SKEW DOMINO SCHENSTED CORRESPONDENCE AND SIGN-IMBALANCE

JANG SOO KIM

ABSTRACT. Using growth diagrams, we define a skew domino Schensted correspondence which is a domino analogue of the skew Robinson-Schensted correspondence due to Sagan and Stanley. The color-to-spin property of Shimozono and White is extended. As an application, we give a simple generating function for the weighted sum of skew domino tableaux, which is a generalization of Stanley’s sign-imbalance formula. The generating function gives a method to calculate the generalized sign-imbalance formula. We also extend Sjöstrand’s theorems on sign-imbalance of skew shapes.

1. INTRODUCTION

The domino Schensted correspondence is a bijection between colored permutations and pairs of domino tableaux of the same shape. It was first developed by Barbasch and Vogan [1] in 1982. Garfinkle [6] described this correspondence in terms of insertion. Van Leeuwen [24] described this correspondence using growth diagrams and extended it in the presence of a nonempty core. Shimozono and White [17] proved that this correspondence has the color-to-spin property. Lam [10] used growth diagrams to prove the color-to-spin property and identities involving colored involutions. Using these properties, Lam [10] obtained enumerative results for domino tableaux and proved Stanley’s sign-imbalance conjectures [23].

For a standard Young tableau (SYT) $T$, the sign of $T$ is defined to be $\text{sign}(T)$, where $\pi$ is the permutation obtained by reading $T$ like a book. For example, if $T = \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \end{array}$ then $\text{sign}(T) = \text{sign}(12435) = -1$. The sign-imbalance $I_{\lambda}$ of a partition $\lambda$ is the sum of $\text{sign}(T)$ for all SYTs $T$ of shape $\lambda$. In [23], Stanley suggested the following interesting sign-imbalance formulas:

$$
\sum_{\lambda \vdash n} x^{v(\lambda)} y^{h(\lambda)} z^{d(\lambda)} I_{\lambda} = (x + y)^{\lfloor \frac{n}{2} \rfloor},
$$

$$
\sum_{\lambda \vdash n} (-1)^{v(\lambda)} I_{\lambda}^2 = 0,
$$

where $v(\lambda)$, $h(\lambda)$ and $d(\lambda)$ denote the maximum numbers of vertical dominoes, horizontal dominoes and $2 \times 2$ rectangles respectively that can be placed in the Young diagram of $\lambda$ without overlaps.

Reifegerste [14] and Sjöstrand [18] independently proved that if $\pi$ corresponds to $(P, Q)$ in the Robinson-Schensted correspondence and $sh(P) = \lambda$ then

$$
\text{sign}(\pi) = (-1)^{v(\lambda)} \text{sign}(P) \text{sign}(Q).
$$

The author was supported by the SRC program of Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MEST) (No. R11-2007-035-01002-0).
Using Eq. (3), Reifegerste [14] and Sjöstrand [18] proved Eq. (2). Sjöstrand [18] also proved Eq. (1) using Chess tableaux.

White [25] observed that sign-imbalance is related to domino tableaux and proved that for a domino tableau $D$,

$$ \text{sign}(D) = (-1)^{ev(D)}, $$

where $ev(D)$ is the number of vertical dominoes of $D$ in even columns.

Lam [10] proved Eq. (1) and Eq. (2) using growth diagrams and Eq. (4).

There are some results about sign-imbalance for skew shapes. Sjöstrand [19] generalized Eq. (3) as follows: If $sh(T) = sh(U) = \alpha/\mu$, $sh(P) = sh(Q) = \lambda/\alpha$ and $\pi$ is an $n$-partial permutation satisfying $(\pi, T, U) \leftrightarrow (P, Q)$ in the skew Robinson-Schensted correspondence of Sagan and Stanley [16], then

$$ (-1)^{v(\lambda)} \text{sign}(P) \text{sign}(Q) = (-1)^{|\alpha|} (-1)^{v(\mu) + |\mu|} \text{sign}(T) \text{sign}(U) \text{sign}(\pi), $$

where $\pi$ is the $n$-permutation extended from $\pi$ with the smallest number of inversions. Using Eq. (5), Sjöstrand [19] generalized Eq. (2) as follows: If $\alpha$ is a fixed partition then

$$ \sum_{\lambda/\alpha \updownarrow n} (-1)^{v(\lambda)} I^\lambda_{\alpha/\alpha} = (-1)^n \sum_{\alpha/\mu \updownarrow n} (-1)^{v(\mu)} I^\alpha_{\mu/\mu} + \frac{1}{2} (-1)^n \sum_{\alpha/\mu \updownarrow n-1} (-1)^{v(\mu)} I^\alpha_{\mu/\mu}. $$

Lam proved Eq. (6) once using signed differential posets [12] and once, when $n$ is even, using the skew domino Cauchy identity [11].

In this paper, inspired by Lam’s work [10], we describe a skew domino Schensted correspondence using growth diagrams, which is a domino analogue of the skew Robinson-Schensted correspondence. This growth diagram approach was used in Roby’s thesis [15] to describe the skew Robinson-Schensted correspondence. Fomin [5] proved the existence of the skew Robinon-Schensted correspondence in a more general context using operators on partitions. The color-to-spin property and Lam’s identities for colored involutions are extended. As an application, we generalize Eq. (1) to skew shapes. We also generalize Eq. (3) to skew tableaux $P$ and $Q$ of shape $\lambda/\alpha$ and $\lambda/\beta$ respectively, and then generalize Eq. (4).

We should note that, in the literature, there are two different definitions of $\text{sign}(T)$ for a SYT $T$ of shape $\lambda/\mu$. In [11, 19], the sign of a SYT $T$ of shape $\lambda/\mu$ does not consider the cells in $\mu$, but in [12], it does. However, if $sh(T) = sh(U)$ then the product $\text{sign}(T) \text{sign}(U)$ is the same in both definitions, and so are Eq. (5) and Eq. (6). In this paper, we use the definition of $\text{sign}(T)$ in [12] and prove that Eq. (4) still holds for skew domino tableaux.

The rest of this paper is organized as follows. In Section 2, we define skew shapes, reversed shapes, domino tableaux and colored permutations. In Section 3, we introduce growth diagrams and a skew domino Schensted correspondence and extend the color-to-spin property and Lam’s identities for colored involutions. We also find a generating function for the weighted sum of domino tableaux which turns out to be closely related to sign-imbalance. In Section 4, we define the sign of a skew tableau and generalize Eq. (1) to skew shapes. The last part of this section is devoted to finding a closed formula for $\sum \lambda/\delta_k \updownarrow n x^{v(\lambda/\delta_k)} y^h(\lambda/\delta_k) z^{d(\lambda/\delta_k)} I_{\lambda/\delta_k}$, where $\delta_k = (k, k-1, \ldots, 1)$. In Section 5, we generalize Eq. (5) and Eq. (6).
2. Preliminaries

2.1. Skew shapes and domino tableaux. For a positive integer \( n \), we denote \([n] = \{1, 2, \ldots, n\}\). A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) of \( n \), denoted by \( \lambda \vdash n \), is a weakly decreasing (possibly empty) sequence of positive integers \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \) summing to \( n \). Each \( \lambda_i \) is called the \( i \)-th part of \( \lambda \). Let \( \ell(\lambda) \) denote the number of parts in \( \lambda \).

A cell is a pair of positive integers. The Young diagram \( Y(\lambda) \) of a partition \( \lambda \) is the set of cells \( (i, j) \) with \( i \leq \ell(\lambda) \) and \( j \leq \lambda_i \). We can draw the Young diagram \( Y(\lambda) \) by placing a square in the \( i \)-th row and \( j \)-th column for each cell \( (i, j) \in Y(\lambda) \).

For example, the drawing of the Young diagram of \((4, 3, 1)\) is \[
\begin{array}{ccc}
\text{1} & \text{2} & \text{3} \\
\text{4} & \text{5} & \\
\text{7} & \text{6} & \text{8}
\end{array}
\]
We will identify a partition \( \lambda \) with its Young diagram \( Y(\lambda) \).

A skew shape \( \lambda/\mu \) is an ordered pair \((\lambda, \mu)\) of partitions satisfying \( \mu \subset \lambda \). The size of \( \lambda/\mu \), denoted by \(|\lambda/\mu|\), is the number of cells in \( \lambda/\mu \). The notation \( \lambda/\mu \vdash n \) means that the size of \( \lambda/\mu \) is \( n \). For example, \((4, 3, 1)/(2, 1) = \[
\begin{array}{ccc}
\text{1} & \text{2} & \text{3} \\
\text{4} & \text{5} & \\
\text{7} & \text{6} & \text{8}
\end{array}
\]
is a skew shape of size 5.

A domino is a horizontal domino or a vertical domino where a horizontal (resp. vertical) domino is a set of two adjacent cells \((i, j)\) and \((i, j+1)\) (resp. \((i, j)\) and \((i+1, j)\)).

A (skew) standard Young tableau (SYT) of shape \( \lambda/\mu \vdash n \) is a bijection \( T \) from the set of cells in \( \lambda/\mu \) to \([n]\) such that \( T((i, j)) \leq T((i', j')) \) whenever \( i \leq i' \) and \( j \leq j' \). For a cell \( c \in \lambda/\mu \), we call the integer \( T(c) \) the entry of \( c \). A (skew) standard domino tableau (SDT) of shape \( \lambda/\mu \vdash 2n \) is a SYT such that two cells with entries \( 2i-1 \) and \( 2i \) make a domino for each \( i = 1, 2, \ldots, n \). Thus we can consider a SDT as a collection of labeled dominoes. For example, \[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & \\
7 & 6 & 8
\end{array}
\]
and \[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & \\
7 & 6 & 8
\end{array}
\]
represent the same SDT. If there is a SDT of shape \( \lambda/\mu \), then we say \( \lambda/\mu \) is a domino-tileable.

Let \( T(\lambda/\mu) \) (resp. \( D(\lambda/\mu) \)) denote the set of all SYTs (resp. SDTs) of shape \( \lambda/\mu \). Let \( f^{\lambda/\mu} = |T(\lambda/\mu)| \) and \( d^{\lambda/\mu} = |D(\lambda/\mu)| \).

For a given partition \( \lambda \), let us take a maximal chain of partitions \( \lambda^{(m)} \subset \lambda^{(m-1)} \subset \cdots \subset \lambda^{(0)} = \lambda \) such that \( \lambda^{(i-1)}/\lambda^{(i)} \) is a domino for \( i = 1, 2, \ldots, m \). Then the partition \( \lambda^{(m)} \) is always the same and is called the 2-core of \( \lambda \). We denote the 2-core of \( \lambda \) by \( \tilde{\lambda} \). Since there is no partition \( \mu \) such that \( \tilde{\lambda}/\mu \) is a domino, \( \tilde{\lambda} \) must be a staircase partition \( \delta_r = (r, r-1, \ldots, 1) \) for some \( r \). We refer the reader to [2; 01; 123] for details of p-cores.

Let \( v(\lambda/\mu) \) (resp. \( h(\lambda/\mu) \)) denote the number of cells in even rows (resp. columns). Let \( d(\lambda/\mu) \) denote the number of cells both in even columns and even rows. It is easy to see that \( h(\lambda) \), \( v(\lambda) \) and \( d(\lambda) \) are the maximum numbers of horizontal dominoes, vertical dominoes and \( 2 \times 2 \) rectangles respectively that can be placed in \( \lambda \) without overlaps.

For a SDT \( D \), let \( oh(D) \), \( eh(D) \), \( ov(D) \) and \( ev(D) \) denote the numbers of horizontal dominoes in odd rows, horizontal dominoes in even rows, vertical dominoes in odd columns and vertical dominoes in even columns respectively. The spin of a SDT is defined to be the number of vertical dominoes divided by 2, that is, \( sp(D) = \frac{1}{2}(ov(D) + ev(D)) \).
Next we prove some relations between statistics of SDTs. We note that these can also be proved by modifying Lam’s results \cite{10} in Proposition 14.

**Lemma 2.1.** If $D \in \mathcal{D}(\lambda/\mu)$ then the following hold.

1. $oh(D) - eh(D) = \frac{1}{2}|\lambda/\mu| - v(\lambda/\mu)$
2. $ov(D) - ev(D) = \frac{1}{2}|\lambda/\mu| - h(\lambda/\mu)$
3. $eh(D) + ev(D) = d(\lambda/\mu)$
4. $v(\lambda/\mu) + h(\lambda/\mu) = \frac{1}{2}|\lambda/\mu| + 2 \cdot d(\lambda/\mu)$

**Proof.** Assign 1 to the cells in odd rows and $-1$ to the cells in even rows in $\lambda/\mu$. Then the sum of all assigned numbers is $|\lambda/\mu| - 2v(\lambda/\mu)$. Each vertical domino contains both 1 and $-1$. Each horizontal domino contains two 1’s or two $-1$’s in accordance with the parity of its row number. Thus the sum is equal to $2oh(D) - 2eh(D)$, which proves the first identity. Similarly we can prove the second identity.

The right hand side of the third equation is the number of cells both in even rows and even columns of $\lambda/\mu$. A domino $d$ contains one of these cells if and only if $d$ is either a horizontal domino in an even row or a vertical domino in an even column. Thus $d(\lambda/\mu) = eh(D) + ev(D)$.

By the first three identities, we get the fourth:

$$v(\lambda/\mu) + h(\lambda/\mu) = |\lambda/\mu| - (oh(D) + ov(D) - eh(D) - ev(D))$$
$$= |\lambda/\mu| - \left(\frac{1}{2}|\lambda/\mu| - 2(oh(D) + ev(D))\right)$$
$$= \frac{1}{2}|\lambda/\mu| + 2 \cdot d(\lambda/\mu). \qed$$

**Remark.** In Lemma 2.1, (4) is not true if $\lambda/\mu$ is not domino-tileable. For example, if $\mu = (1)$ and $\lambda = (2,1)$, then (4) does not hold.

2.2. **Reversed shapes.** Recall that a skew shape $\mu/\lambda$ is a pair $(\mu, \lambda)$ of partitions with $\lambda \subset \mu$. We define a reversed shape $\lambda/\mu$ to be a pair $(\lambda, \mu)$ of partitions with $\mu \subset \lambda$ and denote $\lambda/\mu \vdash |\lambda| - |\mu|$. Thus $\lambda/\mu$ is a reversed shape if and only if $\mu/\lambda$ is a skew shape. We also see that $\lambda/\mu \vdash -n$ is equivalent to $\mu/\lambda \vdash n$. We extend each statistic $\text{stat}$ of skew shapes to reversed shapes by defining $\text{stat}(\lambda/\mu) = -\text{stat}(\mu/\lambda)$, i.e., $|\lambda/\mu| = -|\mu/\lambda|$, $v(\lambda/\mu) = -v(\mu/\lambda)$ and so on. As a shape of a tableau, we will treat $\mu/\lambda$ and $\lambda/\mu$ equally, that is, $T(\lambda/\mu) = T(\mu/\lambda)$ and $\mathcal{D}(\lambda/\mu) = \mathcal{D}(\mu/\lambda)$.

To avoid confusion we will always write a reversed shape with the negative sign, that is, if we write $\lambda/\mu \vdash n$ (resp. $\lambda/\mu \vdash -n$) then it is always assumed that $n \geq 0$ and $\lambda/\mu$ is a skew shape (resp. reversed shape).

The notion of reversed shapes is not essential. However it will give us a simple description for the generalization of Eq. \text{[1]}. In Section 3 we will define the signimbalance $I_{\lambda/\mu}$ of a reversed shape $\lambda/\mu \vdash -2n$.

2.3. **Colored permutations and colored involutions.** A colored permutation $\pi$ of $[n]$ is a permutation of $[n]$ equipped with an assignment of bars to some integers. Let $\pi$ be a colored permutation. The total color $tc(\pi)$ of $\pi$ is the number of barred integers. The permutation matrix of $\pi$ is the matrix $M$ such that $M(i,j)$ is equal to 1 if $\pi_i = j$; $-1$ if $\pi_i = j$ and 0 otherwise. Let $CP_n$ denote the set of colored permutations of $[n]$.

A colored permutation $\pi$ is called an involution if the permutation matrix of $\pi$ is symmetric. We denote the set of involutions in $CP_n$ by $CI_n$. We will consider
the empty word as an involution, thus \( CI_0 = \{\emptyset\} \). We can represent a colored permutation in cycle notation as follows. Given a colored permutation \( \pi \), write the underlying permutation of \( \pi \) in cycle notation, and put a bar over \( i \) if and only if \( i \) is barred in \( \pi \). For example, if \( \pi = 34152 \) then \( \pi = (13)(245) \) in cycle notation.

Let \( \pi \) be a colored involution. Then \( \pi \) has only \( 1 \)-cycles and \( 2 \)-cycles, and moreover, the two integers in a \( 2 \)-cycle of \( \pi \) are both barred or both unbarred. Let \( \sigma_1(\pi), \sigma_2(\pi), \bar{\sigma}_1(\pi) \) and \( \bar{\sigma}_2(\pi) \) denote the numbers of unbarred \( 1 \)-cycles, unbarred \( 2 \)-cycles, barred \( 1 \)-cycles and barred \( 2 \)-cycles in \( \pi \) respectively. For example, if \( \pi = (14)(2)(36)(5)(7) \) then \( \sigma_1(\pi) = 2, \bar{\sigma}_1(\pi) = 1, \sigma_2(\pi) = 1 \) and \( \bar{\sigma}_2(\pi) = 1 \).

We define the weight of a colored involution \( \pi \) by

\[
\text{wt}_\pi = \text{wt}_\pi(x, y, q) = x^{\sigma_1(\pi)} y^{\sigma_2(\pi)} q^{\frac{1}{2}tc(\pi)}. 
\]

Since a colored involution \( \pi \) can be considered as a partition of \([n]\) into 1-subsets and 2-subsets with a possible bar on each subset and \d, we get the following exponential generating function:

\[
\sum_{n \geq 0} \left( \sum_{\pi \in CI_n} \text{wt}_\pi \right) t^n = \exp \left( (x + y\sqrt{q})t + (1 + q)\frac{t^2}{2} \right).
\]

3. Skew Domino Schensted Correspondence

3.1. Definition of a growth diagram. In this section we introduce growth diagrams. Our definition is based on Lam’s [10]. We can define growth diagrams of an arbitrary skew shape. Nevertheless, we will restrict our definition to rectangular shapes for simplicity since we only need that case. The reader is referred to [2, 3, 4] for details of growth diagrams.

For partitions \( \lambda \) and \( \mu \), we write \( \mu \prec_d \lambda \) if \( \lambda/\mu \) is a domino and \( \mu \leq_d \lambda \) if \( \mu = \lambda \) or \( \mu \prec_d \lambda \). A \( d \)-chain is a chain of partitions \( \lambda^{(0)} \prec_d \lambda^{(1)} \prec_d \cdots \prec_d \lambda^{(m)} \) and a \( d \)-multichain is a multichain of partitions \( \lambda^{(0)} \leq_d \lambda^{(1)} \leq_d \cdots \leq_d \lambda^{(m)} \).

An \( n \times m \) growth array \( \Gamma \) is an array of partitions \( \Gamma_{(i,j)} \) for \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \) such that any two adjacent partitions are equal or differ by a domino, i.e., \( \Gamma_{(i-1, j)} \leq_d \Gamma_{(i, j)} \) and \( \Gamma_{(i, j-1)} \leq_d \Gamma_{(i, j)} \) for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).

An \( n \times m \) partial permutation matrix (PPM) \( M \) is an \( n \times m \) matrix whose elements are 1, \(-1\), or 0, and which contains at most one nonzero element in each row and column. For a PPM \( M \), let \( cp(M) \) denote the colored permutation \( \pi \) whose permutation matrix is the matrix obtained from \( M \) by removing the rows and columns consisting of zeroes only.

An \( n \times m \) growth diagram \( G \) is a pair \( (\Gamma, M) \), where \( \Gamma = \Gamma(G) \) is an \( n \times m \) growth array and \( M = M(G) \) is an \( n \times m \) PPM satisfying the following local rules.

Let \( \nu = \Gamma_{(i-1, j-1)} \), \( \mu = \Gamma_{(i-1, j)} \), \( \rho = \Gamma_{(i, j-1)} \) and \( \lambda = \Gamma_{(i, j)} \). Then it must fall into one of the following conditions which determine \( \lambda \):

1. If \( M(i, j) = 1 \) then \( \nu = \mu = \rho \) and \( \lambda \) is the partition obtained from \( \mu \) by adding a horizontal domino to the first row.
2. If \( M(i, j) = -1 \) then \( \nu = \mu = \rho \) and \( \lambda \) is the partition obtained from \( \mu \) by adding a vertical domino to the first column.
3. If \( M(i, j) = 0 \) then there are five cases.
   1. If \( \nu = \mu \) or \( \nu = \rho \) then \( \lambda \) is the maximal partition among \( \nu, \mu \) and \( \rho \).
   2. If \( \nu \prec_d \mu, \nu \prec_d \rho \) and \( \mu \neq \rho \) and \( \mu/\nu \cap \rho/\nu = \emptyset \) then \( \lambda = \mu \cup \rho \).
(c) If \( \nu <_d \mu, \nu <_d \rho, \mu \neq \rho \) and \( \mu/\nu \cap \rho/\nu \neq \emptyset \) then \( \mu/\nu \) and \( \rho/\nu \) share only one cell, say \((p,q)\), and \( \lambda \) is the partition obtained from \( \mu \cup \rho \) by adding the cell \((p+1,q+1)\).

(d) If \( \nu <_d \mu, \nu <_d \rho, \mu = \rho \) and \( \mu/\nu \) is a horizontal domino in \( k \)-th row then \( \lambda \) is the partition obtained from \( \mu \) by adding a horizontal domino to the \((k+1)\)-th row.

(e) If \( \nu <_d \mu, \nu <_d \rho, \mu = \rho \) and \( \mu/\nu \) is a vertical domino in \( k \)-th column then \( \lambda \) is the partition obtained from \( \mu \) by adding a vertical domino to the \((k+1)\)-th column.

For example, see Fig. 1, which represents a growth diagram \( G = (\Gamma, M) \) with

\[
M = \begin{pmatrix}
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

3.2. Skew domino Schensted correspondence. Let \( C = \lambda^{(0)} \leq_d \lambda^{(1)} \leq_d \cdots \leq_d \lambda^{(n)} \) be a \( d \)-multichain. We can naturally construct a SDT from \( C \) as follows. Let \((d_1, d_2, \ldots, d_k)\) be the sequence of dominoes obtained by removing the empty skew shapes (if any) from \((\lambda^{(1)}/\lambda^{(0)}, \lambda^{(2)}/\lambda^{(1)}, \ldots, \lambda^{(n)}/\lambda^{(n-1)})\). Then \( C^{\text{SDT}} \) denotes the SDT of shape \( \lambda^{(n)}/\lambda^{(0)} \) whose domino with entry \( i \) is \( d_i \) for \( i = 1, 2, \ldots, k \).

Let \( G = (\Gamma, M) \) be an \( n \times m \) growth diagram. We define four special \( d \)-multichains of \( G \) as follows:

\[
G_{\text{top}} = \Gamma_{(0,0)} \leq_d \Gamma_{(0,1)} \leq_d \cdots \leq_d \Gamma_{(0,m)},
\]

\[
G_{\text{bottom}} = \Gamma_{(n,0)} \leq_d \Gamma_{(n,1)} \leq_d \cdots \leq_d \Gamma_{(n,m)},
\]

\[
G_{\text{left}} = \Gamma_{(0,0)} \leq_d \Gamma_{(1,0)} \leq_d \cdots \leq_d \Gamma_{(n,0)},
\]

\[
G_{\text{right}} = \Gamma_{(0,m)} \leq_d \Gamma_{(1,m)} \leq_d \cdots \leq_d \Gamma_{(n,m)}.
\]

If both \( G_{\text{bottom}} \) and \( G_{\text{right}} \) are \( d \)-chains, then we call \( G \) a full growth diagram. The local rules say that \( G \) is completely determined by \( G_{\text{top}}, G_{\text{left}} \) and \( M \). On the other hand, one can easily see that the local rules are invertible in the sense that \( \Gamma_{(i-1,j-1)} \) and \( M(i,j) \) are determined by \( \Gamma_{(i-1,j)}, \Gamma_{(i,j-1)} \) and \( \Gamma_{(i,j)} \). Thus a growth diagram \( G \) is also completely determined by \( G_{\text{bottom}} \) and \( G_{\text{right}} \). We define \( \partial^+(G) \) to be the pair \((G_{\text{bottom}}^{\text{SDT}}, G_{\text{right}}^{\text{SDT}})\), and \( \partial^-(G) \) to be the triple \((G_{\text{top}}^{\text{SDT}}, G_{\text{left}}^{\text{SDT}}, M)\). For
example, if \( G \) is the the growth diagram in Fig. 1 then
\[
\partial^+(G) = \begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 2 & & \\
1 & & & \\
\end{pmatrix},
\partial^-(G) = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Let \( \alpha \) and \( \beta \) be partitions. Let \( \mathcal{G}^\alpha_\beta \) denote the set of all \( n \times m \) full growth diagrams \( G = (\Gamma, M) \) satisfying \( \Gamma_{(n,0)} = \alpha \) and \( \Gamma_{(0,m)} = \beta \). Let \( \mathcal{M}^j_{n,m} \) denote the set of all \( n \times m \) PPMs (partial permutation matrices) with \( j \) nonzero elements.

**Lemma 3.1.** Let \( C = C_{(0)} \leq_d C_{(1)} \leq_d \cdots \leq_d C_{(m)} \) and \( C' = C'_{(0)} \leq_d C'_{(1)} \leq_d \cdots \leq_d C'_{(n)} \) be \( d \)-multichains. Let \( M \) be an \( n \times m \) PPM. Then, there is a (necessarily unique) \( n \times m \) full growth diagram \( G = (\Gamma, M) \) such that \( G_{\text{top}} = C \) and \( G_{\text{left}} = C' \) if and only if the following conditions hold:

1. For \( 1 \leq i \leq n \), the \( i \)-th row of \( M \) contains a nonzero element if and only if \( C'_{(i-1)} = C_{(i)} \).
2. For \( 1 \leq j \leq m \), the \( j \)-th column of \( M \) contains a nonzero element if and only if \( C_{(j-1)} = C_{(j)} \).

**Proof.** We can check this easily by the local rules. \( \square \)

**Lemma 3.2.** Let \( \alpha \) and \( \beta \) be partitions. Then the maps \( \partial^+ \) and \( \partial^- \) induce the following bijections:

\[
\partial^+ : \mathcal{G}^\alpha_\beta \to \bigcup_{\lambda/\alpha \vdash 2m} \mathbb{D}(\lambda/\alpha) \times \mathbb{D}(\lambda/\beta),
\partial^- : \mathcal{G}^\alpha_\beta \to \bigcup_{j \geq 0} \left( \bigcup_{\substack{\beta/\mu \vdash 2(m-j) \\ \alpha/\mu \vdash 2(n-j)}} \mathbb{D}(\beta/\mu) \times \mathbb{D}(\alpha/\mu) \times \mathcal{M}^j_{n,m} \right).
\]

**Proof.** By the local rules, a growth diagram \( G = (\Gamma, M) \) is determined by the pair \((G_{\text{bottom}}, G_{\text{right}})\) or the triple \((G_{\text{top}}, G_{\text{left}}, M)\). Moreover, if \( G \) is a full growth diagram, then \((G_{\text{bottom}}, G_{\text{right}})\) and \((G_{\text{top}}, G_{\text{left}}, M)\) are in bijection with \((G_{\text{bottom}}^{\text{SDT}}, G_{\text{right}}^{\text{SDT}})\) and \((G_{\text{top}}^{\text{SDT}}, G_{\text{left}}^{\text{SDT}}, M)\) respectively by **Lemma 3.1**. Thus \( \partial^+ \) and \( \partial^- \) are invertible for full growth diagrams. The surjectiveness of \( \partial^+ \) and \( \partial^- \) follows from the local rules and **Lemma 3.1**. \( \square \)

Now we get a skew domino Schensted correspondence.

**Theorem 3.3.** Let \( \alpha \) and \( \beta \) be fixed partitions and \( n \) and \( m \) be fixed nonnegative integers. Then \( \Phi = \partial^+ \circ (\partial^-)^{-1} \) induces a bijection

\[
\Phi : \bigcup_{j \geq 0} \left( \bigcup_{\substack{\beta/\mu \vdash 2(m-j) \\ \alpha/\mu \vdash 2(n-j)}} \mathbb{D}(\beta/\mu) \times \mathbb{D}(\alpha/\mu) \times \mathcal{M}^j_{n,m} \right) \to \bigcup_{\lambda/\alpha \vdash 2m} \mathbb{D}(\lambda/\alpha) \times \mathbb{D}(\lambda/\beta).
\]
We note that if \( \pi \) corresponds to \((P, Q)\) in the domino Schensted correspondence with the core \( \delta_r \) and \( M \) is the permutation matrix of \( \pi \) then \( \Phi(\emptyset_{\delta_r}, \emptyset_{\delta_r}, M) = (P, Q) \), where \( \emptyset_{\delta_r} \) is the empty SDT of shape \( \delta_r \). The bijection \( \Phi \) is a domino analogue of the skew Robinson-Schensted correspondence, which was first developed using external and internal insertion by Sagan and Stanley \(10\) and was interpreted in terms of growth diagrams, as we did here, by Roby \(15\). Fomin \(5\) proved the existence of the skew Robinson-Schensted correspondence using operators on partitions.

Since the local rules are symmetric we get the following proposition immediately.

**Proposition 3.4.** Let \( \Phi(U, V, M) = (P, Q) \). Then \( \Phi(V, U, M^T) = (Q, P) \).

In the above proposition, if \( U = V \) and \( M \) is symmetric then \( \Phi(U, U, M) = (P, P) \). Let \( \Phi_{sym}(U, M) = P \). Then we get another bijection.

**Corollary 3.5.** Let \( \alpha \) be a fixed partition and \( n \) be a fixed nonnegative integer. Then \( \Phi_{sym} \) induces a bijection

\[
\Phi_{sym} : \bigcup_{j \geq 0} \left( \bigcup_{\alpha/\mu} D(\alpha/\mu) \times \mathcal{S}\mathcal{M}_n^{j} \right) \rightarrow \bigcup_{\lambda/\alpha} D(\lambda/\alpha),
\]

where \( \mathcal{S}\mathcal{M}_n^{j} \) denotes the set of all symmetric \( n \times n \) PPMs with \( j \) nonzero elements.

Shimozono and White \(17\) proved that the domino Schensted correspondence has the color-to-spin property, that is, if \( \pi \) corresponds to \((P, Q)\) then \( tc(\pi) = sp(P) + sp(Q) \). The next proposition generalizes this property. The proof is the same as Lam’s \(10\) Lemma 8).

**Proposition 3.6.** Let \( \Phi(U, V, M) = (P, Q) \) and \( \pi = cp(M) \). Then \( \frac{tc(\pi)}{2} = sp(P) + sp(Q) - sp(U) - sp(V) \).

**Proof.** By the local rules, we can check that the following value is 1 if \( M(i, j) = -1 \) and 0 otherwise: \( sp(\tau(\delta_{i,j})/\tau_{(i-1,j)}) + sp(\tau(\delta_{i,j})/\tau_{(i,j-1)}) - sp(\delta_{(i-1,j)}/\tau_{(i-1,j-1)}) - sp(\delta_{(i,j-1)}/\tau_{(i-1,j-1)}) \). By adding up these for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), we finish the proof. \(\square\)

Lam \(10\) proved that if a colored involution \( \pi \) corresponds to \((D, D)\) in the domino Schensted correspondence then \( \sigma_1(\pi) = ov(D) - ev(D) \) and \( \sigma_2(\pi) = ev(D) \). We can generalize Lam’s results.

**Proposition 3.7.** Let \( M \) be an \( n \times n \) symmetric PPM and \( \pi = cp(M) \). Let \( U \) and \( D \) be SDTs satisfying \( \Phi_{sym}(U, M) = D \). Then we have

\[
\sigma_1(\pi) = (oh(D) - eh(D)) + (oh(U) - eh(U)),
\]
\[
\sigma_1(\pi) = (ov(D) - ev(D)) + (ov(U) - ev(U)),
\]
\[
\sigma_2(\pi) = eh(D) - oh(U),
\]
\[
\sigma_2(\pi) = ev(D) - ov(U).
\]

**Proof.** We will prove the second and the fourth identities. The remaining can be proved similarly. By [Proposition 3.6] we have

\[
\sigma_1(\pi) + 2\sigma_2(\pi) = (ov(D) + ev(D)) - (ov(U) + ev(U)).
\]
Thus, it is sufficient to show that
\[ \bar{s}_1(\pi) = (ov(D) - ev(D)) + (ov(U) - ev(U)). \]

Let \( G = (\Gamma, M) \) be the corresponding \( n \times n \) growth diagram \((\partial^+)^{-1}(D, D)\) and let \( \nu = \Gamma_{(0,0)}, \mu = \Gamma_{(n,0)} = \Gamma_{(0,n)} \) and \( \lambda = \Gamma_{(n,n)} \). Then by Lemma 2.1,
\[ (ov(D) - ev(D)) + (ov(U) - ev(U)) = C_{\lambda/\mu} = \frac{|\lambda/\mu|}{2} - h(\lambda/\mu) + \frac{\mu/\nu}{2} - h(\mu/\nu) \]
\[ = \frac{|\lambda/\mu|}{2} - h(\lambda/\mu). \]

One can check that \( \frac{1}{n!} |\Gamma_{(i,i)}/\Gamma_{(i-1,i-1)}| = h(\Gamma_{(i,i)}/\Gamma_{(i-1,i-1)}) \) is 1 if \( M(i,i) = -1 \) and 0 otherwise. By adding up these for all \( 1 \leq i \leq n \), we finish the proof.

As an application of our skew domino Schensted correspondence, we get some enumerative results. Following Lam’s notation \([10]\), let
\[ f^{\lambda/\mu}_2(q) = \sum_{D \in D(\lambda/\mu)} q^{sp(D)}. \]

**Corollary 3.8.** Let \( \alpha \) and \( \beta \) be fixed partitions and \( n \) and \( m \) be fixed nonnegative integers. Then,
\[ \sum_{\lambda/\alpha \vdash 2m, \lambda/\beta \vdash 2n} f^{\lambda/\alpha}_2(q) f^{\lambda/\beta}_2(q) = \sum_{j \geq 0} \binom{n}{j} \binom{m}{j} (1 + q)^j! \sum_{\beta/\mu \vdash 2(m-j), \alpha/\mu \vdash 2(n-j)} f^{\beta/\mu}_2(q) f^{\alpha/\mu}_2(q). \]

**Proof.** This is immediate from Theorem 3.3 and Proposition 3.6 and the following identity: \( \sum_{\pi \in CP} q^{ev(\pi)} = (1 + q)^j! \). \( \square \)

For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \), let \( 2\lambda \) denote the partition \((2\lambda_1, 2\lambda_2, \ldots, 2\lambda_l)\). We can consider a SYT of shape \( \lambda/\mu \) as a SDT of shape \( 2\lambda/2\mu \) consisting of horizontal dominoes by identifying a cell with a horizontal domino.

There are three interesting specializations of **Corollary 3.8**. When \( q = 0 \) in **Corollary 3.8**, we get the following corollary due to Sagan and Stanley \([16]\). We note that Roby \([15]\) also proved the following corollary using growth diagrams and our proof is essentially the same as Roby’s.

**Corollary 3.9.** \([16]\) Let \( \alpha \) and \( \beta \) be fixed partitions and \( n \) and \( m \) be fixed nonnegative integers. Then,
\[ \sum_{\lambda/\alpha \vdash 2m, \lambda/\beta \vdash 2n} f^{\lambda/\alpha}_2 f^{\lambda/\beta}_2 = \sum_{j \geq 0} \binom{n}{j} \binom{m}{j} j! \sum_{\beta/\mu \vdash m-j, \alpha/\mu \vdash n-j} f^{\beta/\mu}_2 f^{\alpha/\mu}_2. \]

When we set \( q = 1 \) in **Corollary 3.8**, we get a domino analogue.

**Corollary 3.10.** Let \( \alpha \) and \( \beta \) be fixed partitions and \( n \) and \( m \) be fixed nonnegative integers. Then,
\[ \sum_{\lambda/\alpha \vdash 2m, \lambda/\beta \vdash 2n} d^{\lambda/\alpha}_2 d^{\lambda/\beta}_2 = \sum_{j \geq 0} \binom{n}{j} \binom{m}{j} 2^j j! \sum_{\beta/\mu \vdash 2(m-j), \alpha/\mu \vdash 2(n-j)} d^{\beta/\mu}_2 d^{\alpha/\mu}_2. \]

If \( q = -1 \) then, as we will see in the next section, **Corollary 3.8** induces a sign-imbalance formula.
3.3. The weighted sum of domino tableaux. For a SDT $D$, we define the weight $wt_D$ of $D$ by
\[ wt_D = wt_D(x, y, q) = x^{oh(D) - ch(D)} y^{ov(D) - cv(D)} q^{p(D)}. \]

Note that if a colored involution $\pi$ corresponds to $(D, D)$ in the domino Schensted correspondence, then $wt_D = wt_\pi$.

Recall that a reversed shape $\lambda/\mu \vdash -n$ is the one obtained by reversing a skew shape $\mu/\lambda \vdash n$. For a reversed shape $\lambda/\mu \vdash -n$, we define $f_{2}^{\lambda/\mu}(q) = f_{2}^{\mu/\lambda}(q)$.

For a fixed partition $\alpha$ and an integer $n \geq 0$, we define
\[ W_n^\alpha = W_n^\alpha(x, y, q) = \sum_{\lambda/\alpha \vdash 2n} x^{\frac{1}{2}[-\lambda/\alpha - v(\lambda/\alpha)]} y^{\frac{1}{2}[-\lambda/\alpha - h(\lambda/\alpha)]} f_{2}^{\lambda/\alpha}(q), \]
and
\[ W_n^{\alpha} = W_n^{\alpha}(x, y, q) = \sum_{\lambda/\alpha \vdash -2n} x^{\frac{1}{2}[-\lambda/\alpha - v(\lambda/\alpha)]} y^{\frac{1}{2}[-\lambda/\alpha - h(\lambda/\alpha)]} f_{2}^{\lambda/\alpha}(q). \]

Then,
\[ W_n^\alpha = \sum_{\alpha/\lambda \vdash 2n} x^{-n+v(\lambda/\alpha)} y^{-n+h(\lambda/\alpha)} f_{2}^{\lambda/\alpha}(q). \]

Thus $W_n^\alpha = 0$ if $\alpha/\lambda \vdash 2k$ and $n > k$.

By Lemma 2.1 if $n \geq 0$ then $W_n^\alpha$ is the weighted sum of certain SDTs:
\[ W_n^\alpha = \sum_{\lambda/\alpha \vdash 2n} \sum_{D \in D(\lambda/\alpha)} wt_D. \]

We note that $W_n^\alpha$ is a modified generalization of $h_r(n)$ in Lam’s paper [10]:
\[ h_r(n) = \sum_{\lambda/\delta_r \vdash 2n} a^{(o(\lambda) - o(\delta_r))/2} b^{(o(\lambda') - o(\delta_r))/2} c^{d(\lambda) - d(\delta_r)} f_2^{\lambda}(q), \]
where $o(\lambda)$ denotes the number of odd parts in $\lambda$. One can check that $h_r(n) = c^{n} W_n^{k}(bc^{-\frac{1}{2}}, ac^{-\frac{1}{2}}, q)$.

**Theorem 3.11.** Let $\alpha$ be a fixed partition with $\alpha/\lambda \vdash 2k$ and $n \geq 0$. Then
\[ W_n^\alpha = \sum_{j=0}^{k} \binom{n}{j} W_{n-j}^\alpha \sum_{\pi \in CI_{n-j}} wt_\pi. \]

**Proof.** Let $\lambda/\mu \vdash 2n$ and $D \in D(\lambda/\alpha)$. Recall the bijection $\Phi_{sym}$ in Corollary 3.5. Let $(\Phi_{sym})^{-1}(D) = (U, M)$, $sh(U) = \alpha/\mu \vdash 2j$ and $cp(M) = \pi$. Then $j \leq k$ and $\pi \in CI_{n-j}$. By Proposition 3.6, Proposition 3.7 and Lemma 2.1 we have
\[ wt_D = x^{-j+v(\alpha/\mu)} y^{-j+h(\alpha/\mu)} q^{p(U)} wt_\pi. \]

Since $M$ is determined by $\pi$ and choosing $j$ nonzero rows, by Corollary 3.5
\[ W_n^\alpha = \sum_{\lambda/\alpha \vdash 2n} \sum_{D \in D(\lambda/\alpha)} wt_D = \sum_{j=0}^{k} \binom{n}{j} \sum_{\alpha/\mu \vdash 2j} x^{-j+v(\alpha/\mu)} y^{-j+h(\alpha/\mu)} f_{2}^{\alpha/\mu}(q) \sum_{\pi \in CI_{n-j}} wt_\pi = \sum_{j=0}^{k} \binom{n}{j} W_{n-j}^\alpha \sum_{\pi \in CI_{n-j}} wt_\pi. \]
\[ \square \]
Using Theorem 3.11 and Eq. (7), we get a simple generating function for the weighted sum.

**Corollary 3.12.** Let $\alpha$ be a fixed partition. Then
\[
\frac{\sum_{n \geq 0} W_{n}^{\alpha} t^{n}/n!}{\sum_{n \geq 0} W_{n}^{\alpha} t^{n}/n!} = \exp \left( (x + y\sqrt{q})t + (1 + q)t^{2}/2 \right).
\]

If we substitute $\alpha$, $x$, $y$ and $t$ in Corollary 3.12 with $\delta_r$, $bc^{-\frac{1}{2}}$, $ac^{-\frac{1}{2}}$ and $c^\frac{1}{2}t$ respectively then we get Lam’s result [10]:
\[
\sum_{n \geq 0} h_{r}(n) t^{n}/n! = \exp \left( (b + a\sqrt{q})t + c(1 + q)t^{2}/2 \right).
\]

By the argument following Corollary 3.8 if we set $x = 1$ and $y = q = 0$ in Theorem 3.11 then we obtain Sagan and Stanley’s theorem [16] which was reproved by Roby [15], Stanley [20] and Jaggard [8].

**Corollary 3.13.** [16, Sagan and Stanley] Let $\alpha \vdash k$ be a fixed partition. Then
\[
\sum_{\lambda/\alpha \vdash n} f_{\lambda/\alpha} = \sum_{j=0}^{k} \left( \begin{array}{c} n \\ j \end{array} \right) t_{n-j} \sum_{\alpha/\mu \vdash j} f_{\alpha/\mu},
\]
where $t_{m}$ denotes the number of involutions of $[m]$.

If we set $x = y = q = 1$ in Theorem 3.11 we get the following domino analogue.

**Corollary 3.14.** Let $\alpha \vdash \hat{k}$ be a fixed partition with $\alpha/\tilde{\alpha} \vdash 2k$. Then
\[
\sum_{\lambda/\alpha \vdash 2n} d_{\lambda/\alpha} = \sum_{j=0}^{k} \left( \begin{array}{c} n \\ j \end{array} \right) \xi_{n-j} \sum_{\alpha/\mu \vdash 2j} d_{\alpha/\mu},
\]
where $\xi_{m}$ denotes the number of colored involutions of $[m]$.

**Note.** The skew Cauchy identities corresponding to Corollary 3.9 and Corollary 3.13 were obtained by Zelvinsky in the 1985 translation of [13]. The skew domino Cauchy identity corresponding to Corollary 3.8 was introduced by Lam [11].

4. A GENERALIZED SIGN-IMBALANCE FORMULA

4.1. Definition of the sign of a skew SYT. For two cells $a = (i, j)$ and $b = (i', j')$, we write $a \triangleleft b$ if $i < i'$ or $(i = i'$ and $j < j'$). For a SYT $T$, we denote $\text{Inv}(T) = \{(a, b) : a < b, T(a) > T(b)\}$ and $\text{inv}(T) = |\text{Inv}(T)|$.

The sign of a SYT $T$ is defined by $\text{sign}(T) = (-1)^{\text{inv}(T)}$. The sign-imbalance $I_{\lambda}$ of a partition $\lambda$ is defined by
\[
I_{\lambda} = \sum_{T \in \mathcal{T}(\lambda)} \text{sign}(T).
\]

The purpose of this section is to define $I_{\lambda/\mu}$ and generalize Eq. (1) and Eq. (2). In the literature, there are two different definitions for the sign of a skew SYT $T$. We will write them as $\text{sign}_{1}(T)$ and $\text{sign}_{2}(T)$ temporarily. Sjöstrand [19] and Lam [11] used $\text{sign}_{1}(T)$ defined by
\[
\text{sign}_{1}(T) = (-1)^{\text{inv}(T)}.
\]
Thus we have 

\[ D = \text{sign}(T) \]

Then sign(\( T \)) is horizontal then \( #\{ c \in \lambda : a \prec c \prec b \} \mod 2 \).

For example, if \( T_1 = \begin{array}{cc} 0 & 1 \\ 1 & 2 \end{array} \) and \( T_2 = \begin{array}{cc} 2 & 3 \\ 4 & 5 \end{array} \) then \( T_1 \odot T_2 = \begin{array}{cc} 0 & 1 \\ 2 & 3 \\ 4 \end{array} \).

Now we define sign(\( T \)) for a SYT \( T \) of shape \( \lambda / \mu \) by

\[ \text{sign}(T) = \text{sign}(T_0)\text{sign}(T_0 \odot T), \]

where \( T_0 \) is an arbitrary SYT of shape \( \mu \). It is straightforward to show the next proposition which implies that sign(\( T \)) is well-defined.

**Proposition 4.1.** Let \( T \) be a SYT of shape \( \lambda / \mu \). Then sign(\( T_0 \))sign(\( T_0 \odot T \)) is independent of the choice of \( T_0 \in \mathcal{T}(\mu) \). Moreover, sign(\( T_0 \))sign(\( T_0 \odot T \)) = \((-1)^{m}\text{sign}_1(T)\), where \( m = \sum_{j \geq 1} (\lambda_j - \mu_j) \cdot \sum_{j > i} \mu_j \).

We take sign(\( T \)) for the sign of a SYT \( T \). From now on, we will write sign(\( T \)) instead of sign(\( T \)). The sign(\( T \)) has the following product property.

**Proposition 4.2.** Let \( \mu \subset \nu \subset \lambda \), \( T_1 \in \mathcal{T}(\nu / \mu) \) and \( T_2 \in \mathcal{T}(\lambda / \nu) \). Then

\[ \text{sign}(T_1 \odot T_2) = \text{sign}(T_1)\text{sign}(T_2). \]

**Proof.** Let \( T \) be a SYT of shape \( \mu \). Then \( T \odot T_1 \) is a SYT. Thus

\[ \text{sign}(T_1)\text{sign}(T_2) = \text{sign}(T)\text{sign}(T_1)\text{sign}(T_2)\text{sign}(T_1 \odot T_2). \]

The following proposition was proved by White \[25\] and Lam \[10\] for \( \mu = \emptyset \) and \( \mu = (1) \). In our definition of sign(\( D \)), it holds for any \( \mu \). Our proof is similar to Lam’s \[10\] Proposition 21. Recall that a SDT is a SYT with the condition that two cells with entries \( 2i-1 \) and \( 2i \) make a domino. Thus the sign of a SDT is just the sign of a SYT.

**Proposition 4.3.** Let \( D \) be a SDT of shape \( \lambda / \mu \). Then

\[ \text{sign}(D) = (-1)^{ev(D)}. \]

**Proof.** We use induction on \( n \), the number of dominoes in \( D \). It is trivial if \( n = 0 \). Let \( sh(D) = \lambda / \mu \uplus 2n \). Let \( d \) be the domino with entry \( n \) and let \( a \) and \( b \) be the cells in \( d \) with \( a < b \). Let \( D' \) be the SDT obtained from \( D \) by removing \( d \). Let \( T_0 \in \mathcal{T}(\mu) \). Then sign(\( D \)) = sign(\( T_0 \))(-1)\( ^{\text{inv}(T_0 \odot D)} \) and sign(\( D' \)) = sign(\( T_0 \))(-1)\( ^{\text{inv}(T_0 \odot D')} \).

Since \( (T_0 \odot D)(a) \) and \( (T_0 \odot D)(b) \) are greater than any entry of \( T_0 \odot D' \),

\[ \text{inv}(T_0 \odot D) = \text{inv}(T_0 \odot D') \cup \{(a,c) : a < c, c \in \lambda \setminus d\} \cup \{(b,c) : b < c, c \in \lambda \setminus d\}. \]

Thus we have

\[ \text{inv}(T_0 \odot D) \equiv \text{inv}(T_0 \odot D') + \#\{ c \in \lambda : a < c < b \} \mod 2. \]

If \( d \) is horizontal then \( \#\{ c \in \lambda : a < c < b \} = 0 \). If \( d \) is vertical in the \( i \)-th column then \( \#\{ c \in \lambda : a < c < b \} = i - 1 \). Thus

\[ \#\{ c \in \lambda : a < c < b \} \equiv ev(D) - ev(D') \mod 2. \]
Then, using Proposition 4.3, we get Lemma 4.4.

4.2. Sign-imbalance of skew shapes. The sign-imbalance \( I_{\lambda/\mu} \) of a skew shape \( \lambda/\mu \) is defined by

\[
I_{\lambda/\mu} = \sum_{T \in \mathcal{T}(\lambda/\mu)} \text{sign}(T).
\]

Let \( \lambda/\mu \vdash 2n \) and \( T \in \mathcal{T}(\lambda/\mu) \). If \( 2k - 1 \) and \( 2k \) are neither in the same row nor in the same column of \( T \) for some \( k \), let \( T' \) be the SYT obtained from \( T \) by switching the entries \( 2k - 1 \) and \( 2k \) for the smallest such \( k \). Then \( T \mapsto T' \) is a sign reversing involution on \( \mathcal{T}(\lambda/\mu) \setminus \mathcal{D}(\lambda/\mu) \). Thus we only need to consider SDTs. Then, using Proposition 4.3, we get

\[
I_{\lambda/\mu} = \sum_{D \in \mathcal{D}(\lambda/\mu)} \text{sign}(D) = \sum_{D \in \mathcal{D}(\lambda/\mu)} (-1)^{ev(D)}.
\]

The idea of the following lemma is found in the proof of Corollary 24 in Lam’s paper [10].

Lemma 4.4. Let \( n \geq 0 \) and \( \lambda/\mu \vdash 2n \). Then

\[
I_{\lambda/\mu} = (-1)^{-\frac{1}{2}(\frac{1}{2}|\lambda/\mu| - h(\lambda/\mu))} f_{2}^{\lambda/\mu}(-1).
\]

Proof. Using the above argument and Lemma 2.1,

\[
I_{\lambda/\mu} = \sum_{D \in \mathcal{D}(\lambda/\mu)} (-1)^{ev(D)} = \sum_{D \in \mathcal{D}(\lambda/\mu)} (-1)^{-\frac{1}{2}(\text{ov}(D) - ev(D)) + sp(D)}
\]

\[
= (-1)^{-\frac{1}{2}(\frac{1}{2}|\lambda/\mu| - h(\lambda/\mu))} f_{2}^{\lambda/\mu}(-1). \quad \square
\]

Note that, in the above lemma, although both \((-1)^{-\frac{1}{2}(\frac{1}{2}|\lambda/\mu| - h(\lambda/\mu))}\) and \( f_{2}^{\lambda/\mu}(-1) \) lie in \( \mathbb{Z}[\sqrt{-1}] \), it is easy to check that their product is an integer.

Now we get a generalization of Eq. (2) to skew shapes of even size. In Section 5 we prove a stronger theorem which has no restriction on the size of skew shapes.

Corollary 4.5. Let \( \alpha \) and \( \beta \) be fixed partitions and \( n \) and \( m \) be fixed nonnegative integers. Then

\[
\sum_{\lambda/\alpha \vdash 2m, \lambda/\beta \vdash 2n} (-1)^{\nu(\lambda)} I_{\lambda/\alpha} I_{\lambda/\beta} = (-1)^{\nu(\alpha) + \nu(\beta)} \sum_{\beta/\mu \vdash 2m, \alpha/\mu \vdash 2n} (-1)^{\nu(\mu)} I_{\beta/\mu} I_{\alpha/\mu}.
\]

Proof. If \( q = -1 \) in Corollary 3.8 then

\[
\sum_{\lambda/\alpha \vdash 2m, \lambda/\beta \vdash 2n} f_{2}^{\lambda/\alpha}(-1) f_{2}^{\lambda/\beta}(-1) = \sum_{\beta/\mu \vdash 2m, \alpha/\mu \vdash 2n} f_{2}^{\beta/\mu}(-1) f_{2}^{\alpha/\mu}(-1).
\]

Let \( \eta(\lambda) = \frac{1}{2}|\lambda/\lambda| - h(\lambda/\lambda) \). Then for a skew shape \( \lambda/\mu \) with \( \hat{\lambda} = \hat{\mu} \) we have

\[
\frac{|\lambda/\mu|}{2} - h(\lambda/\mu) = \eta(\lambda) - \eta(\mu).
\]
Since we can assume $\tilde{\lambda} = \tilde{\mu} = \tilde{\alpha} = \tilde{\beta}$ (or equivalently, $\lambda/\alpha$, $\lambda/\beta$, $\beta/\mu$ and $\alpha/\mu$ are domino-tileable), by Lemma 4.4 we get
\[
\sum_{\lambda/\alpha \vdash \lambda/\beta} (-1)^{\eta(\lambda)} I_{\lambda/\alpha} I_{\lambda/\beta} = (-1)^{\eta(\alpha)}+\eta(\beta) \sum_{\beta/\mu \vdash \alpha/\mu} (-1)^{\eta(\mu)} I_{\beta/\mu} I_{\alpha/\mu}.
\]
By Lemma 2.4, we have $\eta(\lambda) \equiv v(\lambda/\tilde{\lambda}) \mod 2$, which finishes the proof. \hfill \Box

### 4.3. Definition of a generalized sign-imbalance formula.

Let $\alpha$ be a fixed partition and $n \geq 0$. We define
\[
F^n_\alpha = F^n_\alpha (x, y, z) = \sum_{\lambda/\alpha \vdash \lambda/\alpha} x^{\nu(\lambda/\alpha)} y^{h(\lambda/\alpha)} z^{d(\lambda/\alpha)} I_{\lambda/\alpha}.
\]
Then Eq. (1) can be written as $F^n_\alpha (x, y, z) = (x + y)^\lceil \frac{n}{2} \rceil$.

Let $\alpha^+$ denote the set $\{\lambda : |\lambda| = |\alpha| + 1, \alpha \subset \lambda\}$. For $\lambda \in \alpha^+$, let $u(\lambda, \alpha)$ denote the number of cells $a \in \alpha$ such that $b < a$ for the unique cell $b \in \lambda/\alpha$. For example, if $\alpha = (7, 5, 5, 2)$ and $\lambda = (7, 6, 5, 2)$ then $u(\lambda, \alpha) = 7$.

**Proposition 4.6.** Let $\alpha$ be a fixed partition and $n \geq 0$. Then
\[
F^n_{\alpha+1} = \sum_{\nu \in \alpha^+} (-1)^{u(\nu, \alpha)} \psi_{\nu/\alpha} F^n_{\nu},
\]
where $\psi_{\nu/\alpha} = x^{\nu(\nu/\alpha)} y^{h(\nu/\alpha)} z^{d(\nu/\alpha)}$.

**Proof.** Let $\lambda/\alpha \vdash n + 1$. If $T \in T(\lambda/\alpha)$ then the cell whose entry is 1 must be the unique cell of $\nu/\alpha$ for some $\nu \in \alpha^+$. Since $\nu/\alpha$ contains only one cell, there is a unique SYT of shape $\nu/\alpha$, say $T_{\nu}$. Then sign$(T_{\nu}) = (-1)^{u(\nu, \alpha)}$. Thus $T \in T(\lambda/\alpha)$ if and only if $T = T_{\nu} \circ T'$ for some $\nu \in \alpha^+$ and $T' \in T(\lambda/\nu)$, which implies
\[
I_{\lambda/\alpha} = \sum_{\nu \in \alpha^+} \sum_{T' \in T(\lambda/\nu)} \text{sign}(T_{\nu} \circ T') = \sum_{\nu \in \alpha^+} (-1)^{u(\nu, \alpha)} I_{\lambda/\nu}.
\]
Since $\psi_{\lambda/\alpha} = \psi_{\nu/\alpha} \cdot \psi_{\lambda/\nu}$,
\[
F^n_{\alpha+1} = \sum_{\lambda/\alpha \vdash \lambda/\alpha} \psi_{\lambda/\alpha} \sum_{\nu \in \alpha^+} (-1)^{u(\nu, \alpha)} I_{\lambda/\nu}
= \sum_{\nu \in \alpha^+} (-1)^{u(\nu, \alpha)} \psi_{\nu/\alpha} \sum_{\lambda/\nu \vdash \lambda/\nu} \psi_{\lambda/\nu} I_{\lambda/\nu}
= \sum_{\nu \in \alpha^+} (-1)^{u(\nu, \alpha)} \psi_{\nu/\alpha} F^n_{\nu}.
\]
\hfill \Box

Using Proposition 4.6, we can calculate $F^n_\alpha$ for all $n \geq 0$ if we have $F^n_\alpha$ for all even $n$. Thus we will focus on skew shapes $\lambda/\mu + 2n$ of even size.

We extend the definition of the sign-imbalance $I_{\lambda/\mu}$ to reversed shapes as follows. For a reversed shape $\lambda/\mu \vdash -2n$, define
\[
I_{\lambda/\mu} = (-1)^{-\frac{1}{2}(\frac{1}{2}|\lambda/\mu|-h(\lambda/\mu))} I^{\lambda/\mu}_2 (-1).
\]
Note that the above equation is the same one in Lemma 4.4. We have a relation between $I_{\lambda/\mu}$ and $I_{\mu/\lambda}$.

**Proposition 4.7.** Let $\lambda/\mu \vdash -2n$ be a reversed shape for $n \geq 0$. Then
\[
I_{\lambda/\mu} = (-1)^{v(\mu/\lambda)} I_{\mu/\lambda}.
\]
Proof. If $\mu/\lambda$ is not domino-tileable then $I_{\lambda/\mu} = I_{\mu/\lambda} = 0$. Otherwise, we have $n - h(\mu/\lambda) \equiv v(\mu/\lambda) \mod 2$, by Lemma 2.1. Thus,

$$I_{\lambda/\mu} = (-1)^{-\frac{1}{2}(n - h(\mu/\lambda))} f_{2}^{\mu/\lambda}(-1)$$

$$= (-1)^{n - h(\mu/\lambda)} f_{2}^{\mu/\lambda}(-1)$$

$$= (-1)^{n - h(\mu/\lambda)} I_{\mu/\lambda} = (\lambda/\alpha) I_{\lambda/\alpha}. \quad \square$$

Now we extend the definition of $F_{2n}^{\alpha}$ as follows: for $n \geq 0$, define

$$F_{-2n}^{\alpha} = \sum_{\lambda/\alpha \vdash 2n} x^{v(\lambda/\alpha)} y^{h(\lambda/\alpha)} z^{d(\lambda/\alpha)} I_{\lambda/\alpha}.$$

Then, by Proposition 4.7

$$F_{-2n}^{\alpha} = \sum_{\alpha/\lambda \vdash 2n} (-x)^{-v(\alpha/\lambda)} y^{-h(\alpha/\lambda)} z^{-d(\alpha/\lambda)} I_{\alpha/\lambda}.$$

4.4. A method to obtain a generalized sign-imbalance formula.

Lemma 4.8. Let $\alpha$ be a fixed partition and $n \geq 0$. Then

$$F_{2n}^{\alpha} = W_{n}^{\alpha} ((x\sqrt{z})^{-1}, (y\sqrt{z})^{-1}, 1) \cdot (xy\sqrt{z})^{n},$$

and

$$F_{-2n}^{\alpha} = W_{-n}^{\alpha} ((x\sqrt{z})^{-1}, (y\sqrt{z})^{-1}, 1) \cdot (xy\sqrt{z})^{-n}.$$

Proof. Let $\lambda/\alpha \vdash 2n$ be a domino-tileable skew shape. Then, by Lemma 2.1 we have $\frac{1}{2}(v(\lambda/\alpha) + h(\lambda/\alpha) - \frac{1}{2} |\lambda/\alpha|) = d(\lambda/\alpha)$. Thus

$$W_{n}^{\alpha} ((x\sqrt{z})^{-1}, (y\sqrt{z})^{-1}, 1) \cdot (xy\sqrt{z})^{n}$$

$$= \sum_{\lambda/\alpha \vdash 2n} (x\sqrt{z})^{v(\lambda/\alpha) - \frac{1}{2} |\lambda/\alpha|} (y\sqrt{z})^{h(\lambda/\alpha) - \frac{1}{2} |\lambda/\alpha|} f_{2}^{\lambda/\alpha}(-1) \cdot (xy\sqrt{z})^{d(\lambda/\alpha)}$$

$$= \sum_{\lambda/\alpha \vdash 2n} x^{v(\lambda/\alpha)} y^{h(\lambda/\alpha)} z^{d(\lambda/\alpha)} I_{\lambda/\alpha} = F_{2n}^{\alpha}.$$

Now let $\lambda/\alpha \vdash -2n$ be a reversed shape such that $\alpha/\lambda$ is domino-tileable. Since all the arguments we used here are remaining true when we change $n$ to $-n$, we get the second identity in the lemma as well. \qed

Now we get a generating function for $F_{2n}^{\alpha}$.

Theorem 4.9. Let $\alpha$ be a fixed partition. Then

$$\frac{\sum_{n \geq 0} F_{2n}^{\alpha} z^{n}}{\sum_{n \geq 0} F_{-2n}^{\alpha} z^{n}} = \exp ((x + y)t).$$

Proof. Substitute $x, y, q$ and $t$ in Corollary 3.12 with $(x\sqrt{z})^{-1}, (y\sqrt{z})^{-1}, 1$ and $xy\sqrt{zt}$. Then we get this theorem. \qed

Corollary 4.10. Let $\alpha$ be a fixed partition with $\alpha/\lambda \vdash 2k$. Then, for $n \geq 0$,

$$F_{2n}^{\alpha} = \sum_{j=0}^{k} \binom{n}{j} (x + y)^{n-j} (x^2 y^2 z)^j F_{-2j}^{\alpha}.$$
Table 1. Statistics of $\alpha/\lambda$ for $\alpha = (2, 2)$, $\lambda$ and $j$ with $\alpha/\lambda \vdash 2j$.

| $j$ | 0 | 1 | 2 |
|-----|---|---|---|
| $\lambda$ | (2, 2) | (2) | (1, 1) | $\emptyset$ |
| $\nu(\alpha/\lambda)$ | 0 | 2 | 1 | 2 |
| $h(\alpha/\lambda)$ | 0 | 1 | 2 | 2 |
| $d(\alpha/\lambda)$ | 0 | 1 | 1 | 1 |
| $I_{\alpha/\lambda}$ | 1 | 1 | $-1$ | 0 |

If $\alpha = \delta_r$ then $\alpha/\tilde{\alpha} \vdash 0$. Thus we get the following corollary.

**Corollary 4.11.** For any integers $k \geq 0$ and $n \geq 0$, we have $F_{2n}^{\delta_k} = (x + y)^n$.

The next example shows how to calculate $F_{2n}^{\alpha}$.

**Example 4.12.** Let us find $F_{2n}^{\alpha}$ for $\alpha = (2, 2)$. We have $\tilde{\alpha} = \emptyset$ and $\alpha/\tilde{\alpha} \vdash 4$. Using Table 1 we get

$F_{\alpha/\lambda}^{(2, 2)}(0) = 1$,

$F_{\alpha/\lambda}^{(2, 2)}(1) = -2y^{-1}z^{-1} + (-x)^{-1}y^{-2}z^{-1}(-1) = x^{-2}y^{-2}z^{-1}(x + y)$,

$F_{\alpha/\lambda}^{(2, 2)}(2) = 0$.

Thus $F_{2n}^{(2, 2)} = (n + 1)(x + y)^n$ and

$$\sum_{n \geq 0} F_{2n}^{(2, 2)} \frac{t^n}{n!} = (1 + (x + y)t) \cdot \exp((x + y)t).$$

**4.5. A closed formula for a staircase partition.** Now we can get a closed formula for $F_{2n}^{\delta_k}$.

**Theorem 4.13.** For any integers $k \geq 0$ and $n \geq 0$, we have

$F_{2n}^{\delta_k} = (x + y)^n$,

$$F_{2n+1}^{\delta_k} = \begin{cases} (x + y)^n, & \text{if } k \equiv 0 \mod 4, \\ (x + y)^{n+1}, & \text{if } k \equiv 1 \mod 4, \\ xyz(x + y)^n, & \text{if } k \equiv 2 \mod 4, \\ 0, & \text{if } k \equiv 3 \mod 4. \end{cases}$$

We have already proved the even case in **Corollary 4.10**. For the odd case we need two lemmas. For $0 \leq i \leq k$, let $\delta^i_k$ denote the partition in $\delta^+_k$ obtained from $\delta_k$ by adding the cell $(k + 1 - i, i + 1)$. Recall $\psi_{\lambda/\mu} = x^{\nu(\lambda/\mu)}y^{h(\lambda/\mu)}z^{d(\lambda/\mu)}$, which is used in **Proposition 4.6**.

**Lemma 4.14.** Let $k \geq 0$. Then

$$\sum_{i=0}^{k} (-1)^{\lfloor \frac{i}{2} \rfloor} \psi_{\delta^i_k/\delta_k} = \frac{1 + (-1)^{\lfloor \frac{k}{2} \rfloor}}{2} x^{\frac{k(-(-1)^k)}{2}} + \frac{1 + (-1)^{\lfloor \frac{k-1}{2} \rfloor}}{2} x^{\frac{k(-(-1)^k)}{2}} y^{\frac{k(-(-1)^k)}{2}}.$$
Proof. Since \( \delta_i^k / \delta_k \) contains only one cell \((k-i+1,i+1)\), we have

\[
\sum_{i=0}^{k} (-1)^{\lfloor \frac{i}{2} \rfloor} \psi \delta_i^k / \delta_k = \sum_{i=0}^{k} (-1)^{\lfloor \frac{i}{2} \rfloor} \frac{x^{i+1} y^{i+1}}{2} + \frac{y^{i+1}}{2} yz^{i+1} \sum_{i=0}^{k} (-1)^{\lfloor \frac{i}{2} \rfloor} \frac{y^{i+1}}{2} yz^{i+1}.
\]

Since \( \sum_{i=0}^{m} (-1)^i = \frac{1+(-1)^m}{2} \), we are done. \(\square\)

Lemma 4.15. Let \( j \geq 1 \) and \( \lambda \) be a fixed partition with \( \delta_i^k / \lambda \vdash 2j \) for some \( i = 0, 1, \ldots, k \). Then

\[
\sum_{i=0}^{k} (-1)^{\lfloor \frac{i}{2} \rfloor} I_{\delta_i^k / \lambda} = 0.
\]

Proof. Let \( \mathcal{D} = \bigcup_{k=0}^{\infty} \mathcal{D}(\delta_i^k / \lambda) \). For \( D \in \mathcal{D} \) with \( sh(D) = \delta_i^k / \lambda \), we define \( s(D) = (-1)^{\lfloor \frac{i}{2} \rfloor} I_{\delta_i^k / \lambda} \). Since

\[
\sum_{i=0}^{k} (-1)^{\lfloor \frac{i}{2} \rfloor} I_{\delta_i^k / \lambda} = \sum_{D \in \mathcal{D}} s(D),
\]

it is sufficient to construct an involution \( \omega : \mathcal{D} \rightarrow \mathcal{D} \) satisfying \( s(\omega(D)) = -s(D) \). Let \( D \in \mathcal{D} \) and \( \mathbf{d} \) be the domino of \( D \) with the largest entry. Then \( \mathbf{d} \cap \delta_k \) must have only one cell, say \((a,b)\). Let \( \mathbf{d}' \) be the domino satisfying \( \mathbf{d} \cup \mathbf{d}' = \{(a,b), (a+1,b), (a,b+1)\} \). We define \( \omega(D) \) to be the SDT obtained from \( D \) by changing \( \mathbf{d} \) with \( \mathbf{d}' \), see Fig. 2. It is obvious that \( \omega \) is an involution.

To show \( s(\omega(D)) = -s(D) \), we can assume that \( sh(D) = \delta_i^k / \lambda \) and \( \mathbf{d} \) is a vertical domino. Then \( sh(\omega(D)) = \delta_i^{k+1} / \lambda \) and \( ev(D) - ev(\omega(D)) \) is 1 if \( i \) is odd, and 0 if \( i \) is even, which implies \( (-1)^{ev(D) - ev(\omega(D))} = (-1)^i \). Then we get

\[
\frac{s(D)}{s(\omega(D))} = (-1)^{\lfloor \frac{i}{2} \rfloor + ev(D) - (\lfloor \frac{i+1}{2} \rfloor + i + 1 + ev(\omega(D)))} = (-1)^{\lfloor \frac{i}{2} \rfloor - \lfloor \frac{i+1}{2} \rfloor + i - 1} = -1,
\]

which completes the proof. \(\square\)

Proof of Theorem 4.13. We will show the following equivalent equation:

\[
F_{2n+1} = (x+y)^n \left( \frac{1+(-1)^{\frac{j}{2}}}{2} x^{i-1} + \frac{1+(-1)^{\frac{j-1}{2}}}{2} \frac{y^{i+1}}{2} yz^{i+1} \right).
\]
For the total number of 1’s is \( \Phi \) integers. Then

\[ \text{Theorem 5.1.} \quad \text{Robinson-Schensted correspondence, we can formulate the following theorem.} \]

\[ \text{By Lemma 4.14 it is sufficient to show that, for } j \geq 1, \text{ the following sum is } 0: } \]

\[ \sum_{i=0}^{k} (-1)^{j} \psi_{\delta_k^{(i)}}^{(i)} F_{2j}^{(i)} \]

\[ = \sum_{i=0}^{k} (-1)^{j} \psi_{\delta_k^{(i)}}^{(i)} \sum_{\delta_k^{(i)}} (x-y)^{v(\delta_k^{(i)})} y^{-h(\delta_k^{(i)})} z^{-d(\delta_k^{(i)})} I_{\delta_k^{(i)}}^{(i)} \]

\[ = \sum_{i=0}^{k} (-1)^{j} \sum_{\delta_k^{(i)}} (x-y)^{v(\delta_k^{(i)})} y^{-h(\delta_k^{(i)})} z^{-d(\delta_k^{(i)})} (-1)^{k-i} I_{\delta_k^{(i)}}^{(i)} \]

\[ = \sum_{\lambda} (x-y)^{v(\delta_k^{(i)})} y^{-h(\delta_k^{(i)})} z^{-d(\delta_k^{(i)})} (-1)^{k} \sum_{i=0}^{k} (-1)^{j} I_{\delta_k^{(i)}}^{(i)} \]

where the last sum is over \( \{ \lambda : \delta_k^{(i)} \vdash 2j \text{ for some } i \} \). By Lemma 4.15 we are done. \( \square \)

5. Generalizing Sjöstrand’s theorems

In this section, we only consider (skew) SYTs.

Let \( m_{n,m} \) denote the set of \( n \times m \) matrices whose entries are 0 or 1 such that the total number of 1’s is \( j \) and there is at most one 1 in each row and column. For \( M \in m_{n,m} \), let \( \text{perm}(M) \) denote the permutation whose permutation matrix is obtained from \( M \) by removing the rows and columns without 1’s.

Using the same argument of Theorem 3.3 with the usual local rules for the Robinson-Schensted correspondence, we can formulate the following theorem.

**Theorem 5.1.** Let \( \alpha \) and \( \beta \) be fixed partitions and \( n \) and \( m \) be fixed nonnegative integers. Then \( \Phi = \partial^+ o (\partial^{-})^{-1} \) induces a bijection

\[ \Phi : \bigcup_{j \geq 0} \bigcup_{\delta_k^{(i)}} T(\beta/\mu) \times T(\alpha/\mu) \times m_{n,m} \rightarrow \bigcup_{\lambda/\alpha \vdash m} T(\lambda/\alpha) \times T(\lambda/\beta). \]

The following elegant theorem was proved by Reifegerste [14] and Sjöstrand [18] independently.

**Theorem 5.2.** Let \( \pi \) correspond to \((P, Q)\) in the Robinson-Schensted correspondence and \( sh(P) = \lambda \). Then

\[ \text{sign}(\pi) = (-1)^{v(\lambda)} \text{sign}(P) \text{sign}(Q). \]
By the local rules, the next lemma is an immediate result of Theorem 5.2

**Lemma 5.3.** Let \( P, Q \in T(\lambda) \) and \( M \in \mathfrak{m}_{n,m} \) satisfy \( \Phi(\emptyset_\emptyset, \emptyset_\emptyset, M) = (P, Q) \), where \( \emptyset_\emptyset \) denotes the empty SYT of shape \( \emptyset/\emptyset \). Then

\[
\text{sign}(\text{perm}(M)) = (-1)^{n(\lambda)} \text{sign}(P) \text{sign}(Q).
\]

Let \( k = n + m - j \). For \( M \in \mathfrak{m}_{n,m} \), let \( \overline{M} \) denote the element in \( \mathfrak{m}_{n,m}^{k} \) which can be expressed as \( \binom{A}{B}^{C} \) such that \( A = 0 \), \( \text{perm}(B) = 12 \cdots (m - j) \) and \( \text{perm}(C) = 12 \cdots (n - j) \). It is easy to check that such \( \overline{M} \) exists uniquely. For example, if \( M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \) then \( \overline{M} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \).

For a permutation \( \pi \), let \( \text{inv}(\pi) \) denote the number of inversions, i.e., pairs \((i, j)\) such that \( i < j \) and \( \pi_i > \pi_j \). Let \( \text{inv}(M) = \text{inv}(\text{perm}(\overline{M})) \). The sign of \( M \) is defined by

\[
\text{sign}(M) = (-1)^{\text{inv}(M)}.
\]

For nonnegative integers \( n, k \) and \( a_1, a_2, \ldots, a_r \) such that \( \sum_{i=1}^{r} a_i = n \), we denote \( [n]_q^! = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}) \),

\[
\left[ \begin{array}{c}
 n \\
 a_1, a_2, \ldots, a_r \end{array} \right]_q^! = \frac{[n]_q^!}{[a_1]_q^! [a_2]_q^! \cdots [a_r]_q^!}, \quad \left[ \begin{array}{c}
 n \\
 k, n-k \end{array} \right]_q = \left[ \begin{array}{c}
 n \\
 k \end{array} \right]_q [n]_q^!.
\]

**Proposition 5.4.** Let \( n, m \) and \( j \) be nonnegative integers. Then

\[
\sum_{M \in \mathfrak{m}_{n,m}} q^{\text{inv}(M)} = q^{(n-j)(m-j)} \left[ \begin{array}{c}
 n \\
 j \end{array} \right]_q \left[ \begin{array}{c}
 m \\
 j \end{array} \right]_q [j]_q^!.
\]

**Proof.** Let \( M \in \mathfrak{m}_{n,m} \) and \( \pi = \text{perm}(M) \). Let \( r = r_1 r_2 \cdots r_n \) (resp. \( c = c_1 c_2 \cdots c_m \)) be the \((0, 1)\)-sequence such that \( r_i = 0 \) (resp. \( c_i = 0 \)) if and only if the \( i \)-th row (resp. column) of \( M \) contains 1. Then \( r \in \mathfrak{S}([0^n, 1^{m-j}]) \), \( c \in \mathfrak{S}([0^m, 1^{n-j}]) \) and \( \pi \in \mathfrak{S}([j]) \), where \( \mathfrak{S}(X) \) denotes the set of permutations of \( X \) for a (multi)set \( X \). It is well known, for example see [21, Proposition 1.3.17], that if \( X = \{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\} \) then \( \sum_{\pi \in \mathfrak{S}(X)} q^{\text{inv}(\pi)} = \left[ n \atop a_1, a_2, \ldots, a_n \right]_q^! \). Since \( \text{inv}(M) = (n-j)(m-j) + \text{inv}(r) + \text{inv}(c) + \text{inv}(\pi) \), we are done. \( \Box \)

Substituting \( q = -1 \) in Proposition 5.4 we get the following lemma.

**Lemma 5.5.** Let \( n, m \) and \( j \) be nonnegative integers. Then

\[
\sum_{M \in \mathfrak{m}_{n,m}} \text{sign}(M) = \begin{cases} 
(-1)^{mn}, & \text{if } j = 0, \\
\frac{1}{2}(-1)^{mn}, & \text{if } j = 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Now we can generalize Eq. (5).

**Theorem 5.6.** Let \( U \in T(\beta/\mu) \), \( V \in T(\alpha/\mu) \), \( P \in T(\lambda/\alpha) \), \( Q \in T(\lambda/\beta) \) and a matrix \( M \) satisfy \( \Phi(U, V, M) = (P, Q) \). Then

\[
(-1)^{\nu(\alpha)+\nu(\beta)+\nu(\lambda)} \text{sign}(P) \text{sign}(Q) = (-1)^{\nu(\mu)} \text{sign}(U) \text{sign}(V) \text{sign}(M).
\]
Proof. Let $\lambda/\alpha \vdash m$, $\lambda/\beta \vdash n$ and $\lambda \vdash k$. Let $A \in T(\alpha)$ and $B \in T(\beta)$. Then there is a unique $k \times k$ full growth diagram $G = (\Gamma, N)$ with $\partial^+(G) = (A \diamond P, B \diamond Q)$. It is obvious that $N = (M_{11} M_{12})$ for suitable matrices $M_{11}$, $M_{12}$ and $M_{21}$. We can construct growth diagrams $G_{11}$, $G_{21}$ and $G_{12}$ from $G$ as follows:

\begin{align*}
G_{11} &= \left( (\Gamma_{i,j})_{0 \leq i \leq k-n, 0 \leq j \leq k-m}, M_{11} \right), \\
G_{21} &= \left( (\Gamma_{i,j})_{0 \leq i \leq k, 0 \leq j \leq k-m}, (M_{11} M_{21}) \right), \\
G_{12} &= \left( (\Gamma_{i,j})_{0 \leq i \leq k-n, 0 \leq j \leq k}, (M_{11} M_{12}) \right).
\end{align*}

Let $U_0 = (G_{11})_{\text{bottom}} \in T(\mu)$ and $V_0 = (G_{11})_{\text{right}} \in T(\mu)$, where $C_{\text{SYT}}$ is defined similarly to $C_{\text{SDT}}$ in Section 3. See Fig. 3, which roughly represents $G$ and these SYTs.

Let $\text{perm}(M_{11}) = \gamma$, $\text{cp}(M_{11} M_{21}) = \sigma$, $\text{perm}(M_{11} M_{12}) = \tau$ and $\text{cp}(M_{11} M_{12}^T) = \pi$. Then $\text{sign}(\pi) = \text{sign}(\sigma) \text{sign}(\tau) \text{sign}(\gamma) \text{sign}(M)$, and by Lemma 5.3

\begin{align*}
\text{sign}(\pi) &= (-1)^{v(\lambda)} \text{sign}(A \diamond P) \text{sign}(B \diamond Q), \\
\text{sign}(\sigma) &= (-1)^{v(\alpha)} \text{sign}(A) \text{sign}(V_0 \diamond V), \\
\text{sign}(\tau) &= (-1)^{v(\beta)} \text{sign}(U_0 \diamond U) \text{sign}(B), \\
\text{sign}(\gamma) &= (-1)^{v(\mu)} \text{sign}(U_0) \text{sign}(V_0).
\end{align*}

Multiplying the above five equations, we get this theorem. \hfill \Box

Remark. Sjöstrand’s theorem, which is Eq. (5), is stated in a different way, however, it is not difficult to see that it is equivalent to Theorem 5.6 with $\alpha = \beta$. Also note that, Sjöstrand used sign$_1$ for the sign of a SYT. Despite the different definitions, by Proposition 4.1 if $sh(P) = sh(Q)$ then $\text{sign}_1(P) \text{sign}_1(Q) = \text{sign}(P) \text{sign}(Q)$.

Using Theorem 5.1, Lemma 5.5 and Theorem 5.6, we get the following generalization of Eq. (6).
Theorem 5.7. Let $\alpha$ and $\beta$ be fixed partitions and $n$ and $m$ be fixed nonnegative integers. Then

$$(-1)^{v(\alpha)+v(\beta)} \sum_{\lambda/\alpha \vdash m, \lambda/\beta \vdash n} (-1)^{v(\lambda)} I_{\lambda/\alpha}I_{\lambda/\beta}$$

$$= (-1)^{mn} \sum_{\beta/\mu \vdash m, \alpha/\mu \vdash n} (-1)^{v(\mu)} I_{\beta/\mu}I_{\alpha/\mu} + \frac{1}{2} (-1)^{mn} \sum_{\beta/\mu \vdash m-1, \alpha/\mu \vdash n-1} (-1)^{v(\mu)} I_{\beta/\mu}I_{\alpha/\mu}.$$ 

ACKNOWLEDGMENTS

I would like to thank my advisor, Dongsu Kim, for his encouragement and helpful comments. I would also like to thank Seunghyun Seo for his careful reading of the first version of this paper.

REFERENCES

[1] Dan Barbasch and David Vogan. Primitive ideals and orbital integrals in complex classical groups. *Math. Ann.*, 259(2):153–199, 1982.
[2] S. V. Fomin. The generalized Robinson-Schensted-Knuth correspondence. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 155(Differentialnaya Geometriya, Gruppy Li i Meh. VIII):156–175, 195, 1986.
[3] Sergey Fomin. Duality of graded graphs. *J. Algebraic Combin.*, 3(4):357–404, 1994.
[4] Sergey Fomin. Schensted algorithms for dual graded graphs. *J. Algebraic Combin.*, 4(1):5–45, 1995.
[5] Sergey Fomin. Schur operators and Knuth correspondences. *J. Combin. Theory Ser. A*, 72(2):277–292, 1995.
[6] Devra Garfinkle. On the classification of primitive ideals for complex classical Lie algebra. II. *Compositio Math.*, 81(3):307–336, 1992.
[7] Frank Garvan, Dongsu Kim, and Dennis Stanton. Cranks and $t$-cores. *Invent. Math.*, 101(1):1–17, 1990.
[8] Aaron D. Jaggard. Subsequence containment by involutions. *Electron. J. Combin.*, 12:Research Paper 14, 15 pp. (electronic), 2005.
[9] Gordon James and Adalbert Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.
[10] Thomas Lam. Growth diagrams, domino insertion and sign-imbalance. *J. Combin. Theory Ser. A*, 107(1):87–115, 2004.
[11] Thomas Lam. On Sj¨ ostrand skew sign-imbalance identity. arXiv:math.CO/0607516, 2006.
[12] Thomas Lam. Signed differential posets and sign-imbalance. *J. Combin. Theory Ser. A*, 115(3):466–484, 2008.
[13] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
[14] Astrid Reifegerste. Permutation sign under the Robinson-Schensted correspondence. *Ann. Comb.*, 8(1):103–112, 2004.
[15] Tom Roby. *Applications and Extensions of Fomin’s Generalization of the Robinson-Schensted Correspondence to Differential Posets*. PhD thesis, MIT, 1991.
[16] Bruce E. Sagan and Richard P. Stanley. Robinson-Schensted algorithms for skew tableaux. *J. Combin. Theory Ser. A*, 55(2):161–193, 1990.
[17] Mark Shimozono and Dennis E. White. A color-to-spin domino Schensted algorithm. *Electron. J. Combin.*, 8(1):Research Paper 21, 50 pp. (electronic), 2001.
[18] Jonas Sj¨ ostrand. On the sign-imbalance of partition shapes. *J. Combin. Theory Ser. A*, 111(2):190–203, 2005.
[19] Jonas Sjöstrand. On the sign-imbalance of skew partition shapes. *European J. Combin.*, 28(6):1582–1594, 2007.

[20] R. P. Stanley. On the enumeration of skew Young tableaux. *Adv. in Appl. Math.*, 30(1-2):283–294, 2003. Formal power series and algebraic combinatorics (Scottsdale, AZ, 2001).

[21] Richard P. Stanley. *Enumerative Combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[22] Richard P. Stanley. *Enumerative Combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[23] Richard P. Stanley. Some remarks on sign-balanced and maj-balanced posets. *Adv. in Appl. Math.*, 34(4):880–902, 2005.

[24] Marc A. A. van Leeuwen. The Robinson-Schensted and Schützenberger algorithms, an elementary approach. *Electron. J. Combin.*, 3(2):Research Paper 15, approx. 32 pp. (electronic), 1996. The Foata Festschrift.

[25] Dennis E. White. Sign-balanced posets. *J. Combin. Theory Ser. A*, 95(1):1–38, 2001.

E-mail address: jskim@kaist.ac.kr