ON THE ADDITION AND MULTIPLICATION THEOREMS

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Dedicated with great pleasure to Lev Aronovich Sakhnovich on the occasion of his 80th birthday anniversary

Abstract. We discuss the classes $C$, $M$, and $S$ of analytic functions that can be realized as the Livšic characteristic functions of a symmetric densely defined operator $\hat{A}$ with deficiency indices $(1,1)$, the Weyl-Titchmarsh functions associated with the pair $(\hat{A},A)$ where $A$ is a self-adjoint extension of $\hat{A}$, and the characteristic function of a maximal dissipative extension $\hat{A}$ of $A$, respectively. We show that the class $M$ is a convex set, both of the classes $S$ and $C$ are closed under multiplication and, moreover, $C \subset S$ is a double sided ideal in the sense that $S \cdot C = C \cdot S \subset S$. The goal of this paper is to obtain these analytic results by providing explicit constructions for the corresponding operator realizations. In particular, we introduce the concept of an operator coupling of two unbounded maximal dissipative operators and establish an analog of the Livšic-Potapov multiplication theorem [14] for the operators associated with the function classes $C$ and $S$. We also establish that the modulus of the von Neumann parameter characterizing the domain of $\hat{A}$ is a multiplicative functional with respect to the operator coupling.

1. Introduction

In 1946, M. Livšic [10] introduced fundamental concepts of the characteristic functions of a densely defined symmetric operator $\hat{A}$ with deficiency indices $(1,1)$, and of its maximal non-self-adjoint extension $\hat{A}$. Under the hypothesis that the symmetric operator $\hat{A}$ is prime$^1$ a cornerstone result [10] Theorem 13 (also see [2] and [3]) states that the characteristic function (modulo inessential constant unimodular factor) determines the operator up to unitary equivalence. In 1965, in an attempt to characterize self-adjoint extensions $A$ of a symmetric operator $\hat{A}$, Donoghue [7] introduced the Weyl-Titchmarsh function associated with the pair $(\hat{A},A)$ and showed that the Weyl-Titchmarsh function determines the pair $(\hat{A},A)$ up to unitary equivalence whenever $\hat{A}$ is a prime symmetric operator with deficiency indices $(1,1)$.

In our recent paper [15], we introduced into play an auxiliary self-adjoint (reference) extension $A$ of $\hat{A}$ and suggested to define the characteristic functions of a symmetric operator and of its dissipative extension as the functions associated with the pairs $(\hat{A},A)$ and $(\hat{A},A)$, rather than with the single operators $\hat{A}$ and $\hat{A}$,

\begin{footnote}{1}Recall that a closed symmetric operator $\hat{A}$ is called a prime operator if $\hat{A}$ does not have invariant subspaces where the corresponding restriction of $\hat{A}$ is self-adjoint\end{footnote}
respectively. Honoring M. Livšic’s fundamental contributions to the theory of non-self-adjoint operators and also taking into account the crucial role that the characteristic function of a symmetric operator plays in the theory, we suggested to call the characteristic function associated with the pair \((\hat{A}, A)\) the Livšic function. For a detailed treatment of the aforementioned concepts of the Livšic, Weyl-Titchmarsh, and the characteristic functions including the discussion of their interrelations we refer to [15].

The main goal of this paper is to obtain the following two principal results.

Our first result states that given two Weyl-Titchmarsh functions \(M_1(M_1, A_1)\) and \(M_2(M_2, A_2)\), any convex combination \(pM_1 + qM_2\) can also be realized as the Weyl-Titchmarsh function associated with a pair \((\hat{A}, A_1 \oplus A_2)\), where \(\hat{A}\) stands for some special symmetric extension with deficiency indices \((1, 1)\) of the direct orthogonal sum of \(A_1\) and \(A_2\) (see Theorem 4.1).

Our second result concerns the computation of the characteristic function of an operator coupling \(\hat{A} = \hat{A}_1 \uplus \hat{A}_2\) of two dissipative operators \(\hat{A}_1\) and \(\hat{A}_2\), acting in the Hilbert spaces \(\mathcal{H}_1\) and \(\mathcal{H}_2\), defined as a dissipative extension of \(\hat{A}_1\) outgoing from the Hilbert space \(\mathcal{H}_1\) to the direct sum of the Hilbert space \(\mathcal{H}_1 \oplus \mathcal{H}_2\) satisfying the constraint

\[
\hat{A}\mid_{\text{Dom}(\hat{A}) \cap \text{Dom}(\hat{A})^*} \subset \hat{A}_1 \oplus (\hat{A}_2)^*.
\]

This result, called the multiplication theorem (see Theorem 6.1), states that the product \(S_1 \cdot S_2\) of the characteristic functions \(S_1\) and \(S_2\) associated with the pairs \((\hat{A}_1, A_1)\) and \((\hat{A}_2, A_2)\) coincides with the characteristic function of the operator coupling \(\hat{A} = \hat{A}_1 \uplus \hat{A}_2\) relative to an appropriate reference self-adjoint operator.

It is important to mention that the multiplication theorem substantially relies on the multiplicativity of the absolute value \(\hat{\kappa}(\cdot)\) of the von Neumann extension parameter of a maximal dissipative extension of \(\hat{A}\) established in Theorem 5.4:

\[
(1.1) \quad \hat{\kappa}(\hat{A}_1 \uplus \hat{A}_2) = \hat{\kappa}(\hat{A}_1) \cdot \hat{\kappa}(\hat{A}_2).
\]

Introducing the analytic function classes \(\mathcal{C}\) and \(\mathcal{M}\), elements of which can be realized as the Livšic and Weyl-Titchmarsh functions associated with a pair \((\hat{A}, A)\), respectively, along with the analytic function class \(\mathcal{S}\) consisting of all characteristic functions associated with all possible pairs \((\hat{A}, A)\), as a corollary of our geometric considerations we obtain that

(i) The class \(\mathcal{M}\) is a convex set with respect to addition;
(ii) The class \(\mathcal{S}\) is closed with respect to multiplication, \(\mathcal{S} \cdot \mathcal{S} \subset \mathcal{S}\);
(iii) The subclass \(\mathcal{C} \subset \mathcal{S}\) is a (double sided) ideal in the sense that
\[
\mathcal{C} \cdot \mathcal{S} = \mathcal{S} \cdot \mathcal{C} \subset \mathcal{C};
\]
(iv) The class \(\mathcal{C}\) is closed with respect to multiplication: \(\mathcal{C} \cdot \mathcal{C} \subset \mathcal{C}\).

The closedness of the class \(\mathcal{S}\) under multiplication (ii) is a scalar variant of the multiplication theorem in the unbounded setting. The multiplication theorem for bounded operators was originally obtained in 1950 by M. S. Livšic and

\[\text{We borrow this term from the ring theory. However, it worth perhaps mentioning that the function class \(\mathcal{S}\) as an algebraic structure is not a ring.}\]
V. P. Potapov [14], who in particular established that the product of two characteristic matrix-valued functions of bounded operators coincides with the matrix-valued characteristic function of a bounded operator. After this the result has been extended to the case of operator colligations (systems) [1], [4], [5], [6], [12], [13].

The paper is organized as follows.

In Section 2, we recall the definitions and briefly discuss various properties of the Livšic, Weyl-Titchmarsh and the characteristic functions.

In Section 3, we introduce a coupling of two symmetric operators defined as a symmetric extension with deficiency indices (1, 1) of the direct sum of two symmetric operators \( \tilde{A}_1 \) and \( \tilde{A}_2 \) acting in the Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) and then we explicitly compute the Livšic function of the coupling (see Theorem 3.1).

In Section 4, we prove the Addition Theorem for the Weyl-Titchmarsh functions (see Theorem 4.1).

In Section 5, we develop a variant of the extension theory with constraints, introduce a concept of the operator coupling of two unbounded dissipative operators, discuss its properties, and prove the Multiplicativity of the von Neumann extension parameter (see Theorem 5.4).

In Section 6, we prove the Multiplication Theorem for the characteristic functions (see Theorem 6.1). We also illustrate the corresponding geometric constructions by an example of the differentiation operator on a finite interval (see Example 6.2).

In Appendix A, a differentiation operator on a finite interval is treated in detail (also see [2] for a related exposition).

2. Preliminaries

Throughout this paper we assume the following hypothesis.

**Hypothesis 2.1.** Suppose that \( \tilde{A} \) is a densely defined symmetric operator \( \tilde{A} \) with deficiency indices (1, 1) and \( \tilde{A} \) its self-adjoint extension. Assume that the deficiency elements \( g_+ \in \text{Ker}((\tilde{A})^* - zI) \) are chosen in such a way that \( \|g_+\| = \|g_-\| = 1 \) and that

\[
(2.1) \quad g_+ - g_- \in \text{Dom}(\tilde{A}).
\]

2.1. The Livšic function and the class \( \mathcal{C} \). Under Hypothesis 2.1, introduce the Livšic function \( s = s(\tilde{A}, A) \) of the symmetric operator \( \tilde{A} \) relative to the self-adjoint extension \( A \) by

\[
(2.2) \quad s(\tilde{A}, A)(z) = \frac{z - i}{z + i} \cdot \frac{g_z g_-}{(g_z, g_+)}, \quad z \in \mathbb{C}_+,
\]

where \( g_z, z \in \mathbb{C}_+, \) is an arbitrary deficiency element, \( 0 \neq g_z \in \text{Ker}((\tilde{A})^* - zI) \).

We remark that from the definition it follows that the dependence of the Livšic function \( s(\tilde{A}, A) \) on the reference (self-adjoint) operator \( A \) reduces to multiplication by a \( z \)-independent unimodular factor whenever \( A \) changes. That is,

\[
(2.3) \quad s(\tilde{A}, A_\alpha) = e^{-2i\alpha} s(\tilde{A}, A), \quad \alpha \in [0, \pi),
\]

whenever the self-adjoint reference extension \( A_\alpha \) of \( \tilde{A} \) has the property

\[
(2.4) \quad g_+ - e^{2i\alpha} g_- \in \text{Dom}(A_\alpha).
\]

Denote by \( \mathcal{C} \) the class of all analytic mappings from \( \mathbb{C}_+ \) into the unit disk \( \mathbb{D} \) that can be realized as the Livšic function associated with some pair \( (\tilde{A}, A) \).
The class $\mathcal{C}$ can be characterized as follows (see [10]). An analytic mapping $s$ from the upper-half plane into the unit disk belongs to the class $\mathcal{C}$, $s \in \mathcal{C}$, if and only if
\begin{equation}
(2.5) \quad s(i) = 0 \quad \text{and} \quad \lim_{z \to \infty} z(s(z) - e^{2i\alpha}) = \infty \quad \text{for all} \quad \alpha \in [0, \pi),
\end{equation}
\begin{equation}
0 < \varepsilon \leq \arg(z) \leq \pi - \varepsilon.
\end{equation}

2.2. The Weyl-Titchmarsh function and the class $\mathcal{M}$. Define the Weyl-Titchmarsh function $M(\hat{A}, A)$ associated with the pair $(\hat{A}, A)$ as
\begin{equation}
(2.6) \quad M(\hat{A}, A)(z) = ((Az + I)(A - zI)^{-1} g_+, g_+), \quad z \in \mathbb{C}_+.
\end{equation}

Denote by $\mathcal{M}$ the class of all analytic mapping from $\mathbb{C}_+$ into itself that can be realized as the Weyl-Titchmarsh function $M(\hat{A}, A)$ associated with a pair $(\hat{A}, A)$.

As for the characterization of the class $\mathcal{M}$, we recall that $M \in \mathcal{M}$ if and only if $M$ admits the representation (see [7], [8], [9], [15])
\begin{equation}
(2.7) \quad M(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu,
\end{equation}
where $\mu$ is an infinite Borel measure and
\begin{equation}
(2.8) \quad \int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1, \quad \text{equivalently,} \quad M(i) = i.
\end{equation}

It is worth mentioning (see, e.g., [15]) that the Livšic and Weyl-Titchmarsh functions are related by the Cayley transform
\begin{equation}
(2.9) \quad s(\hat{A}, A)(z) = \frac{M(\hat{A}, A)(z) - i}{M(\hat{A}, A)(z) + i}, \quad z \in \mathbb{C}_+.
\end{equation}

Taking this into account, one can show that the properties (2.5) and (2.7), (2.8) imply one another (see, e.g., [15]).

Combining (2.3), (2.4) and (2.9) shows that the corresponding transformation law for the Weyl-Titchmarsh functions reads as (see [7], [8], [9])
\begin{equation}
(2.10) \quad M(\hat{A}, A_\alpha) = \cos \alpha \frac{M(\hat{A}, A)}{\cos \alpha + \sin \alpha M(\hat{A}, A)}, \quad \alpha \in [0, \pi).
\end{equation}

In view of (2.10), the function classes $\mathcal{C}$ and $\mathcal{M}$ are related by the Cayley transform,
\begin{equation}
\mathcal{C} = K \circ \mathcal{M},
\end{equation}
where
\begin{equation}
K(z) = \frac{z - i}{z + i}, \quad z \in \mathbb{C}.
\end{equation}

That is,
\begin{equation}
\mathcal{C} = \{ K \circ M \mid M \in \mathcal{M} \},
\end{equation}
where $K \circ M$ denotes the composition of the functions $K$ and $M$.

Moreover, the transformation law (2.3) shows that the class $\mathcal{C}$ is closed under multiplication by a unimodular constant,
\begin{equation}
\theta \cdot \mathcal{C} = \mathcal{C}, \quad |\theta| = 1.
\end{equation}

Accordingly, from (2.10) one concludes that the class $\mathcal{M}$ is closed under the action of a one parameter subgroup of $SL(2, \mathbb{R})$ of linear-fractional transformations
\begin{equation}
K_\alpha \circ \mathcal{M} = \mathcal{M}, \quad \mathbb{R} \ni \alpha \mapsto K_\alpha,
\end{equation}
given by
\[ K_\alpha(z) = \frac{\cos \alpha z - \sin \alpha}{\cos \alpha + \sin \alpha z} \]

2.3. The von Neumann extension parameter of a dissipative operator.
Denote by \( D \) the set of all maximal dissipative unbounded operators \( \hat{A} \) such that the restriction \( \hat{A} \) of \( \hat{A} \) onto \( \text{Dom}(\hat{A}) \cap \text{Dom}(\hat{A}^*) \) is a densely defined symmetric operators with indices \((1, 1)\).

Given \( \hat{A} \in D \) and a self-adjoint (reference) extension \( A \) of the underlying symmetric operator \( \hat{A} = \hat{A}|_{\text{Dom}(\hat{A}) \cap \text{Dom}(\hat{A}^*)} \), assume that the pair \((\hat{A}, A)\) satisfies Hypothesis [2,1] with some \( g_\pm \) taken from the corresponding deficiency subspaces, so that
\[ g_+ - g_- \in \text{Dom}(A). \]

In this case,
\[ g_+ - \kappa g_- \in \text{Dom}(\hat{A}) \quad \text{for some} \quad \kappa \in \mathbb{D}. \]

Definition 2.2. We call \( \kappa = \kappa(\hat{A}, A) \) the von Neumann extension parameter of the dissipative operator \( \hat{A} \in D \) relative to the reference self-adjoint operator \( A \).

2.4. The characteristic function of a dissipative operator and the class \( \mathcal{S} \).
Suppose that \( \hat{A} \in D \) is a maximal dissipative operator, \( \hat{A} = \hat{A}|_{\text{Dom}(\hat{A}) \cap \text{Dom}(\hat{A}^*)} \) its symmetric restriction, and \( A \) is a reference self-adjoint extension of \( \hat{A} \). Following [10] (also see [2, 15]) we define the characteristic function \( S = S(\hat{A}, A) \) of the dissipative operator \( \hat{A} \) relative to the reference self-adjoint operator \( A \) as
\[ S(z) = \frac{s(z) - \kappa}{\kappa s(z) - 1}, \quad z \in \mathbb{C}_+, \]
where \( s = s(\hat{A}, A) \) is the Livšic function associated with the pair \((\hat{A}, A)\) and the complex number \( \kappa = \kappa(\hat{A}, A) \) is the von Neumann extension parameter of \( \hat{A} \) (relative to \( A \)).

We stress that for a dissipative operator \( \hat{A} \in D \) one always has that
\[ \text{Dom}(\hat{A}) \neq \text{Dom}(\hat{A}^*) \]
and, moreover, the underlying densely defined symmetric operator \( \hat{A} \) can uniquely be recovered by restricting \( \hat{A} \) on
\[ \text{Dom}(\hat{A}) = \text{Dom}(\hat{A}) \cap \text{Dom}(\hat{A}^*). \]

This explains why it is more natural to associate the characteristic function with the pair \((\hat{A}, A)\) rather than with the triple \((\hat{A}, \hat{A}, A)\) which would perhaps be more pedantic.

The class of all analytic mapping from \( \mathbb{C}_+ \) into the unit disk consisting of all the characteristic functions \( S(\hat{A}, A) \) associated with arbitrary pairs \((\hat{A}, A)\), with \( \hat{A} \in D \) and \( A \) a reference self-adjoint extension of the underlining symmetric operator \( \hat{A} \), will be denoted by \( \mathcal{S} \).

As in the case of the class \( \mathcal{C} \), the class \( \mathcal{S} \) is also closed under multiplication by a constant unimodular factor (cf. (2.11)), that is,
\[ \theta \cdot \mathcal{S} = \mathcal{S}, \quad |\theta| = 1. \]
Indeed, if \( S \in \mathcal{S} \), then
\[
S = \frac{s - \kappa}{\kappa s - 1}
\]
for some \( s \in \mathcal{C} \), \( \kappa \in \mathbb{D} \).

Therefore,
\[
\theta \cdot S = \frac{\theta \cdot s - \theta \kappa}{\theta \kappa \theta \cdot s - 1}, \quad |\theta| = 1.
\]

Since the class \( \mathcal{C} \) is closed under multiplication by a constant unimodular factor, \( \theta \cdot s \in \mathcal{C} \) and since \(|\theta \kappa| < 1\), by definition (2.13), the function \( \theta \cdot S \) belongs to the class \( \mathcal{S} \) as well.

Now it is easy to see that the class \( \mathcal{S} \) coincides with the orbit of the class \( \mathcal{C} \) under the action of the group of automorphisms \( \text{Aut}(\mathbb{D}) \) of the complex unit disk. That is,
\[
(2.15) \quad K \circ \mathcal{C} = \mathcal{S}, \quad K \in \text{Aut}(\mathbb{D}).
\]

In particular, one obtains that
\[
(2.16) \quad \mathcal{C} \subset \mathcal{S}.
\]

From (2.15) and (2.16) follows that the class \( \mathcal{S} \) is closed under the action of the group \( \text{Aut}(\mathbb{D}) \), that is,
\[
K \circ \mathcal{S} = \mathcal{S}, \quad K \in \text{Aut}(\mathbb{D}).
\]

2.5. The unitary invariant \( \hat{\kappa} : \mathcal{D} \to [0,1) \). Combining (2.5) and (2.13) shows that the value of the von Neumann extension parameter \( \kappa(\hat{A}, A) \) can also be recognized as the value of the characteristic function at the point \( z = i \), that is,
\[
(2.17) \quad \kappa = \kappa(\hat{A}, A) = S(\hat{A}, A)(i).
\]

Since by Livšic theorem \([10, \text{Theorem 13}]\) the characteristic function \( S(\hat{A}, A) \) determines the pair \( (\hat{A}, A) \) up to unitary equivalence provided that the underlining symmetric operator \( \hat{A} \) is prime, cf. \([15]\), the parameter \( \kappa \) is a unitary invariant of the pair \( (\hat{A}, A) \).

It is important to notice that the absolute value \( \hat{\kappa}(\hat{A}) = |\kappa(\hat{A}, A)| \) of the von Neumann extension parameter is independent of the choice of the reference self-adjoint extension \( A \). Therefore, the following functional
\[
\hat{\kappa} : \mathcal{D} \to [0,1)
\]
of the form
\[
\hat{\kappa} = \hat{\kappa}(\hat{A}) = |\kappa(\hat{A}, A)|, \quad \hat{A} \in \mathcal{D},
\]
is well defined as one of the geometric unitary invariants of a dissipative operator from the class \( \mathcal{D} \).

The kernel of the functional \( \hat{\kappa} \) can be characterized as follows.

The inclusion (2.13) shows that any Livšic function \( s(\hat{A}, A) \) can be indentified with the characteristic function \( S(\hat{A}, A') \) associated with some pair \( (\hat{A}, A') \) where \( \hat{A} \in \mathcal{D} \) and \( A' \) is an appropriate self-adjoint reference extension of the symmetric operator \( \hat{A} = \hat{A}|_{\text{Dom}(\hat{A}) \cup \text{Dom}((\hat{A})^*)} \).

To be more specific, it suffices to take the maximal dissipative extension \( \hat{A} \) of \( \hat{A} \) with the domain
\[
(2.18) \quad \text{Dom}(\hat{A}) = \text{Dom}(\hat{A}) \cup \text{Ker}((\hat{A})^* - iI)
\]
and to choose the reference self-adjoint operator $A'$ in such a way that
\[ s(\hat{A}, A) = -s(\hat{A}, A'). \]
This is always possible due to (2.3). Since $\hat{\kappa}(\hat{A}) = 0$ (combine (2.12) and (2.18)), it is easy to see that
\[ S(\hat{A}, A') = -s(\hat{A}, A') = s(\hat{A}, A) \]
which proves the claim.

The subclass of maximal dissipative extensions $\hat{A}$ with the property (2.18) will be denoted by $\hat{D}$. That is,
\[ (2.19) \quad \hat{D} = \{ \hat{A} \in \mathcal{D} \mid \hat{\kappa}(\hat{A}) = 0 \} \subset \mathcal{D}, \]
and, therefore,
\[ \hat{D} = \text{Ker}(\hat{\kappa}). \]

3. Symmetric extensions of the direct sum of symmetric operators

Suppose that $\hat{A}_1$ and $\hat{A}_2$ are densely defined symmetric operators with deficiency indices $(1, 1)$ acting in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

In accordance with the von Neumann extensions theory, the set of all symmetric extensions $\hat{A}$ with deficiency indices $(1, 1)$ of the direct sum of the symmetric operators $\hat{A}_1 \oplus \hat{A}_2$ is in one-to-one correspondence with the set of one-dimensional neutral subspaces $\mathcal{L}$ of the quotient space $\text{Dom}(\hat{A}_1 \oplus \hat{A}_2)^*/\text{Dom}(\hat{A}_1 \oplus \hat{A}_2)$ such that the adjoint operator $(\hat{A}_1 \oplus \hat{A}_2)^*$ restricted on $\mathcal{L}$ is symmetric, that is,
\[ \text{Im}((\hat{A}_1 \oplus \hat{A}_2)^* f, f) = 0, \quad f \in \mathcal{L}. \]
The above mentioned correspondence can be established in the following way: given $\mathcal{L}$, the corresponding symmetric operator $\hat{A}$ is determined by the restriction of $(\hat{A}_1 \oplus \hat{A}_2)^*$ on
\[ \text{Dom}(\hat{A}) = \text{Dom}(\hat{A}_1) \oplus \text{Dom}(\hat{A}_2) + \mathcal{L}, \]
and vice versa.

Our main technical result describes the geometry of the deficiency subspaces of the symmetric extensions $\hat{A}$ associated with a two-parameter family of neutral subspaces $\mathcal{L}$. We also explicitly obtain the Livšic function of these symmetric extensions $\hat{A}$ relative to an appropriate self-adjoint extension of $\hat{A}$.

**Theorem 3.1.** Assume that $\hat{A}_k$, $k = 1, 2$, are closed symmetric operators with deficiency indices $(1, 1)$ in the Hilbert spaces $\mathcal{H}_k$, $k = 1, 2$. Suppose that $g_k \in \text{Ker}((\hat{A}_k)^* \mp iI), \|g_k^\pm\| = 1$, $k = 1, 2$.

Introduce the one-dimensional subspace $\mathcal{L} \subset \mathcal{H}_1 \oplus \mathcal{H}_2$ by
\[ \mathcal{L} = \text{lin span} \left\{ (\sin \alpha g_1^+ - \sin \beta g_1^-) \oplus (\cos \alpha g_2^+ - \cos \beta g_2^-) \right\}, \]
\[ \alpha, \beta \in [0, \pi). \]
Then
(i) the linear set $\mathcal{L}$ is a neutral subspace of the quotient space
\[ \text{Dom}((\hat{A}_1 \oplus \hat{A}_2)^*)/\text{Dom}(\hat{A}_1 \oplus \hat{A}_2), \]
(ii) the restriction $\hat{A}$ of the operator $(\hat{A}_1 \oplus \hat{A}_2)^*$ on the domain
\[ \text{Dom}(\hat{A}) = \text{Dom}(\hat{A}_1) \oplus \text{Dom}(\hat{A}_2) \]
is a symmetric operator with deficiency indices $(1,1)$ and the deficiency subspaces of $\hat{A}$ are given by
\[ \text{Ker}((\hat{A}^* \mp iI)) = \text{lin span}\{G_{\pm}\}, \]
where
\[ (3.1) \quad G_+ = \cos \alpha g_1^+ - \sin \beta g_1^{-}\quad \text{and}\quad G_- = \cos \beta g_1^+ - \sin \beta g_2^{-}, \]
\[ \|G_{\pm}\| = 1; \]
(iii) the Livšic function $s=s(\hat{A}, A)$ associated with the pair $(\hat{A}, A)$, where $A$ is a reference self-adjoint extension of $\hat{A}$ such that
\[ G_+ - G_- \in \text{Dom}(A), \]
admits the representation
\[ (3.2) \quad s(z) = \frac{\cos \alpha \cos \beta s_1(z) - s_1(z)s_2(z) + \sin \alpha \sin \beta s_2(z)}{1 - (\sin \alpha \sin \beta s_1(z) + \cos \alpha \cos \beta s_2(z))}, \quad z \in \mathbb{C}_+. \]
Here $s_k = s(\hat{A}_k, A_k)$ are the Livšic functions associated with the pairs $(\hat{A}_k, A_k)$, $k = 1, 2$.

**Proof.** (i). First we note that the element $f \in \mathcal{L} \subset \text{Dom}(\hat{A})$ given by
\[ (3.3) \quad f = (\sin \alpha g_1^+ - \sin \beta g_1^-) + (\cos \alpha g_2^+ - \cos \beta g_2^-) \]
belongs to $\text{Dom}((\hat{A}_1 \oplus \hat{A}_2)^*)$ and that
\[ (3.4) \quad (\hat{A}_1 \oplus \hat{A}_2)^* f = i(\sin \alpha g_1^+ + \sin \beta g_1^- + \cos \alpha g_2^+ + \cos \beta g_2^-). \]
Combining (3.3) and (3.4), one obtains
\[ ((\hat{A}_1 \oplus \hat{A}_2)^* f, f) = i(\sin^2 \alpha - \sin^2 \beta + \cos^2 \alpha - \cos^2 \beta) \]
\[ + i \sin \alpha \sin \beta ((g_1^+, g_1^-) - (g_1^+, g_2^-)) \]
\[ + \cos \alpha \cos \beta ((g_2^+, g_2^-) - (g_2^+, g_2^-)). \]
Hence,
\[ \text{Im}((\hat{A}_1 \oplus \hat{A}_2)^* f, f) = 0, \quad f \in \mathcal{L}, \]
and therefore
\[ \text{Im}((\hat{A}_1 \oplus \hat{A}_2)^* f, f) = 0, \quad f \in \text{Dom}(\hat{A}), \]
which proves that the operator $\hat{A}$ is symmetric and (i) follows.

(ii). Let us show that
\[ \text{Ker}((\hat{A}^* - iI)) = \text{lin span}\{G_+\}. \]
We need to check that
\[ ((\hat{A} + iI) y, G_+) = 0 \quad \text{for all} \quad y \in \text{Dom}(\hat{A}). \]
Take a $y \in \text{Dom}(\hat{A})$. Then $y$ can be decomposed as
\[ y = h_1 + h_2 + Cf, \]
where $h_k \in \text{Dom}(\hat{A}_k)$, $k = 1, 2$, $C \in \mathbb{C}$, and
\[ (3.5) \quad f = (\sin \alpha g_1^+ - \sin \beta g_1^-) + (\cos \alpha g_2^+ - \cos \beta g_2^-) \in \mathcal{L}. \]
Next,
\[(\dot{A} + iI)y, G_+\) = ((\dot{A}_1 + iI)h_1 + (\dot{A}_2 + iI)h_2), G_+) + C((\dot{A} + iI)f, G_+).\]

On the other hand, since \(g_k^1 \in \text{Ker}((\dot{A}_k)^* - iI), k = 1, 2,\)
\[(\dot{A}_1 + iI)h_1 \oplus (\dot{A}_1 + iI)h_2), G_+) = \cos \alpha((\dot{A}_1 + iI)h_1, g_1^1)
- \sin \alpha((\dot{A}_2 + iI)h_2, g_2^1) = 0.\]

Now we can prove that
\[(\dot{A} + iI)f, G_+) = 0, \quad f \in \mathcal{L}.\]

Indeed,
\[(\dot{A} + iI)f = ((\dot{A} + iI)((\sin \alpha g_1^1 - \sin \beta g_1^1) + (\cos \alpha g_2^1 - \cos \beta g_2^1)) = 2i(\sin \alpha g_1^1 + \cos \alpha g_2^1)\]
and since
\[G_+ = \cos \alpha g_1^1 - \sin \alpha g_2^1,\]
we have
\[(\dot{A} + iI)f, G_+) = 2i(\sin \alpha g_1^1 + \cos \alpha g_2^1, \cos \alpha g_1^1 - \sin \alpha g_2^1) = 0.\]

Combining (3.6), (3.7) and (3.8) proves that
\[(\dot{A} + iI)y, G_+) = 0 \quad \text{for all} \quad y \in \text{Dom}(\dot{A}).\]

Therefore,
\[G_+ \in \text{Ker}((\dot{A})^* - iI).\]

In a similar way it follows that \(G_-\) given by
\[(\dot{A} + iI)f, G_-\] = \[\cos \beta g_1^1 - \sin \beta g_2^1\]
generates the deficiency subspace \(\text{Ker}((\dot{A})^* + iI).\)

Since \(\|g_1^1\| = \|g_2^1\| = 1\) and the elements \(g_1^1\) and \(g_2^1\) are orthogonal to each other, (3.9) and (3.11) yield
\[\|(G_\pm)\| = 1.\]

(iii). In order to evaluate the Livšic function associated with the pair \((\dot{A}, A)\), choose nontrivial elements \(g_k^2 \in \text{Ker}((\dot{A}_k)^* - zI), k = 1, 2, z \in \mathbb{C}_+.\)

Suppose that for \(z \in \mathbb{C}_+\) an element \(G_z \neq 0\) belongs to the deficiency subspace \(\text{Ker}((\dot{A})^* - zI).\) Since \(A \subset (\dot{A}_1 \oplus \dot{A}_2)^*\), one gets that
\[G_z = g_1^1 + T(z)g_2^2 \in \text{Ker}((\dot{A}_1 \oplus \dot{A}_2)^* - zI)\]
for some function \(T(z)\) (to be determined later).
Therefore, the Livšic function $s(z) = s(\dot{A}, A)(z)$ associated with the pair $(\dot{A}, A)$ admits the representation

$$(3.12) \quad s(z) = \frac{z - i}{z + i} \left( \frac{G_z, G_-}{G_z, G_+} \right) = \frac{z - i}{z + i} \left( \frac{(g_z^1 + T(z)g_z^2, \cos \beta g_z^1 - \sin \beta g_z^2)}{(g_z^1 + T(z)g_z^2, \cos \alpha g_z^1 - \sin \alpha g_z^2)} \right) = \frac{z - i}{z + i} \left( \frac{\cos \beta(g_z^1, g_z^1) - T(z) \sin \beta(g_z^2, g_z^2)}{\cos \alpha(g_z^1, g_z^1) - T(z) \sin \alpha(g_z^2, g_z^2)} \right).$$

Since $G_z \in \text{Ker}((\dot{A})^* - zI)$ implies that

$$(3.13) \quad (G_z, (\dot{A} - zI)f) = 0,$$

where the element $f \in \mathcal{L}$ is given by (3.5), the equation (3.13) yields the following equation for determining the function $T(z)$:

$$(3.14) \quad \left( g_z^1 + T(z)g_z^2, (\dot{A} - zI) \left( (\sin \alpha g_z^1 - \sin \beta g_z^1) \oplus (\cos \alpha g_z^1 - \cos \beta g_z^1) \right) \right) = 0.$$

Since

$$\dot{A} \subset (\dot{A}_1 + \dot{A}_2)^* \quad \text{and} \quad (\dot{A}_k)^* g_k = \pm ig_k, \quad k = 1, 2,$$

from (3.14) one gets that

$$\begin{align*}
(-i - z) \sin \alpha(g_z^1, g_z^1) - (i - z) \sin \beta(g_z^1, g_z^1) \\
+ T(z) \left[ (-i - z) \cos \alpha(g_z^2, g_z^2) - (i - z) \cos \alpha(g_z^2, g_z^2) \right] = 0.
\end{align*}$$

Solving for $T(z)$, we have

$$(3.15) \quad T(z) = \frac{(-i - z) \sin \alpha(g_z^1, g_z^1) - (i - z) \sin \beta(g_z^1, g_z^1)}{(-i - z) \cos \alpha(g_z^2, g_z^2) - (i - z) \cos \beta(g_z^2, g_z^2)} = \frac{(g_z^1, g_z^1)}{\cos \alpha(g_z^2, g_z^2) - (i - z) \cos \beta(g_z^2, g_z^2)} = \frac{\sin \beta \alpha(g_z^1, g_z^1)}{\cos \alpha(g_z^2, g_z^2) - (i - z) \cos \beta(g_z^2, g_z^2)}.$$

Therefore, taking into account (3.12) and (3.15), one arrives at the repsentation

$$s(z) = \frac{z - i}{z + i} \left( \frac{\cos \beta(\alpha g_z^1, \alpha g_z^1) + \frac{\sin \beta(\alpha g_z^1, \alpha g_z^1)}{\cos \beta(\alpha g_z^1, \alpha g_z^1)} \sin \beta(g_z^2, g_z^2)}{\cos \alpha(g_z^1, g_z^1) + \frac{\sin \beta(\alpha g_z^1, \alpha g_z^1)}{\cos \beta(\alpha g_z^1, \alpha g_z^1)} \sin \alpha(g_z^2, g_z^2)} \right)$$

which, after a direct computation, yields (3.12).

The proof is complete. \hfill \Box

Remark 3.2. A straightforward computation using (3.2) shows that representation (3.2) is a particular case (for $k = 0$) of a more general equality

$$(3.16) \quad s(z) - k = \frac{a_1s_1(z) + a_2s_2(z) - s_1(z)s_2(z) - k}{a_2s_1(z) + a_1s_2(z) - k - s_1(z)s_2(z) - 1}, \quad k \in [0, 1).$$

Here

$$a_1 = \cos \alpha \cos \beta + k \sin \alpha \sin \beta,$$

$$a_2 = \sin \alpha \sin \beta + k \cos \alpha \cos \beta.$$
4. THE ADDITION THEOREM

As the first application of Theorem 3.1 we obtain the following addition theorem for the Weyl-Titchmarsh functions.

**Theorem 4.1 (The Addition Theorem).** Assume the hypotheses of Theorem 3.1 with \( \alpha = \beta \). Suppose that \( \hat{A} \) is the symmetric operator referred to in Theorem 3.1. Then the Weyl-Titchmarsh function \( M \) associated with the pair \((\hat{A}, A_1 \oplus A_2)\) is a convex combination of the Weyl-Titchmarsh functions \( M_k \) associated with the pairs \((\hat{A}_k, A_k)\), \( k = 1, 2 \), which is given by

\[
M(z) = \cos^2 \alpha M_1(z) + \sin^2 \alpha M_2(z), \quad z \in \mathbb{C}_+.
\]

**Proof.** Since by hypothesis \( \alpha = \beta \), one concludes that

\[
G_+ - G_- \in \text{Dom}(A_1 \oplus A_2),
\]

where \( G_\pm \) are the deficiency elements of \( \hat{A} \) from Theorem 3.1 given by (3.1). So, one can apply Theorem 3.1 with the self-adjoint reference operator \( A = A_1 \oplus A_2 \) to conclude that

\[
s(z) = \frac{\cos^2 \alpha (z) - s_1(z)s_2(z) + \sin^2 \alpha s_2(z)}{1 - \left(\sin^2 \alpha s_1(z) + \cos^2 \alpha s_2(z)\right)},
\]

where

\[
s(z) = \frac{M(z) - i}{M(z) + i} \quad \text{and} \quad s_k = \frac{M_k(z) - i}{M_k(z) + i}, \quad k = 1, 2.
\]

Thus, to prove (4.1) it remains to check the equality

\[
\cos^2 \alpha M_1(z) + \sin^2 \alpha M_2(z) - i
\cos^2 \alpha M_1(z) + \sin^2 \alpha M_2(z) + i
\]

\[
= \frac{\cos^2 \alpha \frac{M_1(z) - i}{M_1(z) + i} - \frac{M_1(z) - i}{M_1(z) + i} \frac{M_2(z) - i}{M_2(z) + i} + \sin^2 \alpha \frac{M_2(z) - i}{M_2(z) + i}}{1 - \left(\sin^2 \alpha \frac{M_1(z) - i}{M_1(z) + i} + \cos^2 \alpha \frac{M_2(z) - i}{M_2(z) + i}\right)}
\]

which can be directly verified. \( \square \)

5. AN OPERATOR COUPLING OF DISSIPATIVE OPERATORS

We now introduce the concept of the operator coupling of two dissipative unbounded operators.

**Definition 5.1.** Suppose that \( \hat{A}_1 \in \mathcal{D}(\mathcal{H}_1) \) and \( \hat{A}_2 \in \mathcal{D}(\mathcal{H}_2) \) are maximal dissipative unbounded operators acting in the Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively.

We say that a maximal dissipative operator \( \hat{A} \in \mathcal{D}(\mathcal{H}_1 \oplus \mathcal{H}_2) \) is an operator coupling of \( \hat{A}_1 \) and \( \hat{A}_2 \), in writing,

\[
\hat{A} = \hat{A}_1 \oplus \hat{A}_2,
\]

if

(i) the Hilbert space \( \mathcal{H}_1 \) is invariant for \( \hat{A}_1 \) and the restriction of \( \hat{A} \) on \( \mathcal{H}_1 \) coincides with the dissipative operator \( \hat{A}_1 \), that is,

\[
\text{Dom}(\hat{A}) \cap \mathcal{H}_1 = \text{Dom}(\hat{A}_1),
\]

\[
\hat{A}|_{\mathcal{H}_1 \cap \text{Dom}(\hat{A}_1)} = \hat{A}_1,
\]

(ii) the Hilbert space \( \mathcal{H}_2 \) is invariant for \( \hat{A}_2 \) and the restriction of \( \hat{A} \) on \( \mathcal{H}_2 \) coincides with the dissipative operator \( \hat{A}_2 \), that is,

\[
\text{Dom}(\hat{A}) \cap \mathcal{H}_2 = \text{Dom}(\hat{A}_2),
\]

\[
\hat{A}|_{\mathcal{H}_2 \cap \text{Dom}(\hat{A}_2)} = \hat{A}_2.
\]
and

(ii) the symmetric operator $\hat{A} = \hat{A}|_{\text{Dom}(\hat{A}) \cap \text{Dom}(\hat{A}^*)}$ has the property

$$\hat{A} \subset \hat{A}_1 \oplus (\hat{A}_2)^*.$$

To justify the existence of an operator coupling of two dissipative operators and discuss properties of the concept we proceed with preliminary considerations.

Assume the following hypothesis.

**Hypothesis 5.2.** Suppose that $\hat{A}_1 \in \mathcal{D}(\mathcal{H}_1)$ and $\hat{A}_2 \in \mathcal{D}(\mathcal{H}_2)$ are maximal dissipative unbounded operators acting in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume, in addition, that

$$\hat{A}_j = \hat{A}_j|_{\text{Dom}(\hat{A}_j) \cap \text{Dom}(\hat{A}_j^*)}, \quad j = 1, 2,$$

are the corresponding underlying symmetric operators.

First we show that under Hypothesis 5.2 the following extension problem

with a constraint admits a one-parameter family of solutions.

This problem is:

Find a closed symmetric operator $\hat{A}$ with deficiency indices $(1, 1)$ such that

$$\hat{A}_1 \oplus \hat{A}_2 \subset \hat{A} \quad \text{and} \quad \hat{A} \subset \hat{A}_1 \oplus (\hat{A}_2)^*.$$

The lemma below justifies the solvability of the extension problem with a constraint.

**Lemma 5.3.** Assume Hypothesis 5.2. Then

(i) there exists a one parameter family $[0, 2\pi) \ni \theta \mapsto \hat{A}_\theta$ of symmetric restrictions with deficiency indices $(1, 1)$ of the operator $(\hat{A}_1 \oplus \hat{A}_2)^*$ such that

$$\hat{A}_1 \oplus \hat{A}_2 \subset \hat{A}_\theta \subset \hat{A}_1 \oplus (\hat{A}_2)^*, \quad \theta \in [0, 2\pi);$$

(ii) if $\hat{A}$ is a closed symmetric operator with deficiency indices $(1, 1)$ such that

$$\hat{A}_1 \oplus \hat{A}_2 \subset \hat{A} \subset \hat{A}_1 \oplus (\hat{A}_2)^*,$$

then there exists a $\theta \in [0, 2\pi)$ such that

$$\hat{A} = \hat{A}_\theta.$$

**Proof.** First, introduce the notation. Let $\kappa_j$, $0 \leq \kappa_j < 1$, $j = 1, 2$, stand for the absolute value of the von Neumann parameter of $\hat{A}_j$,

$$\kappa_j = \hat{\kappa}(\hat{A}_j), \quad j = 1, 2.$$

Fix a basis $g^\perp_j \in \text{Ker}((\hat{A}_j)^* \mp iI)$, $\|g^\perp_j\| = 1$, $j = 1, 2$, in the corresponding deficiency subspaces such that

$$g^\perp_j - \kappa_j g^\perp_j \in \text{Dom}(\hat{A}_j), \quad j = 1, 2.$$

(i). To show that there exists at least one symmetric extensions $\hat{A}_0$ with deficiency indices $(1, 1)$ of $\hat{A}_1 \oplus \hat{A}_2$ such that

$$\hat{A}_0 \subset \hat{A}_1 \oplus (\hat{A}_2)^*,$$
suppose that \( \alpha, \beta \in [0, \frac{\pi}{2}] \) are chosen in such a way that

\[
\alpha = \begin{cases} 
\arctan \frac{1}{\kappa_2} \sqrt{\frac{1-\kappa_2^2}{1-\kappa_1^2}}, & \text{if } \kappa_2 \neq 0 \\
\frac{\pi}{2}, & \text{if } \kappa_2 = 0
\end{cases}
\]  

and

\[
\beta = \begin{cases} 
\arctan(\kappa_1 \kappa_2 \tan \alpha), & \text{if } \kappa_2 \neq 0 \\
\frac{\pi}{2} \sqrt{1-\kappa_1^2}, & \text{if } \kappa_2 = 0
\end{cases}
\]

By Theorem 3.1 (i), the one-dimensional subspace

\[(5.4) \quad \mathcal{L}_0 = \text{lin span} \left\{ \sin \alpha g_1^1 - \sin \beta g_1^- \oplus (\cos \alpha g_2^1 - \cos \beta g_2^-) \right\}
\]

is a neutral subspace of the quotient space

\[
\text{Dom}((\hat{A}_1 \oplus \hat{A}_2)^*)/\text{Dom}(\hat{A}_1 \oplus \hat{A}_2).
\]

By Theorem 3.1 (ii), the restriction \( \hat{A}_0 \) of the operator \( (\hat{A}_1 \oplus \hat{A}_2)^* \) on the domain

\[(5.5) \quad \text{Dom}(\hat{A}_0) = \text{Dom}(\hat{A}_1 \oplus \hat{A}_2) + \mathcal{L}_0
\]

is a symmetric operator with deficiency indices \((1,1)\).

Taking into account the relations (see \[5.2\], \[5.3\])

\[
\sin \beta = \kappa_1 \sin \alpha \quad \text{and} \quad \cos \beta = \frac{1}{\kappa_2} \cos \alpha, \quad \kappa_2 \neq 0,
\]

and

\[
\sin \beta = \kappa_1 \quad \text{and} \quad \cos \beta = \sqrt{1-\kappa_1^2}, \quad \kappa_2 = 0,
\]

from \[5.3\] one obtains that the subspace \( \mathcal{L}_0 \) admits the representation

\[
\mathcal{L}_0 = \begin{cases} 
\text{lin span} \left\{ \sin \alpha (g_1^1 - \kappa_1 g_1^-) \oplus \cos \alpha \left(g_2^1 - \frac{1}{\kappa_2} g_2^-\right) \right\}, & \text{if } \kappa_2 \neq 0, \\
\text{lin span} \left\{ (g_1^1 - \kappa_1 g_1^-) \oplus \left(-\sqrt{1-\kappa_1^2} g_2^-\right) \right\}, & \text{if } \kappa_2 = 0.
\end{cases}
\]

It follows that

\[
\mathcal{L}_0 \subset \text{Dom}((\hat{A}_1 \oplus (\hat{A}_2)^*)).
\]

From \[5.5\] one concludes that the symmetric operator \( \hat{A}_0 \) has the property

\[(5.6) \quad \hat{A}_0 \subset (\hat{A}_1 \oplus (\hat{A}_2)^*)^*.
\]

Clearly, for any \( \theta \in [0, 2\pi) \) the subspace

\[
\mathcal{L}_\theta = \begin{cases} 
\text{lin span} \left\{ e^{i\theta} \sin \alpha (g_1^1 - \kappa_1 g_1^-) \oplus \cos \alpha \left(g_2^1 - \frac{1}{\kappa_2} g_2^-\right) \right\}, & \text{if } \kappa_2 \neq 0, \\
\text{lin span} \left\{ e^{i\theta} (g_1^1 - \kappa_1 g_1^-) \oplus \left(-\sqrt{1-\kappa_1^2} g_2^-\right) \right\}, & \text{if } \kappa_2 = 0.
\end{cases}
\]

is also a neutral subspace of the quotient space

\[
\text{Dom}((\hat{A}_1 \oplus \hat{A}_2)^*)/\text{Dom}(\hat{A}_1 \oplus \hat{A}_2).
\]

Therefore, the symmetric operator \( \hat{A}_\theta \) defined as the restrictions of \( (\hat{A}_1 \oplus \hat{A}_2)^* \) on

\[
\text{Dom}(\hat{A}_\theta) = \text{Dom}(\hat{A}_1 \oplus \hat{A}_2) + \mathcal{L}_\theta, \quad \theta \in [0, 2\pi),
\]

has deficiency indices \((1,1)\) and

\[
(\hat{A}_1 \oplus \hat{A}_2) \subset \hat{A}_\theta \subset (\hat{A}_1 \oplus (\hat{A}_2)^*) \subset (\hat{A}_1 \oplus \hat{A}_2)^*, \quad \theta \in [0, 2\pi),
\]

proving the claim (i).
(ii). Introduce the elements

\( f^1 = g_1^1 - \kappa_1 g_1^- \in \text{Dom}(\hat{A}_1) \subset \mathcal{H}_1 \)

and

\( f^2 = g_2^1 - \kappa_2^{-1} g_2^- \in \text{Dom}((\hat{A}_2)^*) \subset \mathcal{H}_2 \quad (\kappa_2 \neq 0). \)

If \( \kappa_2 = 0 \), then we take

\( f^2 = -\sqrt{1 - \kappa_1^2} g_2^- \in \text{Dom}((\hat{A}_2)^*) \subset \mathcal{H}_2. \)

A simple computation shows that

\[ \text{Im}(\hat{A}_1 f^1, f^1) = (1 - \kappa_1^2) > 0 \]

and that

\[ \text{Im}((\hat{A}_2)^* f^2, f^2) = \begin{cases} 
1 - \kappa_2^{-2}, & \text{if } \kappa_2 \neq 0 \\
\kappa_2^2 - 1, & \text{if } \kappa_2 = 0
\end{cases}. \]

Hence, \( \text{Im}((\hat{A}_2)^* f^2, f^2) < 0 \). Therefore, if \( f = af^1 + bf^2, a, b \in \mathbb{C} \), then

\[ \text{Im}((\hat{A}_1 \oplus \hat{A}_2)^* f, f) = \begin{cases} 
|a|^2(1 - \kappa_1^2) + |b|^2(1 - \kappa_2^{-2}), & \kappa_2 \neq 0 \\
|a|^2(1 - \kappa_1^2) - |b|^2(1 - \kappa_2^2), & \kappa_2 = 0
\end{cases}. \]

This means that a one-dimensional subspace

\( \mathcal{L} \subset \text{lin span}\{f^1, f^2\} \subset \text{Dom}(\hat{A}_1) \oplus \text{Dom}((\hat{A}_2)^*) \)

is a neutral (Lagrangian) subspace for the symplectic form

\[ \omega(h, g) = ((\hat{A}_1 \oplus \hat{A}_2)^* h, g) - (h, (\hat{A}_1 \oplus \hat{A}_2)^* g), \quad h, g \in \text{Dom}((\hat{A}_1 \oplus \hat{A}_2)^*) \]

if and only if \( \mathcal{L} \) admits the representation

\[ \mathcal{L} = \text{lin span}\{e^{i\theta} \sin \alpha f^1 \oplus \cos \alpha f^2\}, \]

for some \( \theta \in [0, 2\pi) \) where

\[ \tan \alpha = \frac{\kappa_2^{-2} - 1}{1 - \kappa_1^2} = \frac{1}{\kappa_2} \sqrt{1 - \kappa_1^2}, \]

if \( \kappa_2 \neq 0 \), and

\[ \mathcal{L} = \text{lin span}\{e^{i\theta} f^1 \oplus f^2\}, \]

if \( \kappa_2 = 0 \).

Taking into account (5.8)–(5.10) and comparing (5.13) and (5.14) with (5.7), one concludes that

\[ \mathcal{L} = \mathcal{L}_\theta. \]

By hypothesis (ii), \( \hat{A} \) is a closed symmetric operator with deficiency indices \((1, 1)\) and

\[ \hat{A}_1 \oplus \hat{A}_2 \subset \hat{A} \subset \hat{A}_1 \oplus (\hat{A}_2)^*. \]

Therefore, the subspace

\[ \text{Dom}(\hat{A}) \cap \text{Dom}(\hat{A}_1 \oplus (\hat{A}_2)^*) \]

is a neutral subspace. Hence, by (5.15),

\[ \text{Dom}(\hat{A}) = \text{Dom}(\hat{A}_1 \oplus \hat{A}_2) + \mathcal{L}_\theta \quad \text{for some} \quad \theta \in [0, 2\pi) \]
which means that
\[ \dot{A} = \dot{A}_\theta \]
proving the claim (ii).

The proof is complete. \[ \square \]

Our next result, on the one hand, shows that given a solution \( \dot{A} \) of the extension problem with a constraint \( \text{5.1} \), there exists a unique operator coupling \( \hat{A}_1 \oplus \hat{A}_2 \) of \( \hat{A}_1 \) and \( \hat{A}_2 \) such that
\[ \dot{A} \subset \hat{A}_1 \oplus \hat{A}_2. \]

On the other hand, this result justifies that the functional
\[ \tilde{\kappa} : \mathcal{D} \to [0, 1) \]
introduced in subsection \[ \text{2.5} \] is multiplicative with respect to the operator coupling operation.

**Theorem 5.4 (Multiplicativity of the extension parameter).** Assume Hypothesis \[ \text{5.2} \]. Suppose, in addition, that \( \dot{A} \) is a solution of the extension problem with a constraint \( \text{5.1} \).

Then
\begin{enumerate}[(i)]  
  \item there exists a unique operator coupling \( \hat{A} = \hat{A}_1 \oplus \hat{A}_2 \in \mathcal{D}(\mathcal{H}_1 \oplus \mathcal{H}_2) \) such that
  \[ \hat{A}|_{\text{Dom}(\hat{A}) \cap \text{Dom}(\hat{A}^*)} = \dot{A}; \]
  
  \item for any operator coupling \( \hat{A} \) of \( \hat{A}_1 \) and \( \hat{A}_2 \), the multiplication rule
  \[ \hat{\kappa}(\hat{A}) = \hat{\kappa}(\hat{A}_1) \cdot \hat{\kappa}(\hat{A}_2) \]
  holds. Here \( \hat{\kappa}(\cdot) \) stands for the absolute value of the von Neumann parameter of a dissipative operator.
\end{enumerate}

**Proof.** (i). As in the proof of Lemma \[ \text{5.3} \] start with a basis \( g_\pm^j \in \text{Ker}(\dot{A}_j^* \mp iI), \parallel g_\pm^j \parallel = 1, \ j = 1, 2, \) in the corresponding deficiency subspaces such that
\[ g_\pm^j - \kappa_j g_\mp^j \in \text{Dom}(\dot{A}_j), \ j = 1, 2, \]
where \( \kappa_j \) stands for the absolute value of the von Neumann parameter of \( \dot{A}_j \),
\[ \kappa_j = \tilde{\kappa}(\dot{A}_j), \ j = 1, 2. \]

By Lemma \[ \text{5.3} \] the domain of \( \dot{A} \) admits the representation
\[ \text{Dom}(\dot{A}) = \text{Dom}(\dot{A}_1 \oplus \dot{A}_2) + \mathcal{L}_\theta, \]
where
\[ \mathcal{L}_\theta = \begin{cases} 
\text{lin span} \left\{ e^{i\theta} \sin \alpha \left( g_\pm^1 - \kappa_1 g_\mp^1 \right) \oplus \cos \alpha \left( g_\mp^2 + \frac{1}{\kappa_2} g_\pm^2 \right) \right\}, & \text{if } \kappa_2 \neq 0 \\
\text{lin span} \left\{ e^{i\theta} \left( g_\mp^1 - \kappa_1 g_\pm^1 \right) \oplus \left( -\sqrt{1 - \kappa_1^2} g_\pm^2 \right) \right\}, & \text{if } \kappa_2 = 0 
\end{cases} \]
and
\[ \tan \alpha = \frac{1}{\kappa_2} \sqrt{\frac{1 - \kappa_2^2}{1 - \kappa_1^2}}, \ k_2 \neq 0. \]

Without loss one may assume that \( \theta = 0 \). Indeed, instead of taking the basis \( g_\pm^1 \in \text{Ker}(\dot{A}_1^* \mp iI) \), one can start with the basis \( e^{i\theta} g_\pm^1 \in \text{Ker}(\dot{A}_1^* \mp iI) \) without
changing the von Neumann extension parameter $\kappa_1$ that characterizes the domain of $\hat{A}_1$ (see eq. (5.17)).

Taking into account the relations

$$\sin \beta = \kappa_1 \sin \alpha \quad \text{and} \quad \cos \beta = \frac{1}{\kappa_2} \cos \alpha, \quad \text{if } \kappa_2 \neq 0,$$

and

$$\sin \beta = \kappa_1 \quad \text{and} \quad \cos \beta = \sqrt{1 - \kappa_2^2}, \quad \text{if } \kappa_2 = 0,$$

it is easy to see that

$$\mathcal{L}_0 = \text{lin span} \{ (\sin \alpha g_1^+ - \sin \beta g_1^+) \oplus (\cos \alpha g_2^+ - \cos \beta g_2^+) \}.$$

In accordance with Theorem 3.1, introduce the maximal dissipative extension $\hat{A}$ of $\hat{\mathcal{A}}$ as the restriction of $(\hat{\mathcal{A}}_1 \oplus \hat{\mathcal{A}}_2)^*$ on

$$\text{Dom}(\hat{A}) = \text{Dom}(\hat{\mathcal{A}}) + \text{lin span} \{ G_+ - \kappa_1 \kappa_2 G_- \},$$

where the deficiency elements $G_\pm$ of $\hat{\mathcal{A}}$ are given by (3.1). That is,

$$G_+ = \cos \alpha g_1^+ - \sin \alpha g_2^+,$$

$$G_- = \cos \beta g_1^- - \sin \beta g_2^-.$$

By construction,

$$\hat{\mathcal{A}} = \hat{\mathcal{A}}|_{\text{Dom}(\hat{\mathcal{A}}_1) \cap \text{Dom}(\hat{\mathcal{A}}_2)^*}.$$

Clearly,

$$G_+ - \kappa_1 \kappa_2 G_- = (\cos \alpha g_1^+ - \kappa_1 \kappa_2 \cos \beta g_1^+) \oplus (- \sin \alpha g_2^+ + \kappa_1 \kappa_2 \sin \beta g_2^-)$$

$$= \begin{cases} 
\cos \alpha (g_1^+ - \kappa_1 g_1^-) \oplus (- \sin \alpha) (g_2^+ - \kappa_1 g_2^-), & \text{if } \kappa_2 \neq 0 \\
0 \oplus (-g_2^+), & \text{if } \kappa_2 = 0
\end{cases}.$$}

Therefore,

$$\text{Proj}_{\mathcal{H}_1}(G_+ - \kappa_1 \kappa_2 G_-) \in \text{Dom}(\hat{\mathcal{A}}_1),$$

where $\text{Proj}_{\mathcal{H}_1}$ denotes the orthogonal projection of $\mathcal{H}_1 \oplus \mathcal{H}_2$ onto $\mathcal{H}_1$. Hence, the subspace $\mathcal{H}_1$ is invariant for the dissipative operator $\hat{\mathcal{A}}$ and

$$\hat{\mathcal{A}}|_{\mathcal{H}_1 \cap \text{Dom}(\hat{\mathcal{A}})} = \hat{\mathcal{A}}_1.$$

Combining (5.6), (5.22) and (5.23) shows that the dissipative extension $\hat{A}$ is an operator coupling of $\hat{A}_1$ and $\hat{A}_2$ such that $\hat{A} \subset \hat{A}_1 \oplus \hat{A}_2$, which proves the existence part of the assertion.

To prove the uniqueness of the operator coupling $\hat{A}$ extending $\hat{\mathcal{A}}$ and satisfying the property (5.23), one observes that since $\hat{A} \in \mathcal{D}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, there exists some $|\kappa| < 1$ such that

$$\text{Dom}(\hat{A}) = \text{Dom}(\hat{\mathcal{A}}) + \text{lin span} \{ G_+ - \kappa G_- \}.$$

In particular,

$$G_+ - \kappa G_- \in \text{Dom}(\hat{A}).$$

If $\kappa_2 \neq 0$, from (5.23) it follows that (5.24) holds if and only if

$$\text{Proj}_{\mathcal{H}_1}(G_+ - \kappa G_-) = \cos \alpha g_1^+ - \kappa \cos \beta g_1^- = \cos \alpha \left( g_1^+ - \frac{\kappa}{\kappa_2} g_1^- \right) \in \text{Dom}(\hat{\mathcal{A}}_1)$$

$$= \hat{\mathcal{A}}_1.$$
THE ADDITION AND MULTIPLICATION THEOREMS

which is only possible if

\[ \frac{\kappa}{\kappa_2} = \kappa_1. \]

If \( \kappa_2 = 0 \), and therefore in this case \( G_+ = -g_+^2 \) (see (5.24) with \( \alpha = \frac{\pi}{2} \)), one computes

\[ \text{Proj}_{H_1}(G_+ - \kappa G_-) = -\kappa \cos \beta g_1 \in \text{Dom}(\hat{A}), \]

and hence (5.23) and (5.24) hold if and only if \( \kappa = \kappa_2 = 0 \). In particular, we have shown that in either case

(5.25)

\[ \kappa = \kappa_1 \kappa_2. \]

(ii). By definition of the von Neumann parameter associated with a pair of operators, equality (5.25) means that

\[ \kappa(\hat{A}, A) = \kappa(\hat{A}_1, A_1) \cdot \kappa(\hat{A}_2, A_2), \]

where \( A \) and \( A_j, j = 1, 2 \), are self-adjoint reference extensions of \( \hat{A} \) and \( \hat{A}_j, j = 1, 2 \), such that

\[ G_+ - G_- \in \text{Dom}(A) \]

and

\[ g^j_+ - g^j_- \in \text{Dom}(A_j), \quad j = 1, 2, \]

which proves the remaining assertion (5.10).

The proof is complete. \( \square \)

6. THE MULTIPLICATION THEOREM

Now, we are ready to state the central result of this paper.

**Theorem 6.1 (The Multiplication Theorem).** Suppose that \( \hat{A} = \hat{A}_1 \oplus \hat{A}_2 \) is an operator coupling of two maximal dissipative operators \( \hat{A}_k \in \mathcal{D}(H_k), k = 1, 2 \). Denote by \( \hat{A}, \hat{A}_1 \) and \( \hat{A}_2 \) the corresponding underlying symmetric operators with deficiency indices \((1, 1)\), respectively. That is,

\[ \hat{A} = \hat{A}|_{\text{Dom}(\hat{A}) \cap \text{Dom}(\hat{A}^*)} \]

and

\[ \hat{A}_k = \hat{A}_k|_{\text{Dom}(\hat{A}_k) \cap \text{Dom}(\hat{A}_k^*)}, \quad k = 1, 2. \]

Then there exist self-adjoint reference operators \( A, A_1 \), and \( A_2 \), extending \( \hat{A}, \hat{A}_1 \) and \( \hat{A}_2 \), respectively, such that

(6.1)

\[ S(\hat{A}_1 \oplus \hat{A}_2, A) = S(\hat{A}_1, A_1) \cdot S(\hat{A}_2, A_2). \]

**Proof.** As in the proof of Theorem 5.4, one can always find a basis

\[ g^j_\pm \in \text{Ker}((\hat{A}_j)^* \mp iI), \quad ||g^j_\pm|| = 1, \quad j = 1, 2, \]

such that

\[ g^j_+ - \kappa_j g^j_- \in \text{Dom}(\hat{A}_j), \quad \text{with} \quad \kappa_j = \tilde{\kappa}(\hat{A}_j), \quad j = 1, 2, \]

and that

\[ \text{Dom}(\hat{A}) = \text{Dom}(\hat{A}_1 \oplus \hat{A}_2) + L_0. \]

Here

\[ L_0 = \text{lin span} \{ (\sin \alpha g^1_+ - \sin \beta g^1_-) \oplus (\cos \alpha g^2_+ - \cos \beta g^2_-) \} \]
and
\begin{equation}
\alpha = \arctan \frac{1}{\kappa_2} \sqrt{\frac{1 - \kappa_2^2}{1 - \kappa_1^2}} \left( \alpha = \frac{\pi}{2} \text{ if } \kappa_2 = 0 \right),
\end{equation}

\begin{equation}
\sin \beta = \kappa_1 \begin{cases} \sin \alpha, & \text{if } \kappa_2 \neq 0, \\ 1, & \text{if } \kappa_2 = 0, \end{cases}
\end{equation}

\begin{equation}
\cos \beta = \begin{cases} \frac{1}{\kappa_2} \cos \alpha, & \text{if } \kappa_2 \neq 0, \\ \sqrt{1 - \kappa_1^2}, & \text{if } \kappa_2 = 0. \end{cases}
\end{equation}

By Theorem 3.1, the deficiency elements \( G_\pm \) of \( A \) are given by
\begin{equation}
G_+ = \cos \alpha g_+^1 - \sin \alpha g_+^2, \\
G_- = \cos \beta g_-^1 - \sin \beta g_-^2.
\end{equation}

Introducing self-adjoint reference extensions \( A \) and \( A_j, j = 1, 2, \) of the symmetric operators \( \hat{A} \) and \( \hat{A}_j, j = 1, 2, \) such that
\( G_+ - G_- \in \text{Dom}(A) \) and \( g_+^j - g_-^j \in \text{Dom}(A_j), j = 1, 2, \)

one can apply Theorem 3.1 to conclude that the Livšic function of \( \hat{A} \) relative to \( A \) admits the representation
\begin{equation}
s(z) = s(\hat{A}, A)(z) = \frac{\cos \alpha \cos \beta s_1(z) - s_1(z) s_2(z) + \sin \alpha \sin \beta s_2(z)}{1 - (\sin \alpha \sin \beta s_1(z) + \cos \alpha \cos \beta s_2(z))}.
\end{equation}

Here
\( s_k(z) = s(\hat{A}_k, A_k) \)
are the Livšic functions associated with the pairs \( (\hat{A}_k, A_k), k = 1, 2. \)

Denote the operator coupling \( \hat{A}_1 \uplus \hat{A}_2 \) by \( \hat{A}. \) By Theorem 5.3
\begin{equation}
G_+ - \kappa_1 \kappa_2 G_- \in \text{Dom}(\hat{A}).
\end{equation}

Therefore, from (6.7) it follows that the characteristic function \( S(\hat{A}, A) \) of the dissipative extension \( \hat{A} \) relative to the reference self-adjoint operator \( A \) has the form
\begin{equation}
S(\hat{A}, A)(z) = \frac{s(z) - \kappa_1 \kappa_2}{\kappa_1 \kappa_2 s(z) - 1}.
\end{equation}

By Remark 3.2 with \( \kappa = \kappa_1 \kappa_2, \) one gets that
\[ \frac{s(z) - \kappa_1 \kappa_2}{\kappa_1 \kappa_2 s(z) - 1} = \frac{a_1 s_1(z) + a_2 s_2(z) - s_1(z) s_2(z) - \kappa_1 \kappa_2}{a_2 s_1(z) + a_1 s_2(z) - \kappa_1 \kappa_2 s_1(z) s_2(z) - 1}, \]
where
\[ a_1 = \cos \alpha \cos \beta + \kappa_1 \kappa_2 \sin \alpha \sin \beta, \]
\[ a_2 = \sin \alpha \sin \beta + \kappa_1 \kappa_2 \cos \alpha \cos \beta. \]
From the relations (6.2), (6.3) and (6.4) it follows that \( a_1 = \kappa_2 \) and \( a_2 = \kappa_1 \) and hence
\begin{equation}
\frac{s(z) - \kappa_1 \kappa_2}{\kappa_1 \kappa_2 s(z) - 1} = \frac{\kappa_2 s_1(z) + \kappa_2 s_2(z) - s_1(z) s_2(z) - \kappa_1 \kappa_2}{\kappa_1 s_1(z) + \kappa_2 s_2(z) - \kappa_1 \kappa_2 s_1(z) s_2(z) - 1}
= \frac{s_1(z) - \kappa_1}{\kappa_1 s_1(z) - 1} \frac{s_2(z) - \kappa_2}{\kappa_2 s_2(z) - 1}.
\end{equation}
Thus,

\[
S(\hat{A}, A)(z) = S(\hat{A}_1, A_1)(z) \cdot S(\hat{A}_2, A_2)(z), \quad z \in \mathbb{C}_+.
\]

The proof is complete. \(\square\)

The following example illustrates the Multiplication Theorem 6.1 for a differentiation operator on a finite interval.

**Example 6.2.** For a finite interval \(\delta = [\alpha, \beta]\), denote by \(\hat{D}_\delta\) the first order differentiation operator in the Hilbert space \(L^2(\delta)\) given by the differential expression

\[
\tau = -\frac{1}{i} \frac{d}{dx}
\]
on

\[
\text{Dom}(\hat{D}_\delta) = \{ f \in W^1_2((\alpha, \beta)), \ f(\alpha) = 0 \}.
\]

It is easy to see that if \(\gamma \in (\alpha, \beta)\), and therefore \(\delta = \delta_1 \cup \delta_2\), with \(\delta_1 = [\alpha, \gamma]\) and \(\delta_2 = [\gamma, \beta]\), then

\[
\hat{D}_\delta = \hat{D}_{\delta_1} \cup \hat{D}_{\delta_2} = \hat{D}_{\delta_1} \oplus \hat{D}_{\delta_2},
\]

where \(\hat{D}_{\delta_1} \oplus \hat{D}_{\delta_2}\) stands for the dissipative operator coupling of \(\hat{D}_{\delta_1}\) and \(\hat{D}_{\delta_2}\).

Indeed, by construction, \(\hat{D}_\delta\) is a maximal dissipative extension of \(\hat{D}_{\delta_1}\) outgoing from the Hilbert space \(\mathcal{H}_1 = L^2(\delta_1)\) to the Hilbert space \(\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 = L^2(\delta)\), where \(\mathcal{H}_2 = L^2(\delta_2)\). Moreover, since

\[
\text{Dom}((\hat{D}_{\delta})^*) = \{ f \in W^1_2((\alpha, \beta)), \ f(\alpha) = f(\beta) = 0 \},
\]

the restriction \(\hat{D}_\delta\) of \(\hat{D}_{\delta}^0\) on

\[
\text{Dom}(\hat{D}_\delta) = \text{Dom}(\hat{D}_\delta) \cap \text{Dom}((\hat{D}_{\delta})^*)
\]
is a symmetric operator with deficiency indices \((1, 1)\) given by the same differential expression \(\tau\) on

\[
\text{Dom}(\hat{D}_\delta) = \{ f \in W^1_2((\alpha, \beta)), \ f(\alpha) = f(\beta) = 0 \}.
\]

On the other hand,

\[
\text{Dom}((\hat{D}_{\delta_2})^*) = \{ f \in W^1_2((\gamma, \beta)), \ f(\beta) = 0 \}.
\]

Therefore,

\[
\hat{D}_\delta \subset \hat{D}_{\delta_1} \oplus (\hat{D}_{\delta_2})^*.
\]

Combining (6.11) and (6.12) shows that \(\hat{D}_\delta\) coincides with the dissipative operator coupling of \(\hat{D}_{\delta_1}\) and \(\hat{D}_{\delta_2}\). That is, (6.10) holds.

By Lemma A.1 (see Appendix A), the Livšic function associated with the pair \((\hat{D}_\delta, D_\delta)\) is of the form

\[
S(\hat{D}_\delta, D_\delta)(z) = \exp(i|\delta|z), \quad z \in \mathbb{C}_+,
\]

where \(| \cdot |\) stands for Lebesgue measure of a Borel set and \(D_\delta\) is the self-adjoint reference differentiation operator with antiperiodic boundary conditions defined on

\[
\text{Dom}(D_\delta) = \{ f \in W^1_2((\alpha, \beta)), \ f(\alpha) = -f(\beta) \}.
\]
Therefore, taking into account that
\[ \exp(i|\delta|z) = \exp(i(|\delta_1| + |\delta_2|)z) = \exp(i|\delta_1|z) \cdot \exp(i|\delta_2|z), \]
one obtains that
\[ S(\hat{D}_\delta, D_\delta)(z) = S(\hat{D}_{\delta_1}, D_{\delta_1})(z) \cdot S(\hat{D}_{\delta_2}, D_{\delta_2})(z), \]
which illustrates the statement of Theorem 6.1.

We conclude this section by the following purely analytic result.

**Theorem 6.3.** Let \( \mathcal{M}, \mathcal{C}, \) and \( \mathcal{S} \) be the function classes of Weyl-Titchmarsh, Livšic, and characteristic functions, respectively. Then,

(i) The class \( \mathcal{M} \) is a convex set with respect to addition;
(ii) The class \( \mathcal{S} \) is closed under multiplication,
\[ \mathcal{S} \cdot \mathcal{S} \subset \mathcal{S}; \]
(iii) The subclass \( \mathcal{C} \subset \mathcal{S} \) is a (double sided) ideal under multiplication in the sense that
\[ \mathcal{C} \cdot \mathcal{S} = \mathcal{S} \cdot \mathcal{C} \subset \mathcal{C}; \]
(iv) The class \( \mathcal{C} \) is closed under multiplication:
\[ \mathcal{C} \cdot \mathcal{C} \subset \mathcal{C}. \]

**Proof.** One notices that (i) is a corollary of Theorem 4.1, (ii) follows from Theorem 6.1, and (iv) follows from (iii). Therefore, it remains to prove (iii).

(iii). Suppose that \( S_1 \in \mathcal{C} \) and \( S_2 \in \mathcal{S} \). Since \( \mathcal{C} \subset \mathcal{S} \) and \( \hat{\mathcal{O}} = \ker \hat{\kappa} \), \( S_1 \) is the characteristic function of a dissipative operator \( \hat{A}_1 \) from \( \hat{\mathcal{O}} \) (see (2.19)) relative to some self-adjoint reference operator \( A_1 \). Since \( S_2 \in \mathcal{S} \), the function \( S_2 \) is the characteristic function of a dissipative operator \( \hat{A}_2 \in \hat{\mathcal{O}} \) relative to some self-adjoint reference operator \( A_2 \). By Theorem 6.1, the product \( S_1 \cdot S_2 \) is the characteristic function of an operator coupling \( \hat{A}_1 \uplus \hat{A}_2 \) relative to an appropriate reference self-adjoint operator. Since \( S_1 \in \mathcal{C} \), and therefore \( \hat{\kappa}(\hat{A}_1) = 0 \) and hence the product \( S_1 \cdot S_2 \) belongs to the class \( \mathcal{C} \). \( \square \)

**Remark 6.4.** Recall that the subclass \( \hat{\mathcal{O}} \) of \( \hat{\mathcal{O}} \) has been defined as the set of all dissipative operators form \( \hat{\mathcal{O}} \) with zero value of the corresponding von Neumann parameter (see (2.19)). To express this in a different way, the characteristic functions for the operators from \( \hat{\mathcal{O}} \) are exactly those that belong the class \( \mathcal{C} \).

Having this in mind, a non-commutative version of the “absorption principle” (iii) can be formulated as follows.

Suppose that \( \hat{A} \in \hat{\mathcal{O}}(\mathcal{H}_1) \subset \hat{\mathcal{O}}(\mathcal{H}_1) \) and \( \hat{B} \in \hat{\mathcal{O}}(\mathcal{H}_2) \). Then
\[ \hat{A} \uplus \hat{B} \in \hat{\mathcal{O}}(\mathcal{H}_1 \oplus \mathcal{H}_2) \]
and
\[ \hat{B} \uplus \hat{A} \in \hat{\mathcal{O}}(\mathcal{H}_2 \oplus \mathcal{H}_1). \]
Appendix A. The differentiation on a finite interval

In this Appendix we collect some known results, see, e.g., [2], regarding the maximal and minimal differentiation operators on a finite interval. Here we present them in a version adapted to the notation of the current paper.

Lemma A.1. Let $\hat{D}$ be the first order differentiation operator in the Hilbert space $L^2(0, \ell)$ given by the differential expression

$$\tau = -\frac{1}{i} \frac{d}{dx}$$

on $\text{Dom}(\hat{D}) = \{ f \in W^1_2((0, \ell)), \ f(0) = 0 \}$

and $D$ the self-adjoint realization of $\tau$ on $\text{Dom}(D) = \{ f \in W^1_2((0, \ell)), \ f(\ell) = -f(0) \}$.

Then

(i) the restriction $\dot{D}$ of the operator $\hat{D}$ on

$$\text{Dom}(\dot{D}) = \text{Dom}(\hat{D}) \cap \text{Dom}(\hat{D}^*)$$

is a symmetric operator with deficiency indices $(1, 1)$;

(ii) the Livšic function $s = s(\dot{D}, D)$ of the symmetric operator $\dot{D}$ relative to the self-adjoint reference operator $D$ is of the form

$$s(z) = \frac{e^{iz} - e^{-z}}{e^{-z}e^{iz} - 1};$$

(iii) the von Neumann parameter $\kappa(\dot{D}, \hat{D})$ associated with the pair $(\dot{D}, \hat{D})$ is given by

$$\kappa(\dot{D}, \hat{D}) = e^{-\ell};$$

(iv) the characteristic function $S = S(\hat{D}, D)$ of the dissipative operator $\hat{D}$ relative to $D$ is an inner singular function given by

$$S(z) = e^{iz}, \quad z \in \mathbb{C}_+.$$ 

Proof. It is straightforward to conclude that

$$\text{Dom}(\hat{D}^*) = \{ f \in W^1_2((0, \ell)), \ f(\ell) = 0 \}$$

and therefore

$$\text{Dom}(\dot{D}) = \text{Dom}(\hat{D}) \cap \text{Dom}(\hat{D}^*) = \{ f \in W^1_2((0, \ell)), \ f(0) = f(\ell) = 0 \};$$

Clearly, $\text{Ker}((\dot{D})^* - zI) = \text{lin span}\{g_z\}$, where

$$g_z(x) = e^{-izx}, \quad x \in [0, \ell], \quad z \in \mathbb{C},$$

which proves (i).

To compute the Livšic function, one observes that $\text{Ker}((\dot{D})^* \mp iI) = \text{lin span}\{g_{\pm}\}$, where

$$g_+(x) = \frac{\sqrt{2}}{\sqrt{e^{2\ell} - 1}} e^x \quad \text{and} \quad g_-(x) = \frac{\sqrt{2}}{\sqrt{1 - e^{-2\ell}}} e^{-x}, \quad x \in [0, \ell],$$

Obviously, $\|g_{\pm}\| = 1$. 

Since
\[ g_+(0) - g_-(0) = \frac{\sqrt{2}}{\sqrt{e^{2\ell} - 1}} - \frac{\sqrt{2}}{\sqrt{1 - e^{-2\ell}}} = \frac{\sqrt{2}}{\sqrt{e^{2\ell} - 1}} (1 - e^{\ell}) \]
and
\[ g_+(\ell) - g_-(\ell) = \frac{\sqrt{2}}{\sqrt{e^{2\ell} - 1}} e^{\ell} - \frac{\sqrt{2}}{\sqrt{1 - e^{-2\ell}}} e^{-\ell} = -\frac{\sqrt{2}}{\sqrt{e^{2\ell} - 1}} (1 - e^{\ell}) \]
one observes that \( g_+(0) - g_-(0) = -(g_+(\ell) - g_-(\ell)) \) which proves that \( g_+ - g_- \in \text{Dom}(D) \).

Now, since (A.3) holds, in accordance with definition the Livšic function
\[ s(z) = \frac{z - i}{z + i} \cdot \frac{\langle g_z, g_+ \rangle}{\langle g_z, g_+ \rangle} = \frac{e^{2\ell} - 1}{1 - e^{-2\ell}} \cdot \frac{z - i}{z + i} \cdot \frac{\ell}{\int_0^\ell e^{-i(z-1)x} dx} \]
\[ = \frac{e^{2\ell} - 1}{1 - e^{-2\ell}} \cdot \frac{e^{-(iz-1)\ell} - 1}{e^{-(iz+1)\ell} - 1} = \frac{e^{\ell} e^{-(iz-1)\ell} - 1}{e^{\ell} e^{-(iz+1)\ell} - 1} \]
\[ = \frac{e^{-iz\ell} - e^{\ell}}{e^{\ell} e^{-iz\ell} - 1}, \quad z \in \mathbb{C}_+ \]
Thus,
\[ s(z) = \frac{e^{i\ell z} - e^{-\ell}}{e^{-\ell} e^{i\ell z} - 1}, \quad z \in \mathbb{C}_+ \]
which proves the representation (ii).

Next, since \( g_+(0) = e^{-\ell} g_-(0) \), one also obtains that
\[ g_+ - e^{-\ell} g_- \in \text{Dom}(\hat{D}) \]
which proves the assertion (iii) taking into account (A.3).

Finally, one concludes that the characteristic function
\[ S(z) = \frac{s(z) - e^{-\ell}}{e^{-\ell} s(z) - 1} = \frac{e^{i\ell z} - e^{-\ell}}{e^{-\ell} e^{i\ell z} - e^{-\ell}} \]
\[ = e^{i\ell z}, \quad z \in \mathbb{C}_+ \]
which proves (iv).

The proof is complete. \( \square \)

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