Rigorous KAM results around arbitrary periodic orbits for Hamiltonian systems

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Abstract
We set up a methodology for computer assisted proofs of the existence and the KAM stability of an arbitrary periodic orbit for Hamiltonian systems. We give two examples of application for systems with two and three degrees of freedom. The first example verifies the existence of tiny elliptic islands inside large chaotic domains for a quartic potential. In the 3-body problem we prove the KAM stability of the well-known figure eight orbit and two selected orbits of the so called family of rotating eights. Some additional theoretical and numerical information is also given for the dynamics of both examples.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The KAM theorem (see [2, 16, 27] and also [3]) is a fundamental result for Hamiltonian systems because it ensures the existence of a set, nowhere dense but of positive measure, of points of the phase space which behave in a regular, quasi-periodic way. The main point is that the system should be a perturbation of an integrable system and that a non-degeneracy condition, requiring the invertibility of the actions to frequencies map, has to be satisfied. The standard notation, being a Hamiltonian system given by
\[ q = H_p, \quad p = -H_q, \]

where the Hamiltonian \( H(q, p) : \Omega \to \mathbb{R} \) is a smooth function defined on an open set \( \Omega \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \), that is, with \( n + 1 \) degrees of freedom, will be used.

If we consider the dynamics close to a fixed point the methodology is simple. Assume that the fixed point is totally elliptic or the problem can be reduced to the totally elliptic case—for instance, by restricting the attention to the centre manifold. Then one can proceed to compute the normal form up to a moderate order, say to order 4 in the \((q, p)\) variables. Assuming that no resonances occur up to this order then one can consider the normal form as the integrable Hamiltonian and the remainder as the perturbation, and it is easy to check the non-degeneracy condition. This approach has been used e.g. in the study of the vicinity of the collinear libration points in the general planar three-body problem, restricted to the centre manifold, see [20]. A moderate number of arithmetic operations allows decision on the applicability of the KAM theorem.

The problem is much more involved when we want to apply KAM theorem around an arbitrary totally elliptic periodic orbit which is not known analytically. Even if some analytic expression of the orbit is available, the study of the dynamics on the vicinity at the required order may not be feasible analytically. Usually, one can restrict the problem to the study of the vicinity of a fixed point of a symplectic map on a suitable Poincaré section in dimension 2n.

The goal of this paper is to set up a methodology for rigorous checking of the KAM conditions for the symplectic map (see, e.g. [3]).

We give two examples of application. The first is a simple classical Hamiltonian system with two degrees of freedom and depending on a parameter \( c \). The main feature is that the potential consists only of quartic terms. Changing \( c \), the system can be integrable or display large chaotic domains. In these domains one can guess, by numerical computation of iterates of a Poincaré map, that some tiny islands exist. The problem is to show, rigorously, that there are indeed elliptic periodic points of the Poincaré map inside these islands and that KAM conditions hold.

The second example concerns the well-known figure eight solution of the general three-body problem with equal masses. See [7] for a proof of the existence of that orbit, found numerically by Moore [24]. This is an example of ‘choreography’ (see [32], where the notion of choreographic solution was introduced, and the references therein), that is, a \( T \)-periodic solution of the \( N \)-body problem \((N = 3 \text{ in the present case})\) such that all the bodies move along the same path with time shift \( T/N \) between consecutive bodies. This topic for \( N = 3 \) has been studied by the present authors in [12] where, in particular, it was proved the totally elliptic character of the figure eight on fixed energy levels and remaining at the zero level of angular momentum. Using reductions the problem becomes a Hamiltonian with three degrees of freedom. The related Poincaré map is 4D. In [31] it was claimed, based on a non-rigorous high order computation of a normal form, that the KAM condition is satisfied around the figure eight. The computation of the local expansion of the Poincaré map was done by numerical differentiation using multiple precision and optimal step size for the different orders. In the present paper the validity of the KAM condition for the figure eight orbit is established rigorously.

The paper is organised as follows. First the methodology to be applied is explained, introducing the required notation and emphasizing the rigorous aspects of the computer assisted proofs (CAP). In the next two sections, examples with two and three degrees of freedom are presented and several of their relevant properties are proved or mentioned. In particular, the reduction of the three-body problem in the present case is explicitly carried out, based on [35]. Finally, the results obtained by applying the methods to both examples are shown.
2. Preliminaries and the algorithm

Before going into details we want to emphasize that the algorithm presented in this paper is rigorous. It belongs to the methodology of CAP as it was essentially initiated in the pioneering paper of Lanford concerning the Feigenbaum conjectures [18]. This has been followed by many authors, including Berz, de la Llave, Eckmann, Mrozek, Tucker, Zgliczyński etc, and their co-workers. By rigorous we mean that during all the computations we take into account and bound all possible errors. In this way we get not the exact values but verified estimates of the computed quantities. Therefore the theorems that we apply have assumptions of a special kind (i.e. inequalities, inclusions etc) that can be checked using those estimates. As a result we obtain a computer assisted method to have proofs of the existence and KAM stability of periodic orbits for Hamiltonian systems away from a perturbation problem. In theory all calculations could be done by hand, but in practice the number of operations, even if they are very basic and trivial, exceeds human resources.

2.1. Interval arithmetics

As the precision of the computer is finite we use an interval arithmetic to take care of the round-off errors. All floating point operations are replaced by the corresponding operations on closed intervals, such that we always obtain some representable superset of the true result. This is also extended to all elementary functions.

For the rest of the paper, following [15], we use boldface to denote intervals and objects with interval coefficients. For those objects we use the names interval vector (or box), interval matrix etc to stress their interval nature. For a set $S$, we denote by hull $S$ the interval hull of $S$, i.e. the smallest product of intervals containing $S$. For an interval $a = [g, \bar{a}]$ we define its diameter as $\text{diam} a := \bar{a} - g$. We define the diameter of a box $x = (x_1, x_2, \ldots, x_n)$ as a maximal diameter of its components, i.e. $\text{diam}(x) = \max\{\text{diam}x_k | k = 1, \ldots, n\}$. By int$(x)$ we denote the interior of $x$.

2.2. The rigorous computation of Taylor expansions of Poincaré maps

To rigorously integrate ODEs and to obtain verified enclosures for the partial derivatives with respect to the initial conditions we use $C^1$-Lohner [37] and $C^r$-Lohner [36] algorithms implemented in the CAPD library [5]. Those algorithms are based on the Taylor integrator and set representations proposed by Lohner. For some set of data (say the box $x$) and time step $h$ the $C^1$-Lohner algorithm produces rigorous enclosures for the solution $\varphi(x, h)$ of the ODE and its derivatives $\frac{\partial^|\alpha|\varphi(x, h)}{\partial x^\alpha}$ for all initial points $x \in x$ and all multiindices $|\alpha| \leq r$. The specialized $C^1$-Lohner algorithm is able to compute only first order derivatives.

The ‘naive’ representation of the derivatives as a table of interval vectors leads to a huge overestimation due to the ‘wrapping effect’ (see [19, 25]). Hence, internally during the integration, for each multiindex $\alpha$ the corresponding vector $v = \frac{\partial^{|\alpha|}\varphi(x, h)}{\partial x^\alpha}$ is stored in one of the Lohner representations [19]. For that reason even $C^0$ algorithms need $C^1$ information to properly set coordinates to suppress the “wrapping” error. In the current implementation, to store derivatives we use doubletons $v = x_0 + Cr_0 + Br$, where
where $x_0$ is a point vector (the centre), $C$ and $B$ are matrices of ‘good’ coordinates (usually $C$ is close to the Jacobian matrix and $B$ is its orthogonalization), $r_0$ represents the initial size of vector $v$ and $r$ stores all computational errors.

To compute all the derivatives up to order $r$ for an $n$ dimensional ODE we need to solve $n(r + 1)$ equations. If they were to be solved directly it would lead to the integration in a high dimensional space, which is usually inefficient (most of the rigorous solvers internally need $C^1$ information that squares the dimension). The $C^r$-Lohner algorithm makes use of the special structure of the variational equations to avoid this and as a result can bound derivatives to an arbitrary order $r$ in an efficient way.

On top of those rigorous ODE solvers the CAPD library implements algorithms to rigorously compute Poincaré maps and their derivatives with respect to the initial conditions on an affine section (for details see [36, 37]).

2.3. Notation and definitions related to Hamiltonian systems

We denote by $J$ the Poisson matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where $I_n$ denotes the $n$ dimensional identity matrix. The Poisson bracket of functions $f, g : \mathbb{R}^{2n} \ni M \mapsto \mathbb{R}$ is a new function $\{f, g\} := (\nabla f)^T J \nabla g$.

Let $\mathbb{N}^+$ be the set of all positive integers and $\mathbb{N} := \{0\} \cup \mathbb{N}^+$. An element of $\mathbb{N}^n$ is called a multiindex. For a multiindex $k = (k_1, k_2, \ldots, k_n)\) and a vector $v = (v_1, v_2, \ldots, v_n)$ we define

- $|k| = \sum_{j=1}^{n} |k_j|$
- $v^k = v_1^{k_1} v_2^{k_2} \ldots v_n^{k_n}$.

A vector $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$ satisfies the non-resonant condition up to order $r$ if for all multi-indices $k = (k_1, k_2, \ldots, k_n)$ such that $1 < |k| \leqslant r$ and all $j \in \{1, 2, \ldots, n\}$ we have

$$\lambda_j \neq \lambda^k.$$ (2)

2.4. The algorithm

For a given Hamiltonian system (1) we assume that we have an approximate initial condition $x_0$ of a periodic orbit and a Poincaré section $\Sigma$ that contains $x_0$. For a fixed level of energy $h \approx H(x_0)$ the Poincaré map defines a symplectic map

$$\mathcal{P} : \Sigma \mapsto \Sigma.$$ We will use $x = (q, p) \in \mathbb{R}^{2n}$ to denote local canonically conjugated variables. Let $\tilde{x}_0$ correspond to $x_0$ in these variables.

In this setting, to prove existence and KAM stability of a periodic orbit given by approximate initial conditions $x_0$ it is enough to prove for some $k \in \mathbb{N}^+$ the existence and KAM stability of the unique fixed point of $\mathcal{P}^k$ in some small neighbourhood of $\tilde{x}_0$. The algorithm consists of the following steps:

(1) Proof of the existence of the unique fixed point (a periodic orbit). Rigorous estimates of initial conditions.
(2) Proof of the linear stability.

968
(3) Computation of a rigorous Birkhoff normal form.
(4) Checking an appropriate non-degeneracy condition.

The details for each step will be given in the following sections.

2.5. Proof of the existence

In the first step of the algorithm we prove the existence of an unique periodic orbit close to
\( x_0 \) and obtain bounds for its initial conditions. Therefore the preliminary step is to reduce the
Hamiltonian system (1) by suitable symplectic transformations so that for a given energy level
the periodic orbit is isolated. Section 4 contains a (very classical) example showing how it was
done in the case of the 3-body problem.

For a proof we take a box \( \mathbf{x} \subset \mathbb{R}^2 \) with centre in \( \tilde{x}_0 \), and compute estimates of the interval
Newton operator

\[
N(\tilde{x}_0, \mathbf{x}, F) = \tilde{x}_0 - \text{hull}(DF(\mathbf{x}))^{-1}F(\tilde{x}_0)
\]

for \( F(\mathbf{x}) = \mathcal{P}(\mathbf{x}) - \mathbf{x} \). If we succeed in verifying that \( N(\tilde{x}_0, \mathbf{x}, F) \subset \mathbf{x} \) then the interval Newton
theorem [1, 28] ensures that inside \( \mathbf{x} \) there exists a unique \( k \)-periodic point of \( \mathcal{P} \). Instead of the
interval Newton operator one can also use the interval Krawczyk operator

\[
K(\tilde{x}_0, \zeta, F) = \tilde{x}_0 - CF(\tilde{x}_0) + (I_{2n} - \text{Chull}(DF(\tilde{x}_0)))(\mathbf{x} - \tilde{x}_0)
\]

which does not require the whole interval matrix \( \text{hull}(DF(\mathbf{x})) \) to be invertible. In theory \( C \)
can be any invertible \( 2n \)-dimensional square matrix, but in practice we take the matrix \( C \) to be an
approximation of the inverse of \( DF(\tilde{x}_0) \). According to the interval Krawczyk theorem [14, 17]
to prove the existence of a unique zero of \( F \) in \( \mathbf{x} \) it is enough to show that \( K(\tilde{x}_0, \zeta, F) \subset \text{int}(\mathbf{x}) \).

Remark 1. The problem of proving the existence of zeros of \( F \) when, as in the present
case, this involves the computation of \( \mathcal{P}^k(\mathbf{x}) \) is well suited for the use of the parallel shooting
method. This method, instead of solving \( \mathcal{P}^k(\mathbf{x}) = \mathbf{x} \) directly, solves the system of equations

\[
\mathcal{P}(\mathbf{x}) = x_2 \quad \mathcal{P}(x_2) = x_3 \quad \ldots \quad \mathcal{P}(x_k) = \mathbf{x}.
\]

Each value \( \mathcal{P}(\mathbf{x}), \mathcal{P}(x_2), \ldots, \mathcal{P}(x_k) \) can be computed separately (in parallel) and involves a
numerical integration of an ODE for much shorter time, which helps significantly to decrease
the propagation of errors. In our implementation we make use of it to improve precision and
to speed up computations.

Remark 2. One can improve the estimates of the initial condition of the periodic point by
further iteration of the interval Newton or Krawczyk operator.

2.6. Proof of linear stability

The current step goal is to prove that all the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_{2n} \) of the differential of
the iterated Poincaré map \( D\mathcal{P} = \frac{d\mathcal{P}(\mathbf{x})}{d\mathbf{x}} \), where \( \hat{x} \) is a \( k \) periodic point of \( \mathcal{P} \), lie on the unit circle.
We want also to obtain rigorous estimates \( \lambda_j \) such that \( \lambda_j \notin \mathbb{R} \) for \( j = 1, \ldots, 2n \).

The point \( \hat{x} \) is not known exactly, but from the previous step we have rigorous estimates \( x \)
of it. From estimates \( D\mathcal{P} = \frac{d\mathcal{P}(\mathbf{x})}{d\mathbf{x}} \) using e.g. verified root finding methods applied to the character-
istic polynomial one can obtain estimates of the eigenvalues. Because those estimates
are given by some boxes, parts of them are outside the unit circle. But still the proof of linear
stability is possible, due to the fact that our system is Hamiltonian. It is also worth mentioning
that our method works only if 1 and −1 are no longer eigenvalues of \( DP \). In general this is the case, because to isolate the fixed point we have already had to remove first integrals, set Poincare section and fix energy level.

**Lemma 1.** Let \( A \in \mathbb{R}^{2n \times 2n} \) be a symplectic matrix with eigenvalues \( (\lambda_1, \ldots, \lambda_{2n}) \), and let \( \lambda_j \) be boxes such that \( \lambda_j \in \lambda_j \) for \( j = 1, \ldots, 2n \). If the following hold

\[
\begin{align*}
(A1) & \quad 0 \not\in \text{Im}(\lambda_j) \quad \text{for} \quad j = 1, \ldots, 2n, \\
(A2) & \quad (\lambda_j^{-1} \cap \lambda_k = \emptyset) \implies j = k,
\end{align*}
\]

then all eigenvalues of \( A \) are distinct and lie on the unit circle.

**Proof.** The matrix \( A \) is symplectic, hence if \( \lambda \) is an eigenvalue of \( A \) then \( \lambda \), \( \bar{\lambda} \) are also eigenvalues. But then assumptions (A1) and (A2) ensure that \( \lambda^{-1} = \bar{\lambda} \) and hence \( \|\lambda\| = 1 \). From (A2) we have also that all eigenvalues are distinct. □

This general method requires sharp bounds for the eigenvalues. At least they need to be separated from the real axis and any two of them cannot overlap. This can be a not-so-easy task in general. Another possibility is to translate constraints to the characteristic polynomial. In this case, one first proves that the eigenvalues are on the unit circle and then obtains rigorous enclosures of \( \lambda \) using this fact (see for example [12]).

### 2.7. Computation of Birkhoff normal form

We refer e.g. to [30] for background on Birkhoff normal form for Hamiltonian systems and for 2D symplectic maps, easily generalized to the 2\(n\)-dimensional case. The literature on how to compute Birkhoff normal form is very rich and there are also packages that can do it in a non rigorous way. Here we want to explain how to make this process rigorous. All the computations are done using interval arithmetics. At the end the normal form will have interval coefficients that contain the exact values.

We give here full details for convenience of the readers not familiar with Birkhoff normal form. Readers familiar with it can jump to section 3.

Let \( T(x) = \sum_{k \in \mathbb{Z}^n} c_k x^k \) be the Taylor series of an analytic symplectic map around a totally elliptic fixed point and let \( (\lambda_1, \ldots, \lambda_n, \bar{\lambda}_1, \ldots, \bar{\lambda}_n) \) be the eigenvalues of the linear part of \( T \). Then for \( j \in \{1, 2, ..., 2n\} \) and a multiindex \( k \) such that \( k_j = k_{j \neq n} + 1 \) and \( k_m = k_{m \neq n} \) for \( m \neq j \) the condition (2) is not satisfied and a resonance occurs. This is usually known as an avoidable resonance and we say that the term \( c_j x^k \) corresponds to that resonance. A Taylor series \( T \) is said to be non-resonant if only unavoidable resonances are present.

The goal of this section is to make a symplectic change of variables such that in the new variables the Taylor series \( T \) up to a given order \( r \) reads

\[
z \mapsto \Lambda z + T_r(z),
\]

where \( \Lambda \) is a diagonal matrix, \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n, \bar{\lambda}_1, \ldots, \bar{\lambda}_n) \), and \( T_r(z) \) contains only terms corresponding to unavoidable resonances. This is the so called non-resonant Birkhoff normal form. In what follows, we present an algorithm that computes the non-resonant Birkhoff normal form up to order 3, but it can be easily extended to any given order.

First, using the \( C^r \)-Lohner algorithm [36] we compute rigorous enclosures of the coefficients of the Taylor expansion of \( T^3 \) up to order 3 for all points in \( x \) (an estimate of the fixed point). As a result we obtain a symplectic map (up to order 3)
We recall that from the previous step we know that the eigenvalues \( \lambda = (\lambda_1, \ldots, \lambda_{2n}) \) of the linear part of \( T \) lie on the unit circle.

As a second step we pass the linear part to a diagonal form

\[
\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n}).
\]

To this end we use the linear change, \( B \), given by a matrix formed by eigenvectors corresponding to the above eigenvalues. As we do not know the exact eigenvalues, the estimate of an eigenvector corresponding to \( \lambda_i \) has to be valid for all \( i = 1, \ldots, n \). To ensure that \( B \) is symplectic we use the following lemma and replace the previous eigenvectors by suitable multiples of them.

Lemma 2. Let \( A \in \mathbb{R}^{2n \times 2n} \) be a symplectic matrix with eigenvalues \( (\lambda_1, \ldots, \lambda_n, \lambda_{n+1} = \bar{\lambda}_n, \ldots, \lambda_{2n} = \bar{\lambda}_0) \) such that \( |\lambda_i| = 1 \) and \( \lambda_i = \bar{\lambda}_j \) for \( i, j = 1, \ldots, 2n \) and \( i \neq j \). Let \( (e_1, \ldots, e_{2n}) \) be corresponding eigenvectors. If \( e_i^T J e_{i+n} = 1 \) for \( i = 1, \ldots, n \) then the eigenbasis \( B = (e_1, e_2, \ldots, e_{2n}) \) is symplectic.

Proof. To be symplectic a matrix \( B \) needs to satisfy \( B^T J B = \{ e_i^T J e_j \} = J \). Due to antisymmetry we can assume that \( i < j \). For \( j = i + n \) we already have that \( e_i^T J e_{i+n} = 1 \). For \( j = i \) we have \( \lambda_i \lambda_j = 1 \) and therefore \( e_i^T J e_j = e_j^T \Lambda e_i = \lambda_i \lambda_j e_i^T J e_j = 0 \).

In our implementation initially \( e_i \) and \( e_{i+n} \) are complex conjugate vectors, therefore \( e_i^T J e_{i+n} = \text{ic} \) for some \( c \in \mathbb{R} \). We want to scale those vectors to get \( e_i^T J e_{i+n} = 1 \) and additionally \( e_i = \text{ic} e_{i+n} \). This is possible only if \( c < 0 \). Therefore if \( c > 0 \) we simply interchange the indices of corresponding eigenvalues and eigenvectors. Finally we set

\[
e_i \leftarrow \frac{e_i}{\sqrt{-c}}, \quad e_{i+n} \leftarrow \frac{e_{i+n}}{-1 \sqrt{-c}}.
\]

Let \( u = (u_1, \ldots, u_{2n}) \) be the new coordinates in this basis. The above scaling implies that \( u_i = \text{ic} u_{i+n} \) for \( i = 1, \ldots, n \). The final form of the symplectic transformation up to order 3, before starting the normal form computation, is

\[
S = B^{-1} \circ T \circ B : u \rightarrow \Lambda u + S_2 + S_3,
\]

where \( S_2 \) and \( S_3 \) denote quadratic and cubic terms of \( S(u) \) respectively.

Let us denote as \( U = (U_1, \ldots, U_{2n}) \) the coordinates of the normal form. We achieve a normal form in two steps, by cancelling first all the terms of degree two in \( S \) and then the terms of degree three, except the unavoidable resonances. For the first step we should select, in principle, a transformation of the form

\[
u = N_1(U) = U + Q(U),
\]

where \( Q(U) = (Q_1, \ldots, Q_{2n}) \) are quadratic terms. To cancel the terms of degree two in \( S \) we require that

\[
S \circ N_1 = N_1 \circ \Lambda
\]

holds up to degree two. Let us express this in coordinates. Assume that the quadratic terms of \( S \) and \( N_1 \) are written, respectively, as

\[
(\Sigma_2) = \sum_{|k|=2} c_{ij,k} u^k, \quad Q_i = \sum_{|k|=2} d_{ij,k} U^k, \quad \text{for } j = 1, \ldots, 2n,
\]

where \( k \) is a multiindex. The condition (3) allows us to obtain

\[
\Sigma_2 = Q_1 = \text{ic} \sum_{|k|=2} d_{ij,k} U^k,
\]

where \( c_{ij,k} \) and \( d_{ij,k} \) are constants. The condition (3) allows us to obtain
\[ d_{i,k} = \frac{c_{i,k}}{\lambda_i - \lambda^k} \]

for all the required indices \( i,k \). If there are no resonances of order 2, then all denominators are different from zero. We ensure this using rigorous estimates \( \lambda \).

However, the map \( N_1 \), as it was introduced, is not a symplectic map. Its differential satisfies the symplecticity conditions only to order 1 in \( U \), and we need additional cubic terms to satisfy them to order 2. This suggests defining the symplectic transformation as the time-1 map of some Hamiltonian \( W \)

\[ u = N(U) = \varphi^W_{1}(U). \]

To determine \( W \) we require

\[ \frac{\partial W}{\partial U_{j+n}} = Q_j, \quad -\frac{\partial W}{\partial U_j} = Q_{j+n}, \quad \text{for} \quad j = 1, \ldots, n. \]

As the difference between the map and the identity starts with quadratic terms, the Hamiltonian starts with cubic terms. Then the time-1 map adds terms of degree 3 to the initial ones. Finally we obtain the components of \( N(U) \) as

\[ (N(U))_j = U_j + Q_j + \frac{1}{2} \{ Q_j, W \}, \]

where \( \{ , \} \) denotes the Poisson bracket. This produces the normal form to order two as

\[ S : U \mapsto \Lambda U + S_3(U), \]

where \( S_3(U) \) are the cubic terms.

The last step is to remove all cubic terms except those corresponding to unavoidable resonances. We know that there exists a symplectic, near the identity transformation, that will cancel the non-resonant cubic terms, leaving the resonant terms unchanged. Hence we can simply set to zero all terms in \( S_3(U) \) for which we are able to verify the non-resonant condition.

Finally we obtain the normal form to order three (we use \( z \) as a new variables)

\[ S : z \mapsto \Lambda z + T_3(z), \]

where \( T_3(z) \) are the cubic terms corresponding to unavoidable resonances.

It remains to check the non-degeneracy condition which allows application of the KAM theorem. This is now easy and will be done directly on the examples.

### 3. A family of quartic potentials

As a first example we consider a very simple Hamiltonian

\[ H = \frac{1}{2}(p_x^2 + p_y^2 + x^4 + cx^2y^2 + y^4), \]

where \( c > -2 \) is a real parameter. This is a system widely considered as a paradigm of chaotic dynamics for \( c \) large in the relations between classical and quantum mechanics, see for instance [4, 8] and references therein. The energy should be positive and, due to the homogeneity of the potential, it can be considered equal to a fixed value. We shall consider the level \( H = \frac{1}{2} \). Note that for \( c \leq -2 \) unbounded motion occurs. The system has some obvious symmetries: it is reversible with respect to time and the changes of sign of \( x \) and/or \( y \) leave
the equations invariant. Furthermore, the symplectic change induced by the change of variables \((x, y) \rightarrow (u, v) = \frac{1}{\eta}(x + y, y - x)\) keeps the form of the Hamiltonian with the parameter \(c\) replaced by \(\hat{c} = \frac{12 - 2\eta^2}{2 + \eta^2}\), after a scaling to normalize again the coefficients of \(x^4\) and \(y^4\) to 1 as we had in the initial Hamiltonian. Obviously the map \(c \rightarrow \hat{c}\) is an involution having \(c = 2\) as fixed point. It can also be written as \(\hat{c} + 2 = \frac{16}{c + 2}\). When \(c\) ranges in \((-2, \infty)\) increasing its value, the parameter \(\hat{c}\) ranges in the same interval but decreasing.

It is simple to check that the planes \((x, p_x), (y, p_y), (x = y, p_x = p_y)\) and \((x = -y, p_x = -p_y)\) are invariant and, for the last two, modulo the change \(c \leftrightarrow \hat{c}\), we have the same phase portrait as for the first two.

The first question to be addressed is the integrability of the Hamiltonian. To this end we observe that \(H = T + V\) is a classical Hamiltonian and \(V\) is homogeneous of degree \(k = 4\). We specialize to this degree of homogeneity a theorem due to Morales–Ramis (see theorem 3 in [26]).

**Theorem 1.** Assume \(H = T + V = \frac{1}{2}(p_1^2 + \ldots + p_n^2) + V(x_1, \ldots, x_n)\) where \(V\) is homogeneous of degree 4. Let \(z = (z_1, \ldots, z_n)\) be a non-trivial solution (i.e. \(z = 0\)) of the system of equations \(\dot{z} = \nabla V(z)\) and let \(\lambda_1, \ldots, \lambda_n\) be the eigenvalues of \(\text{Hess} V(z)\). Then, if \(H\) is completely integrable with meromorphic first integrals, the values of \(\lambda_i\) must be equal to numbers of the form \(2m^2 - m, 2m^2 + \frac{1}{3}m + \frac{7}{2}2\) or \(2m^2 + 2m + \frac{1}{3}m\) for \(m \in \mathbb{Z}\).

**Corollary 1.** The Hamiltonian \(H\) is only integrable for \(c = 0, 2, 6\).

**Proof.** The admissible solutions for \(z\) (i.e. for the coordinates \((x, y)\) in the present problem) are \((\pm 1/\sqrt{2}, 0), (0, \pm 1/\sqrt{2})\), \((\pm 1/\sqrt{2} + c, \pm 1/\sqrt{2} + c)\). Beyond the trivial eigenvalue \(\lambda_1 = 3\), due to the homogeneity, one has \(\lambda_2 = c/2\) in the cases in which one of the components of \(z\) is zero and \(\lambda_2 = (6 - c)(c + 2) = \hat{c}/2\) in the other ones. These eigenvalues should be of one of the forms above. Also \((6 - \hat{c})(\hat{c} + 2)\) should be of one of these forms. This reduces the possible values of \(c\) to 0, 2, 6.

On the other hand the case \(c = 0\) is obviously separable; also it is \(\hat{c} = 0\) which corresponds to \(c = 6\). Finally in the case \(c = \hat{c} = 2\) the symplectic change \(x = r \sin \varphi, y = r \cos \varphi, p_x = p_\varphi \cos \varphi r + p_y \sin \varphi, p_y = -p_\varphi \sin \varphi r + p_x \cos \varphi\) converts the Hamiltonian to \(H = \frac{1}{2}(p_\varphi^2 r^2 + p_y^2 + r^4)\), reducible to 1 degree of freedom.

**Remark 3.** We look at non-trivial solutions of \(z = \nabla V(z)\) because then the Hamiltonian has meromorphic solutions obtained from the solutions of a hyperelliptic differential equation. One can look at the normal variational equations around them and at the differential Galois group of these equations. The fact that the identity component of this group must be Abelian if \(H\) is integrable gives the conditions in theorem 1. See [26] for details.

See figure 1 for an illustration of the phase portrait of \(\mathcal{P}\) for values of \(c\) close to the integrable cases. In the left (resp. right) column the value of \(c\) is smaller (resp. larger) than the integrable one. The plots allow to identify easily the bifurcations which occur at the integrable cases.

To understand the dynamics of \(H\) as a function of \(c\) it is useful to consider the Poincaré section \(\Sigma\) through \(y = 0, p_y > 0\), defined for \(p_x^2 + x^4 < 1\). The boundary of that domain is a periodic orbit on the invariant plane \(y = p_y = 0\). The initial data \(x = p_x = 0\) give rise to a periodic orbit \(y = y(t), \dot{y}(t)\), sitting on the \((x, p_x)\) plane, which corresponds to a fixed point of the Poincaré map \(\mathcal{P}\). The normal variational equations \(\xi' = \eta, \eta' = -cy(t)^2 \xi\) give the linear
stability of $(0,0)$. It is easy to check that the trace of $D\mathfrak{L}$ at the origin decreases from $+\infty$ to 2 for $c < 0$ and then, for increasing $c$, it oscillates between $-2$ and 6. In particular it takes the value $+2$ for $c = c_k = k(k+1), k \in \mathbb{N}$. For the values $c = (k - 1/2)(k + 1/2)$ it takes, alternately, the values $+6$ and $-2$ for $k$ even and odd respectively. See figure 2 for an illustration of the behavior of the trace. The set of values of $c$ for which $|\text{Tr}| < 2$ is the union of the intervals $(c_0, c_1) \cup (c_2, c_3) \cup \ldots$, where one has linear stability. The first linear stability intervals are

Figure 1. Phase portrait on $\Sigma$, the Poincaré section $y = 0$, close to the integrable cases. Left column from top to bottom: $c = -0.1, 1.9, 5.9$. Right column $c = 0.1, 2.1, 6.1$. The variables displayed are $(x, p_x)$. 

T Kapela and C Simó
Nonlinearity 30 (2017) 965
(0, 2), (6, 12), (20, 30), ... It is also clear that the periodic orbit at the boundary of $\Sigma$ has the same stability properties as the orbit we have just discussed.

In a similar way, one can consider the initial conditions $x_0 = 0, x_1 = \sqrt{1/2}$ which correspond to a periodic orbit in $\Sigma_{xy}^{pp} = xy$ as Poincaré section. Replacing $c$ by $\tilde{c}$ one has a similar result: for $c \to -2^*$, which implies $c \to \infty$, this fixed point is unstable for $c > 6$. Stability intervals for $\tilde{c}$ are identical to those given before for $c$ in the case of the fixed point at the origin. They give rise to intervals which accumulate to $-2$. The first intervals (in the $c$ parameter) are $(2, 6), (-6/7, 0), (-3/2, -14/11), (-50/29, -18/11), (-42/23, -66/37)$, etc. We shall refer to these two periodic orbits as the basic ones. Of course, symmetries give rise to similar orbits, like the one through $x_0 = 0, x_1 = -\sqrt{1/2}$.

To see the evolution of the phase space away from the integrable cases we have computed an estimate of the ‘fraction of chaotic motion’ in $\Sigma$ as a function of $c$. Due to the symmetries it is enough to do the computations for $x = 0, p_x = 0$. In the domain bounded by $x = 0, p_x = 0$ and $p_x^2 + x^4 = 1$ we have selected ‘pixels’ with centres of form $(i/2000, j/2000)$, $i, j \in \mathbb{N}$, and for each one we have estimated the maximal Lyapunov exponent $\Lambda$. In fact we are not interested in the concrete value but rather in whether one can accept $\Lambda = 0$. Symmetry and some simplifications allow reduction of the computational task. Then the fraction of chaotic motion $\psi(c)$ is estimated as the number of pixels for which one has evidence that $\Lambda > 0$ divided by the total number of pixels in the domain. Several checks have been done using different strategies and maximal number of iterates of the Poincaré map to have reliable information (see, e.g. [29] for details). The results are shown in figure 3.

The interpretation of the figure is clear: close to the integrable cases most of the dynamics is regular. In fact, if $c^*$ is one of the values of $c$ for which one has integrability, the fraction of chaoticity seems to be exponentially small in $|c - c^*|$ for nearby values, in agreement with the exponentially small upper bounds [9]. See also [33] for a variety of examples and [21] for quantitative relations between splitting and measures of the set of points with positive maximal Lyapunov exponent. Between these values of $c$ the value of $\psi(c)$ is below 0.12. Then, to the left of $c = 0$ and to the right of $c = 6$ there is a quick increase of $\psi(c)$. But at the ranges in which one of the basic periodic orbits is linearly stable, we can expect the existence of
islands which decrease the measure of the chaotic domain. The oscillations of the decrease are becoming smaller when the limits, either $c_2/\infty$ or $c/\infty$ are approached.

More concretely, the system—away from the range of values of $c$ where it is close to integrable—has a well defined and repetitive structure. To this end we consider the ‘fraction of regularity’ $1 - \psi(c)$. Figure 4 shows it for the values of the parameter $c$ on the right-hand side of the domain which contains the close-to-integrable dynamics. On the left-hand side of that domain the plot is symmetrical to that shown in figure 4 (left). To better see the scaling properties, the horizontal parameter $r$ is an extension of the index $k$, introduced before, to the real numbers, defined by $r(r + 1) = c$. The different ‘bumps’ are quite similar, and the heights scale as $r_1 \approx 1/r$. The right part of figure 4 shows the behavior for $c \in [c_4, c_6]$, which is similar for all the ranges $[c_{2m}, c_{2m+2}]$, $m \geq 2$. At $c = 20$ the central periodic orbit again becomes elliptic (see figure 2) (remember that the same thing happens for the periodic orbit at the boundary of $\Sigma$). A domain of regular motion starts which is reminiscent of the behavior found

Figure 3. Fraction of chaotic dynamics $\psi(c)$ in the Poincaré section $\Sigma$ as a function of the parameter $c$. To better see the behavior of the function $\psi$ the variable displayed in the horizontal axis is $d(c) = \log(c + 2)/\log(2)$ instead of $c$. The three integrable cases $c = 0, 2, 6$ for which $\psi(c) = 0$ are seen on $d = 1, 2$ and 3 respectively.

Figure 4. Left: the fraction of regularity as a function of $r$, defined by $r(r + 1) = c$. Right: idem for a range of $c$ containing $[c_4, c_6] = [20, 42]$ using $c$ as horizontal variable.
in many other problems. See, for instance, the reference [34] in the case of the Hénon map. However there are important differences due to the many symmetries of the present problem. Anyway the mechanisms to explain the behavior of the plot are those explained in [34]. For instance, around \( c = 25.6 \) the ‘last’ curve surrounding the period 4 islands breaks down and this increases the size of the main chaotic zone. See [22, 23] for relations with standard-like maps and their dynamical consequences.

At \( c = 30 \) the central periodic orbit becomes unstable by a pitchfork bifurcation and its separatrices display the typical figure eight shape, similar to the example displayed in the central plot in figure 5. Each part of the figure eight, skipping the invariant curves surrounding the full figure eight separatrix, is really close to the behavior of the Hénon map (except by non-linear scaling in variables and parameters). The decrease of the regular dynamics in figure 4 right from \( c = 30 \) to \( c \approx 31.1 \) is due to the destruction of the invariant curves surrounding the full figure eight separatrix. Then, at the approximate \( c \) values 32.0, 32.5, 33.5, 35.0 and 36.8 the decrease in the fraction is due to the destruction of curves surrounding period 6, 5, 4 and 3 islands, and to the period doubling, respectively. It is worth mentioning that for the case of period 3 almost no point has regular dynamics. The dynamics in that case, including existence of tiny period 3 islands and other properties, is in perfect agreement with that of the Hénon map as studied in [13]. See also [22, 23].

Our goal is to detect periodic orbits, well inside a chaotic domain, to which we can apply the methodology explained in section 2 to test the KAM conditions. To this end we have selected the value \( c = −0.90 \) (or, equivalently, \( \dot{c} = 138/11 \)), for which the value of \( \psi \) is approximately 0.9528. Iterates of the Poincaré map are easily obtained. Using the Taylor method at order 30 and a maximal relative truncation error of \( 10^{-21} \) they are computed at an average rate of 15 000 iterates per second for that value of \( c \).

Figure 5 shows a global view of the phase portrait and some details. As this value of \( c \) is in one of the instability domains, the periodic orbit through \((0, \sqrt{1/2})\) is unstable, but \( c \) is not too far from the left end of one of the stability intervals, at \( c = −67/7 \). Then the chaotic zone around the figure eight separatrix is still surrounded by invariant curves. Inside that chaotic zone there exist several stable periodic orbits. On the central part of figure 5 one can see a period 16 orbit. We shall test the KAM conditions for that orbit. An approximate initial condition on \( \Sigma \) is \( x = 0, p_x = 0.517 806 657 995 45 \). But also a periodic orbit of period 3, far away

Figure 5. Phase portrait on \( \Sigma \) for \( c = −0.90 \). Left: a global view (in grey) and two periodic orbits (in black). Centre: a magnification of the chaotic zone around the figure eight separatrix associated with the unstable periodic orbit on \((0, \sqrt{1/2})\) and a stable periodic orbit of period 16. Right: the island and some satellite islands near one of the period 3 points, well inside the large chaotic domain. The variables displayed are \((x, p_x)\).
from that separatrix, deeply inside the chaotic domain, has been found. An approximate initial condition for this orbit is \( x_p = 0.352718557335, y_p = 0.547882838499 \), and the applicability of KAM theorem for this orbit will also be checked.

We remark that the largest area inside an invariant curve around that period 3 island is of the order of \( 7 \times 10^{-6} \). It is not easy to capture this periodic orbit, but the previous computation of Lyapunov exponents in a grid with small step size is of great help.

It is also worth remarking that \( \Sigma \) is not a global surface of section, i.e. there are points on \( \Sigma \) that never return to \( \Sigma \). To see this one can consider the left plot in figure 5. Outside the island centred on \( x \approx 0.49235418852187, y = 0 \), but close to it, there is an unstable periodic orbit of period 4 of the Poincaré map related to \( \Sigma \), with one of the points in \( x = x^* \approx 0.328719644899305, y = 0 \). But the full periodic orbit, under the flow, remains on \( x > 0 \). Due to the symmetry, there is a similar orbit, to be denoted as \( O_4 \), which can be seen on the Poincaré section \( \Sigma = \{ x = 0, y > 0 \} \), with initial data \( x = x^*, p_y = 0 \), and which remains on \( y > 0 \). But the invariant manifolds of \( O_4 \) intersect \( \Sigma \). Hence, the Poincaré map associated to \( \Sigma \) is not defined on points in \( W^u(O_4) \cap \Sigma \).

It is well known that inside a ‘chaotic sea’ one can find tiny islands. In this paper we present the methods to prove their existence, even if they are very small. Some islands, even small, can play a role in the diffusive properties of a map—see [23].

4. The problem of the KAM stability of the eight

The figure eight periodic orbit, shown in figure 6, is a remarkable solution of the planar three-body problem with equal masses [7]. The three bodies move on the plane along the same path in solutions of the form \( q(t), q(t + T/3), q(t + 2T/3) \) where \( T \) is the period. In [31] a detailed numerical study of this orbit and an extended vicinity of it was carried out, looking also at the effect of small changes in the masses and the bifurcations that they create. See also [6] for choreographies related to the figure eight, like the satellite and the relative ones. The numerical evidence given in [31] suggests that not only is the orbit linearly stable but also KAM theorem applies around it. The rigorous proof of the linear stability was given in [12] and the proof of the applicability of KAM is studied now.

The figure eight solution, three nearby partially hyperbolic periodic orbits, as well as several 2D and 3D tori around the eight can be seen in [31]. The orbit has zero total angular momentum and due to the homogeneity of the potential and Kepler’s law, one can fix either the level of (negative) energy or the period.

The first step to be done is the reduction of the problem to a three degrees of freedom system. This is a classical result and we follow the exposition that can be found in Whittaker’s treatise [35]. For completeness a short account is given below.
4.1. Reduction of the 3-body problem

We shall assume that the masses are equal to 1. Let \( q_1, q_2, q_3 \) be the positions of the three bodies in \( \mathbb{R}^2 \) and \( p_1, p_2, p_3 \) the corresponding momenta. The Hamiltonian is

\[
H = \frac{1}{2}(|p_1|^2 + |p_2|^2 + |p_3|^2) - \frac{1}{|q_2 - q_3|} - \frac{1}{|q_3 - q_1|} - \frac{1}{|q_1 - q_2|}
\]

and the angular momentum is \( M = q_1 \wedge p_1 + q_2 \wedge p_2 + q_3 \wedge p_3 \).

Let \( q'_1, q'_2, q'_3, p'_1, p'_2, p'_3 \) be new variables introduced by means of the generating function

\[
W = (p_1, q'_1) + (p_2, q'_2) + (p_1 + p_2 + p_3, q'_3),
\]

where \((, )\) denotes scalar product. We recall that

\[
q_j = \frac{\partial W}{\partial p_j}, \quad p'_j = \frac{\partial W}{\partial q'_j}, \quad j = 1, 2, 3
\]

and, hence, the change gives

\[
q_1 = q'_1 + q'_3, \quad q_2 = q'_2 + q'_3, \quad q_3 = q'_3, \quad p_1 = p'_1, \quad p_2 = p'_2, \quad p_3 = p'_3 - p'_1 - p'_2.
\]

Because of the centre of mass integrals, it is not restrictive to assume \( q_1 + q_2 + q_3 = 0 \), \( p_1 + p_2 + p_3 = 0 \), which amounts to \( q'_3 = -(q'_1 + q'_2)/3 \) and \( p'_3 = 0 \). Hence, the new expressions of \( H \) and \( M \), skipping the ' for simplicity, are

\[
H = |p_1|^2 + |p_2|^2 + (p_1, p_2) - \frac{1}{|q_2|} - \frac{1}{|q_3 - q_1|} - \frac{1}{|q_1 - q_2|}, \quad M = q_1 \wedge p_1 + q_2 \wedge p_2,
\]

which reduce the system to four degrees of freedom.

Let \( (q_1, q_2) \) be the components of \( q_1, (q_3, q_4) \) those of \( q_2 \), and define the components of the \( p \) variables in a similar way. We introduce the generating function

\[
W = p_1 q'_1 \cos q'_4 + p_2 q'_1 \sin q'_4 + p_3 (q'_2 \cos q'_4 - q'_1 \sin q'_4) + p_4 (q'_3 \sin q'_4 + q'_5 \cos q'_4),
\]

which gives rise to the transformation

\[
q_1 = q'_1 \cos q'_4, \quad q_2 = q'_1 \sin q'_4, \quad q_3 = q'_2 \cos q'_4 - q'_1 \sin q'_4, \quad q_4 = q'_2 \sin q'_4 + q'_5 \cos q'_4, \quad q_5 = q'_3 \sin q'_4 + q'_5 \cos q'_4,
\]

\[
p_1 = p'_1 \cos q'_4 - (p'_1 - q'_2 p'_3 + q'_1 p'_5) \sin q'_4 / q'_1, \quad p_2 = p'_1 \sin q'_4 + (p'_1 - q'_2 p'_3 + q'_1 p'_5) \cos q'_4 / q'_1, \quad p_3 = p'_2 \cos q'_4 - p'_3 \sin q'_4,
\]

\[
p_4 = p'_2 \sin q'_4 + p'_3 \cos q'_4.
\]

Skipping again the ' one can write the Hamiltonian and angular momentum in the new variables. It turns out that the new \( q_4 \) does not appear in \( H \). It is a cyclic variable. Hence, the conjugated variable \( p_4 \) is constant. But it is obvious that \( M = p_4 \). We still keep its value in the Hamiltonian, despite that in the case of the figure eight \( M = 0 \), because it plays a role in the ‘rotating eights’ solutions, to be studied in the next section. The final reduced Hamiltonian has the form
The new variables have a simple geometrical meaning. Let us denote the positions of the masses as $m_j$, $j = 1, 2, 3$ and as $m_i m_j$ the vector from $m_i$ to $m_j$. Then $q_1$ is the norm of $m_31$ and $p_1$ is the component of the linear momentum of $m_1$ projected along $m_31$; $q_2$ and $q_3$ are the projections of $m_32$ along $m_31$ and orthogonal to it and, in a similar way, $p_2$ and $p_3$ are the projections of the linear momentum of $m_2$; finally $q_4$ is the angle between the $x$-axis and $m_3m_1$. As said, $p_4 = M$.

### 4.2. Rotating eights

If the angular momentum $M$ goes away from zero, the periodic orbit can be continued. It becomes quasi-periodic with two basic frequencies, which will produce a periodic orbit if the ratio of frequencies is rational. But it can be seen again as a periodic solution, even as a choreography, using a rotating frame, a fact noticed by Hénon [10] who also found that the continuation leads to collision orbits. Due to the symmetries of the problem it is enough to consider $M$ increasing to positive values. For most of these ‘rotating eights’ we can apply the same algorithm as for the (non-rotating) eight. Figure 7 shows the values of $\alpha_1 < \alpha_2$ such that the eigenvalues of the Poincaré map associated to the rotating eight are $\exp(\pm 2\pi i \alpha_1)$, $\exp(\pm 2\pi i \alpha_2)$, as a function of $M$. The monotonically decreasing line gives the minimal distance between the bodies along the rotating eight.

$$H = p_1^2 + p_2^2 + p_3^2 + p_1(p_2 - q_1 p_3 - q_2 p_2 - p_4) + \frac{1}{q_1} (q_2 p_3 - q_3 p_2 - p_4)^2$$

$$- \frac{1}{q_1} \frac{1}{(q_2^2 + q_3^2)^{1/2}} - \frac{1}{((q_1 - q_2)^2 + q_3^2)^{1/2}}. \quad (4)$$

The plot changes at $M = M_{\text{crit}} \approx 0.4467$, when $\alpha_2 = 1/2$, and then the orbit loses stability with the appearance of a period doubling.
bifurcation. For $M > M_{pd}$ we plot the logarithm of the modulus of the dominant eigenvalue (divided by 2 to fit in the plot). Low order resonances $k_1 \alpha_1 + k_2 \alpha_2 \in \mathbb{N}$ with $|k_1| + |k_2| \leqslant 4$ are easy to detect for different values of $M$. For them additional terms will appear in the normal form, making our arguments not valid in these cases. Therefore, as an example, we have selected two values of the angular momentum, $M_1 = 0.0048828125$ and $M_2 = 0.1484375$, that are far enough from resonances (at least from low order ones). A non-rigorous exploration of orbits close to the rotating eights for $|M| < M_{pd}$ gives evidence that for most of them the numerical simulation suggests the existence of tori, but for some resonances they seem to be destroyed.

Initially the rotating eights, in coordinates which rotate with suitable angular velocity, look similar to the orbit shown in figure 6, but the left- and right-hand side lobes are no longer symmetric. Later on the orbits in the rotating frame can develop extra loops. As an example figure 8 shows the solution obtained for $M \approx 0.4493$, shortly after the orbits become unstable. The value of $M$ has been selected so that in the fixed frame (middle panel) the orbit is also periodic. On the left panel the orbit is shown in a rotating frame. At some moment, as displayed, one of the bodies is on the rightmost point on the path, on the small loop on the right, and the other two are symmetrical w.r.t. the horizontal axis, on the large loop on the left. When the bodies move they pass very close to collision on the tiny central lobe. The right panel shows a 3D projection, on the variables $q_1, q_2, p_1$, of what is seen in the Poincaré section. The large dot represents the periodic orbit. An initial point, taken by adding $10^{-9}$ to $q_1$, is iterated under the Poincaré map. The points are scattered close to the ‘separatrix’ associated to the unstable/stable directions. Of course, these 1D manifolds do not coincide, but the splitting must be expected to be exponentially small in $M - M_{pd}$ (see again [9]).

5. Results

In this section we present the results of the application of the algorithm to the examples introduced in sections 3 and 4. The computed values are often very thin intervals with lower and upper bound having many common leading digits. To increase readability when printing intervals we put first those common digits and then the remaining digits of lower and upper bound as subscript and superscript respectively, e.g.

$$123.456789_{1234}^{5678} = [123.4567891234, 123.4567895678],$$

$$-123.456789_{1234}^{5678} = [-123.4567895678, -123.4567891234],$$

$$0.0000001234_{0001}^{0001} = [0.0000001001, 0.0000000001].$$

Figure 8. A rotating eight for $M \approx 0.4493$, already on the linear instability domain. See the text for a detailed explanation.
To be rigorous the presented intervals are rounded outwards, e.g. the interval $10^{-10}$ when rounded to three decimal places reads 0.001. This sometimes can suggest that consecutive interval operations are not correct, but in fact they are performed with much higher precision than that displayed, e.g. the result of $1000 \cdot 0.001$ can still be equal to 0.001.

5.1. The quartic potential example

Starting at approximate initial conditions as given in section 3, we carry out the algorithm described in section 2. The computed values are displayed in the table 1. From the first step of the algorithm we obtain a rigorous estimate of the initial condition: $x$. We use this to get enclosures for the eigenvalues: $\lambda_1$ and $\lambda_2$. The Birkhoff normal form of the third and the sixteenth iterate of the Poincaré map, respectively, computed at the fixed point $x \in x$ in both cases is proved to be

$$
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}
\mapsto
\begin{pmatrix}
\lambda_1 z_1 + c_1 z_1^2 z_2 \\
\lambda_2 z_2 + c_2 z_1 z_2^2
\end{pmatrix}
$$

for some $\lambda_j \in \lambda_j$ and $c_j \in c_j$ for $j = 1, 2$. It is enough to work only with the first equation, because $z_1 = i z_2$. Hence the map (up to order 3) reads

$$
z_1 \mapsto z_1 \exp(2\pi i(\alpha + d|z_1|^2)),
$$

where $\lambda_1 = \exp(2\pi i \alpha)$ and $d = \frac{c_1}{\sum_{N}}$. Finally, for the twist map (6) if $d$ (the torsion, or twist coefficient in that case) is different from zero then the fixed point is KAM stable (see [3, 27]). The computed rigorous bounds $d$, shown in table 1, verify that $d$ is not zero for both orbits. Therefore we have

**Theorem 2.** The conditions for the applicability of KAM theorem are satisfied around the periodic orbits of period 3 and 16 given, approximately, in section 3. Hence, there exist two-dimensional tori around these orbits in the phase space.

The proof of the existence of the period 16 orbit using double precision did not succeed, and we were forced to use multiprecision interval arithmetic and the Taylor method of higher order. This significantly increases the computational time.
5.2. Stability of the figure eight orbit

We start with the coordinate system introduced in section 4 in which the Hamiltonian of the planar 3-body problem has the form (4). The variables are \( q_1, q_2, q_3, p_1, p_2, p_3 \). For the figure eight orbit and nearby orbits we select as Poincaré section the passage through \( q_3 = 0 \), that is, when the three bodies are aligned. We recall that this collinear passage happens six times in a full revolution. Then, given \( q_1, q_2, p_1 \) and \( p_2 \) and the value of the energy \( H = h \) we can determine \( p_3 > 0 \). The Poincaré map defines a symplectic 4D map on the fixed level of energy and \( M = 0 \). Local variables which are canonically conjugate are \( q_1, q_2, p_1, p_2 \).

Rough initial conditions in these variables if we start when the bodies are aligned, on the \( x \) axis, with \( m_1 \) to the right and \( m_2 \) at the centre, are

\[
q_1 = 1.990 \, 983 \, 769 \, 729 \, 943 \, 2, \quad q_2 = 0.995 \, 491 \, 884 \, 864 \, 971 \, 6, \quad q_3 = 0, \quad p_1 = -0.347 \, 901 \, 964 \, 979 \, 515 \, 3, \quad p_2 = 0.695 \, 803 \, 929 \, 959 \, 030 \, 7, \quad p_3 = 1.067 \, 859 \, 626 \, 758 \, 401 \, 8.
\]

Of course, \( p_3 \) can be recovered from the level of energy, which was fixed to

\[
h = -1.292 \, 970 \, 857 \, 122.
\]

The corresponding period is close to \( 2\pi \). Of course, by the homogeneity of the potential, any period or any negative value of \( h \) are equivalent.

Now everything is prepared to carry out the algorithm described in section 2. The computations are done using multiprecision intervals with 100 bits of mantissa. For ODE integration we used the Taylor method of order 40. As a result we obtained very sharp rigorous enclosures for the initial conditions of the eight

\[
x = (1.990 \, 983 \, 769 \, 729 \, 943 ^2, \quad 0.995 \, 491 \, 884 \, 864 \, 971 \, 6, \quad -0.347 \, 901 \, 964 \, 979 \, 515 ^2, \quad 0.695 \, 803 \, 929 \, 959 \, 030 ^2, \quad 1.067 \, 859 \, 626 \, 758 \, 401 \, 8)
\]

Next we computed the Taylor expansion of the Poincaré map at the fixed point \( \hat{x} \in x \) and we proved that all eigenvalues of the linear part lie on the unit circle. Their rigorous estimates are:

\[
\lambda_1, \lambda_3 = 0.998 \, 599 \, 982 \, 091 ^{\pm} \pm i \, 0.052 \, 896 \, 840 \, 795 ^{\pm},
\]

\[
\lambda_2, \lambda_4 = -0.297 \, 596 \, 667 \, 515 ^{\pm} \pm i \, 0.954 \, 691 \, 690 \, 274 ^{\pm}.
\]

The final normal form is

\[
\begin{pmatrix}
\lambda_1 z_1 + c_{11} z_1 z_3 + c_{12} z_2 z_4 \\
\lambda_2 z_2 + c_{21} z_1 z_3 + c_{22} z_2 z_4 \\
\lambda_3 z_3 + c_{31} z_1 z_3 + c_{32} z_2 z_4 \\
\lambda_4 z_4 + c_{41} z_1 z_3 + c_{42} z_2 z_4
\end{pmatrix},
\]

where

\[
c_{11} \in -25.05606 + i \, 35.585414, \quad c_{12} \in 22.035 - i \, 11.178455,
\]

\[
c_{21} \in -6.571 + i \, 21.06847, \quad c_{22} \in -0.186 + i \, 0.488383.
\]
In the above expression we let aside, as in all the computations, terms of order 4 or higher. To obtain (9) we take first coefficients and use the fact that the second and fourth variables are related to the unavoidable ones, and using the second and fourth variables are related to the complex conjugates of the first and third ones, we only need to work with the two complex variables \( z_j, z_k \).

At this point the map reads as

\[
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 z_1 + c_{11} |z_1|^2 z_1 + c_{12} |z_2|^2 z_1 \\ \lambda_2 z_2 + c_{21} |z_1|^2 z_2 + c_{22} |z_2|^2 z_2 \end{pmatrix}.
\]  

(8)

Let us write \( \lambda_j = \exp(i2\pi \alpha_j) \) for \( j = 1, 2 \). Then the map can be written in the form

\[
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 \exp(i2\pi(c_{11} |z_1|^2 + d_{12} |z_2|^2)) \\ z_2 \exp(i2\pi(c_{21} |z_1|^2 + d_{22} |z_2|^2)) \end{pmatrix}.
\]  

(9)

where \( d_{jk} = c_{jk}/2\pi\lambda_j \) for \( j = 1, 2 \) and \( k = 1, 2 \). In the above expression we let aside, as in all the computations, terms of order 4 or higher. To obtain (9) we take first coefficients as factors in (8) and compute logarithms. Note that the values of the coefficients \( d_{jk}, j = 1, 2, k = 1, 2 \) must be real, although from rigorous computations we get complex intervals around some real point. It should also hold that the matrix formed by the \( d_{jk} \) coefficients is symmetric, a fact which is compatible with the results obtained in the computations.

For the map (9) the non-degeneracy KAM condition is simply that the determinant of the torsion \( d = d_1 d_{22} - d_2 d_{12} \) must be different from zero. Because for the eight we have $
\begin{align*}
\lambda_1, & λ_3 & = 0.9985126691 ± i054.5284 \quad & λ_2, & λ_4 & = -0.2987454161 ± i0.95433284, \quad & D = (d_{jk}) & = \begin{pmatrix} -36.401 & 3.507 & 0.085 & i10.012 \ 3.507 & 1.001 & 0.011 & i3.280 \ 0.085 & 0.001 & 1.550 & i0.006 \ i10.012 & i3.280 & i0.001 & 5.670 \ \end{pmatrix},
\end{align*}
$ and

\[
d = \det(D) = -19.515^{+6.7} \quad + i^{0.18} \quad -11.61^{+8.3} + i^{0.001}.
\]

Due to the form of the eigenvalues, and to the fact that no resonances appear to order 3, except the unavoidable ones, and using the fact that the second and fourth variables are related to the complex conjugates of the first and third ones, we only need to work with the two complex variables \( z_1, z_2 \).

For the eight orbit we have

\[
\begin{align*}
d_{11} & = -37.624^{+1.015} + i^{0.002}, & d_{12} & = 3.542^{+1.001},
\end{align*}
\]

\[
\begin{align*}
d_{21} & = 3.452^{+1.002}, & d_{22} & = 3.908^{+1.002}.
\end{align*}
\]

For the map (9) the non-degeneracy KAM condition is simply that the determinant of the torsion \( d = d_1 d_{22} - d_2 d_{12} \) must be different from zero. Because for the eight we have

| \( M \) | \( x \) |
|---|---|
| 0.0048828125 | 0.148437500 |
| \( 1.990931659159403^{+0.5} \) | \( 1.95151129135786^{+0.5} \) |
| \( 1.008912014154726^{+0.5} \) | \( 1.375174866932222^{+0.5} \) |
| \( -0.357203564808152^{+0.5} \) | \( -0.70657699542591^{+0.5} \) |
| \( 0.695944468418675^{+0.5} \) | \( 0.835536267808208^{+0.5} \) |

\( \text{diam}(x) = 9.25 \cdot 10^{-22} \) and \( \text{diam}(x) = 6.40 \cdot 10^{-22} \).
we have proved the following result.

**Theorem 3.** The conditions for the applicability of KAM theorem are satisfied around the figure eight periodic orbit of the three-body problem with equal masses and zero angular momentum, as described in section 4. Hence, there exist three-dimensional invariant tori around them on the phase space with fixed values of the first integrals.

5.3. Stability of rotating eights

Exactly the same algorithm as for the figure eight orbit can be used for the rotating Eights introduced in section 4.2. The obtained Birkhoff normal form and torsion condition are the same. Therefore, keeping the same notation as in the previous section, we only list in Table 2 the rigorous estimates that verify KAM stability of rotating eights for the two selected values of angular momentum $M_1 = 0.004 882 8125$ and $M_2 = 0.148 4375$.

6. Implementation

The implementation of the presented algorithm is based on CAPD library [5] with multi-precision provided by MPFR library. It is written in C++. The running times of the proofs on ordinary laptop (Intel i7 processor 2.1 GHz, Ubuntu 13.10) are around 1 min for quartic potential and 110 min for figure eight orbit and rotating eights. The source code, installation instructions and details on performed computations (system configurations, exact outputs) are available on the web page [11].

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