SOME SHARP DIFFERENTIAL SPHERE THEOREMS FOR NONNEGATIVE
SCALAR CURVATURE MANIFOLDS

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ABSTRACT. In this paper, we obtain several new intrinsic and extrinsic differential sphere theorems via Ricci flow. For intrinsic case, we show that a closed simply connected \( n (\geq 4) \)-dimensional Riemannian manifold \( M \) is diffeomorphic to \( S^n \) if one of the following conditions holds point-wisely:

\[
(i) \quad R_0 > \left( 1 - \frac{6}{n(n-1)} \right) K_{\max}; \quad (ii) \quad \frac{\text{Ric}[4]}{4(n-1)} > \left( 1 - \frac{3}{2(n-1)} \right) K_{\max}.
\]

Here \( K_{\max} \), \( \text{Ric}[k] \) and \( R_0 \) stand for the maximal sectional curvature, the \( k \)-th weak Ricci curvature and the normalized scalar curvature. For extrinsic case, i.e., when \( M \) is a closed simply connected \( n (\geq 4) \)-dimensional submanifold immersed in \( \bar{M} \). We prove that \( M \) is diffeomorphic to \( S^n \) if it satisfies some pinching curvature conditions. The only involved extrinsic quantities in our pinching conditions are the maximal sectional curvature \( \bar{K}_{\max} \) and the squared norm of mean curvature vector \( |H|^2 \). More precisely, we show that \( M \) is diffeomorphic to \( S^n \) if one of the following conditions holds:

1. \( R_0 \geq \left( 1 - \frac{2}{n(n-1)} \right) K_{\max} + \frac{4(n-1)}{n(n-2)} |H|^2 \), and strict inequality is achieved at some point;
2. \( \frac{\text{Ric}[2]}{2} \geq (n-2) \bar{K}_{\max} + \frac{4}{n-2} |H|^2 \), and strict inequality is achieved at some point;
3. \( \frac{\text{Ric}[2]}{2} \geq \frac{4(n-3)}{n-2} \left( \bar{K}_{\max} + |H|^2 \right) \), and strict inequality is achieved at some point.

It is worth pointing out that, in the proof of extrinsic case, we apply suitable complex orthonormal frame and simplify the calculations considerably. We also emphasize that all the pinching constants in intrinsic and extrinsic cases are optimal for \( n = 4 \) except condition (1).

Keywords and phrases: sphere theorems, isotropic curvature, positive scalar curvature, submanifold

1. Introduction

It is a basic problem in Riemannian geometry to classify closed Riemannian manifolds in the category of either topology, diffeomorphism, or isometry under some curvature conditions. Among a huge literature on this problem, the uniqueness of sphere under pinched curvatures accounts for a large proportion. One of the reasons for studying uniqueness of sphere is the simpleness of its topology. These uniqueness results are usually called topology sphere theorems (in the homeomorphism sense), differential sphere theorems (in the diffeomorphism sense), and isometry (or rigidity) sphere theorems (in the isometry sense).

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Suppose $M$ is a closed $n$-dimensional Riemannian manifold. If $n = 2$ and $M$ has positive Gaussian curvature, then one can easily see from Gauss-Bonnet formula that $M$ must be a topological sphere. Since the differential structure is unique on a 2-sphere, $M$ must be diffeomorphic to a standard 2-sphere $S^2$. When $n = 3$, the Riemannian curvature tensor is uniquely determined by the Ricci tensor. Hamilton [16] showed that if a closed 3-dimensional manifold has a metric with positive Ricci curvature, then it must be diffeomorphic to a spherical space form. Moreover, if $M$ is simply connected, $M$ must be diffeomorphic to $S^3$. Hamilton [17] classified all closed 3-dimensional Riemannian manifold with nonnegative Ricci curvature. Therefore, in this paper, we focus our attention on the dimension $n \geq 4$ and study sphere theorems with pinched curvatures.

The study of sphere theorems under pinched sectional curvatures goes back to a question of Hopf. In 1951, Rauch [28] showed that a closed simply connected Riemannian manifold with globally $\delta$-pinched ($\delta \approx 0.75$) sectional curvature is homeomorphic to a sphere. Rauch also proposed a question of what the optimal pinching constant should be. Berger [3] and Killingenberg [21] proved that $\delta = \frac{1}{4}$ is the optimal pinching constant. Since on a sphere of arbitrary dimension, the differential structure is not necessarily unique, it is natural to ask that if $\frac{1}{4}$-pinched sectional curvature is necessary for a differential sphere? This question was finally answered by Brendle and Schoen [9] via the Ricci flow.

Another important differential sphere theorem via Ricci flow is due to Böhm and Wilking [4]. They proved that closed manifolds with 2-positive curvature operator are spherical space forms. Moreover, Berger [3] classified all manifolds with weakly $1/4$-pinched curvatures in the homeomorphism sense. Brendle and Schoen [8] provided a classification, up to a diffeomorphism, of all manifolds with weakly $1/4$-pinched curvatures. For more sphere theorems under pinched sectional curvatures, we refer the reader to a good survey book of Brendle [7].

It is well known that the complex projective space $\mathbb{C}P^n$ with Fubini-Study metric has exactly pointwise $\frac{1}{4}$-pinched sectional curvature (see also Example 3.3). Therefore, Brendle-Schoen’s theorem is optimal for even dimension. It is natural to study sphere theorems under other pinched curvature conditions. In 1990’s, Yau collected some open problems and he wrote in Problem 12 ([35]):

“The famous pinching problem says that on a compact simply connected manifold if $K_{\min} > \frac{1}{4}K_{\max}$, then the manifold is homeomorphic to a sphere. If we replace $K_{\max}$ by normalized scalar curvature, can we deduce similar pinching results?”

Classical examples (see [14, Example 1], see also Example 3.3 in this paper) show that the pinching constant is at least $\frac{n-1}{n+2}$. Therefore Yau’s problem can be written in a more concrete way ([14, Yau Conjecture 1]):

**Conjecture** (Yau 1990). Let $(M^n, g)$ be a closed simply connected Riemannian manifold. Denote by $R_0$ the normalized scalar curvature of $M^n$. If $K_{\min} > \frac{n-1}{n+2}R_0$, then $M^n$ is diffeomorphic to a standard sphere $S^n$.

If $K_{\min} > \left(1 - \frac{6}{n-n+6}\right)R_0$, $n \geq 4$, Gu and Xu [14] proved $M$ must be diffeomorphic to a standard sphere, which partially answered Yau’s problem. Moreover, if $M$ is an Einstein manifold, Gu and Xu [33] proved the pinching constant $\frac{n-1}{n+2}$ is optimal and gave an isometric sphere theorem. When the dimension $n = 4$, Costa and Ribeiro Jr. [11] proved Yau’s conjecture. They actually used a weaker assumption by replacing sectional curvature by biorthogonal curvature condition. We
can prove when $K_{\min} > \left(1 - \frac{12}{n^2 - n + 12}\right)R_0$, $M$ must be diffeomorphic to $S^n$. However, when we finish this paper, we know from Professor Hong-Wei Xu that he and his collaborators obtained the same result [15] independently. We would like to thank Professor Hong-Wei Xu for sending their manuscript [15]. For readers’ convenience, we still give a complete proof of this result in Section 3 (see Theorem 3.2).

It is also interesting to study sphere theorems with normalized scalar curvature pinched by $K_{\max}$. Gu and Xu [14, Theorem 1] showed that if $R_0 > \frac{12}{5(n-1)}K_{\max}$, $n \geq 4$, then $M$ is diffeomorphic to a spherical space form. Based on an example of $\mathbb{O}P^2$, the authors also posed a Conjecture (see [14, Conjecture 1]):

**Conjecture.** Let $M^n(n \geq 4)$ be a closed and simply connected Riemannian manifold. If $R_0 > \frac{3}{2}K_{\max}$, then $M$ is diffeomorphic to $S^n$.

We also get a new differential sphere theorem in this direction:

**Theorem 1.1.** Let $M^n(n \geq 4)$ be a closed and simply connected Riemannian manifold. If

$$R_0 > \left(1 - \frac{6}{n(n-1)}\right)K_{\max},$$

then $M$ is diffeomorphic to $S^n$.

**Remark 1.1.** Under the above pinching condition, Gu-Xu-Zhao [15] actually proved $M$ is diffeomorphic to $S^n$ when $n \leq 6$ and homeomorphic to $S^n$ when $n \geq 7$.

For pinched Ricci curvature and sectional curvature, we also have the following sphere theorem.

**Theorem 1.2.** Let $M^n(n \geq 4)$ be a closed and simply connected Riemannian manifold. If

$$\frac{\text{Ric}^4_M}{4(n-1)} > \left(1 - \frac{3}{2(n-1)}\right)K_{\max},$$

then $M$ is diffeomorphic to $S^n$.

**Remark 1.2.**
1. Gu-Xu-Zhao [15] actually proved $M$ is diffeomorphic to $S^n$ when $\text{Ric}_M > \left(1 - \frac{3}{2(n-1)}\right)K_{\max}$.
2. We would like to point out that both of the pinching constants in Theorem 1.1 and Theorem 1.2 are optimal when $n = 4$ (see Example 3.3).

It is also of interest to study sphere theorems for submanifolds. In recent years, many authors investigated related problems and plenty of works were obtained (e.g. [1, 2, 14, 18, 19, 23, 31–34] and therein). We also get sphere theorems for submanifolds corresponding to Theorem 1.1 and Theorem 1.2, see Theorem 4.2, Theorem 4.1, and Theorem 4.3. Besides these results, we use complex orthonormal frames to obtain the following new sphere theorems. The assumptions of these theorems only involve $R_0$, $\text{Ric}^{[2]}$, $\bar{K}_{\max}$ and $|H|^2$.

We prove the following three theorems which are generalizations of Gu-Xu’s results [14, Theorem 3, Theorem 4], Xu-Gu’s result [31, Theorem 1.1], Anderw-Baker’s result [1, Theorem 1], Liu-Xu-Ye-Zhao’s result [23, Corollary 1.2] and Xu-Tian’s result [34, Theorem 1.1].
Theorem 1.3. Suppose \( M^n(n \geq 4) \) is a closed and simply connected submanifold of \( \bar{M}^N \) satisfying
\[
R_0 \geq \left( 1 - \frac{2}{n(n-1)} \right) \bar{K}_{\max} + \frac{n(n-2)}{(n-1)^2} |H|^2,
\]
with strict inequality at some point, then \( M \) is diffeomorphic to \( S^n \).

Theorem 1.4. Suppose \( M^n(n \geq 4) \) is a closed and simply connected submanifold of \( \bar{M}^N \) satisfying
\[
\frac{\text{Ric}_{[2]}^2}{2} \geq (n-2)\bar{K}_{\max} + \frac{n^2}{8} |H|^2,
\]
with strict inequality at some point, then \( M \) is diffeomorphic to \( S^n \).

The pinching condition in Theorem 1.4 is optimal. In fact, when \( \bar{M} \) is the space form \( F^N(c), c > 0 \), Ejiri [12] obtained a rigidity theorem for minimal submanifolds under the pinching condition
\[
\text{Ric}_M > (n-2)c.
\]
Xu-Gu [32] obtained an extension of Ejiri’s results for constant mean curvature submanifolds in the space form \( F^N(c) \) under the condition
\[
\text{Ric}_M > (n-2) \left( c + |H|^2 \right) > 0.
\]
They also obtained a topological sphere theorem for general submanifolds in the space form \( F^N(c), c \geq 0 \) under the same pinching condition mentioned above by using Lawson-Simons theory for stable integral currents [22, 30]. Motivated by these facts, the authors posed the following Conjecture (c.f., [32, Conjecture A]):

Conjecture. Let \( M^n(n \geq 4) \) be a closed and simply connected orientated submanifold in the space form \( F^N(c) \). If \( \text{Ric}_M > (n-2) \left( c + |H|^2 \right) > 0 \), then \( M \) is diffeomorphic to \( S^n \).

Here is a generalization of Gu-Xu’s result [32, Theorem 4.2].

Theorem 1.5. Suppose \( M^n(n \geq 4) \) is a closed and simply connected submanifold of \( \bar{M}^N \) satisfying
\[
\frac{\text{Ric}_{[2]}^2}{2} \geq \frac{n(n-3)}{n-2} \left( \bar{K}_{\max} + |H|^2 \right),
\]
with strict inequality at some point, then \( M \) is diffeomorphic to \( S^n \).

Remark 1.3. The Bonnet-Myers theorem [26] claimed that every complete Riemannian manifold with Ricci curvature bounded from below by a positive constant is compact. For complete noncompact Riemannian manifold with quasi-positive sectional curvature, the soul theorem [10, 13, 27] claimed that such manifold is diffeomorphic to the Euclidean space. Thus, one can consider the sphere theorems for complete Riemannian manifolds with similar curvature pinching conditions in the above theorems.

This paper is organized as follows. In Section 2, we list some notations and known facts. In Section 3, we prove some intrinsic differential sphere theorems with pinched normalized scalar curvatures and pinched Ricci curvatures. In Section 4, we study a Riemannian manifold immersed into another and give several new extrinsic topology sphere theorems and differential sphere theorems.
2. Preliminaries

In this section, we will fix some notations and list several known facts which will be used in next two sections.

Let \((M^n, \langle , \rangle)\) be a Riemannian manifold, \(\nabla\) be the Levi-Civita connection related to \(\langle , \rangle\) and \(R\) be the Riemannian curvature tensor defined by

\[
R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \forall X, Y \in T M.
\]

Denote

\[
K(X, Y) := \langle R(X, Y)V, Z \rangle.
\]

Define

\[
K(X, Y) := R(X, Y, X, Y), \quad \forall X, Y \in T M.
\]

Denote \(K(X, Y)\) by \(K(\pi)\) if \(X, Y\) are orthonormal and \(\pi = \text{span} \{X, Y\}\). By the linearity and symmetry of \(R\), it is easy to check the following identities.

\[
\text{Lemma 2.1. For all } X, Y, Z, W \in T M \text{ and } a, b \in \mathbb{R}, \text{ we have}
\]

\[
\begin{align*}
K(X + Y, X - Y) &= 4K(X, Y), \\
K(X, Y + Z) + K(X, Y - Z) &= 2(K(X, Y) + K(X, Z)), \\
K(aX, bY) &= a^2b^2K(X, Y),
\end{align*}
\]

(2.1) \[4R(X, Y, Z) = K(X, Y + Z) - K(X, Y - Z),\]

(2.2) \[24R(X, Y, Z, W) = K(X + Z, Y + W) + K(X - Z, Y - W) + K(Y + Z, X - W)
+ K(Y - Z, X + W) - K(X + Z, Y - W) - K(X - Z, Y + W)
- K(Y + Z, X + W) - K(Y - Z, X - W).\]

Identities (2.1) and (2.2) actually were first used by Karcher [20] to give a short proof of Berger’s curvature tensor estimate.

Let \(\bar{\left(M^N, \bar{g}\right)}\) be another Riemannian manifold such that there exists an isometrically immersion

\[
f : (M^n, \langle , \rangle) \to \bar{\left(M^N, \bar{g}\right)}.
\]

When we do calculation on the submanifold, we always omit \(f\) and also write \(\bar{g}\) as \(\langle , \rangle\). Let \(\{e_1, \ldots, e_N\}\) be a local orthonormal frame on \(\bar{M}\) such that \(\{e_1, \ldots, e_n\}\) form a local orthonormal frame of \(M\). Let \(\{\omega^1, \ldots, \omega^n\}\) be the coframe of \(\{e_1, \ldots, e_n\}\). Define \(\bar{R}\) and \(\bar{K}\) on \(\bar{M}\) similarly as those on \(M\). In what follows, without special explanation, \(i, j, k, l\) will always range from 1 to \(n\) and \(\alpha, \beta, \gamma\) will always range from \(n + 1\) to \(N\). The second fundamental form is defined to be

\[
B = h^a_\alpha \omega^\beta \otimes \omega^\gamma \otimes e_\alpha.
\]

The squared norm of \(B\) is \(|B|^2 = \sum_{i,j,a} (h^a_{ij})^2\). Write \(H^\alpha = \frac{1}{n} \sum h^\alpha_{ii}\), the mean curvature vector is given by \(H = H^\alpha e_\alpha\), and the (normalized) mean curvature is \(H = \sqrt{\sum_{\alpha} (H^\alpha)^2}\).

The Gauss equation can be written as

\[
R_{ijkl} = \bar{R}_{ijkl} + \sum_{\alpha} \left(h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}\right),
\]
where \( R_{ijkl} = R(e_i, e_j, e_k, e_l) \) and \( \tilde{R}_{ijkl} = \tilde{R}(e_i, e_j, e_k, e_l) \). In tensor language, Gauss equation also can be written as

\[
R = \tilde{R}^T + \frac{1}{2} \sum_\alpha h^\alpha \otimes h^\alpha := \tilde{R}^T + \frac{1}{2} B \otimes B,
\]

where \( \tilde{R}^T \) means the restriction of \( \tilde{R} \) on \( TM \), \( \otimes \) denotes the Kulkarni-Nomizu product of two symmetric (0,2)-tensor \( a \) and \( b \) which defined in local coordinates by

\[
(a \otimes b)_{ijkl} = a_{ik} b_{jl} - a_{il} b_{jk} - a_{jk} b_{il} + a_{jl} b_{ik}.
\]

Fix \( p \in M, X, Y \in T_pM \), the following notations will be used throughout this paper:

\[
K_{\min}(p) = \min_{\pi \in T_pM} K(\pi), \quad K_{\max}(p) = \max_{\pi \in T_pM} K(\pi),
\]

\[
Ric(X, Y) = \sum_i R(X, e_i, Y, e_i), \quad Ric_{jj} = Ric(e_j, e_j), \quad R_0 = \frac{\sum_{i,j} R_{ijj}}{(n(n-1))},
\]

\[
[e_i, \cdots, e_k] = \text{span} \{e_i, \cdots, e_k\}, \quad \forall 1 \leq i_1 < i_2 < \cdots < i_k \leq n,
\]

\[
Ric^k [e_{i_1}, \cdots, e_{i_k}] = \sum_{j=1}^k Ric_{ijj}, \quad Ric_{\min}^k(p) = \min_{[e_{i_1}, \cdots, e_{i_k}] \in T_pM} Ric^k [e_{i_1}, \cdots, e_{i_k}](p),
\]

where \( Ric^k [e_{i_1}, \cdots, e_{i_k}] \) is called \( k \)-th weak Ricci curvature of \( [e_{i_1}, \cdots, e_{i_k}] \) which was first introduced by Gu-Xu in [14]. One can also give similar notations as above on \( M \). Since all our calculations is local (at \( p \)), we will always omit the letter ”\( p \)” in what follows.

Complexify \( TM \) to \( T^\mathbb{C}M \) and assume \( e_1, \cdots, e_n \) is a local orthonormal frame of \( T^\mathbb{C}M \). Extend \( R, \tilde{R}, B \) and \( \langle \cdot, \cdot \rangle \) \( \mathbb{C} \)-linearly and denote by

\[
h^\varphi_{ij} = \langle B(e_i, \bar{e}_j), e_\alpha \rangle, \quad R_{i\bar{j}j\bar{i}} = \tilde{R}(e_i, e_j, \bar{e}_i, \bar{e}_j), \quad Ric_{ij} = \sum_{j=1}^n R_{ijj}.
\]

It is easy to check

\[
h^\varphi_{ij} \in \mathbb{R}, \quad h^\varphi_{ij} = \overline{h^\varphi_{ji}}, \quad R_{i\bar{j}j\bar{i}} \in \mathbb{R}, \quad \sum_{i,j=1}^n R_{i\bar{j}j\bar{i}} = n(n-1)R_0.
\]

A direct computation via the complex linearity gives the following complex Gauss equation, for \( i \neq j \),

\[
R_{i\bar{j}j\bar{i}} = \tilde{R}_{i\bar{j}j\bar{i}} + \sum_\alpha \left( h^\varphi_{ij} h^\varphi_{j\bar{i}} - h^\varphi_{j\bar{i}} h^\varphi_{ij} \right) = \tilde{R}_{i\bar{j}j\bar{i}} + |H|^2 + \sum_\alpha \left( H^\alpha \left( h^\varphi_{ij} + h^\varphi_{j\bar{i}} \right) + \overline{h^\varphi_{ij}} h^\varphi_{j\bar{i}} - \left| h^\varphi_{ij} \right|^2 \right),
\]

where \( \overline{h^\varphi_{ij}} = h^\varphi_{ij} - H^\varphi \delta_{ij} \). Therefore, the complex Ricci curvature is given by

\[
Ric_{ij} = \sum_{j=1}^n R_{i\bar{j}j\bar{i}} + (n-1)|H|^2 + \sum_\alpha \left( (n-2) H^\alpha h^\varphi_{ij} - \sum_{k=1}^n \left| h^\varphi_{ik} \right|^2 \right).
\]
The curvature operator $\mathcal{R} : \Lambda^2 TM \rightarrow \Lambda^2 TM$ is defined as follows:

$$\langle \mathcal{R}(X \wedge Y), Z \wedge W \rangle := R(X, Y, Z, W).$$

A linear subspace $V \in T^C M$ is called totally isotropic if $g(\nu, \nu) = 0$, for all $\nu \in V$. In other words, for all $\nu = X + \sqrt{-1} Y \in V$,

$$|X|^2 - |Y|^2 = \langle X, Y \rangle = 0.$$

To each complex 2-plane $\sigma \in \Lambda^2 T^C M$ the complex sectional curvature $K(\sigma)$ is defined to be

$$K(\sigma) := \frac{\langle \mathcal{R}(z \wedge w), \bar{z} \wedge \bar{w} \rangle}{|z \wedge w|^2},$$

where $\sigma = \text{span}_C \{z, w\}$. It is obvious that $K(\sigma) \in \mathbb{R}$. $K(\sigma)$ is called isotropic curvature if $\sigma$ is totally isotropic. The concept of isotropic curvature was first introduced by Micallef and Moore [25].

It is easy to check that, for every totally isotropic 2-plane, there exists an orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$, such that

$$\sigma = \text{span}_C \{e_1 + \sqrt{-1} e_2, e_3 + \sqrt{-1} e_4\}.$$ 

Moreover, by $\mathbb{C}$-linearity of $\mathcal{R}$ and $\langle \, , \rangle$, we have

$$4K(\sigma) = \langle \mathcal{R}((e_1 + \sqrt{-1} e_2) \wedge (e_3 + \sqrt{-1} e_4)), (e_1 - \sqrt{-1} e_2) \wedge (e_3 - \sqrt{-1} e_4) \rangle$$

$$= \langle \mathcal{R}(e_1 \wedge e_3 - e_2 \wedge e_4 + \sqrt{-1}(e_1 \wedge e_4 + e_2 \wedge e_3)), e_1 \wedge e_3 - e_2 \wedge e_4 - \sqrt{-1}(e_1 \wedge e_4 + e_2 \wedge e_3) \rangle$$

$$= \langle \mathcal{R}(e_1 \wedge e_3 - e_2 \wedge e_4), e_1 \wedge e_3 - e_2 \wedge e_4 \rangle + \langle \mathcal{R}(e_1 \wedge e_4 + e_2 \wedge e_3), e_1 \wedge e_4 + e_2 \wedge e_3 \rangle$$

$$= R_{1313} + R_{2424} - 2R_{1324} + R_{1414} + R_{2323} + 2R_{1423}$$

$$= R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234},$$

where we have used Bianchi identity in the last equality. When $M$ has positive isotropic curvature, Micallef and Moore proved the following theorem.

**Theorem A** (Micallef-Moore [24]). Let $M$ be a closed $n(\geq 4)$-dimensional Riemannian manifold. Assume for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$, the following inequality holds

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0.$$ 

Then $\pi_1(M) = 0$ for $2 \leq k \leq \left[\frac{n}{2}\right]$. In particular, if $M$ is simply connected, then $M$ is homeomorphic to a sphere.

When $M \times \mathbb{R}$ has nonnegative isotropic curvature, i.e., (c.f. [5])

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$, we have the following differential sphere theorem.

**Theorem B** (Brendle [5]). Let $(M, g_0)$ be a closed Riemannian manifold of dimension $n \geq 4$ such that $M \times \mathbb{R}$ has positive isotropic curvature. Then the normalized Ricci flow with initial metric $g_0$ exists for all time and converges to a constant curvature metric as $t \rightarrow \infty$. 
Remark 2.1. Theorem B is also true if one can verify inequality (2.6) for $\lambda \in [0, 1]$. Actually, if inequality (2.6) holds for $\lambda \in [0, 1]$, then for $\mu \in [-1, 0]$, consider orthonormal four-frame \{e_1, e_2, e_3, -e_4\}, we have

$$R_{1313} + \mu^2 R_{1414} + R_{2323} + \mu^2 R_{2424} - 2\mu R_{1234} = R_{1313} + \mu^2 R_{1414} + R_{2323} + \mu^2 R_{2424} - 2(-\mu)K(e_1, e_2, e_3, -e_4) > 0.$$  

Seshadri [29] studied the classification of closed Riemannian manifolds with nonnegative isotropic curvature. When $M \times \mathbb{R}^2$ has nonnegative isotropic curvature, i.e., (c.f. [9])

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} \geq 0,$$

(2.7)

for all points $p \in M$, all orthonormal four-frames \{e_1, e_2, e_3, e_4\} $\subset T_p M$ and all $\lambda, \mu \in [-1, 1]$, or equivalently $M$ has nonnegative complex sectional curvature (c.f. [25, Remark 3.3] or [7, Proposition 17.8]), we have the following classification theorem.

**Theorem C (Brendle-Schoen [8]).** Let $M$ be a closed, locally irreducible Riemannian manifold of dimension $n \geq 4$. If $M \times \mathbb{R}^2$ has nonnegative isotropic curvature, then one of the following statements holds:

(i) $M$ is diffeomorphic to a spherical space form;

(ii) $n = 2m$ and the universal cover of $M$ is a Kähler manifold biholomorphic to $\mathbb{C}P^m$;

(iii) the universal cover of $M$ is isometric to a compact symmetric space.

**Remark 2.2.** Similar to the remark after Theorem B, this classification theorem is true if we can verify the condition (2.7) for all four-frame \{e_1, e_2, e_3, e_4\} and all $\lambda, \mu \in [0, 1]$.

### 3. Sphere theorems for pinched curvatures

In this section, we will prove the intrinsic sphere theorems listed in the introduction. Before we prove these theorems, we give a useful lemma.

**Lemma 3.1.** Let \{e_1, e_2, e_3, e_4\} be any orthonormal four-frame, then we have

$$12R_{1234} = 4 \sum_{1 \leq i < j \leq 4} R_{ijij} - 2 \left( R_{1313} + R_{1414} + R_{2323} + R_{2424} \right) - \left( K(e_1 + e_3, e_2 - e_4) + K(e_1 - e_3, e_2 + e_4) + K(e_2 + e_3, e_1 + e_4) + K(e_2 - e_3, e_1 - e_4) \right).$$

**Proof.** First note that

$$\left\{ \frac{e_1 + e_3}{\sqrt{2}}, \frac{e_1 - e_3}{\sqrt{2}}, \frac{e_2 + e_4}{\sqrt{2}}, \frac{e_2 - e_4}{\sqrt{2}} \right\}, \left\{ \frac{e_1 + e_4}{\sqrt{2}}, \frac{e_1 - e_4}{\sqrt{2}}, \frac{e_2 + e_3}{\sqrt{2}}, \frac{e_2 - e_3}{\sqrt{2}} \right\}.$$
are two orthonormal basises of span \{e_1, e_2, e_3, e_4\}. Therefore, by Lemma 2.1 we have

\begin{align*}
\sum_{1 \leq i < j \leq 4} R_{ijij} &= K(e_1 + e_3, e_1 - e_3) + K(e_1 + e_3, e_2 + e_4) + K(e_1 + e_3, e_2 - e_4) \\
&\quad + K(e_1 - e_3, e_2 + e_4) + K(e_1 - e_3, e_2 - e_4) + K(e_2 + e_4, e_2 - e_4).
\end{align*}

\begin{align*}
\sum_{1 \leq i < j \leq 4} R_{ijij} &= K(e_1 + e_4, e_1 - e_4) + K(e_1 + e_4, e_2 + e_3) + K(e_1 + e_4, e_2 - e_3) \\
&\quad + K(e_1 - e_4, e_2 + e_3) + K(e_1 - e_4, e_2 - e_3).
\end{align*}

Set \(X = e_1, Y = e_2, Z = e_3, W = e_4\) in (3.2), we have

\[24R_{1234} = K(e_1 + e_3, e_2 + e_4) + K(e_1 - e_3, e_2 - e_4) + K(e_2 + e_3, e_1 - e_4) + K(e_2 - e_3, e_1 + e_4) - K(e_1 + e_3, e_2 - e_4) - K(e_1 - e_3, e_2 + e_4) - K(e_2 + e_3, e_1 + e_4) - K(e_2 - e_3, e_1 - e_4) = K(e_1 + e_3, e_1 - e_3) + K(e_1 + e_3, e_2 + e_4) + K(e_1 + e_3, e_2 - e_4) + K(e_1 - e_3, e_2 + e_4) + K(e_1 - e_3, e_2 - e_4) + K(e_2 + e_3, e_2 - e_4) + K(e_1 + e_4, e_1 - e_4) + K(e_1 + e_4, e_2 + e_3) + K(e_1 + e_4, e_2 - e_3) + K(e_1 - e_4, e_2 + e_3) + K(e_1 - e_4, e_2 - e_3) - 2(K(e_1 + e_3, e_2 - e_4) + K(e_1 - e_3, e_2 + e_4) + K(e_2 + e_3, e_1 + e_4) + K(e_2 - e_3, e_1 - e_4)) - K(e_1 + e_3, e_1 - e_3) - K(e_2 + e_4, e_2 - e_4) - K(e_1 + e_4, e_1 - e_4) - K(e_2 + e_3, e_2 - e_3) = 8 \sum_{1 \leq i < j \leq 4} R_{ijij} - 4(R_{1313} + R_{1414} + R_{2323} + R_{2424}) - 2(K(e_1 + e_3, e_2 - e_4) + K(e_1 - e_3, e_2 + e_4) + K(e_2 + e_3, e_1 + e_4) + K(e_2 - e_3, e_1 - e_4)).
\]

In the last equality, we have used (3.1) and (3.2).

\(\square\)

The following theorem obtained by Gu-Xu-Zhao [15] independently. We list a proof here for reader’s convenience.

**Theorem 3.2.** Let \(M^n(n \geq 4)\) be a closed and simply connected Riemannian manifold. Assume the following pinching condition holds,

\[K_{\min} > \left(1 - \frac{12}{n^2 - n + 12}\right)R_0,\]

then \(M\) is diffeomorphic to \(S^n\).
Proof of Theorem [Theorem 3.2] By Theorem B, it is sufficient to prove (2.6) holds for every orthonormal four-frame \( \{e_1, e_2, e_3, e_4\} \) and \( \lambda \in [0, 1] \). By Lemma 3.1, we have

\[
12(R_{1313} + R_{2323} + R_{1234})
\]

\[
= 4 \sum_{1 \leq i < j \leq 4} R_{ijij} - 2(R_{1414} + R_{2424}) + 10(R_{1313} + R_{2323})
\]

\[-(K(e_1 + e_3, e_2 - e_4) + K(e_1 - e_3, e_2 + e_4)) \]

\[-(K(e_2 + e_3, e_1 + e_4) + K(e_2 - e_3, e_1 - e_4)) \]

\[
= 4 \sum_{1 \leq i < j \leq 4} R_{ijij} - 2(R_{1414} + R_{2424}) + 10(R_{1313} + R_{2323})
\]

\[-(4 \sum_{1 \leq i < j \leq 4} R_{ijij} - K(e_1 + e_3, e_2 + e_4) - K(e_1 - e_3, e_2 - e_4) - 4R_{1313} - 4R_{2424}) \]

\[-(4 \sum_{1 \leq i < j \leq 4} R_{ijij} - K(e_2 + e_3, e_1 + e_4) - K(e_2 - e_3, e_1 - e_4) - 4R_{2323} - 4R_{1414}) \]

\[-4 \sum_{1 \leq i < j \leq 4} R_{ijij} + 2(R_{1414} + R_{2424}) + 14(R_{1313} + R_{2323})
\]

\[+ K(e_1 + e_3, e_2 + e_4) + K(e_1 - e_3, e_2 - e_4) + K(e_2 + e_3, e_1 + e_4) + K(e_2 - e_3, e_1 - e_4), \]

where in the second equality, we have used (3.1) and (3.2). Thus,

\[
12(R_{1313} + R_{2323} + R_{1234}) \geq -2 \left(n(n - 1)R_0 - 2 \sum_{i=1}^{4} \sum_{j=5}^{n} R_{ijij} - \sum_{5 \leq i,j \leq n} \sum_{5 \leq i,j \leq n} R_{ijij} \right) + 48K_{min}
\]

\[\geq -2 \left[n(n - 1)R_0 - (2 \times 4(n - 4) + (n - 4)(n - 5))K_{min} \right] + 48K_{min}
\]

\[= 2(-n(n - 1)R_0 - (n(n - 1) + 12)K_{min}). \]

Hence, if

\[K_{min} > 1 - \frac{12}{n^2 - n + 12} \]

we obtain

\[R_{1313} + R_{2323} + R_{1234} > 0.\]

Replace \( e_4 \) by \(-e_4\), we obtain

\[R_{1313} + R_{2323} - R_{1234} > 0.\]

Hence,

\[R_{1313} + R_{2323} - |R_{1234}| > 0.\]

Similarly,

\[R_{1414} + R_{2424} - |R_{1234}| > 0.\]

Therefore,

\[R_{1313} + R_{2323} + \lambda^2(R_{1414} + R_{2424}) > (1 + \lambda^2)|R_{1234}| \geq 2\lambda R_{1234}.\]

Our conclusion follows immediately from Theorem B. \( \square \)
When the dimension \( n = 4 \), the following example indicates that our pinching constant is optimal.

**Example 3.3.** Consider the Fubini-Study metric on \( \mathbb{CP}^n \), then we have
\[
R(X, Y, X, Y) = 1 + 3 \langle JX, Y \rangle^2,
\]
for every orthonormal two-frame \( \{X, Y\} \), where \( J \) is the complex structure. Let \( n = 2m \), consider a local orthonormal frame \( \{e_1, \ldots, e_m, Je_1, \ldots, Je_m\} \), we have
\[
R(e_i, e_j, e_i, e_j) = 1, \quad \forall 1 \leq i \neq j \leq m,
\]
\[
R(e_i, Je_j, e_i, e_j) = 1, \quad \forall 1 \leq i \neq j \leq m,
\]
\[
R(e_i, Je_i, e_i, e_i) = 4, \quad \forall 1 \leq i \leq m,
\]
\[
R(Je_i, Je_j, Je_i, Je_j) = 1, \quad \forall 1 \leq i \neq j \leq m.
\]

Therefore,
\[
s = 4m(m - 1) + 8m = n(n + 2), \quad \text{Ric}_M = 2m + 2 = n + 2, \quad K_{\text{min}} = 1, \quad K_{\text{max}} = 4,
\]
\[
R_0 = \frac{s}{n(n-1)} = \frac{n+2}{n-1} = \frac{\text{Ric}_M}{n-1}, \quad K_{\text{min}} = \frac{n-1}{n+2} R_0 = \frac{n-1 \text{Ric}_M}{n+2} = \frac{n-1}{n+2} R_0 = \frac{\text{Ric}_M}{n-1} = \frac{n+2}{4(n-1)} K_{\text{max}}.
\]

When \( n = 4 \), we have
\[
R_0 = \frac{\text{Ric}_M}{3} = \frac{1}{2} K_{\text{max}}, \quad K_{\text{min}} = \frac{1}{2} R_0.
\]

**Proof of Theorem 1.1** We begin with the following identity:

\[
(3.3) \quad n(n-1) R_0 = \sum_{i,j=5}^n R_{ijj} + 2 \sum_{i=1}^4 \sum_{j=5}^n R_{ijj} + 2 \sum_{1 \leq i < j \leq 4} R_{ijj}.
\]

Notice that, for \( \lambda \in [0, 1] \),

\[
(3.4) \quad \sum_{1 \leq i < j \leq 4} R_{ijj} = \sum_{1 \leq i < j \leq 4} R_{ijj} - \frac{3}{2(1+\lambda^2)} \left( (R_{1313} + R_{2323}) + \lambda^2 (R_{1414} + R_{2424}) - 2\lambda R_{1234} \right)
\]
\[
+ \frac{3}{2(1+\lambda^2)} \left( (R_{1313} + R_{2323}) + \lambda^2 (R_{1414} + R_{2424}) - 2\lambda R_{1234} \right)
\]
\[
= R_{1212} + R_{3434} + \left( 1 - \frac{3}{2(1+\lambda^2)} \right) (R_{1313} + R_{2323}) + \left( 1 - \frac{3\lambda^2}{2(1+\lambda^2)} \right) (R_{1414} + R_{2424}) + \frac{3\lambda}{1+\lambda^2} R_{1234}
\]
\[
+ \frac{3}{2(1+\lambda^2)} \left( (R_{1313} + R_{2323}) + \lambda^2 (R_{1414} + R_{2424}) - 2\lambda R_{1234} \right).
\]

According to Lemma 3.1, replace \( e_4 \) by \( -e_4 \), we obtain

\[
(3.5) \quad 12 R_{1234} = -4 (R_{1212} + R_{3434}) - 2 (R_{1313} + R_{1414} + R_{2323} + R_{2424})
\]
\[
+ (K(e_1 + e_3, e_2 + e_4) + K(e_1 - e_3, e_2 - e_4) + K(e_2 + e_3, e_1 - e_4) + K(e_2 - e_3, e_1 + e_4)).
\]
Therefore,

\[
R_{1212} + R_{3434} + \left(1 - \frac{3}{2(1 + \lambda^2)}\right) (R_{1313} + R_{2323}) + \left(1 - \frac{3\lambda^2}{2(1 + \lambda^2)}\right) (R_{1414} + R_{2424}) + \frac{3\lambda}{1 + \lambda^2} R_{1234}
\]

\[
= \left(1 - \frac{\lambda}{1 + \lambda^2}\right) (R_{1212} + R_{3434}) + \left(1 - \frac{3 + \lambda}{2(1 + \lambda^2)}\right) (R_{1313} + R_{2323}) + \left(1 - \frac{3\lambda^2 + \lambda}{2(1 + \lambda^2)}\right) (R_{1414} + R_{2424})
\]

\[
+ \frac{\lambda}{4(1 + \lambda^2)} (K(e_1 + e_3, e_2 + e_4) + K(e_1 - e_3, e_2 - e_4) + K(e_2 + e_3, e_1 - e_4) + K(e_2 - e_3, e_1 + e_4))
\]

\[
\leq \left(1 - \frac{\lambda}{1 + \lambda^2}\right) \cdot 2K_{\text{max}} + \left(1 - \frac{3 + \lambda}{2(1 + \lambda^2)}\right) \cdot 2K_{\text{max}} + \left(1 - \frac{3\lambda^2 + \lambda}{2(1 + \lambda^2)}\right) \cdot 2K_{\text{max}} + \frac{16\lambda K_{\text{max}}}{4(1 + \lambda^2)}
\]

\[
= 3K_{\text{max}},
\]

where we have used

\[
1 - \frac{\lambda}{1 + \lambda^2} > 0, \quad 1 - \frac{3 + \lambda}{2(1 + \lambda^2)} \geq 0, \quad 1 - \frac{3\lambda^2 + \lambda}{2(1 + \lambda^2)} \geq 0, \quad \forall \lambda \in [0, 1].
\]

Thus, (3.3), (3.4) and (3.6) yield

\[
\frac{3}{1 + \lambda^2} \left( (R_{1313} + R_{2323}) + \lambda^2 (R_{1414} + R_{2424}) - 2\lambda R_{1234} \right)
\]

\[
\geq n(n - 1)R_0 - 6K_{\text{max}} - \left( \sum_{i,j=5}^{n} R_{ijij} + 2 \sum_{i=1}^{4} \sum_{j=5}^{n} R_{ijij} \right)
\]

\[
\geq n(n - 1)R_0 - 6K_{\text{max}} - (n - 4)(n - 5)K_{\text{max}} - 8(n - 4)K_{\text{max}}
\]

\[
= n(n - 1)R_0 - (n^2 - n - 6)K_{\text{max}}.
\]

Consequently, the assumption \(R_0 > \left(1 - \frac{6}{n(n-1)}\right)K_{\text{max}}\) combined with (3.7) imply

\[
(R_{1313} + R_{2323}) + \lambda^2 (R_{1414} + R_{2424}) - 2\lambda R_{1234} > 0.
\]

Our conclusion follows from Theorem B immediately.

\[
\square
\]

Remark 3.1. Example 3.3 also shows that our pinching constant in Theorem 1.1 is optimal when \(n = 4\).

Theorem 1.2 is actually an easy consequence of the following theorem.

Theorem 3.4. Let \(M^n (n \geq 4)\) be a closed Riemannian manifold. If

\[
\frac{\text{Ric}^{[4]}}{4(n-1)} > \left(1 - \frac{3}{2(n-1)}\right) K_{\text{max}},
\]

then \(M\) is diffeomorphic to a spherical space form. In particular, if \(M\) is simply connected, then \(M\) is diffeomorphic to \(S^n\).

Proof. Let \(D\) be a constant satisfying \(\text{Ric}^{[4]} > 4(n-1)D\). Then

\[
4(n - 1)D < \text{Ric}_{11} + \text{Ric}_{22} + \text{Ric}_{33} + \text{Ric}_{44} = \sum_{i=1}^{4} \sum_{j=5}^{n} R_{ijij} + 2 \sum_{1 \leq i < j \leq 4} R_{ijij}.
\]
Check the proof of Theorem 1.1, we actually have proved that for every \( \lambda \in [0, 1] \),
\[
\sum_{1 \leq i < j \leq 4} R_{ijij} \leq 3K_{\text{max}} + \frac{3}{2(1 + \lambda^2)} ((R_{1313} + R_{2323}) + \lambda^2 (R_{1414} + R_{2424}) - 2\lambda R_{1234}).
\]
Combined with (3.8), we obtain
\[
4(n - 1)D < (4(n - 1) - 6) K_{\text{max}} + \frac{3}{1 + \lambda^2} ((R_{1313} + R_{2323}) + \lambda^2 (R_{1414} + R_{2424}) - 2\lambda R_{1234}).
\]
Hence, if
\[
\frac{\text{Ric}^{[4]}}{4(n - 1)} > \left( 1 - \frac{3}{2(n - 1)} \right) K_{\text{max}},
\]
we have
\[
(R_{1313} + R_{2323}) + \lambda^2 (R_{1414} + R_{2424}) - 2\lambda R_{1234} > 0.
\]
We complete our proof. \( \Box \)

Moreover, if \( M \) is Einstein, we obtain the following

**Corollary 3.5.** Let \( M^n (n \geq 4) \) be a closed and simply connected Einstein manifold. If
\[
R_0 > \left( 1 - \frac{3}{2(n - 1)} \right) K_{\text{max}},
\]
then \( M \) is isometric (by scaling) to \( S^n \).

**Proof.** If \( M \) is Einstein, then \( \text{Ric} = c g \) for some positive constant \( c \), the normalized scalar curvature \( R_0 = \frac{\text{Ric}^{[4]}}{4(n - 1)} \). We have actually proved that the isotropic curvature is positive according to proof of Theorem 3.4. Therefore, by Brendle’s Theorem ([6, Theorem 1]) we obtain the conclusion. \( \Box \)

### 4. Submanifolds with pinching curvatures

In this section, we will prove some sphere theorems for a Riemannian manifold isometrically immersed into another with some pinching curvature conditions. It is worth pointing out that our pinching constants in this section also improve Gu-Xu’s corresponding pinching constants in [14] and [33].

Let \( R \) denote an algebraic curvature tensor, for every orthonormal four-frame \( \{e_1, e_2, e_3, e_4\} \) and \( \lambda, \mu \in [-1, 1] \), we give the following notation,
\[
I_{\lambda, \mu}(R) = \frac{1}{(1 + \lambda^2)(1 + \mu^2)} \left( R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda \mu R_{1234} \right),
\]
and we denote \( I_{\lambda, 1}(R) \) briefly by \( I_\lambda(R) \).

Therefore, by Gauss equation (2.3), we have
\[
I_\lambda(R) = I_\lambda(\bar{R}^T) + I_\lambda \left( \frac{1}{2} B \otimes B \right).
\]
Corresponding to Theorem 1.1, we have the following result:
Theorem 4.1. Let $M^n$ be an $n(\geq 4)$-dimensional closed submanifold in an $N$-dimensional Riemannian manifold $\bar{M}^N$.

1. If, pointwisely,
   \[ |B|^2 < \frac{2N(N - 1)}{3} \left[ \bar{R}_0 - \left( 1 - \frac{6}{N(N - 1)} \right) \bar{K}_{\text{max}} \right] + \frac{n^2 |H|^2}{n - 2}, \]
   then $M$ has positive isotropic curvature. Therefore, $\pi_k(M) = 0$ for $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. In particular, if $M$ is simply connected, then $M$ is homeomorphic to a sphere.

2. If, pointwisely,
   \[ |B|^2 < \frac{N(N - 1)}{3} \left[ \bar{R}_0 - \left( 1 - \frac{6}{N(N - 1)} \right) \bar{K}_{\text{max}} \right] + \frac{n^2 |H|^2}{n - 1}, \]
   then $M$ is diffeomorphic to a spherical space form. In particular, if $M$ is simply connected, then $M$ is diffeomorphic to $S^n$.

Proof. Let $\bar{D}$ be a constant satisfying $N(N - 1)\bar{D} < \sum_{i,j=1}^{N} \bar{R}_{ij}$. Then after a similar algebraic argument as the proof of Theorem 1.1 gives a similar inequality as (3.7):

\[ 6I_{\lambda} \left( \bar{R}^T \right) > N(N - 1)\bar{D} - (N^2 - N - 6)\bar{K}_{\text{max}}. \]

In the proof of [14, Lemma 9], the authors give the following estimate

\[ 2I_\lambda \left( \frac{1}{2} B \otimes B \right) \geq \frac{n^2 |H|^2}{n - 1} - |B|^2. \]

Thus, (4.1), (4.2) and (4.3) yield

\[ I_1(R) > \frac{1}{6} \left( N(N - 1)\bar{D} - (N^2 - N - 6)\bar{K}_{\text{max}} \right) + \frac{1}{4} \left( \frac{n^2 |H|^2}{n - 2} - |B|^2 \right) \]
\[ = \frac{1}{4} \left\{ 2N(N - 1) \left[ \bar{D} - \left( 1 - \frac{6}{N(N - 1)} \right) \bar{K}_{\text{max}} \right] + \frac{n^2 |H|^2}{n - 2} - |B|^2 \right\}. \]

Combined with [Theorem A], we complete the proof of Claim (1).

In the proof of [14, Lemma 11], the authors obtain

\[ 2I_{\lambda} \left( \frac{1}{2} B \otimes B \right) \geq \frac{n^2 |H|^2}{n - 1} - |B|^2, \quad \forall \lambda \in [0, 1] \]

Thus, (4.1), (4.2) and (4.4) give

\[ I_{\lambda}(R) > \frac{1}{6} \left( N(N - 1)\bar{D} - (N^2 - N - 6)\bar{K}_{\text{max}} \right) + \frac{1}{2} \left( \frac{n^2 |H|^2}{n - 1} - |B|^2 \right) \]
\[ = \frac{1}{2} \left\{ \frac{N(N - 1)}{3} \left[ \bar{D} - \left( 1 - \frac{6}{N(N - 1)} \right) \bar{K}_{\text{max}} \right] + \frac{n^2 |H|^2}{n - 1} - |B|^2 \right\}. \]

Then Claim (2) follows easily from [Theorem B].

After a similar argument we also have the following two extrinsic sphere theorems corresponding to [Theorem 3.2] and [Theorem 1.2].
Theorem 4.2. Let $M^n$ be an $n(\geq 4)$-dimensional closed submanifold in an $N$-dimensional Riemannian manifold $\bar{M}^N$.

1. If, pointwisely,
   \[ |B|^2 < \frac{N^2 - N + 12}{3} \left( \bar{K}_{\min} - \left( 1 - \frac{12}{N^2 - N + 12} \right) \bar{R}_0 \right) + \frac{n^2|H|^2}{n-2}, \]
   then $M$ has positive isotropic curvature. Therefore, $\pi_k(M) = 0$ for $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. In particular, if $M$ is simply connected, then $M$ is homeomorphic to a sphere.

2. If, pointwisely,
   \[ |B|^2 < \frac{N^2 - N + 12}{6} \left( \bar{K}_{\min} - \left( 1 - \frac{12}{N^2 - N + 12} \right) \bar{R}_0 \right) + \frac{n^2|H|^2}{n-1}, \]
   then $M$ is diffeomorphic to a spherical space form. In particular, if $M$ is simply connected, then $M$ is diffeomorphic to $S^n$.

Theorem 4.3. Let $M^n$ be an $n(\geq 4)$-dimensional closed submanifold in an $N$-dimensional Riemannian manifold $\bar{M}^N$.

1. If, pointwisely,
   \[ |B|^2 < \frac{8(N-1)}{3} \left( \frac{\text{Ric}^{[4]}_{\min}}{4(N-1)} - \left( 1 - \frac{3}{2(N-1)} \right) \bar{K}_{\max} \right) + \frac{n^2H^2}{n-2}, \]
   then $M$ has positive isotropic curvature. Therefore, $\pi_k(M) = 0$ for $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. In particular, if $M$ is simply connected, then $M$ is homeomorphic to a sphere.

2. If, pointwisely,
   \[ |B|^2 < \frac{4(N-1)}{3} \left( \frac{\text{Ric}^{[4]}_{\min}}{4(N-1)} - \left( 1 - \frac{3}{2(N-1)} \right) \bar{K}_{\max} \right) + \frac{n^2H^2}{n-1}, \]
   then $M$ is diffeomorphic to a spherical space form. In particular, if $M$ is simply connected, then $M$ is diffeomorphic to $S^n$.

Also we have the following corollary corresponding to Corollary 3.5.

Corollary 4.4. Let $M^n$ be an $n(\geq 4)$-dimensional closed Einstein submanifold in an $N$-dimensional Riemannian manifold $\bar{M}^N$. If, pointwisely,

\[ |B|^2 < \frac{8(N-1)}{3} \left( \frac{\text{Ric}^{[4]}_{\min}}{4(N-1)} - \left( 1 - \frac{3}{2(N-1)} \right) \bar{K}_{\max} \right) + \frac{n^2H^2}{n-2}, \]

then $M$ is isometric to a spherical space form. In particular, if $M$ is simply connected, then $M$ is isometric to $S^n$ (by scaling).

Remark 4.1. Using a similar method, we also can get a sphere theorem under pinched curvature by $K_{\min}$. But since the pinching constant is the same as Gu-Xu’s result in [14], we omit here.

Next we will use a complex orthonormal frame to state the proofs of Theorem 1.3, Theorem 1.4 and Theorem 1.5. One can verify that in suitable complex orthonormal frame, the calculations will be considerably simplified.
**Proof of Theorem 1.3** Let $e_1, \ldots, e_n$ be a local orthonormal frame of $TM$. For $\lambda, \mu \in [0, 1]$, define
\[ e_1 = \frac{e_1 + \sqrt{1-\lambda}e_2}{\sqrt{1+\lambda^2}}, \quad e_2 = \frac{e_3 + \sqrt{1-\mu}e_4}{\sqrt{1+\mu^2}}, \]
and extend these two vectors to be a local orthonormal frame of $T^\perp M$. Then a direct computation gives
\[ R_{1212} = R(e_1, e_2, \bar{e}_1, \bar{e}_2) = I_{\lambda, \mu}(R). \]

We first claim that
\[ \sum_{i,j=1}^{n} \bar{R}_{ij} \leq (n^2 - n - 2) \bar{K}_{\text{max}} + 2\bar{R}_{1212}. \]

If this is true, then (2.5) and (4.5) give
\[ n(n-1)R_0 = \sum_{i,j=1}^{n} R_{ij} = \sum_{i,j=1}^{n} \bar{R}_{ij} + n(n-1)|H|^2 - \sum_{i,j=1}^{n} \sum_{\alpha} |\bar{h}^\alpha_{ij}|^2 \leq (n^2 - n - 2) \bar{K}_{\text{max}} + 2\bar{R}_{1212} + n(n-1)|H|^2 - \sum_{i,j=1}^{n} \sum_{\alpha} |\bar{h}^\alpha_{ij}|^2 = (n^2 - n - 2) \bar{K}_{\text{max}} + 2R_{1212} + n(n-1)|H|^2 - 2\left(|H|^2 + \sum_{\alpha} \left( H^\alpha (\bar{h}^\alpha_{11} + \bar{h}^\alpha_{22}) + \bar{h}^\alpha_{12} \bar{h}^\alpha_{21} - |\bar{h}^\alpha_{12}|^2 \right) \right) - \sum_{i,j=1}^{n} \sum_{\alpha} |\bar{h}^\alpha_{ij}|^2. \]

On the other hand,
\[ -2\left(|H|^2 + \sum_{\alpha} \left( H^\alpha (\bar{h}^\alpha_{11} + \bar{h}^\alpha_{22}) + \bar{h}^\alpha_{12} \bar{h}^\alpha_{21} - |\bar{h}^\alpha_{12}|^2 \right) \right) - \sum_{i,j=1}^{n} \sum_{\alpha} |\bar{h}^\alpha_{ij}|^2 \leq -2\left(|H|^2 + \sum_{\alpha} \left( H^\alpha (\bar{h}^\alpha_{11} + \bar{h}^\alpha_{22}) + \bar{h}^\alpha_{12} \bar{h}^\alpha_{21} - |\bar{h}^\alpha_{12}|^2 \right) \right) - \sum_{\alpha} \left( |\bar{h}^\alpha_{11}|^2 + |\bar{h}^\alpha_{12}|^2 + 2 |\bar{h}^\alpha_{12}|^2 + \sum_{i,j=3}^{n} |\bar{h}^\alpha_{ij}|^2 \right) \leq -2|H|^2 - 2 \sum_{\alpha} H^\alpha (\bar{h}^\alpha_{11} + \bar{h}^\alpha_{22}) - \sum_{\alpha} \left( |\bar{h}^\alpha_{11}|^2 + |\bar{h}^\alpha_{22}|^2 + 2 |\bar{h}^\alpha_{12}|^2 - \frac{1}{n-2} \sum_{i,j=3}^{n} |\bar{h}^\alpha_{ij}|^2 \right) \leq -2|H|^2 - 2 \sum_{\alpha} H^\alpha (\bar{h}^\alpha_{11} + \bar{h}^\alpha_{22}) - \left( 1 + \frac{1}{n-2} \right) \sum_{\alpha} (\bar{h}^\alpha_{11} + \bar{h}^\alpha_{22})^2 \leq -\frac{n}{n-1} |H|^2, \]
where in the second inequality, we have used
\[ \sum_{i,j=3}^{n} |\bar{h}^\alpha_{ij}|^2 \geq \sum_{i,j=3}^{n} |\bar{h}^\alpha_{ij}|^2 \geq \frac{1}{n-2} \sum_{i,j=3}^{n} \left( \frac{\bar{h}^\alpha_{11} + \bar{h}^\alpha_{22}}{n-2} \right)^2 = \left( \frac{\bar{h}^\alpha_{11} + \bar{h}^\alpha_{22}}{n-2} \right)^2. \]
Therefore, we have

\[ n(n - 1)R_0 \leq (n^2 - n - 2) \bar{K}_{\text{max}} + 2R_{12i2} + \left( n(n - 1) - \frac{n}{n - 1} \right)|H|^2, \]

which implies

\[ 2R_{12i2} \geq n(n - 1) \left[ R_0 - \left( 1 - \frac{2}{n(n - 1)} \right) \bar{K}_{\text{max}} + \frac{n(n - 2)}{(n - 1)^2} |H|^2 \right]. \tag{4.6} \]

Thus, by the assumption of this theorem and (4.6), we have \( R_{12i2} \geq 0 \). Therefore, \( M \times \mathbb{R}^2 \) has nonnegative isotropic curvature (see for example [7, Proposition 17.8]). Also by the assumption, the isotropic curvature of \( M \times \mathbb{R}^2 \) is positive at some point. Consequently, \( M \) has nonnegative isotropic curvature and positive isotropic curvature at some point. Then \( M \) admits a metric with positive isotropic curvature (see [29]). Therefore, \( M \) is a topological sphere by Theorem A. But by the classification theorem of Brendle-Schoen (Theorem C), \( M \) must be diffeomorphic to \( S^n \).

It remains to prove the inequality (4.5). Under the orthonormal frames \( \{e_i\} \), this inequality is equivalent to

\[ \sum_{i,j=1}^{n} \bar{R}_{ijij} \leq (n^2 - n - 2) \bar{K}_{\text{max}} + 2I_{\lambda,\mu} \left( \bar{R}^T \right). \]

Notice that

\[ \sum_{i,j=1}^{n} \bar{R}_{ijij} = 2I_{\lambda,\mu} \left( \bar{R}^T \right) + 2 \left( \sum_{1 \leq i < j \leq 4} \bar{R}_{ijij} - I_{\lambda,\mu} \left( \bar{R}^T \right) \right) + 2 \sum_{i=1}^{4} \sum_{j=5}^{n} \bar{R}_{ijij} + \sum_{i,j=5}^{n} \bar{R}_{ijij} \]

\[ \leq 2I_{\lambda,\mu} \left( \bar{R}^T \right) + 2 \left( \sum_{1 \leq i < j \leq 4} \bar{R}_{ijij} - I_{\lambda,\mu} \left( \bar{R}^T \right) \right) + (n^2 - n - 12) \bar{K}_{\text{max}}. \]

Therefore, it is sufficient to prove

\[ \sum_{1 \leq i < j \leq 4} \bar{R}_{ijij} - I_{\lambda,\mu} \left( \bar{R}^T \right) \leq 5 \bar{K}_{\text{max}}. \]
A direct computation using (3.5) yields

\[
\sum_{1 \leq i < j \leq 4} \bar{R}_{ijij} - I_{k,m}(\bar{R}^T)
\]

\[
= \sum_{1 \leq i < j \leq 4} \bar{R}_{ijij} - \frac{1}{(1 + \lambda^2)(1 + \mu^2)}(\bar{R}_{1313} + \lambda^2 \bar{R}_{1414} + \mu^2 \bar{R}_{2323} + \lambda^2 \mu^2 \bar{R}_{2424} - 2\lambda \mu \bar{R}_{1234})
\]

\[
= (1 - \frac{2\lambda \mu}{3(1 + \lambda^2)(1 + \mu^2)}) \bar{R}_{1212} + \bar{R}_{3434} + (1 - \frac{3 + \lambda \mu}{3(1 + \lambda^2)(1 + \mu^2)}) \bar{R}_{1313} + (1 - \frac{3\lambda^2 + 3\mu}{3(1 + \lambda^2)(1 + \mu^2)}) \bar{R}_{1414} + (1 - \frac{3\mu^2 + \lambda \mu}{3(1 + \lambda^2)(1 + \mu^2)}) \bar{R}_{2323} + (1 - \frac{3\lambda^2 \mu^2 + \lambda \mu}{3(1 + \lambda^2)(1 + \mu^2)}) \bar{R}_{2424} + \lambda \mu (\bar{K}(e_1 + e_3, e_2 + e_4) + \bar{K}(e_1 - e_3, e_2 - e_4) + \bar{K}(e_2 + e_3, e_1 - e_4) + \bar{K}(e_2 - e_3, e_1 + e_4))
\]

\[
\leq (1 - \frac{2\lambda \mu}{3(1 + \lambda^2)(1 + \mu^2)}) \cdot 2\bar{K}_{max} + (1 - \frac{3 + \lambda \mu}{3(1 + \lambda^2)(1 + \mu^2)}) \bar{K}_{max} + (1 - \frac{3\lambda^2 + 3\mu}{3(1 + \lambda^2)(1 + \mu^2)}) \bar{K}_{max} + (1 - \frac{3\mu^2 + \lambda \mu}{3(1 + \lambda^2)(1 + \mu^2)}) \bar{K}_{max} + \frac{16\lambda \mu}{6(1 + \lambda^2)(1 + \mu^2)} \bar{K}_{max}
\]

\[= 5\bar{K}_{max}.
\]

\[\square\]

Theorem 1.4] and [Theorem 1.5] are easy consequences of the following theorem:

**Theorem 4.5.** For fixed \(0 < \varepsilon \leq 1\), set \(\delta(\varepsilon, n) = \frac{((n-4)\varepsilon^2 - 2n^2)}{8(2\varepsilon + (n-4)\varepsilon^2 \varepsilon} \). Suppose \(M^n(n \geq 4)\) is a closed simply connected submanifold of \(\mathbb{M}^N\) satisfying

\[
\frac{\text{Ric}^{[2]}}{2} \geq (n - 1 - \varepsilon)\bar{K}_{max} + \delta(\varepsilon, n) |H|^2,
\]

with strict inequality at some point, then \(M\) is diffeomorphic to \(S^n\).

**Proof.** Let \(\{e_i\}\) be a local orthonormal frame of \(TM\). For \(\lambda, \mu \in \{0, 1\}\), define

\[
\varepsilon_1 = e_1 + \sqrt{1 - \mu}e_2, \quad \varepsilon_2 = e_3 + \sqrt{1 - \lambda}e_4, \quad \varepsilon_3 = \mu e_1 - \sqrt{1 - \varepsilon_2}, \quad \varepsilon_4 = \frac{\lambda e_3 - \sqrt{1 - \varepsilon_4}}{\sqrt{1 + \lambda^2}},
\]

\[\varepsilon_i = e_i, \quad 5 \leq i \leq n.
\]

Then \(\{\varepsilon_i\}\) is a local orthonormal frame of \(T^C M\). Similar as the proof of [Theorem 1.3] it is sufficient to prove \(R_{1212} \geq 0\) and the strict inequality holds for all frame \(\{e_i\}\) and all numbers.
\( \lambda, \mu \in [0, 1] \) at some point. Ricci curvature formula \((2.5)\) gives
\[
(4.7) \quad \frac{1}{2} (\text{Ric}_{11} + \text{Ric}_{22}) \leq \frac{1}{2} \left( \sum_{i=1}^{n} \hat{R}_{i1i1} + \sum_{i \neq 2} \hat{R}_{2i2i} \right) + (n - 1) |H|^2 + \frac{1}{2} \sum_{\alpha} (n - 2) H^\alpha (\hat{h}_{11}^\alpha + \hat{h}_{22}^\alpha) - \sum_{i=1}^{n} \left( |\hat{h}_{11}^\alpha|^2 + |\hat{h}_{22}^\alpha|^2 \right),
\]
\[
(4.8) \quad \frac{1}{n - 2} \sum_{i=3}^{n} \text{Ric}_{ii} \leq \frac{1}{n - 2} \left[ \sum_{i=3}^{n} \sum_{j \neq i} \hat{R}_{ijij} - n \sum_{\alpha} (n - 2) H^\alpha (\hat{h}_{11}^\alpha + \hat{h}_{22}^\alpha) + \sum_{i=3}^{n} \sum_{j=1}^{n} |\hat{h}_{ij}^\alpha|^2 \right] + (n - 1) |H|^2.
\]
Assume
\[
\text{Ric}_{ii} + \text{Ric}_{jj} \geq 2D, \quad \forall 1 \leq i < j \leq n.
\]
Then
\[
D \leq \frac{\text{Ric}_{11} + \text{Ric}_{22}}{2}, \quad D \leq \frac{\sum_{3 \leq j \leq n} (\text{Ric}_{jj} + \text{Ric}_{kk})}{(n - 2)(n - 3)} = \frac{\sum_{i=3}^{n} \text{Ric}_{ii}}{n - 2}.
\]
Hence for every \( 0 < \varepsilon \leq 1 \), by using \((4.7), (4.8)\) and \((2.4)\), we get
\[
D \leq \varepsilon \cdot \frac{\text{Ric}_{11} + \text{Ric}_{22}}{2} + (1 - \varepsilon) \cdot \frac{\sum_{i=3}^{n} \text{Ric}_{ii}}{n - 2}
\leq \varepsilon \sum_{i=3}^{n} \left( \hat{R}_{i11i1} + \hat{R}_{22i2} \right) + \frac{1 - \varepsilon}{n - 2} \sum_{i=3}^{n} \sum_{j=1}^{n} \hat{R}_{ijij} + (n - 1) |H|^2 - \frac{1 - \varepsilon}{n - 2} \sum_{i=3}^{n} \sum_{j=1}^{n} |\hat{h}_{ij}^\alpha|^2
\leq \frac{\varepsilon}{2} \sum_{i=3}^{n} \left( \hat{R}_{i11i1} + \hat{R}_{22i2} \right) + \frac{1 - \varepsilon}{n - 2} \sum_{i=3}^{n} \sum_{j=1}^{n} \hat{R}_{ijij} + \varepsilon R_{12\bar{i}2} + (n - 1 - \varepsilon) |H|^2 - \frac{1 - \varepsilon}{n - 2} \sum_{i=3}^{n} \sum_{j=1}^{n} |\hat{h}_{ij}^\alpha|^2
\leq \frac{\varepsilon}{2} \sum_{i=3}^{n} \left( \hat{R}_{i11i1} + \hat{R}_{22i2} \right) + \frac{1 - \varepsilon}{n - 2} \sum_{i=3}^{n} \sum_{j=1}^{n} \hat{R}_{ijij} + \varepsilon R_{12\bar{i}2} + (n - 1 - \varepsilon) |H|^2 + \varepsilon H^\alpha (\hat{h}_{11}^\alpha + \hat{h}_{22}^\alpha) - \frac{1}{2} \sum_{i=3}^{n} \left( |\hat{h}_{11}^\alpha|^2 + |\hat{h}_{22}^\alpha|^2 \right)
\leq \frac{\varepsilon}{2} \sum_{i=3}^{n} \left( \hat{R}_{i11i1} + \hat{R}_{22i2} \right) + \frac{1 - \varepsilon}{n - 2} \sum_{i=3}^{n} \sum_{j=1}^{n} \hat{R}_{ijij} + \varepsilon R_{12\bar{i}2} + (n - 1 - \varepsilon) |H|^2.
where in the second inequality, we have used
\[
\sum_{i=3}^{n} \sum_{j=1}^{n} |h_{ij}^{a}|^2 \geq \sum_{i=3}^{n} |h_{ii}^{a}|^2 \geq \frac{\left(\sum_{i=3}^{n} h_{ii}^{a}\right)^2}{n-2} = \frac{\left(h_{11}^{a} + h_{22}^{a}\right)^2}{n-2}.
\]
Therefore,
\[
(4.9) \quad \varepsilon R_{1212} \geq D - \left(\frac{\varepsilon}{2} \sum_{i=3}^{n} (\tilde{R}_{1i1i} + \tilde{R}_{2i2i}) + \frac{1-\varepsilon}{n-2} \sum_{i=3}^{n} \sum_{j=1}^{n} \tilde{R}_{ijij} + \delta(\varepsilon, n) |H|^2\right).
\]
We claim that
\[
(4.10) \quad \frac{\varepsilon}{2} \sum_{i=3}^{n} (\tilde{R}_{1i1i} + \tilde{R}_{2i2i}) + \frac{1-\varepsilon}{n-2} \sum_{i=3}^{n} \sum_{j=1}^{n} \tilde{R}_{ijij} \leq (n-1-\varepsilon)\tilde{K}_{\max}.
\]
If this is true, then combined with (4.9), we have
\[
(4.11) \quad \varepsilon R_{1212} \geq D - (n-1-\varepsilon)\tilde{K}_{\max} + \delta(\varepsilon, n) |H|^2.
\]
By the assumption of the theorem, we get
\[
Ric(\varepsilon_1, \bar{e}_1) + Ric(\varepsilon_2, \bar{e}_2) = \frac{Ric(e_1, e_1) + \mu^2 Ric(e_2, e_2)}{1 + \mu^2} + \frac{Ric(e_3, e_3) + \lambda^2 Ric(e_4, e_4)}{1 + \lambda^2}
\]
\[
= \frac{(Ric(e_1, e_1) + Ric(e_3, e_3)) + \lambda^2 (Ric(e_1, e_1) + Ric(e_4, e_4))}{(1 + \lambda^2)(1 + \mu^2)}
\]
\[
+ \frac{\mu^2 (Ric(e_2, e_2) + Ric(e_3, e_3)) + \lambda^2 \mu^2 (Ric(e_2, e_2) + Ric(e_4, e_4))}{(1 + \lambda^2)(1 + \mu^2)}
\]
\[
\geq 2 \left((n-1-\varepsilon)\tilde{K}_{\max} + \delta(\varepsilon, n) |H|^2\right).
\]
Therefore, by the arbitrariness of \(e_1, e_2, e_3, e_4\), we can take
\[
D = (n-1-\varepsilon)\tilde{K}_{\max} + \delta(\varepsilon, n) |H|^2.
\]
Combining the above inequality with (4.11), we have
\[
\varepsilon R(\varepsilon_1, \varepsilon_2, \bar{e}_1, \bar{e}_2) \geq D - \left((n-1-\varepsilon)\tilde{K}_{\max} + \delta(\varepsilon, n) |H|^2\right) = 0.
\]
Therefore we have \(R_{1212} \geq 0\), and strict inequality holds for all frame \(\{e_i\}\) and all numbers \(\lambda, \mu \in [0, 1]\) at some point.
What is left is to prove the inequality \((4.10)\). Under the given basis of \(T^C M\), a direct computation gives

\[
\bar{R}(\bar{e}_1, \bar{e}_2, \bar{e}_1, \bar{e}_2) = \frac{\bar{R}_{1313} + \mu^2 \bar{R}_{2323} + \lambda^2 \bar{R}_{1414} + \lambda^2 \mu^2 \bar{R}_{2424} - 2 \lambda \mu \bar{R}_{1234}}{(1 + \lambda^2)(1 + \mu^2)}
\]

and

\[
\sum_{i=1}^{n} \left( \bar{R}(\bar{e}_1, \bar{e}_2, \bar{e}_1, \bar{e}_2) + \bar{R}(\bar{e}_3, \bar{e}_4, \bar{e}_3, \bar{e}_4) \right) = \sum_{i=1}^{n} \left[ \frac{\mu^2 \bar{R}_{111i} + \lambda^2 \bar{R}_{33i}}{1 - \mu^2} + \frac{\mu^2 \bar{R}_{222i} + \lambda^2 \bar{R}_{44i}}{1 + \lambda^2} \right],
\]

\[
\sum_{j=1}^{n} \bar{R}(\bar{e}_1, \bar{e}_j, \bar{e}_1, \bar{e}_j) = \sum_{j=1}^{n} \bar{R}_{ijij} - \frac{1}{n^2} \leq 5 \leq n.
\]

Note that,

\[
\frac{\epsilon}{2} \sum_{i=1}^{n} \left( \bar{R}(\bar{e}_1, \bar{e}_i, \bar{e}_1, \bar{e}_i) + \bar{R}(\bar{e}_2, \bar{e}_i, \bar{e}_2, \bar{e}_i) \right) + \frac{1 - \epsilon}{n - 2} \sum_{i=3}^{n} \sum_{j=1}^{n} \bar{R}(\bar{e}_i, \bar{e}_j, \bar{e}_i, \bar{e}_j)
\]

\[
= \frac{\epsilon}{2} \sum_{i=1}^{n} \left( \bar{R}(\bar{e}_1, \bar{e}_i, \bar{e}_1, \bar{e}_i) + \bar{R}(\bar{e}_2, \bar{e}_i, \bar{e}_2, \bar{e}_i) \right) - \epsilon \bar{R}(\bar{e}_1, \bar{e}_2, \bar{e}_1, \bar{e}_2) + \frac{1 - \epsilon}{n - 2} \sum_{i=3}^{n} \sum_{j=1}^{n} \bar{R}(\bar{e}_i, \bar{e}_j, \bar{e}_i, \bar{e}_j)
\]

\[
\leq \frac{\epsilon}{2} \sum_{i=1}^{n} \left( \bar{R}(\bar{e}_1, \bar{e}_i, \bar{e}_1, \bar{e_i}) + \bar{R}(\bar{e}_2, \bar{e_i}, \bar{e}_2, \bar{e}_i) \right) - \epsilon \bar{R}(\bar{e}_1, \bar{e}_2, \bar{e}_1, \bar{e}_2) + \frac{1 - \epsilon}{n - 2} \cdot (n - 2)(n - 1) \bar{K}_{\max}.
\]

As a similar argument in the proof of \textbf{Theorem 1.3} we have

\[
\frac{\epsilon}{2} \sum_{i=1}^{n} \left( \bar{R}(\bar{e}_1, \bar{e}_i, \bar{e}_1, \bar{e}_i) + \bar{R}(\bar{e}_2, \bar{e}_i, \bar{e}_2, \bar{e}_i) \right) - \epsilon \bar{R}(\bar{e}_1, \bar{e}_2, \bar{e}_1, \bar{e}_2) \leq \epsilon(n - 2) \bar{K}_{\max}.
\]

Thus,

\[
\frac{\epsilon}{2} \sum_{i=1}^{n} \left( \bar{R}(\bar{e}_1, \bar{e}_i, \bar{e}_1, \bar{e}_i) + \bar{R}(\bar{e}_2, \bar{e}_i, \bar{e}_2, \bar{e}_i) \right) + \frac{1 - \epsilon}{n - 2} \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{R}(\bar{e}_i, \bar{e}_j, \bar{e}_i, \bar{e}_j)
\]

\[
\leq \epsilon(n - 2) \bar{K}_{\max} + \frac{1 - \epsilon}{n - 2} \cdot (n - 2)(n - 1) \bar{K}_{\max}
\]

\[
= (n - 1 - \epsilon) \bar{K}_{\max}.
\]

We complete the proof.

If we take \(\epsilon = 1\) in \textbf{Theorem 4.5} we have \textbf{Theorem 1.4}. If we take \(\epsilon = \frac{2}{n-2}\) in \textbf{Theorem 4.5} we have \textbf{Theorem 1.5}

We also can take \(\epsilon = \frac{2(n^2-6n+10)}{(n-4)(n^2-4n+2)}\) in \textbf{Theorem 4.5} to make the coefficient \(\delta(\epsilon, n)\) to be minimal when \(n \geq 6\).

\textbf{Corollary 4.6.} Suppose \(M^n(n \geq 6)\) is a closed simply connected submanifold of \(\bar{M}^n\) satisfying

\[
\frac{\text{Ric}[2]}{2} \geq \left( n - 1 - \frac{2(n^2-6n+10)}{(n-4)(n^2-4n+2)} \right) \bar{K}_{\max} + \frac{(n - 2)(n - 3)(n - 4)n^2}{(n^2-4n+2)^2} |H|^2,
\]

with strict inequality at some point, then \(M\) is diffeomorphic to \(S^n\).
Remark 4.2. It is easy to check, for \( n \geq 6 \),

\[
    n - 1 - \frac{2(n^2 - 6n + 10)}{(n - 4)(n^2 - 4n + 2)} < \frac{(n - 2)(n - 3)(n - 4)n^2}{(n^2 - 4n + 2)^2} < \frac{n(n - 3)}{n - 2}.
\]

Therefore, when \( n \geq 6 \), Corollary 4.6 implies Theorem 1.5.

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