EQUIVARIANT CARTAN–EILENBERG SUPERGERBERES
FOR THE GREEN–SCHWARZ SUPERBRANES
I. THE SUPER-MINKOWSKIAN CASE

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Abstract. An explicit gerbe-theoretic description of the super-σ-models of the Green–Schwarz type is proposed and its fundamental structural properties, such as equivariance with respect to distinguished isometries of the target supermanifold and κ-symmetry, are studied at length for targets with the structure of a homogeneous space of a Lie supergroup. The programme of (super)geometrisation of the Cartan–Eilenberg super-(p + 2)-cocycles that determine the topological content of the super-p-brane mechanics and ensure its κ-symmetry, motivated by the successes of and guided by the intuitions provided by its bosonic predecessor, is based on the idea of a (super)central extension of a Lie supergroup in the presence of a nontrivial super-2-cocycle in the Chevalley–Eilenberg cohomology of its Lie superalgebra, the gap between the two cohomologies being bridged by a super-variant of the classic Chevalley–Eilenberg construction. A systematic realisation of the programme is herewith begun with a detailed study of the elementary homogeneous space of the super-Poincaré group, the super-Minkowskian spacetime, whose simplicity affords straightforward identification of the supergeometric mechanisms and unobstructed development of formal tools to be employed in more complex circumstances.

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The naturality and adequacy of the language of gerbe theory in the setting of the mechanics of the topologically charged bosonic loop, captured by the two-dimensional non-linear $\sigma$-model, and the efficiency of its higher-geometric, cohomological and categorical methods in the canonical description in Refs. \cite{Gaw88, Sus11a}, symmetry analysis \cite{GSSW08, GSSW11b, GSSW10, GSSW13, GSSW11a, GSSW08, GSSW11a}, and constructive geometric quantisation \cite{Gaw88, Gaw91, GR02, GR03, Gaw01, Sus11a} of field theories from this distinguished class, has, by now, attained the status of a widely documented, albeit clearly insufficiently exploited, fact. Introduced in the guise of the Deligne–Beilinson hypercohomology in the pioneering works of Alvarez \cite{Alv85} and Gawędzki \cite{Gaw88}, the language has found – since the advent of the geometric formulation of gerbe theory worked out by Murray et al. in Refs. \cite{Mur96, MS00, Ste00, BCM02, CJM02, CJM03} – ample structural applications in the study of $\sigma$-models and the associated conformal field theories and string theories, and in particular in a neat cohomological classification of quantum-mechanically consistent field theories of the type indicated (also in the presence of boundaries and defects in the two-dimensional spacetime \cite{FSW08, RS09, Sus11a}), in a concrete formulation of a universal Gauge Principle \cite{GSSW08, GSSW11a, GSSW11b}, going beyond the naive minimal-coupling scheme, and in the resulting classification of obstructions against the gauging of rigid symmetries (or gauge anomalies) and of inequivalent gaugings, and – finally – in a rigorous geometric description of defects and their fusion in the said theories, in which the role of defects in the modelling of symmetries and dualities between theories has been elucidated and turned into a handy field-theoretic tool \cite{FSW08, RS09, Sus11a}. The models that afford the farthest insight and the richest pool of formal methods and constructions are those with a high internal symmetry, reflecting – in consequence of their geometric nature – a high symmetry of the target of propagation of the loop, to wit, the Wess–Zumino–Witten $\sigma$-models of loop dynamics on (compact) Lie groups \cite{Wit84, Gaw91, Gaw99, GTTNB04} and their gauged variants \cite{GKO83, GK90, GK90a, KPSY89, Hor96, Gaw02}, defining that dynamics on the associated homogeneous spaces. The generating nature of these models in the category of rational conformal field theories in two dimensions and – not unrelatedly – their holographic correspondence with the three-dimensional Chern–Simons topological gauge field theory in the presence of Wilson loops, give a measure of the theoretical significance of a good understanding of these models offered by gerbe theory, and simultaneously provide us with numerous and varied means of verification of its field-theoretic predictions. From it, a picture of a coherent and unified higher-geometric and -algebraic description scheme of two-dimensional field theories with a topological charge emerges in which the constructions central to the systematic development of conformal field theory, often beyond the scope of alternative methods, find their manageable geometrisation, e.g., a methodological construction of orbifolds and orientifolds of known $\sigma$-models in terms of gerbes with an equivariant structure resp. a Jandl structure \cite{GR03, SSW07, GSW08, GSW11a}, extending naturally to the formulation of $\sigma$-models on spaces of orbits of the action of continuous groups in what can be thought of as a natural generalisation of the concept of a worldsheet orbifold of Ref. \cite{FFRS09} (going back to the seminal papers \cite{DHVW88, DHVW89} of Dixon, Harvey, Vafa and Witten) using the gauge-symmetry defects of Refs. \cite{Sus11b, Sus12, Sus13} determined by the data of the relevant equivariant structure (cp also Ref. \cite{RS09} for an early instantiation of the idea); explicit equivariant geometric quantisation \cite{Gaw88, Gaw91, GR02, GR03, Gaw01} in terms of the Cheeger–Simons differential characters provided by gerbe theory, leading to a hands-on realisation of Segal’s idea of functorial quantisation advanced in Ref. \cite{Seg04}, and to the discovery of a new species of Dirichlet branes (the so-called non-abelian branes) over fixed points of the action of an orbifold group \cite{Gaw03} (the latter were first noticed by Douglas and Fiol in Refs. \cite{Dou98, DouF99}); and even, somewhat surprisingly, the elucidation of the peculiar structure of the emergent spectral noncommutative geometry of the maximally symmetric D-branes on the target Lie group \cite{RS08}, determined by the loop-mechanical deformation of the Dirac operator, and so also of the associated differential calculus, given by the superconformal current of the relevant super-WZW $\sigma$-model in the spirit of Ref. \cite{FG94}.

Among the phenomena and constructions of the loop mechanics not covered (at least not in all generality) by gerbe theory to date, two stand out as particularly significant and hence pressing:

- a rigorous and exhaustive treatment of purely loop-mechanical dualities, such as T-duality, with view – among other things – to the construction, by means of an adaptation of the aforementioned generalised worldsheet orbifolding procedure, of (classical) geometries modelled on riemannian geometries of fixed topology (of a toroidal principal bundle over a given base) only locally, and with the global structure of an ‘orbifold’ with respect to a suitably defined action of – instead of the standard diffeomorphism group of the model space $\mathbb{R}^n$ – the T-duality ‘group’

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of gerbe-theoretic $\sigma$-model dualities (with the group law captured by fusion of the corresponding T-duality bi-branes) determining the relevant ‘gluing’ data – these constructions, known under the name of T-folds [Hul05, Hul07], would place us outside the paradigm of riemannian geometry;

- an extension of the hitherto successful formalism of gerbe theory to models with supersymmetry.

As for the former issue, we shall not have anything to say in the present work, except for the comment that its proper analysis calls for the application of the methods recently developed in Refs. [GSW13, Sus12] and is a subject of an ongoing research, to be reported shortly. It is the latter point that we want to tackle herein, with the intention of clarifying the fundamental concepts and working out the basic formal tools through a case study focused on a target superspace whose geometric simplicity, as reflected in the (trivial) topology and (high) symmetry, gives hope for a relatively straightforward separation of that which is peculiar to such theories, and hence truly novel, from the standard intricacies and technical complexities of a higher-geometric and -algebraic analysis of a low-dimensional field theory with a topological charge. Thus, our endeavour is meant to be a prelude to more advanced studies in the direction that it sets, preparing the ground for subsequent developments in which the complexity of the geometries considered will no longer obscure the basic mechanisms at play in a supersymmetric $\sigma$-model.

The (pre)history of supersymmetry starts with the works of Miyazawa [Miy66], largely overlooked at an early stage of development of the idea and its associated mathematical formalism, as laid out in the later works of Gervais, Goldan, Volkov and Akulov [GST71, GL71, VA73, AV74], and – in particular – Wess and Zumino [WZ73, WZ74] in which the theoretical concept was rediscovered and boosted in the direction of the at that time much promising and exciting applications in the model-building of high-energy physics. The related theory of supergeometry, based on the notion of a supermanifold, was worked out a little later by Berezin, Leites, Schwarz and Voronov [BL75, Sch84, Vor84], its geometric content clarified by the structure theorem of Gawędzki and Batchelor [Gaw77, Bat79]. These new concepts were assimilated and adapted by the string-theoretic community very early on, and gave rise to a plethora of consistent models free of the pathologies of their purely bosonic counterparts, of which we name only the original breakthrough models of the superstring due to Green and Schwarz [GS84a, GS84b], their higher-dimensional analogues for super-$p$-branes [AETW87], the celebrated anti-de Sitter superstring models of Refs. [MT98, AF08, FG12, DFGT09] and the M-brane models of Ref. [WPPS99, Cla99], as well as the superstring [BST86] and supermembrane [BST87] theories in curved supergravity backgrounds. It borders on impossible to do justice to a vast area of research such as this one and to recapitulate its development over the decades in a concise form adequate for our purposes, and so instead of doing this, we refer the interested Reader to the excellent reviews and introductory materials on the subject, e.g., Refs. [Wei99, Mar97] for an introduction to the physical, even phenomenological, aspects of the idea of supersymmetry, and Refs. [DF99, Fre99, Var04] for the more mathematically oriented mind looking in the same direction, as well as Refs. [DeW99, Rog07] for a gentle introduction to supergeometry. That which renders such a solution all the more apposite is the current somewhat uncertain phenomenological status of supersymmetry as a feature of fundamental interactions, which seems to imply that more weight should be attached to the motivation for the study of field theories exhibiting supersymmetry in an unbroken form than to the standard historical retrospective. In our case, the general motivation is of three-fold nature. On the one hand, there is a purely mathematical argument: Supergeometry, and in particular the theory of Lie supergroups, is a field of a robust mathematical development and it seems only natural to transplant the ideas and methods of (bosonic) higher geometry unto it with view to furthering its progress, especially that this, in the theoretical context in hand, naturally leads to the emergence of a variety of mathematical structures interesting in their own right, such as, e.g., the Lie-$n$-(super)algebras and $L_\infty$-(super)algebras of Baez and Huerta [BHH11, Hue11] that correspond to classes in higher groups of Chevalley–Eilenberg cohomology encountered in the construction of super-$\sigma$-models. On the other hand, there is a physical argument: It is the supersymmetric string theory in the distinguished supergravitational backgrounds of the anti-de Sitter type that forms the basis of one of the very few and important direct applications of string theory in the predictive description of observable phenomena involving strongly interacting elementary coloured particles (outside the perturbative régime) via the conjectural AdS/CFT ‘correspondence’ – the super-$\sigma$-models of loop dynamics of relevance in this context, originally advanced by Metsaev and Tseytlin in Ref. [MT98], are precisely of the distinguished type described above as the corresponding supermanifolds with the anti-de Sitter space as the body are homogeneous spaces of...
certain Lie supergroups, e.g.,

$$\text{AdS}_5 \times S^5 \cong \text{SU}(2, 2|4)/(\text{SO}(1, 4) \times \text{SO}(5)),$$

$$\text{AdS}_4 \times S^7 \cong \text{OSp}(8|4)/(\text{SO}(1, 3) \times \text{SO}(7)), \quad \text{AdS}_7 \times S^4 \cong \text{OSp}(6, 2|4)/(\text{SO}(1, 6) \times \text{SO}(4)),$$

and so developing new geometric tools for these models might shed some light on the fundamental nature of the still incompletely understood correspondence of much physical relevance. Finally, there is a mixed mathematical-physical argument: Given the successes of the gerbe-theoretic paradigm established for the bosonic two-dimensional $\sigma$-model with the topological charge, it is tempting to test its universality by attempting to adapt it to an environment in which cohomological mechanisms altogether different from the previously encountered sheaf-theoretic and purely de Rham ones are at work and demand geometrisation, namely, the Cartan–Eilenberg supersymmetry-invariant cohomology of superdifferential forms on a Lie supergroup resp. its homogeneous space.

Our choice of target supermanifolds to be studied, that is to say homogeneous spaces of Lie supergroups, has far reaching field-theoretic consequences. In the simplest case of the super-Minkowskian spacetime $\text{sMink}^{1,d-1|D_1,d-1}$, the corresponding super-$\sigma$-models are simply super-counterparts of the bosonic WZW $\sigma$-models mentioned earlier [HM85], and more generally they can be thought of as supervariants of gauged WZW $\sigma$-models, in conformity with the findings of Ref. [GK89b]. In the bosonic setting, there are simple geometric mechanisms that effect a quantisation of the topological charge and fix the relative normalisation of the topological and ‘metric’ terms in the action functional. In the former case, and for a compact Lie group, Dirac’s argument is usually adduced, which secretly captures the integrality condition for the periods of the curvature of the gerbe whose holonomy along the embedded worldsheet defines the topological Wess–Zumino term of the $\sigma$-model action functional. In the latter case, it is the requirement of the existence of a (bi-)chiral (centrally extended) loop-group symmetry induced by left- and right-regular translations on the group manifold, and hence also of the (bi-)chiral Virasoro symmetry obtained from it via the standard Sugawara construction, that does the job. In the supergeometric setting at hand, we are confronted with the following obstacles that get in our way if we try to imitate the bosonic scheme: The non-compactness and topological triviality of the target supermanifold, and of the underlying (super)symmetry group, renders Dirac’s argument ineffective, hence no quantisation of the topological charge is observed and the de Rham cohomology behind the topological term is as trivial as that of the bosonic body of the supermanifold. Hence, apparently, the super-$\sigma$-models of interest seem to have no non-trivial gerbe-theoretic content. Furthermore, the local symmetry fixing the relative normalisation of the two terms in the action functional of the super-$\sigma$-model, known as $\kappa$-symmetry [IALS83 SIE83 SIE84], while readily shown to have a simple geometric origin in the linearised (and further constrained) right-regular action of the Lie supergroup on itself, has a rather cumbersome and peculiar field-theoretic realisation in that it necessarily mixes the metric and topological (that is, gerbe-theoretic) components of the standard (Nambu–Goto resp. Polyakov) action functional and – on top of that – requires for the closure of its (commutator) algebra not only an enhancement by worldsheet diffeomorphisms (which is understandable in view of its origin and relation to the chiral symmetries of the bosonic WZW $\sigma$-model – this is simply a super-instantiation of the Sugawara mechanism) but also the imposition of field equations of the super-$\sigma$-model [MC00], which seems to preclude its geometrisation in the form of an equivariant structure on the object geometrising the de Rham super-cocycle that determines the topological term of the action functional. A moment’s thought reveals that both obstacles can and therefore ought to be circumnavigated, and it is the purpose of the present paper to demonstrate how to do it and study the consequences.

The triviality of the de Rham cohomology does not imply – in consequence of the same non-compactness of the supersymmetry group that kills it, but with it also the implications of the Cartan–Eilenberg theorem for the relation between the standard de Rham cohomology and its invariant version – triviality of the supersymmetric (i.e., supersymmetry-invariant) de Rham cohomology, and – indeed – the Green–Schwarz super-$(p + 2)$-cocycles on the super-Minkowskian spacetime defining the Wess–Zumino terms of the respective super-p-brane super-$\sigma$-models bear witness to that. A simple argument due to Rabin and Crane [RC85 Rab87] then shows that the invariant de Rham cohomology actually encodes information on the nontrivial topology of a supermanifold of the same type as $\text{sMink}^{1,d-1|D_1,d-1}$ (i.e., modelled on the same vector bundle in the sense of the Gawędzki–Batchelor Theorem), namely, an orbifold of the super-Minkowskian spacetime by the natural action of the discrete Kostelecký–Rabin

\footnote{1}{The notation will be clarified in the main text.}
\footnote{2}{In general, this follows from a theorem by Kostant [Kos77]. In the cases studied, it can be checked directly.
supersymmetry group constructed in Ref. [KR84] in the context of supersymmetric lattice field theory. This implies that the Green–Schwarz super-σ-model should be understood as a theory of embeddings of the super-\(p\)-brane worldvolume in the topologically nontrivial super-target, and puts the topological term of that model on equal footing with the topological term of the bosonic WZW σ-model with a compact (and topologically nontrivial) Lie-group target. This means, in particular, that we should look for an appropriate geometrisation of the Green–Schwarz super-(\(p+2\))-cocycles that define the topological term. Following this line of reasoning to its logical conclusion, we readily realise that in the present context ‘appropriate’ is equivalent to ‘supersymmetry-(left-)invariant’, which simply means that we may reproduce the geometrisation procedure of cohomological descent that associates a (bosonic) \(p\)-gerbe with a standard de Rham \((p+2)\)-cocycle (to be detailed shortly) as long as we ensure that each supermanifold obtained in the procedure and – as part of it – surjectively submersed onto \(s\text{Mink}^{1,d-1}|D_{1,d-1}\) is equipped with a Lie-supergroup structure that projects, along the surjective submersion, to the original Lie-supergroup structure on \(s\text{Mink}^{1,d-1}|D_{1,d-1}\), and – finally – that the superdifferential forms defined on these supermanifolds and employed in the said procedure are left-invariant with respect to the natural (left) action of the respective Lie supergroups on their support (that is, on themselves). The success of a (super)geometrisation project thus outlined hinges on two classic cohomological results that carry over from the bosonic world to the supergeometric setting (as demonstrated in App. [3]), to wit, the equivalence between the Cartan–Eilenberg invariant cohomology of the Lie (super)group and the Chevalley–Eilenberg cohomology of its Lie (super)algebra with values in the trivial module \(\mathbb{R}\) in conjunction with the correspondence between classes in the second cohomology group of the latter cohomology and equivalence classes of (super)central extensions of the Lie (super)algebra by that module. These results translate the original geometric problem of finding a surjective submersion over the original supermanifold equipped with a Lie-supergroup structure and such that the pullback of the original Cartan–Eilenberg super-cocycle to it trivialises in the corresponding Cartan–Eilenberg cohomology into a purely algebraic one: In a systematic procedure laid out by de Azcárraga et al. in Ref. [CdAIP00], we identify various Cartan–Eilenberg super-2-cocycles engendered by the Green–Schwarz super-(\(p+2\))-cocycles and associate with them supercentral extensions of the underlying super-Minkowskian algebra, subsequently demonstrated to integrate to supercentral extensions of the Lie supergroup \(\mathbb{R}^{1,d-1}|D_{1,d-1} \equiv s\text{Mink}^{1,d-1}|D_{1,d-1}\) on which the pullbacks of the respective super-(\(p+2\))-cocycles trivialise partially, whereupon the procedure can be repeated with respect to these partial (supersymmetric) trivialisations. This leads to a family of so-called extended superspacetimes of the type first considered in Ref. [CdAIP00] which we then take to be the surjective submersions of the gerbe-theoretic geometrisation scheme. This basic idea is then reapplied at higher levels of Murray’s geometrisation ladder [Mur96], ultimately leading to the emergence of a new (super)geometric species – a Green–Schwarz super-\(p\)-gerbe, the central result of the work reported herein (explicitly for \(p \in \{0,1,2\}\)).

At this stage, the structural affinity with the bosonic WZW σ-model becomes a rich source of intuitions concerning anticipated properties of the newly constructed (super)geometric objects – their verification seems to provide the right measure of evidence in support of our claim of naturality of the construction postulated in the paper. The first of these properties is the amenability of a distinguished realisation of the rigid (or global) supersymmetry of the super-σ-model under consideration to gauging, as reflected in the existence of an appropriate supersymmetry-equivariant structure on the associated super-\(p\)-gerbe, in conformity with the findings of Refs. [GSW10, GSW13, Sus11b, Sus12, Sus13]. Here, as before, ‘appropriate’ means ‘supersymmetry-(left-)invariant’ but the concept has to be adapted to the changed circumstances in which the spaces on which the supersymmetry group acts are components of the nerve of the action groupoid of the group subject to gauging. The existing knowledge on the obstructions against gauging of the various possible actions of the maximal symmetry group in the bosonic setting suggest the Lie supergroup \(\mathbb{R}^{1,d-1}|D_{1,d-1}\) in the adjoint realisation as a candidate for the (maximal) gauge group of the super-σ-model on \(s\text{Mink}^{1,d-1}|D_{1,d-1}\) (or, more adequately, as the structure group of the principal bundle that implements the gauge symmetry in the standard manner, cp., Ref. [GSW10, GSW13]), and – indeed – the corresponding Ad-equivariant structure can be consistently defined on the super-\(p\)-gerbe (as has been verified explicitly for \(p \in \{0,1\}\)). We emphasise once more that this is a nontrivial consistency check of our main proposal.

Finally, we come to the second apparent obstacle indicated above: the obstruction to the geometrisation of the gauge supersymmetry of the super-σ-model in the form of a full-fledged standard equivariant structure on the super-\(p\)-gerbe. The relevance of this issue follows from the field-theoretic rôle played by the supersymmetry, which is that of a mechanism effectively removing the spurious (i.e., pure
Ref. [FSS13]. The Author is grateful to Urs Schreiber for kindly drawing his attention to that article.

homogeneous spaces of Lie supergroups.

lagrangean embedding field, of a distinguished super-

metric term in the original (Nambu–Goto resp. Polyakov) action functional with the pullback, along the

ant configuration bundle (or the ‘space of lagrangean fields’) is accompanied by a replacement of the

metric term in the original (Nambu–Goto resp. Polyakov) action functional with the pullback, along the

lagrangean embedding field, of a distinguished super-(p+1)-form on the enlarged target supermanifold, with the topological term left unchanged, i.e., pulled back from the original target supermanifold. The new super-(p + 1)-form being manifestly supersymmetry-invariant, this leads to an extension of the pullback of the previously constructed Green–Schwarz super-p-gerbe by a trivial super-p-gerbe on the enlarged target supermanifold and an effective unification of metric and gerbe-theoretic components of the original supergeometric background. The latter emerge from what may rightly be termed the **extended Green–Schwarz super-p-gerbe** only upon imposition of constraints (corresponding to the field equations for the extra lagrangean fields) that reduce the Hughes–Polchinski action functional to its Nambu–Goto ancestor. Rather conveniently, these constraints, of the type considered long ago by Ivanov and Ogievetsky in Ref. [IO75] in the context of nonlinear realisations of symmetries and recently revived by McArthur [McA00, McA10] and West et al. [Wes00, GKW06b, GKW06a] in the context of (super-)σ-model-building, admit straightforward supergeometrisation, which puts us in a position to enquire in a meaningful manner as to the existence of an equivariant structure on the extended super-p-gerbe reflecting the gauge supersymmetry in the Hughes–Polchinski formulation of the Green–Schwarz super-σ-model. Alas, not all problems with incorporating κ-symmetry in the newly established super-gerbe-theoretic formalism admit an equally satisfactory solution in the Hughes–Polchinski formulation. The construction of a full-blown κ-equivariant structure remains an open question and – more fundamentally – the realisation of global supersymmetry in the presence of the constraints mentioned above, of direct relevance to the very definition of gerbe-theoretic structures on the enlarged target supermanifold, requires further scrutiny. That said, it deserves to be emphasised that a most natural concept of an element-wise realisation of the symmetry under consideration on the extended super-p-gerbe in the form of a partial equivariant structure (termed ‘weak’ in what follows) can be defined and constructed explicitly in the distinguished cases with p ∈ {0, 1}. While far from being fully understood, this construction lends additional and highly nontrivial support to the main claim of the present work, which is that the (super)geometrisation of the Green–Schwarz super-(p + 2)-cocycle postulated hereunder should be regarded as the proper counterpart of the well-established geometrisation scheme for de Rham (p + 2)-cocycles, to be considered in the setting of the supersymmetric supergeometry of homogeneous spaces of Lie supergroups.

**Addendum:** The notion of the supergerbe, understood as a geometrisation of the Green–Schwarz super-(p + 2)-cocycle, was discussed from a formal point of view by Fiorenza, Sati and Schreiber in Ref. [FSS13]. The Author is grateful to Urs Schreiber for kindly drawing his attention to that article.

The paper is organised as follows:

- In Section 2, we recapitulate those elements of gerbe theory that become essential in subsequent analyses, and review the resulting canonical description and geometric quantisation of the bosonic two-dimensional σ-model with a topological term that the gerbe-theoretic approach naturally provides, with special emphasis on the geometric (and cohomological) structures that describe symmetries of the σ-model induced by automorphisms of the target space, and in particular those amenable to gauging; we complement the introductory part with a definition of a bundle 2-gerbe for the sake of handy reference in a later supersymmetric generalisation. 
- In Section 3, we introduce the broad class of supergeometries of direct interest to us in the present work and its planned continuation. These are supermanifolds endowed with the structure of a homogeneous space of a Lie supergroup and a distinguished representative of a class in the corresponding Cartan–Eilenberg cohomology that determines the topological term in the action functional of the supersymmetric σ-model to be studied, that is the Green–Schwarz super-σ-model that describes the geometrodynamics of standard super-p-branes. We recall the two formulations of the super-σ-model used in later considerations: the Nambu–Goto formulation and the Hughes–Polchinski formulation, and subsequently identify sufficient conditions...
Acknowledgements: This work is a humble tribute to a Friend and Teacher, Professor Krzysztof Gawędzki, on the occasion of His seventieth birthday. Any attempt at a concise verbalisation of a meaningful and yet – of necessity – sufficiently formal acknowledgement of His rôle in the scientific and extra-scientific formation of the author is bound to fail short of the sincere intention, and so shall be omitted.

This leaves the author with the pleasurable obligation of expressing a deep and true thankfulness to his Colleagues and Friends at and outside the Department of Mathematical Methods in Physics of the Faculty of Physics at the University of Warsaw for creating and maintaining an inspiring atmosphere of scientific work and human interaction in which the spirit of the late Professor Krzysztof Maurin finds its very fitting incarnation, as well as for their understanding of the author’s other passions, including that for the defense of fundamental civil rights and liberties of his fellow citizens, in which understanding their human sensitivity and decency is congenially reflected – a rare source of satisfaction and relief in these sad times.

Finally, the author cannot but acknowledge, without the least gratitude but with, instead, deepest civil despondency and a poignant awareness of a rapidly growing cultural alienation within a largely indifferent and populism-prone society of the post-truth era, the steadfast and disquietingly methodical efforts on the part of the current pro-authoritarian government of the Republic of Poland, of a truly bewildering intensity and scope and devastating sociological ramifications, to keep him, alongside many other active members of the Polish civil society, as occupied – be it with acts of civil disobedience, stubborn street protests, confrontation with the party-controlled prosecution, the incessant (and sadly ineffective) write-up of petitions and letters of grievance or various activities aimed at raising social consciousness of the government’s heinous wrongdoing and the complex context of the current civilisational devolution – and consequently as withdrawn from research as a passion-driven individual with a non-trivial charge of civil sensitivity and a rich historical memory of the villainy of totalitarian régimes can ever be made by a government with the intelectual deficiencies, cultural ignorance, moral depravity and documented propensity for increasingly frequent abysmal paroxysms of barbarism pure of this one.
2. Recapitulation of the gerbe theory for the bosonic $\sigma$-model

In this opening section, we consider the monophase bosonic two-dimensional non-linear $\sigma$-model with a spacetime $(\Sigma, \gamma)$, termed the worldsheet, given by a closed two-dimensional manifold $\Sigma$ with an intrinsic metric $\gamma$, and a covariant configuration bundle $\Sigma \times M \longrightarrow \Sigma$ whose fibre $M$, termed the target space, is a differentiable manifold of class $C^\infty$. The model is defined by an action functional $S_\sigma$ with domain $C^\infty(\Sigma, M)$ whose stationary points are (generalised) harmonic maps $x : \Sigma \longrightarrow M$. A rigorous formulation of the monophase $\sigma$-model calls for additional structure on $M$, to wit, a metric tensor $g \in \Gamma(T^*M \otimes_{M,B} T^*M)$ (giving rise to the Levi-Civita connection $\nabla_{\text{LC}}$) and an abelian bundle gerbe (with connection and curving) $\mathcal{G}$ of curvature $H \equiv \text{curv}(\mathcal{G}) \in \mathbb{Z}^3_{\text{dr}}(M)$ with periods in $2\pi \mathbb{Z}$. The two tensors $g$ and $H$ are related by the requirement of the vanishing of the Weyl anomaly\footnote{Formulation of the $\sigma$-model requires the target space to be of class $C^2$ only, but we shall assume higher degree of smoothness to keep subsequent formulæ simpler.} of the $\sigma$-model,

$$R_{\mu\nu}(\nabla^\mathcal{G}_{\text{LC}}) - \frac{1}{4} (g^{-1})^{\alpha\gamma} (g^{-1})^{\beta\delta} H_{\mu\alpha\beta} H_{\nu\gamma\delta} + O(\alpha') = 0,$$

a prerequisite of a non-anomalous realisation of the conformal (gauge) symmetry of the classical field theory in the quantum régime. The metric on $M$ determines – through the induction of the first fundamental form $x^*g$ on $\Sigma$ along $x$ – the so-called metric term in $S_\sigma$, which we choose – with hindsight – to write in the Nambu–Goto form\footnote{The anomaly is usually computed and presented as a perturbative series in the string tension $\alpha'$.}

$$S_{\text{metr,NG}}[x] := \int_\Sigma \text{Vol}(\Sigma) \sqrt{|\det(2) (x^*g)|},$$

whereas the gerbe defines the topological Wess–Zumino term that exponentiates to a Cheeger–Simons differential character $\text{Hol}_\mathcal{G}$ termed the (surface) holonomy of gerbe $\mathcal{G}$ (and computed along map $x$), altogether giving rise to a well-defined Dirac–Feynman amplitude (written for $h = 1$)

$$A_{\text{DF}}[x] := \exp(i S_{\sigma}[x]) \equiv \exp(i S_{\text{metr,NG}}[x]) \cdot \text{Hol}_\mathcal{G}(x).$$

The holonomy can most concisely be described as the image of the isoclass of the flat pullback gerbe $x^*\mathcal{G}$ under the composite isomorphism\footnote{There exists an alternative, and classically essentially equivalent form of the metric term, termed the Polyakov form, which, however, will not be employed in the present work.}

$$\mathcal{W}^3(\Sigma; 0) \cong \check{H}^2(\Sigma, \text{U}(1)) \cong \text{U}(1)$$

between the group $\mathcal{W}^3(\Sigma; 0)$ of isoclasses of flat gerbes over $\Sigma$ (with the class of the tensor product of representatives as the group action) and $\text{U}(1)$ (we assume $\Sigma$ to be connected). The intermediate group is the second Čech-cohomology group of $\Sigma$ with values in the sheaf of constant maps to $\text{U}(1)$. It stands to reason that a structural (non-naive) supersymmetrisation of the $\sigma$-model affects the various components $g, \mathcal{G}$ of the geometric background of the loop propagation. While the candidate extension of the tensor $g$ under such supersymmetrisation is not – as shall be elucidated shortly – difficult to conceive and quantify, at least in geometrically simple circumstances, it is not at all clear even how to approach the supergeometric counterpart of $\mathcal{G}$. Therefore, it seems apposite to first present a number of equivalent descriptions and fundamental properties of the gerbe and its field-theoretic guises with view to establishing a vast scope of constructions from which to choose those that generalise to the supergeometric setting naturally and usefully. Below, we demonstrate the many faces of the gerbe $\mathcal{G}$ with a fixed curvature $H \in \mathbb{Z}^3_{\text{dr}}(M)$, to be understood as a geometrisation of the de Rham 3-cocycle $H$ on the base $M$, much in the same manner as a line bundle (with connection) is to be understood as a geometrisation of the de Rham 2-cocycle $F \in \mathbb{Z}^2_{\text{dr}}(M)$ of its curvature.

2.1. Gerbe theory in a nutshell. The point of departure of our recapitulation is the cohomological description of the gerbe. Thus, any local trivialisation of the (co)homology of $M$ yields a presentation of

$$0 \longrightarrow \mathbb{Z}^{2n} \longrightarrow \mathbb{R} \longrightarrow \text{exp}(\cdot) \longrightarrow \text{U}(1) \longrightarrow 0$$

of sheaves of locally constant maps on $\Sigma$.\footnote{The isomorphism can readily be derived by examining a sheaf-theoretic description of the flat gerbe (note that every gerbe over $\Sigma$ is flat for dimensional reasons) and following the long exact sequence in the sheaf cohomology of $\Sigma$ induced by the standard exponential short exact sequence $0 \longrightarrow \mathbb{Z}^{2n} \longrightarrow \mathbb{R} \longrightarrow \text{exp}(\cdot) \longrightarrow \text{U}(1) \longrightarrow 0$ of sheaves of locally constant maps on $\Sigma$.}
in the 2nd real Deligne–Beilinson hypercohomology group \( (2.2) \)
the gerbe is identified with a class \( \left[ (B_i, A_{jk}, g_{lmn}) \right] \)
of a Čech–de Rham 2-cocycle trivialising the de Rham 3-cocycle \( H \) over \( \mathcal{O}_M \), with data \( (B_i, A_{jk}, g_{lmn}) \in \Omega^2(\mathcal{O}_i) \times \Omega^1(\mathcal{O}_{jk}) \times C^\infty(\mathcal{O}_{lmn}, U(1)) \)
defined by the relations
\[
dB_i = H|_{\mathcal{O}_i}, \quad dA_{jk} = (B_k - B_j)|_{\mathcal{O}_{jk}}, \quad id \log g_{lmn} = (A_{mn} - A_{ln} + A_{lm})|_{\mathcal{O}_{lmn}} ,
\]
up to redefinitions, for arbitrary \( (C_i, h_{jk}) \in \Omega^1(\mathcal{O}_i) \times C^\infty(\mathcal{O}_{jk}, U(1)) \),
\[
(B_i, A_{jk}, g_{lmn}) \mapsto (B_i + dC_i, A_{jk} + (C_k - C_j)|_{\mathcal{O}_{jk}} - id \log h_{jk}, g_{lmn} - (h_{mn}^{-1} \cdot h_{ln} \cdot h_{lm}^{-1})|_{\mathcal{O}_{lmn}})
\]
in the 2nd real Deligne–Beilinson hypercohomology group \( \mathbb{H}^2(M, \mathcal{D}(2)\mathbb{C}) \), i.e., the cohomology of the total complex of the bicomplex formed by an extension of the bounded Deligne complex
\[
\mathcal{D}(n)\mathbb{C} = \mathbb{C}\left[ \Omega^1(\mathcal{O}_M) \xrightarrow{id \log} \Omega^2(\mathcal{O}_M) \xrightarrow{d} \Omega^3(\mathcal{O}_M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(\mathcal{O}_M) \right]
\]
of sheaves of locally smooth maps and \( p \)-forms (for \( p \in \mathbb{N} \) with, in the present case, \( n = 2 \) in the direction of the Čech cohomology associated with \( \mathcal{O}_M \), cp. Ref. [Joh02]. Of course, a given gerbe may – just like a line bundle – trivialise over an open cover \( \mathcal{O}_M \) that is not good in the sense specified above – we call the latter a trivialising open cover in the present context.

The gerbe may also, and equivalently, be realised as a purely geometric object
\[
\mathcal{G} = (YM, \pi_{YM}, B, L, \nabla_L, \mu_L),
\]
known also as the bundle gerbe: Given an arbitrary surjective submersion
\[
\pi_{YM} : YM \rightarrow M
\]
on whose total space there exists a globally smooth primitive
\[
B \in \Omega^2(YM)
\]
of the pullback
\[
\pi_{YM}^* H = dB
\]
(terming the curving of the gerbe), we erect, over the double fibred product \( \mathcal{Y}[2]M = YM \times_M YM \) described by the commutative diagram
\[
\begin{array}{ccc}
\mathcal{Y}[2]M & \xrightarrow{pr_1} & YM \\
\downarrow & & \downarrow \pi_{YM} \\
YM & \xrightarrow{\pi_{YM}} & M \\
\end{array}
\]
a principal bundle
\[
\begin{array}{ccc}
C^\infty & \xrightarrow{L} & \mathcal{Y}[2]M \\
\downarrow \pi_L & & \downarrow \\
YM & & \mathcal{Y}[3]M
\end{array}
\]
\footnote{A good open cover is an open cover \( \{ \mathcal{O}_i \}_{i \in \mathcal{I}} \) with all non-empty (finite) multiple intersections \( \mathcal{O}_{i_1} \cap \mathcal{O}_{i_2} \cap \cdots \cap \mathcal{O}_{i_N} = \emptyset \) \( i_1, i_2, \ldots, i_N \in \mathcal{I}, N \in \mathbb{N} \) contractible. In the light of Weil's proof of the Weil–de Rham Theorem, reported in Ref. [Weil2] such a cover always exists on a differentiable manifold of class \( C^2 \), a property implicitly assumed in constructing the \( \sigma \)-model.}
with connection (termed the connection of the gerbe and represented by the covariant derivative) $\nabla_L$ of curvature
\[
\text{curv}(\nabla_L) = \text{pr}_L^*B - \text{pr}_I^*B,
\]
endowed with a fibrewise groupoid structure, i.e., a connection-preserving principal-bundle isomorphism (termed the groupoid structure)
\[
\mu_L : \text{pr}_{1,2,3}^*L \otimes \text{pr}_{2,3,4}^*L \xrightarrow{\sigma} \text{pr}_1^*L
\]
over the triple fibred product $\mathcal{Y}^{[3]}M \equiv \mathcal{Y}M \times_M \mathcal{Y}M \times_M \mathcal{Y}M$ described by the commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y}^{[3]}M & \xrightarrow{\sigma} & \mathcal{Y}M \\
\downarrow & & \downarrow \\
\mathcal{Y}M \times M & \xrightarrow{\pi_{\mathcal{Y}M}} & \mathcal{Y}M
\end{array}
\]

(with its canonical projections $\text{pr}_{i,j} \equiv (\text{pr}_i, \text{pr}_j) : \mathcal{Y}^{[3]}M \to \mathcal{Y}^{[2]}M$, $(i, j) \in \{(1, 2), (2, 3), (1, 3)\}$, subject to the associativity constraint
\[
\pi_{1,2,4,\mu_L} \circ (\text{id}_{\text{pr}_{1,3}^*L} \otimes \text{pr}_{2,3,4}^*\mu_L) = \pi_{1,3,4,\mu_L} \circ (\text{pr}_{1,2,3}^*\mu_L \otimes \text{id}_{\text{pr}_{3,4}^*L})
\]
over the quadruple fibred product $\mathcal{Y}^{[4]}M \equiv \mathcal{Y}M \times_M \mathcal{Y}M \times_M \mathcal{Y}M \times_M \mathcal{Y}M$ described by the commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y}^{[4]}M & \xrightarrow{\sigma} & \mathcal{Y}M \\
\downarrow & & \downarrow \\
\mathcal{Y}M \times M & \xrightarrow{\pi_{\mathcal{Y}M}} & \mathcal{Y}M
\end{array}
\]

(with its canonical projections $\text{pr}_{i,j,k} \equiv (\text{pr}_i, \text{pr}_j, \text{pr}_k) : \mathcal{Y}^{[4]}M \to \mathcal{Y}^{[3]}M$ and $\text{pr}_{i,j} \equiv (\text{pr}_i, \text{pr}_j) : \mathcal{Y}^{[4]}M \to \mathcal{Y}^{[2]}M$, $i, j \in \{1, 2, 3, 4\}$, cp. Refs. [Mum96, MS00, Ste00]. Equivalence between the two pictures: the cohomological and the geometric one is established, on the one hand, with the help of the construction of the nerve of the trivialising open cover $O_M$, whose components become the $M$-fibred powers of the surjective submersion $\bigcup_{i,\mathcal{E}} O_i \to \mathcal{M}$ over which the various collections of local data define smooth geometric objects (in the standard differentiable structure of a disjoint union of manifolds), and, on the other hand, with the help of local sections of the various surjective submersions employed in the geometric description: $\pi_{\mathcal{Y}M}$, $\pi_L$ and those derived from them, providing us with local data of the geometric objects $B, \nabla_L$ and $\mu_L$. Thus, the geometric objects determined by the Čech–Deligne 2-cocycle $(B_i, A_{ij}, g_{ijk})_{i,j,k} \in \mathcal{E}$ are
\[
\pi_{\mathcal{Y}M} : \bigcup_{i,\mathcal{E}} O_i \to M : (x, i) \mapsto x, \quad B|_{O_i} = B_i,
\]
\[
\pi_L = \text{pr}_1 : L = (\bigcup_{(j,k) \in \mathcal{E}^2} O_{jk}) \times \mathcal{X} \to \bigcup_{(j,k) \in \mathcal{E}^2} O_{jk} \equiv \mathcal{Y}^{[2]}M, \quad \nabla_L|_{O_{jk}} = d + \frac{1}{\tau} A_{jk},
\]
\[
\mu_L((x, l, m, z_1), (x, m, n, z_2)) = (x, l, n, g_{lmn}(x) \cdot z_1 \cdot z_2), \quad (x, l, m, n) \in \bigcup_{(j,k) \in \mathcal{E}^2} O_{ijk} \equiv \mathcal{Y}^{[3]}M.
\]
Conversely, given geometric data $(\mathcal{Y}M, \pi_{\mathcal{Y}M}, B, L, \nabla_L, \mu_L)$ and a choice of an open cover $(O_i)_{i,\mathcal{E}}$ of $M$ with local sections $\sigma_i : O_i \to \mathcal{Y}M$ giving rise to local sections $\sigma_{i_1i_2...i_N} \equiv (\sigma_{i_1}, \sigma_{i_2}, ..., \sigma_{i_N}) : O_{i_1i_2...i_N} \to \mathcal{Y}^{[N]}M$ and sufficiently fine for the sets $\sigma_j(O_{ij}) \subset \mathcal{Y}^{[2]}M$ to support flat (unital) local sections $s_{ij} = s_{j}^{-1} \circ \tau : \sigma_{ij}(O_{ij}) \to L$, with $\tau : \mathcal{Y}^{[2]}M \to \mathcal{Y}^{[2]}M : (y_1, y_2) \mapsto (y_2, y_1)$, we define local data
\[
B_i = \sigma_i^*B, \quad A_{ij} \otimes s_{ij} \circ \sigma_{ij} = \sigma_j^*(\nabla_L s_{ij}), \quad \mu(s_{ij} \circ \sigma_{ij}, s_{jk} \circ \sigma_{jk}) = (s_{jk} \circ \sigma_{ik}) \circ g_{ijk}.
\]
\[\text{The tensor product } L_1 \otimes L_2 \text{ of principal } C^\ast \text{-bundles } L_\alpha, \alpha \in \{1, 2\} \text{ is defined, after Ref. [Bry93], as the (principal) bundle } (L_1 \times L_2)/C^\ast \text{ associated with } L_1 \text{ through the defining } C^\ast \text{-action on } L_2, \text{ to be denoted by } \otimes.\]
Under the correspondence, the cohomological equivalence relation behind the definition of the class \([B_i, A_{jk}, g_{imn}]_{i,j,k} \in \mathcal{F}, (j,k) \in \mathcal{F}_2, (i,m,n) \in \mathcal{F}_2\) translates into the notion of an isomorphism between bundle gerbes: Given two such gerbes \(G_\alpha = (Y_i, M, \pi_\alpha, M, B_\alpha, \nabla L_\alpha, \rho L_\alpha), \alpha \in \{1, 2\}\), we call them 1-isomorphic if there exists a quintuple

\[ \Phi = (YY_{1,2}M, \pi YY_{1,2}M, E, \nabla E, \alpha_E) \]

itself termed a 1-isomorphism and composed of a surjective submersion

\[ \pi YY_{1,2}M : YY_{1,2}M \rightarrow Y_1M \times_M Y_2M \equiv Y_{1,2}M \]

over the fibred product \(Y_{1,2}M\), described by the commutative diagram

\[ \begin{array}{ccc}
YY_{1,2}M & \xrightarrow{pr_1} & Y_1M \\
\downarrow \pi YY_{1,2}M & & \downarrow \pi_1M \\
Y_{1,2}M & \xleftarrow{pr_2} & Y_2M \\
\end{array} \]

of a principal bundle

\[ \begin{array}{ccc}
C^* & \xrightarrow{\nabla E} & E \\
\downarrow \pi E & & \downarrow \pi YY_{1,2}M \\
YY_{1,2}M & & \\
\end{array} \]

with connection \(\nabla E\) of curvature

\[ \text{curv}(\nabla E) = \pi_{YY_{1,2}M}^* (pr_2^* B_2 - pr_1^* B_1), \]

and of a connection-preserving principal-bundle isomorphism

\[ \alpha_E: (\pi YY_{1,2}M \times \pi YY_{1,2}M)^* \circ pr_{1,3}^* L_1 \otimes pr_{2,3}^* E \xrightarrow{\cong} pr_1^* E \otimes (\pi YY_{1,2}M \times \pi YY_{1,2}M)^* \circ pr_{2,4}^* L_2 \]

over the fibred product \(Y^{[2]}_{1,2}M = YY_{1,2}M \times_M YY_{1,2}M\) described by the commutative diagram

\[ \begin{array}{ccc}
YY^{[2]}_{1,2}M & \xrightarrow{pr_1} & YY_{1,2}M \\
\downarrow \pi YY^{[2]}_{1,2}M & & \downarrow \pi_{YY_{1,2}M} \\
YY_{1,2}M & \xleftarrow{pr_2} & YY_{1,2}M \\
\end{array} \]

subject to the coherence constraint expressed by the commutative diagram of connection-preserving principal-bundle isomorphisms

\[ \begin{array}{ccc}
\pi_{1,2}^* \circ pr_{1,3}^* L_1 & \otimes & \pi_{2,3}^* \circ pr_{3,5}^* L_1 \otimes pr_3^* E \\
\xrightarrow{id_{\pi_{1,2}^* \circ pr_{1,3}^* L_1} \otimes pr_{3,5}^* \circ \rho L_1 \otimes id_{pr_3^* E}} & & \xrightarrow{pr_{1,3}^* \circ pr_{3,5}^* \circ \rho L_1 \otimes id_{pr_3^* E}} \\
\pi_{1,2}^* \circ pr_{1,3}^* L_1 & \otimes & pr_2^* E \otimes \pi_{2,3}^* \circ pr_{4,6}^* L_2 \\
\xrightarrow{pr_1^* \circ pr_{1,3}^* L_1 \otimes pr_{2,4}^* E \otimes \pi_{2,3}^* \circ pr_{4,6}^* L_2} & & \xrightarrow{pr_{1,3}^* \circ pr_{1,5}^* L_1 \otimes pr_5^* E} \\
pr_1^* E \otimes \pi_{1,2}^* \circ pr_{2,4}^* L_2 & \otimes & \pi_{2,3}^* \circ pr_{4,6}^* L_2 \\
\xrightarrow{pr_1^* E \otimes \pi_{1,2}^* \circ pr_{2,4}^* L_2 \otimes \pi_{2,3}^* \circ pr_{4,6}^* L_2} & & \xrightarrow{id_{pr_1^* E} \otimes \pi_{1,2}^* \circ pr_{2,4}^* \circ \rho L_2} \\
pr_1^* E & \otimes & \pi_{1,3}^* \circ pr_{2,6}^* L_2 \\
\end{array} \]

(2.4)
over the fibred product \( Y^{[3]}Y_{1,2}M \equiv \YY_{1,2}M \times _M \YY_{1,2}M \times _M \YY_{1,2}M \) described by the commutative diagram

\[
\begin{array}{ccc}
\YY_{1,2}M & \xrightarrow{\Phi} & \YY_{1,2}M \\
\xrightarrow{pr_1} & & \xleftarrow{pr_2} \\
\YY_{1,2}M & \xrightarrow{\pi \circ pr_1 \circ pr_2 \circ \pi Y_{1,2}M} & \YY_{1,2}M \\
\xrightarrow{pr_3} & & \xleftarrow{pr_4} \\
\YY_{1,2}M & \xrightarrow{\pi \circ pr_3 \circ \pi Y_{1,2}M} & \YY_{1,2}M
\end{array}
\]

with
\[
\pi_{i,j} = (\pi \circ \pi Y_{1,2}M \times \pi Y_{1,2}M) \circ pr_{i,j}, \quad (i,j) \in \{(1,2), (2,3), (1,3)\},
\]

\[
\pi_{1,2,3} = \pi \circ \pi Y_{1,2}M \times \pi Y_{1,2}M \times \pi Y_{1,2}M.
\]

In view of the results obtained in Ref. [Val07], we may always assume the surjective submersion of the 1-isomorphism to be of the distinguished form \( \pi Y_{1,2}M = \text{id} Y_{1,2}M \), which we do in what follows unless expressly stated otherwise. The situation just described is concisely denoted as

\[
\Phi : G_1 \xrightarrow{\sim} G_2.
\]

In fact, the transformation (2.2) is left unchanged by secondary redefinitions

\[
(C_i, h_{jk}) \mapsto (C_i - \text{id} \log f_i, h_{jk} \cdot (f_k^{-1} \cdot f_j)|_{C_{ij}},
\]

which indicates the existence of isomorphisms between 1-isomorphisms, or **2-isomorphisms**, with local data \([\{(f_i)_{i \in I}\}]\) (defined up to local constants). In the geometric language, and for a given pair of 1-isomorphisms \( \Phi_\beta = (Y^{[2]}Y_{1,2}M, \pi Y_{1,2}M, E_\beta, \nabla_\beta, \alpha_{E_\beta}), \beta \in \{1,2\} \) between bundle gerbes \( G_\alpha = (Y_\alpha, \pi Y_\alpha, M, \pi_\alpha, L_\alpha, L_\alpha, \alpha_{E_\alpha}), \alpha \in \{1,2\} \), a 2-isomorphism is represented by a triple

\[
\varphi = (Y^{[1,2]}Y_{1,2}M, \pi Y_{1,2}M, \beta)
\]

composed of a surjective submersion

\[
\pi Y^{[1,2]}Y_{1,2}M : Y^{[1,2]}Y_{1,2}M \longrightarrow Y^{[1]}Y_{1,2}M \times _{Y^{[1]}Y_{1,2}M} Y^{[2]}Y_{1,2}M \equiv Y^{[1,2]}Y_{1,2}M
\]

and a connection-preserving principal-bundle isomorphism

\[
\beta : (pr_1 \circ \pi Y^{[1,2]}Y_{1,2}M)^* \overset{\sim}{\longrightarrow} (pr_2 \circ \pi Y^{[1,2]}Y_{1,2}M)^* E_1
\]

subject to the coherence constraint expressed by the commutative diagram of connection-preserving principal-bundle isomorphism

\[
\begin{array}{ccc}
p_{1,1}^* L_1 \otimes \pi_{1,2}^* E_1 & \xrightarrow{(\pi_{1,1} \otimes \pi_{1,2})^* \alpha E_1} & \pi_{1,1}^* E_1 \otimes p_{2,1}^* L_2 \\
\text{id}_{1,1} \otimes p_{2,1}^* \beta & \xrightarrow{pr_1^* \otimes pr_2^* \beta} & p_{1,1}^* L_1 \otimes \pi_{2,2}^* E_2 \\
\end{array}
\]

(2.5)

over \( Y^{[1,2]}Y_{1,2}M \times _M Y^{[1,2]}Y_{1,2}M \), with

\[
\pi_i = \text{pr}_i \circ \pi Y^{[1,2]}Y_{1,2}M, \quad \pi_{i,j,k} = \pi_j \circ \text{pr}_k, \quad i,j,k \in \{1,2\},
\]

\[
p_{l,m} = \text{pr}_l \circ \pi Y^{[1,2]}Y_{1,2}M \circ \pi_m \times _M \text{pr}_l \circ \pi Y^{[1,2]}Y_{1,2}M \circ \pi_m, \quad l,m \in \{1,2\}.
\]

We denote the 2-isomorphism as

\[
\varphi : \Phi_1 \xrightarrow{\sim} \Phi_2.
\]

For details of the correspondence indicated above, consult, e.g., Refs. [MS04, GR02] and [Val07].

---

8Strictly speaking, we should consider classes of such triples with respect to the following equivalence relation: \((Y_1 Y^{[1,2]}Y_{1,2}M, \pi Y^{[1,2]}Y_{1,2}M, \beta_1) \sim (Y_2 Y^{[1,2]}Y_{1,2}M, \pi Y^{[1,2]}Y_{1,2}M, \beta_2)\) if there exist surjective submersions \( \pi_\alpha : Z \longrightarrow Y_\alpha Y^{[1,2]}Y_{1,2}M, \alpha \in \{1,2\} \) with the property \( \pi Y^{[1,2]}Y_{1,2}M \circ \pi_1 = \pi Y^{[1,2]}Y_{1,2}M \circ \pi_2 \), and such that \( \pi_1^* \beta_1 = \pi_2^* \beta_2 \).
Our subsequent discussion calls for several additional elementary objects and constructions of the theory of gerbes. The first among them is the trivial gerbe over \( M \) which is none other than a de Rham 3-coboundary \( H = dB \) with a globally smooth primitive \( B \in \Omega^2(M) \), with an obvious cohomological representation (associated with an arbitrary open cover \( O_M \))

\[
I_B = [(B|_{O_1}, 0, 1)_{\mu, \nu}],
\]

and a simple geometrisation

\[
I_B = (M, id_M, B, M \times C^\times, d, \mu)
\]

with the trivial groupoid structure

\[
\mu : (M \times C^\times) \otimes (M \times C^\times) \longrightarrow M \times C^\times : ((x, z_1), (x, z_2)) \mapsto (x, z_1 \cdot z_2).
\]

A trivial 1-isomorphism is defined analogously as a trivial principal \( C^\times \)-bundle. The next concept is that of the tensor product \( G_1 \otimes G_2 \) of (bundle) gerbes \( G_\alpha, \alpha \in \{1, 2\} \) over a common base \( M \). This has a trivial cohomological description over a common trivialising open cover, to wit, given the respective local data \([(B^1_\alpha, A^1_\alpha, g^1_{I_\alpha\beta})_{\mu, \nu}, (j, k) \in \mathcal{S}_2, (l, m, n) \in \mathcal{S}_3]\),

\[
[(B^1_\alpha, A^1_\alpha, g^1_{I_\alpha\beta})_{\mu, \nu}, (j, k) \in \mathcal{S}_2, (l, m, n) \in \mathcal{S}_3] \otimes [(B^2_\alpha, A^2_\alpha, g^2_{I_\alpha\beta})_{\mu, \nu}, (j, k) \in \mathcal{S}_2, (l, m, n) \in \mathcal{S}_3]
\]

\[
= [(B^1_\alpha + B^2_\alpha, A^1_\alpha + A^2_\alpha, g^1_{I_\alpha\beta} \cdot g^2_{I_\alpha\beta})_{\mu, \nu}, (j, k) \in \mathcal{S}_2, (l, m, n) \in \mathcal{S}_3].
\]

The geometric counterpart of this construction, for the choice of respective geometrisations \( G_\alpha = (Y_\alpha M, \pi_\alpha M, B_\alpha, L_\alpha, \nabla_{L_\alpha}, \mu_{L_\alpha}), \alpha \in \{1, 2\} \), is the bundle gerbe

\[
G_1 \otimes G_2 = (Y_1 M, \pi_1 M \circ pr_1, pr_1^* B_1 + pr_2^* B_2, pr_{1,3}^* L_1 \circ pr_{2,4}^* L_2, pr_{1,3}^* \nabla_{L_1} \otimes id_{pr_{2,4}^* L_2} + id_{pr_{1,3}^* L_1} \otimes pr_{2,4}^* \nabla_{L_2} ),
\]

The construction of the tensor product descends naturally to (stable) isomorphisms between (bundle) gerbes: Given gerbes \( G_\alpha, \alpha \in \{1, 2, 3, 4\} \) and isomorphisms \( \Phi_\beta : G_\beta \stackrel{\sim}{\longrightarrow} G_{\beta+2}, \beta \in \{1, 2\} \), we may define a tensor-product isomorphism \( \Phi_1 \otimes \Phi_2 : G_1 \otimes G_2 \stackrel{\sim}{\longrightarrow} G_3 \otimes G_4 \). If the respective local data are \([(C^1_\beta, h^1_{I_\beta\gamma})_{\mu, \nu}, (j, k) \in \mathcal{S}_2]\), we have

\[
[(C^1_\beta, h^1_{I_\beta\gamma})_{\mu, \nu}, (j, k) \in \mathcal{S}_2] \otimes [(C^2_\beta, h^2_{I_\beta\gamma})_{\mu, \nu}, (j, k) \in \mathcal{S}_2] = [(C^1_\beta + C^2_\beta, h^1_{I_\beta\gamma} \cdot h^2_{I_\beta\gamma})_{\mu, \nu}, (j, k) \in \mathcal{S}_2].
\]

When expressed in terms of the respective geometrisations \( \Phi_\beta = (YY_\beta \times M, \pi_{YY_\beta} \times M, E_\beta, \nabla_{E_\beta}, C_{YY_\beta} \), the tensor product takes the form

\[
\Phi_1 \otimes \Phi_2 = (YY, \pi_{YY} \times M \times M, E_1 \otimes E_2, \pi_{YY} \times \nabla_{E_1} \otimes \nabla_{E_2}, pr_{i,3}^* \alpha_{E_1} \otimes pr_{2,4}^* \alpha_{E_2} ),
\]

with the fibred product \( YY \times M \times M \) described by the commutative diagram

\[
\begin{array}{ccc}
YY_1 M 	imes M & \longrightarrow & YY_2 M \\
pr_1 \downarrow & & \downarrow pr_2 \\
YY_1 M & \longrightarrow & YY_2 M \\
\pi_{YY_1 M} \circ pr_1 \circ \pi_{YY_1 M} & \longrightarrow & \pi_{YY_2 M} \circ pr_2 \circ \pi_{YY_2 M} \\
M & \longrightarrow & M
\end{array}
\]

and with

\[
\tau_{YY_1 M, YY_2 M} : Y_1 M \times M Y_2 M \longrightarrow Y_2 M \times M Y_3 M : (y_1, y_2) \mapsto (y_2, y_1).
\]

We may also conceive the tensor product of a pair of 2-isomorphisms \( \varphi_\gamma : \Phi^1_\gamma \stackrel{\sim}{\longrightarrow} \Phi^2_\gamma, \gamma \in \{1, 2\} \) between isomorphisms \( \Phi^\beta : G_\gamma \stackrel{\sim}{\longrightarrow} G_{\gamma+2}, \beta \in \{1, 2\} \). For the (respective) local data \([(f^1_\gamma)_{\mu, \nu}], \) we obtain

\[
[(f^1_\gamma)_{\mu, \nu}] \otimes [(f^2_\gamma)_{\mu, \nu}] = [(f^1_\gamma \cdot f^2_\gamma)_{\mu, \nu}],
\]
whereas in the language of the respective geometrisations \( \varphi_\gamma = (YY^{1,2}_{1,\gamma,2,1}M, \piYY^{1,2}_{1,\gamma,2,1}M, \beta_\gamma) \), the
tensor product is the 2-isomorphism

\[
\varphi_1 \otimes \varphi_2 = (YY^{1,2}_{1,1,3}M \times_M YY^{1,2}_{1,2,4}M,
\]

\[
(id_Y \circ YY^{1,2}_{1,1,3}M \times \piYY^{1,2}_{1,2,4}M \times id_Y \circ YY^{1,2}_{1,2,4}M) \circ \left( \piYY^{1,2}_{1,2,4}M \times \piYY^{1,2}_{1,2,4}M \right),
\]

with the fibred product \( YY^{1,2}_{1,1,3}M \times_M YY^{1,2}_{1,2,4}M \) described by the commutative diagram

and with

\[
\gamma \circ \gamma : (YY^{1,2}_{1,1,3}M \times_M YY^{1,2}_{1,2,4}M \rightarrow YY^{1,2}_{1,2,4}M) \rightarrow YY^{1,2}_{1,2,4}M)
\]

Stable isomorphisms and 2-isomorphisms can be not only tensored, but also composed. Given 1-isomorphisms \( \Phi_\beta : G_\beta \rightarrow G_{\beta+1} \), \( \beta \in \{1,2\} \) between gerbes \( G_\alpha, \alpha \in \{1,2,3\} \), we may define the composite 1-isomorphism \( \Phi_2 \circ \Phi_1 : G_1 \rightarrow G_3 \) with local data

\[
[(C^1, h^1), (j,k) \in \mathcal{F}, (j,k) \in \mathcal{F}_2] = [(C^2, h^2), (j,k) \in \mathcal{F}, (j,k) \in \mathcal{F}_2],
\]

and with a geometrisation

\[
\Phi_2 \circ \Phi_1 = (YY^{1,2}_{1,2,3}M \times_M YY^{1,2}_{1,2,3}M, pr_{1,4} \circ (\piYY^{1,2}_{1,2,3}M \times \piYY^{1,2}_{1,2,3}M), pr_{1}^*E_1 \otimes pr_{2}^*E_2,
\]

\[
pr_{2}^*V_{E_1} \otimes id_{pr_{2}^*E_2} + id_{pr_{1}^*E_1} \otimes pr_{2}^*V_{E_2} \circ (id_{pr_{1}^*E_1} \otimes pr_{2}^*E_2) \circ (pr_{1,3}^* \otimes id_{pr_{2}^*E_2})),
\]

where the fibred product \( YY^{1,2}_{1,2,3}M \times_M YY^{1,2}_{1,2,3}M \) is described by the commutative diagram

In the case of 2-isomorphisms, we encounter two types of composition. Given two pairs of 1-isomorphisms \( \Phi_\gamma^1 : G_\gamma \rightarrow G_{\gamma+1} \), \( \gamma \in \{1,2\} \) between gerbes \( G_\alpha, \alpha \in \{1,2,3\} \) and two 2-isomorphisms \( \varphi_\gamma : \Phi_\gamma \rightarrow \Phi_\gamma^1 \) between the former, we define the horizontal composition \( \varphi_2 \circ \varphi_1 : \Phi_2 \circ \Phi_1 \rightarrow \Phi_2 \circ \Phi_2^1 \) as the 2-isomorphism with local data

\[
[(f^1)^*] \circ [(f^2)^*] = [(f^1)^* \cdot f^2)^*]
\]

and for \( \varphi_\gamma = (YY^{1,2}_{1,\gamma,2,1}M, \piYY^{1,2}_{1,\gamma,2,1}M, \beta_\gamma) \) a geometrisation

\[
\varphi_2 \circ \varphi_1 = ((YY^{1,2}_{1,2,3}M \times_M YY^{1,2}_{1,2,3}M) \times YY^{1,2}_{1,2,3}M, pr_{1,2,5,6}, \pi_{1,2,3,4,6}^*, d_{\Phi_2 \circ \Phi_1}^* \circ (pr_{2}^* \otimes pr_{3}^* \otimes pr_{4}^* \otimes pr_{5}^* \otimes pr_{6}^*)),
\]

written in terms of the surjective submersions

\[
\pi_{1,2,3}^* = (id_{YY^{1,2}_{1,2,3}M} \times_M \piYY^{1,2}_{1,2,3}M, \piYY^{1,2}_{1,2,3}M \times_M \piYY^{1,2}_{1,2,3}M) \circ pr_{1,2,3,4,6},
\]

\[
\pi_{1,2,3}^* = ((pr_{2} \circ \piYY^{1,2}_{1,2,3}M, \piYY^{1,2}_{1,2,3}M \times_M \piYY^{1,2}_{1,2,3}M) \times id_{YY^{1,2}_{1,2,3}M} \times M \piYY^{1,2}_{1,2,3}M) \circ pr_{3,4,5,6},
\]

and of the canonical (connection-preserving principal-bundle) isomorphisms

\[
d_{\Phi_2 \circ \Phi_1}^* : pr_{2}^* (pr_{1}^*E_1^\beta \otimes pr_{2}^*E_2^\beta) \rightarrow (pr_{1}^*E_1^\beta \otimes pr_{2}^*E_2^\beta), \quad \beta \in \{1,2\}
\]
over the respective fibre products \((Y^1Y_{1,2}M \times Y_{2,3}M) \times Y_{1,3}M \times Y_{2,3}M\) derived in Ref. [Wal07]. For any pair \(\varphi_\delta : \Phi_\delta \overset{\delta}{\longrightarrow} \Phi_{\delta+1}\) of 2-isomorphisms between 1-isomorphisms \(\Phi_\delta, \Phi_{\delta+1} : G_1 \overset{\pi}{\longrightarrow} G_2\) between given gerbes \(G_1\) and \(G_2\), on the other hand, we may define their \textbf{vertical composition} \(\varphi_2 \circ \varphi_1 : \Phi_1 \overset{\varphi_2 \circ \varphi_1}{\longrightarrow} \Phi_3\) as the 2-isomorphism with local data

\[
[(f^2_i \circ f^1_i) \circ \varphi_1] \bullet [(f^1_i) \circ \varphi_2] = [(f^2_i \circ f^1_i) \circ \varphi_1 \circ \varphi_2]
\]

and for \(\Phi_\delta = (Y^2Y_{1,2}M, \pi_{YY_{1,2}M}, E, \nabla_{E}, \alpha_E)\) and \(\varphi_\delta = (YY^2Y_{1,2}M, \pi_{YY_{1,2}M}, \alpha_E)\) a geometrisation

\[
\varphi_2 \circ \varphi_1 = (YY^2Y_{1,2}M \times Y_{2,3}M \times \pi_{YY_{1,2}M} \circ \pi_{YY_{1,2}M} \times \pi_{YY_{1,2}M}, \pi_{YY_{1,2}M} \circ \pi_{YY_{1,2}M} \circ \pi_{YY_{1,2}M}, \pi_{YY_{1,2}M} \circ \pi_{YY_{1,2}M} \circ \pi_{YY_{1,2}M})
\]

The above structure can be organised into a (weak) 2-category with (bundle) gerbes as 0-cells (or objects), 1-isomorphisms as 1-cells and 2-isomorphisms as 2-cells, which puts us in the higher-categorical context of the loop (quantum) mechanics.

Finally, we should mention the pullback of the various structures introduced heretofore along smooth maps between their bases. It is completely straightforward to describe it in the local cohomological description. Indeed, let \(f \in C^\infty(M_1, M_2)\) and let

\[
\left[\left(\frac{X^p, X^p_{i_1 i_2}, \ldots, X^p_{i_1 i_2 \ldots i_1 i_2 \ldots i_1 i_2}}{j_1 \ldots j_2}, \frac{Y^p, Y^p_{j_1 j_2}, \ldots, Y^p_{j_1 j_2 \ldots j_1 j_2 \ldots j_1 j_2}}{j_1 \ldots j_2}, \frac{\alpha, \alpha_{\beta_1 \beta_2 \beta_3 \ldots \beta_n}}{\pi, \pi_{\beta_1 \beta_2 \beta_3 \ldots \beta_n}}\right) \right]_{Y^p_{i_1 i_2 \ldots i_1 i_2 \ldots i_1 i_2}}, \quad p \in \{0, 1, 2\}
\]

be local data of an object (a gerbe for \(p = 2\), a 1-isomorphism for \(p = 1\), and a 2-isomorphism for \(p = 0\)) on \(M_2\) associated with an open cover \(O_{M_2} = \{O^2_\delta\}_{\delta \in \mathcal{D}}\) of \(M_2\). In order to define the pullback of that object over \(M_1\) in terms of its local data, we need to fix an open cover \(\{O^1_\delta\}_{\delta \in \mathcal{D}}\) of \(M_1\) together with a map \(\phi : \mathcal{D} \rightarrow O^2\) subordinate to \(f\) in the sense expressed by the condition

\[
\forall \phi \in \mathcal{D} : \phi \circ \varphi_1 = \varphi_2 \circ \phi
\]

(which may require passing to a refinement of \(O_{M_2}\)). Whereupon we define

\[
f^* \left[\left(\frac{X^p, X^p_{i_1 i_2}, \ldots, X^p_{i_1 i_2 \ldots i_1 i_2 \ldots i_1 i_2}}{j_1 \ldots j_2}, \frac{Y^p, Y^p_{j_1 j_2}, \ldots, Y^p_{j_1 j_2 \ldots j_1 j_2 \ldots j_1 j_2}}{j_1 \ldots j_2}, \frac{\alpha, \alpha_{\beta_1 \beta_2 \beta_3 \ldots \beta_n}}{\pi, \pi_{\beta_1 \beta_2 \beta_3 \ldots \beta_n}}\right) \right]_{Y^p_{i_1 i_2 \ldots i_1 i_2 \ldots i_1 i_2}, \quad p \in \{0, 1, 2\}}
\]

by the formulae

\[
\frac{X^p, X^p_{i_1 i_2}, \ldots, X^p_{i_1 i_2 \ldots i_1 i_2 \ldots i_1 i_2}}{j_1 \ldots j_2, \phi(j_1 \ldots j_2 \ldots j_1 j_2 \ldots j_1 j_2), \phi(j_1 \ldots j_2 \ldots j_1 j_2 \ldots j_1 j_2)} = f^* \frac{X^p, X^p_{i_1 i_2}, \ldots, X^p_{i_1 i_2 \ldots i_1 i_2 \ldots i_1 i_2}}{j_1 \ldots j_2, \phi(j_1 \ldots j_2 \ldots j_1 j_2 \ldots j_1 j_2), \phi(j_1 \ldots j_2 \ldots j_1 j_2 \ldots j_1 j_2)}, \quad k \in \overline{0,p}.
\]

We complete our presentation by giving definitions of pullbacks of the geometrisations of the local data that we introduced earlier. Thus, given a gerbe \(G = (YM_2, \pi_{YM_2}, B, L, \nabla_L, \mu_L)\) over the codomain of \(f\), we first erect an arbitrary surjective submersion \(\pi_{YM_1} : YM_1 \rightarrow M_1\) endowed with a smooth map \(\tilde{f} : YM_1 \rightarrow YM_2\) that covers \(f\) in the sense specified by the commutative diagram

\[
YM_1 \quad \tilde{f} \quad YM_2
\]

\[
\pi_{YM_1} \quad \pi_{YM_2}
\]

\[
YM_1 \quad f \quad YM_2
\]

We may, e.g., take \(YM_1 = M_1 \times_MYM_2\) with \(\pi_{YM_1} = \pi_{YM_2} \circ \pi_{YM_2}\) and \(\tilde{f} = \pi_{YM_2}\), and subsequently define

\[
f^* G = (YM_1, \pi_{YM_1}, \pi_{YM_2} \circ \pi_{YM_2}, \nu_{YM_2} \circ \nu_M, \mu_L). \quad \nabla_{YM_1}, (\tilde{f} \times \tilde{f}) \nu_{YM_2} \circ \nu_M, (\tilde{f} \times \tilde{f}) \nu_{YM_2} \circ \nu_M, (\tilde{f} \times \tilde{f}) \nu_{YM_2} \circ \nu_M \circ \mu_L)
\]

Similarly, in order to pull back a 1-isomorphism \(\Phi = (YY_{1,2}M_2, \pi_{YY_{1,2}M_2}, E, \nabla_E, \alpha_E)\) between gerbes \(\mathcal{G}_\alpha = (YM_2, \pi_{YM_2}, B, \nabla_B, \mu_L, \mu_L)\), \(\alpha \in \{1, 2\}\) along \(f\), we choose a surjective submersion \(\pi_{YY_{1,2}M_1} : YY_{1,2}YM_1 \rightarrow YY_{1,2}YM_2\) and a map \(\tilde{f} : YY_{1,2}YM_1 \rightarrow YY_{1,2}YM_2\) satisfying the
condition described by the commutative diagram

\[
\begin{array}{ccc}
\mathbb{Y}Y_{1,2}M_1 & \xrightarrow{f_{1,2}} & \mathbb{Y}Y_{1,2}M_2 \\
\pi_{\mathbb{Y}Y_{1,2}M_1} \downarrow & & \downarrow \pi_{\mathbb{Y}Y_{1,2}M_2} \\
Y_{1,2}M_1 & \xrightarrow{\tilde{f}_1 \times \tilde{f}_2} & Y_{1,2}M_2
\end{array}
\]

(for \(\tilde{f}_a\) the respective covers of \(f\)), whereupon we define

\[
f^* \Phi = \left(\mathbb{Y}Y_{1,2}M_1, \pi_{\mathbb{Y}Y_{1,2}M_1}, \tilde{f}_{1,2}^* E, \tilde{f}_{1,2}^* \nabla_E, (\tilde{f}_{1,2} \times \tilde{f}_{1,2})^* \tilde{\alpha}_{[\mathbb{Y}Y_{1,2}M_1]} \right) : f^* G_1 \xrightarrow{\cong} f^* G_2.
\]

We complete our construction of the pullback functor between the (weak) 2-categories of gerbes over the two manifolds related by the smooth map \(f\) by taking, for any 2-isomorphism \(\varphi = (\mathbb{Y}Y_{1,2}M_2, \pi_{\mathbb{Y}Y_{1,2}M_2}, \beta)\) between 1-isomorphisms \(\Phi_\beta = (\mathbb{Y}Y_{1,2}M_1, \pi_{\mathbb{Y}Y_{1,2}M_1}, \nabla_{E_\beta}, \alpha_{E_\beta})\), \(\beta \in \{1, 2\}\) between gerbes \(G_\alpha = (Y_\alpha M, \pi_{\mathbb{Y}Y_{1,2}M_1}, B_\alpha, L_\alpha, \nabla_{L_\alpha}, \mu_{L_\alpha}), \alpha \in \{1, 2\}\), a surjective submersion \(\pi_{\mathbb{Y}Y_{1,2}M_1} : \mathbb{Y}Y_{1,2}M_1 \rightarrow \mathbb{Y}Y_{1,2}M_1\) together with a map \(\tilde{f}_{1,2}^\beta : \mathbb{Y}Y_{1,2}M_1 \rightarrow \mathbb{Y}Y_{1,2}M_2\) that renders the following diagram commutative,

\[
\begin{array}{ccc}
\mathbb{Y}Y_{1,2}M_1 & \xrightarrow{\tilde{f}_{1,2}^\beta} & \mathbb{Y}Y_{1,2}M_2 \\
\pi_{\mathbb{Y}Y_{1,2}M_1} \downarrow & & \downarrow \pi_{\mathbb{Y}Y_{1,2}M_2} \\
Y_{1,2}M_1 & \xrightarrow{f_{1,2}^\beta \times f_{1,2}^\beta} & Y_{1,2}M_2
\end{array}
\]

(for \(f_{1,2}^\beta\) the respective covers of \(\tilde{f}_1 \times \tilde{f}_2\)), and then write

\[
f^* \varphi = \left(\mathbb{Y}Y_{1,2}M_1, \pi_{\mathbb{Y}Y_{1,2}M_1}, \tilde{f}_{1,2}^\beta \right) : f^* \Phi_1 \xrightarrow{\cong} f^* \Phi_2.
\]

This exhausts the list of rudimentary concepts and constructions of the standard gerbe theory that we shall have a need for in the main part of our subsequent discussion.

Prior to passing to the field-theoretic applications of the formalism recapitulated above, we close this section by giving – after Ref. [Ste01] – one last definition which is going to serve as a reference for our supergeometric constructions. Thus, we consider an object one degree higher in the natural hierarchy of geometrisations of de Rham classes, to wit, a bundle 2-gerbe over a manifold \(M\) with connection of curvature given by a de Rham 3-cocycle with periods in \(2\pi\mathbb{Z}\),

\[
J \in Z^3_{\text{dR}}(M),
\]
to be understood as a quintuple

\[
\mathcal{G}^{(2)} = (YM, \pi_{YM}, \mathcal{C}, \mathcal{G}, \mathcal{M}_G, \mu_\varphi),
\]
composed of a surjective submersion

\[
\pi_{YM} : YM \rightarrow M
\]
supporting a global primitive

\[
\mathcal{C} \in \Omega^3(YM)
\]
of the pullback

\[
\pi_{YM}^* J = d\mathcal{C},
\]
alongside a bundle gerbe \(\mathcal{G}\) over the fibred square \(\mathcal{Y}^{[2]}M\) with connection of curvature

\[
H = (\text{pr}_{2}^* - \text{pr}_{1}^*) \mathcal{C}
\]

together with a 1-isomorphism

\[
\mathcal{M}_G : \text{pr}_{1,2}^* \mathcal{G} \otimes \text{pr}_{2,3}^* \mathcal{G} \xrightarrow{\cong} \text{pr}_{1,3}^* \mathcal{G}
\]
of bundle gerbes over the fibred cube $Y[3]M$ (termed the product of the 2-gerbe) and a 2-isomorphism

$$\text{pr}^*_1 \mathcal{G} \otimes \text{pr}^*_2 \mathcal{G} \otimes \text{pr}^*_3 \mathcal{G} \otimes \text{pr}^*_4 \mathcal{G} \otimes \text{pr}^*_5 \mathcal{G}$$

between the 1-isomorphisms of bundle gerbes over the fourfold fibred product $Y[4]M$ (termed the associator of the 2-gerbe) subject to the coherence constraints expressed by the commutative diagram of 2-isomorphisms (here, $X_{i_1 i_2 \ldots i_k} \equiv \text{pr}^*_{i_1 i_2 \ldots i_k} X$ for any $i_1, i_2, \ldots, i_k \in \{1, 2, 3, 4, 5\}$, $k \in \{2, 3\}$).

2.2. A rigorous definition, canonical description & geometric quantisation of the $\sigma$-model.

The most immediate application of the formalism developed heretofore is an explicit formula for (the logarithm of) the holonomy, determined by local data of the gerbe. The formula calls for an extra property associated with the 1-isomorphisms of bundle gerbes over the fifthfold fibred product $Y[5]M$.

With all the requisite data in place, we have

$$-i \log \text{Hol}_G(x) = \sum_{p \in \mathcal{P}_2(\Sigma)} \left[ \int_p \langle x^*_p \rangle^* B_p + \sum_{e \in \partial p} \left( \int_e \langle x^*_e \rangle^* A_{i_p e} - i \sum_{v \in \partial e} \varepsilon_{pve} \log g_{i_p e} \langle x(v) \rangle \right) \right],$$

where $\varepsilon_{pve} = 1$ if $v$ sits at the end of $e$ with respect to the orientation of the edge induced (as the orientation of the boundary) from that of $p$, and $\varepsilon_{pve} = -1$ otherwise. The above formula is a natural point of departure for the analysis that establishes the nature of geometric objects that complement the monophasic background $(M, g, \mathcal{G})$ in the presence of world-sheet defects, cp. Ref. [RS09].
\[ J^1(\Sigma \times M) \] (written in the standard notation that employs the symbol \( \delta \) for vertical differentials on \( J^1(\Sigma \times M) \))

\[
\Theta_{\sigma}(x^a, \xi^b) \equiv \left( \mathcal{L}_{\sigma}(x^a, \xi^b) - \xi^a \frac{\partial}{\partial x^a}(x^a, \xi^b) \right) \text{Vol}(\Sigma) + \delta x^a \frac{\partial}{\partial \xi^b}(x^a, \xi^b) \land (\partial_t \cup \text{Vol}(\Sigma)).
\]

It is not difficult to see that the extremals of the functional

\[ S_{\Theta_{\sigma}} : \Gamma(J^1(\Sigma \times M)) \to \mathbb{R} : \Psi \mapsto \int_{\Sigma} \Psi^* \Theta_{\sigma} \]

are first jets of extremals of \( S_{\sigma} \), and this observation justifies the definition of a presymplectic form \( \Omega_{\sigma} \) on the space of states (of a single loop) \( P_{\sigma} = T^*LM \) of the \( \sigma \)-model, the space itself being coordinatised by Cauchy data \( \Psi(\ref{eq:cauchy_data}) \equiv (x^a, p_b) \) of extremals \( \Psi \) of \( S_{\Theta_{\sigma}} \) supported on a model Cauchy section, or equitemporal slice, \( \sigma \equiv S^1 \subset \Sigma \). The definition reads

\[
\Omega_{\sigma}[\Psi(\ref{eq:cauchy_data})] = \delta \int_{\mathcal{E}} (\Psi(\ref{eq:cauchy_data}))^\ast \Theta_{\sigma},
\]

and it depends only on the homotopy class of \( \mathcal{E} \) within \( \Sigma \). Through direct computation, we arrive at the explicit form

\[
(P_{\sigma}, \Omega_{\sigma}) = \left( T^*LM, \delta \theta_{T^*LM} + \pi_{T^*LM}^\ast \int_{\mathcal{E}} \text{ev}^\ast H \right),
\]

expressed in terms of the bundle projection \( \pi_{T^*LM} : T^*LM \to LM \), the canonical (action) 1-form \( \theta_{T^*LM} \) on \( T^*LM \) (with local presentation \( \theta_{T^*LM}[x, p] = \int_{\mathcal{E}} \text{Vol}(\mathcal{E}) p_b(\cdot) \delta x^a(\cdot) \)), and the standard evaluation map

\[
\text{ev} : \mathcal{E} \times LM \to M : (\varphi, \gamma) \mapsto \gamma(\varphi).
\]

The 2-form serves to define a Poisson bracket of hamiltonians on \( P_{\sigma} \), i.e. those smooth functionals \( h \) on \( P_{\sigma} \) for which there exist smooth vector fields \( \mathcal{V} \), termed (globally) hamiltonian, satisfying the relation

\[
(2.6) \quad \mathcal{V} \cup \Omega_{\sigma} = -\delta h.
\]

Indeed, for any two such functionals \( h_A, A \in \{1, 2\} \), and the corresponding vector fields \( \mathcal{V}_A \), we may define a bracket

\[
\{ h_1, h_2 \}_{\Omega_{\sigma}}[\Psi(\ref{eq:cauchy_data})] := \mathcal{V}_2 \cup \mathcal{V}_1 \cup \Omega_{\sigma}[\Psi(\ref{eq:cauchy_data})],
\]

and the Jacobi identity follows automatically from the closedness of \( \Omega_{\sigma} \). A detailed discussion of the thus defined canonical description of the \( \sigma \)-model and its adaptations to the multi-phase setting can be found in Refs. [Sus11a, Sus12].

The ultimate confirmation of the naturality and functionality of gerbe theory in the analysis of the bosonic \( \sigma \)-model comes with the derivation of a quantisation scheme from that theory. The latter scheme is based on Gawędzki’s transgression map (extended to the polyphase setting and subsequently employed in the analysis of symmetries and dualities of the \( \sigma \)-model in Refs. [Sus11a, Sus12])

\[
\tau : \mathbb{H}^2(M, D(2)^\ast) \to \mathbb{H}^1(LM, D(1)^\ast)
\]

that canonically associates with (the isomorphism class of) \( \mathcal{G} \) (the isomorphism class of) a principal bundle

\[
\begin{align*}
\mathbb{C}^\times &\xrightarrow{\pi_{\mathcal{L}_G}} \mathcal{L}_G \\
LM &\cong C^\infty(S^1, M)
\end{align*}
\]

over the configuration space \( LM \) of the \( \sigma \)-model, with connection \( \nabla_{\mathcal{L}_G} \) of curvature

\[
\text{curv}(\nabla_{\mathcal{L}_G}) = \int_S \text{ev}^\ast H,
\]

termed the transgression bundle, and thus induces over the phase space \( T^*LM \) of the monophase \( \sigma \)-model a pre-quantum bundle of the \( \sigma \)-model

\[
(2.8) \quad \mathcal{L}_{\sigma} := (T^*LM \times \mathbb{C}^\times) \otimes \pi_{T^*LM}^\ast \mathcal{L}_G,
\]

where the trivial tensor factor is taken to carry the global connection 1-form \( \theta_{T^*LM} \). It ought to be emphasised that the transgression bundle \( \mathcal{L}_G \) can be reconstructed explicitly, on the basis of
Under a gauge transformation of the gerbe $G$ over the pullback along $\pi_{\text{TM}}$ of an overcomplete basis

$$O_i \equiv O_{\Delta(S^1)} \triangleleft \{ \mathbf{X} \in \mathbf{LM} \mid \forall (e, v) \in E_{\Delta(S^1)} \times W_{\Delta(S^1)} : x(e) \in O_{\text{ev}}^M \land x(v) \in O_{\text{ev}}^M \},$$

of the compact-open topology of the Fréchet manifold $\mathbf{LM}$ indexed by pairs $i \equiv (\Delta(S^1), i)$ composed of a tesselisation $\Delta(S^1)$ of the unit circle, with its set of edges $E_{\Delta(S^1)}$ and its set of vertices $W_{\Delta(S^1)}$, and a choice $i : \Delta(S^1) \to \mathcal{F} : \xi \mapsto i_\xi$ of assignment of indices of $O_M$ to elements of $\Delta(S^1)$. By varying these two choices arbitrarily, whereby an index set $J_M$ is formed, we cover all of $\mathbf{LM}$, thus forming an open cover $O_{\mathbf{LM}} = \{ O_i \}_{i \in J_M}$ of the free-loop space $\mathbf{LM}$. It is straightforward to describe intersections of elements of the open cover $O_{\mathbf{LM}}$, cp. Ref. [Gaw88].

Given a pair $O_i$, $\alpha \in \{ 1, 2 \}$ with the respective triangulations $\Delta_\alpha(S^1)$ (consisting of edges $e_\alpha \in E_{\Delta_\alpha(S^1)}$ and vertices $v_\alpha \in E_{\Delta_\alpha(S^1)}$) and index assignments $i^\alpha : \Delta_\alpha(S^1) \to \mathcal{F} : v_\alpha \mapsto i_\alpha(v_\alpha)$, we consider the triangulation $\overline{\Delta}(S^1)$ obtained by intersecting $\Delta_1(S^1)$ with $\Delta_2(S^1)$, by which we mean that the edges $e$ of $\overline{\Delta}(S^1)$ are the non-empty intersections of the edges of the $\Delta_\alpha(S^1)$, and its vertices $v$ are taken from $\overline{W_{\Delta_1(S^1)}} \cup \overline{W_{\Delta_2(S^1)}}$. A non-empty double intersection $O_{1i} \cap O_{2i} = O_{1i2}$ is then labelled by the triangulation $\overline{\Delta}(S^1)$, taken together with the indexing convention such that $\overline{e} \triangleleft i_\alpha$ is the Čech index assigned via $i_\alpha$ – to the edge of $\Delta_\alpha(S^1)$ containing $e \subseteq \Delta_\alpha(S^1)$, and $\overline{v}$ is the Čech index assigned via $i_\alpha$ – to the edge of $\Delta_\alpha(S^1)$ containing $v \subseteq \Delta_\alpha(S^1)$ or the Čech index assigned – also via $i_\alpha$ – to the edge of $\Delta_\alpha(S^1)$ containing $v$ otherwise. With the foregoing description in hand, we may finally write out explicit formulae for local data of the transgression bundle: we begin with local connection 1-forms (written for $x \in O_i$ and $i = (\Delta(S^1), i)$)

$$E_i[x] = - \sum_{e \in E_{\Delta(S^1)}} \int_{e} (x) v \cdot B_{e} - \sum_{v \in W_{\Delta(S^1)}} x^* A_{i_e(v) i_{\overline{v}}(v)},$$

where $e(v)$ and $e_{\overline{v}}(v)$ denote the inclusion and the outgoing edge meeting at $v$, respectively; this leads to and is augmented by the definition of $U(1)$-valued transition maps (written for $y \in O_{ij}$ with $(i, j) \in J_{\mathbf{LM}2}$)

$$G_{ij}[y] = \prod_{\overline{e} \in \overline{E}_{\Delta(S^1)}} e^{\frac{i}{\hbar} \int_{e} (y/v) \cdot A_{e_{\overline{e}}}} \prod_{\overline{v} \in \overline{W}_{\Delta(S^1)}} g_{i_{\overline{v}}(v) i_{\overline{v}}(v)}(y(v)) \cdot g_{i_{\overline{v}}(v) i_{\overline{v}}(v)}(y(v))^{-1},$$

in which the $\overline{e}$ are edges and the $\overline{v}$ are vertices of the triangulation $\overline{\Delta}(S^1)$ described above. As previously, the incoming (resp. outgoing) edge of $\overline{\Delta}(S^1)$ at the vertex $\overline{v}$ is denoted by $\overline{e}(\overline{v})$ (resp. $\overline{e}(\overline{v})$).

The data satisfy the standard cohomological identities (written for $(i, j, l) \in J_{\mathbf{LM}3}$)

$$(E_i - E_l)|_{O_{ij}} = i \delta \log G_{ij}, \quad (G_{it} \cdot G_{it}^{-1} \cdot G_{ij})|_{O_{ij}} = 1.$$  

Under a gauge transformation of the gerbe $G$ with local data $(C_i, h_{ijk})_{i \in \mathcal{F}, (j, k) \in \mathcal{F}_2}$ of Eq. (2.2), the local connection 1-forms undergo the induced gauge transformation

$$E_i \mapsto E_i - i \delta \log H_i,$$

where

$$H_i[x] = \prod_{e \in E_{\Delta(S^1)}} e^{\frac{i}{\hbar} \int_{e} (x) v \cdot C_{e}} \prod_{v \in W_{\Delta(S^1)}} h_{i_{\overline{v}}(v) i_{\overline{v}}(v)}(x(v))^{-1}.$$  

With the help of these data, we define those of the pre-quantum bundle over elements $O_i^* \equiv \pi_{\text{TM}}^* O_i$ of the pullback cover, to wit, the local symplectic potentials

$$\theta_{\Delta} = \theta_{\text{TM}}|_{O_i^*} + \pi_{\text{TM}}^* E_i$$

and the corresponding gluing maps

$$\gamma_{\sigma} : \pi_{\text{TM}}^* G_{ij}.$$  

The construction of the transgression bundle is a key step towards Dirac’s geometric quantisation of the model in what can be regarded as an explicit realisation of Segal’s abstract categorical quantisation paradigm. In it, the Hilbert space

$$\mathcal{H}_\sigma := \Gamma_{\text{pol}}(Z_\sigma)$$

assigned to a loop is the space of suitably polarised sections of the pre-quantum bundle on which hamiltonians are realised as (certain sections of the sheaf of) 1st-order differential operators. The Dirac–Feynman amplitudes for surfaces $\Sigma$ with boundaries are now readily seen to play the rôle of (linear) transport operators between Hilbert spaces assigned to the cobordant loops of $\Sigma - \partial$.  

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Ref. [Gaw88], but also Ref. [Sus11a] for more details. In this picture, a wave functional $\Psi \in \Gamma_{\text{pol}}(\mathcal{L}_\sigma)$ in the position polarisation admits – at least formally – a path-integral presentation

$$\Psi[\phi] = \int_{x_1 \in \Sigma_{\text{in}}} \mathcal{D}x e^{iS_r[x]}$$

written for a worldsheet $\Sigma_{\text{in}} = \mathbb{D}^2$, parameterising the trajectory of an ‘incoming’ state, cp. Ref. [Gaw88] (such formal expressions are also considered in the framework of perturbative quantisation of a lagrangian field theory, cp. Ref. [CMR12]).

The above considerations pave the way to a systematic analysis of symmetries of the $\sigma$-model, both local (or gauge) and global (or rigid). Vector fields on $\mathcal{P}_\sigma$ whose flows realise the former span the kernel of $\Omega_\sigma$, cp. Ref. [Gaw72]. Among those of the latter kind, we find canonical lifts $\tilde{\mathcal{K}} \in \Gamma(\mathcal{T}\mathcal{P}_\sigma)$, from $M$ to $\mathcal{P}_\sigma \equiv \mathcal{T}^* L M$, of fundamental vector fields $\mathcal{K} \in \Gamma(TM)$ associated with (left) automorphisms of the (typical) fibre $M$ of the covariant configuration bundle $\Sigma \times M$ of the $\sigma$-model. As was demonstrated in Ref. [GSW10], they come from Killing vector fields $\mathcal{K}$ of the target-space metric $g$ that satisfy the strong invariance condition \(^{(2.9)}\)

$$\mathcal{K} \cdot \mathcal{H} = -d\kappa$$

for some $\kappa \in \Omega^1(M)$. Vector fields satisfying condition \(^{(2.9)}\) (and its generalisations in which the 3-form $\mathcal{H}$ is replaced by an arbitrary closed $(p+2)$-form on the target space) will be called generalised hamiltonian with respect to $\mathcal{H}$, by analogy with Eq. \(^{(2.4)}\). They span a Lie subalgebra within the Lie algebra $(\Gamma(TM), [, ,])$ of smooth vector fields on $M$ which we denote as

$$\mathfrak{g}_\sigma = \{ \mathcal{K} \in \Gamma(TM) \mid \mathcal{L}_{\mathcal{K}} g = 0 \land \exists_{\kappa \in \Omega^1(M)} : \mathcal{K} \cdot \mathcal{H} = -d\kappa \}_\mathbb{R}$$

in what follows. Their lifts are determined by the strong equivariance condition

$$\mathcal{L}_{\tilde{\mathcal{K}}} \theta_{\mathcal{T}^* L M} = 0,$$

which we may think of as the condition of preservation of the canonical connection 1-form $\theta_{\mathcal{T}^* L M}$ on $\mathcal{T}^* L M$, and so they take the form

$$\tilde{\mathcal{K}}[x, p] = \int_{\mathcal{E}} \text{Vol} (\mathcal{E}) \left( \mathcal{K}^\sigma(x) \left( \frac{d}{dx} \kappa(x) \right) - p_a(\cdot) \left( \frac{d}{dx} \kappa_a(x) \right) \right).$$

We shall denote the $\mathbb{R}$-linear span of all pairs $(\mathcal{K}, \kappa)$ described above as

$$\mathfrak{g}_\sigma = \{ \mathcal{K} \in \Gamma(TM) \mid \mathcal{L}_{\mathcal{K}} g = 0 \land \mathcal{K} \cdot \mathcal{H} = -d\kappa \}_\mathbb{R}.$$

It forms an algebra (over $\mathbb{R}$) with respect to the skew Vinogradov-type bracket, twisted (in the sense of Severa–Weinstein, cp. Ref. [SW01]) by the 3-form $\mathcal{H}$,

$$\langle \cdot, \cdot \rangle^\mathcal{H}_{\mathfrak{g}_\sigma} : \mathfrak{g}_\sigma \times \mathfrak{g}_\sigma \longrightarrow \mathfrak{g}_\sigma$$

$$\langle (\mathcal{K}_1, \kappa_1), (\mathcal{K}_2, \kappa_2) \rangle \longrightarrow \langle (\mathcal{K}_1, \kappa_2), \mathcal{L}_{\mathcal{K}_1} \kappa_2 - \mathcal{L}_{\mathcal{K}_2} \kappa_1 - \frac{1}{2} d (\mathcal{K}_1 \cdot \kappa_2 - \mathcal{K}_2 \cdot \kappa_1) + \mathcal{K}_1 \cdot \mathcal{K}_2 \cdot \mathcal{H} \rangle,$$

a fact first noted in Ref. [AS05] and subsequently generalised (to the polyphase setting) and exploited in Ref. [Sus11b]. This structure may equivalently be understood as coming from the standard (i.e., untwisted) Courant bracket on Hitchin’s generalised tangent bundle $\mathcal{T}^{1,1} M \equiv TM \oplus_{M, R} T^* M \rightarrow M$ twisted by the Čech–de Rham data of the gerbe geometrising $\mathcal{H}$. This interpretation of the Severa–Weinstein twist was originally advanced in Ref. [Hit00] and elaborated in Ref. [Sus12]. The physical meaning of the algebra is revealed through the construction of the Noether currents ($\tilde{\mathcal{I}}$ is the normalised tangent vector field on $S^1$)

$$J_{\mathcal{R}}(x) = \mathcal{K}^\sigma(x) p_a(\cdot) + (x, \tilde{\mathcal{I}})^a(\cdot) \kappa_a(x(\cdot))$$

of the theory. These are (spatial) densities of the standard Noether hamiltonians (or charges) $Q_\mathcal{R}$ of the symmetry,

$$Q_\mathcal{R} = \int_{S^1} \text{Vol}(S^1) \mathcal{L}_{\tilde{\mathcal{I}}} J_{\mathcal{R}}(\cdot) \equiv \tilde{\mathcal{K}} \cdot \mathcal{H} + \tilde{\mathcal{K}},$$

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\(^{10}\) In the case of target manifolds given by homogeneous spaces of Lie groups, the formal construction can be concretised with the help of an invariant Haar measure, cp. , e.g., Refs. [FGK88], [Gaw99], and one may anticipate that an analogous construction works for homogeneous spaces of Lie supergroups, cp. Ref. [WC84].

\(^{11}\) cp. , e.g., Ref. [HM97, Sec. 4B].

\(^{12}\) The condition implies the weaker one: $\mathcal{L}_{\mathcal{K}} \mathcal{H} = 0$, and the latter integrates to the invariance condition $\ell^* \mathcal{H} = \mathcal{H}$ for the action $\ell : G_\sigma \times M \longrightarrow M$ of (the connected component of) the global-symmetry group $G_\sigma \ni g$ of the $\sigma$-model.
defined in terms of the canonical lift (2.10) of the vector field, and that of the 1-form $\kappa$,
$$\vec{\kappa}[x,p] = \int_{\mathbb{R}_1} e^{\gamma^* \kappa},$$
to the space of states $T^*LM$, the former satisfying the standard hamiltonian relation
$$\vec{\kappa} \dual \Omega_\sigma = -\delta Q_R.$$
These furnish an anomalous\[1\] field-theoretic realisation of $\mathfrak{g}_\sigma$, of the simple form ($t$ and $\phi$ are – respectively – the time and space coordinate on $\Sigma$)
$$\{J_{\mathcal{R}_1}(t,\phi), J_{\mathcal{R}_2}(t,\phi')\}_{\Omega_\sigma} = J_{[\mathcal{R}_1, \mathcal{R}_2]}(t,\phi) \delta(\phi - \phi') - 2\mathcal{R}_1 \mathcal{R}_2 \left(t, \frac{1}{2}(\phi + \phi')\right) \delta'(\phi - \phi'),$$
(2.11)
in which
$$\langle \cdot, \cdot \rangle : \Gamma(T^{1,1}M)^{\times 2} \rightarrow T^{1,1}M : \left( (\mathcal{V}_1, \omega_1), (\mathcal{V}_2, \omega_2) \right) \mapsto \frac{1}{2} \left( \mathcal{V}_1 \delta \omega_2 + \mathcal{V}_2 \delta \omega_1 \right)$$
is a natural non-degenerate pairing on $\Gamma(T^{1,1}M)$. Finally, in the geometric quantisation scheme distinguished by the geometric data of the field theory in hand, the canonical lifts (2.10) and the Noether hamiltonians jointly determine their quantum-mechanical counterparts, with restrictions
$$\widetilde{Q}_R|_{\mathcal{O}_1} = -i \mathcal{L}_\vec{\kappa} - \vec{\kappa} \dual \theta_\sigma + Q_R, \quad i \in \mathfrak{g}_{LM}.$$  

Passing from the infinitesimal to the global level of realisation of $\sigma$-model symmetries, we are led to distinguish between the rigid and gauge symmetries. Denote the action of the symmetry Lie group $G_\sigma$ with the Lie algebra $\mathfrak{g}_\sigma$ on the manifold $M$ as
$$\ell : G_\sigma \times M \rightarrow M.$$  
The rigid symmetries are neatly captured by families, indexed by the symmetry group $G_\sigma$, of gerbe 1-isomorphisms
$$\Phi_g : \ell^* G \xrightarrow{\cong} G, \quad g \in G_\sigma$$
that transgress to automorphisms of the (pre)quantum bundle. The 1-isomorphisms can be regarded as geometrisations of the invariance condition (2.9). In the framework of the local(ised) field theory, we are led to demand that a global symmetry of a given field-theoretic model, which can be interpreted passively as invariance of the model under certain distinguished transformations of the reference system (in the space of internal degrees of freedom), and hence also as equivalence between its distinguished classical configurations, be promoted to a local one, or gauged – this is, morally, the content of the universal gauge principle. The rôle of the symmetry algebra $(\mathfrak{g}_\sigma, [\cdot, \cdot]_{\mathcal{V}})$ in the description of the gauging of the global-symmetry group $G_\sigma$ was clarified by the author in Ref. [Sus12], cp. also Ref. [GSW13]. Thus, $G_\sigma$ can be gauged only if the extension of the $H$-twisted Vinogradov bracket $[\cdot, \cdot]_{\mathcal{V}}$ to the $C^\infty(M, \mathbb{R})$-linear span of an arbitrary basis $(\mathcal{R}_A = (\mathcal{K}_A, \mathcal{K}_A))_{A \in \mathfrak{A}_{\text{dim}G_\sigma}}$ of $\mathfrak{g}_\sigma$ determined by a basis
$${\{L_A\}}_{A \in \mathfrak{A}_{\text{dim}G_\sigma}}$$
of the Lie algebra $\mathfrak{g}_\sigma$, for which we have
$$\mathcal{K}_A(x) \equiv \frac{d}{dt} \big|_{t=0} \ell^*_t \mathcal{K}_A(x), \quad x \in M,$$
defines a Lie algebroid over $M$ (with the obvious anchor $\text{pr}_1 : \mathfrak{A}_{\text{dim}G_\sigma} \rightarrow \Gamma(TM)$), which then turns out to be isomorphic with the action algebroid $\mathfrak{g}_\sigma \ltimes M$, that is with the tangent Lie algebroid of the action groupoid $G_\sigma \ltimes M$. The appearance of the latter in the present context is by no means a coincidence – indeed, the groupoid of principal bundles with $G_\sigma \ltimes M$ as the structure groupoid (cp. Ref. [MM03]) was shown in Ref. [Sus12] to naturally quantify the data of the relevant gauged $\sigma$-model, i.e., the choice of the principal bundle $P_{G_\sigma} \rightarrow \Sigma$ with the structure group $G_\sigma$ and a choice of a global section of the associated bundle $P_{G_\sigma \times M} \equiv (P_{G_\sigma} \times M)/G_\sigma$, the latter being identified with a lagrangean field of the gauged $\sigma$-model. Furthermore, it is over the nerve
$$\cdots \xrightarrow{\delta^{(1)}_{-1}} G_\sigma \times M \xrightarrow{\delta^{(2)}} G_\sigma \ltimes M \xrightarrow{\delta^{(3)}} M$$
of the small category $G_\sigma \ltimes M$, with face maps (written for $x \in M, \ g, g_k \in G_\sigma, \ k \in \mathbb{N}$ with $m \in \mathbb{N}$)
$$d^{(1)}_0(g, x) = x \equiv \text{pr}_1(g, x), \quad d^{(1)}_1(g, x) = \ell_g(x),$$
\[1\]Note the purely classical nature of the anomaly in question.
\[d_0^{(m)}(g_m, g_{m-1}, \ldots, g_1, x) = (g_{m-1}, g_{m-2}, \ldots, g_1, x),\]
\[d_1^{(m)}(g_m, g_{m-1}, \ldots, g_1, x) = (g_{m-1}, \ldots, g_2, \ell_{g_i}(x)),\]
\[d_1^{(m)}(g_{m-1}, \ldots, g_1, x) = (g_{m-1}, \ldots, g_{m+2-i}, g_{m+1-i}, g_{m-i}, g_{m-1-i}, \ldots, g_1, x), \quad i \in \mathbb{Z}, m \geq 1,\]

that the full-fledged gauging procedure was developed in Refs. [GSW 10, GSW 13] and ultimately justified, in its structural form proposed by the authors, in terms of a generalised worldsheet gauge-defect construction in Ref. [Sus 12]. The necessary and sufficient condition for the said procedure to work is the existence of a \(G_\s\text{-equivariant structure}\) on the gerbe \(\mathcal{G}\) of the \(\sigma\)-model, composed of a 1-isomorphism

\[\mathcal{Y} : d_1^{(1)} \ast \mathcal{G} \xrightarrow{\sim} d_0^{(1)} \ast \mathcal{G} \otimes \mathcal{I}_{\rho_{\theta_L}},\]

of gerbes over the arrow manifold \(G_\s \times M\) of the action groupoid, written in terms of the distinguished 2-form

\[\rho_{\theta_L} = \text{pr}_2^* \kappa_A \wedge \text{pr}_1^* \theta_L^A - \frac{1}{2} \text{pr}_1^* (\kappa_A \ast \kappa_B) \text{pr}_1^* (\theta_L^A \wedge \theta_L^B)\]

in whose definition \(\theta_L = \theta_L^A \otimes_R t_A\) is the standard \(g_\s\)-valued left-invariant Maurer–Cartan 1-form on \(G_\s\), and of a 2-isomorphism

\[(d_1^{(1)} \circ d_1^{(2)}) \ast \mathcal{G} \xrightarrow{d_2^{(3)} \ast \mathcal{Y}} (d_0^{(1)} \circ d_0^{(2)}) \ast \mathcal{G} \otimes \mathcal{I}_{d_2^{(3)} \ast \rho_{\theta_L}},\]

between the 1-isomorphisms over \(G_\s^2 \times M\), satisfying, over \(G_\s^3 \times M\), the coherence condition

\[d_1^{(3)} \ast \gamma \circ (d_0^{(2)} \ast \text{id}_{\mathcal{I}_{d_2^{(3)} \ast \rho_{\theta_L}}}) \circ (d_1^{(2)} \ast \gamma) = (d_1^{(3)} \ast \gamma \circ \text{id}_{\mathcal{I}_{d_2^{(3)} \ast \rho_{\theta_L}}}) \circ (d_0^{(2)} \ast \text{id}_{\mathcal{I}_{d_2^{(3)} \ast \rho_{\theta_L}}}) \circ (d_1^{(2)} \ast \gamma).\]

(2.12)

The structure ensures that a suitable extension of the pullback gerbe \(\text{pr}_2^* \mathcal{G}\) over \(P_{G_\s} \times M\), determined by the choice of the gauge bundle \(P_{G_\s}\), descends to the covariant configuration bundle \((P_{G_\s} \times M)/G_\s\) of the gauged \(\sigma\)-model and thus enables the definition of the Wess–Zumino term. Alternatively, it provides the necessary and sufficient data for an arbitrary topological gauge-defect embedded in the worldsheet that implements the gauge symmetry, *cf* Ref. [Sus 12]. Whenever the action \(\ell\) is free and proper, so that the orbit space \(M/G_\s\) carries the structure of a manifold, all this implies that the \(\sigma\)-model descends to the orbit manifold \(M/G_\s\) in that it determines a \(\sigma\)-model with the latter as a target space with a metric and a gerbe over it, and equivalence classes of such descended \(\sigma\)-models are essentially enumerated by inequivalent \(G_\s\)-equivariant structures on \(\mathcal{G}\). If the said conditions are not satisfied, on the other hand, it makes sense to regard the gauged \(\sigma\)-model as the definition of the induced loop mechanics on the space \(M/G_\s\) − indeed, it is defined over a manifold directly related to the homotopy \((G_\s)\)-quotient of \(M\).

Under the assumption of the existence of a measure \(\mathcal{L}\) over the space \(x : \Omega \rightarrow M\) invariant under the induced action \((g_x : \Omega) \rightarrow \ell_g \circ x\), the above presentation enables us to discuss quantum lifts of (global) symmetries of the classical theory in an explicit manner. Indeed, let the induced action preserve the lagrangian density of the \(\sigma\)-model up to a total derivative (which is necessary for the action functional for the closed worldsheet to remain invariant under symmetry transformations),

\[\mathcal{L}_\s (\ell_g \circ x, \partial (\ell_g \circ x)) \text{Vol}(\Omega) - \mathcal{L}_\s (x, \partial x) \text{Vol}(\Omega) = dJ_g(x, \partial x),\]

for some \(J_g(x, \partial x) \in \Omega^p(\Omega)\). We then obtain the induced realisation of \(G\) on the quantum space of states in the form

\[(R(g)\Psi)[\phi] := \int_{\Omega} \mathcal{D}(\ell_g \circ x) e^{iS_{\s}\left[\ell_g \circ x\right]} = \int_{\Omega} \mathcal{D} x e^{iS_{\s}\left[ x \right]} e^{i J_g(x, \partial x)} .\]

If, furthermore,

\[J_g = x^* J_g,\]

(2.13)
for some target symmetry current $j_g \in \Omega^1(M)$, then we may rewrite the above definition as

$$R(g)\Psi[\phi] = c_g[\phi] \cdot \Psi[\ell_g^{-1} \circ \phi], \quad c_g[\phi] := e^{i \int_{\Omega_m} (\ell_g^{-1} \circ \phi)^* j_g}.$$  

Thus, to a realisation of the classical (global-)symmetry group on the quantum space of states, there is associated an action \textbf{1-cochain} on $G$ with values in $U(1)$-valued functionals on the classical space of states. The space of such functionals carres the structure of a $G$-module with a (left) $G$-action

$$\langle g_2 \triangleright c_g \rangle[\phi] := c_g[\ell_{g_2^{-1}} \circ \phi].$$

In order to have an actual representation of the symmetry group on quantum states, we must demand that the 1-cochain be a 1-cocycle. Indeed, we have

$$(R(g_1) \circ R(g_2))\Psi[\phi] = c_{g_2}[\phi] \cdot (R(g_2))\Psi[\ell_{g_1^{-1}} \circ \phi] = c_{g_2}[\phi] \cdot c_{g_1}[\ell_{g_1^{-1}} \circ \phi] \cdot \Psi[\ell_{g_2^{-1}} \circ \ell_{g_1^{-1}} \circ \phi]$$

$$= c_{g_1}[\phi] \cdot c_{g_2}[\ell_{g_1^{-1}} \circ \phi] \cdot \Psi[\ell_{(g_1 g_2)^{-1}} \circ \phi]$$

$$= (\delta_G c)_{g_1, g_2}[\phi] \cdot (R(g_1 \cdot g_2))\Psi[\phi]$$

with the homomorphism 2-cocycle

$$(\delta_G c)_{g_1, g_2}[\phi] = c_{g_2}[\ell_{g_1^{-1}} \circ \phi] \cdot c_{g_1}[\ell_{g_2^{-1}} \circ \phi]^{-1} \cdot c_{g_1}[\phi]$$

$$= e^{i \int_{\Omega_m} (\ell_{g_2^{-1}} \circ (\ell_{g_1^{-1}} \circ \phi))^{*} j_{g_2^{-1}} \cdot (\ell_{(g_1 g_2)^{-1}} \circ \phi)^* j_{(g_1 g_2)^{-1}} \cdot (\ell_{g_1^{-1}} \circ \phi)^* j_{g_1}}$$

$$= e^{i \int_{\Omega_m} (\ell_{(g_1 g_2)^{-1}} \circ \phi)^* (j_{g_2^{-1}} - j_{g_1 g_2} + j_{g_1})},$$

the latter being determined by the \textbf{current 2-cocycle}

$$(\delta_G d)_{g_1, g_2, g_3} := d_{g_1 \cdot g_2} \cdot d_{g_1 \cdot g_2 \cdot g_3}^{-1} \cdot d_{g_1 \cdot g_2} \cdot d_{g_2 \cdot g_3}^{-1} \cdot d_{g_2 \cdot g_3} = 1$$

for arbitrary $g_1, g_2, g_3 \in G$. Indeed, given such a 2-cocycle, we may define a standard action of the central extension

$$1 \to U(1) \to \tilde{G} := G \rtimes U(1) \to G \to 1$$

of the symmetry group $G$, with the group operation determined by the 2-cocycle as

$$G \times \tilde{G} \to \tilde{G} : (g_1, u_1), (g_2, u_2) \mapsto (g_1 \cdot g_2, d_{g_1 \cdot g_2} \cdot u_1 \cdot u_2).$$

The action is given by the formula

$$R(g, u)\Psi[\phi] := u \cdot (R(g))\Psi[\phi].$$

These considerations will play an important rôle in the fundamental construction developed in the present article, that is in the (super)geometrisation scheme for \textit{supergroup-invariant} de Rham cohomology of super-$\sigma$-model targets.

3. Tensorial super-$\sigma$-Model Backgrounds – Generalities

We shall be concerned with the by now well-established Green–Schwarz-type models of dynamics of extended supersymmetric objects, also known as \textbf{super-$p$-branes}, whose classical configurations are generalised superharmonic embeddings $\Omega \to \mathcal{M}$ of the \textbf{worldvolume} $\Omega$, a standard manifold of dimension $p+1 \in \mathbb{N}$ (parametrising the history of a charged point-like particle, loop, membrane etc.) equipped with an intrinsic metric $\gamma$, in a \textbf{target supermanifold} $\mathcal{M}$, to be termed the \textbf{supertarget} in what follows. A general (real) supermanifold is a ringed space $\mathcal{M} = (M, \mathcal{O}_M)$ composed of a (second countable Hausdorff) topological space $M$ (termed the \textbf{body} of $M$) and a sheaf $\mathcal{O}_M$ of (real) associative unital superalgebras on $M$ (termed the \textbf{structure sheaf} and to be thought of as a generalisation of the sheaf of real functions on a manifold), locally modelled on $(\mathbb{R}^{\times_m}, C^\infty(\cdot, \mathbb{R}) \otimes_{\mathbb{R}} \Lambda^\bullet \mathbb{R}^n)$ – here, the pair $(m, n)$, constant over the entire $M$, is the so-called superdimension of $M$. The global geometry of
such a structure is identified in the fundamental Batchelor–Gawędzki Theorem of Refs. \cite{Gaw77, Bat79} which states that $\mathcal{M}$ is (globally) isomorphic with the ringed space $(\mathcal{M}, \Gamma(\mathcal{N}^\bullet V))$ for $(\mathcal{V}, \mathcal{M}, \mathcal{N}_\mathcal{V}, \mathbb{R}^{n\times})$ a real vector bundle of rank $n$ over the body. Supermanifolds admit (local) coordinate descriptions, and in this work we shall deal exclusively with supermanifolds with \textit{global} coordinate systems, so that there will be no need for the abstract theory of supermanifolds. The presence of global coordinate systems helps to simplify our treatment of the differential calculus on the supermanifolds of interest, which will be seen to play an instrumental rôle in the field-theoretic constructions. Thus, we shall use the fact that the tangent sheaf $\mathcal{T} \mathcal{M} \equiv s\text{Der}(\mathcal{O}_\mathcal{M})$ of superderivations of the structure sheaf (whose sections are to be thought of as (super)vector fields on $\mathcal{M}$), as well as the dual cotangent sheaf $\mathcal{T}^* \mathcal{M} \equiv \text{Hom}_{\text{Mod}_{\text{Cl}(\mathcal{O}_\mathcal{M})}}(\mathcal{T} \mathcal{M}, \mathcal{O}_\mathcal{M})$ (whose sections are to be thought of as super-1-forms on $\mathcal{M}$) are in general locally, and in our case also globally free, with generators given by – respectively – coordinate superderivations and coordinate differentials. All this will enable us to develop our discussion in a far-reaching structural analogy with the standard \textit{(i.e., commutative-)geometric approach to $\sigma$-models, with the graded nature of the geometry under consideration reflected solely in the elementary sign conventions tabulated in Conv. A.2).

Passing to the supergeometries of interest, we shall further assume the supermanifold to be endowed with a (left) transitive action of a Lie supergroup $G$ \textit{(i.e., a group object in the category of supermanifolds)}, the latter playing the rôle of the (extended) global-(super)symmetry group of the field theory in question. As such the supertarget will be presentable as (or equivariantly superdiffeomorphic with) a supercoset $G/H \equiv \mathcal{M}$ of that supergroup relative to a Lie group $H$ embedded in $G$. Such a presentation of the target supermanifold puts us in the framework of Cartan geometry, which, in turn, affords a neat description of the additional tensorial \textbf{superbackground} (typically degenerate in the ramiﬁcation-theory of the $\sigma$-model) and a left-$G$-invariant de Rham super-$(p + 2)$-cocycle $\chi \in Z_p^{1\times2}(\mathcal{M})^G$. Thus, we construct the action functionals of the models of interest in terms of (left-$G$-equivariant components of the left-invariant Maurer–Cartan 1-form $\theta^\gamma_T$ on $G$ with values in the Lie superalgebra (\textit{cp. App. C}) $\mathfrak{g}$ of that supergroup as well as invarient tensors on the latter, and the lagrangean fields of the theory are identified with the Riemann normal (super)coordinates on $G$ restricted to a section $\gamma \in \Gamma(G)$ of the principal $H$-bundle $G \rightarrow G/H$ modelling the supertarget. In fact, the specific choice of the section and the ensuing treatment of the geometric Goldstone fields coordinatising some of the directions in the complement of the Lie algebra $\mathfrak{h}$ of the Lie group $H$ within the Lie superalgebra $\mathfrak{g}$ has – as has been amply demonstrated in the literature \textit{(cp., \textit{e.g.}, Refs. \cite{McA04, GK06})}, and shall be elaborated in what follows – far-reaching consequences for the geometrisation of the theory \textit{(in the spirit of the previous section)}.

Taking into account the structure of the super-Poincaré algebra that serves as the local model for the geometries under consideration, as well as that of the distinguished anti-de Sitter superbackgrounds to be explored in subsequent studies, we shall restrict our attention to the so-called \textbf{reductive} homogeneous spaces $G/H$, \textit{i.e., those for which the direct-sum (supervector-space) complement $\mathfrak{m}$ of the Lie algebra $\mathfrak{h}$ within ($3.1$)} $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ has the $\mathfrak{h}$-module property

\[
[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.
\]

The prototype of the said structure is an extension of the super-point algebra of anti-commuting \textbf{supercharges} $Q_{\alpha a}, (I, \alpha) \in \mathbb{T} \mathcal{N} \times \mathcal{D}$ (with $N$ denoting the number of supersymmetries, \textit{i.e., of distinct Majorana spinors entering the definition of the relevant GS model, and $D$ the dimension of the respective representation of the underlying Clifford algebra, the two numbers being constrained severely by the requirement of existence of the corresponding GS model}) by the algebra of Grassmann-even translations $P_a$, $a \in \mathbb{N}_{0, d-1}$ ($d$ is the spacetime dimension of the supertarget), further enhanced – as a spinor/vector-module algebra – by the Lorentz algebra $\mathfrak{t}$ with generators $J_{ab} = J_{\{ab\}}$, $a, b \in \mathbb{N}_{0, d-1}$ to form the Lie superalgebra $\mathfrak{g}$ with the defining supercommutation relations

\[
\begin{align*}
{Q}_{I \alpha a}, {Q}_{J \beta b} &= f_{I \alpha a, J \beta b} P_a + f_{I \alpha a, J \beta b} J_{ab}, & \quad [Q_{I \alpha a}, P_b] &= f_{I \alpha a, J \beta b} Q_{J \beta b}, & \quad [P_a, P_b] &= f_{ab, cd} J_{cd}, \\
\end{align*}
\]

\[
[J_{ab}, J_{cd}] = \eta_{ad} J_{bc} - \eta_{ac} J_{bd} + \eta_{bc} J_{ad} - \eta_{bd} J_{ac},
\]

Further enhancements are possible and, indeed, physically relevant, \textit{e.g.,} by generators of dilations and special \textbf{conformal transformations} in the supersymmetric anti-de Sitter setting.
written in terms of the Minkowskian metric $\eta = \text{diag}(1, -1, -1, \ldots, -1)$ on the body of the supertarget, and in terms of the generators $\Gamma_a$, $a \in 0, d-1$ of the corresponding Clifford algebra, cp. App. A. For $N = 1$, the choice of the structure constants

$$f_{\alpha, \beta}^a = \Gamma_{a, \beta}, \quad f_{a, \alpha}^b = \frac{\lambda_a}{R} (\Gamma^{[a})_{\alpha}^{b}, \quad f_{\alpha, a}^\beta = \frac{\lambda_a}{R} (\Gamma_a)^\beta_{\alpha}, \quad f_{a, b}^c = \frac{\lambda_a}{R^2}$$

yields, for certain (normalisation-dependent) values of the numerical constants $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\lambda_3 \in \mathbb{R}_{>0}$, the $d = 4$ super-anti-de Sitter algebra at radius $R \in \mathbb{R}_{>0}$ of Ref. [HVP82] (in the notation of Ref. [FVP92]), and reduces, via the Inönü–Wigner contraction $R \to \infty$, to the standard $(N = 1)$ super-Minkowski algebra

$$f_{\alpha, \beta}^a = \Gamma_{a, \beta}, \quad f_{a, \alpha}^b = \frac{\lambda_a}{R} (\Gamma^{[a})_{\alpha}^{b}, \quad f_{\alpha, a}^\beta = \frac{\lambda_a}{R} (\Gamma_a)^\beta_{\alpha}, \quad f_{a, b}^c = 0.$$

In this setting, $\mathfrak{h}$ is the Lie algebra of a Lie subgroup $H = \text{Spin}(1, p) \times \text{Spin}(d-p-1) \subset \text{Spin}(1, d-1)$ that is preserved in the presence of a classical configuration (embedding) of the extended super-$p$-brane in the supertarget or its (super)extension by the distinguished (super)translations that leave the chosen action functional invariant, cp. below. Given decomposition (3.1) and the corresponding decomposition of the set $\{t_A\}_{a \in \text{dim}_{\mathbb{C}} \mathfrak{g}}$ into subsets: $\{t_A\}_{a \in \text{dim}_{\mathbb{R}} \mathfrak{g}}$ and $\{t_A\}_{a \in \text{dim}_{\mathbb{C}} \mathfrak{g}}$, spanning respectively the $\mathfrak{h}$ and $\mathfrak{m}$, we may write the Maurer–Cartan 1-form as

$$(3.2) \quad \theta_L = \theta_{\mathfrak{h}}^a \otimes_R t_A = \theta_{\mathfrak{h}}^a \otimes_R t_A + \theta_{\mathfrak{m}}^a \otimes_R t_A$$

and subsequently formulate the physical theory of interest in terms of the $\theta_{\mathfrak{h}}^a$.

With all the ingredients in place, we may finally write down the Dirac–Feynman amplitude for the maps $X : \Sigma \to G$ of the field theory of interest, embedding the worldsheet $\Sigma$ within a section of the principal bundle $G \to \text{G/H}$ introduced earlier. Generically, it takes the familiar form

$$(3.3) \quad A_{\text{DF, GS, p}}[X] = \exp(\{i S_{\text{metr, GS, p}}[X]\}) \cdot \exp\left(i \int_\Omega X^* (d^{-1} \chi) \right)$$

in which the first factor $S_{\text{metr, GS}}$ computes the metric $(p + 1)$-volume of the embedded hypersurface $X(\Omega)$ measured in the metric induced from $g$ along $X$ on the worldvolume $\Omega$ and can assume various forms, depending on the choice of the supertarget $\mathcal{M}$ (or, more to the point, on the choice of the embedding of the physical spacetime in $\mathcal{M}$ – cp. below), and in which the second factor, tentatively representing the geometric coupling of the external field $\chi$ to the charge current defined by the propagation of the super-$p$-brane in $\mathcal{M}$, is locally (over $\mathcal{M}$) expressed as the integral

$$(\text{3.4}) \quad S_{\text{top, GS, p}}[\chi] = \int_\Omega X^* (d^{-1} \chi) \biggr|_{\text{loc}} \int_\Omega X^* \chi \biggr|_{(p + 1)}$$

of a (local) de Rham primitive $\beta$ of $\chi$. In the most studied examples, i.e., on supersets of the super-Minkowskian type (body) $\mathbb{R}^{1, d-1}$ (cp. Refs. [BSS94, GS84a, BST86, HLP86, AETW87]) and of the super-anti-de Sitter type $\text{AdS}_{p+2} \times \mathbb{S}^{d-p-2}$, $p \in 0, d-2$, $d \in \{10, 11\}$ (cp. Refs. [MT98, AF08, dWPPS98, Cl99]), the primitives of various physically relevant GS $p+2$-cocycles exist (although in the latter class, they are often given in an implicit integral form), and yet they are typically not supersymmetric (or, in our language, not left-G-invariant), cp. Refs. [CdAIPB00, Sak00] and Refs. [HS04, HS02] for interesting analyses in the super-Minkowskian and super-anti-de Sitter context, respectively.

Hence, in an attempt to grasp the geometric meaning and thereupon properly define the WZ term, we arrive at a crossroads – we are confronted with the choice between the standard de Rham cohomology of the supertarget $\mathcal{M}$ and the G-invariant (or at the very least supersymmetry-invariant) de Rham cohomology of the same space. In many examples of $\sigma$-models with standard (i.e., Graßmann–even) manifolds as targets, there is either no distinguished symmetry group identified in the canonical analysis, or that group is compact, as is the case for the WZW $\sigma$-model on a compact Lie group – the mother of all rational field theories in two dimensions. In the former situation, the question of choice does not even come up, whereas in the latter one, it is answered by the Chevalley–Eilenberg Theorem of Ref. [CE48] that states the equivalence of the two cohomologies (that is, of the existence of an isomorphism of the corresponding cohomology groups). The supergeometric setting of interest does not fall into either category as the symmetry (super)group is built into the definition of the supertarget and the latter group is (assumed) non-compact, which precludes the application of the
Chevalley–Eilenberg Theorem. The universal gauge principle invoked in the previous section prompts us to demand that the global symmetry resp. the supersymmetry be rendered local, or gauged (or, at least, that it be possible to gauge it), so that – in the light of Refs. [GSW13, Sus12] – the super-σ-model may descend to the space of orbits. The problem with this line of reasoning is two-fold: first of all, already the descent to the space of orbits of the all-important supersymmetry transformations (understood as translations in the soul directions) takes us out of the original geometric category of supermanifolds, as illustrated in Ref. [RC85] – this is analogous to the descent to a non-smooth space of orbits of the action of a finite group on a smooth manifold; secondly, the purely geometric meaning of the G- resp. supersymmetry-invariant de Rham cohomology (not to be confused with the G- resp. supersymmetry-equivariant one) is not clear, which undermines the postulate of giving preference to that cohomology over the usual de Rham cohomology. The latter objection would be lifted if we could find a subgroup T \subset G (resp. that of the supersymmetry group) with the following properties:

(i) T-invariance of a differential form on \( \mathcal{M} \) implies its G-invariance (resp. its supersymmetry-invariance);
(ii) the orbit space \( \mathcal{M}/T \) is a supermanifold locally modelled on the Graßmann bundle of the same vector bundle over the body manifold of \( \mathcal{M} \) as that of the supermanifold \( \mathcal{M} \) itself.

The above properties would legitimise thinking of the original super-σ-model with the supertarget \( \mathcal{M} \) as one with the supertarget \( \mathcal{M}/T \) on which the GS super-cocycle defines – by construction – a non-trivial de Rham class, and this would, in turn, mean that the topology of \( \mathcal{M}/T \) encodes the non-trivial Chevalley–Eilenberg cohomology of \( \mathcal{M} \) and justify a geometrisation of the GS super-cocycle in a manner completely analogous with that employed in the Graßmann-even setting.

The existence of the relevant subgroup T \( \subset \text{gM} \) in the case of the super-Minkowski space was demonstrated explicitly by Rabin and Crane in Refs. [RC85, Rab87] and is therefore anticipated (but has to be proven on a case-by-case basis) on a generic supermanifold (of the type under consideration) in the light of the Gawȩdzki–Batchelor Theorem of Refs. [Gaw77, Bat79]. In the former setting, condition (ii) rules out the obvious candidate for T given by the full supersymmetry group – indeed, the resulting quotient is not of the same type as the original supermanifold. It is then readily checked that the Kostelecký–Rabin discrete supersymmetry subgroup of Ref. [KR84], to be thought of as a lattice variant of the continuous supergroup of supertranslations, is a suitable choice – it yields a supermanifold with the fundamental group generated by unital (in the lattice spacing) translations in the Graßmann-odd (or soul) directions, and the only nontrivial supercommutator in the underlying Lie superalgebra (the anticommutator of the supercharges) gives rise to a torsion component in the ensuing homology, cp. Ref. [RC85]. It is also worth noting that Witten’s trick\(^{15}\) does not work in this setting, cp. Ref. [Rab87], which is another reason to look for a geometrisation of the GS cocycle. We shall construct such a geometrisation explicitly in what follows. Since, moreover, we want to study its equivariance under actions of subgroups of G and subalgebras of \( \text{gM} \) induced from the underlying left- and right-regular actions of G on itself upon restriction to the section of the principal bundle \( G \to H \) referred to previously, it will be important to gain a better understanding of the induction scheme first.

Our introductory remarks concerning the general structure of the supertargets of interest essentially determine the nature of the action of the symmetry supergroup G to be considered, and so also – in particular – the implementation of supersymmetries. These will be realised nonlinearly in the scheme originally conceived by Schwinger and Weinberg in Refs. [Sch67, Wei68] in the context of effective field theories with chiral symmetries, and subsequently elaborated in Refs. [CWZ69, CCWZ69, SS69a], to be adapted to the study of spacetime symmetries by Salam, Strathdee and Isham in Refs. [SS69b, ISS71]. The scheme was successfully employed in the setting of a supersymmetric field theory by Akulov and Volkov et al. in Refs. [VA72, VA73, JKK78, LRT79, ZZ82, JS82, SW83, FMW83, BW84], and this is the variant that we encounter below. Thus, we take the Lie superalgebra \( \text{gM} \) of the symmetry supergroup G to decompose as

\[ \text{gM} = \mathfrak{t} \oplus \mathfrak{r} \]

into a Lie algebra \( \mathfrak{r} \subset [\mathfrak{r}, \mathfrak{r}] \) and its super-graded module \( \mathfrak{t} = \mathfrak{t}^{(0)} \oplus \mathfrak{t}^{(1)} \) (not a Lie superalgebra in general),

\[ [\mathfrak{r}, \mathfrak{t}] \subset \mathfrak{t}, \]

\(^{15}\)Defining the Wess–Zumino term for the σ-model on \( \mathcal{M}/T \) as an integral of the GS 3-cocycle over a filling 3-manifold (a solid handlebody) of the worldsheet Σ.
that further splits as
\[ \mathfrak{r} = \mathfrak{d} \oplus \mathfrak{r}_{\text{vac}} \]
into a Lie subalgebra \( \mathfrak{r}_{\text{vac}} \supset [\mathfrak{r}_{\text{vac}}, \mathfrak{r}_{\text{vac}}] \) and its vector-space complement \( \mathfrak{d} \) which together with a subspace \( \mathfrak{e} \subset \mathfrak{t} \) composes the super-vector space
\[ \mathfrak{e} \oplus \mathfrak{d} \equiv \mathfrak{m} \]
mentioned earlier. This leaves us with the direct-sum complement \( \mathfrak{t}_{\text{vac}} \subset \mathfrak{t} \) of \( \mathfrak{e} \) in \( \mathfrak{t} \) as the last ingredient in the definition of the vacuum-symmetry Lie superalgebra
\[ (3.4) \quad \mathfrak{t}_{\text{vac}} \oplus \mathfrak{r}_{\text{vac}} \equiv \mathfrak{h}. \]
and we demand that \( \mathfrak{t}_{\text{vac}} \) be an \( \mathfrak{r}_{\text{vac}} \)-module
\[ [\mathfrak{r}_{\text{vac}}, \mathfrak{t}_{\text{vac}}] \subset \mathfrak{t}_{\text{vac}}. \]
We shall denote the basis vectors (generators) of \( \mathfrak{t} \) as \( \{ t_\mu \}_{\mu \in \text{dim}_\mathfrak{t}}, \) and among them those of \( \mathfrak{t}^{(0)} \) as \( \{ t_I \}_{I \in \text{dim}_\mathfrak{t}^{(0)}}, \) and those of \( \mathfrak{t}^{(1)} \) as \( \{ t_a \}_{a \in \text{dim}_\mathfrak{t}^{(1)}}. \) We further divide the generators of \( \mathfrak{t}^{(0)} \) into those co-spanning \( \mathfrak{t}_{\text{vac}}, \) which we label as \( \{ t_A \}_{A \in [p, p+1]} \), and those co-spanning \( \mathfrak{e}, \) labelled as \( \{ t_\xi \}_{\xi \in \mathfrak{e}}, \) \( \{ t_\phi \}_{\phi \in \mathfrak{e}} \), and those co-spanning \( \mathfrak{r}_{\text{vac}}, \) as \( \{ t_\alpha \}_{\alpha \in \text{dim}_\mathfrak{r}_{\text{vac}}} \). In the specific examples listed above, \( \mathfrak{t} \) is the linear span of supertranslations, and so – in particular – it is promoted to the rank of a sub-Lie superalgebra in the Minkowskian setting.

Taking into account the reasoning presented in Refs. [West00, GKW06d] and the results derived therefrom, we introduce (local) coordinates \( (\xi^\mu, \phi_\alpha)(\mathfrak{g}) \) for \( \mathfrak{g} \) and subsequently parametrise the aforementioned section \( \gamma \in \Gamma(\mathfrak{g}) \) as
\[ \gamma(\xi, \phi) = e^{t_\mu \cdot t_\nu} \cdot e^{\phi_\alpha \cdot t_\alpha}. \]
Accordingly, the lagrangean field of the super-\( \sigma \)-model is of the form
\[ \mathcal{X} \equiv (\xi^\mu, \phi_\alpha)(\cdot) : \Omega \to \mathfrak{g}/\mathfrak{h}. \]
A word is due at this stage concerning the physical meaning of the various components of this field. Thus, the Graßmann-even components \( \{ x^I \equiv \xi^I \}_{I \in \text{dim}_\mathfrak{t}^{(0)}} \) are to be thought of as coordinates on a physical spacetime \( M \subset \mathcal{M} \) in which the super-\( p \)-brane propagates in a manner dictated by the Green–Schwarz super-\( \sigma \)-model – this explains the appearance of coordinates corresponding to vacuum-symmetry directions \( t_{\text{vac}} \) in the above parametrisation of the section. The Graßmann-odd components \( \{ \xi_\alpha \}_{\alpha \in \text{dim}_\mathfrak{r}_{\text{vac}}} \) map the spinorial (super-charge) directions. In both cases, fixing the components in the directions of \( \mathfrak{h} \) explicitly introduced in the parametrisation is tantamount to fixing the gauge of the local symmetry \( \mathfrak{h}. \) The remaining components \( \{ \phi^\alpha \}_{\alpha \in \text{dim}_\mathfrak{g}} \) model certain Goldstone (or pure-gauge) degrees of freedom that are ultimately to be integrated out with the help of their field equations and will be switched on or off depending on the geometric phenomena that we want to capture in our formulation of the dynamics. While purely auxiliary in nature, they serve the important purpose of elucidating the very structure of the super-\( \sigma \)-model and of the geometric mechanism (a variant of the so-called inverse Higgs effect of Ref. [O72]) that leads to the emergence of separate metric and topological contributions to its action functional out of a purely topological (or higher gerbe-theoretic) expression on the larger supermanifold coordinatised by the \( \xi^\mu \) and the \( \phi_\alpha. \) These remarks will be clarified further in Sec. 3.2 and when we pass to the analysis of concrete models.

Parenthetically, and with view to later applications, we note that the pullbacks of the Maurer–Cartan 1-forms can be decomposed in the (local) coordinate basis as
\[ (3.5) \quad \gamma^* \theta^A_\mu(\xi, \phi) = E^A_\mu(\xi, \phi) \, d\xi^\mu + E^A_\alpha(\xi, \phi) \, d\phi_\alpha. \]
The functional coefficients \( E^A_\mu \) and \( E^A_\alpha \) are traditionally termed Vielbeine. The important consequence of our assumptions concerning the structure of the underlying Lie superalgebra \( \mathfrak{g} \) is the general form of the distinguished spacetime components:
\[ \gamma^* \theta^I_\mu(\xi, \phi) = E^I_\mu(\xi, \phi) \, d\xi^\mu. \]
In what follows, we adopt the fairly natural hypothesis that the truncated Vielbein \((E^\mu_j(x, \phi) \equiv \bar{E}^\mu_f(\xi, \phi))\) admits an inverse \((\bar{E}^{-1}_f(\xi, \phi))\) and has no (Graßmann-even) Goldstone degrees of freedom, and so we postpone it to later sections.

The first type of induced action of \(G \supset g\) on the supertarget \(G/H\) can be read off the (local, in general) multiplication rule
\[
g \cdot \gamma(\xi, \phi) =: \gamma(\xi(\xi, \phi, g), \phi(\xi, \phi, g)) \cdot h_l(\xi, \phi, g)
\]
in which \((\xi, \phi) : G/H \times G \rightarrow G/H\) is a certain (non-linear, in general) mapping, and (the inverse of) the last element \(h_l(\xi, \phi, g) \in H\) translates the product \(g \cdot \gamma(\xi, \phi)\) back into the section \(\gamma\), defining therewith an effective (non-linear) transformation
\[
\tilde{\gamma} : G \times \mathcal{M} \rightarrow \mathcal{M} : (g, (\xi, \phi)) \mapsto (\xi(\xi, \phi, g), \phi(\xi, \phi, g))
\]
on the base \(\mathcal{M} \equiv G/H\) of the principal H-bundle. By construction, this action captures the rigid symmetry of the super-\(\sigma\)-model. Besides it, the theory has infinitesimal gauge symmetries that can be modelled – after Refs. [McA00, GKW06a] – on infinitesimal right-regular translations of the lagrangean section in the directions of the subset \(t \subset g\) subject to certain constraints, to be established through a direct calculation later on. In the meantime, we consider unconstrained translations
\[
\gamma(\xi, \phi) \cdot e^{\zeta^\mu \cdot t^\mu} =: \gamma(\xi + \zeta^\mu \delta \xi^\mu(\xi, \phi) + \mathcal{O}(\zeta^2), \phi + \zeta^\mu \delta \phi^\mu(\xi, \phi) + \mathcal{O}(\zeta^2)) \cdot e^{\zeta^\mu \cdot h_l(\xi, \phi, g) t^\mu}
\]
in the supertarget that determines a realisation of the algebraic structure\[16\] generated by the constrained maps \(\tilde{\gamma}^\mu = \zeta^\mu(\cdot) \otimes_{g} t^\mu : \Omega \rightarrow t\), whose set shall be denoted as \(\mathcal{F}(\Omega, t)\) in what follows, on the lagrangean fields of the super-\(\sigma\)-model of the form
\[
\tilde{\gamma} : C^\infty(\Omega, \mathcal{M}) \times \mathcal{F}(\Omega, t) \rightarrow C^\infty(\Omega, \mathcal{M})
\]
\[
((\xi(\cdot, \phi(\cdot)), \tilde{\gamma}(\cdot)) \mapsto (\xi(\cdot) + \zeta^\mu(\cdot) \delta \xi^\mu(\cdot, \phi(\cdot)), \phi(\cdot) + \zeta^\mu(\cdot) \delta \phi^\mu(\cdot, \phi(\cdot)))
\]
While the nature of the assumed rigid symmetry corroborates the idea of formulating the lagrangean super-\(\sigma\)-model in terms of (left-)invariant forms on \(G\), the presence of a gauge symmetry of the type specified turns out to fix the relative normalisation of the metric and topological terms as the latter, defined in terms of a non-invariant de Rham primitive of \(\chi^{(p+2)}\), is not even pseudo-invariant\[17\] under infinitesimal local right translations. An elaboration of these observations requires that a specific superbackground be chosen, and so we postpone it to later sections.

3.1. The Nambu–Goto formulation of the Green–Schwarz super-\(\sigma\)-model. In the most common formulation of the super-\(\sigma\)-model, the worldvolume of the super-\(p\)-brane is embedded entirely in a super-extension of the physical metric spacetime \((M, g)\), with (local) coordinates \(\{\xi^\mu \equiv (\xi^\mu, 0) \equiv \xi^\mu\}_{\mu \in 1, \dim_{\mathbb{M}} t}\), and has no (Graßmann-even) Goldstone degrees of freedom,
\[
\forall a \in \dim_{\mathbb{M}} : \phi^a = 0.
\]
The influence of the background gravitational field on the dynamics of the super-\(p\)-brane is encoded in the action functional giving just the metric volume of the embedded worldvolume \(\Omega\), that is
\[
S^{(NG)}_{\mathcal{M}, \mathcal{G}, p}[\xi] = \int_\Omega \text{Vol}(\Omega) \sqrt{\det_{(p+1)}(g_{IJ}(x) (\partial_i \gamma(\gamma \circ \xi)^{*} \theta^I_L))} (\partial_j \gamma(\gamma \circ \xi)^{*} \theta^I_L))
\]
\[
(3.7)
\]
Thus, the metric term alone favours minimal hypersurfaces. The condition of minimality is deformed in the presence of the topological term
\[
S^{(NG)}_{\mathcal{M}, \mathcal{G}, p}[\xi] = \int_\Omega \xi^a (d^{-1}(p+2)\chi^a)
\]
Together, the two terms yield the Green–Schwarz action functional in the Nambu–Goto form
\[
S^{(NG)}_{\mathcal{G}, p}[\xi] = S^{(NG)}_{\mathcal{M}, \mathcal{G}, p}[\xi] + S^{(NG)}_{\mathcal{M}, \mathcal{G}, p}[\xi]
\]
\[16\]Recall that the pullback 1-forms are to span the cotangent space of the supertarget \(\mathcal{M}\) at each point. The hypothesis is trivially satisfied in the super-Minkowskian setting, cp. Sec. 13.
\[17\]As will be shown later, the structure is far from trivial: it defines a Lie superalgebra only upon restriction to classical field configurations.
\[18\]That is invariant up to an additive de Rham-exact correction.
As a lagrangean field theory, the Green–Schwarz super-σ-model in the above form admits a canonical description in terms of a presymplectic classical space of states and a Poisson algebra of hamiltonians over it. Such a description can be obtained through adaptation to the supersymmetric setting of interest of the formalism of the covariant classical field theory invoked in Sec.\textsuperscript{[2]} We begin by associating with the lagrangean density of the action functional of Eq. (3.9), \( \mathcal{L}_{\text{GS,p}}^{(N)} (\xi^\mu, \partial_i \xi^\nu) \), a function on the first-jet bundle \( J^1 \mathcal{F} \) of its (trivial) covariant configuration bundle \( \mathcal{F} \equiv \Omega \times \mathcal{M} \longrightarrow \Omega \) which in the standard adapted coordinates \( (\xi^\mu, t_i^\nu) \) on the fibre of \( J^1 \mathcal{F} \) is given by \( \mathcal{L}_{\text{GS,p}}^{(N)} (\xi^\mu, t_i^\nu) \). The latter enters the definition of the Poincaré–Cartan form of the model (written in the obvious shorthand notation, with the variational derivatives with respect to Grassmann-odd variables understood to be the left derivatives and so marked accordingly)

\[
\Theta_{\text{GS,p}}^{(N)} (\xi, t) = \left( \mathcal{L}_{\text{GS,p}}^{(N)} (\xi, t) - \mu_i \frac{\delta \mathcal{L}_{\text{GS,p}}^{(N)}}{\delta (\partial_i \xi)} (\xi, t) \right) \text{Vol}(\Omega) + \delta \xi^\mu \frac{\delta \mathcal{L}_{\text{GS,p}}^{(N)}}{\delta (\partial_\mu \xi)} (\xi, t) \wedge (\partial_i \Theta_{\text{GS,p}}^{(N)} (\xi, t)) .
\]

It has the fundamental property (that is straightforward to demonstrate): The extremals of the functional

\[
S_{\Theta_{\text{GS,p}}^{(N)}} : \Gamma (J^1 \mathcal{F}) \longrightarrow \mathbb{R} : \Psi \longrightarrow \int_{\mathcal{M}} \Psi' \Theta
\]

are first jets of extremals of \( S_{\text{GS,p}}^{(N)} \). As in the Grassmann-even setting, we obtain a presymplectic form \( \Omega^{(N)}_{\text{GS,p}} \) on the space of states \( \mathcal{E}_{\text{GS,p}}^{(N)} \) of the super-σ-model

\[
\Omega^{(N)}_{\text{GS,p}} [\Psi] \equiv \int_{\mathcal{M}} (\Psi')^* \Theta_{\text{GS,p}}^{(N)} ,
\]

the space being parameterised by Cauchy data \( \Psi \equiv (\xi^\mu, \pi_I) \) of extremals \( \Psi \) of \( S_{\text{GS,p}}^{(N)} \) supported on a model Cauchy hypersurface \( \mathcal{C} \subset \mathcal{M} \). Here, the \( \pi_I \) correspond to (spacetime) components of the Vielbein-transformed kinetic momentum \( \frac{\delta \mathcal{L}_{\text{GS,p}}^{(N)}}{\delta (\partial_\mu \xi)} (\xi, t) \Gamma^{-1} \xi (\xi) \). The presymplectic form has the universal structure

\[
\Omega^{(N)}_{\text{GS,p}} = \delta \pi + \pi^{*}_{\text{GS,p}} \int_{\mathcal{M}} \text{ev}^*_\pi (\chi) ,
\]

determined by the momentum 1-form

\[
\psi[\xi, \pi] \equiv \int_{\mathcal{M}} \text{Vol}(\mathcal{C}) \pi_I (\cdot) \gamma^* \theta_L^I (\xi (\cdot)) ,
\]

on the (partially symplectically reduced) space of states given by the cotangent bundle

\[
\pi_{\text{GS,p}} : \mathcal{P}_{\text{GS,p}}^{(N)} = T^*_0 \mathcal{M}_{\mathcal{E}} \longrightarrow \mathcal{M}_{\mathcal{E}}
\]

of the space \( \mathcal{M}_{\mathcal{E}} \) of (smooth) maps from the model Cauchy hypersurface \( \mathcal{C} \) to the target supermanifold \( \mathcal{M} \) (generalising the space \( \mathcal{M} \) for \( p = 0 \) and the loop space \( \mathcal{L}_\mathcal{M} \) for \( p = 1 \)) with the Grassmann-odd component of the fibre projected out. Above,

\[
\text{ev} : \mathcal{C} \times \mathcal{M}_{\mathcal{E}} \longrightarrow \mathcal{M} : (\varphi, \xi) \longmapsto \xi (\varphi) .
\]

is the standard evaluation map. As in the Grassmann-even setting, the 2-form enables us to define a Poisson bracket on the algebra of smooth functions on \( \mathcal{E}_{\text{GS,p}}^{(N)} \), and – among other things – analyse the phase-space realisation of symmetries.

To this end, we consider, for every element \( X = X^A t_A \) of the Lie superalgebra \( \mathfrak{g} \), the associated fundamental vector field

\[
\mathcal{K}_X \equiv X^A \mathcal{K}_A \in \Gamma (T \mathcal{M}) ,
\]

expressible in terms of the basis fields \( \mathcal{K}_A \) that act on functions \( f \) on \( \mathcal{M} \) as \( x \) as

\[
(\mathcal{K}_A f)(x) = \frac{d}{dt} \bigg|_{t=0} f (\widetilde{\mathcal{C}}_{e^t A} (x)) .
\]

As mentioned above, in order that \( \mathfrak{g} \) define a global symmetry of \( \mathcal{S}_{\text{GS,p}}^{(N)} \), we must have (as we, indeed, do) a \( G \)-invariant GS \( (p + 2) \)-form with, for every \( X \) as above,

\[
\mathcal{K}_X \cdot \chi = - d \kappa_X
\]

\footnote{The latter component is contained in the characteristic distribution of \( \Omega^{(N)}_{\text{GS,p}} \).}
for some $\kappa_X \in \Omega^p(\mathcal{M})$. We then find the canonical lift of $\mathcal{K}_X$ to $\mathcal{T}\mathcal{F}^{(NG)}_{GS,p}$ as follows. Write the lift as

$$\tilde{\mathcal{K}}_X[\xi, \pi] = \int \mathcal{V}o(\mathcal{C}) \left[ \mathcal{K}^\mu(\xi(\cdot)) \frac{\delta}{\delta \pi^\mu(\cdot)} + \Delta^Y_X(\xi(\cdot), \pi(\cdot)) \frac{\delta}{\delta \pi(\cdot)} \right]$$

to obtain

$$0 = \mathcal{L}_{\tilde{\mathcal{K}}_X} \Phi[\xi, \pi] = \int \mathcal{V}o(\mathcal{C}) \left[ \Delta^Y_X(\xi(\cdot), \pi(\cdot)) \gamma^* \theta_I^1(\xi(\cdot)) + \pi_I(\cdot) \left( \mathcal{L}_{\mathcal{K}_X} \gamma^* \theta_I^1 \right)(\xi(\cdot)) \right],$$

whence, upon invoking the invertibility of the truncated Vielbein, we compute (dropping the implicit dependence on the point $\xi$ for the sake of brevity)

$$\Delta^Y_X(\xi(\cdot), \pi(\cdot)) \gamma^* \theta_I^1(\xi(\cdot)) + \pi_I(\cdot) \left( \mathcal{L}_{\mathcal{K}_X} \gamma^* \theta_I^1 \right)(\xi(\cdot)) = -\pi_J E^{-1}_J \gamma^* \theta_I^1 \left( \mathcal{K}^\mu \frac{\delta}{\delta \pi^\mu} \right)$$

$$+ \pi_J \left( \mathcal{L}_{\mathcal{K}_X} \gamma^* \theta_I^1 \right)(\xi(\cdot)).$$

Clearly, for the above constraints to be solvable thus, we need that

$$\Delta^Y_X(\xi(\cdot), \pi(\cdot)) \gamma^* \theta_I^1(\xi(\cdot)) + \pi_I(\cdot) \left( \mathcal{L}_{\mathcal{K}_X} \gamma^* \theta_I^1 \right)(\xi(\cdot)) = 0.$$

The resulting Noether charge $Q_X$, defined by the relation

$$\tilde{\mathcal{K}}_X \perp \Omega^{(NG)}_{GS,p} = -\delta Q_X,$$

assumes the universal form

$$Q_X[\xi, \pi] = \int \mathcal{V}o(\mathcal{C}) \pi_I(\cdot) \left( \mathcal{K}_X \perp \gamma^* \theta_I^1 \right)(\xi(\cdot)) + \int \mathcal{E}v^* \kappa_X(\xi(\cdot)).$$

The Poisson bracket of two such hamiltonians reads

$$\{Q_{X_1}, Q_{X_2}\}_{\Omega_{GS,p}^{(NG)}}$$

$$= -\tilde{\mathcal{K}}_{X_2} \perp \left[ \int \mathcal{V}o(\mathcal{C}) \left( \left( \mathcal{K}_{X_1} \perp \gamma^* \theta_I^1 \right)(\xi(\cdot)) \delta \pi_I(\cdot) + \pi_I(\cdot) \delta \left( \mathcal{K}_{X_1} \perp \gamma^* \theta_I^1 \right)(\xi(\cdot)) \right) \right]$$

$$+ \pi_{X_2} \mathcal{E}v^* \kappa_{X_2}(\xi(\cdot))$$

$$= Q_{[X_1, X_2]} + \int \mathcal{V}o(\mathcal{C}) \left[ -\Delta^X_{X_2}(\xi(\cdot), \pi(\cdot)) \left( \mathcal{K}_{X_1} \perp \gamma^* \theta_I^1 \right)(\xi(\cdot)) + \pi_I(\cdot) \left( \mathcal{K}_{X_1} \perp \mathcal{L}_{\mathcal{K}_{X_2}} \gamma^* \theta_I^1 \right)(\xi(\cdot)) \right]$$

$$+ \pi_{X_2} \mathcal{E}v^* \left( \mathcal{L}_{\mathcal{K}_{X_1}} \gamma^* \theta_I^1(\xi(\cdot)) \right),$$

and so, in particular,

$$\{Q_{X_1(h_{1(o)},o), X_2(h_{1(o)},o)}\}_{\Omega_{GS,p}^{(NG)}} = Q_{[X_1(h_{1(o)},o), X_2(h_{1(o)},o) + \pi_{X_2} \mathcal{E}v^* \left( \mathcal{L}_{\mathcal{K}_{X_1}} \gamma^* \theta_I^1(\xi(\cdot)) \right)},$$

It is worth pointing out that the latter integrand is closed as

$$-\mathcal{L}_{\mathcal{K}_X} \gamma^* \theta_I^1 = \left( \mathcal{K}_{X_1} \perp \chi \right) = \left[ \mathcal{K}_{X_1}, \mathcal{K}_{X_2} \right] \perp \chi = \mathcal{L}_{\mathcal{K}_{X_1}} \left( \mathcal{K}_{X_2} \perp \chi \right) + \mathcal{L}_{\mathcal{K}_{X_2}} \left( \mathcal{K}_{X_1} \perp \chi \right)$$

$$= \mathcal{L}_{\mathcal{K}_{X_1}} \left( \mathcal{K}_{X_2} \perp \chi \right) - \mathcal{L}_{\mathcal{K}_{X_2}} \left( \mathcal{K}_{X_1} \perp \chi \right) - \mathcal{L}_{\mathcal{K}_{X_1}} \gamma^* \theta_I^1 + \mathcal{L}_{\mathcal{K}_{X_2}} \gamma^* \theta_I^1.$$

Therefore, the realisation of the (left-regular) translational symmetry is hamiltonian iff the p-forms $\kappa_A \equiv \kappa_A^{(\mathcal{M})}$ can be chosen such that the condition

$$\mathcal{L}_{\mathcal{K}_A} \gamma^* \theta_I^1 = f_{AB} \mathcal{C} \kappa_C + \mathcal{D}_{AB}$$

is satisfied for some $D_{AB} \in \Omega^{p-1}(\mathcal{M})$, $A, B \in \mathbb{R}^{\dim \mathcal{M}}$ (as long as $p > 0$). Even then there may occur a classical anomaly structurally identical with the one encountered in the Graßmann-even setting, cp. Eq. (2.11). This suggests a simple geometric model, over the supertarget, of the anomalous Poisson–Lie algebra of Noether hamiltonians associated with left-regular translations of the distinguished type considered, extending (as is clearly necessary) the Lie algebra of vector fields on $\mathcal{M}$. The existence of such a model is to be anticipated on the basis of the symmetry analysis of the bosonic $\sigma$-model
carried out in Ref. [Sus12] and recalled in Sec. 2. Thus, in analogy with the purely Grassmann-even case, consider the fundamental section
\[ R_X := (K_X, \kappa_X) \in \Gamma(\mathcal{L}^{1,p,\mathcal{M}}) \]
of the generalised tangent bundle
\[ \mathcal{L}^{1,p,\mathcal{M}} := T\mathcal{M} \oplus \bigwedge^p T^*\mathcal{M} \]
For any two such Grassmann-homogeneous sections \( R_i := (K_i, \kappa_i), \) \( i \in \{1,2\} \) of the latter space, of the respective parities \( R_i \), we may define a \( \chi \)-twisted (in the sense of Ref. [SW01]) Vinogradov-(Courant)-type superbracket which takes the form
\[
[(K_1, \kappa_1), (K_2, \kappa_2)]^{(p+2)}_{\chi} := \left( [(K_1, K_2), \mathcal{L}_{K_1} K_2] - (-1) \bar{\kappa}_1 \bar{\kappa}_2 \mathcal{L}_{K_2} K_1 \right) - \frac{1}{2} d \left( (K_1 \cup K_2) - (1) \bar{\kappa}_1 \bar{\kappa}_2 K_2 \cup K_1 \right)
\]
with the Lie superbracket of vector fields defined as
\[
[K_1, K_2] := K_1 \circ K_2 - (-1) \bar{\kappa}_1 \bar{\kappa}_2 K_1 \circ K_2,
\]
cp. Refs [Vin90, VC92] and. We then readily find the formula
\[
[R_{X_1}, R_{X_2}]^{(p+2)}_{\chi} = [R_{[X_1,X_2]}] + (0, \alpha_{X_1,X_2}),
\]
with the Lie anomaly super-\( p \)-form
\[
(3.15)
\alpha_{X_1,X_2} := [\mathcal{L}_{K_1} K_2] - \kappa_{[X_1,X_2]} - d (R_{X_1}, R_{X_2}),
\]
written in terms of the following pairing of sections
\[
(R_1, R_2) := \frac{1}{2} \left( (K_1 \cup K_2) - (1) \bar{\kappa}_1 \bar{\kappa}_2 K_2 \cup K_1 \right).
\]
The last (manifestly exact) term in the formula for the anomaly, while absent from Eq. (3.13), becomes visible in an analogous formula for the corresponding Noether currents.

3.2. The Hughes–Polchinski formulation of the Green–Schwarz super-\( \sigma \)-model. There is an alternative to the standard (Nambu–Goto) formulation of the (super-)\( \sigma \)-model with a homogenous space of a (super)group action as a (super)target that was originally conceived in Ref. [HP86] and elaborated significantly in Ref. [GIT90]. Here, we use its full-fledged version and draw on an in-depth geometric understanding thereof worked out, in the context of immediate interest, in Ref. [McA00, McA10] and Refs. [Wes00, GKW06b, GKW06a]. The formulation introduces into the lagrangean density, among other fields, Goldstone fields for (some of) the global spacetime symmetries of the super-\( \sigma \)-model broken by the ‘vacuum’ of the theory, i.e., by the embedding of the membrane in the super-target \( \mathcal{M} \) described by a classical field configuration, and subjects them to the inverse Higgs mechanism of Ref. [OT73] to remove some of them in a manner consistent with the surviving ‘vacuum’ symmetries. In this procedure, the Cartan geometry of the homogeneous space \( \mathcal{M} \) employed in the construction of the action functional proves instrumental. Indeed, the mechanism boils down to the imposition of geometric constraints that restrict the tangents of classical field configurations to the directions within \( G/H \equiv \mathcal{M} \) determined by the unbroken symmetries – these constraints can be expressed as the conditions of the vanishing, on classical field configurations, of the pullbacks along the coset section \( \gamma \) of those components of the Maurer–Cartan 1-form \( (3.2) \) which correspond to the broken (infinitesimal) symmetries in \( \mathfrak{g} \). Technically, this means that the Goldstone fields eliminated in the procedure are expressed in terms of the remaining fields of the theory, and in particular – in terms of the derivatives of the surviving Goldstone fields, whence also the name of the mechanism.

The basic building blocks of the super-\( \sigma \)-model in the Hughes–Polchinski formulation are, as previously, the component Maurer–Cartan 1-forms \( \gamma^\mu [\mathcal{F}], \mu \in \mathbb{1}, \dim_{\mathcal{F}} \mathcal{H} \), however, in this case, we introduce the Goldstone fields \( \phi^a, a \in \mathbb{1}, \dim_{\mathcal{G}} \mathcal{G} \). Consequently, in the standard notation
\[
\text{ad}_\mu(t_I) \equiv [t_I, t_I] = f^{I}{}_{J} t_J, \quad \text{ad}_a(t_a) \equiv [t_a, t_a] = f^{I}{}_{a} t_I,
\]
we obtain the factorised Vielbeine
\[
\gamma^\mu [\mathcal{F}] (\xi, \phi) \otimes \mathbb{R} t_I = e^{I}{}_{\mu} (\xi) \, d\xi^\mu \otimes \mathbb{R} e^{-\phi^a \text{ad}_a(t_I)} = e^{I}{}_{\mu} (\xi) \left( e^{\Lambda(\phi)} \right)^I_J \, d\xi^\mu \otimes \mathbb{R} t_I, \quad \Lambda(\phi) = \phi^a f^{I}{}_{a} \, ,
\]
\[ \gamma^* \theta_L^\alpha (\xi, \phi) \otimes_{\mathbb{R}} t_\alpha = \sigma^\alpha_{\mu}(\xi) d\xi^\mu \otimes_{\mathbb{R}} e^{-\phi^\alpha} \alpha_{\mu}(t_\alpha) = \sigma^\alpha_{\mu}(\xi) (e^{\widetilde{A}(\phi)})^\alpha_{\beta} d\xi^\mu \otimes_{\mathbb{R}} t_\alpha, \]

\[ \widetilde{A}(\phi)^\alpha_{\beta} = \phi^\alpha f_{\alpha\beta}. \]

(3.16)

Prior to writing out the action functional in that formulation, we need to make further algebraic assumptions. Thus, upon introducing the auxiliary matrix

\[ \widetilde{A}(\phi)^\alpha_{\beta} = \phi^\alpha f_{\alpha\beta}, \]

with the obvious decomposition

\[ \widetilde{A}(\phi) = (\Lambda(\phi), \widetilde{A}(\phi)) \in \text{End}_\mathbb{R}(t^{(0)}) \oplus \text{End}_\mathbb{R}(t^{(1)}) \subset \text{End}_\mathbb{R}(t) \]

we presuppose that the Green–Schwarz \((p + 2)\)-form

\[ \chi_{\mu_1\mu_2...\mu_{p+2}} = \chi_{\mu_1\mu_2} \gamma^* \theta^\mu_1 \wedge \gamma^* \theta^\mu_2 \wedge \cdots \wedge \gamma^* \theta^\mu_{p+2}, \]

\[ \chi_{\mu_1\mu_2...\mu_{p+2}} \in \mathbb{R} \]

has the invariance property

\[ \chi_{\mu_1\mu_2...\mu_{p+2}} \Lambda(\phi)^\mu_1_{\nu_1} \Lambda(\phi)^\mu_2_{\nu_2} \cdots \Lambda(\phi)^\mu_{p+2}_{\nu_{p+2}} = \chi_{\mu_1\mu_2...\mu_{p+2}} \]

that implies the identity

\[ \chi_{\mu_1\mu_2...\mu_{p+2}} (\xi, \phi) = \chi_{\mu_1\mu_2...\mu_{p+2}} (\xi, 0). \]

(3.17)

At this stage, it suffices to demand that the Lie-algebra action \((3.4)\) integrate to a unimodular (adjoint) action of the Lie group \(R_{\text{vac}}\) of \(t_{\text{vac}}\) on \(t^{(0)}_{\text{vac}}\), i.e.,

\[ \forall t_{\text{vac}} : \det (T_{t_{\text{vac}}}^{(1)} t_{\text{vac}}(0)) = 1, \]

to be able to define the action functional: We take its topological term to be the same as in the Nambu–Goto formulation (which makes sense in consequence of Eq. \((3.17)\)),

\[ S_{\text{top}, GS, p}^{(\text{HP})}[\xi, \phi] = \int_\Omega (\gamma \circ \chi)^* (d^{-1} \chi_{\text{vac}}) = \int_\Omega (\gamma \circ \chi)^* (d^{-1} \chi_{\text{vac}}) = S_{\text{top}, GS, p}^{(\text{NG})}[\xi] \]

and set – for \( \chi = (\xi, \phi) \)

\[ S_{\text{metr}, GS, p}^{(\text{HP})}[\xi, \phi] = \frac{1}{(p+1)!} \int_\Omega (\gamma \circ \chi)^* \beta^{(\text{HP})}_{(p+1)}, \]

with

\[ \beta^{(\text{HP})}_{(p+1)} = \epsilon_{\nu_1\nu_2...\nu_{p+1}} (\theta^{\nu_1}_{L} \wedge \theta^{\nu_2}_{L} \wedge \cdots \wedge \theta^{\nu_{p+1}}_{L}), \]

written in terms of the standard totally antisymmetric symbol

\[ \epsilon_{\nu_1\nu_2...\nu_{p+1}} = \begin{cases} \text{\text{sign}} \begin{pmatrix} 0 & 1 & \cdots & p \\ A_0 & A_1 & \cdots & A_p \end{pmatrix} & \text{if } \{A_0, A_1, \ldots, A_p\} = 0, \dim \mathfrak{t}^{(0)}_{\text{vac}} = 1 \\ 0 & \text{otherwise} \end{cases} \]

so that – altogether –

\[ S_{GS, p}^{(\text{HP})}[\xi, \phi] = S_{\text{metr}, GS, p}^{(\text{HP})}[\xi, \phi] + S_{\text{top}, GS, p}^{(\text{HP})}[\xi, \phi]. \]

There is a class of super targets for which we may establish a direct relation between the two formulations of the Green–Schwarz super-\(\sigma\)-model, to wit,

**Proposition 3.1.** Let \( G \) be a Lie supergroup with the Lie superalgebra \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{r} \) and let \( \mathcal{H} \subset G \) be its Lie subgroup with the Lie algebra \( \mathfrak{h} \), the two algebras satisfying the relations described at the beginning of Sec. 3. The Green–Schwarz super-\(\sigma\)-model on the homogeneous space \( G/\mathcal{H} \) in the Hughes–Polchinski formulation, determined by the action functional \( S_{GS, p}^{(\text{HP})} \) \((3.21)\) with the metric term \((3.19)\) and the topological term \((3.18)\), is equivalent to the Green–Schwarz super-\(\sigma\)-model on the same super target in the Nambu–Goto formulation, defined by the action functional \((3.3)\) with the metric term \((3.7)\) for the metric \( g = \kappa^{(0)} \coprod_{\text{vac}} \epsilon^{(0)} \) given by the restriction to \( t^{(0)}_{\text{vac}} \oplus \mathfrak{t}^{(0)} \) of the Cartan–Killing metric \( \kappa^{(0)} \) on the Lie algebra \( t^{(0)} \oplus \mathfrak{r} \equiv \mathfrak{g}^{(0)} \) and the topological term \((3.8)\), if the following conditions are satisfied:
\( (E1) \) \( \kappa^{(0)} \) defines an orthogonal decomposition
\[
\mathfrak{f}^{(0)} = \mathfrak{t}_{\text{vac}}^{(0)} \oplus \mathfrak{e}^{(0)} \oplus \mathfrak{f}^{(0)}
\]
(in which \( \mathfrak{f}^{(0)} \) is an orthogonal direct-sum completion of \( \mathfrak{t}_{\text{vac}}^{(0)} \oplus \mathfrak{e}^{(0)} \)) such that \( \kappa^{(0)} \mathfrak{t}_{\text{vac}}^{(0)} \mathfrak{e}^{(0)} \mathfrak{f}^{(0)} \) is non-degenerate;

\( (E2) \) \( S_{\text{GS}, p}^{(\text{HP})} \) is restricted to field configurations satisfying – in the notation introduced at the beginning of Sec. \( 3 \) – the inverse Higgs constraint
\[
\forall \zeta_{S, 1, \sigma - p - 1} : (\gamma \circ X)^* \theta_{\Pi}^S_0 = 0
\]
whose solvability is ensured by the invertibility – in an arbitrary (local) coordinate system \( \{\sigma^i\}_{p} \) on \( \Omega \) – of the (tangent-transport) operator
\[
e^\Delta \phi (\zeta (\sigma)) \frac{\partial \sigma^\mu}{\partial \zeta (\sigma)} = \Delta, \quad \sigma \in \Omega.
\]
The latter constraint is equivalent to the Euler–Lagrange equations of \( S_{\text{GS}, p}^{(\text{HP})} \) obtained by varying the functional in the direction of the Goldstone fields \( \phi^a, \quad a \in 1, \text{dim} \mathfrak{g} \).

**Proof.** In view of the previous remarks, we first have to demonstrate that \( S_{\text{metr}, GS}^{(\text{HP})} \) reduces to \( S_{\text{metr}, GS}^{(\text{NG})} \) whenever conditions \( (E1) \) and \( (E2) \) are satisfied. To this end, we work out explicit formul\( \alpha \) for the relevant components of the Maurer–Cartan 1-form. We have
\[
e^{-\phi^a \text{ad}_a} (t_A) = \sum_{n=0}^\infty \frac{1}{(2n)!} \phi_n^a \phi_n^2 \ldots \phi_n^{a_2n} f_{a_2n} \delta_{2n} f_{a_2n-1} \delta_{2n-1} \ldots f_{a_2n-3} \delta_{2n-2} \ldots f_{a_2} \delta_2 f_{a_1} \delta_1 \uparrow t_A,
\]
and
\[
e^{-\phi^a \text{ad}_a} (t_S) = \sum_{n=0}^\infty \frac{1}{(2n+1)!} \phi_n^a \phi_n^2 \ldots \phi_n^{a_2n+1} f_{a_2n+1} \delta_{2n+1} f_{a_2n} \delta_{2n} \ldots f_{a_2n-3} \delta_{2n-2} \ldots f_{a_2} \delta_2 f_{a_1} \delta_1 \uparrow t_S,
\]
Denote
\[
F(\phi) \overset{\mathbb{A}}{\to} = \phi^a f_{a_2} \delta_{2} f_{a_3} \delta_{3} \ldots f_{a_S} \delta_{S}, \quad \tilde{F}(\phi) \overset{\mathbb{A}}{\to} = \phi^a f_{a_S} \delta_{S},
\]
and
\[
\phi^a \phi^b f_{a_2} \delta_{2} f_{a_3} \delta_{3} \ldots f_{a_S} \delta_{S} =: Q(\phi) \overset{\mathbb{A}}{\to}, \quad \phi^a \phi^b f_{a_2} \delta_{2} f_{a_3} \delta_{3} \ldots f_{a_S} \delta_{S} =: \tilde{Q}(\phi) \overset{\mathbb{A}}{\to}.
\]
Furthermore, for the sake of brevity, use the symbolic notation
\[
L(\phi)^2 := Q(\phi), \quad \tilde{L}(\phi)^2 := \tilde{Q}(\phi)
\]
in (symmetric) functions of \( \phi \) whose dependence on the argument factors through \( Q(\phi) \) or \( \tilde{Q}(\phi) \), e.g.,
\[
e^{-\phi^a \text{ad}_a} (t_A) = (\text{ch} L(\phi)) \overset{\mathbb{A}}{\to} t_A \phi^a f_{a_2} \delta_{2} f_{a_3} \delta_{3} \ldots f_{a_S} \delta_{S} \overset{\mathbb{A}}{\to} t_A.
\]
The above then rewrites as
\[
e^{\lambda(\phi)} = \begin{pmatrix}
\text{ch} L(\phi) & -F(\phi) \cdot \frac{\text{sh} L(\phi)}{L(\phi)} \\
-F(\phi) \cdot \frac{\text{sh} L(\phi)}{L(\phi)} & \text{ch} L(\phi)
\end{pmatrix},
\]
where the blocks correspond (in an obvious manner) to the direct summands in the decomposition
\( t^{(0)} = \mathfrak{t}_{\text{vac}}^{(0)} \oplus \mathfrak{e}^{(0)} \). This can be further decomposed as
\[
e^{\lambda(\phi)} = \begin{pmatrix}
1_{p+1} & -\varphi \\
-\varphi^T & 1_{d-p-1}
\end{pmatrix} \begin{pmatrix}
\text{ch} L(\phi) & 0 \\
0 & \text{ch} L(\phi)
\end{pmatrix},
\]
\[33\]
with
\[ \varphi = F(\phi) \cdot \frac{\text{sh} L(\phi)}{L(\phi) \text{ch} L(\phi)}, \quad \bar{\varphi} = \bar{F}(\phi) \cdot \frac{\text{sh} L(\phi)}{L(\phi) \text{ch} L(\phi)}. \]

In view of the obvious identities
\[ F(\phi) \cdot \bar{Q}(\phi) = Q(\phi) \cdot F(\phi), \quad \bar{F}(\phi) \cdot Q(\phi) = \bar{Q}(\phi) \cdot \bar{F}(\phi), \]

we may rewrite the last definitions in the equivalent form
\[ \varphi = \frac{\text{sh} L(\phi)}{L(\phi) \text{ch} L(\phi)} \cdot F(\phi), \quad \bar{\varphi} = \frac{\text{sh} L(\phi)}{L(\phi) \text{ch} L(\phi)} \cdot \bar{F}(\phi). \]

Furthermore, as
\[ F(\phi) \cdot \bar{F}(\phi) = Q(\phi), \quad \bar{F}(\phi) \cdot F(\phi) = \bar{Q}(\phi), \]

we obtain the relation
\[ \varphi \cdot \bar{\varphi} = \frac{\text{sh} L(\phi)}{L(\phi) \text{ch} L(\phi)} \cdot F(\phi) \cdot \bar{F}(\phi) \cdot \frac{\text{sh} L(\phi)}{L(\phi) \text{ch} L(\phi)} = \left( \frac{\text{sh} L(\phi)}{L(\phi) \text{ch} L(\phi)} \right)^2 = 1_{d+1} - \frac{1}{\text{ch} L(\phi)^2}, \]

and – similarly – the relation
\[ \bar{\varphi} \cdot \varphi = 1_{d-p-1} - \frac{1}{\text{ch} L(\phi)^2}, \]

so that we may ultimately express \( e^{A(\phi)} \) entirely in terms of \( \varphi \) and \( \bar{\varphi} \) as
\[ e^{A(\phi)} = \begin{pmatrix} 1_{p+1} & -\varphi & 0 \\ 0 & 1_{d-p-1} & -\bar{\varphi} \end{pmatrix} \begin{pmatrix} (1_{p+1} - \varphi \cdot \bar{\varphi})^{-\frac{1}{2}} & 0 \\ 0 & (1_{d-p-1} - \bar{\varphi} \cdot \varphi)^{-\frac{1}{2}} \end{pmatrix}. \]

We next use assumption (E1) to relate \( \bar{\varphi} \) to \( \varphi \). To this end, we compute, using the ad-invariance of the Killing form,
\[ \kappa^{(0)} - 1 \frac{\text{ad} f_{a\bar{B}}} {a\bar{B}} \kappa^{(0)}_S T \mathcal{S} = \kappa^{(0)} - 1 \frac{\text{ad} f_{a\bar{B}}} {a\bar{B}} C_{CS}^{(0)} = \kappa^{(0)} - 1 \frac{\text{ad} f_{a\bar{B}}} {a\bar{B}} \kappa^{(0)}([t_a, t_{\bar{B}}], t_S) = -\kappa^{(0)} - 1 \frac{\text{ad} f_{a\bar{B}}} {a\bar{B}} \kappa^{(0)}([t_a, t_S], t_{\bar{B}}) \]
\[ = -\kappa^{(0)} - 1 \frac{\text{ad} f_{a\bar{B}}} {a\bar{B}} C_{CS}^{(0)} = -\frac{\text{ad} A} {aS}, \]

whence also
\[ \kappa^{(0)}_S f_{a\bar{B}}^{\mathcal{T}} \kappa^{(0)}_S - f_{a\bar{B}}^{\mathcal{T}} = -f_{a\bar{B}}^{\mathcal{T}}. \]

Taking into account that \( \bar{\varphi} \) is an odd function of the \( \bar{\phi} \), we then readily establish the fundamental identity
\[ \bar{\varphi} A = -\kappa^{(0)} - 1 \frac{\text{ad} B} {aB} \bar{\varphi} B^{T} \kappa^{(0)}_T. \]

At this stage, we may pass to express (the pullbacks of) the relevant left-invariant Maurer–Cartan 1-forms as functions of \( \xi^a \) and \( \varphi^{A} \), whereupon the imposition of the inverse Higgs constraint becomes straightforward. Thus, taking into account the expressions (3.16) as well as the hitherto results, we find
\[ \gamma^* \theta^A_L(\xi, \phi) = \left( e^B_{\mu}(\xi) - e^S_{\mu}(\xi) \varphi_{B}^{\mathcal{T}}(\phi) \right) \sqrt{1_{p+1} - \varphi \cdot \varphi^{A}}, \]
\[ \gamma^* \theta^B_L(\xi, \phi) = \left( e^T_{\mu}(\xi) - e^A_{\mu}(\xi) \varphi_{A}^{\mathcal{T}}(\phi) \right) \sqrt{1_{d-p-1} - \bar{\varphi} \cdot \bar{\varphi}^{B}}, \]

Denote, for any \( I \in 1, \dim(t^{(0)}) \) and for (local) coordinates \( \{ \iota^i \}_{1^{\mathcal{U}^{(0)}}} \) on \( \Omega \),
\[ \iota^I(\sigma) := e^I_{\mu}(\xi(\sigma)) \frac{\partial e^0_{\mu}} {\partial \sigma^i}(\sigma) \]

and further write
\[ \iota^A \equiv \iota^A, \quad \iota^S \equiv \iota^S \]

for the sake of clarity of the formulae that follow. The solution to the inverse Higgs constraint now reads
\[ \varphi^{A}_L(\phi(\sigma)) = e^{-1} \gamma^* \theta^A_L(\xi, \phi) \iota^A(\sigma), \]

or – in an obvious shorthand notation –
\[ \varphi \circ \phi = e^{-1} \gamma^* \theta^A_L(\xi, \phi) \iota^A(\sigma). \]

Substituting this into the formula for \( X^* \gamma^* \theta^A_L(\xi, \phi) \) and using Eq. (3.24) along the way, we arrive at the expression
\[ \varphi^{A}_L \equiv \varphi^{A}_L d\sigma^i := \left( \xi, \phi(\xi) \right)^* \gamma^* \theta^A_L(\xi, \phi). \]
In order to simplify the above expression and prepare it for subsequent use in the reconstruction of the inverse Higgs-reduced Hughes–Polchinski action functional, let us call

\[ \mathcal{R}_{AB} := \kappa^{(0)}_{AB}, \quad \mathcal{R}_{ST} := \kappa^{(0)}_{ST} \]

and

\[ \mathcal{G}_{ij} := \mathcal{R}_{ST} \epsilon^S_i \epsilon^T_j, \quad \mathcal{G}_{ij} := \mathcal{R}_{AB} \epsilon^A_i \epsilon^B_j, \]

as well as

\[ \mathcal{G}_{ij} := \mathcal{G}_{ij} + \mathcal{G}_{ij} = \mathcal{E}_i \mathcal{E}_j. \]

We then obtain

\[
\varpi_{\mathcal{A}} = \left( B + \frac{\mathcal{G}_{ij}}{\mathcal{E}^{-1}_i \mathcal{E}^{-1}_j} \right) \left( \frac{1}{\sqrt{1 + \mathcal{G}_{ij} \mathcal{E}^{-1}_i \mathcal{E}^{-1}_j}} \right) \frac{A}{B} = \frac{E}{\mathcal{E}} \left( B + \mathcal{G}_{jk} \mathcal{E}^{-1}_j \mathcal{E}^{-1}_k \right) \left( \frac{1}{\sqrt{1 + \mathcal{G}_{jk} \mathcal{E}^{-1}_j \mathcal{E}^{-1}_k}} \right) \frac{A}{B} = \frac{E}{\mathcal{E}} \sqrt{1 + \mathcal{G}_{jk} \mathcal{E}^{-1}_j \mathcal{E}^{-1}_k} \frac{A}{B} = \frac{E}{\mathcal{E}} \sqrt{1 + \mathcal{G}_{jk} \mathcal{E}^{-1}_j \mathcal{E}^{-1}_k} \frac{A}{B}.
\]

At long last, we may now write out the sought-after metric term of the reduced Hughes–Polchinski action functional (in an obvious shorthand notation),

\[
S^{(HP)}_{\text{metr,GS},p}[\xi, \phi(\xi)] = \int_\Omega \text{Vol}(\Omega) \xi_{i_0i_1} \ldots p \varpi^{i_0}_{i_1} \ldots \varpi^p (\cdot) \equiv \int_\Omega \text{Vol}(\Omega) \det(p) (\varpi^\nu) = \int_\Omega \text{Vol}(\Omega) \det(p) \left( \xi \cdot \varpi(\xi) \right),
\]

whence also we finally retrieve the anticipated result

\[
S^{(HP)}_{\text{metr,GS},p}[\xi, \phi(\xi)] = \lambda_p \int_\Omega \text{Vol}(\Omega) \sqrt{\det(p)} \left( \xi \cdot \varpi(\xi) \right),
\]

up to an overall constant \( \lambda_p \) (which we can always set to one by a suitable rescaling of the metric term).

Passing to the closing statement of the proposition, we shall first write out the metric term of the Hughes–Polchinski action functional in a form amenable to further treatment. Taking into account Eq. (3.24), we obtain – in the previously introduced notation –

\[
S^{(HP)}_{\text{metr,GS},p}[\xi, \phi] = \int_\Omega \text{Vol}(\Omega) \det(p) (\varpi) = \int_\Omega \text{Vol}(\Omega) \det(p) M(\xi) \cdot \det(p) \left[ A(\xi, \phi) \cdot B(\xi, \phi)^{-1} \right],
\]

where \( M(\xi) \) is a matrix that does not depend on the \( \phi^p \), and hence does not contribute to the Euler–Lagrange equations for these fields, and where

\[
A(\xi, \phi) = \kappa + \varphi(\phi) \cdot \mathcal{R} \cdot \xi(\xi) \cdot \xi^{-1}(\xi), \quad B(\xi, \phi) = \kappa + \varphi(\phi) \cdot \mathcal{R} \cdot \varphi(\phi)^T.
\]

The said Euler–Lagrange equations read

\[
\frac{\delta \mathcal{E}}{\delta \varphi} \text{tr}(p) A(\xi, \phi)^{-1} \cdot \frac{\delta}{\delta \varphi} A(\xi, \phi) = \frac{1}{2} B(\xi, \phi)^{-1} \cdot \frac{\delta \mathcal{E}}{\delta \varphi} B(\xi, \phi) = 0,
\]

and so using the symmetricity of \( B(\xi, \phi) \), they can be cast in the simple matrix form

\[
\left( \mathcal{R} \cdot \xi(\xi) \cdot \xi^{-1}(\xi) \cdot A(\xi, \phi)^{-1} \right)^T = B(\xi, \phi)^{-1} \cdot \varphi(\phi) \cdot \mathcal{R}.
\]

Upon multiplying both sides of the above equation by \( \varphi(\phi)^T \) and invoking Eq. (3.28), we deduce from the above the identity

\[
A(\xi, \phi)^{-1} = B(\xi, \phi)^{-1}.
\]
which – when used in the original equation – yields the anticipated solution (3.27). \[\square\]

The assumptions of the last proposition exclude important – both mathematically and physically – examples of supertargets such as the super-Minkowski space for which the Killing metric degenerates in the (Graßmann-even) translational directions. At the same time, they suggest very clearly a generalisation that does not – a priori – constrain the structure of the underlying Lie algebra \(\mathfrak{g}^{(0)}\). Thus, we formulate

**Proposition 3.2.** Let \(G\) be a Lie supergroup with the Lie superalgebra \(\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{r}\) and let \(H \subset G\) be its Lie sub(super)group with the Lie superalgebra \(\mathfrak{h}\), the two algebras satisfying the relations described at the beginning of Sec. 3. If condition \((E2)\) of Prop. 3.1 is satisfied in conjunction with condition \((E1')\) there exist non-degenerate bilinear symmetric forms: \(\gamma\) on \(\mathfrak{t}^{(0)}\) and \(\ov{\gamma}\) on \(\mathfrak{e}^{(0)}\) with respective presentations

\[
\gamma = \gamma_{AB} \tau^A \otimes_{\mathbb{R}} \tau^B, \quad \ov{\gamma} = \ov{\gamma}^S_\tau \tau^S \otimes_{\mathbb{R}} \tau^P
\]

in the basis \(\{\tau^A\}_{A \in \text{dim}_{\mathbb{R}} \mathfrak{g}}\) of \(\mathfrak{g}\) dual to \(\{t_A\}_{A \in \text{dim}_{\mathbb{R}} \mathfrak{g}}\),

\[
\tau^A(t_B) = \delta^A_B, \quad A, B \in \text{dim}_{\mathbb{R}} \mathfrak{g},
\]

for which the following identities hold true

\[
(3.29) \quad \gamma^{-1} A_B f_{AB} = -f^A_S\gamma^S_T,
\]

the Green–Schwarz super-\(\sigma\)-model on the homogeneous space \(G/H\) in the Hughes–Polchinski formulation, determined by the action functional \(S^{(HP)}_{\text{GS},p}\) (3.21) with the metric term (3.19) and the topological term (3.18), is equivalent to the Green–Schwarz super-\(\sigma\)-model on the same supertarget in the Nambu–Goto formulation, defined by the action functional (3.9) with the metric \(\gamma = \gamma + \ov{\gamma}\) and the topological term (3.8).

The inverse Higgs constraint is equivalent to the Euler–Lagrange equations of \(S^{(HP)}_{\text{GS},p}\) obtained by varying the functional in the direction of the Goldstone fields \(\phi^a\), \(a \in 1, \text{dim}_{\mathbb{R}} \mathfrak{g}\).

**Proof.** The proof is entirely analogous to that of Prop. 3.1, with identity (3.29) playing the structural rôle of identity (3.23), the latter being satisfied automatically under the assumptions of that proposition. \[\square\]

While we are not going to make essential use of that in what follows, it is to be noted that the canonical description of the Hughes–Polchinski model is highly singular in that the corresponding presymplectic form

\[
\Omega^{(HP)}_{\text{GS},p}(\hat{X}) = \int_{\mathfrak{e}} \ev^*(\delta\beta^{(HP)} + \chi)_{(p+2)}
\]

does not depend on the kinetic momentum. In the light of the above proposition, the latter is reintroduced into the canonical description only through the imposition of the inverse Higgs constraint.

4. THE SUPER-MINKOWSKIAN BACKGROUND

In the present section, we restrict our considerations to one of the simplest supertargets, to wit, the **super-Minkowski spacetime with \(N\) supersymmetries**, and specify its tensorial data necessary for the definition of the relevant super-\(\sigma\)-model.

### 4.1. The Cartan supergeometry of the super-Minkowskian target.

As a supermanifold, the super-Minkowski spacetime with \(N\) supersymmetries is the previously introduced model ringed space

\[
\left(\mathbb{R}^{d}, C^n(\cdot, \cdot) \otimes_{\mathbb{R}} \bigwedge \mathbb{R}^{N_{D_1,d-1}}\right) \equiv \text{sMink}^{1,d-1|N_{D_1,d-1}}, \quad D_{1,d-1} = \dim_{\mathbb{C}} S_{1,d-1}, \quad d \in \{9, 10\},
\]

where \(\dim_{\mathbb{C}} S_{1,d-1}\) denotes the dimension of the Majorana-spinor representation of the spin group \(\text{Spin}(1, d - 1)\) of the Clifford algebra \(\text{Cliff}(1, d - 1)\) of the standard Minkowski (quadratic) space (\(\mathbb{R}^{d, \eta}\), \(\eta = \text{diag}(+,-,-,\ldots,-)\)). The supertarget will be conveniently described as a homogeneous space of the natural action of the \(N\)-extended super-Poincaré Lie supergroup, the latter being given
by the semidirect product of the supertranslation (or supersymmetry) group \( s\mathcal{P}(1,d-1;N) \) with the spin group \( \text{Spin}(1,d-1) \),

\[
s\mathcal{P}(1,d-1;N) = \mathbb{R}^{1,d-1|ND_1,d-1} \times \text{Spin}(1,d-1),
\]

with respect to the standard product supervector representation of \( \text{Spin}(1,d-1) \) on \( \mathbb{R}^{1,d-1|ND_1,d-1} \). The supergroup, and so also the supertarget, admits homogeneous coordinates: the Grassmann-even ones \( g^{(4.1)} \)

\[
S_{\alpha} \] of the Majorana-spinor representation contributes from this representation resummed over the range \( \{ P_l \}_{t\in D_{1,d-1}} \) associated with the left-invariant vector fields \( \{ J_{KL} \}_{K,L\in D_{1,d-1}} \) generating Lorentz transformations, as well as the Grassmann-odd ones \( \{ \theta_\alpha \}_{\alpha \in 1,N} \) associated with left-invariant (super)vector fields \( \{ Q_\alpha \}_{\alpha \in 1,N} \) generating spinorial translations. The Lie-supergroup structure on the above supermanifold is determined by the binary operation

\[
m : \quad s\mathcal{P}(1,d-1;N) \times s\mathcal{P}(1,d-1;N) \rightarrow s\mathcal{P}(1,d-1;N)
\]

\[
\left( \left( x_1^I, \theta_1^\alpha, \phi_1^{KL} \right), \left( x_2^J, \theta_2^\beta, \phi_2^{MN} \right) \right) \mapsto \left( x_1^I + L(\phi_1)^I_J x_2^J, -\frac{1}{2} \theta_1^\alpha C_\alpha^\beta \Gamma^I_J \gamma^J S(\phi_1)_{\gamma}^\gamma \theta_2^\beta, \theta_1^\alpha + S(\phi_1)^\alpha_\beta \theta_2^\beta \right).
\]

written in terms of the vector representation \( L : \text{Spin}(1,d-1) \rightarrow \text{End}_\mathbb{R} (\mathbb{R}^{2d}) \) and of (the \( i \)-th copy of) the Majorana-spinor representation \( S : \text{Spin}(1,d-1) \rightarrow \text{End}_\mathbb{C} (S_1,d-1) \), in which we also take the relevant charge-conjugation matrix and the generators of the Clifford algebra (with contributions from this representation resummed over the range \( i \in 1,N \) in the vectorial Grassmann-even component), and in terms of the standard non-linear group law \( \tilde{\phi} \) for elements of group \( \text{Spin}(1,d-1) \). Upon restriction to \( s\text{Mink}^{1,d-1|ND_1,d-1} \) \( \equiv \mathbb{R}^{1,d-1|ND_1,d-1} \subset s\mathcal{P}(1,d-1;N) \) in the above group law, we recover the natural (left) action of \( s\mathcal{P}(1,d-1;N) \) on the super-Minkowski space (coset),

\[
el : \quad s\mathcal{P}(1,d-1;N) \times s\text{Mink}^{1,d|ND_1,d-1} \rightarrow s\text{Mink}^{1,d|ND_1,d-1}
\]

\[
\left( \left( y^I, \theta_1^\alpha, \psi^{KL} \right), \left( y^J, \theta_2^\beta \right) \right) \mapsto \left( y^I + y^J - \frac{1}{2} \theta_1^\alpha C_\alpha^\beta \Gamma^I_J \gamma^J \psi, \theta_2^\beta + \psi \right).
\]

The right action of the supertranslation group on the super-Minkowski spacetime is defined analogously,

\[
\varphi : \quad s\text{Mink}^{1,d|ND_1,d-1} \times \mathbb{R}^{1,d-1|ND_1,d-1} \rightarrow s\text{Mink}^{1,d|ND_1,d-1}
\]

\[
\left( \left( x^I, \theta_1^\alpha \right), \left( y^J, \theta_2^\beta \right) \right) \mapsto \left( x^I + y^J - \frac{1}{2} \theta_1^\alpha C_\alpha^\beta \Gamma^I_J \gamma^J \xi, \theta_2^\beta + \xi \right).
\]

It is to be noted at this stage that the generators of the Clifford algebra are equivariant with respect to the two representations of \( \text{Spin}(1,d-1) \) introduced above, as expressed by the identities

\[
S(\phi) \cdot \Gamma^I \cdot S(-\phi) = L(-\phi)^I_J \Gamma^J.
\]

Here, \( S(-\phi) \equiv S(\phi)^{-1} \) and, similarly, \( L(-\phi) \equiv L(\phi)^{-1}, \) and we have the defining identity

\[
L(\phi)^K_J \eta_{KL} = L(-\phi)^K_J \eta_{KL}.
\]

In consequence of the symmetry properties of the said generators listed in Conv.5.3, we also obtain the useful equality (writing \( \phi_{IJ} \equiv \phi^{KL} \eta_{KI} \eta_{IJ} \) where necessary)

\[
C \cdot S(\phi) \cdot C^{-1} \equiv C \cdot \exp \left( \frac{1}{2} \phi_{IJ} \Gamma^I \cdot \Gamma^J \right) \cdot C^{-1} = \exp \left( \frac{1}{2} \phi_{IJ} C \cdot \Gamma^I \cdot C^{-1} \cdot C \cdot \Gamma^J \cdot C^{-1} \right)
\]

\[
= \exp \left( \frac{1}{2} \phi_{IJ} \left( (-\Gamma^I)^T \right) \cdot (-\Gamma^J)^T \right) = \exp \left( \frac{1}{2} \phi_{IJ} \left( \Gamma^I \right)^T \cdot \left( \Gamma^J \right)^T \right)
\]

\[
= \exp \left( \frac{1}{2} \phi_{IJ} \left( (\Gamma^I \cdot \Gamma^J) - \Gamma^I \cdot \Gamma^J \right) \right)^T
\]

\[
= \exp \left( \frac{1}{2} \phi_{IJ} \left( 2\eta^{IJ} 1_{D_{1,d-1}} - \Gamma^I \cdot \Gamma^J \right) \right)^T = \exp \left( -\frac{1}{2} \phi_{IJ} \Gamma^I \cdot \Gamma^J \right)^T
\]

\[
S(-\phi)^T = \varphi^{-1}.
\]

We may finally write out the left-invariant supervector fields on \( s\mathcal{P}(1,d-1;N) \):

\[
P_l(\theta, x, \phi) = L(\phi)^l_I \frac{\partial}{\partial x^I}, \quad Q_\alpha^l(\theta, x, \phi) = S(\phi)^\beta_\alpha \left( \frac{\partial}{\partial \theta_\alpha} + \frac{1}{2} \theta_\alpha^\gamma C_\gamma^\delta \Gamma^I_J \delta \frac{\partial}{\partial x^I} \right),
\]

\[\text{We adopt mathematicians’ notation in which the supertranslation group is denoted as } \mathbb{R}^{1,d-1|N}, \text{ while physicists would have it in the form } \mathbb{R}^{1,d-1|N}.\]
These satisfy the familiar super-Poincaré (super)algebra
\[
[P_I, P_J] = 0, \quad [Q^i_\alpha, Q^j_\beta] = \delta^{ij} C_{\alpha \gamma} \Gamma^I_{\gamma \beta} P_I, \quad [P_I, Q^i_\alpha] = 0,
\]
(4.6)
\[
[J_{KL}, J_{MN}] = \eta_{KN} J_{LM} - \eta_{KM} J_{LN} + \eta_{LM} J_{KN} - \eta_{LN} J_{KM}.
\]
We shall also need the right-invariant supervector fields on \(\mathcal{S}(1, d - 1; N)\):
\[
\mathcal{P}_I(\theta, x, \phi) = \frac{\partial}{\partial \theta^I}, \quad Q^i(\theta, x, \phi) = \frac{\partial}{\partial x^i} - \frac{1}{2} \theta^i_\alpha C_{\gamma \beta} \Gamma^I_{\gamma \beta} \frac{\partial}{\partial \theta^I} + \frac{1}{2} \frac{\partial}{\partial \phi_I} \left( \theta^I \phi_I \right),
\]
\[
J_{IJ}(\theta, x, \phi) = x^K \left( \eta_{KJ} \frac{\partial}{\partial x^K} - \eta_{KI} \frac{\partial}{\partial x^K} \right) + \frac{1}{2} \left( \Gamma_{IJ} \right)_\beta^\alpha \phi^\beta_{\gamma} \frac{\partial}{\partial \theta^I} + \frac{1}{2} \frac{\partial}{\partial \phi_I} \left( \theta^I \phi_I \right),
\]
with the corresponding super-Poincaré (super)algebra
\[
[\mathcal{P}_I, \mathcal{P}_J] = 0, \quad [\mathcal{P}_I, Q^1_\beta] = -\delta^{ij} C_{\gamma \beta} \Gamma^I_{\gamma \beta} \mathcal{P}_I, \quad [\mathcal{P}_I, Q^i_\alpha] = 0,
\]
(4.7)
\[
[J_{KL}, J_{MN}] = -\eta_{KN} J_{LM} + \eta_{KM} J_{LN} - \eta_{LM} J_{KN} + \eta_{LN} J_{KM}.
\]
In their derivation, we employed the explicit vector and spinor representations
\[
(J_{KL})_I^J = \delta^J_I \eta_{KL} - \delta^I_J \eta_{KL}, \quad (J_{KL})_\beta^\alpha = \frac{1}{2} (\Gamma_{KL})_\beta^\alpha
\]
of the Lorentz generators.

The above data enable us to describe and manipulate, in a particularly convenient manner, the dual left-invariant Maurer–Cartan 1-forms which are instrumental in defining the super-\(\sigma\)-models. Thus, we parametrise the group as \((t_A = t_A(0, 0, 0))\)
\[
g(\theta, x, \phi) = e^{x^I P_I} \cdot e^{\theta^i_\alpha Q^i_\alpha} \cdot e^{\frac{1}{2} \phi^I_{KL} J_{KL}} \in \mathcal{S}(1, d - 1; N)
\]
and obtain the desired decomposition
\[
g^* \theta_L(\theta, x, \phi) = e^{-\frac{1}{2} \phi^I_{KL} J_{KL}} \cdot e^{-\theta^i_\alpha Q^i_\alpha} \cdot e^{-x^I P_I} \frac{d}{d \theta^I} \left( e^{x^J P_J} \cdot e^{\theta^i_\alpha Q^i_\alpha} \cdot e^{\frac{1}{2} \phi^I_{KL} J_{KL}} \right)
\]
\[
= \frac{d x^I}{\partial \theta^I} \left( e^{x^J P_J} \cdot e^{\theta^i_\alpha Q^i_\alpha} \cdot e^{\frac{1}{2} \phi^I_{KL} J_{KL}} \left( P_I \right) \right) + \left( \text{id}_{Q^i} \otimes \mathcal{P} \right) \left( e^{x^J P_J} \cdot e^{\theta^i_\alpha Q^i_\alpha} \cdot e^{\frac{1}{2} \phi^I_{KL} J_{KL}} \right) \left( \frac{d x^I}{\partial \theta^I} \right)
\]
\[
= \frac{d x^I}{\partial \theta^I} \left( e^{x^J P_J} \cdot e^{\theta^i_\alpha Q^i_\alpha} \cdot e^{\frac{1}{2} \phi^I_{KL} J_{KL}} \left( P_I \right) \right) + \frac{d}{d \theta^I} \cdot e^{\frac{1}{2} \phi^I_{KL} J_{KL}} \left( Q^i \right) \frac{1}{2} \left( \frac{d \left( \eta^{\nu}_{LM} \right)}{\partial \theta^I} \right) \frac{d \left( \phi^I_{KL} \right)}{\partial \theta^I} \left( P_I \right)
\]
\[
= \frac{d x^I}{\partial \theta^I} \left( e^{x^J P_J} \cdot e^{\theta^i_\alpha Q^i_\alpha} \cdot e^{\frac{1}{2} \phi^I_{KL} J_{KL}} \left( P_I \right) \right) + \frac{1}{2} \frac{d \left( \eta^{\nu}_{LM} \right)}{\partial \theta^I} \frac{d \left( \phi^I_{KL} \right)}{\partial \theta^I} \left( P_I \right)
\]
of the Maurer–Cartan 1-form. In its derivation, we have used the following identity (in which we have fixed \(n \in \mathbb{N}^\times\) and suppressed the representation label \(i\) for the sake of transparency):
\[
d(\theta^{\alpha_1} Q_{\alpha_1} \theta^{\alpha_2} Q_{\alpha_2} \ldots \theta^{\alpha_n} Q_{\alpha_n}) = \sum_{k=1}^{n} \theta^{\alpha_1} Q_{\alpha_1} \theta^{\alpha_2} Q_{\alpha_2} \ldots \theta^{\alpha_k} Q_{\alpha_k} \theta^{\alpha_{k+1}} Q_{\alpha_{k+1}} \ldots \theta^{\alpha_n} Q_{\alpha_n}
\]
\[
= \sum_{k=1}^{n} \theta^{\alpha_1} Q_{\alpha_1} \theta^{\alpha_2} Q_{\alpha_2} \ldots \theta^{\alpha_{k-1}} Q_{\alpha_{k-1}} \theta^{\alpha_{k+1}} \left( (Q_{\alpha_k} Q_{\alpha_{k+1}}) - Q_{\alpha_{k+1}} Q_{\alpha_k} \right) \theta^{\alpha_{k+2}} Q_{\alpha_{k+2}} \ldots \theta^{\alpha_n} Q_{\alpha_n} d\theta^{\alpha_k}
\]
\[
= \sum_{k=1}^{n} \theta^{\alpha_1} Q_{\alpha_1} \theta^{\alpha_2} Q_{\alpha_2} \ldots \theta^{\alpha_{k-1}} Q_{\alpha_{k-1}} \theta^{\alpha_{k+1}} Q_{\alpha_{k+1}} \theta^{\alpha_{k+2}} Q_{\alpha_{k+2}} \ldots \theta^{\alpha_n} Q_{\alpha_n} \theta^{\alpha_{k+1}} C_{\alpha \beta} \Gamma^I_{\gamma \beta} d\theta^\gamma P_I
\]
\[
+ \sum_{k=1}^{n} \theta^{\alpha_1} Q_{\alpha_1} \theta^{\alpha_2} Q_{\alpha_2} \ldots \theta^{\alpha_{k-1}} Q_{\alpha_{k-1}} \theta^{\alpha_{k+1}} Q_{\alpha_{k+1}} \theta^{\alpha_{k+2}} \left( (Q_{\alpha_k} Q_{\alpha_{k+1}}) \right)
\]
\[-\theta^{\alpha + \beta} \frac{d \phi^\alpha}{d \phi^\beta} = \theta^\alpha \frac{d \phi^\beta}{d \phi^\alpha}, \]

\[\begin{align*}
- Q_{\alpha_1} \theta^\alpha Q_{\alpha_2} \theta^\alpha & = \cdots \\
= (n-1) (n-2) \cdots 2 \theta^\alpha C_{\alpha \beta} \Gamma^J_{\gamma} d \theta^\gamma P_I \\
+ \theta^\alpha C_{\alpha \beta} \Gamma^I_{\gamma} d \theta^\gamma P_I \\
= (n-1) (n-2) \cdots 2 \theta^\alpha C_{\alpha \beta} \Gamma^J_{\gamma} d \theta^\gamma P_I \\
+ n \theta^\alpha C_{\alpha \beta} \Gamma^I_{\gamma} d \theta^\gamma P_I \end{align*}\]

In this manner, we identify the sought-after component left-invariant 1-forms in the decomposition

\[g^* \theta_L(\theta, x, \phi) = \theta_L^I (\theta, x, \phi) \otimes_R P_I + \theta_{L\alpha}^\sigma (\theta, x, \phi) \otimes_R Q_\sigma^I + \theta_{LKL}^\sigma (\theta, x, \phi) \otimes_R J_{KL}\]

as

\[\theta_L^I (\theta, x, \phi) = L(-\phi)^I_J \left( dx^J + \frac{1}{2} \theta^\alpha C_{\alpha \beta} \Gamma^J_{\gamma} d \theta^\gamma \right), \]

\[\theta_{L\alpha}^\sigma (\theta, x, \phi) = S(-\phi)_{\beta}^\alpha d \theta^\beta, \]

\[\theta_{LKL}^\sigma (\theta, x, \phi) = L(-\phi)^K_M d \theta^M_L (\phi)_{KL}. \]

Their invariance with respect to left translations on \(s \mathcal{P}(1, d-1; N)\) can also be checked directly. They satisfy the Maurer–Cartan equations

\[d \theta_L^I = -\eta_{IK} \theta_L^J \wedge \theta_L^I + \frac{1}{2} L(-\phi)^J_I \left( dx^J + \frac{1}{2} \theta^\alpha C_{\alpha \beta} \Gamma^J_{\gamma} d \theta^\gamma \right), \]

\[d \theta_{L\alpha}^\sigma = d S(-\phi)_{\beta}^\alpha \wedge \theta_{L\alpha}^\sigma = \left( e^{-\frac{1}{2} \phi^{KL}_J} J_{KL}^I \right)_{\beta}^\alpha \wedge \theta_{L\alpha}^\sigma \]

\[d \theta_{LKL}^\sigma = -\eta_{MN} \theta_{L\alpha}^M \wedge \theta_{L\alpha}^N, \]

dictated by the algebra [4.6] in consequence of the standard (Free Differential-Algebraic) relation between the Chevalley–Eilenberg model of the (super-)Lie-algebra cohomology and the Cartan–Eilenberg model of the Lie-(super)group invariant de Rham cohomology. Above, we used Eq. (4.8) to compute the differential

\[d L(-\phi)^I_J = \left( e^{-\frac{1}{2} \phi^{KL}_J} J_{KL}^I \right)_{\beta}^\alpha \wedge \theta_{L\alpha}^\sigma = \left( e^{-\frac{1}{2} \phi^{KL}_J} J_{KL}^I \right)_{\beta}^\alpha \wedge \theta_{L\alpha}^\sigma \]

and its spinorial variant.

Finally, we are ready to specify the (super)group-theoretic form of the lagrangian fields of the two formulations of the Green–Schwarz super-\(\sigma\)-model, regarded as supervariants (of various dimensionality) of the Wess–Zumino–Witten model of Ref. [Wit84]. Thus, in the Nambu–Goto formulation, we take the\(\phi\)-shifted) lagrangian field in the form

\[\gamma \circ \mathcal{X}_{(NG)} : \Omega \rightarrow s \mathrm{Mink}^1, d-1 \rightarrow s \mathcal{P}(1, d-1; N) : \sigma \rightarrow (x^I(\sigma), \theta^\alpha(\sigma)) \rightarrow e^{\xi(\sigma)} P_I \cdot e^{\theta^\alpha(\sigma)} Q_\sigma^I. \]

In the Hughes–Polchinski formulation, on the other hand, we further distinguish among the Grassmann-even coordinates the first \(p+1\) ones, to be denoted as \(\{ x^\perp \} \), which are to be thought of as describing
the embedding of the \((p + 1)\)-dimensional worldvolume of the super-\(p\)-brane in the \(d\)-dimensional target. Local departures from the embedding of flatness are parametrised by additional Goldstone fields

\[ \{ \phi^{i\Delta^i} \} \] associated with generators \( \{ J_{\Delta^i} \} \) of the Lorentz transformations broken by the embedding. Altogether, the lagrangean field of the model takes the form

\[ \gamma \circ X_{(HP)} : \Omega \longrightarrow s \text{Mink}^{1,d\mid N^1_{d-1}} \times (\text{Spin}(1,d-1)/\text{Spin}(1,p) \times \text{Spin}(d-p-1)) \longrightarrow \mathcal{S}(1,d-1; N) \]

\[ : \sigma \longmapsto (x^I(\sigma), \theta^a(\sigma), \phi^{i\Delta^i}(\sigma)) \longrightarrow e^{x^I(\sigma)} P_l \cdot e^{\theta_0^a(\sigma) Q_0} \cdot e^{i\Delta_i(\sigma) J_{\Delta^i}}. \]

(4.10)

From now onwards, we shall, in our analysis, use the shorthand notation (and make the assumptions) of Conv. \( \mathcal{A}, \mathcal{B} \) and consider non-extended supersymmetry,

\[ N = 1, \]

for the sake of clarity, so that the generation index \( i \) can be suppressed. In particular, we denote the corresponding invariant 1-forms as

\[ S(-\phi)^{\alpha_1 \ldots \alpha_{d-1}}(\theta, x) = \Sigma_{(\theta, x, \phi)} = \theta^\alpha(\theta, x, \phi) \]

\[ \text{in order to distinguish them clearly from their spacetime-indexed counterparts} \]

(4.11)

\[ \theta^\alpha_L(x, \phi) = L(-\phi)^{IJ} e^I(\theta, x) \]

in index-free expressions with contracted spinorial indices, such as, e.g., the following one

\[ \theta^\alpha_L \wedge \Gamma_{\alpha \beta} ^{\gamma \delta} \equiv \Sigma_L \wedge \Gamma^{\gamma \delta} \Sigma_L. \]

The field-theoretic relation between the two descriptions of the lagrangean field of the GS super-sigma-model introduced above is made precise in

**Proposition 4.1.** Fix \( d \in \mathbb{N} \) and \( p \in \overline{0, d-1} \) and consider the Minkowski spacetime \((\mathbb{R}^{1,d-1}, \eta)\), regarded as a Lie group. Take the orthogonal decomposition of its Lie algebra

\[ \bigoplus_{l=0}^{d-1} \langle P_l \rangle_R \equiv t(0) \]

induced by \( \eta \),

(4.12)

\[ t^{(0)}_\text{vac} = \bigoplus_{A=0}^{p} \langle P_A \rangle_R, \quad \epsilon^{(0)} = \bigoplus_{S=p+1}^{d-1} \langle P_S \rangle_R. \]

Next, extend the above Lie algebra to the full Poincaré algebra by adjoining the generators of the Lorentz algebra

\[ \text{so}(1, d-1) = \bigoplus_{K=L}^{d-1} \langle J_KL \rangle_R \equiv \mathfrak{r}, \]

further decomposed, relative to the splitting \((4.12)\), into the Lie subalgebra

\[ \bigoplus_{A=0}^{p} \langle J_{\Delta^A} \rangle_R \oplus \bigoplus_{S=p+1}^{d-1} \langle J_{\Delta^S} \rangle_R = \mathfrak{t}_\text{vac} \]

of Lorentz transformations preserving \((4.12)\), and its direct-sum completion

\[ \bigoplus_{A=0}^{p} \langle J_{\Delta^A} \rangle_R \oplus \bigoplus_{S=p+1}^{d-1} \langle J_{\Delta^S} \rangle_R \equiv \mathfrak{d}. \]

Finally, embed the Poincaré group generated by \( t^{(0)} \oplus \mathfrak{r} \) as a Lie subgroup in the super-Poincaré supergroup \( \mathcal{S}(1,d-1; N) \equiv \mathbb{G} \) as in Eq. \((1.11)\). Given a projector \( P \in \text{End}_\mathbb{C} S_1, d-1 \) on the spinor module \( S_1, d-1 \) correlated with the decomposition \((4.12)\) through the relation

\[ \{ P^\gamma_\alpha Q_\gamma, P^\delta_\beta Q_\delta \} = (P^T, \Gamma_{\delta^A}^\beta \cdot P_A \}

and thus determining a Lie superalgebra

\[ t^{(0)}_\text{vac} := t^{(0)}_\text{vac} \oplus t^{(1)}_\text{vac} \]

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with
\[ t^{(1)}_{\text{vac}} := \{ P^\alpha_{\alpha} Q_{\beta} \mid \alpha, \beta \in 1, D_{1,d-1} \} \cong \text{im} P, \]
define a Lie supergroup \( H \subset s\mathcal{D}(1,d-1;1) \) as the semidirect product of the Lorentz group Spin(1,d–1) with the Lie supergroup generated by \( t_{\text{vac}} \), which we shall write symbolically as
\[ H = \exp(t_{\text{vac}}) \times \text{Spin}(1,d-1). \]

The data enumerated above satisfy the assumptions of Prop. 3.2, and so the corresponding Green–Schwarz super-\( \sigma \)-model on \( G/H \) in the Hughes–Polchinski formulation is equivalent to the same model in the Nambu–Goto formulation.

Proof. The proposition is fairly self-evident. Indeed, the restrictions of the Minkowskian metric \( \eta \) to the directions \( \partial_A \eta \in 0, p \) and \( \partial_S \eta \in p + 1, d - T \) in the tangent sheet define – respectively – the non-degenerate bilinear symmetric forms \( \gamma \) and \( \tilde{\gamma} \) mentioned in Prop. 3.2. Furthermore, the action of the Lorentz generators \( \{ J_{AB} \}_{A,B=p,r} \cup \{ \mathcal{J}_{S,T} \}_{S,T=sp(1,d-1)} \) on the momenta \( \{ P_C \}_{C=q,0} \) integrates to a unimodular action of the Lie group \( \text{Spin}(1,p) \times \text{Spin}(d - p - 1) \) on \( t^{(0)}_{\text{vac}} \). Finally, we readily check that the identity (3.28) is trivially satisfied,
\[ \eta^{AB} f_{CSS} S \eta_{F0} \equiv - \eta^{AB} \eta_{BR} \delta^T_S \eta_{F0} = - \delta^A_C \eta_{S0} \equiv - f_{CSS} S. \]

4.2. The \( N = 1 \) GS super-(\( p + 2 \))-cocycles and the ensuing “old branescan”. The super-\( p \)-branes whose dynamics we intend to geometrise carry topological charge, and so their propagation defines a charge current to which a gauge field couples in the usual geometric manner, that is, through pullback (of the gauge potential) to the worldvolume of the charged object. The coupling gives rise to corrections to the condition of minimality of the classical embedding that follows from minimising the metric term of the (super-)\( \sigma \)-model action functional, and the corrections are determined by the field strength of the said gauge field. As was announced at the beginning of Sec. 3 these field strengths are certain distinguished \( s\mathcal{D}(1,d-1;1) \)-invariant de Rham super-(\( p + 2 \))-cocycles that – owing to the topological triviality of the super-Minkowski space – admit global primitives, none of which, however, is \( s\mathcal{D}(1,d-1;1) \)-invariant. The super-(\( p + 2 \))-cocycles that we want to consider take the general form
\[ (\theta, x, \phi) \]
\[ \chi^T(\theta) \wedge S(-\phi)^T \cdot C \cdot \Gamma_{11} \cdot S(-\phi) \cdot \sigma(\theta) = \sigma(\theta) \wedge S(\phi) \cdot \Gamma_{11} \cdot S(-\phi) \cdot \sigma(\theta) \]
\[ \chi = \chi^T(\theta) \wedge S(-\phi)^T \cdot C \cdot \Gamma_{11} \cdot S(-\phi) \cdot \sigma(\theta) \wedge \theta^{L1}_{L2} \cdots \theta^{Lp}_{Lp}, \]
with the sole exception
\[ (\theta, x, \phi) \]
\[ \chi^T(\theta, x, \phi) \]
\[ = \det(10) \cdot (\sigma \wedge \Gamma_{11}) \cdot (\sigma) \equiv \chi^T(\theta, x, 0) \]
and – for \( p > 0 \) –
\[ (\theta, x, \phi) \]
\[ \chi^T(\theta, x, \phi) \]
\[ = \sigma^T(\theta) \wedge S(-\phi)^T \cdot C \cdot \Gamma_{11} \cdot S(-\phi) \cdot \sigma(\theta) \wedge \theta^{L1}_{L2} \cdots \theta^{Lp}_{Lp} \cdot \sigma(\theta, x, \phi) \]
\[ = \sigma^T(\theta) \wedge S(-\phi)^T \cdot C \cdot \Gamma_{11} \cdot S(-\phi) \cdot \sigma(\theta) \wedge \theta^{L1}_{L2} \cdots \theta^{Lp}_{Lp} \cdot \sigma(\theta, x, \phi) \]
\[ = \sigma^T(\theta) \wedge \Gamma_{K1} K2 \cdots Kp \cdot \sigma(\theta) \]
\[ \wedge \eta_{L1} L2 \cdots \eta_{Lp} L(\phi) J1 \cdots Jp L(\phi) L1 \cdots Lp \cdot \theta^{L1}_{L2} \cdots \theta^{Lp}_{Lp} \cdot \sigma(\theta, x, \phi) \]
\[ = \sigma^T(\theta) \wedge \Gamma_{K1} K2 \cdots Kp \cdot \sigma(\theta) \wedge \bigwedge_{\text{all \( p \)}} \eta_{JK} L(\phi) J1 \cdots Jp L(\phi) L1 \cdots Lp \cdot \sigma(\theta, x, \phi) \]
\[ \equiv \chi^T(\theta, x, 0) \]
with 
\[ e^{I_1I_2\ldots I_p} \equiv e^{I_1} \land e^{I_2} \land \ldots \land e^{I_p}, \]
and, for all \( K, L \in 0, d - 1, \)
\[ J_{KL} \cdot \chi_{(p + 2)}(\theta, x, \phi) = 0, \]
which means that the super-\((p + 2)\)-forms are not only rotationally invariant but also horizontal, and so, altogether, basic.

For \( p > 0 \), their closedness,
\[ 0 \overset{!}{=} d_{(p + 2)} \chi_{(p + 2)} = \frac{2}{p} \sum L^\alpha (\Gamma_{I_1I_2\ldots I_p})_{\alpha \beta} \sum L^\beta \land \sum L^\gamma \land (\Gamma_{I^1})_{\gamma \delta} \sum L^\delta \land \theta^{I_1I_2\ldots I_p}, \]
is ensured by a suitable choice of the relevant representation of the Clifford algebra such that the symmetry constraints
\[ (4.15) \]
\[ \Gamma_{\alpha(\beta}(\Gamma_I)_{\gamma\delta)) = 0 \]
implied by the previous condition are obeyed. Note that for \( p = 1 \) the latter reduces to the (more) familiar identity
\[ (4.16) \]
due to the assumed symmetry of the \( \Gamma_I \), cp. Conv. A.3. The admissible pairs \((d, p)\) for which the above constraints can be solved and a super-\(\sigma\)-model with the appropriate supersymmetry (cp. Sec. B) can be written down were found in Ref. [AETW87] and constitute the so-called “old branescan”. Closedness of the GS super-\((p + 2)\)-cocycles implies – in consequence of the (de Rham-)cohomological triviality of their support (which follows directly from the Kostant Theorem of Ref. [Kos77]) – the existence of smooth primitives. These were derived in Refs. [HLP86], albeit in a different convention, and so we rederive them in App. B through an adaptation of the original method to the current algebraic setting.

**Proposition 4.2.** For any \( p > 0 \), the GS super-\((p + 2)\)-cocycle \( \chi_{(p + 2)} \) of Eq. (4.13) admits a manifestly
ISO\((1, d - 1)\)-invariant primitive
\[ (4.17) \]
\[ \beta_{(p + 1)}(\theta, x) = \frac{1}{p + 1} \sum_{k=0}^{p} \vartheta \Gamma_{I_1I_2\ldots I_p} \sigma(\theta) \land d\chi_{I_1} \land dx^{I_2} \land \ldots \land dx^{I_k} \land e^{I_{k+1}I_{k+2}\ldots I_p}(\theta, x). \]
A primitive of the super-2-form \( \chi_{(2)} \) of Eq. (4.14) can be chosen in the form
\[ \beta_{(1)}(\theta, x) = \vartheta \Gamma_{II} \sigma(\theta). \]

**Proof.** Cp. App. [4.1].

The above primitives are manifestly non-supersymmetric. In fact, this cannot be repaired as it was demonstrated in Ref. [DAT85] that the GS super-\((p + 2)\)-cocycles do not admit s.\(\sigma\)(1, \(d - 1)\)-invariant primitives. This result puts us naturally in the framework of the (\(\mathbb{R}\)-valued) Chevalley–Eilenberg cohomology of the super-Poincaré algebra (cp. Ref. [C48]), which we shall exploit in the present treatment, whence a recapitulation thereof in App. [3] in the superalgebraic context of interest.

On the other hand, the manifest left-invariance of the GS super-\((p + 2)\)-cocycle \( \chi_{(p + 2)} \) itself, in conjunction with the triviality of the standard de Rham cohomology of \( s\text{Mink}^{1,d-1}\vert_{D_{1,d-1}} \), ensures that condition Eq. (2.13) is satisfied as the supersymmetry variation of the global primitive \( \beta_{(p + 1)} \) is exact. Indeed, we have the identity
\[ d_{(p + 1)} \beta_{(p + 1)} = \chi_{(p + 2)} = \ell_{(p + 2)}(\chi_{(p + 2)}), \]
which implies the existence of a super-\(p\)-form \( \beta_{(p + 1)}(\epsilon, y) \in \wedge^p \mathcal{T}^*\text{sMink}^{1,d-1}\vert_{D_{1,d-1}} \) satisfying the condition
\[ (\delta_{(p + 1)} \beta_{(p + 1)}(\epsilon, y) = \epsilon \ell_{(p + 2)}(\chi_{(p + 2)}), \]
\[ \beta_{(p + 1)}(\epsilon, y) = \ell_{(p + 2)}(\chi_{(p + 2)}). \]
We shall call $J_{(p)}(\varepsilon, y)$ the **target supercurrent**. The action 1-cochain now takes the explicit form

$$c_{(\varepsilon, y)}[\chi] = e^{i f_{\alpha} \langle \varepsilon_{(y), \gamma} \gamma \eta I_{[\alpha]} \rangle} J_{(\varepsilon, y)}^{(p)}.$$

The very same arguments imply the existence of an extension, to the generalised tangent sheaf

$$\mathcal{E}^{1, p}_{s\text{Mink}}^{1, d-1} | D_{1, d-1} = T_{s\text{Mink}}^{1, d-1} | D_{1, d-1} \oplus \bigwedge^{p} T^{*} s\text{Mink}^{1, d-1} | D_{1, d-1},$$

of the algebra of the (left) supersymmetry generators

$$\varphi(\varepsilon, y):= e^{\beta_{1}} \mathcal{Q} \alpha(\theta, x) + y^{I} \mathcal{P} I(\theta, x), \quad (\varepsilon, y) \in \mathbb{R}^{1, d-1} | D_{1, d-1},$$

with the Lie bracket

$$[\varphi(\varepsilon, y), \varphi(\varepsilon, y)] = \left[\varepsilon_{1}^{\alpha} \mathcal{Q} \alpha, \varepsilon_{2}^{\beta} \mathcal{Q} \beta\right] = -\varepsilon_{1}^{\alpha} \varepsilon_{2}^{\beta} \left\{ \mathcal{Q} \alpha, \mathcal{Q} \beta \right\} = \varphi(0, \varepsilon_{1} \Gamma_{\varepsilon_{2}})$$

readily derived from the elementary ones in Eq. (4.17) and giving us the **right** super-Minkowski supersymmetry algebra

$$[(\varepsilon_{1}, y_{1}), (\varepsilon_{2}, y_{2})] = (0, \varepsilon_{1} \Gamma_{\varepsilon_{2}}).$$

Indeed, we have

**Proposition 4.3.** For any $p \in \mathbb{N}$, the fundamental vector field $\varphi(\varepsilon, y)$ of Eq. (4.18) (defined as above for arbitrary $(\varepsilon, y) \in \mathbb{R}^{1, d-1} | D_{1, d-1}$) is generalised hamiltonian with respect to the super-$(p+2)$-cocycle $\chi$ of Eq. (4.13), that is, there exists a globally smooth super-$p$-form $\theta(\varepsilon, y) \in \Lambda^{p} T^{*} s\text{Mink}^{1, d-1} | D_{1, d-1}$ with the property

$$\varphi(\varepsilon, y) \cdot \chi = -\frac{d}{\varepsilon} \theta(\varepsilon, y),$$

The latter can be chosen in the manifestly ISO(1, d-1)-invariant form

$$\theta_{(p)}(\varepsilon, y)(\theta, x) = -2 \varepsilon_{I_{1}} \Gamma_{\varepsilon_{2}} \theta$$

for $p = 0$, and for $p > 0$

$$\theta_{(p)}(\varepsilon, y)(\theta, x) = -p y^{I} \beta_{I}(\theta, x) - 2(\varepsilon_{I_{1}} \Gamma_{I_{2}} \ldots I_{p}) \theta e_{I_{1}} I_{2} \ldots I_{p}(\theta, x)$$

$$+ \frac{(2p+1)!!}{(2p-1)!!} \sum_{k=1}^{p} \frac{2^{k}(2p+1-2k)!!}{(2p-k)!!} \eta_{I_{k} I_{3} \ldots I_{p}} \wedge d x^{I_{2}} \wedge d x^{I_{3}} \wedge \ldots \wedge d x^{I_{p}} \wedge e_{I_{k+1}} I_{k+3} \ldots I_{p}(\theta, x),$$

written in terms of the super-$p$-forms $\beta_{I}$ from Eq. (3.3) and of the super-$1$-forms $\eta_{I_{1} I_{2} I_{3} \ldots I_{p}}$ from Eq. (3.2).

**Proof.** Cp App. 37.

The extension, defined in terms of the Vinogradov-type bracket of Eq. (3.14), is readily seen to close on pairs of the distinguished fundamental sections

$$\mathcal{B}(\varepsilon, y) = \text{R}(\varepsilon, y) \oplus \theta(\varepsilon, y) \in \mathcal{E}^{1, p}_{s\text{Mink}}^{1, d-1} | D_{1, d-1}$$

of the generalised tangent bundle over $s\text{Mink}^{1, d-1} | D_{1, d-1}$. It turns out that the **right**-regular action of the supersymmetry group $\mathbb{R}^{1, d-1} | D_{1, d-1}$ on $s\text{Mink}^{1, d-1} | D_{1, d-1}$, while not a global symmetry of the super-$\sigma$-model as it stands, cp. Sec. 6.3, exhibits similar properties relative to the GS super-$(p+2)$-cocycles, which justifies our discussion of the corresponding sections of the generalised tangent bundle over $s\text{Mink}^{1, d-1} | D_{1, d-1}$ in the next section. There, we carry out a case-by-case analysis of the relevant current super-2-cocycles

$$\langle \delta J_{(p)}(\varepsilon_{1}, y_{1}), (\varepsilon_{1}, y_{1}) \rangle = \ell_{(p)}^{(\varepsilon_{2}, y_{2})}(\varepsilon_{1}, y_{1}) - J_{(p)}(\varepsilon_{1} + \varepsilon_{2}, y_{1} + y_{2} - \frac{1}{2} \varepsilon_{1} \Gamma_{\varepsilon_{2}}) + J_{(p)}(\varepsilon_{2}, y_{2})$$

and derive the Lie anomaly for the various natural actions of the supersymmetry group that can be constructed out of the two-sided regular actions.

---

\[ Note\] that we (intentionally) consider Graßmann-even supervector fields here.
The Green–Schwarz superstring. Consider, next, the $\mathbb{R}^{1,d|N}$-invariant GS 3-form superfield
\begin{equation}
\chi^{(3)} = E^{\alpha} \wedge \bar{\chi} \wedge \Gamma_{\alpha} \mathcal{E},
\end{equation}
with the $\mathbb{R}^{1,d|0}$-invariant primitive
\begin{equation}
\beta^{(2)}(x, \theta) = \left. \bar{\mathcal{A}} \Gamma_{\alpha} d\theta \wedge E^{\alpha}(x, \theta) \right|_{\mathbb{R}^{1,d|0}} = \bar{\mathcal{A}} \Gamma_{\alpha} d\theta \wedge dx^{\alpha},
\end{equation}
where the last equality is a straightforward consequence of Conv. A.2.

In the canonical description, we readily derive the Cartan–Poincaré form
\begin{equation}
\Theta(x, \theta, \xi, t) = -\mathcal{L}_{\text{GS}}(x, \theta, \xi, t) d\sigma^{0} \wedge d\sigma^{1}
\end{equation}
and gives rise to Noether charges
\begin{equation}
Q_{(y, \varepsilon)}[x, \theta, p] := \int_{S^{1}} \text{Vol}(S^{1}) p_{a} E^{a}(x, \theta).
\end{equation}
These satisfy the algebra
\begin{equation}
\{Q_{(y_{1}, \varepsilon_{1})}, Q_{(y_{2}, \varepsilon_{2})}\} = Q_{(y_{1}, \varepsilon_{1}), (y_{2}, \varepsilon_{2})}[x, \theta, p]
\end{equation}

\begin{equation}
\mathcal{K}_{(y, \varepsilon)}(x, \theta) = \{Q_{(y_{1}, \varepsilon_{1})}, (y_{2}, \varepsilon_{2})\}[x, \theta, p],
\end{equation}
with the vanishing of the last term in the middle expression being ensured by Eq. (4.22). Thus, the Noether charges furnish a Hamiltonian realisation of the Lie (super)algebra $\mathbb{R}^{1,d|N}$ on $P_{\text{GS}}$.

The above algebra is modelled by the Vinogradov-type (or, indeed, $\chi$-twisted Courant) bracket of the fundamental sections
\begin{equation}
\mathcal{R}_{(y, \varepsilon)}(x, \theta) = \mathcal{K}_{(y, \varepsilon)}(x, \theta) \oplus \left[ -y^{\beta} \bar{\mathcal{A}} \Gamma_{\theta} d\theta - (\bar{\varepsilon}) \Gamma_{\theta} \left( 2d\mathcal{A}^{a} - \frac{1}{2} \bar{\mathcal{A}} \Gamma_{\theta} d\theta \right) \right]
\end{equation}
of $\mathcal{E}^{1,1}\text{Mink}^{1,d|N}$, given by
\begin{equation}
\left[ \mathcal{R}_{(y_{1}, \varepsilon_{1})}, \mathcal{R}_{(y_{2}, \varepsilon_{2})} \right]^{(3)}_{\mathcal{V}} = \mathcal{R}_{(y_{1}, \varepsilon_{1}), (y_{2}, \varepsilon_{2})} + 0 \oplus \frac{1}{2} d\left[ y_{1}^{\alpha} (\bar{\varepsilon}_{2} \Gamma_{\theta}) - y_{2}^{\alpha} (\bar{\varepsilon}_{1} \Gamma_{\theta}) + 2(\bar{\varepsilon}_{1} \Gamma_{\varepsilon_{2}}) x^{\alpha} \right]
\end{equation}

\begin{equation}
+ 0 \oplus 2 \left[ (\bar{\varepsilon}_{1} \Gamma_{\varepsilon_{2}}) \bar{\mathcal{A}} \Gamma_{\theta} d\theta + (\bar{\varepsilon}_{2} \Gamma_{\theta}) \bar{\mathcal{A}} \Gamma_{\varepsilon_{2}} d\theta + (\bar{\mathcal{A}} \Gamma_{\theta} \varepsilon_{2}) \right]_{\mathcal{E}^{1,1}\text{Mink}^{1,d|N}}
\end{equation}

\begin{equation}
(\delta \beta_{(2)})(x, \theta) = \bar{\varepsilon} \Gamma_{\theta} d\theta \wedge dx^{\alpha} - \frac{1}{2} \left( \bar{\mathcal{A}} \Gamma_{\theta} d\theta \wedge \bar{\varepsilon} \Gamma_{\theta} d\theta \right) - \frac{1}{2} (\bar{\varepsilon}_{1} \Gamma_{\theta} d\theta) \wedge \left( \bar{\varepsilon}_{1} \Gamma_{\theta} d\theta \right),
\end{equation}
and the last term vanishes as a contraction of the symmetric tensor \( \eta \) with the anticommuting 1-forms. The middle term can be rewritten – with the help of identity (??) – as
\[
(\bar{\Theta}\Gamma_a d\theta) \wedge (\bar{\tau}\Gamma^a d\theta) = \Gamma_{ab\gamma} \Gamma_{\alpha\beta}^a \bar{\Theta}^\alpha d\gamma \wedge \varepsilon^\gamma d\theta^\beta = \frac{1}{2} \left[ \Gamma_{a\alpha\beta} \bar{\Theta}^a \gamma + \Gamma_{a\alpha\beta} \bar{\Theta}^a \gamma \right] \theta^\alpha d\theta^\beta \wedge \varepsilon^\gamma d\theta^\delta
\]
\[
= -\frac{1}{2} \bar{\Theta} \Gamma_{a\alpha\beta} \Delta_{\beta\gamma} \bar{\Theta}^\alpha d\theta^\beta \wedge \varepsilon^\gamma d\theta^\delta \equiv -\frac{1}{2} (\bar{\tau}\Gamma_a d\theta) \bar{\Theta} \Gamma^a d\theta
\]
\[
= d\left( -\frac{1}{2} (\bar{\tau}\Gamma_a d\theta) \bar{\Theta} \Gamma^a d\theta \right) + \frac{1}{2} (\bar{\tau}\Gamma_a d\theta) \wedge (\bar{\Theta} \Gamma^a d\theta)
\]
whence also
\[
(\bar{\Theta}\Gamma_a d\theta) \wedge (\bar{\tau}\Gamma^a d\theta) = d\left( -\frac{1}{2} (\bar{\tau}\Gamma_a d\theta) \bar{\Theta} \Gamma^a d\theta \right).
\]
Thus, the target current associated with our choice of the primitive \( \beta \) takes the form
\[
(4.24) \quad j_{(y,\xi)}(x, \theta) = (\tau\Gamma_a d\theta) \left( dx^a + \frac{1}{6} \bar{\Theta} \Gamma^a d\theta \right)
\]
We compute
\[
(\delta j)_{(1)}(y,\xi_1,\xi_2)(x, \theta) = d \left[ (\tau\Gamma_a \xi_2)(x^a - \frac{1}{3} \bar{\xi}_2 \Gamma^a \theta) \right] - \frac{1}{6} \left[ 2(d\bar{\Theta}\Gamma_a \xi_1)(\bar{\Theta} \Gamma^a \xi_2) + (d\bar{\Theta}\Gamma_a \theta)(\bar{\Theta} \Gamma^a \xi_2) \right]
\]
\[
= d \left[ (\tau\Gamma_a \xi_2)(x^a - \frac{1}{3} \bar{\xi}_2 \Gamma^a \theta) - \frac{1}{6} (\tau\Gamma_a \theta)(\bar{\Theta} \Gamma^a \xi_2) \right]
\]
\[
- \frac{1}{6} \left[ (d\bar{\Theta}\Gamma_a \xi_2)(\bar{\Theta} \Gamma^a \xi_1) + (\tau\Gamma_a \theta)(\bar{\Theta} \Gamma^a \xi_2) + (d\bar{\Theta}\Gamma_a \theta)(\bar{\Theta} \Gamma^a \xi_2) \right].
\]
Taking, again, Eq. (??) into account, we find the relation
\[
(\delta j)_{(1)}(y,\xi_1,\xi_2)(x, \theta) = d \left[ (\tau\Gamma_a \xi_2)(x^a - \frac{1}{3} \bar{\xi}_2 \Gamma^a \theta) - \frac{1}{6} (\tau\Gamma_a \theta)(\bar{\Theta} \Gamma^a \xi_2) \right].
\]
We conclude that the GS superstring admits a non-projective realisation of the classical symmetry group \( \mathbb{R}^{1,d|N} \) on its Hilbert space.

We readily verify that the primitive of the current 2-cocycle
\[
\varphi_{(y,\xi_1,\xi_2)}(x, \theta) := (\tau\Gamma_a \xi_2)(x^a - \frac{1}{3} \bar{\xi}_2 \Gamma^a \theta) - \frac{1}{6} (\tau\Gamma_a \theta)(\bar{\Theta} \Gamma^a \xi_2)
\]
yields a constant 3-cocycle
\[
(\delta\varphi)_{(y,\xi_1,\xi_2,\xi_3)}(x, \theta) = (\tau\Gamma_a \xi_2) y_3^a - \frac{1}{6} \left[ 2(\tau\Gamma_a \xi_2)(\bar{\Theta} \Gamma^a \xi_3) + (\tau\Gamma_a \theta)(\bar{\Theta} \Gamma^a \xi_3) \right]
\]
\[
= \lambda_{(y,\xi_1,\xi_2,\xi_3)}.
\]

The supermembrane. By way of a closing remark, we note that besides the ISO\((1,d-1)\)-invariant super-\(p\)-forms \( \kappa_{(p)} \) of the GS super-\((p+2)\)-cocycles give rise – as revealed by inspection – to a host of supersymmetric super-\(2\)-cocycles that play a fundamental rôle in our geometrisation of the s\(\mathcal{P}(1,d-1)\) invariant cohomology classes of the \( \chi \). These will be obtained through contraction of (certain)
\( \kappa_{(p)} \) \( p \)-tuples of fundamental (right-invariant) vector fields \( \kappa_{A_1} \in \{ \mathcal{D}_a, \mathcal{P}_1 \}_{(a,1)} \mathcal{P}_{1,2,3}\cdots 0, \cdots, 1, 1 \}_p \), \( i \in \mathbb{Z}, p \) (the \( A_1 \) are indices of the supersymmetry algebra) of Eq. (4.18) with the super-\((p+2)\)-cocycle \( \chi \) of Eq. (1.13),
\[
(4.25) \quad b_{\chi} := \lambda^{A_1 A_2 \cdots A_p} \kappa_{A_1} \cdots \kappa_{A_p} \lambda \downarrow \chi_{(p+2)}, \quad \lambda^{A_1 A_2 \cdots A_p} \in \mathbb{R}.
\]
Both, the condition of closedness and the condition of invariance are tantamount to certain linear constraints on the coefficients \( \lambda^{A_1 A_2 \cdots A_p} \) which involve (also linearly) the structure constants of the Lie superalgebra under consideration, and so it is far from obvious that such super-\(2\)-cocycles exist. Specific examples will be examined closely in Sec.
5. Supergerbes for the Nambu–Goto Super-p-branes from Extensions of $sMink^{1,d-1|D_{1,d-1}}$

Our next aim, motivated amply in Sec. 3, is to work out for the GS super-$(p + 2)$-cocycles $\chi^{(p+2)}$ of Sec. 2 a supergeometric analogon of the standard scheme of geometrisation of de Rham cocycles known from the theory of fibre bundles with connection and the theory of bundle $(n)$-gerbes with connection and recalled briefly in Sec. 3 in the context of the 2d bosonic $\sigma$-model with the Wess–Zumino term. The conceptual basis of our construction is the relation between algebra and geometry of the Lie (super)group established – in the manner delineated in Thm. C.7 – by the Chevalley–Eilenberg model of Lie-(super)algebra cohomology (with values in the trivial module $\mathbb{R}$) in conjunction with the interpretation – expressed in Props. C.4 and C.5 – of the second cohomology group in that model in terms of equivalence classes of (super)central extensions of the underlying Lie-(super)algebra. More specifically, the said cohomological results enable us to associate with the super-$(p + 2)$-cocycles $\chi^{(p+2)}$ a tower of supergroup extensions of the Lie supergroup $\mathbb{R}^{1,d-1|D_{1,d-1}} \equiv sMink^{1,d-1|D_{1,d-1}}$ (of the kind originally discovered in Ref. [CdAIPB00]) that are readily verified to play the rôle of the various surjective submersions encountered in the geometric definition of the $(0-, 1$- and $2$-)gerbe and thus give us a natural definition of a supergerbe with curvature $\chi$, conceived along the lines of the fundamental Principle of Categorial Descent of Ref. [Ste00].

Rudimentary aspects of the Lie-superalgebra (to be abbreviated as LSA in what follows) cohomology and its Chevalley–Eilenberg (to be abbreviated as CE) model, as well as the link with the Cartan–Eilenberg (to be abbreviated as CaE) cohomology of supersymmetric differential superforms that are of relevance to the subsequent discussion have been recalled in App. C.

5.1. Geometrisation of Cartan–Eilenberg super-cocycles. By way of preparation for the systematic (super)geometric resolution of the super-$(p + 2)$-cocycles of interest, we should first review – after Refs. [AdA85, CdAIPB00] – the construction of the super-Minkowski spacetime $sMink^{1,d-1|D_{1,d-1}}$ as a central extension of the purely Grassmann-odd superspace $\mathbb{R}^{0|D_{1,d-1}}$ (the so-called superpoint, also known as the odd hyperplane), determined by a canonical 2-cocycle on the supercommutative Lie superalgebra $\mathbb{R}^{0|D_{1,d-1}}$ with values in its trivial module $\mathbb{R}^{1,d-1}$. A natural point of departure for our general discussion is the manifestly closed left-invariant (to be abbreviated as LI in what follows) super-2-form

$$\chi_{(2)}^I := \frac{1}{2} \sigma \wedge \Gamma^I \sigma, \quad I \in 0, d - 1$$

on the supermanifold $\mathcal{M}^{(0)} \equiv \mathbb{R}^{0|D_{1,d-1}}$, with global Grassmann-odd coordinates $\{\theta^\alpha\}_{\alpha \in 1, D_{1,d-1}}$ and the associated LI vector fields

$$\mathcal{D}_\alpha^{(0)}(\theta) = \frac{\partial}{\partial \theta^\alpha}$$

furnishing a realisation of the supercommutative LSA

$$\{\mathcal{D}_\alpha^{(0)}, \mathcal{D}_\beta^{(0)}\} = 0.$$  

The de Rham super-2-cocycle $\chi_{(2)}^I$ does not admit a primitive on $\mathcal{M}^{(0)}$ invariant with respect to the (left) regular action of $\mathbb{R}^{0|D_{1,d-1}}$ on (itself) $\mathcal{M}^{(0)}$,

$$\iota^{(0)} : \mathbb{R}^{0|D_{1,d-1}} \times \mathcal{M}^{(0)} \to \mathcal{M}^{(0)} : (\varepsilon^\alpha, \theta^\alpha) \mapsto \theta^\alpha + \varepsilon^\alpha,$$

and so – arguing along the lines of Appendix C – we are led to consider a (super)central extension $\mathcal{M}^{(1)} := sMink^{1,d-1|D_{1,d-1}}$ of the Lie supergroup $\mathcal{M}^{(0)}$, the former being canonically surjectively submersed onto the latter as a rank-$d$ (real) vector bundle

$$\pi_0 \equiv \text{pr}_1 : \mathcal{M}^{(1)} \to \mathcal{M}^{(0)} : (\theta^\alpha, x^I) \mapsto \theta^\alpha$$

with fibre coordinates $x^I, I \in 0, d - 1$. The pullback of the GS super-2-cocycle $\chi_{(2)}^I$ to $\mathcal{M}^{(1)}$ trivialises in the associated CaE cohomology as (cp. Remark C.7)

$$\pi_0^* \chi_{(2)}^I = de^I,$$

22Strictly speaking, we present an explicit analysis for the cases $p \in (0, 1, 2)$. However, the structural nature of our construction turns it into a tenable proposal for a completely general geometrisation scheme.

23For a detailed account of the fibre-bundle structure on the extended superspacetime(s), consult Refs. [AdA85, CdAIPB00].
for the \( \epsilon^I \) as defined by Eqs. (4.9) and (4.11). The corresponding (super)centrally extended LSA of the equivariant lifts
\[
(5.1) \quad \mathcal{L}^{(1)}_\alpha(\theta, x) := \mathcal{L}^{(0)}_\alpha(\theta) + \frac{1}{2} \Gamma^I_{\alpha \beta} \delta^\beta \frac{\partial}{\partial x^I} 
\]
of the \( \mathcal{L}^{(0)}_\alpha \) and of the coordinate vector fields
\[
(5.2) \quad \mathcal{P}^{(1)}_I(\theta, x) := \frac{\partial}{\partial x^I},
\]
the two families making up a basis of the tangent sheaf dual to that of the cotangent sheaf formed by the LI super-1-forms \( \sigma^\alpha, \alpha \in \Gamma_{1, D_1, d-1} \) and \( \epsilon^I, I \in 0, d-1 \), reads
\[
\{ \mathcal{L}^{(1)}_\alpha, \mathcal{L}^{(1)}_\beta \} = \Gamma^I_{\alpha \beta} \mathcal{P}^{(1)}_I, \quad [ \mathcal{P}^{(1)}_I, \mathcal{P}^{(1)}_J ] = 0, \quad [ \mathcal{P}^{(1)}_I, \mathcal{L}^{(1)}_\alpha ] = 0.
\]

The action of the original supergroup \( \mathbb{R}^{0|D_1,d-1} \) on \( \mathcal{M}^{(1)} \) follows from the demand that it project to \( \ell^{(0)} \) and that the super-1-forms \( \sigma^\alpha \) and \( \epsilon^I \), \( I \in 0, d-1 \) be invariant with respect to it, and we may extend it to a full-blown structure of a Lie supergroup on \( \mathcal{M}^{(1)} \) by requiring that it yield the above supervector fields \( \mathcal{L}^{(1)}_\alpha, \alpha \in \Gamma_{1, D_1, d-1} \) and \( \mathcal{P}^{(1)}_I, I \in 0, d-1 \) as the fundamental left-invariant supervector fields and that it leaves the super-1-forms intact when treated as an action of \( \mathbb{R}^{1,d-1|D_1,d-1} \) on (itself) \( \mathcal{M}^{(1)} \) – this determines the said action in the familiar form
\[
(5.3) \quad \ell^{(1)} \equiv m^{(1)} : \mathbb{R}^{1,d-1|D_1,d-1} \times \mathcal{M}^{(1)} \rightarrow \mathcal{M}^{(1)}
\]
equivalent to the one given in Eq. (2.12) (for \( N = 1 \)). Clearly, we could have equivalently derived it from the LSA (5.3) by exponentiating the generators and computing, with the help of the standard Baker–Campbell–Hausdorff formula,
\[
e^{\sigma^a} \mathcal{L}^{(1)}_\alpha + y^I \mathcal{P}^{(1)}_I, e^{\theta^\beta} \mathcal{L}^{(1)}_\beta + x^I \mathcal{P}^{(1)}_I = e^{(\sigma^a + \theta^\beta)}(\mathcal{L}^{(1)}_\alpha + (y^I + x^I) \mathcal{P}^{(1)}_I) + \frac{1}{2} [e^{\sigma^a} \mathcal{L}^{(1)}_\alpha, e^{\theta^\beta} \mathcal{L}^{(1)}_\beta] = e^{(\sigma^a + \theta^\beta)}(\mathcal{L}^{(1)}_\alpha + (y^I + x^I) \mathcal{P}^{(1)}_I) - \frac{1}{2} e^{\sigma^a} \theta^\beta \mathcal{L}^{(1)}_\alpha \mathcal{L}^{(1)}_\beta = e^{(\sigma^a + \theta^\beta)}(\mathcal{L}^{(1)}_\alpha + (y^I + x^I - \frac{1}{2} \Gamma^I_{\alpha \beta} \theta) \mathcal{P}^{(1)}_I).
\]

Thus, if we take \( \mathcal{M}^{(0)} \) as the basis of our supergeometry for the sake of illustrating the extension principle, the CaE/CE-cohomological trivialisation leads us quite naturally to a surjective submersion over it, with the commutative typical fibre \( \mathbb{R}^{1,d-1} \). In the remainder of this section, we assume, instead, the super-Minkowski spacetime \( \mathcal{M}^{(1)} \) to be the actual basis of subsequent extensions necessitated by the trivialisation of the GS super-(\( p + 2 \))-cocycles. However, in order to indicate the relation between \( \mathcal{M}^{(0)} \) and \( \mathcal{M}^{(1)} \), we pedantically pull back the LI super-1-forms \( \sigma \) to \( \mathcal{M}^{(1)} \) along \( \pi_0 \).

### 5.1.1. The super-0-brane.

The GS super-2-cocycle on the ten-dimensional super-Minkowski spacetime \( \mathcal{M}^{(1)} \) reconstructed above that codeterminates the dynamics of the super-0-brane has the simple form
\[
(5.5) \quad \chi^{(2)} = \Theta \wedge \Gamma_{11} \sigma \equiv \sigma^T \wedge C \cdot \Gamma_{11} \sigma,
\]
implicitly written in a spin representation of the Clifford algebra in which the product of the charge-conjugation matrix and the volume element \( \Gamma_{11} \) is symmetric,
\[
(5.6) \quad (C \cdot \Gamma_{11})^T = C \cdot \Gamma_{11}.
\]
The super-2-form is manifestly LI but does not possess a primitive with this property. Indeed, a global primitive \( \beta \) of \( \chi \) satisfies
\[
\beta^{(1)} (\theta, x) - \Theta \Gamma_{11} \sigma (\theta, x) \in \ker d,
\]
and so – in view of the triviality of the de Rham cohomology of sMink\(^{1,d-1|D_1,d-1}\) – we have
\[
\beta^{(1)} (\theta, x) = \Theta \Gamma_{11} \sigma (\theta) + da (\theta, x)
\]
for a (Graßmann-)even-valued superfunction \( a \) on sMink\(^{1,d-1|D_1,d-1}\). Write
\[
a (\theta, x) = a_0 (x) + \theta^a a_{1,a} (\theta, x)
\]
for some even-valued superfunction $a_0$ and an odd-valued one $a_{1\alpha}$, so that
\[ da(\theta, x) = dx^I \partial_I a_0(x) + \theta^{\alpha} \partial^{\alpha} a_{1\alpha}(\theta, x) + dx^I \theta^{\alpha} \partial_I a_{1\alpha}(\theta, x) - \theta^\beta \partial_I \overrightarrow{\partial}_\beta a_{1\alpha}(\theta, x). \]
Demanding that the above be $\mathbb{R}^{1,d-1}$-invariant at every point $(\theta, x) \in \text{sMink}^{1,d-1|D_1,d-1}$ of its domain implies that the $\partial_I a_0(x)$ are Grassmann-even constants and that the $\partial_I a_{1\alpha}(\theta, x)$ are odd-valued superfunctions of $\theta$, or that
\[ a_{1\alpha}(\theta, x) = c_{1\alpha}(\theta) + x^I c_{1 I\alpha}(\theta) \]
for some odd-valued superfunctions $c_{1\alpha}(\theta)$ and $c_{1 I\alpha}(\theta)$, on which the requirement that
\[ a_{1\alpha}(\theta, x) - \theta^\beta \overrightarrow{\partial}_\beta a_{1\alpha}(\theta, x) = c_{1\alpha}(\theta) - \theta^\beta \overrightarrow{\partial}_\beta c_{1\alpha}(\theta) + x^I \left( c_{1 I\alpha}(\theta) - \theta^\beta \overrightarrow{\partial}_\beta c_{1 I\beta}(\theta) \right) \]
be $\mathbb{R}^{1,d-1}$-invariant immediately yields
\[ c_{1 I\alpha}(\theta) = 0, \]
and so leads to
\[ da(\theta, x) = dx^I c_I + d (\theta^{\alpha} a_{1\alpha}(\theta)) \equiv c_I dx^I + d\overline{a}(\theta) \]
for an (arbitrary) even-valued superfunction $\overline{a}$. Altogether, we obtain
\[ (5.7) \]
\[ \beta(\theta, x) = \overline{\Gamma}_{11} \sigma(\theta) + d\overline{a}(\theta) + c_I dx^I, \]
whence
\[ (5.7) \]
\[ \beta(\theta + \varepsilon, x - \frac{1}{2} \overline{\Gamma} \theta) - \beta(\theta, x) = \overline{\Gamma}_{11} d\theta + d\left( \overline{a}(\theta + \varepsilon) - \overline{a}(\theta) \right), \]
with
\[ \overline{\Gamma}_{11} := \Gamma_{11} - \frac{1}{2} c_I \Gamma^I. \]
However, the term in the expansion of $\overline{a}$ quadratic in the $\theta^\alpha$ is necessarily of the form $\theta^\alpha A \theta$ with $A^T = -A$, which implies that it cannot cancel the first term in the variation containing the (nonzero) symmetric matrix $C \cdot \overline{\Gamma}_{11}$. This infers the necessity to extend $\mathcal{M}^{(1)}$ along the lines of App.3. Prior to proceeding with the extension, we pause to take a closer look at the (super)symmetry properties of the primitive (5.7), with view to understanding the nature of the extension to be constructed. Reasoning along the lines of the (pre-)quantum-symmetry analysis presented at the end of Sec.2.2, we are led to consider the expression
\[ (5.8) \]
\[ \mathcal{J}_{(\varepsilon,y)}(\theta, x) = \overline{\Gamma}_{11} \theta + (\delta \overline{a})(\varepsilon,y)(\theta, x) + c_{(\varepsilon,y)}, \quad c_{(\varepsilon,y)} \in \mathbb{R}, \]
Here, the Grassmann-even constants $c_{(\varepsilon,y)}$ quantify the residual freedom of redefinition of the current. With the latter, we associate, in the manner structurally identical with that discussed in Sec.2.2, the current super-2-cocycle,
\[ (5.9) \]
\[ (\delta \mathcal{J})_{(\varepsilon_1,y_1),(\varepsilon_2,y_2)}(\theta, x) = \overline{\varepsilon}_1 \overline{\Gamma}_{11} (\theta + \varepsilon_2) - (\overline{\varepsilon}_1 + \overline{\varepsilon}_2) \overline{\Gamma}_{11} \theta + \overline{\varepsilon}_2 \overline{\Gamma}_{11} \theta + (\delta^2 \overline{a})_{(\varepsilon_1,y_1),(\varepsilon_2,y_2)}(\theta) \]
\[ + c_{(\varepsilon_1,y_1),(\varepsilon_2,y_2)} - c_{(\varepsilon_1,y_1),(\varepsilon_2,y_2)} + c_{(\varepsilon_2,y_2)} \]
whose nontriviality is readily verified. We begin the proof by rephrasing the question about the triviality of $\delta \mathcal{J}$ – this is tantamount to the existence of a 1-cochain $c$ with the property that
\[ (5.10) \]
\[ (\delta c)_{(\varepsilon_1,0),(\varepsilon_2,0)} = -\overline{\varepsilon}_1 \overline{\Gamma}_{11} \varepsilon_2. \]
Given the nilpotence of the Grassmann-odd coordinates, it makes sense to write the Maclaurin expansion of the parameterised constants,
\[ c_{(\varepsilon,y)} = 2C_0(y) + C_1(y) \varepsilon + \frac{1}{2} \varepsilon C_2(y) \varepsilon + \Delta_3(\varepsilon, y) \]
where $\Delta_3(\varepsilon, y)$ is a rest trilinear in $\varepsilon$, and where, for all $y \in \mathbb{R}^{1,d-1}$,
\[ C_2(y)^T = -C_2(y). \]

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We now obtain
\[(\delta c)_{(\varepsilon_1,0), (\varepsilon_2,0)} \equiv c_{(\varepsilon_1,0)} - c_{(\varepsilon_1 + \varepsilon_2, -\frac{1}{2} \varepsilon_1 \Gamma \varepsilon_2)} + c_{(\varepsilon_2,0)}\]
\[= 4C_0(0) + C_1(0) (\varepsilon_1 + \varepsilon_2) + \frac{1}{2} \varepsilon_1 C_2(0) \varepsilon_1 + \frac{1}{2} \varepsilon_2 C_2(0) \varepsilon_2 + \Delta_3 (\varepsilon_1,0) + \Delta_3 (\varepsilon_2,0)\]
\[= 2C_0(0) + \varepsilon_1 (\partial_1 C_0(0)) \Gamma^I - C_2(0)) \varepsilon_2 + \Delta (\varepsilon_1,\varepsilon_2)\]
in which the last term depends at least cubically on the \(\varepsilon_i\) and hence cannot cancel \(\varepsilon_1 \varepsilon_2\). The relevant equality \((5.10)\) implies
\[\Gamma_1 = (\frac{1}{2} c_I - \partial_1 C_0(0)) \Gamma^I + C_2(0)\]
In view of the assumed symmetricity of the charge-conjugation matrix \(\Gamma_1\) and of the \(\Gamma^I\) (cp. Eq. \((5.3)\)), the above yields the equality
\[C_2(0) = 0\]
and further reduces to
\[\Gamma_1 = (\frac{1}{2} c_I - \partial_1 C_0(0)) \Gamma^I,\]
which admits no solutions. Thus convinced of the nontriviality of the supersymmetry current 2-cocycle, we conclude that the ensuing **homomorphicity super-2-cocycle**
\[d^{(0)}_{(\varepsilon_1, \varepsilon_2)} = \varepsilon_1 \varepsilon_2\]
is also nontrivial, and therefore predicts a projective nature of the realisation of supersymmetry on the Hilbert space of the super-0-brane. This suggests that it is, in fact, the central extension of \(\mathbb{R}^{1,d-1} D_{1,d-1}\) determined by the above super-2-cocycle, and not the supersymmetry group \(\mathbb{R}^{1,d-1} D_{1,d-1}\) itself, that will lift to the extension of \(\mathcal{M}^{(1)}\) that we are about to derive. We may now return to our geometric construction and seek corroboration of our expectations.

Consider a trivial principal \(\mathbb{C}^\times\)-bundle
\[(5.11) \pi_{\mathcal{L}(0)} \equiv \text{pr}_1 : \mathcal{L}(0) := \mathcal{M}^{(1)} \times \mathbb{C}^\times \longrightarrow \mathcal{M}^{(1)} : (\theta^a, x^I, z) \longmapsto (\theta^a, x^I)\]
with a connection
\[(5.12) \nabla_{\mathcal{L}(0)} = d + \frac{1}{2} \beta^{(1)}\]
or – equivalently – a principal \(\mathbb{C}^\times\)-connection 1-form
\[\beta^{(2)} (\theta, x, z) = i \frac{d\theta}{z} + \beta^{(1)} (\theta, x),\]
where we fix the primitive of \(\chi\) to be
\[\beta^{(1)} (\theta, x) = \bar{\theta} \Gamma_1 \sigma (\theta),\]
and demand that a lift of the geometric action \(\ell^{(1)}\) of Eq. \((5.4)\) to the total space \(\mathcal{L}(0)\) be a connection-preserving automorphism. In the light of our analysis of the supersymmetry properties of the primitive \(\beta^{(2)}\) of \(\chi\), it is justified to leave open the possibility of inducing the said lift (through restriction) from the Lie supergroup structure on \(\mathcal{L}(0)\) determined by the binary operation
\[m_0^{(2)} : \mathcal{L}(0) \times \mathcal{L}(0) \longrightarrow \mathcal{L}(0)\]
\[(5.13) \left( \theta_1, x_1 + x_2, z_1 \right), \left( \theta_2, x_1', x_2', z_2 \right) \longmapsto \left( \theta_1^2 + \theta_2^2, x_1 + x_1', x_2 + x_2', \frac{1}{2} \theta_1 \Gamma^I \theta_1, e^{i \lambda (\theta_1, x_1, \theta_2, x_2) \cdot z_1 \cdot z_2} \right),\]
in whose definition \(\lambda\) is a 2-cocycle on \(\mathbb{R}^{1,d-1} D_{1,d-1}\) with values in \(\mathbb{R}/2\pi \mathbb{Z}\), cp. Eq. \((2.14)\). The induced action is then given by the bundle automorphisms
\[\ell^{(1)} : \mathcal{L}(0) \longrightarrow \mathcal{L}(0)\]
\[(5.14) \quad : \quad ((\varepsilon^{\alpha}, y^I), \thinspace (\theta^\beta, \thinspace x^J, \thinspace z)) \mapsto m_0^{(2)}((\varepsilon^{\alpha}, \thinspace y^I, \thinspace 1), \thinspace (\theta^\beta, \thinspace x^J, \thinspace z)) = (\theta^\beta + \varepsilon^{\alpha}, \thinspace x^J + \frac{1}{2} \pi \Gamma^I \theta, \thinspace e^{i \lambda_{(\varepsilon, \theta, x)} \cdot z}).\]

The requirement that it preserve the connection \[(5.12)\] is tantamount to the imposition of the constraints

\[d\lambda_{(\varepsilon, \theta, x)} = d(\pi \Gamma_1 \theta),\]

to which the solution reads, in conformity with our expectations,

\[\lambda_{(\varepsilon, \theta, x)} = \pi \Gamma_1 \theta + \Delta_{(\varepsilon, \theta)},\]

where \(\Delta_{(\varepsilon, \theta)} \in \mathbb{R}/2\pi \mathbb{Z}\) is suitably constrained,

\[\forall (\varepsilon_{1, y_1}, \thinspace \varepsilon_{2, y_2}) \in \mathbb{R}^{1, d-1}D_1, \quad \Delta_{(\varepsilon_{1, y_1}, \thinspace \varepsilon_{2, y_2})} = \Delta_{(\varepsilon_{2, y_2})},\]

so that \(\lambda_{\varepsilon}\) is a 2-cocycle. We shall set

\[\lambda_{(\varepsilon, \theta, x)} = \pi \Gamma_1 \theta,\]

and so also

\[(5.15) \quad d_{(\varepsilon_{1, y_1}, \thinspace \varepsilon_{2, y_2})}^{(0)} = e^{i \pi \Gamma_1 \varepsilon_{2}}.\]

With the phases thus fixed, we arrive at

**Proposition 5.1.** The principal \(\mathbb{C}^\times\)-bundle \(\mathcal{L}^{(0)}\) of Eq. \((5.11)\) equipped with the binary operation

\[m_0^{(2)} : \mathcal{L}^{(0)} \times \mathcal{L}^{(0)} \rightarrow \mathcal{L}^{(0)}\]

\[(5.16) \quad : \quad \left( (\theta^\alpha_1, \thinspace x^I_1, \thinspace z_1), \thinspace (\theta^\alpha_2, \thinspace x^I_2, \thinspace z_2) \right) \mapsto \left( \theta^\alpha_1 + \theta^\alpha_2, \thinspace x^I_1 + x^I_2 - \frac{1}{2} \pi \Gamma^I \theta_2, \thinspace d_{(\varepsilon_{1, y_1}, \varepsilon_{2, y_2})}^{(0)} \right),\]

with the inverse

\[\text{Inv}_{0}^{(2)} : \mathcal{L}^{(0)} \rightarrow \mathcal{L}^{(0)} : \left( \theta^\alpha, \thinspace x^I, \thinspace z \right) \mapsto \left( -\theta^\alpha, \thinspace -x^I, \thinspace z^{-1} \right)\]

and the neutral element

\[e_0^{(2)} = (0, 0, 1)\]

is a Lie supergroup. It is a central extension

\[\mathbb{C}^\times \rightarrow \mathbb{R}^{1, d-1}D_1 \times \mathbb{C}^\times \equiv \mathbb{R}^{1, d-1}D_1 \rightarrow \mathbb{R}^{1, d-1}D_1, \quad \pi_{x^{(0)}} \rightarrow \mathbb{R}^{1, d-1}D_1, \quad 1\]

of the super-Minkowski group \(\mathbb{R}^{1, d-1}D_1\) determined by the homomorphismic super-2-cocycle \(d^{(0)}_{(\varepsilon_{1, y_1}, \varepsilon_{2, y_2})}\) of Eq. \((5.15)\).

**Proof.** Obvious, through inspection \(^{24}\) \(\square\)

Using the above binary operation \(m_0^{(2)}\) in Eq. \((5.11)\), we obtain the composition law of the induced action:

\[\mathcal{L}^{(0)} \cdot \mathcal{L}^{(0)} \quad \left( (\varepsilon_{1, y_1}, \thinspace \varepsilon_{2, y_2}) \right) \circ \mathcal{L}^{(0)} \cdot \mathcal{L}^{(0)} \quad \left( (\varepsilon_{1, y_1}, \thinspace \varepsilon_{2, y_2}) \right) = \left\{ \left( \text{id}_{(\varepsilon)}, \thinspace m \left( e^{i \pi \Gamma_1 \varepsilon_{2}} \right) \right) \circ \mathcal{L}^{(0)} \cdot \mathcal{L}^{(0)} \right\},\]

and so the supergroup structure on \(\mathcal{L}^{(0)}\) defines a projective realisation of supersymmetry on \(\mathcal{L}^{(0)}\). It is only upon defining the action of the full central extension \(\mathbb{R}^{1, d-1}D_1\) that the realisation becomes proper, as anticipated. The definition is deduced from Eq. \((5.16)\) and reads

\[\tilde{\mathcal{L}}^{(0)} \equiv m_0^{(2)} : \mathbb{R}^{1, d-1}D_1 \times \mathcal{L}^{(0)} \rightarrow \mathcal{L}^{(0)}\]

Our discussion leads us naturally to

\(^{24}\)The existence of the structure of a Lie supergroup on this and many other (super)central extensions of Lie supergroups derived from the underlying super-Minkowski group \(\mathbb{R}^{1, d-1}D_1\) through consecutive extensions determined by CaE super-2-cocycles was noted and discussed at great length in Ref. \(\text{CaAIPB00}\) and follows from the general theory of (super)central group extensions, cp. also Ref. \(\text{DAI93}\). Our results, augmented with detailed derivations, should therefore be compared with those obtained in the paper.
Definition 5.3. The Green–Schwarz super-0-gerbe over \( \mathcal{M}^{(1)} \equiv s\text{Mink}^{1,d-1}_{D_1,d-1} \) of curvature \( \chi \) is the triple

\[
G^{(0)}_{\text{GS}} := \left( \mathcal{L}^{(0)}, \pi^{(0)}_L, \beta^{(2)} \right)
\]

constructed in the preceding paragraphs.

We may now restate the results of our analysis in the form of

Proposition 5.3. The Green–Schwarz super-0-gerbe of Definition 5.3 is a \( \mathbb{C}^\ast \)-bundle with connection over the super-Minkowski space \( s\text{Mink}^{1,d-1}_{D_1,d-1} \). The bundle admits the natural projective action \( \mathcal{L}^{(0)}(\ell^{(1)}_L) \) of Eq. (5.14), by connection-preserving principal-bundle automorphisms, of the supersymmetry group \( \mathbb{R}^{1,d-1}_{D_1,d-1} \) induced, through restriction, by the group structure \( m^{(2)}_0 \) of Eq. (5.13) on the total space of the bundle. The said group structure defines, also through restriction, an action of the central extension \( \mathbb{R}^{1,d-1}_{D_1,d-1} \) of \( \mathbb{R}^{1,d-1}_{D_1,d-1} \) determined by the homomorphicity super-2-cocycle \( d^{(0)}_\gamma \) on \( \mathbb{R}^{1,d-1}_{D_1,d-1} \) specified in Eq. (5.17).

With view to subsequent discussion, and to potential future applications, we abstract from the above

Definition 5.4. Let \( G \) be a Lie supergroup with trivial de Rham cohomology. Denote the binary operation on \( G \) as

\[
m_G : G \times G \rightarrow G
\]

and the corresponding left regular action of \( G \) on itself as

\[
\ell_\ast \equiv m_G : G \times G \rightarrow G : (Y, X) \mapsto m_G(Y, X) = \ell_Y(X).
\]

Let \( h \) be a super-2-cocycle on \( G \) representing a class in its (left) CaE cohomology. A Cartan–Eilenberg super-0-gerbe over \( G \) with curvature \( h \) is a triple

\[
G^{(0)}_{\text{CaE}} := \left( L, \pi_L, a_L \right)
\]

composed of

- a trivial principal \( \mathbb{C}^\ast \)-bundle
  \[
  \pi_L \equiv \text{pr}_1 : L := G \times \mathbb{C}^\ast \rightarrow G : (X, z) \mapsto X
  \]
- a principal connection 1-form on it,
  \[
  a_L(X, z) = i \frac{d}{dz} + b^{(1)}(X),
  \]
  and the associated principal connection
  \[
  \nabla_L = d + \frac{1}{i} b^{(1)},
  \]
  determined by a global primitive \( b \) of \( h^{(2)} \),
  \[
  h^{(2)} = d b^{(1)},
  \]
  and with a structure of a Lie supergroup on the total space that lifts that on its base along the Lie-supergroup homomorphism \( \pi_L \) in such a manner that \( a_L \) is an LI super-1-form with respect to it, that is, given a super-0-form \( \lambda \) on \( G^{*2} \) satisfying the identity
  \[
  \ell_Y b^{(1)}(X) - b^{(1)}(X) = d \lambda^{(1)}(Y, X),
  \]
  the binary operation on \( L \) takes the form
  \[
  m_L : L \times L \rightarrow L : ((X_1, z_1), (X_2, z_2)) \mapsto \left( m_G(X_1, X_2), \exp(i \lambda^{(1)}(X_1, X_2)) \cdot z_1 \cdot z_2 \right).
  \]
Given CaE super-0-gerbes $G^{(0)}_{\text{CaE}} = (L_A, \pi_{L_A}, a_{L_A})$, $A \in \{1, 2\}$ over a common base $G$, with the respective principal connections $a_{L_A}(X, z_A) = i \frac{\delta a_{L_A}}{z_A} + b_{L_A}(X)$, an isomorphism between them is a connection-preserving principal-bundle isomorphism

$$\Phi^{(0)}_{\text{CaE}} : G^{(0)}_{\text{CaE}} \xrightarrow{\sim} G^{(0)}_{\text{CaE}}$$

determined by a left-invariant super-0-form $\Phi$ on $G$ satisfying the identity

$$d\Phi = b_2 - b_1,$$

that is, the isomorphism has a coordinate presentation

$$\Phi^{(0)}_{\text{CaE}}(X, z_1) = (X, \exp(\Phi(X)) \cdot z_1).$$

\[\begin{array}{c}
\text{Remark 5.5.} \text{ Note that the tensor product of principal } \mathbb{C}^*\text{-bundles described in the footnote on p. [10] gives rise to a tensor product of super-0-gerbes.}
\end{array}\]

\[\begin{array}{c}
\text{Remark 5.6.} \text{ On the super-Minkowski space and its cartesian powers, the existence of an isomorphism between CaE super-0-gerbes is tantamount to the (strict) equality of the corresponding base components of the principal connection 1-form,}
\end{array}\]

$$b_2 - b_1 = 0.$$

We conclude this part of our discussion of the super-0-brane data with a detailed analysis of various ($\chi$-twisted) algebroidal structures associated with natural actions of the supergroup $\mathbb{R}^{1,d-1|D_1,d-1}$ on the super-Minkowski target. We start with the left-regular action. Here, the fundamental sections $\Theta_{\alpha}(\theta, x) = (\partial_{\alpha}(\theta, x), 0)$, $\Theta_{\alpha}(\theta, x) = (\partial_{\alpha}(\theta, x), -2\Gamma_{\alpha\beta}^{I} \theta^\beta)$, $(I, \alpha) \in 0, d-1 \times I, D_1,d-1$ take the form

$$\Theta_{\alpha}(\theta, x) = (\partial_{\alpha}(\theta, x), 0), \quad \Theta_{\alpha}(\theta, x) = (\partial_{\alpha}(\theta, x), -2\Gamma_{\alpha\beta}^{I} \theta^\beta), \quad (I, \alpha) \in 0, d-1 \times I, D_1,d-1$$

and satisfy the algebra

$$[\Theta_{\alpha}, \Theta_{\beta}] = -2\Gamma_{\alpha\beta}^{I} \Theta_{I} + (0, -2\Gamma_{\alpha\beta}^{I} \theta^\beta).$$

From which we read off the left-regular Lie anomaly super-0-form

$$\alpha_{L}^{\alpha} = 0, \quad \alpha_{L}^{I} = 0 = \alpha_{L}^{I}, \quad \alpha_{L}^{I} = -2\Gamma_{\alpha\beta}^{I} \theta^\beta.$$

Passing to the right-regular action, we find the relevant fundamental sections

$$\Theta_{\alpha}(\theta, x) = (P_{\alpha}(\theta, x), 0), \quad \Theta_{\alpha}(\theta, x) = (Q_{\alpha}(\theta, x), -2\Gamma_{\alpha\beta}^{I} \theta^\beta)$$

and the superbrackets

$$[\Theta_{\alpha}, \Theta_{\beta}] = -2\Gamma_{\alpha\beta}^{I} \Theta_{I} + (0, -2\Gamma_{\alpha\beta}^{I} \theta^\beta).$$

From which we read off the right-regular Lie anomaly super-0-form

$$\alpha_{R}^{\alpha} = 0, \quad \alpha_{R}^{I} = 0 = \alpha_{R}^{I}, \quad \alpha_{R}^{I} = -2\Gamma_{\alpha\beta}^{I} \theta^\beta.$$

Comparison of the two anomalies singles out the adjoint action, with the fundamental sections

$$\Theta_{\alpha}(\theta, x) = (\partial_{\alpha} - P_{\alpha})(\theta, x), 0 = 0, \quad \Theta_{\alpha}(\theta, x) = (\partial_{\alpha} - Q_{\alpha})(\theta, x), 0 = (-2\Gamma_{\alpha\beta}^{I} \theta^\beta \frac{\partial}{\partial x^\alpha}, 0).$$

Given the superbrackets

$$[\Theta_{\alpha}, \Theta_{\beta}] = (0, -2\Gamma_{\alpha\beta}^{I} \theta^\beta),$$

are satisfied.
we obtain a trivial, and hence – in particular – (Lie )anomaly-free superbrackets of the fundamental sections of the adjoint action. Trivially, the fundamental sections for the adjoint action span a Lie superalgebroid.

5.1.2. The Green–Schwarz superstring. At the next level in cohomology, which is where the super-σ-model for the superstring is constructed, we find the GS super-3-cocycle

\[ \chi \in \Gamma(\sigma \wedge \Gamma_{J} \sigma). \]

This is a closed and manifestly LI super-3-form on \( \mathcal{M}^{(1)} \), with no smooth LI primitive on the latter space. The stepwise procedure of its trivialisation in the CaE cohomology through pullback to consecutive (super)central extensions of the underlying Lie supergroup \( \mathbb{R}^{1,d-1}D_{1,d-1} \) begins at the latter space, which is also where we look for LI de Rham super-2-cocycles of the type \((4.25)\). The distinguished members of the family that we shall examine with view to solving the trivialisation problem are

\[ h^{(1)}_{\alpha} = \frac{1}{2} \gamma_{\alpha} \wedge \chi = -\Gamma_{I \alpha \beta} \pi_{0}^{\alpha} \sigma \wedge e^I. \]

Their closedness follows – just as that of the super-3-cocycle \((5.17)\) – directly from the assumed Fierz identity \((4.16)\).

In order to construct a suitable common extension of their support \( \mathcal{M}^{(1)} \) on which their pullbacks trivialise, we first derive their non-LI primitives on \( \mathcal{M}^{(1)} \). To this end, we compute

\[ h_{\alpha}^{(1)}(\theta, x) = -d(\Gamma_{I \alpha \beta} \theta^\beta \, dx^I) - \frac{1}{2} \Gamma_{I \alpha \beta} \Gamma_{I \gamma}^J \theta^\beta \wedge \theta^\gamma \, dx^J. \]

Using identity \((4.16)\), we readily find

\[
\Gamma_{I \alpha \beta} \Gamma_{I \gamma}^J \theta^\beta \wedge \theta^\gamma \, dx^J = d(\Gamma_{I \alpha \beta} \Gamma_{I \gamma}^J \theta^\beta \theta^\gamma \, dx^J) - \Gamma_{I \alpha \beta} \Gamma_{I \gamma}^J \theta^\beta \theta^\gamma \wedge dx^J
\]

so that

\[ h_{\alpha}^{(1)}(\theta, x) = d(-\Gamma_{I \alpha \beta} \theta^\beta \, dx^I + \frac{1}{6} \Gamma_{I \alpha \beta} \Gamma_{I \gamma}^J \theta^\beta \theta^\gamma \, dx^J). \]

Drawing on our hitherto experience, we may now conceive a trivial vector bundle

\[ \pi_{1}^{(2)} \equiv \text{pr}_{1} \circ \mathcal{M}^{(2)} = \mathcal{M}^{(1)} \times \mathbb{R}^{6[D_{1,d-1}] \to \mathcal{M}^{(1)}} \]

with the purely Graßmann-odd fibre \( \mathbb{R}^{6[D_{1,d-1}] \to \mathcal{M}^{(1)}} \) and a Lie supergroup structure that projects to the previously considered supersymmetry-group structure on the base and so lifts the action \((5.4)\) of that supersymmetry group in a manner that we fix by demanding invariance under this lift of the primitives \( \varepsilon_{\alpha}^{(2)} \in \Lambda^1 \mathcal{T}^{*} \mathcal{M}^{(1)}_{\alpha} \) of the distinguished super-2-cocycles,

\[ \pi_{1}^{(2)*} \cdot h_{\alpha}^{(1)} = \varepsilon_{\alpha}^{(2)}. \]

Let us denote the (global) coordinates on \( \mathcal{M}^{(2)} \) as \( \xi_{\alpha} \), \( \alpha \in [D_{1,d-1}] \). We then take

\[ \varepsilon_{\alpha}^{(2)}(\theta, x, \xi) = d\xi_{\alpha} - \Gamma_{I \alpha \beta} \theta^\beta (dx^I + \frac{1}{2} \Gamma_{I} \sigma(\theta)). \]

The supersymmetry variation of the non-LI primitive \( \varepsilon_{\alpha}^{(2)} := \varepsilon_{\alpha}^{(2)} - d\xi_{\alpha} \) of \( h_{\alpha}^{(1)} \) is exact and may be cast in the form

\[
\varepsilon_{\alpha}^{(2)}(\theta + \xi, x + y + \frac{1}{2} \Gamma_{I} \theta) - \varepsilon_{\alpha}^{(2)}(\theta, x) = \frac{1}{3} \Gamma_{I \alpha \beta} \theta^\beta (\Gamma_{I} \sigma(\theta)) - \Gamma_{I \alpha \beta} \varepsilon^\beta (dx^I + \frac{1}{2} \Gamma_{I} \sigma(\theta)) = d(-x^I \Gamma_{I \alpha \beta} \varepsilon^\beta + \frac{1}{6} \Gamma_{I \alpha \beta} (2\varepsilon^\beta + \theta^\beta) \Gamma_{I} \theta)
\]

with the help of the Fierz identity \((4.16)\). In this manner, we arrive at

**Proposition 5.7.** The above-described vector bundle \( \mathcal{M}^{(2)} \) equipped with the binary operation

\[
\begin{align*}
m_{\alpha}^{(2)} : \mathcal{M}_{\alpha}^{(2)} \times \mathcal{M}_{\alpha}^{(2)} & \to \mathcal{M}_{\alpha}^{(2)} \\
((\theta_{1}^{\alpha}, x_{1}^{\alpha}, \xi_{1}^{\alpha}), (\theta_{2}^{\alpha}, x_{2}^{\alpha}, \xi_{2}^{\alpha})) & \mapsto (\theta_{1}^{\alpha} + \theta_{2}^{\alpha}, x_{1}^{I} + x_{2}^{I} - \frac{1}{2} \Gamma_{I} \theta_{2}, \xi_{1}^{\alpha} + \xi_{2}^{\alpha} + \Gamma_{I \alpha \beta} \theta_{1}^{\beta} x_{2}^{I} - \frac{1}{6} \left( \Gamma_{I} \theta_{2} \right) \Gamma_{\alpha \beta}^{J} (2\theta_{2}^{\beta} + \theta_{2}^{\beta})),
\end{align*}
\]


\[ \]
with the inverse
\[
\text{Inv}_1^2 : \mathcal{M}_1^2 \longrightarrow \mathcal{M}_1^2 : (\theta^\alpha, x^I, \xi_\beta) \mapsto (-\theta^\alpha, -x^I, -\xi_\beta + x^I \mathbf{T}_{I, \beta, \gamma} \theta^\gamma)
\]
and the neutral element
\[
e_1^{(2)} = (0, 0, 0)
\]
is a Lie supergroup. It is a (super)central extension
\[
1 \longrightarrow \mathbb{R}^{[0|D_1, d-1]} \longrightarrow \mathbb{R}^{1,d-1|D_1, d-1} \longrightarrow \mathcal{M}_1^{(2)} \longrightarrow 1
\]
of the super-Minkowski group \( \mathbb{R}^{1,d-1|D_1, d-1} \) determined by the family of CE super-2-cocycles corresponding to the CaE super-2-cocycles \( \{ h_\alpha^{(1)} \}_\alpha \) of Eq. (5.18).

**Proof.** Through inspection. In particular, the associativity of \( \mathfrak{m}_1^{(2)} \) hinges upon identity (1.19). \( \square \)

Upon pullback to \( \mathcal{M}_1^{(2)} \), we obtain the sought-after trivialisation
\[
\pi_i^{(2)*} \chi = d \beta^{(2)}_i, \quad \beta^{(2)}_i := \pi_i^{(2)*} \sigma^\alpha \wedge e_\alpha^{(2)},
\]
written in the shorthand notation
\[
e_\alpha^{(2)} := \pi_0 \circ \pi_1^{(2)}
\]
that we adapt in our subsequent considerations. The relation of the above trivialisation to the previously found (in Prop. 4.2) non-LI one on \( \mathcal{M}^{(1)} \) reads
\[
(5.19) \quad \beta^{(2)}_i = \pi_i^{(2)*} \beta^{(2)}_1 + dB, \quad B(\theta, x, \xi) := \theta^\alpha \xi_\alpha.
\]

Structurally, the construction of the (super)central extension
\[
\pi_{Y_1\mathcal{M}^{(1)}} = \pi_1^{(2)} : Y_1\mathcal{M}^{(1)} := \mathcal{M}_1^{(2)} \longrightarrow \mathcal{M}^{(1)} : (\theta^\alpha, x^I, \xi_\beta) \longrightarrow (\theta^\alpha, x^I)
\]
in the present geometric context plays a rôle fully analogous to that of the surjective submersion \( \pi_{YM} : YM \longrightarrow M \) from Section 2, to wit, it yields an epimorphism, in the geometric category of interest, onto the support of a non-trivial 3-cocycle on which, upon pullback along that epimorphism, the 3-cocycle trivialises in the same cohomology (in which it would not, at least in general, trivialise on the base/codomain of the epimorphism). The last observation motivates our subsequent attempt at establishing a (super)geometric realisation of the super-3-cocycle \( \chi^{(3)} \) in the CaE cohomology through a procedure closely imitating the one that defines its analogn \( (YM, \pi_{YM}, B, L, \nabla_L, \mu_L) \) in the standard (purely Grassmann-even) setting.

To this end, let us first consider the fibred square, represented by the commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y}_1^{[2]} \mathcal{M}^{(1)} & \xrightarrow{pr_2} & \mathcal{Y}_1 \mathcal{M}^{(1)} \\
\downarrow \pi_{Y_1\mathcal{M}^{(1)}} & & \downarrow \pi_{Y_1\mathcal{M}^{(1)}} \\
\mathcal{M}^{(1)} & \xrightarrow{\pi_{Y_1\mathcal{M}^{(1)}}} & \mathcal{M}^{(1)}
\end{array}
\]

The difference of the pullbacks of the primitive \( \beta^{(2)}_i \) along the two canonical projections reads
\[
(\text{pr}_2^* - \text{pr}_1^*) \beta^{(2)}_i = (\pi_0 \circ \pi_{Y_1\mathcal{M}^{(1)}} \circ \text{pr}_1)^* \sigma^\alpha \wedge (\text{pr}_2^* - \text{pr}_1^*) e_\alpha^{(2)}.
\]
This is, by construction, an LI super-2-cocycle, and so we may seek to trivialise it, or – if necessary – its pullback to a suitable (super)central extension of \( \mathcal{Y}_1^{[2]} \mathcal{M}^{(1)} \), in the CaE cohomology. Inspection of the coordinate expression
\[
(\text{pr}_2^* - \text{pr}_1^*) \beta^{(2)}_i (\theta, x, \xi^1, \xi^2) = d \bigl( \theta^\alpha \xi_\alpha^{21} \bigr),
\]

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written in terms of the variables $\xi_{a}^{\beta} := \xi_{a}^{\beta} - \xi_{a}^{1}$, $\alpha \in \mathbb{1}, D_{1,d-1}$, convinces us that there is no LI primitive of the above super-2-cocycle on $Y_{1}^{[2]} \mathcal{M}^{(1)}$, and so, invoking our results from the analysis of the GS super-2-cocycle, we are led to associate with it a trivial principal $\mathbb{C}^{\times}$-bundle

(5.20) $\pi_{\mathcal{L}^{(1)}} \equiv \text{pr}_{1} : \mathcal{L}^{(1)} \equiv Y_{1}^{[2]} \mathcal{M}^{(1)} \times \mathbb{C}^{\times} \rightarrow Y_{1}^{[2]} \mathcal{M}^{(1)} : (\theta, x, \xi^{1}, \xi^{2}, z) \mapsto (\theta, x, \xi^{1}, \xi^{2})$

with a principal connection

$$\nabla_{\mathcal{L}^{(1)}} = \text{d} + \frac{1}{\theta} A,$$

or – equivalently – a principal connection 1-form

$$A(\theta, x, \xi^{1}, \xi^{2}, z) = i_{\frac{2\pi}{\theta}} + A((\theta, x, \xi^{1}), (\theta, x, \xi^{2})),$$

with the base component

(5.21) $A((\theta, x, \xi^{1}), (\theta, x, \xi^{2})) \equiv (\text{pr}_{2} - \text{pr}_{1}) B((\theta, x, \xi^{1}), (\theta, x, \xi^{2})) = \theta^{\alpha} d\xi_{a}^{\beta} \equiv 0$,

The fibred product $Y_{1}^{[2]} \mathcal{M}^{(1)}$ of Lie supergroups inherits from $Y_{1} \mathcal{M}^{(1)} \equiv \mathcal{M}^{(2)}$ a Lie-supergroup structure determined by the binary operation

$$m_{1}^{(2)[2]} : Y_{1}^{[2]} \mathcal{M}^{(1)} \times Y_{1}^{[2]} \mathcal{M}^{(1)} \rightarrow Y_{1}^{[2]} \mathcal{M}^{(1)} : \left(\left(\theta^{a}_{1}, x^{1}_{1}, \xi^{1}_{1}, \xi^{2}_{1}, z_{1}\right), \left(\theta^{a}_{2}, x^{2}_{2}, \xi^{1}_{2}, \xi^{2}_{2}, z_{2}\right)\right) \mapsto$$

$$\mapsto \left(\theta^{a}_{1} + \theta^{a}_{2}, x^{1}_{1} + x^{2}_{2} - \frac{1}{6} (\gamma_{1}, \Gamma_{1}, \theta_{2}) \xi^{2}_{1}, \xi^{1}_{2}, \xi^{2}_{1}, z_{1} \mapsto \theta^{a}_{1} \xi^{2}_{1}, z_{2}\right),$$

(5.22) $\xi^{1}_{2} + \xi^{2}_{1} + (\gamma_{1} + \Gamma_{1}, \theta_{2}) \xi^{2}_{1} = (2 \theta^{a}_{1} + \theta^{a}_{2})$, $\xi^{2}_{2}$

In analogy with the case of the superparticle, we endow $\mathcal{L}^{(1)}$ with the structure of a Lie supergroup determined by the requirement of left-invariance of the principal connection 1-form $A$.

**Proposition 5.8.** The principal $\mathbb{C}^{\times}$-bundle $\mathcal{L}^{(1)}$ of Eq. (5.20) equipped with the binary operation

$$n_{1}^{(3)} : \mathcal{L}^{(1)} \times \mathcal{L}^{(1)} \rightarrow \mathcal{L}^{(1)} : \left(\left(\theta^{a}_{1}, x^{1}_{1}, \xi^{1}_{1}, \xi^{2}_{1}, z_{1}\right), \left(\theta^{a}_{2}, x^{2}_{2}, \xi^{1}_{2}, \xi^{2}_{2}, z_{2}\right)\right) \mapsto$$

$$\mapsto \left(\theta^{a}_{1} + \theta^{a}_{2}, x^{1}_{1} + x^{2}_{2} - \frac{1}{6} (\gamma_{1}, \Gamma_{1}, \theta_{2}) \xi^{2}_{1}, \xi^{1}_{2}, \xi^{2}_{1}, z_{1} \mapsto \theta^{a}_{1} \xi^{2}_{1}, z_{2}\right),$$

(5.23) $\xi^{1}_{2} + \xi^{2}_{1} + (\gamma_{1} + \Gamma_{1}, \theta_{2}) \xi^{2}_{1} = (2 \theta^{a}_{1} + \theta^{a}_{2})$, $d^{(1)}$

the latter being defined in terms of the super-2-cocycle

$$d^{(1)}(\theta_{1}, x_{1}, \xi^{1}_{1}, \xi^{2}_{1}, \xi^{1}, \xi^{2}, z_{1}, z_{2}) = \frac{1}{\theta} \xi^{2}_{1},$$

with the inverse

$$\text{Inv}_{1}^{(3)} : \mathcal{L}^{(1)} \rightarrow \mathcal{L}^{(1)}$$

$$: \left(\theta^{a}, x^{1}, \xi^{1}, \xi^{2}, z\right) \mapsto \left(-\theta^{a}, -x^{1}, -\xi^{1}, +x^{1}, \Gamma_{1}, \theta \xi^{2}, -\xi^{2}, +x^{1}, \Gamma_{1}, \theta \xi^{2}, e^{i \theta^{2}} \xi^{2}, z^{-1}\right)$$

and the neutral element

$$\sigma_{1}^{(3)} = (0, 0, 0, 0, 1)$$

is a Lie supergroup. It is a central extension

$$1 \rightarrow \mathbb{C}^{\times} \rightarrow Y_{1}^{[2]} \mathcal{M}^{(1)} \equiv Y_{1}^{[2]} \mathcal{M}^{(1)} \times \mathbb{C}^{\times} \rightarrow Y_{1}^{[2]} \mathcal{M}^{(1)} \rightarrow 1$$

of the Lie supergroup $Y_{1}^{[2]} \mathcal{M}^{(1)}$ of Eq. (5.22) determined by $d^{(1)}$.

**Proof.** Through inspection. □

The above structure can be employed to lift the original action of the supersymmetry group $\mathbb{R}^{1,d-1}$ all the way up to the total space of $\mathcal{L}^{(1)}$ as per

$$\mathcal{L}^{(1)} \equiv n_{1}^{(3)} : Y_{1}^{[2]} \mathcal{M}^{(1)} \times \mathcal{L}^{(1)} \rightarrow \mathcal{L}^{(1)}.$$  

By construction, the lift preserves the connection on $\mathcal{L}^{(1)}$. 

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In the last step, we consider the cartesian cube of the surjective submersion $Y_1 M^{(1)}$ fibred over $M^{(1)}$, with its canonical projections $\text{pr}_{i,j} \equiv (\text{pr}_i, \text{pr}_j)$, $(i, j) \in \{(1, 2), (2, 3), (1, 3)\}$ to $Y_1^2 M^{(1)}$ that render the diagram commutative, and, over it, look for a connection-preserving isomorphism

$$\mu_{\mathcal{L}^{(1)}} : \text{pr}_{1,2}^* \mathcal{L}^{(1)} \otimes \text{pr}_{2,3}^* \mathcal{L}^{(1)} \rightarrow \text{pr}_{1,3}^* \mathcal{L}^{(1)}.$$ 

Comparison of the pullbacks of the (global) connection 1-forms

$$(\text{pr}_{1,2}^* + \text{pr}_{1,3}^* - \text{pr}_{1,3}^*) L((\theta, x, \xi^1), (\theta, x, \xi^2), (\theta, x, \xi^3)) = 0,$$

in conjunction with inspection of the invariance of the relevant combination $z_{1,2} \cdot z_{2,3} \cdot z_{1,3}^{1}$ of the fibre coordinates lead us to set

$$\mu_{\mathcal{L}^{(1)}} \left( \left( \theta, x, \xi^1, \xi^2, z_{1,2} \right) \otimes \left( \theta, x, \xi^2, \xi^3, z_{2,3} \right) \right) := \left( \theta, x, \xi^1, \xi^3, z_{1,2} \cdot z_{2,3} \right),$$

where we have identified

$$\left( \theta, x, \xi^1, \xi^3, z_{1,2} \right) = \left( \left( \theta, x, \xi^1, \xi^2, \xi^3, \xi^4 \right), \left( \theta, x, \xi^1, \xi^3, z_{1,2} \right) \right) \in \text{pr}_{i,j}^* \mathcal{L}^{(1)}.$$

A fibre-bundle map thus defined trivially satisfies the groupoid identity (2.3) over $Y_1^4 M^{(1)}$, and conforms with the previous definition of super-0-gerbe isomorphism.

We conclude our analysis with

**Definition 5.9.** The Green–Schwarz super-1-gerbe over $M^{(1)} \equiv sMink^{1,d-1 \mid D_{1,d-1}}$ of curvature $\chi$ is the sextuple

$$G_{GS}^{(1)} := \left( Y_1 M^{(1)}, \pi_{Y_1 M^{(1)}}, \beta^{(2)} \mathcal{L}^{(1)}, \nabla_{\mathcal{L}^{(1)}}, \mu_{\mathcal{L}^{(1)}} \right)$$

constructed in the preceding paragraphs.

Our discussion is now concisely summarised in

**Proposition 5.10.** The Green–Schwarz super-1-gerbe $G_{GS}^{(1)}$ of Definition 5.3 is an abelian bundle gerbe with connection over the super-Minkowski space $sMink^{1,d-1 \mid D_{1,d-1}}$. The action (5.4) of the supersymmetry group $\mathbb{R}^{1,d-1 \mid D_{1,d-1}}$ on the base of the gerbe lifts to an action, by connection-preserving principal-bundle automorphisms, of the central extension $Y_1^3 M^{(1)}$, detailed in Prop. 5.3 of the Lie supergroup $Y_1^2 M^{(1)}$ defined through Eq. (5.22) and itself being an extension, described in Prop. 5.4 of $\mathbb{R}^{1,d-1 \mid D_{1,d-1}}$.

Following the lower-dimensional example, we formulate

**Definition 5.11.** Adopt the notation of Def 5.4. Let $h$ be a super-3-cocycle on $G$ representing a class in its (left) CaE cohomology. A Cartan–Eilenberg super-1-gerbe over $G$ with curvature $h$ is a sextuple

$$G_{\text{CaE}}^{(1)} := \left( YG, \pi_{YG}, \beta^{(2)} L, a, \mu_L \right)$$

constructed in the preceding paragraphs.
composed of

- a surjective submersion
  \[ \pi_{YG} : YG \rightarrow G \]
  with a structure of a Lie supergroup on its total space that lifts that on \( G \) along the Lie-supergroup homomorphism \( \pi_{YG} \),

- a global primitive \( b \) of the pullback of \( h \) to it,
  \[ \pi_{YG}^* h = d b \]
  which is LI with respect to the induced left-regular action of \( YG \) on itself
  \[ Y \ell. : YG \times YG \rightarrow YG, \]
  lifting \( \ell \) along \( \pi_{YG} \),

- a CaE super-0-gerbe
  \[ \left( L, \pi_L, a_L \right) \]
  over the fibred square \( Y^{[2]}G \equiv YG \times_G YG \) (endowed with the natural (product) Lie-supergroup structure), with a principal connection 1-form \( a_L \) of curvature \( h \),
  
  \[ \pi_L^* h \mid L = d a_L \]
  that satisfies the identity
  \[ h \mid L = (pr_2^* - pr_1^*) b \]

- an isomorphism of CaE super-0-gerbes\(^{25}\)
  \[ \mu_L : \text{pr}_{1,2}^* L \otimes \text{pr}_{2,3}^* L \xrightarrow{=} \text{pr}_{1,3}^* L \]
  over the fibred cube \( Y^{[3]}G \equiv YG \times_G YG \times G YG \) that satisfies the coherence (associativity) condition
  \[ \text{pr}_{1,2,4}^* \mu_L \circ (\text{id}\text{pr}_{1,2}^* L \otimes \text{pr}_{2,3,4}^* \mu_L) = \text{pr}_{1,3,4}^* \mu_L \circ (\text{pr}_{1,2,3}^* \mu_L \otimes \text{id}\text{pr}_{2,3,4}^* L) \]
  over the quadruple fibred product \( Y^{[4]}G \equiv YG \times_G YG \times G YG \times G YG \).

Given CaE super-1-gerbes \( \Phi^{(1)A}_{\text{CaE}} = \left( Y_{A,G}, \pi_{Y_{A,G}}, \mu_{Y_{A,G}}, b_{A,L}, a_{A,L}, \mu_{L_{A,L}} \right), A \in \{1,2\} \) over a common base \( G \), a 1-isomorphism between them is a quintuple

\[ \Phi^{(1)A}_{\text{CaE}} : \left( YY_{1,2}G, \pi_{YY_{1,2}G}, E, a_E, \alpha_E \right) \rightarrow \Phi^{(1)B}_{\text{CaE}} \]

composed of

- a surjective submersion
  \[ \pi_{YY_{1,2}G} : YY_{1,2}G \rightarrow Y_1 G \times_G Y_2 G \equiv Y_{1,2}G \]
  with a structure of a Lie supergroup on its total space that lifts the product Lie-supergroup structure on the fibred product \( Y_{1,2}G \) along the Lie-supergroup homomorphism \( \pi_{YY_{1,2}G} \),

- a CaE super-0-gerbe
  \[ \left( E, \pi_E, a_E \right) \]
  over the total space \( YY_{1,2}G \), with a principal connection 1-form \( a_E \) of curvature \( h_E \),
  
  \[ \pi_E^* h_E = d a_E \]
  that satisfies the identity
  \[ h_E = \pi_{YY_{1,2}G}^* \left( \text{pr}_{2}^* b_2 - \text{pr}_{1}^* b_1 \right) \]

\(^{25}\)Note that pullback along a canonical projection is consistent with the definition of a super-0-gerbe.
• an isomorphism of super-0-gerbes

\[ \alpha_E : (\pi_{YY_{1,2}G} \times \pi_{YY_{1,2}G}) \circ \text{id}_{\ast}^{\ast} \rightarrow \text{id}_{\ast}^{\ast} \]

over the fibred product \( Y^{[2]}Y_{1,2}G \equiv YY_{1,2}G \times YY_{1,2}G \), subject to the coherence constraint expressed by the commutative diagram of isomorphisms of CaE super-0-gerbes

\[ \pi_{1,2} \circ \text{id}_{\ast}^{\ast} \rightarrow \text{id}_{\ast}^{\ast} \]

\[ \pi_{1,2} \circ \text{id}_{\ast}^{\ast} \rightarrow \text{id}_{\ast}^{\ast} \]

\[ \pi_{2,1} \circ \text{id}_{\ast}^{\ast} \rightarrow \text{id}_{\ast}^{\ast} \]

over the fibred product \( Y^{[3]}Y_{1,2}G \equiv YY_{1,2}G \times YY_{1,2}G \), with

\( \pi_{i,j} = (\pi_{YY_{1,2}G} \times \pi_{YY_{1,2}G}) \circ \text{id}_{\ast}^{\ast} \) \( i, j \in \{(1, 2), (2, 3), (1, 3)\} \),

\( \pi_{1,2,3} = \pi_{YY_{1,2}G} \times \pi_{YY_{1,2}G} \times \pi_{YY_{1,2}G} \).

Given a pair of 1-isomorphisms \( \Phi_{\text{CaE}}^{(1)} : (YY_{1,2}G, \pi_{YY_{1,2}G}, B) \rightarrow \Phi_{\text{CaE}}^{(2)} \) between CaE super-1-gerbes \( G_{\text{CaE}}^{(1)} = (Y_A, \pi_{YY_{1,2}G}, B_A, L_A, \mu_{L_A}) \), \( A \in \{1, 2\} \), a 2-isomorphism is represented by a triple

\( \varphi_{\text{CaE}}^{(1)} = (\pi_{YY_{1,2}G, \pi_{YY_{1,2}G}}, \beta) : \Phi_{\text{CaE}}^{(1)} \equiv \Phi_{\text{CaE}}^{(2)} \)

composed of

• a surjective submersion

\[ \pi_{YY_{1,2}G} : YY_{1,2}G \rightarrow YY_{1,2}G \times YY_{1,2}G \]

with a structure of a Lie supergroup on its total space that lifts the product Lie-supergroup structure on the fibred product \( YY_{1,2}G \) along the Lie-supergroup homomorphism \( \pi_{YY_{1,2}G} \),

• an isomorphism of CaE super-0-gerbes

\( \beta : (\pi_{YY_{1,2}G} \circ YY_{1,2}G)^{\ast}E_1 \rightarrow (\pi_{YY_{1,2}G} \circ YY_{1,2}G)^{\ast}E_2 \)

subject to the coherence constraint expressed by the commutative diagram of isomorphisms of CaE super-0-gerbes

\[ p_{1,1}^{\ast}L_1 \otimes p_{1,2}^{\ast}E_1 \rightarrow p_{1,1}^{\ast}E_1 \otimes p_{2,2}^{\ast}L_2 \]

\[ p_{1,1}^{\ast}L_1 \otimes p_{2,2}^{\ast}E_2 \rightarrow p_{2,1}^{\ast}E_2 \otimes p_{2,2}^{\ast}L_2 \]

over \( YY_{1,2}G \), with

\[ p_{i,i} = p_{i} \circ \pi_{YY_{1,2}G} \circ p_{i} \circ \pi_{YY_{1,2}G} \times p_{i} \circ \pi_{YY_{1,2}G} \circ p_{i} \circ \pi_{YY_{1,2}G}, \quad i \in \{1, 2\} \]

\[ p_{j,k} = p_{j} \circ \pi_{YY_{1,2}G} \circ p_{k}, \quad j, k \in \{1, 2\} \].

As in the case of the super-0-brane, we close the section dedicated to the super-1-brane with (Vinogradov-)algebroidal considerations. The first to be discussed are the fundamental sections of \( \mathcal{C}^{1,1}_{\text{sMink}} |_{\mathcal{D}_{1,1,-1}} \) associated with the left-regular action,

\[ \mathcal{R}_i(\theta, x) = (\mathcal{P}_i(\theta, x), -\bar{\theta} \Gamma I \sigma(\theta)), \quad \mathcal{R}_a(\theta, x) = (\mathcal{P}_a(\theta, x), \Gamma_{I, a, \beta} \theta^3 (-2dx^I + \frac{1}{3} \bar{\theta} \Gamma^I \sigma(\theta))) \],

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for which we obtain superbrackets
\[
\{\mathcal{R}_I, \mathcal{R}_J\}^\chi_\gamma_{(3)}(\theta, x) = 0,
\]
\[
\{\mathcal{R}_I, \mathcal{R}_\alpha\}^\chi_\gamma_{(3)}(\theta, x) = (0, \frac{1}{2} \bar{T}_{\alpha\beta} \Gamma_I \alpha \beta \theta^\beta),
\]
\[
\{\mathcal{R}_\alpha, \mathcal{R}_\beta\}^\chi_\gamma_{(3)}(\theta, x) = -\bar{T}_{\alpha\beta} \mathcal{R}_I + (0, -2\bar{T}_{\alpha\beta} dx^I).
\]
These yield the left-regular Lie anomaly super-1-form
\[
\alpha^L_{IJ} = 0, \quad \alpha^L_{I\alpha} = \frac{1}{2} \bar{T}_{\alpha\beta} \Gamma_I \alpha \beta \theta^\beta = -\alpha^L_{\alpha I}, \quad \alpha^L_{\alpha\beta} = -2\bar{T}_{\alpha\beta} dx^I.
\]
For the right-regular action, we find
\[
\mathcal{L}_I(\theta, x) = (P_I(\theta, x), -\mathcal{F}_I \sigma(\theta)), \quad \mathcal{L}_\alpha(\theta, x) = (Q_\alpha(\theta, x), -\bar{T}_{\alpha\beta} \theta^\beta (2dx^I + \frac{1}{3} \bar{T}_I \sigma(\theta)))
\]
and the superbrackets
\[
\{\mathcal{L}_I, \mathcal{L}_J\}^\chi_\gamma_{(3)}(\theta, x) = 0, \quad \{\mathcal{L}_I, \mathcal{L}_\alpha\}^\chi_\gamma_{(3)}(\theta, x) = (0, \frac{1}{2} \bar{T}_{\alpha\beta} \Gamma_I \alpha \beta \theta^\beta),
\]
\[
\{\mathcal{L}_\alpha, \mathcal{L}_\beta\}^\chi_\gamma_{(3)}(\theta, x) = \bar{T}_{\alpha\beta} \mathcal{L}_I + (0, -2\bar{T}_{\alpha\beta} dx^I),
\]
and so also the right-regular Lie anomaly super-1-form
\[
\alpha^R_{IJ} = 0, \quad \alpha^R_{I\alpha} = \frac{1}{2} \bar{T}_{\alpha\beta} \Gamma_I \alpha \beta \theta^\beta = \alpha^R_{\alpha I}, \quad \alpha^R_{\alpha\beta} = -2\bar{T}_{\alpha\beta}.
\]
The mixed superbrackets read
\[
\{\mathcal{R}_I, \mathcal{L}_J\}^\chi_\gamma_{(3)}(\theta, x) = 0, \quad \{\mathcal{R}_I, \mathcal{L}_\alpha\}^\chi_\gamma_{(3)}(\theta, x) = (0, \frac{1}{2} \bar{T}_{\alpha\beta} \Gamma_I \alpha \beta \theta^\beta) = -\{\mathcal{R}_\alpha, \mathcal{L}_I\}^\chi_\gamma_{(3)}(\theta, x),
\]
\[
\{\mathcal{R}_\alpha, \mathcal{L}_\beta\}^\chi_\gamma_{(3)}(\theta, x) = (0, -2\bar{T}_{\alpha\beta} dx^I + \bar{T}_{\alpha\gamma} \bar{T}_{\beta\delta} \Gamma_I \delta \theta^\delta).
\]
Taking these into account alongside the previous ones, we derive a trivial, and so also (Lie) anomaly-free superbrackets of the fundamental sections of the adjoint action
\[
(\mathcal{R}_I - \mathcal{L}_I)(\theta, x) = ((\mathcal{P}_I - P_I)(\theta, x), 0) = 0,
\]
\[
(\mathcal{R}_\alpha - \mathcal{L}_\alpha)(\theta, x) = ((\mathcal{Q}_\alpha - Q_\alpha)(\theta, x), 0) = (-\bar{T}_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^\gamma} + \frac{1}{2} \bar{T}_{\alpha\beta} \theta^\beta \Gamma_I \sigma(\theta)).
\]
Once again, the fundamental sections for the adjoint action span a Lie superalgebroid.

5.1.3. The supermembrane. In order to corroborate our claim as to the structural nature of the proposed geometrization scheme in the setting of the \(\sigma\)-model super-p-brane dynamics, we discuss yet another example of a super-\(n\)-gerbe, to wit, the super-2-gerbe of the super-4-form field that couples to the uniformly charged supermembrane. In so doing, we encounter a slightly more involved extension mechanism than those dealt with heretofore. Thus, we are going to adapt the logic employed – after Refs. [ddAS85, ddAPB00] in the previous paragraph to the problem of trivialisation of the Green–Schwarz super-4-form
\[
\chi = \pi_0^* (\pi_1 \wedge \Gamma_{IJ} \theta^\gamma) \wedge e^I \wedge e^J
\]
whose closedness implies the particular variant
\[
\bar{T}_{(\alpha\beta)} (\Gamma_{IJ})_{\gamma\delta} = 0
\]
of the Fierz identity \([4.15]\), which can be conveniently rewritten as
\[
\bar{T}_{(\alpha\beta)} (\Gamma_{IJ})_{\beta\alpha} = -\bar{T}_{(\alpha\beta)} (\Gamma_{IJ})_{\gamma\delta}.
\]
On the tentative list \([1.24]\) of LI de Rham super-2-cocycles, we now find
\[
\bar{h}_{(2)}^{(1)}_{IJ} = -\frac{1}{2} \mathcal{P}_I \smile \mathcal{P}_J \wedge \chi = \frac{1}{2} \pi_0^* (\pi_1 \wedge \Gamma_{IJ} \theta^\gamma).
\]
Reasoning along the lines of our previous analyses, we erect over \(\mathcal{M}^{(1)} = s\text{Mink}^{1,d-1|D_{1,2}}\) a trivial rank-\(\frac{d(d-1)}{2}\) vector bundle
\[
\mathcal{M}^{(2)}_2 : \mathcal{M}^{(1)} \times \mathbb{R}^{\frac{d(d-1)}{2}} \rightarrow \mathcal{M}^{(1)} : (\theta^\alpha, x^I, \zeta^\ell, -\zeta^K_J) \mapsto (\theta^\alpha, x^I)
\]
\[\text{Note that neither } \mathcal{P}_\alpha \smile \mathcal{P}_\beta \wedge \chi \quad \text{nor} \quad \mathcal{P}_\alpha \smile \mathcal{P}_\beta \wedge \chi \quad \text{are closed.} \]

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endowed with the structure of a Lie supergroup that lifts the same structure on \( \mathbb{R}^{1,d-1|D_1,d-1} \) in a manner that ensures left-invariance of the super-1-forms

\[
e^{(2)}_{I,J}(\theta,x,\zeta) = d\zeta_{IJ} + \frac{1}{2} \overline{\theta} \Gamma_{IJ} \theta,
\]

the latter satisfying the identities

\[
\pi^{(2)}_a * \overline{h}^{(1)}_{IJ} = d e^{(2)}_{IJ}.
\]

The relevant structure is given in

**Proposition 5.12.** The above-described vector bundle \( \mathcal{M}^{(2)}_2 \) equipped with the binary operation

\[
m^{(2)}_2 : \mathcal{M}^{(2)}_2 \times \mathcal{M}^{(2)}_2 \to \mathcal{M}^{(2)}_2
\]

\[
((\theta^\alpha, x^I, \zeta_{JK}), (\theta^\beta, x^L, \zeta_{MN})) \mapsto (\theta^\alpha + \theta^\beta, x^I + x^L - \frac{1}{2} \overline{\theta} \Gamma^I \theta \zeta_{JK} + \zeta_{2JK} - \frac{1}{2} \overline{\theta} \Gamma_{JK} \theta_2)
\]

with the inverse

\[
\operatorname{Inv}^{(2)}_2 : \mathcal{M}^{(2)}_2 \to \mathcal{M}^{(2)}_2
\]

\[
((\theta^\alpha, x^I, \zeta_{JK})) \mapsto (-\theta^\alpha, -x^I, -\zeta_{JK})
\]

and the neutral element

\[
e^{(2)}_0 = (0,0,0)
\]

is a Lie supergroup. It is a central extension

\[
i \to \mathbb{R}^{(d-1)\mathbb{2}} \to \mathbb{R}^{1,d-1|D_1,d-1} \to \mathcal{M}^{(2)}_2 \to 1
\]

of the super-Minkowski group \( \mathbb{R}^{1,d-1|D_1,d-1} \) determined by the family of CE super-2-cocycles corresponding to the \( \text{CaE} \) super-2-cocycles \( \{ h^{(1)}_{(2),IJ} \}_{I,J \in \mathbb{Z}_{0,d-1}} \) of Eq. (5.25).

**Proof.** Through inspection. \( \square \)

In the next step, let us split the super-4-cocycle as

\[
\chi = \lambda_1 \pi^0_0(\overline{\pi} \Lambda \Gamma_{IJ} \sigma) \land e^I \land e^J \lor \lambda_2 \pi^0_0(\overline{\pi} \Lambda \Gamma_{IJ} \sigma) \land e^I \land e^J, \quad \lambda_1 + \lambda_2 = 1
\]

and pull it back to \( \mathcal{M}^{(2)}_2 \), whereupon we judiciously\(^\text{27}\) rewrite it, using the shorthand notation

\[
\pi^{(2)}_{02} \equiv \pi_0 \circ \pi^{(2)}_2
\]

along the way, as

\[
\pi^{(2)}_2 * \chi \equiv d(2\lambda_1 e^{(2)}_{IJ} \land \pi^{(2)}_{02} * e^I \land e^J) + 2\lambda_1 e^{(2)}_{IJ} \land \pi^{(2)}_{02} * (\overline{\pi} \Lambda \Gamma_{IJ} \sigma) \land \pi^{(2)}_{02} * e^J
\]

and subsequently – guided by hindsight once again – we decompose it further as

\[
\pi^{(2)}_2 * \chi \equiv d(2\lambda_1 e^{(2)}_{IJ} \land \pi^{(2)}_{02} * e^I \land e^J) + 2\lambda_1 e^{(2)}_{IJ} \land \pi^{(2)}_{02} * \overline{\pi} \Lambda \Gamma_{IJ} \sigma \land \pi^{(2)}_{02} * e^J
\]

\[
+ 2\lambda_2 e^{(2)}_{IJ} \land \pi^{(2)}_{02} * \overline{\pi} \Lambda \Gamma_{IJ} \sigma \land \pi^{(2)}_{02} * e^J + \lambda_2 \pi^{(2)}_{02} * (\overline{\pi} \Lambda \Gamma_{IJ} \sigma) \land \pi^{(2)}_{02} * e^J,
\]

for \( \lambda_1 + \lambda_2 = 1 \). We may now apply the generating technique that gave us (4.25) to the partially corrected super-4-cocycle

\[
\bar{\chi} = \pi^{(2)}_2 * \chi - d(2\lambda_1 e^{(2)}_{IJ} \land \pi^{(2)}_{02} * e^I \land e^J)
\]

over the supermanifold \( \mathcal{M}^{(2)}_2 \). For that, we need supervector fields dual to the set of LI super-1-forms \( \{ \sigma^\alpha, e^I, e^{JK} \}_{\alpha \in \mathbb{N}, I,J \in \mathbb{Z}_{0,d-1}} \). These are readily found to be given by

\[
\mathcal{D}^{(2)}_\alpha(\theta,x,\zeta) = \mathcal{D}^{(1)}_\alpha(\theta,x) + \frac{1}{2} \overline{\theta} \Gamma_{LM} \alpha \beta \frac{\partial}{\partial \zeta_{LM}},
\]

\[
\mathcal{D}^{(2)}_I(\theta,x,\zeta) = \mathcal{D}^{(1)}_I(\theta,x), \quad \mathcal{D}^{(2)}_{JK}(\theta,x,\zeta) = \frac{\partial}{\partial \zeta_{JK}}.
\]

\( ^{27}\)Trivialising the factor \( \pi^0_0(\overline{\pi} \Lambda \Gamma_{IJ} \sigma) \) within \( \pi^0_0(\overline{\pi} \Lambda \Gamma_{IJ} \sigma) \land e^I \land e^J \) as a whole does not solve our problem.
and furnish the LSA
\[
(\varpi^{(2)}_{\alpha}, \varpi^{(2)}_{\beta}) = \Gamma^I_{\alpha\beta} \delta^I_{\cal F} + \Gamma_{IJ\alpha\beta} \cal F^{(2)}_{IJ}, \quad [\varpi^{(2)}_I, \varpi^{(2)}_J] = 0, \quad [\varpi^{(2)}_{IJ}, \varpi^{(2)}_{KL}] = 0,
\]
\[
[\varpi^{(2)}_I, \varpi^{(2)}_J] = 0, \quad [\varpi^{(2)}_I, \cal F^{(2)}_{IJ}] = 0, \quad [\varpi^{(2)}_I, \cal F^{(2)}_{JK}] = 0.
\]
We may next contract the super-4-cocycle $\tilde{\chi}$ with the vector fields $\varpi^{(2)}_I$ and $\varpi^{(2)}_J$, whereby, for suitably adjusted normalisation constants ($4\lambda_1 = 1 = 4\lambda_2$), we obtain the combination
\[
\tilde{h}_3(I,\alpha) := \varpi^{(2)}_I \wedge \varpi^{(2)}_J \wedge \tilde{\chi} = \Gamma^I_{\alpha\beta} e^{(2)}_I \wedge \pi^{(2)}_{\alpha\beta} + \Gamma_{IJ\alpha\beta} \pi^{(2)}_I e^{(2)}_I \wedge \pi^{(2)}_{\alpha\beta} \theta^\gamma.
\]
The latter is closed in virtue of identity \((5.24)\). Upon rewriting the super-2-cocycle as
\[
(\Gamma^I_{\alpha\beta} e^{(2)}_I(\theta, x, \zeta) + \Gamma_{IJ\alpha\beta} e^{(2)}_I(\theta, x) ) \wedge \sigma^\gamma(\theta) = d\left[\left(\Gamma^I_{\alpha\beta} e^{(2)}_I(\theta, x, \zeta) + \Gamma_{IJ\alpha\beta} e^{(2)}_I(\theta, x) \right) \wedge \sigma^\gamma(\theta)\right]
\]
and taking into account the identity (also following from Eq.\((5.24)\))
\[
-\left(\Gamma^I_{\alpha\beta} \Gamma_{IJ\alpha\beta} + \Gamma_{IJ\alpha\beta} \theta^\gamma \wedge \sigma^\delta(\theta) = 2\left(\Gamma^I_{\alpha\gamma} \Gamma_{IJ\beta\delta} + \Gamma_{IJ\beta\delta} \Gamma_{\alpha\gamma} \theta^\gamma \wedge \sigma^\delta(\theta)\right)\right],
\]
we may write
\[
\tilde{h}_3(I,\alpha) = d\left[\left(\Gamma^I_{\alpha\beta} e^{(2)}_I(\theta, x, \zeta) + \Gamma_{IJ\alpha\beta} e^{(2)}_I(\theta, x) \right) \wedge \sigma^\gamma(\theta)\right] + \frac{1}{4}\left(\Gamma^I_{\alpha\gamma} \Gamma_{IJ\beta\delta} + \Gamma_{IJ\beta\delta} \Gamma_{\alpha\gamma} \theta^\gamma \wedge \sigma^\delta(\theta)\right],
\]
and so we see that the super-2-cocycle admits a manifestly non-LI de Rham primitive. Its trivialisation in the CaE cohomology necessitates the construction of a trivial vector bundle
\[
\pi^{(3)}_2 \equiv \text{pr}_1 : \mathcal{M}^{(3)}_2 := \mathcal{M}^{(2)}_2 \times \mathbb{R}^{0|d_{D_1},d_{-1}} \to \mathcal{M}^{(2)}_2 : (\varphi^\alpha, x^I, \zeta_{JK}, \psi_{L\beta}) \mapsto \left(\varphi^\alpha, x^I, \zeta_{JK}\right)
\]
with the purely Graßmann-odd fibre $\mathbb{R}^{0|d_{D_1},d_{-1}}$ and a Lie-supergroup that extends the previously established structure on its base $\mathcal{M}^{(2)}_2$ in such a way that the super-1-forms
\[
\sigma^{(3)}_{IJ}(\theta, x, \zeta, \psi) = d\psi_{IJ} + \left[\left(\Gamma^I_{\alpha\gamma} e^{(2)}_I(\theta, x, \zeta) - \frac{1}{2} \theta^\gamma \Gamma_{IJ} \sigma(\theta)\right) + \Gamma_{IJ\alpha\beta} e^{(2)}_I(\theta, x) - \frac{1}{2} \theta^\gamma \Gamma_{IJ} \sigma(\theta)\right] \wedge \sigma^\gamma(\theta)
\]
are LI with respect to this extension. We thus obtain

**Proposition 5.13.** The above-described vector bundle $\mathcal{M}^{(3)}_2$ equipped with the binary operation
\[
m^{(3)}_2 : \mathcal{M}^{(3)}_2 \times \mathcal{M}^{(3)}_2 \to \mathcal{M}^{(3)}_2 : (\varphi^\alpha, x^I, \zeta_{JK}, \psi_{L\beta}, \varphi^\alpha, x^I, \zeta_{JK}, \psi_{L\beta}) \mapsto (\varphi^\alpha, x^I, \zeta_{JK}, \psi_{L\beta})
\]
\[
\mapsto \left(\varphi^\alpha + \theta^\alpha, x^I + x^I - \frac{1}{2} \theta_{\cal F} \Gamma_{IJ} \theta_{\zeta} + \zeta_{JK} - \frac{1}{2} \theta_{\cal F} \Gamma_{JK} \theta_{\zeta} + \psi_{L\beta} + \psi_{L\beta}\right)
\]
\[
+ \left(\Gamma^M_{\beta\gamma} \zeta_{ML} + x^M \bar{\Gamma}_{ML\beta\gamma} \right) \theta^\gamma - \frac{1}{8} \left(\Gamma^M_{\beta\gamma} \zeta_{ML} + x^M \bar{\Gamma}_{ML\beta\gamma}\right) \left(2\theta^\gamma + \gamma + \gamma \right)
\]
with the inverse
\[
\text{Inv}^{(3)}_2 : \mathcal{M}^{(3)}_2 \to \mathcal{M}^{(3)}_2 : (\varphi^\alpha, x^I, \zeta_{JK}, \psi_{L\beta}) \mapsto (\varphi^\alpha, x^I, \zeta_{JK}, \psi_{L\beta})
\]
\[
\text{and the neutral element}
\]
\[
\epsilon^{(2)}_2 = (0, 0, 0, 0)
\]
is a Lie supergroup. It is a (super)central extension
\[
1 \to \mathbb{R}^{0|d_{D_1},d_{-1}} \to \mathcal{M}^{(3)}_2 \to \mathcal{M}^{(2)}_2 \to 1
\]
of the Lie supergroup $\mathcal{M}^{(2)}_2$ of Prop.\((5.13)\) determined by the family of CE super-2-cocycles corresponding to the CaE super-2-cocycles \{\(\tilde{h}_3(I,\alpha)\)\} of Eq.\((5.26)\).
When considering the effect of a (lifted) supersymmetry transformation on the super-1-form
\[ \tilde{\sigma}^{(3)}_{I\alpha}(\theta, x, \zeta) = \sigma^{(3)}_{I\alpha}(\theta, x, \zeta, \psi) - d\psi_{I\alpha}, \]
we immediately arrive at the expression
\[
m^{2}_{(2)} \tilde{\sigma}^{(3)}_{I\alpha}((\varepsilon, y, \zeta), (\theta, x, \zeta)) - \tilde{\sigma}^{(3)}_{I\alpha}(\theta, x, \zeta)
= d\left( (\Gamma^{I}_{\alpha\beta} + \zeta^{I}) \Gamma_{IJ\alpha\beta} + \frac{1}{3} (\Gamma^{J}_{\alpha\gamma} \tilde{\Gamma}_{IJ\gamma} \theta + \Gamma_{IJ\alpha\beta} \tilde{\Gamma}^{J}_{\gamma} \theta) \varepsilon^{\gamma} \right)
+ \frac{1}{6} \left( \Gamma^{J}_{\alpha\beta} \Gamma_{IJJ\gamma\delta} + \Gamma^{J}_{\gamma\delta} \Gamma_{IJJ\alpha\beta} + 2 \Gamma^{J}_{\alpha\gamma} \Gamma_{IJ\beta\delta} + 2 \Gamma^{J}_{\beta\delta} \Gamma_{IJJ\gamma\alpha} \right) \varepsilon^{\beta} \varepsilon^{\gamma} d\theta^{\delta},
\]
in which the last term is closed by construction. Hence, we are led to consider the de Rham super-1-cocycle
\[ \eta_{I\alpha}(\theta) = \frac{1}{6} \left( \Gamma^{J}_{\alpha\beta} \Gamma_{IJJ\gamma\delta} + \Gamma^{J}_{\gamma\delta} \Gamma_{IJJ\alpha\beta} + 2 \Gamma^{J}_{\alpha\gamma} \Gamma_{IJ\beta\delta} + 2 \Gamma^{J}_{\beta\delta} \Gamma_{IJJ\gamma\alpha} \right) \varepsilon^{\beta} \varepsilon^{\gamma} d\theta^{\delta}. \]
In consequence of the cohomological triviality of supermanifold under consideration (in fact, the super-1-cocycle descends to the odd hyperplane \( \mathbb{R}^{0|1} D_{1,4-1} \)), and so it is the triviality of the latter that matters here, the super-1-form has a global primitive given by a global section \( F \) of the structure sheaf of \( \mathbb{R}^{0|1} D_{1,4-1} \),

\[ \eta_{I\alpha} = dF_{I\alpha}, \]
which we derive with the help of the standard homotopy argument (the so-called ‘homotopy formula’). Thus, we consider a homotopy
\[ H : [0, 1] \times \mathbb{R}^{0|1} D_{1,4-1} \rightarrow \mathbb{R}^{0|1} D_{1,4-1} : (t, \theta^{\alpha}) \mapsto t\theta^{\alpha} \]
that linearly retracts the odd hyperplane to its distinguished point 0, and write the primitive in the form of the integral over the homotopy fibre
\[ F_{I\alpha}(\theta) = \int_{0}^{1} dt \partial_{t} \int H^{*} \eta_{I\alpha}(t, \theta), \]
to the effect
\[ F_{I\alpha}(\theta) = \frac{1}{6} \int_{0}^{1} dt \partial_{t} \int \left( \Gamma^{J}_{\alpha\beta} \Gamma_{IJJ\gamma\delta} + \Gamma^{J}_{\gamma\delta} \Gamma_{IJJ\alpha\beta} + 2 \Gamma^{J}_{\alpha\gamma} \Gamma_{IJ\beta\delta} + 2 \Gamma^{J}_{\beta\delta} \Gamma_{IJJ\gamma\alpha} \right) \varepsilon^{\beta} \varepsilon^{\gamma} (t d\theta^{\delta} + \theta^{\delta} dt)
= \frac{1}{72} \left( \Gamma^{J}_{\alpha\beta} \Gamma_{IJJ\gamma\delta} + \Gamma^{J}_{\gamma\delta} \Gamma_{IJJ\alpha\beta} + 2 \Gamma^{J}_{\alpha\gamma} \Gamma_{IJ\beta\delta} + 2 \Gamma^{J}_{\beta\delta} \Gamma_{IJJ\gamma\alpha} \right) \varepsilon^{\beta} \varepsilon^{\gamma} \theta^{\delta}, \]
where we used the shorthand notation
\[
\pi^{(2,3)} = \pi^{(2)} \circ \pi^{(3)}, \quad \pi^{(2)} = \pi^{(2)} \circ \pi^{(3)}.
\]
Upon setting \( \lambda_{11} = \lambda_{111} + \lambda_{112} \), we may cast the factor \( \Delta_{(2)\alpha\beta} \) in the form

\[
\Delta_{(2)\alpha\beta} = -\left( \lambda_{111} \Gamma_{\alpha\beta}^{\gamma} \sigma_{I\alpha}^{(3)} + \frac{1}{2} \lambda_{12} \Gamma_{\alpha\gamma}^{\sigma_{I\alpha}} \sigma_{I\alpha}^{(3)} + \frac{1}{2} \lambda_{12} \Gamma_{\beta\gamma}^{\sigma_{I\alpha}} \sigma_{I\alpha}^{(3)} \right) \wedge \pi_{02}^{(2,3)} \sigma^{\gamma} + 2\lambda_{12} \Gamma_{\alpha\beta}^{\gamma} \pi_{2}^{e_I} \Gamma_{I\alpha}^{(2)} \wedge \pi_{2}^{(2,3)} e^{J} \tag{5.28}
\]

and enquire as to the existence of a choice of the parameters for which the latter is closed. Taking into account the definitions of the super-1-forms \( e^{I} \), \( e^{e_{JK}^{(2)}} \), and \( \sigma_{I\alpha}^{(3)} \), we find the exterior derivative of \( \Delta_{(2)\alpha\beta} \) in the form

\[
d\Delta_{(2)\alpha\beta} = (\lambda_{111} - \lambda_{12}) \Gamma_{\alpha\beta}^{\gamma} \pi_{02}^{(3)} \left( e^{(I)} \wedge \pi_{01}^{(3)} (\sigma^{\gamma} \wedge \sigma^{\delta}) \right) + \left[ (\lambda_{111} + \lambda_{12}) \Gamma_{\alpha\beta}^{\gamma} \Gamma_{I\gamma\alpha}^{(2)} + \lambda_{22} \Gamma_{\gamma\delta} \Gamma_{I\alpha\beta} \right.
\]

\[
+ \frac{1}{2} \lambda_{12} \left( \Gamma_{\alpha\gamma}^{\sigma_{I\alpha}} \Gamma_{I\beta\gamma} + \Gamma_{\beta\gamma}^{\sigma_{I\alpha}} \Gamma_{I\alpha\beta} + \Gamma_{\alpha\delta}^{\sigma_{I\alpha}} \Gamma_{I\beta\delta} \right) \pi_{2}^{(2,3)} \left( e^{J} \wedge \pi_{01}^{(3)} (\sigma^{\gamma} \wedge \sigma^{\delta}) \right)
\]

Thus, in order for the derivative to vanish identically in the given representation of the Clifford algebra (that is, with the Fierz identity \([\underline{2.24}]\) in force), we have to impose the constraints

\[
\lambda_{111} - \lambda_{12} \neq 0, \quad 4(\lambda_{111} + \lambda_{12}) = \lambda_{112} \neq \pm 4\lambda_{22}
\]

with the solution

\[
(\lambda_{112}, \lambda_{112}, \lambda_{22}) = (8, 1, 2).
\]

We fix the free coefficient by demanding consistency of the result derived above with the linear relations between the various coefficients, and in particular – with \( \lambda_{1} + \lambda_{2} = 1 \), whereupon we obtain

\[
(\lambda_{111}, \lambda_{112}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (1, 8, 1, 18, 2).
\]

Given the closed super-2-cocycle \( \Delta_{(2)\alpha\beta} \) on \( \mathcal{M}_{2}^{(3)} \), we may – following the same logic as usual – construct a trivial vector bundle

\[
\pi_{2}^{(4)} = \text{pr}_{1} : \mathcal{M}_{2}^{(4)} := \mathcal{M}_{2}^{(3)} \times \mathbb{R}^{3,1,\alpha-1} \longrightarrow \mathcal{M}_{2}^{(3)}, \quad \delta_{1,\alpha-1} = \frac{D_{1,\alpha-1}(D_{1,\alpha-1}+1)}{2}
\]

with the purely Grassmann-even fibre \( \mathbb{R}^{3,1,\alpha-1} \) and a Lie-supergroup structure fixed by the requirement that the super-1-forms \( \sigma_{(4)\alpha\beta}(\theta, x, \zeta, \psi, \nu) = d\nu_{\alpha\beta} - \frac{i}{2} d_{(2)\alpha\beta}(\theta, x, \zeta, \psi) \),

defined in terms of some specific (non-LI) primitives \( d_{(2)\alpha\beta} \) of the respective super-2-forms \( \Delta_{(2)\alpha\beta} \), be LI with respect to this extension. We readily establish

**Proposition 5.14.** The super-2-cocycles \( \Delta_{(2)\alpha\beta} = \Delta_{(2)\alpha\beta} \alpha, \beta \in \mathbb{R}, D_{1,\alpha-1} \) of Eq. \([5.28]\) on \( \mathcal{M}_{2}^{(3)} \) corresponding to the choice of coefficients given in Eq. \([5.29]\) admit primitives

\[
-30 d_{(2)\alpha\beta}(\theta, x, \zeta, \psi) = -\left( \Gamma_{\alpha\beta}^{\gamma} \sigma_{I\alpha}^{(3)} + 4\Gamma_{\alpha\gamma}^{\sigma_{I\alpha}} \sigma_{I\alpha}^{(3)} + 4\Gamma_{\beta\gamma}^{\sigma_{I\alpha}} \sigma_{I\alpha}^{(3)} \right)(\theta, x, \zeta, \psi) \theta^{\gamma} + 2\Gamma_{\alpha\beta}^{\gamma} \pi_{2}^{e_I} \Gamma_{I\alpha}^{(2)}(\theta, x, \zeta) x^{J}
\]

\[
-2\left( 2\Gamma_{\alpha\beta}^{J} \Gamma_{\beta\gamma}^{\theta} \sigma_{I\alpha}^{(3)}(\theta, x, \zeta) + \left( \Gamma_{\alpha\delta}^{J} \Gamma_{I\beta\gamma} + \Gamma_{\alpha\delta}^{I} \Gamma_{I\beta\gamma} \right) e^{J}(\theta, x) \right) \theta^{\gamma} \theta^{\delta}
\]

\[
-2\Gamma_{I\alpha\beta} \theta^{J} e^{J}(\theta, x) - \left( \Gamma_{\alpha\delta}^{I} \Gamma_{I\beta\gamma} + \Gamma_{\alpha\delta}^{I} \Gamma_{I\beta\gamma} \right) x^{J} \theta^{\gamma} \sigma_{I\alpha}^{(3)}(\theta)
\]

\[
-\Delta_{(2)\alpha\beta}(\theta, x, \zeta, \psi) \theta^{\gamma} \theta^{\delta} \sigma_{I\alpha}^{(3)}(\theta),
\]

written in terms of the expressions

\[
\Delta_{(2)\alpha\beta}(\theta, x, \zeta, \psi) = \Gamma_{\alpha\beta}^{\gamma} \Gamma_{I\gamma\alpha}^{(2)} + \left( \Gamma_{\alpha\delta}^{J} \Gamma_{I\beta\gamma} + \Gamma_{\alpha\delta}^{I} \Gamma_{I\beta\gamma} \right) \Gamma_{I\alpha}^{(2)} e^{J}(\theta, x, \zeta, \psi, \nu_{1}\gamma)
\]

These determine, in the manner detailed above, the structure of a Lie supergroup on the vector bundle \( \mathcal{M}_{2}^{(4)} \) with the binary operation

\[
m_{2}^{(4)} : \mathcal{M}_{2}^{(4)} \times \mathcal{M}_{2}^{(4)} \longrightarrow \mathcal{M}_{2}^{(4)} : \left( (\theta_{1}^{\alpha}, x_{1}^{I}, \zeta_{1} J K, \psi_{1} L \beta, \nu_{1}\gamma), (\theta_{2}^{\alpha}, x_{2}^{I}, \zeta_{2} N O, \psi_{2} P \rho, \nu_{2}\lambda) \right) \mapsto (\theta_{1}^{\alpha} + \theta_{2}^{\alpha}, x_{1}^{I} + x_{2}^{I} - \frac{1}{2} \Gamma_{I\alpha}^{J} \Gamma_{I\beta}^{\gamma} \theta_{1}^{\gamma} \zeta_{1} J K + \zeta_{2} J K - \frac{1}{2} \Gamma_{I\alpha}^{J} \Gamma_{I\beta}^{\gamma} \theta_{1}^{\gamma} \psi_{1} L \beta + \psi_{2} L \beta)
\]

\(^{28}\)The normalisation of the super-1-forms involved is arbitrary. We fix it by demanding that the result of the ensuing trivialisation of the GS super-4-cocycle reproduce the one obtained in Ref. [CdAIPB04, Eq. (73)].
\[ + (\Gamma_{\beta\gamma}^M \zeta_{2ML} + x^M \Gamma_{ML,\gamma}) \theta_1^\gamma - \frac{1}{6} (\Gamma_{\beta\gamma}^M \Gamma_{ML,\delta} + \Gamma_{\delta\gamma}^M \Gamma_{ML,\beta}) (2\theta_1^\gamma + \theta_2^\gamma) \theta_2^\delta, \]

\[ u_{1,\gamma} + u_{2,\gamma} + \left( \frac{1}{4} \Gamma_{I\delta}^I \psi_{2,l} + \Gamma_{I\delta}^I \psi_{2,l} + \Gamma_{\delta\gamma}^I \psi_{2,\gamma} \right) \theta_1^I + \frac{1}{4} x_1^I \left( \Gamma_{IJ,\gamma}^I \left( 2x_2^2 - \bar{\theta}_1 \Gamma_{IJ,\gamma}^I \right) \right) \]

\[ + \Gamma_{I\gamma}^I \left( 2\zeta_{2IJ} - \bar{\theta}_1 \Gamma_{IJ,\gamma}^I \right) + \frac{1}{4} x_2^I \left( \Gamma_{IJ,\gamma}^I \Gamma_{IJ,\delta}^I + \Gamma_{IJ,\gamma}^I \Gamma_{IJ,\delta}^I \right) \theta_1^I \theta_2^I \]

\[ + \frac{1}{4} \psi_{2,l} \left( \Gamma_{IJ,\gamma}^I \Gamma_{IJ,\delta}^I \right) \theta_1^I \theta_2^I \]

with the inverse

\[ \text{Inv}^{(4)}_2 \colon \mathcal{M}^{(4)}_2 \longrightarrow \mathcal{M}^{(4)}_2 : (\theta^\alpha, x^I, \zeta_{JK}, \psi_L, \nu_{\gamma,\delta}) \]

\[ \longrightarrow \left( \theta^\alpha - x^I, -\zeta_{JK}, -\psi_L, -\nu_{\gamma,\delta} + \left( \Gamma_{\beta\gamma}^M \zeta_{2ML} + x^M \Gamma_{ML,\beta} \right) \theta^\epsilon, -\nu_{\gamma,\delta} - 4\Gamma_{I\gamma}^I \Gamma_{I\delta}^I \zeta_{I,\gamma} \theta^\beta \theta^\kappa \right) \]

\[ + 2x^I \left( \Gamma_{I\gamma}^I \zeta_{I,\gamma} - \Gamma_{I\gamma}^I \zeta_{I,\delta} \Gamma_{IJ,\gamma}^I \right) \theta^\gamma \theta^\beta \theta^\kappa \]

\[ + \left( \Gamma_{I\gamma}^I \psi_{I,\gamma} + 4\Gamma_{I\gamma}^I \psi_{I,\delta} + 4\Gamma_{I\gamma}^I \psi_{I,\gamma} \right) \theta^\gamma \]

and the neutral element

\[ e^{(4)}_2 = (0, 0, 0, 0). \]

It is a (super)central extension

\[ 1 \longrightarrow \mathbb{R}^{x\beta, l_{\delta, -1}} \longrightarrow \mathcal{M}^{(3)}_2 \longrightarrow \mathcal{M}^{(3)}_2 \longrightarrow 1 \]

of the Lie supergroup \( \mathcal{M}^{(3)}_2 \) of Prop. [5.1.3] determined by the family of CE super-2-cocycles corresponding to the CaE super-2-cocycles \{\Delta_{\alpha\beta}\}_{\alpha, \beta \in \mathbb{T}_1, D_1, d_{-1}}\ of\ Eq.\ (5.28).

**Proof.** Proofs of both statements made in the proposition are rather tedious but otherwise fairly straightforward. The former one requires some ingenuity, therefore, we detail it in App. [D]. \( \square \)

The above analysis gives us an explicit formula for the new LI super-1-form

\[ \sigma_{\alpha\beta}^{(4)}(\theta, x, \zeta, \psi, \nu) = d\nu_{\alpha\beta} - \left( \frac{1}{4} \Gamma_{\alpha\beta}^I \sigma_{I,\gamma}^{(3)} + \Gamma_{I\alpha,\gamma}^I \sigma_{\beta,\gamma}^{(3)} + \Gamma_{I\beta,\gamma}^I \sigma_{I,\alpha}^{(3)}(\theta, x, \zeta, \psi) \right) \theta^\gamma + \frac{1}{2} \Gamma_{I\alpha}^I \left( \psi_{I,\gamma}(\theta, x, \zeta) \right) x^I \]

\[ - \left( \Gamma_{I\alpha,\gamma}^I (x^I) \theta^\gamma \sigma_{\beta,\gamma}^{(3)}(\theta, x, \zeta, \psi) \right) + \frac{1}{2} \Gamma_{I\beta}^I (x^I) \theta^\gamma \sigma_{I,\alpha}^{(3)}(\theta, x, \zeta, \psi) \]

\[ - \frac{1}{4} \Delta_{\alpha\beta;\gamma\delta\epsilon\eta} \theta^\gamma \theta^\beta \sigma^\gamma(\theta, x, \zeta). \]

Altogether, we extract from our hitherto considerations a primitive for (the pullback of) the GS super-4-cocycle:

\[ \pi_{(2, 3, 4)}^{(2)}(e^{(4)}_2) \chi^{(4)}_{(4)} = d \beta^{(4)}_{(3)} \]

given by

\[ \beta^{(4)}_{(3)} = \frac{2}{3} \pi_{(3, 4)}^{(3, 4)} \ast (e^{(2)}_{I, J} \ast \pi_{(2)}^{(2)} \ast (e^I \ast e^J)) - \frac{2}{3} \pi_{(3)}^{(2, 3, 4)} \ast (\sigma_{(2)}^{(2)} \ast \pi_{(2)}^{(2, 3)} \ast e^I \ast e^J \ast \sigma_{(2)}^{(2)} \ast \pi_{(2)}^{(2, 3)}). \]

where we used the self-explanatory shorthand notation

\[ \pi_{(2, 3, 4)}^{(3, 4)} = \pi_{(2)}^{(3, 4)} \circ \pi_{(2)}^{(2)} \circ \pi_{(2)}^{(2, 3, 4)} \circ \pi_{(2)}^{(2, 3, 4)} \circ \sigma_{(2)}^{(2)} \circ \sigma_{(2)}^{(2)}, \]

The primitive is left-invariant with respect to the lift \( \ell^{(1)} \) of the supersymmetry \( \ell^{(1)} \) induced from \( m_{(2)}^{(4)} \) as per

\[ \ell^{(4)} = m_{(2)}^{(4)}, \]
in which the first component of the domain is to be regarded as the extended supersymmetry group. Guided by the intuition developed previously in our analysis of the GS super-2-cocycle, we take the complete extension

\[ \pi_{Y_2 \mathcal{M}^{(1)}} := \pi_{Y_2}^{(2)} \circ \pi_{Y_2}^{(3)} \circ \pi_{Y_2}^{(4)} : Y_2 \mathcal{M}^{(1)} : = \mathcal{M}_2^{(4)} \rightarrow \mathcal{M}^{(1)} : (\theta^\alpha, x^I, \zeta_{JK}, \psi_{L\beta}, \psi_{B\gamma}) \rightarrow (\theta^\alpha, x^I) \]

to be the surjective submersion of a super-geometrisation of the GS super-4-cocycle \( \chi \) that we now work out in detail. As a first step, we compare pullbacks of \( \beta^{(4)} \) to the \( \mathcal{M}^{(1)} \)-fibred square

\[
\begin{array}{ccc}
Y_2 \mathcal{M}^{(1)} & \xrightarrow{\text{pr}_1} & Y_2 \mathcal{M}^{(1)} \\
\downarrow \pi_{Y_2 \mathcal{M}^{(1)}} & & \downarrow \pi_{Y_2 \mathcal{M}^{(1)}} \\
\mathcal{M}^{(1)} & \xrightarrow{\text{pr}_2} & \mathcal{M}^{(1)}
\end{array}
\]

along the two canonical projections to \( Y_2 \mathcal{M}^{(1)} \), whereby we obtain – for \( m_4 \in \{1, 2\} \) and \( \zeta_{IJ}^{(21)} := \zeta_{IJ}^{(21)} - \zeta_{IJ}^{(2)} \), \( \psi_{K\alpha} := \psi_{K\alpha}^{(21)} - \psi_{K\alpha}^{(2)} \) and \( \psi_{\alpha}^{(21)} := \psi_{\alpha}^{(21)} - \psi_{\alpha}^{(2)} \) – the expression

\[
\begin{align*}
\text{pr}_2 - \text{pr}_1 \beta^{(4)}(m_4, m_2) &= \frac{2}{5} d\zeta_{IJ}^{(21)}(e^I \wedge e^J)(\theta, x) - \frac{3}{5}(d\psi_{I\alpha}^{(21)} + \Gamma_{\alpha}\theta^\beta d\zeta_{IJ}^{(21)}) + e^I(\theta, x) \wedge \sigma^\alpha(\theta) \\
&\quad - \frac{2}{5} [d\psi_{I\alpha}^{(21)} - (\frac{1}{4} \Gamma_{I\alpha}^{J} d\psi_{I\gamma}^{(21)} + \frac{1}{4} \Gamma_{I\gamma}^{J} d\psi_{I\alpha}^{(21)}) + \Gamma_{I\alpha}^{J} (d\psi_{I\beta}^{(2)} + \Gamma_{I\beta}^{J} \theta^\delta d\zeta_{IJ}^{(21)})] \\
&\quad + \Gamma_{I\alpha}^{J} (d\psi_{I\alpha}^{(21)} + \Gamma_{I\alpha}^{J} \theta^\delta d\zeta_{IJ}^{(21)}) \theta^\gamma + \frac{1}{2} \Gamma_{I\alpha}^{J} x^J d\zeta_{IJ}^{(21)} \\
&\quad - \Gamma_{I\alpha}^{J} \Gamma_{I\alpha}^{J} \theta^\delta \sigma^\gamma \Gamma_{I\alpha}^{J} \theta^\delta d\zeta_{IJ}^{(21)} \wedge (\sigma^\alpha \wedge \sigma^\beta)(\theta),
\end{align*}
\]

in which the super-1-forms

\[
\begin{align*}
\mathcal{X}_{IJ}(m_4, m_2) &:= d\zeta_{IJ}^{(21)} , \quad \mathcal{Y}_{I\alpha}(m_4, m_2) := d\psi_{I\alpha}^{(21)} + \Gamma_{I\alpha}^{J} \theta^\delta \mathcal{X}_{IJ} \\
\mathcal{X}_{I\alpha}(m_4, m_2) &:= d\psi_{I\alpha}^{(21)} - (\frac{1}{4} \Gamma_{I\alpha}^{J} \mathcal{Y}_{I\gamma} + \frac{1}{4} \Gamma_{I\gamma}^{J} \mathcal{Y}_{I\alpha} + \Gamma_{I\alpha}^{J} \mathcal{Y}_{I\alpha}) (m_4, m_2) \theta^\gamma + \frac{1}{2} \Gamma_{I\alpha}^{J} x^J \mathcal{X}_{IJ}(m_4, m_2)
\end{align*}
\]

are – by construction (as differences of LI super-1-forms) – left-invariant under the diagonal lift of the action \( \ell^{(4)} \) of the Lie supergroup \( \mathcal{M}_2^{(4)} \) to the fibred square \( Y_2 \mathcal{M}^{(1)} \). Following the standard procedure, we seek to trivialise the 3-cocycle in a LI manner by pulling it back to the total space a suitable surjective submersion over (or supercentral extension of) \( Y_2 \mathcal{M}^{(1)} \). To this end, we first consider the collection

\[ (5.30) \quad \mathcal{H}_{(2)} \alpha \beta := pr_1^* \pi_{Y_2}^{(2, 3, 4)*}(\sigma^\alpha \wedge \sigma^\beta) \]

of manifestly LI 2-cocycles and associate with them a trivial vector bundle

\[ \pi_{Y_2}^{(5)} \equiv pr_1 : Y_2 \mathcal{M}^{(1)}(5) := Y_2 \mathcal{M}^{(1)}(5) \times \mathbb{R}^{x\delta_{1, 2-1}} \rightarrow Y_2 \mathcal{M}^{(1)} \]

with the purely Graßmann-even fibre \( \mathbb{R}^{x\delta_{1, 2-1}} \) and a Lie-supergroup structure fixed – as formerly – by the requirement that the super-1-forms

\[ e^{(5)}(m_5) = dX^\alpha \beta + \frac{1}{2} (\theta^\alpha d\theta^\beta + \theta^\beta d\theta^\alpha), \]

be LI with respect to this extension. We have the obvious

**Proposition 5.15.** The above-described vector bundle \( Y_2 \mathcal{M}^{(1)}(5) \) equipped with the binary operation

\[
\begin{align*}
\mathcal{M}_2^{(5)} &:= Y_2 \mathcal{M}^{(1)}(5) \times Y_2 \mathcal{M}^{(1)}(5) \rightarrow Y_2 \mathcal{M}^{(1)}(5) : (m_1, m_2, X^\alpha \beta) \mapsto (m_1, m_2, X^\alpha \beta) 
\end{align*}
\]
\[ \mapsto (m_2^{(3)}(m_4^{(1)}, m_4^{(2)}), m_2^{(3)}(m_4^{(1)}, m_4^{(2)}), X_1^{\alpha\beta} + X_2^{\alpha\beta} - \frac{1}{2}(\theta_1^\alpha \theta_2^\beta + \theta_2^\alpha \theta_1^\beta)) \]

with the inverse
\[ \text{Inv}^{(5)}_2 : Y_2^{[2]}M^{(1)(5)} \rightarrow Y_2^{[2]}M^{(1)(5)} : (m_4^{(1)}, m_4^{(2)}, X^{\alpha\beta}) \mapsto (m_4^{1-1}, m_4^{2-1}, -X^{\alpha\beta}) \]

and the neutral element
\[ c_2^{(5)} = (0, 0, 0) \]

is a Lie supergroup. It is a (super)central extension
\[ 1 \rightarrow \mathbb{R}^{x\beta \cdot d-1} \rightarrow Y_2^{[2]}M^{(1)} \rightarrow \mathbb{R}^{x\beta \cdot d-1} \rightarrow \gamma_2 \rightarrow Y_2^{[2]}M^{(1)} \rightarrow 1 \]

of the (product) Lie supergroup \( Y_2^{[2]}M^{(1)} \), the latter being formed from the Lie supergroup \( Y_2^{[2]}M^{(1)} \) of Prop. 14. The supercentral extension is determined by the family of CE super-2-cocycles corresponding to the CaE super-2-cocycles \( \{ h_{\alpha \beta} \}_{\alpha, \beta \in D_1, \gamma \in D_{-1}} \) of Eq. (2.36).

Proof. Trivial. \( \square \)

The LI 1-forms \( \sigma^{(5)}_{\alpha \beta} = \sigma^{(5)}_{\beta \alpha} \) on the new Lie supergroup, satisfying the identity
\[ d\sigma^{(5)}_{\alpha \beta} = \pi_{(2,3,4,5)}^{(5) *}(\sigma^\alpha \wedge \sigma^\beta), \]

written in the shorthand notation
\[ \pi_{(2,3,4,5)}^{(5) *}(\sigma^\alpha \wedge \sigma^\beta) = \pi_{(2,3,4)} \circ \pi_1 \circ \pi_2^{(5)} \]

(to be adapted to subsequent extensions in an obvious manner), enable us to partially trivialise the super-3-form \( (pr_2 - pr_1)^{(4)}(\gamma_5) \) upon pullback to \( Y_2^{[2]}M^{(1)(5)} \) as
\[ \pi_{(2,3,4,5)}^{(5) *}(\sigma^\alpha \wedge \sigma^\beta)(\gamma_5) \]

\[ = d(\pi_{(2,3,4)}\otimes \sigma^\alpha)(\gamma_5) \]

\[ = d(\pi_{(2,3,4)}\otimes \sigma^\alpha)(\gamma_5) \]

\[ = \pi_{(2,3,4)}(\pi_1^\alpha)(\gamma_5) \]

\[ = \pi_{(2,3,4)}(\pi_1^\alpha)(\gamma_5) \]

\[ = \pi_{(2,3,4)}(\pi_1^\alpha)(\gamma_5) \]

\[ = \pi_{(2,3,4)}(\pi_1^\alpha)(\gamma_5) \]

In the next step, we readily verify that the manifestly LI super-2-form
\[ \tilde{h}_{(2)}^{I \gamma} := 18 \pi_{(2,3,4,5)}^{(5) *}(e^I \wedge \sigma^\alpha)(\gamma_5) - \Gamma_{\beta \gamma} \pi_2^{(5) *}(\sigma^\beta \wedge \sigma^\gamma)(\gamma_5) + 8 \pi_{(2,3,4)}(\pi_{(2,3,4)} \circ \pi_1^\alpha \circ \pi_2^{(5)} \circ \pi_2^{(5)}) \]

is closed, and hence gives rise to yet another (super)central extension. This time, we take the trivial vector bundle
\[ \pi_{(2,3,4,5)}^{(6) *}(\sigma^\alpha)(\gamma_5) \]

\[ = \pi_{(2,3,4,5)}^{(6) *}(\sigma^\alpha)(\gamma_5) \]

\[ = \pi_{(2,3,4,5)}^{(6) *}(\sigma^\alpha)(\gamma_5) \]

\[ = \pi_{(2,3,4,5)}^{(6) *}(\sigma^\alpha)(\gamma_5) \]

\[ = \pi_{(2,3,4,5)}^{(6) *}(\sigma^\alpha)(\gamma_5) \]

\[ = \pi_{(2,3,4,5)}^{(6) *}(\sigma^\alpha)(\gamma_5) \]

are LI with respect to this extension. We have

\[ d\pi_{(2,3,4,5)}^{(6) *}(\sigma^\alpha)(\gamma_5) = \tilde{h}_{(2)}^{I \gamma} \]
Proposition 5.16. The above-described vector bundle $\mathcal{V}^{(2)}_2\mathcal{M}^{(1)}$ equipped with the binary operation

$$\hat{m}^{(2)}_2 : \mathcal{V}^{(2)}_2\mathcal{M}^{(1)} \times \mathcal{V}^{(2)}_2\mathcal{M}^{(1)} \rightarrow \mathcal{V}^{(2)}_2\mathcal{M}^{(1)} = \left( (m^1_4, m^2_4, X^{\alpha\beta}, Y_I^r), \left( \begin{array}{l} (m^1_4, m^2_4, X^{\alpha\beta}, Y_I^r) \\ (m^1_4, m^2_4, X^{\alpha\beta}, Y_I^r) \end{array} \right) \right)$$

$$\rightarrow \left( \begin{array}{l} m^{(2)}_2 \left( m^1_4, m^2_4, X^{\alpha\beta}, Y_I^r \right), m^{(2)}_2 \left( m^1_4, m^2_4, X^{\alpha\beta}, Y_I^r \right) \end{array} \right)$$

$$Z^{\alpha\beta} = -4(\theta_1^\sigma + \theta_2^\sigma) \left( \theta_1^\sigma X^{\alpha\beta} + 8 \theta_1^\sigma X^{\gamma r} \right)$$

with the inverse

$$\hat{m}^{-1}_{2} : \mathcal{V}^{(2)}_2\mathcal{M}^{(1)} \rightarrow \mathcal{V}^{(2)}_2\mathcal{M}^{(1)}$$

is a Lie supergroup. It is a (super)central extension

$$1 \rightarrow \mathbb{R}^{0\lfloor dD_4 \rfloor \rightarrow \cdots \rightarrow \mathbb{R}^{(2)} \mathcal{M}^{(1)} : Y_2^{(2)} \mathcal{M}^{(1)} \rightarrow \mathbb{R}^{(2)} \mathcal{M}^{(1)} \rightarrow 1$$

of the Lie supergroup $\mathcal{Y}_2^{(2)} \mathcal{M}^{(1)}$ of Prop. 5.13. The supercentral extension is determined by the family of CE super-2-cocycles of Eq. (5.31).

Proof. Through inspection. 

Thus, upon pullback to $\mathcal{V}^{(2)}_2\mathcal{M}^{(1)} \ni \hat{m}_6$ along

$$\hat{m}_6 = \hat{m}_6 \circ \hat{m}_6,$$

we obtain

$$\hat{m}^{(2,6)}_2 \circ \hat{m}_6 \circ \hat{m}^{(2)}_6 = \hat{m}_6 \circ \hat{m}^{(2)}_6,$$

and it is easy to check (or deduce from the construction) that the $LI$ super-2-form

$$\hat{m}^{(2)}_6 = \hat{m}^{(2,6)}_2 \circ \hat{m}_6 \circ \hat{m}^{(2)}_6,$$

written in terms of the maps

$$\hat{m}^{(2,6)}_2 \circ \hat{m}_6 \circ \hat{m}^{(2)}_6 = \hat{m}_6 \circ \hat{m}^{(2)}_6 = \hat{m}_6 \circ \hat{m}^{(2,6)}_2 \circ \hat{m}_6 \circ \hat{m}^{(2)}_6,$$

is closed, so that we may finally trivialise the difference of pullbacks in the CaE cohomology by constructing one last (super)central extension. Thus, take the trivial vector bundle

$$\hat{m}^{(2)}_6 \equiv \hat{m}_6 : \mathcal{V}^{(2)}_2\mathcal{M}^{(1)} := \mathcal{V}^{(2)}_2\mathcal{M}^{(1)} \times \mathbb{R}^{d\lfloor dD_4 \rfloor} \rightarrow \mathcal{V}^{(2)}_2\mathcal{M}^{(1)}$$

$$\rightarrow \hat{m}_7 := \left( m^1_4, m^2_4, X^{\alpha\beta}, Y_I^r \right) \rightarrow \left( m^1_4, m^2_4, X^{\alpha\beta}, Y_I^r \right)$$

with the purely Graßmann-even fibre $\mathbb{R}^{d\lfloor dD_4 \rfloor}$ and endow it with the structure of a Lie supergroup that lifts the previously established structure of the same type from its base in such a manner that the super-1-forms

$$\hat{e}^{(2,6)}_2 \circ \hat{m}_7 \circ \hat{e}^{(2)}_2 = \hat{m}_7 \circ \hat{e}^{(2)}_2 = \hat{m}_7 \circ \hat{e}^{(2,6)}_2 \circ \hat{m}_7 \circ \hat{e}^{(2)}_2,$$

satisfying the identities

$$\hat{e}^{(2,6)}_2 \circ \hat{m}_7 \circ \hat{e}^{(2)}_2 = \hat{m}_7 \circ \hat{e}^{(2)}_2 = \hat{m}_7 \circ \hat{e}^{(2,6)}_2 \circ \hat{m}_7 \circ \hat{e}^{(2)}_2,$$
are LI with respect to this new supergroup structure. Yet again, we obtain

**Proposition 5.17.** The above-described vector bundle \( Y_2^{[2]}M(1)(7) \) equipped with the binary operation

\[
\tilde{\eta}_2^{(7)} : Y_2^{[2]}M(1)(7) \times Y_2^{[2]}M(1)(7) \rightarrow Y_2^{[2]}M(1)(7)
\]

\[
\tilde{\eta}_2^{(7)} = ((m_{11}^4, m_{12}^4, X_{1}^{\alpha \beta}, Y_{1}^{I}^J Z_{1}^{JK}), (m_{41}^4, m_{42}^4, X_{2}^{\delta \mu}, Y_{2}^{L}^N Z_{2}^{MN})) \rightarrow (m_{21}^4, m_{22}^4),
\]

with the inverse

\[
\tilde{\eta}_2^{(7)} : Y_2^{[2]}M(1)(7) \rightarrow Y_2^{[2]}M(1)(7)
\]

\[
\tilde{\eta}_2^{(7)} = (m_{11}^{-1}, m_{21}^{-1}, -X_{1}^{\alpha \beta}, -Y_{1}^{I}^J Z_{1}^{JK})
\]

and the neutral element

\[
\tilde{e}_2^{(7)} = (0, 0, 0, 0, 0)
\]

is a Lie supergroup. It is a (super)central extension

\[
1 \rightarrow \mathbb{R}^2 \xrightarrow{\tilde{\eta}_2^{(7)}} Y_2^{[2]}M(1)(6) \xrightarrow{\tilde{\eta}_2^{(7)}} Y_2^{[2]}M(1)(6) \rightarrow 1
\]

of the Lie supergroup \( Y_2^{[2]}M(1)(6) \) of Prop. 5.14. The supercentral extension is determined by the family of CE super-2-cocycles corresponding to the CaE super-2-cocycles \( \{ \tilde{\eta}_2^{(7)} \}_{l, j, \alpha, \beta} \) of Eq. (5.31).

**Proof.** Straightforward, through inspection. □

By the end of the long day, we are left with the desired result

\[
\tilde{\eta}_2^{(5,6,7)} \circ (pr_2 2 - pr_1 2) \circ \beta I J (\tilde{m}_7) = d[\frac{2}{15} \mathcal{Z}_{\alpha \beta}(m_{11}^4, m_{21}^4) \wedge \tilde{e}_2^{(5)}(\tilde{m}_5) + \frac{1}{30} \mathcal{Y}_{I \alpha}(m_{11}^4, m_{21}^4) \wedge \tilde{e}_2^{(6)} I \alpha(\tilde{m}_6)
\]

\[
- \frac{1}{30} \mathcal{I}_{I J}(m_{11}^4, m_{21}^4) \wedge \tilde{e}_2^{(7)} I J (\tilde{m}_7)]
\]

where

\[
\tilde{\eta}_2^{(5,6,7)} = \tilde{\eta}_2^{(5)} \circ \tilde{\eta}_2^{(6)} \circ \tilde{\eta}_2^{(7)}
\]

The above formula suggests that we should take the (super)central extension

\[
\tilde{\eta}_2^{[2]}M(1) : Y_2^{[2]}M(1)(7) \rightarrow Y_2^{[2]}M(1)
\]

\[
\left( m_{11}^4, m_{21}^4, X, Y, Z \right) \rightarrow (m_{21}^4, m_{22}^4)
\]

as the surjective submersion of the super-1-gerbe over \( Y_2^{[2]}M(1) \) with a connection of the LI curvature

\[
\tilde{\eta}_2^{(3,4)} = \frac{2}{15} \mathcal{Z}_{\alpha \beta} \wedge pr_1^2 \mathcal{Z}_{\alpha \beta} + \frac{2}{15} \mathcal{Y}_{I \alpha} \wedge pr_1^2 \mathcal{Y}_{I \alpha} + \frac{2}{15} \mathcal{I}_{I J} \wedge pr_1^2 \mathcal{I}_{I J}
\]

and the LI curving given by the formula

\[
\tilde{\beta}_2^{(2)} = \frac{1}{30} \left( 4 \tilde{\eta}_2^{(5,6,7)} \circ (\tilde{\eta}_2^{(5)} \circ \mathcal{Z}_{\alpha \beta} \wedge \tilde{e}_2^{(5)}(\alpha \beta) + \tilde{\eta}_2^{(7)} \circ (\tilde{\eta}_2^{(5,6,7)} \circ \mathcal{Y}_{I \alpha} \wedge \tilde{e}_2^{(6)} I \alpha) - \tilde{\eta}_2^{(5,6,7)} \circ \mathcal{I}_{I J} \wedge \tilde{e}_2^{(7)} I J \right)
\]

In the next step, we compare pullbacks of that curving along the canonical projections to the \( Y_2^{[2]}M(1) \)-fibred square

\[
\tilde{Y}_2^{[2]}M(1) \times Y_2^{[2]}M(1)(7) \rightarrow Y_2^{[2]}M(1)(7),
\]
whereby we find \( \bar{m}^2_A := (m^1_A, m^2_A, X_A, Y_A, Z_A) \), \( A \in \{1, 2\} \) and \( X^A := X^2 - X^1 \), \( Y^A := Y^2 - Y^1 \) and \( Z^{21} := Z^2 - Z^1 \)

\[
\begin{align*}
&= \frac{1}{2} \bigg[ (4 \bar{X}_{\alpha\beta} (m^1_A, m^2_A) + (4 \bar{X}_{\alpha\beta} (m^1_A, m^2_A) + 8 \bar{Y}_{\alpha\beta}, X^1_{\gamma\delta} (m^1_A, m^2_A) \theta^\gamma + 2 \bar{X}_{IJ} (m^1_A, m^2_A) (2 T^I_{\alpha\beta} T^I_{\gamma\delta} \theta^\gamma \theta^\delta - T^I_{\alpha\beta} X^1_{IJ}) \bigg) \wedge d X^21_{\alpha\beta} \wedge Y^21_{\alpha\beta} - 4 \bar{X}_{IJ} \bigg] \\
&= \frac{1}{2} \bigg[ 4 \bar{d} X^21_{\alpha\beta} \wedge d X^21_{\alpha\beta} + 4 \bar{d} Y^21_{\alpha\beta} - 4 \bar{d} Z^{21}_{IJ} \bigg].
\end{align*}
\]

Thus, just as in the case of the GS super-1-gerbe, we obtain a trivial principal \( \mathbb{C}^* \)-bundle

\[
(5.32) \quad \begin{array}{c}
\pi_{\mathcal{F}} = \text{pr}_1 : \mathcal{F} := \bar{Y}Y^2_{\mathcal{M}}(1) \times \mathbb{C}^* \longrightarrow \bar{Y}Y^2_{\mathcal{M}}(1) : (\bar{m}^1_A, \bar{m}^2_A, \bar{m}^3_A) \longmapsto (\bar{m}^1_A, \bar{m}^2_A)
\end{array}
\]

with a principal connection

\[
\nabla_{\mathcal{F}} = d + \frac{1}{4} \bar{A},
\]

or - equivalently - a principal connection 1-form

\[
\bar{A}(\bar{m}^1_A, \bar{m}^2_A, \bar{m}^3_A) = i_{\mathcal{F}} \bar{A} + \bar{A}(\bar{m}^1_A, \bar{m}^2_A)
\]

with the base component

\[
\bar{A}(\bar{m}^1_A, \bar{m}^2_A) = \frac{1}{2} \bigg[ Z^{21}_{IJ} d \bar{X}^21_{IJ} + Y^21_{\alpha\beta} d \bar{Y}^21_{\alpha\beta} - 4 X^21_{\alpha\beta} d \bar{d}^21_{\alpha\beta} \bigg].
\]

Following the by now well-established procedure, we determine the lift of the Lie-supergroup structure from the base of the bundle to its total space by imposing the requirement that the principal connection 1-form be LI with respect to the rigid lifted supersymmetry induced from the ensuing group law. In order to study its consequences, we first work out in detail how the various coordinate differences entering the definition of \( \bar{A} \) change under a rigid supersymmetry transformation with parameters \( \bar{\theta}_2^A \equiv (\varepsilon^1, y^1, \xi_{JK}, \phi_{I\alpha\beta}, \omega_{\gamma\delta}, \xi^2_{MN}, \phi_{\nu\eta}, \omega_{\nu\eta}, U^{A\lambda\nu}, V^A P^\nu, W^A R^S) \in \bar{Y}Y^2_{\mathcal{M}}(1) \), \( A \in \{1, 2\} \) induced, in the same manner, from \( \bar{m}^1_A \), in which we have taken into account the various fibrations involved in the construction (a point in the \( A \)-th factor of \( \bar{Y}Y^2_{\mathcal{M}}(1) \) is transformed by the corresponding \( \bar{\theta}_2^A \)). We readily find the following transformation laws

\[
\begin{align*}
\zeta^{21}_{IJ} & \longrightarrow \bar{\zeta}^{21}_{IJ}, \\
\psi^{21}_{\alpha\beta} & \longrightarrow \psi^{21}_{\alpha\beta} - \varepsilon^\gamma (\frac{1}{4} T^I_{\alpha\beta} \psi_{I\gamma} + T^I_{\gamma\delta} \psi_{I\beta} + T^I_{\alpha\gamma} \psi_{I\beta} + T^I_{\beta\gamma} \psi_{I\alpha} - \frac{1}{2} y^I T^I_{\gamma\delta} \varepsilon^\gamma + \frac{1}{2} \bar{Y}^21_{\gamma\delta} \varepsilon^\gamma + \frac{1}{2} \bar{X}^21_{\alpha\beta} \varepsilon^\gamma) \zeta^{21}_{IJ}, \\
\psi^{21}_{IJ} & \longrightarrow \psi^{21}_{IJ} - \varepsilon^\gamma (\frac{1}{4} T^I_{\gamma\delta} \psi_{I\alpha\beta} + T^I_{\alpha\gamma} \psi_{I\beta} + T^I_{\beta\gamma} \psi_{I\alpha} + \frac{1}{2} y^I T^I_{\alpha\beta} \varepsilon^\gamma - \frac{1}{2} \bar{Y}^21_{\gamma\delta} \varepsilon^\gamma + \frac{1}{2} \bar{X}^21_{\alpha\beta} \varepsilon^\gamma) \zeta^{21}_{IJ}, \\
X^{21}_{\alpha\beta} & \longrightarrow X^{21}_{\alpha\beta}, \\
Y^{21}_{\alpha\beta} & \longrightarrow Y^{21}_{\alpha\beta}, \\
Z^{21}_{IJ} & \longrightarrow Z^{21}_{IJ} + \frac{1}{2} \varepsilon^\alpha (\bar{T}^I_{\alpha\beta} X^{21}_{\gamma\delta} - \bar{T}^I_{\alpha\beta} Y^{21}_{\gamma\delta}) - (4 T^I_{\alpha\beta} T^I_{\gamma\delta} \varepsilon^\alpha - \gamma^I T^I_{\beta\gamma} + y^I T^I_{\beta\gamma}) X^{21}_{\alpha\beta},
\end{align*}
\]

and so it follows that the base component of the principal connection is actually left-invariant, and not merely quasi-left-invariant as previously,

\[
\bar{A}(\bar{m}^1_A(\bar{\theta}^2_A), \bar{m}^2_A(\bar{\theta}^2_A), \bar{m}^3_A(\bar{\theta}^2_A)) = \bar{A}(\bar{m}^1_A, \bar{m}^2_A).
\]

Accordingly, we may take the lift of the supersymmetry to \( \bar{\mathcal{F}} \) to be trivial, as stated in

**Proposition 5.18.** The principal \( \mathbb{C}^* \)-bundle \( \bar{\mathcal{F}} \) of Eq. (5.32) equipped with the binary operation

\[
\bar{m}^2_A : \bar{\mathcal{F}} \times \bar{\mathcal{F}} \longrightarrow \bar{\mathcal{F}}
\]
with the inverse
\[ \text{Inv}^{(8)}_2 : \mathcal{F} \to \mathcal{F} : (\tilde{m}_1^2, \tilde{m}_2^2, \tilde{z}) \mapsto (\text{Inv}^{(7)}_2(\tilde{m}_1^2), \text{Inv}^{(7)}_2(\tilde{m}_2^2), \tilde{z}^{-1}) \]
and the neutral element
\[ e^{(8)}_2 = (0, 0, 1) \]
is a Lie supergroup. It is a trivial central extension
\[ 1 \to C^\times \to Y^{[2]}_2 M^{(1)} \times C^\times \xrightarrow{\gamma_{\mathcal{F}}} Y^{[2]}_2 M^{(1)} \to 1, \]
that is the direct product of the Lie supergroup \( Y^{[2]}_2 M^{(1)} \) with the structure group \( C^\times \).

**Proof.** Cp. above.

At this stage, we may pass to the fibred cube
\[ Y^{[3]}_2 Y^{[2]}_2 M^{(1)} \equiv Y^{[2]}_2 M^{(1)} \times_{Y^{[2]}_2 M^{(1)}} Y^{[2]}_2 M^{(1)} \times_{Y^{[2]}_2 M^{(1)}} Y^{[2]}_2 M^{(1)}, \]
and look for a connection-preserving isomorphism
\[ \mu_{\mathcal{F}} : \text{pr}_{1,2}^* \mathcal{F} \otimes \text{pr}_{2,3}^* \mathcal{F} \to \text{pr}_{1,3}^* \mathcal{F}. \]
The comparison of the pullbacks of the connection 1-forms
\[ (\text{pr}_{1,2}^* + \text{pr}_{2,3}^* - \text{pr}_{1,3}^*) \Lambda (\tilde{m}_1^2, \tilde{m}_2^2, \tilde{m}_3^2) = 0, \]
in conjunction with Prop. 5.18 immediately suggest the natural choice
\[ \mu_{\mathcal{F}}((\tilde{m}_1^2, \tilde{m}_2^2, \tilde{z}_1^2) \otimes (\tilde{m}_2^2, \tilde{m}_3^2, \tilde{z}_2^2)) := (\tilde{m}_1^2, \tilde{m}_3^2, \tilde{z}_1^2, \cdot \tilde{z}_2^2). \]
A fibre map thus defined trivially satisfies the groupoid identity (2.3) over \( Y^{[3]}_2 Y^{[2]}_2 M^{(1)}. \)

Altogether, then, we establish the existence of a super-1-gerbe
\[ \mathcal{G} = (Y^{[3]}_2 M^{(1)}, \pi_{Y^{[2]}_2 M^{(1)}}, \overline{\mathcal{F}}, \nabla_{\mathcal{G}}, \mu_{\mathcal{F}}) \]
over the fibred square \( Y^{[2]}_2 M^{(1)} \) of the (super)central extension \( Y_2 M^{(1)} \) of the support of the GS super-4-cocycle, in the sense of Def. 5.9. We shall next construct a coherent product on the super-1-gerbe.

To this end, we define the pullback surjective submersions
\[ \begin{array}{ccc}
Y^{[3]}_2 M^{(1)} & \xrightarrow{\pi_{Y^{[3]}_2 M^{(1)}}} & \overline{\mathcal{Y}}^{[2]}_2 M^{(1)} \\
\text{pr}_{i,j} & & \text{pr}_{i,j} \\
Y^{[2]}_2 M^{(1)} & \xrightarrow{\pi_{Y^{[2]}_2 M^{(1)}}} & Y^{[2]}_2 M^{(1)}
\end{array} \]
for \( (i, j) \in \{(1, 2), (2, 3), (1, 3)\} \), with (global) coordinates
\[ \tilde{m}^{(i,j)} = (m_1^1, m_2^1, m_3^1, X^{(i,j)}, Y^{(i,j)}, Z^{(i,j)}) \in \tilde{\mathcal{V}}^{i,j} Y^{[2]}_2 M^{(1)} \]
and projections
\[ \begin{align*}
\text{pr}_{i,j}^* & \left( m_1^1, m_2^1, m_3^1, X^{(i,j)}, Y^{(i,j)}, Z^{(i,j)} \right) := \left( m_1^1, m_2^1, X^{(i,j)}, Y^{(i,j)}, Z^{(i,j)} \right), \\
\pi_{Y^{[2]}_2 M^{(1)}} (m_1^1, m_2^1, m_3^1, X^{(i,j)}, Y^{(i,j)}, Z^{(i,j)}) & := (m_1^1, m_2^1, m_3^1),
\end{align*} \]
and equip them with the obvious Lie-supergroup structure projecting to that of Prop. 5.17 along the respective map \( \text{pr}_{i,j}^* \) and to that of Prop. 5.14 along each of the maps \( \text{pr}_{A} \circ \pi_{Y^{[3]}_2 M^{(1)}}, A \in \{1, 2, 3\} \). Subsequently, we compare the (Deligne) tensor product of the pullback super-1-gerbes (we are dropping some obvious subscripts for the sake of transparency)
\[
\text{pr}_{1,2}^* \mathcal{G} \otimes \text{pr}_{2,3}^* \mathcal{F} = \left( \mathcal{V}^{1,2} Y^{[2]}_2 M^{(1)} \times_{Y^{[2]}_2 M^{(1)}} \mathcal{V}^{2,3} Y^{[3]}_2 M^{(1)}, \pi_{Y^{[2]}_2 M^{(1)}}, \text{pr}_{1,2}^* \mathcal{G} \otimes \text{pr}_{2,3}^* \mathcal{F}, \mu_{\mathcal{F}} \right),
\]

\[ \text{pr}_{1,2}^* \mathcal{G} \otimes \text{pr}_{2,3}^* \mathcal{F} \otimes \text{pr}_{1,3}^* \mathcal{G} \otimes \text{id} \otimes \text{pr}_{2,4}^* \mathcal{F} \otimes \nabla_{\mathcal{G}}. \]
written in terms of the obvious canonical projections (which will be made explicit below), with the pullback super-1-gerbe
\[ \text{pr}_{1,3}^* \mathcal{G} = \left( \mathcal{Y}_{1,3}^1 \mathcal{Y}_{2}^1 \mathcal{M}(1), \pi_{1,3} \mathcal{Y}_{2}^1 \mathcal{M}(1), \text{pr}_{1,3}^* \beta \right), \]

We perform the comparison over the fibred product
\[ \mathcal{Y}_{1,2,3}^1 \mathcal{Y}_{2}^1 \mathcal{M}(1) := \mathcal{Y}_{1,2}^1 \mathcal{Y}_{2}^1 \mathcal{M}(1) \times_{\mathcal{Y}_{2}^1 \mathcal{M}(1)} \mathcal{Y}_{1,2,3}^1 \mathcal{Y}_{2}^1 \mathcal{M}(1) \]
surjectively submersed onto \( \mathcal{Y}_{2}^1 \mathcal{M}(1) \) as per
\[ \pi_{1,2,3} \mathcal{Y}_{2}^1 \mathcal{M}(1) \equiv \pi_{1,2} \mathcal{Y}_{2}^1 \mathcal{M}(1) \circ \text{pr}_1. \]
There, we find
\[
\begin{align*}
\left( \text{pr}_{1,2}^* \pi_{1,3} \mathcal{Y}_{2}^1 \mathcal{M}(1) \right) \mathcal{G} := & \left( \mathcal{Y}_{1,3}^1 \mathcal{Y}_{2}^1 \mathcal{M}(1) \times \mathcal{Y}_{2}^1 \mathcal{M}(1), \pi_{1,3} \mathcal{Y}_{2}^1 \mathcal{M}(1) \times \mathcal{Y}_{2}^1 \mathcal{M}(1), \mathcal{Y}_{1,2,3}^1 \mathcal{Y}_{2}^1 \mathcal{M}(1) \right) \\
& \implies \left( \mathcal{Y}_{1,2}^1 \mathcal{Y}_{2}^1 \mathcal{M}(1), \mathcal{Y}_{2}^1 \mathcal{M}(1), \mathcal{Y}_{1,2,3}^1 \mathcal{Y}_{2}^1 \mathcal{M}(1) \right)
\end{align*}
\]
with an LI principal \( \mathcal{C}^*\)-connection 1-form
\[ \tilde{\alpha} \left( \bar{m}^{(1,2)}_3, \bar{m}^{(2,3)}_3, \bar{m}^{(1,3)}_3, \bar{\zeta}^{1,2,3}_3 \right) \rightarrow \left( \bar{m}^{(1,2)}_3, \bar{m}^{(2,3)}_3, \bar{m}^{(1,3)}_3 \right) \]
with the base component
\[ \tilde{\alpha} \left( \bar{m}^{(1,2)}_3, \bar{m}^{(2,3)}_3, \bar{m}^{(1,3)}_3 \right) = \frac{1}{\mathfrak{m}} \left[ \left( Z_1^{(1,2)} - Z_2^{(2,3)} \right) \pi_2^{(1,2)} \right] \\
+ \left( \pi_2^{(1,2)}, \pi_2^{(2,3)} \right) \mathcal{Y}_{2}^1 \mathcal{M}(1) \times \mathcal{Y}_{2}^1 \mathcal{M}(1) \rightarrow \mathcal{Y}_{2}^1 \mathcal{M}(1) \times \mathcal{Y}_{2}^1 \mathcal{M}(1), \]
and compute
\[
\left( \text{pr}_{1,4}^* + \text{pr}_{2,5}^* \right) \bar{\Lambda} + \text{pr}_{4,5,6}^* \bar{\Lambda} = \text{pr}_{1,2,3}^* \bar{\Lambda} + \text{pr}_{3,6}^* \bar{\Lambda},
\]
whereupon it becomes clear that we have a connection-preserving \( \mathcal{C}^*\)-bundle isomorphism
\[ \tilde{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{G}, \]

The triviality of its form, in conjunction with that of the groupoid structure on \( \mathcal{G} \) established in Eq. (5.3), ensures that it satisfies the usual requirement of compatibility with the respective groupoid structures on \( \text{pr}_{1,2}^* \mathcal{G} \oplus \text{pr}_{2,3}^* \mathcal{G} \) and \( \text{pr}_{1,3}^* \mathcal{G} \). It is also in keeping with our definition of the super-0-gerbe isomorphism. Thus, altogether, we have the desired product 1-isomorphism
\[ \mathcal{G}_{\mathcal{M}} : \text{pr}_{1,2}^* \mathcal{G} \oplus \text{pr}_{2,3}^* \mathcal{G} \rightarrow \text{pr}_{1,3}^* \mathcal{G}. \]
Finally, we verify the existence (over $\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}$) of an associator 2-isomorphism

$$\begin{array}{ccc}
\text{pr}_{1,2}^* \otimes \text{pr}_{2,3}^* \otimes \text{pr}_{3,4}^* & \overset{\text{id}_{\text{pr}_{1,2}^* \otimes \text{pr}_{2,3,4}^* \mathcal{M}_\varphi}}{\longrightarrow} & \text{pr}_{1,3}^* \otimes \text{pr}_{3,4}^* \\
\varphi & \mu^\varphi & \\
\text{pr}_{1,2}^* \otimes \text{pr}_{2,4}^* & \longrightarrow & \text{pr}_{1,4}^* \\
\end{array}$$

(5.36)

For that purpose, we first consider the surjective submersion

$$\begin{aligned}
\mathcal{Y}^{1,2,3}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] \times_{\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}} \mathcal{Y}^{3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] &\cong \mathcal{Y}^{1,3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] \times_{\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}} \mathcal{Y}^{3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] \\
\mathcal{Y}^{1,3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] &\cong \mathcal{Y}^{3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}]
\end{aligned}$$

with projections (defined analogously to the $\tilde{\text{pr}}_{1,3}$ of Diag. (5.33))

$$\begin{aligned}
\mathcal{Y}^{1,2,3}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] &\overset{\text{pr}_{1,3}^*}{\longrightarrow} \mathcal{Y}^{1,3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] \\
\mathcal{Y}^{3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] &\overset{\varphi_{3,4,5}}{\longrightarrow} \mathcal{Y}^{3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}]
\end{aligned}$$

which we use to erect the principal $\mathbb{C}^*$-bundle

$$\tilde{\text{pr}}_{1,3,4}^* \longrightarrow \mathcal{Y}^{1,3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}],$$

pulled back to its base (defined similarly as $\tilde{\mathcal{Y}}^{1,2,3}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}]$ from $\tilde{\mathcal{Y}}^{1,3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}]$ and associated with the super-1-gerbe 1-isomorphism $\text{pr}_{1,3,4}^* \mathcal{M}_\varphi \otimes \text{id}_{\text{pr}_{1,3}^* \mathcal{M}_\varphi}$). Next, we take the principal $\mathbb{C}^*$-bundle

$$\begin{aligned}
\text{pr}_{1,2,3,4}^* (\tilde{\text{pr}}_{1,2,3}^* \otimes \tilde{\text{pr}}_{4,5}^* \mathcal{M}_\varphi) &\otimes \text{pr}_{3,4,5}^* \tilde{\text{pr}}_{1,3,4}^* \\
\end{aligned}$$

over the base

$$\begin{aligned}
\mathcal{Y}^{1,2,3}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] &\times_{\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}} \mathcal{Y}^{3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] \\
\end{aligned}$$

now corresponds to the first composite 1-isomorphism of Diag. (5.36),

$$\text{pr}_{1,3,4}^* \mathcal{M}_\varphi \circ (\text{pr}_{1,2,3}^* \mathcal{M}_\varphi \otimes \text{id}_{\text{pr}_{1,3}^* \mathcal{M}_\varphi}).$$

Analogously, we construct the principal $\mathbb{C}^*$-bundle associated with the other composite 1-isomorphism. Thus, we take the surjective submersion

$$\begin{aligned}
\mathcal{Y}^{1,2,3}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] &\times_{\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}} \mathcal{Y}^{2,3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] \\
\end{aligned}$$

with projections (cp. above)

$$\begin{aligned}
\mathcal{Y}^{1,2,3}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] &\overset{\text{pr}_{1,2}}{\longrightarrow} \mathcal{Y}^{1,2,3}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] \\
\mathcal{Y}^{2,3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}] &\overset{\text{pr}_{3,4,5}}{\longrightarrow} \mathcal{Y}^{2,3,4}[\mathcal{Y}^{[4]}_2\mathcal{M}^{(1)}]
\end{aligned}$$

(5.36)
as the basis of the principal $\mathbb{C}^\ast$-bundle

$$\tilde{\pi}_{1,2}^* \otimes \tilde{\pi}_{3,4,5,6}^* \rightarrow \tilde{\varphi}^{1,2}[2]Y^4_2 M^{(1)} \times Y^4_2 [M^{(1)}] \times Y^2_2 \varphi^{2,3,4}_2 M^{(1)}$$

of the 1-isomorphism $\text{id}_{\tilde{\pi}_{1,2}^*} \otimes \tilde{\pi}_{3,4,5,6}^*$, and then the principal $\mathbb{C}^\ast$-bundle

$$\tilde{\pi}_{1,2,3,4,5,6}^* \rightarrow \tilde{\varphi}^{1,2,4}_2 Y^4_2 M^{(1)}$$

for the 1-isomorphism $\text{pr}_{1,2,3,4,5,6}^* M_{\varphi}$. These two combine to give the bundle

$$\text{pr}_{1,2,3,4,5,6}^* (\tilde{\pi}_{1,2}^* \otimes \tilde{\pi}_{3,4,5,6}^*) \otimes \text{pr}_{3,5,6}^* \tilde{\pi}_{1,2,4,5,6}^*$$

$$\rightarrow \tilde{\varphi}^{1,2}[2]Y^4_2 M^{(1)} \times Y^4_2 [M^{(1)}] \times Y^2_2 \varphi^{2,3,4}_2 M^{(1)} \times Y^4_2 [M^{(1)}] \times Y^4_2 [M^{(1)}]$$

corresponding to

$$\text{pr}_{1,2,4,5,6}^* \circ (\text{id}_{\tilde{\pi}_{1,2}^*} \otimes \text{pr}_{3,4,5,6}^* M_{\varphi})$$.

The sought-after 2-isomorphism is a connection-preserving isomorphism

$$m : \text{pr}_{1,2,3,4,5,6}^* [\tilde{\pi}_{1,2,3,4,5,6}^* (\tilde{\pi}_{1,2}^* \otimes \tilde{\pi}_{3,4,5,6}^*) \otimes \text{pr}_{3,5,6}^* \tilde{\pi}_{1,2,4,5,6}^*]$$

$$\rightarrow \text{pr}_{1,2,4,5,6}^* \circ [\tilde{\pi}_{1,2,3,4,5,6}^* (\tilde{\pi}_{1,2}^* \otimes \tilde{\pi}_{3,4,5,6}^*) \otimes \text{pr}_{3,5,6}^* \tilde{\pi}_{1,2,4,5,6}^*]$$

of principal $\mathbb{C}^\ast$-bundles over

$$\tilde{\varphi}^{1,2,3,4}_2 Y^4_2 M^{(1)} \times Y^4_2 [M^{(1)}] \times Y^2_2 \varphi^{2,3,4}_2 M^{(1)} \times Y^4_2 [M^{(1)}]$$

$$\times_{1,2,4,6} \tilde{\varphi}^{1,2}[2]Y^4_2 M^{(1)} \times Y^4_2 [M^{(1)}] \times Y^2_2 \varphi^{2,3,4}_2 M^{(1)} \times Y^4_2 [M^{(1)}]$$

$$\times_{1,2,4,6} \tilde{\varphi}^{1,2}[2]Y^4_2 M^{(1)} \times Y^4_2 [M^{(1)}] \times Y^2_2 \varphi^{2,3,4}_2 M^{(1)} \times Y^4_2 [M^{(1)}]$$

where the indices on $\times$ indicate the cartesian factors of the component to the left and to the right of the product sign, respectively, that are identified in the fibred product.

The proof of the existence of $m$ is based on the comparison between the relevant connection 1-forms. Their equality, expressed by the formula

$$\tilde{a}(\tilde{m}^{1,2}, \tilde{m}^{2,3}, \tilde{m}^{3,1}) + \tilde{\lambda}(\tilde{m}^{1,3}, \tilde{m}^{2,4}, \tilde{m}^{3,1}) = \tilde{\lambda}(\tilde{m}^{1,2}, \tilde{m}^{2,3}, \tilde{m}^{3,1}) + \tilde{a}(\tilde{m}^{1,3}, \tilde{m}^{2,4}, \tilde{m}^{3,1})$$

(written in an obvious adaptation of the previously employed shorthand notation), leads us to set

$$m((\tilde{m}^{1,2}, \tilde{m}^{2,3}, \tilde{m}^{3,1}), (\tilde{m}^{1,3}, \tilde{m}^{2,4}, \tilde{m}^{3,1}), (\tilde{m}^{1,3}, \tilde{m}^{2,4}, \tilde{m}^{3,1}), (\tilde{m}^{1,3}, \tilde{m}^{2,4}, \tilde{m}^{3,1}), (\tilde{m}^{1,3}, \tilde{m}^{2,4}, \tilde{m}^{3,1}))$$

Clearly, for a super-1-gerbe thus defined, all coherence constraints involving the groupoid structure on $\mu_{\varphi}$ and the product isomorphism $\tilde{\varphi}$ (likewise trivial) are automatically satisfied. Also, once again, we have perfect agreement with our definition of the super-0-gerbe isomorphism.

We conclude our analysis with

**Definition 5.19.** The Green–Schwarz super-2-gerbe of curvature $\chi$ is the quintuple

$$(4) \quad (Y_\ast M^{(1)}, \beta^{4}, \tilde{\varphi}, M_{\varphi}, \mu_{\varphi})$$

constructed in the preceding paragraphs.

Our results are amenable to a straightforward abstraction in the spirit of Defs. 5.4 and 5.11. We leave it to the avid Reader to work out the obvious details of a definition of a Cartan–Eilenberg super-2-gerbe. The same goes for the $\chi$-twisted Vinogradov-type superbrackets of the various fundamental sections of $\mathcal{E}^{1,2} \mathrm{Mink}^{1,d-1|D_{1,d-1}}$ engendered by natural actions of $\mathbb{R}^{1,d-1|D_{1,d-1}}$ on $\mathrm{Mink}^{1,d-1|D_{1,d-1}}$. 
Remark 5.20. In the light of the findings of Refs. [GSW10, GSW13, Sus12], our analysis of the algebroidal structure associated with left- and right-regular as well as adjoint actions of $\mathbb{R}^{1,d-1}|D_{1,d-1}$ on the super-Minkowskian supertarget $s\text{Mink}^{1,d-1}|D_{1,d-1}$ suggests the possibility of the existence of an $\text{Ad}$-equivariant structure on the GS super-$p$-gerbes constructed above. In what follows, we verify this expectation.

5.2. Supersymmetry-equivariance of the Green–Schwarz supergerbe. Our choice of the type of cohomology underlying supergeometric considerations and constructions as well as their field-theoretic applications based on the definition of the GS super-$(p+2)$-cocycles raises the natural question about the existence of a structural realisation (i.e., in categorial terms – by means of suitable morphisms) of the geometric action of the supersymmetry group (on $s\text{Mink}^{1,d-1}|D_{1,d-1}$) on the supergerbe and about the ensuing symmetry content of the super-$\sigma$-model that it (co)determines. An appropriate framework in which such questions can be formulated and answered concretely and rigorously was delineated in Sec. 2.2 and in the literature cited therein. Below, we adapt the formal gerbe-theoretic language of description of $\sigma$-model symmetries to the supergeometric setting in hand, in conformity with the logic of the hitherto discussion.

As recalled in Sec. 2.2 after Refs. [RS09, GSW11a, GSW10, GSW13, Sus11a, Sus12], rigid and gauge symmetries have significantly different gerbe-theoretic emanations: While the former are described by families of gerbe 1-isomorphisms over the original target space of the $\sigma$-model and are directly built into the very construction of the super-gerbes presented in the foregoing sections (which is why we never return to them in the remainder), the geometric data of the latter are bound to scatter over various components of the nerve $\mathbb{N}^\bullet(\mathbb{R}^{1,d-1}|D_{1,d-1}\mathcal{M}(1)) \equiv (\mathbb{R}^{1,d-1}|D_{1,d-1})^{\bullet} \times \mathcal{M}(1)$ of the relevant action groupoid $\mathbb{R}^{1,d-1}|D_{1,d-1}\mathcal{M}(1)$,

and so their analysis in the supergeometric setting unavoidably goes beyond the structural framework of Defs. 5.4 and 5.11, forcing us to abstract a suitable general definition of a supersymmetry-equivariant structure from the study of particular cases, to which we turn next.

When addressing them, we should keep in mind that the choice of the symmetry group alone is not enough in general to decide whether the corresponding rigid symmetry of the (super-)$\sigma$-model is amenable to gauging or not – indeed, in the case of a Lie-group target $G$, we have a variety of representations of the group $G$ (or any of its subgroups, for that matter) on itself that embed in the product $G \times G$ representing the independent left- and right-regular translations, and it has long been known (and confirmed anew in Refs. [GSW10, GSW13], from the gerbe-theoretic vantage point) that in the case of the full group $G$ the adjoint representation admits gauging, whereas the left- and right-regular ones do not (i.e., there appear anomalies). In what follows, we invoke the interpretation of the two-dimensional super-$\sigma$-model as a super-variant of the WZW model as motivation and set out to corroborate the anticipated equivariance pattern in the case of the GS super-0-gerbe and that of the GS super-1-gerbe, leaving the technically much heavier but conceptually fully analogous case of the GS super-2-gerbe (in which we conjecture the very same pattern to be realised) as an exercise for the interested Reader. In so doing, we are guided by the intuition developed through the study, carried out in the previous section, of the algebroidal structures associated with the various actions of the supersymmetry group $\mathbb{R}^{1,d-1}|D_{1,d-1}$ on the super-Minkowskian supertarget $s\text{Mink}^{1,d-1}|D_{1,d-1}$. At the same time, we keep in mind the somewhat involved nature of the right-regular symmetry of the super-$\sigma$-models under consideration, to be recalled and studied in detail in Sec. 6.

By way of a warm-up, and in order to develop the necessary intuitions as to the relevant geometric structure behind supersymmetry-equivariance, we first illustrate the anomalous nature of the left-regular representation of supersymmetry on the example of the super-0-gerbe, and then examine at length its non-anomalous adjoint representation for the super-0-gerbe and the super-1-gerbe. Finally, in Sec. 5.3, we discuss a very special Lie-superalgebraic symmetry of the super-$\sigma$-model and its super-gerbe which is induced from (constrained) right-regular translations.

Prior to launching the proper case-by-case study, let us adapt the notion of left-invariance to the setting of an equivariant structure on a (super-) $p$-gerbe over a (super)manifold $\mathcal{M}$ endowed with a
In fact, we may be slightly more general and consider equivariance with respect to the action of any $G$ components of the nerve fixed in the form $\ell_0 \equiv \ell$. The equivariance of the maps in question ensures that objects which are LI with respect to the original action of $G$ pull back to objects with the same property with respect to the new action. It is clear that the unique choice of the sought-after action on $N^\bullet(G\ltimes \mathcal{M})$ reads

$$\ell^n : G \times N^n(G\ltimes \mathcal{M}) \longrightarrow N^n(G\ltimes \mathcal{M})$$

$$: \quad \left( g, (g_1, g_2, \ldots, g_n, x) \right) \longmapsto \left( Ad_g(g_1), Ad_g(g_2), \ldots, Ad_g(g_n), \ell_g(m) \right),$$

and so it is natural – in the context of left-invariant cohomology and the associated (super)geometric constructions – to demand invariance of the geometric objects (tensors and their geometrisations) over components $N^n(G\ltimes \mathcal{M})$ of the nerve with respect to the respective distinguished extensions $\ell^n$ of $\ell$. In fact, we may be slightly more general and consider equivariance with respect to the action of any normal Lie sub-supergroup $H \subset G$ of the Lie supergroup $G$,

$$\forall g \in G : Ad_g(H) \subset H,$$

with the associated action groupoid $H\ltimes \mathcal{M}$ with the nerve

$$\begin{array}{c}
\xymatrix{
H^3 \times \mathcal{M} \ar[rr]^{d_0} \ar[rr]^{d_1} \ar[rr]^{d_2} & & H^2 \times \mathcal{M} \ar[rr]^{d_0} \ar[rr]^{d_2} & & H \times \mathcal{M} \ar[rr]^{d_0} \ar[rr]^{d_1} & & \mathcal{M},
}\end{array}$$

in which case we shall use the same symbols to denote the corresponding actions

$$\ell^n : G \times N^n(H\ltimes \mathcal{M}) \longrightarrow N^n(H\ltimes \mathcal{M})$$

$$(5.37) \quad : \quad \left( g, (h_1, h_2, \ldots, h_n, x) \right) \longmapsto \left( Ad_g(h_1), Ad_g(h_2), \ldots, Ad_g(h_n), \ell_g(m) \right).$$

In the case of immediate interest, the general scheme specialises to one of the following three possibilities that we shall encounter below: $\ell$ is the left-regular action of $G$ on itself, or the superposition $\rho \circ \text{Inv}$ of the right-regular action of $G$ on itself with the group inverse, or the adjoint action $Ad : G \times G \rightarrow G : (h, g) \mapsto h \cdot g \cdot h^{-1}$. Consequently, we need an explicit form of the adjoint action of the Lie supergroup $\mathcal{M}^{(1)}$ on itself and that of its supercentral extension $\mathcal{M}^{(2)}_1$ of Prop. 6 on itself. As the latter contains the former, we confine ourselves to writing out the latter:

$$\begin{array}{c}
\xymatrix{
\mathcal{M}_1^{(2)} \ar[r]^{Ad^{(2)}} & \mathcal{M}_1^{(2)} \ar[r]^{Ad^{(2)}} & \mathcal{M}_1^{(2)}.
}\end{array}$$

We are now ready to perform the detailed analysis of the invariance resp. equivariance properties of the various objects defined over $N^\bullet(R^{1,d-1|D_1,d-1}G\ltimes \mathcal{M}^{(1)})$.

5.2.1. The GS super-0-gerbe. We begin by looking for an isomorphism

$$\Upsilon^{(0,\ell)}_{GS,\rho} : \ell^*L^{(0)} \xrightarrow{\sim} pr_2^*L^{(0)} \otimes J_{\rho,\ell}^{(c)}$$

of (trivial) principal $C^\infty$-bundles over the supermanifold $R^{1,d-1|D_1,d-1} \times \mathcal{M}^{(1)}$, of which the last one, $J_{\rho,\ell}^{(c)}$, is to be understood as trivial in the sense of invariant cohomology, that is – admitting a connection 1-form $\rho$ on the base which is left-invariant with respect to $\ell^1$ (this enables us to identify it as a super-0-gerbe trivial in the $R^{1,d-1|D_1,d-1}$-invariant cohomology). In so doing, we impose the additional requirement that the isomorphism also be left-invariant with respect to this action, which means that its data (a super-0-form on $R^{1,d-1|D_1,d-1} \times \mathcal{M}^{(1)}$) should have the same property. We compute

$$\begin{array}{c}
\left( \ell^* - pr_2^* \right) \chi \left( (\theta_1, x_1), (\theta_2, x_2) \right) = d(\theta_1 \Gamma_{11} d(\theta_1 + 2\theta_2)),
\end{array}$$
and so we conclude that
\[
\rho^{(\ell)}_{(1)}((\theta_1, x_1), (\theta_2, x_2)) = \nabla_1 \Gamma_{11} d(\theta_1 + 2\theta_2) + d\Delta^{(\ell)}((\theta_1, x_1), (\theta_2, x_2)),
\]
where \(d\Delta\) (written in terms of an super-0-form \(\Delta^{(\ell)}\)) is an admissible de Rham-exact LI correction. In consequence of the (de Rham-)cohomological triviality of \(s_{\text{Mink}}^{1,d-1|D_1,d-1}\), its invariance is tantamount to the left-invariance of the super-0-form \(\Delta^{(\ell)}\) itself. Passing, next, to the level of connection 1-forms, we readily establish the identity
\[
\ell^* \beta + d(F + \Delta^{(\ell)}) = p^*_2 \beta + \rho^{(\ell)}_{(1)}(\theta_1, x_1), (\theta_2, x_2)) = 0,
\]
where
\[
F((\theta_1, x_1), (\theta_2, x_2)) = \nabla_1 \Gamma_{11} \theta_2,
\]
which in conjunction with the non-invariance property of the latter super-0-form,
\[
\ell^*_{(x,y,z)} F((\theta_1, x_1), (\theta_2, x_2)) = F((\theta_1, x_1), (\theta_2, x_2)) + \nabla_1 \theta_1,
\]
infers conclusively that there does not exist an isomorphism of the type sought after. This is to be contrasted with the situation that arises when the left-regular action is replaced by the adjoint action. We have
\[
(\text{Ad}^* - p^*_2) \chi ((\theta_1, x_1), (\theta_2, x_2)) = 0,
\]
and so we may take
\[
\rho^{(\text{Ad})}_{(1)} = d\Delta^{(\text{Ad})},
\]
with \(\Delta^{(\text{Ad})}\) an LI super-0-form. In view of the equality
\[
\text{Ad}^* \beta = p^*_2 \beta,
\]
we ultimately conclude that we may – without any loss of generality – set
\[
\rho^{(\text{Ad})}_{(1)} = 0
\]
and identify the manifestly LI isomorphism in the trivial form
\[
\Gamma^{(0,\text{Ad})}_{(2)} = \text{id}_{p^*_2 \mathcal{L}^{(0)}} : \text{Ad}^* \mathcal{L}^{(0)} \xrightarrow{\cong} p^*_2 \mathcal{L}^{(0)} \otimes \mathcal{J}^{(\text{Ad})}_{(1)} \equiv p^*_2 \mathcal{L}^{(0)}.
\]
The latter satisfies the usual coherence condition
\[
(\mathcal{J}^{(0,\text{Ad})}_{(1)} \otimes \text{id}_{\mathcal{L}^{(2)}}) \circ d^{(2)*}_{(1)} \Gamma^{(0,\text{Ad})}_{(2,0)} = d^{(2)*}_{(1)} \mathcal{L}^{(0,\text{Ad})} = d^{(2)*}_{(1)} \Gamma^{(0,\text{Ad})}_{(2,0)}
\]
(written in terms of the face maps of the nerve \(N^{*}(\mathbb{R}^{1,d-1|D_1,d-1}, \mathcal{M}^{(1)})\) for the adjoint action), to be imposed on data of an equivariant structure on a bundle. This requires that the identity
\[
(\mathcal{J}^{(0,\text{Ad})}_{(1)} \otimes \text{id}_{\mathcal{L}^{(2)}}) \circ d^{(2)*}_{(1)} \Gamma^{(0,\text{Ad})}_{(2,0)} = d^{(2)*}_{(1)} \mathcal{L}^{(0,\text{Ad})} = d^{(2)*}_{(1)} \Gamma^{(0,\text{Ad})}_{(2,0)}
\]
hold true, which it does trivially in the case in hand. We are thus led to postulate

**Definition 5.21.** Adopt the notation of Def.\[5.3\] and let \(H \subset G\) be a normal Lie sub-supergroup of the Lie supergroup \(G\) endowed with a left action
\[
\ell : H \times G \rightarrow G,
\]
the latter determining the action groupoid \(H \bowtie G\) with the nerve \[5.37\] (where \(\mathcal{M} \equiv G\)). A supersymmetric \(H\)-equivariant structure on the Cartan–Eilenberg super-0-gerbe \(G^{(0)}_{\text{Cart}} = (L, \pi_L, a_L)\) with a connection 1-form \(a_L(X, z) = i \frac{d}{dz} + b(X)\), of curvature \(\chi\) over \(G\) is a pair \((T^{(0)}, \rho)\) composed of
Proposition 5.23. The Green–Schwarz super-0-gerbe of Def. 5.2 carries a canonical supersymmetric geometric objects under supersymmetry actions in the next example. This approach developed in the present paper, and – in this sense – provides a natural logical completion of the GS super-0-gerbe, but otherwise constitute its natural generalisation consistent with the invariant-cohomological approach developed in the present paper, and – in this sense – provide a natural logical completion of the earlier part of the definition. We shall encounter instances of such a more general behaviour of geometric objects under supersymmetry actions in the next example.

Remark 5.22. The last two conditions in the above definition (imposed on $\rho$) do not follow directly from our analysis, due to of the triviality of the relevant structures in the case of the GS super-0-gerbe, but otherwise constitute its natural generalisation consistent with the invariant-cohomological approach developed in the present paper, and – in this sense – provide a natural logical completion of the earlier part of the definition. We shall encounter instances of such a more general behaviour of geometric objects under supersymmetry actions in the next example.

We may now summarise our hitherto findings in the form of

Proposition 5.23. The Green–Schwarz super-0-gerbe of Def. 5.2 carries a canonical supersymmetric equivariant structure $\Upsilon_{(0,Ad)}^{(0)}$ with respect to the adjoint action of the Lie supergroup $\mathbb{R}^{1,d-1}|D_{1,d-1}$ on itself, relative to the LI super-1-form $\rho^{(Ad)} = 0$.

5.2.2. The GS super-1-gerbe. Next, we consider the GS super-1-gerbe. We readily convince ourselves that the left-regular representation is anomalous just as in the previous case, and so we pass directly to the adjoint representation, in which we obtain the relation

$$ (\text{Ad}^* - p_{z2}^* \chi) = d \rho^{(Ad)} $$

with

$$ \rho^{(Ad)}((\theta_1, x_1), (\theta_2, x_2)) = (\theta_2 \Gamma^I \theta_1) d \vartheta_2 \wedge \Gamma_I d \vartheta_2. $$

The latter super-1-form is manifestly LI with respect to the action $\ell^1$ of Eq. (5.38) (where $\mathcal{M} \equiv \mathbb{R}^{1,d-1}|D_{1,d-1}$),

$$ \text{Ad}^*_{(z,y)}(\rho^{(Ad)}((\theta_1, x_1), (\theta_2, x_2))) = \rho^{(Ad)}((\text{Ad}(z,y)(\theta_1, x_1), \text{Ad}(z,y)(\theta_2, x_2))) $$

and so we may look for a 1-isomorphism of (super-)gerbes

$$ \Upsilon_{(1,Ad)}^{(1)} : \text{Ad}^* \mathcal{G}_{GS,p}^{(1)} \xrightarrow{\cong} p_{z2}^* \mathcal{G}_{GS,p}^{(1)} \otimes \mathcal{I}_p^{(Ad)}. $$

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over $\mathbb{R}^{1,d-1}/D_{1,d-1} \times \mathcal{M}^{(1)}$ which would be \textit{left-invariant} in a natural manner, suggested in part by Def. \[2.21\]. To this end, we consider the surjective submersion

$$
\begin{align*}
\left(\mathcal{Y}_d^{(1)}\right)^{\times 4} & \quad \left(\mathcal{Y}_d^{(1)}\right)^{\times 3} & \quad \left(\mathcal{Y}_d^{(1)}\right)^{\times 2} & \quad \left(\mathcal{Y}_d^{(1)}\right)^{\times 1} \\
\pi_{\mathcal{Y}_d^{(1)}}(d_0^{(3)}) & \quad \pi_{\mathcal{Y}_d^{(1)}}(d_2^{(3)}) & \quad \pi_{\mathcal{Y}_d^{(1)}}(d_0^{(2)}) & \quad \pi_{\mathcal{Y}_d^{(1)}}(d_2^{(2)}) \\
\mathcal{M}^{(1)} & \quad \mathcal{M}^{(1)} & \quad \mathcal{M}^{(1)} & \quad \mathcal{M}^{(1)} \\
\pi_{\mathcal{M}^{(1)}}(d_0^{(3)}) & \quad \pi_{\mathcal{M}^{(1)}}(d_2^{(3)}) & \quad \pi_{\mathcal{M}^{(1)}}(d_0^{(2)}) & \quad \pi_{\mathcal{M}^{(1)}}(d_2^{(2)}) \\
\mathcal{Y}_d^{(1)} & \quad \mathcal{Y}_d^{(1)} & \quad \mathcal{Y}_d^{(1)} & \quad \mathcal{Y}_d^{(1)}
\end{align*}
$$

\[\cdots\]

over the nerve $\mathbf{N}^\bullet(\mathbb{R}^{1,d-1}/D_{1,d-1} \times \mathcal{M}^{(1)})$, with the important covering property

$$
d_i^{(n)} \circ \pi_{\mathcal{Y}_d^{(1)}}^{\times n+1} = \pi_{\mathcal{M}^{(1)}}^{\times n} \circ \mathcal{Y}_d^{(n)}
$$

that fixes the form of $\mathcal{Y}_d^{(n)}$. Denote

$$
\mathcal{M}^{(1)}_n := \left(\mathbb{R}^{1,d-1}/D_{1,d-1}\right)^{\times n} \times \mathcal{M}^{(1)}
$$

in order to unclutter the formulae that follow. Taking the supermanifold $\left(\mathcal{Y}_d^{(1)}\right)^{\times 2}$ as the common surjective submersion for the pullback gerbes $d_i^{(1)} \mathcal{G}_{d_i^{(1)}}^{(1)}$ and the trivial gerbe $\mathcal{I}_{\rho_2}^{(1)}$, we commence our search for the principal $\mathbb{C}^\times$-bundle of the 1-isomorphism $\mathcal{Y}^{(1,\text{Ad})}$ at the surjective submersion

$$
\mathcal{Y}(\mathcal{M}^{(1)}) := \left(\mathcal{Y}_d^{(1)}\right)^{\times 2} \times_{\mathcal{Y}_d^{(1)}} \left(\mathcal{Y}_d^{(1)}\right)^{\times 2} \cong \left(\left(\left(\theta_1, x_1, \xi_1, \theta_2, x_2, \xi_2, \theta_3, x_3, \xi_3, \theta_4, x_4, \xi_4\right), \left(\theta_1, x_1, \xi_1, \theta_2, x_2, \xi_2\right)\right), \left(\left(\theta_1, x_1, \xi_1, \theta_2, x_2, \xi_2\right), \left(\theta_1, x_1, \xi_1, \theta_2, x_2, \xi_2\right)\right)\right)
$$

where we obtain, in a direct computation invoking \[3.39\], the identity

$$
\text{pr}_{3,4}^\ast \text{pr}_{2}^\ast \beta + \text{pr}_{3,4}^\ast \pi_{\mathcal{Y}_d^{(1)}}^{\ast 2} \rho_2^{(1,\text{Ad})} - \text{pr}_{1,2}^\ast \text{Ad}_{\rho_2}^{(2)} \beta^{(2)} = dE,
$$

in which

$$
E(y_{1,2,1,2}, y_{1,2,3,4}) = \frac{i}{2} \partial_1 \Gamma_I \partial_2 \left(\left(\partial_1 - \partial_2\right) \Gamma_I \text{d} \theta_2\right) + \partial_2^\ast \text{d} \left(\Gamma_I \theta_2^\beta - \Gamma_I \theta_3^\beta - \Gamma_I \theta_2^\beta + \Gamma_I \theta_3^\beta + \xi_4 \alpha - \xi_2 \alpha\right)
$$

is a super-1-form on $\left(\mathcal{Y}_d^{(1)}\right)^{\times 2}$ which we identify as the base component of a principal connection 1-form on a (trivial) principal $\mathbb{C}^\times$-bundle

$$
\pi_{\mathcal{E}} : \mathcal{E} := \mathcal{Y}(\mathcal{M}^{(1)}) \times \mathbb{C}^\times \longrightarrow \mathcal{Y}(\mathcal{M}^{(1)}) : \left(\left(\theta_1, x_1, \xi_1, \theta_2, x_2, \xi_2, \theta_3, x_3, \xi_3, \theta_4, x_4, \xi_4\right), z\right) \longrightarrow \left(\left(y_{1,2,1,2}, y_{1,2,3,4}\right), z\right)
$$

determining the 1-isomorphism sought after. In the above coordinates, the principal connection 1-form reads

$$
a_{\mathcal{E}}(\left(\left(y_{1,2,1,2}, y_{1,2,3,4}\right), z\right)) = i \frac{d\xi}{\xi} + E(y_{1,2,1,2}, y_{1,2,3,4}).
$$

Following the logic overarching our considerations, we inspect the supersymmetry variation of its curvature

$$
H := dE
$$

with respect to the induced action

$$
\text{Ad}_{\rho_2}^{(2)} : \mathbb{R}^{1,d-1}/D_{1,d-1} \times \mathcal{Y}(\mathcal{M}^{(1)}) \longrightarrow \mathcal{Y}(\mathcal{M}^{(1)})
$$

$$
\left(\left(\varepsilon, y, \zeta\right), \left(y_{1,2,1,2}, y_{1,2,3,4}\right)\right) \longrightarrow \left(\text{Ad}_{\rho_2}^{(2)}(\varepsilon, y, \zeta), \left(y_{1,2,1,2}, y_{1,2,3,4}\right)\right),
$$

to the effect:

$$
\left(\text{Ad}_{\rho_2}^{(2)} \times 2\right) H = d \rho_2^{(1,\text{Ad})} H,
$$

where

$$
\rho_2^{(1,\text{Ad})} \left(\left(\varepsilon, y, \zeta\right), \left(y_{1,2,1,2}, y_{1,2,3,4}\right)\right) := \left(\partial_1 \Gamma_I \partial_2 \right) \left(\xi \partial_1 \text{d} \theta_2\right).
$$
The appearance of the latter super-1-form immediately suggests a suitable notion of ‘left-invariance’ for the candidate data \((\mathcal{E}, a, \varrho)\) of the 1-isomorphism, to wit, an equivariant structure in the sense of Def. 5.21 over the nerve \(\mathcal{M}\) of the relevant action groupoid \(\mathbb{R}^{1,d-1}D_1, d-3 \mathcal{Y}(\mathcal{M}^{(1)})\). We now pause to verify this anticipated property. To this end, we first check identity (5.42), in the present setting takes the simple form

\[
\rho^{(1)}(\text{Ad}^{(2)} \cdot x^2) \left( (\varepsilon_1, y_1, \zeta_1), \text{Ad}^{(2)} \cdot x^2 \right)(y_1, y_2, y_3) + \rho^{(1)}(\text{Ad}^{(2)} \cdot x^2) \left( (\varepsilon_2, y_2, \zeta_2), (y_1, y_2, y_3) \right)
\]

\[
- \rho^{(1)}(\text{Ad}^{(2)} \cdot x^2) \left( \mu_1^{(2)} \left( (\varepsilon_1, y_1, \zeta_1), (\varepsilon_2, y_2, \zeta_2) \right), (y_1, y_2, y_3) \right)
\]

\[
= \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) + \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) - \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) = 0.
\]

Motivated by the above, we look for a connection-preserving isomorphism of principal \(\mathbb{C}^*\)-bundles over \(\mathcal{E}\), of the 1-isomorphism, to wit, an equivariant structure in the sense of (5.41). A direct computation

\[
F(\varepsilon, y, \zeta, (y_1, y_2, y_3, y_4)) = (\overline{\theta}_1 \Gamma^I \theta_2)
\]

are data of the isomorphism. These are manifestly LI with respect to the action \(\text{Ad}^{(2)} \cdot x^2\), and so it remains to check the identity (5.41). A direct computation

\[
F((\varepsilon_1, y_1, \zeta_1), \text{Ad}^{(2)} \cdot x^2 \left( (\varepsilon_2, y_2, \zeta_2), (y_1, y_2, y_3, y_4) \right)) + F((\varepsilon_2, y_2, \zeta_2), (y_1, y_2, y_3, y_4))
\]

\[
- F(\mu_1^{(2)} \left( (\varepsilon_1, y_1, \zeta_1), (\varepsilon_2, y_2, \zeta_2) \right), (y_1, y_2, y_3, y_4))
\]

\[
= \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) + \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) - \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) \left( \overline{\theta}_1 \Gamma^I \theta_2 \right) = 0.
\]

convinces us that we have, indeed, an \(\mathbb{R}^{1,d-1}D_1, d-3\)-equivariant structure on \(\mathcal{E}\), as anticipated.

Having established the desired natural realisation of supersymmetry on \(\mathcal{E}\), we may return to the proof of the expectation, based on our earlier findings, that the bundle forms part of an equivariant structure on the GS super-1-gerbe. Thus, upon passing to the fibred product

\[
\mathcal{Y}^{(2)}(\mathcal{M}^{(1)}) \equiv \mathcal{Y}(\mathcal{M}^{(1)}) \times \mathcal{M}^{(1)} \times \mathcal{Y}(\mathcal{M}^{(1)}) \times \left( (y_1, y_2, y_3, y_4) \right) (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)\)
\]

with the canonical projections (written for \((i, j) \in \{(1, 2), (1, 3), (2, 4), (3, 4)\})

\[
\text{pr}_{i,j} : \mathcal{Y}^{(2)}(\mathcal{M}^{(1)}) \longrightarrow (\mathcal{Y}^{(1)}(\mathcal{M}^{(1)}))^{x^2} \times \mathcal{M}^{(1)} : (\mathcal{Y}^{(1)}(\mathcal{M}^{(1)}))^{x^2}
\]

\[
( (y_1, y_2, y_3, y_4), (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) ) \longrightarrow (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8),
\]

we then find the exact equality

\[
\text{pr}_{3,4}^{*} \text{Ad}^{(2)} \cdot x^2 \cdot A + \text{pr}_{3,4}^{*} E = \text{pr}_{1,3}^{*} E + \text{pr}_{2,4}^{*} \text{pr}_{2,4}^{*} E
\]

from which we read off the existence of a trivial (and hence manifestly LI) connection-preserving isomorphism

\[
\alpha_{\mathcal{E}} \equiv \text{id}_{(\text{Ad}^{(2)} \cdot x^2 \cdot \text{pr}_{1,3})^{*} \mathcal{L}^{(1)}} \otimes \text{pr}_{3,4}^{*} \mathcal{E} : (\text{Ad}^{(2)} \cdot x^2 \circ \text{pr}_{1,3})^{*} \mathcal{L}^{(1)} \otimes \text{pr}_{3,4}^{*} \mathcal{E} \longrightarrow \text{pr}_{1,2}^{*} \mathcal{E} \otimes (\text{pr}_{2,4}^{*} \circ \text{pr}_{2,4})^{*} \mathcal{L}^{(1)},
\]

of principal \(\mathbb{C}^*\)-bundles over \(\mathcal{Y}^{(2)}(\mathcal{M}^{(1)})\). Taking into account the triviality of the groupoid structure \(\alpha_{\mathcal{E}}\) on \(\mathcal{L}^{(1)}\), we conclude that \(\alpha_{\mathcal{E}}\) satisfies the desired coherence constraints of (a variant of) Diag. 2.4. Thus, altogether, we obtain a gerbe 1-isomorphism

\[
\mathcal{Y}_{\text{GS,E}}^{(1, \text{Ad})} : \left( (\mathcal{Y}^{(2)}(\mathcal{M}^{(1)}))^{x^2}, \pi_{1,2}^{x^2} \mathcal{Y}^{(2)}(\mathcal{M}^{(1)}), \mathcal{E}, \mathcal{A}_{\mathcal{E}}, \alpha_{\mathcal{E}} \right),
\]

which is LI in a well-defined (and natural) sense.
Having identified the 1-isomorphism over $N^1(\mathbb{R}^{1.d-1|D_{1.d-1}K}\mathcal{M}(1))$ that furnishes the GS super-1-gerbe with a realisation of the supersymmetry group $\mathbb{R}^{1.d-1|D_{1.d-1}}$ in the adjoint, we may now look for the 2-isomorphism over $N^2(\mathbb{R}^{1.d-1|D_{1.d-1}K}\mathcal{M}(1))$ which renders that realisation compatible with the binary operation on the supergroup. To this end, we take the surjective submersion of the supermanifold $N^2(\mathbb{R}^{1.d-1|D_{1.d-1}K}\mathcal{M})$ along the various face maps of the nerve (the $d_1^{(1)} \circ d_2^{(1)}$ and the $d_k^{(2)}$, respectively) and the principal $C^\ast$-bundle $\varepsilon$ to $Y_1\mathcal{M}(1)^{x_3} \times_{\mathcal{M}(3)} Y_1\mathcal{M}(1)^{x_3}$ along the cartesian squares

\[ Y_1^{(2)} d_k^{(2)} := (Y_1 d_k^{(2)}) \times^2 \]

of their extensions $Y_1 d_k^{(2)}$ introduced formerly, we seek to determine a connection-preserving isomorphism

\[ \beta : \text{pr}_{1,2}^\ast (Y_1^{(2)} d_2^{(2)})^\ast \otimes \text{pr}_{2,3}^\ast (Y_1^{(2)} d_0^{(2)})^\ast \otimes \overset{\longrightarrow}{\text{pr}_{1,3}^\ast (Y_1^{(2)} d_1^{(2)})^\ast} \]

of principal $C^\ast$-bundles over the common surjective base

\[ \tilde{Y}_1^{(2)} \mathcal{M}(1)^{x_3} := Y_1\mathcal{M}(1)^{x_3} \times_{\mathcal{M}(3)} Y_1\mathcal{M}(1)^{x_3} \times_{\mathcal{M}(3)} Y_1\mathcal{M}(1)^{x_3} \ni \left((\theta_1, x_1, \xi_1), (\theta_2, x_2, \xi_2), (\theta_3, x_3, \xi_3)\right) \]

\[ =: y_{1,2,3,1,2,3} m_{4,5,6,7,8,9} \]

Inspection of the relevant combination of pullbacks of the connection 1-forms yields the identity

\[ (\text{pr}_{1,2}^\ast (Y_1^{(2)} d_2^{(2)})^\ast + \text{pr}_{2,3}^\ast (Y_1^{(2)} d_0^{(2)})^\ast - \text{pr}_{1,3}^\ast (Y_1^{(2)} d_1^{(2)})^\ast) \mathcal{E}(y_{1,2,3,1,2,3} m_{4,5,6,7,8,9}) \]

\[ = d\left( (\bar{\theta}_3 \Gamma \theta_1) (\bar{\theta}_2 \Gamma \theta_3) \right) + \left( (\bar{\theta}_1 \Gamma_I \theta_2) \theta_3^a + (\bar{\theta}_3 \Gamma_I \theta_1) \theta_2^a + (\bar{\theta}_2 \Gamma_I \theta_3) \theta_1^a \right) \Gamma_{\alpha\beta} \delta_{\beta} \]

obtained with the help of identity (4.16) (in the last line), from which we extract global data of $\beta$, $\tilde{F}(y_{1,2,3,1,2,3} m_{4,5,6,7,8,9}) \in (\bar{\theta}_3 \Gamma \theta_1) (\bar{\theta}_2 \Gamma \theta_3)$. We readily see that an isomorphism with these data is coherent with (suitably tensoerd pullbacks of) $\alpha_\varepsilon$, as expressed in an appropriate adaptation of Diag. (2.7) over $\tilde{Y}_1^{(2)} \mathcal{M}(1)^{x_3} \times_{\mathcal{M}(3)} \tilde{Y}_1^{(2)} \mathcal{M}(1)^{x_3}$, and so we are left with the coherence condition (5.12) to be checked over

\[ Y_1\mathcal{M}(1)^{x_4} \times_{\mathcal{M}(3)} Y_1\mathcal{M}(1)^{x_4} \times_{\mathcal{M}(3)} Y_1\mathcal{M}(1)^{x_4} \times_{\mathcal{M}(3)} Y_1\mathcal{M}(1)^{x_4} \]

This is completely straightforward, and leads us to conclude that there does, indeed, exist a 2-isomorphism

\[ \gamma (1, \mathcal{M}(1)^{x_3}) =: \varepsilon (1, \mathcal{M}(1)^{x_3}) : Y_1^{(2)} d_2^{(2)} \mathcal{E}(1, \mathcal{M}(1)^{x_3}) \otimes Y_1^{(2)} d_0^{(2)} \mathcal{E}(1, \mathcal{M}(1)^{x_3}) \overset{\longrightarrow}{\varepsilon (1, \mathcal{M}(1)^{x_3})} Y_1^{(2)} \mathcal{E}(1, \mathcal{M}(1)^{x_3}). \]

Altogether, from the above analysis, we distill the desired Definition

**Definition 5.24.** Adopt the notation of Defs. 5.11 and 5.21. A supersymmetric H-equivariant structure on the Cartan–Eilenberg super-1-gerbe $\mathcal{G}_{\mathcal{C}_E}^{(1)} = (\mathcal{G}, \pi_{\mathcal{G}}, b, \mathbb{L}, a, \mu_\mathbb{L})$ of curvature $\chi$ over $\mathcal{G}$ is a triple $(\mathcal{G}, \gamma (1), \rho )$ composed of

- a super-2-form $\rho$ on $\mathbb{H} \times \mathcal{G}$ satisfying the identities

\[ \left( d_1^{(1)} - d_0^{(1)} \right) \chi = d\rho \]

and

\[ (d_0^{(2)} + d_2^{(2)} - d_1^{(2)}) \rho = 0, \]

and LI with respect to the action $\ell^1$ of Eq. (5.38) (where $\mathcal{M} \equiv \mathcal{G}$),

\[ \forall \chi \in \mathcal{G} : \ell^1 \times \rho = \rho \]
• a 1-isomorphism
\[
\Upsilon^{(1)} : \ell^* \mathcal{G}_{\text{CaE}}^{(1)} \xrightarrow{\simeq} \text{pr}_2^* \mathcal{G}_{\text{CaE}} \otimes \mathcal{I}_\rho_{(2)}
\]
of gerbes over \(H \times G\) with the following properties:
– the principal \(\mathbb{C}^x\)-bundle \((\mathcal{E}, a_\mathcal{E})\) of \(\Upsilon^{(1)}\) carries a supersymmetric \(H\)-equivariant structure, i.e., admits a connection-preserving isomorphism of principal \(\mathbb{C}^x\)-bundles
\[
\check{\Upsilon}^{(0)} : \check{\ell}^* \mathcal{E} \xrightarrow{\simeq} \text{pr}_2^* \mathcal{E} \otimes \mathcal{J}_{\check{\rho}}_{(1)},
\]
written in terms of an extension \(\check{\ell}^1\) of \(\ell^1\) to its base and of a super-1-form \(\check{\rho}\) satisfying condition
\[
(\check{d}_0^{(2)} + \check{d}_2^{(2)} - \check{d}_1^{(2)}) \check{\rho} = 0,
\]
and with data given by a super-0-form \(\check{F}\) satisfying the identity (written in terms of the base component \(\check{b}\) of \(a_\mathcal{E}\))
\[
(\check{d}_1^{(1)} + \check{d}_0^{(1)}) \check{b} = \check{\rho} + d\check{F},
\]
LI with respect to \(\check{\ell}^1\),
\[
\forall \chi_{\text{gG}} : \check{\ell}_X^* \check{F} = \check{F},
\]
and subject to the coherence constraints
\[
(\check{d}_0^{(2)} + \check{d}_2^{(2)} - \check{d}_1^{(2)}) \check{F} = 0,
\]
written in terms of the face maps \(\check{d}_i^{(2)}\) of the nerve of the action groupoid defined by \(\check{\ell}^1\);
– the connection-preserving \(\mathbb{C}^x\)-bundle isomorphism \(\alpha_\mathcal{E}\) of \(\Upsilon^{(1)}\) has data \(\check{F}\) that are LI with respect to a lift \(\check{\ell}^1\) of \(\ell^1\) to its base,
\[
\forall \chi_{\text{gG}} : \check{\ell}_X^* \check{F} = \check{F},
\]
and subject to the coherence constraints
\[
(\check{d}_0^{(2)} + \check{d}_2^{(2)} - \check{d}_1^{(2)}) \check{F} = 0,
\]
written in terms of the face maps \(\check{d}_i^{(2)}\) of the nerve of the action groupoid defined by \(\check{\ell}^1\),

• a 2-isomorphism
\[
\gamma^{(1)} : d_2^{(2)} \cdot \Upsilon^{(1)} \otimes d_0^{(2)} \cdot \Upsilon^{(1)} \xrightarrow{\simeq} d_1^{(2)} \cdot \Upsilon^{(1)}
\]
with local data \(\check{F}\) that are LI with respect to a lift \(\check{\ell}^2\) of \(\ell^2\) to its base,
\[
\forall \chi_{\text{gG}} : \check{\ell}_X^* \check{F} = \check{F},
\]
and subject to the coherence constraints
\[
(\check{d}_0^{(2)} + \check{d}_2^{(2)} - \check{d}_1^{(2)}) \check{F} = 0,
\]
written in terms of the face maps \(d_i^{(2)}\) of the nerve of the action groupoid defined by \(\check{\ell}^2\), and such that the coherence condition
\[
d_1^{(3)} \cdot \gamma^{(1)} \cdot (\text{id}_{d_2^{(2)} \circ d_0^{(2)}} \cdot \Upsilon^{(1)} \circ d_3^{(3)} \cdot \gamma^{(1)}) = d_2^{(3)} \cdot \gamma^{(1)} \cdot (d_0^{(3)} \circ \text{id}_{d_2^{(2)} \circ d_0^{(2)}} \cdot \Upsilon^{(1)} \circ d_3^{(3)} \cdot \gamma^{(1)})
\]
is obeyed.

Our discussion can then be summarised in

**Proposition 5.25.** The Green–Schwarz super-1-gerbe of Def. 5.4 carries a canonical supersymmetric equivariant structure \((\Upsilon_{\text{GS,p}}^{(1,\text{Ad})} , \gamma_{\text{GS,p}}^{(1,\text{Ad})})\) with respect to the adjoint action of the Lie supergroup \(\mathbb{R}^{1,d-1|D_{1,\text{Ad}}}_{(2)}\) on itself, relative to the LI super-2-form \(\rho^{(\text{Ad})}_{(2)}\).
6. The pure-supergerbe Hughes–Polchinski superbackgrounds & their $\kappa$-symmetry

Our discussion of the higher-geometric structures behind the GS super-$\sigma$-model and its symmetries induced from automorphisms of the supertarget, which we anticipate to be reflected in the equivariance of those structures, brings us to the Hughes–Polchinski formulation recapitulated in Sec. 14. Indeed, a careful canonical analysis of the super-$\sigma$-model in its Nambu–Goto formulation, first performed by de Azcárraga and Lukierski in Ref. [dAL83] and then by Siegel in Ref. [Sie83], reveals the existence of gauge symmetries that engage jointly the metric and the topological term of the super-$\sigma$-model in that neither of them is invariant separately under the corresponding symmetry transformations and it is only a distinguished linear combination of the two, with their relative normalisation thus fixed, that remains intact. The symmetry plays a crucial rôle that justifies devoting the closing paragraphs of this work to its preliminary discussion in the (super)gerbe-theoretic language developed heretofore: It identifies some of the target-space spinorial degrees of freedom of the original model as pure gauge and serves to remove them in the standard gauge-fixing procedure through which actual supersymmetry of the (effective) field content of the physical theory is attained. As the symmetries mix the two terms in the NG action functional, we cannot expect them to geometrise in the supergeometric setting of that formulation in a purely gerbe-theoretic manner analogous to the realisation of the adjoint supersymmetry established in Sec. 5.2. A path to geometrisation opens up – on the firm basis of Prop. 4.1 – only in the HP formulation in which the metric and cohomological structures of the supertarget amalgamate into a purely (super-)gerbe-theoretic structure on a larger supertarget and – as shall be documented in the remainder of this work – in that amalgam the symmetry is encoded in a form resembling closely the previously described one. The local supersymmetry under consideration has its peculiarities, to be detailed below, that preclude the construction of a full-blown equivariant structure on the composite super-gerbe of the HP formulation insofar as there is no obvious way to geometrise the conditions that need to be imposed in order for the supersymmetry to be realised in the super-$\sigma$-model in the first place. Consequently, the analysis to follow provides us with a non-standard geometric instantiation of a field-theoretic symmetry in the presence of a topological charge.

6.1. The Cartan supergeometry of the gauged supersymmetry. The point of departure of our analysis is the HP formulation of the GS super-$\sigma$-model of embeddings of the $(p + 1)$-dimensional riemannian worldvolume $\Omega$ in the supertarget $s\text{Mink}^{1,d-1|D_1,d-1}$ parametrised as in Eq. (4.10). Following the general rules laid out in Sec. 3.2, we may write its action functional in the form

$$S_{\text{GS}}^{(p)}[X_{(p)}] = \int_{\Omega} \left( \theta^\alpha_1, x, \phi(\cdot) \right) + \lambda_0 \theta_1 \sigma(\theta(\cdot)), $$

for $p = 0$, or for $p > 0$

$$S_{\text{GS}}^{(p)}[X_{(p)}] = \int_{\Omega} \left( \frac{1}{\prod_{\alpha=1}^p} \epsilon_{\alpha A_1 \ldots A_p} \left( \theta^A_1 \wedge \theta^A_2 \wedge \ldots \wedge \theta^A_p \right)(\theta, x, \phi(\cdot)),

+ \lambda_p \sum_{k=0}^p \lambda_k \Gamma_{I_1 I_2 \ldots I_p} \sigma(\theta) \wedge dx_{I_1} \wedge dx_{I_2} \wedge \ldots \wedge dx_{I_p} \wedge e^{I_{k+1}, I_{k+2}, \ldots, I_p}(\theta, x(\cdot)), \right),

$$

where we have reinstated a parameter $\lambda_p \in \mathbb{R}^\times$ that quantifies the relative parametrisation of the two terms in the action functional. This parameter passes to the NG formulation upon integrating out the (unphysical) Goldstone degrees of freedom in the HP action functional. There, as in the HP action functional itself, its value does not affect the global symmetries of the super-$\sigma$-model in any qualitative manner, and so it remains arbitrary as long as we consider those symmetries only. Its status changes dramatically in the context of local symmetries for which we look, with hindsight, among infinitesimal (or tangential) shifts of the coordinates $\theta^\alpha, x^I$ and $\phi^\Delta S$ induced by the right-regular translations

$$\varphi : s\mathcal{P}(1, d-1; 1) \times \mathbb{R}^{d-1|D_1,d-1} \rightarrow s\mathcal{P}(1, d-1; 1),$$

$$\left( e^{\theta^\alpha Q_\alpha}, e^{x^I P_I}, e^{\phi^\Delta S} \right) \rightarrow e^{\theta^\alpha Q_\alpha}, e^{x^I P_I}, e^{\phi^\Delta S},$$

$$= e^{(\theta^\alpha + S(\phi)^\alpha_\beta \kappa^\beta)} Q_\alpha, e^{(x^I + L(\phi)^I_\beta \kappa^\beta)} P_I, e^{(\phi^\Delta S)},$$

that is

$$(\theta^\alpha, x^I, \phi^\Delta S) \rightarrow (\theta^\alpha + \widetilde{\kappa}^\alpha(\phi), x^I + \widetilde{y}^I(\phi) - \frac{1}{2} \widetilde{\theta}^I \tilde{\kappa}(\phi), \phi^\Delta S),$$

where $\widetilde{\kappa}^\alpha(\phi) := S(\phi)^\alpha_\beta \kappa^\beta$, $\widetilde{y}^I(\phi) := L(\phi)^I_\beta y^\beta$. 

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These satisfy the algebra
\[
\left[ r((\kappa^2), y_j), r(\kappa^2, y_k) \right] = r(0, \kappa^2, r(\kappa^2, y_j, y_k)).
\]

Note, in particular, that purely Grassmann-odd translations do not form a subalgebra in the latter which implies that closing the algebra of infinitesimal symmetries will require extending the space of such translations by the purely Grassmann-even ones.

The above shifts induce infinitesimal transformations of the relevant Maurer–Cartan super-1-forms:
\[
(\Sigma^\alpha_L, \theta^\alpha_L)(\theta, x, \phi) \rightarrow (\Sigma^\alpha_L, \theta^\alpha_L)(\theta, x, \phi) + (S(-\phi)^\alpha \delta_0 \Sigma^\beta(\phi), L(-\phi)^I \delta_0 \Sigma^I(\theta, x, \phi)) + \theta'(\kappa^2)
\]
that determine the variation of the action functional. Below, we write out and examine the variation in the particular cases of \( p \in \{0, 1\} \) in which we shall subsequently look for a suitable lift of the symmetry to the corresponding super-p-gerbes. The symmetry is discussed in all generality in Refs. (GKW06a, GK06b, cp. also Ref. (HAIM05).

### 6.1.1. The super-\( \theta \)-brane.

The HP action functional for the super-0-brane (in sMink\(^{1,9}|D_{1,0}\)) takes the form
\[
S^{(\text{HP})}_{\text{GS},0}\left[ X^{(\text{HP})} \right] = \int_{\Omega} \Sigma^a_{(1)}(\theta, x, \phi) + \lambda_0 \bar{\theta} \Gamma_{11} \sigma(\theta),
\]
with the integrand given by the pullback of the super-1-form
\[
\bar{\beta}^{(1)}(\theta, x, \phi) = \theta^\alpha_L(\theta, x, \phi) + \lambda_0 \bar{\theta} \Gamma_{11} \sigma(\theta),
\]
whose variation under the Grassmann-odd shift of the lagrangean field reads
\[
r^\alpha_L \bar{\beta}^{(1)}(\theta, x, \phi) = \theta^\alpha_L(\theta, x, \phi) + \lambda_0 \bar{\theta} \Gamma_{11} \sigma(\theta),
\]
\[
= -L(-\phi)^\alpha \bar{\theta} \sigma(\theta) \Gamma_{11} \bar{\kappa}(\phi) - 2\lambda_0 \bar{\theta} \sigma(\theta) \Gamma_{11} \bar{\kappa}(\phi) + d(\lambda_0 \bar{\theta} \Gamma_{11} \bar{\kappa}(\phi))
\]
\[
= -2\Sigma^\alpha_L(\theta, x, \phi) \Gamma^0 \cdot \left( \frac{1}{2} \Sigma^\alpha_L + 2\lambda_0 \Gamma^0 \Gamma_{11} \right) \kappa + d\bar{F}(\lambda_0) \left( (\theta^\alpha, x^I, \phi^A \bar{S}), (\kappa^\beta, 0) \right).
\]
where
\[
\bar{F}(\lambda_0) = \left( (\theta^\alpha, x^I, \phi^A \bar{S}), (\kappa^\beta, 0) \right) = \lambda_0 \bar{\theta} \Gamma_{11} \bar{\kappa}(\phi).
\]
The operator
\[
P^{(0)}_{\lambda_0} = \left[ \frac{1}{2} \right] \left( \Sigma^\alpha_L + 2\lambda_0 \Gamma^0 \Gamma_{11} \right) \in \text{End}_{\mathbb{C}}(S_{1, d-1})
\]
appearing in the first term of the variation is a projector iff
\[
\lambda_0 \in \left\{ -\frac{1}{2}, \frac{1}{2} \right\},
\]
and it then suffices to take
\[
(6.2)
\]
to obtain a symmetry of the action functional. The difference between the two choices is immaterial, hence we set, e.g.,
\[
\lambda_0 = \frac{1}{2}
\]
and proceed with the symmetry analysis of the action functional associated with the super-1-form
\[
(6.3)
\]
Note that the complementary projector \( \mathbf{1}_{D_{1, d-1}} - P^{(0)}_{\lambda_0} \) is precisely of the type described in Prop. 4.1.

\[
\left( \frac{1}{2} \right)^T \cdot \Gamma^I \cdot \left( \frac{1}{2} \right)^T = C', \quad \left( \frac{1}{2} \right)^T \cdot \Gamma^I \cdot \left( \frac{1}{2} \right)^T = C
\]
\[
= \left( \frac{1}{2} \right)^T \cdot \left( \left( \frac{1}{2} \right)^T \cdot \Gamma^I \cdot \left( \frac{1}{2} \right)^T \right) = C
\]
\[
= \delta^{(0)} \cdot \left( \frac{1}{2} \right)^T \cdot \Gamma^I \cdot \left( \frac{1}{2} \right)^T = \delta^{(0)} \cdot \left( \frac{1}{2} \right)^T \cdot \Gamma^I \cdot \left( \frac{1}{2} \right)^T = \delta^{(0)} \cdot \left( \frac{1}{2} \right)^T \cdot \Gamma^I \cdot \left( \frac{1}{2} \right)^T
\]
and in particular the commutator of two Grassmann-odd shifts,
\[ [r(\kappa^\gamma_1, 0), r(\kappa^\gamma_2, 0)] = r(0, \pi, \Gamma_1 \kappa_2), \]

takes the form dictated by the algebra
\[ \pi_1 \Gamma^I \kappa_2 \equiv \left( (1_{D_{1, d}, -} - P^{(0)}_\frac{1}{2}) \kappa_1 \right)^T \Gamma^I \kappa_2 = \pi_1 \Gamma^I \left( \frac{1_{D_{1, d}, -} - (-1)^{I_0} r_{\omega} \Gamma_1 I_1}{2} \right) \kappa_2 = \pi_1 \Gamma^0 \kappa_2 \delta_0^I, \]

or simply
\[ [r(\kappa^\gamma_1, 0), r(\kappa^\gamma_2, 0)] = r(0, \pi, \Gamma_1 \kappa_2) \]

A purely Grassmann-even shift of the lagrangean field of the super-σ-model now yields
\[ r^* \tilde{\beta}^{(\bar{\gamma})}_{(1)} \left( (\theta^\alpha, x^I, \phi A^S), (0, y^J) \right) - \tilde{\beta}^{(\bar{\gamma})}_{(1)} \left( \theta^\alpha, x^I, \phi A^S \right) = L(-\phi)^{(1)} \right) dy^J(\phi) = dy^0 + \theta^0_0(\theta, x, \phi) \eta_I dy^I \equiv \eta_I dy^I \]

and so if we restrict to those Grassmann-even shifts that are required for the closure of the commutator, we obtain the desired result
\[ r^* \tilde{\beta}^{(\bar{\gamma})}_{(1)} \left( (\theta^\alpha, x^I, \phi A^S), (0, y^J_0) \right) - \tilde{\beta}^{(\bar{\gamma})}_{(1)} \left( \theta^\alpha, x^I, \phi A^S \right) = dy^0 \]

without any further manipulations. The distinguished shifts may be seen to engender diffeomorphisms of the super-0-brane worldline in the so-called static gauge, cp. Ref. [GKW06a], and so insisting on their presence among symmetry generators is physically perfectly justified.

At this stage, we still have to reconcile the inverse Higgs constraints, central to the correspondence with the NG formulation, with the newly established symmetries. To this end, we derive the integrability conditions for the differential relation (3.22) in the presence of the symmetries by considering their variation under a general symmetry transformation,
\[ r^* \tilde{\beta}^{(\bar{\gamma})}_{(1)} \left( (\theta^\alpha, x^I, \phi A^S), (0, y^J_0) \right) - \tilde{\beta}^{(\bar{\gamma})}_{(1)} \left( \theta^\alpha, x^I, \phi A^S \right) = dy^0 \]

The demand that the above vanish yields secondary constraints:
\[ (6.4) \quad \theta^0_0(\theta, x, \phi) \equiv 0, \quad \Gamma^S P^{(0)} \Sigma_L(\theta, x, \phi) \equiv 0, \quad \Sigma_L(\theta, x, \phi), \quad \Sigma_L(\theta, x, \phi), \quad \tilde{S} \in \tilde{T}, \]

of which the former were identified in Ref. [GKW06a] as field equations of the super-σ-model under study. Constraints analogous to the latter were encountered in the study of gauge supersymmetries of the GS super-σ-model in Ref. [McA00].

Our findings are summarised in

**Definition 6.1.** The extended super-0-brane κ-symmetry superalgebra (in sMink^{1,0}|D_{1,9}) is the Lie superalgebra of \( \mathbb{R}^{1,0}|D_{1,9} \) with generators
\[ \mathfrak{g} := \left\{ Q^\alpha_{\alpha} := P^{(0)}_{\frac{1}{2}} \alpha Q_{\alpha} \mid \alpha \in \mathbb{R}, \frac{D_{1,9}}{2} \right\}, \]

satisfying the supercommutation relations
\[ \{Q^\alpha_{\alpha}, Q^\beta_{\beta}\} = \left(P^{(0)}_{\frac{1}{2}}, \Gamma^0_{\frac{1}{2}}\right)_{\alpha\beta} P_{0}, \quad \quad [P_0, Q^*_{\alpha}] = 0. \]

Our findings are summarised in

**Proposition 6.2.** The super-0-brane κ-symmetry superalgebra of Def. [6, 7] is a gauge symmetry of the corresponding Green–Schwarz super-σ-model (in the Hughes–Polchinski formulation) and preserves the space of its restricted (classical) field configurations \( D_0 \subset s\mathcal{P}(1,9|1) \) defined by the family of constraints
\[ (6.5) \quad \left(\theta^0_0, \theta^0_0, \frac{P^{(0)}_{\frac{1}{2}}}{\frac{D_{1,9}}{2}} \right) \Sigma_L(\theta, x, \phi) = 0, \quad \Sigma_L(\theta, x, \phi), \quad \tilde{S} \in \tilde{T}. \]

**Remark 6.3.** Note that super-1-form \( \tilde{\beta}^{(\bar{\gamma})}_{(1)} \) and the constraints are invariant with respect to the (right) action of the special orthogonal group \( \text{Spin}(9) \) generated by rotations that leave the distinguished direction \( \partial_0 \) intact. This follows directly from the vectorial nature of the (bare) indices carried by the Maurer–Cartan 1-forms entering the definition of the super-1-form and of the constraints.
6.1.2. The Green–Schwarz superstring. In the case of the superstring, the HP action functional

$$S_{\text{GS,1}}^{(\text{HP})}[X^{(\text{HP})}] = \int_{\Omega} \frac{1}{2} r^* \beta^{(\lambda_1)}_{(2)}$$

is the pullback of the super-2-form

$$\beta^{(\lambda_1)}_{(2)}(\theta, x, \phi) = (\theta^0_L \wedge \theta^1_L)(\theta, x, \phi) + \lambda_1 \, \bar{\sigma} \, \Gamma_I \, \sigma(\theta) \wedge dx^I,$$

and we readily compute, invoking Eqs. (4.3), (4.4) and (4.5) along the way,

$$\begin{align*}
    r^* \beta^{(\lambda_1)}_{(2)}((\theta^a, x^I, \phi^A, \Sigma), (\kappa^\beta, 0)) - \beta^{(\lambda_1)}_{(2)}((\theta^a, x^I, \phi^A, \Sigma)) &= - \bar{\Sigma}_L \Gamma^0 \kappa \wedge \theta^0_L(\theta, x, \phi) + \bar{\Sigma}_L \Gamma^1 \kappa \wedge \theta^0_L(\theta, x, \phi) - 2 \lambda_1 \bar{\Sigma}_L \Gamma_I \kappa \wedge \theta^0_L(\theta, x, \phi) + d(\lambda_1 \, \bar{\sigma} \, \Gamma_I \, \bar{\kappa}(\phi) \, e^I(\theta, x)).
\end{align*}$$

Drawing on the previous observations, we impose the inverse Higgs constraints, whereupon the last formula reduces to

$$r^* \beta^{(\lambda_1)}_{(2)}((\theta^a, x^I, \phi^A, \Sigma), (\kappa^\beta, 0)) - \beta^{(\lambda_1)}_{(2)}((\theta^a, x^I, \phi^A, \Sigma)) = 2(\theta^0_L \wedge \bar{\Sigma}_L \Gamma^0 - \theta^1_L \wedge \bar{\Sigma}_L \Gamma^1)(\theta, x, \phi) \left(\frac{1}{2} \delta^{(\lambda_1)}_{(2)}(\theta^a, x^I, \phi^A, \Sigma, (\theta^a, x^I, \phi^A, \Sigma)) \right) + d(\lambda_1 \, \bar{\sigma} \, \Gamma_I \, \bar{\kappa}(\phi) \, e^I(\theta, x)).$$

Reasoning as in the previous case, we convince ourselves that the operator

$$P^{(1)}_{\lambda_1} := \frac{1}{2} \delta^{(\lambda_1)}_{(2)} \Gamma^0 - \frac{1}{2} \delta^{(\lambda_1)}_{(2)} \Gamma^1 \in \text{End}_C(S_{1, d-1})$$

appearing in the above expression is a projector iff

$$\lambda_1 \in \{\frac{1}{2}, \frac{1}{2} \},$$

and then for

$$\kappa \in \ker P^{(1)}_{\lambda_1}$$

we obtain a symmetry of the space of field configurations subject to the inverse Higgs constraints. Note that in the present case – in contrast with the previous one – field equations have to be invoked already for a single Graßmann-odd variation. Once again, we set

$$\lambda_1 = \frac{1}{2},$$

and continue our analysis for the super-2-form

$$\beta^{(\lambda_1)}_{(2)}((\theta, x, \phi) = (\theta^0_L \wedge \theta^1_L)(\theta, x, \phi) + \frac{1}{2} \bar{\sigma} \, \Gamma_I \, \sigma(\theta) \wedge dx^I.$$
The spinorial super-1-brane $\kappa$-symmetry transformations are the tangential variations of the lagrangean field $X_{(\text{HP})}$ generated by the distinguished linear combinations of the supercharges

$$Q^{\alpha} = \frac{1}{6} \alpha^{\beta} \theta^{\beta}_{1} \circ \varphi, \quad \alpha \in \overline{1, D_{1, d - 1}}.$$
satisfying the supercommutation relations

\[
\{Q^*_\alpha, Q^*_\beta\} = \left(\frac{p^{(1)\top}}{\mathbb{Z}} \cdot \mathcal{T}^0 \cdot \frac{p^{(1)}}{\mathbb{Z}}\right)_{\alpha\beta} F_0 + \left(\frac{p^{(1)\top}}{\mathbb{Z}} \cdot \mathcal{T}_1 \cdot \frac{p^{(1)}}{\mathbb{Z}}\right)_{\alpha\beta} P_1,
\]

\[\{P_A, Q^*_\alpha\} = 0, \quad A \in \{0, 1\}.\]

Finally, we may articulate

**Proposition 6.5.** The spinorial super-1-brane $\kappa$-symmetry transformations of Def. 6.4 are gauge symmetries of the corresponding Green–Schwarz super-$\sigma$-model (in the Hughes–Polchinski formulation) and preserve the space of its restricted (classical) field configurations $\mathcal{D}_1 \subset s\mathcal{P}(1, d - 1|1)$ defined by the family of constraints

\[\theta_L^S, \theta_R^S, \Gamma^S p^{(1)} \Sigma_L \rvert_{\mathcal{T}\mathcal{D}_1} = 0, \quad S \in 2, d - 1.\]

**Remark 6.6.** In analogy with the case of the super-0-brane (and for the very same reasons), the super-2-form $\hat{\beta}^{(2)}$ and the constraints are invariant with respect to the (right) action of the special orthogonal group $\text{Spin}(d - 2)$ generated by rotations that leave the distinguished directions $\partial_0$ and $\partial_1$ intact.

6.2. **The extended Green–Schwarz supergerbe.** Having understood the (super)group-theoretic origin of $\kappa$-symmetry in the framework of Cartan geometry of the homogeneous space of Prop. 1.3, we may next—in the spirit of Sec. 5.1—look for a geometrisation of the HP formulation of the super-$\sigma$-model and the corresponding gerbe-theoretic extension of its gauge-symmetry analysis. This is more than well justified as the relevant action functional $S_{\text{GS}, p}$ has the structure of a gerbe holonomy, with the “metric” term $S_{\text{metr},\text{GS}, p}$ of (3.19) determined by a manifestly LI super-$\sigma$-model and hence defining a trivial super-$p$-gerbe on the extended supertarget. The analysis of the preceding section suggests that the ensuing simple picture of a (Deligne) tensor product of the trivial super-$p$-gerbe defined by the “metric” term with the pullback of the super-$p$-gerbe from $s\text{Mink}^{1,d-1|D_1,d-1}$ to the super-Poincaré supergroup $s\mathcal{P}(1, d - 1|1)$ along the canonical projection\(^3\) be refined through incorporation of the tangential constraints deduced from the $\kappa$-symmetry analysis. In the cases of $p \in \{0, 1\}$, to which we restrict our attention in the remainder of the section, this boils down to the imposition of conditions (6.3) resp. (6.7). Thus, we arrive at

**Definition 6.7.** The extended Green–Schwarz super-0-gerbe is the restriction to the subspace $\mathcal{D}_0$ of Prop. 6.2 of the (Deligne) tensor product\(^2\)

\[\mathcal{G}_{\text{GS}}^{(0)} := \pi_{(0)} \mathcal{G}_{\text{GS}}^{(0)} \otimes \mathcal{I}_2^{\beta^{(\text{HP})}}(1)\]

of the trivial super-0-gerbe defined by the LI super-1-form $2 \beta^{(\text{HP})}$ of Eq. (3.20) with the pullback, along the canonical projection

\[\pi_{(0)} : s\mathcal{P}(1, 9|1) \longrightarrow s\text{Mink}^{1,9|D_1,9},\]

of the Green–Schwarz super-0-gerbe of Def. 5.2.

and the analogous

**Definition 6.8.** The extended Green–Schwarz super-1-gerbe is the restriction to the subspace $\mathcal{D}_1$ of Prop. 6.3 of the (Deligne) tensor product

\[\mathcal{G}_{\text{GS}}^{(1)} := \pi_1^{\star} \mathcal{G}_{\text{GS}}^{(1)} \otimes \mathcal{I}_2^{\beta^{(\text{HP})}}(2)\]

of the trivial super-1-gerbe defined by the LI super-2-form $2 \beta^{(\text{HP})}$ of Eq. (3.20) with the pullback, along the canonical projection

\[\pi_{(1)} : s\mathcal{P}(1, d - 1|1) \longrightarrow s\text{Mink}^{1,d-1|D_1,d-1},\]

\(^3\)Strictly speaking, we should, if anything, pull it back to the HP section coordinatised as in Eq. (4.11).

\(^2\)The numerical factor 2 in front of the global connection super-1-form of the trivial gerbe is an artifact of our initial normalisation of the HP super-1-form $\beta^{(\text{HP})}$.\(\)
of the Green–Schwarz super-1-gerbe of Def. 5.9.

While the above definitions, being formulated in terms of LI (super-)differential objects over \( s\mathcal{P}(1, d−1) \), make perfect sense, they call for a careful revision of the notion of global (left) supersymmetry, a fact of key importance in any subsequent (supersymmetry-)invariance considerations, and so also in the very construction of gerbe-theoretic structures. It seems natural to truncate the original supersymmetry group \( \mathbb{R}^{1,d−1|1} \) (as well as its extensions encountered in this work) to its maximal subgroup composed of elements which preserve the constraints (6.2) and (6.7) derived from the requirement of (spinorial) \( \kappa \)-symmetry. The subgroup will be generated by the one-parameter (sub)groups induced by flows of the right-invariant vector fields on \( s\mathcal{P}(1, d−1) \) from the intersection of the kernels of the left-invariant super-1-forms defining the respective tangent sheaves \( \mathcal{T}_p, p \in \{0, 1\} \). These are natural candidates for global symmetries of the subspaces \( \mathcal{P}_p, p \in \{0, 1\} \) introduced in Props. 6.2 and 6.5. A complete treatment of the supergerbe theory behind the HP formulation of the super-\( \sigma \)-model consistent with the (gauge) supersymmetries present ought to be preceded by an in-depth analysis of the geometric content of such a definition. In this preliminary discussion, we merely point out its necessity and relevance, leaving the clarification of its ramifications to a future thorough investigation.

6.3. Weak \( \kappa \)-equivariance of the extended Green–Schwarz supergerbe. The purely gerbe-theoretic nature of the HP action functional of the GS super-\( \sigma \)-model in conjunction with the presence of a gauge supersymmetry rederived, in a simplified version tailored to our subsequent considerations, at the beginning of the present section, give rise to the expectation, based on former studies reported in Refs. [GW06b, GSW06, Sus12], that the extended GS super-\( p \)-gerbes should be endowed with an equivariant structure of some sort with respect to right-regular (Graßmann-odd) translations on the supertarget. Such an informed guess is confronted with the obvious obstacles: The very existence of the symmetry necessitates the imposition of constraints of the admissible field configurations and the symmetry algebra does not seem to close on non-classical field configurations (cp., e.g., Ref. [McA06]). Luckily, the constraints, enumerated in Props. 6.2 and 6.5, admit a natural geometrisation, i.e., can be treated as linear conditions to be imposed on sections of the tangent sheaf of the supertarget and thus distinguishing its subspace \( \mathcal{P}_p, p \in \{0, 1\} \), whereas the latter obstacle leaves us – in the light of the interpretation given in Ref. [Sus12] (cp. also Ref. [RS09]) to the various components of the full-fledged equivariant structure on the gerbe – with the possibility of having on the extended GS super-\( p \)-gerbes (for \( p \in \{0, 1\} \)) the first component of such a structure, giving an element-wise realisation of the symmetry ‘set’ (or ‘ensemble des opérateurs’ in the sense of Ref. [Bou97, Chap.I § 3.1]). Below, we perform a case-by-case study that provides us with solid arguments in favour of this last expectation.

6.3.1. The extended GS super-0-gerbe. A natural point of departure of our analysis is a lift of the Graßmann-odd shifts of Eq. (6.1) (with \( y^J = 0, J \in 0, d−1 \)) to the total space of the super-0-gerbe of Def. 5.7. The latter is a semidirect product of the total space \( \mathcal{L}^{(0)} \) of the GS super-0-gerbe over \( \text{Spin}(1,9) \), or – equivalently – a principal \( \mathbb{C}^* \)-bundle over \( s\mathcal{P}(1,9) \) endowed with the structure of a central extension of the super-Poincaré group, with an obvious group law (an obvious adaptation of the binary operation on \( \mathcal{L}^{(0)} \) from Prop. 5.1) which we employ to define the sought-after lift in the form

\[
\mathcal{P}^{(0)} : (\mathcal{L}^{(0)} \times \text{Spin}(1,9)) \times \ker P^{(0)}_1 \longrightarrow \mathcal{L}^{(0)} \times \text{Spin}(1,9)
\]

\[
\beta((\theta^\alpha, x^l, z, \phi^{JK})), \kappa^\beta) \mapsto (\theta^\alpha + \bar{\kappa}^\alpha(\phi), x^l - \frac{1}{2} \bar{\sigma}^I \bar{\kappa}(\phi) e^{\bar{\sigma}^I \Gamma_{11} \bar{\kappa}(\phi) \cdot z, \phi^{JK}}.
\]

We now readily verify that the principal \( \mathbb{C}^* \)-connection of the extended GS super-0-gerbe,

\[
\bar{\beta}((\theta, x, z, \phi)) = i \frac{dz}{z} + \beta((\theta, x) + 2 \beta^{(HP)}((\theta, x, \phi),
\]

satisfies, upon restriction to the HP section with coordinates \( (\theta^\alpha, x^l, \phi^{A\Sigma}) \), the expected identity

\[
\bar{\beta}^{(0)}((\theta^\alpha, x^l, z, \phi^{A\Sigma}), \kappa^\beta) - \bar{\beta}((\theta^\alpha, x^l, z, \phi^{A\Sigma}) = -d(\bar{\sigma}^I \Gamma_{11} \bar{\kappa}(\phi)) + \bar{\sigma}^I \Gamma_{11} \bar{d}\kappa(\phi) + \bar{\kappa}(\phi) \Gamma_{11} \sigma(\theta) + \bar{\kappa}^0 \Sigma_{\Sigma}(\theta, x, \phi) = -2\bar{\sigma}(\theta) \bar{\sigma}_{11} \bar{\kappa}(\phi) - 2\Sigma_{\Sigma} \Gamma_{0} \kappa
\]

\[
= -4\Sigma_{\Sigma} \Gamma_{0} P^{(0)}_1 \kappa = 0.
\]
We conclude that there exists over
\[ (\mathcal{M}^{(1)} \times \text{Spin}(1, 9)) \times \ker P_2^{(0)} =: \tilde{\mathcal{M}}^{(1)1} \]
a connection-preserving isomorphism
\[ \Upsilon^{(0)}_\kappa : \pi^{(0)}_* \tilde{\mathcal{G}}^{(0)}_{\text{GS}} \xrightarrow{\cong} \text{pr}_1^* \tilde{\mathcal{G}}^{(0)}_{\text{GS}} \cong \text{pr}_1^* \tilde{\mathcal{G}}^{(0)}_{\text{GS}} \times \mathcal{J}_{F = 0}^{(1)} \]
with data
\[ 2\mathcal{F}(\tilde{t})((\theta^\alpha, x^I, \phi^\beta), (\kappa^\gamma, 0)) = \mathcal{G}_{\Gamma_{11}} \tilde{\kappa}(\phi), \]
determined by the Lie-supergroup structure on the total space \( Z^{(0)} \) of the GS super-0-gerbe \( \mathcal{G}^{(0)}_{\text{GS}} \). Our analysis yields a weak (spinorial) \( \kappa \)-equivariant structure relative to a vanishing super-1-form \( \tilde{\rho} = 0 \), in keeping with the findings of Refs. [GSW10, GSW13], where equivariant structures of this type were identified as the unique ones for which the gerbe descends from its base to the space of orbits of the action of the symmetry group being gauged. It should be noted that the above data appear non-LI, and so, clearly, further study is needed to put the said equivariant structure, implicitly defined by them, on an equal footing with the Ad.-equivariant structure on \( \mathcal{G}^{(0)}_{\text{GS}} \) described in Prop. 5.23. We leave this challenge for future work.

6.3.2. The extended GS super-1-gerbe. In the case of the super-1-gerbe of Def. 3.8, we employ the same ruse as in the previous section, that is, we first judiciously choose the surjective submersion of the extended gerbe \( \tilde{\mathcal{G}}^{(1)}_{\text{GS}} \),
\[ \pi_{Y \tilde{M}^{(1)}} : Y \tilde{M}^{(1)} := Y_1 \mathcal{M}^{(1)} \times \text{Spin}(1, d - 1) \to \mathcal{M}^{(1)} \times \text{Spin}(1, d - 1) \]
\[ : \left( (\theta^\alpha, x^I, \xi_\beta, \phi^{JK}), (\kappa^\gamma) \right) \mapsto \left( \theta^\alpha, x^I, \phi^{JK} \right), \]
and of the pullback gerbes \( r_1^* \tilde{\mathcal{G}}^{(1)}_{\text{GS}} \) and \( \text{pr}_1^* \tilde{\mathcal{G}}^{(0)}_{\text{GS}} \),
\[ \pi_{Y_1 \tilde{M}^{(1)1}} \equiv \pi_{Y_1 \mathcal{M}^{(1)}} \times \text{id}_{\text{Spin}(1, d - 1) \times \ker P_2^{(1)}} : Y_1 \tilde{M}^{(1)1} := Y_1 \tilde{M}^{(1)} \times \ker P_2^{(1)} \to \tilde{M}^{(1)1}, \]
and subsequently induce a lift of the Graßmann-odd shifts to the former with the help of the (suitably adapted) group law of Prop. 5.7, whereby we obtain the transformation law
\[ \pi^{(1)} : \left( Y_1 \mathcal{M}^{(1)} \times \text{Spin}(1, d - 1) \right) \times \ker P_2^{(1)} \to Y_1 \mathcal{M}^{(1)} \times \text{Spin}(1, d - 1) \]
\[ : \left( (\theta^\alpha, x^I, \xi_\beta, \phi^{JK}), (\kappa^\gamma) \right) \mapsto \left( \theta^\alpha + \bar{\kappa}^\alpha(\phi), x^I - \frac{1}{2} \Gamma^I \bar{\kappa}(\phi), \xi_\beta - \frac{1}{6} (\bar{\eta} \Gamma_I \bar{\kappa}(\phi)) \Gamma^I_{\beta\gamma}(2\theta^\gamma + \bar{\kappa}^\gamma(\phi)), \phi^{JK} \right). \]
Following the logic of Sec. 5.2.2, we then pass to the surjective submersion
\[ Y(Y_1 \tilde{M}^{(1)1}) := Y_1 \tilde{M}^{(1)1} \times \tilde{M}^{(1)1}, Y_1 \tilde{M}^{(1)1} : \left( (\theta, x, \xi, \phi, \kappa), (\theta, x, \xi, \phi, \kappa) \right) = (\bar{y}_1, \bar{y}_2), \]
to which we pull back the curvature
\[ \bar{\beta}(\theta, x, \xi, \phi) = \beta^{(2)}(\theta, x, \xi) + 2 \beta^{(H \Phi)}(\theta, x, \phi) \]
of the extended super-1-gerbe from its surjective submersion \( Y_1 \mathcal{M}^{(1)} \times \text{Spin}(1, d - 1) \) along the relevant maps that let us calculate, up to \( \mathcal{O}(\kappa^2) \),
\[ (6.8) \quad \text{pr}_2 \text{pr}_2 \beta_{(2)}^{(2)} - \text{pr}_1^* \beta_{(2)}^{(1)} + \beta = d\bar{E}, \]
where
\[ \bar{E}(\bar{y}_1, \bar{y}_2) = -\bar{\eta} \Gamma_I \bar{\kappa}(\phi) \left( e^I(\theta, x) - \frac{1}{3} \bar{\eta} \Gamma^I \sigma(\theta) \right) - \bar{\kappa}^\alpha(\phi) d\xi_1 + \theta^\alpha d(\xi_2 - \xi_1). \]
Taking into account the commutativity, for arbitrary \((\varepsilon, y, \zeta) \in Y_1 \mathcal{M}^{(1)}\) and
\[ \ell^{(1)} = \eta^{(2)}_1 : Y_1 \mathcal{M}^{(1)} \times Y_1 \mathcal{M}^{(1)} \to Y_1 \mathcal{M}^{(1)} , \]
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of the diagram (expressing none other than the associativity of the binary operation of the Lie supergroup \( Y_1 \mathcal{M}^{(1)} \))

\[
\begin{array}{ccc}
(Y_1 \mathcal{M}^{(1)} \times \text{Spin}(1, d - 1)) \times \ker P^{(1)}_\pm & \xrightarrow{\iota^{(1)}_{(p, q, z)} \circ \text{id}_{\text{Spin}(1, d - 1) \times \ker P^{(1)}_\pm}} & Y_1 \mathcal{M}^{(1)} \times \text{Spin}(1, d - 1) \\
\end{array}
\]

we conclude that the left-hand side of Eq. (6.8), and so also its right-hand side are LI, and so it makes sense to erect over \( Y(\overline{Y_1 \mathcal{M}^{(1)}}) \) a (trivial) principal \( \mathbb{C}^\times \)-bundle

\[
\pi_{\mathcal{E}} \equiv \text{pr}_1 : \mathcal{E} := Y(\overline{Y_1 \mathcal{M}^{(1)}}) \times \mathbb{C}^\times \rightarrow Y(\overline{Y_1 \mathcal{M}^{(1)}}) : (\overline{y}_1, \overline{y}_2, z) \mapsto (\overline{y}_1, \overline{y}_2)
\]

with a principal \( \mathbb{C}^\times \)-connection 1-form

\[
a_{\mathcal{E}}((\overline{y}_1, \overline{y}_2), z) = i \frac{dz}{z} + \tilde{E}(\overline{y}_1, \overline{y}_2)
\]

and assume the latter to be LI, which fixes the data of a connection-preserving principal \( \mathbb{C}^\times \)-bundle automorphism implementing supersymmetry on \( \mathcal{E} \). The bundle is a candidate for the datum of a gerbe 1-isomorphism of the (weak) \( \kappa \)-equivariant structure that we are seeking to reconstruct.

At this stage, it remains to verify our expectations with regard to the status of \( \mathcal{E} \) by establishing a connection-preserving isomorphism

\[
\alpha_{\mathcal{E}} : (\iota^{(1)}_{(\theta, x)} \circ \text{id}_{\mathcal{E}})^* \overline{\mathcal{E}} \cong \text{pr}_{3,4,\mathcal{E}}^* \mathcal{E} \rightarrow \text{pr}_{1,2,\mathcal{E}}^* \mathcal{E} \cong (\text{pr}_{3,4}^* \circ \text{pr}_{2,4})^* \overline{\mathcal{E}}
\]

of principal \( \mathbb{C}^\times \)-bundles over

\[
\begin{aligned}
Y(Y_1 \mathcal{M}^{(1)}) \times \overline{\mathcal{M}^{(1)}}; \quad & Y(Y_1 \mathcal{M}^{(1)}) \equiv Y_1 \mathcal{M}^{(1)} \times \overline{\mathcal{M}^{(1)}}; \quad & Y(Y_1 \mathcal{M}^{(1)}) \equiv Y_1 \mathcal{M}^{(1)} \times \overline{\mathcal{M}^{(1)}}; \quad & Y(Y_1 \mathcal{M}^{(1)}) \equiv Y_1 \mathcal{M}^{(1)} \times \text{Spin}(1, d - 1) \\
\end{aligned}
\]

\[
\equiv \left( \left( \left( \left( \left( (\theta, x, \xi, \alpha) \circ \kappa \right), (\theta, x, \xi, \alpha) \circ \kappa \right), (\theta, x, \xi, \alpha) \circ \kappa \right), (\theta, x, \xi, \alpha) \circ \kappa \right) \right) = (\overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{y}_4)
\]

where

\[
\overline{\mathcal{E}} = \text{pr}_{1,2}^* \mathcal{E} \cong (Y[2] \overline{\mathcal{M}^{(1)}} \times \mathbb{C}^\times) \cong \text{pr}_{1,2}^* \mathcal{E} \rightarrow Y[2] \overline{\mathcal{M}^{(1)}} \equiv Y[2] \overline{\mathcal{M}^{(1)}} \times \text{Spin}(1, d - 1)
\]

is the principal \( \mathbb{C}^\times \)-bundle of the extended GS super-1-gerbe \( \overline{\mathcal{G}}^{(1)}_{\text{GS}} \), in whose definition the trivial tensor factor \( \left( Y[2] \overline{\mathcal{M}^{(1)}} \times \mathbb{C}^\times \right) \) represents the principal \( \mathbb{C}^\times \)-bundle of the trivial super-1-gerbe \( \mathcal{I}^{(2)}_{\beta(\text{MP})} \), with a vanishing connection. To these ends, we calculate the difference of the base components of the relevant principal connection 1-forms (keeping in mind the triviality of the contribution of \( \mathcal{I}^{(2)}_{\beta(\text{MP})} \)),

\[
\text{pr}_{3,4}^* \iota^{(1)}_{(\theta, x)} \circ \text{id}_{\mathcal{E}} - \text{pr}_{1,2}^* \tilde{E} - \text{pr}_{2,4,\mathcal{E}}^* \mathcal{E} = \overline{\Delta},
\]

whereby we obtain the result

\[
\overline{\Delta}(\overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{y}_4) = \tilde{\pi}_1^\alpha(\phi) d(\xi_\alpha - d\xi_\alpha) \equiv \tilde{\pi}_1^\alpha(\phi) \left( \text{pr}_{3,4}^* \pi_1^\alpha e_\alpha^{(2)} - \text{pr}_{1,2}^* \pi_1^\alpha e_\alpha^{(2)} \right)(\overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{y}_4),
\]

in which

\[
\tilde{\pi}_1 : \overline{Y_1 \mathcal{M}^{(1)}} \rightarrow Y_1 \mathcal{M}^{(1)} : (\theta, x, \xi, \alpha) \mapsto (\theta, x, \xi).
\]

Upon recalling the (dual) spinorial nature of the indices carried by the coordinates \( \xi_\alpha, \alpha \in \overline{1,D_{1,d-1}} \) in the fibre of the vector bundle \( Y_1 \mathcal{M}^{(1)} \), we may rewrite the above result in terms of the natural counterparts of the \( e_\alpha^{(2)} \) on the Lie supergroup \( Y_1 \mathcal{M}^{(1)} \), which we choose to denote as \( \theta^{(2)}_{L,\alpha} \). We obtain the simple expression

\[
\overline{\Delta} = \kappa^\alpha \left( \text{pr}_{1,2}^* \theta^{(2)}_{L,\alpha} - \text{pr}_{2,4}^* \theta^{(2)}_{L,\alpha} \right).
\]

Thus, it transpires that if we were to impose the constraints

\[
\theta^{(2)}_{L,\alpha} \left( 1_{\overline{D_{1,d-1}}} - P^{(1)}{\overline{D_{1,d-1}}} \right)_{\alpha} = 0, \quad \alpha \in \overline{1,D_{1,d-1}},
\]

the 1-isomorphism sought after would exist and be, in fact, trivial, which would further imply that it satisfies the coherence conditions (recall the triviality of the groupoid structure on \( \mathcal{L}^{(1)} \)). This would
then ensure the existence of the postulated weak $\kappa$-equivariant structure. That which makes the above constraints plausible, or — indeed — natural is their structural affinity with those derived earlier from the ($\kappa$-)symmetry analysis of the HP action functional. Note also that they play a rôle analogous to that of the $\kappa$-symmetry itself, to wit, they effectively remove part of the Grassmann-odd geometric degrees of freedom. Therefore, we are led to postulate the new constraints as a proper gerbe-theoretic augmentation of those listed in Prop. 6.5.

Of course, a full understanding of the structural observations, reported in this closing section of the paper, that are hoped to have shed some light on the gerbe-theoretic aspect of the $\kappa$-symmetry of the Green–Schwarz super-$\sigma$-model would require a thorough examination of the global supersymmetry in the presence of differential constraints imposed. This we leave to a future work.
In the present paper, we have put forward an essentially complete proposal of a novel geometrisation scheme for a family of super-\((p+2)\)-cocycles representing classes in the Cartan–Eilenberg supersymmetry-invariant cohomology of the super-Minkowskian spacetime \(s\text{Mink}^{1,d-1|D_1,d-1}\) (regarded as a Lie supergroup), of direct relevance to the construction of the Green–Schwarz super-\(\sigma\)-models of super-\(p\)-brane dynamics. The motivation for the geometrisation comes from the construction, due to Rabin and Crane [RC85, Rab87], of an orbifold of the original supertarget \(s\text{Mink}^{1,d-1|D_1,d-1}\) with respect to the natural geometric action of the discrete Kostelecký–Rabin (lattice) supersymmetry group whose nontrivial topology is captured by the said Cartan–Eilenberg cohomology of the topologically trivial super-Minkowskian spacetime. The geometrisation scheme proposed hinges on the relation between the Cartan–Eilenberg cohomology of the Lie supergroup and the Chevalley–Eilenberg cohomology of its Lie superalgebra with values in the trivial module \(\mathbb{R}\), and on the correspondence between the second cohomology group in the latter cohomology and (equivalence classes) of supercentral extensions of the Lie superalgebra, and employs a family of Lie supergroups surjectively submersed over the original supertarget, of the type originally considered by de Azcárraga et al. [dAlPB00], that arise from the supercentral extensions determined by distinguished super-2-cocycles methodically induced from the Green–Schwarz super-\((p+2)\)-cocycles. These extended Lie supergroups were subsequently used as elementary ingredients in a reconstruction of the super-\(p\)-gerbes, carried out explicitly for \(p \in \{0,1,2\}\), along the lines of the standard bosonic geometrisation scheme for de Rham cocycles, due to Murray [Mur96]. The thus obtained Green–Schwarz super-\(p\)-gerbes were shown to possess the expected supersymmetry-(Ad-)equivariant structure, signalling the amenability of the adjoint realisation of the supersymmetry group to gauging in the corresponding super-\(\sigma\)-model. This falls in perfect agreement with the intuitions developed in the bosonic context in the works of Gawędzki et al. [GSW10, GSW13, Sus11b, Sus12, Sus13]. Finally, the geometrisation scheme developed over the supertarget of the Nambu–Goto formulation of the Green–Schwarz super-\(\sigma\)-model was transplanted into the setting of the equivalent, but somewhat nontrivially so from the geometric point of view, Hughes–Polchinski formulation of the same super-\(\sigma\)-model, whereby the extended Green–Schwarz super-\(p\)-gerbe was erected, unifying the metric and topological (gerbe-theoretic) data of the corresponding Nambu–Goto formulation to which it descends upon imposition of certain Cartan-geometric constraints on its covariant configuration bundle (i.e., on the enlarged supertarget). Conditions ensuring equivalence of the two formulations were analysed in considerable detail. The passage to the Hughes–Polchinski formulation opened the possibility for a straightforward (if also, at the same time, incomplete) geometrisation of the all-important gauge supersymmetry of the Green–Schwarz super-\(\sigma\)-model, that is to say, of the \(\kappa\)-symmetry of Refs. [dAL83, Sie83, Sie84], known to effectively implement supersymmetric balance between the bosonic and fermionic degrees of freedom in the field theories under consideration (whence also the necessity to study it closely). The geometrisation assumed the form of an incomplete \(\kappa\)-equivariant structure on the extended super-\(p\)-gerbe, derived explicitly for \(p \in \{0,1\}\) and termed the weak \(\kappa\)-equivariant structure, its existence being, again, in conformity with the bosonic intuition referred to \(\kappa\)-symmetry, the gauge symmetry of the super-\(\sigma\)-model.

The results reported in the present paper prompt a host of natural questions, and actually define a concrete formal context in which these may be formulated. Starting with those of the more fundamental nature, it is certainly tempting to seek an explicit relation between our construction and alternative approaches to supersymmetry in the context of superstring and related models, one such particularly attractive approach being at the heart of the proposal, originally conceived by Killingback [Kil87] and Witten [Wit88], elaborated by Freed [Fre87], recently revived by Freed and Moore [FW04], and ultimately concretised in the higher-geometric language by Bunke [Bun11] (cp. also Ref. [Wall13] for an explicit construction), for a geometrisation of the Pfaffian bundle of the target-space Dirac operator, associated with fermionic contributions to the superstring path integral, in terms of a differential String-structure on the target space. Another, and not entirely unrelated, idea that might – given the role played by the algebra and (super)symmetry arguments in our construction – lead to a deeper understanding of the geometrisation scheme proposed would be to look for an explicit and geometrically meaningful relation between the super-\(p\)-gerbes constructed in the present work, and in particular the towers of supercentral extensions of the Lie supergroups built over the super-Minkowski (resp. super-Poincaré) Lie supergroup, and the Lie-\(n\)-superalgebras and \(L_\infty\)-superalgebras of Baez et al. considered in Refs. [BC04, BH11, Hue11].
On the next level, we find directions in which the study initiated in the present paper could and should be completed. One such question that complements the discourse developed herein concerns the actual (super)geometric and (super)algebraic content of $\kappa$-symmetry, and its full-fledged (super-)gerbe-theoretic realisation – among the issues that have to be settled in order to gain a better understanding in this matter, the compatibility of global supersymmetry with the Cartan-geometric constraints resulting from the $\kappa$-symmetry analysis of the Hughes–Polchinski formulation of the Green–Schwarz super-$\sigma$-model stands out as singularly pressing. Driven by bosonic intuitions, one would also like to enquire about the existence and concrete realisation of a multiplicative structure on the Green–Schwarz super-$p$-gerbe, in keeping with the findings of Refs. [CJM02, Wal10, GW09]. Finally, our construction of the supersymmetry(Ad-)equivariant structure on the super-$p$-gerbe begs for a logical conclusion in the form of a hands-on construction of a gauged Green–Schwarz super-$\sigma$-model (taking into account the nontrivial nature of the right-regular supersymmetry). With the maximal choice of the supersymmetry group to be gauged, i.e., $\mathbb{R}_{1\ldots d-1|D_{1\ldots d-1}}$, one should expect, on the basis of the bosonic experience [Gaw99, GTTNB04, Gaw02], the emergence of a topological field theory of the (super-)Chern–Simons type. A prerequisite for this analysis would be an in-depth study of the maximally (super)symmetric boundary conditions in the proposed formulation – this point in the direction of a systematic study of super-$p$-gerbe (bi-)modules, or – more generally – the reconstruction of the associated higher categories of super-$p$-gerbs over $s\text{Mink}^{1\ldots d-1|D_{1\ldots d-1}}$.

Last but not least, our work paves the way to a variety of natural and interesting applications and extensions. One obvious line of development is the application of the formalism proposed to the super-$\sigma$-models on supertargets with the body of the general type $\text{AdS}_{p+2} \times S^{d-p-2}$ whose exploration has led to remarkable progress in string theory, as seen from the phenomenological but also purely theoretical perspective. Here, the hope is that the ideas and constructions advanced in the present work prove sufficiently universal and technically robust to accommodate the extra complexity of these superbackgrounds whose super-$\sigma$-model description is – after all – structurally akin to that considered above. Another one that can be conceived is an explicit construction of a bosonisation/fermionisation defect (and the associated super-$1$-gerbe bi-brane) in the much tractable super-Minkowskian setting – this promises to shed some light on the geometry behind the correspondence between worldsheet and target-space supersymmetry in superstring theory. We shall certainly return to these ideas in a future work.
**Appendix A. Conventions and Facts**

**Convention A.1.** Fix natural numbers \( m < n \in \mathbb{N} \setminus \{0\} \). Given an arbitrary family \( \{X_{i_1...i_n}\}_{i_1,i_2,...,i_n \in \mathcal{I}} \) of elements of an abelian group, indexed by a set \( \mathcal{I} \), we define the (partial) symmetriser

\[
X_{(i_1i_2...i_m)_{m+1...n}} \equiv \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} X_{i_{\sigma(1)}i_{\sigma(2)}...i_{\sigma(m)}i_{m+1...n}}
\]

and the (partial) antisymmetriser

\[
X_{[i_1i_2...i_m]_{m+1...n}} \equiv \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \text{sign}(\sigma) X_{i_{\sigma(1)}i_{\sigma(2)}...i_{\sigma(m)}i_{m+1...n}}.
\]


**Convention A.2.** For the differential calculus on supermanifolds, we adopt the conventions of Ref. [DF99]. That is, given the standard coordinates \( \{x^1, x^2, \ldots, x^d, \theta^1, \theta^2, \ldots, \theta^N\} \) on the superspace \( \mathbb{R}^{d|N} \) (parameterising locally a given supermanifold), we assign to a differential object \( X \) (superdifferential form, supervector fields etc.) an additive bidegree composed of its Grassmann and de Rham(-cohomology) degrees,

\[
\text{Deg}(X) := (\overline{\text{X}}, \text{deg}_{\text{dr}}(X)).
\]

Thus, for the elementary objects, we have the assignments

\[
\begin{align*}
\text{Deg}(x^I) &= (0, 0), & \text{Deg}(\theta^a) &= (1, 0), \\
\text{Deg}(dx^I) &= (0, 1), & \text{Deg}(d \theta^a) &= (1, 1), \\
\text{Deg}(\frac{\partial}{\partial x^I}) &= (0, -1), & \text{Deg}(\frac{\partial}{\partial \theta^a}) &= (1, -1).
\end{align*}
\]

Upon defining the product of bidegrees

\[
(\text{Deg}(X), \text{Deg}(Y)) := \overline{\text{X}} \cdot \overline{\text{Y}} + \text{deg}_{\text{dr}}(X) \cdot \text{deg}_{\text{dr}}(Y),
\]

we have the bigraded commutativity relations

\[
XY = (-1)^{(\text{Deg}(X), \text{Deg}(Y))} YX.
\]

In particular, we find the following elementary supercommutation relations for the coordinate super-1-forms:

\[
\begin{align*}
x^I \ x^J &= x^J \ x^I, & x^I \ \theta^a &= \theta^a \ x^I, & \theta^a \ \theta^b &= -\theta^b \ \theta^a, \\
x^I \ dx^J &= dx^J \ x^I, & x^I \ d \theta^a &= d \theta^a \ x^I, & \theta^a \ dx^I &= dx^I \ \theta^a, & \theta^a \ d \theta^b &= -d \theta^b \ \theta^a, \\
dx^I \wedge dx^J &= -dx^J \wedge dx^I, & dx^I \wedge d \theta^a &= -d \theta^a \wedge dx^I, & d \theta^a \wedge d \theta^b &= d \theta^b \wedge d \theta^a.
\end{align*}
\]

These assignments are naturally (i.e., additively) extended to objects carrying multiple indices of the elementary type.

**Convention A.3.** For the Clifford algebra

\[
\{\Gamma_I, \Gamma_J\} = 2 \eta_{IJ}
\]

and its Majorana spin representations, we adopt the conventions of Refs. [Wes99, CdAIPB00], lowering resp. raising spacetime indices, wherever necessary, with the help of the Minkowskian metric \( \eta \) resp. its inverse. Thus, in particular, the charge-conjugation matrix \( C = (C_{\alpha\beta})_{\alpha,\beta \in \mathbb{I} \cup \mathbb{N}} \) has the properties

\[
C^{-1} = -C^T, \quad \epsilon \quad C^T = -\epsilon \ C, \quad \epsilon = \begin{cases} -\sqrt{2} \cos \left( \frac{(d+2)\pi}{4} \right) & \text{if } d \in \{1, 3, 5, 7, 9\} \\ -\sqrt{2} \cos \left( \frac{(d+1)\pi}{4} \right) & \text{if } d \in \{2, 6\} \end{cases}
\]

(A.1)

and we simply do not consider supertargets of dimensions other than those listed. Such restrictions ensure that all the matrices \( C \Gamma^{I_1I_2...I_p} \equiv C (\Gamma^{I_1} \Gamma^{I_2}...\Gamma^{I_p}) \) (and so also the matrices \( C \Gamma^{I_1I_2...I_p} = \eta_{I_1J_1} \eta_{I_2J_2}...\eta_{I_pJ_p} C \Gamma^{J_1J_2...J_p} \)) discussed in the main text are symmetric,

(A.2)

\[
(C \Gamma^{I_1I_2...I_p})^T = C \Gamma^{I_1I_2...I_p}, \quad p \in \mathbb{N}.
\]
For the sake of transparency of the formulæ appearing in the article, we shall also use the shorthand notation

\[ \Gamma I_1 I_2 \ldots I_p \equiv C \Gamma I_1 I_2 \ldots I_p, \quad \bar{\Gamma} I_1 I_2 \ldots I_p \equiv C \Gamma I_1 I_2 \ldots I_p. \]

The charge-conjugation matrix defines the fundamental bilinear form on spinors,

\[ (\xi_1, \xi_2) \mapsto \bar{\xi}_1 \xi_2 \equiv \xi_1^{\alpha} C_{\alpha\beta} \xi_2^\beta, \]

with the \( \epsilon \)-symmetry property

\[ \bar{\xi}_2 \xi_1 = \epsilon \bar{\xi}_1 \xi_2. \]

Note also the identity

(A.3) \[ \bar{\xi}_2 \Gamma^I \xi_1 = -\bar{\xi}_1 \Gamma^I \xi_2 \]

that follows from Eq. (A.2) in the cases of interest.

In the distinguished case of \( d = 10 \), we also encounter the volume element of the corresponding Clifford algebra \( \text{Cliff}(\mathbb{R}^{1,9}) \),

\[ \Gamma_{11} := i \Gamma^0 \cdot \Gamma^2 \cdot \ldots \cdot \Gamma^9. \]

It is readily seen to belong to the anticentre of \( \text{Cliff}(\mathbb{R}^{1,9}) \),

\[ \forall I \in \mathbb{R}^9 : \{ \Gamma_{11}, \Gamma^I \} = 0, \]

and satisfy the elementary identities

\[ \bar{\Gamma}^2_{11} = 1_{D_{1,1-1}}, \quad \bar{\Gamma}_{11} \equiv C \cdot \Gamma_{11} = -\Gamma^T_{11} \cdot C. \]

**Appendix B. Elementary properties of the Green–Schwarz \((p + 2)\)-forms**

In this section, we examine closed super-\((p + 2)\)-forms

\[ \chi^{(p+2)}(\theta, x) = d (\bar{\sigma} \Gamma I_1 I_2 \ldots I_p \sigma \wedge \theta_{I_1 I_2 \ldots I_p} (\theta, x)) + \frac{p}{2} \theta \bar{\Gamma} I_1 I_2 \ldots I_p \sigma \wedge (\sigma \wedge \Gamma^I I_1) \sigma \wedge \theta_{I_2 I_3 \ldots I_p} (\theta, x), \]

and so we may use the identity

\[ (\bar{\Gamma} I_1 I_2 \ldots I_p)_{\alpha(\beta} \Gamma^I_{\gamma\delta)} = -\Gamma^I_{\alpha(\beta} (\bar{\Gamma} I_1 I_2 \ldots I_p)_{\gamma\delta)}, \]

following directly from Eqs. (A.12) and (A.2), to rewrite the equality as

\[ \chi^{(p+2)}(\theta, x) = d (\bar{\sigma} \Gamma I_1 I_2 \ldots I_p \sigma \wedge \theta_{I_1 I_2 \ldots I_p} (\theta, x)) + \frac{p}{2} \theta \bar{\Gamma} I_1 I_2 \ldots I_p \sigma \wedge (\sigma \wedge \Gamma^I I_1) \sigma \wedge \theta_{I_2 I_3 \ldots I_p} (\theta, x) \]

\[ = d (\bar{\sigma} \Gamma I_1 I_2 \ldots I_p \sigma \wedge \theta_{I_1 I_2 \ldots I_p} (\theta, x)) + p \bar{\sigma} \Gamma I_1 I_2 \ldots I_p \sigma \wedge \theta_{I_2 I_3 \ldots I_p} (\theta, x). \]

Thus,

\[ \chi^{(p+2)}(\theta, x) = d \left( \frac{1}{p+1} \bar{\sigma} \Gamma I_1 I_2 \ldots I_p \sigma \wedge \theta_{I_1 I_2 \ldots I_p} (\theta, x) \right) + \frac{p}{p+1} d \theta_{I_1} \wedge \chi^{(p+1)} I_1 (\theta, x) \]

with

\[ \chi^{(p+1)} I_1 := \sigma \wedge \Gamma I_1 I_2 \ldots I_p \sigma \wedge \theta_{I_2 I_3 \ldots I_p}, \]

and we may next focus on the latter super-\((p + 1)\)-form. Reasoning as in the previous step, we find

\[ \chi^{(p+1)} I_1 (\theta, x) = d \left( \frac{1}{p+1} \bar{\sigma} \Gamma I_1 I_2 \ldots I_p \sigma \wedge \theta_{I_2 I_3 \ldots I_p} (\theta, x) \right) - \frac{p-1}{p} \bar{\sigma} \Gamma I_2 \sigma \wedge (\sigma \wedge \Gamma I_1 I_2 \ldots I_p \sigma) \wedge \theta_{I_3 I_4 \ldots I_p} (\theta, x) \]

\[ = d (\bar{\sigma} \Gamma I_1 I_2 \ldots I_p \sigma \wedge \theta_{I_2 I_3 \ldots I_p} (\theta, x)) - (p-1) \chi^{(p+1)} I_1 (\theta, x) + (p-1) d \theta_{I_2} \wedge \chi^{(p)} I_1 I_2 (\theta, x), \]

or

\[ \chi^{(p+1)} I_1 (\theta, x) = d \left( \frac{1}{p+1} \bar{\sigma} \Gamma I_1 I_2 \ldots I_p \sigma \wedge \theta_{I_2 I_3 \ldots I_p} (\theta, x) \right) + \frac{p-1}{p} d \theta_{I_2} \wedge \chi^{(p)} I_1 I_2 (\theta, x) \]
with

\[ \chi_{I_1I_2} \equiv \overline{\sigma} \wedge \Gamma_{I_1I_2...I_p} \sigma \wedge e^{I_3I_4...I_p}. \]

Repeating the above reduction procedure \( p \) times, we eventually arrive at the equality

\[ \chi_{I_1} \equiv \frac{d}{(p+1)} \beta_{I_1}, \]

with

(B.1) \[
\beta_{I_1}(\theta, x) = \frac{1}{p} \sum_{k=1}^{p} \bar{\theta} \Gamma_{I_1I_2...I_p} \sigma \wedge dx^{I_2} \wedge ... \wedge dx^{I_k} \wedge e^{I_{k+1}I_{k+2}...I_p}(\theta, x) \]

whence also

\[ \beta_{I_1}(\theta, x) = \frac{1}{p+1} \sum_{k=1}^{p+1} \bar{\theta} \Gamma_{I_1I_2...I_p} \sigma \wedge e^{I_3I_4...I_p}(\theta, x). \]

\[ \square \]

B.2. A proof of Proposition 4.3. Using the elementary identities

\[ R(\varepsilon, x) \perp \sigma^\alpha(\theta, x) = \varepsilon^\alpha, \quad R(\varepsilon, y) \perp e^\ell(\theta, x) = y^\ell - \overline{\sigma} \Gamma^\ell \theta, \]

we compute

\[ R(\varepsilon, y) \perp (p+2)(\theta, x) = \left( p \left( y^{I_1} - \overline{\sigma} \Gamma^{I_1} \theta \right) \left( \overline{\sigma} \wedge \Gamma_{I_1I_2...I_p} \sigma \right) \wedge e^{I_3I_4...I_p}(\theta, x) \right) + \frac{d}{(p+1)} \beta_{I_1}(\theta, x) \]

\[ = \left( \left( \Gamma_{I_1I_2...I_p} \right)_{\alpha \beta} \Gamma_{\gamma \delta} e^{I_1I_2...I_p}(\theta, x) \right) \varepsilon^\alpha \theta^\beta \gamma \wedge \sigma^\delta \]

\[ = - \left( \left( \Gamma_{I_1I_2...I_p} \right)_{\alpha \gamma} \Gamma_{\beta \delta} e^{I_1I_2...I_p}(\theta, x) \right) \varepsilon^\alpha \theta^\beta \gamma \wedge \sigma^\delta \]

\[ = 2 \left( \overline{\sigma} \Gamma_{I_1I_2...I_p} \sigma \right) \wedge \overline{\theta} \Gamma^{I_1} \sigma \]

\[ = 2d \left( \overline{\sigma} \Gamma_{I_1I_2...I_p} \theta \right) \left( \overline{\theta} \Gamma^{I_1} \sigma \right) - 2 \left( \overline{\sigma} \Gamma_{I_1I_2...I_p} \theta \right) \left( \overline{\theta} \Gamma_{I_1I_2...I_p} \sigma \right) \]

so that

\[ \overline{\sigma} \Gamma_{I_1I_2...I_p}(\theta, x) = \frac{2}{3} d \left( \overline{\sigma} \Gamma_{I_1I_2...I_p} \theta \right) \left( \overline{\theta} \Gamma^{I_1} \sigma \right) - \frac{2}{3} \left( \overline{\sigma} \Gamma_{I_1I_2...I_p} \theta \right) \left( \overline{\theta} \Gamma_{I_1I_2...I_p} \sigma \right) \]

Write (B.2)

\[ \eta_{(1)}^{I_1I_2...I_p}(\theta, x) := \left( \overline{\sigma} \Gamma_{I_1I_2...I_p} \theta \right) \left( \overline{\theta} \Gamma^{I_1} \sigma \right) + \left( \overline{\sigma} \Gamma^{I_1} \theta \right) \left( \overline{\theta} \Gamma_{I_1I_2...I_p} \sigma \right) \]

and note the identity

\[ \overline{\sigma} \wedge \Gamma^{I_2} \sigma \wedge \eta_{(1)}^{I_1I_2...I_p}(\theta, x) = - (\overline{\theta} \Gamma^{I_2} \sigma) \wedge \overline{\sigma} \Gamma_{I_1I_2...I_p}(\theta, x). \]

We now obtain

\[ \overline{\sigma} \Gamma_{I_1I_2...I_p} \wedge e^{I_3I_4...I_p}(\theta, x) = \frac{2}{3} d \eta_{(1)}^{I_1I_2...I_p} \wedge e^{I_3I_4...I_p}(\theta, x) \]

\[ = \frac{2}{3} d \left( \eta_{(1)}^{I_1I_2...I_p} \wedge e^{I_3I_4...I_p}(\theta, x) + \frac{1}{3} \left( \overline{\sigma} \wedge \Gamma^{I_2} \sigma \right) \wedge \eta_{(1)}^{I_2I_3...I_p} \wedge e^{I_4I_5...I_p}(\theta, x) \right) \]

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\[
\begin{align*}
\frac{2}{3} d(\eta l_2 l_3 \ldots l_p) (\theta, x) &= -\frac{p-1}{3} (\bar{\theta} l_3 \ldots l_p) \eta l_2 l_3 \ldots l_p (\theta, x) \\
&= \frac{2}{3} d(\eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p) (\theta, x) + \frac{2(p-1)}{3} dx l_2 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x) \\
&\quad - \frac{2(p-1)}{3} \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x),
\end{align*}
\]
and therefore
\[
\begin{align*}
\eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x) &= \frac{2}{3} d(\eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p) (\theta, x) \\
&\quad + \frac{2(p-1)}{3} dx l_2 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x).
\end{align*}
\]

Continuing the reduction as in the previous section, we establish
\[
\begin{align*}
dx l_2 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x) &= \frac{2}{3} dx l_2 \wedge d(\eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p) (\theta, x) \\
&= -\frac{2}{3} d[dx l_2 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x)] + \frac{2}{3} dx l_2 \wedge (\bar{\theta} l_3 \ldots l_p) \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x) \\
&= -\frac{2}{3} d[dx l_2 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x)] \\
&\quad + \frac{2}{3} dx l_2 \wedge (\bar{\theta} l_3 \ldots l_p) \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x) \\
&= -\frac{2}{3} d[dx l_2 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x)] \\
&\quad + \frac{2(p-2)}{3} dx l_2 \wedge dx l_3 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x) \\
&\quad - \frac{2(p-2)}{3} dx l_2 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x),
\end{align*}
\]
whence also
\[
\begin{align*}
dx l_2 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x) &= -\frac{2}{3} d[dx l_2 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x)] \\
&\quad + \frac{2(p-2)}{3} dx l_2 \wedge dx l_3 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x),
\end{align*}
\]
which yields
\[
\begin{align*}
\mathcal{R}(\varepsilon, y) &\cap \chi (\theta, x) = d[p y l_2 \beta I (\theta, x) + 2(\bar{\theta} l_2 \ldots l_p) \theta] e l_2 \ldots l_p (\theta, x) \\
&= \frac{2p}{2p+1} \eta l_2 l_3 \ldots l_p \wedge e l_2 \ldots l_p (\theta, x) \\
&\quad - \frac{2p(p-1)}{(2p+1)(2p-1)} dx l_2 \wedge \eta l_2 l_3 \ldots l_p \wedge e l_3 l_4 \ldots l_p (\theta, x),
\end{align*}
\]
and so, after \( p \) steps, we arrive at the equality
\[
\begin{align*}
\mathcal{R}(\varepsilon, y) &\cap \chi (\theta, x) = \frac{-d \theta \beta I (\theta, x)}{(p+2)},
\end{align*}
\]
with
\[
\frac{-p y l_2 \beta I (\theta, x) - 2(\bar{\theta} l_2 \ldots l_p) \theta] e l_2 \ldots l_p (\theta, x)}{(p+2)} + \sum_{k=1}^{p} \frac{2p(2p+1-2k)!}{(2p+1)!} \eta l_2 l_3 \ldots l_p \wedge dx l_2 \wedge dx l_3 \wedge \ldots \wedge dx l_k \wedge e l_{k+1} \ldots l_p (\theta, x).
\]

\[\square\]

**Appendix C. The Lie-superalgebra cohomology and its Chevalley–Eilenberg model**

In this appendix, we collect basic facts concerning the Lie-superalgebra cohomology that prove useful in algebraic description of supertargets and their differential geometry. In our exposition and discussion, we adopt the conventions of the original articles: [BK70] by Berezin and Kač, and [Lei73] by Leïtes.

We begin with the basic

**Definition C.1. A Lie superalgebra** (to be abbreviated as LSA) over field \( \mathbb{K} \) is a pair \( (\mathfrak{g}, [\cdot, \cdot]) \) composed of a \( \mathbb{K} \)-linear space \( \mathfrak{g} \) endowed with a \( \mathbb{Z}_2 \)-grading \( \gamma: \mathfrak{g} \rightarrow \mathbb{Z}_2 \) that induces a decomposition \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) into a direct sum of homogeneous components, \( \gamma|_{\mathfrak{g}_0} = 0 \), and of a **Lie superbracket** (also termed a **supercommutator**)

\[
[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}: (X_1, X_2) \rightarrow [X_1, X_2] = -(-1)^{\gamma x_1 \gamma x_2} [X_2, X_1],
\]

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that preserves the grading,
\[ [X, Y] = \tilde{X} + \tilde{Y}, \]
and has a vanishing super-Jacobiator (evaluated on arbitrary homogeneous elements \( X_1, X_2, X_3 \in \mathfrak{g} \))

\[ \text{sJac}_g(X_1, X_2, X_3) \]

\[ := (-1)^{\tilde{X}_1 \tilde{X}_3} [[X_1, X_2], X_3] + (-1)^{\tilde{X}_2 \tilde{X}_1} [[X_3, X_1], X_2] + (-1)^{\tilde{X}_2 \tilde{X}_3} [[X_2, X_3], X_1] = 0. \]

Given two LSAs \( (\mathfrak{g}_i, [-, -]), i \in \{1, 2\} \), an **LSA morphism** between them is a \( \mathbb{K} \)-linear map \( \chi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \) that preserves the \( \mathbb{Z}_2 \)-grading, \( \tilde{\chi} \circ \chi = \tilde{\chi} \), and the Lie superbracket,

\[ \chi \circ [-, -]_1 = [-, -]_2 \circ (\chi \times \chi). \]

A **left \( \mathfrak{g} \)-module** is a pair \((V, \ell)\) composed of a \( \mathbb{K} \)-linear superspace with a decomposition \( V = V_0 \oplus V_1 \) into homogeneous components induced by the \( \mathbb{Z}_2 \)-grading, and endowed with a left \( \mathfrak{g} \)-action \( \ell : \mathfrak{g} \times V \rightarrow V : (X, v) \mapsto X \triangleright v \) consistent with the \( \mathbb{Z}_2 \)-gradings \( \tilde{X} \triangleright v = \tilde{X} + \tilde{v} \), and such that for any two homogeneous elements \( X_1, X_2 \in \mathfrak{g} \) and \( v \in V \),

\[ [X_1, X_2] \triangleright v = X_1 \triangleright (X_2 \triangleright v) = (X_1 \triangleright v). \]

The object of our main interest is introduced in

**Definition C.2.** Let \((\mathfrak{g}, [-, -]_{\mathfrak{g}})\) and \((\mathfrak{a}, [-, -]_{\mathfrak{a}})\) be two LSAs over field \( \mathbb{K} \). A **(super)central extension of \( \mathfrak{g} \) by \( \mathfrak{a} \)** is an LSA \((\tilde{\mathfrak{g}}, [-, -]_{\tilde{\mathfrak{g}}})\) over \( \mathbb{K} \) that determines a short exact sequence of LSAs

\[ 0 \longrightarrow \mathfrak{a} \stackrel{j_{\mathfrak{a}}}{\longrightarrow} \tilde{\mathfrak{g}} \stackrel{\pi_{\tilde{\mathfrak{g}}}}{\longrightarrow} \mathfrak{g} \longrightarrow 0, \]

written in terms of an LSA monomorphism \( j_{\mathfrak{a}} \) and of an LSA epimorphism \( \pi_{\tilde{\mathfrak{g}}} \), and such that \( j_{\mathfrak{a}}(\mathfrak{a}) \subseteq \mathfrak{z}(\tilde{\mathfrak{g}}) \) (the centre of \( \tilde{\mathfrak{g}} \)). Hence, in particular, \( \mathfrak{a} \) is necessarily supercommutative, that is \([[-, -]_{\mathfrak{a}} \equiv 0]. \)

Whenever \( \pi_{\tilde{\mathfrak{g}}} \) admits a **section** that is an LSA homomorphism, i.e., there exists

\[ \sigma \in \text{Hom}_{\text{Lie}}(\mathfrak{g}, \tilde{\mathfrak{g}}), \quad \pi_{\tilde{\mathfrak{g}}} \circ \sigma = \text{id}_{\mathfrak{g}}, \]

the (super)central extension is said to **split**.

An equivalence of (super)central extensions \( \tilde{\mathfrak{g}}_{A}, A \in \{1, 2\} \) of \( \mathfrak{g} \) by \( \mathfrak{a} \) is represented by a commutative diagram

\[ \begin{array}{ccc}
0 & \longrightarrow & \mathfrak{a} \\
\downarrow & & \downarrow z \\
\tilde{\mathfrak{g}}_1 & \longrightarrow & \mathfrak{g} \\
\downarrow & & \downarrow \pi_{\tilde{\mathfrak{g}}_2} \\
\mathfrak{g}_2 & \longrightarrow & 0,
\end{array} \]

in which the vertical arrow is an LSA isomorphism.

In close analogy with the purely Graßmann-even case, equivalence classes of (super)central extensions of LSAs are neatly captured by the cohomology of the latter. The relevant cohomology is specified in

**Definition C.3.** Let \((\mathfrak{g}, [-, -])\) be an LSA over field \( \mathbb{K} \) and let \((V, \ell)\) be a \( \mathfrak{g} \)-module. A \( p \)-**cochain on \( \mathfrak{g} \) with values in \( V \)** (also termed a \( p \)-**form on \( \mathfrak{g} \) with values in \( V \)) is a \( p \)-linear map \( \varphi^{(p)} : \mathfrak{g}^p \rightarrow V \) that is totally super-skew-symmetric, i.e., for any homogeneous elements \( X_i \in \mathfrak{g} \), \( i \in \{1, p\} \), it satisfies

\[ \forall \varphi^{(p)}_{(p)} : \varphi^{(p)}(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, X_i, X_{i+2}, X_{i+3}, \ldots, X_p) = (-1)^{\tilde{X}_i \tilde{X}_{i+1}} \varphi^{(p)}(X_1, X_2, \ldots, X_p). \]

Such maps form a \( \mathbb{Z}_2 \)-graded **group of \( p \)-cochains on \( \mathfrak{g} \) with values in \( V \)**, denoted by

\[ C^p(\mathfrak{g}, V) = C^0_0(\mathfrak{g}, V) \oplus C^p_1(\mathfrak{g}, V), \]

with \( \varphi(X_1, X_2, \ldots, X_p) \in V_{\sum_{i=1}^p \tilde{X}_i + n} \) for \( \varphi \in C^p_0(\mathfrak{g}, V) \), composed of **even** \( (n = 0) \) and **odd** \( (n = 1) \) \( p \)-cochains.
The family of these groups indexed by $p \in \mathbb{N}$ forms a semi-bounded complex
\[ C^*(g, V) : C^0(g, V) \xrightarrow{\delta^0} C^1(g, V) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{p-1}} C^p(g, V) \xrightarrow{\delta^p} \cdots \]
with the coboundary operators $\delta^p : C^p_n(g, V) \to C^{p+1}_n(g, V)$ determined by the formulæ (written for homogeneous elements $X, X_i \in g$, $i \in 1, p + 1$ and $\varphi \in C^p(g, V)$)
\[
\left(\delta^p_{(0)} \varphi\right)(X) := (-1)^{\frac{p(p-1)}{2}} X \triangleright \varphi,
\]
\[
\left(\delta^p_{(p)} \varphi\right)(X_1, X_2, \ldots, X_{p+1}) := \sum_{i=1}^{p+1} (-1)^i X_i \triangleright \varphi(X_1, X_2, \ldots, \hat{X_i}, \ldots, X_{p+1}) + \sum_{1 \leq i < j \leq p+1} (-1)^{S(X_i) + S(X_j) + X_i X_j \varphi} \varphi([X_i, X_j], X_1, X_2, \ldots, \hat{X_i}, \hat{X_j}, \ldots, X_{p+1}),
\]
where
\[
(C.3) \quad S(X_i) := \sum_{j=1}^{i-1} X_j + i - 1.
\]

We distinguish the group of $p$-cocycles
\[ Z^p(g, V) := \ker \delta^p_{(0)}, \]
and the group of $p$-coboundaries
\[ B^p(g, V) := \im \delta^{p-1}_{(0)} . \]

The $\mathbb{Z}_2$-graded homology groups of the complex $(C^*, g, V, \delta^*)$ are called the cohomology groups of $g$ with values in $V$ and denoted by
\[ H^p(g, V) := H^p_0(g, V) \oplus H^p_1(g, V), \quad H^p_n(g, V) := \frac{\ker \delta^p_{(0)} | C^p_n(g, V)}{\im \delta^{p-1}_{(0)} | C^{p-1}_n(g, V)} . \]

Let us write out – with view to subsequent applications – the relations defining a 2-cocycle and 2-coboundary on an LSA $g$ with values in a trivial $g$-module $a$. Thus, for any homogeneous elements $X_1, X_2, X_3 \in g$, a 2-cocycle $\Theta \in Z^2(g, a)$ satisfies
\[ (-1)^{X_1 X_3} \Theta([X_1, X_2], X_3) + (-1)^{X_2 X_3} \Theta([X_2, X_1], X_2) + (-1)^{X_1 X_2} \Theta([X_2, X_3], X_1) = 0, \]
and a 2-coboundary obtained from a 1-cochain $\mu \in C^1(g, a)$ evaluates as
\[ (\delta^1_{(1)} \mu)(X_1, X_2) = -\mu([X_1, X_2]). \]

C.1. The algebraic meaning of $H^2_0(g, a)$. We shall now establish a natural correspondence between classes in $H^2(g, a)$ and equivalence classes of (super)central extensions of $g$ by a supercommutative LSA $a$ with a trivial $g$-action. We begin our discussion with

**Proposition C.4.** Let $(g, [\cdot, \cdot]_g)$ be an LSA, and let $a$ be a supercommutative LSA. The equivalence class of (super)central extensions $(\bar{g}, [\cdot, \cdot]_{\bar{g}})$ of $g$ by $a$ canonically determines a class in $H^2_0(g, a)$. This class vanishes iff the short exact sequence determined by the extensions splits.

**Proof.** The short exact sequence $(C.2)$ implies the existence of a $\mathbb{K}$-linear map $\sigma : g \to \bar{g}$ which preserves the $\mathbb{Z}_2$-grading and satisfies the relation $\pi_\bar{g} \circ \sigma = \id_g$, whence the (canonical) isomorphism of $\mathbb{K}$-linear superspaces
\[ \bar{\tau} : \bar{g} \xrightarrow{\cong} a \oplus g : \bar{X} \mapsto j^{-1}_a \left(\bar{X} - \sigma \circ \pi_\bar{g}(\bar{X})\right) \oplus \pi_\bar{g}(\bar{X}). \]

Indeed, the above map is well-defined as $\bar{X} - \sigma \circ \pi_\bar{g}(\bar{X}) \in \ker \pi_\bar{g} = \im j_a$ and $j_a$ is an isomorphism onto its image. Its inverse is explicitly given by
\[ \bar{\tau}^{-1} : a \oplus g \to \bar{g} : A \oplus X \mapsto j_a(A) + \sigma(X). \]

We may, subsequently, promote $\bar{\tau}$ to the rank of an LSA isomorphism by inducing a Lie superbracket on the vector superspace $a \oplus g$ from those on $\bar{g}$ and $g$ as per
\[ [A_1 \oplus X_1, A_2 \oplus X_2]_{a \oplus g} := \bar{\tau}([\bar{\tau}^{-1}(A_1 \oplus X_1), \bar{\tau}^{-1}(A_2 \oplus X_2)])_{\bar{g}} \equiv \tau([\sigma(X_1), \sigma(X_2)])_{g}, \]

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\[ \Theta_{\sigma}(X_2, X_1) = (-1)^{\overline{x}_1 \overline{x}_2} \Theta_{\sigma}(X_1, X_2), \]

and so it is an even 2-cochain on \( \mathfrak{g} \) with values in \( \mathfrak{a} \), the latter being understood as a trivial \( \mathfrak{g} \)-module. Its coboundary reads

\[
\delta^{(2)}_{\mathfrak{g}}(\Theta_{\sigma}(X_1, X_2, X_3)) = (-1)^{\overline{x}_3 \overline{x}_4 + 1} \left[ (-1)^{\overline{x}_1 \overline{x}_2} \Theta_{\sigma}(X_1, X_2, X_3) + (-1)^{\overline{x}_3 \overline{x}_2} \Theta_{\sigma}(X_1, X_2) \right]
+ (-1)^{\overline{x}_2 \overline{x}_1} \Theta_{\sigma}(X_1, X_2, X_3) = (-1)^{\overline{x}_3 \overline{x}_4} \Theta_{\sigma}(X_1, X_2, X_3) + (-1)^{\overline{x}_2 \overline{x}_1} [\sigma([X_3, X_1]_{\mathfrak{g}}), \sigma(X_2)]_{\mathfrak{g}}
+ (-1)^{\overline{x}_3 \overline{x}_2} [\sigma([X_2, X_3]_{\mathfrak{g}}), \sigma(X_1)]_{\mathfrak{g}} = 0.
\]

Let us, next, examine how the 2-cocycle changes when we pass to an equivalent (super)central extension. We now have two LSA monomorphisms \( \pi_{\mathfrak{g}, A} : \mathfrak{g} \rightarrow \mathfrak{g}_{\mathfrak{a}} \), \( A \in \{1, 2\} \) and two LSA epimorphisms \( \pi_{\mathfrak{g}, A} : \mathfrak{g}_{\mathfrak{a}} \rightarrow \mathfrak{g} \) with the corresponding \( \mathbb{K} \)-linear-superspace sections \( \sigma_{\mathfrak{a}, A} : \mathfrak{g} \rightarrow \mathfrak{g}_{\mathfrak{a}} \). Taking into account the commutativity of the diagram

\[
\begin{array}{c}
\mathfrak{g} \\
\pi_{\mathfrak{g}, 1} \downarrow \quad \pi_{\mathfrak{g}, 2} \\
\mathfrak{g}_{\mathfrak{a}} \\
\downarrow \quad \downarrow \\
\mathfrak{g} \\
\end{array}
\]

alongside the identity

\[
\pi_{\mathfrak{g}, 1} \circ (\varepsilon^{-1} \circ \sigma_2 - \sigma_1) = \pi_{\mathfrak{g}, 2} \circ \sigma_2 - \pi_{\mathfrak{g}, 1} \circ \sigma_1 = 0,
\]

the latter implying the existence of a \( \mathbb{K} \)-linear superspace homomorphism \( \mu_{\varepsilon} : \mathfrak{g} \rightarrow \mathfrak{a} \) such that

\[
\varepsilon^{-1} \circ \sigma_2 - \sigma_1 = j_{\mathfrak{a}, 1} \circ \mu_{\varepsilon},
\]

we readily establish, for any homogeneous elements \( X_1, X_2 \in \mathfrak{g} \),

\[
j_{\mathfrak{a}, 1} \circ (\Theta_{\sigma_2} - \Theta_{\sigma_1})(X_1, X_2) = (\varepsilon^{-1} \circ j_{\mathfrak{a}, 2} \circ \Theta_{\sigma_2} - j_{\mathfrak{a}, 1} \circ \Theta_{\sigma_1})(X_1, X_2) = \left[ \varepsilon^{-1} \circ \sigma_2(X_1), \varepsilon^{-1} \circ \sigma_2(X_2) \right]_{\mathfrak{g}} - \left[ \sigma_1(X_1), \sigma_1(X_2) \right]_{\mathfrak{g}} - j_{\mathfrak{a}, 1} \circ \mu_{\varepsilon}([X_1, X_2]_{\mathfrak{g}})
\]

so that, altogether,

\[
\Theta_{\sigma_2} - \Theta_{\sigma_1} = \delta^{(1)}_{\mathfrak{g}} \mu_{\varepsilon} \quad \text{i.e.} \quad [\Theta_{\sigma_2}]_{\mathfrak{g}} = [\Theta_{\sigma_1}]_{\mathfrak{g}}.
\]
Finally, we prove the last statement of the proposition. The vanishing of the (class of) 2-cocycle $\Theta_\sigma$ for $\sigma$ an LSA section is obvious, hence it remains to demonstrate that, conversely, the cohomological triviality of $\Theta_\sigma$ implies the existence of an LSA section. The statement of triviality of the 2-cocycle $\Theta_\sigma$ rewrites neatly as

$$\left[\sigma(X_1),\sigma(X_2)\right]_\Theta = \sigma_\mu([X_1,X_2]_\Theta), \quad \sigma_\mu := \sigma - j_\sigma \circ \mu \in \text{Hom}_K(\mathfrak{g},\mathfrak{g}).$$

In view of the supercommutativity of $j_\sigma(a)$, this yields

$$[\sigma_\mu(X_1),\sigma_\mu(X_2)]_\Theta = \sigma_\mu([X_1,X_2]_\Theta),$$

and so $\sigma_\mu$ can be promoted to the rank of an LSA homomorphism. Since, furthermore, it satisfies the relation

$$\pi_\mathfrak{g} \circ \sigma_\mu = \pi_\mathfrak{g} \circ \sigma - \pi_\mathfrak{g} \circ j_\sigma \circ \mu = \pi_\mathfrak{g} \circ \sigma = \varepsilon_\mathfrak{g},$$

we can identify it as the sought-after LSA section of $\pi_\mathfrak{g}$. \hfill \Box

From the point of view of physical applications in the setting of the GS super-$\sigma$-model, it is of utmost significance that the assignment of classes in $H_2^\mathbb{Z}(\mathfrak{g},\mathfrak{a})$ to (super)central extensions of $\mathfrak{g}$ by a supercommutative LSA $\mathfrak{a}$ detailed above may, in fact, be inverted, as stated in

**Proposition C.5.** Let $(\mathfrak{g},\cdot,\cdot)_\mathfrak{g}$ be an LSA, and let $\mathfrak{a}$ be a supercommutative LSA, regarded as a trivial $\mathfrak{g}$-module. A class in $H_2^\mathbb{Z}(\mathfrak{g},\mathfrak{a})$ canonically induces an equivalence class of (super)central extensions $(\tilde{\mathfrak{g}},\cdot,\cdot)_{\tilde{\mathfrak{g}}}$ of $\mathfrak{g}$ by $\mathfrak{a}$. The extensions split iff the former class vanishes.

**Proof.** Given an arbitrary even 2-cocycle $\Theta \in Z^2_0(\mathfrak{g},\mathfrak{a})$, endow the $\mathbb{Z}_2$-graded space $\mathfrak{a} \oplus \mathfrak{g} =: \tilde{\mathfrak{g}}$ with a $\mathbb{Z}_2$-grading induced from that of its direct summands with a manifestly super-skewsymmetric 2-linear map

$$[\cdot,\cdot]_{\Theta} : \tilde{\mathfrak{g}}^\otimes 2 \longrightarrow \tilde{\mathfrak{g}} : (A_1 \oplus X_1, A_2 \oplus X_2) \longmapsto \Theta(X_1,X_2) \oplus [X_1,X_2]_\Theta.$$

It is ready checked that the map is a Lie superbracket,

$$\text{Jac}_{\tilde{\mathfrak{g}}}(A_1 \oplus X_1, A_2 \oplus X_2, A_3 \oplus X_3) = (-1)^{X_1X_2+1} \Theta(X_1,X_2,X_3) \oplus \text{Jac}_{\mathfrak{g}}(X_1,X_2,X_3) = 0,$

and so $(\tilde{\mathfrak{g}},\cdot,\cdot)_{\tilde{\mathfrak{g}}}$ is an LSA.

The LSA $\mathfrak{a}$ being supercommutative, the standard injection $j_\mathfrak{a} : \mathfrak{a} \hookrightarrow \tilde{\mathfrak{g}} : A \longmapsto A \oplus 0$ is an LSA monomorphism. The $\mathbb{K}$-linear canonical projection $\pi_\mathfrak{g} : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} : A \oplus X \longmapsto X$, on the other hand, is readily seen to be an LSA epimorphism with the obvious property $\ker \pi_\mathfrak{g} = \text{im} j_\mathfrak{a}$, and so we obtain a short exact sequence of LSAs

$$0 \longrightarrow \mathfrak{a} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

that identifies $\tilde{\mathfrak{g}}$ as a (super)central extension of $\mathfrak{g}$ by $\mathfrak{a}$.

For cohomologous 2-cocycles, $\Theta_2 = \Theta_1 + \delta_\mathfrak{g}^{(1)} \mu$, $\mu \in C_0^1(\mathfrak{g},\mathfrak{a})$, the scheme laid out above produces two Lie superbrackets on the (super)central extension $\tilde{\mathfrak{g}} = \mathfrak{a} \oplus \mathfrak{g}$ of $\mathfrak{g}$ by $\mathfrak{a}$, and the $\mathbb{K}$-linear superspace automorphism

$$\varepsilon_\mu : \tilde{\mathfrak{g}} \longrightarrow \tilde{\mathfrak{g}} : A \oplus X \longmapsto (A - \mu(X)) \oplus X$$

is easily verified to isomorphically map $(\tilde{\mathfrak{g}},\cdot,\cdot)_{\Theta_1}$ onto $(\tilde{\mathfrak{g}},\cdot,\cdot)_{\Theta_2}$,

$$[\varepsilon_\mu(A_1 \oplus X_1),\varepsilon_\mu(A_2 \oplus X_2)]_{\Theta_2} = \Theta_2(X_1,X_2) \oplus [X_1,X_2]_\Theta = [\Theta_1(X_1,X_2) - \mu([X_1,X_2]_\Theta)] \oplus [X_1,X_2]_\Theta$$

$$\equiv \varepsilon_\mu([A_1 \oplus X_1, A_2 \oplus X_2]_{\Theta_1}).$$

Whenever $\Theta$ is a 2-coboundary, $\Theta = \delta_\mathfrak{g}^{(1)} \mu$, $\mu \in C_0^1(\mathfrak{g},\mathfrak{a})$, we may inject $\mathfrak{g}$ into $\tilde{\mathfrak{g}}$ by means of a $\mathbb{K}$-linear map

$$\sigma_\mu : \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}} : X \longmapsto -\mu(X) \oplus X$$

that manifestly defines a $\mathbb{K}$-linear superspace section of $\pi_\mathfrak{g}$ and lifts to an LSA monomorphism,

$$[\sigma_\mu(X_1),\sigma_\mu(X_2)]_{\Theta} = [-\mu(X_1) \oplus X_1, -\mu(X_2) \oplus X_2]_{\Theta} = \Theta(X_1,X_2) \oplus [X_1,X_2]_\Theta$$

$$= -\mu([X_1,X_2]_\Theta) \oplus [X_1,X_2]_\Theta = \sigma_\mu([X_1,X_2]_\Theta).$$

In consequence, the associated short exact sequence of LSAs splits.
Conversely, an arbitrary LSA section of $\pi_{\tilde{g}}$ is necessarily of the form

$$\sigma_{\mu} : g \rightarrow \tilde{g} : X \mapsto -\mu(X) \oplus X$$

for some $\mu \in \text{Hom}_{\mathbb{K}}(g,a)$ that preserves the $\mathbb{Z}_2$-grading, and such that

$$\Theta(X_1, X_2) \oplus [X_1, X_2]_g = \left[\sigma_{\mu}(X_1), \sigma_{\mu}(X_2)\right] \oplus \frac{1}{2} \sigma_{\mu}([X_1, X_2]_g) = -\mu([X_1, X_2]_g) \oplus [X_1, X_2]_g,$$

so that $\Theta = \delta_{\tilde{g}}^{(1)} \mu$, as claimed. □

Let us conclude the purely algebraic part of our exposition with the following simple reinterpretation that proves useful shortly.

**Remark C.6.** The existence of an extension of $g$ by $a$ determined by $\Theta$ is tantamount to a trivialisation of the pullback 2-cocycle

$$\tilde{\Theta} := \pi^*_{\tilde{g}} \Theta : \tilde{g}^2 \rightarrow a : (A_1 \oplus X_1, A_2 \oplus X_2) \mapsto \Theta(X_1, X_2)$$

given by

$$\tilde{\Theta} = \delta_{\tilde{g}}^{(1)} \tilde{\mu}, \quad \tilde{\mu} := -\pi_a : \tilde{g} \rightarrow a : A \oplus X \mapsto A.$$

**C.2. The supergeometry of the Chevalley–Eilenberg Lie-superalgebra cohomology.** The LSA cohomology introduced above after Ref. [Lei75] admits various explicit realisations. From the point of view of applications of immediate interest, both physical and geometric, the Chevalley–Eilenberg model of Ref. [CE48] seems most convenient. It is based on the elementary observation: The exterior (super)derivative of a left-invariant super-$p$-form $\omega$ on a Lie (super)group $G$ evaluates on a collection $(L_1, L_2, \ldots, L_{p+1})$ of Grassmann-homogeneous left-invariant vector fields on that (super)group as

$$d^{(p)} \omega(L_1, L_2, \ldots, L_{p+1}) = \sum_{1 \leq i < j \leq p+1} \left(-1\right)^{S(L_i) + S(L_j) + S(L_i) + \sum_{L_j} \omega(\left[L_i, L_j\right], L_1, L_p, \ldots, L_{p+1}),$$

with $S(L_i)$ defined as in Eq. (C.3), which in conjunction with the isomorphism of LSAs (over $\mathbb{R}$) between the LSA $g$ of the Lie supergroup $G$ and the LSA $\mathfrak{X}(G)^L$ of left-invariant vector fields on $G$ (at the group unit),

$$(g, [\cdot, \cdot]_g) \cong (\mathfrak{X}(G)^L, [\cdot, \cdot]),$$

leads to the fundamental

**Theorem C.7.** Let $G$ be a Lie supergroup, and let $(g, [\cdot, \cdot])$ be its LSA (over $\mathbb{R}$), which we take to act trivially on $\mathbb{R}$. Denote by $\Omega^{(p)}(G)^L$ the $\mathbb{R}$-linear superspace of left-invariant $p$-forms on $G$, and by $d^{(p)}$ the restriction of de Rham differential to $\Omega^{(p)}(G)^L$. There exists a canonical bijective cochain map

$$\gamma : \left(C^*(g, \mathbb{R}), \delta_{\tilde{g}}^{(\bullet)}\right) \rightarrow \left(\Omega^*(G, \mathbb{R})^L, d^{(\bullet)}\right)$$

that induces an isomorphism in cohomology,

$$[\gamma] : H^*(g, \mathbb{R}) \xrightarrow{\cong} H^*_{\text{dR}}(G, \mathbb{R})^L \equiv \text{CaE}^*(G),$$

the latter being termed the **Cartan–Eilenberg cohomology** of $G$.

The general theory leads to a correspondence between left-invariant de Rham super-2-cocycles on a Lie supergroup and (super)central extensions of that supergroup on which the pullbacks of those super-2-cocycles along the canonical projection can be trivialised in the Cartan–Eilenberg invariant cohomology. This correspondence is used amply in the construction of (super)geometric realisations of the Green–Schwarz $(p+2)$-forms over $\text{sMink}^{1,d|N\Pi_{d-1}}$, proposed in Section 2.32

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32 That is to say, the pullbacks admit smooth left-invariant primitives on the extended Lie supergroup.
We compute
\[-30 \Delta_{\alpha \beta}(\theta, x, \zeta, \psi, v) = (\Gamma_{\alpha \beta}^{I} \sigma_{\gamma}^{I}) + 4 \Gamma_{\alpha \gamma}^{I} \sigma_{\gamma}^{I} + 4 \Gamma_{\beta \gamma}^{I} \sigma_{\gamma}^{I})(\theta, x, \zeta, \psi) \wedge \sigma^{I}(\theta)\]
\[-2 \Gamma_{\alpha \beta}^{I} e_{(2)}^{I}(\theta, x, \zeta) \wedge e^{I}(\theta, x) - 2 \Gamma_{IJ, \alpha \beta}^{I} (e^{I} \wedge e^{J})(\theta, x)\]
\[= d\left[-(\Gamma_{\alpha \beta}^{I} \sigma_{\gamma}^{I} + 4 \Gamma_{\alpha \gamma}^{I} \sigma_{\gamma}^{I} + 4 \Gamma_{\beta \gamma}^{I} \sigma_{\gamma}^{I})(\theta, x, \zeta, \psi) \theta^{I} - \left[\Gamma_{\alpha \beta}^{I} (\Gamma_{\gamma}^{J} e_{\gamma}^{J}(\theta, x, \zeta) + \Gamma_{IJ, \gamma} e_{\gamma}^{J}(\theta, x))\right] \wedge \sigma^{I}(\theta) \theta^{I} \right.\]
\[+ 4 \Gamma_{\alpha \gamma}^{I} (\Gamma_{\beta \gamma}^{J} e_{\gamma}^{J}(\theta, x, \zeta) + \Gamma_{IJ, \beta \gamma} e_{\gamma}^{J}(\theta, x)) + 4 \Gamma_{\beta \gamma}^{I} (\Gamma_{\alpha \delta}^{J} e_{\delta}^{J}(\theta, x, \zeta) + \Gamma_{IJ, \alpha \delta} e_{\delta}^{J}(\theta, x)) \left.\wedge \sigma^{I}(\theta) \theta^{I}\right]\]
\[-2 \Gamma_{\alpha \beta}^{I} e_{(2)}^{I}(\theta, x, \zeta) \wedge d x^{J} + \Gamma_{\alpha \beta}^{I} \Gamma_{\gamma}^{J} e_{\gamma}^{J}(\theta, x, \zeta) \wedge \sigma^{I}(\theta) \theta^{I} - 2 \Gamma_{IJ, \alpha \beta}^{I} d x^{J} \wedge d x^{J}\]
\[+ 2 \Gamma_{IJ, \alpha \beta}^{I} \Gamma_{\gamma}^{J} \sigma^{I}(\theta) \theta^{I} \wedge d x^{J} - \frac{1}{2} \Gamma_{IJ, \alpha \beta}^{I} \sigma^{I}(\theta) \theta^{I} \wedge \sigma^{I}(\theta)\]
\[= d\left[-(\Gamma_{\alpha \beta}^{I} \sigma_{\gamma}^{I} + 4 \Gamma_{\alpha \gamma}^{I} \sigma_{\gamma}^{I} + 4 \Gamma_{\beta \gamma}^{I} \sigma_{\gamma}^{I})(\theta, x, \zeta, \psi) \theta^{I} + 2 \Gamma_{\alpha \beta}^{I} e_{(2)}^{I}(\theta, x, \zeta) x^{I}\right.\]
\[+ 2 \Gamma_{\alpha \beta}^{I} \Gamma_{\gamma}^{J} e_{\gamma}^{J}(\theta, x, \zeta) + \Gamma_{IJ, \alpha \beta}^{I} \Gamma_{\gamma}^{J} e_{\gamma}^{J}(\theta, x))\]
\[+ 2 \Gamma_{IJ, \alpha \beta}^{I} \Gamma_{\gamma}^{J} \sigma^{I}(\theta) \theta^{I} \wedge \sigma^{I}(\theta)\]
\[+ \frac{1}{2} \Gamma_{IJ, \alpha \beta}^{I} \sigma^{I}(\theta) \theta^{I} \wedge \sigma^{I}(\theta)\].

The last two lines of the above expression define – by construction – a closed super-1-form
\[\eta_{\alpha \beta}(\theta) = (2 \Gamma_{\alpha \beta}^{I} \Gamma_{\gamma}^{J} \Gamma_{IJ, \alpha \beta}^{I} + (\Gamma_{\alpha \gamma}^{I} \Gamma_{\beta \gamma}^{J} + \Gamma_{\beta \gamma}^{I} \Gamma_{\alpha \delta}^{J} \Gamma_{IJ, \alpha \delta}^{I}) e_{\gamma}^{J}(\theta, x)) \theta^{\gamma}\]
\[-2 \Gamma_{IJ, \alpha \beta}^{I} x^{I} \wedge \Gamma_{\gamma}^{J} \sigma^{I}(\theta) \theta^{I} \wedge \sigma^{I}(\theta)\]
\[+ \frac{1}{2} \Gamma_{IJ, \alpha \beta}^{I} \sigma^{I}(\theta) \theta^{I} \wedge \sigma^{I}(\theta)\],

and we derive its global primitive using the formerly advertised homotopy formula, cp. Eq. (2.27).

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