SYNTHESIZABLE SEQUENCE AND PRINCIPLE SUBMODULES IN SCHWARTZ MODULE

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Abstract. We consider a module of entire functions of exponential type and polynomial growth on the real axis, that is, the Schwarz module with a non-metrizable locally convex topology. In relation with the problem of spectral synthesis for the differentiation operator in the space $C^\infty(a; b)$, we study principle submodules in this module. In particular, we find out what functions, apart of products of the polynomials on the generating function, are contained in a principle submodule. The main results of the work is as follows: despite the topology in the Schwarz module is non-metrizable, the principle submodule coincides with a sequential closure of the set of products of its generating function by polynomials. As a corollary of the main result we prove a weight criterion of a weak localizability of the principle submodule. Another corollary concerns a notion of “synthesizable sequence” introduced recently by A. Baranov and Yu. Belov. It follows from a criterion of the synthesizable sequence obtained by these authors that a synthesizable sequence is necessary a zero set of a weakly localizable principle submodule. In the work we give a positive answer to a natural question on the validity of the inverse statement. Namely, we prove that the weak set of a weakly localizable principle submodule is a synthesizable sequence.

Keywords: entire functions, Fourier-Laplace transform, Schwarz space, local description of submodules, spectral synthesis.

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1. Introduction

Given a finite or an infinite interval $(a; b) \subseteq \mathbb{R}$, we denote by $C^\infty(a; b)$ the set of all infinitely differentiable functions equipped with a standard metrizable topology, while its strongly dual space consisting of all distributions compactly supported in $(a; b)$ is denoted by the symbol $\mathcal{E}'(a; b)$.

Let $W \subset \mathcal{E}(a; b)$ be a closed subspace invariant with respect to the differentiation operator $D = \frac{d}{dt}$, or shortly, a $D$-invariant subspace. In work [1], the study of the problem on spectral synthesis was initiated and in particular, it was established that the spectrum $\sigma_W$ of the restriction of the differentiation operator $D : W \rightarrow W$ either coincides with entire complex plane or is discrete, that is, is an infinite of finite, probably, empty sequence of multiple points in $\mathbb{C}$ [1, Thm. 2.1].

For a non-empty relatively closed segment $I \subset (a; b)$, the subspace $W_I$ is defined by the formula

$$W_I = \{ f \in \mathcal{E}(a; b) : f = 0 \text{ on } I \}. \quad (1.1)$$
Each $D$-invariant subspace $W$ possesses a "residual" subspace $W_{res} \subset W$ being the maximal subspace of form $[1.1]$ contained in $W$ [1, Thm. 4.1]. We denote a corresponding segment by $I_W$ and we call it residual segment of the subspace $W$, that is, $W_{res} = W_{I_W}$.

The existence of $D$-invariant subspaces of form $[1.1]$ led the authors of work [1] to the following formulation of the problem on spectral synthesis: to find out under which conditions the $D$-invariant subspace $W$ with a discrete spectrum satisfies the representation

$$W = W_{I_W} + \text{span}(\text{Exp} W).$$

Here $\text{Exp} W$ is the set of all exponential monomials contained in $W$.

It turned out that in the case of a finite (in particular, empty) spectrum $\sigma_W$, the subspace $W$ is always of form $[1.2]$, while if the spectrum $\sigma_W$ is discrete and infinite, then the answer depends on a relation between quantities $|I_W|$ and $2\pi D_{BM}(\Lambda)$, where $|I_W|$ is the length of the residual segment, $D_{BM}(\Lambda)$ is the Beurling-Malliavin density of the set $\Lambda = i\sigma_W$:

1) if $|I_W| < 2\pi D_{BM}(\Lambda)$, then $W = \mathcal{E}(a; b)$, see [2, Rem. 3], [3, Thm. 1.3];

2) if $|I_W| = 2\pi D_{BM}(\Lambda)$, then there exist both $D$-invariant subspaces admitting spectral synthesis in a weak* sense $[1.2]$ [4, 5] and subspaces not possessing this property [3], [6];

3) if $|I_W| > 2\pi D_{BM}(\Lambda)$, then $D$-invariant subspace with a discrete spectrum $\sigma_W = i\Lambda$ and a residual segment $I_W$ admits a weak spectral synthesis $[1.2]$ [2, Cor. 3], [3, Thm. 1.1].

The latter of the three above formulated results can be interpreted as follows: given a complex sequence $\Lambda$ and a relatively closed in $(a; b)$ segment $I$ such that $|I| > 2\pi D_{BM}(\Lambda)$, there exists a unique $D$-invariant subspace $W \subset \mathcal{E}(a; b)$ with a discrete spectrum $\sigma_W = -i\Lambda$ and a residual segment $I_W = I$ and this subspace is of form $[1.2]$.

In view of this interpretation, the authors of work [2] called a sequence $\Lambda \subset \mathbb{C}$ with $D_{BM}(\Lambda) < +\infty$ synthesizable if a $D$-invariant subspace with a spectrum $-i\Lambda$ and a residual segment $[-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)]$ is unique; in this case it is of form $[1.2]$. In that work a complete description of synthesizable sequences was provided. In particular, it was shown that if the system of exponential monomials $\exp_{\Lambda}$ constructed by the sequence $-i\Lambda$ is complete or it has a finite defect in the space $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$, then $\Lambda$ is a synthesizable sequence [7, Prop. 3.2].

If the system $\exp_{\Lambda}$ has an infinite defect in $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$, then the synthesizability of $\Lambda$ is determined by the conditions of the following criterion [7, Thm. 1.3]:

**Theorem A.** A sequence $\Lambda \subset \mathbb{C}$ is synthesizable if and only if it is a zero set of some function $\varphi \in \mathcal{P}_0(\mathbb{R})$ and

$$\dim (\mathcal{H}(\varphi) \otimes H_{pol}) \leq 1.$$
$H_{pol}$ is the closure of the set of polynomials in $\mathcal{H}(\varphi)$.

It is clear that the synthesizability of a sequence $\Lambda$ is a sufficient condition for admitting a weak spectral synthesis by a $D$-invariant subspace with a discrete spectrum $-(i\Lambda)$ and a residual segment of length $2\pi D_{BM}(\Lambda)$. This gives rise to a question: when the synthesizability of a sequence $\Lambda$ is also a necessary condition for admittance of a weak spectral synthesis by a $D$-invariant subspace with the spectrum $-(i\Lambda)$ and a residual segment equalling to $[-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)]$ (or to any other fixed segment of the length $2\pi D_{BM}(\Lambda)$)?

One of the aims of the present work is to answer this question.

Earlier for studying $D$-invariant subspaces we employed effectively the scheme of dual spaces reducing the problem on subspaces to equivalent problems on closed submodules in a special module of entire functions $\mathcal{P}(a; b)$, see [2, 11, 12]. We are going to employ this scheme in the present work and this is why we describe briefly the duality between $D$-invariant subspaces and submodules.

For each element $S \in \mathcal{E}'(a; b)$ we introduce its Fourier-Laplace transform

$$\mathcal{F}(S)(z) = S(e^{-i\pi z}), \quad z \in \mathbb{C},$$

which is an entire function of a completely regular growth at order 1. We denote it by $\varphi$. The indicator diagram of the function $\varphi$ is the segment of the imaginary axis

$$i[c_{\varphi}; d_{\varphi}] \subset i(a; b),$$

where $c_{\varphi} = -h_{\varphi}(-\pi/2)$, $d_{\varphi} = h_{\varphi}(\pi/2)$, and $h_{\varphi}$ is the indicator of the function $\varphi$.

We let $\mathcal{P}(a; b) = \mathcal{F}(\mathcal{E}'(a; b))$. It is well known that $\mathcal{P}(a; b) = \bigcup P_k$, where $\{P_k\}$ is an increasing sequence of Banach spaces, each being the set of all entire functions $\varphi$ with a finite norm

$$||\varphi||_k = \sup_{z \in \mathbb{C}} \frac{||\varphi(z)||}{(1 + |z|)^k \exp(b_k y + a_k z)}, \quad y^\pm = \max\{0, \pm y\}, \quad z = x + iy,$$  \hspace{1cm} (1.4)

$[a_1; b_1] \subset [a_2; b_2] \subset \ldots$ is a sequence of segments exhausting the interval $(a; b)$. Equipping the set $\mathcal{P}(a; b)$ by a locally convex topology of the inductive limit of the sequence $\{P_k\}$, we obtain a space of type $(LN^*)$, see [3], isomorphic to $\mathcal{E}'(a; b)$ [10] Thm. 7.3.1. We note that according to the same theorem, $P_0(\mathbb{R}) = \mathcal{F}(C^\infty_0(\mathbb{R}))$.

In the space $\mathcal{P}(a; b)$, the operation of multiplication by an independent variable $z$ is continuous and this is why $\mathcal{P}(a; b)$ is a topological module over the ring of polynomials $\mathbb{C}[z]$ called Schwartz module.

A closed submodule $J \subset \mathcal{P}(a; b)$ is a closed subspace satisfying also the condition $zJ \subset J$. In what follows, for the sake of brevity, we shall say ”submodule” meaning a closed submodule.

We recall a series of notions characterising submodules, see [11], [12]. An indicator segment of a submodule $J$ is the segment $[c_J; d_J] \subset \mathbb{R}$, where $c_J = \inf_{\varphi \in J} c_{\varphi}$, $d_J = \sup_{\varphi \in J} d_{\varphi}$.

A divisor of a submodule $J \subset \mathcal{P}(a; b)$ is a function $n_J(\lambda) = \min_{\varphi \in J} n_{\varphi}(\lambda)$, $\lambda \in \mathbb{C}$, where $n_{\varphi}(\lambda)$ is a divisor of the function $\varphi \in J$:

$$n_{\varphi}(\lambda) = \begin{cases} 0 & \text{if } \varphi(\lambda) \neq 0, \\ m & \text{if } \lambda \text{ is a zero of } \varphi \text{ of multiplicity } m, \end{cases}$$

and

$$\Lambda_{\varphi} = \{\lambda \in \mathbb{C} : n_{\varphi}(\lambda) > 0\}, \quad \Lambda_J = \{\lambda \in \mathbb{C} : n_J(\lambda) > 0\}$$

are zero sets of the function $\varphi$ and submodule $J$, respectively, and each point $\lambda$ is repeated according its multiplicity.

The submodules of the module $\mathcal{P}(a; b)$ are dual to $D$-invariant subspaces of the space $\mathcal{E}(a; b)$. Namely, there exists an one-to-one correspondence between the set of closed submodules $\{J\}$ of the module $\mathcal{P}(a; b)$ and the of $D$-invariant subspaces $\{W\}$ of the space $\mathcal{E}(a; b)$. This one-to-one
correspondence is defined by the following rule: $J \leftrightarrow W$ if and only if $J = F(W^0)$, where a closed subspace $W^0 \subset \mathcal{E}'(a;b)$ consists of all distributions $S \in \mathcal{E}(a;b)$ annihilating $W$; here
\[
\text{Exp} W = \{v^j e^{-\lambda_j t}, \quad j = 0, \ldots m_k - 1, \quad n_j(\lambda_k) = m_k > 0\},
\]
and the points $c_j$ and $d_j$ serve as boundaries for the residual segment $I_W$, see [2], [12]. The above formulated fact is called the duality principle.

A submodule $J \subset \mathcal{P}(a;b)$ is weakly localizable if for each function $\varphi \in \mathcal{P}(a;b)$ the conditions
\begin{enumerate}
\item $n_\varphi(z) \geq n_J(z)$ for all $z \in \mathbb{C}$,
\item the indicator diagram of the function $\varphi$ is contained in the set $i[c_J; d_J]$;
\end{enumerate}
imply that $\varphi \in J$.

A submodule $J$ is called stable if for each $\lambda \in \mathbb{C}$ an implication holds:
\[
\varphi \in J, \quad n_\varphi(\lambda) > n_J(\lambda) \implies \frac{\varphi}{z-\lambda} \in J.
\]

A $D$-invariant subspace $W$ admits a weak spectral synthesis if and only if its annihilating submodule $J = F(W^0)$ is weakly localizable, see [2], [4].

A $D$-invariant subspace $W$ has a discrete spectrum if and only if its annihilating submodule $J = F(W^0)$ is stable [11, Prop. 3.1], [12, Prop. 2].

A principle submodule $J_\varphi$ generated by a function $\varphi \in \mathcal{P}(a;b)$ is defined as a closure of the set
\[
\text{Pol}_\varphi = \{p\varphi : \quad p \in \mathbb{C}[z]\}
\]
in $\mathcal{P}(a;b)$. A principle submodule is always stable [12].

Let, as above, $\Lambda$ be a complex sequence with a finite Beurling-Malliavin density; $W \subset \mathcal{E}(\mathbb{R})$ be a $D$-invariant subspace with the spectrum $\sigma_W = -i\Lambda$ and the residual segment $I_W = [-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)]$. We observe that if the residual segment $I_W$ is given, the corresponding subspace $W$ can be considered in each space $\mathcal{E}(a;b)$ such that $I_W \subset (a;b)$.

We assume first that the systme Exp$_\Lambda$ is either complete or have a finite defect in the space $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$. It is easy to make sure that this is equivalent to the existence of the function $\varphi \in \mathcal{P}(\mathbb{R}) \setminus \mathcal{P}_0(\mathbb{R})$, with the zero set $\Lambda_\varphi = \Lambda$ and the indicator diagram $[-i\pi D_{BM}(\Lambda); i\pi D_{BM}(\Lambda)]$. By Theorem 2 in work [5], this implies that the annihilator submodule of the subspace $W$ is the principle submodule $J_\varphi$ with generator $\varphi$. Moreover, in this case,
\[
J(\varphi) = J_\varphi = \{p\varphi : \quad p \in \mathbb{C}[z]\},
\]
where the symbol $J(\varphi)$ denotes a weakly localizable submodule with the zero set $\Lambda$ and the indicator segment $[-i\pi D_{BM}(\Lambda); i\pi D_{BM}(\Lambda)]$.

By the duality between $D$-invariant subspaces and submodules and by the said above Theorem A we conclude that
if the system Exp$_\Lambda$ is complete or has a finite defect in the space
\[
L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)),
\]
then $W$ admits a weak spectral synthesis and $\Lambda$ is a synthesizable sequence.

Now we consider the case when the exponential system Exp$_\Lambda$ has an infinite defect in $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$. In this case it follows from Theorem A and the duality that for the synthesizability of $\Lambda$, it is necessary that the space $W$ is of the form
\[
W = W_S = \{f \in \mathcal{E}(\mathbb{R}) : \quad S(f^{(k)}) = 0 \quad \text{for all} \quad k = 0, 1, \ldots \},
\]
where $S \in \mathcal{E}'(\mathbb{R})$, and $\varphi = F(S) \in \mathcal{P}_0(\mathbb{R})$, $\Lambda_\varphi = \Lambda$, while the indicator diagram $\varphi$ is $[-i\pi D_{BM}(\Lambda); i\pi D_{BM}(\Lambda)]$. Then $F(W_S^0) = J_\varphi$ and the admittance of the weak spectral synthesis for $W_S$ is equivalent to the weak localizability of $J_\varphi : J_\varphi = J(\varphi)$. In other words, it
is equivalent to the fact that \( J(\varphi) \) is the closure of the set \( \text{Pol}_\varphi \) in the topology of the space \( \mathcal{P}(\mathbb{R}) \).

On the other hand, in the considered case, if \( \Lambda \) is a synthesizable sequence, then the submodule \( J(\varphi) \) coincides with a sequential closure of the set \( \text{Pol}_\varphi \), that is, with the set of all limits of countable sequences in \( \text{Pol}_\varphi \) converging in the topology of the space \( \mathcal{P}(\mathbb{R}) \); this set is indicated by the symbol \( J_{\varphi, \text{seq}} \). This is implied by Theorem A in view of the remark after Lemma 1, see the next section.

Thus, for the equivalence of the synthesizability of the sequence \( \Lambda \) and the admittance of a weak spectral synthesis by the corresponding \( D \)-invariant subspace \( W \) with the spectrum \( \sigma_W = -i\Lambda \) and the residual segment \( I_W = [-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)] \), it is necessary that \( W = W_S \) and \( J_\varphi = J_{\varphi, \text{seq}} \), where \( \varphi = F(S) \), and \( \Lambda_\varphi = \Lambda \), and the indicator diagram of \( \varphi \) is the segment \( [-i\pi D_{BM}(\Lambda); i\pi D_{BM}(\Lambda)] \).

The space \( \mathcal{P}(a; b) \) is non-metrizable \([9, \text{Cor. 2 of Thm. 1}] \). This is why, generally speaking, the closure of an arbitrary set \( A \subset \mathcal{P}(a; b) \) can not be obtained just by adding the limits of converging countable sequences \( \{\varphi_n\} \subset A \). Therefore, to answer the question on equivalence of synthesizability of the sequence \( \Lambda \) and the weak spectral synthesis for the corresponding subspace of form (1.6) with the spectrum \( \sigma_W = -i\Lambda \), we first need to study whether the identity

\[ J_{\varphi, \text{seq}} = J_\varphi \]  

is possible.

**Theorem 1.** Identity (1.7) holds for all \( \varphi \in \mathcal{P}(a; b) \).

By means of this theorem we prove the equivalence of the synthesizability of the sequence \( \Lambda \) and the admittance of the weak spectral synthesis by a space of form (1.6) with the spectrum \( -i\Lambda \), see Corollary 2. Another important application of Theorem 1 is a convenient weight criterion of the weak localizability of the principle submodule in the module \( \mathcal{P}(a; b) \), see Theorem 2.

The main results of the present work were announced in [13].

2. **SEQUENTIAL DESCRIPTION OF PRINCIPLE SUBMODULES IN THE SCHWARTZ MODULE**

2.1. **Preliminaries.** Let \( [c; d] \subset (a; b) \), \( PW(c; d) = \mathcal{F}(L^2(c; d)) \) be the Paley-Wiener space, \( P_0[c; d] \) be the space of all entire functions \( \psi \) with a finite norm

\[ \|\psi\|_0 = \sup_{z \in \mathbb{C}} \frac{|\psi(z)|}{\exp(dy^+ - cy^-)}, \quad y^+ = \max\{0, \pm y\}, \quad z = x + iy. \]  

(2.1)

**Lemma 1.** If \( \psi \in PW(c; d) \), then \( \psi \in P_0[c; d] \), and

\[ \|\psi\|_0 \leq C_0 \|\psi\|_{PW(c; d)}, \]  

(2.2)

where a positive constant \( C_0 \) is independent of \( c \) and \( d \).

**Proof.** Without loss of generality we can assume that \( c = -d \); then

\[ \psi(z) = \int_{-d}^{d} e^{-ist} f(t) dt, \quad f \in L^2(-d; d), \]

\[ \|\psi\|_0 = \sup_{z \in \mathbb{C}} \frac{|\psi(z)|}{\exp(d|y|)}, \quad z = x + iy. \]

According Plancherel theorem, for a fixed \( y \in \mathbb{R} \) we have

\[ \|\psi(x + iy)\|_{L^2(\mathbb{R})}^2 = 2\pi \|e^{yt} f(t)\|_{L^2(-d; d)}^2. \]
Employing this fact and a subharmonicity of the function $|\psi|^2$, for all $x \in \mathbb{R}$ we obtain the estimates
\[
|\psi(x)|^2 \leq \frac{1}{\pi} \int_{|w-x| \leq 1} |\psi(w)|^2 |dw| \leq \frac{1}{\pi} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} |\psi(s+i\tau)|^2 ds \right) \, d\tau \leq C_1 e^{2d} \|\psi\|^2_{PW(-d,d)},
\]
where $C_1$ is an absolute constant. Inequality (2.2) follows from these estimates and Phragmén-Lindelöf principle.

**Remark 1.** Theorem A and the proven Lemma yield easily that if the zero set $\Lambda_\varphi$ of the function $\varphi \in \mathcal{P}(a;b) \cap \mathcal{P}_0(\mathbb{R})$ is a synthesizable sequence, then
\[
\mathcal{J}(\varphi) = \mathcal{J}_\varphi = \mathcal{J}_{\varphi,\text{seq}}.
\]
Indeed, if $\Phi \in \mathcal{J}(\varphi)$, then $\Phi = \omega \varphi$, where $\omega$ is an entire function of the minimal type and there exists a polynomial $q_\Phi$ such that $\frac{\omega}{q_\Phi} \in \mathcal{H}(\varphi)$. Then, by Theorem A, either $\frac{\omega}{q_\Phi} \in H_{\text{pol}}$ or for an arbitrary fixed point $\lambda_0 \in \Lambda_\omega \setminus \Lambda_\varphi$ there exist numbers $\alpha_1, \alpha_2 \in \mathbb{C}$ such that
\[
\left( \alpha_2 - \frac{\alpha_1}{z - \lambda_0} \right) \cdot \frac{\omega}{q_\Phi} \in H_{\text{pol}}.
\]
In both cases, in view of the intrinsic description of the space $\mathcal{P}(a;b)$ and a sequential convergence in it, [[1]] Cor. 1 from Thm. 2], by the proven lemma we conclude that $\Phi \in \mathcal{J}_{\varphi,\text{seq}}$.

Let $\varphi \in \mathcal{P}_0(\mathbb{R})$, $c_\varphi = h_\varphi(-\pi/2)$, $d_\varphi = h_\varphi(\pi/2)$, where $h_\varphi$ is the indicator of the function $\varphi$. $PW = PW(c_\varphi; d_\varphi)$. We consider the following closed subspaces in $PW$: the subspace $PW(\varphi) = J(\varphi) \cap PW$ and the subspace $PW_{\text{pol}}$ defined as the closure of the set $\text{Pol}_\varphi$ in $PW$.

A one-to-one correspondence
\[
\omega \mapsto \omega \varphi, \quad \omega \in \mathcal{H}(\varphi),
\]
makes an isometry of Hilbert spaces $\mathcal{H}(\varphi)$ and $PW(\varphi)$. The subspace $H_{\text{pol}}$ defined as the closure of the set of polynomials in $\mathcal{H}(\varphi)$ is the pre-image of the subspace $PW_{\text{pol}}$ under this isometry.

We shall need some definitions and facts from the general theory of de Branges spaces [[14]], and also from work [[7]], in which this theory was successfully employed for studying $D$-invariant subspaces in the Schwartz space (in particular, for the proof of Theorem A).

Originally, de Branges space is defined as associated with an entire function $E$ from the Hermite-Biehler class and is the set of all entire functions $F$, such that
\[
\int_{-\infty}^{+\infty} \left| \frac{F(t)}{E(t)} \right|^2 \, dt < +\infty,
\]
and obeying some further restrictions, see [[14] Sects. 19–21, [7 Sect. 2]].

In this work, we restrict ourselves by an exact formulation of an equivalent definition of de Branges space; this definition is an axiomatic description, see [[14] Thm. 23]): a non-trivial Hilbert space of entire functions $\mathcal{H}$ is a de Branges space if and only if the following axioms are satisfied:

(H1) if $F \in \mathcal{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is a zero of the function $F$, then $F_1 = F(z)\frac{z-\lambda}{\bar{z}-\lambda} \in \mathcal{H}$ and the norm of the functions $F$ and $F_1$ are equal;

(H2) for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$, a corresponding linear $\delta_\lambda$-functional acting by the rule $\delta_\lambda(F) = F(\lambda)$, $F \in \mathcal{H}$, is continuous in $\mathcal{H}$;

(H3) for each function $F \in \mathcal{H}$, the function $F^*(z) = \overline{F(\bar{z})}$ belongs to $\mathcal{H}$ and has the same norm as $F$.

By means of this axiomatic description, it was established in [[13], [1 Sect. 2, Thm. 2.7] that $\mathcal{H}(\varphi)$ is a de Branges space. It is also easy to check that axioms (H1)–(H3) holds also for
the subspace $H_{pol}$ regarded as a Hilbert space with scalar product (1.3), that is, $H_{pol}$ is a de Branges space.

We also formulate two results on de Branges space, see [14 Sect. 35], [7 Thm. 2.1] and [14 Sect. 29], respectively.

**Theorem B.** Let $H_1$ and $H_2$ be closed subspaces of the same de Branges space $\mathcal{H}$ being also de Branges spaces with the scalar product induced from $\mathcal{H}$. Then one of the following inclusions holds: $H_1 \subset H_2$ or $H_2 \subset H_1$.

**Theorem C.** Let $\mathcal{H}$ be a de Branges space, $H_k$ be the closure of a linear set $\{ f \in \mathcal{H} : z^j f \in \mathcal{H}, j = 1, \ldots, k \}$, $k \in \mathbb{N}$, in $\mathcal{H}$. Then $\dim (\mathcal{H} \ominus H_k) < +\infty$.

Employing Theorems B and C, it is easy to prove the following lemma.

**Lemma 2.** Assume that in the space $H_{pol}$ there exists a function $\omega_0$ with the following property:

\[ z^{k_0} \omega_0 \in \mathcal{H}(\varphi), \quad z^{k_0} \omega_0 \notin \mathcal{H}(\varphi) \]

for some $k_0 \in \mathbb{N}$. Then

\[ \dim (\mathcal{H}(\varphi) \ominus H_{pol}) \leq 1. \]

**Proof.** For each $k \in \mathbb{N}$, by the symbol $\mathcal{H}_k$ we denote the closure of the set

\[ \{ \omega \in \mathcal{H}(\varphi) : z^k \omega \in \mathcal{H}(\varphi) \} \]

in $\mathcal{H}(\varphi)$.

Since $\mathcal{H}(\varphi)$ is the pre-image of the set $PW \cap J(\varphi)$ under the isometry (2.3), and $J(\varphi)$ is a stable submodule, then $\mathcal{H}_k$ coincides with the subspace $H_k$ from Theorem C. It is also clear that $\mathcal{H}_0 = \mathcal{H}(\varphi)$, $\mathcal{H}_k \subset \mathcal{H}_{k-1}$, $k = 1, 2, \ldots$.

Each $\mathcal{H}_k$ with the scalar product induced by that in $\mathcal{H}(\varphi)$ is a de Branges space, as well as the subspace $H_{pol}$. This is why by Theorem B, either $\mathcal{H}_{k_0} \subset H_{pol}$ or $H_{pol} \subset \mathcal{H}_{k_0}$. But the presence of the function $\omega_0$ in $H_{pol}$ excludes the possibility $H_{pol} \subset \mathcal{H}_{k_0}$; therefore,

\[ \mathcal{H}_{k_0} \subset H_{pol}. \]

In view of Theorem C we have

\[ \dim (\mathcal{H}(\varphi) \ominus H_{pol}) \leq \dim (\mathcal{H}(\varphi) \ominus \mathcal{H}_{k_0}) < +\infty. \]

On the other hand, it is known that the codimension of $H_{pol}$ in $\mathcal{H}(\varphi)$ can take only three possible values: 0, 1, $+\infty$ [7 Thms. 2.1, 2.2, 2.9]. This implies the desired statement. $\square$

### 2.2. Proof of Theorem 1

As it has been already mentioned in the Introduction, by Theorem 2 in [8], the relation $\varphi \in \mathcal{P}(a; b) \setminus \mathcal{P}_0(\mathbb{R})$ is equivalent to the validity of (1.5) and hence, in this case the statement of the theorem is trivial.

Let $\varphi \in \mathcal{P}(a; b) \cap \mathcal{P}_0(\mathbb{R})$. Then, as it has been said in the end of the proof of Lemma 2, the quantity $\dim (\mathcal{H}(\varphi) \ominus H_{pol})$ can take only one of three possible values: 0, 1, $+\infty$.

If $\dim (\mathcal{H}(\varphi) \ominus H_{pol}) = 0$, then

\[ J_{\varphi, seq} = J_{\varphi} = J(\varphi). \tag{2.4} \]

In the case $\dim (\mathcal{H}(\varphi) \ominus H_{pol}) = 1$, identities (2.4) can be proved on the base of Lemma 1 by arguing in the same way as in the remark after this lemma.

We consider the last option:

\[ \dim (\mathcal{H}(\varphi) \ominus H_{pol}) = +\infty. \tag{2.5} \]

We denote by $H_{\varphi}$ the pre-image of the closed subspace $PW_{\varphi} = PW \cap J_{\varphi}$ of the space $PW$ under isometry (2.3) and we let $H_1 = H_{\varphi} \ominus H_{pol}$.
To complete the proof of the theorem, it is sufficient to make sure that

\[ H_1 = \{0\}. \]

First of all we observe that the subspace \( H_1 \) can not contain a non-zero function \( \omega \) satisfying \( \Phi = \omega \varphi \in \mathcal{P}_0(\mathbb{R}) \). Indeed, otherwise

\[ \Phi = \mathcal{F}(s), \quad s \in C_0^\infty(\mathbb{R}) \cap \mathcal{E}'(a;b). \]

And if \( S_\varphi \) is a regular functional belonging to in \( C_0^\infty(a;b) \) obeying the identity \( \varphi = \mathcal{F}(S_\varphi) \), then

\[ \int_a^b S_\varphi^{(k)}(t)s(t)dt = 0, \quad k = 0, 1, 2 \ldots \]

Therefore, \( \tilde{s} \in W_{S_\varphi} \).

On the other hand, \( \Phi \in J_{\varphi} \), since \( \omega \in H_1 \subset H_\varphi \). This is why \( s \in W_{S_\varphi}^0 \) and

\[ 0 = s(\tilde{s}) = \int_a^b s(t)\tilde{s}(t)dt, \]

that is, \( s = 0 \). Thus, if \( \omega \in H_1 \setminus \{0\} \), then there exists a number \( n_\omega \in \mathbb{N} \) such that

\[ z^j\omega \in \mathcal{H}(\varphi), \quad j = 0, \ldots, n_\omega - 1, \quad z^{n_\omega}\omega \notin \mathcal{H}(\varphi). \quad (2.6) \]

Suppose we shall succeed to establish the following fact.

(\( \text{F} \)): in the subspace \( H_{pol} \), there exists a function with property (2.6).

Applying then Lemma \( \text{[2]} \) we conclude that \( \dim (\mathcal{H}(\varphi) \oplus H_{pol}) < +\infty \), and this contradicts relation (2.5). Thus, we have established that in case (2.5) we have

\[ H_1 = \{0\}, \quad J_{\varphi,seq} = J_\varphi \neq J(\varphi), \]

that is, the principle submodule \( J_{\varphi} \) is sequentially generated but is not weakly localizable.

It remains to justify statement (\( \text{F} \)).

Let \( \{\mu_j\} \) be a “sparse” sequence of zeroes a fixed non-zero function \( \omega \in H_1 \), say, such that \( \mu_1 > 1, \mu_j > 8\mu_{j-1}, j = 2, 3, \ldots \) We let

\[ q_m(z) = \prod_{j=1}^m \left(1 - \frac{z}{\mu_j}\right), \quad \tilde{\omega}_m = \frac{\omega}{q_m}. \]

It is clear that \( \tilde{\omega}_m \) satisfies condition (2.6) and by the stability of the submodule \( J_\varphi \) we have \( \tilde{\omega}_m \in H_\varphi \).

Let \( \text{Pr}_{pol} : H_\varphi \to H_{pol} \) and \( \text{Pr}_1 : H_\varphi \to H_1 \) be the projectors on the corresponding subspaces. If \( \text{Pr}_1(\tilde{\omega}_m) = 0 \) for some index \( m \), then statement (\( \text{F} \)) holds. Otherwise \( \text{Pr}_1(\tilde{\omega}_m) \neq 0 \) for all \( m = 1, 2, \ldots \) Employing standard ways for estimating entire functions and for description of bounded sets in locally-convex spaces of type \( (LN^*) \) [Thm. 2], the space \( \mathcal{P}(a;b) \) being one of those, it is easy to confirm that the sequence \( \{\tilde{\omega}_m\varphi\} \) is bounded in the sense of some norm \( \|\cdot\|_{k_0} \), see (1.4). Hence, there exists a subsequence converging in \( \mathcal{P}(a;b) \), more precisely,

\[ \|\tilde{\omega}_{m_j}\varphi - \tilde{\omega}_0\varphi\|_{k_0+1} \to 0, \]

where

\[ \tilde{\omega}_0(z) = \frac{\omega(z)}{\prod_{i=1}^\infty \left(1 - \frac{z}{\mu_i}\right)}. \]

Let \( q \) be some polynomial of degree \( (k_0 + 2) \) with roots in the set \( \Lambda_\omega \setminus \{\mu_i\} \). As in the case of \( \tilde{\omega}_m \), if \( \text{Pr}_1(\tilde{\omega}_mq^{-1}) = 0 \) for some index \( m \), then \( \omega mq^{-1} \in H_{pol} \) satisfies (2.6) and statement (\( \text{F} \)) holds. Otherwise we employ the convergence of the sequence \( \{\tilde{\omega}_mq^{-1}\} \) converges to the function \( \omega_0 = \tilde{\omega}_0q^{-1} \) in the space \( \mathcal{H}(\varphi) \) and \( \omega_0\varphi \in \mathcal{P}_0(\mathbb{R}) \). By the above remark that each
function in \(H_1\) satisfies (2.6), we have \(Pr_{pol}(\omega_0) \neq 0\). If \(Pr_1(\omega_0) \neq 0\), then the function \(Pr_{pol}(\omega_0)\) is the sought one and (F) holds.

It remains to treat the case \(\omega_0 \in H_{pol}\). We observe that multiplying the function \(\omega_0\) by arbitrary rational function \(Q\) such that \(Q\omega_0\) is entire produces a function belonging to \(H_\varphi\) and not satisfying condition (2.6). This is why, if for some rational function \(Q_0\) the inequality \(Pr_1(Q_0\omega_0) \neq 0\) holds, then the function \(Pr_{pol}(Q_0\omega_0)\) satisfies (F).

Finally, let \(Q\omega_0 \in H_{pol}\) for each rational function \(Q\) such that \(Q\omega_0\) is entire. For the principle submodule generated by the function

\[
\Phi = \omega q^{-1}\varphi
\]

relations (1.5) hold since the function \(\omega q^{-1}\) satisfies (2.6). In view of the restrictions determining the choice of the points \(\{\mu_j\}\), now we are under the same conditions as before Theorem 2 in work [8, Sect. 2]. Employing then Lemmata 1–3 of this work, we find a sequence of polynomials \(\{p_j\}\) such that

\[
\lim_{j \to \infty} p_j\omega_0 \varphi = \Phi
\]

in the space \(\mathcal{P}(a; b)\). In view of the description of the sequential convergence in \(\mathcal{P}(a; b)\), see [8, Cor. 1 from Thm. 2], we conclude that there exists a polynomial \(p\) possessing the following property: the sequence \(\{p_j\omega_0 p^{-1}\}\) converges to an entire function \(\nu = \omega q^{-1}p^{-1}\) in the norm of the space \(\mathcal{H}(\varphi)\) and the function \(\nu\) satisfies (2.6). Since \(Pr_1(p_j\omega_0 p^{-1}) = 0\) for all values of the index \(j\), then \(\nu \in H_{pol}\) and this completes the proof.

3. Application of main result

Let \(\Lambda \subset C\), \(2\pi D_{BM}(\Lambda) < b - a\). By Theorem 1 and Theorem A we obtain the following statement.

**Corollary 1.** A stable submodule \(J \subset \mathcal{P}(a; b)\) with a zero set \(\Lambda\) and an indicator segment \(c; d\) \(\subset (a; b)\) of length \(2\pi D_{BM}(\Lambda)\) is unique if and only if it is principle and weakly localizable.

**Proof.** Without loss of generality we assume that

\[
b = -a, \quad d = -c = -\pi D_{BM}(\Lambda).
\]

According to the said in the Introduction for the case, the statement holds when the system \(\text{Exp}_\Lambda\) is complete or has a finite defect in the space \(L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))\). Indeed, this condition for the system \(\text{Exp}_\Lambda\) is equivalent to the fact that the submodule \(J\) is principle and is of the form (1.5).

If the system \(\text{Exp}_\Lambda\) has an infinite defect in \(L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))\), then the part of the statement concerning necessity is implied by Theorem A and the fact that a principle submodule is always stable.

To justify sufficiency we note that if \(J\) is a weakly localizable principle submodule, then

\[
\dim (\mathcal{H}(\varphi) \ominus H_{pol}) \leq 1,
\]

see the proof of Theorem 1 and it remains to apply Theorem A.

The duality principle allows us to provide an equivalent formulation of Corollary 1 in terms of \(D\)-invariant subspaces.

**Corollary 2.** A \(D\)-invariant subspace \(W\) with a given discrete spectrum \((-i\Lambda)\) and a residual segment \(c; d\) \(\subset (a; b)\) of length \(2\pi D_{BM}(\Lambda)\) is unique if and only if it is of form (1.6) and admits a weak spectral synthesis (1.2).
It follows from Theorem 1 that a weak localizability of the principle submodule in the module \( \mathcal{P}(a; b) \) generated by the function \( \varphi \in \mathcal{P}_0(a; b) \) can be studied a possibility of approximating functions \( \Phi \in J(\varphi) \) by countable sequences functions from the set \( \text{Pol}_\Phi \).

To formulate an appropriate criterion, we introduce the following notations: \( u(z) \) is the maximal subharmonic minorant of the function \( (h_{\varphi}(\arg z)|z - \ln |\varphi(z)|) \), where \( h_{\varphi} \) is the indicator function \( \varphi \),

\[
H_u = \{ \omega \in H(C) : \|\omega(z)\|_u = \sup_{z \in \mathbb{C}} |\omega(z)| e^{-u(z)} < +\infty \}.
\]

**Theorem 2.** The principle submodule \( J_{\varphi} \) generated by the function \( \varphi \in \mathcal{P}_0(\mathbb{R}) \) is weakly localizable if and only if each function \( \omega \in H_u \) is approximated by the polynomials in the norm \( \|\cdot\|' = \sup_{z \in \mathbb{C}} |\omega(z)| \exp (-u(z) - 2\ln (2 + |z|)) \).

**Proof.** It is clear we need to prove only necessity.

Let \( \omega \in H_u \) and \( \mu_0 \) be some zero of this function, then \( \frac{\omega}{z - \mu_0} \in \mathcal{H}(\varphi) \). By Corollary 1 and Theorem A, either \( \mathcal{H}(\varphi) = H_{\text{pol}} \) or

\[
\dim (\mathcal{H}(\varphi) \ominus H_{\text{pol}}) = 1.
\]

In the first case for some sequence of polynomials \( \{q_j\} \) the relation holds:

\[
\frac{\omega}{z - \mu_0} = \lim_{j \to \infty} q_j
\]

in the space \( \mathcal{H}(\varphi) \). By Lemma 1

\[
\left\| q_j \varphi - \frac{\omega}{z - \mu_0} \varphi \right\|_0 \to 0,
\]

where \( \| \cdot \|_0 \) is determined by formula (2.1) with \( c = c_{\varphi}, d = d_{\varphi} \). This implies easily the convergence of the polynomials \( \{(z - \mu_0)q_j\} \) to a function \( \omega \) in the norm \( \| \cdot \|' \).

If identity (3.1) holds, then

\[
\left( \frac{\alpha_0 \omega}{z - \mu_0} + \alpha_1 \omega \right) \in H_{\text{pol}},
\]

for some \( \alpha_0, \alpha_1 \in \mathbb{C} \), where \( \mu_1 \neq \mu_0 \) is one more zero of the function \( \omega \). By Lemma 1 some sequence of polynomials \( \{p_j\} \) converges to the function \( ((\alpha_0 + \alpha_1)z - (\alpha_1 \mu_0 + \alpha_0 \mu_1))\omega \) in the norm \( \| \cdot \|' \).

If \( \alpha_0 + \alpha_1 = 0 \), then the statement holds. Otherwise, letting \( \beta = \frac{\alpha_0 \mu_1 + \alpha_1 \mu_0}{\alpha_0 + \alpha_1} \) and taking into consideration the Phragmén-Lindelöf principle and the definition of the function \( u \), we see that the sequence of the polynomials

\[
\hat{p}_j(z) = \frac{p_j(z) - p_j(\beta)}{(\alpha_0 + \alpha_1)z - (\alpha_1 \mu_0 + \alpha_0 \mu_1)}, \quad j = 1, 2, \ldots,
\]

converges to the function \( \omega \) in the norm \( \| \cdot \|' \). \( \Box \)

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