Examining Beam Oscillations in the Space of Technogenic and Seismic Impact Parameters (Part I)

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Abstract. Free transverse oscillations of a compressed beam carrying two evenly distributed loads and two discrete masses in the span are studied. The mathematical model of oscillations is presented as a boundary value problem from the basic equation in hyperbolic partial derivatives of the fourth order in spatial coordinates, of second order in time, boundary conditions and block connection conditions. The technical theory of bending oscillations of rods based on Bernoulli’s hypothesis is used. We consider the spectral problem on the determination of eigenvalues, eigenmodes (Sturm-Liouville problem), which is necessary for the analysis of forced oscillations. It is argued that the solution by analytical methods is inexpedient in view of the large amount of transformations and calculations. The methods of separation of variables and finite differences are used. An algorithm of task solution has been developed and implemented in the Matlab software environment in the form of high precision graphoanalytic calculations. Practical conclusions have been made.

1. Introduction

Nowadays beam oscillations have an extensive bibliography as they are widespread elements of buildings and structures and are used in the equipment of almost all industries. In recent years, the widespread use of robots, automated plants and workshops has resulted in beams experiencing unconventional, non-classical technogenic external influence of dynamic and kinematic character. To date, transverse and longitudinal beam oscillations and rods in general have an extensive bibliography [1-5]. But the scientific and technical revolution of the last decades and the requirements of the digital economy of the last years require more and more accurate structural calculations including beams.

Seismic and technogenic loads are specific and very dangerous for beams because of their random nature very strongly bound to the uncertainties at the place of occurrence of earthquakes and multi-parameter intensity of their course. Research shows that dynamic behaviour of structures essentially depends on parameters of beams and the influences forming a big area in the multidimensional (mathematical!) space of parameters. The multidimensional space of the impact parameters in this paper implies the sum of subspaces of the Euclidian space $\mathbb{R}^n$, which contain the corresponding points of the sources of mechanical oscillations: frequency, dominant frequency, amplitude, phase, force,
direction, point of application, spectral structure, bandwidth, correlation coefficient, mathematical expectation, dispersion, etc. These issues will be addressed in Part II of this paper.

![Figure 1. Beam of flexible resilient supports.](image1.png)  
![Figure 2. Block connection.](image2.png)

2. Mathematical model of oscillations

The substantive formulation of the problem is based on the calculation scheme of a compressed beam depicted in Fig. 1 and bearing two discrete masses. A compression force \( P \) acts on the beam in the axial direction, static forces from the natural and bearing mass \( m \) and \( q \) in the transverse direction, which we will consider evenly distributed. The beam supports are flexible and resilient with stiffness coefficients \( c_1 \) and \( c_2 \). Such a scheme is typical for a system of support beams called a beam cage. In this case, a complex type of beam cage is used, which is applied at high loads and distances between columns.

Proceeding to the mathematical model of oscillations, we will use the technical theory of bending oscillations of rods. Then, the basic equation has the form of a homogeneous hyperbolic partial differential equation [2].

\[
bu'' + Pu'' + ru'' + \mu u'' = 0, \quad b = EJ, \quad r = m + q, \quad x \in (0, l), \quad t > -\infty. \tag{1}
\]

Here, \( u(x,t) \) – is the sought-for function of deflection of the beam in the process of movement, \( b \) – bending stiffness of the beam, \( E \) – modulus of resilience of the beam material, \( J \) – axial moment of inertia of the cross-section, \( r \) – aggregate per-unit mass of the beam, \( \mu \) - specific coefficient of linear viscous internal friction. Dashes in the upper index indicate differentiation of the deflection function by the \( x \) coordinate, dots over the characters – differentiation by time. Static deflection of the beam is not taken into account due to its insignificance compared to dynamic deflections. Its calculation is performed trivially.

The presence of discrete masses \( M_1 \) and \( M_2 \) leads to a discontinuity of the second and third derivatives of the function \( u(x,t) \) at the points of their location, which forces to include block connection conditions of the sites to the left and right of them in the mathematical model of oscillations. Let us consider the situation near the mass \( M_1 \) (Fig. 2). This being said, it is necessary to consider that \( M_1 \) is a material point, the left and right edges of the element actually pass through one point.

The basic equation (1) is joined by boundary conditions corresponding to Fig. 1, and block connection conditions on the left and right of the discrete masses.

Boundary conditions

\[
u''(0,t) = 0, \quad bu''(0,t) - c_1 \mu u(0,t) = 0, \quad u''(l,t) = 0, \quad bu''(l,t) + c_2 u(l,t) = 0, \quad t > -\infty, \tag{2}
\]

\( c_1, c_2 \) – stiffness coefficients of resilient supports.

Some block connection conditions are kinematic and consist in the fact that the function \( u(x,t) \) and its first- and second-order derivatives must be smooth at the location points of discrete masses \( x_0 \).

\[
u(x_0 - 0, t) = u(x_0 + 0, t), \quad u'(x_0 - 0, t) = u'(x_0 + 0, t). \tag{3}
\]

The need to meet the conditions (3) is obvious from the point of view of continuity and integrity of the structure. Other conditions are dynamic and arise due to the presence of force factors at the
junction of adjacent areas (Fig. 2): bending moments on the left and right \( M_1, M_2 \), transverse forces \( O_1, O_2 \), longitudinal forces \( N \), D’Alambert’s inertia force \( D \).

The cross section at the given point is rotated by a small value and it follows that \( M_1 = M_2 \). The mass moment of inertia resulting from the rotation is negligibly small, so we will not consider it. The equality of bending moments in the left and right cross sections corresponds to the smoothness of the second-order derivative function.

\[
bu''(x_0, t) = bu''(x_0 + 0, t). \tag{4}
\]

Transverse forces \( Q_1, Q_2 \) and D’Alambert’s inertia force \( D \) act in a vertical direction and create a progressive mass movement. The movement is

\[
D = Q_1 - Q_r, \tag{5}
\]

where D’Alamberth’s inertia force

\[
D = M_k \ddot{u}(x_k, t).
\]

It follows that equation (5) should be included in the connection conditions. Being more concrete, we will get

\[
M_k \ddot{u}(x_k, t) = b[u''(x_k - 0, t) - u''(x_k + 0, t)], \quad k = 1, 2. \tag{6}
\]

The smoothness of the function \( u(x, t) \) on the third-order derivative is not provided and, this must be taken into account in the forthcoming calculations. Equation (1), boundary conditions (2) and connection conditions (3), (4), (6) form a mathematical model that allows to determine the function \( u(x, t) \).

3. Free oscillations

The study of free oscillations is obligatory before the consideration of forced oscillations because without such results of spectral problem as natural frequencies and forms, any qualified analysis of forced oscillations is impossible.

Then we will use the method of separation of variables and present the function of longitudinal axis deviation of the beam as a product

\[
u(x, t) = X(x)e^{\lambda t}, \tag{7}
\]

where \( X(x) \) – is the sought-for function of natural forms, \( \lambda = -\mu + i\omega \) - a characteristic index, \( \mu, \omega \) - attenuation coefficient and angular frequency of free oscillations, \( i \) – an imaginary unit. The validity of such a representation is that it should describe a process that is an oscillation and, moreover, is fading. The coefficient \( \mu \) will be determined by the bibliographic sources. The frequencies \( \omega \) are to be calculated. By substituting (7) in (1) – (4), (6) and reduction by the total multiplier, we obtain a system of homogeneous linear algebraic equations

\[
B(\lambda)X = 0, \tag{8}
\]

where \( B(\lambda) \) - is a matrix of coefficients which contain the sought-for frequency \( \omega \). Their definition by analytical methods is associated with cumbersome calculations and the need to solve the system of transcendental equations. A more universal and simpler way to determine a spectral pair \( (\omega, X) \), consists in using numerical methods [6] and computational complexes. We apply finite difference method (FDM) and Matlab software complex. The FDM requires transition of the continuous argument region \( 0 \leq x \leq l \) to the discrete argument region (grid)

\[
l_h = \{x_i = (i - 1)h, \quad i = 1, 2, ..., n\}
\]

with the step \( h = l/(n - 1) \). After applying the method of separation of variables, we obtain

\[
bX''''(x) + PX''(x) + AX(x) = 0, \quad A = r\lambda^2 + m\lambda. \tag{9}
\]

\[
X''(0) = 0, \quad bX''(0) - cX(0) = 0, \quad X''(l) = 0, \quad bX''(l) + cX(l) = 0. \tag{10}
\]

\[
b [X''(x_k - 0) - X''(x_k + 0)] - M_i \lambda^2 X = 0, \quad i = 1, 2. \tag{11}
\]
Here in after, \( x_k \) – are the coordinates of connection points. Note that the connection conditions (3), (4) are not specified here because of the smoothness of the function \( X(x) \) itself and its first- and second-order derivatives at the boundary of sections.

We will perform the necessary FDM procedures to replace the derivatives with their finite-difference analogues with the accuracy of \( O(h^2) \) and obtain instead of the basic equation (1)

\[
X_{i-2} + \alpha X_{i-1} + \beta X_i + \alpha X_{i+1} + X_{i+2} = 0.
\]

(12)

Here are the designations for the coefficients

\[
\alpha = -4 + \frac{P h^2}{b}, \quad \beta = 6 - 2 \frac{P h^2}{b} + \frac{A h^4}{b}, \quad \lambda = r^2 + \mu. \lambda.
\]

We will perform similar actions for boundary conditions and the connection condition. Left side:

\[
2X_1 - 5X_2 + 4X_3 - X_4 = 0, \quad \gamma X_3 + 18X_2 - 24X_3 + 14X_4 - 3X_5 = 0, \quad \gamma = -5 - 2h^3c_1 / \beta.
\]

(13)

(14)

Right side:

\[
-X_{n-3} + 4X_{n-2} - 5X_{n-1} + 2X_n = 0, \quad \gamma X_{n-3} + 4X_{n-2} - 18X_{n-1} - 3X_n = 0, \quad \gamma = -5 + 2h^3c_2 / \beta.
\]

(15)

(16)

Connection conditions:

\[
3X_{k-4} - 14X_{k-3} + 24X_{k-2} - 18X_{k-1} - 5X_k - 18X_{k+1} + 24X_{k+2} - 14X_{k+3} + 3X_{k+4} = 0, \quad \epsilon = 10 - 2h^3M^2 / \beta, \quad i = 1, \ 2.
\]

(17)

The mathematical model of free oscillations is now represented by a system of algebraic equations (12) – (17) which we will write in the matrix-vector form

\[
A(\lambda)Y = 0.
\]

(18)

where \( A(\lambda) \) is the matrix of coefficients, \( Y \) is the vector of discrete arguments replacing the continuous eigenfunction \( X(x) \), it should be kept in mind that instead of exact values, the approximate values will be calculated \( Y = [y_1, y_2, ..., y_n] \), \( y_i \approx X(x_i) \), \( i \) – is the number of the finite-difference grid node. The matrix of coefficients has the form

\[
A(\lambda) = \begin{bmatrix}
2 & -5 & 4 & -1 & \gamma \\
18 & -24 & 4 & -3 & 1 \\
\alpha & \beta & \alpha & 1 & \alpha \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \alpha & \beta & \alpha & 1 \\
3 & -14 & 24 & -18 & \epsilon \\
\alpha & \beta & \alpha & 1 & \alpha \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \alpha & \beta & \alpha & 1 \\
3 & -14 & 24 & -18 & \delta \\
-1 & 4 & -5 & 2 & 1
\end{bmatrix}
\]

There are no entries for zero elements. The elements of the main diagonal \( \beta \) contain eigenvalues. They can be determined from the condition of existence of non-zero solutions of the system of equations (18) which consists of the following

\[
\det A(\lambda) = 0.
\]

(19)

This equation is a high order algebraic one and at high \( n \) values, its compilation and solution are very difficult. That is why we have used the Matlab computational complex and with its help performed high precision graphoanalytic calculations. Their main point is that the function \( f(\omega) = \det A(\omega) \) will be calculated at \( \omega \in \Omega \) where \( \Omega \) – is the interval of the assumed location of the first frequencies of free oscillations and is displayed on the plane \( (\omega, f) \) graphically on the monitor screen.
The coordinates of the intersection point of the axis $\omega$ are the sought-for eigenvalues $\omega_k$, $k=1, 2, ...$

Example 1. Input data: $l = 6$ m, $E = 210$ hPa, $J = 2550$ cm$^4$, $m = 24$ kg/m, $M_1 = 8000$ kg, $M_2 = 5000$ kg, $q = 3000$ kg/m, $P = 10$ kN, $c_1 = 200$ kN/m, $c_2 = 100$ kN/m, $\mu = 0.1$ s$^{-1}$, $n=601$.

The result of the solution is displayed on the monitor screen (Fig. 3) as a graph ($\omega$ - det) where the first eigenfrequency is marked with a dot. Three elements of eigenfrequency spectrum are defined $\omega_k = \{21.72, 48.04, 137.66\}$ s$^{-1}$.

These numbers are highly accurate as the monitor screen allows you to increase the size of the graph and view fragments and the surrounding areas of points in great detail.

Figure 3. Frequency of free oscillations.

The use of a normative document (7) shows that the first frequency of free oscillations corresponds to the highest dynamic factor $\beta = 2.5$ (Fig. 4) for structures on soils of all classes. It is advisable to reduce it by changing the parameters of the oscillating system. A qualified solution of such a problem should be based on methods of dynamic oscillation damping, methods of optimal design of dynamic mechanical systems. The arsenal of techniques and special means for such cases is rather extensive and the choice essentially depends on concrete circumstances.

Then, the task is to find the eigenvectors $Y_k$ ($k = 1, 2, 3, ... n$) of matrix A. They can be determined from the system of equations (18) at known eigenvalues $\omega_k$. In this case, the determinant of matrix A becomes zero, and its rank equals $R = n-1$, so that the eigenvectors are determined accurate to the multiplier. It follows that one of the non-zero components can be set equal to an arbitrary number and excluded from the sought-for ones. The system of equations (18) turns from homogenous to heterogenous, from which other components of vector $Y_k$ are defined. The obtained natural forms normalized to the unit and corresponding to the eigenvalues are presented in Fig. 5. It can be seen that with the growth of natural values the waviness of oscillation forms increases.
Figure 5. Forms of free oscillations.

The first natural form almost overlaps the half sine wave. Similar half-waves are also found in other forms. At the same time, they meet the boundary conditions and connection conditions: the edges have flexible supports and therefore have large deviations; the heavy discrete masses $M_1, M_2$ remain low-moving with large oscillations of continuous sections. According to these signs, it can be assumed that the results are integrally reliable in determining frequencies and natural forms.

4. Conclusions
1. Mathematical models have been created to study free oscillations of complex beam structure under seismic and technogenic impacts.
2. A high precision graphoanalytic method for solving the spectral problem for beams has been proposed.
3. An effective numerical method for determining eigenfrequencies and natural forms of free oscillations has been developed.
4. The effectiveness of the mathematical model has been confirmed by numerical examples.
5. It has been proposed to derive the first two eigenfrequencies from the zone of large dynamic factors in accordance with the standards.

5. References
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