Adaptation and Parameter Estimation in Systems with Unstable Target Dynamics and Nonlinear Parametrization

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Abstract

We propose a technique for the design and analysis of adaptation algorithms in dynamical systems. The technique applies both to systems with conventional Lyapunov-stable target dynamics and to ones of which the desired dynamics around the target set is nonequilibrium and in general unstable in the Lyapunov sense. Mathematical models of uncertainties are allowed to be nonlinearly parametrized, smooth, and monotonic functions of linear functionals of the parameters. We illustrate with applications how the proposed method leads to control algorithms. In particular we show that the mere existence of nonlinear operator gains for the desired dynamics guarantees that system solutions are bounded, reach a neighborhood of the target set, and mismatches between the modeled uncertainties and uncertainty compensator vanish with time. The proposed class of algorithms can also serve as parameter identification procedures. In particular, standard persistent excitation suffices to ensure exponential convergence of the estimated to the actual values of the parameters. When a weak, nonlinear version of the persistent excitation condition is satisfied, convergence is asymptotic. The approach extends to a broader class of parameterizations where the monotonicity restriction holds only locally. In this case excitation with oscillations of sufficiently high frequency ensure convergence.

Keywords: nonlinear parametrization, unstable, non-equilibrium dynamics, adaptive control, parameter estimation, (nonlinear) persistent excitation, exponential convergence, monotonic functions

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1 Introduction

Results in adaptive control theory and systems identification are most frequently used in control engineering, but have potentially a much wider significance. In particular these theories are of great potential relevance for sciences such as physics and biology \[55\]. On the other hand, it is in these areas that the current limitations of control theory are most strongly felt. Whereas effective procedures are available in case the system is static \[7\], \[22\], \[66\], \[28\], adequate solutions for dynamical systems have been proposed under conditions that may not be adequate for most scientific applications. These conditions require that systems are linear in their parameters, the target dynamics is stable in the Lyapunov sense, and a Lyapunov function of the target dynamics can be given \[54\], \[43\], \[30\], \[33\], \[16\], \[5\], \[40\]. Each of these restrictions alone is limiting the role of control theory in the scientific arena; together they constitute the ”standard” approach that confines control theory to a limited role, even within the realm of engineering.

Whereas in artificial system design, nonlinear parametrizations could often be avoided, physical and biological models often require the inclusion of nonlinearly parametrized uncertainties \[2\], \[49\], \[6\], \[13\], \[29\]. Proposed solutions to the nonlinear parametrization problem have cemented the standard approach, in that they eliminate any hopes of escaping from the stable target dynamics requirement. Nonlinearity is traditionally solved by invoking dominance of the nonlinear terms \[32\], \[31\]. Dominance inevitably overcompensates the nonlinearity inherent to the system. This is undesirable if the system’s target motions require such nonlinearities. It is in particularly unhelpful, in case the system equations embody certain physical laws or other regularities that necessitate nonlinear parametrization. Terms overcompensated by dominance are likely to be exactly the ones postulated by these laws and regularities. In order to enable nonstable and in a sense more delicate target dynamics, more gentle control is needed: one that enables a system to reach the desired dynamical state by modification of, rather than destroying, its intrinsic motions.

Alternatives to dominance are available, but they face a variety of restrictions that make them appear less satisfactory. Often they apply to a narrowly defined class of parameterizations, e.g. convex functionals as in \[18\]. When a broader class of nonlinearities is considered, for instance Hammerstein (Wiener) models, \[41\], \[54\], \[20\], \[4\], the functions are restricted to static input (output) nonlinearities. Perhaps the most promising approach so far involves local linear (nonlinear) modelling techniques \[26\], \[63\], \[15\]. The resulting models, on the other hand, are not always physically plausible. In case fairly general nonlinear state dependent functions are allowed \[10\], the class of dynamical systems is limited to those modeled by the first-order ordinary differential equations with nonlinear parameterized terms that are Lipschitz in time. The last restriction but not least is that the majority of these methods rely on the assumption of stable target dynamics.

There are physical and biological systems, however, which do not meet the requirement of stable target dynamics \[21\], \[14\]. Multistability and coexistence of multiple attractors \[11\], \[65\], \[12\], \[38\] are well-known
examples where a system could, at best, be only locally stable. In biophysics amplification by oscillatory instability is believed to be a general mechanism of signal detection in sensory systems [8]. Furthermore, as demonstrated in [53], instability (intermittency) offers a solution to the longstanding binding problem in the biology of vision [64]. In fact, also in artificial systems unstable target dynamics are sometimes required [53], [48]. For instance, in [57], the effectiveness of using chaotic dynamics in solving the path finding problems in robotics is shown. In computer science unstable (intermittent) synchrony was shown to be an effective paradigm for solving the image segmentation problems [62].

Control-theoretic motivation and successful solutions to the problem of adaptive regulation to unstable dynamical states are provided in [41], [42] for linear systems with linear parameterization. For nonlinear systems with nonlinear parameterization, and, possibly, unstable target dynamics the problems of adaptive regulation and parameter estimation need further development.

In our present paper we aim to provide a unified tool capable of solving the problems of adaptation and parameter estimation

\begin{itemize}
\item in the presence of nonlinear state-dependent parameterization;
\item with non-trivial target sets (namely, surfaces in the systems’s state space)
\item with potentially unstable target dynamics, and therefore
\item without requiring for knowledge (or existence) of Lyapunov functions of the desired motions\(^1\)
\end{itemize}

Previous efforts to deal with nonlinearity by adopting domination functions [31], [32], or low-order mathematical models [10] have tried to address the most general case. In contrast to this, we restrict ourselves in advance to a certain class of nonlinearities. This class, however, is wide enough to cover a variety of relevant models in physics, mechanics, physiology and neural computation. In particular, it includes models of stiction, slip and surface dependent friction, nonlinearities in dampers, smooth saturation, dead-zones in mechanical systems, and nonlinearities in models of bio-reactors [2], [49], [6], [13], [29].

In order to deal with unstable and non-equilibrium target dynamics, without invoking the knowledge (or even existence) of the corresponding Lyapunov function, we employ operator formalism in the functional spaces rather than conventional tools\(^2\). In particular, we consider desired dynamics in terms of input-output mappings in the specific functional spaces. The only requirement we impose on these mappings is existence of nonlinear operator gains that bound functional norms of the outputs, given norm-bounded inputs. The inputs for these mappings are the mismatches between the modeled uncertainty and a compensator. The outputs are the state \(x\) and a function \(\psi(x, t)\), not necessarily definite in state, which is considered a measure of deviation from the target set. This system-theoretic point of view allows us to formulate the problem of adaptation as a problem of regulation of the mismatches to specific functional spaces followed, if possible,

\(^1\)This problem, as mentioned in [40], was long remained an open theoretical challenge.

\(^2\)We refer here to common practice to fit the derivative of the goal functionals (Lyapunov candidates) to specific algebraic inequalities leading to the property of Lyapunov stability.
by minimization of their functional norm.

We show that the solution to this problem for the given class of parameterizations does not require continuity of the corresponding operator gains. This, in turn, suggests that stability of the desired dynamics, which in many cases is synonymous to continuity of the input-output operators [38], is not a necessary requirement for our approach. Furthermore, given that \( \psi(x,t) \) may not be definite, this new point of view on the adaptation allows us to lift conventional state-space metric restrictions on the goal functionals\(^3\).

Under standard and intuitively clear additional hypotheses (i.e. persistent excitation of a certain functional of state), we show that the proposed adaptation procedures solve the problem of parameter estimation for nonlinearly parameterized uncertainties. In this case convergence is exponential. The estimates of the convergence rates are based on the results of [36] and provided here for consistency. In case the conditions specifying the class of nonlinear in parameter uncertainties hold only locally, we show that sufficiently high frequency of excitation still ensures convergence. For cases where the standard persistent excitation property does not hold, we formulate a new version of nonlinear persistent excitation condition [10]. With this new property it is still possible to show asymptotic convergence of the estimates to the actual values of unknown parameters. Whether the convergence is exponential is not answered in this paper.

The paper is organized as follows. Section 2 describes notations and conventions we are using in the paper; in Section 3 we formulate the problem. For the sake of compact exposition of our results we restrict ourselves to systems that are affine in control, although some non-affine cases are discussed towards the end of the paper. Section 4 contains the main results of the paper. We discuss several immediate extensions of the present results in Section 5. In Section 6 we provide a practically relevant application of our method, and Section 7 concludes the paper.

### 2 Notation

According to the standard convention, symbol \( \mathbb{R} \) defines the field of real numbers and \( \mathbb{R}_{\geq c} = \{ x \in \mathbb{R} | x \geq c \} \), \( \mathbb{R}_+ = \mathbb{R}_{\geq 0} \); symbol \( \mathbb{N} \) defines the set of natural numbers; symbol \( \mathbb{R}^n \) stands for a linear space \( \mathcal{L}(\mathbb{R}) \) over the field of reals with \( \dim \{ \mathcal{L}(\mathbb{R}) \} = n \); \( \| x \| \) denotes the Euclidian norm of \( x \in \mathbb{R}^n \); \( \mathcal{C}^k \) denotes the space of functions that are at least \( k \) times differentiable. Symbol \( \mathcal{K} \) denotes the space of all functions \( \kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that \( \kappa(0) = 0 \), and that \( x' > x'' \), \( x', x'' \in \mathbb{R}_+ \) implies that \( \kappa(x') - \kappa(x'') > 0 \). By symbol \( L^p_{[t_0,T]} \), where \( T > 0, p \geq 1 \) we denote the space of all functions \( f : \mathbb{R}_+ \rightarrow \mathbb{R}^n \) such that

\[
\| f \|_{p,[t_0,T]} = \left( \int_0^T \| f(\tau) \|^p d\tau \right)^{1/p} < \infty
\]

\(^3\)Which are usually defined as positive-definite and radially unbounded functions of state [17], [12], [30]
Symbol $\|f\|_{p,[t_0,T]}$ denotes the $L_p^m[t_0,T]$-norm of vector-function $f(t)$. By $L_\infty^n[t_0,T]$ we denote the space of all functions $f : \mathbb{R}_+ \to \mathbb{R}^n$ such that
\[
\|f\|_{\infty,[t_0,T]} = \text{ess sup}\{\|f(t)\|, t \in [t_0,T]\} < \infty,
\]
and $\|f\|_{\infty,[t_0,T]}$ stands for the $L_\infty^n[t_0,T]$ norm of $f(t)$.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be given. Function $f(x) : \mathbb{R}^n \to \mathbb{R}^m$ is said to be locally bounded if for any $\|x\| < \delta$ there exists constant $D(\delta) > 0$ such that the following holds: $\|f(x)\| \leq D(\delta)$.

Let $\Gamma$ be an $n \times n$ square matrix, then $\Gamma > 0$ denotes a positive definite (symmetric) matrix, and $\Gamma^{-1}$ is the inverse of $\Gamma$. By $\Gamma \geq 0$ we denote a positive semi-definite matrix. Symbols $\lambda_{\min}(\Gamma)$, $\lambda_{\max}(\Gamma)$ stand for the minimal and maximal eigenvalues of $\Gamma$ respectively. By symbol $I$ we denote the identity matrix. We reserve symbol $\|x\|_2^2$ to denote the following quadratic form: $x^T \Gamma x$, $x \in \mathbb{R}^n$. Notation $\| \cdot \|$ stands for the module of a scalar. The solution of a system of differential equations $\dot{x} = f(x, t, \theta, u)$, $x(t_0) = x_0$, $u : \mathbb{R}_+ \to \mathbb{R}^m$, $\theta \in \mathbb{R}^d$ for $t \geq t_0$ will be denoted as $x(t, x_0, t_0, \theta, u)$, or simply as $x(t)$ if it is clear from the context what the values of $x_0$, $\theta$ are and how the function $u(t)$ is defined.

Let $u : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^m$ be a function of state $x$, parameters $\theta$, and time $t$. Let in addition both $x$ and $\theta$ be functions of $t$. Then in case the arguments of $u$ are clearly defined by the context, we will simply write $u(t)$ instead of $u(x(t), \theta(t), t)$.

The (forward complete) system $\dot{x} = f(x, t, \theta, u(t))$, is said to have an $L_p^m[t_0,T] \to L_q^n[t_0,T]$, gain ($T \geq t_0$, $p, q \in \mathbb{R}_{\geq 1} \cup \{\infty\}$) with respect to its input $u(t)$ if and only if $x(t, x_0, t_0, \theta, u(t)) \in L_q^n[t_0,T]$ for any $u(t) \in L_p^m[t_0,T]$ and there exists a function $\gamma_{q,p} : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ such that the following inequality holds:
\[
\|x(t)\|_{q,[t_0,T]} \leq \gamma_{q,p}(x_0, \theta, \|u(t)\|_{p,[t_0,T]})
\]
Function $\gamma_{q,p}(x_0, \theta, \|u(t)\|_{p,[t_0,T]})$ is assumed to be non-decreasing in $\|u(t)\|_{p,[t_0,T]}$, and locally bounded in its arguments.

For notational convenience when dealing with vector fields and partial derivatives we will use the following extended notion of Lie derivative of a function. Let it be the case that $x \in \mathbb{R}^n$ and $x$ can be partitioned as follows $x = x_1 \oplus x_2$, where $x_1 \in \mathbb{R}^q$, $x_1 = (x_{11}, \ldots, x_{1q})^T$, $x_2 \in \mathbb{R}^p$, $x_2 = (x_{21}, \ldots, x_{2p})^T$, $q + p = n$, and $\oplus$ denotes concatenation of two vectors. Define $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x) = f_1(x) \oplus f_2(x)$, where $f_1 : \mathbb{R}^n \to \mathbb{R}^q$, $f_1(\cdot) = (f_{11}(\cdot), \ldots, f_{1q}(\cdot))^T$, $f_2 : \mathbb{R}^n \to \mathbb{R}^p$, $f_2(\cdot) = (f_{21}(\cdot), \ldots, f_{2p}(\cdot))^T$. Then symbol $L_{f_i(x)}\psi(x,t)$, $i \in \{1,2\}$ denotes the Lie derivative of function $\psi(x,t)$ with respect to vector field $f_i(x, \theta)$:
\[
L_{f_i(x)}\psi(x,t) = \sum_{j=1}^{\text{dim } x_i} \frac{\partial \psi(x,t)}{\partial x_{ij}} f_{ij}(x, \theta)
\]
Symbol $\text{sign}(\cdot)$ denotes the signum-function:
\[
\text{sign}(s) = \begin{cases} 
1, & s > 0 \\
0, & s = 0 \\
-1, & s < 0 
\end{cases}
\]
3 Problem Formulation

Let the following system be given:

\[ \begin{align*}
\dot{x}_1 &= f_1(x) + g_1(x)u, \\
\dot{x}_2 &= f_2(x, \theta) + g_2(x)u,
\end{align*} \tag{1} \]

where

\[ \begin{align*}
x_1 &= (x_{11}, \ldots, x_{1q})^T \in \mathbb{R}^q \\
x_2 &= (x_{21}, \ldots, x_{2p})^T \in \mathbb{R}^p \\
x &= (x_{11}, \ldots, x_{1q}, x_{21}, \ldots, x_{2p})^T \in \mathbb{R}^n
\end{align*} \]

\( \theta \in \Omega_\theta \subset \mathbb{R}^d \) is a vector of unknown parameters, and \( \Omega_\theta \) is a closed bounded subset of \( \mathbb{R}^d \); \( u \in \mathbb{R} \) is the control input, and functions \( f_1 : \mathbb{R}^n \to \mathbb{R}^q, f_2 : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^p, g_1 : \mathbb{R}^n \to \mathbb{R}^q, g_2 : \mathbb{R}^n \to \mathbb{R}^p \) are locally bounded. Vector \( x \in \mathbb{R}^n \) is a state vector, and vectors \( x_1, x_2 \) are referred to as uncertainty-independent and uncertainty-dependent partitions of \( x \), respectively.

For the sake of compactness we introduce the following alternative description for (1):

\[ \dot{x} = f(x, \theta) + g(x)u, \tag{2} \]

where

\[ \begin{align*}
g(x) &= (g_{11}(x), \ldots, g_{1q}(x), g_{21}(x), \ldots, g_{2p}(x))^T \\
f(x) &= (f_{11}(x), \ldots, f_{1q}(x), f_{21}(x, \theta), \ldots, f_{2p}(x, \theta))^T
\end{align*} \]

Our goal is to derive both the control function \( u(x, t) \) and estimator \( \hat{\theta}(t) \), such that all trajectories of the system are bounded and state \( x(t) \) converges to the desired domain in \( \mathbb{R}^n \). In addition, we would like to find conditions ensuring that the estimate \( \hat{\theta}(t) \) converges to unknown \( \theta \in \Omega_\theta \) asymptotically. In order to ensure boundedness of the trajectories, we should design an input \( u(x, t) \) that restricts all possible motions of system (2) to an admissible bounded domain \( \Omega \subset \mathbb{R}^n \) in the system state space, and if possible steers trajectories \( x(t) \) to the specific set \( \Omega_0 \subset \Omega \).

As a measure of closeness of trajectories \( x(t) \) to the desired state we introduce the error function \( \psi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R} \), \( \psi \in C^1 \) such that

\[ \Omega_0 = \{ x(t) \in \mathbb{R}^n | \psi(x(t), t) = 0 \} \tag{3} \]

In conventional theories it is usually required that function \( \psi(x, t) \) satisfies (algebraic) metric restrictions:

\[ \nu_1(||x - \xi(t)||) \leq \psi(x, t) \leq \nu_2(||x - \xi(t)||), \quad \nu_1, \nu_2 \in K_\infty, \tag{4} \]
where function $\xi : \mathbb{R}_+ \to \mathbb{R}^n$, $\xi \in C^0$ is, for instance, the reference trajectory. Function $\psi(x, t)$ in this case serves as the Lyapunov candidate for the controlled system under the assumption that $\theta$ is known. The problem, however, is that finding such a Lyapunov candidate is not a trivial task. Furthermore, the desired trajectories $\xi(t)$ as functions of time may only be partially specified. In case no reference function is available (e.g. $\xi(t) = 0$) and the task is to steer the state $x$ of system (1) to a non-trivial set $\Omega_0 \subset \mathbb{R}^n$, it is often difficult to find a goal functional $\psi(x, t)$ such that both (3) and (4) are satisfied. In addition, in physical and nonlinear systems the desired dynamics (e.g. dynamics of convergence of trajectories $x(t)$ to the reference $\xi(t)$) could be unstable in Lyapunov sense \[21, 53\], although it may possess certain degree of attraction \[39\], and bounded deviation from the reference.

In order to tackle these complex, but still possible, phenomena we propose to replace the conventional goal functionals \[4\] with new and less restrictive ones. In particular, we propose to replace the standard norms $\| \cdot \|$ in $\mathbb{R}^n$ in (4) with functional norms $\| x(t) \|_{p, [t_0, T]}$, $T \geq t_0$ in the functional spaces $L_p(t_0, T)$, $T \geq t_0$, $p \in \mathbb{R}_{\geq 1} \cup \infty$. In the other words, we replace algebraic inequality (4) with operator relations. This will allow us to keep function $\psi(x, t)$ as a measure of closeness of trajectories $x(t)$ to the desired set $\Omega_0$ without imposing state-metric restrictions (4) on the function $\psi(x, t)$. On the other hand we will be able to derive bounds for $x(t)$ from the values of functional $L^1_p[t_0, T]$-norms of the function $\psi(x(t), t)$. Let us formally introduce this requirement as follows:

**Assumption 1 (Target operator)** For the given function $\psi(x, t) \in C^1$ the following property holds:

$$\| x(t) \|_{\infty, [t_0, T]} \leq \tilde{\gamma} \left( x_0, \theta, \| \psi(x(t), t) \|_{\infty, [t_0, T]} \right)$$

(5)

where $\tilde{\gamma} \left( x_0, \theta, \| \psi(x(t), t) \|_{\infty, [t_0, T]} \right)$ is a locally bounded and non-negative function of its arguments.

Assumption 1 can be interpreted as a sort of unboundedness observability property \[7\] of system (1) with respect to the “output” function $\psi(x, t)$. It can also be viewed as a bounded input - bounded state assumption for system (1) along the constraint $\psi(x(t, x_0, t_0, \theta, u(x(t), t)), t) = v(t)$, where signal $v(t)$ serves as the new input\(^4\). In order to illustrate this consider the equations of a spring-mass system with nonlinear damping:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= k_0 x_1 + f(x_2, t) + u(t), \quad k_0 < 0
\end{align*}
\]

(6)

where $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, $f(\cdot, \cdot) \in C^1$ is the nonlinear time-varying damping term. Equations of this type arise in broad areas of engineering ranging from active suspension control \[11\] to haptic interfaces \[10\] and identification of the muscle dynamics \[67\]. Let the desired dynamics of (6) be an exponentially fast convergence of $x_1(t)$, $x_2(t)$ to the origin. This requirement is satisfied for the following target set:

$$\Omega_0 = \{ x \in \mathbb{R}^2 | x_1 + \lambda x_2 = 0 \}, \quad \lambda \in \mathbb{R}, \lambda > 0$$

\(^4\)If, however, boundedness of the state is not required explicitly (i.e. it is guaranteed by additional control or follows from the physical properties of the system itself), Assumption 1 can be removed from the statements of our results.
Hence, feedback (9) renders the original system (1) into the well-known nonlinear following class of functions:

\[ \psi(x, t) = x_1 + \lambda x_2. \]

Let \( \psi(x(t), t) \in L^1_{\infty}[t_0, T] \), i.e. \( x_1(t) + \lambda x_2(t) = v(t), v(t) \in L^1_{\infty}[t_0, T] \). An equivalent description of system (9) in accordance with this constraint is given by

\[ \dot{x}_1 = -\lambda^{-1}x_1 + \lambda^{-1}v(t), \quad \lambda x_2(t) + x_1(t) = v(t) \quad (7) \]

It is clear that system (7) has the bounded input - bounded state property with respect to input \( v(t) \) as

\[ ||x_1(t)||_{\infty,[t_0, T]} \leq |x_1(t_0)| + ||v(t)||_{\infty,[t_0, T]} \]  
and

\[ ||x_2(t)||_{\infty,[t_0, T]} \leq \lambda^{-1}(||x_1(t)||_{\infty,[t_0, T]} + ||v(t)||_{\infty,[t_0, T]}). \]

This automatically implies that Assumption 1 holds for system (6) with \( \psi(x, t) = x_1 + \lambda x_2, \lambda > 0 \). In particular, the following estimate holds

\[ ||x(t)||_{\infty,[t_0, T]} \leq (1 + \lambda^{-1})|x_1(t_0)| + (1 + 2\lambda^{-1})||\psi(x(t), t)||_{\infty,[t_0, T]} \]

Let us specify a class of control inputs \( u \) which, in principle, can ensure boundedness of solutions \( x(t, x_0, t_0, \theta, u) \) for every \( \theta \in \Omega_\theta \) and \( x_0 \in \mathbb{R}^n \). According to (5), boundedness of \( x(t, x_0, t_0, \theta, u) \) is ensured if we find a control input \( u \) such that \( \psi(x(t), t) \in L^1_{\infty}[t_0, \infty] \). To this objective consider the dynamics of system (2) with respect to \( \psi(x, t) \):

\[ \dot{\psi} = L_{f(x, \theta)}\psi(x, t) + L_{g(x)}\psi(x, t)u + \frac{\partial \psi(x, t)}{\partial t}, \quad (8) \]

Assuming that the inverse \( (L_{g(x)}\psi(x, t))^{-1} \) exists everywhere, we may choose the control input \( u \) in the following class of functions:

\[ u(x, \hat{\theta}, \omega, t) = (L_{g(x)}\psi(x, t))^{-1} \left( -L_{f(x, \theta)}\psi(x, t) - \varphi(\psi, \omega, t) - \frac{\partial \psi(x, t)}{\partial t} \right) \]

\[ \varphi : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R} \]

where \( \omega \in \Omega_\omega \subset \mathbb{R}^m \) is a vector of known parameters of function \( \varphi(\psi, \omega, t) \). Denoting \( L_{f(x, \theta)}\psi(x, t) = f(x, \theta, t) \) and taking into account (9) we may rewrite equation (5) in the following manner:

\[ \dot{\psi} = f(x, \theta, t) - f(x, \hat{\theta}, t) - \varphi(\psi, \omega, t) \quad (10) \]

Hence, feedback (10) renders the original system (11) into the well-known nonlinear error model form (13)\(^5\).

In practical applications, state \( x \) of original model (11) is hardly ever available. Furthermore, imprecise physical models of the processes and measurement noise often lead to the presence of unmodeled dynamics in (10). Although we do not address these issues in detail in the present article, we do allow additive perturbations that are functions of time from \( L^1_{\infty}[t_0, \infty] \) in the right-hand side of (10). In particular, instead of (10) we consider the following equation:

\[ \dot{\psi} = f(x, \theta, t) - f(x, \hat{\theta}, t) - \varphi(\psi, \omega, t) + \varepsilon(t), \quad (11) \]

\(^5\)The error models (10) have proven to be convenient representations of systems with nonlinear parametrization in the problems of adaptation [22, 23] and parameter estimation [15].

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where, if not stated overwise, the function \( \varepsilon : \mathbb{R} \to \mathbb{R} \), \( \varepsilon \in L^1_2[t_0, \infty] \cap C^0 \). One of the immediate advantages of (11) in comparison with (10) is that it allows us to take the presence of state observers in the system into consideration. This clearly widens the range of possible applications of our results.

Let us now specify the desired properties of function \( \varphi(\psi, \omega, t) \) in (9), (11). The majority of known algorithms for parameter estimation and adaptive control \[54, 43, 30, 40\] assume global (Lyapunov) stability of system (11) for \( \theta \equiv \bar{\theta} \). In our study, however, we would like to refrain from this standard and at the same time restrictive requirement. Instead we propose that finite energy of the signal \( f(x(t), \theta(t), \bar{\theta}(t), t) \), defined for example by its \( L^1_2|t_0, \infty] \) norm with respect to the variable \( t \), results in finite deviation from the target set given by equality \( \psi(x, t) = 0 \). Formally this requirement is introduced in Assumption 2.

**Assumption 2 (Target dynamics operator)** Consider the following system:

\[
\dot{\psi} = -\varphi(\psi, \omega, t) + \zeta(t),
\]

where \( \zeta : \mathbb{R} \to \mathbb{R} \) and \( \varphi(\psi, \omega, t) \) is from (17). Then for every \( \omega \in \Omega_\omega \) system (12) has \( L^1_2|t_0, \infty] \to L^1_\infty|t_0, \infty] \) gain with respect to input \( \zeta(t) \). In the other words,

\[
\zeta(t) \in L^1_2|t_0, \infty] \Rightarrow \psi(t, \psi_0, t_0, \omega) \in L^1_\infty|t_0, \infty], \quad \psi_0 \in \mathbb{R}
\]

and there exists a function \( \gamma_{\infty, 2} \) such that

\[
\|\psi(t)\|_{\infty, [t_0, \infty]} \leq \gamma_{\infty, 2}(\psi_0, \omega, \|\zeta(t)\|_{2,[t_0, t]}, \forall \zeta(t) \in L^1_2[t_0, T]
\]

In contrast to conventional approaches, Assumption 2 does not require global asymptotic stability of the origin of (unperturbed, i.e for \( \zeta(t) = 0 \) system (12). In fact, system (12) is allowed to have Lyapunov-unstable equilibria. Moreover, there may be no equilibria at all in (12), or it can even exhibit chaotic dynamics. Examples of such systems, which potentially inherit chaotic behavior but still satisfy Assumption 2, are the well-known Lorenz \[35\] and Hindmarsh-Rose \[23\] oscillators. The last system models ion current through a membrane in the living cell, and is widely used in artificial neural networks, for instance, for processing of the visual information \[53\].

When the stability of the target dynamics \( \dot{\psi} = -\varphi(\psi, \omega, t) \) is known a-priori, one of the benefits of Assumption 2 is that there is no need to know the particular Lyapunov function of the unperturbed system. Apart from being, in some sense, a more friendly and less invasive concept, this enables us to design adaptive/parameter estimation procedures for systems with externally-driven uncontrolled multistability \[1, 65, 12, 38\].

The differences between conventional restrictions on the goal functionals and alternative requirements formulated in Assumptions 1, 2 are further illustrated with Figure 1. For simplicity it is assumed that

\[6\]In systems with externally driven multistability, i.e. when there are multiple coexistent attractors and trajectories switch from one attractor to another depending on the external perturbation, parameter estimation/control algorithms based on the knowledge of a specific Lyapunov function require additional information about the instant dynamical state (attractors and their allocation) of the system itself. This leads to a necessity to identify current dynamical state of the system prior to control/identification of its parameters.
The function $\psi(x, t)$ does not depend on $t$ explicitly and therefore its zeroes form a (set) surface in $\mathbb{R}^n$. For the conventional approaches this set should additionally satisfy metric conditions (4) in $\mathbb{R}^n$, Fig. 1. a. These conditions often restrict class of the possible target sets to the points in $\mathbb{R}^n$. In case Assumptions 1, 2 are satisfied this restriction does not apply any more. Indeed, given that $\zeta(t) \in L^1_{[t_0, \infty]}$ we can bound $\|\psi(x(t), t)\|_{\infty, [t_0, \infty]} \leq \gamma_{\infty, 2}(\psi_0, \omega, \|\zeta(t)\|_{2, [t_0, \infty]}) = M$. Therefore, according to Assumption 2, the state $x(t)$ is bounded and belongs to the sphere $\Omega_x = \{x \in \mathbb{R}^n | \|x(t)\| \leq \tilde{\gamma}(x_0, \theta, M) = \Delta\}$. On the other hand, the state $x(t)$ belongs to the domain $\Omega_\psi = \{x \in \mathbb{R}^n | x : |\psi(x, t)| \leq M\}$. This implies that the segments of trajectory $x(t, x_0, t_0, \theta, u(t))$, for $t \geq t_0$ will remain in the bounded domain $\Omega_x \cap \Omega_\psi$ (shadowy volume in Fig. 1. b.) for all $t > t_0$.

The Figure 1 also emphasizes the difference between the proposed operator framework and known approaches in adaptive control based on geometrical representations [3]. Indeed, the results based on coordinate transformations around the target manifold (3) are applicable only in a subset of $\mathbb{R}^n$ where $\psi(x, t)$ does not depend explicitly on $t$, and rank of $\psi(x, t)$ is constant. In this respect these results are local. On the other hand, Assumptions 1, 2 do not require constant rank conditions and allow both time-varying $\psi(x, t)$ and $\varphi(\psi, \omega, t)$. This makes Assumptions 1, 2 a suitable replacement to conventional approaches for systems with non-stationary dynamics, or ones which are far away from equilibrium or invariant target manifolds.

So far we have introduced basic assumptions on system dynamics and the class of feedback considered in this article. Let us now specify the class of functions $f(x, \theta, t)$ in (11). Since general parametrization of function $f(x, \theta, t)$ is methodologically difficult to deal with but solutions provided for a restricted class of nonlinearities (for instance to those which allow linear re-parametrization) often yield physically implausible models, we have opted for a new class of parameterizations. This class shall include a sufficiently broad range of physical models, in particular those with nonlinear parametrization; the proposed parameterizations will
also, in principle, be able to handle arbitrary state nonlinearity in the class of functions from $C^1$. As a candidate for such a parametrization we suggest nonlinear functions that satisfy the following assumption:

**Assumption 3 (Monotonicity and Growth Rate in Parameters)** For the given function $f(x, \theta, t)$ in $(11)$ there exists function $\alpha(x, t) : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\alpha(x, t) \in C^1$ and positive constant $D > 0$ such that

$$ (f(x, \hat{\theta}^t, t) - f(x, \theta^t, t))(\alpha(x, t)^T(\hat{\theta} - \theta)) \geq 0 $$

$$ |f(x, \hat{\theta}^t, t) - f(x, \theta^t, t)| \leq D|\alpha(x, t)^T(\hat{\theta} - \theta)| $$

The first inequality in Assumption 3 holds, for example, for every smooth nonlinear function which is monotonic with respect to a linear functional $\phi(x)^T \theta$ over a vector of parameters:

$$ f(x, \theta, t) = f_m(x, \phi(x)^T \theta, t) $$

$$ \text{sign} \left( \frac{\partial f_m(x, \lambda, t)}{\partial \lambda} \right) = \text{const} $$

Hence function $\alpha(x, t)$ satisfying (14) could be chosen in the following form: $\alpha(x, t) = M\phi(x)\kappa(x, t)$, where $\kappa : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\kappa(x, t) \in C^1$.

The second inequality, (15), is satisfied if the function $f(x, \phi(x)^T \theta, t)$ does not grow faster than a linear function in variable $\phi(x)^T \theta$ for every $x \in \mathbb{R}^n$. This requirement holds, for example, for those functions $f(x, \phi(x)^T \theta, t)$ which are globally Lipschitz in $\phi(x)^T \theta$:

$$ |f_m(x, \phi(x)^T \theta, t) - f_m(x, \phi(x)^T \theta', t)| \leq D_\theta(x, t)|\phi(x)^T(\theta - \theta')| $$

In particular, inequalities (14), (15) hold for the function $f(x, \phi(x)^T \theta, t)$ with $\alpha(x, t) = MD_\theta(x, t)\phi(x)$. A graphical illustration of the choice of function $\alpha(x, t)$ is given in Figure 2.
relevant models with nonlinear parametrization. These include effects of stiction forces \[2\], slip and surface dependent friction given by the “magic formula” \[39\] or physics-inspired model of the tyre \[9\], nonlinear processes in dampers for automotive suspension \[29\], smooth saturation, and dead-zones in mechanical systems. It further includes nonlinearities in models of bio-reactors \[6\]. The class of functions \(f(x, \theta, t)\) specified in Assumption \[9\] can also serve as nonlinear replacement of functions that are linear in their parameters in a variety of piecewise approximation models. Last but not least, this set of functions includes sigmoid and Gaussian nonlinearities, which are favored in neuro and fuzzy control and mathematical models of neural processes \[13\]. Table 1 provides some of the parametric nonlinearities that occur in these processes and their corresponding functions \(\alpha(x, t)\).

Table 1: Examples of nonlinearities satisfying Assumption \[9\] Parameters \(\Delta_\theta, \Delta_r\) are positive constants

| physical meaning               | mathematical model of uncertainty \(f(x, \theta, t)\)                                                                 | domain of physical relevance                                                                 | \(\alpha(x, t)\)                  |
|-------------------------------|---------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------|-----------------------------------|
| stiction forces               | \(\theta_0 e^{-x_2^2 \theta_1} = e^{-x_2^2 \theta_1 + \ln(\theta_0)}\)                                                                 | \(\Delta_\theta > \theta_0, \theta_1 > 0\) \(x \in \mathbb{R}^2\)                          | \(\alpha(x, t) = (-x_2^2, 1)^T\)   |
| tyre-road friction \[9\]      | \(F_n \text{sign}(x_2) \frac{\frac{\mu_C}{x_3} - \frac{x_2}{x_3} + G}{\frac{\mu_S - \mu_C}{x_3} e^{-\frac{x_2^2}{x_3}}}\)                      | \(\Delta_\theta > \theta > 0\) \(x_1, x_2 \geq 0\) \(x_3 \in (0, 1)\)                       | \(\alpha(x, t) = \frac{x_3}{1 - x_3}\)    |
| force supported by            | \(\frac{K_S (\theta + \Delta_r x_3 - x_2)}{L (\theta + K_o x_3)}\)                                                                 | \(\Delta_\theta > \theta > 0\) \(x_3 > 0\)                                                      | \(\alpha(x, t) = A_p(x_1 - x_2)\)    |
| hydraulic emulsion in         | \(K_o, A_p, P_o, L > 0\) - parameters                                                                                   |                                                                                                |                                    |
| suspension dampers \[29\]     | \(x = (x_1, x_2, x_3)\)                                                                                                  |                                                                                                |                                    |
| nonlinearities in             |                                                                                                                        |                                                                                                |                                    |
| Monod’s growth model of       |                                                                                                                        |                                                                                                |                                    |
| microorganisms \[9\]          |                                                                                                                        |                                                                                                |                                    |
| blur distortion model         | \(\sum_{i,j=1}^{n} e^{\frac{-|r_{i,j}(t)|}{\epsilon_{c} j_{c} \epsilon_{c} j_{c}}} r_{i,j}(t)\)                      | \(\Delta_\theta > \theta > 0\) \(x_1, x_2 > 0\)                                                      | \(\alpha(x, t) = x_1 x_2 (1, x_1)^T\) |
| in networks for processing of  |                                                                                                                        |                                                                                                |                                    |
| visual information            |                                                                                                                        |                                                                                                |                                    |

Assumption \[9\] bounds the growth rate of the difference \(|f(x, \theta, t) - f(x, \hat{\theta}, t)|\) by the functional \(D|\alpha(x, t)^T (\hat{\theta} - \theta)|\). This will help us to find a parameter estimation algorithm such that the estimates converge to \(\theta\) sufficiently fast for the solutions of \(11\), \(111\) to remain bounded with non-dominating feedback \(6\). On the other hand, parametric error \(\hat{\theta} - \theta\) can be inferred from the changes in the variable \(\psi(x, t)\), according to \(111\), only by means of the difference \(f(x, \theta, t) - f(x, \hat{\theta}, t)\). Therefore, as long as convergence of the estimates \(\hat{\theta}\),
to $\theta$ is expected, it is also useful to have the estimate of $|f(x, \theta, t) - f(x, \hat{\theta}, t)|$ from below, as specified in Assumption 4:

**Assumption 4** For the given function $f(x, \theta, t)$ in (11) and function $\alpha(x, t)$, satisfying Assumption 3, there exists a positive constant $D_1 > 0$ such that

$$|f(x, \hat{\theta}, t) - f(x, \theta, t)| \geq D_1 |\alpha(x, t)^T (\hat{\theta} - \theta)|$$  \hspace{1cm} (16)

In problems of parameter estimation, effectiveness of the algorithms often depends on how "good" the nonlinearity $f(x, \theta, t)$ is, and how predictable locally is the system’s behavior. As a measure of goodness and predictability usually the substitutes as smoothness, boundedness, and Lipschitz conditions are considered. In our study, we distinguish several such specific properties of functions $f(x, \theta, t)$ and $\varphi(\psi, \omega, t)$. These properties are provided below

**H 1** Function $f(x, \theta, t)$ is locally bounded with respect to $x$, $\theta$ uniformly in $t$.

**H 2** Function $f(x, \theta, t) \in C^1$, and $\partial f(x, \theta, t)/\partial t$ is locally bounded with respect to $x$, $\theta$ uniformly in $t$.

**H 3** Function $f(x, \theta, t)$ is globally Lipschitz with respect to $\theta$ uniformly in $x$, $t$:

$$\exists D_0 > 0 : |f(x, \theta, t) - f(x, \hat{\theta}, t)| \leq D_0 \|\theta - \hat{\theta}\|$$

**H 4** Let $U_x \subset \mathbb{R}^n$, $U_\theta \subset \mathbb{R}^d$ be given and $U_x$, $U_\theta$ are bounded. Then there exists constant $D_{U_x, U_\theta} > 0$ such that for every $x \in U_x$ and $\theta, \hat{\theta} \in U_\theta$ Assumption 4 is satisfied with $D_1 = D_{U_x, U_\theta}$.

**H 5** Function $\varphi(\psi, \omega, t)$ is locally bounded in $\psi$, $\omega$ uniformly in $t$.

In the next section we present novel algorithms for adaptive control and parameter estimation in nonlinear dynamical systems [2] which satisfy Assumptions [1][2][3][4]. We show that under an additional structural requirement, which relates properties of function $\alpha(x, t)$ and vector-field $f(x, \theta)$ in (1), [2], the following desired property holds:

$$x(t) \in L^n_\infty[t_0, \infty]; f(x(t), \theta, t) - f(x, \hat{\theta}(t), t) \in L^2_\infty[t_0, \infty]$$ \hspace{1cm} (17)

After boundedness of the solutions is guaranteed, we prove that

$$\lim_{t \to \infty} \psi(x(t), t) = 0$$

In addition, we show that

$$\lim_{t \to \infty} \hat{\theta}(t) = \theta$$ \hspace{1cm} (18)

In particular we demonstrate that the standard persistent excitation condition is sufficient to guarantee the convergence. Furthermore, in the case that Assumptions [3][4] hold only locally (for $x$ from a domain of $\mathbb{R}^n$) we demonstrate that sufficiently high excitation in the system still leads to the desired estimates.

---

1Despite Assumption 4 requires that (10) holds for every $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^d$, and $t \in \mathbb{R}_+$, we will see later that for a variety of problems it is sufficient that it is satisfied only locally.
4 Main Results

Standard approaches in parameter estimation and adaptation problems usually assume feedback and a parameter adjustment algorithm in the following form

$$ u = u(x, \hat{\theta}, t) $$
$$ \dot{\hat{\theta}} = A_{lg}(\psi, x, t) $$

(19)

The most favorite strategy of finding these, known also as certainty-equivalence principle, is a two-stage design prescription. First, construct uncertainty-dependent feedback $u(x, \theta, t)$, $\theta \in \Omega$ which ensures boundedness of the trajectories $x(t)$. Second, replace $\theta$ with $\hat{\theta}$ in $u(x, \theta, t)$ and, given the constraints (e.g., $\dot{x}, \theta$ cannot be measured explicitly, while state $x$ is available), design function $A_{lg} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ which guarantees (17), (18), or/and $\psi(x(t), t) \rightarrow 0$.

With this strategy, the design of the feedback $u(x, \theta, t)$ is generally independent\(^8\) of the specific design of the parameter estimation algorithm $A_{lg}(\psi, x, t)$. This allows the full benefit of contemporary nonlinear control theory [24, 25, 27, 45] in designing feedbacks $u(x, \theta, t)$. On the other hand, this strategy equally benefits from conventional parameter estimation and adaptation theories [33, 16, 43, 40] which provide a list of the ready-to-be-implemented algorithms under the assumption that feedback $u(x, \theta, t)$ ensures stability of the system.

Ironically, the power of the certainty-equivalence principle – simplicity and relative independence of the stages of design – is also its Achilles’ heel. This principle does not take into account the possible interactions between stabilizing control and parameter estimation procedures. It has been reported in [56, 47, 3, 61] that an additional “interaction” term $\hat{\theta}_P(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ added to the parameters $\theta$ in function $u(x, \theta, t)$: $u(x, \theta + \hat{\theta}_P(x, t), t)$ introduces new properties to the system. Unfortunately, straightforward introduction of this "interaction" term as a new variable of the design affects its simplicity, internal order, and so much favored independence of the design stages (control and estimation).

An alternative strategy which introduces a new design paradigm is proposed in [58, 59]. Its main idea is that adaptation algorithms in [19] are initially allowed to depend on unmeasurable variables $\dot{\psi}, \dot{x}, \theta$

$$ \dot{\hat{\theta}} = A_{lg}^*(\psi, \dot{\psi}, x, \dot{x}, \theta, t) $$

(20)

For this reason we refer to such algorithms as virtual algorithms. If the desired properties [17], [18] are ensured with (20) then, taking into account properties of the vector-fields $f(x, \theta)$, $g(x)$ in [11], we convert the unrealizable algorithm (20) into an equivalent representation in integro-differential, or finite, form [19]:

$$ \dot{\theta} = \Gamma(\hat{\theta}_P(x, t) + \theta_I(t)), \quad \Gamma \in \mathbb{R}^{d \times d}, \quad \Gamma > 0 $$

(21)

\(^8\)In particular, it is the standard requirement that function $u(x, \theta, t)$ should guarantee Lyaponov stability of the system for $\theta = \theta$, while parameter adjustment algorithms use this property in order to ensure stability of the whole system. No other properties are required from the function $u(x, \theta, t)$.  

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This approach preserves the convenience of the certainty-equivalence principle, as the feedback $u(x, \theta, t)$ could, in principle, be built independently of the subsequent parameter adjustment procedure. At the same time, it provides the necessary interaction term $\dot{\theta} P(x, t)$ ensuring the required properties. The closed-loop system even if function $f(x, \theta, t)$ in (21) is nonlinear in $\theta$.

In this paper we propose the following class of virtual adaptation algorithms:

$$\dot{\theta} = \Gamma(\psi + \varphi(\psi, \omega, t))\alpha(x, t) + Q(x, \dot{\theta}, t)(\theta - \dot{\theta}), \quad \Gamma \in \mathbb{R}^{d \times d}, \quad \Gamma > 0 \tag{22}$$

where $Q(x, \dot{\theta}, t) : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$, $Q(\cdot) \in C^0$. As a candidate for finite form realization of algorithms we select the following set of equations:

$$\dot{\theta}(x, t) = \Gamma(\dot{\theta} P(x, t) + \dot{\theta} f(t)); \quad \Gamma \in \mathbb{R}^{d \times d}, \quad \Gamma > 0$$

$$\dot{\theta} f = \psi(x, t)\alpha(x, t) - \Psi(x, t) \tag{23}$$

where function $\Psi(x, t) : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies Assumption 5

Assumption 5 There exists function $\Psi(x, t)$ such that

$$\frac{\partial \Psi(x, t)}{\partial x_2} - \psi(x, t) \frac{\partial \alpha(x, t)}{\partial x_2} = B(x, t), \tag{24}$$

where $B(x, t) : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^{d \times p}$ is either zero or, if $f_2(x, \theta)$ is differentiable in $\theta$, satisfies the following:

$$B(x, t)F(x, \theta, \theta') = 0 \quad \forall \theta, \theta' \in \Omega_{\theta}= \mathbb{R}^n$$

$$F(x, \theta, \theta') = \int_0^1 \frac{\partial f_2(x, s(\lambda))}{\partial \theta} \, d\lambda, \quad s(\lambda) = \theta' + (1 - \lambda) \tag{23}$$

Function $R(x, \hat{\theta}, u(x, \hat{\theta}, t), t) : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ in (23) is given as follows:

$$R(x, u(x, \hat{\theta}, t), t) = \partial \Psi(x, t)/\partial t - \psi(x, t)/(\partial \alpha(x, t)/\partial t) -$$

$$\left(\psi(x, t)L_{\theta} L_{\theta'} \alpha(x, t) - L_{\theta} \Psi(x, t)\right) - \left(\psi(x, t)L_{\theta} \alpha(x, t) - L_{\theta} \Psi(x, t)\right)u(x, \hat{\theta}, t)$$

$$+ B(x, t)(f_2(x, \hat{\theta}) + g_2(x)u(x, \hat{\theta}, t)). \tag{25}$$

Functions $\Psi(x, t)$ and $R(x, \hat{\theta}, u(x, \hat{\theta}, t), t)$ are introduced into (23) in order to shape the derivative $\dot{\theta}(x, t)$ to fit equation (23). The role of function $\Psi(x, t)$ in (24) is to compensate for the uncertainty-dependent term $\psi(x, t)L_{\theta} \alpha(x, t)$, and equation (24) is the condition when such compensation is possible. With the function $R(x, \hat{\theta}, u(x, \hat{\theta}, t), t)$ we eliminate the influence of the uncertainty-independent vector fields $f_1(x)$, $g_1(x)$, and $g_2(x)$ on the desired form of the time-derivative $\dot{\theta}(x, t)$. The properties of system (11), together with control (9) and this new adaptation algorithm (23), (25), are summarized in Theorem 11 and Theorem 22.

---

9 Choice of the virtual algorithm in the form of equation (24) is motivated by our previous study of derivative-dependent algorithms for systems with uncertainties that are nonlinear in their parameters.

10 We show below, in the proof of Theorem 11 (see Appendix), that Assumption 5 is indeed sufficient for the function $\dot{\theta}(x, t)$ to be a realization of (23).
Theorem 1 (Boundedness) Let system \([\mathcal{H}], \mathcal{I}, \mathcal{P}, \mathcal{Z}\) be given and Assumptions \([\mathcal{A}], \mathcal{B}, \mathcal{C}\) be satisfied. Then the following properties hold

\begin{enumerate}
  \item Let for the given initial conditions \(x(t_0), \dot{\theta}(t_0)\) and parameters vector \(\theta\), interval \([t_0, T^*]\) be the (maximal) time-interval of existence of solutions in the closed loop system \([\mathcal{H}], \mathcal{I}, \mathcal{P}, \mathcal{Z}\). Then
  \[f(x(t), \theta, t) - f(x(t), \dot{\theta}(t), t) \in L^1_{2}[t_0, T^*]\]
  and
  \[\|f(x(t), \theta, t) - f(x(t), \dot{\theta}(t), t)\|_2[t_0, T^*] \leq D_f(\theta, t_0, \Gamma, \|\varepsilon(t)\|_2[t_0, T^*]);\]
  \[D_f(\theta, t_0, \Gamma, \|\varepsilon(t)\|_2[t_0, T^*]) = \left(\frac{D}{2}\|\theta - \dot{\theta}(t_0)\|_{1-1}^{2}\right)^{0.5} + \frac{D}{2D_1}\|\varepsilon(t)\|_2[t_0, T^*]\]
  (26)

  \end{enumerate}

In addition, if Assumptions \([\mathcal{A}], \mathcal{B}\) are satisfied then

\begin{enumerate}
  \item \(\psi(x(t), t) \in L^1_{2}[t_0, \infty], x(t) \in L^\infty_{n}[t_0, \infty]\) and
  \[\|\psi(x(t), t)\|_{2,t_0,\infty} \leq \gamma_\infty(\psi(x_0, t_0), \omega, D_f(\theta, t_0, \Gamma, \|\varepsilon(t)\|_2[t_0, \infty]) + \|\varepsilon(t)\|_2[t_0, \infty])\]
  (27)

  \end{enumerate}

\(P_3\) if properties \([\mathcal{H}], \mathcal{H}\) hold, and system \([\mathcal{E}]\) has \(L^1_2[t_0, \infty] \rightarrow L^1_2[t_0, \infty], p > 1\) gain with respect to input \(\zeta(t)\) and output \(\psi\) then

\[\varepsilon(t) \in L^1_2[t_0, \infty] \text{ and } L^1_2[t_0, \infty] \Rightarrow \lim_{t \rightarrow \infty} \psi(x(t), t) = 0\]

(28)

If, in addition, property \([\mathcal{E}]\) holds, and functions \(\alpha(x, t), \partial \psi(x, t)/\partial t\) are locally bounded with respect to \(x\) uniformly in \(t\), then

\(P_4\) the following limiting relation holds

\[\lim_{t \rightarrow \infty} f(x(t), \theta, t) - f(x(t), \dot{\theta}(t), t) = 0\]

(29)

Proofs of Theorem \([\mathcal{H}], \mathcal{I}\) and subsequent results are given in the Appendix.

Before we proceed with discussion of the results of Theorem \([\mathcal{H}], \mathcal{I}\) we wish to comment on Assumption \([\mathcal{E}]\) links the possibility to design the parameter adjustment algorithm in the form of equation \([\mathcal{E}]\), with the properties of functions \(\alpha(x, t)\) and \(\psi(x, t)\). These functions depend on the properties of nonlinearity \(f(x, \theta, t)\) itself (function \(\alpha(x, t)\)) and, importantly, on the chosen specification of the desired target set:

\[\{x \in \mathbb{R}^n | \psi(x, t) = 0\} \subseteq \Omega_0\]

given by function \(\psi(x, t)\). Specific properties of the functions \(f(x, \theta, t)\) and \(\psi(x, t)\) are interrelated through the possibility to solve partial differential equation \([\mathcal{E}]\) for the function \(\Psi(x, t)\). Let \(\mathcal{B}(x, t) = \text{col}(\mathcal{B}_1(x, t), \ldots, \mathcal{B}_d(x, t))\), and \(\alpha(x, t) \in C^2, \alpha(x, t) = \text{col}(\alpha_1(x, t), \ldots, \alpha_d(x, t))\), then necessary and sufficient conditions for existence of the function \(\Psi(x, t)\) follow from the Poincaré lemma:

\[\frac{\partial}{\partial x_2} \left(\psi(x, t) \frac{\partial \alpha_i(x, t)}{\partial x_2} + \mathcal{B}_i(x, t)\right) = \left(\frac{\partial}{\partial x_2} \left(\psi(x, t) \frac{\partial \alpha_i(x, t)}{\partial x_2} + \mathcal{B}_i(x, t)\right)\right)^T\]

(30)
This relation, in the form of conditions of existence of the solutions for function $\Psi(x, t)$ in [24], takes into account structural properties of system (11). Indeed, let $B(x, t) = 0$ and consider partial derivatives $\partial \alpha_i(x, t)/\partial x_2$, $\partial \psi(x, t)/\partial x_2$ with respect to vector $x_2 = (x_{21}, \ldots, x_{2p})^T$. Let

$$\begin{align*}
\frac{\partial \psi(x, t)}{\partial x_2} &= (0 0 \cdots 0 * 0 \cdots 0) \\
\frac{\partial \alpha_i(x, t)}{\partial x_2} &= (0 0 \cdots 0 * 0 \cdots 0)
\end{align*}$$

(31)

where symbol * denotes a function of $x$ and $t$. Then condition (31) guarantees that equality (30) (and, subsequently, Assumption 5) holds. Whether or not Assumption 5 holds, depends, roughly speaking, on how large is the part of partition $x_2$ that enters the arguments of functions $\psi(x, t)$, $\alpha(x, t)$. In the case of $\partial \alpha(x_1 \oplus x_2, t)/\partial x_2 = 0$, Assumption 5 holds for arbitrary $\psi(x, t) \in C^1$. If $\psi(x, t)$, $\alpha(x, t)$ depend on just a single component of $x_2$, for instance $x_{2k}$, $k \in \{0, \ldots, p\}$, then conditions (31) hold and function $\Psi(x, t)$ can be derived explicitly by integration

$$\Psi(x, t) = \int \psi(x, t) \frac{\alpha(x, t)}{\partial x_{2k}} \, dx_{2k}$$

(32)

In all other cases, the existence of the required function $\Psi(x, t)$ follows from (30).

The necessity to satisfy Assumption 5 may seem to be a critical restriction, which limits applicability of our approach. However, we notice that it holds in the relevant problem settings\textsuperscript{11} for arbitrary $\alpha(x, t), \psi(x, t) \in C^1$. Consider, for instance [10], where the class of systems is restricted to (33): \textsuperscript{11}

$$\dot{x} = -q(x, u)x + f(\theta, u, x), \quad \psi(x, t) > \varphi_{\min} > 0, \quad x \in \mathbb{R}$$

(33)

The dimension of the state in system (33) coincides with that of the uncertainty-dependent partition and equals to unit (dim $\{x\} = \text{dim} \{x_2\} = 1$). Hence, according to (32), and in case functions $\psi(x, t)$, $\alpha(x, t) \in C^1$, there will always exist a function $\Psi(x, t)$ satisfying equality (24) with $B(x, t) = 0$.

In the general case, when dim $\{x_2\} > 1$, the problems of finding function $\Psi(x, t)$ satisfying condition (24) can be avoided (or converted into one with an already known solutions such as (30), (32)) by the embedding technique proposed in [59]. The main idea of the method is to introduce an auxiliary system

$$\dot{\xi} = f_\xi(x, \xi, t), \quad \xi \in \mathbb{R}^z$$

$$h_\xi = h_\xi(\xi, t), \quad \mathbb{R}^z \times \mathbb{R}_+ \rightarrow \mathbb{R}^h$$

such that

$$f(x(t), \theta, t) - f(x_1(t) \oplus h_\xi(t) \oplus x'_2(t), \theta, t) \in L_2^1[t_0, \infty]$$

(35)

and dim $\{h_\xi\} + \text{dim} \{x'_2\} = p$. Then (11) can be rewritten as follows:

$$\dot{\psi} = f(x_1 \oplus h_\xi \oplus x'_2, \theta, t) - f(x_1 \oplus h_\xi \oplus x'_2, \theta, t) - \varphi(\psi, \omega, t) + \varepsilon(t),$$

(36)

\textsuperscript{11} See, for example, the problem setting in [10] for parameter estimation in the presence of nonlinear state-dependent parametrization. This problem setting, according to our knowledge, is by far one of the most general available in the literature.
where \( \varepsilon(t) \in L^1_{\text{loc}}[t_0, \infty) \), and \( \dim \{ x_2' \} = p - h < p \). In principle, the dimension of \( x_2' \) could be reduced to 1 or 0. As soon as this is ensured, Assumption 5 will be satisfied and the results of Theorem 1 follow. Sufficient conditions ensuring the existence of such an embedding in general case are provided in [59]. For systems in which the parametric uncertainty can be reduced to vector fields with low-triangular structure the embedding is given in [60].

An alternative way to construct system (34) with the desired properties is to use (possible, high-gain, discontinuous) robust observers. In order to illustrate this approach consider the rather general case when function \( f_2(\mathbf{x}, \mathbf{\theta}) \) in (14) is given as \( f_2(\mathbf{x}, \mathbf{\theta}) = \bar{f}_2(\mathbf{x}) + \phi(\mathbf{x}, \mathbf{\theta}) \), and function \( \phi(\mathbf{x}, \mathbf{\theta}) \) is bounded.

Let in addition there exist continuous functions \( h_r : \mathbb{R}^p \rightarrow \mathbb{R}^p \), \( h_x : \mathbb{R}^p \rightarrow \mathbb{R}^p \) such that the following inequality is satisfied

\[
\| h_r(\mathbf{\xi} - \mathbf{x}_2) \| \geq | f(\mathbf{x}_1 \oplus h_x(\mathbf{\xi}), \mathbf{\theta}, t) - f(\mathbf{x}_1 \oplus \mathbf{x}_2, \mathbf{\theta}, t) | \tag{37}
\]

As a candidate for yet unknown tracking system (34) we select the following

\[
\dot{\mathbf{\xi}} = \bar{f}(\mathbf{x}) + f_r(\mathbf{\xi} - \mathbf{x}_2) + g_2(\mathbf{x})\mathbf{u} + \mathbf{v} \tag{38}
\]

where function \( f_r : \mathbb{R}^p \rightarrow \mathbb{R}^p \) and auxiliary input \( \mathbf{v} \in \mathbb{R}^p \) are the design parameters. Subtracting equations for \( \mathbf{x}_2 \) in (14) from (38) yields:

\[
\dot{\mathbf{e}} = f_r(\mathbf{e}) - \phi(\mathbf{x}(t), \mathbf{\theta}) + \mathbf{v} \tag{39}
\]

where \( \mathbf{e} = \mathbf{\xi} - \mathbf{x}_2 \). Let us finally choose the function \( f_r \) in [60] such that the system \( \dot{\mathbf{e}} = f_r(\mathbf{e}) + \mathbf{v} \) is strictly passive with a positive definite storage function \( V(\mathbf{e}, t) \):

\[
\dot{V}(\mathbf{e}, t) \leq \mathbf{y}_r^T \mathbf{v} - \beta \| \mathbf{y}_r \|^2, \quad \beta > 0 \tag{40}
\]

According to [37] inequality (40) guarantees that there always exists input \( \mathbf{v}(t) \) in (38) such that \( h_r(e(t)) \in L^p_{\text{loc}}[t_0, \infty) \). Then taking into account (37) we can conclude that condition (35) holds (with \( x_2' : \dim \{ x_2' \} = 0 \)). This implies that the original error model (11) can be converted into (36), which in our case satisfies Assumption 5 (\( \partial \alpha_i(\mathbf{x}, t) \partial \mathbf{x}_2 = 0 \) for the corresponding \( \alpha_i(\mathbf{x}, t) \) in [60]).

Let us now briefly comment on the results of Theorem 1. The theorem ensures a set of relevant properties for both control (P2, P3) and parameter estimation problems (P1, P4). These properties, as illustrated with (20)–(29), provide conditions for boundedness of the solutions \( \mathbf{x}(t, \mathbf{x}_0, t_0, \mathbf{\theta}, \mathbf{u}(t)) \), reaching the target set \( \Omega_0 \), and exact compensation of the uncertainty term \( f(\mathbf{x}, \mathbf{\theta}, t) \) even in the presence of unknown disturbances \( \varepsilon(t) \in L^1_{\text{loc}}[t_0, \infty] \cap L^1_t[t_0, \infty] \). These characterizations are the consequence of the fact that \( (f(\mathbf{x}(t), \mathbf{\theta}, t) - f(\mathbf{x}(t), \bar{\mathbf{\theta}}(t), t))) \in L^1_{\text{loc}}[t_0, \infty] \), which in turn is guaranteed by properties (14), (15), (16) of the one extra assumption on the function \( f_r(\mathbf{e}) \) in (39) is imposed. In particular it is required that the system \( \dot{\mathbf{e}} = f_r(\mathbf{e}) + \mathbf{v} \) is strongly zero-detectable with respect to inputs \( \mathbf{v} \) and output \( \mathbf{y}_r \). In our case, however, the limiting relations \( \lim_{t \to \infty} \mathbf{y}_r(t) = 0, \lim_{t \to \infty} \mathbf{e}(t) = 0 \) are not necessary. Therefore, as follows from the proof of Theorem 2 in [37], in order to show just \( \| \mathbf{y}_r(t) \| \in L^1_{\text{loc}}[t_0, \infty] \) the assumption of strong zero-detectability can be omitted.
function \( f(\mathbf{x}, \theta, t) \) in Assumptions 3. Among these properties, estimate (10) in Assumption 4 is particularly important for allowing disturbances (potentially unbounded) from \( L^1_0[t_0, \infty] \). When no disturbances are present it is possible to show that P1–P4 hold without involving Assumption 4.

**Corollary 1** Let system (1), (11), (23), (25) be given, Assumptions 1, 3–5 hold, respectively.

Let function \( \zeta \) be given, \( \varepsilon(t) = 0 \), and Assumptions 5, 6 hold. Then

P5) norm \( \|\theta - \hat{\theta}(t)\|^2_{r-1} \) is non-increasing and properties P1–P4 of Theorem 1 hold with \( \varepsilon(t) = 0 \) respectively.

In addition to the fact that \( |f(\mathbf{x}, \theta, t) - f(\mathbf{x}, \hat{\theta}, t)| \) is not required to be bounded from below as in (16), Corollary 1 ensures that \( \|\theta - \hat{\theta}(t)\|^2_{r-1} \) is not growing with time when \( \varepsilon(t) = 0 \). The practical relevance of the corollary is that it will allow us to guarantee desired convergence (18) with a much weaker, local version of Assumption 4. It will also help us to establish conditions for (semi-global) exponential stability in the unperturbed system, which in turn will enable (small) disturbances from \( L^1_\infty[t_0, \infty] \) in the right-hand side of (11).

Another consequence of Theorem 1 concerns the specific case when \( \varepsilon(t) \in L^1_\infty[t_0, \infty] \cap L^2_0[t_0, \infty] \).

**Corollary 2** Let system (1), (11), (23), (25) be given, Assumptions 1, 5, 6 hold, \( \varepsilon(t) \in L^1_\infty[t_0, \infty] \cap L^2_0[t_0, \infty] \), and property H5 holds. Let in addition, system (12) has \( L^1_p[t_0, \infty] \to L^1_\infty[t_0, \infty] \), \( p \geq 2 \) gain. Then

P6) \( \psi(\mathbf{x}(t), t) \in L^1_\infty[t_0, \infty] \), \( \mathbf{x}(t) \in L^2_\infty[t_0, \infty] \);

P7) if properties H1, H5 hold, and system (12) has \( L^1_p[t_0, \infty] \to L^1_q[t_0, \infty] \), \( q > 1 \) gain with respect to input \( \zeta(t) \) and state \( \psi \), then \( \lim_{t \to \infty} \psi(\mathbf{x}(t), t) = 0 \);

If in addition property H4 is satisfied, functions \( \alpha(\mathbf{x}, t) \), \( \partial \psi(\mathbf{x}, t)/\partial t \) are locally bounded with respect to \( \mathbf{x} \) uniformly in \( t \), then limiting relation (24) holds as well.

Corollary 2 extends applicability of algorithms (23), (25) to systems (11) with defined \( L^1_p[t_0, \infty] \to L^1_\infty[t_0, \infty] \) gains for arbitrary \( p \geq 2 \).

Let us formulate conditions ensuring convergence of the estimates \( \hat{\theta}(t) \) to \( \theta \) in the closed loop system (11), (11), (23), (25) and (25). When the mathematical model of the uncertainties is linear in its parameters, i.e. \( f(\mathbf{x}, \theta, t) = \zeta(\mathbf{x}, t)^T \theta \), the usual requirement for convergence is that signal \( \zeta(\mathbf{x}(t), t) \) is persistently exciting (24).

**Definition 1 (Persistent Excitation)** Let function \( \zeta : \mathbb{R}_+ \to \mathbb{R}^k \) be given. Function \( \zeta(t) \) is said to be persistently exciting iff there exist constants \( \delta > 0 \) and \( L > 0 \) such that for all \( t \in \mathbb{R}_+ \) the following holds

\[
\int_t^{t+L} \zeta(\tau)\zeta(\tau)^T d\tau \geq \delta I \tag{41}
\]

\[\text{In this case, however, the bound for } \|\psi(\mathbf{x}(t), t)\|_{\infty} \text{ will be different from the one given by equation (24) in Theorem 1. Its new estimate is given by formula (17) in Appendix 1.}\]
The conventional notion of persistent excitation requires specific properties (i.e., the integral inequality (11)) from signal $\zeta(t)$ as a function of time. In the closed loop system, however, relevant signals in the model of uncertainty $f(x, \theta, t)$ can depend on state $x$ and parameters. In particular, they depend on initial conditions, parameters of the feedback, and initial time $t_0$. In order to address this issue it is suggested in [36] to use the notion of uniform persistent excitation:

**Definition 2 (Uniform Persistent Excitation)** Let function $\zeta : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^k$ be given, and $x(t, x_0, t_0, \theta_0)$ be a solution of (1), where the vector $\theta_0 \in \mathbb{R}^s$ stands for all possible parameters of (1) and feedback (9), (20), (21). Function $\zeta(x(t, x_0, t_0, \theta_0), t)$ is said to be uniformly persistently exciting if there exist constants $\delta > 0$ and $L > 0$ such that for all $t, t_0 \in \mathbb{R}_+, x_0 \in \mathbb{R}^n, \theta_0 \in \mathbb{R}^s$ the following holds:

$$\int_{t}^{t+L} \zeta(x(\tau, x_0, t_0, \theta_0), \tau) \zeta(x(\tau, x_0, t_0, \theta_0), \tau)^T d\tau \geq \delta I$$  \hspace{1cm} (42)

When dealing with nonlinear parameterization, it is also useful to have a characterization which takes into account nonlinearity in the model. In the linear case, persistent excitation of signal $\zeta(x(t), t)$ (inequality (11)) implies that the following property holds:

$$\exists t' \in [t, t + L] : |\zeta(x(t'), t')^T (\theta_1 - \theta_2)| \geq \delta \|\theta_1 - \theta_2\|$$  \hspace{1cm} (43)

In other words, the difference $|\zeta(x(t), t)^T (\theta_1 - \theta_2)|$ is proportional to the distance $\|\theta_1 - \theta_2\|$ in parameter space for some $t' \in [t, t + L]$. In the nonlinear case it is natural to replace the linear term $\zeta(x(t'), t')^T (\theta_1 - \theta_2)$ in (43) with its nonlinear substitute $f(x(t'), \theta_1, t') - f(x(t'), \theta_2, t')$ as has been done, for example, in [10] for systems with convex/concave parametrization. It is also natural to replace the proportion $\delta \|\theta_1 - \theta_2\|$ in the right-hand side of (43) with a nonlinear function. Therefore, as a candidate for the nonlinear persistent excitation condition we propose the following notion:

**Definition 3 (Nonlinear Persistent Excitation)** The function $f(x(t), \theta, t) : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be persistently excited with respect to parameters $\theta \in \Omega_0 \subset \mathbb{R}^d$ if there exist constant $L > 0$ and function $\varrho : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \rho \in K \cap C^0$ such that for all $t \in \mathbb{R}_+, \theta_1, \theta_2 \in \Omega_0$ the following holds:

$$\exists t' \in [t, t + L] : |f(x(t'), \theta_1, t') - f(x(t'), \theta_2, t')| \geq \varrho(\|\theta_1 - \theta_2\|)$$  \hspace{1cm} (44)

Properties (11) and (13) in Definitions 1 and 3 can be considered as alternative characterizations of excitation in dynamical systems. While inequality (11) accounts for specific properties of the signals in the uncertainty, inequality (13) accounts for possibility to detect parametrical difference from the difference $f(x(t), \theta_1, t) - f(x(t), \theta_2, t)$. Taking into account these two equally possible but still rather distinct characterizations of excitation in nonlinear systems, in Theorem 2 below we present a set of alternatives for parameter convergence in system (1), (10), (20), (21).
Theorem 2 (Convergence)  Let system (1), (10), (23), (25) satisfy Assumptions 1–3. Let, in addition, Assumption 5 hold with \( B(x,t) = 0 \). Then \( x(t) \in L_\infty^n[t_0,\infty], \hat{\theta}(t) \in L^d_\infty[t_0,\infty] \). Moreover the limiting relation:

\[
\lim_{t \to \infty} \hat{\theta}(x(t),t) = \theta
\]

is ensured if \( \alpha(x,t) \) is locally bounded in \( x \) uniformly in \( t \), and one of the following alternatives hold:

1) function \( \alpha(x(t),t) \) is persistently exciting, and hypothesis \( H4 \) holds;

2) function \( f(x(t),\theta,t) \) is nonlinearly persistently exciting, i.e. it satisfies condition (44); it satisfies hypotheses \( H1 \) \( H2 \) \( \psi(\omega,t) \) satisfies \( H5 \) \( \partial \psi(x,t)/\partial t \) be locally bounded in \( x \) uniformly in \( t \);

In case alternative 1) is satisfied, the estimates \( \hat{\theta}(x(t),t) \) converge to \( \theta \) exponentially fast. If, in addition, \( \alpha(x(t),t) \) is uniformly persistently exciting and Assumption 4 holds, then convergence is uniform. The rate of convergence can be estimated as follows:

\[
\| \hat{\theta}(t) - \theta \| \leq e^{-\rho t} \| \hat{\theta}(t_0) - \theta \| D_\Gamma
\]

with

\[
\rho = \frac{\delta D_1 \lambda_{\min}(\Gamma)}{2L(1 + \lambda_{\max}(\Gamma)L^2D^2 \alpha_\infty^2)}, \quad D_\Gamma = \left( \frac{\lambda_{\max}(\Gamma)}{\lambda_{\min}(\Gamma)} \right)^{1/2}, \quad \alpha_\infty = \sup_{\|x\| \leq \|x(t)\|_{\infty} \leq \|x(t)\|_{\infty} \leq \|x(t)\|_{\infty}, \ t \geq t_0} \| \alpha(x,t) \|.
\]

Notice that Theorem 2 considers error models (10) where no disturbance term \( \varepsilon(t) \) is present. Despite this, Theorem 2 can be straightforwardly extended to error models with disturbance (11). Indeed, as follows from alternative 1), the parameter estimation subsystem becomes exponentially stable in case function \( \alpha(x(t),t) \) is (uniformly) persistently exciting. This in turn allows (sufficiently small) additive disturbances in the right-hand side of (10). In case the excitation is uniform, convergence of the estimates \( \hat{\theta}(t) \) to a neighborhood of \( \theta \) is guaranteed for every \( \varepsilon(t) \in L_\infty^n[t_0,\infty] \) by inverse Lyapunov stability theorems [27].

In case of alternative 2), nonlinear persistent excitation condition (44) guarantees convergence (18) without invoking Assumption 4 or H4. In this case, however, the convergence may not be robust, which seems to be a natural tradeoff between generality of nonlinear parameterizations \( f(x,\theta,t) \) and robustness with respect to unknown disturbances \( \varepsilon(t) \).

5 Discussion

So far we have shown that, for the class of nonlinearly parameterized systems, there exist a control function and parameter adjustment algorithms, such that solutions of the whole system are bounded and parametric uncertainty is decreasing in time. We have shown also that in case of persistently excited functions \( \alpha(x,t) \) the estimates \( \hat{\theta}(t) \) in (23) converge exponentially fast to vector \( \theta \). These results, however, are not necessarily limited to functions satisfying Assumptions 3 or 4. Due to space limitations, however, we provide just the main ideas of possible extensions, leaving out the technical details. Let us first examine the case where these assumptions hold only in some domains of the system state space.
Nonlinear functions satisfying Assumptions 3, 4 in a domain of $\mathbb{R}^n$. Let, in particular, for the given nonlinear function $f(x, \theta, t)$ there exits the following partition of the state space:

$$
\Omega_x = \Omega_M \cup \Omega_A, \quad \Omega_M = \bigcup_j \Omega_{M,j}, \quad \Omega_A = \Omega_x/\Omega_M
$$

where $\Omega_{M,j}$ are the subsets of $\mathbb{R}^n$ where Assumptions 3, 4 are satisfied for every $\theta \in \Omega_\theta$ with the corresponding functions $\alpha_j(x, t)$ and constants $D_j, D_{1,j}$. Let us also assume that $\Omega_M$ contains an open set.

A typical example of a nonlinear function which satisfies this assumption is $\sin(\theta x)$, where the unknown parameter $\theta$ belongs to a bounded interval. Let, for instance, the system dynamics is given by

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \sin(\theta x_1) + u,
\end{align*}
$$

where parameter $\theta \in \Omega_\theta = [0.6, 1.4]$ is unknown a-priori. For the given bounds of $\Omega_\theta$ the domain $\Omega_M$ can be derived as follows:

$$
\Omega_M = \{ x \mid x_1 \in [-3.38, -2.59] \} \cup \{ x \mid x_1 \in [-1.14, 1.14] \} \cup \{ x \mid x_1 \in [2.59, 3.38] \} = \Omega_{M,1} \cup \Omega_{M,2} \cup \Omega_{M,3}
$$

and the function $\alpha(x, t)$, satisfying Assumptions 3, 4 in $\Omega_M$ is defined as

$$
\alpha(x, t) = \begin{cases} 
x_1, & x \in \Omega_{M,2} \\
-x_1, & x \in \Omega_{M,1} \cup \Omega_{M,3}
\end{cases}
$$

Another example is $x^{\theta}$, $\theta \in [t_0, \infty)$. The last parametrization is widely used in modelling physical “power law” phenomena in nature (see, for example [67], where this function models effects of nonlinear damping in muscles).

The fact that Assumptions 3, 4 hold in the domain $\Omega_M \subset \mathbb{R}^n$, allows us to guarantee decrease of the norm $\| \theta - \hat{\theta}(t) \|_{L_{-1}}^2$ only if the state belongs to $\Omega_M$. Therefore, extra control is needed in order to steer state $x$ back into the domain $\Omega_M$. Let us, for example, pick point $x^* \in \Omega_M$, such that $\text{dist}\{x^*, \Omega_A\} > r$, $r \in \mathbb{R}_+$. 

---

Figure 3: Domain $\Omega_M$ for system (46) with nonlinear parameterized function $\sin(\theta x_1)$, $\theta \in \Omega_\theta$
Let, in addition, there exists control function \( u_0(x,t) \) such that it steers state \( x \) of system (11) from the initial point \( x_0 \) into the \( \delta_0 \)-neighborhood \( U(\delta_0, x^*) \) of \( x^* \) in finite time \( T_0(x_0) \). Suppose also that \( \delta_0 < r \). As follows from Theorems 12, \( \theta(t) \) is bounded for every segment of the solution which starts from \( U(\delta_0, x^*) \) at \( t = t_i \) and leaves the domain \( U(r, x^*) \) at \( t = t_{i+1} \). Furthermore the bound for \( \| \dot{\theta} \|_{\infty, [t_i, t_{i+1}]} \) can be estimated a-priory from the bounds on \( \theta \) and \( \| e(t) \|_{L_2, [t_0, \infty]} \) (see also (20)). Given that the right-hand side of system (11), (23), (25) is locally bounded we can conclude that the time interval \( t_{i+1} - t_i \) will always be separated from zero. Taking into account the results of Theorem 2 equation (45), we may conclude that sufficiently high excitation, defined by the ratio \( \delta/L \), will guarantee that \( \| \dot{\theta}(t_{i+1}) - \theta \| < \kappa \| \dot{\theta}(t_i) - \theta \|, \kappa \in \mathbb{R}, 0 \leq \kappa < 1 \). If the time sequence \( \{t_i\} \) is infinite (i.e. the system always escapes the ball \( U(r, x^*) \)) then convergence is asymptotic. In case the sequence \( \{t_i\} \) is finite (i.e. \( \exists \ t^* > 0 : x(t) \in \Omega_M \forall t > t^* \) convergence is exponential, this follows from Theorem 2.

In principle, the size of \( \Omega_M \) and its location in \( \mathbb{R}^n \) depend on the bounds of \( \Omega_\theta \). In fact, the larger the bounds, the smaller the volume of \( \Omega_M \). Moreover, the size of \( \Omega_M \) as a function of the bounds of \( \Omega_\theta \) depends on specific properties of nonlinearity \( f(x, \theta, t) \). These observations suggest that in order to handle a broader class of nonlinearities (or functions with higher degree of uncertainty in \( \theta \)) within the strategy proposed above, one needs to increase the excitation in functions \( \alpha_j(x, t) \). This is consistent with previously reported results [10] on parameter convergence in nonlinearly parameterized systems. Whether the extension of the class of nonlinearities to more general functions renders it necessary to increase excitation, however, is still an open issue.

Functions \( f(x, \theta, t) \) with nonlinear incremental growth rates in \( \theta \). Another direction to extend the class nonlinear functions suitable for our method is to allow nonlinear bounds for the growth rates in [14], [10] in Assumptions 3, 4. The most straightforward generalizations, which do not change dramatically the machinery of technical proofs of Theorems 12 are provided in Assumptions 6, 7 below:

Assumption 6 For the given function \( f(x, \theta, t) \) in [11] there exist function \( \alpha(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^d, \alpha(x, t) \in C^1 \), function \( \sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d, \sigma(z) = (\sigma_1(z_1), \sigma_2(z_2), \ldots, \sigma_d(z_d))^T \)

\[
\sigma_i(\xi) \xi \geq 0, \lim_{s \rightarrow \infty} \int_0^s \sigma_i(\xi) d\xi = \infty,
\]

and function \( \gamma \in \mathcal{K} \) such that

\[
(f(x, \hat{\theta}, t) - f(x, \theta, t))\alpha(x, t)^T \sigma(\hat{\theta} - \theta) \geq 0 \tag{48}
\]

\[
|f(x, \hat{\theta}, t) - f(x, \theta, t)| \leq \gamma(t) \alpha(x, t)^T \sigma(\hat{\theta} - \theta) \tag{49}
\]

\[\text{An example is given in [10], where nonlinear persistent excitation condition holds for the given parametrization, while linear persistent excitation condition for linear parametrization with respect to the same parameter-independent function is not satisfied.}\]
Assumption 7  
For the given function \( f(x, \theta, t) \) in (7) and function \( \alpha(x, t) \) satisfying Assumption 6 there exists function \( \gamma \in \mathcal{K} \) such that

\[
\gamma(\|\alpha(x, t)^T \sigma(\theta - \hat{\theta})\|) \leq |f(x, \theta, t) - f(x, \hat{\theta}, t)|
\]

Choosing for simplicity \( \Gamma = I \), denoting \( \Delta f(\hat{\theta}, \theta, x, t) = |f(x, \hat{\theta}, t) - f(x, \theta, t)| \), letting \( |\varepsilon(t)| \) in (11) be such that \( \int_0^\infty \gamma_\epsilon(|\varepsilon(\tau)|)d\tau < \infty \), \( \gamma_\epsilon \in \mathcal{K} \) and replacing \( V_\delta(\hat{\theta}, \theta, t) \) in (67) (proof of Theorem 1 Appendix 1) with

\[
V_\delta(\hat{\theta}, \theta, t) = \sum_{i=1}^d \int_0^{\sigma(\theta_i - \hat{\theta}_i)} \sigma(\xi)d\xi + \int_t^\infty \gamma_\epsilon(|\varepsilon(\tau)|)d\tau
\]

we can obtain the following estimate:

\[
\dot{V} = -\sigma(\hat{\theta} - \theta)^T \alpha(x, t)(f(x, \hat{\theta}, t) - f(x, \theta, t)) + \varepsilon(t)\sigma(\hat{\theta} - \theta)^T \alpha(x, t) - \gamma_\epsilon(|\varepsilon(t)|)
\]

\[
\leq -\gamma^{-1}(\Delta f(\hat{\theta}, \theta, x, t))\Delta f(\hat{\theta}, \theta, x, t) + |\varepsilon(t)|\gamma^{-1}(\Delta f(\hat{\theta}, \theta, x, t)) - \gamma_\epsilon(|\varepsilon(t)|) \leq 0
\]

Boundedness of \( \hat{\theta} \) follows from (68) if we can resolve the following inequality for unknown \( \gamma_\epsilon(|\varepsilon(t)|) \):

\[
-\gamma^{-1}(\Delta f(\hat{\theta}, \theta, x, t))\Delta f(\hat{\theta}, \theta, x, t) + |\varepsilon(t)|\gamma^{-1}(\Delta f(\hat{\theta}, \theta, x, t)) - \gamma_\epsilon(|\varepsilon(t)|) \leq 0
\]

If in addition there exists \( \gamma_f \in \mathcal{K} \) such that

\[
-\gamma^{-1}(\Delta f(\hat{\theta}, \theta, x, t))\Delta f(\hat{\theta}, \theta, x, t) + |\varepsilon(t)|\gamma^{-1}(\Delta f(\hat{\theta}, \theta, x, t)) - \gamma_\epsilon(|\varepsilon(t)|) \leq -\gamma_f(|\Delta f(\hat{\theta}, \theta, x, t)|)
\]

then we can guarantee that \( \Delta f(\hat{\theta}(t), \theta, x(t), t) \in L^1_{\gamma_f}[t_0, \infty) \), where \( L^1_{\gamma_f}[t_0, \infty) \) is the space of all functions \( f_0(t) : \mathbb{R}_+ \rightarrow \mathbb{R}^n \) with finite integral \( \int_0^\infty \gamma_f(\|f_0(t)\|)dt \leq \infty \). Therefore, if system (12) has \( L^1_{\gamma_f}[t_0, \infty) \cup L^1_{\gamma_f} \rightarrow L^1_{\gamma_f} \) gain, we can conclude, invoking this new modified Assumption 2 that \( \psi(x(t), t) \in L^1_{\gamma_f} \), \( x(t) \in L^n_{\gamma_f} \).

Notice that letting functions \( \gamma, \gamma_\epsilon, \gamma_f \) linear \( \gamma(|s|) = D|s|, \gamma_\epsilon(|s|) = D_1|s| \), allows us straightforwardly obtain, as in (69), that choice \( \gamma_\epsilon = \frac{D}{4D_1} \varepsilon^2 \) ensures the following inequality:

\[
\dot{V} \leq -\frac{1}{D} \left( |f(x, \hat{\theta}, t) - f(x, \theta, t)| - \frac{D}{2D_1} \varepsilon(t) \right)^2
\]

This implies that properties similar to P1)–P7) can be derived for the case where Assumptions 2–4 are replaced with Assumptions 6–7 and functions \( \gamma_\epsilon(\cdot), \gamma_f(\cdot) \) are linear (for the proofs of Theorem 11 and Corollaries 12 see the Appendix). Parameter convergence in this case can also be deduced from Theorem 2 alternative 2). Notice that, due to the nonlinearities in \( \sigma(\theta - \hat{\theta}) \), convergence in general may not be exponentially fast.

For the nonlinear functions \( \gamma, \gamma_f \) in Assumptions 6–8 the formulation of the results will require the notions of \( L^1_{\gamma_f} \) spaces introduced above. The machinery behind the statements, however, will remain the same. The practical relevance of these results with nonlinear functions \( \gamma, \gamma_f \) is that they enable us to take into account the specific properties of the signal \( \varepsilon(t) \) when designing control, adaptation and parameter estimation procedures. If, for example, disturbance \( \varepsilon(t) \) is due to observers, we might derive requirements...
on convergence rates for the observer-induced errors $\varepsilon(t)$ (i.e. $\varepsilon(t) \in L^1_{r_{\varepsilon}}[t_0, \infty]$, and $\gamma_{r}(\cdot)$ satisfies inequality (41)). Given these rates and the fact that $\Delta f(x(t), \dot{\theta}(t), t) \in L^1_{r_f}[t_0, \infty]$, the target dynamics

$$\dot{\psi} = -\varphi(\psi, \omega, t) + \zeta(t), \quad \zeta(t) \in L^1_{\gamma_{r}}[t_0, \infty] \cup L^1_{r_{\gamma}}[t_0, \infty]$$

should be chosen in order to guarantee boundedness of $\psi(t)$ for all $\zeta(t) \in L^1_{\gamma_{r}}[t_0, \infty] \cup L^1_{r_{\gamma}}[t_0, \infty]$. This will allow synergy at all stages of the design and analysis of adapting systems.

**Singularities in control** (4), and non-affine models. In the problem statement we restricted the class of nonlinear systems of interest models (11) that are affine in control and furthermore, we assumed that inverse $(L_{g(x)}\psi(x,t))^{-1}$ exists everywhere. Even though this restriction holds in wide variety of practically relevant situations, the question is whether the proposed approach could be extended to more general classes of systems. Let us, for instance, assume that either $L_{g(x)}\psi(x,t) = 0$ for some $x \in \mathbb{R}^n$, or the right-hand side of (11) is not affine in control, e.g.

$$\dot{x} = f(x, \theta, u)$$

Obviously, control function (9), which transforms (11) into (10), is not relevant any more. Despite that, it is still possible to transform system (11) into an error model, similar to (10). In order realize this transformation without invoking the use of linearity in the control or taking inverse $(L_{g(x)}\psi(x,t))^{-1}$, it should be possible to find a function $u(x, \theta, \omega, t)$ such that the following invariance condition is satisfied:

$$\frac{\partial \psi(x,t)}{\partial x} f(x, \theta, u(x, \theta, \omega, t)) = -\varphi(\psi(x,t), \omega, t) - \frac{\partial \psi(x,t)}{\partial t}$$

Denoting

$$\frac{\partial \psi(x,t)}{\partial x} f(x, \theta, u(x, \dot{\theta}, \omega, t)) = -f^*(x, \theta, \dot{\theta}, \omega, t)$$

and taking into account (52), (53), and (54) we can calculate derivative $\dot{\psi}$ in the following form

$$\dot{\psi} = -f^*(x, \theta, \dot{\theta}, \omega, t) + f^*(x, \theta, \theta, \omega, t) - f^*(x, \theta, \theta, \omega, t) + \frac{\partial \psi(x,t)}{\partial t}$$

The main difference between error models (55) and (11) is that function $f^*(x, \theta, \dot{\theta}, \omega, t)$ in (55) depends on additional parameters $\theta, \omega$. Despite this difference our approach can still be applied to models (55) if inequalities (14), (16) in Assumption 3 (or/and 4) hold for function $f^*(x, \theta, \dot{\theta}, \omega, t)$ for any $\omega \in \Omega_{\omega}$. Adaptation algorithms in this case can straightforwardly be derived from (51), (55) (in Appendix 1) and will have the form similar to (28), (29).

In the next section we illustrate the application to and main steps in the design of our algorithms for the optimal slip identification problem in brake control systems.
6 Example

Consider the problem of minimizing the braking distance for a single wheel rolling along a surface. The surface properties can vary depending on the current position of the wheel. The wheel dynamics can be given by the following system of differential equations [51]:

\[
\begin{align*}
\dot{x}_1 &= -\frac{1}{m} F_s(F_n, x, \theta), \\
\dot{x}_2 &= \frac{1}{J} (F_s(F_n, x, \theta) r - u) \\
\dot{x}_3 &= -\frac{1}{x_1} \left( \frac{1}{m} (1 - x_3) + \frac{r^2}{J} F_s(F_n, x, \theta) - r \frac{u}{J} \right),
\end{align*}
\] (56)

where

\[ x_1 \text{ is longitudinal velocity, } x_2 \text{ is angular velocity, } x_3 = \frac{x_1 - rx_2}{x_1} \text{ is wheel slip, } \]

\[ x = (x_1, x_2, x_3)^T, \text{ mass of the wheel, } J \text{ is moment of inertia, } r \text{ is radius of the wheel, } u \text{ is control input (brake torque), } \]

\[ F_s(F_n, x, \theta) \text{ is a function specifying the tyre-road friction force depending on the surface-dependent parameter } \theta \text{ and the load force } F_n. \]

This function, for example, can be derived from steady-state behavior of the LuGre tyre-road friction model [9]:

\[
F_s(F_n, x, \theta) = F_n \text{sign}(x_2) \left( \frac{\sigma_0}{L} \frac{x_3}{1-x_3} + g(x_2, x_3, \theta) \right),
\] (57)

\[
g(x_2, x_3, \theta) = \theta (\mu_C + (\mu_S - \mu_C) e^{\frac{-|rx_2 x_3|}{|v_s x_3|}}),
\] (58)

where \( \mu_C, \mu_S \) are Coulomb and static friction coefficients, \( v_s \) is the Stribeck velocity, \( \sigma_0 \) is the normalized rubber longitudinal stiffness, \( L \) is the length of the road contact patch. In order to avoid singularities we assume, as suggested in [51], that the system is turned off when velocity \( x_1 \) reaches a small neighborhood of zero (in our example we stopped simulations as soon as \( x_1 \) becomes less than 5 m/sec). Moreover, given that functions (57), (58) are bounded for the relevant set of the system parameters, it is always possible to design control function \( u(x, t) \) in (56) such that

\[
x_3(t) \in [\delta, 1 - \delta], \quad \delta \in \mathbb{R}, \quad \delta > 0
\] (59)

for all \( t : x_1(t) \geq \delta_1, \delta_1 = 5 \text{ m/sec.} \)

While the majority of the model parameters can be estimated a-priori, the tyre-road parameter \( \theta \) is dependent on the properties of the road surface. Therefore, on-line identification of the parameter \( \theta \) is desirable in order to compute the optimal slip value

\[
x_3^* = \arg \max_{x_3} F_s(F_n, x, \theta)
\] (60)

which ensures the maximum deceleration force and therefore results in the shortest braking distance.

\[\text{[51]}\]

\[\text{[9]}\]

\[\text{[57]}\]

\[\text{[58]}\]

\[\text{[59]}\]

\[\text{[60]}\]

\[\text{[51]}\]

\[\text{[9]}\]

\[\text{[57]}\]

\[\text{[58]}\]

\[\text{[59]}\]

\[\text{[60]}\]

\[\text{[51]}\]
The main loop controller is derived in accordance with the standard certainty-equivalence principle and can be written as follows:

\[ u(x, \hat{\theta}, x_3) = \frac{J}{r} \left( \frac{1}{m} (1 - x_3) + \frac{r^2}{J} F_s(F_n, x, \hat{\theta}) - K_s x_1 (x_3 - \hat{x}_3) \right), \quad K_s > 0 \]

In order to estimate parameter \( \theta \) by measuring the values of variables \( x_1, x_2 \) and \( x_3 \), we construct the following subsystem:

\[ \dot{x}_3 = -\frac{1}{x_1} \left( \frac{1}{m} (1 - x_3) + \frac{r^2}{J} F_s(F_n, x, \hat{\theta}) - \frac{r}{J} u \right) + (x_3 - \hat{x}_3) \]

and consider the dynamics of error function \( \psi(x, t) = \psi(x_3, \hat{x}_3) = x_3 - \hat{x}_3 \):

\[ \dot{\psi} = -\psi + \frac{1}{x_1} \left( \frac{1}{m} (1 - x_3) + \frac{r^2}{J} (F_s(F_n, x, \theta) - F_s(F_n, x, \hat{\theta})) \right) \]

The desired dynamics of system \((61)\) is

\[ \dot{\psi} = -\psi + \xi(t) \]

where \( \xi(t) \) is to be from \( L^1_{\infty}[t_0, \infty] \). Let us check Assumptions \ref{assumption1} \ref{assumption2} for the function \( \psi(x, t) = x_3 - \hat{x}_3(t) \) and system \((62)\). Notice first that state \( x \) of system \((56)\) is bounded according to the physical laws governing the dynamics of \((56)\). In addition, boundedness of \( \psi(x, t) \) implies that \( \dot{x}_3(t) \) is bounded. Hence Assumption \ref{assumption1} holds. System \((62)\), obviously, has \( L^1_{\infty}[t_0, \infty] \Rightarrow L_{\infty}[t_0, \infty] \) gain. We can conclude that Assumption \ref{assumption2} also holds. Let us check Assumptions \ref{assumption4} \ref{assumption5}.

Taking into account \((59)\), we can conclude that function \( \frac{1}{x_1} \left( \frac{1}{m} (1 - x_3) + \frac{r^2}{J} \right) \) in \((61)\) is positive and, furthermore, is separated from zero for all \( x_1 > \delta_1 \). Therefore, taking this into account equations \((57)\), \((58)\), we can conclude that function \( \frac{1}{x_1} \left( \frac{1}{m} (1 - x_3) + \frac{r^2}{J} \right) F_s(F_n, x, \theta) \) in \((61)\) satisfies Assumptions \ref{assumption3} \ref{assumption4} with

\[ \alpha(x, t) = \text{const} = 1, \quad \forall x_1 : \delta_1 < x_1 < x_1(t_0) \]

Therefore, in order to design an estimation scheme satisfying assumptions of Theorem \ref{theorem2} we shall find functions \( \Psi(x, t), B(x, t) \) such that Assumption \ref{assumption1} holds. It is easy to see that this assumption is satisfied with \( \Psi(x, t) = \text{const} \), and \( B(x, t) = 0 \). Let us choose, therefore, \( \Psi(x, t) = 0 \). Then according to \((28)\) and \((31)\), a parameter adjustment algorithm will be given by the following system:

\[ \dot{\hat{\theta}} = \gamma((x_3 - \hat{x}_3) + \hat{\theta}_1), \quad \dot{\hat{\theta}}_1 = (x_3 - \hat{x}_3), \quad \gamma = 100 \]

An important fact about algorithm \((38)\) is that it is a parametric linear proportional-integral scheme. According to Theorem \ref{theorem2} the estimates \((38)\) converge to \( \theta \) exponentially fast in the domain specified by equation \((50)\), and inequality \( x_1(t_0) \geq x_1(t) > \delta_1 \). The last inequality is satisfied as, according to \((51)\), time-derivative of the variable \( x_1(t) \) is non-positive and the system is turned “off” when \( x_1(t) \leq \delta_1 \).

We simulated system \((56) - (63)\) with the following setup of parameters and initial conditions: \( \sigma_0 = 200, L = 0.25, \mu_C = 0.5, \mu_S = 0.9, v_s = 12.5, r = 0.3, m = 200, J = 0.23, F_n = 3000, K_s = 30 \). The effectiveness
of estimation algorithm (63) could be illustrated with Figure 4. Estimates $\hat{\theta}$ approach the actual values of parameter $\theta$ sufficiently fast for the controller to calculate the optimal slip value $x_3^*$ and steer the system toward this point in real braking time. Effectiveness of the proposed identification-based control can be confirmed by comparing the braking distance in the system with on-line estimation of $x_3^*$ according to formula (60) with the one, in which the values of $x_3^*$ were kept constant (in the interval $[0.1, 0.2]$). For model parameters as presently given and road condition given by the piece-wise constant function

$$
\theta(s) = \begin{cases} 
0.3, & s \in [0, 8] \\
1.3, & s \in (8, 16] \\
0.7, & s \in (16, 24] \\
0.4, & s \in (24, 32] \\
1.5, & s \in (32, 40] \\
0.6, & s \in (40, \infty]
\end{cases} , s = \int_0^t x_1(\tau)d\tau
$$

the simulated braking distance obtained with our on-line estimation procedure of $x_3^*$ is 54.95 meters. This result compares favorably with the values obtained for preset values of $x_3^*$, which range between 57.52 and 55.32 (for $x_3^* = 0.1$ and $x_3^* = 0.2$ respectively).

7 Conclusion

In the present article we provided new tool for the design and analysis of adaptive/parameter estimation schemes for dynamic systems with possible Lyapunov-unstable desired dynamics and nonlinear parameterization. In our method we consider adaptation as a process of asymptotic compensation of the uncertainty, or as control in functional spaces, rather than as simply reaching of a control goal. In particular, we wished to achieve that mismatches between the modeled uncertainty and compensator vanish asymptotically with time or belong to specific functional spaces. This understanding of adaptation naturally leads to the possibility to describe the desired dynamics of adapting systems in terms of an operator, which maps these mismatches into error functions, as functions of time from a functional space. Continuity of this target operator is not
required. Hence stability of the desired dynamics, as a substitute of continuity, is not necessary for our approach. The adaptation mechanism itself could be viewed as control in functional spaces. In the other words, the aim of adaptation consists in ensuring that the uncertainty-induced errors of the compensator belong to a specific functional space. This idea leads to classes of adaptive systems, where applications require gentle, non-dominating control and where the desired dynamical state can be unstable.

When the desired motions in the system are known to be Lyapunov stable, our approach allows to design adaptation procedures without knowledge of the particular Lyapunov function. As mentioned in [40], this was one of the open theoretical challenges in the theory of adaptive control.

Another contribution of our present study is that we proposed a new class of parameterizations for nonlinearly parameterized models. Instead of aiming at a general solution for the problem of nonlinearity in the parameters, parametrization was restricted to a set of smooth functions, which are monotonic with respect to a linear functional in the parameters. For this new class, adaptation/estimation algorithms were introduced and analyzed. It was shown that standard linear persistent excitation conditions suffice to ensure exponentially fast convergence of the estimates to the actual values of unknown parameters. If, however, the monotonicity assumption holds only locally in the system state space, excitation with sufficiently high-frequency of oscillations still is able to ensure convergence. In addition to the analysis of the effects of conventional persistent excitation on convergence, we also formulated a much weaker property - nonlinear persistent excitation condition. With this new property we established conditions for asymptotic convergence of the estimates. It is also desirable to notice that in case of linear parametrization the proposed parameter estimation schemes allow to estimate the unknowns in a dynamical system without asking for the usual filtered transformations, thus reducing the number of integrators in the estimator.

An application of our results, which is relevant to the parameter estimation problems for systems with nonlinear parameterization, was provided as an example. In this example we did not cover all solutions to every theoretical problem we were targeting in this article. In particular, it covers only the problem of nonlinear parameterization. The main rationale, however, was to illustrate all steps of our method. Last but not the least, the application presents a practically relevant solution to an important engineering problem. The effectiveness of the solution to this problem leads us to expect that our newly proposed method can successfully be implemented in a variety of other applications.

8 Appendix 1. Proofs of the theorems and auxiliary results

Proof of Theorem 1. Let us first show that property P1) holds. Consider solutions of system (1), (11), (23), (25) passing through the point \( x(t_0), \hat{\theta}(t_0) \) for \( t \in [t_0, T^*] \). According to the theorem formulation, interval \([t_0, T^*]\) is the interval of existence of the solutions.
of function $\dot{\theta}(x, t)$: $\dot{\theta}(x, t) = \Gamma(\hat{\theta}_p + \hat{\theta}_j) = \Gamma(\hat{\psi}\alpha(x, t) + \psi\hat{\alpha}(x, t) - \hat{\Psi}(x, t) + \hat{\theta}_j)$. Notice that
\[
\psi\hat{\alpha}(x, t) - \hat{\Psi}(x, t) + \dot{\theta} = \psi(x, t)\frac{\partial\alpha(x, t)}{\partial x_1}\dot{x}_1 + \psi(x, t)\frac{\partial\alpha(x, t)}{\partial x_2}\dot{x}_2 + \psi(x, t)\frac{\partial\alpha(x, t)}{\partial t} - \dot{\Psi}(x, t) - \dot{\theta}_j.
\]
According to Assumption 5, $\frac{\partial\psi(x, t)}{\partial x_2} = \psi(x, t)\frac{\partial\alpha(x, t)}{\partial x_2} + B(x, t)$. Then taking into account (69), we can obtain
\[
\psi\hat{\alpha}(x, t) - \hat{\Psi}(x, t) + \dot{\theta} = \left(\psi(x, t)\frac{\partial\alpha(x, t)}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\right)\dot{x}_1 + \psi(x, t)\frac{\partial\alpha(x, t)}{\partial x_2}\dot{x}_2 - \frac{\partial\Psi(x, t)}{\partial t} + \dot{\theta}_j
\]
Notice that according to the proposed notation we can rewrite the term $\left(\psi(x, t)\frac{\partial\alpha(x, t)}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\right)\dot{x}_1$ in the following form: $(\psi(x, t)\dot{L}_p\alpha(x, t) - \dot{L}_p\hat{\psi}(x, t)) + (\psi(x, t)\dot{L}_p\alpha(x, t) - \dot{L}_p\hat{\psi}(x, t))u(x, \theta, t)$. Hence it follows from (68) and (65) that $\psi\hat{\alpha}(x, t) - \hat{\Psi}(x, t) + \dot{\theta}_j = \psi(\hat{\psi})\alpha(x, t) - B(x, t)(f_2(x, \theta) - f_2(x, \hat{\theta}))$. Therefore derivative $\dot{\theta}(x, t)$ can be written in the following way:
\[
\dot{\theta} = \Gamma((\psi + \varphi(\psi))\alpha(x, t) - B(x, t)(f_2(x, \theta) - f_2(x, \hat{\theta})))
\]
Consider the following positive-definite function:
\[
V_{\theta}(\theta, \dot{\theta}, t) = \frac{1}{2}\left\|\dot{\theta} - \theta\right\|_{\Gamma}^2 + \frac{D}{4D_1^2}\int_0^\infty \varepsilon^2(\tau)d\tau
\]
Its time-derivative according to equations (66) can be obtained as follows:
\[
\dot{V}_{\theta}(\theta, \dot{\theta}, t) = (\varphi(\psi) + \dot{\psi})(\dot{\theta} - \theta)^T\alpha(x, t) - (\dot{\theta} - \theta)^TB(x, t)(f_2(x, \theta) - f_2(x, \hat{\theta})) - \frac{D}{4D_1^2}\varepsilon^2(t)
\]
Let $B(x, t) \neq 0$, then consider the following difference $f_2(x, \theta) - f_2(x, \hat{\theta})$. Applying Hadamard’s lemma we represent this difference in the following way:
\[
f_2(x, \theta) - f_2(x, \hat{\theta}) = \int_0^1 \frac{\partial f_2(x, \lambda)}{\partial s}d\lambda(\theta - \hat{\theta}), \quad s(\lambda) = \theta_\lambda + \hat{\theta}(1 - \lambda)
\]
Therefore, according to Assumption 5 function $(\dot{\theta} - \theta)^TB(x, t)(f_2(x, \theta) - f_2(x, \hat{\theta}))$ is positive semi-definite, hence using Assumptions 3 and equality (11) we can estimate derivative $\dot{V}_{\theta}$ as follows:
\[
\dot{V}_{\theta}(\theta, \dot{\theta}, t) \leq -(f(x, \hat{\theta}, t) - f(x, \theta, t) + \varepsilon(t))(\dot{\theta} - \theta)^T\alpha(x, t) - \frac{D}{4D_1^2}\varepsilon^2(t)
\]
\[
\leq -\frac{1}{D}(f(x, \hat{\theta}, t) - f(x, \theta, t))^2 + \frac{1}{D_1}||f(x, \hat{\theta}, t) - f(x, \theta, t)||^2 + \frac{D}{4D_1^2}\varepsilon^2(t)
\]
\[
\leq -\frac{1}{D}\left(||f(x, \hat{\theta}, t) - f(x, \theta, t)||^2 - \frac{D}{2D_1^2}\varepsilon(t)\right)^2 \leq 0
\]
It follows immediately from (68), (67) that
\[
\left\|\dot{\theta}(t) - \theta\right\|_{\Gamma}^2 \leq \left\|\dot{\theta}(t_0) - \theta\right\|_{\Gamma}^2 + \frac{D}{2D_1^2}\varepsilon(t)^2 + (t_0, \infty)
\]
In particular, for \( t \in [t_0, T^*] \) we can derive from (67) that
\[
\| \hat{\theta}(t) - \theta \|^2_{\text{f}_1^{-1}} \leq \| \hat{\theta}(t_0) - \theta \|^2_{\text{f}_1^{-1}} + \frac{D}{2D_1^2} \| \varepsilon(t) \|^2_{2,[t_0, T^*]}
\]
Therefore \( \hat{\theta}(t) \in L^2_\infty[t_0, T^*] \). Furthermore \( |f(x(t), \hat{\theta}(t), t) - f(x(t), \theta, t)| - \frac{D}{2D_1} \varepsilon(t) \in L^2_\infty[t_0, T^*] \). In particular
\[
\left\| f(x(t), \hat{\theta}(t), t) - f(x(t), \theta, t) \right\|_{2,[t_0, T^*]}^2 \leq \frac{D}{2} \| \theta - \hat{\theta}(t_0) \|^2_{\text{f}_1^{-1}} + \frac{D^2}{4D_1^2} \| \varepsilon(t) \|^2_{2,[t_0, T^*]}
\]
Hence \( f(x(t), \hat{\theta}(t), t) - f(x(t), \theta, t) \in L^2_\infty[t_0, T^*] \) as a sum of two functions from \( L^2_\infty[t_0, T^*] \). In order to estimate the upper bound of norm \( f(x(t), \hat{\theta}(t), t) - f(x(t), \theta, t) \|_{2,[t_0, T^*]} \) from (71) we use the Minkowski inequality:
\[
\left\| f(x(t), \hat{\theta}(t), t) - f(x(t), \theta, t) \right\|_{2,[t_0, T^*]} \leq \left( \frac{D}{2} \| \theta - \hat{\theta}(t_0) \|^2_{\text{f}_1^{-1}} \right)^{0.5} + \frac{D}{2D_1} \| \varepsilon(t) \|^2_{2,[t_0, T^*]}
\]
and then apply the triangle inequality to the functions from \( L^2_\infty[t_0, T^*] \):
\[
\| f(x(t), \hat{\theta}(t), t) - f(x(t), \theta, t) \|_{2,[t_0, T^*]} \leq \| f(x(t), \hat{\theta}(t), t) - f(x(t), \theta, t) \|_{2,[t_0, T^*]} + \frac{D}{2D_1} \| \varepsilon(t) \|^2_{2,[t_0, T^*]}
\]
Therefore, property P1) is proven.

Let us prove property P2). In order to do this we have to check first if the solutions of the closed loop system are defined for all \( t \in \mathbb{R}_+ \), i.e. they do not reach infinity in finite time. We prove this by a contradiction argument. Indeed, let there exists time instant \( t_s \) such that \( \| x(t_s) \| = \infty \). It follows from P1), however, that \( f(x(t), \hat{\theta}(t), t) - f(x(t), \theta, t) \in L^1_\infty[t_0, t_s] \). Furthermore, according to (72) the norm \( \| f(x(t), \hat{\theta}(t), t) - f(x(t), \theta, t) \|_{2,[t_0, t_s]} \) can be bounded from above by a continuous function of \( \theta, \hat{\theta}(t_0), \Gamma \), and \( \| \varepsilon(t) \|_{2,[t_0, \infty]} \). Let us denote this bound by symbol \( D_f \). Notice that \( D_f \) does not depend on \( t_s \). Consider system (31) for \( t \in [t_0, t_s] \):
\[
\psi = f(x, \theta, t) - f(x, \hat{\theta}, t) - \varphi(\psi, \omega, t) + \varepsilon(t)
\]
Given that both \( f(x(t), \theta, t) - f(x(t), \hat{\theta}(t), t), \varepsilon(t) \in L^1_\infty[t_0, t_s] \) and taking into account Assumption 2 we automatically obtain that \( \psi(x(t), t) \in L^1_\infty[t_0, t_s] \). In particular, using the triangle inequality and the fact that function \( \gamma_{\infty,2} (\psi(x_0, t_0), \omega, M) \) in Assumption 2 is non-decreasing in \( M \), we can estimate the norm \( \| \psi(x(t), t) \|_{\infty,[t_0, t_s]} \) as follows:
\[
\| \psi(x(t), t) \|_{\infty,[t_0, t_s]} \leq \gamma_{\infty,2} \left( \psi(x_0, t_0), \omega, D_f + \| \varepsilon(t) \|^2_{2,[t_0, \infty]} \right)
\]
According to Assumption 11 the following inequality holds:
\[
\| x(t) \|_{\infty,[t_0, t_s]} \leq \hat{\gamma} \left( x_0, \theta, \gamma_{\infty,2} \left( \psi(x_0, t_0), \omega, D_f + \| \varepsilon(t) \|^2_{2,[t_0, \infty]} \right) \right)
\]
Given that a superposition of locally bounded functions is locally bounded, we can conclude that \( \| x(t) \|_{\infty,[t_0, t_s]} \) is bounded. This, however, contradicts to the previous claim that \( \| x(t_s) \| = \infty \). Taking into account
inequality (70) we can derive that both \( \hat{\theta}(x(t), t) \) and \( \hat{\theta}_f(t) \) are bounded for every \( t \in \mathbb{R}_+ \). Moreover, according to (73), (74), (70) these bounds are (locally bounded) functions of initial conditions and parameters. Therefore, \( x(t) \in L^\infty_{x}[t_0, \infty], \hat{\theta}(x(t), t) \in L^4_{\infty}[t_0, \infty] \). Inequality (74) follows immediately from (72), (13), and the triangle inequality. Property P2) is proven.

Let us show that P3) holds. It is assumed that system (12), has \( L^2_{2}[t_0, \infty] \rightarrow L^1_{p}[t_0, \infty], p > 1 \) gain. In addition, we have just shown that \( f(x(t), \theta(t), t) \in L^2_{2}[t_0, \infty] \). Hence, taking into account equation (11) we conclude that \( \psi(x(t), t) \in L^1_{p}[t_0, \infty], p > 1 \). On the other hand, given that \( f(x, \hat{\theta}, t), \varphi(\psi, \omega, t) \) are locally bounded with respect to their first two arguments uniformly in \( t \), and that \( x(t) \in L^\infty_{x}[t_0, \infty], |\psi(x(t), t)| \in L^1_{\omega}[t_0, \infty], \hat{\theta}(t) \in L^4_{\infty}[t_0, \infty], \theta \in \Omega_{\theta} \), signal \( \varphi(\psi(x(t), t), \omega(t)) + f(x(t), \theta(t), t) - f(x(t), \hat{\theta}(t), t) \) is bounded. Then \( \varepsilon(t) \in L^\infty_{x}[t_0, \infty] \) implies that \( \dot{\psi} \) is bounded, and P3) is guaranteed by Barbalat’s lemma.

To complete the proof of the theorem (property P4) consider the time-derivative of function \( f(x, \hat{\theta}, t) \):

\[
\frac{d}{dt}f(x, \hat{\theta}, t) = L_{f(x, \theta) + g(x)(x, \theta, t)}f(x, \hat{\theta}, t) + \frac{\partial f(x, \theta, t)}{\partial \theta} \Gamma(\varphi(\psi, \omega, t) + \dot{\psi}) \alpha(x, t) + \frac{\partial f(x, \theta, t)}{\partial t} d \]

Taking into account that \( f(x, \theta), g(x) \), function \( f(x, \theta, t) \) is continuously differentiable in \( x, \theta \); derivative \( d/dt f(x, \theta, t)/\partial t \) is locally bounded with respect to \( x, \theta \) uniformly in \( t \); functions \( \alpha(x, t), \partial \psi(x, t)/\partial t \) are locally bounded with respect to \( x \) uniformly in \( t \), then \( d/dt(f(x, \theta, t) - f(x, \hat{\theta}, t)) \) is bounded. Then given that \( f(x(t), \theta, t) - f(x(t), \hat{\theta}(t), t) \in L^2_{2}[t_0, \infty] \) by applying Barbalat’s lemma we conclude that \( f(x, \theta, \tau) - f(x, \hat{\theta}, \tau) \to 0 \) as \( t \to \infty \). The theorem is proven.

**Proof of Corollary 1** Let \( \varepsilon(t) = 0 \). Then choosing the function \( V_{\hat{\theta}(\theta, \hat{\theta}, t)} \) as in (67), using (68), and invoking Assumption 3 we obtain that

\[
\dot{V}_{\hat{\theta}(\theta, \hat{\theta}, t)} \leq -(f(x, \theta, t) - f(x, \hat{\theta}, t)) \alpha(x, t) \theta - \hat{\theta} \leq - \frac{1}{D^2}(f(x, \theta, t) - f(x, \hat{\theta}, t))^2 \tag{75}
\]

Equality (75) and the fact that \( \varepsilon(t) = 0 \) in (67) imply that the norm \( \|\hat{\theta} - \theta\|_{p,-1} \) is non-increasing. Furthermore, (75) implies that

\[
\|f(x(t), \theta, t) - f(x(t), \hat{\theta}(t), t)\|_{2,[t_0, T]} \leq \left( \frac{D}{2}\|\hat{\theta}(t_0) - \theta\|^2_{p,-1} \right)^{0.5} \tag{76}
\]

This proves property P1). Taking into account (70) and given that Assumptions 1, 2 are satisfied we can conclude that \( x(t) \in L^\infty_{x}[t_0, \infty], |\psi(x(t), t)| \in L^1_{\omega}[t_0, \infty], \hat{\theta}(t) \in L^4_{\infty}[t_0, \infty], \theta \in \Omega_{\theta} \), and the following estimate holds:

\[
\|\psi(x(t), t)\|_{\infty,[t_0, \infty]} \leq \gamma_{\infty, 2} \left( \psi(x_0, t_0), \omega, \left( \frac{D}{2}\|\hat{\theta}(t_0) - \theta\|^2_{p,-1} \right)^{0.5} \right) \tag{77}
\]

Hence P2) is also proven. Properties P3),P4) follow by the same arguments as in the proof of Theorem 1. Therefore, P5) is proven. The corollary is proven.

**Proof of Corollary 2** Let us show that P6) holds. Without the loss of generality assume that solutions of the system exist over the following time interval \( [t_0, T^*] \). According to Theorem 1 (property P1), the
norm \( \| \theta - \hat{\theta}(t) \| \) is bounded from above by a function of initial conditions \( \hat{\theta}(t_0) \), parameters \( \Gamma, D, D_1 \), and \( \| \varepsilon(t) \|_{2,[t_0,\infty]} \). Let us denote this bound by symbol \( B_\theta \). Notice that \( B_\theta \) does not depend on \( T^* \). On the other hand, according to Hypothesis \( \mathbb{H} \) the following estimate holds:

\[
\exists D_\theta > 0 : |f(x, \theta, t) - f(x, \hat{\theta}, t)| \leq D_\theta \| \theta - \hat{\theta} \|
\]

Hence \( f(x(t), \theta, t) - f(x(t), \hat{\theta}(t), t) \in L_{\infty}^1[t_0, T^*] \) and moreover

\[
\| f(x(t), \theta, t) - f(x, \hat{\theta}(t), t) \|_{\infty,[t_0, T^*]} \leq D_\theta B_\theta \tag{78}
\]

Consider the following signal \( \mu(t) = f(x(t), \theta, t) - f(x(t), \hat{\theta}(t), t) + \varepsilon(t) \). Signal \( \mu(t) \in L_{1}^1[t_0, T^*] \cap L_{\infty}^1[t_0, T^*] \), and let \( M_\infty, M_2 \in \mathbb{R}_+ \) be the bounds for its \( L_{\infty}^1[t_0, T^*] \) and \( L_{2}^1[t_0, T^*] \) norms respectively. According to \( (72) \), \( (78) \), these bounds can be estimated as follows: \( M_\infty = D_\theta B_\theta + \| \varepsilon(t) \|_{\infty,[t_0, t_\infty]} \), \( M_2 = \left( \frac{D_\theta}{D_1} + 1 \right) \| \varepsilon(t) \|_{2,[t_0, t_\infty]} \). Therefore, given \( p \geq 2 \) one can derive that

\[
\int_{t_0}^{\infty} \mu^p(\tau) d\tau = \int_{t_0}^{\infty} \mu^{p-2}(\tau) \mu^2(\tau) d\tau \leq M_{\infty}^{p-2} M_2^2
\]

Hence, \( \mu(t) \in L_{p}^1[t_0, T^*] \) and its \( L_{p}^1[t_0, T^*] \)-norm is bounded from above by \( M_{\infty}^{p-2} M_2^2 \), where the bounds \( M_\infty, M_2 \) both do not depend on \( T^* \). According to the corollary formulation, system \( (12) \) has \( L_{p}^1[t_0, \infty] \to L_{\infty}^1[t_0, \infty] \) gain and therefore \( \psi(x(t), t) \in L_{\infty}^1[t_0, T^*] \). Then applying the same argument as in the proof of property \( \mathbb{P}2 \) of Theorem \( \mathbb{I} \) and using Assumption \( \mathbb{I} \) we can immediately obtain that \( x(t) \in L_{\infty}^n[t_0, \infty] \), \( \psi(x(t), t) \in L_{\infty}^1[t_0, \infty] \), and \( \hat{\theta}(t) \in L_{\infty}^d[t_0, \infty] \). Thus property \( \mathbb{P}6 \) is proven. Property \( \mathbb{P}7 \) can now be proven in the same way as property \( \mathbb{P}3 \) in Theorem \( \mathbb{I} \). The corollary is proven.

**Proof of Theorem \( \mathbb{I} \)** According to the theorem formulation, Assumptions \( \mathbb{I}\mathbb{2} \mathbb{3} \mathbb{4} \mathbb{5} \mathbb{6} \) hold. Hence, applying Corollary \( \mathbb{I} \) we can conclude that \( \hat{\theta}(t) \in L_{\infty}^d[t_0, \infty] \) and \( x(t) \in L_{\infty}^n[t_0, \infty] \).

Let us show that limiting relation \( (18) \) holds in case alternative 1) is satisfied. To this purpose consider derivative \( \hat{\theta} \):

\[
\hat{\theta} = \Gamma(p + \psi(\psi))\alpha(x, t) = \Gamma(f(x, \theta, t) - f(x, \hat{\theta}, t))\alpha(x, t) \tag{79}
\]

Given that \( \hat{\theta}(t) \in L_{\infty}^d[t_0, \infty] \) and \( x(t) \in L_{\infty}^n[t_0, \infty] \), and that Hypothesis \( \mathbb{I} \mathbb{4} \) holds, the function \( f(x, \theta, t) \) satisfies the following inequality for some \( D, D_1, \in \mathbb{R}_+ \):

\[
D_1 |\alpha(x, t)^T (\hat{\theta} - \theta)| \leq |f(x, \theta, t) - f(x, \hat{\theta}, t)| \leq D |\alpha(x, t)^T (\hat{\theta} - \theta)|;
\]

\[
\alpha(x, t)^T (\hat{\theta} - \theta)(f(x, \theta, t) - f(x, \hat{\theta}, t)) \geq 0
\]

Therefore, there exists function \( \kappa : \mathbb{R}_+ \to \mathbb{R}_+ \), \( D_1 \leq \kappa^2(t) \leq D \) such that

\[
\hat{\theta} = -\kappa^2(t) \Gamma\alpha(x, t)^T (\hat{\theta} - \theta)\alpha(x, t) = -\kappa^2(t) \Gamma\alpha(x, t)\alpha(x, t)^T (\hat{\theta} - \theta) \tag{80}
\]
Notice that matrix $\Gamma$ is a positive definite and symmetric matrix. It, therefore, can be factorized as follows $\Gamma = \Gamma_0\Gamma_0^T$: where $\Gamma_0$ is nonsingular $n \times n$ real matrix. Let us define $\hat{\theta} = \Gamma_0^{-1}(\theta - \hat{\theta})$. In these new coordinates, equation (85) will have the following form:

$$\dot{\hat{\theta}} = -\kappa(t)^2\Gamma_0^{-1}\Gamma_0 \Gamma_0^T \alpha(x, t)\alpha(x, t)^T (\theta - \hat{\theta}) = -\kappa(t)^2\Gamma_0^T \alpha(x, t)\alpha(x, t)^T \Gamma_0 \hat{\theta}$$

(81)

Denoting $\kappa(t)\Gamma_0^T \alpha(x, t) = \phi(x, t)$ we can rewrite equation (81) as follows:

$$\dot{\hat{\theta}} = -\phi(x, t)\phi(x, t)^T \hat{\theta}$$

(82)

where function $\phi(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^d$ satisfies equality:

$$\eta^T \int_t^{t+L} \phi(x(\tau), \tau) \phi(x(\tau), \tau) d\tau \eta = \eta^T \Gamma_0^T \left( \int_t^{t+L} \kappa^2(\tau) \alpha(x(\tau), \tau) \alpha(x(\tau), \tau)^T d\tau \right) \eta$$

(83)

for all $\eta \in \mathbb{R}^d$. Taking into account that function $\alpha(x(t), t)$ is persistently exciting, $\Gamma = \Gamma_0^T \Gamma_0$, and that $\kappa^2(t) \geq D_1$ we can obtain the following bound for quadratic form (83):

$$\eta^T \int_t^{t+L} \phi(x(\tau), \tau) \phi(x(\tau), \tau) d\tau \eta \geq \delta D_1 \left\| \Gamma \phi \eta \right\|^2 \geq \delta D_1 \lambda_{\min}(\Gamma) \left\| \eta \right\|^2 = \delta_0 \left\| \eta \right\|^2$$

(84)

Hence, function $\phi(x(t), t)$ is also persistently exciting. Notice also that $\left\| \phi(x, t) \right\|$ is bounded from above:

$$\left\| \phi(x, t) \right\| = \left\| \kappa(t)\alpha(x, t)\Gamma_0 \right\| \leq \lambda_{\max}(\Gamma_0) \left\| \kappa(t)\alpha(x, t) \right\| \leq \lambda_{\max}(\Gamma_0)D_\alpha \alpha_0$$

$$\alpha_\infty = \sup_{\|\alpha\| \leq \|\phi(x)\|, t \geq t_0} \left\| \alpha(x, t) \right\|$$

In order to show that $\lim_{t \to \infty} \hat{\theta}(t) = 0$ exponentially fast we invoke the following useful lemma from [36] (Lemma 5, page 18):

**Lemma 1** Let system (82) be given, condition (81) holds (uniformally), and $\phi(x, t)$ in (82) be bounded $\|\phi(x, t)\| \leq \phi_M$. Then system (82) is (uniformally) exponentially stable and, furthermore:

$$\left\| \hat{\theta}(t) \right\| \leq e^{-Kt} \left\| \hat{\theta}(t_0) \right\|, \quad K = \frac{\delta_0}{L(1 + L\phi_M^2)^2}$$

(85)

According to Lemma 1 solutions of system (82) converge to the origin exponentially fast with a rate of convergence defined by (85), where

$$\delta_0 = \lambda_{\min}(\Gamma)D_1 \delta, \quad \phi_M^2 = \lambda_{\max}(\Gamma)D^2 \alpha_\infty^2$$

(86)

Taking into account equation (81), (83) and observing that $(1 + L\phi_M^2)^2 \leq 2(1 + \phi_M^4 L^2)$, $\lambda_{\max}(\Gamma_0)^2 = \lambda_{\max}(\Gamma)$ we can estimate $\|\hat{\theta}(t)\|$ as follows:

$$\left\| \hat{\theta}(t) \right\| \leq e^{-K_1t} \left\| \hat{\theta}(t_0) \right\|, \quad K_1 = \frac{\delta D_1 \lambda_{\min}(\Gamma)}{2L(1 + \lambda_{\max}(\Gamma)L^2 D^2 \alpha_\infty^2)}$$

(87)

Given that $\|\Gamma_0 \hat{\theta}(t)\| = \|\hat{\theta}(t) - \theta\|$, and using (81) we derive the following bounds for $\|\hat{\theta}(t) - \theta\|$:

$$\|\hat{\theta}(t) - \theta\| \leq \|\Gamma_0\| \|\hat{\theta}(t)\| \leq \lambda_{\max}(\Gamma_0)e^{-K_1t} \|\Gamma_0^{-1}\hat{\theta}(t_0) - \theta\| \leq \left(\frac{\lambda_{\max}(\Gamma_0)}{\lambda_{\min}(\Gamma_0)}\right) e^{-K_1t} \|\theta(t_0) - \theta\|$$
This proves alternative 1) of the theorem.

Let us prove alternative 2). It follows immediately from Corollary 1 of Theorem 1 that

\[ \lim_{t \to \infty} f(x(t), \theta, t) - f(x(t), \dot{\theta}(t), t) = 0 \]  

(88)

Furthermore, given that \( \dot{\theta} = \Gamma(f(x(t), \theta, t) - f(x(t), \dot{\theta}(t), t)) \alpha(x, t) \), \( x(t) \in \mathbb{L}_{\infty}[t_0, \infty] \) and \( \alpha(x, t) \) is locally bounded in \( x \) uniformly in \( t \), we can conclude that \( \dot{\theta} \to 0 \) as \( t \to \infty \). Let us divide the \( \mathbb{R}_+ \) into the following union of subintervals:

\[ \mathbb{R}_+ = \bigcup_{i=1}^{\infty} \Delta_i, \quad \Delta_i = [t_i, t_i + T], \quad t_0 = 0, \quad t_{i+1} = t_i + T, \quad i \in \mathbb{N} \]

The fact that \( \dot{\theta} \to 0 \) as \( t \to \infty \) ensures that

\[ \lim_{i \to \infty} \| \theta(s_i) - \dot{\theta}(\tau_i) \| = 0, \quad \forall \ s_i, \tau_i \in \Delta_i \]  

(89)

In order to show this let us integrate equation (90) and subsequently using the mean value theorem we can obtain the following estimate:

\[ \| \dot{\theta}(s_i) - \dot{\theta}(\tau_i) \| = \| \Gamma \int_{s_i}^{\tau_i} (f(x(\tau), \theta, \tau) - f(x(\tau), \dot{\theta}(\tau), \tau)) \alpha(\theta, \tau) d\tau \| \]  

(90)

Applying the Cauchy-Schwartz inequality to (90) and subsequently using the mean value theorem we can obtain the following estimate:

\[ \| \dot{\theta}(s_i) - \dot{\theta}(\tau_i) \| \leq \int_{s_i}^{\tau_i + T} \| \Gamma \| \cdot | f(x(\tau), \theta, \tau) - f(x(\tau), \dot{\theta}(\tau), \tau) | \cdot \| \alpha(\theta, \tau) \| d\tau \]

\[ = \| \Gamma \| \cdot T \cdot | f(x(\tau_i'), \theta, \tau_i') - f(x(\tau_i'), \dot{\theta}(\tau_i'), \tau_i') | \cdot \| \alpha(\theta, \tau_i') \|, \quad \tau_i' \in \Delta_i \]  

(91)

Given that limiting relation holds, \( x(t) \in \mathbb{L}_{\infty}[t_0, \infty] \), and \( \alpha(x, t) \) is locally bounded uniformly in \( t \) we can conclude from (91) that limiting relation holds.

Let us choose a sequence of points from \( \mathbb{R}_+ : \{ \tau_i \}_{i=1}^{\infty} \) such that \( \tau_i \in \Delta_i, \quad i \in \mathbb{N} \). As follows from the nonlinear persistent excitation condition (inequality (44)), for every \( \theta(\tau_i), \tau_i \in \Delta_i \) there exists a point \( t_i' \in \Delta_i \) such that the following inequality holds

\[ \| f(x(t_i'), \theta(t_i'), t_i') - f(x(t_i), \theta(t_i), t_i') \| \geq \varphi(\| \theta - \dot{\theta}(\tau_i) \|) \geq 0 \]  

(92)

Let us consider the following differences:

\[ f(x(t_i'), \theta(\tau_i), t_i') - f(x(t_i), \theta(t_i), t_i'), \quad \tau_i, \ t_i' \in \Delta_i \]

It follows immediately from (11) (2) and (33) that

\[ \lim_{i \to \infty} f(x(t_i'), \theta(\tau_i), t_i') - f(x(t_i'), \dot{\theta}(t_i'), t_i') = 0, \quad \tau_i, \ t_i' \in \Delta_i \]  

(93)

Taking into account (2) and (33) we can derive that

\[ \lim_{i \to \infty} f(x(t_i'), \theta(t_i'), t_i') - f(x(t_i'), \dot{\theta}(t_i'), t_i') = \lim_{i \to \infty} (f(x(t_i'), \theta(t_i'), t_i') - f(x(t_i'), \dot{\theta}(t_i'), t_i')) + \lim_{i \to \infty} f(x(t_i'), \dot{\theta}(t_i'), t_i') - f(x(t_i'), \dot{\theta}(t_i'), t_i') = 0 \]  

(94)
According to (94) and (92), sequence \( \{ g(\| \theta - \hat{\theta}(\tau_i) \|) \} \) is bounded from above and below by two sequences converging to zero. Hence, \( \lim_{i \to \infty} g(\| \theta - \hat{\theta}(\tau_i) \|) = 0 \). Notice that \( g(\cdot) \in K \cap C^0 \) which implies that
\[
\lim_{i \to \infty} \| \theta - \hat{\theta}(\tau_i) \| = 0 \quad (95)
\]
In order to show that \( \lim_{t \to \infty} (\theta - \hat{\theta}(t)) = 0 \) notice that
\[
\| \theta - \hat{\theta}(t) \| \leq \| \theta - \hat{\theta}(s_i) \|, \quad s_i = \arg \max_{s \in \Delta_i} \| \theta - \hat{\theta}(s) \| \quad \forall \ t \in \Delta_i
\]
Hence, applying the triangle inequality \( \| \theta - \hat{\theta}(s_i) \| \leq \| \theta - \hat{\theta}(\tau_i) \| + \| \hat{\theta}(\tau_i) - \hat{\theta}(s_i) \| \) and using equations (89), (95) we can conclude that \( \| \theta - \hat{\theta}(t) \| \) is bounded from above and below by two functions converging to zero. Hence, \( \| \theta - \hat{\theta}(t) \| \to 0 \) as \( t \to \infty \) and limiting relation (18) holds. The theorem is proven.

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