AMALGAMATION IN CLASSES OF INVOLUTIVE COMMUTATIVE RESIDUATED LATTICES

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Abstract. Amalgamation is investigated in classes of non-divisible, non-integral, and non-idempotent involutive commutative residuated lattices. We demonstrate that several subclasses of totally-ordered, involutive, commutative residuated lattices fail the Amalgamation Property. These include the classes of odd and even ones, sharing the same underlying reason for their failure as observed in the class of discrete linearly ordered abelian groups with positive normal homomorphisms. Conversely, it is proven that three subclasses formed by idempotent-symmetric, totally-ordered, involutive, commutative residuated lattices possess the Amalgamation Property, albeit fail the Strong Amalgamation Property. This failure shares the same underlying reason as observed in the class of linearly ordered abelian groups. Additionally, it is demonstrated that the varieties of semilinear, idempotent-symmetric, odd, involutive, commutative residuated lattices and semilinear, idempotent-symmetric, odd or even, involutive, commutative residuated lattices have the Transferable Injections Property. Finally, it is shown that any variety of semilinear involutive commutative residuated lattices that includes the variety of odd semilinear commutative residuated lattices fails the Amalgamation Property.

1. Introduction

Residuation is a fundamental concept of ordered structures and categories [3]. Residuated mappings are to Galois connections just like covariant functors are to contravariant ones [17]. Residuated binary operations are binary operations such that all of their section mappings are residuated mappings. Residuated lattices were first introduced in the 1930s by Ward and Dilworth [45] [11] to explore the ideal theory of commutative rings with unity, see also Krull [35]. Examples of residuated lattices include Boolean algebras, Heyting algebras [34], complemented semigroups [6], bricks [4], resigation groupoids [7], semiclans [5], Bezout monoids [2], MV-algebras [10], BL-algebras [23], and lattice-ordered groups [1]. In fact, a wide range of other algebraic structures can also be represented as residuated lattices. A more recent development has spurred renewed interest in residuated lattices: varieties of (pointed) residuated lattices serve as algebraic counterparts to substructural logics [16]. The study of substructural logics is a rapidly expanding field and has become a focal point in non-classical logics. Today, it stands not only as a compelling branch of non-classical logic but also as a central area within algebraic logic.

The Amalgamation Property (AP) and its variants are closely linked to various

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syntactic interpolation properties of substructural logics, making their investigation in varieties of residuated lattices particularly intriguing. This paper examines the Amalgamation Property within classes of involutive commutative residuated lattices.

Historically, the Amalgamation Property was first considered for groups through amalgamated free products [40] and has also been studied for varieties such as non-abelian and non-representable ℓ-groups. In the context of residuated lattices, the Amalgamation Property has primarily been investigated in varieties where the algebras are linear, semilinear (also known as representable), conic, or semiconic. The classes investigated in the literature have predominantly been either divisible (often also integral) [37] or idempotent [8, 18]. The scope of this paper is to investigate the Amalgamation Property in varieties of residuated lattices that are neither divisible, integral, nor idempotent. These varieties correspond to involutive substructural logics without the weakening rule.

We mention here a few of the most recent related developments (see [16, 37] and the references therein for a more complete picture). As shown in [20], the varieties of semilinear residuated lattices and semilinear cancellative residuated lattices fail the Amalgamation Property. A comprehensive account of the interrelationships between algebraic properties of varieties and properties of their free algebras and equational consequence relations, including equivalences between different kinds of amalgamation properties and the Robinson property, the congruence extension property, and the deductive interpolation property, can be found in [37]. Additionally, the presence or failure of amalgamation is established for some subvarieties of residuated lattices, notably for all subvarieties of commutative GMV-algebras. As shown in [18], the variety generated by totally-ordered commutative idempotent residuated lattices has the Amalgamation Property, and an example of a non-commutative variety of idempotent residuated lattices that possesses the Amalgamation Property has also been presented.

The Amalgamation Property is indeed quite rare for general varieties. In this paper, we establish the failure of the Amalgamation Property for several classes of involutive (pointed) residuated lattices, including the classes of odd and even involutive totally-ordered commutative residuated lattices. This result is detailed in Theorem 8.1. Additionally, we gain an important insight: within these classes, the failure of amalgamation occurs for the same underlying reason as observed in discrete linearly ordered abelian groups with positive normal homomorphisms [22]. Building upon this observation, we then focus on three subclasses exclusively comprising algebras that are idempotent-symmetric. We demonstrate that these classes possess the Amalgamation Property, as detailed in Theorem 8.3. However, we also establish their failure to satisfy the Strong Amalgamation Property, as indicated in Theorem 8.2. The failure of the Strong Amalgamation Property in these subclasses can be attributed to the same underlying reason observed in the class of linearly ordered abelian groups with positive homomorphisms. Then we now shift our focus from these classes of chains to the semilinear varieties of FLₜ-algebras that they generate. Varieties of semilinear FL-algebras, also known as semilinear pointed residuated lattices, are fundamental in the study of substructural logics. We conclude that every variety of semilinear involutive commutative (pointed) residuated lattices that includes the variety of odd semilinear commutative residuated lattices fails the Amalgamation Property. Furthermore, we demonstrate in Theorem 9.3
that the varieties of idempotent-symmetric, semilinear, odd involutive residuated lattices, as well as idempotent-symmetric, semilinear, odd or even involutive residuated lattices, exhibit the Transferable Injections Property, a strengthening of the Amalgamation Property.

This paper is organized as follows: First, we cover preliminaries on residuated lattices, amalgamation, direct systems, and partially ordered abelian groups in Sections 2–5. The proof of our main results hinges on a categorical equivalence between the category of odd or even involutive commutative residuated chains and the category of bunches (of layer groups), with objects being specific direct systems of abelian \( o \)-groups. This equivalence, which will be presented in Section 6, provides a pathway to simplify the problem of amalgamation in classes of odd or even involutive residuated chains to the amalgamation of particular direct systems of abelian \( o \)-groups. In Section 7, we develop all the necessary machinery for bunches, such as how the injectivity of an FL\(_e\)-algebra homomorphism is reflected in its associated bunch homomorphism. Also, we construct canonical extensions of bunches that are injective objects in a restricted sense and demonstrate that direct systems of torsion-free partially ordered abelian groups over arbitrary chains can be transformed into direct systems of abelian \( o \)-groups by extending only the ordering relations. The main constructions are presented in Section 8. By combining our results with previous findings in the literature, we establish the Amalgamation Property or its failure in varieties of semilinear involutive commutative residuated lattices in Section 9.

2. Involutive FL\(_e\)-algebras

For a partially ordered set (poset) \((X, \leq)\), we define the upper neighbor \( x_u \) of \( x \in X \) as either the unique upper cover of \( x \), if it exists, or \( x \) otherwise. The lower neighbor \( x_l \) is defined dually. A partially ordered algebra with a poset reduct is termed discretely ordered if it satisfies \( x_l < x < x_u \), indicating that each element has a unique upper cover and a unique lower cover.

An algebra equipped with two binary operations, denoted as \( \cdot \) and \( \rightarrow \), where \( \cdot \) is commutative, along with a poset reduct \((X, \leq)\), is termed residuated if it satisfies the condition \( x \cdot y \leq z \) if and only if \( x \rightarrow z \geq y \). Equivalently, the multiplication is order-preserving and for any \( x \) and \( z \), the set \( \{ v \mid xv \leq z \} \) has its greatest element. The residuum of \( x \) and \( z \) is defined as this maximal element: \( x \rightarrow z := \max \{ v \mid xv \leq z \} \), and \( \rightarrow \) is referred to as the residual operation of \( \cdot \). Consequently, the residual operation is unique when it exists.

An \( FL_e \)-algebra is a structure denoted as \( X = (X, \land, \lor, \cdot, \rightarrow, t, f) \) such that \((X, \land, \lor)\) forms a lattice, and \((X, \land, \lor, \cdot, \rightarrow, t)\) constitutes a commutative residuated monoid. Additionally, \( f \) represents an arbitrary constant, referred to as the falsum constant. Commutative residuated lattices are the \( f \)-free reducts of \( FL_e \)-algebras. Other terms used to describe \( FL_e \)-algebras include pointed commutative residuated lattices or pointed commutative residuated lattice-ordered monoids. At times, the lattice operators may be substituted with their induced ordering \( \leq \) in the signature, especially when dealing with \( FL_e \)-chains, i.e., when the order is total.

A residuated structure is inherently lattice-ordered whenever the underlying poset is a lattice, meaning that \( \cdot \) distributes over the join operation. In an \( FL_e \)-algebra, the residual complement operation is defined as \( x' = x \rightarrow f \), and the algebra is termed involutive if it satisfies \( (x')' = x \). In this scenario, \( x \rightarrow y = (xy')' \) and
\( f' = t \) hold true. The latter identity guarantees that homomorphisms of involutive FL\(_e\)-algebras coincide with homomorphisms of their underlying residuated lattice reduct.

Refer to the elements \( x \geq t \) as positive, and term the FL\(_e\)-algebra conic if all elements are comparable with \( t \). An involutive FL\(_e\)-algebra is termed odd if the residual complement operation leaves the unit element fixed, i.e., \( t' = t \), and even if \( x < t \) if and only if \( x \leq f \). The former condition is equivalent to stating that \( f = t \), while the latter condition is equivalent to presuming that \( f \) is the unique lower cover of \( t \), thereby implying, by involutivity, that \( t \) is the unique upper cover of \( f \).

An FL\(_e\)-algebra is called semilinear if it can be described as a subdirect product of FL\(_e\)-chains. In an odd or even involutive FL\(_e\)-algebra, a telegraphic proof shows that the residual complement of every negative (i.e., \( \leq f \)) idempotent element is a positive idempotent element. However, the converse is not true. An odd or even involutive FL\(_e\)-algebra is termed idempotent-symmetric if \( x' \) is idempotent whenever \( x \) is idempotent. This property implies that the set of idempotent elements is symmetric with respect to \( t \), meaning it remains invariant with respect to \( t' \).

In our ensuing discussion, several classes of algebras will play a significant role. Each of these classes is designated by a distinctive notation, as detailed below.

- \( \mathcal{C} \) the class of totally-ordered sets (chains)
- \( \mathcal{A}^c \) the class of abelian o-groups
- \( \mathcal{A}^\ell \) the class of abelian \( \ell \)-groups
- \( \mathcal{A}^{cd} \) the class of discrete abelian o-groups
- \( \mathcal{I} \) the class of involutive FL\(_e\)-algebras
- \( \mathcal{S} \) the class of idempotent-symmetric involutive FL\(_e\)-algebras

Adjunct to \( \mathcal{I} \),
- the superscript \( c \) means restriction to totally-ordered algebras,
- the superscript \( sl \) means restriction to semilinear algebras,
- the subscript \( o \) means restriction to odd algebras,
- the subscript \( e \) means restriction to even algebras,
- the subscript \( ei \) means restriction to even algebras having an idempotent falsum constant,
- the subscript \( en \) means restriction to even algebras having a non-idempotent falsum constant.

When multiple letters appear in the subscript, they denote the union of the corresponding classes. For instance \( \mathcal{S}^{ce} \) refers to the class of idempotent-symmetric involutive FL\(_e\)-chains which are either odd or even with an idempotent falsum constant. We shall denote the trivial (one-element) o-group by \( 1 \).

### 3. Amalgamation

Throughout the paper, algebras are denoted by bold capital letters, while their underlying sets are represented by the corresponding regular letters. Let \( \mathcal{U} \) be a class of algebraic systems.\(^4\) Call \( \langle A, B_1, B_2, \iota_1, \iota_2 \rangle \) a V-formation in \( \mathcal{U} \), if \( A, B_1, B_2 \in \mathcal{U} \) and \( \iota_k \) is an \( \mathcal{U} \)-embedding of \( A \) into \( B_k \) (\( k = 1, 2 \)). The V-formation can be

\(^1\)We use the term in the general sense, which includes relations as well. Consequently, the term “\( \mathcal{U} \)-embedding” below refers to an embedding that preserves both the operations and the relations.
amalgamated in $\mathcal{U}$ if there exists $\langle C, \mu_1, \mu_2 \rangle$, called an amalgam of the V-formation, such that $C \in \mathcal{U}$, $\mu_k$ is an $\mathcal{U}$-embedding of $B_k$ into $C$ ($k = 1, 2$) such that $\mu_1 \circ \iota_1 = \mu_2 \circ \iota_2$, see Fig. 1. If every V-formation in $\mathcal{U}$ can be amalgamated in $\mathcal{U}$, then $\mathcal{U}$ is said to have the Amalgamation Property. $\langle C, \mu_1, \mu_2 \rangle$ is termed a strong amalgam of the V-formation if it amalgamates in such a way that for $b_1 \in B_1$ and $b_2 \in B_2$, if $\mu_1(b_1) = \mu_2(b_2)$, then there exists some $a \in A$ such that $b_1 = \iota_1(a)$ and $b_2 = \iota_2(a)$. If all V-formations in $\mathcal{U}$ can be strongly amalgamated in $\mathcal{U}$, then $\mathcal{U}$ is said to have the Strong Amalgamation Property. To maintain a reasonable level of complexity in the notation, we will assume, without loss of generality, that $\iota_1$ and $\iota_2$ are inclusion maps in the following.

It is known that $\mathcal{A}_c$ and $\mathcal{A}_l$ have the Amalgamation Property [38, 39], and that $\mathcal{C}$ has the Strong Amalgamation Property [39, Lemma 2.2]. One approach to constructing an amalgam involves considering the amalgamated free product, provided it exists. For instance, in the context of $\mathcal{A}_l$, it was shown in [44, Theorem 12.2.1] that given $A, B_1, B_2 \in \mathcal{A}_l$ with embeddings $\iota_1 : A \rightarrow B_1$ and $\iota_2 : A \rightarrow B_2$, there exists the free product in $\mathcal{A}_l$ of $B_1$ and $B_2$ with $A$ amalgamated. This amalgamated product is denoted by $B_1 \ast_A B_2$, it belongs to $\mathcal{A}_l$, and it satisfies the following properties:

(Am1) There exist embeddings $\mu_1$ and $\mu_2$ that make the upper part of the diagram in Fig. 2 commute.

(Am2) For every $P \in \mathcal{A}_l$ and homomorphisms $\varphi_1 : B_1 \rightarrow P$ and $\varphi_2 : B_2 \rightarrow P$ that make the outer square commute, there exists a unique homomorphism $\varphi : B_1 \ast_A B_2 \rightarrow P$ (referred to as the pushout homomorphism of $\varphi_1$ and $\varphi_2$) that makes the two triangles in the lower part of the diagram in Fig. 2 commute.

As with all universal constructions, the amalgamated free product, if it exists, it is unique up to isomorphism.

As shown in [44, Theorem 12.2.2], one way of constructing an amalgam in $\mathcal{A}_c$ of a V-formation in $\mathcal{A}_c$ is to consider its amalgamated free product in $\mathcal{A}_l$ first, and then to extend its lattice order to a total one by the Szpilrajn extension theorem [41]. But there doesn’t exist the amalgamated free product in $\mathcal{A}_c$. This will make our main construction somewhat more involved.

4. DIRECT SYSTEMS AND DIRECT LIMITS

For what follows, it is sufficient to define these categorical notions solely for algebraic systems. A directed partially ordered set is a partially ordered set with the additional property that every pair of elements has an upper bound in the set.
Let $\mathcal{U}$ be a class of algebraic systems of a given type and let $\langle \alpha, \leq \rangle$ be a directed partially ordered set. Let $\{ A_i \in \mathcal{U} : i \in \alpha \}$ be a family from $\mathcal{U}$ along with a family of $\mathcal{U}$-homomorphisms $\varsigma_{i \to j} : A_i \to A_j$ for every $i, j \in \alpha$, $i \leq j$ with the following properties:

1. $\varsigma_{i \to i}$ is the identity of $A_i$, and
2. $\varsigma_{i \to k} = \varsigma_{j \to k} \circ \varsigma_{i \to j}$ for all $i \leq j \leq k$.

The collection $\langle A_i, \varsigma_{i \to j} \rangle$ is termed a direct system in $\mathcal{U}$ over $\alpha$, and the $\varsigma_{i \to j}$’s are termed the transitions of the direct system. The direct limit $\lim \rightarrow \biglim_{\alpha} A_i$ of $\langle A_i, \varsigma_{i \to j} \rangle$ (if exists) is an element in $\mathcal{U}$ together with canonical homomorphisms $\pi_i : A_i \to \lim \rightarrow \biglim_{\alpha} A_i$ such that

1. the universe of $\lim \rightarrow \biglim_{\alpha} A_i$ is the disjoint union of the $A_i$’s modulo the following equivalence relation: for $x_i \in A_i$ and $x_j \in A_j$, $x_i \sim x_j$ if and only if there is some $k \in \alpha$, $k \geq i, j$ such that $\varsigma_{i \to k}(x_i) = \varsigma_{j \to k}(x_j)$,
2. for $i \in \alpha$, the canonical function $\pi_i$ sends each element in $A_i$ to its equivalence class, and
3. the algebraic operations and the relations on $\lim \rightarrow \biglim_{\alpha} A_i$ are defined such that these maps become homomorphisms.

If the direct limit $\lim \rightarrow \biglim_{\alpha} A_i$ exists then it is unique and satisfies the following properties:

1. for all $i \leq j$, $\pi_i = \pi_j \circ \varsigma_{i \to j}$, and
2. if $B \in \mathcal{U}$ and for each $i, j \in \alpha$ with $i \leq j$, there is a homomorphism $\psi_i : A_i \to B$ such that $\psi_i = \psi_j \circ \varsigma_{i \to j}$, then there exists a unique homomorphism $\psi : \lim \rightarrow \biglim_{\alpha} A_i \to B$ such that for every $i \in \alpha$, $\psi_i = \psi \circ \pi_i$ holds (universal property). We shall refer to $\psi$ as the factorizing homomorphism of the family $\{ \psi_i : i \in \alpha \}$.

Sometimes we also write $\langle A_i, \varsigma_{i \to j} \rangle_\alpha$ or $\langle A_i, \varsigma_{i \to j} \rangle_{\langle \alpha, \leq \rangle}$ and $\lim_{\alpha} A_i$ or $\lim_{\alpha} A_i$ to emphasize the index set. The direct limit of $\langle A_i, \varsigma_{i \to j} \rangle_\beta$ coincides with the direct limit of $\langle A_i, \varsigma_{i \to j} \rangle_\alpha$ for any cofinal subset $\alpha$ of $\beta$.

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2Homomorphisms are understood in the general sense of preserving also the relations.
Proposition 4.1. Direct limits exist in \( \mathcal{F} \).

Proof. Let \((A_i, \leq_{i-j})\)\(_\alpha\) be a direct system of totally-ordered abelian groups with positive homomorphisms. The direct limit \(A\) exists in the class of partially ordered abelian groups with positive homomorphisms, as established in [20 Proposition 1.15]. Our objective is to demonstrate that the order on \(A\) is total. Let \(x \notin A^+\).

According to \([\text{Lim}2]\) there exists some \(i \in \alpha\) and \(y \in A_i\) such that \(x = \pi_i(y)\). Consequently, the positivity of \(\pi_i\) implies \(y \notin A_i^+\), as a contrapositive to \(x \notin A^+\).

It follows that \(y^{-1} \in A_i^+\), given the total order of \(A_i\). Therefore, we can conclude that \(x^{-1} = \pi_i(y)^{-1} = \pi_i(y^{-1}) \in A^+\), again leveraging the positivity of \(\pi_i\). \(\square\)

5. Partially ordered abelian groups

An application of Zorn’s lemma in Theorem 7.1 necessitates a basic understanding of partially ordered abelian groups. A partially ordered set (poset) consists of a nonempty set combined with a partial order. \(G = (G, \leq, t)\) is a partially ordered abelian group (a po-group) if \((G, \cdot, t)\) is an abelian group, \((G, \leq)\) is a poset, and \(xz \leq yz\) holds for all \(x \leq y\) and \(z \in G\). The positive cone of \(G\) denotes the set of elements greater than or equal to \(t\), while its strict positive cone encompasses all positive cone elements except \(t\). If \(\leq\) constitutes a partial order on the group reduct of \((G, \cdot, t)\), instead of stating “\(\leq\) is a partial order on \((G, \cdot, t)\) with positive cone \(P\)”, we will succinctly express it as “\(P\) is a partial order on \((G, \cdot, t)\)”, given that a partial order \(\leq\) is uniquely determined by the corresponding positive cone \(P\).

Proposition 5.1. For an abelian group \((G, \cdot, t)\),

- \(P \subseteq G\) is a partial order on \((G, \cdot, t)\) if and only if \(P \cap P^{-1} = \{t\}\) and \(PP \subseteq P\).
- \(Q \subseteq G\) is the strict positive cone of a partial order on \((G, \cdot, t)\) if and only if \(Q \cap Q^{-1} = \emptyset\) and \(QQ \subseteq Q\).

Proof. This result is well-known and can also be easily derived from [12 Theorem 2 on page 13]. \(\square\)

Lemma 5.2. If \(P_1\) and \(P_2\) are two partial orders of the abelian group \((G, \cdot, t)\), then

i. \(P_1 \cap P_2\) contains both \(P_1\) and \(P_2\).
ii. If \(P\) is a partial order on \((G, \cdot, t)\) which contains both \(P_1\) and \(P_2\) then \(P_1 \cap P_2 \subseteq P\).
iii. \(P_1 \cap P_2\) is a partial order on \((G, \cdot, t)\) if and only if \(P_1 \cap P_2^{-1} = \{t\}\).

Proof. We will use Proposition 5.1 directly, without citing any additional references.

i. \(P_1 = P_1 \{t\} \subseteq P_1 \cap P_2 \supseteq \{t\} \subseteq P_2 = P_2\).
ii. \(P_1 \cap P_2 \subseteq PP \subseteq P\).
iii. Assume \(P_1 \cap P_2^{-1} = \{t\}\). Then \(P_1 \cap P_2\) is a partial order on \((G, \cdot, t)\). Indeed,

a) \(P_1 \cap P_2^{-1}\) is a partial order on \((G, \cdot, t)\). Indeed,

b) \(t = tt = (tt)^{-1} \in (P_1 \cap P_2) \cap (P_1 \cap P_2)^{-1}\).

For \(a \in P_1 \cap P_2\), we have \(a = a_1 b_2 = a_1^{-1} b_2^{-1}\) where \(a_1, b_1 \in P_1\) and \(a_2, b_2 \in P_2\). Consequently, \(ab_1 a_2^{-1} = a_1 b_1 = a_2^{-1} b_2^{-1} \in P_1 \cap P_2^{-1}\), implying \(ab_1 a_2^{-1} = t\). Thus, \(a^{-1} = b_1 a_2^{-1} \in P_1 \cap P_2^{-1}\). This further leads to \(a^{-1} = t\), which in turn concludes \(a = t\).
Assume $P_1 P_2$ is a partial order on $(G, \cdot, t)$. Since $P_1$ and $P_2$ are partial orders of $(G, \cdot, t), t \in P_1 \cap P_2^{-1}$ holds. On the other hand, if $a \in P_1 \cap P_2^{-1}$ then $a = at \in P_1 P_2, P_2^{-1} P_1^{-1}$, that is, $a \in P_1 P_2 \cap (P_1 P_2)^{-1}$, hence $a = t$.

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6. A categorical isomorphism between $\mathcal{I}_{\text{oe}}^\ell$ and $\mathcal{B}_G$

This section sets the stage for investigating the amalgamation problem in $\mathcal{I}_{\text{oe}}^\ell$ in a novel way, as detailed in Section 8. We will cite a categorical isomorphism between the category $\mathcal{I}_{\text{oe}}^\ell$ of odd or even involutive FL$_e$-chains (equipped with normal FL-algebra homomorphisms) and the category $\mathcal{B}_G$ of bunches of layer groups, where objects in $\mathcal{B}_G$ are equipped with bunch homomorphisms. This result is established in [29]. We use this categorical isomorphism to facilitate the investigation of the amalgamation problem in $\mathcal{B}_G$ rather than in $\mathcal{I}_{\text{oe}}^\ell$, as the objects in $\mathcal{B}_G$ are better understood than those in $\mathcal{I}_{\text{oe}}^\ell$.

To begin, an original decomposition method, along with the related construction method, has been introduced in [28] for the class of odd or even involutive FL$_e$-chains. The primary approach has been to partition an algebra $X$ by the equivalence classes generated by its local unit function $x \mapsto x \mapsto x$ into a direct system of potentially simpler, “nicer” algebras, referred to as the layer algebras of $X$. These layer algebras are indexed by the positive idempotent elements of $X$, with transitions in the direct system defined by multiplication with a positive idempotent element. This approach is termed the layer algebra decomposition of $X$. Through a process involving Plonka sums and the directed lexicographic order established within this framework, the original algebra can be reconstructed. The impact of the layer algebra decomposition has soon extended beyond its initial application, as it found application in offering structural descriptions of various classes of residuated lattices, see [19, 24, 25, 30, 31, 32, 33, 42]. In these specific contexts, the layer algebras are at once much simpler algebras than the decomposed algebra. However, layer algebras offer only minor improvement over the original algebras in the setting of [28]. To address this additional challenge, we introduced the concept of layer group decomposition. It combines the layer algebra decomposition with a second phase that involves the construction and reconstruction of layer algebras into layer groups. Although the full construction is quite intricate (see [28, Theorem 8.1]), we will present a streamlined version in Theorem 6.6 that suffices for our purposes. This theorem establishes a one-to-one correspondence between odd or even involutive FL$_e$-chains and bunches of layer groups.

In a bunch of layer groups, there are three pairwise disjoint sets: $\kappa_o$, $\kappa_J$, and $\kappa_I$. Their union, denoted $\kappa$, is totally-ordered by $\leq_{\kappa}$ and has a least element $t$. Interestingly, the position of $t$ within these sets determines the type of the corresponding involutive FL$_e$-chain: the chain is odd if $t \in \kappa_o$, even with a non-idempotent falsum constant if $t \in \kappa_J$, or even with an idempotent falsum constant if $t \in \kappa_I$.

**Definition 6.1.** A collection $\mathcal{X} = \langle G, \cdot, H, \kappa_o, \kappa_J, \kappa_I, \leq_{\kappa} \rangle$ is termed a bunch of layer groups (or simply a bunch) if it satisfies the following conditions:

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*As mentioned above, in [28] two different concepts, bunches of layer algebras and bunches of layer groups have been considered. Hence, the use of “bunch” without further context may...*
For all $\upsilon \in \kappa$, $\langle \kappa, \leq_{\kappa} \rangle$ forms a direct system of abelian $o$-groups over the totally-ordered set $\kappa$. The set $\kappa$ has a least element $t$ and is partitioned into $\kappa_{o}$, $\kappa_{J}$, and $\kappa_{I}$ such that

$$(P) \quad \kappa_{o} \subseteq \{t\}.$$ 

**Properties of Layer Groups and Subgroups**

(L1) For $v \in \kappa_{J}$, $\hat{H}_{v}$ is an isomorphic copy of $H_{v} \leq G_{v}$, via the mapping $H_{v} \ni h \mapsto \hat{h} \in \hat{H}_{v}$. Furthermore, if $\kappa \ni u <_{\kappa} v$ then $\varsigma_{u \rightarrow v}$ maps $G_{u}$ into $H_{v}$.

(L2) For $u \in \kappa_{J}$, $G_{u}$ is discrete. Furthermore, if $u <_{\kappa} v \in \kappa$, then

$$\varsigma_{u \rightarrow v}(u) = \varsigma_{u \rightarrow v}(u_{\kappa_{u}}).$$

**Unit Element**: For all $u \in \kappa$, $u$ is the unit element of $G_{u}$.

**Disjointness**: The universes of all $G$'s and $\hat{H}$'s are pairwise disjoint.

Sometimes, for brevity, we will write $\langle G_{u}, \hat{H}_{u}, \varsigma_{u \rightarrow v} \rangle_{\kappa}$. The $G$'s and the $\hat{H}$'s are referred to as the layer groups and layer subgroups of $\mathcal{X}$, respectively. We refer to $\kappa_{o}$, $\kappa_{J}$, and $\kappa_{I}$ as the $o$-component, the $J$-component, and the $I$-component of the skeleton, respectively. We refer to $\langle G_{u}, \varsigma_{u \rightarrow v} \rangle_{\kappa}$ as the underlying direct system of $\mathcal{X}$, and we denote it by $\mathcal{X}_{d}$. We also say that $\mathcal{X}$ forms a bunch structure on its underlying direct system $\mathcal{X}_{d}$. Theorem 6.6 describes a bijection between the class $\mathcal{G}_{\mathcal{X}}$ and the class $\mathcal{B}_{\mathcal{X}}$ of bunches of layer groups. Because of this, if $X$ denotes the odd or even involutive FL$e$-algebra corresponding to the bunch $\mathcal{X}$, then we will freely interchange $\mathcal{X}$ and $X$ in these expressions. For instance, we will say that $G$'s and the $\hat{H}$'s are the layer groups and layer subgroups of $X$, $\langle \kappa, \leq_{\kappa} \rangle$ is the skeleton of $X$, and so on.

**Remark 6.2.** In (L1) and within $\mathcal{X}$, we have stored the isomorphic copies of subgroups associated with $G_{v}$ (for $v \in \kappa_{J}$) along with their corresponding isomorphisms. Consequently, the subgroups $H_{v}$ are uniquely determined. Whenever we mention an element $\hat{h} \in \hat{H}_{v}$, it inherently refers to the element $h \in H_{v}$, and the isomorphism $\cdot$ that maps $h$ to $\hat{h}$. To keep the notation simple, we omit indices on these isomorphisms. This method of defining layer groups incorporates slightly more detail than the approach in [28]. Previously, the structure $\langle G_{u}, H_{u}, \varsigma_{u \rightarrow v} \rangle_{\kappa}$ was utilized, and an isomorphic copy $\hat{H}_{v}$ of $H_{v}$ was introduced during the construction of an odd or even involutive FL$e$-algebra from a bunch of layer groups (refer to [28] footnote 19, p. 917). In contrast, our current framework directly incorporates the $\hat{H}_{v}$'s within $\mathcal{X}$. This enhancement allows us to establish a bijection, demonstrated by $\mathcal{X}(x_{\mathcal{X}}) = \mathcal{X}$ and $X_{\mathcal{X}} = X$, between the classes $\mathcal{C}_{\mathcal{X}}$ and $\mathcal{B}_{\mathcal{X}}$ as detailed in Theorem 6.6. This goes beyond merely stating that $\mathcal{X}(x_{\mathcal{X}}) = \mathcal{X}$ and that $X_{\mathcal{X}}$ is isomorphic to $X$, as was previously concluded in [28] Theorem 8.1.

**Remark 6.3.** If, in the structure $\langle G_{u}, \hat{H}_{u}, \varsigma_{u \rightarrow v} \rangle_{\langle \kappa_{o}, \kappa_{J}, \kappa_{I}, \leq_{\kappa} \rangle}$, all the conditions of Definition 6.1 are satisfied except for the requirement that the universes of all the $G$'s and $\hat{H}$'s are pairwise disjoint, this requirement can be fulfilled by applying the standard set-theoretic approach for constructing a disjoint union.

---

4In this paper we use the term ‘partition’ in a broader sense than usual. Specifically, we allow the involved subsets to be possibly empty, diverging from the traditional definition of a partition that requires all subsets to be nonempty.

5Throughout the paper $1_{u}$ denotes the neighborhood operation in $G_{u}$.
Definition 6.4. Consider two direct systems, $\mathcal{A} = \langle A_i, \xi_{i \rightarrow j} \rangle_{(\alpha, \leq \beta)}$ and $\mathcal{B} = \langle B_i, \xi_{i \rightarrow j} \rangle_{(\beta, \leq \beta)}$, belonging to the same class $\mathcal{U}$ of algebraic systems. A (direct system) homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is defined as a system of $\mathcal{U}$-homomorphisms $\varphi = \{ \varphi_i : A_i \rightarrow B_{\iota_0(i)} \mid i \in \alpha \}$, where

\begin{equation}
\iota_0 : \alpha \rightarrow \beta \text{ is an } o\text{-embedding}
\end{equation}

and for every $i, j \in \alpha$, $i \leq j$ the diagram in Fig. 3 commutes.

![Figure 3. Transitions commute with $\mathcal{U}$-homomorphisms](image)

We say that $\varphi$ is an embedding if for every $i \in \alpha$, $\varphi_i$ is an embedding. Throughout the paper, if $\alpha \subseteq \beta$ and $\leq_{\alpha}, \leq_{\beta}$, then we shall consider $\iota_0$ to denote the inclusion of $\alpha$ into $\beta$.

Every bunch has an underlying direct system. Therefore, in the following definition, we will define a bunch homomorphism as a family of mappings to maintain similarity with the definition of direct system homomorphisms. However, a bunch homomorphism can be investigated more simply if it is regarded as a single function. We will make use of this simplification already in its definition.

Definition 6.5. Let

\begin{equation}
\mathcal{A} = \langle G_u, \hat{H}_u, \varsigma_{u \rightarrow v} \rangle_{\kappa^u} \quad \text{and} \quad \mathcal{B} = \langle G_v, \hat{H}_v, \varsigma_{u \rightarrow v} \rangle_{\kappa^v}
\end{equation}

be bunches of layer groups. A bunch homomorphism

\begin{equation}
\varphi = \{ \varphi_u : u \in \kappa^u \}
\end{equation}

is a direct system homomorphism from $\langle G_u, \varsigma_{u \rightarrow v} \rangle_{\kappa^u(X)}$ to $\langle G_v, \varsigma_{u \rightarrow v} \rangle_{\kappa^v(Y)}$ with the exception of the condition specified in \((\text{Hom1})\), which is relaxed to

\begin{equation}
\varphi : \kappa^u \rightarrow \kappa^v \text{ is an } o\text{-homomorphism}\footnote{That is, an order preserving mapping.}
\end{equation}

while satisfying conditions (\text{Hom1})–(\text{Hom4})\footnote{As an alternative interpretation to viewing a bunch homomorphism as a family of homomorphisms, as in (6.2), it can also be interpreted as a single mapping. Recall that the universes of the layer groups of a bunch are pairwise disjoint. Let us denote: $G^X = \bigcup_{u \in \kappa^u} G^u$ and $G^Y = \bigcup_{u \in \kappa^v} G^u$. In this context we interpret $\varphi$ as a mapping: $\varphi : G^X \rightarrow G^Y$ defined by (6.4) $\varphi(x) = \varphi_u(x)$, where $x \in G_u$.}

\((\text{Hom1})\) $\varphi$ preserves the least element of the skeleton:

$\varphi(t^u) = t^v$,

and respects the partition:

$\varphi(\kappa^u_{\alpha}) \subseteq \kappa^v_{\alpha}$, $\varphi(\kappa^u_{\beta}) \subseteq \kappa^v_{\beta}$, $\varphi(\kappa^u_{\upsilon}) \subseteq \kappa^v_{\upsilon}$, $\varphi(\kappa^u_{\Upsilon}) \subseteq \kappa^v_{\Upsilon}$,
(Hom2) $\varphi$ preserves the subgroups and their complements:
if $u \in \kappa_{t}^\alpha$ and $\varphi(u) \in \kappa_{t}^{\alpha'}$ then
$\varphi_u(H_u) \subseteq H_{\varphi(u)}$ and $\varphi_u((G_u \setminus H_u) \subseteq G_{\varphi(u)} \setminus H_{\varphi(u)}$.

(Hom3) $\varphi$ respects the neighborhood operations

(a) if $u \in \kappa_{t}^\alpha$ and $\varphi(u) \in \kappa_{t}^{\alpha'}$ then for $x \in G_u$, $\varphi(x_{\downarrow u}) = \varphi(x)_{\downarrow \varphi(u)}$;
(b) if $u \in \kappa_{t}^\alpha$ and $\varphi(u) \in \kappa_{t}^{\alpha'}$ then for $x \in G_u$, $\varphi(x_{\uparrow u}) = \varphi(x)$.

(Hom4) $\varphi$ is partially injective: for $u, v \in \kappa_{t}^\alpha$, $x \in G_u, y \in G_v$,
if $\varphi(uv) \in \kappa_{t}^{\alpha'}$ and $\varsigma_{u \rightarrow uv}(y) \prec uv \varsigma_{u \rightarrow uv}(x) \in H_{uv}$ then
$\varphi_{uv}(\varsigma_{u \rightarrow uv}(y)) \prec \varphi_{uv}(\varsigma_{u \rightarrow uv}(x))$.

A bunch homomorphism $\varphi$ is referred to as a (bunch) embedding if $\varphi$ (viewed as a single mapping) is injective and $\varphi(\kappa_{t}^\alpha) \subseteq \kappa_{t}^{\alpha'}$ holds. If $\varphi$ is a bunch embedding, then evidently, for every $u \in \alpha$, $\varphi_u$ is an embedding.

Theorem 6.6 is a condensed version of [28, Theorem 8.1] tailored to the needs of the present paper.

**Theorem 6.6.**

(1) For every odd or even involutive FL-chain

$$X = (X, \leq, \cdot, \to, t, f)$$

one can assign a unique bunch of layer groups

$$\mathcal{X} = (G_u, H_u, \varsigma_{u \rightarrow v})_{(\kappa_\alpha, \kappa, \kappa, \leq, \leq)}$$

where

$$\kappa = \{x \rightarrow x : x \in X\} = \{u \geq t : u \text{ is idempotent}\} \text{ is ordered by } \leq,$$

$$\kappa_{t} \supseteq \{i \in \kappa \setminus \{t\} : u' \text{ is idempotent}\},$$

$$\kappa_{t} \supseteq \{i \in \kappa \setminus \{t\} : u' \text{ is not idempotent}\},$$

and for $u \in \kappa$,

$$G_u \text{ is a subset of } X,$$

and

$$u \text{ is the unit element of } G_u.$$

(2) For every bunch of layer groups $\mathcal{X} = (G_u, H_u, \varsigma_{u \rightarrow v})_{(\kappa_\alpha, \kappa, \kappa, \leq, \leq)}$ one can assign a unique odd or even involutive FL-chain $X_\mathcal{X} = (X, \leq, \cdot, \to, t, f)$ such that

$$X = \bigcup_{i \in \kappa} G_i \cup \bigcup_{i \in \kappa} H_i.$$

In addition, $X_\mathcal{X}$ is odd iff $t \in \kappa_\alpha$, even with a non-idempotent falsum iff $t \in \kappa_{t}$, and even with an idempotent falsum iff $t \in \kappa_{t}$.

(3) The previous two assignments describe a bijection between the classes $\mathcal{X}_{\leq}$ and $\mathcal{B}_\Theta$.

---

$\downarrow$ $u$ is the neighborhood operation of the $u$-layer group.
Figure 4. Illustration: an odd or even involutive FL_e-chain $X$ (left) and its group representation $X_r$ (right) consisting of four layers. We represent neither the partition of the skeleton nor the subgroups in such figures, only the layer groups and the transitions, that is, the underlying direct system of the bunch

Theorem 6.7.\( (1) \) Let $X = (Y, \leq^X, \cdot^X, \rightarrow^X, t^X, f^X)$ and $Y = (Y, \leq^Y, \cdot^Y, \rightarrow^Y, t^Y, f^Y)$ be odd or even involutive FL_e-chains. If $\varphi: X \rightarrow Y$ is a homomorphism from $X$ to $Y$ then

$$\varphi = \varphi|_G^X$$

is a (bunch) homomorphism from $X$ to $Y$.

(2) Let $\mathcal{X} = (G^X_u, \hat{H}^X_u, \zeta^X_u, \nu^X_u)_{u \in G^X}$ and $\mathcal{Y} = (G^Y_u, \hat{H}^Y_u, \zeta^Y_u, \nu^Y_u)_{u \in G^Y}$ be bunches of layer groups. If $\varphi$ is a (bunch) homomorphism from $\mathcal{X}$ to $\mathcal{Y}$ then $\varphi$ extends to a homomorphism $\varphi: \mathcal{X} \rightarrow \mathcal{Y}$ via

$$\varphi(x) = \begin{cases} 
\varphi(x) & \text{if } x \in G^X_u, \\
\varphi(a) & \text{if } x = \hat{a} \in \hat{H}^X_u \text{ and } \varphi(u) \in \kappa^Y_u, \\
\varphi(a) & \text{if } x = \hat{a} \in \hat{H}^X_u \text{ and } \varphi(u) \in \kappa^Y_u.
\end{cases}$$

(3) The previous two items describe a bijective correspondence between homomorphisms of odd or even involutive FL_e-chains and bunch homomorphisms of their corresponding bunch representations.

Theorem 6.8. The functor $\Upsilon: \mathcal{X} \rightarrow \mathcal{Y}$ from the category of odd or even involutive FL_e-chains with FL_e-algebra homomorphisms to the category of bunches of layer groups with bunch homomorphisms given by $\Upsilon X = X$ and $\Upsilon \varphi = \varphi$ (see Theorem 6.7(1) and 6.9) is a categorical isomorphism, its inverse functor $\Upsilon^{-1}$ is given by $\Upsilon^{-1} \mathcal{X} = X_r$ and $\Upsilon^{-1} \varphi = \varphi$ (see Theorem 6.7(2) and 6.10).

Remark 6.9. Notice that any homomorphism $\varphi: X \rightarrow Y$ induces, for every layer $u$ of $X$, a homomorphism $\varphi_u: G^X_u \rightarrow G^Y_u$ via (6.9) and (6.10). We will refer to it as the homomorphism induced on the $u$-layer by $\varphi$.

7. Embedding bunches

Let’s first delve into understanding how the injectivity of an FL_e-algebra homomorphism is reflected in its associated bunch homomorphism.

Lemma 7.1. Adopt the notation of Theorem 6.7. The following statements are equivalent.

(1) $\varphi$ is an embedding of $X$ into $Y$,
(2) $\varphi$ is an (bunch) embedding of $\mathcal{X}$ into $\mathcal{Y}$,
(3) $\varphi$ is an (direct system) embedding of $\mathcal{X}_d$ into $\mathcal{Y}_d$ which
(Emb1) preserves the least element and the partition\(^3\) of the skeleton,
(Emb2) preserves the subgroups and their complements, and
(Emb3) preserves the neighborhood operations: if \(u \in \kappa_j^x\) then for \(x \in G_u^x\),
\[
\varphi(x_{\downarrow u}) = \varphi(x)_{\downarrow \varphi(\iota)}.
\]

Proof. Establishing the equivalence \((1) \iff 2\) entails demonstrating, by definition, that:
\[
(7.1) \quad \varphi \text{ is injective if and only if } \varphi \text{ is injective and satisfies } \varphi(\kappa_j^x) \subseteq \kappa_j^y.
\]

Establishing the injectivity of \(\varphi\) is evident from \((6.9)\), given the injectivity of \(\varphi\). Suppose \(\varphi(\kappa_j^x) \subseteq \kappa_j^y\) does not hold. Then by \([\text{Hom1}]\) there exists \(u \in \kappa_j^x\) such that \(\varphi(u) \in \kappa_j^y\). According to \([\text{P}]\) \(\varphi(u) = t'^y\), and according to \([\text{Hom1}]\) \(\varphi(t'^x) = t'^x\). Since \(\varphi\) is injective, \(u = t'^x\) follows. Consequently, \(t'^x = u \in \kappa_j^x\) and \(t'^y = \varphi(t'^x) = \varphi(u) \in \kappa_j^y\). Therefore, according to Theorem \((6.0)2\), \(X\) is even and \(Y\) is odd. However, this contradicts the injectivity of \(\varphi\), as it implies that the two distinct constants of \(X\) are mapped to the “single” constant of \(Y\).

Regarding the converse, notice that the condition of the second row of \((6.10)\) cannot hold by assumption, since \(\kappa_0^x\) and \(\kappa_j^x\) are disjoint by Definition \((6.1)\). Additionally, it is clear that an element in the first row of \((6.10)\) cannot be equal to an element in the third row of \((6.10)\). This follows from the universes of all the \(G_u^y\)’s and \(\hat{H}_u^y\)’s being pairwise disjoint, as specified in Definition \((6.1)\). Therefore, the injectivity of \(\varphi\) implies the injectivity of \(\varphi\), as required.

The implication \((3) \Rightarrow (2)\) is evidently true since \([\text{Emb1}]\) and \([\text{Emb3}]\) are stronger conditions compared to \([\text{Hom1}]\) and \([\text{Hom3}]\) respectively. Additionally, \([\text{Hom4}]\) clearly holds if \(\varphi\) is injective.

Regarding the converse implication \((2) \Rightarrow (3)\), it’s noteworthy that bunch embeddings are direct system embeddings of their underlying direct systems, as per their definition. Given that \([\text{Emb2}]\) holds by assumption, and that \([\text{Hom3}]\) along with \([\text{Emb1}]\) clearly implies \([\text{Emb3}]\), it suffices to prove \([\text{Emb1}]\). Given that \([\text{Hom1}]\) and \(\varphi(\kappa_j^x) \subseteq \kappa_j^y\) are assumed, proving \([\text{Emb1}]\) amounts for demonstrating, that for every \(u \in \kappa_j^x\), \(\varphi(u) \subseteq \kappa_j^y\). Let \(u \in \kappa_j^x\). By \([\text{Hom1}]\) \(\varphi(u) \in \kappa_j^y \cup \kappa_0^y\). By contradiction, assume \(\varphi(u) \in \kappa_0^y\). Then, \(\kappa_0^y = \{t'^x\}\) follows from \([\text{P}]\) and consequently, \(Y\) is odd and \(\varphi(u) = t'^v\). By \([\text{Hom1}]\) \(\varphi(t'^x) = t'^x\) holds as well, and since \(\varphi\) is injective, \(u = t'^x\) follows. Referring to \((2) \Rightarrow (1)\), it follows that \(\varphi\) is also injective. Consequently, from the fact that \(Y\) is odd, it follows that \(X\) is also odd, contradicting \(t'^x = u \in \kappa_j^x\). \(\square\)

The main construction in the proof of Theorem \((8.3)\) requires a series of theorems, which are outlined in this section. Theorem \((7.3)\) explores the embeddability of direct systems over chains into larger chains. It concerns direct systems of algebras\(^9\) of any type, which include the usual algebraic homomorphisms that preserve operations, and are defined over chains. Additionally, we examine abelian \(\alpha\)-groups, which are equipped with positive homomorphisms that maintain both the operations and the ordering.

---

\(^9\)By preserving the partition we mean \(\varphi(\kappa_0^x) \subseteq \kappa_0^y\), \(\varphi(\kappa_j^x) \subseteq \kappa_j^y\), and \(\varphi(\kappa_j^x \cup \kappa_0^y) \subseteq \kappa_j^y\).

\(^10\)We use the term “algebra” to refer to algebraic systems without any relations.
Definition 7.2. Let $\mathcal{U}$ be a class of algebraic systems. Let $\alpha$ be a coinitial subset of the totally-ordered set $\langle \beta, \leq \rangle$, and $\mathcal{A}_\alpha = \langle A_i, \zeta_{i-j} \rangle_{\langle \alpha, \leq \rangle}$ be a direct system in $\mathcal{U}$. The canonical extension

$$\mathcal{A}_\alpha \to \beta = \langle \hat{A}_i, \hat{\zeta}_{i-j} \rangle_{\langle \beta, \leq \rangle}$$

of $\mathcal{A}_\alpha$ over $\beta$ is defined as follows:

(CEx1) For $j \in \beta$ let

$$\hat{A}_j = \lim_{\overset{\rightarrow}{i \in \alpha, i \leq j}} A_i.$$

(CEx2) Let $i, j \in \beta$ such that $i \leq j$. If $i \in \alpha$ then define $\pi_{i-j}$ as the canonical homomorphism from $A_i$ to $\hat{A}_j$. If $j \in \alpha$ then define $\phi_{i-j}$ as the factorizing homomorphism from $\hat{A}_i$ to $A_j$ of the family of homomorphisms $\{\zeta_{s-j} : s \in \alpha, s \leq i \}$. Finally, let

$$\zeta_{i-j} = \begin{cases} \pi_{w-j} \circ \phi_{i-w} & \text{if } \exists w \in \alpha \text{ such that } i \leq w \leq j \\ \text{id}_{i-j} & \text{if } \nexists w \in \alpha \text{ such that } i \leq w \leq j \end{cases}.$$  

Denote

$$\pi = \{ \pi_{i-i} : i \in \alpha \}.$$

Further, let $\mathcal{X} = \langle A_i, \hat{H}_i, \zeta_{i-j} \rangle_{\langle \alpha, \phi_{i-j}, \leq \rangle}$ be a bunch structure on $\mathcal{A}_\alpha$ with empty $\alpha_j$ component. The canonical extension $\mathcal{X}_\alpha \to \beta$ of $\mathcal{X}$ over $\beta$ is defined as follows:

$$\mathcal{X}_\alpha \to \beta = \langle \hat{A}_i, \hat{\zeta}_{i-j} \rangle_{\langle \beta, \phi_{i-j}, \leq \rangle},$$

where

$$\beta_0 = \alpha_0 \quad \text{and} \quad \beta_I = \beta \setminus \beta_0,$$

and for $i \in \beta_I$, $\hat{L}_i$ is an isomorphic copy of $L_i$ such that the universes of all the $\hat{A}_i$’s and $\hat{L}_i$’s are pairwise disjoint.

Theorem 7.3. Let $\mathcal{U}$ be a class of algebras. Let $\alpha$ be a coinitial subset of the totally-ordered set $\langle \beta, \leq \rangle$, let $\mathcal{A}_\alpha = \langle A_i, \zeta_{i-j} \rangle_{\langle \alpha, \leq \rangle}$ be a direct systems in $\mathcal{U}$. Then

11 The construction works mutatis mutandis in the more general setting where $\langle \alpha, \leq \rangle$ is order-embedded into the totally-ordered set $\langle \beta, \leq \rangle$ such that the image of $\alpha$ under this embedding is a coinitial subset of $\beta$.

12 If $\nexists w \in \alpha$ such that $i \leq w \leq j$, then the sets $\{k \in \alpha, k \leq i\}$ and $\{k \in \alpha, k \leq j\}$ are identical, hence $A_i = A_j$. This allows us to consider the identity mapping $\text{id}_{i-j}$ from $A_i$ to $A_j$. 

(1) (a) $\mathcal{A}_{\alpha \to \beta}$ is a the direct system in $\mathbf{U}$ over $\beta$, in which $\mathcal{A}_{\alpha}$ embeds via the mapping $\pi$.
(b) The restriction of $\mathcal{A}_{\alpha \to \beta}$ to $\alpha$ is isomorphic to $\mathcal{A}_{\alpha}$.
(c) $\mathcal{A}_{\alpha \to \beta}$ is isomorphic to $\mathcal{A}_{\alpha}$.

(2) If the $\mathcal{A}_i$'s are abelian $\alpha$-groups and the $\varsigma_{i \to j}$'s are positive homomorphisms then every $\varsigma_{i \to j}$ and $\pi_{i \to j}$ are also positive.

(3) For every bunch structure $\mathcal{X}$ on $\mathcal{A}_\alpha$ with empty $J$-component,
(a) $\mathcal{X}_{\alpha \to \beta}$ is a bunch structure over $\mathcal{A}_{\alpha \to \beta}$ (also with empty $J$-component) into which $\mathcal{X}$ embeds via $\pi$.
(b) $\pi$ preserves the $I$-component of the skeleton.

Proof. Since $\alpha$ is coinitial in $\beta$, the set $\{i \in \alpha : i \leq j\}$ is nonempty for any $j \in \beta$, and the direct limit exists by Proposition 4.1. Hence $A_j$ is well-defined.

Next we prove that the $\varsigma_{i \to j}$'s are well-defined, that is, their definition in (7.2) does not depend on the particular choice of $w$. By (Lim5) it holds true that

$$\phi_{j \to k} \circ \pi_{i \to j} = \varsigma_{i \to k} \quad (i \leq j \leq k, i, k \in \alpha \text{ and } j \in \beta).$$

Let $i \leq w < v \leq j, v, w \in \alpha$. There exists $\alpha \ni k \leq i$ since $\alpha$ is coinitial in $\beta$.

It holds true that $\pi_{w \to j} \circ \phi_{i \to w} \circ \pi_{k \to i}$ $\quad (\text{Lim4})$

$$\pi_{w \to j} \circ \varsigma_{k \to v} = \pi_{w \to j} \circ \phi_{i \to v} \circ \pi_{k \to i} \quad (\text{Lim4}).$$

Therefore, $\pi_{w \to j} \circ \phi_{i \to w}$ coincides with $\pi_{v \to j} \circ \phi_{i \to v}$ on the range of $\pi_{k \to i}$. Since by (Lim2) the union of these ranges over all $\alpha \ni k \leq i$ is $B_j$, $\pi_{w \to j} \circ \phi_{i \to w} = \pi_{v \to j} \circ \phi_{i \to v}$ follows. Therefore the definition of $\varsigma_{i \to j}$ does not depend on the choice of $w$.

(1) Evidently, the $\mathcal{A}_j$'s are algebras in $\mathbf{U}$, and the $\varsigma_{i \to j}$'s are $\mathbf{U}$-homomorphisms.

To prove (Dir2), let $i \leq j \leq k, i, j, k \in \beta$.

- If there exists $v, w \in \alpha$ such that $i \leq w \leq j \leq v \leq k$ then $\varsigma_{i \to j} \circ \pi_{w \to k} \circ \phi_{i \to v} \circ \pi_{w \to j} \circ \phi_{i \to w}$.

$$\phi_{j \to k} \circ \pi_{i \to j} = \pi_{v \to j} \circ \phi_{i \to v} \circ \pi_{w \to j} \circ \phi_{i \to w} = \pi_{v \to k} \circ \varsigma_{i \to k} \circ \phi_{i \to w} \quad (\text{Lim4}).$$

- If there exists $w \in \alpha$ such that $i \leq w \leq j$ and there does not exist $v \in \alpha$ such that $j \leq v \leq k$ then since $A_j = A_k$ and the direct limit is unique it follows that $id_{j \to k} \circ \pi_{w \to j} = \pi_{w \to k}$, hence $\varsigma_{j \to k} \circ \pi_{i \to j} = \pi_{w \to k} \circ \phi_{i \to w}$.

$$\phi_{j \to k} \circ \pi_{i \to j} = \pi_{j \to k} \circ \phi_{i \to w} \quad (\text{Lim4}).$$

- If there does not exist $v, w \in \alpha$ such that $i \leq w \leq j$ and there exists $v \in \alpha$ such that $j \leq v \leq k$ then since $A_i = A_j$ and $\phi$ is unique in (Lim5) it follows that $\phi_{j \to v} \circ id_{i \to j} = \phi_{i \to v}$, hence $\varsigma_{j \to k} \circ \pi_{i \to j} = \pi_{v \to k} \circ \phi_{i \to w}$.

$$\phi_{j \to k} \circ \pi_{i \to j} = \pi_{v \to k} \circ \phi_{i \to v} \quad (\text{Lim4}).$$

- If there does not exist $v, w \in \alpha$ such that $i \leq w \leq j \leq v \leq k$ then $\varsigma_{j \to k} \circ \pi_{i \to j} = id_{j \to k} \circ id_{i \to j} = id_{i \to k}$.

These confirm (Dir2).
If \( i \in \alpha \) then \( \{ i \} \) is cofinal in \( \{ k \in \alpha : k \leq i \} \), thus

\[
\tilde{A}_i \cong A_i.
\]

Therefore, by [Lim2] for \( i \in \alpha \),

\[
\pi_{i \to i} \quad \text{is an isomorphism.}
\]

Since, \( \pi_{i \to i} \) is onto,

\[
\phi_{i \to j} \circ \pi_{i \to i} = \pi_{j \to i} = \text{id}_{A_j},
\]

implies that \( \phi_{i \to j} \) is also the right-inverse of \( \pi_{i \to j} \), and hence

\[
\zeta_{i \to j} = \pi_{i \to j} \circ \phi_{i \to j} = \text{id}_{A_i}.
\]

This confirms [Dir1] as if \( i \in \beta \setminus \alpha \), then \( \zeta_{i \to j} \) holds, too. We have just demonstrated that \( A_{\alpha \to \beta} \) is the direct system in \( \mathcal{U} \) over \( \beta \). In the light of this, (7.7) and (7.8) confirm both (1b) and (1c). Finally, the rest of (1a) is verified: if \( i, j \in \alpha \), then

\[
\zeta_{i \to j} \circ \pi_{i \to j} = \phi_{j \to i} \circ \phi_{i \to j} \circ \pi_{i \to j} = \pi_{j \to j} \circ \zeta_{i \to j}.
\]

(2) Every \( \zeta_{i \to j} \) is positive, as are all the mappings in (7.2). Referring to from (7.7) and (7.8), the \( \pi_{i \to j} \) mappings are evidently positive as well.

(3) Denote \( \mathcal{X} = (A_i, H_u, \zeta_{i \to j}(\alpha, \emptyset, \alpha, \leq_\alpha) \) and consider its canonical extension \( \mathcal{X}_{\alpha \to \beta} \). We will verify that \( \mathcal{X}_{\alpha \to \beta} \) is indeed a bunch.

- [P] holds for \( \mathcal{X} \), and it implies by (7.3) that either

\[
\alpha_0 = \beta_0 = \{ t \} \text{ or } \alpha_0 = \beta_0 = \emptyset.
\]

Since \( \alpha_0 = \emptyset \), \( \alpha_0 \) and \( \alpha_f \) partitions \( \alpha \). Hence it follows from (7.3) that

\[
\alpha_f = \alpha \setminus \{ t \} \text{ and } \beta_f = \beta \setminus \{ t \} \text{ or } \alpha_f = \alpha \text{ and } \beta_f = \beta,
\]

respectively. Therefore, \( \beta_0 \cup \beta_f = \beta \) and \( \beta_0 \cap \beta_f = \emptyset \).

- Because \( \mathcal{X} \) is a bunch structure on \( \mathcal{A} \), \( \alpha \) possesses a least element, denoted by \( t \). Since \( \alpha \) is coinitial in \( \beta \), it naturally implies that \( t \) serves as the least element of \( \beta \) as well.

Since \( \mathcal{A}_{\alpha \to \beta} \) is the direct system of abelian \( o \)-groups over \( \beta \) by assumption and (1a), these confirm the conditions under the item Underlying Direct System in Definition 6.1.

- Let \( u \in \alpha \). Due to [Lim2] and the isomorphism explained in (7.7), the unit element of \( A_u \) is \( [u] \). Given that \( A_{\alpha \to \beta} \) forms a direct system in \( \mathcal{U} \) over \( \tilde{\beta} \) (see item (1a)), the transitions of this direct system map unit elements to unit elements. Therefore, for \( u \in \beta \setminus \alpha \), \( \zeta_{i \to n}(\{ i \}) \) for some \( (\text{and hence for any, cf. Dir2}) i \in \alpha \) with \( i < u \) (such an \( i \) exists since \( \alpha \) is coinitial in \( \beta \)) is the unit element of \( A_u \). This verifies the Unit Element condition in Definition 6.1.

- Since \( \mathcal{X} \) satisfies [P], (7.3) ensures that \( \mathcal{X}_{\alpha \to \beta} \) also satisfies [P].

- As for (L1), for \( i \in \alpha_f \), \( \pi_{i \to i} \) is an isomorphism by (7.8). Thus from \( H_i \leq \tilde{A}_i \), it follows, as per (7.5), that \( L_i \leq \tilde{A}_i \). For \( i \in \beta_f \setminus \alpha_f \), \( L_i \leq \tilde{A}_i \) is evident as well.

Let \( \beta \ni j \leq i \in \beta_f \). If \( i \in \beta_f \setminus \alpha_f \), then \( \zeta_{j \to i}(A_j) \subseteq A_i = L_i \).
Definition 7.4. Let $\mathcal{A}$ be a coinitial subset of the totally-ordered set $\langle \beta, \leq \rangle$, and consider two direct systems in $\mathcal{U}$: $\mathcal{A}_\alpha = \langle A_i, \xi_{i-j} \rangle_{i \in \alpha}$ and $\mathcal{B}_\beta = \langle B_i, \xi_{i-j} \rangle_{i \in \beta}$. Given a homomorphism $\varphi = \{ \varphi_i : A_i \rightarrow B_i \mid i \in \alpha \}$ from $\mathcal{A}_\alpha$ to $\mathcal{B}_\beta$, we define

$$\varphi_{\alpha \rightarrow \beta} = \{ \psi_i : \tilde{A}_i \rightarrow B_i \mid i \in \beta \}$$

by setting $\psi_i$ to be the factorizing homomorphism of the family

$$\Psi_i = \{ \psi_{k-i} : \alpha \ni k \leq i \}$$

where $\psi_{k-i} : A_k \rightarrow B_i$ is given by

$$\psi_{k-i} = \xi_{k-i} \circ \varphi_k.$$  

(7.13)

Theorem 7.5. Let $\mathcal{U}$ be a class of algebras. Let $\alpha$ be a coinitial subset of the totally-ordered set $\langle \beta, \leq \rangle$ and consider two direct systems in $\mathcal{U}$: $\mathcal{A}_\alpha = \langle A_i, \xi_{i-j} \rangle_{i \in \alpha}$ and $\mathcal{B}_\beta = \langle B_i, \xi_{i-j} \rangle_{i \in \beta}$. Then

1. (a) Every homomorphism $\varphi : \mathcal{A}_\alpha \rightarrow \mathcal{B}_\beta$, factors through $\pi : \mathcal{A}_\alpha \rightarrow \mathcal{A}_\alpha \rightarrow \mathcal{B}_\beta$. In more detail, for every homomorphism $\varphi : \mathcal{A}_\alpha \rightarrow \mathcal{B}_\beta$, $\varphi_{\alpha \rightarrow \beta}$ is a homomorphism from $\mathcal{A}_\alpha \rightarrow \mathcal{B}_\beta$ such that

$$\varphi = \varphi_{\alpha \rightarrow \beta} \circ \pi.$$  

(7.14)

(b) Additionally, if for some $j \in \alpha$, $\varphi_j$ is an embedding, then $\psi_j$ is also an embedding.
(2) If the $A_i$’s are abelian $\alpha$-groups and the $\xi_{i\to j}$’s are positive homomorphisms, then every $\psi_i$ is positive as well.

(3) If $\mathcal{X} = \langle A_i, H_u, \xi_{i\to j} \rangle_{\langle \alpha, 0, 0, 0, \leq \alpha \rangle}$ is a bunch structure on $\mathcal{A}'_\alpha$, $\mathcal{Y} = \langle B_i, L_u, \xi_{i\to j} \rangle_{\langle \beta, 0, 0, 0, \leq \beta \rangle}$ is a bunch structure on $\mathcal{B}_\beta$, and $\varphi : \mathcal{X} \to \mathcal{Y}$ is a bunch homomorphism, then $\varphi_{\alpha\to\beta}$ satisfies Hom2 at indices from $\alpha$. More formally, if $\gamma \in \alpha$ then $\psi_\gamma$ maps $H_u$ to $L_u$, and $A_{\gamma} \setminus H_u$ to $B_{\gamma} \setminus L_u$.

**Proof.** Let $\varphi = \{ \varphi_i : A_i \to B_i \mid i \in \alpha \}$. For $i \in \beta$, the set $\Psi_i$ is nonempty since $\alpha$ is coinitial in $\beta$, hence for $i \in \beta$, there exists $\alpha \ni k \leq i$. The condition of (Lim5) is satisfied since for $j, l \in \alpha$, $j \leq l \leq i$:

$$\psi_{i\to j} \circ \xi_{j\to l} = \xi_{i\to l} \circ \varphi_j \circ \xi_{j\to l} = \xi_{i\to l} \circ \xi_{j\to l} \circ \varphi_j = \xi_{i\to l} \circ \varphi_j \circ \xi_{j\to l} = \psi_{j\to i}. \tag{7.13}$$

Therefore, by (Lim5) it holds true for every $\alpha \ni k \leq i \in \beta$ that

$$\psi_i \circ \pi_{k\to i} = \psi_{k\to i}. \tag{7.15}$$

Setting $k = i$ confirms (Dir1): $\psi_k \circ \pi_{k\to k} = \psi_k \circ \pi_{k\to k} = \xi_{k\to k} \circ \varphi_k = \varphi_k$. To prove that $\varphi_{\alpha\to\beta}$ is a homomorphism, we proceed as follows. By definition, the type of $\psi_i$ is $A_i \to B_i$, hence the condition in (dir1) is clearly satisfied. It remains to prove that the $\psi_i$’s commute with the transitions. For any $k$ such that $\alpha \ni k \leq i \in \beta$,

$$\psi_{k\to i} \circ \xi_{i\to j} = \xi_{k\to j} \circ \varphi_k = \xi_{k\to j} \circ \varphi_k = \xi_{k\to j} \circ \psi_{k\to i} \circ \pi_{k\to k}, \tag{7.16}$$

holds true, that is, $\psi_i \circ \xi_{k\to i} = \xi_{k\to i} \circ \psi_k$ on the range of $\pi_{k\to k}$. Since $\pi_{k\to k}$ is an isomorphism,

$$\psi_i \circ \xi_{k\to i} = \xi_{k\to i} \circ \psi_k \text{ holds for } \alpha \ni k \leq i \in \beta.\tag{7.16}$$

Now let $i, j \in \beta$, $i \leq j$. Let $k \in \alpha$ be arbitrary such that $k \leq i \leq j$; such $k$ exists since $\alpha$ is coinitial in $\beta$. It holds true that $\xi_{i\to j} \circ \psi_i \circ \xi_{k\to i} = \xi_{i\to j} \circ \xi_{k\to i} \circ \psi_k$ hence $\xi_{i\to j} \circ \psi_i \circ \xi_{k\to i}$, for every $k \leq i$. Since by (Lim2) the union of these ranges over all $\alpha \ni k \leq i \in B_i$,

$$\xi_{i\to j} \circ \psi_i = \psi_j \circ \xi_{i\to j} \circ \xi_{k\to i} \text{ holds for all } i, j \in \beta, i \leq j. \tag{7.16}$$

This proves claim $\text{La}$.

Regarding claim $\text{Lb}$ for every $k \in \alpha$,

$$\psi_k = \varphi_k \circ \phi_{k\to k} \tag{7.17}$$
follows from (7.15) by (7.10). Hence, \( \psi_k \) is injective if \( \varphi_k \) is injective, since \( \phi_{k\to k} \) is injective by (7.9).

Regarding claim 2, assume \( \mathcal{U} = \mathfrak{A}^c \). A factorizing homomorphims in \( \mathfrak{A}^c \) is an \( \mathfrak{A}^c \)-homomorphism, so for \( j \in \beta \), \( \varphi_j \) is positive, too. Hence \( \varphi_{\alpha\to\beta} \) is an \( \mathfrak{A}^c \)-homomorphism, as stated.

Claim 3 is evident from (7.7)–(7.9) and (7.17). \( \square \)

**Theorem 7.6.** For any direct system \( \langle L_u, \varsigma_u \to v \rangle_\kappa \) of torsion-free partially ordered abelian groups over an arbitrary chain \( \kappa \), there exists a direct system \( \langle \hat{G}_u, \varsigma_u \to v \rangle_\kappa \) of abelian \( \alpha \)-groups. In this system the abelian group reducts of the \( L_u \)'s and the transitions remain unchanged, while, for every \( u \in \kappa \), the ordering relation of \( \hat{G}_u \) is an extension of the ordering relation of \( L_u \).

**Proof.** Let \( \mathcal{P} = \{ A_u : i \in \kappa \} \), where \( A_u \) is a partial order that extends the partial order on \( L_u \) such that \( \langle L_u, \varsigma_u \to v \rangle_\kappa \), modified by replacing the ordering of \( L_u \) with \( A_u \) for every \( i \in \kappa \), forms a direct system of abelian po-groups. \( \mathcal{P} \) is nonempty since the system of orderings of the \( L_u \)'s is in \( \mathcal{P} \). For \( p, q \in \mathcal{P} \), \( p = \langle A_u : i \in \kappa \rangle \), \( q = \langle B_u : i \in \kappa \rangle \), we set \( p \preceq q \) if for every \( i \in \kappa \), \( A_u \subseteq B_u \). Then \( \preceq \) is a partial ordering of \( \mathcal{P} \), and since the union of chains of partial orders (ordered by inclusion) is a partial order, too, the union of any chain in \( \langle \mathcal{P}, \preceq \rangle \) is in \( \mathcal{P} \). It follows from Zorn’s lemma that \( \langle \mathcal{P}, \preceq \rangle \) has a greatest element

\[ m = \{ \leq_i : i \in \kappa \}. \]

For every \( u \in \kappa \), we denote the positive cone and strict positive cone of \( \leq_u \) by

\[ P_u \text{ and } Q_u, \]

respectively. For \( u \in \kappa \), let us replace the ordering of \( L_u \) by \( \leq_u \), and denote the resulting direct system by

\[ \langle G_u, \varsigma_u \to v \rangle_\kappa. \]

By definition of \( \mathcal{P} \), \( \langle G_u, \varsigma_u \to v \rangle_\kappa \) is a direct system of abelian po-groups. We claim that \( \langle G_u, \varsigma_u \to v \rangle_\kappa \) is, in fact, a direct system of abelian \( \alpha \)-groups. Its proof amounts to proving that for every \( i \in \kappa \), \( \leq_i \) is a total order on \( G_i \). Assume, for reductio ad absurdum, that

there exists \( s \in \kappa \) such that \( \leq_s \) is not a total order.

Our plan is to construct an element of \( \mathcal{P} \) that is larger than \( m \), thus obtaining a contradiction. To begin, we show that in the direct system \( \langle G_u, \varsigma_u \to v \rangle_\kappa \) the preimage of any strictly positive element is strictly positive as well. More formally,

- **Claim 1.** For \( u, v \in \kappa \), \( u \leq_v v \) it holds true that

\[ (\varsigma_u \to v)^{-1}(Q_v) \subseteq Q_u. \]

For \( i \in \kappa \) let

\[ \tilde{Q}_u = \{ x_i \in G_u : \varsigma_u \to v(x_u) \in Q_v \text{ for some } \kappa \ni v \geq \kappa u \}, \]

\[ \tilde{P}_u = \tilde{Q}_u \cup \{ t_u \}. \]

Notice that for all \( u \in \kappa \), since the case \( v = u \) is included in the definition of \( \tilde{Q}_u \), it holds true that

\[ (7.18) \quad P_u \subseteq \tilde{P}_u. \]
If, for \( i \in \kappa \), \( \tilde{P}_u \) renders the group reduct of \( G_u \) a po-group, and the \( \varsigma_{u \rightarrow v} \)'s are positive with respect to the \( \tilde{P}_u \)'s (i.e., for \( u, v \in \kappa, u \leq \kappa v \) implies that \( \varsigma_{u \rightarrow v} \) maps \( \tilde{P}_u \) into \( \tilde{P}_v \)), then, due to (7.18), the group reducts of \( G_u \)'s equipped with the \( \tilde{P}_u \)'s and the \( \varsigma_{u \rightarrow v} \)'s form an element of \( \mathcal{P} \). Therefore, due to the maximality of \( m \), it must be the case that \( P_u = \tilde{P}_u \) for every \( u \in \kappa \). Consequently, \( Q_u = \tilde{Q}_u \) follows, proving that the preimage of any strictly positive element is also strictly positive, confirming Claim 1. Therefore, it is sufficient to prove the following two points.

a) For \( i \in \kappa \), \( \tilde{P}_u \) renders the group reduct of \( G_u \) a po-group. According to Proposition 5.1, it suffices to verify the following two points:

1) \( \tilde{Q}_u \cap \tilde{Q}_u^{-1} = \emptyset \). Assume \( a \in \tilde{Q}_u \cap \tilde{Q}_u^{-1} \). Then \( a, a^{-1} \in \tilde{Q}_u \). There exist \( j, k \geq \kappa u \) such that \( \varsigma_{u \rightarrow j} (a) \) and \( \varsigma_{u \rightarrow k} (a^{-1}) \) are strictly positive (in their respective algebras). Without loss of generality we may assume \( j \leq \kappa k \). It follows that \( \varsigma_{u \rightarrow k} (a) = \varsigma_{j \rightarrow k} (\varsigma_{u \rightarrow j} (a)) \) is positive since \( \varsigma_{u \rightarrow j} (a) \) is positive and \( \varsigma_{j \rightarrow k} \) is a positive homomorphism. Since \( \varsigma_{j \rightarrow k} (a) \) is positive and \( \varsigma_{j \rightarrow k} (a^{-1}) \) is strictly positive, \( \varsigma_{j \rightarrow k} (a^{-1}) = \varsigma_{j \rightarrow k} (a) \varsigma_{j \rightarrow k} (a^{-1}) \) is strictly positive, a contradiction, since unit elements are mapped into unit elements.

2) \( \tilde{Q}_u Q_u \subseteq \tilde{Q}_u \). Let \( a, b \in \tilde{Q}_u \). Then there exist \( j, k \geq \kappa u \) such that \( \varsigma_{u \rightarrow j} (a) \) and \( \varsigma_{u \rightarrow k} (b) \) are strictly positive. Without loss of generality we may assume \( j \leq \kappa k \). It follows that \( \varsigma_{u \rightarrow k} (a) = \varsigma_{u \rightarrow k} (\varsigma_{u \rightarrow j} (a)) \) is positive since \( \varsigma_{u \rightarrow j} (a) \) is positive and \( \varsigma_{u \rightarrow k} \) is a positive homomorphism. Since \( \varsigma_{u \rightarrow k} (a) \) is positive and \( \varsigma_{u \rightarrow k} (b) \) is strictly positive, \( \varsigma_{u \rightarrow k} (ab) = \varsigma_{u \rightarrow k} (a) \varsigma_{u \rightarrow k} (b) \) is strictly positive, ensuring \( ab \in \tilde{Q}_u \).

b) Regarding positivity, we assert that if \( x_i \in \tilde{P}_u \) then \( \varsigma_{u \rightarrow j} (x_i) \in \tilde{P}_v \) holds for \( j \geq \kappa u \). Indeed, given the obvious case for \( j = u \), we can proceed by assuming \( j > \kappa u \). Since \( x_i \in \tilde{P}_u \), either \( x_i = t_u \) holds, in which case the statement is obviously true, or there exists \( k \geq \kappa u \) such that \( \varsigma_{u \rightarrow k} (x_i) \in \tilde{Q}_k \). If \( k \leq \kappa j \) then \( \varsigma_{u \rightarrow j} (x_i) = \varsigma_{k \rightarrow j} (\varsigma_{u \rightarrow k} (x_i)) \in \tilde{Q}_k \subseteq \tilde{Q}_j \) follows, given that \( \langle G_u, \varsigma_{u \rightarrow j} \rangle \subseteq \tilde{P}_j \) is a direct system and \( \tilde{P}_j \subseteq \tilde{P}_j \). If \( k > \kappa j \) then \( \varsigma_{j \rightarrow k} (\varsigma_{u \rightarrow j} (x_i)) = \varsigma_{u \rightarrow k} (x_i) \in \tilde{Q}_k \) implies \( \varsigma_{u \rightarrow j} (x_i) \in \tilde{Q}_j \subseteq \tilde{P}_j \).

The proof of Claim 1 is complete.

Our strategy is to construct a direct system of partially ordered abelian groups by extending some of the orderings in \( \langle G_u, \varsigma_{u \rightarrow v} \rangle \) (refer to (7.19)). We will then use this system to construct an element in \( \mathcal{P} \) that is larger than \( m \) (see (7.20)).

By Corollary 13 on page 39 any partial order on an abelian group can be extended to a total order on the abelian group if and only if the abelian group is torsion-free. Since \( G_u \) is torsion-free by hypothesis, there exists an extension

\[
\tilde{P}_s
\]
of \( P_s \) which makes the group reduct of \( G_u \) an abelian o-group. For \( u \in \kappa \) let

\[
(7.19)\quad \tilde{P}_u = \begin{cases} 
\varsigma_{s \rightarrow u} (\tilde{P}_s) & \text{if } u \geq \kappa s \\
\tilde{P}_u & \text{if } u < \kappa s
\end{cases}
\]

Then the group reducts of the \( G_u \)'s equipped with the \( \tilde{P}_u \)'s form a direct system of partially ordered abelian groups. Indeed, we claim that

a) for \( i \in \kappa \), \( \tilde{P}_u \) renders the group reduct of \( G_u \) a po-group. It suffices to verify it for \( u \geq \kappa s \).
a) Let \( a \in \bar{P}_u \cap (\bar{P}_u)^{-1} \). Then \( \varsigma_{s \rightarrow u}(a_s) = a \) and \( \varsigma_{s \rightarrow u}(b_s) = a^{-1} \), for some \( a_s, b_s \in \bar{P}_s \). Now \( \varsigma_{s \rightarrow u}(b_s)^{-1} = a \) follows, hence \( b_s^{-1} \in \bar{P}_s \), too. Since \( b_s, b_s^{-1} \in \bar{P}_s \) and \( \bar{P}_s \) is a positive cone, it holds true that \( b_s = t_s \). Hence \( a^{-1} = t_u \) and therefore, \( a = t_u \).

a2) Let \( a, b \in \bar{P}_u \). Then \( \varsigma_{s \rightarrow u}(a_s) = a \) and \( \varsigma_{s \rightarrow u}(b_s) = b \), for some \( a_s, b_s \in \bar{P}_s \). Since \( \bar{P}_s \) is a positive cone, it holds true that \( a_s b_s \in \bar{P}_s \) and hence \( ab = \varsigma_{s \rightarrow u}(a_s)\varsigma_{s \rightarrow u}(b_s) = \varsigma_{s \rightarrow u}(a_s b_s) \in \bar{P}_u \).

b) We claim that the \( \varsigma_{u \rightarrow v}'s \) are positive with respect to the \( \bar{P}_u \)'s.

b1) If \( u, v \not< s \) then \( \varsigma_{u \rightarrow v} \) is positive since \( \bar{P}_u = P_u \) and \( \bar{P}_v = P_v \).

b2) Assume \( u \not< s \) and \( v \not< s \). Since \( \bar{P}_s \supseteq P_s \), \( \varsigma_{u \rightarrow s} \) is positive. Hence we may assume \( v > s \). Because of the definition of \( \bar{P}_v \) in (7.19), \( \varsigma_{u \rightarrow v} \) is positive, too. Therefore, \( \varsigma_{u \rightarrow v} = \varsigma_{u \rightarrow v} \circ \varsigma_{u \rightarrow s} \), is positive.

b3) Assume \( u \not< s \). Let \( a \in \bar{P}_u \). There exists \( x_s \in \bar{P}_s \) such that \( a = \varsigma_{s \rightarrow u}(x_s) \) hence \( \varsigma_{u \rightarrow v}(a) = \varsigma_{u \rightarrow v}(\varsigma_{s \rightarrow u}(x_s)) = \varsigma_{s \rightarrow v}(x_s) \) is in \( \bar{P}_v \) by (7.19).

Although we constructed \( \bar{P}_s \) such that \( \bar{P}_s \supseteq P_s \) follows by the indirect assumption, since \( \bar{P}_s \) is total and \( P_s \) is not, \( \bar{P}_s \supseteq P_s \) may not hold for \( v > s \). Therefore, the \( \bar{P}_u \)'s do not necessarily form an element of \( \mathcal{S} \). However, by using them, we are now equipped to construct an element of \( \mathcal{S} \) that is larger than \( m \), as initially intended. For \( u \in \kappa \) let

\[
\hat{\bar{P}}_u = \begin{cases} \bar{P}_u P_u & \text{if } u \geq s \\ P_u & \text{if } u < s \end{cases}.
\]

Then, the group reducts of the \( G_u \)'s equipped with the \( \hat{\bar{P}}_u \)'s form a direct system of partially ordered abelian groups. Indeed,

a) first, we prove that for \( u \geq s \), \( \bar{P}_u P_u \) forms a partial order on the group reduct of \( G_u \). This is evident for \( u = s \) since \( \bar{P}_u P_u = \bar{P}_s \) directly follows from \( \bar{P}_s \supseteq P_s \). For \( u > s \) we need to verify, as per item iii in Lemma 5.52, that \( (P_u)^{-1} \cap P_u = \{t_u\} \). By contradiction, suppose \( \{t_u\} \neq a \in (P_u)^{-1} \cap P_u \). Since \( a^{-1} \in \bar{P}_u \) there exists \( x_s \in \bar{P}_s \) such that \( a^{-1} = \varsigma_{s \rightarrow u}(x_s) \). Therefore, \( a = \varsigma_{s \rightarrow u}(x_s^{-1}) \) holds. Given \( a \in Q_u \) by assumption, and considering that the preimage of any strictly positive element is strictly positive (by Claim 1), it follows that \( x_s^{-1} \in Q_s \). Since \( \bar{P}_s \supseteq P_s \), \( x_s^{-1} \) is also strictly positive in \( \bar{P}_s \), contradicting the fact that \( x_s \in \bar{P}_s \).

b) Second, we claim that the \( \varsigma_{u \rightarrow v}'s \) are positive with respect to the \( \hat{\bar{P}}_u \)'s. Given that the product of two positive elements are also positive, and since the homomorphisms of \( (G_u, \varsigma_{u \rightarrow v})_\kappa \) are positive with respect to the system of their own positive cones (the \( P_u \)'s), as well as with respect to the \( \bar{P}_u \)'s, it is evident that these homomorphisms are positive with respect to the \( \hat{\bar{P}}_u \)'s as well.

Furthermore, the family of the \( \hat{\bar{P}}_u \)'s constitutes an element of \( \mathcal{S} \) since it clearly extends not only the partial orders on the \( L_u \)'s but also the \( P_u \)'s, see (7.20). Hence, the maximality of \( m \) implies \( \hat{\bar{P}}_u = P_u \) for every \( u \in \kappa \), contradicting \( \bar{P}_s = \bar{P}_s \supseteq P_s \). \( \square \)
8. Amalgamation in Classes of $\mathcal{I}_e^c$

We delve into the Amalgamation Property within subclasses of $\mathcal{I}_e^c$. For subclasses where amalgamation or strong amalgamation does not hold, Theorems 8.3 and 8.2 provide a universal method to demonstrate this failure for each specific subclass. Figures 6a through 10 visually depict the key steps involved in the theorem’s core construction. For subclasses admitting amalgamation, the method outlined in Theorem 8.3 establishes the groundwork for this exploration. Figures 7 through 14 illustrate the crucial steps involved in the theorem’s main construction. For subclasses admitting amalgamation, the method outlined in Theorem 8.3 establishes the groundwork for this exploration. Figures 7 through 14 visually illustrate the key steps involved in the theorem’s main construction. For subclasses admitting amalgamation, the method outlined in Theorem 8.3 establishes the groundwork for this exploration. Figures 7 through 14 visually illustrate the key steps involved in the theorem’s main construction.
AMALGAMATION IN CLASSES OF INVOLUTIVE COMMUTATIVE RESIDUATED LATTICES

Figure 5a. A V-formation in \( \mathfrak{A}^\circ \) (left), bunches of layer groups (second from left) along with bunch embeddings (second from right), and their corresponding V-formation of bunches (right)

Figure 5b. Amalgam for the V-formation in \( \mathfrak{I}_c^e \) (left) and the group representation of \( W \) (right)

\( W \) (illustrated in Fig. 5b right). According to Theorem 6.8 and Lemma 7.1, the amalgam \( \langle W, \mu_1, \mu_2 \rangle \) induces amalgams in \( \mathfrak{A}^c \) in every layer of \( X \) (Fig. 5c left). Now, let’s consider the amalgam \( \langle G_u^{w}, \mu_1, \mu_2 \rangle \) in the \( u \)-layer (Fig. 5c right). Given that \( u \in \kappa_f^u \), condition [Emb1] in Lemma 7.1 implies \( u \in \kappa_f^{z} \). Hence, \( G_u^{z} \) is discretely ordered, as indicated by condition [L2] in Definition 6.1. Additionally, by condition [Emb3] in Lemma 7.1, the embeddings \( \mu_1, u \) and \( \mu_2, u \) are normal. Here, we exploit the fact that the preservation of the lower bound is equivalent to the preservation of the upper bound in \( \sigma \)-groups. Consequently, the amalgam \( \langle G_u^{z}, \mu_1, \mu_2 \rangle \) of the V-formation \( \langle G_u^{w}, G_u^{y}, G_u^{z}, \iota_1, u, \iota_2, u \rangle \) in \( \mathfrak{A}_c^\circ \) is not only in \( \mathfrak{A}_c \) but also in \( \mathfrak{A}_c^\circ \), leading to a contradiction.

To prove the statement for \( \mathfrak{I}_e^i \), we can simply modify the previous proof as follows: let \( t \in \kappa_f^{y}, \kappa_f^{x}, \kappa_f^{z} \) (thus ensuring that \( X, Y, Z \in \mathfrak{I}_e^i \)), and we let the subgroups in all copies of \( \mathfrak{I}_e^i \) be trivial. We can then proceed with the proof following the same logic as before.
For $\mathcal{F}_t^c$ and $\mathcal{F}_n^c$, the arguments in the above-detailed proof can be modified as follows. The V-formation $⟨X, Y, Z, ι_1, ι_2⟩$ will correspond directly to the V-formation in $\mathfrak{A}^3$ (depicted in Fig. 5a, left) which is viewed as a V-formation of bunches over singletons. Here we let $u = t$ as belonging to $κ_J, κ_{J'}, κ_{J''}$. Consequently, $X, Y, Z$ will be even with non-idempotent falsum constants. This process is illustrated in Fig. 6.

**Figure 5c.** Induced amalgams in the layers (left) and amalgam for the V-formation in $\mathfrak{A}^3$ (right)

**Figure 6.** Failure of amalgamation in $\mathcal{F}_t^c$ and $\mathcal{F}_n^c$
Finally, $\mathcal{F}_\text{d}$, $\mathcal{F}_\text{e}$, and $\mathcal{F}_\text{f}$ all fail the Amalgamation Property. This is because, for any $V$-formation from $\mathcal{F}_\text{d}$, its amalgam must also be in $\mathcal{F}_\text{d}$, since being odd is preserved by homomorphisms. However, as shown before, $\mathcal{F}_\text{f}$ fails the Amalgamation Property. \hfill $\square$

**Theorem 8.2.** The classes $\mathcal{S}_\text{e}$, $\mathcal{S}_\text{f}$, and $\mathcal{S}_\text{g}$ do not satisfy the Strong Amalgamation Property.

**Proof.** The proof of the failure of strong amalgamation in $\mathcal{S}_\text{e}$ can be reduced to the failure of strong amalgamation in $\mathcal{S}_\text{f}$ in the same manner as the reduction in the case of $\mathcal{F}_\text{d}$ to $\mathcal{F}_\text{e}$ in Theorem 8.1. Therefore, it suffices to prove the statements concerning $\mathcal{S}_\text{e}$ and $\mathcal{S}_\text{f}$. To accomplish this goal, once again, the proof technique employed in Theorem 8.1 specifically the process depicted in Fig 6 can be applied with the following modifications. It is known that the Strong Amalgamation Property fails in $\mathcal{S}_\text{f}$ (cf. Fig 8). Therefore, we consider a $V$-formation in $\mathcal{S}_\text{f}$ that does not possess an amalgam in $\mathcal{S}_\text{f}$ (cf. Fig 8 top-left). To prove the statement for $\mathcal{S}_\text{e}$ we let $t = u \in \kappa_0, \kappa_0', \kappa_0'$ (yielding $X, Y, Z \in \mathcal{S}_\text{e}$), whereas prove the statement for $\mathcal{S}_\text{f}$ we let $t = u \in \kappa_1, \kappa_1', \kappa_1'$ along with $H_1 = H_1' = H_1'' = I$ (yielding $X, Y, Z \in \mathcal{S}_\text{f}$).

In the latter case condition $(\text{Hom}_2)$ is satisfied due to injectivity. As previously, an amalgam is formed in the $u$-layer, thus producing an amalgam of the $V$-formation in $\mathcal{S}_\text{f}$ (Fig 8, bottom-right). In conclusion, we demonstrate that the amalgam is strong. Indeed, let $y \in G_u$ and $z \in G_v$ such that $\mu_1, u(y) = \mu_2, u(z)$. Then $y \in Y$ and $z \in Z$ hold by (6.9), and by (6.10), $\mu_1(y) = \mu_1, u(y) = \mu_2, u(z) = \mu_2(z)$ hold. Since $(W, \mu_1, \mu_2)$ is a strong amalgam for $(X, Y, Z, \iota_1, \iota_2)$, there exists $x \in X$ such that $\iota_1(x) = y$ and $\iota_2(x) = z$. Since $G_u = X$ by construction, $x \in G_{u'}$ follows, and by (6.9), $\iota_1, u(x) = \iota_1(x) = y$ and $\iota_2, u(x) = \iota_2(x) = z$ follow. \hfill $\square$

**Theorem 8.3.** The classes $\mathcal{S}_\text{e}$, $\mathcal{S}_\text{f}$, and $\mathcal{S}_\text{g}$ each satisfy the Amalgamation Property.

**Proof.** Consider a $V$-formation $V = (X, Y, Z, \iota_1, \iota_2)$ in $\mathcal{S}_\text{e}$ and consider the respective group representations

$$
\mathcal{X} = \{ G_u^X, H_u^X, \iota_u^X \} \text{ with } G_u^X = \left( G_u^X, \iota_u^X, \iota_u^X, -1^X, u \right),
$$

$$
\mathcal{Y} = \{ G_v^Y, H_v^Y, \iota_v^Y \} \text{ with } G_v^Y = \left( G_v^Y, \iota_v^Y, -1^Y, u \right),
$$

$$
\mathcal{Z} = \{ G_u^Z, H_u^Z, \iota_u^Z \} \text{ with } G_u^Z = \left( G_u^Z, \iota_u^Z, -1^Z, u \right),
$$

and consider the respective group representatives.

By Theorem 8.1 and Lemma 7.1, $\mathcal{X}$ embeds into $\mathcal{Y}$ and $\mathcal{Z}$ (Fig 7, right). In particular, the squares in Fig 7 (bottom), (cf. Fig 7 right) commute for $u, v \in \kappa^n$, $u \leq \kappa^n$, $v$. Without loss of generality, in order to simplify the notation of the index, we may assume that $\iota_1$ and $\iota_2$ are inclusion maps. Therefore, the induced mappings $\iota_1, u$ and $\iota_2, u$ in the layers (in the bunch embeddings $\iota_1 = \{ \iota_1, u : u \in \kappa^n \}$ and $\iota_2 = \{ \iota_2, u : u \in \kappa^n \}$) are inclusion maps, too.
Figure 7. V-formation of three algebras along with their respective group representations (top-left), embeddings of $X$ into $Y$ and $Z$ (induced by $\iota_1$ and $\iota_2$, respectively, top-right), and embeddings of the layer groups of $X$ (bottom)

We shall construct an amalgam of $V$ in $S\subseteq \mathcal{C}_\mathfrak{a}$ as follows. Recall that by (6.5), we have $\kappa(X) \subseteq X$. Therefore, $\iota_1$ can be restricted to $\kappa(X)$. Since the class $\mathcal{C}$ is known to have the Strong Amalgamation Property, there exists a strong amalgam $(\kappa(W), \nu_1, \nu_2)$ in $\mathcal{C}$ of the V-formation $(\kappa(X), \iota_1|\kappa(X), \kappa(Y), \iota_2|\kappa(X), \kappa(Z))$, see Fig. 8. To simplify notation, we shall assume that $\nu_1$ and $\nu_2$ are inclusion maps, too.

(8.1) $\kappa(X) = \kappa(Y) \cap \kappa(Z)$, $\kappa(W) = \kappa(Y) \cup \kappa(Z)$.

Given the embeddings of $\mathcal{X}$ into $\mathcal{Y}$ and $\mathcal{Z}$, the subsequent step involves embedding both $\mathcal{Y}$ and $\mathcal{Z}$ into their respective canonical extensions $\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{Z}}$ over $\kappa(W)$.

Theorem 7.3 guarantees the existence of such embeddings, denoted by $\pi_Y$ and $\pi_Z$, respectively. Let us denote the compositions of the bunch embeddings, $\pi_Y \circ \iota_1$ and $\pi_Z \circ \iota_2$, by

$\tilde{\iota}_1 = \{\tilde{\iota}_1,u : u \in \kappa(X)\}$ and $\tilde{\iota}_2 = \{\tilde{\iota}_2,u : u \in \kappa(X)\}$

respectively, see Fig. 9. Our overarching objective is to establish all the bunches and
Figure 8. The inclusions $\iota_1|_{\kappa^X}$ and $\iota_2|_{\kappa^X}$ of the skeleton $\kappa^X$ of $X$ into the skeleton $\kappa^Y$ of $Y$ and the skeleton $\kappa^Z$ of $Z$, respectively, and their strong amalgam $\langle \kappa^W, \nu_1, \nu_2 \rangle$.

Figure 9. The embedding of $Y$ and $Z$ into $\tilde{Y}$ and $\tilde{Z}$, respectively, results in embeddings of $X$ into $\tilde{Y}$ and $\tilde{Z}$, respectively ($u, v \in \kappa^X$).

their corresponding bunch embeddings as depicted in Fig. 15. We have now successfully constructed the bunches and embeddings pertaining to the upper portion of that figure.

We will now focus on constructing the elements depicted in the “lower portion” of Fig. 15. In quest of embedding both $\mathcal{Y}$ and $\mathcal{Z}$ into $\mathcal{Y}$ and $\mathcal{Z}$, our plan is to consider amalgamated free products at indices from $\kappa^X$ and free products at indices from $\kappa^W \setminus \kappa^X$ (cf. [83]). These do not exist in the class of abelian $o$-groups; however, they do exist in the class of abelian $\ell$-groups (with positive homomorphisms, see e.g. [43, Theorem 12.2.1]), and the thus constructed abelian $\ell$-groups will serve
as the basis of the construction of the bunch $\mathcal{B}$. In order to define the transitions of $\mathcal{B}_d$, we need an auxiliary direct system $\hat{\mathcal{B}}_d$ over $\kappa^{(w)}$, and homomorphisms from it to $\mathcal{B}_d$ and $\hat{\mathcal{B}}_d$ such that at indices from $\kappa^{(x)}$ the mappings are embeddings.

We proceed as follows. Let $\hat{\mathcal{B}}$ be the canonical extension of $\mathcal{B}$ over $\kappa^{(w)}$. Bunch embeddings are also direct system embeddings, by definition. Therefore, by Theorem 7.5, the embeddings $\hat{\iota}_1$ and $\hat{\iota}_2$ of $\hat{\mathcal{B}}$ into $\mathcal{B}$ and $\hat{\mathcal{B}}$ yield direct system homomorphisms

$$(8.2) \quad (\hat{\iota}_1)_{\kappa^{x} \to \kappa^{w}} = \{ \psi_{1,u} : u \in \kappa^{(w)} \} \quad \text{and} \quad (\hat{\iota}_2)_{\kappa^{x} \to \kappa^{w}} = \{ \psi_{2,u} : u \in \kappa^{(w)} \}$$

of $\hat{\mathcal{B}}_d$ to $\mathcal{B}_d$ and $\hat{\mathcal{B}}_d$, respectively, such that for $u \in \kappa^{(x)}$, $\psi_{1,u}$ and $\psi_{2,u}$ are embeddings, see Fig. 10.

**Figure 10.** Homomorphisms of $\hat{\mathcal{B}}_d$ to $\mathcal{B}_d$ and $\hat{\mathcal{B}}_d$ ($u,v \in \kappa^{(w)}$).

The green arrows in the V-formations are only homomorphisms, the gray ones are embeddings.

Now, consider the following amalgamated free products in $\mathfrak{A}^1$ (cf. Section 3)

$$(8.3) \quad L_{u}^{(w)} = \begin{cases} G_{u}^{(y)} \ast G_{v}^{(x)} \ast G_{d}^{(z)} & \text{if } u \in \kappa^{(x)} \\ G_{u}^{(y)} \ast u \ast G_{d}^{(z)} & \text{if } u \in \kappa^{(w)} \setminus \kappa^{(x)} \end{cases}$$

along with the related (abelian $\ell$-group) embeddings $\mu_{1,u}$ and $\mu_{2,u}$, (see its illustration in Fig. 11 bottom). Let

$$\mu_1 = \{ \mu_{1,u} : u \in \kappa^{(w)} \} \quad \text{and} \quad \mu_2 = \{ \mu_{2,u} : u \in \kappa^{(w)} \}.$$ 

For $u,v \in \kappa$, $u \leq \kappa v$, let $\psi_{u \to v}$ be the pushout homomorphism of $\mu_{1,u} \circ \psi_{u \to v}$ and $\mu_{2,v} \circ \psi_{u \to v}$, (see Fig. 11 top and Fig. 12). The auxiliary direct system $\hat{\mathcal{B}}_d$ ensures that $\psi_{u \to v}$ is well-defined: for $u \in \kappa^{(x)}$, $\mu_{1,u} \circ \psi_{u \to v} \circ \psi_{1,u} = \mu_{1,u} \circ \psi_{1,u} \circ \psi_{u \to v} = \mu_{2,v} \circ \psi_{u \to v} \circ \psi_{2,u}$, hence the condition of $\text{[Am2]}$ is satisfied; note that it is trivially satisfied if $u \in \kappa^{(w)} \setminus \kappa^{(x)}$ since then “only $\mu$ is amalgamated”, and such diagrams always commute. Therefore, by $\text{[Am2]} \quad \psi_{u \to v}$, is an (abelian $\ell$-group) embeddings.

---

$\text{[Am1]}$ Here $\mu$ denotes the trivial one-element subgroup of $G_{u}^{(y)}$. Thus $G_{u}^{(y)} \ast u \ast G_{d}^{(z)}$ is isomorphic to the free product $G_{u}^{(y)} \ast G_{d}^{(z)}$.
homomorphism which makes the two squares in the middle of Fig. 12 commute (cf. the lower part of Fig. 11, top).

Next we prove that the thus defined $\varsigma(\mathfrak{W})$s render $\langle L^w_u, \varsigma^w_u \rangle$ a direct system of abelian $\ell$-groups over $\kappa^w$. As for [Dir1] for $i \in \kappa^w$, $\varsigma^w_{\mathfrak{W}^i}$ is the pushout homomorphism of $\mu_{1,v} \circ \varsigma^x_{\mathfrak{W}^i} u$ and $\mu_{2,v} \circ \varsigma^x_{\mathfrak{W}^i} u$, hence $\varsigma^w_{\mathfrak{W}^i}$ is the identity mapping of $L^w_u$ because of the unicity in (Am2) (cf. Fig. 12). Referring to the unicity in (Am2) to prove (Dir2) that is, $\varsigma^w_{\mathfrak{W}^i} v = \varsigma^w_{\mathfrak{W}^i} v \circ \varsigma^w_{\mathfrak{W}^i} u$ for $u, v, w \in \kappa^w$, $u \leq \kappa^w v \leq \kappa^w w$, it suffices to show that $\varsigma^w_{\mathfrak{W}^i} v \circ \varsigma^w_{\mathfrak{W}^i} u$ makes the lower two squares in Fig. 13 commute. That is, we need to verify $\varsigma^w_{\mathfrak{W}^i} v \circ \varsigma^w_{\mathfrak{W}^i} u \circ \mu_{1,u} = \mu_{1,w} \circ \varsigma^w_{\mathfrak{W}^i} u$ and
κ since their over o of abelianς the skeleton, see Lemma 7.1, both proven analogously. Using that the upper four little squares in Fig. 13 commute,

\[ \begin{array}{c}
\vspace{2mm}
G_u^w & \xrightarrow{\mu_1,u} & L_u^w & \xrightarrow{\mu_2,u} & G_u^z \\
G_v^w & \xrightarrow{\mu_1,v} & L_v^w & \xrightarrow{\mu_2,v} & G_v^z \\
G_w^w & \xrightarrow{\mu_1,w} & L_w^w & \xrightarrow{\mu_2,w} & G_w^z \\
G_x^w & \xrightarrow{\mu_1,x} & L_x^w & \xrightarrow{\mu_2,x} & G_x^z
\end{array} \]

\[ \vspace{2mm} \]

Figure 13. The \( \phi_{u \rightarrow v} \)'s make the \( L_u^w \)'s a direct system (cf. Fig. 11 top)

\( \phi_{u \rightarrow w} \circ \phi_{w \rightarrow v} \circ \mu_{2,u} = \mu_{2,v} \circ \phi_{u \rightarrow w} \). We prove the first one, the other one can be proven analogously. Using that the upper four little squares in Fig. 13 commute,

\[ \phi_{u \rightarrow w} \circ \phi_{w \rightarrow v} \circ \mu_{1,u} = \phi_{v \rightarrow w} \circ \mu_{1,v} \circ \phi_{u \rightarrow v} = \mu_{1,w} \circ \phi_{v \rightarrow w} \circ \phi_{u \rightarrow v} = \mu_{1,w} \circ \phi_{u \rightarrow w} \]

Summing up, we have constructed a direct system \( (L_u^w, \phi_{u \rightarrow w}) \) of abelian \( \ell \)-groups over \( \kappa^w \) along with embeddings \( \mu_1 = \{ \mu_{1,u} : u \in \kappa^w \} \) and \( \mu_2 = \{ \mu_{2,u} : u \in \kappa^w \} \) from \( \tilde{Y}_d \) and \( \tilde{Z}_d \) to it (see Fig. 11 top).

The \( L_u^w \)'s are torsion-free, as abelian \( \ell \)-groups are known to be torsion-free (see e.g., [21, Corollary 0.1.2.]). Therefore, by Theorem 7.6 by only extending the orderings of the \( L_u^w \)'s, \( (L_u^w, \phi_{u \rightarrow w})_{\kappa^w} \) can be transformed into a direct system

\[ \mathcal{W}_d = (G_u^w, \phi_{u \rightarrow w})_{\kappa^w}, \]

of abelian \( o \)-groups (Fig. 11 top). Evidently, for every \( u \in \kappa^w \) the diagram in Fig 14 (bottom) commutes since only the orderings of the \( L_u^w \)'s have been changed with respect to Fig 12. In conclusion, we have constructed the direct system \( \mathcal{W}_d \) of abelian \( o \)-groups, and we have shown that \( \mu_1 \) and \( \mu_2 \) are homomorphisms from \( \tilde{Y}_d \) and \( \tilde{Z}_d \) to \( \mathcal{W}_d \), see the bottom part of Fig. 11.

Our next aim is to enrich \( \mathcal{W}_d \) (represented by the blue part of Fig. 11) to become a bunch \( \mathcal{W} = (G_u^w, H_u^w, \phi_{u \rightarrow v})_{(\kappa^w, \theta, \kappa^{\mu}_w, \kappa^{\mu}_v)} \) (represented by the blue-brown part of Fig. 14) in such a way that \( \mu_1 \) and \( \mu_2 \) become bunch embeddings.

To begin with, recall that \( \tilde{X} \) is embedded into the canonical extensions \( \tilde{Y} \) and \( \tilde{Z} \) of \( \mathcal{W} \) and \( \mathcal{Z} \), respectively, over \( \kappa^w \), see Fig 9. Since bunch embeddings preserve the skeleton, see Lemma 7.1 both \( \kappa^\ell_t \) and \( \kappa^\ell_v \) are equal to \( \kappa^x_\alpha \). Consequently, since their J-components are empty by construction, both \( \kappa^\mu_t \) and \( \kappa^\mu_v \) are equal to \( \kappa^w \setminus \kappa^x_\alpha \). We also define \( \kappa^{\mu}_w = \kappa^{\mu}_x \) and \( \kappa^{\mu}_t = \kappa^w \setminus \kappa^{\mu}_x \). Therefore, referring to
In addition, for $u \in \kappa_I^w$, we define
\begin{align}
(8.5) \quad H_u^w \text{ as the abelian } \alpha\text{-group generated by } \mu_{1,u}(H_u^w) \text{ and } \mu_{2,u}(H_u^z),
\end{align}
and we let $H_u^w$ be a copy of $H_u^w$ according to Remark 6.3.

Next, we verify that $\mathcal{W}$ is a bunch. It is evident that $\kappa_0^w, \kappa_I^w,$ and $\kappa_2^w$ (which is empty) forms a partition, as required. To conclude, we need to verify the nontrivial part of (LI). Let $u \in \kappa_I^w$ and $v \in \kappa_I^w$ with $u <_{\kappa_I^w} v$, and let $x \in G_u^w$. Our construction ensures that $G_u^w$ shares its group structure with the amalgamated free product $L_u^w$ of $G_u^w$ and $G_v^w$. It’s well-known that amalgamated free products are generated by the “embedded copies” of their arguments. Therefore, we can find elements $y \in G_u^w$ and $z \in G_v^w$ such that $x = \mu_{1,u}(y) \cdot_{\kappa_v} \mu_{2,u}(z)$. Therefore, $c_{u \rightarrow v}(x) = c_{u \rightarrow v}((\mu_{1,u}(y)) \cdot_{\kappa_v} \mu_{2,u}(z)) = c_{u \rightarrow v}(\mu_{1,u}(y)) \cdot_{\kappa_v} c_{u \rightarrow v}(\mu_{2,u}(z))$ (Fig. 14)
\begin{align}
\mu_{1,v}(c_{u \rightarrow v}(y)) \cdot_{\kappa_v} \mu_{2,v}(c_{u \rightarrow v}(z)) \leq \mu_{1,v}(H_v^w) \cdot_{\kappa_v} \mu_{2,v}(H_v^z) \leq H_v^w.
\end{align}
Previously we established that $\mu_1 : \mathcal{Y}_d \to \mathcal{W}_d$ and $\mu_2 : \mathcal{Z}_d \to \mathcal{W}_d$ are direct system embeddings. Now we demonstrate that they are also bunch embeddings: $\mu_1 : \mathcal{Y} \to \mathcal{W}$ and $\mu_2 : \mathcal{Z} \to \mathcal{W}$. This is because enriching the direct systems to bunches introduces additional structure, and we can show that these embeddings preserve this extra structure, too. Leveraging Lemma 7.1 we need to verify [Emb1] and [Emb2] as [Emb3] is satisfied because $\kappa^w_j$ is empty. Due to both [8.5] and the fact that by construction any element $u \in \kappa^w_j$ (respectively, $u \in \kappa^w$) is mapped to $u \in \kappa^w$ by $\mu_1$ (resp. $\mu_2$), it is evident that [Emb1] holds.

We will now establish that $\mu_1$ satisfies property [Emb2]. The following proof can be directly applied to demonstrate the same property for $\mu_2$ as well. Let $u \in \kappa^w_j$ and $x \in H^w_u$. Then, $\mu_{1,u}(x) = \mu_{1,u}(x) \cdot u = \mu_{1,u}(x) \cdot u \cdot \mu_{2,u}(u)$ is in $H^w_u$ by [8.5]. This shows that $\mu_1$ preserves the subgroups.

To show that $\mu_1$ preserves the complements of the subgroups, let $x \in G^v_u \setminus H^v_u$. Assume the opposite of the statement, that is, $\mu_{1,u}(x) \in H^w_u$. Then by [8.5], $\mu_{1,u}(x) = \mu_{1,u}(y) \cdot \mu_{2,u}(z)$ for some $y \in H^v_u$ and $z \in H^w_u$. Therefore, $\mu_{2,u}(z) = \mu_{1,u}(y)^{-1} \cdot \mu_{1,u}(x) = \mu_{1,u}(y)^{-1} \cdot x \in H^v_u$ holds along with $c := y^{-1} \cdot x \in H^v_u$.

By construction, $\mathcal{G}_u^{(v)}$ has the same group reduct as the amalgamated free product $L_u^{(w)}$ in [8.3]. Therefore, (8.6) is an immediate contradiction if $u \in \kappa^w \setminus \kappa^x$, since then $L_u^{(w)}$ is a free product, and there are no relations in free products between elements of its embedded arguments (that’s why it is “free”). (8.6) is a contradiction also in the case when $u \in \kappa^x$, since amalgamated free products are known to be certain factor groups of free products; in particular, $\mathcal{L}_u^{(w)}$ is the factor group of the free product $\mathcal{G}_u^{(v)} \ast \mathcal{G}_u^{(v)}$ with respect to the smallest normal subgroup $\mathcal{N}_u^{(w)}$ generated by

$$\{ \mu_{1,u}(\psi_1,u(x)) \cdot \mu_{2,u}(\psi_2,u(x))^{-1} : x \in \mathcal{G}_u^{(v)} \}.$$
AMALGAMATION IN CLASSES OF INVOLUTIVE COMMUTATIVE RESIDUATED LATTICES

Notice that since \( L^w_u \) is commutative, and since all mappings in (8.7) are homomorphisms, in fact,

(8.8) \[ N^{w_u} = \{ x \in G^w_u \} \]

Since \( \iota_1 \) and \( \iota_2 \) are embeddings by assumption, \( \iota_1 \) and \( \iota_2 \) are bunch embeddings by Lemma 7.1. Therefore, by Theorem 7.5, \( \iota_1 \) and \( \iota_2 \) satisfy \( \text{Hom}_2 \) as indices from \( k^{(w)} \): if \( u \in k^{(w)} \) and \( x \in G^w_u \), then \( \psi_1(u)(x) = H^{(w)}_u \) and \( \psi_2(u)(x) \in H^{(w)}_u \) (when \( x \in H^{(w)}_u \)). It contradicts to (8.6).

It is immediate from (8.4) that \( \mu_1 \) and \( \mu_2 \) preserve the \( I \)-components of the skeleton.

We have enriched \( \mathcal{V}_a \) into a bunch \( \mathcal{V} \) such that \( \mu_1 \) and \( \mu_2 \) became bunch embeddings (Fig. 15). Denote by \( W \) the involutive FL\( e \)-algebra corresponding to \( \mathcal{V} \), as guaranteed by Theorem 6.8. Because we constructed \( k^{(w)} \) to be empty (see (8.3)), it follows that \( W \) belongs to \( S^e \). Recall that not only \( \mu_1 \) and \( \mu_2 \), but also \( \pi_Y \) and \( \pi_Z \) preserve the \( I \)-component of the skeleton (Theorem 7.3). Hence the (FL-algebra) homomorphisms corresponding to \( \mu_1 \), \( \mu_2 \), \( \pi_Y \), and \( \pi_Z \), as guaranteed by Theorem 6.8, are embeddings by Lemma 7.1. We denote them by \( \mu_1 \), \( \mu_2 \), \( \pi_Y \), and \( \pi_Z \), respectively. Finally, we prove that \( \langle \mu_1 \circ \pi_Y, \mu_2 \circ \pi_Z \rangle \) is an amalgam of \( \langle X, Y, Z, \iota_1, \iota_2 \rangle \) in \( S^e \), see Fig. 16. Let \( x \in X \). Since bunch embeddings preserve the partition, the second row of (6.10) never applies, and thus

(6.10) \[ F_{10} \]

\[ \begin{cases} 
\left( \mu_1 \circ \pi_Y \circ \iota_1 \right)(x) = \text{if } x \in G^w_u \\
\left( \mu_1 \circ \pi_Y \circ \iota_1 \right)(x) = \text{if } \hat{a} = x \in H^{(w)}_u \\
\left( \mu_2 \circ \pi_Z \circ \iota_2 \right)(x) = \text{if } x \in G^w_u \\
\left( \mu_2 \circ \pi_Z \circ \iota_2 \right)(x) = \text{if } \hat{a} = x \in H^{(w)}_u \\
\end{cases} \]

Figure 16. “Layerwise” amalgamation in \( A^e \) (right), and amalgamation in \( S^e \) (left)

\[ \square \]
9. Amalgamation in the generated semilinear varieties

The investigation in Section 8 demonstrated that the following classes of FLc-chains $\mathcal{I}^c, \mathcal{I}^e, \mathcal{T}_e^n$, as well as every class of involutive FLc-chains which contains $\mathcal{I}_o^c$, fail the Amalgamation Property. Our proof indicates that these classes fail the Amalgamation Property for the same reason as discrete abelian $\alpha$-groups with positive normal homomorphisms. In contrast, the classes $\mathcal{G}^c_i, \mathcal{G}^e_i$, and $\mathcal{G}^c_o$ satisfy the Amalgamation Property, but they fail the Strong Amalgamation Property for the same reason as abelian $\alpha$-groups with positive homomorphisms.

We now shift our focus from these classes of chains to the semilinear varieties of FLc-algebras they generate. Our goal is to transfer the Amalgamation Property, or its failure, from the specific classes of chains to the generated varieties. We accomplish this by utilizing the relevant characterization from the literature, as cited in Lemma 9.2. Theorems 9.3 and 9.5 address the varieties generated by the following classes: $\mathcal{I}_o^c, \mathcal{I}^c, \mathcal{I}^e$, $\mathcal{I}^e_{oe}, \mathcal{G}^c_i, \mathcal{G}^c_o$, and $V(\mathcal{G}^c_i)$. First, we briefly show that the other classes of chains addressed in the previous section—namely, $\mathcal{I}^c, \mathcal{T}_e^n$, $\mathcal{T}_o^n$, and $\mathcal{G}^e_i$—do not generate varieties other than these.

Proposition 9.1. $V(\mathcal{G}^c_i) = V(\mathcal{G}^c_i), V(\mathcal{G}^e_i) = V(\mathcal{G}^e_i), V(\mathcal{T}_e^n) = V(\mathcal{T}_e^n), V(\mathcal{T}_o^n) = V(\mathcal{T}_o^n), V(\mathcal{G}^e_i) = V(\mathcal{G}^e_i), \text{ and } V(\mathcal{G}^c_o) = V(\mathcal{G}^c_o)$.

Proof. According to [28] Theorem 6.3, there is a bijective correspondence between the classes $\mathcal{I}_o^c$ and $\mathcal{I}^c$. Specifically, every odd involutive FLc-chain can be realized as a homomorphic image of an even involutive FLc-chain with an idempotent falsum constant. This realization is achieved through a homomorphism defined by the principal congruence generated by the constants $t$ and $f$. Originally, this result was established through a complex argument intended to support the proof of the representation theorem for odd and even involutive FLc-chains. However, it can be more intuitively understood by employing the categorical isomorphism described in Section 8.

Consider an arbitrary algebra $X$ from the class $\mathcal{I}^e_i$. According to Theorem 6.6, in the bunch $\mathcal{J}$ corresponding to $X$, the least element $t$ of the skeleton is located in $\kappa^X_t$. To transform $\mathcal{J}$, construct a new bunch $\mathcal{Y}$ by repositioning $t$ into $\kappa^Y_t$, while leaving the rest of the structure unchanged. Referring to Definition 6.1 it becomes evident that $\mathcal{Y}$ is a bunch. By Theorem 6.6, the algebra $Y$ corresponding to $\mathcal{Y}$ belongs to the class $\mathcal{I}^c$. Referring to Definition 6.3, it becomes evident that the identity map from $\mathcal{Y}$ to $\mathcal{J}$ acts as a bunch homomorphism. Consequently, $X$ is a homomorphic image of $Y$ via the corresponding homomorphism. This proves

$$V(\mathcal{J}_t^c) = V(\mathcal{J}_t^c), \quad V(\mathcal{J}_t^e) = V(\mathcal{J}_t^e), \quad \text{and also} \quad V(\mathcal{G}^c_i) = V(\mathcal{G}^c_i)$$

if $X$ is chosen to be idempotent symmetric.

The very same construction shows that every idempotent-symmetric odd involutive FLc-chain can be realized as a homomorphic image of an idempotent-symmetric even involutive FLc-chain (hence with an idempotent falsum constant; here, idempotent-symmetry is equivalent to the $\kappa_j$-component being empty). This demonstrates that

$$V(\mathcal{G}^c_i) = V(\mathcal{G}^c_i).$$

Also, a similarly simple transformation demonstrates that every odd involutive FLc-chain $X$ can also be realized as a homomorphic image of an even involutive FLc-chain with a non-idempotent falsum constant, call it $Y$. To achieve this, we
relocate the element $t$ from $\kappa_o^{(x)}$ to $\kappa_p^{(y)}$. Additionally we set $G^{(y)}_{i_1}$ be the lexicographic product of $G^{(x)}_{i_1}$ and $Z$, set the transitions $\zeta^{(y)}_{i_1} = (a, z) \mapsto \zeta^{(x)}_{i_1}(a)$, while keeping the rest of the structure unchanged. With these modifications, $\mathcal{Y}$ is a bunch and the identity map from $\mathcal{Y}$ to $\mathcal{Z}$, when altered at the $t$-level to act as $(a, z) \mapsto a$, functions as a bunch homomorphism. Consequently, this modified identity map ensures that $X$ is a homomorphic image of $Y$. This demonstrates that

$$V(\mathcal{Y}_{\kappa_n}) = V(\mathcal{Y}_{\kappa_m}).$$

\[ \square \]

An extension $A \leq B$ is called essential if the restriction of every non-trivial congruence on $B$ to $A$ is also non-trivial. Equivalently, if a homomorphism from $B$ to $C$ is not injective, then its restriction to $A$ is not injective either. An essential embedding is an embedding $\varphi : A \rightarrow B$ such that $\varphi(A) \leq B$ is essential. A $V$-formation $(A, B_1, B_2, i_1, i_2)$ is called essential if $i_2$ is an essential embedding. A class of algebras $\mathcal{X}$ has the essential Amalgamation Property if every essential $V$-formation in $\mathcal{X}$ has an amalgam in $\mathcal{X}$.

**Lemma 9.2.** ([15] Lemma 3.1 cf. [13] Corollary 3.5] Any variety of semilinear commutative (pointed) residuated lattices has the Amalgamation Property if and only if the class of its finitely subdirectly irreducible members (i.e., the linearly ordered members of the variety) has the essential Amalgamation Property.

A stronger version of the Amalgamation Property is known as the Transferable Injections Property. A class $\mathfrak{U}$ of algebraic structures is said to have the Transferable Injections Property if the following holds: For any structures $A, B_1, B_2 \in \mathfrak{U}$, given an embedding $\iota_1$ of $A$ into $B_1$ and a homomorphism $\iota_2$ of $A$ into $B_2$, there exists an algebra $C \in \mathfrak{U}$, a homomorphism $\mu_1$ from $B_1$ into $C$, and an embedding $\mu_2$ from $B_2$ into $C$ such that $\mu_1 \circ \iota_1 = \mu_2 \circ \iota_2$. The class $2^{sl}$ of semilinear involutive FL$_{oe}$-algebras (and semilinear involutive FL-algebras in general) forms a variety [27 Proposition 3.6.5]). The addition of an axiom equating the two constants axiomatizes the variety $2^{sl}$. Since for involutive FL$_{oe}$-algebras, $\{x \mapsto x : x \in X\}$ is the set of positive idempotent elements ([28 Lemma 3.1(vi)]), idempotence-symmetry can also be captured by the equation $(x \rightarrow x')(x \rightarrow x)' = (x \rightarrow x)'$ or by its simpler equivalent variant $xx'xx' = xx'$. Therefore, the classes $\mathfrak{S}^{sl}$ and $\mathfrak{S}^{e}$ also form varieties.

**Theorem 9.3.** The varieties $\mathfrak{S}^{sl}$ and $V(\mathfrak{S}^{e})$ have the Transferable Injections Property.

**Proof.** $\mathfrak{S}^{sl}$ and $V(\mathfrak{S}^{e})$ are generated by their chains [27 Proposition 3.6.2]. These classes of chains each have the Amalgamation Property as per Theorem 8.3, hence also the weaker essential Amalgamation Property. Therefore, it follows by Lemma 9.2 that $\mathfrak{S}^{sl}$ and $V(\mathfrak{S}^{e})$ have the Amalgamation Property. Since any variety of FL$_{oe}$-algebras that possesses the Amalgamation Property also possesses the Transferable Injections Property ([27 Corollary 44]), the proof is concluded. \[ \square \]

**Remark 9.4.** The axiomatization of $V(\mathfrak{S}^{e})$ remains unclear. Given that this variety possesses such a rare property, its corresponding logic could be of significant interest.
Theorem 9.5. Every variety of semilinear involutive commutative (pointed) residuated lattices that includes the variety of odd semilinear commutative residuated lattices fails the Amalgamation Property.

Proof. Consider any variety of semilinear involutive commutative residuated lattices that contains the variety of odd semilinear commutative residuated lattices. By Lemma 9.2 it suffices to prove that the class $\mathcal{C}$ of its chains fails the essential Amalgamation Property. To this end, consider the V-formation $(X, Y, Z, t_1, t_2)$ in $\mathcal{C}$ of odd involutive FL$_e$-chains constructed in the proof of Theorem 8.1 as shown in Fig. 53 (right). Since this V-formation was shown not to have an amalgam in the class of odd involutive FL$_e$-chains, and since being odd is preserved under homomorphisms, it has no amalgam in $\mathcal{C}$ either. Therefore, by Lemma 9.2 it suffices to prove that this V-formation is essential.

Consider a homomorphism $\psi$ from $Z$ to an involutive FL$_e$-chain $W$. Consider the bunch representation $\mathcal{W}$ of $W$ as per Theorem 6.6 and the bunch homomorphism $\psi$ corresponding to $\psi$ as per Theorem 6.7. By (Hom1), $\psi(u)$ is either in $\kappa^w_0$ or in $\kappa^w_j$.

If $\psi(u) \in \kappa^w_0$, then $\psi$ is injective. Indeed, it is injective over $G_i^x$ because, by (Hom2), homomorphisms preserve the covers if they map a $\kappa_j$-layer to a $\kappa_j$-layer. Additionally, it is injective over $G_i^x$ as well since $G_i^x$ is trivial. Hence, $\psi$ is injective since $G_i^x$ and $G_i^\wedge$ are mapped to different layers of $\mathcal{W}$, given $\psi(u) \in \kappa^w_j$, $\psi(t) \in \kappa^w_0$ by (Hom1) and $\kappa^w_0 \cap \kappa^w_j = \emptyset$.

If $\psi(u) \in \kappa^w_0$, then since $\psi(t) \in \kappa^w_0$ holds by (Hom1), it follows from (P) that $\psi(t) = \psi(u)$. Consequently, both the unit element $u$ of $G_i^x$ and the unit element $t$ of $G_i$ are mapped to the unit element of $G_i^{\psi(u)}$, so we are done. □

Remark 9.6. A specific example of this is the recent proof by W. Fussner and S. Santschi that the variety of semilinear involutive commutative residuated lattices fails the Amalgamation Property [14, Theorem 5.2].

Finally, a last remark: A by-product of our construction in Theorem 8.3 is Corollary 9.7. This corollary is well-known and easy to prove, and it also follows as a corollary through quantifier elimination in first-order theories [36].

Corollary 9.7. The class of odd (cf. [36, Corollary 5.8]) and the class of even (cf. [36, Corollary 5.6]) totally-ordered Sugihara monoids, and also their union has the Amalgamation Property.

Proof. It has been pointed out in [36] that totally-ordered Sugihara monoids are either odd or even. Indeed, since $ff = f$, it follows that $t = f'$ if $f \geq f$. If we had $f < x < t$, then $f = t' < x' < f'$ would hold, which would imply $xx' \geq \min(x, x') \bullet \min(x, x') = \min(x, x') > f$, contradicting residuation. Moreover, in Sugihara monoids, every element is idempotent, so the residual complement of a positive idempotent element is also idempotent. This argument shows that totally-ordered Sugihara monoids belong to either $\mathbb{S}^c_0$ or $\mathbb{S}^c_1$. Since the group representation of totally-ordered even or odd Sugihara monoids is of the form $\langle 1_u, 1_u, s_u \rangle$ or $\langle 1_u, 1_u, s_u \rangle$, respectively [28, Example 8.2], in every layer the groups in the induced V-formation are trivial (cf. Fig. 7). Consequently, direct limits of such direct systems are also trivial, and the free.
product of V-formations with trivial groups is also trivial. Thus, our construction produces a trivial amalgam in every layer. Hence, the amalgamation, as shown in Theorem 8.3, of a V-formation of totally-ordered even or odd Sugihara monoids results in a totally-ordered even or odd Sugihara monoid, respectively. The statement about the union follows from the observation in the last paragraph of the proof of Theorem 8.1

□

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