CANONICAL BASES FOR THE QUANTUM ENVELOPING ALGEBRA OF
\( \mathfrak{gl}(m|1) \) AND ITS MODULES

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Abstract. We construct a crystal basis for the negative half of the quantum enveloping algebra \( \mathcal{U} \) associated to the standard super Cartan datum of \( \mathfrak{gl}(m|1) \), which is compatible with known crystals on Kac modules and simple modules. We show that these crystals admit globalization which produce compatible canonical bases. We then define a braid group action on a family of quantum enveloping algebras including \( \mathcal{U} \), and use this action to show that our canonical basis agrees with those constructed from PBW bases. Finally, we explicitly compute some small-rank examples.

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1. Introduction

1.1. A feature of quantum algebras which has taken on increasing importance is the construction of canonical bases; that is, bases which arise from the algebra in a natural way and which have desirable compatibilities with other features of the algebra. Two standard (and related) examples of such bases are the Kazhdan-Lusztig basis of a Iwahori-Hecke algebra [KL] and the Lusztig-Kashiwara canonical bases of quantum enveloping algebra of a Kac-Moody algebra [Kas, Lus1].

These canonical bases are remarkable in many ways, but one of the most striking is their connection to categorification: canonical bases tend to predict categorifications, and vice versa. Moreover, such categorifications often reveal the canonical basis elements to encode a wealth of geometric data (e.g. Lusztig’s construction of the canonical basis of a quantum enveloping algebra via perverse sheaves, cf. [Lus2, Part 2]) and representation-theoretic data (e.g. as the decategorification of an indecomposable projective module over a KLR algebra, cf. [VV]).

One natural setting to look to construct further examples of canonical bases is the quantum enveloping algebras of Lie superalgebras. These quantum enveloping Lie superalgebras have been gaining significant interest in the years since the foundational work of Yamane [Yam]. For instance, crystal structures have been constructed and studied on their modules [BKK, Kwo1, Kwo2, Kwo3, MZ] and some examples of categorifications have been constructed [Kho, KS, Sar, Tia]. However, despite this attention, it is not known if canonical bases can be defined for many of these quantum algebras or their modules.

Recently, there has been some success in constructing canonical bases, at least on a case-by-case level. Indeed, crystal and canonical bases for some quantum enveloping superalgebras associated to a family of Kac-Moody Lie superalgebras, of which the only finite-type Lie superalgebra is \(osp(1|2n)\), have been constructed by the author in joint work with Hill and Wang [CHW1, CHW2]; this construction is expected to play a role in a higher-representation-theoretic construction of “odd” knot homologies [HW, EL, C].

In another direction, using an approach à la [Lec], it was shown in [CHW3] that any quantum enveloping Lie superalgebra of basic type admits a construction of PBW bases using the combinatorics of Lyndon words and shuffle products. For a few families of Lie superalgebras, including the quantum enveloping algebra of the general linear Lie superalgebra \(U_q(gl(m|n))\), these PBW bases are shown to lead to bar-invariant bases of the half-quantum enveloping algebra \(U_q(gl(m|n))\) associate to the standard positive root system by a standard argument. (We note that [DG] also constructed a bar-invariant basis in the general linear case, using connections to quantum Schur superalgebras.)

1.2. It is natural to want to construct canonical bases for \(U_q(gl(m|n))\), as it is, in many ways, the fundamental example of quantum enveloping superalgebras. In particular, one already encounters many of the structural features complicating such a construction.

Arguably the most fundamental change from the non-super theory is the existence of several non-conjugate bases for the root system in general. This, in turn, leads to non-isomorphic half-quantum enveloping algebras (compare Sections 5.3 and 5.3) with sometimes complicated
defining relations; see [CHW3, Proposition 2.7] for a description of these relations. While the full quantum enveloping algebras are isomorphic (which is not obvious a priori, but, for example, follows from the results of Section 4), one must be careful to indicate which positive root system, and thus half-quantum enveloping algebra, associated to \( \mathfrak{gl}(m|n) \) is being considered. For the most part, we will follow the literature (cf. [BKK, Kwo2, KS]) and work with the standard positive system associated to the Borel subalgebra of upper-triangular matrices. Henceforth, unless otherwise stated, we write \( U_q(\mathfrak{gl}(m|n)) \) (respectively, \( U_q^-(\mathfrak{gl}(m|n)) \)) to mean the (negative part of the) quantum enveloping algebra defined with respect to the standard simple system.

Working over the standard simple system, the results of [CHW3] imply the existence of a bar-invariant basis of \( U_q^-(\mathfrak{gl}(m|n)) \) associated to the standard PBW basis (and indeed, there is a such a bar-invariant basis associated to any PBW basis by a slight generalization of the arguments therein). While these bases satisfies one of the characteristic properties of canonical bases, they fundamentally depend on the choice of PBW basis in general. In essence, this dependence arises because the underlying super-space structure induces a chirality in the quantum parameter \( q \), in the sense that the even subspace having the parameter \( q \) means the odd subspace should have the parameter \( q^{-1} \). In particular, the even subalgebra of \( U_q(\mathfrak{gl}(m|n)) \) is

\[
U_q(\mathfrak{gl}(m|0) \oplus \mathfrak{gl}(0|n)) \cong U_q(\mathfrak{gl}(m)) \otimes U_q^{-1}(\mathfrak{gl}(n)).
\]

This chirality is of fundamental importance when considering the bilinear form and PBW basis (see Example 2.7 for a brief discussion of this phenomenon). This chirality essentially disappears in the case \( m = 1 \) or \( n = 1 \), in which case the bar-invariant bases are also almost-orthogonal (in the sense of Lusztig) and thus the bases agree up to possible sign changes.

The last main complication is that the category of finite-dimensional representations is not semisimple; this causes difficulties with constructing crystal bases of representations, which is one way to approach constructing canonical bases. This can be resolved by restricting to the subcategory of polynomial representations, which is semisimple (cf. [CW, Section 3.2.6] for a proof in the classical case). In fact, the highest weights of irreducible polynomial modules correspond to hook partitions, which means there is a natural combinatorial model for constructing crystals on these modules; see [BKK]. One can also produce crystals on Kac modules [Kwo2]; that is, modules induced from the simple modules of the even subalgebra. However, one should note that in both of these papers, it is crucial that one works with the standard system of simple roots. Indeed, it is observed in [Kwo1] that the combinatorial crystals associated to indecomposable modules are, in general, no longer connected when there is more than one isotropic simple root, which raises substantial difficulties in constructing a crystal basis for the module.

1.3. The main goals of this paper is as follows. We want to construct a canonical (unsigned) basis for the half-quantum enveloping algebra \( U^- \), and show that this canonical basis induces compatible canonical bases on the Kac modules and simple polynomial modules, resolving a conjecture in [CIHW3]. Furthermore, we want to show that this canonical basis is in fact precisely equal to the bar-invariant basis of \( U^- \) constructed in loc. cit. as described above. Our last objective is to get a sense of to what extent the canonical basis is compatible with the simple modules whose highest weight is not polynomial.
To accomplish these goals, we begin by constructing a crystal basis on $U^\sim$. A version of a crystal lattice for $U^\sim$ has been constructed before by Zou [Zou], which is essentially defined using only the $\mathfrak{gl}(m)$-crystal operators and the odd root vectors. However, such a lattice is not closed under all the Kashiwara operators. We will show that under the usual definition of the crystal lattice (i.e. the lattice generated from 1 by all the $\tilde{f}_i$), the lattice is indeed closed under all the Kashiwara operators. See Section 5.1 for a comparison of our lattice and the one in loc. cit. in the $m = 2$ case.

Moreover, we show that this crystal structure is compatible with the crystals on the Kac modules (defined in [Kwo2]) and simple polynomial modules (defined in [BKK]). The latter statement follows naturally from the construction of the lattice, which is a version of Kashiwara’s “Grand Loop” restricted to the statements involving $U^\sim$. Compatibility with Kac modules then follows from the $\mathfrak{gl}(m)$-crystal structure on $U^\sim$ and the definition of the crystal on a Kac module.

We then study some of the implications of this crystal structure, deducing analogues of Kashiwara’s characterizations of the crystal lattice and crystal basis, as well as defining an integral version of the lattice. It follows from these statements and the construction in [CHW3] that the PBW basis, and thus the associated bar-invariant basis, lies in the integral lattice, and that each PBW element (and thus associated bar-invariant basis element) is equivalent modulo $q$ to a crystal basis element, up to a sign. This immediately implies the existence of a globalization as in [Kas, §7], from which we deduce the following theorem.

**Theorem A.** The algebra $U^\sim$ admits a crystal basis $(L(\infty), B(\infty))$. Furthermore, there exists a globalization $G : L(\infty)/qL(\infty) \to \mathfrak{U}^\sim \cap L(\infty) \cap \overline{L}(\infty)$ such that $B = G(B(\infty))$ is a canonical bar-invariant basis of $U^\sim$. Moreover, let $V$ be a simple finite-dimensional polynomial module or a finite-dimensional Kac module with highest weight vector $v$. Then $B(V) = \{ b \in B \mid bv \neq 0 \}$ maps bijectively to a bar-invariant basis of $V$.

As a consequence of the proof of this theorem, we see that $B$ is equal (up to signs) with the bar-invariant bases constructed in [CHW3]. In fact, these bases coincide. The key to proving this in the non-super setting is to utilize Lusztig’s braid automorphisms of the quantum enveloping algebra [Lus1, Sai]. These automorphisms provide a systematic way to construct and compare different PBW bases, and then it is straightforward to show that the PBW bases coincide modulo $q$ to the crystal basis; see [Tin] for a concise exposition of these ideas.

To employ this strategy in our setting, we need braid automorphisms for each simple root. However, one quickly runs into an obvious obstruction for isotropic roots: any reasonable definition of the automorphism would send a non-nilpotent generator to a nilpotent root vector. That such an obstruction occurs is not surprising, as the braid automorphisms are essentially a lift of the Weyl group action on the Cartan part, and there is no reflection in the Weyl group corresponding to isotropic roots. Instead, we should attempt to lift the Weyl groupoid [Ser, HY]; an enlargement of the Weyl group which allows for “odd reflections” at the cost of no longer being a group. This idea has been utilized in [Hec], to produce braid operators for a wide variety of quantum algebras.
In our case, we consider the family of quantum enveloping algebras associated to each possible
generalized Cartan matrix of \( \mathfrak{gl}(m|1) \). These quantum enveloping algebras have braid automor-
phisms corresponding to each of their even simple roots, whereas the odd simple roots yield
isomorphisms between pairs of these algebras. Despite this extra layer of complexity, these
maps still satisfy the type \( A \) braid relations, and we can use them to define PBW bases on \( \mathcal{U}^- \)
which coincide with the bases defined in \([CHW3]\). Then we can use the standard arguments to
prove the following theorem.

**Theorem B.** Let \( < \) be a total order on the simple roots of \( \mathfrak{gl}(m|1) \), and let \( \mathcal{B}(<) \) be the
associated canonical basis on \( \mathcal{U}^- \) as defined in \([CHW3]\). Then \( \mathcal{B}(<) = \mathcal{B} \).

As a consequence, we note that Theorem A and Theorem B proves \([CHW3, Conjecture 8.9]\).
This conjecture was already proven in the polynomial module case by Du and Gu \([DG]\). We
note that our proof improves on the conjecture: not only does the canonical basis of \( \mathcal{U}^- \) induce
canonical bases on the polynomial and Kac modules, but in fact this canonical basis also agrees
with the crystal bases of \([BKK, Kwo2]\).

We finish this paper with a few small examples. These examples are motivated by a desire to
better understand two questions which arise naturally from Theorems A and B. First, there are
finite-dimensional simple modules which are not included in Theorem A, and thus we would like
to understand their compatibility, or lack thereof, with \( \mathcal{B} \). Second, since Theorems A and B are
only proven for \( \mathcal{U}_q^- (\mathfrak{gl}(m|1)) \) associated to the standard positive root system, a natural question
is whether or not these arguments can be similarly applied to the half-quantum enveloping
algebras associated to other positive root systems.

To gain insight into the first question, we consider the canonical basis constructions for
\( \mathcal{U}_q^- (\mathfrak{gl}(2|1)) \) and its finite-dimensional irreducible modules from \([CHW3, Section 7]\) from the
viewpoint of crystal bases. We also explicitly construct the canonical basis for \( \mathcal{U}_q^- (\mathfrak{gl}(3|1)) \), and
compute the images of this basis in some examples of atypical simple modules. In all examples
we compute, we observe that there is indeed an induced canonical basis, despite some cases
having linear dependencies in the image of \( \mathcal{B} \).

For the second question, we consider the case of the non-standard Cartan datum for \( \mathfrak{gl}(2|1) \)
with two isotropic roots in some detail. This case is not included in the statements of Theorems
A and B, and we comment on how in particular the results going into Theorem B fail in this
setting. Nevertheless, we observe that the negative part of the quantum enveloping algebra
still has a sensible definition for a canonical basis which satisfies some compatibility with its
representations.

1.4. Given the results of our paper in combination with those of \([C, CHW2, CHW3, Kwo1]\),
there are many obvious and interesting lines of study to be pursued. We hope that our results
and explicit examples can help to develop further categorification results along the lines of
\([KS]\). In particular, it would be interesting to interpret our canonical basis in terms of the
categorification of loc. cit., and our canonical basis on modules suggests that one look for
categorical representations as well.
We also expect that our construction can be extended to the entire (idempotented) quantum enveloping algebra in the manner of Lusztig (cf. [Lus2, Part IV], and see [EK] for a Schur superalgebras prototype of $\mathcal{U}_q(\mathfrak{gl}(m|n))$), and plan to pursue this in subsequent work. In particular, this should admit a diagrammatic categorification in the sense of Khovanov-Lauda [KL] by building on the construction in [KS]. We note that a geometrically motivated categorification of the idempotented algebra for $\mathfrak{gl}(1|1)$ was constructed in [Tia].

Unfortunately, it is still unclear to us how to uniformly approach the construction of canonical bases for Lie superalgebras at this time. Nevertheless, an ad-hoc approach to constructing canonical bases in specific examples certainly seems viable and should help to clarify the general situation. To that end, the fundamental case to study is the standard datum associated to $\mathfrak{gl}(2|2)$; we make some comments on this in Example 2.7, but it deserves further study. Another further direction would be to try and apply similar techniques in other quantum enveloping algebras of basic Lie superalgebras; for instance, the family $\mathfrak{osp}(2|2n)$ is one of the families for which a signed canonical basis was produced in [CHW3], and one should be able to remove the signs in a similar fashion.

Alternatively, one can hope to find canonical bases associated to non-standard Cartan data for $\mathfrak{gl}(m|n)$. For instance, Section 5.3 addresses this for the non-standard datum for $\mathfrak{gl}(2|1)$, but we plan to extend this work to some higher rank cases in the future. We also expect that these algebras should admit categorifications.

1.5. This paper is organized as follows.

In Section 2, we set our notations and conventions, and recall the definition of the quantum enveloping algebra $\mathcal{U}$. We also recall, and in some cases elaborate on, the results on crystal bases of various $\mathcal{U}$-modules.

In Section 3, we introduce our definition of the crystal lattice and basis for $\mathcal{U}^-$. We then state and prove a truncated version of Kashiwara’s “grand loop” induction argument to construct the crystal basis for $\mathcal{U}^-$ by using the crystals for modules described in [BKK]. Subsequently, we prove that the crystal is characterized by the bilinear form on $\mathcal{U}^-$, and satisfies compatibility with Kwon’s crystal bases of Kac modules. Finally, we introduce an integral form of the lattice on each level ($\mathcal{U}^-$, the Kac modules, and the simple modules), and use a (signed) canonical basis from [CHW3] to quickly deduce the existence of the globalization.

In Section 4, we introduce a family of quantum enveloping algebras associated to $\mathfrak{gl}(m|1)$, which correspond to different choices of Dynkin diagram. We then define braid isomorphisms between these quantum groups which lift the Weyl groupoid action on the simple systems of roots, and use these braid isomorphisms to construct PBW bases, which agree with the PBW bases defined in [CHW3]. In the special case of the standard system, it is shown that these PBW bases span the same $\mathbb{Z}[[q]]$-lattice, and thus that the corresponding canonical bases coincide.

Finally, in Section 5, we compute some small-rank examples. First, we consider the case $m = 2$ and compute the crystal lattice in terms of the canonical basis as defined in [CHW3]; we also compare our crystal lattice to that of [Zou], and define a compatible crystal structure on the atypical finite-dimensional simple modules. Second, we consider the case $m = 3$ and compute the canonical basis. We then consider some examples of atypical modules, and observe
some additional instances of canonical bases in these examples. Finally, we discuss the case of
the unique non-standard simple system of rank 2, and how it is an obstruction to a more general
application of the results of Sections 3 and 4. We observe that there is nevertheless a natural
candidate for canonical basis in this case, and analyze its compatibility with modules.

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2. Standard quantum $\mathfrak{gl}(m|1)$

We begin by introducing our conventions, notations, and definitions for quantum $\mathfrak{gl}(m|1)$, and
recalling some of the essential results in the literature.

2.1. The root data. We write $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$. Throughout, with a $\mathbb{Z}/2\mathbb{Z}$ graded set $X =
X_0 \cup X_1$, we write $p(x) \in \mathbb{Z}/2\mathbb{Z}$ where $x \in X_{p(x)}$. Define the $\mathbb{Z}/2\mathbb{Z}$ graded sets

$$[m|1] = \{\epsilon_1, \ldots, \epsilon_m\} \cup \{\epsilon_{m+1}\}, \quad I = \{1, \ldots, m-1\} \cup \{m\}$$

(2.1)

We define the weight lattice $P = \mathbb{Z}[m|1] = \bigoplus_{k=0}^{m+1} \mathbb{Z} \epsilon_k$ and endow $P$ with the symmetric bilinear
form defined by $\langle \epsilon_i, \epsilon_j \rangle = (-1)^{p(\epsilon_i)} \delta_{ij}$ for $1 \leq i, j \leq m+1$, where here (and throughout) $\delta_{ab}$
is the Kronecker delta with $\delta_{ab} = 1$ if $a = b$ and 0 otherwise. We define the coweight lattice
$P^\vee = \bigoplus_{k=1}^{m+1} \mathbb{Z} \epsilon_k^\vee$ and we have the pairing $\langle \cdot, \cdot \rangle : P^\vee \times P \to \mathbb{Z}$ with $\langle \epsilon_i^\vee, \epsilon_j \rangle = \delta_{ij}$. Note that $P$
and $P^\vee$ are naturally $\mathbb{Z}/2\mathbb{Z}$-graded with

$$p \left( \sum_{i=1}^{m+1} a_i \epsilon_i \right) = p \left( \sum_{i=1}^{m+1} a_i \epsilon_i^\vee \right) = \sum_{i \in [m|1]} a_i p(\epsilon_i).$$

Then we have the root system $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq m+1\}$. This root system has a parity
induced by the parity on $P$, with

$$\Phi_0 = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq m\}, \quad \Phi_1 = \{\epsilon_i - \epsilon_{m+1}, \epsilon_{m+1} - \epsilon_i \mid 1 \leq i \leq m\}.$$ 

Let $\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid i \in I\}$, and note $p(\alpha_i) = p(i)$. Then $\Pi$ is a system of simple roots for $\Phi$
(called the standard simple roots), and we let $\Phi^+$ be the associated set of positive roots. We set
$h_i = \epsilon_i^\vee - (-1)^{p(\epsilon_{i+1})} \epsilon_{i+1}^\vee$ for $i \in I$ to be the simple coroots in $P^\vee$, which satisfy $\langle h_i, \alpha_j \rangle = (\alpha_i, \alpha_j)$. More generally, for $\alpha \in \Phi^+$ with $\alpha = \sum a_i \alpha_i$, we let $h_\alpha = \sum a_i h_i$.

The root lattice is $Q = \sum_{i \in I} \mathbb{Z} \alpha_i \subset P$. We set $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ and $Q^- = -Q^+$. As
usual, we have a partial order on $P$ induced by $Q^+$: we say $\lambda \leq \mu$ if $\mu - \lambda \in Q^+$. A useful
statistic for $Q^+$ is the height of a root defined by $\text{ht}(\sum a_i \alpha_i) = \sum a_i$, and for $l \in \mathbb{Z}_{\geq 0}$, we define
$Q^+(l) = \{\alpha \in Q^+ \mid \text{ht}(\alpha) \leq l\}$.

There are two distinguished vectors in $P$ of use in our discussions. First, we define

$$1_{m|1} = \sum_{k=1}^{m+1} (-1)^{p(\epsilon_k)} \epsilon_k \in P,$$

(2.2)
and note that $\langle h_i, 1_{m|1} \rangle = 0$ for all $i \in I$. Next, the \textit{shifted Weyl vector} $\rho \in P$ associated to this root data is given by

$$\rho = \sum_{k=1}^{m} (m - k + 1) \epsilon_k - \epsilon_{m+1}$$

(2.3)

and satisfies $(\alpha_i, \rho) = \frac{1}{2} (\alpha_i, \alpha_i)$ for all $i \in I$.

A weight $\lambda \in P$ is \textit{\text{gl}(m)-dominant}, or simply \textit{dominant}, if $\langle h_i, \lambda \rangle \geq 0$ for all $i \in I_0$. Explicitly, the set $P^+$ of dominant weights is given by

$$P^+ = \left\{ \sum_{k=1}^{m} a_k \epsilon_k \mid a_k \geq a_{k+1} \text{ for all } 1 \leq k < m \right\}.$$ 

We further say that $\lambda \in P^+$ is \textit{\text{gl}(m|1)-dominant}, or \textit{fully dominant}, if $\langle h_m, \lambda \rangle \geq 0$, and we denote the set of fully dominant weights by $P^{++}$. Explicitly,

$$P^{++} = \left\{ \sum_{k=1}^{m+1} a_k \epsilon_k \mid a_k \geq a_{k+1} \text{ for all } 1 \leq k < m \text{ and } a_m + a_{m+1} \geq 0 \right\}.$$ 

As observed in A weight $\lambda = \sum_{k=1}^{m+1} a_k \epsilon_k$ is \textit{polynomial} if $a_k \geq 0$ for all $k$. We denote the set of polynomial weights by $\tilde{P}$. Finally, we say a weight $\lambda \in P$ is \textit{typical} if $\langle h_\alpha, \lambda + \rho \rangle \neq 0$ for all $\alpha \in \Phi_1^+$ and we denote the set of typical weights by $P_{\text{typ}}$. We also will combine these notations in obvious ways; e.g. $\tilde{P}_{\text{typ}}$ would denote the set of typical fully dominant polynomial weights.

In particular, we note that $P^{++} = \tilde{P}^{++} + Z_{1_{m|1}}$ and $\tilde{P}^{++} = \tilde{P}^+$. We also note that a fully dominant weight $\lambda = \sum a_k \epsilon_k \in P^{++}$ is typical if and only if $\langle h_m, \lambda \rangle > 0$; indeed, we compute that

$$\langle h_{e_{-i} - \epsilon_{m+1}}, \sum a_k \epsilon_k + \rho \rangle = a_i + a_{m+1} + (m - i) \geq a_m + a_{m+1} = \langle h_m, \lambda \rangle \geq 0.$$

2.2. The quantum enveloping algebra. We work over the base field $\mathbb{Q}(q)$, where $q$ is an indeterminant parameter. We will also occasionally work over the following subrings of $\mathbb{Q}(q)$:

- $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$;
- $\mathbb{A}$, the subring of rational functions in $q$ with no pole at 0;
- $\mathbb{A}_\mathbb{Z}$ the $\mathbb{Z}$-subalgebra of $\mathbb{Q}(q)$ generated by $q$ and $\frac{1}{1-q^t}$ for $t \geq 1$.

Recall some standard notation for $q$-integers: for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}; \quad [b]! = \prod_{c=1}^{b} [a]; \quad \left[\begin{array}{c} a \\ b \end{array}\right] = \frac{\prod_{c=0}^{b-1} [a - c]}{[b]!} \in \mathbb{A}. \quad (2.4)$$

As usual, a $\mathbb{Q}(q)$-algebra $A$ has a natural notion of divided powers: if $a \in A$ and $n \in \mathbb{Z}_{\geq 0}$, we set

$$a^{(n)} = \frac{a^n}{[n]!}.$$

Now let us recall the definition of $U_q(\mathfrak{gl}(m|1))$.

\textbf{Definition 2.1.} The algebra $U = U_q(\mathfrak{gl}(m|1))$ is the $\mathbb{Q}(q)$-algebra on generators $E_i, F_i, q^h$ for $i \in I, h \in P^\vee$ with parity grading given by $p(E_i) = p(F_i) = p(i), p(q^h) = 0$; and relations given by

$q^0 = 1, \quad q^h q^{h'} = q^{h+h'}, \quad q^h E_j q^{-h} = q^{(h, h_j)} E_j, \quad q^h F_j q^{-h} = q^{-(h, h_j)} F_j$ for $h, h' \in P^\vee, j \in I;$
We further set $U$ be the algebra $U$ defined as follows:

Lemma 2.2. The algebra $U$ is a Hopf superalgebra with coproduct $\Delta$, antipode $S$, and counit $\varepsilon$ defined as follows:

\begin{align*}
\Delta(E_i) &= E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i, \quad \Delta(q^h) = q^h \otimes q^h; \\
S(E_i) &= -E_i K_i, \quad S(F_i) = -K_i^{-1} F_i, \quad S(q^h) = q^{-h}; \\
\varepsilon(E_i) &= \varepsilon(F_i) = 0, \quad \varepsilon(q^h) = 1.
\end{align*}

Next, recall that $U$ has the weight decomposition

\[ U = \bigoplus_{\zeta \in Q} U_{\zeta}, \quad U_{\zeta} = \left\{ u \in U \mid q^{h} u q^{-h} = q^{(h, \zeta)} u \right\}; \quad (2.5) \]

we write $|u| = \zeta$ if $u \in U_{\zeta}$. It also has the triangular decomposition

\[ U \cong U^- \otimes U^0 \otimes U^+ \cong U^+ \otimes U^0 \otimes U^- . \quad (2.6) \]

Here, $U^+$ is the subalgebra generated by the $E_i$, $U^-$ is the subalgebra generated by the $F_i$, and $U^0$ is the subalgebra generated by $q^h$ for $h \in P^\vee$. We also define the integral form $\tilde{U}$ of $U$ to be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U$ generated by $E_i^{(a)}$, $F_i^{(a)}$, $q^h$ for $i \in I$, $a \in \mathbb{Z}_{\geq 0}$, and $h \in P^\vee$. We further set $U^\pm = U^+ \cap U^- \cap Q$ for $\zeta \in Q$ and $\tilde{U}^\pm = U^\pm \cap \tilde{U}$.

The algebra $U$ has several important involutions:

1. The bar involution $\overline{\cdot}$ defined by

\[ \overline{E_i} = E_i, \quad \overline{F_i} = F_i, \quad \overline{q^h} = q^{-h}, \quad \overline{q} = q^{-1}, \quad \overline{xy} = \overline{y} \overline{x}. \]

2. The anti-involution $\tau$ defined by

\[ \tau(E_i) = E_i, \quad \tau(F_i) = F_i, \quad \tau(q^h) = (-1)^{p(h)} q^{-h}, \quad \tau(xy) = \tau(y) \tau(x). \]

Note that $\tau$ restricts to an anti-involution of $U^\pm$.

3. The anti-involution $\eta$ defined by

\[ \eta(E_i) = q F_i K_i^{-1}, \quad \eta(F_i) = q E_i K_i, \quad \eta(q^h) = q^h, \quad \eta(xy) = \eta(y) \eta(x). \]
Moreover, we recall the following results from [CHW3]. Let \( i \in I \). We define the left-multiplication maps \( f_i : U^- \to U^- \) by \( f_i(x) = F_i x \). We also define the quantum differential \( e_i : U^- \to U^- \) by \( e_i(F_j) = \delta_{ij} \) and, for homogenous \( u, v \in U^- \),

\[
e_i(uv) = e_i(u)v + (-1)^{p(u)p(i)} q^{(\alpha_i,|u|)} u e_i(v).
\]

By twisting with automorphisms, we obtain the following variants

\[
e\tau = e\circ\tau, \quad e_i\tau = \tau \circ e_i \circ \tau,
\]

which satisfy \( e\tau(F_j) = e\tau(F_j) = \delta_{ij} \) and

\[
e\tau(uv) = e\tau(u)v + (-1)^{p(u)p(i)} q^{(\alpha_i,|u|)} u e\tau(v),
\]

\[
e_i\tau(uv) = (-1)^{p(u)p(i)} q^{(\alpha_i,|u|)} e_i\tau(u)v + u e_i\tau(v).
\]

Observe that \( e_i e_j = e_j e_i \).

**Proposition 2.3.** The algebra \( U^- \) is equipped with symmetric nondegenerate bilinear forms \((-,-), (-,-) : U^- \times U^- \to \mathbb{Q}(q)\) satisfying \((1,1) = 1 = 1\), \((f_i(x), y) = (x, e_i(y))\), \((e_i(x), y) = \{x, e_i(y)\}\), \(\{f_i(x), y\} = \{x, e_i(y)\}\), and \(\{x, y\} = (x, y)\). In particular, the subalgebra of \( \text{End}_{\mathbb{Q}(q)}(U^-) \) generated by the \( e_i \) (resp. \( e\tau, f_i \)) is isomorphic to \( U^- \).

We observe the following useful property of the bilinear form.

**Lemma 2.4.** For any \( x, y \in U^- \), \((\tau(x), \tau(y)) = (x, y)\).

**Proof.** Let \( l \geq 0 \) and \( \xi \in Q^+(l) \). We will prove that for all \( x \in U^-_{-\xi} \), \((xF_i, y) = (x, e_i(y))\) and \((\tau(x), \tau(y)) = (x, y)\) for all \( i \in I \) and \( y \in U^- \) by induction on \( l \). If \( l = 0 \) this is obvious. Otherwise, we can write \( x = F_j z \) for some \( j \in I \) and \( z \in U^- \). Then

\[
(F_j x F_i, y) = (z F_i, e_j(y)) = (z, e_i e_j(y)) = (z, e_j e_i(y)) = (x, e_i(y)),
\]

and

\[
(\tau(x), \tau(y)) = (F_j \tau(z), \tau(y)) = (\tau(z), e_j \tau(y)) = (\tau(z), \tau(e_j \tau(y))) = (z, e_j \tau(y)) = (x, y).
\]

\(\square\)

We now recall one of the crucial results from [CHW3] on distinguished bases for \( U^- \). To wit, using quantum shuffles, one can inductively construct PBW bases associated to any total order on the simple roots. These PBW bases then allow us to construct a signed canonical basis.

**Lemma 2.5 ([CHW3]).** Let \( \prec \) be a total order on \( I \). Then there is an induced convex order \( \prec \) on \( \Phi^+ \), so write \( \Phi^+ = \{ \beta_1 \prec \cdots \prec \beta_N \} \). Let \( N = |\Phi^+| \), and let

\[
\mathbb{Z}_{\geq 0}^{\Phi^+} = \{ a = (a_1, \ldots, a_N) \in \mathbb{Z}_{\geq 0}^N \mid a_r \leq 1 \text{ if } p(\beta_r) = 1 \}.
\]

Then \( \mathbb{A} U^- \) has an \( \mathbb{A} \)-basis

\[
\mathbb{B}(\prec) = \left\{ F_{(a_1, \ldots, a_r)} = \prod_{r=1}^{N} F_{(a_r)}^{(a_r)} \mid (a_1, \ldots, a_N) \in \mathbb{Z}_{\geq 0}^{\Phi^+} \right\}.
\]
Here, $F_{<,\beta_r}$ is a root vector of weight $-\beta_r$ depending on the ordering $\prec$, and $F_{<,\beta_r}^{(a)} = F_{<,\beta_r}/[a]!$ as usual. Moreover, for any $a = (a_1, \ldots, a_N)$ and $a' = (a'_1, \ldots, a'_{N'})$ with $a, a' \in \mathbb{Z}_{\geq 0}^{\Phi^+}$, we have
\[
(F_a, F_{a'}) \in \delta_{a,a'} + qA_Z.
\]

(That is, $B(\prec)$ is almost-orthonormal under $(-,-).$) Finally, there exists a homogeneous $\mathbb{A}$-basis $B(\prec) = \{ b_a \mid a \in \mathbb{Z}_{\geq 0}^{\Phi^+} \}$ of $\mathfrak{U}^-$ satisfying
\[
\begin{align*}
(1) & \quad \overline{b}_a = b_a, \quad \text{(bar-invariance)} \\
(2) & \quad b_a - F_a \text{ is in the } q\mathbb{Z}[q]-\text{span of } \mathfrak{PBW}, \quad \text{(q-unitriangularity)} \\
(3) & \quad \text{and } (b_a, b_{a'}) \in \delta_{a,a'} + qA_Z \text{ for all } a, a' \in \mathbb{Z}_{\geq 0}^{\Phi^+}. \quad \text{(almost-orthogonality)}
\end{align*}
\]

Proof. These statements follow from [CHW3] in the standard ordering case. In a non-standard ordering $\prec$, it is only shown that the PBW basis exists for $\mathfrak{U}^-$ and is orthonormal. However, direct computation using the formulas in loc. cit. shows that the PBW basis still has the claimed properties; alternatively, this will follow from the construction in Section 4. The existence of the canonical basis then follows from similar arguments to loc. cit. Section 7.

Note that to prove $(F_a, F_{a'}) \in \delta_{a,a'} + qA_Z$, Theorem 5.7 in loc. cit. implies it suffices to show this in the case of a divided power of a single root vector. For root vectors of even parity, this follows from the same calculation as in $\mathfrak{gl}(m)$; for the odd root vectors, it suffices to show that the norm of the root vector lies in $1 + q\mathbb{Z}[q]$ but this is easy to verify directly using the fact that root vectors for $\beta \in \Phi^+ \setminus \Pi$ are always of the form $F_\beta = F_i F_{\beta-\alpha_i} - qF_{\beta-\alpha_i} F_i$ for some $i \in I$. \qed

We call $B(\prec)$ the $\prec$-canonical basis of $\mathfrak{U}^-$. 

Remark 2.6. It is well-known that the conditions on $B(\prec)$ imply it is unique up to a sign (cf. [Lus2, Theorem 14.2.3]), so the signed basis $B(\prec) \cup -B(\prec)$ does not depend on $\prec$. One of our goals will be to show that $B(\prec)$ itself is independent of $\prec$.

Example 2.7. A version of Lemma 2.5 is proven in [CHW3, Section 7] for the standard data associated to some families of basic Lie superalgebras. In particular, $\mathfrak{U}_q(\mathfrak{gl}(m|n))$ is observed to have a pseudo-canonical basis; that is, a basis $B$ which is bar-invariant and $q$-unitriangular with respect to the standard PBW basis, but which is not almost-orthogonal under the bilinear form. (Another construction of this basis is given in [DG], motivated by connections to quantum Schur superalgebras.) However, it is easy to see this basis crucially depends on the choice of PBW basis. Indeed, in the case of $m = n = 2$, the standard Cartan datum has $I = \{1 < 2 < 3\}$ with $I_1 = \{2\}$ and the GCM $\Lambda = \begin{bmatrix} 2 & -1 & 0 \\
-1 & 0 & 1 \\
0 & 1 & -2 \end{bmatrix}$. The PBW vectors are then $F_{12} = F_2 F_1 - qF_1 F_2$, $F_{23} = F_3 F_2 - q^{-1}F_2 F_3$, and $F_{123} = F_3 F_2 F_1 - qF_1 F_3 F_2 - q^{-1}F_2 F_1 F_3 + F_1 F_2 F_3$. On the other hand, applying $\tau$ gives us the PBW vectors for the opposite order; that is, the order $I = \{3 <_{\text{op}} 2 <_{\text{op}} 1\}$. Then, for instance, it is easy to see that we have $F_2 F_3, F_3 F_2 - [2] F_2 F_3 \in B(\prec)$, but $F_3 F_2, F_2 F_3 - [2] F_3 F_2 \in B(\prec_{\text{op}})$. 

It is unclear how to resolve this sort of incompatibility, but morally one would want the canonical basis of $U_q(\mathfrak{gl}(m|n))$ to be compatible with its Levi subalgebras. In particular, this means that the root spaces corresponding to the subalgebras $U_q(\mathfrak{gl}(2|1))$ and $U_q(\mathfrak{gl}(1|2)) \cong U_q^{-1}(\mathfrak{gl}(2|1))$ should have the canonical bases given in [CHW3, §8]. For this to be possible, we see that both almost-orthogonality and $q$-unitriangularity fail, as necessarily one subalgebra will satisfy the conditions in $q$ while the other satisfies the conditions in $q^{-1}$. We do not yet see how to glue these conditions in such a way to have a well-defined canonical basis on the remaining root spaces, hence we restrict our attention to the $n = 1$ case.

2.3. Weight modules. Throughout, a $U$-module is a $\mathbb{Z}/2\mathbb{Z}$-graded weight module; that is, a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Q}(q)$-vector space $M = M^0 \oplus M^1$ with a decomposition $M^t = \bigoplus_{\lambda \in P} M^t_\lambda$ for $t = 0, 1$ such that $M^t_\lambda$ is finite-dimensional and $q^h|_{M^t_\lambda} = q^{(h,\lambda)}$. We write $M_\lambda = M^0_\lambda \oplus M^1_\lambda$, and for $m \in M^t_\lambda$ we write $p(m) = t$ and $|m| = \lambda$.

Remark 2.8. A $U$-module homomorphism is always assumed to be homogeneous with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading. In particular, we generally consider $M = M^0 \oplus M^1$ and its parity-shift $\tilde{M} = M^0 \oplus M^1$ with $\tilde{M}^t = M^{1-t}$ (with the same $U$-action) to be non-isomorphic. However, in general we will ignore this facet of the representation theory, since we will essentially exclusively work with a family of modules with a canonical choice of parity, and the parity is non-essential except in tensor products.

Given modules $M, N$, we define the module $M \otimes N = M \otimes_{\mathbb{Q}(q)} N$ with the action of $U$ using the Hopf superalgebra structure; in other words, for homogeneous $x \in M$ and $y \in N$, we define $u \cdot x \otimes y = \sum (-1)^{p(u_2)p(x)} u_1 x \otimes u_2 y$, where $\Delta(u) = \sum u_1 \otimes u_2$.

Let us now recall some properties $U$-modules. We say that $M$ is polynomial if $M_\lambda \neq 0$ implies $\lambda$ is polynomial. As usual, we say that $M$ is a highest weight module there is a $\lambda \in P$ such that $\dim_{\mathbb{Q}(q)} M_\lambda = 1$, $M_\mu \neq 0$ only if $\mu \leq \lambda$, and $M = U M_\lambda$. Finally, we say $M$ has a polarization if there is a bilinear form $(\cdot, \cdot) : M \times M \to \mathbb{Q}(q)$ (which we call the polarization on $M$) satisfying $(uv, w) = (v, \eta(u)w)$.

Lemma 2.9 ([BKK]). Let $M$ and $N$ be $U$-modules with polarizations. Then $M \otimes N$ has a polarization defined by

$$(v \otimes w, v' \otimes w')_{M \otimes N} = (v, v')_M(w, w')_N.$$
Henceforth, let $k_\lambda$ denote a choice of highest weight vector of $K(\lambda)$ and let $v_\lambda$ be the image of $k_\lambda$ in $V(\lambda)$ for all $\lambda \in P$. We define $\pi^K_\lambda : U^- \to K(\lambda)$ to be the $U^-$-module homomorphism with $\pi^K_\lambda(x) = xk_\lambda$, $\pi^K_{\lambda, V} : K(\lambda) \to V(\lambda)$ to and write $\pi_\lambda = \pi^K_{\lambda, V} \circ \pi^K_\lambda : U^- \to V(\lambda)$.

These modules satisfy the following properties.

Lemma 2.10. We observe the following properties of Kac modules and irreducible modules.

1. Let $\lambda \in P$. Then the following are equivalent:
   (a) $\dim_{Q(q)} K(\lambda) < \infty$;
   (b) $\dim_{Q(q)} V_\lambda(\lambda) < \infty$;
   (c) $\dim_{Q(q)} V(\lambda) < \infty$.

2. Let $\lambda \in P^+$. As $U^-$-modules, $K(\lambda) \cong U^-/I_\lambda$ where $I_\lambda$ is the ideal generated by $F_i^{(h_i, \lambda)+1}$ for $i \in I_0$.

3. If $\lambda \in P^+$ is typical, $K(\lambda) = V(\lambda)$.

4. Let $\lambda \in P^+$, and let $M$ be a weight module and $v \neq 0 \in M_\lambda$ such that $E_i v = 0$ for all $i \in I$ and $F_i^{(h_i, \lambda)+1} v = 0$ for $i \in I_0$. Then there is a unique $U$-module homomorphism $K(\lambda) \to M$ (or from the parity shift $\widehat{K(\lambda)} \to M$) such that $k_\lambda \mapsto v$.

5. If $M$ is a finite-dimensional irreducible module, then $M \cong V(\lambda)$ (or its parity shift $\widehat{V(\lambda)}$) for some $\lambda \in P^+$.

6. If $\lambda \in P^+$, then $V(\lambda)$ carries a unique polarization with $(v_\lambda, v_\lambda) = 1$.

Proof. We observe that (1), (2) and (3) can essentially be found in [Kwo2], but we will recall the broad strokes here. First, note that (1) follows directly from the PBW theorem and triangular decomposition in a way entirely analogous to the classical case (cf. [CW, Proposition 2.1]). Indeed, we note that the implications (a) $\Rightarrow$ (c) $\Rightarrow$ (b) are clear, and the remaining implication follows from observing that, as vector spaces, $K(\lambda) \cong \wedge(\Phi^-) \otimes V_\lambda(\lambda)$ (where $\wedge(\Phi^-)$ is the exterior algebra of the formal vector space on the set $\Phi^-_\lambda$). For (2), note the action of $U^-$ on $K(\lambda)$ induces a surjection $U^-/I_\lambda \to K(\lambda)$ sending the image of 1 to $k_\lambda$. On the other hand, it is easy to see from the PBW basis with respect to the opposite standard ordering (i.e. where $m$ is minimal and the odd root vectors are left factors) that these vector spaces have the same dimension, so the surjection must be an isomorphism. We can take a classical limit of $U$ and $K(\lambda)$ to deduce (3) from the classical case.

Observe that (4) now follows immediately from (2), by observing that the $U^-$-module projection $U^- \to M$ given by $x \mapsto xv$ factors through $K(\lambda)$. It is easy to see that this map preserves the weights and the action of the $E_i$ for $i \in I$.

Next, for (5) note that since $M$ is finite-dimensional, it has a highest weight space (that is, a weight space $M_\lambda \neq 0$ such that $U^+M_\lambda = 0$). Then by (4), given a nonzero vector $v \in M_\lambda$, we have a non-zero $U$-module homomorphism $K(\lambda) \to M$. Since $M$ is irreducible, this map is surjective, hence it must be that $M \cong V(\lambda)$.

Finally, (6) is proven in [BKK] when $\lambda \in P^{++}$ by restricting the polarization on tensor powers of the standard representation (possibly with an additional 1-dimensional tensor factor twisting the weight by $a1_{m|1}$), but we note this is easily proven directly for all $\lambda \in P^+$ using standard
arguments. (In particular, that $V(\lambda) \cong V(\lambda)^*$ where the action of $U$ on the dual is defined using $\eta$.) □

Let $\xi \in Q^+$. We will say $\lambda \gg \xi$ if $\langle h_i, \lambda - \xi \rangle > 0$ for all $i \neq m$. Note that if $\lambda \gg \xi$, then $U^{-\xi} \cong K(\lambda)_{\lambda-\xi}$ as vector spaces. In particular, when $\lambda \in P^+_{\text{typ}}$ and $\lambda \gg \xi$, $U^{-\xi} \cong V(\lambda)_{\lambda-\xi}$

2.4. Crystal bases of modules. We shall now recall the definition and some facts about crystal bases as defined in [BKK, Kwo2]. For $i \in I$, let $U(i)$ be the subalgebra of $U$ generated by $E_i$, $F_i$, and $K_i$.

**Definition 2.11** ([BKK, Definition 2.2]). The category $\mathcal{O}_{\text{int}}$ is the full subcategory of $U$-weight modules such that if $M \in \mathcal{O}_{\text{int}}$:

1. $M$ is locally $U(i)$-finite; that is, $\dim(U(i)v) < \infty$ for any $v \in M$.
2. If $M_\mu \neq 0$ for some $\mu \in P$, then $\langle h_m, \mu \rangle \geq 0$.
3. If $v \in M_\mu$ such that $\langle h_m, \mu \rangle = 0$, then $E_m v = F_m v = 0$.

We note that $V(\lambda) \in \mathcal{O}_{\text{int}}$ if and only if $\lambda \in P^{++}$, but that $K(\lambda) \notin \mathcal{O}_{\text{int}}$ for $\lambda \notin P^{++}$.

If $M \in \mathcal{O}_{\text{int}}$, then we may define Kashiwara operators on $M$ as follows. Let $v \in M_\lambda$ for some $\lambda \in P$. If $i \neq m$, then there exists a unique family of $v_n \in M_{\lambda+n\alpha_i}$ with $\langle h_i, \lambda + n\alpha_i \rangle \geq n \geq 0$ such that $v = \sum_{n \in \mathbb{N}} F_i^{(n)} v_n$, and we set

$$\tilde{e}_i v = \sum_{n \in \mathbb{N}} F_i^{(n-1)} v_n, \quad \tilde{f}_i v = \sum_{n \in \mathbb{N}} F_i^{(n+1)} v_n.$$ (2.8)

(Here, we denote $F_i^{(-1)} = 0$ for convenience.) On the other hand, if $i = m$, then we set

$$\tilde{e}_m v = q^{-1} K_m E_m v, \quad \tilde{f}_m v = F_m v.$$ (2.9)

Now recall that $A$ is the subalgebra of $\mathbb{Q}(q)$ of rational functions with no poles at $0$.

**Definition 2.12** ([BKK, Definitions 2.3, 2.4, 2.10]). Let $M \in \mathcal{O}_{\text{int}}$. We say that an $A$-submodule $L$ of $M$ is a crystal lattice of $M$ if:

1. $M = \mathbb{Q}(q) \otimes_A L$;
2. $L = \bigoplus_{\lambda}(L_\lambda^0 \oplus L_\lambda^1)$ where $L_\lambda^i = L \cap M_\lambda^i$;
3. $\tilde{e}_i L \subseteq L$ and $\tilde{f}_i L \subseteq L$ for all $i \in I$.

Suppose $L$ is a crystal lattice of $M$, and let $B \subseteq L/qL$. We say that $(L, B)$ is a signed crystal basis of $M$ if:

1. $B$ is a signed $\mathbb{Q}$-basis of $L/qL$ (that is, $B = B' \cup (-B')$ for some $\mathbb{Q}$-basis of $L/qL$);
2. $B = \bigcup_{\lambda}(B_\lambda^0 \cup B_\lambda^1)$ where $B_\lambda^i = B \cap (L_\lambda^i/qL_\lambda^i)$;
3. $\tilde{e}_i B \subseteq B \cup \{0\}$ and $\tilde{f}_i B \subseteq B \cup \{0\}$ for all $i \in I$.
4. If $b, b' \in B$ and $i \in I$, then $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b = b$.

If $B$ satisfies (5)-(7) and is a $\mathbb{Q}$-basis of $L/qL$, then we say $(L, B)$ is an unsigned crystal basis of $M$, or simply a crystal basis of $M$. Finally, we say $(L, B)$ is polarizable if $M$ has a polarization $(\cdot, \cdot)$ such that $(L, L) \subseteq A$ and the induced bilinear form $(\cdot, \cdot)_0$ on $L/qL$ satisfies $(b, b) = 1$ and $(b, b') = 0$ for all $b \neq \pm b' \in B$. Note that by the same proof as in [Kas], if a crystal basis is polarizable, then $L = \{x \in M \mid (x, x) \in A\}$. 

Let $M$ be a module with a (signed) crystal basis $(L, B)$. For $i \in I_0$, let

$$\phi_i(b) = \max \left\{ t \in \mathbb{N} \mid \tilde{e}_i^t b \neq 0 \right\}, \quad \epsilon_i(b) = \max \left\{ t \in \mathbb{N} \mid \tilde{e}_i^t b \neq 0 \right\}.$$ 

Then note we have $\phi_i(b) - \epsilon_i(b) = \langle h_i, |b| \rangle$. Then crystal bases of modules in $\mathcal{O}_{\text{int}}$ satisfy the following tensor product rule.

**Proposition 2.13 ([BKK, Proposition 2.8, Lemma 2.11, and Theorem 2.12]).** Let $M_1, M_2$ be modules with (signed) crystal bases $(L_1, B_1)$ and $(L_2, B_2)$. Set $M = M_1 \otimes M_2$, $L = L_1 \otimes L_2$, and $B = B_1 \otimes B_2 \subset (L_1/qL_1) \otimes (L_2/qL_2)=L/qL$. Then $(L, B)$ is a signed crystal basis of $M$. Moreover, for $b_1 \in B_1$ and $b_2 \in B_2$, we have for $i \in I_0$:

$$
\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{e}_i(b_1) \otimes b_2 & \text{if } \phi_i(b_1) \geq \epsilon_i(b_2), \\
 b_1 \otimes \tilde{e}_i(b_2) & \text{otherwise}.
\end{cases}
$$

$$
\tilde{e}_m(b_1 \otimes b_2) = \begin{cases} 
\tilde{e}_m(b_1) \otimes b_2 & \text{if } \langle h_m, |b| \rangle > 0 \\
(-1)^{p(b_1)}b_1 \otimes \tilde{e}_m(b_2) & \text{otherwise}.
\end{cases}
$$

$$
\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_i(b_1) \otimes b_2 & \text{if } \phi_i(b_1) > \epsilon_i(b_2), \\
b_1 \otimes \tilde{f}_i(b_2) & \text{otherwise}.
\end{cases}
$$

$$
\tilde{f}_m(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_m(b_1) \otimes b_2 & \text{if } \langle h_m, |b| \rangle > 0 \\
(-1)^{p(b_1)}b_1 \otimes \tilde{f}_m(b_2) & \text{otherwise}.
\end{cases}
$$

Finally, if $(L_1, B_1)$ and $(L_2, B_2)$ are polarizable, then so is $(L, B)$, and $M_1 \otimes M_2$ is completely reducible.

The following theorem is a special case of the main result in [BKK].

**Theorem 2.14.** Let $\lambda \in P^{++}$. Then $V(\lambda)$ has a polarizable signed crystal basis $(L(\lambda), B(\lambda) \cup -B(\lambda))$, where

$$L(\lambda) = \sum A\tilde{x}_{i_1} \ldots \tilde{x}_{i_t} v_\lambda, \quad \text{with the sum being over } t \geq 0, x \in \{e, f\}, i_1, \ldots, i_t \in I;$$

$$B(\lambda) = \{\tilde{x}_{i_1} \ldots \tilde{x}_{i_t} v_\lambda + qL(\lambda) \mid t \geq 0, x \in \{e, f\}, i_1, \ldots, i_t \in I\} \setminus 0.$$

In the context of $\mathfrak{gl}(m|1)$, this result can be improved in two ways. First, note that in loc. cit. it was shown that, for $\mathfrak{gl}(m|n)$ with $n \geq 2$, there may exist $x \in L(\lambda)$ such that $\tilde{e}_i x \in qL$ for all $i \in I$ but $x \notin L(\lambda)_\lambda$. Consequently, we may need to apply both $\tilde{e}$ and $\tilde{f}$ operators to reach every element of the crystal.

Second, Theorem 2.14 is proved by realizing $V(\lambda)$ as a summand of a tensor power of the standard representation $V(e_1)$ (up to a twist by a one-dimensional module of weight $a_1|_{m|1}$). Then we can identify the crystal $B(\lambda) \cup -B(\lambda)$ in a tensor power of $B(e_1) \cup -B(e_1)$. However, since the crystal operators on $B(\lambda)$ are determined by the tensor product rule, there may be signs introduced by commuting the odd Kashiwara operators past odd-parity vectors. As such, despite $B(e_1) \cup \{0\}$ being closed under the Kashiwara operators, it is not necessarily true that $B(\lambda) \cup \{0\}$ is closed hence $V(\lambda)$ only has a signed crystal basis in general.
However, in the case of $\mathfrak{gl}(m|1)$, we can remove both of these ambiguities as shown by the following lemma.

**Lemma 2.15.** Let $\lambda \in P^{++}$. Then
\[
B(\lambda) = \left\{ \tilde{f}_1 \ldots \tilde{f}_i v_\lambda + qL(\lambda) \mid t \geq 0, i_1, \ldots, i_t \in I \right\}
\]
then a basis of $L(\lambda)/qL(\lambda)$, hence
\[
L(\lambda) = \sum \mathcal{A} \tilde{f}_1 \ldots \tilde{f}_i v_\lambda, \quad \text{with the sum being over } t \geq 0, i_1, \ldots, i_t \in I;
\]
and $(L(\lambda), B(\lambda))$ is a crystal basis of $V(\lambda)$.

**Proof.** We prove this using the realization of $V(\lambda)$ as a direct summand of $V(\epsilon_1)^{\otimes t} \otimes S$ for some $t \in \mathbb{Z}_{\geq 0}$ and one-dimensional $U$-modules $S$ of weight $a1_{m|1}$; since the factor $S$ will make no difference with respect to the action of crystal operators, it suffices to prove this in the case $S$ is trivial and $\lambda \in \hat{P}^{++}$. Then we can freely identify the crystal with the (signed) crystal of semistandard Young tableaux in the super alphabet $\{1, \ldots, m\} \cup \{m+1\}$; we refer the reader to [BKK, Sections 3.2 and 4] for details and examples.

First, let us show that we don’t need a signed basis. To do this, note that the signed crystal $B(\lambda) \cup -B(\lambda)$ is isomorphic to a subcrystal of $V(\epsilon_1)^{\otimes t}$. It suffices to prove that $B(\epsilon_1)^{\otimes t}$ (which is unsigned) is closed under the action of $\tilde{e}_i$ and $\tilde{f}_i$. Clearly it is closed when $i \in I_0$, so we only need to show this in the case $i = m$. This is easy, since $\tilde{e}_m$ and $\tilde{f}_m$ kill all boxes (i.e. elements of $B(\epsilon_1)$) other than those colored by $m$ and $m-1$ respectively; in particular, we only move them past boxes of even parity (i.e. those colored by $n \leq m-1$) so we don’t have signs appearing. This shows that $B(\lambda)$, which is generated by an element of $B(\epsilon_1)^{\otimes t}$, is closed under $\tilde{e}_i$ and $\tilde{f}_i$ and hence itself a(n unsigned) basis of $L(\lambda)/qL(\lambda)$.

Next, we wish to show that $L(\lambda)$ and $B(\lambda)$ are actually generated from the highest weight vector by applying sequences of $\tilde{f}_i$ for $i \in I$. This is immediate if there is no “fake highest weight vector”; that is, no $b \in B(\lambda)$ such that $\tilde{e}_ib = 0$ for all $i$ yet $b \neq v_\lambda + qL(\lambda)$. Indeed, starting from any $b' \in B(\lambda)$, by weight considerations there is some element $b = \tilde{e}_{i_m} \ldots \tilde{e}_i b' \in B(\lambda)$ such that $\tilde{e}_j b = 0$ for all $j \in I$. If there is no fake highest weight vector, then $b = v_\lambda + qL(\lambda)$ hence $b' = \tilde{f}_{i_1} \ldots \tilde{f}_{i_m} v_\lambda + qL(\lambda)$. In the case of $\mathfrak{gl}(m|1)$, there is no fake highest weight vector as noted in the proof of [BKK, Theorem 4.8]; for the sake of completeness, we will prove that claim here.

Now to prove that there is no fake highest weight vectors, suppose $T$ is a tableaux such that $\tilde{e}_iT = 0$ for all $i$. Since $T$ is semistandard, let $T'$ be the semistandard tableaux in the alphabet $J_0$ obtained by deleting the boxes colored by $m+1$. Note that by the definition of semistandard, each row of $T/T'$ contains at most one box. Now, suppose $k$ is the first nonempty row in the skew diagram $T/T'$, or $k = \infty$ if $T/T'$ is empty. Then since $e_iT = 0$ for all $i \in I_0$, $T'$ must be a highest weight tableaux with respect to $\mathfrak{gl}(m|1)$. If $k = m + 1$ or $k = \infty$, then $T$ is the genuine highest weight vector, so assume $k \leq m$. Since $\tilde{e}_mT = 0$, in any admissible reading of $T$ the first box read which contains $m$ or $m+1$ must contain $m$, which implies that there is a box in $T'$ containing $m$ in the first $k - 1 \leq m - 1$ rows. But the highest weight tableaux with respect to $\mathfrak{gl}(m)$ necessarily contain only $j$ in boxes in the $j$-th row, which is a contradiction.  

Now let us observe some more properties of the crystals $(L(\lambda), B(\lambda))$.  


Lemma 2.16. Let \( \lambda \in P^{++}. \)

1. \( \{ u \in L(\lambda)/qL(\lambda) \mid \tilde{e}_i u = 0 \text{ for all } i \in I \} = qv_\lambda + qL(\lambda) \)
2. \( \{ u \in V(\lambda) \mid \tilde{e}_i u \in L(\lambda) \text{ for all } i \in I \} = L(\lambda) + V(\lambda) \)

Proof. (1) follows from the fact there are no fake highest weight vectors. For (2), suppose \( \tilde{e}_i u \in L(\lambda) \) for all \( i \in I. \) There is some minimal \( t \in \mathbb{N} \) such that \( u \in q^{-t}L(\lambda). \) If \( t = 0, \) then \( u \in L(\lambda). \) If \( t > 1, \) then \( \tilde{e}_i (q^tu) \in q^tL(\lambda) \) for all \( i \in I, \) so \( \tilde{e}_i (q^tu + qL(\lambda)) = 0. \) But then by (1), \( u \in V(\lambda) \).

Let \( \lambda, \mu \in P^{++}. \) Then since \( V(\lambda) \) and \( V(\mu) \) have polarizable crystal bases, \( V(\lambda) \otimes V(\mu) \) is completely reducible. In particular, there exist unique \( U \)-module homomorphisms

\[
\begin{align*}
S_{\lambda, \mu} : V(\lambda + \mu) &\rightarrow V(\lambda) \otimes V(\mu), \\
G_{\lambda, \mu} : V(\lambda) \otimes V(\mu) &\rightarrow V(\lambda + \mu),
\end{align*}
\]

Then since \( \tilde{e}_i u \in L(\lambda) \) for all \( i \in I, \) there is some minimal \( t \in \mathbb{N} \) such that \( u \in q^{-t}L(\lambda). \) If \( t = 0, \) then \( u \in L(\lambda). \) If \( t > 1, \) then \( \tilde{e}_i (q^tu) \in q^tL(\lambda) \) for all \( i \in I, \) so \( \tilde{e}_i (q^tu + qL(\lambda)) = 0. \) But then by (1), \( u \in V(\lambda) \).

Let \( \lambda, \mu \in P^{++}. \) Then since \( V(\lambda) \) and \( V(\mu) \) have polarizable crystal bases, \( V(\lambda) \otimes V(\mu) \) is completely reducible. In particular, there exist unique \( U \)-module homomorphisms

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\begin{align*}
S_{\lambda, \mu} : V(\lambda + \mu) &\rightarrow V(\lambda) \otimes V(\mu), \\
G_{\lambda, \mu} : V(\lambda) \otimes V(\mu) &\rightarrow V(\lambda + \mu),
\end{align*}
\]

These maps satisfy \( G_{\lambda, \mu} S_{\lambda, \mu} = 1_{V(\lambda + \mu)} \) and \( (G_{\lambda, \mu} (x), y) = (x, S_{\lambda, \mu} (y)) \text{ for all } x, y \in V(\lambda) \otimes V(\mu) \) and \( y \in V(\lambda + \mu), \) where \( \langle \cdot, \cdot \rangle \) denotes the polarization on each module. We also define the \( U^- \)-linear map

\[
P_{\lambda, \mu} : V(\lambda) \otimes V(\mu) \rightarrow V(\lambda), \quad P_{\lambda, \mu} (w \otimes v) = (v_\mu, v) w.
\]

Corollary 2.17. Let \( \lambda, \mu \in P^{++}. \)

1. If \( b \in B(\lambda) \) and \( b' \in B(\mu) \) such that \( \tilde{e}_i (b \otimes b') = 0 \) for any \( i \in I, \) then \( b = v_\lambda + qL(\lambda). \)
2. If \( b \in B(\lambda) \) and \( i \in I, \) then \( \tilde{e}_i (b \otimes v_\mu) = (\tilde{f}_i b) \otimes v_\mu \) or \( \tilde{f}_i b = 0. \)
3. \( S_{\lambda, \mu} (L(\lambda + \mu)) \subset L(\lambda) \otimes L(\mu) \) and \( G_{\lambda, \mu} (L(\lambda) \otimes L(\mu)) = L(\lambda + \mu) \) and the induced map

\[
G_{\lambda, \mu} : (L(\lambda) \otimes L(\mu))/qL(\lambda) \otimes L(\mu)) &\rightarrow L(\lambda + \mu)/qL(\lambda + \mu)
\]

satisfies \( G_{\lambda, \mu} (B(\lambda) \otimes B(\mu)) = B(\lambda + \mu) \cup \{0\}. \)
4. \( P_{\lambda, \mu} (L(\lambda) \otimes L(\mu)) = L(\lambda) \) and the induced map

\[
P_{\lambda, \mu} : (L(\lambda) \otimes L(\mu))/qL(\lambda) \otimes L(\mu)) &\rightarrow L(\lambda)/qL(\lambda)
\]

satisfies \( P_{\lambda, \mu} \tilde{f}_i \) for all \( i \in I. \)

Proof. For (1), first note that if \( \tilde{e}_i (b \otimes b') = 0, \) then \( \tilde{e}_i (b) = 0 \) (since \( \tilde{e}_i (b \otimes b') = b \otimes \tilde{e}_i (b') \) would imply \( \varepsilon_i (b') \geq \phi_i (b) \geq 0 \) if \( i \in I_0 \) or \( \langle h_m, |b| \rangle = 0 \) if \( i = m, \) hence in either case \( \tilde{e}_i (b') \neq 0). \) Since \( V(\lambda) \) has no fake highest weight vectors, \( \tilde{e}_i (b) = 0 \) for any \( i \) implies that \( b = v_\lambda. \)

For (2), note that if \( \tilde{e}_i (b \otimes v_\mu) = b \otimes \tilde{f}_i v_\mu, \) then \( \tilde{f}_i (b) \leq \varepsilon_i (v_\mu) = 0 \) if \( i \in I_0 \) and \( \langle h_m, |b| \rangle = 0 \) if \( i = m, \) hence \( \tilde{f}_i (b) = 0 \) in either case.

For (3), note that \( S_{\lambda, \mu} \) and \( G_{\lambda, \mu} \) are \( U \)-module homomorphisms, hence they preserve the actions of \( \tilde{e}_i \) and \( \tilde{f}_i \) for \( i \in I. \) Moreover, recall that

\[
S_{\lambda, \mu} (v_{\lambda + \mu}) = v_\lambda \otimes v_\mu,
\]

\[
G_{\lambda, \mu} (v_\lambda \otimes v_\mu) = v_{\lambda + \mu}.
\]

Then \( G_{\lambda, \mu} ((L(\lambda) \otimes L(\mu))_{\lambda + \mu}) = L(\lambda + \mu)_{\lambda + \mu}. \) In particular, assume \( G_{\lambda, \mu} ((L(\lambda) \otimes L(\mu))_{\lambda + \mu - \zeta}) \subset L(\lambda + \mu) \text{ for } h \zeta < l \text{ with } l > 0. \) Then note that if \( h \zeta = l, \)

\[
\tilde{e}_i G_{\lambda, \mu} ((L(\lambda) \otimes L(\mu))_{\lambda + \mu - \zeta}) \subset G_{\lambda, \mu} ((L(\lambda) \otimes L(\mu))_{\lambda + \mu - \zeta + \alpha_i}) \subset L(\lambda + \mu).
\]
By Corollary 2.16 (2) and weight space considerations, we must have \( G_{\lambda,\mu}((L(\lambda)\otimes L(\mu))_{\lambda+\mu-\zeta}) \subset L(\lambda+\mu) \). Since \( L(\lambda+\mu) \) is generated from \( \mathcal{A}v_{\lambda+\mu} \) by the \( \hat{f}_i \)'s, the result follows from the tensor product rule.

For (4), note that for \( x \otimes y \in L(\lambda) \otimes L(\mu) \), \( P_{\lambda,\mu}(x \otimes y) = (v_{\mu},y)x \in L(\lambda) \) since \( x \in L(\lambda) \) and \( (v_{\mu},y) \in \mathcal{A} \). Moreover, \( P_{\lambda,\mu} \) is easily seen to be surjective since \( P_{\lambda,\mu}(x \otimes v_{\mu}) = x \). Now \( \hat{P}_{\lambda,\mu} \) commutes with all \( \hat{f}_i \) for \( i \in I_0 \) by (2) and the definition of \( P_{\lambda,\mu} \).

Finally, let us recall the results of [Kwo2]. Note that, in general, \( K(\lambda) \notin \mathcal{O}_{\text{int}} \). Nevertheless, we can define a crystal structure on \( K(\lambda) \). To do this, note that \( K(\lambda) \) is naturally still a weight \( U_q(\mathfrak{gl}(m)) \)-module, hence the Kashiwara operators \( \hat{e}_i \) and \( \hat{f}_i \) for \( i \in I_0 \) are well-defined on \( K(\lambda) \). On the other hand, \( \hat{e}_m \) is not a good choice of crystal operator for \( K(\lambda) \) in general, since in general it will not be an “inverse at \( q = 0 \)” to \( \hat{f}_m \).

We can fix this by using the quantum differential from (2.7). Indeed, note that in Lemma 2.10 (2), we trivially have \( e_m(I_\lambda) \subset I_\lambda \). In particular, \( e_m \) descends to a map on \( K(\lambda) \), so for \( x \in K(\lambda) \) we define
\[
\tilde{f}_i^K x = \hat{f}_i x, \quad \tilde{e}_i^K x = \hat{e}_i x \quad \text{for} \quad i \in I_0;
\]
\[
\tilde{f}_m^K x = \hat{f}_m x = F_m x, \quad \tilde{e}_m^K x = e_m(x).
\]

**Proposition 2.18** ([Kwo2, Theorems 4.7-4.11]). Let \( \lambda \in P^+ \). Then \( K(\lambda) \) has a signed crystal basis (with respect to the crystal operators \( \hat{e}_i^K, \tilde{f}_i^K \) for \( i \in I \)) given by \( (L^K(\lambda), B^K(\lambda) \cup -B^K(\lambda)) \), where
\[
L^K(\lambda) = \sum \mathcal{A} \tilde{x}_1^K \ldots \tilde{x}_t^K x_{\lambda}, \quad \text{with the sum being over} \quad t \geq 0, x \in \{e,f\}, i_1, \ldots, i_t \in I;
\]
\[
B^K(\lambda) = \{ \tilde{x}_1^K \ldots \tilde{x}_t^K x_{\lambda} + qL(\lambda) \mid t \geq 0, x \in \{e,f\}, i_1, \ldots, i_t \in I \} \setminus 0.
\]

Moreover, if \( \lambda \in \hat{P}^+ \), then the projection \( \pi_{\lambda,V} : K(\lambda) \rightarrow V(\lambda) \) given by \( k_{\lambda} \mapsto v_{\lambda} \) satisfies
\[
(1) \quad \pi_{\lambda,V}^K(L^K(\lambda)) = L(\lambda), \quad \text{hence} \quad \pi_{\lambda,V}^K \quad \text{induces a projection}
\]
\[
\frac{\pi_{\lambda,V}^K}{\pi_{\lambda,V}} : L^K(\lambda)/qL^K(\lambda) \rightarrow L(\lambda)/qL(\lambda);
\]
\[
(2) \quad \pi_{\lambda,V}^K(B^K(\lambda)) = B(\lambda) \cup \{0\}; \quad \text{and}
\]
\[
(3) \quad \pi_{\lambda,V}^K \quad \text{restricts to a weight-preserving bijection between} \quad \{ b \in B^K(\lambda) \mid \pi_{\lambda,V}^K(b) \neq 0 \} \quad \text{and}
\]
\[
B(\lambda).
\]

3. The crystal \( L(\infty) \) and globalization

We will now construct a crystal basis on \( U^- \) which is compatible with those on representations as described above. Subsequently, we will construct a canonical basis on \( U^- \) and the representations by “globalizing” the crystal bases in the sense of Kashiwara [Kas].

3.1. On the quantum differentials. The differentials \( e_i, \bar{e}_i \) on \( U^- \) are closely tied to the multiplicative structure of \( U \) as follows.

**Lemma 3.1.** Let \( x \in U^- \) and \( i \in I \). Then
\[
E_i x - (-1)^{p(i)p(x)} x E_i = \frac{K_i \bar{e}_i(x) - K_i^{-1} e_i(x)}{q - q^{-1}}
\]
Proof. We proceed by induction on $|x|$. If $x \in \mathbb{Q}(q)$ or $|x| = -j$ for $j \in I$, then this is trivial. Now suppose $x = uv$ for some homogeneous $u, v \in U^-$ with $|u|, |v| \in Q^- \setminus 0$. Then by induction

$$E_i x = E_i uv = (-1)^{p(i)p(u)} u E_i v + \frac{K_i e_i(u) v - K_i^{-1} e_i(u) v}{q - q^{-1}} = (-1)^{p(u)p(u)} u v E_i + \frac{K_i e_i(u) v - K_i^{-1} e_i(u) v}{q - q^{-1}} + (-1)^{p(i)p(u)} u K_i e_i(v) - u K_i^{-1} e_i(v)$$

$$= (-1)^{p(i)p(x)} x E_i + \frac{K_i e_i(x) - K_i^{-1} e_i(x)}{q - q^{-1}}.$$

\[\square\]

Lemma 3.2. Let $i \in I$ and $u \in U^-_{-}$ with $e_i'(u) = 0$. Let $M$ be a $U$-module and $\lambda \in P$. Then for any $x \in M_{\lambda}$ with $E_i x = 0$, we have

$$K_i^t E_i^t u x = \frac{q^{t(2(h_i, \lambda - \zeta) + (1 - \delta_i m)(3t + 1))}}{(q - q^{-1})^t} ((e_i')^t(u)x).$$

Proof. For $i \in I_0$, the proof is the same as that of [Kas, Cor. 3.4.6]. For $i = m$, note that both sides are zero if $t > 1$, and for $t = 1$ this trivially follows from the previous lemma. \[\square\]

Finally, we note that the differentials $e_i$, together with the left-multiplication maps $f_i$, generate a subalgebra of $\text{End}_{\mathbb{Q}(q)}(U^-)$ which is a version of Kashiwara’s boson algebra. In particular, many of the results of [Kas, Section 3.4] generalize to our setting; see [Zou] for some details on this.

3.2. The $L(\infty)$ crystal, and a less grand loop. For each $i \in I$, we define operators $\tilde{e}_i, \tilde{f}_i$ on $U^-$ as follows. Write $u = \sum F_i^{(t)} u_t$ with $e_i(u_t) = 0$; for $i \in I_0$, the existence and uniqueness of such a decomposition follows from [Kas, Section 3.4], whereas for $i = m$, this follows from [Kwo2, Lemma 4.5]. Then we set

$$\tilde{e}_i u = \sum F_i^{(t-1)} u_t, \quad \tilde{f}_i u = \sum F_i^{(t+1)} u_t.$$

Note that $\tilde{f}_m u = F_m u$, and $\tilde{e}_m(u) = e_m(u)$.

Definition 3.3. We say that an $A$-submodule $L$ of $U^-$ is a crystal lattice of $U^-$ if:

1. $U^- = \mathbb{Q}(q) \otimes_A L$;
2. $L = \bigoplus \llbracket L_{\lambda}^0 \oplus L_{\lambda}^1 \rrbracket$ where $L_{\lambda}^0 = L \cap M_{\lambda}^0$;
3. $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for all $i \in I$.

Suppose $L$ is a crystal lattice of $U^-$, and let $B \subset L/qL$. We say that $(L, B)$ is a signed crystal basis of $M$ if:

4. $B$ is a signed $\mathbb{Q}$-basis of $L/qL$ (that is, $B = B' \cup (-B')$ for some $\mathbb{Q}$-basis of $L/qL$);
5. $B = \bigcup \llbracket B_{\lambda}^0 \cup B_{\lambda}^1 \rrbracket$ where $B_{\lambda}^0 = B \cap (L_{\lambda}^0/qL_{\lambda}^1)$;
6. $\tilde{e}_i B \subset B \cup \{0\}$ and $\tilde{f}_i B \subset B \cup \{0\}$ for all $i \in I$.
7. If $b, b' \in B$ and $i \in I$, then $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b' = b$.

If $B$ satisfies (5)-(7) and is a $\mathbb{Q}$-basis of $L/qL$, then we say $(L, B)$ is an unsigned crystal basis of $U^-$, or simply a crystal basis of $U^-$. 
Theorem 3.4. Let $L(\infty) = \sum A \tilde{f}_{i_1} \cdots \tilde{f}_{i_n} 1$ and $B(\infty) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_n} 1 \mid i_1, \ldots, i_n \in I \} \backslash \emptyset$. Then $(L(\infty), B(\infty))$ is a crystal basis of $U^-$. Moreover, for $\lambda \in P^{++}$, the projection $\pi_\lambda : U^- \rightarrow V(\lambda)$ satisfies $\pi_\lambda(L(\infty)) = L(\lambda)$. Finally, the induced projection $\hat{\pi}_\lambda : L(\infty)/qL(\infty) \rightarrow L(\lambda)/qL(\lambda)$ satisfies $\hat{\pi}_\lambda(B(\infty)) = B(\lambda) \cup 0$ and induces a bijection 

$$\{ b \in B(\infty) \mid \pi_\lambda(b) \neq 0 \} \leftrightarrow B(\lambda).$$

For the remainder of this subsection, we will prove the theorem by induction on the weight grading. We prove the following statements, which form a truncated version of Kashiwara’s grand loop [Kas].

(S1) For $\xi \in Q^+(l)$, $\tilde{e}_i L(\infty)_{-\xi} \subset L(\infty)$.

(S2) For $\xi \in Q^+(l)$, $\pi_\lambda L(\infty)_{-\xi} \subset L(\lambda)_{\lambda - \xi}$ for $\lambda \in P^{++}$, so we have the induced projection $\hat{\pi}_\lambda : L(\infty)_{-\xi}/qL(\infty)_{-\xi} \rightarrow L(\lambda)_{\lambda - \xi}/qL(\lambda)_{\lambda - \xi}$.

(S3) For $\xi \in Q^+(l)$, $B(\infty)_{-\xi}$ is a basis for $L(\infty)_{-\xi}/qL(\infty)_{-\xi}$.

(S4) For $\xi \in Q^+(l-1)$ and $\lambda \in P^{++}$, $\tilde{f}_i(xv_\lambda) \equiv (\tilde{f}_i x)v_\lambda + qL(\lambda)_{\lambda - \xi}$ for $x \in L(\infty)_{-\xi}$.

(S5) For $\xi \in Q^+(l)$, $\tilde{e}_i B(\infty)_{-\xi} \subset B(\infty) \cup \{0\}$.

(S6) For $\xi \in Q^+(l)$ and $\lambda \in P^{++}$, $\hat{\pi}_\lambda$ induces a bijection between $\{ b \in B(\infty)_{-\xi} \mid \hat{\pi}_\lambda(b) \neq 0 \}$ and $B(\lambda)_{\lambda - \xi}$.

(S7) For $\xi \in Q^+(l)$ and $\lambda \in P^{++}$, if $\hat{\pi}_\lambda(b) \neq 0$ for some $b \in B(\infty)_{-\xi}$, then $\tilde{e}_i \hat{\pi}_\lambda(b) = \tilde{\pi}_\lambda(\tilde{e}_i b)$.

(S8) For $\xi \in Q^+(l)$ (resp. $\xi \in Q^+(l-1)$) and for $b \in B(\infty)_{-\xi}$, if $\tilde{e}_i b \neq 0$ (resp. $\tilde{f}_i b \neq 0$) then $\tilde{f}_i \tilde{e}_i b = b$ (resp. $\tilde{e}_i \tilde{f}_i b = b$).

Assume (S1)-(S8) for $l \geq 2$ (since they are obvious for $l = 0, 1$).

Lemma 3.5 (S4l). Let $i \in I$, $\xi \in Q^+(l-1)$ and $x \in U^-_{-\xi}$. Suppose $\lambda \in P^{++}$. Then $\tilde{f}_i(xv_\lambda) \equiv \tilde{f}_i(xv_\lambda)$ modulo $qL(\lambda)$. If we further assume that $\langle h_i, \lambda \rangle \gg 0$, then $(\tilde{e}_i x)v_\lambda \equiv \tilde{e}_i(xv_\lambda)$ modulo $qL(\lambda)$.

Proof. If $i = 0$, this follows from the same proof as [Kas, Lemma 4.4.1, Proposition 4.4.2]. Suppose $i = m$. Then $x = x_0 + F_m x_1$ with $e_m(x_0) = e_m(x_1) = 0$. Since $\tilde{f}_i(x) = F_m x_1$ and $\tilde{f}_i(xv_\lambda) = F_m x v_\lambda$ by definition, obviously $\tilde{f}_i(xv_\lambda) = (\tilde{f}_i x)v_\lambda$.

For $\tilde{e}_m$, it suffices to prove the statement when $x = x_0$ or $x = F_m x_1$. If $x = x_0$, then $\tilde{e}_m x = 0$. On the other hand,

$$\tilde{e}_m(xv_\lambda) = q^{-1} K_m E_m x v_\lambda = \frac{q^{2(h_m, \lambda - \xi)}}{q^2 - 1} \tau_m(x)v_\lambda.$$ 

Then since $\tau_m(x) \in q^{-N} L(\infty)$ for some $N$, and thus by (S2l-1) we have $\tau_m(x) \in q^{-N} L(\lambda)$ for any $\lambda$, we see that $q^{2(h_m, \lambda - \xi)} \tau_m(x)v_\lambda \in qL(\lambda)$ for $\langle h_m, \lambda \rangle \gg 0$, hence $\tilde{e}_m(xv_\lambda) \equiv 0 = (\tilde{e}_m x)v_\lambda$ modulo $qL(\lambda)$ in that case.

If $x = F_m x_1$, then $\tilde{e}_m x = x_1$. On the other hand,

$$\tilde{e}_m(xv_\lambda) = q^{-1} K_m E_m x v_\lambda = \frac{q^{2(h_m, \lambda - \xi)}}{q^2 - 1} x_1 v_\lambda - \frac{q^{2(h_m, \lambda - \xi)}}{q^2 - 1} F_m \tau_m(x_1)v_\lambda,$$

hence in particular if $\langle h_m, \lambda \rangle \gg 0$, then

$$\tilde{e}_m(xv_\lambda) - (\tilde{e}_m x)v_\lambda = \frac{q^{2(h_m, \lambda - \xi)}}{q^2 - 1} x_1 v_\lambda - \frac{q^{2(h_m, \lambda - \xi)}}{q^2 - 1} F_m \tau_m(x_1)v_\lambda \in qL(\lambda).$$
Corollary 3.6 (S2, and S3). For any $\lambda \in P^{++}$ and $\xi \in Q^+(l)$, we have $\pi_\lambda(L(\infty)_{-\xi}) = L(\lambda)_{\lambda-\xi}$. Moreover, the induced map

$$\hat{\pi}_\lambda : L(\infty)_{-\xi}/qL(\infty)_{-\xi} \to L(\lambda)_{\lambda+\xi}/qL(\lambda)_{\lambda-\xi}$$

satisfies $\hat{\pi}_\lambda(B(\infty)_{-\xi}) \setminus 0 = B(\lambda)_{\lambda-\xi}$. Finally, if $\lambda \gg \xi$, then $\pi_\lambda$ restricts to an isomorphism $L(\infty)_{-\xi} \cong L(\lambda)_{\lambda-\xi}$ and $\hat{\pi}_\lambda$ restricts to a bijection $B(\infty)_{-\xi} \leftrightarrow B(\lambda)_{\lambda-\xi}$. Consequently, $B(\infty)_{-\xi}$ is a basis for $L(\infty)_{-\xi}/qL(\infty)_{-\xi}$.

Proof. By Lemma 3.5 and (S2$\,_{l-1}$), $\pi_\lambda(L(\infty)_{-\xi}) \subset L(\lambda)$ and $L(\lambda)_{\lambda-\xi} \subset \pi_\lambda(L(\infty)_{-\xi}) + qL(\lambda)_{\lambda-\xi}$, hence $\pi_\lambda(L(\infty)_{-\xi}) = L(\lambda)_{\lambda-\xi}$ by Nakayama’s lemma. Then $\hat{\pi}_\lambda$ commutes with $\bar{f}_i$ by Lemma 3.5, hence $\hat{\pi}_\lambda(B(\infty)_{-\xi}) \setminus 0 = B(\lambda)_{\lambda-\xi}$. The final statements follow from the fact that if $\lambda \gg \xi$, then $U_{\lambda-\xi} \cong V(\lambda)_{\lambda-\xi}$ and that $B(\lambda)$ is a basis for $L(\lambda)/qL(\lambda)$.

Corollary 3.7 (S1$\,_l$ and S5$\,_l$). For $\xi \in Q^+(l)$, we have

$$\check{e}_iL(\infty)_{-\xi} \subset L(\infty) \quad \text{and} \quad \check{e}_iB(\infty)_{-\xi} \subset B(\infty) \cup \{0\}.$$  

Proof. Let $\lambda \in \bar{P}^+$ with $\lambda \gg \xi$ and $\langle h_i, \lambda \rangle \gg 0$. Let $x \in L(\infty)$.

Then $\pi_\lambda(\check{e}_ix) \equiv \check{e}_ixv_\lambda$ modulo $qL(\lambda)$ by Lemma 3.5. Since $x\tau(\lambda) = L(\lambda)$, we have $\pi_\lambda(\check{e}_ix) \equiv \check{e}_ixv_\lambda$ modulo $qL(\lambda)$. By the proof of loc. cit. is valid for $i = I_0$, and for $i = m$ observe that $\check{e}_m(x \otimes v_\lambda) = (q^{-1}K_mE_mx) \otimes v_\lambda$.

Now suppose $b = x + qL(\infty) \in B(\infty)$. Then applying $\check{\pi}_\lambda$, we have $\check{e}_ib \in B(\infty)_{-\lambda+\alpha_i} \cup \{0\}$. Moreover, by Lemma 3.5 we have $\check{\pi}_\lambda(\check{e}_ib) = \check{e}_ib$. Since $\lambda \gg \xi - \alpha_i$, $\check{\pi}_\lambda$ is a bijection $B(\infty)_{-\lambda+\alpha_i} \cup \{0\}$. We consider the case

$$\check{\pi}_\lambda(\check{e}_ib) = \check{e}_i\pi_\lambda(b)$$

for all $\lambda \in \bar{P}^+$.

Lemma 3.8 (S7$\,_l$). Let $\xi \in Q^-(l)$. If $b \in B(\infty)_{-\xi}$ with $\check{e}_ib \neq 0$, then $\pi_\lambda(\check{e}_ib) = \check{e}_i\pi(\lambda)(b)$ for all $\lambda \in \bar{P}^+$.

Proof. This is proved exactly as in [Kas, §4.6]. Namely, observe that for any $\lambda, \mu \in P^{++}$ and $x \in L(\lambda)$, we have $\check{e}_i(x \otimes v_\mu) \equiv \check{e}_ix \otimes v_\mu$ modulo $qL(\lambda) \otimes L(\mu)$; the proof in loc. cit. is valid for $i = I_0$, and for $i = m$ observe that $\check{e}_m(x \otimes v_\lambda) = (q^{-1}K_mE_mx) \otimes v_\mu$.

Now fix $\lambda \in P^{++}$, and pick a $\mu \in P^{++}$ such that $\lambda + \mu \gg \xi$. In $L(\lambda + \mu)$, $\check{e}_i$ commutes with $\pi_{\lambda+\mu}$ modulo $q$ as in the proof of (S1$\,_l$). Note that we can push this congruence to $L(\lambda) \otimes L(\mu)$ by applying $S_{\lambda,\mu}$, and then to $L(\lambda)$ by applying $P_{\lambda,\mu}$.

In particular, suppose $b \in B(\infty)_{-\xi}$ is given by $b = \check{f}_i1 + qL(\infty)$. It suffices to show $$(\check{e}_i\check{f}_i1) \equiv (\check{e}_i\check{f}_i1) \otimes v_\lambda \equiv \check{e}_i(\check{f}_i1) \otimes v_\lambda$$ modulo $qL(\lambda)$. Well, observe that $e_i(\check{f}_i1, \check{f}_i1, v_\lambda) \otimes v_\mu \equiv e_i\check{f}_i1 \otimes v_\mu$. Applying $S_{\lambda,\mu}$ to the congruence $e_i\check{f}_i1 \otimes v_\mu$ modulo $qL(\lambda + \mu)$, we have $e_i\check{f}_i1 \otimes v_\mu$ modulo $qL(\lambda) \otimes L(\mu)$. But then applying $P_{\lambda,\mu}$, we obtain the desired congruence.

Corollary 3.9 (S8$\,_l$). Let $\xi \in Q^+(l)$ (resp. $\xi \in Q^+(l - 1)$) and $b \in B(\infty)_{-\xi}$. Suppose $\check{e}_ib \neq 0$ (resp. $\check{f}_ib \neq 0$). Then $\check{f}_i\check{e}_ib = b$ (resp. $\check{e}_i\check{f}_ib = b$).

Proof. Let $b \in B(\infty)_{-\xi}$. Pick $\lambda \in \bar{P}^+$ with $\lambda \gg \xi$. We consider the case $\xi \in Q^+(l)$ and $\check{e}_ib \neq 0$, as the other case follows by a similar argument. Then since $\check{\pi}_\lambda$ is a bijection, $\check{\pi}_\lambda(\check{e}_ib) = \check{e}_i\check{\pi}(b) \neq 0$, and so $\check{\pi}_\lambda(b) = \check{\pi}(\check{f}_i\check{e}_ib)$ thus $\check{f}_i\check{e}_ib = b$. □
Corollary 3.10 (S6ℓ). Let ξ ∈ Q^+(l). Then for any λ ∈ P^{++}, \hat{\pi}_\lambda induces a bijection between \{b ∈ B(∞)_{-ξ} | \pi_\lambda(b) \neq 0\} and B(λ)_{-ξ}.

Proof. Let λ ∈ P^{++}. We already know that \hat{\pi}_\lambda(B(∞)_{-ξ}) \setminus 0 = B(λ)_{-ξ}. Now suppose b, b' ∈ B(∞)_{-ξ} such that \hat{\pi}_\lambda(b) = \hat{\pi}_\lambda(b') \neq 0. Clearly, there is some i ∈ I such that \tilde{e}_ib \neq 0, hence \hat{\pi}_\lambda(\tilde{e}_ib) = \hat{\pi}_\lambda(\tilde{e}_ib') \neq 0. But then by (S6ℓ-1), \tilde{e}_ib = \tilde{e}_ib', so b = \tilde{f}_i\tilde{e}_ib = \tilde{f}_i\tilde{e}_ib' = b'. □

This finishes the induction, and thus Theorem 3.4 is proven.

3.3. Some further properties of L(∞). We can now deduce some properties of the crystal on U^−. First, we note that L(∞) enjoys many favorable properties with respect to the bilinear form on U^−.

Proposition 3.11. We have the following.

(1) (L(∞),L(∞)) ⊂ A and hence (−,−) descends to a Q-valued bilinear form (−,−)_0 on L(∞)/qL(∞).
(2) (\tilde{e}_iu,v)_0 = (u,\tilde{f}_iv)_0 for u,v ∈ L(∞)/qL(∞).
(3) For any b,b' ∈ B(∞), (b,b')_0 = δ_{b,b'}. In particular, (−,−)_0 is positive definite.
(4) L(∞) = \{x ∈ U^− | (x,L(∞)) ∈ A\} = \{x ∈ U^− | (x,x) ∈ A\}.
(5) τ(L(∞)) = L(∞).

Proof. First, let us prove (L(∞)_{-ξ},L(∞)_{-ξ}) ⊂ A for ξ ∈ Q^+(l) by induction on l. In particular, since L(∞)_{-ξ} = \sum \tilde{f}_iL(∞)_{-ξ+\alpha_i}, it suffices to show that (\tilde{f}_iu,v) ≡ (u,\tilde{e}_iv) modulo qL(∞) for u ∈ L(∞)_{-ξ+\alpha_i} and v ∈ L(∞)_{-ξ}. We may further assume u = F^{(x)}_i u_0 and v = F^{(y)}_i v_0 for some x,y ∈ N and u_0,v_0 ∈ U^− with e'_i(u_0) = e'_i(v_0) = 0. Then the case i ∈ I_0 is virtually the same as in [Kas]. For i = m, it is even simpler: (\tilde{f}_mu,v) = (F_mu,v) = (u,e'_m(v)) = (u,\tilde{e}_m v). This proves (1) and (2). Then (3) and (4) follow exactly as proven in [Kas]. Finally, (5) follows from (4) and the τ-invariance of (−,−). □

Now let us compare the crystal on U^− with the crystal on K(λ). To facilitate this, we need to refer to the odd PBW vectors. To that end, let I = \{1 < \ldots < m\} be the standard ordering on I. Then for χ = {α_1 < \ldots < α_k} ∈ \Phi_0^+, let F_χ = F_{<,α_1} \ldots F_{<,α_k} and observe that e'_i(F_χ) = 0 for all i ∈ I_0.

In particular, let W be Kashiwara’s Boson algebra for \text{gl}(m); that is, the \Q(q)-subalgebra of End\Q(q)(U^−) generated by e_i, f_i with i ∈ I_0. Let us write \((L_{\text{gl}(m)}(∞),B_{\text{gl}(m)}(∞))\) for the crystal basis of U^−_{q}(\text{gl}(m)). Then by [Kas, Remarks 3.4.10 and 3.5.1], U^− is a direct sum of W-modules isomorphic to U^−_{q}(\text{gl}(m)) and and \((L(∞),B(∞))\) is a direct sum of crystals isomorphic to \((L_{\text{gl}(m)}(∞),B_{\text{gl}(m)}(∞))\). In fact, we can be more specific: we see that as W-modules, U^− = \bigoplus_{χ \in \Phi_0^+} U^−_{q}(\text{gl}(m))F_χ, and with respect to this decomposition\footnote{Note that in [Zou], while a similar decomposition of the lattice is claimed, the PBW vectors there are taken with respect to the opposite order. As a result, F_i,F_{m,1} is not included in the lattice, causing it to not be closed under all the operators. This is not a problem here, and as we shall see any PBW basis maps onto the crystal basis modulo q.} L(∞) = \bigoplus L_χ where L_χ is the sublattice generated by F_χ under \tilde{f}_i for i ∈ I_0.
Theorem 3.12. Let $\lambda \in P^+$, and let $\pi^K_\lambda : U^- \to K(\lambda)$ be the projection map. Then $\pi^K_\lambda(L(\infty)) = L^K(\lambda)$. Moreover, let $\hat{\pi}^K_\lambda : L(\infty)/qL(\infty) \to L^K(\lambda)/qL^K(\lambda)$ be the induced projection modulo $q$. Then $\hat{\pi}^K_\lambda$ induces a bijection between $\{ b \mid b \in B(\infty) \text{ such that } \pi^K_\lambda(b) \neq 0 \}$ and $B^K(\lambda)$.

Proof. Observe that since $\pi^K_\lambda(1) = k_\lambda$, if $\pi^K_\lambda(\tilde{f}_i y) \equiv \tilde{f}_i \pi^K_\lambda(y)$ modulo $qL(\lambda)$ for $i \in I$ and $y \in U^-$ then $\pi^K_\lambda(L(\infty)) = L^K(\lambda)$. This is obvious from the definition when $i = m$. When $i \in I_0$, suppose $y \in U^-$. Without loss of generality, we can assume $y = x F_\chi \in U^- (\mathfrak{gl}(m)) F_\chi$ for some $\chi \in \Phi^+_{\mathfrak{h}}$ and $x \in U^- (\mathfrak{gl}(m))$. Then $\tilde{f}_i(x F_\chi) = (\tilde{f}_i x) F_\chi$ hence $\pi^K_\lambda(\tilde{f}_i(x F_\chi)) = (\tilde{f}_i x) \pi^K_\lambda(F_\chi)$.

Now, as a $U_q(\mathfrak{gl}(m))$-module we have a decomposition into irreducibles $K(\lambda) \cong \bigoplus_{s=1}^n V(\lambda; \lambda')$ and by construction and the tensor product rule, we have corresponding decompositions of the crystal basis: $L^K(\lambda) \cong \bigoplus L(\lambda')$ and $B^K(\lambda) = \bigsqcup B(\lambda')$. Let $\pi^K_\lambda$ be the composition of $\pi^K_\lambda$ with projection on the $s$th component of this direct sum. Then by [Kas, (C1.6)], $(\tilde{f}_i x) \pi^K_\lambda(F_\chi) \cong \tilde{f}_i \pi^K_\lambda(x F_\chi)$ modulo $qL(\lambda')$ for each $s$. In particular, $\pi^K_\lambda(\tilde{f}_i(x F_\chi)) \equiv \tilde{f}_i \pi^K_\lambda(x F_\chi)$ modulo $qL^K(\lambda)$.

This proves that $\pi^K_\lambda$ preserves the lattice. It is then easy to see that $\pi^K_\lambda$ induces the desired bijection on the bases, as it does so on each $\mathfrak{gl}(m)$-component of the crystals. \hfill \Box

Note that Theorem 3.12 implies a slight refinement of Proposition 2.18.

Corollary 3.13. Let $\lambda \in P^+$. Then we have

$$L^K(\lambda) = \sum A^{\tilde{f}_i k_\lambda, \ldots, \tilde{f}_i k_\lambda} \text{ with the sum being over } t \geq 0, i_1, \ldots, i_t \in I;$$

$$B^K(\lambda) = \left\{ \tilde{f}^{\tilde{f}_i k_\lambda, \ldots, \tilde{f}_i k_\lambda + qL(\lambda)} | t \geq 0, i_1, \ldots, i_t \in I \right\} \setminus 0.$$

Moreover, Proposition 2.18 (1)-(3) also hold for $\lambda \in P^{++}$.

3.4. The integral form and globalization. Finally, we turn to the construction of canonical bases. For this, we first need to produce an integral form of our lattices.

Recall the integral form $\mathbb{A}U$ defined in Section 2.2. Let $L_{\mathbb{A}}(\infty) = L(\infty) \cap \mathbb{A}U^-$. Note that $\mathbb{A}U^-$ is closed under $e_i(x)$ for all $i \in I$; indeed, in the case $i \in I_0$ this follows from properties of Kashiwara’s boson algebra, and if $i = m$ this is trivial to verify. In particular, let $i \in I$ and write $u = \sum F_i^{(t)} u_t \in \mathbb{A}U^-$ where $u_t \in U^-$ with $e_i'(u_t) = 0$. Then it is easy to verify that all $u_t \in \mathbb{A}U^-$ as well, hence $\mathbb{A}U^-$ and thus $L_{\mathbb{A}}(\infty)$ is closed under $\tilde{e}_i$ and $\tilde{f}_i$ for all $i \in I$.

Now observe

$$B(\infty) \subset L_{\mathbb{A}}(\infty)/qL_{\mathbb{A}}(\infty) \subset L(\infty)/qL(\infty). \quad (3.1)$$

Recall $A_Z$ is the $\mathbb{Z}$-subalgebra of $\mathbb{Q}(q)$ generated by $q$ and $\frac{1}{1-q^t}$ for $t \geq 1$. Let $K_Z$ be the subalgebra generated by $A_Z$ and $q^{-1}$. Then we have $A_Z = \mathbb{A} \cap K_Z$. On the other hand, $(qU^-, \mathbb{A}U^-) \subset K_Z$ hence $(L_{\mathbb{A}}(\infty), L_{\mathbb{A}}(\infty)) \subset A_Z$. Therefore, since $A_Z/qA_Z = \mathbb{Z}$, we see that the specialization of the bilinear form $(-, -)_0$ is $\mathbb{Z}$-valued on $L_{\mathbb{A}}(\infty)$.

Then the following lemma immediately follows from this discussion and Proposition 3.11.

Lemma 3.14. We have that $L_{\mathbb{A}}(\infty)/qL_{\mathbb{A}}(\infty)$ is a free $\mathbb{Z}$-module with basis $B(\infty)$. Moreover,

$$B(\infty) \cup -B(\infty) = \{ u \in L_{\mathbb{A}}(\infty)/qL_{\mathbb{A}}(\infty) | (u, u)_0 = 1 \}.$$

Note that Lemma 2.5 immediately implies the following.
Corollary 3.15. Let \( \prec \) be a total order on \( I \). Then \( \mathcal{B}(\prec) \subset L_\lambda(\infty) \), and moreover

\[
\mathcal{B}(\prec) + qL_\lambda(\infty) \subset B(\infty) \cup -B(\infty).
\]

Now let \( \lambda \in P^{++} \) and \( \mu \in P^+ \). Set \( V_\lambda(\mu) = _\lambda U^- v_\lambda \) and \( K_\lambda(\mu) = _\lambda U^- k_\mu \). Then \( V_\lambda(\mu) \) (resp. \( K_\lambda(\mu) \)) is a \( _\lambda U^- \)-submodule of \( V(\lambda) \) (resp. \( K(\mu) \)). Furthermore, define \( L_\lambda(\lambda) = V_\lambda(\lambda) \cap L(\lambda) \) and \( L_\lambda^K(\mu) = K_\lambda(\mu) \cap L^K(\mu) \). We define a bar involution on \( V(\lambda) \) (resp. \( K(\mu) \)) by \( \overline{v_\lambda} = v_\lambda \) (resp. \( \overline{u k_\mu} = \overline{u} k_\mu \)) for \( u \in U^- \). Then \( K_\lambda(\mu) \) and \( V_\lambda(\lambda) \) are stable under the bar involution. Note that \( \pi_\lambda(L_\lambda(\infty)) = L_\lambda(\lambda) \) (resp. \( \pi^K_\mu(L_\lambda(\mu)) = L^K_\lambda(\mu) \)) and \( B(\lambda) \subset L_\lambda(\lambda)/qL_\lambda(\lambda) \) (resp. \( B^K(\mu) \subset L^K_\lambda(\mu)/qL^K_\lambda(\mu) \)).

We now have the necessary ingredients to construct Kashiwara’s globalization.

Theorem 3.16. For every \( \xi \in Q^+, \mu \in P^+, \) and \( \lambda \in P^{++} \), there are isomorphisms

\[
G : L_\lambda(\infty)_{-\xi}/qL_\lambda(\infty)_{-\xi} \rightarrow _\lambda U^- \cap L_\lambda(\infty) \cap L_\lambda(\infty).
\]

\[
G^K_\mu : L^K_\lambda(\mu)_{-\xi}/qL^K_\lambda(\mu)_{-\xi} \rightarrow K_\lambda(\mu)_{-\xi} \cap L^K_\lambda(\mu) \cap L^K_\lambda(\mu).
\]

\[
G_\lambda : L_\lambda(\lambda)_{-\xi}/qL_\lambda(\lambda)_{-\xi} \rightarrow V_\lambda(\lambda)_{-\xi} \cap L_\lambda(\lambda) \cap L_\lambda(\lambda).
\]

We call \( \mathcal{B} = G(B(\infty)) \) (resp. \( \mathcal{B}(\lambda) = G(\lambda)(B(\lambda)), \mathcal{B}^K(\mu) = G^K_\mu(B^K(\mu)) \)) the canonical basis of \( U^- \) (resp. the canonical basis of \( V(\lambda) \)). Furthermore, we observe that:

1. \( _\lambda U^- \cap L_\lambda(\infty) = \bigoplus_{b \in B(\infty)_{-\xi}} Z[q]G(b) \) and \( _\lambda U^- = \bigoplus_{b \in B(\infty)_{-\xi}} _\lambda U^-(b) \);
2. \( K_\lambda(\mu)_{-\xi} \cap L^K_\lambda(\mu) = \bigoplus_{b \in B(\mu)_{-\xi}} Z[q]G_\mu(b) \) and \( K_\lambda(\mu)_{-\xi} = \bigoplus_{b \in B(\mu)_{-\xi}} K_\lambda(b) \);
3. \( V_\lambda(\lambda)_{-\xi} \cap L_\lambda(\lambda) = \bigoplus_{b \in B(\lambda)_{-\xi}} Z[q]G_\lambda(b) \) and \( V_\lambda(\lambda)_{-\xi} = \bigoplus_{b \in B(\lambda)_{-\xi}} V_\lambda(b) \);
4. \( G_\lambda(\pi_\lambda(b)) = \pi_\lambda(G(b)) \), \( G^K_\mu(\pi^K_\mu(b)) = \pi^K(G(b)) \), and \( G_\lambda(\pi^K_\lambda(b)) = \pi^K_\lambda(G_\lambda(b)) \);
5. \( G(b) = G(b) \) for any \( b \in L_\lambda(\infty)/qL_\lambda(\infty) \).

Proof. Observe that for any total order \( \prec \) on \( I \), the \( \prec \)-canonical basis generates a \( \mathbb{Z} \)-submodule \( Z = \mathbb{Z}\mathcal{B}(\prec)_{-\xi} \) of \( _\lambda U^- \) such that \( AZ = _\lambda U^- \mathcal{B} \). Moreover, by Corollary 3.15 and because \( \mathcal{B}(\prec)_{-\xi} \) is a bar-invariant basis of \( _\lambda U^- \), clearly \( Z \subset _\lambda U^- \cap L_\lambda(\infty) \cap L_\lambda(\infty) \) and we see that the projections \( Z \rightarrow L_\lambda(\infty)_{-\xi}/qL_\lambda(\infty)_{-\xi} \) and \( Z \rightarrow \overline{L_\lambda(\infty)_{-\xi}/qL_\lambda(\infty)_{-\xi}} \) are injective. Then applying [Kas, Lemma 7.1.1 (ii)], we obtain the isomorphism \( G \). An entirely similar argument applies to the modules, using instead the fact that the nonzero images of \( \mathcal{B}(\prec)_{-\xi} \) under the projection is equal to the crystal basis modulo \( q \), up to possible signs, and hence lifts to a basis of the lattice, and thus the module, by Nakayama’s lemma. The statements (1)-(5) follow from the same arguments as in [Kas, Section 7]. \( \square \)

Remark 3.17. We remark that the result can be proven independently of the results of [CHW3] by mimicking the proof in [Kas, Section 7]. In particular, we can also show directly that if \( G(b) = \overline{f^\mu_i b} \), then \( G(b) \in F_i^{(n)}U^- \). However, this will not be necessary here (and indeed, will follow from the PBW realization of the crystal and canonical bases).
4. Braid Operators

We now aim to show that the canonical basis arising from the crystal structure on $U^-$ is compatible with the PBW bases of [CHW3]. To do this, we need to reinterpret the PBW bases using analogues of Lusztig’s braid operators. As mentioned in the introduction, this requires some shifting of perspective, as now these are not necessarily automorphisms but rather a network of isomorphisms between quantum enveloping algebras associated to different choices of simple roots.

4.1. Perspective. To clarify the definitions in the following sections, it helps to view Lusztig’s braid operators in a particular way (which no doubt is well known to experts, but nevertheless we find to be obscured in the standard definition of these maps). To be concrete, let’s take the example of $U_q(\mathfrak{gl}(3))$. Usually, we define this algebra with generators $E_1, E_2, F_1, F_2, q^h$ for $h \in P^\vee$ subject to the usual relations, which one might call the *agnost*ic presentation as we make no definitive choice of simple roots. We then define the braid operator $T_2$ as an automorphism lifting the action of the reflection $s_2 = s_{\alpha_2}$ on the weight data; that is, it is an automorphism which sends the $\mu$-weight space to the $s_2(\mu)$ weight space, and the $q^h$ to $q^{s_2(h)}$. Explicitly, we have e.g.

$$T_2(E_2) = -K_2^{-1}F_2, \quad T_2(E_1) = E_1E_2 - q^{-1}E_2E_1, \quad T_1(q^{\alpha_2}) = q^{-\alpha_2}, \quad T_1(q^{\alpha_1}) = q^{\alpha_1 + \alpha_2}.$$  \hfill (4.1)

For the sake of argument let us consider two formally different *agnost*ic versions of this algebra which commit to a choice of simple roots:

1. $U_1$ is the $\mathbb{Q}(q)$ algebra on generators

$$X_{\epsilon_1-\epsilon_2}, \quad X_{\epsilon_2-\epsilon_1}, \quad X_{\epsilon_2-\epsilon_3}, \quad X_{\epsilon_3-\epsilon_2}, \quad q^h$$

for $h \in P^\vee$,

satisfying the relations of $U_q(\mathfrak{gl}(3))$ with the replacements $E_1 \leftrightarrow X_{\epsilon_1-\epsilon_2}, E_2 \leftrightarrow X_{\epsilon_2-\epsilon_3}, F_1 \leftrightarrow X_{\epsilon_2-\epsilon_1}, F_2 \leftrightarrow X_{\epsilon_3-\epsilon_2}$. In particular, note that the weights satisfy $|X_\alpha| = \alpha$.

2. $U_2$ is the $\mathbb{Q}(q)$ algebra on generators

$$X_{\epsilon_1-\epsilon_3}, \quad X_{\epsilon_3-\epsilon_1}, \quad X_{\epsilon_3-\epsilon_2}, \quad X_{\epsilon_2-\epsilon_3}, \quad q^h$$

for $h \in P^\vee$,

satisfying the relations of $U_q(\mathfrak{gl}(3))$ with the replacements $E_1 \leftrightarrow X_{\epsilon_1-\epsilon_3}, E_2 \leftrightarrow X_{\epsilon_3-\epsilon_2}, F_1 \leftrightarrow X_{\epsilon_3-\epsilon_1}, F_2 \leftrightarrow X_{\epsilon_2-\epsilon_3}$. Again, $|X_\alpha| = \alpha$.

Then we can think of $T_2$ as a weight-preserving isomorphism $U_1 \rightarrow U_2$ which is the identity on $q^h$ for $h \in P^\vee$: translating (4.1) into this notation, we see that

$$T_2(X_{\epsilon_2-\epsilon_3}) = -q^{\epsilon_2-\epsilon_3}X_{\epsilon_2-\epsilon_3}, \quad T_2(X_{\epsilon_1-\epsilon_2}) = X_{\epsilon_1-\epsilon_3}X_{\epsilon_2-\epsilon_3} - q^{-1}X_{\epsilon_3-\epsilon_2}X_{\epsilon_1-\epsilon_3},$$

$$T_2(q^{\epsilon_2-\epsilon_3}) = q^{-(\epsilon_2-\epsilon_3)} = q^{\epsilon_2-\epsilon_3}, \quad T_1(q^{\epsilon_1-\epsilon_2}) = q^{\epsilon_1-\epsilon_3}q^{\epsilon_2-\epsilon_3} = q^{\epsilon_1-\epsilon_2}.$$ 

Now in the case of $U_q(\mathfrak{gl}(m|1))$, the situation is similar. The crucial difference is that we cannot formally identify different agnostic Chevalley-Serre-Yamane presentations of the algebra like we can in the classical case: indeed, interpreting (4.1) in terms of $U_q(\mathfrak{gl}(2|1))$, an obvious problem is that $E_1^2$ is nonzero and yet $(E_1E_2 - q^{-1}E_2E_1)^2 = 0$. Instead, we must treat $T_2$ in this example (and more generally, any odd braid operator) as an isomorphism to a different
presentation of the same quantum enveloping algebra: one with generators $E_1, E_2$ satisfying $E_1^2 = E_2^2 = 0$, coming from the choice of simple roots where both are isotropic.

In the following, we will make several simplifications for the sake of controlling notation and to make the construction as similar to Lusztig’s as possible. To that end, we will work with the quantum enveloping algebras associated to $\mathfrak{sl}(m|1)$ rather than $\mathfrak{gl}(m|1)$ and we will stick to the agnostic presentations; this will allow us to avoid some tedious notation. In particular, note that the different quantum enveloping algebras will have different sets of simple roots, despite the common labeling of generators, so it may be helpful to think of them as being different algebras rather than different presentations of the same algebra.

4.2. The Cartan data orbit. To that end, we say an $I \times I$ matrix $A$ is a generalized Cartan matrix (or GCM for short) of type $\mathfrak{gl}(m|1)$ if $A = [A_{ij}]_{i,j \in I}$ satisfies

1. $A_{ii} \in \{0, 2\}$ for all $i \in I$;
2. There exists $i \in \{0, \ldots, m\}$ such that $A_{ii} = A_{i+1,i+1} = 0$ and $A_{jj} = 2$ for all $j \in I$ with $j \neq i, i+1$. (Here, we include $a_{00} = 0$ and $a_{m+1,m+1} = 0$ for convenience.)
3. $A_{ij} = A_{ji} = 0$ for $j \neq i \pm 1$;
4. If $i, i+1 \in I$, then $A_{i,i+1} = A_{i+1,i} = -1$ if $A_{i,i} = 2$ or $A_{i+1,i+1} = 2$; otherwise, $A_{i,i+1} = A_{i+1,i} = 1$.

Note that any such matrix $A$ is associated to a positive system of roots $\Phi^+(A) \subset \Phi$ and choice of simple roots $\Pi(A) \subset \Phi^+(A)$ in the sense that, writing $\Pi(A) = \{\alpha_i^A \mid i \in I\}$, $(\alpha_i^A, \alpha_j^A) = A_{ij}$. Associated to $A \in \mathcal{A}$ is a parity function $p_A : I \to \mathbb{Z}/2\mathbb{Z}$, explicitly given by $p_A(i) = p(\alpha_i^A) = 1 - A_{ii}/2$. In particular, we define $I_{A,1} = \{i \in I \mid A_{ii} = 0\}$ to be the odd roots relative to $A$. Finally, observe that $I_{A,1}$ completely determines the matrix; indeed, for $i, j, k \in I$ with $j = i \pm 1$ and $k \neq i, i \pm 1$, we have

$$X_{ii} = 1 + (-1)^{p(i)}, \quad X_i = (-1)^{1-p(i)}p(j), \quad X_{ik} = 0.$$  \hspace{1cm} (4.2)

It will be convenient in the following to introduce the following shorthand notation. We will say $i, j \in I$ are connected if $i = j \pm 1$, and write $i \sim j$. Likewise, we say not connected if $i \neq j, j \pm 1$, and write $i \not\sim j$. Note that given $i, j \in I$, we either have $i \sim j, i \not\sim j$, or $i = j$.

Let $\mathcal{A}$ be the collection of GCMs of type $\mathfrak{gl}(m|1)$. We can write $\mathcal{A} = \{A^t \mid 0 \leq t \leq m\}$ where $A^t$ is the unique GCM with $a_{tt} = a_{t+1,t+1} = 0$; in particular, note that $A^m$ is the GCM associated to the standard root system $\Pi$ defined in Section 2.1. There is an action of $F_2(I)$ on $\mathcal{A}$ given by $i \cdot A^t = A^t$ for $i \neq t, t+1$, $t \cdot A^t = A^{t-1}$, $(t+1) \cdot A^t = A^{t+1}$. Note that this action satisfies

$$i \cdot j \cdot A^t = j \cdot i \cdot A^t$$
$$i \cdot (i+1) \cdot i \cdot A^t = (i+1) \cdot i \cdot (i+1) \cdot A^t$$

for all $j \neq i, i \pm 1$ and $0 \leq t \leq m$. In particular, the $F_2(I)$-action factors through $S_{m+1}$, the symmetric group on $m + 1$ letters.

Example 4.1. In the case $m = 4$, then $\mathcal{A}$ with its $S_5$-action is described in Figure 1.

Remark 4.2. Essentially, the $S_{m+1}$-action on $\mathcal{A}$ corresponds to the $S_{m+1}$-action on $\Phi$ induced by $\sigma \cdot \epsilon_j = \epsilon_{\sigma(j)}$ as in the $\mathfrak{gl}(m+1)$ case. The crucial difference between the $\mathfrak{gl}(m+1)$ and
for \( i, j, k \) determine the matrix entries. Indeed, let \( E \) naturally on words in \( I \). Then \( E \) to \( k \) observe that \( \sigma \in \varsigma \). We also observe the following fact.

We also observe the following fact.

**Lemma 4.3.** Recall that \( A^m \) is the GCM with \( A^m_{ii} = 2 - 2\delta_{i,m} \). Let \( \sigma \in S_{m+1} \). Then \( \sigma \cdot A^m = A^t \) for \( t < m \) if and only if \( l(s_{l+1}\sigma) < l(\sigma) \).

**Proof.** We can restate the problem as follows. To each \( A^t \), we associate its \( \epsilon\)-\( \delta \) sequence (cf. [CW, Section 1.3.2]); that is, a word \( w_t = a_1 \ldots a_{m+1} \) in the letters \( \{\epsilon, \delta\} \) such that \( a_{t+1} = \epsilon \) and \( a_s = \delta \) for all \( s \neq t + 1 \). Let \( \mathcal{E}\mathcal{D} \) be the set of these sequences. The symmetric group \( S_{m+1} \) acts naturally on words in \( \mathcal{E}\mathcal{D} \) via \( \sigma \cdot a_1 \ldots a_{m+1} = a_{\sigma(1)} \ldots a_{\sigma(m+1)} \); and note that for each \( w \in \mathcal{E}\mathcal{D} \), \( \sigma \cdot w \in \mathcal{E}\mathcal{D} \). In particular, we see that as \( S_{m+1} \)-sets, \( \mathcal{E}\mathcal{D} \) is isomorphic to \( \mathcal{A} \) via \( w_t \leftrightarrow A^t \).

On the other hand, note that any \( \sigma \in S_{m+1} \) can be written in the form \( \sigma = s_is_{i+1} \ldots s_{m} \) where \( \varsigma \in S_m \) and \( s_i = (i, i + 1) \) are the simple reflections. In particular, since \( S_m = \text{Stab}_{S_{m+1}}(w_{m}) \), observe that \( \sigma(w_m) = w_i \) with \( i < m \) if and only if it has a reduced expression with \( s_{i+1} \) as a left factor; in other words, if \( \sigma(w_m) = w_i \), then \( l(s_{i+1}\sigma) < l(\sigma) \).

Translating back to \( A \), we see that if \( \sigma \cdot A^m = A^t \), then \( l(s_{l+1}\sigma) < l(\sigma) \).

**Corollary 4.4.** Let \( \sigma \in S_{m+1} \) such that \( l(\sigma s_{i+1}) > l(\sigma) \) for some \( i \in I \). Set \( X = \sigma^{-1} \cdot A^m \). Then \( I_{X,1} \neq \{i, i + 1\} \).

**Proof.** By the previous lemma, \( X \neq A^t \), hence the result follows by definition.

Lastly, it will be helpful in several places to note that if \( i \) and \( j \) are connected, then

\[
\frac{q^{-A_{ij}} - q^{A_{ij}}}{q - q^{-1}} = (-1)^{p_A(i) + p_A(j)}
\]

(4.4)

4.3. A family of quantum enveloping algebras. Now for \( X \in \mathcal{A} \), we will now associate a quantum enveloping algebra \( \mathbf{U}(X) \) with generators \( E_i = E_{X,i}, F_i = F_{X,i}, \) and \( K_i^{\pm 1} = K_{X,i}^{\pm 1} \) for \( i \in I \), and have parity \( p = p_X \) given by \( p(E_i) = p(F_i) = p_X(i) \) and \( p(K_i) = 0 \). (For ease
of reading, we will drop the $X$ subscript when the $X$ is clear from context.) These generators satisfy the relations

\begin{align*}
K_i K_j &= K_j K_i, \quad K_i K_i^{-1} = 1 \quad \text{for } i, j \in I; \\
K_i E_j K_i^{-1} &= q^{X_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-X_{ij}} F_j \quad \text{for } i, j \in I; \\
E_i F_j - (1) \delta_{pq} F_j E_i &= \frac{K_i - K_i^{-1}}{q - q^{-1}}, \quad \text{for } i, j \in I \\
E_i^2 &= F_i^2 = 0 \quad \text{if } i \in I_1; \\
E_i E_j &= E_j E_i, \quad F_i F_j = F_j F_i \quad \text{if } i \sim j \in I; \\
E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 &= 0 \quad \text{if } i \sim j \in I \text{ and } p(i) = 0; \\
F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 &= 0 \quad \text{if } i \sim j \in I \text{ and } p(i) = 0; \\
E_i E_j E_k E_i &= (-1)^{\delta_{pq}} (E_i E_j E_j + E_j E_i E_k E_i) \quad \text{if } i, j, k \in I \text{ such that } j \sim i, \\
&\quad + (-1)^{\delta_{pq}} (E_k E_j E_i E_i + E_i E_j E_k E_i) \quad i \sim k, j \neq k, \text{ and } p(i) = 1; \\
F_i F_j F_k F_i &= (-1)^{\delta_{pq}} (F_i F_j F_j F_i + F_j F_i F_k F_i) \quad \text{if } i, j, k \in I \text{ such that } j \sim i, \\
&\quad + (-1)^{\delta_{pq}} (F_k F_j F_i F_i + F_i F_j F_k F_i) \quad i \sim k, j \neq k, \text{ and } p(i) = 1.
\end{align*}

Note that $U(X)$ has a natural $Q$-grading given by $|E_i| = \alpha_i^X = -|F_i|, |K_i| = 0$. We note that $(\alpha_i^X, \alpha_i^Y) = X_{ij}$, so $K_i u K_i^{-1} = q^{(\alpha_i^X, |u|) \cdot X} u$.

The algebra $U(X)$ has several standard properties of quantum enveloping algebras. In particular, it has a triangular decomposition

$$U(X) = U^-(X) \otimes U^0(X) \otimes U^+(X) = U^+(X) \otimes U^0(X) \otimes U^-(X),$$

where $U^-(X)$ (resp. $U^0(X), U^+(X)$) is the subalgebra generated by the $F_i$ ($K_i, E_i$) for $i \in I$. We also define $U^{\geq 0}(X) = U^0(X) U^+(X)$ and $U^{\leq 0} = U^0(X) U^-(X)$. Moreover, $U^-(X)$ has quantum differentials $e_i = e_{X, i}, \bar{e}_i = \tau_{X, i} = -\circ e_i \circ - : U^-(X) \to U^-(X)$ for $i \in I$ satisfying

$$e_i (F_j) = \delta_{ij}, \quad e_i(x y) = e_i(x) y + (-1)^{\delta_{pq}} p(x) q^{(\alpha_i^X, |x|)} x e_i(y),$$

$$E_i x - (1)^{\delta_{pq}} x E_i = \frac{K_i \bar{e}_i(x) - K_i^{-1} e_i(x)}{q - q^{-1}}.$$

We also observe that there are a number of important automorphisms of $U(X)$. Indeed, we define maps $\tau, \bar{\tau}$ satisfying

$$\tau(E_i) = E_i, \quad \tau(F_i) = F_i, \quad \tau(K_i) = (1)^{\delta_{pq}} K_i^{-1}, \quad \tau(x y) = \tau(y) \tau(x).$$

$$\rho(E_i) = F_i, \quad \rho(F_i) = E_i, \quad \rho(K_i) = K_i, \quad \rho(x y) = \rho(y) \rho(x).$$

$$E_i = E_i, \quad F_i = F_i, \quad K_i = K_i^{-1}, \quad q = q^{-1} \quad \bar{\tau} = \tau \bar{\tau}.$$

Equipped with this family of algebras, we now define analogues of Lusztig’s braid operators.
Theorem 4.5. Let \( i \in I \) and \( X \in A \), \( e = \pm 1 \), and set \( Y = i \cdot X \). There exists a \( \mathbb{Q}(q) \)-linear algebra isomorphisms \( T'_{i,e}, T''_{i,e} : U(X) \to U(Y) \) satisfying

\[
T'_{i,e}(E_{X,j}) = \begin{cases} 
-(1)^{\rho Y(i)} K^{-1}_{Y,i} F_{Y,i} & \text{if } j = i; \\
E_{Y,j} E_{Y,i} - (1)^{\rho Y(i) \rho Y(j)} q^{e Y(i)} E_{Y,i} E_{Y,j} & \text{if } j \sim i; \\
E_{Y,j} & \text{otherwise;}
\end{cases}
\]

\[
T''_{i,e}(F_{X,j}) = \begin{cases} 
-(1)^{\rho Y(i)} E_{Y,i} K^{e}_{Y,i} & \text{if } j = i; \\
F_{Y,i} F_{Y,j} - (1)^{\rho Y(i) \rho Y(j)} q^{e Y(i)} F_{Y,i} F_{Y,j} & \text{if } j \sim i; \\
F_{Y,j} & \text{otherwise;}
\end{cases}
\]

\[
T'_{i,e}(K_{X,j}) = \begin{cases} 
(1)^{\rho Y(i)} K_{Y,i}^{-1} & \text{if } j = i; \\
(1)^{\rho Y(i) \rho Y(j)} K_{Y,i} K_{Y,j} & \text{if } j \sim i; \\
K_{Y,j} & \text{otherwise;}
\end{cases}
\]

and

\[
T''_{i,e}(E_{X,j}) = \begin{cases} 
-F_{Y,i} K^{e}_{Y,i} & \text{if } j = i; \\
E_{Y,i} E_{Y,j} - (1)^{\rho Y(i) \rho Y(j)} q^{e Y(i)} E_{Y,i} E_{Y,j} & \text{if } j \sim i; \\
E_{Y,j} & \text{otherwise;}
\end{cases}
\]

\[
T''_{i,e}(F_{X,j}) = \begin{cases} 
-K^{-e}_{Y,i} E_{Y,i} & \text{if } j = i; \\
F_{Y,i} F_{Y,j} - (1)^{\rho Y(i) \rho Y(j)} q^{-e Y(i)} F_{Y,i} F_{Y,j} & \text{if } j \sim i; \\
F_{Y,j} & \text{otherwise;}
\end{cases}
\]

\[
T''_{i,e}(K_{X,j}) = \begin{cases} 
(1)^{\rho Y(i)} K_{Y,i}^{-1} & \text{if } j = i; \\
(1)^{\rho Y(i) \rho Y(j)} K_{Y,i} K_{Y,j} & \text{if } j \sim i; \\
K_{Y,j} & \text{otherwise;}
\end{cases}
\]

Remark 4.6. Let us make two observations about this theorem.

1. Note that the maps \( T'_{i,e} \) and \( T''_{i,e} \) are parity-preserving. In fact, they are also weight-preserving in the sense of Section 4.1. This will be convenient to keep in mind for transporting coefficients between the algebras associated to different \( X \)'s.

2. We note that Theorem 4.5 and the proof can be generalized to \( \text{gl}(m|n) \); one only has to check a few additional cases. However, we don’t need this generality, and as the proof for our special case is already quite involved, we do not do this here.

In the remainder of this subsection, we will prove Theorem 4.5 and then verify that the maps satisfy the braid relations in Lemmas 4.13 and 4.15. To do so succinctly, first observe that

\[
T'_{i,e} = \tau T''_{i,e} \tau, \quad T'_{i,-e} = - T'_{i,e} \rho = \rho T'_{i,e} \rho, \quad T''_{i,e} = (T'_{i,e})^{-1}; \quad (4.14)
\]

it is easy to verify these identities on the generators (for the \( j \) connected to \( i \) case, see Lemma 4.14). In particular, to prove the desired results in the case of \( T'_{i,1} \), as all the remaining cases follow immediately from (4.14)
We need to verify that the images of the generators of $U(X)$ under $T'_{i,1}$ satisfy the relations (4.5)-(4.13). In the case of (4.5), this is trivial. The relation (4.6) essentially follows from the construction, but in any case it is easy to verify directly. In order to keep the proof of the remaining relations digestible, we will break up the verification into lemmas.

Since the calculations get quite involved, we will drop the subscripts on the generators of $U(Y)$ for the sake of readability. First, we check the relation (4.8).

**Lemma 4.7.** If $i, j \in I$ such that $p_X(j) = 1$, then

$$T'_{i,1}(E_{X,j})^2 = T'_{i,1}(F_{X,j})^2 = 0.$$  

**Proof.** This is trivial if $i$ is not connected to $j$, so suppose $i$ is connected to $j$. In this case, (4.3) implies $p(j) = 1 + p_X(i) = 1 + p(i)$ hence $p(i)p(j) = 0$ and $Y_{ij} = -1$. Then $T'_{i,1}(E_{X,j}) = E_i E_j - q^{Y_{ij}} E_i E_j$, and so

$$T'_{i,1}(E_{X,j})^2 = (E_i E_j - q^{-1} E_i E_j)^2$$

$$= E_i E_j E_i E_j - q^{-1} E_i E_j E_i E_j - q^{-1} E_i E_j E_i E_j + q^{-2} E_i E_j E_i E_j.$$

Now observe that either $E_j^2 = 0$ or $E_i^2 = 0$; let’s assume $E_i^2 = 0$ as the computation is similar in the other case. Then using the identity $E_i E_j E_i E_j = E_i E_j^{(2)} E_i$ (which follows from the Serre relation for $i, j$)

$$T'_{i,1}(E_{X,j})^2 = E_i E_j^{(2)} E_i - q^{-1}(q + q^{-1}) E_i E_j^{(2)} E_i + q^{-2} E_i E_j^{(2)} E_i = 0.$$  

The statement for $F_{X,j}$ is proved similarly. □

Next, we verify the commutation relation (4.7) in two steps.

**Lemma 4.8.** If $i, j, k \in I$ with $j \neq k$, then

$$T'_{i,1}(E_{X,j}) T'_{i,1}(F_{X,k}) = (-1)^{p_X(j)p_X(k)} T'_{i,1}(F_{X,k}) T'_{i,1}(E_{X,j}).$$  

**Proof.** Let $c_{jk} = T'_{i,1}(E_{X,j}) T'_{i,1}(F_{X,k}) - (-1)^{p_X(j)p_X(k)} T'_{i,1}(F_{X,k}) T'_{i,1}(E_{X,j})$. We want to prove that $c_{jk} = 0$ for all $j \neq k$. First, observe that if either $j$ or $k$ is not connected to $i$, then the statement is trivially true; indeed, if $j$ is not connected to $i$, then $T'_{i,1}(E_{X,j}) = E_j$ and $p(j) = p_X(j)$. On the other hand, $T'_{i,1}(F_{X,k})$ is a polynomial in the elements $K_i$, $E_i$, $F_k$, and $F_i$ with $p(T'_{i,1}(F_{X,k})) = p_X(k)$. Since $E_j$ supercommutes with all of these elements, the statement in this case follows.

The remaining cases are when $j$ and $k$ are both either equal to or connected to $i$.

First, suppose $k$ is connected to $j = i$. Then we have

$$c_{ik} = (-K_i^{-1} F_i)(F_i F_k - (-1)^{p(i)p(k)} q^{-Y_{ik}} F_k F_i)$$

$$- (-1)^{p_X(i)p_X(k)} (F_i F_k - (-1)^{p(i)p(k)} q^{-Y_{ik}} F_j F_i)(-K_i^{-1} F_i)$$

$$= -K_i^{-1}(F_i^2 F_k - (-1)^{p(i)p(k)} q^{-Y_{ik}} F_i F_k F_i$$

$$- (-1)^{p(i)p(k)} q^{-Y_{ik}} F_i F_k F_i + (-1)^{p(i)} q^{-2Y_{ik}} F_i F_k F_i) F_k F_i$$

Now there are two subcases. If $p_X(i) = p(i) = 1$, then $Y_{ii} = 0$ and $F_{Y_{ii}} = 0$ whence

$$c_{ik} = (-1)^{p(k)} K_i^{-1}(q^{-Y_{ik}} - q^{-Y_{i_k}} F_i F_k F_i = 0.$$ 


If $p_X(i) = p(i) = 0$, then $Y_{ii} = 2$ and $Y_{ik} = -1$, so we see that
\[ c_{ik} = K^{-1}_i (E^2_i F_k - [2] F_i F_k F_i + F_k F_i^2) = 0. \]

We note the case $j$ is connected to $k = i$ is entirely similar.

Now suppose $j$ and $k$ are connected to $i$. Note that then $p_X(j)p_X(k) = p(j)p(k) = 0$; moreover, if $p(i) = 1$, then either $p(j) = 1$ or $p(k) = 1$, hence $p(i)p(j) + p(i)p(k) = p(i)$. Then we have
\[ c_{ik} = (E_j E_i - (-1)^{p(i)p(j)} q^{Y_{ij}} E_i E_j)(F_i F_k - (-1)^{p(i)p(k)} q^{-Y_{ik}} F_k F_i) \]
\[ - (F_i F_k - (-1)^{p(i)p(k)} q^{-Y_{ik}} F_i F_k)(E_j E_i - (-1)^{p(i)p(j)} q^{Y_{ij}} E_i E_j) \]
\[ = E_j E_i F_k - (-1)^{p(i)p(k)} q^{-Y_{ik}} E_i F_k q^{Y_{ij}} (E_i F_i - (-1)^{p(i)} F_i E_i) E_j F_k \]
\[ - q^{-Y_{ik}} E_j F_k (E_i F_i - (-1)^{p(i)} F_i E_i) q^{Y_{ij}} (E_i F_i - (-1)^{p(i)} F_i E_i) E_j \]
\[ = \frac{1}{q-1} E_j ((1 - q^{2Y_{ij}} - 1 + q^{2Y_{ij}}) K_i - (1 - 1 - q^{-2Y_{ik}} + q^{-2Y_{ik}}) K^{-1}_i) F_k = 0. \]

This finishes the proof. \qed

Next, we check (4.7) in the case $j = k$.

**Lemma 4.9.** If $i, j \in I$, then
\[ T_{i,1}^j (E_{X,j}) T_{i,1}^j (F_{X,j}) - (-1)^{p_X(j)} T_{i,1}^j (F_{X,j}) T_{i,1}^j (E_{X,j}) = \frac{T_{i,1}^j (K_{X,j}) - T_{i,1}^j (K_{X,j})^{-1}}{q - q^{-1}}. \]

**Proof.** Observe that the statement is trivially true for $j$ not connected or equal to $i$, and if $j = i$ then the statement is easy to verify. Therefore, let us assume that $j$ is connected to $i$.

Let $c_{jj} = T_{i,1}^j (E_{X,j}) T_{i,1}^j (F_{X,j}) - (-1)^{p_X(j)p_X(j)} T_{i,1}^j (F_{X,j}) T_{i,1}^j (E_{X,j})$. Then the statement follows by verifying that $c_{jj} = (-1)^{p_X(j)} \frac{K_i K_j - K^{-1}_i K^{-1}_j}{q - q^{-1}}$. To that end, we first do some preliminary computations. First, recall that $p_X(j) = p_Y(i) + p_Y(j)$. Then
\[ z_1 = E_j E_i F_j F_i - (-1)^{p_X(j)} F_j F_i E_j E_i \]
\[ = E_j E_i q^{Y_{ij}} K_i - q^{Y_{ij}} K^{-1}_i + (-1)^{p_Y(i)} F_j E_i \frac{q^{Y_{ij}} K_j - q^{-Y_{ij}} K^{-1}_j}{q - q^{-1}} \]
\[ z_2 = E_j E_i F_j F_i - (-1)^{p_X(j)} F_j F_i E_j E_i \]
\[ = (-1)^{p_Y(i)} F_j E_i K_i - K^{-1}_i + (-1)^{p_Y(i)} F_j E_i \frac{K_j - K^{-1}_j}{q - q^{-1}} \]
\[ z_3 = E_j E_i F_j F_i - (-1)^{p_X(j)} F_j F_i E_j E_i \]
\[ = (-1)^{p_Y(i)} F_j E_i K_j - K^{-1}_j + (-1)^{p_Y(j)} F_j E_i \frac{K_i - K^{-1}_i}{q - q^{-1}} \]
\[ z_4 = E_j E_i F_j F_i - (-1)^{p_X(j)} F_j F_i E_j E_i \]
\[ = E_j E_i q^{Y_{ij}} K_j - q^{Y_{ij}} K^{-1}_j + (-1)^{p_Y(j)} F_j E_i \frac{q^{Y_{ij}} K_i - q^{-Y_{ij}} K^{-1}_i}{q - q^{-1}} \]
Now observe that
\[
c_{jj} = z_1 - (-1)^{p_1(j)p_1(j)} q^{-Y_{ij}} z_2 - (-1)^{p_1(j)p_1(j)} q^{Y_{jj}} z_3 + z_4
\]
\[
= E_j F_j \frac{q^{-Y_{ij}} - q^{Y_{ij}}}{q - q^{-1}} K_i^{-1} - (-1)^{p_1(j)} F_j E_j \frac{q^{-Y_{ij}} - q^{Y_{ij}}}{q - q^{-1}} K_i^{-1}
\]
\[
+ E_i F_i \frac{q^{-Y_{ij}} - q^{Y_{ij}}}{q - q^{-1}} K_j - (-1)^{p_1(i)} F_i E_i \frac{q^{-Y_{ij}} - q^{Y_{ij}}}{q - q^{-1}} K_j
\]
\[
= (-1)^{p_1(i)} q^{Y_{ij}} \frac{K_j K_i - K_j^{-1} K_i^{-1}}{q - q^{-1}}.
\]
This finishes the proof. \(\square\)

Now we need to verify the various Serre relations. The next lemma checks the relation (4.9).

Lemma 4.10. If \(i, j, k \in I\) such that \(j \sim k\), then
\[
T'_{i,1}(E_{X,j}) T'_{i,1}(E_{X,k}) = T'_{i,1}(E_{X,k}) T'_{i,1}(E_{X,j}).
\]

Proof. Again, as the \(E\) and \(F\) cases are similar, we will prove this for \(E\). We need to verify that \(T'_{i,1}(E_{X,j}) T'_{i,1}(E_{X,k}) = T'_{i,1}(E_{X,k}) T'_{i,1}(E_{X,j})\) when \(k \sim j\). Well, if neither \(j\) nor \(k\) is connected to \(i\), this is trivial, so suppose \(j \sim i \sim k\). Moreover, observe that if \(p_X(i) = 0\), then one or less of \(j, k\) are odd, and so the calculation is formally identical to the same identity for \(U_q(\mathfrak{sl}(4))\); therefore, we may as well assume \(p_X(i) = 1\). Then either \(p_X(j) = 0\) or \(p_X(k) = 0\), and as the roles of \(j\) and \(k\) are symmetric, we can assume \(p_X(j) = 0\) without loss of generality.

Then note that \(p_X(k) = 1\), and so \(p(j) = 1 = p(i)\) and \(p(k) = 0\). Therefore we see that it suffices to show that \(T'_{i,1}(E_{X,j}) = E_j E_i + q E_i E_j\) and \(T'_{i,1}(E_{X,k}) = E_k E_i - q^{-1} E_i E_k\) commute. Well,
\[
(E_j E_i + q E_i E_j)(E_k E_i - q^{-1} E_i E_k) = E_j E_i E_k E_i + q E_i E_j E_k E_i - E_i E_j E_k E_i
\]
\[
(E_k E_i - q^{-1} E_i E_k)(E_j E_i + q E_i E_j) = E_k E_i E_j E_i - q^{-1} E_i E_j E_k E_i - E_i E_k E_i E_j.
\]
The difference of these two equations is zero by (4.12). \(\square\)

Next up are (4.10) and (4.11).

Lemma 4.11. If \(i, j, k \in I\) such that \(j \sim k\) and \(p_X(j) = 0\). Then
\[
T'_{i,1}(E_{X,j})^2 T'_{i,1}(E_{X,k}) - [2] T'_{i,1}(E_{X,j}) T'_{i,1}(E_{X,k}) T'_{i,1}(E_{X,j}) + T'_{i,1}(E_{X,k}) T'_{i,1}(E_{X,j})^2 = 0,
\]
\[
T'_{i,1}(E_{X,j})^2 T'_{i,1}(E_{X,k}) - [2] T'_{i,1}(F_{X,j}) T'_{i,1}(F_{X,k}) T'_{i,1}(F_{X,j}) + T'_{i,1}(F_{X,k}) T'_{i,1}(F_{X,j})^2 = 0.
\]

Proof. Again, due to similarity of the arguments, we will prove this only for the \(E\)’s. If neither \(j\) nor \(k\) is connected to \(i\), this is trivial. Otherwise, we must check case by case. Suppose first that \(j \sim k \sim i\) and \(j \neq i\). Then observe that \(T'_{i,1}(E_{X,j}) = E_j\) and \(T'_{i,1}(E_{X,k}) = E_k E_i - (-1)^{p_1(i)p_1(k)} q^{Y_{jk}} E_i E_k\). Then the Serre relation follows by observing that \(E_j\) and \(E_i\) commute, so if \(S_{jk} = E_j^2 E_k - [2] E_j E_k E_j + E_k E_j^2 = 0\), then
\[
T'_{i,1}(E_{j})^2 T'_{i,1}(E_{k}) - [2] T'_{i,1}(E_{j}) T'_{i,1}(E_{k}) T'_{i,1}(E_{j}) + T'_{i,1}(E_{k}) T'_{i,1}(E_{j})^2
\]
\[
= S_{jk} E_i - (-1)^{p_1(i)p_1(k)} q^{Y_{jk+1}} E_i S_{jk} = 0
\]
The remaining cases are when we have $k = i$ or $j = i$. It will be convenient to observe the following commutation relation: if $\ell \sim i$, then using (4.7) and (4.4), we observe that

$$T_{i,1}'(E_{X,\ell})T_{i,1}'(E_{X,i}) = q^{Y_{i\ell} + Y_{i\ell}'}(-1)^{p(i)+p(\ell)} T_{i,1}'(E_{X,i})T_{i,1}'(E_{X,\ell}) - (-1)^{p(i)+p(\ell)}q^{-Y_{i\ell} + Y_{i\ell}'} E_{X,\ell}.$$ 

First, suppose $k = i$. Note that either $p_X(i) = 0$ and hence $p(j) = p(i) = 0$, or $p_X(i) = 1 = p(i) = p(j)$. The first case is a $U_q(\mathfrak{sl}(3))$ calculation, which is known. Therefore we may assume $p(i) = p(j) = 1$, which implies that $Y_{i\ell} = 0$, and $Y_{ij} = 1$. Note that

$$T_{i,1}'(E_{X,i})E_j = q^{-1}E_j T_{i,1}'(E_{X,i}).$$

Then we compute that

$$T_{i,1}'(E_{X,j})^2 T_{i,1}'(E_{X,i}) = q T_{i,1}'(E_{X,j}) T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,j}) - E_j T_{i,1}'(E_{X,i})$$

$$= q T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,j})^2 - (q^2 + 1) E_j T_{i,1}'(E_{X,j}).$$

$$T_{i,1}'(E_{X,j})^2 T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,j}) = q T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,j}) T_{i,1}'(E_{X,j}) - E_j T_{i,1}'(E_{X,j}).$$

It is now easy to verify that

$$T_{i,1}'(E_{X,j})^2 T_{i,1}'(E_{X,i}) - (q + q^{-1}) T_{i,1}'(E_{X,j}) T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,j}) + T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,j})^2 = 0.$$ 

The last case is when $j = i$; in particular, note that $p(i) = 0$ hence $Y_{i\ell} = 2$ and $Y_{ik} = -1$. Now we observe that

$$T_{i,1}'(E_{X,i}) E_k = q E_k T_{i,1}'(E_{X,i}).$$

In this case,

$$T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,k}) = q T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,k}) - T_{i,1}'(E_{X,i}) E_k$$

$$= q T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,k}) - (q^2 + 1) T_{i,1}'(E_{X,i}) E_k$$

$$T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,i}) = q T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,i}) - E_k T_{i,1}'(E_{X,i}).$$

Then we see that

$$T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,k}) - (q + q^{-1}) T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,i}) + T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,i}) T_{i,1}'(E_{X,i}) = 0.$$ 

Finally, we must verify (4.12) and (4.13). As these relations are so long to state, it will help to introduce the following notation. For $t \in \{0, 1\}$, let $S_t(x_1, x_2, x_3) \in \mathbb{Q}(q) \langle x_1, x_2, x_3 \rangle$ be the polynomial in three non-commuting variables given by

$$S_t(x_1, x_2, x_3) = [2] x_2 x_1 x_3 x_2 - (-1)^t (x_2 x_3 x_2 x_1 + x_1 x_2 x_3 x_2 - x_2 x_1 x_2 x_3 - x_3 x_2 x_1 x_2).$$

Then for instance, we see that (4.12) is the statement that $S_{p(i)}(E_i, E_j, E_k) = 0$ for $i \sim j \sim k$ with $i \neq k$. 

Lemma 4.12. Let $i, j, k, \ell \in I$ with $j \sim k \sim \ell$, $j \neq \ell$, and $p_X(k) = 1$. Then
\[
S_{p_X(j)}(T'_{i,1}(E_{X,j}), T'_{i,1}(E_{X,k}), T'_{i,1}(E_{X,\ell})) = S_{p_X(j)}(T'_{i,1}(F_{X,j}), T'_{i,1}(F_{X,k}), T'_{i,1}(F_{X,\ell})) = 0.
\]

Proof. Again, the $E$ and $F$ cases are similar, so we will only prove the $E$ case. We also note that necessarily $p(j)p(\ell) = 0$. Furthermore, if none of $j, k, \ell$ are connected or equal to $i$, then this is trivial to verify.

First suppose $k$ is not connected or equal to $i$, but $j$ or $\ell$ is connected to $i$; without loss of generality, we can assume $j$ is connected to $i$. Since $p_X(k) = p(k) = 1$, in this case we necessarily have $p(i) = 0$ and $Y_{ij} = -1$. Then $T'_{i,1}(E_{k,X}) = E_{k}, T'_{i,1}(E_{\ell}) = E_{\ell}, T'_{i,1}(E_{X,j}) = E_j E_i - q^{-1} E_i E_j$. Since $E_i$ commutes with $E_k, E_{\ell}$, we see that
\[
S_{p(j)}(T'_{i,1}(E_{j}), T'_{i,1}(E_{k}), T'_{i,1}(E_{\ell})) = S_{p(j)}(E_j, E_k, E_{\ell})E_i - q^{-1} E_i S_{p(j)}(E_j, E_k, E_{\ell}) = 0.
\]

Next, suppose $k \sim i$ and assume wlog that $j = i$. Observe that since $p(k) = 1 + p(i)$, $p(i)p(k) = 0$ and $Y_{ik} = -1$. Then $T'_{i,1}(E_{k,X}) = E_k E_i - q^{-1} E_i E_k, T'_{i,1}(E_{\ell}) = E_{\ell},$ and $T'_{i,1}(E_{X,i}) = -( -1)^{p(i)} K^{-1}_i F_i$. Note the identies
\[
T'_{i,1}(E_{X,i})T'_{i,1}(E_{X,\ell}) = T'_{i,1}(E_{X,\ell})T'_{i,1}(E_{X,i}),
\]
\[
T'_{i,1}(E_{X,k})T'_{i,1}(E_{X,i}) = (-1)^{p(i)} q^{Y_{ii} - 1} T'_{i,1}(E_{X,i})T'_{i,1}(E_{X,k}) - (-1)^{p(i)} q^{Y_{ii} - 1} E_j.
\]

Now let $x_{abcd} = T'_{i,1}(E_{X,a})T'_{i,1}(E_{X,b})T'_{i,1}(E_{X,c})T'_{i,1}(E_{X,d})$. Then
\[
x_{k\ell ki} = (-1)^{p(i)} q^{Y_{ii} - 1} x_{k\ell ik} - (-1)^{p(i)} q^{Y_{ii} - 1} T'_{i,1}(E_{X,k})E_i E_k.
\]
\[
x_{ik\ell k} = (-1)^{p(i)} q^{1 - Y_{ii}} x_{k\ell ik} + E_j E_i T'_{i,1}(E_{X,k}).
\]
\[
x_{ik\ell k} = T'_{i,1}(E_{X,k})E_k E_{\ell}.
\]
\[
x_{\ell k\ell k} = E_{\ell} T'_{i,1}(E_{X,j})E_j
\]
We want to show that $x_{k\ell ki} + x_{ik\ell k} - x_{k\ell ik} - x_{\ell k\ell k} = (-1)^{p(i)} |2| x_{k\ell ik}$. Now the calculation proceeds slightly differently depending on $p(i)$, so we split into subcases. Suppose first $p(i) = 0$, and so $p(k) = 1, Y_{ii} = 2$, and $Y_{ik} = -1$. Then we have
\[
x_{k\ell ki} = q x_{k\ell ik} - q E_k E_i E_{\ell} E_k + E_i E_k E_{\ell} E_k.
\]
\[
x_{ij\ell j} = q^{-1} x_{k\ell ik} + E_k E_i E_{\ell} E_k - q^{-1} E_k E_i E_{\ell} E_k.
\]
\[
x_{ik\ell k} = E_k E_i E_k E_{\ell}.
\]
\[
x_{\ell k\ell k} = E_{\ell} E_k E_k
\]
In particular, we see that
\[
x_{k\ell ki} + x_{ik\ell k} - x_{k\ell ik} - x_{\ell k\ell k} = [2] x_{k\ell ik} - S_{p(i)}(E_i, E_{\ell}) = [2] x_{k\ell k}.
\]
Now suppose that $p(i) = 1$, in which case $p(k) = 0$, $Y_{ii} = 0$, and $Y_{ik} = -1$. Then we have
\[ x_{k\ell ki} = q^{-1}x_{ki\ell k} + q^{-1}E_k E_i E_\ell E_k - q^{-2}E_i E_k E_\ell E_k. \]
\[ x_{i\ell k\ell} = q^{-1}x_{ki\ell k} + E_k E_\ell E_k - q^{-1}E_k E_\ell E_k. \]
\[ x_{k\ell k\ell} = E_k E_i E_\ell E_k - q^{-1}E_i E_k E_\ell E_k. \]
\[ x_{\ell k i\ell} = E_\ell E_k E_i E_\ell - q^{-1}E_i E_k E_\ell E_k. \]
Now observe that using the Serre relation (4.10) repeatedly,
\[ E_k E_\ell E_k E_i - E_\ell E_k E_i E_k - E_k E_i E_k E_\ell = -E_i E_k^{(2)} E_\ell - E_i E_\ell E_k^{(2)} = -E_i E_k E_\ell E_k; \]
Then in particular,
\[ x_{k\ell ki} + x_{ki\ell k} - x_{k\ell k\ell} - x_{\ell k i\ell} = -[2]x_{ki\ell k} - q^{-1}E_i (E_k^2 E_\ell - [2]E_k E_\ell E_k + E_\ell E_k^2) = -[2]x_{ki\ell k}. \]
Finally, suppose $k = i$. We can assume without loss of generality that $p(j) = 0 = p_X(\ell)$ and $p(\ell) = 1 = p_X(j)$, hence $Y_{ii} = 0$, $Y_{ij} = -1$, and $Y_{i\ell} = 1$. Then $T_{i,1}^j(E_{X,j}) = E_j E_i - q^{-1}E_i E_j$, $T_{i,1}^\ell(E_{X,\ell}) = E_\ell E_i + qE_i E_\ell$, and $T_{i,1}^j(E_{X,i}) = K_i^{-1}F_i$, and we have the identities
\[ T_{i,1}^j(E_{X,j})T_{i,1}^\ell(E_{X,i}) = -q^{-1}T_{i,1}^j(E_{X,i})T_{i,1}^\ell(E_{X,j}) + q^{-1}E_j. \]
\[ T_{i,1}^\ell(E_{X,\ell})T_{i,1}^j(E_{X,i}) = qT_{i,1}^\ell(E_{X,i})T_{i,1}^j(E_{X,\ell}) - qE_\ell. \]
\[ E_i T_{i,1}^j(E_{X,i}) = qT_{i,1}^j(E_{X,i})E_\ell \]
\[ E_j T_{i,1}^\ell(E_{X,i}) = q^{-1}T_{i,1}^\ell(E_{X,i})E_j \]
Again, let $x_{abcd} = T_{i,1}^j(E_{X,a})T_{i,1}^\ell(E_{X,b})T_{i,1}^j(E_{X,c})T_{i,1}^\ell(E_{X,d})$. Then
\[ x_{ijij} = K_i^{-1}F_i(-qE_j E_\ell E_i + E_\ell E_i E_j), \]
\[ x_{ji\ell i} = -qT_{i,1}^j(E_{X,j})T_{i,1}^\ell(E_{X,i})E_\ell = K_i^{-1}F_i(E_j E_\ell E_i - q^{-1}E_i E_j E_\ell) - E_j E_\ell, \]
\[ x_{ijij} = K_i^{-1}F_i(q^{-1}E_j E_i E_\ell + E_j E_i E_\ell), \]
\[ x_{ijji} = q^{-1}T_{i,1}^\ell(E_{X,\ell})T_{i,1}^j(E_{X,i})E_j = K_i^{-1}F_i(E_\ell E_j E_i + qE_i E_j E_\ell) - E_j E_\ell, \]
\[ x_{ijij} = qx_{ijji} - qK_i^{-1}F_i(E_j E_\ell E_i - q^{-1}E_i E_j E_\ell)E_\ell = K_i^{-1}F_i(E_j E_\ell E_i + E_i E_j E_\ell). \]
Then we compute that
\[ x_{i\ell ij} + x_{ji\ell i} - x_{ijji} - x_{\ell iji} = K_i^{-1}F_i((-q - q^{-1})E_j E_\ell E_i + (-q - q^{-1})E_i E_j E_\ell) \]
\[ = -[2]x_{ji\ell i}. \]
This finishes the proof.

We have now finished the proof that the maps $T_{i,e}$ and $T_{i,e}''$ are algebra isomorphisms for all $i \in I$ and $e = \pm 1$. Finally, we will show that they satisfy the braid relations of type $A$.

Lemma 4.13. Let $i \prec j \in I$, and $X \in \mathcal{A}$. Let $Y = i \cdot X$, $Y' = j \cdot X$, and $Z = j \cdot Y = i \cdot Y'$. Then as maps $\mathbf{U}(X) \to \mathbf{U}(Z)$, $T_{i,e} T_{j,e} = T_{j,e} T_{i,e}$ and $T_{j,e}'' T_{i,e} = T_{j,e}'' T_{i,e}$.
Proof. Again, it suffices to prove this for $T'_{i,1}$, as the other cases follow from this one. To verify that $T'_{i,1}T'_{j,1}(u) = T'_{j,1}T'_{i,1}(u)$ for all $u \in \mathbf{U}(X)$, it suffices to check in the cases $u = E_{X,k}, F_{X,k}, K_{X,k}$ for some $k \in \Lambda$.

Let $k \in \Lambda$. Note that if $k$ is not connected to $i$ or $j$, then the statement is obvious. Suppose that $k$ is connected or equal to exactly one of $i$ or $j$; without loss of generality, let’s suppose $k$ is connected or equal to $j$. Then for $L \in \{E, F, K\}$, note that $T'_{j,1}(L_k)$ is a polynomial in $K^{\pm 1}_k$, $E_k$, $F_k$, $E_j$, and $F_j$, all of which are “fixed” (i.e. mapped to the corresponding generator) by $T'_{i,1}$, hence $T'_{i,1}T'_{j,1}(L_k) = T'_{j,1}(L_k) = T'_{j,1}T'_{i,1}(L_k)$.

Finally, suppose $k$ is connected to both $i$ and $j$. Then

$$T'_{i,1}T'_{j,1}(E_k) = T'_{i,1}(E_k E_j) = (-1)^{p_Y(j)p_Y(k)} q^{-z_{ik}} E_j E_k$$

$$= E_k E_i E_j + (-1)^{p_Y(j)p_Y(k)} q^{-z_{ik}} E_i E_k E_j$$

$$T'_{j,1}T'_{i,1}(E_k) = T'_{j,1}(E_k E_i)$$

$$= E_k E_i E_j + (-1)^{p_Y(j)p_Y(k)} q^{-z_{ik}} E_i E_k E_j$$

Now, observe that $p_Y(i) = p_Z(i)$, $p_Y(k) = p_Z(k) + p_Z(j)$, and $p_Z(i)p_Z(j) = 0$. Then one finds that $p_Y(i)p_Y(k) = p_Z(i)p_Z(k)$ and hence $Y_{ik} = Z_{ik}$. Replacing $i$ with $j$ and $Y$ with $Y'$, we find $p_Y'(j)p_Y'(k) = p_Z(j)p_Z(k)$ and $Y_{jk}' = Z_{jk}$, and thus $T'_{i,1}T'_{j,1}(E_k) = T'_{j,1}T'_{i,1}(E_k)$. Similar arguments prove that $T'_{i,1}T'_{j,1}(F_k) = T'_{j,1}T'_{i,1}(F_k)$ and $T'_{i,1}T'_{j,1}(K_k) = T'_{j,1}T'_{i,1}(K_k)$. 

\[\square\]

Lemma 4.14. Let $i \sim j \in \Lambda$. Then

$$T'_{i,e}T'_{j,e}(E_{X,i}) = T'_{i,e}T'_{j,e}(E_{X,i}) = E_{Z,j},$$

$$T'_{i,e}T'_{j,e}(F_{X,i}) = T'_{i,e}T'_{j,e}(F_{X,i}) = F_{Z,j}, \text{ and}$$

$$T'_{i,e}T'_{j,e}(K_{X,i}) = T'_{i,e}T'_{j,e}(K_{X,i}) = K_{Z,j}.$$

Proof. We have $T'_{i,1}(E_{X,i}) = E_{Y,i} E_{Y,j} - (-1)^{p_Y(j)p_Y(i)} q^{-z_{ij}} E_{Y,j} E_{Y,i}$, so

$$T'_{i,1}T'_{j,1}(E_{X,i}) = (-1)^{pz(i)} K^{-1}_{Z,i} F_{Z,i} (E_{Z,j} E_{Z,i}$$

$$+ (-1)^{pz(j)}E_{Z,i} q^{-z_{ij}} (E_{Z,j} E_{Z,i} - (-1)^{pz(j)}E_{Z,i} q^{-z_{ij}} E_{Z,j}) (K^{-1}_{Z,i} F_{Z,i})$$

$$= (-1)^{pz(i)} E_{Z,j} F_{Z,i} - (-1)^{pz(i)} E_{Z,i} F_{Z,j}$$

$$= (-1)^{pz(i)} E_{Z,j} F_{Z,i} - (-1)^{pz(i)} E_{Z,i} F_{Z,j}$$

$$= (-1)^{pz(i)} E_{Z,j} F_{Z,i} - (-1)^{pz(i)} E_{Z,i} F_{Z,j}$$

$$= (-1)^{pz(i)} E_{Z,j} F_{Z,i} - (-1)^{pz(i)} E_{Z,i} F_{Z,j}$$

$$= (-1)^{pz(i)} E_{Z,j} F_{Z,i} - (-1)^{pz(i)} E_{Z,i} F_{Z,j}$$

$$= (-1)^{pz(i)} q^{-z_{ij}} E_{Z,j} = E_{Z,j}$$
Similarly, we have \( T'_{i,1} T'_{j,1}(F_{X,i}) = F_{Y,j} F_{Y,i} - (-1)^{p_Y(j) p_Y(i)} q^{-Y_{ij}} F_{Y,i} E_{F,j} \), so

\[
T'_{i,1} T'_{j,1}(F_{X,i}) = (F_{Z,i} F_{Z,j} - (-1)^{p_Z(i) p_Z(j)} q^{-Z_{ij}} F_{Z,j} F_{Z,i}) (-(-1)^{p_X(i) E_{Z,i} K_{Z,i}}
+ (-1)^{p_Z(i) p_Z(j)} q^{-Z_{ij} + Z_{ij}} (E_{Z,i} K_{Z,i}) (F_{Z,i} F_{Z,j} - (-1)^{p_Z(i) p_Z(j)} q^{-Z_{ij}} F_{Z,j} F_{Z,i})
\]

\[
= (-1)^{p_Z(i) p_Z(j)} ((E_{Z,i} F_{Z,i} - (-1)^{p_Z(i) F_{Z,i} E_{Z,i}}) F_{Z,j}
- q^{-Z_{ij}} F_{Z,j} (E_{Z,i} F_{Z,i} - (-1)^{p(i) F_{Z,i} E_{Z,i}}) K_{Z,i})
\]

\[
= (-1)^{p_Z(i) p_Z(j)} F_{Z,j} (q^{-Z_{ij} K_{Z,i}} - q^{-Z_{ij}} K_{Z,i}^{-1} - q^{-Z_{ij}} K_{Z,i} - K_{Z,i}^{-1}) K_{Z,i}
\]

\[
= (-1)^{p_Z(i) p_Z(j)} q^{-Z_{ij}} - q^{-Z_{ij}} F_{Z,j} = F_{Z,j}
\]

Finally, note that we have \( T'_{i,1} T'_{j,1}(K_{X,i}) = (-1)^{p_Y(i) p_Y(j) + p_Z(i) p_Z(j)} K_Y j \), and observe that \( p_Y(i) p_Y(j) = p_Z(i) + p_Z(i) p_Z(j) \) since \( Y = i \cdot Z \).

**Lemma 4.15.** Let \( i, j \in I \) with \( j = i \pm 1 \) and \( W \in \mathcal{A} \), and set \( X = i \cdot W \) and \( X' = j \cdot W \), \( Y = j \cdot X \) and \( Y' = i \cdot X' \), and \( Z = i \cdot Y = j \cdot Y' \). Then as maps \( U(W) \to U(Z) \), we have \( T'_{i,e} T'_{j,e} T'_{i,e} = T'_{j,e} T'_{i,e} T'_{j,e} \) and \( T'_{i,e} T'_{j,e} T'_{i,e} = T'_{j,e} T'_{i,e} T'_{i,e} \).

**Proof.** Note that it suffices to check this on the generators. To reduce the clutter of notation in the proof, we will drop the \( W, X, Y, Z \) subscripts on the generators whenever the ambient space of the elements is clear from context. We will also freely use the identities given by Lemma 4.14 in the following computations.

First observe that if \( k \) is not connected to \( i \) or \( j \), \( T'_{i,1} T'_{j,1} T'_{i,1} \) and \( T'_{j,1} T'_{i,1} T'_{j,1} \) map \( E_k \mapsto E_k \), \( F_k \mapsto F_k \), and \( K_k \mapsto K_k \) hence they agree on these generators.

Now suppose that \( k \notin \{i, j\} \) but is connected to \( i \) or \( j \); without loss of generality, assume \( k \) is connected to \( j \) but not \( i \). Then using (4.3) we note the coincidences

\[
p_X(j) p_X(k) = p_Z(i) p_Z(j) + p_Z(i) p_Z(k) = p_Z(i) p_Z(j),
\]

\[
p_Y(j) p_Y(k) = p_Z(i) p_Z(k) + p_Z(j) p_Z(k) = p_Z(j) p_Z(k).
\]

Using (4.2), this immediately implies

\[
X'_{jk} = Z_{ij}, \quad Y_{jk} = Z_{jk}.
\]

First, let us compare the images of \( K_k \). We compute that

\[
T'_{i,1} T'_{j,1} T'_{i,1}(K_k) = T'_{i,1} T'_{j,1}(K_k) = T'_{i,1}((-1)^{p_Y(j) p_Y(k)} K_j K_k) = (-1)^{p_Z(i) p_Z(j) + p_Y(j) p_Y(k)} K_i K_j K_k,
\]

\[
T'_{j,1} T'_{i,1} T'_{j,1}(K_k) = (-1)^{p_X(j) p_X(k)} T'_{j,1} T'_{i,1}(K_j) T'_{j,1} T'_{i,1}(K_k) = (-1)^{p_X(j) p_X(k) + p_Z(j) p_Z(k)} K_i K_j K_k.
\]
Comparing in view of the above coincidences, we see that \( T'_{i,1} T'_{j,1} T'_{i,1}(K_k) = T'_{j,1} T'_{i,1} T'_{j,1}(K_k) \). Next, we consider the images of \( E_k \). We compute

\[
T'_{i,1} T'_{j,1} T'_{i,1}(E_k) = T'_{i,1} T'_{j,1}(E_k) \\
= E_k(E_{ij} E_i - (-1)^{pZ(i)pZ(j)} qZ_{ij} E_i E_j) \\
- (-1)^{pY(j)pY(k)} qY_{jk}(E_{ij} E_i - qZ_{ij} (-1)^{pZ(i)pZ(j)} E_i E_j)E_k \\
= E_k E_{ij} E_i - (-1)^{pZ(i)pZ(j)} qZ_{ij} E_i E_j - (-1)^{pY(j)pY(k)} qY_{jk} E_i E_j E_k \\
+ (-1)^{pY(j)pY(k)} + pZ(i)pZ(j) qY_{jk} + Z_{ij} E_i E_j E_k.
\]

On the other hand,

\[
T'_{j,1} T'_{i,1} T'_{j,1}(E_k) = T'_{j,1} T'_{i,1} T'_{i,1}(E_k) \\
= T'_{j,1} T'_{i,1}(E_k) E_i - qX_{jk} (-1)^{pX'(j)pX'(k)} T'_{j,1} T'_{i,1}(E_k) E_i T'_{i,1}(E_k) \\
= (E_{ij} E_i - (-1)^{pZ(i)pZ(k)} qZ_{jk} E_i E_j)E_i \\
- qX_{jk} (-1)^{pX'(j)pX'(k)} E_i E_j E_i - (-1)^{pZ(j)pZ(k)} qZ_{jk} E_i E_j E_k \\
= E_k E_{ij} E_i - (-1)^{pZ(j)pZ(k)} qZ_{jk} E_j E_i E_k - (-1)^{pX'(j)pX'(k)} qX_{jk} E_i E_k E_j \\
+ (-1)^{pX'(j)pX'(k)} + pZ(j)pZ(k) qX_{jk} E_i E_j E_k.
\]

Comparing coefficients in view of the above coincidences, we see that

\[
T'_{i,1} T'_{j,1} T'_{i,1}(E_k) = T'_{j,1} T'_{i,1} T'_{j,1}(E_k).
\]

A similar computation proves

\[
T'_{i,1} T'_{j,1} T'_{i,1}(F_k) = T'_{j,1} T'_{i,1} T'_{j,1}(F_k).
\]

Finally, the last case is for \( k = i, j \). We will prove the case \( k = i \), as the \( k = j \) case follows from reversing the roles of \( i \) and \( j \) in the following arguments. First, note that

\[
T'_{i,1} T'_{j,1} T'_{i,1}(K_i) = (-1)^{pX(i)} T'_{i,1} T'_{j,1}(K_i^{-1}) = (-1)^{pX(i)} K_i^{-1}.
\]

\[
T'_{j,1} T'_{i,1} T'_{j,1}(K_i) = T'_{j,1}(K_j) = (-1)^{pZ(j)} K_j^{-1}.
\]

Note that (4.3) implies \( pX(i) = pY(i) + pY(j) = pZ(j) \), thus \( T'_{i,1} T'_{j,1} T'_{i,1}(K_i) = T'_{j,1} T'_{i,1} T'_{i,1}(K_i) \). Next, let us verify that \( T'_{i,1} T'_{j,1} T'_{i,1}(E_i) = T'_{j,1} T'_{i,1} T'_{j,1}(E_i) \). Well, by the previous lemma,

\[
T'_{i,1} T'_{j,1}(E_i) = E_j, \text{ hence } T'_{j,1} T'_{i,1} T'_{j,1}(E_i) = (-1)^{pZ(j)} K_j^{-1} F_j.
\]

On the other hand, \( T'_{i,1}(E_i) = (-1)^{pX(i)} K_i^{-1} F_i \), and so

\[
T'_{i,1} T'_{j,1} T'_{i,1}(E_i) = -T'_{i,1} T'_{j,1}(K_i^{-1}) T'_{i,1} T'_{j,1}(E_i) = -(-1)^{pX(i)} K_i^{-1} F_i.
\]

Then as before, \( pX(i) = pZ(j) \) hence \( T'_{i,1} T'_{j,1} T'_{i,1}(E_i) = T'_{i,1} T'_{j,1} T'_{i,1}(E_i) \). Proving the identity \( T'_{i,1} T'_{j,1} T'_{i,1}(F_i) = T'_{j,1} T'_{i,1} T'_{j,1}(F_i) \) proceeds similarly.
4.4. Constructing PBW bases. Now we turn to the problem of constructing PBW bases using our braid operators. To that end, let us now fix \( T_i = T_{i,1}'' \) and \( T_i^{-1} = T_{i,1}' \).

Let \( \omega_0 \in S_{m+1} \) be the longest element and let \( I \) be the set of \( i = (i_1, \ldots, i_N) \in I^N \) such that \( \omega_0 = s_{i_1} \cdots s_{i_N} \) is a reduced expression. There is an associated convex order on \( \Phi^+ (X) = \{ \beta_1 < \ldots < \beta_N \} \) where \( \beta_i = s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}^+) \) (here, the action is as in the classical case, as described in Remark 4.2). In particular, let \( X_{i,t} = i_{t-1} \cdots i_2 \cdot i_1 \cdot X \) (so \( X_{i,1} = X \)). Define the root vectors \( F_{i,\beta_i} = T_{i_1} \cdots T_{i_{t-1}}(F_{X_{i,t},i_1}) \). The we define the set of PBW vectors (relative to \( i \)) to be

\[
B_i = \left\{ F_{i,\beta_1} \cdots F_{i,\beta_N} \mid a_1, \ldots, a_n \in \mathbb{N}, a_s < 2 \text{ if } p(\beta_s) = 1 \right\}.
\]

(4.15)

Lemma 4.16. Let \( i = (i_1, \ldots, i_N) \in I \).

1. Suppose there is an \( 1 < t \leq N \) such that \( i_{t-1} \) and \( i_t \) are not connected. Let \( j = (j_1, \ldots, j_N) \) where \( j_s = i_s \) for \( s \neq t-1, t \), \( j_{t-1} = i_{t-1} \) and \( j_t = i_t \). Let \( \sigma = (t-1, t) \in S_{N+1} \). Then \( j \in I \), and if \( F_{i,\beta_j} = F_{i,\beta_{\sigma_j}(\beta)} \) for \( 1 \leq s \leq N \), then \( F_{i,\beta_j} = F_{j,\beta_j} \) for all \( \beta \in \Phi^+ \). Moreover, \( F_{i,\beta_{t-1}} F_{i,\beta_{i_t}} = F_{i,\beta_{t-1}} F_{i,\beta_{i_t+1}} \) and \( B_i = B_j \).

2. Suppose there is an \( 1 < t < N \) such that \( i_t = i \) and \( i_{t+1} = i_{t-1} = j \) connected to \( i \). Let \( j = (j_1, \ldots, j_N) \) where \( j_s = i_s \) for \( s \neq t-1, t \), \( j_{t+1} = j \) and \( j_{t+1} = i_t \). Let \( \sigma = (t-1, t+1) \in S_{N+1} \). Then \( j \in I \) and for \( s \neq \sigma \), \( F_{i,\beta_s} = F_{i,\beta_{\sigma_s}(\beta)} \) and thus \( F_{i,\beta_j} = F_{j,\beta_j} \) for all \( \beta \neq \beta_j \). Furthermore, \( \beta_{t-1} + \beta_{t+1} = \beta_t \). Setting \( Z = X_{i,\sigma_i} \), we have

\[
F_{i,\beta_j} = F_{i,\beta_j} F_{i,\beta_{t-1}} - (-1)^{p(\beta_{t-1})p(\beta_{t+1})} q^{p(\beta_{t-1})p(\beta_{t+1})} F_{i,\beta_{t-1}} F_{i,\beta_{t+1}}.
\]

3. For any \( i \in I \), \( F_{i,\alpha_i} X = F_i \).

4. For any \( \beta \in \Phi^+ \), \( F_{\beta} \in U^-(X) \), the subalgebra generated by the \( F_i \).

In the cases of (1) and (2) of the lemma, we will say \( i \) and \( j \) are braid-connected. It is well-known that in any Coxeter group, any reduced expression can be obtained from a given one using braid moves. In particular, note that between any \( i, j \in I \), there is a sequence \( i_0 = i, i_1, \ldots, i_t = j \) of \( i_s \in I \) such that \( i_s \) and \( i_{s+1} \) are braided.

Proof. For (1) and (2), it is well known that \( j \in I \), and the fact that the braid operators respect the braid relations proves that the root vectors are the same for most roots. The remaining statements are easily proved by applying \( T_{i_{t-2}}^{-1} \cdots T_{i_1}^{-1} \) and observing that the claims follow from elementary rank 2 calculations. In particular, the coefficient \( (-1)^{p(\beta_{t-1})p(\beta_{t+1})} q^{p(\beta_{t-1})p(\beta_{t+1})} \) can be expressed thusly from the observation that the maps \( T_i \) are parity-preserving and “weight-preserving” in the sense of Remark 4.6 and Section 4.1.

In particular, note that (2) implies that if \( i \) and \( j \) are braid connected, then \( F_{i,\alpha_i} = F_{j,\alpha_i} \) since \( \alpha_i \) cannot be written as a sum of positive roots. Then (3) immediately follows from (2), as there is a \( j \in I \) with \( j = (i, j_2, \ldots, j_N) \) which is connected to \( i \) by a sequence of braid moves, and \( F_{i,\alpha_i} = F_{j,\alpha_i} = F_j \) by definition.

Finally, note that (4) follows by induction on the height of \( \beta \) exactly as in [Tin, Lemma 3.2].

\[ \square \]

Corollary 4.17. The set \( B_i \) is a basis of \( U^-(X) \).
Proof. This follows essentially the same proof as [Tin, Lemma 3.4]. However, note that we essentially prove by simultaneous induction for the entire family of algebras \( \{ U(X) \}_{X \in \mathcal{A}} \). \( \square \)

**Lemma 4.18.** Let \( \mathbf{i} = (i_1, \ldots, i_N) \in \mathbf{I} \). Let \( 1 \leq \tau < s \leq N \). Then \( T_{i_s}^{-1} T_{i_{s-1}}^{-1} \ldots T_{i_1}^{-1} (F_{\beta_s}) \in U^0(X_{i:s+1}) \).

**Proof.** Let \( Y = X_{i:s+1} \). Note that \( T_{i_s}^{-1} T_{i_{s-1}}^{-1} \ldots T_{i_1}^{-1} (F_{\beta_s}) = -(-1)^{p(i_s)} E_{i_s} K_{i_s} = -\omega(F_{i_s}) K_{i_s} \), where \( \omega = -\circ \rho \circ \tau \). Then

\[
T_{i_s}^{-1} \ldots T_{i_{\tau+1}}^{-1}(F_i) = \omega(T_{i_s} \ldots T_{i_{\tau+1}} (F_{i_s}) = \omega(F_{\beta_{s-\tau}}),
\]

where \( j = (i_s, \ldots, i_1, i_N, \ldots, i_{s+1}) \in \mathbf{I} \). In particular, since Lemma 4.16 (4) says \( F_{\beta_{s-\tau}} \in U^- (Y) \), it follows that \( \sigma(F_{\beta_{s-\tau}}) \in U^+ (Y) \). Then the result follows by observing that \( T_{i_s}^{-1} (U^0(Z)) = U^0(i \cdot Z) \) for all \( i \in I \) and \( Z \in \mathcal{A} \), and because the \( T_{i_s}^{-1} \) are algebra isomorphisms. \( \square \)

**Corollary 4.19.** Let \( \mathbf{i} \in \mathbf{I} \) and \( 1 \leq r < s \leq N \). Write \( F_{i_1} \beta_1 F_{i_2} \beta_2 = \sum c_a F_a^i \), where \( \mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{N} \) with \( a_s < 2 \) if \( p(\beta_s) = 1 \), \( c_a \in \mathbb{Q} (q) \), and \( F_i^a = F_{i_1}^{(a_1)} \ldots F_{i_N}^{(a_N)} \). Then \( c_a = 0 \) unless \( a_t = 0 \) for \( t < r \) or \( t > s \). Moreover, the coefficient of \( F_{i_1} \beta_1 F_{i_2} \beta_2 \) is \( (-1)^{p(\beta_r)} p(\beta_s) q^{-p(\beta_r, \beta_s)} \).

**Proof.** This follows from a similar proof as in [Tin, Lemma 3.5]; namely, observe that from the proof of the previous lemma,

\[
T_{i_1}^{-1} \ldots T_{i_1}^{-1} (F_i^a) = \omega(F_{i_1}^{(a_1)} \ldots F_{i_N}^{(a_N)}) f_{a_1, \ldots, a_t} (K_i; i \in I) F_{i_1}^{(a_1)} \ldots F_{i_N}^{(a_N)} ,
\]

where \( f_{a_1, \ldots, a_t} \) is some polynomial in \( |I| \) variables with \( f_{0, \ldots, 0} = 1 \) Furthermore, observe that the \( \omega(F_{i_1}^{(a_1)} \ldots F_{i_N}^{(a_N)}) \) are linearly independent vectors in \( U^+ (X_{i:t+1}) \).

By Lemma 4.16 (4),

\[
x = T_{i_r}^{-1} \ldots T_{i_1}^{-1} (F_i) \in U^- (X_{i:t}) .
\]

On the other hand, applying (a) to the right-hand side of \( F_{i_1} \beta_1 F_{i_2} \beta_2 = \sum c_a F_i^a \) and using the triangular decomposition \( U(X_{i:t}) \cong U^+(X_{i:t}) \otimes U^0(X_{i:t}) \otimes U^-(X_{i:t}) \), we see that

\[
x = \sum c_a \omega(F_{i_1}^{(a_1)} \ldots F_{i_N}^{(a_N)}) \otimes f_{a_1, \ldots, a_t} (K_i; i \in I) \otimes F_{i_1}^{(a_1)} \ldots F_{i_N}^{(a_N)} .
\]

In particular, the only way we can have \( c_a \neq 0 \) given (b) and (c) is if \( a_1 = \ldots = a_r-1 = 0 \). A similar argument proves that \( c_a \neq 0 \) only if \( a_{s-1} = \ldots = a_N = 0 \).

Lastly, let \( c \) be the coefficient of \( F_{i_1} \beta_1 F_{i_2} \beta_2 \). Then by the conditions on when \( c_a \neq 0 \),

\[
z = T_{i_r}^{-1} \ldots T_{i_1}^{-1} (F_i) \in U^- (X_{i:t+1}) .
\]

On the other hand, we have that

\[
y = T_{i_r}^{-1} \ldots T_{i_1}^{-1} (F_i) \in U^- (X_{i:t+1}) \text{ and } T_{i_r}^{-1} \ldots T_{i_1}^{-1} (F_i) = -K_{i_r}^{-1} E_{i_r} ,
\]

so

\[
z = c K_{i_r}^{-1} E_{i_r} y - y K_{i_r}^{-1} E_{i_r} = c \frac{\mathbf{e}_{i_r} (y) - K_{i_r}^{-1} e_{i_r} (y)}{q - q^{-1}} + ((-1)^{p(i_r)} p(y) c - q^{(i_r, |y|)}) K_{i_r}^{-1} y E_{i_r} .
\]

In particular, applying the triangular decomposition and the fact that \( z \in U^- (X_{i:t+1}) \), we see that \( e_{i_r} (y) = 0 \) and \( c = (-1)^{p(i_r)} p(y) q^{(i_r, |y|)} \). Since the maps \( T_i \) are parity- and weight-preserving the result follows. \( \square \)

Note that we extract the following corollary from the above argument.

**Corollary 4.20.** If \( \mathbf{i} = (i_1, \ldots, i_N) \in \mathbf{I} \), then \( e'_{i_1} (F_{i_1} \beta_r) = 0 \) for all \( r > 1 \).
Lastly, let us note that the PBW bases constructed here agree with those constructed via quantum shuffles.

**Lemma 4.21.** Let \( i \in I \) and let \( \leq i \leq \) be the associated total order on \( I \) induced by the convex order on \( \Phi^+ \). Let \( F_{<;\beta} \) be the root vector defined with respect to \( < \) as in [CHW3]. Then \( F_{<;\beta} = F_{i;\beta} \).

**Proof.** This follows from a similar proof to that in [Lec]. To wit, first note that \( F_{<;\alpha_i} = F_i \). More generally, suppose \( \beta = \beta_r + \beta_s \) such that \( \beta_r \) is maximal, and assume that by induction on the height, \( F_i;\beta_r = F_{<;\beta_r} \) and \( F_i;\beta_s = F_{<;\beta_s} \). Then on one hand, letting \( l_1 = l(\beta_r) \) and \( l_2 = l(\beta_s) \) be the Lyndon words associated to the roots, as in [Lec, Theorem 28], we deduce that \( l_1 l_2 = l(\beta) \) is the costandard factorization, and thus by definition (cf. [CHW3, Proposition 4.11 and (5.1)])

\[
F_{<;\beta} = F_{<;\beta_r} F_{<;\beta_r} - (-1)^{p(\beta_r)p(\beta_s)} q^{-(\beta_r,\beta_s)} F_{<;\beta_r} F_{<;\beta_s}.
\]

On the other hand, Corollary 4.19 together with the maximality of \( \beta_r \) implies that

\[
F_{i;\beta_r + \beta_s} = F_{i;\beta_r} F_{i;\beta_s} - (-1)^{p(\beta_r)p(\beta_s)} q^{-(\beta_r,\beta_s)} F_{i;\beta_r} F_{i;\beta_s} = F_{<;\beta_r} F_{<;\beta_r} - (-1)^{p(\beta_r)p(\beta_s)} q^{-(\beta_r,\beta_s)} F_{<;\beta_r} F_{<;\beta_s} = F_{<;\beta}.
\]

\[\square\]

**4.5. Comparison to the lattice.** Now let us consider the case of \( X = A^m \), the unique GCM with \( X_{n,m} = 0 \), which corresponds to the standard Dynkin diagram for \( \mathfrak{gl}(m|1) \). In this case, \( U(X) \) is a subalgebra of \( U \) as defined in §2, but \( U^-(X) = U^- \).

**Lemma 4.22.** Let \( i,j \in I \) be connected by a sequence of braid moves. Then \( \mathbb{Z}[q]B_i = \mathbb{Z}[q]B_j \). Moreover, for any \( b_1 \in B_i \), there is a \( b_2 \in B_j \) such that \( b_1 \equiv b_2 \) modulo \( q\mathbb{Z}[q]B_j \).

**Proof.** It suffices to prove this in the case that \( i \) and \( j \) differ by a single braid move. If it is a braid move of the form \( i,j \mapsto j,i \), then there is nothing to say by Lemma 4.16 (1). Suppose it is a braid move of the form \( i_{t-1} i_{t+1} \mapsto i_t i_{t-2} \), where \( i_{t+1} = i \) and \( i_t = j \); Then 4.16 (2) implies that the only root vectors that change are the ones corresponding to the braid move, and applying \( T_{i_{t-2}}^{-1} \ldots T_{i_t}^{-1} \) it suffices to prove this in the rank 2 case; that is, to prove that in \( U(\sigma^{-1} A^m) \) where \( \sigma = s_{i_1} \ldots s_{i_{t-2}} \), we have

\[
\sum \mathbb{Z}[q]F_i^{(a)} T_i(F_j)^{(b)} T_i T_j(F_j)^{(c)} = \sum \mathbb{Z}[q]F_j^{(a)} T_j(F_i)^{(b)} T_i T_j(F_j)^{(c)}. \tag{*}
\]

Well, first observe that since \( i = (i_1, \ldots, i_{t-2}, j, i, i, \ldots) \) and \( j = (i_1, \ldots, i_{t-2}, j, i, j, \ldots) \) correspond to a reduced expressions of the longest element of \( S_{m+1} \), \( l(\sigma i_1) = l(\sigma j) > l(\sigma) \). Then by Corollary 4.4, at least one of \( i \) or \( j \) must be even, so without loss of generality we can assume that \( i \) is even. If \( j \) is also even, this follows from [Lus2, 42.1.5]. Otherwise, this follows easily from [CHW3, §8]; indeed, if we take the PBW basis with the opposite ordering, it is easy to see that it’s \( \mathbb{Z}[q] \) lattice agrees with the \( \mathbb{Z}[q] \) lattice of the canonical basis therein by applying the anti-involution \( \tau \).

\[\square\]

**Theorem 4.23.** For any \( i \in I \), \( B_i + q\mathcal{L} = B(\infty) \), thus in particular the canonical bases defined in [CHW3] coincide and are equal to \( B \).
Proof. It suffices to show that $B_1 + qL$ is closed under the action of the $\tilde{f}_i$, since $1 + qL \in B_1 + qL$. For this, fix $i \in I$ and note that $i$ appears in $i$. If $i = (i_1, \ldots, i_N)$ and $i_1 = i$, then by Corollary 4.20 it follows that $\tilde{f}_i F_i^{(a_1, \ldots, a_N)} = F_i^{(a_1 + 1, \ldots, a_N)}$. Otherwise, there is some $j = (j_1, \ldots, j_N)$ connected to $i$ by a sequence of braid moves such that $j_1 = i$. Then for any $b_1 \in B_1$, there is a $b_2 \in B_j$ such that $b_1 \equiv b_2 \in B_j$, and thus $\tilde{f}_i b_1 \equiv \tilde{f}_i b_2 \in B_i + qL$.

Note that the anti-involution $\tau$ maps any PBW basis to the PBW basis associated to the opposite ordering on simple roots, hence in particular we have the following straightforward consequence of the theorem.

Corollary 4.24. The canonical basis $B(\infty)$ is invariant under the anti-involution $\tau$.

5. Examples

5.1. Canonical bases for standard quantum $\mathfrak{gl}(2|1)$. Let us consider our construction in the special case of $U_q^-(\mathfrak{gl}(2|1))$. First, we should compare the crystal lattice constructed here to the one in [Zou]. Therein, the author constructs a partial crystal structure on $U_q^{-}(\mathfrak{gl}(m|1))$. To do this, essentially they use the even Kashiwara operators to construct a lattice by descending from the nilpotent PBW elements. However, while this lattice is closed under $\tilde{e}_m$, it is not closed under $\tilde{f}_m$. (An upper crystal basis is also constructed which is closed under $\tilde{f}_m$, but not $\tilde{e}_m$.)

We observe that the lattice defined therein when $m = 2$ is $\left(\bigoplus_t A F_1^{(t)} \right) \oplus \left( \bigoplus_t A F_1^{(t)} F_2 \right) \oplus \left( \bigoplus_t q A F_2 F_1^{(t)} \right) \oplus \left( \bigoplus_t A F_2 F_1^{(t)} F_2 \right)$. Now let us relate our crystal basis to the canonical basis from [CHW3]. This canonical basis is constructed from the PBW bases, as described in the following proposition.

Proposition 5.1. $U^{-}$ admits the following canonical basis:

$$b_1 r = F_1^{(r)}, \quad b_1 r_2 = F_1^{(r)} F_2, \quad b_2 1 r = F_2 F_1^{(r)}, \quad b_2 1 r_2 = F_2 F_1^{(r+1)} F_2 \quad (\forall r \geq 0).$$

Here, the subscripts are words in the alphabet \{1, 2\} and for $r \in \mathbb{Z}$, we set $1^r = \overbrace{1 \cdots 1}^r$.

Let us now consider how this canonical basis relates to the crystals constructed in this section. Computing the action of the Kashiwara operators on the canonical basis, one easily sees that the canonical basis lies in the crystal lattice (hence spans it). Indeed, we compute that for $r \geq 0$, the action of the Kashiwara operators on the canonical basis elements of height $r$ are given by

$$\tilde{f}_1 b_1 r = b_1 r+1; \quad \tilde{f}_2 b_1 r = b_2 1 r;$$

$$\tilde{e}_1 b_1 r = b_1 r-1; \quad \tilde{e}_2 b_1 r = 0;$$

$$\tilde{f}_1 b_1 r_{-1} = b_1 1 r; \quad \tilde{f}_2 b_1 r_{-1} = b_2 1 r_{-1};$$

$$\tilde{e}_1 b_1 r_{-1} = b_1 1 r_{-2}; \quad \tilde{e}_2 b_1 r_{-1} = q^{-1} b_1 r_{-1};$$

$$\tilde{f}_1 b_2 1 r = b_2 1 r + (q^{-1} - q^r) b_1 1 r; \quad \tilde{f}_2 b_2 1 r = 0;$$

$$\tilde{e}_1 b_2 1 r = b_2 1 r_{-2} + (q - q^{-1}) b_1 1 r_{-2}; \quad \tilde{e}_2 b_2 1 r = b_1 r_{-1};$$

$$\tilde{f}_1 b_2 1 r_{-2} = b_2 1 r_{-2}; \quad \tilde{f}_2 b_2 1 r_{-2} = 0;$$

$$\tilde{e}_1 b_2 1 r_{-2} = b_2 1 r_{-3}; \quad \tilde{e}_2 b_2 1 r_{-2} = b_1 r_{-2} - q^{-2} b_2 1 r_{-2}.$$
Furthermore, let us comment on the canonical basis of modules in this case. Let $\lambda = a\epsilon_1 + b\epsilon_2 + c\epsilon_3 \in P^+$, so $a, b, c \in \mathbb{Z}$ with $a \geq b$. Then there is a unique finite-dimensional simple module $V(\lambda)$ of highest weight $\lambda$. The compatibility of the canonical basis with these modules is considered in [CHW3, §8]. Specifically, for $\lambda \in P^+$, let $\mathcal{B}(\lambda) = \{uv_\lambda \mid u \in \mathcal{B}\} \setminus \{0\}$; note that for $\lambda \in \tilde{P}^+$, this agrees with the definition in Theorem 3.16. Then in loc. cit., it is shown that

1. $\mathcal{B}(\lambda)$ is a basis for $V(\lambda)$ when $\lambda$ is typical;
2. $\mathcal{B}(\lambda)$ is a basis for $V(\lambda)$ when $b = -c$;
3. $\mathcal{B}(\lambda)$ is a basis for $V(\lambda)$ when $a = -c - 1 = b$; and
4. $\mathcal{B}(\lambda)$ is linearly dependent in $V(\lambda)$ when $a = -c - 1 > b$.

First, note that $K(\lambda) = V(\lambda)$ whenever $\lambda$ is typical, so (1) follows from Theorem 3.16 for $K(\lambda)$. For (2), note that $V(\lambda) \in \mathcal{O}_{\text{int}}$ so the result follows from Theorem 3.16 for $V(\lambda)$ in this case.

It remains to comment on (3) and (4). Let $n = \langle h_1, \lambda \rangle = a - b$, and note that $\langle h_2, \lambda \rangle = b + c = b - a - 1 = -(n + 1)$. Then as observed in [CHW3],

$$\mathcal{B}(\lambda) = \left\{F_1^{(r)}v_\lambda, \quad F_1^{(r+1)}F_2v_\lambda, \quad F_2F_1^{(r)}v_\lambda \mid 0 \leq r \leq n\right\}$$

is a spanning set, but we have the linear dependence $F_2F_1^{(r)}v_\lambda = \frac{[n+1-r]}{[n+1]}F_1^{(r)}F_2v_\lambda$ for $1 \leq r \leq n$ if $n > 0$. However, if we consider the lattice $L = A\mathcal{B}(\lambda)$, since $\frac{[n+1-r]}{[n+1]} \in q^rA$, we observe that we can canonically pick a basis for $V(\lambda)$; that is, the vectors

$$\mathcal{B}'(\lambda) = \{uv_\lambda \mid u \in \mathcal{B} \text{ such that } uv_\lambda \notin qL\} = \left\{F_1^{(r)}v_\lambda, F_2v_\lambda, F_1^{(r+1)}F_2v_\lambda \mid 0 \leq r \leq n\right\}.$$ 

We cannot define a crystal structure on $L$ as we have above. Indeed, note that $L$ is not closed under the operator $\tilde{e}_2$:

$$\tilde{e}_2F_1^{(r)}F_2v_\lambda = q^{-1}K_2E_2F_1^{(r)}F_2v_\lambda = -q^{-n-2+r}[n + 1]F_1^{(r)}v_\lambda \in q^{r-2n-2}L.$$
However, if we define \( \tilde{e}'_2 = -q^{2(n+1)} \tilde{e}_2 \), then we see that \( (L, B'(\lambda) + qL) \) is a crystal lattice under \( \tilde{e}_1, \tilde{f}_1, \tilde{e}'_2, \tilde{f}_2 \). Moreover this crystal is compatible with the one on \( U^- \) modulo \( q \); see Figure 3 and compare to Figure 2.

5.2. Canonical bases for standard quantum \( gl(3|1) \). Next, let us explicitly construct the canonical basis in the case \( m = 3 \). We also want to produce some examples of canonical bases on atypical modules. Both of these tasks are made easier by comparing canonical basis elements to their corresponding PBW vectors modulo \( q \), hence we will write \( x \equiv q y \) if \( x - y \in qL \). On the other hand, each task is better suited by a particular PBW basis: either that associated to the standard order \( I = \{ 1 < 2 < 3 \} \), or the opposite order \( I = \{ 3 <^{\text{op}} 2 <^{\text{op}} 1 \} \). To that end, we recall the associated PBW vectors

\[
F_{12} = F_2 F_1 - q F_1 F_2, \quad F_{23} = F_3 F_2 - q F_2 F_3; \quad F_{123} = F_3 F_{12} - q F_{12} F_3 = F_{23} F_1 - q F_1 F_{23},
\]

with respect to the standard order, whereas the opposite order yields the PBW vectors

\[
F_{21} = F_1 F_2 - q F_2 F_1, \quad F_{32} = F_2 F_3 - q F_3 F_2; \quad F_{321} = F_{21} F_3 - q F_3 F_{21} = F_1 F_{32} - q F_{32} F_1.
\]

Note that

\[
F_{21} = \tau(F_{12}), \quad F_{32} = \tau(F_{23}), \quad F_{321} = \tau(F_{123}),
\]

and indeed in general we have \( B(\langle \rangle) = \tau(B(\langle^{\text{op}} \rangle)) \). It will be useful to note the identities

\[
F_i^{(a)} F_j^{(b)} = \left[ a + b - 1 \atop b \right] F_i^{(a+b)} F_j + \left[ a + b - 1 \atop b \right] F_j F_i^{(a+b)} \quad i \in I_0, \quad j = i \pm 1;
\]

\[
F_3 F_2^{(a)} = F_2^{(a-1)} F_3; \quad F_{21} F_{23} = F_{23} F_{21} \quad F_3 F_2^{(a)} = q F_2^{(a-1)} F_3; \quad F_3 F_{23} = -q F_{23} F_3.
\]

We also observe that \( F_2^{(b)} F_3 F_2 F_3 F_2^{(a)} = F_3 F_2^{(a+b+1)} F_3 \), and \( F_3 F_2 F_1 F_3 F_2 F_3 \) is central in \( U^- \).
Theorem 5.2. For $x, y, z \in \{0, 1\}$ and $a, b, c \in \mathbb{Z}_{>0}$, let $u = u(x, y, z, a, b, c) \in \mathcal{B}$ be the unique element equal to the PBW vector $F_3^a F_2^b F_1^c$ modulo $q$. Then

$$u = \begin{cases} F_3^a F_2^b F_1^c & \text{if } c \geq a, \\ F_3^a F_2^b F_1^c & \text{if } a > c \text{ and } y \leq x, \\ \sum_{t=0}^{b} (-1)^t \begin{pmatrix} a - c - 1 + t \\ t \end{pmatrix} F_2^{(a+1)t} F_1^{(b+c+z)} F_3^{(b+z)} F_2^{(a+b+z)} F_3^{(a+b+c+z)} \equiv F_3^{(a+b+c+z)} & \text{otherwise}. \end{cases}$$

Note that, in this theorem, the canonical basis elements are being indexed by PBW vectors corresponding to the opposite order; this is the most useful description for determining canonical basis elements for Kac modules. Of course, applying $\tau$ produces a similar description in terms of the standard PBW basis, and it turns out that this is the easiest PBW basis for constructing the canonical basis elements.

Proof. We consider elements of weight $\mu = -i \alpha_1 - j \alpha_2 - k \alpha_3$ for $i, j, k \in \mathbb{N}$. Note that the $\mu$ weight space is nonzero only if $k \leq 3$. This makes it convenient to describe the canonical basis on a case-by-case basis in terms of $k$. Furthermore, observe that for $k \leq 1$, we can formally identify $\mathcal{B}_{\mu}$ with $\mathcal{U}_{q}^{-}(\mathfrak{sl}(4))_{\mu}$ (or more precisely, with the quotient $(\mathcal{U}_{q}^{-}(\mathfrak{sl}(4))/(F_3^3))_{\mu}$). In particular, for $k = 0$ we have

$$\mathcal{B}_{\mu} = \begin{cases} \{ F_1^{(r)} F_2^{(j)} F_1^{(s)} \equiv q F_1^{(r)} F_2^{(j-s)} F_3, \} & \text{if } j \geq i; \\ \{ F_1^{(i)} F_2^{(r)} \equiv q F_1^{(i-r)} F_3, \} & \text{if } j \leq i; \end{cases}$$

and for $k = 1$, from essentially the same calculations as in [Xi] we see that $\mathcal{B}_{\mu}$ contains the elements

$$F_1^{(r)} F_2^{(j)} F_1^{(s)} \equiv q F_1^{(r)} F_2^{(j-s)} F_3, \quad j \geq r + s = i \quad (5.1)$$

$$F_3 F_1^{(r)} F_2^{(j)} F_1^{(s)} \equiv q F_1^{(r-j)} F_2^{(j)} F_3, \quad j \geq r + s = i \quad (5.2)$$

$$F_2^{(s)} F_1^{(i)} F_2^{(r)} F_3 \equiv q F_1^{(i-r)} F_3, \quad i \geq r + s = j \quad (5.3)$$

$$F_3 F_1^{(i)} F_2^{(r)} F_2^{(s)} \equiv q F_1^{(i-j)} F_2^{(j-r)} F_3, \quad i \geq r + s = j \quad (5.4)$$

$$F_2^{(s)} F_1^{(i)} F_3 F_2^{(r)} \equiv q F_1^{(i-j)} F_2^{(r-j)} F_3, \quad i \geq r + s = j \quad (5.5)$$

$$\sum_{t=0}^{s} (-1)^t \begin{pmatrix} j - i - 2 + t \\ t \end{pmatrix} F_2^{(s-t)} F_1^{(i)} F_3 F_2^{(j-s+t)} \equiv q F_1^{(i-s)} F_1^{(i)} F_2^{(j-s-1)}, \quad \text{if } j > i + 1 \text{ and } 1 \leq s \leq i \quad (5.6)$$
Now consider the case $k = 2$. Observe that applying $\tilde{f}_3$ to the $k = 1$ canonical basis elements, we obtain the families

$$F_3 F_1^{(r)} F_2^{(j)} F_3 \equiv q F_1^{(r)} F_2^{(j-1)} F_3 F_2^{(j-s)} F_3 \quad \text{if } j \geq r + s = i \text{ and } s \geq 1,$$

$$F_2 F_1^{(i)} F_3 \equiv q F_1^{(i)} F_2^{(j)} F_3 F_2^{(j-1)} F_3 \quad \text{if } j \geq i = r \text{ and } j > s = 0. \quad (5.7)$$

$$F_3 F_2^{(s)} F_1^{(i)} F_2^{(r)} F_3 \equiv q F_1^{(i-s)} F_2^{(s-1)} F_3 F_2^{(r)} F_3 \quad \text{if } i \geq r + s = j \text{ and } s \geq 1,$$

$$F_2 F_1^{(i)} F_3 F_2^{(r)} F_3 \equiv q F_1^{(i-s)} F_2^{(s-1)} F_3 F_2^{(r-1)} F_3 \quad \text{if } i + 1 \geq r + s = j \text{ and } r, s \geq 1 \quad (5.8)$$

$$\sum_{t=0}^{s} (-1)^t \left[ j - i - 2 + t \right] F_3 F_2^{(s-t)} F_1^{(i)} F_2^{(j-s+t)} \equiv q F_1^{(i-s)} F_2^{(s-1)} F_3 F_2^{(j-s-1)} F_3 \quad \text{if } j > i + 1 \text{ and } 1 \leq s \leq i \quad (5.9)$$

This accounts for most of the canonical basis elements, except for most of those whose corresponding PBW elements modulo $q$ has both $F_{23}$ and $F_3$ as factors. However, since $F_{23} F_3 = F_3 F_2 F_3$, it is clear that if $v$ is a canonical basis element from the $k = 0$ case, then $v F_3 F_2 F_3$ is a canonical basis element; indeed, note that if $v = \sum a_i F^a$ is the expression for $v$ in terms of the PBW basis, then we obtain the PBW expansion $v F_3 F_2 F_3 = \sum a_i F^a F_3 F_2 F_3$, and the coefficients satisfy the same constraints. Therefore, we obtain the families

$$F_1^{(r)} F_2^{(j-1)} F_3 F_2 F_3 \equiv q F_1^{(r)} F_2^{(j-1-s)} F_3 F_2^{(j-s)} F_3 \quad \text{if } j - 1 \geq i = r + s, \quad (5.11)$$

$$F_2^{(s)} F_1^{(i)} F_3 F_2^{(r)} F_3 \equiv q F_1^{(i-s)} F_2^{(s-1)} F_3 F_2^{(r-1)} F_3, \quad \text{if } i + 1 \geq r + s = j. \quad (5.12)$$

Finally, for the case $k = 3$, observe that $F_{123} F_3 F_3 = F_3 F_2 F_1 F_3 F_2 F_3$, which is central in $U^-$. Since $F_2$ and $F_{123}$ commute, we again note that multiplying $k = 0$ canonical basis elements by $F_{123} F_3 F_3$ on the right produces canonical basis elements, and thus the canonical basis elements are

$$F_1^{(r)} F_2^{(j-2)} F_3 F_2 F_1 F_3 F_2 F_3 \equiv q F_1^{(r)} F_2^{(j-1-s)} F_3 F_2^{(j-s)} F_3 \quad \text{if } j - 2 \geq r + s = i - 1, \quad (5.13)$$

$$F_2^{(s)} F_1^{(i-1)} F_3 F_2 F_1 F_3 F_2 F_3 \equiv q F_1^{(i-1-s)} F_2^{(s-1)} F_3 F_2^{(r-1)} F_3 \quad \text{if } i - 1 \geq r + s = j - 2 \text{.} \quad (5.14)$$

This finishes the construction of the canonical basis elements. To obtain the description in the statement of the theorem, we apply $\tau$ and reinterpret $r, s, i, j, k$ in terms of the powers of the root vectors.

With this explicit description in hand, we can consider the compatibility of this basis with the irreducible finite-dimensional modules $V(\lambda)$ for $\lambda \in P^+$. Of course, this is answered for $\lambda$ typical or fully dominant by Theorem 3.16, so it remains to consider the case when $\lambda$ is a dominant atypical weight which is not fully dominant; explicitly, those weights $\lambda = a e_1 + b e_2 + c e_3 + d e_4$ such that $a \geq b \geq c$ and $d = -b - 1$ or $d = -a - 2$. At present, we do not know of a way to do this systematically, but we will explicitly work out some examples.
where again $0 \leq a \leq b \leq n$, and we deduce that $K(\lambda)$ has the canonical basis elements

$$F_2^{(a)} F_1^{(b)} 1_\lambda, \quad F_3 F_2^{(a)} F_1^{(b)} 1_\lambda, \quad F_2^{(a+1)} F_1^{(b)} F_3 1_\lambda, \quad F_2^{(a)} F_1^{(b+1)} F_2 F_3 1_\lambda$$

$$F_3 F_2^{(a+1)} F_1^{(b)} F_3 1_\lambda, \quad F_3 F_2^{(a)} F_1^{(b+1)} F_3 F_1 1_\lambda, \quad F_2^{(a+1)} F_1^{(b+1)} F_2 F_3 F_1 1_\lambda,$$

where again $0 \leq a \leq b \leq n$. On the other hand, one can compute that $\dim V(\lambda) = 2n^2 + 8n + 7$ (cf. [VdJ] for the formula in the classical limit\(^2\)). In particular, we have $\dim N(\lambda) = 2n^2 + 4n + 1$ basis vectors to remove to get a basis of $V(\lambda)$.

Well, first one finds that we have the linear dependencies

$$F_3 F_2^{(a+1)} F_1^{(b+1)} v_\lambda = \frac{n + 1 - a}{n + 2} F_2^{(a+1)} F_1^{(b+1)} F_3 v_\lambda - \frac{n - b}{n + 2} F_2^{(a)} F_1^{(b+1)} F_2 F_3 v_\lambda$$

for $0 \leq a \leq b \leq n - 1$. Consequently, we deduce that

$$F_3 F_2^{(a+1)} F_1^{(b+1)} F_3 v_\lambda = \frac{n - b}{n + 1 - a} F_3 F_2^{(a)} F_1^{(b+1)} F_2 F_3 v_\lambda$$

\(^2\)This formula still holds in the quantum case by arguments similar to those in [Kwo2, Section 2.6]; to wit, it makes sense to take a classical limit of a Kac module to get the Kac module of $gl(m|1)$, and in this limit the maximal submodule of $K(\lambda)$ is still a submodule. This gives us an upper bound $2n^2 + 4n + 1$ on the dimension of this submodule, and on the other hand we shall produce $2n^2 + 4n + 1$ linearly independent vectors in this submodule.
for $0 \leq a \leq b \leq n - 1$. On the other hand, observe that

$$F_3 F_2^{(a+1)} F_1^{(b+1)} F_3 v_\lambda = F_2^{(a)} F_3 F_2 F_1^{(b+1)} F_3 v_\lambda = \frac{n - b}{n + 1} F_2^{(a)} F_1^{(b+1)} F_3 F_2 F_3 v_\lambda,$$

hence

$$F_3 F_2^{(c+1)} F_1^{(c)} F_3 v_\lambda = \frac{n + 1 - c}{n + 1} F_2^{(c)} F_1^{(c)} F_3 F_2 F_3 v_\lambda,$$

$$F_3 F_2^{(a)} F_1^{(b+1)} F_3 F_2 F_3 v_\lambda = \frac{n + 1 - a}{n + 1} F_2^{(a)} F_1^{(b+1)} F_3 F_2 F_3 v_\lambda$$

for $1 \leq a \leq b \leq n$ and $1 \leq c \leq n$ (the case $b = n$ being a similar computation). We also note that $F_2^{(a)} F_1^{(b)} F_3 F_2 F_1 F_3 F_2 F_3 v_\lambda = 0$ for $0 \leq a \leq b \leq n$.

In particular, note that we have shown that $3 \binom{n+1}{2} + \binom{n+2}{2} + n = 2n^2 + 4n + 1$ of the canonical basis vectors in $K(\lambda)$ are $qA$-linear combinations of canonical basis elements, so we have the canonical basis

$$B(\lambda) = \{ b v_\lambda \mid b \in B, \ b v_\lambda \notin qA B v_\lambda \}$$

$$= \left\{ \begin{array}{ll}
F_2^{(a)} F_1^{(b)} v_\lambda, & 0 \leq a \leq b \leq n \\
F_2^{(a+1)} F_1^{(b)} F_3 v_\lambda, & 0 \leq c \leq n \\
F_2^{(a)} F_1^{(b+1)} F_3 F_2 F_3 v_\lambda, & 0 \leq d \leq n + 1
\end{array} \right\}$$

We note that this basis seems compatible with the crystal structure on $K(\lambda)$, though as in the $\mathfrak{gl}(2|1)$ case one would need to modify the operator $\tilde{e}_3$ to obtain an actual crystal structure; for instance, if $L = A B(\lambda)$ and $B = B(\lambda) + qL$, then $(L, B)$ is a crystal basis of $V(\lambda)$ where we replace $\tilde{e}_3$ with $\tilde{e}_3' = -q^{-2\epsilon_3'} \tilde{e}_3$. See Figure 5 for an example in the case $n = 1$.

**Example 5.4.** Consider the weight $\mu = n \epsilon_1 - \epsilon_4$, in which case we have $\dim V(\mu) = 3n^2 + 8n + 4$. By similar considerations to the previous example, we can show that $F_3 F_2 F_3 1_\mu$ is singular and generates $N(\mu)$, and we have the canonical basis

$$B(\mu) = \{ b v_\mu \mid b \in B, \ b v_\mu \neq 0 \}$$

$$= \left\{ \begin{array}{ll}
F_2^{(a)} F_1^{(b)} v_\mu, & 0 \leq a \leq b \leq n \\
F_2^{(a+1)} F_1^{(b)} F_3 v_\mu, & 0 \leq c \leq d \leq n, \ d > 0 \\
F_3 F_2^{(c+1)} F_1^{(d)} F_3 v_\mu, & 1 \leq e \leq f \leq n
\end{array} \right\}.$$

We note that $E_3 F_3 F_2 F_1 v_\mu = 0$ and thus $\tilde{e}_3 \tilde{f}_3 F_3 F_2 F_1 v_\mu = 0$, so once again we fail to have a crystal basis with respect to the previously defined Kashiwara operators.

**Example 5.5.** Let us consider one more example. Let $\lambda_1 = \epsilon_1 + \epsilon_2 - 2\epsilon_4$ and $\lambda_2 = \epsilon_1 + \epsilon_2 - 3\epsilon_4$. Then for $\lambda = \lambda_1$ or $\lambda_2$, $K(\lambda)$ is a 24-dimensional $U$-module. In particular, letting $1_\lambda$ denote the highest weight vector of $K(\lambda)$, we can explicitly describe the crystal and canonical bases; see Figure 6.

First, let us compute a basis for $V(\lambda_1)$; we compute, again using [VdJ], that $\dim V(\lambda_1) = 9$. Now in $K(\lambda_1)$, observe that $F_2 F_3 1_{\lambda_1} - 2 F_3 F_2 1_{\lambda_1}$ is a singular vector. In particular, in $V(\lambda_1)$
we have the identity $F_3 F_2 v_{\lambda_1} = \frac{q}{1 + q^2} F_2 F_3 v_{\lambda_1}$, which further implies

$$F_1 F_3 F_2 v_{\lambda_1} = \frac{q}{1 + q^2} F_1 F_2 F_3 v_{\lambda_1}, \quad F_2 F_1 F_3 F_2 v_{\lambda_1} = \frac{q}{1 + q^2} F_1 F_2^{(2)} F_3 v_{\lambda_1},$$

$$F_3 F_2 F_3 v_{\lambda_1} = 0, \quad F_3 F_2 F_1 F_3 F_2 v_{\lambda_1} = 0,$$

Thus, in this case, we see that have the canonical basis

$$\mathcal{B}(\lambda_1) = \{ x v_{\lambda_1} \mid x \in \mathcal{B}(\infty) \text{ and } x v_{\lambda_1} \notin q \mathcal{L}(\lambda_1) \} = \begin{pmatrix} v_{\lambda_1}, & F_2 v_{\lambda_1}, & F_3 v_{\lambda_1}, & F_1 F_2 v_{\lambda_1}, & F_2 F_3 v_{\lambda_1}, & F_1 F_2^{(2)} F_3 v_{\lambda_1}, & F_1 F_2^{(2)} F_3 v_{\lambda_1} \\ F_2 v_{\lambda_1}, & F_3 v_{\lambda_1}, & F_1 F_2 v_{\lambda_1}, & F_2 F_3 v_{\lambda_1}, & F_1 F_2^{(2)} F_3 v_{\lambda_1}, & F_1 F_2^{(2)} F_3 v_{\lambda_1}, & F_1 F_2^{(2)} F_3 v_{\lambda_1} \\ F_3 v_{\lambda_1}, & F_1 F_2 v_{\lambda_1}, & F_2 F_3 v_{\lambda_1}, & F_1 F_2^{(2)} F_3 v_{\lambda_1}, & F_1 F_2^{(2)} F_3 v_{\lambda_1}, & F_1 F_2^{(2)} F_3 v_{\lambda_1}, & F_1 F_2^{(2)} F_3 v_{\lambda_1} \\
\end{pmatrix}$$
Similarly, we compute that \( \dim V(\lambda_2) = 20 \). Well, in \( K(\lambda_2) \), we have the unique singular vector

\[
F_2 F_1 F_3 F_2 F_3 1_{\lambda_2} - [2] F_1 F_3 F_2^{(2)} F_3 1_{\lambda_2} + [3] F_3 F_2 F_1 F_3 F_2 1_{\lambda_2},
\]

hence in particular in \( V(\lambda_2) \) we have the identity

\[
F_3 F_2 F_1 F_3 F_2 v_{\lambda_2} = \frac{q + q^3}{1 + q^2 + q^4} F_3 F_2^{(2)} F_3 v_{\lambda_2} - \frac{q^2}{1 + q^2 + q^4} F_2 F_1 F_3 F_2 F_3 v_{\lambda_2}.
\]

This further implies \( F_3 F_2 F_1 F_3 F_2 F_3 v_{\lambda_2} = 0 \), and we find we have the basis

\[
\mathcal{B}(\lambda_2) = \{ x v_{\lambda_2} \mid x \in \mathcal{B}(\infty) \text{ and } x v_{\lambda_2} \notin q\lambda(\lambda_2) \}
\]

5.3. The other rank 2 case. In Theorem 4.23, we assume that we are working with the standard Borel associated to the GCM \( A^m \), but what about a canonical basis in other cases? The main difference is that Corollary 4.20 doesn’t apply, so to relate the lattices spanned by different PBW bases, there is an additional rank 2 case of comparing the lattices spanned by the two PBW bases is associated to the GCM \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \); note that \( U^-(A) \cong \mathbb{Q}(q) \langle F_1, F_2 \rangle / (F_1^2, F_2^2) \).

In this case, we have the two reduced expressions \( i = (1, 2, 1) \) and \( j = (2, 1, 2) \), which yield the root vectors \( F_{1,\alpha_1+\alpha_2} = T_2(F_1) = F_2 F_1 + q^{-1} F_1 F_2 \) and \( F_{3,\alpha_1+\alpha_2} = T_1(F_2) = F_1 F_2 + q^{-1} F_2 F_1 \), thus the PBW bases are given by

\[
\mathcal{B}_i = \left\{ F_2^{(a)}(F_2 F_1 + q^{-1} F_1 F_2)^{(b)} F_2^{(c)} \mid a, c \in \{0, 1\}, b \geq 0 \right\}
\]

\[
\mathcal{B}_j = \left\{ F_2^{(a)}(F_2 F_1 + q^{-1} F_1 F_2)^{(b)} F_1^{(c)} \mid a, c \in \{0, 1\}, b \geq 0 \right\}.
\]

In particular, observe that it is more natural to consider the \( \mathbb{Z}[q^{-1}] \)-span of these bases then the \( \mathbb{Z}[q] \)-span; indeed, \( F_2 F_1 \notin \mathbb{Z}[q] \mathcal{B}_i \), so Lemma 4.22 doesn’t hold in this case. Of course, note that using the \( \mathbb{Z}[q^{-1}] \) span for this case also invalidates Lemma 4.22 for the full rank case; this is the simplest instance of the obstruction caused by the chirality in \( q \) mentioned in §1.

One might hope to still construct a canonical basis by reinterpretting the lemma in this rank 2 case with \( q \) replaced everywhere by \( q^{-1} \). This is not unreasonable, since it is easy to verify that the \( \mathbb{Z}[q, q^{-1}] \)-span of the PBW bases agree and satisfy triangularity under the bar involution (cf. [CHW3, Lemmas 7.6, 7.7] or [Tin, Theorem 5.1]) so standard bar-invariant bases exist. However, even the \( \mathbb{Z}[q^{-1}] \)-span of the PBW bases are different: for instance, note that

\[
(\mathcal{B}_i)_{2\alpha_1+2\alpha_2} = \left\{ F_1 F_2 F_1 F_2, \frac{F_2 F_1 F_2 F_1}{[2]} + q^{-2} \frac{F_1 F_2 F_1 F_2}{[2]} \right\}.
\]
This finishes the proof of the classification of singular vectors.

Nevertheless, there is a natural choice of basis for $U^-$:

$$B = \left\{ F_i^a (F_2 F_1)^b \; | \; a, c \in \{0, 1\}, b \in \mathbb{Z}_{\geq 0} \right\} = \left\{ F_i^a (F_2 F_1)^b \; | \; a, c \in \{0, 1\}, b \in \mathbb{Z}_{\geq 0} \right\}.$$

This is a bar-invariant basis of $U^-$ which is also trivially a basis of the Lusztig integral form, since for any $i \in I$, we have $F_i^{(a)}(n) = \delta_{1,n} F_i$. Moreover, $B$ has a natural crystal structure (where $\tilde{f}_i x = F_i x$ and $\varepsilon_i x = e_i(x)$), and is congruent modulo $q$ to the unnormalized PBW bases

$$\left\{ F_i^a (F_2 F_1 + q F_i F_j)^b \; | \; a, c \in \{0, 1\}, b \geq 0 \right\} \text{ for } \{i, j\} = I.$$

Moreover, $B$ even satisfies some compatibilities with finite-dimensional weight modules. Note that in this case, the simple roots are $\alpha_1 = \epsilon_1 - \epsilon_3$ and $\alpha_2 = \epsilon_3 - \epsilon_2$, so the coroots are $h_1 = \epsilon_1^\vee + \epsilon_3^\vee$ and $h_2 = -\epsilon_3^\vee - \epsilon_2^\vee$. For $\lambda = a \epsilon_1 + b \epsilon_2 + c \epsilon_3 \in P$, define $M(\lambda)$ to be the Verma module of highest weight $\lambda$ as usual, and let $\lambda_1$ denote a highest weight vector.

**Lemma 5.6.** Let $\lambda = a \epsilon_1 + b \epsilon_2 + c \epsilon_3 \in P$ and define $\lambda(i) = \langle h_i, \lambda \rangle$. The singular vectors (up to constant multiple) in $M(\lambda)$, other than $1$, are as follows.

1. $F_1 \lambda_1$ if and only if $\lambda(1) = 0$;
2. $F_2 \lambda_1$ if and only if $\lambda(2) = 0$;
3. $[\lambda(1)](F_1 F_2)^{\lambda(1) + \lambda(2)} \lambda_1 - [\lambda(2)](F_2 F_1)^{\lambda(1) + \lambda(2)} \lambda_1$ if and only if $\lambda(1) + \lambda(2) > 0$.

In particular, the unique simple quotient $V(\lambda)$ of $M(\lambda)$ is finite dimensional if and only if $\lambda(1) = \lambda(2) = 0$ or $\lambda(1) + \lambda(2) > 0$.

**Proof.** Observe that

$$E_1 F_1^\tau (F_2 F_1)^y F_2^\lambda = \delta_{x,1} [\lambda(1) - y - z] (F_2 F_1)^y F_2^\lambda 1 + \delta_{x,0} (-1)^{x+1} [\lambda(1)] F_1^\tau (F_2 F_1)^y F_2^\lambda$$

$$E_2 F_1^\tau (F_2 F_1)^y F_2^\lambda = \delta_{x,1} [\lambda(2) - y] F_1 (F_2 F_1)^y F_2^\lambda 1 + \delta_{x,1} (-1)^{y} [\lambda(2)] F_1^\tau (F_2 F_1)^y F_2^\lambda.$$

Observe that $(F_2 F_1)^y F_2 \lambda_1$ is singular if and only if $\lambda(2) = 0$ and $y = 0$, and likewise $F_1 (F_2 F_1)^y F_2 \lambda_1$ is singular if and only if $\lambda(1) = 0$ and $y = 0$. Furthermore,

$$c((F_2 F_1)^y F_2 \lambda_1 + d((F_2 F_1)^y F_2 \lambda_1$$

is singular if and only if there is some $y \geq 0$ such that

$$c[\lambda(1) - y] = d[\lambda(1)] \text{ and } c[\lambda(2)] = d[\lambda(2) - y].$$

In particular, these equations are satisfied when $c = d = 0; c \neq 0, d = 0 = \lambda(2) \text{ and } y = \lambda(1) \geq 0; d \neq 0, c = 0 = \lambda(1), \text{ and } y = \lambda(2) \geq 0; \text{ or } c \neq 0 \text{ and } d \neq 0, \text{ and}\n
$$[\lambda(1) - y][\lambda(2) - y] = [\lambda(1)][\lambda(2)],$$

which holds if and only if $y = \lambda(1) + \lambda(2) \geq 0$. In the case $c \neq 0$ and $d \neq 0$, we see that

$$c[\lambda(2)] = -d[\lambda(1)].$$

This finishes the proof of the classification of singular vectors.

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3Here, for consistency with earlier discussions of canonical/crystal bases, we switch to the bar-conjugate version of the PBW bases.
Recall that $N(\lambda)$ is the maximal submodule of $M(\lambda)$, and $V(\lambda) = M(\lambda)/N(\lambda)$ is the simple quotient. Let us denote by $v_\lambda$ the image of $1_\lambda$ in $V(\lambda)$. Note that $N(\lambda)$ contains, and is generated by, all singular vectors other than (nonzero multiples of) $1_\lambda$. We see that when $\lambda(1) = \lambda(2) = 0$, $F_1 1_\lambda, F_2 1_\lambda \in N(\lambda)$ hence $V(\lambda) = M(\lambda)/N(\lambda) = \mathbb{Q}(q)v_\lambda$ is one-dimensional. Further, note that if $\lambda(1) + \lambda(2) < 0$ or $\lambda(1) = -\lambda(2)$, then $1_\lambda$ is the only singular vector, hence $N(\lambda) = 0$ and $V(\lambda)$ is infinite-dimensional. Finally, if $\lambda(1) + \lambda(2) > 0$, then we have
\[
V(\lambda) = \bigoplus_{0 \leq r \leq 1; 0 \leq y < \lambda(1)+\lambda(2)} \mathbb{Q}(q)F_1^r(F_2F_1)^yF_2^yv_\lambda \text{ where } I = \{i, j\} \text{ and } \lambda(i) \neq 0. \quad (5.15)
\]

In particular, we observe the following corollary.

**Corollary 5.7.** Let $L(\lambda) = L(\infty)v_\lambda$, where $L(\infty)$ is the crystal lattice of $U^-$. Then $V(\lambda)$ has the canonical basis
\[
\mathcal{B}(\lambda) = \{bv_\lambda \mid b \in \mathcal{B} \text{ such that } bv_\lambda \notin qL(\lambda)\}.
\]

We note that in the case $\lambda(1) = \lambda(2) > 0$, $(F_1F_2)^{\lambda(1)+\lambda(2)}v_\lambda = (F_2F_1)^{\lambda(1)+\lambda(2)}v_\lambda$; in particular, in contrast to the results of Section 5.1, the projection $\pi_\lambda : U^- \to V(\lambda)$ does not restrict to a bijection between $\{b \in \mathcal{B} \mid bv_\lambda \notin qL(\lambda)\}$ and $\mathcal{B}(\lambda)$.

**Remark 5.8.** As a final remark, let us observe a trend in the results of Section 5. It is unfortunately not the case in general that the finite-dimensional irreducible modules inherit a basis from the canonical basis of a half quantum $\mathfrak{gl}(m|1)$, since the modules with atypical highest weight generally will have linear dependencies. Nevertheless, we can often canonically remove the redundant elements by taking only the nonzero elements in the quotient $L/qL$, where $L$ is the $\mathcal{A}$-lattice generated by the images of canonical basis elements. It would be interesting to realize this as some sort of crystal basis construction, though, as we have noted in the examples, one would need a different definition of Kashiwara operators.

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