Spiked eigenvalues of noncentral Fisher matrix with applications

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Abstract: In this paper, we investigate the asymptotic behavior of spiked eigenvalues of the noncentral Fisher matrix defined by $F_p = C_n(S_N)^{-1}$, where $C_n$ is a noncentral sample covariance matrix defined by $(\Xi + X)(\Xi + X)^*/n$ and $S_N = YY^*/N$. The matrices $X$ and $Y$ are two independent Gaussian arrays, with respective $p \times n$ and $p \times N$ and the Gaussian entries of them are independent and identically distributed (i.i.d.) with mean 0 and variance 1. When $p$, $n$, and $N$ grow to infinity proportionally, we establish a phase transition of the spiked eigenvalues of $F_p$. Furthermore, we derive the central limiting theorem (CLT) for the spiked eigenvalues of $F_p$. As an accessory to the proof of the above results, the fluctuations of the spiked eigenvalues of $C_n$ are studied, which should have its own interests. Besides, we develop the limits and CLT for the sample canonical correlation coefficients by the results of the spiked noncentral Fisher matrix and give three consistent estimators, including the population spiked eigenvalues and the population canonical correlation coefficients.

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1. Introduction

Fisher matrix is one of the most classical and important tools in multivariate statistic analysis (for details see [1], [29], and [30]). [22] provided a remarked five-way classification of the distribution theory and introduced some representative applications, such as signal detection in noise and testing equality of group means under unknown covariance matrix and so on. Among these applications, some statistics can be transformed into a Fisher matrix while others can be studied by a noncentral Fisher matrix. So it is natural to study the spectral properties of the Fisher matrix and noncentral Fisher matrix.

There have been many works focusing on the Fisher matrix. [34] derived the limiting spectral distribution (LSD) of Fisher matrix, which is the celebrated Wachter distribution. [19] proved the largest eigenvalue of Fisher matrix follows Tracy-Widom (T-W) law, see [33], [38] was devoted to the CLT for linear spectral statistics (LSS) of the Fisher matrix. [40] studied the LSD and CLT of LSS of
the so-called general Fisher matrix. In fact, these above works all focus on the central Fisher matrix. Before introducing the concept of the noncentral Fisher matrix, it is necessary to know the large dimensional information-plus-noise-type matrix,

$$C_n = \frac{1}{n}(\Xi + X)(\Xi + X)^*$$

(1)

where $X$ is a $p \times n$ matrix containing i.i.d. entries with mean 0 and variance 1. $\Xi^*/n$ is a deterministic and it is assumed to have a LSD. Here and subsequently, $*$ denotes conjugate transpose, and $T$ stands for transpose on real matrix and vector. In [17, 18, 14, 5], a lot of spectral properties of $C_n$ have been researched. Actually, the matrix $C_n$ is a noncentral sample covariance matrix and $\Xi^*/n$ is called the noncentral parameter matrix, whose eigenvalues are arranged as a descending order:

$$l_{\Xi}^1 \geq l_{\Xi}^2 \geq \cdots \geq l_{\Xi}^p.$$  

(2)

However, many problems, such as signal detection in noise and testing equality of group means under unknown covariance matrix always involve the noncentral Fisher matrix, which is constructed based on the matrix $C_n$,

$$F_p = C_n S_N^{-1},$$  

(3)

where $S_N = YY^*/N$ and $Y$ is independent of $X$. The entries $\{Y_{ij}, 1 \leq i \leq p, 1 \leq j \leq N\}$ are i.i.d. with mean 0 and variance 1. To the best of our knowledge, there are only a handful of works devoted to the noncentral Fisher matrices. Under the Gaussian assumption, [27] developed an approximation to the distribution of the largest eigenvalue of the noncentral Fisher matrix and [12] derived the CLT for the LSS of the large dimensional noncentral Fisher matrix. In this paper, we concentrate on the outlier eigenvalues of the noncentral Fisher matrix defined in (3). Specially, we will work with the following assumption,

**Assumption a:** Assume that $\Xi$ is a $p \times n$ nonrandom matrix and the empirical spectral distribution (ESD) of $\Xi^*/n$ satisfies $H_n \overset{w}{\rightarrow} H$, ($w$ denoting weakly convergence), where $H$ is a non-random probability measure. In addition, the eigenvalues of $\Xi^*/n$ are subject to the condition

$$l_{\Xi}^{j_k+1} = l_{\Xi}^{j_k+2} = \cdots = l_{\Xi}^{j_k+m_k} = a_k, \quad k \in \{1, \cdots, K\}$$  

(4)

and $a_k$ satisfies the separation condition, that is,

$$\min_{k \neq j} \left| \frac{a_k}{a_j} - 1 \right| > d,$$  

(5)

where $d$ is positive constant and independent of $n$ and $a_k, k \in \{1, \cdots, K\}$ is allowed to grow at an order $o(\sqrt{n})$. In addition, $J_k = \{j_k+1, \cdots, j_k+m_k\}$ denotes the set of ranks of $a_k$, where $m_k$ is the multiplicities of $a_k$ satisfying $m_1 + \cdots + m_K = M$, a fixed integer.
Remark 1.1. Note that $a_k, k \in \{1, \cdots, K\}$ can be located in any gap between the supports of $H$, which means that $a_k$ is not just the extreme eigenvalues of $\Xi \Xi ^t/n$.

The eigenvalues $a_k, k \in \{1, \cdots, K\}$ are called as population spiked eigenvalues of the noncentral sample covariance matrix (1) and the noncentral Fisher matrix (3). We call these two matrices satisfying (4) the spiked noncentral sample covariance matrix and the spiked noncentral Fisher matrix, respectively. In fact, the spiked eigenvalues $a_k, k \in \{1, \cdots, K\}$ should have allowed to diverge at any rate, but under the restriction of our studying method, we have to assume them at a rate of $o(\sqrt{n})$. In this paper, we are devoted to exploring the properties of the limits of the sample spiked eigenvalues (corresponding to $a_k, k \in \{1, \cdots, K\}$) of the noncentral Fisher matrix. From now on, we call the sample spiked eigenvalues simply spiked eigenvalues when no confusion can arise.

To study the principal component analysis (PCA), [26] proposed the spiked model based on the covariance matrix. The spiked model has been studied much further and extended in various random matrices such as Fisher matrix, sample canonical correlation matrix, separable covariance matrix, see [11, 16] for more details. The main emphasis of the research of the spiked model is on the limits and the fluctuations of the spiked eigenvalues of these kinds of random matrices. [8, 31, 6, 7, 2, 13, 23] focused on the spiked sample covariance matrix and [35, 25, 24] concentrated on the central spiked Fisher matrices.

The main contribution of the paper is the establishment of the limits and fluctuations of the spiked eigenvalues of the noncentral Fisher matrix under the Gaussian population assumption. Even better, we apply the above theoretical results to the canonical correlation analysis (CCA) and derive the limits and fluctuations of sample canonical correlation coefficients and give three consistent estimators, including the population spiked eigenvalues and the population canonical correlation coefficients. In addition, we study the properties of sample spiked eigenvalues of the noncentral sample covariance matrix, which should have its own interest.

The rest of the paper is organized as follows. In Section 2, we define some notations and we present the LSD of some random matrices. In Section 3, we study the limits and fluctuations of spiked eigenvalues of the noncentral sample covariance matrix and noncentral Fisher matrix. In Section 4, we present the limits and fluctuations of spiked eigenvalues of sample canonical correlation matrix are investigated, give three estimators of the population spiked eigenvalues, and conduct the actual data analysis about climate and geography by CCA. To show the correctness and rationality of theorems intuitively, we design a series of simulations in Section 5. In Section 6, we summarize the main conclusions and the outlook. Section 7 presents technical proof.
2. Preliminaries

In this section, we collect some notations and preliminary results or assumptions, which will be throughout the paper. Although some notations have been mentioned above, we still provide the precise definitions here.

2.1. Basic notions

For any $n \times n$ matrix $A_n$ with only real eigenvalues, let $F_n$ be the empirical spectral distribution (ESD) function of $A_n$, that is,

$$F_n(x) = \frac{1}{n} \text{Card}\{i; \lambda_i^{A_n} \leq x\},$$

where $\lambda_i^{A_n}$ denotes the $i$-th largest eigenvalue of $A_n$. If $F^{\lambda^{A_n}}$ has a limiting distribution $F$, then we call it the limiting spectral distribution (LSD) of sequence $\{A_n\}$. For any function of bounded variation $G$ on the real line, its Stieltjes transform (ST) is defined by

$$m(z) = \int \frac{1}{\lambda - z} dG(\lambda), \quad z \in \mathbb{C}^+.$$

2.2. Symbols and Assumptions

In this paper, we study the spiked eigenvalues of the noncentral spiked sample covariance matrix $C_n$ defined in (1) and the noncentral spiked Fisher matrix $F_p$ defined in (3) with the matrix $\Xi^*/n$ satisfying (4). In order to distinguish the symbols of these three matrices clearly, we show the notations of eigenvalues and ST of the matrices in Table 1.

| Matrix   | $\Xi^*/n$ | $C_n$ | $F_p$ |
|----------|----------|-------|-------|
| LSD      | $H$      | $F^C$ | $F$   |
| ST       | $m_1$    | $m_2$ | $m_3$ |
| Population spiked eigenvalue | $a_k$ | $a_k$ | $a_k$ |
| Sample eigenvalue | $l_i^{C_n}$ | $l_{st}$ |
| Limit    | $\lambda_i^{C} = \psi_C(a_k)$ | $\lambda_k = \psi(a_k) = \psi_F(\psi_C(a_k))$ |

Throughout the paper, we consider the following assumptions about the high-dimensional setting and the moment conditions.

**Assumption b**: Assume that $p < n$, $p < N$ with $p/n = c_1n \rightarrow c_1 \in (0, 1)$, $p/N = c_2N \rightarrow c_2 \in (0, 1)$, as $\min(p, n, N) \rightarrow \infty$.

**Assumption c**: Assume that the matrix $X_n$ and $Y_N$ are two independent arrays of independent standard Gaussian distribution variables $\{X_{jk}: 1 \leq j \leq p, 1 \leq k \leq n\}$ and $\{Y_{jk}: 1 \leq j \leq p, 1 \leq k \leq N\}$, respectively. If $X_{ij}$ and $Y_{ij}$ are complex, $EX_{ij}^2 = 0$, $EY_{ij}^2 = 0$, $E|X_{ij}|^2 = 1$ and $E|Y_{ij}|^2 = 1$ are required.
2.3. LSD for $C_n$ and $F_p$

To introduce the necessary conclusions and symbols in the following sections, we provide the LSD for $C_n$ and $F_p$ by the results in [17, 39]. Note that similar results are obtained in [12]. According to Theorem 1 of [17], the ST $m_2(z)$ of the LSD of $C_n$ is the unique solution to the equation

$$m_2 = \int \frac{dH(t)}{1+c_1m_2} - \frac{(1+c_1m_2)z+1-c_1}{1+c_1m_2} \]$$

where $m_1(z)$ is the ST of $H$. For simplicity, we write $m_2(z)$ as $m_2$ in (6). By Theorem 2.1 of [39], the ST $m_3(z)$ of the LSD of $F_p$ satisfies the following equation:

$$m_3 = \int \frac{dH(t)}{1+(c_1+c_2z)m_3} - \frac{1-c_1}{1+c_2zm_3} - \frac{z(1+(c_1+c_2z)m_3)}{1+c_2zm_3}.$$  

From (6.12) in [4], we have

$$\frac{m_3(z)}{1+c_2zm_3(z)} = m_2(z(1+c_2zm_3(z))).$$

3. Main Results

In this section, we state our main results and briefly summarize our proof strategy. Our main results include the limits and the CLT of the spiked eigenvalues of the noncentral sample covariance matrix $C_n$ and the noncentral Fisher matrix $F_p$.

3.1. Limits and fluctuations for the matrix $C_n$

The noncentral sample covariance matrix $C_n$ and its eigenvalues are arranged as a descending order as

$$l_1^{C_n} \geq l_2^{C_n} \geq \cdots \geq l_p^{C_n}. $$

**Theorem 3.1.** If Assumption $[a]-[c]$ hold and the population spiked eigenvalues $a_k, k \in \{1,\cdots,K\}$ satisfies $\psi_C'(a_k) > 0$, for $1 \leq k \leq K$. Then we have

$$\frac{\psi_C'(a_k)}{\psi_C(a_k)} - 1 \xrightarrow{a.s.} 0, \quad j \in J_k$$

where

$$\psi_C(a_k) = a_k \left(1 - c_1n \int \frac{1}{t-a_k}dH_n(t) \right)^2 + (1-c_1n) \left(1 - c_1n \int \frac{1}{t-a_k}dH_n(t) \right),$$

$$\frac{1}{l-a_k}dH_n(t) \right)^2 + (1-c_1n) \left(1 - c_1n \int \frac{1}{t-a_k}dH_n(t) \right),$$

$$\frac{1}{l-a_k}dH_n(t) \right)^2 + (1-c_1n) \left(1 - c_1n \int \frac{1}{t-a_k}dH_n(t) \right),$$

$$\frac{1}{l-a_k}dH_n(t) \right)^2 + (1-c_1n) \left(1 - c_1n \int \frac{1}{t-a_k}dH_n(t) \right),$$

$$\frac{1}{l-a_k}dH_n(t) \right)^2 + (1-c_1n) \left(1 - c_1n \int \frac{1}{t-a_k}dH_n(t) \right),$$
Remark 3.1. Considering that the convergence \( H_n \to H \) may be slow, in \( \psi_C(a_k) \), we use \( H_n \) and \( c_{1n} \) instead of \( H \) and \( c_1 \), respectively. In following theorems related to the limits of spiked eigenvalues, we will take the same treatment.

Having disposed of the limits of the sample spiked eigenvalues, we are now in a position to show the CLT for the sample spiked eigenvalues.

**Theorem 3.2.** Suppose the Assumption \([a] - [c]\) hold, the \( m_k \)-dimensional random vector
\[
\gamma_n^{C_n} = \sqrt{n}\left\{ \left( \frac{l_{j_n}^{C_n}}{\psi_C(a_k)} - 1 \right) \frac{1}{\sqrt{\beta \theta_1}}, \ j \in J_k \right\}
\]
converges weakly to the joint distribution of the \( m_k \) eigenvalues of Gaussian random matrix \( \Omega \), where \( \Omega \) is a \( m_k \)-dimensional standard GOE (GUE) matrix. If the samples are either real, \( \beta = 2 \); or complex, \( \beta = 1 \), and \( \theta_1 = 1 - \frac{1}{\lambda_c^{C_n}} + c_1 m_2(\lambda_c^{C_n}) \).

\[
\theta_1 = \frac{\lambda_c^{C_n} m_2'}{\lambda_c^{C_n}(1 + c_1 m_2')},
\]
where \( m_2 \) and \( m_2' \) are the derivatives of \( m_2 \) and \( m_2' \) at the point \( \lambda_c^{C_n} \), respectively, and
\[
m_2(\lambda_c^{C_n}) = -\frac{1 - c_1}{\lambda_c^{C_n}} + c_1 m_2(\lambda_c^{C_n}).
\]

Remark 3.2. It is worth pointing out that \( \theta_1 \) is equal to the (half of) variance of \( \sqrt{n}(l_{j_n}^{C_n}/\psi_C(a_k) - 1) \) when \( a_k \) is a single eigenvalue. When \( a_k \) is multiple, the limiting distribution of \( \sqrt{n}(l_{j_n}^{C_n}/\psi_C(a_k) - 1) \) is related to that of the eigenvalues of a GOE (GUE) matrix, the variance of whose diagonal elements is equal to \( 2\theta_1 \) (\( \theta_1 \)). In what follows, we also call \( \theta_1 \) as the scale parameter of the GOE (GUE) matrix.

Remark 3.3. [15] and [10] focused on the noncentral spiked sample covariance matrix. [15] studied the limits and rates for the spiked eigenvalues and vectors of the noncentral sample covariance matrix. [10] was devoted to the fluctuation of the spiked vectors of the noncentral sample covariance matrix. Note that both of the works are under the condition of finite rank, especially, the rank of the \( \Xi \) is finite. Compared with [15] and [10], our assumptions about \( \Xi \) is more general.

### 3.2. Limits and fluctuations for the noncentral Fisher matrix \( F_p \)

Having disposed of the noncentral spiked sample covariance matrix \( C_n \), we can now return to the noncentral Fisher matrix \( F_p \). The eigenvalues of noncentral Fisher matrix \( F_p \) are sorted in descending order as
\[
l_1 \geq l_2 \geq \cdots \geq l_p.
\]
Theorem 3.3. Let the Assumption \([a] - [c]\) hold, the noncentral Fisher matrix \(F_p\) is defined in (3). If \(a_k\) satisfies \(\psi_F'(\psi_C(a_k)) > 0\) and \(\psi_C'(a_k) > 0\), for \(1 \leq k \leq K\), then we have

\[
\frac{l_j}{\psi_F(\psi_C(a_k))} - 1 \overset{a.s.}{\longrightarrow} 0, \quad j \in J_k
\]

where

\[
\psi_F(x) = \frac{x}{1 + c_{2n} \cdot x \cdot m_2'(x)}, \quad (14)
\]

\(\psi_F'(\cdot)\) is the derivative of \(\psi_F(\cdot)\), \(m_2\) and \(m_2\) are same, but \(c_1, c_2\) and \(H\) replaced by \(c_{1n}, c_{2n}\) and \(H_n\), respectively.

Theorem 3.4. Suppose that the Assumption \([a] - [c]\) hold, the \(m_k\) dimensional random vector

\[
\gamma_k \triangleq \sqrt{n} \left\{ \left( \frac{l_j - \psi_F(\psi_C(a_k))}{\psi_F(\psi_C(a_k))} \right) \frac{1}{\sqrt{\theta_2}}, \quad j \in J_k \right\},
\]

converges weakly to the joint distribution of the eigenvalues of Gaussian random matrix \(\Omega\) where

\[
\theta_2 = \frac{c_2}{c_1 \cdot \vartheta} + \left[ \frac{1 - c_2(\lambda_k C)^2 m_2'(\lambda_k C)}{1 + c_2 \lambda_k m_3(\lambda_k)} \right]^2 \theta_1, \quad (15)
\]

\(\theta_1\) is defined in Theorem 3.2 and \(\vartheta\) satisfies

\[
\vartheta = 1 + 2 \lambda_k c_2 m_3(\lambda_k) + c_2^2 \lambda_k m_3'(\lambda_k)
\]

4. Applications

In this section, we discuss some applications of our results in limiting properties of sample canonical correlation coefficients, estimators of the population spiked eigenvalues and the population canonical correlations coefficients. At the end, we present an experiment on a real environmental variables for world countries data.

4.1. Limits and fluctuations for the sample canonical correlation matrix

The CCA is the general and favorable method to investigate the relationship between two random vectors. Under the high-dimensional setting and the Gaussian assumption, (11) studied the limiting properties of sample canonical correlation
coefficients. Under the sharp moment condition, [36, 28] prove the largest eigenvalue of sample canonical correlation matrix converges to T-W law to test the independence of random vectors with two different structures. Moreover, [37] shows the limiting distribution of the spiked eigenvalues depends on the fourth cumulants of the population distribution. Note that [3, 36, 28, 37] only focused on the finite rank case, where the number of the positive population canonical correlation coefficients of two groups of high-dimensional Gaussian vectors are finite. However, we popularize finite rank case to infinite rank case, in other words, we get the limits and fluctuations of the sample canonical correlation coefficients under the infinite rank case. In the following, we will introduce the application about limiting properties of sample canonical correlations coefficients in great detail.

Let $z_i = (x_i^T, y_i^T)^T$, $i = 1, \cdots, n$, be independent observations from a $(p+q)$-dimensional Gaussian distribution with mean zero and covariance matrix

$$
\Sigma = \begin{pmatrix}
\Sigma_{xx} & \Sigma_{xy} \\
\Sigma_{yx} & \Sigma_{yy}
\end{pmatrix},
$$

where $x_i$ and $y_i$ are $p$-dimensional and $q$-dimensional vectors with the population covariance matrices $\Sigma_{xx}$ and $\Sigma_{yy}$, respectively. Without loss of generality, we assume that $p \leq q$. Define the corresponding sample covariance matrix as

$$
S_n = \frac{1}{n} \sum_{i=1}^{n} z_i z_i^T,
$$

(17)

which can be formed as

$$
S_n = \begin{pmatrix}
S_{xx} & S_{xy} \\
S_{yx} & S_{yy}
\end{pmatrix} = \frac{1}{n} \begin{pmatrix}
XX^T & XY^T \\
YX^T & YY^T
\end{pmatrix}
$$

with

$$
X = (x_1, \cdots, x_n)_{p \times n}, \quad Y = (y_1, \cdots, y_n)_{q \times n},
$$

In the sequel, $\Sigma_{xx}^{-\frac{1}{2}} \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} \Sigma_{yx}$ is called as the population canonical correlation matrix and its eigenvalues are denoted by

$$
1 > \rho_1^2 \geq \rho_2^2 \geq \cdots \geq \rho_p^2.
$$

(18)

By the singular value decomposition, we have that

$$
\Sigma_{xx}^{-\frac{1}{2}} \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} \Sigma_{yx} = P_1 \Lambda P_2^T
$$

(19)

where

$$
\Lambda = \begin{pmatrix}
\Lambda_{11} & 0_{12} \\
0_{12} & 0_{22}
\end{pmatrix},
$$

(20)

$\Lambda_{11} = \text{diag}(\rho_1, \rho_2, \cdots, \rho_p)$, $0_{12}$ is a $p \times (q-p)$ zero matrix, $P_1$ and $P_2$ are orthogonal matrix with size $p \times p$ and $q \times q$, respectively. It follows that $\rho_1^2, \rho_2^2, \cdots, \rho_p^2$
are also the eigenvalues of the diagonal matrix $\Lambda \Lambda^T$. According to Theorem 12.2.1 of [1], the nonnegative square roots $\rho_1, \cdots, \rho_p$ are the population canonical correlation coefficients. Correspondingly, $S^{-1}_{xx} S_{xy} S^{-1}_{yy} S_{yx}$ is called as the sample canonical correlation matrix and its eigenvalues are denoted by $\lambda^2_1 \geq \lambda^2_2 \geq \cdots \geq \lambda^2_p$.

The following theorem describes the function relation between sample canonical correlation coefficients and the eigenvalues of a special noncentral Fisher matrix.

**Theorem 4.1.** [Theorem 1 in [3]] Suppose that $\lambda^2_i, i = 1, \cdots, p,$ is the ordered eigenvalue of the sample canonical correlation matrix $S^{-1}_{xx} S_{xy} S^{-1}_{yy} S_{yx}$. Then, there exists a noncentral Fisher matrix $F(\Xi)$ whose eigenvalue $l_i$ satisfies

$$l_i = g(\lambda_i) \triangleq \frac{(n-q)\lambda^2_i}{q(1-\lambda^2_i)}, \quad i = 1, \cdots, p$$

and $\Xi$ is the noncentral parameter matrix,

$$\Xi = \frac{n}{q} T (n^{-1} \tilde{Y} \tilde{Y}^T) T^T$$

where $TT^T = \text{diag}(\rho^2_1/(1-\rho^2_1), \rho^2_2/(1-\rho^2_2), \cdots, \rho^2_p/(1-\rho^2_p))$ and $\tilde{Y}$ is a $p \times n$ matrix and contains i.i.d. elements with standard Gaussian distribution.

Combining Theorem 4.1 with the properties of the noncentral Fisher matrix, we obtain the limits and fluctuations of the sample canonical correlation coefficients under the following assumption.

**Assumption d**: The empirical spectral distribution (ESD) of $\Sigma^{-1}_{xx} \Sigma_{xy} \Sigma^{-1}_{yy} \Sigma_{yx}$, $\mathcal{H}_n$, tends to proper probability measure $\mathcal{H}$, if $\min(p, q, n) \to \infty$. Assume that $\alpha_k, i = 1, \cdots, p$ is subject to the condition

$$\alpha_k = \rho^2_{m_k-1+1} = \cdots = \rho^2_{m_k-1+m_k}, \quad k \in \{1, \cdots, K\},$$

where $\alpha_k$ is out of the support of $\mathcal{H}$ and satisfies the separation condition defined in (5). $M = \sum_{i=1}^K m_i$ is a fixed positive integer with convention $m_0 = 0$. In addition, $\alpha_1$ is allowed to be with the order $1 - o(n^{-1/2})$.

**Remark 4.1.** Note that the noncentral parameter matrix (22) is random, so the assumption about multiple roots in (23) is not reasonable. In the following Assumption d’, we replace the condition of multiple roots by single root. However, we think the results of the limits and fluctuations for the multiple roots case are correct and our guess will be verified by some simulations in the following section.

**Assumption d’**: The assumptions are same as Assumption d, the additional assumption $m_i = 1, \quad i \in \{1, \cdots, K\}$.

**Theorem 4.2.** Under the conditions stated in Theorem 4.1, if moreover Assumption d’ holds, and $\alpha_k$ satisfies $\psi'_\Xi(f(\alpha_k)) > 0$, $\Psi'_C(\alpha_k) > 0$, and $\Psi'(\alpha_k) > 0$ for $1 \leq k \leq K$, then we have

$$\frac{\lambda^2_i}{f(\alpha_k)} - 1 \overset{a.s.}{\to} 0, \quad i \in J_k$$

(24)
where
\[ t(x) = g^{-1} \circ \Psi(x), \quad \Psi(x) = \psi_F \circ \psi_C \circ \psi_{\Xi} \circ f(x), \]
\[ \Psi_C(x) = \psi_C \circ \psi_{\Xi} \circ f(x), \quad f(x) = \frac{n}{q} \frac{x}{1-x}, \quad \psi_F(x) = \frac{x}{1 + \frac{p}{n-q} \cdot x \cdot m_C(x)}; \]
\[ \psi_{\Xi}(x) = x \left( 1 + \frac{p}{q} \int \frac{t}{x-t} d\tilde{H}(t) \right), \quad \tilde{H}(x) = H_n \left( \frac{qx}{n+qx} \right), \]
\[ \psi_C(x) = x \left( 1 - \frac{p}{q} \int \frac{1}{t-x} dF_{mp/n,\tilde{H}}(t) \right)^2 - \left( 1 - \frac{p}{q} \right) \left( 1 - \frac{p}{q} \right) \int \frac{1}{t-x} dF_{mp/n,\tilde{H}}(t), \]
and \( g^{-1} \circ \Psi \) stands for the composition of \( \Psi(\cdot) \) and \( g^{-1}(\cdot) \). \( F_{mp/n,\tilde{H}} \) denotes M-P law with the parameter \( p/n \) and \( \tilde{H} \). Moreover, \( m_C(\cdot) \) stands for the unique solution (6) with \( c_1, H \) replaced by \( p/q, F_{mp/n,\tilde{H}} \), respectively.

It is worth noting that we can conclude the results in Theorem 1.8 of [11] by Theorem 4.2 or Theorem 3.1 in [35], when the LSD \( \mathcal{H} \) of \( \Sigma^{-1}_{xx} \Sigma_{xy} \Sigma^{-1}_{yy} \Sigma_{yx} \) degenerates to \( \delta_{\{0\}} \).

**Corollary 4.2.** Let the Assumption \([c]\) and \([d']\) hold, furthermore, the LSD \( \mathcal{H} \) degenerates to \( \delta_{\{0\}} \) and \( \alpha_k(k = 1, \cdots, K) \) satisfies \( \alpha_k > \alpha_r \), then we have
\[ \frac{\lambda_i^2}{\phi(\alpha_k)} - 1 \xrightarrow{a.s.} 0, \quad i \in \mathcal{J}_k, \tag{25} \]
where
\[ \phi(\alpha_k) = \frac{[\alpha_k(1-r_1) + r_1][\alpha_k(1-r_2) + r_2]}{\alpha_k} \]
\[ \alpha_r = \sqrt{\frac{r_1 r_2}{(1-r_1)(1-r_2)}}, \quad r_1 = p/n, \quad r_2 = q/n. \]

Having disposed of the results of limits of the square of sample canonical correlation coefficients \( \lambda_i^2, i \in \mathcal{J}_k \) associated to the \( \alpha_k(k = 1, \cdots, K) \), we can now return to show the CLT for them.

**Theorem 4.3.** Let the Assumption \([c]\) and \([d']\) hold, and \( \lambda_i^2 \) is the square of eigenvalues of sample canonical correlation matrix. We set \( p/q \to c_3 \) and \( p/(n-q) \to c_4 \) as \( n \) tends to infinity, then the \( m_k \)-dimensional random vector
\[ \gamma_k = \sqrt{q} \left\{ \frac{\lambda_i^2 - t(\alpha_k)}{t(\alpha_k)} \right\} \frac{1}{\sqrt{\beta \eta}}, \quad i \in \mathcal{J}_k \]
converges weakly to the joint distribution of the eigenvalues of Gaussian random
Spiked eigenvalues of noncentral Fisher matrix with applications

matrix $\Omega$, and

\[
\eta = \left[ \frac{c_4}{(c_3 \cdot \eta_2)} + \left( \frac{1 - c_4 \Psi_C(\lambda_k)^2}{1 + c_4 \Psi_C(\lambda_k)} \right)^2 \eta_1 + \eta_3 \right] \times \frac{c_3^2 \Psi(\lambda_k)^2}{(c_3 + c_4 \Psi(\lambda_k))^4 |t(\lambda_k)|^2},
\]

\[
\eta_1 = \left[ \Psi_C(\lambda_k) m_C' + \frac{\psi_\Psi(f(\lambda_k))(1 + c_3 m_C + c_3 \Psi_C(\lambda_k) m_C')}{\Psi_C(\lambda_k)(1 + c_3 m_C)^2} \right]^{-2} \times \left( \frac{m_C' + \frac{(\psi_\Psi(f(\lambda_k)))^2 c_3 m_C'}{\Psi_C(\lambda_k)^2(1 + c_3 m_C)^2} + \frac{2 \psi_\Psi(f(\lambda_k))(1 + c_3 m_C + \Psi_C(\lambda_k) m_C')}{(\Psi_C(\lambda_k))^2(1 + c_3 m_C)^2} \right),
\]

\[
\eta_2 = 1 + 2 \Psi_C(\lambda_k) c_4 m_F(\Psi_C(\lambda_k)) + c_4 (\Psi_C(\lambda_k))^2 m_F(\Psi_C(\lambda_k)),
\]

\[
\eta_3 = \frac{\tilde{q}}{\tilde{p}} \frac{\psi_F'(\Psi_C(\lambda_k))^2 (\psi_\Psi(f(\lambda_k)))^2}{\psi_\Psi(f(\lambda_k)) m_C'(\psi_\Psi(f(\lambda_k)))^2 \Psi_\Psi(f(\lambda_k))} \Psi_C(\lambda_k)^2,
\]

where $m_C$ and $m'_C$ denote the value and derivative at the point $\Psi_C(\lambda_k)$, respectively, $m_F$ stands for the unique solution (7) with $c_1$, $c_2$, $H$ replaced by $p/q$, $p/(n - q)$, $F_{np}^{p/n,H}$, respectively.

The proof of this theorem can be drawn by the Delta method, Theorem 3.4 and Theorem 4.1 and it will be postponed in subsection 7.5.

4.2. Estimators of the population spiked eigenvalues

In this section, we develop two consistent estimators of population spiked eigenvalues $a_k$ defined in (23), which are derived by the results of the noncentral sample covariance matrix and the noncentral Fisher matrix, respectively. At first, we proceed to show the estimator of population spiked eigenvalues for the noncentral sample covariance matrix. From the conclusion of Theorem 3.3, we have

\[
\lambda_i^C = a_k \left( 1 - c_1 m_1(a_k) \right)^2 + (1 - c_1) \left( 1 - c_1 m_1(a_k) \right),
\]

and

\[
a_k = \lambda_k^C \left( 1 + c_1 m_2(\lambda_k^C) \right)^2 - (1 - c_1) \left( 1 + c_1 m_2(\lambda_k^C) \right).
\]

It is sufficient to consider the estimators of $\lambda_i^C$ and $m_2(\lambda_k^C)$, which are denoted by $\tilde{\lambda}_k^C$ and $\tilde{m}_2(\tilde{\lambda}_k^C)$, respectively. We adopt an approach similar to that in [23] to estimate $\tilde{m}_2(\tilde{\lambda}_k^C)$. Define $r_{ik} = |\lambda_i^C - \tilde{\lambda}_k^C|/|\tilde{\lambda}_k^C|$ and the set $\mathcal{J}_k = \{ i \in (1, \cdots, p) : r_{ik} \leq 0.2 \}$ and $\tilde{c}_1 = (p - |\mathcal{J}_k|)/n$; then,

\[
\tilde{m}_2(\tilde{\lambda}_k^C) = \frac{1}{p - |\mathcal{J}_k|} \sum_{i \notin \mathcal{J}_k} (\lambda_i^C - \tilde{\lambda}_k^C)^{-1}
\]
is a good estimator of \(m_2(\lambda_k^C)\), where the set \(\mathcal{J}_k\) is selected to avoid the effect of multiple roots and to make the estimator more accurate. Then we have

\[
\hat{a}_k = \lambda_k^C \left( 1 + \hat{c}_1 \hat{m}_2(\lambda_k^C) \right)^2 - (1 - \hat{c}_1) \left( 1 + \hat{c}_1 \hat{m}_2(\lambda_k^C) \right).
\]  

(27)

We now turn to give the other estimator for the population spiked eigenvalues by the results of the noncentral Fisher matrix. From the conclusion of Theorem 3.3, we have

\[
\lambda_i = \psi_C(a_k) \left[ 1 + c_2 \psi_C(a_k) m_3(\lambda_i) \right],
\]

then,

\[
\psi_C(a_k) = \lambda_i (1 + c_2 \lambda_i m_3(\lambda_i)).
\]  

(28)

According to Theorem 3.1, we know

\[
\psi_C(a_k) = a_k (1 - c_1 m_1(a_k))^2 + (1 - c_1) (1 - c_1 m_1(a_k)),
\]

then,

\[
a_k = \psi_C(a_k) \left[ 1 + c_1 m_2 \psi_C(a_k) \right]^2 - (1 - c_1) [1 + c_1 m_2 \psi_C(a_k)].
\]

Note that (28) and \(\hat{\lambda}_k\) is the natural estimator of \(\lambda_k\), we set \(\hat{a}_k\) as the estimator of \(\psi_C(a_k)\) and have

\[
\hat{a}_k = \hat{\lambda}_k (1 + \hat{c}_2 \hat{\lambda}_k \hat{m}_3(\hat{\lambda}_k)),
\]

where \(\hat{m}_3(\hat{\lambda}_k)\) are the estimators of \(m_3(\lambda_k)\). Similar considerations in (26) are applied here, we have \(r_{ik} = |\hat{\lambda}_i - \hat{\lambda}_k|/|\hat{\lambda}_k|\) and the set \(\mathcal{J}_k = \{i \in \{1, \cdots, p\} : r_{ik} \leq 0.2\}\) and \(\hat{c}_2 = (p - |\mathcal{J}_k|)/N\); then,

\[
\hat{m}_3(\hat{\lambda}_k) = \frac{1}{p - |\mathcal{J}_k|} \sum_{i \notin \mathcal{J}_k} (\hat{\lambda}_i - \hat{\lambda}_k)^{-1}
\]  

(29)

is a good estimator of \(m_3(\lambda_k)\). Then we have,

\[
\hat{a}_k = \left[ \hat{a}_k (1 + c_1 \hat{m}_2(\hat{a}_k)) + (1 - c_1) (1 + c_1 \hat{m}_2(\hat{a}_k)) \right],
\]  

(30)

where \(\hat{m}_2(\hat{a}_k)\) is the estimator of \(m_2\) at \(\psi_C(a_k)\), by (8) we have

\[
\hat{m}_2(\hat{a}_k) = \hat{m}_3/(1 + \hat{c}_2 \hat{\lambda}_k \hat{m}_3).
\]  

(31)

4.3. The environmental variables for world countries data

To illustrate the application of canonical correlation, we apply our result to the environmental variables for world countries data\(^1\). Deleting the samples missing

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\(^1\)The data can be download from: https://www.kaggle.com/zanderventer/environmental-variables-for-world-countries.
value and the variables related to others, then, we get 188 samples with 18 variables. We divide the 18 variables into two groups, one \((x, p = 7)\) contains the elevation above sea level, percentage of country covered by cropland, percentage cover by trees > 5m in height and so on, the other one \((y, q = 11)\) contains annual precipitation, temperature mean annual, mean wind speed, average cloudy days per year and so on. We want to explore the relationship between the geographical conditions \(x\) and climatic conditions \(y\), so we test their first canonical correlation coefficient being zero or not, i.e.,

\[
H_0 : \rho_1^2 = 0 \quad \text{v.s.} \quad H_1 : \rho_1^2 > 0.
\]

Therefore, we take the largest eigenvalue \(\lambda_1^2\) of the sample canonical correlation matrix as the test statistic. By the formula (2.1) in [11], the normalized largest eigenvalues of the sample CCA matrix tends to the T-W law under the null hypothesis. The eigenvalues of \(S_{xx}^{-1}S_{xy}S_{yy}^{-1}S_{yx}\) are as follows:

| \(\lambda_1^2\) | \(\lambda_2^2\) | \(\lambda_3^2\) | \(\lambda_4^2\) | \(\lambda_5^2\) | \(\lambda_6^2\) | \(\lambda_7^2\) |
|-----------|---------|---------|---------|---------|---------|---------|
| 0.9152    | 0.7755  | 0.4560  | 0.4034  | 0.2548  | 0.2247  | 0.0492  |

According to the data, we obtain that \(p\)-value approaches zero. Thus we have strong evidence to reject the null hypothesis and conclude that the geographical conditions relate to climatic conditions. Moreover, we use Algorithm 1 to give an estimator of the population canonical correlation coefficients,

\[
\hat{\rho}_i^2 = \frac{q/n * \hat{a}_i}{1 + q/n * \hat{a}_i}, \quad i = 1, \ldots, p.
\] (32)

By (32), we get the population canonical correlation coefficients \(\hat{\rho}_1^2 = 0.9064\), which implies that the correlation between geographical conditions and climatic conditions is strong.

**Algorithm 1** estimate population canonical correlation coefficients

| Input: | the samples \(x_i, y_i, i = 1, \ldots, n\) |
|--------|----------------------------------|
| Output: | \(\hat{\rho}_i^2, i = 1, \ldots, p\) |

1: \(\{l_{SCC}^{SCC} \leq \cdots \leq l_{SCC}^{SCC}\} = \text{Eigenvalue}(S_{xx}^{-1}S_{xy}S_{yy}^{-1}S_{yx})\);
2: \(l_i^{SCC} = l_i^{SCC}/(1 - l_i^{SCC}) \ast (n - q)/q \quad i = 1, \ldots, p\);
3: Use step 2 and (30) to get the estimator for the population spiked eigenvalues of the noncentral Fisher matrix
4: \(\hat{\rho}_i^2 = \frac{q/n * \hat{a}_i}{1 + q/n * \hat{a}_i}, \quad i = 1, \ldots, p\)
5: return result \(\hat{\rho}_i^2\)

**5. Simulation**

We conduct simulations that support the theoretical results and illustrate the accuracy of the estimators. The simulations are divided into two sections, one
is to verify the accuracy of Theorem 3.2, 3.4 and 4.3, and another is confirm the performance of the estimators in (27), (30) and (32).

5.1. Simulations for the asymptotic normality

In this section, we assume the eigenvalues of $\Xi^*/n$ satisfy

$$\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_p - 2, \lambda_1,$$ (33)

in this situation, we set $\Lambda$ defined in (20) satisfying

$$\Lambda = (\sqrt{10/11}, \sqrt{15/17}, \sqrt{1/2}, \cdots, \sqrt{1/2}).$$ (34)

In Figure 1-6, we compare the empirical density (the blue histogram) of the two of largest eigenvalues of $C_n$, $F_p$ and the CCA matrix with the standard normal density curve (the red line) with 2000 repetitions. To be more convincing, we put the Q-Q plots together with the histograms. According to the histograms and Q-Q plots, we can conclude that the Theorem 3.2, 3.4 and 4.3 are reasonable.

![Figure 1](image1.png)

**Fig 1.** The asymptotic normality of the largest eigenvalue of noncentral sample covariance matrix with $(p, n) = (200, 2000)$.

5.2. Simulations for the estimators

We conduct the following simulations to verify the accuracy of the estimators in (27), (30) and (32). Unlike (33), in this subsection, the eigenvalues of $\Xi^*/n$ are set as

$$\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_p - 3, \lambda_1,$$ (35)

where $a_1 = 10$, $a_2 = a_3 = 7.5$, and $H = \delta_{\{1\}}$. Note that we set the second eigenvalue as a multiple eigenvalue. According to the single root condition in Assumption $d'$, we set the eigenvalues of $\Lambda$ satisfying the model (34).
Remark 5.1. Compared with the assumption for the eigenvalues in (35), (33) is set without multiple population spiked eigenvalues, which is considered that the joint distribution of the eigenvalues of multiple dimensional GOE matrix can not be visually displayed.

We consider the estimator $\hat{a}_1$ and $\hat{a}_2$ with $p = 100, 200$ and $400$, respectively. Then the frequency histograms of the estimators are present in Figure 7-10 with...
5000 repetitions. In Figure 11-12, we show the accuracy of the estimator $\hat{\rho}_i$ for the two of largest population canonical correlation coefficients. We conclude that the estimators become accurate with $p$ increasing as the horizontal axis is more concentrated among three estimators.

To a certain extent, the histograms present the accuracy of estimates of the spiked eigenvalues, however, which is not convincing enough. So we use the Mean Square Errors (MSE) criteria to show the accuracy of estimates in Table
Spiked eigenvalues of noncentral Fisher matrix with applications

Fig 8. The estimator of $a_2$ ($a_2 = 7.5$) by the results of the noncentral sample covariance matrix with $p = 100$, $200$, and $400$.

Fig 9. The estimator of $a_1$ ($a_1 = 10$) by results of the noncentral Fisher matrix with $p = 100$, $200$, and $400$.

Fig 10. The estimator of $a_2$ ($a_2 = 7.5$) by results of the noncentral Fisher matrix with $p = 100$, $200$, and $400$.

Fig 11. The estimator of $\rho_1^2$ ($\rho_1^2 = 10/11 \approx 0.9091$) by results of the CCA matrix with $p = 100$, $200$, and $400$.

3.
The estimator of $\rho_2^2$ ($\rho_2^2 = 15/17 \approx 0.8824$) by results of the CCA matrix with $p = 100$, $200$, and $400$.

|       | $a_1 = 10$ ($\rho_1 = \sqrt{10/11}$) | $a_2 = 7.5$ ($\rho_2 = \sqrt{15/17}$) |
|-------|-------------------------------------|-------------------------------------|
|       | $p$  | 100 | 200 | 400 | 100 | 200 | 400 |
|       | $S$  | 1.2928 | 0.6183 | 0.3176 | 0.4017 | 0.2194 | 0.1144 |
|       | $F$  | 0.2077 | 0.1064 | 0.0540 | 0.0821 | 0.0421 | 0.0222 |
|       | CCA  | 4.8001e-05 | 2.5296e-05 | 1.2784e-05 | 7.7782e-05 | 4.4870e-05 | 2.3276e-05 |

The notation $S$, $F$ and CCA in Table 3 are short for the noncentral sample covariance matrix, the noncentral Fisher matrix and CCA matrix, respectively. According to Table 3, we find that the MSE decreases as the dimensionality $p$ increases, which is consistent with the above mentioned simulation results. Compared with the MSE of the two estimators in Table 3, the estimator derived by the noncentral Fisher matrix is more consistent than the noncentral sample covariance matrix.

5.3. Simulation for multiple roots case

Both Theorem (4.2) and Theorem (4.3) need satisfy the Assumption $d'$, specially, the spiked eigenvalue is a single root. So we set $\Lambda$ meet (34) in subsection 5.1. However, we guess our theories and estimator for the single root are effective under the multiple roots condition. In this subsection, we will present some simulations to show the correctness and rationality estimating population canonical correlation coefficient under the multiple roots condition. We assume

$$\Lambda = (\sqrt{10/11}, \sqrt{15/17}, \sqrt{15/17}, \sqrt{1/2}, \cdots, \sqrt{1/2}),$$

set the ratio among $(p : n : N)$ as $(1 : 3 : 9)$ and 1000 repetitions. According to Fig 13 and Table 4, the performance looks good.

6. Conclusion and discussion

In this work, we study the limiting properties of the sample spiked eigenvalues of the noncentral Fisher matrix under high-dimensional setting and Gaussian
assumptions. Similar to the spiked model work, we find a phase transition for the limiting properties of the sample spiked eigenvalues of the noncentral Fisher matrix. In addition, we present the CLT for the sample spiked eigenvalues. As an accessory to the proof of the results of the limiting properties of the sample spiked eigenvalues of the noncentral Fisher matrix, the fluctuations of the spiked eigenvalues of noncentral sample covariance matrix $\mathbf{C}_n$ are studied, which should have its own interests.

General distribution. It would be natural to ask if our results in the current work can be extended from Gaussian to more general distribution of the matrix entries. Our future work will focus on the limiting properties of the sample spiked eigenvalues of the noncentral Fisher matrix under general distribution of the matrix entries.

7. Appendix

7.1. Proof of Theorem 3.1

There is no loss of generality in assuming

$$\Xi = \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} = \begin{pmatrix} \mathbf{D}_{11} & 0 \\ 0 & \mathbf{D}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{D}_{11} & 0 & 0 \\ 0 & \mathbf{D}_{22} & 0 \end{pmatrix},$$

(37)

where $\Xi_1$ is the first $M$ rows of $\Xi$, the diagonal of $\mathbf{D}_{11}$ is composed of the $M$ spiked eigenvalues while the diagonal of $\mathbf{D}_{22}$ is composed of the non-spiked bounded eigenvalues, and $\mathbf{D}_{22} = (\mathbf{D}_{22}, 0)$. According to the structure of $\Xi$ in (37), we decompose $\mathbf{X}$ as following,

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix},$$
where $X_1$ denotes the first $M$ rows of $X$, and $X_2$ stands for the remaining rows of $X$. Then the sample eigenvalues of noncentral sample covariance matrix $C_n$ are sorted in descending order as

$$I_1^{C_n} \geq I_2^{C_n} \geq \cdots \geq I_p^{C_n}.$$ 

If we only consider the sample spiked eigenvalues of $C_n$, $I_i^{C_n}$, $i \in J_k$, $k = 1, \ldots, K$, then the eigen-equation is

$$0 = |I_i^{C_n}I_p - \frac{1}{n}(\Xi + X)(\Xi + X)^*|$$

$$= |I_i^{C_n}I_p - \frac{1}{n}(\Xi_1 + X_1)(\Xi_1 + X_1)^*| \quad (38)$$

$$= |I_i^{C_n}I_M - \frac{1}{n}(\Xi_1 + X_1)(\Xi_1 + X_1)^* - \frac{1}{n}(\Xi_1 + X_1)(\Xi_2 + X_2)^*|$$

$$- \frac{1}{n}(\Xi_2 + X_2)(\Xi_1 + X_1)^* - \frac{1}{n}(\Xi_2 + X_2)(\Xi_2 + X_2)^*|.$$ 

Because $M$ is fixed, $\Xi_3, \Xi_4/n$ and $\Xi_5, \Xi_6/n$ share the same the LSD, then $I_i^{C_n}$ is an outlier for large $n$, i.e., $|I_i^{C_n}I_p - \frac{1}{n}(\Xi_2 + X_2)(\Xi_2 + X_2)^*| \neq 0$. Rewrite (38) by the inverse of a partitioned matrix ([20], Section 0.7.3) and the in-out exchange formula, we have

$$0 = |I_i^{C_n}I_M - \frac{1}{n}(\Xi_1 + X_1)(\Xi_1 + X_1)^*|$$

$$\times \left[I_i^{C_n}I_p - \frac{1}{n}(\Xi_2 + X_2)(\Xi_2 + X_2)^*\right]^{-1} \frac{1}{n}(\Xi_2 + X_2)(\Xi_1 + X_1)^*|$$

$$\iff 0 = \left|I_M - \frac{1}{n}(\Xi_1 + X_1)\left[I_i^{C_n}I_p - \frac{1}{n}(\Xi_2 + X_2)(\Xi_2 + X_2)^*\right]^{-1} (\Xi_1 + X_1)^*\right|.$$ 

If $I_i^{C_n} \neq 0$. For simplicity, we denote $\frac{1}{n}(\Xi_2 + X_2)^*(\Xi_2 + X_2) - I_i^{C_n}I_n^{-1}$ briefly by $A_n(I_i^{C_n})$. When there is no ambiguity, we also rewrite it as $A$. Then

$$\Omega_n^{C_n} \triangleq I_M + \frac{1}{n}(\Xi_1 + X_1)A_n(I_i^{C_n})(\Xi_1 + X_1)^*$$

$$= I_M + \frac{1}{n}(\text{tr} A_n(I_i^{C_n}))I_M - \frac{1}{\tilde{l}_i^{C_n}(1 + c_{1n}m_{2n}(I_i^{C_n}))} \frac{1}{n} \Xi_1 (\Xi_1 + X_1)^* + \Omega_0^{C_n}, \quad (39)$$

where

$$\Omega_0^{C_n}(I_i^{C_n}) = \frac{1}{n} X_1 A X_1^* - \frac{1}{n} (\text{tr} A)I_M + \frac{1}{n} X_1 A (\Xi_1 + X_1)^* + X_1 A (\Xi_1 + X_1)^*$$

$$+ \frac{1}{n} D_{11} \left[A_{11} + \frac{I}{\tilde{l}_i^{C_n}(1 + c_{1n}m_{2n}(I_i^{C_n}))}\right] D_{11}, \quad (40)$$

$$m_{2n}(I_i^{C_n}) = \frac{1}{p - M} \text{tr} \left[\frac{1}{n}(X_{22} + D_{22})(X_{22} + D_{22})^* - I_i^{C_n}I_{p-M}\right]^{-1} \quad (41)$$
\( c_{1n} = (p - M)/(n - M) \), and \( A_{11} \) is the first \( M \times M \) major diagonal submatrix of \( A_n(t_{i_1}^C) \).

Here we announce that the following results of almost sure convergence are true without proofs and the detailed proofs are postponed in Subsection 7.1.1.

\[
A_{11}(t_{i_1}^C) + \frac{1}{t_{i_1}^C + l_{i_1}^C (c_{1n} m_{2n}(t_{i_1}^C))} I_M \xrightarrow{a.s.} 0_{M \times M},
\]

\[
\frac{1}{n} X_1 A_n(t_{i_1}^C) X_1^\top - \frac{1}{n} (\text{tr} A_n(t_{i_1}^C)) I_M \xrightarrow{a.s.} 0_{M \times M},
\]

\[
\frac{1}{n} X_1 A_n(t_{i_1}^C) \Xi_1 + \frac{1}{n} \Xi_1 A_n(t_{i_1}^C) X_1^\top \xrightarrow{a.s.} 0_{M \times M}.
\]

According to (39), we have \( \Omega_0^C \xrightarrow{a.s.} 0_{M \times M} \). By the formula (1.3) in [17], we arrive at

\[
\frac{1}{n} \text{tr} A_n(z) \xrightarrow{a.s.} m_{2n}(z) \quad \text{uniformly for } z.
\]

From

\[
0 = \left| I_M + \frac{1}{n} \text{tr} A_n(t_{i_1}^C) I - \frac{1}{n} D_{11} D_{11}^\top I_{l_{i_1}^C + l_{i_1}^C (c_{1n} m_{2n}(t_{i_1}^C))} + \Omega_0^C(t_{i_1}^C) \right|
\]

we can obtain

\[
\left| I_M + m_{2n}^0(t_{i_1}^C) I_M - \frac{1}{n} D_{11}^{-1} D_{11}^- I_{l_{i_1}^C (1 + c_{1n} m_{2n}^0(t_{i_1}^C))} \right| \xrightarrow{a.s.} 0.
\]

For arbitrary \( k \), let

\[
\lambda_{nk}^C \triangleq a_k \left( 1 - c_{1n} m_{1n}^0(a_k) \right)^2 + (1 - c_{1n}) \left( 1 - c_{1n} m_{1n}^0(a_k) \right)
\]

where \( c_{1n} = (p - M)/(n - M) \), and \( m_{1n}^0 \) denotes the ST of ESD of \( p - M \) bulk eigenvalues of \( \Xi \Xi^\top/n \). An easy calculation shows that

\[
1 + m_{2n}^0(\lambda_{nk}^C) - \frac{a_k}{\lambda_{nk}^C (1 + c_{1n} m_{2n}^0(\lambda_{nk}^C))} = 0,
\]

where \( m_{2n}^0 \) is the ST of LSD of \((\Xi_2 + X_2)(\Xi_2 + X_2)^\top/n\) with \( c_1 \) and \( H \) replaced by \( p - M/(n - M) \) and ESD of \( p - M \) bulk eigenvalues of \( \Xi \Xi^\top/n \). Combining (46) and the fact that the dimension of matrix is finite, there exists \( j' \) (assume \( j \in J_k \)) such that the \( j \)-th diagonal element convergence almost surely to zero. For this \( j' \) we have

\[
\left| I_M + m_{2n}^0(t_{i_1}^C) I_M - \frac{1}{n} D_{11}^{-1} D_{11}^- I_{l_{i_1}^C (1 + c_{1n} m_{2n}^0(t_{i_1}^C))} \right| \xrightarrow{a.s.} 0.
\]

Then subtracting the \( j \)-th one from all the diagonal elements of the matrix in the above determinant, we find the difference has a lower bound except the \( k \)-th block containing \( j \)-th location, i.e.

\[
\frac{a_s - a_k}{l_{i_1}^C (1 + c_{1n} m_{2n}^0(t_{i_1}^C))}, \quad s \notin J_k
\]
is with the lower bound as a result of (5) in Assumption A. So for the diagonal elements of $k$-th block, we have

$$m^0_{2n}(t^C_n) - \frac{a_k}{t^C_n(1 + c_1m^0_{2n}(t^C_n))} - m^0_{2n}(\lambda_{nk}) + \frac{a_k}{\lambda_{nk}(1 + c_1m^0_{2n}(\lambda_{nk}))} \xrightarrow{a.s.} 0$$

or

$$\left( t^C_n - \frac{\lambda_{nk}}{\lambda^C_{nk}} \right) \left[ \lambda^C_{nk}(m^0_{2n})'(\xi_1) + \frac{a_k}{t^C_n(1 + c_1m^0_{2n}(t^C_n))(1 + c_1m^0_{2n}(\lambda_{nk}))} \right] \xrightarrow{a.s.} 0.$$  

By the factor

$$\lambda^C_{nk}(m^0_{2n})'(\xi_1) + \frac{a_k}{t^C_n(1 + c_1m^0_{2n}(t^C_n))} \left[ 1 + c_1m^0_{2n}(\lambda_{nk}) + t^C_n(m^0_{2n})'(\xi_2) \right]$$

has lower bound, we get $(t^C_n - \lambda_{nk})/\lambda^C_{nk} \xrightarrow{a.s.} 0$. If $a_k$ is bounded, the limit $\lambda^C_{nk}$ of $\lambda^C_{nk}$ satisfies

$$0 = 1 - \frac{1 - c_1}{\lambda^C_k} + c_1m_2(\lambda^C_k) + \frac{a_k}{-\lambda^C_k - c_1m_2(\lambda^C_k)}$$

$$\iff a_k = \lambda^C_k \left( 1 + c_1m_2(\lambda^C_k) \right)^2 - \left( 1 - c_1 \right) \left( 1 + c_1m_2(\lambda^C_k) \right)$$

$$\iff a_k = \left[ \lambda^C_k \left( 1 + c_1 \frac{m_1(a_k)}{1 - c_1m_1(a_k)} \right) - (1 - c_1) \right] \left[ 1 + c_1 \frac{m_1(a_k)}{1 - c_1m_1(a_k)} \right]$$

$$\iff \psi(a_k) \triangleq \lambda^C_k - a_k \left( 1 - c_1m_1(a_k) \right)^2 + (1 - c_1) \left( 1 - c_1m_1(a_k) \right),$$

where $m_1$ is the Stieljes transform of the LSD $H$ of $\Xi\Xi^T/n$. The second equivalence relation is a consequence of (6). The proof of Theorem 3.1 is completed.

**Remark 7.1.** If $a_k$ tends to infinite as stated in Assumption A, we only need to multiply by $\sqrt{n} \cdot t^C_{nk}^{-1}$ from both side of (39), and similar arguments above applying to the infinite case, we can get the same conclusion with only describing as: $(t^C_n - \lambda_{nk})/\lambda^C_{nk} \xrightarrow{a.s.} 0$. Therefore, we will not repeat the situation that $a_k$ tending infinite in following proof.

### 7.1.1. Proofs of (42), (43) and (44)

(44) can be obtain by Kolmogorov’s law of large numbers straightly. The proof of (42) and (43) are similar, so we take the proof of (43) as example. We consider the following series for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left( \left\| \frac{1}{n}X_1A_nX_1^* - \frac{1}{n}(\text{tr}A)I_M \right\|_K > \varepsilon \right)$$

$$= \sum_{n=1}^{\infty} P\left( \left\| \frac{1}{n}X_1A_nX_1^* - \frac{1}{n}(\text{tr}A)I_M \right\|_K > \varepsilon, \mathcal{A} \right)$$

$$+ \sum_{n=1}^{\infty} P\left( \left\| \frac{1}{n}X_1A_nX_1^* - \frac{1}{n}(\text{tr}A)I_M \right\|_K > \varepsilon, \mathcal{A}^c \right)$$
where the event $A$ means the spectral norm of $A$ is bounded, i.e., $\|A\| \leq C$ and $\|\cdot\|_K$ means Kolmogorov norm, defined as the largest absolute value of all the entries. Then we have

$$P(\| \frac{1}{n}X_1A_nX_1^* - \frac{1}{n}(\text{tr}A)I_M\|_K > \varepsilon, A^c) \leq P(A^c) = o(n^{-t}),$$

where the last equation is a consequence of the exact separation result of information-plus-noise type matrices in [5]. For the first term, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(\| \frac{1}{n}X_1A_nX_1^* - \frac{1}{n}(\text{tr}A)I_M\|_K > \varepsilon \cap A) = \sum_{n=1}^{\infty} \mathbb{E}[\mathbb{P}(\| \frac{1}{n}X_1A_nX_1^* - \frac{1}{n}(\text{tr}A)I_M\|_K > \varepsilon \cap A) | A]] \leq \sum_{n=1}^{\infty} \mathbb{E} \left[ \left| \frac{1}{\varepsilon r} \mathbb{E} \left| \frac{1}{n}X_1A_nX_1^* - \frac{1}{n}(\text{tr}A)I_M\|_K > \varepsilon \cap A \right| \right|_A \right] ,$$

which implies (48) is summable when $r = 1$. By the Borel-Cantelli lemma, we have

$$\frac{1}{n}X_1A_nX_1^* - \frac{1}{n}(\text{tr}A)I_M \overset{a.s.}{\to} 0_{M \times M}.$$  

7.2. Proof of Theorem 3.2

In this section, we will consider the random vector

$$\gamma_{C_n}^k = \sqrt{n}\{c_n^i / \lambda_{nk}^C - 1, i \in J_k\},$$

where $\lambda_{nk}^C$ is defined as (47). The reason of using $\lambda_{nk}^C$ rather than its limit $\lambda_k^C$ lies in the fact that the convergence may be very slow. The following proof is based on (39) and (40), then

$$0 = I_M + \frac{m_2}{m_0} \frac{1}{\lambda_{nk}^C} + \frac{1}{\lambda_{nk}^C (1 + c_1m_2 / (\lambda_{nk}^C))} + \Omega_0^C(\lambda_{nk}^C) + \varepsilon_1 I_M + \varepsilon_2 \frac{1}{n} D_{11} D_{11}^* + \varepsilon_3 ,$$
where $m_0^2_{nk}$ is the Stieltjes transform of $F^C$ with parameter $H$ and $c_1$ replaced by $H_n$ and $c_{1n}$, and

\[
\varepsilon_1 = \frac{1}{n} \text{tr} A_n(l_i^C) - m_0^2(l_i^C) 
\]

\[
\varepsilon_2 = \frac{l_i^C}{2} - \frac{1}{m_i^C} - m_2^0(l_i^C) - \lambda_{nk}^C + \lambda_{nk}^C c_{1n} m_2^0(\lambda_{nk}^C),
\]

\[
\varepsilon_3 = \Omega_0^C(l_i^C) - \Omega_0^C(\lambda_{nk}^C).
\]

Here we give the estimators of $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ and the detailed proof is postponed in the following part,

\[
\varepsilon_1 = \frac{\lambda_{nk}^C}{\sqrt{n}} \left[ m_0^2(\lambda_{nk}^C) + o_p(1) \right],
\]

\[
\varepsilon_2 = \frac{\lambda_{nk}^C}{\sqrt{n}} \left[ 1 + c_1 m_2(\lambda_{nk}^C) + c_1 \lambda_{nk}^C m_2(\lambda_{nk}^C) \right],
\]

\[
\varepsilon_3 = o_p \left( \frac{1}{\sqrt{n}} \right) 11'.
\]

According to the definition (47), if $i \in J_k$, then we obtain

\[
1 + m_0^2(\lambda_{nk}^C) - \frac{a_k}{\lambda_{nk}^C(1 + c_{1n} m_2^0(\lambda_{nk}^C))} = 0.
\]

We rewrite the $k$-th block matrix of $\Omega_n^C$ as

\[
[\Omega_n^C]_{kk} = \left[ \Omega_0^C(\lambda_{nk}^C) \right]_{kk} + \varepsilon_1 I_{m_k} + \varepsilon_2 a_k I_{m_k} + o_p \left( \frac{1}{\sqrt{n}} \right).
\]

By the discussion of the limiting distribution of $\Omega_0^C(\lambda_{nk}^C)$ and Skorokhod strong representation theorem, for more details, see [32] or [21], on an appropriate probability space, one may redefine the random variables such that $\Omega_0^C$ tends to the normal variables with probability one. Then, the eigen-equation of (39) becomes

\[
0 = \begin{pmatrix}
\frac{a_k(1 - \frac{r}{2})}{\lambda_{nk}^C(1 + \frac{r}{2})} + O(n^{-1/2}) & O(n^{-1/2}) & O(n^{-1/2}) \\
O(n^{-1/2}) & O(n^{-1/2}) & \ddots & \ddots \\
O(n^{-1/2}) & \Omega_0^C |_{kk} + \varepsilon_1 I_{m_k} + \varepsilon_2 a_k I_{m_k} & \ddots & \ddots \\
O(n^{-1/2}) & \ddots & \ddots & O(n^{-1/2}) \\
O(n^{-1/2}) & \frac{a_k(1 - \frac{r}{2})}{\lambda_{nk}^C(1 + \frac{r}{2})} + O(n^{-1/2})
\end{pmatrix}
\]

where $b(\lambda_{nk}^C) = 1 + c_{1n} m_0^2(\lambda_{nk}^C)$ and $[\Omega_0^C]_{kk}$ is $k$-th diagonal block of $\Omega_0^C$. Then multiplying $n^{1/4}$ to the $k$-th block row and column of the determinant of the eigen-equation above, and making $n \to \infty$. Then we have

\[
\sqrt{n} \left( [\Omega_0^C]_{kk} + \varepsilon_1 I_{m_k} + \varepsilon_2 a_k I_{m_k} \right) \xrightarrow{a.s.} 0.
\]
Simplifying the above, we have the random vector \( \gamma^C_{nk} \) tends to a random vector consists of the ordered eigenvalues of GOE (GUE) matrix under real (complex) case with the scale parameter

\[
\theta_1 = \frac{1}{[\lambda^C_{nk} m'_2 + a_k(1+c_1m_2+c_1\lambda^C_{nk} m'_2)]^2} \times \left( m'_2 + \frac{a_k^c c_1 m'_2}{(\lambda^C_{nk})^2 (1+c_1m_2)^4} + \frac{2a_k (1+m_2 + \lambda^C_{nk} m'_2)}{(\lambda^C_{nk})^2 (1+c_1m_2)^2} \right),
\]

where \( m_2 \) and \( m'_2 \) are defined in (6) and (12), \( m'_2 \) and \( m''_2 \) stand for their derivatives at \( \lambda^C_{nk} \), respectively. We shall have establish the proof of theorem 3.2 if we prove the limiting distribution of \( \Omega^C_0 \) and the limits of \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \). In the following, we will put these proofs.

7.2.1. Limiting distribution of \( \Omega^C_0 \)

In this section, we proceed to show the limiting distribution of \( \Omega^C_0 \). According to the definition of \( \Omega^C_0 \) in (40), the proof will be divided into three parts, where

\[
\frac{1}{\sqrt{n}} X_1^* A X_1^* - \frac{1}{\sqrt{n}} (\text{tr} A) I_M / n, \quad (X_1^* \Xi_1^+ + \Xi_1^+ A X_1^*) / n
\]

and

\[
\frac{1}{n} D_{11}^* \left[ A_{11} + \frac{I}{\lambda^C_{nk} (1+c_1m_2n(\lambda^C_{nk}))} \right] D_{11}.
\]

By Theorem 7.1 in [6], it is easy to obtain that the \( k \)-th block of \( M \times M \) matrix \( \frac{1}{\sqrt{n}} X_1^* A X_1^* - \frac{1}{\sqrt{n}} (\text{tr} A(\lambda^C_{nk})) I_M \) tends to \( m_k \)-dimensional GOE (GUE) matrix with scale parameter \( m'_2(\lambda^C_{nk}) \) under real (complex) case.

Having disposed of \( \frac{1}{\sqrt{n}} X_1^* A X_1^* - \frac{1}{\sqrt{n}} (\text{tr} A(\lambda^C_{nk})) I_M \), we can now turn to the proof of (57). To complete the proof of (57), we consider the limiting distribution of

\[
A_{11}(\lambda^C_{nk}) + \frac{1}{\lambda^C_{nk} (1+c_1m_2n(\lambda^C_{nk}))} I_M.
\]

Rewrite \( \Xi_2 \) as \( \Xi_2 = (0_{p-M,M}, (D_{22})_{p-M,n-M}) \) and similar considerations apply to \( X_2, X_2' = ((X_21)_{p-M,M}, (X_22)_{p-M,n-M}) \). Then, we have

\[
A = \left( \frac{1}{n} X_{21}^* X_{21} - \lambda^C_{nk} I_M, \frac{1}{n} X_{21}^* (X_{22} + D_{22})^* X_{21}, \frac{1}{n} (X_{22} + D_{22})^* (X_{22} + D_{22}) - \lambda^C_{nk} I_{n-M} \right)^{-1}.
\]
Then, the first $M \times M$ major diagonal submatrix is

$$A_{11} = \left( \frac{1}{n} X_{21}^* X_{21} - \lambda_{nk} C I_M - \frac{1}{n} X_{21}^* (X_{22} + \bar{D}_{22}) A_{22} - \frac{1}{n} (X_{22} + \bar{D}_{22})^* X_{21} \right)^{-1}$$

$$= \left( -\lambda_{nk} C I_M - \frac{1}{n} X_{21}^* \left[ I_{p-M} - \frac{1}{n} (D_{22} + X_{22}) A_{22} (D_{22} + X_{22})^* \right] X_{21} \right)^{-1}$$

$$= \left( -\lambda_{nk} C I_M - \frac{\lambda_{nk} C}{n} X_{21}^* \bar{A}_{22} X_{21} \right)^{-1}$$

$$= \left( -\lambda_{nk} C I_M - \frac{\lambda_{nk} C}{n} (\text{tr} \bar{A}_{22}) I_M - \Omega^n_{1} (\lambda_{nk}, X_{21}) \right)^{-1},$$

where

$$A_{22} = \left[ \frac{1}{n} (X_{22} + \bar{D}_{22})^* (X_{22} + \bar{D}_{22}) - \lambda_{nk} C I_{n-M} \right]^{-1},$$

$$\hat{A}_{22} = \left[ \frac{1}{n} (X_{22} + \bar{D}_{22}) (X_{22} + \bar{D}_{22})^* - \lambda_{nk} C I_{p-M} \right]^{-1},$$

$$\Omega^n_{1} (\lambda_{nk}, X_{21}) = \frac{\lambda_{nk}}{n} \left[ X_{21}^* \hat{A}_{22} X_{21} - (\text{tr} \hat{A}_{22}) I_M \right].$$

We emphasize that both $A_{22}$ and $A$ are noncentral sample covariance matrices with the same limiting noncentral parameter matrix. So the Stieltjes transform of LSD of $A_{22}$ is $m_{21}(\cdot)$. Similar arguments apply to $\hat{A}_{22}$, we conclude that

$$\frac{1}{p-M} \text{tr} \hat{A}_{22}$$

tends to $m_{21}(\lambda_{nk} C)$ with probability one. By the CLT of the quadratic form, we have

$$[\Omega^n_{1} (\lambda_{nk}, X_{21})]_{ij} = O_p \left( \frac{\lambda_{nk}}{\sqrt{n}} \right),$$

and the $k$-th block of

$$\frac{1}{\sqrt{p-M}} (X_{21}^* \hat{A}_{22} X_{21} - (\text{tr} \hat{A}_{22}) I_M)$$

tends to $m_{k}$-dimensional GOE (GUE) matrix with scale parameter $m_{2}^{'}(\lambda_{nk} C)$ under real (complex) case. Moreover,

$$A_{11} = \frac{-1}{\lambda_{nk} C + \lambda_{nk} C \frac{p-M}{n} m_{2n}(\lambda_{nk})} I_M$$

$$= \frac{1}{\lambda_{nk} C + \lambda_{nk} C \frac{p-M}{n} m_{2n}(\lambda_{nk})} \Omega^n_{1} (\lambda_{nk}, X_{21}) A_{11}$$

$$= \frac{\Omega^n_{1} (\lambda_{nk}, X_{21})}{[\lambda_{nk} C + \lambda_{nk} C \frac{p-M}{n} m_{2n}(\lambda_{nk})]^2} + \frac{[\Omega^n_{1} (\lambda_{nk}, X_{21})]^2 A_{11}}{[\lambda_{nk} C + \lambda_{nk} C \frac{p-M}{n} m_{2n}(\lambda_{nk})]^2} \quad \text{(59)}$$

where

$$\frac{[\Omega^n_{1} (\lambda_{nk}, X_{21})]^2 A_{11}}{[\lambda_{nk} C + \lambda_{nk} C \frac{p-M}{n} m_{2n}(\lambda_{nk})]^2} = O_p \left( \frac{1}{n} \right) \| A_{11} \| I_{11}'.$$

By (59) and classical CLT, it is easy to see that the corresponding block of

$$\frac{1}{\sqrt{p-M}} D_{11} \left( A_{11} + \bar{C}_{nk} \left( 1 + \frac{1}{p-M} \right) \sqrt{n} \right) \left( A_{11} + \bar{C}_{nk} \left( 1 + \frac{1}{p-M} \right) \sqrt{n} \right)^* I_{M} D_{11}$$

tends to the GOE (GUE) matrix under
Spiked eigenvalues of noncentral Fisher matrix with applications

real (complex) case with scale parameter

\[
\frac{a_k^2 c_1 m_2' (\lambda_k^C)}{(\lambda_k^C)^2 (1 + c_1 m_2(\lambda_k^C))^4}.
\]

We shall have established the limiting distribution of \( \Omega_0^C \) if we give the limiting distribution of

\[
\frac{1}{\sqrt{n}} X_1 A \Xi_1^* + \frac{1}{\sqrt{n}} \Xi_1 A X_1^*. \quad (60)
\]

It is easily seen that the elements \((s, t) \) \((s \leq M, t \leq M)\) of \((60)\) is \( 1/\sqrt{d} x_s a_t + 1/\sqrt{d} a_s^* x_t^*\), where \( x_s \) is the \( s \)-th row of \( X_1 \) and \( a_t \) is the \( t \)-th column of \( A \). Since \( X_1 \) is independent of \( A \), the limiting distribution of \( 1/\sqrt{d} x_s a_t + 1/\sqrt{d} a_s^* x_t^*\) is Gaussian under given \( A \). The mean of the Gaussian is zero and its variance is present under two different situations. Under the real samples situation, the variance is equal to

\[
\sigma_n^2(s, t) = \begin{cases} 4 a_s E a_s^T a_s^*, & \text{if } s = t \\ a_s E a_s^T a_s + a_t E a_t^* a_t, & \text{if } s \neq t \end{cases} \quad (61)
\]

and under the complex samples situation, the variance is equal to

\[
\sigma_n^2(s, t) = a_s E a_s^* a_s^* + a_t E a_t^* a_t. \quad (62)
\]

What is left is to compute the limits of \((61)\) and \((62)\). By the inverse blockwise matrix formula, we know that the vector consisted of the first \( M \) components of \( a_t \) is equal to the \( t \)-th column \( A_{11t} \) of \( A_{11} \) and the vector consisted of next \( n - M \) components is equal to the \( j \)-th column of \( A_{22} \frac{1}{n} (X_{22} + \tilde{D}_{22}) X_{21} A_{11t} \).

By \((58)\), we have

\[
E a_s^* a_t = E A_{11t} A_{11t} + E A_{11t} X_{21} \frac{1}{n} (X_{22} + \tilde{D}_{22}) A_{22}^2 \frac{1}{n} (X_{22} + \tilde{D}_{22})^* X_{21} A_{11t}
\]

\[
= E A_{11t} \left[ I_M + \frac{1}{n} X_{21} \frac{1}{n} (X_{22} + \tilde{D}_{22}) A_{22}^2 (X_{22} + \tilde{D}_{22})^* X_{21} \right] A_{11t}
\]

\[
= E A_{11t} \left[ I_M + \frac{1}{n} \text{tr} \left[ \frac{1}{n} (X_{22} + \tilde{D}_{22}) A_{22}^2 (X_{22} + \tilde{D}_{22})^* \right] I_M + O_p \left( \frac{1}{\sqrt{n}} \right) \right] A_{11t},
\]

where

\[
\left( 1 + \frac{1}{n} \text{tr} \frac{1}{n} (X_{22} + \tilde{D}_{22}) A_{22}^2 (X_{22} + \tilde{D}_{22})^* \right)
\]

\[
= \left( 1 + \frac{1}{n} \text{tr} A_{22}^2 \frac{1}{n} (X_{22} + \tilde{D}_{22})^* (X_{22} + \tilde{D}_{22}) \right)
\]

\[
= \left( 1 + \frac{1}{n} \text{tr} A_{22} + \frac{\lambda_n^C}{n} \text{tr} A_{22}^2 \right) = 1 + m_{2n}(\lambda_n^C) + \lambda_n^C m_{2n}'(\lambda_n^C).
\]
where the last equality is the consequence of Theorem 1 in \[9\].

As the scale parameter \(\Omega\) among them. We conclude that the limiting distribution of the \(k\) bounded a.s. and Theorem 3.1, we have

\[
Ea_s^*a_t = \left(1 + m_{2n}(\lambda_m^C) + \lambda_m^C m'_2(\lambda_m^C)\right) \left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right)
\]

\[
= \frac{1}{(\lambda_m^C)^2(1 + c_1 m_2(\lambda_m^C))^2} (1 + m_2(\lambda_m^C) + \lambda_m^C m'_2(\lambda_m^C)),
\]

(63)

In the same manner we can see that

\[
Ea_s^*a_t \to 0, \text{ for } s \neq t,
\]

(64)

which implies the variables in asymmetric positions are asymptotic independent.

From the above analysis, we obtain the corresponding block of \(X^*_1A\Xi_1^* + \frac{1}{\sqrt{n}}\Xi_1AX_1^*\) tends to a \(m_k \times m_k\) GOE (GUE) matrix under real (complex) case with scale parameter

\[
\frac{2a_k(1 + m_2(\lambda_m^C) + \lambda_m^C m'_2(\lambda_m^C))}{(\lambda_m^C)^2(1 + c_1 m_2(\lambda_m^C))^2}.
\]

The third moment of normal population is zero so that the limiting distribution of \(\frac{1}{n}X_1^*A\Xi_1^* - \frac{1}{n}(\text{tr}A)I_m \frac{1}{n}(X_1^*A\Xi_1^* + \Xi_1AX_1^*)\) and \(57\) are independent among them. We conclude that the limiting distribution of the \(k\)-th block of \(\Omega_0^C\) equals the \(m_k \times m_k\) GOE (GUE) matrix under real (complex) matrix with the scale parameter

\[
m'_2(\lambda_m^C) + \frac{a_k^2 c_1 m'_2(\lambda_m^C)}{(\lambda_m^C)^2(1 + c_1 m_2(\lambda_m^C))^2} + \frac{2a_k(1 + m_2(\lambda_m^C) + \lambda_m^C m'_2(\lambda_m^C))}{(\lambda_m^C)^2(1 + c_1 m_2(\lambda_m^C))^2}.
\]

(65)

7.2.2. Limits of \(\varepsilon_1, \varepsilon_2 \text{ and } \varepsilon_3\)

In this section, we proceed to show the limits of \(\varepsilon_1, \varepsilon_2 \text{ and } \varepsilon_3\) defined in \((49)-(51)\).

To deal with \(\varepsilon_1\), we note that

\[
\varepsilon_1 = \frac{\gamma_k^C \lambda_k^C}{\sqrt{n}} \frac{1}{n} \text{tr}A_n(\lambda^C_m)A_n(l^C_m) + \frac{1}{n} \text{tr}A_n(\lambda^C_m) - \frac{m_0}{n} \lambda^C_m
\]

\[
= \frac{\gamma_k^C \lambda_k^C}{\sqrt{n}} \frac{1}{n} \text{tr}A_n(\lambda^C_m) + \frac{\gamma_k^C}{\sqrt{n}} \lambda_k^C \frac{1}{n} \text{tr}A_n(\lambda^C_m)[A_n(l^C_m) - A_n(\lambda^C_m)] + O_p\left(\frac{1}{n}\right),
\]

where the last equality is the consequence of Theorem 1 in \[9\]. As \(\|A_n(l^C_m)\|\) is bounded a.s. and Theorem 3.1, we have

\[
\frac{1}{n} \text{tr}A_n(\lambda^C_m) \xrightarrow{a.s.} m'_2(\lambda^C_m),
\]

\[
\frac{\gamma_k^C \lambda_k^C}{\sqrt{n}} \frac{1}{n} \text{tr}A_n(\lambda^C_m)[A_n(l^C_m) - A_n(\lambda^C_m)] = o_p\left(\frac{\gamma_k^C}{\sqrt{n}} \lambda_k^C\right).
\]
The task is now to consider the limit of $\varepsilon_2$.

$$
\varepsilon_2 = \frac{-1}{l_i^2 + l_i^2 \frac{p-M}{n} m_{2n}(l_i^2) - \lambda_{nk}^C + \lambda_{nk}^C c_{1n} m_{2n}(\lambda_{nk}^C)} \varepsilon_2
$$

$$
= \frac{l_i^2 (1 + \frac{p-M}{n} m_{2n}(l_i^2) - \lambda_{nk}^C (1 + c_{1n} m_{2n}(\lambda_{nk}^C))}{[\lambda_{nk}^C + \lambda_{nk}^C c_{1n} m_{2n}(\lambda_{nk}^C)]^2}
+ \frac{(l_i^2 - \lambda_{nk}^C) (1 + c_{1n} m_{2n}(l_i^2) + \lambda_{nk}^C c_{1n} (m_{2n}^0 (\lambda_{nk}^C))^2 + o_p(\frac{1}{n})}{[\lambda_{nk}^C + \lambda_{nk}^C c_{1n} m_{2n}(\lambda_{nk}^C)]^2 (l_i^2 + l_i^2 \frac{p-M}{n} m_{2n}(l_i^2))}
+ \frac{\gamma_{k}^C}{\sqrt{n}} \left[ 1 + c_{1n} (l_i^2 - \lambda_{nk}^C) + c_{1n}^2 (l_i^2 - \lambda_{nk}^C) \right] + o_p(1),
$$

the last equality being a consequence of

$$
\frac{(l_i^2 - \lambda_{nk}^C) (1 + c_{1n} m_{2n}(l_i^2) + \lambda_{nk}^C c_{1n} (m_{2n}^0 (\lambda_{nk}^C))^2 + o_p(\frac{1}{n})}{[\lambda_{nk}^C + \lambda_{nk}^C c_{1n} m_{2n}(\lambda_{nk}^C)]^2 (l_i^2 + l_i^2 \frac{p-M}{n} m_{2n}(l_i^2))} = o_p \left( \frac{\gamma_{k}^C}{\sqrt{n}} \right).
$$

It remains to show the limit of $\varepsilon_3$. Here, we focus on $\frac{1}{n} X_1 A X_1^* \frac{1}{n} - \frac{1}{n} \text{tr}(A) I_M$ and other terms can be handle in a similar way. According to $a_k$, we have

$$
\frac{1}{n} X_1 A (l_i^2)^* X_1^* - \frac{1}{n} \text{tr}(A) (l_i^2)^* I_M = \frac{1}{n} X_1 A (l_i^2)^* X_1^* - \frac{1}{n} \text{tr}(A) (l_i^2)^* I_M.
$$

$$
= O_p \left( \frac{\gamma_{k}^C}{\sqrt{n}} \right) I_M = o_p \left( \frac{\gamma_{k}^C}{\sqrt{n}} \right) I_M.
$$

Note that the assumption the rate of $a_k$ diverging to infinity in Assumption A is used here. To be specific, We find that if the rate of $a_k$ diverging to infinity is more than $\sqrt{n}$, $\lambda_{nk}^C / \sqrt{n}$ will tend infinity.

Combining the limiting distribution of $\Omega_0 C^*$, we obtain the limiting distribution of the random vector $\{\gamma_{k}^C\}$ and its key scale parameter is defined in (56).

### 7.3. Proof of Theorem 3.3

At first, we consider the first order limit of the spiked eigenvalues of $F_p$ under given the matrix sequence $\{C_n\}$. Recall

$$
C_n = \frac{1}{n} (\Xi + X)(\Xi + X)^* = U \left( \begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right) U^*,
$$

$$
S_N = \frac{1}{N} Y_1 Y_1^* = \frac{1}{N} \left( \begin{array}{cc} Y_1 & Y_2 \end{array} \right) \left( \begin{array}{cc} Y_1^* & Y_2^* \end{array} \right) = \frac{1}{N} \left( \begin{array}{cccc} Y_1 Y_1^* & Y_1 Y_2^* \\ Y_2 Y_1^* & Y_2 Y_2^* \end{array} \right).
$$
where $\Sigma_1$ is an $M \times M$ diagonal matrix and $Y_1$ denotes the first $M$ rows of $Y$. The matrix $C_n$ can be seen as a general non-negative definite matrix with the eigenvalues formed in descending order,

$$l_{1}^{C_n} \geq l_{2}^{C_n} \geq \cdots \geq l_{p}^{C_n}. \quad (66)$$

We consider the eigen-equation

$$|C_n S_N^{-1} - \lambda I| = 0 \iff |C_n - \lambda S_N| = 0$$

$$\iff |U \left( \begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right) U^* - \frac{\lambda}{N} Y_N Y_N^*| = 0$$

$$\iff \left| \left( \begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right) - \frac{\lambda}{N} U^* Y_N Y_N^* U \right| = 0.$$

We denote $U^* Y_N Y_N^* U / N$ as $\tilde{Y}_N \tilde{Y}_N^* / N$. The entries of $Y_N$ are the standard normal so that both $\tilde{Y}_N \tilde{Y}_N^* / N$ and $Y_N Y_N^* / N$ have the same distribution. If there is no confusion, we will still write the notation $Y_N$. Then the eigen-equation becomes

$$\left| \left( \begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right) - \frac{\lambda}{N} Y_N Y_N^* \right| = 0$$

$$\iff \left| \left( \begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right) - \left( \frac{\lambda}{N} Y_1 Y_1^* \frac{1}{N} Y_1 Y_2^* \frac{1}{N} Y_2 Y_1^* \frac{1}{N} Y_2 Y_2^* \right) \right| = 0.$$

According to the sample spiked eigenvalues $l_i, i \in J_k, k = 1, \cdots, K$ of $F_p = C_n S_N^{-1}$, we have $|\Sigma_2 - l_i \frac{1}{N} Y_2 Y_2^*| \neq 0$ almost surely, then

$$|\Sigma_1 - l_i \frac{1}{N} Y_1 Y_1^* - l_i^2 \frac{1}{N} Y_1 Y_2^* \left( \Sigma_2 - l_i \frac{1}{N} Y_2 Y_2^* \right)^{-1} \frac{1}{N} Y_2 Y_1^*| = 0$$

$$|\Sigma_1 - l_i \frac{1}{N} Y_1 \left[ I_N + l_i \frac{1}{N} Y_2 \left( \Sigma_2 - l_i \frac{1}{N} Y_2 Y_2^* \right)^{-1} Y_2 \right] Y_1^*| = 0$$

$$|\Sigma_1 - l_i \frac{1}{N} \text{tr} \left[ I_N + l_i \frac{1}{N} Y_2 \left( \Sigma_2 - l_i \frac{1}{N} Y_2 Y_2^* \right)^{-1} Y_2 \right] I_M + \Omega_q^F (l_i)| = 0$$

$$|\Sigma_1 - l_i \left( 1 + l_i \frac{N}{p-M} \frac{1}{p-M} \text{tr} \left[ \left( \Sigma_2 - l_i \frac{1}{N} Y_2 Y_2^* \right)^{-1} \frac{1}{N} Y_2 Y_2^* \right] \right) I_M + \Omega_q^F (l_i)| = 0$$

$$|\Sigma_1 - l_i \left( 1 + l_i \frac{N}{p-M} \frac{1}{p-M} \text{tr} \left[ \left( \Sigma_2 \left( \frac{1}{N} Y_2 Y_2^* \right)^{-1} - l_i I \right)^{-1} \right] \right) I + \Omega_q^F (l_i)| = 0,$$

where

$$\Omega_q^F (l_i) = \frac{l_i}{N} Y_1 \left[ I_N + l_i \frac{1}{N} Y_2 \left( \Sigma_2 - l_i \frac{1}{N} Y_2 Y_2^* \right)^{-1} Y_2 \right] Y_1^* \quad (67)$$

$$-\frac{l_i}{N} \text{tr} \left[ I_N + l_i \frac{1}{N} Y_2 \left( \Sigma_2 - l_i \frac{1}{N} Y_2 Y_2^* \right)^{-1} Y_2 \right] I_M. \quad (68)$$
In this section, we consider the CLT of the random vector \( \gamma \).

### 7.4.1. The conditional limiting distribution of \( \gamma \)

According to (8), we have

\[
\lambda \quad \text{where} \quad \gamma
\]

At first, we give the conditional limiting distribution of corresponding empirical parameters, and which will be divided into two steps.

According to (8), we have

\[
z m_3(z) = (z + c_2 z^2 m_3(z)) m_2(z + c_2 z^2 m_3(z))
\]

\[\iff \lambda_k m_3(\lambda_k) = \lambda_k (1 + c_2 \lambda_3 m_3(\lambda_k)) m_2(\lambda_k (1 + c_2 \lambda_3 m_3(\lambda_k))) = \lambda_k^c m_2(\lambda_k^c),\]

i.e.,

\[
\lambda_k^c = \lambda_k (1 + c_2 \lambda_k^c m_2(\lambda_k^c))
\]

\[
\psi_F(\lambda_k^c) \triangleq \lambda_k = \frac{\lambda_k^c}{1 + c_2 \lambda_k^c m_2(\lambda_k^c)} = \frac{\lambda_k^c}{1 + c_2 \lambda_3 m_3(\lambda_k)}.
\]

From what has already been proved, we conclude that the first order limit of \( \lambda_k \) is independent of \( \{C_n\} \), only related to the limit of their spiked eigenvalues. An easy computation shows the relationship between \( \lambda_k \) and \( a_k \):

\[
\lambda_k = \frac{\lambda_k^c}{1 + c_2 \lambda_k^c m_2(\lambda_k^c)}, \quad \lambda_k^c = a_k (1 - c_1 m_1(a_k))^2 + (1 - c_1) (1 - c_1 m_1(a_k)).
\]

## 7.4. Proof of Theorem 3.4

We are now in a position to study the asymptotic distribution of the random vector

\[
\gamma_{Nk} = (\sqrt{N} (l_i - \lambda_{Nk}) / \lambda_{Nk}, i \in J_k)
\]

where \( \lambda_{Nk} = \psi_F(\lambda_{nk}^c) \) with the parameters in function \( \psi_F \) replaced by the corresponding empirical parameters, and which will be divided into two steps. At first, we give the conditional limiting distribution of \( \gamma_{Nk} | C_n \), then we will find the limiting distribution are independent with the choice of conditioning \( C_n \). Secondly, combining the above theorems and the following subsection, we can complete the proof of Theorem 3.4.

### 7.4.1. The conditional limiting distribution of \( \gamma_{Nk} | C_n \)

In this section, we consider the CLT of the random vector \( \gamma_{Nk} | C_n = \{\sqrt{N} (l_i - \psi_F(l_i^C_n)) / \psi_F(l_i^C_n), i \in J_k\} \). Recall the eigen-equation

\[
\left| \Sigma_1 - l_i \left( 1 + l_i \frac{M}{N} \right) \text{tr} \left( \Sigma_2 \left( \frac{1}{N} Y Y^* \right)^{-1} - l_i \mathbf{I}_{M-\mathbf{I}} \right) \mathbf{I} M + \Psi_{N}(l_i) \right| = 0
\]

\[
\left| \Sigma_1 - \psi_F(l_i^C_n) \left( 1 + \frac{M}{N} \psi_F(l_i^C_n) m_3 N \psi_F(l_i^C_n) \right) \mathbf{I} M + \Psi_{N}(\psi_F(l_i^C_n)) + \epsilon_1 \right| = 0
\]
where
\[
m_{3N}(\psi_F(l_i^{C_n})) = \frac{1}{p-M} \text{tr} \left( \Sigma_2 \left( \frac{1}{N} Y_2^* Y_2 \right)^{-1} - \psi_F(l_i^{C_n}) I_{p-M} \right)^{-1}
\]
\[
\varepsilon_1 = \psi_F(l_i^{C_n}) \left( 1 + \psi_F(l_i^{C_n}) \frac{p-M}{N} m_{3N}(\psi_F(l_i^{C_n})) \right) I_M
- l_i \left( 1 + l_i \frac{p-M}{N} m_{3N}(l_i) \right) I_M + \Omega_F^\varepsilon(l_i) - \Omega_F^\varepsilon(\psi_F(l_i^{C_n})).
\]

By simple calculation, we set \( c_{2N} = (p-M)/N \) and have
\[
\varepsilon_1 = - \left( l_i - \psi_F(l_i^{C_n}) - (l_i^2 - (\psi_F(l_i^{C_n}))^2) c_{2N} m_{3N}(l_i) \right)
- (\psi_F(l_i^{C_n}))^2 c_{2N} m_{3N}(\psi_F(l_i^{C_n}))(l_i - \psi_F(l_i^{C_n}))(1 + o(1)) I_M
+ \Omega_F^\varepsilon(l_i) - \Omega_F^\varepsilon(\psi_F(l_i^{C_n})) \]
i.e.
\[
\varepsilon_1 = - \frac{\gamma_{Nk}|C_n|}{\sqrt{N}} \lambda_k \left[ 1 + 2 \lambda_k c_2 m_3(\lambda_k) + \lambda_k^2 c_3 m_3(\lambda_k) \right] (1 + o_p(1)) I + o_p \left( \frac{l_i}{\sqrt{N}} \right) I I'.
\]

By (71), we have
\[
l_i^{C_n} = \psi_F(l_i^{C_n})(1 + c_2 \psi_F(l_i^{C_n}) m_3(l_i^{C_n})).
\]

We recall
\[
\left| \Sigma_1 - \psi_F(l_i^{C_n}) \left( 1 + \psi_F(l_i^{C_n}) c_{2N} m_{3N}(\psi_F(l_i^{C_n})) \right) I_M + \Omega_F^\varepsilon(\psi_F(l_i^{C_n})) + \varepsilon_1 I_M \right| = 0
\]
becomes
\[
0 = \begin{vmatrix}
\lambda_F - l_i^{C_n} + O \left( \frac{\psi_F(l_i^{C_n})}{\sqrt{N}} \right) & \cdots & O \left( \frac{\psi_F(l_i^{C_n})}{\sqrt{N}} \right) & \cdots & O \left( \frac{\psi_F(l_i^{C_n})}{\sqrt{N}} \right) \\
O \left( \frac{\psi_F(l_i^{C_n})}{\sqrt{N}} \right) & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & |\Omega_F^\varepsilon|_{kk} + \varepsilon_1 I_{m_k} & \cdots & \cdots \\
O \left( \frac{\psi_F(l_i^{C_n})}{\sqrt{N}} \right) & \cdots & \cdots & \cdots & O \left( \frac{\psi_F(l_i^{C_n})}{\sqrt{N}} \right) \\
O \left( \frac{\psi_F(l_i^{C_n})}{\sqrt{N}} \right) & \cdots & \cdots & \cdots & \lambda_M^{C_n} - l_i^{C_n} + O \left( \frac{\psi_F(l_i^{C_n})}{\sqrt{N}} \right)
\end{vmatrix}
\]
(74)

where \(|\Omega_F^\varepsilon|_{kk}\) is k-th diagonal block of \(\Omega_F^\varepsilon(\psi_F(l_i^{C_n}))\).

By Skorokhod strong representation theorem (for more details, see [32] or [21]), on an appropriate probability space, one may redefine the random variables such that \(\Omega_F^\varepsilon\) tends to the Gaussian variables with probability one. Multiplying \((\psi_F(l_i^{C_n}))^{-1/2} N^{1/4}\) to the k-th block row and column of the determinant in (74), \(p \to \infty\) and \(N \to \infty\). It is easily seen that all non-diagonal elements tend to zero and all the diagonal entries except the k-th are bounded away from zero as \(p \to \infty\) or \(N \to \infty\). Therefore,
\[
|\sqrt{N} \Omega_F^\varepsilon|_{kk} - (\gamma_{Nk}|C_n|) \lambda_k \theta(\lambda_k) I_{m_k} \xrightarrow{a.s.} 0,
\]
where

\[ \vartheta(\lambda_k) = 1 + 2\lambda_k c_2 m_3(\lambda_k) + c_2 \lambda_k^2 m_3'(\lambda_k). \] (75)

By classical CLT, we have \([\sqrt{N}\Omega^F]_{kk}\) tends to an \(m_k\)-dimensional GOE (GUE) matrix under real (complex) case with the scale parameter \(\lambda_k^2 \vartheta\). In fact, the scale parameter is the limit of

\[
\frac{l_i^2}{N} \text{tr} \left[ I + l_i \frac{1}{N} Y_2^* \left( \Sigma_2 - l_i \frac{1}{N} Y_2 Y_2^* \right)^{-1} Y_2 \right]^2 \\
= \frac{l_i^2}{N} \text{tr} I_N + 2 \frac{l_i^3}{N} \text{tr} \left( \Sigma_2 \left( \frac{1}{N} Y_2 Y_2^* \right)^{-1} - l_i I \right)^{-1} + \frac{l_i^4}{N} \text{tr} \left( \Sigma_2 \left( \frac{1}{N} Y_2 Y_2^* \right)^{-1} - l_i I \right)^{-2}
\]

a.s. \(\rightarrow \lambda_k^2 (1 + 2c_2 \lambda_k m_3(\lambda_k) + \lambda_k^2 c_2 m_3'(\lambda_k)) = \lambda_k^2 \vartheta(\lambda_k).\)

Then we conclude that the conditional limiting distribution of \(\gamma_{Nk}|C_n\) equals the joint distribution of the eigenvalues of GOE (GUE) matrix with the scale parameter \(1/\vartheta\).

7.4.2. The limiting distribution of \(\gamma_{Nk}\)

In this part, we will give the asymptotic distribution of \(\gamma_{Nk} = (\sqrt{n} (l_i/\lambda_{Nk} - 1), i \in J_k)\). It is worth pointing out that the asymptotic distribution of \(\gamma_{Nk} = (\sqrt{n} (l_i/\lambda_{Nk} - 1), i \in J_k)\) is without the condition \(C_n\). According to (72), we have

\[
\gamma_{Nk} = \sqrt{n} l_i - \lambda_{Nk} \lambda_{Nk} \lambda_{Nk} = \sqrt{n} l_i - \psi_F(l_i^{C_n}) + \psi_F(l_i^{C_n}) - \psi_F(\lambda_{Nk}^{C_n}) \\
= \sqrt{n} \frac{l_i - \lambda_{Nk}^{C_n}}{\psi_F(l_i^{C_n})} + \psi_F(l_i^{C_n}) - \psi_F(\lambda_{Nk}^{C_n}) \lambda_{Nk}^{C_n} \psi_F(\lambda_{Nk}^{C_n}) (1 + o(1)),
\]

where the condition limiting distribution of

\[ \sqrt{n} \frac{l_i - \lambda_{Nk}^{C_n}}{\psi_F(l_i^{C_n})} \]

is independent with the condition, so the asymptotic distribution of the first term of (76) is the joint distribution of the order eigenvalues of a GOE (GUE) matrix with parameter \(c_2/(c_1 \ast \vartheta)\). According to Subsection 7.2, it follows that the limiting distribution of

\[ \sqrt{n} \frac{l_i^{C_n} - \lambda_{Nk}^{C_n}}{\lambda_{Nk}^{C_n}} \]

equals the joint distribution of the order eigenvalues of a GOE (GUE) matrix with parameter \(\theta_1\) defined in (56). Combining (70) with (71), we obtain

\[ \frac{\lambda_{Nk}^{C_n}}{\psi_F(\lambda_{Nk}^{C_n})} = \lambda_{Nk}^{C_n} \xrightarrow{a.s.} \lambda_{Nk} \xrightarrow{a.s.} 1 + c_2 \lambda_k m_3(\lambda_k), \]
and
\[ \psi_F'(\lambda_{nk}^C) \xrightarrow{a.s.} \frac{1 - c_2(\lambda_{nk}^C)^2 m_2'(\lambda_{nk}^C)}{(1 + c_2 \lambda_{nk} m_3(\lambda_{nk}))^2}. \]

Then the asymptotic distribution of the second term of (76) is the same as the eigenvalues of a GOE (GUE) matrix with parameter
\[ \left[ 1 - c_2(\lambda_{nk}^C)^2 m_2'(\lambda_{nk}^C) \right]^2 \theta_1. \]

In summary, the limiting distribution of \( \gamma_{Nk} \) is related to that of eigenvalues of the GOE (GUE) matrix with parameter
\[ \frac{c_2}{c_1 \cdot \theta} + \left[ 1 - c_2(\lambda_{nk}^C)^2 m_2'(\lambda_{nk}^C) \right]^2 \theta_1. \]

### 7.5. Proof of Theorem 4.3

According to Theorem 4.1, we know that there exists a function relation between sample canonical correlation coefficients and the eigenvalues of a special noncentral Fisher matrix. The noncentral parameter matrix defined in (22) is a random matrix, so the proof of Theorem 4.3 cannot be obtained directly by Theorem 3.4 and 4.1. Now we will present the details of the proof.

Consider the random variable
\[ \gamma_0^k = \sqrt{q} \left( \frac{l_i - \Psi(\alpha_k)}{\Psi(\alpha_k)} \right), \quad \text{for } i \in J_k, \]
we have
\[ \gamma_k^0 = \sqrt{q} \left( \frac{l_i - \psi_F \circ \psi_C(l_i^\Xi)}{\psi_F \circ \psi_C(l_i^\Xi)} \frac{\psi_F \circ \psi_C(l_i^\Xi) - \Psi(\alpha_k)}{\Psi(\alpha_k)} \right). \]

Under the Assumption d’ and given \( \hat{Y} \), the limiting distribution of first term in \( \gamma_k^0 \) can be obtained by Theorem 3.4 and its covariance satisfies (15), which is independent of selection \( \hat{Y} \). We apply the Mean value theorem to the second term in \( \gamma_k^0 \), i.e.,
\[ \sqrt{q} \frac{\psi_F \circ \psi_C(l_i^\Xi) - \Psi(\alpha_k)}{\Psi(\alpha_k)} = \sqrt{n} \frac{l_i^\Xi - \psi(\alpha_k)}{\sqrt{n} \Psi(\alpha_k)} \frac{\sqrt{q} \psi(\alpha_k)}{\sqrt{n} \Psi(\alpha_k)} \psi_F'(\xi_1) \psi_C'(\xi_2), \]
where \( \xi_1 \in (\Psi_C(\alpha_k), \psi_C(l_i^\Xi)) \) or \( (\psi_C(l_i^\Xi), \Psi_C(\alpha_k)) \), and \( \xi_2 \in (\psi(\alpha_k), l_i^\Xi) \) or \( (l_i^\Xi, \psi(\alpha_k)) \). By Theorem 3.1 and 3.3, we have
\[ \psi_F'(\xi_1) \xrightarrow{a.s.} \psi_F'(\psi_C(\alpha_k)) \]
\[ \psi_C'(\xi_2) - \psi_C'(\psi(\alpha_k)) \xrightarrow{a.s.} 0. \]
where
\[
\psi_F(\Psi(\alpha_k)) = \frac{1 - c_4 \Psi_C(\alpha_k) m'_C(\Psi(\alpha_k))}{[1 + c_4 \Psi_C(\alpha_k) m_C(\Psi(\alpha_k))]^2},
\]
\[
\psi_C(\psi(f(\alpha_k))) \triangleq \left(1 - c_3 \int \frac{dF_{mp/n}^{p/n,H}(t)}{t - \psi(f(\alpha_k))}\right)^2
- 2 \psi(f(\alpha_k)) \left(1 - c_3 \int \frac{dF_{mp/n}^{p/n,H}(t)}{t - \psi(f(\alpha_k))}\right) c_3 \int \frac{dF_{mp/n}^{p/n,H}(t)}{(t - \psi(f(\alpha_k)))^2}
- (1 - c_3) c_3 \int \frac{dF_{mp/n}^{p/n,H}(t)}{(t - \psi(f(\alpha_k)))^2}.
\]
\[
\text{And}
\]
\[
\sqrt{n} \frac{\phi(f(\alpha_k)) - \psi(f(\alpha_k))}{\psi(f(\alpha_k))} \times \left(\frac{2}{\psi(f(\alpha_k)) m'(\psi(f(\alpha_k)))}\right) \xrightarrow{d} N(0, 1),
\]
where
\[
m(\psi(f(\alpha_k))) = -\frac{1 - p/n}{\psi(f(\alpha_k))} + p/n \int \frac{dF_{mp/n}^{p/n,H}(t)}{t - \psi(f(\alpha_k))}
\]
Then the covariance function of \(\gamma_k^0\) satisfies
\[
\eta_3 + \frac{c_2}{c_1 \cdot \eta_2} \left[\frac{1 - c_2 (\lambda_C^0 m_2') m_3(\lambda_C^0)}{1 + c_2 \lambda_C m_3(\lambda_C)}\right]^2 \eta_1.
\]
(77)
where
\[
\eta_3 = \frac{q \left(\psi_F(\Psi(\alpha_k)))^2(\psi_C(\psi(f(\alpha_k))))^2 \psi(f(\alpha_k)) \psi(f(\alpha_k)) \right)}{\psi(f(\alpha_k)) m'(\psi(f(\alpha_k))) \Psi(\alpha_k)}
\]
According to Delta method, we rewrite
\[
\gamma_k = \sqrt{q} \frac{\lambda_k^2}{t(\alpha_k)} = \sqrt{q} \frac{\lambda_k^2}{\Psi(\alpha_k)} \frac{1 - \gamma^{-1}(\Psi(\alpha_k))}{\Psi(\alpha_k)}
\]
then the covariance function of \(\gamma_k\) will be
\[
(77) \ast \left(\frac{c_4}{1 + c_4 \Psi(\alpha_k)^2}\right)^2 \Psi(\alpha_k) / t^2(\alpha_k)
\]
References

[1] Theodore W. Anderson (2003). An Introduction to Multivariate Statistical Analysis, Third edition ed. Wiley.
[2] Bai, Z. and Ding, X. (2012). Estimation of spiked eigenvalues in spiked models. *Random Matrices: Theory and Applications* **01** 1150011. MR2934717

[3] Bai, Z., Hou, Z., Hu, J., Jiang, D. and Zhang, X. (2021). Limiting canonical distribution of two large dimensional random vectors. *In Advances on Methodology and Applications of Statistics - A Volume in Honor of C.R. Rao on the Occasion of his 100th Birthday. Edited by Carlos A. Coelho, N. Balakrishnan and Barry C. Arnold.*

[4] Bai, Z. and Silverstein, J. W. (2010). *Spectral Analysis of Large Dimensional Random Matrices*, second ed. *Springer Series in Statistics*. Springer, New York. MR2567175

[5] Bai, Z. and Silverstein, J. W. (2012). No eigenvalues outside the support of the limiting spectral distribution of Information-plus-Noise type matrices. *Random Matrices: Theory and Applications* **01** 1150004. MR2930382

[6] Bannai, M., Najim, J. and Yao, J. (2020). A CLT for linear spectral statistics of large random Information-plus-Noise matrices. *Stochastic Processes and their Applications* **130** 2250–2281.

[7] Baik, J., Arous, G. B. and Péché, S. (2005). Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *Annals of Probability* **33** 1643–1697. MR2165575

[8] Baik, J., Rous, G. B. and Péché, S. (2008). Central limit theorems for eigenvalues in a spiked population model. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques* **44** 447–474. MR2451053

[9] Bao, Z., Ding, X. and Wang, K. (2021). Singular vector and singular subspace distribution for the matrix denoising model. *The Annals of Statistics* **49** 370-392.

[10] Bao, Z., Hu, J., Pan, G. and Zhou, W. (2019). Canonical correlation coefficients of high-dimensional Gaussian vectors: finite rank case. *The Annals of Statistics* **47** 612–640. MR3909944

[11] Böttcher, T., Dette, H. and Parolya, N. (2019). Testing for independence of large dimensional vectors. *The Annals of Statistics* **47** 2977–3008. MR3988779

[12] Capitaine, M. (2014). Exact separation phenomenon for the eigenvalues of large Information-plus-Noise type matrices. *Annals of Statistics* **48** 1255–1280. MR4124322

[13] Ding, X. (2020). High dimensional deformed rectangular matrices with applications in matrix denoising. *Bernoulli* **17** 387-417.

[14] Ding, X. and Yang, F. (2019). Spiked separable covariance matrices and principal components. *arXiv:1905.13060.*
[17] Dozier, R. B. and Silverstein, J. W. (2007). On the empirical distribution of eigenvalues of large dimensional Information-plus-Noise-type matrices. *Journal of Multivariate Analysis* **98** 678–694.

[18] Dozier, R. B. and Silverstein, J. W. (2007). Analysis of the limiting spectral distribution of large dimensional Information-plus-Noise type matrices. *Journal of Multivariate Analysis* **98** 1099–1122.

[19] Han, X., Pan, G. and Zhang, B. (2016). The Tracy-Widom law for the largest eigenvalue of F type matrices. *The Annals of Statistics* **44** 1564–1592. MR3519933

[20] Horn, R. A. and Johnson, C. R. (2012). *Matrix Analysis*. Cambridge university press.

[21] Hu, J. and Bai, Z. (2014). Strong representation of weak convergence. *Science China Mathematics* **57** 2399–2406. MR3266500

[22] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Annals of Mathematical Statistics* **35** 475–501. MR181057

[23] Jiang, D. and Bai, Z. (2021). Generalized four moment theorem and an application to CLT for spiked eigenvalues of high-dimensional covariance matrices. *Bernoulli* **27** 274–294. MR4177370

[24] Jiang, D., Hou, Z. and Bai, Z. (2019). Generalized Four Moment Theorem with an application to the CLT for the spiked eigenvalues of high-dimensional general Fisher-matrices. *arXiv:1904.09236*.

[25] Jiang, D., Hou, Z. and Hu, J. (2021). The limits of the sample spiked eigenvalues for a high-dimensional generalized Fisher matrix and its applications. *Journal of Statistical Planning and Inference. Accept*. MR1863961

[26] Johnstone, I. M. (2001). On the distribution of the largest eigenvalue in principal components analysis. *The Annals of Statistics* **29** 295–327. MR1863961

[27] Johnstone, I. M. and Nadler, B. (2017). Roy’s largest root test under rank-one alternatives. *Biometrika* **104** 181–193.

[28] Ma, Z. and Yang, F. (2021). Sample canonical correlation coefficients of high-dimensional random vectors with finite rank correlations. *arXiv:2102.03297*.

[29] Mardia, K. V., Kent, J. T. and Bibby, J. M. (1979). *Multivariate analysis*. Academic press London.

[30] Robb J. Muirhead (1982). *Aspects of Multivariate Statistical Theory*. Wiley.

[31] Paul, D. (2007). Asymptotics of sample eigenstructure for a large dimensional spiked covariance model. *Statistica Sinica* **17** 1617–1642.

[32] Skorokhod, A. V. (1956). Limit theorems for stochastic processes. *Theory of Probability & Its Applications* **1** 261–290.

[33] Tracy, C. A. and Widom, H. (1996). On orthogonal and symplectic matrix ensembles. *Communications in Mathematical Physics* **177** 727–754.

[34] Wachter, K. W. (1980). The limiting empirical measure of multiple discriminant ratios. *The Annals of Statistics* 937–957.

[35] Wang, Q. and Yao, J. (2017). Extreme eigenvalues of large-dimensional
spiked Fisher matrices with application. *The Annals of Statistics* **45** 415–460. MR3611497

[36] **Yang, F.** (2020). Sample canonical correlation coefficients of high-dimensional random vectors: local law and Tracy-Widom limit. *arXiv:2002.09643*.

[37] **Yang, F.** (2021). Limiting Distribution of the Sample Canonical Correlation Coefficients of High-Dimensional Random Vectors. *arXiv:2103.08014*.

[38] **Zheng, S.** (2012). Central limit theorems for linear spectral statistics of large dimensional F-matrices. *Annales de l’I.H.P. Probabilités et statistiques* **48** 444–476.

[39] **Zheng, S., Bai, Z. and Yao, J.** (2015). CLT for linear spectral statistics of a rescaled sample precision matrix. *Random Matrices: Theory and Applications* **04** 1550014. MR3418843

[40] **Zheng, S., Bai, Z. and Yao, J.** (2017). CLT for eigenvalue statistics of large-dimensional general Fisher matrices with applications. *Bernoulli* **23** 1130–1178. MR3600762