The rigorous solution of the scattering problem for a finite cone embedded in a dielectric sphere surrounded by the dielectric medium

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Abstract

Wave scattering from a finite hollow cone with perfectly conducting boundaries embedded in a dielectric sphere is considered. The structure is excited axially symmetrically by the radial electric dipole. The scattering problem is formulated in the spherical coordinate system as the boundary value problem for the Helmholtz equation. The diffracted field is given by expansion in the series of eigenfunctions. Owing to the enforcement of the conditions of continuity together with the orthogonality properties of the Legendre functions the diffraction problem is reduced to infinite system of linear algebraic equations (ISLAE) of the first kind. The usage of the analytical regularization approach transforms the ISLAE of the first kind to the second one and allows one to justify the truncation method for obtaining the numerical solution in the required class of sequences. This system is proved to be regularized by a pair of operators, which consist of the convolution type operator and the corresponding inverse one. The inverse operator is found analytically using the factorization technique. The numerical examples are presented. The static and the low-frequency approximations as well as the transition to the limiting case when the cone degenerates into the disc are considered.

1 | INTRODUCTION

The analysis of wave diffraction problems from a penetrable sphere with different insets has attracted attention of scientists due to its numerous applications in radioscience, optics and nanotechnologies. One of the most applicable geometries for such an analysis employs layered dielectric spheres. The above mentioned geometries allow for obtaining the solutions of wave diffraction problem in the analytical form. To this class of the scatterers belongs also the dielectric hemisphere placed on the perfectly conducting plane [1, 2], as well as the dielectric spherical sectors formed by the semi-infinite perfectly conducting conical and wedge insertions [3–5].

The wide class of the spherical scatterers that allows for the rigorous analysis is formed by the metallic sphere with a circular hole embedded in a dielectric sphere and vice versa. The effective solution for these problems is obtained in [6, 7] using the analytical regularization technique which is based on the Abel integral transformation.

The above mentioned spherical structures are often used for rarefaction of the frequency spectra and the increase of the quality factor of the resonators [8–11], as well as for modelling of the field distribution of antennas and the scattering properties of the nanoparticles in the modern physics [12–15].

Here, we consider the scattering from a finite hollow cone with perfectly conducting boundaries totally embedded in a dielectric sphere. The cone is supposed to be with an infinitely thin boundary and forms in the spherical volume two different conical regions which are joint with the spherical region. Therefore, we encounter the problem of wave diffraction from the bifurcated dielectric sphere. The proposed model allows for an analysis of the complicated scattering processes between a finite cone and dielectric boundary which, as far as we know, has not been studied previously. From this point of view, the considered scatterer is of a great physical interest for the study...
of the influence of the insets surface singularities on the spherical resonators characteristics. This structure, depending on the geometrical parameters, can also be applied for the analysis of radiation through concave dielectric surface from the conical horn antenna, which is important for media diagnostics as well as for modelling the radiation patterns of the conical, dipole and disc antennas that are placed in the layered media.

In the present paper, we study some scattering phenomena in sphere-conical structures. Today there are many possibilities to study this problem using modern software packages. Our aim here is to discuss this problem on a rigorous level. For this purpose, we simplify the problem and consider the axially symmetric TM excitation of the sphere-conical structure produced by the radial electric dipole. To find the solution we apply the analytical regularization procedure that we have used earlier for the rigorous analysis of the conical scatterers in homogeneous media [16–21].

Our analysis includes the mode matching, reduction of the problem to the infinite system of linear algebraic equations (ISLAE) of the first kind, inversion of its singular part, and finally, formulation of the problem in terms of the second kind ISLAE. Note, we have represented the short analysis of the above mentioned problem for the first time in [22]. Here we return to this problem since it is of importance for the modern diffraction theory. We study the left-side and right-side regularization procedures, the transition to the disc insets and to the low-frequency approximation. We also discuss resonance excitation of the spherical resonators with the conical insets, radiation from the horn antenna into the concave dielectric, and the transition to the problem solution if the multi-layered medium surrounds a dielectric sphere.

In [23], the problem of wave diffraction by a finite cone in metallic spherical resonator was analysed by the Wiener-Hopf technique without the numerical analysis. The method of analytical regularization, which was applied for the analysis of wave diffraction from waveguide discontinuities, is shown in [24]. In [25], the Riemann-Hilbert method was used for the wave diffraction analysis of the radially slotted semi-infinite cone. The survey of the analytical regularization methods is considered in [26].

2 | BASIC EQUATIONS OF THE PROBLEM

In this section we reduce the problem to the ISLAE using the method of separation of variables, eigen function series for field representation, and the mode matching technique.

2.1 | Statement of the problem

Let us consider a homogeneous and isotropic dielectric sphere $S$ with the relative scalar permittivity $\varepsilon_2$, and permeability $\mu_2$, which is surrounded by the homogeneous isotropic medium with the relative scalar permittivity $\varepsilon_1$, and permeability $\mu_1$. Spherical coordinates $(r, \theta, \varphi)$ are centred at the sphere origin. The radius of the sphere $S$ is denoted as $a$. Let the finite circular hollow cone $Q$ formed by the infinitely thin surface with perfectly conducting boundaries be totally embedded in a dielectric sphere $S$, where $Q : \{r \in (0, c), \ c < a; \ \theta = \gamma; \ \varphi \in [0, 2\pi]\}$, $\gamma$ is the conical opening angle, $c$ is the radial coordinate of its circular rib (see Figure 1).

![FIGURE 1] Geometry scheme

Let $Q$ be excited by the transverse magnetic (TM) wave with respect to the coordinate $r$, produced by radial electric dipole which is located on the axis of symmetry at $P(r_0, \theta_0)$ with $r = r_0, \theta_0 = 0$. Time factor is assumed to have harmonic variations $e^{-jwt}$ and is suppressed throughout this paper. The problem is to find the distribution of the field components established in the presence of the scattering structure. In view of the axial symmetry of the problem, the field components do not depend on $\varphi$ with $E_r, E_\theta, H_\varphi \neq 0$. The problem is formulated in terms of Debye scalar potential $U = U(r, \theta)$ that satisfies the Helmholtz equation. The components of the field are expressed as follows:

$$
E_r = -(r \sin \theta)^{-1} \partial_\theta (\sin \theta \partial_\theta U_q),
$$
$$
E_\theta = r^{-1} \partial_\varphi (r U_q), \ H_\varphi = iomega \partial_\varphi U_q,
$$

where $q = 1, 2$ denotes the number of the medium.

In view of Equation (1), the wave diffraction problem is reduced to the boundary value problem proceeding from the Helmholtz equations

$$
\Delta U_q + k_q^2 U_q = 0,
$$

where

$$
\Delta = \partial_r + 2r^{-1} \partial_r + (r^2 \sin \theta)^{-1} \partial_\theta (\sin \theta \partial_\theta)
$$
with boundary conditions

\[
(U \cdot \nabla \theta)(\sin \theta \partial_\theta U_i^j) = 0, \quad r \in (0, c), \quad \theta = \gamma \pm 0, \quad \theta = \pi \pm 0, \quad \theta = \frac{\pi}{2} \pm 0, \quad \theta = \frac{3\pi}{2} \pm 0,
\]

\[
\partial_\theta (r U_i^j) = \partial_\theta (r U_n^j), \quad r = a, \theta \in [0, \pi],
\]

\[
e^{i\nu_\theta} \partial_\theta U_i^j = e^{i\nu_\theta} U_n^j, \quad r = a, \theta \in [0, \pi],
\]

Here, \( k_0 = k_0 \sqrt{\varepsilon_0 m_0} \) is the wave number for each medium, \( k_q = k_q + ik_q, k_q > 0, k_q \geq 0, k_0 = \omega \varepsilon_0 \mu_0; \varepsilon_0, \mu_0 \) are the dielectric constants of vacuum; \( \tilde{U}_q = U_q(r, \theta) \) is the potential of the total field. Without decreasing the level of generalization, let us assume that the dipole is located at the conical area with \( r_0 < c \), therefore \( U_1^i(r, \theta) = U_1(r, \theta) + U_1^2(r, \theta) \) and \( U_2^i(r, \theta) = U_2(r, \theta) \), where \( U_1(r, \theta) \) denotes the diffracted field and \( U_2^2(r, \theta) \) is the incident field. We search for the solution of the mixed boundary value problem (Equations 2–6) in the class of functions that satisfy the Silver-Müller radiation condition in form

\[
\lim_{r \to \infty} \left( \tilde{U}, \overrightarrow{H} + Z_\gamma \overrightarrow{E} \right) = \tilde{I}_x \cdot 0
\]

as well as energy bounded condition as

\[
f \left( \varepsilon_0 |\overrightarrow{E}|^2 + \mu_0 |\overrightarrow{H}|^2 \right) dV < \infty.
\]

Here, \( Z_\gamma = \sqrt{\mu_0 / \varepsilon_0} \) is the medium impedance and \( dV \) is the element of any finite volume \( V \).

### 2.2 Mode-matching series equations

We seek the solution for the formulated above problem using the method of separation of variables, and represent the diffracted field for each of the regions \( D_1 : \{ r \in (0, c); \theta \in [0, \gamma) \} \), \( D_2 : \{ r \in (0, c); \theta \in [\gamma, \pi] \} \), \( D_3 : \{ r \in (c, a); \theta \in [0, \pi] \} \) and \( D_4 : \{ r \in (a, \infty); \theta \in [0, \pi] \} \) as

\[
U_i^j = \frac{1}{\sqrt{S_1 \gamma}} \times
\]

\[
= \sum_{p=1}^{\infty} \left( P_{\nu_{\gamma} - 2}(s_1 \gamma) \frac{I_{\nu_p}^{(1)}(s_1 r)}{I_{\nu_p}^{(1)}(s_1 c)} + P_{\nu_{\gamma} - 2}^{(1)}(s_1 \gamma) \frac{I_{\nu_p}^{(2)}(s_1 r)}{I_{\nu_p}^{(2)}(s_1 c)} \right),
\]

\[
\sum_{n=1}^{\infty} P_{\nu_\gamma - 2}(s_1 \gamma) \left( \frac{K_{\nu_\gamma}^{(1)}(s_1 r)}{K_{\nu_\gamma}^{(1)}(s_1 c)} + \frac{I_{\nu_p}^{(2)}(s_1 \gamma)}{I_{\nu_p}^{(2)}(s_1 c)} \right),
\]

\[
; r, \theta \in D_3,
\]

and

\[
U_2^2 = \frac{1}{\sqrt{S_2 \gamma}} \sum_{n=1}^{\infty} x_n^{(3)} P_{\nu_{\gamma} - 2}(s_1 \gamma) \frac{K_{\nu_\gamma}^{(1)}(s_1 r)}{K_{\nu_\gamma}^{(1)}(s_1 a)},
\]

\[
r, \theta \in D_4.
\]

Here, \( x_n^{(1)}, y_n^{(1)}, y_n^{(2)} \) are unknown expansion coefficients, \( l = 1, 2, 3, s_q = -ik_q \) is the Legendre function; \( K_{\nu_\gamma}(\cdot) \) and \( I_{\nu_p}(\cdot) \) are the Macdonald and the modified Bessel functions; \( x = \cos \gamma; z_n = n + 1/2, n = 1, 2, 3, \ldots; y_p \) and \( \mu_\gamma \) are positive growing sequences of the simple roots dependent on the cone opening angle \( \gamma (\gamma \neq \pi/2) \) of the transcendental equations as

\[
P_{\nu_{\gamma} - 2}(\cos \gamma) = 0, P_{\nu_{\gamma} - 2}(-\cos \gamma) = 0.
\]

For our convenience without decreasing the level of generalization, we suppose that the radial electric dipole is located in the region \( D_1 : \{ r \in (0, c); \theta \in [0, \gamma) \} \) and the incident field is represented as the total field of this dipole in the semi-infinite conical region \( \{ 0 < r < \infty, 0 \leq \theta \leq \gamma' \} \), bounded by the perfectly conducting surface [16]:

\[
U_i = \frac{A_0^c}{\pi \sqrt{S_1 \gamma}} \sum_{p=1}^{\infty} \frac{\mu_\gamma \nu_p}{\sin \nu_p \gamma} \frac{I_{\nu_p}^{(2)}(s_1 r)}{I_{\nu_p}^{(2)}(s_1 c)} \times
\]

\[
\left\{ \begin{array}{ll}
K_{\nu_p}(s_1 r_0) & r_0 \geq r \\
I_{\nu_p}(s_1 r_0) & r_0 \leq r.
\end{array} \right.
\]

Here, \( A_0^c = P_0 Z_0/s_1 \) and \( P_0 \) is the moment of the dipole. Because of the relation

\[
(sin \theta)^{-1} \partial_\theta \overrightarrow{E}(\sin \theta \partial_\theta P_{\nu_{\gamma} - 1}(\pm \cos \theta)) =
\]

\[
= - (x^2 - 1/4) P_{\nu_{\gamma} - 1}(\pm \cos \theta)
\]

and Equation (11), Equations (9) and (12) ensure the fulfilment of the boundary conditions (Equation 4) at the finite cone \( Q \) as well as the energy-bounded condition (Equation 8), at the tip of the cone. Using the asymptotic of the modified Bessel functions for small arguments, we find from Equations (9), (12) and (1) that \( E_r, E_\theta = O \left( r^{\nu_{\gamma} - 3/2} \right), H_\theta = O \left( r^{\nu_{\gamma} - 1/2} \right), \) if \( r \to 0 \), where \( \eta_1 = \min(\mu_1, \mu_2) \). If \( \gamma \to \pi \), it is correct that \( \nu_1 \to 1/2 + 2[\ln(2/\pi - \gamma)]^{-1} \) [27]. As follows from these estimations the electric field admits the integrable singularities at the tip of the sharp cone. The unknown coefficients for the expressions (Equation 9) will be found in the class of sequences which provide the absolute and uniform convergence of series, as well as their first-order derivatives with respect to \( r \) or \( \theta \) variables. The second-order derivatives involved in electromagnetic field components, which are normal to the edge, admit the integrable singularity: \( E_r, E_\theta = O \left( \rho^{\nu_{\gamma} - 1/2} \right) \) for \( \rho \to 0 \), where \( \rho \) is the distance to the edge of the cone in the local coordinates. To
satisfy the Meixner condition at the edge of a finite cone we search for the unknown coefficients of the series Equation (9) in the class of sequences \( \lambda_n = O(n^{-2}) \) for \( n \to \infty \), where \( \lambda_n = x_n^{(1)}(\eta_1^{(1)}, \eta_2^{(2)}) \). This guarantees the satisfaction of the energy-bounded condition (Equation 8). The radiation condition (Equation 7) is also satisfied in view of Equation (10). Using the boundary conditions (Equations 5 and 6), and representations (Equations 9 and 10), we arrive at the equations

\[
\begin{align*}
\chi_n^{(1)} A_n^{(1)}(s_1a) + K_{n}(s_1a) K_{n}(s_1c) - \chi_n^{(2)} B_n^{(1)}(s_1a) = \chi_n^{(3)} \sqrt{\frac{s_1}{s_2}} A_n^{(1)}(s_2a), \\
\chi_n^{(1)} K_{n}(s_1a) + \chi_n^{(2)} K_{n}(s_1c) = \chi_n^{(3)} \frac{c_{2s}}{c_{2sr}} \sqrt{\frac{s_1}{s_2}}.
\end{align*}
\]

(13a)

Here,

\[
A_n^{(1)}(s_1a) = 1 + 2sa \frac{K_{n}(s_1a)}{K_{n}(s_1c)},
\]

(14a)

\[
B_n^{(1)}(s_1a) = 1 + 2sa \frac{I_{n}(s_1a)}{I_{n}(s_1c)}.
\]

(14b)

where the prime sign denotes the derivative with respect to the relevant argument. From Equations (13a) and (13b), we get the correlation between the coefficients as

\[
\begin{align*}
\chi_n^{(2)} &= \chi_n^{(1)}(\omega) \frac{K_{n}(s_1a)}{K_{n}(s_1c)}, \\
\chi_n^{(3)} &= \chi_n^{(1)}(\omega) \frac{c_{2s}}{c_{2sr}} \sqrt{\frac{s_1}{s_2}} K_{n}(s_1a) K_{n}(s_1c).
\end{align*}
\]

(15a)

(15b)

where

\[
\gamma_n^{(1)}(\omega) = \frac{c_{2s} - A_n^{(1)}(s_1a)/A_n^{(1)}(s_2a)}{c_{2s} + B_n^{(1)}(s_1a)/A_n^{(1)}(s_2a)}.
\]

(16a)

Note, if the finite hollow cone is surrounded by the metallic perfectly conducting spherical surface instead of a dielectric one, then the boundary condition

\[
\partial_{\theta}(r U_{n}) = 0, \text{ if } r = a, \theta \in [0, \pi]
\]

must be used instead of Equations (5) and (6). Then Equation (16a) is significantly simplified and takes the form as

\[
\gamma_n^{(1)}(\omega) = \frac{A_n^{(1)}(s_1a)}{B_n^{(1)}(s_1a)}.
\]

(16b)

The unknown coefficients in Equation (9) can be found using the continuity conditions of the tangential field components on the virtual spherical surface \( r = c \) containing the circular aperture of a finite cone. Since, the behaviour of the \( E_\theta \) at the circular rib has singularity, we represent these equations by way of limiting passing as

\[
\lim_{n \to \infty} \sum_{n=1}^{N} \chi_n^{(1)} P_{n-1}^{(1)}(x) \left[ 1 + \gamma_n^{(1)}(\omega) \frac{K_{n}(s_1a)}{K_{n}(s_1c)} I_{n}(s_1c) \right] = 0
\]

(17a)

the result of which is

\[
\begin{align*}
\lim_{n \to \infty} \sum_{n=1}^{P} P_{n-1}^{(1)}(x) \left[ \frac{\gamma_n^{(1)} + A_p}{I_p^{(1)}(s_1c)} \right] = 0 \\
\lim_{n \to \infty} \sum_{n=1}^{P} P_{n-1}^{(1)}(x) \left[ \frac{\gamma_n^{(2)} + A_p}{I_p^{(1)}(s_1c)} \right] = 0.
\end{align*}
\]

(17b)

Here, \( N = P + K \); \( x = \cos \theta \); the upper lines correspond to \( \theta \in [0, \gamma] \) and the lower lines to \( \theta \in (\gamma, \pi] \); \( P_{n-1}^{(1)}(\pm \cos \gamma) = \pm \partial_{\gamma}[P_{n-1}^{(1)}(\pm \cos \gamma)] \) is the associate Legendre function [27];

\[
A_p = \frac{(\omega / \sqrt{s_1s_2}) \cos \gamma I_{p-1}(s_1r_0) K_{p}(s_1c)}{\cos(\pi\theta) \partial_{\gamma}[P_{p-1}(\cos \gamma)]}.
\]

(18)

Equations (17a) and (17b) are the desirable series equations of our problem. It is well known that there are different ways to reduce these equations to the ISLAE using the orthogonality properties of the Legendre functions and, as a result, we arrive at the different kinds of the ISLAE. In order to carry out the analytical regularization, we choose the ISLAE acceptable for separating the singular operator.

### 2.3 Reduction to the ISLAE of the first kind

In order to reduce the series Equations (17a) and (17b) to the ISLAE, we use the property of orthogonality of the Legendre functions, which leads to

\[
P_{n-1}^{(1)}(\cos \theta) = q(z_n, \gamma) \times
\]

\[
\lim_{n \to P(K)} \sum_{p=1}^{P(K)} \eta_p \frac{q(z_n, \gamma)}{\eta_p - q(z_n, \gamma)} P_{n-1}^{(1)}(\pm \cos \theta).
\]

(19)
Here, the upper and lower signs correspond to the angle regions \( \theta \in [0, \gamma) \) with \( \eta_p = \nu_p \) and \( \theta \in (\gamma, \pi) \) with \( \eta_p = \mu_p \), respectively;

\[
q(z_n, \gamma) = (z_n^2 - 1/4) P_{z_n - 1}(\cos \gamma),
\]

(20a)

\[
\alpha^\pm(\eta_p, \gamma) = \mp 2 \left( \eta_p^2 - \frac{1}{4} \right) P_{z_n - 1}(\pm \cos \gamma) \right\}^{-1}.
\]

(20b)

Let us substitute Equation (19) into the left-hand side of the series Equations (17a) and (17b). Further, let us equate the terms with the same Legendre functions. Limiting the finite number of unknowns, we obtain the following finite system of linear algebraic equations:

\[
\sum_{n=1}^{N} \frac{x_n}{\nu_p^2 - z_n^2} \left[ 1 + \gamma_n^{(1)}(\omega) \frac{K_n(s_1 d) I_n(s_1 c)}{I_n(s_1 a) K_n(s_1 c)} \right] = \frac{1}{\nu_p \alpha^+(\nu_p, \gamma)} \left[ \gamma^{(1)}_p + A_p \right], \quad p = 1, P,
\]

\[
\sum_{n=1}^{N} \frac{x_n}{\nu_p^2 - z_n^2} \left[ \frac{K_n'(s_1 c)}{K_n(s_1 c)} + \gamma_n^{(1)}(\omega) \frac{K_n(s_1 d) I_n'(s_1 c)}{I_n(s_1 a) K_n(s_1 c)} \right] = \frac{1}{\nu_p \alpha^+(\nu_p, \gamma)} \left[ \gamma^{(1)}_p + A_p \right], \quad p = 1, P,
\]

\[
\sum_{n=1}^{N} \frac{x_n}{\mu_k^2 - z_n^2} \left[ 1 + \gamma_n^{(1)}(\omega) \frac{K_n(s_1 d) I_n(s_1 c)}{I_n(s_1 a) K_n(s_1 c)} \right] = -\frac{\gamma^{(2)}_k}{\mu_k \alpha^-(\mu_k, \gamma)}, \quad k = 1, K,
\]

\[
\sum_{n=1}^{N} \frac{x_n}{\mu_k^2 - z_n^2} \left[ \frac{K_n'(s_1 c)}{K_n(s_1 c)} + \gamma_n^{(1)}(\omega) \frac{K_n(s_1 d) I_n'(s_1 c)}{I_n(s_1 a) K_n(s_1 c)} \right] = -\frac{\gamma^{(2)}_k}{\mu_k \alpha^-(\mu_k, \gamma)} \left[ I_n(s_1 c) \right], \quad k = 1, K;
\]

where \( x_n = q(z_n, \gamma)x_n^{(1)}, N = P + K; \quad P(K) = 1, 2, 3, \ldots, P(K) \). Eliminating the unknowns \( \gamma^{(1)}_p, \gamma^{(2)}_k \) we arrive at

\[
\sum_{n=1}^{N} \frac{x_n}{\nu_p^2 - z_n^2} \left[ \frac{s_1 c W[K_n I_{\nu_p}]}{K_n(s_1 c) I_{\nu_p}(s_1 c)} + \gamma_n^{(1)}(\omega) \frac{s_1 c W[I_n I_{\nu_p}]}{I_n(s_1 c) I_{\nu_p}(s_1 c)} \right] = \phi^{(1)}_p,
\]

(22a)

\[
\sum_{n=1}^{N} \frac{x_n}{\mu_k^2 - z_n^2} \left[ \frac{s_1 c W[K_n I_{\mu_k}]}{K_n(s_1 c) I_{\mu_k}(s_1 c)} + \gamma_n^{(1)}(\omega) \frac{s_1 c W[I_n I_{\mu_k}]}{I_n(s_1 c) I_{\mu_k}(s_1 c)} \right] = \phi^{(2)}_k.
\]

(22b)

Here, \( p = \mp \frac{1}{P}; \quad k = 1, K; \quad W[\cdot] \) is the Wronskian, \( W[\phi_1 \phi_2] = f(\theta)g(\theta) - f(\theta)g(\theta) \),

\[
\phi^{(1)}_p = -\frac{\left( \frac{A_p}{\sqrt{s_1 \gamma_0}} \right) \left( \nu_p^2 - \frac{1}{4} \right) P_{z_n - 1}(\pm \cos \gamma)}{2 \cos(\pi \nu_p)} I_{\nu_p}(s_1 \gamma_0),
\]

(23)

\[
\phi^{(2)}_k = 0.
\]

Let us introduce the growing sequence formed with the positive roots of the transcendental Equation (11):

\[
\xi_{q}^{(1)} N_{q=1}^{P} = \left\{ \nu_p \right\}_{p=1}^{P} \cup \left\{ \mu_k \right\}_{k=1}^{K}.
\]

(24)

Next, we rearrange Equations (22a) and (22b) in accordance with the sequence (Equation 24). Then, passing to the limit \( (P, K, N) \to \infty, (N = P + K) \), we arrive at the ISLAE, the matrix form of which looks as

\[
[A]_{x_{n+1}} + B[X] = F_{1}.
\]

(25)

Here, \( X = \{x_n\}_{n=1}^{\infty} \) is the unknown vector;

\[
A_{II} = \left\{ a_{\nu \nu}^{(1)} = \frac{s_1 c W[K_n I_{\nu}]}{\Delta_{\nu} K_n(s_1 c) I_{\nu}(s_1 c)} \right\},
\]

(26)

\[
B = \left\{ b_{\nu \nu} = \frac{\gamma_n^{(1)}(\omega) s_1 c W[I_n I_{\nu}]}{\Delta_{\nu} I_n(s_1 c) I_{\nu}(s_1 c)} K_n(s_1 a) I_n(s_1 c) I_{\nu}(s_1 a) \right\},
\]

(27)

the known vector \( F_{1} = \{f_{q}^{(1)}\}_{q=1}^{\infty} \), where \( f_{q}^{(1)} = \phi_{q}^{(1)}, \) if \( \xi_{q} = \nu_{q} \) and \( f_{q}^{(1)} = \phi_{q}^{(2)}, \) if \( \xi_{q} = \mu_{q} \),

\[
\Delta_{\nu} = \xi_{q}^2 - s_2^2.
\]

(28)

Note that the matrix operator \( A_{II} \) in Equation (26) corresponds to the diffraction problem from a finite cone in the homogeneous medium and the operator \( B \) describes the influence of the spherical medium interface.

### 3 | Analytical Regularization Technique

In this section, we analyse the asymptotic properties of the matrix operators (Equations 26 and 27), derive the
regularization operators and formulate the initial problem in terms of the ISLAE of the second kind.

3.1 | Construction of the regularizing operators

Taking into account the asymptotic properties of the modified Bessel and Macdonald functions, we find from Equations (26) and (27) that

\[
a_{qn}^{(1)} = \frac{1}{\xi_q - z_n} + \begin{cases} O\left(\left\{ \xi_q^{2n}(\xi_q - z_n) \right\}^{-1} \right), & \text{if } z_n, \xi_q \gg |s_1|, \\
O\left(\left(\frac{s_1}{2}\right)^2\right), & \text{if } |s_1| \to 0,
\end{cases}
\]

(29)

if \( z_n, \xi_q \gg \max\{ |s_1|, |s_2|a | \} \) and

\[
b_{qn} \sim \frac{\delta^{2\nu_2}(e_{2r} - e_{1r})}{(\xi_q + z_n)(e_{2r} + e_{1r})},
\]

(30a)

if \( |s_1|, |s_2|a | \to 0, \) where \( \delta = \frac{\xi}{a} < 1. \)

Let us introduce the operator formed by the main parts of the asymptotic Equation (29) and the inverse operator as

\[
A : \{ a_{qn} = (\xi_q - z_n)^{-1}\}^\infty_{q,n=1},
\]

(31)

\[
A^{-1} : \{ e_{qn} = (M^-_{-1}(\xi_q))M^-_{-1}(z_k)(z_k - \xi_q)\}^\infty_{k,q=1}.
\]

(32)

Here, it is correct that [16–19].

\[
A^{-1}A = I,
\]

(33a)

\[
AA^{-1} = I,
\]

(33b)

\[
I \text{ is the identity matrix; } [M^-_{-1}(\xi_q)]' = \partial_{\xi}M^-_{-1}(\xi_q), \quad M^-_{-1}(z_k) = \partial_z[M^-_{-1}(z_k)], \quad M_{-1}(\nu) \text{ is determined by the factorization of the even meromorphic function } M(\nu), \text{ which is regular in the strip } \Pi : \{|\text{Re}(\nu)| < 1/2\} \text{ with simple zeroes and poles at } \nu = \pm z_k, \nu = \pm \xi_q \text{ that are located on the real axis outside of the } \Pi;
\]

\[
M(\nu) = \frac{\cos(\pi \nu)}{(\nu^2 - \frac{1}{4})P_{\nu-\frac{1}{2}}(-\cos \gamma)P_{\nu-\frac{1}{2}}(\cos \gamma)}.
\]

(34)

where \( M(\nu) = M_{-1}(\nu)M_{-1}(\nu); \quad M_{-1}(\nu), M_{-1}(\nu) \) are split functions, regular in the right (\( \text{Re}(\nu) > -1/2 \)) and in the left (\( \text{Re}(\nu) < 1/2 \)) half-planes respectively; \( M(\nu) = O(\nu^{-1}) \) and \( M_{-1}(\nu) = M_{-1}(\nu) = O(\nu^{-1/2}) \), if \( |\nu| \to \infty \) in the regularity region;

\[
M_{-1}(\nu) = \left\{ B_0 \left( \frac{1}{2} - \nu \right) \Gamma\left( \frac{1}{2} - \nu \right) e^{-q_1 \nu} \prod_{p=1}^\infty \left( 1 - \frac{\nu}{q_p} \right)^{-\frac{1}{q_p}} \right\}^{-1}.
\]

(35)

Here,

\[
B_0 = i \pi^{\frac{1}{2}} \frac{P_{\nu-\frac{1}{2}}(\cos \gamma)P_{\nu-\frac{1}{2}}(-\cos \gamma)}{2},
\]

\[
\chi = \frac{\gamma}{\pi} \ln \frac{\gamma - \gamma}{\pi} - \frac{\gamma - \gamma}{\pi} - \frac{\gamma}{4} - S(\gamma) - S(\gamma),
\]

\[
S(\gamma) = \sum_{n=1}^\infty \left( \frac{\gamma}{\pi(n-1/4)} - 1 \right) \mu_n,
\]

\[
S(\gamma - \gamma) = \sum_{n=1}^\infty \left( \frac{\gamma - \gamma}{\pi(n-1/4)} - 1 \right) \mu_n,
\]

\( \Gamma(\cdot) \) and \( \psi(\cdot) \) are the gamma function and its logarithmic derivative, respectively.

Note, that the simple formulas to calculate the derivatives of the split function \( M_{-1}(\nu) \) (Equation 35) at zeros and poles can be written as

\[
M'(z_k) = -\pi \left( \frac{z_k^2 - 1}{4} \right) \frac{P_{\nu-\frac{1}{2}}(\cos \gamma)}{M_{-1}(z_k)} \right)^{-1},
\]

(36)

\[
[M^-_{-1}(\xi_q)]' = \left( \frac{\xi_q^2 - 1/4}{4} \right) \frac{M_{-1}(\xi_q)}{\cos(\pi \xi_q)} \times
\]

\[
\left\{ P_{\nu-\frac{1}{2}}(-\cos \gamma)\partial_\nu P_{\nu-\frac{1}{2}}(\cos \gamma), \text{ if } \xi_q = \nu_q \right\}
\]

\[
\times P_{\nu-\frac{1}{2}}(\cos \gamma)\partial_\nu P_{\nu-\frac{1}{2}}(-\cos \gamma), \text{ if } \xi_q = \mu_q.
\]

In order to prove the equalities (Equations 33a and 33b), let us represent Equation (33a) as

\[
\sum_{q=1}^\infty \tau_{bn}a_{qn} = \delta_{bn},
\]

(38)

where \( \delta_{bn} \) is the Kronecker symbol. Let us introduce the integral as follows:

\[
I_{bn} = \frac{1}{2\pi M_{-1}(z_k)} \frac{M_{-1}(t)dt}{(z_k - t)(t - z_n)}.
\]
Here, the integration path in the complex plane \( t \) is the circle \( C_R \), the points \( t = 0 \) and \( R \) are the centre and the radius of the circle respectively (see Figure 2); \( C_R \) outline encompasses the simple poles of the integrand at \( t = \xi_q \) \((q = 1, 2, 3, \ldots)\) and \( t = z_k\) if \( k = n\). For \( |t| \to \infty \) the integrand as a function of \( t \) tends to zero not slower than \( t^{-3/2} \), therefore, \( J_{kn} \to 0, \) if \( R \to \infty \). Then, applying the residues theorem, we arrive at Equations (38) and (33a).

Next we represent Equation (33b) in the form as

\[
\sum_{k=1}^{\infty} d_{pk} \xi_{kn} = \delta_{pn}.
\]

Then, introducing the integral as

\[
J_{pn} = \frac{1}{2\pi i} [M^{-1}(\xi_n)]' \oint_{C_R} M_-(t)(t-\xi_n)(\xi_p-t) dt
\]

we prove the relation (Equation 39) using the technique that we have applied in the previous case.

### 3.2 Left-side regularization

Next, using the ISLAE of the first kind (Equation 25) and taking into account the properties of the operators (Equations 31 and 32), we formulate the original diffraction problem via the ISLAE of the second kind as follows:

\[
X = A^{-1}(A - A_{11})X - A^{-1}BX + A^{-1}F_1.
\]

The ISLAE (40) is valid for \( \gamma \neq \pi/2 \). The technique described above is elaborated in [16–20] and called the analytical regularization procedure. The ISLAE (40) admits the solution in the class of sequences \( b(\sigma) : \{ ||X|| = \sup_{n \to \infty} |x_n|, \lim_{n \to \infty} |x_n\sigma^n| \to 0 \text{ for } 0 \leq \sigma < 1/2 \} \). This fulfils the necessary conditions for the existence of a unique solution of the ISLAE (40), including the Meixner condition on the edge. The proof of these statements is based on the use of asymptotic estimates (Equations 29 and 30a 30b) as well as on the estimate of Equation (32):

\[
\tau_{kp} = O\left(\xi_p^{-1/2}\xi_k^{1/2}/(z_k - \xi_p)\right),
\]

if \( k, \rho \to \infty \).

### 3.3 Right-side regularization

Let us introduce the new unknown vector \( Y \) as

\[
X = A^{-1}Y.
\]

Taking this into account, let us rewrite the ISLAE (25) in the form of

\[
Y = (A - A_{11})A^{-1}Y - BA^{-1}Y + F_1.
\]

Equation (42) is the desirable ISLAE of the second kind, the solution of which is \( \{y_n\}_{n=1}^{\infty} \in \ell_2, \ell_2 : \{y_n \in C^d; \sum_{k=1}^{\infty} |y_k| < \infty \} \). Therefore, the ISLAE (42) as well as the ISLAE (40) allow for obtaining of the solution with the given accuracy for any geometrical parameters and frequency; these solutions guarantee the fulfilment of all the necessary conditions for the field components including the edge condition. The transition to the solution of the wave diffraction problem from a disc inset in a dielectric sphere as well as a summation of the special series of \( \Gamma \)-functions are considered in Appendix.

Note, our theory allows for the simple generalization: If the finite cone \( Q \) is embedded in the sphere \( S \), which is surrounded by the \( N \) concentric homogeneous and isotropic spherical dielectric layers the scattering problem is reduced to the solution of Equations (40) and (42), where the operator \( B \) (Equation 27) contains the function \( \gamma_n^{(N)} \) instead of \( \gamma_n^{(1)} \); \( \gamma_n^{(N)} \) is determined from the solution of the wave diffraction problem for a multilayered sphere without a conical inset.

### 4 LOW-FREQUENCY SOLUTION

In this section we apply the obtained equations to simplify the solutions in the low-frequency range; we find the exact solution for the problem in static limit and derive the approximate formula to determine the shifts of the resonance frequencies caused by a small-size conical inset in a dielectric resonator.

#### 4.1 Static approximation

Let us suppose that \( |s_{12}a| \to 0 \) and using the static parts of the matrix operators \( A_{11} \) and \( B \) we simplify the ISLAE (40) and (42) as
1. Left-side regularization:

\[ x_q = \tilde{g}_q^{(1)} - \sum_{k=1}^{\infty} \delta^{2q} \beta_k \kappa_{qk}(\gamma) x_k; \quad (43a) \]

1. Alright-side regularization:

\[ y_q = f_q^{(1)} - \sum_{n=-\infty}^{\infty} \chi_{qn}(\gamma) y_n. \quad (43b) \]

Here, we introduce the new notations

\[
\kappa_{qk}(\gamma) = \sum_{n=1}^{\infty} \tau_{qn} x_n + z_k, \quad (44a)
\]

\[
\tilde{g}_q^{(1)} = \sum_{k=1}^{\infty} \tau_{qk} f_q^{(1)},
\]

\[
\chi_{qn}(\gamma) = \sum_{k=1}^{\infty} \delta^{2q} \beta_k \tau_{qn} x_n + z_n, \quad (44b)
\]

\[
\beta_k(\varepsilon_{1r}, \varepsilon_{2r}) = \frac{(1 - 2z_k(\varepsilon_{1r} - \varepsilon_{2r}))}{(1 + 2z_k)(\varepsilon_{2r} - (1 - 2z_k)\varepsilon_{1r}).}
\]

The ISLAE (43a) and (43b) are acceptable for derivation of the static and the low-frequency approximations of the problem solution. For the solution of the obtained system for \( x_q \) we apply the iteration method and begin our calculation with \( x_q = \tilde{g}_q^{(1)}. \) Thus in the first iteration

\[
x_q = \tilde{g}_q^{(1)} - \sum_{k=1}^{\infty} \delta^{2q} \beta_k \kappa_{qk}(\gamma) \tilde{g}_q^{(1)},
\]

in the second iteration

\[
x_q = \tilde{g}_q^{(1)} - \sum_{k=1}^{\infty} \delta^{2q} \beta_k \kappa_{qk}(\gamma) \tilde{g}_q^{(1)}
\]

\[
+ \sum_{i_1,i_2=1}^{\infty} \delta^{2(i_1+i_2)} \beta_{i_1,i_2} \kappa_{i_1,i_2}(\gamma) \beta_{i_1,i_2}(i_1) \tilde{g}_q^{(1)};
\]

and after \( N \) repeats, the final solution takes the form

\[
x_q = a_q^{(0)} + \sum_{i_1=1}^{\infty} a_{q,i_1}^{(1)} \delta^{2i_1} + \sum_{i_1,i_2=1}^{\infty} a_{q,i_1,i_2}^{(2)} \delta^{2(i_1+i_2)}
\]

\[
+ \ldots + \sum_{i_1,i_2,\ldots,i_N=1}^{\infty} a_{q,i_1,i_2,\ldots,i_N}^{(N)} \delta^{2(i_1+i_2+\ldots+i_N)} + \ldots,
\]

where

\[
a_q^{(0)} = \tilde{g}_q^{(1)},
\]

\[
a_{q,i_1}^{(1)} = -\beta_{i_1} \kappa_{q,i_1}(\gamma) a_{q,i_1}^{(0)};
\]

\[
a_{q,i_1,i_2}^{(2)} = -\beta_{i_1,i_2} \kappa_{q,i_1,i_2}(\gamma) a_{q,i_1,i_2}^{(1)};
\]

\[
a_{q,i_1,i_2,\ldots,i_N}^{(N)} = -\beta_{i_1,i_2,\ldots,i_N} \kappa_{q,i_1,i_2,\ldots,i_N}(\gamma) a_{q,i_1,i_2,\ldots,i_N}^{(N-1)};
\]

Note that \( \beta_k = \beta_k(\varepsilon_{1r}, \varepsilon_{2r}) = (\varepsilon_{2r} - \varepsilon_{1r})/(\varepsilon_{2r} + \varepsilon_{1r}) \) for \( z_k \gg 1. \) This simplifies the calculation of the coefficients \( \beta_k \) in Equation (46) for a large index \( k. \) Let us turn to Equation (44a) and find that

\[
\kappa_{qk}(\gamma) = \frac{M_+(z_k)}{M_+(z_q)(z_q + z_k)}, \quad (47)
\]

In order to obtain the summation formula (Equation 47), let us introduce the integral as

\[
J_{qk} = \frac{1}{2\pi i} \oint \frac{M_+(t)dt}{(z_q - t)(t + z_k)}. \quad (48)
\]

Here, the integration path in the complex plane \( t \) is the circle \( C_R, \) the points \( t = 0 \) and \( R \) are the centre and the radius of the circle respectively; \( C_R \) outline encompasses the simple poles of the integrand at \( t = \xi_n (n = 1, 2, 3, \ldots) \) and \( t = -z_k. \)

For \( |t| \to \infty \) the integrand as a function of \( t \) tends to zero not slower than \( t^{-5/2}, \) therefore, \( J_{qk} \to 0, \) if \( R \to \infty. \) Then, applying the residues theorem, we arrive at

\[
\sum_{n=1}^{\infty} \frac{1}{[M_+(\xi_n)](\xi_q - \xi_n)(\xi_q + z_k)} = \frac{M_+(z_k)}{z_q + z_k}. \quad (49)
\]

Directly from the equality (Equation 49) and the representations (Equations 44a and 32) follows the relation (Equation 47). Equation (46) allows one to find the arbitrary coefficients in the solution of the ISLAE by the series (Equation 45) which is absolutely convergent for any \( \delta < 1 \) and gives the explicit solution for our problem in the static case. Consequently, Equations (45), (21a 21b), (9) and (10) give an approximate solution for the diffraction problem in the low-frequency range.

4.2 | Small-size conical inset in a dielectric/metallic sphere

Let us estimate the influence of the small-size conical inset \( (|\xi_T| \ll 1) \) on the spectrum of the dielectric spherical resonator surrounded by the dielectric medium; the dielectric sphere parameters \( s_{1,2,3,4} \) are free from the mentioned restriction. To do this, let us substitute the modified Bessel and Macdonald functions, whose arguments contain the dimensionless parameter of a small conical inset for Equations (26) and (27), by the corresponding asymptotic formulas passing from Equation (40) to the ISLAE in the form

\[
x_q + \sum_{k=1}^{\infty} \left( \frac{\alpha_k}{2} \right)^{2q} d_{qk}(\omega, \gamma) x_k = \tilde{g}_q^{(1)}. \quad (50)
\]
Here, $q = 1, 2, 3, \ldots; \rho = s_1 c$;

$$d_{qk}(\omega, \gamma) = \Theta_{qk}(\gamma) \Omega_k(\omega),$$

$$\Theta_{qk}(\gamma) = \frac{2k_{qk}(\gamma)}{\Gamma(2k_0)\Gamma(2k_0 + 1)}.$$

(51)

$$\Omega_k(\omega) = \left( \frac{E_{1r}^{q}}{E_{1r}^{q'}} \right) K_{2q}(\rho_1) \sqrt{2} \left[ \sqrt{2} K_2(q) \right] - K_2(q) \sqrt{2} \left[ \sqrt{2} K_2(q) \right]' - K_2(q) \sqrt{2} \left[ \sqrt{2} K_2(q) \right]' + K_2(q) \sqrt{2} \left[ \sqrt{2} K_2(q) \right]' \right),$$

$$\left( \frac{E_{1r}^{q}}{E_{1r}^{q'}} \right) I_{s_{qk}}(\rho_1) \sqrt{2} \left[ \sqrt{2} K_2(q) \right] + K_2(q) \sqrt{2} \left[ \sqrt{2} K_2(q) \right]' \right) \right),$$

(52)

where $\rho_{1(2)} = s_{1(2)} a$. Note that the simple poles of the functions (Equation 52) correspond to resonance oscillation of the spherical dielectric resonator. Therefore, the small parameter $\rho_c$ in Equation (50) is kept at the terms (Equation 52).

Let us apply Equation (50) to determine the resonance frequencies perturbation of a dielectric sphere, stipulated by the presence of small-size conical insets. To do so, let us consider the equation

$$\det \left( \delta_{qk} + \Theta_{qk}(\gamma) \Omega_k(\omega) \left( \frac{\rho_1}{2} \right)^{2q} \right) = 0.$$  

(53)

Here, $\delta$ denotes the matrix operator (Equation 50); $q, k = 1, 2, 3, \ldots$

Because of the small parameter $|\rho_c|$, let us take into account only the diagonal elements of the matrix (Equation 53) and reduce the problem to the solution of the following equations:

$$1 + \Theta_{kk}(\gamma) \Omega_k(\omega) \left( \frac{\rho_1}{2} \right)^{2q} = 0.$$  

(54)

From Equation (54), we find the formula for determination of the shifts of the resonance frequencies, which we write as

$$\omega - \omega_{cp} = -\Theta_{kk}(\gamma) \left( \frac{\rho_1}{2} \right)^{2q} \frac{\Omega_k(\omega)}{\Gamma_{\omega} \Omega_k^{-1}(\omega)}.$$  

(55)

where $\omega_{cp} = \text{Re}(\omega_{cp}) + i \text{Im}(\omega_{cp})$ is the complex resonance frequency of $E_{1r}^{q}$ oscillation of a spherical dielectric resonator without a conical inset; $\text{Re}(\omega_{cp}) > 0, \text{Im}(\omega_{cp}) \leq 0$; $\rho_{c}(cp) = \text{Re}(\omega_{cp})C_{1r}^{q} + \text{Im}(\omega_{cp})\text{Re}(\omega_{cp}) e^{-i\gamma/2}$.

From Equation (55) it follows that the geometrical parameters of a small-size conical inset influence the spectra of a dielectric sphere independently.

Let us represent the approximate expressions of the kernel function (Equation 52) for the two particular cases:

1. The dielectric spherical resonator with large dielectric permittivity $|\epsilon_{1r}| \gg 1$ and small radius $|s_{1}a| < \ll 1$.

$$\Omega_k(\omega) = \left( \frac{c}{\epsilon_0} \right) \left[ 1 - 2z \right] K_{2q}(\rho_1) - 2\sqrt{\rho_1 \left[ \sqrt{\rho_1} K_{2q}(\rho_1) \right]'} - \left( \frac{c}{\epsilon_0} \right) \left[ 1 - 2z \right] I_{s_{qk}}(\rho_1) + 2\sqrt{\rho_1 \left[ \sqrt{\rho_1} I_{s_{qk}}(\rho_1) \right]'};$$

(56a)

Note, Equation (55) generalizes the known result, obtained in [23] for determination of the shifts of resonance frequencies, stipulated by small conical insets in metallic spherical resonator. This problem has been solved by the Wiener-Hopf method with the use of the Kontorovich-Lebedev integral transformation on the finite interval.

5 | NUMERICAL ANALYSIS

In this section, we study numerically the influence of a finite conical inset on the properties of the metallic and dielectric spherical resonators and the field radiation from the conical horn antenna into the concave dielectric.

For further convenience let us derive the $H_\varphi$ total field representation in the spherical layer $D_s : \{ r \in (\epsilon, a); \theta \in [0, \pi] \}$ from Equations (1), (9) and (15a):

$$H_\varphi = -\sqrt{\frac{s_{1}}{rZ_1}} \sum_{n=1}^{\infty} \left[ \frac{p^{(1)}_{n} \left( s_{1} r \right)}{K_{n+1}(s_{1} r)} + \frac{\gamma_{n+1}^{(1)}(n) K_{n}(s_{1} r)}{K_{n+1}(s_{1} r) I_{n}(s_{1} r)} \right] \times$$

$$\left[ K_{n}(s_{1} r) \right] \left[ I_{n}(s_{1} r) \right].$$

In order to simplify the numerical analysis, we assume that the relative dielectric permittivity $\epsilon_{1r}^{(2)}$ and permeability $\mu_{1r}^{(2)}$ are real, and electric constants of vacuum accept the unit values. We analyse all the scattered characteristics by the numerical solution of the reduced ISLAE (40). To Equation (40) we apply the coefficients (Equation 16a) for calculation of the matrix elements $B_{mn}$ in Equation (27) if the cone is embedded in a dielectric sphere surrounded by the dielectric medium, and (Equation 16b) if the cone is placed into the closed hollow metallic resonator which is not connected to the outer space. The order of reduction has been chosen from the condition $N = [k_{1c}] + [q]$ with $4 \leq q \leq 10, [x]$ is the whole number of $x$.

5.1 | Finite cone in the metallic spherical resonator

Here, we consider the metallic spherical resonator with a conical inset and analyse the behaviour of the magnitude of the magnetic field component at the concaved spherical surface
with the dimensionless radius $k_c$. Let the cone $Q$ be placed into the perfectly conducting spherical resonator which is filled with medium $\varepsilon_r$, $\mu_r = 1$, and this cone is excited by the dipole with $P_0 = 1/(4\pi)$ [\AA].

Let us consider the $E_{101}$-resonance oscillation in the empty metallic spherical resonator. In order to limit the resonance mode amplitude, the dimensionless radius of the resonator $k_r = 2.743$ is taken somewhat smaller than the resonance one; the corresponding resonance radius is the first root of the denominator (Equation 56b) for $z_1 = 2/3$. Let us introduce a finite cone into a resonator and analyse the influence of its geometrical parameters on the $H_\varphi$ - total field amplitude. The curves 1–5 in Figure 3 show the influence of the aperture angle of a small conical inset on the magnetic field distribution at the inner metallic spherical surface. As the angular dependence of the $H_\varphi$ - component of the $E_{101}$-vibration is described by the function $P_1(\cos \theta) = -\sin \theta$, we observe the half sine curves in this figure. The comparison of curves 1–3 shows that the presence of a small conical inset can cause an increase of the amplitude of $H_\varphi$ - field at $\theta = \pi/2$; this effect essentially depends on the aperture angle. This property can be used for ‘tuning’ the resonator with the help of conical insets for regulating the amplitude of the resonance mode. The dependencies of $|H_\varphi(\pi/2)|$ on cone dimensionless length at different aperture angles $\gamma$ are shown in Figure 4. The peaks in this figure are formed by the resonance $E_{101}$-mode. We have found the curves here with the step $\Delta(k_c) = 0.02$; the level of the peaks has not been reached exactly.

Let us note that the change in cone aperture angle $\gamma \rightarrow \pi - \gamma$ does not change the positions of the peaks. In Figure 4 we can see that, if the aperture parameter $\gamma$ tends to $\pi/2$, the resonance peaks are formed for the conical insets of greater lengths. This property of the conical insets can be observed in more detail in Figure 5. The curve in this figure shows how the height of the cone $h = k_c \cos(\pi - \gamma)$, at which we observe the peaks $|H_\varphi(\pi/2)|$, depends on the aperture angle. As can be seen from this figure, the effect of the resonator ‘tuning’ due to a conical inset is caused by small deviations of the cone from the disc; we can see that $\max h < 0.04\lambda$, where $\lambda$ is the dimensionless wavelength.

For comparison, we have considered the influence of a conical inset on $E_{102}$ resonance oscillation of the spherical metallic resonator; we analyse the resonator with $k_r = 6.117$ which is somewhat smaller than the resonance one. In contrast to the previous case, the rise of a conical inset in length quickly damps the resonance mode. This can be clearly seen from the comparison of the curves 1–3 in Figure 6. For the aperture angles close to $\gamma = \pi/2$, and at $3.4 < k_c < 4.2$ we observe the formation of the plateau, where values $|H_\varphi(\pi/2)|$ on the inner surface of the metallic sphere remain practically the same (see curves 4, 5). The further tending of the cone aperture angle to $\gamma = \pi/2$ preserves the value of $|H_\varphi(\pi/2)|$ practically unchanged. This value remains close to that in a spherical resonator without a conical inset, almost on the whole range of

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**Figure 3** Magnetic field pattern of the $E_{101}$-resonance mode on the metallic surface at $k_r = 2.743$, $k_c = 0.5$, $k_r = 0.05$, $1 - \gamma = 89^\circ$, $2 - \gamma = 100^\circ$, $3 - \gamma = 110^\circ$, $4 - \gamma = 115^\circ$, $5 - \gamma = 150^\circ$.

**Figure 4** Magnetic field of the $E_{101}$-resonance mode at $\theta = \pi/2$ on the inner metallic surface as a function of $k_c$ at $k_r = 2.743$, $k_r = 0.05$, $1 - \gamma = 130^\circ$, $2 - \gamma = 110^\circ$, $3 - \gamma = 100^\circ$, $4 - \gamma = 95^\circ$, $5 - \gamma = 93^\circ$, $6 - \gamma = 91^\circ$.

**Figure 5** Dependence of the height of the conical inset $h = k_c \cos(\pi - \gamma)$ at which the peaks of $|H_\varphi(\pi/2)|$ appear on the aperture angle at $k_r = 2.743$, $k_r = 0.05$.

**Figure 6** $|H_\varphi(\pi/2)|$ as a function of $k_c$ for $E_{102}$-resonance mode at $k_r = 6.117$, $k_r = 0.05$, $1 - \gamma = 100^\circ$, $2 - \gamma = 95^\circ$, $3 - \gamma = 92^\circ$, $4 - \gamma = 91^\circ$, $5 - \gamma = 90.5^\circ$, $6 - \gamma = 90.1^\circ$. 
changes of the generating length. The behaviour of the curve 6 in Figure 6 displays this phenomenon, which can be explained by the fact that the $E_{101}$ oscillation is also resonance for the “semi-spherical” resonator closed by the disc.

### 5.2 Finite cone in a dielectric sphere

Let us consider a finite cone completely submerged into a dielectric sphere, whose relative dielectric permittivity is lower than that of the outer medium $\varepsilon_1 > \varepsilon_2$ and $\mu_{1(2)} = 1$. At $\gamma < \pi/2$ and $\varepsilon_1 = 1$ our structure can be considered a model of radiation from the conical horn that is placed in the empty spherical cavity into dielectric environment. This model can be useful for diagnostics of gases, liquids and plasma.

Figure 7 shows the total field distribution on spherical interface, radiated from the conical horn which aperture approaches the surface of the spherical void. The curves are plotted here for $0^\circ < \theta < 180^\circ$ in view of the problem symmetry. In order to obtain the effective response, the geometric parameters (sphere radius $k_{\text{rad}} = 7.7537$ and cone aperture angle $\gamma = 50^\circ$) are taken using the conditions under which the $E_{101}$ resonance oscillation can be excited in the corresponding closed metallic sphere-conical resonator. Here $\nu_1 = 2.74004$ is the first positive root of the first Equation (11) for $\gamma = 50^\circ$ and $k_{\text{rad}} = 7.7537$ is the second root of the transcendental equation: $\gamma_1(\varepsilon_1) + 2\gamma I_0'\left(\nu_1\right) = 0$. Curves in Figure 7 illustrate the character of influence of permittivity $\varepsilon_2$ on $|H_\varphi|$. We observe a strong growth of the main lobe of the field distribution, if the dielectric parameter $\varepsilon_2$ is growing, and we see a deep shadow outside the aperture of the conical horn.

Let us consider the spherical dielectric resonator with the dielectric permittivity $\varepsilon_1 \gg 1$. It is well known that spherical dielectric resonators with large dielectric permittivity are widely used in different electronic devices because they support the high quality resonance oscillations.

Their complex resonance frequencies are determined as the zeros of the denominator (52). It is also known that the metallic inclusion can improve their characteristics. Here, we study the influence of a finite metallic conical inset on the module of the magnetic component of the lowest $E_{101}$ resonance oscillation at the point of maximum at $\theta = \pi/2$. The plots that illustrate these dependences are presented in Figure 8. The real parts of the resonance frequencies of a dielectric sphere with $\varepsilon_1 \gg 1$ are determined approximately from the Equation $\gamma_{1/2}(\varepsilon_1) = 0$ [4]; $\chi \approx 4.4248$ is the first positive root. Analysing the curves in Figure 8 we can see that the conical inlets closed in the shape of the disc ($\gamma = 91^\circ$) with the rise in the length of the cone keep the oscillation amplitude practically constant. This occurs because this oscillation is resonance for the open resonator formed by the dielectric hemisphere on metallic plane as well. Therefore, the rise of the $k_{\text{rad}}$ parameter in the cones close to the disc can support $E_{101}$-type of resonance oscillation, as it occurred in metallic spherical resonators. This phenomenon is observed at small deviations of the cone aperture angle from $\gamma = \pi/2$. The peculiarity of this effect in dielectric resonators is that this type of oscillations is supported by insets, whose generating length does not exceed a certain value. When this value is exceeded, quick damps of the resonance oscillations occurs (see curves 2 and 3 in Figure 8). The formation of sharper short cones in the dielectric resonator leads to the abrupt rise in the $E_{101}$-oscillation amplitude, as can be seen from comparison of curves 4 and 5 in this figure.

### 6 CONCLUSIONS

The analytical regularization technique for the rigorous analysis of wave diffraction from a finite cone with perfectly conducting boundaries totally embedded in a dielectric sphere that is surrounded by the dielectric medium is developed. The governing Equations (40) and (42) of the second kind are obtained for the problem solution. The analytical form solution of the problem is obtained for the static limit in Equations (45) and (46). The approximate formula (Equation 55) is obtained to determine the shifts of resonance frequencies for the spherical dielectric resonators by the small-size conical inlets. It is shown that the geometrical parameters of such a cone (the length and the opening angle) independently perturb the spectra of a dielectric sphere. The transition to the solution for the limiting case, when the cone degenerates into the disc, is obtained. The influence of the conical inlets on the resonance
properties of the spherical metallic and dielectric resonators are analysed numerically: it is found that the presence of a conical inset in the metallic spherical resonator can cause increase of the amplitude $E_{101}$-oscillations; this effect is observed for the conical insets with small deviations from the disc. The effect of the essential growth of the $E_{101}$-oscillations is found for dielectric spherical resonator with large dielectric permittivity $\varepsilon_r > 1$ caused by the short sharp conical insets. The model of the wave radiation from the conical horn into the concave spherical dielectric is considered and a strong growth of the main lobe of the radiation field is found, if $\varepsilon_r$ is growing.

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**APPENDIX**

Here we represent all the necessary relations for the transition to the problem solution for the disc inset ($\gamma = \pi/2$) in a dielectric sphere.

**Disc inset in a dielectric sphere**

From Equation (11) and the representation of the Legendre function for $\gamma = \pi/2$

$$P_{\nu-\frac{1}{2}}(0) = \frac{\sqrt{\pi}}{\Gamma(\nu/2 + 3/4)\Gamma(-\nu/2 + 3/4)},$$

we find that $\nu_0 = \mu_0 = 2n - 1/2$ with $n = 1, 2, 3, \ldots$. Taking this into account we find that the indexes of the Legendre functions in our basic series field Equations (9), (10) and (12) in the case of a disc inset in a dielectric sphere accept the positive integer values: odd, if $\gamma < \pi$ and even, if $\gamma > \pi$ [28]. Then using the orthogonality properties of the Legendre functions with

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the odd and even indexes and applying the steps that have been used before for the general problem we arrive at the ISLAE (25) with indexes $\xi_q = 2q - 1/2$ and $z_n = 2n + 1/2$. Then the matrix elements of the asymptotic operator (Equation 31) looks as

$$a_{qm} = \frac{1}{2(q - n - 1/2)}. \quad (58)$$

In order to determine the matrix elements for the inverse operator (Equation 32) in the disc-inset case let us use the Legendre function Equation (57) and the well-known formula

$$\cos(\pi \nu) = \frac{\pi}{\Gamma(\nu + 1/2)\Gamma(-\nu + 1/2)}.$$

we find from Equation (34) that

$$M(\nu) = \frac{\Gamma^2\left(\frac{\nu}{2} + \frac{1}{4}\right)\Gamma^2\left(-\frac{\nu}{2} + \frac{3}{4}\right)}{(\nu^2 - 1/4)\Gamma(\nu + 1/2)\Gamma(-\nu + 1/2)}. \quad (59)$$

Now we represent the split functions as follows:

$$M_{\pm}(\nu) = \frac{i 2^{\pm \nu} \Gamma(\pm \nu + 1/2)}{(1/2 \pm \nu) \Gamma(\pm \nu + 1/2)}. \quad (60)$$

Here, $M_{\pm}(\nu) = O(\nu^{-1})$, if $|\nu| \to \infty$. Therefore, using Equations (36) and (37) as well as Equations (59) and (60) we arrive at

$$M'_k(z_k) = \frac{i \pi \sqrt{\pi/2} \Gamma(k)}{2 \Gamma(k + 1/2)}, \quad \text{if } z_k = 2k + 1/2, \quad (61a)$$

$$[M^{-1}(\xi_q)]' = \frac{i \sqrt{\pi/2} (q - 1/2) \Gamma(q)}{\Gamma(q + 1/2)}, \quad \text{if } \xi_q = 2q - 1/2. \quad (61b)$$

Taking this into account, we finally arrive at the matrix elements of the inverse operator (Equation 32) in form

$$\tau_{kq} = \frac{2\Gamma(q + 1/2)\Gamma(k + 1/2)}{\pi^2(1/2 - q)(k - q + 1/2)\Gamma(q)\Gamma(k)}. \quad (62)$$

Equations (58, 59 60, 61a, 61b, 62) allow us to obtain all the necessary equations and their solutions for the case of a disc inset in a dielectric sphere using the solutions for conical insets.

**Special series of gamma functions**

From Equations (60, 61a, 61b, 62) and the relations (38) and (39) directly follow the new formulas for exact summation of the gamma functions series as

$$\sum_{q=1}^{\infty} \frac{\Gamma(q - 1/2)}{(n - q + 1/2)(k - q + 1/2)\Gamma(q)} = \begin{cases} 0, & \text{if } k \neq n \\ \frac{\pi^2 \Gamma(k)}{\Gamma(k + 1/2)}, & \text{if } k = n \end{cases} \quad (63)$$

and

$$\sum_{k=1}^{\infty} \frac{\Gamma(k + 1/2)}{(n - k - 1/2)(p - k - 1/2)\Gamma(k)} = \begin{cases} 0, & \text{if } p \neq n \\ \frac{\pi^2 \Gamma(n)}{\Gamma(n - 1/2)}, & \text{if } p = n \end{cases}. \quad (64)$$

The series (Equations 63 and 64) are the alternative representations of Equations (33a) and (33b), respectively, for the case of the disc.