The replacements of signed graphs and Kauffman brackets of links\textsuperscript{1}

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Abstract

Let $G$ be a signed graph. Let $\hat{G}$ be the graph obtained from $G$ by replacing each edge $e$ by a chain or a sheaf. We first establish a relation between the $Q$-polynomial of $\hat{G}$ and the $W$-polynomial of $G$. Two special dual cases are derived from the relation, one of which has been studied in [8]. Based on the one to one correspondence between signed plane graphs and link diagrams, and the correspondence between the $Q$-polynomial of signed plane graph and the Kauffman bracket of link diagram, we can compute the Kauffman bracket of link diagram corresponding to $\hat{G}$ by means of the $W$-polynomial of $G$. By this way we use transfer matrix approach to compute the Kauffman bracket of rational links, and obtain their closed-form formulae. Finally we provide an example to point out that the relation we built can be used to deal with a wide type of link family.

Keywords: Kauffman bracket, Graph polynomial, signed graph, rational link

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1 Introduction

The Kauffman square bracket \([D] \in \mathbb{Z}[A, B, d]\) of a link diagram \(D\) is defined by the following two rules [1]:

1. The kauffman square bracket of a diagram consisting of \(n\) disjoint simple closed curves in the plane is \(d^{n-1}\);

2. \([D] = A[D^A_v] + B[D^B_v]\) for any crossing \(v\) of link diagram \(D\), where \(D^A_v\) and \(D^B_v\) are link diagrams obtained from \(D\) by opening the \(A\)-channel and \(B\)-channel respectively.

Kauffman bracket \(<D> \in \mathbb{Z}[A]\) of a link diagram \(D\) is related to Kauffman square bracket \([D]\) by

\[
< D > = [D]|_{B=A^{-1}, d=-A^2-A^{-2}}
\]

(1)

The Kauffman bracket of link diagram is a regular isotopy invariant of (unoriented) links, the famous Jones polynomial \(V_L(t) \in \mathbb{Z}[t]\) of an oriented link \(L\) is related to the Kauffman bracket by

\[
V_L(t) = (-A^3)^{-w(D)} < D > |_{A=t^{-1/4}},
\]

(2)

where \(D\) is the diagram of \(L\), \(w(D)\) and \(<D>\) are the writhe and the Kauffman bracket of \(D\) respectively.

The Kauffman brackets of some link families have been computed, for example, a recursive formula and a closed-form formula of Kauffman brackets of pretzel links are obtained in [3] and [4] respectively.

It is a well-known fact that link diagrams are in one-to-one correspondence with signed plane graphs. Based on this correspondence, Kauffman associate the signed graph \(G\) a polynomial \(Q[G] \in \mathbb{Z}[A, B, d]\) in three variables \(A, B\) and \(d\) [5,6], which specializes to Kauffman square bracket of the link diagram when \(G\) is planar.

Let \(D\) be a link diagram corresponding to signed plane graph \(G\). In [7,8], the authors of this paper studied the Kauffman bracket of the link diagram family \(\{D_c\}\) corresponding to homeomorphism class \(\{G_c\}\) of \(G\) obtained from \(G\) by replacing each edge of \(G\) by a chain (path) with its sign reserved, and established a relation between the \(Q\)-polynomial of \(G_c\) (the Kauffman bracket of \(D_c\)) and the chain polynomial of \(G\). Dually, the relation between the Kauffman bracket of the link diagram family \(\{G_s\}\) corresponding to anallamorphism class \(\{G_s\}\) of \(G\) obtained from \(G\) by replacing each edge of \(G\) by a sheaf (multiple edges) with its sign reserved and the sheaf polynomial of \(G\) can also be obtained easily.

In this paper we compute the Kauffman bracket of the link diagram family \(\{\hat{D}\}\) corresponding to graphs \(\{\hat{G}\}\) obtained from \(G\) by replacing some edges by chains and some edges by sheaves which includes the two special subfamilies \(\{D_c\}\) and \(\{D_s\}\). We establish a relation between the \(Q\)-polynomial of \(\{\hat{D}\}\) and the Tutte polynomial of \(G\) defined by Bollobás [9], which generalize the result in [7,8].

Rational links are a family of links obtained from rational tangles [10,11] by joining with simple arcs the two upper ends and the two lower ends. In section 5, we apply the relation to compute the Kauffman brackets of rational links. By transfer matrix approach the explicit formula of Kauffman brackets of rational links are obtained.
The graph $G = (V(G), E(G))$ in this paper allows loops and multiple edges. We denote by $k(G)$ the number of connected components of $G$, and by $n(G) = |E(G)| - |V(G)| + k(G)$ the nullity (cyclomatic number) of $G$. We denote by $E_n$ the graph with $n$ vertices and no edges. For $F \subset E(G)$ we write $< F >$ for the spanning subgraph of $G$ with edge set $F$, and $k < F >$ and $n < F >$ for the number of components and the nullity of this graph respectively. We use $G - F$ and $G/F$ to denote the graphs obtained from $G$ by deleting and contracting (that is, deleting the edges and identifying the ends of each edges in $F$) the edges in $F$ respectively. In particular, for $e \in E$, $G - e$ and $G/e$ denote the graphs obtained from $G$ by deleting and contracting the edge $e$ respectively. We refer to [10] for more graph theory.

2 Link diagram and signed plane graph

We first describe the classical correspondence between link diagrams and signed plane graphs. A signed graph is a graph with each edge labelled with a sign (+ or −). We denote by $s(e)$ the sign of the edge $e$.

The medial graph $M(G)$ of a connected non-trivial plane graph $G$ is a 4-regular plane graph obtained by inserting a vertex on every edge of $G$, and joining two new vertices by an edge lying in a face of $G$ if the vertices are on adjacent edges of the face; if $G$ is trivial (that is, it is an isolated vertex), its medial graph is a simple closed curve surrounding the vertex (strictly, it is not a graph); if $G$ is not connected, its medial graph $M(G)$ is the disjoint union of the medial graphs of its connected components.

Given a signed plane graph $G$ with medial graph $M(G)$, to turn $M(G)$ into a link diagram $D = D(G)$, we turn the vertices of $M(G)$ into crossings by defining a crossing to be over or under according to the sign of the edge as shown in Figure 1. Conversely, given a link diagram $D$, shade it checker-boardly firstly so that the unbounded face is unshaded, then we associate $D$ with a signed plane graph $G(D)$ as follows: For each shaded face $F$, take a vertex $V_F$ in $F$, and for each crossing at which $F_1$ and $F_2$ meet, take an edge $V_{F_1}V_{F_2}$, furthermore, give each edge $V_{F_1}V_{F_2}$ a sign according to the type of the crossing as the rules shown in Figure 1. In Figure 1 and Figure 2 below, the broken line is the edge of graph, and the solid line is the arc of link diagram.

Figure 1: The correspondence between a signed edge and a crossing

Let $G$ be a signed plane graph. We define the dual signed plane graph $G^*$ of $G$ as follows: Neglecting the signs of $G$, we first obtain the dual plane graph of $G$, then give each edge $e^*$ in the dual plane graph a sign + (resp. −) if the sign of the corresponding edge $e$ in $G$ is − (resp. +). Note that $G$ and $G^*$ correspond to the same link diagram when they are both connected.

A chain is a graph which is a path and a sheaf is a graph with two vertices (not necessarily distinct) connected by some parallel edges. Given a signed plane graph $G$. 
Let $\hat{G}$ be the graph obtained from $G$ by replacing each edge $e$ by a chain $c_e$ or a sheaf $s_e$. By the second Reidemeister Move shown in Figure 2, the adjacent two edges with different signs in a chain or a sheaf can cancel each other, without loss of generality we assume that the signs of the edges in a chain or a sheaf are the same as that of the edge it replaces. We identify sign + with +1, sign $-$ with $-1$, and define the length of $c_e$ (resp. the width of $s_e$) as the sum of the signs of the edges in the chain (resp. the sheaf).

![Figure 2: The second Reidemeister Move](image)

Let $G$ be a signed plane graph. Since an edge $e$ can be replaced by a chain or a sheaf and the length of a chain and the width of a sheaf can be any non-zero integers, we obtain a family of signed plane graphs $\{\hat{G}\}$ and a link diagram family $\{\hat{D}\}$ accordingly. We call $G$ the associated graph with the link diagram family $\{\hat{D}\}$. For example, to the graph $G$ with two parallel edges, we replace one edge by a chain and the other by a sheaf, then obtain a rational link family (also called generalized twist link and when $m_2 = 2$, it is called twist knot). Figure 3 shows the rational link $m_1m_2$ with $m_1 > 0$ and $m_2 > 0$ and its associated graph. When $m_i < 0$, the link diagram is obtained by replacing the corresponding half-twists by its mirror image.

![Figure 3: Rational link $m_1m_2$ and its associated graph](image)

### 3 Graph polynomials of Tutte type

In this section, we give three graph polynomials of Tutte type which are all generalizations of Tutte polynomial [13]. They are $Q$-polynomial for signed graphs, $W$-polynomial
for colored graphs, and chain and sheaf polynomials introduced by Kauffman [6], Bollobás and Riordan[9], and Read and Whitehead [14] respectively.

3.1 Q-polynomial

In [6] Kauffman introduce the Tutte polynomial \(Q[G] = Q[G](A, B, d) \in \mathbb{Z}[A, B, d]\) for signed graph \(G\). Hereafter we assume that \(X = A + Bd\) and \(Y = Ad + B\). We can redefine the \(Q\)-polynomial by the following recursive rules:

1. \(Q[E_n] = d^{n-1}\). \(\quad(3)\)

2. (a) If \(e\) is a loop, then
   
   \(Q[G] = XQ[G - e]\) if \(s(e) = -\),
   
   \(Q[G] = YQ[G - e]\) if \(s(e) = +\). \(\quad(4)\)

(b) If \(e\) is not a loop, then

   \(Q[G] = AQ[G - e] + BQ[G/e]\) if \(s(e) = -\),
   
   \(Q[G] = BQ[G - e] + AQ[G/e]\) if \(s(e) = +\). \(\quad(5)\)

Theorem 1[6] Let \(G\) be a signed plane graph. Let \(D(G)\) be the corresponding link diagram. Then \(Q[G] = [D(G)]\).

Theorem 1 can be obtained by comparing the rules defining Kauffman square bracket of link diagram with rules (3)-(5) defining \(Q\)-polynomial of the corresponding signed plane graph. On the other hand, since \(G\) and \(G^*\) correspond to the same link diagram, Theorem 1 suggests that \(Q[G] = Q[G^*]\).

Note1: When \(B = A^{-1}\) and \(d = -A^2 - A^{-2}\), we have \(X = -A^{-3}\) and \(Y = -A^{3}\).

3.2 W-polynomial

A colored graph is a graph \(G\) together with a function \(c\) from \(E(G)\) to an arbitrary set \(\Lambda\) of colors, which is a generalization of signed graph. In [9] Bollobás and Riordan introduced a Tutte polynomial for colored graphs. We consider a variant for our purpose, and denoted it by \(W(G)\). The polynomial \(W(G) = W(G)(t, z_1, z_2)\) for a colored graph \(G\) is defined by the following recursive rules:

1. \(W(E_n) = t^{n-1}\). \(\quad(6)\)

2. Let \(c(e) = \lambda\). Then

   (a) If \(e\) is a bridge, then

   \(W(G) = (x_\lambda + z_1 y_\lambda)W(G/e)\). \(\quad(7)\)
(b) If $e$ is a loop, then
\[ W(G) = (x_λz_2 + y_λ)W(G - e). \]  
(8)

(c) If $e$ is neither a bride nor a loop, then
\[ W(G) = x_λF(G/e) + y_λW(G - e). \]  
(9)

$W$-polynomial is a naturally generalization of $Q$-polynomial with $Λ = \{+, −\}$. The well-definedness of $W(G)$ is based on the fact that it can be expressed as the sum

**Theorem 2** [9]

\[ W(G) = t^{k(G) - 1} \sum_{S \subseteq E(G)} \left\{ \prod_{e \in S} x_{c(e)} \right\} \left\{ \prod_{e \notin S} y_{c(e)} \right\} \frac{z_{k-S}}{z_{k-G} - k(G)} \frac{y_{z_2}}{y_{z_2 - z_{n+S}}}. \]

We denote by $G_1 \cup G_2$ the union of two disjoint graphs $G_1$ and $G_2$, and by $G_1 \cdot G_2$ the union of two graphs with only one vertex in common. By Theorem 2, we have

**Theorem 3**

1. $W(G_1 \cup G_2) = tW(G_1)W(G_2)$;
2. $W(G_1 \cdot G_2) = W(G_1)W(G_2)$.

A graph $F = (V', E')$ is spanning forest of $G = (V, E)$ if $V' = V, E' \subseteq E$, and each component of $F$ is a spanning tree of a component of $G$. Let $G$ be a colored graph and let us consider an order on its edge set. Let $F$ be a spanning forest of $G$. Call an edge $e \in E(F)$ internally active if it is the smallest edge in the unique cut induced by deleting $e$ in $F$ with respect to the given order and internally inactive otherwise. Similarly, call an edge $e \notin E(F)$ externally active if it is the smallest edge in the unique cycle induced by adding $e$ in $F$ with respect to the given order and externally inactive otherwise. $W(G)$ also has the spanning forest expansion as the various (weighted) graph polynomials of Tutte type [6,13,15].

**Theorem 4** [9] For any given order on the edge set of $G$, we have

\[ W(G) = t^{k(G) - 1} \sum_F \left\{ \prod_{IA} (x_{c(e)} + z_1y_{c(e)}) \right\} \left\{ \prod_{EA} (x_{c(e)}z_2 + y_{c(e)}) \right\} \left\{ \prod_{II} x_{c(e)} \right\} \left\{ \prod_{EI} y_{c(e)} \right\}, \]

where the sum is over all spanning forests of $G$, and the products are over the edges of internal active, external active, internal inactive, external inactive respect to $F$ respectively.

### 3.3 Chain and sheaf polynomials

Chain and sheaf polynomials are introduced by Read and Whitehead in [14] for studying the chromatic polynomial of homeomorphism class and the flow polynomial of amalgamation class of graphs. They are both the weighted versions of Tutte polynomial [16]. These two dual polynomials are both defined on the graph whose edges have been labelled with $a, b, c \ldots \; ^1$. We denoted the two polynomials by $Ch[G]$ and $Sh[G]$ respectively.

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1We usually use the label of an edge to represent this edge.
Before defining the chain and sheaf polynomials, we first give the definitions of the flow and tension polynomials \[13\]. Let \( G \) be a graph and let \( \overrightarrow{G} \) be an orientation of \( G \) obtained by assigning an arbitrary but fixed orientation \( \overrightarrow{e} \) to each edge \( e \) of \( G \). Let \( \overrightarrow{E} \) denote the set of oriented edges of \( \overrightarrow{G} \). A map \( f : \overrightarrow{E} \rightarrow \mathbb{Z}_q \) is called a \( q \)-flow if, for each vertex \( v \in V(G) \), \( \sum_+ f(\overrightarrow{e}) = \sum_- f(\overrightarrow{e}) \), where \( \sum_+ \) denotes summation over all edges \( \overrightarrow{e} \) incident with \( v \) and oriented towards \( v \), and \( \sum_- \) denotes summation over all edges \( \overrightarrow{e} \) incident with \( v \) and oriented away from \( v \). A map \( f : \overrightarrow{E} \rightarrow \mathbb{Z}_q \) is called a \( q \)-tension if there is a map \( g : V \rightarrow \mathbb{Z}_q \) called potential function such that, for each edge \( \overrightarrow{e} = (u, v) \in \overrightarrow{E} \) oriented from \( u \) to \( v \), \( f(\overrightarrow{e}) = g(u) - g(v) \). \( f \) is called nowhere zero if, for each \( \overrightarrow{e} \in \overrightarrow{E} \), \( f(\overrightarrow{e}) \neq 0 \).

The flow polynomial \( F[G](q) \) gives, for each positive integer \( q \), the number of nowhere-zero \( q \)-flows on \( \overrightarrow{G} \). The tension polynomial \( T[G](q) \) gives, for each positive integer \( q \), the number of nowhere-zero \( q \)-tensions on \( \overrightarrow{G} \). It is not difficult to see that the flow and tension polynomials do not depend on the orientation of \( G \).

The chain and sheaf polynomials can be defined as follows:

\[
Ch[G] = \sum_Y F[G - Y](1 - w)\epsilon(Y)
\]

and

\[
Sh[G] = \sum_Y T[G/Y](1 - w)\epsilon(Y),
\]

where the summations are both over all subsets of \( E(G) \); \( F[G - Y] \) and \( T[G/Y] \) are the flow polynomial of \( G - Y \) and the tension polynomial of \( G/Y \); \( \epsilon(Y) \) is the product of labels of the edges in \( Y \).

We substitute \( 1 - w \) for the variable of the flow and tension polynomials in the above definition in order that our definition is consistent with the original definition by Read and Whitehead. The following theorem is implicit in \[14\] and explicit in \[16\] and can also be proved directly by dividing the summand in the definition into two parts, one is over all the \( Y \)'s which contain the edge \( a \), the other is over all the \( Y \)'s which do not contain the edge \( a \).

**Theorem 5**[14,16] The chain polynomial satisfies the following recursive rules:

1. \( Ch[E_n] = 1 \). \hspace{1cm} (10)

2. If \( a \) is a loop of \( G \), then

\[
Ch[G] = (a - w)Ch[G - a],
\]

(11)

If \( a \) is not a loop, then

\[
Ch[G] = (a - 1)Ch[G - a] + Ch[G/a].
\]

(12)

Dually, the sheaf polynomial satisfies the following recursive rules:
1. \[ \text{Sh}[E_n] = 1. \] (13)

2. If \( a \) is a bridge of \( G \), then
   \[ \text{Sh}[G] = (a - w)\text{Sh}[G/a] \] (14)
   If \( a \) is not a bridge, then
   \[ \text{Sh}[G] = (a - 1)\text{Sh}[G/a] + \text{Sh}[G - a]. \] (15)

4. The relation between \( Q[\hat{G}] \) and \( W(G) \)

   Let \( G \) be a signed graph. Let \( \hat{G} \) be the graph obtained from \( G \) by replacing each edge \( e \) by a chain \( c_e \) or a sheaf \( s_e \). In this section we establish a relation between the \( Q \)-polynomial of \( \hat{G} \) (hence Kauffman bracket of \( D = D(\hat{G}) \)) and the \( W \)-polynomial of \( G \). Two special cases are also derived.

   **Lemma 6** [8] If \( e \in E(G) \) is replaced by a chain with length \( n \), then
   \[ Q[\hat{G}] = \frac{X^n - A^n}{d} Q[\hat{G} - e] + A^n \delta Q[\hat{G}/e], \] (16)
   where \( \delta = d \) if \( a \) is a loop of \( G \), and \( \delta = 1 \) otherwise.

   **Lemma 7** If \( e \in E(G) \) is replaced by a sheaf with width \( n \), then
   \[ Q[\hat{G}] = B^n Q[\hat{G} - e] + \frac{Y^n - B^n}{d} \delta Q[\hat{G}/e], \] (17)
   where \( \delta = d \) if \( e \) is a loop of \( G \), and \( \delta = 1 \) otherwise.

   **Proof.** We suppose \( s(e) = + \) firstly.
   **Case 1.** If \( e \) is a loop of \( G \), by Equation (4) then
   \[ Q[\hat{G}] = Y^n Q[\hat{G} - e] \]
   \[ = B^n Q[\hat{G} - e] + \frac{Y^n - B^n}{d} d Q[\hat{G}/e]. \]

   **Case 2.** If \( e \) is not a loop of \( G \).
   Suppose that the edges in \( s_e \) are \( e_1, e_2, \ldots, e_n \) successively, by Equations (4) and (5) then
   \[ Q[\hat{G}] = BQ[\hat{G} - e_1] + AQ[\hat{G}/e_1] \]
   \[ = BQ[\hat{G} - e_1] + AY^{n-1} Q[\hat{G}/e] \]
   \[ = B^2 Q[\hat{G} - e_1 - e_2] + (AY^{n-1} + BAY^{n-2}) Q[\hat{G}/e] \]
   \[ = \ldots \]
   \[ = B^n a Q[\hat{G} - e_1 - e_2 - \cdots - e_n] + (AY^{n-1} + BAY^{n-2} + \cdots + B^{n-2} AY + B^{n-1} A) Q[\hat{G}/e] \]
   \[ = B^n Q[\hat{G} - e] + \frac{Y^n - B^n}{d} d Q[\hat{G}/e]. \]
The argument, if \( s(e) = - \) is virtually identical to that above, with \( A \) and \( B \), \( X \) and \( Y \), \( n \) and \(-n\) interchanged simultaneously. Notice that \( B = A^{-1}, Y = X^{-1} \), the lemma also holds.

Let \( c_n \) be the color of the edge \( e \in E(G) \) representing the corresponding chain with length \( n \) in \( \hat{G} \) and \( s_n \) the color of the edge \( e \in E(G) \) representing the corresponding sheaf with width \( n \) in \( \hat{G} \).

**Theorem 8** Let \( \hat{G} \) and \( G \) be signed graphs explained above. In \( W(G) \), if we set

\[
x_{cn} = A^n, \quad y_{cn} = \frac{X^n - A^n}{d}
\]

and

\[
x_{sn} = \frac{Y^n - B^n}{d}, \quad y_{sn} = B^n,
\]

then

\[
Q[\hat{G}] = W(G)(d, d, d).
\]

**Proof:** By Equations (3) and (6), the theorem holds when \( G = E_n \). If \( z_1 = t \), by Theorem 3, Equation (7) can be replaced by Equation (9). Comparing the coefficients (16) and (17) with (8) and (9), the theorem is proved.

Now we study two special cases of Theorem 8. Let \( G \) be a signed graph. Let \( G_c \) be the graph obtained from \( G \) by replacing each edge \( e \) by a chain \( c_e \) and \( G_s \) be the graph obtained from \( G \) by replacing each edge \( e \) by a sheaf \( s_e \).

By Theorem 8, we have

\[
Q[G_c] = W(G)(d, d, d),
\]

where

\[
x_{cn} = A^n, \quad y_{cn} = \frac{X^n - A^n}{d}.
\]

Let

\[
\tilde{W}(G) = d^{1-p(G)}W(G)(d, d, d),
\]

where \( p(G) \) is the number of vertices of \( G \). We have

1.

\[
\tilde{W}(E_n) = 1.
\]

2. If \( e \) is a loop of \( G \), then

\[
\tilde{W}(G) = \frac{A^n}{d}(((X/A)^n - (1 - d^2))\tilde{W}(G - e).
\]

If \( e \) is not a loop, then

\[
\tilde{W}(G) = \frac{A^n}{d}(((X/A)^n - 1)\tilde{W}(G - e) + \tilde{W}(G/e)).
\]
Comparing Equations (22)-(24) with Equations (10)-(12), we obtain

**Corollary 9** In $Ch[G]$, if we replace $w$ by $1 - d^2$, and replace $a$ by $(X/A)^{n_a}$ for every chain $a$, where $n_a$ is the length of the chain $a$, then we have

$$Q[G] = \frac{A^{\sum_{e \in E(G)} n_a}}{d^{q-p+1}} Ch[G],$$

where $p$ and $q$ are the number of vertices and edges of graph $G$, respectively.

Similarly, by Theorem 8, we have

$$Q[G_s] = W(G)(d, d, d)$$

where

$$x_{s_n} = \frac{Y^n - B^n}{d}, \quad y_{s_n} = B^n.$$

Let

$$\tilde{W}(G) = d^{1-2k(G)+p(G)} W(G)(d, d, d).$$

We have

1. $$\tilde{W}(E_n) = 1.$$ (27)

2. If $e$ is a bridge of $G$, then

$$\tilde{W}(G) = B^n ((Y/B)^n - (1 - d^2)) \tilde{W}(G/e).$$ (28)

If $e$ is not a bridge, then

$$\tilde{W}(G) = B^n (((Y/B)^n - 1) \tilde{W}(G - e) + \tilde{W}(G/e)).$$ (29)

Comparing Equations (27)-(29) with Equations (13)-(15), we obtain

**Corollary 10** In $Sh[G]$, if we replace $w$ by $1 - d^2$, and replace $a$ by $(Y/B)^{n_a}$ for every chain $a$, where $n_a$ is the width of the sheaf $a$, then we have

$$Q[G_s] = \frac{B^{\sum_{e \in E(G)} n_a}}{d^{p-2k+1}} Sh[G],$$

where $p$ is the number of vertices of $G$, and $k$ is the number of the connected components of $G$.

**Note 2** When $t = z_1 = z_2 = d$,

$$W(G)(d, d, d) = d^{k(G)-1} \sum_{S \subseteq E(G)} \prod_{e \in S} x_{c(e)} \left\{ \prod_{e \notin S} y_{c(e)} \right\} d^{k < S > - k(G)} d^{m < S >}$$

$$= d^{k(G)-1} \sum_{S \subseteq E(G)} \prod_{e \in S} x_{c(e)} \left\{ \prod_{e \notin S} y_{c(e)} \right\} d^{k < S > - k(G)} d^{(|S|-|V(G)|+k < S >)}$$

$$= d^{(-k(G)-|V(G)|)} \sum_{S \subseteq E(G)} \prod_{e \in S} x_{c(e)} \left\{ \prod_{e \notin S} y_{c(e)} \right\} d^{(|S|+2k < S >)}.$$
5 Kauffman brackets of rational links

In this section we use the relation built in the last section and transfer matrix approach to provide an explicit formula for the Kauffman brackets of general rational links $m_1m_2\cdots m_k$, where $m_1, m_2, \ldots, m_k$ are all non-zero integers. There are two kinds of rational links according to the parity of $k$. When $k$ is odd we call $m_1m_2\cdots m_k$ horizontal rational link, and when $k$ is even we call $m_1m_2\cdots m_k$ vertical rational link. Horizontal rational link $m_1m_2\cdots m_6$ and vertical rational link $m_1m_2\cdots m_7$ are shown in Figure 4 (upper) respectively. The horizontal and vertical boxes containing integers $m_i$ represent $m_i$ half-twists shown in Figure 4 (lower).

![Figure 4: Two kinds of rational links](image)

Horizontal and vertical rational link diagrams both end with a horizontal box, and begin with a horizontal box and a vertical box respectively. The graphs associated with the two kinds of rational links are shown in Figure 5. We denote them by $HF_n$ and $VF_n$ respectively.

**Theorem 11** Let $m_1m_2\cdots m_{2n}m_{2n+1}$ ($n \geq 1$) be a horizontal rational link. Let $B = A^{-1}$, $d = -A^2 - A^{-2}$, $X = -A^{-3}$ and $Y = -A^3$. Then

$$< m_1m_2\cdots m_{2n}m_{2n+1} > = d^{-n-1} X_0^T \left\{ \prod_{i=1}^{n} A_i \right\} J,$$

where $X_0 = (B^{m_1}d^2, Y^{m_1} - B^{m_1})$,

$$A_i = \begin{pmatrix} X^{m_{2i}}B^{m_{2i+1}}d \\ (X^{m_{2i}} - A^{m_{2i}})B^{m_{2i+1}}d \ -X^{m_{2i}}(Y^{m_{2i+1}} - B^{m_{2i+1}})d^{-1} & (X^{m_{2i}} - A^{m_{2i}})(Y^{m_{2i+1}} - B^{m_{2i+1}})d^{-1} + A^{m_{2i}}Y^{m_{2i+1}}d \end{pmatrix}$$
and $J = (1, 1)^T$.

**Proof:** For simplicity, we use $W(G)$ to denote $\sum_{S \subseteq E(G)} \{ \prod_{e \in S} x_{c(e)} \} \{ \prod_{e \notin S} y_{c(e)} \} d^{|S| + 2k < S>}$. We denote by $\Lambda_n$ the graph induced by the two edges $c_{m2n}$ and $s_{m2n+1}$, and view $HF_n$ as the union of $HF_{n-1}$ and $\Lambda_n$. Note that each $S \subseteq E(HF_n)$ can be written as $S = B \cup C$ with $B \subseteq E(HF_{n-1})$ and $C \subseteq E(\Lambda_n)$. We have

$$W(HF_n) = \sum_{S \subseteq E(HF_n)} \{ \prod_{e \in S} x_{c(e)} \} \{ \prod_{e \notin S} y_{c(e)} \} d^{|S| + 2k < S>}$$

$$= \sum_{B \cup C} \{ \prod_{e \in B} x_{c(e)} \} \{ \prod_{e \in C} x_{c(e)} \} \{ \prod_{e \notin B} y_{c(e)} \} \{ \prod_{e \notin C} y_{c(e)} \} d^{|B| + |C| + 2(k < B > + \delta(B,C))} \tag{31}$$

where $B \subseteq E(HF_{n-1})$, $C \subseteq E(\Lambda_n)$, $e \notin B$ means $e \in E(HF_{n-1}) - B$, $e \notin C$ means $e \in E(\Lambda_n) - C$, $k < B >$ is the number of connected components of $< B >$, the spanning subgraph with edge set $B$ of $HF_{n-1}$, and $\delta(B,C) = k < B \cup C > - k < B >$, where $k < B \cup C >$ is the number of connected components of $< B \cup C >$, the spanning subgraph with edge set $B \cup C$ of $HF_n$.

Note that $S \subseteq E(HF_n)$ and $B \subseteq E(HF_{n-1})$, we denote by $s_1$ the state of $S$ (resp. $B$) if $u$ and $v_n$ (resp. $v_{n-1}$) are disconnected in $< S >$ (resp. $< B >$), and by $s_2$ the state of $S$ (resp. $B$) if $u$ and $v_n$ (resp. $v_{n-1}$) are connected in $< S >$ (resp. $< B >$). It is not difficult to see that $\delta(B,C)$ can be computed from the knowledge of the state of $B$ and $C$. We list them in Table 1. The last two columns of the table are the state of $S = B \cup C$ and the contribution to the second sum of Equation (31).

Let $A_n$ be $2 \times 2$ matrix with $(i,j)$-entry $a^n_{i,j}$, where $a^n_{i,j}$ is the sum of contributions
the state of $B$ & $C$ & $\delta(B, C)$ & the state of $S$ & contribution
\hline
$s_1$ & $\emptyset$ & 1 & $s_1$ & $y_{cm_{2n}}y_{sm_{2n+1}}d^2$
$s_1$ & $\{s_{m_{2n+1}}\}$ & 0 & $s_2$ & $y_{cm_{2n}}x_{sm_{2n+1}}d$
$s_1$ & $\{c_{m_{2n}}\}$ & 0 & $s_1$ & $x_{cm_{2n}}y_{sm_{2n+1}}d$
$s_1$ & $\{s_{m_{2n+1}}, c_{m_{2n}}\}$ & -1 & $s_2$ & $x_{cm_{2n}}x_{sm_{2n+1}}$
$s_2$ & $\emptyset$ & 1 & $s_1$ & $y_{cm_{2n}}y_{sm_{2n+1}}d^2$
$s_2$ & $\{s_{m_{2n+1}}\}$ & 0 & $s_2$ & $y_{cm_{2n}}x_{sm_{2n+1}}d$
$s_2$ & $\{c_{m_{2n}}\}$ & 0 & $s_2$ & $x_{cm_{2n}}y_{sm_{2n+1}}d$
$s_2$ & $\{s_{m_{2n+1}}, c_{m_{2n}}\}$ & 0 & $s_2$ & $x_{cm_{2n}}x_{sm_{2n+1}}d^2$
\hline
Table 1: The states of $B$ and $S$ and the contributions

from initial state (the state of $B$) $s_i$ to final state $s_j$ (the state of $S$), that is,

$$a_{1,1}^n = y_{cm_{2n}}y_{sm_{2n+1}}d^2 + x_{cm_{2n}}y_{sm_{2n+1}}d,$$

$$a_{1,2}^n = y_{cm_{2n}}x_{sm_{2n+1}}d + x_{cm_{2n}}x_{sm_{2n+1}},$$

$$a_{2,1}^n = y_{cm_{2n}}y_{sm_{2n+1}}d^2 \text{ and}$$

$$a_{2,2}^n = x_{cm_{2n}}y_{sm_{2n+1}}d + y_{cm_{2n}}x_{sm_{2n+1}}d + x_{cm_{2n}}x_{sm_{2n+1}}d^2.$$ 

We divide $B'$s into two subclasses: one is the $B'$s with state $s_1$ denoted by $B : s_1$, and the other is the $B'$s with state $s_2$ denoted by $B : s_2$. Then we have

$$W(HF_n) = \sum_{B, e \in B} \{ \prod_{e \in B} x_{c(e)} \} \{ \prod_{e \notin B} y_{c(e)} \} d^{\mid B \mid + 2k < B} \sum_{C, e \in C} \{ \prod_{e \notin C} x_{c(e)} \} \{ \prod_{e \in C} y_{c(e)} \} d^{\mid C \mid + 2\delta(B,C)}$$

$$= \sum_{B : s_1} \{ \prod_{e \in B} x_{c(e)} \} \{ \prod_{e \notin B} y_{c(e)} \} d^{\mid B \mid + 2k < B} \sum_{C, e \in C} \{ \prod_{e \notin C} x_{c(e)} \} \{ \prod_{e \in C} y_{c(e)} \} d^{\mid C \mid + 2\delta(B,C)}$$

$$+ \sum_{B : s_2} \{ \prod_{e \in B} x_{c(e)} \} \{ \prod_{e \notin B} y_{c(e)} \} d^{\mid B \mid + 2k < B} \sum_{C, e \in C} \{ \prod_{e \notin C} x_{c(e)} \} \{ \prod_{e \in C} y_{c(e)} \} d^{\mid C \mid + 2\delta(B,C)}$$

$$= \sum_{B : s_1} \{ \prod_{e \in B} x_{c(e)} \} \{ \prod_{e \notin B} y_{c(e)} \} d^{\mid B \mid + 2k < B} (a_{1,1}^n + a_{1,2}^n)$$

$$+ \sum_{B : s_2} \{ \prod_{e \in B} x_{c(e)} \} \{ \prod_{e \notin B} y_{c(e)} \} d^{\mid B \mid + 2k < B} (a_{2,1}^n + a_{2,2}^n).$$

Note that the above two summands. We denoted them by $s_1^{n-1}$ and $s_2^{n-1}$ respectively, that is,

$$s_1^{n-1} = \sum_{B : s_1} \{ \prod_{e \in B} x_{c(e)} \} \{ \prod_{e \notin B} y_{c(e)} \} d^{\mid B \mid + 2k < B}$$

$$s_2^{n-1} = \sum_{B : s_2} \{ \prod_{e \in B} x_{c(e)} \} \{ \prod_{e \notin B} y_{c(e)} \} d^{\mid B \mid + 2k < B}.$$
On the other hand,

\[
W(HF_n) = \sum_{S \subset E(HF_n)} \{ \prod_{e \in S} x_{c(e)} \} \{ \prod_{e \not\in S} y_{c(e)} \} d^{|S|+2k<\mathcal{S}>}
\]

\[
= \sum_{S:s_1} \{ \prod_{e \in S} x_{c(e)} \} \{ \prod_{e \not\in S} y_{c(e)} \} d^{|S|+2k<\mathcal{S}>} + \sum_{S:s_2} \{ \prod_{e \in S} x_{c(e)} \} \{ \prod_{e \not\in S} y_{c(e)} \} d^{|S|+2k<\mathcal{S}>}.
\]

If we denote by \(s_1^n\) and \(s_2^n\) the above two summands, then \(s_1^n = s_{1}^{n-1} a_{1,1} + s_{2}^{n-1} a_{2,1}\) and \(s_2^n = s_{1}^{n-1} a_{1,2} + s_{2}^{n-1} a_{2,2}\). Thus we have

\[
W(HF_n) = s_1^n + s_2^n
\]

\[
= (s_1^n, s_2^n)J
\]

\[
= (s_1^{n-1}, s_2^{n-1}) A_n J
\]

\[
= (s_1^{n-2}, s_2^{n-2}) A_{n-1} J
\]

\[
= \ldots
\]

\[
= (s_1^0, s_2^0) \prod_{i=1}^{n} A_i J,
\]

where

\[
s_1^0 = \sum_{S \subset E(HF_0):s_1} \{ \prod_{e \in S} x_{c(e)} \} \{ \prod_{e \not\in S} y_{c(e)} \} d^{|S|+2k<\mathcal{S}>},
\]

\[
s_2^0 = \sum_{S \subset E(HF_0):s_2} \{ \prod_{e \in S} x_{c(e)} \} \{ \prod_{e \not\in S} y_{c(e)} \} d^{|S|+2k<\mathcal{S}>},
\]

\[
A_i = \begin{pmatrix} a_{1,1}^i & a_{1,2}^i \\ a_{2,1}^i & a_{2,2}^i \end{pmatrix}
\]

with

\[
a_{1,1}^i = y_{cm_{2i}} y_{sm_{2i+1}} d^2 + x_{cm_{2i}} y_{sm_{2i+1}} d,
\]

\[
a_{1,2}^i = y_{cm_{2i}} x_{sm_{2i+1}} d + x_{cm_{2i}} x_{sm_{2i+1}},
\]

\[
a_{2,1}^i = y_{cm_{2i}} y_{sm_{2i+1}} d^2 \quad \text{and}
\]

\[
a_{2,2}^i = x_{cm_{2i}} y_{sm_{2i+1}} d + y_{cm_{2i}} x_{sm_{2i+1}} d + x_{cm_{2i}} x_{sm_{2i+1}} d^2,
\]

and \(J = (1, 1)^T\).

Note that \(HF_0\) is the graph induced by the edge \(s_{m_1}\); the subset of \(E(HF_0)\) with state \(s_1\) is \(\emptyset\) and the subset of \(E(HF_0)\) with state \(s_2\) is \(\{s_{m_1}\}\). We obtain \((s_1^0, s_2^0) = (y_{sm_1} d^4, x_{sm_1} d^3)\).

By Theorem 8, let \(x_{cn} = A^n, y_{cn} = \frac{X^n - A^n}{d} x_{sn} = \frac{Y^n - B^n}{d}\) and \(y_{sn} = B^n\), we obtain \((s_1^0, s_2^0) = (B^m d^4, (Y^{m_1} - B^{m_1}) d^2) = d^2 (B^m d^2, Y^{m_1} - B^{m_1})\) and

\[
a_{1,1}^i = X^{m_{2i}} B^{m_{2i+1}} d,
\]

\[
a_{1,2}^i = X^{m_{2i}} (Y^{m_{2i+1}} - B^{m_{2i+1}}) d^{-1},
\]

\[
a_{2,1}^i = (X^{m_{2i}} - A^{m_{2i}}) B^{m_{2i+1}} d \quad \text{and}
\]

\[
a_{2,2}^i = (X^{m_{2i}} - A^{m_{2i}}) (Y^{m_{2i+1}} - B^{m_{2i+1}}) d^{-1} + A^{m_{2i}} Y^{m_{2i+1}} d.
\]
The theorem is proved $\square$

**Theorem 12** Let $m_1m_2 \cdots m_{2n+1}m_{2n+2}$ ($n \geq 1$) be a vertical rational link. Let $B = A^{-1}, d = -A^2 - A^{-2}, X = -A^{-3}$ and $Y = -A^3$. Then

$$< m_1m_2 \cdots m_{2n+1}m_{2n+2} > = d^{-n-2}X_0^T \prod_{i=1}^n A_i J,$$

where $X_0^T = ((X^{m_1} - A^{m_1})B^{m_2}d^2, A^{m_1}Y^{m_2}d^2 + (X^{m_1} - A^{m_1})(Y^{m_2} - B^{m_2}))$,

$$A_i = \left( \begin{array}{c} X^{m_2i+1}B^{m_2i+2}d \\ (X^{m_2i+1} - A^{m_2i+2})B^{m_2i+2}d \\ (X^{m_2i+1} - A^{m_2i+1})(Y^{m_2i+2} - B^{m_2i+2})d^{-1} \\ (X^{m_2i+1} - A^{m_2i+1})(Y^{m_2i+2} - B^{m_2i+2})d^{-1} + A^{m_2i+1}Y^{m_2i+2}d \end{array} \right)$$

and $J = (1, 1)^T$.

**Proof:** The proof is similar to that of Theorem 11. The only difference is that $V F_0$ is the graph induced by two edges $c_{m_1}$ and $s_{m_2}$, the subset of $E(V F_0)$ with state $s_1$ is $\emptyset$ and the subset of $E(H F_0)$ with state $s_2$ is $\{c_{m_1}\}, \{s_{m_2}\}$ and $\{c_{m_1}, s_{m_2}\}$. Thus

$$s_1^0 = y_{c_{m_1}} y_{s_{m_2}} d^4,$$

$$s_2^0 = x_{c_{m_1}} y_{s_{m_2}} d^3 + y_{c_{m_1}} x_{s_{m_2}} d^3 + x_{c_{m_1}} x_{s_{m_2}} d^4.$$

After replacement, we have

$$s_1^0 = (X^{m_1} - A^{m_1})B^{m_2}d^3,$$

$$s_2^0 = A^{m_1}Y^{m_2}d^3 + (X^{m_1} - A^{m_1})(Y^{m_2} - B^{m_2})d.$$

$\square$

There are only $m_1m_2$ (i.e. generalized twist link) shown in Figure 3 and $m_1$ (i.e. $(m_1,2)$-torus link) shown in Figure 6 left for us to compute.

![Figure 6: $(m_1,2)$-torus link and its associated graph](image)

1. **Kauffman bracket of rational link $m_1$.**

The graph associated with $m_1$ is the graph induced by an edge $e_1$, which represents a sheaf with width $m_1$ and its $W$-polynomial is $x_{c(e_1)} + z_1 y_{c(e_1)}$. By Theorem 8, the Kauffman bracket of $m_1$ is

$$\frac{Y^{m_1} - B^{m_1}}{d} + B^{m_1}d = \frac{1}{-A^2 - A^{-2}} \{(-A^3)^{m_1} - A^{-m_1} + A^{-m_1}(-A^2 - A^{-2})\}.$$
2. Kauffman bracket of rational link $m_1m_2$.

The graph associated with $m_1m_2$ is the graph with two vertices by two parallel edges, say, $e_1$ and $e_2$ (see Figure 3, right). By the definition or Theorem 4, the $W$-polynomial of this graph is

$$
(x_{c(e_1)} + z_1 y_{c(e_1)}) y_{c(e_2)} + (x_{c(e_1)} z_2 + y_{c(e_1)}) x_{c(e_2)}.
$$

Note that the length of the chain $e_1$ is $m_1$ and the width of the sheaf $e_2$ is $m_2$, by Theorem 8, the Kauffman bracket of $m_1m_2$ is

$$
(A^{m_1} + d \frac{X^{m_1} - A^{m_1}}{d}) B^{m_2} + (A^{m_1} d + \frac{X^{m_1} - A^{m_1}}{d}) Y^{m_2} - B^{m_2}.
$$

After reduction, we obtain

$$
<m_1m_2> = A^{m_1-m_2} \left\{ (-A^{-4})^{m_1} + (-A^4)^{m_2} - 1 \right\} +
\frac{A^{m_1-m_2}}{A^{-4} + 2 + A^4} \left\{ (-A^{-4})^{m_1-m_2} - (-A^{-4})^{m_1} - (-A^4)^{m_2} + 1 \right\}.
$$

(32)

6 Further examples

It is clear we can also use Theorem 8 to compute the Kauffman brackets of other link families of rational links type. For example, using Corollary 9 the explicit formulae of Kauffman brackets of pretzel links are obtained by the present author in [8]. In this section we provide another example. We denote by $L(m_1, m_2, m_3)$ the link shown in Figure 7 when $m_i > 0$ for $i = 1, 2, 3$. When $m_i < 0$, the link diagram is obtained by replacing the corresponding half-twists by its mirror image.

![Figure 7](image)

Figure 7: The link $L(m_1, m_2, m_3)$ and its associated graph $\Theta$

The graph $\Theta$ associated with $L(m_1, m_2, m_3)$ has three spanning trees with edge sets \{c$_{m_1}$\}, \{s$_{m_2}$\} and \{c$_{m_3}$\} respectively. By Theorem 4, we have

$$
W(\Theta)(d, d, d) = \sum_F \prod_{IA} (x_{c(e)} + dy_{c(e)}) \prod_{EA} (x_{c(e)} d + y_{c(e)}) \prod_{H} x_{c(e)} \prod_{EI} y_{c(e)}
$$

$$
= (x_{c m_1} + dy_{c m_1}) y_{s m_2} y_{c m_3} + (x_{c m_1} d + y_{c m_1}) x_{s m_2} y_{c m_3}
$$

$$
+ (x_{c m_1} d + y_{c m_1}) (x_{s m_2} d + y_{s m_2}) x_{c m_3}.
$$

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By Theorem 8, we obtain the Kauffman brackets of \( L(m_1, m_2, m_3) \)

\[
< L(m_1, m_2, m_3) > = X^{m_1} B^{m_2} (X^{m_3} - A^{m_3}) / d + \\
(A^{m_1} d + (X^{m_1} - A^{m_1}) / d)(Y^{m_2} - B^{m_2}) / d (X^{m_3} - A^{m_3}) / d + \\
((d^2 - 1) A^{m_1} + X^{m_1}) Y^{m_2} A^{m_3}.
\]

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