LOCAL AND GLOBAL ANALYTICITY FOR 
\( \mu \)-CAMASSA-HOLM EQUATIONS

Hideshi Yamane

Department of Mathematical Sciences
Kwansei Gakuin University
Gakuen 2-1 Sanda, Hyogo 669-1337, Japan

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Abstract. We solve Cauchy problems for some \( \mu \)-Camassa-Holm integro-
partial differential equations in the analytic category. The equations to be con-
sidered are \( \mu \)CH of Khesin-Lenells-Misiolek, \( \mu \)DP of Lenells-Misiolek-Tiglay,
the higher-order \( \mu \)CH of Wang-Li-Qiao and the non-quasilinear version of Qu-
Fu-Liu. We prove the unique local solvability of the Cauchy problems and
provide an estimate of the lifespan of the solutions. Moreover, we show the
existence of a unique global-in-time analytic solution for \( \mu \)CH, \( \mu \)DP and the
higher-order \( \mu \)CH. The present work is the first result of such a global nature
for these equations.

Introduction. We consider a functional equation
\[
\mu(u_t) - u_{txx} = -2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx}, \quad x \in S^1 = \mathbb{R}/\mathbb{Z},
\]
called the \( \mu \)-Camassa-Holm equation (\( \mu \)CH), and its variants in the complex-analytic
or real-analytic category. Here \( \mu(u) = \int_{S^1} u \, dx \). Multiplying by the inverse of \( \mu - \partial_x^2 \),
we get an evolution equation
\[
u_t + u_x + \partial_x(\mu - \partial_x^2)^{-1} \left[ 2\mu(u)u + \frac{1}{2} u_x^2 \right] = 0.
\]
It motivates one to consider Cauchy problems, not only in Sobolev spaces but also
in spaces of analytic functions. In the latter case, the solutions are analytic in both
t and x. Recall that solutions to the KdV equation can be analytic in x but not in
t. This is because it is not ‘Kowalevskian’, which means the first-order derivative \( u_t \)
equals a quantity involving higher derivatives. Our evolution equation mentioned
above is ‘Kowalevskian’ in a generalized sense due to the presence of the negative
order pseudodifferential operator \( (\mu - \partial_x^2)^{-1} \).

Because of the nonlocal nature of \( (\mu - \partial_x^2)^{-1} \), our consideration is always global
in x. So we will work with the Sobolev space \( H^m(S^1) \) or \( A(\delta) \), the space of analytic
functions on \( S^1 \ni x \) which admits analytic continuation to \( |y| < \delta \). On the other
hand, we can work either locally or globally in t. Our local study will be given in
Section 2 and Appendix. It is based on the Ovsyannikov theorem used in [1] and
[12]. It is a kind of abstract Cauchy-Kowalevsky theorem about a scale of Banach

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spaces and enables us to obtain local-in-time solutions which are analytic in both $t$ and $x$. Our global study will be given in Sections 3 and 4. Since global-in-time solutions are known to exist in Sobolev spaces, what remains to be done here is to prove their analyticity. We carry out this task by using the method of [13] following [3]. In the final part of the proof, we quote a result in [16], which gives a useful criterion of real analyticity.

Now we explain some background and history. In the course of it, we will introduce some equations that will be studied in the present paper. All the equations mentioned below are integrable in some sense.

The original Camassa-Holm equation

$$u_t - u_{txx} = -3uu_x + 2u_xu_{xx} + uu_{xxx} \quad (1)$$

was introduced in [10] (hereditary symmetries) and [4] (shallow water wave). Another aspect of this equation is that it is a re-expression of geodesic flow on the diffeomorphism group of the line ([6]). It is known that (1) is completely integrable and admits peaked soliton (peakon) solutions.

The Cauchy problem for (1) can be formulated by introducing a pseudodifferential operator. Indeed, (1) can be written in the form

$$u_t + uu_x + \partial_x(1 - \partial_x^2)^{-1} \left[ u^2 + \frac{1}{2} u_x^2 \right] = 0. \quad (2)$$

Since (2) is Kowalevskian in a generalized sense, it is natural to solve this equation in the analytic setting as in the classical Cauchy-Kowalevsky theorem. Moreover, the analyticity of the solutions is of great interest since the equation models water waves (See [7]). In [1], the authors introduced a kind of Sobolev spaces with exponential weights consisting of holomorphic functions in a strip of the type $|y| < \text{const}$. Since these spaces form a scale of Banach spaces, an Ovsyannikov type argument can be applicable. It leads to the unique solvability and an estimate of the lifespan of the solution in the periodic and non-periodic cases.

There are a lot of works about solutions of the Cauchy problem for (1) or (2) in Sobolev spaces. See the references in [1] and [23]. Local well-posedness and blowup mechanism are major topics. In [1], the local unique solvability in the analytic category was proved. Moreover, there is a result about the global-in-time solvability in [3]. Indeed, according to [3], if the initial value is in $H^s(\mathbb{R}), s > 5/2$, and the McKean quantity $m_0 = (1 - \partial_x^2)u_0$ does not change sign, then the Cauchy problem for (a generalization of) (2) has a unique global-in-time solution $u \in C([0, \infty); H^s(\mathbb{R}))$. See [19] for the necessity and sufficiency of the no-change-of-sign condition. Moreover, in [3], it is proved that this solution is analytic in both $t$ and $x$ if the initial value is in the space of analytic functions mentioned above. In the present paper, we follow [1] for local theory and the analyticity part of [3] for global theory.

In [15], the $\mu$-version of (1), namely

$$\mu(u_t) - u_{txx} = -2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx}, \quad x \in S^1 = \mathbb{R}/\mathbb{Z}, \quad (3)$$

was introduced. The authors call this equation $\mu\text{HS}$ (HS is for Hunter-Saxton), while it is called $\mu\text{CH}$ in [18]. We have $\mu(u_t) = 0$, but we keep $\mu(u_t)$ because this formulation facilitates later calculation. The interest of (3) lies, for example, in the fact that it describes evolution of rotators in liquid crystals with external magnetic field and self-interaction, and it is related to the diffeomorphism groups of the circle
with a natural metric. Set
\[ A(\varphi) = \mu(\varphi) - \varphi_{xx}. \]
Then it is invertible for a suitable choice of function spaces and commutes with \( \partial_x \).
The equation (3) can be written in the following form ([15, (5.1)]):
\[ u_t + uu_x + \partial_x A^{-1} \left[ 2\mu(u)u + \frac{1}{2}u_x^2 \right] = 0. \tag{4} \]
In [15], the local well-posedness and the global existence in Sobolev spaces is demonstrated. In the global problem, the \( \mu \)-McKean quantity \((\mu - \partial_x^2)u_0(x)\) is assumed to be free from change of sign.

There are similar \( \mu \)-equations. In [18, (5.3)], the following equation, called \( \mu \)DP,
\[ u_t + uu_x + \partial_x A^{-1} \left[ 3\mu(u)u \right] = 0. \tag{5} \]
It is a \( \mu \)-version of the Degasperis-Procesi equation
\[ u_t - uu_{txx} = -4uu_x + 3u_xu_{xxx} + uu_{xxxx}, \]
which has similar properties to those of (1) including a relation to water waves and a geometric interpretation ([9]). The local well-posedness for (5) in \( H^s(S^1), s > 3/2 \), and the global existence in \( H^s(S^1), s > 3 \), was proved in [18].

In [5], a family of higher-order Camassa-Holm equations depending on \( k = 2, 3, \ldots \) was introduced. It is related to diffeomorphisms of the unit circle. In [21], the \( \mu \)-version of the case \( k = 2 \) was formulated. The local well-posedness and the global existence in \( H^s(S^1), s > 7/2 \), was proved in [21]. Notice that the no-change-of-sign condition is not imposed in this work. In [22], a different version
\[ u_t + uu_x + \partial_x A^{-2} \left[ 2\mu(u)u + \frac{1}{2}u_x^2 - 3u_xu_{xxx} - \frac{7}{2}u_{xx}^2 \right] = 0 \tag{7} \]
is studied. The global existence is proved there.

The modified \( \mu \)-Camassa-Holm equation (modified \( \mu \)CH) with non-quasilinear terms
\[ u_t + 2\mu(u)uu_x - \frac{1}{3}u_x^3 + \partial_x A^{-1} \left[ 2\mu^2(u)u + \mu(u)u_x^2 + \gamma u \right] + \frac{1}{3}\mu(u_x^3) = 0 \tag{8} \]
was introduced in [20, (3.2)] \((\gamma = 0)\) and [20, (2.7)], [11]. The global existence in Sobolev spaces remains open, as opposed to that of the other equations mentioned above, so it is not possible to show the global existence of analytic solutions by using the method in the present article. Because of this exceptional nature of the equation, we treat it in Appendix separately from the others. Notice that local theory for (8) is developed in a certain space of analytic functions as well as in Besov spaces in [11].

The outline of this article is as follows. In Section 1, we introduce some function spaces and operators and investigate their properties. In Section 2, we prove the local existence of analytic solutions of (4), (5), (6) and (7). These results are used in the proofs of the global existence theorems about (4), (5) and (6) in Sections 3 and 4. In Appendix, the local existence of analytic solutions of (8) is proved.
1. Function spaces and operators. In the present paper, \( L^2(S^1) \) consists of \textit{real-valued} square-integrable functions on \( S^1 = \mathbb{R}/\mathbb{Z} \). We sometimes identify an element of it with a function on \( \mathbb{R} \) with period 1. For a function on \( S^1 \), we set \( \hat{\varphi}(k) = \int_{S^1} \varphi(x)e^{-2k\pi ix} \, dx \). We introduce a family of Hilbert spaces \( G^{\delta,s} (\delta \geq 0, s \geq 0) \) by

\[
G^{\delta,s} = \{ \varphi \in L^2(S^1); \| \varphi \|_{\delta,s} < \infty \}, \quad \| \varphi \|_{\delta,s}^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} e^{4\pi i|k|} |\hat{\varphi}(k)|^2,
\]

where \( \langle k \rangle = (1 + k^2)^{1/2} \) (Japanese bracket). Notice that our definition is not exactly the same as that in [1, 2]. In particular, the base space is \( \mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z}) \) in [1, 2]. It is easy to see that we have continuous injections \( G^{\delta,s} \rightarrow G^{\delta',s'} \) and \( G^{\delta,s} \rightarrow G^{\delta',s''} \) if \( 0 \leq \delta' < \delta, 0 \leq s' < s \). Their norms are 1. We have a continuous injection \( G^{\delta,s} \rightarrow G^{\delta',s'} \) under the weaker assumption \( 0 \leq \delta' < \delta \). We recover the usual Sobolev spaces

\[
H^s = G^{0,s}, \quad H^\infty = \cap_{s \geq 0} H^s
\]

and we set \( \| \varphi \|_s = \| \varphi \|_{0,s} \). Notice that \( \| \cdot \|_0 \) is the \( L^2 \) norm. The corresponding inner product is denoted by \( \langle \cdot, \cdot \rangle_0 \). Set \( \Lambda^2 = 1 - (2\pi)^{-2}\partial_x^2 \). Then we have

\[
\| \varphi \|^2 = \| \Lambda^2 \varphi \|^2_0 = \| \varphi \|^2_0 + \frac{1}{2\pi^2} \| \varphi' \|^2_0 + \frac{1}{(2\pi)^2} \| \varphi'' \|^2_0,
\]

because \( \int_{S^1} \varphi \varphi'' \, dx = -\int_{S^1} (\varphi')^2 \, dx \).

When \( \delta > 0 \), set

\[
S(\delta) = \{ z = x + iy \in \mathbb{C}; |y| < \delta \},
\]

\[
A(\delta) = \{ f: S^1 = \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}; f \text{ has an analytic continuation to } S(\delta) \}.
\]

Remark 1. If a function is analytic on \( S^1 \), then it belong to \( A(\delta) \) for some \( \delta > 0 \). See Proposition 2 below. Notice that an analogous statement does not hold true if \( S^1 \) is replaced with \( \mathbb{R} \).

We identify an element of \( A(\delta) \) with its analytic continuation. For \( f \in A(\delta) \), set

\[
\| f \|^2_{(\sigma,s)} = \sum_{j=0}^{\infty} \frac{e^{4\pi j\sigma}}{j!^2} \| f^{(j)} \|^2_0 \quad (e^{2\pi \sigma} < \delta).
\]

This norm will be used in the global theory. Do not confuse \( \| \cdot \|^2_{(\sigma,s)} \) with \( \| \cdot \|_{\delta,s} \).

**Proposition 1.** For \( \delta > 0, s \geq 0 \), the space \( G^{\delta,s} \) is continuously embedded in \( A(\delta) \).

**Proof.** Assume \( \varphi \in G^{\delta,s} \). The series \( \varphi(z) = \sum \varphi(k)e^{2k\pi i z} = \sum \varphi(k)e^{-2k\pi y}e^{2k\pi ix} \) converges locally uniformly in \( |y| < \delta \), because

\[
|\varphi(z)|^2 \leq \sum \langle k \rangle^{2s} e^{4\pi i|k|} |\hat{\varphi}(k)|^2 \times \sum \langle k \rangle^{-2s} e^{-4\pi i|k|} e^{-4k\pi y}
\]

\[
= \| \varphi \|^2_{\delta,s} \sum \langle k \rangle^{-2s} e^{-4\pi i|k|} e^{-4k\pi y}.
\]

Therefore \( G^{\delta,s} \subset A(\delta) \) and this embedding is continuous from \( G^{\delta,s} \) to \( A(\delta) \) with the topology (i) in Proposition 7 below. \( \square \)
Remark 2. Proposition 1 is a variant of the Sobolev embedding theorem: if \( s > 1/2 \), then there is a continuous embedding \( H^s = \mathcal{G}^0, s \to C^0(S^1) \) as is proved by

\[
|\varphi(x)|^2 \leq \left| \sum \hat{\varphi}(k)e^{2k\pi ix} \right|^2 \leq \sum \langle k \rangle^{2s}|\hat{\varphi}(k)|^2 \times \sum \langle k \rangle^{-2s}|e^{2k\pi ix}|^2 \\
\leq \|\varphi\|^2_s \sum \langle k \rangle^{-2s}.
\]

Proposition 2. If \( \varphi \) is a real-analytic function on \( S^1 \), then there exists \( \delta > 0 \) such that \( \varphi \in \mathcal{G}^{\delta,s} \) for any \( s \). More precisely, if \( \varphi \in A(\delta) \), then \( \varphi \in \mathcal{G}^{\delta',s} \) for any \( \delta' \in [0,\delta] \) and any \( s \).

Proof. (This proposition is given in \([1]\) without proof.) By the periodicity, contour deformation gives

\[
\hat{\varphi}(k) = \int_0^1 \varphi(x \pm \delta') \exp(-2k\pi i [x \pm \delta']i) \, dx.
\]

Set

\[
a_k = \int_0^1 \varphi(x - \text{sgn}(k)\delta') e^{-2k\pi ix} \, dx = e^{2\pi i |k|\delta'} \hat{\varphi}(k), \quad k \neq 0.
\]

These are the Fourier coefficients of \( \varphi(x \pm i\delta') \) and we have \( \sum_{k \in \mathbb{Z} \setminus \{0\}} \langle k \rangle^{2s}|a_k|^2 < \infty \) for any \( s \). Therefore the series \( \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |e^{2\pi i |k|\delta'}|\hat{\varphi}(k)|^2 \) converges. \( \square \)

Proposition 3. We have the following three estimates about products of functions.

(i) Assume \( s > 1/2, \delta \geq 0 \). Then \( \mathcal{G}^{\delta,s} \) is closed under pointwise multiplication and we have

\[
\|\varphi\psi\|_{\delta,s} \leq c_s \|\varphi\|_{\delta,s} \|\psi\|_{\delta,s}, \quad c_s = \left[ 2(1 + s^2) \sum_{k=0}^\infty \langle k \rangle^{-2s} \right]^{1/2}.
\]

(ii) There exists a positive constant \( d_s \) such that we have

\[
\|\varphi\psi\|_0 \leq d_s \|\varphi\|_0 \|\psi\|_s
\]

for any \( \varphi \in H^0 = L^2(S^1) \) and any \( \psi \in H^s \) \( (s \geq 1) \).

(iii) There exists a constant \( \gamma \) such that we have

\[
\|\varphi\psi\|_2 \leq \gamma (\|\varphi\|_1 \|\psi\|_2 + \|\varphi\|_2 \|\psi\|_1)
\]

for any \( \varphi, \psi \in H^2 \).

Proof. The proof of (i) is given in \([2]\). Although different formulations are used in \([2]\) and the present article, the constant \( c_s \) is the same. This is because \( \varphi\psi(k) = \sum_n \hat{\varphi}(n)\hat{\psi}(k - n) \) holds in either situations. The difference of the base spaces are offset by that of conventions, namely the presence or the absence of the \( 1/(2\pi) \) factor in the definition of the Fourier coefficients. The other three estimates follow from the boundedness of \( H^1 \to C^0(S^1) \) (and the Leibniz rule). \( \square \)

Remark 3. A better estimate than Proposition 3 (iii) can be found in \([14]\), which implies that \( \|\cdot\|_1 \) can be replaced with \( \|\cdot\|_{L^\infty} \). It is used in \([21]\), but (iii) is good enough in the present paper.

Proposition 4. ([1, Lemma 2]) If \( 0 \leq \delta' < \delta, s \geq 0 \) and \( \varphi \in \mathcal{G}^{\delta,s} \), then

\[
\|\varphi_x\|_{\delta',s} \leq \frac{e^{-1}}{\delta - \delta'} \|\varphi\|_{\delta,s}, \\
\|\varphi_x\|_{\delta,s} \leq 2\pi \|\varphi\|_{\delta,s+1}.
\]
A derivative can be estimated in two ways: ‘larger $\delta$’ or ‘larger $s$’.

Proof. The second inequality is easy to prove. We give a proof of the first in order to clarify that the assumption $\delta \leq 1$ in [1, 2] is superfluous. The present author thinks that the authors of [1, 2] wrote $\delta \leq 1$ not because they really needed it for the omitted proof but for the sole reason that they were interested only in $0 < \delta \leq 1$.

Set $f(x) = x^2 \exp[2(-\delta + \delta')x]$, $x \geq 0$. We have

$$\|\varphi\|_{\delta,s}^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^2 e^{4\pi |k|} |\hat{\varphi}(k)|^2,$$

$$\|\varphi_x\|_{\delta,s}^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^2 e^{4\pi |k|} (2k\pi)^2 |\hat{\varphi}(k)|^2,$$

$$= \sum_{k \in \mathbb{Z}} \langle k \rangle^2 e^{4\pi |k|} f(2\pi |k|) |\hat{\varphi}(k)|^2.$$

Since $0 \leq f(x) \leq \left( e^{-1}/(\delta - \delta') \right)^2$, we get $\|\varphi_x\|_{\delta,s}^2 \leq \left( e^{-1}/(\delta - \delta') \right)^2 \|\varphi\|_{\delta,s}^2$. □

We set $A(\varphi) = \mu(\varphi) - \varphi_{xx}$, $B(\varphi) = \mu(\varphi) + (\partial_x^2 + \partial_{xx}^4)\varphi$. For $\varphi = \sum_{k \in \mathbb{Z}} a_k e^{2k\pi ix} \in G^\delta,s$, we have

$$\mu(\varphi) = a_0,$$

$$A(\varphi) = a_0 + \sum_{k \neq 0} (2\pi k)^2 a_k e^{2k\pi ix},$$

$$A^{-1}(\varphi) = a_0 + \sum_{k \neq 0} \frac{a_k}{(2\pi k)^2} e^{2k\pi ix},$$

$$B(\varphi) = a_0 + \sum_{k \neq 0} \left( (2\pi k)^2 + (2\pi k)^4 \right) a_k e^{2k\pi ix},$$

$$B^{-1}(\varphi) = a_0 + \sum_{k \neq 0} \frac{a_k}{(2\pi k)^2 + (2\pi k)^4} e^{2k\pi ix}.$$

It follows that $A$ is a bounded operator from $G^\delta,s+2$ to $G^\delta,s$. It is a bijection and its inverse $A^{-1}$ is a pseudodifferential operator of order $-2$. Therefore it is bounded from $G^\delta,s$ to $G^\delta,s+2$. On the other hand, $B^{-1}$ is a pseudodifferential operator of order $-4$. Notice that $A^{-1}$ and $B^{-1}$ commute with $\partial_x$.

The following proposition is easy to prove.

Proposition 5. We have

$$|\mu(\varphi)| \leq \|\varphi\|_{\delta,s},$$

$$\|\partial_x A^{-1}(\varphi)\|_{\delta,s+2-j} \leq \|\varphi\|_{\delta,s} (0 \leq j \leq 2),$$

$$\|\partial_x B^{-1}(\varphi)\|_{\delta,s+4-j} \leq \|\varphi\|_{\delta,s} (0 \leq j \leq 4)$$

for $\varphi \in G^\delta,s$. 
If $\varphi \in G^{\delta,s}$, then by Propositions 4 and 5, we have the following estimates of the larger $\delta$, smaller $s$ type:

$$
\|\partial_x A^{-1}(\varphi)\|_{s',s+1} \leq \frac{e^{-1}}{\delta - \delta'} \|\varphi\|_{\delta,s}, \quad 0 \leq \delta' < \delta \leq 1,
$$

(10)

$$
\|\partial_x B^{-1}(\varphi)\|_{s',s+3} \leq \frac{e^{-1}}{\delta - \delta'} \|\varphi\|_{\delta,s}, \quad 0 \leq \delta' < \delta \leq 1.
$$

(11)

In these estimates, the left-hand sides can be replaced with $\|\partial_x A^{-1}(\varphi)\|_{s',s+2}$ and $\|\partial_x B^{-1}(\varphi)\|_{s',s+4}$ respectively, but (10) and (11) are good enough.

In later sections, we will use the following estimates repeatedly. Let $R > 0$ and $u_0 \in G^{\delta,s+1}$ be given. If $\|u_j - u_0\|_{s,s+1} < R$ for $u_j \in G^{\delta,s+1} \subset G^{\delta,s} (j = 1, 2)$, then we have

$$
\|u_j\|_{\delta,s} \leq \|u_j\|_{\delta,s+1} \leq \|u_0\|_{\delta,s+1} + R,
$$

(12)

$$
\|u_1 + u_2\|_{\delta,s} \leq \|u_1 + u_2\|_{\delta,s+1} \leq 2 (\|u_0\|_{\delta,s+1} + R).
$$

(13)

**Proposition 6.** (cf. [13, Lem 2.2]) If $f \in H^\infty$ satisfies $\|f\|_{(\sigma,s)} < \infty$ for some $s \geq 0$ and for any $\sigma$ with $e^{2\pi \sigma} < \delta$, then $f \in A(\delta)$.

Conversely, $f \in A(\delta)$ implies $\|f\|_{(\sigma,s)} < \infty$ for any $\sigma$, $s$ with $e^{2\pi \sigma} < \delta$ and $s \geq 0$.

**Proof.** In the proof of the first part, we may assume $s = 0$. Let

$$
M^2 = \|f\|^2_{(\sigma,0)} = \sum_{j=0}^\infty (j!)^{-2} e^{4\pi j^2} \|f^{(j)}\|^2_0.
$$

Then $\|f^{(j)}\|_0 \leq j! e^{-2\pi j^2} M$ and

$$
\|f^{(j)}\|_0 + \|f^{(j+1)}\|_0 \leq M [j! e^{-2\pi j^2} + (j + 1)! e^{-2\pi (j+1)^2}].
$$

Since there exists $C > 0$ such that $\|\varphi\|_{C_0} \leq C (\|\varphi\|_{0} + \|\varphi'\|_{0})$ for any $\varphi \in H^1$ by the Sobolev embedding, we have

$$
|f^{(j)}(x)| \leq CM \left[j! e^{-2\pi j^2} + (j + 1)! e^{-2\pi (j+1)^2}\right]
$$

(14)

for any $x$. Therefore $f(x + iy) = \sum_{j=0}^\infty f^{(j)}(x)(iy)^j/j!$ is holomorphic in $|y| < e^{2\pi \sigma}$ for any $\sigma$ with $e^{2\pi \sigma} < \delta$. Hence $f \in A(\delta)$.

Next, we show the second part. Assume $0 < \delta' < \delta$. It is enough to prove $\|f\|^2_{(\sigma,s)} < \infty$ for any $\sigma$, $s$ with $e^{2\pi \sigma} \leq \delta'$. Let $S = \sup_{S(y')} |f| < \infty$. Set $L_{\pm} = \{t \pm iy; 0 \leq t \leq 1\}$. If $0 < |y| \leq \delta'$, the periodicity of $f$ and Goursat’s formula yield

$$
f^{(j)}(x) = \pm \frac{j!}{2\pi i} \left(\int_{L_-} - \int_{L_+}\right) \frac{f(\zeta)}{(x - \zeta)^{j+1}} d\zeta, \quad 0 \leq x \leq 1.
$$

We have $|f^{(j)}(x)| \leq \pi^{-1} j! |S|/y^{j+1}$. Then $y^{2(j+1)} \|f^{(j)}\|^2_{0} \leq \pi^{-2} j!^2 S^2$ in $|y| \leq \delta'$. Integrating in $y \in [0, \delta']$, we get

$$
(\delta')^{2j+2} \|f^{(j)}\|^2_{0} \leq \pi^{-2} (2j + 3) j!^2 S^2.
$$

It follows that

$$
\|f\|^2_{(\sigma,0)} = \sum_{j=0}^\infty (j!)^{-2} e^{4\pi j^2} \|f^{(j)}\|^2_{0} \leq \sum_{j=0}^\infty \pi^{-2} (2j + 3) (\delta')^{-2j+2} e^{4\pi j^2} S^2 < \infty.
$$

(15)
For $s > 0$, we can prove that

$$
\|f\|_{(\sigma,s)} \leq \text{const.} \|f\|_{(\sigma',0)} < \infty,
$$

(16)

where $e^{2\pi \sigma} < e^{2\pi \sigma'} < \delta'$. To see this, we may assume that $s$ is a positive integer in view of $\|f\|_{(\sigma,s_1)} \leq \|f\|_{(\sigma,s_2)} (s_1 \leq s_2)$. For simplicity, we explain the case of $s = 1$ only. (The general case follows the same line of proof, the only additional tool being the binomial expansion of powers of the Japanese bracket.) We have

$$
\|f\|^2_{(\sigma,1)} = \sum_{j=0}^{\infty} (j!)^{-2} e^{4\pi j \sigma} \|f(j)\|^2_1 = \sum_{j=0}^{\infty} (j!)^{-2} e^{4\pi j \sigma} \sum_{k \in \mathbb{Z}} (1 + k^2)(2\pi k)^{2j} |\hat{f}(k)|^2
$$

= $I_0 + I_1$,

where $I_p = \sum_{j=0}^{\infty} (j!)^{-2} e^{4\pi j \sigma} \sum_{k \in \mathbb{Z}} (2\pi k)^{2j} |\hat{f}(k)|^2 (p = 0, 1)$. Obviously $I_0 = \|f\|^2_{(\sigma,0)} \leq \|f\|^2_{(\sigma',0)}$. On the other hand, setting $\ell = j + 1$, we get

$$
I_1 \leq \sum_{\ell=1}^{\infty} \frac{\ell^2 e^{-4\pi \sigma} e^{4\pi \ell (\sigma - \sigma')}}{\ell^2} e^{4\pi \ell \sigma'} \sum_{k \in \mathbb{Z}} (2\pi k)^{2\ell} |\hat{f}(k)|^2
$$

\leq \sum_{\ell=0}^{\infty} \text{const.} \frac{e^{4\pi \ell \sigma'}}{\ell^2} \sum_{k \in \mathbb{Z}} (2\pi k)^{2\ell} |\hat{f}(k)|^2 = \sum_{\ell=0}^{\infty} \text{const.} \frac{e^{4\pi \ell \sigma'}}{\ell^2} \|f^{(\ell)}\|^2_0 = \text{const.} \|f\|^2_{(\sigma',0)}.

The proof of the second part is over.

Lemma 1.1. If $\sigma < \sigma'$, we have

$$
\|f\|_{(\sigma,s)} \leq \text{const.} \|f\|_{(\sigma',s')}
$$

for any $s, s' \geq 0$, where the constant depends on $s, s', \sigma, \sigma'$.

Proof. This estimate follows from (16) immediately.

Proposition 7. The following four families of norms on $A(\delta)$ determine the same topology as a Fréchet space.

(i) $\sup_{z \in S(\delta')} |f(z)| (0 < \delta' < \delta$), (ii) $\|\cdot\|_{(\sigma,s)} (e^{2\pi \sigma} < \delta, s \geq 0)$,

(iii) $\|\cdot\|_{(\sigma,2)} (e^{2\pi \sigma} < \delta)$, (iv) $\|\cdot\|_{(\sigma,0)} (e^{2\pi \sigma} < \delta)$.

With this topology, $A(\delta)$ is continuously embedded in $H^\infty(S^1) = \cap_{n \geq 0} H^s(S^1)$.

Proof. It is trivial that (ii) is stronger (not weaker) than (iv) and that (iii) is between (ii) and (iv). On the other hand, (16) implies (iv) is stronger than (ii).

The estimate (14) and the Taylor expansion imply that (iv) is stronger than (i). On the other hand, (15) implies (i) is stronger than (iv).

Proposition 8. (cf. [13, Lemma 2.4]) Let $f_n \in A(\delta) (n = 0, 1, 2, \ldots)$ be a sequence with $\|f_n\|_{(\sigma,s)}$ bounded, where $e^{2\pi \sigma} < \delta$, and assume $f_\infty \in A(\delta)$. If $f_n \to f_\infty$ in $H^\infty$ as $n \to \infty$, then $\|f_n\|_{(\sigma',s)} \to f_\infty$ for each $\sigma' < \sigma$.

Proof. We may assume $f_\infty = 0$. Let $M^2 = \sup_n \|f_n\|_{(\sigma,s)}^2$. Then $j!^{-2} e^{4\pi j \sigma} \|f_n^{(j)}\|_s^2 \leq M e^{4\pi (j - \sigma)} j!$. Since $\lim_{n \to \infty} \|f_n^{(j)}\|_s = 0$ for each $j$ and $\sum_{j \geq 0} M^2 e^{4\pi (j - \sigma)} < \infty$, we can apply (the sum version of) Lebesgue’s dominated convergence theorem to $\|f_n\|_{(\sigma',s')} = \sum_{j \geq 0} j!^{-2} e^{4\pi j \sigma'} \|f_n^{(j)}\|_s^2 (n = 0, 1, 2, \ldots)$.
2. Local-in-time solutions.

2.1. Autonomous Ovsyannikov theorem. We recall some basic facts about the autonomous Ovsyannikov theorem. Among many versions, we adopt the one in [1, 2]. Let \( \{X_\delta, \| \cdot \|_\delta\}_{0 < \delta \leq 1} \) be a (decreasing) scale of Banach spaces, i.e. each \( X_\delta \) is a Banach space and \( X_\delta \subset X_{\delta'} \), \( \| \cdot \|_{\delta'} \leq \| \cdot \|_\delta \) for any \( 0 < \delta' < \delta \leq 1 \). (For \( s \) fixed, \( \{G^{s, \delta}, \| \cdot \|_{s, \delta}\}_{0 < \delta \leq 1} \) is a scale of Banach spaces.) Assume that \( F: X_\delta \to X_{\delta'} \) is a mapping satisfying the following conditions.

(a) For any \( u_0 \in X_1 \) and \( R > 0 \), there exist \( L = L(u_0, R) > 0, M = M(u_0, R) > 0 \) such that we have

\[
\|F(u_0)\|_\delta \leq \frac{M}{1-\delta} \tag{17}
\]

if \( 0 < \delta < 1 \) and

\[
\|F(u) - F(v)\|_{\delta'} \leq \frac{L}{\delta - \delta'} \|u - v\|_\delta \tag{18}
\]

if \( 0 < \delta' < \delta \leq 1 \) and \( u, v \in X_\delta \) satisfies \( \|u - u_0\|_\delta < R, \|v - u_0\|_\delta < R \).

(b) If \( u(t) \) is holomorphic on the disk \( D(0, a(1-\delta)) = \{t \in \mathbb{C}: |t| < a(1-\delta)\} \) with values in \( X_\delta \) for \( a > 0, 0 < \delta < 1 \) satisfying \( \sup_{|t| < a(1-\delta)} \|u(t) - u_0\|_\delta < R \), then the composite function \( F(u(t)) \) is a holomorphic function on \( D(0, a(1-\delta)) \) with values in \( X_{\delta'} \) for any \( 0 < \delta' < \delta \).

The autonomous Ovsyannikov theorem below is our main tool. For the proof, see [1].

**Theorem 2.1.** Assume that the mapping \( F \) satisfies the conditions (a) and (b). For any \( u_0 \in X_1 \) and \( R > 0 \), set

\[
T = \frac{R}{16LR + 8M}. \tag{19}
\]

Then, for any \( \delta \in ]0, 1[ \), the Cauchy problem

\[
\frac{du}{dt} = F(u), \quad u(0) = u_0 \tag{20}
\]

has a unique holomorphic solution \( u(t) \) in the disk \( D(0, T(1-\delta)) \) with values in \( X_\delta \) satisfying

\[
\sup_{|t| < T(1-\delta)} \|u(t) - u_0\|_\delta < R.
\]

2.2. \( \mu \text{CH and } \mu \text{DP equations.} \) First we consider the analytic Cauchy problem for the \( \mu \text{CH equation (4),} \) namely.

\[
\begin{cases}
    u_t + uu_x + \partial_x A^{-1} \left[ 2\mu(u)u + \frac{1}{2}u_x^2 \right] = 0, \\
    u(0, x) = u_0(x).
\end{cases} \tag{21}
\]

**Theorem 2.2.** Let \( s > 1/2 \). If \( u_0 \in G^{1,s+1} \), then there exists a positive time \( T = T(u_0, s) \) such that for every \( \delta \in ]0, 1[ \), the Cauchy problem (21) has a unique solution which is a holomorphic function valued in \( G^{s,s+1} \) in the disk \( D(0, T(1-\delta)) \). Furthermore, the analytic lifespan \( T \) satisfies

\[
T = \text{const.} \frac{\|u_0\|_{1,s+1}}{\|u_0\|_{1,s+1}}. 
\]
Proof. Assume \( \|u - u_0\|_{\delta,s+1} < R, \|v - u_0\|_{\delta,s+1} < R \). By Proposition 3 (i), the first inequality in Proposition 4 and (13),

\[
\|(u^2)_x - (v^2)_x\|_{\delta,s+1} \leq \frac{e^{-1}}{\delta - \delta'} \|u^2 - v^2\|_{\delta,s+1}
\]

\[
\leq \frac{e^{-1}c_{s+1}}{\delta - \delta'} \|u + v\|_{\delta,s+1} \|u - v\|_{\delta,s+1}
\]

(22)

\[
\leq \frac{2e^{-1}c_{s+1}}{\delta - \delta'} (\|u_0\|_{\delta,s+1} + R) \|u - v\|_{\delta,s+1}.
\]

On the other hand, since we have \( \mu(u)u - \mu(v)v = \mu(u - v)u + \mu(v)(u - v) \), Proposition 5 and (12) imply

\[
\|\mu(u)u - \mu(v)v\|_{\delta,s} \leq |\mu(u - v)|\|u\|_{\delta,s} + |\mu(v)|\|v\|_{\delta,s}
\]

\[
\leq (\|u\|_{\delta,s} + \|v\|_{\delta,s})\|u - v\|_{\delta,s}
\]

\[
\leq 2 (\|u_0\|_{\delta,s+1} + R) \|u - v\|_{\delta,s+1}.
\]

Therefore (10) yields

\[
\|\partial_x A^{-1} [\mu(u)u - \mu(v)v]\|_{\delta,s+1} \leq \frac{e^{-1}}{\delta - \delta'} \|\mu(u)u - \mu(v)v\|_{\delta,s}
\]

(23)

\[
\leq \frac{2e^{-1}c_s}{\delta - \delta'} (\|u_0\|_{\delta,s+1} + R) \|u - v\|_{\delta,s+1}.
\]

Next, by Proposition 3 (i) and the second inequality in Proposition 4,

\[
\|u_x^2 - v_x^2\|_{\delta,s} \leq c_s \|u_x + v_x\|_{\delta,s} \|u_x - v_x\|_{\delta,s} \leq 4\pi^2 c_s \|u + v\|_{\delta,s+1} \|u - v\|_{\delta,s+1}
\]

\[
\leq 8\pi^2 c_s (\|u_0\|_{\delta,s+1} + R) \|u - v\|_{\delta,s+1}.
\]

Hence (10) gives

\[
\|\partial_x A^{-1} (u_x^2 - v_x^2)\|_{\delta,s+1} \leq \frac{e^{-1}c_s}{\delta - \delta'} \|u_x^2 - v_x^2\|_{\delta,s}
\]

(24)

\[
\leq \frac{8\pi^2 e^{-1}c_s}{\delta - \delta'} (\|u_0\|_{\delta,s+1} + R) \|u - v\|_{\delta,s+1}.
\]

Now set

\[
F_{\mu}(u) = -uu_x - \partial_x A^{-1} \left[ 2\mu(u)u + \frac{1}{2}u_x^2 \right]
\]

\[
= -\frac{1}{2} (u_x^2)_x - \partial_x A^{-1} \left[ 2\mu(u)u + \frac{1}{2}u_x^2 \right].
\]

(25)

Then (22), (23) and (24) give the Lipschitz continuity of \( F_{\mu} \):

\[
\|F_{\mu}(u) - F_{\mu}(v)\|_{\delta,s+1} \leq \frac{L}{\delta - \delta'} \|u - v\|_{\delta,s+1},
\]

(26)

where \( L = C (\|u_0\|_{1,s+1} + R), C = e^{-1}(c_{s+1} + 4 + 4\pi^2 c_s) \).

Next we will derive an estimate of \( \|F_{\mu}(u_0)\|_{\delta,s+1} \). Since

\[
\|(u_x^2)_x\|_{\delta,s+1} \leq \frac{e^{-1}}{1 - \delta} \|u_x^2\|_{1,s+1} \leq \frac{e^{-1}c_{s+1}}{1 - \delta} \|u_0\|_{2,s+1}^2,
\]

\[
\|\partial_x A^{-1}[\mu(u_0)]\|_{\delta,s+1} \leq \frac{e^{-1}}{1 - \delta} \|\mu(u_0)u_0\|_{1,s} \leq \frac{e^{-1}}{1 - \delta} \|u_0\|_{1,s+1}^2.
\]
\[ \| \partial_x A^{-1}(\partial_x u_0)^2 \|_{d,s+1} \leq \frac{e^{-1}}{1-\delta} \| (\partial_x u_0)^2 \|_{1,s} \leq \frac{e^{-1}c_s}{1-\delta} \| \partial_x u_0 \|_{1,s}^2 \]

\[ \leq \frac{(2\pi)^2 e^{-1}c_s}{1-\delta} \| u_0 \|_{1,s+1}^2, \]

we have

\[ \| F_\mu(u_0) \|_{\delta,s+1} \leq \frac{M}{1-\delta}, \quad M = \frac{C}{2} \| u_0 \|_{1,s+1}^2. \]

We set

\[ T = \frac{R}{16LR + 8M} = \frac{R}{4C \left[ 4R (\| u_0 \|_{1,s+1} + R) + \| u_0 \|_{1,s+1}^2 \right]}. \]

Because of Theorem 2.1, there exists a unique solution \( u = u(t) \) to (21) which is a holomorphic mapping from \( D(0, T(1-\delta)) \) to \( G^{d,s+1} \) and

\[ \sup_{\| t \| < T(1-\delta)} \| u(t) - u_0 \|_{\delta,s+1} < R. \]

If we set \( R = \| u_0 \|_{1,s+1} \), we have

\[ T = \frac{1}{36e^{-1}(c_{s+1} + 4 + 4\pi^2 c_s)\| u_0 \|_{1,s+1}^2}. \]

In Theorem 2.2, we assumed the initial value \( u_0 \) was in \( G^{1,s+1} \). We can relax this assumption as in the following theorem.

**Theorem 2.3.** If \( u_0 \) is a real-analytic function on \( S^1 \), then the Cauchy problem (21) has a holomorphic solution near \( t = 0 \). More precisely, we have the following:

(i) There exists \( \Delta > 0 \) such that \( u_0 \in G^{\Delta,s+1} \subset A(\Delta) \) for any \( s \).

(ii) If \( s > 1/2 \), there exists a positive time \( T_\Delta = T(u_0, s, \Delta) \) such that for every \( d \in [0, 1] \), the Cauchy problem (21) has a unique solution which is a holomorphic function valued in \( G^{\Delta d,s+1} \) in the disk \( D(0, T_\Delta(1-d)) \). Furthermore, the analytic lifespan \( T_\Delta \) satisfies

\[ T_\Delta = \text{const.} \frac{1}{\| u_0 \|_{\Delta,s+1}} \]

when \( \Delta \) is fixed.

**Proof.** The first statement is nothing but Proposition 2.

Set \( X_d = G^{\Delta d,s+1}, || \cdot ||_{\Delta,s+1} = || \cdot ||_{\Delta d,s+1} \). Then \( \{ X_d, || \cdot ||_{\Delta d,s+1} \}_{0< d \leq 1} \) is a (decreasing) scale of Banach spaces and \( u_0 \in X_1 \).

Assume \( \| u - u_0 \|_{\Delta,s+1} < R, \| v - u_0 \|_{\Delta,d',s+1} < R \) and \( 0 < d' < d \leq 1 \). Then (22), (23) and (24) give \( (\Delta = \Delta d, \delta' = \Delta d') \) the following counterpart of (26):

\[ \| F_\mu(u) - F_\mu(v) \|_{\Delta,d',s+1} \leq \frac{L_\Delta}{d - d'} \| u - v \|_{\Delta,d',s+1}, \] (27)

where

\[ L_\Delta = \Delta^{-1} C \left( \| u_0 \|_{\Delta,s+1} + R \right) = C \left( \| u_0 \|_{\Delta,s+1} + R \right), \]

\[ C = e^{-1} (c_{s+1} + 4 + 4\pi^2 c_s). \]

Simpler estimates give

\[ \| F_\mu(u_0) \|_{\Delta,d',s+1} \leq \frac{M_\Delta}{1-d}, \quad M_\Delta = \frac{C}{2\Delta} \left( \| u_0 \|_{\Delta,s+1} \right)^2. \]
We set
\[ T_\Delta = \frac{R}{16L_\Delta R + 8M_\Delta} = \frac{R\Delta}{4C \left[ 4R \left( \|u_0\|_{\Delta,s+1} + R \right) + \|u_0\|_{\Delta,s+1}^2 \right]}. \]
Because of Theorem 2.1, there exists a unique solution \( u = u(t) \) to (21) which is a holomorphic mapping from \( D(0, T(1 - \delta)) \) to \( X_d = G^{\Delta, s+1} \) and
\[ \sup_{|t|<T(1-\delta)} \|u(t) - u_0\|_{\Delta,s+1} < R. \]
If we set \( R = \|u_0\|_{1,s+1} = \|u_0\|_{\Delta,s+1} \), we have
\[ T_\Delta = \frac{\Delta}{36e^{-1}(c_s+1 + 4 + 4\pi^2c_s)}\|u_0\|_{\Delta,s+1}. \]

We can study the following Cauchy problem for the \( \mu \)DP equation (5) by using the same estimates (22) and (23).
\[
\begin{cases}
  u_t + uu_x + \partial_x A^{-1} [3\mu(u)u] = 0, \\
  u(0, x) = u_0(x).
\end{cases}
\]
\( (28) \)

**Theorem 2.4.** If \( u_0 \) is a real-analytic function on \( S^1 \), then the Cauchy problem (28) has a holomorphic solution near \( t = 0 \). More precisely, we have the following:

(i) There exists \( \Delta > 0 \) such that \( u_0 \in G^{\Delta, s+1} \subset A(\Delta) \) for any \( s \).

(ii) If \( s > 1/2 \), there exists a positive time \( T_\Delta = T(u_0, s, \Delta) \) such that for every \( d \in [0, 1] \), the Cauchy problem (28) has a unique solution which is a holomorphic function valued in \( G^{\Delta, s+1} \) in the disk \( D(0, T_\Delta(1 - d)) \). Furthermore, the analytic lifespan \( T_\Delta \) satisfies
\[ T_\Delta = \text{const.} \quad \frac{\Delta}{\|u_0\|_{\Delta,s+1}} \]
when \( \Delta \) is fixed.

2.3. **Higher-order \( \mu \)CH equation.** We consider the analytic Cauchy problem for the higher-order \( \mu \)CH equation (6), namely.
\[
\begin{cases}
  u_t + uu_x + \partial_x B^{-1} [2\mu(u)u + \frac{1}{2} u_x^2 - 3u_x u_{xxx} - \frac{7}{2} u_x^3] = 0, \\
  u(0, x) = u_0(x).
\end{cases}
\]
\( (29) \)

**Theorem 2.5.** Let \( s > 1/2 \). If \( u_0 \in G^{1,s+3} \), then there exists a positive time \( T = T(u_0, s) \) such that for every \( \delta \in [0, 1] \), the Cauchy problem (29) has a unique solution which is a holomorphic function valued in \( G^{\delta, s+3} \) in the disk \( D(0, T(1 - \delta)) \). Furthermore, the analytic lifespan \( T \) satisfies
\[ T = \text{const.} \quad \frac{\text{const.}}{\|u_0\|_{1,s+3}} \]

**Proof.** Assume \( u_0 \in G^{\Delta, s+3} \) and \( \|u - u_0\|_{\delta,s+3} < R, \|v - u_0\|_{\delta,s+3} < R \). We follow the proofs of (22), (23) and (24) with (11) instead of (10) to obtain
\[
\begin{align*}
\|\partial_x B^{-1}[\mu(u)u - \mu(v)v]\|_{\delta,s+3} &\leq \frac{2e^{-1}c_s}{\delta - \delta'} (\|u_0\|_{\delta,s+3} + R) \|u - v\|_{\delta,s+3}, \\
\|\partial_x (u_x^2 - v_x^2)\|_{\delta,s+3} &\leq \frac{8\pi^2 e^{-1} c_s}{\delta - \delta'} (\|u_0\|_{\delta,s+3} + R) \|u - v\|_{\delta,s+3}.
\end{align*}
\]
Next we study the difference associated with $\partial_x B^{-1}(u_x u_{xxx})$. Since $u_x u_{xxx} - v_x v_{xxx} = (u_x - v_x)u_{xxx} + v_x(u_{xxx} - v_{xxx})$, we have
\[
\|u_x u_{xxx} - v_x v_{xxx}\|_{\delta,s} \leq c_s \left(\|u_x - v_x\|_{\delta,s} \|u_{xxx}\|_{\delta,s} + \|v_x\|_{\delta,s} \|u_{xxx} - v_{xxx}\|_{\delta,s}\right).
\]
Since
\[
\|u_x - v_x\|_{\delta,s} \leq 2\pi \|u - v\|_{\delta,s+3},
\]
\[
\|u_{xxx}\|_{\delta,s} \leq (2\pi)^3 \|u\|_{\delta,s+3} \leq (2\pi)^3 (\|u_0\|_{\delta,s+3} + R),
\]
\[
\|v_x\|_{\delta,s} \leq 2\pi (\|u_0\|_{\delta,s+3} + R),
\]
\[
\|u_{xxx} - v_{xxx}\|_{\delta,s} \leq (2\pi)^3 \|u - v\|_{\delta,s+3},
\]
we have
\[
\|u_x u_{xxx} - v_x v_{xxx}\|_{\delta,s} \leq 2^5 \pi^4 c_s (\|u_0\|_{\delta,s+3} + R) \|u - v\|_{\delta,s+3}.
\]
Therefore by (11)
\[
\|\partial_x B^{-1}[u_x u_{xxx} - v_x v_{xxx}]\|_{\delta',s+3} \leq \frac{2^5 \pi^4 e^{-1} c_s}{\delta - \delta'} (\|u_0\|_{\delta,s+3} + R) \|u - v\|_{\delta,s+3}. \tag{33}
\]
Next we study the difference associated with $\partial_x B^{-1}(u_{xx}^2)$. We have $u_{xx}^2 - v_{xx}^2 = (u_{xx} + v_{xx})(u_{xx} - v_{xx})$ and
\[
\|u_{xx}^2 - v_{xx}^2\|_{\delta,s} \leq c_s \|u_{xx} + v_{xx}\|_{\delta,s} \|u_{xx} - v_{xx}\|_{\delta,s}.
\]
Since
\[
\|u_{xx} + v_{xx}\|_{\delta,s} \leq (2\pi)^2 \|u + v\|_{\delta,s+2} \leq (2\pi)^2 \|u + v\|_{\delta,s+3}
\]
\[
\leq 2^3 \pi^2 (\|u_0\|_{\delta,s+3} + R),
\]
\[
\|u_{xx} - v_{xx}\|_{\delta,s} \leq (2\pi)^2 \|u - v\|_{\delta,s+2} \leq 2^2 \pi^2 \|u - v\|_{\delta,s+3},
\]
we have
\[
\|u_{xx}^2 - v_{xx}^2\|_{\delta,s} \leq 2^5 \pi^4 c_s (\|u_0\|_{\delta,s+3} + R) \|u - v\|_{\delta,s+3}
\]
and
\[
\|\partial_x B^{-1}[u_{xx}^2 - v_{xx}^2]\|_{\delta',s+3} \leq \frac{e^{-1}}{\delta - \delta'} B^{-1}[u_{xx}^2 - v_{xx}^2]\|_{\delta,s}
\]
\[
\leq \frac{2^5 \pi^4 e^{-1} c_s}{\delta - \delta'} (\|u_0\|_{\delta,s+3} + R) \|u - v\|_{\delta,s+3}. \tag{34}
\]
Now we set
\[
G(u) = -\frac{1}{2} (u^2)_x - \partial_x B^{-1} \left[2\mu(u)u + \frac{1}{2} u_{xx}^2 - 3u_x u_{xxx} - \frac{7}{2} u_{xx}^2\right]. \tag{35}
\]
Then by (30), (31), (32), (33) and (34), we obtain
\[
\|G(u) - G(v)\|_{\delta',s+3} \leq \frac{L_G}{\delta - \delta'} \|u - v\|_{\delta,s+3}, \tag{36}
\]
\[
L_G = e^{-1} [c_{s+3} + 4 + (4\pi^2 + 208\pi^4) c_s] (\|u_0\|_{\delta,s+3} + R). \tag{37}
\]
We need an estimate of $G(u_0)$. By using
\[
\| (u_0^2)_x \|_{\delta,s+3} \leq \frac{e^{-1}}{1 - \delta} \| u_0^2 \|_{1,s+3} \leq \frac{e^{-1} \cdot c_{s+3}}{1 - \delta} \| u_0 \|_{1,s+3},
\]
\[
\| \partial_x B^{-1}[\mu(u_0)] \|_{\delta,s+3} \leq \frac{e^{-1}}{1 - \delta} \| u_0 \|_{1,s+3},
\]
\[
\| \partial_x B^{-1}[(\partial_x u_0)^2] \|_{\delta,s+3} \leq \frac{e^{-1}}{1 - \delta} \| \partial_x u_0 \|_{1,s} \leq \frac{e^{-1} \cdot c_x}{1 - \delta} \| u_0 \|_{1,s+3},
\]
\[
\| \partial_x B^{-1}[(u_0)_x \cdot (u_0)_{xxx}] \|_{\delta,s+3} \leq \frac{e^{-1}}{1 - \delta} \| u_0 \|_{1,s+3},
\]
\[
\| \partial_x B^{-1}[(u_0^2)_{xxx}] \|_{\delta,s+3} \leq \frac{e^{-1}}{1 - \delta} \| u_0 \|_{1,s+3},
\]
we obtain
\[
\| G(u_0) \|_{\delta,s+3} \leq \frac{M_G}{1 - \delta}, \tag{38}
\]
\[
M_G = e^{-1} \left[ \frac{1}{2} c_{s+3} + 2 + (2\pi^2 + 104\pi^4)c_x \right] \| u_0 \|_{1,s+3}. \tag{39}
\]

We set
\[
T = \frac{R}{16L_G R + 8M_G}.
\]

Because of Theorem 2.1, there exists a unique solution $u = u(t)$ to (29) which is a holomorphic mapping from $D(0, T(1 - \delta))$ to $G^{\delta,s+3}$ and
\[
\sup_{|t| < T(1 - \delta)} \| u(t) - u_0 \|_{\delta,s+3} < R.
\]

If we set $R = \| u_0 \|_{1,s+3}$, we have
\[
T = \frac{1}{160 e^{-1} |c_{s+3}/4 + 1 + (\pi^2 + 52\pi^4)c_x|} \| u_0 \|_{1,s+3}. \tag{40}
\]

Similarly, we have the following theorem about the other higher order $\mu i\mathcal{H}$ (7).

**Theorem 2.6.** Let $s > 1/2$. If $u_0 \in G^{1,s+3}$, then there exists a positive time $T = T(u_0, s)$ such that for every $\delta \in (0, 1)$, the Cauchy problem
\[
\begin{align*}
&u_t + uu_x + \partial_x A^{-2} \left[ 2\mu(u)u + \frac{1}{2} u_x^2 - 3u_x u_{xxx} - \frac{7}{2} u_{xxx}^2 \right] = 0, \\
&u(0, x) = u_0(x)
\end{align*}
\]
has a unique solution which is a holomorphic function valued in $G^{\delta,s+3}$ in the disk $D(0, T(1 - \delta))$. Furthermore, the analytic lifespan $T$ satisfies
\[
T = \frac{\text{const.}}{\| u_0 \|_{1,s+3}}.
\]

**Proof.** The proof is almost the same as for Theorem 2.5. Indeed, $A^{-2}$ has the same properties as those of $B^{-1}$.

Global-in-time analytic solutions of (21), (28), (29) and (7) will be studied in the following sections. The proofs will rely on known results about the global existence in Sobolev spaces. On the other hand, that kind of existence for the non-quasilinear
equation (8) is unknown. Therefore, the argument given below is not valid for it. The local theory of (8) will be given in Appendix.

3. Global-in-time solutions.

3.1. Statement of the main results. We recall known results about global-in-time solutions to $\mu$CH and $\mu$DP in Sobolev spaces. First, we have

**Theorem 3.1.** ([15, Theorem 5.1, 5.5]) Let $s > 5/2$. Assume that $u_0 \in H^s(S^1)$ has non-zero mean and satisfies the condition

$$(\mu - \partial_x^2)u_0 \geq 0 \ (\text{or} \leq 0).$$

Then (21) has a unique global-in-time solution in $C(\mathbb{R}, H^s(S^1)) \cap C^1(\mathbb{R}, H^{s-1}(S^1))$.

Moreover, local well-posedness (in particular, uniqueness) holds.

The quantity $(\mu - \partial_x^2)u_0$ is the $\mu$-version of the McKean quantity $(1 - \partial_x^2)u_0$ ([19]). There is an analogous result about $\mu$DP.

**Theorem 3.2.** ([18, Theorem 5.1, 5.4]) Let $s > 3$. Assume that $u_0 \in H^s(S^1)$ has non-zero mean and satisfies the condition

$$(\mu - \partial_x^2)u_0 \geq 0 \ (\text{or} \leq 0).$$

Then (28) has a unique global-in-time solution in $C(\mathbb{R}, H^s(S^1)) \cap C^1(\mathbb{R}, H^{s-1}(S^1))$.

Moreover, local well-posedness holds.

We will discuss global-in-time analytic solutions. These solutions are analytic in both the time and space variables. Notice that in the KdV and other cases treated in [13], solutions are analytic in the space variable only. This is due to the absence of Cauchy-Kowalevsky type theorems. Our main result about the $\mu$-Camassa-Holm equation is the following:

**Theorem 3.3.** Assume that a real-analytic function $u_0$ on $S^1$ has non-zero mean and satisfies the condition

$$(\mu - \partial_x^2)u_0 \geq 0 \ (\text{or} \leq 0).$$

Then the Cauchy problem (21) has a unique solution $u \in C^\omega(\mathbb{R} \times S^1)$.

We have the following estimate of the radius of analyticity. Let $u_0 \in A(r_0)$. Fix $\sigma_0 < (\log r_0)/(2\pi)$ and set

$$K = (20\pi d_1 + 8 + 4\pi^2 c_1) \left[1 + \max \{\|u(t)\|_2; t \in [-T,T]\}\right],$$

$$\sigma(t) = \sigma_0 - \frac{\sqrt{6\pi\gamma}}{K}\|u_0\|_{(\sigma_0,2)}(e^{K|t|/2} - 1).$$

Then, for any fixed $T > 0$, we have $u(\cdot,t) \in A(e^{2\pi\sigma(t)})$ for $t \in [-T,T]$.

There is an analogue about the $\mu$DP equation which reads as follows:

**Theorem 3.4.** Assume that a real-analytic function $u_0$ on $S^1$ has non-zero mean and satisfies the condition

$$(\mu - \partial_x^2)u_0 \geq 0 \ (\text{or} \leq 0).$$

Then the Cauchy problem (28) has a unique solution $u \in C^\omega(\mathbb{R} \times S^1)$.
We have the following estimate of the radius of analyticity. Let \( u_0 \in A(r_0) \). Fix \( \sigma_0 < (\log r_0)/(2\pi) \) and set
\[
K' = (20\pi d_1 + 12)[1 + \max \{ \|u(t)\|_2; \ t \in [-T,T]\}],
\]
\[
\sigma'(t) = \sigma_0 - \frac{2\sqrt{2\pi \sigma}}{\sqrt{3K'}} \|u_0\|_{(\sigma_0,2)}(e^{K'|t|/2} - 1).
\]
Then, for any fixed \( T > 0 \), we have \( u(\cdot,t) \in A(e^{2\pi \sigma'(t)}) \) for \( t \in [-T,T] \).

The proof of Theorem 3.3 will be given in the later subsections. First we will prove the analyticity in \( x \) and establish the lower bound of \( \sigma(t) \). Next we will establish the analyticity in \((t,x)\). The proof of Theorem 3.4 is essentially contained in that of Theorem 3.3. We can employ Remark 4 instead of Proposition 9.

3.2. Regularity theorem by Kato-Masuda. In [13], the authors used their theory of Liapnov families to prove a regularity result about the KdV and other equations. Later, it was applied to a generalized Camassa-Holm equation in [3]. Here we recall the abstract theorem in [13] in a weaker, more concrete form. It is good enough for our purpose. The inequality (41) is a special case of that in [13]. Notice that \( v \) and \( u(t,\cdot) \) are elements of \( O \) in (41) and (42). In applications, making a suitable choice of the subset \( O \) of \( Z \) is essential.

**Theorem 3.5.** Let \( X \) and \( Z \) be Banach spaces. Assume that \( Z \) is a dense subspace of \( X \). Let \( O \) be an open subset of \( Z \) and \( F \) be a continuous mapping from \( Z \) to \( X \). Let \( \{ \Phi_s; -\infty < s < \infty \} \) be a family of real-valued functions on \( Z \) satisfying the conditions (a), (b) and (c) below.

(a) The Fréchet partial derivative of \( \Phi_s(v) \) in \( v \in Z \) exists not only in \( L(Z;\mathbb{R}) \) but also in \( L(X;\mathbb{R}) \). It is denoted by \( D\Phi_s(v) \). This statement makes sense because \( L(X;\mathbb{R}) \subset L(Z;\mathbb{R}) \) by the canonical identification. [(a) follows from (b) below.]

(b) The Fréchet derivative of \( \Phi_s(v) \) in \( (s,v) \) exists not only in \( L(\mathbb{R} \times Z;\mathbb{R}) \) but also in \( L(\mathbb{R} \times X;\mathbb{R}) \) and is continuous from \( \mathbb{R} \times Z \) to \( L(\mathbb{R} \times X;\mathbb{R}) \). This statement makes sense because \( L(\mathbb{R} \times X;\mathbb{R}) \subset L(\mathbb{R} \times Z;\mathbb{R}) \) by the canonical identification.

(c) There exist positive constants \( K \) and \( L \) such that
\[
|\langle F(v), D\Phi_s(v) \rangle| \leq K \Phi_s(v) + L \Phi_s(v)^{1/2} \partial_s \Phi_s(v)
\]
holds for any \( v \in O \). Here \( \langle \cdot, \cdot \rangle \) (no subscript) is the pairing of \( X \) and \( L(X;\mathbb{R}) \).

Let \( u \in C([0,T]; O) \cap C^1([0,T]; X) \) be the solution to the Cauchy problem
\[
\begin{cases}
\frac{du}{dt} = F(u), \\
u(0,x) = u_0(x).
\end{cases}
\]
(42)

Moreover, for a fixed constant \( s_0 \), set
\[
r(t) = \Phi_{s_0}(u_0)e^{Kt},
\]
\[
s(t) = s_0 - \int_0^t Lr(\tau)^{1/2} d\tau = s_0 - \frac{2L\Phi_{s_0}(u_0)^{1/2}}{K}(e^{Kt/2} - 1),
\]
for \( t \in [0,T] \). Then we have
\[
\Phi_{s(t)}(u(t)) \leq r(t), \ t \in [0,T].
\]

Roughly speaking, this theorem means that the regularity of \( u(t) \) for \( t \in [0,T] \) follows from that of \( u_0 \). Later we will use this when \( X = H^{m+2}, Z = H^{m+5} \) and \( \Phi_s \) is related to some variant of the Sobolev norms.
We can extend Theorem 3.5 to $t \leq 0$.

**Corollary 1.** Let $u \in C([-T, T]; \mathcal{O}) \cap C^1([-T, T]; X)$ be the solution to the Cauchy problem (42). Extend the definition of $\rho$ and $\sigma$ so that $r(t) = r(|t|), s(t) = s(|t|)$ by

$$
r(t) = \Phi_{s_0}(u_0) e^{K|t|},
$$

$$
s(t) = s_0 - \int_0^{|t|} L\rho(\tau)^{1/2} d\tau = s_0 - \frac{2L\Phi_{s_0}(u_0)^{1/2}}{K}(e^{K|t|/2} - 1),
$$

for $t \in [-T, T]$. Then we have

$$
\Phi_{s(t)}(u(t)) \leq r(t), \quad t \in [-T, T].
$$

**Proof.** For $t \leq 0$, set $t' = -t, \tilde{u}(t') = u(-t') = u(t)$. Then $\tilde{u}$ satisfies $d\tilde{u}/dt' = -F(\tilde{u}), \tilde{u}(0, x) = u_0(x)$. Notice that $-F$ satisfies (41). By Theorem 3.5, we have $\Phi_{s(t')}(\tilde{u}(t')) \leq r(t')$ for $t' \in [0, T]$. It means $\Phi_{s(t)}(u(t)) \leq r(t)$ for $t \in [-T, 0]$. □

### 3.3. Pairing and estimates.

Recall the norm $\|\cdot\|_{(\sigma, s)}$ in Section 1. When $s = 2$, we have

$$
\|v\|_{(\sigma, 2)}^2 = \sum_{j=0}^{\infty} \frac{1}{j!^2} e^{4\pi\sigma j} \|v^{(j)}\|_2^2.
$$

It is approximated by the finite sum

$$
\|v\|_{(\sigma, 2, m)}^2 = \sum_{j=0}^{m} \frac{1}{j!^2} e^{4\pi\sigma j} \|v^{(j)}\|_2^2.
$$

Set $\Psi_j(v) = \frac{1}{2} \|v^{(j)}\|_2^2 = \frac{1}{2} \|\Lambda^2 v^{(j)}\|_0^2 = \frac{1}{2} \int_{S^1} \left(\Lambda^2 v^{(j)}\right)^2 dx$ for $j = 0, 1, \ldots, m$. We introduce

$$
\Phi_{\sigma, m}(v) = \frac{1}{2} \|v\|_{(\sigma, 2, m)}^2 = \sum_{j=0}^{m} \frac{1}{j!^2} e^{4\pi\sigma j} \frac{\|v^{(j)}\|_2^2}{2} = \sum_{j=0}^{m} \frac{1}{j!^2} e^{4\pi\sigma j} \Psi_j(v).
$$

Later $\{\Phi_{\sigma, m}\}$ will play the role of $\{\Phi_{s}\}$ in Corollary 1. Assume $w \in H^{m+2}, v \in H^{m+5}$. Let $D\Psi_j$ be the Fréchet derivative of $\Psi_j$ and $\langle w, D\Psi_j(v) \rangle$ be the pairing of $w \in H^{m+2}$ and $D\Psi_j(v) \in (H^{m+2})^* \subset (H^{m+5})^*$. Indeed, $D\Psi_j(v)$ is an element of $(H^{m+2})^*$, because

$$
\langle w, D\Psi_j(v) \rangle = D\Psi_j(v)w = \frac{d}{d\tau} \Psi_j(v + \tau w) \bigg|_{\tau=0} = \int_{S^1} \Lambda^2 v^{(j)} \Lambda^2 w^{(j)} dx = \langle v^{(j)}, w^{(j)} \rangle_2,
$$

where $\langle \cdot, \cdot \rangle_2$ is the $H^2$ inner product. Recall

$$
F_\mu(u) = -uu_x - \partial_x A^{-1} \left[ 2\mu(u)u + \frac{1}{2} u_x^2 \right].
$$

It is easy to see that $F_\mu$ is continuous from $H^{m+5}$ to $H^{m+2}$.

**Proposition 9.** We have

$$
|\langle F_\mu(v), D\Phi_{\sigma, m}(v) \rangle| \leq (20\pi d_1 + 8 + 4\pi^2 c_1)\|v\|_2 \Phi_{\sigma, m}(v) + \sqrt{3}\pi \gamma \Phi_{\sigma, m}(v)^{1/2} \partial_x \Phi_{\sigma, m}(v)
$$

for $v \in H^{m+5}$, where $D\Phi_{\sigma, m}$ is the Fréchet derivative of $\Phi_{\sigma, m}$.
Proof. We have

\[
\langle F_\mu(v), D\Phi_{\sigma,m}(v) \rangle = \sum_{j=0}^{m} \frac{1}{j!^2} e^{4\pi \sigma j} \langle F_\mu(v), D\Psi_j(v) \rangle.
\]  

(46)

By (44), we have

\[
\langle F_\mu(v), D\Psi_j(v) \rangle = \langle v^{(j)}, \partial_x^j F_\mu(v) \rangle_2
\]

\[
= -\langle v^{(j)}, \partial_x^j (v\nu_x) \rangle_2 - 2\langle v^{(j)}, \partial_x^{j+1} A^{-1} [\mu(v)v] \rangle_2 - \frac{1}{2} \langle v^{(j)}, \partial_x^{j+1} A^{-1} (v^2) \rangle_2.
\]  

(47)

By using (46), (47) and the estimates (61), (62) and (65) below, we obtain (45). In the following subsections, we will calculate the three terms in (47) separately.

Remark 4. The $\mu$DP equation can be studied by using the following estimate. Set

\[
F^{\text{DP}}_\mu(u) = -wu_x - \partial_x A^{-1} [3\mu(u)u].
\]

Then we have

\[
|\langle F^{\text{DP}}_\mu(v), D\Phi_{\sigma,m}(v) \rangle| \leq (20\pi d_1 + 12)\|v\|_2 \Phi_{\sigma,m}(v) + \frac{2\pi \gamma}{\sqrt{3}} \Phi_{\sigma,m}(v)^{1/2} \partial_\sigma \Phi_{\sigma,m}(v).
\]

3.3.1. Estimate of $\langle v^{(j)}, \partial_x^j (v\nu_x) \rangle_2$. We have

\[
\langle v^{(j)}, \partial_x^j (v\nu_x) \rangle_2 = P_j + Q_j,
\]

(48)

\[
P_j = \langle v^{(j)}, v\nu^{(j+1)} \rangle_2,
\]

(49)

\[
Q_j = \sum_{\ell=1}^{j} \frac{j!}{\ell!} \langle v^{(j)}, v^{(\ell)} v^{(j-\ell+1)} \rangle_2 (j \geq 1), \quad Q_0 = 0.
\]

(50)

Because of (9), we have

\[
P_j = I_0 + \frac{1}{2\pi^2} I_1 + \frac{1}{(2\pi)^4} I_2,
\]

(51)

where $I_i = \langle v^{(j+i)}, \partial_x^i (v\nu^{(j+1)}) \rangle_0$. Set $w = v^{(j)}$ for brevity. Then we have

\[
I_i = \langle w^{(j)}, \partial_x^i (v\nu_x) \rangle_0.
\]

Integrating $\partial_x(vw^2/2) = vw_x + vw_x$, we obtain

\[
I_0 = \langle w, vw_x \rangle_0 = -\frac{1}{2} \int_{S^1} v_x w_x^2 \, dx.
\]

Therefore the Schwarz inequality and Proposition 3 (ii) imply

\[
|I_0| \leq \frac{1}{2} \|v_x\|_0 \|w_x^2\|_0 \leq \pi d_1 \|v\|_1 \|w\|_0 \|w_x\|_1 \leq \pi d_1 \|v\|_2 \|w_x\|_1^2.
\]  

(52)

Next, we have $I_1 = \int_{S^1} w_x \partial_x (vw_x) \, dx = -\int_{S^1} vvw_xw_x \, dx$ by integration by parts. Integrating $\partial_x(vw^2_x/2) = vvw_xw_x + v_x w_x^2/2$, we obtain

\[
I_1 = -\frac{1}{2} \int_{S^1} v_x w_x^2 \, dx.
\]

Therefore we get, by the Schwarz inequality, Proposition 3 (ii) and Proposition 4,

\[
|I_1| \leq \frac{1}{2} \|v_x\|_0 \|w_x^2\|_0 \leq \pi d_1 \|v\|_1 \|w_x\|_0 \|w_x\|_1 \leq 4\pi^3 d_1 \|v\|_2 \|w_x\|_1^2.
\]  

(53)
The estimate of \( I_2 \) is a bit complicated. We divide it into a sum of three terms as in
\[
I_2 = \langle w_{xx}, \partial_x^2(vw_x) \rangle_0 = I_{21} + I_{22} + I_{23},
\]
where
\[
I_{21} = \langle w_{xx}, v_{xx}w_x \rangle_0, \quad I_{22} = 2\langle w_{xx}, v_ww_x \rangle_0, \quad I_{23} = \langle w_{xx}, vw_{xxx} \rangle_0.
\]
We have
\[
|I_{21}| \leq \|w_{xx}\|_0 \|v_{xx}w_x\|_0 \leq d_1 \|w_{xx}\|_0 \|v_x\|_0 \|w_x\|_1 \leq 32\pi^5 d_1 \|v\|_2 \|w\|_2^2, \quad (54)
\]
\[
|I_{22}| \leq 2 \|w_{xx}\|_0 \|v_xw_x\|_0 \leq 2d_1 \|w_{xx}\|_0 \|v_x\|_1 \|w_x\|_0 \leq 64\pi^5 d_1 \|v\|_2 \|w\|_2^2. \quad (55)
\]
Integrating \( \partial_x(vw_x^2/2) = vw_xw_{xxx} + v_xw_x^2/2 \), we obtain
\[
I_{23} = \int_{S^1} vw_xw_{xxx} \, dx = -\frac{1}{2} \int_{S^1} v_xw_x^2 \, dx = -\frac{1}{4} I_{22}.
\]
Hence
\[
|I_{23}| \leq 16\pi^5 d_1 \|v\|_2 \|w\|_2^2. \quad (56)
\]
The three inequalities (54), (55) and (56) give
\[
|I_2| \leq 112\pi^5 d_1 \|v\|_2 \|w\|_2^2. \quad (57)
\]
Combining (51), (52), (53) and (57), we obtain
\[
|P_j| \leq 10\pi d_1 \|v\|_2 \|w_j\|_2^2 = 10\pi d_1 \|v\|_2 \|v^{(j)}\|_2^2 \quad (58)
\]
and
\[
\left| \sum_{j=0}^{m} \frac{1}{j!^2} e^{4\pi\sigma j} P_j \right| \leq 10\pi d_1 \|v\|_2 \sum_{j=0}^{m} \frac{1}{j!^2} e^{4\pi\sigma j} \|v^{(j)}\|_2^2 = 20\pi d_1 \|v\|_2 \Phi_{\sigma,m}(v). \quad (59)
\]
Next we calculate \( \sum_{j=1}^{m} j^{-2} e^{4\pi\sigma j} Q_j \). Recall
\[
Q_j = \sum_{\ell=1}^{j} \left( \begin{array}{c} j \\ \ell \end{array} \right) \langle v^{(\ell)}, v^{(\ell-\ell+1)} \rangle_2 \quad (j \geq 1), \quad Q_0 = 0.
\]
By Proposition 3 (iii), we have
\[
\|v^{(\ell)}v^{(j-\ell+1)}\|_2 \leq \gamma \left( \|v^{(\ell)}\|_2 \|v^{(j-\ell+1)}\|_1 + \|v^{(j)}\|_1 \|v^{(j-\ell+1)}\|_2 \right)
\]
\[
\leq 2\pi \gamma \left( \|v^{(\ell)}\|_2 \|v^{(j-\ell)}\|_2 + \|v^{(\ell-1)}\|_2 \|v^{(j-\ell+1)}\|_2 \right).
\]
Combining this estimate with the Schwarz inequality, we get
\[
|Q_j| \leq 2\pi \gamma (Q_{j,1} + Q_{j,2}),
\]
\[
Q_{j,1} = \|v^{(j)}\|_2 \sum_{\ell=1}^{j} \left( \begin{array}{c} j \\ \ell \end{array} \right) \|v^{(\ell)}\|_2 \|v^{(j-\ell)}\|_2;
\]
\[
Q_{j,2} = \|v^{(j)}\|_2 \sum_{\ell=1}^{j} \left( \begin{array}{c} j \\ \ell \end{array} \right) \|v^{(\ell-1)}\|_2 \|v^{(j-\ell+1)}\|_2.
\]
Now we set $b_k = k^{-1}e^{2\pi\sigma k\|v^{(k)}\|_2}$ $(k = 0, 1, \ldots, j)$. Then we have

$$\frac{1}{j^2} e^{4\pi\sigma j} Q_{j,1} \leq \sum_{\ell=1}^{j} b_{\ell} b_{\ell-j}$$

$$\frac{1}{j^2} e^{4\pi\sigma j} Q_{j,2} \leq \sum_{\ell=1}^{j} b_{\ell} b_{\ell-j} (j - \ell + 1) b_{\ell-j+1}$$

Set $B = \left(\sum_{j=0}^{m} j^2\right)^{1/2}$, $\tilde{B} = \left(\sum_{j=1}^{m} j b_j^2\right)^{1/2}$. Notice that $B^2 = \|v\|^2_{(\sigma,2,\mu)} = 2\Phi_{\sigma,m}(v)$, $\tilde{B}^2 = (2\pi)^{-1} \partial_\sigma \Phi_{\sigma,m}(v)$. The repeated use of the Schwarz inequality gives ([13, Lemma 3.1])

$$\sum_{j=1}^{m} \frac{1}{j^2} e^{4\pi\sigma j} Q_{j,1} \leq \sum_{j=1}^{m} \sum_{\ell=1}^{j} b_{\ell} b_{\ell-j} \leq \sum_{\ell=1}^{m} \frac{b_{\ell}}{\sqrt{\ell}} \sum_{j=1}^{m} \sqrt{\ell} b_j b_{\ell-j}$$

$$\leq B\tilde{B} \sum_{\ell=1}^{m} \frac{\sqrt{\ell} b_{\ell}}{\ell} \leq B\tilde{B} \left(\sum_{\ell=1}^{m} \frac{1}{\ell^2}\right)^{1/2} \leq \frac{\pi}{\sqrt{6}} B\tilde{B}^2,$$

$$\sum_{j=1}^{m} \frac{1}{j^2} e^{4\pi\sigma j} Q_{j,2} \leq \sum_{j=1}^{m} \sum_{\ell=1}^{j} b_{\ell} b_{\ell-j} (j - \ell + 1) b_{\ell-j+1}$$

$$\leq \sum_{\ell=1}^{m} \frac{b_{\ell-1}}{\ell} \sum_{j=\ell}^{m} \sqrt{\ell} b_j \sqrt{j - \ell + 1} b_{j-\ell+1}$$

$$\leq \tilde{B}^2 \sum_{\ell=1}^{m} \frac{b_{\ell-1}}{\ell} \leq \tilde{B}^2 B \left(\sum_{\ell=1}^{m} \frac{1}{\ell^2}\right)^{1/2} \leq \frac{\pi}{\sqrt{6}} B\tilde{B}^2.$$

Recalling $Q_0 = 0$, we obtain

$$\left|\sum_{j=0}^{m} \frac{1}{j^2} e^{4\pi\sigma j} Q_j\right| \leq \frac{(2\pi)^2 \gamma}{\sqrt{6}} B\tilde{B}^2 = \frac{2\pi \gamma}{\sqrt{3}} \Phi_{\sigma,m}(v)^{1/2} \partial_\sigma \Phi_{\sigma,m}(v). \quad (60)$$

The inequalities (59) and (60) give, together with (48),

$$\left|\sum_{j=0}^{m} \frac{1}{j^2} e^{4\pi\sigma j} (v^{(j)}, \partial_x^{j+1}A^{-1}[\mu(v)v])_2\right| \leq 20\pi d_1 \|v\|_2 \Phi_{\sigma,m}(v) + \frac{2\pi \gamma}{\sqrt{3}} \Phi_{\sigma,m}(v)^{1/2} \partial_\sigma \Phi_{\sigma,m}(v). \quad (61)$$

3.3.2. Estimate of $\langle v^{(j)}, \partial_x^{j+1}A^{-1}[\mu(v)v]\rangle_2$. We consider the second term of the right-hand side of (47). By (9), we have

$$\langle v^{(j)}, \partial_x^{j+1}A^{-1}[\mu(v)v]\rangle_2 = \mu(v)\langle v^{(j)}, A^{-1}v^{(j+1)}\rangle_2,$$

$$= \mu(v) \left(J_0 + \frac{1}{2\pi^2} J_1 + \frac{1}{(2\pi)^4} J_2\right),$$

where $J_i = \langle v^{(j+i)}, A^{-1}v^{(j+i+1)}\rangle_0$ $(i = 0, 1, 2)$. Set $w = v^{(j)}$ for brevity. We have

$$J_0 = \langle w, A^{-1}\partial_x w\rangle_0, \quad J_1 = \langle w_x, A^{-1}\partial_x^2 w\rangle_0, \quad J_2 = \langle w_{xx}, A^{-1}\partial_x^2 w_x\rangle_0.$$
Since $A^{-1} \partial_x^i : H^0 \to H^0$ ($i = 0, 1, 2$) are bounded operators whose norms are equal to 1, we have

$$
|\mu(v) J_i| \leq \|v\|_2 \|w\|_2 \|A^{-1} \partial_x w\|_0 \leq \|v\|_2 \|w\|_2^2,
$$

$$
|\mu(v) J_1| \leq \|v\|_2 \|w_x\|_0 \|A^{-1} \partial_x^2 w\|_0 \leq 2\pi \|v\|_2 \|w\|_2^2,
$$

$$
|\mu(v) J_2| \leq \|v\|_2 \|w_{xx}\|_0 \|A^{-1} \partial_x^2 w_x\|_0 \leq (2\pi)^3 \|v\|_2 \|w\|_2^2.
$$

Therefore we get

$$
\left| \langle v^{(j)}, \partial_x^{j+1} A^{-1} [\mu(v) v] \rangle \right|_2 \leq \left[ 1 + 1/\pi + 1/(2\pi) \right] \|v\|_2 \|v^{(j)}\|_2^2 \\
\leq 2 \|v\|_2 \|v^{(j)}\|_2^2
$$

and

$$
\sum_{j=0}^m \frac{1}{j!} 2 e^{4\pi j} \langle v^{(j)}, \partial_x^{j+1} A^{-1} [\mu(v) v] \rangle \leq 4 \|v\|_2 \Phi_{\sigma,m}(v). \quad (62)
$$

3.3.3. Estimate of $\langle v^{(j)}, \partial_x^{j+1} A^{-1}(v_x^2) \rangle_2$. We consider the third term of the right-hand side of (47).

First, assume $j = 0$. Since the norm of $\partial_x A^{-1} : H^1 \to H^2$ does not exceed 1, we have

$$
\langle v^{(j)}, \partial_x A^{-1}(v_x^2) \rangle_2 \leq \|v\|_2 \|\partial_x A^{-1}(v_x^2)\|_2 \leq \|v\|_2 \|v_x^2\|_1 \\
\leq c_1 \|v\|_2 \|v_x\|_2 \|v_x\|_1 \leq (2\pi)^2 c_1 \|v\|^3_2 \leq 8\pi^2 c_1 \|v\|_2 \Phi_{\sigma,m}(v). \quad (63)
$$

Next we assume $j \geq 1$. We have

$$
\langle v^{(j)}, \partial_x^{j+1} A^{-1}(v_x^2) \rangle_2 = \langle v^{(j)}, (\partial_x^2 A^{-1})^{j+1}(v_x^2) \rangle_2 \\
= \sum_{\ell=0}^{j-1} \binom{j-1}{\ell} \langle v^{(j)}, (\partial_x^2 A^{-1})^{(j+1)}(v^{(j-1)} v^{(j-\ell)}) \rangle_2 \\
= \sum_{\ell=1}^j \binom{j-1}{\ell-1} \langle v^{(j)}, (\partial_x^2 A^{-1})^{(j-1)}(v^{(\ell)} v^{(j-\ell+1)}) \rangle_2. \quad (64)
$$

This is similar to $Q_j$ in (50). Since the norm of $\partial_x^2 A^{-1} : H^2 \to H^2$ is 1, this operator can be neglected in estimating $\langle v^{(j)}, (\partial_x^2 A^{-1})^{(j+1)}(v^{(j-\ell+1)}) \rangle_2$. Moreover we have

$$
\binom{j-1}{\ell-1} \leq \binom{j}{\ell} \quad (j \geq 1).
$$

We can follow (60) for $j \geq 1$ and employ (63) for $j = 0$ (Recall $Q_0 = 0$).

Finally we obtain

$$
\sum_{j=0}^m \frac{1}{j!} 2 e^{4\pi j} \langle v^{(j)}, \partial_x^{j+1} A^{-1}(v_x^2) \rangle_2 \\
\leq 8\pi^2 c_1 \|v\|_2 \Phi_{\sigma,m}(v) + \frac{2\pi^2}{\sqrt{3}} \Phi_{\sigma,m}(v)^{1/2} \partial_x \Phi_{\sigma,m}(v). \quad (65)
$$

It completes the proof of Proposition 9.
3.4. Analyticity in the space variable. In this subsection, we prove a part of Theorem 3.3. We assume that $u_0 \in A(\tau_0)$ satisfies the conditions in Theorem 3.3. Then we can apply Theorem 3.1 with arbitrarily large $s$. We will prove the analyticity of $u(t)$ in $x$ for each fixed $t$.

**Proposition 10.** Let $T > 0$ be given. We have $u(t) \in A(e^{2\pi\sigma(t)}) \subset A(e^{2\pi\sigma(T)})$ for $t \in [-T, T]$, where the function $\sigma(t)$ is defined in Theorem 3.3.

**Proof.** Theorem 3.1 implies $u(t) \in H^\infty$. Set
\[
\mu_0 = 1 + \max \{ \|u(t)\|_2; t \in [-T, T] \},
\]
\[
\mathcal{O} = \left\{ v \in H^{m+5}; \|v\|_2 < \mu_0 \right\},
\]
\[
K = (20\pi d_1 + 8 + 4\pi^2 c_1)\mu_0.
\]
Then $u(t) \in \mathcal{O}$ for $t \in [-T, T]$. Proposition 9 implies that
\[
|\langle F_\mu(v), D\Phi_{\sigma,m}(v) \rangle| \leq K\Phi_{\sigma,m}(v) + \sqrt{3}\pi\gamma\Phi_{\sigma,m}(v)^{1/2}\partial_x\Phi_{\sigma,m}(v)
\] (66)
holds for any $v \in \mathcal{O}$. Fix $\sigma_0$ with $e^{2\pi\sigma_0} < \tau_0$. Then $u_0 \in A(\tau_0)$ implies $\|u_0\|_{(\sigma_0,2)} < \infty$ by Proposition 6.

Set
\[
\rho_m(t) = \frac{1}{2}\|u_0\|^2_{(\sigma_0,2,m)}e^{K|t|} = \Phi_{\sigma_0,m}(u_0)e^{K|t|},
\]
\[
\sigma_m(t) = \sigma_0 - \int_0^{|t|} \sqrt{3}\pi\gamma\rho_m(\tau)^{1/2} d\tau
\]
\[
= \sigma_0 - \sqrt{6}\pi\gamma/K\|u_0\|_{(\sigma_0,2,m)}(e^{K|t|/2} - 1).
\]
These functions correspond to $r(t)$ and $s(t)$ in Corollary 1. Similarly, set
\[
\rho(t) = \frac{1}{2}\|u_0\|^2_{(\sigma_0,2)}e^{K|t|},
\]
\[
\sigma(t) = \sigma_0 - \int_0^{|t|} \sqrt{3}\pi\gamma\rho(t)^{1/2} d\tau
\]
\[
= \sigma_0 - \sqrt{6}\pi\gamma/K\|u_0\|_{(\sigma_0,2)}(e^{K|t|/2} - 1).
\]
We have $\rho_m(t) \leq \rho_{m+1}(t) \leq \rho(t)$, $\rho_m(t) \to \rho(t)$ (as $m \to \infty$) and $\sigma_m(t) \geq \sigma_{m+1}(t) \geq \sigma(t)$, $\sigma_m(t) \to \sigma(t)$ (as $m \to \infty$). We can apply Corollary 1 and obtain
\[
\Phi_{\sigma_m(t),m}(u(t)) \leq \rho_m(t) \leq \rho(t), \quad t \in [-T, T].
\] (67)
By Fatou’s Lemma,
\[
\|u(t)\|_{(\sigma(T),2)} = \sum_{j=0}^{\infty} \frac{1}{j!^2} e^{4\pi j \sigma(t)} \|\partial_x^j u(t)\|^2_2 \leq \liminf_{m \to \infty} \sum_{j=0}^{m} \frac{1}{j!^2} e^{4\pi j \sigma_m(t)} \|\partial_x^j u(t)\|^2_2
\]
\[
= 2\liminf_{m \to \infty} \Phi_{\sigma_m(t),m}(u(t)) \leq 2\rho(t) < \infty.
\]
Therefore $u(t) \in A(e^{2\pi\sigma(t)}) \subset A(e^{2\pi\sigma(T)})$ for $t \in [-T, T]$ by Proposition 6.

**Proposition 11.** The mapping $[-T, T] \to A(e^{2\pi\sigma(T)})$, $t \mapsto u(t)$ is continuous.

**Proof.** Let $\{t_n\} \subset [-T, T]$ be a sequence converging to $t_\infty \in [-T, T]$. We have $u(t_n) \to u(t_\infty)$ in $H^\infty$. On the other hand, $\|u(t_n)\|_{(\sigma(T),2)} (n \geq 0)$ is bounded since
\[
\|u(t)\|_{(\sigma(T),2)} \leq \|u(t)\|_{(\sigma(t),2)} \leq \rho(t) \leq \rho(T).
\]
Proposition 8 implies that \( u(t_n) \) converges to \( u(t_\infty) \) with respect to \( \| \cdot \|_{(\sigma',2)} (\sigma' < \sigma(T)) \). By Proposition 7, this means convergence in \( A(e^{2\pi\sigma(T)}) \). \( \square \)

3.5. **Analyticity in the space and time variables.** We continue the proof of Theorem 3.3. In the previous subsection, we have established the analyticity in the space variable. Here, we will prove the analyticity in the space and time variables.

**Proposition 12.** Under the situation of Theorem 3.3, for any \( T > 0 \), there exists \( \delta_T > 0 \) such that we have \( u \in C^\omega([-T,T], A(\delta_T)) \).

**Proof.** We have \( u_0 \in G^\Delta,s+1 \) for any \( \Delta < r_0 \). Let \( s > 1/2 \). By Theorem 2.3, there exists \( \tilde{T} > 0 \) such that the Cauchy problem (21) has a unique solution \( \tilde{u} \in C^\omega(|t| \leq \tilde{T}(1-d), G^{\Delta,d,s+1}) \) for \( 0 < d < 1 \). We have \( \tilde{u} = u \) by the local uniqueness, where \( u \) is the solution in Theorem 3.1. Set \( d = 1/2, \tilde{T} = \tilde{T}/2 \). Then \( u = \tilde{u} \in C^\omega(|t| \leq \tilde{T}, G^{\Delta/2,s+1}) \). By Proposition 1, a convergent series in \( G^{\Delta/2,s+1} \) is convergent in \( A(\Delta/2) \). We have \( u \in C^\omega(|t| \leq \tilde{T}, A(\Delta/2)) \).

We have shown that \( u \) is analytic in \( t \) at least locally. Our next step is to show that \( u \) is analytic in \( t \) globally. Set

\[
S = \{ T > 0; u \in C^\omega([-T,T], A(\delta_T)) \text{ for } \exists \delta_T > 0 \} \ni \tilde{T},
\]

\[
T^* = \sup S = \tilde{T}.
\]

We prove \( T^* = \infty \) by contradiction. Assume \( T^* < \infty \). By Proposition 10, \( u(T^*) \) is well-defined and there exists \( \delta^* > 0 \) such that

\[
u(T^*) \in A(\delta^*) \subset G^{\delta',s+1} (0 < \delta' < \delta^*).
\]

By Theorem 2.3 (with \( t \) replaced with \( t - T^* \)), there exists \( \varepsilon > 0 \) and \( \hat{u} \in C^\omega([T^*-\varepsilon,T^*+\varepsilon], G^{\delta'/2,s+1}) \) such that

\[
\hat{u}_t + \frac{1}{2}(\hat{u}^2)_x + \partial_x A^{-1} \left[ 2 \mu(\hat{u}) u + \frac{1}{2} \hat{u}_x^2 \right] = 0,
\]

\[
\hat{u}|_{t=T^*} = u(T^*).
\]

By the local uniqueness, we have \( \hat{u} = u \). Namely, \( \hat{u} \) is an extension of \( u \) up to \( |t| \leq T^* + \varepsilon \) (valued in \( G^{\delta'/2,s+1} \subset A(\delta'/2) \)). Therefore \( T^* + \varepsilon \in S \). This is a contradiction. \( \square \)

**Proposition 13.** Under the situation of Theorem 3.3, the Cauchy problem (21) has a unique solution \( u \in C^\omega(\mathbb{R}_t \times S_x^1) \).

**Proof.** The uniqueness in \( H^\infty \) implies the uniqueness in the real-analytic category.

Let \( T \) be fixed. For \( r > 0 \) sufficiently small, we have

\[
\partial_t^j u(t) = \frac{j!}{2\pi i} \int_{|\tau - \epsilon| = r} \frac{u(\tau)}{(\tau - t)^{j+1}} \, d\tau, \quad t \in [-T,T].
\]

The integral is performed in \( A(\delta_T) \) and converges with respect to \( \| \cdot \|_{(\sigma,0)} (e^{2\pi\sigma} < \delta_T) \). By Cauchy’s estimate, there exists \( C_0 > 1/r \) such that

\[
\| \partial_t^j u(t) \|_{(\sigma,0)} \leq C_0 j! r^{-j} \leq C_0^j + 1! j!.
\]

Therefore we have

\[
\| \partial_x^k \partial_t^j u(\cdot,t) \|_0 \leq C_0^{j+1} e^{-2\pi k \sigma} j! k!.
\]
and there exists $C > 0$ such that
\[
\left\| \partial_x^j \partial_t^k u \right\|_{L^2(S^1 \times [-T,T])} \leq \sqrt{2T} C^{j+k+1} (j+k)!. \tag{68}
\]
Set $\Delta = \partial_x^2 + \partial_t^2$. The binomial expansion of $\Delta^\ell (\ell = 0, 1, 2, \ldots)$ and (68) yield
\[
\left\| \Delta^\ell u \right\|_{L^2(S^1 \times [-T,T])} \leq \sqrt{2T} C^{2\ell+1} (2\ell)! \sum_{p=0}^{\ell} \binom{\ell}{p} \leq \sqrt{2T} C^{2\ell+1} (2\ell)!.
\]
This estimates implies the real-analyticity of $u$ due to Theorem 3.6 by Komatsu given below.

In the last step of the proof of Proposition 13, we have used the following theorem.

\textbf{Theorem 3.6.} ([16]) Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $P = P(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ be an elliptic partial differential operator of order $m$ with constant coefficients. Then, for a function $f \in L^2_{\text{loc}}(\Omega)$ to be analytic in $\Omega$, it is (necessary and) sufficient that
1) for every $\ell \in \mathbb{Z}_+$, $P^\ell f$ (in the sense of distributions) belongs to $L^2_{\text{loc}}(\Omega)$, and that
2) for every compact subset $K \subset \Omega$, there exist positive constants $M$ and $A$ such that
\[
\left\| P^\ell f \right\|_{L^2(K)} \leq M (A\ell)^m.
\]

This better known result in this direction is [17] which concerns an elliptic operator of order $m$ with analytic coefficients and the right-hand side of (69) is replaced with $M^{\ell+1}(m\ell)!$. We can employ the result of [17] instead of Theorem 3.6.

4. Global-in-time solutions: Higher-order case. In this section we consider (29). Global-in-time solutions in Sobolev spaces have been studied in [21]. Notice that the non-zero mean and the no-change-of-sign conditions are not imposed.

4.1. Statement of the main results.

\textbf{Theorem 4.1.} ([21, Theorem 2.1, 3.5]) Let $u_0 \in H^s(S^1), s > 7/2$. Then the Cauchy problem (29) has a unique global solution $u$ in $C(\mathbb{R}, H^s) \cap C^1(\mathbb{R}, H^{s-1})$. Moreover, local uniqueness holds.

\textbf{Remark 5.} In [21], the authors solve the Cauchy problem for $t > 0$ only. Since the equation is invariant under $(t, u) \mapsto (-t, -u)$, the result for $t < 0$ follows immediately.

There is an analogous result about (40).

\textbf{Theorem 4.2.} ([22, Theorem 2.1, 3.2]) Let $u_0 \in H^s(S^1), s > 7/2$. Then the Cauchy problem (40) has a unique global solution $u$ in $C(\mathbb{R}, H^s) \cap C^1(\mathbb{R}, H^{s-1})$. Moreover, local uniqueness holds.

Our main result about the higher order $\mu$CH is the following theorem.

\textbf{Theorem 4.3.} If $u_0$ is a real-analytic function on $S^1$, then the Cauchy problem (29) has a unique solution $u \in C^\omega(\mathbb{R}_t \times S^1_\omega)$. 
We have the following estimate of the radius of analyticity. Assume \( u_0 \in A(r_0) \). Fix \( \sigma_0 < (\log r_0)/(2\pi) \) and set

\[
\tilde{\sigma}(t) = \sigma_0 - \frac{\sqrt{2}\gamma_2}{\tilde{K}} ||u_0||_{(\sigma_0,2)} (e^{\tilde{K}t/2} - 1),
\]
\[
\tilde{K} = \gamma_1 [1 + \max \{||u(t)||_4; t \in [-T,T] \}],
\]
\[
\gamma_1 = 8 + 18\pi^2 c_1 + (20\pi + 24\pi^3 + 104\pi^4) d_1,
\]
\[
\gamma_2 = \frac{16\pi \gamma}{\sqrt{3}}.
\]

Then, for any fixed \( T > 0 \), we have \( u(\cdot, t) \in A(\tilde{\sigma}(t)) \) for \( t \in [-T,T] \).

**Proof.** Theorem 4.1 implies \( u(t) \in H^\infty \) if \( u_0 \in H^\infty \). Set

\[
\tilde{\mu}_0 = 1 + \max \{||u(t)||_4; t \in [-T,T] \} \ (H^4 \text{ norm, not } H^2),
\]
\[
\tilde{O} = \{v \in H^{m+5}; ||v||_4 < \tilde{\mu}_0 \} \ (H^4 \text{ norm, not } H^2),
\]
\[
\tilde{K} = \gamma_1 \tilde{\mu}_0,
\]
\[
\tilde{\rho}(t) = \frac{1}{2} ||u_0||_{(\sigma_0,2)}^2 e^{\tilde{K}t},
\]
\[
\tilde{\sigma}(t) = \sigma_0 - \int_0^{|t|} \gamma_2 \rho(\tau)^{1/2} d\tau.
\]

Then the proof is almost the same as that of Theorem 3.3 and follows from Proposition 14 below. It is an analogue of Proposition 9. Notice that \( ||v||_2 \) in (45) has been replaced with \( ||v||_4 \).

There is an analogous result about (40).

**Theorem 4.4.** If \( u_0 \) is a real-analytic function on \( S^1 \), then the Cauchy problem (40) has a unique solution \( u \in C^\omega(R_t \times S^1_x) \). The estimate of the radius of analyticity is the same as in Theorem 4.3.

The proof of Theorem 4.4 is almost the same as that of Theorem 4.3. One has only to replace \( B^{-1} \) with \( A^{-2} \). The rest of this section is devoted to the proof of Theorem 4.3. It is enough to prove Proposition 14 below.

**Proposition 14.** We have

\[
|\langle G(v), D\Phi_{\sigma,m}(v) \rangle| \leq \gamma_1 ||v||_4 \Phi_{\sigma,m}(v) + \gamma_2 \Phi_{\sigma,m}(v)^{1/2} \partial_\sigma \Phi_{\sigma,m}(v),
\]

where \( \gamma_1 \) and \( \gamma_2 \) are given in Theorem 4.3.

**Proof.** Recall (35), namely

\[
G(u) = -w w_x - \partial_x B^{-1} \left[ 2\mu(u) u + \frac{1}{2} u_x^2 - 3u_x u_{xxx} - \frac{7}{2} u_{xx}^2 \right].
\]

We have

\[
\langle G(v), D\Phi_{\sigma,m}(v) \rangle = \sum_{j=0}^m \frac{1}{j!} e^{4\pi j/2} \langle G(v), D\Psi_j(v) \rangle.
\]
By (44), we have
\[ \langle G(v), D\Psi_j(v) \rangle = \langle v^{(j)}, \partial_x^j G(v) \rangle_2 \]
\[ = -\langle v^{(j)}, \partial_x^j (v v_x) \rangle_2 - 2 \langle v^{(j)}, \partial_x^{j+1} B^{-1} [\mu(v)] \rangle_2 - \frac{1}{2} \langle v^{(j)}, \partial_x^{j+1} B^{-1} (v^2_x) \rangle_2 \]
\[ + 3 \langle v^{(j)}, \partial_x^{j+1} B^{-1} [v_x v_{xxx}] \rangle_2 + \frac{7}{2} \langle v^{(j)}, \partial_x^{j+1} B^{-1} (v^2_x) \rangle_2. \]  \hspace{1cm} (72)

We compare it with (47). We encountered \( \langle v^{(j)}, \partial_x^j (v v_x) \rangle_2 \) in (47). The following two terms \( -2 \langle v^{(j)}, \partial_x^{j+1} B^{-1} [\mu(v)] \rangle_2 \) and \( -\frac{1}{2} \langle v^{(j)}, \partial_x^{j+1} B^{-1} (v^2_x) \rangle_2 \) have better estimates than \( -2 \langle v^{(j)}, \partial_x^{j+1} A^{-1} [\mu(v)] \rangle_2 \) and \( -\frac{1}{2} \langle v^{(j)}, \partial_x^{j+1} A^{-1} (v^2_x) \rangle_2 \). Therefore we can employ (61) and analogues of (62) and (65). Here we replace \( \langle v \rangle_2 \) with \( (\|v\|_2 \leq \|v\|_4) \). Our remaining task is to estimate \( 3 \langle v^{(j)}, \partial_x^{j+1} B^{-1} [v_x v_{xxx}] \rangle_2 \) and \( \frac{7}{2} \langle v^{(j)}, \partial_x^{j+1} B^{-1} (v^2_x) \rangle_2 \). The results will be given as (77) and (82) in the following subsection.

\[ \square \]

4.2. Estimates: Higher-order case.

4.2.1. Estimate of \( \langle v^{(j)}, \partial_x^{j+1} B^{-1} [v_x v_{xxx}] \rangle_2 \). First we assume \( j \geq 3 \). Since
\[ \langle v^{(j)}, \partial_x^{j+1} B^{-1} [v_x v_{xxx}] \rangle_2 = \langle v^{(j)}, (\partial_x^3 B^{-1}) \partial_x^{-3} [v_x v_{xxx}] \rangle_2 \]
\[ = \sum_{\ell=0}^{j-3} \binom{j-3}{\ell} \langle v^{(j)}, (\partial_x^3 B^{-1}) v^{(j-\ell)} \rangle_2 \]
\[ = \sum_{\ell=1}^{j-2} \binom{j-3}{\ell-1} \langle v^{(j)}, (\partial_x^3 B^{-1}) v^{(j-\ell+1)} \rangle_2. \]

This is better than (50). Indeed, \( \partial_x^3 B^{-1}: H^2 \to H^2 \) is as good as \( \partial_x^3 A^{-1}: H^2 \to H^2 \) and the binomial coefficients have become smaller. We follow (60) with \( j \geq 3 \) instead of \( j \geq 0 \) and get
\[ \sum_{j=3}^{m} \frac{1}{j!} e^{4\pi j} \langle v^{(j)}, \partial_x^{j+1} B^{-1} (v_x v_{xxx}) \rangle_2 \leq \frac{2 \pi \gamma}{\sqrt{3}} \Phi_{\sigma,m}(v)^{1/2} \partial_x \Phi_{\sigma,m}(v). \]  \hspace{1cm} (73)

Next we consider the case \( j = 0 \). We have
\[ \|\langle \nu, \partial_x B^{-1} [v_x v_{xxx}] \rangle_2 \|_2 \leq \|v\|_2 \|\partial_x B^{-1} [v_x v_{xxx}] \|_2 \]
and, by Proposition 3 (ii),
\[ \|\partial_x B^{-1} [v_x v_{xxx}] \|_2 \leq \|v_x v_{xxx}\|_0 \leq d_1 \|v_x\|_1 \|v_{xxx}\|_0 \]
\[ \leq (2\pi)^4 d_1 \|v\|_2 \|v\|_3. \]

These two inequalities yield
\[ \|\langle \nu, \partial_x B^{-1} [v_x v_{xxx}] \rangle_2 \|_2 \leq (2\pi)^4 d_1 \|v\|_3 \|v\|_2^2 \leq (2\pi)^4 d_1 \|v\|_4 \|v\|_2^2. \]  \hspace{1cm} (74)

We have used \( \|v\|_3 \leq \|v\|_4 \), because \( \|v\|_4 \) will inevitably appear later. We deal with it by modifying the definition of \( \mathcal{O} \), so that \( \|v\|_3 \) and \( \|v\|_4 \) are bounded there. Proposition 9 should be modified accordingly.

Next assume \( j = 1 \). We have
\[ \|\langle v^{(1)}, \partial_x^2 B^{-1} [v_x v_{xxx}] \rangle_2 \|_2 \leq \|v^{(1)}\|_2 \|\partial_x^2 B^{-1} [v_x v_{xxx}] \|_2 \]
and, by Proposition 3 (ii) again,
\[
\|\partial_x B^{-1}[v_x v_{xxx}]\|_2 \leq \|v_x v_{xxx}\|_0 \leq d_1 \|v_x\|_1 \|v_{xxx}\|_0 \\
\leq (2\pi)^3 d_1 \|v\|_2 \|v^{(1)}\|_2.
\]
Therefore
\[
\left| e^{4\pi \sigma} \langle v^{(1)}, \partial_x B^{-1}[v_x v_{xxx}] \rangle_2 \right| \leq (2\pi)^3 d_1 \|v\|_2 e^{4\pi \sigma} \|v^{(1)}\|_2 \\
\leq (2\pi)^3 d_1 \|v\|_4 e^{4\pi \sigma} \|v^{(1)}\|_2.
\]

Next we assume \( j = 2 \). We have
\[
|\langle v^{(2)}, \partial_x B^{-1}[v_x v_{xxx}] \rangle_2| \leq \|v^{(2)}\|_2 \|\partial_x B^{-1}[v_x v_{xxx}]\|_2 \leq (2\pi)^2 \|v\|_4 \|\partial_x B^{-1}[v_x v_{xxx}]\|_2
\]
and, by Proposition 3 (ii),
\[
\|\partial_x B^{-1}[v_x v_{xxx}]\|_2 \leq \|\partial_x(v_x v_{xxx})\|_0 \leq \|v^{(2)}\|_4 + \|v^{(1)}\|_4 \|v^{(5)}\|_0 \\
\leq d_1 (\|v^{(2)}\|_1 \|v^{(3)}\|_0 + \|v^{(4)}\|_0 \|v^{(1)}\|_1 \leq (2\pi + 4\pi^2)d_1 \|v^{(2)}\|_2
\]
Therefore
\[
\left| e^{8\pi \sigma} \langle v^{(2)}, \partial_x B^{-1}[v_x v_{xxx}] \rangle_2 \right| \leq (8\pi^3 + 16\pi^4)d_1 \|v\|_4 e^{8\pi \sigma} \|v^{(2)}\|_2.
\]

By (73), (74), (75) and (76), we obtain
\[
\sum_{j=0}^{m} \frac{1}{12} e^{4\pi \sigma j} \langle v^{(j)}, \partial_x^{j+1} B^{-1}(v_x v_{xxx}) \rangle_2 \\
\leq (8\pi^3 + 16\pi^4)d_1 \|v\|_4 \Phi_{\sigma,m}(v) + \frac{2\pi \gamma}{\sqrt{3}} \Phi_{\sigma,m}(v)^{1/2} \partial_x \Phi_{\sigma,m}(v).
\]

4.2.2. Estimate of \( \langle v^{(j)}, \partial_x^{j} B^{-1}[v_x v_{xxx}] \rangle_2 \). First we assume \( j \geq 3 \). Since
\[
\langle v^{(j)}, \partial_x^{j+1} B^{-1}[v_x v_{xxx}] \rangle_2 = \langle v^{(j)}, (\partial_x B^{-1}) \partial_x^{j-3} [v_x v_{xxx}] \rangle_2 \\
= \sum_{\ell=0}^{j-3} \binom{j-3}{\ell} \langle v^{(j)}, (\partial_x B^{-1})^{\ell+2} v^{(j-\ell-1)} \rangle_2 \\
= \sum_{\ell=2}^{j-1} \binom{j-3}{\ell-2} \langle v^{(j)}, (\partial_x B^{-1})^{\ell} v^{(j-\ell+1)} \rangle_2.
\]
This is better than (50). We follow (60) with \( j \geq 3 \) instead of \( j \geq 0 \) and get
\[
\sum_{j=3}^{m} \frac{1}{j!} e^{4\pi \sigma j} \langle v^{(j)}, \partial_x^{j+1} B^{-1}[v_x v_{xxx}] \rangle_2 \\
\leq \frac{2\pi \gamma}{\sqrt{3}} \Phi_{\sigma,m}(v)^{1/2} \partial_x \Phi_{\sigma,m}(v).
\]

Next we consider the case \( j = 0 \). We have
\[
|\langle v, \partial_x B^{-1}[v_x v_{xxx}] \rangle_2| \leq \|v\|_2 \|\partial_x B^{-1}[v_x v_{xxx}]\|_2
\]
and, by Proposition 3 (ii),
\[
\|\partial_x B^{-1}[v_x v_{xxx}]\|_2 \leq \|v_x v_{xxx}\|_0 \leq d_1 \|v_{xxx}\|_1 \|v_{xxx}\|_0 \\
\leq (2\pi)^4 d_1 \|v|_3 \|v\|_2.
\]
These two inequalities yield
\[
|\langle v, \partial_x B^{-1}[v_x v_{xxx}] \rangle_2| \leq (2\pi)^4 d_1 \|v|_3 \|v\|_2 \leq (2\pi)^4 d_1 \|v\|_4 \|v\|_2.
\]
Next assume \( j = 1 \). We have
\[
|\langle v^{(1)}, \partial_x^2 B^{-1} [v_{xx}^2] \rangle| \leq \|v^{(1)}\|_2 \| \partial_x^2 B^{-1} [v_{xx}^2] \|_2
\]
and, by Proposition 3 (ii),
\[
\| \partial_x^2 B^{-1} [v_{xx}^2] \|_2 \leq \|v_{xx}^2\|_0 \leq d_1 \|v_{xx}\|_1 \|v_{xx}\|_0 \\
\leq (2\pi)^3 d_1 \|v^{(1)}\|_2 \|v\|_2.
\]
Therefore
\[
e^{4\pi \sigma} \langle v^{(1)}, \partial_x^2 B^{-1} [v_{xx}^2] \rangle_2 \leq (2\pi)^3 d_1 \|v\|_2 e^{4\pi \sigma} \|v^{(1)}\|_2^2 \\
\leq (2\pi)^3 d_1 \|v\|_4 e^{4\pi \sigma} \|v^{(1)}\|_2^2.
\]
(80)

Next assume \( j = 2 \). We have
\[
|\langle v^{(2)}, \partial_x^3 B^{-1} [v_{xx}^2] \rangle| \leq \|v^{(2)}\|_2 \| (\partial_x^3 B^{-1}) [v_{xx}^2] \|_2 \\
\leq (2\pi)^2 \|v\|_4 \| (\partial_x^3 B^{-1}) [v_{xx}^2] \|_2
\]
and
\[
\| (\partial_x^3 B^{-1}) [v_{xx}^2] \|_2 \leq \|v_{xx}^2\|_1 \leq c_1 \|v_{xx}\|_1^2 \leq c_1 \|v^{(2)}\|_2.
\]
Therefore
\[
e^{8\pi \sigma} \langle v^{(2)}, \partial_x^3 B^{-1} [v_{xx}^2] \rangle_2 \leq (2\pi)^2 c_1 \|v\|_4 e^{8\pi \sigma} \|v^{(2)}\|_2^2.
\]
(81)

By using (78), (79), (80) and (81), we obtain
\[
\left| \sum_{j=0}^{m} \frac{1}{j!} e^{4\pi \sigma} \langle v^{(j)}, \partial_x^{j+1} B^{-1} [v_{xx}^2] \rangle_2 \right| \\
\leq (16\pi^4 d_1 + 4\pi^2 c_1) \|v\|_4 \Phi_{\sigma,m}(v) + \frac{2\pi\gamma}{\sqrt{3}} \Phi_{\sigma,m}(v)^{1/2} \partial_\sigma \Phi_{\sigma,m}(v).
\]
(82)

**Appendix: Local study of the non-quasilinear modified \( \mu \)-CH equation.**

The difficulty of (8) lies in the presence of the non-quasilinear terms \(-\frac{1}{2}u^3\) and \(-\frac{1}{4} \mu(u^3)\). In [2], the authors employed the power series method to deal with the non-quasilinear term \( av^k u_x^3\) of the \( k\)-abc-equation. In the present paper, we overcome the difficulty of non-quasilinearity by a \( \mu \)-version of a classical trick used in the proof of the Cauchy-Kowalevsky theorem ([8]) following [12] and [11]. We set \( v = u_x \) and differentiate (8) in \( x \). It can be proved that (8) is equivalent to the following quasi-linear modified \( \mu \)-CH system:
\[
\begin{cases}
u_t + 2\mu(u)uv - \frac{\nu^3}{3} + \partial_x A^{-1} \left[ 2\mu^2(u)u + \mu(u)v^2 + \gamma u \right] + \frac{\mu(\nu^3)}{3} = 0, \\
u_t + 2\mu(u)(uv)_x - \frac{(\nu^3)_x}{3} + \partial_x A^{-1} \left[ 2\mu^2(u)v + \mu(u)(v^2)_x + \gamma v \right] = 0.
\end{cases}
\]
(83)

Of course, this trick works for the \( k\)-abc-equation as well. We can prove unique solvability of the Cauchy problem for (83) and (8). The Cauchy problem (8) for the non-quasilinear modified \( \mu \)-CH equation can be written in the following form:
\[
\begin{cases}
u_t + 2\mu(u)uv - \frac{\nu^3}{3} + \partial_x A^{-1} \left[ 2\mu^2(u)u + \mu(u)v^2 + \gamma u \right] + \frac{1}{4} \mu(\nu^3) = 0, \\
u(0,x) = u_0(x).
\end{cases}
\]
(84)
We introduce the system below.

\[
\begin{align*}
    u_t + 2\mu(u)u_x &- \frac{1}{3} v^3 + \partial_x A^{-1} \left[ 2\mu^2(u)u + \mu(u)v^2 + \gamma u \right] + \frac{1}{3} \mu(v^3) = 0, \\
v_t + 2\mu(u)(v)_x &- \frac{1}{3} (v^3)_x + \partial_x A^{-1} \left[ 2\mu^2(u)v + \mu(u)(v^2)_x + \gamma v \right] = 0, \\
u(0, x) &= u_0(x), v(0, x) = v_0(x).
\end{align*}
\]

**Theorem 4.5.** The Cauchy problems (84) and (85) are equivalent to each other if \( v_0(x) = \partial_x u_0(x) \). In particular, (85) implies \( v = \partial_x u = u_x \) if \( v_0(x) = \partial_x u_0(x) \).

**Proof.** By differentiation with respect to \( x \), we get the second equation in (85) from (84).

To show the converse, differentiate both sides of the first equation of (85) in \( x \).

By comparing it with the second equation, we get

\[(v - u_x)_t + \partial_x A^{-1} \left[ 2\mu^2(u)(v - u_x) + \gamma (v - u_x) \right] = 0.\]

It is enough to prove that \( w_t + \partial_x A^{-1}[a(t)w] = 0 \) and \( w(0, x) = 0 \) imply \( w = 0 \), where \( a(t) \) is a continuous function in \( t \). Set \( w = \sum_{k \in \mathbb{Z}} w_k(t)e^{2k\pi i x} \). Then we have

\[w'_0(t) = 0, \quad w'_k(t) - \frac{a(t)w_k(t)}{2k\pi i} = 0 (k \neq 0).\]

Since \( w_k(0) = 0 \), we have \( w_k(t) = 0 (k \in \mathbb{Z}) \) for any \( t \). It implies \( w = 0 \) and \( v = u_x \).

**Remark 6.** The trick of setting \( v = u_x \) has been used in [12] (non-\( \mu \) equation) and [11] in a different function space rather formally, i.e. without a discussion corresponding to Theorem 4.5. In [12], this trick is applied to a quasilinear equation.

**Theorem 4.6.** If \( u_0 \) and \( v_0 \) are real-analytic functions on \( S^1 \), then the Cauchy problem (85) has a holomorphic solution near \( t = 0 \). More precisely, we have the following:

(i) There exists \( \Delta > 0 \) such that \( u_0, v_0 \in G^{\Delta, s+1} \subset A(\Delta) \) for any \( s \).

(ii) If \( s > 1/2 \), there exists a positive time \( T_\Delta = T(u_0, v_0, s, \Delta) \) such that for every \( d \in (0, 1) \), the Cauchy problem (85) has a unique solution which is a holomorphic function valued in \( G^{\Delta d, s+1} \) in the disk \( D(0, T_\Delta(1 - d)) \). Furthermore, if \( \gamma = 0 \), the analytic lifespan \( T_\Delta \) satisfies

\[T_\Delta \approx \frac{\text{const.}}{\|(u_0, u'_0)\|_{2, s+1}^2},\]

when \( \Delta \) is fixed. On the other hand, if \( \gamma \neq 0 \), we have the asymptotic behavior

\[T_\Delta \approx \frac{\text{const.}}{\|(u_0, v_0)\|_{s, s+1}^2} \text{ (large initial values), } \quad T_\Delta \approx \text{const.} \text{ (small initial values).}\]

**Proof.** We give a detailed proof assuming \( \Delta = 1 \). The general case can be proved in the same way as for Theorem 2.3. The norm on \( \oplus^2 G^{\delta, s+1} \) is defined by

\[\|(u, v)\|_{\delta, s+1} = \|u\|_{\delta, s+1} + \|v\|_{\delta, s+1}.\]

We use the same notation \( \| \cdot \|_{\delta, s+1} \) for \( G^{\delta, s+1} \) and \( \oplus^2 G^{\delta, s+1} \). Assume

\[\|(u, v) - (u_0, v_0)\|_{\delta, s+1} < R, \quad \|(u', v') - (u_0, v_0)\|_{\delta, s+1} < R.\]

Set \( R_{s+1} = R + \max \left( \|u_0\|_{1, s+1}, \|v_0\|_{1, s+1} \right) \). Assume \( 0 < \delta' < \delta \leq 1 \). Then by (12), we have

\[\|w\|_{\delta, s+1} < R_{s+1}, \quad |\mu(w)| < R_{s+1},\]

(86)
where \( w \) is any of \( u, v, u', v' \). Let us consider differences concerning \( \mu(u)wv \) and \( \mu(u)(wv) \). Since \( uv - u'v' = u(v - v') + (u - u')v' \), we get
\[
\|uv - u'v'\|_{\delta,s+1} \leq \|u - u'\|_{\delta,s+1} \|v'\|_{\delta,s+1} \\
\leq c_{s+1}(\|u\|_{\delta,s+1} + \|u - u'\|_{\delta,s+1} + \|v'\|_{\delta,s+1}) \\
\leq c_{s+1}R_{s+1}(u, v) - (u', v')\|_{\delta,s+1}. \tag{87}
\]
Combining (87) with Proposition 4, we get
\[
\|uv\|_{\delta,s+1} - (u'v')_{\delta,s+1} \leq \frac{e^{-1}}{\delta - \delta'} \|uv\|_{\delta,s+1} - (u'v')_{\delta,s+1} \tag{88}
\]
On the other hand, we have
\[
\|u'v'\|_{\delta,s+1} \leq \frac{e^{-1}c_{s+1}R_{s+1}^2}{\delta - \delta'} \|uv\|_{\delta,s+1} \tag{89}
\]
Combining \( \mu(u)wv = \mu(u)(wv) + [\mu(u) - \mu(u')] v' \) with (86) and (87), we obtain
\[
\|\mu(u)wv - \mu(u')v'\|_{\delta,s+1} \leq |\mu(u)||uv - u'v'||_{\delta,s+1} + \|u - u'\|_{\delta,s+1} \|v'\|_{\delta,s+1} \\
\leq R_{s+1}(uv - u'v')_{\delta,s+1} + c_{s+1}R_{s+1}^2 \|u - u'\|_{\delta,s+1} \\
\leq 2c_{s+1}R_{s+1}^2(u, v) - (u', v')\|_{\delta,s+1} \tag{90}
\]
Next, combining \( \mu(u)(wv)_{\delta,s+1} - \mu(u')(wv)_{\delta,s+1} = \mu(u)[(wv)_{\delta,s+1} - (wv')_{\delta,s+1}] + [\mu(u) - \mu(u')] (wv')_{\delta,s+1} \)
with (88) and (90), we obtain
\[
\|\mu(u)(wv)_{\delta,s+1} - \mu(u')(wv')_{\delta,s+1}\|_{\delta,s+1} \leq 2e^{-1}c_{s+1}R_{s+1}^2 \|u - (u', v')\|_{\delta,s+1} \tag{91}
\]
The next step is to consider \( -v^3/3, -(v^3)_{\delta,s+1} \) and \( \mu(v^3) \). The factorization
\( v^3 - v'^3 = (v^2 + vv' + v'^2)(v - v') \) implies
\[
\|v^3 - v'^3\|_{\delta,s+1} \leq c_{s+1}\|v^2 + vv' + v'^2\|_{\delta,s+1} \|v - v'\|_{\delta,s+1}. \]
Here we have
\[
\|v^2 + vv' + v'^2\|_{\delta,s+1} \leq c_{s+1}(\|v\|^2_{\delta,s+1} + \|v\|_{\delta,s+1} \|v'\|_{\delta,s+1} + \|v'\|^2_{\delta,s+1}) \leq 3c_{s+1}R_{s+1}^2. \\
\]
Therefore
\[
\|v^3 - v'^3\|_{\delta,s+1} \leq 3c_{s+1}R_{s+1}^2 \|v - v'\|_{\delta,s+1}, \tag{92}
\]
and it immediately gives
\[
\|v^3 - v'^3\|_{\delta,s+1} \leq 3c_{s+1}R_{s+1}^2 \|v - v'\|_{\delta,s+1}. \tag{93}
\]
Combining (92) with Proposition 4, we get
\[
\| (v^3)_x - (v'^3)_x \|_{\delta', s+1} \leq \frac{e^{-1}}{\delta - \delta'} \| v^3 - v'^3 \|_{\delta, s+1} \\
\leq \frac{3e^{-1}c_{s+1}^2 R_{s+1}^2}{\delta - \delta'} \| v - v' \|_{\delta, s+1}.
\] (94)

An immediate consequence of (93) is
\[
\| \mu(v^3) - \mu(v'^3) \|_{\delta', s+1} = \| v^3 - v'^3 \|_{\delta', s+1} \leq \frac{3e^{-1}c_{s+1}^2 R_{s+1}^2}{\delta - \delta'} \| v - v' \|_{\delta, s+1}.
\] (95)

The last step is to consider the images of \( \mu^2(u)v, \mu(u)v^2 \) and \( \mu(u)(v^2)_x \) under \( \partial_x A^{-1} \). Since
\[
\begin{align*}
\mu^2(u)v - \mu^2(u')v' &= \mu^2(u)(v - v') + [\mu(u) + \mu(u')] [\mu(u) - \mu(u')] v' \\
\mu^2(u)v^2 - \mu^2(u')(v^2)' &= \mu^2(u)(v^2 - v'^2) + [\mu(u) + \mu(u')] [\mu(u) - \mu(u')] v'^2, \\
\mu(u)(v^2)_x - \mu(u')(v'^2)_x &= \mu(u)(v^2 - v'^2)_x + [\mu(u) - \mu(u')] (v'^2)_x,
\end{align*}
\]
we have, by \( c_s \leq c_{s+1} \), \( \| \cdot \|_{\delta, s} \leq \| \cdot \|_{\delta, s+1} \) and \( R_{s} \leq R_{s+1} \),
\[
\begin{align*}
\| \mu^2(u)v - \mu^2(u')v' \|_{\delta, s} &\leq \| \mu^2(u)(v - v') \|_{\delta, s} + \| [\mu(u) + \mu(u')] [\mu(u) - \mu(u')] v' \|_{\delta, s} \\
&\leq 3R_{s+1} \| u - u' \|_{\delta, s+1}, \\
\| \mu^2(u)v^2 - \mu^2(u')(v^2)' \|_{\delta, s} &\leq \| \mu^2(u)(v^2 - v'^2) \|_{\delta, s} + \| [\mu(u) + \mu(u')] [\mu(u) - \mu(u')] v'^2 \|_{\delta, s} \\
&\leq 3R_{s+1} \| (u, v) - (u', v') \|_{\delta, s+1}, \\
\| \mu(u)(v^2)_x - \mu(u')(v'^2)_x \|_{\delta, s} &\leq 2\pi \| \mu(u)(v^2 - v'^2) \|_{\delta, s+1} + 2\pi \| [\mu(u) - \mu(u')] (v'^2)_x \|_{\delta, s+1} \\
&\leq 6\pi c_{s+1} R_{s+1} \| (u, v) - (u', v') \|_{\delta, s+1}.
\end{align*}
\]
To derive the last one, we have used the second inequality of Proposition 4. We employ (10) to obtain
\[
\begin{align*}
\| \partial_x A^{-1} \left[ \mu^2(u)v - \mu^2(u')v' \right] \|_{\delta', s+1} &\leq \frac{3e^{-1}R_{s+1}^2}{\delta - \delta'} \| u - u' \|_{\delta, s+1}, \\
\| \partial_x A^{-1} \left[ \mu^2(u)v^2 - \mu^2(u')(v^2)' \right] \|_{\delta', s+1} &\leq \frac{3e^{-1}R_{s+1}^2}{\delta - \delta'} \| (u, v) - (u', v') \|_{\delta, s+1}, \\
\| \partial_x A^{-1} \left[ \mu(u)(v^2)_x - \mu(u')(v'^2)_x \right] \|_{\delta', s+1} &\leq \frac{6\pi e^{-1}c_{s+1}R_{s+1}^2}{\delta - \delta'} \| (u, v) - (u', v') \|_{\delta, s+1}.
\end{align*}
\]
Similarly we have

$$\| \partial_x A^{-1} [\gamma u - \gamma u'] \|_{\delta',s+1} \leq \frac{e^{-1} \gamma}{\delta - \delta'} \| u - u' \|_{\delta,s+1}, \tag{100}$$

$$\| \partial_x A^{-1} [\gamma v - \gamma v'] \|_{\delta',s+1} \leq \frac{e^{-1} \gamma}{\delta - \delta'} \| v - v' \|_{\delta,s+1}. \tag{101}$$

Set

$$F_{\mu,1}(u, v) = -2\mu(u)uv + \frac{1}{3}v^3 - \partial_x A^{-1} \left[ 2\mu^2(u)u + \mu(u)v^2 + \gamma u \right] - \frac{1}{3}\mu(v^3),$$

$$F_{\mu,2}(u, v) = -2\mu(u)(uv)_x + \frac{1}{3}(v^3)_x - \partial_x A^{-1} \left[ 2\mu^2(u)v + \mu(u)(v^2)_x + \gamma v \right].$$

Then by (90), (92), (96), (98), (95) and (100), we have

$$\| F_{\mu,1}(u, v) - F_{\mu,1}(u', v') \|_{\delta',s+1} \leq \frac{R^2}{\delta - \delta'} (4c_{s+1} + 2c_{s+1} + 6e^{-1} + 3e^{-1}c_{s+1}) + e^{-1}\gamma \| (u, v) - (u', v') \|_{\delta,s+1}. \tag{102}$$

By (91), (94), (97), (99) and (101), we have

$$\| F_{\mu,2}(u, v) - F_{\mu,2}(u', v') \|_{\delta',s+1} \leq \frac{e^{-1}R^2}{\delta - \delta'} (4c_{s+1} + c_{s+1}^2 + 6 + 6\pi c_{s+1}) + e^{-1}\gamma \| (u, v) - (u', v') \|_{\delta,s+1}. \tag{103}$$

We have obtained two inequalities of the Lipschitz type.

Next we estimate $F_j(u_0, v_0) (j = 1, 2)$. We have

$$\| \mu(u_0)u_0v_0 \|_{\delta,s+1} \leq c_{s+1} \| u_0 \|_{1,s+1}^2 \| v_0 \|_{1,s+1},$$

$$\| v_0^3 \|_{\delta,s+1} \leq c_{s+1}^2 \| v_0 \|_{1,s+1}^3,$$

$$\| \partial_x A^{-1} [\mu^2(u_0)] \|_{\delta,s+1} \leq \frac{e^{-1}}{1 - \delta} \| u_0 \|_{1,s+1}^3,$$

$$\| \partial_x A^{-1} [\mu(u_0)v_0^2] \|_{\delta,s+1} \leq \frac{e^{-1} c_{s+1}}{1 - \delta} \| u_0 \|_{1,s+1} \| v_0 \|_{1,s+1}^2,$$

$$\| \mu(v_0^3) \|_{\delta,s+1} \leq \| v_0 \|_{1,s+1}^3,$$

$$\| \partial_x A^{-1} [\gamma u_0] \|_{\delta,s+1} \leq \frac{e^{-1} \gamma}{1 - \delta} \| u_0 \|_{1,s+1}.$$

Since $X^2Y \leq (X + Y)^3/3, X^3 \leq (X + Y)^3$ for $X, Y \geq 0$, we get

$$\| F_{\mu,1}(u_0, v_0) \|_{\delta,s+1} \leq \frac{2c_{s+1} + c_{s+1}^2 + 6e^{-1} + e^{-1}c_{s+1} + 1}{3(1 - \delta)} \| (u_0, v_0) \|_{1,s+1}^3 + \frac{e^{-1} \gamma}{1 - \delta} \| (u_0, v_0) \|_{1,s+1}.$$

Similarly we have

$$\| \mu(u_0)(u_0v_0)_x \|_{\delta,s+1} \leq \frac{e^{-1} c_{s+1}}{1 - \delta} \| u_0 \|_{1,s+1}^2 \| v_0 \|_{1,s+1},$$

$$\| (v_0^3)_x \|_{\delta,s+1} \leq \frac{e^{-1} c_{s+1}^2}{1 - \delta} \| v_0 \|_{1,s+1}^3,$$
\[
\| \partial_x A^{-1} \left[ \mu^2(u_0)v_0 \right] \|_{\delta,s+1} \leq \frac{e^{-1}}{1-\delta} \| u_0 \|_{1,s+1}^2 \| v_0 \|_{1,s+1},
\]
\[
\| \partial_x A^{-1} \left[ \mu(u_0)(v_0^2) \right] \|_{\delta,s+1} \leq \frac{2\pi e^{-1} c_{s+1}}{1-\delta} \| u_0 \|_{1,s+1} \| v_0 \|_{1,s+1}^2,
\]
\[
\| \partial_x A^{-1} [\gamma] v_0 \|_{\delta,s+1} \leq \frac{e^{-1}}{1-\delta} \| v_0 \|_{1,s+1}.
\]

Therefore
\[
\| F_{\mu,2}(u_0, v_0) \|_{\delta,s+1} \leq \frac{e^{-1}(2c_{s+1} + c_{s+1}^2 + 2 + 2\pi c_{s+1})}{3(1-\delta)} \| (u_0, v_0) \|_{1,s+1}^3 + \frac{e^{-1}\gamma}{1-\delta} \| (u_0, v_0) \|_{1,s+1}.
\]

We set \( R = \| (u_0, v_0) \|_{1,s+1} \). If \( \gamma = 0 \), the constants corresponding to \( L \) and \( M \) in (17) and (18) are of degrees 2 and 3 respectively. Therefore \( T \) equals a constant multiple of \( \| (u_0, v_0) \|_{1,s+1}^2 \) if \( \gamma = 0 \). If \( \gamma \neq 0 \), we have to consider two cases separately: large or small initial values. When the initial values are large, larger order terms are dominant and \( T \) is approximated by a constant multiple of \( \| (u_0, v_0) \|_{1,s+1}^2 \), while \( T \) approaches a constant as the initial values approach 0. Theorem 4.5 allows us to get a result about the original Cauchy problem (84). We assume \( u_0 \in G^{D,s+2} \) so that \( v_0 = \partial_x u_0 \) belongs to \( G^{D,s+1} \).

**Theorem 4.7.** If \( u_0 \) is a real-analytic function on \( S^1 \), then the Cauchy problem (84) has a holomorphic solution near \( t = 0 \). More precisely, we have the following:

(i) There exists \( \Delta > 0 \) such that \( u_0 \in G^{D,s+2} \) for any \( s \).

(ii) If \( s > 1/2 \), there exists a positive time \( T_\Delta = T(u_0, s, \Delta) \) such that for every \( d \in (0,1) \), the Cauchy problem (84) has a unique solution which is a holomorphic function valued in \( G^{D,d,s+2} \) in the disk \( D(0, T_\Delta (1-d)) \). Furthermore, if \( \gamma = 0 \), the analytic lifespan \( T_\Delta \) satisfies

\[
T_\Delta = \frac{\text{const.}}{\| (u_0, u_0') \|_{D,s+1}^2},
\]

when \( \Delta \) is fixed. On the other hand, if \( \gamma \neq 0 \), we have the asymptotic behavior

\[
T_\Delta \approx \frac{\text{const.}}{\| (u_0, u_0') \|_{D,s+1}^2} \text{ (large initial values), \quad } T_\Delta \approx \text{const.} \text{ (small initial values).}
\]

**Remark 7.** In [11], the author solved (84) in a space of analytic functions following [12]. What is new in the present paper is precise estimates of the lifespan.

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E-mail address: yamane@kwansei.ac.jp