Multiscale Analysis of Semilinear Damped Stochastic Wave Equations

Aurelien FOUETIO
Department of Mathematics, Higher Teacher Training College, University of Ngaoundere,
P. O. Box 652, Bertoua, Cameroon
E-mail: aurelienfouetio@yahoo.fr

Gabriel NGUETSENG
Department of Mathematics, University of Yaounde I, P. O. Box 812, Yaounde, Cameroon
E-mail: nguetsengg@yahoo.fr

Jean Louis WOUKENG
Department of Mathematics and Computer Science, University of Dschang, P. O. Box 67,
Dschang, Cameroon
E-mail: jwoukeng@yahoo.fr

Abstract In this paper we proceed with the multiscale analysis of semilinear damped stochastic wave motions. The analysis is made by combining the well-known sigma convergence method with its stochastic counterpart, associated to some compactness results such as the Prokhorov and Skorokhod theorems. We derive the equivalent model, which is of the same type as the micro-model. One of the novelties of the work is that the corrector problem is solved in the classical sense of distributions, thereby allowing numerical computations of the homogenized coefficients.

Keywords Hyperbolic stochastic equations, Wiener Process, sigma convergence, tightness of probability measures

MR(2010) Subject Classification 35B40, 60H15, 46J10

1 Introduction and the main result

An important class of physical phenomena is the one represented by wave phenomena. In continuous media, the most frequent form of energy transmission is through waves. As clearly known, wave nature is apparent in very important phenomena such as the sound propagation, the electromagnetic radiation’s propagation, just to cite a few. It appears in many fields of science such as the acoustics, optics, geophysics, radiophysics and mechanics. However, in order to take into account some physical data such as turbulence, the most accurate models are known to be stochastic ones. Indeed, damping may occur through wave scattering due to the inhomogeneity of the medium and the surface (size of inhomogeneity in relation to wavelength, inhomogeneities distributed in a continuous manner or in the form of a random/deterministic
discrete scattering elements). Hence, because of damping, scattered waves may cause fluctuations of the amplitude, leading to the justification of the use of stochastic models. In the quest of mastering wave motion, it is very important to study the phenomenon starting from a micro-model (like in (1.1) below) in order to predict the overall behavior on the large scale model. Hence the importance of multiscale analysis in such a study.

To be more precise, we study waves motion through an inhomogeneous medium made of microstructures of small size (representing the inhomogeneities). Because of what we said above, we assume that the microstructures are deterministically distributed in the medium, and their distribution is represented by an assumption made on the fast spatial variable \( y = x/\varepsilon \) covering a wide range of behaviors such as the uniform (or periodic) distribution, the almost periodic distribution and the asymptotic almost periodic one. The study also takes into account an assumption made on the fast time variable \( \tau = t/\varepsilon \), which allows us to predict the long time behavior of the waves. The micro-model is presented in the following lines.

Let \( D \) be a Lipschitz domain in \( \mathbb{R}^d \), \( T \) and \( \varepsilon \) be positive real numbers with \( 0 < \varepsilon < 1 \), \( D_T = D \times (0, T) \) and let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space on which is defined an infinite sequence of independent standard 1-d Brownian motion \( (W_k)_{k \geq 1} \). We equip the probability space by the natural filtration, denoted by \( \mathcal{F}^t \), of \( W_k \). Now let \( \mathcal{U} \) be a fixed Hilbert space with orthonormal basis \( \{e_k : k \geq 1\} \). We may define a cylindrical Wiener process \( W \) by setting \( W = \sum_{k=1}^{\infty} W_k e_k \) (see [8]). Let us denote by \( L_2(\mathcal{U}, X) \) the space of Hilbert–Schmidt operators from \( \mathcal{U} \) to the Hilbert space \( X \),

\[ L_2(\mathcal{U}, X) = \left\{ g \in L(\mathcal{U}, X) : \sum_{k=1}^{\infty} |ge_k|^2_X < \infty \right\}, \]

and we define the Hilbert space \( \mathcal{U}_0 \subset \mathcal{U} \) by

\[ \mathcal{U}_0 = \left\{ r = \sum_{k=1}^{\infty} \alpha_k e_k : \sum_{k=1}^{\infty} \frac{\alpha_k^2}{k^2} < \infty \right\}. \]

We endow \( \mathcal{U}_0 \) with the norm

\[ |r|_{\mathcal{U}_0}^2 = \sum_{k=1}^{\infty} \frac{\alpha_k^2}{k^2}. \]

It is a well-known fact that there exists \( \Omega' \in \mathcal{F} \) with \( \mathbb{P}(\Omega') = 1 \) such that \( W(\omega) \in C(0, T, \mathcal{U}_0) \), for any \( \omega \in \Omega' \) (see [8]). For any given function \( h \in L^2(\Omega; L^2(0, T; L_2(\mathcal{U}, X))) \) such that \( h(t) \) is \( \mathcal{F}_t \)-adapted we may define the stochastic integral by

\[ \int_0^t h dW = \sum_{k=1}^{\infty} \int_0^t h e_k dW_k, \]

as an element of the space of \( X \)-valued square integrable martingale. Moreover, we have

\[ \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t h dW \right|^p \leq C \mathbb{E} \left( \int_0^T |h|^2_{L_2(\mathcal{U}, X)} \right)^{\frac{p}{2}}, \]

for any \( p \geq 1 \).

We consider the linear elliptic partial differential operator

\[ P^\varepsilon = - \text{div}(a^\varepsilon \nabla) \equiv - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( a^\varepsilon_{ij}(x) \frac{\partial}{\partial x_j} \right) \]
and we aim at studying the asymptotic behavior (as $0 < \varepsilon \to 0$) of the sequence of solutions $u_\varepsilon$ of the following initial-boundary value problem

$$
\begin{aligned}
\begin{cases}
\partial_t u_\varepsilon + (P^\varepsilon u_\varepsilon - \Delta u_\varepsilon')dt = f^\varepsilon(t, \cdot, u_\varepsilon')dt + g^\varepsilon(t, \cdot, u_\varepsilon')dW & \text{in } D_T;
\quad \\
u_\varepsilon = 0 & \text{on } \partial D \times (0, T);
\quad \\
u_\varepsilon(x, 0) = u^0(x) & \text{and } u_\varepsilon'(x, 0) = u^1(x) \text{ in } D,
\end{cases}
\end{aligned}
$$

(1.1)

where $u'_\varepsilon = \frac{\partial u_\varepsilon}{\partial t}$ and the functions $f^\varepsilon(t, \cdot, u_\varepsilon')$ and $g^\varepsilon(t, \cdot, u_\varepsilon')$ are defined as follows:

$$
f^\varepsilon(t, \cdot, u_\varepsilon')(x, t) = f\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon'(x, t)\right)
$$

and a similar definition for $g(\cdot, \cdot, u_\varepsilon')$, the dependence on $\omega \in \Omega$ being implicit. We impose on the coefficients of (1.1) the following constraints:

(A1) The matrix $a^\varepsilon = (a^\varepsilon_{ij})_{1 \leq i, j \leq d}$ is defined by $a^\varepsilon_{ij}(x) = a_{ij}(x, \tan \frac{x}{\varepsilon})$ ($x \in D$), where

1. $a_{ij} \in C(\overline{D}, L^\infty(\mathbb{R}^d))$ ($1 \leq i, j \leq d$) with $a_{ij} = a_{ji},$
2. there exists a constant $\alpha > 0$ such that

$$
\sum_{i, j=1}^d a_{ij}(x, y)\xi_i \xi_j \equiv a(x, y)\xi \cdot \xi \geq \alpha|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d, \ x \in \overline{D} \text{ and a.e. } y \in \mathbb{R}^d.
$$

(A2) We define the maps $f$, $g$ and the functions $u^0$, $u^1$ such that

1. The function $f : (y, \tau, v) \mapsto f(y, \tau, v)$ from $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ satisfies the properties:

   - (i) $f$ is measurable,
   - (ii) there exists a constant $c_1 > 0$ such that $|f(y, \tau, v)|^2 \leq c_1(1 + |v|^2)$ for a.e. $(y, \tau) \in \mathbb{R}^d \times \mathbb{R}$ and for all $v \in \mathbb{R}$,
   - (iii) there exists a constant $c_2 > 0$ such that $|f(y, \tau, v_1) - f(y, \tau, v_2)| \leq c_2|v_1 - v_2|$ for a.e. $(y, \tau) \in \mathbb{R}^d \times \mathbb{R}$ and for all $v_1, v_2 \in \mathbb{R}$;

2. The function $g : (y, \tau, v) \mapsto g(y, \tau, v)$ from $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}^m$ satisfies the properties:

   - (i) $g$ is measurable,
   - (ii) there exists a constant $c_3 > 0$ such that $|g(y, \tau, v)|^2 \leq c_3(1 + |v|^2)$ for a.e. $(y, \tau) \in \mathbb{R}^d \times \mathbb{R}$ and for all $v \in \mathbb{R}$,
   - (iii) there exists a constant $c_4 > 0$ such that $|g(y, \tau, v_1) - g(y, \tau, v_2)| \leq c_4|v_1 - v_2|$ for a.e. $(y, \tau) \in \mathbb{R}^d \times \mathbb{R}$ and for all $v_1, v_2 \in \mathbb{R}$;

3. $u^0 \in H^1_0(D)$ and $u^1 \in L^2(D)$.

Under the above conditions, it is easy to see that if $u_\varepsilon$ and $v_\varepsilon$ are two solutions to (1.1) on the same stochastic system with the same initial condition then $u_\varepsilon(t) = v_\varepsilon(t)$ in $H^1_0(D)$ almost surely for any $t$. Thanks to this fact together with Yamada–Watanabe’s Theorem [26] and the existence Theorem in [27, Theorem 2.2], we see that Problem (1.1) (for any fixed $\varepsilon > 0$) possesses a unique strong solution $u_\varepsilon$ satisfying:

1. $u_\varepsilon$ and $u'_\varepsilon$ are continuous with respect to time in $H^1_0(D)$ and $L^2(D)$ respectively,
2. $u_\varepsilon$ and $u'_\varepsilon$ are $\mathcal{F}_t$-measurable,
3. $u_\varepsilon \in L^2(\Omega; L^\infty(0, T; H^1_0(D)))$, $u'_\varepsilon \in L^2(\Omega; L^\infty(0, T; L^2(D))) \cap L^2(\Omega; L^2(0, T; H^1_0(D)))$,
4. $u_\varepsilon$ satisfies

$$
(u'_\varepsilon(t), \phi) + \int_0^t [(P^\varepsilon u_\varepsilon(\tau), \phi) - (\Delta u'_\varepsilon(\tau), \phi)]d\tau
$$
We easily prove as in [18, Lemma 7] that the family of probability measures \( \pi \) for all \( u \), where the hypothesis \((A3)\) below.

Theorem 1.1 For each \( \varepsilon > 0 \), let \( u_\varepsilon \) be the unique solution of (1.1) on a given stochastic system \( (\Omega, \mathcal{F}, \mathbb{P}, W, \mathcal{F}_t) \). Under the assumptions \((A1), (A2)\) and \((A3)\), the sequence \( (u_\varepsilon)_{\varepsilon > 0} \) converges in law to the unique solution \( u_0 \) of the problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
du_0 - (\text{div}(\hat{\alpha} \nabla u_0) + \Delta u_0) dt = \hat{f}(u_0) dt + \hat{g}(u_0) dW \\
u_0 = 0 \quad \text{on } \partial D \times (0, T);
\end{array} \right. \\
u_0(x, 0) = u^0(x) \quad \text{and} \quad u_0(x, 0) = u^1(x) \quad \text{in } D,
\end{aligned}
\]

where \( \hat{\alpha}, \hat{f} \) and \( \hat{g} \) are the homogenized coefficients defined by \( \hat{\alpha}(x) = M(a(x, \cdot)(I_d + \nabla \chi(x, \cdot))) \) for \( x \in \overline{D}, \hat{g}(r) = (\hat{g}_k(r))_{k \geq 1} \) with \( \hat{g}_k(r) = (M(|g_k(\cdot, \cdot, r)|^2))^{\frac{1}{2}} \) and \( \hat{f}(r) = M(f(\cdot, \cdot, r)) \) for \( r \in \mathbb{R} \). Here \( I_d \) is the identity matrix of order \( d \), \( \chi = (\chi_j)_{1 \leq j \leq d} \in \mathcal{C}(\overline{D}; B^1_{\| \cdot \|_d}(\mathbb{R}^d)^d) \), where
\( \chi_j(x, \cdot) \) is the unique solution (up to an additive function of \( x \)) in the sense of distributions in \( \mathbb{R}^d \), of the corrector problem

\[
- \text{div}_y(\mathbf{a}(x, \cdot)(e_j + \nabla_y \chi_j(x, \cdot))) = 0 \quad \text{in } \mathbb{R}^d,
\]

(1.6)
e_j being the \( j \)th vector of the canonical basis of \( \mathbb{R}^d \).

In Theorem 1.1, the homogenized operator \( - \text{div}(\widehat{\mathbf{a}} \nabla u_0) \) depends on the homogenized matrix \( \widehat{\mathbf{a}} \) which in turn, depends on the corrector functions \( \chi_j \) as usual. However, one fact deserves to be highlighted: the corrector function \( \chi_j \) is here solution of a partial differential equation stated in the usual sense of distributions, which was not the case in the previous work dealing with the homogenization beyond the periodic setting, of such equations; see e.g. \([9, 10, 24, 25]\). This allows numerical computations of the homogenized coefficients, and provides also a way of computing the rates of convergence in the approximation \( u_\varepsilon(x, t, \omega) \approx u_0(x, t, \omega) + \varepsilon u_1(x, t, x/\varepsilon, \omega) \). This is an advance as far as the multiscale analysis of such kind of problems is concerned, and is one of our next objectives.

The assumption (A3) is stated above and the homogenized operators \( \widehat{P}, \widehat{f} \) and \( \widehat{g} \) are given in Section 3. One of the motivation of this work lies on the presence of the damping term \( -\Delta u_\varepsilon' \). Indeed as seen in Lemma 2.4, it allows to choose the corrector term to be independent of the fast time variable, although the micro-problem (1.1) depends on the fast time variable \( \tau = t/\varepsilon \). This simplifies the homogenization process as seen in the passage to the limit step.

The homogenization of hyperbolic stochastic partial differential equations with rapidly oscillating coefficients has been considered for the first time in \([18]\) under a periodic assumption on the coefficients. It is also important to note that the coefficients in \([18]\) do not oscillate in time \( \tau = t/\varepsilon \), which is one of the main features of the work under consideration here. Because of the oscillation in time, the limit passage in the stochastic term is more involved as seen in the proof of Lemma 3.2.

Since we consider a general deterministic behavior with respect to the oscillations in space and time, our study therefore falls within the scope of the sigma-convergence concept introduced in 2003 in \([19]\) by the second author. In order to get the homogenized limit, we combine two different concepts of sigma-convergence: the sigma-convergence for stochastic processes (see Definition 2.1), which helps to pass to the limit in the stochastic term, and the usual sigma-convergence method, which is used to study the terms not involving the stochastic integral; see the proof of Proposition 3.3. It is worth noticing that the linear counterpart of Problem (1.1) has been considered in \([10]\).

The layout of the paper is as follows. In Section 2, we present an overview of the fundamentals of the sigma-convergence method, which has been introduced in \([19]\). Section 3 deals with the homogenization result. In Section 4, we provide some applications of the homogenization result. We end the work with two appendices in which we provide some key estimates of the sequence of solutions to (1.1) together with a criterion under which the limit of stochastic integrals driven by convergent semimartingales is the stochastic integral of the limits.

2 Sigma-convergence Concept

We begin this section by recalling some notion in connection with the well known concept of algebras with mean value.
2.1 Generalized Besicovitch Spaces

We are concerned here with the main features of the sigma-convergence method, which we define. The reader may find more details in [30, 31].

We first recall that by an algebra with mean value $A$ on $\mathbb{R}^d$ is meant any Banach algebra consisting of bounded uniformly continuous functions on $\mathbb{R}^d$ satisfying

(i) $A$ contains the constants;

(ii) $u(-x) \in A$ for any $u \in A$ and $x \in \mathbb{R}^d$;

(iii) For any $u \in A$, the limit $M(u) = \lim_{R \to \infty} \int_{B_R} u(y) dy$ exists and is called the mean value of $u$.

In (iii) above, $\int_{B_R}$ is the integral mean over the open ball $B_R$ centered at 0 and of radius $R$: $\int_{B_R} = |B_R|^{-1} \int_{B_R}$.

We consider an algebra with mean value $A$ on $\mathbb{R}^d$ and $m \in \mathbb{N}$. We define

$$A^\infty = \{ u \in C^\infty(\mathbb{R}^d) : D_y^\alpha u \in A, \forall \alpha \in \mathbb{N}^d, |\alpha| \leq m \},$$

a Fréchet space when endowed with the locally convex topology defined by the family of norms $\| \cdot \|_m$.

We consider $A_y$ (resp., $A_\tau$) an algebra with mean value on $\mathbb{R}^d_y$ (that corresponds to the fast spacial variable) (resp., $\mathbb{R}_\tau$ (that corresponds to the fast time variable)), and $\text{BUC}(\mathbb{R}^d; \mathbb{R})$ the Banach space of bounded uniformly continuous functions $u : \mathbb{R}^d \to \mathbb{R}$. We will use in this work some tools as:

1. The product algebra with mean value $A_y \odot A_\tau$ which is the closure in $\text{BUC}(\mathbb{R}^{d+1})$ of the tensor product $A_y \otimes A_\tau = \{ \sum_{\text{finite}} u_i \otimes v_i ; u_i \in A_y, v_i \in A_\tau \},$ which is an algebra with mean value on $\mathbb{R}^{d+1}$.

2. The vector-valued algebra with mean value $A(\mathbb{R}^d; \mathbb{R})$, which is the closure of $A \odot \mathbb{R}$ in $\text{BUC}(\mathbb{R}^d; \mathbb{R})$, where $A \otimes \mathbb{R}$ is the space of functions of the form $A \otimes \mathbb{R} = \sum_{\text{finite}} u_i \otimes e_i , u_i \in A, e_i \in \mathbb{R}$, with $(u_i \otimes e_i)(y) = u_i(y)e_i$ for $y \in \mathbb{R}^d$. It is an easy exercise to show that $A_y \otimes A_\tau = A_y(\mathbb{R}^d, A_\tau) = A_\tau(\mathbb{R}, A_y)$.

Now, let $f \in A(\mathbb{R}^d) = A(\mathbb{R}^d; \mathbb{R})$ (integer $d \geq 1$), we define the Besicovitch seminorm on $A(\mathbb{R}^d)$ as follows: for $1 \leq p < \infty$, we define the Marcinkiewicz-type space $\mathfrak{M}^p(\mathbb{R}^d)$ to be the vector space of functions $u \in L^p_{\text{loc}}(\mathbb{R}^d)$ such that

$$\|u\|_p = \left( \limsup_{R \to \infty} \int_{B_R} |u(y)|^p dy \right)^{\frac{1}{p}} < \infty$$

where $B_R$ is the open ball in $\mathbb{R}^d$ centered at the origin and of radius $R$. Under the seminorm $\| \cdot \|_p$, $\mathfrak{M}^p(\mathbb{R}^d)$ is a complete seminormed space with the property that $A(\mathbb{R}^d) \subset \mathfrak{M}^p(\mathbb{R}^d)$ since $\|u\|_p < \infty$ for any $u \in A(\mathbb{R}^d)$. We therefore define the generalized Besicovitch space $B^p_A(\mathbb{R}^d) = B^p_A(\mathbb{R}^d; \mathbb{R})$ as the closure of $A(\mathbb{R}^d)$ in $\mathfrak{M}^p(\mathbb{R}^d)$. The following hold true:

1. The space $B^p_A(\mathbb{R}^d) = B^p_A(\mathbb{R}^d)/\mathcal{N}$ (where $\mathcal{N} = \{ u \in B^p_A(\mathbb{R}^d) : \|u\|_p = 0 \}$) is a Banach space under the norm $\|u + \mathcal{N}\|_p = \|u\|_p$ for $u \in B^p_A(\mathbb{R}^d)$.

2. The mean value $M : A(\mathbb{R}^d) \to \mathbb{R}$ extends by continuity to a continuous linear mapping (still denoted by $M$) on $B^p_A(\mathbb{R}^d)$. 

Moreover, for \( u \in B^p_A(\mathbb{R}^d) \) we have
\[
\| u \|_p = (M(|u|^p))^{1/p} \equiv \left( \lim_{R \to \infty} \int_{B_R} |u(y)|^p \, dy \right)^{1/p}
\]
and for \( u \in \mathcal{N} \) one has \( M(u) = 0 \).

It is to be noted that \( B^2_A(\mathbb{R}^d; H) \) (when \( H \) is a Hilbert space) is a Hilbert space with inner product
\[
(u, v)_2 = M[(u, v)_H] \quad \text{for } u, v \in B^2_A(\mathbb{R}^d; H)
\]
where \((\cdot, \cdot)_H\) stands for the inner product in \( H \) and \((u, v)_H \) the function \( y \mapsto (u(y), v(y))_H \) from \( \mathbb{R}^d \) to \( \mathbb{R} \), which belongs to \( B^1_A(\mathbb{R}^d) \).

We have \( B^p_A(\mathbb{R}^d) \subset B^p_A(\mathbb{R}^d) \) for \( 1 \leq p \leq q < \infty \). From this last property one may naturally define the space \( B^\infty_A(\mathbb{R}^d) \) as follows:
\[
B^\infty_A(\mathbb{R}^d) = \left\{ f \in \bigcap_{1 \leq p < \infty} B^p_A(\mathbb{R}^d) : \sup_{1 \leq p < \infty} \| f \|_p < \infty \right\}.
\]
We endow \( B^\infty_A(\mathbb{R}^d) \) with the seminorm \( [f]_\infty = \sup_{1 \leq p < \infty} \| f \|_p \), which makes it a complete seminormed space.

In this regard, we consider the space \( B^1_A(\mathbb{R}^d) = \{ u \in B^p_A(\mathbb{R}^d) : \nabla_y u \in (B^p_A(\mathbb{R}^d))^d \} \) endowed with the seminorm
\[
\| u \|_{1,p} = (\| u \|^p_p + \| \nabla_y u \|^p_p)^{1/p},
\]
which is a complete seminormed space. The Banach counterpart of the previous spaces are defined as follows. We define \( B^{1,p}_A(\mathbb{R}^d) \) mutatis mutandis: replace \( B^p_A(\mathbb{R}^d) \) by \( B^p_A(\mathbb{R}^d) \) and \( \partial / \partial y_i \) by \( \overline{\partial} / \partial y_i \), where \( \overline{\partial} / \partial y_i \) is defined by
\[
\frac{\overline{\partial}}{\partial y_i} (u + \mathcal{N}) := \frac{\partial u}{\partial y_i} + \mathcal{N} \quad \text{for } u \in B^{1,p}_A(\mathbb{R}^d).
\]
It is important to note that \( \overline{\partial} / \partial y_i \) is also defined as the infinitesimal generator in the \( i \)th direction coordinate of the strongly continuous group \( T(y) : B^p_A(\mathbb{R}^d) \to B^p_A(\mathbb{R}^d) \); \( T(y)(u + \mathcal{N}) = u(\cdot + y) + \mathcal{N} \). Let us denote by \( \varrho : B^{1,p}_A(\mathbb{R}^d) \to B^{1,p}_A(\mathbb{R}^d) \), \( \varrho(u) = u + \mathcal{N} \), the canonical surjection. We remark that if \( u \in B^{1,p}_A(\mathbb{R}^d) \) then \( \varrho(u) \in B^{1,p}_A(\mathbb{R}^d) \) with further
\[
\frac{\overline{\partial} \varrho(u)}{\partial y_i} = \varrho \left( \frac{\partial u}{\partial y_i} \right),
\]
as seen above in (2.2).

We assume in the sequel that the algebra \( A \) is ergodic, that is, any \( u \in B^p_A(\mathbb{R}^d) \) that is invariant under \( (T(y))_{y \in \mathbb{R}^d} \) is a constant in \( B^p_A(\mathbb{R}^d) \), i.e., if \( \| T(y)u - u \|_p = 0 \) for every \( y \in \mathbb{R}^d \), then \( \| u - c \|_p = 0 \), where \( c \) is a constant. As in [6] we observe that if the algebra with mean value \( A \) is ergodic, then \( u \in B^{1,p}_A(\mathbb{R}^d) \) is invariant if and only if \( \overline{\partial} u / \partial y_i = 0 \) for all \( 1 \leq i \leq d \). We denote by \( I^p_A(\mathbb{R}^d) \) the space of invariant functions in \( B^{1,p}_A(\mathbb{R}^d) \). Let us also recall the following property [30, 31].

(iii) The mean value \( M \) viewed as defined on \( A \), extends by continuity to a positive continuous linear form (still denoted by \( M \)) on \( B^p_A(\mathbb{R}^d) \). For each \( u \in B^p_A(\mathbb{R}^d) \) and all \( a \in \mathbb{R}^d \), we have \( M(u(\cdot + a)) = M(u) \), and \( \| u \|_p = (M(|u|^p))^{1/p} \).
To the space \( B_A^p(\mathbb{R}^d) \) we also attach the following **corrector** space

\[
B_A^{1,p}(\mathbb{R}^d) = \{ u \in W_{\text{loc}}^{1,p}(\mathbb{R}^d) : \nabla u \in B_A^p(\mathbb{R}^d) \text{ and } M(\nabla u) = 0 \}.
\]

We also define the space \( \nabla B_A^{1,p}(\mathbb{R}^d) = \{ \nabla u : u \in B_A^{1,p}(\mathbb{R}^d) \} \). Identifying an element of \( \nabla B_A^{1,p}(\mathbb{R}^d) \) with its class in \( (B_A^p(\mathbb{R}^d))^d \), \( \nabla B_A^{1,p}(\mathbb{R}^d) \) will be considered as a subspace of \( (B_A^p(\mathbb{R}^d))^d \).

Moreover we identify two elements of \( B_A^{1,p}(\mathbb{R}^d) \) by their gradients: \( u = v \) in \( B_A^{1,p}(\mathbb{R}^d) \) iff \( \nabla(u - v) = 0 \), i.e., \( \|\nabla(u - v)\|_p = 0 \). We may therefore equip \( B_A^{1,p}(\mathbb{R}^d) \) with the gradient norm \( \|u\|_{\#p} = \|\nabla u\|_p \). This defines a Banach space [7, Theorem 3.12] (actually \( \nabla B_A^{1,p}(\mathbb{R}^d) \) is closed in \( B_A^p(\mathbb{R}^d) \)), so that \( B_A^{1,p}(\mathbb{R}^d) \) is a Banach space containing \( B_A^{1,p}(\mathbb{R}^d) \) as a subspace.

For \( u \in B_A^p(\mathbb{R}^d) \) (resp., \( v = (v_1, \ldots, v_d) \in (B_A^p(\mathbb{R}^d))^d \)), we define the gradient operator \( \nabla_y \) and the divergence operator \( \nabla_y' \) by

\[
\nabla_y u := \left( \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_d} \right) \quad \text{and} \quad \nabla_y' v \equiv \overline{\text{div}}_y v := \sum_{i=1}^d \frac{\partial v_i}{\partial y_i}.
\]

Then the divergence operator sends continuously and linearly \((B_A^p(\mathbb{R}^d))^d \) into \((B_A^{1,p}(\mathbb{R}^d))^d\)' and satisfies

\[
\langle \overline{\text{div}}_y u, v \rangle = -\langle u, \nabla_y v \rangle \quad \text{for } v \in B_A^{1,p}(\mathbb{R}^d) \quad \text{and} \quad u = (u_i)_{1 \leq i \leq d} \in (B_A^p(\mathbb{R}^d))^d,
\]

where \( \langle u, \nabla_y v \rangle := M(u \cdot \nabla_y v) \).

### 2.2 Sigma-convergence

We are now able to recall the definition and main properties of \( \Sigma \)-converge method. To this end, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with expectation \( \mathbb{E} \). We denote by \( B(\Omega) \) the Banach space of bounded functions \( f : \Omega \to \mathbb{R} \), and by \( B(\Omega) \) the closure in \( F(\Omega) \) of the vector space \( H(\Omega) \) consisting of all finite linear combinations of characteristic functions \( 1_X \) of sets \( X \in \mathcal{F} \).

Then \( B(\Omega) \) is the Banach space of bounded \( \mathcal{F} \)-measurable functions. We also define the space \( B(\Omega, Z) \) of bounded \((\mathcal{F}, B(Z))\)-measurable functions \( f : \Omega \to Z \), where \( Z \) is a Banach space equipped with the Borelians \( B(Z) \). For a Banach space \( F \), the space of \( F \)-valued random variables \( u \) such that \( \|u\|_F \in L^p(\Omega; \mathcal{F}, \mathbb{P}) \) will be denoted by \( L^p(\Omega, \mathcal{F}, \mathbb{P}; F) \) (or merely \( L^p(\Omega; F) \), if there is no danger of confusion). In all what follows, random variables will be considered on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The letter \( E \) will throughout denote any ordinary sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) with \( 0 < \varepsilon_n \leq 1 \) and \( \varepsilon_n \to 0 \) as \( n \to \infty \).

For a function \( u \in L^p(\Omega; L^p(D_T; B_A^{1,p}(\mathbb{R}^{d+1}))) \) we denote by \( u(x, t, \cdot, \omega) \) (for fixed \((x, t, \omega) \in D_T \times \Omega\)) the element of \( B_A^{1,p}(\mathbb{R}^{d+1}) \) defined by

\[
u(x, t, \cdot, \omega)(y, \tau) = u(x, t, y, \tau, \omega) \quad \text{for } (y, \tau) \in \mathbb{R}^{d+1}.
\]

**Definition 2.1** A **sequence of random variables** \((u_\varepsilon)_{\varepsilon > 0} \subset L^p(\Omega; L^p(D_T)) \) \((1 \leq p < \infty)\) is said to:

(i) **weakly \( \Sigma \)-converge** in \( L^p(D_T \times \Omega) \) to some random variable \( u_0 \in L^p(\Omega; L^p(D_T; B_A^{1,p}(\mathbb{R}^{d+1}))) \) if as \( \varepsilon \to 0 \), we have

\[
\int \int_{D_T \times \Omega} u_\varepsilon(x, t, \omega) f \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) dx dt d\mathbb{P} \\
- \int \int_{D_T \times \Omega} M(u_0(x, t, \cdot, \omega) f(x, t, \cdot, \omega)) dx dt d\mathbb{P}
\]  

(2.4)
for every \( f \in L^p'(\Omega; L^{p'}(D_T; A)) \), \( \frac{1}{p} + \frac{1}{p'} = 1 \). We express this by writing \( u_\varepsilon \to u_0 \) in \( L^p(D_T \times \Omega) \)-weak \( \Sigma \);

(ii) strongly \( \Sigma \)-converge in \( L^p(D_T \times \Omega) \) to \( u_0 \in L^p(\Omega; L^p(D_T; B^p_A(\mathbb{R}^{d+1}))) \) if (2.4) holds and further

\[
\|u_\varepsilon\|_{L^p(D_T \times \Omega)} \to \|u_0\|_{L^p(D_T \times \Omega; B^p_A(\mathbb{R}^{d+1}))}. \tag{2.5}
\]

We express this by writing \( u_\varepsilon \to u_0 \) in \( L^p(D_T \times \Omega) \)-strong \( \Sigma \).

**Remark 2.2** The convergence (2.4) still holds true for \( f \in B(\Omega; C(D_T; B^{p',\infty}_A(\mathbb{R}^{d+1}))) \), where

\[
B^{p',\infty}_A(\mathbb{R}^{d+1}) = B^p_A(\mathbb{R}^{d+1}) \cap L^{\infty}(\mathbb{R}^{d+1}), \quad \frac{1}{p} + \frac{1}{p'} = 1.
\]

**Remark 2.3** If in Definition 2.1, we choose test functions independent of \( \omega \) and we do not integrate over \( \Omega \), we get the usual \( \Sigma \)-convergence method, that is, \( u_\varepsilon \to u_0 \) in \( L^p(D_T) \)-weak \( \Sigma \) if

\[
\int_{D_T} u_\varepsilon(x, t, \omega) f(x, t, \frac{\varepsilon}{\varepsilon'}) \, dx dt \to \int_{D_T} M(u_0(x, t, \cdot), \omega) f(x, t, \cdot) \, dx dt \quad \mathbb{P}\text{-a.s.}
\]

for every \( f \in L^p'(D_T; A) \). We define the strong \( \Sigma \)-convergence method accordingly.

The main properties of the sigma-convergence for stochastic processes can be found in [30] (see also [24, 25]). They read as follows.

**(SC)\_1** For \( 1 < p < \infty \), any sequence of random variables which is bounded in \( L^p(\Omega; L^p(D_T)) \) possesses a weakly \( \Sigma \)-convergent subsequence.

**(SC)\_2** Let \( 1 < p < \infty \). Let \( (u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega; L^p(0,T; W_0^{1,p}(D))) \) be a sequence of random variables which satisfies the following estimate

\[
\sup_{\varepsilon \in E} \mathbb{E}\|u_\varepsilon\|_{L^p(0,T; W_0^{1,p}(D))}^p < \infty.
\]

Then there exist a subsequence \( E' \) from \( E \) and a couple of random variables \( (u_0, u_1) \) with \( u_0 \in L^p(\Omega; L^p(0,T; W_0^{1,p}(D))) \) and \( u_1 \in L^p(\Omega; L^p(D_T; B_A^p(\mathbb{R}_\tau; B^{1,p}_A(\mathbb{R}^d))) \) such that as \( E' \ni \varepsilon \to 0 \),

\[
u_\varepsilon \to u_0 \quad \text{in } L^p(\Omega; L^p(0,T; W_0^{1,p}(D)))-\text{weak}
\]

and

\[
\frac{\partial u_\varepsilon}{\partial x_i} - \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \quad \text{in } L^p(D_T \times \Omega)-\text{weak}, \quad 1 \leq i \leq d. \tag{2.6}
\]

**(SC)\_3** Assume that the hypotheses of (SC)\_2 above are satisfied. Finally suppose further that \( p \geq 2 \) and that there exists a subsequence \( E' \) from \( E \) and a random variable

\[
u_\varepsilon \to u_0 \quad \text{in } L^2(D_T \times \Omega)-\text{strong}. \tag{2.7}
\]

Then there exists a subsequence of \( E' \) (not relabeled) and a \( L^p(D_T; B^p_A(\mathbb{R}_\tau; B^{1,p}_A(\mathbb{R}^d))) \)-valued stochastic process \( u_1 \in L^p(\Omega; L^p(D_T; B^p_A(\mathbb{R}_\tau; B^{1,p}_A(\mathbb{R}^d))) \) such that (2.6) holds when \( E' \ni \varepsilon \to 0 \).

**(SC)\_4** Let \( 1 < p, q < \infty \) and \( r \geq 1 \) be such that \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1 \). Assume that \( (u_\varepsilon)_{\varepsilon > 0} \subset L^q(D_T \times \Omega) \) is weakly \( \Sigma \)-convergent in \( L^q(D_T \times \Omega) \) to some \( u_0 \in L^q(D_T \times \Omega; B^q_A(\mathbb{R}^{d+1})) \) and \( (v_\varepsilon)_{\varepsilon > 0} \subset L^p(D_T \times \Omega) \) is strongly \( \Sigma \)-convergent in \( L^p(D_T \times \Omega) \) to some \( v_0 \in L^p(D_T \times \Omega; B^p_A(\mathbb{R}^{d+1})) \). Then the sequence \( (u_\varepsilon v_\varepsilon)_{\varepsilon > 0} \) is weakly \( \Sigma \)-convergent in \( L^r(D_T \times \Omega) \) to \( u_0 v_0 \).
The above properties are the random counterpart of the same properties already derived in [19] (see also [21]) in the deterministic setting.

It is a well-known fact that if $A = A_y \cap A_\tau$ is an algebra with mean value on $\mathbb{R}^{d+1}$, then the natural choice of test functions in the homogenization process is given by $\Phi_\varepsilon(x, t, \omega) = \psi_0(x, t, \omega) + \varepsilon \psi_1(x, t, x/\varepsilon, t/\varepsilon, \omega)$, $(x, t, \omega) \in D_T \times \Omega$. However, due to the following result, although the coefficients in our problem depend on the fast time variable $\tau = t/\varepsilon$, the function $\psi_1$ will not depend on $\tau = t/\varepsilon$. This is a consequence of the control of the damping term.

**Lemma 2.4** Let $(u_0, u_1)$ be a couple of random variables satisfying (2.6) and $(u_\varepsilon)_{\varepsilon \in E}$ be a bounded sequence in $L^2(\Omega; L^2(0, T; H^1_0(D)))$ such that

$$\sup_{\varepsilon \in E} \|\nabla u'_\varepsilon\|^2_{L^2(D_T)} \leq C, \quad \text{a.e. } \omega \in \Omega,$$

where the constant $c > 0$ is independent of $\varepsilon$ and $\omega \in \Omega$.

Then we have $u_1 \in L^2(D_T; B^{1,2}_{\#A_y}(\mathbb{R}^d))$.

**Proof** For $\varphi \in C_0^\infty(D_T)$, $\psi \in A^\infty_y$, $\phi \in A^\infty$, and $1 \leq i \leq d$, one has

$$\varepsilon \int_{D_T} \frac{\partial^2 u_\varepsilon}{\partial t \partial x_i}(x, t) \varphi(x, t) \psi\left(\frac{x}{\varepsilon}\right) \phi\left(\frac{t}{\varepsilon}\right) dx dt + \varepsilon \int_{D_T} \frac{\partial u_\varepsilon}{\partial x_i}(x, t) \frac{\partial \varphi}{\partial t}(x, t) \psi\left(\frac{x}{\varepsilon}\right) \phi\left(\frac{t}{\varepsilon}\right) dx dt = - \int_{D_T} \frac{\partial u_\varepsilon}{\partial x_i}(x, t) \varphi(x, t) \psi\left(\frac{x}{\varepsilon}\right) \frac{d\phi}{d\tau}\left(\frac{t}{\varepsilon}\right) dx dt. \quad (2.8)$$

Since $(\nabla u_\varepsilon)_{\varepsilon \in E}$ and $(\nabla u'_\varepsilon)_{\varepsilon \in E}$ are bounded in $L^2(D_T)$, passing to the limit in (2.8), as $E' \ni \varepsilon \to 0$ (the subsequence in (SC)$_2$ above), the left hand-side of the above equality goes to zero. So using the arbitrariness of $\varphi$ and $\psi$, we have

$$M(\left(\frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i}\right) \frac{d\phi}{d\tau}) = 0 \quad \text{in } D_T \times \mathbb{R}^d$$

for all $\phi \in A^\infty_y$ (1 \leq i \leq d). This implies that $\frac{\partial u_0}{\partial y_i}$ does not depend on $\tau$, hence $\frac{\partial u_0}{\partial y_i} \in L^2(D_T; B^{1,2}_{\#A_y}(\mathbb{R}^d))$. Letting $v_1 = M(\nabla u_1) \in L^2(D_T; B^{1,2}_{\#A_y}(\mathbb{R}^d))$ we have $\nabla_y v_1 = M(\nabla_y u_1) = \nabla_y u_1$, so that $\nabla_y (u_1 - v_1) = 0$. Since in $B^{1,2}_{\#A_y}(\mathbb{R}^d)$ two elements are identified by their gradients, we obtain readily $u_1 = v_1$, so that $u_1 \in L^2(D_T; B^{1,2}_{\#A_y}(\mathbb{R}^d))$.

**Remark 2.5** Lemma 2.4 allows us to choose the function $u_1$ in $L^2(D_T; B^{1,2}_{\#A_y}(\mathbb{R}^d))$. Indeed, as seen in the proof of that lemma, we may replace $u_1$ by $v_1 = M(\nabla u_1)$ so that $\nabla_y v_1 = \nabla_y u_1$. As a result, we will choose the test functions (in the homogenization process) that are independent of $\tau$.

### 3 Homogenization Results

#### 3.1 Passage to the Limit

The passage to the limit will be made under the assumption (A3) defined in Section 1, characterizing essentially the distribution of the microstructures in the medium $D$ and the behavior on the fast time scale. It reads as follows:

Starting with the sequence of probability measures $(\pi^\varepsilon)$ defined by (1.4), since it is tight, the Prokhorov’ theorem [23] yields the existence of a subsequence $(\pi^\varepsilon_n)$ of $(\pi^\varepsilon)$ that weakly
converges to a probability measure \( \pi \) in \( S \). Moreover, by Skorokhod’s theorem \([29]\), there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and random variables \((W_{\varepsilon_n}, u_{\varepsilon_n}, u'_{\varepsilon_n})\) and \((\mathcal{W}, u_0, u'_0)\) with values in \( S \) such that:

(C1) \( \pi^{\varepsilon_n} \) and \( \pi \) are respectively the probability law of \((W_{\varepsilon_n}, u_{\varepsilon_n}, u'_{\varepsilon_n})\) and \((\mathcal{W}, u_0, u'_0)\),
(C2) \( u_{\varepsilon_n} \to u_0 \) in \( L^2(0,T; L^2(D)) \cap C(0,T; H^{-1}(D)) \) \( \mathbb{P} \)-a.s.,
(C3) \( u'_{\varepsilon_n} \to u'_0 \) in \( C(0,T; L^2(D)) \) \( \mathbb{P} \)-a.s.,
(C4) \( W_{\varepsilon_n} \to \mathcal{W} \) in \( C(0,T; u_0) \) \( \mathbb{P} \)-a.s.

With this in mind, we deduce that \( \{W_{\varepsilon_n}\} \) is a sequence of \( m \)-dimensional Wiener process. Now, set

\[
\mathcal{F}_t = \sigma\{\mathcal{W}(s), u_0(s), u'_0(s)\}_{s \in [0,t]}.
\]

Then arguing as in \([27]\), we may prove that \( \mathcal{W}(t) \) is a \( \mathcal{F}_t \)-standard Wiener process. By the same way of proceeding as in \([2]\), we can show that \( u_{\varepsilon_n} \) (obtained above) satisfies

\[
(u'_{\varepsilon_n}(t), \phi) + \int_0^t [\langle P^{\varepsilon_n} u_{\varepsilon_n}(\tau), \phi \rangle - \langle \nabla u'_{\varepsilon_n}(\tau), \nabla \phi \rangle]d\tau = (u_1, \phi) + \int_0^t \left( f\left(\varepsilon_n, \frac{\tau}{\varepsilon_n}, u'_{\varepsilon_n}\right), \phi \right) d\tau + \left( \int_0^t g\left(\varepsilon_n, \frac{\tau}{\varepsilon_n}, u'_{\varepsilon_n}\right) dW_{\varepsilon_n}(\tau), \phi \right)
\]

for any \( \phi \in H_0^1(D) \) and for almost all \((\sigma, t) \in \Omega \times [0,T]\).

Since \( u_{\varepsilon_n} \) satisfies (3.1), then we obtain from the application of Prokhorov’s and Skhorokhod’s compactness results that \( u_{\varepsilon_n} \) and \( u'_{\varepsilon_n} \) satisfy the a priori estimates

\[
\mathbb{E} \sup_{0 \leq t \leq T} \|u_{\varepsilon_n}(t)\|_{H^1_0(D)}^4 \leq C, \quad \mathbb{E} \sup_{0 \leq t \leq T} \|u'_{\varepsilon_n}(t)\|_{L^2(D)}^4 \leq C, \quad (3.2)
\]

\[
\mathbb{E} \sup_{|\theta| \leq \delta} \int_0^T \|u'_{\varepsilon_n}(t + \theta) - u'_{\varepsilon_n}(t)\|_{H^{-1}(D)}^2 dt \leq C\delta. \quad (3.3)
\]

We infer from these estimates that

\[
\begin{align*}
u_{\varepsilon_n} & \to u_0 \quad \text{in} \ L^2(\Omega; L^\infty(0,T; H^1_0(D))) \text{-weak *}, \\
u_{\varepsilon_n} & \to u_0 \quad \text{in} \ L^2(\Omega; L^\infty(0,T; L^2(D))) \text{-weak *}, \\
u'_{\varepsilon_n} & \to u'_0 = \frac{\partial u_0}{\partial t} \quad \text{in} \ L^2(\Omega; L^\infty(0,T; L^2(D))) \text{-weak *}.
\end{align*}
\]

Therefore, the combination of (3.2), (3.3) and Vitali’s theorem yield

\[
\begin{align*}
u_{\varepsilon_n} & \to u_0 \quad \text{in} \ L^2(\Omega; L^\infty(0,T; L^2(D))) \text{-strong}, \\
u'_{\varepsilon_n} & \to u'_0 \quad \text{in} \ L^2(\Omega; L^\infty(0,T; L^2(D))) \text{-strong}.
\end{align*}
\]

Hence for almost all \((\omega, t) \in \Omega \times [0,T]\), we obtain

\[
\begin{align*}
u_{\varepsilon_n} & \to u_0 \quad \text{in} \ L^2(D) \text{-strong} \quad \text{and} \quad \nu'_{\varepsilon_n} \to u'_0 \quad \text{in} \ L^2(D) \text{-strong} \quad (3.5)
\end{align*}
\]

with respect to the measure \( d\mathbb{P} \otimes dt \).

**Lemma 3.1** Let \( (u_\varepsilon)_\varepsilon \) be a sequence in \( L^2(D_T \times \Omega) \) such that \( u'_\varepsilon \to u'_0 \) in \( L^2(D_T \times \Omega) \) as \( \varepsilon \to 0 \), where \( u'_0 \in L^2(D_T \times \Omega) \). Then, as \( \varepsilon \to 0 \), the following holds true:

\[
f^\varepsilon(\cdot, \cdot, u'_\varepsilon) \to f(\cdot, \cdot, u'_0) \quad \text{in} \ L^2(D_T) \text{-strong} \quad \mathbb{P} \text{-a.s.} \quad (3.6)
\]

and, for each \( k \geq 1 \),

\[
g_k^\varepsilon(\cdot, \cdot, u'_\varepsilon) \to g_k(\cdot, \cdot, u'_0) \quad \text{in} \ L^2(D_T \times \Omega) \text{-strong} \quad \Sigma. \quad (3.7)
\]
Proof. Let us first prove (3.7). It amounts in checking (i) and (ii) below:

(i) \( g^\varepsilon_k(\cdot, \cdot, u'_\varepsilon) \to g_k(\cdot, \cdot, u'_0) \) in \( L^2(D_T \times \Omega) \)-weak \( \Sigma \),
(ii) \( \|g^\varepsilon_k(\cdot, \cdot, u'_\varepsilon)\|_{L^2(D_T \times \Omega)} \to \|g_k(\cdot, \cdot, u'_0)\|_{L^2(D_T \times \Omega; B^2_A(\mathbb{R}^{d+1}))} \).

We first consider (i). Let \( u \in B(\Omega; \mathcal{C}(\overline{D_T})) \); then the function \( (x, t, y, \tau, \omega) \mapsto g_k(y, \tau, u(x, t, \omega)) \) lies in \( B(\Omega; \mathcal{C}(\overline{D_T}; B^2_A(\mathbb{R}^{d+1})) \)), so that we have \( g^\varepsilon_k(\cdot, \cdot, u) \to g_k(\cdot, \cdot, u) \) in \( L^2(D_T \times \Omega) \)-weak \( \Sigma \) as \( \varepsilon \to 0 \). Using the density of \( B(\Omega; \mathcal{C}(\overline{D_T})) \) in \( L^2(D_T \times \Omega) \) (which allows us to approximate \( u'_0 \) in \( L^2(D_T \times \Omega) \) by a strongly convergent sequence extracted from \( B(\Omega; \mathcal{C}(\overline{D_T})) \)), we obtain the following result:

\[
g^\varepsilon_k(\cdot, \cdot, u'_\varepsilon) \to g_k(\cdot, \cdot, u'_0) \quad \text{in } L^2(D_T \times \Omega)-\text{weak } \Sigma \text{ as } \varepsilon \to 0. \tag{3.8}\]

Next, for \( h \in L^2(\Omega; L^2(D_T; A)) \), we have

\[
\int_{D_T \times \Omega} g^\varepsilon_k(\cdot, \cdot, u'_\varepsilon)h^\varepsilon \, dx \, dt \, dp - \int_{D_T \times \Omega} M(g_k(\cdot, \cdot, u'_0)h) \, dx \, dt \, dp
= \int_{D_T \times \Omega} (g^\varepsilon_k(\cdot, \cdot, u'_\varepsilon) - g^\varepsilon_k(\cdot, \cdot, u'_0))h^\varepsilon \, dx \, dt \, dp
+ \left( \int_{D_T \times \Omega} g^\varepsilon_k(\cdot, \cdot, u'_0)h^\varepsilon \, dx \, dt \, dp - \int_{D_T \times \Omega} M(g_k(\cdot, \cdot, u'_0)h) \, dx \, dt \, dp \right)
= I_1 + I_2
\]

where

\[
|I_1| = \left| \int_{D_T \times \Omega} (g^\varepsilon_k(\cdot, \cdot, u'_\varepsilon) - g^\varepsilon_k(\cdot, \cdot, u'_0))h^\varepsilon \, dx \, dt \, dp \right| \leq C \|u'_\varepsilon - u'_0\|_{L^2(D_T \times \Omega)} \|h^\varepsilon\|_{L^2(D_T \times \Omega)} \tag{3.9}
\]

and

\[
I_2 = \int_{D_T \times \Omega} g^\varepsilon_k(\cdot, \cdot, u'_0)h^\varepsilon \, dx \, dt \, dp - \int_{D_T \times \Omega} M(g_k(\cdot, \cdot, u'_0)h) \, dx \, dt \, dp.
\]

Since \( u'_\varepsilon \to u'_0 \) in \( L^2(D_T \times \Omega) \)-strong as \( \varepsilon \to 0 \), and using (3.8) and (3.9) we obtain (i). Now, for (ii), we have to show that

\[
\int_{D_T \times \Omega} |g^\varepsilon_k(\cdot, \cdot, u'_\varepsilon)|^2 \, dx \, dt \, dp \to \int_{D_T \times \Omega} M(|g_k(\cdot, \cdot, u'_0)|^2) \, dx \, dt \, dp. \tag{3.10}
\]

As above, if we choose \( u \in B(\Omega; \mathcal{C}(\overline{D_T})) \), then the function

\[
g_k(\cdot, \cdot, u) : (x, t, y, \tau, \omega) \mapsto g_k(y, \tau, u(x, t, \omega))
\]

is an element of \( B(\Omega; \mathcal{C}(\overline{D_T}; B^2_A(\mathbb{R}^{d+1})) \)), so that

\[
g^\varepsilon_k(\cdot, \cdot, u) \to g_k(\cdot, \cdot, u) \quad \text{in } L^2(D_T \times \Omega)-\text{strong } \Sigma.
\]

Now, we choose a sequence \((v_n)_n \subset B(\Omega; \mathcal{C}(\overline{D_T})) \) satisfying

\[
v_n \to u'_0 \quad \text{in } L^2(D_T \times \Omega) \text{ as } n \to \infty. \tag{3.11}
\]

We have

\[
\int_{D_T \times \Omega} |g^\varepsilon_k(\cdot, \cdot, u'_\varepsilon)|^2 \, dx \, dt \, dp = \int_{D_T \times \Omega} g^\varepsilon_k(\cdot, \cdot, u'_\varepsilon)(g^\varepsilon_k(\cdot, \cdot, u'_\varepsilon) - g^\varepsilon_k(\cdot, \cdot, u'_0)) \, dx \, dt \, dp
+ \int_{D_T \times \Omega} g^\varepsilon_k(\cdot, \cdot, u'_0)(g^\varepsilon_k(\cdot, \cdot, u'_0) - g^\varepsilon_k(\cdot, \cdot, v_n)) \, dx \, dt \, dp
\]
\[
\psi = J_1 + J_2 + J_3.
\]

As above, using the boundedness of the sequence \((g_k^\varepsilon(\cdot, u_\varepsilon'))\) in \(L^2(D_T \times \Omega)\) and the Lipschitz property of \(g_k\) associated to the strong convergence of \(u_\varepsilon'\) towards \(u_0'\) in \(L^2(D_T \times \Omega)\), we get that \(J_1 \to 0\) as \(\varepsilon \to 0\). The same argument as above associated to the strong convergence \((3.11)\) yield \(J_2 \to 0\). As for \(J_3\), we use the weak \(\Sigma\)-convergence (i) with the test function \(g_k(\cdot, v_n)\) to get

\[
J_3 \to \int_{D_T \times \Omega} M(g_k(\cdot, u_0')g_k(\cdot, v_n))dxdtd\mathbb{P} \quad \text{when } \varepsilon \to 0.
\]

In the above convergence result, we let finally \(n \to \infty\) to get

\[
J_3 \to \int_{D_T \times \Omega} M(|g_k(\cdot, u_0')|^2)dxdtd\mathbb{P}.
\]

This shows (ii), and hence \((3.7)\). Noticing that up to a subsequence, we have \(u_\varepsilon' \to u_0'\) in \(L^2(D_T)\)-strong \(\mathbb{P}\)-a.s., we may hence proceed as above to obtain \((3.6)\), thereby completing the proof. \(\square\)

The next result deals with the convergence of the stochastic term.

**Lemma 3.2** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Let \(E\) be a fundamental sequence, and let \((u_\varepsilon)_{\varepsilon \in E}\) be a sequence in \(L^2(D_T \times \Omega)\) satisfying

\[
u_\varepsilon' \to u_0' \quad \text{in } L^2(D_T \times \Omega)-\text{strong as } E \ni \varepsilon \to 0. \quad (3.12)
\]

Let \((W^\varepsilon)_{\varepsilon \in E}\) be a sequence of \(m\)-dimensional Wiener process on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying

\[
W^\varepsilon \to W \quad \text{in } \mathcal{C}(0,T;\mathbb{R}^m) \text{ as } E \ni \varepsilon \to 0 \text{ } \mathbb{P}\text{-a.s.} \quad (3.13)
\]

where \(W\) is an \(m\)-dimensional Wiener process on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Then, for any \(\psi_0 \in \mathcal{C}_0^\infty(D_T)\) we have, as \(E \ni \varepsilon \to 0\),

\[
\int_{D_T} g^\varepsilon(\cdot, u_\varepsilon')\psi_0 dW^\varepsilon \to \int_{D_T} \tilde{g}(u_0')\psi_0 dW \quad \text{in law} \quad (3.14)
\]

in the space of continuous paths \(\mathcal{C}(D_T)\), where \(\tilde{g}(u_0') = (\tilde{g}_k(u_0'))_{k \geq 1}\) is defined by \(\tilde{g}_k(u_0') = (M(|g_k(\cdot, u_0')|^2))^{1/2}\).

**Proof** We proceed in three steps.

**Step 1** Let us first show that

\[
\int_{D_T} g^\varepsilon(\cdot, u_\varepsilon')\psi_0 dW^\varepsilon \to \int_{D_T} \tilde{g}(u_0')\psi_0 dW \quad \text{in law}.
\]

We have

\[
\int_{D_T} g^\varepsilon(\cdot, u_\varepsilon')\psi_0 dW^\varepsilon = \int_{D_T} (g^\varepsilon(\cdot, u_\varepsilon') - g^\varepsilon(\cdot, u_0'))\psi_0 dW^\varepsilon + \int_{D_T} g^\varepsilon(\cdot, u_0')\psi_0 dW^\varepsilon
\]

\[
= I_1 + I_2.
\]

\[\]
Concerning $I_1^\varepsilon$ we have
\[
\mathbb{E}|I_1^\varepsilon|^2 = \int_{D_T} \mathbb{E}(|g^\varepsilon(\cdot, u_\varepsilon' \cdot) - g^\varepsilon(\cdot, u_0')|^2)\psi_0^2 dxdt
\leq C \int_{D_T} \mathbb{E}(|u_\varepsilon' - u_0'|^2)\psi_0^2 dxdt
\leq C\|\psi_0\|^2_{\infty} \int_{D_T} \mathbb{E}(|u_\varepsilon' - u_0'|^2)dxdt \to 0 \quad \text{as } E \ni \varepsilon \to 0,
\]
where in the last inequality above, we have used the convergence result (3.12). It follows that, up to a subsequence of $E$ not relabeled, $I_1^\varepsilon \to 0 \ P\text{-a.s.} \text{ when } E \ni \varepsilon \to 0$.

Now, as for $I_2^\varepsilon$, we treat each term of the sum separately, that is, we consider each
\[
I_{2,k}^\varepsilon = \int_{D_T} g_k^\varepsilon(\cdot, u_0')\psi_0 dxW_k, \quad 1 \leq k \leq m.
\]
Each of these integrals is a Gaussian $\mathcal{N}(0, \sigma_k^2)$ where
\[
\sigma_k^2 = \mathbb{E}|I_{2,k}^\varepsilon|^2 = \int_{D_T} \mathbb{E}(|g_k^\varepsilon(\cdot, u_0')|^2)\psi_0^2 dxdt.
\]
Using the convergence result (3.7) in Lemma 3.1, we get that
\[
\sigma_k^2 \to \int_{D_T} \mathbb{E}(|\hat{g}(u_0')|^2)\psi_0^2 dxdt \quad \text{as } E \ni \varepsilon \to 0.
\]
It follows from the martingale representation theorem that
\[
\int_{D_T} g_k^\varepsilon(\cdot, u_0')\psi_0 dxW_k \to \int_{D_T} \hat{g}(u_0')\psi_0 dxW_k \quad \text{in law.} \quad (3.15)
\]
We recall that a sequence of Gaussian $\mathcal{N}(m, \sigma^2)$ converges in law to the Gaussian $\mathcal{N}(m, \sigma^2)$ if and only if $m_\varepsilon \to m$ and $\sigma_\varepsilon^2 \to \sigma^2$. This can be verified by using the characteristic function $\Phi_{m_\varepsilon, \sigma_\varepsilon^2}(t) = \exp(im_\varepsilon t - \sigma_\varepsilon^2 t^2/2)$ of $\mathcal{N}(m, \sigma^2)$. We therefore infer from (3.15) that, as $E \ni \varepsilon \to 0$,
\[
I_2^\varepsilon \to \int_{D_T} \hat{g}(u_0')\psi_0 dxW \quad \text{in law, where } \hat{g}(u_0') = (\hat{g}(u_0'))_{1 \leq k \leq m}.
\]
It follows that
\[
\int_{D_T} g^\varepsilon(\cdot, u_\varepsilon')\psi_0 dxW \to \int_{D_T} \hat{g}(u_0')\psi_0 dxW \quad \text{in law} \quad (3.16)
\]
in the space of continuous paths $C(D_T)$.

**Step 2** We focus at this level on the convergence of the stochastic integral
\[
J_\varepsilon = \int_{D_T} g^\varepsilon(\cdot, u_\varepsilon')\psi_0 dxW^\varepsilon.
\]
We have
\[
J_\varepsilon = \int_{D_T} (g^\varepsilon(\cdot, u_\varepsilon') - g^\varepsilon(\cdot, u_0'))\psi_0 dxW^\varepsilon + \int_{D_T} g^\varepsilon(\cdot, u_0')\psi_0 dxW^\varepsilon
\]
\[
= J_{1,\varepsilon} + J_{2,\varepsilon}.
\]
We proceed as in the Step 1 to show that $J_{1,\varepsilon} \to 0$ as $E \ni \varepsilon \to 0$. Regarding $J_{2,\varepsilon}$, we need to show that, as $E \ni \varepsilon \to 0$,
\[
\int_{D_T} g^\varepsilon(\cdot, u_0')\psi_0 dxW^\varepsilon \to \int_{D_T} \hat{g}(u_0')\psi_0 dxW \quad P\text{-a.s.} \quad (3.17)
\]
We consider each term separately as in Step 1, that is, we need to show that
\[
\int_{D_T} g_k(\cdot, \cdot, u_0) \psi_0 dx dW_k^\varepsilon \to \int_{D_T} \tilde{g}_k(u_0) \psi_0 dx dW_k \quad \mathbb{P}\text{-a.s.}
\] (3.18)

We first observe that the function \( g_k(\cdot, \cdot, u) : (x, t, y, \tau, \omega) \mapsto g_k(y, \tau, u(x, t, \omega)) \) belongs to \( B(\Omega; C(D_T; B_\infty^2(\mathbb{R}^{d+1}))) \) and the latter space is dense in \( L^2(D_T \times \Omega) \). So it is sufficient to check (3.18) by replacing \( g_k(\cdot, \cdot, u_0) \) by any element of \( B(\Omega; C(D_T; B_\infty^2(\mathbb{R}^{d+1}))) \). However, as \( \psi_0 \) lies in \( C_0^\infty(D_T) \), it suffices to replace \( g_k(\cdot, \cdot, u_0) \psi_0 \) by an element of
\[
B(\Omega; K(D_T; B_\infty^2(\mathbb{R}^{d+1}))).
\]

But, as in [20, Lemma 3.1 and Proposition 3.3] (see also [19, Proposition 4.5]) where it has been shown (using a density argument) that we may replace the space \( L^2(D_T; A) \) by \( K(D_T; B_\infty^2(\mathbb{R}^{d+1})) \) in the definition of the \( \Sigma \)-convergence, we can proceed in the same way to show that showing (3.18) reduces in proving (using another density argument) that, as \( E \ni \varepsilon \to 0 \),
\[
\int_{D_T} \psi(x, t, x, \tau, \frac{t}{\varepsilon}) dx dW_k^\varepsilon \to \int_{D_T} M(\psi(x, t, \cdot, \cdot)) dx dW_k \to 0 \quad \mathbb{P}\text{-a.s.}
\] (3.19)

for any \( \psi(x, t, y, \tau) = \varphi(x) \phi(y) \chi(t) \theta(\tau) \) with \( \varphi \in K(D) \), \( \phi \in A_y \), \( \chi \in K(0, T) \) and \( \theta \in A_\tau \). But for \( \psi \) as above, we have
\[
\int_{D_T} \psi^\varepsilon dx dW_k^\varepsilon = \int_D \varphi(x) \phi \left( \frac{x}{\varepsilon} \right) dx \int_0^T \chi(t) \theta \left( \frac{t}{\varepsilon} \right) dW_k^\varepsilon.
\]

At this level we consider the process
\[
M_\varepsilon(t) = \int_0^t \chi(s) \theta \left( \frac{s}{\varepsilon} \right) dW_k^\varepsilon(s).
\]

We know that, setting \( \Phi(t, \tau) = \chi(t) \theta(\tau) \) so that \( \Phi^\varepsilon(t) = \Phi(t, t/\varepsilon) \), the sequence of processes \( (\Phi^\varepsilon, W_k^\varepsilon) \) converges in law to \( (M_\varepsilon, \Phi(t, \cdot), W_k) \) in \( S_0 = L^2(0, T) \times C([0, T]) \). So following [16, Proposition 2], we need to show that the sequence \( (M_\varepsilon)_{\varepsilon > 0} \) is a good sequence (see Appendix A2 for the definition and characterization of good sequences). Indeed, in view of Theorem 6.4 associated to Definition 6.2, we have that the quadratic variation \( \langle M_\varepsilon, M_\varepsilon \rangle(t) \) of \( M_\varepsilon \) is bounded independently of \( \varepsilon \); indeed
\[
\langle M_\varepsilon, M_\varepsilon \rangle(t) = \int_0^t |\Phi^\varepsilon(s)|^2 ds \leq t ||\Phi||^2_{\infty},
\]
so that the sequence \( (M_\varepsilon)_{\varepsilon > 0} \) satisfies the condition UCV (see Definition 6.2) and is hence a good sequence of semimartingales. It follows readily that
\[
\int_0^T \Phi^\varepsilon(t) dW_k^\varepsilon(t) \to \int_0^T M_\tau(\Phi(t, \cdot)) dW_k(t).
\]

Using the convergence result
\[
\int_D \varphi(x) \phi \left( \frac{x}{\varepsilon} \right) dx \to \int_D \varphi(x) M_\varphi(\phi) dx,
\]
we readily get
\[
\int_{D_T} \psi^\varepsilon dx dW_k^\varepsilon \to \int_{D_T} M(\psi(x, t, \cdot, \cdot)) dx dW_k.
\]
This completes the Step 2.

**Step 3** Putting together the results obtained in the previous steps, we are led at once at (3.14).

Now, we use the previous convergence results (see especially (C2)) to deduce that the sequence \((u_{\varepsilon_n})_n\) is bounded in \(L^2(0, T; H^1_0(D))\) \(\mathbb{P}\)-a.s. Hence there exists a subsequence of \((u_{\varepsilon_n})_n\) (not relabeled) which converges weakly in \(L^2(0, T; H^1_0(D))\) to \(u_0\) determined by (C2). It follows from (SC) \(3\) and Lemma 2.4 associated to the Remark 2.5 that there exists \(u_1 \in L^2(D_T; B^{1,2}_{#A_y}(\mathbb{R}^d))\) such that

\[
\frac{\partial u_{\varepsilon_n}}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \quad \text{in } L^2(D_T)\text{-weak } \Sigma, \mathbb{P}\text{-a.s.}, \quad 1 \leq i \leq d
\]

(3.20)

when \(\varepsilon_n \to 0\). So we have that \((u_0, u_1) \in \mathbb{F}_0^1\) where

\[
\mathbb{F}_0^1 = L^2(0, T; H^1_0(D)) \times L^2(D_T; B^{1,2}_{#A_y}(\mathbb{R}^d)).
\]

It is an easy exercise in showing that the space \(\mathcal{F}_0^\infty = C^\infty_0(D_T) \times (C^\infty_0(D_T) \otimes (A^\infty_0/\mathbb{R}))\) is a dense subspace of \(\mathbb{F}_0^1\).

For \(v = (v_0, v_1) \in \mathbb{F}_0^1\) we set \(\mathbb{D}_i v = \frac{\partial v_0}{\partial x_i} + \frac{\partial v_1}{\partial y_i}\) and \(\mathbb{D} v = (\mathbb{D}_i v)_{1 \leq i \leq d} \equiv \nabla v_0 + \nabla y v_1\). We define \(\mathbb{D} \Phi\) for \(\Phi = (\psi_0, \psi_1) \in \mathcal{F}_0^\infty\) mutatis mutandis. With this in mind, we consider the following linear functional:

\[
q(u, v) = \sum_{i,j=1}^d \int_{D_T} M(a_{ij} \mathbb{D}_j u \mathbb{D}_i v) dx dt \quad \text{for } u = (u_0, u_1), v = (v_0, v_1) \in \mathbb{F}_0^1.
\]

We also set \(\widehat{f}(u_0') = M(f(\cdot, \cdot, u_0'))\). Then the following global homogenization result holds.

**Proposition 3.3** Let \(E\) be a fundamental sequence. There exists a subsequence \(E'\) from \(E\) such that the sequences \((u_\varepsilon)_{\varepsilon \in E'}\) and \((\nabla u_\varepsilon)_{\varepsilon \in E'}\) converge in law to \(u_0\) and \(\mathbb{D} u\) in \(L^2(D_T)\)-strong and in \(L^2(D_T)^d\)-weak \(\Sigma\) respectively, where \(u = (u_0, u_1)\) solves the following variational problem:

\[
\begin{aligned}
\left\{ \begin{array}{l}
- \int_{D_T} u_0' \psi_0' dx dt + q(u, \Phi) + \int_{D_T} M(\mathbb{D} u' \cdot \mathbb{D} \Phi) dx dt \\
= \int_{D_T} \widehat{f}(u_0') \psi_0 dx dt + \int_{D_T} \widehat{g}(u_0') \psi_0 dx dW(3.21)
\end{array} \right.
\end{aligned}
\]

for all \(\Phi = (\psi_0, \psi_1) \in \mathcal{F}_0^\infty\).

**Proof** In what follows we will write \(\varepsilon\) instead of \(\varepsilon_n\). By virtue of Lemma 2.4, we may choose the test functions of the form

\[
\Phi_\varepsilon(x, t) = \psi_0(x, t) + \varepsilon \psi_1 \left( x, t, \frac{x}{\varepsilon} \right) \quad ((x, t) \in D_T),
\]

where \(\psi_0 \in C^\infty_0(D_T)\) and \(\psi_1 \in C^\infty_0(D_T) \otimes A^\infty_0\). Using \(\Phi_\varepsilon\) as a test function in the variational form of (1.1) yields

\[
\begin{aligned}
\left\{ \begin{array}{l}
- \int_{D_T} u_\varepsilon' \frac{\partial \Phi_\varepsilon}{\partial t} dx dt + \int_{D_T} \alpha^\varepsilon \nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon dx dt + \int_{D_T} \nabla u_\varepsilon' \cdot \nabla \Phi_\varepsilon dx dt \\
= \int_{D_T} f(\cdot, \cdot, u_\varepsilon') \Phi_\varepsilon dx dt + \int_{D_T} g(\cdot, \cdot, u_\varepsilon') \Phi_\varepsilon dx dW(\varepsilon). (3.22)
\end{array} \right.
\end{aligned}
\]
Our aim is to pass to the limit in (3.22). First, we have
\[
\frac{\partial \Phi_\varepsilon}{\partial t} = \frac{\partial \psi_0}{\partial t} + \varepsilon \left( \frac{\partial \psi_1}{\partial t} \right)^\varepsilon, \quad \nabla \Phi_\varepsilon = \nabla \psi_0 + \varepsilon (\nabla \psi_1)^\varepsilon + (\nabla_y \psi_1)^\varepsilon, \\
\nabla \Phi_\varepsilon' = \nabla \psi_0' + \varepsilon (\nabla \psi_1')^\varepsilon + (\nabla_y \psi_1')^\varepsilon.
\]

Now, we use the usual \( \Sigma \)-convergence method (see Remark 2.3) to obtain, as \( \varepsilon \to 0 \),
\[
\frac{\partial \Phi_\varepsilon}{\partial t} \to \frac{\partial \psi_0}{\partial t} \quad \text{in } L^2(0, T; H^{-1}(D))-\text{weak} \tag{3.23}
\]
\[
\nabla \Phi_\varepsilon \to \nabla \psi_0 + \nabla_y \psi_1 \quad \text{in } L^2(D_T)^d-\text{strong } \Sigma \tag{3.24}
\]
\[
\nabla \Phi_\varepsilon' \to \nabla \psi_0' + \nabla_y \psi_1' \quad \text{in } L^2(D_T)^d-\text{strong } \Sigma. \tag{3.25}
\]
\[
\Phi_\varepsilon \to \psi_0 \quad \text{in } L^2(D_T)-\text{strong}. \tag{3.26}
\]

It follows readily from (3.20) and (3.24) that
\[
\nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon \to \mathbb{D} u \cdot \mathbb{D} \Phi \quad \text{in } L^1(D_T)-\text{weak } \Sigma \text{ a.s.}
\]

Using the strong convergence of \( u_\varepsilon' \) towards \( u_0' \) in \( L^2(D_T) \) together with (3.23) we obtain
\[
\int_{D_T} u_\varepsilon' \frac{\partial \Phi_\varepsilon}{\partial t} dxdt \to \int_{D_T} u_0' \psi_0' dxdt. \tag{3.27}
\]

Next, since \( a_{ij} \in C(\overline{D}; B_{A_y}^{2,\infty}(\mathbb{R}^d)) \subset C(\overline{D_T}; B_{A_y}^{2,\infty}(\mathbb{R}^d)) \), we use \( a_{ij} \) as test function to get
\[
\int_{D_T} a^\varepsilon \nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon dxdt \to q(u, \Phi). \tag{3.28}
\]

Considering the next term we have \( \int_{D_T} \nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon dxdt = \int_{D_T} \nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon dxdt \), so that, from (3.20) we obtain
\[
\int_{D_T} \nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon dxdt \to \int_{D_T} M((\nabla u_0 + \nabla_y u_1)(\nabla \psi_0' + \nabla_y \psi_1')) dxdt.
\]

However
\[
\int_{D_T} M((\nabla u_0 + \nabla_y u_1)(\nabla \psi_0' + \nabla_y \psi_1')) dxdt = \int_{D_T} \mathbb{M} u \cdot \mathbb{D} \Phi dxdt.
\]

Thus
\[
\int_{D_T} \nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon dxdt \to \int_{D_T} \mathbb{M} u \cdot \mathbb{D} \Phi dxdt. \tag{3.29}
\]

Let us now deal with the stochastic term. We have
\[
\int_{D_T} g^\varepsilon(\cdot, \cdot, u_\varepsilon') \Phi_\varepsilon dxdW^\varepsilon = \int_{D_T} g^\varepsilon(\cdot, \cdot, u_\varepsilon') \psi_0 dxdW^\varepsilon + \varepsilon \int_{D_T} g^\varepsilon(\cdot, \cdot, u_\varepsilon') \psi_1 dxdW^\varepsilon
\]
\[
= I_1 + I_2.
\]

Appealing to Lemma 3.2, we get that
\[
I_1 \to \int_{D_T} \tilde{g}(u_0') \psi_0 dxdW \quad \text{in law} \tag{3.30}
\]
in the space of continuous paths \( \mathcal{C}(0, T; \mathbb{R}^m) \). Concerning \( I_2 \), we proceed as in [18] (using the Burkhölder–Davis–Gundy’s inequality) to show that \( I_2 \to 0 \). Thus
\[
\int_{D_T} g^\varepsilon(\cdot, \cdot, u_\varepsilon') \Phi_\varepsilon dxdW^\varepsilon \to \int_{D_T} \tilde{g}(u_0') \psi_0 dxdW \quad \text{in law} \tag{3.31}
\]
in the space of continuous paths $\mathcal{C}(0,T;\mathbb{R}^m)$. Finally we use (3.6) in Lemma 3.1 to get
\[
\int_{D_T} f^\varepsilon(\cdot,\cdot,u'_\varepsilon) \Phi_\varepsilon dxdt \to \int_{D_T} \widehat{f}(u'_0) \psi_0 dxdt. \tag{3.32}
\]
On the other hand, we use the equality
\[
\int_{D_T} g^\varepsilon(\cdot,\cdot,u'_\varepsilon) \Phi_\varepsilon dx \to D^\varepsilon \to D^0
\]
and (3.33) to derive
\[
\int_{D_T} g^\varepsilon(\cdot,\cdot,u'_\varepsilon) \Phi_\varepsilon dxW^\varepsilon = - \int_{D_T} u'_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt + \int_{D_T} a^\varepsilon \nabla u'_\varepsilon \cdot \nabla \Phi_\varepsilon dxdt + \int_{D_T} \nabla u'_\varepsilon \cdot \nabla \Phi_\varepsilon dxdt
\]
\[
- \int_{D_T} f^\varepsilon(\cdot,\cdot,u'_\varepsilon) \Phi_\varepsilon dxdt
\]
together with (3.22), (3.27), (3.28), (3.29) and (3.32) to derive
\[
\int_{D_T} g^\varepsilon(\cdot,\cdot,u'_\varepsilon) \Phi_\varepsilon dxW^\varepsilon \to - \int_{D_T} u'_0 \psi'_0 dxdt + q(u, \Phi) + \int_{D_T} M(Du' \cdot D\Phi) dxdt
\]
\[
- \int_{D_T} M(f(\cdot,\cdot,u'_0)) \psi_0 dxdt \quad \text{F-a.s.} \tag{3.33}
\]
So the convergence (3.33) also holds in law. By virtue of the uniqueness of the limit law, we infer from (3.33) and (3.30) that
\[
\int_{D_T} \widehat{g}(u'_0) \psi_0 dxW = - \int_{D_T} u'_0 \psi'_0 dxdt + q(u, \Phi) + \int_{D_T} M(Du' \cdot D\Phi) dxdt
\]
\[
- \int_{D_T} M(f(\cdot,\cdot,u'_0)) \psi_0 dxdt \quad \text{in law.}
\]
This ends the proof.

The variational problem (3.21) is the global homogenized problem for (1.1).

3.2 Homogenized Problem: Proof of Theorem 1.1

In order to derive the homogenized problem we need to deal with an equivalent expression of problem (3.21). As we can see, this problem is equivalent to the following system (3.34)–(3.35) reading as
\[
\int_{D_T} M(aDu + Du') \cdot \nabla \psi_1 dxdt = 0 \quad \text{for all } \psi_1 \in \mathcal{C}_0^\infty(D_T) \otimes A_y^\infty; \tag{3.34}
\]
\[
\left\{ \begin{array}{l}
- \int_{D_T} u'_0 \psi'_0 dxdt + q(u, \psi_0, 0) + \int_{D_T} M(Du' \cdot D\psi_0) dxdt \\
= \int_{D_T} \widehat{f}(u'_0) \psi_0 dxdt + \int_{D_T} \widehat{g}(u'_0) \psi_0 dxW 
\end{array} \right. \quad \text{for all } \psi_0 \in \mathcal{C}_0^\infty(D_T). \tag{3.35}
\]
We first deal with (3.34). Choosing in (3.34)
\[
\psi_1(x,t,y,\tau) = \varphi(x,t)\phi(y) \quad \text{for } (y,\tau) \in \mathbb{R}^{d+1} \tag{3.36}
\]
with $\varphi \in \mathcal{C}_0^\infty(D_T)$ and $\phi \in A_y^\infty$, it follows that
\[
M_y((aDu + Du') \cdot \nabla \phi) = 0 \quad \text{for all } \phi \in A_y^\infty. \tag{3.37}
\]
But $M_y(Du' \cdot \nabla \phi) = 0$, so that (3.34) becomes
\[
M_y((aDu) \cdot \nabla \phi) = 0 \quad \text{for all } \phi \in A_y^\infty. \tag{3.38}
\]
In view of Formula (2.3), Equality (3.38) is the variational form of the equation
\[
- \text{div}_y(a(x,\cdot)(\nabla u_0(x,t,\omega) + \nabla_y u_1(x,t,\cdot,\omega))) = 0 \quad \text{in } \mathbb{R}^d
\]
for a.e. \((x, t, \omega)\). So for \(\xi \in \mathbb{R}^d\) be freely fixed, we consider the corrector problem

\[
\begin{cases}
\text{Find } \pi_\xi(x, \cdot) \in B^{1,2}_{\mathbb{A}_y}(\mathbb{R}^d) \text{ such that: } \\
- \text{div}(a(x, \cdot)(\xi + \nabla_y \pi_\xi(x, \cdot))) = 0 \text{ in } \mathbb{R}^d.
\end{cases}
\]  \(3.39\)

Then appealing to [12, Theorem 1.2] and in view of the properties of \(a\), there exists a function \(\pi_\xi \in C(\mathbb{D}; B^{1,2}_{\mathbb{A}_y}(\mathbb{R}^d))\) whose gradient \(\nabla_y \pi_\xi\) is unique and which is such that, for any \(x \in \mathbb{D}\), \(\pi_\xi(x, \cdot) \in B^{1,2}_{\mathbb{A}_y}(\mathbb{R}^d)\) solves uniquely (3.39) in the usual sense of distributions in \(\mathbb{R}^d\). Now choosing \(\xi = \nabla u_0(x, t, \omega)\) in (3.39) and using the uniqueness of the gradient of the solution of (3.39), we infer \(u_1(x, t, y, \omega) = \pi \nabla u_0(x, t, \omega)(x, y)\) for a.e. \((x, t, \omega) \in D_T \times \Omega\). This defines \(u_1 \in L^2(0, T; C(\mathbb{D}; B^{1,2}_{\mathbb{A}_y}(\mathbb{R}^d)))\). Now, choosing \(\xi = e_j\) (the \(j\)th vector of the canonical basis in \(\mathbb{R}^d\)) and denoting by \(\chi_{j}(x, \cdot, \cdot)\) the corresponding solution of (3.39), we easily see that

\[u_1(x, t, y, \omega) = \nabla u_0(x, t, \omega) \cdot \chi(x, y),\]

where \(\chi(x, \cdot, \cdot) = (\chi_{j}(x, \cdot, \cdot))_{1 \leq j \leq d} \).

This being so, we define the homogenized matrix \(\tilde{a}\) and the homogenized operator \(\tilde{P}\) as follows:

\[
\tilde{a}(x) = M(a(x, \cdot)(I_d + \nabla_y \chi(x, \cdot))) \quad \text{for } x \in \mathbb{D} \text{ and } \tilde{P} = -\text{div}(\tilde{a} \nabla)
\]

where

\[
\text{div}(\tilde{a} \nabla) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( \tilde{a}_{ij} \frac{\partial}{\partial x_j} \right)
\]

with \(\tilde{a} = (\tilde{a}_{ij})_{1 \leq i, j \leq d}, \tilde{a}_{ij}(x, \cdot) = M(a_{ij}(x, \cdot) (\delta_{ij} + \frac{\partial x_i}{\partial y_j}(x, \cdot))), I_d\) being the \(d \times d\) identity matrix.

We recall the definition of homogenized functions \(\tilde{f}\) and \(\tilde{g}\):

\[
\tilde{g}(r) = (\tilde{g}_k(r))_{1 \leq k \leq m} \quad \text{with } \tilde{g}_k(r) = (M(|g_k(\cdot, \cdot, r)|^2))^{\frac{1}{2}}
\]

\[
\tilde{f}(r) = M(f(\cdot, \cdot, r)) \quad \text{for } r \in \mathbb{R}.
\]

Remark 3.4 It is an easy exercise in showing that the functions \(\tilde{f}\) and \(\tilde{g}\) (as functions of the argument \(r \in \mathbb{R}\)) satisfy assumptions similar to those of \(f\) and \(g\), and that \(\tilde{P}\) is uniformly elliptic.

The following result provides us with the homogenized model whose \(u_0\) is solution.

**Proposition 3.5** The function \(u_0\) is solution of the boundary value problem:

\[
\begin{cases}
\frac{du_0}{dt} + (\tilde{P}u_0 - \Delta u_0)dt = \tilde{f}(u_0)dt + \tilde{g}(u_0)d\mathbb{W} \quad \text{in } D_T \\
u_0 = 0 \quad \text{on } \partial D \times (0, T) \\
u_0(x, 0) = u^0 \text{ and } u_0(x, 0) = u^1 \quad \text{in } D.
\end{cases}
\]  \(3.41\)

**Proof** Substituting in (3.35) \(u_1\) by its expression given by (3.40) and choosing there \(\psi_0(x, t) = \varphi(x, t)\), with \(\varphi \in C_0^\infty(D_T)\), we arrive at (3.41).

The next result establishes the uniqueness of the solution of (3.41) on the same probability system.

**Proposition 3.6** Let \(u_0\) and \(u_0^\prime\) be two solutions of (3.41) on the same probability system \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{W}, \mathbb{F}^7)\) with the same initial conditions \(u^0\) and \(u^1\). Then \(u_0 = u_0^\prime \mathbb{P}\)-almost surely.
Proof Choosing \( w_0 = u_0 - u_0^* \), we have
\[
\begin{align*}
\begin{cases}
   \,dw_0 + \hat{P} w_0 dt - \Delta w_0 dt &= (\hat{f}(u_0) - \hat{f}(u_0^*)) dt + (\hat{g}(u_0) - \hat{g}(u_0^*)) d\hat{W} \\
   \,w_0(x,0) = 0, \quad w_0'(x,0) = 0.
\end{cases}
\end{align*}
\]

Applying Itô’s formula and integrating over \([0,t]\) yields
\[
\begin{align*}
\|w_0(t)\|^2_{H^1_0(D)} + \|w_0'(t)\|^2_{L^2(D)} + \int_0^t \|w_0'(t)\|^2_{H^1_0(D)} dt \\
\leq \int_0^t \|\hat{g}(u_0'(s)) - \hat{g}(u_0^*(s))\|^2_{L^2(D)} ds + 2 \int_0^t (\hat{f}(u_0'(s)) - \hat{f}(u_0^*(s)), w_0'(s)) ds \\
+ 2 \int_0^t (\hat{g}(u_0'(s)) - \hat{g}(u_0^*(s)), w_0'(s)) d\hat{W}(s). \tag{3.42}
\end{align*}
\]

Hence
\[
2 \int_0^t (\hat{f}(u_0'(s)) - \hat{f}(u_0^*(s)), w_0'(s)) ds \leq C \int_0^t (\|\hat{f}(u_0'(s)) - \hat{f}(u_0^*(s))\|^2_{L^2(D)} + \|w_0'(s)\|^2_{L^2(D)}) ds \leq C \int_0^t (\|w_0(s)\|^2_{H^1_0(D)} + \|w_0'(s)\|^2_{L^2(D)}) ds. \tag{3.43}
\]

It follows that
\[
\int_0^t \|\hat{g}(u_0'(s)) - \hat{g}(u_0^*(s))\|^2_{L^2(D)} ds \leq C \int_0^t (\|w_0(s)\|^2_{H^1_0(D)} + \|w_0'(s)\|^2_{L^2(D)}) ds \tag{3.44}
\]

Taking the mathematical expectation in (3.42), we appeal to the assumptions on \( g \) and combine (3.43) with (3.44) to get
\[
\mathbb{E}(\|w_0(t)\|^2_{H^1_0(D)} + \|w_0'(t)\|^2_{L^2(D)}) \leq C \mathbb{E} \int_0^t (\|w_0(s)\|^2_{H^1_0(D)} + \|w_0'(s)\|^2_{L^2(D)}) ds.
\]

It emerges from Gronwall’s lemma that \( w_0 = 0 \) \( \mathbb{P} \)-almost surely. \( \square \)

Remark 3.7 The pathwise uniqueness result in Proposition 3.6 and Yamada–Watanabe’s Theorem [26] yields the existence of unique strong probabilistic solution of (3.41) on a prescribed probabilistic system \((\Omega, \mathcal{F}, \mathbb{P}, W, \mathcal{F}^t)\).

The aim of the rest of this section is to prove Theorem 1.1. Its proof uses the pathwise uniqueness for (3.41) in conjunction with the following two lemmas.

Lemma 3.8 ([14]) Let \( X \) be a polish space equipped with its Borel \( \sigma \)-algebra. A sequence of \( X \)-valued random variables \( \{Y_n, n \in \mathbb{N}\} \) converges in probability if and only if for every joint laws \( \{\mu_{n_k}, k \in \mathbb{N}\} \), there exists a further subsequence which converges weakly to a probability measure \( \mu \) such that
\[
\mu((x,y) \in X \times X : x = y) = 1.
\]

Consider \( X = L^2(0,T;L^2(D)) \cap C(0,T;H^{-1}(D)) \times C(0,T;H^{-1}(D)), X_1 = C(0,T;\mathbb{R}^m), \)
\( X_2 = X \times X \times X_1 \). For \( S \in \mathcal{B}(X) \), we set \( \pi^\varepsilon(S) = \mathbb{P}((u_\varepsilon, u'_\varepsilon) \in S) \). For \( S \in \mathcal{B}(X_1) \), we set \( \pi^W = \mathbb{P}(W \in S) \).

We define the joint probability laws as
\[
\pi^{\varepsilon,\varepsilon'} = \pi^\varepsilon \times \pi^{\varepsilon'},
\nu^{\varepsilon,\varepsilon'} = \pi^\varepsilon \times \pi^{\varepsilon'} \times \pi^W.
\]
Lemma 3.9  The family \( \{\nu^{\varepsilon,\varepsilon'} : \varepsilon, \varepsilon' > 0\} \) is tight on \((X_2, \mathcal{B}(X_2))\).

Proof  The proof of this lemma is similar to that of [18, Lemma 7].

We now have all the ingredients to prove the main result of the work, viz. Theorem 1.1.

Proof of Theorem 1.1  Thanks to Lemma 3.9 and Skorokhod’s theorem [29] we infer the existence of a subsequence from \( \{\nu^{\varepsilon,\varepsilon'} : \varepsilon, \varepsilon' > 0\} \) which converges to a probability measure \( \nu \) on \((X_2, \mathcal{B}(X_2))\) and there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which the sequence \(((u_{\varepsilon,j}, u'_{\varepsilon,j}), (u'_{\varepsilon,j}, W^j))\) is defined and converges almost surely in \(X_2\) to a couple of random variables \(((u_0, u'_0), (v_0, v'_0), \mathbb{W})\). We also have \(\mathcal{L}((u_{\varepsilon,j}, u'_{\varepsilon,j}), (u'_{\varepsilon,j}, W^j)) = \nu^{\varepsilon,j,\varepsilon'}\) and \(\mathcal{L}((u_0, u'_0), (v_0, v'_0), \mathbb{W}) = \nu\). We set

\[
Z_j^{u_{\varepsilon,j}, u'_{\varepsilon,j}} = ((u_{\varepsilon,j}, u'_{\varepsilon,j}), W^j) \quad \text{and} \quad \bar{Z}_j^{u_{\varepsilon,j}, u'_{\varepsilon,j}} = ((u'_{\varepsilon,j}, u_{\varepsilon,j}), W^j),
\]

\[
Z^((u_0, u'_0), \mathbb{W}) \quad \text{and} \quad \bar{Z}^{(v_0, v'_0)} = ((v_0, v'_0), \mathbb{W}).
\]

It follows from the above argument that \(\pi^{\varepsilon,j,\varepsilon'}\) converges to a measure \(\pi\) such that

\[
\pi((\cdot, \cdot), (\cdot', \cdot')) \in X \times X : (x, y) = (x', y') = \mathbb{P}((u_0, u'_0) = (v_0, v'_0) \in X) = 1. \tag{3.45}
\]

Thanks to (3.45) and Lemma 3.9 we conclude that the original sequence \((u_{\varepsilon,j}, u'_{\varepsilon,j})\) defined on the original probability space \((\Omega, \mathcal{F}, \mathbb{P}, W, \mathcal{F})\) converges in law to \((u_0, u'_0)\) in the topology of \(X\), which implies that the sequence \((u_{\varepsilon,j})\) converges in law to \(u_0\) in \(L^2(D_T)\) and \((u'_{\varepsilon,j})\) converges in law to \(u'_0\) in \(L^2(0,T; H^{-1}(D))\), where \(u_0\) is the unique solution of (1.5).

\[
\Box
\]

4 Some Applications of Theorem 1.1

We have made a fundamental assumption (A3) under which the multiscale analysis of (1.1) has been made possible. Here we give some concrete situations in which this holds true.

Problem 4.1  (Periodic setting)  We assume here that \(a(x, \cdot), f(\cdot, \cdot, v)\) and \(g(\cdot, \cdot, v)\) are periodic with period 1 in each coordinate for each \(x \in \overline{D}\) and \(v \in \mathbb{R}\). The appropriate algebras with mean value here are \(A_y = C_{\text{per}}(Y)\) and \(A_\tau = C_{\text{per}}(T)\) and so \(\mathcal{A} = C_{\text{per}}(Y \times T)\), where \(Y = (0,1)^d\) and \(T = (0,1)\), where \(C_{\text{per}}(Y)\) (resp., \(C_{\text{per}}(T)\) and \(C_{\text{per}}(Y \times T)\)) is the Banach algebra of continuous \(Y\)-periodic functions defined on \(\mathbb{R}^d\) (resp., on \(\mathbb{R}\) and \(\mathbb{R}^{d+1}\)). We therefore have \(B^p_{A_y}(\mathbb{R}^d) = L^p_{\text{per}}(Y), B^p_{A}(\mathbb{R}^{d+1}) = L^p_{\text{per}}(Y \times T)\) for \(1 \leq p \leq \infty\) and \(B^{1,2}_{A_y}(\mathbb{R}^d) = H^1_{\#}(Y) = \{u \in H^1_{\text{per}}(Y) : \int_Y u dy = 0\}\). The homogenized coefficients are defined as follows:

\[
\tilde{a}(x) = \int_Y a(x, y)(I_d + \nabla_y \chi(x, y))dy \quad \text{and} \quad \tilde{P} = -\text{div}(\tilde{a} \nabla)
\]

\[
\tilde{f}(r) = \int_{Y \times T} f(y, \tau, r)d\tau dy \quad \text{and} \quad \tilde{g}(r) = \left(\int_{Y \times T} |g(y, \tau, r)|^2 d\tau dy\right)^{1/2}
\]
where \( I_d \) is the \( d \times d \) identity matrix, and \( \chi = (\chi_j)_{1 \leq j \leq d} \) is the solution of the cell problem

\[
\begin{cases}
\chi_j(x, \cdot) \in H^1_\#(Y), \\
- \text{div}_y(a(x, \cdot)(e_j + \nabla_y \chi_j(x, \cdot))) = 0 \quad \text{in } Y.
\end{cases}
\]

This problem has been considered in [18], but with a linear operator not involving the damping term.

**Problem 4.2** (Almost periodic setting) We assume that the function \( a(x, \cdot), f(\cdot, \cdot, v) \) and \( g(\cdot, \cdot, v) \) are Besicovitch almost periodic [3] for any \( x \in \overline{D} \) and \( v \in \mathbb{R} \). Then we perform the homogenization of (1.1) with \( A_y = AP(\mathbb{R}^d), \ A_r = AP(\mathbb{R}) \) and so \( A = AP(\mathbb{R}^{d+1}) \) where \( AP(\mathbb{R}^d) \) [4] is the algebra of Bohr almost periodic functions on \( \mathbb{R}^d \). Let us remind that the mean value of a function \( u \in AP(\mathbb{R}^d) \) is the unique constant belonging to the close convex hull of the family of the translates \((u(\cdot + a))_{a \in \mathbb{R}^d}\) of \( u \).

**Problem 4.3** We consider the space \( B_\infty(\mathbb{R}; X) \) of all continuous functions \( \psi \in C(\mathbb{R}; X) \) such that \( \psi(\zeta) \) has a limit in \( X \) as \( |\zeta| \to +\infty \), where \( X \) is a Banach space. The space \( B_\infty(\mathbb{R}^d; \mathbb{R}) = B_\infty(\mathbb{R}^d) \) is an algebra with mean value on \( \mathbb{R}^d \). Then we perform the homogenization of (1.1) under the assumption that: \( a(x, \cdot) \in L^1_{\text{per}}(Y)^{d \times d} \) for any \( x \in \overline{D} \) and \( f(\cdot, \cdot, v), g(\cdot, \cdot, v) \in B_\infty(\mathbb{R}_r; L^2_{\text{per}}(Y)) \) for all \( v \in \mathbb{R} \), where \( A_y = C_{\text{per}}(Y) \) and \( A_r = B_\infty(\mathbb{R}_r) \), hence \( A = B_\infty(\mathbb{R}_r) \cap C_{\text{per}}(Y) = B_\infty(\mathbb{R}_r; C_{\text{per}}(Y)). \)

## 5 Appendix A1

**Lemma 5.1** Under the assumptions (A1) and (A2), the solution \( u_\varepsilon \) of Problem (1.1) satisfies the following estimates

\[
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq T} \|u_\varepsilon(t)\|^2_{H^1_\#(D)} + \mathbb{E} \sup_{0 \leq t \leq T} \|u_\varepsilon'(t)\|^2_{L^2(D)} + \mathbb{E} \int_0^T \|u_\varepsilon'(t)\|^2_{H^1_\#(D)} dt & \leq C, \\
\mathbb{E} \sup_{0 \leq t \leq T} \|u_\varepsilon(t)\|^4_{H^1_\#(D)} + \mathbb{E} \sup_{0 \leq t \leq T} \|u_\varepsilon'(t)\|^4_{L^2(D)} & \leq C,
\end{align*}
\]

where \( C \) is a positive constant independent of \( \varepsilon \).

**Proof** By Itô’s formula and integrating over \([0, t]\) with \( 0 \leq t \leq T \), we obtain

\[
\begin{align*}
\|u_\varepsilon'(t)\|^2_{L^2(D)} & + 2 \int_0^t (P^x u_\varepsilon(s), u_\varepsilon'(s)) ds - 2 \int_0^t (\Delta u_\varepsilon(s), u_\varepsilon'(s)) ds \\
& = \|u_\varepsilon\|^2_{L^2(D)} + 2 \int_0^t \left( f \left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon'(s) \right), u_\varepsilon'(s) \right) ds + 2 \int_0^t \left( g \left( \frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon'(s) \right), u_\varepsilon'(s) \right) dW(s) \\
& \quad + \int_0^t \left\| g \left( \frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon'(s) \right) \right\|^2_{L^2(D)^m} ds.
\end{align*}
\]

So we obtain

\[
\begin{align*}
\|u_\varepsilon(t)\|^2_{H^1_\#(D)} + \|u_\varepsilon'(t)\|^2_{L^2(D)} & + 2 \int_0^t \|u_\varepsilon'(s)\|^2_{H^1_\#(D)} ds \\
& \leq C(\|u_\varepsilon\|^2_{L^2(D)} + \|u_\varepsilon^0\|^2_{H^1_\#(D)}) + 2 \int_0^t \left( f \left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon'(s) \right), u_\varepsilon'(s) \right) ds \\
& \quad + \int_0^t \left\| g \left( \frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon'(s) \right) \right\|^2_{L^2(D)^m} ds + 2 \int_0^t \left( g \left( \frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon'(s) \right), u_\varepsilon'(s) \right) dW(s)
\end{align*}
\]

(5.4)
Taking the supremum in $t \in [0, T]$, and the mathematical expectation in both side of (5.4), and using also the assumption (A1) we have

$$
\mathbb{E} \sup_{0 \leq t \leq T} (\|u_\varepsilon(t)\|_{H_0^1(D)}^2 + \|u_\varepsilon'(t)\|_{L^2(D)}^2) + 2\mathbb{E} \int_0^T \|u_\varepsilon'(s)\|_{H_0^1(D)}^2 dt \\
\leq C(\|u^1\|_{L^2(D)}^2 + \|u^0\|_{H_0^1(D)}^2) + 2\mathbb{E} \int_0^T \left( f \left( \frac{\cdot}{\varepsilon} \varepsilon, u_\varepsilon'(s) \right), u_\varepsilon'(s) \right) ds \\
+ \mathbb{E} \int_0^T \left\| g \left( \frac{\cdot}{\varepsilon} \varepsilon, u_\varepsilon'(s) \right) \right\|_{L^2(D)}^2 ds + 2 \sup_{0 \leq t \leq T} \mathbb{E} \int_0^T \left( g \left( \frac{\cdot}{\varepsilon} \varepsilon, u_\varepsilon'(s) \right), u_\varepsilon'(s) \right) dW(s).
$$

Making use of Hölder’s inequality, Young’s inequality and the assumption on $f$, we obtain

$$
2\mathbb{E} \int_0^T \left( f \left( \frac{\cdot}{\varepsilon} \varepsilon, u_\varepsilon'(s) \right), u_\varepsilon'(s) \right) ds \\
\leq \mathbb{E} \int_0^T \left\| f \left( \frac{\cdot}{\varepsilon} \varepsilon, u_\varepsilon'(s) \right) \right\|_{L^2(D)}^2 + \|u_\varepsilon'(s)\|_{L^2(D)}^2 ds \\
\leq C \mathbb{E} \int_0^T (1 + \|u_\varepsilon(s)\|_{H_0^1(D)}^2 + \|u_\varepsilon'(s)\|_{L^2(D)}^2) ds.
$$

Using the assumption on $g$, we get

$$
\mathbb{E} \int_0^T \left\| g \left( \frac{\cdot}{\varepsilon} \varepsilon, u_\varepsilon'(s) \right) \right\|_{L^2(D)}^2 ds \leq C \mathbb{E} \int_0^T (1 + \|u_\varepsilon(s)\|_{H_0^1(D)}^2 + \|u_\varepsilon'(s)\|_{L^2(D)}^2) ds.
$$

Applying Burkholder–Davis–Gundy’s inequality, we get

$$
\mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \left( g \left( \frac{\cdot}{\varepsilon} \varepsilon, u_\varepsilon'(s) \right), u_\varepsilon'(s) \right) dW(s) \\
\leq C \mathbb{E} \left( \int_0^T \left( g \left( \frac{\cdot}{\varepsilon} \varepsilon, u_\varepsilon'(s) \right), u_\varepsilon'(s) \right)^2 ds \right)^{\frac{1}{2}} \\
\leq C \mathbb{E} \left( \int_0^T \left\| g \left( \frac{\cdot}{\varepsilon} \varepsilon, u_\varepsilon'(s) \right) \right\|_{L^2(D)}^2 \|u_\varepsilon'(s)\|_{L^2(D)}^2 ds \right)^{\frac{1}{2}} \\
\leq C \mathbb{E} \left( \int_0^T (1 + \|u_\varepsilon(s)\|_{H_0^1(D)}^2 + \|u_\varepsilon'(s)\|_{L^2(D)}^2) \|u_\varepsilon'(s)\|_{L^2(D)}^2 ds \right)^{\frac{1}{2}} \\
\leq C \mathbb{E} \sup_{0 \leq t \leq T} (1 + \|u_\varepsilon(s)\|_{H_0^1(D)}^2 + \|u_\varepsilon'(s)\|_{L^2(D)}^2) \left( \int_0^T \|u_\varepsilon'(s)\|_{L^2(D)}^2 ds \right)^{\frac{1}{2}} \\
\leq \frac{1}{2} C \mathbb{E} \sup_{0 \leq t \leq T} (1 + \|u_\varepsilon(s)\|_{H_0^1(D)}^2 + \|u_\varepsilon'(s)\|_{L^2(D)}^2) + \frac{1}{2} C \mathbb{E} \int_0^T \|u_\varepsilon'(s)\|_{L^2(D)}^2 ds
$$

Combining (5.5) and (5.6), we obtain

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|u_\varepsilon(t)\|_{H_0^1(D)}^2 + \mathbb{E} \sup_{0 \leq t \leq T} \|u_\varepsilon'(t)\|_{L^2(D)}^2 + 2\mathbb{E} \int_0^T \|u_\varepsilon'(t)\|_{H_0^1(D)}^2 dt \\
\leq C(\|u^1\|_{L^2(D)}^2 + \|u^0\|_{H_0^1(D)}^2) + C(1 + \|u_\varepsilon(s)\|_{H_0^1(D)}^2 + \|u_\varepsilon'(s)\|_{L^2(D)}^2) \\
+ C \mathbb{E} \int_0^T \|u_\varepsilon'(s)\|_{L^2(D)}^2 ds.
$$
The Gronwall inequality then gives
\[ E \sup_{0 \leq t \leq T} \| u_\varepsilon(t) \|^2_{H^1_0(D)} + E \sup_{0 \leq t \leq T} \| u'_\varepsilon(t) \|^2_{L^2(D)} + E \int_0^T \| u'_\varepsilon(t) \|^2_{H^1_0(D)} dt \leq C. \]

We can now prove the last estimate. Taking the square in both sides of the inequality (5.4) we have
\[ \| u_\varepsilon(t) \|^4_{H^1_0(D)} + \| u'_\varepsilon(t) \|^4_{L^2(D)} \leq C(\| u_\varepsilon \|^4_{L^2(D)} + \| u_\varepsilon \|^4_{L^2(D)}) + C \left( \int_0^t \left( f \left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon(s) \right), u'_\varepsilon(s) \right) ds \right)^2 + 4 \left( \int_0^t \left( g \left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon(s) \right), u'_\varepsilon(s) \right) dW(s) \right)^2. \] (5.7)

Taking the supremum with respect to \( t \in [0, T] \) and using the assumption on \( f \), we get
\[ E \left( \int_0^T \left( f \left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u'_\varepsilon(s) \right), u'_\varepsilon(s) \right) ds \right)^2 \leq C E \left( \int_0^T \left( \left\| f \left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u'_\varepsilon(s) \right) \right\|^2_{L^2(D)} + \| u'_\varepsilon(s) \|^2_{L^2(D)} \right) ds \right)^2 \leq C E \int_0^T (1 + \| u_\varepsilon(s) \|^4_{H^1_0(D)} + \| u'_\varepsilon(s) \|^4_{L^2(D)}) ds. \] (5.8)

Next, using again the Burkholder–Davis–Gundy inequality, we obtain
\[ E \sup_{0 \leq t \leq T} \left( \int_0^t \left( g \left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u'_\varepsilon(s) \right), u'_\varepsilon(s) \right) dW(s) \right)^2 \leq E \left( \int_0^T \left( \left\| g \left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u'_\varepsilon(s) \right) \right\|^2_{L^2(D)} + \| u'_\varepsilon(s) \|^2_{L^2(D)} \right) ds \right)^2 \leq C E \int_0^T (1 + \| u_\varepsilon(s) \|^4_{H^1_0(D)} + \| u'_\varepsilon(s) \|^4_{L^2(D)}) ds. \] (5.9)

Collecting the inequalities (5.7)–(5.9) and using Gronwall’s inequality we are led to
\[ E \sup_{0 \leq t \leq T} \| u_\varepsilon(t) \|^4_{H^1_0(D)} + E \sup_{0 \leq t \leq T} \| u'_\varepsilon(t) \|^4_{L^2(D)} \leq C. \]

This ends the proof.

**Lemma 5.2** Under the assumptions of Lemma 5.1, we have
\[ E \sup_{|\theta| \leq \delta} \int_0^T \| u'_\varepsilon(t + \theta) - u'_\varepsilon(t) \|^2_{H^{-1}(D)} dt \leq C\delta \] (5.10)

where \( \delta > 0 \) is a small parameter and \( C \) is a positive constant independent of \( \varepsilon \) and \( \delta \).

**Proof** From (1.1) we can write, for \( \theta \geq 0 \),
\[ u'_\varepsilon(t + \theta) - u'_\varepsilon(t) = \int_t^{t+\theta} P^x u_\varepsilon ds + \int_t^{t+\theta} \Delta u_\varepsilon ds + \int_t^{t+\theta} f \left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon(s) \right) ds \]
\[ + \int_t^{t+\theta} g \left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon(s) \right) dW(s). \]

Thus
\[ \| u'_\varepsilon(t + \theta) - u'_\varepsilon(t) \|^2_{H^{-1}(D)} \leq \left( \int_t^{t+\theta} \| u_\varepsilon(s) \|^2_{H^1_0(D)} ds \right)^2 + \left( \int_t^{t+\theta} \| \nabla u_\varepsilon(s) \|^2_{L^2(D)} ds \right)^2. \]
\[ + C \left( \int_t^{t+\theta} \left\| f\left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon'(s) \right) \right\|_{L^2(D)}^2 \, ds \right) + \left| \int_t^{t+\theta} g\left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon'(s) \right) dW(s) \right|^2. \]  

(5.11)

Integrating between 0 and \( T \) and taking the mathematical expectation, we have

\[
\mathbb{E} \sup_{0 \leq \theta \leq \delta} \left| \int_0^{T-\theta} \| u_\varepsilon'(t + \theta) - u_\varepsilon'(t) \|_{H^{-1}(D)}^2 \, dt \right|
\leq C \delta^2 \left( \mathbb{E} \sup_{0 \leq s \leq T} \| u_\varepsilon(s) \|_{H_0^1(D)}^2 + \mathbb{E} \int_0^T \| \nabla u_\varepsilon'(s) \|_{L^2(D)}^2 \, ds \right)
+ C \delta^2 \left( 1 + \mathbb{E} \sup_{0 \leq s \leq T} (\| u_\varepsilon(s) \|_{H_0^1(D)}^2 + \| u_\varepsilon'(s) \|_{L^2(D)}^2) \right)
+ \mathbb{E} \int_0^{T-\theta} \sup_{0 \leq \theta \leq \delta} \left| \int_t^{t+\theta} g\left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon'(s) \right) dW(s) \right|^2 \, dt.
\]

(5.12)

Next using Burkhölder–Davis–Gundy’s inequality we obtain

\[
\mathbb{E} \int_0^{T-\theta} \sup_{0 \leq \theta \leq \delta} \left| \int_t^{t+\theta} g\left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon'(s) \right) dW(s) \right|^2 \, dt
\leq \mathbb{E} \int_0^T \left( \int_t^{t+\theta} \left\| g\left( \frac{x}{\varepsilon}, \frac{s}{\varepsilon}, u_\varepsilon'(s) \right) \right\|^2 \, ds \right) \, dt
\leq C \mathbb{E} \int_0^T \left( \int_t^{t+\theta} (1 + \| u_\varepsilon(s) \|_{H_0^1(D)}^2 + \| u_\varepsilon'(s) \|_{L^2(D)}^2) \, ds \right) \, dt
\leq C \delta^2,
\]

(5.13)

where we have used the assumption on \( g \). Collecting the results and making the same reasoning with \( \theta < 0 \), we finally obtain

\[
\mathbb{E} \sup_{|\theta| \leq \delta} \int_0^T \| u_\varepsilon'(t + \theta) - u_\varepsilon'(t) \|_{H^{-1}(D)}^2 \, dt \leq C \delta,
\]

whence the lemma.  \( \square \)

6 Appendix A2

Let \( (\Omega, \mathcal{A}, \mathbb{P}) \) be a given probability space with expectation \( \mathbb{E} \). We recall here some facts about the convergence of the integral \( \int_0^t H_\varepsilon dX_\varepsilon(s) \) where \( (X_\varepsilon)_{\varepsilon>0} \) is a sequence of semimartingales and \( (H_\varepsilon)_{\varepsilon>0} \) is a sequence of càdlàg (right-continuous with left limit) processes such that \( H_\varepsilon \) is adapted to the filtration generated by \( X_\varepsilon \). Here we follow the presentation of [16] which is borrowed from [15]. We begin with some definitions borrowed from the preceding references.

**Definition 6.1** (Good sequence) A sequence of càdlàg semimartingales \( (X_\varepsilon)_{\varepsilon>0} \) is said to be a good sequence if \( (X_\varepsilon)_{\varepsilon>0} \) converges in law to a process \( X \), and for any sequence \( (H_\varepsilon)_{\varepsilon>0} \) of càdlàg processes such that \( H_\varepsilon \) is adapted to the filtration generated by \( X_\varepsilon \) and \( (H_\varepsilon,X_\varepsilon) \) converges in law to \( (H,X) \), then \( X \) is a semimartingale with respect to the smallest filtration \( \mathcal{H} = (\mathcal{H}(t))_{t \geq 0} \) generated by \( (H,X) \) satisfying the usual hypotheses, and, when all the involved stochastic integrals are defined, \( \int_0^t H_\varepsilon(s) dX_\varepsilon(s) \rightarrow \int_0^t H(s) dX(s) \) \( \mathbb{P} \)-a.s.
From now on, we restrict ourselves to the sequence of semimartingales defined by $X^\varepsilon(t) \equiv M^\varepsilon(t) = \int_0^t H^\varepsilon(s) dB^\varepsilon(s)$ where $(B^\varepsilon(t))_{\varepsilon > 0}$ is a sequence of 1-dimensional Brownian motions defined on $(\Omega, \mathcal{A}, \mathbb{P})$.

**Definition 6.2** (Condition UCV; [15, Definition 7.5, p. 23]) A sequence of continuous semimartingales $(X^\varepsilon)_{\varepsilon > 0}$ is said to satisfy condition UCV (that is, $(X^\varepsilon)_{\varepsilon > 0}$ is said to have Uniformly Controlled Variations) if for each $\eta > 0$ and each $\varepsilon > 0$, there exists a stopping time $T_{\varepsilon, \eta}$ such that $\mathbb{P}(T_{\varepsilon, \eta} \leq \eta) \leq \frac{1}{\eta}$ and the quadratic variation $\langle M^\varepsilon, M^\varepsilon \rangle(t)$ of $M^\varepsilon$ is such that

$$
\sup_{\varepsilon > 0} \mathbb{E}[\langle M^\varepsilon, M^\varepsilon \rangle(\min(1, T_{\varepsilon, \eta}))] = \sup_{\varepsilon > 0} \int_0^{\min(1, T_{\varepsilon, \eta})} \mathbb{E}|H^\varepsilon(s)|^2 ds < \infty. \quad (6.1)
$$

**Remark 6.3** Condition (6.1) in the above definition is given in [16] for a more general sequence of processes as the one above. Here we adapt it to our setting.

The next result provides us with a characterization of good sequences.

**Theorem 6.4** ([16, Theorem 1]) Let $(X^\varepsilon)_{\varepsilon > 0}$ be a sequence of semimartingales converging in law to some process $X$. Then, the sequence $(X^\varepsilon)_{\varepsilon > 0}$ is good if and only if it satisfies the condition UCV.

**References**

[1] Bensoussan, A.: Some existence results for stochastic partial differential equations, In Partial Differential Equations and Applications (Trento 1990), Pitman Res. Notes Math. Ser., Vol. 268, Longman Scientific and Technical, Harlow, UK, 1992, 37–53

[2] Bensoussan, A.: Stochastic Navier–Stokes Equations. *Acta Appl. Math.*, 38, 267–304 (1995)

[3] Besicovitch, A. S.: Almost Periodic Functions, Dover Publications, Cambridge, 1954

[4] Bohr, H.: Almost Periodic Functions, Chelsea, New York, 1947

[5] Billingsley, P.: Convergence of Probability Measures, Second Edition, John Wiley, New York, 1999

[6] Bourgeat, A., Mikelić, A., Wright, S.: Stochastic two-scale convergence in the mean and applications. *J. Reine Angew. Math.*, 456, 19–51 (1994)

[7] Casado Díaz, J., Gayte, I.: The two-scale convergence method applied to generalized Besicovitch spaces. *Proc. R. Soc. Lond. A*, 458, 2925–2946 (2002)

[8] Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge, UK, 1992

[9] Deugoue, G., Woukeng, J. L.: Sigma-convergence of semilinear stochastic wave equations. *Nonlinear Differ. Equ. Appl.*, 25, 1–29 (2018)

[10] Fouetio, A., Woukeng, J. L.: Homogenization of hyperbolic damped stochastic wave equations. *Acta Math. Sinica, Engl. Ser.*, 34, 233–254 (2018)

[11] Gikhman, I. I., Skorokhod, A. V.: Stochastic Differential Equations, Ergebnisse der Mathematik und ihrer Grenzgebiete, 72, Springer-Verlag, Berlin, 1972

[12] Jäger, W., Tambue, A., Woukeng, J. L.: Approximation of homogenized coefficients in deterministic homogenization and convergence rates in the asymptotic almost periodic setting, arXiv:1906.11501, 2019

[13] Jiang, Y. X., Wang, W., Feng, Z. S.: Spatial homogenization of stochastic wave equations with large interaction. *Canad. Math. Bull.*, 59, 542–552 (2016)

[14] Krylov, N. V., Rozovskii, B. L.: Stochastic evolution equations. *J. Soviet Math.*, 14, 1233–1277 (1981)

[15] Kurtz, T. G., Protter, P.: Weak convergence of stochastic integrals and differential equations, In: (D. Talay and L. Tubaro, editors), Probability Models for Nonlinear Partial Differential Equations, Montecatini Terme, 1995, Lecture Notes in Mathematics, Vol. 1627, Springer-Verlag, Berlin, 1996, 1–41

[16] Lejay, A.: On the convergence of stochastic integrals driven by processes converging on account of a homogenization property. *Electron. J. Probability*, 7, 1–18 (2002)

[17] Lions, J. L.: Quelques méthodes de résolution des problèmes aux limites non linéaires (in French), Dunod, Paris, 1969
Mohammed, M., Sango, M.: Homogenization of linear hyperbolic stochastic partial differential equation with rapidly oscillating coefficients: the two-scale convergence method. *Asymp. Anal.*, 91, 341–371 (2015)

Ngouetseng, G.: Homogenization structures and applications I. *Z. Anal. Anwen.*, 22, 73–107 (2003)

Ngouetseng, G.: Almost periodic homogenization: asymptotic analysis of a second order elliptic equation, Preprint, 2000

Ngouetseng, G., Woukeng, J. L.: Deterministic homogenization of parabolic monotone operators with time dependent coefficients. *Electr. J. Differ. Eq.*, 2004, 1–23 (2004)

Pardoux, E.: Equations aux dérivées partielles stochastiques monotones (in French), Thèse de Doctorat, Université Paris-Sud, 1975

Prohorov, Y. V.: Convergence of random processes and limit theorems in probability theory (in Russian). *Teor. Veroyatnost. i Primenen.*, 1, 177–238 (1956)

Razafimandimby, P., Sango, M., Woukeng, J. L.: Homogenization of a stochastic nonlinear reaction-diffusion equation with a large reaction term: the almost periodic framework. *J. Math. Anal. Appl.*, 394, 186–212 (2012)

Razafimandimby, P., Woukeng, J. L.: Homogenization of nonlinear stochastic partial differential equations in a general ergodic environment. *Stochastic Anal. Appl.*, 31, 755–784 (2013)

Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, Grundlehren der mathematischen Wissenschaften, Vol. 293, Springer-Verlag, Berlin, 1999

Sango, M.: Splitting-up scheme for nonlinear stochastic hyperbolic equations. *Forum Math.*, 25, 931–965 (2013)

Simon, J.: Compact sets in the space $L_p(0,T,B)$. *Ann. Mat. Pura Appl.*, 146, 65–96 (1987)

Skorokhod, A. V.: Limit theorems for stochastic processes (in Russian). *Teor. Veroyatnost. i Primenen.*, 1, 289–319 (1956)

Woukeng, J. L.: Homogenization in algebras with mean value. *Banach J. Math. Anal.*, 9, 142–182 (2015)

Woukeng, J. L.: Introverted algebras with mean value and applications. *Nonlinear Anal. TMA*, 99, 190–215 (2014)