A Cold Atomic Fermi Gas with a Spatially Modulated Interaction

Hao Fu\textsuperscript{1} and Alberto G. Rojo\textsuperscript{2}

\textsuperscript{1}MCTP, FOCUS Center, and Physics Department, University of Michigan, Ann Arbor, Michigan 48109-1040
\textsuperscript{2}Department of Physics, Oakland University, Rochester, MI 48309
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Abstract

We study an ultra-cold atomic Fermi Gas with the atom-atom interaction modulated periodically in space. A novel ground state with cooper pairs occupying non-zero center of mass momenta is found. Such a state is closely related to the state proposed by Fulde, Ferrell, Larkin, and Ovchinnikov (FFLO). The resultant single particle excitations with momenta along the direction of the modulation shows multiple-gap structures. Such a system can be realized in experiment with a spatially modulated Feshbach resonance. Experimental signatures of such a state are discussed.
In several recent experiments, polarized Fermi gases have been realized. These progresses drive a new wave of studies of the so called FFLO state. Such a state is of great interest because of its implications in many systems, including color superconducting quark matter, dense nuclear matter, and superconducting systems in solid states. The observation of such a state, however, is still elusive because it exists only in a very narrow parameter regime in cold atomic gases. In addition, it is not completely clear whether such a state favors occupation of multiple or single center of mass (COM) states. In this paper, we propose to study a ground state closely related to the FFLO state. At zero temperature in a homogeneous system, it is the only ground state of the system and the study of such a state should shed light on the study of the FFLO state.

We consider a fermion gas with the atom-atom interaction varying in space. For an ultra cold atomic gas, the effective atom-atom interaction is, \( V(\rho) = 4\pi a_s \hbar^2 \delta(\rho) / m \) where \( \rho \) is relative coordinate between two particles, \( m \) is the mass of the atom, and \( a_s \) is the s-wave scattering length. Near a Feshbach resonance, the scattering length can be described by \( a_{\text{eff}} = a_{bg} - \frac{m}{4\pi\hbar^2 v_0} |g_0|^2 \), where \( a_{bg} \) is the background scattering length, and the second term is the contribution from the Feshbach resonance nearby. The constants \( g_0 \) and \( v_0 \) are the coupling strength and energy detuning between the scattering channel and the molecular channel, respectively. In magnetic Feshbach resonances, the detuning \( v_0 \) can be controlled by an external magnetic field. In an optical Feshbach resonance, \( v_0 \) can be tuned by laser frequencies, and the free bound coupling \( g_0 \) can be tuned by the laser intensity. A spatially dependent scattering length can be achieved by applying spatially varying external fields. This gives an effective interaction, \( V(r, \rho) = 4\pi a_s(r) \hbar^2 \delta(\rho) / m \), that depends on both the relative coordinate and the center of mass (COM) coordinate \( r \). This direct substitution of the scattering length locally requires that the scattering length to vary only in a scale much larger than the local scattering length. In particular, we consider the simplest form of such an interaction,

\[
V(r, \rho) = [g_0 + g \cos (k_0 \cdot r)] \delta(\rho)
\]

The constant \( g_0 \) term, with \( g_0 < g < 0 \), is added to guarantee that the system stays in the BCS side of the resonance. Such a system, which to the best of our knowledge have not been studied before, is only feasible with recent advances in atomic physics. A relatively obvious phenomenon that arises from such a spatially modulated interaction would be the
modulation of the atomic density. Atoms are prone to congregate in locations with maximum attractions. A much more interesting phenomenon resides in the pairing of atoms. In a typical BCS theory, the ground state wave function consists of Cooper pairs with their COM momenta equal to zero. The interaction we propose creates coherences between pairs with COM momenta differing by \( k_0 \) in order to lower the free energy of the system. The ground state thus includes Cooper pairs with zero, as well as \( \pm k_0, \pm 2k_0 \) COM momenta. However, we find that the probability of occupying higher COM momenta is low due to disadvantage in kinetic energies and the constant attractive interaction \( g_0 \). Our ground state is closely related to the FFLO state discussed in the literature. In the FFLO state, due to the mismatch of the chemical potential for spin up and spin down particles, particles can pair with non-zero COM momenta to lower the energy. Before proceeding with our analysis, we emphasize that although our ground state is similar to the FFLO one, the spatial symmetry is spontaneously broken in the FFLO case, while in our case the spatial symmetry is broken by an external field.

We work in a grand canonical ensemble and use a single channel Hamiltonian which is appropriate for the case of a broad Feshbach resonances,

\[
K = \int d\mathbf{r} \sum_\alpha \psi_\alpha^\dagger(\mathbf{r}) \left( -\frac{1}{2m} \nabla^2 - \mu \right) \psi_\alpha(\mathbf{r}) + \frac{1}{2} \int d\mathbf{r} \sum_{\alpha,\beta} \tilde{V}(\mathbf{r}) \psi_\alpha^\dagger(\mathbf{r}) \psi_\beta^\dagger(\mathbf{r}) \psi_\beta(\mathbf{r}) \psi_\alpha(\mathbf{r})
\]

where \( \mu \) is the chemical potential, and \( \psi_\alpha(\mathbf{r}), \psi_\alpha^\dagger(\mathbf{r}) \) are the fermionic annihilation and creation operators with spin index \( \alpha \) at position \( \mathbf{r} \). In the interaction part, the relative coordinate is already integrated out. The \( \tilde{V}(\mathbf{r}) \) is the standard renormalized interaction from our bare interaction. We solve this system in a mean field level at zero temperature using the Bogoliubov de Gennes (BdG) transformation.

\[
\begin{align*}
(\varepsilon_k - \mu) u_k &+ \tilde{U}_k u_{k-k'} + \Delta_k v_{k-k'} = E v_k \\
-(\varepsilon_k - \mu) v_k &- \tilde{U}_k v_{k-k'} + \Delta_k^* u_{k+k'} = E u_k \\
\frac{1}{V} v_{k-k'}^{(n)} v_{k+k'}^{(n)} &+ \tilde{g}_{k''} = \bar{n}_k \\
\frac{1}{V} v_{k-k'}^{(n)} v_{k+k'}^{(n)} &+ \bar{g}_{k''} = \Delta_k \\
\frac{2}{V} v_{k}^{(n)} v_{k}^{(n)} &= \bar{n}
\end{align*}
\]
A summation convention is used for any momentum that appears twice on the left. We take the gap as $\Delta_{\beta\alpha}(\mathbf{r}) = -g(\mathbf{r}) \langle \hat{\psi}_\beta(\mathbf{r}) \hat{\psi}_\alpha(\mathbf{r}) \rangle$ and Hatree potential as $U_{\alpha\alpha}(\mathbf{r}) = g(\mathbf{r}) \langle \hat{\psi}_\alpha^+(\mathbf{r}) \hat{\psi}_\alpha(\mathbf{r}) \rangle$. The spin index can be neglected with no confusion due to the symmetry between spin up and down. In the following we take $k_0$ along $z$ direction. With this choice, $\Delta(\mathbf{r})$, $U(\mathbf{r})$ are functions of $z$ only. In momentum space, we can solve the single particle problem in a set of subspaces corresponding to constant $k_x$ and $k_y$. The Fourier transforms of $u(\mathbf{r}), v(\mathbf{r}), \tilde{V}(\mathbf{r}), \Delta(\mathbf{r})$ and $U(\mathbf{r})$ are defined as $u_k, v_k, g_k, \Delta_k$ and $U_k$ respectively. We can take $u(\mathbf{r}), v(\mathbf{r})$ and $\Delta_k$ to be real without loss of generality. Since we are dealing with a problem invariant under reflection $\mathbf{r} \to -\mathbf{r}$, we can also take $a_k, b_k, \Delta_k$ and $U_k$ to be real. $\epsilon_k = \hbar^2 k^2 / 2m$ is the kinetic energy of a particle with wave vector $k$, and $V$ is the quantization volume. We note that near a Feshbach resonance the chemical potential, $\mu$, need to be self-consistently determined by an average density of the Fermi gas $\bar{n}$ in equation (7).

We solve the BdG equations self-consistently by numerical iteration. The self-consistent gap is found to have only three components in momentum space, namely, $k_z = 0, \pm k_0$. Due to the form of interaction we chose, the zero momentum component gap is always larger than the component of momenta $\pm k_0$ and therefore the gap is nonzero everywhere in real space. This is different from the pairing gap discussed in the FFLO papers, where the gap has only $k_0$ or $\pm k_0$ components. The pairing of the many body ground state is given by $\langle a_{k_0}^+ a_{-k_0} \rangle$. Note that in the $k_x - k_y$ plane, the pairing occurs with opposite momenta. In the $z$ direction, however, there are non zero COM $q$ values. We study the pairing of atoms with their $k_x = k_y = 0$ without loss of generality. It can be shown that $\langle a_{k_z}^+ a_{q-k_z} \rangle = \sum u_{k_z}^{(n)} v_{k_z-q}^{(n)}$ and the numerical result is presented in figure (1a).

We find that only states with COM momenta $q = 0, \pm k_0$ are occupied. The higher COM momentum states with integer multiples of $k_0$ have negligible weight. Given the magnitude of the coupling between different COM states components, it is surprising that atom pairs do not occupy states with COM momenta being higher harmonics of $k_0$. Actually this result can be explained by the nonlinear property of the BdG equations. Atom pairs tend to occupy as few COM states as possible because the interaction energy is proportional to the square of $\langle a_{k_z}^+ a_{q-k_z} \rangle$. By occupying only these three COM states, the pairing energy is maximized in magnitude. The pairing occurs mostly for atoms near the Fermi surface. For a particular $k_z$, the pairing amplitudes for the COM $q = \pm k_0$ are generally not equivalent,
FIG. 1: Pairing amplitudes $a_{k_z^+}a_{-k_z^+}$ for $k_x = k_y = 0$. Plot (a) corresponds to the ground state, (b) to the first excited state, (c) to states right before the second gap, and (d) to the state right after the second gap. Only momenta close to the fermi surface are shown. The arrows are used as guide to the eye to pair amplitudes where significant changes taken place. We have used a units with fermi momentum and fermi energy to be one. The numerical values for the parameters are: $g_0 = 20$, $g = 15$, and $k_0 = 0.2$. The self-consistant gaps and chemical potential are found to be, $\Delta_0 = 0.13$, $\Delta_{\pm k_0} = 0.06$ and $\mu = 0.85$.

i.e. $\langle a_{k_z^+}a_{-k_0-k_z^+}\rangle \neq \langle a_{k_z^+}a_{-k_0-k_z^+}\rangle$. This is because for $k_z \neq 0$, either $k_0 - k_z$ or $-k_0 - k_z$ is closer to the Fermi surface. Since pairs can occupy several COM states coherently, one particular atom, say $a_{k_z^+}^\dagger$, can form pairs simultaneously with several particles, say $a_{-k_z^+}$, $a_{k_0-k_z^+}$, and $a_{-k_0-k_z^+}$. Right at the Fermi surface, pairings with opposite momenta dominate,
and therefore the amplitudes \( \langle a_k^{\uparrow} a_{-k}^{\downarrow} \rangle \) and \( \langle a_k^{\uparrow} a_{-k_0}^{\downarrow} \rangle \) are small. Slightly away from the Fermi surface, the pairing with opposite COM momenta is not as strong and one starts to see pairings with nonzero COM. This feature of the ground state pairing is closely related to the single particle excitation spectrum discussed below.

The eigenenergies that we found by solving (3)(4) are the single particle excitation spectrum of the many body state. The excitation is always gapped as opposed to the state studied by FFLO [2][3]. The quantity \( \Delta(r) \) is positive everywhere in real space with its minimum at \( \Delta_0 - \Delta_{k_0} \). This minimum is the lower bound of the excitation gap. Above the gap, the excitation spectrum is continuous due to the continuous distribution of \( k_x \) and \( k_y \).

For a particular set of \( k_x \) and \( k_y \), for example \( k_x = k_y = 0 \), additional gaps emerge due to the periodic modulation of the interaction along \( z \) direction [see Fig(2)].

The modulation therefore induces band gap structures in quasi-particle excitations. More interestingly, we can understand the excitation spectrum in a pair breaking picture. The pairing amplitudes in an excited state \( |n\rangle \) are, \( \langle n | a_{k_q}^{\uparrow} a_{-k_q}^{\downarrow} | n\rangle = \langle 0 | a_{k_q}^{\uparrow} a_{-k_q}^{\downarrow} | 0\rangle - u_k^{(n)} v_{k-q}^{(n)} \). The pairing amplitudes for relevant states are shown in Fig (1b, 1c, 1d). The pair breaking happens simultaneously around both \( -k_F \) and \( k_F \). The energy lowered by this superposition is a small effect, and in the following we focus on one side of the Fermi surface. For the lowest excited state \( |1\rangle \), the change in the pairing amplitudes occurs right at the Fermi surface. The amplitude \( a_{k_F}^{\uparrow} a_{-k_F}^{\downarrow} \) is almost completely destroyed in state \( |1\rangle \). One the other hand, recall that in the ground state, there are no pairing amplitudes for \( a_{k_F}^{\uparrow} a_{-k_F}^{\downarrow} \) and \( a_{k_F}^{\uparrow} a_{-k_0-k_F}^{\downarrow} \), due to the dominant pairing of \( a_{k_F}^{\uparrow} a_{-k_F}^{\downarrow} \). Now that the \( a_{k_F}^{\uparrow} a_{-k_F}^{\downarrow} \) pair is broken, the particle \( a_{k_F}^{\uparrow} \) is free to pair with particles \( a_{\pm k_0 - k_F}^{\downarrow} \) and this decrease the energy by a amount of \( \Delta_{k_0} \). Since the excitation is close to the Fermi surface, the kinetic energy contribution can be neglected. Furthermore, the Hartree term is almost the same for the ground and excited states and it can be neglected too. Therefore the emergence of gaps can be attribute to the pairing energy only, namely \( \sum_k \left[ 2\Delta_0 u_k^{(n)} v_k^{(n)} + 2\Delta_{k_0} u_{\pm k_0 + k} v_k \right] \). The gap energy can then be estimated to be 0.09, close to the observed value 0.10. The excitation energy increases continuously as the excitation gets further and further away from the fermi surface. The states with energies right below and above the second gap have sudden flips of the signs for the pairing amplitude \( a_{k'}^{\uparrow} a_{\pm k_0 - k'}^{\downarrow} \). Here the \( k' \) are the particular wavevectors where pairing amplitudes \( a_{k'}^{\uparrow} a_{\pm k_0 - k'}^{\downarrow} \) undergo significant changes with respect to the ground state. We can repeat the analysis above and estimate the energy cost for the sign flipping.
FIG. 2: Plot of the single particle excitation energy. Here we only look at the eigen–excitations with $k_x = k_y = 0$. The $n$ labels the eigenstates. We have used same numerical values for relevent parameters as figure(1).

to be 0.09, close to the observed value 0.11. Therefore, the phase flips of the pairs with nonzero COM momenta give rise to the second gap. Similar observations can be made to locate gaps at higher energies. However, as the kinetic energy increases, the gap structure gets more and more obscure. Actually, with parameters used in the calculation, only two gaps are observed. We would like to emphasis that, after summation over the contribution from different $k_x$ and $k_y$, the additional gap we discussed above, is not a true second gap in the excitation spectrum. However, it nevertheless induces a sharp change of density of states for energies close to the gap. Such effects, arising from the rich structures of the pair...
wavefunction, can be observed in experiments.

The parameters we use in the calculation are based on recent $^6$Li experiments \cite{7,5}. We take the fermi energy $E_F = (3\pi^2n)^{1/3} \sim 3\mu k$ and momentum $k_F$ to be the unit energy and wave vector, respectively. In this units, the background scattering length $a_{bg} \sim -0.5$ and the modulation of the interaction \cite{11} $k_0 \sim 20$. Here we estimate $k_0$ to be the order of laser wavevector. It is smaller than $k_F$ and this choice automatically satisfies the condition $k_0 \ll |2\pi/a|$, necessary for the validity of equation \cite{11}. The constant coupling strength is about $g_0 \sim -20$. The coupling between different COM $g$ is chosen to be 15, smaller in magnitude than $g_0$. This coupling can be realized in experiment both by external magnetic and optical fields. However, it is not easy to generate spatial variations of the magnetic field in a scale as small as the sample size. Optical Feshbach resonance, as demonstrated in the experiment \cite{12}, is a more promising way to generate the spatial modulation of the interaction. By coupling the incident channel of two atom scattering to the molecular excited state close channel \cite{11}, one can induce the periodical spatial modulation by an optical lattice. The laser field frequency needs to be chosen to be far away from the resonance of a single atom to minimize the induced single particle effects.

The observation of this novel pairing in the ground state involves measurements of the COM momenta of pairs. This can be done by pair-wise projecting Cooper pairs into molecules by a fast sweep from BCS side of the resonance to the BEC side. After this procedure, the molecules of the condensate should coherently occupy momentum states $\mathbf{q} = 0$, $\mathbf{q} = \pm k_0$, which can be measured by a time of flight image of molecules. Such measurements have already been used in several BEC-BCS cross over experiments \cite{5}. Another way of observing the pairing correlation, recently demonstrated experimentally \cite{15}, is to measure the shot noise correlations. For the excited states properties, one needs to measure the radio frequency (RF) spectrum. RF spectra have been proposed \cite{13} and recently used to measure the fermi gas pairing gap \cite{6}. In experiments, two of the hyperfine levels of atoms, $|1\rangle$, $|2\rangle$ are identified as the spin up and spin down states. A probing RF field couples one of the hyperfine levels, say $|2\rangle$, to a third hyperfine level $|3\rangle$. The transition rate can be evaluated directly from our BdG solution of the quasiparticle excitations. However, such calculation turns out to be numerically challenging. This requires to discretize the momentum space in a very small grid. Here, we adopt a Local Density Approximation (LDA) approach. The calculation should be valid for $k_0 \ll k_F$, which is exactly the situation we are interested
in. In LDA, the RF spectrum is calculated locally and the final result is a summation of local contributions \[13\], \( R(\omega) = \frac{\pi}{h} \Omega^2 \int d^3r D(\varepsilon_k) \frac{\Delta^2(r)}{\hbar^2 \omega^2} \Theta[\varepsilon_k(\omega)] . \) Note that the local gap and chemical potential are given by our self-consistent BdG solutions. Here \( \omega \) is the detuning of the laser frequency from the frequency difference in \(|2\rangle\) and \(|3\rangle\), and \( \Omega \) is the coupling between them. \( D(\varepsilon_k) \) is the free particle density of states and it is proportional to \( \sqrt{\varepsilon_k} \). Inside the step function, the \( \varepsilon_k(\omega) = \omega^2 - \Delta(r)^2 + \mu(r) \) should be greater than zero, which gives the threshold of the excitation. The chemical potential of state \(|3\rangle\) is taken to be zero and the RF spectrum is shown in figure (3).

We note two features of the spectrum. First, the threshold is shifted to a lower energy comparing with the uniform case. This corresponds to a local gap \( \Delta(r) = \Delta_0 - \Delta_{k_0} \), which is consistent with the observation that the lowest excitation states correspond to break a COM zero pair and increase the non-zero COM pairing. Second, there is an extra peak at the higher energy side of the spectrum, which corresponds to a local gap \( \Delta(r) = \Delta_0 + \Delta_{k_0} \). This is consistent with our analysis that the second gap originates from sigh flips of the non-zero COM pairs and the magnitude is proportional to the \( \Delta_{k_0} \) only.

In conclusion, we studied a fermi gas with spatially modulated interaction in a mean field level at zero temperature. We also discussed its experimental realization and detection. Such a state has a periodical modulation of the order parameter similar to the FFLO states. Even though we considered the spatially varying interaction along only one direction, our analysis can be easily generalized to a system where the interaction is modulated in three directions. In this case the second gaps along three directions can overlap and produce true additional gaps of the quasiparticle excitation. This should produce more pronounce signals in the RF spectrum.

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FIG. 3: (color online) The RF spectrum is plotted for a homogenously interacting system and our spatially modulated interacting system. Notice that there is an extra peak in the RF signal for the system with a spatially modulated interaction. We have again used the numerical values as Figure[1] for various parameters.

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