PRICING EUROPEAN OPTION WITH THE SHORT RATE UNDER SUBDIFFUSIVE FRACTIONAL BROWNIAN MOTION REGIME

FOAD SHOKROLLAHI

Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, FIN-65101 Vaasa, FINLAND

Abstract. The purpose of this paper is to analyze the problem of option pricing when the short rate follows subdiffusive fractional Merton model. We incorporate the stochastic nature of the short rate in our option valuation model and derive explicit formula for call and put option and discuss the corresponding fractional Black-Scholes equation. We present some properties of this pricing model for the cases of $\alpha$ and $H$. Moreover, the numerical simulations illustrate that our model is flexible and easy to implement.

1. Introduction

Nowadays, the Black–Scholes (BS) model [1] is still classical and most popular model of the market. However, empirical research shows that it cannot capture many of the characteristic features of prices, such as: long-range correlations, heavy-tailed and skewed marginal distributions, lack of scale invariance, periods of constant values, etc. Therefore, improvements of the BS model itself did not stand still either. Since fractional Brownian motion (FBM) has two important properties called self-similarity and long-range dependence, it has the ability to capture the typical tail behavior of stock prices or indexes [15, 14, 2, 13]. The FBM model is an improvement of the BS model, by replacing the FBM with the Brownian motion in the standard BS model. That is

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB^H(t),$$

(1.1)

here $\mu, \sigma$ are constants, and $B^H$ is a FBM with Hurst parameter $H \in [\frac{1}{2}, 1)$.

Subdiffusive Brownian motion is an another generalization of the BS model, which is introduced by Magdziarz [9]. In order to describe properly financial data exhibiting periods of constant values, he put forward the subdiffusive strategy based on the geometric Brownian motion to describe financial data with the periods of the constant prices. He replaced the physical time $t$ with inverse $\alpha$-stable subordinator $T_\alpha(t)$ in the standard BS model where $\alpha \in (0, 1)$. Magdziarz showed that the considered model is arbitrage-free but incomplete, and obtained the corresponding
subdiffusive BS formula for the fair prices of European options. Moreover, Hui Gua et al. [4] applied subdiffusive FBM regime
\begin{equation}
X_\alpha(t) = X(T_\alpha(t)),
\end{equation}
as the model of asset prices exhibiting subdiffusive dynamics. Here the parent process $X(\tau)$ is the FBM defined in Equation (1.1). $T_\alpha(t)$ is the inverse $\alpha$-stable subordinator with $\alpha \in (0, 1)$. Later, many scholars made some improvements of this model [4, 16, 6].

Constant short rate during the life of the option is the assumption at all above studies. This assumption is clearly at odds with reality because, as a matter of fact, the short rate $r(t)$ is evolving random of time. Hence, in this study, we combine the stochastic nature into our option pricing model. Specifically, we will consider the option pricing of the European options under the Merton short rate model [12] in a subdiffusive FBM regime. That is, $r(t) = X(T_\alpha(t))$ in which $X(\tau)$ follows
\begin{equation}
dX(\tau) = \mu r d\tau + \sigma r dB_{H_1}(\tau),
\end{equation}
and the stock price $S(t) = \hat{X}(T_\alpha(t))$ in which $\hat{X}(\tau)$ follows
\begin{equation}
d\hat{X}(\tau) = \mu_s \hat{X}(\tau) d\tau + \sigma_s \hat{X}(\tau) dB_{H_2}(\tau),
\end{equation}
where $\mu_r, \sigma_r, \mu_s, \sigma_s$, are constant, $B_{H_1}(\tau)$ and $B_{H_2}(\tau)$ are two FBM with Hurst parameter $H \in [\frac{1}{2}, 1]$ and correlation coefficient $\rho$. $T_\alpha(t)$ is the inverse $\alpha$-stable subordinator with $\alpha \in (0, 1)$ defined as follows
\begin{equation}
T_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\},
\end{equation}
$\{U_\alpha(\tau)\}_{\tau \geq 0}$ is a $\alpha$-stable Levy process with nonnegative increments and Laplace transform: $E(e^{-uU_\alpha(\tau)}) = e^{-\tau u^\alpha}$.

Fig. 2 shows typically the differences and relationships between the sample paths of the stock price in the FBM model and the subdiffusive FBM model.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sample_paths.png}
\caption{Comparison of the sample paths of the stock price in the FBM model (left) and the subdiffusive FBM model (right) for $r = 0.01, \alpha = 0.9, H = 0.8, \sigma = 0.1, S_0 = 1$.}
\end{figure}
Here, assume that $T_\alpha(t)$ is independent of $B^H_1(\tau)$ and $B^H_2(\tau)$. Specially, when $H = \frac{1}{2}$, it is a subdiffusion process mentioned in Refs. [10, 11] and when $\alpha \uparrow 1$, $T_\alpha(t)$ reduces to physical time $t$. In this study, we apply the subdiffusive mechanism of trapping events in order to describe financial data exhibiting periods of constant values.

This paper is organized as follows. In Section 2, we derive the formula for the price of a riskless zero-coupon bond paying $1$ at maturity. In Section 3, we obtain the corresponding BS equation by using delta hedging argument and discuss some special cases of this equation. In Section 4 we present an analytic pricing formula for the European call and put options. In Section 5, we study some special properties of this pricing formula. Furthermore, we show how to use our model to price options by numerical simulations. The comparison of our model and traditional models is undertaken in this section. Finally, Section 6 draws the concluding remarks.

2. Pricing formula for zero-coupon bond

The purpose of this section is to derive the pricing formula for zero-coupon bond $P(r, t, T)$. Here, $P(r, T; T) = 1$, that is, the zero-coupon bond will pay for 1 dollar at the expiry date $T$.

We assume that the short rate $r(t)$ satisfy Equation (1.3), $\alpha \in (\frac{1}{2}, 1)$ and $2\alpha - \alpha H > 1$, then by applying the Taylor series expansion to $P(r, t, T)$ we obtain that

$$P(r + \Delta r, t + \Delta t) = P(r, t, T) + \frac{\partial P}{\partial r} \Delta r + \frac{\partial P}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} (\Delta r)^2 + \frac{1}{2} \frac{\partial^2 P}{\partial r \partial t} \Delta r (\Delta t) + \frac{1}{2} \frac{\partial^2 P}{\partial t^2} (\Delta t)^2 + O(\Delta t).$$

(2.1)

From Equation (1.3) and [16], we have

$$\Delta r = \mu_r(\Delta T_\alpha(t)) + \sigma_r B^H_1(T_\alpha(t))$$

(2.2)

$$\Delta r = \mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} (\Delta t)^{2H} + \sigma_r \Delta B^H_1(T_\alpha(t)) + O((\Delta t)^{2H}).$$

(2.3)

$$\Delta r = \sigma_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} (\Delta t)^{2H} + O((\Delta t)^{2H}).$$

(2.4)

Then from the Lemma 1 in [16], we can get

$$dP(r, t, T) = \left[ \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \left( \mu_r \frac{\partial P}{\partial r} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 P}{\partial r^2} \right) 2H t^{2H-1} + \frac{\partial P}{\partial t} \right] dt + \sigma_r \frac{\partial P}{\partial t} dB^H_1(T_\alpha(t)).$$

(2.5)

Assuming
\[
\mu = \frac{1}{P} \left[ \left( \frac{t}{\Gamma(\alpha)} \right)^{2H} \left( \mu_r \frac{\partial P}{\partial r} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 P}{\partial r^2} \right) 2Ht^{2H-1} + \frac{\partial P}{\partial t} \right],
\]
(2.6)

\[
\sigma = \frac{1}{P} \left( \frac{\partial P}{\partial r} \right),
\]
(2.7)

and letting the local expectations hypothesis holds for the term structure of interest rates (i.e. \(\mu = r\)), we obtain

\[
\frac{\partial P}{\partial t} + 2Ht^{2H-1} \mu_r \left( \frac{t}{\Gamma(\alpha)} \right)^{2H} \frac{\partial P}{\partial r} + Ht^{2H-1} \sigma_r^2 \left( \frac{t}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 P}{\partial r^2} - rP = 0.
\]
(2.8)

To solve Equation (2.8) for \(P(r, t, T)\), let \(\tau = T - t, P(r, t, T) = \exp \{ f_1(\tau) - \mu r f_2(\tau) \} \), then we have

\[
\frac{\partial P}{\partial t} = P \left( -\frac{\partial f_1(\tau)}{\partial \tau} + r \frac{\partial f_2(\tau)}{\partial \tau} \right),
\]
(2.9)

\[
\frac{\partial P}{\partial r} = -Pf_2(\tau),
\]
(2.10)

\[
\frac{\partial^2 P}{\partial r^2} = Pf_2(\tau)^2.
\]
(2.11)

Substituting Equations (2.10) and (2.11) into Equation (2.9) and simplifying Equation (2.8) becomes

\[
P \left[ Ht^{2H-1} \sigma_r^2 f_2(\tau)^2 \left( \frac{t}{\Gamma(\alpha)} \right)^{2H} - 2Ht^{2H-1} \mu_r f_2(\tau) \left( \frac{t}{\Gamma(\alpha)} \right)^{2H} \right] - \frac{\partial f_1(\tau)}{\partial \tau} + r \left( \frac{\partial f_2(\tau)}{\partial t} - 1 \right) = 0.
\]
(2.12)

From Equation (2.12), we have

\[
\frac{\partial f_1(\tau)}{\partial \tau} = Ht^{2H-1} \left( \frac{t}{\Gamma(\alpha)} \right)^{2H} \left( \sigma_r^2 f_2(\tau)^2 - 2\mu_r f_2(\tau) \right),
\]
(2.13)

\[
\frac{\partial f_2(\tau)}{\partial \tau} = 1.
\]

Then,
\[ f_1(\tau) = \frac{H \sigma_r^2}{(\Gamma(\alpha))^{2H}} \int_0^\tau (T - s)^{(\alpha-1)2H+2H-1}s^2 ds - \frac{2H \mu_r}{(\Gamma(\alpha))^{2H}} \int_0^\tau (T - s)^{(\alpha-1)2H+2H-1}s ds, \]
\[ f_2(\tau) = \tau. \]

Hence, we obtain a formula for the price at time \( t \) of a riskless zero-coupon bond which pay \$1 at maturity \( T \) is given by
\[ P(r, t, T) = e^{-r\tau + f_1(\tau)}. \]

Corollary 2.1. When \( \alpha \uparrow 1 \), Equations (1.3) and (1.4) reduce to the FBM, we obtain
\[ f_1(\tau) = H \sigma_r^2 \int_0^\tau (T - s)^{2H-1}s^2 ds - 2H \mu_r \int_0^\tau (T - s)^{2H-1}s ds, \]
specially, if \( t = 0 \)
\[ f_1(\tau) = \sigma_r^2 \frac{T^{2H+2}}{(2H + 1)(2H + 2)} - \frac{\mu_r T^{2H+1}}{2H + 1}, \]
then
\[ P(r, t, T) = \exp \left\{ -rT + \sigma_r^2 \frac{T^{2H+2}}{(2H + 1)(2H + 2)} - \frac{\mu_r T^{2H+1}}{2H + 1} \right\}. \]

Corollary 2.2. If \( H = \frac{1}{2} \), from Equation (2.14), we obtain
\[ f_1(\tau) = \frac{1}{2} \frac{\sigma_r^2}{\Gamma(\alpha)} \int_0^\tau (T - s)^{\alpha-1}s^2 ds\]
\[ - \frac{\mu_r}{\Gamma(\alpha)} \int_0^\tau (T - s)^{\alpha-1}s ds, \]
then the result is consistent with the result in [5].
Further, if \( \alpha \uparrow 1 \) and \( H = \frac{1}{2} \), Equations (1.3) and (1.4) reduce to the geometric Brownian motion, then we have
\[ f_1(\tau) = \frac{1}{6} \sigma_r^2 \tau^3 - \frac{1}{2} \mu_r \tau^2, \]
then
\[ P(r, t, T) = e^{-r\tau + \frac{1}{6} \sigma_r^2 \tau^3 - \frac{1}{2} \mu_r \tau^2}. \]
which is consistent with the result in [7, 3].
3. Fractional BS Equation

The purpose of this section is to derive the fractional BS equation for European options when the short rate \( r(t) \) and stock price \( S(T) = S(T_\alpha(t)) \) satisfy Equations (3.3) and (3.4), respectively. We assume that \( B^H_t(T_\alpha(t)) \) and \( B^H_t(T_\alpha(t)) \) are two FBM with Hurst parameter \( H \in ]\frac{1}{2}, 1[ \) and correlation coefficient \( \rho \).

Let \( C = C(S, r, t) \) be the price of a European call option at time \( t \) with a strike price \( K \) that matures at time \( T \). Then we have.

**Theorem 3.1.** Assume that the stock price short rate \( r(t) \) and \( S(t) \) satisfy Equations (3.3) and (3.4), respectively. Then, \( C(S, r, t) \) satisfies the following fractional BS equation

\[
\frac{\partial C}{\partial t} + \tilde{\sigma}_s^2(t)S^2 \frac{\partial^2 C}{\partial S^2} + \tilde{\sigma}_r^2(t) \frac{\partial^2 C}{\partial r^2} + 2\rho \tilde{\sigma}_r(t)\tilde{\sigma}_s(t) \frac{\partial^2 C}{\partial S \partial r} \\
+ 2Ht^{2H-1}\mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial C}{\partial r} + rS \frac{\partial C}{\partial S} - rC = 0,
\]

where

\[
\tilde{\sigma}_s^2(t) = Ht^{2H-1}\sigma_s^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H},
\]

\[
\tilde{\sigma}_r^2(t) = Ht^{2H-1}\sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H}.
\]

\( \sigma_s, \sigma_r, \mu_s, \mu_r, \) are constant, \( H \in \left[\frac{1}{2}, 1\right) \) and \( \alpha \in (\frac{1}{2}, 1) \) and \( 2\alpha - \alpha H > 1 \).

**Proof:** We consider a portfolio with \( D_{1t} \) units of stock and \( D_{2t} \) units of zero-coupon bond \( P(r, t, T) \) and one unit of \( C = C(r, t, T) \). Then, the value of the portfolio at current time \( t \) is

\[
\Pi_t = C - D_{1t}S_t - D_{2t}P_t.
\]

Then, from (3.4) we have

\[
d\Pi_t = C_t - D_{1t}dS_t - D_{2t}dP_t = \left[ \frac{\partial C}{\partial t} dt + Ht^{2H-1}\sigma_s^2 S^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 C}{\partial S^2} + Ht^{2H-1}\sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 C}{\partial r^2} \\
+ 2Ht^{2H-1}\rho \sigma_r \sigma_s S \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 C}{\partial S \partial r} \right] dt + \left[ \frac{\partial C}{\partial t} - D_{1t} \right] dS_t \\
+ \left[ \frac{\partial C}{\partial r} - D_{2t} \frac{\partial P}{\partial r} \right] dr + D_{2t} \left[ \frac{\partial P}{\partial t} + Ht^{2H-1}\sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 P}{\partial r^2} \right] dt.
\]

By setting \( D_{1t} = \frac{\partial C}{\partial S}, \ D_{2t} = \frac{\partial P}{\partial r} \), to eliminate the stochastic noise, then

\[
d\Pi_t = \left[ \frac{\partial C}{\partial t} + Ht^{2H-1} \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \left( \sigma_s^2 S^2 \frac{\partial^2 C}{\partial S^2} + \sigma_r^2 \frac{\partial^2 C}{\partial r^2} + 2\rho \sigma_r \sigma_s S \frac{\partial^2 C}{\partial S \partial r} \right) \right] dt \\
- \frac{\partial C}{\partial P} \left[ rP - 2Ht^{2H-1}\mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial P}{\partial r} \right] dt.
\]
The return of an amount $\Pi_t$ invested in bank account is equal to $r(t)\Pi_t dt$ at time $dt$, $E(a\Pi_t) = r(t)\Pi_t dt = r(t) (C - D_1 S_t - D_2 P_t)$, hence from Equation (3.6) we have

$$\frac{\partial C}{\partial t} + H t^{2H-1} \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \left( \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \sigma^2_r \frac{\partial^2 C}{\partial r^2} + 2 \rho \sigma_r \sigma_s \frac{\partial^2 C}{\partial S \partial r} \right)$$

$$\quad + 2 H t^{2H-1} \mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial C}{\partial r} + r S \frac{\partial C}{\partial S} - r C = 0.$$  

Let

$$\tilde{\sigma}_d^2(t) = H t^{2H-1} \sigma_d^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H},$$

$$\tilde{\sigma}_r^2(t) = H t^{2H-1} \sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H}.$$  

Then

$$\frac{\partial C}{\partial t} + \tilde{\sigma}_d^2(t) S^2 \frac{\partial^2 C}{\partial S^2} + \tilde{\sigma}_r^2(t) \frac{\partial^2 C}{\partial r^2} + 2 \rho \tilde{\sigma}_r(t) \tilde{\sigma}_d(t) \frac{\partial^2 C}{\partial S \partial r}$$

$$\quad + 2 H t^{2H-1} \mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial C}{\partial r} + r S \frac{\partial C}{\partial S} - r C = 0,$$

proof is completed.

From Theorem (3.1), we can get the following corollaries

**Corollary 3.1.** If $\rho = 0$ and $r(t)$ be a constant, then the European call option $C = C(S, r, T)$ satisfies

$$\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C = 0,$$

which is a fractional BS equation considered in [8].

**Corollary 3.2.** When $\alpha \uparrow 1$, we obtain

$$\frac{\partial C}{\partial t} + H t^{2H-1} \sigma_d^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 C}{\partial S^2} + H t^{2H-1} \sigma_r^2 \frac{\partial^2 C}{\partial r^2} + 2 \rho \sigma_r \sigma_s \frac{\partial^2 C}{\partial S \partial r}$$

$$\quad + 2 H t^{2H-1} \mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial C}{\partial r} + r S \frac{\partial C}{\partial S} - r C = 0,$$

Further, if $\rho = 0$, $H = \frac{1}{2}$, and $r(t)$ be a constant, from Equation (3.12) we have the celebrated BS equation

$$\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C = 0,$$
4. PRICING FORMULA UNDER SUBDIFFUSIVE FRACTIONAL MERTON SHORT RATE MODEL

In this section, we propose an explicit formula for European call option when its value satisfy the partial differential equation \[ 3.1 \] with boundary condition \[ C(S, r, T) = (S_T - K)^+ \]. Then, we can get

**Theorem 4.1.** Let \( r(t) \) satisfies Equation \[ 1.3 \] and \( S(t) \) satisfies Equation \[ 1.4 \], then the price of European call and put options with strike price \( K \) and maturity \( T \) are given by

\[
\begin{align*}
C(S, r, t) &= S\phi(d_1) - KP(r, t, T)\phi(-d_2) - \phi(-d_1), \\
P(S, r, t) &= KP(r, t, T)\phi(-d_2) - \phi(-d_1).
\end{align*}
\]

where

\[
\begin{align*}
d_1 &= \frac{\ln \frac{S}{K} - \ln P(r, t, T) + \frac{H}{(\Gamma(\alpha))^2H} \int_t^T \tilde{\sigma}^2(s)(s^{(\alpha-1)2H+2H-1}) ds}{\sqrt{2\frac{H}{(\Gamma(\alpha))^2H} \int_t^T \tilde{\sigma}^2(s)(s^{(\alpha-1)2H+2H-1}) ds}}, \\
d_2 &= d_1 - \sqrt{2\frac{H}{(\Gamma(\alpha))^2H} \int_t^T \tilde{\sigma}^2(s)(s^{(\alpha-1)2H+2H-1}) ds}, \\
\tilde{\sigma}^2(t) &= \sigma_s^2 + 2\rho\sigma_s\sigma_r(T-t) + \sigma_r^2(T-t)^2.
\end{align*}
\]

\( P(r, t, T) \) is given by Equation \[ 2.16 \] and \( \phi(.) \) is the cumulative normal distribution function.

**Proof:**

Consider the partial differential equation \[ 3.1 \] of the European call option with boundary condition \( C(S, r, T) = (S_T - K)^+ \)

\[
\begin{align*}
\frac{\partial C}{\partial t} + \tilde{\sigma}_s^2(t)S_t^2 \frac{\partial^2 C}{\partial S^2} + \tilde{\sigma}_r^2(t) \frac{\partial^2 C}{\partial r^2} + 2\rho\tilde{\sigma}_s(t)\tilde{\sigma}_r(t) \frac{\partial^2 C}{\partial S \partial r} \\
+ 2Ht^{2H-1} \mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial C}{\partial r} + rS \frac{\partial C}{\partial S} - rC = 0.
\end{align*}
\]

Denote

\[
z = \frac{S}{P(r, t, T)}, \quad \Theta(z, t) = \frac{C(S, r, t)}{P(r, t, T)},
\]

therefore by computing, we get
\[
\begin{align*}
\frac{\partial C}{\partial t} &= \Theta \frac{\partial P}{\partial t} + P \frac{\partial \Theta}{\partial t} - z \frac{\partial \Theta}{\partial z} \frac{\partial P}{\partial t}, \\
\frac{\partial C}{\partial r} &= \Theta \frac{\partial P}{\partial r} - z \frac{\partial \Theta}{\partial z} \frac{\partial P}{\partial r}, \\
\frac{\partial C}{\partial S} &= \frac{\partial \Theta}{\partial z}, \\
\frac{\partial^2 C}{\partial r^2} &= \Theta \frac{\partial^2 P}{\partial r^2} - z \frac{\partial \Theta}{\partial z} \frac{\partial^2 P}{\partial r^2} + z^2 \frac{\partial^2 \Theta}{\partial r^2} \left( \frac{\partial P}{\partial r} \right)^2, \\
\frac{\partial^2 C}{\partial r \partial S} &= -z \frac{\partial \Theta}{\partial z} \frac{\partial P}{\partial r}, \\
\frac{\partial^2 C}{\partial S^2} &= \frac{1}{P} \frac{\partial^2 \Theta}{\partial z^2}.
\end{align*}
\]

(4.8)

Inserting Equation (4.8) into Equation (4.6)

\[
\begin{align*}
\frac{\partial \Theta}{\partial t} + \frac{\partial^2 \Theta}{\partial z^2} &+ \frac{\sigma^2(t) S^2}{P^2} \frac{\partial^2 P}{\partial r^2} + 2 \rho \sigma_r(t) \sigma_s(t) \frac{1}{P} \frac{\partial P}{\partial r} + \sigma_s^2(t) z^2 \left( \frac{1}{P} \frac{\partial P}{\partial r} \right)^2 \\
&- \frac{z}{P} \left[ \frac{\partial P}{\partial t} + \sigma_r^2(t) \frac{\partial^2 P}{\partial r^2} + 2 H t^{2H-1} \mu_r \left( \frac{t + \Gamma(\alpha)}{\Gamma(\alpha)} \right)^{2H} \frac{\partial P}{\partial r} - t \frac{S}{z} \right] \\
&+ \frac{\Theta}{P} \left[ \frac{\partial P}{\partial t} + \sigma_r^2(t) \frac{\partial^2 P}{\partial r^2} + 2 H t^{2H-1} \mu_r \left( \frac{t + \Gamma(\alpha)}{\Gamma(\alpha)} \right)^{2H} \frac{\partial P}{\partial r} - t P \right] = 0.
\end{align*}
\]

(4.9)

From Equation (2.8), we can obtain

\[
\frac{\partial \Theta}{\partial t} + \sigma^2(t) z^2 \frac{\partial^2 \Theta}{\partial z^2} = 0,
\]

with boundary condition \( \Theta(z, T) = (z - K)^+ \),

where

\[
\sigma^2(t) = \sigma^2_s(t) + 2 \rho \sigma_r(t) \sigma_s(t) (T - t) + \sigma_r(t)^2 (T - t)^2.
\]

The solution of partial differential Equation (4.10) with boundary condition \( \Theta(z, T) = (z - K)^+ \), is given by

\[
\Theta(z, t) = z \phi(\hat{d}_1) - K \phi(\hat{d}_2),
\]

(4.12)

here

\[
\hat{d}_1 = \frac{\ln \frac{z}{K} + \int_t^T \sigma^2(s) ds}{\sqrt{2 \int_t^T \sigma^2(s) ds}},
\]

(4.13)

\[
\hat{d}_2 = \hat{d}_1 - \sqrt{2 \int_t^T \sigma^2(s) ds}.
\]

(4.14)

Thus, from Equation (4.7) and (4.12)-(4.14) we obtain

\[
C(S, r, t) = S \phi(\hat{d}_1) - KP(r, t, T) \phi(\hat{d}_2),
\]

(4.15)
where

\begin{align*}
(4.16) \quad d_1 &= \frac{\ln \frac{S}{K} - \ln P(r, t, T) + \frac{H}{\Gamma(\alpha)} \int_t^T \tilde{\sigma}^2(s)s^{(\alpha-1)2H+2H-1}ds}{\sqrt{\frac{2H}{\Gamma(\alpha)} \int_t^T \tilde{\sigma}^2(s)s^{(\alpha-1)2H+2H-1}ds}}, \\
(4.17) \quad d_2 &= d_1 - \sqrt{\frac{2H}{\Gamma(\alpha)}} \frac{1}{2} \int_t^T \tilde{\sigma}^2(s)s^{2H-1}ds.
\end{align*}

Letting \( \alpha \uparrow 1 \), from Theorem 4.1, we obtain

**Corollary 4.1.** Suppose that the short rate \( r(t) \) satisfies Equation (1.3) and the stock price \( S(t) \) satisfies Equation (1.4), then the price of European call and put options with strike price \( K \) and maturity \( T \) are given by

\begin{align*}
(4.18) \quad C(S, r, T) &= S\phi(d_1) - K P(r, t, T)\phi(d_2), \\
(4.19) \quad P(S, r, T) &= K P(r, t, T)\phi(-d_2) - S\phi(-d_1).
\end{align*}

where

\begin{align*}
(4.20) \quad d_1 &= \frac{\ln \frac{S}{K} - \ln P(r, t, T) + H \int_t^T \tilde{\sigma}^2(s)s^{2H-1}ds}{\sqrt{2H \int_t^T \tilde{\sigma}^2(s)s^{2H-1}ds}}, \\
(4.21) \quad d_2 &= d_1 - \sqrt{2H \int_t^T \tilde{\sigma}^2(s)s^{2H-1}ds}, \\
(4.22) \quad \tilde{\sigma}^2(t) &= \sigma_s^2 + 2\rho \sigma_r \sigma_s (T-t) + \sigma_r^2 (T-t)^2, \\
(4.23) \quad P(r, t, T) &= \exp \left\{ -r \tau + H \sigma_r^2 \int_0^\tau (T-s)^{2H-1}s^2ds \right\}, \tau = T-t.
\end{align*}

More specifically, if \( H = \frac{1}{2} \), we have

\begin{align*}
(4.24) \quad d_1 &= \frac{\ln \frac{S}{K} - \ln P(r, t, T) + \frac{1}{2} \phi(t, T)}{\sqrt{\phi(t, T)}}, \\
(4.25) \quad d_2 &= d_1 - \sqrt{\phi(t, T)}, \\
(4.26) \quad \phi(t, T) &= \sigma_s^2 (T-t) + \rho \sigma_r \sigma_s (T-t)^2 + \frac{1}{3} \sigma_r^2 (T-t)^3, \\
(4.27) \quad P(r, t, T) &= \exp \left\{ -r (T-t) - \frac{1}{2} \mu_r (T-t)^2 + \frac{1}{6} \sigma_r^2 (T-t)^3 \right\}.
\end{align*}

which is consistent with result in [3].

Letting \( H = \frac{1}{2} \), from Theorem 4.1, we can get

**Corollary 4.2.** Suppose that the short rate \( r(t) \) satisfies Equation (1.3) and the stock price \( S(t) \) satisfies Equation (1.4), then the price of European call and put options with strike price \( K \) and maturity \( T \) are given by

\begin{align*}
(4.28) \quad C(S, r, T) &= S\phi(d_1) - K P(r, t, T)\phi(d_2), \\
(4.29) \quad P(S, r, T) &= K P(r, t, T)\phi(-d_2) - \phi(-d_1).
\end{align*}
where

\begin{align}
\frac{4.30}{d_1} &= \frac{\ln \frac{S}{K} - \ln P(r, t, T) + \frac{1}{2\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s)s^{\alpha-1}ds}{\sqrt{\frac{1}{\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s)s^{\alpha-1}ds}}, \\
\frac{4.31}{d_2} &= d_1 - \sqrt{\frac{1}{\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s)s^{\alpha-1}ds}, \\
\hat{\sigma}^2(t) &= \sigma_s^2 + 2\rho\sigma_s\sigma_s(T - t) + \sigma_r^2(T - t)^2, \\
P(r, t, T) &= \exp \left\{ -r\tau + \frac{\sigma_r^2}{2\Gamma(\alpha)} \int_0^\tau (T - s)^{\alpha-1}s^{2}ds \right\}.
\end{align}

Specially, if \( \rho = 0 \) from Equations \( 4.28 - 4.33 \), we have

\begin{align}
\frac{4.34}{d_1} &= \frac{\ln \frac{S}{K} - \ln P(r, t, T) + \frac{1}{2\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s)s^{\alpha-1}ds}{\sqrt{\frac{1}{\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s)s^{\alpha-1}ds}}, \\
\frac{4.35}{d_2} &= d_1 - \sqrt{\frac{1}{\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s)s^{\alpha-1}ds}, \\
\hat{\sigma}^2(t) &= \sigma_s^2 + \sigma_r^2(T - t)^2, \\
P(r, t, T) &= \exp \left\{ -r\tau + \frac{\sigma_r^2}{2\Gamma(\alpha)} \int_0^\tau (T - s)^{\alpha-1}s^{2}ds \right\}.
\end{align}

which is similar with results mentioned in [5].

5. Simulation studies

Let us first discuss about the implied volatility of the subdiffusive FBM model, then we will show some simulation findings.

**Corollary 5.1.** If \( t = 0 \), the value of European call option \( \mathcal{C}(K, T) \) and put option \( \mathcal{P}(K, T) \) can be written as

\begin{align}
\mathcal{C}(K, T) &= S_0\phi(d_1) - KP_0\phi(d_2), \\
\mathcal{P}(K, T) &= KP_0\phi(-d_2) - S_0\phi(-d_1).
\end{align}
where

\[ P_0 = \exp \left\{ -r_0 T + \frac{2HT(\alpha-1)^{2H+2H+1}}{(\Gamma(\alpha))^{2H}((\alpha-1)^{2H+2H})((\alpha-1)^{2H+2H+1})} \right\} \]

\[ d_1 = \ln \frac{S_0}{K} + \frac{r_0 T + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}}, \]

\[ d_2 = d_1 - \sigma \sqrt{T}, \]

\[ \bar{P} = \frac{2HT(\alpha-1)^{2H+2H}}{(\Gamma(\alpha))^{2H}((\alpha-1)^{2H+2H})((\alpha-1)^{2H+2H+1})} \left( \mu_r - \frac{\sigma_r^2 T}{(\alpha-1)^{2H+2H+2}} \right), \]

\[ \sigma^2 = \frac{2HT(\alpha-1)^{2H+2H-1}}{(\Gamma(\alpha))^{2H}((\alpha-1)^{2H+2H})}\left( \sigma_s^2 + \frac{\rho \sigma_r \sigma_s T}{(\alpha-1)^{2H+2H+1}} \right) \]

\[ \sigma^2 T^2 \left( ((\alpha-1)^{2H+2H+1})((\alpha-1)^{2H+2H+2}) \right). \]

and \( \phi(.) \) is the cumulative normal distribution function.

Now, for an illustration of the differences among these models: the Merton, sub-diffusive Merton and our fractional Merton (FM) and subdiffusive fractional Merton (SF M) models, we report the theoretical prices of some hypothetical options using different methods. The prices computed by different models are presented in Table 1, where \( S_0 \) denotes the stock price, \( P_M \) denotes the prices computed by the Merton model, \( P_{SM} \) denotes the price simulated by the subdiffusive Merton model, \( P_{FM} \) shows the price obtained by the FM model and \( P_{SF M} \) denotes the price computed according to SFM model.

| \( S \) | \( T = 0.2 \) | \( T = 1 \) |
|-------|-------------|-------------|
| \( P_M \) | \( P_{SM} \) | \( P_{FM} \) | \( P_{SF M} \) | \( P_M \) | \( P_{SM} \) | \( P_{FM} \) | \( P_{SF M} \) |
| 2 | 0.0174 | 0.0334 | 0.0012 | 0.0036 | 1.8826 | 1.9129 | 1.7986 | 1.8347 |
| 2.25 | 0.0638 | 0.0979 | 0.0122 | 0.0236 | 2.1326 | 2.1629 | 2.0486 | 2.0847 |
| 2.5 | 0.1598 | 0.2126 | 0.0587 | 0.0859 | 2.3826 | 2.4129 | 2.2986 | 2.3347 |
| 2.75 | 0.3094 | 0.3754 | 0.1687 | 0.2094 | 2.6326 | 2.6629 | 2.5486 | 2.5847 |
| 3 | 0.5023 | 0.5752 | 0.3440 | 0.3900 | 2.8826 | 2.1929 | 2.7986 | 2.8347 |
| 3.25 | 0.7235 | 0.7988 | 0.5630 | 0.6086 | 3.1326 | 3.1629 | 3.0486 | 3.0847 |
| 3.5 | 0.9604 | 1.0360 | 0.8026 | 0.8466 | 3.3826 | 3.4129 | 3.2986 | 3.3347 |
| 3.75 | 1.2094 | 1.2801 | 1.0498 | 1.0926 | 3.6326 | 3.6629 | 3.5486 | 3.5847 |
| 4 | 1.4527 | 1.5275 | 1.2991 | 1.3414 | 3.8826 | 3.9129 | 3.7986 | 3.8347 |

By comparing columns \( P_M, P_{SM}, P_{FM} \) and \( P_{SF M} \) in Table 1 we have the conclusion that the call option prices obtained by the valuation models are close
to each other in the both in-the-money and out-of-the-money cases with low and high maturities. Meanwhile, we can see that the prices given by the our FM and SFM models are smaller than the prices given by the Merton and subdiffusive Merton models $[3, 5]$. 

**Figure 2.** The European call option under SFM. Where $r_0 = 0.1, \alpha = 0.9, H = 0.8, \sigma_r = 0.3, \sigma_s = 0.4, S_0 = 3, \mu_r = 0.2, \rho = 0.2$. 

**Figure 3.** The difference between the price of the European call option under SFM, subdiffusive Merton and Merton models. Where $r_0 = 0.1, \alpha = 0.9, H = 0.8, \sigma_r = 0.3, \sigma_s = 0.4, S_0 = 3, \mu_r = 0.2, t = 0, \rho = 0.3$. 
We give three figures of the prices of European call options in the SFM model for different parameters (see Figs. 2, 3 and 4). From Equations (5.1)-(5.7), it is easy to see that $\sigma_{im}$ is just the implied volatility of the classical BS model.

6. Conclusion

Previous option pricing research typically assumes that the risk-free rate or the short rate is constant during the life of the option. Since fractional Brownian motion is a well-developed mathematical model of strongly correlated stochastic processes, in this paper, we incorporate the fractional version of the Merton model with the subdiffusive mechanism to get better subdiffusive characteristic of financial markets. Then, we obtain pricing formula for call and put options when the short rate follows the subdiffusive fractional Merton model short rate and present some simulation results.

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