Quantum-Optical set-up for the Monty Hall problem and a secure communication protocol

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A quantization scheme for the Monty Hall problem is proposed inspired by an experimentally-feasible, quantum-optical set-up that resembles the classical game. The expected payoff of the player is studied from a frequentist perspective by analyzing the classical expectation values of the obtained quantum probabilities. Results are examined considering both entanglement and non-entanglement between player and host, and using two different approaches: random and strategy-based. A potential application to secure communications of this quantization scheme and its results is also briefly discussed.

I. Introduction

Inspired by the applications of game theory to the study of classical information problems, quantum information theorists began to include quantum probability amplitudes and quantum entanglement into classical game theory \[1\], creating what is now known as quantum game theory. The main motivation behind this new area was, and still is, the applications it has in secure quantum communications, as quantum eavesdropping can be treated as a game in which the spy’s goal is to extract the maximum amount of information from a quantum communication channel.

In 1999, Meyer provided the techniques to include quantum strategies into classical game theory \[1\], showing that the use of these actually increases the expected payoffs of the players. In the same year, motivated by the lack of an unconditionally secure remote gambling, Goldberg et al. presented a protocol that allows two remote parties to play a quantum fair-gambling game \[2\]. Later on, a quantum version of the famous game, the prisoner’s dilemma, was discussed by Eisert et al. and Benjamin and Hayden in Refs. \[3\]–\[5\]. Collective quantum games, in which more than two players take part, were also studied by Benjamin and Hayden in Ref. \[6\], concluding that quantum entanglement enables different kinds of cooperative behavior, preventing players from betraying one another.

One interesting problem that arises in quantum game theory, is the relation between the classical and quantum results. This issue is briefly discussed by van Enk and Pike in Ref. \[7\]. In their paper, they analyze to what extent the quantum solution of the prisoner’s dilemma solves the classical problem.

In this work we focus on describing a quantization scheme of the Monty Hall problem inspired by an experimentally-feasible, quantum-optical set-up that resembles the classical game with the addition of some quantum features. We also analyze the expected payoff of the player by studying the classical expectation values of the quantum probabilities, allowing us to compare the obtained results with the classical ones. A potential application to secure communications of this quantization scheme and its results is also briefly discussed.

Classical Monty Hall problem

The Monty Hall problem is a famous, seemingly paradoxical problem in probability \[8\]–\[10\]. It describes a contest in which a player is asked to choose between three boxes. Inside one of the boxes, a prize was randomly placed beforehand. There are two main characters in this contest: the host (Monty Hall), who knows in which box the prize is, and the player, who does not have any information about its location.

The contest begins with the player choosing (but not opening) one of the boxes. If the chosen box is the one with the prize inside, the host, who knows where the prize hides, randomly opens one of the two empty boxes. On the other hand, if the player chooses one of the empty boxes, the host opens the other remaining empty box. In both cases the host shares this information with the player. Lastly, the host asks the player if she wants to open her initial choice or prefers to open the other box that remains closed. The apparent paradox results from the fact that, when doing the calculations, it is found that the probability of the player finding the prize in the box she initially chose is \(\frac{1}{3}\), while the probability of finding the prize if she decides to open the other box is \(\frac{2}{3}\).

The above result can be thought of as follows: the location of the prize, with a probability of \(\frac{1}{3}\) for each box, and the initial choice of the player, also with a probability of \(\frac{1}{3}\) for each box, are independent events; therefore, the probability of the prize being in box \(j\) and the player initially choosing box \(i\) is \(P(i,j) = \frac{1}{9}\) for all \(i,j = 1,2,3\).

Table 1 shows the elements \((i,j)\) of the corresponding sample space. The cases in which the player initially chooses the box in which the prize is located, that is, the

\[\begin{array}{|c|c|c|}
\hline
\text{i} & \text{j} & P(i,j) \\
\hline
1 & 1 & \frac{1}{9} \\
1 & 2 & \frac{1}{9} \\
1 & 3 & \frac{1}{9} \\
2 & 1 & \frac{1}{9} \\
2 & 2 & \frac{1}{9} \\
2 & 3 & \frac{1}{9} \\
3 & 1 & \frac{1}{9} \\
3 & 2 & \frac{1}{9} \\
3 & 3 & \frac{1}{9} \\
\hline
\end{array}\]
Analogously, the probability

\[ P_{ns} = P(1, 1) + P(2, 2) + P(3, 3) = \frac{1}{3}, \]

(1)

Analogously, the probability \( P_s \) of the player winning by switching her initial choice, is:

\[ P_s = P(1, 2) + P(1, 3) + P(2, 1) \]
\[ + P(2, 3) + P(3, 1) + P(3, 2) = \frac{2}{3}. \]  

(2)

Quantum Monty Hall game

When the term “quantization” is mentioned, it usually refers to the mathematical procedure in which a Hilbert space is constructed from a Hamiltonian representation of a classical system. However, in the description of the Monty Hall problem there is no Hamiltonian to quantize. In this case, the quantization procedure becomes an entirely subjective task, usually referring to considering the different elements of the classical game (the location of the prize, the opening of a box by the host, etc.) as elements present in the study of a quantum system (superpositions, projective measurements, etc.). Different approaches and quantization schemes of the Monty Hall problem have already been proposed \[11\,15\], and there is even a quantum algorithm developed so that two persons can play a version of the Monty Hall quantum game on a quantum computer \[17\]. Some of the most interesting quantizations schemes are those of Flitney and Abbott \[12\], and D’Ariano et al. \[13\].

In Ref. \[12\], Flitney and Abbott describe a quantization scheme in which the location of the prize, the initially chosen box by the player and the box opened by the host, are each represented by a state in a three dimensional space: \( \mathcal{H}_a \), \( \mathcal{H}_b \) and \( \mathcal{H}_o \) respectively. The full system is initialized in an arbitrary state \( |\psi\rangle \in \mathcal{H} = \mathcal{H}_o \otimes \mathcal{H}_b \otimes \mathcal{H}_a \), a feature that does not have a classical analogy. After this initial preparation, the game begins: the host hides the prize by acting a unitary operator on \( \mathcal{H}_a \), the act of the player choosing a box is also implemented by acting a unitary operator on \( \mathcal{H}_b \) and the opening of a box by the host is performed too by acting a unitary operator on the full space \( \mathcal{H} \). Lastly, the decision of switching between boxes is made by acting a superposition of two operators, a switching operator and a not-switching operator, on the full space \( \mathcal{H} \), allowing a non-classical feature: a superposition of switching and not switching between boxes.

In Ref. \[13\], D’Ariano et al. present a scheme in which they represent the location of the prize as a state in a three dimensional space \( \mathcal{H} \), information about the state is given to the host classically or via measurements on an ancillary system entangled with the prize. The initial choice of the player is in this case, not an operation, but a simple choice of a state \( |p_i\rangle \in \mathcal{H}_p \). The opening of an empty box by the host, is performed as a projective measurement, reducing the space \( \mathcal{H} \) to a two-dimensional space \( \mathcal{H}_p \) spanned by \( |p_i\rangle \) and the prize state. Lastly, the player can choose to stay with her initial choice or to change to a different state \( |p_f\rangle \in \mathcal{H}_p \).

In this work we construct a quantization scheme inspired by a quantum-optical set-up that resembles the classical game. A non-linear crystal allows us to have entanglement between the prize’s location and the initial choice of the player, while the use of polarized beam splitters and polarization rotators enable the superposition feature of both the prize’s location and the initial choice of the player. In our quantization scheme, however, the decision of switching between boxes is binary and based on the classical sample space in table I.

### Proposed Quantization Scheme

The system under study will be modeled by two Hilbert spaces \( \mathcal{H}_a \) and \( \mathcal{H}_b \), each with a respective orthonormal basis \( \{|1_a\rangle, |2_a\rangle, |3_a\rangle\} \) and \( \{|1_b\rangle, |2_b\rangle, |3_b\rangle\} \). States in \( \mathcal{H}_a \) correspond to the initially chosen box by the player (which we will refer to as Alice). Notice that the initial choice of Alice can be a superposition of boxes. On the other hand, states in \( \mathcal{H}_b \) correspond to the prize’s location, initially prepared by the host (which we will refer to as Bob). Analogously, the prize can also be in a superposition of boxes.

The game begins with Bob preparing the prize in a state

\[ |\psi_b\rangle = \sum_{i=1}^{3} b_i |i_b\rangle, \]  

(3)

which Alice will have no information about. She then proceeds to choose a box, i.e. a state in \( \mathcal{H}_a \):

\[ |\psi_a\rangle = \sum_{i=1}^{3} a_i |i_a\rangle. \]  

(4)

Table I. Elements of the sample space of the Monty Hall problem. The prize being in box \( i \) and the player initially choosing box \( j \) is represented as \((i, j)\).

\begin{align*}
(1, 1) & \quad (1, 2) & \quad (1, 3) \\
(2, 1) & \quad (2, 2) & \quad (2, 3) \\
(3, 1) & \quad (3, 2) & \quad (3, 3)
\end{align*}
The state that describes both the prize and Alice’s initial choice is thus
\[ |\psi_b\rangle = |\psi_a\rangle \otimes |\psi_b\rangle \in \mathcal{H}_a \otimes \mathcal{H}_b. \] (5)

Once Alice has chosen a box, Bob has to do something analogous to open one of them. A useful way (for our proposed experimental set-up) of mathematically doing this, is to apply the door-opening operator
\[ \hat{D}_o := \cos(\varphi_1) |1_b\rangle \langle 1_b| + \sin(\varphi_2) |2_b\rangle \langle 2_b| + \sin(\varphi_3) |3_b\rangle \langle 3_b| \] (6)
to \(|\psi_b\rangle\), where \(\varphi_1, \varphi_2, \varphi_3 \in [0, \pi]\). The door-opening operator must also satisfy a two-doors-remained-closed condition, which can be modeled as
\[ \cos^2(\varphi_1) + \sin^2(\varphi_2) + \sin^2(\varphi_3) = 2. \] (7)

There are two main things worth noticing here. The first is that the door-opening operator acts as a projection onto the two-dimensional subspace generated by \(\{|2_b\rangle, |3_b\rangle\}, \{|1_b\rangle, |3_b\rangle\}\) and \(\{|1_b\rangle, |2_b\rangle\}\) when \((\varphi_1 = \frac{\pi}{2}, \varphi_2 = \frac{\pi}{4}, \varphi_3 = \frac{\pi}{2}\), \((\varphi_1 = 0, \varphi_2 = 0, \varphi_3 = \frac{\pi}{2}\) and \((\varphi_1 = \frac{\pi}{2}, \varphi_2 = \frac{\pi}{2}, \varphi_3 = 0\) respectively. The second is that the application of \(\hat{D}_o\) on \(|\psi_b\rangle\) de-normalizes it. Thus, in order to maintain the interpretation of the inner product as a probability amplitude, the resulting state must be renormalized, leading to define
\[ \sum_{i=1}^{3} \beta_i |i_b\rangle = \frac{\hat{D}_o |\psi_b\rangle}{\sqrt{\langle \psi_b | \hat{D}_o \hat{D}_o^\dagger | \psi_b \rangle}}, \] (8)
for some \(\beta_i \in \mathbb{C}\) such that \(|\beta_1| + |\beta_2| + |\beta_3| = 1\). The composite state of Alice’s initial choice and the prize’s location thus becomes
\[ |\psi\rangle = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i \beta_j |i_a, j_b\rangle. \] (9)

In analogy with the classical case, and as it was shown in table I, the states \(\{|1_a, 1_b\}, |2_a, 2_b\}, |3_a, 3_b\}\) correspond to Alice winning with her initial choice, while the states \(\{|1_a, 2_b\}, |1_a, 3_b\}, |2_a, 1_b\}, |2_a, 3_b\}, |3_a, 1_b\}, |3_a, 2_b\}\) correspond to Alice winning by switching her initial choice. Therefore, the probability of Alice winning by not switching and the probability of her winning by switching are respectively
\[ P_{ns} = \sum_{i=1}^{3} |a_i \beta_i|^2, \] (10)
\[ P_{s} = \sum_{i,j=1}^{3} |a_i \beta_j|^2. \] (11)

The classical Monty Hall problem may be thought of using this quantization scheme by restricting Alice and Bob to only use “classical” states \(\{|1_a\}, |2_a\}, |3_a\}\) and \(\{|1_b\}, |2_b\}, |3_b\}\) respectively. Nevertheless, expressions (10) and (11) would only yield the objective quantum probability (i.e. 1 or 0) of Alice winning given her choices and the state of the prize. In order to calculate the subjective classical probability (i.e. the one due to the lack of information from Alice) and obtain the same results as in (1) and (2), one must perform a classical probability calculation (using table I for example).

There is, however, a selection of quantum states by Alice and Bob that actually resembles the classical problem, a semi-classical case: First, Bob prepares the prize in the state with \(b_i = \frac{1}{\sqrt{3}}\) (i.e. equally distributed over the three boxes), then Alice chooses the state with \(a_1 = \frac{1}{\sqrt{3}}\) (i.e. the same confidence in each box) and finally, Bob applies the door-opening operator \(\hat{D}_o\) with \((\varphi_1 = \frac{\pi}{2}, \varphi_2 = \frac{\pi}{4}, \varphi_3 = \frac{\pi}{2}\) or \((\varphi_1 = 0, \varphi_2 = 0, \varphi_3 = \frac{\pi}{2}\) and \((\varphi_1 = \frac{\pi}{2}, \varphi_2 = \frac{\pi}{2}, \varphi_3 = 0\) respectively. In this case, equations (10) and (11) give respectively \(\frac{1}{2}\) and \(\frac{1}{3}\).

The semi-classical case discussed above points out to a classical interpretation of the amplitudes \(a_i\) and \(b_i\) in our scheme, being \(|a_i|^2\) the probability of Alice initially choosing box \(i\) and \(|b_i|^2\) the probability of the prize being placed in box \(i\). This means that our quantization scheme, until now, can be entirely reproduced in classical probability theory by the classical Monty Hall problem or by analyzing a non-symmetric case: the host is more inclined to hide the prize in certain box and the player has some kind of bias towards initially choosing one of the boxes. However, it is worth mentioning that expression (9) assumes \(|\psi_0\rangle\) is a separable state in \(\mathcal{H}_a \otimes \mathcal{H}_b\) (equation (5)). This is a fully classical restriction, one we do not have to follow in the quantum realm. For a general state
\[ |\psi\rangle = \sum_{i,j=1}^{3} \gamma_{ij} |i_a, j_b\rangle \in \mathcal{H}_a \otimes \mathcal{H}_b, \] (12)
the probability of Alice winning by not switching and the probability of her winning by switching are respectively
\[ P_{c,ns} = \sum_{i=1}^{3} |\gamma_{ii}|^2, \] (13)
\[ P_{c,s} = \sum_{i,j=1}^{3} |\gamma_{ij}|^2. \] (14)
The consideration of entanglement in equation [12] between the prize’s location and the player’s initial choice would make the game unfair. It is here considered for the purpose of extending our framework to construct a secure communication protocol, as will be discussed below.

In the next section we describe the quantum-optical set-up in which our quantization scheme is inspired, and use it to analyze both a separable and an entangled initial state $|φ₀⟩$.

### Experimental Realization

Figure [1] shows a diagram of a quantum-optical approach to an experimental realization of our proposed quantization scheme for the Monty Hall problem. In it, detectors A1, A2 and A3 represent Alice’s initial choice of a box, while detectors B1, B2 and B3 represent the boxes where the prize is hidden.

The game starts with a pair of polarization-entangled photons produced via spontaneous parametric down-conversion in a Beta Barium Borate (BBO) type II crystal. Photon A, represented by the vertical output of the BBO crystal in the diagram, corresponds to the system available to Alice and modeled in $H_a$. While photon B, represented by the horizontal output of the BBO crystal in the diagram, corresponds to the system available to Bob and modeled in $H_b$. The state of the composite system at this stage is given by the entangled state

$$|φ₀⟩ = \frac{1}{\sqrt{2}} (|V₁, H₁⟩ + |H₁, V₁⟩) ∈ H_a ⊗ H_b,$$  \hspace{1cm} (15)

where $V$ and $H$ stand for vertical and horizontal polarization respectively, and the subindex 1 represents the label of the detector to which the photon is heading ($A₁$ for photon A and $B₁$ for photon B).

Firstly, a vertical polarizer (Pol V) and an horizontal polarizer (Pol H) are respectively placed in photon A’s and photon B’s path. These polarizers allow to control entanglement between the two subsystems. If the polarizers are present, the entanglement between photons A and B is lost, as the state $|φ₀⟩$ reduces to $|V₁, H₁⟩$. On the other hand, if the polarizers are removed, then the system remains entangled in the state $|φ₀⟩$.

Both entangled and non-entangled cases are analyzed, but for the sake of simplicity, let us describe just the case where the polarizers are placed and the initial state of the system is described by

$$|φ₀⟩ = |V₁, H₁⟩.$$ \hspace{1cm} (16)

In order to obtain the expressions corresponding to the entangled case, the operations we will be describing must also be applied to the term $|H₁, V₁⟩$.

The next device placed in the photons path is a polarization rotator ($\text{Rot}(θ)$) with rotation angles $θ_{a₁}$ and $θ_{b₁}$ for photon A and photon B respectively. By choosing an angle $θ_{b₁}$, Bob is fixing the probability amplitude $b₃$ in [3]. Analogously, by choosing an angle $θ_{a₁}$, Alice is fixing the probability amplitude $a₃$ in eq. (4).

The polarization-rotator linear operator performs the operations

$$\text{Rot}(θ) |V⟩ = \cos θ |V⟩ − \sin θ |H⟩$$ \hspace{1cm} (17)

$$\text{Rot}(θ) |H⟩ = \cos θ |H⟩ + \sin θ |V⟩,$$ \hspace{1cm} (18)

changing the initial state $|φ₀⟩$ to

$$|φ₁⟩ = (\cos θ_{a₁} |V₁⟩ − \sin θ_{a₁} |H₁⟩)$$

$$\otimes (\cos θ_{b₁} |H₁⟩ + \sin θ_{b₁} |V₁⟩).$$ \hspace{1cm} (19)

Next, each of the photons encounters a polarized beam splitter (PBS), positioned to reflect the vertical component of the polarization and transmit the horizontal one. These first PBS’s reflect the vertical component of the polarization of photons A and B towards detectors A3 and B3 respectively, performing the operation

$$\text{PBS} (α |V₁⟩ + β |H₁⟩) = α |V₃⟩ + β |H₁⟩.$$ \hspace{1cm} (20)

The state of the system after these first PBS’s is then

$$|φ₂⟩ = (\cos θ_{a₁} |V₃⟩ − \sin θ_{a₁} |H₁⟩)$$

$$\otimes (\cos θ_{b₁} |H₁⟩ + \sin θ_{b₁} |V₃⟩).$$ \hspace{1cm} (21)

In order for Alice and Bob to fix amplitudes $a₁$, $a₂$ in [4] and $b₁$, $b₂$ in [5], another polarization rotators with angles $θ_{a₂}$ and $θ_{b₂}$ are respectively placed in the path of the horizontal component of photons A and B. Applying the polarization-rotator operator (equations (17) and (22)) to state $|φ₂⟩$ with angles $θ_{a₂}$ and $θ_{b₂}$, the state of the system becomes

$$|φ₃⟩ = (\cos θ_{a₁} |V₃⟩ − \sin θ_{a₁} \sin θ_{a₂} |V₁⟩$$

$$− \sin θ_{a₁} \cos θ_{a₂} |H₁⟩) \otimes (\cos θ_{b₁} |V₃⟩ + \sin θ_{b₁} \cos θ_{b₂} |H₁⟩).$$ \hspace{1cm} (22)

Then, a polarized beam splitter (PBS) is placed in both photons paths. These last PBS’s are positioned to reflect the vertical component of the polarization towards detectors A2 and B2. So the state of the system becomes

$$|φ₄⟩ = (\cos θ_{a₁} |V₄⟩ − \sin θ_{a₁} \sin θ_{a₂} |V₂⟩$$

$$− \sin θ_{a₁} \cos θ_{a₂} |H₁⟩) \otimes (\cos θ_{b₁} |V₃⟩ + \sin θ_{b₁} \cos θ_{b₂} |H₁⟩).$$ \hspace{1cm} (23)

Notice equation (3) is analogous to equation (23) with
Figure 1. Diagram of the experimental set-up proposed for the Monty Hall problem. The BBO type II crystal produces a pair of entangled photons in both position and polarization. The vertical arm of the set-up corresponds to Alice’s subsystem, while the horizontal one corresponds to Bob’s. “Pol V” and “Pol H” represent vertical and horizontal polarizers respectively. “Rot(θ)” represents a polarization rotator of an angle θ, “PBS” represents a polarized beam splitter, “Pol H(ϕ)” represents a polarizer at an angle ϕ respect to the horizontal polarization. Detectors A1, A2 and A3 correspond to Alice’s initial choice of a box, while the detectors B1, B2 and B3 correspond to the doors in which the prize is prepared.

\[ a_1 = -\sin \theta_1 \cos \theta_2, \]  
\[ a_2 = -\sin \theta_1 \sin \theta_2, \]  
\[ a_3 = \cos \theta_1, \]  
\[ b_1 = \cos \theta_1 \cos \theta_2, \]  
\[ b_2 = \cos \theta_1 \sin \theta_2, \]  
\[ b_3 = \sin \theta_1. \]  

The process described so far has been to prepare the state of the system as in expression (5). The next step in the Monty Hall problem is for Bob to open one box. In order to model this door-opening procedure in our experimental set-up, three polarizers (Pol H(ϕ_i)) at angles ϕ_i respect to the horizontal polarization, are placed before the detectors B_i. Strictly speaking, a polarizer performs a dissipative operation, since it absorbs part of the radiation that arrives to it. However, we are only interested in photons that account for coincidences between detectors A and B, i.e. photons that do reach detectors B1, B2 or B3. This is because these photons are the ones that allow us to measure the probabilities in an analogous way as we did classically with table I. Therefore, we can include the effect of the polarizers by changing the probability amplitudes \( b_i \) in Bob’s subsystem as

\[ b'_1 = b_1 \cos \varphi_1, \]  
\[ b'_2 = b_2 \sin \varphi_2, \]  
\[ b'_3 = b_3 \sin \varphi_3, \]

which is equivalent to applying the door-opening operator (6) to Bob’s subsystem, using the two-doors-remained-closed condition expressed in (7) as we only need to block one third of the photons.

Then, as it was described in the previous section, we must renormalize the state, which is experimentally justified by the fact that we are calculating the probability of measuring the state \( \mid i_a, j_b \rangle \) as

\[ P_{i,j} = \frac{C_{i,j}}{\sum_{m,n=1}^{3} C_{m,n}}, \]

where \( C_{m,n} \) stands for the number of coincidences between detectors Am and Bn.
Renormalization of the state leads to the new probability amplitudes $\beta_i$ (defined in expression \(8\)) for Bob’s subsystem:

$$\beta_i = \frac{b'_i}{\sqrt{(b'_1)^2 + (b'_2)^2 + (b'_3)^2}}. \quad (34)$$

In the case when the first polarizers (Pol V and Pol H) are removed, and the initial state $|\phi_0\rangle$ is as in equation \(15\), the non-normalized amplitudes with respect to the basis $\{|i_a, j_b\rangle\}_{j=1,2,3}$; after applying the door-opening operator to the resulting state, turn out to be

$$c_{11} = \frac{-1}{\sqrt{2}} \cos \theta_{a2} \cos \theta_{b2} \sin (\theta_{a1} + \theta_{b1}) \cos \varphi_1,$$

$$c_{12} = \frac{-1}{\sqrt{2}} \cos \theta_{a2} \sin \theta_{b2} \sin (\theta_{a1} + \theta_{b1}) \sin \varphi_2,$$

$$c_{13} = \frac{-1}{\sqrt{2}} \cos \theta_{a2} \cos \theta_{b2} \sin (\theta_{a1} + \theta_{b1}) \sin \varphi_3,$$

$$c_{21} = \frac{-1}{\sqrt{2}} \sin \theta_{a2} \cos \theta_{b2} \sin (\theta_{a1} + \theta_{b1}) \cos \varphi_1,$$

$$c_{22} = \frac{-1}{\sqrt{2}} \sin \theta_{a2} \sin \theta_{b2} \sin (\theta_{a1} + \theta_{b1}) \sin \varphi_2,$$

$$c_{23} = \frac{1}{\sqrt{2}} \sin \theta_{a2} \cos (\theta_{a1} + \theta_{b1}) \sin \varphi_3,$$

$$c_{31} = \frac{1}{\sqrt{2}} \cos \theta_{b2} \cos (\theta_{a1} + \theta_{b1}) \cos \varphi_1,$$

$$c_{32} = \frac{1}{\sqrt{2}} \sin \theta_{b2} \cos (\theta_{a1} + \theta_{b1}) \sin \varphi_2,$$

$$c_{33} = \frac{1}{\sqrt{2}} \sin (\theta_{a1} + \theta_{b1}) \sin \varphi_3.$$  \(43\)

Thus, the amplitudes $\gamma_{ij}$ necessary to calculate $P_{e,ns}$ and $P_{e,s}$ as in equations \(13\) and \(14\) respectively, are

$$\gamma_{ij} = \frac{c_{ij}}{\sqrt{\sum_{i,j=1}^{3} |c_{ij}|^2}}. \quad (44)$$

**Results**

In this section we analyze the expected payoff of Alice (the player) from a frequentist perspective. Namely, if the Monty Hall experimental set-up is played multiple times, does Alice have a better chance of winning the game by not switching (bet for a coincidence between detectors A1/B1, A2/B2 and A3/B3) or by switching (bet for a coincidence between detectors A1/B2, A1/B3, A2/B1, A2/B3, A3/B1 and A3/B2)? We answer this question using two different approaches: random and strategy-based.

**Random game**

In the random approach, the parameters of the experiment are considered as random variables with a constant joint probability density function $\rho$. Angles $\theta_{a1}$, $\theta_{a2}$, $\theta_{b1}$ and $\theta_{b2}$ are considered as independent random variables, while the angles associated with the door-opening operator, $\varphi_1$, $\varphi_2$ and $\varphi_3$, are considered as random variables subject to the two-doors-remained-closed condition \(7\), which restricts its values to the region

$$0 < \cos \varphi_1 < 1,$$

$$\sin \varphi_1 < \sin \varphi_2 < 1,$$

$$\sin \varphi_3 = \sqrt{\sin^2 \varphi_1 + \cos^2 \varphi_2},$$

defining the joint probability density function $\rho$ through

$$\frac{1}{\rho} = \frac{128}{\pi^6}.$$  \(49\)

The probabilities in equations \(10\), \(11\), \(13\) and \(14\), are then functions of these random variables. We use their expectation values with respect to the probability density function $\rho$, as the expected payoff of Alice in each case.

When entanglement is not considered, as in the state described in eq. \(15\), the expectation values of the probability of winning by not switching and by switching, are respectively

$$\langle P_{e,ns} \rangle_{ran} \approx 0.3664, \quad (50)$$

$$\langle P_{e,s} \rangle_{ran} \approx 0.6336. \quad (51)$$

Analogously, when entanglement is considered, as in the state described in eq. \(15\), the expectation values of the probability of winning by not switching and by switching, are respectively

$$\langle P_{e,ns} \rangle_{ran} \approx 0.5083, \quad (52)$$

$$\langle P_{e,s} \rangle_{ran} \approx 0.4917. \quad (53)$$

**Strategy-based game**

In the strategy-based approach, the angles $\theta_{a1}$, $\theta_{a2}$, $\theta_{b1}$ and $\theta_{b2}$ are also considered as independent (in analogy with the classical game) random variables. However, the door-opening parameters, $\varphi_1$, $\varphi_2$ and $\varphi_3$, are left free to be tuned in favor of a certain strategy from Bob (the
In this approach, the joint probability density function is just

$$\varrho = \frac{16}{\pi^4}. \quad (54)$$

making the expectation values of the probabilities in equations (10), (11), (13) and (14), to be functions of the door-opening parameters \(\varphi_1\) and \(\varphi_2\).

Figures 2 and 3 show respectively the graphics of the expectation values of the probability of Alice winning by switching and by not switching, as functions of \(\varphi_1\) and \(\varphi_2\), for both the entangled \((\langle P_{e,s} \rangle, \langle P_{e,ns} \rangle)\) and non-entangled cases \((\langle P_s \rangle, \langle P_{ns} \rangle)\).

Table II shows the approximate maximum and minimum values of \(\langle P_s \rangle, \langle P_{ns} \rangle, \langle P_{e,s} \rangle\) and \(\langle P_{e,ns} \rangle\) along with the corresponding \(\varphi_1\) and \(\varphi_2\) where that maximum or minimum is reached.

Table II. Numerically obtained maximum and minimum values of \(\langle P_s \rangle, \langle P_{ns} \rangle, \langle P_{e,s} \rangle\) and \(\langle P_{e,ns} \rangle\) along with the corresponding \(\varphi_1\) and \(\varphi_2\) where that maximum or minimum is reached.

| | \(\varphi_1\) | \(\varphi_2\) | \(\varphi_1\) | \(\varphi_2\) |
|---|---|---|---|---|
| \(\langle P_s \rangle\) | \(\frac{\pi}{2}\) | \(\frac{\pi}{2}\) | \(0\) | \(\frac{\pi}{2}\) | 0.5908 | 0.75 |
| \(\langle P_{ns} \rangle\) | \(0\) | \(\frac{\pi}{2}\) | \(0\) | \(0\) | 0.25 | 0.4092 |
| \(\langle P_{e,s} \rangle\) | \(\frac{\pi}{2}\) | \(\frac{\pi}{2}\) | \(0\) | \(\frac{\pi}{2}\) | 0.4003 | 0.6487 |
| \(\langle P_{e,ns} \rangle\) | \(0\) | \(\frac{\pi}{2}\) | \(0\) | \(0\) | 0.3513 | 0.5997 |

In the classical Monty Hall problem, the host, opening one of the empty boxes, helps the player by creating an imbalance between the probabilities of winning by switching and by not switching, allowing her to make a rational decision. Motivated by this, we define the absolute value of the difference between \(\langle P_{ns} \rangle\) and \(\langle P_s \rangle\) for the non-entangled case as
Figure 4. Absolute value of the difference between expectation values of the probabilities of Alice winning by not switching and by switching, as a function of the door-opening parameters $\varphi_1$ and $\varphi_2$. **Left:** Non-entangled-case. **Right:** Entangled case.

\[
P_{\text{abs}} = |\langle P_{\text{ns}} \rangle - \langle P_s \rangle|.
\]

(55)

Analogously, we also define the absolute value of the difference between $\langle P_{e,\text{ns}} \rangle$ and $\langle P_{e,s} \rangle$ for the entangled case as

\[
P_{e,\text{abs}} = |\langle P_{e,\text{ns}} \rangle - \langle P_{e,s} \rangle|.
\]

(56)

Figure 4 shows the graphics of $P_{\text{abs}}$ and $P_{e,\text{abs}}$ as functions of $\varphi_1$ and $\varphi_2$, while Table III shows their approximate maximum and minimum values as well as the approximate corresponding $\varphi_1$ and $\varphi_2$ where that maximum or minimum is reached. Notice from this table that it is Bob who can use entanglement to his advantage, either to help or to affect Alice.

|       | $\varphi_1$ | $\varphi_2$ | min  | $\varphi_1$ | $\varphi_2$ | max  |
|-------|-------------|-------------|------|-------------|-------------|------|
| $P_{\text{abs}}$ | $\frac{\pi}{2}$ | $\frac{\pi}{2}$ | 0.1817 | 0 | $\frac{\pi}{2}$ | 0.5 |
| $P_{e,\text{abs}}$ | $\frac{\pi}{6}$ | $\frac{\pi}{6}$ | 0.0001 | 0 | $\frac{\pi}{2}$ | 0.2973 |

Table III. Numerically obtained maximum and minimum values of $P_{\text{abs}}$ and $P_{e,\text{abs}}$ along with the corresponding $\varphi_1$ and $\varphi_2$ where that maximum or minimum is reached.

The experimental realization here proposed, along with the obtained results, may be helpful to construct a secure classical communication protocol by considering the cases of Alice winning by switching and by not switching as classical bits. Particularly, when entanglement is considered, the random approach of the game shows an almost exact balance between the expectation values of the switching and not-switching probabilities, so that any deviation from the predicted balance would mean an eavesdrop in the communication channel. This approach would not need any kind of active selection of the parameters, as these would be randomly chosen in the devices (both subsystems of the experimental set-up).

The strategy-based approach may be an even more secure option, as Bob’s choice of helping Alice or affecting her could be randomly selected for each pair of photons, a feature that could be exploited to share a private key between Alice and Bob, filtering the desired results afterwards in an information-reconciliation stage. It is worth mentioning, however, that this shared private key will not be encoded in quantum bits, but in classical ones (the photon detection signal), even though the ability to share it is due to the entanglement between the subsystems.

**Discussion and Conclusions**

In the random approach, when no entanglement is considered, the expectation values of the probabilities of Alice (the player) winning by not switching and by switching, given by equations (50) and (51) respectively, differ from the classical probabilities by just 0.033, concluding that, in average, Alice has a better chance of winning by switching, approximately 1.73 times better, a slightly worse result than in the classical case. When entanglement is considered, as in equations (52) and (53), the results show the opposite conclusion. In this case, Alice has a better chance of winning by not switching, but just approximately 1.03 times better, meaning that this kind of correlation between the prize’s location and Alice’s initial choice, is actually bad for Alice, not allowing her to make a switching choice as meaningful as in the classical case or the quantum non-entangled case.

In the strategy-based approach, Bob (the host) can freely choose the door-opening parameters, allowing him to increase or decrease (in average) the chances of Alice winning by switching and by not switching. From the results presented in tables II and III when entanglement is
not considered, we notice that Bob can increase the imbalance between the switching and not-switching cases to a maximum of 0.5, with \(\langle P_s \rangle = 0.75\) and \(\langle P_{ns} \rangle = 0.25\), making the switching decision three times better than the not-switching one, being this the best strategy possible, in average, if Bob wishes to help Alice win the prize. When entanglement is considered and if Bob wants again to help Alice, he can increase the imbalance between the switching and not-switching cases to a maximum of 0.2973, with \(\langle P_{e,s} \rangle = 0.6487\) and \(\langle P_{e,ns} \rangle = 0.3513\), making the switching decision approximately 1.85 times better than the not-switching one, a very similar result to the classical case.

If Bob does not want Alice to win the prize, his strategy would be to minimize the imbalance between the switching and not-switching cases, decreasing the advantage that she gets from it. Results in tables II and III show that, when entanglement is not considered, Bob can decrease this imbalance to a minimum of 0.1817 with \(\langle P_s \rangle = 0.5908\) and \(\langle P_{ns} \rangle = 0.4092\), making the switching decision 1.4438 times better than the not-switching one, still allowing Alice to make a rational choice. However, when entanglement is considered, Bob can decrease the imbalance to a minimum of 0 at both \(\langle P_{e,s} \rangle\) and \(\langle P_{e,ns} \rangle\) have the same value of 0.5 at approximately \(\varphi_1 = \frac{\pi}{20}\) and \(\varphi_2 = \frac{\pi}{4}\), leaving Alice with no other option but to randomly decide whether she bets for switching or not switching.

In conclusion, we have presented a quantization scheme of the Monty Hall problem, based on a quantum-optical set-up that is experimentally feasible. This set-up allows for a secure communication protocol, which exploits the entanglement between the host and the player choices either in a random or in a strategy-based scenario.

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