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Dynamics of an epidemic model with spatial diffusion

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HIGHLIGHTS
- We obtained a spatial epidemic model with logistic growth.
- Using multiple-scale analysis, we present amplitude equations.
- There are different types of stationary patterns.
- Reaction diffusion epidemic systems have rich dynamics.

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ABSTRACT
Mathematical models are very useful in analyzing the spread and control of infectious diseases which can be used to predict the developing tendency of the infectious disease, determine the key factors and to seek the optimum strategies of disease control. As a result, we investigated the pattern dynamics of a spatial epidemic model with logistic growth. By using amplitude equation, we found that there were different types of stationary patterns including spotted, mixed, and stripe patterns, which mean that spatial motion of individuals can form high density of diseases. The obtained results can be extended in other related fields, such as vegetation patterns in ecosystems.

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1. Introduction

With the development of human civilization, infectious diseases have been effectively controlled. However, in some areas, especially in developing countries these infectious cases often appear. We take SARS (Severe Acute Respiratory Syndrome) as an example. From November in 2002 to May in 2003, the firstly familial aggregation case occurred in China and then the SARS showed the tendency of a rapid spread. SARS cases were found in more than 30 countries and regions around the world which threaten globally human health [1]. In February 2013, a new type of avian influenza, H7N9, appeared in Mainland China which caused more than 100 people to be infected [2]. As a result, it is also necessary to investigate the mechanisms of disease spreading.

Mathematical models have been important methods in disease control which can be used to determine the spread and seek the best strategies to prevent and control its spread. The transmission or other interactions formation of many important epidemiological phenomenon are largely affected by space interaction [3,4] and thus many studies have proposed spatial epidemic models. In these studies, the reaction–diffusion equations have been used which belong to temporal and spatial forms. The reaction term is the process of individuals changes or the interaction among species in the absence of diffusion, and the diffusion term describes the spatial movement of individuals [5,6].

Pattern structures are non-uniform macro-structures having a regularity in space or time which widely exist in the real life. Viewed from the perspective of thermodynamics, it can be divided into two categories. The first category is presented under thermodynamic equilibrium conditions, such as the crystal structure in inorganic chemistry, and self-organized pattern...
formation in organic polymers. The second category is not in thermodynamic equilibrium conditions. For the first one, its mechanism is more systematic and well understood. Such pattern formation can be explained by the equilibrium thermodynamics and statistical physics [7,8].

In this article, we will use the standard multi-scale analysis to study pattern selection of a spatial epidemic model, where the parameter controlled method is used to study a small parameter $\varepsilon$ and the Fredholm solvability condition. In the vicinity of the bifurcation point (such as the Turing and Hopf bifurcations) the critical amplitude $A_j (j = 1, 2, 3)$ follows the general form, and its standard form can be derived by standard technically analysis and symmetry breaking theory [9–11]. Normal forms of the critical amplitude can be well applied to the study of pattern formation and the subtle changes of the pattern formation are derived through appropriately spatial symmetry terms [12].

2. A spatial epidemic model

To begin this section, we firstly give some assumptions, which are as follows.

(i) The population, in which a pathogenic agent is active, comprises two subgroups: the healthy individuals who are susceptible (S) to infection and the already infected individuals (I) who can transmit the disease to the healthy ones.

(ii) Infected populations do not migrate and the disease-related death rate from the infected is $d$.

(iii) All the parameters are constant which means they do not depend on the space and time.

Denote $S + I = N$ and the model is as follows:

$$\begin{align*}
\frac{dS}{dt} &= r N \left( 1 - \frac{N}{K} \right) - \beta SI \frac{1}{N} - (\mu + m)S, \\
\frac{dI}{dt} &= \beta SI \frac{1}{N} - (\mu + d)I,
\end{align*}$$

where $r$ is the intrinsic growth rate, $K$ is the carrying capacity, $\beta$ denotes the contact transmission rate, $\mu$ is the natural mortality, $d$ denotes the disease-induced mortality, $m$ is the per-capita emigration rate of the susceptible. Here, $\frac{dN}{dt} = r N \left( 1 - \frac{N}{K} \right)$ means population $N$ satisfies logistic growth. More details about this model can be found in Ref. [13].

Define the basic reproductive number $R_0$:

$$R_0 = \frac{\beta}{\mu + d}. \quad (2)$$

In the fields of mathematical biology, when $R_0 > 1$, the infectious disease will spread; when $R_0 < 1$, the infectious diseases will disappear. Therefore, $R_0$ is a quantity which determines whether an infectious disease will outbreak.

Define the basic demographic reproductive number $R_d$:

$$R_d = \frac{r}{\mu + m}. \quad (3)$$

$R_d \geq 1$ the population grows and $R_d \leq 1$ implies that the population does not survive.

Let $S \rightarrow \frac{1}{r} I, \quad I \rightarrow \frac{1}{r} S, \quad t \rightarrow \frac{1}{\mu + d}$ and we have:

$$\begin{align*}
\frac{\partial S}{\partial t} &= v R_0 (S + I) \left( 1 - (S + I) \right) - R_0 \frac{SI}{S + I} - vS + d_1 \nabla^2 S, \\
\frac{\partial I}{\partial t} &= R_0 \frac{SI}{S + I} - I + d_2 \nabla^2 I,
\end{align*}$$

where $v = \frac{\mu + m}{\mu + d}$.

Supposing that the susceptible and the infectious individuals move randomly in space, we use a simply spatial model describing the model (4). Let $dt \rightarrow (S + I) dt$ and the simplified reaction–diffusion equation is as follows:

$$\begin{align*}
\frac{\partial S}{\partial t} &= v R_0 (S + I)^2 \left[ 1 - (S + I) \right] - R_0 SI + d_1 \nabla^2 S, \\
\frac{\partial I}{\partial t} &= R_0 SI - I(S + I) + d_2 \nabla^2 I.
\end{align*}$$

In the two dimension space consisting of x and y, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denotes the Laplace operator in two-dimension space, $d_1$ and $d_2$ are the diffusion coefficients of $S$ and $I$, respectively. In this paper, all the parameters are positive.

The initial condition of system (5) is

$$S(r, 0) = S_0(r) \geq 0, \quad I(r, 0) = I_0(r) \geq 0, \quad r = (x, y) \in \Omega = [0, L] \times [0, L].$$

The boundary condition is

$$\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0,$$

where $n$ is space vector, $(x, y) \in \partial \Omega$ and $\Omega$ is the space domain.
3. Dynamical behaviors of the system (5)

3.1. Linear analysis

In order to obtain the Turing instability of the reaction–diffusion system, it is very important to consider the local dynamics of system (5). The corresponding non-diffusion model is as follows:

\[
\begin{align*}
\frac{dS}{dt} &= vR_d(S + I)^2[1 - (S + I)] - R_0SI = f(S, I), \\
\frac{dI}{dt} &= R_0SI - I(S + I) = g(S, I).
\end{align*}
\]

(6)

In the domain \( S \geq 0, I \geq 0 \), it holds. Let \( f(S, I) = 0, g(S, I) = 0 \), then system (6) has two equilibria \( E_0 \) and \( E^*(R_d > \frac{R_0 + R_0 + 1}{R_0}, R_0 > 1) \):

1. \( E_0 = (S_0, I_0) = (1 - \frac{1}{R_d}, 0) \);
2. \( E^* = (S^*, I^*) = (\frac{R_0R_d - R_0 + 1}{R_0R_d}, R_0 - 1)S^* \).

We only need to investigate the local characteristics of equilibria \( E^* \) and the Jacobian matrix at \( E^* \) is

\[
J = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix},
\]

where

\[
\begin{align*}
a_{11} &= 2vR_d(S^* + I^*) (1 - S^* - I^*) - vR_d(S^* + I^*)^2 - R_0I^*, \\
a_{12} &= 2vR_d(S^* + I^*) (1 - S^* - I^*) - vR_d(S^* + I^*)^2 - R_0S^*, \\
a_{21} &= (R_0 - 1)I^*, \\
a_{22} &= (R_0 - 1)S^* - 2I^*.
\end{align*}
\]

Diffusion-driven instability requires that the stable, homogeneous steady state is driven unstable by the interaction of the dynamics and diffusion of the species [14]. As a result, we need to obtain the conditions that the equilibrium \( E^*(S^*, I^*) \) in the corresponding ordinary differential equation (i.e., system (6)) is stable, but in the partial differential equation (i.e., system (5)) is not stable.

Firstly adding the heterogeneous disturbance term, then we will process time stability of the equilibrium. The reaction–diffusion equation is analyzed by standard linear analysis [15]:

\[
\begin{pmatrix}
S \\
I
\end{pmatrix} = \begin{pmatrix}
S^* \\
I^*
\end{pmatrix} + \varepsilon \begin{pmatrix}
S_k \\
I_k
\end{pmatrix} e^{\lambda t + ik} + cc + O(\varepsilon^2),
\]

where \( \lambda \) is disturbance growth rate of \( t \) moment, \( i \) is the imaginary unit, \( k \) represents wavenumber, \( \mathbf{r} = (X, Y) \) is a factor two-dimensional in the complex conjugate plane. Inserting the above equation into the system (5), we obtain the characteristic equation about \( \lambda \) dependent on the determinant \( \det \ A \), where

\[
A = \begin{pmatrix}
a_{11} - d_1k^2 - \lambda & a_{12} \\
a_{21} & a_{22} - d_2k^2 - \lambda
\end{pmatrix}.
\]

(9)

The characteristic equation is

\[
\lambda_k^2 + tr_k\lambda_k + \Delta_k = 0.
\]

(10)

Then the characteristic value \( \lambda_k \) is

\[
\lambda_k = -\frac{tr_k \pm \sqrt{(tr_k)^2 - 4\Delta_k}}{2},
\]

(11)

where

\[
tr_k = a_{11} + a_{22} - k^2(d_1 + d_2),
\]

\[
\Delta_k = a_{11}a_{22} - a_{12}a_{21} - k^2(a_{11}d_2 + a_{22}d_1) + k^4d_1d_2.
\]

Hopf bifurcation occurs when \( \text{Im}(\lambda_k) \neq 0, \text{Re}(\lambda_k) = 0 \) with \( k = 0 \) which equals \( a_{11} + a_{22} = 0 \). The parameter of Hopf bifurcation \( \beta \) is given by

\[
\beta = \frac{R_0 - 1}{R_0R_d}.
\]
When \( \text{Im}(\lambda_k) \neq 0, \text{Re}(\lambda_k) = 0, \) and \( k = k_T \neq 0, \) Turing bifurcation will take place, and

\[
k_T^2 = \frac{(\nu R_0 R_d - R_0 + 1)(4d_2 R_0 - d_2 R_0^2 - d_2 \nu R_0 R_d - d_1 R_0 + d_1 + 3d_2)}{2\nu d_1 d_2 R_0^2 R_d}.
\]

Inserting \( k \) into \( \Delta_k = 0, \) we can obtain the critical value of Turing bifurcation, and parameter \( \nu \) equals

\[
\nu_T = \frac{2d_1 R_0 - d_2 R_0 - d_1 + 3d_2 - 2\sqrt{d_1^2 R_0^2 - d_1 d_2 R_0^2 - d_2^2 R_0 + 2d_1 d_2 R_0}}{d^2 R_0 R_d} (R_0 - 1).
\]

With the parameters \( R_0 \) and \( \nu \) changing, the parameter domain and bifurcation are represented in Fig. 1. In the space marked by \( T, \) stationary inhomogeneous patterns can be observed.

### 3.2. Amplitude equation

Amplitude equations can be derived by the standard multi-scale analysis [16–18]. When parameter \( \nu \) is close to the critical value \( \nu_T, \) eigenvalues of the critical mode are close to zero. When the mode deviated from the critical state changes dramatically, the change of critical state mode is very small. As a result, we only need to consider the disturbance term \( k \) located near \( k_T. \) The whole dynamics of system (5) can be reduced to slow mode. The stability of system (5) depends on the amplitude equation of the active mode. Turing Pattern can be represented by three pairs of active resonant modes \( (k_j, -k_j) (j = 1, 2, 3) \) and the angle between each of the active mode is \( \frac{2\pi}{3} \) with \( |k_j| = k_T. \)

In order to derive the amplitude equation, we firstly give the linear part of system (5) at \( E^* \):

\[
\frac{\partial x}{\partial t} = a_{11} x + a_{12} y + (1 - 3S^* - 3l^*) \nu R_d x^2 + (1 - 3S^* - 3l^*) \nu R_d y^2
- ((2 - 6S^* - 6l^*) \nu R_d - R_0) xy - \nu R_d x^3 - \nu R_d y^3 - 3 \nu R_d x^2 y - 3 \nu R_d x y^2 + d_1 \nabla^2 x,
\]

\[
\frac{\partial y}{\partial t} = a_{21} x + a_{22} y - y^2 + (R_0 - 1) xy + d_2 \nabla^2 y.
\]

In the point \( \nu = \nu_T, \) the model can be expanded to the following form:

\[
U = U_o + \sum_{j=1}^3 U_0 [A_j \exp(ik_j \cdot r) + \bar{A}_j \exp(-ik_j \cdot r)].
\]

It also can be expanded as

\[
U^0 = \sum_{j=1}^3 U_0 [A_j \exp(ik_j \cdot r) + \bar{A}_j \exp(-ik_j \cdot r)],
\]

where \( U_o \) represents unique stable point, \( U_0 = (a_{11}^* d_2 + a_{22}^* d_1)/(2a_{21}^* d_1), \) \( 1 \) is the eigenvector of linear operator. \( U_0 \) represents the direction of eigenmodes of compared space (the comparison of \( x \) and \( y), A_j \) and its conjugate \( \bar{A}_j \) represent the amplitude associated with \( k_j \) and \( -k_j. \) By solving the third derivative of disturbance factor, the amplitude of spatiotemporal
evolution can be described by the following equation:

\[
\begin{align*}
\tau_0 \frac{dA_1}{dt} &= \mu A_1 + hA_2A_3 - (g_1|A_1|^2 + g_2(|A_2|^2 + |A_3|^2))A_1, \\
\tau_0 \frac{dA_2}{dt} &= \mu A_2 + hA_1A_3 - (g_1|A_1|^2 + g_2(|A_1|^2 + |A_3|^2))A_2, \\
\tau_0 \frac{dA_3}{dt} &= \mu A_3 + hA_1A_2 - (g_1|A_3|^2 + g_2(|A_1|^2 + |A_2|^2))A_3,
\end{align*}
\]

(15)

where \( \mu = (v_T - v)/v_T \) is the normalized distance, \( \tau_0 \) is a typical relaxation time. Obviously when the perturbation parameter \( v \) reduces, the distance \( \mu \) of amplitude model of (12) increases. The form of Eq. (15) for the Turing branch is relatively common, but the exact expression of its coefficient is very special. Here we need to find the expression of coefficients \( \tau_0, h, g_1, g_2 \) which can be found in Appendix A.

From Appendix B, we can obtain the stability of five kinds of solutions.

4. Pattern formation

In this section, we show the spatial system (5) in two-dimensional space by a mount of numerical simulations. All simulations are studied in 200 × 200 lattices with zero-flux conditions (\( \partial S/\partial n = \partial I/\partial n = 0 \), and \( n \) is space vector). We set \( R_1 = 1.8, R_0 = 1.5, d_1 = 0.01, d_2 = 0.25, \) and \( v \) is a varied parameter. The spatial step is \( dx = 1 \) and the temporal step is \( dt = 0.01 \). The numerical simulations will stop if the solutions reach a fixed situation or their dynamical behaviors do not undergo any further changes. In the present paper, we want to know the distribution of the infectious (I), so we only need to analyze the pattern formation of I.

Fig. 2 shows the evolution of the spatial pattern of infected at 0, 500, 2000, 10 000, 20 000 and 100 000 iterations with \( v = 0.17 \). It can be concluded from this figure that random distribution can result in regular spotted patterns. The parameter values satisfy \( \mu_2 < \mu < \mu_3 \) which means \( H_0 \) spotted pattern will emerge.

Fig. 3 shows the spatial pattern of infected at 0, 1000, 20 000, and 100 000 iterations with \( v = 0.19 \). These parameters set is corresponding to that \( \mu > \mu_4 \) which means stripe-like patterns will emerge.

Fig. 4 shows the spatial pattern of infected at 0, 10 000, 30 000, and 50 000 iterations with \( v = 0.22 \). In that case, we obtain that \( \mu \in (\mu_3, \mu_4) \) which means coexistence of spotted and stripe patterns will emerge.
5. Discussion and conclusion

In this paper, we have investigated the pattern dynamics in an epidemic model by giving the conditions on the emergence of spatial patterns. By standard multiple-scale analysis, we give the pattern selection of the epidemic model. As can be found from this paper, infectious diseases may have rich pattern structures, including spotted, mixed and stripe patterns.

It should be noted that our results were obtained under the assumption that the motion of individuals of a given population is random and isotropic. That is to say that the populations walk in the space without any preferred direction. Moreover, individuals can exhibit a correlated motion towards certain directions instead of random motion which was referred to as migration [19–21]. We need to investigate the pattern dynamics of epidemic models with migration in the future work.

The real world is more complex than models and laboratories and thus the deterministic environment cannot be widely found. In fact, natural environments are random environments. In this sense, noise sources should be included in the epidemic models which need further investigation and may give more realistic results from the physical point of view.

Appendix A

Let \( X = (x, y)^T, N = (N_1, N_2)^T \), then the amplitude model of (12) can be translated as:

\[
\frac{\partial X}{\partial t} = LX + N,
\]

where

\[
L = \begin{pmatrix}
a_{11} + a_{12} \nabla^2 & a_{12} \\
a_{21} & a_{22} + a_{23} \nabla^2
\end{pmatrix}
\]
Fig. 4. (Color online) Snapshots of contour pictures of the time evolution of infected populations at different instants with $v = 0.22$. (A): 0 iteration; (B): 10 000 iterations; (C): 30 000 iterations; (D): 50 000 iterations.

and

$$N = \begin{pmatrix} \frac{(1 - 3S - 3I^*)vR_0(x^2 + y^2)}{R_0 - 1}xy - vR_d(x^2y + xy^2) \\ ((2 - 6S - 6I^*)vR_D - R_0)xy - vR_d(x^2y^2 + y^3) - 3vR_d(x^3y + xy^2) \end{pmatrix}.$$ (17)

We only need to analyze the property of the parameters near the critical point $v = v_T$. So we can expand $v$ as following:

$$v_T - v = \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + o(\varepsilon^4),$$

where $\varepsilon$ is a very small parameter. Let the variable $X$ and nonlinear term $N$ expand into the series form associated with $\varepsilon$:

$$X = \frac{x}{y} = \varepsilon \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right) + \varepsilon^2 \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right) + \varepsilon^3 \left( \begin{array}{c} x_3 \\ y_3 \end{array} \right) + o(\varepsilon^4),$$

$$N = \varepsilon^2 n^2 + \varepsilon^3 n^2 + o(\varepsilon^4),$$

where the nonlinear term $N$ is the second derivative and the third derivative associated with $\varepsilon$, linear operator $L$ can be expanded as the following form:

$$L = L_T + (v_T - v)M,$$

where

$$L_T = \begin{pmatrix} a_{11}^* + d_1 \nabla^2 & a_{12}^* \\ a_{21}^* & a_{22}^* + d_2 \nabla^2 \end{pmatrix}, \quad M = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$
The main idea of standard multiple analysis is to separate the dynamical system according to different spatial and temporal scales. We will divide \((16)\) by temporal scales (namely \(T_0 = t, T_1 = \varepsilon t, T_2 = \varepsilon^2 t\)) and every temporal scale \(T_i\) can serve as an independent variable. \(T_i\) is a dynamical form of variable of \(\varepsilon^i\). Therefore, \(T_i\) with the time derivative can be converted:

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + o(\varepsilon^3).
\]

(21)

The basic solution formula of \((14)\) is unrelated to time and amplitude is a variable of slow changes, so the derivative \(\frac{\partial}{\partial T_0}\) with respect to time has no effect on amplitude. Then we can have:

\[
\frac{\partial A}{\partial t} = \varepsilon \frac{\partial A}{\partial T_1} + \varepsilon^2 \frac{\partial A}{\partial T_2} + o(\varepsilon^3).
\]

(22)

Inserting \((21), (22)\) into \((16)\) and expanding the series form of \(\varepsilon\), we can obtain the linear term associated with \(\varepsilon\):

\[
L_T \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0.
\]

The quadratic term with respect to \(\varepsilon\):

\[
L_T \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{\partial}{\partial T_1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - v_1 M \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - h_2.
\]

The cubic term with respect to \(\varepsilon\):

\[
L_T \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \frac{\partial}{\partial T_1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \frac{\partial}{\partial T_2} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - v_1 M \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - v_2 M \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - h_3.
\]

Since \(L_T\) is linear operator at initial point, \((x, y)^T\) is the linear combination of the eigenvector of zero eigenvalue with

\[
\tau \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0,
\]

(23)

then we can get

\[
\begin{aligned}
(a_{11}^* + d_1 \nabla^2) x_1 + a_{12}^* y_1 &= 0, \\
a_{21}^* x_1 + (a_{22}^* + d_2 \nabla^2) y_1 &= 0.
\end{aligned}
\]

(24)

Assuming \(y_1 = 1\), we obtain \(x_1 = \frac{a_{11}^* d_2 - a_{22}^* d_1}{2a_{21}^* d_1}\). Then

\[
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \left(\frac{a_{11}^* d_2 - a_{22}^* d_1}{2a_{21}^* d_1}\right) (W_1 \exp(ik_1 r) + (W_2 \exp(ik_2 r)) + (W_3 \exp(ik_3 r)) + c \cdot c,
\]

(25)

where \(|k_j| = k_j^*\). \(W_j\) is the module of \(\exp(ik_j r)\) of the first-order disturbance term. Its form can be determined by the higher order of disturbance term.

For the second-order differential equation about \(\varepsilon\), let

\[
L_T \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{\partial}{\partial T_1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - v_1 M \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - (1 - 3S^* - 3l^*) v_R d (x_1^2 + y_1^2) + ((2 - 6S^* - 6l^*) v_R d - R_0) x_1 y_1.
\]

(26)

According to the Fredholm solvability condition, in order to ensure the existence of non-trivial solution of the equation, we must make that the vector function of right of \((26)\) and zero eigenvectors of operator are orthogonal. The zero eigenvector of the operator \(L_T^*\) is in this system:

\[
\begin{pmatrix} 1 \\ -2a_{21}^* d_2/a_{11}^* d_2 - a_{22}^* d_1 \end{pmatrix} \exp(-ik_j r) + c \cdot c(j = 1, 2, 3).
\]

(27)

By the orthogonality, we have:

\[
\begin{pmatrix} 1 - 2a_{21}^* d_2/a_{11}^* d_2 - a_{22}^* d_1 \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix} = 0,
\]

(28)
where $F_x^i$ and $F_y^i$ represent the coefficients. For $\exp(-ik_1r)$, we can get:

$$
\left( \begin{array}{c}
F_x^1 \\
F_y^1
\end{array} \right) = \left( \begin{array}{c}
\frac{\partial W_1}{\partial T_1} \\
\frac{\partial W_1}{\partial T_1}
\end{array} \right) - v_1 \left( \begin{array}{c}
b_{11}W_1 + b_{12}W_2 \\
b_{21}W_1 + b_{22}W_2
\end{array} \right) - \left( \begin{array}{c}
4\varepsilon^2 vR_d(1 - 3\epsilon^* - 3\epsilon^*)\bar{W}_1\bar{W}_3 + 2l((2 - 6\epsilon^* - 6\epsilon^*)vR_d - R_0)]\bar{W}_3\bar{W}_3 \\
2!l(R_0 - 1)W_2\bar{W}_3 - 2W_2\bar{W}_3
\end{array} \right).
$$

From Fredholm solvable conditions:

$$
\left( 1 - \frac{d_1}{d_2} \right) \frac{dW_1}{dT_1} = \beta_1 \left( lb_{11} + b_{12} - \frac{d_1}{d_2}l(b_{21} + b_{22}) \right) W_1
- \left( 1 + \frac{d_1}{d_2} \right) (1 - 3\epsilon^* - 3\epsilon^*)((4\varepsilon^2 vR_d + 2lv_0R_d - R_0)\bar{W}_3\bar{W}_3.
$$

At the same time, let

$$
\begin{align*}
\left( \begin{array}{c}
x_2 \\
y_2
\end{array} \right) &= \left( \begin{array}{c}
x_0 \\
y_0
\end{array} \right) + \sum_{j=1}^3 \left( \begin{array}{c}
x_j \\
y_j
\end{array} \right) \exp(ik_jr) + \sum_{j=1}^3 \left( \begin{array}{c}
x_j \\
y_j
\end{array} \right) \exp(i2k_jr) \\
&+ \left( \begin{array}{c}
x_{12} \\
y_{12}
\end{array} \right) \exp(ik_1 - k_2) + \left( \begin{array}{c}
x_{23} \\
y_{23}
\end{array} \right) \exp(i(k_2 - k_3)r) + \left( \begin{array}{c}
x_{31} \\
y_{31}
\end{array} \right) \exp(i(k_3 - k_1)r) + c.c.
\end{align*}
$$

The coefficients of (31) can be derived by solving the system of linear equation associated with the set of the module $\exp(0)$, $\exp(ik_jr)$, $\exp(i2k_jr)$, $\exp(i(k_j - k_3)r)$:

$$
\left( \begin{array}{c}
x_0 \\
y_0
\end{array} \right) = \left( \begin{array}{c}
x_0 \\
y_0
\end{array} \right) \left( |W_1|^2 + |W_2|^2 + |W_3|^2 \right), \quad X_i = iY_i,
$$

and

$$
\left( \begin{array}{c}
x_i \\
y_i
\end{array} \right) = \left( \begin{array}{c}
x_{i1} \\
y_{i1}
\end{array} \right) W_i^2, \quad \left( \begin{array}{c}
x_{ik} \\
y_{ik}
\end{array} \right) = \left( \begin{array}{c}
x_{ik} \\
y_{ik}
\end{array} \right) W_i\bar{W}_k.
$$

Solving the third order differential equation on $\epsilon$:

$$
L_T \left( \begin{array}{c}
x_3 \\
y_3
\end{array} \right) = \frac{\partial}{\partial T_1} \left( \begin{array}{c}
x_2 \\
y_2
\end{array} \right) + \frac{\partial}{\partial T_2} \left( \begin{array}{c}
x_1 \\
y_1
\end{array} \right) - v_1M \left( \begin{array}{c}
x_2 \\
y_2
\end{array} \right) - v_2M \left( \begin{array}{c}
x_1 \\
y_1
\end{array} \right)
- \left( (2 - 6\epsilon^* - 6\epsilon^*)vR_d(x_1x_2 + y_1y_2) + u_0R_d(x_1y_2 + y_1x_2) + u_0R_d(x_2^2 + y_2^2) \right) + \left( 3vR_d(x_1y_2 + y_1x_2) \right).
$$

By Fredholm solvable conditions, we have:

$$
\frac{d_2 - d_1}{d_2} \frac{dW_1}{dT_1} + \frac{d_2 - d_1}{d_2} \frac{dW_1}{dT_2} = v_2 \left( lb_{11} + b_{12} - \frac{d_1}{d_2}l(b_{21} + b_{22}) \right) W_1 + v_1 \left( lb_{11} + b_{12} - \frac{d_1}{d_2}l(b_{21} + b_{22}) \right) Y_1
- \left( 1 + \frac{d_1}{d_2} \right) (1 - 3\epsilon^* - 3\epsilon^*)((4\varepsilon^2 vR_d + 2lv_0R_d - R_0)(\bar{W}_3\bar{W}_2 + \bar{W}_2\bar{W}_3).
$$

Similarly, we can get the other two equations. The amplitude $A_i$ can be expanded as follows:

$$
A_i = \varepsilon W_i + \varepsilon^2 V_i + o(\varepsilon^3).
$$

Then we can get the amplitude equation on $A_i$:

$$
\frac{\partial A_1}{\partial t} = \mu A_1 + h\bar{A}_2\bar{A}_3 - (g_1|A_1|^2 + g_1(|A_2|^2 + |A_3|^2))A_1.
$$
Appendix B

By using substitutions, we have:

\[
\begin{align*}
\frac{d\phi}{dt} &= -h \frac{\rho_1^2 \rho_2^2 + \rho_1^3 \rho_3^2 + \rho_2^3 \rho_3^2}{\rho_1 \rho_2 \rho_3} \sin \phi, \\
\frac{d\rho_1}{dt} &= \xi \rho_1 + h \rho_2 \rho_3 \cos \phi - g_1 \rho_1^3 - g_2 (\rho_1^2 \rho_3^2) \rho_1, \\
\frac{d\rho_2}{dt} &= \xi \rho_2 + h \rho_1 \rho_3 \cos \phi - g_1 \rho_2^3 - g_2 (\rho_1^2 \rho_2^2) \rho_2, \\
\frac{d\rho_3}{dt} &= \xi \rho_3 + h \rho_1 \rho_2 \cos \phi - g_1 \rho_3^3 - g_2 (\rho_1^2 \rho_2^2) \rho_3,
\end{align*}
\]

(37)

where, \(\phi = \phi_1 + \phi_2 + \phi_3\).

The dynamical system (36) possesses five kinds of solutions [9].

1. The stationary state (O), given by

\[\rho_1 = \rho_2 = \rho_3 = 0,\]

is stable for \(\xi < \xi_2 = 0\), and unstable for \(\xi > \xi_2\).

2. Stripe patterns (S), given by

\[\rho_1 = \sqrt[1]{\xi \frac{g_1}{g_1}} \neq 0, \quad \rho_2 = \rho_3 = 0,\]

(38)

are stable for \(\xi > \xi_3 = \frac{h^2 g_1}{(g_2 - g_1)^2}\), and unstable for \(\xi < \xi_3\).

3. Hexagon patterns \((H_0, H_0')\) are given by

\[\rho_1 = \rho_2 = \rho_3 = \frac{|h| + \sqrt{h^2 + 4(g_1 + 2g_2 \xi)}}{2(g_1 + 2g_2)},\]

(39)

with \(\phi = 0 \text{ or } \pi\), and exist when

\[\xi > \xi_1 = \frac{-h^2}{4(g_1 + 2g_2)}\].

(40)

The solution \(\rho^+ = \frac{|h| + \sqrt{h^2 + 4(g_1 + 2g_2 \xi)}}{2(g_1 + 2g_2)}\) is stable only for

\[\xi < \xi_4 = \frac{2g_1 + g_2}{(g_2 - g_1)^2} h^2,\]

(41)

and \(\rho^- = \frac{|h| - \sqrt{h^2 + 4(g_1 + 2g_2 \xi)}}{2(g_1 + 2g_2)}\) is always unstable.
The mixed states are given by

\[ \rho_1 = \frac{|h|}{g_2 - g_1}, \quad \rho_2 = \rho_3 = \frac{\xi - g_1 \rho_1^2}{g_1 + g_2}, \]

with \( g_2 > g_1 \). They exist when \( \xi > \xi_3 \) and are always unstable.

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