Phase transition in an exactly solvable reaction-diffusion process

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We study a non-conserved one-dimensional stochastic process which involves two species of particles \(A\) and \(B\). The particles diffuse asymmetrically and react in pairs as \(A\emptyset \leftrightarrow AA \leftrightarrow BA \leftrightarrow A\emptyset\) and \(B\emptyset \leftrightarrow BB \leftrightarrow AB \leftrightarrow B\emptyset\). We show that the stationary state of the model can be calculated exactly by using matrix product techniques. The model exhibits a phase transition at a particular point in the phase diagram which can be related to a condensation transition in a particular zero-range process. We determine the corresponding critical exponents and provide a heuristic explanation for the unusually strong corrections to scaling seen in the vicinity of the critical point.

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I. INTRODUCTION

One-dimensional driven-diffusion systems have been a subject of study in recent years because they exhibit interesting properties such as non-equilibrium phase transitions \([1]\). These systems have many applications in different fields of physics and biology \([2, 3]\). A well-known example is the Asymmetric Simple Exclusion Process (ASEP), which is studied experimentally by optical tweezers \([4, 5]\).

Various approaches have been developed in order to solve such systems exactly, including for example the matrix product method. With the matrix product method, the steady-state weight of a configuration is written as the trace of a product of operators corresponding to the local state of each lattice site. The operators obey certain algebraic rules which are derived from the dynamics of the model \([6]\). The algebraic relations among these operators might have finite or infinite dimensional matrix representations \([7, 8]\). Recently the matrix product method with quadratic algebras attracted renewed attention as it can also be applied to dissipative quantum systems \([9, 10]\).

It is well known that one-dimensional systems with open boundary conditions, in which the particle number is not conserved at the boundaries, can exhibit a phase transition \([11]\). On the other hand a phase transition may also take place in systems with non-conserving dynamics in the bulk \([12, 13]\). For example, in Ref. \([13]\) the authors have studied a three-states model on a lattice with periodic boundary conditions with two particle species which evolve by diffusion, creation and annihilation. By changing the annihilation rate of the particles, this model displays a transition from a maximal current phase to a fluid phase.

As shown in \([14]\) it is possible to map a one-dimensional driven-diffusive system defined on a periodic lattice onto a so-called zero-range process (ZRP). Recently this mapping was used to study various models which have an exact solution in the steady state \([15, 16]\). It was shown that a phase transition in the original model corresponds to a condensation transition in the corresponding ZRP.

In present work, we introduce and study an exactly solvable one-dimensional driven-diffusive model with non-conserved dynamics which exhibits an interesting type of phase transition. The model is defined on a ring of \(L\) sites which can be either empty (denoted by a vacancy \(\emptyset\)) or occupied by a one particle of type \(A\) or type \(B\). The system evolves random-sequentially according to a set of two-site processes which can be written in the most general form as

\[
I\emptyset \xrightarrow{\alpha_+} I\emptyset, \quad IK \xrightarrow{\beta_+} JK, \quad IJ \xrightarrow{\omega_+} J\emptyset, \quad (1)
\]

where \(I, J, K \in \{A, B\}\). In what follows we study a special case of this model defined by the processes

\[
A\emptyset \xrightarrow{\alpha} AA, \quad B\emptyset \xrightarrow{\alpha} BB
\]

\[
AB \xrightarrow{p_+} BB, \quad AA \xrightarrow{p_-} BA
\]

\[
AB \xrightarrow{\alpha} BB, \quad A\emptyset \xrightarrow{\alpha} BA
\]

\[
A\emptyset \xrightarrow{\alpha} AA, \quad B\emptyset \xrightarrow{\alpha} BB
\]

where the rates \(\alpha\) and \(p\) are given by the ratios

\[
\alpha = \frac{\alpha_+}{\alpha_-}, \quad p = \frac{p_+}{p_-}. \quad (3)
\]

As we will see below, for this particular choice the model turns out to be exactly solvable. Obviously, this defines a non-conserved dynamics, allowing the number of particles \((N_A\) and \(N_B)\) and vacancies \((N_\emptyset)\) to fluctuate under the constraint \(L = N_A + N_B + N_\emptyset\). Moreover, the model is a driven system since diffusion and reaction processes are not left-right symmetric. The dynamical rules \([1]\) is extensible to an exactly solvable model with the various types of particles, in which a phase transition is accessible. A generalized model consisting of three species of particles is presented in the Appendix A.

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In this paper we demonstrate that the model defined in (2) exhibits a phase transition and that its stationary state can be determined exactly by means of the matrix product method. In Sect. IV we show that our model can be mapped onto a ZRP and that the phase transition corresponds to a condensation transition in the ZRP. In Sect. V we study the dynamical behavior, which is not part of the exact solution, by numerical simulations. It turns out that the dynamical behavior near the critical point in the final result. We find that the algebra (6) to 0, indicating first-order behavior, while $\rho_B$ changes continuously as in a second-order phase transition.

To give a first impression how the process behaves in different parts of the phase diagram, we show various typical snapshots of the space-time evolution in Fig. 2. For $p = 0$ the density of $B$-particles (blue pixels) is very low while the $A$-particles (red pixels) form fluctuating domains with a high density. As we will see in the last section, these sharply bounded domains are important for a qualitative understanding of the phase transition.

For $\alpha < 2$ the $A$-particles eventually fill the entire system while for $\alpha > 2$ the $A$-domains almost disappear, leaving diffusing $B$-particles behind. For $p > 0$ one can see that $B$-particles are continuously generated. Thus the parameter $\alpha$ controls the domain size of $A$-particles while the parameter $p$ controls the creation and therewith the density of $B$-particles.

III. EXACT RESULTS

The matrix product method is an important analytical tool developed in the 90’s to compute the steady-state of driven diffusive systems exactly [6]. Let us now investigate the stationary state of the model by using this method. We consider a configuration $C = \{\tau_1, \ldots, \tau_L\}$ with $\tau_i \in \{\emptyset, A, B\}$ on a discrete lattice of length $L$ with periodic boundary condition. According to this method, the stationary state weight of a configuration $C$ is given by the trace of a product of non-commuting operators $X_i$:

$$W(C) = \text{Tr} \left[ \prod_{i=1}^{L} X_i \right].$$  \hspace{1cm} (5)

Note that this method differs from the well-known transfer matrix method in so far as different matrices are used depending on the actual configuration of the lattice sites, i.e. the choice of the operator $X_i$ at site $i$ depends on its local state. In our model, the operator $X_i = E$ stands for a vacancy while $X_i = A(B)$ represents a particle of type $A (B)$. Depending on the dynamical rules, these operators should satisfy a certain set of algebraic relations. For the dynamical rules listed in (2) one obtains a quadratic algebra of the form

$$p_- BA + AE = (1 + p_+)AA$$
$$p_- BB + \alpha_- BE = (\alpha_+ + p_+)AB$$
$$p_+ AA + p_+ AE = 2p_- BA$$
$$p_+ AB + pBE = (p_- + \alpha)BB$$
$$p_- BA + AA - (\alpha_+ + p_+)AE = A\bar{E}$$
$$\alpha_+ AB + \alpha BB - (p + 2\alpha_+ - 1)BE = B\bar{E}$$
$$\alpha_+ AE - EA = -E\bar{A}$$
$$\alpha_- BE - EB = -E\bar{B}$$
$$E\bar{E} - E\bar{E} = 0$$

where $\bar{E}$ is an auxiliary matrix which is expected to cancel out in the final result. We find that the algebra [9]...
has a two-dimensional matrix representation given by the following matrices

\[
A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = p \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}
\] (7)

and \( \bar{E} = E - \alpha \cdot I \), where \( I \) is an identity 2 \( \times \) 2 matrix.

We note that the algebra (6) and its representation (7) were studied for the first time by Basu and Mohanty in Ref. [17] in the context of a different model. It differs from our one in so far as it evolves only according to the processes in the first two lines of (2), where the \( A \) and \( B \)-particles hop with different rates and can also transform into each other, meaning that the total number of particles is conserved. The authors calculated the spatial correlations exactly and mapped their model to a ZRP. However, as the particle number is conserved in their model, a phase transition does not occur by changing the rates. In other words, although the matrix algebra already contains information about the phase transition, their model could not access the part of the phase diagram where the transition takes place. The model presented here is an extension of their model with the same matrix representation but with a non-conserved dynamics and an extended parameter space, in which the phase transition becomes accessible.

To compute the partition sum of the system, we first note that according to (2) a configuration without a particle of type \( A \) or \( B \) is not dynamically accessible. Therefore, the partition function, defined as a sum of the weights of all available configurations with at least one particle, is given by

\[
Z_L = \text{Tr}[(A + B + E)^L - E^L].
\] (8)

With this partition sum the stationary density of the \( A \) and \( B \)-particles can be written as

\[
\rho_A^{\text{stat}} = \frac{\text{Tr}[A(A + B + E)^{L-1}]}{Z_L},
\] (9)

\[
\rho_B^{\text{stat}} = \frac{\text{Tr}[B(A + B + E)^{L-1}]}{Z_L}.
\] (10)

We can also compute the density of the vacancies using \( \rho_0^{\text{stat}} = 1 - (\rho_A^{\text{stat}} + \rho_B^{\text{stat}}) \). Using the representation (7) the equations (8)-(10) can be calculated exactly. In the thermodynamic limit \( L \to \infty \), where high powers of matrices are dominated by their largest eigenvalue, the density of the \( A \) and \( B \)-particles is given by (see Fig. 1)

\[
\rho_A^{\text{stat}} = \frac{(2 - \alpha)(\alpha + p) + \alpha \sqrt{4 - 4\alpha + (p + \alpha)^2}}{2(2\alpha + p)\sqrt{4 - 4\alpha + (p + \alpha)^2}},
\] (11)

\[
\rho_B^{\text{stat}} = \frac{p(p + 3\alpha - 2 + \sqrt{4 - 4\alpha + (p + \alpha)^2})}{2(2\alpha + p)\sqrt{4 - 4\alpha + (p + \alpha)^2}}.
\] (12)

Approaching the critical point at \( p = 0 \) and \( \alpha_c = 2 \), we find a discontinuous behavior

\[
\rho_A^{\text{stat}} = \begin{cases} 1/2 & \text{for } \alpha < \alpha_c \\ 0 & \text{for } \alpha > \alpha_c \end{cases},
\] (13)

\[
\rho_B^{\text{stat}} = \begin{cases} 1/2 & \text{for } \alpha < \alpha_c \\ 1 & \text{for } \alpha > \alpha_c \end{cases},
\] (14)

while \( \rho_0^{\text{stat}} = 0 \). In fact, it is clear from (2) that for \( p = 0 \), the \( B \)-particles can only transform into \( A \)-particles or vacancies but they are not created. Hence, in the steady state in the thermodynamic limit, the \( B \)-particles will disappear.

We also observe that the density of the \( B \)-particles in the vicinity of the critical point changes discontinuously in a particular limit. This can be seen already in the snapshots of Fig. 2a and 2c: For \( \alpha < 2 \) and \( p = 0 \) the density of \( B \)-particles vanishes rapidly on an exponentially short time scale, while for \( \alpha > 2 \) one observes some kind of annihilating random walk with a slow algebraic decay. Therefore, for a small value of \( p > 0 \), i.e. when switching on the creation of \( B \)-particles at a small rate, it is plausible that the system will respond differently in both cases. In fact, expanding (10) around \( p = 0 \) to first
order in $p$ in the two phases $\alpha > \alpha_c$ or $\alpha = \alpha_c + \epsilon$ and
$\alpha < \alpha_c$ or $\alpha = \alpha_c - \epsilon$, where $\epsilon$ is very small, we find a
band gap as

$$\Delta = \rho_{B}^{\text{stat}, \alpha > \alpha_c} - \rho_{B}^{\text{stat}, \alpha < \alpha_c} \approx \frac{L^2 p \epsilon}{8}$$ (15)

which is valid for $1 \ll L \ll L_{\text{max}}$ where $L_{\text{max}} = (p \epsilon)^{-1/2}$.

IV. RELATION TO A ZERO-RANGE PROCESS

A zero range process (ZRP) is defined as a system of $L$ boxes where each box can be empty or occupied by an
arbitrary number of particles. The particles hop between
neighboring boxes with a rate that can depend on the
number of particles in the box of departure [15]. The
stationary state of the ZRP factorizes, meaning that the
steady-state weight of any configuration is given by a
product of factors associated with each of the boxes.

It is known that various driven-diffusive systems can be
mapped onto a ZRP [15]. This is usually done by interpreting
the vacancies (particles) in the driven-diffusive systems as particles (boxes) in the ZRP. Following the
same line we find that our model can be mapped onto
a non-conserving ZRP with two different types of boxes.
More specifically, the $n$ vacancies to the right of an $(A(B)-$
particle are regarded as an $(A(B)$-box containing $n$
particles in the ZRP denoted as $A_n(B_n)$. The total number of particles distributed among the boxes is denoted
as $N_0$ while number of boxes of type $(A(B)$ is denoted as
$N_A(N_B)$. By definition, the sum $N_A + N_B + N_0 = L$
is conserved. However, the individual numbers are not con-
served and change according to the following dynamical rules:

(i) Particles from an $(A(B)$-box hop to the neighboring
left box with rate $\alpha_+ (\alpha_-)$:

$$X_m A_n \xrightarrow{\alpha_+} X_{m+1} A_{n-1} \quad (X = A, B)$$ (16)

$$X_m B_n \xrightarrow{\alpha_-} X_{m+1} B_{n-1}$$

(ii) An empty $(A(B)$-box transforms into an empty
$B(A)$ box with the rate $p_+ (p_-)$:

$$A_0 \xrightarrow{p_+} B_0 \quad (p_-)$$ (17)

(iii) An $(A(B)$-box with $n$ particles together with an
adjacent empty $B(A)$-box on the left side transforms
into a single $(A(B)$-box containing $n + 1$ particles
with rate $p_- (\alpha_+)$. The reversed process is also
possible and takes place with rate $p_+ (\alpha_-)$:

$$B_0 A_n \xrightarrow{p_-} A_{n+1}, \quad A_0 B_n \xrightarrow{\alpha_+} B_{n+1}$$ (18)

(iv) An $(A(B)$-box containing $n$ particles and a neigh-
boring empty $(A(B)$-box on the left side transform
into an $(A(B)$-box with $n + 1$ particles with the rate
$1 (\alpha)$. The reversed process is also possible and
takes place with rate $1 (p)$:

$$A_0 A_n \xrightarrow{1} A_{n+1}, \quad B_0 B_n \xrightarrow{\alpha} B_{n+1}$$ (19)

With these dynamical rules, we can show that the weights
of configurations in the ZRP can be expressed as factor-
ized forms. We consider a configuration consisting of
$\delta = N_A + N_B$ boxes with $N_0$ particles distributed in the
boxes. Defining $n_k$ as the number of particles in $k$th
box of type $\tau_k \in \{A, B\}$, where $\sum_{k=1}^{\infty} n_k = N_0$, the weight of the
configuration can be written as

$$W_{\text{ZRP}}\{\{ n_1 \tau_1, \ldots, n_3 \tau_3 \} \} = \prod_{k=1}^{\delta} f_{\tau_k}(n_k),$$ (20)

where $f_A(n)$ ($f_B(n)$) is the weight of an $(A(B)$-box con-
taining $n$ particles. In order to compute $f_A(n)$ and $f_B(n)$,
let us define the vectors $|a_1\rangle$, $|a_2\rangle$, $|b_1\rangle$ and $|b_2\rangle$ by

$$|a_1\rangle = |b_1\rangle = |1\rangle + |2\rangle, \quad |a_2\rangle = |1\rangle, \quad |b_2\rangle = p|2\rangle, \quad (21)$$

where we used the basis vectors

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (22)$$

Then the operators $A$ and $B$ in the matrix representa-
tion [7] can be rewritten as

$$A = |a_1\rangle \langle a_2|, \quad B = |b_1\rangle \langle b_2|.$$ (23)

Using Eqs. [21]-[23] and [7] we obtain

$$f_A(n) = \langle a_2 | E^n | a_1 \rangle = \langle a_2 | E^n | b_1 \rangle = 1$$ (24)

$$f_B(n) = \langle b_2 | E^n | b_1 \rangle = \langle b_2 | E^n | a_1 \rangle = p a^n.$$ (25)

We can show that Eq. (20) satisfy the pairwise balance
condition [15], therefore it is the stationary state for the
dynamics specified by [16]-[19].

Let us finally turn to the case $p = 0$. It is clear from
Eqs. [20], [24] and [25] that the stationary state weight
of the ZRP consists only of the weights of $A$-boxes con-
taining particles. Defining $\langle N_A \rangle$ as the average number
of $A$-boxes and $\langle n \rangle$ as the average number of particles
in an $A$-box, and noticing the dynamical rules of the
non-conserving ZRP, [13] and [14], we observe different behaviors for $\langle N_A \rangle$ and $\langle n \rangle$, namely

- for $p = 0$, $\alpha < \alpha_c$, $\langle N_A \rangle$ and $\langle n \rangle$ are finite.
- for $p = 0$, $\alpha > \alpha_c$, $\langle N_A \rangle = O(1)$ and $\langle n \rangle = O(L)$.

Therefore, we have a condensation transition where a
large number of particles accumulate in a single $A$-box.
V. NUMERICAL RESULTS

Since all stationary properties of the model defined in [2] can be computed exactly, our numerical simulations focus on its dynamical evolution. As we will see, the dynamical behavior is affected by strong scaling corrections which will be explained heuristically in Sect. VI.

A. Decay of $\rho_A$ and $\rho_B$ at the critical point

At the critical point $p = 0, \alpha = 2$ we have $p_+/p_- = p = 0$ and $\alpha_+ / \alpha_- = 2$, implying $p_+ = 0$, meaning that at this point the model controlled by two parameters $\alpha_+$ and $p_-$. In Fig. 3 we measured the time dependence of both order parameters for $p_- = 1$ and various values of $\alpha_+$, starting with a random initial state with $\rho_A(0) = \rho_B(0) = 1/3$. The behavior turns out to be qualitatively similar in all cases: While the density $\rho_A(t)$ seems to increase slightly, the density $\rho_B(t)$ shows a decay reminding of a power law $\rho_B(t) \sim t^{-\delta}$. However, if we first estimate the exponent $\delta \approx 0.57$ and then divide the data by $t^{-\delta}$ one observes a significant curvature of the data: The effective exponent $\delta_{\text{eff}}$ decreases from 0.6 down 0.57 without having reached a stable value in the numerically accessible regime, indicating strong scaling corrections.

It turns out that the effective exponent depends strongly on the particle densities in the initial state. This freedom can be used to reduce the influence of the scaling corrections. Choosing for example a random initial configuration with $\rho_A(0) = 0.9$ and $\rho_B(0) = 0.1$ one obtains a less pronounced curvature of $\rho_B(t)$ with an effective exponent of only $\delta \approx 0.51$. This suggests that the asymptotic exponent is $\delta = 1/2$.

B. Finite-size scaling

Using the initial condition $\rho_A(0) = 0.9$ and $\rho_B(0) = 0.1$ we repeated the simulation in finite systems. The results are plotted in the left panel of Fig. 4, where we divided $\rho_B(t)$ by the expected power law $t^{-1/2}$ so that an infinite system should produce an asymptotically horizontal line. As can be seen, a finite system size leads to a sudden breakdown of $\rho_B(t)$ while there is no change in $\rho_A(t)$. Plotting the same data against $t/L^z$ (right panel), where $z = \nu_{\parallel} / \nu_{\perp}$ is the dynamical exponent, the best data collapse is obtained for $z = 2$. This is plausible since so far all systems, which have been solved by means of matrix product methods, are essentially diffusive with a dynamical exponent $z = 2$.

C. Off-critical simulations

Finally we investigate the two-dimensional vicinity of the critical point where

$$\Delta \alpha = \alpha - \alpha_c = \alpha - 2$$

as well as $p$ are small. First we choose $\Delta \alpha = 0$ and study the model for $p > 0$. In this case the order parameter $\rho_B(t)$ first decays as if the system was critical until it saturates at a constant value, as shown in the inset of Fig. 5.

Surprisingly, $\rho_B(t)$ first goes through a local minimum and then increases again before it reaches the plateau. This phenomenon of undershooting has also been observed in conserved sandpile models [19] and may indicate that the system has a long-time memory for specific correlations in the initial state. Plotting $\rho_B(t)t^{1/2}$ against $tp^\nu$ one finds an excellent data collapse for $\nu_{\parallel} = 1.00(5)$, indicating that $\nu_{\parallel} = 1$.

Next we keep $p = 0$ fixed and vary $\Delta \alpha$. For $\Delta \alpha < 0$ one finds that the density $\rho_B(t)$ crosses over to an exponential decay. For $\Delta \alpha > 0$, where one expects supercritical behavior, $\rho_B(t)$ does not saturate at a constant, instead it first decreases as $t^{-1/2}$ followed by a short period of a decelerated decay until it continues to decay as $t^{-1/2}$. This means that $\alpha > 2$ causes an increase of the amplitude but not a crossover to a different type of decay. To our knowledge this is the first example of a power law to the same power law but with a different amplitude.

Plotting $\rho_B(t)t^{1/2}$ against $tp^\nu_{\parallel}$ the data collapse is unsatisfactory due to the scaling corrections discussed above. However, the best compromise is obtained for $\eta_{\parallel} = 1.9(2)$, which is compatible with $\eta_{\parallel} = 2$. 
D. Phenomenological scaling properties

Apart from the scaling corrections which will be discussed in the following section, the collected numerical results suggest that the process in the vicinity of the critical point is invariant under scale transformations of the form

\[ t \to \Lambda^\nu t, \quad L \to \Lambda^{\nu_L} L, \quad \rho_B \to \Lambda^\beta \rho_B \]

\[ p \to \Lambda^\theta p, \quad \Delta \alpha \to \Lambda^\theta \Delta \alpha, \]

where \( \theta = \nu_\parallel/\eta_\parallel \) is the crossover exponent between the two control parameters.

Assuming that the critical behavior is described by simple rational exponents, our findings suggest that the universality class of the process is characterized by four exponents \( \beta = 1/2, \nu_\parallel = 1, \nu_\perp = 1/2, \theta = 1/2 \) together with the scaling relations

\[ \delta = \frac{\beta}{\nu_\parallel} = \frac{1}{2} \]

\[ z = \frac{\nu_\parallel}{\nu_\perp} = \frac{\eta_\parallel}{\eta_\perp} = 2 \]

\[ \theta = \frac{\nu_\parallel}{\eta_\parallel} = 1/2. \]

The values of the exponents are listed in Table I. Regarding the stationary properties for \( p > 0 \), these exponents are in full agreement with the exact solution in Sect. III

The scaling scheme (27) implies various scaling relations. For example, it allows us to predict that the stationary density of \( B \)-particles in the vicinity of the critical point should scale as

\[ \rho_B^{\text{stat}} = p^\beta F\left(\frac{(\Delta \alpha)^2}{p}\right), \]

where \( F \) is a universal scaling function. Comparing this form with the exact result (12) we find that

\[ F(\xi) = \frac{1}{2\sqrt{4 + \xi}}. \]
rate of spontaneous pair creation and therefore plays a similar role as $p$ in the original model.

The reduced process starts with an alternating initial configuration $+-+-+-+...$, where $\rho_+(0) = \rho_-(0) = 1/2$. As time evolves, particles are created and annihilated in pairs, meaning that the two densities

$$\rho_+(t) = \rho_-(t)$$

are exactly equal. These densities are expected to play the same role as the order parameter $\rho_B(t)$ in the original model.

**B. Numerical results for the reduced model**

The reduced model has the advantage that it can be implemented very efficiently on a computer by storing the coordinates of the kinks in a dynamically generated list. Simulating the model we find the following results:

- $q > 0$: The model evolves into a stationary state with a constant density $\rho_+ = \rho_-$, qualitatively reproducing the corresponding results for the full model shown in the right panel of Fig. 1.

- $q = 0, \lambda > 1$: Positive charges move to the right and negative charges move to the left until they form bound $+-$ pairs which perform a slow unbiased random walk. If two such pairs collide they coagulate into a single one by the effective reaction $+-+- \rightarrow +-$. Therefore, one expects the density of particles to decay as $t^{-1/2}$ in the same way as in a coagulation-diffusion process [21].

- $q = 0, \lambda = 1$: At the critical point the particle density seems to decay somewhat faster than $t^{-1/2}$. The origin of these scaling corrections will be discussed below.

- $q = 0, \lambda < 1$: In this case the negative charges diffuse to the right while positive charges diffuse to the left. When they meet they quickly annihilate in pairs, reaching an empty absorbing state in exponentially short time.

Therefore, the reduced model exhibits the same type of critical behavior as the full model. Moreover, repeating the standard simulations of Sect. V (not shown here) we obtain similar estimates of the critical exponents.

**C. Explaining the scaling corrections heuristically**

Performing extensive numerical simulations of the reduced model at the critical point over seven decades in time (see Fig. 7) one can see a clear curvature in the double-logarithmic plot. Unlike initial transients in other model, this curvature seems to persist over the whole temporal range. To confirm this observation, we plotted the corresponding local exponent $\delta_{\mathrm{eff}}$ against $1/\ln(t)$, interpreted as an effective critical exponent $-\delta_{\mathrm{eff}}$. A visual extrapolation along the red dashes line to $t \rightarrow \infty$ is consistent with the expected asymptotic exponent $\delta = 0.5$.

Where do the slow scaling corrections come from? This question is of general interest because various other nonequilibrium phase transitions, where the universal properties are not yet fully understood, show similar corrections. For example, the diffusive pair contact process [21] and fixed-energy sandpiles [19, 22] both exhibit a similar slow curvature of the particle decay at the criti-
cal point. Here we have a particularly simple system with an exactly known critical point, where the origin of the slow scaling corrections can be identified much easier.

To explain the scaling corrections heuristically, let us consider the pair annihilation process defined in [23] at the critical point starting with an alternating initial configuration (+−+−+−...). We first note that this process has the special property that pairs of particles which eventually annihilate must have been nearest neighbors in the initial configuration. In so far this process differs significantly from the usual annihilation process $2A \rightarrow \emptyset$, where in principle any pair can annihilate.

If the process had started with only a single +− pair, both particles would perform a simple random walk until they collide and annihilate. In this case the annihilation probability would be related to the first-return probability of a random walk [23]. Since the first-return probability would be related to the first-return probability would be impossible in 1D because in this case the minority islands could not have surface tension. Although this was originally restricted to two-state models, the claim was shown by one of the authors [24, 25] that first-order phase transitions in non-conserving systems with fluctuating domains should be impossible in one dimension. In [24] it was argued that a first-order transition needs a robust mechanism in order to eliminate spontaneously generated minority islands of the opposite phase, but this would be impossible in 1D because in this case the minority islands do not have surface tension. Although this claim was originally restricted to two-state models, the question arises why we find the contrary in the present case.

VII. CONCLUSIONS

In this work we have introduced and studied a two-species reaction-diffusion process on a one-dimensional periodic lattice which exhibits a nonequilibrium phase transition. Its stationary state can be determined exactly by means of the matrix product method. Together with numerical studies of the dynamics we have identified the critical exponents which are listed in Table I.

The transition can be explained qualitatively by relating the model to a reduced process (see Sect. VII). This relation also provides a heuristic explanation of the unusual corrections to scaling observed in this model.

Our findings seem to be in contradiction with a previous claim by one of the authors [24, 25] that first-order phase transitions in non-conserving systems with fluctuating domains would be impossible in one dimension. In [24] it was argued that a first-order transition needs a robust mechanism in order to eliminate spontaneously generated minority islands of the opposite phase, but this would be impossible in 1D because in this case the minori-
Again the caricature of the reduced process sketched in Fig. 6 provides a possible explanation: As can be seen there are two types of white patches, namely, large islands with a blue $B$-particle at the left boundary, and small islands without. This means that the $B$-particles are used for marking two different types of vacant islands, giving them different dynamical properties. Only the large islands containing a $B$-particle are minority islands in the sense discussed in [24], while the small islands without $B$-particles inside the $A$-domains are biased to shrink by themselves.

Therefore, we arrive at the conclusion that first-order phase transitions in non-conserving 1D systems with fluctuating domains are indeed possible in certain models with several particle species if one of the species is used for marking different types of minority islands.

Appendix A: An exactly solvable three species model

In this appendix, we show that a similar type of phase transition can also exist in four-state models. We introduce an exactly solvable one-dimensional driven-diffusive model with non-conserved dynamics consisting of three species of particles. The system evolves randomly sequentially according to the dynamical rules (1) where $I, J, K \in \{A, B, C\}$. This system is defined by the processes

\[
\begin{align*}
A0 & \mathop{\rightarrow}^\lambda_+ 0A, \quad B0 & \mathop{\rightarrow}^\lambda_+ 0B, \quad C0 & \mathop{\rightarrow}^\beta 0C \\
AA & \mathop{\rightarrow}^{p_+} BA, \quad AC & \mathop{\rightarrow}^{p_+} BC, \quad AB & \mathop{\rightarrow}^{p_+} BB \\
AB & \mathop{\rightarrow}^{q} CB, \quad AA & \mathop{\rightarrow}^{q} CA, \quad AC & \mathop{\rightarrow}^{q} CC \\
BA & \mathop{\rightarrow}^{q} CA, \quad BB & \mathop{\rightarrow}^{q} CB, \quad BC & \mathop{\rightarrow}^{q} CC \\
AB & \mathop{\rightarrow}^{\lambda_{+}} B\emptyset, \quad AC & \mathop{\rightarrow}^{\lambda_{+}} C\emptyset, \quad CB & \mathop{\rightarrow}^{\lambda_{+}} B\emptyset \\
BB & \mathop{\rightarrow}^{\lambda_{+}} A\emptyset, \quad BC & \mathop{\rightarrow}^{\lambda_{+}} C\emptyset, \quad CC & \mathop{\rightarrow}^{\lambda_{+}} C\emptyset \\
A0 & \mathop{\rightarrow}^{p_-} BA, \quad A0 & \mathop{\rightarrow}^{p_-} AA, \quad A0 & \mathop{\rightarrow}^{q_-} CA
\end{align*}
(A1)
\]

where the rates $\alpha$, $\beta$, $p$ and $q$ are given by the ratios

\[
\alpha = \frac{\lambda_+}{\lambda_+}, \quad \beta = \frac{\lambda_+}{\beta_+}, \quad p = \frac{p_+}{p_-}, \quad q = \frac{q_+}{q_-}.
\]

The first four lines of (A1) have been studied in Ref. [17] where the phase transition is not accessible. We have found that the matrix algebra of the dynamical rules (A1) has a three-dimensional matrix representation given by the following matrices

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad B = p \begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad C = q \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad E = \begin{pmatrix}
0 & \alpha & 0 \\
0 & 0 & \beta
\end{pmatrix}.
(A2)
\]

The representation (A2) is the same as the matrix representation represented in Ref. [17]. The partition function defined as a sum of the weights of all available configurations with at least one particle, is given by

\[
Z_L = \text{Tr}[(A + B + C + E)^L - E^L].
(A3)
\]

The stationary density of the A, B and C-particles can be written as

\[
\rho_A^{\text{stat}} = \frac{\text{Tr}[A(A + B + C + E)^{L-1}]}{Z_L},
(A4)
\]

\[
\rho_B^{\text{stat}} = \frac{\text{Tr}[B(A + B + C + E)^{L-1}]}{Z_L},
(A5)
\]

\[
\rho_C^{\text{stat}} = \frac{\text{Tr}[C(A + B + C + E)^{L-1}]}{Z_L}.
(A6)
\]

We can compute the density of the vacancies using $\rho_\emptyset^{\text{stat}} = 1 - (\rho_A^{\text{stat}} + \rho_B^{\text{stat}} + \rho_C^{\text{stat}})$. Using the representation (A2), the equations (A3)-(A6) can be calculated exactly. In the thermodynamic limit $L \rightarrow \infty$, the density of the A-particles and the vacancies vary discontinuously approaching the critical point, namely

(i) For $\beta \leq 2$ and $p = q = 0$, we find a discontinuous behavior as

\[
\rho_A^{\text{stat}} = \begin{cases}
\frac{1}{2} & \text{for } \alpha < 2 \\
0 & \text{for } \alpha > 2,
\end{cases}
\]

(ii) For $\alpha \leq 2$ and $p = q = 0$, we find a discontinuous behavior as

\[
\rho_A^{\text{stat}} = \begin{cases}
\frac{1}{2} & \text{for } \beta < 2 \\
0 & \text{for } \beta > 2,
\end{cases}
\]

and $\rho_\emptyset^{\text{stat}} = \rho_C^{\text{stat}} = 0$. 

\[
\rho_A^{\text{stat}} = \begin{cases}
\frac{1}{2} & \text{for } \beta < 2 \\
0 & \text{for } \beta > 2,
\end{cases}
\]
\[ \rho^\text{stat}_0 = \begin{cases} 
  \frac{1}{2} & \text{for } \beta < 2 
  1 & \text{for } \beta > 2, 
\end{cases} 
\]
and \( \rho^\text{stat}_B = \rho^\text{stat}_C = 0. \)

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