The Blaschke-Lebesgue problem for constant width bodies of revolution

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Abstract
We prove that among all constant width bodies of revolution, the minimum of the ratio of the volume to the cubed width is attained by the constant width body obtained by rotation of the Reuleaux triangle about an axis of symmetry.

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Introduction
The width of a convex body $B$ in $n$-dimensional Euclidean space in the direction $\mathbf{u}$ is the distance between the two supporting planes of $B$ which are orthogonal to $\mathbf{u}$. When this distance is independent of $\mathbf{u}$, $B$ is said to have constant width.

The ratio $I(B)$ of the volume of a constant width body to the volume of the ball of the same width is homothety invariant, as is the isoperimetric ratio. Moreover the maximum of $I(B)$ is attained by round spheres, just as the minimum of the isoperimetric ratio. However, while the latter is not bounded from above, the infimum of $I$ is strictly positive, since compactness properties of the space of convex sets ensures the existence of a minimizer. It is known since the works of Blaschke and Lebesgue that the Reuleaux triangle, obtained by taking the intersection of three discs centered at the vertices of an equilateral triangle, minimizes $I$ in dimension $n = 2$. The determination of the minimizer of $I$ in any dimension is the Blaschke-Lebesgue problem. Recently several simpler solutions of the problem in dimension 2 have been given (cf [Ba],[Ha]), however the Blaschke-Lebesgue problem in dimension $n = 3$ appears to be very difficult to solve and remains open.

In this paper we prove:

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**Main Theorem:** Amongst all constant width bodies of revolution in Euclidean 3-space, the minimum of the ratio the volume to the cubed width is attained by the constant width body $B_{Reul}$ obtained by rotation of the Reuleaux triangle about an axis of symmetry.

This result has been proved in [CCG] by a geometric argument which relies on a direct comparison between the volume of an arbitrary convex body of revolution of constant width, and the rotated Reuleaux triangle of the same width. Our proof is more analytical in nature as it uses calculus of variations. Moreover, some of the observations we make do not depend on the assumption of rotational symmetry.

It is known that there exists a convex body $B_{Meis}$, called Meissner’s tetrahedron, satisfying $I(B_{Meis}) \simeq 0.8019$ (cf [CG],[GK], [Ba]), while $I(B_{Reul}) \simeq 0.8584$. Thus the main theorem implies the following:

**Corollary:** The solution of the Blaschke-Lebesgue problem is not a body of revolution.

As in [Ba] and [Ha], our proof is based on the analysis of the support function $s$ which characterizes a convex body of constant width $2w$. A crucial point is the following observation, stated in [GK]: flowing the boundary of a convex body along its inward unit normal vector field preserves the constant width condition, as long as the evolving surface remains convex. Moreover, the ratio $I$ decreases along the flow, so the minimizer of $I$ must occur at the latest time such that convexity holds, and therefore must be singular. This issue is easily controlled by introducing the function $h = s - w$, which is invariant along the normal flow, while the width $2w$ decreases linearly. Thus, there exists a positive number $w_0(h)$ such that for any $w \geq w_0(h)$, the function $s = h + w$ is the support function of some convex body of constant width $2w$. Hence, we can restrict the minimization process to the class of support functions of the form $s = h + w_0(h)$, while all the necessary information is carried by the function $h$. The assumption of rotational symmetry made in the present article, since it reduces all the involved calculus to one variable, simplifies considerably the exposition; however, all these facts hold for arbitrary constant width bodies in Euclidean 3-space.

The next step in the proof of our main theorem consists of using the second order condition of minimization (i.e. stability) to prove that the map $|h'' + h|$ must be constant. It follows that the value of $I(h)$ is completely determined by the set of the discontinuities of $h'' + h$. The rotated Reuleaux triangle $B_{Reul}$ corresponds to the case of $h'' + h$ having the least possible number of discontinuities. We then
show that, unless the number of discontinuities of $h'' + h$ is minimal, one can always reduce the ratio $I$, which completes the proof.

As a final comment, we point out that one can prove the fact that the Reuleaux triangle minimizes $I$ among the constant width bodies of the plane by a slight modification of our argument.

1 Preliminaries: constant width bodies of revolution

Let $B$ be a convex body of revolution in $\mathbb{R}^3$, i.e. it is invariant under rotation around some axis. We may assume without loss of generality that this axis is vertical. Therefore the boundary of $B$ can be parametrized by

$$X: [a,b] \times S^1 \rightarrow \mathbb{R}^3$$

$$\ (\phi, \theta) \mapsto (x(\phi) \cos \theta, x(\phi) \sin \theta, y(\phi)),$$

where $\gamma(\phi) = (x(\phi), y(\phi))$ is a parametrized curve such that $x(\phi) \geq 0$ and $x(a) = x(b) = 0$. It is known (see [Ho], [Ba]) that if $B$ has constant width it must be strictly convex. It follows that the generating curve $\gamma$ is also strictly convex, which allows us to reparametrize it by the angle $t$ made by its unit outward normal vector $\vec{n}(t) = (\cos t, \sin t)$ with the horizontal plane: $\gamma(t) = (x(t), y(t))$, with $t \in [-\pi/2, \pi/2]$. This parametrization holds even if the curve $\gamma$ is not regular.

Next we express the constant width assumption of $B$ in terms of the curve $\gamma$. For this purpose it is convenient to consider the union of $\gamma$ with its image under reflection through the vertical axis, which gives a strictly convex, closed, planar curve. This closed curve is parametrized by

$$\gamma(t) = (x(t), y(t)) := (-x(\pi - t), y(\pi - t)), \forall t \in [-\pi, -\pi/2] \cup [\pi/2, \pi],$$

and it is then possible to parametrize $\partial B$ by

$$X: (S^1 \times S^1)/\sim \rightarrow \mathbb{R}^3$$

$$\ (t, \theta) \mapsto (x(t) \cos \theta, x(t) \sin \theta, y(t)),$$

where $\sim$ denotes the equivalence relation defined on the torus $S^1 \times S^1$ by $(t, \theta) \sim (\pi - t, \theta + \pi)$. In particular, the antipodal point of $(t, \theta)$ is $(-t, \theta + \pi) \sim (t + \pi, \theta)$.

Next the support function of $\partial B$ at the point $X(t, \theta)$ is defined to be

$$s_X(t, \theta) := \langle X(t, \theta), \vec{N}(t, \theta) \rangle,$$
where $\vec{N}$ is the unit outward vector of $\partial B$ at the point $X(t, \theta)$. The width of $B$ in the direction $\pm \vec{N}$ is equal to the sum of the support function evaluated at the two antipodal points corresponding to this direction:

$$2w(t, \theta) = s_X(t, \theta) + s_X(-t, \theta + \pi) = s_X(t, \theta) + s_X(t + \pi, \theta).$$

Similarly, the support function of the curve $\gamma$ is $s_\gamma(t) := \langle \gamma(t), \vec{n}(t) \rangle = \langle \gamma(t), (\cos t, \sin t) \rangle$. An easy computation, using the fact that $\vec{N} = (\cos t \cos \theta, \cos t \sin \theta, \sin t)$, gives:

$$2w(t, \theta) = \langle X(t, \theta), \vec{N}(t, \theta) \rangle + \langle X(t + \pi, \theta), \vec{N}(t + \pi, \theta) \rangle$$

$$= \langle (x(t), y(t)), (\cos t, \sin t) \rangle + \langle (x(t + \pi), y(t + \pi)), (\cos(t + \pi), \sin(t + \pi)) \rangle$$

$$= s_\gamma(t) + s_\gamma(t + \pi).$$

The last expression is nothing but the width of the curve $\gamma$ in the direction $(\cos t, \sin t)$, so we have proved:

**Lemma 1** $B$ has constant width if and only if $\gamma$ has constant width.

From now on we focus on the curve $\gamma$ and use complex notation in the Euclidean plane $\{(x, y) \simeq x + iy\}$. The curve $\gamma$ can be reconstructed from its support function $s_\gamma$:

$$\gamma(t) = s_\gamma(t)e^{it} + s'_\gamma(t)ie^{it}.$$

Differentiating this expression yields $\gamma'(t) = (s''_\gamma + s_\gamma)ie^{it}$ and thus the curve is regular if and only if $s''_\gamma + s_\gamma > 0$. Moreover, it changes its convexity with the sign of $s''_\gamma + s_\gamma$. As the curve may not be regular everywhere, but must remain strictly convex, we are left with the condition $s''_\gamma + s_\gamma \geq 0$. This quantity is nothing but the radius of curvature of $\gamma$.

Next set $w := \frac{1}{2\pi} \int_{\mathbb{S}} s_\gamma(t)dt$ and $h := s_\gamma - w$. Hence, the curve $\gamma$ has constant width if and only if

$$h(t) + h(t + \pi) = 0,$$

and in this case the width is exactly $2w$. On the other hand, the symmetry of the curve with respect to the vertical axis, i.e. $x(\pi - t) + iy(\pi - t) = -x(t) + iy(t)$ implies that $s_\gamma(t) - s_\gamma(\pi - t) = 0$ and thus

$$h(t) - h(\pi - t) = 0.$$
By Equations (1) and (2) it is enough to define $h$ on the interval $[0, \pi/2]$ and to extend it to $S^1$ by the symmetries (1) and (2). Furthermore we have $h(0) = 0$ and $h'(\pi/2) = 0$. On the other hand, it is proven in [Ho] that the support function of a constant width body is $C^{1,1}$ so we conclude that the functional space of the $h$ corresponding to constant width curves with axial symmetry (and thus to constant width bodies of revolution) is

$$E := \{ h \in C^{1,1}([0, \pi/2]), h(0) = 0, h'(\pi/2) = 0 \}.$$  

Finally, a given pair $(h, w)$ corresponds to a curve $\gamma$ if the support function $s_\gamma = h + w$ satisfies the condition $s''_\gamma + s_\gamma = h'' + h + w \geq 0$. By the Rademacher theorem, the fact $h \in C^{1,1}$ implies that $h''$ exists a.e. and belongs to $L^\infty(0, \pi/2)$. Thus we must have $w \geq -(h'' + h)$, almost everywhere in $S^1$. As $h$ is odd, it is equivalent to require that $w \geq h'' + h$, a.e. in $S^1$. Hence, for any given $h \in E$, we define

$$w_0(h) := ||h(t) + h''(t)||_{L^\infty(0,\pi/2)}.$$  

Summing up, we have proven:

**Proposition 1** There is a one-to-one correspondence between the convex bodies $B$ of revolution which have constant width $2w$, and the set of pairs $(h, w)$, $h \in E$, $w \geq w_0(h)$, where

$$E := \{ h \in C^{1,1}([0, \pi/2]), h(0) = 0, h'(\pi/2) = 0 \},$$  

and

$$w_0(h) := ||h(t) + h''(t)||_{L^\infty(0,\pi/2)}.$$  

**Example 1** If $h = c \sin t$, where $c$ is some real constant, the curve $\gamma$ is a circle centered in the vertical axis and thus the corresponding body is a ball.

## 2 The Blaschke-Lebesgue problem

We now compute the volume of a constant width body of revolution $B$ in terms of $h$ and $w$. We start by calculating the first derivatives of the immersion:

$$X_\theta = (x' \cos \theta, x' \sin \theta, y'), \quad X_\theta = (-x \sin \theta, x \cos \theta, 0).$$  

As $\det(X, X_\theta, X_t) > 0$ the volume of $B$ is

$$V(B) = \frac{1}{3} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \det(X, X_\theta, X_t) dt d\theta$$  

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\[
\frac{2\pi}{3} \int_{-\pi/2}^{\pi/2} x(y'x - yx')dt.
\]

The integrand turns out to be a polynomial in \(w\):

\[
x(y'x - yx') = ((w + h) \cos t - h' \sin t)(w + h + h'')(w + h) = (w \cos t + h \cos t - h' \sin t)(w^2 + (2h + h'')w + h(h'' + h)) = \cos tw^3 + (h \cos t - h' \sin t + (2h + h'') \cos t) w^2 + (2h'' h \cos t - h'' h' \sin t + 3h^2 \cos t - 2hh' \sin t)w + (h \cos t - h' \sin t)(h'' + h)h.
\]

Next, as \(h\) is odd, the functions \(h \cos t, h' \sin t\) and \((h + h'') \cos t\) are odd as well, thus the integrals of both the term in \(w^2\), and the constant term, vanish and we are left to compute of the \(w\) term. Using the fact that

\[
\int_{0}^{\pi/2} (2h'' h' \sin t + (h')^2 \cos t) dt = \int_{0}^{\pi/2} ((h')^2 \sin t)' dt,
\]

and

\[
\int_{0}^{\pi/2} (h'' h \cos t - h' h \sin t + (h')^2 \cos t) = \int_{0}^{\pi/2} (hh' \cos t)' dt,
\]

vanish by the boundary conditions, we get

\[
\int_{-\pi/2}^{\pi/2} (2h'' h \cos t + h'' h' \sin t + 3h^2 \cos t - 2hh' \sin t)dt = 2 \int_{0}^{\pi/2} (2h'' h \cos t - h'' h' \sin t + 3h^2 \cos t - 2hh' \sin t)dt
\]

\[
= 2 \int_{0}^{\pi/2} (3h^2 \cos t - \frac{3}{2} (h')^2 \cos t)dt.
\]

Thus we obtain

\[
\mathcal{V}(B) = 4\pi \left( \frac{w^3}{3} + w \int_{0}^{\pi/2} \left(h^2 - \frac{(h')^2}{2}\right) \cos t dt \right).
\]
Proposition 2 Let \((h, w)\) with \(h \in E\) and \(w \geq w_0(h)\) and let \(B\) be the corresponding constant width body of revolution (see Proposition 1). Then

\[
\mathcal{V}(B) = 4\pi \left( \frac{w^3}{3} + w\mathcal{F}(h) \right)
\]

where the functional \(\mathcal{F}\) is defined to be

\[
\mathcal{F}(h) := \int_0^{\pi/2} \left( h^2 - \frac{1}{2} (h')^2 \right) \cos t dt.
\]

Remark 1 We could also have computed the area of \(\partial B\) and used the Blaschke formula (see [GK]) for bodies of constant width: \(\mathcal{V}(B) = A(\partial B)w - \frac{8\pi}{3} w^3\).

It is then easy to express the ratio \(I(B)\) in terms of \(h\) and \(w\):

\[
I(B) := \frac{\mathcal{V}(B)}{4\pi w^3 / 3} = 1 + \frac{3\mathcal{F}(h)}{w^2}.
\]

The next proposition, which may be seen as a weighted version of the classical Wirtinger inequality, shows that the last term in the expression above is negative:

**Proposition 3 (weighted Wirtinger inequality)** Let \(h \in E\). Then the following inequality holds,

\[
\mathcal{F}(h) = \int_0^{\pi/2} \left( h^2 - \frac{1}{2} (h')^2 \right) \cos t dt \leq 0,
\]

and the equality is attained if and only if \(h = c \sin t\) for some real constant \(c\).

*Proof.* Introduce the function \(g := h \cos t - h' \sin t\). It is easy to check that \(g' = -\sin t(h + h'')\). The boundary conditions \(h(0) = 0\) and \(h'(\pi/2) = 0\) imply that \(g(0) = g(\pi/2) = 0\), and, moreover,

\[
\lim_{\epsilon \to 0} \frac{g(t)}{\sin t} = \lim_{\epsilon \to 0} (h(t) \cot t - h'(t)) = h'(0) - h'(0) = 0.
\]

Hence

\[
\mathcal{F}(h) = \lim_{\epsilon \to 0} \int_\epsilon^{\pi/2} (h + h'')(h \cos t - h' \sin t) dt
\]

\[
= \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi/2} \frac{gg'}{\sin t} dt = \lim_{\epsilon \to 0} \left( \int_{\epsilon}^{\pi/2} \frac{g^2(t)}{2} \frac{1}{\sin t} dt + \left[ \frac{g^2(t)}{2 \sin t} \right]_{\epsilon}^{\pi/2} \right) - \frac{\mathcal{F}(h)}{w^2}.
\]
\[
= \lim_{\epsilon \to 0} \left( \frac{1}{2} \int_{\epsilon}^{\pi/2} \frac{g^2(t) \cos t}{\sin^2 t} - \frac{g^2(\epsilon)}{2 \sin \epsilon} \right) = \lim_{\epsilon \to 0} \frac{1}{2} \int_{\epsilon}^{\pi/2} \frac{g^2(t) \cos t}{\sin^2 t} \leq 0.
\]

The last inequality shows that if $F(h)$ vanishes, so does $g$. We thus have $h \cos t = h' \sin t$. It is easy to check that the only solutions of this linear equation with initial conditions $h(0) = 0$ and $h'(\pi/2) = 0$ are $h = c \sin t$, where $c$ is some real constant.

From this lemma we recover the fact that the ratio $I$ achieves its maximum for $h = c \sin t$, which corresponds to $B$ being a ball. Moreover, it follows that the ratio $I(B)$ is increasing with respect to $w$. So by Proposition 1 we get:

**Corollary 1** Let $(h, w)$ be a minimizer of $I(h, w)$. Then $w = w_0(h)$.

We end this section using again the weighted Wirtinger inequality to prove that $|h + h''|$ must be constant.

**Proposition 4** Let $(h, w_0(h))$ be a minimizer of $I(h, w)$. Then the quantity $|h'' + h|$ is constant almost everywhere in $[0, \pi/2]$.

**Proof.** We proceed by contradiction assuming that there is a non-empty interval $[a, b]$ included in $[0, \pi/2]$ such that $|h(t) + h''(t)| < w_0(h)$ a. e. in $[a, b]$. Consider a map $V$ of $E$ whose support is contained in $[a, b]$ and which is not of the form $V = c \sin t$, and define the deformation $h^\epsilon := h + \epsilon V$ of $h$. For small $\epsilon$,

$$w_0(h^\epsilon) = \|(h^\epsilon)' + h^\epsilon\|_{L^\infty(0,\pi/2)} = \|h + h''\|_{L^\infty(0,\pi/2)} = w_0(h),$$

hence

$$\frac{\mathcal{F}(h^\epsilon)}{w_0^2(h^\epsilon)} = \frac{\mathcal{F}(h)}{w_0^2(h)} + \epsilon \frac{\delta \mathcal{F}(h, V)}{w_0^2(h)} + \frac{\epsilon^2}{2} \frac{\delta^2 \mathcal{F}(h, V)}{w_0^2(h)} + o(\epsilon^2).$$

As $h$ is a minimizer of $I$, and thus, of $\mathcal{F}(h)/w_0^2(h)$, we must have both $\delta \mathcal{F}(h, V) = 0$ and $\delta^2 \mathcal{F}(h, V) \geq 0$. On the other hand the functional $\mathcal{F}$ is quadratic, so that $\delta^2 \mathcal{F}(h, V) = \mathcal{F}(V, V)$, which is strictly negative by Proposition 3.

**Remark 2** The quantity $h'' + h + w$ being the radius of curvature of the curve $\gamma$, the geometric interpretation of the previous proposition is the following: when $h + h'' = -w$, i.e. the radius of curvature vanish, we are at a singularity (vertex) of the curve $\gamma$, and when $h + h'' = w$, the radius of curvature is constant and the corresponding portion of curve is an arc of circle of radius $2w$. 

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3 Proof of the main theorem

Let \((h, w_0(h))\) be a minimizer of \(\mathcal{I}\). By Proposition 4, we know that \(|h'' + h|\) is constant, and we may assume without loss of generality that this constant is one. Moreover, \(h'' + h\) is characterized, up to multiplication by \(-1\) (which does not change the value of \(\mathcal{F}\)) by its set of discontinuities. The symmetry conditions (1) and (2) imply that there is a discontinuity at 0, and not at \(\pi/2\). Moreover, there must be at least one discontinuity in the open interval \((0, \pi/2)\): otherwise \(h = A \cos t + B \sin t \pm 1\), and the boundary conditions imply \(A + (\pm 1) = 0\) and \(A = 0\), a contradiction. The remainder of the proof of the main theorem is organized as follows: we first observe that if \(h\) has only one discontinuity in the interval \((0, \pi/2)\), then the corresponding curve \(\gamma\) is the Reuleaux triangle. Then we prove that if \(h\) has at least two singularities in \((0, \pi/2)\), there exists a \(h^* \in \mathcal{E}\) such that \(\mathcal{F}(h^*) < \mathcal{F}(h)\). Therefore there is no minimizer with at least two singularities and so the only possible one is the rotated Reuleaux triangle.

3.1 Case of one interior discontinuity

Let \(t_1 \in (0, \pi/2)\) be the unique interior discontinuity of \(h'' + h\). Thus

\[
\begin{align*}
    h \big|_{[0,t_1]} &= A_0 \cos t + B_0 \sin t + 1 \\
    h \big|_{[t_1, \pi/2]} &= A_1 \cos t + B_1 \sin t - 1.
\end{align*}
\]

The conditions \(h(0) = 0\) and \(h'(\pi/2) = 0\) imply \(A_0 = -1\) and \(A_1 = 0\). On the other hand, the continuity of \(x(t) = (h(t) + w) \cos t - h'(t) \sin t\) at \(t_1\) yields

\[
\begin{cases}
    A_1 = A_0 + 2 \cos t_1 \\
    B_1 = B_0,
\end{cases}
\]

so in particular \(t_1 = \pi/3\). Changing the constant \(B_0\) (and thus \(B_1\)) amounts to making a vertical translation of the curve \(\gamma\) and does not affect the geometry of the problem. Therefore the corresponding curve is unique and is nothing but the Reuleaux triangle. From the computations of Section 2, it is easy to compute the volume of the rotated Reuleaux triangle: we have

\[
\mathcal{F}(h_{Reul}) = \int_0^{\pi/3} (\cos t - 1) dt + \int_{\pi/3}^{\pi/2} \cos t dt = 1 - \pi/3.
\]

Therefore,

\[
\mathcal{I}(B_{Reul}) = 1 + 3(1 - \frac{\pi}{3}) = 4 - \pi \simeq 0.858407346.
\]
3.2 General case

Let $0 \leq t_0 < t_1 < t_2 < t_3 \leq \pi/2$ and let $h \in E$ such that

- $|h'' + h| = 1$;
- $(t_0, t_1, t_2)$ are three successive discontinuities of $h'' + h$;
- $t_3$ is either the next discontinuity after $t_2$, or $t_3 = \pi/2$.

In particular the case of two interior discontinuities $(t_0, t_3) = (0, \pi/2)$ is covered. We thus have the following expressions for $h$:

$h \mid_{[t_0, t_1]} = A_0 \cos t + B_0 \sin t + 1$,

$h \mid_{[t_1, t_2]} = A_1 \cos t + B_1 \sin t - 1$,

$h \mid_{[t_2, t_3]} = A_2 \cos t + B_2 \sin t + 1$,

where $A_i$ and $B_i$ are real constants. As will become clear later, the values of the constants $B_0, B_1$ and $B_2$ do not affect the problem. Next observe that the continuity of $x(t) = (h(t) + w) \cos t - h'(t) \sin t$ at the points $t_1$ and $t_2$ yield the following relations:

$A_0 + 2 \cos t_1 = A_1 = A_2 + 2 \cos t_2$

and thus

$\cos t_1 = \frac{A_1 - A_0}{2} \quad \cos t_2 = \frac{A_1 - A_2}{2}$.

From now on we set $x := -A_0, y := A_1$ and $z := -A_2$, so that $x, y$ and $z$ are three positive constants, and by the assumption $t_1 < t_2$, we have $z < x$. We are going to show that $h$ is not a minimizer of $F$, dividing the proof in three different cases.

3.2.1 The case $z < y < x$

We construct explicitly a map $h^* \in E$ which has one less singularity than $h$, as follows:

$|(h^*)'' + h^*| = 1, \forall t \in [0, \pi/2],$

$h^*(t) = h(t), \forall t \in [0, t_0],$

$h^*(t) = -h(t), \forall t \in [t_3, \pi/2],$

and $(h^*)'' + h^*$ has exactly one discontinuity at $t^* \in (t_0, t_3)$. Thus

$h^* \mid_{[t_0, t^*]} = A^* \cos t + B^* \sin t + 1,$
\[ h^* \big|_{[t^*, t_3]} = A^{**} \cos t + B^{**} \sin t - 1. \]

Moreover, as we have \( h^*(t_1) = h(t_1) \) and \( h^*(t_3) = -h(t_3) \), and a similar relation for the first derivatives, we deduce that \( A^* = A_0 = -x \) and \( A^{**} = -A_2 = z \). Finally, the \( C^1 \) assumption at \( t^* \) implies that

\[ A^{**} = A^* + 2 \cos t^*, \]

so that

\[ \cos t^* = \frac{A^{**} - A^*}{2} = -\frac{A_0 + A_2}{2} = \frac{x + z}{2}. \]

**Remark 3** If \( t_0 = 0 \) and \( t_3 = \pi/2 \), i.e. the case of two singularities, one can check that \( t^* = \pi/3 \), so that \( h^* \) corresponds to the Reuleaux triangle.

Next we compute

\[
\mathcal{F}(h^*) - \mathcal{F}(h) = \int_0^{\pi/2} ((h^*)'' + h^*)(h^* \cos t - (h^*)' \sin t) dt - \int_0^{\pi/2} (h'' + h)(h \cos t - h' \sin t) dt
\]

\[
= \int_{t_0}^{t_3} ((h^*)'' + h^*)(h^* \cos t - (h^*)' \sin t) dt - \int_{t_0}^{t_3} (h'' + h)(h \cos t - h' \sin t) dt
\]

\[
= (A^*(t^* - t_0) - A^{**}(t_3 - t^*)) - \left( A_0(t_1 - t_0) - A_1(t_2 - t_1) + A_2(t_3 - t_2) \right)
\]

\[
= A_0(t^* - t_1) + A_1(t_2 - t_1) + A_2(t_2 - t^*)
\]

\[
= (z - x) \arccos \left( \frac{x + z}{2} \right) + (x - y) \arccos \left( \frac{x + y}{2} \right) + (y - z) \arccos \left( \frac{y + z}{2} \right).
\]

In order to prove that the latter is negative, we first introduce the coefficients \( a_n \) of the power series of the function \( \arcsin \). It is well known that \( a_n > 0, \forall n \geq 1 \) and that the radius of convergence of the series is 1. Moreover we have

\[ \arcsin X = \frac{\pi}{2} - \sum_{n=1}^{\infty} a_n X^n. \]

Next we define the positive map

\[ b_n(a, b, c) := \frac{(\frac{a+b}{2})^n - (\frac{a+c}{2})^n}{\frac{b-c}{2}} = \sum_{i=0}^{n-1} \left( \frac{a+b}{2} \right)^i \left( \frac{a+c}{2} \right)^{n-1-i} \].
Finally we conclude:

\[
\mathcal{F}(h^*) - \mathcal{F}(h) = (z-x) \arccos \left( \frac{x + z}{2} \right) + (y-z) \arccos \left( \frac{y + z}{2} \right)
\]

\[
= (z-x) \arccos \left( \frac{x + z}{2} \right) + (x-z+y) \arccos \left( \frac{x + y}{2} \right) + (y-z) \arccos \left( \frac{y + z}{2} \right)
\]

\[
= (z-x) \left( \arccos \left( \frac{x + z}{2} \right) - \arccos \left( \frac{x + y}{2} \right) \right) + (y-z) \left( \arccos \left( \frac{y + z}{2} \right) - \arccos \left( \frac{x + y}{2} \right) \right)
\]

\[
= (z-x) \sum_{n=1}^{\infty} a_n \frac{z - y}{2} b_n(x, z, y) + (y-z) \sum_{n=1}^{\infty} a_n \frac{z - x}{2} b_n(y, z, x)
\]

\[
= \frac{(x-z)(y-z)}{2} \sum_{n=1}^{\infty} a_n (b_n(x, z, y) - b_n(y, z, x)) < 0,
\]

since \( b_n(x, z, y) - b_n(y, z, x) \) has the same sign as \( y - x \).

### 3.2.2 The case \( x \leq y \)

We consider an infinitesimal variation \( h^\varepsilon \) of \( h \) such that \( y^\varepsilon = y + \varepsilon \) and we shall prove that \( \frac{\partial \mathcal{F}}{\partial \varepsilon}(h) < 0 \). Therefore, by choosing negative \( \varepsilon \) such that \( |\varepsilon| \) is small enough, we obtain a map \( h^{\varepsilon*} \) such that \( \mathcal{F}(h^{\varepsilon*}) < \mathcal{F}(h) \).

From the expressions \( \cos t_1 = \frac{z+y}{2} \) and \( \cos t_2 = \frac{y+z}{2} \), we get

\[
\cos t_1^\varepsilon = \cos t_1 + \frac{\varepsilon}{2} + o(\varepsilon) \quad \text{and} \quad \cos t_2^\varepsilon = \cos t_2 + \frac{\varepsilon}{2} + o(\varepsilon),
\]

which implies

\[
t_1^\varepsilon - t_1 = -\frac{1}{2 \sin t_1} + o(\varepsilon) \quad \text{and} \quad t_2^\varepsilon - t_2 = -\frac{1}{2 \sin t_2} + o(\varepsilon),
\]

while \( \cos t_0^\varepsilon = \cos t_0 \) and \( \cos t_3^\varepsilon = \cos t_3 \). In particular, \( h^\varepsilon \) and \( h \) coincide outside the interval \((t_0, t_3)\). By a straightforward computation we deduce that

\[
\mathcal{F}(h^\varepsilon) - \mathcal{F}(h) = x(t_1 - t_1^\varepsilon) - (y + \varepsilon)(t_2^\varepsilon - t_1^\varepsilon) + y(t_2 - t_1) + z(t_2^\varepsilon - t_2)
\]

\[
= \varepsilon \left( \frac{x}{2 \sin t_1} + (t_1 - t_2) + y \left( \frac{1}{2 \sin t_2} - \frac{1}{2 \sin t_1} \right) - \frac{z}{2 \sin t_2} \right) + o(\varepsilon),
\]

hence
\[
\frac{\partial F}{\partial \epsilon}(h) = \arccos \left( \frac{x+y}{2} \right) - \arccos \left( \frac{y+z}{2} \right) + \frac{x-y}{\sqrt{4 - (x+y)^2}} + \frac{y-z}{\sqrt{4 - (y+z)^2}}.
\]

On the other hand,

\[
\arccos \left( \frac{x+y}{2} \right) - \arccos \left( \frac{y+z}{2} \right) = \int_{x}^{z} \frac{\partial \arccos \left( \frac{\xi+y}{2} \right)}{\partial \xi} \, d\xi < -\frac{x-z}{\sqrt{4 - (y+z)^2}}
\]

since the map \( \xi \mapsto -\frac{1}{\sqrt{4 - (y+\xi)^2}} \) is decreasing. Therefore we have

\[
\frac{\partial F}{\partial \epsilon}(h) < \frac{z-x}{\sqrt{4 - (y+z)^2}} + \frac{x-y}{\sqrt{4 - (x+y)^2}} + \frac{y-z}{\sqrt{4 - (y+z)^2}}
\]

\[
= \frac{y-x}{\sqrt{4 - (y+z)^2}} + \frac{x-y}{\sqrt{4 - (x+y)^2}}
\]

\[
= (y-x) \left( \frac{1}{\sqrt{4 - (y+z)^2}} - \frac{1}{\sqrt{4 - (x+y)^2}} \right) \leq 0,
\]

since \( x \leq y \) and using again the fact that \( \xi \mapsto -\frac{1}{\sqrt{4 - (y+\xi)^2}} \) is decreasing; therefore

\[
\frac{\partial F}{\partial \epsilon}(h) < 0.
\]

3.2.3 The case \( y \leq z \)

As in the previous one, we perform an infinitesimal deformation of \( h^\epsilon \) of \( h \) by setting \( y^\epsilon = y + \epsilon \). Here we prove that \( \frac{\partial F}{\partial \epsilon}(h) > 0 \) and the conclusion follows as above. Since the situation is very similar to the previous case, the details are left to the Reader.

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