NON-TRIVIAL SELF-CONCORDANCES 
AND A RECENT CONJECTURE BY BOTVINNIK

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Abstract. The goal of this note is to construct, on many manifolds, non-trivial concordances from the identity to itself. This produces counterexamples to a recent conjecture by Botvinnik.

1. Statement of the results

Recall that a (smooth) concordance on a smooth manifold $M$ is a diffeomorphism of $M \times I$ which is the identity in a neighborhood of $M \times 0 \cup \partial M \times I$. For a concordance $H$ of $M$, denote by $e(H)$ the induced diffeomorphism on $M \times 1$. We say that $H$ is trivial if it is isotopic to the identity, via an isotopy that fixes a neighborhood of $M \times 0 \cup \partial M \times I$.

Concordances can be described, in a stable range, by algebraic $K$-theory. In this note we explain how to use this relationship to prove:

**Theorem 1.1.** For $n \geq 9$ there exists a non-trivial concordance $H$ of $S^1 \times D^{n-1}$ such that $e(H) = id$.

In fact, more generally we have:

**Theorem 1.2.** On any smooth compact orientable manifold $M$ of dimension $n \geq 9$ such that $\pi_1(M) = \mathbb{Z}$, there is a non-trivial concordance $H$ such that $e(H) = id$.

Recently, in his impressive preprint [1], Botvinnik has proposed the following “Topological Conjecture”.

**Conjecture 1.3.** Let $M$ be a closed manifold equipped with a metric $g$ of positive scalar curvature. If $H$ is a non-trivial concordance on $M$, then $g$ and $e(H)^*g$ are non-isotopic as metrics of positive scalar curvature.

**Corollary 1.4.** The Topological Conjecture does not hold.

2. Proof of Theorem 1.1

Write $M = S^1 \times D^{n-1}$ and denote by $C(M)$ the group of concordances modulo isotopy. Let $h \in C(M)$. Shrinking the interval $I = [0, 1]$ to $[0, \frac{1}{2}]$, we may consider $h$ as a self-diffeomorphism of $M \times [0, \frac{1}{2}]$ which we may extend again to the whole of $M \times I$ by flipping:

$H_{|M \times [\frac{1}{2}, 1]} := i \circ h \circ i$

where $i$ is induced by reflection of $I$ at $\frac{1}{2}$. Clearly $e(H) = id$.

**Remark 2.1.** A further analysis using the Hatcher spectral sequence and surgery theory shows that on the manifold $M$ any concordance from the identity to itself is isotopic to one of this form.

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Note that in the abelian group $C(M)$, we have $H = h + \tau h$ where $\tau$ denotes the canonical involution on $C(M)$. To prove Theorem 1.1 we therefore need to show that $\tau \neq -\text{id}$.

By the stable parametrized $h$-cobordism theorem [8] there is a short exact sequence

$$0 \to \pi^2_1(M \amalg \{\ast\}) \to \pi_2 A(M) \xrightarrow{\tau} C(M) \to 0$$

provided $\dim(M) \geq 9$ (this “stable range” is due to [5]). Here $A(M)$ denotes Waldhausen’s $K$-theory of spaces [7]; by homotopy invariance we have $\pi_2 A(M) \cong \pi_2 A(S^1)$. By [6] the functor $A(-)$ carries a canonical involution $T$ so that the map $\pi$ is equivariant up to the sign $(-1)^n$. (Here we use that $M$ is parallelizable.)

By the fundamental theorem [3]

$$\pi_2 A(S^1) \cong \pi_2 A(\ast) \oplus \pi_1 A(\ast) \oplus \pi_2 NA(\ast) \oplus \pi_2 NA(\ast);$$

the involution $T$ interchanges the two copies of $\pi_2 NA(\ast)$ by [4]. Moreover $\pi_k A(\ast) \cong \pi^k_1$ for $k \leq 2$ in a way that

$$\pi_2 A(\ast) \oplus \pi_1 A(\ast) = \pi^2_1 (S^1 \amalg \{\ast\})$$

as subgroups of $\pi_2 A(S^1)$. So

$$C(M) \cong \pi_2 NA(\ast) \oplus \pi_2 NA(\ast)$$

and $\tau$ acts, up to sign, by interchanging the summands. In particular $\tau \neq -\text{id}$ (with $\pi_2 C(M)$ being non-zero by [2]).

3. Proof of Theorem 1.2

We may assume that $M$ is connected. Let $H$ be a non-trivial concordance on $S^1 \times D^{n-1}$ as given by Theorem 1.1. Let $i : S^1 \to M$ be an embedding representing the generator of $\pi_1 (M)$. Since $M$ is oriented, $i$ has a trivial normal bundle and induces an embedding $\tilde{i} : S^1 \times D^{n-1} \to M$. So we may extend $H$ by the identity to a concordance $\tilde{H}$ on $M$, such that $\epsilon(\tilde{H}) = \text{id}$.

In the stable range $\dim(M) \geq 9$, the assignment $\tau \to C(M)$ is a homotopy functor [2]. Thus, if $\rho : M \to S^1 \times D^{n-1}$ classifies (up to homotopy) the universal covering of $M$, then the composite homomorphism

$$C(S^1 \times D^{n-1}) \xrightarrow{i_*} C(M) \xrightarrow{\rho_*} C(S^1 \times D^{n-1})$$

is the identity. Hence $\tilde{H} = \tilde{i}_*(H) \neq 0$.

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