A LINEAR TIME ALGORITHM FOR THE 3-NEIGHBOUR TRAVELLING SALESMAN PROBLEM ON HALIN GRAPHS AND EXTENSIONS

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Abstract. The quadratic travelling salesman problem (QTSP) is to find a least cost Hamiltonian cycle in an edge-weighted graph. We define a restricted version of QTSP, the \( k \)-neighbour TSP (TSP\((k)\)), and give a linear time algorithm to solve TSP\((k)\) on a Halin graph for \( k \leq 3 \). This algorithm can be extended to solve TSP\((k)\) on any fully reducible class of graphs for any fixed \( k \) in polynomial time. This result generalizes corresponding results for the standard TSP. Further, these results are useful in establishing approximation bounds based on domination analysis. TSP\((k)\) can be used to model various machine scheduling problems including an optimal routing problem for unmanned aerial vehicles (UAVs).

1. Introduction

The Travelling Salesman Problem (TSP) is to find a least cost Hamiltonian cycle in an edge weight graph. It has been one of the most widely studied combinatorial optimization problems and is well-known to be NP-complete. The TSP model has been used to formulate a wide variety of applications. For details we refer the reader to the well-known books [23, 18, 9, 2, 27] as well as the following papers [21, 22, 5, 14].

For some applications, more than linear combinations of distances between consecutive nodes are desirable in formulating an objective function. Consider determining an optimal routing for a unmanned aerial vehicle (UAV) which has a list of targets at specific locations. This problem can be modelled as a TSP which requires a tour that minimizes the distance travelled, however, this neglects to factor in the physical limitations of the vehicle, such as turn radius or momentum. To illustrate this idea, in Figure 1 we give a Hamilton path, in Figure 2 we give the corresponding flight path, and Figure 3 shows a route which is longer but can be travelled at a greater speed. To model the traversal time, we can introduce
penalties for pairs of edges to force a smooth curve for its traversal. In this paper we consider a generalization of the TSP which can be used to model similar situations and contains many variations of the TSP, such as the angle-metric TSP \[1\], Dubins TSP \[24\] and precedence-constrained TSP \[6\] as special cases.

![Figure 1. Optimal TSP tour.](image1)

![Figure 2. Smoothing of the optimal TSP tour.](image2)

![Figure 3. A route which can be travelled more quickly.](image3)

Let \( G = (V, E) \) be an undirected graph on the node set \( V = \{0, 1, \ldots, n - 1\} \) with the convention that all indices used hereafter are taken modulo \( n \). For each edge \((i, j) \in E\) a nonnegative cost \( c_{ij} \) is prescribed. A Hamiltonian cycle or tour in \( G \) is a cycle containing every node of \( G \). Let \( \tau = (v_0, v_1, \ldots, v_{n-1}, v_0) \) be a tour in \( G \) and let \( \mathcal{F} \) be the set of all tours in \( G \). The edges \( e = (v_i, v_{i+1}) \) and \( f = (v_j, v_{j+1}) \) are \( k \)-neighbours on \( \tau \) if and only if a shortest path in terms of the number of edges between \( e \) and \( f \) on \( \tau \) containing these edges has exactly \( k \) edges, for \( k \geq 1 \). Here the shortest path refers to the path with the least number of edges, rather than the minimum cost path. Thus \( e \) and \( f \) are 2-neighbours in \( \tau \) if and only if they share a common node in \( \tau \).

Let \( q(e, f) \) be the cost of the pair \((e, f)\) of edges and \( \delta(k, \tau) = \{(e, f) : e, f \in \tau \text{ and } e \text{ and } f \text{ are } p \text{-neighbours on } \tau \text{ for } 2 \leq p \leq k\} \). Assume that \( q(e, f) = q(f, e) \) for every pair of edges \((e, f) \in G\). Then the \( k \)-neighbour TSP (TSP\((k)\)) is defined as \[28\]

\[
\text{TSP}(k) : \quad \text{Minimize} \quad \sum_{(e,f) \in \delta(k,\tau)} q(e,f) + \sum_{e \in \tau} c(e) \\
\text{Subject to} \quad \tau \in \mathcal{F}.
\]

A closely related problem, the quadratic TSP (QTSP), is defined as follows:
\[ \text{QTSP}: \quad \text{Minimize} \quad \sum_{(e,f) \in \tau \otimes \tau} q(e,f) + \sum_{e \in \tau} c(e) \]

Subject to \[ \tau \in \mathcal{F}. \]

where \( \tau \otimes \tau = \tau \times \tau \setminus \{(e,e) : e \in \tau\} \). Note:

\[
\tau \otimes \tau = \begin{cases} 
\delta(n/2, \tau) & \text{if } n \text{ is even} \\
\delta((n+1)/2, \tau) & \text{if } n \text{ is odd.}
\end{cases}
\]

Thus when \( k \geq n/2 \) (for \( n \) even) or \( k \geq (n+1)/2 \) (for \( n \) odd), the \( k \)-neighbour TSP reduces to the quadratic TSP. We note TSP(1) is simply the original TSP, while TSP(\( k \)) is QTSP for fixed \( k \geq \lceil n/2 \rceil \).

The bottleneck version of TSP(\( k \)) was introduced by Arkin et al. in [3, 4], denoted as the \( k \)-neighbour maximum scatter TSP. Jager and Molitor [19] encountered TSP(2) while studying the Permutated Variable Length Markov model. Several heuristics are proposed and compared in [19, 16] as well as a branch and bound algorithm for TSP(2) in [16]. Polyhedral results for TSP(2) were obtained by Fischer and Helmberg [17] and Fischer [15]. The \( k \)-neighbour TSP is also related to the \( k \)-peripatetic salesman problem [20, 13] and the watchman problem [8]. To the best of our knowledge, no other works in the literature address TSP(\( k \)).

Referring to the UAV example discussed earlier, it is clear that the flight subpaths depend on both the angle and distances between successive nodes. By precalculating these and assigning costs to \( q(e,f) \) for some \( e, f \in E(G) \), we see that QTSP is a natural model for this problem. In fact, the flight paths may be affected by edges further downstream. Thus we can get successively better models by considering TSP(1), TSP(2),..., TSP(\( k \)) in turn. In practice we expect diminishing returns to take hold quickly.

In this paper we show that QTSP is strongly NP-complete even if the costs are restricted to 0-1 values and the underlying graph is Halin. Halin graphs are an important class of graphs described in the next section. In contrast, TSP on a Halin graph can be solved in \( O(n) \) time [10]. Interestingly, we show that TSP(2) and TSP(3) can also be solved on a Halin graph in \( O(n) \) time. In fact, our approach can be extended to any fixed \( k \). We note that while Halin graphs have treewidth 3, the results on graphs with bounded treewidth (e.g. [12, 7]) are not easily extended to graph problems with quadratic objective functions.

The paper is organized as follows. In Section 2 we introduce some preliminary results and notations for the problem. The complexity result for QTSP on Halin graphs is given in Section 3. A \( O(n) \) algorithm to solve TSP(3) on Halin graphs is given in Section 4.1 and we discuss briefly how it can be extended to TSP(\( k \)) for fixed \( k \) on any fully reducible class of graphs in Section 5.
2. Notations and definitions

A Halin graph $H = T \cup C$ is obtained by embedding a tree with no nodes of degree two in the plane and connecting the leaf nodes of $T$ in a cycle $C$ so that the resulting graph remains planar. The non-leaf nodes belonging to $T$ are referred to as tree or internal nodes and the nodes in $C$ are referred to as cycle or outer nodes of $H$. A Halin graph with exactly one internal node is called a wheel. If $T$ has at least two internal nodes and $w$ is an internal node of $T$ which is adjacent to exactly one other internal node, then $w$ is adjacent to a set of consecutive nodes of $C$, which we denote by $C(w)$. Note that $|C(w)| \geq 2$. The subgraph of $H$ induced by $\{w\} \cup C(w)$ is referred to as a fan, and we call $w$ the centre of the fan. See Figure 4.

Figure 4. A Halin graph with 3 fans.

Lemma 2.1. \[10\] Every Halin graph which is not a wheel has at least two fans.

Let $G = (V, E)$ be a graph and let $S \subseteq V$ be a connected subgraph of $G$. Let $\varphi(S)$ be the cutset of $S$, that is, the smallest set of edges whose removal disconnect $S$ from $G$. Let $G(S)$ be the graph obtained by contracting $S$ to a single node, called a ‘pseudo-node’ and denoted by $v_S$ \[10\]. The edges in $G(S)$ are obtained as follows:

1. An edge with both ends in $S$ is deleted;
2. An edge with both ends in $G - S$ remains unchanged;
3. An edge $(v_1, v_2)$ with $v_1 \in G - S$, $v_2 \in S$ is replace by the edge $(v_1, v_s)$

Lemma 2.2. \[10\] If $F$ is a fan in a Halin graph $H$, then $H(F)$ is a Halin graph.

Note that each time a fan is contracted using the graph operation $H(F)$, the number of non-leaf nodes of the underlying tree is reduced by one. That is, after fewer than $\lceil (n - 1)/2 \rceil$ fan contractions, a Halin graph will be reduced to a wheel.
Let \( w \) be the centre of a fan \( F \), and label the outer nodes in \( F \) in the order they appear in \( C \) as, \( u_1, u_2, \ldots, u_r \) \((r \geq 2)\). Let \( \{j, k, l\} \) be the 3-edge cutset \( \varphi(F) \) which disconnect \( F \) from \( G \) such that \( j \) is adjacent to \( u_1 \), \( k \) is adjacent to \( w \) but not adjacent to \( u_i \) for any \( i \), \( 1 \leq i \leq r \) and \( l \) is adjacent to \( u_r \) (See Figure 5, \( r = 4 \)).

Note that any Hamilton cycle \( \tau \) in \( H \) contains exactly two edges of \( \{j, k, l\} \). The pair of edges chosen gives us a small number of possibilities for traversing \( F \) in a tour \( \tau \). For example, if \( \tau \) uses \( k \) and \( l \), it contains the subsequence \( v, u_1, u_2, \ldots, u_r \) (call this a left-traversal of \( F \)), if \( \tau \) uses \( j \) and \( k \) it contains the subsequence \( u_1, u_2, \ldots, u_r, v \) (call this a right-traversal of \( F \)) and if \( \tau \) uses \( j \) and \( l \), it contains a subsequence of the form \( u_1, u_2, \ldots, u_i, v, u_{i+1}, \ldots, u_r \), for some \( i \in \{1, 2, \ldots, r - 1\} \) as it must detour through the centre of \( F \) (call this a centre-traversal of \( F \)).

![Figure 5. A Halin graph with 3-edge cutset \( \{j, k, l\} \).](image)

### 3. Complexity of QTSP on Halin graphs

Many optimization problems that are NP-Hard on a general graph are solvable in polynomial time on a Halin graph \([26, 10, 25]\). Unlike these special cases, we show that QTSP is strongly NP-hard on Halin graphs. The decision version of QTSP on a Halin graph, denoted by RQTSP, can be stated as follows:

“Given a Halin graph \( H \) and a constant \( \theta \), does there exist a tour \( \tau \) in \( H \) such that \( \sum_{e \in E(\tau)} c(e) + \sum_{e, f \in E(\tau)} q(e, f) \leq \theta \)?”

**Theorem 3.1.** \([28]\) RQTSP is strongly NP-complete even if the values \( c(e) \in \{0, 1\} \) and \( q(e, f) \in \{0, 1\} \) for \( e, f \in H \).

**Proof.** RQTSP is clearly in NP. We now show that the 3-SAT problem can be reduced to RQTSP. The 3-SAT problem can be stated as follows: “Given a Boolean formula \( R \) in CNF containing a finite number of clauses \( C_1, C_2, \ldots, C_h \) on variables \( x_1, x_2, \ldots, x_t \) such that each
clause contains exactly three literals \((L_1, \ldots, L_{3h})\) where for each \(i\), \(L_i = x_j\) or \(L_i = \neg x_j\) for some \(1 \leq j \leq t\), does there exists a truth assignment such that \(R\) yields a value ‘true’?

From a given instance of 3-SAT, we will construct an instance of RQTSP. The basic building block of our construction is a 4-fan gadget obtained as follows. Embed a star on 5 nodes with center \(v\) and two specified nodes \(\ell\) and \(r\) on the plane and add a path \(P\) from \(\ell\) to \(r\) covering each of the pendant nodes so that the resulting graph is planar (See Figure 6). Call this special graph a 4-fan gadget.

![Figure 6. 4-fan gadget.](image)

The nodes on path \(P\) of this gadget are called outer nodes and edges on \(P\) are called outer edges. Let \(\mu_1, \mu_2, \mu_3\) be edges with distinct end points in \(P\). Note that any \(\ell\)-\(r\) Hamiltonian path of the gadget must contain all the outer edges except one which is skipped to detour through \(v\). We will refer to an \(\ell\)-\(r\) Hamiltonian path in a 4-fan gadget as a center-traversal as before.

We will construct a Halin graph \(H\) using one copy of the gadget for each clause and let \(\mu_1, \mu_2, \mu_3\) correspond to literals contained in that clause. We will assign costs to pairs of edges such that every Hamilton cycle with cost 0 must contain a centre-traversal for each clause. To relate a Hamilton cycle to a truth assignment, a centre-traversal which does not contain edge \(\mu_i\) corresponds to an assignment of a true value to literal \(L_i\).

Now construct \(H\) as follows. For each clause \(C_1, \ldots, C_h\), create a copy of the 4-fan gadget. The \(r, \ell, \) and \(v\) nodes of the 4-fan gadget corresponding to the clause \(C_i\) are denoted by \(r_i, \ell_i\) and \(v_i\) respectively. Connect the node \(r_i\) to the node \(\ell_{i+1}, i = 1, 2, \ldots n - 1\). Introduce nodes \(v_x\) and \(v_y\) and the edges \((\ell_1, v_x), (v_x, v_y), (v_y, r_n)\). Also introduce a new node \(w\) and connect it to \(v_x, v_y\) and \(v_i\) for \(i = 1, 2, \ldots h\). The resulting graph is the required Halin graph \(H\). See Figure [7].

Assign the cost \(c(e) = 0\) for every edge in \(H\). Let \(x = (v_x, w)\) and \(y = (v_y, w)\). Refer to the edges adjacent to \(w\) as “spokes”. Note that every tour which contains edges \(x\) and \(y\) traverses every gadget using a centre-traversal. For each gadget: assign paired cost \(q(e, f) = 1\) for pairs of edges which are neither outer edges nor both adjacent to the same literal edge \(\mu_1, \mu_2\)
or \( \mu_3 \), and for all other pairs of edges within the gadget assign cost 0. For each variable \( x_l \), \( l = 1, \ldots, n \), and all literals \( L_m, L_q \) (\( m \neq q \)) if \( x_l = L_m = \neg L_q \), assign cost \( q(\mu'_m, \mu'_q) = 1 \)
where \( \mu'_m \) and \( \mu'_q \) are edges connecting \( \mu_m \) (and \( \mu_q \)) to the respective 4-fan gadget centre \( v \).

All other paired costs are assumed to be 0.

Suppose \( B \) is a valid truth assignment. Then in each clause there exists at least one true literal. Consider a tour \( \tau \) in \( H \) which contains the edges \( x \) and \( y \) and traverses every gadget such that \( \tau \) detours around exactly one literal edge which corresponds to a literal which is true in \( B \). Since the truth assignment is valid, such a \( \tau \) exists. Clearly \( \tau \) has cost 0, since no costs are incurred by pairs of edges contained in a single gadget, nor are costs incurred of the form \( q(\mu'_a, \mu'_b) \) where \( L_a = \neg L_b \). The latter must be true because in any truth assignment, the variable corresponding to \( L_a \), say \( x_a \) must be either assigned a value of true or false. Suppose a cost of 1 is incurred by \( q(\mu'_a, \mu'_b) \) and hence \( L_a = \neg L_b \). If \( x_a \) is true, and \( x_a = L_a \), then \( L_b \) clearly must be false, so \( \tau \) cannot detour to miss both \( \mu'_a \) and \( \mu'_b \). The same contradiction arises, if \( x_a \) is false. Hence a yes instance of 3-SAT can be used to construct a yes instance for RQTSP.

Now suppose there is an optimal tour which solves RQTSP with \( k = 0 \). Suppose \( \tau' \) is such a tour. Clearly it must use edges \( x \) and \( y \), and hence must traverse every gadget via a
centre-traversal. Such a detour must skip a literal edge in every gadget, otherwise a cost of 1 is incurred. Suppose $D = \{L_1, \ldots, L_s\}$ is the set of literals which are skipped. $L_i \neq \neg L_j$ for any $i, j$, otherwise a cost of 1 is incurred. This implies that a truth assignment which results in every literal in $D$ being $true$ is a valid truth assignment to the variables $x_1, \ldots, x_n$. That is, for each literal edge which is skipped in $\tau'$, assign $true$ or $false$ to the corresponding variable such that the literal evaluates to $true$ (if $L_i = x_j$, set $x_j = true$ and if $L_i = \neg x_j$, set $x_j = false$). The truth values for any remaining variables can be assigned arbitrarily. This truth assignment returns $true$ for each clause since exactly one literal in each clause is detoured, and evaluates to $true$. Hence this truth assignment is a valid assignment for 3-SAT.

4. Complexity of $k$-neighbour TSP on Halin graphs

We now examine the case when $k$ is fixed. Note that for fixed $k$, only a constant number of pairs of edges are relevant in evaluating a tour. Define edges $e = (u, x)$ and $f = (x, v)$ in a planar graph $G$ to be consecutive at $x$ in a planar embedding of $G$ if there exists a face which contains both $e$ and $f$. This may include the unbounded outer face.

**Theorem 4.1.** \[28\] Given a planar embedding of $H$, no tour in $H = T \cup C$ contains edges $e = (u, x)$ and $f = (x, v)$ which are non-consecutive at $x$.

**Proof.** Proceed by contraction. Suppose that there exists a planar embedding of $H$ and a tour $\tau$ in $H$ such that $e = (u, x)$ and $f = (x, v)$ are nonconsecutive edges at $x$ and $e, f \in \tau$. By our previous discussion regarding fan traversals, we only need to consider $x \notin C$. Since $e$ and $f$ are non-consecutive at $x$, there are edges in the planar embedding of $H$, say $g = (y, x)$ and $h = (x, z)$, such that the clockwise order of edges around $x$ is $f, \ldots, h, \ldots, e, \ldots, g$ (See Figure 8). Suppose $T$ is rooted at $x$, then $x$ has (at minimum), subtrees rooted at $u, v, y$ and $z$. Since $\tau$ is a tour which contains edge $e$, it must contain a path through the subtree rooted at $u$ from $u$ to $u_c \in C$, which we denote by $P_1$. Similarly, it must contain $P_2$ in the subtree rooted at $v$ from $v$ to $v_c \in C$ (See Figure 8). Now we note that $H \setminus P$ where $P = P_1 \cup P_2 \cup \{e, f\}$ has two components, one containing $y$ and one containing $z$. Restricting to $\tau$, which also contains $P$ as a path, we see that $\tau - P$ contains at least 2 components, which is a contradiction.

We will refer to paths that may be contained in a tour as candidate paths and $k$-candidate paths when the length $k$ is specified. These must satisfy the condition of Theorem 4.1.

**Corollary 4.2.** For fixed $k$, $H$, given tour $\tau$ and $e \in \tau$, $1 \leq k \leq \lfloor n/2 \rfloor$, there are at most $k \cdot 2^{k-2}$ $k$-candidate paths containing $e$ which may belong to $\tau$. 
From Corollary 4.2, it is clear that the number of quadratic costs which are relevant is bounded above by $2^{k-2} \cdot |V(G)| = O(n)$ for any fixed $k$.

Since the interior nodes of $H$ form a tree, it is clear that any face of $H$ must contain an outer edge. Moreover, the following corollary will prove useful.

**Corollary 4.3.** If $H$ is embedded in the plane such that it is planar and $C$ defines the outer face, for any outer edge $e$ which is contained in the outer face and face $F_e$, every tour which does not contain $e$ must contain all other edges in $F_e$.

4.1. **TSP(3) on Halin graphs.** TSP(1) is the same as TSP, which is solvable in linear time on Halin graphs [10]. TSP(2) can also be solved in linear time by minor modifications of the algorithm in [26]. However, for $k \geq 3$, such minor changes appear not to be a viable option. We now develop a linear time algorithm to handle TSP($k$), $k \geq 3$ and fixed $k$.

The following modification is applied to the cost function to simplify the problem. Note that we can restrict our attention to the $O(n)$ candidate 3-paths as a result of Corollary 4.2.

Let $P_3(G) = \{(e, f, g) : (e, f, g) \text{ is a candidate path in } G\}$. For each candidate triple $(e, f, g)$, let

$$q(e, f, g) = q(e, g) + \frac{q(e, f) + q(f, g)}{2} + \frac{c(e) + c(f) + c(g)}{3}$$

Consider the problem:

$$STSP(3) : \text{ Minimize } \sum_{(e, f, g) \in P_3(\tau)} q(e, f, g)$$

Subject to $\tau \in \mathcal{F}$.
Theorem 4.4. Any optimal solution to the STSP(3) is also optimal for TSP(3).

The proof of Theorem 4.4 follows from routine algebra. In view of Theorem 4.4, we can restrict our attention to STSP(3).

We note that the approach used in Cornuejols et al. [10] where the costs of paths are assigned to the newly created arcs by solving a linear system of equations cannot be extended for any $k$-neighbour TSP for $k \geq 3$ as it leads to an over-determined system of equations which may be infeasible. Instead, we extend the penalty approach used in Phillips et al. [26]. The idea here is to introduce a vertex-weighted version of the problem where we define a penalty function on the nodes of $C$. We iteratively contract the fans in $H$, storing the values of the best subpaths which traverse each fan in the penalty function. Once we are left with a wheel, we can find the optimal tour. We reverse the order of contraction and uncontract, maintaining optimality, until we have an optimal tour in the original graph.

Let us first discuss the case where $H$ is a Halin graph that contains a fan. In this case, $H$ will have at least two fans.

Let $F$ be an arbitrary fan in $H$ with $v$ as the centre. Label the outer nodes in the order they appear in $C$, say, $u_1, u_2, \ldots, u_r$ ($r \geq 2$). Let $\{j, k, l\}$ be the 3-edge cutset $\varphi(F)$ which disconnects $F$ from $H$ such that $j$ is adjacent to $u_1$, $k$ is adjacent to $v$ and $l$ is adjacent to $u_r$. Let $j = (u_1, u_0)$ and $l = (u_r, u_{r+1})$. There are exactly two edges not connected to $F$, incident on $u_0$, sharing a face with $F$. These edges are labelled $\alpha_1, \alpha_2$. Likewise, there are exactly two edges not connected to $F$, incident on $u_{r+1}$ but sharing a face with $F$. These edges are labelled $\alpha_3, \alpha_4$. (See Figure 9) Without loss of generality $\alpha_1, \alpha_3$ are in $C$ and $\alpha_2, \alpha_4$ are in $T$.

To complete a fan contraction operation, we consider the 3 types of traversals of $F$. For any left or right traversal of $F$, there is a single path through $F$, the cost of which can be stored in the new variables created as a result of the contraction operation.

We define a penalty function stored at nodes of $C$ which contains the cost of the minimum centre traversals of $F$ to be used when $\tau$ contains $j$ and $l$. Any tour which includes $j$ and $l$ must also pass through one edge in of each of the pairs of edges incident with $j$ (or $l$) lying outside $F$. That is, every $\tau$ containing $j$ and $l$ must contain one of the subpaths from the set $\{\alpha_1 - j - l - \alpha_3, \alpha_1 - j - l - \alpha_4, \alpha_2 - j - l - \alpha_3, \alpha_2 - j - l - \alpha_4\}$. We denote these as types 1 through 4 respectively. We refer to a centre-traversal of $F$ which bypasses $y_1 \in F \cap C$ as a left path, one which bypasses $y_s \in F \cap C$ for some $s \in [2, r-2]$ as a middle path, and one which bypasses $y_{r-1} \in F \cap C$ as a right path. For $a \in \{L, M, R\}$ and $b \in \{1, 2, 3, 4\}$, let $\beta_a(b, j, l)$ be the penalty incurred if $\tau$ uses the edges $j, l$ and is of type $b$. The subscripts L,M, and R denote left, middle and right paths prior to the contraction of $F$. 


Define:

\[
P_i(\tau) = \begin{cases} 
\beta_L(u_i, 1, j, l) & \text{if } j, l, \alpha_1, \alpha_3 \in \tau \text{ and } y_1 \notin \tau, \\
\beta_M(u_i, 1, j, l) & \text{if } j, l, \alpha_1, \alpha_3 \in \tau \text{ and } y_s \notin \tau \text{ for some } s \in \{2, \ldots, r-2\}, \\
\beta_R(u_i, 1, j, l) & \text{if } j, l, \alpha_1, \alpha_3 \in \tau \text{ and } y_{r-1} \notin \tau, \\
\beta_L(u_i, 2, j, l) & \text{if } j, l, \alpha_1, \alpha_4 \in \tau \text{ and } y_1 \notin \tau, \\
\beta_M(u_i, 2, j, l) & \text{if } j, l, \alpha_1, \alpha_4 \in \tau \text{ and } y_s \notin \tau \text{ for some } s \in \{2, \ldots, r-2\}, \\
\beta_R(u_i, 2, j, l) & \text{if } j, l, \alpha_1, \alpha_4 \in \tau \text{ and } y_{r-1} \notin \tau, \\
\beta_L(u_i, 3, j, l) & \text{if } j, l, \alpha_2, \alpha_3 \in \tau \text{ and } y_1 \notin \tau, \\
\beta_M(u_i, 3, j, l) & \text{if } j, l, \alpha_2, \alpha_3 \in \tau \text{ and } y_s \notin \tau \text{ for some } s \in \{2, \ldots, r-2\}, \\
\beta_R(u_i, 3, j, l) & \text{if } j, l, \alpha_2, \alpha_3 \in \tau \text{ and } y_{r-1} \notin \tau, \\
\beta_L(u_i, 4, j, l) & \text{if } j, l, \alpha_2, \alpha_4 \in \tau \text{ and } y_1 \notin \tau, \\
\beta_M(u_i, 4, j, l) & \text{if } j, l, \alpha_2, \alpha_4 \in \tau \text{ and } y_s \notin \tau \text{ for some } s \in \{2, \ldots, r-2\}, \\
\beta_R(u_i, 4, j, l) & \text{if } j, l, \alpha_2, \alpha_4 \in \tau \text{ and } y_{r-1} \notin \tau, \\
0 & \text{Otherwise.}
\end{cases}
\]

The problem now contains a cost for every triplet of consecutive edges in tour \(\tau\) and additionally, a penalty at each outer node \(v\). Consider the modified 3-Neighbor TSP on a Halin graph defined as follows:

\[
MTSP(3): \quad \text{Minimize} \quad \sum_{(e,f,g) \in P_3(\tau)} q(e,f,g) + \sum_{i \in C} P_i(\tau)
\]

Subject to \(\tau \in \mathcal{F}\).
Define the penalty associated with pseudo-fan of type \( \{ \) induced by the nodes \( v \) to pseudo-node \( K \) by the edges in centre-traversal of \( F \). Let \( K \) represents the cost incurred by selecting the edges in node \( q \) for \( (K,q) = y \) be represent the centre-traversal of \( (K,q) = y \) be represent the centre-traversal of \( F \). Let \( K \) be the centre-traversal of \( F \) which does not contain \( y_i \). Then \( q(K(y_i)) \) represents the cost incurred by the edges in \( K(y_i) \). That is,

\[
q(K(y_1)) = q(K) + q(j, t_1, t_2) + q(t_1, t_2, y_2) + q(t_2, y_2, y_3) - q(j, y_1, y_2) - q(y_1, y_2, y_3)
\]

\[
q(K(y_p)) = q(K) + q(y_{p-2}, y_{p-1}, t_p) + q(y_{p-1}, t_p, t_{p+1}) + q(t_p, t_{p+1}, y_{p+1}) + q(t_{p+1}, y_{p+1}, y_{p+2})
- q(y_{p-2}, y_{p-1}, y_p) - q(y_{p-1}, y_p, y_{p+1}) - q(y_p, y_{p+1}, y_{p+2})
\]

for \( p \in \{2, \ldots, r-2\} \), and

\[
q(K(y_{r-1})) = q(K) + q(y_{r-3}, y_{r-2}, t_{r-1}) + q(y_{r-2}, t_{r-1}, t_r) + q(t_{r-1}, t_r, l)
- q(y_{r-3}, y_{r-2}, y_{r-1}) - q(y_{r-2}, y_{r-1}, l)
\]

When fan \( F \) is contracted to pseudo-node \( v_F \), we perform the following update.

\[
q(\alpha_1, j, k) = q(\alpha_1, j, y_1) + q(K) - q(y_{r-2}, y_{r-1}, l) + q(y_{r-2}, y_{r-1}, t_r)
+ q(y_{r-1}, t_r, k) + \min_d \{PF_{1,r-1}(1,d)\}
\]

\[
q(\alpha_2, j, k) = q(\alpha_2, j, y_1) + q(K) - q(y_{r-2}, y_{r-1}, l) + q(y_{r-2}, y_{r-1}, t_r)
+ q(y_{r-1}, t_r, k) + \min_d \{PF_{1,r-1}(3,d)\}
\]

\[
q(k, l, \alpha_3) = q(k, t_1, y_1) + q(t_1, y_1, y_2) + q(K) - q(j, y_1, y_2)
+ q(y_{r-1}, l, \alpha_3) + \min_d \{PF_{2,r}(1,d)\}
\]
\[
q(k, l, \alpha_4) = q(k, t_1, y_1) + q(t_1, y_1, y_2) + q(K) - q(j, y_1, y_2) + q(y_{r-1}, l, \alpha_4) + \min_d \{PF_{2,r}(2, d)\}
\]

\[
\beta_L(v_F, c, j, l) = q(\alpha_e, j, t_1) + q(K(y_1)) + q(y_{r-1}, l, \alpha_{f+2}) + \min_d \{PF_{3,r}(e, d)\}
\]

\[
\beta_M(v_F, c, j, l) = \min_{g \in \{2, \ldots, r-2\}} \{q(K(y_g)) + \min_d \{PF_{1,g-1}(e - 1, d)\} + \min_d \{PF_{g+2,r}(f, d)\}\}
\]

\[
\beta_R(v_F, c, j, l) = q(\alpha_{e/2}, j, y_1) + q(K(y_{r-1})) + q(t_r, l, \alpha_{f+2}) + \min_d \{PF_{3,r}(e - 1, d)\}
\]

where \(e = 2\lceil c/2 \rceil\) and \(f = |c \mod 2 - 2|\). All other costs and \(\beta\) values containing \(v_F\) are set to 0.

The necessary values for \(PF_{a,b}(c, d)\) can be calculated using the following recursion for \(n \in \{1, \ldots, r\}\).

\[
PF_{1,n}(c, L) = \min \left\{ PF_{1,n-1}(e, L) + \beta_L(u_n, f, j, l), \right. \\
PF_{1,n-1}(e, M) + \beta_L(u_n, f, j, l), \\
PF_{1,n-1}(e, R) + \beta_L(u_n, f + 2, j, l) \right\}
\]

\[
PF_{1,n}(c, M) = \min \left\{ PF_{1,n-1}(e - 1, L) + \beta_M(u_n, f, j, l), \right. \\
PF_{1,n-1}(e - 1, M) + \beta_M(u_n, f, j, l), \\
PF_{1,n-1}(e - 1, R) + \beta_M(u_n, f + 2, j, l) \right\}
\]

\[
PF_{1,n}(c, R) = \min \left\{ PF_{1,n-1}(e - 1, L) + \beta_R(u_n, f, j, l), \right. \\
PF_{1,n-1}(e - 1, M) + \beta_R(u_n, f, j, l), \\
PF_{1,n-1}(e - 1, R) + \beta_R(u_n, f + 2, j, l) \right\}
\]

\[
PF_{n-1,r}(c, L) = \min \left\{ PF_{n,r}(f, L) + \beta_L(u_n, e, j, l), \right. \\
PF_{n,r}(f, M) + \beta_L(u_n, e - 1, j, l), \\
PF_{n,r}(f, R) + \beta_L(u_n, e - 1, j, l) \right\}
\]
\[ PF_{n-1,r}(c, M) = \min \begin{cases} 
PF_{n,r}(f, L) + \beta_M(u_n, e, j, l), \\
PF_{n,r}(f, M) + \beta_M(u_n, e - 1, j, l), \\
PF_{n,r}(f, R) + \beta_M(u_n, e - 1, j, l) 
\end{cases} \] (4.5)

\[ PF_{n-1,r}(c, R) = \min \begin{cases} 
PF_{n,r}(f + 2, L) + \beta_R(u_n, e, j, l), \\
PF_{n,r}(f + 2, M) + \beta_R(u_n, e - 1, j, l), \\
PF_{n,r}(f + 2, R) + \beta_R(u_n, e - 1, j, l). 
\end{cases} \] (4.6)

We iteratively perform the fan contraction operation, updating costs and penalties until we are left with a wheel. The optimal tour in \( H \) skirts the cycle \( C \) and detours exactly once through centre \( w \), skipping exactly one edge of \( C \). Orient the cycle \( C \) in the clockwise direction. \( \tau \) contains all edges in \( C \) except for the skipped edge, say \( c_i = (u_i, u_{i+1}) \), together with the two edges which detour around \( r \). Define function \( \phi(c_i) \) for each edge \( c_i = (u_i, u_{i+1}) \in E(C) \). Let \( t_i \) and \( t_{i+1} \) be the tree edges adjacent to \( u_i \) and \( u_{i+1} \), respectively.

\[ \phi(c_i) \equiv q(c_{i-2}, c_{i-1}, t_i) + q(c_{i-1}, c_i, t_{i+1}) + q(t_i, t_{i+1}, c_{i+1}) \\
+ q(t_{i+1}, c_{i+1}, c_{i+2}) - q(c_{i-2}, c_{i-1}, c_i) - q(c_{i-1}, c_i, c_{i+1}) \\
- q(c_i, c_{i+1}, c_{i+2}). \]

Suppose that we fix an edge \( c_i \) in \( \tau \). Then \( H \) can be considered to be a fan as shown in Figure 11.

**Figure 10.** A tour \( \tau \) in a wheel, which skips edge \( c_i \).

**Figure 11.** Wheel \( H \) with center \( w \) considered as a fan.
Fix edge $c_r$ in $\tau$ and consider $H$ to be a fan $F_{c_r}$. Then the minimum tour in $H$ can be determined by calculating the minimum of the minimum traversal of $F_{c_r}$ and the tour which bypasses $c_r$ (using, say, fan $F_{c_1}$). That is,

$$\tau^* = \min \begin{cases} \phi(c_1) + \min_d\{PF_{3,r}(1,d)\}, \\ \min_{p \in \{2, \ldots, r-2\}} \{\phi(c_p) + \min_d\{PF_{1,p-1}(1,d)\} + \min_d\{PF_{p+2,r}(1,d)\}\}, \\ \phi(c_{r-1}) + \min_d\{PF_{1,r-2}(1,d)\}, \\ \phi(c_r) + \min_d\{PF_{2,r-1}(1,d)\} \end{cases}$$

The preceding discussion yields the following algorithm.

**Algorithm 1 HalinTSP(3)(H)**

**Require:** Halin graph $H$
  
  if $H$ is a wheel then
  Use the wheel procedure to find an optimal tour $\tau$ in $H$
  else
  Let $F$ be a fan in $H$
  Contract $F$ to a single vertex $v_F$, using the Case 1 procedure. That is, assign penalties $\beta$ to $v_F$, assign cost 0 to all triples in $H(F)$ which include edges $j$ and $l$ and assign costs $q'$ to all remaining triples which are 3-neighbours in $H$.
  HalinTSP3($H(F)$)
  end if
  Expand all pseudo-nodes in reverse order and update $\tau$
  return $\tau$

As a result of Corollary 4.3, the preceding algorithm can be extended to solve TSP($k$) in $O(n)$ time by extending the penalty functions at outer nodes to accommodate subpaths of length $2^{(k+1)/2}$.
The method presented here may be extended to TSP\((k)\) for any fixed \(k\), however the complexity increases by a factor of \(2^{\lceil (k+1)/2 \rceil}\), which is constant for fixed \(k\) and \(O(\sqrt{n})\) when \(k = \log n\).

5. TSP\((k)\) on fully reducible graph classes

We say that a class \(\mathcal{C}\) of 3-connected graphs is fully reducible if it satisfies the following:

1. If \(G \in \mathcal{C}\) has a 3-edge cutset with shores \(S\) and \(\bar{S}\), then both \(G(S)\) and \(G(\bar{S})\) are in \(\mathcal{C}\) and we call \(G\) a reducible graph in \(\mathcal{C}\); and
2. TSP can be solved in polynomial time for the graphs in \(\mathcal{C}\) that do not have non-trivial 3-edge cutsets. We call such graphs irreducible.

For instance, Halin graphs can be understood as graphs built up from irreducible fans connected to the remainder of the graph via 3-edge cutsets. Cornuejols et al. [11] show that the ability to solve TSP in polynomial time on irreducible graphs in \(\mathcal{C}\) allows to solve TSP in polynomial time on all of \(\mathcal{C}\) using facts about the TSP polyhedron.

We remark that the algorithm of Section 4 can be used to show a similar statement for TSP\((k)\). Here we consider a graph class \(\mathcal{C}\) that is fully \(k\)-reducible in the sense that either it can be subdivided into irreducible graphs via 3-edge cutsets, or it is irreducible and it is possible to solve the \(k\)-neighbour Hamilton path problem in polynomial time.

This requires solving the following problem:

\[
MTSP(k) : \quad \text{Minimize} \sum_{(e_1, \ldots, e_k) \in P_k(\tau)} q(e_1, \ldots, e_k) + \sum_{i \in V} P_i(\tau) \\
\text{Subject to} \quad \tau \in \mathcal{F}.
\]

where \(P_i(\tau)\) is a penalty function for the pseudonode which depends on how tour \(\tau\) traverses \(i\), analogous to the construction for the 3-neighbour TSP of section [1].
Algorithm 2 Solving MTSP\((k)(G)\)

Require: A graph \(G\) from a class of fully reducible graphs.

if \(G\) is irreducible then
    HP\((k)\) and hence MTSP\((k)\) can be solved in polynomial time on \(G\). Let \(\tau\) be the minimum cost tour in \(G\).

else
    Let \(S\) be an irreducible subgraph in \(G\).
    Contract \(S\) to a single vertex \(v_S\), assigning penalties \(P_i(\tau)\) to \(v_S\). Call the resulting graph \(G(S)\).
    Assign cost 0 to all \(k\)-tuples in \(G(S)\) which correspond to \(k\)-paths containing \(v_S\) as an interior node. All other costs remain the same in \(G(S)\) as in \(G\).
    MTSP\((k)(G(S))\)
end if

Expand all pseudo-nodes in reverse order and update \(\tau\)
return \(\tau\)

We recursively perform the contraction operation on the irreducible subgraphs of \(G\), storing the necessary tour information in the penalty at the resulting pseudo-node. For any fixed \(k\), this requires a polynomial number of penalties and the least cost traversals of \(S\) can be computed in polynomial time using a generalization of the pseudo-fan strategy above.

Suppose the contraction operation is performed on a subgraph of size \(r\) in time \(O(P(r))\), where \(P(r)\) is a polynomial in \(r\). Each time this operation is performed, the number of nodes in the graph is reduced by \(r\). This operation is performed at most \(n\) times and it follows that the entire algorithm can be performed in polynomial time.

6. Conclusions

In this paper, we have shown that QTSP is NP-complete even when the costs are restricted to taking 0-1 values on Halin graphs. We have presented a polynomial time algorithm to solve a restriction of QTSP, denoted TSP\((k)\) on any fully reducible graph for any fixed \(k\). To illustrate this, we have given an algorithm which solves TSP\((3)\) on a Halin graph in \(O(n)\) time. It may be noted that the approach that is presented can directly be extended to the corresponding bottleneck versions on any fully reducible graph class by replacing the ‘sum’ operations with ‘max’ with the same time complexity.

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