Noncommutative geodesics and the KSGNS construction

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Abstract

We study parallel transport and geodesics in noncommutative geometry by means of bimodule connections and completely positive maps using the Kasparov, Stinespring, Gel’fand, Naimark & Segal (KSGNS) construction. This is motivated from classical geometry, and we also consider the algebras $M_2(\mathbb{C})$, $C(\mathbb{Z}_n)$ and first order deformations of classical manifolds.

1 Introduction

In classical geometry we frequently consider flows on manifolds due to vector fields (e.g. Morse theory) or vector fields as velocities along paths (e.g. geodesics). While a good definition of vector fields in noncommutative geometry has been around for some time [10, 21, 2] it has been difficult to apply them to the two classical applications above.

One fundamental decision is just what sort of maps to take between $C^*$-algebras or dense subalgebras. For various purposes the class of maps has been extended beyond $*$-algebra maps. For example Connes & Higson [14] introduced asymptotic $*$-algebra maps (called asymptotic morphisms) for $E$-theory, and these maps were also used by Dadarlat [15] and Manuilov & Thomsen [24] for noncommutative shape theory. Connes introduced the idea of correspondences (bimodules) to study von Neumann algebras [13]. Completely positive maps received attention from many authors, several of which (Kasparov, Stinespring, Gel’fand, Naimark & Segal) appear in the name of the KSGNS construction. For this construction and the theory of Hilbert $C^*$-bimodules we refer to the textbook [22].

The purpose of this paper is to use the KSGNS construction for noncommutative geodesics. The natural interpretation of these geodesics will be as paths in the state space of the algebra. Classically, evaluation at a point of a space $X$ is a state on $C(X)$, and moving the point along a path moves the state. I shall use ‘paths’ as completely positive maps from an algebra to $C^\infty(\mathbb{R})$. Given the general setting of the noncommutative construction, it is likely that this restriction to the ‘time algebra’ being $C^\infty(\mathbb{R})$ is unnecessary, but we shall stick to ‘real commutative time’ here.

The KSGNS construction represents a completely positive map between $C^*$-algebras as $\phi(a) = \langle ma, m \rangle$ in terms of an element $m$ of a Hilbert $C^*$-bimodule. The noncommutative theory of connections on bimodules has been studied for some time. We simply put these ingredients together, and look at examples. The critical result is that geodesics in classical differential geometry are precisely recovered as a special case.
The KSGNS construction is also well adapted to dealing with quantum theory, indeed it contains the usual theory of Hilbert spaces and observables (with some extension to unbounded operators). Paths in classical differential geometry have, at a given time, a precise position and a precise velocity. It is somewhat obvious that this idea will have to be modified in quantum theory, as there position and velocity (or rather momentum) obey the Heisenberg uncertainty principle. But why should parallel transport or geodesics make sense in quantum theory? The answer is simply that we can observe geodesic motion in the real world, so if the real world is governed by quantum theory, then to some extent geodesics must still make sense.

Quantum theory, by quantising the stress-energy tensor source term for gravity, \textit{de facto} quantises geometry, and at a scale conceivably much larger than the Planck length. There is no reason to expect that observations of quantum gravity will necessarily first take place in measurements of momentum eigenstates, in other words, in the normal domain of the perturbation theory solution methods of quantum field theory. Given the local nature of gravitational fields, it is likely that measurements of position will be involved. To back up any such observations it would be necessary to have a theory allowing the calculation of positions in quantum gravity, and as quantum gravity may well manifest itself, at least to ‘first order’, as noncommutative geometry, that may mean a physical theory of paths or world lines in noncommutative geometry.

As the reader will see, there is often great flexibility in extending classical ideas to noncommutative geometry. Saying that a particular way is \textit{the} way is not something to say lightly. The purpose of this paper is simply to show that a way of addressing geodesics in noncommutative geometry exists, and that it can be applied to many examples. It proposes that for a $C^\infty(\mathbb{R})$-$A$-bimodule $M$ the equation $\nabla(\sigma_M) = 0$ is a reasonable and calculable extension of the equations for vector fields in classical geodesics, and that $\nabla_M(m) = 0$ for $m \in M$ in the KSGNS construction above gives the corresponding time evolution on the state space.

There are other matters which we do not address, such as the constant speed of a classical geodesic. To do so would involve additional structure, such as hermitian inner products on the vector fields, and this paper is long enough. This paper is phrased in terms of bimodule connections as its basic object, and classically we might take the Levi Civita connection once we have a metric. By starting with bimodule connections we give a potentially more general discussion, and avoid another problem about just what connection to take for what sort of noncommutative metric.

Again, while many things can be done, the bigger question is ‘what is a useful working noncommutative definition of parallel transport or geodesics’. This question can only be answered through further theory and examples, and (as geometry is fundamentally an applied subject) models of physics.

In [1] there is a construction of quantum stochastic parallel transport processes which are possibly related to the methods in this paper.

The numerical equation solving and graphics were done on Mathematica.
2 Preliminaries

We give a very brief introduction to some ideas from the calculus of algebras, for a more comprehensive account see [5, 4]. Suppose that $A$ is a unital possibly noncommutative algebra over the field $\mathbb{C}$ (taking this choice to link with $C^\ast$-algebras later). We think of $A$ as the $C$ valued functions on a hypothetical noncommutative manifold. An $A$-module will correspond to a vector bundle on this hypothetical manifold. The tensor product of vector bundles corresponds to taking $E \otimes_A N$ where $E$ is a right and $N$ a left $A$-module. Here $E \otimes_A N$ has elements $e \otimes n$ for $e \in E$ and $n \in N$ where we set $e.a \otimes n = e \otimes a.n$ for all $a \in A$.

**Definition 2.1** A first order differential calculus $(\Omega^1, d)$ over $A$ means

1. $\Omega^1_A$, an $A$-bimodule.
2. A linear map $d : A \rightarrow \Omega^1_A$ (the exterior derivative) with $d(ab) = (da)b + a db$ for all $a,b \in A$.
3. $\Omega^1_A = \text{span}\{a db \mid a,b \in A\}$ (the surjectivity condition).

**Example 2.2** For the usual calculus on $\mathbb{R}^n$, the algebra $A = C^\infty(\mathbb{R}^n)$ (functions on $\mathbb{R}^n$ which are differentiable infinitely many times) has $\Omega^1_A$ the usual 1-forms on $\mathbb{R}^n$, i.e. $\xi_i dx^i$ for coordinates $x^1, \ldots, x^n$ and $\xi_i \in C^\infty(\mathbb{R}^n)$. We will just call this $\Omega^1(\mathbb{R}^n)$ to avoid writing $C^\infty(\mathbb{R}^n)$ as a subscript.

We also define vector fields.

**Definition 2.3** A (right) vector field on $A$, notation $v \in \mathfrak{X}_A$, is a right module map $v : \Omega^1_A \rightarrow A$.

**Example 2.4** For the usual calculus on $\mathbb{R}^n$, the vector fields $\mathfrak{X}(\mathbb{R}^n)$ are of the form $v = v^i \frac{\partial}{\partial x^i}$. They are maps from $\Omega^1(\mathbb{R}^n)$ to $A = C^\infty(\mathbb{R}^n)$ via the evaluation $\text{ev} : \mathfrak{X}(\mathbb{R}^n) \otimes_A \Omega^1(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n)$ which is $\text{ev}(v^i \frac{\partial}{\partial x^i} \otimes \xi_j dx^j) = v^i \xi_j$. We also have a dual basis of vector fields, which is expressed as a single element $\text{coev}(1) = dx^i \otimes \frac{\partial}{\partial x^i} \in \Omega^1(\mathbb{R}^n) \otimes_A \mathfrak{X}(\mathbb{R}^n)$ or more categorically by the coevaluation bimodule map $\text{coev} : A \rightarrow \Omega^1_A \otimes_A \mathfrak{X}_A$. This has the property that $(\text{id} \otimes \text{ev})(\text{coev}(1) \otimes \xi) = \xi$ for all $\xi \in \Omega^1(\mathbb{R}^n)$, which is easily verified by

\[
(id \otimes ev)(dx^i \otimes \frac{\partial}{\partial x^i} \otimes \xi_j dx^j) = dx^i \delta_{i,j} \xi_j = \xi_j dx^j.
\]

We now give two examples of noncommutative calculi, the first on a noncommutative algebra and the second a noncommutative calculus on a commutative algebra.

**Example 2.5** Set $A = M_2(\mathbb{C})$, the $2 \times 2$ complex matrices. This is given a calculus where $\Omega^1_A$ is freely generated by two central generators $s^1$ and $s^2$, with $da = s^1 [E_{12}, a] + s^2 [E_{21}, a]$ where $E_{ij} \in M_2(\mathbb{C})$ has zero entries except for 1 in the $ij$ position. We take $e_1, e_2$ to be the (central) dual basis of vector fields to $s^1, s^2$.\[\]
Example 2.6 Take the finite group \((\mathbb{Z}/n, +)\) and the algebra \(A = \mathbb{C}(\mathbb{Z}/n)\) of functions \(f : \mathbb{Z}/n \to \mathbb{C}\) with basis \(\delta_i\) for \(0 \leq i \leq n-1\), which is the function \(\delta_i(j) = \delta_{i,j}\). This has a calculus where \(\Omega^1_A\) has two non-central generators \(e_+\) and \(e_-\), where

\[
e_+ f = R_a(f) e_a, \quad df = e_+(f - R_-(f)) + e_-(f - R_+(f)),
\]

and \(R_a(f)(i) = f(i + a) \text{ (mod } n\text{ arithmetic)}\). (This is a Hopf algebra with bicovariant calculus.) Take \(\kappa_+\) and \(\kappa_-\) to be the dual basis of vector fields to \(e_+\) and \(e_-\).

Bimodule connections were introduced in [16] [17] [25] and extensively used in [23]. However, we need to use them in the more unusual context of mixed bimodules (different algebras on the left and right), and for that we refer to [3]. A \(B-A\)-bimodule \(M\) is a left \(B\)-module and a right \(A\)-module, with the compatibility condition \((b.m).a = b.(m.a)\). The idea of bimodules strictly generalises the idea of the usual left or right modules for algebras over a field \(K\), as a \(B-K\)-bimodule is simply a left \(B\)-module and a \(K-A\)-bimodule is simply a right \(A\)-module. We write \(M \in B M_A\) as the category whose objects are \(B-A\)-bimodules, and whose morphisms are bimodule maps (we will not refer to the morphisms anyway). Now suppose that \(A\) and \(B\) have differential calculi \(\Omega_A\) and \(\Omega_B\).

Definition 2.7 A left \(B-A\)-bimodule connection on \(M \in B M_A\) means

1. A linear map \(\nabla_M : M \to \Omega^1_B \otimes_B M\) satisfying the left Liebniz rule

\[
\nabla_M(b.m) = db \otimes m + b.\nabla_M(m), \quad b \in B, \ m \in M.
\]

2. A \(B-A\)-bimodule map \(\sigma_M : M \otimes_A \Omega^1_A \to \Omega^1_B \otimes_B M\) such that

\[
\nabla_M(m.a) = \nabla_M(m).a + \sigma_M(m \otimes da), \quad a \in A, \ m \in M.
\]

An example of a left \(C^\infty(\mathbb{R}^n)-C^\infty(\mathbb{R}^n)\)-bimodule connection is a usual connection on the tangent space to \(\mathbb{R}^n\). For \(v = v^i \frac{\partial}{\partial x^i}\) we have

\[
\nabla(v^i \frac{\partial}{\partial x^i}) = dx^k \otimes (v^i \frac{\partial}{\partial x^i} + v^j \Gamma^i_{kj} \frac{\partial}{\partial x^j}) = dx^i \otimes \sigma_i.
\]  

(1)

Here the \(\Gamma^i_{kj}\) are the usual Christoffel symbols, and note the common use of the subscript \(k\) for a partial derivative with respect to \(x^k\).

As originally pointed out in [11] for \(A-A\)-bimodules, but easily generalising to the current case, if we have \((Q, \nabla_Q, \sigma_Q)\) a left \(C-B\)-bimodule connection and \((M, \nabla_M, \sigma_M)\) a left \(B-A\)-bimodule connection, then we obtain a left \(C-A\)-bimodule connection on \(Q \otimes_B M\) by

\[
\nabla_{Q \otimes_B M}(q \otimes m) = \nabla_Q q \otimes m + (\sigma_Q \otimes \text{id})(q \otimes \nabla_M m), \quad \sigma_{Q \otimes_B M} = (\sigma_Q \otimes \text{id})(\text{id} \otimes \sigma_M).
\]

(2)

If \((M, \nabla_M, \sigma_M)\) and \((N, \nabla_N, \sigma_N)\) are left \(B-A\)-bimodule connections then given a left module map \(\theta : M \to N\) we define its derivative

\[
\nabla(\theta) = \nabla_N \theta - (\text{id} \otimes \theta) \nabla_M : M \to \Omega^1_B \otimes_B N.
\]

(3)

The following result is an easy generalisation to the mixed context from [8].
Proposition 2.8 For θ and ∇ in [3], ∇(θ) is a left module map. Further supposing that θ is a bimodule map, then ∇(θ) is a bimodule map if and only if \( σ_N \circ (θ \otimes id) = (id \otimes θ) \circ σ_M \).

Proof: Firstly, for \( b ∈ B \) and \( m ∈ M \),

\[
∇(θ)(bm) = ∇_N(bθ(m)) - (id \otimes θ) ∇_M(bm) = db \otimes θ(m) + b.∇_Nθ(m) - db \otimes θ(m) - b.(id \otimes θ) ∇_M(m) .
\]

Secondly, for \( a ∈ A \),

\[
∇(θ)(ma) = ∇_N(θ(m))a + (id \otimes θ) ∇_M(ma) = σ_N(θ(m) \otimes da) + ∇_N(θ(m)).a - (id \otimes θ)σ_M(m \otimes da) - (id \otimes θ)(∇_M(m)).a
\]

As we shall be concerned with positivity we shall deal exclusively with star algebras, and to avoid confusion we will be quite explicit about conjugate modules of these algebras. If \( M \) is a \( B-A \)-bimodule then its conjugate \( M^2 \) is an \( A-B \)-bimodule. Writing elements \( m ∈ M \) where \( m \in M \) we have

\[
λm + μm = λ^\ast m + μ^\ast m , \quad a.m = m.a^\ast , \quad m.b = b^\ast .m
\]

for all \( m, n ∈ M \), \( λ, μ ∈ ℂ \), \( a ∈ A \) and \( b ∈ B \). There is a theory of calculi on star algebras and connections on conjugate modules, e.g. [4], however we shall not require this for our current purpose.

3 Classical parallel transport and bimodule covariant derivatives

In this section we shall consider the basics of geodesic motion, the velocity of a parameterized path and the derivative of the velocity being zero. We begin by giving a rather unusual description of geodesics in classical differential geometry, but one suited to our purpose later. We use \( ℜ^3 \) to be explicit, but we could equally well take a coordinate chart of a manifold.

Example 3.1 Give \( ℜ^n \) coordinates \( (x^1, \ldots, x^n) \) and \( ℜ \) a coordinate \( t \), and consider a path \( γ : ℜ → ℜ^n \) written in coordinates as \( (γ^1, \ldots, γ^n) \). There is an induced algebra map \( \tilde{γ} : C^∞(ℜ^n) → C^∞(ℜ) \) given by \( \tilde{γ}(a) = a \circ γ \), so in particular \( \tilde{γ}(x^i) = γ^i \). Now \( M = C^∞(ℜ) \tilde{γ} \) is a \( C^∞(ℜ) \)-\( C^∞(ℜ^n) \) bimodule, which is simply \( C^∞(ℜ) \) as a vector space with left action by \( C^∞(ℜ) \) the usual product. The right action is via the map \( \tilde{γ} \) and given by \( f ∘ a = f \tilde{γ}(a) \), so in particular \( f ∘ x^i = f γ^i \). We give \( M \) a left \( C^∞(ℜ) \)-\( C^∞(ℜ^n) \)-bimodule connection by \( ∇_M(f) = f'(t) dt \otimes 1 ∈ Ω^1(ℜ) ⊗ C^∞(ℜ) M \) and

\[
σ_M(1 \otimes dx^i) = ∇_M(1 \ll x^i) - ∇_M(1) \ll x^i = ∇_M(γ^i) = \frac{dγ^i(t)}{dt} dt \otimes 1.
\]

(4)
Next give $\mathfrak{X}(\mathbb{R}^n)$, the vector fields on $\mathbb{R}^n$, the connection in [7]. Following [4], there is a tensor product connection on $M \otimes_{\mathcal{C}^{\infty}(\mathbb{R}^n)} \mathfrak{X}(\mathbb{R}^n)$ given by

$$\nabla_M \otimes \mathfrak{X}(\mathbb{R}^n)(f \otimes \frac{\partial}{\partial x^i}) = f'(t) \frac{dt}{dt} \otimes 1 \otimes \frac{\partial}{\partial x^i} + \sigma_M(f \otimes \Gamma^i_{jk} dx^j) \otimes \frac{\partial}{\partial x^i}$$

$$= f'(t) \frac{dt}{dt} \otimes 1 \otimes \frac{\partial}{\partial x^i} + f(\Gamma^i_{jk} \circ \gamma) \sigma_M(1 \otimes dx^j) \otimes \frac{\partial}{\partial x^i}$$

$$= f'(t) \frac{dt}{dt} \otimes 1 \otimes \frac{\partial}{\partial x^i} + f(\Gamma^i_{jk} \circ \gamma) \frac{d\gamma^j(t)}{dt} \frac{dt}{dt} \otimes 1 \otimes \frac{\partial}{\partial x^i}.$$

Now to construct an element of $M \otimes_{\mathcal{C}^{\infty}(\mathbb{R}^n)} \mathfrak{X}(\mathbb{R}^n)$ to apply the connection to. Choose a preferred translation invariant element $\frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R})$, and use the coevaluation $\text{coev}(1) = \sum_k dx^k \otimes \frac{\partial}{\partial x^i} \in \Omega^1(\mathbb{R}^n) \otimes_{\mathcal{C}^{\infty}(\mathbb{R}^n)} \mathfrak{X}(\mathbb{R}^n)$ to define

$$\mathcal{V} = (\text{ev.id} \otimes \text{id})(\frac{d}{dt} \otimes (\sigma \otimes \text{id})(1 \otimes \text{coev}(1))) \in M \otimes_{\mathcal{C}^{\infty}(\mathbb{R}^n)} \mathfrak{X}(\mathbb{R}^n)$$

which comes out as

$$\mathcal{V} = (\text{ev.id} \otimes \text{id})(\frac{d}{dt} \otimes \sigma(1 \otimes dx^k) \otimes \frac{\partial}{\partial x^i}) = \frac{d\gamma^k(t)}{dt} \frac{dt}{dt} \otimes 1 \otimes \frac{\partial}{\partial x^i}.$$

This shall be our definition of the velocity along the path $\gamma$. Then

$$\nabla_M \otimes \mathfrak{X}(\mathbb{R}^n)(\mathcal{V}) = \left(\frac{d^2\gamma^i(t)}{dt^2} + (\Gamma^i_{jk} \circ \gamma) \frac{d\gamma^j(t)}{dt} \frac{dt}{dt} \otimes 1 \otimes \frac{\partial}{\partial x^i}\right) \frac{dt}{dt} \otimes 1 \otimes \frac{\partial}{\partial x^i},$$

so $\nabla_M \otimes \mathfrak{X}(\mathbb{R}^n)(\mathcal{V}) = 0$ is just the usual equation of a geodesic.

We have already used some of the language of bimodule connections, but now we put the equation of parallel transport of the velocity into a particularly simple form. This can be seen on arguing about the derivative of evaluation and coevaluation maps from the previous exercise where everything was written in terms of vector fields, but it is probably simpler to do the calculation for differential forms from first principles.

**Proposition 3.2** For the bimodule connection $(M, \nabla_M, \sigma_M)$ of Example [3.3] the equation $\nabla_M(\sigma_M) = 0$ is equivalent to the parallel transport of the velocity vector $\mathcal{V}$.

**Proof:** As $\nabla_M(\sigma_M) : M \otimes_{\mathcal{C}^{\infty}(\mathbb{R}^n)} \Omega^1(\mathbb{R}^n) \rightarrow \Omega^1(\mathbb{R}) \otimes_{\mathcal{C}^{\infty}(\mathbb{R})} \Omega^1(\mathbb{R}) \otimes_{\mathcal{C}^{\infty}(\mathbb{R})} M$ is a left module map, we only have to evaluate it on a left basis $1 \otimes dx^i$ of the module. Now

$$\nabla_M(\sigma_M)(1 \otimes dx^i) = \nabla_{\Omega^1(\mathbb{R})} \otimes M \sigma_M(1 \otimes dx^i) - (\text{id} \otimes \sigma_M) \nabla_M \otimes \Omega^1(\mathbb{R}^n)(1 \otimes dx^i).$$

We break this up, calculating using [4],

$$(\text{id} \otimes \sigma_M)\nabla_M \otimes \Omega^1(\mathbb{R}^n)(1 \otimes dx^i) = -(\text{id} \otimes \sigma_M)(\sigma_M(1 \otimes \Gamma^i_{jk} dx^j) \otimes dx^k)$$

$$= -\gamma_i(\Gamma^i_{jk}) (\text{id} \otimes \sigma_M)(\sigma_M(1 \otimes dx^j) \otimes dx^k)$$

$$= -\gamma_i(\Gamma^i_{jk}) \frac{d\gamma^j(t)}{dt} \frac{dt}{dt} \otimes \sigma_M(1 \otimes dx^k) = -\gamma_i(\Gamma^i_{jk}) \frac{d\gamma^j(t)}{dt} \frac{dt}{dt} \otimes 1 \otimes 1,$$

and

$$\nabla_{\Omega^1(\mathbb{R})} \otimes M \sigma_M(1 \otimes dx^i) = \nabla_{\Omega^1(\mathbb{R})} \otimes M \left(\frac{d\gamma^i(t)}{dt} \frac{dt}{dt} \otimes 1 \right) = \frac{d^2\gamma^i(t)}{dt^2} \frac{dt}{dt} \otimes 1 \otimes 1.$$
We can express the rate of change of a section of a general bundle with connection along a curve as follows, using $\gamma$ and $M$ from Example 3.1.

**Proposition 3.3** Given a vector bundle on $\mathbb{R}^n$ with sections $E$ and connection $\nabla_E$ we write $\nabla_E(e) = dx^i \otimes \nabla E_i(e)$. Then the rate of change of $e \in E$ along $\gamma$ is

$$\nabla_M \otimes E(1 \otimes e) = \frac{d\gamma^i(t)}{dt} dt \otimes 1 \otimes \nabla E_i(e).$$

Further, if we take a basis $e_j$ of $E$ and write $\nabla_E(e_j) = \Gamma^k_{Ei,j} e_k$ then

$$\nabla_M \otimes E(1 \otimes e_j) = \frac{d\gamma^i(t)}{dt} (\Gamma^k_{Ei,j} \circ \gamma) dt \otimes 1 \otimes e^k \in \Omega^1(\mathbb{R}) \otimes_{C^\infty(\mathbb{R})} M \otimes_{C^\infty(\mathbb{R}^n)} E.$$ 

**Proof:** Use the formula for $\sigma_M$ from [1] and the formula

$$\nabla_M \otimes E(1 \otimes e) = (\sigma_M \otimes \text{id})(1 \otimes dx^i \otimes \nabla E_i(e)) = \frac{d\gamma^i(t)}{dt} dt \otimes 1 \otimes \nabla E_i(e).$$

Note that here $1 \otimes e \in M \otimes_{C^\infty(\mathbb{R}^n)} E$ is the section $e \in E$ restricted to the curve $\gamma(t)$, and in particular $1 \otimes e^k \in M \otimes_{C^\infty(\mathbb{R}^n)} E$ is the basis of $E$ restricted to the curve $\gamma(t)$. One way to see this is that $M \otimes_{C^\infty(\mathbb{R}^n)} E$ is a left $C^\infty(\mathbb{R})$ module, and because in Example 3.1 $M = C^\infty(\mathbb{R}) \gamma$, its elements are effectively functions of $t \in \mathbb{R}$ into $E$ at the point $\gamma(t)$. The equation for $f(t) \otimes e \in E$ (sum implicit) being parallel transported along $\gamma$ simply becomes $\nabla_M \otimes E(f(t) \otimes e) = 0$.

4 Noncommutative paths and the KSGNS construction

In Section 3 we constructed the bimodule $M$ from a path $\gamma : \mathbb{R} \to \mathbb{R}^n$. Now we replace $C^\infty(\mathbb{R}^n)$ by the possibly noncommutative algebra $A$, but still keep $T = C^\infty(\mathbb{R})$ as the other algebra, and so still keep the idea of ‘time evolution’. Our preferred differential form $dt$ or vector field $\frac{\partial}{\partial v}$ corresponds to a choice of measuring time along the path, and is translation invariant for the additive group structure on $\mathbb{R}$. But how do we generalise the induced algebra map $\hat{\gamma} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R})$ from Example 3.1 and will we still have a natural associated bimodule? The KSGNS construction gives the answer to both these questions. For our purposes we give a very weak form of the construction (also swapping sides to be convenient), for the general theory of Hilbert $C^*$-bimodules including a proper description of the KSGNS construction we refer to [22]. The algebras we consider are algebras of ‘differentiable functions’ rather than $C^*$-algebras, but they are frequently dense $*$-subalgebras of $C^*$-algebras (or even local $C^*$-algebras as in [1]).

From the point of view of quantum mechanics, the Heisenberg uncertainty principle makes it likely that an idea of a geodesic as a single path will have to be replaced by a more uncertain or ‘probabilistic’ idea, as we cannot precisely measure both position and velocity (momentum) at the same time, and the geodesic depends on both these
quantities. From the Schrödinger picture we recall that the expected value of a function \( f(x, y, z) \) of position for a particle with wave function \( \psi(x, y, z) \) for \( (x, y, z) \in \mathbb{R}^3 \) is given by

\[
\mathbb{E}(f(x, y, z)\psi) = \int_{\mathbb{R}^3} \psi(x, y, z)^* f(x, y, z) \psi(x, y, z) \, dx \, dy \, dz .
\]

(Note that we use \( * \) for conjugation in \( \mathbb{C} \) in line with other \( * \)-algebras and because we do not want to cause confusion with the bar for conjugate modules.) This fits the KSGNS construction where we take \( \psi \in L^2(\mathbb{R}^3, \mathbb{C}) \) (the square integrable complex valued functions) and a hermitian inner product on \( L^2(\mathbb{R}^3, \mathbb{C}) \)

\[
\langle \kappa, \psi \rangle = \int_{\mathbb{R}^3} \kappa(x, y, z)^* \psi(x, y, z) \, dx \, dy \, dz \quad \kappa, \psi \in L^2(\mathbb{R}^3, \mathbb{C}) .
\]

However, for convenience we take the conjugate on the other side as we would prefer to have left covariant derivatives on modules.

**Definition 4.1** For \( * \)-algebras \( A \) and \( B \) and a \( B \)-\( A \)-bimodule \( M \), a hermitian metric on \( M \) is a bimodule map \( \langle , \rangle : M \otimes A \overline{M} \to B \) which obeys \( \langle m, \overline{m}' \rangle^* = \langle m', \overline{m} \rangle \) for all \( m, m' \in M \). For dense subalgebras of \( C^* \)-algebras we can also consider positive (semi-)inner products where \( \langle m, \overline{m} \rangle \) is positive in \( B \) for all \( m \in M \).

I shall only briefly describe the KSGNS construction, as getting into details of the \( C^* \)-algebra construction is not required. Basically it says that completely positive maps \( \phi : A \to B \) for \( C^* \)-algebras \( A \) and \( B \) are all given by \( B \)-\( A \)-bimodules \( M \) with positive inner products using the formula \( \phi(a) = \langle m, a, \overline{m} \rangle \) for some \( m \in M \).

**Example 4.2** Now we explain what Example 3.1 has to do with the KSGNS construction. First define a positive inner product \( \langle , \rangle : M \otimes \overline{M} \to C^\infty(\mathbb{R}) \) by \( \langle f, \overline{g} \rangle = \int_{\mathbb{R}} f y^* \). Given the usual right \( C^\infty(\mathbb{R}) \) action on the conjugate bimodule, \( \overline{g} \ll k = \overline{k} g \), the inner product is a \( C^\infty(\mathbb{R}) \)-bimodule map. Using the usual left \( C^\infty(\mathbb{R}^n) \) action on the conjugate \( \overline{M} \) corresponding to the right \( C^\infty(\mathbb{R}^n) \) action on \( M \), we can check that we get a well defined map \( \langle , \rangle : M \otimes C^\infty(\mathbb{R}^n) \overline{M} \to C^\infty(\mathbb{R}) \). To do this we use the following, restricting to the case \( x^i \in C^\infty(\mathbb{R}^n) \) simply to avoid explicitly writing compositions

\[
\langle f \ll x^i, \overline{f} \rangle = \gamma^i \langle f, \overline{f} \rangle, \quad \langle f, x^i \gg \overline{f} \rangle = \langle f, \overline{k} x^i \rangle = \gamma^i \langle f, \overline{f} \rangle
\]

by the reality of the \( x^i \) and \( \gamma^i \). (Complex valued functions would also work, being conjugated twice in the second calculation.)

Taking motivation from the Schrödinger picture we set \( N = A \) as a left \( A \)-module, and take a function \( \phi : A \to \mathbb{C} \). We suppose that \( A \) is a unital algebra and that \( \phi \) is normalised so that \( \phi(1) = 1 \), which means that we are not in the case above where \( \phi \) would be the integral over all of \( \mathbb{R}^3 \). Rather we should think of a state on a \( C^* \) algebra, characterised by the properties that \( \phi(1) = 1 \) and \( \phi(aa^*) \geq 0 \) for all \( a \in A \). To avoid confusion, it is simplest to think of \( N \) as a free right \( A \)-module with generator \( n \in N \), so that elements of \( N \) can be uniquely written in the form \( na \) for \( a \in A \). We suppose that \( \langle , \rangle_N : N \otimes A \overline{N} \to \mathbb{C} \) is an inner product, for example we could have \( \langle na, \overline{nb} \rangle_N = \phi(ab^*) \).
for a state on a \( C^* \)-algebra containing the \(*\)-algebra \( A \). Now define an inner product on \( C^\infty(\mathbb{R}) \otimes N \) by
\[
\langle f(t) \otimes na, g(t) \otimes nb \rangle = f(t)g(t)^* \langle na, nb \rangle_N .
\]

This is best thought of by considering \( C^\infty(\mathbb{R}) \otimes N \subset C^\infty(\mathbb{R}, N) \), where we shall loosely interpret \( C^\infty(\mathbb{R}, N) \) as some sensible completion where solutions of differential equations may live. (If \( A \) is finite dimensional, we get equality.) In this case we take the pointwise inner product at a given time \( t \). In fact, we shall take advantage of the fact that we have restricted ourselves to normal commutative real valued time to abuse notation by writing various quantities as functions of \( t \) without further comment or concern about where the \( t \) occurs.

5 Noncommutative parallel transport

Suppose that \( T = C^\infty(\mathbb{R}) \) has its classical calculus \( \Omega_\mathbb{R} \) and that the unital noncommutative algebra \( A \) has calculus \( \Omega_A \).

**Proposition 5.1** Let \( N \) be a free right \( A \)-module, with generator \( n \in N \), and set \( M = C^\infty(\mathbb{R}) \otimes N \) regarded as a \( C^\infty(\mathbb{R}) \)-\( A \)-bimodule. Then a general left bimodule connection on \( M \) is of the form, for \( c \in C^\infty(\mathbb{R}) \otimes A \) and \( \xi \in \Omega_\mathbb{A}^1 \)
\[
\nabla_M(nc) = dt \otimes n(bc + \frac{\partial c}{\partial t} + K(dc)) , \quad \sigma_M(n \otimes \xi) = dt \otimes nK(\xi)
\]

for some \( b \in C^\infty(\mathbb{R}) \otimes A \) and \( K \in C^\infty(\mathbb{R}) \otimes \mathcal{X}_A \). Note that explicitly including time evaluation we have \( K(\eta)(t) = K(t)(\eta(t)) \) for \( \eta \in C^\infty(\mathbb{R}) \otimes \Omega_A^1 \), and that we regard \( M = C^\infty(\mathbb{R}) \otimes N \) as a \( C^\infty(\mathbb{R}) \)-\( A \)-bimodule in the trivial way.

**Proof:** The \( C^\infty(\mathbb{R}) \)-\( A \)-bimodule map \( \sigma_M : M \otimes_A \Omega_A^1 \to \Omega_\mathbb{R}^1 \otimes C^\infty(\mathbb{R}) M \) is uniquely specified by its value \( \sigma_M(n \otimes \xi) \in \Omega_\mathbb{R}^1 \otimes C^\infty(\mathbb{R}) M \) for \( \xi \in \Omega_A^1 \), and we write \( \sigma_M(n \otimes \xi) = dt \otimes nK(\xi) \) where \( K : \Omega_A^1 \to C^\infty(\mathbb{R}) \otimes A \) is a right \( A \)-module map, because \( \sigma_M \) is a right \( A \) module map. Next as \( \nabla_M(n) \in \Omega_\mathbb{R}^1 \otimes C^\infty(\mathbb{R}) M \) it must be of the form \( dt \otimes nb \) for some \( b \in C^\infty(\mathbb{R}) \otimes A \). Now we put \( c(t) = f(t) a \) for \( f \in C^\infty(\mathbb{R}) \) and \( a \in A \) and calculate
\[
\nabla_M(n a) = \nabla_M(fna) = \frac{\partial f}{\partial t} dt \otimes na + f \nabla_M(na)
\]
\[
= \frac{\partial f}{\partial t} dt \otimes na + f \nabla_M(na) + f \sigma_M(n \otimes da)
\]
giving the required answer. \( \square \)

Now we examine the equation \( \nabla(\sigma_M) = 0 \). However this is for a different bimodule to that in Proposition 3.2 and so for we have not linked this equation to geodesics.

**Proposition 5.2** Take \( N, A \) and \((M, \nabla_M, \sigma_M)\) as in Proposition 5.1. Also take the trivial connection on \( \Omega_A^1 \) (i.e. \( \nabla_M(\xi) = \frac{\partial f}{\partial t} dt \otimes \xi + \sigma_M(\xi) \)) and a left bimodule connection \( \nabla_A \) on the \(\mathbb{R}\)-\( A \)-bimodule \( \Omega_A^1 \) with invertible \( \sigma_A : \Omega_A^1 \otimes_A \Omega_A^1 \to \Omega_A^1 \otimes_A \Omega_A^1 \). Then \( \nabla(\sigma_M) = 0 \) if and only if both \( K(K \otimes \text{id}) \sigma_A = K(K \otimes \text{id}) \) and
\[
\frac{\partial K(\xi)}{\partial t} = K(b \xi) - bK(\xi) + K(K \otimes \text{id}) \sigma_A^{-1} \nabla_A(\xi) - K(dK(\xi)) . \tag{7}
\]
Further, if $K$ satisfies (7) for $\xi \in \Omega^1_A\otimes_A \Omega^1_A$ then it also satisfies it for $\xi \in \Omega_A^1$ for all $a \in A$. Hence it is only necessary to verify (7) for a collection of right generators of $\Omega_A^1$.

**Proof:** The equation $\nabla(\sigma_M) = 0$ reduces to
\[
\nabla_{\Omega^1_A \otimes_M \sigma_M}(n \otimes \xi) = \nabla_{\Omega^1_A \otimes_M}(dt \otimes n \cdot K(\xi)) = \langle id \otimes \sigma_M \rangle \nabla_{M \otimes \Omega^1}(n \otimes \xi)
\]  
(8) for $\xi \in \Omega^1_A$, and as $\nabla_R(dt) = 0$ the LHS of (8) is
\[
(\sigma_R \otimes id)(dt \otimes (dt \otimes nbK(\xi) + dt \otimes n \cdot \partial K(\xi) + \sigma_M(n \otimes dK(\xi))))
\] 
\[
= \sigma_R(dt \otimes dt) \otimes n(bK(\xi) + \partial K(\xi))/\partial t + K(dK(\xi))
\]
whereas if we write $\nabla_A(\xi) = \eta \otimes \kappa$ then the RHS is
\[
(id \otimes \sigma_M)(dt \otimes nb \otimes \xi + \sigma_M(n \otimes \eta) \otimes \kappa)
\]
\[
= dt \otimes \sigma_M(nb \otimes \xi + nK(\eta) \otimes \kappa) = dt \otimes dt \otimes n(K(b\xi) + K(\eta)\kappa)
\]
so as $\sigma_R(dt \otimes dt) = dt \otimes dt$ we get $\nabla(\sigma_M) = 0$ reducing to
\[
bK(\xi) + \partial K(\xi)/\partial t + K(dK(\xi)) = K(b\xi) + K(\eta)\kappa)\n\]
(9)
Now if $\nabla(\sigma_M) = 0$ from Proposition 2.8 we also have $\sigma_{\Omega^1_A \otimes_M}(\sigma_M \otimes id) = \langle id \otimes \sigma_M \rangle \sigma_M \otimes_1 A$, which is just the braiding relation
\[
(\sigma_R \otimes id)(id \otimes \sigma_M)(\sigma_M \otimes id) = (id \otimes \sigma_M)(\sigma_M \otimes id)(id \otimes \sigma_A)
\]
(10) and this gives the equation $K(\kappa \otimes id)\sigma_A = K(\kappa \otimes id)$. Using this and the invertibility of $\sigma_A$ in (9) gives (7). To check the right multiplication property for (7) we look at
\[
K(\kappa \otimes id)\sigma_A^{-1}\nabla_A(\xi \cdot a) - K(\kappa \otimes id)\nabla_A(\xi \cdot a) = K(\kappa \otimes id)\sigma_A^{-1}\nabla_A(\xi \cdot a) + K(\kappa \otimes id)(\xi \otimes da) - K(\kappa \otimes id)(\xi \otimes da) - K(\kappa \otimes id)(\xi \otimes da).
\]
(7)

The right multiplication property of (7) given in Proposition 5.2 makes it much easier to solve than the original equation $\nabla(\sigma_M) = 0$, and as they are equivalent on classical manifolds (i.e. the braiding relation (10) is true), it is very tempting to simply take it as the defining equation for vector fields generating geodesics. However, we shall now consider the additional equation $K(\kappa \otimes id)(\sigma_A - id) = 0$ from Proposition 5.2. As (7) is in the form of a time evolution for $K$ as a function of $t$, it is natural to specify $K$ at time zero and then try to solve (7) for some time interval including $t = 0$. We must then assume that $K(\kappa \otimes id)(\sigma_A - id) = 0$ at time zero. But does this remain true under the time evolution given by (7)?

**Corollary 5.3** If we set $G = K(\kappa \otimes id)(\sigma_A - id) : \Omega^1_A \otimes_A \Omega^1_A \rightarrow A$ then
\[
\frac{\partial G}{\partial t} = GL_b - L_bG + K(\kappa \otimes id)(K \otimes id \otimes id)\nabla(\sigma_A) + G(K \otimes id \otimes id)\nabla_A(\sigma_A) - KdG
\]
\[
- K(G\sigma_A^{-1} \otimes id)\nabla_A(\sigma_A - id) - G\sigma_A^{-1} \otimes id - G\sigma_A^{-1} \otimes id
\]
where $L_b$ is the left multiply by $b$ operation, we use $\nabla_A(\sigma_A)$ to stand for the tensor product connection on $\Omega^1_A \otimes_A \Omega^1_A$ and $\nabla(\sigma_A) = \nabla_A(\sigma_A) - (id \otimes \sigma_A)\nabla_A$. 

10
\[ \frac{\partial K(K \otimes \text{id})(\xi \otimes \mu)}{\partial t} = K \left( (K(b\xi) - bK(\xi)) + K(K \otimes \text{id})(\sigma_{A}^{-1} \nabla_{A}(\xi) - K(dK(\xi)) \otimes \mu) \right) \]

\[ + K(bK(\xi) \otimes \mu) + K(K \otimes \text{id})(\sigma_{A}^{-1} \nabla_{A}(\xi) \otimes \mu) - K(dK(\xi) \otimes \mu) \]

\[ = K(K(b\xi)) + K(K \otimes \text{id})(\sigma_{A}^{-1} \nabla_{A}(\xi) \otimes \mu) - K(K \otimes \text{id})(dK(\xi) \otimes \mu) \]

If we put \( S = K(K \otimes \text{id}) \) then we find

\[ \frac{\partial S(\xi \otimes \mu)}{\partial t} = S(b\xi \otimes \mu) - bS(\xi \otimes \mu) + K(S\sigma_{A}^{-1} \otimes \text{id}) \nabla_{A} - KdS(\xi \otimes \mu) \]

or in terms of operators,

\[ \frac{\partial S}{\partial t} = SL_b - L_bS + S(K \otimes \text{id} \otimes \text{id}) \nabla_{A} - KdS \]

and then compose this with \((\sigma_A - \text{id})\).

In the cases where \( \nabla(\sigma_A) = 0 \) (such as classical geometry) or where we can otherwise ensure the vanishing of \( K(K \otimes \text{id})(\sigma_A - \text{id}) \nabla(\sigma_A) \) we see that \( G = 0 \) is a solution of the equation on the interval where \( K \) is defined. Thus, if we have uniqueness of solution of the equation we would have \( K(K \otimes \text{id})(\sigma_A - \text{id}) = 0 \).

Now we need to justify why the equation \( \nabla(\sigma_A) = 0 \) in Proposition 5.2 has anything to do with geodesics.

**Example 5.4** We consider the classical case with algebra \( A = C^\infty(\mathbb{R}^n) \). In Section 3 we took \( M = C^\infty(\mathbb{R}) \) as a \( C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R}^n) \) bimodule, for some path \( \gamma: \mathbb{R} \to \mathbb{R}^n \), and showed that our approach gave the parallel transport equations. Now we have switched to considering a bimodule which in this case is is \( C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R}^n) \), and we need to examine how these different approaches fit together. The equation (7) for the time dependent vector field \( K \) on \( \mathbb{R}^n \) becomes

\[ \frac{\partial K^i}{\partial t} + K^s K^i_{\cdot s} + K^k K^j_{\cdot k} \Gamma^i_{jk} = 0 \]  

where \( \Gamma^i_{jk} \) are the Christoffel symbols for the connection \( \nabla_{\mathbb{R}^n} \). Now, suppose that we start a point at \( x(0) \in \mathbb{R}^n \) for \( t = 0 \) and move it according to the vector field \( \frac{dx}{dt} = K(x) \). As the point moves, the ‘convective derivative’ from fluid mechanics gives \( \frac{dK(x)}{dt} = \frac{\partial K^i}{\partial t} + K^s K^i_{\cdot s} \) and so (12) becomes \( \frac{dK^i(x)}{dt} + K^k K^j_{\cdot k} \Gamma^i_{jk} = 0 \), which is the usual equation for the velocity being parallel transported. Thus the vector field approach here is actually
Example 2.6 We set the initial condition. Now \( \nabla \) \( \nabla \) Figure 1. The entries of \( K \)

We take the initial conditions \( K \)

together. Then, by the comments after Corollary 5.3, we would expect (subject to the

\( K(\dot{K})\) \( K(\dot{K})\) \( K(\dot{K})\) evaluated on the generators becomes

\[
\frac{\partial K(s^i)}{\partial t} - K(dK(s^i)) = -K(s^1 [E_{12}, K_i]) - K(s^2 [E_{21}, K_i])
\]

(13)

We take the initial conditions

\[
K_1(0) = \left( \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right), \quad K_2(0) = \left( \begin{array}{cc} -3 & 2 \\ 3 & 0 \end{array} \right)
\]

(14)

and solve these coupled nonlinear equations numerically, with the solution shown in

Figure 1. The entries of \( K_1(t) \) and \( K_2(t) \) in (a) and (b) respectively can easily be

identified by the starting values at \( t = 0 \). In this case \( \sigma_A(s^i \otimes s^j) = s^i \otimes s^j \) so we have

\( \nabla : (\sigma_A) = 0 \). Then, by the comment after Corollary 5.3 we would expect (subject to the

uniqueness of solution) to have \( K(\dot{K} \otimes \text{id}) (\sigma_A - \text{id}) = 0 \) on the interval if it is true for

the initial condition. Now

\[
K(\dot{K} \otimes \text{id}) (\sigma_A - \text{id})(s^1 \otimes s^2) = K(\dot{K} \otimes \text{id})(s^2 \otimes s^3 - s^1 \otimes s^2)
\]

\[
= K(K(s^2) s^1 - K(s^1) s^2) = K(s^1 K(s^2) - s^2 K(s^1)) = [K_1, K_2]
\]

and the condition \([K_1(0), K_2(0)] = 0\) is true for (14). Figure 1 part (c) shows that the

condition is true to within \( 10^{-14} \) for the numerical solution in the interval \( t \in [0, 3] \). ⊗

Example 5.5 As an example of Proposition 5.2 we take \( A = M_2(\mathbb{C}) \) with calculus as described in Example 2.5. We give \( \Omega^1 \) a connection with \( \nabla_A(s^i) = 0 \) and set \( b = 0 \).

Then we set \( K(s^i) = K_1 \in M_2 \) and \( \nabla \) evaluated on the generators becomes

\[
\frac{\partial K(s^i)}{\partial t} - K(dK(s^i)) = -K(s^1 [E_{12}, K_i]) - K(s^2 [E_{21}, K_i])
\]

(13)

We take the initial conditions

\[
K_1(0) = \left( \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right), \quad K_2(0) = \left( \begin{array}{cc} -3 & 2 \\ 3 & 0 \end{array} \right)
\]

(14)

and solve these coupled nonlinear equations numerically, with the solution shown in

Figure 1. The entries of \( K_1(t) \) and \( K_2(t) \) in (a) and (b) respectively can easily be

identified by the starting values at \( t = 0 \). In this case \( \sigma_A(s^i \otimes s^j) = s^i \otimes s^j \) so we have

\( \nabla : (\sigma_A) = 0 \). Then, by the comment after Corollary 5.3 we would expect (subject to the

uniqueness of solution) to have \( K(\dot{K} \otimes \text{id}) (\sigma_A - \text{id}) = 0 \) on the interval if it is true for

the initial condition. Now

\[
K(\dot{K} \otimes \text{id}) (\sigma_A - \text{id})(s^1 \otimes s^2) = K(\dot{K} \otimes \text{id})(s^2 \otimes s^3 - s^1 \otimes s^2)
\]

\[
= K(K(s^2) s^1 - K(s^1) s^2) = K(s^1 K(s^2) - s^2 K(s^1)) = [K_1, K_2]
\]

and the condition \([K_1(0), K_2(0)] = 0\) is true for (14). Figure 1 part (c) shows that the

condition is true to within \( 10^{-14} \) for the numerical solution in the interval \( t \in [0, 3] \). ⊗
where $K_\pm = K(e_{\pm 1}) \in \mathbb{C}(\mathbb{Z}_n)$. In the case $n = 3$ we get six equations

$$\frac{\partial K_\pm(i)}{\partial t} = -K_+(i)(K_+(i) - K_+(i - 1)) - K_-(i)(K_-(i) - K_+(i + 1))$$  \hfill (16)

for the six functions $K_\pm(i)(t)$ with $0 \leq i \leq 2$. We solve this numerically with the initial conditions at $t = 0$

$$K_-(1)(0) = 1, \ K_-(2)(0) = 2, \ K_+(0)(0) = 3, \ K_+(1)(0) = 4, \ K_+(2)(0) = -3, \ K_+(0)(0) = 2,$$

(17)

and the results for $K_-(t)$ and $K_+(t)$ are shown in Figure 2 (a) and (b) respectively. The domain $t \in [0, 0.4]$ is chosen for the good reason that the solution likely becomes singular between $t = 0.4$ and $t = 0.5$.

Similarly to Example 5.5, $\sigma_A(e_{a} \otimes e_{b}) = e_{b} \otimes e_{a}$, and as

$$K(K \otimes \text{id})(e_{+1} \otimes e_{-1}) = K(K_{+}e_{-1}) = K(e_{-1}R_{+1}(K_{+})) = K_{-}R_{+1}(K_{+})$$

and similarly for $e_{-1} \otimes e_{+1}$, the extra $K(K \otimes \text{id})(\sigma_{A} - \text{id}) = 0$ condition corresponds to

$$K_{-}R_{+1}(K_{+}) = K_{+}R_{-1}(K_{-}), \ i.e. K_{-}(i)K_{+}(i + 1) = K_{-}(i - 1)K_{+}(i) \text{ for } 0 \leq i \leq 2,$$

which reduces to $K_{-}(0)K_{+}(1) = K_{-}(1)K_{+}(2) = K_{-}(2)K_{+}(0)$. Figure 2 (c) plots

$$K_{-}(0)K_{+}(1) - K_{-}(1)K_{+}(2), \ K_{+}(2)K_{+}(0) - K_{-}(1)K_{+}(2)$$  \hfill (18)

for the initial conditions (17). The vanishing of the expressions in (18) gives the condition $K(K \otimes \text{id})(\sigma_{A} - \text{id}) = 0$, and as $\nabla(\sigma_{A}) = 0$ we would expect the extra condition.

---

**Figure 2**: Numerical solution to (16) with initial conditions (17) for $t \in [0, 0.4]$.

**Figure 3**: Numerical solution to (16) with initial conditions (19) for $t \in [0, 0.3]$. 

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13
to remain true if it was true initially. To test this, we use new initial conditions
\[ K_-(1)(0) = 4, \ K_-(2)(0) = \frac{1}{2}, \ K_-(0)(0) = 3, \ K_+(1)(0) = \frac{1}{3}, \ K_+(2)(0) = \frac{1}{4}, \ K_+(0)(0) = 2, \]
and plot the result in Figure 3, where (c) is consistent with the extra condition holding
to an accuracy of $10^{-7}$.

Note in Examples 5.5 and 5.6 we do not attempt to make any general statements about the stability or asymptotic behaviour of the systems, the numerical calculations are merely for interest.

6 First order deformation of classical geometry

Here we consider the deformation of functions on a classical manifold due to an antisymmetric bivector $\omega$.

A Poisson tensor.

This is described in [6] and to avoid a very long discussion we shall not repeat the
description in detail here. Note that in the presence of curvature the calculus fails to
be associative at $O(\lambda^2)$, so there is no point in looking at higher orders in the
case of associative algebras, and therefore in particular we do not assume that $\omega^{ij}$
is a Poisson tensor.

The basic thing to remember to keep life simple is that the noncommutative cal-
culations are relatively easy, they are just noncommutative algebra, and the classical
calculations are relatively easy also. The really messy stage is converting between the
noncommutative picture and the classical picture. To minimise the complexity we shall
work noncommutatively as far as possible, and then convert to our explicit semiclassical
construction at the last minute. We have generators $e^i = dx^i$ of the 1-forms, and
applying \[7\] to these with $b = 0$ we find
\[
\frac{\partial K^i}{\partial t} = K(K \otimes \text{id})\sigma_A^{-1}\nabla_A(e^i) - K(dK^i). \tag{21}
\]
where we have set $K^i = K(e^i)$. Now $\sigma_A^{-1}\nabla_A$ is actually a right connection on $\Omega_A^1$, and
we write $\sigma_A^{-1}\nabla_A(e^i) = -e^j \otimes e^k \Gamma_{ij}^{\phantom{ij}k} (\text{there is no reason to believe that the covariant}
\text{derivative relevant to the geodesics is the quantising connection}), and then
\[
K(K \otimes \text{id})\sigma_A^{-1}\nabla_A(e^i) = -K(K^j e^k \Gamma_{ijk}) = -K(K^j e^k) \Gamma_{ijk}^i
\]
\[
= -K\left(K^j e^k\right) \Gamma_{ijk}^i - K(e^k K^j) \Gamma_{ijk}^i
\]
\[
= -K\left(K^j e^k\right) \Gamma_{ijk}^i - K^k K^j \Gamma_{ijk}^i
\]
remarking that $K$ is a right $A$-module map. Now we rewrite (21) using entirely the noncommutative approach

$$\frac{\partial K^i}{\partial t} = -K([K^j,e^k]) \tilde{\Gamma}^i_{jk} - K^k K^j \tilde{\Gamma}^i_{jk} - K(dK^i).$$

Now we convert the RHS of (22) to classical operations with an explicit deformation parameter term by term. First $[K^j,e^k] = \lambda \omega^{pq} K^j_{,p} V_q (e^k) = -\lambda \omega^{pq} K^j_{,p} \Gamma^k_{qs} e^s$ so we get the first term being $\lambda \omega^{pq} K^j_{,p} \Gamma^k_{qs} K^s \tilde{\Gamma}^i_{jk}$ with the classical product. As $d$ is undeformed we get $dK^i = dx^s K^i_{,s}$, but to use the right $A$-module property of $K$ we need to convert this to the deformed product, which we temporarily write as $\bullet$.

$$K(dK^i) = K(dx^s K^i_{,s}) = K(dx^s \bullet K^i_{,s}) - \frac{1}{2} \lambda K(\omega^{pq} \nabla_p (dx^s) K^i_{,sq})$$

$$= K^s \bullet K^i_{,s} + \frac{1}{2} \lambda \omega^{pq} K^s_{,sq} \Gamma^i_{ps} K^j$$

$$= K^s K^i_{,s} + \frac{1}{2} \lambda \omega^{pq} K^s_{,sq} K^i_{ps} \Gamma^i_{ps} K^j.$$

We write $\tilde{\Gamma}^i_{jk} = \bar{\Gamma}^i_{jk} + \lambda \bar{\Gamma}^i_{jk}$ where $\bar{\Gamma}^i_{jk}$ are the Christoffel symbols of the classical connection which has been deformed. The product of 3 elements can be written using $\omega^{pq}$ fairly simply. The result is

$$\frac{\partial K^i}{\partial t} + K^s K^i_{,s} + K^k K^j \bar{\Gamma}^i_{jk} = \lambda \omega^{pq} K^k_{,p} \Gamma^k_{qs} K^i_{ps} - \lambda K^k K^j \bar{\Gamma}^i_{jk}$$

$$- \frac{1}{2} \lambda \omega^{pq} (K^k_{,p} \bar{\Gamma}^i_{jk} + K^k_{,j} \bar{\Gamma}^i_{jk} + K^k_{,q} K^i_{jk})$$

$$- \frac{1}{2} \lambda \omega^{pq} K^s_{,sq} K^i_{ps} \Gamma^i_{ps} K^j.$$

Now we put $K^i = V^i + \lambda W^i$ where the classical vector field $V^i$ obeys (12) (i.e. is the velocity field of the classical geodesics), and then

$$\frac{\partial V^i}{\partial t} + V^s W^i_{,s} + W^s V^i_{,s} + V^k W^j \bar{\Gamma}^i_{jk} + W^k V^j \bar{\Gamma}^i_{jk} = \omega^{pq} V^j_{,p} \Gamma^k_{qs} V^s \bar{\Gamma}^i_{jk} - V^k V^j \bar{\Gamma}^i_{jk}$$

$$- \frac{1}{2} \omega^{pq} (V^k_{,p} \bar{\Gamma}^i_{jk} + V^k_{,j} \bar{\Gamma}^i_{jk} + V^k_{,q} \bar{\Gamma}^i_{jk})$$

$$- \frac{1}{2} \omega^{pq} V^s_{,sq} V^i_{ps} - \frac{1}{2} \omega^{pq} V^i_{sq} \Gamma^i_{ps} V^j.$$

On assuming that the classical connection is torsion free (i.e. $\bar{\Gamma}^i_{jk} = \bar{\Gamma}^i_{kj}$) we have

$$\omega^{pq} V^k_{,p} V^j_{,q} \bar{\Gamma}^i_{jk} = -\omega^{pq} V^k_{,q} V^j_{,p} \bar{\Gamma}^i_{jk}$$

so the previous equation simplifies to

$$\frac{\partial V^i}{\partial t} + V^s W^i_{,s} + W^s V^i_{,s} + 2 V^k W^j \bar{\Gamma}^i_{jk} = \omega^{pq} V^j_{,p} \Gamma^k_{qs} V^s \bar{\Gamma}^i_{jk} - V^k V^j \bar{\Gamma}^i_{jk}$$

$$- \omega^{pq} V^k_{,p} V^j \bar{\Gamma}^i_{jk} - \frac{1}{2} \omega^{pq} V^s_{,sq} V^i_{ps} - \frac{1}{2} \omega^{pq} V^i_{sq} \Gamma^i_{ps} V^j.$$

Thus the $O(\lambda)$ correction $W$ to the velocity field obeys a linear differential equation with source terms specified by the classical velocity field $V$, and the $O(\lambda)$ correction to the connection $\tilde{\Gamma}^i_{jk}$ only appears in one source term.
7 Vector fields and flows on the state space

Finally we come to the heart of the matter, we have to get vector fields to generate paths. Having motivated previous calculations by bimodules and inner products, we continue by specifying paths by positive maps and the KSGNS construction. However, there is one ‘obvious’ but bad choice which we must highlight: We could take a vector field $v$ and make it act directly on elements of the algebra, correcting to preserve hermitian elements, for example

$$\frac{da}{dt} = v(da) + v(da^*)^* . \tag{24}$$

Reasonable though this may seem, even considering simple cases like Examples 7.3 & 7.4 shows that though hermiticity of $a(t)$ is preserved by brute force in (24), positivity is definitely not.

**Example 7.1** In Example 3.1 the equation $\nabla_M(m) = 0$ simply means that $m \in C^\infty(\mathbb{R})$ is constant. Taking this constant to be 1 we get \( (mf, m) = f(\gamma(t)) \) for $f \in C^\infty(\mathbb{R}^n)$. This is exactly what we want for a state valued function of time given by evaluation at a point $\gamma(t) \in \mathbb{R}^n$.

Unfortunately things are not always so simple.

**Example 7.2** We continue from Example 3.1 and check whether the equation $\nabla_M(m) = 0$ really corresponds to the classical situation. We take a positive function on $C^\infty(\mathbb{R}^3)$ given by

$$\phi(t)(f) = \int_{\mathbb{R}^3} |m(t)|^2 f \, d^3x$$

for the usual Lebesgue measure on $\mathbb{R}^3$ and a time dependent rapidly decreasing function (or rather density) $m(t) \in L^2(\mathbb{R}^3, \mathbb{C})$. We use the connection $\nabla_M(m) = mb + K(dm) + \frac{\partial m}{\partial t}$ from Proposition 5.1 and set $b = 0$ (more on this later), and then for $m(t)$ we have $\frac{\partial m}{\partial t} = -K(dm)$, or for real $K$ (the case classically), $\frac{\partial |m|^2}{\partial t} = -K(d(|m|^2))$. To simplify the description we temporarily suppose that the vector field $K$ is constant in time, and then it gives an action of $(\mathbb{R}, +)$ on the manifold $X$ by the flow $F : \mathbb{R} \times X \to X$ given by $\frac{\partial F(t,x)}{\partial t} = K(F(t,x))$ (i.e. a tangent vector at $F(t,x)$).

We can solve the equation for $|m|^2(t)$ by setting $|m|^2(t)(x) = |m|^2(0)(F(-t,x))$. If at time 0 we have $|m|^2$ concentrated at the point $x_0$ then at time $t$ it is concentrated at $x_t$ where $F(-t,x_0) = x_0$, i.e. $x_t = F(t,x_0)$. Thus, as expected, we get the point where $|m|^2$ is concentrated moving with the flow generated by the vector field $K$. We have not mentioned delta functions as there is a little problem with the normalisation, to which we now turn.

Now in the connection above we suppose that $b$ may be nonzero and that $K$ may depend on time. If we wanted the equation $\nabla_M(m) = 0$ to give a time evolving positive function which preserved normalisation (i.e. $\phi(t)(1) = 1$) then we would require

$$\int_{\mathbb{R}^3} \left( |m(t)|^2 (b + b^*) + K^i \frac{\partial |m|^2}{\partial x^i} \right) \, d^3x = 0 ,$$
or using integration by parts,

\[ \int_{\mathbb{R}^3} |m(t)|^2 \left( b + b^* - K^i \cdot i \right) \, dx = 0. \]

Of course we can always set \( b + b^* = K^i \cdot i \) (the divergence of the vector field \( K \)) and then \( 7 \) in the classical case (not in general as \( K \) might not be a left module map) shows that the time evolution of \( K \) would not be changed for the geodesic.

In various noncommutative examples we might try similar modifications to preserve the normalisation of positive functions using \( \nabla_M(m) = 0 \) (or even introduce a nonzero right hand side). For example, the divergence of \( K \) makes perfect sense in a noncommutative context, as we simply apply a right covariant derivative and then an evaluation to \( K \). However, it is not at all obvious that it is necessary or desirable to do this in all cases. In the presence of particle creation or annihilation in quantum mechanics we might well have positive functions with varying normalisation. As an example related to geometry, but far beyond the scope of this paper, Hawking radiation \( 10 \) produces particles at the expense of the intrinsic energy of space-time - which is what the negative energy states disappearing into the black hole and reducing its mass effectively amount to. In the spirit of providing simple examples to show that the constructions work we shall proceed without worrying about preserving the normalisation.

**Example 7.3** As an example we take \( M_2(\mathbb{C}) \) with calculus as in Example 2.5. We take the right \( M_2(\mathbb{C}) \)-module \( N = M_2(\mathbb{C}) \) itself, with \( n = I_2 \) and inner product \( (p, q) = \text{trace}(pq^*) \). There is a central metric on the calculus given by \( g = e_1 \otimes e_1 - e_2 \otimes e_2 \) and this gives a gradient by \( \text{grad} \left( \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \right) = (\text{id} \otimes \text{ev}) (g \otimes d(a_k \otimes b_k)) \), which works out as

\[
\text{grad} \left( \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \right) = \begin{pmatrix} b_k & 0 \\ 0 & c_k \end{pmatrix} e_1 + \begin{pmatrix} c_k & d_k - a_k \\ 0 & -c_k \end{pmatrix} e_2.
\]

As before we take \( M = C^\infty(\mathbb{R}) \otimes N \) and then take the left connection on \( M \) as in Proposition 5.1 with \( b = 0 \) and \( K = -\text{grad} \left( \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \right) \). Now we solve \( \nabla_M(m) = 0 \), which is just \( \frac{dm}{dt} = \text{grad} \left( \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \right)(dm) \). This has solutions for the matrix \( m(t) \)

\[
\alpha I_2 + \beta e^{2k} \begin{pmatrix} a_k - d_k & 2b_k \\ 2c_k & d_k - a_k \end{pmatrix} + \gamma e^k \begin{pmatrix} c_k & a_k - d_k \\ 0 & c_k \end{pmatrix} + \delta e^k \begin{pmatrix} 0 & b_k \\ c_k & 0 \end{pmatrix}
\]

where \( k = (d_k - a_k)t \) and \( \alpha, \beta, \gamma, \delta \) are constants.

The pure states on \( M_2(\mathbb{C}) \) are given by \( 12 \)

\[
\phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = (\lambda \mu^*) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda^* & 0 \\ 0 & \mu^* \end{pmatrix} = |\lambda|^2 a + \lambda \mu^* b + \lambda^* \mu c + |\mu|^2 d \tag{25}
\]

for \( \lambda, \mu \in \mathbb{C} \) with \( |\lambda|^2 + |\mu|^2 = 1 \) chosen up to a multiple, so they are in 1-1 correspondence with \( \mathbb{C}P^1 \). If we set \( x + iy = \lambda \mu^* \) then the pure states can be written as

\[
\phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \left( \frac{1}{2} - s \right) a + (x + iy)b + (x - iy)c + \left( \frac{1}{2} + s \right)d \tag{26}
\]

17
where \((s, x, y) \in \mathbb{R}^3\) with \(x^2 + y^2 + s^2 = \frac{1}{4}\). The set of all normalised states is the points in the closed solid ball \(x^2 + y^2 + s^2 \leq \frac{1}{4}\). We can find \((s, x, y)\) from \(\phi\) by

\[
  s = \frac{1}{2\phi(I_2)} \phi\left(\begin{array}{c}
-1 \\
0
\end{array}\right), \quad x = \frac{1}{2\phi(I_2)} \phi\left(\begin{array}{c}
0 \\
1
\end{array}\right), \quad y = \frac{1}{2\phi(I_2)} \phi\left(\begin{array}{c}
0 \\
-1
\end{array}\right).
\]

Now \(\phi(r) = \langle m(t)r, \bar{m}(t) \rangle = \text{trace}(mrm^*)\) for \(k \in M_2(\mathbb{C})\) gives a time dependent positive function from \(M_2(\mathbb{C})\) to \(\mathbb{C}\). To visualise this we take the particular gradient vector field given by the height element \(\left(\begin{array}{c}
a \ b \\
0 \\
b \
0
\end{array}\right) = \left(\begin{array}{c}
-1/2 \\
1/2 \\
1/2 \\
1/2
\end{array}\right)\) and initial conditions specified by \(\alpha = 1\), \(\beta = 3\), \(\gamma = 1\) and \(\delta = -3\). Then \(\phi\) is a multiple \(2 + 4e^t + 27e^{2t} - 84e^{3t} + 90e^{4t}\) of the state illustrated in Figure 4. For these values \(v = 0\) and the plot is for \((s, x)\), and the pure state space intersection with that plane is given as a circle. The origin \((0, 0)\) corresponds to both \(t \to \infty\) and \(t \to -\infty\).

**Example 7.4** For another example of a vector field which is constant in time we take \(A = \mathbb{C}(\mathbb{Z}_n)\) with calculus as in Example 2.6. We use the vector field \(v = v_{+1} \kappa_{+1} + v_{-1} \kappa_{-1}\) for some \(v_{+1}, v_{-1} \in \mathbb{C}(\mathbb{Z}_n)\), and then from (24) for \(f \in \mathbb{C}(\mathbb{Z}_n)\)

\[
  v(df) = (v_{+1} \kappa_{+1} + v_{-1} \kappa_{-1})(e_{+1}(f - R_{-1}(f)) + e_{-1}(f - R_{+1}(f)))
  = v_{+1} (f - R_{-1}(f)) + v_{-1} (f - R_{+1}(f)).
\]

We use the values \(v_{+1} = i(\delta_0 + \delta_2 - \delta_1)\) and \(v_{-1} = -i(\delta_1 + \delta_0)\).

We take the right \(\mathbb{C}(\mathbb{Z}_n)\)-module \(N = \mathbb{C}(\mathbb{Z}_n)\) itself, with \(n = 1\) and inner product \(\langle p, \bar{q} \rangle = \frac{1}{3} \sum_i p(i)q(i)^*\), and set \(M = C^\infty(\mathbb{R}) \otimes N\) as before. We take the left connection
on $M$ as in Proposition 5.4 with $b = 0$ and $K = v$. Now we solve $\nabla_M(m) = 0$, which is just \( \frac{dm}{dt} = -v(\text{div}) \). This has solution

\[
m(t) = (c - 3ae^{2it})\delta_1 + (-ae^{2it} - be^{it} + c)\delta_2 + (ae^{2it} + be^{it} + c)\delta_0
\]

for constants $a, b, c$. For a specific case we set $a = 1$, $b = 2$ and $c = 3$. In Figure 5 (a) the three graphs represent the state evolving in time, with the lower graph being $\phi(\delta_1)/\phi(1)$, the middle being $\phi(\delta_1 + \delta_2)/\phi(1)$ and the upper $\phi(\delta_1 + \delta_2 + \delta_0)/\phi(1) = 1$. Figure 5 (b) gives the multiple $\phi(1)$ of the state. All the values are periodic.

Now we look at an example of geodesic flow.

**Example 7.5** Now we consider the flow due to the non-constant vector fields in Example 5.6 and Figure 3. Recall that for this example we solve $\nabla_M(\sigma_M) = 0$, which in the classical case as given in Example 5.4 would give geodesics.

Now for this vector field we solve $\nabla_M(m) = 0$ as in Example 7.4. The initial state $\phi(0)$ is given by $\phi(0)(\delta_0) = \phi(0)(\delta_2) = \frac{1}{3}$ and $\phi(0)(\delta_1) = 0$, and the state $\phi(t)/\phi(t)(1)$ and the value $\phi(t)(1)$ are plotted in Figure 6. In Figure 6 (a) the three graphs represent the state evolving in time, with the lower graph being $\phi(\delta_1)/\phi(1)$, the middle being $\phi(\delta_1 + \delta_2)/\phi(1)$ and the upper $\phi(\delta_1 + \delta_2 + \delta_0)/\phi(1) = 1$. Note that the state seems to be evolving towards a limiting translation invariant state where all points have weight $\frac{1}{3}$.

Note that in Example 7.3 specifying the gradient flow with velocity $-\text{grad} (a \frac{\partial h}{\partial t}, b \frac{\partial h}{\partial t}, c \frac{\partial h}{\partial t})$ does not specify a unique flow on the space of states. The state at time zero lives in a 3D real space, but we have 4 complex values $\alpha, \beta, \gamma, \delta$ (up to a multiple) to choose to specify $m(t)$, so in general $m(t)$ contains additional information beyond the initial state, and that will alter the path in future time. From the point of view of quantum theory this should be no surprise – the Schrödinger wave function $\psi(x, y, z)$ contains much more information than just $|\psi(x, y, z)|^2 \in L^1(\mathbb{R}^3)$, which is sufficient to determine the initial state on the algebra of functions on $\mathbb{R}^3$. (Of course we could extend the algebra to include momentum etc., but that is another story.) We consider another, smaller, right module $N$ for $M_2(\mathbb{C})$, and then set $M = C^\infty(\mathbb{R}) \otimes N$. We then consider the equation $\nabla_M(\sigma_M) = 0$ and the corresponding flow, although this has no direct classical geodesic justification for this module.
Example 7.6 For the algebra \( A = \text{M}_2(\mathbb{C}) \) set the right \( A \)-module \( N = \text{Row}^2(\mathbb{C}) \) (the two dimensional row vectors), and then the inner product \( \langle n', n \rangle = n' n^* \in \mathbb{C} \) gives the pure states, as setting \( n = (\lambda, \mu) \) (if non-zero) gives \( \phi(r) = \langle nr, n \rangle \) for \( r \in \text{M}_2(\mathbb{C}) \) in \([23]\). Set \( M = C^\infty(\mathbb{R}) \otimes N \), and then the possible bimodule maps \( \sigma_M : M \otimes_A \Omega^1_A \rightarrow \Omega^1_R \otimes C^\infty(\mathbb{R}) \) are given by

\[
\sigma_M(w \otimes s_i) = dt \otimes Q_i w
\]

for \( w \in \text{Row}^2(\mathbb{C}) \) and \( Q_i \in C^\infty(\mathbb{R}) \). Then for \( f \in C^\infty(\mathbb{R}) \) we can take

\[
\nabla_M(f w) = \frac{df}{dt} dt \otimes w - dt f \otimes (Q_1 E_{12} + Q_2 E_{21} + Q_0 I_2)
\]

for some \( Q_0 \in C^\infty(\mathbb{R}) \). On \( \Omega^1_A \) we use the connection as in Example 5.3 with \( \nabla_A(s^i) = 0 \). We check the braid relation \([10]\) by calculating

\[
(\sigma_R \otimes \text{id})(\sigma_M \otimes \text{id})(f w \otimes s_i \otimes s_j) = (\sigma_R \otimes \text{id})(\text{id} \otimes \sigma_M)(dt \otimes Q_i f w \otimes s_j)
\]

\[
= dt \otimes dt \otimes Q_j Q_i f w
\]

and

\[
(\text{id} \otimes \sigma_M)(\sigma_M \otimes \text{id})(\sigma_M \otimes \text{id})(f w \otimes s_i \otimes s_j) = (\text{id} \otimes \sigma_M)(\sigma_M \otimes \text{id})(f w \otimes s_j \otimes s_i)
\]

\[
= dt \otimes dt \otimes Q_j Q_i f w.
\]

Now Proposition 2.8 shows that \( \nabla(\sigma_M) \) is a bimodule map, so to show that \( \nabla(\sigma_M) \) vanishes it is only necessary to show that it vanishes on generators. Now

\[
(\text{id} \otimes \sigma_M)\nabla_M \otimes \Omega^1_A(w \otimes s_i) = -dt \otimes \sigma_M(w(Q_1 E_{12} + Q_2 E_{21} + Q_0 I_2) \otimes s_i)
\]

\[
= -dt \otimes dt \otimes w(Q_1 E_{12} + Q_2 E_{21} + Q_0 I_2)Q_i .
\]

and

\[
\nabla_{\Omega^1_R \otimes M}\sigma_M(w \otimes s_i) = \nabla_{\Omega^1_R \otimes M}(Q_i dt \otimes w)
\]

\[
= \frac{dQ_i}{dt} dt \otimes dt \otimes w - dt \otimes Q_i dt \otimes w(Q_1 E_{12} + Q_2 E_{21} + Q_0 I_2)
\]

so \( \nabla(\sigma_M) = 0 \) implies that \( Q_1 \) and \( Q_2 \) are constant and \( Q_0 \) is arbitrary. Now set \( m = (\lambda, \mu)(t) \) and then \( \nabla_M(m) = 0 \) gives

\[
\left( \frac{d\lambda}{dt}, \frac{d\mu}{dt} \right) = (Q_0 \lambda + Q_2 \mu, Q_0 \mu + Q_1 \lambda)
\]

Putting \( z = \frac{\lambda}{\mu} \) we get \( \frac{dz}{dt} = Q_2 - Q_1 z^2 \). In terms of the action of \( SL_2(\mathbb{C}) \) on the Riemann sphere \( \mathbb{C}_\infty \) of pure states by Möbius transformations

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \end{pmatrix} = \begin{pmatrix} a z + b \\ c z + d \end{pmatrix}
\]

the time action is given by the one parameter group given by \( \exp(t \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix}) \).
References

[1] D. Applebaum, Stochastic Evolution of Yang-Mills Connections on the Noncommutative Two-Torus, Letters in Mathematical Physics 16 (1988) 93-99.

[2] P. Aschieri and P. Schupp, Vector fields on Quantum Groups, Int. J. Mod. Phys. A, 11 (1996) 1077-1100

[3] E.J. Beggs and T. Brzeziński, The Serre spectral sequence of a noncommutative fibration for de Rham cohomology, Acta Math. 195 (2005) 155–196

[4] E.J. Beggs and S. Majid, *-compatible connections in noncommutative Riemannian geometry, J. Geom. Phys. 61 (2011) 95–124

[5] E.J. Beggs and S. Majid, Quantum Riemannian Geometry, draft manuscript (in consideration by a publisher).

[6] E.J. Beggs and S. Majid, Poisson-Riemannian geometry, J. Geom. Phys. 114 (2017) 450–491

[7] E.J. Beggs and S. Majid, Semiclassical differential structures, Pac. J. Math. 224 (2006) 1–44

[8] E.J. Beggs and S. Majid, Quantum Bianchi identities via DG categories, J. Geom. Phys. 124 (2018) 350–370

[9] B. Blackadar, K-theory for operator algebras, MSRI Publications, Berkeley, 1986.

[10] A. Borowiec, Vector fields and differential operators: noncommutative case, Czech. J. Phys. 47 (1997) 1093–1100

[11] K. Bresser, F. Müller-Hoissen, A. Dimakis and A. Sitarz, Noncommutative geometry of finite groups. J. Phys. A, 29 (1996) 2705–2735

[12] M.-D. Choi, Completely positive linear maps on complex matrices, Lin. Algebra Applic. 10 (1975) 285–290

[13] A. Connes, Noncommutative Geometry, Academic Press, Inc., San Diego, CA, 1994

[14] A. Connes and N. Higson, Déformations, morphismes asymptotiques et K-théorie bivariante, C.R. Acad. Sci. Paris Sér. I Math. 311 (1990) 101–106

[15] M. Dădărlat, Shape theory and asymptotic morphisms for $C^*$-algebras, Duke Math. J. 73 (1994) 687–711

[16] M. Dubois-Violette and T. Masson, On the first-order operators in bimodules, Lett. Math. Phys. 37 (1996) 467–474

[17] M. Dubois-Violette and P.W. Michor, Connections on central bimodules in noncommutative differential geometry, J. Geom. Phys. 20 (1996) 218–232

[18] G. Fiore and J. Madore, Leibniz rules and reality conditions, Eur. Phys. J. C Part. Fields 17 (2000) 359–366

[19] S.W. Hawking, Particle creation by black holes, Commun. Math. Phys. 43, 199–220 (1975)
[20] E. Hawkins, Noncommutative rigidity, Comm. Math. Phys. 246 (2004) 211–235

[21] P. Jara and D. Llena, Lie bracket of vector fields in noncommutative geometry, Czech. J. Phys. 53 (2003) 743–758

[22] E.C. Lance, Hilbert $C^*$-modules, A toolkit for operator algebraists, LMS. Lecture Note Series 210, C.U.P. 1995

[23] J. Madore, An introduction to noncommutative differential geometry and its physical applications, LMS Lecture Note Series, 257, C.U.P. 1999.

[24] V. Manuilov and K. Thomsen, Shape theory and extensions of $C^*$-algebras, J. London Math. Soc. 84 (2011) 183–203

[25] J. Mourad, Linear connections in noncommutative geometry, Class. Quant. Grav. 12 (1995) 965–974