Exterior Algebras Related to the Quantum Group $\mathcal{O}(O_q(3))$

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Abstract

For the 9-dimensional bicovariant differential calculi on the quantum group $\mathcal{O}(O_q(3))$ several kinds of exterior algebras are examined. The corresponding dimensions, bicovariant subbimodules and eigenvalues of the antisymmetrizer are given. Exactly one of the exterior algebras studied by the authors has a unique left invariant form with maximal degree.

1 Introduction

A general framework of bicovariant differential calculus on quantum groups was given by the pioneering work of Woronowicz [1]. Bicovariant first order differential calculi on $q$-deformed simple Lie groups were constructed, studied, and classified by many authors, see [1, 2, 3, 4]. Recently several problems connected with higher order differential forms (exterior algebras) were studied. But only in case of the simplest examples like $GL_q(N)$ and $SL_q(N)$ the structures of the left-invariant and of the bi-invariant exterior algebras are known, see [3, 4, 5]. For the orthogonal and symplectic quantum groups even the existence of an analogue of the volume form, i.e. a single form of maximal degree, was open. There are two purposes of this paper. Firstly we show that for the $N^2$-dimensional bicovariant first order differential calculi on $O_q(N)$ Woronowicz’ left-invariant external algebra is infinite dimensional.

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In other words, for each positive integer $k$ there exists a nonzero $k$-form. Secondly, in case of $O_q(3)$ we discuss alternative constructions of exterior algebras which yield a finite differential complex. For several choices of the ideal of symmetric forms in $\Gamma^\otimes$, we compute the dimensions of spaces of left-invariant $k$-forms. Exactly one of those exterior algebras has a unique left invariant form of maximal degree.

Let $A$ be the Hopf algebra $O(O_q(N))$ as defined in [8]. The fundamental matrix corepresentation of $A$ is denoted by $\rho$. We use the symbol $\Delta$ for the comultiplication and Sweedler’s notation $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$. The two-sided ideal of an algebra generated by a set $\{a_i \mid i \in I\}$ is denoted by $\langle a_i \mid i \in I \rangle$. Let $v$ be a corepresentation of $A$. As usual $v^c$ denotes the contragredient corepresentation of $v$. The space of intertwiners of corepresentations $v$ and $w$ is $\text{Mor}(v, w)$. We write $\text{Mor}(v)$ for $\text{Mor}(v, v)$. Lower indices of a matrix $A$ always refer to the components of a tensor product where $A$ acts (‘leg numbering’). The unit matrix is denoted by $I$. As usual $\hat{R}$ and $C$ stand for the corresponding $R$-matrix and the metric, see [8].

2 Exterior Algebras

Let $A$ be a coquasitriangular Hopf algebra [9] with universal $r$-form $r$ and $v = (v^i_j)$ an arbitrary $n$-dimensional corepresentation of $A$. Let $\Gamma := \Gamma(v)$ be the corresponding bicovariant bimodule such that $\{\theta_{ij} \mid i, j = 1, \ldots, n\}$ is a basis of the vector space of left invariant 1-forms and

$$\Delta_R(\theta_{ij}) = \sum_{k,l=1}^n \theta_{kl} \otimes v^k_l (v^c)^l_j,$$

$$\theta_{ij} a = \sum_{k,l=1}^n a_{(1)} (v^k_i, a_{(2)}) r(a_{(3)}, v^j_l) \theta_{kl}.$$

Let $\Gamma^\otimes k$ denote the $k$-fold algebraic tensor product $\Gamma \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma$ ($k$ factors) of $\Gamma$ and $\Gamma^\otimes := \sum_{k=0}^\infty \Gamma^\otimes k$. Let $S$ be a bicovariant subbimodule and two-sided ideal of $\Gamma^\otimes$. Then $\Gamma^\wedge := \Gamma^\otimes / S$ is called an exterior algebra of $\Gamma$. Since $\Gamma^\otimes$ has a $\mathbb{Z}$-gradation, we require $S = \bigoplus_{k=0}^\infty S^k$. (Traditionally, the space $S$ of symmetric forms has some additional properties but we don’t want to consider them at the moment.) The general theory of bicovariant bimodules gives $\Gamma^\otimes k = A \Gamma^\otimes L$ and $S^k = A S^k_L$, where $\Gamma^\otimes L = \{\rho \in \Gamma^\otimes k \mid \Delta_L(\rho) = 1 \otimes \rho\}$.
and $S^k_L = \{ \rho \in S^k | \Delta_L(\rho) = 1 \otimes \rho \}$ are the corresponding left-invariant subspaces. In the nontrivial cases we have $S^0_L = S^1_L = \{0\}$.

Let $\sigma$ be the canonical braiding of $\Gamma \otimes \pi \Gamma$. Recall that $\sigma$ is a homomorphism of bicovariant bimodules. Let $\hat{R}^- = (\hat{R}^-_{ij})$ be the complex matrix with entries $\hat{R}^-_{ij} = r(v^i_k, v^c_j)$ and let $\hat{R}^-_i$ be this matrix acting on the $i$-th and $(i+1)$-th component of a vector. Let us define a $k$-twist $b_k$ by the formula

$$b_k := (\hat{R}^-_2 \hat{R}^-_4 \cdots \hat{R}^-_{2k-2})(\hat{R}^-_3 \cdots \hat{R}^-_{2k-3}) \cdots \cdots (\hat{R}^-_{k-1} \hat{R}^-_{k+1})\hat{R}^-_k.$$  \hspace{1cm} (1)

As an example we draw the corresponding pictures for $\hat{R}^-$ and $b_4$ in the language of braids.

\begin{center}
\includegraphics[width=\textwidth]{braids.png}
\end{center}

Since $\hat{R}^- \in \text{Mor}(v \otimes v^c, v^c \otimes v)$, we conclude that $b_k \in \text{Mor}(v^\otimes k v^c \otimes k, (v^c v^{}^\otimes)^\otimes k)$. Moreover, both $\hat{R}^-$ and $b_k$ are invertible.

Let $P$ be a projection onto a subspace of $\Gamma^\otimes k$. This subspace is $\Delta_R$-invariant iff $P \in \text{Mor}((v v^c)^\otimes k)$. For an endomorphism $T$ of the bicovariant bimodule $\Gamma^\otimes k$ let

$$\dot{T} := b^{-1}_k T b_k.$$  \hspace{1cm} (2)

If there are projections $P' \in \text{Mor}(v^\otimes k)$ and $P'' \in \text{Mor}((v^c)^\otimes k)$ such that $\hat{P} = P' \circ P''$ then $A \cdot P(\Gamma^\otimes k)$ is a bicovariant subbimodule of $\Gamma^\otimes k$ where

$$(P' \circ P'')^i_{j_1, \ldots, j_{2k}} = (P')^i_{j_1, \ldots, j_{k}}(P'')^i_{j_{k+1}, \ldots, j_{2k}}.$$

This method gives a large class (in some cases all) of bicovariant sub-bimodules of $\Gamma^\otimes k$ and the corresponding bimodule homomorphisms. From now on we suppose that

$$S^k = A b_k(\sum_i P'_i \circ P''_i)b^{-1}_k(\Gamma^\otimes k),$$

where $P'_i \in \text{Mor}(v^\otimes k)$ and $P''_i \in \text{Mor}(v^c \otimes k)$ are arbitrary projections.
3 Exterior Algebras for $\mathcal{O}(SL_q(N))$

Let us recall some results of [6]. If $A = \mathcal{O}(SL_q(N))$ and $v = u$ is the fundamental corepresentation, then there are two canonical methods to construct exterior algebras fitting in the above concept:

a) $S = \langle \ker(I - \sigma) \rangle$,

b) $S = \langle \ker A_k \mid k \geq 2 \rangle$.

Here $A_k$ denotes the $k$-th antisymmetrizer on $\Gamma^\otimes k$ defined in [1].

Both definitions give the same algebra $\Gamma^\wedge$. The dimension of the vector space $\Gamma^\wedge_k$ is $\binom{N^2}{k}$, hence there is a unique left invariant form with maximal degree. The elements of $\Gamma^\wedge_k$ can be given by

$$Ab_k(\sum_i P_i \odot \tilde{P}_i)b_k^{-1}(\Gamma^\otimes L),$$

where the projections $(P_i \odot \tilde{P}_i)$ have multiplicity 1 and $\tilde{P}_i$ is the projection ‘conjugated’ to $P_i$.

4 Woronowicz’ external algebra for $\mathcal{O}(O_q(3))$

Throughout the section we fix a positive integer $k$. Set $\epsilon = 0$ if $k$ is even and $\epsilon = 1$ if $k$ is odd.

**Theorem.** Let $A$ be the quantum group $O_q(N)$, $\Gamma$ one of the $N^2$-dimensional bicovariant first order differential calculi $\Gamma_+$ or $\Gamma_-$ on $A$ and $q$ be transcendental. Let $\Gamma^\wedge$ denote Woronowicz’ external algebra over $\Gamma$.

Then for each positive integer $k$ there exists a nonzero $k$-form in $\Gamma^\wedge$. In other words $\Gamma^\wedge$ is an infinite differential complex.

We sketch the main steps of the proof. From [4] and [2] and for $k = 3$ we obtain for example

$$\hat{\sigma}_{12} = \left\langle \begin{array}{c} \circ \ \odot \ \circ \end{array} \right\rangle, \quad \hat{\sigma}_{23} = \left\langle \begin{array}{c} \circ \ \odot \ \circ \ \odot \end{array} \right\rangle.$$

Similarly, Woronowicz’ formula for $A_k$ gives

$$A_k := \sum_{w \in S_k} (-1)^{i(w)} T_w \odot T_w^c,$$

where $T_w = \hat{R}_{i_1} \hat{R}_{i_2} \cdots \hat{R}_{i_m}$, $T_w^c = \hat{R}_{i_1}^c \cdots \hat{R}_{i_m}^c$, and $w = s_{i_1} \cdots s_{i_m}$ is an expression of the permutation $w$ of length $m$ into a product of $m$ nearest neighbour
transpositions $s_j$. Note that $\hat{R}^- \in \text{Mor}(u^c \otimes u^c)$, $\hat{R}^-_{rs} = \hat{R}^{-1}_{sr}$. We construct a nonzero vector $t_k$ in $V^{\otimes 2k}$, $V = \mathbb{C}^N$, with $\hat{A}_kt_k = \tau_k t_k$, where $\tau_k$ is a nonzero eigenvalue of $\hat{A}_k$. Using the decomposition of $u^{\otimes k}$ (resp. of $u^{c\otimes k}$) into irreducible subcorepresentations we get a decomposition of the right coaction of $\mathcal{A}$ on $\Gamma^{\otimes k}$ into smaller components $\pi_{\lambda} \otimes \tilde{\pi}_\mu$. Here $\lambda$ is a partition of $k - 2f$, $f = 0, \ldots, [k/2]$, and $\pi_{\lambda}$ (resp. $\tilde{\pi}_\mu$) stands for the irreducible subcorepresentation of $u^{\otimes k}$ (resp. of $u^{c\otimes k}$) determined by the Young diagram $\lambda$ (resp. $\mu$).

**Step 1.** Throughout we only consider the case $\lambda = (e)$ and $\mu = (k)$. The corresponding minimal central idempotents of $\text{Mor}(u^{\otimes k})$ (resp. of $\text{Mor}(u^{c\otimes k})$) are denoted by $z_{(e)}$ (resp. by $\tilde{z}_{(k)}$). Using $\tilde{z}_{(k)} \hat{R}^- = q^{-1} \tilde{z}_{(k)}$, the antisymmetrizer $\hat{A}_k$ reduces to

$$\hat{A}_k(z_{(e)} \otimes \tilde{z}_{(k)}) = (z_{(e)} \otimes \tilde{z}_{(k)}) \sum_{w \in S_k} (-q)^{-\ell(w)} T_w \otimes 1.$$ 

Hence it suffices to work with $\overline{A}_k := z_{(e)} \sum_{w \in S_k} (-q)^{-\ell(w)} T_w$ in the simple component $z_{(e)} \text{Mor}(u^{\otimes k})$ of $\text{Mor}(u^{\otimes k})$. Obviously, $\overline{A}_k$ acts on $M_k := \text{Mor}(u^e, u^{\otimes k})$ by composition (here $u^0 = 1$ denotes the trivial corepresentation).

**Step 2.** We define elements $e_m \in \text{Mor}(u^e, u^{\otimes k})$ and $e^m \in \text{Mor}(u^{\otimes k}, u^e)$, $m = 1, \ldots, k$ by $e^1 = e_0 = I$, $e_2 = (C^{ab})$, $e^2 = ((C^{-1})_{ab})$ and by recursion formulae $e_{k+2} = e_k \otimes e_2$ and $e^{k+2} = e^{k+2} \otimes e^2$. Set $\alpha_1 = 1$, $\alpha_2 = 1 - r^{-1}q^{-1}$, and $r = q^{N-1}$. Further, we abbreviate $\alpha_k = \beta_{k-1} \alpha_{k-1}$, where $\beta_{k-1} = (1 + \cdots + q^{-2k+2}) \alpha_2$ for $k$ even and $\alpha_k = \alpha_{k-1}$ for $k$ odd. Let $t_k = \alpha_k^{-1} \overline{A}_k e_k$.

**Lemma.** (i) The antisymmetrizer $\overline{A}_k$ has rank one on the module $M_k$. The unique up to scalars image is $\tau_k$. The minimal polynomial of $\overline{A}_k$ on $M_k$ is

$$\overline{A}_k(\overline{A}_k - \tau_k) = 0.$$

(ii) We have

$$(e^2)_{i,i+1}t_k = \gamma_k t_{k-2}, \quad i = 1, \ldots, k-1,$$

where $\gamma_k = \alpha_2(q - q^{-1})^{-1}(q + q^{-k+1}r)$ for $k$ odd and $\gamma_k = \gamma_{k-1}$ for $k$ even.

From (ii) it follows that $e^k t_k = \gamma_k \gamma_{k-2} \cdots \gamma_e$ is nonzero. Consequently, $t_k$ is a nonzero vector. Note that for the quantum group $Sp_q(N)$ this is no
longer true since $r = -q^{N+1}$. Then we have $\gamma_N = 0$ and moreover $t_N = 0$.

Step 3. It remains to prove that the eigenvalue $\tau_k$ of $A_k$ to the eigenvector $t_k$ is nonzero: we compute for each $k$ a polynomial $p_k(x, y)$ with integer coefficients such that $\tau_k = p_k(q^{-1}, r^{-1})$, $p_k(0, 0) = 1$, and $p_k(1, 1) = 0$. Since $q$ is transcendental, $\tau_k \neq 0$. The explicit values for small $k$ are

$$
\begin{align*}
\tau_2 &= 1 - q^{-1}r^{-1}, \\
\tau_3 &= 1 + 2q^{-2} - 2q^{-1}r^{-1} - q^{-3}r^{-1}, \\
\tau_4 &= (1 + q^{-2})(1 - q^{-1}r^{-1})\tau_3, \\
\tau_5 &= (1 + q^{-2})(1 + 3q^{-2} + 6q^{-4} + 5q^{-6}) \\
&\quad - r^{-1}(4q^{-1} + 11q^{-3} + 11q^{-5} + q^{-7}) + r^{-2}(5q^{-2} + 6q^{-4} + 3q^{-6} + q^{-8}).
\end{align*}
$$

5 Further Bicovariant Bimodules for $O(O_q(3))$

Now let $\mathcal{A} = O(O_q(3))$ and $v = u$. Further, we suppose that $q$ is a transcendental complex number. The matrix of the braiding $\sigma$ with respect to the basis given above is of the form $\sigma = b_2(\hat{R} \odot \hat{R}^{-1})b_2^{-1}$. It has 7 eigenvalues (see also [3]): $1, q^3, q^{-3}, -q^2, -q^{-2}, -q$, and $-q^{-1}$. We have $u^c \cong u$ and $\text{Mor}(u^\otimes k)$ is a factor algebra $B_k$ of the Birman-Wenzl-Murakami algebra. The algebra $B_2$ has three projections: $P_+, P_-$, and $P_0$. Let us first consider Woronowicz’ external algebra.

a) $S_{L,1} := (\ker A_k \mid k \geq 2)$. From the previous section we have $\dim \Gamma^\wedge_{L,1} = \infty$.

The following table (which is valid for $O(O_q(N))$ and $O(Sp_q(N))$ as well) gives the projections onto nonzero bicovariant subbimodules of $\Gamma^\wedge_{L,1}$ and the corresponding eigenvalues of $A_3$. 

6
Here we used the $q$-numbers $[2] = q + 1/q$, $[3] = q^2 + 1 + q^{-2}$ and the abbreviations $Q = q - 1/q$, $r = q^{N-1}$ for $O(O_q(N))$ and $r = -q^{N+1}$ for $O(Sp_q(N))$. The complex numbers $\lambda_i$, $i = 1, 2, 3, 4$ are zeros of the equations

\[
\lambda_i^2 - 2((r - 1/r)^2 + Q(Q^2 + 1)(r - 1/r) + 2Q^2) = 0,
\]

\[
\lambda_{3,4}^2 - 2([3] - Q(r - 1/r))\lambda_{3,4} - 2(r - 1/r)((r - 1/r) + Q[3]) = 0.
\]

In particular if we only require that $q$ is not a root of unity the vector spaces $S_{L,1}^3$ may have different dimensions for different $q$'s.

b) $S_{L,2} := \langle \ker(I - \sigma) \rangle$.

With our formulation it means

$S_{L,2} = \langle b_k(P_+ \circ P_+ + P_- \circ P_+ + P_0 \circ P_0)b_k^{-1}(\Gamma_L^{\otimes 2}) \rangle$.

The dimensions of $\Gamma_{L,2}^{\wedge k} = \Gamma^{\otimes k}/S_{L,2}^k$, $k \leq 6$, are

| $k$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
| dim | 1   | 9   | 46  | 183 | 628 | 1938| 5514|

7
The computations were carried out with the help of the computer algebra program FELIX \cite{10}. Using the table in the case $a$ it is easy to see that $\dim \Gamma_{L,2}^\wedge = \dim \Gamma_{L,1}^\wedge$ for $k \leq 3$. Obviously, $\dim \Gamma_{L,2}^\wedge \geq \dim \Gamma_{L,1}^\wedge = \infty$ by the previous section.

Let us give some motivation for the next definition.

The bicovariant bimodules for a coquasitriangular Hopf algebra described above admit a second braiding: \[
\tilde{\sigma} = b_2(\hat{R} \circ \check{R})b_2^{-1},
\]
where $\hat{R}_{kl}^{ij} = r(u_k^j, u_l^i)$ and $\check{R}_{kl}^{ij} = r(u_c^j_k, u_c^i_l)$. For $A = \mathcal{O}(SL_q(N))$ the matrix $\tilde{\sigma}$ has eigenvalues $-1$, $q^2$, and $q^{-2}$ and
\[
\ker(I - \sigma) = \text{im}(I + \tilde{\sigma}).
\]

For $A = \mathcal{O}(O_q(3))$ the operator $\tilde{\sigma}$ commutes with $\sigma$ and its eigenvalues are $q^2$, $q^{-2}$, $q^{-4}$, $q^{-1}$, $-1$, $-q^{-3}$.

$c) \quad \mathcal{S}_{L,3} := \langle \text{im}(I + \tilde{\sigma}) \rangle$. This definition is equivalent to $\mathcal{S}_{L,3} = \langle \ker(I - \sigma), \ker(q^3 - \sigma), \ker(q^{-3} - \sigma), \ker(q + \sigma), \ker(1/q - \sigma) \rangle$. The dimension of the vector space $\Gamma_{L,3}^\wedge = \Gamma_{L,3}^\otimes$ is given by the following table:

| $k$  | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| dim | 1 | 9 | 30 | 39 | 0 |

The corresponding projections are

$k = 0 : \emptyset \circ \emptyset$

$k = 1 : \Box \circ \Box$

$k = 2 : \Box \circ \Box + \Box \circ \Box$

$k = 3 : \Box \circ \Box + \Box \circ \Box + \Box \circ \Box$

It was already suggested by Carow-Watamura et al. \cite{2} to prefer the following choice:

$d) \quad \mathcal{S}_{L,4} := \langle \ker(I - \sigma) \oplus \ker(q^3 - \sigma) \oplus \ker(q^{-3} - \sigma) \rangle.$
Then the dimension of $\Gamma^{\wedge k}_{L,4} = \Gamma^{\otimes k}_{L}/S^k_{L,4}$ becomes

| $k$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|-----|-----|-----|-----|-----|-----|-----|
| dim | $1$ | $9$ | $36$ | $54$ | $1$ | $0$ |

and the corresponding projections are

$k = 0 : \emptyset \otimes \emptyset$

$k = 1 : \begin{array}{c} \square \otimes \square \end{array}$

$k = 2 : \begin{array}{c} \square \square \otimes \square + \square \otimes \square \square \otimes \emptyset + \emptyset \otimes \emptyset \otimes \square \end{array}$

$k = 3 : \begin{array}{c} \square \square \square \otimes \square + \square \square \otimes \square \square + \square \otimes \square \square \square + \emptyset \otimes \emptyset \otimes \square + \square \otimes \emptyset \otimes \square \end{array}$

$k = 4 : \emptyset \otimes \emptyset$

The radical $\mathcal{R}$ of $\Gamma^{\wedge}_{L,4}$,

$$\mathcal{R} = \{ \rho \in \Gamma^{\wedge}_{L,4} \mid \rho \wedge \theta_{ij} = 0 \text{ for all } i, j = 1, 2, 3 \}$$

is non-trivial: we have $\mathcal{R} \subset \Gamma^{\wedge 3}_{L,4}$ and $\dim \mathcal{R} = 45$. Hence we can define $\tilde{\Gamma}_L := \Gamma^{\wedge}_{L,4}/\mathcal{R}$ and obtain the dimensions

| $k$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|-----|-----|-----|-----|-----|-----|-----|
| dim | $1$ | $9$ | $36$ | $9$ | $1$ | $0$ |

Recall that for the $4D_{4r}$-calculi on $\mathcal{O}(SL_q(2))$ the left-invariant external algebra due to Woronowicz is finite dimensional and has a unique left-invariant form of maximal degree $4$ as well. On the other hand, $\mathcal{O}(O_q(3))$ is isomorphic to a subalgebra of $\mathcal{O}(SL_q(2))$ and the fundamental bicovariant bimodule on $\mathcal{O}(O_q(3))$ examined in this paper can be obtained from the $9$-dimensional one on $\mathcal{O}(SL_q(2))$ (determined by the $3$-dimensional corepresentation of the latter, see [4]). Moreover, the corresponding exterior algebras have the same dimensions, since the bicovariant comodules of left-invariant symmetric $k$-forms are isomorphic for all $k$. Is this the reason, why the volume form has degree $4$? What happens for $\mathcal{A} = \mathcal{O}(SL_q(2))$ and the other bicovariant bimodules (differential calculi)? Is there a unique form of
maximal degree for other quantum groups such as $\mathcal{O}(O_q(N))$ or $\mathcal{O}(Sp_q(N))$, $N \geq 4$?

Let us conclude with two conjectures: firstly, Woronowicz’ left-invariant external algebra is infinite dimensional for the $N^2$-dimensional bicovariant differential calculi on $\mathcal{O}(Sp_q(N))$. Secondly, let $\Gamma$ be an $N^2$-dimensional bicovariant differential calculus on $\mathcal{O}(O_q(N))$ or $\mathcal{O}(Sp_q(N))$, $N \geq 3$. Then for transcendental values of $q$ the two-sided ideals $\langle \ker(I - \sigma) \rangle$ and $\langle \ker A_k | k \geq 2 \rangle$ of the algebra $\Gamma^\otimes$ coincide.

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