Characteristic foliations — A survey

Fabrizio Anella\textsuperscript{1} | Daniel Huybrechts\textsuperscript{2}

Abstract

This is a survey article, with essentially complete proofs, of a series of recent results concerning the geometry of the characteristic foliation on smooth divisors in compact hyperkähler manifolds, starting with work by Hwang–Viehweg, but also covering articles by Amerik–Campana and Abugaliev. The restriction of the holomorphic symplectic form on a hyperkähler manifold $X$ to a smooth hypersurface $D \subset X$ leads to a regular foliation $\mathcal{F} \subset \mathcal{T}_D$ of rank 1, the characteristic foliation. The picture is complete in dimension 4 and shows that the behaviour of the leaves of $\mathcal{F}$ on $D$ is determined by the Beauville–Bogomolov square $q(D)$ of $D$. In higher dimensions, some of the results depend on the abundance conjecture for $D$.

MSC 2020

14J42 (primary)

1 | MAIN THEOREM AND MOTIVATION

Throughout, $D \subset X$ denotes a smooth connected hypersurface in a compact hyperkähler manifold $X$ of complex dimension $2n$, that is, $X$ is a simply connected, compact Kähler manifold such that $H^0(X, \Omega^2_X)$ is spanned by a holomorphic symplectic form $\sigma$. Usually, $X$ will be in addition assumed to be projective, although one expects all results to hold in general.

The symplectic form $\sigma$ induces a regular foliation of rank 1 on $D$, that is, a line sub-bundle $\mathcal{F} \subset \mathcal{T}_D$. We shall denote a generic leaf of the foliation by $L$ and its Zariski closure by $\bar{L}$. The space of leaves will be denoted as $D/\mathcal{F}$. These notions will all be recalled in Sections 2 and 3.
We will discuss the following table. The ultimate goal, only partially met at the moment, would be to establish the equivalence of all assertions in each row. We will throughout assume $n > 1$, but see Remark 1.1.

| (i) | (ii) | (iii) | (iv) |
|-----|------|------|------|
| (1) | dim $\tilde{L} = 1$ | $L = \tilde{L} \cong \mathbb{P}^1$ | $q(D) < 0$ | $D$ is uniruled |
| (2) | dim $\tilde{L} = n$ | $\tilde{L}$ Lagr. torus | $q(D) = 0$ | $D = f^{-1}H \subset X \xrightarrow{f} B$ Lagr. fibration |
| (3) | dim $\tilde{L} = 2n - 1$ | $\tilde{L} = D$ | $q(D) > 0$ | $D$ is of general type |

The table was originally proposed by Campana, cf. the work of Amerik–Guseva cf. [4].

Essentially, in each row, the four conditions are known or at least expected to be equivalent to each other. The assumption on $D$ to be smooth is essential, see Section 8.2. The following serves as a guide for what will be discussed in the subsequent sections, where precise references will be provided.

**Case (1): Closed leaves**, cf. [3, 7, 18].

\[(1) : \quad (i) \xrightarrow{\S 4.1} (iv) \xrightarrow{\S 4.2} (iii) \xrightarrow{\S 4.3} (iv) \xrightarrow{\S 4.4} (i).\]

Additionally, we observe the easy equivalence $(i) \xleftarrow{\S 4.5} (ii)$.

**Case (2): Lagrangian fibrations**, cf. [1, 4, 21].

\[(2) : \quad (i) \xrightarrow{\S 5.1} (iii) \xrightarrow{\S 5.2} (iv) \xrightarrow{\S 5.3} (i).\]

The implication $(iii) \Rightarrow (iv)$ is currently only proved assuming the abundance conjecture for $D$, so the proof is only complete for $n = 2$, cf. [4].

We also address $(iv) \Rightarrow (ii) \Rightarrow (i)$ in Section 5.5 under the assumption that $\mathcal{O}(D)$ is base point free, which is satisfied, for instance, if $B$ is smooth.

**Case (3): Dense leaves**, cf. [2, 3, 22].

\[(3) : \quad (i) \xrightarrow{\S 6.1} (iii) \xrightarrow{\S 6.2} (iv) \xrightarrow{\S 6.3} (iii) \xrightarrow{\S 6.4} (i).\]

The implication $(i) \nRightarrow (iii)$ is proved assuming that $(iii) \Rightarrow (iv)$ of Case (2) holds. Note that the equivalence $(i) \xleftarrow{\S 6.5} (ii)$ is clear in this case.

Additionally, we will also provide a direct argument for $(iv) \xrightarrow{\S 6.5} (i)$.

**Remark** 1.1. Let us consider the case $n = 1$, that is, $X$ a K3 surface. Then, a smooth hypersurface $D \subset X$ is just a smooth curve. Clearly, in this case, $(i)$ holds in all three Cases (1)–(3). The equivalence of the other conditions $(ii)$–$(iv)$ in (1) and (2) and of $(iii)$ and $(iv)$ in (3) is well known.

### 2 PREPARATIONS I: LINEAR ALGEBRA OF THE CHARACTERISTIC FOLIATION

We collect some linear algebra results and discuss applications to the geometry of the leaves of a foliation.
2.1 We begin with discussing some easy linear algebra results that will be used throughout the later sections.

Let $W$ be a vector space together with a symplectic structure $\sigma$, that is, $\sigma \in \bigwedge^2 W^*$ such that the induced map $\sigma : W \rightarrow W^*$ is an isomorphism. In this situation, the dimension of $W$ is even, so $\dim W = 2n$.

**Lemma 2.1.** Assume $V \subset W$ is a subspace of codimension 1. Then the subspace

$$F := \ker \left( \sigma|_V : V \subseteq W \xrightarrow{\sigma} W^* \rightarrow V^* \right) \subset V$$

is of dimension 1.

Similarly, if $U \subset W$ is of codimension 2, then either $\dim \ker(\sigma|_U : U \rightarrow U^*) = 2$ or $\sigma|_U \in \bigwedge^2 U^*$ is non-degenerate, that is, $\ker(\sigma|_U) = 0$.

**Proof.** Since $W \rightarrow W^* \rightarrow V$ has a one-dimensional kernel, we have $\dim F \leq 1$. Furthermore, since $\dim V = 2n - 1$ is odd, the alternating form $\sigma|_V \in \bigwedge^2 V^*$ cannot be non-degenerate, that is, $\ker(\sigma|_V) \neq 0$. Hence, $\dim F = 1$. The proof of the second assertion is analogous. \qed

**Lemma 2.2.** Assume $V \subset W$ is of codimension 1 and let $F = \ker(\sigma|_V) \subset V$. Then, $\sigma$ naturally induces a symplectic structure $\bar{\sigma}$ on $V/F$.

**Proof.** By definition of $F$, the restriction $\sigma|_V : V \rightarrow V^*$ factors through $V \rightarrow V/F \xrightarrow{\sigma} V^*$. Furthermore, since $\sigma$ is alternating, $\sigma|_V$ takes values in $(V/F)^* \subset V^*$. Hence, for dimension reasons, $\bar{\sigma} : V/F \rightarrow (V/F)^*$. \qed

Here are a few more concepts from linear algebra: A subspace $U \subset W$ of codimension $c$ of a symplectic vector space $(W, \sigma)$ is called isotropic if $\sigma|_U \in \bigwedge^2 U^*$ is trivial or, equivalently, if

$$U \subset U^\perp := \ker(W \rightarrow W^* \rightarrow U^*) = \{w \in W \mid \sigma(U, w) = 0\}.$$

If $U^\perp \subset U$, then the subspace $U$ is called coisotropic. Since by definition $U^\perp$ is of dimension $c$, a subspace $U \subset W$ is coisotropic if and only if $\dim \ker(\sigma|_U : U \rightarrow U^*) = c$. Finally, $U \subset W$ is Lagrangian if $U$ is simultaneously isotropic and coisotropic, that is, $U = U^\perp$.

**Lemma 2.3.** Assume $V \subset W$ is of codimension 1 and let $F = \ker(\sigma|_V) \subset V$.

(i) Then for any Lagrangian subspace $U \subset W$ that is contained in $V$ one has $F \subset U$.

(ii) If a subspace of codimension 2 $U \subset W$ is contained in $V$ and $F \subset U$, then $U$ is coisotropic.

**Proof.** The first claim follows from the commutativity of the diagram

and the assumption that $U$ is Lagrangian, which implies that $U = \ker(W \rightarrow W^* \rightarrow U^*)$. 
For the second assertion, apply Lemma 2.1. Since $F \subset U \subset V$, the restriction $\sigma|_U$ is degenerate, and therefore, $\dim \ker(\sigma|_U) = 2$. □

2.2 A regular foliation of a smooth variety (or complex manifold) $D$ is a locally free subsheaf $F \subset \mathcal{T}_D$ with a locally free quotient and such that $F$ is integrable, that is, $[F, F] \subset F$. Note that the integrability condition is automatically satisfied if $\text{rk}(F) = 1$, which is the case of interest to us.

A leaf of a foliation is a maximal connected and immersed complex submanifold $L \subset D$ (more precisely, a complex manifold together with an injective immersion into $D$) with $F|_L = \mathcal{T}_L$ as subsheaves of $\mathcal{T}_D|_L$. The integrability condition ensures that there exists a (unique) leaf through any point of $D$. This is the Frobenius integrability theorem, cf. [26, §5, Thm. 2]. A submanifold $Z \subset D$ is invariant under the foliation if $F|_Z \subset \mathcal{T}_Z$ as subsheaves of $\mathcal{T}_D|_Z$. If $Z$ is a singular subvariety of $D$, then we call $Z$ invariant if its smooth locus is invariant. It is not hard to see that the Zariski closure of an invariant complex submanifold is invariant. Also note that every leaf $L$ intersecting an invariant submanifold $Z \subset D$ is contained in its closure.

A leaf $L \subset D$ is typically not closed. Its Zariski closure $\overline{L} \subset D$ can be identified with the smallest subvariety containing $L$ that is invariant under the foliation.

Consider now the case of a smooth hypersurface $D \subset X$ of a compact hyperkähler manifold. By virtue of Lemma 2.1, the kernel

$$F := \ker\left(\sigma|_D : \mathcal{T}_D \overset{\sigma}{\longrightarrow} \mathcal{T}_X|_D \overset{\sigma}{\longrightarrow} \Omega^*_X|_D \longrightarrow \Omega^*_D\right) \subset \mathcal{T}_D$$

is a sub-line bundle with locally free kernel. It is called the characteristic foliation of the hypersurface $D \subset X$ and was first studied by Hwang and Viehweg [22].

**Lemma 2.4.** The normal bundle of the characteristic foliation $\mathcal{N}_F := \mathcal{T}_D/F$ is naturally endowed with a symplectic structure and

$$\mathcal{N}_F \simeq \omega_D^\ast.$$ 

In particular, any local transverse section $\Sigma$ of a leaf $L \subset D$ has a natural symplectic structure.

**Proof.** The first assertion follows from Lemma 2.2. The existence of a symplectic structure on $\mathcal{N}_F$ implies $\det(\mathcal{N}_F) \simeq \mathcal{O}_D$ and hence $F \simeq \det(\mathcal{T}_D) \simeq \omega_D^\ast$. □

**Remark 2.5.** For foliations, in general $\det(F)^\ast$ is often called the canonical bundle $\omega_F$ of the foliation. For the characteristic foliation, we thus have $\omega_F \simeq \omega_D$. Note that for $n = 1$, this becomes $F \simeq \mathcal{T}_D$ which is not interesting so that we usually assume $n > 1$. Then, according to [12, Thm. 1.1], $\det(F) \simeq \omega_D^\ast$ cannot be big and nef.

The geometric versions of isotropic, coisotropic and Lagrangian for subspaces of a symplectic vector space are readily defined: For example, a subvariety $Z \subset X$ is coisotropic if the rank of $\sigma|_Z : \mathcal{T}_Z \longrightarrow \Omega_Z$ (say over the smooth locus of $Z$) is $2\dim(Z) - \dim(X)$ or, equivalently, if $\text{rk}(\ker(\sigma|_Z)) = \text{codim}(Z \subset X)$.

The geometric analogue of Lemma 2.3 is the following.
Corollary 2.6. Assume $D \subset X$ is a smooth hypersurface of a compact hyperkähler manifold.

(i) If $T \subset X$ is a smooth Lagrangian submanifold that is contained in $D$, then $T$ is covered by leaves or, equivalently, every leaf $L \subset D$ intersecting $T$ is contained in $T$.

(ii) Furthermore, any invariant subvariety $Z \subset X$ of codimension 2 that is contained in $D \subset X$ is coisotropic.

3 | PREPARATIONS II: SPACE OF LEAVES

There is no standard text on foliations on complex manifolds or algebraic varieties, but see, for example, [8]. The arguments typically rely very much on the differentiable theory. The holomorphic version of Reeb’s classical stability theorem, cf. [22, 23], is one example. We recommend [3, Sec. 2.1] for further comments.

3.1 Consider a foliation $\mathcal{F}$ (of rank 1) on a compact complex manifold $D$. The space of leaves is the quotient $D/\mathcal{F}$ by the equivalence relation that identifies two points if they are contained in the same leaf. The quotient topology is often complicated and frequently non-Hausdorff, but the projection $\pi : D \rightarrow D/\mathcal{F}$ is open, that is, for any open set $U \subset D$, the union of all leaves intersecting $U$ (its saturation) is again an open subset. For more information, see [8, Ch. III].

A typical example is that of a $\mathbb{P}^1$-bundle $\pi : D = \mathbb{P}(\mathcal{E}) \rightarrow Z$ with $\mathcal{F} = T\pi$. In this situation, $D/\mathcal{F} = Z$. We will come back to the local structure of $D/\mathcal{F}$ and the map $\pi : D \rightarrow D/\mathcal{F}$ in the case that the foliation is algebraically integrable, that is, when every leaf is compact.

3.2 Let $L = \overline{L} \subset D$ be a compact leaf. For a fixed point $x \in L$, we pick a small transversal section $x \in \Sigma_x \subset D$ (think of it as a germ of a transversal section). Consider a closed loop $\gamma : [0,1] \rightarrow L$ with $\gamma(0) = \gamma(1) = x$ and pick a point $y \in \Sigma_x$ close to $x$. Then, there exists a differentiable map $\Phi : \Sigma_x \times [0,1] \rightarrow X$ such that $\Phi(y,0) = \Phi(y,1), \Phi(0,t) = \gamma(t)$, and $\Phi(y,0) = y$.

The pull-back of the foliation $\mathcal{F}$ defines a real foliation of rank 1 on $\Sigma_x \times S^1$.

Starting with a point $(y,0)$ and integrating defines a path $\gamma_y : [0,1] \rightarrow \Sigma_x \times S^1$ satisfying $\gamma_y(t) = (\rho_{\gamma,y}(t), t)$ and $\gamma_y(0) = (y,0)$. Note, however, that this path is not necessarily closed, so possibly $\gamma_y(1) \neq (y,0)$.

It turns out that the map $y \mapsto \rho_{\gamma,y}(1)$ only depends on the homotopy class of $\gamma$, which gives rise to the following.

Definition 3.1. The holonomy of a compact leaf $L \subset D$ is the group homomorphism

$$\rho : \pi_1(L,x) \rightarrow \text{Diff}(\Sigma_x), \quad \gamma \mapsto (\rho_{\gamma} : y \mapsto \rho_{\gamma,y}(1)).$$

The leaf has finite holonomy if the image of $\rho$, the holonomy group

$$G_L := \text{Im}(\rho) \subset \text{Diff}(\Sigma_x),$$

is finite.

† Or, equivalently, when it admits one compact leaf with finite holonomy [30, Thm. 1], cf. Definition 3.1.
Note that the image of \( \rho \) only depends on \( x \) up to conjugation. In particular, the property of a leaf to have finite holonomy does not depend on the chosen base point.

Since we are interested in foliations of rank 1, a compact leaf will be a compact complex curve. If this curve is rational, that is, \( L = \bar{L} \simeq \mathbb{P}^1 \), then it has automatically finite (and, in fact, trivial) holonomy. Also note that due to a result of Holmann [16, Prop. 4.2], one knows that if all leaves of a foliation on a Kähler manifold are compact, then they all have finite holonomy.†

**Theorem 3.2.** Assume that a foliation \( \mathcal{F} \) of rank 1 on a smooth projective variety \( D \) has one leaf isomorphic to \( \mathbb{P}^1 \). Then, \( \mathcal{F} \) is algebraically integrable and all leaves are curves isomorphic to \( \mathbb{P}^1 \).

**Proof.** According to a result of Pereira [30, Thm. 1], for a foliation on a compact Kähler manifold, the existence of one compact leaf with finite holonomy implies that all leaves are compact with finite holonomy. This proves that \( \mathcal{F} \) is algebraically integrable. Reeb stability [22, Prop. 2.5] then implies that all leaves are isomorphic to each other.‡

### 3.3 Let us come back to the space of leaves \( D / \mathcal{F} \) and the map \( \pi : D \longrightarrow D / \mathcal{F} \) for an algebraically integrable foliation (of rank 1) on a smooth projective variety \( D \). We collect the facts that will be used at various places later.añ

- The map \( \pi : x \longrightarrow |G_{L_x}| \cdot [L_x] \) defines a holomorphic map from \( D \) to the Chow variety (or Barlet space), cf. [22, Prop. 2.5] of [29]. Here, \( L_x \) is the unique leaf through \( x \) and \( G_{L_x} \) is its holonomy group.
- The quotient \( D / \mathcal{F} \) can be identified with (the normalisation of) the image of \( \pi \). In particular, \( D / \mathcal{F} \) is an algebraic variety and \( \pi : D \longrightarrow D / \mathcal{F} \) is a proper morphism. The map is in general not flat. Indeed, by ‘miracle flatness’, the flatness of the equidimensional morphism \( \pi \) is equivalent to the smoothness of the leaf space \( D / \mathcal{F} \), which is discussed next.
- Assume \( x \in \Sigma_x \subset D \) is a transversal section of a leaf \( x \in L \) as in Section 3.2. Then, locally, \( \Sigma_x / G_x \) is a chart of \( D / \mathcal{F} \) at the point corresponding to the leaf \( L_x \), cf. [22, Thm. 2.4]. In particular, \( D / \mathcal{F} \) has quotient singularities.
- Assume that the fibres of \( D \longrightarrow D / \mathcal{F} \) are rational curves. Then the description of local charts shows that \( D / \mathcal{F} \) is a smooth projective variety, for in this case \( \pi_1(L_x) = \{1\} \) and hence \( G_{L_x} = \{1\} \). In the differentiable setting, this is [15] and for a discussion in our setting, see, for example, [31, Lem. 5]. If, furthermore, \( \mathcal{F} \) is the characteristic foliation of a smooth hypersurface in a projective hyperkähler manifold, then \( D / \mathcal{F} \) comes with a natural symplectic structure.
- For a dense open subset of points \( x \in D \), the leaf \( L_x \) through \( x \) has trivial holonomy \( |G_{L_x}| = 1 \), cf. [14].
- The scheme-theoretic fibre of \( \pi : D \longrightarrow D / \mathcal{F} \) over a point \( [L] \in D / \mathcal{F} \) corresponding to a leaf with trivial holonomy \( |G_L| = 1 \) is the leaf \( L \). The fibre is non-reduced over points with non-trivial holonomy; more precisely, it is a multiple fibre with multiplicity \( |G_L| \neq 1 \).

---

*†* In fact, Holman [16, Prop. 4.2] proved that for a holomorphic foliation with only compact leaves stability is equivalent to finite holonomy. Earlier results in this direction are due to Epstein [13].

*‡* A word on the name ‘Reeb stability’: A leaf \( L \) is stable if every open neighbourhood \( L \subset U \) of it contains an invariant open neighbourhood \( L \subset U' \subset U \). The foliation is stable if all leaves are stable. Reeb stability in the holomorphic context as in [22, 23] can be viewed as saying that compact leaves with finite holonomy are stable.

*añ* These results seem well known to the experts but we could not find a source with complete proofs. Thanks to J.-B. Bost for an instructive email exchange.
3.4 It is easy to prove that a smooth curve $C \subset S$ in a K3 surface with $(C.C) \geq 0$ is nef. The following is the hyperkähler analogue of this fact. \(^\dagger\)

**Proposition 3.3.** Let $D \subset X$ be a smooth hypersurface in a projective hyperkähler manifold $X$. If $q(D) \geq 0$, then $D$ is nef.

**Proof.** Assume $D$ is not nef. Then, there exists an irreducible curve $C \subset X$ with $D \cdot C < 0$. The latter implies $C \subset D$ and $\deg(\omega_D|_C) < 0$, which by Lemma 2.4 shows $\deg(F|_C) > 0$, that is, $F|_C$ is ample.

However, the latter implies that the foliation $F$ is algebraic, that is, all leaves are compact or, equivalently, algebraic curves. This is a consequence of a much more general result that was original proved by Bogomolov–McQuillan \([5]\) and Bost \([6]\) with details provided by Kebekus, Solá Conde and Toma \([23, \text{Thm. 1 \& 2}]\): If the restriction of a foliation to some complete curve $C$ is an ample vector bundle, then the leaf through any point of $C$ is algebraic. Moreover, the leaf through a general point of $C$ is rationally connected and, in fact, all leaves are rationally connected. In fact, according to Theorem 3.2, all we need is one compact rational leaf. \(^\dagger\)

In our situation, the result means that all leaves of $F$ are smooth rational curves and, in particular, $D$ is uniruled. Then, the discussion in Section 4.2, which is independent of this proposition, leads to the contradiction $q(D) < 0$. \(\square\)

4 | **CASE (1): CLOSED LEAVES**

We are proving the equivalence of the conditions (i)–(iv) in Case (1). All results are unconditional. The main reference for this section is \([3]\) with priori work \([7, 18]\).

4.1 (i) $\Rightarrow$ (iv): We assume that the leaves of the foliation have one-dimensional closures and want to show that $D$ is then uniruled. \(^\#\)

First of all, since the boundary $L \setminus L$ is invariant, it is a union of leaves. However, under our assumption, all leaves are one-dimensional, and therefore, all leaves are, in fact, closed $L = L$. Thus, all leaves are algebraic curves, that is, $F$ is algebraically integrable, and the natural projection

$$
\pi : D \longrightarrow D/F
$$

is a proper morphism between algebraic varieties.

The proof proceeds in three steps.

**Step 1:** Prove that $\pi$ has no multiple fibres in codimension 1 and that the canonical divisor of $D/F$ is trivial.

---

\(^\dagger\) We wish to thank R. Abugaliev for communicating this result to us. It seems known to some experts, but has not been written down anywhere.

\(^\dagger\) Note that in this sense, Reeb stability shows that the ampleness along $C$ determines the behaviour of the foliation globally.

\(^\#\) This part is the most technical one of all of this survey and we will have to be sketchy at points.
Step 2: Deduce the isotriviality of $\pi$, combining results of Miyaoka and Hwang–Viehweg, and consider a finite quasi-étale cover of $D$ that splits into a product.

Step 3: Reach a contradiction by considering the numerical dimension of $\omega_D$.

Step 1. Intuitively, the morphism $\pi : D \rightarrow D/F$ induced by the algebraically integrable foliation $F$ contracts all curves with tangent space contained in the kernel of $\sigma|_D$. Therefore, $\sigma|_D$ should descend to a non-degenerate two-form on the a priori singular space $D/F$, and so, we expect (the smooth part of) $D/F$ to have trivial canonical bundle.

For the open set covered by leaves with trivial holonomy, this can be made rigorous by taking a locally transverse section $\Sigma$ to each leaf of the foliation, which can be taken as a local model for $D/F$ near the point corresponding to the leaf. Then, by Lemma 2.4, $\sigma|_\Sigma$ is symplectic. These symplectic forms glue to a global symplectic form on the open subset of $D/F$ corresponding to leaves with trivial holonomy, see [31, Lem. 6] for some more details.

Looking at the local behaviour of the symplectic form around the multiple fibres, Amerik and Campana [3] proved the following.

Lemma 4.1 (Amerik–Campana). The map $\pi$ has no multiple fibres in codimension 1. Moreover, some multiple of the canonical bundle of $D/F$ is trivial.

Proof. Suppose by contradiction that there exists a divisor $V \subset D/F$ such that the fibres over $V$ are multiple of order $m > 1$. The statement is local around a generic point $0 \in V$. We can take a local multisection $W$ over 0 that meets transversally the non-reduced fibres. We choose coordinates $(z, t)$ around $0 \in V$ such that $z$ are coordinates for $V$ and $t$ parametrises the normal direction. Thus, we can choose coordinates $(u, s, w)$ around $W$ in such a way that $W$ is given by the equation $w = 0$ and $\pi(u, s, w) = (u, s^m)$.

By the discussion above, $\sigma^{n-1} = \pi^* \alpha$ for some form of top degree on the base at least over the complement of $V$. Locally, $\alpha = G(z, t) \cdot dz \wedge dt$, where $dz$ is an $n-2$ form, and then $|G(z, t)| = e^{g(z, t)} \cdot |t|^{-c}$ for some real-valued bounded function $g$. We claim that $c = 1 - 1/m$. Assuming for the moment that this is true, then the meromorphic function $G(z, t)$ has poles of order strictly less than one around $t = 0$, which is absurd.

To prove the claim, we denote by $\pi_0$ the restriction of $\pi : D \rightarrow D/F$ to $W$. In coordinates, $\pi_0(u, s, w) = (u, s^m)$. By the base change formula, we see that the restriction of $\sigma^{n-1}$ to $W$ is

$$\sigma^{n-1}|_W = \pi_0^* \alpha = G(u, s^m) \cdot m \cdot s^{m-1} \cdot du \wedge ds = h(u, s) \cdot du \wedge ds$$

for some function $h(u, s)$ that does not vanish when $s = 0$. Thus, we can write

$$|G(z, t)| = |G(u, s^m)| = \frac{|h(u, s)|}{m} |s|^{1-m} = e^{g(u, s)} |s|^{1-m} = e^{g(z, t)} |t|^{-1+1/m},$$

which proves the claim. $\square$

The singular fibres of $\pi : D \rightarrow D/F$ are simply multiples of smooth curves. By the above lemma, we can assume that $\pi$ is smooth over the complement $D^0/F \subset D/F$ of a closed set of codimension 2 and we may assume that $D^0/F$ is smooth. Denote by $\pi^0 : D^0 \rightarrow D^0/F$ the restriction of $\pi$. If one leaf is rational, then by Reeb stability, see Theorem 3.2, all the leaves are rational.
curves and we are done.† So, we can assume that all the leaves have positive genus and singular, that is, multiple, fibres appear in codimension at least two.

**Step 2.** We want to prove that \( \pi^o : D^o \to D^o / F \) is isotrivial. There are two possibilities depending on the genus \( g \) of the general leaf: If \( g = 1 \), then the fibration has to be isotrivial, for otherwise one of the fibres of \( \pi \) would be rational, in which case we are done already, cf. Theorem 3.2.

The isotriviality is less trivial for \( g > 1 \). It follows from the observation that the following results of Miyaoka–Mori and Hwang–Viehweg contradict each other.

- For any coherent subsheaf \( H \subset \Omega_{D^o / F} \), one has \( \chi(D^o / F, \det(H)) \leq 0 \). Indeed, according to [27, Cor. 8.6] or [28, Thm. 1], the restriction of \( \Omega_{D^o / F} \) to a generic complete intersection curve is semi-positive. At the same time, its determinant is trivial. Hence, all sub-sheaves of \( \Omega_{D^o / F} \) have non-positive degree, which leads to the assertion.
- Assuming \( g > 1 \), there exists a coherent subsheaf \( H \subset \Omega_{D^o / F} \) such that \( \text{var}(\pi^o) = \chi(D^o / F, H) \), cf. [22, Thm. 3.2 & Prop. 4.4]. Roughly, the relative cotangent sheaf of the natural map \( D^o \to M_g \) provides this subsheaf. (Strictly speaking, this is only true after passing to a finite cover of \( D^o \) which does not affect the argument.)

Once isotriviality for \( g = 1 \) and \( g > 1 \) has been established, one can use the fact that the moduli space \( M_g(N) \) of curves with a level \( N \)-structures, \( N \geq 3 \), is fine to show that there exists a finite étale cover \( \Delta \to D^o / F \) such that pull-back \( \hat D := D^o \times_{D^o / F} \Delta \) splits as \( \hat D \simeq \Delta \times C \), where \( C \) is the generic fibre of \( \pi \). Indeed, there exists a finite étale cover \( \Delta \to D^o / F \) such that the finite cohomology groups \( H^t(D^o / \mathbb{Z} / NZ) \), \( t \in \Delta \), of the fibres of the pull-back family \( \pi^o : \hat D := D^o \times_{D^o / F} \Delta \to \Delta \) form a trivial local system. The induced morphism \( \Delta \to M_g(N) \) has the property that the pull-back of the universal curve over \( M_g(N) \) is isomorphic to \( \hat \pi^o \). However, the isotriviality implies that \( \Delta \to M_g(N) \) is constant, and therefore, \( \hat D \) splits as claimed.

**Step 3.** Since \( D^o / F \) has trivial canonical bundle, the same holds for \( \Delta \). Hence,

\[
\nu(\hat D, \omega_D) = \chi(\hat D) = \begin{cases} 
0 & \text{if } g = 1 \\
1 & \text{if } g > 1.
\end{cases}
\]

As the numerical (and also the Kodaira) dimension is preserved under étale maps, one finds

\[
\nu(D, \omega_D) = \begin{cases} 
0 & \text{if } g = 1 \\
1 & \text{if } g > 1.
\end{cases}
\]

Since \( \omega_D = \mathcal{O}_X(D)|_D \), we have \( \nu(D, \omega_D) = \nu(X, D) - 1 \). However, the numerical dimension of a nef divisor in a hyperkähler manifold can be \( 0, n \) or \( 2n \). Since \( n > 1 \), the only possibility is that \( n = 2 \) and \( g > 1 \), which is excluded as follows: A fibre \( S \) of the canonical map is equivalent as a cycle, up to a multiple, to \( D \cdot D \). This means that \( S \) is Lagrangian, for \( \int_S \sigma \bar{\sigma} = q(D) = 0 \). Hence, by Corollary 2.6, the leaves of the characteristic foliation must be contained in \( S \) and induce a fibration on \( S \) of curves of genus at least two. This contradicts the fact that the canonical bundle of \( S \) is trivial.

† One can avoid using Reeb stability here: Instead of showing (i) \( \Rightarrow \) (iv), one shows (i) \( \Rightarrow \) (iii) and then uses Section 4.3 to complete by (iii) \( \Rightarrow \) (iv). Indeed, \( q(D) < 0 \) is equivalent to \( \int_D c_1(F)H^{2n-2} > 0 \) for some ample divisor \( H \) on \( D \). The latter follows if one can show \( \int_D c_1(F)x^n H^{2n-2} > 0 \) for some ample divisor \( H_0 \) on \( D / F \), which, in turn, would follow from \( \int_L c_1(F)|_L > 0 \), that is, \( g(L) = 0 \).
4.2 (iv) \Rightarrow (iii): We assume that $D$ is uniruled and will show $q(D) < 0$ by excluding $q(D) > 0$ and $q(D) = 0$.

Suppose $q(D) > 0$. Then $D$ is contained in the interior of the positive cone and, therefore, also in the interior of the pseudo-effective cone. Hence, $D$ is big [24, Lem. 2.2.3], that is, $h^0(X, O(kD)) \sim k^{2n}$, which implies $h^0(D, \omega_D^k) \sim k^{2n-1}$ contradicting the assumption that $D$ is uniruled.

Suppose $q(D) = 0$. If $D$ is nef, then $\omega_D$ is nef too, which again would contradict the assumption that $D$ uniruled. If $D$ is not contained in the closure of the movable cone, then by Boucksom’s duality of movable and pseudo-effective cone [7], it is contained in the interior of the pseudo-effective cone and one argues as above. If $D$ is contained in the boundary of the movable cone, $D$ is the limit of movable divisors and hence its restriction to $D$ is still a limit of effective divisors. However, this implies that $\omega_D \simeq O(D)|_D$ is pseudo-effective which contradicts $D$ uniruled.

† The discussion should be compared to the result [25, Thm. 3.7] asserting in the general setting that $D$ is uniruled if and only if $\omega_F$ is not pseudo-effective. The above discussion can be interpreted as saying that any smooth divisor $D \subset X$ with $q(D) \geq 0$ has a pseudo-effective $\omega_D$; thus, $D$ cannot be uniruled.

4.3 (iii) \Rightarrow (iv): We assume $q(D) < 0$ and want to show that $D$ is then uniruled. (The smoothness of $D$ is not essential.) We offer two proofs.

Firstly, it is known that prime exceptional divisors are uniruled [18, Prop. 5.4] or [7, Prop. 4.7 & Thm. 4.3]. Indeed, since the positive cone is self-dual, it contains a class $\alpha$ such that $q(\alpha, D) < 0$. Hence, there exists a bimeromorphic map between hyperkähler manifolds $f : X \to X'$ such that $f_* \alpha = \omega' + \sum a_i D'_i$ for some uniruled divisors $D'_i$, positive real numbers $a_i$ and a Kähler class $\omega'$, cf. [18] or [7, Thm. 4.3 (iii)]. Since the quadratic form is preserved by $f$, we have $0 > q(\alpha, D) = q(\omega' + \sum a_i D'_i, f_* D) > \sum a_i q(D'_i, f_* D)$, and hence, for some $i$, we have $q(D'_i, f_* D) < 0$. This implies that $f_* D$ and $D_i$ coincide and that in particular $D$ is uniruled since its push-forward in $X'$ is.

Here is a more direct proof relying on the criterion for uniruledness by Miyaoka and Mori [27, 28]: A smooth projective variety $Z$ of dimension $d$ is uniruled if $\int_Z c_1(\omega_Z) \cdot H^{d-1} < 0$ for an ample divisor $H_Z$ on $Z$. Applied to $Z = D$ and observing that $q(D) < 0$ implies $\int_D c_1(D) \cdot H|_D^{2n-2} = \int_X [D]^2 \cdot H^{2n-2} < 0$ for any ample divisor $H$ on $X$, it implies the result.

4.4 (iv) \Rightarrow (i): We assume that $D$ is uniruled and want to prove that the leaves are closed.

By assumption, there exists a dominant map $\varphi : \mathbb{P}^1 \times V \dashrightarrow D$ with $\dim(V) = 2n - 2$. Since $\mathbb{P}^1$ admits no non-trivial forms of degree 1 or 2, the pull-back of $\sigma$ to $\mathbb{P}^1 \times V$ is the pull-back of a holomorphic form on $V$. This readily shows that all $\varphi_*(\mathbb{P}^1) \subset D$ are invariant with respect to the foliation. Hence, the generic leaf is of this form, which proves the claim. See [11, Prop. 4.5] for a generalisation to singular uniruled divisors.

4.5 (i) $\iff$ (ii): Clearly, (ii) implies (i). The converse is part of the discussion in Section 4.1.

† We wish to thank R. Abugaliev for his help with this argument.
5 | CASE (2): LAGRANGIAN FIBRATIONS

The equivalence of the conditions (i)–(iv) in Case (2) is shown in dimension 4. In higher dimensions, the proof assumes that abundance holds for $D$. The direction $(iv) \Rightarrow (i)$ is due to Abigail and is the main result of [1].

5.1 $(i) \Rightarrow (iii)$: We assume $\dim L = n$ and want to exclude $q(D) < 0$ and $q(D) > 0$.

According to Section 4, $q(D) < 0$ implies $\dim L = 1$. To exclude $q(D) > 0$, use that according to Section 6.4,\footnote{We leave it to the reader to check that the argument is not circular.} it would imply that the leaves are dense.

5.2 $(iii) \Rightarrow (iv)$: We assume $q(D) = 0$. Then, by Proposition 3.3, $D$ and hence $\omega_D \simeq \mathcal{O}(D)|_D$ are nef. Assuming the abundance conjecture for $D$, we know that $\omega_D$ is semi-ample. Hence, by [10, Cor. 1.8] also $D$ is semi-ample,\footnote{This is a highly non-trivial statement asserting that $H^0(X, \mathcal{O}(kD)) \rightarrow H^0(D, \mathcal{O}(kD)|_D)$ is surjective for sufficiently divisible $k$. For an alternative, algebraic argument, see [3, Cor. 5.2].} that is, some power $\mathcal{O}(kD)$ defines a Lagrangian fibration $f : X \rightarrow B$, and therefore, $kD$ is the pull-back of a divisor in $B$. Hence, $D$ is vertical.

Remark 5.1. The implication $(i) \Rightarrow (iv)$ in dimension 4 was first proved by Amerik and Guseva [4].

5.3 $(iv) \Rightarrow (iii)$: We assume now that there exists a Lagrangian fibration $X \fib B$ such that $D$ is the pre-image (as a set) of a hypersurface $H \subset B$. Then, $[D]$ and $f^*[H]$ are proportional. Therefore, since $[H]^{n+1} = 0$, also $[D]^{n+1} = 0$ and hence $q(D) = 0$.

5.4 $(iv) \Rightarrow (i)$: We assume that $X$ comes with a Lagrangian fibration $f : X \fib B$ such that $D = f^{-1}(H)$ (as sets) for some hypersurface $H \subset B$ and want to show that the closure of the generic leaf is of dimension $n$ (and, in fact, a torus).

Assume first that $H$ is contained in the discriminant locus $\Delta \subset B$. Then $D$ is algebraically integrable by a result of Hwang and Oguiso [21, Thm. 1.2]. By the results of Section 4, the latter implies that $D$ is uniruled and hence $q(D) < 0$, which contradicts $(iii)$ that we proved already in Section 5.3.\footnote{The argument shows that for any component of the discriminant divisor $H \subset B$, the reduction of $f^{-1}(H)$ cannot be smooth. Either it consists of more than one component, with possibly each component individually smooth, or it is irreducible but singular.}

Assume now that $H$ is not contained in the discriminant divisor. Then, since $D$ is smooth, the generic fibre of $f|_D : D \fib H = f(D)$ is a smooth Lagrangian torus. By Corollary 2.6, the generic leaf is contained in a fibre of $D \fib f(D)$. We have to show that it is dense in there. Note that for $n = 2$, the result is immediate. Indeed, if the generic leaf is not dense in the fibre, then by Section 4, the foliation is algebraically integrable and the leaves are rational curves, which, however, do not exist in a torus.

Let $T := f^{-1}(t), t \in f(D)$, be a generic fibre. The foliation $F \subset T_D$ induces a foliation $F|_T \subset T_T$ of the abelian variety $T$. It is well known that the closure of a leaf of a foliation on an abelian
variety is a translate of an abelian subvariety. Indeed, observing $O_F \simeq F|_T \subset T \simeq O_F^\mathbb{P}^n$ and writing $T = \mathbb{C}^n / \Gamma$, one finds that the leaves of the foliation $F$ are given by the images under the natural projection $\mathbb{C}^n \longrightarrow T$ of the translates of the line $\mathbb{C} \subset \mathbb{C}^n$ corresponding to $F|_T \subset T$. The closure of the leaf through the origin then corresponds to the smallest linear subspace $\mathbb{C}^m \subset \mathbb{C}^n$ containing the given line and such that $\Gamma \cap \mathbb{C}^m \subset \mathbb{C}^m$ is a lattice.

Thus, if the abelian variety $T$ is known to be simple, which is frequently the case, then the assertion is immediate.

If $T$ is not simple, then Abugaliev proceeds in two steps. The first is a result of general interest [1, Thm. 0.5].

**Lemma 5.2** (Abugaliev). Let $f : X \longrightarrow B$ be a Lagrangian fibration of a projective hyperkähler manifold and let $H \subset B$ be a very ample hypersurface not contained in the discriminant divisor of $f$.

If $D = f^{-1}(H) \subset X$ is smooth, then for the generic fibre $T = f^{-1}(t), t \in H$:

$$\text{Im} \left( H^*(X, \mathbb{Q}) \xrightarrow{\text{res}_{X,T}} H^*(T, \mathbb{Q}) \right) = \text{Im} \left( H^*(D, \mathbb{Q}) \xrightarrow{\text{res}_{D,T}} H^*(T, \mathbb{Q}) \right).$$

Note that the left-hand side is known to be isomorphic to $H^* (\mathbb{P}^n, \mathbb{Q})$ according to results by Matsushita, Oguiso, Voisin and Shen–Yin, see the survey [20, Thm. 2.1] for references. In particular, it is of dimension 1 in each even degree.

**Proof.** The assertion is invariant under small deformations of $H$, which preserve the smoothness of $D$. One may assume that the intersection $H \cap \Delta$ with the discriminant locus is sufficiently generic such that $\pi_1 (H \setminus \Delta) \longrightarrow \pi_1 (B \setminus \Delta)$ is surjective (and, in fact, an isomorphism for $n > 2$) by [9, Lem. 1.4] applied to $B \setminus \Delta$. In particular, the monodromy invariant parts of $H^* (T, \mathbb{Q})$ for the two families $X \longrightarrow B$ and $D \longrightarrow H$ coincide. Thus, Deligne’s invariant cycle theorem implies

$$\text{Im} (\text{res}_{X,T}) = H^*(T, \mathbb{Q})^{\pi_1 (B \setminus \Delta)} = H^*(T, \mathbb{Q})^{\pi_1 (H \setminus \Delta)} = \text{Im} (\text{res}_{D,T}),$$

which concludes the proof.

The idea of the second step is that the family of tori obtained as closures of leaves $L \subset \bar{L} \subset T$ contained in a fixed generic fibre $T$ is distinguished and hence invariant under monodromy. This gives a cohomology class in $H^{2k} (T, \mathbb{Q})$ that is invariant under monodromy of the family $D \longrightarrow H$. However, classes that are invariant under the monodromy of the family $X \longrightarrow B$ are all powers of the polarisation, and therefore, cannot be realised by proper subtori.

**5.5** (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i): Assume first (iv) holds. By Corollary 2.6, the generic leaf $L$ is contained in a fibre of $D \longrightarrow f(D) \subset B$. If we allow ourselves to use (iv) $\Rightarrow$ (i) in Section 5.4, then the closure $\bar{L}$ is the generic fibre which is a torus. The second implication (ii) $\Rightarrow$ (i) is clear.

**5.6** Let $T = f^{-1}(t) \subset X$ be a smooth fibre of a Lagrangian fibration $f : X \longrightarrow B$. Then $T$ is isomorphic to an abelian variety and picking a point $x \in X$ allows one to write $T$ as a torus $T = T_x T / \Gamma$.

\footnote{The reader will observe that the result actually holds without assuming that $X$ is hyperkähler or that $f$ is a Lagrangian fibration.}
For a hypersurface $t \in H \subset B$, which we assume to be smooth at $t$, we let $D := f^{-1}(H)$ be its pre-image. Since the symplectic structure of $X$ provides an isomorphism $T_xT \simeq T^*_yB$, the tangent space $T_tH \subset T_tB$, viewed as a line in $T^*_tB$, corresponds to a line $\ell_H \subset T_xT$. The image of this line under $T_xT \to T$ gives the leaf through $x \in D$ of the characteristic foliation on $D$. If $H$ is chosen very general, then the line $\ell_H \subset T_xT$ is very general, and therefore, its image in the quotient $T = T_xT/\Gamma$ is dense. This proves the following.

**Proposition 5.3.** For a fixed smooth fibre $T = f^{-1}(t) \subset X$ of a Lagrangian fibration $f : X \to B$ and a very general smooth hypersurface $t \in H \subset B$, the leaf of the characteristic foliation of $D = f^{-1}(H)$ through a point $x \in T$ is dense in the fibre $T$.

6 | CASE (3): DENSE LEAVES

Again, the equivalence of the conditions (i)–(iv) holds in dimension 4, but assumes that the abundance conjecture holds for $D$. Hwang and Viehweg [22] showed that if $D$ is of general type, the foliation is not algebraically integrable. In the converse direction, in dimension 4, Amerik and Campana [3] proved that the foliation is algebraically integrable if and only if $D$ is uniruled. The assertion that $D$ being nef and big implies density of the leaves is due to Abugaliev and the main result of [2].

6.1 (i) ⇒ (iii): We assume that $\dim L = 2n - 1$ and want to exclude that $q(D) < 0$ or $q(D) = 0$.

First, by the results of Section 4, we know that the three conditions $q(D) < 0$, $D$ uniruled, and $\dim L = 1$ are all equivalent. Hence, $q(D) < 0$ is excluded for $\dim L > 1$.

Next suppose $q(D) = 0$. Then, by Proposition 3.3, $D$ is nef and hence also $\omega_D \simeq \mathcal{O}(D)|_D$ is. Assuming the abundance conjecture for $D$, we conclude that $\omega_D$, and therefore, $\mathcal{O}(D)$ are semiample, cf. the argument in Section 5.2. Hence, $X$ comes with a Lagrangian fibration $X \to B$ such that some multiple of $D$ is the pre-image of a divisor in $B$. However, then the generic leaf is not dense, as a leaf passing through a smooth fibre stays in this fibre, cf. the discussion in Section 5.4.

6.2 (iii) ⇒ (iv): We assume $q(D) > 0$. Clearly, if $D$ is ample, then by adjunction, $\omega_D \simeq \mathcal{O}(D)|_D$ is ample as well, and therefore, $D$ is of general type. If $D$ is only nef, then a priori also $\omega_D$ is only nef. However, $q(D) > 0$ implies $\int_D c_1(\omega_D)^{2n-1} > 0$, that is, $\omega_D$ is big and nef. By the Kawamata–Viehweg vanishing theorem $H^i(D, \omega_D^k) = 0$ for $k > 1$ and $i > 0$ and by the Hirzebruch–Riemann–Roch formula $h^0(D, \omega_D^k) \sim k^{2n-1}$. Hence, $D$ is of general type. Since by Proposition 3.3, any smooth hypersurface $D \subset X$ with $q(D) \geq 0$ is nef, this concludes the proof.

Here is an alternative argument not relying on Proposition 3.3: Since $D$ is contained in the interior of the positive cone, it is also contained in the interior of the pseudo-effective cone and, therefore, big by [24, Prop. 2.2.6], that is, $h^0(X, \mathcal{O}(kD)) \sim k^{2n}$. Using $\omega_D \simeq \mathcal{O}(D)|_D$ eventually shows that $\omega_D$ is big, that is, that $D$ is of general type.

6.3 (iv) ⇒ (iii): We assume that $D$ is of general type and want to prove $q(D) > 0$ by excluding the other two possibilities $q(D) < 0$ and $q(D) = 0$. 

Suppose \( q(D) < 0 \). Then, by virtue of Section 4.3, for example, by applying the Miyaoka–Mori numerical criterion for uniruledness \([27, 28]\), we know that \( D \) is uniruled, so in particular not of general type.

Next, suppose that \( q(D) = 0 \), which implies \( \int_D c_1(\omega_D)^{2n-1} = (2n) = 0 \). Now use again Proposition 3.3 to conclude that \( \mathcal{O}(D) \) and hence \( \omega_D \simeq \mathcal{O}(D)|_D \) are nef. However, a nef divisor \( E \) on a projective variety \( Z \) of dimension \( m \) is big, that is, \( h^0(Z, \mathcal{O}(kE)) \sim k^m \), if and only if \( (E)^m > 0 \), see \([24, \text{Thm. 2.2.16}]\). Since \( D \) is assumed to be of general type and so \( h^0(D, \mathcal{O}(kD)|_D) \sim k^{2n-1} \), this is a contradiction.

\[\text{Lemma 6.1. Assume } D \subset X \text{ is a smooth hypersurface of a hyperkähler manifold that is big and nef. Then, the restriction }\]
\[H^i(X, \mathbb{Q}) \rightarrow H^i(D, \mathbb{Q})\]
\[\text{is an isomorphism for } i < \dim(D) = 2n - 1.\]

**Proof.** For an ample hypersurface, this is the content of the Lefschetz hyperplane theorem \([24, \text{Thm. 3.1.17}]\). If \( D \) is just big and nef, Kawamata–Viehweg vanishing still shows that all higher cohomology groups \( H^i(X, \mathcal{O}(kD)) \), \( i > 0 \), are trivial. Hence, \( D \) deforms sideways with \( X \) in any family \( \mathcal{X} \rightarrow \Delta \) for which the line bundle \( \mathcal{O}(D) \) deforms. However, the very general fibre \( \mathcal{X}_t \) of the universal such deformation has Picard number one. Therefore, a generic deformation \( D_t \subset \mathcal{X}_t \) of \( D \subset X \) is ample \([17, \text{Thm. 3.11}]\). Hence, \( H^i(\mathcal{X}_t, \mathbb{Q}) \rightarrow H^i(D_t, \mathbb{Q}) \) for \( i < \dim(D_t) \) by the classical Lefschetz hyperplane theorem. Since the assertion is topological, this suffices to conclude. \(\square\)

The key step is the following result \([2, \text{Prop. 4.1}]\).

**Proposition 6.2 (Abugaliev).** A smooth hypersurface \( D \subset X \) which is big and nef (or, equivalently, a smooth hypersurface with \( q(D) > 0 \), cf. Proposition 3.3) cannot be covered by coisotropic varieties of codimension 2 in \( X \).

**Proof.** Recall that a subvariety \( Z \subset X \) of codimension 2 is called coisotropic if the kernel of \( \sigma|_Z : T_Z \rightarrow \Omega_Z \) (over the smooth locus) is a sheaf of rank 2, see Section 2.2.

First observe that by Lemma 6.1, that for any subvariety \( Z \subset D \), there exists a class \( \alpha \in H^2(X, \mathbb{Q}) \) with \( \alpha|_D = [Z] \in H^2(D, \mathbb{Q}) \). Clearly, the class \( \alpha \) is of type (1,1). On the other hand, if \( Z \subset X \) is a coisotropic subvariety of codimension 2, then \( 0 = [Z] \cap \sigma^{n-1} \in H^{2n+2}(X, \mathbb{C}) \). So, if \( Z \subset D \) is coisotropic and we write \( [Z] = \alpha|_D \in H^2(D, \mathbb{Q}) \), then \( 0 = [D] \cap \alpha \cap \sigma^{n-1} \in H^{2n+2}(X, \mathbb{C}) \), which implies \( \int_X [D] \cap \alpha \cap \sigma^{n-1} \cap \sigma^{n-1} = 0 \) and, therefore, \( q(D, \alpha) = 0 \). Now use the well-known

\(\dagger\) The original proof in \([2]\) uses the Kodaira–Akizuki–Nakano vanishing theorem. The above argument is quicker, but uses deformation theory and the projectivity criterion for hyperkähler manifolds.
formula

\[ q(\gamma_1, \gamma_2) \cdot \int_X \gamma_1^{2n} = 2q(\gamma_1) \cdot \int_X \gamma_1^{2n-1} \wedge \gamma_2, \]  

(6.1)

cf. [19, Exer. 23.2] to deduce from \( q(D) > 0 \) and \( q(D, \alpha) = 0 \), that \( 0 = \int_X [D]^{2n-1} \wedge \alpha = \int_Z [D]^{2n-2}_Z \). If \( D \) is ample, this is absurd. So, we proved the stronger statement that an ample, smooth hypersurface \( D \subset X \) does not contain any coisotropic subvariety of codimension 2.† If \( D \) is only big and nef, then \( q(D, \alpha) = 0 \) still implies \( q(\alpha) < 0 \) by the Hodge index theorem, which, in turn, by a formula [2, Lem. 4.2] similar to (6.1), gives \( \int_X [D]^{2n-2} \wedge \alpha \wedge \alpha < 0 \). However, if \( D \) can be covered by coisotropic varieties of codimension 2, then there exist two such \( Z_1, Z_2 \subset D \) realising the same class \( \alpha|_D = [Z_1] = [Z_2] \) and then \( Y := Z_1 \cap Z_2 \subset D \) is empty or of codimension 2 in \( D \), which leads to the contradiction \( 0 \leq \int_Y [D]^{2n-3}_Y = \int_X [D]^{2n-2} \wedge \alpha \wedge \alpha < 0 \). □

Recall that a subvariety \( Z \subset D \) is called invariant under the characteristic foliation of the smooth hypersurface \( D \) if the leaf through any \( x \in Z \) is contained in \( Z \) or, equivalently, if \( \mathcal{F}|_Z \subset \mathcal{T}_D|_Z \) is contained in \( \mathcal{T}_Z \subset \mathcal{T}_D|_Z \) (over the smooth locus of \( Z \)).

The following result [2, Thm. 0.5] is now a consequence of the above discussion. It concludes the proof of (iii) \( \Rightarrow \) (i) in Case (3).

**Theorem 6.3** (Abugaliev). Assume \( D \subset X \) is a smooth hypersurface of a hyperkähler manifold \( X \) satisfying \( q(D) > 0 \). Then the generic leaf of the characteristic foliation on \( D \) is Zariski dense.

**Proof.** If the generic leaf \( L \subset D \) is not Zariski dense, then its Zariski closure \( \overline{L} \subset D \) defines a proper closed subvariety \( Z \subset D \). The family of all such leaves gives a covering family \( \{Z_t\} \) of \( D \). Assume first that \( Z_t \subset D \) is of codimension 2 in \( X \). Since \( Z_t = \overline{L} \) is clearly invariant under the characteristic foliation and hence coisotropic by Corollary 2.6 (ii), this is a contradiction to Proposition 6.2.

If the subvarieties \( Z_t \) are of higher codimension, taking unions produces a covering family \( \{Z'_s\} \) of \( D \) consisting of subvarieties of codimension 2 in \( X \) and such that each \( Z'_s \) is a union of \( Z_t = \overline{L} \). In particular, again, by Corollary 2.6 (ii), each \( Z'_s \) is coisotropic and one can conclude as before. □

**6.5** (iv) \( \Rightarrow \) (i): Of course, this direction is a consequence of the implications proved before, but we wish to mention a weaker statement due to Hwang–Viehweg that motivated much of the later work on characteristic foliations. They proved [22, Thm. 1.2] that the characteristic foliation of a smooth hypersurface \( D \subset X \) cannot be algebraic or, in other words, that \( \dim \overline{L} > 1 \).

7 | **ALTERNATIVE SUMMARY**

We think that it is instructive to present the discussion concerning the equivalence of the two conditions (iii) and (iv) in a somewhat differently structured way, making it more evident where and how foliations are used.

† In dimension 4, this says that a smooth ample hypersurface does not contain any smooth Lagrangian surface, see Section 8.2.
7.1 (iii) ⇒ (iv): For a smooth hypersurface $D \subset X$, one wants to show that the sign of $q(D)$ largely determines the geometry of $D$.

This part only involves more or less classical results and Proposition 3.3, that is, the nefness of $D$ if $q(D) \geq 0$. Recall that the proof of Proposition 3.3 used foliations in an essential way.

- Assume $q(D) < 0$. Then $\int_D c_1(\omega_D) \cdot H^{2n-2} < 0$ and by [27, 28] $D$ is uniruled.
- Assume $q(D) = 0$. Then, $D$ and hence $\omega_D$ are nef by Proposition 3.3. Using abundance conjecture for $D$ combined with [10], see footnote on page 12, one finds that $D$ is semi-ample. Therefore, $\mathcal{O}(kD)$ defines a Lagrangian fibration $f : X \to B$ for some $k > 0$ and hence $D = f^{-1}(f(D))$.
- Assume $q(D) > 0$. In this case, $D$ is of general type for which we presented two proofs: The one not using Proposition 3.3 just observed that under these assumptions, $D$ is in the interior of the pseudo-effective cone and hence big.

7.2 (iv) ⇒ (iii): The geometry of $D$ determines the sign of $q(D)$. Again, only Proposition 3.3 is used.

- Assume that $D$ is uniruled. Then, $q(D) < 0$ is proved by excluding $q(D) > 0$ and $q(D) = 0$. If $q(D) > 0$, then $D$ is of general type as explained above. To exclude $q(D) = 0$, one distinguishes two cases: Firstly, if $D$ is in the boundary of the movable cone, then $\omega_D = \mathcal{O}(D)|D$ is a limit of effective divisors and, therefore, pseudo-effective which contradicts the assumption that $D$ is uniruled. Secondly, if $D$ is not contained in the boundary of the movable cone, then $D$ is in the interior of the pseudo-effective cone. Hence, $D$ and $\omega_D$ are big, contradicting again the assumption on $D$. Alternatively, one could apply Proposition 3.3 to see that $D$ and hence $\omega_D$ are nef, but the latter clearly contradicts $D$ being uniruled.
- Assume $D = f^{-1}(H)$ is the set theoretic pre-image of a hypersurface $H \subset B$ in the base of a Lagrangian fibration $f : X \to B$. Then, the classes $[D], f^*[H] \in H^2(X, \mathbb{Z})$ are proportional. Since $[H]^{n+1} = 0$, also $[D]^{n+1} = 0$ in $H^{2n+2}(X, \mathbb{Z})$, and therefore, $q(D) = 0$.
- Assume that $D$ is of general type. Then $q(D) > 0$ is proved by excluding $q(D) < 0$ and $q(D) = 0$. Indeed, the former would imply that $D$ is uniruled as explained before. The latter is excluded by observing that $\omega_D$ is nef by Proposition 3.3 and big, for $D$ is of general type. However, this implies $\int_D c_1(\omega_D)^{2n-1} > 0$ which excludes $q(D) = 0$.

8 | EXAMPLES

8.1 We provide examples of divisors for the first two situations in the case that $X$ is the Hilbert scheme $X = S^{[2]}$ of a K3 surface $S$.

(i) The natural example for Case (1) is the exceptional divisor $D = E$ of the Hilbert–Chow morphism $\pi : S^{[2]} \to S^{(2)}$. It is well known that $q(E) = -2$ and it is a $\mathbb{P}^1$ bundle over the diagonal $S \subset S^{(2)}$. More explicitly:

- The divisor $E$ is naturally isomorphic to $\mathbb{P}(\Omega_S)$.
- The restriction of the symplectic form of $S^{[2]}$ to $E$ is the pullback of the symplectic form on $S$ via the projection $\mathbb{P}(\Omega_S) \to S$.
- The characteristic foliation $F$ is the relative tangent bundle $T_\pi$ of the map $\pi : \mathbb{P}(\Omega_S) \to S$.
- The leaves are the $\mathbb{P}^1$ contracted by $\pi$, which via the identification with $\mathbb{P}(\Omega_S)$ is just the projection to $S$. Hence, the diagonal $S \subset X^{(2)}$ is the space of leaves $D/F$. 
(ii) Assume that $S$ admits a genus one fibration. Then $S^{[2]}$ comes with a natural Lagrangian fibration $\pi : S^{[2]} \to \mathbb{P}^2 \cong (\mathbb{P}^1)^{[2]}$ over the Hilbert scheme of two points on $\mathbb{P}^1$. The pre-image $D$ of a generic line $\ell \subset \mathbb{P}^2$ is a hypersurface which is smooth by Bertini and satisfies $q(D) = 0$. The leaves of the characteristic foliation on $D$ are contained in the fibres of $D \to \ell$ but the very general ones are not compact, that is, they are dense in the fibres. This follows from the implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) proved in Sections 5.2 and 5.4.

The situation changes if $\ell \subset \mathbb{P}^2$ is special. For example, if $S_0$ is a smooth fibre of $S \to \mathbb{P}^1$, then the pre-image of $\ell_0 := \{(0, t) \mid t \in \mathbb{P}^1\} \subset (\mathbb{P}^1)^{[2]}$ is the hypersurface $D_0 := \pi^{-1}(\ell_0) = \{(p, q) \mid p \in S_0\}$. It still satisfies $q(D_0) = 0$, but it is not smooth. In fact, it is not even normal, and its normalisation is the natural map $\text{Bl}_\Delta(S_0 \times S) \to D_0$ from the blow-up in the diagonal in $S_0 \times S_0$ which restricts to the degree two map $S_0^{[2]} \to S^{[2]} \subset D_0 \subset S^{[2]}$ and is injective on the complement. The fibre of $\pi : D_0 \to \ell_0$ over a point $\{0, t\}$, $t \neq 0$, is the surface $S_0 \times S_t$, which is smooth for all but finitely many $t$. The singularities of $D_0$ have an effect on the characteristic foliation (of the smooth part) of $D_0$: The leaves in the generic fibre $\pi^{-1}(\{0, t\})$ are the curves $S_0 \times x$, $x \in S_t$, which, in particular, are not dense in the smooth(!) fibre $S_0 \times S_t$.

8.2 As we have just seen, if the hypersurface $D \subset X$ is not smooth, then typically, the conditions (i)–(iv) are not equivalent.

(i) Let us first discuss this in Case (2). Consider the pre-image $f^{-1}(\Delta) \subset X$ of the discriminant divisor of a Lagrangian fibration $f : X \to B$. Note that even for $\Delta$ irreducible its pre-image may be reducible. By [21, Thm. 1.2], the characteristic foliation of any irreducible component of $f^{-1}(\Delta)$ is algebraically integrable. Assume that there is a component of $D$ of $f^{-1}(\Delta)$ such that $D = f^{-1}(f(D))$. This happens, for instance, when $X$ is general among the hyperkähler manifolds with a Lagrangian fibration. Then, $q(D) = 0$ but its characteristic foliation is algebraically integrable. This divisor satisfies (iii) and (iv) of Case (2) but not (i).

(ii) We turn to Case (3). Consider a smooth cubic fourfold $Y \subset \mathbb{P}^5$ and its Fano variety of lines $X := F(Y)$, which is a hyperkähler fourfold. The set of lines contained in a hyperplane section $Y \cap H$ is a Lagrangian surface $F(Y \cap H) \subset X$ which for generic $H$ is smooth and of general type. For a one-dimensional family $\{Y \cap H_t\}$ of hyperplane sections, these Lagrangian surfaces sweep out a hypersurface $D \subset X$. Then, $q(D) > 0$, since for a generic cubic fourfold, the Picard number of $X = F(Y)$ is one.

According to Corollary 2.6 (i), any leaf that intersects a generic $F(Y \cap H_t)$ is contained in it. However, if $D$ were smooth, then the results if Section 6.4 would imply that the generic leaf is dense. Contradiction. Hence, for no one-dimensional family of hyperplane section, $\{Y \cap H_t\}$ can the associated hypersurface $D$ be smooth.

In particular, this is an example of a singular divisor that satisfies (iii) and (iv) of Case (3) but not (i).

More abstractly, a smooth ample hypersurface $D \subset X$ in a hyperkähler fourfold does not contain any Lagrangian surface, cf. the proof of Proposition 6.2. In particular, for a general cubic fourfold $Y$, a smooth Lagrangian surface $F(Y \cap H_t)$ cannot be contained in any smooth divisor of $X = F(Y)$.

† We wish to thank the referee for this observation.
ACKNOWLEDGEMENTS
We would like to thank R. Abugaliev and J.-B. Bost for discussions, for answering our questions and for their help with the literature. We are particularly grateful to the referee for numerous corrections, instructive comments and helpful suggestions.

We do not claim any originality for the results presented in this survey, although we sometimes give alternative arguments or provide more details. We hope that the survey contributes to the dissemination of these results.

Open access funding enabled and organized by Projekt DEAL.

JOURNAL INFORMATION
The Bulletin of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID
Daniel Huybrechts https://orcid.org/0000-0003-4397-4836

REFERENCES
1. R. Abugaliev, Characteristic foliation on vertical hypersurfaces on holomorphic symplectic manifolds with Lagrangian fibration, arXiv:1909.07260.
2. R. Abugaliev, Characteristic foliation on hypersurfaces with positive Beauville–Bogomolov–Fujiki square, Int. Math. Res. Not. IMRN (2024), no. 5, 3690–3705.
3. E. Amerik and F. Campana, Characteristic foliation on non-uniruled smooth divisors on projective hyperkähler manifolds, J. London Math. Soc. 95 (2014), 115–127.
4. E. Amerik and L. Guseva, On the characteristic foliation on a smooth hypersurface in a holomorphic symplectic fourfold, Moscow Math. J. 18 (2018), 193–204.
5. F. Bogomolov and M. McQuillan, Rational curves on foliated varieties, Foliation theory in algebraic geometry, Simons Symp., Springer, Cham, 2016, pp. 21–51.
6. J.-B. Bost, Algebraic leaves of algebraic foliations over number fields, Publ. Math. Inst. Hautes Études Sci. 93 (2001), 161–221.
7. S. Boucksom, Divisorial Zariski decompositions on compact complex manifolds, Ann. Sci. Éc. Norm. Supér. 37 (2004), 45–76.
8. C. Camacho and A. Nito, Geometric theory of foliations, Springer, New York, 1985.
9. P. Deligne, Le groupe fondamental du complément d’une courbe plane n’ayant que des point doubles ordinaires est abélien (d’après W. Fulton), Bourbaki Seminar, Vol. 1979/80, Springer, Berlin, 1981, pp. 1–10.
10. J.-P. Demailly, C. Hacon, and M. Păun, Extension theorems, non-vanishing and the existence of good minimal models, Acta Math. 210 (2013), 203–259.
11. S. Druel, Quelques remarques sur la décomposition de Zariski divisoriale sur les variétés dont la première classe de Chern est nulle, Math. Z. 267 (2011), 413–423.
12. S. Druel, On foliations with nef anti-canonical bundle, Trans. Amer. Math. Soc. 369 (2017), 7765–7787.
13. D. Epstein, Foliations with all leaves compact, Ann. Inst. Fourier, Grenoble 26 (1976), 265–282.
14. D. Epstein, K. Millet, and D. Tischler, Leaves without holonomy, J. London Math. Soc. 16 (1977), 548–552.
15. R. Hermann, On the differential geometry of foliations, Ann. Math. 72 (1960), 445–457.
16. H. Holmann, On the stability of holomorphic foliations, Springer LNM 798 (1980), 192–202.
17. D. Huybrechts, Compact hyperkähler manifolds: basic results, Invent. Math. 135 (1999), 63–113.
18. D. Huybrechts, The Kähler cone of a compact hyperkähler manifold, Math. Ann. 326 (2003), 499–513.
19. D. Huybrechts, Compact hyperkähler manifolds, Calabi–Yau manifolds and related geometries, Springer, Berlin, 2002.
20. D. Huybrechts and M. Mauri, *Lagrangian fibrations*, Milan J. Math. 90 (2022), 459–483.
21. J. M. Hwang and K. Oguiso, *Characteristic foliation on the discriminant hypersurface of a holomorphic Lagrangian fibration*, Amer. J. Math. 134 (2009), 981–1007.
22. J.-M. Hwang and E. Viehweg, *Characteristic foliation on a hypersurface of general type in a projective symplectic manifold*, Compos. Math. 146 (2010), 497–506.
23. S. Kebekus, L. Solá Conde, and T. Matei, *Rationally connected foliations after Bogomolov and McQuillan*, J. Alg. Geom. 16 (2007), 65–81.
24. R. Lazarsfeld, *Positivity in algebraic geometry I*, Springer, Berlin, 2004.
25. F. Loray, J. Pereira, and F. Touzet, *Singular foliations with trivial canonical class*, Invent. Math. 213 (2018), 1327–1380.
26. Y. Manin, *Gauge field theory and complex geometry*, Grundlehren der mathematischen Wissenschaften, vol. 289, Springer, Berlin, 1997.
27. Y. Miyaoka, *Deformation of a morphism along a foliation and applications*, Proc. Symp. Pure Math. 46 (1987), 245–268.
28. Y. Miyaoka and S. Mori, *A numerical criterion for uniruledness*, Ann. Math. 124 (1986), 65–69.
29. I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge University Press, Cambridge, 2003.
30. J. Pereira, *Global stability for holomorphic foliations on Kähler manifolds*, Qual. Theory Dyn. Syst. 2 (2001), no. 2, 381–384.
31. J. Sawon, *Foliations on hypersurfaces in holomorphic symplectic manifolds*, IMRN 23 (2009), 4496–4545.