Obtaining Measure Concentration from Markov Contraction

Leonid (Aryeh) Kontorovich*
Department of Mathematics
Weizmann Institute of Science
Rehovot, Israel

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Abstract

Concentration bounds for non-product, non-Haar measures are fairly recent: the first such result was obtained for contracting Markov chains by Marton in 1996. Since then, several other such results have been proved; with few exceptions, these rely on coupling techniques. Though coupling is of unquestionable utility as a theoretical tool, it appears to have some limitations. Coupling has yet to be used to obtain bounds for more general Markov-type processes: hidden (or partially observed) Markov chains, Markov trees, etc. As an alternative to coupling, we apply the elementary Markov contraction lemma to obtain simple, useful, and apparently novel concentration results for the various Markov-type processes. Our technique consists of expressing probabilities as matrix products and applying Markov contraction to these expressions; thus it is fairly general and holds the potential to yield numerous results in this vein.

1 Introduction

1.1 Background

In 1996 Marton [20] published a concentration inequality for contracting Markov chains – apparently, the first such result for a non-product, non-Haar measure. In the decade that followed, Marton and others continued to distill and expand a key insight: analogues of the Azuma-Hoeffding-McDiarmid inequality [2, 10, 25] for independent random variables may be obtained for dependent ones, provided a strong mixing condition holds.

To recall, the aforementioned inequality implies that if \( \mu \) is a product distribution on \( \Omega^n \) and \( f : \Omega^n \to \mathbb{R} \) satisfies \( \|f\|_{\text{Lip}} \leq n^{-1} \) under the Hamming metric, we have

\[
\mu \{|f - \mu f| > t\} \leq 2 \exp(-2nt^2).
\]

In [20], Marton pioneered the transportation method for proving concentration inequalities. This technique is in principle applicable to arbitrary nonproduct measures, and when applied to Markov chains \( \mu \) with contraction coefficient \( \theta < 1 \), it yields

\[
\mu \{|f - M_f| > t\} \leq 2 \exp \left[ -2n \left( t(1 - \theta) - \sqrt{\frac{\log 2}{2n}} \right)^2 \right],
\]

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where $M_f$ is a $\mu$-median of $f$. Since product distributions are degenerate cases of Markov chains (with $\theta = 0$), Marton’s result is a powerful generalization of (1).

The Markov contractivity condition $\theta < 1$ implies strong mixing, and in a series of papers [21, 22, 23], Marton gave other concentration results for dependent variables under various metrics and types of mixing. In particular, Theorem 2 of [21] gives a generic mixing condition which implies a transportation inequality and therefore concentration.

Further progress in obtaining concentration from mixing was made, among others, in [21, 22, 23]. Using Stein’s method for exchangeable pairs, Chatterjee [5] obtained an elegant concentration inequality in terms of a Dobrushin-Shlosman type contractivity condition. Samson [29] was apparently the first to use explicit mixing coefficients in a concentration result. Since these are central to this paper we define them without further delay; the (standard) notation is clarified in Section 1.3.

Let $\mu$ be the joint distribution of $(X_1, \ldots, X_n)$, $X_i \in \Omega$. For $1 \leq i < j \leq n$ and $x \in \Omega^i$, we denote by

$$
\mu((X_j, \ldots, X_n) | (X_1, \ldots, X_i) = x)
$$

the distribution of $(X_j, \ldots, X_n)$ conditioned on $(X_1, \ldots, X_i) = x$. For $y \in \Omega^{i-1}$ and $w, w' \in \Omega$, define

$$
\eta_{ij}(y, w, w') = \|\mu((X_j, \ldots, X_n) | (X_1, \ldots, X_i) = yw) - \mu((X_j, \ldots, X_n) | (X_1, \ldots, X_i) = yw')\|_{TV},
$$

and

$$
\bar{\eta}_{ij} = \sup_{y \in \Omega^{i-1}, w, w' \in \Omega} \eta_{ij}(y, w, w').
$$

(3)

The coefficients $\bar{\eta}_{ij}$, termed $\eta$-mixing coefficients\(^1\) in [15], play a key role in several recent concentration results. Define $\Gamma$ and $\Delta$ to be upper-triangular $n \times n$ matrices, with $\Gamma_{ii} = \Delta_{ii} = 1$ and

$$
\Gamma_{ij} = \sqrt{\bar{\eta}_{ij}}, \quad \Delta_{ij} = \bar{\eta}_{ij}
$$

for $1 \leq i < j \leq n$.

In 2000, Samson [29] proved that any distribution $\mu$ on $[0,1]^n$ and any convex $f : [0,1]^n \to \mathbb{R}$ with $\|f\|_{\text{Lip}} \leq 1$ (with respect to $\ell_2$) satisfy

$$
\mu\{|f - \mu f| > t\} \leq 2 \exp\left( -\frac{t^2}{2 \|\Gamma\|_2^2} \right)
$$

(4)

where $\|\Gamma\|_2$ is the $\ell_2$ operator norm.

In 2007, almost synchronously and using different techniques, Chazottes et al. [6] and the author with K. Ramanan [15] showed that any distribution $\mu$ on $\Omega^n$ and any $f : \Omega^n \to \mathbb{R}$ with $\|f\|_{\text{Lip}} \leq n^{-1/2}$ (with respect to the Hamming metric) satisfy

$$
\mu\{|f - \mu f| > t\} \leq 2 \exp\left( -\frac{t^2}{2 \|\Delta\|_{\infty}^2} \right)
$$

(5)

\(^1\)That choice of terminology is perhaps suboptimal in light of the unrelated notion of $\eta$-weak dependence of Doukhan et al. [8], but the sufficiently distinct contexts should prevent confusion.
where \( \| \Delta \|_\infty \) is the \( \ell_\infty \) operator norm (\( \| \Delta \|_\infty \) may be replaced by \( \| \Delta \|_2 \) and [6] achieves a better constant in the exponent).

The results (4) and (5) are not readily comparable as they hold in different spaces for different metrics with different normalization, and the former requires convexity. They share the feature of establishing concentration for a wide class of measures, in terms of the natural mixing coefficients \( \tilde{\eta}_{ij} \). Indeed, since

\[
\| \Delta \|_\infty = \max_{1 \leq i < n} (1 + \tilde{\eta}_{i,i} + \tilde{\eta}_{i,i+1} + \ldots + \tilde{\eta}_{i,n})
\]

and by the Geršgorin disc theorem [11]

\[
\| \Gamma \|_2^2 = \lambda_{\text{max}}(\Gamma^T \Gamma) \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} (\Gamma^T \Gamma)_{ij},
\]

suitable upper estimates on \( \tilde{\eta}_{ij} \) provide bounds for \( \| \Gamma \|_2 \) and \( \| \Delta \|_\infty \).

Aside from the straightforward observation (due to Samson) that the \( \eta \)-mixing coefficients are bounded by the \( \phi \)-mixing ones (see [4]), we are only aware of a few of cases where simple, readily computed estimates on \( \tilde{\eta}_{ij} \) are given. In particular, Samson [29] controls \( \tilde{\eta}_{ij} \) by the contraction coefficients of a Markov chain, and Chazottes et al. [6] give some estimates on \( \tilde{\eta}_{ij} \) for various temperature regimes of Gibbs random fields. The estimates quoted above are obtained via the coupling method – which, while powerful, often requires some ingenuity to construct the requisite joint distribution, even in the simple case of a Markov chain [20, 29]. In some cases, the coupling may even elude explicit construction [6].

As the random processes of interest become more complex, it becomes progressively more difficult to obtain estimates on mixing coefficients via coupling. We are particularly interested in examining the \( \eta \)-mixing of several Markov-type processes, motivated by statistical and computer science applications. Hidden Markov Models (HMMs) have been used in natural language processing [18, 27] and signal processing [24] for decades, with considerable success. Concentration bounds for Markov Chains (and more generally, HMMs) have implications in machine learning and empirical process theory [9, 13]. A Markov-type process called the Markov marginal process (MMP) in [14] underlies adaptive Markov Chain Monte Carlo simulations [1]; these evolve according to an inhomogeneous Markov kernel, which in addition to time also depends on the path history. In a forthcoming work, A. Brockwell and the author give strong laws of large numbers for MMPs in terms of the \( \eta \)-mixing coefficients. Random processes indexed by trees have been attracting the attention of probability theorists for some time [3, 26], and the principal technical contribution of this paper is a bound on \( \tilde{\eta}_{ij} \) for these types of processes.

Our results do not invoke the coupling method but rather rely on the Markov contraction lemma (Lemma 2.1). The technique provides novel concentration bounds for the processes listed above – results which the coupling method has yet to yield or reproduce.

Remark 1.1. On some level, the distinction between our technique and the coupling method is semantic. From conversations with experts it appears that what we call here “Markov contraction” is commonly referred to as “coupling”. The novelty of our method lies in (i) avoiding any constructions (implicit or explicit) of joint distributions (ii) rewriting complicated sums as simple(r) matrix and tensor products (iii) applying Lemma 2.1 to the latter expressions. Thus it seems that our method is sufficiently different from classical coupling techniques, both in execution and results obtained, to merit the terminological distinction.
1.2 Main results

In this paper we present estimates on the $\eta$-mixing coefficients $\bar{\eta}_{ij}$ defined in (3), for the various Markov-type processes mentioned above. These bounds immediately imply concentration inequalities for a wide class of metrics and measures, via (4) and (5).

The precise statements of the results require preliminary definitions and are postponed until later sections. The main technical contribution of this paper is Theorem 4.1, which bounds $\eta$-mixing coefficients for Markov-tree processes, yielding what appears to be the first concentration of measure result for these. However, we give equal priority to the goal of presenting Markov contraction as a versatile new method for bounding $\bar{\eta}_{ij}$ – a global function of the distribution $\mu$ – by some local, easily computed contraction coefficients of $\mu$. For example, let $\mu$ be an inhomogeneous Markov chain defined by the transition kernels $\{p_i : 0 \leq i < n\}$, which induces a density on $\Omega^n$ by

$$\mu(x) = p_0(x_1) \prod_{i=1}^{n-1} p_i(x_{i+1} | x_i), \quad x \in \Omega^n.$$ 

Define the $i$th contraction coefficient:

$$\theta_i = \sup_{y, y' \in \Omega} \|p_i(\cdot | y) - p_i(\cdot | y')\|_{TV}, \quad 1 \leq i < n. \quad (6)$$

This quantity turns out to control the $\eta$-mixing coefficients for $\mu$:

$$\bar{\eta}_{ij} \leq \theta_i \theta_{i+1} \cdots \theta_{j-1}$$

– a fact which is proved in [29] using coupling. In [15] we gave an (arguably simpler) alternative proof, which paves the way for the several new results presented here.

This paper is organized as follows. In Section 1.3 we summarise some basic notation used throughout the paper. Some auxiliary lemmas are given in Section 2. The remaining three sections deal with bounding $\bar{\eta}_{ij}$ for Markov chains, Markov tree processes, and Markov marginal processes, respectively.

1.3 Notation and definitions

Since the contribution of this paper is not measure-theoretic in nature, we henceforth take $\Omega$ to be a finite set. Extensions to the countable case are quite straightforward [15] and the continuous case, under mild assumptions, is not much more difficult [13, 14].

We use the terms measure, density and distribution interchangeably; all measures are probabilities unless noted otherwise. If $\mu$ is a measure on $\Omega^n$ and $f : \Omega^n \to \mathbb{R}$, we use the standard notation

$$\mu f = \int_{\Omega^n} f d\mu$$

and write

$$\mu \{|f - \mu f| > t\}$$
as a shorthand for
\[ \mu \left( \{ x \in \Omega^n : |f(x) - \mu f| > t \} \right). \]

The (unnormalized) Hamming metric on \( \Omega^n \) is defined by
\[ d(x, y) = \sum_{i=1}^{n} \mathbb{1}_{\{x_i \neq y_i \}}, \quad x, y \in \Omega^n, \]
where the indicator variable \( \mathbb{1}_{\{ \cdot \}} \) assigns 0-1 truth values to the predicate in \( \{ \cdot \} \).

The Lipschitz constant of a function, with respect to some metric \( d \), is defined by
\[ \| f \|_{\text{Lip}} = \sup_{x \neq y \in \Omega^n} \frac{|f(x) - f(y)|}{d(x, y)}. \]

Random variables are capitalized (\( X \)), specified sequences are written in lowercase (\( x \in \Omega^n \)), the shorthand \( X^j_i = (X_i, \ldots, X_j) \) is used for all sequences, and sequence concatenation is denoted multiplicatively: \( x^j_i x^k_{j+1} = x^k_i \). Sums will range over the entire space of the summation variable; thus \( \sum_{x_i^j} f(x_i^j) \) stands for \( \sum_{x_i^j \in \Omega^{j-i+1}} f(x_i^j) \).

By convention, when \( i > j \), we define
\[ \sum_{x_i^j} f(x_i^j) \equiv f(\varepsilon) \]
where \( \varepsilon \) is the null sequence. Products of spaces and measures are denoted by \( \otimes \).

The total variation norm of a signed measure \( \nu \) on \( \Omega^n \) (i.e., vector \( \nu \in \mathbb{R}^{\Omega^n} \)) is defined by
\[ \| \nu \|_{\text{TV}} = \frac{1}{2} \| \nu \|_1 = \frac{1}{2} \sum_{x \in \Omega^n} |\nu(x)| \]
(the factor of 1/2 is not entirely standard). For readability, we will drop the subscript TV from the norm; thus everywhere in the sequel, \( \| \cdot \| \) will mean \( \| \cdot \|_{\text{TV}} \).

A signed measure \( \nu \) on a set \( \mathcal{X} \) is called balanced if \( \nu(\mathcal{X}) = 0 \). Departing from standard convention, our stochastic matrices will be column- (as opposed to row-) stochastic.

## 2 Contraction and tensorization

Our method for bounding \( \eta \)-mixing coefficients rests on the following simple result:

**Lemma 2.1.** Let \( P : \mathbb{R}^\Omega \rightarrow \mathbb{R}^\Omega \) be a Markov operator:
\[ (P\nu)(x) = \sum_{y \in \Omega} P(x | y) \nu(y), \]
where $P(x | y) \geq 0$ and $\sum_{x \in \Omega} P(x | y) = 1$. Define the contraction coefficient of $P$ as above:

$$\theta = \max_{y, y' \in \Omega} \| P(\cdot | y) - P(\cdot | y') \|.$$

Then

$$\| P \nu \| \leq \theta \| \nu \|$$

for any balanced signed measure $\nu$ on $\Omega$ (i.e., $\nu \in \mathbb{R}^\Omega$ with $\sum_{x \in \Omega} \nu(x) = 0$).

This result is sometimes credited to Dobrushin [7]; the quantity $\theta$ has been referred to in the literature, alternatively, as the Doeblin contraction or Dobrushin ergodicity coefficient. However, the observation apparently goes as far back as Markov himself [19] (see [15] for a proof), so it seems appropriate to refer to the result above as the Markov contraction Lemma.

Another important property of the total variation norm is that it tensorizes, in the following way:

**Lemma 2.2.** Consider two finite sets $\mathcal{X}, \mathcal{Y}$, with probability measures $p,p'$ on $\mathcal{X}$ and $q,q'$ on $\mathcal{Y}$. Then

$$\| p \otimes q - p' \otimes q' \| \leq \| p - p' \| + \| q - q' \| - \| p - p' \| \cdot \| q - q' \|.$$

This fact seems to be folklore knowledge; we were not able to locate it in published literature. A proof using coupling is straightforward, and a non-coupling proof is given in [13].

### 3 Markov chains

#### 3.1 Directed

Technically, this section might be considered superfluous, since this result has already appeared in [15], and is strictly generalized in later sections. However, we find it instructive to work out the simple Markov case as it provides the cleanest illustration of our technique.

Let $\mu$ be an inhomogeneous Markov measure on $\Omega^n$, induced by the kernels $p_0$ and $p_i(\cdot | \cdot)$, $1 \leq i < n$. Thus,

$$\mu(x) = p_0(x_1) \prod_{i=1}^{n-1} p_i(x_{i+1} | x_i).$$

The $i^{th}$ contraction coefficient, $\theta_i$ is defined as in (6). As stated in the Introduction, Markov contraction provides an estimate on estimate $\eta$-mixing:

**Theorem.**

$$\bar{\eta}_{ij} \leq \theta_i \theta_{i+1} \ldots \theta_{j-1}.$$

**Proof.** Fix $1 \leq i < j \leq n$ and $y_{i-1}^1 \in \Omega^{i-1}$, $w_i, w_{i}' \in \Omega$. Then

$$\eta_{ij}(y, w, w') = \frac{1}{2} \sum_{x_j^n} \left| \mu(x_j^n | y_{i-1}^1 w_i) - \mu(x_j^n | y_{i-1}^1 w_i') \right|$$

$$= \frac{1}{2} \sum_{x_j^n} \pi(x_j^n) | \zeta(x_j) |$$
where
\[
\pi(u_\ell^k) = \prod_{t=k}^{\ell-1} p_t(u_{t+1} \mid u_t)
\]
and
\[
\zeta(x_j) = \begin{cases} 
\sum_{j_i+1} p_{j-1}(x_j \mid z_{j-1}) \pi(z_{j+1}^{j-1}) \left( p_i(z_{i+1} \mid w_i) - p_i(z_{i+1} \mid w'_i) \right), & j - i > 1 \\
p_i(x_j \mid w_i) - p_i(x_j \mid w'_i), & j - i = 1.
\end{cases}
\]

Define \( h \in \mathbb{R}^\Omega \) by \( h_v = p_i(v \mid w_i) - p_i(v \mid w'_i) \) and \( P(k) \in \mathbb{R}^{\Omega \times \Omega} \) by \( P^{(k)} = p_h(u \mid v) \). Likewise, define \( z \in \mathbb{R}^\Omega \) by \( z_v = \zeta(v) \). It follows that
\[
z = P(j-1)P(j-2) \ldots P(i+2)P(i+1)h.
\]

The claim follows by (repeated applications of) the Markov contraction lemma.

The reader may wish to compare this proof with Samson’s [29].

### 3.2 Undirected

In this section we analyze Markov chains under a different parametrization, in an “undirected graphical model” setting [17]. For any graph \( G = (V,E) \), where \( |V| = n \) and the maximal cliques have size 2 (i.e., are edges), we can define a measure on \( \Omega^V = \Omega^n \) as follows
\[
\mu(x) = \frac{\prod_{(i,j) \in E} \psi_{ij}(x_i, x_j)}{\sum_{x' \in \Omega^n} \prod_{(i,j) \in E} \psi_{ij}(x'_i, x'_j)}, \quad x \in \Omega^n
\]
for some for some nonnegative “potential functions” \( \psi_{ij} \).

Consider the very simple case of chain graphs; any such measure is a Markov measure on \( \Omega^n \). We can relate the induced Markov transition kernel \( p_i(\cdot \mid \cdot) \) to the random field measure \( \mu \) as follows:
\[
p_i(x \mid y) = \frac{\sum_{v' \in \Omega} \sum_{z_i=2}^{v_i-1} \mu(y'yxz)}{\sum_{x' \in \Omega} \sum_{v' \in \Omega} \sum_{z_i=2}^{v_i-1} \mu(y'yx'z')}, \quad x, y \in \Omega.
\]

Our goal is to bound the \( i^{th} \) contraction coefficient \( \theta_i \) of the Markov chain in terms of \( \psi_{ij} \). We claim a simple relationship between \( \theta_i \) and \( \psi_{ij} \):
Theorem 3.1.

\[ \theta_i \leq \frac{R_i - r_i}{R_i + r_i}, \quad 1 \leq i < n \]  

(9)

where

\[ R_i = \max_{x,y \in \Omega} \psi_{i,i+1}(x, y) \]

and

\[ r_i = \min_{x,y \in \Omega} \psi_{i,i+1}(x, y). \]

First we prove a simple lemma:

Lemma 3.2. Let \( \alpha, \beta, \gamma \in \mathbb{R}^{k+1}_+ \) and \( r, R \in \mathbb{R} \) be such that \( 0 \leq r \leq \alpha_i, \beta_i \leq R \), for \( 1 \leq i \leq k + 1 \). Then

\[ \frac{1}{2} \sum_{i=1}^{k+1} \left| \frac{\alpha_i \gamma_i}{\sum_{j=1}^{k+1} \alpha_j \gamma_j} - \frac{\beta_i \gamma_i}{\sum_{j=1}^{k+1} \beta_j \gamma_j} \right| \leq \frac{R - r}{R + r}. \]

(10)

Proof. When \( p, q \in \mathbb{R}^{k+1}_+ \) are two distributions satisfying \( 0 < r \leq p_i, q_i \), it is straightforward to verify that \( \|p - q\|_1 \) may be maximized, with value \( d \), by choosing \( a \in [r, (1 - d)/k] \), \( b = a + d/k \) and setting \( p_i = a, q_i = b \) for \( 1 \leq i \leq k \) and \( p_{k+1} = 1 - ka, q_{k+1} = 1 - kb \). Applying this principle to (9), we obtain

\[ \sum_{i=1}^{k+1} \left| \frac{\alpha_i \gamma_i}{\sum_{j=1}^{k+1} \alpha_j \gamma_j} - \frac{\beta_i \gamma_i}{\sum_{j=1}^{k+1} \beta_j \gamma_j} \right| \leq \frac{gR - g'r}{gR + g'r} \leq \frac{2g''(R^2 - r^2)}{(R + g''kr)(g''kR + r)} \]

where \( g = \sum_{i=1}^k \gamma_i, g' = \gamma_{k+1} \) and \( g'' = g/g' \).

Define \( f : \mathbb{R}_+ \to \mathbb{R} \) by

\[ f(x) = \frac{2(R^2 - r^2)x}{(R + rx)(Rx + r)}; \]

elementary calculus verifies that \( f \) is maximized at \( x = 1 \).

Proof of Theorem 3.1. Let us define the shorthand notation:

\[ \pi(u_k^l) = \prod_{t=k}^{l-1} \psi_{t,t+1}(u_t, u_{t+1}) \]

Then we expand

\[
p_i(x | y) = \frac{\sum_{v_{i-1}^1} \sum_{z_{i+2}^n} \pi(v_{i-1}^{i-2}) \psi_{i-1,i}(v_{i-1}, y) \psi_{i,i+1}(y, x) \psi_{i+1,i+2}(x, z_{i+2}) \pi(z_{i+2}^n)}{\sum_{x' \in \Omega} \sum_{v_{i-1}^1} \sum_{z_{i+2}^n} \pi(v_{i-1}^{i-2}) \psi_{i-1,i}(v_{i-1}, y) \psi_{i,i+1}(y, x') \psi_{i+1,i+2}(x', z_{i+2}^n) \pi(z_{i+2}^n)}
\]

\[ = \frac{\psi_{i,i+1}(y, x) a_{yx}}{\sum_{x' \in \Omega} \psi_{i,i+1}(y, x') a_{yx'}} \]

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where
\[
a_{yx} = \sum_{v_{i-1}^{i+2}} \pi(v_{i-1}^{i+2})\psi_{i-1, i}(v_{i-1}, y)\psi_{i+1, i+2}(x, z_{i+2})\pi(z_{i+2})
\]

(we take the natural convention that \(\psi_{i, j}(\cdot, \cdot) = 1\) whenever \((i, j) \notin E\).

Fix \(y, y' \in \Omega\). Define the quantities, for each \(x \in \Omega\):
\[
\begin{align*}
\alpha_x &= \psi_{i, i+1}(y, x) \\
\beta_x &= \psi_{i, i+1}(y', x) \\
\gamma_x &= a_{yx} \\
\gamma'_x &= a_{y'x}.
\end{align*}
\]

Then
\[
\sum_{x \in \Omega} |p_i(x \mid y) - p_i(x \mid y')| = \sum_{x \in \Omega} \left| \frac{\alpha_x \gamma_x}{\sum_{x' \in \Omega} \alpha_{x'} \gamma_{x'}} - \frac{\beta_x \gamma'_x}{\sum_{x' \in \Omega} \beta_{x'} \gamma'_{x'}} \right| \tag{11}
\]
\[
= \sum_{x \in \Omega} \left| \frac{\alpha_x \gamma_x}{\sum_{x' \in \Omega} \alpha_{x'} \gamma_{x'}} - \frac{\beta_x \gamma_x}{\sum_{x' \in \Omega} \beta_{x'} \gamma'_{x'}} \right| \tag{12}
\]

the last equality follows since \(\gamma'_x = c \gamma_x\), where \(c = \frac{\psi_{i, i+1}(v_{i-1}, y)}{\psi_{i, i+1}(v_{i-1}, y')}\). Now Lemma 3.2 can be applied to establish the claim. \(\square\)

\section{Markov tree processes}

\subsection{Preliminaries}

We begin by defining some notation specific to this section. A collection of variables may be indexed by subset: if \(x \in \Omega^V\) and \(I \subseteq V\) with \(I = \{i_1, i_2, \ldots, i_m\}\), then we write \(x_I \equiv x[I] = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}\); we will write \(x_I\) and \(x[I]\) interchangeably, as dictated by convenience. To avoid cumbersome subscripts, we will also occasionally use the bracket notation for vector components. Thus, if \(u \in \mathbb{R}^\Omega\), then
\[
u_x I \equiv u_{x[I]} \equiv u[x_I] \equiv u[x[I]] = u_{(x_{i_1}, x_{i_2}, \ldots, x_{i_m})} \in \mathbb{R}
\]
for each \(x[I] \in \Omega^I\). A similar bracket notation will apply for matrices. If \(A\) is a matrix then \(A_{*, j} = A[*, j]\) will denote its \(j^{th}\) column. We will use \(|\cdot|\) to denote set cardinalities, and write \([n]\) for the set \(\{1, \ldots, n\}\). Probabilities are denoted by \(P\) in this section.

If \(G = (V, E)\) is a graph, we will frequently abuse notation and write \(u \in G\) instead of \(u \in V\), blurring the distinction between a graph and its vertex set. This notation will carry over to set-theoretic operations \(G = G_1 \cap G_2\) and indexing of variables (e.g., \(X_G\)).

\subsection{Graph theory}

Consider a directed acyclic graph \(G = (V, E)\), and define a partial order \(\prec_G\) on \(G\) by the transitive closure of the relation
\[
u \prec_G v \quad \text{if} \quad (u, v) \in E.
\]
We define the *parents* and *children* of \( v \in V \) in the natural way:

\[
\text{parents}(v) = \{ u \in V : (u, v) \in E \}
\]

and

\[
\text{children}(v) = \{ w \in V : (v, w) \in E \}.
\]

If \( G \) is connected and each \( v \in V \) has at most one parent, \( G \) is called a *(directed)* tree. In a tree, whenever \( u \prec_T v \) there is a unique directed path from \( u \) to \( v \). A tree \( T \) always has a unique minimal (with respect to \( \prec_T \)) element \( r_0 \in V \), called its *root*. Thus, for every \( v \in V \) there is a unique directed path \( r_0 \prec_T r_1 \prec_T \ldots \prec_T r_d = v \); define the *depth* of \( v \), \( \text{dep}_T(v) = d \), to be the length (i.e., number of edges) of this path. Note that \( \text{dep}_T(r_0) = 0 \). We define the depth of the tree by

\[
\text{dep}(T) = \sup_{v \in T} \text{dep}_T(v).
\]

For \( d = 0, 1, \ldots \) define the \( d \)th *level* of the tree \( T \) by

\[
\text{lev}_T(d) = \{ v \in V : \text{dep}_T(v) = d \};
\]

note that the levels induce a disjoint partition on \( V \):

\[
V = \bigcup_{d=1}^{\text{dep}(T)} \text{lev}_T(d).
\]

We define the *width*\(^2\) of a tree as the greatest number of nodes in any level:

\[
\text{wid}(T) = \sup_{1 \leq d \leq \text{dep}(T)} |\text{lev}_T(d)|. \tag{13}
\]

We will consistently take \(|V| = n\) for finite \( V \). An ordering \( J : V \to \mathbb{N} \) of the nodes is said to be *breadth-first* if

\[
\text{dep}_T(u) < \text{dep}_T(v) \implies J(u) < J(v). \tag{14}
\]

Since every finite directed tree \( T = (V, E) \) has some breadth-first ordering,\(^3\) we will henceforth blur the distinction between \( v \in V \) and \( J(v) \), simply taking \( V = [n] \) (or \( V = \mathbb{N} \)) and assuming that \( \text{dep}_T(u) < \text{dep}_T(v) \implies u < v \) holds. This will allow us to write \( \Omega^V \) simply as \( \Omega^n \) for any set \( \Omega \).

Note that we have two orders on \( V \): the partial order \( \prec_T \), induced by the tree edges, and the total order \( < \), given by the breadth-first enumeration. Observe that \( i \prec_T j \) implies \( i < j \) but not vice versa.

If \( T = (V, E) \) is a tree and \( u \in V \), we define the *subtree* induced by \( u \), \( T_u = (V_u, E_u) \) by \( V_u = \{ v \in V : u \preceq_T v \} \), \( E_u = \{ (v, w) \in E : v, w \in V_u \} \).

### 4.3 Markov tree measure

If \( \Omega \) is a finite set, a *Markov tree measure* \( \mu \) is defined on \( \Omega^n \) by a tree \( T = (V, E) \) and transition kernels \( p_{ij} = \{ p_{ij} : (i, j) \in E \} \). Continuing our convention above, we have a breadth-first order < and

\(^2\)This definition is nonstandard.

\(^3\)One can easily construct a breadth-first ordering on a given tree by ordering the nodes arbitrarily within each level and listing the levels in ascending order: \( \text{lev}_T(1), \text{lev}_T(2), \ldots \).
the total order \( \prec_T \) on \( V \), and take \( V = \{1, \ldots, n\} \). Together, the edges of \( T \) and the transition kernels determine the distribution \( \mu \) on \( \Omega^n \):

\[
\mu(x) = p_0(x_1) \prod_{(i,j) \in E} p_{ij}(x_j | x_i), \quad x \in \Omega^n. \tag{15}
\]

A measure on \( \Omega^n \) satisfying (15) for some \( T \) and \( \{p_{ij}\} \) is said to be compatible with tree \( T \); a measure is a Markov tree measure if it is compatible with some tree.

Suppose \( \Omega \) is a finite set and \( (X_i)_{i \in \mathbb{N}}, X_i \in \Omega \) is a random process defined on \( (\Omega^N, \mathbf{P}) \). If for each \( n > 0 \) there is a tree \( T^{(n)} = ([n], E^{(n)}) \) and a Markov tree measure \( \mu_n \) compatible with \( T^{(n)} \) such that for all \( x \in \Omega^n \) we have

\[
\mathbf{P}\{X^n_1 = x\} = \mu_n(x)
\]

then we call \( X \) a Markov tree process. The trees \( \{T^{(n)}\} \) are easily seen to be consistent in the sense that \( T^{(n)} \) is an induced subgraph of \( T^{(n+1)} \). So corresponding to any Markov tree process is the unique infinite tree \( T = (\mathbb{N}, E) \). The uniqueness of \( T \) is easy to see, since for \( v > 1 \), the parent of \( v \) is the smallest \( u \in \mathbb{N} \) such that

\[
\mathbf{P}\{X_v = x_v | X_u = y\} = \mathbf{P}\{X_v = x_v | X_u = y\} \mathbf{P}\{X_{T_v'} = x' | X_u = y\}.
\]

In other words, the subtrees induced by the children are conditionally independent given the parent; this follows directly from the definition of the Markov tree measure in (15).

### 4.4 Statement of result

**Theorem 4.1.** Let \( \Omega \) be a finite set and let \( (X_i)_{1 \leq i \leq n}, X_i \in \Omega \) be a Markov tree process, defined by a tree \( T = (V, E) \) and transition kernels \( p_0, \{p_{uv}(\cdot | \cdot)\}_{(u,v) \in E} \). Define the \((u,v)\)-contraction coefficient \( \theta_{uv} \) by

\[
\theta_{uv} = \max_{y,y' \in \Omega} \| p_{uv}(\cdot | y) - p_{uv}(\cdot | y') \|.
\]

Suppose \( \max_{(u,v) \in E} \theta_{uv} \leq \theta < 1 \) for some \( \theta \) and \( \text{wid}(T) \leq L \). Then for the Markov tree process \( X \) we have

\[
\tilde{\eta}_{ij} \leq (1 - (1 - \theta)^L)^{(j-i)/L}, \tag{17}
\]

for \( 1 \leq i < j \leq n \).

To cast (17) in more usable form, we first note that for \( k, L \in \mathbb{N} \) with \( k \geq L \), we have

\[
\frac{k}{L} \geq \frac{k}{2L - 1}. \tag{18}
\]
(we omit the elementary number-theoretic proof). Using (18), we have
\[
\tilde{\eta}_{ij} \leq \tilde{\theta}^{j-i}, \quad \text{for } j \geq i + L
\]  
where
\[
\tilde{\theta} = (1 - (1 - \theta)^L)^{1/(2L-1)};
\]
this implies the dimension-free bound
\[
\|\Delta\| \leq L - 1 + (1 - \tilde{\theta})^{-1}.
\]
In the (degenerate) case where the Markov tree is a chain, we have \(L = 1\) and therefore \(\tilde{\theta} = \theta\); thus we recover Theorem 3.1.

4.5 Proof of Theorem 4.1

The proof of Theorem 4.1 is combination of elementary graph theory and tensor algebra. We start with a graph-theoretic lemma:

**Lemma 4.2.** Let \(T = ([n], E)\) be a tree and fix \(1 \leq i < j \leq n\). Suppose \((X_i)_{1 \leq i \leq n}\) is a Markov tree process whose distribution \(P\) on \(\Omega^n\) is compatible with \(T\) (in the sense of Section 4.3). Define the set
\[
T_i^j = T_i \cap \{j, j+1, \ldots, n\},
\]
consisting of those nodes in the subtree \(T_i\) whose breadth-first numbering does not precede \(j\). Then, for \(y \in \Omega^{i-1}\) and \(w, w' \in \Omega\), we have
\[
\eta_{ij}(y, w, w') = \begin{cases} 
0, & T_i^j = \emptyset, \\
\eta_{ij_0}(y, w, w'), & \text{otherwise}, 
\end{cases}
\]
where \(j_0\) is the minimum (with respect to \(<\)) element of \(T_i^j\).

**Remark 4.3.** This lemma tells us that when computing \(\eta_{ij}\) it is sufficient to restrict our attention to the subtree induced by \(i\).

**Proof.** The case \(j \in T_i\) implies \(j_0 = j\) and is trivial; thus we assume \(j \notin T_i\). In this case, the subtrees \(T_i\) and \(T_j\) are disjoint. Putting \(\bar{T}_i = T_i \setminus \{i\}\), we have by the Markov property,
\[
P\{X_{\bar{T}_i} = x_{\bar{T}_i}, X_{T_j} = x_{T_j} | X_i^i = yw\} = P\{X_{\bar{T}_i} = x_{\bar{T}_i} | X_i = w\} P\{X_{T_j} = x_{T_j} | X_i^{i-1} = y\}.
\]

Then from the definition of \(\eta_{ij}\) and by marginalizing out the \(X_{T_j}\), we have
\[
\eta_{ij}(y, w, w') = \frac{1}{2} \sum_{x_j} \left| P\{X_j^n = x_j^n | X_1^i = yw\} - P\{X_j^n = x_j^n | X_1^i = yw'\} \right|
\]
\[
= \frac{1}{2} \sum_{x_{T_j}} \left| P\{X_{T_j} = x_{T_j} | X_i = w\} - P\{X_{T_j} = x_{T_j} | X_i = w'\} \right|.
\]

If \(T_i^j = \emptyset\) then obviously \(\eta_{ij} = 0\); otherwise, \(\eta_{ij} = \eta_{ij_0}\), since \(j_0\) is the “first” element of \(T_i^j\). □
Next we develop some basic results for tensor norms. If $A$ is an $M \times N$ column-stochastic matrix (i.e., $A_{ij} \geq 0$ for $1 \leq i \leq M$, $1 \leq j \leq N$ and $\sum_{i=1}^{M} A_{ij} = 1$ for all $1 \leq j \leq N$) and $u \in \mathbb{R}^N$ is balanced in the sense that $\sum_{j=1}^{N} u_j = 0$, we have, by Lemma 2.1
\begin{equation}
\|Au\| \leq \|A\| \|u\|,
\end{equation}
where
\begin{equation}
\|A\| = \max_{1 \leq j,j' \leq N} \|A_{*,j} - A_{*,j'}\|,
\end{equation}
and $A_{*,j} \equiv A[*,j]$ denotes the $j^{\text{th}}$ column of $A$. An immediate consequence of (21) is that $\|\cdot\|$ satisfies
\begin{equation}
\|AB\| \leq \|A\| \|B\|
\end{equation}
for column-stochastic matrices $A \in \mathbb{R}^{M \times N}$ and $B \in \mathbb{R}^{N \times P}$.

Remark 4.4. Note that if $A$ is a column-stochastic matrix then $\|A\| \leq 1$, and if additionally $u$ is balanced then $Au$ is also balanced.

If $u \in \mathbb{R}^M$ and $v \in \mathbb{R}^N$, define their tensor product $w = v \otimes u$ by
\begin{equation}
w_{(i,j)} = u_i v_j,
\end{equation}
where the notation $(v \otimes u)_{(i,j)}$ is used to distinguish the 2-tensor $w$ from an $M \times N$ matrix. The tensor $w$ is a vector in $\mathbb{R}^{MN}$ indexed by pairs $(i,j) \in [M] \times [N]$; its norm is naturally defined to be
\begin{equation}
\|w\| = \frac{1}{2} \sum_{(i,j) \in [M] \times [N]} |w_{(i,j)}|.
\end{equation}

To develop a convenient tensor notation, we will fix the index set $V = \{1, \ldots, n\}$. For $I \subset V$, a tensor indexed by $I$ is a vector $u \in \mathbb{R}^{\Omega^I}$. A special case of such an $I$-tensor is the product $u = \bigotimes_{i \in I} v^{(i)}$, where $v^{(i)} \in \mathbb{R}^{\Omega}$ and
\begin{equation}
u[x_I] = \prod_{i \in I} v^{(i)}[x_i]
\end{equation}
for each $x_I \in \Omega^I$. To gain more familiarity with the notation, let us write the total variation norm of an $I$-tensor:
\begin{equation}
\|u\| = \frac{1}{2} \sum_{x_I \in \Omega^I} |u[x_I]|.
\end{equation}

In order to extend Lemma 2.2 to product tensors, we will need to define the function $\alpha_k : \mathbb{R}^k \to \mathbb{R}$ and state some of its properties:

Lemma 4.5. Define $\alpha_k : \mathbb{R}^k \to \mathbb{R}$ recursively as $\alpha_1(x) = x$ and
\begin{equation}
\alpha_{k+1}(x_1, x_2, \ldots, x_{k+1}) = x_{k+1} + (1 - x_{k+1}) \alpha_k(x_1, x_2, \ldots, x_k).
\end{equation}

Then
(a) \( \alpha_k \) is symmetric in its \( k \) arguments, so it is well-defined as a mapping

\[ \alpha : \{ x_i : 1 \leq i \leq k \} \mapsto \mathbb{R} \]

from finite real sets to the reals

(b) \( \alpha_k \) takes \([0, 1]^k\) to \([0, 1]\) and is monotonically increasing in each argument on \([0, 1]^k\)

(c) If \( B \subset C \subset [0, 1] \) are finite sets then \( \alpha(B) \leq \alpha(C) \)

(d) \( \alpha_k(x, x, \ldots, x) = 1 - (1 - x)^k \)

(e) if \( B \) is finite and \( 1 \in B \subset [0, 1] \) then \( \alpha(B) = 1 \).

(f) if \( B \subset [0, 1] \) is a finite set then \( \alpha(B) \leq \sum_{x \in B} x \).

Remark 4.6. In light of (a), we will use the notation \( \alpha_k(x_1, x_2, \ldots, x_k) \) and \( \alpha(\{x_i : 1 \leq i \leq k\}) \) interchangeably, as dictated by convenience.

Proof. Claims (a), (b), (e), (f) are straightforward to verify from the recursive definition of \( \alpha \) and induction. Claim (c) follows from (b) since

\[ \alpha_{k+1}(x_1, x_2, \ldots, x_k, 0) = \alpha_k(x_1, x_2, \ldots, x_k) \]

and (d) is easily derived from the binomial expansion of \((1 - x)^k\).

The function \( \alpha_k \) is the natural generalization of \( \alpha_2(x_1, x_2) = x_1 + x_2 - x_1 x_2 \) to \( k \) variables, and it is what we need for the analog of Lemma 2.2 for a product of \( k \) tensors:

**Corollary 4.7.** Let \( \{u^{(i)}\}_{i \in I} \) and \( \{v^{(i)}\}_{i \in I} \) be two sets of tensors and assume that each of \( u^{(i)}, v^{(i)} \) is a probability measure on \( \Omega \). Then we have

\[
\left\| \bigotimes_{i \in I} u^{(i)} - \bigotimes_{i \in I} v^{(i)} \right\| \leq \alpha \left\{ \left\| u^{(i)} - v^{(i)} \right\| : i \in I \right\}. \tag{27}
\]

Proof. Pick an \( i_0 \in I \) and let \( p = u^{(i_0)}, q = v^{(i_0)} \),

\[ p' = \bigotimes_{i \in I \setminus i_0} u^{(i)}, \quad q' = \bigotimes_{i \neq i_0} v^{(i)}. \]

Apply Lemma 2.2 to \( \| p \otimes q - p' \otimes q' \| \) and proceed by induction.

Our final generalization concerns linear operators over \( I \)-tensors. For \( I, J \subseteq V \), an \( I, J \)-matrix \( A \) has dimensions \( |\Omega^I| \times |\Omega^J| \) and takes an \( I \)-tensor \( u \) to a \( J \)-tensor \( v \): for each \( y_J \in \Omega^J \), we have

\[ v[y_J] = \sum_{x_I \in \Omega^I} A[y_J, x_I] u[x_I], \tag{28} \]

which we write as \( Au = v \). If \( A \) is an \( I, J \)-matrix and \( B \) is a \( J, K \)-matrix, the matrix product \( BA \) is defined analogously to (28).
As a special case, an $I,J$-matrix might factorize as a tensor product of $|\Omega| \times |\Omega|$ matrices $A^{(i,j)} \in \mathbb{R}^{\Omega \times \Omega}$. We will write such a factorization in terms of a bipartite graph\footnote{Our notation for bipartite graphs is standard; it is equivalent to $G = (I \cup J, E)$ where $I$ and $J$ are always assumed to be disjoint.} $G = (I + J, E)$, where $E \subset I \times J$ and the factors $A^{(i,j)}$ are indexed by $(i, j) \in E$:

$$A = \bigotimes_{(i,j) \in E} A^{(i,j)},$$

where

$$A[y_J, x_I] = \prod_{(i,j) \in E} A^{(i,j)}_{y_j x_i}$$

for all $x_I \in \Omega^I$ and $y_J \in \Omega^J$. The norm of an $I,J$-matrix is a natural generalization of the matrix norm defined in (22):

$$\|A\| = \max_{x_I, x'_I \in \Omega^I} \|A[* , x_I] - A[* , x'_I]\|$$

(30)

where $u = A[* , x_I]$ is the $J$-tensor given by

$$u[y_J] = A[y_J, x_I];$$

(30) is well-defined via the tensor norm in (25). Since $I, J$ matrices act on $I$-tensors by ordinary matrix multiplication, $\|Au\| \leq \|A\| \|u\|$ continues to hold when $A$ is a column-stochastic $I,J$-matrix and $u$ is a balanced $I$-tensor; if, additionally, $B$ is a column-stochastic $J,K$-matrix, $\|BA\| \leq \|B\| \|A\|$ also holds. Likewise, since another way of writing (29) is

$$A[* , x_I] = \bigotimes_{(i,j) \in E} A^{(i,j)}[*, x_i],$$

Corollary 4.7 extends to tensor products of matrices:

**Lemma 4.8.** Fix index sets $I, J$ and a bipartite graph $(I + J, E)$. Let $\{A^{(i,j)}\}_{(i,j) \in E}$ be a collection of column-stochastic $|\Omega| \times |\Omega|$ matrices, whose tensor product is the $I,J$ matrix

$$A = \bigotimes_{(i,j) \in E} A^{(i,j)}.$$ 

Then

$$\|A\| \leq \alpha \left\{ \|A^{(i,j)}\| : (i, j) \in E \right\}.$$ 

We are now in a position to state the main technical lemma, from which Theorem 4.1 will follow straightforwardly:

**Lemma 4.9.** Let $\Omega$ be a finite set and let $(X_i)_{1 \leq i \leq n}, X_i \in \Omega$ be a Markov tree process, defined by a tree $T = (V, E)$ and transition kernels $p_0, \{p_{uv}(\cdot | \cdot)\}_{(u,v) \in E}$. Let the $(u,v)$-contraction coefficient $\theta_{uv}$ be as defined in (16).
Fix $1 \leq i < j \leq n$ and let $j_0 = j_0(i, j)$ be as defined in Lemma 4.2 (we are assuming its existence, for otherwise $\eta_{ij} = 0$). Then we have

$$\eta_{ij} \leq \prod_{d=\text{dep}_{T}(j_0)}^{\text{dep}_{T}(j_0)} \alpha \{\theta_{uv} : v \in \text{lev}_{T}(d)\}. \quad (31)$$

**Proof.** For $y \in \Omega^{i-1}$ and $w, w' \in \Omega$, we have

$$\eta_{ij}(y, w, w') = \frac{1}{2} \sum_{x_{j}} \left| \sum_{x_{[C]}} \left( \mathbb{P}\{X_{j} = x_{j}^n | X_{1} = yw\} - \mathbb{P}\{X_{j} = x_{j}^n | X_{1} = yw'\} \right) \right| \quad (32)$$

$$= \frac{1}{2} \sum_{x_{j}} \left| \sum_{x_{i+1}} \left( \mathbb{P}\{X_{i+1} = z_{i+1} x_{j}^n | X_{1} = yw\} - \mathbb{P}\{X_{i+1} = z_{i+1} x_{j}^n | X_{1} = yw'\} \right) \right|. \quad (33)$$

Let $T_i$ be the subtree induced by $i$ and

$$Z = T_i \cap \{i+1, \ldots, j_0 - 1\} \quad \text{and} \quad C = \{v \in T_i : (u, v) \in E, u < j_0, v \geq j_0\}. \quad (34)$$

Then by Lemma 4.2 and the Markov property, we get

$$\eta_{ij}(y, w, w') = \frac{1}{2} \sum_{x_{[C]}} \left| \sum_{x_{[Z]}} \left( \mathbb{P}\{X_{C \cup Z} = x_{C \cup Z} | X_{1} = w\} - \mathbb{P}\{X_{C \cup Z} = x_{C \cup Z} | X_{1} = w'\} \right) \right| \quad (35)$$

(the sum indexed by $\{j_0, \ldots, n\} \setminus C$ marginalizes out).

Define $D = \{d_k : k = 0, \ldots, |D|\}$ with $d_0 = \text{dep}_{T}(i)$, $d_{|D|} = \text{dep}_{T}(j_0)$ and $d_{k+1} = d_k + 1$ for $0 \leq k < |D|$. For $d \in D$, let $I_d = T_i \cap \text{lev}_{T}(d)$ and $G_d = (I_{d-1} + I_d, E_d)$ be the bipartite graph consisting of the nodes in $I_{d-1}$ and $I_d$, and the edges in $E$ joining them (note that $I_{d_0} = \{i\}$).

For $(u, v) \in E$, let $A^{(u,v)}$ be the $|\Omega| \times |\Omega|$ matrix given by

$$A_{x,x'}^{(u,v)} = p_{uv}(x | x')$$

and note that $\|A^{(u,v)}\| = \theta_{uv}$. Then by the Markov property, for each $z[I_d] \in \Omega^{I_d}$ and $x[I_{d-1}] \in \Omega^{I_{d-1}}$, $d \in D \setminus \{d_0\}$, we have

$$\mathbb{P}\{X_{I_d} = z_{I_d} | X_{I_{d-1}} = x_{I_{d-1}}\} = A^{(d)}[z_{I_d}, x_{I_{d-1}}],$$

where

$$A^{(d)} = \bigotimes_{(u,v) \in E_d} A^{(u,v)}.$$
Likewise, for \( d \in D \setminus \{d_0\} \),

\[
\mathbf{P}\{X_{I_d} = x_{I_d} \mid X_i = w\} = \sum\sum\cdots\sum_{x'_{I_1}, x'_{I_2}, \ldots, x'_{I_{d-1}}} \mathbf{P}\{X_{I_1} = x'_{I_1} \mid X_i = w\} \mathbf{P}\{X_{I_2} = x'_{I_2} \mid X_{I_1} = x'_{I_1}\} \cdots \mathbf{P}\{X_{I_{d-1}} = x_{I_{d-1}} \mid X_{I_{d-2}} = x_{I_{d-2}}\}.
\]

(36)

Define the (balanced) \( I_{d_1} \)-tensor

\[
h = \mathbf{A}^{(d_1)}[*, w] - \mathbf{A}^{(d_1)}[*, w'],
\]

(37)

the \( I_{d(D)} \)-tensor

\[
f = \mathbf{A}^{(d(D))} \mathbf{A}^{(d(D-1))} \cdots \mathbf{A}^{(d_2)} h,
\]

(38)

and \( C_0, C_1, Z_0 \subset \{1, \ldots, n\} \):

\[
C_0 = C \cap I_{\text{dep}(j_0)}, \quad C_1 = C \setminus C_0, \quad Z_0 = I_{\text{dep}(j_0)} \setminus C_0,
\]

(39)

where \( C \) and \( Z \) are defined in (34). For readability we will write \( \mathbf{P}(x_U \mid \cdot) \) instead of \( \mathbf{P}\{X_U = x_U \mid \cdot\} \) below; no ambiguity should arise. Combining (35) and (36), we have

\[
\eta_{ij}(y, w, w') = \frac{1}{2} \sum_{x_C} \left| \sum_{x_Z} \left( \mathbf{P}(x[C \cup Z] \mid X_i = w) - \mathbf{P}(x[C \cup Z] \mid X_i = w') \right) \right|
\]

(40)

\[
= \frac{1}{2} \sum_{x_{C_0} x_{C_1}} \left| \sum_{x_{Z_0}} \mathbf{P}(x[C_1] \mid x[Z_0]) f[C_0 \cup Z_0] \right|
\]

(41)

\[
= \| \mathbf{B} f \|
\]

(42)

where \( \mathbf{B} \) is the \( |\Omega_{C_0 \cup C_1}| \times |\Omega_{C_0 \cup Z_0}| \) column-stochastic matrix given by

\[
\mathbf{B}[x_{C_0} \cup x_{C_1}, x'_{C_0} \cup x_{Z_0}] = \mathbf{1}_{\{x_{C_0} = x'_{C_0}\}} \mathbf{P}(x_{C_1} \mid x_{Z_0})
\]

with the convention that \( \mathbf{P}(x_{C_1} \mid x_{Z_0}) = 1 \) if either of \( Z_0 \) or \( C_1 \) is empty. The claim now follows by reading off the results previously obtained:

\[
\| \mathbf{B} f \| \leq \| \mathbf{B} \| \| f \| \quad \text{Eq. (21)}
\]

\[
\leq \| f \| \quad \text{Remark 4.4}
\]

\[
\leq \| h \| \prod_{k=2}^{D} \left\| \mathbf{A}^{(d_k)} \right\| \quad \text{Eqs. (23,38)}
\]

\[
\leq \prod_{k=1}^{D} \alpha \{ \| \mathbf{A}^{(u,v)} \| : (u,v) \in E_{d_k} \} \quad \text{Lemma 4.8}
\]

\]
Proof of Theorem 4.1. We will borrow the definitions from the proof of Lemma 4.9. To upper-bound \( \bar{\eta}_{ij} \) we first bound \( \alpha\{\|A^{(u,v)}\| : (u,v) \in E_{d_k}\} \). Since

\[
|E_{d_k}| \leq \text{wid}(T) \leq L
\]

(because every node in \( I_{d_k} \) has exactly one parent in \( I_{d_k-1} \)) and

\[
\|A^{(u,v)}\| = \theta_{uv} \leq \theta < 1,
\]

we appeal to Lemma 4.5 to obtain

\[
\alpha\{\|A^{(u,v)}\| : (u,v) \in E_{d_k}\} \leq 1 - (1 - \theta)^L. \tag{43}
\]

Now we must lower-bound the quantity \( h = \text{dep}_T(j_0) - \text{dep}_T(i) \). Since every level can have up to \( L \) nodes, we have

\[
j_0 - i \leq hL
\]

and so \( h \geq [(j_0 - i)/L] \geq [(j - i)/L] \).

The calculations in Lemma 4.9 yield considerably more information than the simple bound in (17). For example, suppose the tree \( T \) has levels \( \{I_d : d = 0, 1, \ldots\} \) with the property that the levels are growing at most linearly:

\[
|I_d| \leq cd
\]

for some \( c > 0 \). Let \( d_i = \text{dep}_T(i) \), \( d_j = \text{dep}_T(j_0) \), and \( h = d_j - d_i \). Then

\[
j - i \leq j_0 - i \leq c \sum_{d_i+1}^{d_j} k = \frac{c}{2}(d_j(d_j + 1) - d_i(d_i + 1)) < \frac{c}{2}((d_j + 1)^2 - d_i^2) < \frac{c}{2}(d_i + h + 1)^2
\]

so

\[
h > \sqrt{2(j - i)/c - d_i - 1},
\]

which yields the bound, via Lemma 4.5(f),

\[
\bar{\eta}_{ij} \leq \prod_{k=1}^{h} \sum_{(u,v) \in E_k} \theta_{uv}. \tag{44}
\]

Let \( \theta_k = \max\{\theta_{uv} : (u,v) \in E_k\} \); then if \( ck\theta_k \leq \beta \) holds for some \( \beta \in \mathbb{R} \), this becomes

\[
\bar{\eta}_{ij} \leq \prod_{k=1}^{h} (ck\theta_k) \leq \sqrt{2(j - i)/c - d_i - 1} \prod_{k=1}^{h} (ck\theta_k) \leq \beta\sqrt{2(j - i)/c - d_i - 1}. \tag{45}
\]
This is a non-trivial bound for trees with linearly growing levels: recall that to bound \( \| \Delta \|_\infty \), we must bound the series
\[
\sum_{j=i+1}^{\infty} \bar{\eta}_{ij}.
\]
By the limit comparison test with the series \( \sum_{j=1}^{\infty} 1/j^2 \), we have that
\[
\sum_{j=i+1}^{\infty} \beta \sqrt{2(j-1)/c - d - 1}
\]
converges for \( \beta < 1 \). Similar techniques may be applied when the level growth is bounded by other slowly increasing functions. It is hoped that this method will be extended to obtain concentration bounds for larger classes of directed acyclic graphical models.

5 Markov marginal processes

In this section, we define a random process that strictly generalizes Markov and hidden Markov chains. It was first defined in the author’s forthcoming work with A. Brockwell [14], where it is termed a Markov marginal process and applied to the analysis of adaptive Markov Chain Monte Carlo algorithms.

Consider two finite sets, \( \hat{\Omega} \) (the “hidden state” space) and \( \bar{\Omega} \) (the “observed state” space). Let \( \mu \) be a Markov measure on \( (\bar{\Omega} \times \hat{\Omega})^n \) defined by the initial distribution \( p_0 \) and the kernels \( \{ K_i(\cdot | \cdot) \}_{1 \leq i \leq n} \):
\[
\mu \left( \hat{x}_1^n, \bar{x}_1^n \right) = p_0 \left( \hat{x}_1^n \right) \prod_{i=1}^{n-1} K_i \left( x_{i+1} \mid x_i \right), \tag{46}
\]
where for readability we use the stacked notation \( \left( \hat{x}, \bar{x} \right) \) – instead of the more standard \( (\hat{x}, \bar{x}) \) – for elements of \( \hat{\Omega} \times \bar{\Omega} \).

A Markov marginal processes (MMP) is a measure \( \rho \) on \( \hat{\Omega}^n \) defined by
\[
\rho(\hat{x}_1^n) = \sum_{\hat{x}_1^n} \mu \left( \hat{x}_1^n, \bar{x}_1^n \right). \tag{47}
\]

Let us define the \( i^{th} \) contraction coefficient \( \theta_i \) of a MMP with kernels \( \{ K_i(\cdot | \cdot) \}_{1 \leq i \leq n} \) by
\[
\theta_i = \max_{\hat{x}, \hat{x}' \in \hat{\Omega}} \max_{\bar{x}, \bar{x}' \in \bar{\Omega}} \left\| K_i \left( \cdot \mid \hat{x} \right) - K_i \left( \cdot \mid \bar{x} \right) \right\|. \tag{48}
\]
It is shown in [14] that the \( \eta \)-mixing coefficients of a MMP may be controlled by its contraction coefficients:

**Theorem 5.1.** A MMP \( \rho \) on \( \hat{\Omega}^n \), as defined above, satisfies
\[
\bar{\eta}_{ij} = \theta_i \theta_{i+1} \ldots \theta_{j-1}.
\]
Note that this result subsumes the bound for Markov chains and hidden Markov chains [12, 15]. As [14] is still in preparation, we find it instructive to reproduce the proof here:

**Proof.** Fix an \( n > 0, 1 \leq i < j \leq n, \hat{y}_i^{i-1} \in \hat{\Omega}^{i-1} \) and \( \hat{w}_i, \hat{w}_i' \in \hat{\Omega} \). We will use the generic notation \( P(x) \) and \( P(x \mid y) \) for the probabilities induced by \( \mu \) and \( \rho \), consistently indicating observed sequences by a dot (\( \cdot \)) and hidden ones by a hat (\( \hat{\cdot} \)); no confusion should arise. We will occasionally drop subscripts and superscripts for readability. In this proof, whenever \( K_t(a \mid b) \) appears in an expression where \( t \) takes on the value 0, it is to be interpreted as \( p_0(a) \). Empty products evaluate to unity by convention. We expand

\[
\eta_{ij}(\hat{y}_i^{i-1}, \hat{w}_i, \hat{w}_i') = \frac{1}{2} \sum_{\hat{x}_j^n} \left| P(\hat{x}_j^n \mid \hat{y}_i^{i-1} \hat{w}_i) - P(\hat{x}_j^n \mid \hat{y}_i^{i-1} \hat{w}_i') \right|
= \frac{1}{2} \sum_{\hat{x}_j^n} \left| \sum_{\hat{z}_j^{t+1}} \left( P(\hat{z}_j^{t+1} \hat{x}_j^n \mid \hat{y}_i^{i-1} \hat{w}_i) - P(\hat{z}_j^{t+1} \hat{x}_j^n \mid \hat{y}_i^{i-1} \hat{w}_i') \right) \right|
= \frac{1}{2} \sum_{\hat{x}_j^n} \left| \sum_{\hat{z}_j^{t+1}} \sum_{\hat{y}_i'} \left[ P\left( \hat{y} \hat{w} \hat{z} \hat{x} \hat{y} \hat{z} \hat{x} \right) / P(\hat{y} \hat{w}) \right. \right.
- \left. \left. P\left( \hat{y} \hat{w}' \hat{z} \hat{x} \hat{y} \hat{z} \hat{x} \right) / P(\hat{y} \hat{w}') \right] \right|
\leq \frac{1}{2} \sum_{\hat{x}_j^n} \sum_{\hat{y}_i'} \left| \sum_{\hat{z}_j^{t+1}} \sum_{\hat{x}_j^n} \left[ P\left( \hat{y} \hat{w} \hat{z} \hat{x} \hat{y} \hat{z} \hat{x} \right) / P(\hat{y} \hat{w}) \right. \right.
- \left. \left. P\left( \hat{y} \hat{w}' \hat{z} \hat{x} \hat{y} \hat{z} \hat{x} \right) / P(\hat{y} \hat{w}') \right] \right|. \tag{49}
\]

To make the above shorthand quite explicit, let us elaborate:

\[
P(\hat{y} \hat{w}) \equiv P\left\{ \hat{X}_1^i = \hat{y}_i^{i-1} \hat{w}_i \right\}
\]

\[
P(\hat{z}_j^{t+1} \hat{x}_j^n \mid \hat{y}_i^{i-1} \hat{w}_i) \equiv P\left\{ \hat{X}_1^{t+1} = \hat{z}_j^{t+1} \hat{x}_j^n \mid \hat{X}_1^i = \hat{y}_i^{i-1} \hat{w}_i \right\}
\]

\[
P\left( \hat{y} \hat{w} \hat{z} \hat{x} \hat{y} \hat{z} \hat{x} \right) \equiv P\left\{ \hat{X}_1^n = \hat{y}_i^{i-1} \hat{w}_i \hat{z}_j^{t+1} \hat{x}_j^n, \hat{X}_1^n = \hat{y}_i^{i-1} \hat{w}_i' \hat{z}_j^{t+1} \hat{x}_j^n \right\}.
\]

Using the definitions in (46) and (47), we rewrite (49) in matrix notation as

\[
\text{r.h.s. of (49)} = \frac{1}{2} \sum_{\hat{x}_j, \hat{z}_j} \sum_{\hat{z}_j, \hat{z}_j} \prod_{t=j}^{n-1} K_t \left( \hat{x}_{t+1} \mid \hat{x}_t \right) \left| \mathbf{z}_{\hat{x}_j, \hat{x}_j} \right|
= \frac{1}{2} \sum_{\hat{x}_j, \hat{z}_j} \left| \mathbf{z}_{\hat{x}_j, \hat{x}_j} \right| \sum_{\hat{z}_j, \hat{z}_j} \prod_{t=j}^{n-1} K_t \left( \hat{x}_{t+1} \mid \hat{x}_t \right)
= \left\| \mathbf{z} \right\|, \tag{50}
\]
where $z \in \mathbb{R}^{\hat{\Omega} \times \hat{\Omega}}$ is a vector given by
\[
z = K^{(j-1)}K^{(j-2)}\cdots K^{(i+1)}h
\] (51)
with $h \in \mathbb{R}^{\hat{\Omega} \times \hat{\Omega}}$ given by
\[
h_{\hat{\imath},\hat{\jmath}} = \sum_{\hat{\imath}_1}^{i-2} \prod_{t=1}^{\hat{\imath}_1-1} K_t \left( \begin{array}{c} \hat{y}_{t+1} \\ \hat{y}_t \end{array} \right) \left[ K_{i-1} \left( \begin{array}{c} \hat{w}_i \\ \hat{y}_{i-1} \end{array} \right) K_{i-1} \left( \begin{array}{c} \hat{\imath} \\ \hat{y}_i \end{array} \right) + P(\hat{y}_{i-1}^{-1}\hat{w}_i) \right]
\]
and $K^{(t)}$ is a $|\hat{\Omega} \times \hat{\Omega}| \times |\hat{\Omega} \times \hat{\Omega}|$ column-stochastic matrix given by
\[
[K^{(t)}]_{(\hat{\imath},\hat{\jmath}),({\hat{\imath}}',{\hat{\jmath}}')} = K_t \left( \begin{array}{c} \hat{\imath} \\ \hat{\imath}' \end{array} \right) \left( \begin{array}{c} \hat{\jmath} \\ \hat{\jmath}' \end{array} \right).
\]
Let us bound $||h||$. Since
\[
P(\hat{y}_{j-1}\hat{w}_j) = \sum_{\hat{\imath}_1}^{i-2} K_{j-1} \left( \begin{array}{c} \hat{w}_i \\ \hat{y}_{i-1} \end{array} \right) \prod_{t=1}^{\hat{\imath}_1-1} K_t \left( \begin{array}{c} \hat{y}_{t+1} \\ \hat{y}_t \end{array} \right)
\]
(and similarly for $P(\hat{y}_{j-1}\hat{w}'_j)$) we have that $h$ is a difference of convex combinations of conditional distributions:
\[
h_{\hat{\imath},\hat{\jmath}} = \sum_{\hat{\imath}_1}^{i-2} \alpha_\hat{\imath} K_{i-1} \left( \begin{array}{c} \hat{w}_i \\ \hat{\imath} \end{array} \right) - \sum_{\hat{\imath}_1}^{i-2} \alpha_\hat{\imath}' K_{i-1} \left( \begin{array}{c} \hat{w}'_i \\ \hat{\imath}' \end{array} \right)
\]
where $\alpha, \alpha' \geq 0$ and $\sum \alpha = \sum \alpha' = 1$. Since the function $f(x, y) = ||x - y||$ is convex in both arguments, we have
\[
||h|| \leq \theta_i.
\] (52)
The claim follows by applying the Markov contraction lemma to (52) and (51).

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