ON THE GEOMETRIC ZILBER–PINK THEOREM AND THE LAWRENCE–VENKATESH METHOD

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Abstract. Using our recent results on the algebraicity of the Hodge locus for variations of Hodge structures of level at least 3, we improve the results of Lawrence-Venkatesh in direction of the refined Bombieri–Lang conjecture.

1. Results

The aim of this short note is to explain how the Geometric Zilber–Pink conjecture for variations of Hodge structures, recently established by the authors [2], can be used to improve the main result of Lawrence and Venkatesh [7], giving a special case of the refined Bombieri–Lang conjecture.

Let \( U_{n,d} \) be the Hilbert scheme of smooth hypersurfaces of degree \( d \) in \( \mathbb{P}^{n+1} \), this is a smooth scheme over \( \mathbb{Z} \). In a recent breakthrough, Lawrence and Venkatesh proved the following:

**Theorem 1.1** ([7, Theorem 10.1, Proposition 10.2]). There exist \( n_0 \in \mathbb{N}_{\geq 3} \) and a function \( d_0 : \mathbb{N} \to \mathbb{N} \) such that,

\[
\text{(1.0.1)} \quad \text{for every } n \geq n_0 \text{ and } d \geq d_0(n),
\]

the set \( U_{n,d}(\mathbb{Z}[S^{-1}]) \) is not Zariski dense in \( U_{n,d} \), for every finite set of primes \( S \).

Consider \( f_{n,d} : X_{n,d} \to U_{n,d} \) the universal family of smooth degree \( d \) hypersurfaces in \( \mathbb{P}^{n+1} \). We denote by \( V \) the polarized \( \mathbb{Z} \)-variation of Hodge structure \( (R^n f_{n,d,C}^\an, Z)_{\text{prim}} \) on \( U_{n,d,C} \) and by \( \Phi : U_{n,d,C}^\an \to \Gamma \backslash D \) the associated period map. An irreducible algebraic subvariety \( Y \subset S \) is said to be of positive period dimension if \( \Phi(Y_{C}^\an) \) has positive dimension. We prove the following reinforcement of **Theorem 1.1**:

**Theorem 1.2.** As long as \( \text{(1.0.1)} \) is satisfied, there exists a closed strict subscheme \( E \subset U_{n,d} \) such that, for all finite set of primes \( S \), we have

\[
\overline{U_{n,d}(\mathbb{Z}[S^{-1}])}_{\text{pos}} \subset E,
\]

where \( \overline{U_{n,d}(\mathbb{Z}[S^{-1}])}_{\text{pos}} \) denotes the union of the irreducible components of the Zariski closure of \( U_{n,d}(\mathbb{Z}[S^{-1}]) \) in \( U_{n,d} \) of positive period dimension. That is: the Zariski closure of \( U_{n,d}(\mathbb{Z}[S^{-1}]) - E(\mathbb{Z}[S^{-1}]) \) has period dimension zero.

**Remark 1.3.** The complement of \( U_{n,d} \) in \( \mathbb{P}^{N(n,d)} \) is a hypersurface. Hence \( U_{n,d} \) is an open affine subvariety, stable under the natural \( \text{PGL}(n+2) \)-action on \( \mathbb{P}^{N(n,d)} \). Let \( \mathcal{M}_{n,d} := [\text{PGL}(n + 2) \backslash U_{n,d}] \) be the stack of smooth hypersurfaces in \( \mathbb{P}^{n+1} \) of degree \( d \). This is a finite type separated Deligne-Mumford algebraic stack over \( \mathbb{Z} \) with affine coarse space, see [3]. The period map \( \Phi : U_{n,d,C}^\an \to \Gamma \backslash D \) factorizes through \( \mathcal{M}_{n,d,C}^\an \). Notice moreover that the Torelli theorem (see e.g. [8] and references therein) assures that the period map \( \mathcal{M}_{n,d,C}^\an \to \Gamma \backslash D \) is quasi-finite for \( (n, d) \neq (2, 3) \).

The moduli stack \( \mathcal{M}_{n,d,C}^\an \) is thus Brody hyperbolic. A famous conjecture of Bombieri-Lang (see for instance [5, Chapter F.5.2]) thus predicts that \( \mathcal{M}_{n,d}(\mathbb{Z}[S^{-1}]) \) is finite. In particular \( E \) in **Theorem 1.2** should be empty.

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We first recall in Section 2 (a special case of) the Geometric Zilber–Pink conjecture mentioned above, which is a purely geometric result; and then in Section 3 the Lawrence-Venkatesh method, which is of arithmetic nature. In Section 4 we explain what can be obtained by combining the two results.

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2. The geometry of the Hodge locus

Let \( f : X \to S \) be a smooth projective morphism of smooth irreducible complex quasi-projective varieties, of relative dimension \( n \). The primitive Betti cohomology \( H^n(^X S, Z)_{\text{prim}} \) of the fibres \( X_s \), \( s \in S(\mathbb{C}) \), form a polarized \( \mathbb{Z} \)-variation of Hodge structures \( V \) on the complex manifold \( S^{an} \), described by a complex analytic period map \( \Phi : S^{an} \to \Gamma \backslash D \) (we refer for instance to [2] for more details on period maps). Motivated by the study of the Hodge conjecture for the fibres of \( f \), one defines the Hodge locus \( \text{HL}(S, V^\otimes) \) as the locus of points \( s \in S^{an} \) for which the Hodge structure \( H^n(^X S, \mathbb{Q})_{\text{prim}} \) admits more Hodge tensors than the primitive cohomology of the very general fibre. Here a Hodge class of a pure \( Z \)-Hodge structure \( V \) is a class in \( V_\mathbb{Q} \) whose image in \( V_\mathbb{C} \) lies in the zeroth piece \( F^0V_\mathbb{C} \) of the Hodge filtration, or equivalently a morphism of Hodge structures \( \mathbb{Q}(0) \to V_\mathbb{Q} \); and a Hodge tensor for \( V \) is a Hodge class in \( V^\otimes := \bigoplus_{a,b \in \mathbb{N}} V^{\otimes a} \otimes (V^\vee)^{\otimes b} \), where \( V^\vee \) denotes the Hodge structure dual to \( V \). Cattani, Deligne and Kaplan [4, Theorem 1.1] proved in particular that the Hodge locus \( \text{HL}(S, V^\otimes) \) is a countable union of irreducible algebraic subvarieties of \( S \), called the special subvarieties of \( S \) or \( \text{HL}(S, V^\otimes)_{\text{pos}} \) the Hodge locus of positive period dimension, that is the union of the special subvarieties whose image under \( \Phi \) has positive dimension in \( \Gamma \backslash D \).

Let \( P_Q^{N(n,d)} := P(H^0(P_Q^{n+1}, \mathcal{O}_{P_Q^{n+1}}(d))) \) be the Hilbert scheme of hypersurfaces \( X \) of \( P_Q^{n+1} \) of degree \( d \) (where \( N(n,d) = (n+d+1) \choose d \) - 1). Let \( U_{n,d} \subset P_Q^{N(n,d)} \) be the Zariski-open Hilbert scheme of smooth hypersurfaces \( X \) and consider \( f_{n,d} : X_{n,d} \to U_{n,d} \) the universal family of smooth degree \( d \) hypersurfaces in \( P_Q^{n+1} \). We denote by \( V \) the polarized \( \mathbb{Z} \)-variation of Hodge structure \( (R^n f_{n,d,\mathbb{C}}^* \mathbb{Z})_{\text{prim}} \) on \( U_{n,d,\mathbb{C}} \). We write \( \text{HL}(U_{n,d,\mathbb{C}}, V^\otimes) \) for its Hodge locus and \( \text{HL}(U_{n,d,\mathbb{C}}, V^\otimes)_{\text{pos}} \) for its Hodge locus of positive period dimension.

In our previous paper we have established the following as a particular case of our main result:

**Theorem 2.1** ([2, Corollary 2.7]). If \( n \geq 3, d \geq 5 \) and \( (n,d) \neq (4,5) \) then the Hodge locus \( \text{HL}(U_{n,d,\mathbb{C}}, V^\otimes)_{\text{pos}} \) of positive period dimension is a closed (not necessarily irreducible) algebraic subvariety of \( U_{n,d,\mathbb{C}} \). That is, there are only finitely many (rather than countably many) maximal strict special subvarieties of \( U_{n,d,\mathbb{C}} \) for \( V \) of positive period dimension.

**Remark 2.2.** For what follows, the easier [2, Theorem 5.1] would actually be enough (that is the Geometric Part of Zilber-Pink, for weakly-special subvarieties).

3. Non-denseness of integral points and the Lawrence-Venkatesh method

The following elucidation of the Lawrence-Venkatesh method for proving Theorem 1.1 will be crucial for us. Lawrence and Venkatesh actually prove that (quoting the third paragraph of [7, Section 1.1]) the monodromy for the universal family of hypersurfaces must drop over each component of the Zariski closure of the integral points (see also the last three lines of [7, Theorem 10.1]): for any \( S \), there exists a closed subscheme \( V_S \) of \( U_{n,d} \) (over \( \mathbb{Z}[S^{-1}] \)) whose irreducible components are of positive period dimension and not monodromy generic, such that \( U_{n,d}(\mathbb{Z}[S^{-1}]) - V_S(\mathbb{Z}[S^{-1}]) \) is finite. By the Deligne-André monodromy theorem (see for example
Remark 3.1. The Lawrence-Venkatesh method requires the choice of an auxiliary prime number $p$, and the choice of an identification between $C$ and $\overline{Q}_p$. Indeed, to prove that the $\mathbb{Z}[S^{-1}]$-points of $U_{n,d}$ are not Zariski dense, Lawrence and Venkatesh prove that some $p$-adic period map sending $x \in U_{n,d}(\mathbb{Z}[S^{-1}])$ to some $p$-adic representation of the absolute Galois group of $\mathbb{Q}_p$ has fibers that are not Zariski dense in $U_{n,d}$. This is done by working on a residue disk in $U_{n,d}(\mathbb{Q}_p)$ and the $p$-adic and complex period maps are then related by a study of the Gauss-Manin connection \cite[Lemma 3.2]{7}. What their proof actually shows, with respect to our fixed embedding $\overline{C} \subset C$ that for each $S$, there exists an automorphism $\iota_p$ of $C$ such that $V_S$ is contained in $HL(U_{n,d}, V^\otimes)^\iota_p \subset U_{n,d,C}$. What allows us to say that $V_S$ lies in $HL(U_{n,d}, V^\otimes)_{\text{pos}}$ is the fact that $HL(U_{n,d}, V^\otimes)_{\text{pos}}$ is actually defined over some number field, as one sees by combining Theorem 2.1 and \cite[Theorem 1.10]{6}.

Remark 3.2. Let us emphasize that both \cite[Theorem 10.1]{7} and Theorem 2.1 build on the Ax-Schanuel theorem \cite{1}, a deep and general theorem establishing strong functional transcendence properties of period maps. Actually, in Lawrence-Venkatesh, a $p$-adic version of such a result is used, see indeed \cite[Lemma 9.3]{7}.

4. PROOF OF THE MAIN RESULT

The proof is essentially a combination of Theorem 1.1 and Theorem 2.1.

It follows from Theorem 2.1 that $HL(U_{n,d}, V^\otimes)_{\text{pos}}$ is a (closed, strict) algebraic subvariety of $U_{n,d}$ and, thanks to the elucidation of Theorem 1.1, we have

$$\bigcup_S U_{n,d}(\mathbb{Z}[S^{-1}])_{\text{pos}} = \bigcup_S V_S \subset HL(U_{n,d}, V^\otimes)_{\text{pos}},$$

where the union ranges over all finite set of primes $S$. It follows from Theorem 2.1 that

$$E' := \bigcup_S V_S^{\text{Zar}} \subset HL(U_{n,d}, V^\otimes)_{\text{pos}}.$$

We remark here that the above inclusion may happen to be strict. Therefore we obtained a closed $\mathbb{Q}$-subvariety $E' \subset HL(U_{n,d}, V^\otimes)_{\text{pos}}$ containing all $V_S$ (seen as $\overline{\mathbb{Q}}$-varieties). The Zariski closure $E$ in $\mathbb{P}^N_{\mathbb{Z}}$ of $E'$ enjoys the desired property. The proof of the Theorem is concluded.

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\footnote{In fact, and to justify Remark 2.2, their proof actually shows that each $V_S$ is contained in the atypical Hodge locus of positive period dimension. Such subspace of the Hodge locus is proven to be non-Zariski dense in $U_{n,d}$ in \cite{2} as a first step towards Theorem 2.1, but it holds true for any variety supporting any variation of Hodge structures whose adjoint generic Mumford-Tate group is simple.}
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