LINEAR AND SUBLINEAR CONVERGENCE RATES FOR
A SUBDIFFERENTIABLE DISTRIBUTED DETERMINISTIC
ASYNCHRONOUS DYKSTRA’S ALGORITHM

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Abstract. In [Pan18a, Pan18b], we designed a distributed deterministic asynchronous algorithm for minimizing the sum of subdifferentiable and proximable functions and a regularizing quadratic on time-varying graphs based on Dykstra’s algorithm, or block coordinate dual ascent. Each node in the distributed optimization problem is the sum of a known regularizing quadratic and a function to be minimized. In this paper, we prove sublinear convergence rates for the general algorithm, and a linear rate of convergence if the function on each node is smooth with Lipschitz gradient. Our numerical experiments also verify these rates.

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1. INTRODUCTION

Let $V$ and $\bar{E}$ be finite sets. Define the set $\mathbf{X} := X_1 \times \cdots \times X_{|V|}$, where each $X_i$ is a finite dimensional Hilbert space. For each $i \in V$, let $f_i : X_i \to \mathbb{R} \cup \{\infty\}$ be a closed convex function, and let $f_i : \mathbf{X} \to \mathbb{R} \cup \{\infty\}$ be defined by $f_i(x) = f_i([x]_i)$. Let $\delta_C(\cdot)$ be the indicator function for a closed convex set $C$. For each $\alpha \in \bar{E}$, let $H_\alpha \subset \mathbf{X}$ be a linear subspace, and define $f_\alpha : \mathbf{X} \to \mathbb{R}$ by $f_\alpha(x) = \delta_{H_\alpha}(x)$. The

Date: August 23, 2018.
2010 Mathematics Subject Classification. 68W15, 90C25, 90C30, 65K05.
Key words and phrases. Distributed optimization, subdifferentiable functions, Dykstra’s algorithm, time-varying graphs.
C.H.J. Pang acknowledges grant R-146-000-214-112 from the Faculty of Science, National University of Singapore.
The (Fenchel) dual of (1.1) can be found to be

\[ \min_{\bar{x} \in \mathbb{X}} \frac{1}{2} \| x - \bar{x} \|^2 + \sum_{i \in V} f_i(x) + \sum_{\alpha \in E} \sigma H_{\alpha}(x). \] (1.1)

Note that the last two sums in (1.1) can be written as \( \sum_{\alpha \in V \cup E} f_\alpha(x) \). Typically, the hyperplanes \( \{ H_{\alpha} \}_{\alpha \in E} \) are overdetermined (see Definition 2.1 later). Partition the set \( V \) as the disjoint union \( V = V_1 \cup V_2 \cup V_3 \cup V_4 \) so that

- \( f_i(\cdot) \) are proximable functions for all \( i \in V_1 \).
- \( f_i(\cdot) \) are indicator functions of closed convex sets for all \( i \in V_2 \).
- \( f_i(\cdot) \) are proximable functions such that \( \text{dom}(f_i) = \mathbb{X}_i \) for all \( i \in V_3 \).
- \( f_i(\cdot) \) are subdifferentiable functions (i.e., a subgradient is easy to obtain) such that \( \text{dom}(f_i) = \mathbb{X}_i \) for all \( i \in V_4 \). (In Sections 3 and 4 we shall assume \( f_i(\cdot) \) are smooth with \( \nabla f_i(\cdot) \) is Lipschitz with modulus \( \frac{1}{\sigma} \) for all \( i \in V_4 \).)

The (Fenchel) dual of (1.1) can be found to be

\[ \max_{z_\alpha \in \mathbb{X} : \alpha \in V \cup E} F(\{ z_\alpha \}_{\alpha \in E \cup V}), \] (1.2)

where

\[ F(\{ z_\alpha \}_{\alpha \in E \cup V}) := -\frac{1}{2} \| \bar{x} - \sum_{\alpha \in E \cup V} z_\alpha \|^2 + \frac{1}{2} \| \bar{x} \|^2 - \sum_{\alpha \in E \cup V} f'_\alpha(z_\alpha). \] (1.3)

We now explain that the problem (1.1) includes the general case of the distributed Dykstra’s algorithm in [Pan18a, Pan18b].

**Example 1.1.** [Pan18a, Pan18b] (Distributed Dykstra’s algorithm is a special case of (1.1)) Let \( G = (V, E) \) be an undirected connected graph. Suppose each \( \mathbb{X}_i = \mathbb{R}^m \) for all \( i \in V \), and let \( E := E \times \{1, \ldots, m\} \). For each \( x \in \mathbb{X} = [\mathbb{R}^m]^{|V|} \), we let \( [x]_i \in \mathbb{R}^m \) be the \( i \)-th component, and we let \([x]_{i,k}\) be the \( k \)-th component of \([x]_i\). For each \((i,j), k \in E \), let the linear subspace \( H_{((i,j),k)} \subset \mathbb{X} \) of codimension 1 be defined to be

\[ H_{((i,j),k)} := \{ x \in \mathbb{X} : [x]_i |_k = [x]_j |_k \}. \] (1.4)

Then the problem (1.1) is equivalent to

\[ \min_{x \in \mathbb{R}^m} \sum_{i \in V} [\frac{1}{2} \| x - [x]_i \|^2 + f_i(x)]. \] (1.5)

We elaborate on distributed optimization algorithms and the features of the distributed Dykstra’s algorithm in Example 1.1 in the following subsections.

1.1. **Distributed optimization.** Since this paper builds on [Pan18a, Pan18b], we shall give a brief introduction. Our algorithm is for the case when the edges are undirected. But we remark on the directed case. A notable paper based on the directed case using the subgradient algorithm is [SLWY15], and surveys are [Ned15] and [Ned17]. The papers [NO15] and [NOS17] further touch on the case of time-varying graphs. The algorithms in [BCS17, VHDG11] address the averaged consensus problem for the case of directed graphs with unreliable and reliable communications respectively. Based on [BCS17], [BCN+17] uses a Newton-Raphson method to design a distributed algorithm for directed graphs. Naturally, the communication requirements for directed graphs need to be more stringent that the requirements for undirected graphs.
From here on, we discuss only algorithms for undirected graphs. A product space formulation on the ADMM leads to a distributed algorithm \cite[Chapter 7]{BPC+10}. Such an algorithm is decentralized and distributed, but is not asynchronous and so can get slowed down by slow vertices. An approach based on \cite{CE18} allows for asynchronous operation, but is not decentralized.

Moving beyond deterministic algorithms, distributed decentralized asynchronous algorithms were proposed, but many of them involve some sort of randomization. For example, the work \cite{IBCH13, BHI14, WO13} and the generalization \cite{PXYY16} are based on monotone operator theory (see for example the textbook \cite{BC11}), and require the computations in the nodes to follow specific probability distributions.

We now look at asynchronous distributed algorithm with deterministic convergence (rather than probabilistic convergence). We mention that incremental aggregated gradient algorithm like \cite{GOP17, AFJ16} is an algorithm for strongly convex problems that is primal in nature, so it can’t have more than one proximal term, and hence can’t handle more than one constraint set. (We consider such algorithms distinct from what we do in this paper as they do not need the function on each node to be strongly convex, but the algorithms need a central node.) The method in \cite{AH16} is distributed and deterministic and the averaging operation can be performed asynchronously, but some parts of the algorithm are still required to be synchronous.

### 1.2. Distributed Dykstra’s algorithm.

We now recall some history of Dykstra’s algorithm \cite{Dyk83}. Dykstra’s algorithm originally solves

\[
\min_{x \in X} \frac{1}{2} \|x - \bar{x}\|^2 + \sum_{i \in V} \delta_{C_i}(x)
\]  

for closed convex sets $C_i$. It was also recognized to be block coordinate minimization of the dual problem (similar to \eqref{eq:1.2}) in \cite{Han88}, where convergence is proved under a constraint qualification ensuring the existence of a dual minimizer. The convergence of Dykstra’s algorithm to the primal minimizer even when a dual minimizer does not exist is sometimes known as the Boyle-Dykstra theorem \cite{BD85}. This proof is written through duality in \cite{GM89}. Other interesting properties of Dykstra’s algorithm related to this paper are that Dykstra’s algorithm converges even when the functions are sampled in a non-cyclic order \cite{HD97}.

Based on these previous works, we made a few extensions of Dykstra’s algorithm in \cite{Pan18a} and further extended it in \cite{Pan18b}. We call our algorithm the distributed Dykstra’s algorithm. Its favorable properties are:

1. distributed (with communications occurring only between adjacent agents $i$ and $j$ connected by an edge).
2. decentralized (i.e., there is no central node coordinating calculations).
3. asynchronous (contrast this to synchronous algorithms, where the faster agents would wait for slower agents before their next calculations).
4. deterministic (i.e., not using any probabilistic methods, like stochastic gradient methods).
5. able to incorporate more than one proximable function naturally. (This largely rules out primal-only methods since they usually allow just one proximal term.) Hence, the algorithm would be able to allow for constrained optimization, where the feasible region is the intersection of several sets.
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(6) able to allow for time-varying graphs in the sense of [NO15, NOS17] (to be robust against failures of communication between two agents).
(7) able to use simpler subproblems for subdifferentiable functions.
(8) able to use simpler subproblems for smooth functions.
(9) able to allow for partial communication of data.

Since Dykstra’s algorithm is also dual block coordinate ascent, the following property is obtained:
(10) choosing a large number of dual variables to be maximized over gives a greedier increase of the dual objective value.

We are not aware of other algorithms that satisfy properties 1-5 at the same time. We note that the approach in [AFJ16] is essentially a primal algorithm that allows for one proximal term (and hence one constrained set). Due to technical difficulties (see Remark 4.3), a dual or primal-dual method seems necessary to handle the case of more than one constrained set. Algorithms derived from the primal dual algorithm [CP11, CP16], like [AH16], are very much different from what we study in this paper. The most notable difference is that they study ergodic convergence rates, which is not directly comparable with our results.

1.2.1. Convergence rates. Since the subproblems in our case are strongly convex, standard techniques for block coordinate minimization, like [BT13, Bec15], can be used to prove the $O(1/k)$ convergence rate when a dual solution exists and all functions are treated as proximable functions.

We also showed in [Pan18b] that if $|V| = 1$, $|E| = 0$ and the function $f_1(\cdot)$ is subdifferentiable, then a $O(1/k)$ rate of convergence of the dual objective value can be proved with a method somewhat similar to the bundle method, and it improves to linear convergence if $f_1(\cdot)$ is smooth. (See Lemma 2.9 for more details.) The following question remains:

What is the convergence rate for the subdifferentiable distributed Dykstra’s algorithm when $|V| > 1$ and $|E| > 0$? (1.7)

A particular case of the original Dykstra’s algorithm having a linear convergence rate is when the sets $C_i$ in (1.6) are linear subspaces. It is well known that in such a case, Dykstra’s algorithm reduces to the method of alternating projections. The case of alternating projections for the case of linear subspaces is rather old, so we refer to the references [BB96, Deu01b, Deu01a, ER11] for example.

1.3. Contributions of this paper. Given that the distributed Dykstra’s algorithm has some desirable properties as listed in Subsection 1.2, we wish to find out how its convergence rates compare to other well-known distributed optimization algorithms.

In Section 3, we prove the linear convergence (of the dual objective function (1.3)) of this method when $V_1 = V_2 = V_3 = \emptyset$ and the functions $f_i(\cdot)$ are smooth for all $i \in V_4$ (instead of just being subdifferentiable).

In Section 4 we prove that in the case when $f_i(\cdot)$ is smooth for all $i \in V_4$ (instead of just being subdifferentiable), a dual minimizer exists, the dual iterates are bounded, and $V_1$, $V_2$ and $V_3$ are not necessarily empty, the convergence rate is $O(1/k)$. This convergence rate is the best we can expect with the distributed Dykstra’s algorithm because block coordinate minimization has a convergence rate of at best $O(1/k)$ [BT13, Bec15].
In Section 5, we establish a $O(1/k^{1/3})$ convergence rate for the distributed Dykstra’s algorithm in the general case when a dual minimizer exists and the dual iterates are bounded, addressing the question in [17]. When there are no subdifferentiable functions, the common rate of $O(1/k)$ is obtained. While the $O(1/k^{1/3})$ rate is slower than the subgradient algorithm, our experimental results suggest a $O(1/k)$ rate for our set of problems. And as mentioned, we are not aware of any other distributed optimization algorithms with properties (1) to (5). Our algorithm is also not easily comparable to the subgradient algorithm because the subgradient algorithm does not include problems whose domain is the intersection of more than one convex set (see the issues for such problems in Remark 4.3). We hope our work can lead to subsequent research for distributed problems.

1.4. Notation. The functions $f_\alpha(\cdot)$ are indexed by $\alpha \in V \cup \bar{E}$, and sometimes $\beta \in V \cup \bar{E}$. We use $f_i(\cdot)$ when we want to index with $i \in V$, and we sometimes use $j \in V$. We usually reserve bold variables like $x, z_\alpha$, and $s_\alpha$ to be variables in $X$, but we also use $z, s$ for vectors in $X^{V \cup \bar{E}}$. We use $[x]_i \in X_i$ in order to index the $i$-th component of $x \in X$. We usually use $x, s$ to represent vectors in $X_i$.

2. ALGORITHM STATEMENT AND PRELIMINARIES

In this section, we list down the preliminaries and description of the distributed Dykstra’s algorithm studied in [Pan18a, Pan18b]. We do not claim originality in this section, and we recall some results useful for subsequent proofs.

For all $n \geq 1$ and $w \in \{1, \ldots, \hat{w}\}$, define $f_{\alpha,n,w} : X \to \mathbb{R}$ by

$$f_{\alpha,n,w}(\cdot) = f_\alpha(\cdot) \text{ for all } \alpha \in [\bar{E} \cup V] \setminus V_4$$

and

$$f_{\alpha,n,w}(\cdot) \leq f_\alpha(\cdot) \text{ for all } \alpha \in V_4.$$  \hfill (2.1b)

Define the function $F^{n,w} : X^{V \cup \bar{E}} \to \mathbb{R} \cup \{\infty\}$ to be

$$F^{n,w}(\{z_\alpha\}_{\alpha \in \bar{E} \cup V}) := -\frac{1}{2} \left\| \bar{x} - \sum_{\alpha \in \bar{E} \cup V} z_\alpha \right\|^2 + \frac{1}{2} \left\| \bar{x} \right\|^2 - \sum_{\alpha \in \bar{E} \cup V} \bar{f}_{\alpha,n,w}^*(z_\alpha).$$  \hfill (2.2)

Based on our original motivation in Example 1.1 from [Pan18a, Pan18b], we make the following definition.

Definition 2.1. Let $D := \bigcap_{\alpha \in \bar{E}} H_\alpha$. We say that a subset $E' \subset \bar{E}$ connects $V$ if

$$\bigcap_{\alpha \in E'} H_\alpha = D.$$  \hfill (2.3)

Since $H_\alpha$ were assumed to be linear subspaces, it is clear that condition (2.3) on $E'$ is equivalent to

$$\sum_{\alpha \in E'} H_\alpha^\perp = D^\perp.$$  \hfill (2.4)

The following simple result does not play much of a role here. But it plays a role in [Pan18a, Pan18b] to show the convergence for time-varying graphs when a dual minimizer may not exist. This explains line 5 in Algorithm 2.3.

Lemma 2.2. There is a constant $C_{\text{reg}} > 0$ such that for any $v \in D^\perp$ and any $E' \subset \bar{E}$ such that $E'$ connects $V$, we can write $v = \sum_{\alpha \in E'} v_\alpha$ so that $v_\alpha \in H_\alpha^\perp$ and $\|v_\alpha\| \leq C_{\text{reg}}\|v\|$ for all $\alpha \in E'$.
Proof. The proof of a slightly weaker result was in [Pan18a] to accommodate the case when \( X \) is not necessarily \( \mathbb{R}^{|V|} \), but we include this proof for completeness. Since we have assumed finite dimensionality of all \( X_i \)'s, for all \( \alpha \in \bar{E} \), we can write \( H_\alpha^\perp \subset X \) as \( H_\alpha^\perp := \text{span}\{v_{\alpha,1}, \ldots, v_{\alpha,d_\alpha}\} \), where \( \{v_{\alpha,1}, \ldots, v_{\alpha,d_\alpha}\} \) is a set of \( d_\alpha \) linearly independent vectors and \( d_\alpha \) is the dimension of \( H_\alpha^\perp \). For all \( E' \subset \bar{E} \), we can find a subset of the \( S' := \cup_{\alpha \in E'} \cup_{d_\alpha=1}^{d_\alpha} \{v_{\alpha,i}\} \) so that \( S' \) is a set of linearly independent vectors that span \( D^\perp \). From basic linear algebra, we can find a constant \( C_{E'} \), such that for any \( v \in H_\perp \), we can write \( v \) as a linear combination of the vectors in \( S' \) so that each component has norm less than \( C_{E'} \|v\| \). The result follows readily. \( \square \)

To simplify calculations, we let the vectors \( v_A, v_H \) and \( x \) in \( X \) be denoted by

\[
v_H = \sum_{\alpha \in E} z_\alpha \quad (2.5a)
\]

\[
v_A = v_H + \sum_{i \in V} z_i \quad (2.5b)
\]

\[
x = \tilde{x} - v_A. \quad (2.5c)
\]

We now state Algorithm 2.3 on the following page. Algorithm 2.3 calls on Algorithm 2.4 on page 8 as a subalgorithm.

Remark 2.5. (Intuition behind Algorithms 2.3 and 2.4) We summarize the intuition behind Algorithms 2.3 and 2.4. Dykstra’s algorithm is block coordinate ascent on the dual (1.2), and this is reflected in lines 7-14 of Algorithm 2.3. That is, find \( z \in X^{\bar{E} \cup V} \) that tries to improve the objective value of (1.3). As explained in [Pan18a], one only needs to keep track of \( x_i \) and \( |z_i| \), for all \( i \in V \), and not all the variables. Line 5 corrects \( \{z_\alpha\}_{\alpha \in \bar{E}} \) so that the dual objective value remains the same, and this consideration is needed when we try to prove that the algorithm works for time-varying graphs. Lastly, to explain Algorithm 2.4, note that if \( f_i(\cdot) \) is subdifferentiable, then we can form affine minorants \( \tilde{f}_i(\cdot) \) of \( f_i(\cdot) \) so that \( \tilde{f}_i(\cdot) \leq f_i(\cdot) \) and \( \tilde{f}_i^*(\cdot) \geq f_i^*(\cdot) \). This process is similar to the bundle method. By iteratively updating \( \tilde{f}_i^*(\cdot) \) and maximizing

\[-\frac{1}{2}\|\tilde{x} - \sum_{\alpha \in \bar{E}_i \cup V} z_\alpha\|^2 + \frac{1}{2}\|\tilde{x}\|^2 - \sum_{\alpha \in \bar{E}_i} \delta_{H_\alpha}^z(z_\alpha) - \sum_{i \in V} \tilde{f}_i^*(z_i),\]

we can converge to the optimal dual objective value.

The following result is essential for showing that the distributed Dykstra’s algorithm is asynchronous, and will be useful in some proofs in this paper.

Proposition 2.6. (Sparsity of \( z_\alpha \)) We have \( z_i^{n,w} = 0 \) for all \( j \in V \setminus \{i\} \), \( n \geq 1 \) and \( w \in \{0, 1, \ldots, \bar{w}\} \). Furthermore, in the case of Example 1.1, where \( X_i = \mathbb{R}^m \) for all \( i \in V \) and \( H_{(e,k)} \) are defined as in (1.4) for all \( (e, k) \in E \times \{1, \ldots, m\} \), the vector \( z_i^{n,w} \in \mathbb{R}^{|V|} \) satisfies \( [z_i^{n,w}]_{j'} = 0 \) unless \( k = k' \) and \( i \) is an endpoint of \( e \).

Sketch of proof. The proof of this result is similar to the corresponding result in [Pan18a]. The claim for \( z_i^{n,w} \) relies on the fact that \( f_{i,n,w}(\cdot) \) depends only on the \( i \)-th component, and the claim for \( z_i^{n,w} \) relies on the fact that \( f_{i,k}(\cdot) = \delta_{H_{(e,k)}}(\cdot) \), with \( H_{(e,k)} \) containing vectors that are zero in all but 2 coordinates. \( \square \)
Algorithm 2.3. (Distributed Dykstra’s algorithm) Consider the problem (1.1) along with the associated dual problem (1.2).

Let \( w \) be a positive integer. Let \( C_{reg} > 0 \) satisfy Lemma 2.2. For each \( \alpha \in [E \cup V] \setminus V_4 \), \( n \geq 1 \) and \( w \in \{1, \ldots, \bar{w}\} \), let \( f_{\alpha,n,w} : X \to \mathbb{R} \) be as defined in (2.1).

Our distributed Dykstra’s algorithm is as follows:

01 Let

- \( z_i^{1,0} \in X \) be a starting dual vector for \( f_i(\cdot) \) for each \( i \in V \) so that \( [z_i^{1,0}]_j = 0 \in X_j \) for all \( j \in V \setminus \{i\} \).
- \( v_H^{1,0} \in D^\perp \) be a starting dual vector.
  - Note: \( \{z_{\alpha}^{n,0}\}_{\alpha \in \bar{E}} \) is defined through \( v_H^{n,0} \) in (2.6).
- Let \( x_i^{1,0} = \bar{x} - v_H^{1,0} - \sum_{i \in V} z_i^{1,0} \).

02 For each \( i \in V_4 \), let \( f_{i,1,0} : X \to \mathbb{R} \) be a function such that \( f_{i,1,0}(\cdot) \leq f_i(\cdot) \).

03 For \( n = 1, 2, \ldots \)

04 Let \( \bar{E}_n \subset \bar{E} \) be such that \( \bar{E}_n \) connects \( V \) in the sense of Definition 2.1.

05 Define \( \{z_{\alpha}^{n,0}\}_{\alpha \in \bar{E}} \) so that:

\[
\begin{align*}
  z_{\alpha}^{n,0} &= 0 \text{ for all } \alpha \notin \bar{E}_n \quad (2.6a) \\
  z_{\alpha}^{n,0} &\in H_{\alpha}^+ \text{ for all } \alpha \in \bar{E} \quad (2.6b) \\
  ||z_{\alpha}^{n,0}|| &\leq C_{reg} ||v_H^{n,0}|| \text{ for all } \alpha \in \bar{E} \quad (2.6c)
\end{align*}
\]

and

\[
\sum_{\alpha \in \bar{E}} z_{\alpha}^{n,0} = v_H^{n,0}. \quad (2.6d)
\]

(This is possible by Lemma 2.2)

06 For \( w = 1, 2, \ldots, \bar{w} \)

07 Choose a set \( S_{n,w} \subset \bar{E}_n \cup V \) such that \( S_{n,w} \neq \emptyset \).

08 If \( S_{n,w} \subset V_4 \), then

09 Apply Algorithm 2.4.

10 else

11 Set \( f_{i,n,w}(\cdot) := f_{i,n,w-1}(\cdot) \) for all \( i \in V_4 \).

12 Define \( \{z_{\alpha}^{n,w}\}_{\alpha \in S_{n,w}} \) by

\[
\{z_{\alpha}^{n,w}\}_{\alpha \in S_{n,w}} = \arg \min \left\{ \frac{1}{2} ||\bar{x} - \sum_{\alpha \notin S_{n,w}} z_{\alpha}^{n,w-1} - \sum_{\alpha \in S_{n,w}} z_{\alpha} ||^2 + \sum_{\alpha \in S_{n,w}} f_{\alpha,n,w}(z_{\alpha}) \right\}.
\]

13 end if

14 Set \( z_{\alpha}^{n,w} := z_{\alpha}^{n,w-1} \) for all \( \alpha \notin S_{n,w} \).

15 End For

16 Let \( z_{i}^{n+1,0} = z_{i}^{n,\bar{w}} \) for all \( i \in V \) and \( v_H^{n+1,0} = v_H^{n,\bar{w}} = \sum_{\alpha \in \bar{E}} z_{\alpha}^{n,\bar{w}} \).

17 Let \( f_{i,n+1,0}(\cdot) = f_{i,n,\bar{w}}(\cdot) \) for all \( i \in V_4 \).

18 End For

We will come back to the setting of Example 1.1 to prove specific results. Another fact we will use later is that under the setting in Example 1.1 the set \( D \) defined
Algorithm 2.4. (Subalgorithm for subdifferentiable functions) This algorithm is run when line 9 of Algorithm 2.3 is reached. Note that to get to this subalgorithm, $S_{n,w} \subset V_4$. Suppose Assumption 2.7 holds.

01 For each $i \in S_{n,w}$
02 For $f_{i,n,w-1} : X_i \to \mathbb{R}$ defined by
03 Let the primal and dual solutions of (2.9) be $x_i^+$ and $z_i^+$.  
04 Define $f_{i,n,w} : X_i \to \mathbb{R}$ to be the affine function
05 In other words, $f_{i,n,w}(\cdot)$ is chosen such that the primal and dual optimizers to (2.9) coincide with that of
06 Define the function $f_{i,n,w} : X \to \mathbb{R}$ and the dual vector $z_{i,n,w} \in X$ to be
07 End for
08 For all $i \in V_4 \setminus S_{n,w}$, $f_{i,n,w}(\cdot) = f_{i,n,w-1}(\cdot)$.

through (2.3) has the simplifications

$$ D = \{x \in [\mathbb{R}^m]^{\mid V \mid} : x = (x, x, \ldots, x) \text{ for some } x \in \mathbb{R}^m \} \quad (2.13a) $$

$$ D^\perp = \{x \in [\mathbb{R}^m]^{\mid V \mid} : \sum_{i \in V} [x]_i = 0 \}. \quad (2.13b) $$

We state some notation necessary for further discussions. For any $\alpha \in \overline{E} \cup V$ and $n \in \{1, 2, \ldots \}$, let $p(n, \alpha)$ be

$$ p(n, \alpha) := \max \{w' : w' \leq \bar{w}, \alpha \in S_{n,w'} \}. \quad (2.14) $$

In other words, $p(n, \alpha)$ is the index $w'$ such that $\alpha \in S_{n,w'}$ but $\alpha \notin S_{n,k}$ for all $k \in \{w'+1, \ldots, \bar{w}\}$. We make three assumptions listed below.

Assumption 2.7. (Start of Algorithm 2.4) Recall that at the start of Algorithm 2.4 $S_{n,w} \subset V_4$. We make three assumptions.

1. Whenever $(n, w)$ is such that $w > 1$ and $S_{n,w} \subset V_4$ so that Algorithm 2.4 is invoked, each $z_{i,n,w-1} \in X$, where $i \in V_4$, is such that $z_{i,n,w-1}$ is the optimizer to the problem

$$ \min_{z \in X_i} \frac{1}{2} \|\bar{x} - v_{H,i}^{n,w-1} - z\|^2 + f_{i,n,w-1}(z). \quad (2.15) $$
In other words, suppose \( w_i \geq 1 \) is the largest \( w' \) such that \( i \in S_{n,w'} \) and \( i \notin S_{n,w} \) for all \( \tilde{w} \in \{ w' + 1, w' + 2, \ldots, w - 1 \} \). Then for all \( \tilde{w} \in \{ w_i + 1, \ldots, w - 1 \} \), and \( \alpha \in S_{n,\tilde{w}} \), the condition \( v \in H_\alpha \) implies \( \|v\| = 0 \).

(2) Suppose that for all \( i \in V_4 \), \( \tilde{w} \in \{ p(n,i) + 1, \ldots, \tilde{w} \} \) and \( \alpha \in S_{n,\tilde{w}} \), the condition \( v \in H_\alpha \) implies \( \|v\| = 0 \). (This implies \( x_{i,n,p(n,i)} = x_{i,n,\tilde{w}} \).)

(3) Suppose that \( S_{n,1} = V_4 \) for all \( n > 1 \).

With these assumptions, we are able to prove the following. Even though the proof in [Pan18b] for the analogue of Theorem 2.8 below was for the case of Example 1.1, the proofs can be carried over in a straightforward manner.

**Theorem 2.8.** [Pan18b] (Convergence to primal minimizer) Consider Algorithm 2.3. Assume that the problem (1.1) is feasible, and for all \( n \geq 1 \), \( \mathcal{E}_n = [\bigcup_{w=1}^n S_{n,w}] \cap E \), and \( [\mathcal{e}_{n,w}^t S_{n,w}] \supset V \). Suppose also that Assumption 2.7 holds.

For the sequence \( \{z_{n,w}^t\}_{t \in \mathbb{N}} \subseteq X \) for each \( \alpha \in \mathcal{E} \cup V \), generated by Algorithm 2.3 and the sequences \( \{v_{n,w}^t\}_{t \in \mathbb{N}} \subseteq X \) and \( \{v_{n,w}^t\}_{t \in \mathbb{N}} \subseteq X \) thus derived, we have:

(i) For all \( n \geq 1 \) and \( w_1, w_2 \in \{1, \ldots, \tilde{w}\} \) such that \( w_1 \leq w_2 \),

\[
F_{n,w_2}(z_{n,w_1}) \geq F_{n,w_1}(z_{n,w_1}) + \frac{1}{2} \sum_{w' = w_1 + 1}^{w_2} \|v_{A,w_2} - v_{A,w_1}\|^2.
\]

Hence the sum \( \sum_{t=1}^{\infty} \sum_{w=1}^{\tilde{w}} \|v_{A,w} - v_{A,w-1}\|^2 \) is finite and \( \{F_{n,\tilde{w}}(\{z_{n,\tilde{w}}^t\}_{t \in \mathcal{E}_n})\}_{n=1}^\infty \) is nondecreasing.

(ii) There is a constant \( C \) such that \( \|v_{A,w}^t\|^2 \leq C \) for all \( n \in \mathbb{N} \) and \( w \in \{1, \ldots, \tilde{w}\} \).

(iii) For all \( i \in V_3 \cup V_4 \), \( n \geq 1 \) and \( w \in \{1, \ldots, \tilde{w}\} \), the vectors \( z_{i,n,w} \) are bounded.

We now list down a result that will be useful for showing the decrease of the dual objective value in terms of \( f_i((x_{\tilde{w}}^n)_i) - f_{i,n,w}((x_{\tilde{w}}^n)_i) \).

**Lemma 2.9.** [Pan18b] Suppose \( f : X \to \mathbb{R} \) is a closed convex subdifferentiable function such that \( \text{dom}(f) = X \). Consider the problem

\[
\min_x f(x) + \frac{1}{2} \|x - \bar{x}\|^2,
\]

which has (Fenchel) dual

\[
\max_x -f^*(z) + \frac{1}{2} \|\bar{x}\| - \frac{1}{2} \|z - \bar{x}\|^2.
\]

Strong duality is satisfied for this primal dual pair. Let the common objective value be \( v^* \). Let \( f_1 : X \to \mathbb{R} \) be an affine function \( f_1(x) := a_1^T x + b_1 \) such that \( f_1(\cdot) \leq f(\cdot) \). We have \( f_1(\cdot) \geq f^*(\cdot) \). Let \( z_1 \) be the maximizer of \( \max_x -f_1^*(z) + \frac{1}{2} \|\bar{x}\|^2 - \frac{1}{2} \|x - \bar{x}\|^2 \), and let the corresponding solution to the primal problem \( \min_x f_1(x) + \frac{1}{2} \|x - \bar{x}\|^2 \) be \( x_1 \). Define \( \tilde{f}_1 : X \to \mathbb{R} \) to be an affine minorant of \( f(\cdot) \) at \( x_1 \), i.e., \( \tilde{f}_1(x) = f(x) + s_1^T (x - x_1) \) for some \( s_1 \in \partial f(x_1) \). Let \( x_2 \) be the minimizer to the problem

\[
\min_x \{\max \{f_1(x), \tilde{f}_1(x)\} + \frac{1}{2} \|x - \bar{x}\|^2\},
\]

and let \( z_2 \) be the dual solution. Let \( f_2 : X \to \mathbb{R} \) be the affine function such that the problem

\[
\min_x f_2(x) + \frac{1}{2} \|x - \bar{x}\|^2
\]
has the same primal and dual solutions \( x_2 \) and \( z_2 \). Let
\[
\alpha_i = v^* - \left[ -f_i^*(z_i) + \frac{1}{2} \| \tilde{x} \|^2 - \frac{1}{2} \| z_i - \tilde{x} \|^2 \right] \text{ for } i = 1, 2.
\]
One can see that \( \alpha_i \geq 0 \), and \( \alpha_i \) is the measure of the gap between the estimate of the dual objective value (2.17) and its true value \( v^* \). We have the following:

1. Let \( L \) be the Lipschitz constant of \( f(\cdot) \). Then
   \[
   \alpha_2 \leq \alpha_1 - \frac{1}{2}t^2, \quad \text{where} \quad \frac{1}{(2L + 1)t^2} t^2 + t \geq [f(x_1) - f_1(x_1)].
   \]

2. If \( f(\cdot) \) is smooth and \( \nabla f(\cdot) \) is Lipschitz with constant \( \Lambda' \), then
   \[
   \frac{1}{4(L^2 + 1)} \left( \alpha_2 \right)^2 + \frac{3}{4} \leq 1.
   \]

Note that \( f(x_1) - f_1(x_1) \) in (2.19) plays the role of \( f_i(\{x^n_{w_1} \})_i - f_{i,w_1}(\{x^n_{w_1} \})_i \) in the proofs in Section 3.

Recall the definition of \( p(\cdot, \cdot) \) in (2.14). It follows from line 14 in Algorithm 2.3 that
\[
z^{n,p(n,\alpha)}_\alpha = z^{n,p(n,\alpha)+1}_\alpha = \ldots = z^{n,w}_\alpha \text{ for all } \alpha \in \bar{E} \cup V.
\]
Moreover, \( \alpha \notin \bar{E}_n \) implies \( \alpha \notin S_{n,w} \) for all \( w \in \{1, \ldots, \bar{w} \} \), so
\[
0 \overset{(2.6a)}{=} z^{n,0}_\alpha = z^{n,1}_\alpha = \ldots = z^{n,w}_\alpha \text{ for all } \alpha \in \bar{E} \setminus \bar{E}_n.
\]

**Remark 2.10.** (On the condition \( S_{n,1} = V_4 \)) Throughout this paper, we assumed \( S_{n,1} = V_4 \) in Assumption 2.7. Algorithm 2.3 with this condition would not be truly asynchronous, but it is relatively easy to enforce this condition. One way to enforce this condition is to use a global clock. Another way to enforce this condition is to use the sparsity of \( z_\alpha \) in Proposition 2.6. We limit ourselves to the special case in Example 1.1. Suppose that \( \{S_{n,w}\}_{w=1}^W \) is such that for all \( i \in V_4 \), \( S_{n,w_i} = \{i\} \) for some \( w_i \in \{1, \ldots, \bar{w}_i\} \). Suppose also that for all \( i, j \in V_4 \) such that \( w_i < w_j \):

\[ (* \text{ ) There are no } (e, k) \in \bar{E} \text{ such that } i \text{ and } j \text{ are the two endpoints of } e \text{ and } (e, k) \in S_{n,w'} \text{ for some } w' \text{ such that } w_i < w' < w_j. \]

If condition (*) holds for some \( i, j \in V_4 \), then the sparsity of \( z^{n,w}_\alpha \) implies that if we changed from \( S_{n,w_i} = \{i\} \) and \( S_{n,w_j} = \{j\} \) to \( S_{n,w_i} = \{i, j\} \) and \( S_{n,w_j} = \emptyset \), then the iterates \( \{x^{n,w}_w\}\) obtained will remain equivalent. It is possible to ensure (*) for all \( i, j \in V_4 \) using a signal from a fixed node in \( V \) propagated as computations in the algorithm are carried out.

**Remark 2.11.** (The dual objective value) As we have discussed in Pan18a, Pan18b, strong duality holds for the problem (1.1). In particular, for any primal feasible \( x \) and dual feasible \( z \), the duality gap satisfies
\[
\frac{1}{2} \| \bar{x} - x \|^2 + \sum_{\alpha \in \bar{E} \cup V} f_\alpha(x) - F(\{z_\alpha\}_{\alpha \in \bar{E} \cup V}) \overset{\text{Lipschitz}}{\geq} \frac{1}{2} || \bar{x} - x ||^2 + \sum_{\alpha \in \bar{E} \cup V} f_\alpha(x) + f^*_\alpha(z_\alpha) - \left\langle x, \sum_{\alpha \in \bar{E} \cup V} z_\alpha \right\rangle + \frac{1}{2} \left\| \sum_{\alpha \in \bar{E} \cup V} z_\alpha \right\|^2
\]
Fenchel duality
\[
\overset{\geq}{\geq} \frac{1}{2} || \bar{x} - x ||^2 + \left\langle x, \sum_{\alpha \in \bar{E} \cup V} z_\alpha \right\rangle - \left\langle \bar{x}, \sum_{\alpha \in \bar{E} \cup V} z_\alpha \right\rangle + \frac{1}{2} \left\| \sum_{\alpha \in \bar{E} \cup V} z_\alpha \right\|^2
= \frac{1}{2} \left\| \bar{x} - x - \sum_{\alpha \in \bar{E} \cup V} z_\alpha \right\|^2 \geq 0.
\]
Since the estimate of the primal solution \( \mathbf{x}^{n,w} \) is \( \mathbf{z} - \sum_{\alpha \in V \cup \hat{E}} \mathbf{z}_{\alpha}^{n,w} \) by (2.5), substituting \( \mathbf{x} \) in (2.22) to be the primal optimal solution \( \mathbf{x}^* \) shows that the difference of the dual objective value and its optimal value bounds the distance \( \frac{1}{2} \| \mathbf{x}^{n,w} - \mathbf{x}^* \|^2 \).

For the rest of this paper, we shall be looking at the rate of convergence of the dual objective value (2.2) to its optimum value. This remark justifies the usefulness of our results.

3. Linear convergence when all functions are smooth

Throughout this section, we make the following assumption.

**Assumption 3.1.** For the problem (1.1), we make the following assumptions:

1. \( V = V_4 \). In other words, \( V_1 = V_2 = V_3 = \emptyset \).
2. The sets \( X_i = \mathbb{R}^m \), \( E = E \times \{1, \ldots , m\} \) and \( \{H_\alpha\}_{\alpha \in \hat{E}} \) are as described in Example 1.1.
3. For all \( i \in V_4 \), \( f_i^*(\cdot) \) is strongly convex with modulus \( \sigma > 0 \), which is equivalent to \( \nabla f_i^*(\cdot) \) being Lipschitz continuous with constant \( \frac{1}{\sigma} \). [Note that in general, \( f_i^*(\cdot) \) are subdifferentiable for all \( i \in V_4 \), but we now limit to only smooth \( f_i^*(\cdot) \).]

For the problem (1.5) in Example 1.1 satisfying Assumption 3.1, a primal method, for example \( \text{DPA} \) [17], can achieve linear convergence. Since Algorithm 2.3 has the features explained in Subsection 1.2, it is of interest to find out whether Algorithm 2.3 also has linear convergence under Assumption 3.1. We shall prove such a result in this section.

We write down the function \( F_S : X^{|V \cup \hat{E}|} \to \mathbb{R} \cup \{\infty\} \) to be minimized.

\[
\min_{\mathbf{z}_\alpha \in X, \alpha \in V \cup \hat{E}} F_S(\{\mathbf{z}_\alpha\}_{\alpha \in V \cup \hat{E}}) := \sum_{i \in V} f_i^*(\mathbf{z}_i) + \sum_{\alpha \in \hat{E}} \delta_{H_\alpha}(\mathbf{z}_\alpha) + \frac{1}{2} \| \mathbf{x} - \sum_{\alpha \in V \cup \hat{E}} \mathbf{z}_\alpha \|^2. \tag{3.1}
\]

Note that \( F_S(\cdot) \) is related to the dual function \( F(\cdot) \) in (1.3) by a sign change and a constant.

For a set of dual variables \( \mathbf{z}^0 \in X^{|V \cup \hat{E}|} \), let \( \mathbf{x}^0 \), \( \mathbf{v}_H^0 \) and \( \mathbf{v}_A^0 \) be related via (2.5). Note that \( \mathbf{v}_H^0 = \sum_{\alpha \in \hat{E}} \mathbf{z}^0_\alpha \in \sum_{\alpha \in \hat{E}} H_\alpha \subset D_\perp \). We write \( \mathbf{z}^0 \in \mathbb{R}^m \) as

\[
\mathbf{z}^0 := -\frac{1}{|V|} \sum_{i \in V} [\mathbf{x}^0]_i, \quad \mathbf{v}_H^0 = -\frac{1}{|V|} \sum_{i \in V} [\mathbf{x} - \mathbf{v}_H^0 - \sum_{j \in V} \mathbf{z}^0_j]_i \quad \text{(2.6)}
\]

\[
\mathbf{z}^0_\alpha = -\frac{1}{|V|} \sum_{i \in V} [\mathbf{x} - \mathbf{v}_H^0 - \mathbf{z}^0_j]_i \quad \mathbf{v}_A^0 = \frac{1}{|V|} \sum_{i \in V} [\mathbf{z}^0_\alpha - \mathbf{x}]. \tag{3.2}
\]

In other words, the vector \( \mathbf{z}^0 := (\mathbf{z}^0_0, \ldots , \mathbf{z}^0) \) is the projection of \( -\mathbf{x} \) onto \( D \) as defined in (2.13a). Define \( e \in \mathbb{R}^m \) by \( [e]_i := [\mathbf{x} - \mathbf{v}_H^0 - \mathbf{z}^0_i]_i + [\mathbf{z}^0_i]_i \), so that \([\mathbf{x} - \mathbf{v}_H^0 - \mathbf{z}^0_i]_i = [-\mathbf{z}^0 + e]_i \). The value \( F_S(\{\mathbf{z}^0_\alpha\}_{\alpha \in V \cup \hat{E}}) \) can be written as

\[
F_S(\{\mathbf{z}^0_\alpha\}_{\alpha \in V \cup \hat{E}}) = \sum_{i \in V} f_i^*(\mathbf{z}^0_i) + \frac{1}{2} \| \mathbf{z}^0 - \mathbf{z}^0_\alpha + [e]_i \|_2^2. \tag{3.3}
\]

Let \( \mathbf{z}^* \) be a minimizer of \( F_S(\cdot) \). The strong convexity of \( f_i^*(\cdot) \) from Assumption 3.1 ensures that \( \mathbf{z}^*_\alpha \) is unique if \( i \in V \). (Though \( \mathbf{z}^*_\alpha \) need not be unique if \( \alpha \in \hat{E} \).)

The unique solution has the value

\[
F_S^* := F_S(\mathbf{z}^*) = \sum_{i \in V} f_i^*(\mathbf{z}^*_i) + \frac{1}{2} \| \mathbf{z}^* - \mathbf{z}^*_\alpha + [e]_i \|_2^2.
\]
where \( \hat{\beta}^* \) is defined to be \( \hat{\beta}^* = -\frac{1}{m} \sum_{i \in V} [\hat{x} - \hat{v}^*_H - z^*_i]_i = \frac{1}{m} \sum_{i \in V} [z^*_i - \bar{x}]_i \). Note that \((-\hat{\beta}^*, \ldots, -\hat{\beta}^*)\) is the projection of \( \bar{x}^* = \tilde{x} - \tilde{v}^*_H - \sum_{i \in V} z^*_i \) onto \( D \). For \( \tilde{v}^*_H \) to be optimal, we need all the components of \( \tilde{x} - \tilde{v}^*_H - \sum_{i \in V} z^*_i \) to have the same components so that \( \frac{1}{2} \| \bar{x} - \tilde{v}^*_H - \sum_{i \in V} z^*_i \|^2 \), which appears in the dual objective function (2.2) through (2.5a), has minimum norm. This leads to \( x^* \) having all components being \(-\hat{\beta}^*\).

**Lemma 3.2.** Suppose Assumption 3.1(2) and (3) hold. Suppose \( \{z^*_0\}_{\alpha \in V \cup \tilde{E}} \) is a dual variable, and let the derived variables \( x^0, \tilde{v}^0, \tilde{v}^0_A \) and \( e \) be as defined in the above commentary. Let \( x^*_i \in [\mathbb{R}^m]_{1} \) be defined so that \([z^*_i]_j = 0 \) when \( i \neq j \) and

\[
[z^*_i]_i = \arg\min_{z \in \mathbb{R}^m} f^*_i(z) + \frac{1}{2} \| z^0 + z \|_2^2 - z^0 + \| z \|_2^2 \text{ for all } i \in V. \tag{3.3}
\]

For all \( i \in V \), let \( f^*_i : [\mathbb{R}^m] \to \mathbb{R} \) and \( \tilde{f}^*_i : [\mathbb{R}^m]_1 \to \mathbb{R} \) be related through \( f^*_i(x) = \tilde{f}^*_i(x) \). Assume also that \( f^*_i(x) \leq \tilde{f}^*_i(x) \), which is equivalent to \( \tilde{f}^*_i(x) \geq f^*_i(x) \). Let \( \tilde{F}_S : [\mathbb{R}^n]_1 [\mathbb{R}^n]_1 \to \mathbb{R} \cup \{\infty\} \) be defined in a manner similar to (3.1) as

\[
\tilde{F}_S(\{z_\alpha\}_{\alpha \in V \cup \tilde{E}}) := \sum_{\alpha \in V \cup \tilde{E}} \tilde{f}^*_i(z_i) + \sum_{\alpha \in E} \delta_i(z_\alpha) + \frac{1}{2} \left\| \bar{x} - \sum_{\alpha \in E} z_\alpha \right\|_2^2. \tag{3.4}
\]

Then one can check that

\[
F_S(\{z_i\}_{i \in V}, \{z^0_\alpha\}_{\alpha \in \tilde{E}}) = \sum_{i \in V} \left[ f^*_i([z_i]_i) + \frac{1}{2} \| z^0 + [z_i + e]_i \|_2 \right]
\]

and

\[
\tilde{F}_S(\{z_i\}_{i \in V}, \{z^0_\alpha\}_{\alpha \in \tilde{E}}) = \sum_{i \in V} \left[ \tilde{f}^*_i([z_i]_i) + \frac{1}{2} \| z^0 + [z_i + e]_i \|_2 \right].
\]

Let \( z^* \) be a minimizer of \( F_S(\cdot) \), and let \( z^*_i = z^*_i \) for all \( i \in \tilde{E} \). Then there exists constants \( \gamma \in (0, 1) \) and \( M > 0 \) such that if \( z^0 \) and \( z^+ \) are related as described, then

\[
F_S(z^+) - F_S(z^*) \leq \gamma \left[ F_S(z^0) - F_S(z^*) \right] + M \sum_{i \in V} \| e_i \|^2 \tag{3.5}
\]

Proof. The second inequality of (3.5) is obvious from \( \tilde{f}^*_i(\cdot) \geq f^*_i(\cdot) \). We prove the first inequality. By (3.3) and Assumption 3.1(3), we have, for all \( i \in V \),

\[
f^*_i([z^*_i]_i) + \frac{1}{2} \| z^0 + [e]_i \|^2 \tag{3.6}
\]

\[
\geq f^*_i([z^*_i]_i) + \frac{1}{2} \| z^0 \|^2 + \frac{1}{2} \| e \|^2 + \frac{1}{2} \| [z^*_i - z^*_i] \|^2 \tag{3.7}
\]

Also, the optimality condition of (3.3) implies that \(-z^0 + [z^0 + z^*_i + e]_i \in \partial f_i([z^*_i]_i)\), so together with Assumption 3.1(3), we have

\[
f^*_i([z^*_i]_i) + \frac{1}{2} \| z^0 \|^2 + \frac{1}{2} \| [z^0 - z^*_i] \|^2 + \frac{1}{2} \| [e]_i \|^2 + \frac{1}{2} \| [z^*_i - z^*_i] \|^2
\]

\[
\geq f^*_i([z^*_i]_i) + \frac{1}{2} \| z^0 + [z^0 - z^*_i + e]_i \|^2 + \frac{1}{2} \| [z^*_i - z^*_i] \|^2
\]

\[
-\frac{1}{2} \| [z^0 - z^*_i] \|^2 + \frac{1}{2} \| [z^0 + z^*_i + e]_i \|^2 + \frac{1}{2} \| [z^0 - z^*_i] \|^2
\]

\[
+ \left( -\frac{1}{2} \| [z^0]_i \|^2 + \frac{1}{2} \| [z^0 - z^*_i] \|^2 + \frac{1}{2} \| [e]_i \|^2
\]

\[
+ (-z^0, [z^0 - z^*_i]_i) + \langle [e]_i, -z^0 \rangle + \langle [e]_i, [z^0 - z^*_i]_i \rangle). \]
For the terms not involving \([e]_i\) in the last formula of (3.7), we have
\[
\sum_{i \in V} \langle [z]_i - [z]_i, [z]_i \rangle \geq \sum_{i \in V} \left[ -\frac{1}{2\epsilon} \|[z]_i^0 - [z]_i\|_2^2 - \frac{\epsilon}{2} \|[z]_i^0 - [z]_i\|_2^2 \right],
\] (3.8)
and
\[
\sum_{i \in V} \left[ \langle [z]_i - [z]_i^0, [z]_i^0 \rangle + \langle [z]_i - [z]_i, [z]_i \rangle \right] + \frac{1}{2\epsilon} \|[\hat{z}]_i - [\hat{z}]_i^\ast\|_2^2 - \frac{\epsilon}{2} \|[\hat{z}]_i - [\hat{z}]_i^\ast\|_2^2 \right]
\]
\[
= \sum_{i \in V} \left[ \langle [z]_i^0 - [z]_i, [z]_i \rangle \right] + \frac{1}{2\epsilon} \|[\hat{z}]_i - [\hat{z}]_i^\ast\|_2^2 - \frac{\epsilon}{2} \|[\hat{z}]_i - [\hat{z}]_i^\ast\|_2^2 \right]
\]
(3.9)
\[
= |V| \langle [z]_i^0, [z]_i^0 - [z]_i^\ast \rangle + \frac{1}{\epsilon} \|[\hat{z}]_i - [\hat{z}]_i^\ast\|_2^2 - \frac{1}{\epsilon} \|[\hat{z}]_i - [\hat{z}]_i^\ast\|_2^2 = \frac{1}{\epsilon} \|[\hat{z}]_i^0 - [\hat{z}]_i^\ast\|_2^2 \geq 0.
\]
For the terms involving \([e]_i\) in the last formula in (3.7), we have
\[
\sum_{i \in V} \langle [e]_i, [z]_i^0 - [z]_i^\ast \rangle \geq \sum_{i \in V} \left[ -\frac{1}{2\epsilon} \|[e]_i\|_2^2 - \frac{\epsilon}{2} \|[z]_i^0 - [z]_i^\ast\|_2^2 \right],
\]
(3.10)
\[
\sum_{i \in V} \langle [e]_i, [z]_i^0 - [z]_i^\ast \rangle = \left\langle \sum_{i \in V} [e]_i, -[\hat{z}]_i \right\rangle = 0.
\]
(3.11)
Summing up the right hand sides of (3.8), (3.9) and (3.10) and \(\sum_{i \in V} \|[z]_i - [z]_i^\ast\|_2^2 \geq \frac{1}{2\epsilon} \|[e]_i\|_2^2 - \frac{1}{2} \|[z]_i^0 - [z]_i^\ast\|_2^2 \) and setting \(\epsilon = \sigma/2\) gives
\[
\sum_{i \in V} \left[ \langle [z]_i - [z]_i^0, [z]_i^0 \rangle - \frac{\epsilon}{2} \|[z]_i^0 - [z]_i^\ast\|_2^2 \right]
\]
\[
= \sum_{i \in V} \left[ -\frac{1}{\sigma} \|[z]_i^0 - [z]_i^\ast\|_2^2 - \frac{1}{2} \|[z]_i^0 - [z]_i^\ast\|_2^2 \right].
\]
(3.12)
Summing the formulas in (3.7) to (3.11), we have
\[
\sum_{i \in V} \left[ f_i^* ([z]_i^\ast) + \frac{1}{\sigma} \|[z]_i^\ast\|_2^2 \right]
\]
\[
= \sum_{i \in V} \left[ f_i^* ([z]_i^\ast) + \frac{1}{\sigma} \|[z]_i^\ast\|_2^2 - \frac{\epsilon}{2} \|[\hat{z}]_i - [\hat{z}]_i^\ast\|_2^2 \right]
\]
\[
= \left( \frac{4+\sigma}{\sigma(1+\sigma)} \right) \sum_{i \in V} \left[ f_i^* ([z]_i^\ast) + \frac{1}{\sigma} \|[z]_i^\ast\|_2^2 + \sum_{i \in V} \left[ f_i^* ([z]_i^\ast) + \frac{1}{\sigma} \|[z]_i^\ast\|_2^2 \right] \right.
\]
\[
\geq \left( \frac{4+\sigma}{\sigma(1+\sigma)} + 1 \right) \sum_{i \in V} \left[ f_i^* ([z]_i^\ast) + \frac{1}{\sigma} \|[z]_i^\ast\|_2^2 + \sum_{i \in V} \left[ f_i^* ([z]_i^\ast) + \frac{1}{\sigma} \|[z]_i^\ast\|_2^2 \right] \right.
\]
\[
- \frac{1}{\sigma} \left[ \frac{1}{\sigma} + \frac{\sigma}{4} \right] \sum_{i \in V} \|[e]_i\|_2^2.
\]
(3.13)
Letting \(\gamma = \left( \frac{4+\sigma}{\sigma(1+\sigma)} + 1 \right) \), we can rearrange (3.13) to get the first inequality in (3.5) with \(M = (1 - \gamma) \left[ \frac{1}{\sigma} + \frac{\sigma}{4} \right] \). This concludes the proof. \(\square\)
We now proceed to prove the linear convergence result. Let $\hat{F}_S^{n,w}(\cdot)$ be defined by

$$\hat{F}_S^{n,w}(\{z_\alpha\}_{\alpha \in V \cup E}) := \sum_{i \in V} f^*_i(z_i) + \sum_{\alpha \in E} \delta^*_\alpha(z_\alpha) + \frac{1}{2} \|\bar{x} - \sum_{\alpha \in V \cup E} z_\alpha\|^2. \quad (3.14)$$

**Theorem 3.3.** (Linear convergence) Suppose Assumption 3.1 holds. Consider Algorithm 2.3 being applied to solve (1.1). Suppose $S_{n,1} = V$ and $z^*$ is a minimizer of $F_S(\cdot)$. Suppose also that $\bigcup_{w=1}^{\bar{w}} S_{n,w} \cap E = \bar{E}_n$. Then there is some $c \in (0, 1)$ such that

$$\hat{F}_S^{n+1,0}(z^{n+1,0}) - F_S(z^*) \leq c[\hat{F}_S^{n,0}(z^{n,0}) - F_S(z^*)].$$

**Proof.** Let $z^+ \in X^{V \cup E}$ satisfy (3.3), where $z^0 = z^{n,0}$ and $z^0 \in \bigcup_{i \in V} [z_i^0 - \bar{x}]$. Then there is some $\gamma \in (0, 1)$ such that

$$F_S(z^+) - F_S(z^0) \leq \gamma[\hat{F}_S^{n,0}(z^{n,0}) - F_S(z^*)] + M \sum_{i \in V} \|e_i\|^2.$$

Let $p_i = -z_i^0 + [z_i^0 + e_i]_i$. We make use of Lemma 3.2 to see that there is a constant $\gamma_2 \in [0, 1)$ such that, for all $i \in V$,

$$\frac{1}{2}\|p_i\|^2 \leq \gamma_2 [f^*_i(z_i^{n,0}) + \frac{1}{2}\|p_i\|^2 - [f^*_i(z_i^n)]_i - \frac{1}{2}\|p_i - [z_i^n]_i\|^2]. \quad (3.15)$$

Summing (3.15) over all $i \in V$ applied to (3.14) gives

$$\hat{F}_S^{n+1}(z^{n+1}) - F_S(z^+) \leq \gamma_2[\hat{F}_S^{n,0}(z^{n,0}) - F_S(z^*)]. \quad (3.16)$$

Then we have

$$\hat{F}_S^{n+1}(z^{n,1}) - F_S(z^*) = \hat{F}_S^{n+1}(z^{n,1}) - F_S(z^+) + F_S(z^+) - F_S(z^*) \leq \gamma_2[\hat{F}_S^{n,0}(z^{n,0}) - F_S(z^*)] + F_S(z^+) - F_S(z^*) \leq \gamma_2[\hat{F}_S^{n,0}(z^{n,0}) - F_S(z^*)] + (1 - \gamma_2)\|F_S(z^*) - F_S(z^*)\|

\begin{align*}
\gamma_3 & \leq [1 - (1 - \gamma_2)(1 - \gamma_2)] [\hat{F}_S^{n,0}(z^{n,0}) - F_S(z^*)] + (1 - \gamma_2)M \sum_{i \in V} \|e_i\|^2.
\end{align*}

Since $\gamma < 1$ and $\gamma_2 < 1$, the $\gamma_3$ as marked above satisfies $\gamma_3 \in [0, 1)$. We now consider 2 cases.

**Case 1:** $(1 - \gamma_2)M \sum_{i \in V} \|e_i\|^2 \leq \frac{1 - \gamma_2}{2}[\hat{F}_S^{n,0}(z^{n,0}) - F_S(z^*)]$. In this case, we use Theorem 2.3 to get

$$\hat{F}_S^{n+1,0}(z^{n+1,0}) - F_S(z^*) \leq \hat{F}_S^{n+1,0}(z^{n+1}) - F_S(z^*) \leq \frac{1 + \gamma_3}{2}[\hat{F}_S^{n,0}(z^{n,0}) - F_S(z^*)]. \quad (3.17)$$

**Case 2:** $(1 - \gamma_2)M \sum_{i \in V} \|e_i\|^2 \geq \frac{1 - \gamma_2}{2}[\hat{F}_S^{n,0}(z^{n,0}) - F_S(z^*)]$. We recall that $x^{n,w} = \bar{x} - \sum_{\alpha \in V \cup E} z_{n,w}^\alpha$. The value $\sum_{i \in V} \|e_i\|^2$ equals $d(x^{n,0}, D)^2$, where $D$ is the diagonal set $D(\alpha_0 \in E_n \sum_{i \in V} \|e_i\|^2$. By the Hoffman lemma and the fact that $\bar{E}_n$ connects $V$, there is a constant $\kappa$ such that $\kappa d(x, D) \leq \max_{\alpha \in E_n} d(x, H_\alpha)$. Let $\alpha^* \in \bar{E}_n$ be such that $\kappa d(x^{n,0}, D) \leq d(x^{n,0}, H_{\alpha^*})$. Let $w_{\alpha^*}$ be in $\{1, \ldots, \bar{w}\}$

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Lemma 4.1. \( \frac{1}{2} \sum_{i \in V} ||e_i||^2 \leq d(x_0, D)^2 \) (3.18)

Next,

\[
\frac{1}{2} \sum_{u=0}^{w_*} ||x_n - x_{n+1}||^2 \geq \frac{1}{2w_*} \left[ \sum_{u=0}^{w_*} ||x_n - x_{n+1}||^2 \right] \geq \frac{\kappa^2(1-\gamma)}{4M(1-\gamma)} \left[ \tilde{F}_S^{n,0}(z^{n,0}) - F_S(z^*) \right].
\] (3.19)

So

\[
\tilde{F}_S^{n+1,0}(z^{n+1,0}) - F_S(z^*) \leq \frac{\kappa^2(1-\gamma)}{4M(1-\gamma)} \left[ \tilde{F}_S^{n,0}(z^{n,0}) - F_S(z^*) \right].
\] (3.20)

This completes the proof of linear convergence.

Remark 3.4. The linear convergence rates in (3.18) can plausibly be refined with the study of effective resistances in [AG17].

4. O(1/k) CONVERGENCE WHEN ALL FUNCTIONS ARE EITHER SMOOTH OR PROXIMABLE

In this section, we prove the O(1/k) convergence rate of the dual objective value when all \( f_i(\cdot) \) are smooth for all \( i \in V_4 \). We begin by discussing conditions ensuring the boundedness of \( \{z^{n,w}\} \) before our main result.

4.1. On the boundedness of \( \{z^{n,w}\} \). In this subsection, we discuss a standard constraint qualification that ensures the boundedness of \( \{z^{n,w}\} \).

We write down a lemma on functions whose domain is not the entire space.

Lemma 4.1. Suppose \( X \) is a finite dimensional Hilbert space, and \( f : X \rightarrow \mathbb{R} \) is a closed convex function. Suppose \( \{x_i\}_{i=1}^{\infty} \) and \( \{z_i\}_{i=1}^{\infty} \) are sequences such that \( z_i \in \partial f(x_i) \) for all \( i \geq 1 \). Suppose that \( \lim_{i \rightarrow \infty} x_i \) exists, say \( x^* \), and \( \lim_{i \rightarrow \infty} z_i = \infty \). Then a cluster point of \( \left\{ \frac{z_i}{||z_i||} \right\} \) lies in \( N_{\text{cl}(\text{dom}(f))}(x^*) \).

Proof. We make use of the fact that \( z \in \partial f(x) \) if and only if \( (z, -1) \in N_{\text{epi}(f)}(x, f(x)) \). By choosing subsequences, we can assume that \( \lim_{i \rightarrow \infty} f(x_i) \) exists as either a finite number or \( \infty \). We can assume that \( \lim_{i \rightarrow \infty} \frac{z_i}{||z_i||} \) exists and equals \( z^* \).

Suppose \( \lim_{i \rightarrow \infty} f(x_i) \) is a finite number, say \( f^* \). In this case, \( (x^*, f^*) \) lies in the epigraph of \( f(\cdot) \). By the closedness of the normal cones, \( \lim_{i \rightarrow \infty} \frac{(z_i, -1)}{||z_i||} = (z^*, 0) \) is a normal vector of \( \text{epi}(f) \) at \( (x^*, f^*) \). This implies that \( z^* \) lies in \( N_{\text{cl}(\text{dom}(f))}(x^*) \).

Suppose \( \lim_{i \rightarrow \infty} f(x_i) \) is infinity. Take any \( x' \in \text{dom}(f) \), Let \( I \) be large enough so that \( f(x_i) > f(x') \) for all \( i > I \). Then the point \((x', f(x_i))\) would lie in \( \text{epi}(f) \). Then

\[
\left( \frac{(z_i, -1)}{||z_i||}, [(x_i, f(x_i)) - (x', f(x_i))] \right) \leq 0 \text{ for all } i > I,
\]
which gives $\langle \frac{x_i}{\|x_i\|}, x_i - x' \rangle \leq 0$ for all $i > I$. Taking limits gives $\langle z^*, x^* - x' \rangle \leq 0$, which implies that $z^* \in N_{\text{cl}(\text{dom}(f))}(x^*)$.

With the above lemma, we can prove the following result on the boundedness of the iterates $\{x_\alpha\}_{\alpha \in V_1 \cup V_2}$ under a constraint qualification.

**Theorem 4.2.** Suppose we are under the setting of Example 1.1 and let $x^*$ be the primal minimizer to (1.5). Suppose that the condition

$$\exists \lim_{i \to \infty} \{ \alpha \in \text{dom}(f) \}$$

holds. Then the iterates $\{x_\alpha^{n,w}\}_{n,w}$ of Algorithm 2.3 are bounded.

**Proof.** Seeking a contradiction, suppose $\{x_\alpha^{n,w}\}_{n,w}$ is not bounded for some $i \in V$. This means that we can find a subsequence $\{n_k\}$ such that

$$\lim_{k \to \infty} \max_{j \in V} \|z_{j,n_k}^{n_k,w}\| = \infty. \quad (4.2)$$

Theorem 2.8(ii), formula (2.5) and the finite dimensionality of $X$ ensures that $\{x_{n_k}^{n_k,w}\}_{n_k=1}^\infty$ has a cluster point. We can choose a further subsequence if necessary so that $\lim_{k \to \infty} x_{n_k}^{n_k,w} = \hat{x}$ for some $\hat{x} \in X$. Theorem 2.8(i) implies that $\lim_{k \to \infty} \|x_{n_k}^{n_k,w} - x_{n_k,p(n_k,\alpha)}\| = 0$ for all $\alpha \in V \cup E$, so $\lim_{k \to \infty} x_{n_k}^{n_k,p(n_k,\alpha)} = \hat{x}$. Since $x_{n_k}^{n_k,p(n_k,\alpha)} \in H_\alpha$ for all $\alpha \in E$, going back to the primal problem associated with (2.7) gives us $\hat{x} \in \bigcap_{\alpha \in E} H_\alpha$. Let $\hat{x} \in \mathbb{R}^m$ be any component of $\hat{x} \in [\mathbb{R}^m]^{\vert V \vert}$. (Note that all components in $\hat{x} \in [\mathbb{R}^m]^{\vert V \vert}$ are equal since $\hat{x} \in \bigcap_{\alpha \in E} H_\alpha$.)

Assume, by taking further subsequences if necessary, that for all $i \in V$.

$$\hat{z}_i := \lim_{k \to \infty} \frac{x_{i,n_k}^{n_k,w}}{\max_{j \in V} \|x_{j,n_k}^{n_k,w}\|} \quad (4.3)$$

exists as a limit. From Lemma 4.1 we have $[\hat{z}_i]_\alpha \in N_{\text{cl}(\text{dom}(f_i))}(\hat{x})$ for all $i \in V$. Recall that $\{V_\alpha^{n,w}\}$ is a bounded set by Theorem 2.8(ii), so by (2.5), $\{\alpha \in V \cup E \}^{\infty}_{\alpha=1}$ is bounded. Note that for all $\alpha \in E, z_\alpha^{n,w} \in H_\alpha^{\perp} \subset D^{\perp}$. We then have

$$\sum_{i \in V} \left[ \sum_{\alpha \in V \cup E} z_{i,n_k}^{n_k,w} \right]_{j} = \sum_{i \in V} \sum_{\alpha \in V \cup E} [z_{i,n_k}^{n_k,w}]_{j} = \sum_{i \in V} [z_{i,n_k}^{n_k,w}]_{j}, \quad (2.13)_{i}$$

so $\sum_{i \in V} [z_{i,n_k}^{n_k,w}]_{j}$ is bounded. The formulas (4.2) and (4.3) imply that $\sum_{i \in V} [\hat{z}_i]_i$ equals zero.

Let $x^* \in \mathbb{R}^m$ be any component of the primal optimal solution $x^*$. We now show that $[\hat{z}_i]_i \in N_{\text{cl}(\text{dom}(f_i))}(x^*)$. On the one hand, we have

$$\langle \sum_{i \in V} [\hat{z}_i]_i, \hat{x} - x^* \rangle = \langle 0, \hat{x} - x^* \rangle = 0. \quad (4.4)$$

On the other hand since $[\hat{z}_i]_i \in N_{\text{cl}(\text{dom}(f_i))}(\hat{x})$ and $x^* \in \bigcap_{i \in V} \text{cl}(\text{dom}(f_i))$, we have

$$\langle [\hat{z}_i]_i, \hat{x} - x^* \rangle \leq 0. \quad (4.5)$$

One can easily check that (4.4) and (4.5) implies that $\langle [\hat{z}_i]_i, \hat{x} - x^* \rangle = 0$. Next, for any $x \in \text{dom}(f_i)$, we have

$$\langle [\hat{z}_i]_i, x^* - x \rangle = \langle [\hat{z}_i]_i, x^* - \hat{x} \rangle + \langle [\hat{z}_i]_i, \hat{x} - x \rangle \leq 0.$$
Thus $[z_i]_i \in N_{c_i(\text{dom}(f_i))}(x^*)$. This means that the constraint qualification (4.1) fails at $x^*$, which is a contradiction.

\[\]

**Remark 4.3.** (Constraint qualifications on intersections of sets) The assumption in Theorem 4.2 cannot be easily weakened because the constraint qualification (4.1) is well known to be related to sensitivity analysis issues related to the intersection of convex sets. For the convex case, we refer to [Kru06]. Such results have been extended to the nonconvex case in, for example, [BBL99] and the references mentioned within.

The theory on Dykstra’s algorithm says that for any $w \in \{1, \ldots, \bar{w}\}$, $\lim_{n \to \infty} x^{n,w}$ exists and is $x^*$, the minimizer of the primal problem (1.5). Hence under the constraint qualification (4.1), the iterates $\{(z_i^{n,w})_{i}(n,w)\}$ are bounded for all $i \in V$.

4.2. The $O(1/k)$ convergence. Throughout this section, we make an assumption similar to Assumption 3.1.

**Assumption 4.4.** For the problem (1.1), suppose Assumption 3.1(2) and (3) hold.

Recall the definition of $F_S(\cdot)$ in (3.1). Just like in Lemma 3.2, we define $z_0^a \in [\mathbb{R}^m]^{1 \times V}$ and $z_+^a \in [\mathbb{R}^m]^{1 \times V}$ for all $a \in V \cup E$ so that $z_i^+ = z_0^a$ for all $e \in E$, and $z_i^+ = z_0^e$ for all $e \in E$. Denote $z_i^{++}$ defined through $\{z_i^{a \pm}\}_{a \in V \cup E}$ by (3.3).

Now, define $z_i^{++} \in [\mathbb{R}^m]^{1 \times V}$ so that $z_i^{++} = z_i^+$ for all $i \in V$, and for all $e \in E$, $z_i^{++}$ are defined so that $v_H^{++} = \sum_{a \in E} z_i^{a \pm}$ also satisfies

$$v_H^{++} = \arg \min_{v_H \in X} \frac{1}{2} \left\| \sum_{i \in V} z_i^+ + v_H - \bar{x} \right\|^2.$$  (4.6)

Note that $z^+$ and $z^{++}$ are defined by a block coordinate minimization of the function $F_S(\cdot)$ starting from the dual iterate $z^0$. Consider $F_V : X^2 \to \mathbb{R}$ defined by

$$F_V(z, v_H) = \sum_{i \in V} f_i^*(\{z_i\}) + \delta_D(v_H) + \frac{1}{2} \|z + v_H - \bar{x}\|^2.$$  (4.7)

Note that

$$F_S(z) = F_V \left( \sum_{i \in V} z_i, \sum_{e \in E} z_e \right).$$  (4.8)

The iterates $z^+$ and $z^{++}$ mentioned earlier can be checked to be related to iterates produced by alternating minimization on $F_V(\cdot)$. More precisely,

$$\sum_{i \in V} z_i^+ = \arg \min_{z \in X} F_V \left( z, \sum_{e \in E} z_0^e \right)$$  (4.9a)

$$v_H^{++} = \arg \min_{v_H \in X} F_V \left( \sum_{i \in V} z_i^+, v_H \right).$$  (4.9b)

We first show the following:

**Lemma 4.5.** Suppose $z^+$ and $z^{++}$ are defined via $z^0$ being set to be $z^{+0}$. Then there is some constant $c_1 > 0$ such that

$$\tilde{F}_S^n(\bar{z}^{n,0}) - F_S^n(z^{n,0}) \geq c_1 [\tilde{F}_S^n(z^{n,0}) - F_S^n(z^{++})].$$  (4.10)
Proof. Let $c_3 < \frac{1}{3}$ be any constant. We divide into two cases.

Case 1: $\tilde{F}^{n,0}_S(z^{n,0}) - F_S(z^+) \geq c_3[F^{n,0}_S(z^{n,0}) - F_S(z^{++})]$.

By Lemma 2.9(2) and the definition of $z^+$, there is a constant $c_2 > 0$ such that $\tilde{F}^{n,0}_S(z^{n,0}) - F^{n,1}_S(z^{n,1}) \geq c_2[F^{n,0}_S(z^{n,0}) - F_S(z^+)]$. We have

$$\tilde{F}^{n,0}_S(z^{n,0}) - \tilde{F}^{n,0}_S(z^{n,0}) \geq \tilde{F}^{n,1}_S(z^{n,1}) \geq c_2[F^{n,0}_S(z^{n,0}) - F_S(z^+)].$$

From the inequality in 2.8(i) and the definition of $z^+$, we have

$$c_3[F^{n,0}_S(z^{n,0}) - F_S(z^{++})] \geq F^{n,0}_S(z^{n,0}) - F_S(z^+) \geq F^{n,0}_S(z^{n,0}) - F_S(z^{++}).$$

Case 2: $\tilde{F}^{n,0}_S(z^{n,0}) - F_S(z^+) \leq c_3[F^{n,0}_S(z^{n,0}) - F_S(z^{++})]$.

From the inequality in 2.8(i) and the definition of $z^+$, we have

$$c_3[F^{n,0}_S(z^{n,0}) - F_S(z^{++})] \geq F^{n,0}_S(z^{n,0}) - F_S(z^+) \geq F^{n,0}_S(z^{n,0}) - F_S(z^{++}).$$

(4.11)

Since $z^{++}$ was defined so that $\bar{x} - \sum_{\alpha \in V \cup \tilde{E}} z^{++}_\alpha$ is the projection of $\bar{x} - \sum_{\alpha \in V \cup \tilde{E}} z^+\alpha$ onto $D$ as defined in (2.13a), we have

$$d\left(\bar{x} - \sum_{\alpha \in V \cup \tilde{E}} z^{n,0}_\alpha, D\right)^2 = F_S(z^+) - F_S(z^{++}) \geq F_S(z^{n,0}) - F_S(z^{++})\tag{4.12}$$

Case 2

$$c_3[F^{n,0}_S(z^{n,0}) - F_S(z^{++})] \geq (1 - c_3)[F^{n,0}_S(z^{n,0}) - F_S(z^{++})].$$

So

$$d\left(\bar{x} - \sum_{\alpha \in V \cup \tilde{E}} z^{n,0}_\alpha, D\right) \geq d\left(\bar{x} - \sum_{\alpha \in V \cup \tilde{E}} z^{+}_\alpha, D\right) - \left\| \sum_{\alpha \in V \cup \tilde{E}} z^+_\alpha - \sum_{\alpha \in V \cup \tilde{E}} z^{n,0}_\alpha \right\| \geq \sqrt{1 - c_3} \sqrt{2[F^{n,0}_S(z^{n,0}) - F_S(z^{++})]}.$$
Hence we have

\[ \text{Theorem 4.6.} \]

Let

\[ \sum_{\alpha \in V \cup E} z_{\alpha}^{n,w} \notin H_{\alpha} \]

This concludes the proof.

\[ \square \]

**Theorem 4.6.** Let \( h^n = \hat{F}_S^n(z^{n,0}) - F_S(z^*) \). The values \( \{h^n\}_{n=1}^{\infty} \) converge to zero at the rate of \( O(1/n) \).

**Proof.** From the form of \( F_V(\cdot) \) in (4.7), we have

\begin{align*}
0 & \overset{(4.9a)}{=} \partial \left[ \sum_{i \in V} f_i(\cdot) \right] \left( \sum_{i \in V} z_{i}^{+} + \sum_{i \in V} z_{i}^{-} + v_{H}^{0} - \bar{x} \right) \\
0 & \overset{(4.9a)}{=} \partial \delta_D(v_{H}^{++}) + \sum_{i \in V} z_{i}^{+} + v_{H}^{++} - \bar{x}.
\end{align*}

We then have \((v_{H}^{++} - v_{H}^{0}, 0) \in \partial F_V(\sum_{i \in V} z_{i}^{+}, v_{H}^{++})\). Let an optimal solution of \( F_V(\cdot) \) be \((\sum_{i \in V} z_{i}^{*}, v_{H}^{*})\). Then

\[ F_S(z^{++}) - F_S(z^*) \overset{4.8}{=} F_V \left( \sum_{i \in V} z_{i}^{+}, v_{H}^{++} \right) - F_V \left( \sum_{i \in V} z_{i}^{*}, v_{H}^{*} \right) \overset{4.15}{=} - \left( (v_{H}^{++} - v_{H}^{0}, 0), \left( \sum_{i \in V} z_{i}^{*}, v_{H}^{*} \right) - \left( \sum_{i \in V} z_{i}^{+}, v_{H}^{++} \right) \right) \leq \|v_{H}^{++} - v_{H}^{0}\| \left\| \sum_{i \in V} z_{i}^{*} - \sum_{i \in V} z_{i}^{+} \right\|.
\]

Since the \( z_{i}^{+} \) would be bounded if the constraint qualification (4.1) is satisfied, there is some \( C > 0 \) such that \( \left\| \sum_{i \in V} z_{i}^{*} - \sum_{i \in V} z_{i}^{+} \right\| \leq C \). We also have

\[ \|v_{H}^{++} - v_{H}^{0}\| \overset{4.9a}{=} \sqrt{2 \left[ F_V \left( \sum_{i \in V} z_{i}^{+}, v_{H}^{++} \right) - F_V \left( \sum_{i \in V} z_{i}^{+}, v_{H}^{0} \right) \right]^{2}} \leq \sqrt{2} [F_S(z^{++}) - F_S(z^*)]\]

Hence we have

\[ F_S(z^{++}) - F_S(z^*) \overset{4.15}{\leq} C \|v_{H}^{++} - v_{H}^{0}\| \overset{4.16}{\leq} \sqrt{2} C \sqrt{F_S(z^{++}) - F_S(z^*)}, \]

\[ \text{(4.17)} \]
so

$$
\tilde{F}_{S,n,0}(z) - F_S(z^*) \geq F_S(z) - F_S(z^*) \geq \frac{\|F_S(z^+) - F_S(z^*)\|^2}{2c_4}\geq \frac{1}{2c_4}[F_S(z^+) - F_S(z^*)] + \frac{1}{2c_4}[F_S(z^+) - F_S(z^*)]^2.
$$

Let \( h^{++} := F_S(z^{++}) - F_S(z^*) \). We have

$$
\frac{1}{h^{++}} - \frac{1}{n^{++}} = \frac{h^{++} - h^{+++1}}{h^{++}} \geq \frac{c_1}{h^{++}} \frac{h^{+++1}}{C_n^1} \geq \frac{c_1}{h^{++}} \left(1 - \frac{h^{++}}{h^{++}}\right) \geq \frac{c_4}{h^{++}} \left(1 - \frac{h^{++}}{h^{++}}\right).
$$

We can check from simple calculus that there is a constant \( c_4 > 0 \) such that \( \max\{\frac{c_1}{h^{+}} \left( \frac{h^{++}}{h^{++}} \right)^2, \frac{c_1}{h^{+}} \left(1 - \frac{h^{++}}{h^{++}}\right)\} > c_4 \), which implies that \( \frac{1}{n^{++}} - \frac{1}{n^{++}} > c_4 \). This implies the \( O(1/n) \) convergence of \( \{h^{++}\}_{n=1}^\infty \) as needed.

5. \( O(1/k^{1/3}) \) CONVERGENCE WHEN SOME FUNCTIONS ARE SUBDIFFERENTIABLE BUT NOT SMOOTH

In this section, we prove the \( O(1/n^{1/3}) \) convergence rate in the general case. Here, the sets \( V, E \) and \( X \) need not take the form in Example 1.1.

For \( i \in V_4 \), let \( \hat{z}_i^{n,p(n,i)} \in X \) be the minimizer of

$$
\min_{\hat{z}_i \in X} \frac{1}{2} \left\| \hat{z}_i - \left[ x - \sum_{\beta \neq i} z_{\beta}^{n,p(n,i)} \right] \right\|^2 + f_\ast^i(\hat{z}_i).
$$

(5.1)

Note that by (2.7), \( z^{n,p(n,i)}_i \) is the minimizer of a similar problem as (5.1) but with \( f_\ast^i(\cdot) \) replaced by \( f_\ast^{i,n,p(n,i)}(\cdot) \). So \( \hat{z}_i^{n,p(n,i)} \) is distinct from \( z_i^{n,p(n,i)} \). We also define \( \hat{z}_i^{n,p(n,i)} \) to be \( \hat{z}_i^{n,p(n,i)} = z_i^{n,p(n,i)} \). Since \( f_\ast(\cdot) \) depends only on the \( i \)-th coordinate of its input \( x \), one can check that \( \hat{z}_i \in \partial f_i(x) \) for some \( x \in X \), which implies that \( [\hat{z}_i]_j = 0 \) for all \( j \neq i \). The problem (5.1) is equivalent to the problem of finding \( z_i^{n,p(n,i)} \in X_i \) that minimizes

$$
\min_{\hat{z}_i \in X_i} \frac{1}{2} \left\| \hat{z}_i - \left[ [x]_i - \sum_{\beta \neq i} [z_{\beta}^{n,p(n,i)}]_i \right] \right\|^2 + f_\ast^i(\hat{z}_i),
$$

(5.2)

where \( z_i^{n,p(n,i)} \) and \( z_i^{n,p(n,i)} \) are related by the formula \( [z_i^{n,p(n,i)}]_j = 0 \) if \( j \neq i \), and \( [z_i^{n,p(n,i)}]_i = z_i^{n,p(n,i)} \). Let \( p_i \) be the prox center as marked in (5.2). The dual of (5.2) is, up to a constant and a change of sign,

$$
\min_{\hat{z}_i \in X_i} \frac{1}{2} \left\| \hat{z}_i - p_i \right\|^2 + f_i(\hat{z}_i).
$$

(5.3)

(A more accurate primal–dual pair is (2.16) and (2.17), but this form is equivalent up to a sign change and constant.) By the Moreau decomposition theorem, the \( \hat{x}_i^{n,p(n,i)} \), \( z_i^{n,p(n,i)} \), \( [x]^{n,p(n,i)}_i \), and \( [z_i^{n,p(n,i)}]_i \) satisfy

$$
\hat{x}_i^{n,p(n,i)} + z_i^{n,p(n,i)} = p_i \text{ and } [x]^{n,p(n,i)}_i + [z_i^{n,p(n,i)}]_i = p_i.
$$

(5.3)
From optimality conditions of (5.1) and (2.7), we have

\[ 0 \in \sum_{\beta \neq i} z_{\beta}^{n,p(i)} + z_i^{n,p(i)} - \bar{x} + \partial f_{\alpha}^*(z_i^{n,p(i)}) \quad \text{for all } i \in V_4, \quad \text{and} \quad (5.4) \]

\[ 0 \in \sum_{\beta \in V \cup \bar{E}} z_{\beta}^{n,p(\alpha)} - \bar{x} + \partial f_{\alpha,n,p(\alpha)}^*(z_{\alpha}) \quad \text{for all } \alpha \in V \cup \bar{E}. \quad (5.5) \]

If \( \alpha \in V_1 \cup V_2 \cup V_3 \cup \bar{E} \), then \( z_i^{n,p(i)} = z_i^{n,p(n,i)} \). Next, define \( z_i \in X_i \) by

\[ \Delta z_i := [z_i^{n,p(i)}]_i - [z_i^{n,p(n,i)}]_i \quad \text{for all } i \in \cup_{\alpha \in V \cup \bar{E}} \partial_v \circ (\hat{z}_n) \]

We have

\[ \langle \langle x^{n,p(i)} \rangle \rangle_i - [z_i^{n,p(\alpha)}]_i - [z_i^{n,p(n,i)}]_i, \Delta z_i \rangle + \frac{1}{2} \| \Delta z_i \|^2. \quad (5.6) \]

We also have that \( z_i^{n,p(i)} \in \partial f_{\alpha}^*(z_i^{n,p(i)}) \). Recall (1.3). At the point \( \hat{z}_n \in X \), due to the separability of the non-quadratic term, we have, for each \( \alpha \in V \cup \bar{E} \), the partial subdifferential of \( -F \) in the \( \alpha \)-th coordinate is

\[ \partial(-F)\alpha(\hat{z}_n) = \sum_{\beta \in V \cup \bar{E}} \hat{z}_\beta^{n,\bar{w}} - \bar{x} + \partial f_{\alpha}^*(\hat{z}_n^{n,p(\alpha)}) \quad (5.7) \]

Let \( \hat{t}_\alpha \) and \( t_\alpha \) be as marked above. Let \( s \in X^{V \cup \bar{E}} \) be defined by \( s_\alpha := \hat{t}_\alpha - t_\alpha \). In view of (5.8), we have \( s \in \partial(-F)(\hat{z}_n^{n,\bar{w}}) \). Define \( t_\alpha \in X \) to be

\[ \hat{t}_\alpha := \sum_{\beta \neq \alpha} z_\beta^{n,\bar{w}}. \quad (5.9) \]

Let \( z^* \) be an optimizer to (1.2), which we assume to exist. Since \( s \in \partial(-F)(\hat{z}_n^{n,\bar{w}}) \), we have

\[ -F(\hat{z}_n^{n,\bar{w}}) - [F(z^*)] \leq -\langle s, z^* - \hat{z}_n^{n,\bar{w}} \rangle = \sum_{\alpha \in V \cup \bar{E}} -\langle \hat{t}_\alpha - t_\alpha, z^*_\alpha - \hat{z}_\alpha^{n,\bar{w}} \rangle \leq \sum_{\alpha \in V \cup \bar{E}} \left[ \| \hat{t}_\alpha - t_\alpha \| + \| t_\alpha - \hat{t}_\alpha \| \right] \| z^*_\alpha - \hat{z}_\alpha^{n,\bar{w}} \|. \quad (5.10) \]

Since \( x^{n,\bar{w}} \) equals \( x - V_{\bar{A}}^{n,\bar{w}} \) is bounded by Theorem 2.8(ii), for all \( i \in V_4 \), the subdifferentiable function \( f_i(\cdot) \) is Lipschitz with some constant \( L_i \) in the domain of interest. Define \( V_{\bar{A}}^{n,\bar{w}} \in X \) like in (2.5) to be

\[ V_{\bar{A}}^{n,\bar{w}} := \sum_{\alpha \in V \cup \bar{E}} \hat{z}_\alpha^{n,\bar{w}}. \quad (5.11) \]
From the fact that $\tilde{z}_i^{n,w} = z_i^{n,w}$ for all $i \in V_1 \cup V_2 \cup V_3$, we have

$$-[F^{n,\tilde{w}}(z^{n,\tilde{w}}) - F(\tilde{z}^{n,\tilde{w}})] \quad (5.12)$$

$$\sum_{i \in V_4} [f_i^{n,\tilde{w}}(z_i^{n,\tilde{w}}) - f_i^{n,\tilde{w}}(\tilde{z}_i^{n,\tilde{w}})] + \frac{1}{2} \| \tilde{x} \| - \sum_{\beta \in \mathcal{V} \cup \tilde{E}} \tilde{z}_\beta^{n,\tilde{w}} \| \tilde{x} - \sum_{\beta \in \mathcal{V} \cup \tilde{E}} \tilde{z}_\beta^{n,\tilde{w}} \| \quad (5.11)$$

$$\sum_{i \in V_4} [f_i^{n,\tilde{w}}(z_i^{n,\tilde{w}})] - f_i^{n,\tilde{w}}(\tilde{z}_i^{n,\tilde{w}})] + \frac{1}{2} \| \tilde{x} - \tilde{V}_A^{n,\tilde{w}} \| - \frac{1}{2} \| \tilde{x} - V_A^{n,\tilde{w}} \| \quad (2.20)$$

Define $\Delta f_i \in \mathbb{R}$ to be

$$\Delta f_i : = f_i([x_i^{n,p(n,i)}], - f_i, n, p, (i)([x_i^{n,p(n,i)}]). \quad (5.13)$$

Since $f_i, n, w(\cdot) \leq f_i(\cdot)$, we have $\Delta f_i \geq 0$.

**Lemma 5.1.** Recall the formulas of $\Delta z_i$ in (2.12) and $\Delta f_i$. For all $i \in V_4$, we have

$$\| \Delta z_i \| \leq \sqrt{\Delta f_i}. \quad (5.14)$$

**Proof.** Since $[x_i^{n,p(n,i)}]$ is the minimizer of $f_i, n, p, (i)(\cdot) + \frac{1}{2} \| p_i \|$, where $p_i$ is as in (5.2), and $x_i^{n,p(n,i)}$ is the minimizer of $f_i(\cdot) + \frac{1}{2} \| p_i \|$, we have

$$f_i, n, p, (i)([x_i^{n,p(n,i)}]) \leq f_i, n, p, (i)(\tilde{x}_i^{n,p(n,i)}) + \frac{1}{2} \| x_i^{n,p(n,i)} - \tilde{x}_i^{n,p(n,i)} \| \leq f_i, n, p, (i)(\tilde{x}_i^{n,p(n,i)}) + \frac{1}{2} \| x_i^{n,p(n,i)} - \tilde{x}_i^{n,p(n,i)} \|$$

Rearranging inequality (5.15) and using (5.6) gives us what we need. \hfill $\square$

In view of Lemma 5.1 and the fact that $z_\alpha^{n,p(n,\alpha)} = \tilde{z}_\alpha^{n,p(n,\alpha)}$ for all $\alpha \in V_1 \cup V_2 \cup V_3 \cup \tilde{E}$, we have

$$\tilde{t}_\alpha - \hat{t}_\alpha \quad (5.16a) \quad (5.16a)$$

and

$$\|V_A^{n,\tilde{w}} - V_A^{n,\tilde{w}} \| \leq \sum_{i \in V_4} \| z_i^{n,\tilde{w}} - \tilde{z}_i^{n,\tilde{w}} \|. \quad (5.11)$$
Moreover,
\[
\sum_{i \in V_4} \| z_{n,i}^w - \hat{z}_{n,i}^w \| \leq \sum_{i \in V_4} \| \Delta z_i \| \leq \sum_{i \in V_4} \sqrt{\Delta f_i}. \tag{5.17}
\]

If \( \Delta f_i \) were arbitrarily large, then Lemma 2.9(1) would contradict the fact that the dual objective value is monotonically nonincreasing. Lemma 5.1 then shows that \( \Delta z_i \) is bounded.

Next, Theorem 2.8(iii) shows that \( z_{n,p_i}^{n,p_i} \) is bounded for all \( i \in V_3 \cup V_4 \). Along with the fact that \( \Delta z_i \) is bounded for all \( i \in V_4 \), we have \( z_{n,p_i}^{n,p_i} \) being bounded for all \( i \in V_4 \).

Let \( h^{n,w} = -F^n(\hat{z}^{n,0}) - (-F(z^*)) \), and let \( h^n \) be defined by
\[
h^n := h^{n,0} = -F^n(\hat{z}^{n,0}) - (-F(z^*)). \tag{5.18}
\]

Note that \( h^n \geq 0 \). From the fact that \( z_{n,p_i}^{n,p_i} \) are bounded, we have
\[
\| t_\alpha - \hat{t}_\alpha \| \leq \sum_{\beta \in V \cup E} \| z_{n,\beta}^{n,p(\alpha)} - z_{n,\beta}^{n,w} \| \leq \sum_{\beta \in V \cup E} \| z_{n,\beta}^{n,w-1} - z_{n,\beta}^{n,w} \| \leq \sqrt{w} \sum_{w=1}^{\tilde{w}} \| v_{A,n,w-1} - v_{A,n,w} \| ^2 \leq \sqrt{w} h^n - h^{n+1}.
\]

**Theorem 5.2.** Suppose that a dual optimizer \( z^* \) exists for (1.2). Suppose that the dual iterates \( \{z_{n,w}^\alpha\} \) are bounded for all \( \alpha \in V \cup \tilde{E} \), \( w \in \{1, \ldots, \tilde{w}\} \) and \( n \geq 1 \). Then we have the recurrence
\[
h^{n+2} \leq h^n - \gamma [h^{n+2}]^4.
\]

for some \( \gamma > 0 \), which together with Lemma 5.3 shows that the dual function value converges at a rate of \( O(1/n^{1/3}) \). Moreover, if \( V_2 = 0 \), we have the recurrence \( h^{n+1} \leq C_1 \sqrt{w} h^n - h^{n+1} \) for some \( C_1 > 0 \), which shows that the dual function value converges at a rate of \( O(1/n) \).

**Proof.** We make the following claim:

**Claim:** Throughout Algorithm 2.3, the quantities \( v_{A,n,w} \), \( x^{n,w} \) and \( \hat{v}_{A,n,w} \) are bounded.

We recall that Theorem 2.8(ii) implies that \( \{x^{n,w}\} \) is bounded. Since \( x^{n,w} \) is also bounded, from Lemma 2.9(1) and the nonincreasingsness of \( \{ -F(x^{n,w}) \} \) through Theorem 2.8(i), we deduce that \( \Delta f_i \) is bounded. From (5.17) and (5.16b), we can deduce that \( \{v_{A,n,w}\} \) is bounded.

Note that \( \{z_{n,w}^\alpha\} \) is assumed to be bounded for all \( \alpha \in V \cup \tilde{E} \) in the theorem statement. Combining with the claim above with (5.16), (5.17) and (5.19) shows us that there are nonnegative constants \( C_1, C_2 \) and \( C_3 \) such that
\[
h^{n+1} \leq -[F^{n,w}(z_{n,w}^\alpha) - F(\hat{z}_{n,w}^\alpha)] - [F(\hat{z}_{n,w}^\alpha) - F(z^*)] \leq C_1 \sqrt{w} h^n - h^{n+1} + \sum_{i \in V_4} [C_2 \Delta f_i + C_3 \sqrt{\Delta f_i}]. \tag{5.20}
\]
We now address the last statement of the theorem first. When \( V_4 = \emptyset \), the formula (5.20) is reduced to \( h^{n+1} \leq C_1 \sqrt{\bar{w}} \sqrt{h^n - h^{n+1}} \), which shows that \( h^n \geq h^{n+1} + C_1^2 \bar{w} [h^{n+1}]^2 \). This recurrence would give us an \( O(1/n) \) convergence rate from [BT13, Lemma 3.5].

We now continue to proving our main result. Let \( L = \max_{i \in V_4} L_i \). From Lemma 2.9(1) being applied to all coordinates \( i \) in \( V_4 \), we have the constraints

\[
h^{n+1,1} \leq h^{n+1,0} - \frac{1}{2} \sum_{i \in V_4} \theta_i^2,
\]

and

\[
\frac{1}{2(L+1)^2} \theta_i^2 + \theta_i \geq f_i([x^{n+1,0}], f_{i,n,w}([x^{n+1,0}], i)) \text{ for all } i \in V_4.
\]

Due to Assumption 2.7(2) and (5.13), the right hand side of (5.22) is \( C \). This gives

\[
\sum_{i \in V_4} \Delta f_i \leq \sqrt{\Delta f_i} (C \sqrt{C_6} + C_3).
\]

Also,

\[
h^{n+2} \leq h^{n+1} - \frac{1}{2} \sum_{i \in V_4} \theta_i^2 \leq h^{n+1} - \frac{1}{2(L+1)^2} \sum_{i \in V_4} \{|\Delta f_i|^2 \leq h^{n+1} - \frac{1}{2C_1^2 |V_4|} \sum_{i \in V_4} \Delta f_i^2}.
\]

Let \( C_7 = C_2 \sqrt{C_6} + C_3 \). We have,

\[
h^{n+1} \leq C_1 \sqrt{\bar{w}} \sqrt{h^n - h^{n+1}} + \sum_{i \in V_4} C_7 \sqrt{\Delta f_i} \leq C_1 \sqrt{\bar{w}} \sqrt{h^n - h^{n+1}} + C_7 \sqrt{|V_4|} \sqrt{\sum_{i \in V_4} \Delta f_i}.
\]

We now split into two cases to find a recurrence.

**Case 1:** If \( h^{n+1} \leq 2C_1 \sqrt{\bar{w}} \sqrt{h^n - h^{n+1}} \), then \( h^n \geq h^{n+1} + \frac{1}{4C_1^2 \bar{w}} [h^{n+1}]^2 \). There is some \( \bar{h} \) such that \( h^n \geq \bar{h} \) for all \( n \), which gives

\[
h^n \geq \bar{h}^{n+1} + \frac{1}{4C_1^2 \bar{w}} [\bar{h}^{n+1}]^2 \geq \bar{h}^{n+2} + \frac{1}{4C_1^2 \bar{w}} [\bar{h}^{n+2}]^4.
\]

**Case 2:** If \( h^{n+1} \geq 2C_1 \sqrt{\bar{w}} \sqrt{h^n - h^{n+1}} \), then

\[
\sqrt{\sum_{i \in V_4} \Delta f_i} \geq \frac{1}{C_7 \sqrt{|V_4|}} [h^{n+1} - C_1 \sqrt{\bar{w}} \sqrt{h^n - h^{n+1}}] \geq \frac{1}{2C_1 \sqrt{|V_4|}} h^{n+1}.
\]

This gives

\[
h^{n+2} \leq h^{n+1} - \frac{1}{32C_1^2 |V_4|^2 C_7^2} [h^{n+1}]^4 \leq h^n - \frac{1}{32C_1^2 |V_4|^2 C_7^2} [h^{n+2}]^4.
\]

The recurrences (5.27) and (5.29) ensure that the conclusion holds.

The following result is adapted from the techniques in [BT13, Bec15].

**Lemma 5.3.** Suppose that a nonnegative sequence \( \{a_k\}_{k=1}^\infty \) has the recurrence \( a_k \geq a_{k+1} + \gamma a_{k+1}^4 \). Then \( a_k \leq \left( \frac{1}{\gamma^4} + (k - 1)3\gamma \right)^{1/3} \) for all \( k \geq 1 \), which means that \( \{a_k\}_k \) has a \( O(1/k^{1/3}) \) rate of convergence.
Proof. We have

\[
\frac{1}{a_k^2} - \frac{1}{a_{k+1}^2} = \frac{a_{k+1}^2 - a_k^2}{a_k^2 a_{k+1}^2} = \frac{(a_{k+1} - a_k)(a_{k+1}^2 + a_k a_{k+1}^2)}{a_k^2 a_{k+1}^2} \\
\geq \frac{\gamma a_{k+1}^2 - \gamma a_k^2}{a_k^2 a_{k+1}^2} \geq 3\gamma \frac{a_{k+1}^2 - a_k^2}{a_k^2}.
\]

Another bound is

\[
\frac{1}{a_k^2} - \frac{1}{a_{k+1}^2} = \frac{1}{a_k^2} \left(1 - \frac{a_{k+1}^2}{a_k^2}\right) \geq \frac{1}{a_k^2} \left(1 - \frac{a_{k+1}^2}{a_k^2}\right).
\]

It is elementary to calculate that \(\max \left\{3\gamma \frac{a_{k+1}^2}{a_k^2}, \frac{1}{a_k^2} \left(1 - \frac{a_{k+1}^2}{a_k^2}\right)\right\}\) has a minimum value of \((3\gamma a_k^2 + 1)^{-1} 3\gamma\) attained at \(a_{k+1}^2 = (3\gamma a_k^2 + 1)^{-1}\). Then

\[
\frac{1}{a_k^2} \geq \frac{1}{a_k^2} + (k - 1)3\gamma (3\gamma a_k^2 + 1)^{-1},
\]

which gives us the required conclusion.

As a corollary of Theorem 4.2, we have the following.

Corollary 5.4. Suppose that the condition (4.1) holds. Then the iterates \(z_i^{n,w}\) of Algorithm 2.3 are bounded, which implies that Theorem 5.2 can be applied to show the \(O(1/n^{1/5})\) convergence rate of Algorithm 2.3.

6. Numerical experiments

We present our numerical experiments. Since a distributed optimization algorithm is designed to handle the distributed nature of the data and keeping the communications between the nodes low, a distributed algorithm would converge less quickly than a comparable centralized algorithm. So we aim only to verify the theoretical rates obtained in this paper.

Since the distributed Dykstra’s algorithm extends the averaged consensus algorithm, the kind of graph that the distributed Dykstra’s algorithm does best in is one where the degree of each node is relatively high so that each node can actively seek neighbors to average their primal variable with (which occurs when \(S_{n,m}\) is the edge connecting the two nodes). Nevertheless, we are keeping our experiments simple by looking at the graph where \(|V| = 5\) and \(E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}\}\).

We look at the setting of Example 1.1 where \(X_i = \mathbb{R}^m\) and \(m = 4\) for all \(i \in V\), and look at halfspaces of the form

\[
H_{(i,j)} = \{x \in X : [x]_i = [x]_j\}
\]

instead of the halfspaces \(H_{(i,j,k)}\) defined in (1.4) to simplify computations. Let \(e\) be \(\text{ones}(m, 1)\). First, we find \(\{v_i\}_{i \in V}\) and \(\bar{x}\) such that \(\sum_{i \in V} v_i + |V|(e - \bar{x}) = 0\). We then find closed convex functions \(f_i(\cdot)\) such that \(v_i \in \partial f_i(e)\). It is clear from the KKT conditions that \(e\) is the primal optimum solution to (1.5) if \([x]_i\) are all equal to \(\bar{x}\) for all \(i \in V\).

The \(f_i(\cdot)\) can be defined as either smooth or non-smooth functions, or as the indicator functions of level sets of smooth or non-smooth functions. They are described using some Matlab functions below.

\[
(F-S) \quad f_i(x) \coloneqq \frac{1}{2} x^T A_i x + b_i^T x + c_i,
\]

where \(A_i\) is of the form \(vv^T + rI\), where \(v\) is generated by \(\text{rand}(m, 1)\), \(r\) is generated by \(\text{rand}(1)\). \(b_i\) is chosen to be such that \(v_i = \nabla f(e)\), and \(c_i = 0\).
Our code is equivalent to $\bar{w} = 8$ with 

$$S_{n,1} = \{(1, 2)\}, S_{n,2} = \{1, 2\}, S_{n,3} = \{(1, 3)\}, S_{n,4} = \{1, 3\},$$

$$S_{n,5} = \{(1, 4)\}, S_{n,6} = \{1, 4\}, S_{n,7} = \{(1, 5)\}, \text{ and } S_{n,8} = \{1, 5\}.$$ 

From the analysis in [Pan18a, Pan18b] (which traces its origins to [GM89]), the duality gap is bounded from below by 

$$0 \leq \frac{1}{2} \|x^{n,w} - x^*\|^2 \leq \frac{1}{2} \|x^* - x\|^2 + \sum_{i \in V} f_i(x) - F^{n,w}((z_\alpha)_{\alpha \in E \cup V}). \quad (6.1)$$

We will keep track of the values of $\frac{1}{2} \|x^{n,w} - x^*\|^2$ and the duality gap as marked. Note that the duality gap is monotonically nonincreasing.

We now report on the results of the numerical experiments, starting with the case of smooth functions and see the effect of treating the smooth functions $f_i(\cdot)$ as subdifferentiable functions (i.e., being in $V_1$) and as proximable functions (i.e., being in $V_1$). The theory in our paper suggests linear convergence, which was observed. One might expect that if we treat the $f_i(\cdot)$ as proximable functions, the dual objective value in (1.3) or its lower estimate (2.2) converges to the optimal value faster. While this is mostly true, we have encountered settings where treating the smooth functions as subdifferentiable can give faster decrease in the dual objective value. We ran our experiments 40000 times, and in 5002 times, treating the smooth function as a subdifferentiable function results in a lower duality gap faster. While this is mostly true, we have encountered settings where treating the smooth functions as subdifferentiable can give faster decrease in the dual objective value. We ran our experiments 40000 times, and in 5002 times, treating the smooth function as a subdifferentiable function results in a lower duality gap by the 200th iteration. This is illustrated in the first diagram in Figure 6.1.

We now look at the nonsmooth case. For the case when we treat the functions as subdifferentiable functions, a plot of the duality gap over the iterations shows that the convergence rate of the duality gap to zero is $O(1/n)$, which coincides with the theory. (See Figure 6.1, 4th diagram.) We make two observations that cannot be predicted by our theory so far. The first observation we see is that $\frac{1}{2} \|x^{n,w} - x^*\|^2$ apparently converges to zero at the rate of $O(1/n^2)$. (Equivalently, $\|x^{n,w} - x^*\|$ converges to zero at the rate of $O(1/n)$.) See Figure 6.1, 5th diagram.) The next observation is that when we treat the functions as proximable, the duality gap and the distance to the optimal solution converges linearly to zero. We ran more than 300 experiments, and found that this linear convergence always holds. (See Figure 6.1, 2nd diagram.)

7. Conclusion

We proved what we have set out to do in Subsection 1.3. The linear convergence and $O(1/k)$ rates in Sections 3 and 4 cannot be improved to a faster rate, but it is unclear whether the $O(1/k^{1/3})$ rate in Section 3 is optimal. Indeed, our numerical experiments suggest a rate of $O(1/k)$, and there might be reasonable conditions leading to the linear convergence observed for the nonsmooth proximable case. These require further investigation.
Figure 6.1. This figure illustrates a sample run of our numerical experiments. The first diagram shows the rate of linear convergence for the smooth case when all (smooth) functions are treated as subdifferentiable functions and when all functions are treated as proximable functions. In this case, we see the anomalous case of when the algorithm actually runs faster when the functions are treated as subdifferentiable. (The last diagram shows what usually happens.) The second and third diagrams are for the nonsmooth case, and illustrate the linear convergence rate (at a much slower rate than the smooth case) when all functions are treated as proximable, and the sublinear convergence rate for when all functions are treated as subdifferentiable. The diagrams suggest that the duality gap converges at a rate of $O(1/n)$ (as is suggested by the theory) and $\frac{1}{2}\|x - x^*\|^2$ converges at a rate of $O(1/n^2)$ (which is not covered by the theory).

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