A note on Hardy-type inequalities in variable exponent Lebesgue spaces

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Abstract

We present new estimate for Hardy-type inequality in variable exponent Lebesgue spaces. More precisely, by imposing regularity assumptions on the exponent, we prove that the estimations can be reduced to the fixed exponents.

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Key Words and Phrases: Hardy inequality, variable exponent.

1 Introduction

It is well known that Hardy-type inequalities play an important role in Harmonic Analysis. For instance, they appear in the interpolation of spaces \([1]\), in the study of hyperbolic partial differential equations and for studying the decay of linear waves on black hole back \([7]\). We can find some interesting applications of Hardy-type inequalities in \([15]\) and references therein.

The classical Hardy inequalities says that

\[
\left\| t^s \int_t^\infty \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau} \right\|_{L^q((0,\infty),\mathbb{L})} + \left\| t^{-s} \int_0^t \tau^s \varepsilon_\tau \frac{d\tau}{\tau} \right\|_{L^q((0,\infty),\mathbb{L})} \lesssim \left\| \varepsilon_\tau \right\|_{L^p((0,\infty),\mathbb{L})}
\]

for any \( s > 0 \) and any \( 1 \leq q < \infty \). This statement in variable exponent Lebesgue spaces was first proved by V. Kokilashvili and S. Samko \([17]\) and by L. Diening and S. Samko \([9]\) under the assumption that \( q \) is log-Hölder continuous both at the origin and at infinity. More results for Hardy-type inequalities in variable exponent Lebesgue spaces can be found in S. Boza and J. Soria \([2]\), Cruz-Uribe and Mamedov \([5]\), P. Harjulehto, P. Hästö, and M. Koskinoja \([14]\), F. I. Mamedov and A. Harman \([18]\), and references therein. Here under the same assumptions we prove that \( \left\| \varepsilon_\tau \right\|_{L^p((0,\infty),\mathbb{L})} \) can be replaced by

\[
\left( \int_0^1 \varepsilon_t^{q_t(0)} \frac{dt}{t} \right)^{\frac{1}{q_t(0)}} + \left( \int_1^\infty \varepsilon_t^{q_t(\infty)} \frac{dt}{t} \right)^{\frac{1}{q_t(\infty)}}.
\]

2 Preliminaries

As usual, we denote by \( \mathbb{R} \) the reals, \( \mathbb{N} \) the collection of all natural numbers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). The letter \( \mathbb{Z} \) stands for the set of all integer numbers. The expression \( f \lesssim g \)
means that $f \leq cg$ for some independent constant $c$ (and non-negative functions $f$ and $g$), and $f \approx g$ means $f \lesssim g \lesssim f$.

If $E \subseteq \mathbb{R}$ is a measurable set, then $\chi_E$ denotes the characteristic function of the set $E$.

By $c$ we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g. $c(p)$ means that $c$ depends on $p$, etc.). Further notation will be properly introduced whenever needed.

The variable exponents that we consider are always measurable functions $p$ on $\mathbb{R}$ with range in $[1, \infty]$. We denote the set of such functions by $\mathcal{P}(\mathbb{R})$. We use the standard notation $p^- := \text{ess-inf}_{x \in \mathbb{R}} p(x)$, $p^+ := \text{ess-sup}_{x \in \mathbb{R}} p(x)$.

The variable exponent modular is defined by

$$
\varrho_{p(.)}(f) := \int_{\mathbb{R}} \varrho_{p(x)}(\|f(x)\|)dx,
$$

where $\varrho_p(t) = t^p$. The variable exponent Lebesgue space $L^{p(.)}$ consists of measurable functions $f$ on $\mathbb{R}$ such that $\varrho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg norm on this space by the formula

$$
\|f\|_{p(.)} := \inf \left\{ \lambda > 0 : \varrho_{p(.)} \left( \frac{f}{\lambda} \right) \leq 1 \right\}.
$$

A useful property is that $\|f\|_{p(.)} \leq 1$ if and only if $\varrho_{p(.)}(f) \leq 1$, see [11], Lemma 3.2.4. Let $p, q \in \mathcal{P}(\mathbb{R})$. The mixed Lebesgue-sequence space $l^{p(.)}(L^{q(.)})$ is defined on sequences of $L^{p(.)}$-functions by the modular

$$
\varrho_{l^{p(.)}(L^{q(.)})}((f_v)_v) := \sum_{v = -\infty}^{\infty} \inf \left\{ \lambda_v > 0 : \varrho_{p(.)} \left( \frac{f_v}{\lambda_v^{1/q(.)}} \right) \leq 1 \right\}.
$$

The (quasi)-norm is defined from this as usual:

$$
\|(f_v)_v\|_{l^{p(.)}(L^{q(.)})} := \inf \left\{ \mu > 0 : \varrho_{l^{p(.)}(L^{q(.)})} \left( \frac{1}{\mu}(f_v)_v \right) \leq 1 \right\}. \quad (1)
$$

If $q^+ < \infty$, then we can replace (1) by the simpler expression $\varrho_{l^{p(.)}(L^{q(.)})}((f_v)_v) := \sum_{v = -\infty}^{\infty} \|f_v\|_{l^{p(.)}(L^{q(.)})}$.

We say that a function $g : \mathbb{R} \to \mathbb{R}$ is log-Hölder continuous at the origin (or has a log decay at the origin), if there exists a constant $c_{log}(g) > 0$ such that

$$
|g(x) - g(0)| \leq \frac{c_{log}(g)}{\ln(e + 1/|x|)}
$$

for all $x \in \mathbb{R}$. If, for some $g_{\infty} \in \mathbb{R}$ and $c_{log} > 0$, there holds

$$
|g(x) - g_{\infty}| \leq \frac{c_{log}}{\ln(e + |x|)}
$$

for all $x \in \mathbb{R}$, then we say that $g$ is log-Hölder continuous at infinity (or has a log decay at infinity). The constants $c_{log}(g)$ and $c_{log}$ are called the locally log-Hölder constant and the log-Hölder decay constant, respectively. We refer to the recent monograph [6] for further properties, historical remarks and references on variable exponent spaces.
2.1 Technical lemmas

In this subsection we present some results which are useful for us. The following lemma is a Hardy-type inequality which is easy to prove.

**Lemma 1** Let $0 < a < 1$, $\sigma \geq 0$ and $0 < p \leq \infty$. Let $\{\varepsilon_k\}_k$ be a sequences of positive real numbers and denote $\delta_k = \sum_{j=-\infty}^{\infty} |k-j|^\sigma a^{k-j}\varepsilon_j$. Then there exists constant $c > 0$ depending only on $a$ and $p$ such that

$$\left(\sum_{k=-\infty}^{\infty} \delta_k^p\right)^{1/p} \leq c \left(\sum_{k=-\infty}^{\infty} \varepsilon_k^p\right)^{1/p}.$$ 

We will make use of the following statement, see [12], Lemma 3.3 for $w := 1$.

**Lemma 2** Let $Q = (a, b) \subset \mathbb{R}$ with $0 < a < b < \infty$. Let $p \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous at the origin and $w : \mathbb{R} \to \mathbb{R}^+$ be a weight function. Then for every $m > 0$ there exists $\gamma = e^{-4mc\log(1/p)} \in (0, 1)$ such that

$$\left(\frac{\gamma}{w(Q)} \int_Q |f(y)| w(y) dy\right)^{p(y)} \leq \max \left(1, (w(Q))^{1-p(y)/p}\right) \frac{1}{w(Q)} \int_Q |f(y)|^{p(y,0)} w(y) dy + \omega(m, b) \left(\frac{1}{w(Q)} \int_Q g(x, y) w(y) dy\right)$$

hold if $0 < w(Q) < \infty$, all $x \in Q \subset \mathbb{R}$ and all $f \in L^p(w) + L^\infty$ with $\|fw^{1/p(\cdot)}\|_{p(\cdot)} + \|f\|_{\infty} \leq 1$, where

$$\omega(m, b) = \min(b^m, 1), \quad p(y, 0) = p(y) \quad \text{and} \quad g(x, y) = (e + \frac{1}{x})^{-m} + (e + \frac{1}{y})^{-m}$$

or

$$\omega(m, b) = \min(b^m, 1), \quad p(y, 0) = p(0) \quad \text{and} \quad g(x, y) = (e + \frac{1}{x})^{-m} \chi_{\{|x|p(x) < p(0)\}}(x).$$

In addition we have the same estimate, where

$$\omega(m, b) = 1, \quad p(y, 0) = p_\infty \quad \text{and} \quad g(x, y) = (e + x)^{-m} \chi_{\{|x|p(x) < p_\infty\}}(x),$$

if $p \in \mathcal{P}(\mathbb{R})$ satisfies the log-Hölder decay condition, where we take $\gamma = e^{-4mc\log}$.

Notice that in the proof of this theorem we need only that

$$\int_Q |f(y)|^{p(y)} w(y) dy \leq 1$$

and/or $\|f\|_{\infty} \leq 1$. The proof of this lemma is given in [13].
3 Main results

Various important results have been proved in the space $L^{p(\cdot)}$ under some assumptions on $p$ such as the boundedness of the maximal operator in $L^{p(\cdot)}$ spaces on bounded domains. This fact was first realized by L. Diening [8]. This statement was then extended to the unbounded case by D. Cruz-Uribe, A. Fiorenza and C. Neugebauer [4]. Estimates for potential type operators in variable $L^{p(\cdot)}$ spaces were first considered by Samko [19]. Fractional maximal operators were first studied in this setting by Kokilashvili and Samko [16]. We refer to [11] for further contributions and historical remarks in the study of singular integral and fractional integral operators in variable exponent spaces. As mentioned in the introduction we present some new estimate for Hardy operators $\int_1^\infty \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau}$ and $\int_0^\infty \tau^s \varepsilon_\tau \frac{d\tau}{\tau}$, $t > 0$ in variable exponent Lebesgue spaces. More precisely, we have the following results:

**Theorem 1** Let $s > 0$. Let $p \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous both at the origin and at infinity with $1 \leq p^- \leq p^+ < \infty$. Let $\{\varepsilon_t\}_t$ be a sequence of positive measurable functions. Let

$$\eta_t = t^s \int_t^\infty \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau} \quad \text{and} \quad \lambda_t = t^{-s} \int_0^t \tau^s \varepsilon_\tau \frac{d\tau}{\tau}. $$

Then there exists constant $c > 0$ depending only on $s$, $p^-$, $\alpha_{\log}(p)$ and $p^+$ such that

$$\|\eta_t\|_{L^{p(\cdot)}((0,\infty), \frac{dt}{t})} \approx \left( \int_0^1 \eta_t^{p(0)} \frac{dt}{t} \right)^{\frac{1}{p(0)}} + \left( \int_1^\infty \eta_t^{p(\infty)} \frac{dt}{t} \right)^{\frac{1}{p(\infty)}}, \quad (2)$$

and

$$\|\lambda_t\|_{L^{p(\cdot)}((0,\infty), \frac{dt}{t})} \approx \left( \int_0^1 \lambda_t^{p(0)} \frac{dt}{t} \right)^{\frac{1}{p(0)}} + \left( \int_1^\infty \lambda_t^{p(\infty)} \frac{dt}{t} \right)^{\frac{1}{p(\infty)}}, \quad (3)$$

Moreover,

$$\|\eta_t\|_{L^{p(\cdot)}((0,\infty), \frac{dt}{t})} + \|\lambda_t\|_{L^{p(\cdot)}((0,\infty), \frac{dt}{t})} \lesssim \left( \int_0^1 \varepsilon_t^{p(0)} \frac{dt}{t} \right)^{\frac{1}{p(0)}} + \left( \int_1^\infty \varepsilon_t^{p(\infty)} \frac{dt}{t} \right)^{\frac{1}{p(\infty)}}. \quad \text{Proof.}$$

We will do the proof in several steps.

**Step 1.** We prove that

$$\|\eta_t\|_{L^{p(\cdot)}((0,\infty), \frac{dt}{t})} \lesssim \left( \int_0^1 \eta_t^{p(0)} \frac{dt}{t} \right)^{\frac{1}{p(0)}} + \left( \int_1^\infty \eta_t^{p(\infty)} \frac{dt}{t} \right)^{\frac{1}{p(\infty)}}. $$

We suppose that the right-hand side is less than or equal one. Notice that

$$\|\eta_t\|_{L^{p(\cdot)}((0,\infty), \frac{dt}{t})} \approx \left\| \left( t^{-\frac{1}{p(\cdot)}} \eta_t \chi_{[2^v, 2^{v+1}]^c} \right) \right\|_{L^{p(\cdot)}(\mathbb{R})}. $$

We see that

$$\eta_t = \frac{t^s}{\log 2} \int_2^t t^{-s} \eta_t \frac{d\tau}{\tau} \lesssim \frac{t^s}{\log 2} \int_2^t \tau^{-s} \eta_t \frac{d\tau}{\tau} \leq \frac{t^s}{\log 2} \int_{2^{v-1}}^\infty \tau^{-s} \eta_t \frac{d\tau}{\tau},$$

for any $v \in \mathbb{Z}$ and any $t \in [2^v, 2^{v+1}]$. We write,

$$t^s \int_{2^{v-1}}^\infty \tau^{-s} \eta_t \frac{d\tau}{\tau} = \sum_{j=v-1}^{\infty} t^s \int_{2^j}^{2^{j+1}} \tau^{-s} \eta_t \frac{d\tau}{\tau}, \quad v \in \mathbb{Z}.$$
Substep 1.1. \( v \leq 0 \). For any \( t \in [2^v, 2^{v+1}] \), we write

\[
\eta_{t,v} = \sum_{j=v-1}^{\infty} t^j \int_{2^j}^{2^{j+1}} \tau^{-s} \eta_{\tau} \frac{d\tau}{\tau} = \sum_{j=v-1}^{-1} t^j \cdots + \sum_{j=0}^{\infty} \cdots = \eta_{t,1,v} + \eta_{t,2,v}.
\]

**Estimation of \( \eta_{t,1,v} \).** Let \( \sigma > 0 \) be such that \( p^+ < \sigma \). We have

\[
\left( \sum_{j=v-1}^{-1} t^j \int_{2^j}^{2^{j+1}} \tau^{-s} \eta_{\tau} \frac{d\tau}{\tau} \right)^{p(t)/\sigma} \leq \sum_{j=v-1}^{-1} \left( \int_{2^j}^{2^{j+1}} \tau^{-s} \eta_{\tau} \frac{d\tau}{\tau} \right)^{p(t)/\sigma} \leq \sum_{j=v-1}^{-1} 2^{-jsp(t)/\sigma} \left( \int_{2^j}^{2^{j+1}} \eta_{\tau} \frac{d\tau}{\tau} \right)^{p(t)/\sigma} = 2^{-vsp(t)/\sigma} \sum_{j=v-1}^{-1} 2^{(v-j)sp(t)/\sigma} \left( \int_{2^j}^{2^{j+1}} \eta_{\tau} \frac{d\tau}{\tau} \right)^{p(t)/\sigma}.
\]

By Hölder’s inequality, we estimate this expression by

\[
c2^{-vsp(t)/\sigma} \left( \sum_{j=v-1}^{-1} 2^{(v-j)sp(t)/\sigma} \left( \int_{2^j}^{2^{j+1}} \eta_{\tau} \frac{d\tau}{\tau} \right)^{p(t)/\sigma} \right)^{1/\sigma},
\]

where \( c > 0 \) is independent of \( v \) and \( j \). By Lemma 2 we find \( m > 0 \) such that

\[
\left( \frac{1}{(j - v + 3) \log 2} \int_{2^j}^{2^{j+2}} \eta_{\tau} \chi_{[2^j, 2^{j+1}]}(\tau) \frac{d\tau}{\tau} \right)^{p(t)} \lesssim \frac{1}{j - v + 3} \int_{2^j}^{2^{j+2}} \eta_{\tau}^{p(0)} \chi_{[2^j, 2^{j+1}]}(\tau) \frac{d\tau}{\tau} + 2^{jm} \chi_{\{t: q(t) < q(0)\}}(t)
\]

\[
\lesssim \frac{1}{j - v + 3} \int_{2^j}^{2^{j+2}} \eta_{\tau}^{p(0)} \frac{d\tau}{\tau} + 2^{jm} \chi_{\{t: q(t) < q(0)\}}(t)
\]

for any \( v - 1 \leq j \leq -1 \) and any \( t \in [2^v, 2^{v+1}] \subset [2^{v-1}, 2^{j+2}] \). Therefore,

\[
\eta_{t,1,v}^{p(t)} \lesssim \sum_{j=v-1}^{-1} 2^{(v-j)sp(-)/\sigma} (j - v + 3)^{p+1} \frac{1}{j - v + 3} \int_{2^j}^{2^{j+2}} \eta_{\tau}^{p(0)} \frac{d\tau}{\tau} + h_v
\]

for any \( t \in [2^v, 2^{v+1}] \), where

\[
h_v = \sum_{j=v-1}^{-1} 2^{(v-j)sp(-)/\sigma} (j - v + 3)^{p+1} 2^{jm}.
\]

We have \( \int_{2^v}^{2^{v+1}} \frac{dt}{\tau} \lesssim 1 \). Therefore,

\[
\int_{2^v}^{2^{v+1}} \eta_{t,1,v}^{p(t)} \frac{dt}{t} \lesssim \sum_{j=v-1}^{-1} 2^{(v-j)p(-)/\sigma} (j - v + 3)^{p+1} \frac{1}{j - v + 3} \int_{2^j}^{2^{j+2}} \eta_{\tau}^{p(0)} \frac{d\tau}{\tau} + h_v.
\]
Applying Lemma 1 we get
\[ \sum_{v=-\infty}^{0} \int_{2^v}^{2^{v+1}} \frac{\eta_{t,1,v}^p(t)}{t} dt \lesssim \sum_{j=-\infty}^{-1} \int_{2^j}^{2^{j+1}} \eta_{\tau}^p(0) \frac{d\tau}{\tau} + c \lesssim \int_{0}^{1} \eta_{\tau}^p(0) \frac{d\tau}{\tau} + c \lesssim 1, \]
by taking \( m \) large enough such that \( m > 0 \).

Estimation of \( \eta_{t,2,v} \). Let \( \sigma > 0 \) be such that \( p^+ < \sigma \). Again, we have
\[ \left( \sum_{j=0}^{\infty} \int_{2^j}^{2^{j+1}} \tau^{-s} \eta_{\tau}^p d\tau \right)^{p(t)/\sigma} \leq 2^{-v \sigma(p(t))/\sigma} \sum_{j=0}^{\infty} 2^{(v-j)s} p(t)/\sigma \left( \int_{2^j}^{2^{j+1}} \eta_{\tau}^p d\tau \right)^{p(t)/\sigma}. \]

By Hölder’s inequality, we estimate this expression by
\[ c 2^{-v \sigma(p(t))/\sigma} \left( \sum_{j=0}^{\infty} 2^{(v-j)s} p(t)/\sigma \left( \int_{2^j}^{2^{j+1}} \eta_{\tau}^p d\tau \right)^{p(t)/\sigma} \right)^{1/\sigma}. \]

Again, by Lemma 2 we find \( m > 0 \) such that
\[ \left( \frac{1}{(j-v+1) \log 2} \int_{2^v}^{2^{j+1}} \eta_{\tau}^p \chi_{[2^j,2^{j+1}]}(\tau) \frac{d\tau}{\tau} \right)^{p(t)} \lesssim \frac{1}{j-v+1} \int_{2^v}^{2^{j+1}} \eta_{\tau}^p \chi_{[2^j,2^{j+1}]}(\tau) \frac{d\tau}{\tau} + 1 \]
\[ \lesssim \frac{1}{j-v+1} \int_{2^j}^{2^{j+1}} \eta_{\tau}^p d\tau + 1 \]
for any \( j \geq 0 \) and any \( t \in [2^v,2^{v+1}] \subset [2^v,2^{j+1}] \). Therefore,
\[ \eta_{t,2,v}^p \lesssim \sum_{j=0}^{\infty} 2^{(v-j)s} p^+/(s-\sigma)(j-v+1)^{p^+} \int_{2^j}^{2^{j+1}} \eta_{\tau}^p d\tau + h_v, \]
for any \( t \in [2^v,2^{v+1}] \), where
\[ h_v = \sum_{j=0}^{\infty} 2^{(v-j)s} p^+/(s-\sigma)(j-v+1)^{p^+}. \]

We have \( \int_{2^v}^{2^{v+1}} \frac{dt}{t} \lesssim 1 \). Observe that
\[ h_v \leq 2^{\frac{w^{-1}}{2\sigma}} \sum_{j=0}^{\infty} 2^{(v-j)s} p^+/(s-\sigma)(j-v+1)^{p^+} \lesssim 2^{\frac{w^{-1}}{2\sigma}}, \quad v \leq 0. \]

Therefore,
\[ \int_{2^v}^{2^{v+1}} \frac{\eta_{t,2,v}^p(t)}{t} dt \lesssim \sum_{j=0}^{\infty} 2^{(v-j)s} p^+/(s-\sigma)(j-v+1)^{p^+} \int_{2^j}^{2^{j+1}} \eta_{\tau}^p d\tau + 2^{\frac{w^{-1}}{2\sigma}}. \]
Again, by Lemma 1 we get
\[
\sum_{v=-\infty}^{-1} \int_{2^v}^{2^{v+1}} \eta_{\tau,2,v}^{p(t)} \frac{dt}{t} \lesssim \sum_{j=0}^{\infty} \int_{2j}^{2j+1} \eta_{\tau}^{p_{\infty}} \frac{d\tau}{\tau} + c \lesssim \int_{1}^{\infty} \eta_{\tau}^{p_{\infty}} \frac{d\tau}{\tau} + c \lesssim 1.
\]

**Substep 1.2.** \(v > 0\). Let \(\sigma > 0\) be such that \(p^+ < \sigma\). We have
\[
\left( \sum_{j=v-1}^{\infty} \int_{2^j}^{2^{j+1}} \tau^{-s} \eta_{\tau} \frac{d\tau}{\tau} \right)^{p(t)/\sigma} \lesssim 2^{-v p(t)/\sigma} \sum_{j=v-1}^{\infty} 2^{(v-j) sp(t)/\sigma} \left( \int_{2^j}^{2^{j+1}} \eta_{\tau} \frac{d\tau}{\tau} \right)^{p(t)/\sigma}.
\]

By Hölder’s inequality, we estimate this expression by
\[
c 2^{-v p(t)/\sigma} \left( \sum_{j=v-1}^{\infty} 2^{(v-j) sp(t)/\sigma} \left( \int_{2^j}^{2^{j+1}} \eta_{\tau} \frac{d\tau}{\tau} \right)^{p(t)/\sigma} \right)^{1/\sigma}.
\]

Applying Lemma 2 we find \(m > 0\) such that
\[
\left( \frac{1}{(j-v+3) \log 2} \int_{2^{v-1}}^{2^{j+2}} \eta_{\tau} \chi_{[2^v,2^{j+1}]}(\tau) \frac{d\tau}{\tau} \right)^{p(t)} \lesssim \frac{1}{j-v+3} \int_{2^{v-1}}^{2^{j+2}} \eta_{\tau}^{p_{\infty}} \chi_{[2^v,2^{j+1}]}(\tau) \frac{d\tau}{\tau} + 2^{-vm}
\]
\[
\lesssim \frac{1}{j-v+3} \int_{2^j}^{2^{j+1}} \eta_{\tau}^{p_{\infty}} \frac{d\tau}{\tau} + 2^{-vm}
\]
for any \(j \geq v-1\) and any \(t \in [2^v, 2^{v+1}] \subset [2^{v-1}, 2^{j+2}]\). Therefore,
\[
\eta_{\tau,v}^{p(t)} \lesssim \sum_{j=v-1}^{\infty} 2^{(v-j) sp(t)/\sigma} (j-v+2)^{p^+/\sigma} \int_{2^j}^{2^{j+1}} \eta_{\tau}^{p_{\infty}} \frac{d\tau}{\tau} + h_v
\]
for any \(t \in [2^v, 2^{v+1}]\), where
\[
h_v = \sum_{j=v-1}^{\infty} 2^{(v-j) sp(t)/\sigma} (j-v+2)^{p^+/\sigma} 2^{-vm}.
\]

Therefore,
\[
\int_{2^v}^{2^{v+1}} \eta_{\tau,v}^{p(t)} \frac{dt}{t} \lesssim \sum_{j=v-1}^{\infty} 2^{(v-j) sp(t)/\sigma} (j-v+2)^{p^+/\sigma} \int_{2^j}^{2^{j+1}} \eta_{\tau}^{p_{\infty}} \frac{d\tau}{\tau} + h_v.
\]

Applying Lemma 1 we get
\[
\sum_{v=1}^{\infty} \int_{2^v}^{2^{v+1}} \eta_{\tau,v}^{p(t)} \frac{dt}{t} \lesssim \sum_{v=1}^{\infty} \int_{2^v}^{2^{v+1}} \eta_{\tau}^{p_{\infty}} \frac{d\tau}{\tau} + c \lesssim \int_{1}^{\infty} \eta_{\tau}^{p_{\infty}} \frac{d\tau}{\tau} + c \lesssim 1,
\]
by taking \(m\) large enough such that \(m > 0\). The proof is completed by the scaling argument.
Step 2. We prove that
\[
\left( \int_0^1 \eta_t^{p(0)} \frac{dt}{t} \right)^{\frac{1}{p(0)}} + \left( \int_1^{\infty} \eta_t^{p(\infty)} \frac{dt}{t} \right)^{\frac{1}{p(\infty)}} \lesssim \| \eta_t \|_{L^p((0, \infty), \frac{dt}{t})} .
\] (4)

We suppose that the right-hand side is less than or equal one. We will prove that
\[
\sum_{v=1}^{\infty} \int_{2^{-v}}^{2^{1-v}} \eta_t^{p(0)} \frac{dt}{t} \lesssim 1.
\]
Clearly follows from the inequality
\[
\eta_t^{p(0)} \lesssim \int_{2^{-v-1}}^{2^{1-v}} \eta_\tau^{p(\tau)} \frac{d\tau}{\tau} + 2^{-v} = \delta
\]
for any \( v \in \mathbb{N} \) and any any \( t \in [2^{-v}, 2^{1-v}] \). This claim can be reformulated as showing that
\[
\left( \delta^{-\frac{1}{p(0)}} \eta_t \right)^{p(0)} \lesssim \left( \frac{1}{\log 2} \int_{2^{-v-1}}^{2^{1-v}} \delta^{-\frac{1}{p(0)}} \eta_\tau^{p(\tau)} \frac{d\tau}{\tau} \right)^{p(0)} \lesssim 1.
\]
By Lemma [2]
\[
\left( \frac{\gamma}{\log 2} \int_{2^{-v-1}}^{2^{1-v}} \delta^{-\frac{1}{p(0)}} \eta_\tau^{p(\tau)} \frac{d\tau}{\tau} \right)^{p(0)} \lesssim \int_{2^{-v-1}}^{2^{1-v}} \delta^{-\frac{1}{p(0)}} \eta_\tau^{p(\tau)} \frac{d\tau}{\tau} + 1,
\]
where \( \gamma = e^{-4mc\log(1/p)} \) and \( m > 0 \). We use the log-Hölder continuity of \( p \) at the origin to show that
\[
\delta^{-\frac{1}{p(0)}} \approx \delta^{-1}, \quad \tau \in [2^{-v-1}, 2^{1-v}], v \in \mathbb{N}.
\]
Therefore, from the definition of \( \delta \), we find that
\[
\int_{2^{-v-1}}^{2^{1-v}} \delta^{-1} \eta_\tau^{p(\tau)} \frac{d\tau}{\tau} \lesssim 1
\]
for any \( v \in \mathbb{N} \) and this implies that
\[
\left( \delta^{-\frac{1}{p(0)}} \eta_t \right)^{p(0)} \lesssim 1
\]
for any \( v \in \mathbb{N} \) and any any \( t \in [2^{-v}, 2^{1-v}] \).

Now, we will prove that
\[
\sum_{v=1}^{\infty} \int_{2^v}^{2^{v+1}} \eta_t^{p(\infty)} \frac{dt}{t} \lesssim 1.
\]
Clearly follows from the inequality
\[
\eta_t^{p(\infty)} \lesssim \int_{2^v-1}^{2^{v+1}} \eta_\tau^{p(\tau)} \frac{d\tau}{\tau} + 2^{-v} = \delta
\]
for any \( v \in \mathbb{N} \) and any any \( t \in [2^v, 2^{v+1}] \). This claim can be reformulated as showing that
\[
\left( \delta^{-\frac{1}{p(\infty)}} \eta_t \right)^{p(\infty)} \lesssim \left( \frac{1}{\log 2} \int_{2^v-1}^{2^{v+1}} \delta^{-\frac{1}{p(\infty)}} \eta_\tau^{p(\tau)} \frac{d\tau}{\tau} \right)^{p(\infty)} \lesssim 1.
\]
By Lemma 2, 

\[
\left( \frac{\gamma}{\log 2} \int_{2^{v-1}}^{2^{v+1}} \delta^{-p_{\infty}} \eta_{\tau} \frac{d\tau}{\tau} \right)^{p(t)} \lesssim \int_{2^{v-1}}^{2^{v+1}} \delta^{-p_{\infty}} \eta_{\tau} \frac{d\tau}{\tau} + 1,
\]

where \( \gamma = e^{-4mc_{\log}} \) and \( m > 0 \). We use the logarithmic decay condition on \( q \) at infinity to show that

\[
\delta^{-p_{\infty}} \approx \delta^{-1}, \quad \tau \in [2^{v-1}, 2^{v+1}], \quad v \in \mathbb{N}.
\]

Therefore, from the definition of \( \delta \), we find that

\[
\int_{2^{v-1}}^{2^{v+1}} \delta^{-1} \eta_{\tau} \frac{d\tau}{\tau} \lesssim 1
\]

for any \( v \in \mathbb{N} \) and this implies that for any \( v \in \mathbb{N} \) any any \( t \in [2^v, 2^{v+1}] \),

\[
(\delta^{-p_{\infty}} \eta_{\tau})^{p_{\infty}} \lesssim 1,
\]

which completes the proof of (4), by the scaling argument.

**Step 3.** We prove that

\[
\|\lambda_t\|_{L^{p(t)}(\mathbb{R}^d)} \lesssim \left( \int_0^1 \lambda_t^{p(0)} \frac{dt}{t} \right)^{1/p(0)} + \left( \int_1^{\infty} \lambda_t^{p_{\infty}} \frac{dt}{t} \right)^{1/p_{\infty}}.
\]  

(5)

We suppose that the right-hand side is less than or equal one. Notice that

\[
\|\lambda_t\|_{L^{p(t)}((0, \infty), \frac{dt}{t})} \approx \left\| \left( t^{-\frac{1}{p(t)}} \lambda_t \chi_{[2^v, 2^{v+1}]} \right) \right\|_{L^{p(t)}(\mathbb{R}^d)}.
\]

We see that

\[
\lambda_t = \frac{1}{\log 2} \int_t^{2t} \lambda_t \frac{d\tau}{\tau} \leq \frac{1}{\log 2} \int_t^{2t} \lambda_t \frac{d\tau}{\tau} \lesssim t^{-s} \int_0^{2^{v+2}} \tau^{s} \lambda_t \frac{d\tau}{\tau}
\]

\[
\leq \sum_{v=-\infty}^{\infty} t^{-s} \int_{2^2}^{2^{2^{v+2}}} \tau^{s} \lambda_t \frac{d\tau}{\tau} = \sum_{j=-v}^{\infty} t^{-s} \int_{2^{2j-2}}^{2^{2j}} \tau^{s} \lambda_t \frac{d\tau}{\tau}
\]

for any \( v \leq 0 \) any any \( t \in [2^v, 2^{v+1}] \). Let \( \sigma > 0 \) be such that \( p^+ < \sigma \). We have

\[
\left( \frac{\sum_{j=-v}^{\infty} \int_{2^{2j-2}}^{2^{2j}} \tau^{s} \lambda_t \frac{d\tau}{\tau}}{\lambda_t} \right)^{p(t)/\sigma} \leq 2^{vsp(t)/\sigma} \sum_{j=-v}^{\infty} 2^{-(v+j)s} \lambda_t \frac{d\tau}{\tau} \left( \int_{2^{2j-2}}^{2^{2j}} \lambda_t \frac{d\tau}{\tau} \right)^{p(t)/\sigma}.
\]

Again, by Hölder’s inequality, we estimate this expression by

\[
c^{2vsp(t)/\sigma} \left( \sum_{j=-v}^{\infty} 2^{-(v+j)s} \lambda_t \frac{d\tau}{\tau} \left( \int_{2^{2j-2}}^{2^{2j}} \lambda_t \frac{d\tau}{\tau} \right)^{p(t)/\sigma} \right)^{1/\sigma}.
\]

\[\text{9}\]
Applying again Lemma 2 we get

\[
\left( \frac{1}{(j + v + 2) \log 2} \int_{2^{-j-1}}^{2^{v+2}} \lambda_\tau \chi_{[2^{-j}, 2^{2-j}]}(\tau) \frac{d\tau}{\tau} \right)^{p(t)} \lesssim \frac{1}{j + v + 2} \int_{2^{-j-1}}^{2^{v+2}} \lambda_\tau^{p(0)} \chi_{[2^{-j}, 2^{2-j}]}(\tau) \frac{d\tau}{\tau} + 2^vm.
\]

Therefore,

\[
\lambda_t^{p(t)} \lesssim \sum_{j=-v}^{\infty} 2^{-(v+j)sp(t)/\sigma} \int_{2^{-j-1}}^{2^{v+2}} \lambda_\tau^{p(0)} \frac{d\tau}{\tau} + f_v
\]

for \( v \leq 0 \) and any \( t \in [2^v, 2^{v+1}] \subset [2^{-j}, 2^{v+2}] \) where

\[
f_v = 2^vm.
\]

Therefore,

\[
\int_{2^v}^{2^{v+1}} \lambda_t^{p(t)} \frac{dt}{t} \lesssim \sum_{j=-v}^{\infty} \int_{2^{-j-1}}^{2^{v+2}} \lambda_\tau^{p(0)} \frac{d\tau}{\tau} + f_v.
\]

By taking \( m \) large enough such that \( m > 0 \) and again by Lemma 1 we get

\[
\sum_{v=-\infty}^{0} \int_{2^v}^{2^{v+1}} \lambda_t^{p(t)} \frac{dt}{t} \lesssim \sum_{j=1}^{\infty} \int_{2^{-j-1}}^{2^{v+2}} \lambda_\tau^{p(0)} \frac{d\tau}{\tau} + c \lesssim 1.
\]

Now we see that

\[
\lambda_t = \int_t^{2^2} \lambda_\tau \frac{d\tau}{\tau} \leq \int_t^{2^2} \lambda_\tau \frac{d\tau}{\tau} \lesssim t^{-s} \int_1^{2^2} \tau^s \lambda_\tau \frac{d\tau}{\tau} \leq \sum_{j=0}^{v} t^{-s} \int_{2^j}^{2^{j+2}} \tau^s \lambda_\tau \frac{d\tau}{\tau}
\]

for any \( v > 0 \) any \( t \in [2^v, 2^{v+1}] \). Let \( \sigma > 0 \) be such that \( p^+ < \sigma \). We have

\[
\left( \sum_{j=0}^{v} \int_{2^j}^{2^{j+2}} \tau^s \lambda_\tau \frac{d\tau}{\tau} \right)^{p(t)/\sigma} \leq \sum_{j=0}^{v} \left( \int_{2^j}^{2^{j+2}} \tau^s \lambda_\tau \frac{d\tau}{\tau} \right)^{p(t)/\sigma}
\]

\[
\leq \sum_{j=0}^{v} 2^{jsp(t)/\sigma} \left( \int_{2^j}^{2^{j+2}} \lambda_\tau \frac{d\tau}{\tau} \right)^{p(t)/\sigma}
\]

\[
= 2^{sp(t)/\sigma} \sum_{j=0}^{v} 2^{(j-v)sp(t)/\sigma} \left( \int_{2^j}^{2^{j+2}} \lambda_\tau \frac{d\tau}{\tau} \right)^{p(t)/\sigma}.
\]

Again, by Hölder’s inequality, we estimate this expression by

\[
c 2^{sp(t)/\sigma} \left( \sum_{j=0}^{v} 2^{(j-v)sp(t)/\sigma} \left( \int_{2^j}^{2^{j+2}} \lambda_\tau \frac{d\tau}{\tau} \right)^{p(t)} \right)^{1/\sigma}.
\]
Applying again Lemma 2 we get

\[
\left(\frac{1}{(v - j + 2) \log 2} \int_{2^j}^{2^{v+2}} \lambda_{\tau} \chi_{[2^j, 2^{v+1}]}(\tau) \frac{d\tau}{\tau}\right)^{p(t)} \lesssim \frac{1}{v - j + 2} \int_{2^j}^{2^{v+2}} \lambda_{\tau}^{p(\infty)} \chi_{[2^j, 2^{v+1}]}(\tau) \frac{d\tau}{\tau} + 2^{-jm}
\]

for \( v > 0 \) and any \( t \in [2^v, 2^{v+1}] \subset [2^j, 2^{v+1}] \). Therefore,

\[
\lambda_{t}^{p(t)} \lesssim \sum_{j=0}^{v} 2^{(j-v)sp(t)/\sigma} (v - j + 2)^{p^+ - 1} \int_{2^j}^{2^{v+2}} \lambda_{\tau}^{p(\infty)} \frac{d\tau}{\tau} + f_v
\]

for \( v > 0 \) and any \( t \in [2^v, 2^{v+1}] \), where

\[
f_v = \sum_{j=0}^{v} 2^{(j-v)sp^-/\sigma} (v - j + 2)^{p^+} 2^{-jm}.
\]

Therefore,

\[
\int_{2^v}^{2^{v+1}} \lambda_{t}^{p(t)} \frac{dt}{t} \lesssim \sum_{j=0}^{v} \int_{2^j}^{2^{v+2}} \lambda_{\tau}^{p(\infty)} \frac{d\tau}{\tau} + f_v.
\]

By taking \( m \) large enough such that \( m > 0 \) and again by Lemma 1 we get

\[
\sum_{v=1}^{\infty} \int_{2^v}^{2^{v+1}} \lambda_{t}^{p(t)} \frac{dt}{t} \lesssim \sum_{j=0}^{\infty} \int_{2^j}^{2^{v+2}} \lambda_{\tau}^{p(\infty)} \frac{d\tau}{\tau} + c \lesssim 1.
\]

The proof of (5) is completed by the scaling argument.

**Step 4.** We prove that

\[
\left( \int_{0}^{1} \lambda_{t}^{p(0)} \frac{dt}{t} \right)^{\frac{1}{p(0)}} + \left( \int_{1}^{\infty} \lambda_{t}^{p(\infty)} \frac{dt}{t} \right)^{\frac{1}{p(\infty)}} \lesssim \|\lambda_{t}\|_{L^{p(\cdot)}((0, \infty), \frac{dt}{t})}.
\]

We omit the proofs of this estimate, since they are essentially similar to the proof of (4). □

We would like to mention that the estimates (2) and (3) are true if we assume that

\[
\eta_t \leq \eta_{\tau}, \quad 0 < \tau \leq t
\]

and

\[
\lambda_t \leq \lambda_{\tau}, \quad 0 < t \leq \tau
\]

respectively. Also we find that

\[
\|\eta_t\|_{L^{p(\cdot)}((0, \infty), \frac{dt}{t})} \approx \|\eta_{\tau}\|_{L^{p(\cdot)}((0, \infty), \frac{dt}{t})}
\]

and

\[
\|\lambda_t\|_{L^{p(\cdot)}((0, \infty), \frac{dt}{t})} \approx \|\lambda_{\tau}\|_{L^{p(\cdot)}((0, \infty), \frac{dt}{t})}
\]

for any \( p, q \in \mathcal{P}(\mathbb{R}) \) are log-Hölder continuous both at the origin and at infinity with

\[
1 \leq q^- \leq q^+ < \infty, \quad 1 \leq p^- \leq p^+ < \infty,
\]

\[
p(0) = q(0) \quad \text{and} \quad p_{\infty} = q_{\infty}.
\]

By the technical of Theorem \( \square \) we immediately arrive at the following result.
Theorem 2 Let $s > 0$. Let $p \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous both at the origin with $1 \leq p^- \leq p^+ < \infty$. Let $\{\varepsilon_t\}_t$ be a sequence of positive measurable functions. Let 
\[ \eta_t = t^s \int_t^1 \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau} \quad \text{and} \quad \lambda_t = t^{-s} \int_0^t \tau^s \varepsilon_\tau \frac{d\tau}{\tau}. \]

Then there exists constant $c > 0$ depending only on $s$, $p^-$, $\alpha_{\log}(p)$ and $p^+$ such that
\[ \|\eta_t\|_{L^p((0,1], \frac{dt}{t})} \approx \left( \int_0^1 \eta_t^{p(0)} \frac{dt}{t} \right)^{\frac{1}{p(0)}} \]  
and
\[ \|\lambda_t\|_{L^p((0,1], \frac{dt}{t})} \approx \left( \int_0^1 \lambda_t^{p(0)} \frac{dt}{t} \right)^{\frac{1}{p(0)}}. \]  
Moreover,
\[ \|\eta_t\|_{L^p((0,1], \frac{dt}{t})} + \|\lambda_t\|_{L^p((0,1], \frac{dt}{t})} \lesssim \left( \int_0^1 \varepsilon_t^{p(0)} \frac{dt}{t} \right)^{\frac{1}{p(0)}}. \]

Again, we would like to mention that the estimates (6) and (7) are true if we assume that 
\[ \eta_t \leq \eta_\tau, \quad 0 < \tau \leq t \leq 1 \]
and 
\[ \lambda_t \leq \lambda_\tau, \quad 0 < t \leq \tau \leq 1, \]
respectively. Also we find that 
\[ \|\eta_t\|_{L^q((0,1], \frac{dt}{t})} \approx \|\eta_t\|_{L^p((0,1], \frac{dt}{t})} \]
and 
\[ \|\lambda_t\|_{L^q((0,1], \frac{dt}{t})} \approx \|\lambda_t\|_{L^p((0,1], \frac{dt}{t})} \]
for any $p, q \in \mathcal{P}(\mathbb{R})$ are log-Hölder continuous at the origin with $1 \leq q^- \leq q^+ < \infty$, $1 \leq p^- \leq p^+ < \infty$ and 
\[ p(0) = q(0). \]

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