Compatibility Relations between the Reduced and Global Density Matrixes

Yong-Jian Han, Yong-Sheng Zhang, Guang-Can Guo

Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei 230026, China

It is a hard and important problem to find the criterion of the set of positive-definite matrices which can be written as reduced density operators of a multi-partite quantum state. This problem is closely related to the study of many-body quantum entanglement which is one of the focuses of current quantum information theory. We give several results on the necessary compatibility relations between a set of reduced density matrixes, including: (i) compatibility conditions for the one-party reduced density matrixes of any $N_A \times N_B$ dimensional bi-partite mixed quantum state, (ii) compatibility conditions for the one-party and two-party reduced density matrixes of any $N_A \times N_B \times N_C$ dimensional tri-partite mixed quantum state, and (iii) compatibility conditions for the one-party reduced density matrixes of any $M$-partite pure quantum state with the dimension $N^\otimes M$.

PACS number(s): 03.67.-a, 03.65.Ta

I. INTRODUCTION

Any quantum states (pure or mixed) in finite dimensional Hilbert space can be represented by positive-definite matrixes. If the Hilbert space $C$ has a tensor product structure $C = C_1 \otimes C_2 \otimes \cdots \otimes C_M$, given a known global density matrix $\rho_{12\ldots M}$ in the space $C$ for the composite system, it is straightforward to calculate the reduced density matrixes $\rho_{\alpha}$ ($\alpha = 1, 2, \cdots, M$) for each subsystem $\alpha$. However, the reverse of this problem becomes much more involved. Given some positive-definite matrixes $\rho_{\alpha}$, it is hard to determine whether they can be written as reductions of some global quantum state. This is what the compatibility problem concerns about. We can list, for instance, the following comparability problems:

- Find out the criteria for the one-party density matrixes $\rho_{\alpha}$ ($\alpha = 1, 2, \cdots, M$) so that they can be written as reduced density matrixes of some global quantum state $\rho_{12\ldots M}$ in the whole Hilbert space $C$. In this case, we typically put some restrictions on the global state $\rho_{12\ldots M}$, for instance, we may require $\rho_{12\ldots M}$ to be pure or to have a known spectrum (eigenvalues). Otherwise, the trivial assignment $\rho_{12\ldots M} = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_M$ makes the above question meaningless. Pure states actually correspond to a special class of the known-spectrum states $\rho_{12\ldots M}$ with the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \cdots = 0$ ($\lambda_i (i = 1, 2, \cdots)$ are arranged in the decreasing order)

- Find out the criteria for the two-party (or multi-party) density matrixes $\rho_{\alpha\beta}$ ($\alpha, \beta = 1, 2, \cdots, M$) so that they can be expressed as reduced density matrix of some global quantum state $\rho_{12\ldots M}$. This problem is much more involved than the first one, and also much more important for various applications for which we will mention several examples. In this case, even without any requirement on the global state $\rho_{12\ldots M}$, in general there is no trivial solution to the above problem.

The investigation of the compatibility conditions between the local and the global quantum states has some important implications: Firstly, this problem is closely related to the study of multi-partite entanglement, which is one of the focuses of current quantum information theory [1–6]. The complexity of the compatibility conditions really comes from the many-body entanglement inherent in the global quantum state $\rho_{12\ldots M}$. If we restrict the global state $\rho_{12\ldots M}$ to be separable states, the compatibility condition would be very simple. On the other hand, some general understanding of the compatibility conditions will shed new light on the properties of multi-partite entanglement. Although we have achieved remarkable understanding of bi-partite entanglement through consideration of local operations and classical communications, we still know little about multi-partite entanglement. Secondly and more importantly, the understanding of the compatibility conditions will have significant applications in computational many-body physics [8]. In general, the physical interactions are sufficiently local, so the interaction energy typically only depends on some reduced local density matrixes. If we find out some general compatibility relations between the reduced and the global quantum states, we could design some powerful variational approaches to solve various many-body problems. Finally, the study of the compatibility conditions would also help us to get a better understanding of the basic structure of quantum mechanics and the associated Hilbert space [7,10,11].

What do we know now about the compatibility relations between the local and the global quantum states? There are several interesting results concerning the compatibility conditions in some special cases. First, the well-known GHJW theorem in quantum information theory [1,9] can be considered as a compatibility condition with the global
state restricted to be a pure bi-partite quantum state in the Hilbert space $C_A^N \otimes C_B^N$, where $N$ denotes the dimension of each subsystem. The compatibility criterion in this case is that the two reduced density matrices should have the same spectrum (eigenvalues). There are some generalizations of this result. In particular, the compatibility criteria between the one-party reduced density matrices have been found in some recent works [12,13] if the global quantum state is restricted to be a pure state in the Hilbert spaces $(C^2)^{\otimes M}$, $C^2 \otimes C^2 \otimes C^4$, or $C^3 \otimes C^3$. 

In this paper, we provide several new results concerning the compatibility conditions in some more general cases. We derive pretty strong necessary conditions for compatibility between the one-party reduced density matrices for the cases that the global quantum state is a bi-partite mixed state in the Hilbert space $(C^2)^{\otimes M}$. Note that the previously known compatibility conditions for the Hilbert spaces $(C^2)^{\otimes M}$ and $C^2 \otimes C^2 \otimes C^4$ (a pure state in $C^2 \otimes C^2 \otimes C^4$ corresponds to a mixed state in $C^2 \otimes C^2$) can be considered as special cases of the results derived here. We also consider the compatibility conditions between the two-party density matrices for the first time, and derive some necessary compatibility conditions when the global state is a general tri-partite mixed state in the Hilbert space $C^{N_A} \otimes C^{N_B} \otimes C^{N_C}$ (the dimensions $N_A$, $N_B$, and $N_C$ are arbitrary) with a known eigenvalue spectrum $\{\lambda_i^{ABC}\}$. All these results are derived from a unified mathematical method.

The paper is arranged as follows: in Sec. II, after mention of some mathematical results which are critical for our derivation, we derive the necessary compatibility conditions for the reduced density matrices from mixed bi-partite quantum states; then in Sec III, we generalize this result and derive the necessary compatibility conditions for the one-party and two-party reduced density matrices from general mixed tri-partite quantum states; and finally, in Sec. IV, we derive the necessary compatibility conditions for the one-party reduced density matrices from any $M$-partite pure quantum state with the dimension $N^{\otimes M}$.

II. COMPATIBILITY RELATIONS OF REDUCED DENSITY MATRIXES FROM MIXED BI-PARTITE QUANTUM STATES

Our methods for derivation of the compatibility relations extensively use two mathematical lemmas. First let us summarize these two lemmas:

In matrix analysis, there is an important theorem which connects the minimization of a matrix with the matrix' eigenvalues. That is the content of the following lemma [14]

**Lemma 1:** Let $A$ denote an $n \times n$ Hermitian matrix, and $U$ denote any $n \times r$ matrix with $U^*U = I_r$ ($1 \leq r \leq n$, and $U^*$ is the adjoint of the matrix $U$). Then the minimization

$$\min_U(tr(U^*AU)) = \lambda_1^A(A) + \lambda_2^A(A) + \cdots + \lambda_r^A(A), \quad (1')$$

where the eigenvalues $\lambda_1^A(A), \lambda_2^A(A), \cdots, \lambda_r^A(A)$ are arranged in the increasing order.

In some cases, not all the column vectors of the matrix $U$ are orthogonal to each other. We can get another more convenient lemma.

**Lemma 2:** Let $A$ denote an $n \times n$ Hermitian matrix and $U \in M_{n\times r}$, each column of the matrix $U$ are normalized. The columns of $U$ can be divided into two groups, and the columns in the same group are orthogonal each other. Suppose the linear dependent number $\kappa = \sum_{i,j} |u_i^a u_j^b|^2$ is an integer (where $u_i^a$ and $u_j^b$ are the columns in the first and the second group, respectively). Then we have the following conclusion

$$\min_U(tr(U^*AU)) \geq \sum_{i=1}^\kappa \lambda_i^A(A) + \sum_{i=1}^{s-\kappa} \lambda_i^A(A) \quad (1)$$

where $\lambda_1^A(A), \lambda_2^A(A), \cdots, \lambda_s^A(A)$ are the eigenvalues of the matrix $A$ and arranged in increasing order.

Proof: Without loss of generality, we suppose that the vectors $u_1^a, u_2^a, \cdots, u_\kappa^a$ are the columns in the group $I$ and the vectors $u_1^b, u_2^b, \cdots, u_{s-\kappa}^b$ are the columns in the group $II$. We suppose the eigenvectors of the matrix $A$ are $v_1, v_2, \cdots, v_n$, corresponding to $\lambda_1^A(A), \lambda_2^A(A), \cdots, \lambda_n^A(A)$, respectively. So all of the vectors in the group $I$ and group $II$ can be expanded by the eigenvectors, that is

$$u_1^a = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n,$$

$$u_2^a = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n,$$

$$\cdots$$

$$u_\kappa^a = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$
and

\[ u^b_i = \beta_{11} v_1 + \beta_{12} v_2 + \cdots + \beta_{1n} v_n, \]
\[ u^b_i = \beta_{21} v_1 + \beta_{22} v_2 + \cdots + \beta_{2n} v_n, \]
\[ \vdots \]
\[ u^b_{s-l} = \beta_{s-l,1} v_1 + \beta_{s-l,2} v_2 + \cdots + \beta_{s-l,n} v_n. \]

Since the vectors in the same group are normalized and orthogonal to each other, the indexes satisfy \( l \leq n \) and \( s - l \leq n \). We can add some vectors into each group to make this group a complete basis of the space. So the coefficients must satisfy the requirements \( \sum_{i=1}^{s-l} |\alpha_{ij}|^2 \leq 1 \) and \( \sum_{i=1}^{s-l} |\beta_{ij}|^2 \leq 1 \). Now we can get the formula

\[
\min_U (\text{tr}(U^*AU)) = \min_U \left( \sum_{i=1}^{s-l} \sum_{j=a}^{b} |u^b_i|^2 \right)
= \min_U \left( \sum_{j=1}^{n} \sum_{i=1}^{l} |\alpha_{ij}|^2 + \sum_{i=1}^{s-l} |\beta_{ij}|^2 \lambda^2_j(A) \right)
\]

To make the former function \( \sum_{j=1}^{n} (\sum_{i=1}^{l} |\alpha_{ij}|^2 + \sum_{i=1}^{s-l} |\beta_{ij}|^2 \lambda^2_j(A)) \) smaller, we must make the coefficients before the smaller eigenvalues more bigger. Since the constraints \( \sum_{j=1}^{l} |\alpha_{ij}|^2 \leq 1 \) and \( \sum_{i=1}^{s-l} |\beta_{ij}|^2 \leq 1 \), the coefficients before the eigenvalues are not more than 2. However, there is another constraint between the coefficients \( \alpha_{ij} \) and \( \beta_{ij} \), that is, \( \kappa = \sum_{j=1}^{s-l} |u^a_i|^2 |u^b_j|^2 \). It means that the number of the coefficients 2 before the eigenvalues is not more than \( \kappa \). Under these constraints, we can get the lower bound of this function, that is,

\[
\min_U (\text{tr}(U^*AU)) \geq \sum_{i=1}^{\kappa} \lambda^2_i(A) + \sum_{i=1}^{s-l} \lambda^2_j(A).
\]

This is the end of the proof.

It is need emphasis that \( s - \kappa \) is equal to the least number of the linear independent columns when the parameter \( \kappa \) is an integer. When the number \( \kappa \) is not a integer, the similar results can be found easily. We can find at the following that the integer situation is enough for our propose.

The situation that the matrix \( U \) can be divided into more than two groups (the columns in the same group are orthogonal to each other) is more difficult to deal with. But, for principle, we can get the similar results by carefully calculating the linear dependent number \( \kappa \). Though the general case is very complex, We will give a special case in the Sec IV..

Now we use these lemmas to a multi-partite density matrix to get some relations between the reduced density matrix and the global matrix. When the density matrix is a two particle density matrix, and each particle are qubit, the original density matrix of the two qubit can be written as

\[
\rho_{AB} = \begin{bmatrix}
  a_{00,00} & a_{00,01} & a_{00,10} & a_{00,11} \\
  a_{01,00} & a_{01,01} & a_{01,10} & a_{01,11} \\
  a_{10,00} & a_{10,01} & a_{10,10} & a_{10,11} \\
  a_{11,00} & a_{11,01} & a_{11,10} & a_{11,11}
\end{bmatrix}
\]

We can get the reduced density matrix of \( \rho_A \) as

\[
\rho_A = \begin{bmatrix}
  a_{00,00} & a_{00,10} \\
  a_{10,00} & a_{10,10}
\end{bmatrix} + \begin{bmatrix}
  a_{01,01} & a_{01,11} \\
  a_{11,01} & a_{11,11}
\end{bmatrix}
= A_0 + A_1
\]

For this case we have the following theorem

**Theorem 1.** For the two-qubit density matrix \( \rho_{AB} \), the eigenvalues between \( \rho_A, \rho_B \) and \( \rho_{AB} \) have the following relations

\[
\lambda^2_i(A) \geq \lambda^2_i(AB) + \lambda^2_j(AB) \quad (2.1)
\]
\[
\lambda^2_j(B) \geq \lambda^2_i(AB) + \lambda^2_j(AB) \quad (2.2)
\]
where $\lambda_i^1(A)$ and $\lambda_i^1(B)$ are the smaller eigenvalue of $\rho_A$ and $\rho_B$, respectively; $\lambda_1^1(AB)$, $\lambda_2^1(AB)$, $\lambda_3^1(AB)$, $\lambda_4^1(AB)$ are the eigenvalues of $\rho_{AB}$, and they are arranged in increasing order.

Proof: With the lemma 1, we can get the smaller eigenvalue of the local density matrix $\rho_A$

$$\lambda_i^1(A) = \min_U [tr(U^* \rho_A U)]$$
$$= \min_U [tr(U^* (A_0 + A_1) U)]$$
$$= \min_U [tr(U^* A_0 U) + tr(U^* A_1 U)]$$
$$= \min_U [tr([u_1^+ u_2^+] a_{00,00} a_{00,10} a_{10,00} a_{10,10}) [u_1^+ u_2^+] + tr([u_1^+ u_2^+] a_{01,01} a_{01,11} a_{11,01} a_{11,11}) [u_1^+ u_2^+]]$$
$$= \min_U [tr([u_1^+ 0 u_2^+] a_{00,00} a_{00,10} a_{10,00} a_{10,10}) [u_1^+ 0 u_2^+] + tr([0 u_1^+ 0 u_2^+] a_{01,01} a_{01,11} a_{11,01} a_{11,11}) [0 u_1^+ 0 u_2^+]]$$

where $\%$ means that arbitrary number will make the equality hold, $u_i^+$ means the conjugate of $u_i$. We notice that the position of the elements $a_{i,j,m}(i,j,l,m = 0,1)$ are the same as the position they are in matrix $\rho_{AB}$. So we choose the proper numbers to make the middle matrix is just equal to the density matrix $\rho_{AB}$. Then

$$\lambda_i^1(A) = \min_U [tr([u_1^+ 0 u_2^+] \rho_{AB} [u_1^+ 0 u_2^+]) + tr([0 u_1^+ 0 u_2^+] \rho_{AB} [0 u_1^+ 0 u_2^+])$$
$$= \min_U [tr([u_1^+ 0 u_2^+] \rho_{AB} [u_1^+ 0 u_2^+])$$
$$\geq \min_U [tr([u_1^+ u_2^+ u_3^+ u_4^+] \rho_{AB} [u_1^+ u_2^+ u_3^+ u_4^+])$$
$$= \lambda_i^1(AB) + \lambda_i^1(AB)$$

where vectors $\{u_{11}, u_{12}, u_{13}, u_{14}\}$ and $\{u_{21}, u_{22}, u_{23}, u_{24}\}$ are orthogonal.

With the same reason, we can obtain $\lambda_1^1(B) \geq \lambda_1^1(AB) + \lambda_2^1(AB)$. It is more important that we can use this method to calculate the relations between the smaller eigenvalues of matrix $\rho_A$ and $\rho_B$, that is

$$\lambda_i^1(A) + \lambda_i^1(B) = \min_U [tr(U^* \rho_{AB} U)]$$

Since the variables in $U_1$ are independent on the variables in $U_2$, we can combine these two matrixes into one matrix $U$, that is

$$\lambda_i^1(A) + \lambda_i^1(B) = \min_U [tr(U^* \rho_{AB} U)]$$
where
\[
U = \begin{bmatrix}
u_{11} & 0 & u_{21} & 0 \\
0 & u_{11} & u_{22} & 0 \\
u_{12} & 0 & 0 & u_{21} \\
0 & u_{12} & 0 & u_{22}
\end{bmatrix}.
\] (1)

Now using the lemma 2, we need to calculate the maximal linear dependent number \( \kappa \) between unitary \( U_1 \) and \( U_2 \). Here the linear dependent number \( \kappa \) is a constant 1. On the other hand, we can find that this matrix at least has three linear independent columns. Then we get the relation
\[
\lambda^1_1(A) + \lambda^1_1(B) \geq 2\lambda^1_1(AB) + \lambda^1_2(AB) + \lambda^1_3(AB).
\]

QED

These conditions can be viewed as the necessary conditions for the problem whether the single-qubit reduced density matrices are compatible with a two qubit density matrix with the eigenvalues \( \{\lambda^1_1(AB), \lambda^2_1(AB), \lambda^3_1(AB)\} \). These conditions have already received by Bravyi [13] in another way. Unfortunately, these conditions are not sufficient, and the sufficient conditions need another condition
\[
\left|\lambda^1_1(A) - \lambda^1_1(B)\right| \leq \min\{\lambda^3_1(AB) - \lambda^1_1(AB), \lambda^1_2(AB) - \lambda^1_2(AB)\}.
\]

This condition can not find by our method easily. Now we turn to consider the general case of the two particle situation.

Suppose the dimensions of particle \( A \) and particle \( B \) are \( L \) and \( N \), respectively. Let \( \{\lambda^1_1(A), \lambda^2_1(A), \cdots, \lambda^L_1(A)\} \), \( \{\lambda^1_1(B), \lambda^2_1(B), \cdots, \lambda^N_1(B)\} \) and \( \{\lambda^1_1(AB), \lambda^2_1(AB), \cdots, \lambda^{LN}_1(AB)\} \) be the eigenvalues of the density matrix \( \rho_A, \rho_B \) and \( \rho_{AB} \), respectively, and they are arranged in increasing order. Before giving the following theorem, we need define majorization relation between two vectors. Let \( x = \{x_1^\uparrow, x_2^\uparrow, \cdots, x_n^\uparrow\} \) and \( y = \{y_1^\uparrow, y_2^\uparrow, \cdots, y_n^\uparrow\} \) are \( n \)-dimensional vectors and the elements are arranged in increasing order. Then we call the vector \( x \) is majorized by vector \( y \) [15], denoted by \( y \succ x \), if for each \( k \) \((k = 1, 2, \cdots, n) \) the following inequality hold
\[
\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i
\]
and the equality hold when \( k = n \). The majorization relation have already been extensively used in quantum information [16]. If we define the vector \( \lambda_A = \{\lambda^1_1(A), \lambda^2_1(A), \cdots, \lambda^L_1(A)\} \), \( \lambda_B = \{\lambda^1_1(B), \lambda^2_1(B), \cdots, \lambda^N_1(B)\} \), \( \lambda_{AB} = \{\sum_{j=1}^{N} \lambda^j_1(AB), \sum_{j=N+1}^{2N} \lambda^j_1(AB), \cdots, \sum_{j=(L-1)N+1}^{LN} \lambda^j_1(AB)\} \) and \( \lambda^2_{AB} = \{\sum_{j=1}^{L} \lambda^j_1(AB), \sum_{j=L+1}^{2L} \lambda^j_1(AB), \cdots, \sum_{j=(N-1)L+1}^{NL} \lambda^j_1(AB)\} \). Using these definition, we can get the theorem for the general bi-partite case as the following

**Theorem 2.** As the note before, we get the following relations between the eigenvalues of \( \rho_A, \rho_B \) and \( \rho_{AB} \):
\[
\lambda^A_{AB} \succ \lambda_A; \quad \lambda^B_{AB} \succ \lambda_B; \quad (3.1)
\]
\[
\sum_{i=1}^{k} \lambda^i_1(A) + \sum_{j=1}^{l} \lambda^j_1(B) \geq \sum_{i=1}^{kN+(L-k)L} \lambda^i_1(AB) + \sum_{j=1}^{kl} \lambda^j_1(AB), k = 1, 2, \cdots, L - 1; l = 1, 2, \cdots, N - 1. \quad (3.3)
\]

Before giving the proof, We need some discussions about these conditions. The majorization relations (3.1), (3.2) between the one-party reduced density matrix and the bi-partite density matrix are not just hold for this special case. We can see in the following, the majorization is a universal relations between the eigenvalues of reduced density matrices and the multi-partite density matrix. This can be viewed as one of the reasons why the majorization relations play an important role in the quantum information. The relations (3.3) tell us that some equalities in the former two relations can not be hold at the same time except for some special situations. This fact can be viewed as the correlation between the different reduced density matrices. When \( L = N = 2 \), this theorem is reduced to the theorem 1.
Proof Using the similar method as the qubit case, we can get

\[
\sum_{i=1}^{k} \lambda_i^r(A) = \min_U [\text{tr}(U^* \rho_A U)] \\
= \min_U [\text{tr}(U^* A_0 U) + \text{tr}(U^* A_1 U) + \cdots + \text{tr}(U^* A_N U)] \\
= \min_U [\text{tr}(U^* \rho_{AB} U_1)] \\
\]

where the unitary matrix \(U \in M_{L \times k}, U^* U = I \in M_k\) and \(A_i\) is equal to \(\langle 0| \rho_{AB} |0 \rangle\). Using the same method which used to construct the matrix \((I)\), we can get a unitary matrix \(U_1 \in M_{LN \times kN}\), we divide this matrix into \(k\) blocks, and each block is a \(LN \times N\) matrix which has the following form

\[
\begin{pmatrix}
  u_{1i} & u_{1i} & \cdots & u_{1i} \\
  u_{N+1,i} & u_{N+1,i} & \cdots & u_{N+1,i} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{(L-1)N+1,i} & u_{(L-1)N+1,i} & \cdots & u_{(L-1)N+1,i} \\
\end{pmatrix}
\]

where only the elements \((pN + q, q)(p = 0, 1, \cdots, L - 1; q = 1, 2, \cdots, N)\) in this block are nonzero and \(i\) denotes the \(i\)th block. The first columns in the different blocks are orthogonal and normalized. So all of the columns in the matrix \(U_1\) are orthogonal and normalized, that is

\[
\sum_{i=1}^{k} \lambda_i^r(A) \geq \sum_{i=1}^{kN} \lambda_i^r(AB). 
\]

As the same reason,

\[
\sum_{i=1}^{l} \lambda_i^r(B) = \min_U [\text{tr}(U^* \rho_B U)] \\
= \min_U [\text{tr}(U^* B_0 U) + \text{tr}(U^* B_1 U) + \cdots + \text{tr}(U^* B_L U)] \\
= \min_{U_2} [\text{tr}(U_2^* \rho_{AB} U_2)]
\]

where the unitary matrix \(U_2 \in M_{LN \times Ll}\). We divide the matrix \(U_2\) into \(l\) blocks, and each block is a \(LN \times L\) matrix which has the following form
where only the elements \(((p-1)N+q,p)\,(p=1,\cdots,L;q=1,2,\cdots,N)\) in the \(i\)th block are nonzero. The first columns in the different blocks are also orthogonal and normalized. So we can get

\[
\sum_{i=1}^{l} \lambda_i^i(B) \geq \sum_{i=1}^{lL} \lambda_i^L(AB).
\]

We also need to find the relations between the eigenvalues of \(\rho_A\) and \(\rho_B\)

\[
\sum_{i=1}^{k} \lambda_i^i(A) + \sum_{j=1}^{l} \lambda_j^j(B) = \min_{U_1} [\text{tr}(U_1^* \rho_A U_1)] + \min_{U_2} [\text{tr}(U_2^* \rho_B U_2)].
\]

As the lemma 2, we need to calculate the linear dependent number between matrix \(U_1\) and \(U_2\). Here it is easy to get the linear dependent number \(\kappa\), where \(\kappa = \sum_{i=1}^{l} \sum_{j=1}^{k} |\langle V_i^2 | V_j^1 \rangle|^2\) (\(V_i^2\) is the \(i\)th column in the unitary matrix \(U_1\) and \(V_j^1\) is the \(j\)th column in the unitary matrix \(U_2\)) is equal to \(kl\). So we get the result

\[
\sum_{i=1}^{k} \lambda_i^i(A) + \sum_{j=1}^{l} \lambda_j^j(B) \geq \sum_{i=1}^{kN+lL-kl} \lambda_i^L(AB) + \sum_{j=1}^{kl} \lambda_j^L(AB).
\]

QED

We can also find that there are at least \(kN + lL - kl\) linear independent columns in matrix \(U_1\) and \(U_2\). For convenience, we let \(k \geq l\), we find in the following that at most \(kl\) columns in unitary \(U_1\) are linear dependent on the columns in \(U_2\). At first, we point out that in each block of unitary \(U_1\) at most \(k\) columns are linear dependent on the columns in unitary \(U_2\). Suppose the \(j\)th column of the \(i\)th block in the unitary \(U_1\) is linear dependent on the columns in unitary \(U_2\), then only the first columns of the block in unitary \(U_2\) contribute to the first \(N\) elements of the unitary \(U_1\). So we can get

\[
\alpha_{j1}V_1 + \alpha_{j2}V_2 + \cdots + \alpha_{jl}V_l = W_j,
\]

where the vector \(V_j\) is the first column of the \(i\)th block in the unitary \(U_2\) and the vector \(W_j\) is a \(LN\) vector \(\{0,\cdots,0,u_{ij},0,\cdots,0\}^T\) (\(u_{ij}\) is the \(j\)th element in this vector and is equal to the first nonzero element of the \(j\)th column of \(i\)th block and \(T\) means transpose). Since \(V_i\) for \(i=1,2,\cdots,l\) are orthogonal each other for different \(i\), and they span a \(l\)-dimensional space. Then at most \(l\) orthogonal \(V_i\) can be get from \(V_i\) by liner combination. That is, at most \(l\) orthogonal \(W_i\) can be get from \(V_i\) by liner combination. Then there are \(k\) blocks, then there at most \(kl\) columns in the unitary \(U_1\) are linear dependent on the columns in the unitary \(U_2\). Since there are \(k\) blocks, then there at most \(kl\) columns in the unitary \(U_1\) are linear dependent on the columns in the unitary \(U_2\). We will show that this situation can be reached by letting the element \(u_{i(-1),N+1,i}\) in the \(i\)th block of the unitary \(U_1\) be equal to 1 and the other elements zeros. So we can also get the conditions

\[
\sum_{i=1}^{k} \lambda_i^i(B) + \sum_{j=1}^{l} \lambda_j^j(B) \geq \sum_{i=1}^{kN+lL-kl} \lambda_i^L(AB) + \sum_{j=1}^{kl} \lambda_j^L(AB).
\]
These conditions also can be viewed as the necessary conditions to the problem whether the one-party reduced density matrices are compatible with the multi-partite density matrix. The method we used to find these conditions is very simple, and it is dependent on neither the number of the particles nor the dimensions of the particles. During the calculation, the most important thing is to get the maximal linear dependent number \( \kappa \) between the different unitary matrices.

III. COMPATIBILITY RELATIONS BETWEEN ONE-PARTY AND TWO-PARTY DENSITY MATRIXES FROM TRI-PARTITE MIXED QUANTUM STATES

There is a density matrix \( \rho_{ABC} \), where the particle \( A, B \) and \( C \) are in \( L \)-dimension, \( M \)-dimension and \( N \)-dimension Hilbert space, respectively. Let \( \{ \lambda_1^A(AB), \lambda_2^A(AB), \cdots, \lambda_{LM}^A(AB) \} \), \( \{ \lambda_1^B(BC), \lambda_2^B(BC), \cdots, \lambda_{MN}^B(BC) \} \), \( \{ \lambda_1^C(ABC), \lambda_2^C(ABC), \cdots, \lambda_{LMN}^C(ABC) \} \) be the eigenvalues of the density matrix \( \rho_{AB}, \rho_{BC}, \rho_{B} \) and \( \rho_{ABC} \), respectively, and they are arranged in increasing order. For convenience, We define the following vectors \( \{ \lambda_1^A, \lambda_2^A, \cdots, \lambda_M^A \} \), \( \{ \lambda_1^B, \lambda_2^B, \cdots, \lambda_N^B \} \), \( \{ \lambda_1^C, \lambda_2^C, \cdots, \lambda_{LMN}^C \} \) and \( \lambda_{AB} = \{ \lambda_1^A(AB), \lambda_2^A(ABC), \cdots, \lambda_{LMN}^A(ABC) \} \) and \( \lambda_{BC} = \{ \lambda_1^B(BC), \lambda_2^B(ABC), \cdots, \lambda_{LMN}^B(ABC) \} \) and \( \lambda_{AB} = \{ \lambda_1^A(AB), \lambda_2^A(ABC), \cdots, \lambda_{LMN}^A(ABC) \} \) and \( \lambda_{BC} = \{ \lambda_1^B(BC), \lambda_2^B(ABC), \cdots, \lambda_{LMN}^B(ABC) \} \) and \( \lambda_{ABC} = \{ \lambda_1^A(AB), \lambda_2^A(ABC), \cdots, \lambda_{LMN}^A(ABC) \} \). For this situation, we can get the following theorem.

**Theorem 3.** Using the notes defined before, we can get the relations between the eigenvalues of \( \rho_{BC}, \rho_{AB}, \rho_{B} \) and \( \rho_{ABC} \) as

\[
\lambda_{ABC}^A \succ \lambda_{AB} \quad (4.1)
\]

\[
\lambda_{ABC}^B \succ \lambda_{BC} \quad (4.2)
\]

\[
\lambda_{ABC}^B \succ \lambda_{B} \quad (4.3.1)
\]

\[
\lambda_{ABC}^B \succ \lambda_{B} \quad (4.3.2)
\]

\[
\sum_{i=1}^{\mu L+r} \lambda_i^A(AB) + \sum_{j=1}^{\mu N+s} \lambda_j^B(BC) \geq \sum_{k=1}^{\mu LN+r+L_s-r_s} \lambda_k^A(ABC) + \sum_{i=1}^{\mu LN+r} \lambda_i^B(ABC) , \mu < M, 0 \leq r < L, 0 \leq s < N. \quad (4.4)
\]

The relation (4.1), (4.2), (4.3) make sure the universality of the majorization relations between the eigenvalues of the reduced density matrix and the multi-partite density matrix. The conditions (4.1), (4.2) and (4.3) can obtain as the same as in the bi-partite case easily. The relations (4.4) are new relations, and they include more informations than the relations (3.3). We can see in the following that we can get some further results of this relations. We only need proof the conditions of (4.4).

**Proof:** Using the method used in the bi-partite case

\[
\sum_{i=1}^{R=L+r} \lambda_i^A(AB) + \sum_{j=1}^{S=M+s} \lambda_j^B(BC) = \min_{U_1}[\text{tr}(U_1^* \rho_{ABC} U_1)] + \min_{U_2}[\text{tr}(U_2^* \rho_{ABC} U_2)].
\]

where \( U_1 \) is a \( LMN \times RN \) matrix, every \( N \) columns can be viewed as a block. So matrix \( U_1 \) is divided into \( R \) blocks. The \( l \)th block has the form
where only the elements \(((k - 1)N + j, j)\) \((k = 1, 2, \ldots, LM; j = 1, 2, \ldots, N)\) are nonzero in the \(i\)th block, the first columns in the different blocks are orthogonal and normalized. The form of this matrix is the same as the matrix \(U_1\) used in the general bi-partite case. But together with the following unitary matrix \(U_2\), we can find that this situation is different from the situation in the bi-partite case. In the bi-partite case, there is only one position that both the columns in the blocks of \(U_2\) and the columns in \(U_1\) are nonzero. But in this case, it is not. This can substantially effect the linear dependent number \(\xi\) between the unitary matrices \(U_1\) and \(U_2\).

The unitary matrix \(U_2 \in M_{LMN \times LS}\), we divide this matrix into \(S\) blocks, and each block is a \(LMN \times L\) matrix which has the following form

\[
\begin{bmatrix}
  u_{1,i} & & & & & u_{1,i} & & & & \\
  & u_{1,i} & & & & & u_{1,i} & & & \\
  & & \ddots & & & & & \ddots & & \\
  & & & u_{N+1,i} & & & & & u_{N+1,i} & \\
  & & & & u_{N+1,i} & & & & & u_{N+1,i} \\
  & & & & & \ddots & & & & \ddots \\
  & & & & & & u_{(LM-1)N+1,i} & & & & u_{(LM-1)N+1,i} \\
  & & & & & & & u_{(LM-1)N+1,i} & & & \\
  & & & & & & & & u_{(LM-1)N+1,i} & \\
\end{bmatrix}
\]

\[(IV)\]

where only the elements \(((p - 1)N + q, p)(p = 1, \ldots, L; q = 1, 2, \ldots, MN)\) in the \(i\)th block are nonzero and the first columns in the different blocks are also orthogonal and normalized. Now we calculate the maximal linear dependent number \(\xi\) between \(U_1\) and \(U_2\). We need to point out two facts about these two unitary matrices.

The first, if \(R < L\) and \(S < N\), we can find at most \(RS\) columns in the unitary matrix \(U_2\) are linear dependent on the columns in the unitary matrix \(U_1\) as the same discussion in the bi-partite case. The second, when \(R = \mu L\) and \(S = \mu N\), all of the columns in the unitary matrix \(U_2\) can be linear dependent on the columns in the unitary matrix \(U_1\). This can be reached by letting the elements \(((j - 1)MN + iN + 1, 1)\) in the \([(i - 1)L + j]\)th \((i = 0, 1, 2, \ldots, \mu - 1; j = 1, 2, \ldots, L)\) block of unitary \(U_1\) be 1 and the other elements zeros; at same time letting the elements \(((i - 1)N + j, k)\) \((k = (i - 1)N + 1, (i - 1)N + 2, \ldots, iN - 1)\) in the \([(i - 1)N + j]\)th \((i = 1, 2, \ldots, \mu; j = 1, 2, \ldots, N)\) be nonzero and the others zeros. Using these two facts, when \(R = \mu L + r, S = \mu N + s\), there are \(\mu LN\) columns in \(U_2\) linear dependent on \(\mu LN\) columns in \(U_1\). Since the columns in the same unitary matrix are orthogonal to each other, we need only consider the rest columns in the unitary matrix \(U_1\) and \(U_2\). So at most \(\mu LN + rs\) columns in the unitary \(U_2\) are linear dependent on the columns in the unitary matrix \(U_1\). QED

When some eigenvalues of the density matrices are zeros, we can get some stronger relations between the reduced density matrix and the global density matrix.

**Theorem 4:** Suppose \(\text{rank}(\rho_{ABC}) = LMN - Ls\), \(\text{rank}(\rho_{BC}) = MN - s\), \(\text{rank}(\rho_{AB}) = LM - r\) and \(\text{rank}(\rho_B) = M - t\), if \(r\) and \(s\) satisfy the condition \(Nr \leq Ls\), then
are not dependent on \( j \).

In this sub-block, only the elements \((iMN)\) the same division. The matrix

\[ U \]

is the same as the matrix \( U \) in the proof of the theorem 3 and made the same division. We viewed each \( s \) columns as a block. So the matrix \( U \) is divided into \( L \) blocks. Each block has the form as a \( LMN \times s \) matrix. In the \((i+1)\)th block, only the elements \((iMN+j,k) (j = 1, 2, \ldots, MN; k = 1, 2, \ldots, s)\) are nonzero. We can note that the nonzero elements are not dependent on the index \( i \), that is, the element \((iMN+j,k) = v_{jk} \).

The form of the \((i+1)\)th block is

\[
\begin{pmatrix}
v_{iMN+1,1} & v_{iMN+1,2} & \cdots & v_{iMN+1,s} \\
v_{iMN+2,1} & v_{iMN+2,2} & \cdots & v_{iMN+2,s} \\
\vdots & \vdots & \cdots & \vdots \\
v_{iMN+MN,1} & v_{iMN+MN,2} & \cdots & v_{iMN+MN,s}
\end{pmatrix}
\]

The columns in the same block are normalized and orthogonal to each other.

The matrix \( C \) can be divided into many blocks, each block is a \( LMN \times LN \) matrix, further more, we can divide each block into \( L \) sub-blocks. The \((i+1)\)th sub-block has the following form

\[
\begin{pmatrix}
w_{iMN,1} & w_{iMN,1} & \cdots & w_{iMN,1} \\
w_{iMN+N,1} & w_{iMN+N,1} & \cdots & w_{iMN+N,1} \\
w_{iMN+(M-1)N,1} & w_{iMN+(M-1)N,1} & \cdots & w_{iMN+(M-1)N,1}
\end{pmatrix}
\]

In this sub-block, only the elements \((iMN+kN+j,k) (k = 0, 1, \ldots, M-1; j = 0, 1, \ldots, N)\) are nonzero and they are not dependent on \( j \). For each block, the value of the nonzero element are not dependent on the index \( i \). The first columns of the different blocks are orthogonal and normalized. Using the lemma 1, we can get the relations of the eigenvalues

\[
\sum_{j=1}^{r} \lambda_{j}^{i}(AB) = \min_{A} tr[A^{*} \rho_{ABC} A],
\]

\[
\sum_{k=1}^{s} \lambda_{k}^{i}(BC) = \min_{B} tr[B^{*} \rho_{ABC} B],
\]
\[
\sum_{j=1}^{t} \lambda_j^i(B) = \min_A \text{tr}[C^* \rho_{ABC} C].
\]

Since \( \sum_{k=1}^{s} \lambda_k^i(BC) = \sum_{k=1}^{Ls} \lambda_k^i(ABC) = 0 \), there is a unitary transformation between the \( Ls \) minimal eigenvectors and the columns of the matrix \( B \). Because of \( \sum_{j=1}^{r} \lambda_j^i(AB) = 0 \) and the density matrix \( \rho_{ABC} \) has only the \( Ls \) zero eigenvalues, we can see that the columns in matrix \( A \) must be a linear combination of the \( Ls \) minimal eigenvectors. So all of the columns in the matrix \( A \) must be a linear combinations of the columns of the matrix \( B \). For convenience, we divide each block of the matrix \( A \) into \( L \) sub-blocks, each sub-block is a \( MN \times N \) matrix, the \((i + 1)\)th sub-block has the following form

\[
\begin{bmatrix}
  u_1^{MN,1} & u_1^{MN,1} & \cdots & u_1^{MN,1} \\
  u_2^{MN,1} & u_2^{MN,1} & \cdots & u_2^{MN,1} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{MN+(M-1)N,1} & u_{MN+(M-1)N,1} & \cdots & u_{MN+(M-1)N,1}
\end{bmatrix}^{MN \times N},
\]

\[(VIII)\]

Now we can find that each columns in the \((i + 1)\)th sub-block of the matrix \( A \) must be the linear combination of the columns of the \((i + 1)\)th block of the matrix \( B \). And we take out the first columns of all of the sub-block and to find how many columns are linear independent. As the following proof, there are at least \( \lceil \frac{Ls}{2} \rceil + 1 \) columns are linear independent. At first, we denote the column of the \((i + 1)\)th sub-block in the \((j + 1)\)th block as \( V_{ij} \). We write all these vectors in a matrix form as

\[
\begin{bmatrix}
  V_{01} & V_{02} & \cdots & V_{0r} \\
  V_{11} & V_{12} & \cdots & V_{1r} \\
  \vdots & \vdots & \ddots & \vdots \\
  V_{(L-2),1} & \cdots & V_{(L-2),r} \\
  V_{(L-1),1} & V_{(L-1),2} & \cdots & V_{(L-1),r}
\end{bmatrix}^{L \times r},
\]

\[(IX)\]

The different columns in this matrix are orthogonal and normalized. At first, we consider the columns. There are at least one linear independent vector in each column, that is, every elements in the same columns are equal to the same vector multiply by a scalar. So we can rewrite the matrix (IX) as the following matrix

\[
\begin{bmatrix}
  \alpha_{1,1}V_1 & \alpha_{2,1}V_2 & \cdots & \alpha_{r,1}V_r \\
  \alpha_{1,2}V_1 & \alpha_{2,2}V_2 & \cdots & \alpha_{r,2}V_r \\
  \vdots & \vdots & \ddots & \vdots \\
  \alpha_{1,1L-1}V_1 & \cdots & \alpha_{r,L-1}V_r \\
  \alpha_{1,1L}V_1 & \alpha_{2,1L}V_2 & \cdots & \alpha_{r,1L}V_r
\end{bmatrix}^{L \times r},
\]

\[(X)\]

where \( \alpha_{i,j} (i = 1, 2, \cdots, r; j = 1, 2, \cdots, L) \) are complex number. Now we use the orthogonal conditions that the different columns are orthogonal each other. That is,

\[
\langle \alpha_i | \alpha_j \rangle \langle V_i | V_j \rangle = 0 \quad i, j = 1, 2, \cdots, r
\]

where the vector \( |\alpha_j\rangle \) is \( (\alpha_{1,1}, \alpha_{1,2}, \cdots, \alpha_{1,L})^T \). From this equations we know that if the vectors \( V_i \) and \( V_j \) are linear dependent, then the vectors \( |\alpha_i\rangle \) and \( |\alpha_j\rangle \) must be orthogonal. Since the vector \( |\alpha_j\rangle \) is in \( L \) Dimension space, there are at most \( L \) \( V_i \) are linear dependent to the same vector \( \alpha_i \) and the vectors \( |\alpha_j\rangle \) are orthogonal each other. So there are at least \( \lceil \frac{Ls}{2} \rceil + 1 \) linear independent \( u_p \). Using the Schmidt method, we can get \( \lceil \frac{Ls}{2} \rceil + 1 \) orthogonal and normalized vectors \( w_i (i = 1, 2, \cdots, L) \). All this vectors can be expressed as the linear combination of the columns of the matrix \( B \).

If we let the first column of the \((i + 1)\)th sub-block of the matrix \( C \) be equal to one of the vectors \( w_i \). Then the columns of the \((i + 1)\)th sub-block also can be expressed as the linear combinations of the columns of the \((i + 1)\)th
block of the matrix $B$. Since the nonzero elements of the matrix $B$ and $C$ are not dependent on the index of $i$, the columns in the block which the sub-block belongs to can be expressed as the linear combination of the columns of the matrix $B$. So the matrix $\rho_B$ must have at least \( \lceil \frac{r-1}{2} \rceil + 1 \) zero eigenvalues.

QED

From the proof of this theorem, we can find that the most important fact is that the columns in the matrix $A$ must be the linear combinations of the columns of the matrix $B$. Since the symmetry of the particles, exchange the role of the density matrix of $\rho_{AB}$ and $\rho_{BC}$, the former theorem is hold too. This theorem is the character of the multi-partite density matrix, the bi-partite situation has no theorem similar as this theorem. This can be viewed as a new correlation between the particles. This theorem is the further results of the relations (4.4). The parameters $r$ and $s$ must satisfy the constraining of the inequality (4.4).

IV. COMPATIBILITY RELATIONS BETWEEN ONE-PARTY DENSITY MATRIXES FROM N-PARTITE PURE QUANTUM STATES

When using our method to the multi-partite density matrix, we can get much more complicated relations between the reduced density matrices and the multi-partite density matrix. The majorization relations between them are hold now.

**Theorem 6:** For a $N$-partite pure state, if every particle are in the $M$-dimensional Hilbert space, the eigenvalues of the one-party reduced density matrices satisfy the following relations

\[
\sum_{j=1, j \neq k, l}^{N} \sum_{i=1}^{M-1} \lambda_{ij}(j) + \sum_{i=1}^{p} \lambda_{i}^{*}(k) \geq \sum_{i=1}^{p} \lambda_{i}^{*}(l), \quad p = 1, 2, \ldots, M-1, \quad k \neq l = 1, 2, \ldots, N, \tag{6}
\]

where $\lambda_{ij}(j), \lambda_{i}^{*}(j), \ldots, \lambda_{M-1}^{*}(j)$ are the eigenvalues of the partial density matrix $\rho_{j}$ and they are arranged in increasing order.

If $N=2$, we can get the necessary and sufficient conditions for the single particle partial density matrices compatible with the bi-partite pure state. When $M=2$, it can also give the necessary and sufficient compatibility conditions between the set of one-party reduced density matrices and the $N$-partite density matrix [12].

**Proof** We consider the $(N−1)$-partite density matrix $\rho_{123\ldots N−1}$. From this density matrix, we can get the one-party reduced density matrices $\rho_i (i = 1, 2, \ldots, N−1)$. Now we use the method before to get

\[
\sum_{j=1}^{N−2} \sum_{i=1}^{M−1} \lambda_{ij}(j) + \sum_{i=1}^{p} \lambda_{i}^{*}(N−1) = \sum_{i=1}^{N−2} \min_{U_i} [\text{tr}(U_i^* \rho_{12\ldots N−1} U_i)] + \min_{U_{N−1}} [\text{tr}(U_{N−1}^* \rho_{12\ldots N−1} U_{N−1})]
\]

where the unitary matrix $U_{i} (i = 1, 2, \ldots, N−2)$ can be divided into $M−1$ blocks, each block is a $M^{N−1} \times M^{N−2}$ matrix. The unitary matrix $U_{N−1}$ can be divided into $p$ blocks, each block is also a $M^{N−1} \times M^{N−2}$ matrix. The form of the block (the positions of the nonzero elements in the block) is independent of the block in the same unitary matrix $U_{i}$. The first column in different blocks of the same unitary matrix are orthogonal and normalized. Furthermore, we can divide the blocks into some sub-blocks. For the block in the unitary matrix $U_{i}$, we can divide it into $M^{i−1}$ sub-blocks, the form of the $j$th sub-block of the $k$th block in $U_{i}$ is the following
This is the end of the proof of the theorem 6. QED

where only the elements \((j - 1)M^{N-1} + (l - 1)M^{N-i-1} + k, k) (l = 1, 2, \cdots, M; k = 1, 2, \cdots, M)\) are nonzero. The nonzero elements of the first columns in the different sub-blocks are equal to each other for the same index \(l\). We can find there are some self-similar property on the position of the nonzero elements in unitary matrices \(U_i\) and \(U_{i+1}\). In this special case, our main task is to find the number of the least linear independent columns in the unitary matrices \(U_i(i = 2, 3, \cdots, N - 1)\). Using the same discussion in the bi-party case, we find that there are at most \((M - 1)M^{N-2}\) columns in the unitary matrices \(U_i(i = 2, 3, \cdots, N - 1)\) are linear dependent on the columns in the unitary \(U_1\), and at most \(pM^{N-2}\) columns in matrix \(U_{N-1}\) are linear dependent on the columns in the matrix \(U_1\). This can be reached by letting the elements \(((k - 1)M^{N-2} + j, j)(j = 1, 2, \cdots, M^{N-2})\) be equal to 1 in the \(k\)th block of the unitary matrix \(U_1\). We use the same method to consider the columns remaining in each unitary until the remaining columns only in the unitary \(U_{N-1}\). Then we sum up all of the number of the linear independent columns in each unitary matrix, we get

\[
(M - 1)M^{N-2} + (M - 1)M^{N-3} + \cdots + (M - 1)M + p = M^{N-1} - M + p.
\]

Since the \(N\)-partite state is a pure state, the nonzero eigenvalues of the density matrix \(\rho_N\) is the same as the nonzero eigenvalues of the density matrix \(\rho_{12\cdots N-1}\) as the Schmidt theorem. So the density matrix \(\rho_{12\cdots N-1}\) has at most \(M\) nonzero eigenvalues and the first \(M^{N-1} - M\) eigenvalues are zeroes. The relations between the eigenvalues of the one-party density matrices \(\rho_i(i = 1, 2, \cdots, N - 1)\) and the global density matrix \(\rho_{12\cdots N-1}\) become the relations between the eigenvalues of the one-party reduced density matrices \(\rho_i(i = 1, 2, \cdots, N - 1)\) and \(\rho_N\). So the relations between the eigenvalues of the one party density matrix \(\rho_i(i = 1, 2, \cdots, N)\) are

\[
\sum_{j=1}^{N-2} \sum_{i=1}^{M-1} \lambda_i^j(j) + \sum_{i=1}^{p} \lambda_i^j(N - 1) \geq \sum_{i=1}^{p} \lambda_i^j(N).
\]

This is the end of the proof of the theorem 6. QED

From this proof, we can find that the nontrivial relations between the one-party reduced density matrices of the multi-party pure state in this situation must include at least \((M - 1)(N - 2) + 1\) eigenvalues (the same eigenvalues are calculated repeatedly) at the left of the inequality. Further more, using our method we can find almost all of the linear relations between the eigenvalues of the one-party reduced density matrixes, only need carefully consider the number of the orthogonal vectors needed to express all of the columns. We conjecture that the necessary and sufficient conditions of the compatibility problem between the one-party density matrices and a pure state can be expressed by the linear relations between these eigenvalues of the local one-party density matrix, that is, the necessary and sufficient conditions can form a polytope in the eigenvalues space. If this conjecture is true, we can find the necessary and sufficient conditions for the compatibility problem by our method. The necessary and sufficient conditions of the compatibility between the single qutrit density matrices and a pure state in \(C^3 \otimes C^3 \otimes C^3\) is

- \(\lambda_1^1(A) + \lambda_2^1(B) + \lambda_3^1(C) \leq \lambda_1^1(B) + \lambda_2^1(C) + \lambda_3^1(A)\)
- \(\lambda_1^2(A) + \lambda_2^2(B) + \lambda_3^2(C) \leq \lambda_1^2(B) + \lambda_2^2(C) + \lambda_3^2(A)\)
- \(\lambda_1^3(A) + \lambda_2^3(B) + \lambda_3^3(C) \leq \lambda_1^3(B) + \lambda_2^3(C) + \lambda_3^3(A)\)
\begin{equation}
\lambda^1_1(A) + 2\lambda^1_2(A) \leq \lambda^1_1(B) + 2\lambda^1_2(B) + \lambda^1_1(C) + 2\lambda^1_2(C)
\end{equation}
\begin{equation}
2\lambda^1_1(A) + \lambda^1_2(A) \leq \lambda^1_1(B) + 2\lambda^1_2(B) + 2\lambda^1_2(C) + \lambda^1_2(C)
\end{equation}
\begin{equation}
2\lambda^2_1(A) + \lambda^2_3(A) \leq \lambda^2_1(B) + 2\lambda^2_2(B) + 2\lambda^2_2(C) + \lambda^2_3(C)
\end{equation}
\begin{equation}
2\lambda^3_1(A) + \lambda^3_2(A) \leq 2\lambda^1_1(B) + \lambda^2_2(B) + \lambda^2_2(C) + 2\lambda^3_3(C)
\end{equation}

and the conditions permutation $A,B$ and $C$. These conditions which are found by Higuchi [12] can be found in our method more conveniently. But it is very difficult to prove that this conditions are sufficient. This necessary and sufficient conditions are linear and this support our conjecture. When the number of the particle increasing, the simplexes of this polytope increase rapidly, the proof used by Higuchi in the three qutrit case is not convenient. We must need another method to proof the convex property of this set.

\section*{V. SUMMARY}

In this paper we use a theorem of the analysis matrix to give a simple method to find the relations between the reduced density matrix and the multi-partite density matrix. We find hat the majorization relations are the universal relations between the eigenvalues of reduced density matrices and the multi-partite density matrix. We also give some relations of the eigenvalues between the different reduced density matrixes. All of the relations received in this paper can be viewed as the necessary conditions of the problem whether the reduced density matrix is compatible with a multi-partite density matrix. What is the necessary and sufficient conditions for the compatibility problem between a arbitrary set of density matrices and a multi-partite is far from completely solved, even the special problem whether a set of one-party reduced density matrixes is compatible with a pure multi-partite state is very difficult. But the method used in this paper give us a possible way to solve these problems, especially for the compatibility problem of the pure state. In this paper we only analyzed the number of the linear independent vectors in the N-partite case, if we analyze more carefully such as the coefficients of the linear combination, we may find more relations. This is one of our further work. On the other hand, if we add some symmetry such as the translation invariant on the particles, we can get more constrains on the eigenvalues of the density matrixes, then we may find the necessary and sufficient conditions of the compatibility problem. This symmetry is very useful in condensed matter and the quantum phase transition. This problem will also be investigate in the future.

The author would like to thank for invaluable discussion with L. M. Duan. This work was funded by the National Fundamental Research Program (2001CB309300), the Innovation Funds from Chinese Academy of Sciences (CAS), the outstanding Ph. D thesis award (L.M.D) and the CAS’s talented scientist award (L.M.D.).