Probabilistic Voting Models
with Varying Speeds of Correlation Decay

Gabor Toth

Abstract
We model voting behaviour in the multi-group setting of a two-tier voting system using sequences of de Finetti measures. Our model is defined by using the de Finetti representation of a probability measure (i.e. as a mixture of conditionally independent probability measures) describing voting behaviour. The de Finetti measure describes the interaction between voters and possible outside influences on them. We assume that for each population size there is a (potentially) different de Finetti measure, and as the population grows, the sequence of de Finetti measures converges weakly to the Dirac measure at the origin, representing a tendency toward weakening social cohesion as the population grows large. The resulting model covers a wide variety of behaviour, ranging from independent voting in the limit under fast convergence, a critical convergence speed with its own pattern of behaviour, to a subcritical convergence speed which yields a model in line with empirical evidence of real-world voting data, contrary to previous probabilistic models used in the study of voting. These models can be used, e.g., to study the problem of optimal voting weights in two-tier voting systems.

Keywords: probabilistic voting models, spin models, de Finetti representation, limit theorems, phase transitions

2020 Mathematics Subject Classification: 60F05, 82B20, 91B12

1 Introduction
Spin models from statistical mechanics have been used to model voting behaviour (see e.g. [2, 4, 14, 15, 25, 34, 24]). These stochastic models of voting describe the typical voting behaviour in terms of a probability distribution over the set of all possible voting outcomes for a population of \( n \in \mathbb{N} \) voters that decide on a binary matter. A complete record of all individual decisions is called a voting configuration \((x_1, \ldots, x_n) \in \{-1,1\}^n\). These binary choices, with values encoded as \(\pm 1\), cover both referenda and elections for public office with two candidates. The individual votes are random variables \(X_i, i = 1, 2, \ldots, n\), with joint distribution \(P\). In the literature, it is customary to impose the symmetry condition

\[
P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = -x_1, \ldots, X_n = -x_n)
\]

for all voting configurations \((x_1, \ldots, x_n)\). The justification for this is that there is no fundamental distinction between the two alternatives. We assume that each voter has fixed preferences concerning all possible issues that can be put to vote. However, the way the question is phrased in the referendum, or the order in which the candidates appear on the ballot, is assumed to be random, so it is reasonable to expect any given voter to choose \(+1\) with the same probability as \(-1\).

Definition 1. A probability measure on \(\{-1,1\}^n\) that satisfies (1) is called a voting measure.

With only two alternatives, we can apply the majority rule to determine the winning alternative, and measure the outcome of the vote by adding all votes to obtain the voting margin \(S_n := \sum_{i=1}^{n} X_i\). If the voting margin
is positive, the decision is in favour; otherwise, it is against the proposal. The expected per capita absolute voting margin is $E\left(|S_n|/n\right)$, and it captures the essence of voting behaviour under the model described by $\mathbb{P}$.

We would like to note that even though the motivation for the study of the models in this article is voting theory, all of these models can also be considered spin models in the context of statistical physics, where there is a tendency for the spins to align.

The focus of this article is the asymptotic behaviour of sequences of voting measures as the population size $n$ goes to infinity. Empirical data of elections give us an idea of the properties these models should have to reflect real-world voting. Penrose’s square root law [26] has been the gold standard in the field of voting theory. Penrose investigated the question of how to equalise the influence of each voter on the decision in a two-tier voting system. A two-tier voting system consists of several groups or constituencies of possibly different sizes and a council that makes decisions on behalf of the union of these groups. Each group sends a representative to the council who votes according to the preferences of the group. The voting weights in the council have to take into account the potentially different group sizes. According to Penrose’s square root law, the optimal weights in the council should be proportional to the square root of the size of each group. Aside from equalising the influence of each voter on the council vote, other criteria have been proposed. This is the problem of optimal voting weights, a topic treated by a variety of authors [28, 29, 27, 30, 14, 15, 31, 34, 20, 21]. It is but one of the applications for voting models with real-world consequences.

A different but related subject is the study of dynamic models of binary voting which can also be interpreted as models of opinion dynamics. One work which shares some overlap with the present article is [35], in which we specify a dynamic model of binary voting, where the population is subdivided into a large number of groups (such as work groups or friendship groups) that discuss the issue to be voted on and influence each other. After discussion takes place, there is a general tendency for everybody to adopt the majority opinion prevalent in the group. However, we allow for contrarian tendencies which manifest in a refusal to adopt said opinion. This is triggered randomly for each member of the group and the probability is what we call a ‘flip parameter’. Several rounds of these discussions take place sequentially, updating opinions across the population in each round. This type of model is very general in that a wide variety of behaviours can be observed, depending on the choice of the flip parameters. Therefore, in [35], we study how the general class of these models behaves if we suppose the flip parameters are randomly chosen with a certain correlation structure inherent in the joint distribution of all flip parameters. Thus, the main differences to the present work lie in the dynamic aspect of the model considered in [35], the small size of groups, where instead of holding the number of groups fixed, we instead study groups of fixed sizes, as well as the modelling approach of selecting the parameters at random. In the present article, following the approach in the works cited in the last paragraph, we study static models of voting with fixed parameters.

A basic assumption in [26] is the independence of all voters. Intuitively speaking, the assumption that all voters are independent seems unrealistic. Gelman et al. [12, 11] criticised the square root law based on statistical evidence. $E\left(|S_n|/n\right)$ is a characteristic of the distribution of voting configurations under a voting measure which can be statistically estimated. Gelman et al. pointed out that under the assumption of independence of all voters, which leads to a binomial distribution of the sum of all votes, $E\left(|S_n|/n\right)$ should be of order $1/\sqrt{n}$ as $n$ goes to infinity. This would imply that larger countries or constituencies should on average have far closer elections, i.e. smaller per capita voting margins, than smaller countries. Gelman et al. argued that although this can be observed in the data on U.S. and European elections to some extent, the per capita voting margins decrease as a far lower power of $n$ than the power $1/2$ specified by the square root law. Indeed, the typical magnitude of $E\left(|S_n|/n\right)$ seems to be of order $n^{-\alpha}$ with $0.1 \leq \alpha \leq 0.2$. This has implications for the determination of the optimal voting weights in two-tier voting systems.

Models of voting behaviour are often adapted from statistical physics. Two categories are the collective bias model and the Curie-Weiss model. See Section 4 for a discussion of these models and how they fit into the framework introduced in Definition 4 below.

In this article, we will deal with the multivariate (i.e. multi-group) setting. This is due to the application of these models to two-tier voting systems, in which the overall population is subdivided into a fixed number of groups such as e.g. states of a federal republic. We could describe each group’s voters by a separate voting model, but this would preclude us from considering correlated voting across group boundaries. Therefore, we choose to analyse multi-group models. The general framework is a set of binary random variables double-
indexed by the group and individual \(X_{\lambda i} \in \{-1, 1\}\), where \(\lambda = 1, \ldots, M\) indicates the group and \(i = 1, \ldots, n_\lambda\), the individual voter. The group sizes \(n_\lambda\) sum to \(n\), the overall population size. We will assume that all \(M\) groups grow without bound as the overall population goes to infinity. Each \(n_\lambda\) is thus actually a sequence \((n_\lambda(n))_{n \in \mathbb{N}}\) that goes to infinity as \(n \to \infty\). So for each value of the overall population \(n\), we have a vector of group sizes \((n_1, \ldots, n_M)\).

Our aim in this article is to study sequences of de Finetti measures \((\mu_n)\) with certain concentration properties. Aside from the case of each \(\mu_n\) living on the compact set \([-1, 1]^M\), we will also consider more general measures with supports in \(\mathbb{R}^M\). All these sequences will have in common that they converge weakly to \(\delta_0\), the Dirac measure at the origin. This convergence assumption represents a tendency towards lower social cohesion or lower central influence as the size of the population increases.

We will use the Rademacher distribution in the definition of de Finetti voting models below.

**Definition 2.** The Rademacher distribution with parameter \(s \in [-1, 1]\) is a probability measure \(P_s\) on \([-1, 1]\) given by \(P_s\{1\} := \frac{1+s}{2}\).

**Notation 3.** We will write for all \(t \in [-1, 1]^M\) and all \((x_{11}, \ldots, x_{Mn_M}) \in \{-1, 1\}^n\)
\[
P_t^{\otimes n} (x_{11}, \ldots, x_{Mn_M}) := \prod_{\lambda=1}^{M} \prod_{i=1}^{n_\lambda} P_t \{x_{\lambda i}\}.
\]

Let \(E_t\) stand for the expectation with respect to the product measure \(P_t^{\otimes n}\).

Recall that \(M \in \mathbb{N}\) is the number of groups and \(n = \sum_{\lambda=1}^{M} n_\lambda \in \mathbb{N}\) is the size of the overall population.

**Definition 4.** Let \((\mu_n)\) be a sequence of probability measures on \(\mathbb{R}^M\) which are symmetric with respect to the origin, i.e. \(\mu_n(A) = \mu_n(-A)\) holds for all measurable sets \(A \subset \mathbb{R}^M\) and all \(n \in \mathbb{N}\). The voting measure \(P_n\) is defined by
\[
P_n (X_{11} = x_{11}, \ldots, X_{Mn_M} = x_{Mn_M}) := \int_{\mathbb{R}^M} P^{\otimes n} (x_{11}, \ldots, x_{Mn_M}) \mu_n (dm)
\]
for all voting configurations \((x_{11}, \ldots, x_{Mn_M}) \in \{-1, 1\}^n\) and all \(n \in \mathbb{N}\). The expression \(\bar{m}\) is some function \(m \in \mathbb{R}^M \mapsto \bar{m} \in \mathbb{R}^M\) with the following properties:

- \(m \mapsto \bar{m}\) is increasing in each component of \(m\) and its range lies in \([-1, 1]^M\).
- For each \(\lambda\), \(\lim_{m \to 0} \frac{\bar{m}_\lambda}{m_\lambda} = 1\).
- The limit of \(\bar{m}\) taken where each component of \(m\) goes to \(\infty\) is \(1\), and the limit where each component goes to \(-\infty\) is \(-1\).

We will call such sequences of voting measures \((P_n)\) de Finetti voting models with de Finetti sequence \((\mu_n)\). We will refer to \((\mu_n)\) as the sequence of de Finetti measures of \((P_n)\).

**Remark 5.** Note that the assumed symmetry of each probability measure \(\mu_n\) implies the symmetry condition [1] for each \(P_n\).

** Remark 6.** We will concentrate on the case where \((\mu_n)\) is supported on \([-1, 1]^M\) with \(\bar{m} = m\). The reason we consider the more general setup with supports not contained in \([-1, 1]^M\) is that some voting models such as the Curie-Weiss model defined in Section [2] fall into this category. However, there are no significant mathematical differences between the more restrictive assumption of a compact support and the general case.

For large population sizes \(n\), Definition [3] covers all models with exchangeable random variables \(X_{\lambda i}\) within each group \(\lambda\). (See [4] for the result that states that for finite numbers \(n\) of exchangeable random variables, the total variation distance between the joint distribution of the random variables and the distribution of a mixture as in the formula above is at most of order \(1/n\).) In the language of voting theory, these voting systems are called neutral, as the identity of the individual voters (at least within each group) does not affect the outcome.
**Definition 7.** Suppose \((\mathbb{P}_n)_n\) is a de Finetti voting model. We define the *group voting margins* for each group \(\lambda\):

\[
S_{n,\lambda} := \sum_{i=1}^{n_\lambda} X_{\lambda i}.
\]

Instead of the voting margin \(S_n\) of a single-group model, we will be interested in the joint distribution of the group voting margins \(S_{n,1}, \ldots, S_{n,M}\); more precisely, we will study the asymptotic behaviour of the sums

\[
S_n := \left(\frac{S_{n,1}}{\gamma_{n,1}}, \ldots, \frac{S_{n,M}}{\gamma_{n,M}}\right),
\]

normalised by sequences \((\gamma_{n,\lambda})_{n \in \mathbb{N}}\) to ensure \(S_n\) converges to some limiting distribution as \(n\) goes to infinity. We will stipulate that \((\mu_n)_n\) converges weakly to \(\delta_0\). This represents an underlying tendency towards lower social cohesion which is reflected in the convergence to 0 of the correlation \(E(X_{\lambda 1} X_{\nu 2})\) between two votes belonging any groups \(\lambda\) and \(\nu\). Note that this does not necessarily imply that the limiting distribution of \(S_n\) features independent entries. We will return to the topic of the correlations \(E(X_{\lambda 1} X_{\nu 2})\) in Proposition 18 and in Section 4.

In Section 2, we present limit theorems for \(S_n\) given sequences \((\mu_n)_n\) that converge weakly to \(\delta_0\) and show that the speed of convergence is crucial for the limiting distribution. The main results are limit Theorems 11 and 13 and Corollary 17 concerning the behaviour of the expected per capita voting margins. Section 3 contains the proofs of these results. Section 4 discusses some voting models featured prominently in the literature, how they fit into the framework introduced in Definition 4 and in Example 31 provides a de Finetti voting model which fits the empirical evidence. Finally, Section 5 presents the conclusions.

### 2 Results

**Notation 8.** In the following, let for any vectors \(x, y \in \mathbb{R}^d\) the expression \(x \circ y \in \mathbb{R}^d\) stand for the componentwise multiplication of \(x\) and \(y\). We will use the symbol ‘\(\ast\)’ to denote the convolution of two measures. ‘\(\rightarrow_{n \to \infty}\)’ stands for convergence in distribution as \(n\) goes to infinity. We will write \(\mathcal{N}(0, C)\) for a centred multivariate normal distribution with covariance matrix \(C\), and \(\delta_x\) for the Dirac measure at \(x \in \mathbb{R}^d\). \(I_k\) will stand for the identity matrix of dimension \(k \in \mathbb{N}\). For \(a, b \in \mathbb{R}^M\) with \(a \leq b\) componentwise, we will use \([a, b]\) to denote \(\prod_{\lambda=1}^{M} [a_\lambda, b_\lambda]\). Let \(\|\cdot\|_{\infty}\) stand for the sup norm on \(\mathbb{R}^d\) for any \(d \in \mathbb{N}\) as well as on the set of all functions \(f : \mathbb{R}^d \rightarrow \mathbb{R}\).

We will be working with sequences of vectors \((\varepsilon_n)_n\) in \((0, \infty)^M\) such that \(\|\varepsilon_n\|_{\infty} \xrightarrow[n \to \infty]{} 0\).

**Lemma 9.** Let \((\mu_n)_n\) be a sequence of probability measures on \(\mathbb{R}^M\). Then the following statements are equivalent:

(i) \((\mu_n)_n\) converges weakly to \(\delta_0\).

(ii) For all \(\delta > 0\), we have \(\mu_n \left(\prod_{\lambda=1}^{M} [-\delta, \delta]\right) \xrightarrow[n \to \infty]{} 1\).

(iii) There is a sequence \((\varepsilon_n)_n\) in \((0, \infty)^M\) with \(\|\varepsilon_n\|_{\infty} \xrightarrow[n \to \infty]{} 0\) such that \(\mu_n ([-\varepsilon_n, \varepsilon_n]) \xrightarrow[n \to \infty]{} 1\).

This lemma states that by considering all sequences \((\varepsilon_n)_n\) as in (iii) above, we are covering all sequences of probability measures \((\mu_n)_n\) that converge weakly to \(\delta_0\).

We will use the following notation for the asymptotic behaviour of sequences:

**Notation 10.** For any complex-valued sequences \((f_n)_n, (g_n)_n\), we will write \(f_n = \Theta (g_n)\) if

\[
0 < \liminf_{n \to \infty} \frac{|f_n|}{|g_n|} \leq \limsup_{n \to \infty} \frac{|f_n|}{|g_n|} < \infty,
\]
\[ f_n = O(g_n) \text{ if} \]
\[ \limsup_{n \to \infty} \left| \frac{f_n}{g_n} \right| < \infty, \]
and \[ f_n = o(g_n) \text{ if} \]
\[ \lim_{n \to \infty} \frac{f_n}{g_n} = 0. \]

The convergence speed of \((\mu_n)_n\) turns out to be crucial for the limiting distribution of the suitably normalised sums \(S_n\). First we present a theorem that says that under ‘fast convergence’, a term we will give formal meaning to in Definition 15, we obtain a universal limit for the normalised sums which is independent of the specific sequence of de Finetti measures, with only the convergence speed mattering.

**Theorem 11.** Let \((\P_n)_n\) be a de Finetti voting model with group sizes \(n_\lambda, \lambda = 1, \ldots, M\), and de Finetti measures \((\mu_n)_n\). Let \((\epsilon_n)_n\) be a sequence in \((0, \infty)^M\) with \(\epsilon_{n,\lambda} = o\left(\frac{1}{\sqrt{n_\lambda}}\right)\) for each coordinate \(\lambda\). If \(\mu_{\lambda}([-\epsilon_n, \epsilon_n]) \xrightarrow{n \to \infty} 1\), then
\[
\left(\frac{S_{n,1}}{\sqrt{n_1}}, \ldots, \frac{S_{n,M}}{\sqrt{n_M}}\right) \xrightarrow{d} N(0, I_M).
\]

This theorem says that if convergence is fast enough – that is, faster than \(1/\sqrt{n_\lambda}\) in each component – then the specifics of the sequence of distributions \((\mu_n)_n\) are lost in the limit. The groups become independent in the limit, and each normalised group voting margin \(S_{n,\lambda}/\sqrt{n_\lambda}\) converges to a standard normal distribution. As we will next see, this universality is lost if the convergence speed is not faster than the critical speed \(1/\sqrt{n_\lambda}\). Then, the specifics of the sequence \((\mu_n)_n\) affect the limiting distribution of the normalised sums. For this reason, we assume that each \(\mu_n\) is a contraction of some fixed probability measure \(\mu\) on \(\mathbb{R}^M\).

For the next theorem, we define the notation for a contraction:

**Notation 12.** If \(\mu\) is a probability measure on \(\mathbb{R}^d\) and \(h \in (0, \infty)^d\), then we denote by \(h \circ \mu\) the rescaled measure of \(\mu\) by \(h\) given by \(h \circ \mu(A) := \mu(h \circ A)\) for all measurable sets \(A \subset \mathbb{R}^d\).

**Theorem 13.** Let \(\mu\) be a probability measure on \(\mathbb{R}^M\) and \((\epsilon_n)_n\) a sequence in \((0, \infty)^M\) with \(\|\epsilon_n\|_\infty \xrightarrow{n \to \infty} 0\).
We define for each \(n \in \mathbb{N}\) the probability measure \(\mu_n := \left(\frac{1}{\epsilon_{n,1}}, \ldots, \frac{1}{\epsilon_{n,M}}\right) \circ \mu\). Let \((\P_n)_n\) be a de Finetti voting model with group sizes \(n_\lambda, \lambda = 1, \ldots, M\), and de Finetti measures \((\mu_n)_n\).

1. If for each component \(\lambda \epsilon_{n,\lambda} = o\left(\frac{1}{\sqrt{n_\lambda}}\right)\), then
\[
\left(\frac{S_{n,1}}{\sqrt{n_1}}, \ldots, \frac{S_{n,M}}{\sqrt{n_M}}\right) \xrightarrow{d} N(0, I_M).
\]

2. If for each component \(\lambda \lim_{n \to \infty} \epsilon_{n,\lambda} \sqrt{n_\lambda} = h_\lambda > 0\), we set \(h := (h_1, \ldots, h_M)\), and then
\[
\left(\frac{S_{n,1}}{\sqrt{n_1}}, \ldots, \frac{S_{n,M}}{\sqrt{n_M}}\right) \xrightarrow{d} N(0, I_M) * (h \circ \mu).
\]

3. If for each component \(\lambda \frac{1}{\sqrt{n_\lambda}} = o(\epsilon_{n,\lambda})\), we set for each \(n\) and each \(\lambda \gamma_{n,\lambda} := \epsilon_{n,\lambda} n_\lambda\), and then
\[
\left(\frac{S_{n,1}}{\gamma_{n,1}}, \ldots, \frac{S_{n,M}}{\gamma_{n,M}}\right) \xrightarrow{d} \mu.
\]

Under fast contraction, i.e. \(\epsilon_{n,\lambda} = o(1/\sqrt{n_\lambda})\) for each \(\lambda\), we obtain the universal limiting distribution \(N(0, I_M)\) as in Theorem 11. However, when the contraction speed is critical, i.e. \(\lim_{n \to \infty} \epsilon_{n,\lambda} \sqrt{n_\lambda} = h_\lambda > 0\) for each \(\lambda\), then the limiting distribution for the normalised sums is the convolution of the rescaled version
$h \circ \mu$ of the measure $\mu$, which is used to define the contracting sequence $(\mu_n)_n$, and a multivariate normal noise with independent components and standard normal marginal distributions. Lastly, if the contraction speed is subcritical, then the sums must be normalised differently than in the supercritical and critical cases. The normalisation factor is $\gamma_{n,\lambda} = \varepsilon_{n,\lambda} n \lambda$ for each component $\lambda$. The assumption $1 / \sqrt{n \lambda} = o (\varepsilon_{n,\lambda})$ implies that $\gamma_{n,\lambda}$ goes to infinity faster than $\sqrt{n \lambda}$. The limiting distribution of the normalised sums in the subcritical case is the measure $\mu$.

From Theorem 13 the corresponding result for single-group models (with $M = 1$ and $n_1 = n$) follows:

**Corollary 14.** Let $\mu$ be a probability measure on $\mathbb{R}$ and $(\varepsilon_n)_n$ a sequence in $(0, \infty)$ with $\varepsilon_n \xrightarrow{n \to \infty} 0$. We define for each $n \in \mathbb{N}$ the probability measure $\mu_n := \frac{1}{\varepsilon_n} \circ \mu$. Let $(\mathbb{P}_n)_n$ be a de Finetti voting model with de Finetti measures $(\mu_n)_n$.

1. If $\varepsilon_n = o\left(\frac{1}{\sqrt{n}}\right)$, then
   \[\frac{S_n}{\sqrt{n}} \xrightarrow{d, n \to \infty} \mathcal{N}(0, 1)\.\]

2. If $\lim_{n \to \infty} \varepsilon_n \sqrt{n} = h > 0$, then
   \[\frac{S_n}{\sqrt{n}} \xrightarrow{d, n \to \infty} \mathcal{N}(0, 1) \ast (h \circ \mu)\.\]

3. If $\frac{1}{\sqrt{n \lambda}} = o (\varepsilon_n)$, we set for each $n \gamma_n := \varepsilon_n n$. Then
   \[\frac{S_n}{\gamma_n} \xrightarrow{d, n \to \infty} \mu\.\]

The next definition formalises the different convergence speeds in Theorem 13. We will classify some voting models that fit the de Finetti framework in Section 4 according to the speed of convergence to $\delta_0$ of their de Finetti measures $(\mu_n)_n$.

**Definition 15.** Let $(\mathbb{P}_n)_n$ be a de Finetti voting model with group sizes $n_\lambda$, $\lambda = 1, \ldots, M$, de Finetti measures $(\mu_n)_n$, and $(\varepsilon_n)_n$ a sequence in $(0, \infty)^M$ with $\|\varepsilon_n\|_\infty \xrightarrow{n \to \infty} 0$. Assume $\mu_n ([0, \varepsilon_n]) \xrightarrow{n \to \infty} 1$ holds.

1. If for each component $\lambda \varepsilon_{n,\lambda} = o\left(\frac{1}{\sqrt{n \lambda}}\right)$, we will say the model exhibits fast (or supercritical) convergence.
2. If for each component $\lambda \lim_{n \to \infty} \varepsilon_{n,\lambda} \sqrt{n \lambda} = h_\lambda > 0$, we will say the model has critical convergence speed.
3. If for each component $\lambda \frac{1}{\sqrt{n \lambda}} = o(\varepsilon_{n,\lambda})$, we will say the model exhibits slow (or subcritical) convergence.

Theorem 13 states that there is a phase transition in the space of convergence speeds with a critical speed of order $1 / \sqrt{n \lambda}$, as well as supercritical and subcritical regimes.

Next we state that different components (or groups) $\lambda$ can be in different regimes depending on the contraction speed $\varepsilon_{n,\lambda}$ for each group.

**Theorem 16.** Let $(\mathbb{P}_n)_n$ be a de Finetti voting model with group sizes $n_\lambda$, group voting margins $S_{n,\lambda}$, $\lambda = 1, \ldots, M$, de Finetti measures $(\mu_n)_n$, defined as a contraction of a probability measure $\mu$ on $\mathbb{R}^M$ as in Theorem 13 and $(\varepsilon_n)_n$ a sequence in $(0, \infty)^M$ with $\|\varepsilon_n\|_\infty \xrightarrow{n \to \infty} 0$. Let the $M$ groups be partitioned into three clusters $C_1$, $C_2$, and $C_3$, each of which comprises $M_1 > 0$ of the $M$ groups, respectively. Let the contraction speeds of $(\varepsilon_n)_n$ and the definition of the sequence $(\gamma_n)_n$ be those given in the following display:

\[
\varepsilon_{n,\lambda} = o\left(\frac{1}{\sqrt{n \lambda}}\right), \quad \gamma_{n,\lambda} := \sqrt{n \lambda}, \quad \lambda \in C_1,
\]

\[
\lim_{n \to \infty} \varepsilon_{n,\lambda} \sqrt{n \lambda} = h_\lambda > 0, \quad \gamma_{n,\lambda} := \sqrt{n \lambda}, \quad \lambda \in C_2,
\]

\[
\frac{1}{\sqrt{n \lambda}} = o(\varepsilon_{n,\lambda}), \quad \gamma_{n,\lambda} := \varepsilon_{n,\lambda} n \lambda, \quad \lambda \in C_3
\]
We set \( h := (h_{M_1+1}, \ldots, h_{M_1+M_2+1}, \ldots, 1) \in \mathbb{R}^{M_2+M_3} \). Then
\[
\left( S_{n,1} \gamma_{n,1}, \ldots, S_{n,M} \gamma_{n,M} \right) \overset{d}{\longrightarrow} \left( N(0, I_{M_1}), (N(0, I_{M_2}), \delta_0) * (h \circ \mu_{C_2, C_3}) \right),
\]
where \( \mu_{C_2, C_3} \) is the marginal distribution of the coordinates belonging to \( C_2 \) and \( C_3 \) under the measure \( \mu \).

Note that the groups in \( C_1 \) are asymptotically independent both of each other as well as of the groups belonging to \( C_2 \) and \( C_3 \), whereas the groups in \( C_2 \) and \( C_3 \) preserve the dependence given by the probability measure \( \mu \) with some additive noise introduced in \( C_2 \) due to the critical nature of the contraction speed for that cluster.

We turn to the behaviour of the expected per capita voting margins \( E(|S_{n,\lambda}|/n_\lambda) \), as these can be used to assess the goodness of fit of the model to the data presented in [12, 11].

**Corollary 17.** Let \( (\mathbb{P}_n)_n \) be a de Finetti voting model with group sizes \( n_\lambda \), group voting margins \( S_{n,\lambda} \), \( \lambda = 1, \ldots, M \), de Finetti measures \( (\mu_n)_n \) defined as a contraction of a probability measure \( \mu \) on \( \mathbb{R}^M \) as in Theorem 73 and \( (\varepsilon_n)_n \) a sequence in \( (0, \infty)^M \) with \( \|\varepsilon_n\|_\infty \overset{n \to \infty}{\longrightarrow} 0 \).

1. If for some \( \lambda \varepsilon_{n,\lambda} = o\left(\frac{1}{\sqrt{n_\lambda}}\right) \), then
   \[
   E\left(\frac{|S_{n,\lambda}|}{n_\lambda}\right) = \Theta\left(n_\lambda^{-1/2}\right).
   \]
2. If for some \( \lambda \lim_{n \to \infty} \varepsilon_{n,\lambda} \sqrt{n_\lambda} = h_\lambda > 0 \), then
   \[
   E\left(\frac{|S_{n,\lambda}|}{n_\lambda}\right) = \Theta\left(n_\lambda^{-1/2}\right).
   \]
3. If for some \( \lambda \frac{1}{\sqrt{n_\lambda}} = o(\varepsilon_{n,\lambda}) \), then
   \[
   E\left(\frac{|S_{n,\lambda}|}{n_\lambda}\right) = \Theta(\varepsilon_{n,\lambda}).
   \]

Hence, if \( \varepsilon_{n,\lambda} = n_\lambda^{-\alpha} \) with \( 0.1 \leq \alpha \leq 0.2 \), then we land in the empirical range given in [12, 11] for the group voting margins \( E(|S_{n,\lambda}|/n_\lambda) \). In accordance with Definition 15 the empirical evidence suggests that de Finetti voting models with subcritical convergence speeds best describe real-world voting behaviour. So it seems that the assumption of a decay of social cohesion as the population grows is correct; however, this decay is fairly slow. See Example 31 in Section 4 for a simple example of such a de Finetti voting model where the parameters can be adjusted to fit the empirical per capita voting margins presented in [12, 11].

Another way to assess the asymptotic loss of social cohesion is to look at the behaviour of correlations \( E(X_{\lambda i} X_{\nu j}) \) between two votes for any groups \( \lambda \) and \( \nu \). Note that due to the exchangeability of the random variables \( \{X_M | i = 1, \ldots, n_\lambda\} \) for any group \( \lambda \), it does not matter which two votes we pick for our correlation as long as they are two different votes:

\[
E(X_{\lambda i} X_{\nu j}) = E(X_{\lambda i'} X_{\nu j'}), \quad i, i', j, j' = 1, \ldots, n_\lambda, i \neq j, i' \neq j',
\]
\[
E(X_{\lambda i} X_{\nu j}) = E(X_{\lambda i'} X_{\nu j'}), \quad \lambda \neq \nu, i, i' = 1, \ldots, n_\lambda, j, j' = 1, \ldots, n_\nu.
\]

Recall the transformation \( m \mapsto \bar{m} \) from Definition 4.

**Proposition 18.** Let \( (\mathbb{P}_n)_n \) be a de Finetti voting model and \( (\varepsilon_n)_n \) a sequence in \( (0, \infty)^M \) with \( \|\varepsilon_n\|_\infty \overset{n \to \infty}{\longrightarrow} 0 \). If \( \mu_n([-\varepsilon_n, \varepsilon_n]) \overset{n \to \infty}{\longrightarrow} 1 \), then
\[
E(X_{\lambda i} X_{\nu j}) = \int_{\mathbb{R}^M} \bar{m}_\lambda \bar{m}_\nu \mu_n(dm) \overset{n \to \infty}{\longrightarrow} 0, \quad \lambda, \nu = 1, \ldots, M.
\]
The proposition says that as a consequence of the loss of social cohesion represented by the assumption \( \|\varepsilon_n\|_\infty \to 0 \) and \( \mu_n([-\varepsilon_n, \varepsilon_n]) \to 1 \), individual votes become asymptotically uncorrelated, even if both votes belong to the same group. This is an interesting contrast to the results above which state that, provided that convergence is not supercritical, the group voting margins do not, in general, become independent in the large population limit.

Remark 19. Note that we cannot say anything about the speed of convergence of \( \mu_n([-\varepsilon_n, \varepsilon_n]) \to 1 \) unless in addition to the speed of convergence of each \( \varepsilon_n, \lambda \to 0 \) we also make an assumption about the speed of convergence of \( \mu_n([-\varepsilon_n, \varepsilon_n]) \to 1 \).

Finally, we give a local limit theorem for the fast convergence case. In this regime, the convergence in distribution can be strengthened to local convergence, i.e. convergence of the scaled point probabilities for the normalised sums to the Lebesgue density function \( \phi \) of \( \mathcal{N}(0, I_M) \). For this, we need to make an assumption about the tails of \( (\mu_n)_n \).

Theorem 20. Let \( (\mathbb{P}_n)_n \) be a de Finetti voting model with group sizes \( n_\lambda \), group voting margins \( S_{n,\lambda} \), \( \lambda = 1, \ldots, M \), and de Finetti measures \( (\mu_n)_n \). Let \( (\varepsilon_n)_n \) be a sequence in \( (0, \infty)^M \) with the property that for each component \( \lambda \varepsilon_{n,\lambda} = a \left( \frac{1}{\sqrt{n_\lambda}} \right) \) if \( \mu_n([-\varepsilon_n, \varepsilon_n]) \to 1 \) and there is a constant \( \tau \in (0, 1)^M \) such that

\[
\sum_{n=1}^{\infty} n^{M-1} \mu_n \left( \mathbb{R}^M \setminus [-\tau, \tau] \right) < \infty, \tag{3}
\]

then

\[
\sup_{x \in \mathcal{L}_n} \left| \frac{\prod_{\lambda=1}^{M} \sqrt{n_\lambda}}{2^M} \mathbb{P} \left( \left( \frac{S_{n,1}}{\sqrt{n_1}}, \ldots, \frac{S_{n,M}}{\sqrt{n_M}} \right) = x \right) - \phi(x) \right| \to 0 \quad \text{as} \quad n \to \infty,
\]

where \( \mathcal{L}_n \) is the lattice on which the vector \( \left( \frac{S_{n,1}}{\sqrt{n_1}}, \ldots, \frac{S_{n,M}}{\sqrt{n_M}} \right) \) lives.

Note that the summability condition \( \sup_{x \in \mathcal{L}_n} \left| \frac{\prod_{\lambda=1}^{M} \sqrt{n_\lambda}}{2^M} \mathbb{P} \left( \left( \frac{S_{n,1}}{\sqrt{n_1}}, \ldots, \frac{S_{n,M}}{\sqrt{n_M}} \right) = x \right) - \phi(x) \right| \to 0 \quad \text{as} \quad n \to \infty \) is a weaker assumption regarding the convergence speed of \( \mu_n([-\varepsilon_n, \varepsilon_n]) \to 1 \) than an exponential concentration property \( \sup_{x \in \mathcal{L}_n} \left| \frac{\prod_{\lambda=1}^{M} \sqrt{n_\lambda}}{2^M} \mathbb{P} \left( \left( \frac{S_{n,1}}{\sqrt{n_1}}, \ldots, \frac{S_{n,M}}{\sqrt{n_M}} \right) = x \right) - \phi(x) \right| \to 0 \quad \text{as} \quad n \to \infty \) in which for each \( \delta > 0 \) there are \( C, D > 0 \) such that

\[
\mu_n \left( \left[ -1, 1 \right]^M \setminus [-\delta, \delta]^M \right) < C \exp \left( -Dn \right), \quad n \in \mathbb{N}.
\]

3 Proofs

3.1 Proof of Lemma 9

We show the implications \( (i) \implies (ii), (ii) \implies (iii), \) and \( (iii) \implies (i) \).

3.1.1 \( (i) \implies (ii) \)

Let \( \delta > 0 \) and \( f : \mathbb{R}^M \to \mathbb{R} \) be a continuous and bounded function with the properties

\[
f \geq 0, \quad f(0) = 0, \quad \text{and} \quad f(x) = 1, x \in \mathbb{R}^M \setminus \prod_{\lambda=1}^{M} [-\delta, \delta].
\]

The weak convergence of \( (\mu_n)_n \) to \( \delta_0 \) implies

\[
\int_{\mathbb{R}^M} f \, d\mu_n \to \int_{\mathbb{R}^M} f \, d\delta_0 = f(0) = 0.
\]

\footnote{The Curie-Weiss model defined in Section 1 satisfies such an exponential concentration condition \( \sup_{x \in \mathcal{L}_n} \left| \frac{\prod_{\lambda=1}^{M} \sqrt{n_\lambda}}{2^M} \mathbb{P} \left( \left( \frac{S_{n,1}}{\sqrt{n_1}}, \ldots, \frac{S_{n,M}}{\sqrt{n_M}} \right) = x \right) - \phi(x) \right| \to 0 \quad \text{as} \quad n \to \infty \) in its high temperature regime.}
Since \( f \geq 0 \) and \( f \) equals 1 on \( \mathbb{R}^M \setminus \prod_{\lambda=1}^{M} [-\delta, \delta] \), we have

\[
\mu_n \left( \mathbb{R}^M \setminus \prod_{\lambda=1}^{M} [-\delta, \delta] \right) \rightarrow 0.
\]

### 3.1.2 \((ii) \implies (iii)\)

By \((ii)\), we have for all \( m \in \mathbb{N} \)

\[
\mu_n \left( \mathbb{R}^M \setminus \prod_{\lambda=1}^{M} \left[ \frac{-1}{m}, \frac{1}{m} \right] \right) \rightarrow 0.
\]

(4)

We define the natural numbers \( n_1, n_2, \ldots \) as follows:

\[
n_m := \min \left\{ k \in \mathbb{N} \mid \mu_n \left( \prod_{\lambda=1}^{M} \left[ \frac{-1}{m}, \frac{1}{m} \right] \right) \geq 1 - \frac{1}{m}, n \geq k \right\}, \quad m \in \mathbb{N}.
\]

Note that the sets in the display are non-empty for each \( m \) due to (4). Set

\[
\epsilon_{n,\lambda} := \frac{1}{m}, \quad \lambda = 1, \ldots, M, \quad n_m \leq n < n_{m+1}, \quad m \in \mathbb{N}.
\]

By construction, \( \|\epsilon_n\|_{\infty} \rightarrow 0 \) and \( \mu_n ([\epsilon_n, \epsilon_n]) \rightarrow 1 \).

### 3.1.3 \((iii) \implies (i)\)

Let \( f : \mathbb{R}^M \rightarrow \mathbb{R} \) be a continuous and bounded function. Let \( \eta > 0 \). Since \( f \) is continuous, there is a \( \delta > 0 \) such that, for all \( x \in \mathbb{R}^M \),

\[
\|x\|_{\infty} < \delta \implies |f(x) - f(0)| < \eta.
\]

Then

\[
\left| \int_{\mathbb{R}^M} f \, d\mu_n - \int_{\mathbb{R}^M} f \, d\delta_0 \right| \leq \int_{\mathbb{R}^M} |f(x) - f(0)| \, \mu_n (dx).
\]

(5)

Let \( n_0 \in \mathbb{N} \) be large enough that

\[
\|\epsilon_n\|_{\infty} < \delta \quad \text{and} \quad \mu_n ([\epsilon_n, \epsilon_n]) > 1 - \eta, \quad n \geq n_0.
\]

We continue with our calculation:

\[
\frac{3}{5} = \int_{\epsilon_n, \epsilon_n} |f(x) - f(0)| \, \mu_n (dx) + \int_{\mathbb{R}^M \setminus \epsilon_n, \epsilon_n} |f(x) - f(0)| \, \mu_n (dx)
\]

\[
\leq \eta \mu_n ([\epsilon_n, \epsilon_n]) + 2 \|f\|_{\infty} \eta,
\]

which proves the weak convergence of \((\mu_n)\) to \( \delta_0 \).

### 3.2 Proof of Theorem \( \text{[11]} \)

The proofs of the theorems in Section \( \text{[2]} \) use characteristic functions. Thus, we have to show pointwise convergence of the sequence of characteristic functions of the normalised sums to the characteristic function of the limiting distribution. In the case of Theorem \( \text{[11]} \) we define \( \varphi_n \) to be the characteristic function of the distribution of

\[
S_n = \left( \frac{S_{n,1}}{\sqrt{n_1}}, \ldots, \frac{S_{n,M}}{\sqrt{n_M}} \right)
\]

and \( \varphi_{N(0, I_M)} \) to be the characteristic function of \( N(0, I_M) \). Then our task is to show \( \varphi_n(t) \xrightarrow{n \to \infty} \varphi_{N(0, I_M)}(t) \) for all \( t \in \mathbb{R}^M \). We will only prove the result for sequences of de Finetti measures on \([-1, 1]^M \). The results for de Finetti sequences on \( \mathbb{R}^M \) can be shown analogously.

We will use the following notation for ‘asymptotic equivalence’:
Notation 21. Complex-valued sequences \((f_n)_n, (g_n)_n\) are called asymptotically equal (as \(n \to \infty\)), in short \(f_n \approx g_n\), if
\[
\lim_{n \to \infty} \frac{f_n}{g_n} = 1.
\]

Recall Definition 2 and Notation 3. By Definition 4, \(\phi(t)\) can be expressed as
\[
\mathbb{E} \exp(i t \cdot S_n) = \mathbb{E} \left( \exp \left( i \left( t_1 \frac{S_{n,1}}{\sqrt{m_1}} + \cdots + t_M \frac{S_{n,M}}{\sqrt{m_M}} \right) \right) \right)
= \int_{[-1,1]^M} E_m \exp \left( i \left( t_1 \frac{S_{n,1}}{\sqrt{m_1}} + \cdots + t_M \frac{S_{n,M}}{\sqrt{m_M}} \right) \right) \mu_n (dm). \tag{6}
\]

We first note that due to the boundedness of the integrand in (6) and the concentration property
\[
\mu_n \left[ -\varepsilon_n, \varepsilon_n \right] \xrightarrow{n \to \infty} 1,
\]
we obtain
\[
(6) \approx \int_{[-\varepsilon_n, \varepsilon_n]} E_m \exp \left( i \left( t_1 \frac{S_{n,1}}{\sqrt{m_1}} + \cdots + t_M \frac{S_{n,M}}{\sqrt{m_M}} \right) \right) \mu_n (dm). \tag{7}
\]
This step follows from the next lemma, provided that (7) converges to a non-zero limit, which we will show in (14).

Lemma 22. Let \(f_n : [-1,1]^M \to \mathbb{C}\) be measurable functions with \(\|f_n\|_\infty \leq 1, n \in \mathbb{N}\), and \((\mu_n)_n\) the de Finetti sequence in the statement of Theorem 17. Then
\[
\left| \int_{[-1,1]^M} f_n \, d\mu_n - \int_{[-\varepsilon_n, \varepsilon_n]} f_n \, d\mu_n \right| \xrightarrow{n \to \infty} 0.
\]

Proof. We estimate
\[
\left| \int_{[-1,1]^M} f_n \, d\mu_n - \int_{[-\varepsilon_n, \varepsilon_n]} f_n \, d\mu_n \right| = \left| \int_{\mathbb{R}^M \setminus [-\varepsilon_n, \varepsilon_n]} f_n \, d\mu_n \right| \leq \mu_n \left( \mathbb{R}^M \setminus [-\varepsilon_n, \varepsilon_n] \right) \xrightarrow{n \to \infty} 0.
\]

Next, we have to calculate the expectation \(E_m \exp \left( i \left( t_1 S_{n,1} / \sqrt{m_1} + \cdots + t_M S_{n,M} / \sqrt{m_M} \right) \right)\). Under \(P_{m,n}^\otimes\), all random variables \(X_{\lambda i}, \lambda = 1, \ldots, M, i = 1, \ldots, n_{\lambda}\) are independent. Within each group \(\lambda\), the \(X_{\lambda i}\) are even i.i.d. We subtract the expected value \(m_\lambda\) of each \(X_{\lambda i}\) under \(P_{m,n}^\otimes\). Note that for \(t_\lambda = 0\),
\[
E_m \exp \left( i \left( t_\lambda \frac{\sum_j (X_{\lambda j} - m_\lambda)}{\sqrt{n_\lambda}} \right) \right) = 1.
\]

Now assume \(t_\lambda \neq 0\). Using a Taylor expansion yields
\[
E_m \exp \left( i \left( t_\lambda \frac{\sum_j (X_{\lambda j} - m_\lambda)}{\sqrt{n_\lambda}} \right) \right) = \left( 1 - (1 - m_\lambda^2) \frac{t_\lambda^2}{2n_\lambda} + O \left( \frac{1}{n_\lambda^{3/2}} \right) \right)^{n_\lambda}.
\]

Thus, the conditional expectation in the integral in (7) can be written as
\[
\prod_{\lambda : t_\lambda \neq 0} \exp (im_\lambda t_\lambda \sqrt{n_\lambda}) \left( 1 - (1 - m_\lambda^2) \frac{t_\lambda^2}{2n_\lambda} + O \left( \frac{1}{n_\lambda^{3/2}} \right) \right)^{n_\lambda}.
\]
Set
\[ a_{n,\lambda} := \exp \left( i m \lambda \sqrt{n \lambda} \right), \]
\[ b_{n,\lambda} := \left( 1 - (1 - m^2 \lambda^2) \frac{t^2}{2 n \lambda} + O \left( \frac{1}{n^{3/2}} \right) \right)^{n \lambda}, \]
\[ c_{n,\lambda} := a_{n,\lambda} b_{n,\lambda} \]
for each \( n \) and each \( \lambda \).

First, we analyse the asymptotic behaviour of \((a_{n,\lambda})_n\).

**Lemma 23.** We have for all \( n \in \mathbb{N} \) and all \( \lambda = 1, \ldots, M \)
\[ a_{n,\lambda} \in A_n := \left\{ \exp \left( i \omega \right) \mid \omega \in [-t_{\lambda} |\varepsilon_n,\lambda\sqrt{n\lambda}|, t_{\lambda} |\varepsilon_n,\lambda\sqrt{n\lambda}|] \right\}, \]
and
\[ A_n \setminus \{1\} \quad \text{as} \quad n \to \infty. \]

**Proof.** Since \( t_{\lambda} \neq 0 \) and \( m \) takes values on the interval \([-\varepsilon_n, \varepsilon_n]\), \( a_{n,\lambda} \) lies on the unit circle in the complex plane, specifically in the section defined by \( A_n \). The convergence speed \( \varepsilon_{n,\lambda} = o \left( \frac{1}{\sqrt{n \lambda}} \right) \) implies \( \pm |t_{\lambda}| \varepsilon_{n,\lambda}\sqrt{n \lambda} \xrightarrow{n \to \infty} 0 \) and therefore \( A_n \setminus \{1\} \) as \( n \to \infty. \)

Next, we deal with the sequence \((b_{n,\lambda})_n\). We estimate
\[ \left| b_{n,\lambda} - \exp \left( -\frac{t^2}{2} \right) \right| \leq \left| b_{n,\lambda} - \left( 1 - (1 - m^2 \lambda^2) \frac{t^2}{2 n \lambda} \right)^{n \lambda} \right| + \left| \left( 1 - (1 - m^2 \lambda^2) \frac{t^2}{2 n \lambda} \right)^{n \lambda} - \exp \left( - (1 - m^2 \lambda^2) \frac{t^2}{2} \right) \right|, \]
and we will find upper bounds for each of the summands in (8).

**Lemma 24.** There is a constant \( C > 0 \) such that for all \( n \in \mathbb{N} \) we have
\[ \left| b_{n,\lambda} - \left( 1 - (1 - m^2 \lambda^2) \frac{t^2}{2 n \lambda} \right)^{n \lambda} \right| \leq \frac{C}{n^{3/2}}. \]

**Proof.** We define \( r_{n,\lambda} := (1 - m^2 \lambda^2) \frac{t^2}{2 n \lambda} \) and \( s_{n,\lambda} \) to be the \( O \left( \frac{1}{n \lambda} \right) \) term in the definition of \( b_{n,\lambda} \). Then we have
\[ b_{n,\lambda} = \left( 1 - \frac{r_{n,\lambda}}{n \lambda} + s_{n,\lambda} \right)^{n \lambda} = \sum_{k=0}^{n \lambda} \binom{n \lambda}{k} \left( 1 - \frac{r_{n,\lambda}}{n \lambda} \right)^{n \lambda-k} \left( 1 - \frac{s_{n,\lambda}}{n \lambda} \right)^{k}. \]

We calculate
\[ \left| b_{n,\lambda} - \left( 1 - (1 - m^2 \lambda^2) \frac{t^2}{2 n \lambda} \right)^{n \lambda} \right| \leq \left| \left( 1 - \frac{r_{n,\lambda}}{n \lambda} \right)^{n \lambda} \sum_{k=0}^{n \lambda} \binom{n \lambda}{k} \left( \frac{s_{n,\lambda}}{1 - r_{n,\lambda}/n \lambda} \right)^k \left( 1 - \frac{r_{n,\lambda}}{n \lambda} \right)^{n \lambda} \right| 
\]
\[ = \left| 1 - \frac{r_{n,\lambda}}{n \lambda} \right|^{n \lambda} \sum_{k=0}^{n \lambda} \binom{n \lambda}{k} \left( \frac{s_{n,\lambda}}{1 - r_{n,\lambda}/n \lambda} \right)^k \left( 1 - \frac{r_{n,\lambda}}{n \lambda} \right)^{-1} 
\]
\[ = \left| 1 - \frac{r_{n,\lambda}}{n \lambda} \right|^{n \lambda} \left( \frac{s_{n,\lambda}}{1 - r_{n,\lambda}/n \lambda} \right)^{n \lambda+1} \left( 1 - \frac{s_{n,\lambda}}{1 - r_{n,\lambda}/n \lambda} \right)^{-1}, \]
where in the last step we applied the formula for the value of a geometric series with complex summands. We note that \( r_{n,\lambda} \xrightarrow{\lambda \to \infty} \frac{1}{2} \) and hence

\[
\frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \xrightarrow{\lambda \to \infty} 0 \quad \text{and} \quad \left| 1 - \frac{r_{n,\lambda}}{n,\lambda} \right|^{n,\lambda} \xrightarrow{\lambda \to \infty} \exp\left(-\frac{t^2}{2}\right).
\]

There is a constant \( C_1 > 0 \) such that

\[
\left| 1 - \frac{r_{n,\lambda}}{n,\lambda} \right|^{n,\lambda} \leq C_1, \quad n,\lambda \in \mathbb{N}.
\]

We continue with

\[
\left| 1 - \left( \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \right)^{n,\lambda+1} \right| = \left| 1 - \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \right| - 1 = \left| 1 - \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \right|^{n,\lambda} = \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \left| 1 - \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \right|^{n,\lambda}.
\]

Now we note that

\[
1 - \frac{r_{n,\lambda}}{n,\lambda} \xrightarrow{\lambda \to \infty} 1, \quad 1 - \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \xrightarrow{\lambda \to \infty} 1, \quad \text{and} \quad 1 - \left( \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \right)^{n,\lambda} \xrightarrow{\lambda \to \infty} 1.
\]

The above implies

\[
1 - \frac{r_{n,\lambda}}{n,\lambda} \xrightarrow{\lambda \to \infty} 1, \quad \text{and} \quad 1 - \left( \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \right)^{n,\lambda} = \Theta(1)
\]

and the existence of a constant \( C_2 > 0 \) such that

\[
\left| \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \right| \left| 1 - \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \right|^{n,\lambda} \leq C_2 \left| \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \right|, \quad n,\lambda \in \mathbb{N}.
\]

Since \( 1 - \frac{r_{n,\lambda}}{n,\lambda} \xrightarrow{\lambda \to \infty} 1 \), there is a constant \( C_3 > 0 \) such that

\[
\left| \frac{s_{n,\lambda}}{1 - \frac{s_{n,\lambda}}{n,\lambda}} \right| \leq C_3 |s_{n,\lambda}|, \quad n,\lambda \in \mathbb{N}.
\]

Finally, since \( |s_{n,\lambda}| = O\left(\frac{1}{n,\lambda}\right) \), there is a constant \( C_4 > 0 \) such that

\[
|s_{n,\lambda}| \leq \frac{C_4}{n,\lambda^{3/2}}, \quad n,\lambda \in \mathbb{N}.
\]

The claim follows by setting \( C := C_1C_2C_3C_4 > 0 \).

Next, we will find an upper bound for the second summand in (8).
Lemma 25. There is a constant $D > 0$ such that for all $n \in \mathbb{N}$ we have

$$\left| \left( 1 - \left( 1 - m_{\lambda}^2 \right) \frac{t_{\lambda}^2}{2m_{\lambda}} \right)^{n_{\lambda}} - \exp \left( - \left( 1 - m_{\lambda}^2 \right) \frac{t_{\lambda}^2}{2} \right) \right| \leq \frac{D}{n_{\lambda}}.$$ 

Proof. We again define $r_{n_{\lambda}} := \left( 1 - m_{\lambda}^2 \right) \frac{t_{\lambda}^2}{2}$. Recall that $r_{n_{\lambda}} \xrightarrow{n_{\lambda} \to \infty} \frac{t_{\lambda}^2}{2}$. Choose $m_0 \in \mathbb{N}$ large enough that

$$\left| \frac{r_{n_{\lambda}}}{m_{\lambda}} \right| < \frac{1}{2} \quad \text{and} \quad \left| r_{n_{\lambda}} - \frac{t_{\lambda}^2}{2} \right| + \frac{r_{n_{\lambda}}^2}{m_{\lambda}} < 1, \quad m_{\lambda} \geq m_0. \quad (9)$$

We use Taylor expansions for the function $\exp$ and $\ln$:

$$\exp z = 1 + z + O \left( z^2 \right), \quad z \in \mathbb{R}, |z| \leq 1,$$

$$\ln (1 + z) = z + O \left( z^2 \right), \quad z \in \mathbb{R}, |z| \leq \frac{1}{2},$$

More precisely, we have the bounds

$$|\exp z - (1 + z)| \leq \frac{e}{2} z^2, \quad z \in \mathbb{R}, |z| \leq 1, \quad (10)$$

$$|\ln (1 + z) - z| \leq 2z^2, \quad z \in \mathbb{R}, |z| \leq \frac{1}{2}, \quad (11)$$

on the remainder terms of order 2, as is easy to see from the Lagrange form of said remainder terms. The constant $m_0$ in (9) is chosen such that the value of $z$ in the Taylor expansions will be small enough in our calculations.

Then we have for all $n_{\lambda} \geq m_0$

$$\left( 1 - \left( 1 - m_{\lambda}^2 \right) \frac{t_{\lambda}^2}{2m_{\lambda}} \right)^{n_{\lambda}} = \left( 1 - \frac{r_{n_{\lambda}}}{m_{\lambda}} \right)^{n_{\lambda}} = \exp \left[ n_{\lambda} \ln \left( 1 - \frac{r_{n_{\lambda}}}{m_{\lambda}} \right) \right]$$

$$= \exp \left[ n_{\lambda} \left( - \frac{r_{n_{\lambda}}}{n_{\lambda}} + s_{n_{\lambda}} \right) \right] = \exp \left[ -r_{n_{\lambda}} + n_{\lambda} s_{n_{\lambda}} \right],$$

where the remainder term $s_{n_{\lambda}}$ satisfies $|s_{n_{\lambda}}| \leq \left( \frac{r_{n_{\lambda}}}{n_{\lambda}} \right)^2 = O \left( \frac{1}{n_{\lambda}^2} \right)$ by (11). We continue with our calculation:

$$\exp (-r_{n_{\lambda}} + n_{\lambda} s_{n_{\lambda}}) = \exp \left[ -\frac{t_{\lambda}^2}{2} - \left( r_{n_{\lambda}} - \frac{t_{\lambda}^2}{2} \right) + n_{\lambda} s_{n_{\lambda}} \right]$$

$$= \exp \left( -\frac{t_{\lambda}^2}{2} \right) \exp \left[ - \left( r_{n_{\lambda}} - \frac{t_{\lambda}^2}{2} \right) + n_{\lambda} s_{n_{\lambda}} \right]$$

$$= \exp \left( -\frac{t_{\lambda}^2}{2} \right) \left[ 1 - \left( r_{n_{\lambda}} - \frac{t_{\lambda}^2}{2} \right) + n_{\lambda} s_{n_{\lambda}} + u_{n_{\lambda}} \right],$$

where the remainder term $u_{n_{\lambda}}$ satisfies

$$|u_{n_{\lambda}}| \leq \frac{e}{2} \left( - \left( r_{n_{\lambda}} - \frac{t_{\lambda}^2}{2} \right) + n_{\lambda} s_{n_{\lambda}} \right)^2$$

$$= O \left( - \left( r_{n_{\lambda}} - \frac{t_{\lambda}^2}{2} \right) + n_{\lambda} s_{n_{\lambda}} \right)^2$$

$$= O \left( r_{n_{\lambda}} - \frac{t_{\lambda}^2}{2} \right)^2 + O \left( n_{\lambda} s_{n_{\lambda}} \right)^2 = O \left( r_{n_{\lambda}} - \frac{t_{\lambda}^2}{2} \right)^2 + O \left( \frac{1}{n_{\lambda}^2} \right), \quad (12)$$

13
by (10). We used $O \left( \left( n_\lambda s_{n_\lambda} \right)^2 \right) = O \left( \frac{n_\lambda^2}{n_\lambda} \right) = O \left( \frac{1}{n_\lambda} \right)$ in the last step above. Now we estimate the difference

$$\left( 1 - (1 - m_\lambda^2 \frac{t_\lambda^2}{2n_\lambda}) \right)^{n_\lambda} - \exp \left( - (1 - m_\lambda^2 \frac{t_\lambda^2}{2}) \right) = \exp \left( \frac{t_\lambda^2}{2} \right) - \left( 1 - m_\lambda^2 \frac{t_\lambda^2}{2} + n_\lambda s_{n_\lambda} + u_{n_\lambda} - 1 \right) \right|$$

$$\leq \exp \left( - \frac{t_\lambda^2}{2} \right) \left[ r_{n_\lambda} - \frac{t_\lambda^2}{2} + |n_\lambda s_{n_\lambda}| + |u_{n_\lambda}| \right] = O \left( r_{n_\lambda} - \frac{t_\lambda^2}{2} \right) + O \left( \frac{1}{n_\lambda} \right) + O \left( \frac{1}{n_\lambda} \right) \right|^2 + O \left( \frac{1}{n_\lambda} \right)$$

$\exp (\frac{t_\lambda^2}{2}) \leq e^\frac{t_\lambda^2}{2} \leq \frac{1}{n_\lambda}$. 

Taking into account $n_\lambda s_{n_\lambda} = O \left( \frac{1}{n_\lambda} \right)$ and (12), we have

$$\left( 1 - (1 - m_\lambda^2 \frac{t_\lambda^2}{2n_\lambda}) \right)^{n_\lambda} - \exp \left( - (1 - m_\lambda^2 \frac{t_\lambda^2}{2}) \right) = O \left( \frac{1}{n_\lambda} \right)$$

and hence the statement of the lemma is proved. \(\square\)

Lemmas 24 and 25 imply the following corollary:

**Corollary 26.** The range of $b_{n,\lambda}$ over all possible values of $m_\lambda \in [-\varepsilon_{n,\lambda}, \varepsilon_{n,\lambda}]$ lies in a $\frac{D}{n_\lambda}$-neighbourhood $B_n$ of the set $\left\{ \exp \left( - \frac{t_\lambda^2}{2} \right), \exp \left( - (1 - \varepsilon_{n,\lambda}^2 \frac{t_\lambda^2}{2}) \right) \right\}$ in $\mathbb{C}$. The set sequence $B_n$ converges:

$$B_n \setminus \left\{ \exp \left( - \frac{t_\lambda^2}{2} \right) \right\} \quad \text{as } n \to \infty.$$ 

Since $\mu_n$ is symmetric, once we integrate $c_{n,\lambda}$ with respect to $\mu_n$, only the integral of the real part $\text{Re} \left( c_{n,\lambda} \right)$ remains. This real part is the radius of the circle on which all values of $c_{n,\lambda}$ lie, and these radii are the values that comprise the set $B_n$. As each $\mu_n$ is a probability measure and we assumed $\mu_n \left[ -\varepsilon_n, \varepsilon_n \right] \xrightarrow{n \to \infty} 1$, we have

$$\left( \prod_{\lambda=1}^{M} \exp \left( - \frac{t_\lambda^2}{2} \right) \right) = \varphi_N(0, I_M)(t). \quad (14)$$

As we see, $\prod_{\lambda=1}^{M} \exp \left( - \frac{t_\lambda^2}{2} \right)$ converges to a non-zero limit as $n \to \infty$, thereby proving the asymptotic equivalence of $\prod_{\lambda=1}^{M} \exp \left( - \frac{t_\lambda^2}{2} \right)$ to $\varphi_n(t)$.

This concludes the proof of Theorem 11.
3.3 Proof of Theorem 13

We now show the statements of Theorem 13. The first statement is a corollary of Theorem 11. To prove the second statement, we once again have to calculate $\varphi_n(t) = \mathbb{E}\exp(it \cdot \mathbf{S}_n)$. The normalisation $\gamma_{n,\lambda} = \sqrt{n\lambda}$ is the same as in Theorem 11. Since $\mu_n$ is the contraction of $\mu$, which has support in $[-1, 1]^M$, the support of $\mu_n$ is a subset of $[-\varepsilon_n, \varepsilon_n]$ for each $n \in \mathbb{N}$, and hence
\[
\varphi_n(t) = \int_{[-\varepsilon_n, \varepsilon_n]} E_m \exp\left( i \left( t_1 \frac{S_{n,1}}{\gamma_{n,1}} + \cdots + t_M \frac{S_{n,M}}{\gamma_{n,M}} \right) \right) \mu_n(dm). \tag{15}
\]
At this point, we change variables by setting $s := \left( \frac{1}{\varepsilon_n}, \ldots, \frac{1}{\gamma_{n,M}} \right) \circ m$. Then
\[
E_{\varepsilon_n \circ s} \exp\left( i \left( t_1 \frac{S_{n,1}}{\gamma_{n,1}} + \cdots + t_M \frac{S_{n,M}}{\gamma_{n,M}} \right) \right) \mu_n(ds) = \int_{[-1,1]^M} E_{\varepsilon_n \circ s} \exp\left( i \left( t_1 \frac{S_{n,1}}{\gamma_{n,1}} + \cdots + t_M \frac{S_{n,M}}{\gamma_{n,M}} \right) \right) \mu(ds), \tag{16}
\]
where we used that, by definition, $\mu_n(\varepsilon_n \circ A) = \mu(A)$ for all measurable sets $A$. The conditional expectations $E_{\varepsilon_n \circ s} \exp\left( it \sqrt{n\lambda} \right)$ can be expressed via a Taylor expansion as
\[
E_{\varepsilon_n \circ s} \exp\left( it \sqrt{n\lambda} \right) = \exp(i\varepsilon_n \lambda s \lambda t \lambda \gamma_{n,\lambda}) \left( 1 - \left( 1 - \varepsilon_n^2 n\lambda s \lambda^2 \right) \frac{t^2}{2n\lambda} + O\left( \frac{1}{n\lambda^{3/2}} \right) \right)^n \lambda \xrightarrow{n \to \infty} \exp(it \lambda \varepsilon_n \lambda s \lambda) \exp\left( -\frac{t^2}{2n\lambda} \right).
\]
By dominated convergence, we obtain
\[
\text{(16)} \xrightarrow{n \to \infty} \int_{[-1,1]^M} \prod_{\lambda=1}^M \exp(it \lambda \varepsilon_n \lambda s \lambda) \exp\left( -\frac{t^2}{2n\lambda} \right) \mu(ds) = \varphi_\mu(h \circ t) \varphi_{\lambda(0,tM)}(t),
\]
where $\varphi_\mu$ is the characteristic function of $\mu$.

Since $\varphi_\mu(h \circ t) = \varphi_{\lambda \mu}(t)$, this concludes the proof of the second statement of Theorem 13.

In statement 3, we have the normalisation sequences $\gamma_{n,\lambda} := \varepsilon_n \lambda n\lambda$. Let $\varphi_n$ be the characteristic function of the distribution of
\[
\mathbf{S}_n = \left( \frac{S_{n,1}}{\gamma_{n,1}}, \ldots, \frac{S_{n,M}}{\gamma_{n,M}} \right).
\]
The proof of statement 3 proceeds as outlined above and yields
\[
\varphi_n(t) = \int_{[-1,1]^M} E_{\varepsilon_n \circ s} \exp\left( i \left( t_1 \frac{S_{n,1}}{\gamma_{n,1}} + \cdots + t_M \frac{S_{n,M}}{\gamma_{n,M}} \right) \right) \mu(ds). \tag{17}
\]
We once again expand the conditional expectation in the integral above:
\[
E_{\varepsilon_n \circ s} \exp\left( it \sqrt{n\lambda} \right) = \exp\left( i\varepsilon_n \lambda s \lambda t \lambda \gamma_{n,\lambda} \right) \left( 1 - \left( 1 - \varepsilon_n^2 n\lambda s \lambda^2 \right) \frac{t^2}{2n\lambda} + O\left( \frac{1}{n\lambda^{3/2}} \right) \right)^n \lambda .
\]
Recall that we assumed $1/\sqrt{n\lambda} = o(\varepsilon_n \lambda)$. It follows that $1/n\lambda = o(\varepsilon_n^2)$ and $n\lambda \varepsilon_n^2 n\lambda \xrightarrow{n \to \infty} \infty$. Hence, $n\lambda = o\left( \varepsilon_n^2 n\lambda \right)$, and due to $\gamma_{n,\lambda}^2 = \varepsilon_n^2 n\lambda^2$ we have
\[
E_{\varepsilon_n \circ s} \exp\left( it \sqrt{n\lambda} \right) \approx \exp\left( i\varepsilon_n \lambda s \lambda t \lambda \gamma_{n,\lambda} \right) = \exp(it \lambda \varepsilon_n \lambda s \lambda) .
\]
By dominated convergence,

\[ \varphi_n(t) \xrightarrow{n \to \infty} \int_{[-1,1]^M} \prod_{\lambda=1}^M \exp (is\lambda t_\lambda) \mu (ds) = \varphi_\mu (t), \]

and our proof is complete.

We omit the proof of Theorem 16 as it amounts to a repetition of the arguments presented above, carefully keeping track of the coordinates belonging to each of the three clusters. Corollaries 17 and 14 follow immediately from Theorem 16.

### 3.4 Proof of Proposition 18

We calculate the correlation \( \mathbb{E} (X_{\lambda_1} X_{\nu_2}) \) using Definition 4

\[
\mathbb{E} (X_{\lambda_1} X_{\nu_2}) = \int_{\mathbb{R}^M} E_{\bar{m}} (X_{\lambda_1} X_{\nu_2}) \mu_n (dm) = \int_{\mathbb{R}^M} (P_{\bar{m}}^{\otimes n} (X_{\lambda_1} = X_{\nu_2}) - P_{\bar{m}}^{\otimes n} (X_{\lambda_1} \neq X_{\nu_2})) \mu_n (dm)
\]

\[ = 2 \int_{\mathbb{R}^M} P_{\bar{m}}^{\otimes n} (X_{\lambda_1} = X_{\nu_2}) \mu_n (dm) - 1. \]

We express \( P_{\bar{m}}^{\otimes n} (X_{\lambda_1} = X_{\nu_2}) \) in terms of \( \bar{m} \):

\[
P_{\bar{m}}^{\otimes n} (X_{\lambda_1} = X_{\nu_2}) = P_{\bar{m}}^{\otimes n} (X_{\lambda_1} = X_{\nu_2} = 1) + P_{\bar{m}}^{\otimes n} (X_{\lambda_1} = X_{\nu_2} = -1) = \frac{1 + \bar{m}_1 \bar{m}_\nu}{2} + \frac{1 - \bar{m}_\lambda - \bar{m}_\nu}{2} = \frac{1}{2} \left( 1 + \bar{m}_\lambda \bar{m}_\nu \right),
\]

where in the second step we used that all \( X_{\kappa_i} \) are independent random variables under \( P_{\bar{m}}^{\otimes n} \). Hence,

\[
\mathbb{E} (X_{\lambda_1} X_{\nu_2}) = \int_{\mathbb{R}^M} (1 + \bar{m}_\lambda \bar{m}_\nu) \mu_n (dm) - 1 = \int_{\mathbb{R}^M} \bar{m}_\lambda \bar{m}_\nu \mu_n (dm).
\]

We have

\[
\int_{\mathbb{R}^M} \bar{m}_\lambda \bar{m}_\nu \mu_n (dm) = \int_{[-\varepsilon_n, \varepsilon_n]} \bar{m}_\lambda \bar{m}_\nu \mu_n (dm) + \int_{\mathbb{R}^M \setminus [-\varepsilon_n, \varepsilon_n]} \bar{m}_\lambda \bar{m}_\nu \mu_n (dm),
\]

and we upper bound each summand in turn:

\[
\left| \int_{[-\varepsilon_n, \varepsilon_n]} \bar{m}_\lambda \bar{m}_\nu \mu_n (dm) \right| \leq \sup_{m \in [-\varepsilon_n, \varepsilon_n]} |\bar{m}_\lambda \bar{m}_\nu| = O (\varepsilon_n \lambda \varepsilon_{n, \nu}) \xrightarrow{n \to \infty} 0,
\]

where we used the fact that \( \lim_{m \to 0} \frac{\bar{m}_\nu}{\bar{m}_n} = 1 \) for all \( \kappa = 1, \ldots, M \), and

\[
\left| \int_{\mathbb{R}^M \setminus [-\varepsilon_n, \varepsilon_n]} \bar{m}_\lambda \bar{m}_\nu \mu_n (dm) \right| \leq \mu_n (\mathbb{R}^M \setminus [-\varepsilon_n, \varepsilon_n]) \xrightarrow{n \to \infty} 0,
\]

where we used that \( \|\bar{m}_\lambda \bar{m}_\nu\| \leq 1 \).

### 3.5 Proof of Theorem 20

For our proof of the local limit Theorem 20 we will need the following auxiliary lemmas:

**Lemma 27.** Let \( Y := (Y_1, \ldots, Y_d) \) be a random vector with values on the lattice \( \prod_{\lambda=1}^d (v_\lambda + w_\lambda \mathbb{Z}) \), where \( v_\lambda \in \mathbb{R} \) and \( w_\lambda > 0 \), and let \( \varphi \) be the characteristic function of the distribution of \( Y \). Then we have for all \( k_1, \ldots, k_d \in \mathbb{Z} \)

\[
|\varphi \left( 2\pi \left( \frac{k_1}{w_1}, \ldots, \frac{k_d}{w_d} \right) \right)| = 1,
\]

and for all \( t \in \mathbb{R}^d \) such that \( 0 < t_\lambda < 2\pi w_\lambda \) for at least one component \( \lambda \), we have

\[
|\varphi (t)| < 1.
\]
Proof. The lemma follows from the proof of Theorem 3.5.2 on page 140 in [7].

Lemma 27 gives an upper bound for the characteristic function of a random vector on a lattice (such as $S_n$), which we shall use in our calculations later on. We will use the following inversion formulas to recover distributions from their characteristic functions:

Lemma 28. Let $(Y_1, \ldots, Y_d)$ be a random vector as in Lemma 27 with characteristic function $\varphi$. Then, for all $x \in \prod_{\lambda=1}^d (\nu_\lambda + w_\lambda \mathbb{Z})$,

$$
\mathbb{P} \left( (Y_1, \ldots, Y_d) = x \right) = \frac{\prod_{\lambda=1}^d w_\lambda}{(2\pi)^d} \int_{\Pi_\lambda} \left[ \frac{n - \mathbf{e}_\lambda}{n - \mathbf{e}_\lambda} \right] e^{-i t \cdot x} \varphi(t) dt.
$$

Proof. See e.g. Section 3.10 in [7].

Lemma 29. Let $\varphi$ be the characteristic function of some probability distribution on $\mathbb{R}^d$ such that $\varphi$ is Lebesgue integrable. Then

$$
f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i t \cdot x} \varphi(t) dt
$$

for all $x \in \mathbb{R}^d$ defines a continuous Lebesgue density function $f$ for said probability distribution.

Proof. This is Theorem 5.5 in [39] (alternatively, see Theorem 3.3.14 in [7]).

Now we start the proof proper of Theorem 20. Let $\mathbb{I}_A$ be the indicator function of a measurable set $A$. The lattice on which $S_n$ lives is $\mathcal{L}_n := \prod_{\lambda=1}^M \left( \sqrt{n_\lambda} + \frac{\mathbf{e}_\lambda}{\sqrt{n_\lambda}} \mathbb{Z} \right)$. Let $x \in \mathcal{L}_n$. By Lemmas 28 and 29 we have

$$
\left| \frac{\prod_{\lambda=1}^M \sqrt{n_\lambda}}{2^M} \mathbb{P}(S_n = x) - \phi(x) \right| = \frac{1}{(2\pi)^M} \left| \int_{\Pi_\lambda} \left[ \frac{\mathbf{e}_\lambda}{\sqrt{n_\lambda} + \frac{\mathbf{e}_\lambda}{\sqrt{n_\lambda}}} \right] e^{-i t \cdot x} \varphi_n(t) dt - \int_{\mathbb{R}^M} e^{-i t \cdot x} \varphi_{N(0,1_M)}(t) dt \right|
\leq \frac{1}{(2\pi)^M} \int_{\mathbb{R}^M} \prod_{\lambda} \left[ \frac{n_\lambda - \delta \mathbf{e}_\lambda}{n_\lambda - \delta \mathbf{e}_\lambda} \right] (t) \left| \varphi_n(t) - \varphi_{N(0,1_M)}(t) \right| dt
+ \frac{1}{(2\pi)^M} \int_{\mathbb{R}^M \setminus \prod_{\lambda} \left[ \frac{n_\lambda - \delta \mathbf{e}_\lambda}{n_\lambda - \delta \mathbf{e}_\lambda} \right]} \left| \varphi_{N(0,1_M)}(t) \right| dt.
$$

(18)

(19)

Note that both (18) and (19) are independent of the point $x \in \mathcal{L}_n$. The term (19) converges to 0 as $n \to \infty$, since $|\varphi_{N(0,1_M)}|$ is integrable. Thus, our remaining task is to show that (18) converges to 0. By Theorem 21 which we have already shown, $\varphi_n(t) \xrightarrow{n \to \infty} \varphi_{N(0,1_M)}(t)$ holds for all $t \in \mathbb{R}^M$. Hence, we can prove that (18) converges to 0 by finding an appropriate integrable majorant and applying the theorem of dominated convergence.

We pick some $0 < \delta < \pi/2$ and partition the set $\prod_{\lambda} \left[ \frac{n_\lambda - \delta \mathbf{e}_\lambda}{n_\lambda - \delta \mathbf{e}_\lambda} \right]$ into the disjoint sets

$$
A_n := \prod_{\lambda} \left[ \frac{n_\lambda - \delta \mathbf{e}_\lambda}{n_\lambda - \delta \mathbf{e}_\lambda}, \frac{n_\lambda + \delta \mathbf{e}_\lambda}{n_\lambda + \delta \mathbf{e}_\lambda} \right] \quad \text{and} \quad B_n := \prod_{\lambda} \left[ \frac{n_\lambda - \delta \mathbf{e}_\lambda}{2}, \frac{n_\lambda + \delta \mathbf{e}_\lambda}{2} \right] \setminus A_n \quad \text{for each} \quad n \in \mathbb{N}.
$$

Let for all $s \in [-1,1]$ $\varphi_{R(s)}$ be the characteristic function of the Rademacher distribution on $\{-1,1\}$ (see Definition 2). We construct an integrable majorant first over $A_n$ and then $B_n$. The following upper bound holds over $A_n$:

$$
\mathbb{I}_{A_n}(t) |\varphi_n(t)| \leq \mathbb{I}_{A_n}(t) \int_{[-1,1]^M} \prod_{\lambda} \left| \varphi_{R(s_\lambda)} \left( \frac{s_\lambda}{\sqrt{n_\lambda}} \right) \right|^{n_\lambda} \mu_n(dm).
$$

(20)

The next lemma will provide an upper bound for the characteristic function of the Rademacher distribution.

Lemma 30. For any $x \in \mathbb{R}$ and any $n \in \mathbb{N}$, the upper bound

$$
\left| \exp(ix) - \sum_{k=0}^{n} \frac{(ix)^k}{k!} \right| \leq \min \left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^{n}}{n!} \right\}
$$

holds.
\textbf{Proof.} We use Taylor expansions with the remainder term in Legendre form. First we expand to order \( n \),
\[
\exp(ix) - \sum_{k=0}^{n} (ix)^k \frac{k!}{k!} = \frac{(ix)^{n+1} \exp(i\xi_{n+1})}{(n+1)!},
\]
and then to order \( n-1 \),
\[
\exp(ix) - \sum_{k=0}^{n-1} (ix)^k \frac{k!}{k!} = \frac{(ix)^n \exp(i\xi_n)}{n!}.
\]
The variables \( \xi_n \) and \( \xi_{n+1} \) each lie between 0 and \( x \). Combining the two displays above yields
\[
\exp(ix) - \sum_{k=0}^{n} (ix)^k \frac{k!}{k!} = \exp(ix) - \sum_{k=0}^{n-1} (ix)^k \frac{k!}{k!} - (ix)^n \frac{n!}{n!} = \frac{(ix)^n \exp(i\xi_n)}{n!} - (ix)^n \frac{n!}{n!}.
\]
Taking into account \( |i^n| = |i^{n+1}| = |\exp(i\xi_n)| = 1 \), we obtain the desired result by applying upper bounds to the absolute value of (21) and (22).

Applying the lemma to \( x = X_{\lambda 1} - s \) and \( n = 2 \) and taking the expectation \( E_s \), we obtain
\[
|E_s \exp(iu(X_{\lambda 1} - s)) - E_s \left( \sum_{k=0}^{2} (iu)^k (X_{\lambda 1} - s)^k \frac{k!}{k!} \right)| \leq u^2 E_s \min \left\{ \left| u \right| |X_{\lambda 1} - s|^3, |X_{\lambda 1} - s|^2 \right\} \leq u^2 \min \left\{ \left| u \right| E_s |X_{\lambda 1} - s|^3, E_s |X_{\lambda 1} - s|^2 \right\}.
\]

We calculate the expectations in the upper bound above:
\[
E_s |X_{\lambda 1} - s|^2 = 1 - s^2 \quad \text{and} \quad E_s |X_{\lambda 1} - s|^3 = 2 |s| (1 - s^2).
\]

We calculate an upper bound for the Rademacher characteristic function:
\[
|\varphi_R(u)| = |E_s \exp(iu(X_{\lambda 1} - s))| |\exp(ius)| \leq 1 - (1 - s^2) \frac{u^2}{2} + u^2 (1 - s^2) \min \left\{ |u| |s|, 1 \right\} \leq 1 - (1 - s^2) \frac{u^2}{2} + (1 - s^2) \frac{u^2}{4} \leq \exp \left( - (1 - s^2) \frac{u^2}{4} \right).
\]

Above, in the first inequality, we used a Taylor expansion of order 2 with remainder term upper bounded by (23). The second inequality holds for all \( u \in \mathbb{R} \) such that \( |u| |s| \leq 1/4 \), for which we have
\[
|u|^3 (1 - s^2) |s| \leq (1 - s^2) \frac{u^2}{4}.
\]

The third inequality follows from \( 1 - x \leq \exp(-x) \) for all \( x \in \mathbb{R} \).

Therefore,
\[
|\varphi_R(u)| \leq \exp \left( - (1 - m^2_{\lambda}) \frac{t^2}{4n_\lambda} \right)
\]
holds for all \( n \in \mathbb{N} \) large enough.
By assumption, there is some \( \tau \in (0,1)^M \) with the property (3). We continue with our calculation:

\[
\indic{A_n}(t) \int_{[-1,1]^M} \exp \left( -\frac{1}{4} \sum_{\lambda} (1 - m_{\lambda}^2) t_{\lambda}^2 \right) \mu_n(dm) \\
\leq \int_{[-\tau,\tau]} \exp \left( -\frac{1}{4} \sum_{\lambda} (1 - m_{\lambda}^2) t_{\lambda}^2 \right) \mu_n(dm) + \indic{A_n}(t) \mu_n \left( [-1,1]^M \setminus [-\tau,\tau] \right) \\
\leq \exp \left( -\frac{1}{4} \sum_{\lambda} (1 - \tau_{\lambda}^2) t_{\lambda}^2 \right) + \indic{A_n}(t) \mu_n \left( [-1,1]^M \setminus [-\tau,\tau] \right).
\]

(24)

It is clear that the first summand in (24) is integrable. For the second summand, we have

\[
\indic{A_n}(t) \mu_n \left( [-1,1]^M \setminus [-\tau,\tau] \right) \leq \indic{A_1}(t) \mu_1 \left( [-1,1]^M \setminus [-\tau,\tau] \right) + \sum_{k=1}^{\infty} \indic{A_{k+1}\setminus A_k}(t) \mu_k \left( [-1,1]^M \setminus [-\tau,\tau] \right) =: f(t).
\]

Let \( \lambda^M \) be the Lebesgue measure on \( \mathbb{R}^M \). We show that the function \( f \) on the right hand side is an integrable majorant for all \( \indic{A_n}(t) \mu_n \left( [-1,1]^M \setminus [-\tau,\tau] \right) \), \( n \in \mathbb{N} \):

\[
\int_{\mathbb{R}^M} f(t)dt = \lambda^M \left( A_1 \right) \mu_1 \left( [-1,1]^M \setminus [-\tau,\tau] \right) + \sum_{k=1}^{\infty} \lambda^M \left( A_{k+1}\setminus A_k \right) \mu_k \left( [-1,1]^M \setminus [-\tau,\tau] \right).
\]

Each summand in the series above can be bounded above by

\[
\lambda^M \left( A_{k+1}\setminus A_k \right) \mu_k \left( [-1,1]^M \setminus [-\tau,\tau] \right) \leq O \left( \left( \sqrt{k+1} - \sqrt{k} \right) \left( \sqrt{k+1} \right)^{M-1} \right) \mu_k \left( [-1,1]^M \setminus [-\tau,\tau] \right),
\]

which is summable in \( k \) by (3).

As \( |\varphi_{\mathcal{N}(0,1,M)}(t)| \) is integrable as well, we have thus found an integrable majorant for \( \indic{A_n}(t) |\varphi_n(t) - \varphi_{\mathcal{N}(0,1,M)}(t)| \).

By Theorem 11, \( |\varphi_n(t) - \varphi_{\mathcal{N}(0,1,M)}(t)| \xrightarrow{n \to \infty} 0 \) pointwise, so we conclude that the integral of \( \indic{A_n}(t) |\varphi_n(t) - \varphi_{\mathcal{N}(0,1,M)}(t)| \) over \( \mathbb{R}^M \) converges to 0 as \( n \to \infty \).

We proceed with the integrand over the set \( B_n \):

\[
\indic{B_n}(t) |\varphi_n(t)| \leq \indic{B_n}(t) \int_{[-1,1]^M} \prod_{\lambda} \varphi_{\mathcal{R}(m_{\lambda})} \left( \frac{t}{\sqrt{n_{\lambda}}} \right)^{n_{\lambda}} \mu_n(dm) \\
\leq \indic{B_n}(t) \int_{[-1,1]^M} (\theta(m))^{n_{\lambda}} \mu_n(dm),
\]

(25)

where the existence of

\[
\theta(m) = \max_{t \in B_n} \lambda = 1,\ldots,M \left| \varphi_{\mathcal{R}(m_{\lambda})}(t_{\lambda}) \right| < 1
\]

is a consequence of Lemma 27. We continue with the calculation of an upper bound using the constant \( \tau \) from (3):

\[
\indic{B_n}(t) \int_{[-\tau,\tau]} (\theta(m))^{n_{\lambda}} \mu_n(dm) + \indic{B_n}(t) \mu_n \left( [-1,1]^M \setminus [-\tau,\tau] \right)
\]

On the interval \( [-\tau,\tau] \), \( \theta \) is bounded away from 1:

\[
s := \sup_{m \in [-\tau,\tau]} \theta(m) < 1.
\]

This leads us to our final upper bound

\[
\indic{B_n}(t) |\varphi_n(t)| \leq \indic{B_n}(t) \left( s^{n_{\lambda}} + \mu_n \left( [-1,1]^M \setminus [-\tau,\tau] \right) \right).
\]

For the last expression, we can construct an integrable majorant in the same manner as for the second summand in (24). This concludes the proof of Theorem 20.
4 Examples of de Finetti Voting Models

We discuss two classes of voting models featured prominently in the literature that are de Finetti voting models according to Definition 4. As mentioned in the Introduction, models of voting behaviour are often adapted from statistical physics. Two categories are the collective bias model (CBM) and the Curie-Weiss model (CWM). The single-group CBM first appeared in Straffin’s work on voting systems [32] with a uniform de Finetti measure. It was later generalised in the works [14, 15]. The CWM in its single-group version has a long history in statistical physics and was first defined by Husimi [13] and Temperley [33]. It has seen a multitude of applications to the social sciences (see [2] for the first instance the CWM was used in such a context). The multi-group version of the CWM was defined independently in [5] and [1]. Subsequently, the model has received a lot of attention from other authors (see [8, 3, 9, 23, 17, 18, 19, 22]).

In the CWM, the voters tend to align with each other and there is no external influence on them. In the CBM, the voters do not care about others’ opinions, but there is some central influence (such as cultural or religious institutions) that induces correlation between the voters. In this sense, the two models are opposite. However, mathematically, they have more in common than a cursory glance would suggest. A multi-group CBM is defined by the voting measure

$$P_{CBM}(X_{11} = x_{11}, \ldots, X_{Mn_M} = x_{Mn_M}) := \int_{[-1,1]^M} P_{m}^\otimes n (x_{11}, \ldots, x_{Mn_M}) \mu(dm)$$

for all voting configurations $({x}_{11}, \ldots, {x}_{Mn_M})$. In the integral above, $\mu$ is a symmetric probability measure on $[-1,1]^M$ and recall the probability measure $P_{m}^\otimes n$ from Notation 3. By (26), the CBM is a de Finetti voting model according to Definition 4 with a constant sequence of de Finetti measures $\mu_n := \mu$, $n \in \mathbb{N}$.

The variable $m$ in the integral (26) measures the prevalent bias in the population. A positive bias means the voters are more likely to be in favour of the proposal. For a fixed bias $m$, the voters are conditionally independent in their decision. In a CBM,

$$\left(\frac{S_{n,1}}{n_1}, \ldots, \frac{S_{n,M}}{n_M}\right) \xrightarrow{d} \mu.$$  

The above limit theorem provides useful information on the large population behaviour of the CBM with the single exception of the (degenerate) case $\mu = \delta_0$, in which all voters are independent and we have the central limit theorem

$$\left(\frac{S_{n,1}}{\sqrt{n_1}}, \ldots, \frac{S_{n,M}}{\sqrt{n_M}}\right) \xrightarrow{d} \mathcal{N}(0, I_M).$$  

See [20] for these results.

While a CBM is a de Finetti voting model, it does not satisfy the assumption of decreasing social cohesion embodied by the weak convergence of $\mu_n$ to $\delta_0$, with the exception of $\mu = \delta_0$. For more on single-group CBMs in the context of voting theory, see [14], and for an in-depth discussion of multi-group versions of the CBM, see [20].

The single-group CWM is defined for all voting configurations $({x}_{1}, \ldots, {x}_{n})$ by

$$P_{CWM}(X_1 = x_1, \ldots, X_n = x_n) := Z_{\beta,n}^{-1} \exp \left(\frac{\beta}{2n} \sum_{i=1}^{n} x_i^2\right),$$

where $Z_{\beta,n}$ is a normalisation constant which depends on $\beta$ and $n$. The parameter $\beta \geq 0$ is the inverse temperature in the physical context of the CWM as a model of ferromagnetism. As a voting model, $\beta$ measures the degree of influence the voters exert over each other. As we see, the most probable voting configurations are those with unanimous votes in favour of or against the proposal. However, there are only two of these extreme configurations, whereas there is a multitude of low probability configurations with roughly equal numbers of votes for and against. This is the ‘conflict between energy and entropy’. Which one of these pseudo forces dominates depends on the regime the model is in, determined by the value of the
parameter $\beta \geq 0$. Using the Hubbard-Stratonovich transformation, the CWM can also be expressed in the following way (see e.g. Chapter 2 of [10] or Theorem 5.6 in the monograph [16]):

$$
\mathbb{P}_{\text{CWM}}(X_1 = x_1, \ldots, X_n = x_n) = (Z'_{\beta,n})^{-1} \int_{[-1,1]} \mathcal{P}^\otimes_n (x_{11}, \ldots, x_{MnM}) \frac{\exp(-nF(m))}{1 - m^2} \, dm,
$$

(29)

for all voting configurations $(x_1, \ldots, x_n)$.

$$
Z'_{\beta,n} = \int_{[-1,1]} \frac{\exp(-nF(m))}{1 - m^2} \, dm
$$

is a normalising constant and the function $F$ above is given by $F(m) := \frac{1}{2} \left( \frac{1}{2} \ln \frac{1+m}{1-m} \right)^2 + \ln (1 - m^2)$, $m \in [-1,1]$. The representation (29) of the CWM is called ‘de Finetti representation’. We note that for fixed $n$ we can define

$$
\mu_n := (Z'_{\beta,n})^{-1} \exp(-nF(m)) \frac{1}{1 - m^2} \mathbb{1}_{[-1,1]} \lambda,
$$

(30)

where $\lambda$ is the Lebesgue measure, and the CWM is a CBM. However, this is only true for fixed $n$. Whereas in (29) the de Finetti measure $\mu$ is static, i.e. independent of $n$, in (30) we see that there is a different de Finetti measure $\mu_n$ for each model of size $n$. Hence, taken as a class of models with any $n \in \mathbb{N}$, the CWM is not a special case of the CBM.

The CWM has three distinct ‘regimes’ in which the voting margin $S_n$ behaves differently. For $\beta < 1$, the so-called ‘high temperature regime’, $S_n/\sqrt{n}$ converges in distribution to a centred normal distribution. The sequence of de Finetti measures defined in (30) converges weakly to the Dirac measure at 0. Moreover, it has a concentration property: for any $\delta > 0$, there are constants $C,D > 0$ such that

$$
\mu_n ([-1,1] \setminus [-\delta, \delta]) < C \exp(-Dn), \quad n \in \mathbb{N}.
$$

(31)

This exponential decay is key to analysing the asymptotic behaviour of the CWM. In the ‘critical regime’, $\beta = 1$, a limit theorem holds for $S_n/n^{3/4}$ with a limiting distribution which is not normal. Instead, the distribution has a density proportional to $\exp(-x^4/12)$, $x \in \mathbb{R}$. In the ‘low temperature regime’, $\beta > 1$, $S_n/n$ converges to the convex combination of two Dirac measures, $1/2 (\delta_{-m(\beta)} + \delta_{m(\beta)})$, where $0 < m(\beta) < 1$ is the positive solution of the Curie-Weiss equation $x = \tanh(\beta x)$.

The multi-group version of the CWM defined in (29) is given by a positive semi-definite coupling matrix $J = (J_{\lambda \nu})_{\lambda,\nu = 1, \ldots, M} \in \mathbb{R}^{M \times M}$ and the Gibbs measure

$$
\mathbb{P}_{\text{CWM}}(X_{11} = x_{11}, \ldots, X_{MnM} = x_{MnM}) := Z_{J,n}^{-1} \exp \left( \frac{1}{2} \sum_{\lambda,\nu = 1}^M J_{\lambda \nu} V_{M\lambda V_{M\nu}} \sum_{i=1}^{n_\lambda} \sum_{j=1}^{n_\nu} x_{\lambda i} x_{\nu j} \right).
$$

(32)

for all voting configurations. The normalisation constant $Z_{J,n}$ depends on the group sizes $n_\lambda = n_\lambda(n)$ and the coupling matrix $J$. The entries $J_{\lambda \nu}$ of the coupling matrix give the strength of interaction between any pair of voters, one of which belongs to group $\lambda$ and the other to group $\nu$. For this probability measure, there is a de Finetti representation given by

$$
\mathbb{P}_{\text{CWM}}(X_1 = x_1, \ldots, X_{MnM} = x_{MnM}) = \int_{\mathbb{R}^M} \mathcal{P}^\otimes_n (x_{11}, \ldots, x_{MnM}) \exp(-nF_{J,n}(x)) \, dx.
$$

Above, the mapping $m \mapsto \tilde{m}$ is $\tilde{m} := (\tanh m_\lambda)_{\lambda = 1, \ldots, M}$. The CWM has a sequence of de Finetti measures given by

$$
\mu_n := \exp(-nF_{J,n}) \lambda^M, \quad n \in \mathbb{N},
$$

(33)

with $\lambda^M$ being the Lebesgue measure on $\mathbb{R}^M$. See [19] display (12) for the form of $F_{J,n}$ and [19] Section 2 for the three regimes of the model. An alternative representation with compactly supported de Finetti measures can be obtained by a change of variables in the integral in (32) setting $t := (\tanh x_\lambda)_{\lambda = 1, \ldots, M}$, yielding de
Critical regime

High temperature regime with \( J \neq 0 \)

Critical regime

Low temperature regime

---

Table 1: De Finetti Regimes of the CBM and the CWM

Finetti measures \( \mu_n \) whose support belongs to \([-1, 1]^M\). The multi-group CWM thus fits into the framework outlined in Definition 4.

The parameter space of the multi-group CWM is

\[
\Phi := \left\{ J \in \mathbb{R}^{M \times M} \mid J \text{ is positive definite or } J = (\beta)_{\lambda, \nu = 1, \ldots, M} \text{ for some } \beta \geq 0 \right\}
\]

Analogously to the single-group model, the multi-group CWM has three regimes. If \( J \) is positive definite, the high temperature regime is the region in \( \Phi \) where \( C := (I_M - J)^{-1} \) is positive definite. If \( J = (\beta)_{\lambda, \nu = 1, \ldots, M} \), \( \beta < 1 \) characterises the high temperature regime. In both cases, this regime features weak interactions between voters. The sequence \( (\mu_n) \), given in (26), exhibits a critical convergence speed according to Definition 15 and the limiting distribution of \( (S_{n,1}/\sqrt{n_1}, \ldots, S_{n,M}/\sqrt{n_M}) \) is \( \mathcal{N}(0, C) \) with a (generally) non-diagonal covariance matrix \( C \). As such, the limiting distribution of \( (S_{n,1}/\sqrt{n_1}, \ldots, S_{n,M}/\sqrt{n_M}) \) can be expressed as the convolution of two normal distributions:

\[
\left( \frac{S_{n,1}}{\sqrt{n_1}}, \ldots, \frac{S_{n,M}}{\sqrt{n_M}} \right) \xrightarrow{d} \mathcal{N}(0, C) * \mathcal{N}(0, \Sigma),
\]

where \( \Sigma = (J^{-1} - I_M)^{-1} \) if the coupling matrix \( J \) is positive definite, and \( \Sigma = (\beta/(1-\beta))_{\lambda, \nu = 1, \ldots, M} \) is singular if \( J \) is a homogeneous matrix with all its entries equal to some \( 0 \leq \beta < 1 \). The critical regime of the CWM fits into the subcritical regime according to Definition 15. Lastly, the low temperature regime of the CWM features a sequence \( (\mu_n) \) defined in (33) which does not converge weakly to \( \delta_0 \). See [19] Sections 3 and 5.2 for more details.

In the CWM, all groups must be in the same regime of convergence speeds given in Definition 15 unless there are clusters of groups which are already independent for finite \( n \) due to the coupling constants being equal to 0 between groups belonging to different clusters. This setup is realised by choosing the coupling matrix \( J \) to be a block diagonal matrix. For CBMs, since the de Finetti sequence \( (\mu_n) \) is constant and equal to \( \mu \) given in (26), \( \mu_n \) does not converge to \( \delta_0 \) at all, except in the degenerate case \( \mu = \delta_0 \) noted above.

No version of either the CBM or the CWM exhibits behaviour in line with the empirical evidence provided in [12, 11] and discussed in the Introduction. Depending on the de Finetti measure \( \mu \), the CBM produces expected per capita absolute voting margins for each group \( \lambda \), \( \mathbb{E}(\left| S_{n,\lambda} \right| / n_\lambda) \), of order \( 1/\sqrt{n_\lambda} \) or 1, as follows from the limit theorems given in (24) and (28), with nothing in between. The CWM features three distinct regimes with \( \mathbb{E}(\left| S_{n,\lambda} \right| / n_\lambda) \) of order \( 1/\sqrt{n_\lambda}, 1/n_\lambda^{1/4} \), and 1, respectively. (See [19] Section 3 for these results.)

We sum up the convergence behaviour of the CBM and the CWM in Table 1.

We next take a look at the asymptotic behaviour of the correlations between two votes \( \mathbb{E}(X_{\lambda_1}X_{\nu_2}) \). By Proposition 15, de Finetti voting models with an asymptotic loss of social cohesion, represented by the weak convergence

\[\mu = \delta_0, \quad J = 0^2, \quad \text{Fast convergence}\]

\[\mu \neq \delta_0, \quad \text{Low temperature regime} \quad \mu_n \rightarrow \delta_0\]

\[\mu = \delta_0, \quad J = 0, \quad \text{Critical regime}\]

\[\mu \neq \delta_0, \quad \text{High temperature regime with } J \neq 0, \quad \text{Critical convergence}\]

\[\mu \neq \delta_0, \quad \text{Low temperature regime} \quad \mu_n \rightarrow \delta_0, \quad \text{Slow convergence}\]
convergence of the de Finetti measures \((\mu_n)_n\) to \(\delta_0\), feature decaying correlations, i.e. \(\mathbb{E}(X_{\lambda_1}X_{\nu_2}) \xrightarrow{n \to \infty} 0\). Given the de Finetti measure \(\mu\) of the CBM defined in (26), we have the equality

\[
\mathbb{E}(X_{\lambda_1}X_{\nu_2}) = \int_{[-1,1]^M} m_{\lambda} m_{\nu} \mu(dm), \quad \lambda, \nu = 1, \ldots, M.
\]

So correlations do not depend on the size of the population, and, except for some special cases such as \(\mu = \delta_0\), correlations are non-zero in a CBM.

In the CWM, the asymptotic behaviour of the correlations \(\mathbb{E}(X_{\lambda_1}X_{\nu_2})\) depends on the regime the model is in. We have (cf. [19, Section 5.2])

\[
\mathbb{E}(X_{\lambda_1}X_{\nu_2}) = \begin{cases} 
\Theta \left( \frac{1}{n} \right) & \text{in the high temperature regime}, \\
\Theta \left( \frac{1}{\sqrt{n}} \right) & \text{in the critical regime}, \\
\Theta(1) & \text{in the low temperature regime}.
\end{cases}
\]

So we see that in a CWM the loss of social cohesion in the high temperature and the critical regime is reflected in the decay of correlations between pairs of votes. In the low temperature regime, where social cohesion does not decay, neither do we have a decay of \(\mathbb{E}(X_{\lambda_1}X_{\nu_2})\).

Finally, we provide a concrete example of a de Finetti voting model in line with the cited empirical evidence:

**Example 31.** Let \(0.1 \leq \alpha_1, \ldots, \alpha_M \leq 0.2\). Let \(\mu\) be a probability measure on the space \(\{-1,1\}^M\) that is symmetric with respect to the origin. We set

\[
\mu_n := \sum_{x \in \{-1,1\}^M} \mu \{x\} \delta \left( \frac{1}{n_{\alpha_1}^\lambda}, \ldots, \frac{1}{n_{\alpha_M}^\lambda}, \frac{1}{n_{\lambda}^\lambda} \right), \quad n \in \mathbb{N},
\]

and let \((\mathbb{P}_n)_n\) be the de Finetti voting model with de Finetti sequence \((\mu_n)_n\). This model falls into the contraction pattern defined in Theorem [13]. Since \(0.1 \leq \alpha_1, \ldots, \alpha_M \leq 0.2\), its convergence speed is subcritical according to Definition [15]. By Theorem [13] we have the limit theorem

\[
\left( \frac{S_{n,1}^{\lambda}}{n_{\lambda}^{1-\alpha_1}}, \ldots, \frac{S_{n,M}^{\lambda}}{n_{\lambda}^{M-\alpha_M}} \right) \xrightarrow{d} \mu.
\]

Corollary [17] states that the expected per capita voting margins exhibit the following behaviour:

\[
\mathbb{E}\left( \frac{|S_{n,\lambda}|}{n_{\lambda}} \right) = \Theta \left( n_{\lambda}^{-\alpha_\lambda} \right), \quad \lambda = 1, \ldots, M.
\]

Thus, choosing the parameters \(\alpha_\lambda, \lambda = 1, \ldots, M\), allows us to obtain a good fit of the model to the data in [12, 11]. Note that any possible correlation structure between the groups can be realised by choosing the probability measure \(\mu\) appropriately. Hence, this choice can make the model fit the empirical correlations between the voting margins \(S_\lambda\) and \(S_\nu\) of different groups \(\lambda\) and \(\nu\). Due to both the discrete support and the symmetry of \(\mu\) with respect to the origin, this is a fairly simple statistical model of binary voting. If desired, the introduction of further symmetries in \(\mu\) can lead to a reduction of the number of parameters. See [21] for a paper about optimal voting weights in a two-tier voting system under a CWM with a reduced set of parameters.

## 5 Conclusion

We discussed existing multi-group voting models such as the CBM and the CWM. None of these models exhibits voting behaviour in line with empirical evidence presented in [12, 11]. Therefore, we introduced a general framework of probabilistic voting models defined by a de Finetti representation. When the de Finetti measures converge weakly to the Dirac measure at the origin, these models satisfy limit theorems for the vector of group voting margins. The convergence speed of the de Finetti measures falls into one of three regimes:
• If the convergence speed is fast (supercritical), which means faster than $1/\sqrt{n}$, where $n$ is the population, then we obtain for the normalised group voting margins a universal limiting distribution which is multivariate normal with independent standard normal entries.

• If the convergence speed is critical, which means of order $1/\sqrt{n}$, and the de Finetti measures are a contraction of some underlying probability measure $\mu$, then we obtain as the limiting distribution of the normalised group voting margins a rescaled version of $\mu$ plus an additive multivariate normal noise with independent entries.

• Slow (subcritical) convergence speeds require normalisation of the group voting margins by sequences which go to infinity faster than $\sqrt{n}$, the normalisation factor in the other two regimes. The limiting distribution of the normalised group voting margins is the underlying probability measure $\mu$.

The subcritical regime can be used to model real-world voting behaviour by assuming a convergence speed of order $n^{-\alpha}$, with $0.1 \leq \alpha \leq 0.2$, such as in Example [11]. By adjusting $\alpha$ to the data, we can specify voting models in terms of some underlying probability distribution $\mu$. This distribution can then be estimated to obtain a statistical model of the voting data.

References

[1] Berthet, Quentin; Rigollet, Philippe; Srivastava, Piyush: Exact Recovery in the Ising Blockmodel, Ann. Statist. 47 (4) 1805 - 1834, August 2019.

[2] Brock, William A.; Durlauf, Steven N.: Discrete Choice with Social Interactions, Review of Economic Studies, Oxford University Press, vol. 68(2), pages 235-260. 2001.

[3] Collet, Francesca: Macroscopic Limit of a Bipartite Curie-Weiss Model: A Dynamical Approach, J. Stat. Phys., 157(6), pp. 1301-1319 (2014)

[4] Contucci, Pierluigi, and Ghirlanda, S.: Modeling Society with Statistical Mechanics: An Application to Cultural Contact and Immigration. Quality and Quantity, 41, 569-578 (2007)

[5] Contucci, Pierluigi, Gallo, Ignacio: Bipartite Mean Field Spin Systems. Existence and Solution, Math. Phys. Elec. Jou. Vol 14, N.1, 1-22 (2008)

[6] Diaconis, Persi; Freedman, David: Finite Exchangeable Sequences, Ann. Probab. 8(4), 745-764, (August, 1980)

[7] Durrett, Rick: Probability Theory and Examples, Fifth Edition, Cambridge University Press (2019)

[8] Fedele, Micaela; Contucci, Pierluigi: Scaling Limits for Multi-species Statistical Mechanics Mean-Field Models, J. Stat. Phys. 144:1186–1205 (2011)

[9] Fedele, Micaela: Rescaled Magnetization for Critical Bipartite Mean-Fields Models, J. Stat. Phys. 155:223–226 (2014)

[10] Friedli, Sacha and Velenik, Yvan: Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction, Cambridge University Press, 2017

[11] Gelman, Andrew; Katz, Jonathan N.; Bafumi, Joseph: Standard Voting Power Indexes Do Not Work: An Empirical Analysis, B. J. Pol. S. 34, 657–674 (2004)

[12] Gelman, Andrew; Katz, Jonathan N.; Tuerlinckx, Francis: The Mathematics and Statistics of Voting Power. Statistical Science17, 420–435. (2002)

[13] Husimi, K.: Statistical Mechanics of Condensation, Proceedings of the International Conference of Theoretical Physics, pp. 531-533, Science Council of Japan, Tokyo (1953)
[34] Toth, Gabor: Correlated Voting in Multipopulation Models, Two-Tier Voting Systems, and the Democracy Deficit, PhD Thesis, Fernuniversität in Hagen. (2020) https://doi.org/10.18445/20200505-103735-0

[35] Toth, Gabor: Models of Opinion Dynamics with Random Parametrisation, arXiv:2211.13601

[36] Wengenroth, Jochen: Wahrscheinlichkeitstheorie, Walter de Gruyter (2008)