The classification of $\mathbb{Z}$–graded modules of the intermediate series over the $q$-analog Virasoro-like algebra

Yina Wu and Weiqiang Lin

1. Department of Mathematics, Zhangzhou Teacher’s College, Zhangzhou 363000, Fujian, China.
2. Department of Mathematics, University of Science and Technology of China, Hefei 230026, Anhui, China.

Abstract

In this paper, we complete the classification of the $\mathbb{Z}$-graded modules of the intermediate series over the $q$-analog Virasoro-like algebra $L$. We first construct four classes of irreducible $\mathbb{Z}$-graded $L$-modules of the intermediate series. Then we prove that any $\mathbb{Z}$-graded $L$-modules of the intermediate series must be the direct sum of some trivial $L$-modules or one of the modules constructed by us.

Keywords: modules of the intermediate series, $\mathbb{Z}$-graded $L$-module, the $q$-analog Virasoro-like algebra.

MSC: 17B68; 17B65; 17B10.

1 Introduction

The classification of irreducible graded modules with finite dimensional homogeneous subspaces over a graded Lie algebra is one of the main subject in the study of Lie theory. Meanwhile, the irreducible graded modules with finite dimensional homogeneous subspaces for some infinite dimensional Lie algebras, such as the Heisenberg Lie algebra and the Virasoro algebra, have important applications in the study of the vertex operator algebras and theoretical physics. In this paper, we study the classification of $\mathbb{Z}$-graded modules of the intermediate series over the $q$-analog Virasoro-like algebra. The $q$-analog Virasoro-like algebra is introduced by Kirkman etc.

---

* Supported by the National Science Foundation of China (No. 10671160) and the China Postdoctoral Science Foundation (No. 20060390693). To appear in Algebra colloquium.
† e-mail: linwq83@yahoo.com.cn
in [1]. It can be realized as the universal central extension of the inner derivation Lie algebra of the quantum torus $C_q[x_1^{\pm1}, x_2^{\pm1}]$ (see [2] or [3]), where $q$ is generic. Quantum torus is one of the main objects in noncommutative geometry, and plays an important role in the classification of extended affine Lie algebras ([2]). Meanwhile, the $q$-analog Virasoro-like algebra can be regarded as a $q$ deformation of the Virasoro-like algebra introduced and studied by Arnold, Wit, etc when they tried to generalize the Virasoro algebra ([4], [5] and [1]). There are some papers devoted to the study of the structure and representations of the $q$-analog Virasoro algebra. Jiang and Meng studied its derivation Lie algebra and the automorphism group of its derivation Lie algebra ([3]). Chen, Lin, etc. studied the structure of its automorphism group ([6]). Zhao and Rao constructed a class of highest weight irreducible $Z$-graded modules over the $q$-analog Virasoro-like algebra, and gave a sufficient and necessary condition for such a module with finite dimensional homogeneous subspaces ([7]). Gao constructed a class of principal vertex representations for the extended affine Lie algebras coordinatized by certain quantum tori by using the representation of the $q$-analog Virasoro-like algebra([8]). The classification of the irreducible graded modules with finite dimensional homogeneous subspaces and nontrivial centers over the $q$-analog Virasoro-like algebra has been completed in [9]. Thus, we only consider the classification of the $Z$-graded modules of the intermediate series over the centerless $q$-analog Virasoro-like algebra $L$ in this paper. In section 2, we first recall some notations about the centerless $q$-analog Virasoro-like algebra $L$ and its $Z$-graded modules of the intermediate series. Then we construct four classes of $Z$-graded $L$-modules of the intermediate series, and show that they are irreducible. In section 3, we complete the classification of the $Z$-graded $L$-modules of the intermediate series.

**2 The $q$-analog Virasoro-like algebra and its $Z$-graded modules of the intermediate series**

Throughout this paper we use $Z$, $Z^*$, $C$, $C^*$ and $N$ to denote the sets of integers, nonzero integers, complex number, nonzero complex number and positive integers respectively. All spaces are over $C$.

In this paper, we require $q \in C$ to be a fixed nonzero non-root of unity. Let $L$ be a vector space
spanned by $\{t_1^ht_2^j | (h,j) \in \mathbb{Z}^2 \setminus \{(0,0)\}\}$. We denote it by $L = \langle t_1^ht_2^j | (h,j) \in \mathbb{Z}^2 \setminus \{(0,0)\}\rangle$.

Define a commutator in $L$ as follows:

$$[t_1^ht_2^j, t_1^mh_2^n] = (q^{jn} - q^{hn})t_1^{h+m+1}t_2^{j+n}, \quad (2.1)$$

then $L$ is called the centerless $q$-analog Virasoro-like algebra. And it is easy to check that $L = \bigoplus_{u \in \mathbb{Z}} L_u$, where $L_u = \langle t_1^u t_2^j | j \in \mathbb{Z} \rangle$, is a $\mathbb{Z}$-graded Lie algebra.

Next, we recall the definition of the $\mathbb{Z}$-graded $L$-modules of the intermediate series. If a vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ satisfies:

(1) $V$ is a $L$-module,

(2) Regarding to $L = \bigoplus_{u \in \mathbb{Z}} L_u$, there are $L_u.V_n \subseteq V_{n+u}$ for any $n, u \in \mathbb{Z}$,

(3) $\dim V_n \leq 1, \forall n \in \mathbb{Z}$,

then $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is called a $\mathbb{Z}$-graded $L$-module of the intermediate series.

In this paper, we study this type of $\mathbb{Z}$-graded $L$-module and its classification. First we construct four classes of $\mathbb{Z}$-graded $L$-modules.

**Proposition 2.1:** Set $V = \langle v_j | j \in \mathbb{Z} \rangle$. For any $a \in \mathbb{C}^*$, define the action of the elements in $L$ on $V$ by linearly extending the following maps respectively:

$\langle \alpha \rangle$. $(t_1^m t_2^n).v_k = (aq^k)^n v_{k+m}$, for all $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$

$\langle \beta \rangle$. $(t_1^m t_2^n).v_k = (-1)^m(aq^k)^n v_{k+m}$, for all $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$

$\langle \gamma \rangle$. $(t_1^m t_2^n).v_k = (-1)^{m+n+1}(aq^{-k-m})^n v_{k+m}$, for all $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$

$\langle \delta \rangle$. $(t_1^m t_2^n).v_k = (-1)^{n+1}(aq^{-k-m})^n v_{k+m}$, for all $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$

Then $V$ becomes a $L$-module with the actions defined in $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 3 \rangle$ or $\langle 4 \rangle$ respectively.

**Proof.** We take $\langle 1 \rangle$ and $\langle 3 \rangle$ for examples to show that $V$ is a $L$-module with respect to the action defined in $\langle 1 \rangle$or$\langle 3 \rangle$ respectively.

For the operators defined in $\langle 1 \rangle$, we have

$$[t_1^m t_2^n, t_1^h t_2^j].v_k = (q^{jn} - q^{hn})(t_1^m t_2^n).v_k = (q^{jn} - q^{hn})(aq^k)^n v_{k+m+j}.$$ 

Meanwhile,

$$(t_1^m t_2^n). (t_1^h t_2^j).v_k - (t_1^h t_2^j). (t_1^m t_2^n).v_k = (aq^j h t_1^m t_2^n).v_{k+j} - (aq^k n t_1^h t_2^j).v_{k+m}$$
The classification of $Z$-graded $L$-

We will first prove two Lemmas.

In this section, we discuss the classification of the

where $V$ is a $Z$-module. Hence,

Thus

$[t_1^m t_2^n, t_1^j t_2^h].v_k = (t_1^m t_2^n)(t_1^j t_2^h).v_k = (t_1^m t_2^n)(t_1^j t_2^h).v_k - (t_1^j t_2^h)(t_1^m t_2^n).v_k$.

Therefore, $V$ is a $L$-module.

For the operators defined in (3), we have

$[t_1^m t_2^n, t_1^j t_2^h].v_k = (q^{jn} - q^{hm})(t_1^m t_2^n)(t_1^j t_2^h).v_k = (q^{jn} - q^{hm})(-1)^{m+j+n+h+1}(aq^{-k-m-j})^{n+h}v_{k+m+j}$,

and

$(t_1^m t_2^n)(t_1^j t_2^h).v_k - (t_1^j t_2^h)(t_1^m t_2^n).v_k$

$= (-1)^{j+h+1}(aq^{-k-j})^{h}(t_1^m t_2^n).v_{k+j} - (-1)^{m+n+1}(aq^{-k-m})^{n}(t_1^j t_2^h).v_{k+m}$

$= (q^{jn} - q^{hm})(-1)^{m+j+n+h+1}(aq^{-k-m-j})^{n+h}v_{k+m+j}$.

Thus

$[t_1^m t_2^n, t_1^j t_2^h].v_k = (t_1^m t_2^n)(t_1^j t_2^h).v_k - (t_1^j t_2^h)(t_1^m t_2^n).v_k$.

Hence, $V$ is a $L$-module.

Similarly, one can check that $V$ is a $L$-module with the operators defined in (2) or (4) respectively.

**Remark 2.2:** One can easily see that the modules defined above are irreducible $Z$-graded $L$-modules of the intermediate series with respect to the linear space decomposition $V = \bigoplus_{n \in Z} V_n$, where $V_n = C v_n$. We will use $V(a, I)$, $V(a, II)$, $V(a, III)$ and $V(a, IV)$ to denote the corresponding $Z$-graded $L$-modules defined by (1), (2), (3), and (4) respectively.

### 3 The classification of $Z$-graded $L$-modules of the intermediate series

In this section, we discuss the classification of the $Z$-graded $L$-modules of the intermediate series.

We will first prove two Lemmas.
Lemma 3.1: If $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is a $\mathbb{Z}$-graded $L$-modules of the intermediate series and the action of $t_1^1t_1^{-1}$ is degenerate, then $V$ can be decomposed into the direct sum of some trivial $L$-submodules.

Proof. Without loss of generality, we can assume $V \neq 0$. Since the action of $t_1^1t_1^{-1}$ is degenerate, there exists a nonzero vector $v_j \in V_j$ such that $t_1^1t_1^{-1}v_j = 0$. Thus $t_1^1t_1^{-1}v_j = t_1^{-1}t_1v_j = 0$ since $[t_1^1, t_1^{-1}] = 0$ by the definition of the Lie algebra $L$. Therefore, we obtain that

$$
\begin{align*}
&\begin{cases}
  t_1^1V_j = 0; \\
  t_1^{-1}V_j = 0,
\end{cases} & \begin{cases}
  t_1^{-1}V_j = 0; \\
  t_1^{-1}V_{j+1} = 0,
\end{cases} & \begin{cases}
  t_1^1V_{j-1} = 0; \\
  t_1^1V_j = 0,
\end{cases} & \begin{cases}
  t_1^1V_{j-1} = 0; \\
  t_1^{-1}V_{j+1} = 0,
\end{cases}
\end{align*}
$$

(3.1)

since $\dim V_k \leq 1$ for all $k \in \mathbb{Z}$. Considering that $t_1^1t_1^{-1}v = t_1^{-1}t_1v$ for any $v \in V$ and $\dim V_n \leq 1, \forall n \in \mathbb{Z}$, we deduce that, for any $n \in \mathbb{Z}$,

$$
(1) \begin{cases}
  t_1^1V_n = 0; \\
  t_1^{-1}V_n = 0.
\end{cases} \quad (2) \begin{cases}
  t_1^{-1}V_n = 0; \\
  t_1^{-1}V_{n+1} = 0.
\end{cases} \quad (3) \begin{cases}
  t_1^1V_{n-1} = 0; \\
  t_1^1V_n = 0.
\end{cases} \quad (4) \begin{cases}
  t_1^1V_{n-1} = 0; \\
  t_1^{-1}V_{n+1} = 0.
\end{cases}
$$

(3.2)

Now we prove that $V_n$ is a trivial $L$-module. We will first prove that $t_2^k$ acts trivially on $V_n$ for any $k \in \mathbb{Z}^*$, $n \in \mathbb{Z}$. We only give the proof of this claim for case (1) here. The proofs of this claim for the other three cases are similar.

Suppose that $t_1^1V_n = 0$ and $t_1^{-1}V_n = 0$. By the definition of $\mathbb{Z}$-graded $L$-module, we have that

$$
t_2^kV_n \subseteq V_n \text{ for all } k \in \mathbb{Z}^*.
$$

(3.3)

Thus we deduce that

$$
[[t_1^1, t_2^k], t_1^{-1}]V_n = (t_1^1t_2^k, t_1^{-1}.V_n - t_2^k, t_1^{-1}.V_n) - t_1^{-1}.(t_1^1t_2^k, V_n - t_2^k, t_1V_n) = 0.
$$

Therefore $t_2^k$ acts trivially on $V_n$ since

$$
[[t_1^1, t_2^k], t_1^{-1}]V_n = (1 - q^k)(q^{-k} - 1)t_2^kV_n.
$$

Next, we show that $t_2^j$ acts trivially on $V_n$ for any $j \in \mathbb{Z}^*$. On one hand, we have that

$$
[[t_1^1, t_2^j], t_2^{-1}]V_n = (1 - q^j)(1 - q^{-j})t_1^1V_n.
$$

On the other hand, for any $j \in \mathbb{Z}^*$, we have that

$$
[[t_1^1, t_2^j], t_2^{-1}]V_n = (t_1^1t_2^j, t_2^{-1}.V_n - t_2^j, t_2^{-1}.V_n) - t_2^{-1}.(t_1^1t_2^j, V_n - t_2^j, t_2V_n) = 0.
$$
since \( t^k_2 \) acts trivially on \( V_n, \forall k \in \mathbb{Z}^* \), \( n \in \mathbb{Z} \). Hence, \( t^j_1.V_n = 0 \) for any \( j \in \mathbb{Z}^* \).

Finally, for any \( k, j \in \mathbb{Z}^* \), we have that

\[
(1 - q^{jk})t^j_1.t^k_2.V_n = [t^j_1, t^k_2]_V = t^j_1.t^k_2.V_n - t^k_2.t^j_1.V_n = 0.
\]

Therefore, \( V_n \) is a trivial \( L \)-submodules for any \( n \in \mathbb{Z} \). Thus \( V \) can be decomposed into the direct sum of some trivial \( L \)-submodules. \( \blacksquare \)

Lemma 3.2: If \( V = \bigoplus_{n \in \mathbb{Z}} V_n \) is a \( \mathbb{Z} \)-graded \( L \)-modules of the intermediate series and the action of \( t^1_1.t^{-1}_1 \) is nondegenerate, then \( V \) must be isomorphic to \( V(a, I), V(a, II), V(a, III) \) or \( V(a, IV) \) for some \( a \in \mathbb{C}^* \). (ref. Remark 2.2)

Proof. Since the action of \( t^1_1.t^{-1}_1 \) is nondegenerate, the action of \( t^{\pm 1}_1 \) is nondegenerate. Together with \( L_u.V_n \subseteq V_{n+u} \) and \( \dim V_n \leq 1 \) for any \( u \in \mathbb{Z}^* \) and \( n \in \mathbb{Z} \), there must be \( t^{\pm 1}_1.V_n = V_{n \pm 1} \). Thus \( \dim V_n = 1 \) for any \( n \in \mathbb{Z} \).

We first show that there exists a base \( \{ v_j \in V_j | j \in \mathbb{Z} \} \) of \( V \) such that \( t^{\pm 1}_1.v_j = \lambda v_{j+1} \) for any \( j \in \mathbb{Z} \). Suppose \( \omega_0 \in V_0 \) with \( \omega_0 \neq 0 \) and set \( \omega_n = t^1_1.\omega_{n-1} \in V_n \). Since the action of \( t^{\pm 1}_1 \) is nondegenerate and \( \dim V_n = 1 \) for all \( n \in \mathbb{Z} \), we have that \( \{ \omega_n \in V_n | n \in \mathbb{Z} \} \) forms a base of \( V \) and \( t^1_1.\omega_n = \omega_{n+1} \). Denote \( t^{\pm 1}_1.\omega_k = \phi(k)\omega_{k-1} \). By (2.1), we have \( [t^1_1, t^{-1}_1] = 0 \). Thus \( t^1_1.(t^{\pm 1}_1.\omega_k) = t^{\pm 1}_1.(t^1_1.\omega_k) \) for all \( k \in \mathbb{Z} \), which implies \( \phi(k) = \phi(k+1) \) for any \( k \in \mathbb{Z} \). Thus there exists \( p \in \mathbb{C}^* \) such that \( t^{\pm 1}_1.\omega_k = p\omega_{k-1} \) for any \( k \in \mathbb{Z} \). Set \( \lambda = \sqrt{p} \neq 0 \) and \( v_k = \frac{\omega_k}{\lambda^k} \), then \( t^{\pm 1}_1.v_k = \lambda v_{k+1} \) for any \( k \in \mathbb{Z} \).

Set \( (t^h_1t^i_2).v_k = f(h, j, k)v_{k+h} \). Then \( f(1, 0, k) = \lambda \) for any \( k \in \mathbb{Z} \) by the choice of \( v_k \). Now, we prove that \( f(m, 0, k) = f(m, 0, 0) \) for any \( k \in \mathbb{Z} \). Since \( [t^m_1, t^1_1] = 0 \), we have that \( t^{m_1}_1.(t^1_1.v_k) = t^{m_1}_1.(t^{m_1}_1.v_k) \) which implies

\[
f(m, 0, k) = f(m, 0, k + 1) \text{ for any } m \in \mathbb{Z}^*, k \in \mathbb{Z}.
\]

Therefore,

\[
f(m, 0, k) = f(m, 0, 0) \text{ for any } m \in \mathbb{Z}^*, k \in \mathbb{Z}.
\]

(3.4)

Next we prove the following claim.

Claim: \( f(0, 1, k) = ab^k \), \( f(0, -1, k) = \frac{(1-qb)(1-q^{-1})}{a(2-b-b^{-1})}b^{-k} \) for some \( a, b \in \mathbb{C}^* \setminus \{1\} \).
Since \([t_1^1, t_2^\pm{1}], v_k = (1 - q)(q^{-1} - 1) t_2^\pm{1}.v_k\), we obtain that

\[(2\lambda^2 + (1 - q)(q^{-1} - 1)) f(0, \pm 1, k) = \lambda^2 (f(0, \pm 1, k - 1) + f(0, \pm 1, k + 1)), \forall k \in \mathbb{Z}. \quad (3.5)\]

We have that the characteristic equation of (3.5) is as follow

\[\lambda^2 x^2 - (2\lambda^2 + (1 - q)(q^{-1} - 1)) x + \lambda^2 = 0. \quad (3.6)\]

Now we divide our proof of the claim into two cases according to whether the equation (3.6) has different roots or not.

Case I. The equation (3.6) has not different roots. Then

\[\Delta = 4\lambda^2 (1 - q)(q^{-1} - 1) + (1 - q)^2(q^{-1} - 1)^2 = 0,\]

which implies \((1 - q)(q^{-1} - 1) = -4\lambda^2\). Applying this result to (3.6), we have \(x^2 + 2x + 1 = 0\), since \(\lambda \neq 0\). Thus the root of equation (3.6) is \(-1\). By a result in [10], we obtain that there exist \(\lambda_1(\pm 1), \lambda_2(\pm 1) \in \mathbb{C}\) such that

\[f(0, \pm 1, k) = (-1)^k (\lambda_1(\pm 1)) + k\lambda_2(\pm 1), \forall k \in \mathbb{Z}. \quad (3.7)\]

Since \([t_1^1, t_2^\pm{1}], v_k = (1 - q)(1 - q^{-1}) t_1^1.v_k\), we have

\[(1 - q)(1 - q^{-1}) = (f(0, 1, k) - f(0, 1, k + 1))(f(0, -1, k) - f(0, -1, k + 1)). \quad (3.8)\]

Applying (3.7) to (3.8), we have that

\[(1 - q)(1 - q^{-1}) = 4\lambda_1(1) \lambda_1(-1) + (2k + 1)(2\lambda_2(1) \lambda_1(-1) + 2\lambda_1(1) \lambda_2(-1) + (2k + 1)\lambda_2(1)\lambda_2(-1))\]

for all \(k \in \mathbb{Z}\), which implies that

\[\lambda_2(1)\lambda_2(-1) = 0, \quad \lambda_2(1)\lambda_1(-1) + \lambda_1(1)\lambda_2(-1) = 0,\]

and

\[4\lambda_1(1)\lambda_1(-1) = (1 - q)(1 - q^{-1}).\]

Therefore \(\lambda_2(1)\lambda_1(-1) = \lambda_1(1)\lambda_2(-1) = 0\). Thus we obtain that

\[\lambda_2(1) = \lambda_2(-1) = 0 \text{ but } \lambda_1(1)\lambda_1(-1) \neq 0.\]
Hence \( f(0, 1, k) = \lambda_1(1)(-1)^k \) and \( f(0, -1, k) = \frac{(1-q)(1-q^{-1})}{4\lambda_1(1)}(-1)^k \). Thus the claim holds in this case with \( a = \lambda(1) \) and \( b = -1 \).

Case II. The equation (3.6) has different roots. Then

\[
\triangle := 4\lambda^2(1-q)(q^{-1}-1) + (1-q)^2(q^{-1}-1)^2 \neq 0.
\]

Since \( \lambda \neq 0 \), the root of equation (3.6) cannot be zero. Thus we can assume the roots of (3.6) are \( x \) and \( x^{-1} \) respectively. By a result in [10], we have that there exist \( \lambda_1(\pm 1), \lambda_2(\pm 1) \in \mathbb{C} \) such that

\[
f(0, \pm 1, k) = \lambda_1(\pm 1)x^k + \lambda_2(\pm 1)x^{-k}, \quad \forall \ k \in \mathbb{Z}. \tag{3.9}
\]

Substituting (3.9) into (3.8), we obtain that

\[
(1-q)(1-q^{-1}) = \lambda_1(-1)\lambda_1(1)(1-2x+x^2)x^{2k} + \lambda_2(-1)\lambda_2(1)(1-2x^{-1}+x^{-2})x^{-2k}

- (\lambda_1(-1)\lambda_2(1) + \lambda_2(-1)\lambda_1(1))(x+x^{-1}-2), \tag{3.10}
\]

for all \( k \in \mathbb{Z} \).

Now we prove that

\[
\lambda_1(-1)\lambda_1(1) = \lambda_2(-1)\lambda_2(1) = 0.
\]

We will divide our proof into two cases according to whether \( |x| = 1 \) or not.

Subcase 1. \(|x| = 1\).

Since the equation (3.6) has different roots, we deduce that \( x \neq \pm 1 \). If \( x^2 \neq -1 \), as the equation (3.10) holds for all \( k \in \mathbb{Z} \), then it is easy to obtain \( \lambda_1(-1)\lambda_1(1) = \lambda_2(-1)\lambda_2(1) = 0 \) by the geometry significance. If \( x^2 = -1 \), then \( x = \pm i \). Without loss of generality, we can assume \( x = i \). Substituting it into (3.9), we have

\[
f(0, \pm 1, k) = \lambda_1(\pm 1)i^k + \lambda_2(\pm 1)(-i)^k. \tag{3.11}
\]

Substituting \( x = i \) into (3.10), we have

\[
(1-q)(1-q^{-1}) = (-1)^k(2i)(-\lambda_1(-1)\lambda_1(1) + \lambda_2(-1)\lambda_2(1)) - 2(\lambda_1(-1)\lambda_2(1) + \lambda_2(-1)\lambda_1(1)), \tag{3.12}
\]
where $k \in \mathbb{Z}$. Notice that (3.12) holds for any $k \in \mathbb{Z}$, we deduce that

$$\lambda_1(-1)\lambda_1(1) = \lambda_2(-1)\lambda_2(1).$$

If $\lambda_1(-1)\lambda_1(1) = \lambda_2(-1)\lambda_2(1) \neq 0$, we can assume $\frac{\lambda_1(1)}{\lambda_2(1)} = \frac{\lambda_2(-1)}{\lambda_1(-1)} = \eta$. Then

$$f(0, 1, k) = \lambda_2(1) \left(\eta^k + (-i)^k\right), \quad f(0, -1, k) = \lambda_1(-1) \left(i^k + \eta(-i)^k\right) \quad (3.13)$$

In this condition, we can get that $f(2,0,0) \neq 0$. In fact, since $[[t_1^1, t_2^{-1}], t_1^1].v_k = \frac{q^{-1} - 1}{1 + q}f(2,0,0) \left(f(0, -1, k) - f(0, -1, k + 2)\right)$, we have

$$\lambda^2 \left((f(0, -1, k + 1) - f(0, -1, k + 2)) - (f(0, -1, k) - f(0, -1, k + 1))\right) = \frac{q^{-1} - 1}{1 + q}f(2,0,0) \left(f(0, -1, k) - f(0, -1, k + 2)\right), \quad (3.14)$$

by (3.4). Substituting (3.13) into (3.14), we have that

$$\lambda^2 f(0, -1, k + 1) = \frac{q^{-1} - 1}{1 + q}f(2,0,0) f(0, -1, k), \quad \forall k \in \mathbb{Z},$$

which implies $f(2,0,0) \neq 0$. Since $[[t_1^2, t_2^{-1}], t_1^{-1}].v_k = (1 - q^2)(1 - q^{-2})t_1^2.v_k$, we can obtain

$$(1 - q^2)(1 - q^{-2}) = (f(0, 1, k) - f(0, 1, k + 2)) \left(f(0, -1, k) - f(0, -1, k + 2)\right), \quad (3.15)$$

by (3.4). Substituting (3.11) into (3.15), we have that

$$(1 - q^2)(1 - q^{-2}) = 4(-1)^k(\lambda_1(1)\lambda_1(-1) + \lambda_2(1)\lambda_2(-1)) + 4(\lambda_1(1)\lambda_2(-1) + \lambda_2(1)\lambda_1(-1)),$$

for all $k \in \mathbb{Z}$. Thus $\lambda_1(1)\lambda_1(-1) + \lambda_2(1)\lambda_2(-1) = 0$. This together with $\lambda_1(1)\lambda_1(-1) = \lambda_2(1)\lambda_2(-1)$, we have $\lambda_1(1)\lambda_1(-1) = \lambda_2(1)\lambda_2(-1) = 0$, which is a contradiction. Therefore, we have proved that if $|x| = 1$ then $\lambda_1(1)\lambda_1(-1) = \lambda_2(1)\lambda_2(-1) = 0$.

Subcase 2. $|x| \neq 1$.

Without loss of generality, we can suppose $|x| > 1$. Then $\lim_{k \to +\infty} |x|^k = \infty$, $\lim_{k \to -\infty} |x|^k = 0$, which implies $\lim_{k \to +\infty} x^k = \infty$, $\lim_{k \to -\infty} x^k = 0$. If $\lambda_1(-1)\lambda_1(1) \neq 0$, we get that

$$\lim_{k \to +\infty} \left(\lambda_1(-1)\lambda_1(1)(1 - 2x + x^2)x^{2k} + \lambda_2(-1)\lambda_2(1)(1 - 2x^{-1} + x^{-2})x^{-2k}
- (\lambda_1(-1)\lambda_2(1) + \lambda_2(-1)\lambda_1(1))(x + x^{-1} - 2)\right) = \infty,$$
but \( \lim_{k \to +\infty} (1-q)(1-q^{-1}) \) is a scalar, which is a contradiction to (3.10). Hence, \( \lambda_1(-1)\lambda_1(1) = 0. \) One can deduce that \( \lambda_2(-1)\lambda_2(1) = 0 \) similarly.

All in all, the result that \( \lambda_1(-1)\lambda_1(1) = \lambda_2(-1)\lambda_2(1) = 0 \) is obtained.

Applying the result above to (3.10), we obtain

\[
(1-q)(1-q^{-1}) = (\lambda_1(-1)\lambda_2(1) + \lambda_2(-1)\lambda_1(1))(2-x-x^{-1}). \tag{3.16}
\]

If \( \lambda_1(1) = \lambda_2(1) = 0 \) or \( \lambda_1(-1) = \lambda_2(-1) = 0 \), substituting this result to (3.16), we have

\[
(1-q)(1-q^{-1}) = 0, \quad \text{which is absurd.}
\]

Thus

\[
\begin{cases}
\lambda_1(1) = \lambda_2(-1) = 0, \\
\lambda_1(-1) \neq 0, \lambda_2(1) \neq 0
\end{cases}
\quad \text{or} \quad
\begin{cases}
\lambda_1(-1) = \lambda_2(1) = 0, \\
\lambda_1(1) \neq 0, \lambda_2(-1) \neq 0
\end{cases}
\]

Without loss of generality, we can assume that \( \lambda_1(-1) = \lambda_2(1) = 0, \lambda_1(1) \neq 0 \) and \( \lambda_2(-1) \neq 0 \).

Substituting it into (3.16) and (3.9) respectively, we have

\[
(1-q)(1-q^{-1}) = \lambda_2(-1)\lambda_1(1)(2-x-x^{-1}), \quad \tag{3.17}
\]

\[
f(0,1,k) = \lambda_1(1)x^k; \quad f(0,-1,k) = \lambda_2(-1)x^{-k}. \tag{3.18}
\]

Setting \( \lambda_1(1) = a \), from (3.17) and (3.18) we obtain

\[
f(0,1,k) = ax^k; \quad f(0,-1,k) = \frac{(1-q)(1-q^{-1})}{a(2-x-x^{-1})} x^{-k}, \quad \forall k \in \mathbb{Z}.
\]

In a word, we have proved the claim that

\[
f(0,1,k) = ab^k \quad \text{and} \quad f(0,-1,k) = \frac{(1-q)(1-q^{-1})}{a(2-b-b^{-1})} b^{-k}
\]

for some \( a \in C^* \) where \( b^{\pm 1} \in C^* \setminus \{1\} \) are roots of the equation (3.6).

Considering that \( f(\pm 1,0,k) = \lambda \) for any \( k \in \mathbb{Z} \), by using (3.4) and the equation

\[
[[t_1^m, t_2^1], t_1^1].v_0 = \frac{(1-q^m)(q-1)}{1-q^{m+1}}[t_1^m+1, t_2^1].v_0,
\]

where \( m \neq 0, -1 \), we have

\[
\frac{(1-q^m)(q-1)}{1-q^{m+1}} (f(0,1,0) - f(0,1,m+1)) f(m+1,0,0)
\]

\[
= \lambda (f(0,1,1) - f(0,1,m+1) - f(0,1,0) + f(0,1,m)) f(m,0,0). \tag{3.20}
\]
Substituting (3.19) into (3.20), we have that
\[
\frac{1 - b^{m+1}}{1 - q^{m+1}} f(m + 1, 0, 0) = \frac{\lambda(1 - b)}{1 - q} \frac{1 - b^m}{1 - q^m} f(m, 0, 0) \quad \text{for all } m \neq 0, -1.
\] (3.21)
Thus \(b^m \neq 1\) for all \(m \in \mathbb{Z}^*\) since \(f(\pm 1, 0, 0) = \lambda \neq 0\) and \(b \neq 1\). (In fact, one can easily see that Case I discussed in page 7 would not occur from this result.) Therefore, by the equation (3.21), we deduce that
\[
f(m, 0, 0) = \begin{cases} 
\left( \frac{\lambda(1-b)}{1-q} \right)^m \frac{1-q^m}{1-b^m}, & m \geq 1; \\
\frac{q}{b} \left( \frac{\lambda(1-b)}{1-q} \right)^{m+2} \frac{1-q^m}{1-b^m}, & m \leq -1.
\end{cases}
\]
(3.22)
From (3.4) and (3.22) we get
\[
f(m, 0, k) = \begin{cases} 
\left( \frac{\lambda(1-b)}{1-q} \right)^m \frac{1-q^m}{1-b^m}, & m \geq 1; \\
\frac{q}{b} \left( \frac{\lambda(1-b)}{1-q} \right)^{m+2} \frac{1-q^m}{1-b^m}, & m \leq -1,
\end{cases}
\]
(3.23)
for any \(k \in \mathbb{Z}\).

Applying (3.19) to the equation
\[
(1 - q^m)(t_1^m t_2^1) . v_k = [t_1^m t_2^{j+1}, t_2^1] . v_k,
\]
where \(m \neq 0\), \(j \neq 1\), we have
\[
f(m, j, k) = \frac{ab^k(1 - b^m)}{1 - q^m} f(m, j - 1, k),
\]
(3.24)
where \(m \neq 0\), \(j \neq 1\). And the equation (3.24) implies
\[
f(m, j, k) = \begin{cases} 
\left( \frac{ab^k(1-b^m)}{1-q^m} \right)^{j-1} f(m, 1, k), & j \geq 1, \\
\left( \frac{ab^k(1-b^m)}{1-q^m} \right)^{j} f(m, 0, k), & j \leq 0,
\end{cases}
\]
(3.25)
where \(m \neq 0\). Using (3.19) and \([t_1^m t_2^1] . v_k = (1 - q^m)(t_1^m t_2^1) . v_k\), we obtain that
\[
f(m, 1, k) = \frac{ab^k(1 - b^m)}{1 - q^m} f(m, 0, k) \quad \text{for all } m \in \mathbb{Z}^*.
\]
(3.26)
Substituting (3.26) into (3.25) and considering \(f(m, 0, k) = f(m, 0, 0)\) for all \(k \in \mathbb{Z}\), we have
\[
f(m, j, k) = \left( \frac{ab^k(1-b^m)}{1-q^m} \right)^j f(m, 0, 0).
\]
(3.27)
Since \([t_1^j t_2^j, t_1^{-1}]_1 v_k = (q^{-j} - 1)t_2^j v_k\), we deduce that

\[
\lambda (f(1, j, k - 1) - f(1, j, k)) = (q^{-j} - 1)f(0, j, k).
\]  

(3.28)

Applying (3.27) to (3.28), we have

\[
f(0, j, k) = \frac{\lambda^2 (1 - b^j)}{q^{-j} - 1} \left( \frac{a b^{k+1}(1-b)}{1 - q} \right)^j, \quad \forall \ j \in \mathbb{Z}^*.
\]  

(3.29)

Using (3.27) and (3.29), we obtain

\[
f(m, j, k) = \begin{cases} 
\frac{\lambda^2 (1 - b)}{q^{-j} - 1} \left( \frac{a b^{k+1}(1-b)}{1 - q} \right)^j, & \text{where } m = 0, \ j \in \mathbb{Z}^*; \\
\left( \frac{a b^{k+1}(1-b^{mn})}{1 - q^m} \right)^j f(m, 0, 0), & \forall \ m \in \mathbb{Z}^*, \ j, k \in \mathbb{Z},
\end{cases}
\]  

(3.30)

where \(f(m, 0, 0)\) is given by (3.22).

From the equation \([t_1^h t_2^j, t_1^m t_2^n]_1 v_k = (q^{jm} - q^{nh})(q^{h+m+j+n} t_2^j t_2^n) v_k\) we can obtain

\[
f(m, n, k) f(h, j, k + m) - f(h, j, k) f(m, n, h + k) = (q^{jm} - q^{nh}) f(h + m, j + n, k).
\]  

(3.31)

Setting \(m = h = 1, j = 0\) and \(n = 3\) in (3.31) and using (3.22) and (3.30), we obtain

\[
\frac{(1 + b)^2}{b} = \frac{(1 + q)^2}{q},
\]  

(3.32)

which implies \(b = q\) or \(q^{-1}\). Thus the equation (3.6) has different roots \(q\) and \(q^{-1}\) by (3.19), which implies \(q + q^{-1} = \frac{2 \lambda^2 + (1-q)(q^{-1} - 1)}{\lambda^2 - 1}\). Therefore, we have \(\lambda = \pm 1\).

Substituting \(b = q\) and \(\lambda = 1\) into (3.22) and (3.30), we obtain

\[
f(m, j, k) = (aq^k)^j, \quad \forall \ (m, j) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \ k \in \mathbb{Z},
\]  

(3.33)

which deduces that the \(L\)-module \(V\) be isomorphic to \(V(a, I)\).

Substituting \(b = q\) and \(\lambda = -1\) into (3.22) and (3.30), we obtain

\[
f(m, j, k) = (-1)^m (aq^k)^j, \quad \forall \ (m, j) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \ k \in \mathbb{Z}.
\]  

(3.34)

Thus \(V\) be isomorphic to \(V(a, II)\).

Substituting \(b = q^{-1}\) and \(\lambda = 1\) into (3.22) and (3.30), we obtain

\[
f(m, j, k) = (-1)^{m+j+1} (aq^{-k-m})^j, \quad \forall \ (m, j) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \ k \in \mathbb{Z},
\]  

(3.35)
which deduces that $V$ be isomorphic to $V(a, III)$.

Substituting $b = q^{-1}$ and $\lambda = -1$ into (3.22) and (3.30), we obtain

$$f(m, j, k) = (-1)^{j+1}(aq^{-k-m})^j, \quad \forall (m, j) \in \mathbb{Z}^2\{(0, 0)\}, k \in \mathbb{Z}. \quad (3.36)$$

Therefore, $V$ be isomorphic to $V(a, IV)$.

In a word, $L$-module $V$ is isomorphic to one of the four classes of $\mathbb{Z}$-graded $L$-modules constructed in Proposition 2.1.

From Lemma 3.1 and Lemma 3.2, we obtain our main result in this paper.

**Theorem 3.3**: If $V$ is a $\mathbb{Z}$-graded $L$-module of the intermediate series, then $V$ is isomorphic to $V(a, I)$, $V(a, II)$, $V(a, III)$, $V(a, IV)$, or the direct sum of some trivial $L$-modules.

**References**

[1] E. Kirkman, C. Procesi and L. Small, A q-analog for the Virasoro algebra, Communi. Algebra, 22(1994), 3755-3774.

[2] S. Berman, Y. Gao and Y.S. Krylyuk, Quantum Tori and the structure of Elliptic Quasisimple Lie Algebras, J. Functional Analysis, 135(1996), 339-389.

[3] C. Jiang, D. Meng, The derivation algebra of the associative algebra $\mathbb{C}[X, Y, X^{-1}, Y^{-1}]$, Communi. Algebra, 26(1998), 1723-1736.

[4] V. I. Arnold, Mathematical Methods of Classical Mechanics, Springer, New York, 1978.

[5] B. de Wit, U. Marquard and H. Nicolai, Area-preserving Diffeomorphisms and Supermembrane Lorentz Invariance, Phys. Lett. B, 237(1988), 201.

[6] Y. Chen, M. Xue, W. Lin, S. Tan, The isomorphism and the automorphism group of a class of derivation Lie algebra over the quantum torus, Chinese Annals of Math., 26A(2005), 755-764.

[7] S. Eswara Rao, K. Zhao, Highest weight irreducible representations of the rank 2 quantum tori, Math. Res. Lett., 11(2004), 615-628.

[8] Y. Gao, Representations of extended affine Lie algebra coordinatized by certain quantum tori, Compositio Math., 123(2000), 1-25.

[9] Weiqiang Lin, Shaobin Tan, Graded modules over the $q$-analog Virasoro-like algebra, preprint.

[10] Rucheng Cao, Combinatorics, South China technical University press, 1999, 80-87.