SPLITTING CRITERIA FOR VECTOR BUNDLES INDUCED BY RESTRICTIONS TO DIVISORS

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Abstract. In this note I investigate the splitting and triviality of vector bundles by restricting them to \( q \)-ample divisors; it generalizes work of Bakhtary, who considered restrictions to ample divisors. In particular, I study the splitting of vector bundles on the total space of fibre bundles. All the statements are illustrated with examples.

The triviality criterion is particularly suited to Frobenius split varieties, whose splitting is defined by the power of a section in the anti-canonical line bundle. As an application, I prove that a vector bundle on a smooth toric variety \( X \), whose anti-canonical bundle is \( (\dim X - 3) \)-ample, is trivial precisely when its restrictions to the invariant divisors are trivial, with trivializations compatible along the various intersections.

Introduction

Horrocks’ criterion [13] states that a vector bundle on \( \mathbb{P}^n \), \( n \geq 3 \), splits if and only if its restriction to some hyperplane \( D \cong \mathbb{P}^{n-1} \) splits. This was generalized in [2], where the author restricts vector bundles on ‘Horrocks varieties’ to ample divisors. The ampleness assumption excludes several natural situations, such as the case of morphisms, where one wishes to restrict vector bundles either to pre-images of ample divisors or to relatively ample ones.

The goal of this note is to generalize the splitting criterion in op. cit. to include \( q \)-ample (respectively \( q \)-positive) divisors; this covers the case of morphisms mentioned above. Also, in some cases, the ‘Horrocks variety’ assumption, which is a rather restrictive cohomological property, is weakened. The note contains two types of results: splitting and triviality criteria.

Theorem. (splitting criteria). Let \((X, \mathcal{O}_X(1))\) be a smooth, complex projective variety. Let \( \mathcal{V} \) be an arbitrary vector bundle on \( X \), \( \mathcal{E} := \text{End}(\mathcal{V}) \) the bundle of endomorphisms, \( \mathcal{L} \in \text{Pic}(X) \) be \( q \)-ample, and \( D \in |d\mathcal{L}| \).

The equivalence \( [\mathcal{V} \text{ splits} \iff \mathcal{V}_D \text{ splits}] \) holds in any of the following cases:

(a) If \( q \leq \dim X - 3 \), and \( H^1(\mathcal{E}_D \otimes \mathcal{L}_D^{-a}) = 0 \) for all \( a \geq d \). The parameter \( d \) is bounded from below by a linear function in the regularity of \( \mathcal{E} \) with respect to \( \mathcal{O}_X(1) \).

(b) If \( X \) satisfies (2-split), \( q \leq \dim X - 4 \), and \( D \) is smooth.

(c) If \( X \) satisfies (1-split), \( q \leq \dim X - 5 \), \( \mathcal{L}^d \) is globally generated, and \( D \) is very general.

The conditions (1-, 2-split) are defined in [2,1] they are respectively the notions of ‘splitting’ and ‘Horrocks variety’ in [2]. Bakhtary’s result corresponds to the case (b) above with \( q = 0 \).

I illustrate the advantage of allowing partially ample line bundles by simplifying the cohomological splitting criteria for vector bundles on multi-projective spaces [5] and on products of projective spaces and quadrics [3]. Both involve a large number of cohomological tests; by restricting them to ‘sub-products’, the number of these tests is massively reduced.

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Trivializable vector bundles are particular cases of split ones, therefore the triviality criteria hold in greater generality. Notably, one can eliminate the (1-, 2-split) assumptions.

**Theorem. (triviality criteria).** Let $X, \mathcal{L}, \mathcal{V}$ be as above and $D \in |\mathcal{L}|$.

The equivalence $[\mathcal{V} \text{ is trivial} \iff \mathcal{V}_D \text{ is trivial}]$ holds in any of the following cases:

- (a) If $\mathcal{L} \in \text{Pic}(X)$ is globally generated and $(\dim X - 3)$-ample.
- (b) If $\mathcal{L}$ is relatively ample for a smooth, surjective morphism $f : X \to Y$ of relative dimension at least three.
- (c) If the anti-canonical bundle $\kappa_X^{-1}$ is $(\dim X - 3)$-ample, $X$ is Frobenius split by a power of a section $\sigma$ in $\kappa_X^{-1}$, and $D = \text{divisor}(\sigma)$.

The article ends with two brief appendices where I recall the relevant definitions and properties concerning $q$-ampleness and Frobenius splitting.

1. **The general splitting principle**

Throughout this article $X$ stands for a smooth projective variety over $\mathbb{C}$ of dimension at least three.

**Definition 1.1** Let $T$ be a scheme defined over $\mathbb{C}$ and $S$ a closed subscheme of it; we assume that $\Gamma(\mathcal{O}_S) = \Gamma(\mathcal{O}_T) = \mathbb{C}$. For a locally free sheaf (vector bundle) $\mathcal{V}$ of rank $r$ on $T$, let $\mathcal{E}_S := \mathcal{E} \otimes_{\mathcal{O}_T} \mathcal{O}_S$; we denote by $\mathcal{E} := \text{End}(\mathcal{V})$ the vector bundle of endomorphisms.

An eigenvalue of an endomorphism $h_S \in \Gamma(\mathcal{E}_S)$ is a complex root of the polynomial

$$p_{h_S} := \det(t \mathbb{I} - h_S) \in \Gamma(\text{End}(\det(\mathcal{V}_S))[t] = \Gamma(\mathcal{O}_S)[t] = \mathbb{C}[t].$$

We say that $\mathcal{V}$ splits if it is isomorphic to a direct sum of $r$ invertible sheaves (line bundles).

Let $S, h_S$ be as above. Note that, if $\varepsilon \in \mathbb{C}$ is an eigenvalue of $p_{h_S}$, then $\text{Ker}(\varepsilon \mathbb{I} - h_S) \subset \mathcal{V}_S$ is a non-zero $\mathcal{O}_S$-module. Indeed, let $x \in S_{\text{red}} \subset S$ be a closed point, with maximal ideal $m_x \subset \mathcal{O}_S$. Then $\varepsilon$ is an (usual) eigenvalue of $h_S \otimes \mathcal{O}_S \rightarrow \mathcal{O}_S$. We denote by $\mathcal{E} := \text{End}(\mathcal{V})$ the vector bundle of endomorphisms.

**Lemma 1.2** Let $S$ be a closed subscheme of a scheme $T$ defined over $\mathbb{C}$, and assume that $\Gamma(\mathcal{O}_S) = \Gamma(\mathcal{O}_T) = \mathbb{C}$.

(i) $\mathcal{V}_S$ splits if and only there is $h_S \in \Gamma(\mathcal{E}_S)$ with $r$ pairwise distinct eigenvalues in $\mathbb{C}$.

(ii) Let $\mathcal{V}_T$ be a locally free sheaf on $T$ such that $\Gamma(\mathcal{E}_T) \rightarrow \Gamma(\mathcal{E}_S)$ is surjective. Then $\mathcal{V}_T$ splits if and only if $\mathcal{V}_S$ splits.

**Proof.** (i) If $h_S \in \Gamma(\mathcal{E}_S)$ has distinct eigenvalues $\varepsilon_1, \ldots, \varepsilon_r \in \mathbb{C}$, then $\ell_j := \text{Ker}(\varepsilon_j \mathbb{I} - h_S) \neq 0$. The (polynomial) identity

$$1 = \sum_{j=1}^r c_j p_j(t), \quad \text{with } c_j := \left( \prod_{k \neq j} (\varepsilon_j - \varepsilon_k) \right)^{-1} \in \mathbb{C}, \quad p_j(t) := \prod_{k \neq j} (t - \varepsilon_k) \in \mathbb{C}[t],$$

yields $\mathcal{V}_S = \sum_{j=1}^r \text{Im}(p_j(h_S))$; by the Cayley-Hamilton theorem (over commutative rings),

$$\text{Im}(p_j(h_S)) \subset \ell_j \Rightarrow \mathcal{V}_S = \sum_{j=1}^r \ell_j.$$
The identity also implies that the sum is direct. Indeed, if \( v_j \in \ell_j \) for all \( j \), then holds:

\[
v_1 + v_2 + \ldots + v_r = 0 \quad \implies \quad p_1(h_S)(v_1) = 0,
\]

\[
v_1 = \sum_{j=2}^{r} c_j \cdot p_j(h_S)(v_1) v_1 \in \ker(h_S) 0, \text{ etc.}
\]

We deduce that \( \ell_j, j = 1, \ldots, r \), are (locally) projective \( \mathcal{O}_S \)-modules, so they are locally free (cf. [5, Ch. II, §5.2, Théorème 1]) of rank one.

(ii) If \( \mathcal{V}_S \) splits, there is \( h_S \in \Gamma(\mathcal{E}_S) \) with \( r \) pairwise distinct complex eigenvalues. This extends to \( h_T \in \Gamma(\mathcal{E}_T) \) which has the same eigenvalues, because \( p_{h_T} = \det(tI - h_T) \in \mathbb{C}[t] \), so \( p_{h_T} = p_{h_S} \). \( \square \)

**Definition 1.3** Let \( \mathcal{L} \) be a locally free sheaf on \( X \) and \( D \in |d\mathcal{L}| \) an effective divisor. For \( m \geq 0 \), the \( m \)-th order thickening \( D_m \) of \( D \) is the subscheme of \( X \) defined by the ideal \( \mathcal{I}^m \), where \( \mathcal{I}_D = \mathcal{O}_X(-D) \cong \mathcal{L}^{-d} \).

The structure sheaves of successive thickenings fit into the exact sequences

\[
0 \to \mathcal{L}_D^{-dm} \to \mathcal{O}_{D_m} \to \mathcal{O}_{D_m-1} \to 0, \quad m \geq 1.
\] (1.1)

**Lemma 1.4** Let \( \mathcal{L} \) be a \( q \)-ample line bundle on \( X \), with \( q \leq \dim X - 2 \). Consider \( D \in |\mathcal{L}| \) which is either reduced or irreducible. Then the following hold:

(i) \( \Gamma(\mathcal{O}_{D_m}) = \mathbb{C} \), for all \( m \geq 1 \);

(ii) \( \mathcal{V}_X \) splits if and only if its restriction \( \mathcal{V}_{D_m} \) splits, for \( m \geq 0 \).

**Proof.** (i) The \((\dim X - 2)\)-amplitude of \( \mathcal{L} \) implies that \( \mathbb{C} = \Gamma(\mathcal{O}_X) \to \Gamma(\mathcal{O}_{D_m}) \) is an isomorphism for \( m \gg 0 \), so \( D \) is connected.

If \( D \) is reduced, then \( \Gamma(\mathcal{O}_D) = \mathbb{C} \). As \( \Gamma(\mathcal{L}^{-m}) = 0 \) for \( m \gg 0 \), the same holds for all \( m \geq 1 \).

The conclusion follows from (1.1).

If \( D \) is irreducible, then \( D = m_0D_{\text{red}} \), for some \( m_0 \geq 1 \), and the previous reasoning shows that \( \Gamma(\mathcal{O}_{D_{\text{red},m}}) = \mathbb{C} \) for all \( m \geq 1 \).

(ii) We apply lemma 1.2: \( \Gamma(\mathcal{E}_X) \to \Gamma(\mathcal{E}_{D_m}) \) is surjective if \( H^1(\mathcal{E}_X \otimes \mathcal{L}^{-d(m+1)}) = 0 \). This is indeed the case, for large \( m \). \( \square \)

Now we state the splitting principle for vector bundles, obtained by restricting them to zero loci of sections of partially positive line bundles.

**Proposition 1.5** Assume \( \mathcal{L} \in \text{Pic}(X) \) is \( q \)-ample, with \( q \leq \dim X - 2 \), and let \( D \in |d\mathcal{L}| \) be an effective divisor which is either reduced or irreducible. If

\[
H^1(D, \mathcal{E}_D \otimes \mathcal{L}^{-a}) = 0, \quad \forall a \geq c,
\] (1.2)

then the following properties hold:

(i) \( H^1(X, \mathcal{E} \otimes \mathcal{L}^{-a}) = 0 \), for all \( a \geq c \).

(ii) If moreover \( d \geq c \) and \( \mathcal{V}_D \) splits, then \( \mathcal{V} \) splits too.

**Proof.** (i) Denote

\[
a_0 := \max\{a \mid H^1(X, \mathcal{E} \otimes \mathcal{L}^{-a}) \neq 0\} < \infty.
\]

The sequence \( 0 \to \mathcal{L}^{-d} \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \) yields

\[
\ldots \to H^1(\mathcal{E} \otimes \mathcal{L}^{-d-a_0}) \to H^1(\mathcal{E} \otimes \mathcal{L}^{-a_0}) \to H^1(\mathcal{E}_D \otimes \mathcal{L}_D^{-a_0}) \to \ldots ,
\]
with $-d - a_0 \leq -(a_0 + 1)$, so the leftmost term vanishes. If $a_0 \geq c$, then the rightmost and the middle terms vanish too. This contradicts the definition of $a_0$.

(ii) We have $\Gamma(\mathcal{O}_D) = \mathbb{C}$, by (4.14) as $d > c$, $\Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E}_D)$ is surjective. \hfill \Box

**Remark 1.6** (i) The uniform q-ampleness property [17, Theorem 6.4] implies that there is a linear function $l(r) = \lambda r + \mu$, with $\lambda, \mu$ depending on $\mathcal{L}$ only, such that (1.5(i)) holds for all $a \geq l(\text{reg}(\mathcal{E}))$, where reg($\mathcal{E}$) stands for the regularity of $\mathcal{E}$ with respect to a (fixed) ample line bundle $\mathcal{O}_X(1)$.

(ii) The condition (1.2) is formulated for $\mathcal{E}_D$, which splits by assumption. This choice is made because it is easier to decide the vanishing of the cohomology of line bundles, rather than of vector bundles. Also, for $q \leq \dim X - 3$, the condition (1.2) holds indeed for $c \gg 0$.

**Corollary 1.7** Let $f : X \rightarrow Y$ be a smooth, surjective morphism of relative dimension $\delta \geq 3$, $\mathcal{V}$ a vector bundle on $X$, and $\mathcal{L} \in \text{Pic}(X)$ be $f$-relatively ample such that $\mathcal{E} \otimes \mathcal{L}$ is relatively ample. If both $D \in \vert \mathcal{L} \vert$, $d \geq 1$, and $D \rightarrow Y$ are smooth, then holds: $[(\mathcal{V} \text{ splits } \iff \mathcal{V}_D \text{ splits})]$.

**Proof.** By [A.4], $\mathcal{L}$ is $\dim Y$-positive. As $f$ is smooth, the Kodaira vanishing and Grauert’s theorem [12, Ch. III, Corollary 12.9] imply $R^i f_* (\mathcal{E}_D \otimes \mathcal{L}_D^{-a}) = 0$, for $i = 0, \ldots, \delta - 2$ and $a \geq 1$ (use $\mathcal{E} = \mathcal{E}^\vee$). The Leray spectral sequence yields $H^1 (\mathcal{E}_D \otimes \mathcal{L}_D^{-a}) = 0$, so (1.2) is satisfied. \hfill \Box

2. Splitting along divisors: a ‘deterministic’ approach

**Definition 2.1** For $s \geq 1$, we say that a variety $X$ is $(s\text{-split})$ if $H^j (X, \ell) = 0$, for $j = 1, \ldots, s$, $\forall \ell \in \text{Pic}(X)$. $(s\text{-split})$

**Remark 2.2** For $s = 1, 2$ one gets respectively the ‘splitting’ and ‘Horrocks scheme’ notions introduced in [2]. Examples of varieties satisfying $(s\text{-split})$ are as follows:

(a) arithmetically Cohen-Macaulay varieties $X$ (e.g. homogeneous spaces, complete intersections) with cyclic Picard group (here $s = \dim X - 1$);

(b) projectivized bundles; if $Y$ satisfies $(s\text{-split})$, $M_1, \ldots, M_r$, $r \geq s + 2$, are line bundles on $Y$, then $X := \mathbb{P}(M_1 \oplus \ldots \oplus M_r)$ still satisfies $(s\text{-split})$ (cf. [2, Example 4.9]);

(c) Let $D$ be an effective $q$-ample divisor on $X$, $q \leq \dim X - 3$, which satisfies (1-split). Then $X$ satisfies (2-split). The proof is ad litteram [2, Proposition 4.13].

Thus, the conditions (1-split) for $D$ and (2-split) for $X$ are equivalent, as soon as $\text{Pic}(X) \overset{\cong}{\rightarrow} \text{Pic}(D)$.

**Theorem 2.3** If $D$ is (1-split) and $(\dim X - 2)$-ample, then holds: $[(\mathcal{V} \text{ splits } \iff \mathcal{V}_D \text{ splits})]$. If $X$ is $s$-split, $\mathcal{L}$ is $(\dim X - 4)$-positive, and $D \in \vert \mathcal{L} \vert$ is smooth, then $D$ is $(s - 1)$-split.

The statement is [2, Proposition 4.13] with the difference that here we consider $q$-ample instead of ample divisors, a considerably weaker condition.

**Proof.** Since $\mathcal{V}_D$ splits, $H^1 (\mathcal{E}_D \otimes \mathcal{L}_D^{-a}) = 0$ for all $a \geq 1$. The conclusion follows from (4.15)

It remains to prove that (2-split) for $X$ implies (1-split) for $D$. By [A.5], $\text{Pic}(X) \rightarrow \text{Pic}(D)$ is an isomorphism, so for all $\ell \in \text{Pic}(D)$ there is a unique $\hat{\ell} \in \text{Pic}(X)$ such that $\hat{\ell}_D = \ell$. Now take the cohomology of the sequence $0 \rightarrow \mathcal{L}^{-1} \otimes \hat{\ell} \rightarrow \hat{\ell} \rightarrow \ell \rightarrow 0$. \hfill \Box
The interest in allowing partially positive line bundles in \cite{2} is that one can apply the result to morphisms (fibre bundles). This is beyond the reach of \cite{2}.

**Corollary 2.4** Let \( f : X \to Y \) be a smooth morphism between smooth projective varieties, and assume that \( X \) is (2-split). In any of the following cases, the splitting of \( \mathcal{V}_D \) implies the splitting of \( \mathcal{V} \): 

(i) \( M \in \text{Pic}(Y) \) is \((\dim Y - 4)\)-positive, \( D_Y \in |M| \) smooth, \( D := f^{-1}(D_Y) \);

(ii) \( \mathcal{L} \in \text{Pic}(X) \) is \( f \)-relatively ample, \( D \in |\mathcal{L}| \) is smooth, and \( \dim X - \dim Y \geq 4 \).

**Proof.** In the case (i), \( \mathcal{L} := f^*M \) is \((\dim X - 4)\)-positive; in the case (ii), lemma \ref{A} implies that \( \mathcal{L} \) is \( \text{dim} Y \)-positive. We conclude by \ref{2}.

**Example 2.5** Let \( Y \) be a smooth projective variety, \( M, M_1, \ldots, M_r \in \text{Pic}(Y), r \geq 3 \); define \( X := \mathbb{P}(M_1 \oplus \cdots \oplus M_r) \). The vector bundle \( \mathcal{V} \) on \( X \) splits if the following conditions hold:

- \( M \) is \((\dim Y - 3)\)-ample;
- \( D_Y \in |M| \) is (1-split), e.g. \( M \) is \((\dim Y - 4)\)-positive and \( Y \) is (2-split);
- \( \mathcal{V}_{f^{-1}(D_Y)} \) splits.

**Example 2.6** (i) A straightforward consequence of \ref{2} is the following. Let \( G/P \) be a homogeneous variety with \( P \subset G \) a maximal parabolic subgroup. A vector bundle on \( G/P \) splits if and only if its restriction to some complete intersection surface \( S \subset G/P \) splits.

(ii) (Vector bundles on multi-projective spaces). A splitting criterion for vector bundles \( \mathcal{V} \) on \( X := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \) is obtained in \cite{3} Theorem 4.7. It generalizes Horrocks’ criterion, and involves the vanishing of \((n_1 + 1) \cdot \ldots \cdot (n_r + 1)\) cohomology groups. By repeatedly applying \ref{2} for the pull-back of \( \mathcal{O}_{\mathbb{P}^n}(1) \), we deduce that \( \mathcal{V} \) splits if and only if it does so along an arbitrary \( Z := \mathbb{P}^2 \times \cdots \times \mathbb{P}^2 \). This reduces the number of cohomological tests to \( 3^r \).

(iii) (Vector bundles on products of projective spaces and quadrics). The results of \cite{3} have been extended in \cite{3} Theorem 2.14, 2.15 to vector bundles on the product \( X_1 \times X_2 \), where \( X_1 \) is as above and \( X_2 \) is a product of hyper-quadrics \( Q_{r_1} \subset \mathbb{P}^{n+r} \). The splitting criterion involves a very large number of cohomological conditions. By \ref{2} a vector bundle on \( X_1 \times X_2 \) splits if and only if it splits when restricted to some \( X'_1 \times X'_2 \subset X_1 \times X_2 \), where \( X'_1 \) is a product of projective planes \( \mathbb{P}^2 \) and \( X'_2 \) is a product of copies of \( Q_3 \). Again, the number of necessary cohomological tests is dramatically reduced.

### 3. Splitting along divisors: a ‘probabilistic’ approach

Here we obtain splitting criteria for vector bundles by restricting them to zero loci of generic sections of globally generated, partially positive line bundles. The global generation allows to replace the (2-split) by the weaker (1-split) condition. Note that, if \( \mathcal{L} \) is a \( q \)-ample line bundle on \( X \) such that \( L^d \) is globally generated for some \( d \geq 1 \), then \( \mathcal{L} \) is \( q \)-positive (cf. \cite{14} Theorem 1.4)); henceforth we replace \( L^d \) by \( \mathcal{L} \). Also, the fibres of the morphism \( f : X \to |\mathcal{L}| \) are at most \( q \)-dimensional (cf. ibid.), so \( \dim(\text{Im}(f)) \geq \dim X - q \).

Let the situation be as above. We start with general considerations: the equations defining \( X, \mathcal{L}, \mathcal{V} \) involve finitely many coefficients in \( \mathbb{C} \). By adjoining them to \( \mathbb{Q} \), we obtain a field extension \( \mathbb{Q} \hookrightarrow \mathbb{k} \) of finite type (which depends on \( \mathcal{V} \)); its algebraic closure is then a countable subfield of \( \mathbb{C} \). After replacing \( \mathbb{k} \) by \( \bar{\mathbb{k}} \), we may assume that \( X, \mathcal{L}, \mathcal{V} \) are defined over a countable, algebraically closed subfield \( \mathbb{k} \hookrightarrow \mathbb{C} \); we denote by \( X_k, \mathcal{L}_k, \mathcal{V}_k \) these objects.
The sheaf $\mathcal{G} := \text{Ker}(\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L})$ is locally free and the incidence variety

$$\mathcal{Y} := \{(s, x) \mid s(x) = 0\} \subset |\mathcal{L}| \times X$$

is naturally isomorphic to the projective bundle $\mathbb{P}(\mathcal{G})$ over $X$. Denote by $\pi, \rho$ the projections of $\mathcal{Y}$ onto $|\mathcal{L}|, X$ respectively. All these objects are defined over $k$ and are denoted by $\mathcal{L}_k, \mathcal{Y}_k, \pi_k, \rho_k$. For any open $S \subset |\mathcal{L}|$, let $\mathcal{Y}_S := \pi^{-1}(S)$; for $s \in |\mathcal{L}|$, let $D_s := \{s = 0\}$.

**Definition 3.1** We say that a property (in our case, the splitting of $\mathcal{Y}_{D_s}$) holds for a very general point in some parameter space $(s \in |\mathcal{L}|$ in our case), if it holds in the complement of countably many proper subvarieties of the parameter space.

**Lemma 3.2** Denote $K_C := \mathbb{C}(|\mathcal{L}|)$, the function field of the projective space $|\mathcal{L}|$.

1. If $\mathcal{Y}_{D_s}$ splits for a very general $s \in |\mathcal{L}|$, then $\mathcal{Y} \otimes K_C$ splits.
2. If $\mathcal{Y} \otimes K_C$ splits, then there is an (analytic) open ball $\mathbb{B} \subset |\mathcal{L}|$ such that $D_s$ is smooth, for all $s \in \mathbb{B}$, and $\mathcal{Y}_\mathbb{B}$ splits over $\mathcal{Y}_\mathbb{B}$.

**Proof.** (i) Let $\tau : |\mathcal{L}| \to |\mathcal{L}_k|$ be the trace morphism. Since $k$ is countable and $\mathbb{C}$ is not, $\tau(s)$ is the generic point of $|\mathcal{L}_k|$, for $s \in |\mathcal{L}|$ very general. Hence $k(\tau(s)) = k(|\mathcal{L}_k|) =: K_k$, and we obtain the Cartesian diagram below:

$$\begin{array}{ccc}
Y_s & \longrightarrow & \mathcal{Y}_k \\
\downarrow & & \downarrow \\
\text{Spec}(\mathbb{C}) & \longrightarrow & \text{Spec}(K_k) \\
\downarrow & & \downarrow \\
|\mathcal{L}_k| & . & .
\end{array}$$

For varieties defined over algebraically closed fields (C and $K_k$ in our case), the splitting of a vector bundle commutes with base change. The previous discussion implies that $\mathcal{Y}_k$ splits on $Y_{K_k}$, thus the same holds for $\mathcal{Y}_K$.

(ii) We observe that $\mathcal{Y}_{K_C}$ actually splits over an intermediate field $K_C \to K' \to K_C$ finitely generated (and also algebraic) over $K_C$, so there is an open affine $S \subset |\mathcal{L}|$, an affine variety $S'$ over $\mathbb{C}$, and a finite morphism $S' \to S$ such that the direct summands of $\mathcal{Y}_{K_C}$ are defined over $\mathbb{C}[S']$. After shrinking $S$ further, we may assume that $Y_s$ is smooth, for all $s \in S$, by Bertini’s theorem. Finally, there are open balls $\mathbb{B}' \subset S'$ and $\mathbb{B} \subset S$ such that $\sigma : \mathbb{B}' \to \mathbb{B}$ is an analytic isomorphism. Then the splitting of $\mathcal{Y}_s$ descends to $\mathcal{Y}_s$ on $\mathcal{Y}_\mathbb{B}$. \qed

**Theorem 3.3** Let $\mathcal{L}$ be globally generated, $(\dim X - 5)$-positive, and $D \in |\mathcal{L}|$ very general (thus smooth). If $X$ satisfies (1-split), then holds: [$\mathcal{Y}$ splits $\Leftrightarrow \mathcal{Y}_{D_s}$ splits].

The result, on one hand, is stronger than theorem [23] because it allows varieties satisfying the weaker condition (1-split) rather than (2-split); on the other hand, it requires very general divisors, instead of arbitrary ones. The previous lemma also precises the meaning of a very general point $s \in |\mathcal{L}|$: the coordinates of $s$ should be algebraically independent over the definition field $k$ of $X, \mathcal{L}, \mathcal{Y}$.

**Proof.** By [15, Proposition 5.1], the cohomological dimension $\text{cd}(X \setminus D_s) \leq \dim X - 5$, so the intersections $D_{st} := D_s \cap D_t$ and $D_{ost} := D_o \cap D_s \cap D_t$ are non-empty and connected, for all $o, s, t \in \mathbb{B}$; since $\mathcal{L}$ is globally generated, these intersections are generically transverse.
Moreover, theorem \[\text{A.5}\] implies \(\text{Pic}(X) \xrightarrow{\cong} \text{Pic}(D_s)\), for all \(s \in \mathbb{B}\). In the diagram below all the arrows are isomorphisms whenever \(D_{st}\) is a transverse intersection:

\[
\begin{array}{c}
\text{Pic}(X) \\
\text{res}^X_{D_s} \quad \text{res}^X_{D_{st}} \\
\text{Pic}(D_s) \quad \text{Pic}(D_{st}) \quad \text{Pic}(D_t) \\
\text{res}^D_{D_s} \quad \text{res}^D_{D_{st}} \quad \text{res}^D_{D_t}
\end{array}
\]

\[\text{(3.1)}\]

Claim \(\rho^* : \text{Pic}(X) \to \text{Pic}(\mathcal{Y}_\mathbb{B})\) is an isomorphism. Indeed, fix \(a \in \mathbb{B}\). The composition \(\text{Pic}(X) \xrightarrow{\rho^*} \text{Pic}(\mathcal{Y}_\mathbb{B}) \xrightarrow{\text{res}_{\mathcal{Y}_\mathbb{B}}} \text{Pic}(\mathcal{Y}_0)\) is bijective, so \(\rho^*\) is injective. For the surjectivity, take \(\ell \in \text{Pic}(\mathcal{Y})\). If \(\ell_{Y_0} \cong O_{Y_0}\), then \(\{s \in \mathbb{B} \mid \ell_{Y_s} \not\cong O_{Y_s}\} = \{s \in \mathbb{B} \mid h^0(\ell_{Y_s}) = 0\}\) is open in \(\mathbb{B}\), so \(\{s \in S \mid \ell_{Y_s} \cong O_{Y_s}\}\) is closed. On the other hand, by restricting to \(Y_{os}\), \[\text{(3.1)}\] implies that this set is dense, so it is the whole \(\mathbb{B}\), hence \(\ell \cong O_{Y_0}\). If \(\ell \in \text{Pic}(\mathcal{Y}_\mathbb{B})\) is arbitrary, take \(\mathcal{L} \in \text{Pic}(X)\) such that \(\ell_{Y_s} \cong O_{Y_s}\), so \((\rho^*\ell^{-1})|_{Y_0}\) is trivial.

By using that \(\mathcal{Y}_\mathbb{B}\) splits and the previous claim, we deduce that \(\rho^*\mathcal{Y} \cong \rho^*(\bigoplus_{j \in J} \mathcal{L}_j^{\oplus d_j})\), with \(\mathcal{L}_j \in \text{Pic}(X)\) pairwise non-isomorphic. We consider the following partial order on line bundles:

\[\mathcal{L} \prec M \iff \mathcal{L} \neq M \text{ and } \Gamma(\mathcal{L}^{-1} \otimes M) \neq 0.\]

For \(s \in \mathbb{B}\), let \(J_{s, \text{max}} \subset J\) be the set of maximal elements for \(\prec\) on \(\text{Pic}(D_s) \cong \text{Pic}(X)\). By semi-continuity, the set \(\{t \in \mathbb{B} \mid J_{s, \text{max}} \subset J_{t, \text{max}}\}\) is open. Thus, after shrinking \(\mathbb{B}\), we may assume that \(J_{s, \text{max}} \subset J\) is independent of \(s\); we denote it by \(\text{J}_{\text{max}}\).

The maximality property implies that there is a natural, pointwise injective homomorphism

\[
h : \bigoplus_{\mu \in J_{\text{max}}} \rho^* \mathcal{L}_\mu \otimes \pi^* \pi_* \left(\rho^* (\mathcal{L}_\mu^{-1} \mathcal{Y})\right) \rightarrow \left(\rho^* \mathcal{Y}\right)_{\mathbb{B}}.
\]

\[\text{(3.2)}\]

Claim \(h\) descends to \(X\) after suitable changes of bases in \(\mathcal{O}^{\oplus d_\mu}_{\mathcal{Y}_\mathbb{B}}\) by some map \(\mathbb{B} \to \prod_{\mu \in J_{\text{max}}} \text{GL}(d_\mu)\).

Indeed, fix \(s \in \mathbb{B}\), and for each \(\mu \in J_{\text{max}}\) choose a basis in \(\Gamma(D_s, \mathcal{L}_\mu^{-1} \mathcal{Y}) \cong \mathbb{C}^{\oplus d_\mu}\). For any \(s \in \mathbb{B}\), \(D_{os}\) is non-empty, connected, so there is a (unique) basis in \(\Gamma(D_s, \mathcal{L}_\mu^{-1} \mathcal{Y}) \cong \mathbb{C}^{\oplus d_\mu}\), whose restriction to \(D_{os}\) coincides with the restriction of the basis along \(D_0\). We observe that, for any \(s, t \in \mathbb{B}\), the restrictions of these bases from \(D_s, D_t\) to \(D_{st}\) coincide: it is enough to check this on the triple intersections \(D_{ost} = D_o \cap D_{st}\) (which are non-empty, connected), where both bases are induced from \(D_o\). After this reparameterization \(h\) descends, as claimed, to the open set \(\mathcal{U} := \rho(\mathcal{Y}_\mathbb{B}) \subset X\). Indeed, define

\[
\bar{h} : \left( \bigoplus_{\mu \in J_{\text{max}}} \mathcal{L}_\mu^{\oplus d_\mu} \right) \otimes \mathcal{O}_{\mathcal{U}} \rightarrow \mathcal{Y}_{\mathcal{U}}\text{, }\bar{h}(x) := h_a(x)\text{ for some }s \in \mathbb{B}\text{ such that }x \in D_s.
\]

\[\text{(3.3)}\]

The remark above implies that \(\bar{h}(x)\) is independent of \(s \in \mathbb{B}\).

The homomorphism \[\text{(3.3)}\] yields the extension of locally free sheaves on \(\mathcal{U}\):

\[
0 \rightarrow \left( \bigoplus_{\mu \in J_{\text{max}}} \mathcal{L}_\mu^{\oplus d_\mu} \right) \otimes \mathcal{O}_{\mathcal{U}} \rightarrow \mathcal{Y}_{\mathcal{U}} \rightarrow \mathcal{W}_{\mathcal{U}} \rightarrow 0, \quad \rho^* \mathcal{W}_{\mathcal{U}} \cong \rho^* \left( \bigoplus_{\mu \in J_{\text{max}}} \mathcal{L}_\mu^{\oplus d_\mu} \right) \otimes \mathcal{O}_{\mathcal{U}}.
\]

But \(\mathcal{U}\) is an analytic neighborhood of \(D_o\), so it determines extensions on the thickenings \((D_o)_m, m \geq 1\), of \(D_o\) (see \[\text{I.3}\]). Since \(X\) is (1-split) and \(D_o\) is \((\dim X - 5)\)-ample,

\[
0 = \text{Ext}^1(\mathcal{L}, \mathcal{M}) \rightarrow \text{Ext}^1(\mathcal{L}(D_o)_m, \mathcal{M}(D_o)_m)
\]

\[\text{(3.4)}\]
is an isomorphism, for all \( \mathcal{L}, \mathcal{M} \in \text{Pic}(X) \), \( m \gg 0 \). Recursively, we deduce that \( \mathcal{V} \) splits on some \((D_0)_m\), \( m \gg 0 \), so \( \mathcal{V} \) splits on \( X \), by lemma \ref{lemma:splitting1}.

**Remark 3.4** The hypothesis that \( \mathcal{L} \) is \((\dim X - 5)\)-ample allows to conclude that the arrows in \( \ref{table:splitting1} \) are isomorphisms. If one knows \textit{a priori} that \( \ref{table:splitting1} \) holds, then is enough to assume that \( \mathcal{L} \) is \((\dim X - 3)\)-ample, which is necessary for the isomorphism \( \ref{table:splitting1} \).

**Example 3.5** Homogeneous varieties are an abundant source of \((1\text{-split})\) varieties (with non-cyclic Picard group). Consider a connected, reductive, linear algebraic group \( G \) and the parabolic subgroup \( P_I \) generated by the set \( I \) of simple roots of \( G \). In \cite{[H]
[6.1]} is proved that

\[ G/P_I \text{ is \((1\text{-split})\) } \iff \text{there are no simple roots of } G \text{ orthogonal to all } I. \]

Let \( I \) be as above. Take a maximal parabolic \( P \supset P_I \); then the morphism \( f : G/P_I \to G/P \) is a smooth fibration. If \( \dim G/P \geq 5 \), then \( \mathcal{L} := f^*\mathcal{O}_{G/P}(1) \) is globally generated and \((\dim X - 5)\)-positive, so one can apply theorem \ref{thm:splitting3} in combination with \ref{thm:splitting1}(i).

4. Triviality criteria

Finally, in this section we restrict our discussion to the case of trivializable vector bundles. The motivation is, first, that the ‘effective splitting criterion’ \ref{thm:splitting5} which is valid for arbitrary varieties, is not explicit enough (cf. remark \ref{rem:splitting6}). Second, it is desirable to remove the conditions (1-, 2-split) in the sections \ref{thm:splitting2} \ref{thm:splitting3} imposed precisely to ensure the vanishing \ref{thm:splitting2}.

An effective bound was pointed out in \ref{thm:splitting7} in the case when \( \mathcal{L} \) is relatively ample. Unfortunately, the Kodaira vanishing (built into the proof of \ref{thm:splitting7}) does not hold for \( q \)-ample line bundles. Thus, to obtain effective results in this situation, one must search for appropriate conditions which imply the Kodaira vanishing. These lines of thought lead to the triviality criteria below.

**Lemma 4.1** Assume \( \mathcal{L} \in \text{Pic}(X) \) is \((\dim X - 2)\)-ample and satisfies \( H^i(X, \mathcal{L}^{-a}) = 0 \), for all \( a \geq 1 \) and \( i = 0, 1, 2 \). For \( D \in |\mathcal{L}| \) and an arbitrary vector bundle \( \mathcal{V} \) on \( X \), the following statements hold:

(i) The restriction \( \text{Pic}(X) \to \text{Pic}(D) \) is injective;

(ii) One has the equivalence: \( \mathcal{V} \cong \mathcal{O}^\oplus_X \iff \mathcal{V}_D \cong \mathcal{O}^\oplus_D \).

**Proof.** (i) The exact sequence \( 0 \to \mathcal{L}^{-1} \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \) implies \( H^i(\mathcal{L}^{-a}_D) = 0 \), for all \( a \geq 1 \) and \( i = 0, 1 \). By plugging this into \( 0 \to \mathcal{L}^{-a} \to \mathcal{O}_D^\times \to \mathcal{O}_{D,a}^\times \to 0 \), we deduce that \( \text{Pic}(D_a) \to \text{Pic}(D_{a-1}) \) is injective, for all \( a \geq 1 \).

Take \( M \in \text{Ker}(\text{Pic}(X) \to \text{Pic}(D)) \), that is \( \mathcal{M}_D \cong \mathcal{O}_D \); it follows \( \mathcal{M}_{D,a} \cong \mathcal{O}_{D,a} \), for all \( a \geq 0 \). But the restrictions of \( \Gamma(\mathcal{M}), \Gamma(\mathcal{M}^{-1}) \to \Gamma(\mathcal{M}_{D,a}) = \mathbb{C} \) are isomorphisms, for \( a \gg 0 \). We conclude that \( \mathcal{M} \cong \mathcal{O}_X \).

(ii) We already noticed that \( H^i(\mathcal{L}^{-a}_D) = 0 \), for all \( a \geq 1 \) and \( i = 0, 1 \); also \( \Gamma(\mathcal{O}_{D,a}) = \mathbb{C} \), for \( a \geq 0 \). The proposition \ref{thm:splitting5} implies that \( \Gamma(\mathcal{E}') \to \Gamma(\mathcal{E}_D) = \text{End}(\mathbb{C}^r) \) is an isomorphism; so \( \mathcal{V} \) splits, actually \( \mathcal{V} \cong \mathcal{M}^\oplus r \) for some \( M \in \text{Pic}(X) \). As \( \mathcal{V}_D = \mathcal{O}^\oplus_D \), we deduce that both \( \mathcal{M}_D \) and \( \mathcal{M}^{-1}_D \) admit non-trivial sections, so \( \mathcal{M}_D \cong \mathcal{O}_D \). By (i) above, \( \mathcal{M} \) is trivial. \( \square \)

**Theorem 4.2** Consider an arbitrary vector bundle \( \mathcal{V} \) on \( X \), \( \mathcal{L} \in \text{Pic}(X) \), and \( D \in |\mathcal{L}| \). In any of the following cases one has the equivalence: \( \mathcal{V} \cong \mathcal{O}^\oplus_X \iff \mathcal{V}_D \cong \mathcal{O}^\oplus_D \).

(i) If \( \mathcal{L} \in \text{Pic}(X) \) is globally generated and \((\dim X - 3)\)-ample.
(ii) If \(L\) is relatively ample for a smooth, surjective morphism \(f : X \to Y\) of relative dimension at least three.

Proof. (i) Since \(L\) is \((\dim X - 3)\)-ample, theorem A.6 implies \(H^i(L^{-a}) = 0\) for \(a \geq 1, i = 0, 1, 2\). The conclusion follows from 4.1.

(ii) Since \(L\) is relatively ample, \(L\) is dim \(Y\)-positive. The Leray spectral sequence yields \(H^i(L^{-a}) = 0\), for \(a \geq 1\) and \(i = 0, 1, 2\), and the conclusion follows again from 4.1. □

4.1. The case of Frobenius split (F-split) varieties. These objects are ubiquitous, especially in representation theory. Examples of F-split varieties (defined in characteristic zero) include Fano varieties (cf. [7, Exercise 1.6E(5)]), spherical varieties, in particular projective homogeneous varieties and toric varieties (cf. [16, Section 31] and references therein). The notions and properties relevant for this note are summarized in the appendix B.

Theorem 4.3 Let \(D\) be a \((\dim X - 3)\)-ample, effective divisor, which is F-split. Then holds:

\[ V \cong \mathcal{O}_X^{\oplus r} \iff V_D \cong \mathcal{O}_D^{\oplus r}. \]

Proof. Since \(\mathcal{O}_D(D)\) is \((\dim D - 2)\)-ample, theorem B.3 implies \(H^1(D, \mathcal{O}_D(-aD)) = 0\), for all \(a \geq 1\). The conclusion follows from the proposition 1.5. □

This criterion allows to handle more ‘exotic’ situations than before. Many examples arise from varieties whose F-splitting is compatible with respect to a divisor (cf. appendix B).

Corollary 4.4 Let \(X\) be a projective variety such that the dualizing sheaf \(\kappa^{-1}_X\) of \(X\) is \((\dim X - 3)\)-ample. Assume that \(X\) is F-split by \(\sigma \in \Gamma(\kappa^{-1}_X)\), and denote \(D := \text{divisor}(\sigma)\). Then holds: \([V \cong \mathcal{O}_X^{\oplus r} \iff V_D \cong \mathcal{O}_D^{\oplus r}]\).

Proof. In this case \(D\) is F-split, compatibly with the splitting defined by \(\sigma\). □

One of the simplest applications is the case of a smooth projective toric variety \(X_\Sigma\) defined by a regular fan \(\Sigma\). Then \(X_\Sigma\) is F-split, compatibly with the invariant divisors \(D_\rho, \rho \in \Sigma(1)\) and their intersections (cf. [7, Exercise 1.3E(6)]). The previous corollary, applied to the torus-invariant anti-canonical divisor \(\sum_{\rho \in \Sigma(1)} D_\rho\), yields the following:

Proposition 4.5 Assume \(\kappa^{-1}_{X_\Sigma}\) is \((\dim X - 3)\)-ample. Then a vector bundle \(V\) on \(X_\Sigma\) is trivializable if and only if the restrictions \(V_{D_\rho}\) are trivializable, and these trivializations are compatible with the various intersections between the invariant divisors \(D_\rho, \rho \in \Sigma(1)\).

---

1 One might wonder if it is possible to have a splitting criterion for toric varieties which involves an irreducible, torus-invariant, \((\dim X - 2)\)-ample divisor. The answer is no in general, one has to consider reducible divisors for the following reason. If \(D\) was such an irreducible divisor, then \(\text{cd}(X \setminus D) \leq \dim X - 2\), so \(D\) intersects all the other torus-invariant divisors. Hence \(\Sigma\) has the following property: if \(\xi_D \in \Sigma(1)\) defines \(D\), then \(\xi_D, \xi\) span a cone of \(\Sigma\), for all \(\xi \in \Sigma(1) \setminus \{\xi_D\}\). This condition is clearly not satisfied in general.

Concerning the \((\dim X - 3)\)-amplitude of \(\kappa^{-1}_X\): this is likely to be a weak assumption, which is satisfied by ‘generic’ fans. There are explicit formulae, depending on the combinatorics of \(\Sigma\), which express the dimension of the cohomology groups of line bundles on \(X_\Sigma\).
Although it seems surprising, the issue concerning the bare existence of non-trivial vector bundles on toric varieties is not yet settled in general (cf. [10]).

More generally, spherical varieties (which are still described by combinatorial data) are Frobenius split, compatible with the anti-canonical divisor (cf. [6] and [16, §31.4]). Therefore, a similar proposition holds in this (much) more general case.

**APPENDIX A. ABOUT q-AMPLE AND q-POSITIVE LINE BUNDLES**

In this section we summarize the notions and results about partial positivity for line bundles which are used in this note. Throughout this section, \( X \) stands for a smooth projective variety defined over \( \mathbb{C} \).

**Definition A.1**

(i) (cf. [17]) The line bundle \( \mathcal{L} \) on \( X \) is called \( q \)-ample if for any coherent sheaf \( \mathcal{F} \) on \( X \) there is \( m_\mathcal{F} > 0 \) such that \( H^i(X, \mathcal{F} \otimes \mathcal{L}^m) = 0 \), for all \( m \geq m_\mathcal{F} \) and \( i > q \).

(ii) (cf. [1, 9]) The line bundle \( \mathcal{L} \) on \( X \) is called \( q \)-positive if it admits a Hermitian metric such that, at each \( x \in X \), the curvature is positive definite on a subspace of \( T_{X,x} \) of dimension at least \( \dim X - q \). (Equivalently, the curvature has at each point \( x \in X \) at most \( q \) negative or zero eigenvalues.)

**Remark A.2**

(i) Any \( q \)-positive line bundle is \( q \)-ample (cf. [1, Proposition 28], [9, Proposition 2.1]), but the converse is false (cf. [15, Theorem 10.3]).

(ii) If \( \ell, \mathcal{L} \in \text{Pic}(X) \) and \( \mathcal{L} \) is \( q \)-ample, then \( \ell \otimes \mathcal{L}^m \) is \( q \)-ample for all \( m \gg 0 \). Indeed, this is a direct consequence of the uniform \( q \)-ampleness property [17, Theorem 6.4].

(iii) The definition [A.1.i] makes sense for any projective scheme, not necessarily a smooth variety. This more general definition will be used in [A.3].

(iv) If \( \mathcal{L} \) is \( q \)-ample (positive), then it is also \( q' \)-ample (positive), for any \( q' \geq q \). Thus, the larger the value of \( q \) the weaker the restriction on \( \mathcal{L} \).

**Theorem A.3**

(cf. [14, Theorem 1.4]) If \( \mathcal{L} \) is semi-ample (that is, a tensor power is globally generated) and \( q \)-ample, then \( \mathcal{L} \) is \( q \)-positive.

**Lemma A.4** Let \( X, Y \) be smooth projective varieties, and \( f : X \to Y \) be a smooth, surjective morphism of relative dimension \( \delta \). Then the following implications hold:

(i) If \( M \in \text{Pic}(Y) \) is \( q \)-ample, then \( \mathcal{L} := f^*M \) is \( (\delta + q) \)-ample;

(ii) If \( M \in \text{Pic}(Y) \) is \( q \)-positive, then \( \mathcal{L} := f^*M \) is \( (\delta + q) \)-positive;

(iii) If \( \mathcal{L} \in \text{Pic}(X) \) is \( f \)-relatively ample, then \( \mathcal{L} \) is \( \dim Y \)-positive.

**Proof.** Leray’s spectral sequence implies (i); for (ii), the pull-back metric on \( \mathcal{L} \) satisfies [A.1] (iii) Note that \( \mathcal{L}' := \mathcal{L} \otimes f^*A \) is ample, for \( A \in \text{Pic}(Y) \) sufficiently ample. Then \( m\mathcal{L}', m \gg 0 \), defines an embedding \( \iota : X \to \mathbb{P} \) into some projective space; the morphism \( (f, \iota) : X \to Y \times \mathbb{P} \) is an embedding too, and \( \mathcal{L}^m = (f, \iota)^*(f^*A^{-m} \otimes O_{\mathbb{P}}(1)) \). The restriction to \( X \) of the product metric on the line bundle on the right hand side is positive definite on \( \ker(df) \). \( \square \)

**Theorem A.5** Let \( D \subset X \) be a smooth \( q \)-positive divisor, \( q \leq \dim X - 4 \). Then the restriction \( \text{Pic}(X) \to \text{Pic}(D) \) is an isomorphism.

**Proof.** The \( q \)-positivity of \( \mathcal{L} \) implies that \( H^i(X; \mathbb{Z}) \to H^i(D; \mathbb{Z}) \), \( i \leq 2 \), are isomorphisms (cf. [31, Theorem III], [15, Lemma 10.1]); hence the same holds with \( \mathbb{C} \)-coefficients. The Hodge
decomposition for $X, D$ implies $H^i(X; \mathcal{O}_X) \cong H^i(D; \mathcal{O}_D)$, for $i \leq 2$, and the exponential sequence yields $\text{Pic}(X) \cong \text{Pic}(D)$.

**Theorem A.6** Assume that $\mathcal{L} \in \text{Pic}(X)$ is globally generated and $q$-ample, $q \leq \dim X - 1$. Then holds $H^i(X, \mathcal{L}^{-a}) = 0$, $\forall i \leq \dim X - q - 1$, $\forall a \geq 1$.

**Proof.** Consider the morphism $X \to |\mathcal{L}|$ and note that $\dim |\mathcal{L}| \geq 1$. Otherwise, $\mathcal{L} \cong \mathcal{O}_X$ is $(\dim X - 1)$-ample, so $\dim X = 0$, a contradiction. Now, Bertini’s theorem implies that the generic divisor $D \in |\mathcal{L}|$ is smooth.

By [15, Corollary 5.2], $H^i(X; \mathcal{C}) \to H^i(D; \mathcal{C})$ is an isomorphism for $i \leq \dim X - q - 2$, and it is injective for $i = \dim X - q - 1$. The same statement holds for $H^i(X; \mathcal{O}_X) \to H^i(D; \mathcal{O}_D)$, for $i \leq \dim X - q - 1$, by the Hodge decomposition. This yields the conclusion. \qed

**APPENDIX B. ABOUT FROBENIUS SPLIT (F-SPLIT) VARIETIES**

We recall the relevant definitions; the reference for the concept of Frobenius splitting is the book [7], and also [16, Section 31] for applications.

**Definition B.1** (cf. [7, Definition 1.1.3 and Section 1.6]) Let $X_p$ be a projective variety over $\overline{\mathbb{F}_p}$ (the algebraic closure of the field $\mathbb{Z}/p\mathbb{Z}$). The absolute Frobenius morphism $F$ of $X_p$ determines the sheaf homomorphism $F^\sharp : \mathcal{O}_{X_p} \to F_* \mathcal{O}_{X_p}$. One says that $X_p$ is $F$-split if there is an $\mathcal{O}_{X_p}$-linear homomorphism

$$\varphi : F_* \mathcal{O}_{X_p} \to \mathcal{O}_{X_p} \text{ such that } \varphi \circ F^\sharp = 1_{\mathcal{O}_{X_p}}.$$

A closed subscheme $Y \subset X$ defined by the sheaf of ideals $I_Y$ is compatibly split, if $\varphi(I_Y) = I_Y$.

If $X$ is a smooth projective variety defined over a field of characteristic zero, there is a finite set $s$ of primes, a finitely generated $\mathbb{Z}[s^{-1}]$-algebra $R$, and a smooth $\text{Spec}(R)$-scheme $\mathcal{X}$ such that $X = \mathcal{X} \times_R \mathbb{C}$. If $\mathcal{L} \in \text{Pic}(X)$, one may choose $R$ in such a way that $\mathcal{L}$ also extends over $\text{Spec}(R)$.

**Definition B.2** For a maximal ideal $m \in \text{Spec}(R)$, the residue field $k(m)$ is a finite extension of $\mathbb{F}_p$, with $p \not\in s$. The variety $X_p := \mathcal{X} \times_R k(m)$ is called a reduction modulo $p$ of $X$. (Note that $k(m) \cong \overline{\mathbb{F}_p}$)

We say that $X$ (defined in characteristic zero) is $F$-split if $X_p$ is so, at infinitely many $m \in \text{Spec}(R)$. (Note that such a subset is automatically dense in $\text{Spec}(R)$.)

In our context, the importance of this notion is captured in the following

**Theorem B.3** Let $Y$ be an equidimensional, $F$-split, Cohen-Macaulay (not necessarily irreducible) projective variety over $\mathbb{C}$ and let $\mathcal{L} \in \text{Pic}(Y)$ be $q$-ample. Then holds $H^i(Y, \mathcal{L}^{-1}) = 0$, for all $i < \dim Y - q$.

In this note, the result will be applied in the case when $Y$ is a compatibly split, normal crossing divisor of a variety $X$.

**Proof.** Consider $\mathcal{Y} \to \text{Spec}(R)$ as above, such that $\mathcal{L}$ extends to $\mathcal{L} \to \mathcal{Y}$. Then for all primes $p$ large enough, $\mathcal{L}_p \in \text{Pic}(Y_p)$ is still $q$-ample (cf. [17, Theorem 8.1]), so $H^i(Y_p, \mathcal{L}_p^{-m}) = 0$, for $i < \dim Y - q$ and $m \gg 0$, by Serre duality (cf. [12, Ch. III, Corollary 7.7]). The $F$-splitting property implies that $H^i(Y_p, \mathcal{L}_p^{-1}) = 0$ (cf. [7, Lemma 1.2.7]). Finally, the generic rank of
the coherent sheaf $R^i\pi_*\mathcal{L}^{-1}$ on Spec($R$) is constant. The conclusion follows from the fact that this vanishing holds for infinitely many primes $p$. □

The $F$-splitting of a non-singular variety $X_p$ (defined in characteristic $p$) is defined by an element in $\Gamma(X_p,\kappa_X^{-p})$, where $\kappa$ stands for the canonical sheaf, satisfying a certain algebraic equation (cf. [7, Theorem 1.3.8]). In characteristic zero, an important source of $F$-splittings arise from varieties $X$ which have the property that their reduction $X_p$ modulo $p$ is split by the $(p - 1)$-st power of (the mod $p$ reduction of) a section $\sigma \in \Gamma(X,\kappa_X^{-1})$; in this case $D := \text{divisor}(\sigma)$ is a compatibly split subvariety of $X$. By abuse of language, we say that $X$ is $F$-split by $\sigma \in \Gamma(X,\kappa_X^{-1})$. This latter category includes spherical varieties, in particular projective homogeneous varieties and toric varieties. These are of interest in this article.

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