Quenched many-body quantum dynamics with \( k \)-body interactions using \( q \)-Hermite polynomials

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Abstract. In an \( m \) particle quantum system, the rank of interactions and the nature of particles (fermions or bosons) can strongly affect the dynamics of the system. To explore this, we study the nonequilibrium dynamics with particles in a one-body mean-field and quenched by an interaction of body-rank \( k = 2, 3, \ldots, m \). Using fermionic embedded Gaussian orthogonal ensembles (EGOE\( s \)) and bosonic embedded Gaussian orthogonal ensembles (BEGOE\( s \)) of one plus \( k \)-body interactions (also the unitary variants EGUE and BEGUE), it is seen that the short time decay of the survival probability of many-particle systems is given by the Fourier transform of the generating function \( v(E|q) \) of the \( q \)-Hermite polynomials. Deriving the formulas for \( q \) for both fermion and boson systems as a function of \( m, k \) and the number of single particle states \( N \), we have verified that the Fourier transform of \( v(E|q) \) agrees very well with the numerical ensemble calculations for both fermion and boson systems. These results bridge the gap between the known results for \( k = 2 \) and \( k = m \).

Keywords: quantum quenches, random matrix theory and extensions, quantum disordered systems, quantum Monte Carlo simulations

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1. Introduction

In the last decade, there has been special focus on the statistical properties of isolated finite many-particle quantum systems such as atomic nuclei, atoms, mesoscopic systems (quantum dots, small metallic grains), interacting spin systems modeling the quantum computing core, ultra-cold atoms and quantum black holes with the SYK model, and so on [1–5]. A route to investigating the statistical properties of isolated finite many-particle quantum systems is to employ the classical Gaussian orthogonal (GOE) or unitary (GUE) or symplectic (GSE) random matrix ensembles with various deformations [1, 6]. However, in most of the isolated finite many-particle quantum systems, their constituents predominantly interact via few-particle interactions and one refinement of the classical ensembles which retains the basic stochastic approach but allows for this feature consists of the use of embedded random matrix ensembles [1, 4, 7–10].

Representing an isolated finite interacting quantum system, say with m particles (fermions or bosons) in N single particle (sp) states, from random matrix models generated by random k-body interactions and propagating the information in the interaction to many particle spaces, we have random interaction matrix models for m-particle systems. In the simplest version, the k-particle Hamiltonian ($H$) of a spinless fermion (or boson) system is represented by GOE/GUE/GSE (all three classical ensembles combined are referred to as GE, Gaussian ensembles), and then the m particle $H$
matrix is generated using the $m$-particle Hilbert space geometry. As a GOE/GUE/GSE random matrix ensemble in $k$-particle spaces is embedded in the $m$-particle $H$ matrix, these ensembles are generically called $k$-body embedded ensembles (EE)\textsuperscript{(k)}\cite{11,12}. Then, with GOE embedding, we have embedded the Gaussian orthogonal ensemble of the $k$-body interactions (EGOE\textsuperscript{(k)}) and similarly with GUE embedding EGUE\textsuperscript{(k)}.

Following our previous works (see \cite{1,9} and references therein), we will use the notation EGOE\textsuperscript{(k)} and EGUE\textsuperscript{(k)} for fermionic systems and BEGOE\textsuperscript{(k)} and BEGUE\textsuperscript{(k)} for bosonic systems (here, B denotes bosons). For common reference, we will use EE\textsuperscript{(k)}, which includes all four mentioned cases.

Following the seminal paper of Mon and French\textsuperscript{(13)} and many numerical calculations \cite{1,4,7,14}, it is well known that the EGOE\textsuperscript{(k)} and EGUE\textsuperscript{(k)} spectral density for a system of $m$ spinless fermions (also for BEGOE\textsuperscript{(k)} and BEGUE\textsuperscript{(k)} for a system of $m$ spinless bosons) in $N$ spin states changes from the Gaussian to the semi-circle of the classical RMT as the body rank $k$ of the interaction changes from $k = 1$ to $k = m$. This is also proved later by evaluating the lower order moments of the spectral density by many other groups using different methods \cite{1,8,15,16}. However, the most recent study of the spectral density of the so-called SYK model \cite{17,18} and quantum spin glasses \cite{19} employs $q$-Hermite polynomials. Following these studies, we show that the generating function for $q$-Hermite polynomials describes the semi-circle to Gaussian transition in spectral densities and the local density of states (LDOS) (also known as strength functions) of EGOE\textsuperscript{(k)} and BEGOE\textsuperscript{(k)} (also the unitary variants EGUE\textsuperscript{(k)} and BEGUE\textsuperscript{(k)}) as a function of the rank of interactions $k$. The LDOS gives the spread of the basis states over the eigenstates. It is important to mention that the spectral densities also exhibit a transition from semi-circle to Gaussian form as one increases the number of particles $m$ with a given $k$; see \cite{1,7} for more details. However, we do not deal with this situation in the present paper.

The thermalization of isolated finite interacting quantum systems is a topic of great current interest \cite{3}. In this context, the study of the time evolution of a many-body quantum system quenched far from equilibrium has attracted considerable attention, and the survival probability decay is a fundamental quantity of interest in all investigations \cite{14,20,21}. In \cite{14}, a first attempt was made to obtain the analytical and numerical results for survival probability decay for fermions in a mean-field quenched by a random $k$-body interaction with $k$ changing from 1 to $m$. The Hamiltonian with a mean-field one-body part and a random $k$-body interaction represented by EGOE\textsuperscript{(k)} with some strength $\lambda$ forms EGOE(1 + $k$) and for a sufficiently large value of $\lambda$, the EGOE(1 + $k$) properties go over to those of EGOE\textsuperscript{(k)} \cite{1,14}. For chaotic systems, the analytical results for the survival probability decay using EGOE\textsuperscript{(k)} are derived \cite{14} in the two extreme limits of the Gaussian LDOS (here $k \ll m$ applies) and semi-circle form for LDOS (then the full random matrix or GOE and equivalently EGOE\textsuperscript{(k)} with $k = m$ applies). We demonstrate that the numerical Fourier transform of the generating function of $q$-Hermite polynomials describes the short time decay of the survival probability in EGOE(1 + $k$) and BEGOE(1 + $k$) as $k$ varies. We derive the formula for $q$ for EGOE\textsuperscript{(k)}, EGUE\textsuperscript{(k)} and BEGUE\textsuperscript{(k)} (also for BEGUE\textsuperscript{(k)}) that governs the behavior of the spectral density, the LDOS and the survival probability. Now, we will give a preview.

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In section 2, some basic properties of \( q \)-Hermite polynomials and their generating function are described. We also define the embedded ensembles for finite interacting quantum systems. Using these, formulas for the parameter \( q \) are presented for EGOE(\( k \)), EGUE(\( k \)) and BEGUE(\( k \)) in section 3. These results are tested for spectral densities using numerical examples for EGOE(\( k \)), EGUE(\( k \)), BEGOE(\( k \)) and BEGUE(\( k \)) in section 4. Section 5 gives results for the LDOS and the survival probability decay in EGOE(\( 1+k \)) and BEGOE(\( 1+k \)), which are compared to the generating function of \( q \)-Hermite polynomials and its numerical Fourier transform respectively. Finally, section 6 gives conclusions.

2. Preliminaries

2.1. \( q \)-Hermite polynomials

The \( q \)-Hermite polynomials were introduced by Rogers, who used them to prove the Rogers–Ramanujan identities [22]. It is well known in mathematics literature that \( q \)-Hermite polynomials are orthogonal with respect to a function that takes the Gaussian form for \( q = 1 \) and semi-circle form for \( q = 0 \) [22, 23]. We will restrict our discussion to \( q \) real. In this section, we collect some basic properties of \( q \)-Hermite polynomials and then use them to describe the spectral density, the LDOS and survival probability of EE(\( k \)).

Let us begin with the \( q \) number \([n]_q\) defined by

\[
[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots + q^{n-1}.
\]

(1)

Note that \([n]_{q\to 1} = n\). Similarly \([n]_q! = \prod_{j=1}^{n} [j]_q\) with \([0]_q! = 1\). Now, \( q \)-Hermite polynomials \( H_n(x|q) \) are defined by the recursion relation [22, 23],

\[
x H_n(x|q) = H_{n+1}(x|q) + [n]_q H_{n-1}(x|q)
\]

(2)

with \( H_0(x|q) = 1 \) and \( H_{-1}(x|q) = 0\). Note that for \( q = 1 \), the \( q \)-Hermite polynomials reduce to normal Hermite polynomials (related to the Gaussian) and for \( q = 0 \) they will reduce to Chebyshev polynomials (related to the semi-circle). More importantly, \( q \)-Hermite polynomials are orthogonal within the limits \( \pm 2/\sqrt{1-q} \), with the weight function \( v(x|q) \) (see equation (2.14) of [22]),

\[
\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} H_n(x|q) H_m(x|q) v(x|q) \, dx = [n]_q! \delta_{mn}.
\]

(3)

The explicit form of \( v(x|q) \) is given by equation (2.15) of [22]. After some simplifications of this equation, it is easy to see that

\[
v(x|q) = N_q \sqrt{1 - \frac{x^2}{x_0^2}} \prod_{n=1}^{\infty} \left[ 1 - \frac{4(x^2/x_0^2)}{2 + q^n + q^{-n}} \right];
\]

\[
x_0^2 = \frac{4}{1-q}.
\]

(4)
Here, $x$ is the standardized variable (has zero mean and variance unity) with
$$-\frac{2}{\sqrt{1-q}} \leq x \leq \frac{2}{\sqrt{1-q}}$$
and $N_q$ is a normalization constant such that
$$\int_{-\frac{2}{\sqrt{1-q}}}^{\frac{2}{\sqrt{1-q}}} v(x|q) \, dx = 1.$$ It is seen that in the limit $q \rightarrow 1$, $v(x|q)$ will take the Gaussian form and in the $q = 0$ limit, $v(x|q)$ will be a semi-circle. Thus, $v(x|q)$ interpolates Gaussian and semi-circle forms. It is also shown in [22] (see proposition 4.1 and its proof) that the even order reduced moments of $v(x|q)$ are,
\[
\mu_{2n}(q) = \int_{-\frac{2}{\sqrt{1-q}}}^{\frac{2}{\sqrt{1-q}}} x^{2n} v(x|q) \, dx \\
= (1-q)^{-n} \sum_{r=-n}^{n} \binom{2n}{n+r} (-1)^r q^{r(r-1)/2}.
\]
(5)
Note that all the odd order moments vanish. Simplifying equation (5) for $n = 2, 3$ and 4 will give the following important formulas (it is easily seen that $\mu_2 = 1$ as we are using a standardized variable),
\[
\mu_4(q) = 2 + q,
\mu_6(q) = 5 + 6q + 3q^2 + q^3,
\mu_8(q) = 14 + 28q + 28q^2 + 20q^3 + 10q^4 + 4q^5 + q^6.
\]
(6)
Using these formulas and the moments to order 8 for EGUE($k$), EGOE($k$), BEGUE($k$), and BEGOE($k$), it is seen that the spectral densities and LDOS will be close to $v(x|q)$, which generates $q$-Hermite polynomials. With this, the formulas for $q$ are identified in the next section.

2.2. Embedded ensembles

The constituents of finite many-body quantum systems such as nuclei, atoms, molecules, small metallic grains, quantum dots, arrays of ultracold atoms, and so on, interact via few-body (mainly two-body) interactions. As is well-known, the classical random matrix ensembles (GOE/GUE) incorporate many-body interactions. Embedded ensembles take into account the few-body nature of interactions and hence, they are more appropriate for analyzing the various statistical properties of finite quantum systems.

Given a system of $m$ particles (fermions or bosons) distributed in $N$ degenerate levels interacting via the $k$-body ($1 \leq k \leq m$) interactions, embedded ensembles are generated by representing the few-particle ($k$) Hamiltonian by a classical GOE/GUE and then the many-particle Hamiltonian ($m > k$) is generated by the Hilbert space geometry. In other words, the $k$-particle Hamiltonian is embedded in the $m$-particle Hamiltonian and the non-zero $m$-particle Hamiltonian matrix elements are appropriate linear combinations of the $k$-particle matrix elements. Due to the $k$-body selection rules, many matrix elements of the $m$-particle Hamiltonian will be zero, unlike in a GOE.

The random $k$-body Hamiltonian in the second quantized form for a EGOE/BEGOE ($\beta = 1$) and EGUE/BEGUE ($\beta = 2$) is [10, 24],
Here, α and γ are k-particle configuration states in an occupation number basis. Distributing k particles (fermions in agreement with Pauli’s exclusion principle or bosons) in N sp states will generate the complete set of these distinct configurations. The total number of these configurations is \( \binom{N}{k} \) for fermions and \( \binom{N + k - 1}{k} \) for bosons. In the occupation number basis, we order the sp levels (denoted by \( \mu_i \)) in increasing order, \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_N \). The operators \( \psi^\dagger(k; \alpha) \) and \( \psi(k; \gamma) \) respectively are k-particle creation and annihilation operators for fermions or bosons, i.e. \( \psi^\dagger(k; \alpha) = \prod_{i=1}^{k} a^\dagger_{\mu_i} \) for fermions and \( \psi^\dagger(k; \alpha) = \mathcal{N}_\alpha \prod_{i=1}^{k} b^\dagger_{\mu_i} \) for bosons. Here, \( \mathcal{N}_\alpha \) and \( \mathcal{N}_\gamma \) are the factors that guarantee the unit normalization of k-particle bosonic states. The sum in equation (7) stands for summing over a subset of k-particle creation and annihilation operators. These k-particle operators obey the usual anti-commutation (commutation) relations for fermions (bosons).

In equation (7), \( v_{k,\beta}^{\alpha,\gamma} \) is chosen to be a \( \binom{N}{k} \) \( \binom{N + k - 1}{k} \) dimensional GOE/GUE (depending on the \( \beta \) value) in the k-particle spaces. This means \( v_{k,\beta}^{\alpha,\gamma} \) are antisymmetrized (symmetrized) few-body matrix elements for fermions (bosons) chosen to be randomly distributed independent Gaussian variables with zero mean and variance

\[
\overline{v_{k,\beta}^{\alpha,\gamma}} \overline{v_{k,\beta}^{\alpha',\gamma'}} = v^2 \left( \delta_{\alpha,\gamma'} \delta_{\alpha',\gamma} + \delta_{\beta,1} \delta_{\alpha,\alpha'} \delta_{\gamma',\gamma} \right).
\]

Here, the bar denotes ensemble averaging and we choose \( v = 1 \) without loss of generality.

Distributing the m fermions (bosons) in all possible ways in N levels generates the many-particle basis states defining \( d_F(N, m) = \binom{N}{m} \) \( d_B(N, m) = \binom{N + m - 1}{m} \) dimensional Hilbert space. The action of the Hamiltonian operator \( V(k, \beta) \) defined by equation (7) on the many-particle states generates the EGOE(k)/EGUE(k)/BEGOE(k)/BEGUE(k) ensemble in m-particle spaces.

3. Formulas of \( q \)

3.1. EGUE(k) and EGOE(k)

In the dilute limit defined by \( N \to \infty, m \to \infty, k \to \infty, m/N \to 0 \) and \( k/m \to 0 \), the moments \( \langle H^p \rangle^m \) of the spectral density generated by EGOE(k) and EGUE(k) will be those of a Gaussian. This result is easily derived and well known [1, 8, 13, 16]. The reduced moments to order eight with \( N \to \infty, m \to \infty \) but \( k/m \) finite are derived in [13] for EGOE(k) and in [16] for EGUE(k). For example, for EGUE(k), the reduced moments up to order eight are [16],

\[
V(k, \beta) = \sum_{\alpha, \gamma} v_{k,\beta}^{\alpha,\gamma} \psi^\dagger(k; \alpha) \psi(k; \gamma).
\]
$\mu_4(m, k) = 2 + G(m, k, 1),$

$\mu_6(m, k) = 5 + 6G(m, k, 1) + 3 [G(m, k, 1)]^2$

$+ [G(m, k, 2)]^2 G(m, k, 1),$

$\mu_8(m, k) = 14 + 28G(m, k, 1) + 28 [G(m, k, 1)]^2$

$+ 12 [G(m, k, 1)]^3 + 8G(m, k, 2)G(m, k, 1)$

$+ 4G(m, k, 1) [G(m, k, 2)]^2$

$+ 8 [G(m, k, 1)]^2 G(m, k, 2)$

$+ [G(m, k, 1)G(m, k, 2)G(m, k, 3)$

$+ 2 [G(m, k, 1)]^3 \sum_{\alpha} \binom{k}{\alpha} \binom{m-2k}{k-\alpha};$

$G(m, k, r) = \binom{m-\nu k}{k} \binom{m}{k}.$

Comparing these with the EGOE($k$) formulas given in [13], it is seen that the moments to order six for EGOE($k$) are the same as those given in equation (9) and for $\mu_8$ only the last term is different. Comparing equations (9) and (6), it is seen that the lower order reduced moments of EGUE($k$) and EGOE($k$) will be essentially the same as those of the generating function of the $q$-Hermite polynomials with $q$ given by

$q \sim G(m, k, 1) = \binom{m-k}{k} \binom{m}{k} = \mu_4 - 2.$

Using finite-$N$ corrections to $\mu_4$ as given in [1, 8, 25], a better approximation for $q$ for EGUE($k$) is

$q \sim \left( \begin{array}{c} N \\ m \end{array} \right)^{-1} \sum_{\nu=0}^{\nu_{\text{max}}} \frac{\Lambda^\nu(N, m, m-k) \Lambda^\nu(N, m, k) d(g_\nu)}{[\Lambda^0(N, m, k)]^2};$

$\Lambda^\nu(N, m, r) = \binom{m-\nu}{r} \binom{N-m+r-\nu}{r},$

$d(g_\nu) = \binom{N}{\nu}^2 - \binom{N}{\nu-1}^2.$
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Here \( \nu_{\text{max}} = \min\{k, m-k\} \) in the summation. Note that the formula for \( \mu_4 \) that is used to write equation (11), was derived using the underlying \( U(N) \) algebra and, within the binary correlation approximation, it is exact (see \cite{8, 25}; for the binary correlation approximation, see appendix A and \cite{13}). On the other hand, using finite \( N \) corrections to \( \mu_4 \) as given in \cite{1, 9} for EGOE\( (k) \), a better formula for \( q \) for EGOE\( (k) \) is (see appendix for the derivation),

\[
q \sim F(N, m, k)/ [T(N, m, k)]^2
\]

\[
T(N, m, k) = \binom{m}{k} \left[ \left( \binom{N-m+k}{k} \right) + 1 \right],
\]

\[
F(N, m, k) = \binom{m}{k}^2 + \sum_{s=0}^{k} \binom{m-s}{k-s}^2 \binom{N-m+k-s}{k}
\times \binom{m-s}{k} \binom{N-m}{s} \binom{m}{s}
\times \left[ \frac{N-2s+1}{N-s+1} \right] \left( \binom{N-s}{k} \right)^{-1} \left( \binom{k}{s} \right)^{-1} \left\{ 2 + \binom{N+1}{s} \right\}.
\]

(12)

It is important to note that equation (12) is valid in the dilute limit with finite-\( N \) corrections. Here, the so-called binary correlation approximation and some other approximations are used as described in the appendix. We are interested in \( m \ll N \) systems as considered in \cite{8, 13}, and therefore restrict ourselves to these particular cases (see also the discussion in section 3.3 of \cite{8}). In practice, it is seen that equation (12) is good even for \( m = N/2 \) (see next section for examples). Table 1 gives some numerical values for \( q \) as a function of \( k \) for EGUE\( (k) \) and EGOE\( (k) \) obtained using equations (11) and (12) respectively. As we see from the table, the \( q \) values become zero rapidly with increasing \( k \) even for a very large system.

3.2. BEGUE\( (k) \) and BEGOE\( (k) \)

The formula for \( \mu_4 \) for BEGUE\( (k) \), with finite \( N \) corrections, follows easily from equation (11) by using the so-called \( N \rightarrow -N \) law as described in \cite{25}. This law implies that for the fermion results replace \( N \) by \( -N \) in all the terms and then take the absolute value of each term to obtain the corresponding final result for the bosons. Applying this rule, equation (11) will go over to equation (13) below \cite{25}. This equation was also derived explicitly without using the \( N \rightarrow -N \) rule in \cite{15}. Using these, the formula for \( q \) for BEGUE\( (k) \) is

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\[ q \sim \left( \frac{N + m - 1}{m} \right)^{-1} \sum_{\nu=0}^{\nu_{\text{max}}} \frac{X(N, m, k, \nu) \, d_B(g_{\nu})}{[\Lambda_B^0(N, m, k)]^2} \]

\[ X(N, m, k, \nu) = \Lambda_B^\nu(N, m, m - k) \, \Lambda_B^0(N, m, k); \]

\[ \Lambda_B^\nu(N, m, r) = \binom{m - \nu}{r} \binom{N + m + \nu - 1}{r}, \]

\[ d_B(g_{\nu}) = \left( \binom{N + \nu - 1}{\nu} \right)^2 - \left( \binom{N + \nu - 2}{\nu - 1} \right)^2. \quad (13) \]

It is important to mention that, in general, we are interested in the dense limit for boson systems with the dense limit defined by \( N \to \infty \), \( m \to \infty \) and \( m/N \to \infty \) with \( k \) fixed. Therefore, in all the numerical examples in this paper, \( m \ll N \). Some numerical results for \( q \) calculated using equation (13) are given in table 2. Unlike the situation with EGOE(\( k \)), the formula for \( \mu_4 \) for BEGOE(\( k \)), for a general \( k \) value, is not available \[15\]. Deriving the required moments for the BEGOE(\( k \)) case is beyond the scope of the present paper. The application of the simple \( N \to -N \) law to equation (12) will not be appropriate as this formula uses several approximations besides the binary correlation approximation \[9\]. However, as demonstrated using the numerics in this paper, all the quantities studied for BEGOE(\( k \)) are very well described by the \( q \) values obtained for the corresponding BEGUE(\( k \)) case.

4. Gaussian to semi-circle transition in the spectral density

Normalizing the eigenvalues \( E \) with the centroid \( E_c \) and the spectrum width \( \sigma \), we have \( E = (E - E_c)/\sigma \). Then the spectral density \( \rho(E) \, dE \) for the four EE(\( k \)) ensembles considered, from the above results, is given by

\[ \text{Table 1. Values of parameter } q \text{ for EGUE}(k) \text{ and EGOE}(k) \text{ for various } (N, m) \text{ as a function of the rank of interaction } k \text{ computed using equations (11) and (12)} \text{ respectively.} \]

| \( N \) | \( m \) | \( k \) | \( q \) | \( N \) | \( m \) | \( k \) | \( q \) | \( N \) | \( m \) | \( k \) | \( q \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 6 | 1 | 0.7347 | 0.7656 | 20 | 8 | 1 | 0.8136 | 0.8239 | 50 | 10 | 1 | 0.8786 | 0.8798 |
| 2 | 0.2869 | 0.3044 | 2 | 0.4169 | 0.4209 | 2 | 0.5672 | 0.5674 |
| 3 | 0.0567 | 0.0610 | 3 | 0.1188 | 0.1196 | 3 | 0.2397 | 0.2397 |
| 4 | 0.0051 | 0.0058 | 4 | 0.0149 | 0.0150 | 4 | 0.0526 | 0.0526 |
| 5 | 0.0000 | 0.0000 | 5 | 0.0001 | 0.0001 | 5 | 0.0032 | 0.0032 |
| 6 | 0.0000 | 0.0000 | 6 | 0.0000 | 0.0000 | 6 | 0.0000 | 0.0000 |
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\[ \rho(E) dE = dE N_q \frac{1}{\sigma} \sqrt{1 - \frac{(E - E_c)^2}{E_0^2}} \times \prod_{\kappa=1}^{\infty} \left[ 1 - \frac{4 \{(E - E_c)^2 / E_0^2\}}{2 + q^{\kappa} + q^{-\kappa}} \right] ; \]

\[ E_0^2 = \frac{4\sigma^2}{1 - q} \quad \frac{E_c - 2\sigma}{\sqrt{1 - q}} \leq E \leq \frac{E_c + 2\sigma}{\sqrt{1 - q}}. \]

The limits in this equation show that the density will go to zero at \( E_c \pm 2\sigma/\sqrt{1 - q} \). This gives correctly the known results for the Gaussian \((q = 1)\) and semi-circle \((q = 0)\). The infinite product in equation (14) can be simplified in some situations \([5, 18]\). Also, it is easy to write the formulas for the spectral variance \(\sigma^2\) for the four ensembles \([1]\). Equation (14) along with the formulas for \(q\) is tested in some numerical calculations and the results are shown in figures 1 and 2.

Table 2. The values of parameter \(q\) for BEGUE(k) for various \((N, m)\) as a function of the rank of interaction \(k\) computed using equation (13). These values are also applicable for BEGOE(k) as demonstrated by the numerical calculations for the eigenvalue densities and LDOS.

| \(N\) | \(m\) | \(k\) | \(q\) | \(N\) | \(m\) | \(k\) | \(q\) |
|------|------|------|------|------|------|------|------|
| 5    | 10   | 1    | 0.9694 | 10   | 20   | 1    | 0.9839 |
| 2    | 0.8612 | 2    | 0.9318 |
| 3    | 0.6636 | 3    | 0.8402 |
| 4    | 0.4053 | 4    | 0.7116 |
| 5    | 0.1717 | 5    | 0.5561 |
| 6    | 0.0450 | 6    | 0.3921 |
| 7    | 0.0075 | 7    | 0.2423 |
| 8    | 0.0008 | 8    | 0.1266 |
| \(\geq 9\) | 0.0000 | 9    | 0.0537 |
| 10   |      |      | 0.0178 |
| 11   |      |      | 0.0045 |
| 12   |      |      | 0.0009 |
| \(\geq 13\) | 0.0000 |      |      |

Figure 1 shows the ensemble averaged spectral density of a 1000 member EGOE(k) and EGUE(k) (histograms) with \(N = 12\) and \(m = 6\) as a function of \(k\). The corresponding Hamiltonians are 924 dimensional. Smooth curves are obtained using equations (11), (12) and (14) with the corresponding values of \(q\) given in table 1. As seen from the figure, the results for EGUE(k) and EGUE(k) are identical for all \(k\) values. The eigenvalue density is Gaussian for \(k = 1\) and 2 and makes a transition to a semi-circle at \(k = 3\). The agreement between the theory given by the \(q\)-Hermite polynomials and numerics is excellent.

Similarly, figure 2 shows the transition in the spectral density of a 1000 member BEGOE(k) and BEGUE(k) (histograms) with \(N = 5\) and \(m = 10\) as a function of \(k\). The corresponding Hamiltonians are 1001 dimensional. Smooth curves are obtained using equations (13) and (14) with the corresponding values of \(q\) given in table 2. Here,
the results for BEGOE\((k)\) and BEGUE\((k)\) are the same except for some deviations for \(k = 1\). The eigenvalue density is Gaussian till \(k = 4\) and makes the transition to a semicircle at \(k = 5\). The agreement between the theory given by \(q\)-Hermite polynomials and numerics is excellent.

5. LDOS and survival probability decay in EGOE\((1 + k)\) and BEGOE\((1 + k)\)

Understanding the nonequilibrium dynamics of interacting many-body quantum systems is fundamental for many branches of physics \cite{2, 26, 27}. The unitary evolution of quantum systems is investigated experimentally using cold atoms, ion traps and nuclear magnetic resonance \cite{28–31}. In order to characterize system evolution, we analyze the relaxation dynamics of the survival probabilities. The survival probability \(F(t)\) is the probability of finding the system still in the initial state \(|\psi(0)\rangle\) after time \(t\),

\[
F(t) = \langle |\psi(0)\rangle \exp(-iHt)|\psi(0)\rangle \rangle^2.
\]

The system is prepared in an eigenstate \(|\psi(0)\rangle\) of the unperturbed mean-field Hamiltonian \(h(1)\). The dynamics starts with a sudden change in the parameter \(\lambda\) (denoting the strength of the perturbation) in a time interval much shorter than any of the characteristic time scales of the model. With a quench \(V(k)\) (we consider only \(\beta = 1\) and hence, drop it from now on) of strength \(\lambda\), this results in a final (perturbed) Hamiltonian

\[
H = h(1) + \lambda V(k)
\]
with eigenvalues $E$ and eigenstates $|E\rangle \neq |\psi(0)\rangle$. The initial state $|\psi(0)\rangle$ unitarily changes after time $t$ as $\psi(t) = \exp(-iHt)|\psi(0)\rangle$.

Expanding the mean-field initial state $|\psi(0)\rangle$ over the eigenstates $|E\rangle$, the LDOS is defined as

$$\text{LDOS} = \sum_{E'} |C_{\psi(0)}^{E'}|^2 \delta(E - E'),$$

with $C_{\psi(0)}^{E'} = \langle E|\psi(0)\rangle$ being the overlaps. Thus, the LDOS gives the spread of the basis states over the eigenstates.

Equation (16) denotes EGOE$(1 + k)$ or BEGOE$(1 + k)$ depending on the choice of particles (fermions or bosons) with $V(k)$ defined by equation (7). The mean-field Hamiltonian $h(1)$ is defined by the fixed sp energies $\epsilon_i = i + 1/i$. Note that in the situation where one deals with random $k$-body interactions in the presence of a mean-field, the EGOE and BEGOE notations [1, 7, 8, 13, 15] are extended to EGOE$(1 + k)$ and BEGOE$(1 + k)$, respectively, with $2 \leq k \leq m$. We choose $\lambda = 0.5$ close to the region of thermalization [1, 33]. We construct a 1000 member EGOE$(1 + k)$ ensemble with $N = 12$, $m = 6$ and $k$ varying from 2–6 and a 1000 member BEGOE$(1 + k)$ ensemble.
with $N = 5$, $m = 10$ and $k$ varying from 2–10. The corresponding matrix dimensions are $d_F(12,6) = 924$ and $d_B(5,10) = 1001$. The LDOS is then computed as follows. First of all, the basis state energies $e_b(m)$ are the diagonal elements of the $H$ matrix in the many-particle Fock-space basis giving $e_b(m) = \langle b|H(1) + \lambda V(k)|b\rangle$. Note that the centroids of the $e_b(m)$ energies are the same as that of the eigenvalue ($E$) spectra but their widths are different. For each member of the ensemble, the energies $E$ and $e_b(m)$ are zero centered and scaled by the spectrum width $\sigma_H(m)$. For each member of the ensemble, $\left|C_{\psi(0)}^E\right|^2$ are summed over the basis states in the energy windows $e_{\psi(0)}(m) \pm \delta$ and then the ensemble averaged LDOS are constructed as histograms as a function of energy. We choose $e_{\psi(0)}(m) = 0$ and $\delta = 0.2$. The results are shown in figures 3 and 4. For sufficiently strong $k$-body interactions, the LDOS change from Gaussian to semi-circle form just as the spectral density. Importantly, the LDOS follows $v(x|q)$ as $k$ changes from 2 to $m$. The agreement between the numerical histograms and solid curves obtained using equation (4) is excellent.

Figure 3. The ensemble averaged LDOS of a 1000 member EGOE(1 + $k$) (histograms) with $N = 12$ and $m = 6$ and $k$ changing from 2 to 6. The smooth (red) curves are obtained using equation (4) with the corresponding values of $q$ given in table 1.
Using equation (15), we easily see that the survival probability is essentially given by the Fourier transform of the LDOS,

\[
F(t) = \left| \sum_E \left| C^E_{\psi(0)} \right|^2 \exp(-iEt) \right|^2
\]

\[
= \left| \int \text{LDOS} \exp(-iEt) \, dE \right|^2. \tag{18}
\]

The survival probabilities are computed as follows. For each member at a given time \(t\), \(\sum_E C^E_{\psi(0)} C^E_{\psi(0)} \) are summed over the basis states \( |\psi(0)\rangle \) in the energy window \( e_{\psi(0)}(m) \pm \delta_1 \). We choose \( \delta_1 = 0.01 \) and \( e_{\psi(0)}(m) = 0 \). Then, the ensemble averaged survival probability for a fixed initial mean-field basis state is obtained by binning.

The Monte-Carlo results of the survival probability decays for EGOE\((1 + k)\) and BEGOE\((1 + k)\) as a function of \(k\) are shown in figure 5. The decay becomes faster with increasing \(k\) for both EGOE\((1 + k)\) and BEGOE\((1 + k)\). The survival probability decay shows oscillations for all \(k\) values in EGOE\((1 + k)\) while for BEGOE\((1 + k)\), oscillations...
are seen only for $k \geq 4$ and they become more pronounced with increasing $k$. This can be understood as follows: for small $k$ values, the LDOS is Gaussian. Thus, the survival probability decay given by the Fourier transform of the LDOS is also Gaussian and does not show any oscillations. As the LDOS becomes semi-circular with an increasing $k$ parameter, the survival probability decay shows oscillations as the Fourier transform of the semi-circle is a Bessel function of the first kind [14, 20]. Analytical results for the Fourier transform of $v(x|q)$ are not yet available (here, the results in [32] may prove to be useful). Therefore, we have numerically computed the Fourier transform of equation (14) and compared it with the Monte-Carlo results for the survival probabilities in figures 6 and 7.

For a 1000 member EGOE($1+k$) ensemble with $N = 12$, $m = 6$, we compare the Monte-Carlo results for the survival probability decay (solid circles) with the numerical Fourier transform of equation (14) in figure 6. The crossover to the region of thermalization will be faster with an increasing $k$; see the discussion in appendix G of [1] for a comparison between EGOE($1+2$) and EGOE($1+3$). Due to this, the results for the survival probability for $k = 4-6$ are the same. As seen from the figure, the results for the survival probability show oscillations with an increasing time $t$. The Fourier transform of equation (14) describes the short-time behavior accurately and the agreement gets better with increasing $k$. It also captures the positions of the oscillations. It is important to note that the oscillations are damped for Monte-Carlo calculations for two reasons: (i) we approximate the LDOS appearing in equation (18) by $v(E|q)$, which represents only the smoothed part of the LDOS; and (ii) there is an averaging of the LDOS in the window of width $\delta = 0.2$ around $e_{v_0}(m) = 0$. Further investigations of this will be reported elsewhere.
Similarly, figure 7 shows a comparison of the Monte-Carlo results for the survival probability decay (solid circles) with the numerical Fourier transform of equation (14) for a 1000 member BEGOE(1 + k) ensemble with $N = 5$, $m = 10$ and $\lambda = 0.5$. As the crossover to the region of thermalization is faster in bosonic systems compared to fermionic ones [1], the results for $k = 7 - 10$ are the same. The decay shows oscillations with increasing $t$ and these become prominent with increasing $k$. The short time dynamics and the positions of oscillations are well captured by the Fourier transform of equation (14).

6. Conclusions

We have shown that the generating function of $q$-Hermite polynomials describes the Gaussian to semi-circle transition in the spectral densities and LDOS of EGOE($k$)/EGUE($k$)/BEGOE($k$)/BEGUE($k$) as a function of the rank of interactions $k$ by deriving the formula for the $q$ parameter for these few-body ensembles. The survival
probability decay in EGOE\((1 + k)\) and BEGOE\((1 + k)\) is explained by the numerical Fourier transform of the generating function of \(q\)-Hermite polynomials. The dynamics of nonequilibrium quantum systems depends strongly on the nature of particles (fermions or bosons) and the rank of interactions \(k\). Although two-body interactions are dominant, three and higher body interactions may become prominent in strongly interacting quantum systems [34, 35]. The present results establish that \(q\)-Hermite polynomials (which may also be other \(q\)-special functions) play an important role in the embedded ensembles in explaining the dependence of the spectral density, LDOS and survival probabilities on \(k\) and the nature of particles. In the future, it will be good to explore bivariate \(q\)-Hermite polynomials and their generating function as they may prove to be useful in understanding the two-point correlation functions that determine level fluctuations and also the transition strengths generated by a transition operator [1, 36].

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Appendix. Derivation of the fourth moment for EGOE(k) using binary correlation approximation method

Here, for completeness we will describe briefly the derivation of the formula for \( \mu_4 \) for EGOE(k) that includes finite \( N \) corrections. All the results in this appendix are from [9, 37, 38].

We denote a \( k_H \)-body operator as,

\[
H(k_H) = \sum_{\alpha, \beta} v^{\alpha \beta}_{H} \alpha (k_H)^\dagger \beta(k_H).
\]  (A.1)

Here, \( \alpha (k_H) \) is the \( k_H \) particle creation operator and \( \beta(k_H) \) is the \( k_H \) particle annihilation operator. Similarly, \( v^{\alpha \beta}_{H} \) are matrix elements of the operator \( H \) in \( k_H \) particle space, i.e. \( v^{\alpha \beta}_{H} = \langle k_H \beta | H | k_H \alpha \rangle \). The following basic traces will be used throughout [13, 37, 38],

\[
\sum_\alpha \alpha (k_H) \alpha (k_H) = \left( \hat{n}_k \right) \Rightarrow \left( \sum_\alpha \alpha (k_H) \alpha (k_H) \right)^m = \left( \frac{m}{k} \right).
\]  (A.2)

\[
\sum_\alpha \alpha (k_H) \alpha ^\dagger (k_H) = \left( N - \hat{n}_k \right) \Rightarrow \left( \sum_\alpha \alpha (k_H) \alpha ^\dagger (k_H) \right)^m = \left( \frac{m}{k} \right); \quad \tilde{m} = N - m.
\]  (A.3)

\[
\sum_\alpha \alpha ^\dagger (k_H) B(k') \alpha (k_H) = \left( \hat{n}_k - k' \right) B(k')
\Rightarrow \left( \sum_\alpha \alpha ^\dagger (k_H) B(k') \alpha (k_H) \right)^m = \left( \frac{m - k'}{k} \right) B(k').
\]  (A.4)

\[
\sum_\alpha \alpha (k_H) B(k') \alpha ^\dagger (k_H) = \left( N - \hat{n}_k - k' \right) B(k')
\Rightarrow \left( \sum_\alpha \alpha (k_H) B(k') \alpha ^\dagger (k_H) \right)^m = \left( \frac{\tilde{m} - k'}{k} \right) B(k').
\]  (A.5)

Equation (A.2) follows from the fact that the average should be zero for \( m < k \) and one for \( m = k \) and similarly, equation (A.3) follows from the same argument except that the particles are replaced by holes. Equation (A.4) follows first by writing the \( k' \)-body operator \( B(k') \) in operator form using equation (A.1), i.e.

\[
B(k') = \sum_{\beta, \gamma} v^{\beta \gamma}_{B} \beta ^\dagger (k') \gamma(k')
\]  (A.6)

and then applying the commutation relations for the fermion creation and annihilation operators. This gives \( \sum_{\beta, \gamma} v^{\beta \gamma}_{B} \beta ^\dagger (k') \sum_\alpha \alpha ^\dagger (k_H) \alpha (k_H) \gamma(k') \). Now applying equation (A.2)
to the sum involving $\alpha$ gives equation (A.4). Equation (A.5) follows from the same arguments except one has to assume that $B(k')$ is a fully irreducible $\nu = k'$ operator and therefore, it has particle–hole symmetry. For a general $B(k')$ operator, this is valid only in the $N \to \infty$ limit. Therefore, this equation has to be applied with caution.

Using the definition of the $H$ operator in equation (A.1), we have

$$
\langle H(k_H)H(k_H) \rangle^m = \sum_{\alpha, \beta} \left\{ v_H^{\alpha \beta} \right\}^2 \langle \alpha^\dagger(k_H)\beta(k_H)\beta^\dagger(k_H)\alpha(k_H) \rangle^m
$$

$$
= v_H^2 \left\langle \sum_{\alpha} \alpha^\dagger(k_H) \left\{ \sum_{\beta} \beta(k_H)\beta^\dagger(k_H) \right\} \alpha(k_H) \right\rangle^m
$$

$$
= v_H^2 T(m, N, k_H). \tag{A.7}
$$

Here, $H$ is taken as EGOE($k_H$) with all the $k_H$ particle matrix elements being independent Gaussian variables with a zero center and the same variance for off-diagonal matrix elements (twice for the diagonal matrix elements). This makes $(v_H^{\alpha \beta})^2 = v_H^2$ independent of the $\alpha$, $\beta$ labels. It is important to note that in the dilute limit, the diagonal terms ($\alpha = \beta$ in equation (A.7)) in the averages are neglected (as they are smaller by at least one power of $1/N$) and the individual $H$s are unitarily irreducible. These assumptions are no longer valid for finite-$N$ systems and hence, the evaluation of averages is more complicated. In the dilute limit, we have

$$
T(m, N, k_H) = \left\langle \sum_{\alpha} \alpha^\dagger(k_H) \left\{ \sum_{\beta} \beta(k_H)\beta^\dagger(k_H) \right\} \alpha(k_H) \right\rangle^m
$$

$$
= \left( \frac{\tilde{m} + k_H}{k_H} \right) \left\langle \sum_{\alpha} \alpha^\dagger(k_H)\alpha(k_H) \right\rangle^m = \left( \frac{\tilde{m} + k_H}{k_H} \right) \left( \begin{array}{c} m \\ k_H \end{array} \right). \tag{A.8}
$$

Note that we have used equation (A.3) to evaluate the summation over $\beta$ and equation (A.2) to evaluate the summation over $\alpha$ in equation (A.8). In order to incorporate the finite-$N$ corrections, we have to consider the contribution of the diagonal terms. Then, we have,
\[
T(m, N, k_H) = \left( \sum_{\alpha \neq \beta} \alpha^\dagger(k_H)\beta(k_H)\beta^\dagger(k_H)\alpha(k_H) \right)^m + 2 \left( \sum_{\alpha} \alpha^\dagger(k_H)\alpha(k_H)\alpha^\dagger(k_H)\alpha(k_H) \right)^m
\]

\[
= \left( \sum_{\alpha} \alpha^\dagger(k_H) \left( \sum_{\beta} \beta(k_H)\beta^\dagger(k_H) \right)\alpha(k_H) \right)^m + \left( \sum_{\alpha} \alpha^\dagger(k_H)\alpha(k_H)\alpha^\dagger(k_H)\alpha(k_H) \right)^m
\]

\[
= \left( \sum_{\alpha} \alpha^\dagger(k_H) \alpha(k_H)\alpha^\dagger(k_H)\alpha(k_H) \right)^m \left( \sum_{\alpha} \alpha^\dagger(k_H)\alpha(k_H)\alpha^\dagger(k_H)\alpha(k_H) \right)^m
\]

Note that prefactor \(2\) in the second term of the first line in equation (A.9) comes because the variance of the diagonal terms is twice that of the off-diagonal terms. Also, the trace \(\sum_{\alpha} \alpha^\dagger(k_H)\alpha(k_H)\alpha^\dagger(k_H)\alpha(k_H) = \sum_{\alpha} \alpha^\dagger(k_H)\alpha(k_H)\alpha^\dagger(k_H)\alpha(k_H)\) conserves the number of particles. Now we turn to evaluating fourth-order averages.

For averages involving the product of four operators of the form

\[
\langle H(k_H)G(k_G)H(k_H)G(k_G) \rangle^m,
\]

with operators \(H\) and \(G\) independent and of body ranks \(k_H\) and \(k_G\) respectively, there are two possible ways of evaluating this trace. Either (a) first contract the \(H\) operators across the \(G\) operator using equation (A.5) and then contract the \(G\) operators using equation (A.4), or (b) first contract the \(G\) operators across the \(H\) operator using equation (A.5) and then contract the \(H\) operators using equation (A.4). Following (a), in the dilute limit, we get

\[
\langle H(k_H)G(k_G)H(k_H)G(k_G) \rangle^m = v_H^2 v_G^2 \left( \frac{\tilde{m} + k_H - k_G}{k_H} \right) \left( \frac{m - k_G}{k_H} \right) \left( \frac{\tilde{m} + k_G}{k_H} \right) \left( \frac{m}{k_G} \right). (A.10)
\]

Similarly, following (b), in the dilute limit, we get

\[
\langle H(k_H)G(k_G)H(k_H)G(k_G) \rangle^m = v_H^2 v_G^2 \left( \frac{\tilde{m} + k_G - k_H}{k_G} \right) \left( \frac{m - k_H}{k_G} \right) \left( \frac{\tilde{m} + k_H}{k_G} \right) \left( \frac{m}{k_H} \right). (A.11)
\]

The result should be independent of the preference. In other words, the average should have the \(k_H \leftrightarrow k_G\) symmetry. As seen from equations (A.10) and (A.11), this symmetry is violated except for the trivial case of \(k_H = k_G\). However, the \(k_H \leftrightarrow k_G\) symmetry is valid for the ‘strict’ \(N \rightarrow \infty\) result and also for the result incorporating finite \(N\) corrections as discussed below. In general, the final result can be expressed as

\[
\langle H(k_H)G(k_G)H(k_H)G(k_G) \rangle^m = v_H^2 v_G^2 F(m, N, k_H, k_G). (A.12)
\]

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In the ‘strict’ dilute limit, both equations (A.10) and (A.11) reduce to give result
\[ F(m, N, k_H, k_G) = \left( \frac{m - k_H}{k_G} \right) \left( \frac{m}{k_H} \right) \left( \frac{N - k_H}{k_G} \right) \left( \frac{N}{k_H} \right). \]
In order to obtain finite-N corrections to \( F(\cdots) \), we have to contract over operators whose lower symmetry parts cannot be ignored. The operator \( H(k_h) \) contains irreducible symmetry parts \( F(s) \) denoted by \( s = 0, 1, 2, \ldots, k_H \) with respect to the unitary group \( SU(N) \) decomposition of the operator. For a \( k_H \)-body number conserving operator \([13, 37]\),
\[ H(k_H) = \sum_{s=0}^{k_H} \left( \frac{m - s}{k_H - s} \right) F(s). \] 
Here, the \( F(s) \) are orthogonal with respect to the \( m \)-particle averages, i.e. \( \langle F(s) F(s') \rangle^m = \delta_{ss'} \). Now, the \( m \)-particle trace in equation (A.10) with binary correlations will have four parts,
\[ \langle H(k_H) G(k_G) H(k_H) G(k_G) \rangle^m \]
\[ = v_H^2 v_G^2 \sum_{\alpha, \beta, \gamma, \delta} \left( \alpha^\dagger(k_H) \beta(k_H) \gamma(k_G) \delta(k_G) \right) \left( \alpha(k_H) \beta^\dagger(k_H) \gamma^\dagger(k_G) \delta^\dagger(k_G) \right)^m \]
\[ + v_H^2 v_G^2 \sum_{\alpha, \gamma, \delta} \left( \alpha^\dagger(k_H) \alpha(k_H) \gamma^\dagger(k_G) \delta(k_G) \right) \left( \alpha(k_H) \delta^\dagger(k_G) \gamma(k_G) \right)^m \]
\[ + v_H^2 v_G^2 \sum_{\alpha, \beta, \gamma} \left( \alpha^\dagger(k_H) \beta(k_H) \gamma^\dagger(k_G) \gamma(k_h) \right) \left( \alpha(k_H) \beta^\dagger(k_H) \gamma^\dagger(k_G) \gamma(k_G) \right)^m \]
\[ + v_H^2 v_G^2 \sum_{\alpha, \beta, \delta} \left( \alpha^\dagger(k_H) \alpha(k_H) \delta^\dagger(k_G) \delta(k_G) \right) \left( \alpha(k_H) \delta^\dagger(k_G) \right)^m \]
\[ = X + Y_1 + Y_2 + Z. \] 
Note that we have decomposed each operator into diagonal and off-diagonal parts. We have used the condition that the variance of the diagonal matrix elements is twice that of the off-diagonal matrix elements in the defining spaces to convert the restricted summations into unrestricted summations appropriately and obtain the four terms in the rhs of equation (A.15). Following Mon’s thesis [37] and applying unitary decomposition to \( \gamma^\dagger \) (also \( \delta^\dagger \)) in the first two terms and \( \alpha^\dagger \) (also \( \beta^\dagger \)) in the third term we get \( X, Y_1 \) and \( Y_2 \). Formulas for the \( X, Y_1, Y_2 \) and \( Z \) terms above are obtained as follows.
Applying unitary decomposition to the operators \( \gamma^\dagger(k_G) \delta(k_G) \) and \( \gamma(k_G) \delta^\dagger(k_G) \) using equation (A.13), we have
\[ X = v_H^2 v_G^2 \sum_{\alpha, \beta, \gamma, \delta} \sum_{s=0}^{k_G} \left( \frac{m - s}{k_G - s} \right)^2 \left( \alpha^\dagger(k_H) \beta(k_H) F_{\sigma}(s) \beta^\dagger(k_H) \alpha(k_H) F_{\delta}(s) \right)^m. \]
Contracting the operators \( \beta \beta^\dagger \) across \( F \) using equation (A.5) and operators \( \alpha^\dagger \alpha \) across \( F \) using equation (A.4) gives,

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\[ X = v_H^2 v_G^2 \sum_{s=0}^{k_G} \left( \frac{m-s}{k_G-s} \right)^2 \left( \frac{\bar{m} + k_H - s}{k_H} \right) \left( \frac{m-s}{k_H} \right) \sum_{\gamma, \delta} \left( \mathcal{F}_{\gamma \delta}(s) \mathcal{F}_{\gamma \delta}(s) \right)^m. \]

The inversion of the equation,

\[ \sum_{\gamma, \delta} \left( \gamma^\dagger(k_G) \delta(k_G) \delta^\dagger(k_G) \gamma(k_G) \right)^m = Q(m) = \sum_{s=0}^{k_G} \left( \frac{m-s}{k_G-s} \right)^2 \sum_{\gamma, \delta} \left( \mathcal{F}_{\gamma \delta}(s) \mathcal{F}_{\gamma \delta}(s) \right)^m, \]

gives,

\[ \left( \frac{m-s}{k_G-s} \right)^2 \sum_{\gamma, \delta} \left( \mathcal{F}_{\gamma \delta}(s) \mathcal{F}_{\gamma \delta}(s) \right)^m = \left( \frac{m-s}{k_G-s} \right)^2 \left( \frac{N-m}{s} \right) \left( \frac{m}{s} \right) \left[ (k_G-s)!s! \right]^2 (N-2s+1) \]

\[ \times \sum_{t=0}^{s} \frac{(-1)^{t-s}([(N-t-k_G)!(N-t-k_G)!])}{(s-t)!(N-s-t+1)!t!(N-t)!} Q(N-t). \]

It is important to mention that there are errors in the equation given in Mon’s thesis and we have verified equation (A.18) using Mathematica (Mon = equation (A.18)/[(N-2s)!s!]2). For the average required in equation (A.17), we have

\[ Q(m) = \sum_{\gamma, \delta} \left( \gamma^\dagger(k_G) \delta(k_G) \delta^\dagger(k_G) \gamma(k_G) \right)^m = \left( \frac{\bar{m} + k_G}{k_G} \right) \left( \frac{m}{k_G} \right). \]

Simplifying equation (A.18) using equation (A.19) and using the result in equation (A.16) along with the series summation

\[ \sum_{t=0}^{s} \frac{(-1)^{t-s}(N-t-k_G)! (k_G-t)!}{(s-t)!(t)![(N-t-k_G)!](N-t)!} = \frac{k_G!(N-k_G-s)!}{(N+1-s)!} \left( \frac{t}{k_G} \right) \left( \frac{N+1}{s} \right), \]

the expression for \( X \) is,

\[ X = v_H^2 v_G^2 \mathcal{F}(m, N, k_H, k_G); \]

\[ \mathcal{F}(m, N, k_H, k_G) = \sum_{s=0}^{k_G} \left( \frac{m-s}{k_G-s} \right)^2 \left( \frac{\bar{m} + k_H - s}{k_H} \right) \left( \frac{m-s}{k_H} \right) \left( \frac{\bar{m}}{s} \right) \left( \frac{m}{s} \right) \left( \frac{N+1}{s} \right) \]

\[ \times \frac{N-2s+1}{N-s+1} \left( \frac{N-s}{k_G} \right)^{-1} \left( \frac{k_G}{s} \right)^{-1}. \]
Although not obvious, $X$ has $k_H \leftrightarrow k_G$ symmetry and this can be verified easily for $k_H, k_G \leq 2$. Applying the same procedure that led to equation (A.21), we have for $Y_1, Y_2$ and $Z$ in equation (A.15),

$$Y_1 = v_H^2 v_G^2 G_1(m_N, k_H, k_G), \quad Y_2 = v_H^2 v_G^2 G_2(m_N, k_H, k_G), \quad Z = v_H^2 v_G^2 \left( \frac{m}{k} \right)^2. \tag{A.22}$$

Here, $G_1(m_N, k_H, k_G)$ is the same as $\mathcal{F}(m_N, k_H, k_G)$ in equation (A.21) but without the $\left( \frac{N+1}{s} \right)$ factor. Similarly, $G_2(m_N, k_H, k_G)$ is the same as $\mathcal{F}(m_N, k_G, k_H)$ in equation (A.21) but without the $\left( \frac{N+1}{s} \right)$ factor. Combining equations (A.21) and (A.22) will give the formula for $F(m_N, k_H, k_G)$ in equation (A.12) with finite $N$ corrections.

Turning to the fourth order average of interest $\langle H^4(k_H) \rangle^m$, it is easy to see that there will be three different correlation patterns that will contribute to this average in the binary correlation approximation (we must correlate in pairs the operators for all moments of order $>2$),

$$\langle H^4(k_H) \rangle^m = \langle H(k_H) H(k_H) H(k_H) H(k_H) \rangle^m$$

$$+ \langle H(k_H) H(k_H) H(k_H) H(k_H) \rangle^m$$

$$+ \langle H(k_H) H(k_H) H(k_H) H(k_H) \rangle^m. \tag{A.23}$$

In equation (A.23), we denote the correlated pairs of operators by the same color in each pattern. The first two terms on the rhs of equation (A.23) are equal due to cyclic invariance and follow easily from equation (A.7),

$$\langle H(k_H) H(k_H) H(k_H) H(k_H) \rangle^m = \langle H(k_H) H(k_H) H(k_H) H(k_H) \rangle^m$$

$$= \left[ \langle H^2(k_H) \rangle^m \right]^2. \tag{A.24}$$

Similarly, the third term on the rhs of equation (A.23) will be,

$$\langle H(k_H) H(k_H) H(k_H) H(k_H) \rangle^m = v_H^4 F(m_N, k_H, k_H). \tag{A.25}$$

Finally, $\langle H^4(k_H) \rangle^m$ is given by

$$\langle H^4(k_H) \rangle^m = v_H^4 \left[ 2 \left\{ T(m_N, k_H) \right\}^2 + F(m_N, k_H, k_H) \right]. \tag{A.26}$$

Equation (A.26) along with equations (A.21) and (A.22) will give equation (12) in the text.

References

[1] Kota V K B 2014 *Embedded Random Matrix Ensembles in Quantum Physics* (Heidelberg: Springer)
[2] Borgonovi F, Izrailev F M, Santos L F and Zelevinsky V G 2016 Phys. Rep. 626 1
[3] D’Alessio L, Kafri Y, Polkovnikov A and Rigol M 2016 Adv. Phys. 65 239
[4] Kota V K B and Chavda N D 2018 Int. J. Mod. Phys. E 27 1830001

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