UNIQUENESS OF $\mathbb{CP}^n$

VALENTINO TOSATTI

Abstract. We give an exposition of a theorem of Hirzebruch, Kodaira and Yau which proves the uniqueness of the Kähler structure of complex projective space, and of Yau’s resolution of the Severi Conjecture.

1. Introduction

It is a classical result in complex analysis that every simply connected closed Riemann surface is biholomorphic to the projective line $\mathbb{CP}^1$. The purpose of this note is to explain in detail two higher-dimensional generalizations of this fact.

Theorem 1.1 (Hirzebruch, Kodaira [4], Yau [14]). If a Kähler manifold $M$ is homeomorphic to $\mathbb{CP}^n$ then $M$ is biholomorphic to it.

More precisely, Hirzebruch and Kodaira proved this for all $n$ odd, leaving open the case of $n$ even which was finally solved by Yau. Also, Hirzebruch and Kodaira assumed that $M$ is diffeomorphic to $\mathbb{CP}^n$, and this was relaxed to homeomorphic after work of Novikov. When $n = 2$, a stronger result holds, which was known as the Severi Conjecture [13], and was solved by Yau:

Theorem 1.2 (Yau [14]). If a compact complex surface $M$ is homotopy equivalent to $\mathbb{CP}^2$ then it is biholomorphic to it.

A brief outline of the proofs of these theorems is the following. From the assumptions, using the Hirzebruch-Riemann-Roch theorem, one deduces that either $M$ is Fano (i.e. $c_1(M)$ can be represented by a Kähler metric) or else the canonical bundle $K_M$ is positive (i.e. $-c_1(M)$ can be represented by a Kähler metric). The second case can only arise when $n$ is even. When $M$ is Fano a geometric argument shows that $M$ is biholomorphic to $\mathbb{CP}^n$, which settles the case when $n$ is odd. On the other hand, when $K_M$ is positive then a key inequality between Chern numbers holds, as shown by Yau. Furthermore, in our case we have that equality holds, and this implies that $M$ is biholomorphic to the unit ball in $\mathbb{C}^n$, which is absurd because $M$ is compact.

The details are presented in Section 2, mostly following the original sources (together with a small simplification of part of the argument from

Supported in part by a Sloan Research Fellowship and NSF grant DMS-1308988. I am grateful to Yuguang Zhang and to the referee for helpful comments.
[7], and in Section 3 we discuss a natural conjectural extension of these theorems, and how it is related to another well-known open problem.

2. Proofs of the main results

Proof of Theorem 1.1. The fact that $M$ is Kähler gives us the Hodge decomposition on cohomology, which we will use repeatedly. From the hypothesis we see that

$$H^2(M, \mathbb{Z}) \cong \mathbb{Z}, \quad H^1(M, \mathbb{C}) \cong 0 \cong H^{0,1}(M),$$

$$H^2(M, \mathbb{C}) \cong \mathbb{C} \cong H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M),$$

and since $H^{2,0}(M) \cong H^{0,2}(M)$, we see that they are both zero, while $H^{1,1}(M) \cong \mathbb{C}$. Thanks to the vanishing of $H^{0,1}(M)$ and $H^{0,2}(M)$, the exponential exact sequence gives that the first Chern class map

$$c_1 : \text{Pic}(M) \to H^2(M, \mathbb{Z}) \cong \mathbb{Z},$$

is an isomorphism, where as usual the Picard group $\text{Pic}(M)$ is the group of isomorphism classes of holomorphic line bundles on $M$.

Lemma 2.1. $M$ is projective and its holomorphic Euler characteristic satisfies

$$\chi(M, \mathcal{O}) := \sum_{p=0}^{n} (-1)^p \dim H^{0,p}(M) = 1.$$

Proof. Choose a Kähler form $\tilde{\omega}$ on $M$. Its cohomology class $[\tilde{\omega}]$ lies in $H^2(M, \mathbb{R}) \cong \mathbb{R}$ so we can rescale $\tilde{\omega}$ to get another Kähler form $\omega$ whose cohomology class generates $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$. We have that $\int_M \omega^n > 0$ because this equals $n!$ times the total volume of $M$ measured using the Kähler metric $\omega$. On $\mathbb{C}P^n$ a generator $\alpha$ of $H^2(M, \mathbb{Z})$ satisfies $\langle \alpha^{-n}, [\mathbb{C}P^n] \rangle = \pm 1$, and since $\omega$ is Kähler we have that $\int_M \omega^n = 1$. Since $c_1$ is an isomorphism, there exists $L \to M$ a holomorphic line bundle whose first Chern class is $[\omega]$. If $h$ is a smooth Hermitian metric on the fibers of $L$ then its curvature form $\gamma$ is a closed real $(1, 1)$ form cohomologous to $c_1(L) = [\omega]$. By the $\partial\bar{\partial}$-Lemma, which holds because $M$ is Kähler, there is a smooth real-valued function $\psi$ on $M$ such that $\omega = \gamma + \sqrt{-1} \partial\bar{\partial}\psi$. The Hermitian metric $\tilde{h} = e^{-\psi} h$ on $L$ then has curvature form equal to $\omega$, and so $L$ is a positive line bundle. Thanks to the Kodaira Embedding Theorem [5] Proposition 5.3.1, $L$ is ample and the manifold $M$ is projective.

Since $\int_M \omega^n \neq 0$, it follows that the classes $[\omega^k] \in H^{k,k}(M)$ are nonzero for $1 \leq k \leq n$, and as above the Hodge decomposition implies that $H^{p,q}(M) = 0$ if $p \neq q$. This gives that the holomorphic Euler characteristic of $M$ satisfies $\chi(M, \mathcal{O}) = 1$. \hfill \Box

Recall the following definition: if $F \to M$ is a real vector bundle, then its Pontrjagin classes are defined to be $p_k(F) = (-1)^i c_{2i}(F \otimes \mathbb{C}) \in H^{4i}(M, \mathbb{Z})$, where $c_{2i}$ denotes the $(2i)^{th}$ Chern class of the complex vector bundle $F \otimes \mathbb{C}$. 

\[\]
If $F = TM$ we just write $p_i(M)$. Now we need the following theorem, which we will quote without proof.

**Theorem 2.2** (Novikov [12]). The rational Pontrjagin classes of a closed smooth manifold are invariant under homeomorphism.

Here the rational Pontrjagin classes are just the images of $p_i(M)$ under the natural map $H^{4i}(M, \mathbb{Z}) \to H^{4i}(M, \mathbb{Q})$. Since our manifold $M$ has torsion-free integral cohomology, we obtain in our case the invariance of the integral Pontrjagin classes. In particular if $f : M \to \mathbb{C}P^n$ is the given homeomorphism, then $f^*p_i(\mathbb{C}P^n) = p_i(M)$ for all $i$. Notice that if $f$ is assumed to be a diffeomorphism then this is obvious since $f^*(T\mathbb{C}P^n) \cong TM$ is an isomorphism of real vector bundles, which induces an isomorphism of complex vector bundles $f^*(T\mathbb{C}P^n \otimes \mathbb{C}) \cong TM \otimes \mathbb{C}$ which therefore preserves the Chern classes, so we do not need Novikov’s theorem in that case. On the other hand, it is in general false that $f^*c_i(\mathbb{C}P^n) \cong c_i(M)$ when $f$ is a diffeomorphism, which is why we are forced to work with Pontrjagin classes instead of Chern classes.

**Lemma 2.3.** The holomorphic Euler characteristic of $M$ satisfies

\begin{equation}
\chi(M, \mathcal{O}) = \int_M e^{\frac{c_1(M)}{2}} \left( \frac{\omega/2}{\sinh(\omega/2)} \right)^{n+1}.
\end{equation}

*Proof.* If $H$ denotes the hyperplane class on $\mathbb{C}P^n$, then it is well-known (see e.g. [11, Example 15.6]) that

$$p_i(\mathbb{C}P^n) = \binom{n+1}{i} H^{2i},$$

for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Moreover the fact that $f$ is a homeomorphism implies that $f^*H$ is a generator of $H^2(M, \mathbb{Z})$ and so $f^*H = \pm [\omega]$. Putting these together we get

\begin{equation}
p_i(M) = \binom{n+1}{i} [\omega^{2i}],
\end{equation}

for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. The Hirzebruch-Riemann-Roch Theorem [3, Theorem 20.3.2] says that for any holomorphic line bundle $F$ on $M$ we have

$$\chi(M, F) := \sum_{p \geq 0} (-1)^p \dim H^p(M, F) = \int_M e^{c_1(F)} \text{Td}(M),$$

where $\text{Td}(M)$ is the Todd genus of $M$. This is defined in terms of the Chern classes of $M$, but since in our case we only know the Pontrjagin classes of $M$, we need to express $\text{Td}(M)$ as much as possible in terms of these. To do this, we use the identity [3, p.150, (6*)]

$$\text{Td}(M) = e^{\frac{c_1(M)}{2}} \hat{A}(M),$$
where the $\hat{A}$ genus of $M$ is defined as follows (see [3] for details). We formally write
\[
\sum_{j \geq 0} p_j(M)x^j = \prod_{j \geq 1} (1 + \gamma_j x),
\]
for some symbols $\gamma_j$, and let
\[
\hat{A}(M) = \prod_{j \geq 0} \frac{\sqrt{\gamma_j/2}}{\sinh(\sqrt{\gamma_j}/2)},
\]
which is therefore a polynomial in the Pontrjagin classes $p_j(M)$. Taking $F = O$ in the Hirzebruch-Riemann-Roch formula (where $O$ is the trivial line bundle) gives
\[
\chi(M, O) = \int_M e^{\frac{c_1(M)}{2}} \hat{A}(M).
\]
Now thanks to (2.2) we have
\[
\sum_{j \geq 0} p_j(M)x^j = (1 + [\omega^2]x)^{n+1},
\]
which gives $\gamma_1 = \cdots = \gamma_{n+1} = [\omega^2]$ and $\gamma_j = 0$ for $j > n+1$. Thus, we obtain the key identity (2.1).

In order to proceed with the proof, we need to determine $c_1(M)$.

**Lemma 2.4.** We have that $c_1(M)$ equals either $(n + 1)[\omega]$ or $-(n + 1)[\omega]$, with the latter only possibly occurring when $n$ is even.

**Proof.** The reduction mod 2 of $c_1(M)$ is the second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$, which is a topological invariant. Hence it is equal to $w_2(\mathbb{CP}^n)$ which is $c_1(\mathbb{CP}^n)$ mod 2, that is $n + 1$ mod 2. On the other hand since $c_1(M)$ and $[\omega]$ both belong to $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$, we have $c_1(M) = \lambda [\omega]$ for some $\lambda \in \mathbb{Z}$, and so $\lambda = n + 1 + 2s$ for some $s \in \mathbb{Z}$. From Lemma 2.3 we get
\[
\chi(M, O) = \int_M e^{\frac{n+1+2s}{2}\omega} \left( \frac{\omega/2}{\sinh(\omega/2)} \right)^{n+1} = \int_M e^{s\omega} \left( \frac{\omega}{1 - e^{-\omega}} \right)^{n+1},
\]
using the identity
\[
\frac{x}{1 - e^{-x}} = e^{\frac{x}{2}} \frac{x/2}{\sinh(x/2)}.
\]
Since $\int_M \omega^n = 1$, and the integrals over $M$ of all other powers of $\omega$ are zero by definition, this means that $\chi(M, O)$ equals the coefficient of $x^n$ in the power series expansion of
\[
e^{sx} \left( \frac{x}{1 - e^{-x}} \right)^{n+1}.
\]
Following [4] we give two different ways of calculating this coefficient. The first method uses residues, and more precisely the fact that if we define a holomorphic function $F$ by

$$F(z) = e^{sz} \left( \frac{z}{1 - e^{-z}} \right)^{n+1},$$

then Cauchy’s integral formula shows that the coefficient that we are interested in equals the contour integral

$$\frac{1}{2\pi i} \int \frac{F(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \int \frac{e^{sz}}{(1 - e^{-z})^{n+1}} dz,$$

where the contour is a small circle around the origin, with counterclockwise orientation. Since the power series expansion of $1 - e^{-z}$ at $z = 0$ starts with $z$, this function is a local biholomorphism near the origin, so we can change variable $y = 1 - e^{-z}$ near 0 and rewrite our contour integral as

$$\frac{1}{2\pi i} \int \frac{1}{(1 - y)(s+1)y^{n+1}} dz,$$

where the contour is again a small circle around the origin. By the Residue theorem this integral equals the residue of the function $\frac{1}{(1-y)(s+1)y^{n+1}}$ at 0, which is the coefficient of $y^n$ in the Taylor expansion of $(1-y)^{-s-1}$ at 0. Expanding this function, we finally obtain that our desired coefficient equals

$$\binom{n + s}{n} = \frac{(n + s)(n + s - 1) \cdots (s + 1)}{n!},$$

where we allows $s < 0$.

The second way to calculate this coefficient is as follows: by Hirzebruch-Riemann-Roch again, this coefficient equals

$$\int_{\mathbb{C}P^n} e^{sH} \left( \frac{H}{1 - e^{-H}} \right)^{n+1} d\tilde{A} = \int_{\mathbb{C}P^n} e^{sH} e^{\frac{n+1}{2}H} \left( \frac{H/2}{\sinh(H/2)} \right)^{n+1}$$

$$= \int_{\mathbb{C}P^n} e^{c_1(\mathcal{O}(s))} e^{\frac{c_1(\mathcal{O}(s))}{2}} \tilde{A}(\mathbb{C}P^n)$$

$$= \chi(\mathbb{C}P^n, \mathcal{O}(s)),$$

and it is well-known (see e.g. [5, Example 5.2.5]) that $\chi(\mathbb{C}P^n, \mathcal{O}(s))$ equals $\binom{n + s}{n}$. So, using either of the two methods, we conclude that

$$\chi(M, \mathcal{O}) = \binom{n + s}{n}.$$

Since $\chi(M, \mathcal{O}) = 1$ by Lemma 2.1 we get that $\binom{n + s}{n} = 1$, which can be rewritten as

$$n! = (s + n) \cdots (s + 1).$$

If $n$ is odd this implies that $s = 0$, while if $n$ is even, $s$ is either 0 or $-n - 1$. But we saw that $c_1(M) = (n + 1 + 2s)[\omega]$ and so if $n$ is odd we...
get $c_1(M) = (n+1)[\omega]$, while if $n$ is even, $c_1(M)$ is either $(n+1)[\omega]$ or $-(n+1)[\omega]$. □

Assume first that $c_1(M) = (n+1)[\omega]$, which implies that $M$ is a Fano manifold (i.e. there is a Kähler metric in $c_1(M)$). Then $c_1(K_M) = -c_1(M) = -(n+1)c_1(L)$ and so $K_M = -(n+1)L$, since the map $c_1$ is an isomorphism. Then Serre duality gives $H^k(M, L) \cong H^{n-k}(M, K_M - L)$ and $K_M - L = -(n+2)L$ is negative, so $H^k(M, L) = 0$ if $k > 0$ by Kodaira vanishing. Hence using Hirzebruch-Riemann-Roch again we get

$$\dim H^0(M, L) = \chi(M, L) = \int_M e^{c_1(L) + \frac{c_1(L)}{2}} \left( \frac{\omega/2}{\sinh(\omega/2)} \right)^{n+1} = \int_M e^\omega \left( \frac{\omega}{1 - e^{-\omega}} \right)^{n+1} = n + 1,$$

using again the calculation from earlier of the coefficient in the power series expansion. Then the following lemma, whose proof we postpone, gives that $M$ is biholomorphic to $\mathbb{C}P^n$.

Lemma 2.5 (Theorem 1.1 in [7]). If $M$ is a compact Kähler manifold and $L$ is a positive line bundle on $M$ with $\int_M c_1^2(L) = 1$ and $\dim H^0(M, L) = n + 1$ then $M$ is biholomorphic to $\mathbb{C}P^n$.

We can then assume that $n$ is even (so $n \geq 2$) and that $c_1(M) = -(n+1)[\omega]$, which says that $K_M$ is positive. By a theorem due independently to Yau [15] and Aubin [1] we know that $M$ then admits a unique Kähler-Einstein metric with constant Ricci curvature equal to $-1$, that is a Kähler metric $\omega_{KE}$ such that

$$\text{Ric}(\omega_{KE}) = -\omega_{KE}. \tag{2.3}$$

Recall here that the Riemann curvature tensor of a Kähler metric $\omega = \sqrt{-1} g_{ij} dz^i \wedge d\bar{z}^j$ in local holomorphic coordinates has components given by

$$R_{\bar{k}z^k} = -\frac{\partial^2 g_{\bar{k}z^k}}{\partial z^i \partial \bar{z}^j} + g^p_{z^i} \frac{\partial g_{\bar{k}z^k}}{\partial z^j} \frac{\partial g_{\bar{z}^p}}{\partial \bar{z}^j},$$

the Ricci curvature tensor is its trace

$$R_{\bar{j}} = g^{k\bar{k}} R_{\bar{k}z^k} = -\frac{\partial^2 \log \det(g_{\bar{k}\bar{k}})}{\partial z^i \partial \bar{z}^j},$$

and the Ricci form is defined by

$$\text{Ric}(\omega) = \sqrt{-1} R_{\bar{j}} dz^i \wedge d\bar{z}^j,$$

so that the Kähler-Einstein condition (2.3) is equivalent to

$$R_{\bar{j}} = -g_{\bar{j}}.$$

With this in mind, we have the following:
Lemma 2.6. If \((M, \omega)\) is a Kähler-Einstein manifold of complex dimension \(n \geq 2\), so that \(\text{Ric}(\omega) = \lambda \omega\) for some \(\lambda \in \mathbb{R}\), then we have
\[
\left(\frac{2(n+1)}{n} c_2(M) - c_1^2(M)\right) \cdot [\omega]^{n-2} \geq 0,
\]
with equality iff \(\omega\) has constant holomorphic sectional curvature.

Proof. The tensor
\[ R^0_{ijkl} = R^0_{ijkl} - \frac{\lambda}{n+1}(g_{ij}g_{kl} + g_{ik}g_{jl}) \]
vanishes iff \(\omega\) has constant holomorphic sectional curvature (see e.g. [6, Proposition IX.7.6]). Its tensorial norm square is easily computed as
\[
|Rm|_0^2 = |Rm|^2 + \frac{\lambda^2}{(n+1)^2} g^{ij} g^{kl} (g_{ij}g_{kl} + g_{ik}g_{jl})(g_{ij}g_{kl} + g_{ik}g_{jl}) R^0_{ijkl} R^0_{ijkl}
\]
\[
= |Rm|^2 + \frac{\lambda^2}{(n+1)^2} (2n^2 + 2n) - 4\lambda R,
\]
where \(R\) denotes the scalar curvature. The assumption \(R_{ij} = \lambda g_{ij}\) gives \(R = \lambda n\) and \(|\text{Ric}|^2 = \lambda^2 n^2\). Then
\[
|Rm|_0^2 = |Rm|^2 - \frac{2\lambda^2 n}{n+1}.
\]
On the other hand if \(\Omega_i^j = \sqrt{-1} R^j_{ikl} dz^k \wedge d\bar{z}^l\) denote the curvature forms, then Chern-Weil theory says that
\[
\frac{1}{2\pi} \text{Ric}(\omega) = \frac{1}{2\pi} \sum_i \Omega_i^j = \frac{\sqrt{-1}}{2\pi} R_{k\bar{l}} dz^k \wedge d\bar{z}^l,
\]
is a closed form that represents \(c_1(M)\) in \(H^2(M, \mathbb{R})\), while the form
\[
\frac{1}{4\pi^2} \text{tr}(\Omega \wedge \Omega) = \frac{1}{4\pi^2} \sum_{k,i} \Omega^k_i \wedge \Omega^k_i = \frac{(\sqrt{-1})^2}{4\pi^2} \sum_{k,i} R^k_{ij\bar{q}r} R^i_{k\bar{q}r} dz^p \wedge d\bar{z}^q \wedge dz^r \wedge d\bar{z}^s,
\]
represents \(c_2(M) = 2c_2(M)\). Since (2.4) is an integral inequality, we can ignore torsion in integral cohomology, and so we can use Chern-Weil forms to prove (2.4). Given a point \(p \in M\) we choose local holomorphic coordinates so that \(p\) we have \(g_{ij} = \delta_{ij}\), and so also
\[
\omega^n = n!(\sqrt{-1})^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n,
\]
\[
\omega^{n-2} = (n-2)! (\sqrt{-1})^{n-2} \sum_{i<j} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^j \wedge d\bar{z}^j \wedge \cdots \wedge dz^n \wedge d\bar{z}^n,
\]
and it follows that at \( p \) we have
\[
n(n-1)\text{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2} = \sum_{k,i} \sum_{p \neq r} (R^k_{ijpq} R^i_{kpq} - R^k_{ijpq} R^i_{kpq}) \omega^n
\]
\[
= \sum_{k,i,p,r} (R^k_{ijpq} R^i_{kpq} - R^k_{ijpq} R^i_{kpq}) \omega^n
\]
\[
= (|\text{Ric}|^2 - |\text{Rm}|^2) \omega^n = (\lambda^2 n - |\text{Rm}|^2) \omega^n.
\]
Hence this holds at all points, and so
\[
\frac{|\text{Rm}|^2}{n(n-1)} \omega^n = -\text{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2} + \lambda^2 \left( \frac{1}{n-1} - \frac{2}{(n+1)(n-1)} \right)
\]
\[
= -\text{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2} + \lambda^2 \frac{n}{n+1}.
\]
Now notice that
\[
\frac{1}{4\pi^2} \int_M \lambda^2 \omega^n = \frac{1}{4\pi^2} \int_M (\lambda \omega)^2 \wedge \omega^{n-2} = c_1^2(M) \cdot [\omega]^{n-2},
\]
and so
\[
\frac{1}{n(n-1)4\pi^2} \int_M |\text{Rm}|^2 \omega^n = \left( 2c_2(M) - \left( 1 - \frac{1}{n+1} \right) c_1^2(M) \right) \cdot [\omega]^{n-2},
\]
which implies what we want.

We claim that equality in (2.4) does in fact hold in our case. This will finish the proof of Theorem 1.1 since then \( M \) would have constant negative holomorphic sectional curvature, and since it is also simply connected it would be biholomorphic to the unit ball in \( \mathbb{C}^n \) (see e.g. [6, Theorem IX.7.9]), which is impossible.

We already know that \( c_1^2(M) = (n+1)^2[\omega^2] \). To compute \( c_2(M) \) we notice that by definition \( p_1(M) = p_1(TM) = -c_2(TM \otimes \mathbb{C}) \). But \( TM \otimes \mathbb{C} \cong TM \oplus TM \) and the Chern classes satisfy \( c_k(TM) = (-1)^k c_k(TM) \), so
\[
(2.5)
\]
\[
p_1(M) = -c_2(TM \oplus TM) = -c_2(TM) - c_2(TM) - c_1(TM) \cdot c_1(TM)
\]
\[
= -2c_2(M) + c_1^2(M).
\]
Putting this together with (2.2) we get
\[
2c_2(M) = (n+1)^2[\omega^2] - (n+1)\omega^2 = n(n+1)[\omega^2],
\]
and thus equality holds in (2.4). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let us denote by \( \tau(M) \) the signature of \( M \), which is the difference between the number of positive and negative eigenvalues for the intersection form
\[
H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \to \mathbb{Z}.
\]
The signature is a topological invariant (up to sign), and so
\[ \tau(M) = \pm \tau(\mathbb{CP}^2) = \pm 1. \]

Hirzebruch’s Signature Theorem [5, p.235] gives
\[ \tau(M) = \frac{1}{3} \int_M p_1(M). \]

But from (2.5) we get
\[ \frac{1}{3} \int_M (c_1^2(M) - 2c_2(M)) = \pm 1, \]
and Chern-Gauss-Bonnet’s Theorem [5, p.235] gives
\[ \int_M c_2(M) = \chi(M) = \chi(\mathbb{CP}^2) = 3, \]
and so
\[ \int_M c_1^2(M) = 3(2 \pm 1) > 0. \]

A theorem of Kodaira [8] then says that \( M \) is projective. As before we see that \( \chi(M, \mathcal{O}) = 1 \) and then Riemann-Roch (see [5, p.233]) gives
\[ \chi(M, \mathcal{O}) = \frac{K_M^2 + \chi(M)}{12} = \frac{K_M^2 + 3}{12}, \]
which gives \( \int_M c_1^2(M) = K_M^2 = 9 \) (so in fact \( \tau(M) = 1 \)). Let \( \omega \) be as before, then \( c_1(M) = \lambda[\omega] \) for some \( \lambda \in \mathbb{Z} \). Then we have that \( \lambda = \pm 3, \) and these are exactly the same cases as in Theorem 1.1. If \( \lambda = 3, \) we need to check that \( \dim H^0(M, L) = 3. \) But we have \( K_M = -3L \) and \( K_M \cdot L = -3 \) so Riemann Roch [5, p.233] gives
\[ \chi(M, L) = \frac{L^2 - K_M \cdot L}{2} = 3. \]

Serre duality and Kodaira vanishing give
\[ H^1(M, L) \cong H^1(M, K_M - L) = 0, \]
because \( K_M - L = -4L \) is negative, and also
\[ H^2(M, L) \cong H^0(M, K_M - L) = 0. \]

So \( \chi(M, L) = \dim H^0(M, L) = 3. \) Then the proof continues as in Theorem 1.1. \( \square \)

**Proof of Lemma 2.5.** Let \( (\varphi_1, \ldots, \varphi_{n+1}) \) be a basis of \( H^0(M, L) \) and let \( D_j = \{ \varphi_j = 0 \} \) be the corresponding divisors (they are nonempty, because otherwise \( L \) would be trivial, and so it would have \( \dim H^0(M, L) = 1 \)). Define \( V_n = M \) and
\[ V_{n-k} = D_1 \cap \cdots \cap D_k \]
for \( 1 \leq k \leq n. \)

**Lemma 2.7.** For each \( 0 \leq r \leq n \) we have that
(1) $V_{n-r}$ is irreducible, of dimension $n - r$ and Poincaré dual to $c_1^r(L)$

(2) The sequence

$$0 \to \text{Span}(\varphi_1, \ldots, \varphi_r) \to H^0(M, L) \to H^0(V_{n-r}, L)$$

is exact, where the last map is given by restriction.

**Proof.** The proof is by induction on $r$, the case $r = 0$ being obvious. Assuming that (1) and (2) hold for $r - 1$, we see that $V_{n-r+1}$ is irreducible and that $\varphi_r$ is not identically zero on it. Hence $V_{n-r} = \{ x \in V_{n-r+1} \mid \varphi_r(x) = 0 \}$ is an effective divisor on $V_{n-r+1}$ and so it can be expressed as a sum of irreducible subvarieties of dimension $n - r$. Since $c_1^{r-1}(L)$ is dual to $V_{n-r+1}$ and $c_1(L)$ is dual to $D_r$ we see that $c_1^r(L)$ is dual to $V_{n-r}$. If $V_{n-r}$ were reducible, then $V_{n-r} = V' + V''$ and so

$$1 = \int_M c_1^r(L) = \int_M c_1^r(L) \cdot c_1^{n-r}(L) = \int_{V_{n-r}} c_1^{n-r}(L)$$

$$= \int_{V'} c_1^{n-r}(L) + \int_{V''} c_1^{n-r}(L).$$

But since $L$ is positive, the last two terms are both positive integers, and this is a contradiction. Thus (1) is proved. As for (2), the restriction exact sequence

$$0 \to \mathcal{O}_{V_{n-r+1}} \to \mathcal{O}_{V_{n-r+1}}(L) \to \mathcal{O}_{V_{n-r}}(L) \to 0,$$

gives

$$0 \to H^0(V_{n-r+1}, \mathcal{O}) \to H^0(V_{n-r+1}, L) \to H^0(V_{n-r}, L),$$

where the first map is given by multiplication by $\varphi_r$. This means that the kernel of the restriction map $H^0(V_{n-r+1}, L) \to H^0(V_{n-r}, L)$ is spanned by $\varphi_r$. This together with the statement in (2) for $r - 1$ proves (2) for $r$. \qed

Now we apply Lemma 2.7 with $r = n$ and see that $V_0$ is a single point and that $\varphi_{n+1}$ does not vanish there. So given any point of $M$ there is a section of $L$ that does not vanish there (i.e. $L$ is base-point-free). Then we can define a holomorphic map $f : M \to \mathbb{CP}^n$ by sending $x$ to $\{ \varphi \in H^0(M, L) \mid \varphi(x) = 0 \}$. This is a hyperplane in $H^0(M, L) \cong \mathbb{C}^{n+1}$ and so gives a point in $\mathbb{CP}^n$. If $y \in \mathbb{CP}^n$ corresponds to a hyperplane, which is spanned by some sections $(\varphi_1, \ldots, \varphi_n)$, then $f(x) = y$ iff $\varphi_1(x) = \cdots = \varphi_n(x) = 0$. Again Lemma 2.7 with $r = n$ says that $x = V_0$ exists and is unique, and so $f$ is a bijection. \qed

### 3. Closing remarks

As a partial generalization of Theorems 1.1 and 1.2, Libgober-Wood [10] proved that a compact Kähler manifold of complex dimension $n \leq 6$ which is homotopy equivalent to $\mathbb{CP}^n$ must be biholomorphic to it.

A natural question is whether the Kähler hypothesis is really necessary in Theorem 1.1, and so one can ask whether a compact complex manifold diffeomorphic to $\mathbb{CP}^n$ must be biholomorphic to it. This is a well-known open problem (see e.g. [10]), and it is known that if this is true when $n = 3$
then there is no complex manifold diffeomorphic to $S^6$ (another famous open problem, see e.g. [9]):

**Proposition 3.1.** If there exists a compact complex manifold $M$ diffeomorphic to $S^6$, then there exists a compact complex manifold $\tilde{M}$ diffeomorphic to $\mathbb{CP}^3$ but not biholomorphic to it.

This well-known fact was remarked already in [2, p.223].

**Proof.** Let $M$ be a compact complex manifold diffeomorphic to $S^6$, and let $\tilde{M}$ be its blowup at one point $p \in M$. This is a compact complex manifold which is diffeomorphic to the connected sum $S^6 \sharp \mathbb{CP}^3$, see e.g. [5, Proposition 2.5.8]. This is of course diffeomorphic to $\mathbb{CP}^3$, and so also to $\mathbb{CP}^3$ (in fact, it is even oriented-diffeomorphic to $\mathbb{CP}^3$, since this manifold has the explicit orientation-reversing diffeomorphism $[z_0 : \cdots : z_3] \mapsto [\overline{z_0} : \cdots : \overline{z_3}]$). So $\tilde{M}$ is diffeomorphic to $\mathbb{CP}^3$, and if $\tilde{M}$ was biholomorphic to $\mathbb{CP}^3$ we would have

$$\int_{\tilde{M}} c_1(\tilde{M})^3 = \int_{\mathbb{CP}^3} c_1(\mathbb{CP}^3)^3 = 64.$$  

But if we let $\pi : \tilde{M} \to M$ be the blowup map and $E = \pi^{-1}(p)$ is its exceptional divisor (which is biholomorphic to $\mathbb{CP}^2$), then we have (see [5, Proposition 2.5.5])

$$c_1(\tilde{M}) = \pi^* c_1(M) - 2[E],$$

where $[E]$ denotes the Poincaré dual of $E$. Since $b_2(M) = 0$ we have $c_1(M) = 0$, and so

$$\int_{\tilde{M}} c_1(\tilde{M})^3 = -8 \int_{\tilde{M}} [E]^3 = -8 \int_{E} [E]^2 = -8 \int_{\mathbb{CP}^2} c_1(O(-1))^2 = -8,$$

since $[E]|_E$ equals the first Chern class of the tautological bundle $O(-1)$ over $\mathbb{CP}^2$ (see [5, Corollary 2.5.6]). Therefore $\tilde{M}$ is not biholomorphic to $\mathbb{CP}^3$, as claimed. 

**References**

[1] Aubin, T. *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, C. R. Acad. Sci. Paris Sér. A-B 283 (1976), no. 3, Aiii, A119–A121.

[2] Hirzebruch, F. *Some problems on differentiable and complex manifolds*, Ann. of Math. (2) 60 (1954), 213–236.

[3] Hirzebruch, F. *Topological methods in algebraic geometry*, Springer-Verlag, Berlin, 1995.

[4] Hirzebruch, F., Kodaira, K. *On the complex projective spaces*, J. Math. Pures Appl. 36 (1957), 201–216.

[5] Huybrechts, D. *Complex geometry. An introduction*, Springer-Verlag, Berlin, 2005.

[6] Kobayashi, S., Nomizu, K. *Foundations of differential geometry, Vol. II*, John Wiley & Sons, 1969.

[7] Kobayashi, S., Ochiai, T. *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ. 13 (1973), 31–47.

[8] Kodaira, K. *On the structure of compact complex analytic surfaces. I*, Amer. J. Math. 86 (1964), 751–798.
[9] LeBrun, C. Orthogonal complex structures on $S^6$, Proc. Amer. Math. Soc. 101 (1987), no.1, 136–138.
[10] Libgober, A.S., Wood, J. W. Uniqueness of the complex structure on Kähler manifolds of certain homotopy types, J. Differential Geom. 32 (1990), no. 1, 139–154.
[11] Milnor, J.W., Stasheff, J.D. Characteristic classes, Princeton University Press, 1974.
[12] Novikov, S.P. Rational Pontrjagin classes. Homeomorphism and homotopy type of closed manifolds. I, Izv. Akad. Nauk SSSR Ser. Mat. 29 (1965), 1373–1388.
[13] Severi, F. Some remarks on the topological characterization of algebraic surfaces, 1954.
[14] Yau, S.-T. Calabi’s conjecture and some new results in algebraic geometry, Proc. Nat. Acad. Sci. U.S.A. 74 (1977), no. 5, 1798–1799.
[15] Yau, S.-T. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure Appl. Math. 31 (1978), 339–411.

Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208
E-mail address: tosatti@math.northwestern.edu