Model of black hole evolution

J.G. Russo

Theory Division, CERN
CH-1211 Geneva 23, Switzerland

Abstract

From the postulate that a black hole can be replaced by a boundary on the apparent horizon with suitable boundary conditions, an unconventional scenario for the evolution emerges. Only an insignificant fraction of energy of order $(mG)^{-1}$ is radiated out. The outgoing wave carries a very small part of the quantum mechanical information of the collapsed body, the bulk of the information remaining in the final stable black hole geometry.

February 1996
It has been argued that, due to back-reaction effects, the Hawking model of black hole evaporation \[1\] may break down long before the evaporation is complete \[2,3\]. Because of the exponential redshift, the outgoing modes arise from a reservoir of transplanckian energies, with frequencies even higher than the total black hole mass. If a Planck-scale cutoff is imposed before the horizon, it seems that there would be only a scarce amount of outgoing modes, and black holes would lose an insignificant mass by evaporation \[3\]. Lacking the fundamental short-distance theory, by the time the outgoing modes arise with Planck frequencies, some extra assumption is needed. Extrapolating the Hawking radiation into this region leads to paradoxes, e.g. loss of quantum coherence. However, a concrete alternative scenario to the Hawking model has been elusive so far.

Recently, there have been some indications on how the problem should be formulated \[4,5\]. The idea is that the description of physics which is appropriate to external observers may require imposing a phenomenological boundary on a surface (the ‘stretched’ horizon \[6\]) located about one Planck unit away from the event horizon, where gravitational self-interactions become very strong \[4\]. It can be seen, in particular, that the apparent horizon is always inside the stretched horizon and it coincides with it once the supply of collapsing energy-density flux is over \[4\].

In this letter a novel theory of black hole evaporation will be constructed and examined. It will be assumed that the adequate framework for outside observers is based on a quantum theory with a boundary on the apparent horizon. The outgoing flux of energy in this model will coincide with the one predicted by the Hawking model only in the region which is not causally connected with the apparent horizon. By that retarded time the Hawking radiation flux is still negligible. In the region in causal contact with the boundary the total flux will be very small and it will exponentially go to zero. As a result, the final state will contain a stable geometry with approximately the same mass as the ADM mass of the original configuration. Only an energy of order \((mG)^{-1}\) will be evaporated.\[1\]

A scenario where the Hawking radiation stops leaving a macroscopic black hole was contemplated by Giddings as a possible solution of the information problem. In ref. \[8\] it was suggested that the radiation should stop when a certain bound on the information content is saturated. Although this is not what the present model predicts, the way the information paradox is resolved is similar, the Hawking process terminates and the information remains stored in the final black hole geometry.

Let us restrict our attention to spherically symmetric configurations,

\[
ds^2 = g_{ij}(x^0, x^1)dx^i dx^j + r^2(x^0, x^1)d\Omega^2, \quad i, j = 0, 1.
\]

\[1\] There is another well-known disturbing feature of the Hawking model, namely that at the endpoint of the evaporation the curvature singularity remains exposed to outside observers. This problem is thus absent in the model described here.
In this spherically symmetric space-time, the location of the apparent horizon is determined by \( g^{ij} \partial_i r \partial_j r = 0 \) (see e.g. ref. [7]). In the conformal gauge, \( g_{ij}(x^0, x^1)dx^idx^j = e^{2\rho(U,V)}dUdV \), this equation takes the form \( \partial_U r \partial_V r = 0 \). For the part of \( U, V \) space which is physically relevant in the process of gravitational collapse, the apparent horizon will be simply given by the equation \( r = 0 \). Below we will first determine the apparent horizon curve in terms of the infalling matter, and then calculate the outgoing energy-density fluxes by implementing suitable boundary conditions on the apparent horizon.

Let \( r = r(U, V) \), \( \rho = \rho(U, V) \) be the classical solution of the Einstein equations for a given infalling spherically symmetric configuration \( T_{\mu\nu} \). For simplicity only massless matter will be considered. In the conformal gauge, the classical Einstein equations for the \( g_{UU} \) and \( g_{VV} \) components are given by

\[
\partial_U^2 r - 2\partial_U \rho \partial_U r = -4\pi G r T_{UU} \; , \quad \partial_V^2 r - 2\partial_V \rho \partial_V r = -4\pi G r T_{VV} \; ,
\]

(2)

where \( T_{VV} \) and \( T_{UU} \) represent incoming and outgoing energy-density fluxes. Let the apparent horizon curve be classically given by \( U = -P(V) \). It is easy to obtain \( r(U, V) \) in the neighborhood of the apparent horizon. Expanding around \( U = -P(V) \), we have \( \partial_V^2 r \equiv f(U, V) = -F(V)(U + P(V)) + O((U + P(V))^2) \). By a conformal transformation one can always set \( F(V)dV \rightarrow \text{const.}dV \), so that the equation simply becomes \( \partial_V^2 r = -\text{const.}(U + P(V)) + \ldots \) It is convenient to choose the multiplicative constant equal to \( 2e^{-1} \) (cf. eqs. (6), (8)). By integration we obtain

\[
r^2(U, V) = (2M(V)G)^2 - 2e^{-1}V(U + P(V)) + O((U + P(V))^2) \; ,
\]

(3)

where a possible additive function \( f(U) = c(U + P(V)) + \ldots \) is removed by a shift of \( V \), and a function \( M(V) \) was introduced, defined by

\[
2eG^2 \frac{dM^2(V)}{dV} = V \frac{dP(V)}{dV} \; .
\]

(4)

Using eqs. (2), (3) and (4), the functions \( M(V), P(V) \) can be related to the incoming energy momentum tensor. In particular, evaluating the \( VV \)-constraint (2) near the apparent horizon, the second term can be dropped, and one finds

\[
\frac{dP(V)}{dV} \approx \frac{T_{VV}}{T(V)} \; , \quad T(V) \equiv (16\pi eG^3 M^2(V))^{-1} \; .
\]

(5)

To fix the notation, let us consider the static Schwarzschild geometry. The standard connection with Kruskal coordinates \( U, V \) is given by

\[
2mG(r - 2mG)e^{\frac{2mG}{r}} = -V(U + p) \; , \quad p = 2mG \; ,
\]

(6)
\( U + p = -2mGe^{-\frac{\sqrt{m}}{r}} \), \( V = 2mGe^{\frac{\sqrt{m}}{r}} \), \( v, u = t \pm r^* \), \( r^* = r + 2mG \log(r - 2mG) \). \( (7) \)

In this case the apparent horizon coincides with the event horizon. The solution of \( \partial_V r = 0 \) is \( U = -p \). Expanding \( r(U, V) \) in eq. (6) near \( U = -p \) one obtains

\[ r^2 \approx (2mG)^2 - 2e^{-1}V(U + p) + O((U + p)^2) \]. \( (8) \)

For a dynamically formed black hole, assuming that \( T_{VV} \) vanishes for \( V > V_1 \), \( m = M(V_1) \) will represent the total ADM mass of the collapsing body, and \( p = P(V_1) \) will be associated with the total infalling Kruskal momentum.

The equation of the apparent horizon in the absence of incoming fluxes was determined in [7]. It is easy to generalize this calculation to incorporate infalling matter. The equation \( \frac{\partial r(U, V)}{\partial V} = 0 \) can be written in terms of the total derivative on the apparent horizon curve, \( r = r_{AH}(U, V(U)) \),

\[ 0 = \frac{dr_{AH}}{dU} - \frac{\partial r_{AH}}{\partial U} \approx \frac{dr_{AH}}{dU} + \frac{1}{2eMG}V \]

\( (9) \)

For a large Schwarzschild black hole, \( r_{AH} \approx 2MG \), so that \( -\frac{V}{2eMG} = \frac{dr_{AH}}{dU} \approx 2G\frac{dM}{dU} \). In the vicinity of the horizon, a black hole loses mass at a rate as dictated by the Stefan-Boltzmann law, and it gains mass in accordance to the incoming energy-density flux,

\[ \frac{dM}{dv} = -\left( -N\frac{\pi^2}{30}T_H^4 + T_{vv} \right) \left( 4\pi r_s^2 \right), \quad r_s = 2M(V)G \]

\( (10) \)

where \( N \) represents the number of scalar field degrees of freedom. Using \( \frac{dM}{dv} = \frac{dM}{dv} \frac{dV(U)}{dU} \), and eqs. (7), (9), (10), one obtains

\[ \frac{dV}{dU} \left[ -\frac{NeGm^2}{480\pi M^2 V^2} + \frac{T_{VV}}{T(V)} \right] = -1 \]. \( (11) \)

From eq. (11) (see also eq. (10)) we see that there is a critical value of the incident energy-density flux for which \( dV/dU \) changes sign: for lower \( T_{VV} \) the apparent horizon is time-like; for larger \( T_{VV} \), it is space-like. Note that a space-like apparent horizon necessarily involves a black hole geometry, since it implies that the curve \( r(U, V) = 0 \) is space-like. In Minkowski coordinates:

\[ T_{vv}^{cr} \bigg|_{AH} = \frac{N\frac{\pi^2}{30}T_H^4}{M} = \frac{N}{122880\pi^2G^4M^4} \]. \( (12) \)

Equation (11) can be easily integrated when \( V \) is close to \( V_1 \), where \( M(V) \approx m \). In this region the apparent horizon curve takes the simple form

\[ V(U + P(V)) \approx -kG, \quad k = \frac{Ne}{480\pi} \]. \( (13) \)
Let us first discuss a situation where the incoming energy density flux is less than $T_{t\nu}^{\text{cr}}$ in the vicinity of the apparent horizon, so that it is time-like (see fig. 1). It may be assumed that this subcritical incident flux is striking on the apparent horizon of an already formed black hole. The simplest boundary condition is that this low energy-density matter is just reflected on the time-like apparent horizon. A classical reflection on a boundary $V(U)$ is a relation of the form: $T_{U\nu}^R = T_{V\nu} \left( \frac{dV}{dU} \right)^2$. Quantum mechanically, there is an additional contribution, which depends on the normal ordering subtraction of the composite operators $T_{U\nu}, T_{V\nu}$:

$$\left( T_{U\nu}^R - t_{U\nu} \right) = \left( \frac{dV}{dU} \right)^2 \left( T_{V\nu} - t_{V\nu} \right).$$

(14)

The calculation given here will not depend on the explicit form of $t_{U\nu}, t_{V\nu}$. In region (i) the outgoing energy-density flux $T_{U\nu}^H$ can be obtained from the constraint equation,

$$T_{U\nu}^H = -(4\pi G r)^{-1} \left( \frac{\partial^2}{\partial r^2} - 2 \frac{\partial U}{\partial u} \frac{\partial r}{\partial u} \right) + t_{U\nu}, \quad U < U_0 .$$

(15)

For $V \to \infty$ the solution in region (i) approaches the classical Schwarzschild solution, so that the first term in eq. (15) vanishes, and one obtains:

$$T_{U\nu}^H = t_{U\nu}, \quad U < U_0 ,$$

(16)

which represents the standard Hawking flux. The total energies radiated in regions (i) and (ii) will be given by (we use $4mGdU = -(U + p)du$):

$$E_{\text{out}}^{(i)} = 4\pi \int_{-\infty}^{U_0} du \ r^2 T_{uu} = -\frac{\pi}{mG} \int_{-\infty}^{U_0} dU (U + p) r^2 T_{U\nu}^H ,$$

(17)

$$E_{\text{out}}^{(ii)} = -\frac{\pi}{mG} \int_{U_0}^{U_1} dU (U + p) r^2 T_{U\nu}^R .$$

(18)

---

2 A similar expression can be derived from the reflection condition (14) which gives $T_{U\nu}^H = t_{U\nu} - \left( \frac{dV}{dU} \right)^2 t_{V\nu}$. The second term is a small correction to eq. (16) which can be neglected near a black hole horizon.
Let us now gradually increase $T_{v_0}$ above $T_{v_1}^{cr}$ so that a part of the apparent horizon becomes space-like, as in fig. 2. In this process a part of region (i) ends up superposing with region (ii), giving rise to the region (b) of fig. 2. In this region the two contributions $T_{UU}^R$ and $T_{UU}^H$ are thus superposed. The correct outgoing $T_{UU}^{(b)}$ can be obtained by carefully continuing the previous formulas. Now $U_0 > U_1$, so that

\[
\int_{-\infty}^{U_0} dU (U + p)T_{UU}^H = \int_{-\infty}^{U_1} dU (U + p)T_{UU}^H + \int_{U_1}^{U_0} dU (U + p)T_{UU}^H ,
\]

\[
\int_{U_0}^{U_1} dU (U + p)T_{UU}^R = \int_{U_1}^{U_0} dU (U + p)(-T_{UU}^R) .
\]

Therefore, the total energy radiated between $U_1$ and $U_0$ is

\[
E_{\text{out}}^{(b)} = -\frac{\pi}{mG} \int_{U_1}^{U_0} dU (U + p)^2 T_{UU}^{(b)} , \quad T_{UU}^{(b)} = T_{UU}^H - T_{UU}^R = -(\frac{dV}{dU})^2 (T_{VV} - t_{VV}) . \quad (19)
\]

Thus when the apparent horizon is space-like $T_{UU}^R$ contributes with the reverse sign. An extra contribution in region (b) is not a surprise, since the geometry in region (b) is expected to undergo some modification, being in causal contact with the boundary line. The flip of sign can be physically understood as follows. For each given $U'$, the geometry at $V'$ is determined in terms of the energy that has crossed $U'$ at earlier $V < V'$. In the presence of the reflecting space-like wall at $U > U_1$, the energy-momentum flux crossing $U_1$ cannot be felt by the geometry in region (b). The net effect is that the geometry in region (b) is changed in such a way that the flux $T_{UU}^R$ must be subtracted from the outgoing flux. This interpretation is confirmed in ref. [9] for an exactly solvable two-dimensional model [10], where the full time-dependent geometry, including the geometry in region (b), can be explicitly obtained.
Fig. 3: A macroscopic black hole geometry. The thick line represents the apparent horizon.

It should be noted that only subcritical matter reflects off the (time-like) apparent horizon. The critical energy density \( \rho_c \) at the horizon of a massive black hole is extremely low (e.g. \( 10^{-64} \text{g/cm}^3 \), for a solar mass black hole). For infalling objects with energy-density greater than critical, the apparent horizon will be space-like, and they will just go inside the black hole increasing its mass. As shown below, only a very minor part of their energy and of their information will be emitted.

Let us consider the evolution of a macroscopic black hole geometry, i.e. with total mass \( m \gg m_P \), \( m_P = 1/\sqrt{G} \). For convenience we will assume that \( T_{VV} \) vanishes for \( V < V_0 \) and \( V > V_1 \). For \( V > V_1 \) the geometry is approximately static and given by the Schwarzschild geometry with \( m = M(V_1) \gg m_P \), \( p = P(V_1) \gg |U_0| \). For \( V > V_1 \) the apparent horizon curve will be given by the equation \( V(U+p) = -kG \). The geometry is shown in fig. 3. Using eq. (19), and neglecting \( t_{VV} \) as compared with the classical collapsing matter contribution \( T_{VV} \), we obtain

\[
T_{UU}^{(b)} \approx -T_{VV} \left( \frac{dV}{dU} \right)^2.
\]

We notice that this outgoing energy-density flux is negative. It will soon be clear that the amount of negative energy radiated in region (b) is a tiny Planck-scale quantity. (In quantum theory the energy density is not positive definite, and global energy positivity will not be violated. It is the tail of the outgoing wave that carries off this bit of negative energy).

First, let us estimate the total (positive) energy radiated in region (a). As is well known, the Hawking radiation flux is significant only near \( U = -p \) (more precisely, for \( U \)
exponentially close to \( -p, U+p \sim \exp[-\text{const}.Gm^2] \), where it has the form \( T_{uu}^H \sim \frac{1}{r^2(U+p)^2} \), or \( T_{UU}^H \sim \frac{1}{r^2(U+p)^2} \), and it can be neglected for \( U < U_0' \equiv -p - kG/V_0 \). Thus

\[
E_{\text{out}}^{(a)} \approx -\frac{\pi}{mG} \int_{U_0'}^{U_1} dU (U+p)r^2 T_{UU}^H \approx -\frac{k}{4emG} \log \frac{U_1 + p}{U_0' + p} = \frac{k}{4emG} \log \frac{V_1}{V_0}, \tag{21}
\]

which, indeed, is a small amount of energy. This can be more explicitly seen by relating \( \log \frac{V_1}{V_0} \) to the physical parameters characterizing the incoming energy-density flux, such as the total energy \( m \). In particular, consider an approximately constant (\( v \)-independent) flux \( T_{vv} \), which is such that \( T_{vv} \sim 1/(Gm^2) \). The total mass will be given by \( m \approx 4\pi r_s^2 \mathcal{E}(v_1 - v_0) \sim (mG)^3 \mathcal{E} \log \frac{V_1}{V_0} \). We find

\[
E_{\text{out}}^{(a)} \approx \frac{k}{16} (mG)^{-1} \mathcal{E}_{\text{cr}} \mathcal{E}, \quad \mathcal{E}_{\text{cr}} \equiv (16\pi eG^3 m^2)^{-1} = \mathcal{T}(V_1). \tag{22}
\]

The parameter \( \mathcal{E}_{\text{cr}} \) is roughly equal to the critical density at which a uniform spherical body would lie within its Schwarzschild radius (note that \( T_{vv}^{(\text{cr})} \) is much smaller than \( \mathcal{E}_{\text{cr}}, \mathcal{T}_{vv}^{(\text{cr})} \approx \mathcal{E}_{\text{cr}} m^2/(mG) \)).

Next, we calculate the (negative) energy received in region (b). Let \((V_2, U_2 \equiv -p)\) be the point at the intersection between the apparent horizon and the null line \( U = -p \), i.e. \( V_2(-p + P(V_2)) = -kG \). Let us note that for \( m \gg m_P \), \( V_2 \) and \( V_1 \) differ by a small quantity (it should be remembered that the splitting between the time-like part of the apparent horizon and the horizon \( U = -p \) is a quantum effect). In particular, for a constant density flux one has \( \frac{V_2}{V_1} \approx 1 - \frac{k \mathcal{E}_{\text{cr}}}{16Gm^2} \). The outgoing energy momentum tensor in region (b) is given by eq. (20). Since we are only interested in the leading order in \( m_P/m \), we can use \( \frac{dU}{dV} \approx -P'(V) \). Inserting eq. (3) into eq. (20), one obtains

\[
T_{UU}^{(b)} \approx \mathcal{T}(V) \frac{dV}{dU} = -\frac{\mathcal{T}^2(V)}{\mathcal{T}_{VV}}. \tag{23}
\]

\( T_{UU}^{(b)} \) carries out information about the small fraction of the infalling matter that arrived at the apparent horizon between \( V_2 \) and \( V_1 \). In Minkowski coordinates,

\[
T_{uu}^{(b)} \approx -\frac{V_2^2 \mathcal{T}^2(V)}{(8mG)^2 \mathcal{T}_{vv}} \exp[-\frac{u}{2mG}], \tag{24}
\]

Hence \( E_{\text{out}}^{(b)} \approx 4\pi r_s^2 \Delta u T_{uu}^{(b)}(u_1), \quad \Delta u \sim 2mG \),

\[
E_{\text{out}}^{(b)} \approx -\frac{\pi \mathcal{E}_{\text{cr}}^2}{2\mathcal{E}} mGV_1^2 e^{-\frac{\alpha}{2\mathcal{E}} mG} \approx -a(G^2 m^3)^{-1}, \quad a = \frac{k^2 \mathcal{E}_{\text{cr}}}{128e\mathcal{E}} < 1, \tag{25}
\]
where we have used $e^{-mG/k} = 2mV_1/k$. Thus the emitted negative energy is smaller than $m_P/m^3$ in absolute value. Since $m \gg m_P$, this is a tiny energy (e.g. for a solar mass black hole, $E_{\text{out}}^{(b)} \sim -10^{-114} m_P$). From eqs. (22) and (25) one finds

$$\frac{|E_{\text{out}}^{(b)}|}{E_{\text{out}}^{(a)}} \approx \frac{m_P^2}{m^2} \ll 1.$$  \hspace{1cm} (26)

Thus the total radiated energy $E_{\text{out}}^{(a)} + E_{\text{out}}^{(b)}$ is positive and of order $E_{\text{out}}^{(a)} \sim (mG)^{-1}$ (see eq. (22)).

To summarize, a simple theory of black hole evolution based on reflecting boundary conditions on the apparent horizon was described. The departure from Hawking theory occurs precisely by the time the outgoing modes arise with Planckian frequencies from the vicinity of the horizon (further discussions on the problem of Planck frequencies can be found in refs. [11,12]). The sudden fall of the subsequent outgoing flux is caused by a contribution from the expanding trapped surface. The total radiated energy is a small (positive) Planckian quantity. The final configuration is a stable black hole geometry, which has retained most of its mass together with the quantum mechanical information of the original configuration.

The stability of the final geometry can be understood in different ways. It is known that in order to have zero fluxes at infinity (in the present case, in region (b)), the gravitational field must be greatly modified near the line $U = -p$. This picture is sometimes referred to as the Boulware vacuum choice, defined in terms of the Schwarzschild Killing vector (here the geometry has settled down to this situation dynamically having started from the Unruh vacuum). Accordingly, the geometry in region (b) will be given by the Schwarzschild metric only at far distances from $U = -p$, viz. for $-V(U + p) \gg \exp[-\text{const.}Gm^2]$. This condition is satisfied in the whole of region (b) where $-V(U + p) > kG$, and therefore the corrections to the Schwarzschild metric will be exponentially small in the allowed space-time. The boundary does not imply that inertial infalling observers will encounter a barrier at the apparent horizon; their description of physics is different (e.g. they do not see Hawking radiation) and it may be complementary in the usual sense of quantum mechanics.

The author wishes to thank D. Amati for useful discussions and collaboration in the 1+1 dimensional analog [9], and E. Verlinde for helpful remarks.
References

[1] S. Hawking, Commun. Math. Phys. 43 (1975) 199.
[2] G. 't Hooft, Nucl. Phys. B256 (1985) 727; Nucl. Phys. B335 (1990) 138.
[3] T. Jacobson, Phys. Rev. D44 (1991) 1731.
[4] L. Susskind, L. Thorlacius and J. Uglum, Phys. Rev. D48 (1993) 3743.
[5] C. Stephens, G. 't Hooft and B.F. Whiting, Class. Quant. Grav. 11 (1994) 621.
[6] K. Thorne, R. Price and D. MacDonald, *Black holes: the membrane paradigm* (Yale
Univ. Press, New Haven, CT, 1986).
[7] J.G. Russo, Phys. Lett. B359 (1995) 69.
[8] S. Giddings, Phys. Rev. D46 (1992) 1347.
[9] D. Amati and J.G. Russo, *Black holes by analytic continuation*, hep-th/9602125.
[10] J.G. Russo, L. Susskind and L. Thorlacius, Phys. Rev. D46 (1992) 3444; Phys. Rev.
D47 (1993) 533.
[11] Y. Kiem, H. Verlinde and E. Verlinde, Phys. Rev. D52 (1995) 7053.
[12] T. Jacobson, preprint Utrecht THU-96/01, hep-th/9601064.