \sim e^{-az} \sum_{k=1}^\infty (-1)^k \frac{1}{\lambda_k (\lambda_k + z^2)} \]
as \( \tau \to \infty \) modulo exponentially decaying.

Moreover we see that
\[ e^{-az} \int_0^{T'} e^{-z^2 t} dt \sim e^{-az} \frac{1}{z^2} \]
and
\[ e^{-az} \int_0^{T'} t e^{-z^2 t} dt \sim e^{-az} \frac{1}{z^2} \]
as \( \tau \to \infty \) modulo exponentially decaying.

Using these and (2.4), we get
\[ e^{-az} \int_0^{T'} u(a, t) e^{-z^2 t} dt \sim e^{-az} \left\{ \frac{a}{6z^2} - \frac{1}{az^4} + \frac{2}{az^2} \sum_{k=1}^\infty (-1)^k \frac{1}{\lambda_k} - \frac{2}{az^2} \sum_{k=1}^\infty (-1)^k \frac{1}{\lambda_k + z^2} \right\} \]
\[ = e^{-az} \left\{ \frac{a}{6z^2} - \frac{1}{az^4} + \frac{2}{az^2} \sum_{k=1}^\infty (-1)^k \frac{1}{k^2} + \frac{2}{az^2} \left( \sum_{k=1}^\infty (-1)^{k-1} \frac{1}{k^2} - \frac{1}{2z^2} \right) \right\} . \]

Here we cite the formula
\[ \sum_{k=1}^\infty (-1)^{k-1} \frac{1}{k^2} = \frac{\pi^2}{12} . \]

This yields
\[ e^{-az} \int_0^{T'} u(a, t) e^{-z^2 t} dt \sim \frac{2e^{-az}}{az^2} \left( \sum_{k=1}^\infty (-1)^{k-1} \frac{1}{\lambda_k + z^2} - \frac{1}{2z^2} \right) . \]

(2.5)

Therefore the problem is the asymptotic behaviour of the function \( f(w) \) defined by the formula
\[ f(w) = \sum_{k=1}^\infty (-1)^{k-1} \frac{1}{\lambda_k + w^2}, \quad \text{Re} \ w > 0 . \]

We know that Olver (see p. 302 of [9]) has already studied the asymptotic behaviour of the function \( S_1 \) given by (1.8). Using the function one can write
\[ f(w) = -\left( \frac{a}{\pi} \right)^2 S_1 \left( \frac{aw}{\pi} \right) + \frac{1}{w^2} \]
\[ = \frac{1}{2w^2} - \left( \frac{a}{\pi} \right)^2 R_1 \left( \frac{aw}{\pi} \right) \]
11
where
\[ R_1(w) = S_1(w) - \frac{1}{2w^2}. \]

Then (2.5) becomes
\[ e^{-az} \int_0^{T'} u(a, t)e^{-z^2 t} dt \sim -\frac{2ae^{-az}}{\pi^2 z^2} R_1\left(-\frac{az}{\pi}\right). \]  

(2.6)

Here we make use of the formula (p. 304, (7.04) of [9]) for an arbitrary fixed \( \delta > 0 \) and \( w \to \infty \) in \( |\arg w| \leq \frac{\pi}{2} - \delta \):
\[ S_1(w) = \frac{1}{2w^2} + \frac{2\pi}{\Gamma(1)w^{1/2}} \sum_{j=0}^{\infty} K_{-1/2}((2j+1)\pi w) \frac{1}{(j+\frac{1}{2})^{-1/2}} \]  

(2.7)

where \( K_{-1/2} \) is the Macdonald function and has the asymptotic form
\[ K_{-1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left(1 + O\left(\frac{1}{z}\right)\right), \quad z \to \infty \quad \text{in} \quad |\arg z| \leq \frac{3\pi}{2} - \delta. \]  

(2.8)

See p. 250 and p. 238 of [9].

One implication of (2.7) and (2.8) is the asymptotic formula
\[ R_1(w) = \frac{\pi e^{-\pi w}}{w} \left(1 + O\left(\frac{1}{w}\right)\right). \]

This yields
\[ e^{-az} \int_0^{T'} u(a, t)e^{-z^2 t} dt = \frac{2}{z^3} \left(1 + O\left(\frac{1}{z}\right)\right). \]  

(2.9)

This is the desired conclusion.

\( \square \)

3 Proof of Theorem 1.1 and Corollary 1.2

Consider the case when
\[ u_x(0, t) = \frac{t^m}{m!}, \quad 0 < t < T'. \]

We make use of the simple formulae:
\[ u_m(x, t) = \int_0^t u_{m-1}(x, t) dt, \quad m = 1, 2, \ldots, \quad 0 < t < T' \]

where \( u_m, \ m = 0, 1, \cdots \) are the solutions corresponding to the condition \( u_x(0, t) = \frac{t^m}{m!}, 0 < t < T' \).

Using this formulae and the initial condition \( u_m(x, 0) = 0 \), we get
\[ \int_0^{T'} u_m(a, t)e^{-z^2 t} dt = -\frac{u_m(a, t)}{z^2} e^{-z^2 T'} + \frac{1}{z^2} \int_0^{T'} (u_m)_t(a, t)e^{-z^2 t} dt \]
\[ = -\frac{u_m(a, T')}{z^2} e^{-z^2 T'} + \frac{1}{z^2} \int_0^{T'} u_{m-1}(a, t)e^{-z^2 t} dt \]

\[ \int_0^{T'} u_m(a, t)e^{-z^2 t} dt \]

\[ = -\frac{u_m(a, T')}{z^2} e^{-z^2 T'} + \frac{1}{z^2} \int_0^{T'} u_{m-1}(a, t)e^{-z^2 t} dt \]
and this thus yields the recurrence formula
\[ e^{-az} \int_0^{T'} u_m(a, t) e^{-z^2 t} dt = \frac{e^{-az}}{z^2} \int_0^{T'} u_{m-1}(a, t) e^{-z^2 t} dt + O \left( e^{-\tau(T' - ca)} \right). \]

This immediately yields
\[ e^{-az} \int_0^{T'} u_m(a, t) e^{-z^2 t} dt \sim \frac{e^{-az}}{(z^2)^m} \int_0^{T'} u_0(a, t) e^{-z^2 t} dt. \] (3.1)

Now we are ready to prove Theorem 1.1. By the assumption, one may assume that \( u_x(0, t) \) has the form
\[ u_x(0, t) = \sum_{m=0}^l \gamma_m m! t^m, \quad 0 < t < T'. \]

Then the uniqueness of the direct problem gives
\[ u(x, t) = \sum_{m=0}^l \gamma_m u_m(x, t), \quad 0 < t < T'. \]

From (3.1) we obtain
\[ e^{-az} \int_0^{T'} u(a, t) e^{-z^2 t} dt = \sum_{m=0}^l \gamma_m e^{-az} \int_0^{T'} u_m(a, t) e^{-z^2 t} dt \]
\[ \sim \left( \sum_{m=0}^l \frac{\gamma_m}{(z^2)^m} \right) e^{-az} \int_0^{T'} u_0(a, t) e^{-z^2 t} dt. \] (3.2)

Let \( j = \min \{ m \mid \gamma_m \neq 0 \} \). The number \( j \) satisfy \( \gamma_j \neq 0 \) and \( \gamma_m = 0 \) if \( m < j \).

A combination of (2.9) and (3.2) gives
\[ \lim_{\tau \to \infty} z^{2j+3} e^{-az} \int_0^{T'} u(a, t) e^{-z^2 t} dt = 2\gamma_j (\neq 0) \]
and from (2.3) we obtain the asymptotic formula
\[ \lim_{\tau \to \infty} z^{2(j+1)} e^{2iac\tau} \sqrt{1 - \frac{1}{c^2 \tau}} |I_c(\tau; s)|_{s=2ca} = -2\gamma_j. \]

From this formula one knows that: there exist positive numbers \( \tau_0 \) and \( A \) such that, for all \( \tau \geq \tau_0 \)
\[ \frac{3A}{2} \geq \tau^{2(j+1)} e^{2ca\tau} |I_c(\tau; 0)| \] (3.3)
and
\[ \tau^{2(j+1)} e^{2ca\tau} |I_c(\tau; 0)| \geq \frac{A}{2}. \] (3.4)

Now all the conclusions in Theorem 1.1 follow from (3.3) and (3.4). In particular, we have
\[ \frac{\log |I_c(\tau; 0)|}{\tau} = -2ca + 2(j + 1) \frac{\log \tau}{\tau} + O \left( \frac{1}{\tau} \right). \] (3.5)
Finally we give a proof of Corollary 1.2. We have

\[ I_c(\tau; E_1, E_2) - I_c(\tau; 0) = \int_0^T (-zE_1(t) + E_2(t)) e^{-z^2 t} dt. \]

This yields the estimate

\[ |I_c(\tau; E_1, E_2) - I_c(\tau; 0)| \leq \left( c\sqrt{\tau} + \frac{1}{\sqrt{2\tau}} \right) \delta. \]  \hspace{1cm} (3.6)

A combination of (3.4) and (3.6) gives

\[ |I_c(\tau; E_1, E_2)| \geq |I_c(\tau; 0)| \left( 1 - \frac{2}{A} e^{2ca\tau} \tau^{2(j+1)} (c\sqrt{\tau} + \frac{1}{\sqrt{2\tau}}) \delta \right) \]  \hspace{1cm} (3.7)

and

\[ |I_c(\tau; E_1, E_2)| \leq |I_c(\tau; 0)| \left( 1 + \frac{2}{A} e^{2ca\tau} \tau^{2(j+1)} (c\sqrt{\tau} + \frac{1}{\sqrt{2\tau}}) \delta \right). \]  \hspace{1cm} (3.8)

Now choose \( \tau \) depending on \( \delta \) in such a way that

\[ \frac{2}{A} e^{2ca\tau} \tau^{2(j+1)} (c\sqrt{\tau} + \frac{1}{\sqrt{2\tau}}) \delta \to 0 \]  \hspace{1cm} (3.9)

as \( \delta \to 0 \). Since we have assumed that \( T > 2ca \), one possible choice of such \( \tau \) is to solve the equation

\[ e^{\tau T} = \delta^{-\sigma}. \]

The solution is just the \( \tau \) in Corollary 1.2. In this case, the estimation of the convergence rate of (3.9) is given by

\[ O \left( |\log \delta|^{2(j+1)+1/2} \delta^{1-\sigma} \right). \]  \hspace{1cm} (3.10)

Now from (3.5) and (3.7) to (3.10) we obtain the desired formula (1.10).

\[ \square \]

4 A generalization. Stationary approach

The aim of this section is to

• present an alternative simpler method for the proof of Theorem 1.1
• give a generalization of Theorem 1.1.

First we describe the problem. Let \( a > 0 \). Let \( u = u(x, t) \) be an arbitrary solution of the problem:

\[ u_t = u_{xx} \text{ in } ]0, a[ \times ]0, T[, \]

\[ u_x(a, t) + \rho u(a, t) = 0 \text{ for } t \in ]0, T[, \]  \hspace{1cm} (4.1)

\[ u(x, 0) = 0 \text{ in } ]0, a[ \]

here \( \rho \) is an arbitrary fixed constant.
Then the problem is: assume that both of $\rho$ and $a$ are unknown. extract $a$ from $u(0,t)$ and $u_x(0,t)$ for $0 < t < T$.

In this section the notion defined below plays the central role.

**Definition 4.1.** Let $c > 0$. We say that a function $f \in L^1(0,T)$ satisfies the condition $c$ if there exist positive constant $C_1$, $C_2$ and real numbers $\tau_0(\geq c^{-2})$, $\mu_1$, $\mu_2$ such that, for all $\tau \geq \tau_0$

$$C_1\tau^{\mu_1} \leq \left| \int_0^T f(t)e^{-z^2t}dt \right| \leq C_2\tau^{\mu_2}$$

where

$$z^2 = \tau + i2c^2\tau^2\sqrt{1 - \frac{1}{c^2}}.$$  

It is easy to see that if $f(t)$ is given by a polynomial of $t$ on an interval $]0, T'[\subset]0, T[$ that is not identically zero, then for all $c > 0$ $f(t)$ satisfies the condition $c$. Moreover, it is not difficult to see that if $f(t)$ is smooth on $]0, T'$ and $t = 0$ is not a zero of $f(t)$ with infinite order, then $f(t)$ satisfies the condition $c$.

**Definition 4.2.** Given $c > 0$ define the indicator function $I_c(\tau)$ by the formula

$$I_c(\tau) = \int_0^T (-v_x(0,t)u(0,t) + u_x(0,t)v(0,t))dt, \ \tau > c^{-2}$$

where $u$ satisfies (4.1) and $v$ is the function given by (1.5).

The following gives the answer to the problem mentioned above and generalizes Theorem 1.1.

**Theorem 4.1.** Assume that we know a positive number $M$ such that $M \geq 2a$. Let $c$ be an arbitrary positive number satisfying $Mc < T$. Let the $u_x(0,t)$ satisfy the condition $c$. Then the formula

$$\lim_{\tau \to \infty} \frac{\log |I_c(\tau)|}{\tau} = -2ca,$$  

(4.2)

is valid.

**Proof.** Introduce a new function $w$ by the formula

$$w(x) = \int_0^T u(x,t)e^{-z^2t}dt, \ 0 < x < a.$$  

Note that, for simplicity of description we omitted indicating the dependence of $w$ on $\tau$ and $c$. This function satisfies

$$w'' - z^2w = e^{-z^2T}u(x,T) \text{ in }]0, a[,$$

$$w'(a) + \rho w(a) = 0$$

and the indicator function has the expression

$$I_c(\tau) = -zw(0) + w'(0).$$

Then the integration by parts gives another expression

$$I_c(\tau) = -(z + \rho)w(a)e^{az} - e^{-z^2T} \int_0^a u(\xi, T)d\xi.$$
This yields

\[ I_c(\tau) e^{-2az} = -(z + \rho)w(a)e^{-az} + O(e^{-\tau(T-2ca)}). \]  

(4.3)

Now we write \( w(a) \) by using \( w'(0) \). It is easy to see that \( w \) has the form

\[ w(x) = Ae^{xz} + Be^{-xz} + \frac{e^{-2xT}}{2z} \left( e^{xz} \int_0^x u(\xi, T)e^{-\xi z}d\xi - e^{-xz} \int_0^x u(\xi, T)e^{\xi z}d\xi \right) \]

where \( A \) and \( B \) are constants to be determined later. Then the boundary conditions at \( x = 0 \) and \( x = a \) give the system of equations

\[
zA - zB = w'(0),
\]

\[
(z + \rho)Ae^{az} - (z - \rho)Be^{-az}
\]

\[
= -\frac{e^{-2xT}}{2} \left( (1 + \frac{\rho}{z})e^{az} \int_0^a u(\xi, T)e^{-\xi z}d\xi + (1 - \frac{\rho}{z})e^{-az} \int_0^a u(\xi, T)e^{\xi z}d\xi \right).
\]

Solving this system of equations, we obtain

\[
Ae^{az} = \frac{1}{z \left( (1 - \frac{\rho}{z})e^{-az} - (1 + \frac{\rho}{z})e^{az} \right)} \times \left( (1 - \frac{\rho}{z})w'(0) + \frac{e^{-2zT}}{2} \left( (1 + \frac{\rho}{z})e^{2az} \int_0^a u(\xi, T)e^{-\xi z}d\xi + (1 - \frac{\rho}{z}) \int_0^a u(\xi, T)e^{\xi z}d\xi \right) \right)
\]

\[
Be^{-az} = \frac{1}{z \left( (1 - \frac{\rho}{z})e^{-az} - (1 + \frac{\rho}{z})e^{az} \right)} \times \left( (1 + \frac{\rho}{z})w'(0) + \frac{e^{-2zT}}{2} \left( (1 + \frac{\rho}{z}) \int_0^a u(\xi, T)e^{-\xi z}d\xi + (1 - \frac{\rho}{z})e^{2az} \int_0^a u(\xi, T)e^{\xi z}d\xi \right) \right).
\]

A direct computation yields

\[
w(a) = \frac{2w'(0)}{z \left( (1 - \frac{\rho}{z})e^{-az} - (1 + \frac{\rho}{z})e^{az} \right)} + \frac{e^{-2zT}}{z \left( (1 - \frac{\rho}{z})e^{-az} - (1 + \frac{\rho}{z})e^{az} \right)} \left( \int_0^a u(\xi, T)e^{\xi z}d\xi + \int_0^a u(\xi, T)e^{-\xi z}d\xi \right)
\]

and thus this yields

\[
(z + \rho)w(a)e^{-az} \sim 2 \left( 1 + \frac{2\rho}{z} + \frac{1}{\tau^2} \right) w'(0)
\]

(4.4)

modulo exponentially decaying as \( \tau \to \infty \). From the assumption on \( u_x(0, t) \), (4.3) and (4.4) one concludes that, there exist positive constants \( C'_1 \), \( C'_2 \) and real numbers \( \tau'_0(\geq c^{-2}) \), \( \mu_1 \) and \( \mu_2 \) such that for all \( \tau \geq \tau'_0 \)

\[
C'_1 \tau^{\mu_1} \leq |I_c(\tau)|e^{2c^2a} \leq C'_2 \tau^{\mu_2}.
\]
This gives (4.2).

Remark 4.1. If once \( a \) is known, then one can extract \( \rho \) by the formula:

\[
\lim_{\tau \to \infty} \frac{z(I_c(\tau)e^{-2az} + 2w'(0))}{2w'(0)} = -2\rho.
\]

5 Remark. Application to the wave equation

Finally we give a comment on an application to the wave equation. Let \( a > 0 \) and \( c > 0 \). Let \( u = u(x, t) \) be an arbitrary solution of the problem:

\[
\begin{align*}
  u_{tt} &= \frac{1}{c^2} u_{xx} \quad \text{in } ]0, a[ \times ]0, T[, \\
  \frac{1}{c} u_x(a, t) + \rho u(a, t) &= 0 \quad \text{for } t \in ]0, T[, \\
  u(x, 0) &= 0 \quad \text{in } ]0, a[, \\
  u_t(x, 0) &= 0 \quad \text{in } ]0, a[
\end{align*}
\]

(5.1)

here \( \rho \) is an arbitrary fixed constant. The quantity \( 1/c \) denotes the propagation speed of the signal governed by the equation. In contrast to the previous sections, one can not change \( c \).

Then the problem is: assume that both of \( \rho \) and \( a \) are unknown. extract \( a \) from \( u(0, t) \) and \( u_x(0, t) \) for \( 0 < t < T \).

Introduce the function \( w \) by the formula

\[
w(x) = \int_0^T u(x, t)e^{-\tau t}dt, \quad 0 < x < a.
\]

Again we omitted indicating the dependence of \( w \) on \( \tau \). Then \( w \) satisfies

\[
\frac{1}{c^2} w'' - \tau^2 w = e^{-\tau T}(u_t(x, T) + \tau u(x, T)) \quad \text{in } ]0, a[, \\
\frac{1}{c} w'(a) + \rho w(a) = 0.
\]

Now it is easy to see that

\[
w(a) = -\frac{2w'(0)}{e^{-\tau T}c \left(1 + \frac{\rho}{\tau}\right) e^{ca\tau} - \left(1 - \frac{\rho}{\tau}\right)e^{-ca\tau}}
\]

\[
-\frac{e^{-\tau T}}{\tau \left(1 + \frac{\rho}{\tau}\right) e^{ca\tau} - \left(1 - \frac{\rho}{\tau}\right)e^{-ca\tau}}
\]

\[
\times \left( \int_0^a (u_t(\xi, T) + \tau u(\xi, T))e^{-\xi \tau}d\xi + \int_0^a (u_t(\xi, T) + \tau u(\xi, T))e^{\xi \tau}d\xi \right).
\]

(5.2)
**Definition 5.1.** Define the *indicator function* $I(\tau)$ by the formula

$$I(\tau) = \int_0^T \left( -\frac{1}{c} u_x(0,t)u(0,t) + \frac{1}{c} u_x(0,t)v(0,t) \right) dt, \quad \tau > 0$$

where $u$ satisfies (5.1) and $v$ is the function given by

$$v(x,t) = e^{-\tau(cx+t)}.$$

Integration by parts gives the expression

$$I(\tau)e^{2ca\tau} = (\tau - \rho)w(a)e^{ca\tau} - ce^{-\tau(T-2ca)} \int_0^a (u_t(\xi,T) + \tau u(\xi,T))e^{-c\xi\tau} d\xi.$$

Hereafter, using a similar arguments as done in Section 4 and (5.2), we obtain the following

**Theorem 5.1.** Assume that we know a positive number $M$ such that $M \geq 2a$. Let $u_x(0,t)$ satisfy the condition: there exist positive constants $C_1, C_2$ and real numbers $\tau_0(>0)$, $\mu_1$, $\mu_2$ such that, for all $\tau \geq \tau_0$

$$C_1 \tau^{\mu_1} \leq \left| \int_0^T u_x(0,t)e^{-\tau t} dt \right| \leq C_2 \tau^{\mu_2}. \quad (5.3)$$

Let $T > Mc$. Then the formula

$$\lim_{\tau \to \infty} \frac{\log |I(\tau)|}{\tau} = -2ca, \quad (5.4)$$

is valid.

Needless to say, if once $a$ is known, then one can extract $\rho$ by the formula:

$$\lim_{\tau \to \infty} \frac{c\tau(I(\tau)e^{2ca\tau} + \frac{2}{e} w'(0))}{2w'(0)} = 2\rho.$$

The quantity $2ca$ coincides with the travel time of a signal governed by the wave equation with propagation speed $1/c$ which starts at the boundary $x = 0$ and initial time $t = 0$, reflects another boundary $x = a$ and returns to $x = 0$. The restriction $T > Mc$ is quite reasonable and does not against the well known fact: the wave equation has the *finite propagation property*. The condition (5.3) ensures that $u_x(0,t)$ can not be identically zero in an interval $]0, T'[, T[. Therefore surely a signal occurs at the initial time. However, it should be emphasized that the formula (5.4) makes use of the *averaged value* of the measured data with an *exponential weight* over the observation time. This is a completely different idea from the well known approach in nondestructive evaluation by sound wave: monitoring of the first *arrival time* of the echo, one knows the travel time.

**Acknowledgement**

This research was partially supported by Grant-in-Aid for Scientific Research (C)(No. 18540160) of Japan Society for the Promotion of Science. The author would like to thank Prof. Belishev for remarks about the relationship between the response operator and the our observation data for the heat equation in one space dimension and comments on our results.
References

[1] Avdonin, S., Belishev, M. and Rozhkov, Yu., The BC-method in the inverse problem for the heat equation, J. Inv. Ill-Posed Probl., 5(1997), 309-322.

[2] Bryan, K. and Caudill, F. Jr., Uniqueness for a boundary identification problem in thermal imaging, Differential Equations and Computational Simulations III, Electric Journal of Differential Equations, Conference 01, 1997, 23-39.

[3] Canuto, B., Rosset, E. and Vessella, S., Quantitative estimate of unique continuation for parabolic equations and inverse initial-boundary value problems with unknown boundaries, Trans. Amer. Math. Soc., 354(2002), 491-535.

[4] Ikehata, M., Enclosing a polygonal cavity in a two-dimensional bounded domain from Cauchy data, Inverse Problems, 15(1999), 1231-1241.

[5] Ikehata, M., A regularized extraction formula in the enclosure method, Inverse Problems, 18(2002), 435-440.

[6] Ikehata, M., Inverse scattering problems and the enclosure method, Inverse Problems, 20(2004), 533-551.

[7] Ikehata, M. and Ohe, T., A numerical method for finding the convex hull of polygonal cavities using the enclosure method, Inverse Problems, 18(2002), 111-124.

[8] Jost, J., Partial differential equations, Graduate texts in mathematics, 214, Springer, 2002.

[9] Olver, F. W., Asymptotics and special functions, Academic Press, New York and London, 1974.

e-mail address
ikehata@math.sci.gunma-u.ac.jp