Decay estimates for a class of wave equations on the Heisenberg group

Manli Song\(^1,2\) · Jiale Yang\(^2\)

Received: 18 August 2022 / Accepted: 24 March 2023 / Published online: 20 April 2023
© Fondazione Annali di Matematica Pura ed Applicata and Springer-Verlag GmbH Germany, part of Springer
Nature 2023

Abstract
In this paper, we study a class of dispersive wave equations on the Heisenberg group \(H^n\). Based on the group Fourier transform on \(H^n\), the properties of the Laguerre functions and the stationary phase lemma, we establish the decay estimates for a class of dispersive semigroup on \(H^n\) given by \(e^{it\phi(L)}\), where \(\phi : \mathbb{R}^+ \to \mathbb{R}\) is smooth, and \(L\) is the sub-Laplacian on \(H^n\). Finally, using the duality arguments, we apply the obtained results to derive the Strichartz inequalities for the solutions of some specific equations, such as the fractional Schrödinger equation, the fractional wave equation and the fourth-order Schrödinger equation.

Keywords Decay estimates · Heisenberg group · Sub-Laplacian

Mathematics Subject Classification 22E25 · 33C45 · 35H20 · 35B40

1 Introduction

The aim of this paper is to investigate the decay estimates for a class of dispersive equations on the Heisenberg group \(H^n\)

\[
\begin{aligned}
    i\partial_t u + \phi(L) u &= f, \\
    u|_{t=0} &= u_0,
\end{aligned}
\]  

where \(\phi : \mathbb{R}^+ \to \mathbb{R}\) is smooth, and \(L\) is the sub-Laplacian on the Heisenberg group \(H^n\).

In the past decades, Strichartz estimates have been very useful in the study of nonlinear partial differential equations. These estimates in the Euclidean setting have been proved for many dispersive equations, such as the wave equation and Schrödinger equation (see [9, 12, 19]). To obtain Strichartz estimates, it involves basically two types of ingredients. The first
one consists of estimating the decay in time on the evolution group associated with the free equation (i.e., with \( f = 0 \)). The second one consists of abstract arguments, which are mainly duality arguments. Therefore, the decay estimate plays a crucial role.

In 2008, Guo, Peng and Wang [10] used a unified way to study the decay for a class of dispersive semigroup on the Euclidean space \( \mathbb{R}^n \) given by \( e^{it\phi(\Delta)} \), where \( \Delta = -\sum_{j=1}^{n} \partial^2_{x_j} \) is the Laplacian on \( \mathbb{R}^n \). Many dispersive wave equations reduce to this type, for instance, the Schrödinger equation (\( \phi(\Delta) = \frac{1}{2} \)), the wave equation (\( \phi(r) = r^2 \)), the fractional Schrödinger equation (\( \phi(r) = r^\alpha \) (0 < \( \alpha < 1 \)), the fourth-order Schrödinger equation (\( \phi(r) = r^2 + r \)), etc. When \( \phi \) is a homogeneous function of order \( m \), namely, \( \phi(\lambda r) = \lambda^m \phi(r) \), \( \forall \lambda > 0 \), the dispersive estimate can be easily obtained by a theorem of Littman and dyadic decomposition. However, it becomes very complicated when \( \phi \) is not homogeneous since the scaling constants cannot be effectively separated from the time. To overcome the difficulty, they applied frequency localization by separating \( \phi \) between high and low frequency in different scales. We shall assume that \( \phi : \mathbb{R}^+ \to \mathbb{R} \) is smooth and satisfies:

**H1** There exists \( m_1 > 0 \), such that for any \( \alpha \geq 2 \) and \( \alpha \in \mathbb{N} \),

\[
|\phi'(r)| \sim r^{m_1-1}, \quad |\phi^{(\alpha)}(r)| \lesssim r^{m_1-\alpha}, \quad r \geq 1.
\]

**H2** There exists \( m_2 > 0 \), such that for any \( \alpha \geq 2 \) and \( \alpha \in \mathbb{N} \),

\[
|\phi'(r)| \sim r^{m_2-1}, \quad |\phi^{(\alpha)}(r)| \lesssim r^{m_2-\alpha}, \quad 0 < r < 1.
\]

**H3** There exists \( m_1 > 0 \), such that

\[
|\phi''(r)| \sim r^{\alpha_1-2}, \quad r \geq 1.
\]

**H4** There exists \( m_2 > 0 \), such that

\[
|\phi''(r)| \sim r^{\alpha_2-2}, \quad 0 < r < 1.
\]

(H1) and (H3) represent the homogeneous order of \( \phi \) in high frequency. Similarly, the homogeneous order of \( \phi \) in low frequency is described by (H2) and (H4).

**Remark 1.1** Throughout this paper, \( A \lesssim B \) means that \( A \leq CB \), and \( A \sim B \) stands for \( C_1B \leq A \leq C_2B \), where \( C, C_1, C_2 \) denote positive universal constants.

Choose a Littlewood–Paley dyadic partition of unity \( \{ \Phi_j \}_{j \in \mathbb{Z}} \) on \( \mathbb{R}^n \) and define the Littlewood–Paley projector by \( \Delta_j = \mathcal{F} \Phi_j(\xi) \mathcal{F}^{-1} \), where \( \mathcal{F} \) is the Fourier transform on \( \mathbb{R}^n \). Guo, Peng and Wang [10] obtained the following decay estimates in time for the semigroup \( e^{it\phi(\Delta)} \):

1. For \( j \geq 0 \), \( \phi \) satisfies (H1), then

\[
\| e^{it\phi(\sqrt{-\Delta})} \Delta_j u_0 \|_{L^\infty(\mathbb{R}^n)} \lesssim |t|^{-\theta} 2^{j(n-m_1\theta)} \| u_0 \|_{L^1(\mathbb{R}^n)}, \quad 0 \leq \theta \leq \frac{n-1}{2};
\]

In addition, if \( \phi \) satisfies (H3), then

\[
\| e^{it\phi(\sqrt{-\Delta})} \Delta_j u_0 \|_{L^\infty(\mathbb{R}^n)} \lesssim |t|^{-\frac{n-1+\theta}{2}} 2^{j(n-m_1(n-1+\theta)/2 - \theta(n-m_1)/2)} \| u_0 \|_{L^1(\mathbb{R}^n)}, \quad 0 \leq \theta \leq 1.
\]

2. For \( j < 0 \), \( \phi \) satisfies (H2), then

\[
\| e^{it\phi(\sqrt{-\Delta})} \Delta_j u_0 \|_{L^\infty(\mathbb{R}^n)} \lesssim |t|^{-\theta} 2^{j(n-m_2\theta)} \| u_0 \|_{L^1(\mathbb{R}^n)}, \quad 0 \leq \theta \leq \frac{n-1}{2};
\]

\( \bigcirc \) Springer
In addition, if $\phi$ satisfies (H4), then
\[
\|e^{it\phi(\sqrt{-\Delta})} \Delta_j u_0\|_{L^\infty(\mathbb{R}^n)} \lesssim |t|^{-\frac{n-1+\theta}{2}} 2^j (\frac{n-2m_1}{2} - \frac{\theta(n_2-m_2)}{2}) \|u_0\|_{L^1(\mathbb{R}^n)}, \quad 0 \leq \theta \leq 1.
\]

Besides, by using standard duality arguments, they applied the above estimates to some concrete equations and established the corresponding Strichartz estimates. Their results could not only cover known results but also make some improvements and provide simple proof for some equations.

Many authors are also interested in adapting the well-known Strichartz estimates from the Euclidean setting to a more abstract setting, such as the Heisenberg type groups and metric measure spaces. In 2000, Bahouri, Gérard and Xu [2] initially discussed the Strichartz estimates with the sub-Laplacian on the Heisenberg group $H^n$, by means of Besov spaces defined by a Littlewood–Paley decomposition related to the spectral of the sub-Laplacian. In their work, they showed such estimates existed for the wave equation, while surprisingly failed for the Schrödinger equation. To avoid the particular behavior of the Schrödinger operator, one way is to replace the Heisenberg group by a bigger space, like the Heisenberg type group whose center dimension is bigger than 1, on which it was shown in [11] that the Schrödinger equation is dispersive. For more results on H-type groups and more generality of step 2 stratified Lie groups, see [3, 16, 17]. Another way is to adapt the Fourier transform restriction analysis initiated by Strichartz [19] in the Euclidean space. Using this approach, Bahouri, Barilari and Gallagher [1] obtain an anisotropic Strichartz estimate for the Schrödinger operator on $H^n$. For more results on the Heisenberg group, see [7, 8]. The case in the metric measure spaces is also considered, such as [4, 5, 13–15].

Our purpose is to establish the decay estimates for a class of dispersive equations (1.1) on the Heisenberg group. The proof originates from the combination of Bahouri, Gérard and Xu [2] and Guo, Peng and Wang [10]. When proving the dispersive estimates, the main difficulty is that the Fourier transform on the Heisenberg group is more complicated than in the Euclidean case. Our main results are in the following.

**Theorem 1.1** Assume $\phi : \mathbb{R}^+ \to \mathbb{R}$ is smooth. $U_t = e^{it\phi(L^2)}$, $\varphi_j$ is Littlewood–Paley function introduced in Sect. 2, and $N = 2n + 2$ is the homogeneous dimension of $H^n$. Then, we have the following results.

1. For $j \geq 0$, $\phi$ satisfies (H1), then
\[
\|U_t \varphi_j\|_{L^\infty(H^n)} \leq C_n 2^{Nj}.
\]
In addition, if $\phi$ satisfies (H3), then
\[
\|U_t \varphi_j\|_{L^\infty(H^n)} \leq C_n |t|^{-\frac{\theta}{2}} 2^j (N-\alpha_1 \theta), \quad 0 \leq \theta \leq 1.
\]

2. For $j < 0$, $\phi$ satisfies (H2), then
\[
\|U_t \varphi_j\|_{L^\infty(H^n)} \leq C_n 2^{Nj}.
\]
In addition, if $\phi$ satisfies (H4), then
\[
\|U_t \varphi_j\|_{L^\infty(H^n)} \leq C_n |t|^{-\frac{\theta}{2}} 2^j (N-\alpha_2 \theta), \quad 0 \leq \theta \leq 1.
\]

This paper is organized as follows: In Sect. 2, we introduce some basic notations and definitions on the Heisenberg group. We also recall the properties for spherical Fourier transform and the homogeneous Besov spaces on the Heisenberg group. Section 3 is dedicated
to describe the useful technical lemmas for proving the decay estimates and Strichartz estimates. In Sect. 4, we prove our main results Theorem 1.1. In Sect. 5, we exploit Theorem 1.1 to obtain the Strichartz estimates for some concrete equations, such as the fractional Schrödinger equation, the fractional wave equation and the fourth-order Schrödinger equation. It is worth noticing that the list of applications is not exhaustive since we just intend to show the generality our approach.

2 Preliminaries

2.1 Heisenberg groups

In this section, we recall some fundamental conceptions and notations on the Heisenberg group. These facts are familiar and can be easily found in many references.

The Heisenberg group $H^n$ is a non-commutative nilpotent Lie group with the underlying manifold $\mathbb{C}^n \times \mathbb{R}$. The element of $H^n$ is denoted by $(z, s)$ where $z = x + iy \in \mathbb{C}^n$ and $s \in \mathbb{R}$. The group law is given by

$$(z, s) \cdot (z', s') = (z + z', s + s' + 2Im(z \cdot \overline{z'})),$$

where $z \cdot \overline{z'} = z_1\overline{z'_1} + \cdots + z_n\overline{z'_n}$ is the Hermitian product on $\mathbb{C}^n$. The Haar measure on $H^n$ coincides with the Lebesgue measure $dz\,dt$ on $\mathbb{C}^n \times \mathbb{R}$. It is easily to check that the inverse of the element $(z, s)$ is $(z, s)^{-1} = (-z, -s)$ and the unitary element is $(0, 0)$.

The non-isotropic dilations on $H^n$ are defined by

$$\delta_a(z, s) = (az, a^2 s), \quad (a > 0).$$

The Haar measure on $H^n$ satisfies $d\delta_a(z, t) = a^N dz\,dt$, where $N = 2n + 2$ is the homogeneous dimension of $H^n$.

The Schwartz class of rapidly decreasing smooth function on $H^n$ is denoted by $\mathcal{S}(H^n)$, and its dual, the space of tempered distribution on $H^n$, by $\mathcal{S}'(H^n)$. For $\Omega \subseteq H^n$, the space $C_c^\infty(\Omega)$ consists of smooth functions compactly supported on $\Omega$. We say a function $f$ is radial on $H^n$ if the value of $f(z, s)$ depends only on $|z|$ and $s$. We denote by $\mathcal{S}_{rad}(H^n)$ and $L^p_{rad}(H^n)$, $1 \leq p \leq \infty$, the space of radial functions in $\mathcal{S}(H^n)$ and $L^p(H^n)$, respectively. In particular, the set of $L^1_{rad}(H^n)$ endowed with the convolution product

$$f \ast g(w) = \int_{H^n} f(wv^{-1})g(v)dv, \quad w \in H^n$$

is a commutative algebra.

The left invariant vector fields on the Heisenberg group $H^n$ are defined by

$$X_j = \partial_{x_j} + 2y_j \partial_s, \quad Y_j = \partial_{y_j} - 2x_j \partial_s,$$

where $j = 1, 2, \cdots, n$. In terms of these vector fields, we introduce the sub-Laplacian $\mathcal{L}$ by

$$\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2)$$

$$= -\Delta_z - 4|z|^2 \partial_s^2 + 4\partial_s \sum_{j=1}^n (x_j \partial_{y_j} - y_j \partial_{x_j}).$$
2.2 Spherical Fourier transform

In this section, we discuss the spherical Fourier transform on the Heisenberg group \( H^n \).

The Laguerre polynomials of type \( d \in \mathbb{N} \) and degree \( m \in \mathbb{N} \) are defined by

\[
L_m^{(d)}(t) = \sum_{k=0}^{m} (-1)^k \left( \frac{m + d}{m - k} \right) \frac{t^k}{k!}, \quad t \geq 0.
\]

**Definition 2.1** Let \( f \in L^1_{\text{rad}}(H^n) \). For any \( m \in \mathbb{N} \) and \( \lambda \in \mathbb{R} \), we define the spherical Fourier transform of \( f \) by

\[
\hat{f}(m, \lambda) = \left( \frac{m + n - 1}{m} \right)^{-1} \int_{H^n} e^{i\lambda \cdot s} f(z, s) L_m^{(n-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} \, dz \, ds.
\]

By a direct computation, we have \( \hat{f} \ast g = \hat{f} \cdot \hat{g} \) for any \( f, g \in L^1_{\text{rad}}(H^n) \).

The following inversion Fourier formula also holds.

**Proposition 2.1** For all \( f \in \mathcal{S}_{\text{rad}}(H^n) \) such that

\[
\sum_{m \in \mathbb{N}} \left( \frac{m + n - 1}{m} \right) \int_{\mathbb{R}} |\hat{f} (\lambda, m)| |\lambda|^d \, d\lambda < \infty,
\]

we have

\[
f(z, s) = \frac{2^{n-1}}{\pi^{n+1}} \sum_{m \in \mathbb{N}} \int_{\mathbb{R}} e^{-i\lambda \cdot s} \hat{f}(m, \lambda) L_m^{(n-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} |\lambda|^n \, d\lambda,
\]

where the sum is convergent in \( L^\infty(H^n) \).

Moreover, if \( f \in \mathcal{S}_{\text{rad}}(H^n) \), the functions \( \mathcal{L} f \) are also in \( \mathcal{S}_{\text{rad}}(H^n) \) and its spherical Fourier transform is given by

\[
\hat{\mathcal{L} f}(m, \lambda) = b_m(\lambda) \hat{f}(m, \lambda),
\]

where \( b_m(\lambda) = 4(2m+n)|\lambda| \). The sub-Laplacian \( \mathcal{L} \) is a positive self-adjoint operator densely defined on \( L^2(H^n) \). So by the spectral theorem, for any bounded Borel function \( h \) on \( \mathbb{R}^+ \), we have

\[
h(\mathcal{L} f)(m, \lambda) = h(b_m(\lambda)) \hat{f}(m, \lambda).
\]

2.3 Homogeneous Besov spaces

In this section, we shall recall the homogeneous Besov spaces related to the sub-Laplacian \( \mathcal{L} \) on the Heisenberg group \( H^n \) given in [2].

Let \( R^* \in C_c^\infty(C_0) \) with \( C_0 = \{ \tau \in \mathbb{R} : \frac{1}{2} \leq |\tau| \leq 4 \} \) be an even and positive function such that

\[
\sum_{j \in \mathbb{Z}} R^*(2^{-2j} \tau) = 1, \quad \forall \tau \neq 0.
\]

Since the restriction of \( R^* \) on \( \mathbb{R}^+ \) is in \( \mathcal{S}(\mathbb{R}) \), by [6], we have the kernel \( \varphi \) of the operator \( R^*(\mathcal{L}) \in \mathcal{S}_{\text{rad}}(H^n) \). In particular,

\[
\hat{\varphi}(m, \lambda) = R_m^*(\lambda) := R^*((2m+n)\lambda).
\]
By the inversion Fourier transform (2.1), we also have
\[
\varphi(z, s) = \frac{2^{n-1}}{\pi^{n+1}} \sum_{m=0}^{\infty} \int_{\mathbb{R}} e^{-i\lambda s} R_m^*(\lambda) L_n^2(\lambda)(2|\lambda||z|^2)e^{-|\lambda||z|^2} |\lambda|^n d\lambda.
\]

For \( j \in \mathbb{Z} \), we denote by \( \varphi_j \) the kernel of the operator \( R^*(2^{-j} \mathcal{L}) \). It can be easily calculated that \( \varphi_j(z, s) = 2^{Nj} \varphi(\delta_{2^j}(z, s)) \), \( \forall (z, t) \in H^n \). The Littlewood–Paley projector on the Heisenberg group is define by
\[
\Delta_j f = f \ast \varphi_j, \quad \forall f \in \mathcal{S}'(H^n).
\]
Obviously \( \widehat{\varphi}_j(m, \lambda) = R_m^*(2^{-2j}|\lambda|) := R_m^j(\lambda) \). Let \( \tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1} \). We have \( f \ast \varphi_j = f \ast \varphi_j \ast \tilde{\varphi}_j \) holds true for all \( f \in \mathcal{S}'(H^n) \).

By the spectral theorem, for any \( f \in L^2(H^n) \), the following homogeneous Littlewood–Paley decomposition holds:
\[
f = \sum_{j \in \mathbb{Z}} \Delta_j f \quad \text{in} \quad L^2(H^n).
\]
So
\[
\|f\|_{L^\infty(H^n)} \leq \sum_{j \in \mathbb{Z}} \|\Delta_j f\|_{L^\infty(H^n)}, \quad f \in L^2(H^n), \tag{2.2}
\]
where both sides of (2.2) are allowed to be infinite.

In addition, it has been proved in [6] that for any \( \sigma \in \mathbb{R}, j \in \mathbb{Z}, 1 \leq p \leq \infty \) and \( f \in \mathcal{S}'(H^n) \), then
\[
\left\| \mathcal{L}^{\frac{\sigma}{2}} \Delta_j f \right\|_{L^p(H^n)} \leq C2^{j\sigma} \|\Delta_j f\|_{L^p(H^n)}. \tag{2.3}
\]

**Definition 2.2** Let \( 1 \leq p, r \leq \infty, \rho < N/p \). The homogeneous Besov space \( \dot{B}^\rho_{p,r}(H^n) \) is defined as the set of distributions \( f \in \mathcal{S}'(H^n) \) such that
\[
\|f\|_{\dot{B}^\rho_{p,r}(H^n)} = \left( \sum_{j \in \mathbb{Z}} 2^{j\rho r} \|\Delta_j f\|_p \right)^{\frac{1}{r}} < \infty,
\]
and \( f = \sum_{j \in \mathbb{Z}} \Delta_j f \) in \( \mathcal{S}'(H^n) \).

**Definition 2.3** Let \( \rho < N/2 \). The homogeneous Sobolev space \( \dot{H}^\rho(H^n) \) is
\[
\dot{H}^\rho(H^n) = \dot{B}^0_{2,2}(H^n),
\]
which is equivalent to
\[
u \in \dot{H}^\rho(H^n) \Leftrightarrow \Delta^{\rho/2} \nu \in L^2(H^n).
\]
And the associated norms are of course equivalent.

According to [2], we list some properties of the spaces \( \dot{B}^\rho_{p,r}(H^n) \) in the following proposition.

**Proposition 2.2** Let \( p, r \in [1, \infty] \) and \( \rho < N/p \).

(i) The space \( \dot{B}^\rho_{p,r}(H^n) \) is a Banach space with the norm \( \| \cdot \|_{\dot{B}^\rho_{p,r}(H^n)} \);

(ii) the definition of \( \dot{B}^\rho_{p,r}(H^n) \) does not depend on the choice of the function \( R^* \) in the Littlewood–Paley decomposition;

(iii) for \( -\frac{N}{p} < \rho < \frac{N}{p} \), the dual space of \( \dot{B}^\rho_{p,r}(H^n) \) is \( \dot{B}^{-\rho}_{p',r'} \).
(iv) for any $u \in \mathcal{L}^\sigma(H^n)$ and $\sigma > 0$, then $u \in \dot{B}^{\rho}_{p,r}(H^n)$ if and only if $L^{\sigma/2}u \in \dot{B}^{\rho-\sigma}_{p,r}(H^n)$;
(v) for any $p_1, p_2 \in [1, \infty]$, the continuous inclusion holds
\[
\dot{B}^{\rho_1}_{p_1,r}(H^n) \subseteq \dot{B}^{\rho_2}_{p_2,r}(H^n), \quad \frac{1}{p_1} - \frac{\rho_1}{N} = \frac{1}{p_2} - \frac{\rho_2}{N}, \quad \rho_1 \geq \rho_2;
\]
(vi) for all $p \in [2, \infty]$, we have the continuous inclusion $\dot{B}^{0}_{p,2}(H^n) \subseteq L^p(H^n)$;
(vii) $\dot{B}^{0}_{2,2}(H^n) = L^2(H^n)$.

3 Technical lemmas

By the inversion Fourier formula (2.1), we may write $U_t \varphi_j$ explicitly into a sum of a list of oscillatory integrals. In order to estimate the oscillatory integrals, we recall the stationary phase lemma.

Lemma 3.1 (see [18]) Let $g \in C^\infty([a, b])$ be real-valued such that $|g''(x)| \geq \delta$ for any $x \in [a, b]$ with $\delta > 0$. Then, for any function $\psi \in C^\infty([a, b])$, there exists a constant $C$ which does not depend on $\delta, a, b, g$ or $\psi$, such that
\[
\left| \int_a^b e^{itg(x)}\psi(x) \, dx \right| \leq C t^{-1/2} (\|\psi\|_\infty + \|\psi'\|_1).
\]

In order to prove the sharpness of the time decay dispersion of the solutions of some concrete equations in Sect. 5, we describe the asymptotic expansion of oscillating integrals.

Lemma 3.2 (see [18]) Suppose $k \geq 2$, and $\phi(x_0) = \phi'(x_0) = \cdots = \phi^{k-1}(x_0)$, while $\phi^k(x_0) \neq 0$. If $\psi$ is supported in a sufficiently neighborhood of $x_0$, then
\[
I(t) = \left| \int_{\mathbb{R}} e^{it\phi(x)}\psi(x) \, dx \right| \sim |t|^{-1/k} \sum_{j=0}^{\infty} a_j |t|^{-j/k}, \quad \text{as } t \to \infty.
\]

Besides, it will involve the Laguerre functions when we handle the oscillatory integrals. We need the following estimates.

Lemma 3.3 (see [11]) Consider the following:
\[
\left| (\tau \frac{d}{d\tau})^\alpha (L_m^{(d-1)}(\tau)e^{-\tau/2}) \right| \leq C_{\alpha,d}(2m + d)^{d-1/4}
\]
for all $0 \leq \alpha \leq d$.

Remark 3.1 In fact, from the proof of Lemma 3.2 in [11], we have a better estimate
\[
\left| (\tau \frac{d}{d\tau})^\alpha (L_m^{(d-1)}(\tau)e^{-\tau/2}) \right| \leq C_{\alpha,d}(2m + d)^{d-1},
\]
for $0 \leq \alpha \leq d - 1$.

Finally, we apply the following duality arguments to prove the Strichartz estimates for some concrete equations on the Heisenberg group.
Lemma 3.4 (see [9]) Let $H$ be a Hilbert space, $X$ a Banach space, $X^*$ the dual of $X$, and $D$ a vector space densely contained in $X$. Let $A \in \mathcal{L}_a(D, H)$ and let $A^* \in \mathcal{L}_a(H, D^*_a)$ be its adjoint, defined by

$$(A^* v, f)_D = \langle v, Af \rangle_H, \quad \forall f \in D, \quad \forall v \in H,$$

where $\mathcal{L}_a(Y, Z)$ is the space of linear maps from a vector space $Y$ to a vector space $Z$. $D^*_a$ is the algebraic dual of $D$. $(\varphi, f)_D$ is the pairing between $D^*_a$ and $D$, and $(,)_H$ is the scalar product in $H$. Then, the following three conditions are equivalent:

1. There exists $a, 0 \leq a \leq \infty$ such that for all $f \in D$,

$$\|Af\| \leq a\|f\|_X.$$

2. $\mathcal{R}(A^*) \subset X^*$, and there exists $a, 0 \leq a \leq \infty$, such that for all $v \in H$,

$$\|A^* v\|_{X^*} \leq a\|v\|.$$

3. $\mathcal{R}(A^* A) \subset X^*$, and there exists $a, 0 \leq a \leq \infty$, such that for all $f \in D$,

$$\|A^* Af\|_{X^*} \leq a^2\|f\|_X,$$

where $\|\cdot\|$ denote the norm in $H$. The constant $a$ is the same in all three parts. If one of those conditions is satisfied, the operators $A$ and $A^* A$ extend by continuity to bounded operators from $X$ to $H$ and from $X$ to $X^*$, respectively.

Lemma 3.5 (see [9]) Let $H, D$ and two triplets $(X_i, A_i, a_i), i = 1, 2$, satisfy any of the conditions of Lemma 3.4. Then, for all choices of $i, j = 1, 2, \mathcal{R}(A_i^* A_j) \subset X_i^*$, and for all $f \in D$,

$$\|A_i^* A_j f\|_{X_i^*} \leq a_i a_j \|f\|_{X_j}.$$

Lemma 3.6 (see [9]) Let $H$ be a Hilbert space, let $I$ be an interval of $\mathbb{R}$, let $X \subset S'(I \times \mathbb{R})$ be a Banach space, let $X$ be stable under time restriction, let $X$ and $A$ satisfy (any of) the conditions of Lemma 3.4. Then, the operator $A^* A$ is a bounded operator from $L^1_t(I, H)$ to $X^*$ and from $X$ to $L^\infty_t(I, H)$.

4 Decay estimates

In this section, we will prove Theorem 1.1.

**Proof of Theorem 1.1:** First, we prove (1). According to $\psi_j$ introduced in section 2, we have

$$U_t \psi_j(z, s) = \frac{2^{n-1}}{\pi^{n+1}} \sum_{m=0}^{\infty} \int_{-\infty}^{+\infty} e^{-i\lambda s} e^{it\phi(4(2m+n)|\lambda|)} \times R_m^j(\lambda) L_m^{(n-1)} (2|\lambda||z|^2) e^{-|\lambda||z|^2|\lambda|^n} d\lambda.$$

By the transform of variable $\lambda' = 2^{-j} \lambda$, we get

$$U_t \psi_j(z, s) = \frac{2^{n-1}}{\pi^{n+1}} \cdot 2^{N_j} \sum_{m=0}^{\infty} \int_{-\infty}^{+\infty} e^{-i2^j \lambda' s} e^{it\phi(2m+n)2^j|\lambda'|} \times R_m^j(\lambda') L_m^{(n-1)} (2^{2j+1}|\lambda'||z|^2) e^{-2^j|\lambda'||z|^2|\lambda'|^n} \lambda'^n d\lambda'$$

$$= \frac{2^{n-1}}{\pi^{n+1}} \cdot 2^{N_j} \sum_{m=0}^{\infty} I_m(t, z, s).$$

© Springer
Denote
\[ I_m(t, z, s) = \int_{-\infty}^{+\infty} e^{-it2^j\lambda's} e^{i\phi((2m+n)2^j+2|\lambda'|)} \times R^*_m(\lambda') L_m^{(n-1)}(2^j+1|\lambda'||z|^2) e^{-2^j|\lambda'||z|^n} d\lambda'. \]

Let \( s = ts' \), and \( x = (2m+n)\lambda' \). We have
\[
I_m(t, z, s) = \int_{C_0} e^{i\phi(2^j+2|x|) - \frac{2^j|x'|^2}{2m+n}} R^*_m(x) \times L_m^{(n-1)}(2^j+1|x||z|^2) e^{-\frac{2^j|x||z|^2}{2m+n}} \frac{|x|^n}{(2m+n)^n+1} dx.
\]

\[
|I_m(t, z, s)| \leq (2m+n)^{-n-1} \int_{C_0} |L_m^{(n-1)}(2^j+1|x||z|^2) e^{-\frac{2^j|x||z|^2}{2m+n}} R^*_m(x)|x|^n dx
\]
\[
\leq C_n (2m+n)^{-n-1} \cdot (2m+n)^{n-1} \int_{C_0} R^*_m(x)|x|^n dx
\]
\[
\leq C_n (2m+n)^{-2},
\]
where the second inequality follows from Remark 3.1 for the case \( \alpha = 0 \).

So
\[
|U_t \varphi_j(z, s)| \leq C_n 2^{Nj} \sum_{m=0}^{\infty} (2m+n)^{-2} \leq C_n 2^{Nj}.
\]

Finally, we get
\[
\|U_t \varphi_j\|_{L^{\infty}(H^n)} \leq C_n 2^{Nj},
\]
which is the result of (1.2), as desired.

Now, we assume (H3) also holds. Let \( g(x) = \phi(2^j+2|x|) - \frac{2^j|x|s'}{2m+n} \). By a direct computation, we obtain
\[
g''(x) = 2^j+4 \phi''(2^j+2|x|).
\]
then \(|g''(x)| \gtrsim 2^j \alpha_1| \) on the support of \( R^* \). According to Lemma 3.1, we have
\[
|I_m(t, z, s)| \leq |t \cdot 2^{j\alpha_1}|^{-\frac{1}{2}} \left( \left\| R^*_m(x) L_m^{(n-1)}(2^j+1|x||z|^2) e^{-\frac{2^j|x||z|^2}{2m+n}} \frac{|x|^n}{(2m+n)^n+1} \right\|_{L^\infty(C_0)} \right)
\]
\[
+ \int_{C_0} \left| \frac{d}{dx} \left( R^*_m(x) L_m^{(n-1)}(2^j+1|x||z|^2) e^{-\frac{2^j|x||z|^2}{2m+n}} \frac{|x|^n}{(2m+n)^n+1} \right) \right| dx.
\]
Since \( R^* \in C^\infty_c(C_0) \), by Lemma 3.3, it yields that
\[
\left\| R^*_m(x) L_m^{(n-1)}(2^j+1|x||z|^2) e^{-\frac{2^j|x||z|^2}{2m+n}} \frac{|x|^n}{(2m+n)^n+1} \right\|_{L^\infty(C_0)} \leq C_n (2m+n)^{-2},
\]
\[
\left| \left( \frac{d}{dx} R^*_m(x) L_m^{(n-1)}(2^j+1|x||z|^2) e^{-\frac{2^j|x||z|^2}{2m+n}} \frac{|x|^n}{(2m+n)^n+1} \right) \right| \leq C_n (2m+n)^{-2},
\]
\[
\left| R^*_m(x) L_m^{(n-1)}(2^j+1|x||z|^2) e^{-\frac{2^j|x||z|^2}{2m+n}} \frac{1}{(2m+n)^n+1} \frac{d}{dx}(|x|^n) \right| \leq C_n (2m+n)^{-2}.
\]
\[
R^* (x) \frac{d}{dx} \left( L_{(m-1)} \left( \frac{2^{2j+1} |x||z|^2}{2m+n} \right) e^{-\frac{2^{2j} |x| |z|^2}{2m+n}} \right) \left| \frac{|x|^n}{(2m+n)^{n+1}} \right|
\]
\[
= R^* (x) \cdot \frac{d}{dy} \left( L_{(m-1)} \left( 2ye^{-y} \right) \right) \left| y=\frac{2^{2j} |x| |z|^2}{2m+n} \right| \cdot \left| \frac{|x|^n}{(2m+n)^{n+1}} \right|
\]
\[
\leq C_n (2m+n)^{-2}.
\]

So we get

\[
|I_m (t, z, s)| \leq C_n |t|^{-\frac{1}{2}} \cdot 2^{-j\alpha_1} (2m+n)^{-2},
\]

and

\[
|U_t \varphi_j (z, s)| \leq C_n |t|^{-\frac{1}{2}} \cdot 2^{(N-\alpha_1)j} \sum_{m=0}^{\infty} (2m+n)^{-2} \leq C_n |t|^{-\frac{1}{2}} \cdot 2^{(N-\alpha_1)j}.
\]

Then,

\[
\|U_t \varphi_j \|_{L^\infty (H^n)} \leq C_n |t|^{-\frac{1}{2}} \cdot 2^{(N-\alpha_1)j}.
\]

(4.2)

By an interpolation between (4.1) and (4.2), we obtain

\[
\|U_t \varphi_j \|_{\infty} \leq C_n |t|^{-\frac{\theta}{2}} \cdot 2^{(N-\alpha_1)j}, \quad \text{for } 0 \leq \theta \leq 1,
\]

which completes the proof of (1).

The proof of (2) is similar to (1), and we omit the details.

\[
\text{\square}
\]

5 Applications

In this section, we prove the Strichartz estimates for some concrete equations on the Heisenberg group by using Theorem 1.1.

5.1 The fractional Schrödinger equation

First, we consider the fractional Schrödinger equation \(0 < \alpha < 1\)

\[
\begin{align*}
\begin{cases}
i \partial_t u + \mathcal{L}^\alpha u = f, \\
u \big|_{t=0} = u_0.
\end{cases}
\end{align*}
\]

(5.1)

The solution is given by

\[
u (t) = S_t u_0 - i \int_0^t S_{t-\tau} f (\tau) \, d\tau,
\]

where \(S_t = e^{it\mathcal{L}^\alpha}\) and it corresponds to the case when \(\phi (r) = r^\alpha\). By a simple calculation,

\[
\phi' (\lambda) = \alpha r^{\alpha-1}, \quad \phi'' (\lambda) = \alpha (\alpha - 1) r^{\alpha-2}.
\]

We see that \(\phi\) satisfies (H1)-(H4) with \(m_1 = \alpha_1 = m_2 = \alpha_2 = \alpha\). So according to Theorem 1.1, we know

\[
\|S_t \varphi_j \|_{L^\infty (H^n)} \leq C_n |t|^{-\frac{\theta}{2}} 2^{j(N-\alpha\theta)}, \quad \forall j \in \mathbb{Z}.
\]

We notice that when \(\theta = 1\) the fractional Schrödinger semigroup has the best decay in time

\[
\|S_t \varphi_j \|_{L^\infty (H^n)} \leq C_n |t|^{-\frac{1}{2}} 2^{j(N-\alpha)}, \quad \forall j \in \mathbb{Z}.
\]

(5.2)
Theorem 5.1 If $u$ is the solution of the free Schrödinger equation (5.1) (i.e., with $f = 0$). For $0 < \alpha < 1$, there exists a constant $C > 0$ which does not depend on $u_0$ or $t$ such that
\[
\|u(t)\|_{L^\infty(H^n)} \leq C|t|^{-1/2}\|u_0\|_{\dot{B}^{{N-\alpha}}_{1,1}(H^n)},
\]
and the time decay is sharp in time.

Proof According to the property of the Fourier transform, for any $f \in S(H^n)$, we have
\[
\Delta_j(S_t f) = S_t(\Delta_j f) = S_t(f * \varphi_j) = f * \varphi_j * (S_t \tilde{\varphi}_j).
\]
By the Young inequality and (5.2), we get
\[
\|\Delta_j(S_t f)\|_{L^\infty(H^n)} \leq \|f * \varphi_j\|_{L^1(H^n)} \cdot \|S_t \tilde{\varphi}_j\|_{L^\infty(H^n)} \leq C_n|t|^{-\frac{1}{2}}2^{j(N-\alpha)}\|\Delta_j f\|_{L^1(H^n)}.
\]
Finally, we have the following results:
\[
\|u(t)\|_{L^\infty(H^n)} = \|S_t u_0\|_{L^\infty(H^n)} = \left\| \sum_{j \in \mathbb{Z}} \Delta_j(S_t u_0) \right\|_{L^\infty(H^n)} \leq \sum_{j \in \mathbb{Z}} \|\Delta_j(S_t u_0)\|_{L^\infty(H^n)} \leq \sum_{j \in \mathbb{Z}} C|t|^{-\frac{1}{2}}2^{j(N-\alpha)}\|\Delta_j u_0\|_{L^1(H^n)} = C|t|^{-\frac{1}{2}}\|u_0\|_{\dot{B}^{{N-\alpha}}_{1,1}(H^n)}.
\]
Now, let us give an example to explain the sharpness of the decay in time $t$. Let $Q \in C_0^\infty(D_0)$ with $Q(1) = 1$, where $D_0$ is a small neighborhood of 1 such that $0 \notin D_0$. Then,
\[
\tilde{u}_0(m, \lambda) = Q(|\lambda|)\delta_{m0},
\]
\[
u_0(z, s) = \int_{D_0} e^{-i\lambda s}Q(\lambda)\lambda^n d\lambda.
\]
Consider the free fractional Schrödinger equation (5.1). We have
\[
u(z, s, t) = e^{it\mathcal{L}_\alpha}u_0 = C \int_{D_0} e^{-i\lambda s} e^{it(4\lambda n)^\alpha}Q(\lambda)e^{-\lambda|z|^2\lambda^n} d\lambda.
\]
In particular,
\[
u(0, ts, t) = C \int_{D_0} e^{it(-\lambda s + 4\alpha \lambda^n n^\alpha)}Q(\lambda)\lambda^n d\lambda.
\]
This oscillating integral has a phase $\Psi(\lambda) = -\lambda s + 4\alpha \lambda^n n^\alpha$ with a unique critical point $\lambda_0 = \left(\frac{s}{\alpha 4^n n^\alpha}\right)^{\frac{1}{4+\alpha}}$, which is not degenerate. By Lemma 3.2, for $s_0 = \alpha 4^n n^\alpha$, we get
\[
u(0, ts_0, t) \sim C|t|^{-1/2}.
\]
\[\square\]
Proposition 5.1 Let \(0 < \alpha < 1\). For \(i = 1, 2\), let \(p_i, r_i \in [2, \infty)\) and \(\rho_i \in \mathbb{R}\) such that

(i) \(\frac{2}{p_i} + \frac{1}{r_i} \leq \frac{1}{2}\),
(ii) \(N \frac{2\alpha}{r_i} + \frac{p_i - \rho_i}{p_i} = \frac{N}{2}\).

Let \(\bar{p}_i, \bar{r}_i\) denote the conjugate exponent of \(p_i, r_i\) for \(i = 1, 2\). Then, the following estimates are satisfied:

\[
\| S_t u_0 \|_{L^{p_1}(\mathbb{R}, \dot{B}^{\rho_1}_{r_1, 2}(H^n))} \leq C \| u_0 \|_{L^2(H^n)}, \tag{5.4}
\]

\[
\left\| \int_0^t S_{t-\tau} f(\tau) \, d\tau \right\|_{L^{p_1}(0, T), \dot{B}^{\rho_1}_{r_1, 2}(H^n)} \leq C \| f \|_{L^{p_2}(0, T), \dot{B}^{\rho_2}_{r_2, 2}(H^n)}, \tag{5.5}
\]

where the constant \(C > 0\) does not depend on \(u_0, f\) or \(T\).

**Proof** Firstly, according to Plancherel theorem, it is easy to obtain

\[
\| \Delta_j(S_t f) \|_{L^2(H^n)} = \| S_t(\Delta_j f) \|_{L^2(H^n)} = \| \Delta_j f \|_{L^2(H^n)}. \]

By an interpolation between the above equation and (5.3), for any \(2 \leq r \leq \infty\), we have

\[
\| \Delta_j(S_t f) \|_{L^r(H^n)} \leq C \|\| t \|^{-\gamma(r)(1/2-1/r)} 2^{j(N-\alpha)(1/2-1/r)} \Delta_j f \|_{L^r(H^n)}. \]

Let \(\gamma(r) = 1/2 - 1/r, \beta(r) = (N-\alpha)(1/2 - 1/r)\). We have

\[
\| \Delta_j(S_t f) \|_{L^r(H^n)} \leq C \| \| t \|^{-\beta(r)} 2^{j\beta(r)} \Delta_j f \|_{L^2(H^n)}. \]

Then,

\[
\| S_t f \|_{\dot{B}^{\rho_1}_{r_2, 2}} = \left( \sum_{j=0} \| 2^{-2j\beta(r)} \left( \| \Delta_j(S_t f) \|_{L^r(H^n)} \right)^2 \right)^{1/2} \leq C \left( \sum_{j=0} 2^{2j\beta(r)} |t|^{-2\gamma(r)} \| \Delta_j f \|_{L^2(H^n)}^2 \right)^{1/2} \leq C |t|^{-\gamma(r)} \| f \|_{\dot{B}^{\rho_1}_{r_2, 2}}. \tag{5.6}
\]

To prove (5.4), it is sufficient to prove it for any \(g \in L^2(H^n), \psi \in L^{\tilde{p}_1}(\mathbb{R}, \dot{B}^{-\tilde{\rho}_1}_{\tilde{r}_1, 2}(H^n))\), the following holds:

\[
\left| \langle S_t g, \psi \rangle_{L^2(\mathbb{R}, L^2(H^n))} \right| \leq C \| g \|_{L^2(H^n)} \| \psi \|_{L^{\tilde{p}_1}(\mathbb{R}, \dot{B}^{-\tilde{\rho}_1}_{\tilde{r}_1, 2})}.
\]

By density, we only need to prove for any \(g \in L^2(H^n), (5.1)\) holds for any \(\psi \in \mathscr{S}(\mathbb{R}, L^2(H^n))\).

First, we have

\[
\int_{\mathbb{R} \times H^n} S_t g(z, s) \psi(t, z, s) \, dz \, ds = \langle g, \int_{\mathbb{R}} S_t \psi(t) \, dt \rangle_{L^2(H^n)} \leq \| g \|_{L^2(H^n)} \int_{\mathbb{R}} \| S_t \psi(t) \|_{L^2(H^n)} \, dt \tag{5.7}
\]

\[
= \| g \|_{L^2(H^n)} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \langle S_{t-\tau} \psi(t), \psi(\tau) \rangle_{L^2(H^n)} \, d\tau \right)^{1/2}.
\]
We can easily obtain
\[ \langle S_{t-\tau} \psi(t), \psi(\tau) \rangle_{L^2(H^n)} \leq \| S_{t-\tau} \psi(t) \|_{\dot{B}^{-\beta(r)}_{r,2}(H^n)} \| \psi(\tau) \|_{\dot{B}^{\beta(r)}_{r,2}(H^n)}. \]

By (5.6), it yields that
\[ \| S_{t-\tau} \psi(t) \|_{\dot{B}^{-\beta(r)}_{r,2}(H^n)} \leq C |t - \tau|^{-\gamma(r)} \| \psi(t) \|_{\dot{B}^{\beta(r)}_{r,2}(H^n)}. \]

Thus, it holds
\[ \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \langle S_{t-\tau} \psi(t), \psi(\tau) \rangle_{L^2(H^n)} \frac{d\tau}{|t - \tau|^{\gamma(r)}} \right)^{1/2} \leq \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\| \psi(t) \|_{\dot{B}^{\beta(r)}_{r,2}(H^n)} \| \psi(\tau) \|_{\dot{B}^{\beta(r)}_{r,2}(H^n)}}{|t - \tau|^{\gamma(r)}} d\tau d\tau \right)^{1/2}. \]

By the Hardy–Littlewood–Sobolev inequality, we have
\[ \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\| \psi(t) \|_{\dot{B}^{\beta(r)}_{r,2}(H^n)} \| \psi(\tau) \|_{\dot{B}^{\beta(r)}_{r,2}(H^n)}}{|t - \tau|^{\gamma(r)}} d\tau d\tau \right)^{1/2} \leq C \| \psi(t) \|_{L^{p_1}(\mathbb{R}, \dot{B}^{\beta(r)}_{r,2}(H^n))}. \]

where \( \frac{2}{p_1} + \frac{1}{r} = \frac{1}{2}, \rho = -\beta(r). \) It follows from the above equation and (5.7) that
\[ \langle S_t g, \psi \rangle_{L^2(\mathbb{R}, L^2(H^n))} \leq C \| g \|_{L^2(H^n)} \| \psi \|_{L^{p_1}(\mathbb{R}, \dot{B}^{\beta(r)}_{r,1,2}(H^n))}. \]

Thus, (5.4) holds.

So far, we have proved (5.4) for the extreme case when \( \frac{2}{p_1} + \frac{1}{r} = \frac{1}{2}, \frac{N}{r} + \frac{2\alpha}{p_1} - \rho = \frac{N}{2}. \)

For \( \frac{2}{p_1} + \frac{1}{r} \leq \frac{1}{2} \) and \( \frac{N}{r_1} + \frac{2\alpha}{p_1} - \rho_1 = \frac{N}{2}, \) we have \( r_1 \geq r \) and \( \rho_1 \leq \rho. \) By Proposition 2.2, we have \( \dot{B}^{\rho}_{r,2}(H^n) \subset \dot{B}^{\rho_1}_{r_1,1,2}(H^n) \) and (5.4) also holds true.

According to (5.4), Lemma 3.4 and Lemma 3.5 we can immediately get
\[ \left\| \int_0^t S_{t-\tau} f(\tau) d\tau \right\|_{L^{p_1}(0,T), \dot{B}^{\rho_1}_{r_1,1,2}(H^n)} \leq C \| f \|_{L^{p_2}(0,T), \dot{B}^{-\rho_2}_{r_2,2}(H^n)}. \]

Thus, (5.5) holds. In addition, by Lemma 3.6, we also have
\[ \left\| \int_0^t S_{t-\tau} f(\tau) d\tau \right\|_{L^{p_1}(I, \dot{B}^{\rho_1}_{r_1,1,2}(H^n))} \leq C \| f \|_{L^1(I, L^2(H^n))}. \]
\[ \left\| \int_0^t S_{t-\tau} f(\tau) d\tau \right\|_{L^\infty(I, L^2(H^n))} \leq C \left\| \int_0^t S_{t-\tau} f(\tau) d\tau \right\|_{L^{p_1}(I, \dot{B}^{-\rho_1}_{r_1,1,2}(H^n))}. \]
Theorem 5.2 Under the same hypotheses as in Theorem 5.1, the solution \( u \) of the fractional Schrödinger equation (5.1) satisfies the following estimate
\[
\| u \|_{L^p([0,T],\dot{B}^{\rho}_{r,2}(H^n))} \leq C \left( \| u_0 \|_{L^2(H^n)} + \| f \|_{L^p([0,T],\dot{B}^{-\rho}_{r,2}(H^n))} \right),
\]
where the constant \( C > 0 \) does not depend on \( u_0, f \) or \( T \).

Next, by the embedding relation between the homogeneous Besov space and Lebesgue space, we prove the Strichartz inequalities on Lebesgue spaces.

Corollary 5.1 Let \( 0 < \alpha < 1 \) and \( u \) be the solution of the fractional Schrödinger equation (5.1). Suppose \( p \in [2N-2\alpha, +\infty) \) and \( q \) satisfy
\[
\frac{2\alpha}{p} + \frac{N}{q} = \frac{N}{2} - 1.
\]
Then, we have the following estimate
\[
\| u \|_{L^p([0,T],L^q(H^n))} \leq C \left( \| u_0 \|_{\dot{H}^1(H^n)} + \| f \|_{L^1([0,T],\dot{H}^1(H^n))} \right),
\]
where the constant \( C > 0 \) does not depend on \( u_0, f \) or \( T \).

**Proof** Under the same hypotheses as in Proposition 5.1 and according to (5.4) and (5.8), we have
\[
\| u \|_{L^p([0,T],\dot{B}^{\rho+1}_{r,2}(H^n))}
\leq \| S_T u_0 \|_{L^p([0,T],\dot{B}^{\rho+1}_{r,2}(H^n))} + \left\| \int_0^T S_{t-s} f(s) ds \right\|_{L^p([0,T],\dot{B}^{\rho+1}_{r,2}(H^n))},
\]
\[
\leq \| u_0 \|_{\dot{H}^1(H^n)} + \| f \|_{L^1([0,T],\dot{H}^1(H^n))}.
\]
Combining with the fact \( \dot{B}^{\rho+1}_{r,2}(H^n) \subset L^q(H^n) \) where \( 0 \leq \rho + 1 < N/r \) and \( \frac{1}{q} = \frac{1}{r} - \frac{\rho + 1}{N} \), we have
\[
\| u \|_{L^p([0,T],L^q(H^n))} \leq C \left( \| u_0 \|_{\dot{H}^1(H^n)} + \| f \|_{L^1([0,T],\dot{H}^1(H^n))} \right).
\]
\( \square \)

**Remark 5.1** It has been proved in [2] that the Schrödinger equation involving the sub-Laplacian on the Heisenberg group is not dispersive, and for this reason, the classical method to obtain Strichartz inequalities from dispersive estimates does not work. Our approach inspired by [10] enables us to obtain dispersive estimates and also Strichartz inequalities for fractional Schrödinger equations.

### 5.2 The fractional wave equation

We now consider the fractional wave equation \( 0 < \alpha < 2 \)
\[
\begin{cases}
\partial_t^2 u + \mathcal{L}^\alpha u = f \in L^1([0, T], L^2(H^n)), \\
u|_{t=0} = u_0 \in \dot{H}^\alpha(H^n), \\
\partial_t u|_{t=0} = u_1 \in L^2(H^n),
\end{cases}
(5.9)
\]
The general solution of (5.9) has the form
\[
u = v + w.
\]
where \( v \) is a solution of (5.9) with \( f = 0 \) and \( w \) is the solution of (5.9) with \( u_0 = u_1 = 0 \). They are classically given by

\[
\begin{align*}
  v(t) &= \frac{dA_t}{dt}u_0 + A_tu_1, \\
  w(t) &= \int_0^t A_{t-\tau}f(\tau)d\tau,
\end{align*}
\]

where

\[
A_t = \frac{\sin(t\mathcal{L}^{\frac{\alpha}{2}})}{\mathcal{L}^{\frac{\alpha}{2}}}, \quad \frac{dA_t}{dt} = \cos(t\mathcal{L}^{\frac{\alpha}{2}}).
\]

So we naturally introduce the operator \( W_t = e^{it\mathcal{L}^{\frac{\alpha}{2}}} \), which corresponds to the case when \( \phi(r) = r^{\frac{\alpha}{2}} \). By a simple calculation,

\[
\begin{align*}
  \phi'(r) &= \frac{\alpha}{2} \cdot r^{\frac{\alpha}{2}}, \quad \phi''(\lambda) = \frac{\alpha}{2} (\frac{\alpha}{2} - 1) r^{\frac{\alpha}{2} - 2}.
\end{align*}
\]

It yields that \( \phi \) satisfies (H1)-(H4) with \( m_1 = \alpha_1 = m_2 = \alpha_2 = \frac{\alpha}{2} \). According to Theorem 1.1, we have

\[
\|W_t\varphi_j\|_{L^\infty(H^n)} \leq C \|t\|^{-\frac{1}{2}} 2^j (N - \frac{\alpha}{2}), \quad \forall j \in \mathbb{Z}. \tag{5.12}
\]

**Theorem 5.3** Let \( u_0 \in \dot{B}^{N - \frac{\alpha}{2}}_{1,1}(H^n) \) and \( u_1 \in \dot{B}^{N - \frac{3}{2} \alpha}_{1,1}(H^n) \). Suppose \( v \) is the solution of the free fractional wave equation (5.9) (i.e., with \( f = 0 \)). Then, there exists a constant \( C > 0 \), which does not depend on \( u_0 \) or \( u_1 \) such that

\[
\|v(t)\|_{L^\infty(H^n)} \leq C |t|^{-1/2} \left( \|u_0\|_{\dot{B}^{N - \frac{\alpha}{2}}_{1,1}(H^n)} + \|u_1\|_{\dot{B}^{N - \frac{3}{2} \alpha}_{1,1}(H^n)} \right), \tag{5.13}
\]

and the time decay is sharp in time.

**Proof** Because of (5.10), we can write

\[
v(t) = \frac{dA_t}{dt}u_0 + A_tu_1 = \frac{W_t + W_{-t}}{2}u_0 + \mathcal{L}^{\frac{\alpha}{2}} W_t - W_{-t}u_1.
\]

Therefore, it suffices to prove that \( \|W_tu_0\|_{L^\infty(H^n)} \) and \( \|\mathcal{L}^{\frac{\alpha}{2}} W_tu_1\|_{L^\infty(H^n)} \) can be controlled by the right side of (5.13).
According to (5.12) and the Young inequality, we get
\[
\|W_t u_0\|_{L^\infty(\mathbb{R}^n)} = \left\| \sum_{j \in \mathbb{Z}} \Delta_j (W_t u_0) \right\|_{L^\infty(\mathbb{R}^n)} \leq \sum_{j \in \mathbb{Z}} \left\| \Delta_j (W_t u_0) \right\|_{L^\infty(\mathbb{R}^n)} = \sum_{j \in \mathbb{Z}} \left\| W_t (\Delta_j u_0) \right\|_{L^\infty(\mathbb{R}^n)} = \sum_{j \in \mathbb{Z}} \left\| W_t (u_0 * \varphi_j) \right\|_{L^\infty(\mathbb{R}^n)} = \sum_{j \in \mathbb{Z}} \left\| u_0 * \varphi_j * W_t \varphi_j \right\|_{L^\infty(\mathbb{R}^n)} \leq \sum_{j \in \mathbb{Z}} \left\| u_0 * \varphi_j \right\|_{L^1(\mathbb{R}^n)} \| W_t \varphi_j \|_{L^\infty(\mathbb{R}^n)} \leq \sum_{j \in \mathbb{Z}} C|t|^{-\frac{1}{2}} j^{(N-\frac{q}{2})} \| \Delta_j u_0 \|_{L^1(\mathbb{R}^n)} = C\|u_0\|_{B_{1,1}^{-\frac{N}{2}}(\mathbb{R}^n)}^{\frac{N}{2}}.
\]

It follows from (2.3) that
\[
\| \Delta_j (\mathcal{L}_{\mathbb{R}}^{\frac{q}{2}} u_1) \|_{L^1(\mathbb{R}^n)} = \left\| (\mathcal{L}_{\mathbb{R}}^{\frac{q}{2}} (\Delta_j u_1)) \right\|_{L^1(\mathbb{R}^n)} \leq C \| \Delta_j u_1 \|_{L^1(\mathbb{R}^n)}.
\]

Hence, we have
\[
\| \mathcal{L}_{\mathbb{R}}^{\frac{q}{2}} W_t u_1 \|_{L^\infty(\mathbb{R}^n)} \leq C|t|^{-1/2} \| u_1 \|_{B_{1,1}^{-\frac{N}{2}}(\mathbb{R}^n)}.
\]

Next, we also give an example to illustrate the sharpness of the decay in time $t$. For the same $Q$, $D_0$ and $u_0$ as in the case of the fractional Schrödinger equation, we consider the fractional wave equation (5.9) when $u_1 = 0$, $f = 0$. We have
\[
u(z, s, t) = e^{it\mathcal{L}_{\mathbb{R}}^{\frac{q}{2}}} u_0 = C \int_{D_0} e^{-i\lambda s} e^{it(4\lambda n)^{\frac{q}{2}}} Q(\lambda) e^{-\lambda |z|^2} \lambda^n d\lambda.
\]
In particular,
\[
u(0, ts, t) = C \int_{D_0} e^{it(-\lambda s + (4\lambda n)^{\frac{q}{2}})} Q(\lambda) \lambda^n d\lambda.
\]

This oscillating integral has a phase $\Psi(\lambda) = -\lambda s + (4\lambda n)^{\frac{q}{2}}$ with a unique critical point $\lambda_0 = \frac{1}{4n} \left( \frac{2an}{s} \right)^{\frac{2}{q-2}}$, which is not degenerate. By Lemma 3.2, for $s_0 = \frac{a (4n)^{\frac{q}{2}}}{2}$, it yields that
\[
u(0, ts_0, t) \sim C|t|^{-1/2}.
\]

As Proposition 5.1, we have the following Strichartz estimates.

\[\square\] Springer
Proposition 5.2 Suppose $0 < \alpha < 2$. For $i = 1, 2$, let $p_i, r_i \in [2, \infty)$ and $\rho_i \in \mathbb{R}$ such that
\[
(i) \quad \frac{2}{p_i} + \frac{1}{r_i} \leq \frac{1}{2},
(ii) \quad \frac{N}{r_i} + \frac{\alpha}{p_i} - \rho_i = \frac{N}{2},
\]
Then, the following estimates are satisfied:
\[
\| W_t g \|_{L^{p_i}(\mathbb{R}, \dot{H}^{\rho_i}_{1,2}(H^n))} \leq C \| g \|_{L^2(H^n)}, \tag{5.14}
\]
\[
\left\| \int_0^t W_{t-\tau} f(\tau) \, d\tau \right\|_{L^{p_i}([0,T], \dot{H}^{\rho_i}_{1,2}(H^n))} \leq C \| f \|_{L^{p_2}([0,T], \dot{H}^{-\rho_i}_{2,2}(H^n))}, \tag{5.15}
\]
where the constant $C > 0$ does not depend on $u_0$, $f$ or $T$.

Now, we describe the Strichartz inequalities for the solution of the fractional wave equation (5.9).

Theorem 5.4 Suppose $0 < \alpha < 2$. For $i = 1, 2$, let $p_i, r_i \in [2, \infty)$ and $\rho_i \in \mathbb{R}$ such that
\[
(i) \quad \frac{2}{p_i} + \frac{1}{r_i} \leq \frac{1}{2},
(ii) \quad \frac{N}{r_1} + \frac{\alpha}{p_1} - \rho_1 = \frac{N}{2} - \alpha,
(iii) \quad \frac{N}{r_2} + \frac{\alpha}{p_2} - \rho_2 = \frac{N}{2},
\]
Then, the following estimates are satisfied:
\[
\| v \|_{L^{p_1}(\mathbb{R}, \dot{H}^{\rho_1}_{1,2}(H^n))} + \| \partial_t v \|_{L^{p_1}(\mathbb{R}, \dot{H}^{\rho_1-\alpha}_{1,2}(H^n))} \leq C \left( \| u_0 \|_{\dot{H}^\alpha(H^n)} + \| u_1 \|_{L^2(H^n)} \right) \tag{5.16}
\]
\[
\| w \|_{L^{p_1}([0,T], \dot{H}^{\rho_1}_{1,2}(H^n))} + \| \partial_t w \|_{L^{p_1}([0,T], \dot{H}^{\rho_1-\alpha}_{1,2}(H^n))} \leq C \| f \|_{L^{p_2}([0,T], \dot{H}^{-\rho_2}_{2,2}(H^n))}, \tag{5.17}
\]
where the constant $C > 0$ does not depend on $u_0$, $f$ or $T$.

Proof First, we can write
\[
v = \frac{dA_t}{dt} u_0 + A_t u_1 = \frac{W_t + W_{-t}}{2} u_0 + L^{1/2} \frac{W_t - W_{-t}}{2i} u_1,
\]
and
\[
\partial_t v = A_t \mathcal{L}^{\alpha} u_0 + \frac{dA_t}{dt} u_1 = L^{1/2} \frac{W_t - W_{-t}}{2i} u_0 + \frac{W_t + W_{-t}}{2} u_1.
\]
By (2.3), we know
\[
\| \Delta_j v \|_{L^1(H^n)} = \left\| L^{-\alpha} \mathcal{L}^{\alpha} \Delta_j v \right\|_{L^1(H^n)} \leq C 2^{-j\alpha} \| \mathcal{L}^{\alpha} \Delta_j v \|_{L^1(H^n)}.
\]
Thus, we have
\[
\| v \|_{L^{p_1}(\mathbb{R}, \dot{H}^{\rho_1}_{1,2}(H^n))} \leq C \| L^{\frac{\alpha}{2}} v \|_{L^{p_1}(\mathbb{R}, \dot{H}^{\rho_1-\alpha}_{1,2}(H^n))},
\]

\[\text{Springer}\]
Then,
\[
\|v\|_{L^p([0,T],\dot{B}^{\rho_1}_{r_1,2}(H^n))} + \|\partial_t v\|_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))} \\
\leq C \left( \left\| W_t \mathcal{L}_t^\alpha u_0 \right\|_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))} + \left\| W_{-t} \mathcal{L}_t^\alpha u_0 \right\|_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))} \\
+ \|W_t u_1\|_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))} + \|W_{-t} u_1\|_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))} \right).
\]

According to (5.14), we get
\[
\|W_t \mathcal{L}_t^\alpha u_0\|_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))} \leq \|\mathcal{L}_t^\alpha u_0\|_{L^2} \leq \|u_0\|_{\dot{H}^\alpha(H^n)},
\]
\[
\|W_t u_1\|_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))} \leq \|u_1\|_{L^2(H^n)}.
\]

The same discussion can apply to \(W_{-t}\). Now, we obtain (5.16).

It remains to prove (5.17). We write
\[
w = \int_0^t A_{t-\tau} f(\tau) d\tau = \int_0^t \mathcal{L}_t^\alpha W_{t-\tau} - W_{t-\tau} f(\tau) d\tau,
\]
Similarly, we have
\[
\partial_t w = \int_0^t \frac{dA_{t-\tau}}{d\tau} f(\tau) d\tau = \int_0^t W_{t-\tau} - \frac{W_{t-\tau}}{2} f(\tau) d\tau.
\]

Therefore, we obtain
\[
\|w\|_{L^p([0,T],\dot{B}^{\rho_1}_{r_1,2}(H^n))} \leq C \left( \left\| \mathcal{L}_t^\alpha \int_0^t W_{t-\tau} f(\tau) d\tau \right\|_{L^p([0,T],\dot{B}^{\rho_1}_{r_1,2}(H^n))} \\
+ \left\| \mathcal{L}_t^\alpha \int_0^t W_{t-\tau} f(\tau) d\tau \right\|_{L^p([0,T],\dot{B}^{\rho_1}_{r_1,2}(H^n))} \right)
\]
\[
\leq C \left( \left\| \int_0^t W_{t-\tau} f(\tau) d\tau \right\|_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))} \\
+ \int_0^t W_{t-\tau} f(\tau) d\tau \right)_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))}.
\]

and
\[
\|\partial_t w\|_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))} \\
\leq C \left( \left\| \int_0^t W_{t-\tau} f(\tau) d\tau \right\|_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))} + \int_0^t W_{t-\tau} f(\tau) d\tau \right)_{L^p([0,T],\dot{B}^{\rho_1-\alpha}_{r_1,2}(H^n))}.
\]

According to (5.15), then
\[
\int_0^t W_{t-\tau} f(\tau) d\tau \leq C \|f\|_{L^p([0,T],\dot{B}^{\rho_1}_{r_1,2}(H^n))},
\]

Thus, (5.17) holds. \(\square\)
As Corollary 5.1, we obtain the Strichartz inequalities on the Lebesgue space by the embedding relation between the homogeneous Besov space and Lebesgue space on the Heisenberg group.

**Corollary 5.2** Let \( 0 < \alpha < 2 \) and \( u \) be the solution of the fractional wave equation (5.9). Suppose \( p \in [2N - 2\alpha, +\infty) \) and \( q \) satisfy
\[
\frac{\alpha}{p} + \frac{N}{q} = \frac{N}{2} - \alpha.
\]
Then, the following estimate holds true
\[
\|u\|_{L^p([0,T],L^q(H^n))} \leq C \left( \|u_0\|_{\dot{B}^{\alpha}_{2,1}(H^n)} + \|u_1\|_{L^2} + \|f\|_{L^1([0,T],L^2(H^n))} \right),
\]
where the constant \( C > 0 \) does not depend on \( u_0, u_1, f \) or \( T \).

**Remark 5.2** When \( \alpha = 1 \), it corresponds to the results in [2]. Therefore, our work is more general.

### 5.3 The fourth-order Schrödinger equation

Finally, we consider the fourth-order Schrödinger equation
\[
\begin{aligned}
i \partial_t u + L^2 u + Lu &= f, \\
u|_{t=0} &= u_0.
\end{aligned}
\tag{5.18}
\]
The solution is given by
\[
u(t) = U_t u_0 - i \int_0^t U_{t-\tau} f(\tau) \, d\tau,
\]
where \( U_t = e^{it(L^2 + L)} \) and it corresponds to the case when \( \phi(r) = r^2 + r \). By a simple calculation,
\[
\phi'(\lambda) = 2r + 1, \quad \phi''(\lambda) = 2.
\]
We know \( \phi \) satisfies (H1)-(H4) with \( m_1 = \alpha_1 = \alpha_2 = 2, m_2 = 1 \). According to Theorem 1.1, we get
\[
\|U_t \phi\|_{L^\infty(H^n)} \leq C_0 |t|^{-\frac{1}{2}j(N-2)}.
\]

**Theorem 5.5** If \( u \) is the solution of the free fourth-order Schrödinger equation (5.1) (i.e., with \( f = 0 \)), then there exists a constant \( C > 0 \) which does not depend on \( u_0 \) or \( t \) such that
\[
\|u(t)\|_{L^\infty(H^n)} \leq C |t|^{-1/2} \|u_0\|_{\dot{B}^1_{2,1}},
\]
and the decay is sharp in time.

Since the proof of the inequality is similar to the case of the fractional Schrödinger equation, we only show the sharpness of the decay. For the same \( Q, D_0 \) and \( u_0 \), and if \( u \) is the solution of (5.18) when \( f = 0 \), we have
\[
u(z, s, t) = e^{i(\frac{z^2}{2} + z)} u_0 = C \int_{D_0} e^{-i\lambda s} e^{i((4n\lambda)^2 + 4n\lambda)} Q(\lambda) e^{-\lambda |\xi|^2} \lambda^n \, d\lambda.
\]
In particular,
\[
u(0, ts, t) = C \int_{D_0} e^{i(\frac{ts}{2} + (4n\lambda)^2 + 4n\lambda)} Q(\lambda) \lambda^n \, d\lambda.
\]
This oscillating integral has a phase \( \Psi(\lambda) = -\lambda s + (4n\lambda)^2 + 4n\lambda \) with a unique critical point \( \lambda_0 = \frac{s - 4n}{32n^2} \), which is not degenerate. Applying Lemma 3.2, for \( s_0 = 32n^2 + 4n \), we have
\[
u(0, ts_0, t) \sim C|t|^{-1/2}.
\]

**Proposition 5.3** For \( i = 1, 2, \) let \( p_i, r_i \in [2, \infty] \) and \( \rho_i \in \mathbb{R} \) such that
\[
(i) \quad \frac{2}{p_i} + \frac{1}{r_i} \leq \frac{1}{2},
(ii) \quad \frac{N}{r_i} + \frac{4}{p_i} - \rho_i = \frac{N}{2},
\]
then, the following estimates are satisfied:
\[
\|U_t u_0\|_{L^p([0,T], \dot{B}^{\rho_1}_{r_1,2}(H^n))} \leq C\|u_0\|_{L^2(H^n)},
\]
\[
\int_0^T \|U_t f(\tau)\|_{L^p([0,T], \dot{B}^{\rho_2}_{r_2,2}(H^n))} d\tau \leq C\|f\|_{L^\infty([0,T], \dot{B}^{-\rho_2}_{r_2,2}(H^n))},
\]
where the constant \( C > 0 \) does not depend on \( u_0, f \) or \( T \).

**Theorem 5.6** Under the same hypotheses as in Theorem 5.3, the solution of the fourth-order Schrödinger equation (5.18) \( u \) satisfies the following estimate
\[
\|u\|_{L^p([0,T], \dot{B}^{\rho_1}_{r_1,2}(H^n))} \leq C\left(\|u_0\|_{L^2(H^n)} + \|f\|_{L^\infty([0,T], \dot{B}^{-\rho_2}_{r_2,2}(H^n))}\right),
\]
where the constant \( C > 0 \) does not depend on \( u_0, f \) or \( T \).

**Corollary 5.3** Suppose \( p \in [2N - 4, +\infty] \) and \( q \) satisfy
\[
\frac{4}{p} + \frac{N}{q} = \frac{N}{2} - 1
\]
If \( u \) is a solution of the fourth-order Schrödinger equation (5.18), then the following inequality holds true
\[
\|u\|_{L^p([0,T], L^q(H^n))} \leq C\left(\|u_0\|_{\dot{H}^{1}(H^n)} + \|f\|_{L^1([0,T], \dot{H}^{1}(H^n))}\right),
\]
where the constant \( C > 0 \) does not depend on \( u_0, f \) or \( T \).

The proofs of Proposition 5.3, Theorem 5.6, Corollary 5.3 are similar to those in the case of the fractional Schrödinger case. We omit it here.

**Acknowledgements** The work is supported by the National Natural Science Foundation of China (Grant No. 11701452), the China Postdoctoral Science Foundation (Grant No. 2017M623230) and the Natural Science Foundation of Shaanxi Province (Grant No. 2020JQ-112) and Guangdong Basic and Applied Basic Research Foundation (No. 2023A1515010656).

**Declarations**

**Conflict of interest** The author declares to have no conflict of interest.
References

1. Bahouri, H., Barilari, D., Gallagher, I.: Strichartz estimates and Fourier restriction theorems on the Heisenberg group. J. Fourier Anal. Appl. 27(2), 1–41 (2021)
2. Bahouri, H., Gérard, P., Xu, C.J.: Espaces de Besov et estimations de Strichartz généralisées sur le groupe de Heisenberg. J. Anal. Math. 82, 93–118 (2000)
3. Bahouri, H., Kammerer, C.F., Gallagher, I.: Dispersive estimates for the Schrödinger operator on step 2 stratified Lie groups. Anal. PDE 9, 545–574 (2016)
4. Bui, T.A., D’Ancona, P., Xuan, T.D., Müller, D.: On the flows associated to self-adjoint operators on metric measure spaces. Math. Ann. 375, 1393–1426 (2019)
5. D’Ancona, P., Pierfelice, V., Ricci, F.: On the wave equation associated to the Hermite and the twisted Laplacian. J. Fourier Anal. Appl. 16(2), 294–310 (2010)
6. Furioli, G., Melzi, C., Veneruso, A.: Littlewood-Paley decompositions and Besov spaces on Lie groups of polynomial growth. Math. Nachr. 279, 1028–1040 (2006)
7. Furioli, G., Melzi, C., Veneruso, A.: Strichartz inequalities for the wave equation with the full Laplacian on the Heisenberg group. Can. J. Math. 59(6), 1301–1322 (2007)
8. Furioli, G., Veneruso, A.: Strichartz inequalities for the Schrödinger equation with the full Laplacian on the Heisenberg group. Stud. Math. 160, 157–178 (2004)
9. Ginibre, J., Velo, G.: Generalized Strichartz inequalities for the wave equation. J. Funct. Anal. 133, 50–68 (1995)
10. Guo, Z., Peng, L., Wang, B.: Decay estimates for a class of wave equations. J. Funct. Anal. 254(6), 1642–1660 (2008)
11. Hierro, M.D.: Dispersive and Strichartz estimates on H-type groups. Stud. Math. 169, 1–20 (2005)
12. Keel, M., Tao, T.: Endpoints Strichartz estimates. Am. J. Math. 120, 955–980 (1998)
13. Nandakumaran, A.K., Ratnakumar, P.K.: Schrödinger equation and the oscillatory semi group for the Hermite operator. J. Funct. Anal. 224(2), 371–385 (2021)
14. Ratnakumar, P.K.: On Schrödinger Propagator for the Special Hermite Operator. J. Fourier Anal. Appl. 14(2), 286–300 (2008)
15. Sohani, V.K.: Strichartz estimates for the Schrödinger propagator for the Laguerre operator. Proc. Math. Sci. 123, 525–537 (2013)
16. Song, M.: Decay estimates for fractional wave equations on H-type groups. J. Inequal. Appl. 2016(246), 12 (2016)
17. Song, N., Zhao, J.: Strichartz estimates on the quaternion Heisenberg group. Bull. Sci. Math. 138(2), 293–315 (2014)
18. Stein, E.M.: Harmonic analysis: real-variable methods, orthogonality and oscillatory integrals. Princeton University Press, New Jersey (1993)
19. Strichartz, R.S.: Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. Duke Math. J. 44, 705–714 (1977)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.