BLOW UP OF SOLUTIONS OF SEMILINEAR HEAT EQUATIONS IN NON RADIAL DOMAINS OF $\mathbb{R}^2$

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Abstract. We consider the semilinear heat equation

$$\begin{cases} v_t - \Delta v = |v|^{p-1}v & \text{in } \Omega \times (0, T) \\ v = 0 & \text{on } \partial \Omega \times (0, T) \\ v(0) = v_0 & \text{in } \Omega \end{cases} \quad (P_p)$$

where $p > 1$, $\Omega$ is a smooth bounded domain of $\mathbb{R}^2$, $T \in (0, +\infty]$ and $v_0$ belongs to a suitable space. We give general conditions for a family $u_p$ of sign-changing stationary solutions of $(P_p)$, under which the solution of $(P_p)$ with initial value $v_0 = \lambda u_p$ blows up in finite time if $|\lambda - 1| > 0$ is sufficiently small and $p$ is sufficiently large. Since for $\lambda = 1$ the solution is global, this shows that, in general, the set of the initial conditions for which the solution is global is not star-shaped with respect to the origin. In [4] this phenomenon has been previously observed in the case when the domain is a ball and the sign changing stationary solution is radially symmetric. Our conditions are more general and we provide examples of stationary solutions $u_p$ which are not radial and exhibit the same behavior.

1. Introduction

We consider the nonlinear heat equation

$$\begin{cases} v_t - \Delta v = |v|^{p-1}v & \text{in } \Omega \times (0, T) \\ v = 0 & \text{on } \partial \Omega \times (0, T) \\ v(0) = v_0 & \text{in } \Omega \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, $N \geq 1$ is a smooth bounded domain, $p > 1$, $T \in (0, +\infty]$ and $v_0 \in C_0(\Omega)$, where

$$C_0(\Omega) = \{v \in C(\bar{\Omega}), \ v = 0 \text{ on } \partial \Omega\}.$$ 

It is well known that the initial value problem (1.1) is locally well-posed in $C_0(\Omega)$ and admits both nontrivial global solutions and blow-up solutions. Denoting with $T_{v_0}$ the maximal existence time of the solution of (1.1), we define the set of initial conditions for which the corresponding solution is global, i.e.

$$\mathcal{G} = \{v_0 \in C_0(\Omega), \ T_{v_0} = +\infty\}.$$ 

2010 Mathematics Subject classification: 35K91, 35B35, 35B44, 35J91.

Keywords: semilinear heat equation, finite-time blow-up, sign-changing stationary solutions, asymptotic behavior.

Research partially supported by FIRB project: Analysis and Beyond and PRIN project 2012: 74FYK7-005.
and its complementary set of initial conditions for which the corresponding solution blows-up in finite time:

$$\mathcal{B} = \{ v_0 \in C_0(\Omega), \ T_{v_0} < +\infty \}. $$

For a fixed $w \in C_0(\Omega), \ w \neq 0$ and $v_0 = \lambda w, \ \lambda \in \mathbb{R}$, it is known that if $|\lambda|$ is small enough then $v_0 \in \mathcal{G}$, while if $|\lambda|$ is sufficiently large then $v_0 \in \mathcal{B}$.

Moreover, for any $N \geq 1$, considering nonnegative initial data, it can be easily proved that the set $\mathcal{G}^+ = \{ v_0 \in \mathcal{G}, \ v_0 \geq 0 \}$ is star-shaped with respect to 0 (indeed it is convex, see [8]). On the contrary when the initial condition changes sign $\mathcal{G}$ may not be star-shaped.

Indeed for $N \geq 3$, in [1] and [7] it has been shown that there exists $p^* < p_S$, with $p_S = \frac{N+2}{N-2}$, such that $\forall \ p \in (p^*, p_S)$ the elliptic problem

$$\begin{cases}
-\Delta u_p = |u_p|^{p-1}u_p & \text{in } \Omega \\
 u_p = 0 & \text{on } \partial \Omega,
\end{cases} \tag{1.2}$$

admits a sign a changing solution $u_p$ for which there exists $\epsilon > 0$ such that if $0 < |1-\lambda| < \epsilon$ then $\lambda u_p \in \mathcal{B}$. More precisely this result has been proved first in [1] when $\Omega$ is the unit ball and $u_p$ is any sign-changing radial solution of (1.2), and then in [7] for general domains $\Omega$ and sign-changing solutions $u_p$ of (1.2) (assuming w.l.g. that $\|u^+_p\|_\infty = \|u^+_p\|_\infty$), satisfying the following conditions

$$\int_{\Omega} |\nabla u_p|^2 dx \to 2S^\frac{2}{N} \quad (a)$$

$$\lim_{R \to +\infty} \lim_{p \to +\infty} S_{p,R} = 0 \quad (b)$$

as $p \to p^*$, where $S$ is the best Sobolev constant for the embedding of $H^1_0(\Omega)$ into $L^{2^*}(\Omega)$.

When $N = 1$ such a result does not hold since it is easy to see that for any sign changing solution $u_p$ of (1.2), $\lambda u_p \in \mathcal{G}$ for $|\lambda| < 1$ and $\lambda u_p \in \mathcal{B}$ for $|\lambda| > 1$.

The case $N = 2$ was left open in the papers [1] and [7], mainly because there is not a critical Sobolev exponent in this dimension so that the conditions and results of these papers are meaningless. Recently inspired by [5], [6], Dickstein, Pacella and Sciunzi in [4] succeeded to prove a blow up theorem similar to the one in [1], considering radial sign changing stationary solutions $u_p$ of (1.2) in the unit ball for large exponents $p$. Indeed, the condition $p \to +\infty$ in dimension $N = 2$ can be considered to be the natural extension of the condition $p \to p_S$ for $N \geq 3$.

In this paper we consider again the case $N = 2$ but the bounded domain $\Omega$ is not necessarily a ball. We deal with sign-changing solutions $u_p$ of (1.2) with the following two properties:

$$\exists C > 0, \text{ such that } \ p \int_{\Omega} |\nabla u_p|^2 dx \leq C, \quad (A)$$

$$\lim_{R \to +\infty} \lim_{p \to +\infty} S_{p,R} = 0, \quad (B)$$
where, for $R > 0$, 
\[
S_{p,R} := \sup \left\{ \frac{|u_p(y)|^{p-1}}{|u_p(x_p^+)|} : y \in \Omega, |y - x_p^+| > R \mu_p^+ \right\},
\]
for $x_p^+$ such that $|u_p(x_p^+)| = \|u_p\|_{\infty}$ and $\mu_p^+ := \frac{1}{\sqrt[p]{|u_p(x_p^+)|^{p-1}}}$.

Our main result is the following

**Theorem 1.1.** Let $N = 2$ and $u_p$ be a family of sign-changing solutions of problem (1.2) satisfying (A) and (B). Then, up to a subsequence, there exists $p^* > 1$ such that for $p > p^*$ there exists $\epsilon = \epsilon(p) > 0$ such that if $0 < |1 - \lambda| < \epsilon$, then
\[
\lambda u_p \in \mathcal{B}.
\]

A few comments on conditions (A) and (B) are needed. The first one is an estimate of the asymptotic behavior, as $p \to +\infty$, of the energy of the solutions. It is satisfied by different kinds of sign changing solutions (see [2], [3], [5]), in particular by the radial ones in the ball (see [4], [6]). The condition (B) is more peculiar and essentially concerns the asymptotic behavior of $|u_p(y)|^{p-1}$ for points $y$ which are not too close to $x_p^+$; note that $p|u_p(y)|^{p-1}$ can also be divergent since $\liminf_{p \to +\infty} \|u_p\|_{\infty} \geq 1$. It is satisfied, in particular, by sign changing radial solutions $u_{p,K}$ of (1.2) having any fixed number $K$ of nodal regions (see Section 7 for details). But it also holds for a class of sign changing solutions in more general domains as we show in the next theorem.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, containing the origin, invariant under the action of a cyclic group $G$ of rotations about the origin with
\[
|G| \geq me
\]
($|G|$ is the order of $G$), for a certain $m > 0$.

Let $(u_p)$ be a family of sign changing $G$-symmetric solutions of (1.2) such that
\[
p \int_{\Omega} |\nabla u_p|^2 dx \leq \alpha 8\pi e, \quad \text{for } p \text{ large}
\]
and for some $\alpha < m + 1$. Then $u_p$ satisfies (A) and (B) up to a subsequence.

**Remark 1.3.** The existence of sign-changing solutions of (1.2) satisfying the assumption (1.4) of Theorem 1.2 has been proved in [2] for $m \geq 4$ and for large $p$.

Putting together Theorem 1.1 and Theorem 1.2 one gets the extension of the blow-up result of [4] to other symmetric domains. Note that this in particular includes the case of sign changing radial solutions in a ball, providing so a different proof of the result of [4] which strongly relied on the radial symmetry.
The proof of Theorem 1.1 follows the same strategy as the analogous results in [1, 7, 4] being a consequence of the following proposition, which is a particular case of [1, Theorem 2.3]

Proposition 1.4. Let $u$ be a sign changing solution of (1.2) and let $\varphi_1$ be a positive eigenvector of the self-adjoint operator $L$ given by $L\varphi = -\Delta \varphi - p|u|^{p-1}\varphi$, for $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$. Assume that
\[
\int_{\Omega} u\varphi_1 \neq 0.
\]
Then there exists $\epsilon > 0$ such that if $0 < |1 - \lambda| < \epsilon$, then $\lambda u \in B$.

More precisely we will show that under the assumptions (A) and (B) condition (1.5) is satisfied for $p$ large (see Theorem 4.1 in the following). This proof is based on rescaling arguments about the maximum point of $u_p$: using the properties of the solution of the limit problem, we analyze the asymptotic behavior as $p \to +\infty$ of the rescaled solutions and of the rescaled first eigenfunction of the linearized operator at $u_p$. In this analysis the assumption (B) is crucial.

To get Theorem 1.2 we prove a more general result which shows that condition (B) holds for a quite general class of solutions $u_p$ of (1.2) (see Lemma 5.1 and Theorem 5.3).

The paper is organized as follows.
In Section 2 we collect some properties satisfied by any family of solutions $u_p$ under condition (A) and we give a characterization of condition (B).
In Section 3 we carry out an asymptotic spectral analysis under assumption (A) and (B), studying the asymptotic behavior of the first eigenvalue and of the first eigenfunction of the linearized operator at $u_p$.
Section 4 is devoted to the proof of Theorem 1.1 via rescaling arguments.
In Section 5 we find a sufficient condition which ensures the validity of property (B) (see Lemma 5.1) and we select a general class of solutions to (1.2) which satisfies this sufficient condition (see Theorem 5.3).
In Section 6 we prove Theorem 1.2.
Finally in Section 7 we show that also the sign-changing radial solutions in the unit ball satisfy the assumptions of Theorem 1.2.

2. Preliminary results

We fix some notation. For a given family $(u_p)$ of sign-changing stationary solutions of (1.1) we denote by
- $NL_p$ the nodal line of $u_p$,
- $x^+_p$ a maximum/minimum point in $\Omega$ of $u_p$, i.e. $u_p(x^+_p) = \pm ||u^+_p||_\infty$,
- $N^+_p := \{x \in \Omega : u^+_p(x) \neq 0\} \subset \Omega$ denote the positive/negative domain of $u_p$. 

\( \mu_p^\pm := \frac{1}{\sqrt{p|u_p(x_p^\pm)|^{p-1}}} \),
\( \tilde{\Omega}_p := \frac{\Omega}{\mu_p^+} = \left\{ x \in \mathbb{R}^2 : x^+ + \mu_p^+ x \in \Omega \right\} \),
\( \tilde{N}_p^+ := \frac{N_p^+ - x^+}{\mu_p^+} = \left\{ x \in \mathbb{R}^2 : x^+ + \mu_p^+ x \in N_p^+ \right\} \),
\( d(x, A) := \text{dist}(x, A) \), for any \( x \in \mathbb{R}^2, A \subset \mathbb{R}^2 \).

We assume w.l.o.g. that \( \|u_p\|_\infty = \|u_p^+\|_\infty \).

In the next two lemmas we collect some useful properties holding under condition (A).

**Lemma 2.1.** Let \( (u_p) \) be a family of solutions to (1.2) and assume that (A) holds. Then
\[
\|u_p\|_{L^\infty(\Omega)} \leq C \\
\liminf_{p \to +\infty} u_p(x_p^\pm) \geq 1 \\
\mu_p^\pm \to 0 \text{ as } p \to +\infty
\] (2.1)

**Proof.** It is well known, see for instance [3, Lemma 2.1] or [5]. \(\square\)

**Lemma 2.2.** Let \( (u_p) \) be a family of solutions to (1.2) and assume that (A) holds. Then, up to a subsequence, the rescaled function
\[
v_p^+(x) := \frac{p}{u_p(x_p^+)}(u_p(x_p^+ + \mu_p^+ x) - u_p(x_p^+))
\] (2.3)
defined on \( \tilde{\Omega}_p \) converges to \( U \) in \( C^1_{\text{loc}}(\mathbb{R}^2) \), where
\[
U(x) = \log \left( \frac{1}{1 + \frac{1}{8}|x|^2} \right)^2
\] (2.4)
is the solution of the Liouville problem
\[
\begin{cases}
-\Delta U = e^U \text{ in } \mathbb{R}^2 \\
U \leq 0, U(0) = 0 \text{ and } \int_{\mathbb{R}^2} e^U = 8\pi.
\end{cases}
\] (2.5)

Moreover
\[
\frac{d(x_p^+, NL_p)}{\mu_p^+} \to +\infty \text{ as } p \to +\infty
\] (2.6)
and
\[
\frac{d(x_p^+, \partial \Omega)}{\mu_p^+} \to +\infty \text{ as } p \to +\infty.
\] (2.7)

**Proof.** It is well known, see for instance [3, Proposition 2.2 & Corollary 2.4] \(\square\)

Observe that, under condition (A), by (2.7), for any \( R > 0 \) there exists \( p_R > 1 \) such that the set \( \left\{ y \in \Omega, |y - x_p^+| > R\mu_p^+ \right\} \neq \emptyset \) for \( p \geq p_R \). As a consequence for any \( R > 0 \) the value
\[
S_{p,R} := \sup \left\{ \frac{|u_p(y)|}{u_p(x_p^+)}^{p-1} : y \in \Omega, |y - x_p^+| > R\mu_p^+ \right\}
\]
in the definition of condition \((B)\) is well-defined for \(p \geq p_R\).

Next we give a characterization of condition \((B)\):

**Proposition 2.3.** Assume that \(u_p\) is a family of sign-changing solutions to (1.2) which satisfies condition \((A)\). Then for any \(R > 0\) there exists \(p_R > 1\) such that the set \(\{y \in N_p^+, \ |y - x_p^+| > R\mu_p^+\} \neq \emptyset\) for \(p \geq p_R\) and so

\[
\mathcal{M}_{p,R} := \sup \left\{ \left| \frac{u_p(y)}{u_p(x_p^+)} \right|^{p-1} : y \in N_p^+, \ |y - x_p^+| > R\mu_p^+ \right\}
\]  

(2.8)
is well defined.

Moreover condition \((B)\) is equivalent to

\[
\begin{cases}
\lim_{p \to +\infty} \frac{\|u_p\|^{p-1}_{L^\infty(\Omega)}}{\|u_p^+\|^{p-1}_{L^\infty(\Omega)}} = 0 & (B1) \\
\lim_{R \to +\infty} \lim_{p \to +\infty} \mathcal{M}_{p,R} = 0 & (B2)
\end{cases}
\]

**Proof.** For \(R > 0\) and \(p > 1\) define the set

\[
\Omega_{p,R} := \{y \in \Omega, \ |y - x_p^+| > R\mu_p^+\} \ (\subseteq \Omega).
\]

In order to prove the equivalence it is enough to show that for any \(R > 0\) there exists \(p_R > 1\) such that

\[
\Omega_{p,R} = (\Omega_{p,R} \cap N_p^+) \cup N_p^- \cup NL_p
\]

(2.9)
for \(p \geq p_R\) and the union is disjoint. Indeed (2.9) implies that

\[
S_{p,R} = \max \left\{ \mathcal{M}_{p,R}, \frac{\|u_p\|^{p-1}_{L^\infty(\Omega)}}{\|u_p^+\|^{p-1}_{L^\infty(\Omega)}} \right\}
\]

and so the conclusion. By definition \(x_p^+ \in N_p^+\), moreover by (2.6) and (2.7) \(\frac{d(x_p^+, \partial \Omega)}{\mu_p^+} \to +\infty\) as \(p \to +\infty\) and so for any \(R > 0\) there exists \(p_R > 1\) such that \(\frac{d(x_p^+, \partial \Omega)}{\mu_p^+} \geq 2R\) and \(\frac{d(x_p^+, \partial \Omega)}{\mu_p^+} \geq 2R\) for \(p \geq p_R\), which implies that \(B_{R\mu_p^+}(x_p^+) \subset N_p^+\) for \(p \geq p_R\). As a consequence \((\Omega_{p,R} \cap N_p^-) = N_p^-, (\Omega_{p,R} \cap NL_p) = NL_p\) and the set \((\Omega_{p,R} \cap N_p^+) \neq \emptyset\) for \(p \geq p_R\). Hence (2.9) follows from the fact that

\[
\Omega_{p,R} = (\Omega_{p,R} \cap N_p^+) \cup (\Omega_{p,R} \cap N_p^-) \cup (\Omega_{p,R} \cap NL_p)
\]

and the union is disjoint. \(\Box\)

**Remark 2.4.** Condition \((B1)\) can be equivalently written as

\[
\lim_{p \to +\infty} \frac{\mu_p^+}{\mu_p^-} = 0.
\]
3. Asymptotic spectral analysis

3.1. Linearization of the limit problem. In [4] the linearization at $U$ of the Liouville problem (2.5) has been studied. In the following we recall the main results.

For $v \in H^2(\mathbb{R}^2)$ we define the linearized operator by

$$L^*(v) := -\Delta v - e^U v.$$ 

We consider the Rayleigh functional $R^*: H^1(\mathbb{R}^2) \to \mathbb{R}$

$$R^*(w) := \int_{\mathbb{R}^2} (|\nabla w|^2 - e^U w^2) \, dx$$

and define

$$\lambda^*_1 := \inf_{\|w\|_{L^2(\mathbb{R}^2)} = 1} R^*(w).$$ \hfill (3.1)

**Proposition 3.1.** We have the following

i) $(-\infty) < \lambda^*_1 < 0$;

ii) every minimizing sequence of (3.1) has a subsequence strongly converging in $L^2(\mathbb{R}^2)$ to a minimizer;

iii) there exists a unique positive minimizer $\varphi^*_1$ to (3.1), which is radial and radially non-increasing;

iv) $\lambda^*_1$ is an eigenvalue of $L^*$ and $\varphi^*_1$ is an eigenvector associated to $\lambda^*_1$. Moreover $\varphi^*_1 \in L^\infty(\mathbb{R}^2)$.

**Proof.** See [4, Proposition 3.1]. \hfill \Box

3.2. Linearization of the Lane-Emden problem. In this section we consider the linearization of the Lane-Emden problem and study its connections with the linearization $L^*$ of the Liouville problem.

We define the linearized operator at $u_p$ of the Lane-Emden problem in $\Omega$

$$L_p(v) := -\Delta v - p |u_p|^{p-1} v, \quad v \in H^2(\Omega) \cap H^1_0(\Omega).$$

Let $\lambda_{1,p}$ and $\varphi_{1,p}$ be respectively the first eigenvalue and the first eigenfunction (normalized in $L^2$) of the operator $L_p$. It is well known that $\lambda_{1,p} < 0$, $\forall \ p > 1$ and that $\varphi_{1,p} \geq 0$, moreover $\|\varphi_{1,p}\|_{L^2(\Omega)} = 1$.

Rescaling

$$\tilde{\varphi}_{1,p}(x) := \mu_p^+ \varphi_{1,p}(x_p^+ + \mu_p^+ x), \quad x \in \tilde{\Omega}_p$$

and setting

$$\tilde{\lambda}_{1,p} := (\mu_p^+)^2 \lambda_{1,p}$$
then, it is easy to see that \( \tilde{\varphi}_{1,p} \) and \( \tilde{\lambda}_{1,p} \) are respectively the first eigenfunction and first eigenvalue of the linear operator \( \tilde{L}_p \) in \( \tilde{\Omega}_p \) with homogeneous Dirichlet boundary conditions, defined as

\[
\tilde{L}_p v := -\Delta v - V_p(x) v, \quad v \in H^2(\tilde{\Omega}_p) \cap H^1_0(\tilde{\Omega}_p),
\]

where

\[
V_p(x) := \left| \frac{u_p(x^+ + \mu_p x^+)}{u_p(x^+)} \right|^{p-1} = 1 + \frac{v^+_p(x)}{p} \quad (v^+_p \text{ is the function defined in Lemma 2.2}).
\]

Observe that \( \|\tilde{\varphi}_{1,p}\|_{L^2(\tilde{\Omega}_p)} = \|\varphi_{1,p}\|_{L^2(\Omega)} = 1 \).

**Lemma 3.2.** Up to a subsequence

\[
\int_{\tilde{\Omega}_p} (e^U - V_p) \tilde{\varphi}_{1,p}^2 \to 0 \quad \text{as} \quad p \to +\infty.
\]

**Proof.** For \( R > 0 \) we divide the integral in the following way

\[
\int_{\tilde{\Omega}_p} (e^U - V_p) \tilde{\varphi}_{1,p}^2 = \underbrace{\int_{\tilde{\Omega}_p \cap \{|x| \leq R\}} (e^U - V_p) \tilde{\varphi}_{1,p}^2}_{A_{p,R}} + \underbrace{\int_{\tilde{\Omega}_p \cap \{|x| > R\}} (e^U - V_p) \tilde{\varphi}_{1,p}^2}_{B_{p,R}}.
\]

Now

\[
A_{p,R} = \int_{\tilde{\Omega}_p \cap \{|x| \leq R\}} (e^U - V_p) \tilde{\varphi}_{1,p}^2 \to 0 \quad \text{as} \quad p \to +\infty, \quad \text{for all} \quad R > 0, \quad (3.2)
\]

since, up to a subsequence, \( v^+_p \to U \) in \( C^1_{\text{loc}}(\mathbb{R}^2) \) as \( p \to +\infty \) (see Lemma 2.2) and so \( V_p \to e^U \) uniformly in \( B_R(0) \), up to a subsequence.

On the other hand

\[
B_{p,R} = \int_{\tilde{\Omega}_p \cap \{|x| > R\}} (e^U - V_p) \tilde{\varphi}_{1,p}^2 = \underbrace{\int_{\tilde{\Omega}_p \cap \{|x| > R\}} e^U \tilde{\varphi}_{1,p}^2}_{C_{p,R}} - \underbrace{\int_{\tilde{\Omega}_p \cap \{|x| > R\}} V_p \tilde{\varphi}_{1,p}^2}_{D_{p,R}}. \quad (3.3)
\]

Following [4] we estimate

\[
(0 \leq) \quad C_{p,R} = \int_{\tilde{\Omega}_p \cap \{|x| > R\}} e^U \tilde{\varphi}_{1,p}^2 \leq \sup_{|x| > R} e^U(x) \int_{\tilde{\Omega}_p} \tilde{\varphi}_{1,p}^2 = \sup_{|x| > R} e^U(x) = \sup_{|x| > R} \left( 1 + \frac{|x|^2}{8} \right)^2 \leq \frac{64}{R^2}.
\]
Last we estimate
\[
(0 \leq) \quad D_{p,R} = \int_{\tilde{\Omega}_p \cap \{|x| > R\}} V_p \tilde{\varphi}_{1,p}^2 \leq \sup_{\tilde{\Omega}_p \cap \{|x| > R\}} V_p(x) \int_{\tilde{\Omega}_p} \tilde{\varphi}_{1,p}^2
\]
\[
= \sup_{\tilde{\Omega}_p \cap \{|x| > R\}} \int_{\tilde{\Omega}_p} \frac{|u_p(x^+_p + \mu^+_p x)|^{p-1}}{u_p(x^+_p)}
\]
\[
= \sup_{\Omega \cap \{|y - x^+_p | > R \mu^+_p\}} \left| \frac{u_p(y)}{u_p(x^+_p)} \right|^{p-1}
\]
\[
= S_{p,R},
\]
so assumption (B) implies that
\[
\lim_{R \to +\infty} \lim_{p \to +\infty} D_{p,R} = 0,
\]
and this concludes the proof. \(\Box\)

**Theorem 3.3.** We have, up to a subsequence, that
\[
\tilde{\lambda}_{1,p} \to \lambda^*_1 \quad \text{as} \quad p \to +\infty.
\] (3.4)

**Proof.** We divide the proof into two steps.

**Step 1.** We show that, up to a subsequence, for \(\epsilon > 0\)
\[
\lambda_1^* \leq \tilde{\lambda}_{1,p} + \epsilon \quad \text{for } p \text{ sufficiently large.} \tag{3.5}
\]
\[
\lambda_1^* \leq \int_{\mathbb{R}^2} (|\nabla \tilde{\varphi}_{1,p}|^2 - e^U \tilde{\varphi}_{1,p}^2)
\]
\[
= \int_{\tilde{\Omega}_p} V_p \tilde{\varphi}_{1,p}^2 + \tilde{\lambda}_{1,p} \int_{\tilde{\Omega}_p} \tilde{\varphi}_{1,p}^2 - \int_{\tilde{\Omega}_p} e^U \tilde{\varphi}_{1,p}^2
\]
\[
= \tilde{\lambda}_{1,p} - \int_{\tilde{\Omega}_p} (e^U - V_p) \tilde{\varphi}_{1,p}^2 \tag{3.6}
\]
and so the conclusion follows by Lemma 3.2.

**Step 2.** We show that, up to a subsequence, for \(\epsilon > 0\)
\[
\tilde{\lambda}_{1,p} \leq \lambda_1^* + \epsilon \quad \text{for } p \text{ sufficiently large.} \tag{3.7}
\]
The proof is similar to the one in [4], we repeat it for completeness. For \( R > 0 \), let us consider a cut-off regular function \( \psi_R(x) = \psi_R(r) \) such that
\[
\begin{cases}
0 \leq \psi_R \leq 1 \\
\psi_R = 1 & \text{for } r \leq R \\
\psi_R = 0 & \text{for } r \geq 2R \\
|\nabla \psi_R| \leq 2/R
\end{cases}
\]
and let us set
\[
w_R := \frac{\psi_R \varphi_1^*}{\|\psi_R \varphi_1^*\|_{L^2(\mathbb{R}^2)}}.
\]
Hence, from the variational characterization of \( \widetilde{\lambda}_{1,p} \), we deduce that
\[
\tilde{\lambda}_{1,p} \leq \int_{\mathbb{R}^2} \left( |\nabla w_R|^2 - V_p(x) w_R^2 \right) dx = \int_{\mathbb{R}^2} \left( |\nabla w_R|^2 - e^{U(x)} w_R^2 \right) dx + \int_{\mathbb{R}^2} \left( e^{U(x)} - V_p(x) \right) w_R^2 dx.
\]
(3.8)
Since \( w_R \to \varphi_1^* \) in \( H^1(\mathbb{R}^2) \) as \( R \to +\infty \), it is easy to see that given \( \epsilon > 0 \) we can fix \( R > 0 \) such that
\[
\int_{\mathbb{R}^2} \left( |\nabla w_R|^2 - e^{U(x)} w_R^2 \right) dx \leq \lambda_1^* + \frac{\epsilon}{2}.
\]
(3.9)
For such a fixed value of \( R \) we can argue similarly as in the proof of (3.2) in Lemma 3.2 to obtain that, up to a subsequence in \( p \),
\[
\int_{\mathbb{R}^2} \left( e^{U(x)} - V_p(x) \right) w_R^2 dx \leq \frac{\epsilon}{2}
\]
(3.10)
for \( p \) large enough. Hence the proof of Step 2 follows from (3.8), (3.9) and (3.10). \( \square \)

**Corollary 3.4.** Up to a subsequence
\[
\tilde{\varphi}_{1,p} \to \varphi_1^* \text{ in } L^2(\mathbb{R}^2) \text{ as } p \to +\infty.
\]
**Proof.** By Lemma 3.2 and Theorem 3.3 we have that, passing to a subsequence
\[
\lambda_1^* - \mathcal{R}^*(\tilde{\varphi}_{1,p}) = (\lambda_1^* - \tilde{\lambda}_{1,p}) + \tilde{\lambda}_{1,p} - \mathcal{R}^*(\tilde{\varphi}_{1,p}) = (\lambda_1^* - \tilde{\lambda}_{1,p}) + \int_{\mathbb{R}^2} \left( e^{U(x)} - V_p(x) \right) \tilde{\varphi}_{1,p}^2 \to 0
\]
as \( p \to +\infty \), namely \( \tilde{\varphi}_{1,p} \) is a minimizing sequence for (3.1) and so the result follows from points ii) and iii) of Proposition 3.1. \( \square \)
4. Proof of Theorem 1.1

The proof of Theorem 1.1 follows the same strategy as in [1, 7, 4] and it is a consequence of Proposition 1.4, which is a particular case of Theorem 2.3 in [1]. Hence, to obtain Theorem 1.1 it is enough to prove the following:

**Theorem 4.1.** Let $u_p$ be a family sign changing solutions to (1.2) which satisfies conditions (A) and (B). Then there exists $p^* > 1$ such that up to a subsequence, for $p > p^*$

$$
\int_{\Omega} u_p \varphi_{1,p} > 0,
$$

where $\varphi_{1,p}$ is the first positive eigenfunction of the linearized operator $L_p$ at $u_p$.

**Proof.** Since by an easy computation

$$
\int_{\Omega} u_p \varphi_{1,p} = \frac{p - 1}{-\lambda_{1,p}} \int_{\Omega} |u_p|^{p-1} u_p \varphi_{1,p}
$$

(see [4, pg. 14]), we can study the sign of

$$
\int_{\Omega} |u_p|^{p-1} u_p \varphi_{1,p},
$$

which is the same as the sign of

$$
\frac{1}{u_p(x_p^+)^p \mu_p^+} \int_{\Omega} |u_p|^{p-1} u_p \varphi_{1,p}.
$$

We show that, up to a subsequence,

$$
\frac{1}{u_p(x_p^+)^p \mu_p^+} \int_{\Omega} |u_p|^{p-1} u_p \varphi_{1,p} \to \int_{\mathbb{R}^2} e^{U^*_1} \varphi^*_1 > 0 \quad \text{as} \quad p \to +\infty \quad (4.1)
$$

from which the conclusion follows.

In order to prove (4.1) we change the variable and, for any $R > 0$, we split the integral in the following way

$$
\frac{1}{u_p(x_p^+)^p \mu_p^+} \int_{\Omega} |u_p|^{p-1} u_p \varphi_{1,p} = \frac{1}{u_p(x_p^+)^p} \int_{\tilde{\Omega}_p} |u_p(x_p^+ + \mu_p^+ x)|^{p-1} u_p(x_p^+ + \mu_p^+ x) \varphi_{1,p}(x) dx
$$

$$
= \frac{1}{u_p(x_p^+)^p} \int_{\tilde{\Omega}_p \cap \{|x| \leq R\}} |u_p(x_p^+ + \mu_p^+ x)|^{p-1} u_p(x_p^+ + \mu_p^+ x) \varphi_{1,p}(x) dx
$$

$$
+ \frac{1}{u_p(x_p^+)^p} \int_{\tilde{\Omega}_p \cap \{|x| > R\}} |u_p(x_p^+ + \mu_p^+ x)|^{p-1} u_p(x_p^+ + \mu_p^+ x) \varphi_{1,p}(x) dx
$$

$$
= E_{p,R} + E_{p,R}
$$
By Hölder inequality, the convergence of $v^+_p$ to $U$ in $C^1_{loc}(\mathbb{R}^2)$ up to a subsequence (see Lemma 2.2) and Corollary 3.4 we have, for $R > 0$ fixed:

$$
E_{p,R} - \int_{\{x|\leq R\}} e^U \varphi^*_1 \leq \left| \int_{\Omega \cap \{|x| \leq R\}} \frac{u_p(x^+_p + \mu^+_p x)}{u_p(x^+_p)} |\tilde{\varphi}_{1,p}(x) - \varphi^*_1(x)| \right| \ dx
$$

$$
+ \int_{\{x|\leq R\}} \varphi^*_1(x) \left| 1 + \frac{v^+_p(x)}{p} \right|^{p-1} \left( 1 + \frac{v^+_p(x)}{p} \right) - e^U(x) \right| \ dx
$$

$$
\leq \|\tilde{\varphi}_{1,p} - \varphi^*_1\|_{L^2(\mathbb{R}^2)} \left[ \int_{\Omega \cap \{|x| \leq R\}} \left| 1 + \frac{\tilde{\varphi}^*_1(x)}{p} \right|^{2p} \right]^{\frac{1}{2}}
$$

$$
+ \sup_{\{x|\leq R\}} \left| 1 + \frac{\tilde{\varphi}^*_1(x)}{p} \right|^{p-1} \left( 1 + \frac{\tilde{\varphi}^*_1(x)}{p} \right) - e^U(x) \int_{\{x|\leq R\}} \varphi^*_1(x) \ dx
$$

$$
\rightarrow 0,
$$

as $p \rightarrow +\infty$, up to a subsequence. For $R$ sufficiently large the term

$$
\int_{\{x|> R\}} e^U \varphi^*_1 \ dx
$$

may be made arbitrary small since $e^U \in L^1(\mathbb{R}^2)$ and $\varphi^*_1$ is bounded (Proposition 3.1 iv)).

Using Hölder inequality, $\|\tilde{\varphi}_{1,p}\|_{L^2(\tilde{\Omega}_p)} = 1$, assumption (A) and (2.1) we have

$$
|F_{p,R}|^2 \leq \|\tilde{\varphi}_{1,p}\|^2_{L^2(\tilde{\Omega}_p)} \left( \int_{\tilde{\Omega}_p \cap \{|x| > R\}} \left| \frac{u_p(x^+_p + \mu^+_p x)}{u_p(x^+_p)} \right|^2 \ dx \right)
$$

$$
= \int_{\tilde{\Omega}_p \cap \{|x| > R\}} \left| \frac{u_p(x^+_p + \mu^+_p x)}{u_p(x^+_p)} \right|^2 \ dx
$$

$$
= \frac{1}{u_p(x^+_p)^2} \int_{\tilde{\Omega} \cap \{|y-x^+_p| > R\mu^+_p\}} p |u_p(y)|^{2p} \ dy
$$

$$
\leq \frac{p \int_{\Omega} |u_p(y)|^{p+1}}{u_p(x^+_p)^2} \sup_{\tilde{\Omega} \cap \{|y-x^+_p| > R\mu^+_p\}} \left| \frac{u_p(y)}{u_p(x^+_p)} \right|^{p-1}
$$

(A)+(2.1)

$$
\leq CS_{p,R}.
$$

And so by assumption (B) we have

$$\lim_{R \rightarrow +\infty} \lim_{p \rightarrow +\infty} F_{p,R} = 0.$$
5. A SUFFICIENT CONDITION FOR (B)

Next property is a sufficient condition for (B):

**Lemma 5.1.** Assume that there exists $C > 0$ such that

$$|x - x_p^+|^2 p |u_p(x)|^{p-1} \leq C$$

(5.1)

for all $p$ sufficiently large and all $x \in \Omega$. Then condition (B) holds true up to a subsequence in $p$.

**Proof.** Let $R > 0$ fixed and let $y \in \Omega$, $|y - x_p^+| > R \mu_p^+$, then for $p$ large, by (5.1)

$$\frac{|u_p(y)|^{p-1}}{u_p(x_p^+)^{p-1}} = \frac{p |u_p(y)|^{p-1}}{p |u_p(x_p^+)|^{p-1}} \leq \frac{C}{|y - x_p^+|^2} \frac{1}{p |u_p(x_p^+)|^{p-1}} \leq \frac{C}{R^2} \frac{1}{p |u_p(x_p^+)|^{p-1}} = \frac{C}{R^2}$$

(5.2)

hence

$$0 \leq \limsup_{p \to +\infty} S_{p,R} \leq \frac{C}{R^2}$$

and (B) follows, up to a subsequence in $p$, passing to the limit as $R \to +\infty$. □

Condition (5.1) is a special case of a more general result that has been proved in [3] for any family $(u_p)$ of solutions to (1.2) under condition (A) and which we recall here. Given $n \in \mathbb{N} \setminus \{0\}$ families of points $(x_{i,p})$, $i = 1, \ldots, n$ in $\Omega$ such that

$$p |u_p(x_{i,p})|^{p-1} \to +\infty \text{ as } p \to +\infty,$$

(5.3)

which implies in particular

$$\liminf_{p \to +\infty} u_p(x_{i,p}) \geq 1,$$

(5.4)

we define the parameters $\mu_{i,p}$ by

$$\mu_{i,p}^{-2} = p |u_p(x_{i,p})|^{p-1}, \text{ for all } i = 1, \ldots, n,$$

(5.5)

and we introduce the following properties:

(P1) For any $i, j \in \{1, \ldots, n\}$, $i \neq j$,

$$\lim_{p \to +\infty} \frac{|x_{i,p} - x_{j,p}|}{\mu_{i,p}} = +\infty.$$

(P2) For any $i = 1, \ldots, n$,

$$v_{i,p}(x) := \frac{p}{u_p(x_{i,p})} \left( u_p(x_{i,p} + \mu_{i,p} x) - u_p(x_{i,p}) \right) \to U(x)$$

in $C^{1}_{loc}(\mathbb{R}^2)$ as $p \to +\infty$, where

$$U(x) = \log \left( \frac{1}{1 + \frac{1}{8} |x|^2} \right)^2$$

(5.6)
is the solution of \(-\Delta U = e^U\) in \(\mathbb{R}^2\), \(U \leq 0\), \(U(0) = 0\) and \(\int_{\mathbb{R}^2} e^U = 8\pi\).

Moreover
\[
\frac{d(x_{i,p}, \partial \Omega)}{\mu_{i,p}} \to +\infty \quad \text{and} \quad \frac{d(x_{i,p}, NL_p)}{\mu_{i,p}} \to +\infty \quad \text{as} \quad p \to +\infty.
\]

(\(P^n_k\)) There exists \(C > 0\) such that
\[
p \min_{i=1, \ldots, n} |x - x_{i,p}|^2 |u_p(x)|^{p-1} \leq C
\]
for all \(p\) sufficiently large and all \(x \in \Omega\).

Proposition 5.2 (\cite{3 Proposition 2.2}). Let \((u_p)\) be a family of solutions to \((1.2)\) and assume that (A) holds. Then there exist \(k \in \mathbb{N} \setminus \{0\}\) and \(k\) families of points \((x_{i,p})\) in \(\Omega\) \(i = 1, \ldots, k\) such that \(x_{1,p} = x_0^+\) and, after passing to a sequence, \((P^1_k), (P^2_k),\) and \((P^3_k)\) hold. Moreover, given any family of points \(x_{k+1,p}\), it is impossible to extract a new sequence from the previous one such that \((P^{k+1}_1), (P^{k+1}_2),\) and \((P^{k+1}_3)\) hold with the sequences \((x_{i,p}), i = 1, \ldots, k + 1\). At last, we have
\[
\sqrt{p} u_p \to 0 \quad \text{in} \quad C^1_{\text{loc}}(\Omega \setminus \lim_{p} x_{i,p}, i = 1, \ldots, k) \quad \text{as} \quad p \to +\infty.
\]

Theorem 5.3. Let \((u_p)\) be a family of solutions to \((1.2)\) which satisfies condition (A). Let \(k \in \mathbb{N} \setminus \{0\}\) be the maximal number of families of points \((x_{i,p}), i = 1, \ldots, k,\) for which \((P^1_k), (P^2_k)\) and \((P^3_k)\) hold up to a subsequence (as in Proposition 5.2).

If \(k = 1\) then condition (B) holds true up to a subsequence in \(p\).

Proof. By (A) Proposition 5.2 applies. Just observe that \(x_{1,p} = x_0^+\) and so when \(k = 1\) property \((P^3_k)\) reduces to \((5.1)\) and so the conclusion follows by Lemma 5.1. \(\square\)

In the following we use Theorem 5.3 to obtain condition (B) for suitable classes of solutions.

6. PROOF OF THEOREM 1.2

Before proving Theorem 1.2 we observe that the existence of sign changing stationary solutions \(u_p\) to \((1.1)\) satisfying assumptions \((1.3)\) and \((1.4)\) has been proved for \(m \geq 4\) in \cite{2} for \(p\) large. The proof uses the fact that the energy is decreasing along non constant solutions, and relies on constructing a suitable initial condition \(v_0\) for problem \((1.1)\) such that any stationary solution in the corresponding \(\omega\)-limit set satisfies the energy estimate \((1.4)\). This construction can be done for \(p\) large even without any symmetry assumption on \(\Omega\) (see \cite{2} for details). Anyway when \(\Omega\) is a simply connected \(G\)-symmetric smooth bounded domain with \(|G| \geq m\) also some qualitative properties of \(u_p\) under condition
may be obtained (for instance the nodal line does not touch \( \partial \Omega \), it does not pass through the origin, etc, as shown in [2]).

Then, in [3] a deeper asymptotic analysis of \( u_p \) as \( p \to +\infty \) has been done, showing concentration in the origin and a bubble tower behavior, when \( \Omega \) is a simply connected \( G \)-symmetric smooth bounded domain with \( |G| \geq me \).

Here we do not require \( \Omega \) to be simply connected.

Clearly assumption (1.4) is a special case of condition (A), hence in particular Proposition 5.2 holds. As before we assume w.l.o.g. that \( \|u_p\|_\infty = \|u^+_p\|_\infty \).

The proof of Theorem 1.2 follows then from the following

**Proposition 6.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a smooth bounded domain, \( O \in \Omega \), invariant under the action of a cyclic group \( G \) of rotations about the origin which satisfies (1.3) for a certain \( m > 0 \). Let \( (u_p) \) be a family of sign changing \( G \)-symmetric stationary solutions of (1.1) which satisfies (1.4). Then condition (B) is satisfied up to a subsequence.

As we will see Proposition 6.1 is a consequence of the general sufficient condition in Theorem 5.3.

Hence in order to prove it we only need to show that \( k = 1 \), where the number \( k \) is the maximal number of families of points \( (x_{i,p}) \), \( i = 1, \ldots, k \), for which \( (P^+_1) \), \( (P^+_2) \) and \( (P^+_3) \) hold, up to a subsequence, as in Proposition 5.2. When \( m = 4 \) the result has been already proved in [3, Proposition 3.6]. Here we show the general case (see also [3, Remark 4.6]).

We start with the following:

**Lemma 6.2.** Let \( \Omega \subset \mathbb{R}^2 \) be a smooth bounded domain, \( O \in \Omega \), invariant under the action of a cyclic group \( G \) of rotations about the origin which satisfies (1.3) for a certain \( m > 0 \). Let \( (u_p) \) be a family of sign changing \( G \)-symmetric stationary solutions of (1.1) which satisfies (1.4).

Let \( k, x_{i,p} \) and \( \mu_{i,p} \) for \( i = 1, \ldots, k \) be as in Proposition 5.2. Then

\[
\frac{|x_{i,p}|}{\mu_{i,p}} \text{ is bounded.}
\]

**Proof.** The proof is similar to the one of [3, Proposition 3.3].

Let us fix \( i \in \{1, \ldots, k\} \). In order to simplify the notation we drop the dependence on \( i \) namely we set \( x_p := x_{i,p} \) and \( \mu_p := \mu_{i,p} \).

Without loss of generality we can assume that either \( (x_p)_p \subset \mathcal{N}^+_p \) or \( (x_p)_p \subset \mathcal{N}^-_p \). We prove the result in the case \( (x_p)_p \subset \mathcal{N}^+_p \), the other case being similar.

Let \( h := |G| \), \( (\mathbb{N} \setminus \{0\} \ni h \geq me) \) and let us denote by \( g^j, j = 0, \ldots, h - 1 \), the elements of \( G \). We consider the rescaled nodal domains

\[
\mathcal{N}^{i,j}_p := \{ x \in \mathbb{R}^2 : \mu_p x + g^j x_p \in \mathcal{N}^+_p \}, \quad j = 0, \ldots, h - 1,
\]
and the rescaled functions $z_p^{j+}(x) : \tilde{\mathcal{N}}^+_p \to \mathbb{R}$ defined by

$$z_p^{j+}(x) := \frac{p}{u_p(x_p)} \left( u_p(\mu_p x + g^j x_p) - u_p^+(x_p) \right), \quad j = 0, \ldots, h - 1. \quad (6.1)$$

Observe that, since $\Omega$ is $G$-invariant, $g^j x_p \in \Omega$ for any $j = 0, \ldots, h - 1$. Moreover $u_p$ is $G$-symmetric and $x_p$ satisfies (5.7), hence it’s not difficult to see from $(P_k^2)$ that each function $z_p^{j+}$ converges to $U$ in $C^1_{\text{loc}}(\mathbb{R}^2)$, as $p \to \infty$ and $8\pi = \int_{\mathbb{R}^2} e^U dx$ (see also [3, Corollary 2.4]).

Assume by contradiction that there exists a sequence $p_n \to +\infty$ such that $\frac{|x_{p_n}|}{\mu_{p_n}} \to +\infty$. Let

$$d_n := |g^j x_{p_n} - g^{j+1} x_{p_n}|, \quad j = 0, \ldots, h - 1. \quad (6.2)$$

Then, since the $h$ distinct points $g^j x_{p_n}$, $j = 0, \ldots, h - 1$, are the vertices of a regular polygon centered in $O$, $d_n = 2\tilde{d}_n \sin \frac{\pi}{h}$, where $\tilde{d}_n := |g^j x_{p_n}| \equiv |x_{p_n}|$, $j = 0, \ldots, h - 1$. Hence

$$\frac{d_n}{\mu_{p_n}} \to +\infty. \quad (6.3)$$

Let

$$R_n := \min \left\{ \frac{d_n}{3}, \frac{d(x_{p_n}, \partial \Omega)}{2}, \frac{d(x_{p_n}, NL_{p_n})}{2} \right\}, \quad (6.4)$$

then by (6.2) and (5.7)

$$\frac{R_n}{\mu_{p_n}} \to +\infty, \quad (6.5)$$

moreover, by construction,

$$B_{R_n}(g^j x_{p_n}) \subseteq \mathcal{N}^+_p, \quad \text{for} \quad j = 0, \ldots, h - 1 \quad (6.5)$$

$$B_{R_n}(g^j x_{p_n}) \cap B_{R_n}(g^l x_{p_n}) = \emptyset, \quad \text{for} \quad j \neq l. \quad (6.6)$$
Using (6.4), the convergence of $z_{pn}^{j, +}$ to $U$, (5.4) and Fatou’s lemma, we have

$$8\pi = \int_{\mathbb{R}^2} e^{U} \, dx$$

Fatou + conv. of $v_{pn} + (6.4)$$
\leq \lim_n \int_{B_{R_n}^{(0)}} \frac{z_{pn}^{j, +} + (p_n + 1)}{e^{z_{pn}^{j, +} + (p_n + 1)}} \, dx$$

$$= \lim_n \int_{B_{R_n}^{(0)}} \frac{1 + \frac{z_{pn}^{j, +}(x)}{p_n}}{(p_n + 1)} \, dx$$

$$= \lim_n \int_{B_{R_n}^{(0)}} \frac{u_{pn}^{j, +}(x) + g^j x_{pn}}{u_{pn}^{j, +}(x_{pn})} \, dx$$

$$= \lim_n \int_{B_{R_n}^{(0)}} \frac{u_{pn}^{j, +}(x)}{(p_n + 1)} \, dx$$

$$\leq \lim_n \int_{B_{R_n}(x_{pn})} \frac{u_{pn}^{j, +}(x)}{(p_n + 1)} \, dx.$$ 

(6.7)

Summing on $j = 0, \ldots, h - 1$, using (6.6), (6.5) and assumption (1.4) we get:

$$h \cdot 8\pi \leq \lim_n p_n \sum_{j=0}^{h-1} \int_{B_{R_n}(x_{pn})} \frac{u_{pn}^{j, +}(x)}{(p_n + 1)} \, dx$$

$$\leq \lim_n p_n \int_{N_{pn}} \frac{\left|u_{pn}(x)\right|}{(p_n + 1)} \, dx$$

$$= \lim_n \left(p_n \int_{\Omega} \frac{\left|u_{pn}(x)\right|}{(p_n + 1)} \, dx - p_n \int_{N_{pn}} \frac{\left|u_{pn}(x)\right|}{(p_n + 1)} \, dx\right)$$

$$\leq \lim_n p_n \int_{\Omega} \frac{\left|u_{pn}(x)\right|}{(p_n + 1)} \, dx - 8\pi e$$

$$\leq (\alpha - 1) 8\pi e$$

$$< m 8\pi e$$

which gives a contradiction with (1.3).

Last using Lemma 6.2 we can prove that the number $k$ in Proposition 5.2 is equal to one.

**Lemma 6.3.** Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $O \in \Omega$, invariant under the action of a cyclic group $G$ of rotations about the origin which satisfies (1.3) for a certain $m > 0$. Let $(u_{\mu})$ be a family of sign changing $G$-symmetric stationary solutions of (1.1)
which satisfies (1.4). Let $k$ be, as in Proposition 5.2, the maximal number of families of points $(x_i,p)$, $i = 1, \ldots, k$, for which, after passing to a subsequence, $(P_i^1)$, $(P_i^2)$ and $(P_i^3)$ hold. Then

$k = 1$.

Proof. The proof is the same as in [3, Proposition 3.6], we repeat it for completeness. Let us assume by contradiction that $k > 1$ and set $x_p^+ = x_{1,p}$. For a family $(x_j,p)$, $j \in \{2, \ldots, k\}$ by Lemma 6.2 there exists $C > 0$ such that

$$\frac{|x_{1,p}|}{\mu_{1,p}} \leq C \quad \text{and} \quad \frac{|x_{j,p}|}{\mu_{j,p}} \leq C.$$ 

Thus, since by definition $\mu_p^+ = \mu_{1,p} \leq \mu_{j,p}$, also

$$\frac{|x_{1,p}|}{\mu_{j,p}} \leq C.$$ 

Hence

$$\frac{|x_{1,p} - x_{j,p}|}{\mu_{j,p}} \leq \frac{|x_{1,p}| + |x_{j,p}|}{\mu_{j,p}} \leq C \quad \text{as } p \to +\infty,$$

which contradicts $(P_i^k)$. \qed

7. A special case in Theorem 1.2: the radial solutions

In this section we show that, when the domain $\Omega$ is the unit ball in $\mathbb{R}^2$, the unique (up to a sign) radial solution $u_{p,K}$ of (1.2) with $K \geq 2$ nodal regions satisfies conditions (A) and (B).

Thus Theorem 1.1 applies to $u_{p,K}$, namely we re-obtain the result already known from [4] through a different proof which does not rely on radial arguments.

Let us fix the number of nodal regions $K \geq 2$. As before we assume w.l.o.g. that $\|u_{p,K}\|_{\infty} = \|u_{p,K}^+\|_{\infty}$.

The main result is the following:

**Proposition 7.1.** Let $\Omega$ be the unit ball in $\mathbb{R}^2$ and for $K \geq 2$ let $u_{p,K}$ be the unique radial solution of (1.2) with $K$ nodal domains. Then there exists $m(= m(K)) > 0$ for which the assumptions of Theorem 1.2 are satisfied.

**Proof.** In [4, Proposition 2.1] it has been proved that $u_{p,K}$ satisfies assumption (A) (by extending the arguments employed in [2] for the case with two nodal regions). Hence there exists $C(= C(K))$ such that

$$p \int_{\Omega} |u_{p,K}|^{p+1} dx \leq C.$$
Let us define
\[ \alpha := \frac{C}{8\pi e} \]
and let \( \bar{m} > 0 \) be such that
\[ \bar{m} > \frac{C}{8\pi e} - 1. \]
Let \( G \) be a cyclic group of rotations about the origin such that \( |G| \geq \bar{m}e \). Of course the unit ball is \( G \)-invariant, moreover, since \( u_{p,K} \) is radial, it is in particular \( G \)-symmetric and so we have proved that (1.3) and (1.4) hold true with \( m = \bar{m} \).

\[ \square \]

We conclude the section with some more consideration on condition (\( B \)) in the radial case.

It is easy to show that (see [4, Proposition 2.4 - i]) for the proof \( \|u_{p,K}\|_\infty = u_{p,K}(0) (> 0) \), namely \( x_1^p \equiv 0 \). Hence condition (\( B \)) in this radial case reads as follows:

\[ \lim_{R \to +\infty} \lim_{p \to +\infty} S_{p,R} = 0, \]  

(\( B \))

where, for \( R > 0 \),

\[ S_{p,R} := \sup \left\{ \frac{|u_{p,K}(r)|^{p-1}}{|u_{p,K}(0)|} : R\mu_p^+ < r < 1 \right\} \]

and \( u_{p,K}(r) = u_{p,K}(|x|), r = |x| \).

In addition to the general characterization in Proposition 2.3, it is easy to prove in the radial case, also the following characterization of condition (\( B \)):

**Proposition 7.2.** Let \( \Omega \) be the unit ball in \( \mathbb{R}^2 \) and for \( K \geq 2 \) let \( u_{p,K} \) be the unique radial solution of (1.2) with \( K \) nodal domains. Set \( 0 < r_{p,1} < r_{p,2} < \ldots < r_{p,K-1} < 1 \) the nodal radii of \( u_{p,K}(r) \).

Then for any \( R > 0 \) there exists \( p_R > 1 \) such that the set \( \{R\mu_p^+ < r < r_{p,1}\} \neq \emptyset \) for \( p \geq p_R \) and so

\[ \mathcal{M}'_{p,R} := \sup \left\{ \frac{|u_{p,K}(r)|^{p-1}}{|u_{p,K}(0)|} : R\mu_p^+ < r < r_{p,1} \right\} \]  

(7.1)

is well defined.

Moreover condition (\( B \)) is equivalent to

\[ \lim_{p \to +\infty} \sup_{\{r_{p,1} < r < 1\}} \frac{|u_{p,K}(r)|^{p-1}}{u_{p}(0)^{p-1}} = 0 \]  

(\( B1' \))

\[ \lim_{R \to +\infty} \lim_{p \to +\infty} \mathcal{M}'_{p,R} = 0 \]  

(\( B2' \))
Remark 7.3. Observe that $(B1')$ was already known, indeed in [4, Proposition 2.4 - iv)] the authors proved that

$$\sup_{\{r_p,1<r<1\}} |u_{p,K}(r)| \rightarrow \alpha < \frac{1}{2} \quad \text{as} \quad p \to +\infty.$$ 

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