Spontaneous Breakdown of the Lorentz Invariance

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We re-examine three-dimensional gauge theory with a Chern-Simons term in which the Lorentz invariance is spontaneously broken by dynamical generation of a magnetic field. A non-vanishing magnetic field leads, through the Nambu-Goldstone theorem, to the decrease of zero-point energies of photons, which accounts for a major part of the mechanism. The asymmetric spectral flow plays an important role. The instability in pure Chern-Simons theory is also noted.

In the previous paper [1] we have shown that in a class of three-dimensional gauge theories described by

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \sum_a \frac{1}{2} \left[ \bar{\psi}_a , (\gamma^\mu_i i \partial_\mu + q_a A_\mu) - m_a \psi_a \right] , \]

(1)

the Lorentz invariance is spontaneously broken by dynamical generation of a magnetic field \( \hat{B} \). In this paper we shall give additional arguments in support of this conclusion.

In (1) \( \psi_a \) is a two-component Dirac spinor, and Dirac matrices are characterized by their signature \( \eta_a = \frac{1}{2} \text{Tr} \gamma_a^0 \gamma_a^1 = \pm 1 \). In terms of \( \gamma_a^\mu = - \gamma_a^{\mu*} \), the fermion part of the Lagrangian takes the same form as the original one except for the change in the sign of the mass terms \( m_a > 0 \). Hence \( \eta_a = +, m_a > 0 \) is equivalent to \( \eta_a = -, m_a < 0 \). Both descriptions are useful. The sign is called “chirality”. The model is invariant under charge conjugation so that one can take \( q_a > 0 \) without loss of generality.

In the perturbative vacuum \( \langle F_{12}(x) \rangle = 0 \). In the previous paper we have shown that in models of a variational ground state, in which \( \langle F_{12}(x) \rangle = - \hat{B} \neq 0 \), has a lower energy density than the perturbative vacuum, and therefore the Lorentz invariance is spontaneously broken.

In the variational ground state the energy spectrum of Dirac particles is characterized by Landau levels:

\[ E = \begin{cases} + \epsilon(\eta B) \cdot \omega_n & (n \geq 0) \\ - \epsilon(\eta B) \cdot \omega_n & (n \geq 1) \end{cases} \]

(2)

where \( \omega_n = (m^2_a + 2nq_a|B|)^{1/2} \). There is asymmetry in the \( n=0 \) modes (zero modes). They exist in either positive or negative energy states, or in other words, only for either particles or anti-particles [2]. We have considered variational ground states in which these lowest Landau levels \( n=0 \) are either empty or completely filled. Accordingly a filling factor \( \nu_a = 0 \) or 1 is assigned. A variational ground state is denoted as \( \Psi_{g.a.}(\hat{B}, \{\nu_a\}) \).

We fix Dirac matrices \( \gamma_a^\mu \) for the moment. To evaluate physical quantities one may continuously change the value of the fermion mass \( m_a \) from positive to negative. Then except for zero-modes \( n=0 \) in the spectrum (2) all positive (negative) frequencies remain positive (negative). However, for \( \eta B > 0 \) instance, the positive frequency zero-modes \( E = \omega_0 = m_a \) become negative zero-modes. There appears crossing in the spectrum. (See Fig. 1.) Yet physical quantities must be continuous functions of \( m_a \).

Suppose that \( \eta_a = + \) and \( \nu_a = 0 \). In expanding the Dirac field operator \( \psi_a(x) \) in terms of Landau level eigenstates with energy eigenvalues (2), annihilation operators \( \alpha_{np} \) of particles (creation operators \( \beta^\dagger_{np} \) of anti-particles) are associated with positive (negative) frequency eigenstates. (Here \( p \) is an additional index to

Fig. 1. The energy spectrum as a function of a mass \( m \), given by Eq. (2) for a positive \( \eta B \) in arbitrary units. The \( n=0 \) mode exhibits crossing in the spectral flow.
specify eigenstates.) The fact that \( n = 0 \) positive frequency becomes negative frequency implies that the annihilation operator \( a_\alpha \) is transformed into the creation operator \( b_\alpha^\dagger \) under the change of \( m_\alpha \). Hence
\[
a_\alpha^\dagger a_\alpha \rightarrow b_\alpha^\dagger b_\alpha = 1 - b_\alpha^\dagger b_\alpha .
\]
In other words an empty state \( \nu_\alpha = 0 \) with positive \( m_\alpha \) becomes a completely filled state \( \nu_\alpha = 1 \) with negative \( m_\alpha \).

Since fermions with \( (\eta_\alpha, m_\alpha < 0) \) are equivalent to those with \( (\eta_\alpha, m_\alpha \) \), a continuous change \( m_\alpha \rightarrow -m_\alpha \) results in the transformation of \( (\eta_\alpha, \nu_\alpha, |m_\alpha|) \rightarrow (\eta_\alpha, 1 - \nu_\alpha, |m_\alpha|) \). Therefore any physical quantities, \( R \), must satisfy
\[
R(\eta_\alpha, \nu_\alpha, m_\alpha^2) = R(-\eta_\alpha, 1 - \nu_\alpha, m_\alpha^2) .
\]
(3)

The charge density \( \langle j^0 \rangle \equiv J^0 \) is a physical quantity. It has been shown that \( J^0 \) is non-vanishing in the presence of a magnetic field [1,3]:
\[
J^0(x) = \frac{1}{2\pi} \sum_a \eta_\alpha q_\alpha^2 (\nu_\alpha - \frac{1}{2}) \cdot B ,
\]
(4)
which satisfies (3). The factor \( \eta_\alpha (\nu_\alpha - \frac{1}{2}) \) reflects the asymmetric spectral flow.

Combined with the Euler equation, Eq. (4) leads to a consistency condition for having \( B \neq 0 \):
\[
\kappa = \frac{1}{2\pi} \sum_a \eta_\alpha q_\alpha^2 (\nu_\alpha - \frac{1}{2}) .
\]
(5)

With a given bare Chern-Simons coefficient, filling factors \( \{\nu_\alpha\} \) are not arbitrary. It is a necessary condition for having \( B \neq 0 \). It is the purpose of this paper to show that in a wide class of models the condition (5) also serves as a sufficient condition for having \( B \neq 0 \).

Let us quote some of the results in ref. [1] relevant for our discussion below.

To find the difference, \( \Delta \mathcal{E} = \mathcal{E}_{g.s.} - \mathcal{E}_{p.v.} \), in the energy densities of the variational ground state and perturbative vacuum, we split the gauge coupling \( q_\alpha A_\mu \) into two parts, \( q_\alpha A_\mu^{(0)} + \alpha q_\alpha A_\mu^{(1)} \), where \( A_\mu^{(0)} \) corresponds to a dynamically generated \( B \). The auxiliary parameter \( \alpha \), ranging from 0 to 1, has been introduced in the coupling of fluctuations, defining \( \Delta \mathcal{E}(\alpha) \). The primary interest is \( \Delta \mathcal{E}(1) \).

It is easy to see that \( \Delta \mathcal{E}(0) = \frac{1}{2} B^2 + \Delta \mathcal{E}_f \) where the first term is the Maxwell energy, while the second term represents the shift in fermion zero-point energies due to a constant magnetic field \( B \). In the massless fermion limit \( \Delta \mathcal{E}_f \) is positive and is \( O(|B|^3/2) \). At \( \alpha = 0 \) there arises no change in the photon spectrum.

\( \Delta \mathcal{E}(1) - \Delta \mathcal{E}(0) \) is found by adiabatically switching \( \alpha \) on from 0 to 1. It can be expressed in terms of photon propagators:
\[
\Delta \mathcal{E}(1) - \Delta \mathcal{E}(0) = i \int_0^1 \frac{d\alpha}{\alpha} \int \frac{d^3p}{(2\pi)^3} \text{tr} D_0^{-1}(p) \left\{ D(p)_{g.s.} - D(p)_{p.v.} \right\}.
\]
(6)

Here \( D_\mu^\nu(p) \) is the bare photon propagator, the same in both variational ground state and perturbative vacuum. \( D_\mu^\nu(p)_{g.s.} \) and \( D_\mu^\nu(p)_{p.v.} \) are the full photon propagators with a given \( \alpha \) in the variational ground state and perturbative vacuum, respectively. Notice that Eq. (6) is exact.

There are at most three invariant functions \( \Pi_k(p_0^2, \vec{p}^2) \) \( (k = 1 \sim 3) \) to parametrize the sum of one-particle irreducible diagrams \( \Gamma = D_0^{-1} - D^{-1} \) [5,6].

\[
\begin{align*}
\Gamma & = (p_\mu p_\nu - p^2 g_\mu^\nu) \Pi_0 + \frac{i}{\alpha} \epsilon_\mu^\nu \rho_\rho \Pi_1 \\
& + (1 - \delta^{\mu 0})(1 - \delta^{\nu 0})(p_\mu p_\nu - \vec{p}^2 \delta^{\mu \nu})(\Pi_2 - \Pi_0) .
\end{align*}
\]
(7)

Then det \( D^{-1} = (p^2)^2 \cdot S(p) \) where
\[
S(p) = (1 + \Pi_0)(p^2 + p_0^2 \Pi_0 - \vec{p}^2 \Pi_2) - (\kappa - \Pi_1)^2 .
\]
(8)

The photon spectrum is determined by \( S(p) = 0 \).

So far all expressions are exact. At this stage we introduce an approximation in which all \( \Pi_k \)'s are evaluated to \( O(\alpha^2) \). In this approximation the \( \alpha \) integral in (6) can be readily performed, yielding
\[
\Delta \mathcal{E}(1) - \Delta \mathcal{E}(0) = -i \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \ln \frac{S(p)_{g.s.}}{S(p)_{p.v.}}.
\]
(9)

The one-loop approximation has been adopted to \( \Pi_k(p) \)'s, but not to \( \Delta \mathcal{E} \).

\( -\Pi_1(p=0) \) is the induced Chern-Simons coefficient. It was evaluated by explicit computations and by employing the Nambu-Goldstone theorem. One-loop computations in the variational ground state have yielded [1]
\[
\Pi_1(0)_{g.s.} = \frac{1}{2\pi} \sum_a \eta_\alpha q_\alpha^2 (\nu_\alpha - \frac{1}{2})
\]
(10)
which satisfies (3). The consistency condition (5) may be written as
\[
\kappa = \Pi_1(0)_{g.s.} .
\]
(11)

In the perturbative vacuum [3] one finds \( \Pi_1(0)_{p.v.} = -\sum_a \eta_\alpha q_\alpha^2/4\pi \).

On the other hand the Nambu-Goldstone theorem associated with the spontaneous breaking of the Lorentz invariance due to a non-vanishing \(- F_{12}(0) \) implies that a photon becomes the Nambu-Goldstone boson in the sense that its spectrum satisfies [7]
\[
\lim_{\vec{p} \rightarrow 0} p_0(\vec{p}) = 0 .
\]
(12)
Suppose that fermion masses are small, but finite. Then Π₀(0), Π₂(0) ≠ 0. Since the photon spectrum is given by \( S(p) = 0 \) with (8), the Nambu-Goldstone boson nature of a photon implies the relation (11) to all order. In other words the necessary condition (5) for having \( B ≠ 0 \) is also a consequence of \( B ≠ 0 \).

Now consider a chirally symmetric model consisting of \( N_f \) pairs of \( \eta_a = + \) and \( \eta_a = - \) fermions with the same mass \( m_a > 0 \) and charge \( q_a > 0 \). In this model one has \( \sum_a \eta_a q_a^2 = 0 \), and \( \Pi_l(p)_{p.v.} = 0 \) exactly in the perturbative vacuum. We suppose that the bare Chern-Simons coefficient and filling factors \( \{\nu_a\} \) of the variational ground state satisfy the condition (5):

\[ \kappa = \sum_a \eta_a \nu_a q_a^2 / 2\pi. \]

For small \( |B| \), in the limit \( m_a \rightarrow 0 \),

\[ \Pi_k(p)_{p.v.} \sim \Pi_k(p)_{g.s.} \quad (k = 0, 2) \]

\[ = \frac{\eta_a q_a^2}{16\sqrt{-\vec{p}^2} + O(B^2)} \]

\[ \Pi_1(p)_{p.v.} = 0 \]

\[ \Pi_1(p)_{g.s.} = \frac{\eta_a \nu_a q_a^2}{\pi(2 - \vec{p}^2 / l_{ave}^2)} + O(B^2) \]

where \( l_{ave}^2 = q_a |B| \).

The shift in the energy density \( \Delta \mathcal{E}(1) \) is found by inserting (13) into (9). The result is

\[ \Delta \mathcal{E}(1) = -\frac{\sum \eta_a \nu_a q_a^2}{2\pi^4} \cdot \tan^{-1} \frac{1}{2} \sum \eta_a \nu_a q_a^2 \cdot |B| + O(|B|^{3/2}) \]

If the coefficient of the linear term (\( \propto |B| \)) is negative, the energy density is minimized at \( B ≠ 0 \), henceforth the Lorentz invariance is spontaneously broken. In the previous paper we have considered a special case with all \( q_a = e \), in which the coefficient is negative. A wide class of models yield a negative coefficient.

The decrease of the energy density by a non-vanishing \( B \) is understood in terms of zero-point energies. Return to (8) and (9). If \( \Pi_0 \) and \( \Pi_2 \) were constant, the \( p_0 \)-integral in (9) would give the difference between the sums of zero-point energies of photons in the variational ground state and perturbative vacuum. In perturbation theory a photon is topologically massive with a mass given by the bare Chern-Simons coefficient \( \kappa \). In the variational ground state, as explained above, \( B ≠ 0 \) implies that a photon becomes the Nambu-Goldstone boson satisfying (12). As a momentum \( |\vec{p}| \) increases, all \( \Pi_l \)'s approach to zero. The crossover takes place, as deduced from (13), around \( |\vec{p}| = l_{ave}^{-1} \) where

\[ \frac{1}{l_{ave}^2} = \left| \frac{\sum \eta_a \nu_a q_a^2 / l_{ave}^2}{\sum \eta_a \nu_a q_a^2} \right|^2 = \left| \frac{\sum \eta_a \nu_a q_a^2}{\sum \eta_a \nu_a q_a^2} \right|^2 \cdot |B|, \]

provided that the denominator is non-vanishing, or \( \kappa ≠ 0 \).

Hence in the variational ground state we have, as a rough estimate, \( p_0 \sim |\vec{p}| \) for \( |\vec{p}| < l_{ave}^{-1} \), and \( p_0 \sim (|\vec{p}|^2 + \kappa^2)^{1/2} \) for \( |\vec{p}| > l_{ave}^{-1} \). The shift in zero-point energies is thus

\[ \Delta \mathcal{E} \sim \int_{l_{ave}^{-1}}^{l_{ave}^{-1}} \frac{dp^2}{(2\pi)^2} \cdot \frac{1}{2} \left\{ \sqrt{\vec{p}^2} - \sqrt{\vec{p}^2 + \kappa^2} \right\} \]

\[ = \frac{1}{8\pi l_{ave}^2} \left[ \kappa \right] + O \left( \frac{1}{l_{ave}^2} \right) \]

(16)

Comparing (14) and (16), one finds that in a typical model the shift in zero-point energies of photons explains about 50% of the effect.

What we are observing here is the following self-consistent cycle of arguments. Suppose that a photon is topologically massive in perturbation theory, i.e. \( \kappa ≠ 0 \). (i) \( B ≠ 0 \) implies the spontaneous breakdown of the Lorentz invariance. (ii) The photon is the Nambu-Goldstone boson associated with the spontaneous symmetry breaking. In particular, the energy spectrum of photons is significantly lowered for a momentum \( |\vec{p}| < l_{ave}^{-1} \) where \( l_{ave}^{-1} \propto |B|^{1/2} \). (iii) Then zero-point energies of photons are decreased by an amount \( \propto |B| \) for small \( |B| \). (iv) Hence the energy is minimized at \( B ≠ 0 \).

With this perception at hand we recognize that the condition (5) or (11) is not merely a necessary condition for having \( B ≠ 0 \). It is also a sufficient condition for lowering the energy density, since it represents the Nambu-Goldstone boson nature of photons.

A few comments are in order. First, a photon, as the Nambu-Goldstone boson, has only one degree of freedom, whereas there are two broken Lorentz-boost generators, \( L_k \) \( k = 1, 2 \). The mismatch in the numbers is traced back to the facts that \( L_k \) is the first moment of the energy-momentum tensor, and that a photon is a vector. A photon couples to both generators. Secondly, \( L_k \) does not commute with the Hamiltonian. The Lorentz-boosted variational ground state, \( \Psi_{g.s.} \), does not have the same energy density as the original one, \( \Psi_{g.s.} \). Also in the boosted state the current density \( J^k \), as well as \( J^0 \), is non-vanishing. \( \Psi_{g.s.} \) is expected to be a stable state among various states with a constant non-vanishing current density.

One can generalize the above analysis to the case of pure Chern-Simons gauge fields. It is most convenient to consider

\[ \mathcal{L} = -\frac{g}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{2} \varepsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu A_\rho + \cdots \]
and take the limit \( g \to 0 \) at the end. The formula (9) still holds with the replacement

\[
S(p) = (g + \Pi_0)(gp^2 + \rho_0^2\Pi_0 - p^2\Pi_2) - (\kappa - \Pi_1)^2.
\]  

(18)

The \( p \)-integral in (9) can be easily done as before. In (14) the coefficient of the linear term is multiplied by \( g^{-1} \), while the Maxwell energy becomes \( \frac{1}{2}gB^2 \). Therefore the minimum of the energy density is located around \( |B_{\text{min}}| \sim g^{-2} \sum \eta_a \nu_a q_a^3 \). This also follows from the scaling argument. If one redefines, in (17), \( A' = g^{1/2}A \) and \( q_a' = g^{-1/2}q_a \), then \( |B'_{\text{min}}| \sim q^3 \), or equivalently \( |B_{\text{min}}| \sim g^{-2}q^3 \).

The implication to pure Chern-Simons theory is rather puzzling. As \( g \to 0 \), \( |B_{\text{min}}| \to \infty \). The theory is totally unstable. At \( g=0 \) with massless fermions the Lagrangian does not contain any dimensional parameters after redefining \( A' = |\kappa|^{-1/2}A \). Hence the minimum must occur either at \( B'=0 \) or at \( |B'| = \infty \). Our analysis indicates the latter.

In the literature it is often said that the effect of gauge interactions in pure Chern-Simons theory is to merely change the statistics of matter fields, making them anyons. As a special case let us consider a model consisting of only one pair of fermions with the same charge \( e \) and mass, but with opposite chirality. If \( \kappa = e^2/2\pi \), the consistency condition (5) is satisfied. According to the folklore [8] Dirac particles are supposed to be transformed to free hard-core bosons with spin either 0 or 1.

Our analysis indicates something more drastic, i.e. the gauge interactions induce the instability. However, a reservation must be made that the \( g \to 0 \) limit corresponds to a strong coupling limit so that the analysis may need refinement.

In this paper we have shown that the spontaneous breaking of the Lorentz invariance in a wide class of models is not only the source, but also a consequence, of the Nambu-Goldstone theorem. We shall come back for more details in separate publications.

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