Introduction—The Ising model [1] is one of the most fundamental models in statistical physics and condensed matter, and has great influence in almost every branch of modern physics. When introduced in 1925, the Ising model was shown to have no finite-temperature phase transition in one dimension (1D) [2]. In 1944, a milestone was achieved by Onsager [3, 4] who obtained the exact free energy density on the square lattice without external field. In 1952, Yang derived that, close to the upper critical dimension of percolation, the Ising model simultaneously exhibits two upper critical dimensions at \( d_c = 4, d_p = 6 \), and critical clusters for \( d \geq d_p \), except the largest one, are governed by exponents from percolation universality. We predict a rich variety of geometric properties and then provide strong evidence in dimensions from 4 to 7 and on complete graphs. Our findings significantly advance the understanding of the Ising model, which is a fundamental system in many branches of physics.

The upper critical dimension of the Ising model is known to be \( d_c = 4 \), above which critical behavior is regarded as trivial. We hereby argue from extensive simulations that, in the random-cluster representation, the Ising model simultaneously exhibits two upper critical dimensions at \((d_c = 4, d_p = 6)\), and critical clusters for \( d \geq d_p \), except the largest one, are governed by exponents from percolation universality. We predict a rich variety of geometric properties and then provide strong evidence in dimensions from 4 to 7 and on complete graphs. Our findings significantly advance the understanding of the Ising model, which is a fundamental system in many branches of physics.
FIG. 1. Evidence for the two upper critical dimensions, with power-law scaling illustrated by approximately straight lines in the log-log scale. (a), evidence for $d_c = 4$ from finite-size scaling. The largest-cluster size $C_1$, the rescaled second-largest-cluster size $\tilde{C}_2$ and the magnetization $M$ are plotted versus system volume $V$. Up to non-universal constants, data of $C_1$ and $M$ collapse well onto a line with slope 3/4 for $d = 4, 5, 6, 7$ and on CGs, and data of $\tilde{C}_2$ collapse onto a line of slope 1/2. (b), evidence for $d_p = 6$ from geometric fractals. Size $s$ of medium clusters is shown versus gyration radius $R$, and the fractal dimension $D_{c2}$ is $1 + d/2$ for $4 \leq d < 6$ and $4$ for $d \geq 6$ (percolation universality). (c), evidence for $d_r = 6$ from the largest cluster, which has fractal dimension $D_{c1} = 3d/4$ for $4 \leq d < 6$ and $D_{c1} = 9/2$ for $d \geq 6$.

| $4 \leq d < 6$ | $d \geq 6$ |
|-----------------|-----------------|
| $D_{c1}$ | 3d/4 $\Leftrightarrow$ same |
| $D_{c2}$ (CG-Ising asy.) | 9/2 |
| $D_{c2}$ | 1 + d/2 $\Leftrightarrow$ same |
| $D_{c2}$ (Gaussian f.p.) | 4 (percolation) |

TABLE I. Conjectured exact fractal dimensions for $4 \leq d < 6$ and $d \geq 6$, as inspired by CG-Ising asymptotics, Gaussian fixed point and results for high-$d$ percolation.

other hand, critical clusters exhibit different geometric structures for $4 \leq d < 6$ and $d \geq 6$. Consider the thermodynamic fractal dimension $D_F$, as defined from the asymptotic power-law dependence of the size of a cluster on its gyration radius. For $4 \leq d < 6$, one has $D_{F1} = 3d/4$ for the largest cluster and $D_{F2} = 1 + d/2$ for all the remaining ones. However, for $d \geq 6$, one has $D_{F1} = 9/2$ and $D_{F2} = 4$; the latter is from high-$d$ percolation universality. This is summarized in Table I. A variety of other critical behaviors are observed. For instance, for $d \geq 6$, the number of spanning clusters and the winding number of the largest cluster are both divergent as $L$ increases, while they are of $O(1)$ in lower dimensions. Further, there exist two scaling windows: the leading one is of CG-Ising type, and the other is of Gaussian type for $4 \leq d < 6$ and of percolation type for $d \geq 6$, align with rigorous result for CGs.

Models—The Hamiltonian of the Ising model reads

$$\mathcal{H} = -K \sum_{i \neq j} S_i S_j, \quad (S_i = \pm 1)$$

(1)

where $K > 0$ represents the ferromagnetic coupling and the summation is over all neighboring pairs. By the FK transformation, it can be mapped onto the $Q$-state RC model with $Q = 2$, with the partition function

$$Z = \sum_{A \subseteq G} p^{|A|}(1 - p)^{|E \setminus A|}Q^{c(A)},$$

(2)

where the lattice is denoted as $G \equiv (V, E)$, the summation is over all spanning subgraphs $A \subseteq G$, $|A|$ and $c(A)$ respectively represent the number of occupied bonds and of clusters, and the bond probability is $p = 1 - e^{-2K}$.

We simulate the FK-Ising model on $d$-dimensional tori with $4 \leq d \leq 7$ and on CGs, at and near the critical points as in Refs. [39, 40]. A combination of the Wolff and Swendsen-Wang algorithms [30, 31] is applied, and the latter is mainly used to generate FK-bond configurations. The maximum system volume is $V = 48^4, 51^5, 24^6, 16^7, 2^{22}$ for $d = 4, 5, 6, 7$ and CGs, respectively. We also simulate bond percolation in 7D at criticality [41].

We sample the number $n(s, V)$ of clusters of size $s$ per site, and the sizes of the largest- and the second-largest clusters as $C_1 = \langle C_1 \rangle$ and $C_2$, respectively, with $\langle \cdot \rangle$ for ensemble average. Further, to study geometric fractal structures, we use the breadth-first search method to grow FK clusters and measure their gyration radius in an unwrapped way, effectively taking into account periodic boundary effects [38]. Given a cluster $C$, we randomly choose a seed site and assign it a $d$-dimensional zero coordinate $(x \equiv 0)$, and each newly included site $v$ is assigned an unwrapped coordinate as $x_v = x_u + e_i (\equiv e_1)$, if $v$ is grown from $u$ along (against) the $i$th direction, with $e_i$ the corresponding unit vector. The unwrapped gyration radius is calculated as $R = \sqrt{(|x_u|^2 - |x_v|^2)}$, where the average is over all sites in cluster $C$. We also measure the unwrapped expansion distance $U$ along the first-coordinate direction for each cluster.

Evidence for $d_c = 4$ from finite-size scaling—In the spin representation, $d_c = 4$ is widely known for the Ising model. Nevertheless, finite-size scaling behavior for $d \geq 4$ has been a long-standing debate [38, 42, 43]. It is now believed [38, 39, 40] that the critical free energy on high-$d$ tori contain two scaling terms, having
RG exponents \((y_t = 2, y_h = 1 + d/2)\) from the GFP and \((y_t' = d/2, y_h' = 3d/4)\) from the CG-Ising asymptotics. An important consequence is that the critical two-point function behaves as \(G(x, L) \approx \|x\|^{2-d} L^{-d/2}\), algebraically decaying with distance \(\|x\|\), with exponent \(2 - d\) from GFP, and then saturating to a plateau of height \(L^{-d/2}\) from CG-Ising asymptotics \([47, 49]\). This implies that the magnetic susceptibility, which is exactly the average cluster size in the FK representation, scales as \(L^{d/2}\).

Figure 1(a) shows the critical magnetization \(M \equiv \langle \sum_i S_i \rangle\) versus volume \(V\). The good data collapsing for \(d = 4, 5, 6, 7\) and on CGs, displaying \(M \sim V^{3/4}\). Moreover, the \(C_1\) data collapse well onto those for \(M\). This confirms the conventional upper dimension \(d_c = 4\), and demonstrates the uniform scaling \(\sim V^{3/4}\) for \(d \geq d_c\), which can be proved for CGs \([36, 50]\).

From extensive simulations and results in Ref. \([53]\), we conjecture that, for \(d > 4\), the FSS of \(C_2\) behaves as \(C_2 \sim L^{1+d/2}/\sqrt{V^{1/d}}\), corresponding to the GFP. This is seemingly consistent with the scaling \(C_2 \sim \sqrt{V} \ln V\) for CGs \([30]\), where \(\ln V\) might relate to the term \(V^{1/d}\) for finite-\(d\). Rescaled quantities are then defined as \(\tilde{C}_2 \equiv C_2/L\) for \(d \geq 4\) and \(\tilde{C}_2 \equiv C_2/\ln V\) for CGs. Indeed, the \(\tilde{C}_2\) data for \(d = 4, 5, 6, 7\) and on CGs collapse well on a line with slope \(1/2\), shown in Fig. 1(a).

At the upper critical dimensions, logarithmic corrections are usually expected. For the Ising model in the spin representation, field theory predicts the form of logarithmic corrections for many quantities at \(d_c = 4\) \([51, 52]\), such as the magnetization \(M \sim L^3(\ln L)^{1/4}\), and the susceptibility \(\chi \sim L^2(\ln L)^{1/2}\). We now examine the effect of logarithmic corrections to \(C_1\) and \(C_2\). In Fig. 2 we plot in log-log scale \(C_1\) and \(C_2\), rescaled by their expected power-law scaling, versus \(\ln L\). Our data suggest that \(C_1 \sim L^3(\ln L)^{1/4}\), consistent with the field-theory prediction for \(M\), and \(C_2 \sim L^3(\ln L)^{-1/4}\) which has no direct counterpart in the spin representation.

**Evidence for \(d_p = 6\) from geometric fractals**— In comparison with FSS, intrinsic geometric properties of clusters are better characterized by the power-law dependence of cluster size on gyration radius as \(s \sim R^{D_p}\), which is shown in Fig. 1(b) for medium-size clusters—i.e., clusters with size \(1 \ll s \ll C_1\). Distinct fractal structures are revealed: the fractal dimension \(D_{p2}\) is \(1 + d/2\) for \(4 \leq d < 6\), and becomes constant \(4\) for \(d \geq 6\). While the former is from the GFP, the latter is consistent with percolation universality \([53]\), as well illustrated by the 7D-percolation data in Fig. 1(b).

Actually, the largest cluster also has different fractal dimensions below and above \(d_p = 6\). The plot of the \(C_1\) data against the gyration radius \(R_1\) in Fig. 1(c) gives \(D_{p1} = 3d/4\) for \(4 \leq d < 6\) and \(9/2\) for \(d \geq 6\), with \(9/2\) calculated from \(3d/4\) with \(d = 6\).

Therefore, we conclude that \(d_p = 6\) is also an upper critical dimension for the FK-Ising model.

**Evidence for \(d_p = 6\) from topological properties**— The essential assumption of the standard FSS theory is that the divergent correlation length—e.g., as characterized by \(R_1\)—is cut off as \(O(L)\), resulting in that the number of percolating clusters is of \(O(1)\). This has been widely used as a powerful tool in numerical study of critical phenomena.

We first look at the cluster-number density \(n(s, L) \sim s^{-\tau} \tilde{n}(s/L^{D_{11}})\), where \(\tau\) is the Fisher exponent, \(D_{11}\) is the finite-size fractal dimension and \(\tilde{n}\) is a universal function.
The hyperscaling relation, $\tau = 1 + d/D_{c1}$, is further believed to hold, giving $\tau = 7/3$ for $d \geq 4$. As shown in Fig. 3(a), while being indeed true for 4D, the hyperscaling relation is broken for 7D, which has $\tau \approx 5/2$. From the data collapsing for the FK-Ising and percolation models in 7D, it can be restored by using $D_x = 2d/3$ for percolation universality.

The scaling window for $d \geq 6$. Within window $t \sim O(V^{-1/3})$, the largest cluster scales percolation-like as $C_1 \sim V^{2/3}$. The data points of various shapes are for different system sizes $V$, and the colors are for 6D (blue), 7D (red) and CGs (green). The inset demonstrates the CG-Ising scaling window of $O(V^{-1/2})$, in which $C_1(t, V) \sim V^{3/4}C_1(tV^{1/2})$.

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the CP\(^1\) model \[54\].

In FK-Ising clusters for \(d \geq 4\), the thermodynamic and finite-size scaling behaviors are surprisingly rich, partially summarized in Tables I and II. Two pronounced features can be seen: 1) as long as \(d \geq 4\), there exist two-scale properties, two scaling windows and two configuration sectors; and 2), for \(d > 6\), the scaling behaviors of all clusters, except the largest one, are in percolation universality, unexpected from the first sight. Interestingly, while the geometric properties are very sophisticated, critical behaviors in the spin representation are much simpler: no percolation-like behaviors exist and the upper critical dimension \(d_p = 6\) cannot be seen.

Several open questions arise. First, what are the precise forms of logarithmic corrections in critical FK clusters at \(d_c = 4\) and \(d_p = 6\)? In this work, we study the logarithmic corrections of sizes of the largest two clusters, but it will be interesting to carry out a systematic study of the effect of logarithmic corrections to various geometric quantities, especially at \(d_p = 6\). Second, in the loop representation of the Ising model, which is another geometric representation and can be coupled to the RC model via the loop-cluster joint model \[33\], what would be geometric effects for \(d \geq 4\)? Finally, most of the exact exponents in Tables I and II are conjectured and rigorous proofs remain elusive.

**Acknowledgements** This work has been supported by the National Natural Science Foundation of China (under Grant No. 11625522), the Science and Technology Committee of Shanghai (under grant No. 20DZ2210100), the National Key R&D Program of China (under Grant No. 2018YFA0306501). We thank Eren M. Elço, Jens Grimm, Timothy Garoni, Martin Weigel and Jonathan Machta for valuable discussions, in particular for Jesper Jacobsen. When finalizing the collection and analysis of our Monte Carlo data, we learned from private communications that Jesper Jacobsen and Kay Wiese (ENS, Paris) are working on the same topic using a field-theoretical approach and propose another scenario—i.e., for \(4 \leq d < 6\), the scaling behavior of some geometric observables could be described by non-trivial critical exponents other than those from the CG-Ising asymptotic and from the GFP. Taking into account the logarithmic correction for the scaling of the second-largest cluster, which is conjectured solely based on simulations, this interesting scenario cannot be ruled out.

We dedicate this work to Professor Henk W.J. Blöte, who passed away on June 10, 2022. Blöte was internationally renowned for his numerous contributions to statistical mechanics, holding official positions at Delft University of Technology and Leiden University until his retirement in 2008, as well as a lifetime service to physics. Since his first paper on the specific heat singularities of Ising antiferromagnets in 1967, Blöte has maintained a particular passion for the Ising model among his research interests in different physical topics. As this work demonstrates, he has successfully conveyed his spirit to his students (Y.D) and his second-generation students (S.F and Z.Z). Blöte has maintained a very close relationship with China over the past few decades, even learning to speak the Chinese language. Blöte gave enormous guidance to the students and researchers he supervised, treating them as his children. His research fellows, especially his two Chinese PhD students (Youjin Deng and Xiaofeng Qian) and his Chinese postdoc Wen’an Guo are so grateful for having had Blöte as their supervisor. Blöte was very generous, kind, and always ready to provide us with support and love. Blöte was our physics mentor and remains our lifetime mentor. The seed of physics he sowed in China has grown into academic trees of several generations; the seed of love he planted in China has grown into a sea of sunflowers that warms the hearts of countless people.

\[| \]$$\begin{array}{|c|c|c|}
\hline
\tau - 1 & 4 \leq d < 6 & d \geq 6 \\
\hline
R_1 & d/(1 + d/2) & 3/2 \\
N_s & \sim L \sim L^{3/6} & \sim L^{d-6} \\
\hline
\end{array}$$

**TABLE II. Some scaling behaviors for \(4 \leq d < 6\) and \(d \geq 6\), including Fisher exponent \(\tau\), finite-size scaling of the gyration radius \(R_1\) and the number \(N_s\) of spanning clusters, and two scaling windows.**
