Chern band insulators in a magnetic field

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Abstract

The effect of a magnetic field on a two-dimensional Chern band insulator is discussed. It is shown that, unlike the trivial insulator, an anomalous Hall insulator with Chern number \( C \) becomes a metal when a magnetic field is applied at constant particle density, for any \( C > 0 \). For a time-reversal invariant topological insulator with a spin Chern resolved number, \( C_↑ = -C_↓ = C \), the magnetic field induces a spin polarized spin Hall insulator. We consider also the effect of a superlattice potential and extend previous results for the quantization of the Hall conductance of filled Hofstadter bands to this problem.

Keywords: topological insulators, Chern number, Quantum Hall effects, Hofstadter spectrum, edge states

((Some figures may appear in colour only in the online journal)

1. Introduction

Two-dimensional Bloch bands with non-trivial topology have recently become a topic of intense research activity [1, 2]. For spinless fermions, the most important topological index is the Chern number, \( C \), of the filled band. Non-zero \( C \) implies the existence of chiral states along the system’s boundary, the number of which is given by the bulk-boundary correspondence, \( N_R - N_L = \delta C \), where \( N_{R(L)} \) counts the number of right- (left-) moving states along the boundary between two regions where the Chern number differs by \( \delta C \). If the system is in contact with a trivial insulator (the vacuum, for instance), the edge states’ chirality depends on the sign of \( C \), as shown in figure 1(a). Let us consider now that a magnetic field is applied perpendicular to a two-dimensional system. We do not consider here the Zeeman coupling to the spin, only the effect of the minimal coupling to orbital degrees of freedom. The magnetic field’s vector potential introduces a chirality which manifests itself in the Hall effect (figure 1(b)). The effect of a magnetic field on a two-dimensional Bloch band has long been established: it splits the original band into sub-bands, the so-called Hofstadter butterfly spectrum, as shown explicitly for the square [3, 4], hexagonal [5] and honeycomb [6] lattices. The Hall conductance is quantized when the sub-bands are filled.

![Figure 1](image-url)

Figure 1. The chirality of the edge states for: (a) a 2D topological system; (b) a 2D system under an applied perpendicular magnetic field; (c) the geometry considered in figure 4. The chiralities in (a) and (b) are reversed for opposite signs of \( C \) or \( B_z \).

While the topological properties of 2D systems have been intensively studied, the question of how an applied magnetic field modifies a topological system has received less attention [7, 8]. Two questions immediately arise, as follows. (i) How does the chirality introduced by the magnetic field...
interfere with that from the underlying band’s topology? Also, given that the periodic potential responsible for the topologically non-trivial band produces a Hofstadter spectrum under a magnetic field, with a quantized charge Hall conductance when the chemical potential lies in the band gaps, then (ii) how does the non-trivial topology of the underlying lattice modify this quantization of the Hall conductance?

The effect of a weak magnetic field in a two-band \( C = 1 \) Chern insulator has been considered by Haldane in his seminal paper [9]. The pair of \( n = 0 \) Landau levels, one per Dirac cone, is degenerate in this model, in contrast to a trivial system. This fact was used by Haldane to prove the existence of an anomalous Hall insulator (AHI) in the limit of zero magnetic field. Although Haldane’s original construction was devised for the honeycomb lattice, we here consider its general application to any time-reversal invariant lattice model with arbitrary number of Dirac cones and refer to it as ‘Haldane’s AHI’. It is the effect of a perpendicular magnetic field applied to this generalized Haldane AHI that we address here.

In section 2 we review Haldane’s construction of the AHI in a way that can be generalized to models with arbitrary Chern number. In section 3 the effect of a weak magnetic field on the AHI is considered and explicit lattice models are presented. The effect of the magnetic field on the Kane–Mele \( \mathbb{Z}_2 \)-topological insulator is also addressed. In section 4 the Hofstadter spectrum for the AHI in a magnetic field is presented. A final summary is given in section 5.

2. Generalized Haldane AHI

To model a spinless topological Bloch band, a two-sublattice system, at least, is needed [10]. The Hamiltonian that contains the minimal ingredients can be written as

\[
\hat{H} = \hbar \langle k \rangle \cdot \tau
\]  

where the Pauli matrices \( \tau_\mu \) (\( \mu = 1, 2, 3 \)) act on the sublattice (‘pseudo-spin’) space and \( \langle k \rangle = (k_x, k_y) \) runs over the Brillouin zone (BZ). There are points in the BZ where the gap closes when a topological transition occurs. Right at the transition, the spectrum at such points is a Dirac cone [11], and the opening of a gap is due to a finite Dirac mass, \( \hbar_c \). Suppose that at some point \( \mathbf{K} \) in the BZ the Hamiltonian can be linearized as

\[
\hat{H} \approx (i \hbar \tau_3 \partial_\xi - i \hbar \tau_1 \partial_\eta, \hat{h}_z) \cdot \tau.
\]

The contribution of this Dirac point to the Chern invariant of the lower band,

\[
C = \frac{1}{4\pi} \int \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} \frac{\partial \hat{h}_x}{\partial k_x} \times \frac{\partial \hat{h}_y}{\partial k_y},
\]

is given by \( \Delta C = \frac{1}{2} \text{sgn}[\hbar_z(\mathbf{K})] \). Time-reversal symmetry (TRS) requires both \( \hbar_{\pm z}(\mathbf{K}) \) to be even functions of \( \mathbf{k} \) and \( \hbar_z \) to be odd. Therefore, another Dirac point must exist at point \( \mathbf{K}' \) related to that at point \( \mathbf{K} \) by TRS. In the vicinity of \( \mathbf{K}' \) the Hamiltonian can be linearized as

\[
\hat{H} \approx (i \hbar \tau_3 \partial_\xi + i \hbar \tau_1 \partial_\eta, \hat{h}_z) \cdot \tau.
\]

which gives the contribution \( -\frac{1}{2} \text{sgn}[\hbar_z(\mathbf{K})] \) to the Chern invariant (3). Since TRS imposes the Dirac masses to be the same, \( \hbar_z(\mathbf{K}) = -\hbar_z(\mathbf{K}') \), the total Chern number is \( C = 0 \), and a topologically trivial insulator is realized.

The above pair of Dirac cones embodies the fermion’s doubling theorem [12]. In order to have non-zero \( C \), TRS must be broken. Following a procedure analogous to that of Haldane’s [9], we may opt to break TRS by choosing the Dirac masses as \( \hbar_z(\mathbf{K}) = -\hbar_z(\mathbf{K}') \) and the lower band’s Chern number is then \( C = \text{sgn}[\hbar_z(\mathbf{K})] \). The resulting system is the Haldane’s AHI. In a finite system, edge states have the chirality shown in figure 1(a). The Dirac masses determine the sign of \( C \), hence the edge states’ chirality.

Model bands with arbitrarily higher \( C \) may be constructed, having \( |C| \) such pairs of Dirac cones in the BZ.

3. Weak field

3.1. Low energy, continuum description

We now consider the effect of an applied perpendicular magnetic field, for spinless particles, from the minimal coupling to the vector potential \(-i\hbar \nabla \rightarrow -i\hbar \nabla - e \mathbf{A} \), with \( e < 0 \), in equations (2) and (4), where \( \nabla \times \mathbf{A} = B \mathbf{z} \). Denoting the off-diagonal elements of the Hamiltonian matrix by \( \hat{O} = \mp i \hbar \tau_3 \partial_\xi - e A_x - \hbar \tau_1 \partial_\eta, + i e A_y \), for cones \( \mathbf{K} \) and \( \mathbf{K}' \), the Hamiltonian matrix now has the form

\[
\hat{H} = (\hbar_z \hat{O'}, -\hbar_z),
\]

where \( [\hat{O}, \hat{O}'] = -2\hbar \tau_3 e B \) at cone \( \mathbf{K} \) and \( [\hat{O}, \hat{O}'] = 2\hbar \tau_3 e B \) at cone \( \mathbf{K}' \). The spectrum consists of Landau levels (LLs), with energies given by \( E_n = \text{sgn}(n)\sqrt{\hbar^2 z^2 + |n| 2\hbar^2 z^2 |eB|} \) where the relative integer \( n \neq 0 \). For \( n \neq 0 \) the LLs distribute themselves symmetrically around zero energy. But the position of the \( n = 0 \) LL depends on the cone’s chirality and the sign of \( B \), according to

\[
\begin{align*}
[\hat{O}, \hat{O}'] > 0 & \Rightarrow E_0 = -\hbar_z, \quad (6) \\
[\hat{O}, \hat{O}'] < 0 & \Rightarrow E_0 = \hbar_z. \quad (7)
\end{align*}
\]

We still use \( C \) to denote the Chern number before the field is applied. As usual, each LL contains \( |eB| / h \) states per unit area. The interplay between the magnetic field and topology manifests itself in the position of the \( n = 0 \) LL in a single Dirac cone, which is \( E_0 = -|h_z| \text{sgn}(C \cdot B) \), as shown in figure 2 (top). The Hall response of Dirac fermions has been studied in detail in the context of graphene [13, 14]. It is known that each filled LL in a single Dirac cone gives a contribution \( \text{sgn}(B) \frac{1}{2} e^2 / h \) to the charge Hall conductivity, \( \sigma_{xy} \). If the \( n = 0 \) LL is empty, then \( \sigma_{xy} = -\text{sgn}(B) \frac{1}{2} e^2 / h \) for a single cone.

For the trivial insulator with \( C = 0 \) the pair of Dirac cones at \( \mathbf{K} \) and \( \mathbf{K}' \), which are related by TRS, have opposite chiralities. When \( B \neq 0 \) their \( n = 0 \) LLs shift in opposite directions, as is shown in figure 2 (top). In thermal equilibrium the fermions migrate to the cone with LL energy \( E_0 = -|h_z| \),
which becomes completely filled. The contribution of such a filled cone to the charge Hall conductivity, $\sigma_{yx}$, is $\frac{e^2}{h}$. The other cone, which has an empty $n=0$ LL, contributes $-\frac{e^2}{h}$. The total $\sigma_{yx}=0$, so the system remains a trivial insulator.

In the case of Haldane’s AHI, the two cones at $K$ and $K'$ have the same chirality. Their $n=0$ LLs shift in the same direction: the cones are perfectly degenerate in a magnetic field. The $n=0$ LL is half-filled in each cone, and the system is a metal. The Hall conductivity is not quantized. Without considering other effects, we then reach the conclusion that in this clean, ideal situation, Haldane’s AHI becomes metallic under a perpendicular weak magnetic field, at constant particle density. Other effects, such as disorder and interactions, can play a decisive role, however. If the Dirac masses are not exactly symmetrical, one of the LLs becomes full and the other empty, in which case the combined $\sigma_{yx}=0$. This result may also be obtained if we consider the role of disorder and appeal to the scaling theory of localization in the integer quantum Hall effect [15–17]. Generalizing this theory to a single Dirac cone, the Hall conductance scales to half-integer multiples of $e^2/h$, as shown in figure 2 (bottom).

In the present case, if the two $n=0$ LLs have different fillings, one expects the Hall conductance of one Dirac cone to scale to $\frac{1}{2} e^2/h$ while the other scales to $-\frac{1}{2} e^2/h$. Surprisingly, such an insulating state arises independently of whether the chirality of $B$ is the same or opposite to the system’s original chirality (figure 1). Electron repulsion could also play an important role, since the $n=0$ LL in a single cone occupies only one of the sublattices.

3.2. Specific lattice models

To illustrate what has been discussed in terms of the low energy effective description (pairs of Dirac cones), we consider in this work two-band models on the square lattice with $C\geq 1$. For comparison we also show results for the original Haldane model [9] proposed for the honeycomb lattice (see section 4).

To be specific, we introduce a $C=1$ model on the square lattice where the vector $h(k)$ in equation (1) reads (in units where the lattice constant $a=1$)

$$h_x(k) = -1 + 2 \cos(k_x) + 2 \cos(k_y)$$

$$h_y(k) = 2 \sin(k_x + k_y)$$

$$h_z(k) = -\frac{1}{2} \sin(k_z)$$

(8)

The $K$ Dirac cone is located at $K_x = -K_y = \cos^{-1}(1/4)$ and $K' = -K$.

We provide the vector $h(k)$ in equation (1) for the Haldane model in the honeycomb lattice, which we use below for the sake of comparison,

$$h_x(k) = 1 + 2 \cos(k_x/2) \cos(\sqrt{3}k_y/2)$$

$$h_y(k) = 2 \cos(k_x/2) \sin(\sqrt{3}k_y/2)$$

$$h_z(k) = 2t_2[2 \sin(k_x/2) \cos(\sqrt{3}k_y/2) - \sin(k_z)]$$

As is well known, the Haldane model has $C=1$ for $t_2 \neq 0$.

We point out that one may also construct spinless Chern insulators not following Haldane’s procedure of breaking TRS at the level of $h_z$. As an alternative to equations (2) and (4), consider, for instance, that in the vicinity of momenta $K$ and $-K$ the Hamiltonian takes the linearized form

$$K : \hat{H} \approx \left( -i\hbar v_F \partial_x, -i\hbar v_F \partial_y, h_z \right) \cdot \tau$$

$$-K : \hat{H} \approx \left( i\hbar v_F \partial_x, i\hbar v_F \partial_y, h_z \right) \cdot \tau$$

(10)

where $h_z(K') = h_z(-K)$. The term $h_z$ breaks TRS in (10) and $C = \text{sgn}(h_z)$ in the lower band. In both Dirac cones the $n=0$ LL has energy $E_0 = -h_z$ and all of the previous discussion remains valid.

Such a model was proposed in [18], for the square lattice, and can be tuned between $C=1$ and 2 by varying hopping parameters. The vector $h(k)$ in equation (1) reads

$$h_x(k) = \sqrt{2} \cos(k_x + \cos k_y)$$

$$h_y(k) = \sqrt{2} \cos(k_x - \cos k_y)$$

$$h_z(k) = \frac{1}{4} \sin k_x \sin k_y + \frac{1}{2} \sin k_x + \sin k_y$$

(11)

3 The spectrum is given by $h(k) \pm i|h(k)|$ and the direct band gaps occur at $K$ and $K'$. 

Figure 2. Top: the position of the $n=0$ LL (thick horizontal bar) with respect to the massive Dirac cones if $B > 0$. The Chern number of the lower band of a single cone (in the absence of magnetic field) is also indicated. If $B < 0$ the LL’s positions are interchanged. Bottom: renormalization flow of conductivities as the system size increases. This graph proposes a generalization of the previous scaling theory of localization [15, 16] to a single Dirac cone.
For $t'_1 < 1/4$ we have $C = 2$, while $t'_1 > 1/4$ implies $C = 1$. There are two pairs of Dirac cones. One of such pairs has the cones located at $K_1 = \pm (\pi/2, \pi/2)$ and the other pair at $K_2 = \pm (\pi/2, -\pi/2)$.

Case study. As an example let us examine the $C = 2$ lattice model (11). The band structure for $B = 0$ is shown in figure 3 (top), both for $C = 2$ (left panel) and for $C = 1$ (right panel).

For finite magnetic field each Dirac cone originates a single $n = 0$ Landau level. For $C = 2$ all four $n = 0$ Landau levels are degenerate and, as anticipated above, this Haldane’s AHI is metallic at half-filling. The position of the $n = 0$ Landau levels for a particular choice of the magnetic field sign is shown in the left panel of figure 3 (top).

The system may become a $C = 1$ Chern insulator after band inversion of the $K'_1$ cone. As shown in the right panel of figure 3 (top) the two $n = 0$ Landau levels from $K_1$ and $K'_1$ now move in opposite directions. At half-filling the $n = 0$ Landau level at $K'_1$ has the highest energy and becomes empty, while the one at $K_1$, with the lowest energy, becomes fully occupied; note that the gap (Dirac mass) at $K_1$ increases when we close and reverse the gap at $K'_1$ by tuning $t'_1$. So, as in the $C = 2$ case, this $C = 1$ Haldane’s AHI is metallic, and only for $C = 0$ would the system become a trivial insulator.

A quantitative description can be obtained in the ribbon geometry shown in figure 1(c). In figures 3(a) and (c) (bottom panel) we show the spectrum at $B = 0$ for $C = 2$ and $C = 1$, respectively. The number of edge states running at each edge (same velocity) is precisely $C$, as it should be for a Chern insulator. The $B \neq 0$ case is shown in figures 3(b) and (d) (bottom panel), respectively for $C = 2$ and $C = 1$. The magnetic flux per square lattice unit cell, $\phi$, is set to $\phi/\phi_0 \approx 3.7 \times 10^{-3}$, where $\phi_0 = h/e$ denotes the flux quantum. It is clear that at half-filling for $C = 2$ there are four $n = 0$ Landau levels crossing the Fermi level (horizontal dashed line), while there are only two for $C = 1$. Note also that the Hall conductivity as obtained from the Laughlin–Halperin [19, 20] argument fully agrees with the contribution expected from Dirac cones in a low energy description. For $C = 2$ the Hall conductivity is $\sigma_{yx} = 2e^2/h$ if the four $n = 0$ Landau levels are full and $\sigma_{yx} = -2e^2/h$ if they are empty—the new Fermi level always crosses two edge states per edge. In the $C = 1$ case the Hall conductivity is $\sigma_{yx} = e^2/h$ if the two degenerate $n = 0$ Landau levels are full and $\sigma_{yx} = -e^2/h$ if they are empty—the new Fermi level only crosses one edge state per edge. This is nothing but $\sigma_{yx} = \pm |C|e^2/h$, as expected.
3.3. Kane–Mele $\mathbb{Z}_2$-topological insulator

3.3.1. Low energy, continuum description. It is interesting now to consider Kane–Mele’s [21] construction of the topological insulator (TI). Endowing the fermions with spin, the spin $s_z$ particles see the Dirac cones at $K$ and $K'$ with $n = 0$ LL energy $E_0 = -|h_z|$ as in figure 2 (top left), while the $-s_z$ particles see the two Dirac cones with the energy level $E_0 = |h_z|$ shown in figure 2 (top right). These $n = 0$ LLs at $E_0 = \pm |h_z|$ are then initially half-filled. The thermal equilibrium configuration is achieved when the electrons migrate to the cone with lowest $n = 0$ LL energy, $E_0 = -|h_z|$, which becomes completely filled with spin $s_z$. Thus the total system becomes spin polarized, with total spin density $2s_z |CeB| / h$ since there are $|C|$ pairs of cones. Transitions between such spin polarized LLs by optical absorption were discussed very recently for the particular case of silicene [8]—the experimental realization [22] of Kane–Mele’s TI originally proposed for graphene [21]. The total spin $s_z = 0$ but there is a finite spin Hall conductance since the $n = 0$ LL is filled with $s_z$ electrons while the $-s_z$ electrons fill the $n = -1$ LL of the other cone. Under a magnetic field and at constant electron density, Kane–Mele’s topological insulator then becomes a spin polarized quantum spin Hall insulator. This conclusion is valid under the assumption that $s_z$ is conserved, so that the Chern number matrix [23] is diagonal and $C_{\uparrow} = -C_{\downarrow} = C$. Such a state is stable against potential disorder, but unstable against spin-flip perturbations, in which case it would become a trivial insulator; a similar situation occurs for a quantum spin Hall insulator in a parallel (in plane) magnetic field, which breaks both TRS and $s_z$ conservation [24]. Note that the spin polarization is achieved without considering the Zeeman coupling to spin and stems from the $s_z$-preserving spin–orbit coupling that originated the TI in the first place. When TRS is present, the TI’s $\mathbb{Z}_2$ index $v$ is given by the parity of $|C|$. Equivalently, one can use the spin Chern number [23], $C_{\nu} = C_{\uparrow} - C_{\downarrow} = 2C$, with $\nu = (C_\text{mod} 4)/2$ [25, 24]. The magnetic field breaks TRS, restoring the $\mathbb{Z}$ index, $C$, which counts the number of edge states for each spin projection running in a given edge. Therefore, if $|C| > 1$, the spin Hall conductance $\sigma_{xy}^{s_z} = |C|2s_z e/h$.

3.3.2. Specific lattice model. Here we consider model (8) generalized to include spin using the Kane–Mele [21] construction, by replacing $h_z$ in (8) with $h_z, (k) = -\text{sgn}(s_z) \frac{1}{2} \sin(k_x)$. A weak magnetic field is applied perpendicularly. Figure 4 shows the LLs and edge states for the ribbon geometry in figure 1(c). It can be seen that counter-propagating states with spins $s_z$ and $-s_z$ exist when the chemical potential lies just above the $n = 0$ LL of the $s_z$ subsystem. The Laughlin–Halperin [19, 20] argument clearly implies the spin Hall conductance $\sigma_{xy}^{s_z} = 2s_z e / h$, consistent with the result of section 3.3.1 for $C = 1$. It is also clear that the edge states are not robust with respect to a spin-flip perturbation, even if such a perturbation is time-reversal invariant (such as a spin–orbit term). This is easily seen from the fact that the edge states level crossing (marked in figure 4(c)) occurs at non-zero momentum, a non-time reversal invariant momentum. Such edge states, in the presence of the magnetic field, are not Kramers pairs and can, therefore, be coupled by a time-reversal invariant perturbation. Panel (b) can also be seen as the Chern insulator with $C = +1$, as the $n = 0$ LL is full. Panel (a) shows the trivial insulator obtained by replacing $h_z$ in (8) with $h_z = -1/2$, for comparison.

3.4. Superlattice potential effect

Now consider that the above Haldane’s AHI in a perpendicular magnetic field $B > 0$ is also subjected to a weak square superlattice potential with rational flux $\phi / \phi_0 = p / q$ per unit cell. The superlattice potential is diagonal in the pseudo-spin index and cannot therefore produce intercone scattering. The $n = 0$ LL of each single cone splits into $p$ sub-bands. The Hall conductance for filled sub-bands obeys a Diophantine equation [26]. When the chemical potential lies in the $r$th gap of the split $n = 0$ LL, the quantized $\sigma_{xy}$ for a single Dirac cone is given by $\sigma_{xy} = \frac{e^2}{h} (-\frac{1}{2} + t_r)$, where $t_r$ obeys the Diophantine equation [26]

$$r = sq + tp,$$  \hspace{1cm} (12)

where $|s| \leq p/2$.

Consider now the trivial insulator with a pair of Dirac cones with LLs at energies $E_0 = \pm h_z$, as shown in figure 2 (top). If only the lower cone is filled, then the Hall conductance is

$$\sigma_{xy} = \frac{e^2}{h} (t_r - 1).$$  \hspace{1cm} (13)

In the case where the Dirac mass vanishes (as in spinless graphene), the $n = 0$ LLs are degenerate at $E_0 = 0$ and $\sigma_{xy} = (2t_r - 1) e^2 / h$.

Consider now Haldane’s AHI where $C'$ pairs of Dirac cones have degenerate $n = 0$ LLs at energy $E_0 = -|h_z|$ which
are partially filled, and $C''$ pairs of cones have $E_0 = |h_z|$. The Hall conductance is

$$\sigma_{xy} = \frac{e^2}{h} \left[ C'' (2t_r - 1) - C' \right],$$

while the total Chern number in the absence of magnetic field and superlattice potential is $C = C' - C''$.

4. Strong field

It is well known that the spectrum of fermions in a magnetic field and periodic potential consists of Hofstadter bands [3]. In order to study the interplay between a strong magnetic field’s gauge potential and a band’s topology, we use the two-band model in equation (8) for the square lattice which has unit Chern number in the lower band. The Hofstadter spectrum is displayed in figure 5, top panel, as a function of the flux, $\phi$, per unit cell. The spectrum is invariant under the transformation $\phi \rightarrow \phi + 1$, and symmetrical with respect to $\phi = 0.5$. The half-filled band case is shown. The Chern numbers in the gaps have been calculated with the method given in [14].

It is seen in the top panel of figure 5 that, for increasing flux, the $n = 0$ LL opens and closes a gap in the middle, with zero Hall conductance in the gap. The system goes through metallic and trivial band insulator regimes as the flux per unit cell is increased. We note, however, that this feature is model dependent. For the original Haldane model proposed in the honeycomb lattice [9], equation (9), no such splitting of the $n = 0$ LL occurs as a function of flux. This can be seen in the middle panel of figure 5. The half-filled system remains always metallic, as discussed in section 3.4. We attribute the different behaviour of the two models to the different underlying lattices and the way the two basis atoms hybridize in the lattice. While in model (8) the two basis atoms hybridize at the same square lattice site, in model (9) the two atoms are spatially separated. As a consequence, there is a Peierls phase for hoppings connecting the two basis atoms for the latter case. We have verified that when this Peierls factor is artificially suppressed, a gap opens at half-filling also for model (9). We may further illustrate this point with the following model for a $C = 2$ Chern insulator:

$$h_x(k) = 2 \sin(k_x) - b$$
$$h_y(k) = 2 \sin(k_y)$$
$$h_z(k) = 0.2 \cos(k_x) \cos(k_y).$$

The parameter $b \neq 0$ couples two orbitals at the same lattice site and the $n = 0$ LL is split at moderate flux, as figure 5 (bottom left) shows. If $b = 0$ the model couples only spatially separated orbitals. Then the magnetic field does not split the $n = 0$ LL, as figure 5 (bottom right) shows. It is therefore expected that Haldane’s AHI in a non-Bravais lattice under a magnetic field remains metallic for all values of the flux.

A final remark is in order. It has been assumed above that the direct band gap between the bands is located at Dirac points. This may not be the case for some nearly flat band models [27–29]. In a nearly flat band, the Haldane mass in the Dirac cone becomes large, equal to the nearly uniform

Figure 5. Top: the Hofstadter spectrum of model (8) against flux ($\phi$) per unit cell. The occupied bands for a half-filled system are shown, as well as the Chern number ($C$) in some of the gaps, which gives the Hall conductance $\sigma_{xy}$ in units of $e^2/h$. Middle: the same for model (9). Bottom: model (15) with $b \neq 0$ (left) and $b = 0$ (right).
(across the BZ) gap between the bands. The $k$ point at which
the direct band gap, $2|\hbar(k)|$, is minimum may happen not to
be a Dirac cone, hence the low energy spectrum is not of the
form discussed (equations (2) and (4)) and the corresponding
analysis of the $n = 0$ LL no longer applies. Then the lowest
LLs may behave as topologically trivial LLs do, in which case
the AHI placed under a magnetic field may turn out to remain
a band insulator.

5. Summary

In summary, we have studied the effect of a magnetic gauge
field on a topological fermionic band insulator by considering
a generalization of Haldane’s model of the anomalous Hall
insulator to models with an arbitrary Chern number. We
have shown that a spinless system becomes metallic under
a weak magnetic field, unless other physical effects, such as
disorder, are taken into account. However, in some model
systems a stronger magnetic field can induce an insulating
phase. We have also addressed the effect of a weak square
superlattice potential on the LL splitting and Hall conductance
of high Chern number systems. In the case of the $Z_2$ quantum
spin Hall insulator with $s_z$ conservation, the magnetic field’s
vector potential induces a finite magnetization even without
considering the Zeeman coupling. This magnetized quantum
Hall state is unstable with respect to spin-flip perturbations
(including spin–orbit terms).

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