How to Make a New Logic

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Abstract

We discuss about how to make a new logic with considering a slogan: take the inversion of what you know. Suppose that we have a Gentzen-style logical system $S$. The operation to make a new logic from $S$ is the following: for all the axioms and rules of $S$, change the direction of all the arrows occurred in the sequents of the axiom or the rule to the opposite side. We call this operation Stahlization. We consider certain logics in this respect.

Keywords: Gentzen’s sequent calculus, Stahlization, contradiction, anti-intuitionistic calculus, symmetry, paraconsistent logic, Vasiliev’s logic.

1 The slogan for how to make a new logic

When we look at past revolutional works especially for philosophy, art, mathematics and science, which gave us totally different perspectives over individual field, such as Kant’s Copernican revolution, J. S. Bach’s ‘Die Kunst der Fuge’ (BWV 1080), homological algebra, H. M. Friedman’s reverse mathematics, the Copernican system, Einstein’s special theory of relativity and
so on, we will be soon aware of some similarity comprised in them. The similarity which we mean can be expressed as the following slogan:

Take the inversion of what you know.

This is a very simple principle. However, the simple operation has already provided us with incredibly great novelties in spite of its simplicity.

There actually are many ways to make a new logic. We shall roughly classify the possible ways to two categories.

The first category: a modification of a given logic, in a broad sense, such as the symmetrization of the logic, some addition of new axioms or rules to the logic, taking a meaningful subsystem of the logic and so forth.

The second category: a creation of a totally new logic from a own philosophical point of view, or as a systematic description of some logical structure appeared in particular phenomena which we see. These two categories are not exclusive. For example, intuitionistic logic whose creating process would belong to the second category is nothing but a subsystem of classical logic as the result of the axiomatization of what L. E. J. Brouwer intended.

2 Stahlization

We think that we would have been enough prefaced with the above paragraphs. The aim of this essay is to make the readers realize that a simple operation to be mentioned below, satisfying the slogan introduced in the first paragraph and the first category just mentioned, is a very promising way to create a new logic. Suppose that we have a Gentzen-style logical system $S$. The operation to make a new logic from $S$ is the following: for all the axioms and rules of $S$, change the direction of all the arrows occurred in the sequents of the axiom or the rule to the opposite side. We shall call this operation Stahlization after the name of a philosopher Gerold Stahl, mimicking 'Gödelization' in the terminology of recursion theory.

We can find the example of Stahlization in two papers, i.e. Goodman [9] and Inoué [11]. Though [9] is ornamented by many considerations about his

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4 Rather technical examples are e.g. Lebesgue integration, the Boyer-Moore string-search algorithm, Inverse kinematics and so on.

5 In this respect, B. C. Fraassen’s book [26] is interesting.

6 We may call the operation dualization instead of Stahlization. However, this terminology is not much fun.
subject, it is, in a word, a study of the Stahlization of intuitionistic propositional calculus in Gentzen-style (for that, see Kleene [15]). He called the Stahlization of the logic anti-intuitionistic propositional calculus. According to him, the choice of a logic in order to express his motivation and intention was due to F. W. Lawvere, one of great figures in the area of category theory. Goodman’s motivation lies on a study in logic subject to vagueness or indeterminateness in the scope of his way and interest. His attempt is however still in the stage of try and error. We shall mention Goodman’s contribution again in the sequel. Now, we shall proceed to another example of Stahlization. Stahl [24] proposed some propositional calculi to derive only the negation of a thesis in classical propositional logic. Such a system is called Stahl’s opposite system and we call a thesis in such a system a contradiction. The real novelty of Stahl’s formulation from the standpoint of axiomatic rejection consists in the axiomatization of all the contradictions free from the notion of provability. In our use of words, Inoué [11] observed that the idea of Stahl is nothing but the Stahlization of classical propositional logic in Gentzen-style. His observation is based on the inversion principle after P. Lorenzen with respect to negation in Gentzen’s sequent calculus LK. He could extend Stahl’s logic to the first-order predicate level on the basis of the Stahlization (see [11]). It should be mentioned that I, the author of [11], also was stimulated by category theory to get the idea.

We remark that Goodman’s anti-intuitionistic propositional calculus and my systems $SP$ and $SC$, in the notion of [11], which are thus the Stahlization of classical propositional and predicate logics, since the former is a consistent system in which not all but some contradictions are derived, and the later is the extreme case of paraconsistent logic, namely the logic in which all theses are inconsistent.

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7Better to say, in the area of the foundation of mathematics, we do not mean however that category theory is the foundation of mathematics.
8Some systems equivalent to Stahl’s opposite system were independently studied by Morgan [16] and Bunder [5].
9Skura [21] obtained some general result in their respect. See also Inoué [12].
10The inversion principle with respect to negation reads that if $\Gamma_1, \Gamma_2 \rightarrow \Theta_1, \neg A, \Theta_2$ holds in LK, then so does $\Gamma_1, A, \Gamma_2 \rightarrow \Theta_1, \Theta_2$.
11Goodman’s system is a consistent in the sense that not every sentence is derivable in the system.
12I was also inspired by studying the method of axiomatic rejection. For axiomatic rejection, see for example Inoué [12] and Inoué, Ishimoto and Kobayashi [14]. Also refer to Carnielli and Pulcini [6], and Goranko, Pulcini and Skura [10].
3 Paraconsistent logic, Vasiliev’s logic and beyond

It is not a wonder that some of the current interests in philosophy and logic are in something vague or inconsistent such as vagueness, fuzzy logic, paraconsistent logic\(^{13}\) and so on. Also we know a tradition which goes back to Heraclitus, who insisted that the world is not consistent\(^{14}\). We also have to consider the influence of computer science to philosophy and logic. Computer science needs much more knowledge of inconsistent reasoning such as that of paraconsistent logic. Bodies of data stored in a computer memory may be not consistent, and if the stock of data is very large, it is not practical (thus not economical) to check the consistency of it at all\(^{15}\).

In connection with this essay, it is very important that G. Priest pointed out the similarity between the interpretation of Vasiliev’s neo-syllogistic proposed by Smirnov \([23]\) and that of anti-intuitionistic propositional calculus in Goodman \([9]\) in his review Priest \([17]\) of \([23]\). N. A. Vasiliev considered in the early years of this century atomic propositions of the form: \(S\) is and is not \(P\)\(^{16}\). In the interpretation presented in Smirnov \([23]\), such statements are true if all the points in the set \(S\) are on the topological boundary of the set \(P\). From this fact, I am tempted to say that there is a meaningful syllogistic and Vasiliev’s logic is the Stahlization of the logic. To find such an original logic is kind of inverse problem. From the definition of Stahlization, if we have a Genzen-style formulation of Vasiliev’s neo-syllogistic, we immediately obtain the original logic by taking the Stahlization of the neo-syllogistic. We will obtain much more about the neo-syllogistic by the simple operation.

If we are given a logic, the Stahlization of it will give us new perspective over the original logic, if the Gentzen-style formulation of the logic is possible. This means the creation of a new logic in other words. This is what I wanted to write in this essay, that is, one of my paradigm over logic.

\(^{13}\)As a recent literature, see Ba¸skent and Ferguson \([4]\).
\(^{14}\)See for example Rescher and Brandom \([18]\) pp. 1-2.
\(^{15}\)See da Costa and Marconi \([7]\) p. 24].
\(^{16}\)For Vasiliev’s works, see especially Smirnov \([22, 23]\) and Arruda \([1, 2, 3]\).
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