Combinatorial coherent states via normal ordering of bosons

Pawel Blasiak†‡, Karol A. Penson† and Allan I. Solomon†§

† Université Pierre et Marie Curie
Laboratoire de Physique Théorique des Liquides, CNRS UMR 7600
Tour 16, 5ème étage, 4, place Jussieu, F 75252 Paris, Cedex 05, France
e-mail: penson@lptl.jussieu.fr

‡ H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Science
Department of Theoretical Physics
ul. Radzikowskiego 152, PL 31-342 Kraków, Poland
e-mail: Pawel.Blasiak@ifj.edu.pl

§ The Open University
Physics and Astronomy Department
Milton Keynes MK7 6AA
e-mail: A.I.Solomon@open.ac.uk

Abstract. We construct and analyze a family of coherent states built on sequences of integers originating from the solution of the boson normal ordering problem. These sequences generalize the conventional combinatorial Bell numbers and are shown to be moments of positive functions. Consequently, the resulting coherent states automatically satisfy the resolution of unity condition. In addition they display such non-classical fluctuation properties as super-Poissonian statistics and squeezing.

Keywords: Coherent states, combinatorics, normal order.

Mathematical Subject Classifications (2000): 81R30, 11B73.

Since their introduction in quantum optics many generalizations of standard coherent states (CS) have been proposed.

The main purpose of such generalizations is to account for a full description of interacting quantum systems. The conventional CS provide a correct description of a typical non-interacting system, the harmonic oscillator. One formal approach to this problem is to redefine the standard boson creation \( \hat{a} \) and annihilation \( \hat{a}^\dagger \) operators, satisfying \([\hat{a}, \hat{a}^\dagger] = 1\), to \( A = a f(a^\dagger a) \), where the function \( f(n) \), \( n = a^\dagger a \), is chosen to adequately describe the interacting problem. Any deviation of \( f(x) \) from \( f(x) = \text{const} \) describes a non-linearity in the system. This amounts to introducing the modified (deformed) commutation relations [1] [2] [3]

\[
[A, A^\dagger] = [n + 1] - [n],
\]

where the “box” function \([n]\) is defined as \([n] = n f^2(n) > 0\). Such a way of generalizing the boson commutator naturally leads to generalized
“nonlinear” CS in the form \([n]! = [0][1] \ldots [n], [0] = 1\)

\[
|z\rangle = \mathcal{N}^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}} |n\rangle,
\]

(2)

which are eigenstates of the “deformed” boson annihilation operator \(A\)

\[
A|z\rangle = z|z\rangle.
\]

(3)

The \(|n\rangle\)'s are normalized kets in a Fock space.

It is worth pointing out that such nonlinear CS have been successfully applied to a large class of physical problems in quantum optics [4, 5]. The comprehensive treatment of CS of the form of Eq. (2) can be found in [1, 2, 3].

An essential ingredient in the definition of CS is the completeness property (or the resolution of unity condition) [6, 7, 8]. A guideline for the construction of CS in general has been put forward in [6] as a minimal set of conditions. Apart from the conditions of normalizability and continuity in the complex label \(z\), this set reduces to satisfaction of the resolution of unity condition. This implies the existence of a positive function \(\tilde{W}(|z|^2)\) satisfying

\[
\int \int_{\mathbb{C}} d^2z \ |z\rangle \tilde{W}(|z|^2) \langle z| = I = \sum_{n=0}^{\infty} |n\rangle \langle n|,
\]

(4)

which reflects the completeness of the set \(\{|z\rangle\}\).

In Eq. (4) \(I\) is the unit operator and \(|n\rangle\) is a complete set of orthonormal eigenvectors. In a general approach one chooses strictly positive parameters \(\rho(n), n = 0, 1, \ldots\) such that the state \(|z\rangle\) which is normalized, \(\langle z|z\rangle = 1\), is given by

\[
|z\rangle = \mathcal{N}^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}} |n\rangle,
\]

(5)

with normalization

\[
\mathcal{N}(|z|^2) = \sum_{n=0}^{\infty} |z|^{2n} \rho(n) > 0,
\]

(6)

which in this note we assume to be a convergent series in \(|z|^2\) for all \(z \in \mathbb{C}\). In view of Eqs. (11) and (24) this corresponds to \(\rho(n) = [n]!\) or \([n] = \rho(n)/\rho(n-1)\) for \(n = 1, 2, \ldots\).

Condition (4) can be shown to be equivalent to the following infinite set of equations [7]:

\[
\int_{0}^{\infty} x^n \left[ \frac{\tilde{W}(x)}{\mathcal{N}(x)} \right] dx = \rho(n), \quad n = 0, 1, \ldots,
\]

(7)
a Stieltjes moment problem for $W(x) = \pi \tilde{W}(x)$.

Recently considerable progress was made in finding explicit solutions of Eq. (7) for a large set of $\rho(n)$'s, generalizing the conventional choice $|z\rangle_c$ for which $\rho_c(n) = n!$ with $\mathcal{N}_c(x) = e^x$ (see Refs. [7, 8, 9, 10, 11, 12] and references therein), thereby extending the known families of CS. This progress was facilitated by the observation that when the moments form certain combinatorial sequences a solution of the associated Stieltjes moment problem may be obtained explicitly [13].

In this work we make contact with the combinatorial sequences appearing in the solution of the boson normal ordering problem [15, 16]. These sequences have the very desirable property of being moments of Stieltjes-type measures and so automatically fulfill the resolution of unity requirement. It is therefore natural to use these sequences for the CS construction, thereby providing a link between the quantum states and the combinatorial structures.

The normal ordering problem for canonical bosons $[a, a^\dagger] = 1$ is related to certain combinatorial numbers $S(n, k)$ called Stirling numbers of the second kind through [14]

$$ (a^\dagger a)^n = \sum_{k=1}^{n} S(n, k)(a^\dagger)^k a^k, \quad (8) $$

with corresponding numbers $B(n) = \sum_{k=1}^{n} S(n, k)$ called Bell numbers.

For integers $n, r, s > 0$ we define generalized Stirling numbers of the second kind $S_{r,s}(n, k)$ through ($r \geq s$):

$$ [(a^\dagger)^r a^s]^n = (a^\dagger)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n, k)(a^\dagger)^k a^k, \quad (9) $$

as well as generalized Bell numbers $B_{r,s}(n)$

$$ B_{r,s}(n) = \sum_{k=s}^{ns} S_{r,s}(n, k). \quad (10) $$

For both $S_{r,s}(n, k)$ and $B_{r,s}(n)$ exact and explicit formulas have been found [15, 16].

In this study we shall only be interested in the subset $B_{r,1}(n)$, for which a convenient infinite series representation may be given [15, 16]:

$$ B_{r,1}(n) = \frac{(r - 1)^{n-1}}{e} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(n + \frac{k+1}{r-1})}{\Gamma(1 + \frac{k}{r-1})}, \quad r > 1. \quad (11) $$

This is a generalization of the celebrated Dobiński relation

$$ B_{1,1}(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}. \quad (12) $$
for conventional Bell numbers $B_{1,1}(n)$, see [18],[19]. For reference we quote $B_{1,1}(n) = 1, 2, 5, 15, 52, 203, \ldots$.

From Eqs. (9) and (10) one sees that $B_{r,1}(n)$ are integers. Eq. (11) gives explicitly $B_{2,1}(n) = 1, 3, 13, 73, 501, 4051 \ldots$; $B_{3,1}(n) = 1, 4, 25, 211, 2236, 28471, \ldots$; $B_{4,1}(n) = 1, 5, 41, 6721, 117941 \ldots$ etc. Through transformations of Eq. (11) one finds that $B_{r,1}(n)$ for $r > 1$ can be expressed as special values of hypergeometric functions of type $_1F_{r-1}$:

$$B_{2,1}(n) = \frac{n!}{e} \, _1F_1(n + 1; 2; 1),$$

$$B_{3,1}(n) = \frac{2^{n-1}}{e} \left( \frac{2\Gamma(n + \frac{1}{2})}{\sqrt{\pi}} \right) \, _1F_2(n + \frac{1}{2}; \frac{3}{2}; \frac{1}{4}) + n! \, _1F_2(n + 1; \frac{3}{2}; \frac{1}{4}),$$

$$B_{4,1}(n) = \frac{2^{n-1}}{e} \left( \frac{2\Gamma(n + \frac{1}{2})}{\sqrt{\pi}} \right) \, _1F_3(n + \frac{1}{2}; \frac{3}{2}; \frac{3}{4}) + \ldots$$

etc.

It is essential for our purposes to observe that the integer $B_{r,1}(n+1)$, $n = 0, 1, \ldots$ is the $n$-th moment of a positive function $W_{r,1}(x)$ on the positive half-axis. For $r = 1$ we see from Eq. (12) that $B_{1,1}(n+1)$ is the $n$-th moment of a discrete distribution $W_{1,1}(x)$ located at positive

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure1}
\caption{The weight functions $W_{r,1}(x)$, $(x = |z|^2)$, in the resolution of unity for $r = 2, 3, 4$ (continuous curves) and for $r = 1$, a Dirac’s comb (in the inset), as a function of $x$.}
\end{figure}
Combinatorial coherent states via normal ordering of bosons

**Integers, a so-called Dirac comb:**

\[
B_{1,1}(n + 1) = \int_0^\infty x^n \left[ \frac{1}{e} \sum_{k=1}^\infty \frac{\delta(x - k)}{(k - 1)!} \right] \, dx. \quad (15)
\]

For every \( r > 1 \) a continuous distribution \( W_{r,1}(x) \) will be obtained by excising \((r - 1)^n \Gamma(n + k + 1)\) from Eq.(11), performing the inverse Mellin transform on it and inserting the result back in the sum of Eq.(11). (Note also that \( B_{r,1}(0) = \frac{1}{r-1}, r = 2, 3, \ldots \) is no longer integral). In this way we obtain

\[
B_{r,1}(n + 1) = \int_0^\infty x^n W_{r,1}(x) \, dx, \quad (16)
\]

which yields for \( r = 2, 3, 4 \):

\[
W_{2,1}(x) = e^{-x - 1} \sqrt{x} I_1(2 \sqrt{x}), \quad (17)
\]

\[
W_{3,1}(x) = \frac{1}{2} \sqrt{\frac{x}{2}} e^{-\frac{x}{2}} \left( \frac{2}{\sqrt{\pi}} \; _0F_2\left(\frac{1}{2}; \frac{3}{2}; \frac{x}{8}\right) + \frac{x}{\sqrt{2}} \; _0F_2\left(\frac{3}{2}; 2; \frac{x}{8}\right) \right), \quad (18)
\]

\[
W_{4,1}(x) = \frac{1}{18 \pi \Gamma\left(\frac{2}{3}\right)} e^{-\frac{2}{3} x} \left( \frac{3 \pi}{2} \Gamma\left(\frac{2}{3}\right) x^\frac{1}{3} \; _0F_3\left(\frac{1}{3}; 2; \frac{4}{3}; \frac{x}{81}\right) + \frac{3 \pi}{4} x^\frac{2}{3} \; _0F_3\left(\frac{2}{3}; 3; \frac{5}{3}; \frac{x}{81}\right) + \pi \Gamma\left(\frac{2}{3}\right) x \; _0F_3\left(\frac{4}{3}; 5; 2; \frac{x}{81}\right) \right). \quad (19)
\]

In Eqs.(17), (18), (20) \( I_\nu(y) \) and \( _0F_p(\ldots; y) \) are modified Bessel and hypergeometric functions, respectively. Other \( W_{r,1}(x) > 0 \) for \( r > 4 \) can be generated by essentially the same procedure.

In Figure 1 we display the weight functions \( W_{r,1}(x) \) for \( r = 1 \ldots 4 \); all of them are normalized to one. In the inset the height of the vertical line at \( x = k \) symbolizes the strength of the delta function \( \delta(x - k) \), see Eq.(15). For further properties of \( W_{1,1}(x) \) and more generally of \( W_{r,r}(x) \) associated with Eq.(10), see [17].

A comparison of Eqs.(5), (6) and (16) indicates that the normalized states defined through \( \rho(n) = B_{r,1}(n + 1) \) as

\[
|z\rangle_r = \mathcal{N}_r^{-1/2}(|z|^2) \sum_{n=0}^\infty \frac{z^n}{\sqrt{B_{r,1}(n + 1)}} |n\rangle, \quad (20)
\]

with normalization

\[
\mathcal{N}_r(x) = \sum_{n=0}^\infty \frac{x^n}{B_{r,1}(n + 1)} > 0, \quad (21)
\]
Figure 2. Mandel parameters $Q_r(x)$, for $r = 1 \ldots 4$, as a function of $x = |z|^2$, see Eq. (23).

automatically satisfy the resolution of unity condition of Eq. (4), since for $W_{r,1}(x) = \frac{n \tilde{W}_{r,1}(x)}{N_{r}(x)}$,

$$
\int \int_C d^2 z \, |z\rangle_r \tilde{W}_{r,1}(|z|^2)_r \langle z | = I = \sum_{n=0}^{\infty} |n\rangle \langle n|.
$$

(22)

Note that Eq. (20) is equivalent to Eq. (2) with the definition $[n]_r = B_{r,1}(n + 1)/B_{r,1}(n), n = 0, 1, 2, \ldots$.

Having satisfied the completeness condition with the functions $W_{r,1}(x), r = 1, 2, \ldots$ we now proceed to examine the quantum-optical fluctuation properties of the states $|z\rangle_r$. From now on we consider the $|n\rangle$'s to be eigenfunctions of the boson number operator $N = a^\dagger a$, i.e. $N|n\rangle = n|n\rangle$.

The Mandel parameter

$$
Q_r(x) = x \left( \frac{N''_r(x)}{N'_r(x)} - \frac{N'_r(x)}{N_r(x)} \right),
$$

(23)

allows one to distinguish between the sub-Poissonian (antibunching effect, $Q_r < 0$) and super-Poissonian (bunching effect, $Q_r > 0$) statistics of the beam. In Figure 2 we display $Q_r(x)$ for $r = 1 \ldots 4$. It can be seen that all the states $|z\rangle_r$ in question are super-Poissonian in nature, with
Figure 3. The squeezing parameters of Eqs.\(^{(24)}\) and \(^{(25)}\) for the coordinate \(Q\) (three upper curves) and for the momentum \(P\) (three lower curves) for different \(r\), as a function of \(\Re(z)\), for \(r = 1, 2, 3\).

The deviation from \(Q_r = 0\), which characterises the conventional CS, diminishing for \(r\) increasing.

In Figure 3 we show the behavior of

\[
S_{Q,r}(z) = \frac{r \langle z | (\Delta Q)^2 | z \rangle_r}{2}, \tag{24}
\]

and

\[
S_{P,r}(z) = \frac{r \langle z | (\Delta P)^2 | z \rangle_r}{2}, \tag{25}
\]

which are the measures of squeezing in the coordinate and momentum quadratures respectively. In the display we have chosen the section along \(\Re(z)\). All the states \(|z\rangle_r\) are squeezed in the momentum \(P\) and dilated in the coordinate \(Q\). The degree of squeezing and dilation diminishes with increasing \(r\). By introducing the imaginary part in \(z\) the curves of \(S_Q(z)\) and \(S_P(z)\) smoothly transform into one another, with the identification \(S_Q(i\alpha) = S_P(\alpha)\) and \(S_P(i\alpha) = S_Q(\alpha)\) for any positive \(\alpha\).

In Figure 4 we show the signal-to-quantum noise ratio \(^{(20)}\) relative to \(4|\langle z | N | z \rangle_c| = 4|z|^2\), its value in conventional coherent states; i.e. the
Figure 4. The signal-to-quantum noise ratio relative to its value in the standard coherent states $\bar{\sigma}_r$, see Eq. (26), as a function of $Re(z)$, for $r = 1 \ldots 4$.

Figure 5. Metric factor $\omega_r(x)$, calculated with Eq. (27) as a function of $x = |z|^2$, for $r = 1 \ldots 4$. 
quantity \( \bar{\sigma}_r = \sigma_r - 4[c\langle z|N|z\rangle_c] \), where

\[
\sigma_r = \frac{\left[ r_\langle z|Q|z\rangle_r \right]^2}{(\Delta Q)^2}, \tag{26}
\]

with \((\Delta Q)^2 = r_\langle z|Q^2|z\rangle_r - [r_\langle z|Q|z\rangle_r]^2\). Again only the section \( Re(z) \) is shown. We conclude from Figure 4 that the states \( |z\rangle_r \) are more “noisy” than the standard CS with \( \rho_c(n) = n! \).

In Figure 5 we give the metric factors

\[
\omega_r(x) = \left[ x \frac{N'_{r'}(x)}{N_r(x)} \right]' \tag{27}
\]

which describe the geometrical properties of embedding the surface of coherent states in Hilbert space, or equivalently a measure of a distortion of the complex plane induced by the CS [7]. Here, as far as \( r \) is concerned, the state \( |z\rangle_1 \) appears to be most distant from the \( |z\rangle_c \) CS for which \( \omega_c = 1 \).

The use of sequences \( B_{r,1}(n) \) to construct CS is not limited to the case exemplified by Eq. (20). In fact, any sequence of the form \( B_{r,1}(n+p) \), \( p = 0, 1, \ldots \) will also define a set of CS, as then their respective weight functions will be \( V_{r,1}^{(p)}(x) = x^{p-1}W_{r,1}(x) > 0 \). Will the physical properties of CS defined with \( \rho_p(n) = B_{r,1}(n+p) \) depend sensitively on \( p \)? Our first guess was that by varying \( p \) the salient features of the physical results will not change. It was based on a study of \( \rho(n) = (n+p)! \) in [7] as a function of \( p \) in a somewhat similar situation. However, this was not confirmed by actual calculations (and in fact could have been envisaged from the outset as \( V_{r,1}^{(0)}(x) \) looks quite different from \( W_{r,1}(x) \)). A case in point is that of \( p = 0 \) for which qualitative differences from Figure 2 \( (p = 1) \) appear. In Figure 6 we present the Mandel parameter for these states. Whereas for \( r = 1 \) the state is still super-Poissonian, for \( r = 2, 3, 4 \) one observes novel behaviour, namely a crossover from sub- to super-Poissonian statistics for finite values of \( x \). Also, as indicated in Figure 7 we note a cross-over between squeezing and dilating behaviour for different \( r \). These curious features merit further investigation.

In conclusion, we have used the combinatorial numbers \( B_{r,1}(n) \) to construct and analyze families of coherent states which automatically satisfy the resolution of unity property. This study will be extended to other \( B_{r,s}(n) \) arising in the problem of boson normal ordering, since such sequences, being moments of positive functions [15] [16], are ideally suited for the construction of coherent states.
Figure 6. Mandel parameters $Q_r(x)$ for $r = 1\ldots4$, as a function of $x = |z|^2$, for states with $\rho(n) = B_{r,1}(n)$.

Figure 7. Squeezing parameters for the states with $\rho(n) = B_{r,1}(n)$ for $r = 1, 2, 3$, as a function of $x = |z|^2$. The subscripts $P$ and $Q$ refer to momentum and coordinate variables respectively.
Acknowledgements

We thank A. Horzela, M. Mendez, J. Pitman, C. Quesne and A. Vor-udas for important discussions.

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