RATIONAL RUIJSENAARS-SCHNEIDER HIERARCHY AND BISPECTRAL DIFFERENCE OPERATORS

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Abstract. We show that a monic polynomial in a discrete variable $n$, with coefficients depending on time variables $t_1, t_2, \ldots$ is a $\tau$-function for the discrete Kadomtsev-Petviashvili hierarchy if and only if the motion of its zeros is governed by a hierarchy of Ruijsenaars-Schneider systems. These $\tau$-functions were considered in [12], where it was proved that they parametrize rank one solutions to a difference-differential version of the bispectral problem.

1. Introduction

In [4], Airault, McKean and Moser discovered a mysterious connection between equations of KdV type and the Calogero-Moser system. They showed that the motion of the poles of a rational solution to the KdV or Boussinesq equation that vanishes at infinity is described by the Calogero-Moser system [6], with some constraint on the configuration of poles. Krichever [16] observed that the poles of the rational solutions to the KP equation that vanish at $x = \infty$, move according to the Calogero-Moser system with no constraint. Shiota [22] extended this phenomenon to the whole KP hierarchy, which combined with the work of Adler [1] led to a simple explicit formula for the $\tau$-function.

A surprising link to the above theory was observed by Duistermaat and Grünbaum [8] in connection with a problem in limited angle tomography [10], known as the bispectral problem. As originally formulated, this problem asks for which ordinary differential operators $L(x, d/dx)$ there exists a family of eigenfunctions $\Psi(x, z)$ that are also eigenfunctions for another differential operator $B(z, d/dz)$ in the “spectral parameter” $z$. In the case when the operator $L(x, d/dx)$ belongs to a rank one commutative ring of differential operators (i.e. $L$ commutes with an operator of odd order), the solution of the bispectral problem (up to translations and rescalings of $x$ and $z$) are precisely the operators which can be obtained by finitely many rational Darboux transformations from $L_0 = d^2/dx^2$. This combined with work of Adler and Moser [2] shows that the rank one solutions of the bispectral problem are exactly the rational solutions discovered in [4]. Wilson [26] proposed to extend the problem to commutative rings of differential operators. Such a ring is called bispectral when there is a joint eigenfunction of the operators in the ring that is also a joint eigenfunction of a ring of differential operators in the spectral variable. An important invariant of such a ring is

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its rank, meaning the dimension of the common space of eigenfunctions to
the operators belonging to the ring. He proved that the bispectral maximal
rank one commutative rings of differential operators are parametrized by a
sub-Grassmannian $\text{Gr}^{\text{ad}}$ of Sato’s Grassmannian $[21]$, which corresponds to
the rational solutions of the KP equation studied by Krichever $[16]$. Moreover, in a subsequent paper $[27]$, Wilson gave a beautiful explanation of
the bispectral property based on the connection with Calogero-Moser sys-
tems and their geometric description $[15]$. He also deepened the mystery
by showing that the correspondence between the Calogero-Moser and the
KP systems extends even to the locus where the particles collide with each
other. For a very nice characterization of the Grassmannian $\text{Gr}^{\text{ad}}$ in terms of
representation theory see the recent work of Horozov $[13]$. For an intriguing
connection of the above theory to noncommutative geometry see $[5]$.

In $[12]$, jointly with Luc Haine, we constructed rank one commutative
rings of difference operators in a discrete variable $n \in \mathbb{Z}$, corresponding to
a flag of nested subspaces, each of which belongs to $\text{Gr}^{\text{ad}}$. We showed that
the common eigenfunction of the operators in the ring is also the common
eigenfunction of a maximal rank one commutative ring of differential op-
erators in the spectral variable, i.e. they provide rank one solutions to a
difference-differential version of the bispectral problem. The corresponding
$\tau$-functions $\tau(n; t)$ are polynomials in $n$ and give rational solutions of the
discrete KP hierarchy.

In the present paper, we investigate the motion of the zeros of polynomial
(in $n$) $\tau$-functions of the discrete KP hierarchy. We show that a monic
polynomial in $n$ of degree $N$ is a $\tau$-function for the discrete KP hierarchy
if and only if the motion of its roots $\{x_i\}_{i=1}^N$ is governed by a hierarchy
of Ruijsenaars-Schneider systems. We restrict our attention to the generic
situation when the roots satisfy the constraints $x_i - x_j \notin \{0, 1\}$ for $i \neq
j$. This condition means that the rational solution to the simplest zero
curvature (Zakharov-Shabat) equation for the discrete KP hierarchy has $2N$
distinct poles. It is a challenging problem to investigate the more general
case allowing collisions of the poles.

The paper can be thought of as a discrete analog of Shiota’s paper $[22]$. In
particular, from the proof, we can easily write an explicit formula for the
$\tau$-functions in terms of the Ruijsenaars-Schneider hierarchy, which implies
that they parametrize the rank one solutions of the difference-differential
version of the bispectral problem constructed in $[12]$.

We note that there is a related work of van Diejen $[24]$ in the case of
second-order difference operators, where the dynamics of the zeros of the
solitonic Baker-Akhiezer function in the spectral variable $z$ is studied. For
soliton solutions of KP and 2D Toda equations, see Ruijsenaars $[19]$ and
van Diejen-Puschmann $[25]$, and for elliptic generalizations see Krichever-
Zabrodin $[17]$. For a $q$-deformation of the KP hierarchy and connections
with the bispectral problem see $[14]$.

The paper is organized as follows. In the next section, we briefly introduce
the necessary ingredients of the discrete KP hierarchy. The approach follows
closely [12], which leads to \( \tau \)-functions that differ by an exponential factor from the ones constructed in [3, 23]. In Section 3 we formulate the main result of the paper and its connection to the bispectral problem. For a very nice account on the difference-differential version of the bispectral problem and its relations to orthogonal polynomials and the Toda lattice see [11]. Section 4 is devoted to the proof of the main result.

2. THE DISCRETE KP HIERARCHY AND \( \tau \)-FUNCTION

We denote by \( \Delta \) and \( \nabla \) the customary forward and backward difference operators acting on functions of a discrete variable \( n \in \mathbb{Z} \) by

\[
\Delta f(n) = f(n + 1) - f(n) \quad \text{and} \quad \nabla f(n) = f(n) - f(n - 1).
\]

The formal adjoint to \( \Delta \) is \( \Delta^* = -\nabla \). If we define

\[
\Delta^j \cdot f(n) = \sum_{i=0}^{\infty} \binom{j}{i} (\Delta^i f)(n + j - i) \Delta^{j-i}, \quad \text{for all } j \in \mathbb{Z},
\]

we obtain an associative ring of formal pseudo-difference operators

\[
R\{\Delta\} = \left\{ X = \sum_{j=-\infty}^{d} a_j(n) \Delta^j \right\}.
\]

We denote by \( X_+ = \sum_{j=0}^{d} a_j(n) \Delta^j \) the positive difference part of \( X \) and by \( X_- = \sum_{j=-\infty}^{-1} a_j(n) \Delta^j \), the Volterra part of \( X \).

The discrete Kadomtsev-Petviashvili hierarchy (in short KP) is the family of evolution equations in infinitely many time variables \( t = (t_1, t_2, t_3, \ldots) \) given by the Lax equations

\[
\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \quad (2.1)
\]

where \( L \) is a general formal pseudo-difference operator of the form

\[
L = \Delta + \sum_{j=0}^{\infty} a_j(n) \Delta^{-j}.
\]

A \( \tau \)-function for the hierarchy \( (2.1) \) can be defined as follows. First, we define a wave operator

\[
W(n; t) = 1 + \sum_{j=1}^{\infty} w_j(n; t) \Delta^{-j},
\]

which conjugates \( L \) to \( \Delta \), that is

\[
L = W \Delta W^{-1}. \quad (2.2)
\]

The vector fields \( (2.1) \) can be extended by

\[
\frac{\partial W}{\partial t_k} = -(L^k)_- W. \quad (2.3)
\]
For simplicity, we denote by \( \text{Exp}(n; t, z) \) the exponential function

\[
\text{Exp}(n; t, z) = (1 + z)^n \exp \left( \sum_{i=1}^{\infty} t_i z^i \right).
\]

The wave function \( w(n; t, z) \) and the adjoint wave function \( w^*(n; t, z) \) of the discrete KP hierarchy (2.1) are defined by

\[
w(n; t, z) = W(n; t) \text{Exp}(n; t, z) = \left( 1 + \frac{w_1(n; t)}{z} + \frac{w_2(n; t)}{z^2} + \cdots \right) \text{Exp}(n; t, z)
\]

(2.4a)

and

\[
w^*(n; t, z) = (W^{-1}(n - 1; t))^* \text{Exp}^{-1}(n; t, z) = \left( 1 + \frac{w^*_1(n; t)}{z} + \frac{w^*_2(n; t)}{z^2} + \cdots \right) \text{Exp}^{-1}(n; t, z).
\]

(2.4b)

The functions \( w(n; t, z) \) and \( w^*(n; t, z) \) can be written in terms of a \( \tau \)-function as follows

\[
w(n; t, z) = \frac{\tau(n; t - [z^{-1}])}{\tau(n; t)} \text{Exp}(n; t, z),
\]

(2.5a)

and

\[
w^*(n; t, z) = \frac{\tau(n; t + [z^{-1}])}{\tau(n; t)} \text{Exp}^{-1}(n; t, z),
\]

(2.5b)

where \([z] = (z, z^2/2, z^3/3, \ldots)\). We refer the reader to [12] for more details and proofs of the above construction.

**Remark 2.1.** It is well known (see for instance [7, Proposition 5.1.4, p. 75]) that the Lax equations (2.1) imply the zero curvature (Zakharov-Shabat) equations

\[
\partial_t (L^k) - \partial_m (L^m) = [(L^m)_+, (L^k)_+],
\]

(2.6)

where \( k, m \in \mathbb{N} \). In the differential case, the first flow \( t_1 \) corresponds to a translation in the spatial variable and therefore the simplest (nontrivial) zero curvature equation can be obtained for \( m = 2 \) and \( k = 3 \). This leads to the KP equation, which gave the name of the whole hierarchy. In the discrete case, the first flow is no longer trivial and the simplest zero curvature equation will correspond to the choice \( m = 2 \) and \( k = 1 \). In the example below we carry out this computation explicitly, which leads to a nonlinear partial differential-difference equation for the function \( a_0(n; t) \).

**Example 2.2.** Let us take \( m = 2 \) and \( k = 1 \) in (2.6). Clearly, \( (L)_+ = \Delta + a_0(n; t) \) and a short computation shows that

\[
(L^2)_+ = \Delta^2 + (a_0(n; t) + a_0(n + 1, t)) \Delta + a_2(n; t) + a_1(n, t) + a_1(n + 1, t).
\]

Next, we see that

\[
[(L^2)_+, (L)_+] = \left( \Delta^2 a_0(n, t) - \Delta (a_1(n; t) + a_1(n + 1; t)) \right) (\Delta + 1).
\]
Using the relations above and comparing the coefficients of $\Delta^i$ for $i = 0, 1$ in (2.6) with $m = 2$ and $k = 1$ we get the system
\[
- \frac{\partial}{\partial t_2} \left( a_0(n; t) + a_0(n + 1; t) \right) = \Delta^2 a_0(n; t) - \Delta (a_1(n; t) + a_1(n + 1; t))
\]
\[
\frac{\partial a_0(n; t)}{\partial t_2} = \frac{\partial}{\partial t_1} \left( a_0^2(n; t) + a_1(n; t) + a_1(n + 1; t) \right)
\]
\[
= \Delta^2 a_0(n; t) - \Delta (a_1(n; t) + a_1(n + 1; t)).
\]
Eliminating $a_1(n; t)$ we obtain the following equation for $a_0 = a_0(n; t)$
\[
\frac{\partial}{\partial t_2} \Delta a_0 = \frac{\partial}{\partial t_1} (\Delta a_0^2 - 2\Delta a_0) + \frac{\partial^2}{\partial t_1^2} (\Delta a_0 + 2a_0).
\] 

(2.7)

3. Polynomial $\tau$-functions and the dynamics of their zeros

The main result of the paper is the following theorem.

**Theorem 3.1.** Let $x_1(t), x_2(t), \ldots, x_N(t)$ be smooth functions of $t = (t_1, t_2, \ldots)$ such that $x_i(t) - x_j(t) \notin \{0, 1\}$ for $i \neq j$ and $\partial x_i(t)/\partial t_1 \neq 0$ in a neighborhood of $t = 0$. Let us define functions $y_1(t), y_2(t), \ldots, y_N(t)$ by the following relation
\[
e^{-y_i(t)} = -\frac{\partial x_i(t)}{\partial t_1} \prod_{s=1 \atop s \neq i}^N \frac{x_i(t) - x_s(t)}{x_i(t) - x_s(t) + 1}.
\] 

(3.1)

Then the following conditions are equivalent.

(i) The function
\[
\tau(n; t) = \prod_{i=1}^N (n - x_i(t)),
\] 

(3.2)
is a $\tau$-function for the discrete KP hierarchy (2.1).

(ii) The motion of $\{x_i(t), y_i(t)\}_{i=1}^N$ is governed by the Ruijsenaars-Schneider hierarchy of Hamiltonian systems

\[
\frac{\partial}{\partial t_k} \begin{pmatrix} x_i \\ y_i \end{pmatrix} = (-1)^k \begin{pmatrix} \partial H_k/\partial y_i \\ -\partial H_k/\partial x_i \end{pmatrix}, \quad k = 1, 2, \ldots,
\] 

(3.3)
where $H_k = \text{tr}(Y^k)$, and $Y$ is an $N \times N$ matrix with entries
\[
Y_{ij} = \delta_{ij} + \frac{e^{-y_i}}{x_i - x_j} - 1 \prod_{s=1 \atop s \neq i}^N \frac{x_i - x_s + 1}{x_i - x_s}.
\] 

(3.4)

**Remark 3.2.** Although the Hamiltonian system above differs slightly from the standard rational Ruijsenaars-Schneider system, one can easily connect
Indeed, the rational Ruijsenaars-Schneider model is a dynamical system, whose equations of motion can be written in the following form

\[ \ddot{q}_j = 2 \sum_{k=1 \atop k \neq j}^N \frac{\gamma^2}{(\gamma^2 + (q_j - q_k)^2)(q_j - q_k)}, \quad j = 1, 2, \ldots, N, \quad (3.5) \]

see [20, formulas (B22)-(B23), p. 402]. If we now put \( q_j = i\gamma x_j \), then (3.5) gives precisely the dynamical system (3.3) for the first flow \( \partial/\partial t_1 \), see equation (4.7).

Remark 3.3. The solutions of the KP equation

\[ \frac{3}{4} u_{yy} = \left\{ u_t - \frac{1}{4}(u_{xxx} + 6uu_x) \right\}_x \]

that are rational in \( x \) and vanish as \( x \to \infty \) have the form

\[ u(x, y, t) = -2 \sum_{j=1}^N \frac{1}{(x - x_j(y, t))^2}. \]

When all \( x_j \) are distinct their motion is governed by the Calogero-Moser system, as shown in [16]. The case discussed in the present paper is similar in the following sense: the solutions described in Theorem 3.1 provide rational solutions of equation (2.7), which vanish as \( n \to \infty \). These solutions have simple poles (as functions of \( n \)) at the points \( \{x_j, x_j - 1\}_{j=1}^N \), see formula (4.3). The condition \( x_i - x_j \notin \{0, 1\} \) for \( i \neq j \) simply means that all these poles are distinct.

As a consequence of the proof of Theorem 3.1 we also obtain an explicit formula for the \( \tau \)-function in terms of \( \{x_i, y_i\}_{i=1}^N \) at \( t_1 = t_2 = \cdots = 0 \). Let us denote by \( X \) the diagonal matrix with entries \( x_i(t) \), i.e.

\[ X = \text{diag}(x_1(t), x_2(t), \ldots, x_N(t)). \quad (3.6) \]

Corollary 3.4. Let \( X^0 \) and \( Y^0 \) be the matrices \( X \) and \( Y \), defined by (3.6) and (3.4) at \( t_1 = t_2 = \cdots = 0 \). Then the \( \tau \)-function in equation (3.2) can be computed from the following formula

\[ \tau(n; t) = \det \left( nI - X^0 + \sum_{j=1}^\infty jt_j(I - Y^0)(-Y^0)^{j-1} \right), \quad (3.7) \]

where \( I \) is the identity \( N \times N \) matrix.

From formula (3.7) it is easy to see that \( \tau(n; t) = \tau(0; t_1 + n, t_2 - n/2, t_3 + n/3, \ldots) \), where \( \tau(0; t) \) is a \( \tau \)-function for the (continuous) KP hierarchy, corresponding to a plane in Wilson’s adelic Grassmannian. Thus, the results in [12] imply that the functions \( \tau(n; t) \) described in Theorem 3.1 parametrize rank-one solutions to a difference-differential version of the bispectral problem. More precisely, there exist a rank-one commutative ring \( A \) of difference operators in the variable \( n \), and a rank-one commutative ring \( A' \) of
differential operators in $z$, such that

$$L w(n; t, z) = f_L(z)w(n; t, z), \quad \forall L \in A$$

$$B w(n; t, z) = g_B(n)w(n; t, z), \quad \forall B \in A'$$

where $f_L(z)$ and $g_B(n)$ are functions of $z$ and $n$, respectively, and $w(n; t, z)$ is the wave function defined by \((2.5a)\).

4. Proof of Theorem 3.1

The strategy of the proof is as follows. For the implication (i)$\Rightarrow$(ii), we investigate the motion of the poles with respect to the first flow $\partial / \partial t_1$ and we write the corresponding dynamical system in an appropriate Lax form. This represents a discrete analog of some of the results in [16, 18], except that in the continuous case the first nontrivial flow is $\partial / \partial t_2$. Next, we adapt the approach in [22] to establish the Hamiltonian equations for the higher flows $\partial / \partial t_k, \ k \geq 2$. The opposite direction can be deduced by using the connection between polynomial (in $t_1$) $\tau$-functions of the KP hierarchy and polynomial (in $n$) $\tau$-functions of the discrete KP hierarchy [12].

Let us start with the implication (i)$\Rightarrow$(ii). From equations \((2.4), (2.5)\) and \((3.2)\) it is clear that we can write $w_k(n; t)$ and $w_k^*(n; t)$ as

$$w_k(n; t) = \sum_{i=1}^{N} \frac{w_{k,i}(t)}{n - x_i(t)} \tag{4.1a}$$

$$w_k^*(n; t) = \sum_{i=1}^{N} \frac{w_{k,i}^*(t)}{n - x_i(t)} \tag{4.1b}$$

In particular, for $k = 1$ we see that $w_{1,i}(t) = \partial x_i(t) / \partial t_1$ and $w_{1,i}^*(t) = -\partial x_i(t) / \partial t_1$. From \((2.2), (2.3)\) and \((2.4a)\) it follows that

$$\frac{\partial w(n; t, z)}{\partial t_1} = (\Delta + a_0(n; t))w(n; t, z). \tag{4.2}$$

Writing \((2.2)\) as $LW = W\Delta$ and comparing the coefficients of $\Delta^0$ on both sides gives

$$a_0(n; t) = -w_1(n + 1; t) + w_1(n; t)$$

$$= \sum_{i=1}^{N} \frac{1}{(n - x_i(t))(n + 1 - x_i(t))} \frac{\partial x_i(t)}{\partial t_1} \tag{4.3}$$
where in the last equality we used (4.1a) for \( k = 1 \). Plugging the last formula for \( a_0(n; t) \) in (4.2) and using (4.1a) we get the following identity
\[
\sum_{i=1}^{N} \left( \frac{w_{k+1,i}(t)}{n-x_i(t)} + \frac{1}{n-x_i(t)} \frac{\partial w_{k,i}(t)}{\partial t_1} + \frac{w_{k,i}(t)}{(n-x_i(t))^2} \frac{\partial x_i(t)}{\partial t_1} \right)
= \sum_{i=1}^{N} \left( \frac{w_{k+1,i}(t)}{n+1-x_i(t)} + \frac{w_{k,i}(t)}{n+1-x_i(t)} \frac{\partial x_i(t)}{\partial t_1} - \frac{w_{k,i}(t)}{n-x_i(t)} \right) + \left( \sum_{i=1}^{N} \frac{1}{n-x_i(t)} \frac{\partial x_i(t)}{\partial t_1} \right) \left( \sum_{i=1}^{N} \frac{w_{k,i}(t)}{n-x_i(t)} \right).
\] (4.4)

Notice that (4.4) can be rewritten as a polynomial identity in \( n \), which is true for every \( n \in \mathbb{Z} \) and therefore, it will be true for every \( n \in \mathbb{C} \). Computing the residue at \( n = x_i(t) - 1 \) we obtain
\[
-w_{k+1,i}(t) = w_{k,i} - \sum_{j=1}^{N} \frac{w_{k,j}(t)}{x_i(t) - x_j(t) - 1} \frac{\partial x_i(t)}{\partial t_1}.
\] (4.5)

If we denote \( \vec{w}(t) = (w_{k,1}(t), w_{k,2}(t), \ldots, w_{k,N}(t))^t, \vec{e} = (1, 1, \ldots, 1)^t \), then the last formula can be rewritten in vector notations as \( \vec{w}_{k+1}(t) = (-Y)\vec{w}_k(t) \), where \( Y \) is the matrix defined in Theorem 3.1. Thus we see that \( \vec{w}_k(t) = (-Y)^{k-1} \frac{\partial X}{\partial t_1} \vec{e}, \) (4.6)

where \( X \) is the diagonal matrix given in equation (3.6).

Computing also the residue of equation (4.4) at \( n = x_i(t) \) we obtain
\[
w_{k+1,i}(t) + \frac{\partial w_{k,i}(t)}{\partial t_1} = -w_{k,i}(t) + \frac{\partial x_i(t)}{\partial t_1} \sum_{j=1}^{N} \frac{w_{k,j}(t)}{x_i(t) - x_j(t)}
+ w_{k,i}(t) \left( -\frac{\partial x_i(t)}{\partial t_1} + \sum_{j=1}^{N} \frac{1}{(x_i(t) - x_j(t))(x_i(t) + 1 - x_j(t))} \frac{\partial x_j(t)}{\partial t_1} \right).
\]

Using the last identity and (4.5) we can eliminate \( w_{k+1,i}(t) \). For \( k = 1 \) this leads to the following second-order differential equation for \( x_i(t) \)
\[
\frac{\partial^2 x_i(t)}{\partial t_1^2} = -2\frac{\partial x_i(t)}{\partial t_1}
\sum_{j=1}^{N} \frac{1}{(x_i(t) - x_j(t))(x_i(t) - x_j(t) + 1)(x_i(t) - x_j(t) - 1)} \frac{\partial x_j(t)}{\partial t_1},
\] (4.7)

which will be needed later.

Similarly, if we work with the adjoint wave function \( w^*(n; t, z) \) we can show that it satisfies the following equation
\[
\frac{\partial w^*(n; t, z)}{\partial t_1} = (\nabla - a_0(n - 1; t))w^*(n; t, z).
\]
Denoting $\vec{w}_k^*(t) = (w_{k,1}^*(t), w_{k,2}^*(t), \ldots, w_{k,N}^*(t))^t$ we obtain as above that
\[
\vec{w}_k^*(t) = -\frac{\partial X}{\partial t_1}(-Y^t)^{k-1} \vec{e}.
\] (4.8)

Next, we use (2.3). From equations (2.2) and (2.4) we deduce that
\[
L_k = \Sigma_{i,j=0}^{\infty} w_{k,j}^*(n,t) \Delta^{k-j} \cdot w_j^*(n+1,t),
\]
where $w_0(n,t) = w_0^*(n,t) = 1$. This shows that
\[
(L_k) - = \Sigma_{j=0}^{k+1} w_{k+1-j}(n,t) w_j^*(n,t) \Delta^{-1} + O(\Delta^{-2}).
\]

On the other hand
\[
W(n,t) = 1 + \left( \Sigma_{i=1}^{N} \frac{1}{n-x_i(t)} \frac{\partial x_i(t)}{\partial t_1} \right) \Delta^{-1} + O(\Delta^{-2}).
\]

Plugging the last two formulas in (2.3), and equating the coefficients of $\Delta^{-1}$ on both sides we get
\[
\Sigma_{i=1}^{N} \left( \frac{1}{n-x_i(t)} \frac{\partial^2 x_i(t)}{\partial t_1 \partial t_k} + \frac{1}{(n-x_i(t))^2} \frac{\partial x_i(t)}{\partial t_1} \frac{\partial x_i(t)}{\partial t_k} \right)
= - \Sigma_{j=0}^{k+1} w_{k+1-j}(n,t) w_j^*(n,t).
\]

The last equality holds for every $n \in \mathbb{Z}$ and therefore it must hold for every $n \in \mathbb{C}$. Comparing the coefficients of $(n-x_i(t))^{-2}$ gives
\[
\frac{\partial x_i(t)}{\partial t_1} \frac{\partial x_i(t)}{\partial t_k} = - \Sigma_{j=1}^{k} w_{k+1-j,i}(t) w_j^*(t).
\]

Let us denote by $I_i$ the elementary $N \times N$ matrix having 1 at entry $(i,i)$ and 0 everywhere else. Using the last identity, (4.6) and (4.8) we obtain
\[
\frac{\partial x_i(t)}{\partial t_1} \frac{\partial x_i(t)}{\partial t_k} = - \Sigma_{j=1}^{k} \vec{w}_{k+1-j,i}(t) I_i \vec{w}_j^*(t)
= (-1)^{k+1} \Sigma_{j=1}^{k} e^{tY-j^{-1}} I_i Y^{k-j} \frac{\partial X}{\partial t_1} \vec{e}.
\]

Notice that $\partial X/\partial t_1 I_i = \partial x_i(t) / \partial t_1 I_i$ and therefore, we can cancel $\partial x_i(t) / \partial t_1$ and the last formula reduces to
\[
\frac{\partial x_i(t)}{\partial t_k} = (-1)^{k+1} \Sigma_{j=1}^{k} e^{tY-j^{-1}} I_i Y^{k-j} \frac{\partial X}{\partial t_1} \vec{e}.
\]
For every $N \times N$ matrix $A$ we have $\vec{e}^t A \vec{e} = \text{tr}(A \vec{e} \vec{e}^t)$, and thus we can rewrite the last formula for $\frac{\partial x_i(t)}{\partial t_k}$ as follows

$$\frac{\partial x_i(t)}{\partial t_k} = (-1)^{k+1} \text{tr} \left( \sum_{j=1}^{k} Y^{j-1} I_i Y^{k-j} \frac{\partial X}{\partial t_1} \right).$$

From the definitions of matrices $X$ and $Y$ it is easy to see that $\frac{\partial X}{\partial t_1} \vec{e} \vec{e}^t = -(XY - YX - Y + I)$. Making this substitution and using the fact that $\text{tr}(AB) = \text{tr}(BA)$ we get

$$\frac{\partial x_i(t)}{\partial t_k} = (-1)^k \text{tr} \left( \sum_{j=1}^{k} I_i Y^{k-j} (XY - YX - Y + I) \right)$$

$$= (-1)^k \text{tr} \left( I_i Y^{k-j} (XY - YX - Y + I) \right)$$

$$= (-1)^k \text{tr}(I_i (XY^k - Y^{k+1}X - kY^k + kY^{k-1})).$$

However, $I_i X = XI_i$ and therefore $\text{tr}(I_i XY^k) = \text{tr}(I_i Y^k X)$. This shows that

$$\frac{\partial x_i(t)}{\partial t_k} = k(-1)^k \text{tr}((I_i - I_i Y) Y^{k-1}). \quad (4.9)$$

On the other hand it is easy to see that

$$\frac{\partial Y}{\partial y_i} = I_i - I_i Y.$$

Thus

$$\frac{\partial}{\partial y_i} \text{tr}(Y^k) = \text{tr} \left( \sum_{j=1}^{k} Y^{j-1} \frac{\partial Y}{\partial y_i} Y^{k-j} \right) = \text{tr} \left( \sum_{j=1}^{k} \frac{\partial Y}{\partial y_i} Y^{k-j} \right)$$

$$= k \text{tr} \left( \frac{\partial Y}{\partial y_i} Y^{k-1} \right) = k \text{tr}((I_i - I_i Y) Y^{k-1}).$$

The last formula combined with (4.9) gives the first equation in (3.3). In order to prove that the second equation holds, we first notice that (1.7) is equivalent to the Lax equation

$$\frac{\partial Y}{\partial t_1} = [Y, M], \quad (4.10)$$

where $M$ is an $N \times N$ matrix with entries

$$M_{i,j} = \frac{1}{x_i(t) - x_j(t)} \frac{\partial x_i(t)}{\partial t_1} \quad \text{for} \quad i \neq j$$

$$M_{i,i} = \sum_{k=1}^{N} \frac{1}{x_i(t) - x_k(t) + 1} \frac{\partial x_k(t)}{\partial t_1} - \sum_{k=1}^{N} \frac{1}{x_i(t) - x_k(t)} \frac{\partial x_k(t)}{\partial t_1}.$$
Differentiating (4.1) with respect to $t_k$ we get

$$\frac{\partial y_i(t)}{\partial t_k} = - \frac{\partial}{\partial t_k} \log \left( \frac{\partial x_i(t)}{\partial t_1} \right) + \frac{\partial}{\partial t_k} \log \left( \prod_{s=1}^{N} \frac{x_i(t) - x_s(t) + 1}{x_i(t) - x_s(t)} \right). \quad (4.11)$$

The derivative in the second term on the right-hand side of (4.11) can be evaluated using (4.9)

$$\frac{\partial}{\partial t_k} \log \left( \prod_{s=1}^{N} \frac{x_i(t) - x_s(t) + 1}{x_i(t) - x_s(t)} \right) = k(-1)^k \text{tr} \left( \sum_{j=1}^{N} \left( \frac{\partial}{\partial x_j} \log \prod_{s=1}^{N} \frac{x_i(t) - x_s(t) + 1}{x_i(t) - x_s(t)} \right) I_j (I - Y) Y^{k-1} \right).$$

For the first term we use both (4.9) and (4.10):

$$\frac{\partial}{\partial t_k} \log \left( \frac{\partial x_i(t)}{\partial t_1} \right) = k(-1)^k \left( \frac{\partial x_i(t)}{\partial t_1} \right)^{-1} \text{tr} \left( I_i Y^{k-1} - I_i Y^k \right)$$

$$= k(-1)^k \left( \frac{\partial x_i(t)}{\partial t_1} \right)^{-1} \text{tr} \left( \sum_{j=1}^{k-1} I_i Y^{j-1}[Y, M] Y^{k-1-j} - \sum_{j=1}^{k} I_i Y^{j-1}[Y, M] Y^{k-j} \right)$$

$$= k(-1)^k \left( \frac{\partial x_i(t)}{\partial t_1} \right)^{-1} \text{tr} \left( (MI_i(I - Y) - (I - Y)I_iM) Y^{k-1} \right).$$

To simplify the formulas, let us denote $\hat{Y} = I - Y$ and $(\hat{M})_{i,j} = (1 - \delta_{i,j})M_{i,j}$ (i.e. $\hat{M}$ is the matrix obtained from $M$ by replacing the diagonal entries with zeros). Then, the last two formulas combined with (4.11) show that

$$\frac{\partial y_i(t)}{\partial t_k} = k(-1)^{k+1} \text{tr}(BY^{k-1}), \quad (4.12)$$

where

$$B = \left( \frac{\partial x_i(t)}{\partial t_1} \right)^{-1} (\hat{M}I_i \hat{Y} - \hat{Y}I_i \hat{M}) - \sum_{j=1}^{N} \left( \frac{\partial}{\partial x_j} \log \prod_{s=1}^{N} \frac{x_i(t) - x_s(t) + 1}{x_i(t) - x_s(t)} \right) I_j \hat{Y}.$$  

A straightforward computation now shows that

$$B = \frac{\partial Y}{\partial x_i} + \left[ \sum_{j=1}^{N} \frac{1}{x_j - x_i + 1} I_j - \sum_{j=1}^{N} \frac{1}{x_j - x_i} I_j, Y \right],$$

which combined with (4.12) gives

$$\frac{\partial y_i(t)}{\partial t_k} = k(-1)^{k+1} \text{tr} \left( \frac{\partial Y}{\partial x_i} Y^{k-1} \right) = (-1)^{k+1} \frac{\partial}{\partial x_i} \text{tr}(Y^k),$$
completing the proof of (3.3).

Conversely, assume now that (3.3) holds. Let us consider \( \{x_i(t), y_i(t)\}_{i=1}^N \) and the corresponding matrices \( X \) and \( Y \) at the initial time \( t_1 = t_2 = \cdots = 0 \) and let us denote \( x_i^0 = x_i(0), y_i^0 = y_i(0) \), \( X^0 = X|_{t=0} \), \( Y^0 = Y|_{t=0} \). Notice that
\[
\text{rank}(X^0 Y^0 - Y^0 X^0 + I - Y^0) = 1.
\]

Using the Cauchy determinant formula we see that
\[
\det(I - Y^0) = e^{-\sum_{i=1}^N y_i^0} \neq 0.
\]
Thus, if we denote \( \tilde{X}^0 = X^0(I - Y^0)^{-1} \) we have
\[
\text{rank}([\tilde{X}^0, Y^0] + I) = 1.
\]

Therefore the pair \( (\tilde{X}^0, Y^0) \) defines a plane in Wilson’s adelic Grassmannian \( \text{Gr}_{ad}^{\text{ad}} \), see [27]. The corresponding \( \tau \)-function can be computed by Shiotō’s formula
\[
\tilde{\tau}^0(t) = \det \left( -\tilde{X}^0 + \sum_{j=1}^{\infty} j t_j (-Y^0)^{j-1} \right),
\]
see [22] Corollary 1, p. 5845. Applying [12] Theorem 2.4, p. 290 we deduce that \( \tilde{\tau}^0(t_1 + n, t_2 - n/2, t_3 + n/3, \ldots) \) is a \( \tau \)-function for the discrete KP hierarchy (2.11). Multiplying by the nonzero constant factor \( \det(I - Y^0) \) we see that
\[
\tilde{\tau}(n; t) = \det(I - Y^0)^{\tilde{\tau}^0(t_1 + n, t_2 - n/2, t_3 + n/3, \ldots)}
\]
\[
= \det(n I - X^0 + \sum_{j=1}^{\infty} j t_j (I - Y^0)(-Y^0)^{j-1})
\]
is a \( \tau \)-function for the discrete KP hierarchy. Clearly, \( \tilde{\tau}(n; t) \) is a monic polynomial in \( n \), and therefore, by the first part of theorem, its roots \( \tilde{x}_j(t) \) and the corresponding \( \tilde{y}_j(t) \) will satisfy the Hamiltonian systems (3.3). To complete the proof we show that \( \tau(n; t) \) given by (3.2) coincides with \( \tilde{\tau}(n; t) \) defined above. Since the roots of \( \tau(n; t) \) and \( \tilde{\tau}(n; t) \) satisfy the same systems (3.3), it is enough to show that \( x_j^0 = \tilde{x}_j(0) \) and \( y_j^0 = \tilde{y}_j(0) \). This follows easily from the explicit formula for \( \tilde{\tau}(n; t) \):
\[
\tilde{\tau}(n; t_1, 0, 0, \ldots) = \det(n I - X^0 + t_1 (I - Y^0))
\]
\[
= \prod_{j=1}^{N} \left( n - x_j^0 + t_1 e^{-y_j^0} \prod_{s=1, s \neq j}^{N} \frac{x_j^0 - x_s^0 + 1}{x_j^0 - x_s^0} \right) + O(t_1^2).
\]

Remark 4.1. It would be interesting to see if one can use the explicit formulas for \( \tau \)-functions of KP hierarchy in terms of matrices satisfying rank one conditions [9] and the construction of \( \tau \)-functions for \( q \)-KP hierarchy from classical ones to extend the above proof and to show that every solution of the \( q \)-deformed Calogero-Moser hierarchy described in [14] Theorem 6.1] leads to a \( \tau \)-function for \( q \)-KP. This would give a one to one correspondence between rational solutions to the \( q \)-KP hierarchy (which also parametrize
rank one solutions to a bispectral problem for $q$-difference operators) and $q$-deformed Calogero-Moser type systems.

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