A Cantor–Bendixson dichotomy of domatic partitions

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Abstract

Let $\Gamma = \prod_{i \in \mathbb{N}} \Gamma_i$ be an infinite product of nontrivial finite groups, or let $\Gamma = (\mathbb{R}/\mathbb{Z})^n$ be a finite-dimensional torus. For a countably infinite set $S \subseteq \Gamma$, an $\aleph_0$-domatic partition is a partial $\aleph_0$-coloring on $\Gamma$ such that the $S$-neighborhood $S \cdot x$ of every vertex $x \in \Gamma$ contains at least one instance of each color. We show that an open $\aleph_0$-domatic partition exists, iff a Baire measurable $\aleph_0$-domatic partition exists, iff the topological closure of $S$ is uncountable. We also investigate domatic partitions in the general descriptive combinatorics setting.

1 Introduction

1.1 Notations

We mostly adopt standard terminology about graphs and descriptive combinatorics [see 7]. We define graphs over a vertex set $X$ to be subsets $G \subseteq X^2$, thus graphs may be directed or may contain self-loops, unless stated otherwise. We write $E_G \subseteq X^2$ to be the connectedness equivalence relation of $G$, and say a set $A \subseteq X$ is $G$-invariant if $A$ is closed under $E_G$-equivalence. We define colorings on $X$ to be any function over the domain $X$, and we call colorings with no monochromatic $G$-edges proper colorings on $G$.

With $G$ a graph on $X$, we define the neighborhood $N_G : X \to P(X)$ to be the out-neighborhood $N_G(x) = \{y : (x,y) \in G\}$. Similarly, we define degree and regularity based on out-degrees of vertices.

A set $D \subseteq X$ is dominating if it intersects every neighborhood of $G$\(^\dagger\). For $\kappa \leq \aleph_0$ a countable cardinal, a $\kappa$-domatic partition for $G$ is a sequence of $\kappa$ pairwise disjoint dominating sets. A (partial) coloring $f : X \to \kappa$ is $\kappa$-domatic at vertex $x$ if $f[N_G(x)] = \kappa$, and $f$ is $\kappa$-domatic if it’s $\kappa$-domatic everywhere. Thus $\kappa$-domatic partitions and $\kappa$-domatic colorings are interchangeable notions.

\(^\dagger\)A more usual definition requires $D$ to intersect only the neighborhoods of vertices not in $D$. This can be recovered from our definition if the graph $G$ has self-loops at every vertex, which we allow to happen.
We note here the monotonicity of domaticity. If \( G \subseteq H \) are graphs on a same vertex set \( X \), then \( G \) admitting a \( \kappa \)-domatic partition implies that \( H \) also admits a \( \kappa \)-domatic partition. Similarly if \( \kappa \leq \lambda \) are countable cardinals, then a graph \( G \) admitting a \( \lambda \)-domatic partition implies that it admits a \( \kappa \)-domatic partition. Intuitively, domaticity is positively correlated with the “richness” of the graph.

Let \( \Gamma \curvearrowright X \) be a group action and let \( S \subseteq \Gamma \) be a marked subset. The Schreier graph \( G(S) \) on \( X \) is defined via \((x, y) \in G(S) \) iff \( y \in S \cdot x \), thus \( N_{G(S)}(x) = S \cdot x \) for all \( x \in X \).

### 1.2 Overview

We are motivated by the following finitary result.

**Theorem 1.1 (Folklore).** For all \( n \in \omega \), the standard hypercube graph \( Q_n \) on \( 2^n \) vertices admits an \( n \)-domatic partition iff \( n \) is a power of two.

The subject of descriptive combinatorics has provided new views of phenomena from classical combinatorics in countless instances, for a survey see Kechris–Marks [8]. In Section 2, we will prove the main dichotomy result for hypercube-like compact Polish groups, as an infinitary analog of the previous finitary fact:

**Theorem 1.2 (Corollary 2.8).** Let \( \Gamma = \prod_{i \in \omega} \Gamma_i \) for groups \( 2 \leq |\Gamma_i| < \omega \), or let \( \Gamma = (\mathbb{R}/\mathbb{Z})^n \) for some \( 1 \leq n < \omega \). Let \( S \subseteq \Gamma \) be countable, inducing the Schreier graph \( G(S) \) on \( \Gamma \) via the left multiplication action \( \Gamma \curvearrowright \Gamma \). Exactly one of the following holds:

1. The closure \( \overline{S} \) is countable, and any Baire measurable \( \omega \)-coloring of \( G(S) \) has a comeager set of vertices with finitely-colored neighborhoods;

2. The closure \( \overline{S} \) is uncountable, and there exists an \( \omega \)-domatic partition of \( G(S) \) by open sets.

Conditioning the two cases on the countability of the closure \( \overline{S} \) is in the spirit of the original Cantor–Bendixson dichotomy [7, Theorem 6.4]. The suggested analogy to the finitary result comes from comparing the uncountable case, i.e. \( |\overline{S}| = 2^\aleph_0 \), to the power-of-2 case, and comparing the countable case, i.e. \( |\overline{S}| = \aleph_0 \) to the non-power-of-2 case.

In proving Baire measurable anti-domaticity, we use the common technique of combining generic continuity of Baire measurable colorings with the countable compactness of the acting set \( \overline{S} \).

In constructing open domaticity, we develop the notion which we call the open pair property (see Definition 2.2) among compact Polish groups, which allows us to use both compactness and a Cantor scheme argument to lift finite domaticity up to \( \omega \)-domaticity. We witness the open pair property in the selected Polish groups by both an argument using the Lovász local lemma and another argument using linear independence.

In Sections 3 and 4, we gather examples of domaticity and non-domaticity, separately in the two cases \( \kappa = \aleph_0 \) and \( \kappa < \aleph_0 \), where we’re often able to directly apply preexisting methods. Section 3
contains applications of the greedy algorithm, Lusin–Novikov uniformization, and a construction of measurable $\omega$-edge-grabbing (see Lemma 3.3). Section 4 contains applications of measurable Lovász local lemmas [1, 3], Baire measurable path decomposition [2], the Ramsey degree of $[\mathbb{N}]^\omega$ [4], and Borel determinacy constructions [9]. For an index of results see Table 1.

| Method                  | Graph                          | Borel   | Measure | Category                  | Section |
|-------------------------|-------------------------------|---------|---------|---------------------------|---------|
| Baire category          | Countable compact Schreier    | $\mathbb{N}_0$ | ?       | $\mathbb{N}_0$ & finite   | 2.1     |
| Open pair property      | $\prod_i \Gamma_i \& (\mathbb{R}/\mathbb{Z})^d$ | $\mathbb{N}_0$ open | —       | —                         | 2.2     |
| Greedy                  | Smooth                        | $\mathbb{N}_0$ | —       | —                         | 3.1     |
| Lusin–Novikov           | Function graphs               | Finite & $\mathbb{N}_0$ | —       | —                         | 3.2     |
| Edge-grabbing           | Edge coloring                | $\mathbb{N}_0$-regular | —       | Finite                    | 3.3     |
| Randomness              |                                | —       | —       | $\mathbb{N}_0$            | 4.1     |
| Lovász local lemma      | Schreier                      | Finite for subexponential growth | Finite | Finite                    | 4.2     |
| Path decomposition      | Locally finite trees          | —       | —       | Finite                    | 4.3     |
| Ramsey degree           | Reverse cofinite inclusion on $[\mathbb{N}]^\omega$ | Any $n \in \mathbb{N}$ | $\mathbb{N}_0$ | $\mathbb{N}_0$            | 4.4     |
| Determinacy             | Bernoulli shift (edge coloring) | Any $n \in \mathbb{N}$ | Finite | Finite                    | 4.5     |

a If $\Gamma = \prod_i \Gamma_i$ is an infinite product of finite groups, then the existence of all finite domatic partitions follows from the Lovász local lemma; see the proof of Theorem 2.5.

b The behavior of $\Gamma = \mathbb{R}^n$ is similar; see the remark at the end of Section 2.4.

c The behavior at $\mathbb{N}_0$ is also unknown.

d For any $1 \leq n \in \mathbb{N}$, there exists a graph $G_n$ with a Borel $n$-domatic partition, but with no Borel $(n+1)$-domatic partition. The graph is not degenerate, that is, it has measure-and-category-theoretic $\mathbb{N}_0$-domatic partitions.

Table 1: Index of results. The word “finite” refers to arbitrarily large finite cardinalities; the use of a “$<$” sign indicates a negative result; a question mark indicates unknown behavior at the cardinality $\kappa$ of interest; a dash indicates behavior not of main interest.

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2 Main result

In this section, we consider continuous actions of (typically compact) Polish groups $\Gamma$ on (typically compact) Polish spaces $X$. Let $S \subseteq \Gamma$ be a countable marked set, and we ask whether its Schreier graph $G(S)$ admits an $\omega$-domatic partition.

2.1 Countable compactness implies anti-domaticity

**Theorem 2.1.** Let a Polish group $\Gamma$ continuously act on a Polish space $X$. Let $S \subseteq \Gamma$ be countable compact, with its Schreier graph $G(S)$ on $X$. For any Baire measurable function $f : X \to \omega$, there is a comeager set of $x \in X$ for which $f[N_{G(S)}(x)]$ is finite. In particular, $f$ is not domatic at any such vertex $x$.

**Proof.** We’ll show for a comeager set $A'$ of $x \in X$, there is an open neighborhood $U \ni x$, a subset $C \subseteq U$ that is comeager in $U$, and a finite set $\Phi \subseteq \omega$, such that $f[N_{G(S)}(y)] \subseteq \Phi$ for all $y \in C$. This implies the desired by localizing.

By Kechris [1, Theorem 8.38], fix some comeager $G_\delta$ set $A \subseteq X$ such that $f \upharpoonright A$ is continuous. Since each $\gamma \in \Gamma$ acts on $X$ by a homeomorphism and hence preserves comeager-ness, the intersection $A' = \bigcap_{\gamma \in \langle S \rangle} \gamma \cdot A \subseteq A$ over the countable subgroup $\langle S \rangle$ generated by $S$ is also comeager $G_\delta$. Then $f \upharpoonright A'$ is also continuous, and $A'$ is $G(S)$-invariant.

Fix an arbitrary $x \in A'$ and pick an arbitrary $s \in S$. Since $s \cdot x$ is in the set $f^{-1}\{f(s \cdot x)\}$ which is relatively open in $A'$, and since the group action restricted to $S \times A' \to A'$ is continuous, we may use continuity to pick some open neighborhoods $Y_s \ni s$ in $\Gamma$ and $U_x \ni x$ in $X$ such that

$$(Y_s \cap S) \cdot (U_x \cap A') \subseteq f^{-1}\{f(s \cdot x)\}$$

Now $\{Y_s : s \in S\}$ forms an open cover of $S$, so by compactness pick some finite $F \subseteq S$ where $\{Y_\phi : \phi \in F\}$ is still an open cover of $S$. Let $U = \bigcap_{\phi \in F} U_\phi$ which is an open neighborhood of $x$, let $C = A' \cap U$, and let $\Phi = \{f(\phi \cdot x) : \phi \in F\}$ which is a finite subset of $\omega$. By the remark at the beginning of the proof, it suffices to see for all $t \in S$ and $y \in C$, that $f(t \cdot y) \in \Phi$. We can find some $\phi \in F$ such that $t \in Y_\phi \cap S$, and also $y \in C = U \cap A' \subseteq U_\phi \cap A'$ by choice, so by definitions of $Y_\phi, U_\phi$ we have $f(t \cdot y) = f(\phi \cdot x) \in \Phi$ as desired. \hfill $\Box$

Since nonexistence of domatic partitions can be passed to subgraphs, the same result will hold if the marked $S$ is only assumed to have countable compact closure in $\Gamma$.

2.2 Open pair property implies domaticity

**Definition 2.2.** Let $\Gamma$ be an infinite compact Polish group. We say that $\Gamma$ has the open pair property iff for every finite collection of non-empty perfect sets $P_0, \ldots, P_{k-1} \subseteq \Gamma$, there exist a pair of disjoint open sets $A_0, A_1 \subseteq \Gamma$ such that for all $i \in k$ and $\gamma \in \Gamma$, both intersections $(P_i \cdot \gamma) \cap A_0$ and $(P_i \cdot \gamma) \cap A_1$ are non-empty.
We’ll use this last definition in two ways: In this subsection we’ll use the fact that the function $f : \Gamma \to 2$ defined via $f \upharpoonright A_i : x \mapsto i$ is indeed domatic with topological regulations, which will nicely lead to $\omega$-domaticity. In the next subsection we’ll think about right translations of perfect sets, in order to show that hypercube-like groups have the open pair property.

**Lemma 2.3.** Let $\Gamma$ be a compact Polish group with the open pair property, and let $P_0, \ldots, P_{k-1}$ and $A_0, A_1$ be any fixed sequence witnessing its open pair property. Then there are finite sets $F_0, \ldots, F_{k-1}$ such that: Each finite set $F_i$ contains pairs of the form $(x, U)$ with $x \in P_i$ and $U \ni x$ an open neighborhood; for all $i \in k$ and $\gamma \in \Gamma$, there exist $(x, U), (y, V) \in F_i$ such that $U \cdot \gamma \subseteq A_0$, $V \cdot \gamma \subseteq A_1$.

**Proof.** Fix a compatible metric $d$ with $\Gamma$’s compact Polish topology. For any $x \in \Gamma$ and $r > 0$, notate the open ball $B(x, r) = \{y \in \Gamma : d(x, y) < r\}$. For any point $x \in \Gamma$ and non-empty compact subset $K \subseteq \Gamma$, notate the distance $d(x, K) = \min\{d(x, y) : y \in K\}$.

Fix some $i \in k$ and $j \in 2$ throughout, and we will aim to construct a finite set $F_i^j$ such that for all $\gamma \in \Gamma$ there is $(x, U) \in F_i^j$ with $x \in P_i$ and $U \ni x$ an open neighborhood such that $U \cdot \gamma \subseteq A_j$. By the open pair property, the collection $\{x^{-1} \cdot A_j : x \in P_i\}$ is an open cover of $\Gamma$, so by compactness we can fix some finite subset $E_i \subseteq P_i$ such that $\{x^{-1} \cdot A_j : x \in E_i\}$ is a finite open subcover of $\Gamma$.

Define $f : \Gamma \to [0, \infty)$ via

$$f(\gamma) = \max_{x \in E_i} d(x \cdot \gamma, \Gamma \setminus A_j)$$

Note that for all $\gamma \in \Gamma$, by construction of $E_i$ there exists some $z \in E_i$ with $z \cdot \gamma \in A_j$, so then openness of $A_j$ implies $f(\gamma) \geq d(z \cdot \gamma, \Gamma \setminus A_j) > 0$ is strictly positive. Note also for all $x \in E_i$ that $\gamma \mapsto d(x \cdot \gamma, \Gamma \setminus A_j)$ is continuous, so $f$ as a maximum of $|E_i| < \omega$ many continuous functions is also continuous. By the compactness of $\Gamma$, the range $f[\Gamma]$ of $f$ is a compact subset of $(0, \infty)$, so we fix some $r > 0$ such that $f(\gamma) > r$, for all $\gamma \in \Gamma$.

By the Heine–Cantor theorem, the group operation $\Gamma \times \Gamma \to \Gamma$ being a continuous function over a compact domain implies that it is uniformly continuous. We may then fix some $q > 0$ such that for all $w, x, y, z \in \Gamma$, $d(w, x) < q$ and $d(y, z) < q$ implies $d(wy, xz) < r$.

We put $F_i^j = \{(x, B(x, q)) : x \in E_i\}$, and claim for all $\gamma \in \Gamma$ that there is some $(x, U) \in F_i^j$ such that $U \cdot \gamma \subseteq A_j$. Since $f(\gamma) > r$, there is some $x \in E_i$ such that $B(x \cdot \gamma, r) \subseteq A_j$. The construction of $q$ implies $B(x \cdot \gamma, r) \subseteq B(x \cdot \gamma, r) \subseteq A_j$, so $(x, B(x, q)) \in F_i^j$ meets the claim of the paragraph.

Finally put $F_i = F_i^0 \cup F_i^1$, where the last paragraph’s claim implies the lemma. $\square$

**Theorem 2.4.** Let $\Gamma$ be a compact Polish group with the open pair property. Let $S_0, \ldots, S_{\ell-1} \subseteq \Gamma$ be finitely many sets such that for all $k \in \ell$, $S_k$ has uncountable closure in $\Gamma$. Then there are pairwise-disjoint open sets $\{D_n : 1 \leq n < \omega\}$ in $\Gamma$, such that for all $k \in \ell$, $x \in \Gamma$, and $1 \leq n < \omega$, there exists $\gamma \in S_k$ such that $\gamma \cdot x \in D_n$. Consequently if $S_k$ is countable, then $\{D_n : 1 \leq n < \omega\}$ forms an open $\omega$-domatic partition for $G(S_k)$, the Schreier graph on $\Gamma$ induced by the left multiplication $\Gamma \curvearrowright \Gamma$.  

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Then the group \( \Gamma \) is uncountable, so by the Cantor–Bendixson theorem and an argument similar to Kechris \(^7\), we may fix a Cantor scheme \( \{ U^k_s : s \in 2^{<\omega} \} \) of open subsets \( U^k_s \subseteq \Gamma \) such that: For all \( s \in 2^{<\omega} \), \( U^k_{s+1} \subseteq U^k_s \) are disjoint; for all \( \sigma \in 2^{<\omega} \) there exists a (unique) \( p^\sigma_k \in \mathbb{S}_k \) such that \( \{ U^k_{n+1} : n \in \omega \} \) is a neighborhood basis of \( p^\sigma_k \). We fix the resulting homeomorphism \( f^k : 2^{<\omega} \to P^k \subseteq \mathbb{S}_k \) via \( f^k(\sigma) = p^\sigma_k \).

We will inductively on \( n \in \omega \) define a pair of disjoint open sets \( A^0_n, A^1_n \subseteq \Gamma \) and a finite “level” set \( L_n \subseteq 2^{<\omega} \) such that: We put \( L_0 = \{ \varnothing \} \) and \( A^0_0, A^1_0 \) undefined; for all \( n \in \omega \), \( s \in L_n \), \( \gamma \in \Gamma \), and \( k \in \ell \), there exist \( t_0, t_1 \in L_{n+1} \) such that \( t_0, t_1 \supseteq s \), and \( U^k_{t_0} \gamma \subseteq A^0_{n+1}, U^k_{t_1} \gamma \subseteq A^1_{n+1} \).

Assume we’ve constructed the finite set \( L_n \subseteq 2^{<\omega} \), and we want to construct next \( L_{n+1}, A^0_{n+1}, A^1_{n+1} \).

For each \( s \in L_n \) and \( k \in \ell \), fix the non-empty perfect set \( P^k_s = f^k[N_s] = P^k \cap U^k_s \). We fix \( A^0_{n+1}, A^1_{n+1} \) according to the open pair property of \( \Gamma \) applied to the finite collection of perfect sets \( \{ P^k_s : s \in L_n, k \in \ell \} \). By the last lemma, there are finite sets \( \{ F^k_s : s \in L_n, k \in \ell \} \) such that for all \( s \in L_n, k \in \ell, \gamma \in \Gamma \), and \( j \in 2 \), there is \( (y, V) \in F^k_s \) such that \( y \in P^k_s \), \( V \supseteq y \) open, and \( V \cdot \gamma \subseteq A^j_{n+1} \). Since every \( (y, V) \in F^k_s \) satisfies \( y \in P^k_s \subseteq P^k \) and \( V, U^k_s \supseteq y \) are open neighborhoods, we may use the neighborhood basis condition of the Cantor scheme to fix some \( t = t^k_s(y, V) \in 2^{<\omega} \) such that \( y \in U^k_t \subseteq V \cap U^k_s \) is a smaller open neighborhood. We put

\[
L_{n+1} = \bigcup_{s \in L_n \cap k \in \ell} \left\{ t^k_s(y, V) : (y, V) \in F^k_s \right\} \subseteq 2^{<\omega}
\]

We’ll show \( L_{n+1}, A^0_{n+1}, A^1_{n+1} \) are as desired before. The \( L_{n+1} \subseteq 2^{<\omega} \) is a finite set since \( L_n, \ell \) and each \( F^k_s \) are finite. For any \( \gamma \in \Gamma \), \( s \in L_n \), \( k \in \ell \), and \( j \in 2 \), the last lemma gives some \( (y, V) \in F^k_s \) with \( V \cdot \gamma \subseteq A^j_{n+1} \). So \( t = t^k_s(y, V) \in L_{n+1} \) is such that \( t \supseteq s \) by \( U^k_t \subseteq U^k_s \), and \( U^k_t \gamma \subseteq V \cdot \gamma \subseteq A^j_{n+1} \). Thus the inductive construction is complete.

For each \( 1 \leq n < \omega \) put \( D_n = A^0_n \cap \bigcap_{0 < m < n} A^0_m \) an open set. All \( D_n \) are pairwise-disjoint since every pair \( A^0_m, A^1_m \) is disjoint. For any \( k \in \ell, x \in \Gamma \), and \( 1 \leq n < \omega \), we can inductively find an increasing sequence \( s_0 \subseteq \ldots \subseteq s_n \) in \( 2^{<\omega} \) such that: \( s_0 = \varnothing \subseteq L_0 \); for all \( 0 < m < n, s_m \in L_m \), and \( U^k_{s_m} x \subseteq A^0_m, s_n \in L_n \), and \( U^k_{s_n} x \subseteq A^1_n \). Since \( U^k_0 \supseteq \ldots \supseteq U^k_{s_n} \), we see that \( U^k_{s_n} x \subseteq D_n \). Since \( U^k_{s_n} \cap P^k \neq \varnothing \) is non-empty and \( P^k \) is a set of closure points of \( S_k \), there is some \( \gamma \in S_k \) such that \( \gamma \in U^k_{s_n} \). Thus \( \gamma \cdot x \in U^k_{s_n} \cap D_n \), completing the proof.

For more discussion on colorings which are simultaneously domatic for multiple graphs, see the remarks at the end of Section 4.2.

### 2.3 Open pair property in hypercube-like groups

**Theorem 2.5.** Let \( \{ \Gamma_i : i < \omega \} \) be a sequence of finite groups such that \( 2 \leq |\Gamma_i| < \omega \) for all \( i < \omega \). Then the group \( \Gamma = \prod_{i < \omega} \Gamma_i \) has the open pair property.

**Proof.** Fix some arbitrary finite collection of non-empty perfect sets \( P_0, \ldots, P_{k-1} \subseteq \Gamma \). Let \( n \gg k \) be a large enough natural number such that \( 2ek[(kn)^2 + 1] < 2^n \). Inductively, from each set
Proof. We say a triplet of points $q_j^0, \ldots, q_j^{n-1} \in P_j$ such that the $kn$-many points $\{q_j^m : j \in k, m \in n\}$ are all distinct.

Fix $\ell < \omega$ such that the elements $\{q_j^m \upharpoonright \ell : j \in k, m \in n\}$ are still all distinct, by using finiteness of $k, n$. Let $\Gamma' = \prod_{i < \ell} \Gamma_i$ a finite group, and let $\pi : \Gamma \to \Gamma'$ be the canonical projection.

We'll next apply the Lovász local lemma on uniformly random colorings $f : \Gamma' \to 2$. For each $x \in \Gamma'$, we'll define the bad event $B_x$ over the domain

$$\text{dom}(B_x) = \{ \pi(q_j^m) \cdot x : j \in k, m \in n \} \subseteq \Gamma'$$

We say that a function $f : \Gamma' \to 2$ falls into the bad event $B_x$ whenever for some $j \in k$, the set $\{ \pi(q_j^m) \cdot x : m \in n \} \subseteq \text{dom}(B_x)$ is $f$-monochromatic.

It's not hard to see that the maximum degree of dependency of an event $B_x$ is $\leq (kn)^2$, and that the probability of any event $B_x$ is $\leq k \cdot 2^{-n}$. By the construction of $n$, the condition in the Lovász local lemma is satisfied, so we get a coloring $f : \Gamma' \to 2$ which avoids falling into any bad event $B_x$.

We claim that the pair of disjoint sets $A_i = (f \circ \pi)^{-1} \{i\} \subseteq \Gamma$ for $i \in 2$ witnesses the open pair property of $\Gamma$. Both $A_0, A_1$ are open since $f \circ \pi : \Gamma \to 2$ is continuous. Let $\gamma \in \Gamma$ and $j \in k$ be arbitrary, and let $x = \pi(\gamma) \in \Gamma'$. Since $f$ does not fall into the bad event $B_x$, the set $\{ \pi(q_j^m) \cdot x : m \in n \} \subseteq \pi[P_j \cdot \gamma]$ is not $f$-monochromatic, so we get that $f \circ \pi[P_j \cdot \gamma] = 2$ achieves full range, and thus $P_j \cdot \gamma$ intersects both $A_0, A_1$ as desired. \qed

**Lemma 2.6.** The group $\Gamma = \mathbb{R}/\mathbb{Z}$ has the open pair property.

**Proof.** We say a triplet of points $x, y, z \in \Gamma$ is in **acute position** if in some order, they partition the circumference-1 circle $\Gamma$ into 3 arcs of length $< 1/2$ each. Thus they form the vertices of an inscribed acute triangle in the circle $\Gamma$. We put $U \subseteq \Gamma^3$ to be the set of all ordered triples in acute position, so that $U$ is non-empty open.

Fix some finite collection $P_0, \ldots, P_{k-1} \subseteq \Gamma$ of non-empty perfect sets. Since each $P_i$ is uncountable, we can inductively on $i \in k$ fix elements $y_i^0, y_i^1, y_i^2 \in P_i$ such that the $3k$-many points $\{y_i^j \in \Gamma : i < k, j < 3\}$ together with 1 are linearly independent over $\mathbb{Q}$. More precisely, these $3k$ points admit no nontrivial integer relations in $\Gamma$.

Let $p = (y_0^0, y_0^1, \ldots, y_{k-1}^2) \in \Gamma^{3k}$ be the tuple formed by all points $y_i^j \in \Gamma$. Then Kronecker’s theorem on Diophantine approximation [5, Theorem 443] states that, since all coordinates of $p$ with 1 are linearly independent over $\mathbb{Q}$, the cyclic subgroup $\langle p \rangle = \{ z \cdot p : z \in \mathbb{Z} \}$ is dense in $\Gamma^{3k}$. In particular, it intersects the non-empty open set $U^k \subseteq \Gamma^{3k}$. Thus we get some integer $n \in \mathbb{Z}$ such that for all $i \in k$, the triplet $ny_i^0, ny_i^1, ny_i^2 \in \Gamma$ is in acute position.

Fix $f : \Gamma \to \Gamma$ via $f(\gamma) = n \cdot \gamma$. Put $A_0 = f^{-1}[\{0, 1/2\}]$ and $A_1 = f^{-1}[\{1/2, 1\}]$ a pair of disjoint open sets in $\Gamma$. For all $\gamma \in \Gamma$ and $i \in k$, the points $f(y_i^0 + \gamma), f(y_i^1 + \gamma), f(y_i^2 + \gamma)$ are in acute position, so a translation by $f(\gamma)$ sending them to $f(y_i^0 + \gamma), f(y_i^1 + \gamma), f(y_i^2 + \gamma)$ preserves their acute-ness.

As every inscribed acute triangle in a circle $\Gamma$ has vertices intersecting both open semicircular arcs $(0, 1/2), (1/2, 1)$, there are $a, b \in 3$ such that $f(y_i^a + \gamma) \in (0, 1/2), f(y_i^b + \gamma) \in (1/2, 1)$. Consequently $y_i^a, y_i^b \in P_i$ are such that $y_i^a + \gamma \in A_0, y_i^b + \gamma \in A_1$, witnessing the open pair property of $\Gamma$ at $\gamma \in \Gamma$. \qed
Theorem 2.7. For all $1 \leq n < \omega$, the group $\Gamma = (\mathbb{R}/\mathbb{Z})^n$ has the open pair property.

Proof. For any vector $v \in \mathbb{Z}^n$, define $f_v : \Gamma \to \mathbb{R}/\mathbb{Z}$ via $f_v(x) = v \cdot x = \sum_{m \in \mathbb{N}} v_m x_m$. For a non-empty perfect set $P \subseteq \Gamma$, we say $v \in \mathbb{Z}^n$ is $P$-bad if the image $f_v[P] \subseteq \mathbb{R}/\mathbb{Z}$ is countable. Otherwise say $v \in \mathbb{Z}^n$ is $P$-good if $f_v[P]$ is uncountable.

Take any $v_0, \ldots, v_{n-1} \in \mathbb{Z}^n$ which form a $\mathbb{Q}$-linear basis of $\mathbb{Q}^n$, and put $g : \Gamma \to \Gamma$ via $g(x) = (f_{v_0}(x), \ldots, f_{v_{n-1}}(x))$. We claim that $g$ is countable-to-one. Put $A \in \mathbb{Z}^{n \times n}$ to be the matrix where the $i$-th row vector of $A$ is $v_i$, so $A$ is invertible. If $x \in \mathbb{R}^n$ is such that $x + \mathbb{Z}^n \in \ker(g)$, then we must have $A[x + \mathbb{Z}^n] \subseteq \mathbb{Z}^n$ or $x \in A^{-1}[\mathbb{Z}^n]$, which is satisfied by only countably many $x$, by invertibility of $A$.

With $v_0, \ldots, v_{n-1} \in \mathbb{Z}^n$ as above, if moreover $P \subseteq \Gamma$ is non-empty perfect, then $g[P] \subseteq \Gamma$ is also uncountable. Also $g[P] \subseteq \prod_{i \in \mathbb{N}} f_{v_i}[P]$ by definition of $g$, so there exists some $i \in \mathbb{N}$ such that $f_{v_i}[P]$ is uncountable, so that $v_i$ is $P$-good.

Fix any arbitrary finite collection $P_0, \ldots, P_{k-1} \subseteq \Gamma$ of non-empty perfect sets. For all $j \in \omega$ fix the vector $b_j = (1, j, j^2, \ldots, j^{n-1}) \in \mathbb{Z}^n$. Note for all distinct $j_0 < j_1 < \cdots < j_{n-1}$, the Vandermonde determinant implies the vectors $b_{j_0}, \ldots, b_{j_{n-1}}$ are $\mathbb{Q}$-linearly independent. For any $i \in \mathbb{k}$, the last paragraph implies that every set of $n$-many distinct $j \in \omega$ contains some $j'$ such that $b_{j'}$ is $P_i$-good, so there can be at most $n-1$ many $j \in \omega$ such that $b_j$ is $P_i$-bad.

Then there are at most $(n-1)k$ many $j \in \omega$ that is $P_i$-bad for some $i \in \mathbb{k}$, and we may fix $r \in \omega$ such that $b_r$ is $P_i$-good for all $i \in \mathbb{k}$. Let $f = f_{b_r} : \Gamma \to \mathbb{R}/\mathbb{Z}$.

For all $i \in \mathbb{k}$, $f[P_i] \subseteq \mathbb{R}/\mathbb{Z}$ is uncountable compact which contains a non-empty perfect subset, so by the last lemma on the open pair property of $\mathbb{R}/\mathbb{Z}$, there are disjoint open sets $E_0, E_1 \subseteq \mathbb{R}/\mathbb{Z}$ witnessing the open pair property for $\{f[P_i] : i \in \mathbb{k}\}$. Put $A_0 = f^{-1}[E_0]$ and $A_1 = f^{-1}[E_1]$. Then for all $i \in \mathbb{k}$, $\gamma \in \Gamma$, and $j \in 2$, $(f[P_i] + f(\gamma)) \cap E_j \neq \emptyset$ implies that $(P_i + \gamma) \cap A_j \neq \emptyset$, witnessing the open pair property of $\Gamma$.

\[\square\]

2.4 Main theorem

Combining previous theorems and remarks, we arrive at the main theorem of this paper.

Corollary 2.8. Let $\Gamma = \prod_{i \in \omega} \Gamma_i$ for groups $2 \leq |\Gamma_i| < \omega$, or let $\Gamma = (\mathbb{R}/\mathbb{Z})^n$ for some $1 \leq n < \omega$. Let $S \subseteq \Gamma$ be countable, inducing the Schreier graph $G(S)$ on $\Gamma$ via the left multiplication action $\Gamma \curvearrowright \Gamma$. Exactly one of the following holds:

1) The closure $\overline{S}$ is countable, and any Baire measurable $\omega$-coloring of $G(S)$ has a comeager set of vertices with finitely-colored neighborhoods;

2) The closure $\overline{S}$ is uncountable, and there exists an $\omega$-domatic partition of $G(S)$ by open sets.

This leaves open the question of further completing the dichotomy phenomenon. For example, can anything be said of case (1) if the coloring were $\mu$-measurable for some Borel probability measure $\mu$? Can anything be said of the open pair property among general compact Polish groups?
Although not much is known in general about non-compact Polish groups, the analysis on $\mathbb{R}^n$ can mostly follow from the analysis on $(\mathbb{R}/\mathbb{Z})^n$. For a countably infinite set $S \subseteq \mathbb{R}^n$, the same of $\omega$-domaticity for $G(S)$ holds in $\mathbb{R}^n$ as that in $(\mathbb{R}/\mathbb{Z})^n$, in the cases when $S$ has either countable compact closure or uncountable closure. When $S$ is unbounded, we may use diagonalization to get a rapidly increasing sequence of positive radii $0 = R_0 < R_1 < \ldots$ such that any shift of $S$ eventually intersects with every far enough spherical shell between these radii. Then an infinitely redundant $\omega$-coloring of the spherical shells will form an open $\omega$-domatic partition for $G(S)$. We may conclude that for $S \subseteq \mathbb{R}^n$, an open $\omega$-domatic partition exists, iff a Baire measurable $\omega$-domatic partition exists, iff the topological closure of $S$ is not countable compact.

3 More examples of infinite domaticity

As mentioned in the introduction, the following examples of $\omega$-domaticity typically follow from a greedy algorithm and/or Lusin–Novikov uniformization [7, Theorem 18.10].

3.1 Smooth graphs via the greedy algorithm

Let $G \subseteq X^2$ be a Borel graph such that $E_G \subseteq X^2$ is a Borel equivalence relation. We say $E_G$ is smooth iff there is some Borel map $f : X \rightarrow Y$ between standard Borel spaces such that for all $x, y \in X$, $(x, y) \in E_G$ iff $f(x) = f(y)$. If moreover $E_G$ is locally countable, then it follows from Lusin–Novikov uniformization applied to the inverse $f^{-1} \subseteq Y \times X$ that the equivalence $E_G$ admits a Borel transversal set.

**Theorem 3.1.** Let $G \subseteq X^2$ be an $\omega$-regular graph such that $E_G$ is smooth. Then $G$ admits a Borel $\omega$-domatic partition.

**Proof.** Fix a Borel transversal set $B \subseteq X$ of $E_G$. By the Lusin–Novikov theorem partitioning $E_G \cap (B \times X)$ into $\omega$ many uniformizing Borel partial functions, we can fix a Borel function $\nu : X \rightarrow \omega$ which is injective on each single $E_G$-class.

We will build a sequence $\{A_n : n \in \omega\}$ of Borel sets $A_n \subseteq X$ such that: The sets $A_n$ are pairwise-disjoint; for all $n \in \omega$, $\nu^{-1}\{n\} \subseteq A_0 \cup \ldots \cup A_n$; for all $n \in \omega$, the set $A_n$ intersects each $E_G$-class finitely many times; for all $n \in \omega$ and $m < n$, every point $x \in A_m$ has its neighborhood $N_G(x)$ intersecting $A_n$.

Pick $A_0 = \nu^{-1}\{0\}$, and we inductively from $A_0, \ldots, A_n$ construct $A_{n+1}$. Put $E_n = A_0 \cup \ldots \cup A_n$. By that $G$ is $\omega$-regular, and that $E_n$ intersects each $E_G$-class finitely many times by induction, each non-empty vertical section of $G \cap (E_n \times (X \setminus E_n))$ has size $\omega$. By the Lusin–Novikov theorem, we can fix a uniformizing Borel function $f : E_n \rightarrow X \setminus E_n$ such that $f \subseteq G$ as subsets of $X^2$. The function $f$ is finite-to-one since each $f^{-1}\{x\} \subseteq [x]_{E_G} \cap E_n$ is finite, so we recall the fact that every finite-to-one Borel image of a Borel set is Borel, giving us the Borel set $C_n = f[E_n] \subseteq X \setminus E_n$.

We put $A_{n+1} = C_n \cup (\nu^{-1}\{n+1\} \setminus E_n)$ which is Borel, completing the construction. We’ll next show that the construction matches with the claim before. We easily see that each
A_{n+1} is disjoint from \( E_n = A_0 \cup \ldots \cup A_n \) by choice. It’s also by choice that \( \nu^{-1}\{n+1\} \subseteq A_0 \cup \ldots \cup A_{n+1} \). Next, each \( C_n \) intersects every \( E_G \)-class \( [x]_{E_G} \) at most \( |E_n \cap [x]_{E_G}| < \omega \) many times, and \( \nu^{-1}\{n+1\} \) intersects every \( E_G \)-class at most once, so \( A_{n+1} \) also intersects every \( E_G \)-class finitely many times. Finally for all \( m < n+1 \) and \( x \in A_m \subseteq E_n \), we know that \( f(x) \in N_G(x) \cap C_n \) by choice, so \( N_G(x) \) intersects \( A_{n+1} \supseteq C_n \).

Fix a surjection \( \pi : \omega \to \omega \) where every \( i \in \omega \) has \( |\pi^{-1}\{i\}| = \omega \). We put \( D_i = \bigcup_{n \in \omega^{-1}\{i\}} A_n \) for all \( i \in \omega \), and claim that \( \{D_i : i \in \omega \} \) is a Borel \( \omega \)-domatic partition of \( G \). Clearly all \( D_i \) are Borel and pairwise-disjoint, by pairwise-disjointness of \( A_n \). Fix some arbitrary \( x \in X \) and \( i \in \omega \). Since \( x \in A_0 \cup \ldots \cup A_{\nu(x)} \), we see that \( N_G(x) \) intersects every \( A_n \) with \( n > \nu(x) \).

By the infinitude of \( \pi^{-1}\{i\} \), there is some \( n > \nu(x) \) with \( \pi(n) = i \), so \( N_G(x) \) also intersects \( D_i \supseteq A_n \). This proves the domaticity of \( \{D_i : i \in \omega \} \). \( \square \)

### 3.2 Function graphs via Lusin–Novikov

Let \( f : X \to X \) be a Borel function on a standard Borel space \( X \). We define the inverse function graph \( f^{-1} \subseteq X^2 \) on \( X \) via \( (x,y) \in f^{-1} \) iff \( x = f(y) \). Thus every \( N_{f^{-1}}(x) = f^{-1}\{x\} \).

**Theorem 3.2.** Let \( k \leq \omega \), and let \( f : X \to X \) be a Borel function such that \( \forall x \in X, k \leq |f^{-1}\{x\}| \leq \omega \). Then \( f^{-1} \) has a Borel \( k \)-domatic partition.

**Proof.** By the Lusin–Novikov theorem in the form of Kechris [7, Exercise 18.15] applied to \( f^{-1} \), we fix some uniformizing Borel partial functions \( g_n : X \to X \) for \( n \in \omega \) such that for all \( x \in X \), \( g_n(x) \in f^{-1}\{x\} \), and \( \{g_n(x) : n < |f^{-1}\{x\}|\} \) forms an enumeration of the set \( f^{-1}\{x\} \).

We define a function \( d : X \to \omega \) via requiring for all \( x \in X \), \( x = g_{d(x)}(f(x)) \). The function \( d \) is Borel since for all \( n \in \omega \), \( d^{-1}\{n\} \subseteq X \) is Borel. We claim that \( d \upharpoonright d^{-1}\{k\} \) is a \( k \)-domatic partial coloring for \( G = f^{-1} \). For all \( x \in X \) and \( i \in k \), we have \( y = g_i(x) \in f^{-1}\{x\} = N_G(x) \), so \( d(y) = i \) is a witness of domaticity of \( d \upharpoonright d^{-1}\{k\} \) at \( x \). \( \square \)

Note that since domaticity can be passed to supergraphs, the coloring is also \( k \)-domatic for the undirected function graph \( G_f^{\pm 1} = f \cup f^{-1} \subseteq X^2 \).

### 3.3 Edge colorings via greedy edge-grabbing

In this subsection, all graphs are assumed to be undirected with no self-loops.

For an undirected \( \omega \)-regular graph \( G \subseteq [X]^2 \) whose edges are represented by unordered pairs \( \{x,y\} \subseteq X \), we say that an \( \omega \)-edge-coloring \( f : G \to \omega \) is \( \omega \)-edge-domatic, iff every vertex \( x \in X \) is incident to edges of every color. In relation to vertex colorings, we can form the bipartite incidence graph \( G_\epsilon \subseteq (X \sqcup G)^2 \) where \( (x,e) \in G_\epsilon \) iff \( x \in e \), so then the \( \omega \)-coloring \( f : G \to \omega \) is \( \omega \)-edge-domatic for \( G \) iff \( f \) is \( \omega \)-domatic at \( X \) for \( G_\epsilon \).

We will construct measurable \( \omega \)-edge-domatic colorings by finding orientations of the \( \omega \)-regular graph \( G \) that preserve \( \omega \)-regularity almost everywhere.
Lemma 3.3. Let $G \subseteq X^2$ be a Borel $\omega$-regular graph on a Borel probability space $(X, \mu)$. Then there exists a Borel orientation $\eta: G \to X$ of $G$ such that: For all $(x,y) \in G$, $\eta(x,y) = \eta(y,x) \in \{x,y\}$; for $\mu$-almost every $x \in X$, there exist infinitely many $y \in N_G(x)$ such that $\eta(x,y) = x$.

Proof. By Kechris–Solecki–Todorcevic [6, Proposition 4.10], we may fix a Borel proper edge-coloring $f: G \to \omega$ of $G$ such that $f(x,y) = f(y,x)$ and every two edges sharing a common vertex receive different colors.

Note that every $x \in X$ is incident to edges of infinitely many colors. For $n \in \omega$, we let $A_n$ be the Borel set of vertices $x \in X$ incident to some edge of color $n$. For all $m \in \omega$ fixed, $\bigcup_{n \geq m} A_n = X$ implies for all $\varepsilon > 0$ that there exists some $\sigma(m, \varepsilon) \in \omega$ such that $\mu(\bigcup_{m \leq n < \sigma(m, \varepsilon)} A_n) > 1 - \varepsilon$. Fix any such function $\sigma : \omega \times (0,1) \to \omega$.

Inductively on $i \in \omega$, fix some natural number $n_i \in \omega$ such that: $n_0 = 0$; for all $i \in \omega$, we pick $n_{i+1} = \sigma(n_i, 2^{-i-1}) > n_i$. Next, fix the subgraph $G_i = f^{-1}[n_i, n_{i+1}] \subseteq X^2$ of $G$, and we get that: For all $x \in X$, the degree of $x$ in $G_i$ is at most $n_{i+1} - n_i < \omega$; if for $i \in \omega$ we set $M_i = \bigcup_{m \leq n < n_{i+1}} A_n$, then $\mu(M_i) > 1 - 2^{-i-1}$, and every $x \in M_i$ has positive $G_i$-degree.

By Kechris–Solecki–Todorcevic [6, Proposition 4.2], the locally finite Borel graph $G_i$ admits a Borel maximal independent set $B_{i}^0 \subseteq X$. Let $B_{i}^1 = X \setminus B_i^0$, giving a bipartition $X = B_{i}^0 \sqcup B_{i}^1$. Then for all $j \geq 2$ and $x \in B_{i}^j \cap M_i$, since $x$ is of positive $G_i$-degree and since $B_{i}^j$ is maximal $G_i$-independent, it holds that $B_{i}^{j-1} \cap N_{G_i}(x) \neq \emptyset$.

We fix any surjection $p : \omega \to \omega$ such that for all $i \in \omega$, $p(i) \leq i$ and $|p^{-1}\{i\}| = \omega$.

Now we give an informal description of the combinatorics that is about to take place. The underlying space for the combinatorics will be $X \times \omega$. For every $i \in \omega$, we will orient all bipartite $G_i$-edges between $B_{i}^0$ and $B_{i}^1$ to point towards one of the two parts, call it $C_i \in \{B_{i}^0, B_{i}^1\}$. Thus all points in $C_i \cap M_i$ are incident to a correctly oriented $G_i$-edge, which we mark with the subspace $(C_i \cap M_i) \times [0, i] \subseteq X \times \omega$. In order to arrive at a correct orientation $\eta$, we want all such marked subspaces to cover a $\mu$-conull set of $X \times \omega$, due to Fubini’s theorem. Thus at each individual level $X \times \{i\}$ we would inductively cover a big enough portion of the uncovered space by tactically choosing $C_m$, for all $m$ such that $p(m) = i$. By the $\omega$-to-one redundancy of the function $p$ we will succeed.

To formalize the setting, we will inductively on $i \in \omega$ fix a Borel set $C_i \subseteq X$ such that: For all $i \in \omega$, either $C_i = B_{i}^0$ or $C_i = B_{i}^1$; for all $i \in \omega$, $C_i$ occupies at least half the measure of $D_i = X \setminus \bigcup_{p(i) \leq n < i} (C_n \cap M_n)$. To attain this, observe that every $D_i$ is relatively bipartitioned by $B_{i}^0 \sqcup B_{i}^1$, so for some $j \geq 2$ there is $\mu(B_{i}^j \cap D_i) \geq \frac{1}{2} \mu(D_i)$ and we set $C_i = B_{i}^j$.

We claim that for all $i \in \omega$ fixed, the set $\bigcup_{n \geq i} (C_n \cap M_n)$ is $\mu$-conull. Fix an increasing enumeration $\{m_k : k \in \omega\}$ of $p^{-1}\{i\}$, and for all $m \in \omega$ put $E_m = \bigcup_{i \leq n < m} (C_n \cap M_n)$. We see that $\mu(E_m)$ is non-decreasing with $m$. For all $k \in \omega$, $D_{m_k} = X \setminus E_{m_k}$, and $C_{m_k}$ takes up at least half the measure of $D_{m_k}$, so

$$
\mu(E_{m_{k+1}}) \geq \mu(E_{m_k+1}) = \mu(E_{m_k}) + \mu((C_{m_k} \cap M_{m_k}) \setminus E_{m_k}) \\
= \mu(E_{m_k}) + \mu(C_{m_k} \cap M_{m_k} \cap D_{m_k}) \\
\geq \mu(E_{m_k}) + \mu(C_{m_k} \cap D_{m_k}) - 2^{-m_k-1}
$$
Theorem 3.4. Let $G \subseteq X^2$ be a Borel $\omega$-regular graph on a Borel probability space $(X, \mu)$. Then there is a Borel $\omega$-edge-coloring of $G$ such that $\mu$-almost every $x \in X$ is incident to all colors in $\omega$.

Proof. Fix the Borel orientation $\eta : G \to X$ and its corresponding $\mu$-conull set $Y \subseteq X$ of witnesses of $\omega$-regularity, both given by the previous lemma.

Following the remarks at the beginning of this section, we may view $G$ as a set $G \subseteq [X]^2$ of pairs $\{x, y\} \subseteq X$, view $\eta$ as a function over this subset $G \subseteq [X]^2$, and then consider the incidence graph $G_\ominus \subseteq (X \sqcap G)^2$. By repeating the proof of Theorem 3.2 viewing $G_\ominus$ as a supergraph of an inverse function $G_\ominus \supseteq \eta^{-1}$, we attain a Borel coloring $d : G \to \omega$ that is $\omega$-domatic at $Y$ for $G_\ominus$. This $\omega$-edge-coloring $d$ on $G$ is as desired.

Theorem 3.5. Let $G \subseteq X^2$ be a Borel $\omega$-regular graph on a Polish space $X$. Then there exist:

1. A Borel orientation $\eta : G \to X$ such that $\eta(x, y) = \eta(y, x) \in \{x, y\}$, and for a comeager set of $x \in X$, there exist infinitely many $y \in N_G(x)$ such that $\eta(x, y) = x$;

2. A Borel $\omega$-edge-coloring of $G$ such that a comeager set of $x \in X$ are incident to edges of all colors in $\omega$.

Proof. The proof closely follows the proofs of the previous lemma and theorem. For sake of conciseness, we only point out the necessary modifications to be made.

Fix an enumeration $\{U_k : k \in \omega \}$ of a countable topological base on $X$. 

\[
\begin{align*}
\geq \mu(E_{m_k}) + \frac{1}{2}\mu(D_{m_k}) - 2^{-m_k-1} \\
= \mu(E_{m_k}) + \frac{1}{2}(1 - \mu(E_{m_k})) - 2^{-m_k-1} \\
= \frac{1}{2}(1 + \mu(E_{m_k}) - 2^{-m_k}) \\
\geq \frac{1}{2}(1 + \mu(E_{m_k}) - 2^{-k})
\end{align*}
\]
In picking the sequence \( \{(G_i, M_i) : i \in \omega \} \), we arrange so that every set \( M_i \) is relatively non-meager in each of the open sets \( U_0, \ldots, U_i \). To do so, we first use \( \omega \)-regularity as we did before, to fix a function \( \tau : \omega \times \omega \to \omega \) such that for all \( m, k \in \omega \), the set \( \bigcup_{m \leq n < \tau(m,k)} A_n \) is relatively non-meager in \( U_k \). Then construct \( \{n_i : i \in \omega \} \) by \( n_0 = 0 \) and \( n_{i+1} = \tau_i \circ \ldots \circ \tau_0(n_i) \) where \( \tau_k(n) = \tau(n,k) \). Finally define \( G_i = f^{-1}[n_i, n_{i+1}) \) and \( M_i = \bigcup_{n_i \leq n < n_{i+1}} A_n \) as we did before.

Next is to pick \( \{C_m : m \in \omega \} \). Fix an increasing enumeration \( \{m_k : k \in \omega \} \) of \( p^{-1}\{p(m)\} \), and fix \( k \in \omega \) such that \( m = m_k \). Note that as \( k \leq m_k = m \), \( M_m \) is relatively non-meager in \( U_k \) by the construction above. Thus we can pick \( C_m \in \{B^0_m, B^1_m\} \) such that \( C_m \cap M_m \) is relatively non-meager in \( U_k \).

It remains to show that for all \( i \in \omega \), \( \bigcup_{n \geq i}(C_n \cap M_n) \) is comeager. Let \( \{m_k : k \in \omega \} \) be an increasing enumeration of \( p^{-1}\{i\} \). For all \( k \in \omega \), since \( m_k \geq i \) witnesses that \( C_{m_k} \cap M_{m_k} \) is relatively non-meager in \( U_k \) by the previous construction, it also witnesses that \( \bigcup_{n \geq i}(C_n \cap M_n) \) is relatively non-meager in \( U_k \). Thus \( \bigcup_{n \geq i}(C_n \cap M_n) \) is relatively non-meager in all basic open sets \( U_k \), which implies its comeager-ness.

Therefore the set \( Y = \bigcap_{i \in \omega} \bigcup_{n \geq i}(C_n \cap M_n) \subseteq X \) is comeager, and the proof can be finished analogously to the measurable case. \( \square \)

### 4 More examples of finite domaticity

The main goal of this section is to survey a variety of preexisting methods.

#### 4.1 Omega-regular graphs admit approximate domaticity

**Theorem 4.1.** Let \( G \subseteq X^2 \) be a Borel \( \omega \)-regular graph on a Borel probability space \((X, \mu)\). For every \( k < \omega \) and \( \varepsilon > 0 \), there exists a Borel \( k \)-coloring that is \( k \)-domatic at a set of vertices of \( \mu \)-measure at least \( 1 - \varepsilon \).

**Proof.** Fix \( N \in \omega \) a large enough number such that \( k \cdot \left(1 - \frac{1}{k}\right)^N < \varepsilon/3 \).

By Lusin–Novikov uniformization applied to \( G \), we can choose \( N \) points from each neighborhood \( N_G(x) \) in a Borel manner. Connecting every \( x \in X \) to such \( N \) many adjacent vertices forms a Borel \( N \)-regular subgraph \( H \subseteq G \).

Fix a compatible Polish metric \( d \) on \( X \). For each \( x \in X \), fix \( r(x) \) to be the minimal distance \( d(y, z) > 0 \) among all finitely many pairs of distinct \( y, z \in N_H(x) \). We note that \( x \mapsto r(x) \) is Borel. Since every \( r(x) > 0 \), we can use \( \lim_{\rho \to 0^+} \mu \{x \in X : r(x) > \rho\} = 1 \) to fix some \( r_0 > 0 \) such that \( \mu \{x \in X : r(x) > r_0\} > 1 - \varepsilon/3 \), and put \( A = \{x \in X : r(x) > r_0\} \). As \( d \) is Polish, we may fix a Borel partition \( X = \bigsqcup_{n \in \omega} C_n \) such that for all \( n \in \omega \), the diameter \( \text{diam}(C_n) < r_0 \). Thus \( \{C_n : n \in \omega\} \) is such that for all \( x \in A \), there exist no pair of distinct \( y, z \in N_H(x) \) both belonging to a same part \( C_n \).
Consider all colorings $\chi : X \to k$ such that $\chi$ is constant restricted to every $C_n$. We identify the space of all such colorings with $k^{\omega}$, where the $i$-th position of an infinite $k$-ary string denotes the constant color of $\chi$ at $C_i$. Consider the canonical Borel probability measure $\nu$ on $k^{\omega}$, defined as the $\omega$-fold product of the uniform distribution on $k$. Then for all $x \in A$, the induced local distribution $\chi \upharpoonright N_H(x)$ is also uniform, as we calculate

$$
\nu(\{\chi \in k^{\omega} : \chi \text{ is } H\text{-domatic at } x\}) \geq 1 - k \left(1 - \frac{1}{k}\right)^N > 1 - \frac{\varepsilon}{3}
$$

For $\mu \times \nu$ the product probability measure on $X \times k^{\omega}$, $\mu(A) > 1 - \varepsilon/3$ implies

$$(\mu \times \nu)(\{(x, \chi) : \chi \text{ is } H\text{-domatic at } x\}) \geq \int_A \left(1 - \frac{\varepsilon}{3}\right) d\mu \geq 1 - \frac{2\varepsilon}{3}
$$

By Fubini’s theorem, there must exist some $\chi \in k^{\omega}$ such that

$$
\mu(\{x \in X : \chi \text{ is } H\text{-domatic at } x\}) > 1 - \varepsilon
$$

Such a $\chi \in k^{\omega}$, realized as a coloring $\chi : X \to k$ is what we wanted. □

### 4.2 Finite-regularity and group actions via the Lovász local lemma

As surveyed by Bernshteyn [1, Section 2.C.3], there are many flavors of the Lovász local lemma in descriptive combinatorics. Here are some direct applications.

**Theorem 4.2.** For every $k, c \in \omega$ there exists some $N \in \omega$ such that for all $N \leq n \in \omega$, and for any Borel $n$-regular graph $G \subseteq X^2$ on a standard Borel space $X$ such that every vertex has in-degree at most $c \cdot n$:

1. If $\mu$ is a Borel probability measure on $X$, then $G$ has a $\mu$-measurable $k$-domatic partition;
2. If $X$ is a Polish space, then $G$ has a Baire measurable $k$-domatic partition;
3. If the symmetrization $G^{\pm 1} = G \cup G^{-1}$ has uniform subexponential growth$^\dagger$ then $G$ has a Borel $k$-domatic partition.

**Proof.** Let $N$ be large enough such that for all $n \geq N$, it holds that $k(1 - \frac{1}{k})^n(cn^2 + 1)^8 \leq 2^{-15}$.

Fix a graph $G$ of the given form. For all vertices $x \in X$, define the bad event $B_x$ over the domain $N_G(x)$, such that a function $f : X \to k$ falls into the bad event $B_x$ iff $f$ is not domatic at $x$. To use the Lovász local lemma, we compute that the probability of a bad event is at most $p = k(1 - \frac{1}{k})^n$, and that the dependence degree of a bad event is at most $d = cn^2$. By construction, we get $p(d + 1)^8 \leq 2^{-15}$ and also the weaker $p(d + 1) < e^{-1}$.

$^\dagger$That is, there exists a function $g : \omega \to \omega$ such that for every vertex $v \in X$, the number of vertices at $G^{\pm 1}$-distance $\leq n$ away from $v$ is at most $g(n)$, and moreover $g$ is of subexponential growth, i.e. $\lim_{n \to \infty} g(n)^{1/n} = 1$. 

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Then cases (1) and (2) follow from Bernshteyn [1, Theorem 2.20], and case (3) follows from Csóka–Grabowski–Máthé–Pikhurko–Tyros [3, Theorem 1.3]. We used that if the function \( f : X \to k \) does not fall into any bad event \( B_x \), as guaranteed by the respective Lovász local lemmas, \( f \) is \( k \)-domatic by definition.

Corollary 4.3. Let \( k \in \omega \), and let \( \Gamma \rhd X \) be a free Borel action of a countably infinite group \( \Gamma \) on a standard Borel space \( X \). Fix an infinite subset \( S \subseteq \Gamma \) and its Schreier graph \( G(S) \) on \( X \).

(1) If \( \mu \) is a Borel probability measure on \( X \), then \( G(S) \) has a \( \mu \)-measurable \( k \)-domatic partition;

(2) If \( X \) is a Polish space, then \( G(S) \) has a Baire measurable \( k \)-domatic partition;

(3) If every finitely generated subgroup of \( \Gamma \) is of subexponential growth, then \( G(S) \) has a Borel \( k \)-domatic partition.

Proof. Fix \( N \in \omega \) given by the last theorem with \( c = 1 \), and let \( S_0 \subseteq S \) be an \( N \)-element subset. Then apply the last theorem to the subgraph \( G(S_0) \) of \( G(S) \), and use the general fact that the domaticity of a given coloring can be passed to supergraphs.

We remark on an application of the Lovász local lemma to simultaneous domaticity. Given graphs \( G_0, \ldots, G_{k-1} \subseteq X^2 \) defined on a same standard Borel space \( X \), a coloring \( f : X \to k \) is simultaneously \( k \)-domatic if it’s domatic for every graph \( G_i \). Previously we saw simultaneous \( \omega \)-domaticity in the form of Theorem 2.4, where it was inductively built up from a similar simultaneity condition from the open pair property. In the finite coloring case, simultaneity can be achieved by choosing “locally random” colorings, given by the Lovász local lemmas on neighborhoods of the union graph \( G = G_0 \cup \ldots \cup G_{k-1} \). The details are similar to the last theorem, where we saw local randomness of the Lovász local lemmas implied domaticity for a single graph.

For example, let \( \mathbb{Z}^n \rhd X \) be a free Borel action of \( \mathbb{Z}^n \) \((1 \leq n < \omega)\) on a standard Borel space \( X \). For each \( i < n \), consider the subgroup action \( \mathbb{Z} \rhd_i X \) induced by the \( i \)-th coordinate of \( \mathbb{Z}^n \), and let \( G_i = G_i(\mathbb{Z}) \) be the Schreier graph of this subgroup action where the marked set is all of \( \mathbb{Z} \). Then by the polynomial growth rate of \( \mathbb{Z}^n \), Csóka–Grabowski–Máthé–Pikhurko–Tyros [3, Theorem 1.3] implies that there is a Borel coloring of \( X \) which is simultaneously finitely domatic for all of the graphs \( G_0, \ldots, G_{n-1} \).

4.3 Small-degree trees via Baire measurable path decomposition

Theorem 4.4. Let \( 2 \leq k \in \omega \), and let \( G \subseteq X^2 \) be an undirected locally finite acyclic Borel graph (with possible self-loops) on a Polish space \( X \), where every vertex has degree at least \( k + 1 \). Then \( G \) has a Baire measurable \( k \)-domatic partition.

Proof. We will use the technique and terminology of Baire measurable path decomposition by Conley–Marks–Unger [2, Lemmas 3.4 and 3.5]. By their results, we may fix a \( G \)-invariant
comeager Borel set \( D \subseteq X \) and a path decomposition \( P_0, P_1, \ldots \) of \( G \upharpoonright D \) of length at least 6 and at most 12 (temporarily neglecting any self-loop that \( G \) may have).

We inductively \( k \)-color the vertices on each set of paths \( P_i \). For every path \( p \in P_i \), we want to color it such that: For any endpoint \( v \) of \( p \), its unique neighbor in \( p \) takes on any color that’s not yet in the neighborhood of \( v \), if possible; every non-endpoint vertex \( w \) of \( p \) has a non-endpoint neighbor \( w' \) also in \( p \) such that the colors of \( w, w' \) are distinct. To show that the latter condition is attainable, consider for example if \( p \)’s vertices are labeled 0, 1, 2, 3, 4, 5, 6 in order, and if the colors of 0, 1, 5, 6 have been fixed by the first condition, then color the rest in the order of \( 2 \rightarrow 3 \rightarrow 4 \) such that each pair \((1,2), (2,3), (4,5)\) receives distinct colors. Then, since a desirable coloring of \( p \) always exists, we choose to color \( p \) by the lexicographically least such coloring.

It remains to show validity of our coloring on \( D \). Let \( v \in D \) be any arbitrary vertex, and let \( i \in \omega \) be the index of the stage during which \( v \) is colored, so equivalently \( i \in \omega \) is the least such that \( P_i \) is incident to \( v \). Then \( v \) is a vertex of some path \( p \in P_i \), and so by the coloring of \( p \), the colored neighborhood of \( v \) as of stage \( i \) can receive at most 1 redundant color (even if \( G \upharpoonright D \) has a self-loop at \( p \)). By the path decomposition, the rest of \( N_G(v) \) are colored by prioritizing filling in the missing colors of \( N_G(v) \) if possible, so since \( v \) has \( G \)-degree at least \( k + 1 \), the resulting coloring is \( k \)-domatic at \( v \).

We also note that Conley–Marks–Unger [2] has identified conditions under which Borel and/or \( \mu \)-measurable path decompositions exist. See their paper for more details.

### 4.4 The shift graph and Ramsey theory

Let \([\omega]^{\omega}\) be the set of all infinite subsets of \( \omega \), with a Polish space structure viewed as being either the closed subset of all increasing sequences in \( \omega^{\omega} \), or the \( G_\delta \) subset of all their characteristic functions in \( 2^{\omega} \). We’ll use the two identical induced topologies on \([\omega]^{\omega}\) interchangeably.

The *shift function* \( S : [\omega]^{\omega} \to [\omega]^{\omega} \) is defined as the finite-to-one continuous function such that \( S(A) = A \setminus \{\min(A)\} \) for all infinite \( A \subseteq \omega \). The *shift graph* is the corresponding function graph \( S \subseteq X^2 \) on \( X = [\omega]^{\omega} \).

For any Polish space \( X \), we say that a set \( A \subseteq X \) is *universally measurable* if \( A \) is \( \mu \)-measurable for every Borel probability measure \( \mu \) on \( X \); we say that a set \( B \subseteq X \) is *globally Baire* if \( \pi^{-1}[B] \) has the Baire property for every continuous function \( \pi : Y \to X \) between Polish spaces. Both these properties of sets form \( \sigma \)-algebras, so they also describe the respective classes of functions they measure.

We have a following example of distinction between Borel and measurable domaticities.

**Example 4.5.** Define the reverse inclusion graph \( G_2 \subseteq ([\omega]^{\omega})^2 \), where for all pair of infinite sets \( A, B \in [\omega]^{\omega} \), put \( (A, B) \in G_2 \) iff \( A \supseteq B \). For each \( A \in [\omega]^{\omega} \), its forward neighborhood is precisely \( N_2(A) = [A]^{\omega} \). This graph, despite being \( 2^{\aleph_0} \)-regular, does not admit any Borel 2-domatic partition, by the Galvin–Prikry theorem [7, Theorem 19.11]. In the measurable cases, Miller [10, Theorem B] showed that if \( \text{add}(\text{null}) = 2^{\aleph_0} \), then the shift graph \( S \) admits proper 3-colorings that can be
Theorem 4.7. Let $G \supseteq W$.

We'll prove the lemma via a straightforward collapsing argument to collapse the proof.

By the finitistic nature of the argument by Di Prisco–Todorcevic [4, Theorem 3], we can strengthen the example above. Define the reverse cofinite inclusion graph $G \supseteq \omega$.

Lemma 4.6. Let $\Sigma$ be any $\sigma$-algebra on $[\omega]^\omega$ that is closed under continuous preimages along any $\pi : [\omega]^\omega \rightarrow [\omega]^\omega$. Let $k, m \in \omega$, and let $f : [\omega]^\omega \rightarrow m$ be a $\Sigma$-measurable coloring such that for every $A \in [\omega]^\omega$, the neighborhood $N_\omega(A)$ contains at least $k$ colors, so $|f[N_\omega(A)]| \geq k$. Then the graph $G_\omega$ admits a $\Sigma$-measurable $k$-domatic partition, i.e. we're free to assume $m = k$.

Proof. We'll prove the lemma via a straightforward collapsing argument to collapse the $m$-colors down to $k$, along each $G_\omega$-connected component.

Define the tail equivalence $E_i \subseteq ([\omega]^\omega)^2$ such that $(x, y) \in E_i$ iff $\exists i, j \in \omega$, $S^i(x) = S^j(y)$, where $S^n$ denotes functional composition. Thus $E_i$ is the same as the connectedness equivalence relation of $G_\omega$.

Note that for every vertex $x \in [\omega]^\omega$, there is $N_\omega(Sx) \subseteq N_\omega(x)$, so that the function $S : [\omega]^\omega \rightarrow [\omega]^\omega$ shrinks $G_\omega$-neighborhoods. Using this, we define the essential vision function $\text{ess} : [\omega]^\omega \rightarrow P(m)$ via $\text{ess}(x) = \bigcap_{n \in \omega} f[N_\omega(S^n x)]$. Thus for all $x \in [\omega]^\omega$ and $i \in m$, $i \in \text{ess}(x)$ iff for all $n \in \omega$, there exists $y \preceq S^n(x)$ a cofinite subset such that $f(y) = i$. This last condition implies that $x \mapsto \text{ess}(x)$ is $\Sigma$-measurable.

For all $x \in [\omega]^\omega$, since $S$ shrinks $G_\omega$-neighborhoods, $\text{ess}(x)$ is a decreasing intersection of subsets of $m$ of size at least $k$, so we get $|\text{ess}(x)| \geq k$. Similarly by shrinking, we see that $\text{ess}(x) = \text{ess}(S^i x)$ for all $i \in \omega$, which implies that $\text{ess}(x) = \text{ess}(y)$ whenever $(x, y) \in E_i$.

Define the $k$-coloring $g : [\omega]^\omega \rightarrow k$ such that whenever $f(x) \in \text{ess}(x)$, $g(x)$ is equal to the index of $f(x)$ in an increasing enumeration of $\text{ess}(x)$, if $g(x) < k$ and otherwise undefined. Then by the above established properties of $\text{ess}(x)$, we may verify that this $g$ is a $\Sigma$-measurable $k$-domatic partition for $G_\omega$ as desired.

Theorem 4.7. Let $G_\omega \subseteq ([\omega]^\omega)^2$ be the Borel $\omega$-regular graph as defined above.

1. If $\text{add}(\text{null}) = 2^{\aleph_0}$, then $G_\omega$ admits universally measurable and globally Baire $\omega$-domatic partitions;

2. In general, for any Borel probability measure $\mu$ on $[\omega]^\omega$, $G_\omega$ has a $\mu$-measurable $\omega$-domatic partition. Furthermore, with respect to the canonical Polish topology on $[\omega]^\omega$, $G_\omega$ has a Baire measurable $\omega$-domatic partition.
Proof. First assume \( \text{add} (\text{null}) = 2^{\aleph_0} \), so by Miller [10, Theorem B], there exist universally measurable and globally Baire proper 3-colorings on \( S \).

Following the proof from Di Prisco–Todorcevic [4, Theorem 3] while making corresponding changes whenever necessary, one can in fact show the stronger intermediate statement, that for all \( k \in \omega \) there exists \( m_k \in \omega \) (explicitly \( m_k = (3^n)^k \) in their notation), such that there exists a coloring \( d : [\omega]^{\omega} \to m \) where for all \( B \subseteq [\omega]^{\omega} \), \( |d[N_{\omega}(B)]| \geq k \). For example, in their Case 2 choosing a cofinite subset \( A \subseteq B \) with large gaps in \( B \setminus A \), it suffices to require sufficiently large gaps in a bounded-length prefix of \( B \subseteq \omega \) so as to accommodate all the potential finitely many choices of the filler set \( F \). One may verify that the resulting function \( d \) can be either universally measurable or globally Baire according to the given proper 3-coloring of \( S \).

Next, by the last lemma, we get a \( k \)-domatic partition \( d' \) for \( G_{\omega} \) with the same measurabilities as \( d \), for every \( k \in \omega \). We may sew together the various domatic partitions, by following the same E. M. Kleinberg argument presented by Di Prisco–Todorcevic [4, Section 1]. We finally get an \( \omega \)-domatic partition for \( G_{\omega} \), that can be either universally measurable or globally Baire.

We proceed to part (2). Fix any Borel probability measure \( \mu \) on \( [\omega]^{\omega} \). In order for the resulting \( \omega \)-domatic partition we got from proving part (1) to be \( \mu \)-measurable, we find by chasing back the proof, that it suffices for the starting proper 3-coloring on \( S \) to be measurable by countably many continuously pushed-forward measures on \( [\omega]^{\omega} \). Then using the countable additivity of null sets in place of \( \text{add} (\text{null}) = 2^{\aleph_0} \), we may follow the proof of Miller [10, Theorem 2.8] to produce a proper 3-coloring on \( S \) with the desired measurabilities. The Baire measurable case is similar, where one uses the countable additivity of meager sets in place of \( \text{add} (\text{mgr}) = 2^{\aleph_0} \).

Finally, we remark on extending the application of the Galvin–Prikry theorem to the Borel case. For any \( 1 \leq n < \omega \), let \( G_n = K_n \square G_{\omega} \) be defined as a Cartesian product graph. That is, the vertex set of \( G_n \) is \( n \times [\omega]^{\omega} \), and \( \langle (i, x), (j, y) \rangle \in G_n \) is a \( G_n \)-edge iff either \( i = j \) and \( x \supseteq y \) or \( i \neq j \) and \( x = y \). This graph has both (\( \mu \) and Baire) measurable \( \omega \)-domatic partitions, which are the \( n \)-fold copies of the two measurable \( \omega \)-domatic partitions given by the previous theorem. Moreover, this graph has a Borel \( n \)-domatic partition, defined via \( (i, x) \mapsto i \). However we can show that \( G_n \) admits no Borel \( (n + 1) \)-domatic partition. Given any Borel \( f : n \times [\omega]^{\omega} \to (n + 1) \), the Galvin–Prikry theorem implies the existence of some \( x \in [\omega]^{\omega} \) such that \( N_{\omega}(0, x) = \{ 0 \} \times [x]^{\omega} \) is \( f \)-monochromatic. Since the \( G_n \)-neighborhood of \( (0, x) \) consists of \( \{ 0 \} \times [x]^{\omega} \) and only \( (n - 1) \)-many other \( (i, x) \) with \( i \neq 0 \), we find that \( (0, x) \) sees \( \leq n \) colors in its \( G_n \)-neighborhood. Thus \( f \) cannot be Borel \( (n + 1) \)-domatic.

### 4.5 Edge colorings via Marks’ determinacy approach

All graphs in this subsection are assumed to be undirected with no self-loops. Recall the concept of edge-domatic colorings from Section 3.3.

**Theorem 4.8.** For every \( 2 \leq n < \omega \), there exists a Borel \( n \)-regular acyclic graph \( G \subseteq Y^2 \) on a standard Borel space \( Y \), such that there exists no Borel \( 2 \)-edge-coloring \( f : G \to 2 \) such that \( f(x, y) = f(y, x) \) and every vertex \( x \in Y \) is incident to edges of both colors.
Proof. We’ll use the technique and terminology of Marks’ lemma [9, Lemma 2.1]. Let the graph \( G \subseteq Y^2 \) be the same graph as defined in Marks [9, Theorems 3.8 and 3.9]. For any Borel 2-edge-coloring \( f : G \to 2 \), define an associated \( A \subseteq \text{Free}(\mathbb{N}^\Gamma \ast \Delta) \) similarly as by Marks. If \( f \) is a coloring of the desired form, then \( A \) and its complement form disjoint complete sections of \( E_\Gamma \) and \( E_\Delta \), contradicting Marks [9, Theorem 3.7].

We note that by inductively iterating on Marks’ lemma [9, Lemma 2.1], we attain a reformulation of the lemma in terms of Borel colorings.

**Lemma 4.9** (essentially Marks). For every \( n \in \omega \), if \( \Gamma_0, \ldots, \Gamma_{n-1} \) are countable groups, \( \Gamma = \Gamma_0 \ast \ldots \ast \Gamma_{n-1} \) is their free product, and if \( c : \text{Free}(\mathbb{N}^\Gamma) \to n \) is a Borel \( n \)-coloring, then there exists some \( i < n \) and some \( \Gamma_i \)-equivariant continuous embedding \( f : \text{Free}(\mathbb{N}^{\Gamma_i}) \to \text{Free}(\mathbb{N}^\Gamma) \) such that \( c \circ f \) is the monochromatic coloring of constant color \( i \).

The purpose of this is to be able to replace the role of the Galvin–Prikry theorem in the final example from the previous subsection. We leave to the motivated readers to construct Borel graphs with a domatic threshold at any arbitrary \( n \in \omega \). If one wishes to get rid of \( n \)-uniform hyperedges on the resulting graph \( G \), one may work with suitable versions of the incidence graph \( G \in \) earlier defined in Section 3.3.

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