AVERAGING PRINCIPLES FOR THE SWIFT-HOHENBERG EQUATION

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Abstract. This work studies the effects of rapid oscillations (with respect to time) of the forcing term on the long-time behaviour of the solutions of the Swift-Hohenberg equation. More precisely, we establish three kinds of averaging principles for the Swift-Hohenberg equation, they are averaging principle in a time-periodic problem, averaging principle on a finite time interval and averaging principle on the entire axis.

1. Introduction. In studies of pattern formation, the Swift–Hohenberg equation
\[ z_t = \alpha z - (1 + \partial_{xx})^2 z - z^3 \]
plays a central role. It is a widely accepted model for the thermal convection in a thin layer of fluid heated from below, it describes the pattern formation in fluid layers confined between horizontal well-conducting boundaries. Proposed in 1977 by Swift and Hohenberg [35] in connection with Rayleigh-Bénard convection, it has since featured in a variety of problems, such as Taylor-Couette flow [19, 30], and in the study of lasers [25]. We also refer to the surveys given in [6, 7] and the recent review [3].

During the past years, many authors have paid much attention to the Swift-Hohenberg equation: the global attractor [24], the stability of stationary solutions [32], the bifurcation problem [26, 33], the pattern selections of solutions [31], slow and fast dynamics [16], optimal control [37], local exact controllability to the trajectories [13] ·····

In this paper, we consider the averaging principle for the Swift-Hohenberg equation. More precisely, we study the following problem:

\[
\begin{align*}
\begin{cases}
    u_t + u_{xxxx} + 2u_{xx} + \beta u + u^3 &= f(x, \frac{t}{\varepsilon}) & \text{in } Q, \\
    u(0, t) = 0 &= u(1, t) & \text{in } (0, T), \\
    u_{xx}(0, t) = 0 &= u_{xx}(1, t) & \text{in } (0, T), \\
    u(x, 0) &= u_0(x) & \text{in } I,
\end{cases}
\end{align*}
\]

where \( \varepsilon, \beta \) are positive constants, \( I = (0, 1), \ T > 0, \ Q = I \times (0, T) \).

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The study of stability of nonautonomous nonlinear systems is known as a hard problem. One of the techniques used to simplify the consideration is the averaging method consisting in replacing the original system by an averaged autonomous one which has similar property but its analysis is easier. The averaging of a partial differential equation is a process which consists in showing the convergence of the solution of an equation with rapidly varying coefficients towards an equation with simpler (e.g. constant) coefficients.

Starting from the fundamental work of Bogolyubov[2], the averaging theory for ordinary differential equations has been developed and generalized in a large number of works (see [5, 11, 29] and the references therein). Bogolyubov’s main theorems have been generalized in [10] to the case of differential equations with bounded operator-valued coefficients. Some problems of averaging of differential equations with unbounded operator-valued coefficients have been considered in [10, 20, 28, 36] in the framework of abstract parabolic equations. [8] obtains a general version of the Bogolyubov averaging lemma for a time-varying differential equation. The works [10, 21, 22] are devoted to generalization of method of averaging for dissipative partial differential equations. [8] uses the concept of pullback attractors of such systems to establish the existence of almost periodic (quasi-periodic, almost automorphic, recurrent, pseudo recurrent) solutions corresponding to time dependent coefficients of these types and a global averaging principle is derived. [17] establishes averaging principle for quasi-geostrophic motion. [18] derives an averaging principle for the 2D quasi-geostrophic flow.

During the recent years, many authors have paid much attention to the averaging principles for the partial differential equations: Dissipative partial differential equations: [9, 21, 22]; Quasigeostrophic motion: [17]; 2D quasigeostrophic flow: [18]; Schrödinger equations: [12]; Three-dimensional primitive equations: [34]; Navier-Stokes equations: [21]; Non-linear dissipative hyperbolic equation: [21]; ·····

Motivated by previous research and from both physical and mathematical standpoints, the following mathematical questions arise naturally which are important from the point of view of dynamical systems:

- Does the averaging principle in a time-periodic problem for (1) hold?
- Does the averaging principle on a finite time interval for (1) hold?
- Does the averaging principle on the entire axis for (1) hold?

In this paper we will answer these questions.

1.1. Averaging principle in a time-periodic problem. We first study the behavior of the set of time-periodic solutions of the Swift-Hohenberg equation

\[
\begin{align*}
\begin{cases}
\frac{du}{dt} + u_{xxxx} + 2u_{xx} + \beta u + u^3 = f(x, t) \\
u(x, 0) = u(x, T) \\
u_{xx}(x, 0) = u_{xx}(x, T)
\end{cases}
\end{align*}
\]

as the frequency of the oscillations of the right-hand side tends to infinity, where \( f \in C(\mathbb{R}, H^1_0(I)) \) and \( f(x, t + T) = f(x, t) \).

It is established that the set of periodic solutions tends to the solution set of the averaged stationary equation

\[
\begin{align*}
\begin{cases}
u_{xxxx} + 2u_{xx} + \beta u + u^3 = f_0(x) \\
u(0) = u(1) \\
u_{xx}(0) = u_{xx}(1)
\end{cases}
\end{align*}
\]
where \( f_0(x) = \frac{1}{T} \int_0^T f(x,t) \, dt \).

Denote by \( U_\varepsilon \) the collection of \( \varepsilon T \)-periodic solutions of (2) for a fixed \( \varepsilon > 0 \). Let \( U_0 \) be a collection of solutions of (3).

The first averaging principle for the Swift-Hohenberg equation is as follows:

**Theorem 1.1** (Averaging principle in a time-periodic problem). Let \( f \in C(\mathbb{R},H_0^1(I)) \) and

\[
\frac{f(x, t + T)}{f(x, t)} = f(x, t).
\]

If we take \( \beta \) large enough, then the semideviation

\[
d_t(U_\varepsilon, U_0) \equiv \sup_{u_\varepsilon \in U_\varepsilon} \inf_{u^0 \in U_0} \|u_\varepsilon - u^0\| \to 0 \text{ as } \varepsilon \to 0
\]

uniformly in \( t \in \mathbb{R} \).

**Remark 1.** 1. Theorem 1.1 can be seen as The Bogolyubov Lemma for Swift-Hohenberg equation: time-periodic case.

The classical Bogolyubov Lemma is established for the almost periodic solution for ordinary differential equations, such as in [4, P20].

2. The same problem has been established for Navier-Stokes equation in [23].

1.2. **Averaging principle on a finite time interval.** If we set \( Au = -(u_{xxxx} + 2u_{xx} + \beta u) \), \( N(u) = -u^3 \), then (1) becomes

\[
\begin{align*}
    u_t &= Au + N(u) + f(x, t) \quad \text{in } Q, \\
    u(0, t) &= u(1, t) \quad \text{in } (0, T), \\
    u_{xx} (0, t) &= u_{xx} (1, t) \quad \text{in } (0, T), \\
    u(x, 0) &= u_0(x) \quad \text{in } I.
\end{align*}
\]

The averaged equation of (4) is

\[
\begin{align*}
    \bar{u}_t &= A \bar{u} + N(\bar{u}) + f_0(x) \quad \text{in } Q, \\
    \bar{u}(0, t) &= \bar{u}(1, t) \quad \text{in } (0, T), \\
    \bar{u}_{xx} (0, t) &= \bar{u}_{xx} (1, t) \quad \text{in } (0, T), \\
    \bar{u}(x, 0) &= u_0(x) \quad \text{in } I.
\end{align*}
\]

We set \( \tau = \frac{t}{\varepsilon} \), (4) and (5) become

\[
\begin{align*}
    u_t &= \varepsilon [Au + N(u) + f(x, \tau)] \\
    u(0, \tau) &= u(1, \tau) \\
    u_{xx} (0, \tau) &= u_{xx} (1, \tau) \\
    u(x, 0) &= u_0(x)
\end{align*}
\]

and

\[
\begin{align*}
    \bar{u}_t &= \varepsilon [A \bar{u} + N(\bar{u}) + f_0(x)] \\
    \bar{u}(0, \tau) &= \bar{u}(1, \tau) \\
    \bar{u}_{xx} (0, \tau) &= \bar{u}_{xx} (1, \tau) \\
    \bar{u}(x, 0) &= u_0(x).
\end{align*}
\]

We make the following assumption (**H1**):

* (averaged condition) There exists function \( f_0 \in H_0^1(I) \) such that

\[
\frac{1}{\tau} \int_0^\tau \|f(t) - f_0\|_{H_0^1(I)} \, dt \leq \min (K, \sigma(\tau)),
\]

where \( K > 0 \) is a positive constant and \( \sigma(\tau) \to 0 \) as \( \tau \to \infty \).
• \( \beta \) is large enough. (7) has a solution \( \bar{u} \in C_b([0, +\infty); H^3(I)) \).

We prove the averaging principle on a finite time interval, the so-called first Bogolyubov theorem:

**Theorem 1.2** (Averaging principle on a finite time interval). Let \( T > 0 \) be arbitrary and fixed, \( u \) and \( \bar{u} \) be the solutions of (6) and (7), \( B_{H^1(I)}(R_0) \) be an absorbing ball of (6) in \( H^1(I) \) with radius \( R_0 \). Suppose that the hypothesis \((H1)\) holds, if \( u(0) = \bar{u}(0) \in B_{H^1(I)}(R_0) \), for \( \tau \in [0, T] \) we have
\[
\|u(\tau) - \bar{u}(\tau)\|_{H^1(I)} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

**Remark 2.** This theorem gives comparison estimate, stability estimate and convergence result (as \( \varepsilon \to 0 \)) between (6) and (7), on finite but large time intervals.

1.3. **Averaging principle on the entire axis.** We make the following assumption \((H2)\):

• (averaged condition) \( f \in L^1_{loc}(\mathbb{R}, H^5(I) \cap H^1_0(I)) \) and there exists a function \( f_0 \in H^5(I) \cap H^1_0(I) \) such that
\[
\frac{1}{\tau} \int_0^\tau \|f(t) - f_0\|_{H^5(I)} dt \leq \min(K, \sigma(\tau)),
\]
where \( K > 0 \) is a positive constant and \( \sigma(\tau) \to 0 \) as \( \tau \to \infty \).

• \( \beta \) is large enough. There exists a function \( u_0 \in H^4(I) \) such that
\[
\begin{cases}
Au_0 + N(u_0) + f_0 = 0 & \text{in } I, \\
u_0(0) = 0 = u_0(1) \\
u_{0xx}(0) = 0 = u_{0xx}(1)
\end{cases}
\]
and \( \|u_0\|_{H^1(I)} \leq \delta_0 \), where \( \delta_0 > 0 \) is sufficiently small.

The result in this section is the averaging principle on the entire real axis, the so-called second Bogolyubov theorem:

**Theorem 1.3** (Averaging principle on the entire axis). Suppose that the hypothesis \((H2)\) holds, if \( \varepsilon \) is small enough, equation (6) has the following properties:

1. In a small neighbourhood of the stationary point \( u_0 \), equation (6) has a unique solution \( u^*(\tau) \), which is bounded on the entire axis and satisfies:
\[
\|u^*(\tau) - u_0\|_{H^1(I)} \leq \delta(\varepsilon) \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]

2. If the function \( f : \mathbb{R} \to H^4(I) \) is an almost periodic function, then the solution \( u^* \) is almost periodic with frequency basis contained in that of \( f \).

**Remark 3.** This theorem shows that if there exists a stationary solution \( u_0 \) of the averaged equation and this stationary point is small enough, then in a small neighbourhood of this point there exists a solution \( u^*(\tau) \) of the original equation that is bounded on the entire real axis. Moreover, this solution is almost periodic if \( f \) is almost periodic.

This paper is organized as follows. In Sec. 2, we present the framework and some preliminary results. In Sec. 3-5, we prove Theorem 1.1-Theorem 1.3.
2. Preliminaries.

2.1. Mathematical setting. We introduce the following mathematical setting:

- We denote by $L^2(I)$ the space of all Lebesgue square integrable real-valued functions on $I$. The inner product on $L^2(I)$ is $(u, v) = \int_I uvdu$, for any $u, v \in L^2(I)$. The norm on $L^2(I)$ is $||u|| = (u, u)^{\frac{1}{2}}$, for any $u \in L^2(I)$. $H^s(I)$ is the classical Sobolev spaces of real-valued functions on $I$. The definition of $H^s(I)$ can be found in [27], the norm on $H^s(I)$ is $|| \cdot ||_{H^s(I)}$.

- The letter $C$ with or without subscripts denotes positive constants whose value may change in different occasions. We will write the dependence of constant on parameters explicitly if it is essential.

- $\lambda_0 > 0$ is the smallest constant such that the following inequality holds

$$||u_x||^2 \geq \lambda_0 ||u||^2,$$

where $u \in H^1_0(I)$.

2.2. Some auxiliary results. In this section, we give some auxiliary results, which will be used in the proof of main results.

We frequently use the Agmon inequality in one dimension:

$$||u||_{L^\infty(I)} \leq C||u||_{H^1(I)},$$

for any $u \in H^1(I)$, where $C$ is a positive constant depending only on $I$.

We also use the following property of $H^1(I)$:

$$||u_1u_2||_{H^1(I)} \leq C||u_1||_{H^1(I)}||u_2||_{H^1(I)},$$

for any $u_1, u_2 \in H^1(I)$, where $C$ is a positive constant depending only on $I$.

By using the classical energy method, we can obtain the following results.

**Proposition 1.** Let $u$ be the solution of

$$\begin{cases}
  u_t + u_{xxxx} + 2u_{xx} + \beta u = 0 & \text{in } Q, \\
  u(0, t) = 0 = u(1, t) & \text{in } (0, T), \\
  u_{xx}(0, t) = 0 = u_{xx}(1, t) & \text{in } (0, T), \\
  u(x, 0) = u_0(x) & \text{in } I,
\end{cases}$$

then $u$ satisfies the following identical equation

$$\frac{1}{2} \frac{d}{dt} ||u||^2 + ||u_{xx}||^2 - 2||u_x||^2 + \beta ||u||^2 = 0.$$

Let $S(t) = e^{At}$ be the semigroup corresponding to the equation

$$\begin{cases}
  u_t + u_{xxxx} + 2u_{xx} + \beta u = 0 & \text{in } I \times (0, +\infty) \\
  u(0, t) = 0 = u(1, t) & \text{in } I \times (0, +\infty) \\
  u_{xx}(0, t) = 0 = u_{xx}(1, t) & \text{in } (0, +\infty) \\
  u(x, 0) = u_0(x) & \text{in } I.
\end{cases}$$

Then, we have

**Proposition 2.** If we take $\beta$ large enough, there exists $\gamma > 0$ such that

$$||S(t)||_{L^2(H^s(I); H^s(I))} \leq C_k e^{-\gamma t},$$

for any $k \in \mathbb{N}$.

**Proof.** The case $k = 0$ can be obtained from Proposition 1. The case $k \geq 1$ can be obtained by the same method as in [12, Proposition 2].
Then, we have the following result:

**Proposition 3.** The inhomogeneous equation $y_t = Ay + f(t)$ with $f \in L^\infty(\mathbb{R}; H^0_0(I))$ has a unique solution $y \in C(\mathbb{R}; H^1_0(I))$ bounded on the entire axis:

$$y(t) = \int_{-\infty}^{t} S(t-s)f(s)ds.$$  

Moreover, there exists a positive constant $K(\gamma)$ such that

$$\|y\|_{C(\mathbb{R};H^1_0(I))} \leq K(\gamma) \|f\|_{L^\infty(\mathbb{R};H^1_0(I))}.$$  

**Proof.** It follows from Proposition 2 that

$$\|y(t)\|_{H^1_0(I)} \leq \left\| \int_{-\infty}^{t} S(t-s)f(s)ds \right\|_{H^1_0(I)} \leq K(\gamma) \|f\|_{L^\infty(\mathbb{R};H^1_0(I))}.$$  

**Proposition 4.** [23, P761 Lemma 2] Denote by $\chi_\varepsilon(t)$ a periodic function with period $\varepsilon T$ defined by the formula $\frac{1}{2} - \frac{t}{\varepsilon T}$ for $0 \leq t \leq \varepsilon T$ and extended as an $\varepsilon T$-periodic function to the whole real axis. For any absolutely continuous (on $[0, \varepsilon T]$) and $\varepsilon T$-periodic numerical function $\eta(t)$ the following equality holds:

$$\int_{0}^{\varepsilon T} \chi_\varepsilon(t-s)\dot{\eta}(s)ds = \eta(t) - \eta^0.$$  

Here $\eta^0$ is the mean value of the function $\eta(t)$ defined by the equality

$$\eta^0 = \frac{1}{\varepsilon T} \int_{0}^{\varepsilon T} \eta(t)dt$$  

and $\dot{\eta}$ denotes the derivative with respect to $s$.

3. **Proof of Theorem 1.1.** The proof of Theorem 1.1 is divided into several steps.

**Step 1.** Existence of periodic solutions $\{u_\varepsilon\}_{\varepsilon > 0}$.

By the same method as in [15, 14], we can obtain the following proposition:

**Proposition 5.** If we take $\beta$ large enough, let $h \in C(\mathbb{R}, H^1_0(I))$ and $h(x, t + T) = h(x, t)$. There exists a positive constant $\delta > 0$ such that $\|h\|_{C(\mathbb{R};H^1_0(I))} < \delta$, then

$$\begin{cases}
  u_t + u_{xxxx} + 2u_{xx} + \beta u + u^3 = h(x, t) & \text{in } I \times \mathbb{R}, \\
  u(0, t) = 0 = u(1, t) & \text{in } \mathbb{R}, \\
  u_{xx}(0, t) = 0 = u_{xx}(1, t) & \text{in } \mathbb{R}
\end{cases}$$

admits a unique $T$-periodic solution.

According to Proposition 5, the following result holds.

**Corollary 1.** Let $f \in C(\mathbb{R}, H^1_0(I))$ and $f(x, t + T) = f(x, t)$. Then there exists a positive constant $\varepsilon_0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, system (2) admits a unique $\varepsilon T$-periodic solution.

**Step 2.** A priori estimates of periodic solutions $\{u_\varepsilon\}_{\varepsilon > 0}$.

In all subsequent arguments, the following proposition plays a crucial role.
Proposition 6. If we take $\beta$ large enough and $u_\varepsilon$ is the $\varepsilon T$-periodic solution to (2), then $u_\varepsilon$ satisfies

$$\sup_{t \in \mathbb{R}} \|u_\varepsilon(t)\| \leq C, \quad \frac{1}{\varepsilon T} \int_0^{\varepsilon T} u_\varepsilon^4 \, dx \, dt \leq C, \quad \frac{1}{\varepsilon T} \int_0^{\varepsilon T} \|u_\varepsilon(t)\|_{H^2(I)}^2 \, dt \leq C,$$

$$\sup_{t \in \mathbb{R}} \|u_\varepsilon(t)\|_{H^1(I)} \leq C, \quad \frac{1}{\varepsilon T} \int_0^{\varepsilon T} \|u_\varepsilon(t)\|_{H^3(I)}^2 \, dt \leq C, \quad \frac{1}{\varepsilon T} \int_0^{\varepsilon T} \|u_\varepsilon\|^2 \, dt \leq C,$$

where $C$ is a constant independent of $\varepsilon$. There exists $\varepsilon_0$ such that when $0 < \varepsilon < \varepsilon_0$, it holds that

$$\sup_{t \in \mathbb{R}} \|u_{\varepsilon xx}(t)\| \leq C, \quad \frac{1}{\varepsilon T} \int_0^{\varepsilon T} \|u_\varepsilon(t)\|_{H^1(I)}^2 \, dt \leq C,$$

where $C$ is a constant independent of $\varepsilon$.

Proof. • Multiplying (2) by $u_\varepsilon$ and integrating it over $I$, we get

$$\frac{1}{2} \frac{d}{dt} \|u_\varepsilon(t)\|^2 + \|u_{\varepsilon xx}(t)\|^2 + \beta \|u_\varepsilon(t)\|^2 + \int_I u_\varepsilon^4 \, dx$$

$$= 2(-u_{\varepsilon xx}, u_\varepsilon) + (f\left(\frac{t}{\varepsilon}\right), u_\varepsilon)$$

$$\leq \frac{1}{2} \|u_{\varepsilon xx}(t)\|^2 + 2\|u_\varepsilon(t)\|^2 \leq \|f\left(\frac{t}{\varepsilon}\right)\| \|u_\varepsilon(t)\|,$$

it is easy to see that

$$\frac{1}{2} \frac{d}{dt} \|u_\varepsilon(t)\|^2 + \frac{1}{2} \|u_{\varepsilon xx}(t)\|^2 + (\beta - 2) \|u_\varepsilon(t)\|^2 + \int_I u_\varepsilon^4 \, dx \leq \|f\left(\frac{t}{\varepsilon}\right)\| \|u_\varepsilon(t)\|,$$

namely, it holds that

$$\frac{d}{dt} \|u_\varepsilon(t)\|^2 + 2(\beta - 2) \|u_\varepsilon(t)\|^2 \leq 2\|f\left(\frac{t}{\varepsilon}\right)\| \|u_\varepsilon(t)\| \leq 2M \|u_\varepsilon(t)\|.$$

According to [15, Lemma 2.5], we obtain $\|u_\varepsilon(t)\| \leq \frac{M}{\beta - 2}$.

In view of the $\varepsilon T$-periodicity of $u_\varepsilon$, we can obtain

$$\int_0^{\varepsilon T} \|u_{\varepsilon xx}(t)\|^2 \, dt + 2\int_0^{\varepsilon T} \int_I u_\varepsilon^4 \, dx \, dt \leq 2\int_0^{\varepsilon T} \|f\left(\frac{t}{\varepsilon}\right)\| \|u_\varepsilon(t)\| \, dt \leq C\varepsilon T.$$

• Multiplying (2) by $-u_{\varepsilon xx}$ and integrating it over $I$, we get

$$\frac{1}{2} \frac{d}{dt} \|u_{\varepsilon xx}(t)\|^2 + \|u_{\varepsilon xx}(t)\|^2 \leq \beta \|u_{\varepsilon xx}(t)\|^2 - \int_I u_{\varepsilon xx} u_\varepsilon^3 \, dx = 2\|u_{\varepsilon xx}(t)\|^2 + (f\left(\frac{t}{\varepsilon}\right), -u_{\varepsilon xx}).$$

According to the interpolation inequality, it holds that

$$- \int_I u_{\varepsilon xx} u_\varepsilon^3 \, dx = \int_I 3u_{xx}^2 u_\varepsilon^2 \, dx \geq 0,$$

$$2\|u_{\varepsilon xx}(t)\|^2 \leq \frac{1}{2} \|u_{\varepsilon xx}(t)\|^2 + C\|u_\varepsilon(t)\|^2,$$

$$(f\left(\frac{t}{\varepsilon}\right), -u_{\varepsilon xx}) = (f\left(\frac{t}{\varepsilon}\right), u_{\varepsilon xx}) \leq \|f\left(\frac{t}{\varepsilon}\right)\| \|u_{\varepsilon xx}\|,$$

thus,

$$\frac{1}{2} \frac{d}{dt} \|u_{\varepsilon xx}(t)\|^2 + (\beta - C) \|u_{\varepsilon xx}(t)\|^2 \leq \|f\left(\frac{t}{\varepsilon}\right)\| \|u_{\varepsilon xx}\| \leq M \|u_{\varepsilon xx}\|.$$

According to [15, Lemma 2.5], we obtain $\|u_{\varepsilon xx}(t)\| \leq \frac{M}{\beta - C}$.
In view of the $\varepsilon T$-periodicity of $u_\varepsilon$, we can obtain

$$\int_0^{\varepsilon T} \|u_{xxxx}(t)\|^2 dt + \int_0^{\varepsilon T} \|u_x(t)\|^2 dt - \int_0^{\varepsilon T} \int_I u_{xxxx} u_x^3 dx dt$$

$$= 2 \int_0^{\varepsilon T} \|u_{xxxx}(t)\|^2 dt + \int_0^{\varepsilon T} (f(\frac{t}{\varepsilon}) - u_{xxxx}) dt,$$

thus,

$$\int_0^{\varepsilon T} \|u_{xxxx}(t)\|^2 dt + \int_0^{\varepsilon T} \|u_x(t)\|^2 dt - \int_0^{\varepsilon T} \int_I u_{xxxx} u_x^3 dx dt$$

$$\leq C \int_0^{\varepsilon T} \|u_{xxxx}(t)\|^2 dt + \int_0^{\varepsilon T} \|f(\frac{t}{\varepsilon})\|^2 dt \leq C \varepsilon T,$$

namely, we have $\frac{1}{\varepsilon T} \int_0^{\varepsilon T} \|u_{xxxx}(t)\|^2 dt \leq C$.

- Multiplying (2) by $u_{xx}$ and integrating it over $I$, we get

$$\|u_{xx}\|^2 + \frac{d}{dt} \left( \frac{1}{2} \|u_{xxx}\|^2 - \frac{1}{2} \|u_x(t)\|^2 + \frac{\beta}{2} \|u_{xx}(t)\|^2 + \frac{1}{4} \int_I u_x^4 dx \right) = (f(\frac{t}{\varepsilon}), u_{xx}).$$

In view of the $\varepsilon T$-periodicity of $u_\varepsilon$, we can obtain

$$\int_0^{\varepsilon T} \|u_{xx}(t)\|^2 dt = \int_0^{\varepsilon T} (f(\frac{t}{\varepsilon}), u_{xx}) dt \leq \frac{1}{2} \int_0^{\varepsilon T} \|f(\frac{t}{\varepsilon})\|^2 dt + \frac{1}{2} \int_0^{\varepsilon T} \|u_{xx}(t)\|^2 dt,$$

namely, we have $\int_0^{\varepsilon T} \|u_{xx}(t)\|^2 dt \leq \int_0^{\varepsilon T} \|f(\frac{t}{\varepsilon})\|^2 dt \leq \varepsilon TM$.

- Multiplying (2) by $u_{xxxx}$ and integrating it over $I$ and noting that

$$\int_I u_{xxxx} u_x^3 dx = \int_I 3u_{xxx}^2 u_x^2 dx + 6 \int_I u_{xx} u_x u_{xx} dx,$$

we have

$$\frac{1}{\varepsilon T} \frac{d}{dt} \|u_{xx}(t)\|^2 + \|u_{xxxx}(t)\|^2 - 2\|u_{xxx}(t)\|^2 + 2\|u_{xx}(t)\|^2 + \frac{\beta}{2} \|u_{xx}(t)\|^2$$

$$+ 3 \int_I u_{xxx}^2 u_x^2 dx + 6 \int_I u_{xx} u_x u_{xx}^2 dx$$

$$= (f(\frac{t}{\varepsilon}), u_{xxxx}),$$

namely, it holds that

$$\frac{1}{\varepsilon T} \frac{d}{dt} \|u_{xx}(t)\|^2 + \|u_{xxxx}(t)\|^2 + \beta \|u_{xx}(t)\|^2 + 3 \int_I u_{xx}^2 u_x^2 dx$$

$$= -6 \int_I u_{xxx} u_x u_{xx} dx + 2\|u_{xxx}(t)\|^2 + (f(\frac{t}{\varepsilon}), u_{xxxx})$$

$$\leq 2 \int_I u_{xx}^2 u_x^2 dx + C \int_I u_{xx}^4 dx + \frac{1}{4} \|u_{xxxx}(t)\|^2$$

$$+ C \|u_{xx}(t)\|^2 + \|f(\frac{t}{\varepsilon})\|^2 + \frac{1}{4} \|u_{xxxx}(t)\|^2.$$

Thus, we have

$$\frac{1}{\varepsilon T} \frac{d}{dt} \|u_{xx}(t)\|^2 + \frac{1}{2} \|u_{xxxx}(t)\|^2 + (\beta - C) \|u_{xx}(t)\|^2 + \int_I u_{xx}^2 u_x^2 dx$$

$$\leq C \int_I u_{xx}^4 dx + \|f(\frac{t}{\varepsilon})\|^2.$$
According to Gagliardo-Nirenberg inequality, we have \( \|u_x\|_{L^4(I)} \leq C\|u_{xx}\|^\frac{2}{3}\|u\|^\frac{1}{3} \), thus, we have \( \|u_x\|^4_{L^4(I)} \leq C\|u_{xx}\|^\frac{2}{3}\|u\|^\frac{1}{3} \). It follows from the estimate of \( \|u_\varepsilon\| \) that
\[
\int_I u_{xx}^4dx \leq C\|u_{xx}\|^\frac{2}{3}.
\]
Then, we have
\[
\frac{d}{dt}\|u_{xxx}(t)\|^2 + \|u_{xxxxx}(t)\|^2 + (\beta - C)\|u_{xxx}(t)\|^2 + \int_I u_{xx}^2 u_{x}^2 dx
\leq C\|u_{xxx}\|^\frac{2}{3} + \|f(t)\|^2.
\]
By taking \( \beta >> 1 \) and using the Young inequality, we have
\[
\frac{d}{dt}\|u_{xxx}(t)\|^2 + \|u_{xxxxx}(t)\|^2 \leq C\|u_{xxx}(t)\|^\frac{2}{3} + M^2. \tag{8}
\]
We set \( y = \|u_{xxx}(t)\|^2 \), it holds that
\[
\frac{d}{dt}y \leq Cy^\frac{2}{3} + M^2 \leq y^2 + C_0.
\]
It follows from the comparison principle of ODE that
\[
y \leq \sqrt{C_0}\tan(\sqrt{C_0}t + \arctan \frac{y(0)}{\sqrt{C_0}}),
\]
for \( t \) satisfies \( \sqrt{C_0}t + \arctan \frac{y(0)}{\sqrt{C_0}} < \frac{\pi}{2} \).

If we take \( 0 < \varepsilon << 1 \) such that
\[
|\sqrt{C_0}(\varepsilon T) + \arctan \frac{y(0)}{\sqrt{C_0}} | < \frac{\pi}{2},
\]
we obtain an estimate of \( y(t) \) which is uniform in \([0, \varepsilon T]\), namely,
\[
\|u_{xxx}(t)\|^2 \leq \sqrt{C_0}\tan(\sqrt{C_0}t + \arctan \frac{y(0)}{\sqrt{C_0}}), ~ \forall t \in [0, \varepsilon T].
\]
By the \( \varepsilon T \)-periodicity of the function \( y(t) \), we have an estimate uniform on the whole axis \( \mathbb{R} \),
\[
\sup_{t \in \mathbb{R}} \|u_{xxx}(t)\| \leq C.
\]

In view of (8) and the \( \varepsilon T \)-periodicity of \( u_\varepsilon \), we can obtain
\[
\int_0^{\varepsilon T} \|u_{xxxxx}(t)\|^2 dt \leq C \int_0^{\varepsilon T} \|u_{xxx}\|^\frac{5}{3} dt + M^2 \varepsilon T \leq \varepsilon TC.
\]

**Step 3.** The convergence of \( u_\varepsilon - (u_\varepsilon)^0 \), where \( (u_\varepsilon)^0 = \frac{1}{\varepsilon T} \int_0^{\varepsilon T} u_\varepsilon(t)dt \).

**Proposition 7.** There exists \( \varepsilon_0 \) such that if \( 0 < \varepsilon << \varepsilon_0 \), it holds that
\[
\|(u_\varepsilon)^0\|^2_{L^4(I)} \leq C, \tag{9}
\]
where \( C \) is a constant independent of \( \varepsilon \).
Proof. Noting the facts that
\[ \| (u_\varepsilon)^0 \|^2 = \frac{1}{\varepsilon T} \int_0^{\varepsilon T} u_\varepsilon(t) \, dt \leq \frac{1}{\varepsilon T} \int_0^{\varepsilon T} \| u_\varepsilon(t) \| \, dt \]
\[ \leq \frac{1}{\varepsilon T} (\varepsilon T)^{\frac{3}{2}} (\int_0^{\varepsilon T} \| u_\varepsilon(t) \|^2 \, dt)^{\frac{1}{2}} \leq C \frac{1}{\varepsilon T} (\varepsilon T)^{\frac{3}{2}} (\varepsilon T)^{\frac{1}{2}} = C, \]
we have
\[ \| (u_\varepsilon)^0 \|^2 = \frac{1}{\varepsilon T} \int_0^{\varepsilon T} (u_\varepsilon(t))_{xxxx} \, dt \leq \frac{1}{\varepsilon T} \int_0^{\varepsilon T} \| u_\varepsilon(t) \|_{H^1(I)} \, dt \]
\[ \leq \frac{1}{\varepsilon T} (\varepsilon T)^{\frac{3}{2}} (\int_0^{\varepsilon T} \| u_\varepsilon(t) \|^2_{H^1(I)} \, dt)^{\frac{1}{2}} \leq C \frac{1}{\varepsilon T} (\varepsilon T)^{\frac{3}{2}} (\varepsilon T)^{\frac{1}{2}} = C, \]
and according to the interpolation inequality, we can obtain (9).

**Proposition 8.** There exists a constant \( C > 0 \) such that the time-periodic solutions \( u_\varepsilon \) satisfy
\[ \sup_{t \in \mathbb{R}} \| u_\varepsilon(t) - (u_\varepsilon)^0 \| \leq C\varepsilon T. \]

Proof. Now let us multiply both sides of (1) by the function \( \chi_\varepsilon(t-s) \) used in Proposition 4 and integrate the result over \( s \) from 0 to \( \varepsilon T \). Then, in view of the \( \varepsilon T \)-periodicity of the function \( u_\varepsilon \) and by Proposition 4, we have
\[ u_\varepsilon(t) - (u_\varepsilon)^0 = \int_0^{\varepsilon T} \chi_\varepsilon(t-s) [f(\frac{s}{\varepsilon}) - u_{xxxx}(s) - 2u_{xxx}(s) - \beta u_x(s) - u_\varepsilon(s)] \, ds. \]
Since \( |\chi_\varepsilon(t-s)| \leq \frac{1}{2} \), we have
\[ 2\| u_\varepsilon(t) - (u_\varepsilon)^0 \| \leq \int_0^{\varepsilon T} \| f(\frac{t}{\varepsilon}) \| \, dt + \int_0^{\varepsilon T} \| u_{xxxx}(t) \| \, dt + 2 \int_0^{\varepsilon T} \| u_{xxx}(t) \| \, dt + \beta \int_0^{\varepsilon T} \| u_\varepsilon(t) \| \, dt + \int_0^{\varepsilon T} \| u_\varepsilon(t)^3 \| \, dt. \]
According to Proposition 6, it holds that
\[ \int_0^{\varepsilon T} \| f(\frac{t}{\varepsilon}) \| \, dt \leq M\varepsilon T, \quad \int_0^{\varepsilon T} \| u_{xxxx}(t) \| \, dt \leq (\varepsilon T)^{\frac{3}{2}} (\int_0^{\varepsilon T} \| u_{xxxx} \|^2 \, dt)^{\frac{1}{2}} \leq C\varepsilon T, \]
\[ \int_0^{\varepsilon T} \| u_{xxx}(t) \| \, dt \leq C\varepsilon T, \quad \int_0^{\varepsilon T} \| u_\varepsilon(t)^3 \| \, dt \leq C\varepsilon T, \quad \int_0^{\varepsilon T} \| u_\varepsilon(t) \| \, dt \leq C\varepsilon T, \]
thus, we have \( \sup_{t \in \mathbb{R}} \| u_\varepsilon(t) - (u_\varepsilon)^0 \| \leq C\varepsilon T. \)

**Step 4.** The convergence subsequence of \( \{(u_\varepsilon)^0\}_{\varepsilon > 0} \).

**Proposition 9.** Any sequence of solutions \( \{u_\varepsilon\}_{\varepsilon > 0} \) of (2) contains a subsequence \( \{(u_{\varepsilon_n})^0\}_{n \in \mathbb{N}} \) converging in \( L^2(I) \) to some solution \( u^0 \) of (3) uniformly on \( \mathbb{R} \).

Proof. It follows from (2) that \( (u_\varepsilon)^0_{xxxx} + \frac{1}{\varepsilon T} \int_0^{\varepsilon T} (2u_{xxx} + \beta u_x + u_\varepsilon^3) \, dt = f_0, \) if we set
\[ Lu = u_{xxxx}, \quad g(u) = 2u_{xx} + \beta u + u^3, \]
it holds that \( (u_\varepsilon)^0_{xxxx} + \frac{1}{\varepsilon T} \int_0^{\varepsilon T} g(u_\varepsilon) \, dt = f_0, \) namely,
\[ (u_\varepsilon)^0 + \frac{1}{\varepsilon T} \int_0^{\varepsilon T} L^{-1} g(u_\varepsilon) \, dt = L^{-1} f_0, \]
then, we have
\[(u_\varepsilon)^0 - L^{-1}f_0 = -\frac{1}{\varepsilon T} \int_0^{\varepsilon T} L^{-1}g(u_\varepsilon)dt\]
\[= -L^{-1}g((u_\varepsilon)^0) + L^{-1}g((u_\varepsilon)^0) - \frac{1}{\varepsilon T} \int_0^{\varepsilon T} L^{-1}g(u_\varepsilon)dt\]
\[= -L^{-1}g((u_\varepsilon)^0) + \frac{1}{\varepsilon T} \int_0^{\varepsilon T} L^{-1}[g((u_\varepsilon)^0) - g(u_\varepsilon)]dt.\]
Thus, it holds that
\[(u_\varepsilon)^0 [L^{-1}g((u_\varepsilon)^0) - L^{-1}f_0] = \frac{1}{\varepsilon T} \int_0^{\varepsilon T} L^{-1}[g((u_\varepsilon)^0) - g(u_\varepsilon)]dt.\]

Let us multiply both sides of the resulting equality scalarly in \(L^2(I)\) by an arbitrary element \(\varphi \in L^2(I)\), then
\[(u_\varepsilon)^0, \varphi) + \int (L^{-1}g((u_\varepsilon)^0) - L^{-1}f_0), \varphi) \frac{1}{\varepsilon T} \int_0^{\varepsilon T} (L^{-1}g((u_\varepsilon)^0) - g(u_\varepsilon)), \varphi)dt.\]

It is not difficult to deduce that
\[(L^{-1}[((u_\varepsilon)^0)_{xx} - u_{xx}], \varphi) = (L^{-\frac{1}{2}}[(u_\varepsilon)^0)_{xx} - u_{xx}], L^{-\frac{1}{2}}\varphi) = ((u_\varepsilon)^0 - u_\varepsilon, L^{-\frac{1}{2}}\varphi)\]
\[\leq C||(u_\varepsilon)^0 - u_\varepsilon|| ||\varphi||,\]
\[(L^{-1}[(u_\varepsilon)^0 - u_\varepsilon], \varphi) \leq C||(u_\varepsilon)^0 - u_\varepsilon|| ||\varphi||,\]

(10)
it follows from Proposition 6 and Proposition 7 that
\[(L^{-1}[((u_\varepsilon)^0)^3 - u_\varepsilon^3], \varphi) = ((u_\varepsilon)^0)^3 - u_\varepsilon^3, L^{-1}\varphi)\]
\[= ((u_\varepsilon)^0 - u_\varepsilon)((u_\varepsilon)^0)^2 + (u_\varepsilon)^0u_\varepsilon + u_\varepsilon^2, L^{-1}\varphi)\]
\[\leq C||(u_\varepsilon)^0 - u_\varepsilon|| ||\varphi||,\]

(11)
according to the above estimates (10)-(11), we have
\[(L^{-1}[g((u_\varepsilon)^0) - g(u_\varepsilon)], \varphi) \leq C||(u_\varepsilon)^0 - u_\varepsilon|| ||\varphi||,\]

(12)
thus, it holds that
\[\left| \frac{1}{\varepsilon T} \int_0^{\varepsilon T} (L^{-1}[g((u_\varepsilon)^0) - g(u_\varepsilon)], \varphi)dt \right| \leq C \sup_{t \in \mathbb{R}} ||(u_\varepsilon)^0 - u_\varepsilon|| ||\varphi||.\]

On the other hand, it follows from Proposition 7 that \{(u_\varepsilon)^0\}_{\varepsilon > 0} is compact in \(H^2(I)\). Therefore, each sequence of this family contains a convergent (in \(H^2(I)\)) subsequence \{(u_{\varepsilon_n})^0\}_{n \in \mathbb{N}}, we denote its limit by \(u^0\). By the same method as in (12), we have
\[||(L^{-1}[g((u_{\varepsilon_n})^0) - g(u^0)], \varphi)|| \leq C||(u_{\varepsilon_n})^0 - u^0|| ||\varphi||.\]

By passing to the limit in
\[(u_{\varepsilon_n}, \varphi) + (L^{-1}g((u_{\varepsilon_n})^0) - L^{-1}f_0, \varphi) = \frac{1}{\varepsilon_n T} \int_0^{\varepsilon_n T} (L^{-1}[g((u_{\varepsilon_n})^0) - g(u_{\varepsilon_n})], \varphi)dt\]
as \(n \to \infty\), we conclude that \((u^0, \varphi) + (L^{-1}g(u^0) - L^{-1}f_0, \varphi) = 0\), for any \(\varphi \in L^2(I)\).
Namely, we have \((u^0 + [L^{-1}g(u^0) - L^{-1}f_0], \varphi) = 0\), thus, it holds that
\[Lu^0 + g(u^0) - f_0 = 0.\]
Therefore, \(u^0\) is a solution of (3) and \(\|(u_{\varepsilon_n})^0 - u^0\| \rightarrow 0\) as \(n \rightarrow \infty\). Together with Proposition 8, this implies
\[
\sup_{t \in \mathbb{R}} \|u_{\varepsilon_n}(t) - u^0\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
\[
\square
\]

**Step 5.** Proof of Theorem 1.1.
Suppose that the assertion of Theorem 1.1 does not hold. Then there exist numbers \(\delta_0, \varepsilon_n \rightarrow 0\), and \(t_n \in \mathbb{R}\) such that
\[
d_{t_n}(U_{\varepsilon_n}, U_0) > \delta_0.
\]
Therefore, for some subsequence of functions \(u_{\varepsilon_{n_k}}\) and for numbers \(t_{n_k} \in \mathbb{R}\), we have
\[
\inf_{u^0 \in U_0} \|u_{\varepsilon_{n_k}}(t_{n_k}) - u^0\| \geq \frac{\delta_0}{2}.
\]
But this contradicts the fact (see Proposition 9) that the sequence \(u_{\varepsilon_{n_k}}\) contains a subsequence converging to an element from \(U_0\). Thus, Theorem 1.1 is proved.

4. **Proof of Theorem 1.2.**

**Proof of Theorem 1.2.** It was shown in [16, 24], that the nonlinear semigroup \(F(t)\) corresponding to equation (4) possesses absorbing sets in the space \(H^1(I)\), in particular, the ball \(B_{H^1(I)}(R_0)\), where \(R_0\) is large enough. This means that for every bounded set \(B\) in \(H^1(I)\), there exists \(t(B, R_0) > 0\) such that
\[
F(t)B \subset B_{H^1(I)}(R_0), \quad t > t(B, R_0).
\]
In addition, the semigroup is uniformly bounded in this space, that is, given any ball, in particular, the ball \(B_{H^1(I)}(R_0)\), there exists a ball \(B_{H^1(I)}(R)\) such that
\[
F(t)B_{H^1(I)}(R_0) \subset B_{H^1(I)}(R), \quad t > 0.
\]
By increasing \(R\) we may assume that
\[
F(t)B_{H^1(I)}(R_0) \subset B_{H^1(I)}(R - \rho), \quad t > 0, \rho > 0.
\]
We suppose that the trajectory of (6) with \(u(0) \in B_{H^1(I)}(R_0)\) stays in the ball \(B_{H^1(I)}(R)\) on the interval \([0, \frac{T}{\varepsilon}]\), namely, \(u(\tau) \in B_{H^1(I)}(R)\) for any \(\tau \in [0, \frac{T}{\varepsilon}]\). This will be proved in the end of this section.

Given a point \(u_0\) in \(B_{H^1(I)}(R_0)\), let trajectories of systems (6) and (7) start from this point. Set \(z = u - \bar{u}\), then \(z\) satisfies
\[
z_\tau = \varepsilon[Az + N(u) - N(\bar{u}) + f - f_0].
\]
By the semigroup theory, we can obtain the following equivalent integral equation
\[
z(\tau) = \varepsilon \int_0^\tau e^{\varepsilon A(\tau-s)} [N(u) - N(\bar{u}) + f(s) - f_0] ds.
\]
It is easy to check that
\[
\| \int_0^\tau e^{\varepsilon A(\tau-s)} [N(u) - N(\bar{u})] ds \|_{H^1(I)} \leq C \int_0^\tau e^{-\varepsilon \gamma(\tau-s)} R^2 \|z(s)\|_{H^1(I)} ds.
\]
Now, we estimate the term $\int_0^T e^{\varepsilon A(t-s)}(f - f_0)ds$, integrating by parts, by the same method as in [12, P2158], we have
\[
\| \int_0^T e^{\varepsilon A(t-s)}(f - f_0)ds \|_{H^1(I)} = \| e^{\varepsilon A}\int_0^T (f - f_0)ds - \varepsilon \int_0^T A e^{\varepsilon A(t-s)} \int_s^T (f - f_0)dt ds \|_{H^1(I)} \\
\leq e^{-\varepsilon \gamma T} \| \frac{1}{T} \int_0^T (f - f_0)ds \|_{H^1(I)} + \| \varepsilon \| \int_0^T A e^{\varepsilon A(t-s)} \int_s^T (f - f_0)dt ds \|_{H^1(I)} \\
\leq e^{-\varepsilon \gamma T} \min (K, \sigma(\tau)) + \varepsilon \int_0^T e^{-\varepsilon \gamma T} \min (K, \sigma(r)) dr.
\]
By the same method as in [21, P659], according to the above inequalities, we have
\[
\| z(\tau) \|_{H^1(I)} \leq \varepsilon R^2 \int_0^T \| z(s) \|_{H^1(I)} ds + G(\varepsilon),
\]
where $G(\varepsilon) \to 0$ as $\varepsilon \to 0$. It follows from Gronwall inequality that
\[
\| z(\tau) \|_{H^1(I)} \leq e^{\varepsilon R^2} G(\varepsilon), \quad \tau \in \left[ 0, \frac{T}{\varepsilon} \right].
\]  
(13)

Namely, by assuming that the trajectory $u(t)$ with initial value $u(0) \in B_{H^1(I)}(R_0)$ stays in the ball $B_{H^1(I)}(R)$ on the interval $[0, \frac{T}{\varepsilon}]$, we have proved the proximity of solutions of (6) and (7) in $H^1(I)$.

Next, we prove the trajectory $u(t)$ with initial condition $u(0) \in B_{H^1(I)}(R_0)$ stays in the ball $B_{H^1(I)}(R)$ on the interval $[0, \frac{T}{\varepsilon}]$.

Indeed, let $\varepsilon$ be so small that the right-hand side of (13) is less than $\frac{\rho}{2}$, where $\rho$ is defined earlier in this section when we discuss absorbing sets. Suppose that the trajectory $u(t)$ leaves the ball $B(R)$ during the interval $[0, \frac{T}{\varepsilon}]$ and let $\tau^*$ be the first moment when $\| u(\tau^*) \|_{H^1(I)} = R$. However, on the interval $\tau \in [0, \tau^*]$ both trajectories of (6) and (7) stay in the ball $B_{H^1(I)}(R)$ and what we have proved so far shows that the inequality
\[
\| u(\tau) - \bar{u}(\tau) \|_{H^1(I)} \leq \frac{\rho}{2}
\]
is valid. In particular, it is valid for $\tau = \tau^*$. This together with the inequality $\| \bar{u}(\tau^*) \|_{H^1(I)} \leq R - \rho$, which holds by the hypothesis of the theorem, gives the contradiction:
\[
\| u(\tau^*) \|_{H^1(I)} \leq \| u(\tau^*) - \bar{u}(\tau^*) \|_{H^1(I)} + \| \bar{u}(\tau^*) \|_{H^1(I)} \leq R - \frac{\rho}{2}.
\]  

5. **Proof of Theorem 1.3.** Note the fact $Au_0 + N(u_0) + f_0 = 0$. We change the variable in the equation $u = u_0 + h - \varepsilon v$ in (6), where $v$ and $h$ satisfy
\[
\begin{cases}
v_\tau = \varepsilon Av + f - f_0, \\
v(0, \tau) = v(1, \tau), \\
v_{xx}(0, \tau) = v_{xx}(1, \tau)
\end{cases}
\]  
(14)

and
\[
\begin{cases}
h_\tau = \varepsilon [Ah + N(u_0 + h - \varepsilon v) - N(u_0)], \\
h(0, \tau) = h(1, \tau), \\
h_{xx}(0, \tau) = h_{xx}(1, \tau),
\end{cases}
\]  
(15)

we set $F(h, \varepsilon, \tau) = N(u_0 + h - \varepsilon v) - N(u_0)$. 

\[\square\]
5.1. Study of $v$. We consider equation (14).

**Proposition 10.** Equation (14) has a unique solution $v(\tau, \varepsilon)$ bounded in $H^1(I)$ uniformly in $\tau \in \mathbb{R}$. Moreover

$$\lim_{\varepsilon \to 0} \sup_{\tau \in \mathbb{R}} \|v(\tau, \varepsilon)\|_{H^1(I)} = 0.$$

If $f$ is almost periodic with value in $H^1(I)$, then $v$ is almost periodic in $H^1(I)$ with frequency basis contained in that of $f$.

**Proof.** The desired solution is given by the formula

$$v(\tau, \varepsilon) = \int_{-\infty}^{\tau} e^{\varepsilon A(\tau-s)}(f(s) - f_0)ds,$$

thus,

$$\|v(\tau, \varepsilon)\|_{H^1(I)} = \|\varepsilon \int_{-\infty}^{\tau} e^{\varepsilon A(\tau-s)}(f(s) - f_0)ds\|_{H^1(I)}$$

$$= \|\varepsilon \int_{0}^{\infty} se^{\varepsilon A t} A \frac{1}{s} \int_{0}^{s} (f(t) - f_0)dt ds\|_{H^1(I)}$$

$$\leq \varepsilon \int_{0}^{\infty} se^{-\gamma s} \|A \frac{1}{s} \int_{0}^{s} (f(t) - f_0)dt\|_{H^1(I)} ds$$

$$= \varepsilon \int_{0}^{\infty} se^{-\gamma s} \|A(0(\tau - t) - f_0)\|_{H^1(I)} dt ds$$

$$\leq \varepsilon \int_{0}^{\infty} se^{-\gamma s} \|f(t) - f_0\|_{H^1(I)} dt ds$$

$$\leq \varepsilon \int_{0}^{\infty} se^{-\gamma s} \frac{1}{s} \int_{0}^{s} se^{-\gamma t} ds.$$

Letting $\delta \to 0$ and then $\varepsilon \to 0$ we obtain $\lim_{\varepsilon \to 0} \sup_{\tau \in \mathbb{R}} \|v(\tau, \varepsilon)\|_{H^1(I)} = 0$.

Let us prove the last statement of the proposition.

It is easy to check that

$$v(\tau, \varepsilon) = \int_{0}^{+\infty} e^{\varepsilon A t}(f(\tau - t) - f_0)dt,$$

thus,

$$\sup_{\tau \in \mathbb{R}} \|v(\tau + \tau_m) - v(\tau)\| = \sup_{\tau \in \mathbb{R}} \|\int_{0}^{\infty} e^{\varepsilon A t}(f(\tau + \tau_m - t) - f(\tau - t))dt\|_{H^1(I)}$$

$$\leq \sup_{\tau \in \mathbb{R}} \int_{0}^{\infty} \|e^{\varepsilon A t}(f(\tau + \tau_m - t) - f(\tau - t))\|_{H^1(I)} dt$$

$$\leq \sup_{\tau \in \mathbb{R}} \int_{0}^{\infty} e^{-\varepsilon \gamma t} \|(f(\tau + \tau_m - t) - f(\tau - t))\|_{H^1(I)} dt$$

$$\leq \sup_{\tau \in \mathbb{R}} \int_{0}^{\infty} e^{-\varepsilon \gamma t} \sup_{\xi \in \mathbb{R}} \|f(\xi + \tau_m) - f(\xi)\|_{H^1(I)} dt$$

$$= \sup_{\xi \in \mathbb{R}} \|f(\xi + \tau_m) - f(\xi)\|_{H^1(I)} \int_{0}^{\infty} e^{-\varepsilon \gamma t} dt,$$

it is sufficient to show that every $m$-recurrent sequence $\{\tau_m\}$ is also $v$-recurrent. □
5.2. Study of $h$. We revert to the original time $t$ in (15)

$$h_t = Ah + Q(h, \varepsilon, t),$$

(16)

where $Q(h, \varepsilon, t) = F(h, \varepsilon, t)$.

**Proposition 11.** Let $\varepsilon, \delta_0$ be small enough, equation (16) has a unique bounded solution $h^*$ with the following properties:

1) $\|h^*\|_{C^0_b(\mathbb{R}, H^1(I))} \leq \delta(\varepsilon) \to 0$, $\varepsilon \to 0$.

2) If the function $Q(h, \varepsilon, \cdot) : \mathbb{R} \to H^1(I)$ is almost periodic, then the solution $h^*$ is almost periodic with frequency basis contained in that of $Q$.

**Proof.** 1) Some properties of the function $Q$ are given below. A simple calculation yields

$$\|Q(h_1, \varepsilon, t) - Q(h_2, \varepsilon, t)\|_{H^1(I)} = \|N(u_0 + h_1 - \varepsilon v) - N(u_0 + h_2 - \varepsilon v)\|_{H^1(I)}$$

$$\leq \frac{3}{2}\|u_0 + h_1 - \varepsilon v\|_{H^1(I)}^2 + \|u_0 + h_2 - \varepsilon v\|_{H^1(I)}^2\|h_1 - h_2\|_{H^1(I)}$$

$$\leq C\|u_0\|_{H^1(I)}^2 + \|h_1\|_{H^1(I)}^2 + \|h_2\|_{H^1(I)}^2 + \varepsilon^2\|v\|_{H^1(I)}^2\|h_1 - h_2\|_{H^1(I)},$$

$$\|Q(h, \varepsilon, t)\|_{H^1(I)} = \|N(u_0 + h - \varepsilon v) - N(u_0)\|_{H^1(I)}$$

$$\leq C\|u_0 + h - \varepsilon v\|_{H^1(I)} + \|u_0\|_{H^1(I)}\|h - \varepsilon v\|_{H^1(I)}.$$  

We define $\mathcal{F}(h)(t) = \int_{t}^{1} S(t - s)Q(h(s), \varepsilon, s)ds$. It is easy to see that

$$\|\mathcal{F}(h)\|_{C^0_b(\mathbb{R}, H^1(I))}$$

$$= \sup_{t \in \mathbb{R}} \| \int_{t}^{1} S(t - s)Q(h(s), \varepsilon, s)ds\|_{H^1(I)}$$

$$= \sup_{t \in \mathbb{R}} \| \int_{-\infty}^{t} S(t - s)Q(0, \varepsilon, s) + Q(h(s), \varepsilon, s) - Q(0, \varepsilon, s)ds\|_{H^1(I)}$$

$$\leq C(\gamma)(\|u_0\|_{C^0_b(\mathbb{R}, H^1(I))} + \|u_0\|_{H^1(I)}\|v\|_{C^0_b(\mathbb{R}, H^1(I))})$$

$$+ (\|u_0\|_{H^1(I)}^2 + \|h_1\|_{C^0_b(\mathbb{R}, H^1(I))}^2 + \varepsilon^2\|v\|_{C^0_b(\mathbb{R}, H^1(I))}^2)\|h\|_{C^0_b(\mathbb{R}, H^1(I))},$$

$$\|\mathcal{F}(h_1) - \mathcal{F}(h_2)\|_{C^0_b(\mathbb{R}, H^1(I))}$$

$$= \sup_{t \in \mathbb{R}} \| \int_{t}^{1} S(t - s)(Q(h_1(s), \varepsilon, s) - Q(h_2(s), \varepsilon, s))ds\|_{H^1(I)}$$

$$\leq C\gamma(\|u_0\|_{H^1(I)}^2 + \|h_1\|_{C^0_b(\mathbb{R}, H^1(I))}^2 + \|h_2\|_{C^0_b(\mathbb{R}, H^1(I))}^2)$$

$$+ \varepsilon^2\|v\|_{C^0_b(\mathbb{R}, H^1(I))}^2\|h_1 - h_2\|_{C^0_b(\mathbb{R}, H^1(I))}.$$  

If $h_1, h_2 \in B_{H^1(I)}(\rho)$ and choosing $\rho, \varepsilon, \|u_0\|_{H^1(I)}, \|m\|_{C^0_b(\mathbb{R}, H^1(I))}$ small enough, $\mathcal{F}$ is a contraction map taking a ball in $C^0_b(\mathbb{R}, H^1(I))$ of radius $\rho$ into itself. By the Banach contraction principle, $\mathcal{F}$ has a unique fixed point

$$\mathcal{F}h^* = h^*,$$

in other words, $h^*$ is a $H^1(I)$-bounded solution of equation (16).
2) Let \( \{ \tau_m \} \) be a \( Q \)-recurrent sequence, then
\[
\sup_{\tau \in \mathbb{R}} \| h^*(\tau + \tau_m) - h^*(\tau) \|_{H^1(I)}
\]
\[
= \sup_{\tau \in \mathbb{R}} \| \mathcal{F}(h^*)(\tau + \tau_m) - \mathcal{F}(h^*)(\tau) \|_{H^1(I)}
\]
\[
= \sup_{\tau \in \mathbb{R}} \| \int_{-\infty}^{\tau} S(\tau + \tau_m - s)Q(h(s), \varepsilon, s)ds - \int_{-\infty}^{\tau} S(\tau - s)Q(h(s), \varepsilon, s)ds \|_{H^1(I)}
\]
\[
= \sup_{\tau \in \mathbb{R}} \| \int_{-\infty}^{\tau} S(\tau - s)Q(h^*(s + \tau_m), \varepsilon, s + \tau_m)ds
\]
\[
- \int_{-\infty}^{\tau} S(\tau - s)Q(h^*(s), \varepsilon, s)ds \|_{H^1(I)}
\]
\[
\leq \mathcal{K}(\gamma) \sup_{s \in \mathbb{R}} \| Q(h^*(s + \tau_m), \varepsilon, s + \tau_m) - Q(h^*(s + \tau_m), \varepsilon, s) \|_{H^1(I)}
\]
\[
+ \frac{1}{2} \sup_{\tau \in \mathbb{R}} \| h^*(\tau + \tau_m) - h^*(\tau) \|_{H^1(I)},
\]
thus,
\[
\sup_{\tau \in \mathbb{R}} \| h^*(\tau + \tau_m) - h^*(\tau) \|_{H^1(I)}
\]
\[
\leq 2\mathcal{K}(\gamma) \sup_{s \in \mathbb{R}} \| Q(h^*(s + \tau_m), \varepsilon, s + \tau_m) - Q(h^*(s + \tau_m), \varepsilon, s) \|_{H^1(I)}
\]
\[
\leq 2\mathcal{K}(\gamma)\varepsilon_m,
\]
that is, the sequence \( \{ \tau_m \} \) is \( h^* \)-recurrent.

\( \square \)

**Proof of Theorem 1.3.** According to Proposition 10 and Proposition 11, noting the representation \( u = u_0 + h - \varepsilon v \), we can prove Theorem 1.3.

\( \square \)

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