TORIC DEGENERATIONS OF TORIC VARIETIES AND TROPICAL CURVES

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Abstract. We show that the counting of rational curves on a complete toric variety that are in general position to the toric prime divisors coincides with the counting of certain tropical curves. The proof is algebraic-geometric and relies on degeneration techniques and log deformation theory.

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Introduction.

The advent of mirror symmetry has rekindled the interest in the classical enumerative question of counting algebraic curves on a smooth projective variety. In fact, the unexpected relation of this problem with period computations for the mirror family [COGP] really caused the great mathematical interest in mirror symmetry. While meanwhile verified mathematically by direct computation [Gi] (see also [LLY]) no geometric explanation for the validity of these computations have been found yet.

In connection with the mirror symmetry program with M. Gross [GrSi1, GrSi2], the second author has long suspected that this counting problem should have a purely combinatorial counterpart in integral affine geometry (cf. also [PaOh], [KoSa], and the work on Gromov-Witten invariants for degenerations [LiARu, JoPa, LiJ2, Si]). Moreover, the Legendre dual interpretation of the combinatorial counting problem should be related to period

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computations on the mirror, thus eventually giving a satisfactory explanation for the mirror computation in \[\text{COGP}\].

Recently G. Mikhalkin used certain piecewise linear, one-dimensional objects in \(\mathbb{Q}^d\), called tropical curves, to give a count of algebraic curves on toric surfaces \([\text{Mi}]\). As tropical curves make perfectly sense in the integral affine context, it is then natural to look at the counting of tropical curves from the degeneration point of view of \([\text{GrSi2}]\). The pleasant surprise was that not only do tropical curves come up very naturally as dual intersection graphs of certain transverse curves on the central fiber, but also that our logarithmic techniques give more transparent and robust proofs in the toric case. In particular, as logarithmic deformation theory replaces the hypersurface method of patchworking used in \([\text{Mi}]\), our results are not limited to two dimensions. We also believe that this technical robustness will be essential in generalizations to the Calabi-Yau case. The purpose of this paper is thus not only a generalization of the results of \([\text{Mi}]\) to arbitrary dimensions, but also to lay the foundations for a treatment of the Calabi-Yau case in the framework of \([\text{GrSi2}]\).

The main result is Theorem 8.3. It says how exactly tropical curves count algebraic curves on an arbitrary complete toric variety. The formulation of this theorem involves some concepts introduced in the first three sections. We therefore devote only Section 8 to statement and discussion of this theorem. This section also ties together the various strands of proof given in the other sections.

During the preparation of this paper we learned that G. Mikhalkin has also announced a generalization of his work to higher dimensions, using symplectic techniques. We apologize for any duplication of results. We also thank him for pointing out an elucidating example concerning the multiplicity definition for tropical curves in higher dimensions, which had not been worked out properly by us at that time.

**Conventions.** We work in the category of schemes of finite type over an algebraically closed field \(k\) of characteristic 0. Throughout the paper \(N\) is a free abelian group of rank \(n \geq 2\) and \(N_\mathbb{Q} = N \otimes \mathbb{Z} \mathbb{Q}, \ N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}\). For toric geometry fans will be defined in \(N_\mathbb{Q}\) or in \(N_\mathbb{Q} \times \mathbb{Q}\). If \(\Sigma\) is a fan in \(N\) then \(X(\Sigma)\) is the associated toric \(k\)-variety with big torus \(\text{int} X(\Sigma) \simeq \mathbb{G}(N) \subset X(\Sigma)\). Its complement is sometimes referred to as the *toric boundary* of \(X(\Sigma)\). For a polyhedral cone \(\sigma \subset N_\mathbb{Q}\) the union of proper faces of \(\sigma\) is denoted \(\partial \sigma\), and \(\text{int} \sigma = \sigma \setminus \partial \sigma\) is the relative interior. Dually we have \(M = \text{Hom}(N, \mathbb{Z})\) and \(M_\mathbb{Q} = M \otimes \mathbb{Z} \mathbb{Q}\), and the dual pairing \(M \times N \to \mathbb{Q}\) is denoted by brackets \(\langle \ , \ \rangle\). The notation for the algebraic torus \(\text{Spec} k[M]\) is \(\mathbb{G}(N)\). Then \(N\) and \(M\) can be identified with the space of one-parameter subgroups \(\mathbb{G}_m \to \mathbb{G}(N)\) and of characters \(\mathbb{G}(N) \to \mathbb{G}_m\), respectively. This motivates the notation \(\chi^m\) for the monomial in \(k[M]\), or in any subring, corresponding to \(m \in M\). If \(\Xi \subset N_\mathbb{Q}\) is a subset \(L(\Xi) \subset N_\mathbb{Q}\) denotes the linear subspace spanned by differences \(v - w\) for \(v,w \in \Xi\), and \(C(\Xi) \subset N_\mathbb{Q} \times \mathbb{Q}\) is the closure of the convex hull of \(\mathbb{Q} \geq 0 \cdot (A \times \{1\})\). In particular, if \(A\) is an affine subspace then \(L(A) \subset N_\mathbb{Q}\) is the associated linear space and \(LC(A) := L(C(A)) \subset N_\mathbb{Q} \times \mathbb{Q}\) is the linear closure of \(A \times \{1\}\). The natural numbers \(\mathbb{N}\) include 0.
1. Tropical curves

Recall the definition of tropical curve given in [Mi], Definition 2.2. Let $\Gamma$ be (the geometric realization of) a weighted, connected finite graph without divalent vertices. Its sets of vertices and edges are denoted $\Gamma[0]$ and $\Gamma[1]$ respectively, and $w_{\Gamma}: \Gamma[1] \to \mathbb{N}\setminus\{0\}$ is the weight function. An edge $E \in \Gamma[1]$ has adjacent vertices $\partial E = \{V_1,V_2\}$. Let $\Gamma[0]_{\infty} \subset \Gamma[0]$ be the set of one-valent vertices. We set $\Gamma = \Gamma \setminus \Gamma[0]_{\infty}$.

By abuse of notation let us refer to such an object as a (weighted) open graph with vertices, edges and weight function $\Gamma[0], \Gamma[1], w_{\Gamma}$. Now some edges may be non-compact, and these are called unbounded edges. Write $\Gamma[1]_{\infty} \subset \Gamma[1]$ for the subset of unbounded edges. The product of all weights of bounded edges is the total inner weight of $\Gamma$

$$w(\Gamma) := \prod_{E \in \Gamma[1] \setminus \Gamma[1]_{\infty}} w_{\Gamma}(E).$$

The set of flags of $\Gamma$ is $F\Gamma = \{(V,E) \mid V \in \partial E\}$. Because every unbounded edge has only one adjacent vertex we sometimes consider $\Gamma[1]_{\infty}$ as subset of $F\Gamma$.

**Definition 1.1.** A parameterized tropical curve in $\mathbb{N}_Q$ is a proper map $h : \Gamma \to \mathbb{N}_R = \mathbb{N}_Q \otimes_{\mathbb{Q}} \mathbb{R}$ satisfying the following conditions.

(i) For every edge $E \subset \Gamma$ the restriction $h|_E$ is an embedding with image $h(E)$ contained in an affine line with rational slope.

(ii) For every vertex $V \in \Gamma$ we have $h(V) \in \mathbb{N}_Q$ and the following balancing condition holds. Let $E_1, \ldots, E_m \in \Gamma[1]$ be the edges adjacent to $V$, and let $u_i \in \mathbb{N}$ be the primitive integral vector emanating from $h(V)$ in the direction of $h(E_i)$. Then

$$\sum_{j=1}^{m} w(E_j) u_j = 0. \tag{1}$$

An isomorphism of tropical curves $h : \Gamma \to \mathbb{N}_R$ and $h' : \Gamma' \to \mathbb{N}_R$ is a homeomorphism $\Phi : \Gamma \to \Gamma'$ respecting the weights of the edges and such that $h = h' \circ \Phi$. A tropical curve is an isomorphism class of parameterized tropical curves. Note that the isomorphism class of a tropical curve is determined by the weighted graph $\Gamma$, a map $\Gamma[0] \to \mathbb{N}_Q$ telling the images of the vertices, and a map $\Gamma[1]_{\infty} \to \mathbb{N}_Q$ for the slopes of the unbounded edges. The genus of a tropical curve $h : \Gamma \to \mathbb{N}_R$ is the first Betti number of $\Gamma$. A rational tropical curve is a tropical curve of genus 0. \hfill \Box

**Remark 1.2.** From a systematic point of view one might also want to allow contracted edges, as in the latest version of [Mi]. In strict analogy with stable maps one should then also introduce a stability condition involving marked edges. For the unobstructed cases treated in this paper contracted edges are irrelevant.

Implicit in (ii) is the definition of a map $u : F\Gamma \to \mathbb{N}$ sending a flag $(V,E)$ to the primitive integral vector $u_{(V,E)} \in \mathbb{N}$ emanating from $V$ in the direction of $h(E)$. Note that $u_{(V_1,E)} =$
where $e$ spanning tree, subject to linear relations coming from the remaining edges. In particular, for

Local coordinates are given by the position of one vertex and the lengths of the edges of a

graph: $\text{of } \Gamma$ ([Mi], Proposition 2.14). The overvalence measures the difference from $\Gamma$ to a trivalent

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Let $l \in \mathbb{N}$. An $l$-marked tropical curve is a tropical curve $h : \Gamma \to N_\mathbb{R}$ together with a choice of

edges $\mathbf{E} = (E_1, \ldots, E_l) \subset (\Gamma^{[1]})^l$. In this definition we do not assume the $E_i$ to be pairwise

distinct. The notation is $(\Gamma, \mathbf{E}, h)$. The total marked weight of the marked weighted graph

$(\Gamma, \mathbf{E})$ is

$$w(\Gamma, \mathbf{E}) := w(\Gamma) \cdot \prod_{i=1}^l w(E_i).$$

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type of $(\Gamma, \mathbf{E}, h)$ is the marked graph $(\Gamma, \mathbf{E})$ together with the map $u : FT \Gamma \to N$. For given type $(\Gamma, \mathbf{E}, u)$ denote by $\mathcal{T}(\Gamma, \mathbf{E}, u) \simeq \mathcal{T}(\Gamma, u)$ the space of isomorphism classes of (marked) tropical curves of this type. If non-empty it is a manifold of dimension

$$\dim \mathcal{T}(\Gamma, u) \geq e + (n - 3)(1 - g) - \text{ov}(\Gamma),$$

where $e$ is the number of unbounded edges, $g$ is the genus of $\Gamma$ and $\text{ov}(\Gamma)$ is the overvalence

of $\Gamma$ ([Mi], Proposition 2.14). The overvalence measures the difference from $\Gamma$ to a trivalent graph:

$$\text{ov}(\Gamma) = \sum_{V \in \Gamma^{[0]}} \left| \left\{ E \in \Gamma^{[1]} \mid (V, E) \in FT \Gamma \right\} - 3 \right|.$$  

Local coordinates are given by the position of one vertex and the lengths of the edges of a spanning tree, subject to linear relations coming from the remaining edges. In particular, for $g = 0$

$$\dim \mathcal{T}(\Gamma, u) = e + n - 3 - \text{ov}(\Gamma) \leq e + n - 3,$$

with equality iff $\Gamma$ is trivalent, and $e - 3 - \text{ov}(\Gamma)$ equals the number of bounded edges of $\Gamma$.

**Definition 1.3.** For $\mathbf{d} = (d_1, \ldots, d_l) \in \mathbb{N}^l$ an affine constraint of codimension $\mathbf{d}$ is an $l$-tuple

$\mathbf{A} = (A_1, \ldots, A_l)$ of affine subspaces $A_i \subset N_\mathbb{Q}$ with $\dim A_i = n - d_i - 1$. An $l$-marked tropical curve $(\Gamma, \mathbf{E}, h)$ matches the affine constraint $\mathbf{A}$ if

$$h(E_i) \cap A_i \neq \emptyset, \quad i = 1, \ldots, l.$$  

The space of $l$-marked tropical curves of type $(\Gamma, \mathbf{E}, u)$ matching $\mathbf{A}$ is denoted $\mathcal{T}(\Gamma, \mathbf{E}, u)(\mathbf{A})$.

The degree of a type $(\Gamma, \mathbf{E}, u)$ of tropical curves is the function $N \setminus \{0\} \to \mathbb{N}$ with finite support defined by

$$\Delta(\Gamma, \mathbf{E}, u)(v) = \Delta(\Gamma, u)(v) := \# \left\{ (V, E) \in \Gamma^{[1]}_\infty \mid w(E) \cdot u(V, E) = v \right\},$$

where we consider $\Gamma^{[1]}_\infty$ as a subset of $FT \Gamma$. In other words, it is the abstract set of directions of unbounded edges together with their weights, with repetitions allowed. The degree of a (marked) tropical curve is the degree of its type. For $g \in \mathbb{N}$ and $\Delta \in \text{Map}(N \setminus \{0\}, \mathbb{N})$ the set of $l$-marked tropical curves of genus $g$ and degree $\Delta$ is denoted $\mathcal{T}_{g,l,\Delta}$. For the subset matching an affine constraint $\mathbf{A}$ the notation is $\mathcal{T}_{g,l,\Delta}(\mathbf{A})$. For $\Delta \in \text{Map}(N \setminus \{0\}, \mathbb{N})$ with
finite support define $|\Delta| = \sum_{v \in N \setminus \{0\}} \Delta(v)$. A tropical curve of degree $\Delta$ has $|\Delta|$ unbounded edges (unweighted count).

Note that by the balancing condition there are no tropical curves of degree $\Delta \in \text{Map}(N \setminus \{0\}, N)$ unless $\sum_{v \in N \setminus \{0\}} \Delta(v) \cdot v = 0$. The enumerative count on the affine side is of tropical curves of fixed degree and matching a given general affine constraint. To make this precise the next section discusses transversality in this context. Subsequent sections treat the toric side, with the degree selecting the homology class of curves to be considered, and the affine constraints corresponding to orbits of subgroups to be intersected by the algebraic curves.

2. Affine transversality

Proposition 2.1. For any $\Delta \in \text{Map}(N \setminus \{0\}, N)$ and any $g \in \mathbb{N}$ there are only finitely many types of tropical curves of degree $\Delta$ and genus $g$, that is, the set

$$\{(\Gamma, u) \text{ type of tropical curve} \mid \Delta(\Gamma, u) = \Delta, \; g(\Gamma, u) = g, \; \mathcal{T}_{(\Gamma, u)} \neq \emptyset\}$$

is finite.

Proof. Let $(\Gamma, u)$ have degree $\Delta$ and genus $g$. Then the number of unbounded edges of $\Gamma$ is equal to $|\Delta|$. Hence $\Delta$ must have finite support. Moreover, for each $e$ there are only finitely many graphs $\overline{\Gamma}$ of genus $g$ with $e$ one-valent vertices. It therefore suffices to show, for a given $\Gamma$, finiteness of the set

$$\{u_{(V,E)} \in N \mid (V,E) \in F\Gamma, \; (\Gamma, u) \text{ type of tropical curve, } \Delta(\Gamma, u) = \Delta\},$$

of possible slopes.

Choose a basis $\lambda_1, \ldots, \lambda_n \in \text{Hom}(N, \mathbb{Z})$ and define projections $\pi_i = (\lambda_i, \lambda_n) : N_{\mathbb{Q}} \to \mathbb{Q}^2$, $i = 1, \ldots, n - 1$. If $h : \Gamma \to N_{\mathbb{R}}$ is a tropical curve of degree $\Delta$ then $\pi_i \circ h$ is a plane tropical curve of degree $\pi_i(\Delta)$, possibly after removing contracted edges from the domain. By [Mi], Corollary 3.16 and Remark 3.17 any plane tropical curve is dual to an integral subdivision of an integral polygon associated to its degree. This shows that for any $i$ there are only finitely many slopes possible for the 1-cells of $\pi_i \circ h(\Gamma)$. After replacing $\lambda_n$ by $\lambda_n + \sum_{i=1}^{n-1} \varepsilon_i \lambda_i$ for general $\varepsilon_i \in \mathbb{Z}$, we may assume that vertical slopes do not occur. In other words, $\lambda_n \circ h$ is non-constant on any $E \in \Gamma^{[1]}$.

Now for non-vertical affine lines $L_i \subset \mathbb{Q}^2$, $i = 1, \ldots, n-1$, the hyperplanes $\pi_i^{-1}(L_i)$ intersect transversally and hence $\bigcap_{i=1}^{n-1} \pi_i^{-1}(L_i) \subset N_{\mathbb{Q}}$ is a line. Therefore

$$\bigcap_{i=1}^{n-1} \pi_i^{-1}(\pi_i \circ h(\Gamma))$$

is a one-dimensional cell complex containing $h(\Gamma)$. But by what we said before there are only finitely many slopes possible for the edges of $\pi_i(h(\Gamma))$. Therefore only finitely many slopes may occur in $h(\Gamma)$ for any of the tropical curves considered. \qed
Remark 2.2. The projection method employed in the proof also shows that any tropical curve is contained in a complete intersection of \( n - 1 \) tropical hypersurfaces that are pull-backs from tropical curves on \( \mathbb{Q}^2 \) of known degree. As the types of the latter tropical curves can be effectively enumerated by integral subdivisions of an integral polygon this gives an algorithm to list all types of tropical curves of given degree in any dimension. This is certainly a rather crude method, but the only one known to the authors at the time of writing.

Let us now turn to the prime transversality result for tropical curves.

Definition 2.3. Let \( \Delta \in \text{Map}(\mathbb{N} \setminus \{0\}, \mathbb{N}) \) be a degree and \( e := |\Delta| \). An affine constraint \( A = (A_1, \ldots, A_l) \) of codimension \( d = (d_1, \ldots, d_l) \) is \textit{general} for \( \Delta \) if \( \sum_i d_i = e + n - 3 \) and if for any rational \( l \)-marked tropical curve \( (\Gamma, E, h) \) of degree \( \Delta \) and matching \( A \) the following holds:

(i) \( \Gamma \) is trivalent.
(ii) \( h(\Gamma^{[0]}) \cap \bigcup_i A_i = \emptyset \).
(iii) \( h \) is injective for \( n > 2 \). For \( n = 2 \) it is at least injective on the subset of vertices, and all fibers are finite.

Otherwise it is called \textit{non-general}.

Proposition 2.4. Let \( \Delta \in \text{Map}(\mathbb{N} \setminus \{0\}, \mathbb{N}) \) and let \( A \) be an affine constraint of codimension \( d = (d_1, \ldots, d_l) \in \mathbb{N}^l \) with \( \sum_i d_i = |\Delta| + n - 3 \). Denote by \( \mathfrak{A} := \prod_{i=1}^l \mathbb{Q}/L(A_i) \) the space of affine constraints that are parallel to \( A \). Then the subset

\[
\mathfrak{J} := \left\{ A' \in \mathfrak{A} \mid A' \text{ is non-general for } \Delta \right\}
\]

of \( \mathfrak{A} \) is nowhere dense.

Moreover, for any \( A' \in \mathfrak{A} \setminus \mathfrak{J} \) and any \( l \)-marked type \( (\Gamma, E, u) \) of genus 0 and degree \( \Delta \) there is at most one tropical curve \( (\Gamma, E, h) \) of type \( (\Gamma, E, u) \) matching \( A' \).

Proof. (Cf. [Mi], Proposition 4.11.) By Lemma 2.1 we may as well fix the type \( (\Gamma, E, u) \) of the (rational) tropical curves \( (\Gamma, E, h) \) in the definition of \( \mathfrak{J} \). Then \( e := |\Delta| \) is the number of unbounded edges of \( \Gamma \). Define the incidence variety

\[
\mathfrak{I} := \left\{ ((\Gamma, E, h), A') \in \mathfrak{T}_{(\Gamma, E, u)} \times \mathfrak{A} \mid \forall i : h(E_i) \cap A'_i \neq \emptyset \right\},
\]

and consider the diagram of canonical projections

\[
\begin{array}{ccc}
\mathfrak{I} & \xrightarrow{\rho} & \mathfrak{T}_{(\Gamma, E, u)} \\
\pi \downarrow & & \\
\mathfrak{A} & & 
\end{array}
\]

(3)

Let \( \mathfrak{I}_{\text{ng}} \subset \mathfrak{I} \) be the subset of pairs \( ((\Gamma, E, h), A') \) with \( (\Gamma, E, h) \) non-general for \( A' \). We are interested in \( \mathfrak{J} = \pi(\mathfrak{I}_{\text{ng}}) \subset \mathfrak{A} \) and in the fibers of \( \pi \) over the complement of this set.

We claim that the maps in Diagram 3 can be described by affine maps. As discussed after the dimension formula 2.1 there is an embedding of \( \mathfrak{T}_{(\Gamma, E, u)} \) into \( \mathbb{Q}_0 \times \mathbb{Q}^{e-3} \) as open subset. The factors are the position of one vertex and the lengths of the bounded edges, respectively.
Any point from $N_Q \times Q^{e-3}$ still corresponds uniquely to a map $h : \Gamma \to N_Q$, but some edges might be contracted or point in the opposite directions than indicated by the type. Now the incidence condition of the $i$-th edge with the $i$-th affine subspace is expressed by an affine equation. This proves the claim. Note that our discussion expresses $\mathcal{I}$ as the intersection of $N_Q \times Q_{\leq 0}^{e-3} \times \mathfrak{A}$ with an affine subspace. Thus $\mathcal{I}$ is naturally the interior of a convex polytope in a $Q$-vector space.

To show that $\pi$ is either an isomorphism or has image contained in a proper submanifold it remains to prove $\dim \mathcal{I} \leq \dim \mathfrak{A}$. For fixed $(\Gamma, E, h) \in \mathcal{I}(\Gamma, E, u)$ the set of $A_i' \in N_Q/L(A_i)$ with $h(E_i) \cap A_i' \neq \emptyset$ is of codimension $\geq d_i$ inside $N_Q/L(A_i)$. Thus the fiber of $\rho$ over $(\Gamma, E, h)$ has codimension at least $\sum_i d_i$ inside $\{(\Gamma, E, h)\} \times \mathfrak{A}$. By the dimension formula $\dim(\Gamma, E, h)$ varies in a $(e + n - 3 - \text{ov}(\Gamma))$-dimensional family, and $e + n - 3 = \sum_i d_i$ by assumption. Hence the incidence variety on the upper left of Diagram 3 has at most the dimension of $\mathfrak{A}$, as claimed.

Finally, we look at the genericity conditions Definition 2.3 (i)–(iii). (i) If $\Gamma$ has a vertex of valency larger than three then $\dim \mathcal{I}(\Gamma, E, u) < e + n - 3$. (ii) For the subset of tropical curves with a vertex $V \in \partial E_i$ mapping to $A_i$ the codimension of the fiber of $\rho$ inside $\{(\Gamma, E, h)\} \times \mathfrak{A}$ is strictly larger than $\sum_i d_i$. (iii) The conditions that two vertices of $\Gamma$ map to the same point, that the images of two non-adjacent edges are contained in the same line or, for $n > 2$, that two non-adjacent edges intersect, are given by proper affine subspaces inside $N_Q \times Q^{e-3}$.

Finally, let $h : \Gamma \to N_R$ be a trivalent tropical curve with $h(E_1) \cap h(E_2) \neq \{V\}$ for two adjacent edges $E_1, E_2$. Then $u(E_1, V) = u(E_2, V) = -u(E_3, V)$ for $E_3$ the third edge emanating from $V$, and either $h(E_1) \subset h(E_2)$ or $h(E_2) \subset h(E_1)$, say the former. In this case the fiber of $\pi$ through the given $((\Gamma, E, h), \mathfrak{A})$ is at least one-dimensional, for we can move the image of $V$ along the line containing $h(E_i)$. Hence for such types $(\Gamma, E, u)$ of tropical curves $\text{im}(\pi)$ is contained in a proper submanifold of $\mathfrak{A}$.

Thus any of these exceptional tropical curves describe subspaces of $\mathcal{I}$ with image in $\mathfrak{A}$ contained in a proper submanifold. Hence $\pi(\mathcal{I}^{\text{reg}}) \subset \mathfrak{A}$ is contained in a finite union of proper submanifolds.

To summarize, two cases may occur. (1) The affine extension of $\pi$ is an isomorphism. Then $\pi(\mathcal{I}^{\text{reg}}) \subset \mathfrak{A}$ is closed and nowhere dense. Moreover, for any $\mathfrak{A}' \in \mathfrak{A}$ there is a unique point in $N_Q \times Q^{e-3}$ fulfilling the extended incidence condition. This point corresponds to a tropical curve $(\Gamma, E, h) \in \mathcal{I}(\Gamma, E, u)(\mathfrak{A}')$ iff it lies in $N_Q \times Q_{\leq 0}^{e-3}$. (2) $\pi$ is not an isomorphism. Then $\text{im}(\pi) \subset \mathfrak{A}$ is closed and nowhere dense, and for $\mathfrak{A}'$ in the complement of this set there is no tropical curve of type $(\Gamma, E, u)$ matching $\mathfrak{A}'$. \hfill \Box

General affine constraints imply a certain infinitesimal rigidity as follows. Let $(\Gamma, E, u)$ be a type of $l$-marked rational tropical curves and let $d \in \mathbb{N}^l$ with $\sum_i d_i = |\Delta(\Gamma, u)| + n - 3$. Orient the bounded edges of $\Gamma$ arbitrarily, thus defining two maps $\partial^\pm : \Gamma^{[1]} \setminus \Gamma^{[0]} \to \Gamma^{[0]}$ with $\partial E = \{\partial^- E, \partial^+ E\}$. If $E \in \Gamma^{[1]}$ is unbounded then $\partial^- E$ denotes the unique vertex adjacent...
to $E$. For an affine constraint $A$ of codimension $d$ consider the affine map

$$(4) \quad \Phi : \text{Map}(\Gamma[0], N_Q) \rightarrow \prod_{E \in \Gamma[0]} \frac{N_Q/\mathbb{Q}u(\partial^+E,E) \times \prod_{i=1}^l N_Q/(u(\partial^-E_i, E_i) + L(A_i))}{},$$

$$h \mapsto (h(\partial^+E) - h(\partial^-E))_E, (h(\partial^+E_i) - h(\partial^-E_i))_i.$$

Then $h \in \text{Map}(\Gamma[0], N_Q)$ is the restriction to $\Gamma[0]$ of a tropical curve $\tilde{h} : \Gamma \rightarrow N_R$ of type $(\Gamma, E, u)$ matching $A$ iff the following conditions are satisfied:

(i) $\Phi(h) = 0$.

(ii) For any $E \in \Gamma[1] \setminus \Gamma[\infty]$ the proportionality factor $\lambda \in \mathbb{Q}$ with $\partial^+E - \partial^-E = \lambda \cdot u(\partial^-E, E)$ existing by (1) is strictly positive.

(iii) For any $i$ the line passing through $h(E_i)$ intersects $A_i$ between $h(\partial^+E_i)$ and $h(\partial^-E_i)$.

The proposition now implies the following.

**Corollary 2.5.** If the affine constraint $A$ is general for $\Delta$ and $\mathcal{T}(\Gamma, E, u)(A) \neq \emptyset$ then the affine map $\Phi$ in [4] is an isomorphism.

**Proof.** By the proposition there is at most one tropical curve of type $(\Gamma, E, u)$ and matching $A$. Thus $\Phi$ is injective because $\Phi^{-1}(0) \neq \emptyset$ by assumption. The proof is finished by comparing dimensions: $\Gamma$ being rational, connected and trivalent implies $\sharp \Gamma[0] = e - 2$, with $e = \sharp \Gamma[\infty]$ the number of unbounded edges. Hence the dimension of the left-hand side is $n(e - 2)$. For the right-hand side observe $\sharp((\Gamma[1] \setminus \Gamma[\infty])) = \sharp \Gamma[0] - 1 = e - 3$ by rationality and connectedness, via an Euler characteristic count. Taking into account $\sum d_i = e + n - 3$ now also gives

$$(e - 3)(n - 1) + \sum_i (n - (n - d_i - 1 + 1)) = n(e - 2).$$

□

**Remark 2.6.** In higher genus (and $n > 2$) there are families of tropical curves of larger than the expected dimension, and a result analogous to Proposition [24] does not hold. A treatment of these cases therefore has to implement some kind of virtual intersection theory on the moduli space of tropical curves. Such spaces should itself be tropical varieties, that is, a cell complex of integral affine polyhedra together with weights and compatible fan structures at the vertices. The interior of each cell parametrizes tropical curves of exactly one type. The balancing condition in higher dimensions should come from the Minkowski weight condition formulated in [FuSt].

3. Polyhedral decompositions and degenerations

Here we consider degenerations of toric varieties given by toric morphisms to $\mathbb{A}^1$. They have the property that all fibers, except the one over the origin, are isomorphic to a fixed toric variety. We begin with some definitions.

In this paper a (rational) polyhedron is the solution set in $N_Q \simeq \mathbb{Q}^n$ of finitely many linear inequalities $\langle m, . \rangle \geq \text{const}$, $m \in M_Q$. As an intersection of finitely many closed halfplanes it is always closed and convex, but not necessarily bounded and it may have any dimension $\leq n$. The sets where some of the defining inequalities are equalities define the faces, which
are itself lower dimensional polyhedra. A vertex is a zero-dimensional face. A polyhedron is strongly convex if it has at least one vertex.

**Definition 3.1.** A (semi-infinite) polyhedral decomposition of \( \mathbb{N}_Q \) is a covering \( \mathcal{P} = \{ \Xi \} \) of \( \mathbb{N}_Q \) by a finite number of strongly convex polyhedra satisfying the following properties:

(i) If \( \Xi \in \mathcal{P} \) and \( \Xi' \subset \Xi \) is a face, then \( \Xi' \in \mathcal{P} \).

(ii) If \( \Xi, \Xi' \in \mathcal{P} \), then \( \Xi \cap \Xi' \) is a common face of \( \Xi \) and \( \Xi' \).

The unbounded elements of a polyhedral decomposition \( \mathcal{P} \) define a fan \( \Sigma_{\mathcal{P}} \) by rescaling by \( a \in \mathbb{Q} > 0 \) and letting \( a \) tend to 0. In fact, for each \( \Xi \in \mathcal{P} \) the limit \( \lim_{a \to 0} a \Xi \) exists in the Hausdorff sense. Note that for every bounded \( \Xi \) this limit is just \( 0 \in \mathbb{N}_Q \). We call \( \Sigma_{\mathcal{P}} := \left\{ \lim_{a \to 0} a \Xi \subset \mathbb{N}_Q \mid \Xi \in \mathcal{P} \right\} \) the asymptotic fan of \( \mathcal{P} \). The terminology is justified by the following lemma.

**Lemma 3.2.** \( \Sigma_{\mathcal{P}} \) is a complete fan.

*Proof.* If \( \Xi \in \mathcal{P} \) is the solution set to \( m_i \geq c_i \) for \( m_i \in \mathbb{M}_Q, c_i \in \mathbb{Q} \), then

\[
\lim_{a \to 0} a \Xi = \left\{ n \in \mathbb{N}_Q \mid \forall i (m_i, n) \geq 0 \right\}.
\]

(5)

This is a strongly convex polyhedral cone. The intersection of any two cones is again of this form because of (ii) in Definition 3.1. Completeness is clear. \( \square \)

A polyhedral decomposition defines a degenerating family of toric varieties as follows. First, we extend the fan \( \Sigma_{\mathcal{P}} \) in \( \mathbb{N}_Q \) to a fan \( \widetilde{\Sigma}_{\mathcal{P}} \) in \( \mathbb{N}_Q \times \mathbb{Q} \) covering the half-space \( \mathbb{N}_Q \times \mathbb{Q}_\geq 0 \):

For each \( \Xi \in \mathcal{P} \) let \( C(\Xi) \) be the closure of the cone spanned by \( \Xi \times \{1\} \) in \( \mathbb{N}_Q \times \mathbb{Q} \):

\[
C(\Xi) = \left\{ (a, (n, 1)) \mid a \geq 0, n \in \Xi \right\}.
\]

If \( \Xi \) is defined by inequalities \( m_i \geq c_i \) then

\[
C(\Xi) = \left\{ (n, b) \in \mathbb{N}_Q \times \mathbb{Q}_\geq 0 \mid (m_i, n) - b \cdot c_i \geq 0 \right\}.
\]

(6)

Thus any \( C(\Xi) \) is a strongly convex polyhedral cone and

\[
\widetilde{\Sigma}_{\mathcal{P}} := \{ \sigma \subset C(\Xi) \text{ face} \mid \Xi \in \mathcal{P} \}
\]

is a fan covering \( \mathbb{N}_Q \times \mathbb{Q}_\geq 0 \). For later reference we note:

**Lemma 3.3.** If we identify \( \mathbb{N}_Q \) with \( \mathbb{N}_Q \times \{0\} \subset \mathbb{N}_Q \times \mathbb{Q} \) then

\[
\Sigma_{\mathcal{P}} = \left\{ \sigma \cap (\mathbb{N}_Q \times \{0\}) \mid \sigma \in \widetilde{\Sigma}_{\mathcal{P}} \right\}.
\]

*Proof.* For \( \Xi \in \mathcal{P} \) the description of the elements of \( \Sigma_{\mathcal{P}} \) in [5] and of \( \widetilde{\Sigma}_{\mathcal{P}} \) in [6] by inequalities readily shows \( \lim_{a \to 0} a \Xi = C(\Xi) \cap (\mathbb{N}_Q \times \{0\}) \). \( \square \)

The projection \( \mathbb{N}_Q \times \mathbb{Q} \to (\mathbb{N}_Q \times \mathbb{Q})/\mathbb{N}_Q = \mathbb{Q} \) onto the second factor defines a non-constant map of fans from \( \widetilde{\Sigma}_{\mathcal{P}} \) to the fan \( \{0, \mathbb{Q}_\geq 0\} \) of \( \mathbb{A}^1 \). Thus there is a non-constant, hence flat, toric morphism

\[
\pi : X(\widetilde{\Sigma}_{\mathcal{P}}) \longrightarrow \mathbb{A}^1.
\]
It is equivariant for the morphism of algebraic tori $G_{m}^{n+1} \simeq G(N \times \mathbb{Z}) \to G(\mathbb{Z}) \simeq G_{m}$. Because $G(\mathbb{Z})$ acts transitively on the closed points of $A^{1} \setminus \{0\}$ the closed fibers of $\pi$ over $A^{1} \setminus \{0\}$ are all pairwise isomorphic.

**Lemma 3.4.** For a closed point $t \in A^{1} \setminus \{0\}$ the fiber $\pi^{-1}(t) \subset X(\tilde{\Sigma}_{\mathcal{P}})$ with the action of $G(N) \subset G(N \times \mathbb{Z})$ is torically isomorphic to $X(\Sigma_{\mathcal{P}})$.

**Proof.** Because $N \subset N \times \mathbb{Z}$ is the kernel of the projection to the second factor, $G(N) \subset G(N \times \mathbb{Z})$ induces a trivial action on $A^{1}$, and hence respects the fibers of $\pi$.

The elements of $\tilde{\Sigma}_{\mathcal{P}}$ contained in $N_{Q} \subset N_{Q} \times \mathbb{Q}$ form the fan of the open part $X(\tilde{\Sigma}_{\mathcal{P}}) \setminus \pi^{-1}(0) = \pi^{-1}(A^{1} \setminus \{0\})$. Hence Lemma 5.3 shows

$$\pi^{-1}(A^{1} \setminus \{0\}) = (A^{1} \setminus \{0\}) \times X(\Sigma_{\mathcal{P}}),$$

with $\pi$ the projection to the first factor. $\square$

Note also that $\tilde{\Sigma}_{\mathcal{P}}$ and the product fan $\Sigma_{\mathcal{P}} \times \{0, \mathbb{Q}_{\geq 0}\}$ have common refinements that agree on $N \times \{0\}$. Subdivisions of fans lead to toric birational morphisms. Hence there is a birational transformation

$$X(\tilde{\Sigma}_{\mathcal{P}}) \cdots \to X(\Sigma_{\mathcal{P}}) \times A^{1},$$

with centers on the central fiber. The central fiber can also be easily read off from $\mathcal{P}$ provided that $\mathcal{P}$ is integral, that is, $\mathcal{P}[0] \subset N$. (Integrality is a necessary and sufficient condition for $\pi^{-1}(0)$ to be reduced.) For every vertex $v \in \mathcal{P}$ the star of $\mathcal{P}$ at $v$ defines a complete fan

$$\Sigma_{v} := \{Q_{\geq 0} \cdot (\Xi - v) \mid \Xi \in \Sigma_{\mathcal{P}}, \Xi \ni v\}.$$

More generally, for $\Xi \in \mathcal{P}$ the rays emanating from $\Xi$ through adjacent $\Xi' \in \mathcal{P}$ define a complete fan $\Sigma_{\Xi}$ in $N_{Q}/L(\Xi)$:

$$\Sigma_{\Xi} := \{Q_{\geq 0} \cdot (\Xi' - \Xi) \subset N_{Q}/L(\Xi) \mid \Xi' \in \mathcal{P}, \Xi \subset \Xi'\}.$$

For shortness write $X_{\Xi} := X(\Sigma_{\Xi})$. In particular, $X_{v} = X(\Sigma_{v})$ is a toric divisor in $X(\tilde{\Sigma}_{\mathcal{P}})$. For any vertex $v \in \Xi$ let $\Sigma_{v}(\Xi) \subset \Sigma_{v}$ be the subfan of cones intersecting $\Xi - v$ non-trivially. Then $N_{Q} \to N_{Q}/L(\Xi)$ defines a map of fans $\Sigma_{v}(\Xi) \to \Sigma_{\Xi}$, hence a toric morphism between the open subset $X(\Sigma_{v}(\Xi)) \subset X_{v}$ to $X_{\Xi}$. This map induces an isomorphism between the (codim $\Xi$)-dimensional toric stratum in $X_{v}$ corresponding to $\Xi$ and $X_{\Xi}$. Thus for any $\Xi, \Xi' \in \mathcal{P}$ with $\Xi' \subset \Xi$ there is a closed embedding $X_{\Xi} \to X_{\Xi'}$ that is equivariant with respect to $G(N)$ and compatible with compositions. While this is standard in toric geometry we wish to view the $X_{\Xi}$ together with these closed embeddings as a directed system of schemes with $G(N)$-action.

**Proposition 3.5.** Assume $\mathcal{P}[0] \subset N$. Then there exists a system of closed embeddings $X_{\Xi} \to \pi^{-1}(0)$, $\Xi \in \mathcal{P}$, compatible with the directed system, inducing an isomorphism $\pi^{-1}(0) \simeq \lim_{\Xi \in \mathcal{P}} X_{\Xi}$.

**Proof.** This is a trivial special case of the discussion in [GrSi2], §2.2. (The boundedness of the cells of $\mathcal{P}$ that we assumed in this paper is irrelevant for this part.) Let us just show here how
to get the standard toric embeddings $X_v \to \pi^{-1}(0)$, leaving the straightforward verifications of compatibility with the directed system and the universal property to the reader.

The total space $X(\Sigma, \mathcal{P})$ of the degeneration $\pi$ is glued from affine sets $\text{Spec} k[C(\Xi)^\vee \cap (M \times \mathbb{Z})]$ for $\Xi \in \mathcal{P}$. The monomial $\chi^{(0,1)}$ is the image of the affine coordinate on $\mathbb{A}^1$, so this generates the ideal of the central fiber. On the other hand, if $\mathcal{P}$ is integral this equals the ideal generated by $\chi^{(m,k)}$ for all $(m,k) \in \text{int}(C(\Xi)^\vee \cap (M \times \mathbb{Z}))$. In fact, $(m,k) \in M \times \mathbb{Z}$ lies in $\text{int}(C(\Xi)^\vee)$ iff $(m,v) + k > 0$ for all $v \in \Xi$. If all vertices of $\Xi$ are integral it suffices to test this inequality for $v \in \Xi \cap N$, and then it holds $(m,v) + k \geq 1$. This implies $(m,k-1) \in C(\Xi)^\vee \cap (M \times \mathbb{Z})$ and hence $\chi^{(m,k)} \in (\chi^{(0,1)})$. Thus $\pi^{-1}(0)$ is covered by $\text{Spec} k[\partial C(\Xi)^\vee \cap (M \times \mathbb{Z})]$, where this notation means

$$\chi^{(m_1,a_1)} \cdot \chi^{(m_2,a_2)} := \begin{cases} 
\chi^{(m_1+m_2,a_1+a_2)}, & \text{if } (m_1+m_2,a_1+a_2) \in \partial C(\Xi)^\vee, \\
0, & \text{otherwise}.
\end{cases}$$

This space has irreducible components $\text{Spec} k[C_v \cap (M \times \mathbb{Z})]$ with $C_v \subset C(\Xi)^\vee$ the $n$-dimensional face dual to $\mathbb{Q}_{\geq 0} \cdot (v,1)$, for $v$ a vertex of $\Xi$. Now the projection $M \times \mathbb{Z} \to M$ induces an isomorphism of $C_v \cap (M \times \mathbb{Z})$ with the integral points of $\mathbb{Q}_{> 0}(\Xi - v)^\vee$. Hence there is a canonical isomorphism of a toric affine patch of an irreducible component of $\pi^{-1}(0)$ with an affine patch of $X_v$. The identification is compatible with the gluing maps and thus gives the claimed closed embedding $X_v \to \pi^{-1}(0)$. \hfill \Box

Now we describe how an affine constraint in $N_\mathbb{Q}$ yields a family of incidence conditions in the degeneration $X(\Sigma, \mathcal{P}) \to \mathbb{A}^1$. An affine subspace $A \subset N_\mathbb{Q}$ spans the linear subspace $LC(A) \subset N_\mathbb{Q} \times \mathbb{Q}$ where the fan $\tilde{\Sigma} \mathcal{P}$ lives. For any closed point $P$ in the big torus of $X(\Sigma, \mathcal{P})$ the closure of the orbit $G(LC(A) \cap (N \times \mathbb{Z})).P$ defines a subvariety $Z \subset X(\Sigma, \mathcal{P})$ projecting onto $\mathbb{A}^1$. Our incidence condition is non-trivial intersection with $Z$. Most of the relevant properties of this subvariety generalize to any pair (fan in $\mathbb{Q}$-vector space, linear subspace). For simplicity we therefore revert to the notation $\Sigma$ for the fan and $N_\mathbb{Q}$ for the $\mathbb{Q}$-vector space during the following discussion.

**Proposition 3.6.** Let $\Sigma$ be a fan in $N_\mathbb{Q}$, $P \in X(\Sigma)$ a closed point in the big torus and $L \subset N_\mathbb{Q}$ a linear subspace. For $\sigma \in \Sigma$ denote by $X_\sigma \subset X(\Sigma)$ the corresponding toric subvariety. Then the following holds.

1. $\overline{G(L \cap N)}.P \cap \text{int}(X_\sigma) = \emptyset \iff L \cap \text{int} = \emptyset.$

2. If $L \cap \text{int} \neq \emptyset$ then $G(L \cap N)$ acts transitively on $\overline{G(L \cap N)}.P \cap \text{int}(X_\sigma)$.

**Proof.** As the statements are local along $\text{int}(X_\sigma)$ we may assume $\Sigma$ is the fan of faces of $\sigma$. Then $X(\Sigma) = \text{Spec} k[\sigma^\vee \cap M]$ and $X_\sigma \simeq G(N/L(\sigma) \cap N)$. In a first step we reduce to the case $L(\sigma) = N_\mathbb{Q}$.

Choose a complement $Q_\mathbb{Q} \subset N_\mathbb{Q}$ of $L(\sigma)$ with $L = (L \cap Q_\mathbb{Q}) + (L \cap L(\sigma))$ and write $Q = Q_\mathbb{Q} \cap N$. Then the natural map

$$Q \times (L(\sigma) \cap N) \longrightarrow N$$
is the inclusion of a sublattice of finite index. Let $\Sigma'$ be the fan of faces of the preimage of $\sigma$ in $Q_\mathbb{Q} \times L(\sigma)$. Then the toric morphism $X(\Sigma') \to X(\Sigma)$ is a finite surjection that is equivariant for $G(Q) \times G(L(\sigma) \cap N) \to G(N)$. Under this homomorphism the action of $G(L \cap N) \times G(L \cap L(\sigma) \cap N) \subset G(Q) \times G(L(\sigma) \cap N)$ covers the action of $G(L \cap N) \subset G(N)$. Hence the closure of a $G(L \cap N)$-orbit is the image of the closure of a $G(L \cap Q) \times G(L \cap L(\sigma) \cap N)$-orbit and all our statements can be checked on $X(\Sigma')$. We may thus assume $N = Q \times (L(\sigma) \cap N)$.

Let $\Sigma$ be the fan in $L(\sigma)$ of faces of $\sigma$. Then the splitting $N = Q \times (L(\sigma) \cap N)$ defines a decomposition $X(\Sigma) = G(Q) \times X(\Sigma)$ that is compatible with the product $G(N) = G(Q) \times G(L(\sigma) \cap N)$, and $X_{\sigma} = G(Q) \times 0$ for $0 \subset X(\Sigma)$ the unique zero-dimensional torus orbit. In particular, $G(L \cap N).P \cap X_{\sigma} = \emptyset$ if and only if $G(L \cap L(\sigma) \cap N).\Psi(P) \cap 0 = \emptyset$ where $\Psi : X(\Sigma) \to X(\Sigma)$ is the projection onto the second factor. Thus we may verify the conclusions of the proposition after dividing out $G(Q)$. This amounts to going over from $\Sigma$ to $\Sigma$, from $N_{\mathbb{Q}}$ to $L(\sigma)$ and from $L$ to $L \cap L(\sigma)$. Hence we may assume $L(\sigma) = N_{\mathbb{Q}}$. Then (2) follows trivially from (1) because $X_{\sigma} = 0$ is just a closed point.

For (1) assume that $0 \not\in G(L \cap N).P$. Then there exists $f = \sum_{m \in \sigma' \cap M} a_m \chi^m \in \mathcal{O}(X(\Sigma))$ with

$$f(0) = 0, \quad f|_{G(L \cap N).P} = 1.$$ 

The $G(L \cap N)$-invariant part $\sum_{m \in \sigma' \cap L \cap M} a_m \chi^m$ of $f$ has the same properties, and is thus non-constant. Hence there exists $m \in M$ with $m(L) = 0$, $m|_{\text{int}(\sigma)} > 0$. This implies $L \cap \text{int}(\sigma) = \emptyset$. Conversely, if $L \cap \text{int}(\sigma) = \emptyset$ there exists $m \in L \cap M$ with $m|_{\text{int}(\sigma)} > 0$, and then $f = \chi^m$ provides a $G(L \cap N)$-invariant function separating 0 from $G(L \cap N).P$. \hfill \Box

Note that the intersection of $G(L \cap N).P$ with the toric boundary of $X(\Sigma)$ generally need not be reduced. Consider for example $N_{\mathbb{Q}} = \mathbb{Q}^2$, $L = \mathbb{Q} \cdot (2,3)$ and $\Sigma$ the fan of faces of $\sigma = \mathbb{Q}_{\geq 0}^2$. Then $G(L \cap N).(1,1)$ is the Neil parabola and the intersection with the toric boundary is a point of multiplicity 5. However, in the proof of the proposition we have indeed already shown:

**Corollary 3.7.** In the situation of the proposition assume that $\sigma \in \Sigma$ is contained in $L$. Let $Q$ be a complement to $L(\sigma) \cap N$ in $N$ with $L \subset (L \cap Q_\mathbb{Q}) + (L \cap L(\sigma))$, and let $\Sigma'$ be the fan in $L(\sigma)$ of faces of $\sigma$. Then locally around the big torus $G_{\sigma} \subset X_{\sigma}$ there is a toric isomorphism of $X(\Sigma)$ with $G(Q) \times X(\Sigma)$ mapping $G(L \cap N).P$ to $G(L \cap Q) \times G(L(\sigma) \cap N).P'$ for some $P' \in G_{\sigma}$.

Reverting to our situation of a toric degeneration associated to a polyhedral decomposition $\mathcal{P}$ we obtain the following.

**Corollary 3.8.** Let $\mathcal{P}$ be an integral polyhedral decomposition, $A \subset N_{\mathbb{Q}}$ an affine subspace and $v \in A \cap N$ a vertex of $\mathcal{P}$. Consider the associated toric degeneration $\pi : X(\Sigma_{\mathcal{P}}) \to \mathbb{A}^1$ with reduced central fiber $\pi^{-1}(0) = \bigcup_{v \in \mathcal{P}[0]} X_v$ and let $P \in \text{int}(X(\Sigma_{\mathcal{P}}))$ be a closed point. Then locally around $\text{int}(X_v)$ in $X(\Sigma_{\mathcal{P}})$ the orbit closure $Z = G(LC(A) \cap (N \times \mathbb{Z})).P \subset X(\Sigma_{\mathcal{P}})$ projects smoothly to $\mathbb{A}^1$ with fiber over 0 $\in \mathbb{A}^1$ a $G(L(A) \cap N)$-orbit. Moreover, the
map \( L(A) \otimes \mathbb{Q} \mathcal{O}_Z \rightarrow \Theta_{Z/A^1} \) induced by the fiberwise action of \( \mathbb{G}(L(A) \cap N) \subset \mathbb{G}(N \times \{0\}) \) on \( Z \rightarrow A^1 \) is an isomorphism.

Proof. Because \( v \) is integral we have the splitting \( N \times Z = (N \times \{0\}) \oplus (\mathbb{Z} \cdot (v, 1)) \). This induces an isomorphism of the toric affine open set \( U \simeq A^1 \times \text{int}(X_v) \subset X(\Sigma_\partial) \) corresponding to \( \mathbb{Q}_{\geq 0}(v, 1) \in \Sigma_\partial \) with \( \mathbb{G}(N) \times A^1 \), and such that \( \pi \) is the projection to the second factor. By Corollary 3.7 the orbit closure \( \mathbb{G}(L(A) \cap (N \times \mathbb{Z})) \cdot P \) maps to \( \mathbb{G}(L(A) \cap N) \cdot P' \times A^1 \) under this isomorphism, for some \( P' \in \text{int}(X_v) \). From this the statements are evident. \( \square \)

The final topic of this section concerns refinements of polyhedral decompositions.

**Proposition 3.9.** Let \( h : \Gamma \rightarrow N_\mathbb{R} \) be a tropical curve and \( S = \{ \mathbb{Q}_{\geq 0}(h(E) - h(\partial E)) \mid E \in \Gamma^{[\Sigma]} \} \) the set of directions of unbounded edges. Then for any fan \( \Sigma \) on \( N_\mathbb{Q} \) with \( S \subset \Sigma^{[\Sigma]} \) there exists a polyhedral decomposition \( \mathcal{P} \) of \( N_\mathbb{Q} \) with asymptotic fan \( \Sigma \) and such that

\[
\bigcup_{b \in \Gamma^{[\mu]}} h(b) \subset \bigcup_{\Xi \in \mathcal{P}^{[\mu]}} \Xi, \quad \mu = 0, 1.
\]

Moreover, \( \mathcal{P}^{[0]} \) can be chosen to contain any finite subset of \( N_\mathbb{Q} \).

**Proof.** If \( \mathcal{P} \) is a polyhedral decomposition of \( N_\mathbb{Q} \) and \( \Xi \subset N_\mathbb{Q} \) is a polyhedron then \( \{ \Xi \cap \Xi' \mid \Xi' \in \mathcal{P} \} \) is a face-fitting decomposition of \( \Xi \) into subpolyhedra. Thus if \( \mathcal{P}, \mathcal{P}' \) are two polyhedral decompositions of \( N_\mathbb{Q} \) then \( \{ \Xi \cap \Xi' \mid \Xi \in \mathcal{P}, \Xi' \in \mathcal{P}' \} \) is a refinement of both \( \mathcal{P} \) and \( \mathcal{P}' \). Moreover, if \( \mathcal{P}, \mathcal{P}' \) have the same asymptotic fan \( \Sigma \) then so does this common refinement. It remains to construct, for every edge \( E \in \Gamma^{[\Sigma]} \) a polyhedral decomposition \( \mathcal{P} \) with \( h(E) \in \mathcal{P}^{[\Sigma]} \). If \( E \) is an unbounded edge with \( \partial E = V \) then by assumption \( h(V) + \Sigma \) is such a decomposition. We may therefore assume \( E \) to be bounded. Denote by \( V_1, V_2 \) the adjacent vertices.

Let \( \Sigma' \subset \Sigma \) be the subset of \( n \)-dimensional cones containing the ray \( e := \mathbb{Q}_{\geq 0}(h(E) - h(V_1)) \) and \( B_1 = \bigcup_{\sigma \in \Sigma'} \sigma \). We claim that \( \partial B_1 \) divides \( N_\mathbb{Q} \) into two connected components. In fact, \( \Sigma \) consists of cones over cells of the polyhedral decomposition \( \{ \sigma \cap S^{n-1} \mid \sigma \in \Sigma \} \) of the unit sphere in \( N_\mathbb{Q} \). From this point of view \( \Sigma' \) is obtained by taking cones over the union of \( (n-1) \)-cells containing the one-point set \( e \cap S^{n-1} \). This is nothing but the closed star of the minimal cell containing \( e \cap S^{n-1} \), which is thus a cell. Therefore its boundary divides \( S^{n-1} \) into two connected components. Taking cones gives the desired statement for \( N_\mathbb{Q} \).

We have thus divided \( N_\mathbb{Q} \) into the two regions \( B_1 \) and \( B_2 = N_\mathbb{Q} \setminus \text{int} B_1 = \bigcup_{\sigma \in \Sigma \setminus \Sigma'} \sigma \) intersecting along a union of \( (n-1) \)-faces of \( \Sigma \). Any maximal cone of \( \Sigma \) is contained in one of the two regions, and \( e \setminus \{0\} \subset \text{int}(B_1) \), \( -e \setminus \{0\} \subset \text{int}(B_2) \). Now define \( \mathcal{P} \) as the set of polyhedra of the following types: (I) \( h(V_1) + \sigma, \sigma \in \Sigma, \sigma \subset B_2 \) (II) \( h(V_2) + \sigma, \sigma \in \Sigma, \sigma \subset B_1 \) (III) \( h(E) + \tau, \tau \in \Sigma, \tau \subset B_1 \cap B_2 \). Polyhedra of types I and II give polyhedral decompositions of the disjoint regions \( h(V_2) + B_1 \) and \( h(V_1) + B_2 \), respectively. The closure of the complement of their union is covered by polyhedra of type III. Note that such a polyhedron \( h(E) + \tau \) has boundary

\[
\partial(h(E) + \tau) = (h(E) + \partial \tau) \cup (h(V_1) + \tau) \cup (h(V_2) + \tau),
\]
which is thus covered by polyhedra of types III, I, and II, in the order of appearance. This shows that $\mathcal{P}$ is a polyhedral decomposition. Moreover, in the above notation the limits under rescaling $\lim_{a \to 0} a \cdot \Xi$ for $\Xi \in \mathcal{P}$ of types I, II and III are $\sigma$, $\sigma$ and $\tau$, respectively. Hence the asymptotic fan of $\mathcal{P}$ is $\Sigma$. \hfill $\square$

4. Dual intersection graphs and tropical curves

In the last section we saw how a polyhedral decomposition $\mathcal{P}$ with asymptotic fan $\Sigma_\mathcal{P}$ gives rise to a toric degeneration $X \to \mathbb{A}^1$ with general fibers $X(\Sigma_\mathcal{P})$. The central fiber is a union of toric varieties, one for each vertex of $\mathcal{P}$, glued along toric divisors: $X_0 = \bigcup_{v \in \mathcal{P}[0]} X_v$.

The topic of this section is the basic correspondence between tropical curves factoring over the 1-skeleton of $\mathcal{P}$ and stable maps to $X_0$ of a certain primitive kind on each irreducible component and with image disjoint from toric strata of codimension at least 2. This last property is very essential in what follows and therefore deserves a name.

Definition 4.1. Let $X$ be a toric variety. An algebraic curve $C \subset X$ is **torically transverse** if it is disjoint from all toric strata of codimension $> 1$.

A stable map $\varphi : C \to X$ defined over a scheme $S$ is torically transverse if the following holds for the restriction $\varphi_s$ of $\varphi$ to every geometric point $s \to S$: $\varphi_s^{-1}(\text{int } X) \subset C_s$ is dense and $\varphi_s(C_s) \subset X$ is a torically transverse curve.

Note that if $C \subset X$ is torically transverse then no irreducible component of $C$ lies in the toric boundary; similarly, a torically transverse stable map defined over $k$ does not contract any irreducible component to the toric boundary.

The tropical curve associated to an algebraic curve $C \subset X_0$ will be its dual intersection graph, viewed as subcomplex of $\mathcal{P}$. For the balancing condition recall that if $X$ is a non-singular toric variety and $u_1, \ldots, u_k \in N$ are the primitive generators for the rays of the defining fan $\Sigma$ then

$$H_2(X, \mathbb{Z}) = \left\{ (w_1, \ldots, w_k) \in \mathbb{Z}^k \mid \sum_i w_i u_i = 0 \right\}.$$  

The $w_i$ are the intersection numbers of a singular cycle with the toric divisors $D_i$ corresponding to $u_i$. Thus a homology class $(w_1, \ldots, w_k) \in H_2(X, \mathbb{Z})$ with all $w_i \geq 0$ can be viewed as the tropical curve $h : \Gamma \to \mathbb{R}_N$ with edges the rays $\mathbb{Q}_{\geq 0} u_i$ of $\Sigma$, weighted by $w_i$, for all $i$ with $w_i > 0$. While for general $\Sigma$ a homological interpretation of the right-hand side does not seem possible, one can still define intersection numbers $w_i$ for transverse curves because a toric variety is non-singular in codimension one. Thus the following result suffices for our purposes.

Lemma 4.2. Let $X$ be a proper toric variety and $u_1, \ldots, u_k \in N$ the primitive generators of the rays of the defining fan. Assume $\varphi : C \to X$ is a torically transverse stable map from a complete, normal algebraic curve. Then $\sum_{i=1}^k w_i u_i = 0$ for $w_i = \deg \varphi^*(D_i)$ the intersection number of $C$ with the toric divisor corresponding to $u_i$.
Proof. It suffices to prove the claimed identity after pairing with any element \( m \in M = \text{Hom}(N, \mathbb{Z}) \). Note that \( \langle m, u_i \rangle \) is the order of zero (or minus the pole order) of the monomial rational function \( \chi^m \) along \( D_i \). Pulling back by \( \varphi \) brings in a factor \( w_i \). Now \( \chi^m \) being monomial all zeroes and poles are along toric divisors. Thus the identity \( \sum_i \langle m, w_i u_i \rangle = 0 \) is nothing but the vanishing of the sum of orders of zeros and poles of \( \varphi^* (\chi^m) \) on the complete, non-singular curve \( C \). \( \square \)

Having clarified this point we are in position to produce tropical curves from certain stable maps to \( X_0 \). We need the following refinement of Definition 4.1.

**Definition 4.3.** Let \( X_0 = \bigcup_{v \in \mathcal{P}} X_v \) be the central fiber of the toric degeneration \( X \rightarrow \mathbb{A}^1 \) defined by an integral polyhedral decomposition \( \mathcal{P} \) of \( N_\mathbb{Q} \). A **pre-log curve** on \( X_0 \) is a stable map \( \varphi : C \rightarrow X_0 \) with the following properties:

(i) For any \( v \) the projection \( C \times_{X_0} X_v \rightarrow X_v \) is a torically transverse stable map.

(ii) Let \( P \in C \) map to the singular locus of \( X_0 \). Then \( C \) has a node at \( P \), and \( \varphi \) maps the two branches \( (C', P), \ (C'', P) \) of \( C \) at \( P \) to different irreducible components \( X_{v'}, X_{v''} \subset X_0 \). Moreover, if \( w' \) is the intersection index with the toric boundary \( D' \subset X_{v'} \) of the restriction \( (C', P) \rightarrow (X_{v'}, D') \), and \( w'' \) accordingly for \( (C'', P) \rightarrow (X_{v''}, D'') \), then \( w' = w'' \).

Note that (ii) implies that if an irreducible component of \( C \) hits the intersection of two irreducible components \( X_{v'}, X_{v''} \subset X_0 \) then \( C \) has a node at this intersection with the two branches entering both \( X_{v'} \) and \( X_{v''} \). There are no “loose ends”, so to speak. In the proof of the Main Theorem in Section 8 we will see that (ii) is indeed a necessary condition for a torically transverse curve to deform in the family \( \pi \). The necessity of such a condition in a similar context was first pointed out by Tian [4], and it occurs at prominent place in Jun Li’s work on Gromov-Witten theory for semistable degenerations [7,11].

Now let us see how a pre-log curve \( \varphi : C \rightarrow X_0 \) gives rise to a tropical curve.

**Construction 4.4.** We say two irreducible components of \( C \) are **indistinguishable** if they intersect in a node **not** mapping to the singular locus of \( X \). Now define a weighted open graph \( \tilde{\Gamma} \) together with a map \( h : \tilde{\Gamma} \rightarrow N_\mathbb{R} \) as follows. Its set of vertices equals the quotient of the set of irreducible components of \( C \) modulo identification of indistinguishable ones. If \( C_V \subset C \) denotes the irreducible component indexed by a vertex \( V \) then \( h(V) = v \) for the unique \( v \in \mathcal{P}[0] \) with \( \varphi(C_V) \subset X_v \) (Definition 4.3 (i)). The set of bounded edges of \( \tilde{\Gamma} \) is the set of nodes of \( C \), with \( P_E \in C \) denoting the nodal point corresponding to \( E \in \tilde{\Gamma}[1] \setminus \tilde{\Gamma}[\infty] \). The map \( h \) identifies \( E \) with the line segment joining \( h(V'), h(V'') \) if \( P_E \in C_{V'} \cap C_{V''} \). If \( D \subset X_0 \) denotes the union of toric prime divisors of the \( X_v \) **not** contained in the non-normal locus of \( X_0 \) then the set of unbounded edges is \( \varphi^{-1}(D) \). An unbounded edge \( E \) labeled by \( P_E \in \varphi^{-1}(D) \) attaches to \( V \in \tilde{\Gamma}[0] \) if \( P_E \in C_V \). If \( D_e \subset X_{h(V)}, e \in \Sigma[1], \) is the toric prime divisor with \( \varphi(P_E) \subset D_e \) then \( h \) maps \( E \) homeomorphically to \( h(V) + e \subset N_\mathbb{Q} \). Finally define the weights of the edges at a vertex \( V \) by the intersection numbers of \( \varphi|_{C_V} \) with the toric prime divisors of \( X_{h(V)} \). This is well-defined by Definition 4.3 (ii).
While $\tilde{\Gamma}$ may have divalent vertices Definition 4.3 (ii) assures that the two weights at such a vertex agree. We may thus remove any divalent vertex by joining the adjacent edges into one edge. The resulting weighted open graph $\Gamma$ has the same topological realization as $\tilde{\Gamma}$ and hence $h$ can be interpreted as a map $h : \Gamma \rightarrow N_\mathbb{Q}$. This is a tropical curve for the balancing condition holds by Lemma 4.2.

Note that if $g(C) = 0$ then $\Gamma$ must be a tree, and more generally $b_1(\Gamma) \leq g(C)$. In higher genus $\Gamma$ may have multiple edges.

5. Maximally degenerate algebraic curves

The main result of this section is a description of the space of pre-log curves with fixed associated tropical curve $h : \Gamma \rightarrow N_\mathbb{Q}$. This is a partial converse of Construction 4.4. To make this work it is necessary to assume the tropical curve to be trivalent and of the correct genus. In the rational case this can be achieved by imposing a general affine constraint (Definition 2.3). Then there exist indeed only finitely many algebraic curves with given tropical dual intersection graph, and their number can be readily computed.

On the side of algebraic curves trivalence amounts to requiring that $\varphi : C \rightarrow X_0$ is of the following form componentwise.

Definition 5.1. Let $X$ be a complete toric variety and $D \subset X$ the toric boundary. A line on $X$ is a non-constant, torically transverse map $\varphi : \mathbb{P}^1 \rightarrow X$ such that $\sharp \varphi^{-1}(D) \leq 3$.

A line intersects either 2 or 3 toric prime divisors. We refer to these cases as *divalent* and *trivalent* respectively. For the following discussion of lines we fix the following notation. Let $u_i \in N$, $i = 1, 2$ or $i = 1, 2, 3$ be the primitive generators of the rays corresponding to the divisors being intersected, and $w_i$ the intersection numbers with $\varphi$. Then $\sum_i w_i u_i = 0$ (Lemma 4.2). Conversely, for any $(u, w) = (u_i, w_i) \in N^a \times (\mathbb{N} \setminus \{0\})^a$, $a \in \{2, 3\}$, with $u_i \in N$ primitive and $\sum_i w_i u_i = 0$ there exists a moduli space of lines $\mathcal{L}_{(u, w)}$ of type $(u, w)$. It is an open subspace of an appropriate space of stable maps. Clearly, $\mathbb{G}(N)$ acts on $\mathcal{L}_{(u, w)}$ by composition with the action on $X$. Let $E = (\sum_i Qu_i) \cap N$. Beware this need not be the same as the lattice generated by the $u_i$.

Lemma 5.2. Any line of type $(u, w)$ is contained in the closure of a fiber of the canonical map $\mathbb{G}(N) \rightarrow \mathbb{G}(N/E)$. The map

$$\mathcal{L}_{(u, w)} \longrightarrow \mathbb{G}(N/E)$$

thus defined is a morphism that is equivariant under $\mathbb{G}(N) \rightarrow \mathbb{G}(N/E)$.

Proof. Let $S$ be the one- or two-dimensional toric variety defined by the complete fan in $E$ with rays $Qu_i$. Then up to a toric birational transformation $X$ is a product $S \times Y$ with $Y$ a complete toric variety defined by a fan in $(N/E)_\mathbb{Q}$. Moreover, we can take this transformation to be an isomorphism at the generic points of the divisors $D_i \subset X$ corresponding to the $u_i$. Then it is an isomorphism in a neighborhood of each line of the considered type. This identifies $\mathcal{L}_{(u, w)}$ with the space of lines $\tilde{\varphi} : \mathbb{P}^1 \rightarrow S \times Y$ intersecting only the strict transforms.
of $D_i$. In particular, $\varphi$ is disjoint from toric divisors of $S \times Y$ projecting onto $S$. Hence the composition of $\varphi$ with the projection $S \times Y \to Y$ is constant. The other statements are then clear. □

Our discussion is thus reduced to $\dim X \leq 2$. The next lemma deals with the two-dimensional (trivalent) case. It is the toric analogue of the discussion in [Mi], Section 6.3. Let $\Sigma_{\mathbb{P}^2}$ be the fan in $\mathbb{P}^2$ with rays $\mathbb{Q} \cdot (1,0)$, $\mathbb{Q} \cdot (0,1)$, $\mathbb{Q} \cdot (-1,-1)$.

**Lemma 5.3.** Let $\Lambda$ be a free abelian group of rank 2 and $(u, w) = (u_i, w_i) \in \Lambda^2 \times (\mathbb{N} \setminus \{0\})^3$ with $u_i$ primitive and $\sum u_i w_i = 0$. Let $S$ be the toric surface associated to the complete fan $\Sigma_S$ with rays $\mathbb{Q} u_i$ and denote by $f_{(u, w)} : \mathbb{P}^2 \to S$ the covering defined by the map of fans

$$\Sigma_{\mathbb{P}^2} \to \Sigma_S, \quad (a, b) \mapsto aw_1 \cdot u_1 + bw_2 \cdot u_2.$$ 

Then any line $\varphi : \mathbb{P}^1 \to S$ of type $(u, w)$ is isomorphic to the composition of a linear embedding $\mathbb{P}^1 \to \mathbb{P}^2$ with $f_{(u, w)}$.

**Proof.** Let $C$ be the normalization of an irreducible component of $\mathbb{P}^1 \times S \mathbb{P}^2$ and $\tilde{\varphi} : C \to \mathbb{P}^2$ the projection. It suffices to show that $\tilde{\varphi}$ is the embedding of a line in $\mathbb{P}^2$ and the projection $C \to \mathbb{P}^1$ has degree 1.

The degree of $f_{(u, w)}$ equals the index $\delta$ of the sublattice of $\Lambda$ generated by the $w_i u_i$. Now an exercise in toric geometry shows that over the $i$-th toric divisor $D_i \subset S$ the covering $f_{(u, w)}$ has $\delta / w_i$ branches, each of which totally branched of order $w_i$ over $D_i$. This order agrees with the intersection index of $\varphi$ with $D_i$ at the unique point of intersection. Thus the composition $C \to \mathbb{P}^1 \times_S \mathbb{P}^2 \to \mathbb{P}^1$ is an unbranched cover, hence an isomorphism, and there is a unique point of intersection of $\varphi$ with the $i$-th toric prime divisor in $\mathbb{P}^2$, with intersection number 1. Hence $\varphi$ is the embedding of a curve of degree 1. □

**Remark 5.4.**
1) From the construction in the lemma it follows that in the trivalent case a line in $X(\Sigma)$ is the normalization of a rational curve with at most nodes. It touches each of the three divisors $D_i$ in only one point in $\text{int}(D_i)$ of order $w_i$. In particular, a trivalent line in $\mathbb{P}^2$ can have arbitrary degree.
2) A divalent line is just a multiple cover of degree $w_1 = w_2$ of an orbit closure for $G(\mathbb{Z} u_i)$, intersecting $D_1, D_2$ transversally.

The two previous lemmas imply the following result on the structure of the space of lines of fixed type.

**Proposition 5.5.** Let $X$ be a toric variety. Then the action of the big torus $G(N)$ on the space of lines $\mathcal{L}_{(u, w)}$ of fixed type is transitive. In the trivalent case this action is simply transitive, while in the divalent case the action factors over a simply transitive action of $G(N/\mathbb{Z} u_1) = G(N/\mathbb{Z} u_2)$.

**Proof.** Lemma 5.2 reduces to the case $\dim X \leq 2$. In the one-dimensional (divalent) case there is only one isomorphism class of stable maps $\mathbb{P}^1 \to X$ that is totally branched over 0 and $\infty$. Its image is the closure of a $G(\mathbb{Z} u_i)$-orbit.
In the trivalent case the toric birational morphism $X \to S$ to the toric surface from Lemma 5.3 reduces further to the case $X = S$. We retain the notations from this lemma. Any line $\varphi : \mathbb{P}^1 \to S$ lifts to at most $\delta = \deg f_{(u,w)}$ distinct lines in $\mathbb{P}^2$. On the other hand, $\delta$ is also the order of the kernel of the homomorphism $G(\mathbb{Z}^2) \to G(\Lambda)$ underlying $f_{(u,w)}$, and $G(\mathbb{Z}^2)$ acts transitively on the set of lines in $\mathbb{P}^2$. Hence $\ker (G(\mathbb{Z}^2) \to G(\Lambda))$ acts transitively on the set of lifts of $\varphi$. This proves that the action of $G(\Lambda)$ on the space of lines on $S$ of considered type is simply transitive. \hfill \Box

Having understood lines on toric varieties we are now in position to discuss maximally degenerate curves on unions of toric varieties.

**Definition 5.6.** Let $X_0 = \bigcup_{v \in \mathcal{P}[0]} X_v$ be the central fiber of a toric degeneration defined by an integral polyhedral decomposition $\mathcal{P}$. A pre-log curve (Definition 1.13) $\varphi : C \to X_0$ is called **maximally degenerate** if for any $v \in \mathcal{P}[0]$ the projection $C \times_{X_0} X_v \to X_v$ is a line, or, for $n = 2$, the disjoint union of two divalent lines intersecting disjoint toric divisors.

Thus a maximally degenerate curve is a collection of lines, at most one ($n = 2$: two) for each irreducible component of $X_0$, which match in the sense that they glue to a pre-log curve. The matching condition involves both incidence of the intersections with the toric divisors and equality of the intersection numbers for glued branches.

To realize a rational tropical curve $h : \Gamma \to N_\mathbb{R}$ as the dual intersection graph of a maximally degenerate curve we assume that $\Gamma$ factors over the 1-skeleton of a polyhedral decomposition.

**Proposition 5.7.** 1) Let $\Delta \in \text{Map}(N \setminus \{0\}, \mathbb{N})$ be a degree and $A = (A_1, \ldots, A_l)$ an affine constraint that is general for $\Delta$. If $\Xi_{(\Gamma,E,u)}(A) \neq \emptyset$ for an $l$-marked tree $(\Gamma, E)$ then the map

\begin{equation}
\text{Map}(\Gamma[0], N) \to \prod_{E \in \Gamma[0] \setminus \Gamma_\infty} N/\mathbb{Z}u_{(\partial^+ E, E)} \times \prod_{i=1}^l N/(\mathbb{Q}u_{(\partial^- E_i, E_i)} + L(A_i)) \cap N,
\end{equation}

is an inclusion of lattices of finite index $\Delta$. Here $\partial^\pm : \Gamma[1] \setminus \Gamma_\infty \to \Gamma[0]$ is an arbitrarily chosen orientation of the bounded edges, that is, $\partial E = \{\partial^+ E, \partial^- E\}$ for any $E \in \Gamma[1] \setminus \Gamma_\infty$. If $E \in \Gamma_\infty$ then $\partial^- E$ denotes the unique vertex adjacent to $E$.

2) Assume that $\mathcal{P}$ is an integral polyhedral decomposition of $N_\mathbb{Q}$ with

\[ h(\Gamma[\mu]) \subset \bigcup_{\Xi \in \mathcal{P}[\mu]} \Xi, \quad \mu = 0, 1, \quad \text{and} \quad h(\Gamma) \cap A_j \subset \mathcal{P}[0], \quad j = 1, \ldots, l, \]

and associated toric degeneration $X(\bar{\Sigma}_\mathcal{P}) \to \mathbb{A}^1$ with central fiber $X_0$, and let $P_j, j = 1, \ldots, l$ be closed points in the big torus of $X(\bar{\Sigma}_\mathcal{P})$. Then $\Delta$ equals the number of isomorphism classes of maximally degenerate curves in $X_0$ with associated tropical curve $\Gamma$ and intersecting

\[ Z_i := \overline{G(\text{LC}(A_i)).P_i} \subset X(\bar{\Sigma}_\mathcal{P}), \]

the closure of the orbit through $P_i$ for the subgroup $G(\text{LC}(A_i)) \subset G(N \times \mathbb{Z})$ acting on $X(\bar{\Sigma}_\mathcal{P})$. 

Proof. Tensored with \( \mathbb{Q} \) the map agrees with the linear part of \( \Phi \) in Corollary 2.5. By this corollary and since \( \mathbf{A} \) is general \( \Phi \) is an isomorphism. This proves (1).

Going over to algebraic tori thus leads to a covering of degree \( \mathcal{D} \). Now a maximally degenerate curve with associated tropical curve \( \Gamma \) is glued from a number of lines. To make this precise let \( \tilde{\Gamma} \) denote the open graph obtained from \( \Gamma \) by inserting vertices at all points of \( h^{-1}([0]) \setminus \Gamma[0] \). Conversely, \( \Gamma \) is obtained from \( \tilde{\Gamma} \) by merging all edges meeting in a divalent vertex (cf. Construction 4.4). Now for each vertex \( V \in \tilde{\Gamma}[0] \) there is a moduli space of lines \( \mathcal{L}(u(V),w(V)) \) on the component \( X_{h(V)} \subset X_0 \). Here \( u(V),w(V) \) are the tuples of directions \( u(V,E) \) and weights \( w(E) \) for edges \( E \) adjacent to \( V \), respectively. A trivalent vertex \( V \) corresponds to a trivalent line, and in this case \( \mathcal{L}(u(V),w(V)) \) is a torsor under \( \mathbb{G}(N) \) (Proposition 5.5). Hence \( \prod_{V \in \tilde{\Gamma}[0]} \mathcal{L}(u(V),w(V)) \) is a torsor under \( \text{Map}(\Gamma[0],\mathbb{G}(N)) \). If \( V \) is divalent with adjacent edges \( E^\pm(V) \) then \( \mathcal{L}(u(V),w(V)) \) is only a torsor under \( \mathbb{G}(N/\mathbb{Z}u(V,E^-(V))) = \mathbb{G}(N/\mathbb{Z}u(V,E^+(V))) \). Taken together \( \prod_{V \in \tilde{\Gamma}[0]} \mathcal{L}(u(V),w(V)) \) is a torsor under the algebraic group belonging to the left-hand side of (17) times \( \mathcal{G} := \prod_{V \in \tilde{\Gamma}[0]\setminus \Gamma[0]} \mathbb{G}(N/\mathbb{Z}u(V,E^-(V))) \).

Now the matching and incidence conditions are also a torsor, but under the algebraic torus action defined by the right-hand side of (17), times \( \mathcal{G} \). In fact, if a bounded edge \( E \) of \( \Gamma \) connects two trivalent vertices then the lines \( L^- \in \mathcal{L}(u(\partial^- E),w(\partial^- E)), L^+ \in \mathcal{L}(u(\partial^+ E),w(\partial^+ E)) \) intersect the \( (n-1) \)-dimensional toric variety \( X_{h(E)} \) in closed points \( P^\pm \) in its big torus. This big torus is a torsor under \( \mathbb{G}(N/\mathbb{Z}u(\partial^- E,E)) = \mathbb{G}(N/\mathbb{Z}u(\partial^+ E,E)) \), and \( L^\pm \) glue iff \( P^- = P^+ \). The gluing on the schem-theoretic level works by proposition 3.5. The other case is that one of the vertices is divalent, say \( \partial^- E \). Then \( \mathcal{L}(u(\partial^- E),w(\partial^- E)) \) is a torsor under \( \mathbb{G}(N/\mathbb{Z}u(\partial^- E,E)) \subset \mathcal{G} \) and hence \( L^- \) is determined uniquely by \( L^- \cap X_{h(E)} \). If \( E' \) is the other edge adjacent to \( \partial^- E \) then this has the effect of transferring the matching condition on \( X_{h(E)} \) to \( X_{h(E')} \). In other words, the matchings at divalent vertices form a torsor under \( \mathcal{G} \).

Finally, if \( h(E_i) \) intersects \( A_i \) in \( V \in \tilde{\Gamma}[0] \) then the image of any \( L^- \in \mathcal{L}(u(V),w(V)) \) is the closure of a \( \mathbb{G}(\mathbb{Z}u(\partial^- E_i,E_i)) \)-orbit, while by Corollary 3.8 \( Z_i \cap X_{h(V)} \) is the closure of a \( \mathbb{G}(L(A_i) \cap N) \)-orbit in int(\( X_{h(V)} \)). Choose a closed point \( P_i \in Z_i \cap \text{int}(X_{h(V)}) \), and put \( L_0^- := \mathbb{G}(\mathbb{Z}u(\partial^- E_i,E_i))P_i \). Then for any \( \mathbb{k} \)-rational point \( \lambda \) of \( \mathbb{G}(N) \)

\[
\lambda \cdot L_0^- \cap Z_i \neq \emptyset \iff \lambda \in \mathbb{G}\left((\mathbb{Q}u(\partial^- E_i,E_i) + L(A_i)) \cap N\right)(\mathbb{k}).
\]

In fact, by the definition of \( L_0^- \), \( \lambda \cdot L_0^- \cap Z_i \neq \emptyset \) iff there exists \( \lambda' \in \mathbb{G}(\mathbb{Z}u(\partial^- E_i,E_i))(\mathbb{k}) \) with \( \lambda'P_i \in Z_i \). Since \( Z_i = \mathbb{G}(L(A_i) \cap N)P_i \) it follows that \( \lambda' \) is a \( \mathbb{k} \)-rational point of \( \mathbb{G}(L(A_i) \cap N) \). Thus \( \lambda \) is a \( \mathbb{k} \)-rational point of the subtorus of \( \mathbb{G}(N) \) generated by \( \mathbb{G}(\mathbb{Z}u(\partial^- E_i,E_i)) \) and \( \mathbb{G}(L(A_i) \cap N) \), which is the claimed \( \mathbb{G}(\mathbb{Q}u(\partial^- E_i,E_i) + L(A_i)) \cap N) \). Conversely, for any \( \mathbb{k} \)-rational point \( \lambda \) of this subtorus \( \lambda L_0^- \cap Z_i \neq \emptyset \). Hence \( \mathbb{G}(N/(\mathbb{Q}u(\partial^- E_i,E_i) + L(A_i)) \cap N) \) acts simply transitively on the incidence condition with \( Z_i \).

Taken together the problem of finding maximally degenerate curves of the considered type is governed by a map of torsors that is equivariant for a map of algebraic tori of degree \( \mathcal{D} \). Hence there are exactly \( \mathcal{D} \) solutions. \( \square \)

Remark 5.8. 1) The points \( P_i \) in the proposition need not be general. For example, they could all be equal to the distinguished point \( 1 \in \mathbb{G}(N \times \mathbb{Z}) \subset X(\Sigma_{\Phi})\). The point is that we
only care about genericity of the degeneration of the $G(LC(A_i))$-orbits $Z_i$. This is achieved by genericity of the incidence conditions $A_i$.

2) The proof also determines the number of intersection points between $Z_i$ and the image $C_0 \subset X_0$ of the constructed maximally degenerate curves that intersect $Z_i$. In fact, they intersect only in the big torus of one component of $X_0$, namely $X_v$ if $A_i \cap \text{im}(h) = \{v\}$.

The proof establishes a bijection between $Z_i \cap C_0$ and the intersection of the two subtori $G(Z_u(\partial - E_i, E_i))$ and $G(L(A_i) \cap N)$ in $G(N)$. This latter number of intersection points is the covering degree of $G(Z_u(\partial - E_i, E_i)) \times G(L(A_i) \cap N) \rightarrow G((Qu(\partial - E_i, E_i) + L(A_i)) \cap N)$, which equals the index $[Z_u(\partial - E_i, E_i) + L(A_i)] \cap N : (Qu(\partial - E_i, E_i) + L(A_i)) \cap N]$. 

3) In dimension two a tropical curve may have two edges with interiors of their images intersecting. This corresponds to two divalent lines on the same component of $X_0$ with intersecting images. However, as we construct stable maps $C_0 \rightarrow X_0$ rather than embedded curves intersection points of such lines are not images of nodes of $C_0$. Hence this phenomenon is irrelevant to our treatment. □

6. Transversality of degenerating families

In the previous section we described rational curves in degenerate toric varieties that are transverse with respect to the toric strata. In this section we show that for any finite number of one-parameter degenerating families of curves we can always achieve this kind of transversality by toric blow-up of the central fiber, possibly after base change.

**Lemma 6.1.** Let $X$ be a toric variety and $W \subset X$ a proper subset without irreducible components contained in the toric boundary. Then there exists a toric blow-up $\Upsilon : \tilde{X} \rightarrow X$ such that the strict transform $\tilde{W}$ of $W$ under $\Upsilon$ does not contain any 0-dimensional toric stratum.

**Proof.** Let $\Sigma$ be the fan defining $X$. Any toric blow-up is defined by a subdivision of $\Sigma$. The conclusion of the theorem is stable under further toric blow-up, and combinatorially this translates into a freedom of further subdivision. Now subdivisions of subfans of $\Sigma$ are subfans of a common subdivision of $\Sigma$. We may therefore assume that $\Sigma$ is the fan defined by the faces of a single $n$-dimensional cone $\sigma \subset \mathbb{N}_Q$. (If $\dim \sigma < n$ there are no 0-dimensional toric strata.) Then $X = \text{Spec} \mathbb{K}[\sigma^\vee \cap M]$ is affine and taking a general element of the ideal defining $W$ reduces to the case that $W$ is a hypersurface.

Let $f = \sum_{p \in \sigma^\vee \cap M} a_p x^p \in \mathbb{K}[\sigma^\vee \cap M]$ be the regular function defining $W$. Denote by $\Delta_f = \text{conv} \left( \bigcup_{\{p \in \sigma^\vee \cap M | a_p \neq 0\}} \{p\} + (\sigma^\vee \cap M) \right) \subset \mathbb{M}_Q$ the Newton polyhedron of $f$. Let $\tilde{\Sigma}$ be the normal fan of $\Delta_f$. This fan has rays $\mathbb{Q} \cdot u$ with $\langle u, p \rangle \geq 0$ for all $p \in \sigma^\vee \cap M$ and such that $u^\perp$ is parallel to a facet of $\Delta_f$. In particular, it
is a subdivision of $\Sigma$, and hence it defines a toric blow-up $\Upsilon : \tilde{X} \to X$. We claim that $\Upsilon$ has the requested properties.

Let $x \in \tilde{X}$ be a 0-dimensional toric stratum and $\tau \in \bar{\Sigma}$ be the corresponding $n$-dimensional cone. Then $\tau$ defines the affine toric neighborhood $\text{Spec} \, k[\tau^\vee \cap M]$ of $x$, and $\Upsilon$ is given locally by the inclusion of rings $\phi : k[\tau^\vee \cap M] \to k[\tau^\vee \cap M]$. Since $\bar{\Sigma}$ is the normal fan of $\Delta_f$ there exists a $p_r \in \sigma^\vee \cap M$ with $a_{p_r} \neq 0$ and $\Delta_f \subset p_r + (\tau^\vee \cap M)$. In particular, every monomial in $\phi(f)$ is divisible by $\chi^{p_r}$:

$$\phi(f) = \chi^{p_r} \cdot \sum_{p \in \sigma^\vee \cap M \subset \tau^\vee \cap M} a_p \chi^{p-p_r}.$$ 

Then $f_r = \sum_{p \in \sigma^\vee \cap M} a_{p_r+p} \chi^p$ is a function vanishing on the strict transform $\tilde{W}$ of $W$. This last argument uses the assumption that $W$ is not contained in the toric boundary. But $f_r(x) = a_{p_r} \neq 0 \in \mathcal{O}_{\tilde{X},x}/m_x = k$, and hence $x \notin \tilde{W}$. \hfill $\square$

**Proposition 6.2.** Let $X$ be a toric variety and $W \subset X$ a closed subset of codimension $> c$. We assume that no component of $W$ is contained in the toric boundary. Then there exists a toric blow-up $\Upsilon : \tilde{X} \to X$ such that the strict transform $\tilde{W}$ of $W$ under $\Upsilon$ is disjoint from any toric stratum of dimension $\leq c$.

**Proof.** By induction on $c$ we may assume that $W$ is already disjoint from toric strata of dimension less than $c$. Any $c$-dimensional torus orbit in $X$ has a toric neighborhood of the form $\text{Spec} \, k[\tau^\vee \cap M]$ for $\tau \subset N_{Q}$ an $(n-c)$-dimensional strongly convex cone. As in the proof of the lemma the existence of common refinements of partial subdivisions thus reduces to the case that the fan $\Sigma$ defining $X$ is the fan of faces of such a $\tau$. Let $\bar{\Sigma}$ be the fan in $L(\tau)$ of faces of $\tau$. A linear projection $\pi : N_{Q} \to L(\tau)$ defines a map of fans $\Sigma \to \bar{\Sigma}$. Let $\Psi : X \to Y$ be the corresponding toric morphism. A subdivision $\Sigma_\tau$ of $\bar{\Sigma}$ induces the subdivision

$$\pi_\tau^{-1}\Sigma_\tau := \{ \tau \cap \pi_\tau^{-1}(\omega) \mid \omega \in \Sigma_\tau \}$$

of $\Sigma$. We are going to construct $\Sigma_\tau$ such that the strict transform of $W$ for the toric blow-up $\tilde{X} \to X$ defined by this subdivision is disjoint from all $c$-dimensional toric strata of $\tilde{X}$.

Now the unique $c$-dimensional toric stratum of $X$ is mapped to the 0-dimensional toric stratum of $Y$. Because $\dim \Psi(W) \leq \dim W < n - c = \dim Y$ and because no irreducible component of $W$ is contained in the toric boundary of $X$, $\Psi(W) \subset Y$ is a proper subset without irreducible components in the toric boundary of $Y$. Thus Lemma 6.1 provides a toric blow-up of $Y$ such that the strict transform of $\Psi(W)$ does not contain 0-dimensional toric strata. The desired fan $\Sigma_\tau$ is the fan declaring this toric blow-up. \hfill $\square$

**Proposition 6.3.** Let $X \to \mathbb{A}^1$ be a toric degeneration with special fiber $X_0 \subset X$, let $R$ be a discrete valuation ring with residue field $k$ and quotient field $K$ and $\text{Spec} \, R \to \mathbb{A}^1$ a dominant morphism mapping the closed point to $0 \in \mathbb{A}^1$. Let $(C^* \to X \setminus X_0, (x_1^*, \ldots, x_l^*))$ be a torically transverse stable map with $l$ marked points $x_1^*, \ldots, x_l^* : \text{Spec} \, K \to C^*$, defined over $\text{Spec} \, K$.

Then possibly after base change $\mathbb{A}^1 \to \mathbb{A}^1$, $t \mapsto t^b$ there exists a toric blow-up $\tilde{X} \to \mathbb{A}^1$ with centers in $X_0$ with the following property: $C^*$ extends to a stable map $(C \to \tilde{X}, (x_1, \ldots, x_l))$
over Spec $R$ such that for every irreducible component $\tilde{X}_v \subset \tilde{X}_0$ the projection $C \times_{\tilde{X}_0} \tilde{X}_v \to \tilde{X}_v$ is a torically transverse stable map.

Proof. Let $\check{C}^* \to C^*$ be the normalization of $C$. By Abhyankar’s Lemma ([SGA1, Exposé X, Lemma 3.6]) there exists a totally ramified base change $t \mapsto t^b$ such that the conductor locus of this normalization is the image of additional sections $y_1^*, \ldots, y_l^*$. Then $C^*$ is obtained by pairwise identification of the images of these sections. Let $\tilde{x}_i^*$ be the pull-back of $x_i^*$ to $\check{C}^*$. Then for any irreducible component $\check{C}_\mu^* \subset \check{C}^*$ the composition $\check{C}_\mu^* \to \check{C}^* \to C^* \to X$ marked by those $\tilde{x}_i^*$ and $y_j^*$ with image contained in $\check{C}_\mu^*$ is also a marked stable map fulfilling the hypothesis. If any of these stable maps extend after a base change then we obtain an extension of the original stable map by pairwise identification of the extensions $y_j$ of $y_j^*$. This argument reduces to the case $C^*$ irreducible.

Let $W$ be the closure of the image of $C^* \to X$. Then $W$ is one- or two-dimensional. In the one-dimensional case, by Proposition 6.2 we may assume that $W$ is disjoint from any toric stratum of $X_0$ of dimension less than $n = \dim X_0 = \dim X - 1$. Then $W \to \mathbb{A}^1$ is a proper and dominant map of curves, hence a finite surjection. The geometric fibers of $C^* \to \text{Spec } K$ being connected, $C^* \to W$ thus factors over $\text{Spec } K$ (apply Stein factorization to $C^* \to \mathbb{A}^1$). It therefore just remains to extend $(C^* \to \text{Spec } K, (x_1^*, \ldots, x_l^*))$ as a marked stable curve. This is possible by [Kn] after another totally ramified base change.

In the two-dimensional case Proposition 6.2 merely achieves that $W$ is disjoint from toric strata of dimension less than $n - 1$. Let $X_\tau \subset X$ be an irreducible component of $X_0$ intersecting $W$. There is a neighbourhood $U_\tau \subset X$ of $\text{int}(X_\tau) \simeq G_m \times G_m$ (non-canonically) isomorphic to $\text{int}(X_\tau) \times V_e (e = e(\tau))$ with

\[ V_k = \text{Spec } k[x, y, t]/(xy - t^e). \]

Let $\check{U}_\tau^*$ be the preimage of $U_\tau$ in $C^*$. Consider the composition of $\check{U}_\tau^* \to U_\tau$ with the projection $U_\tau \to V_e$. This is a dominant map of reduced $k$-schemes of the same dimension, hence it is étale away from a nowhere dense closed subset. Let $Z_\tau \subset V_e$ be this closed subset union the closure of the images of the marked points, intersected with $V_e$. By another application of Proposition 6.2 we may assume that $Z_\tau$ does not contain the zero-dimensional torus orbit of $V_e$. Do this for any $\tau$ with $X_\tau \cap W \neq \emptyset$.

Now apply the stable reduction theorem for stable maps ([FuPa, Proposition 6]). This gives, after another base change, an extension of $(C^* \to X, (x_1^*, \ldots, x_l^*))$ to a marked stable map $(\varphi: C \to X, (x_1, \ldots, x_l))$ over $\text{Spec } R$. By construction the image of this map is a family of torically transverse curves. It remains to show that the restriction $\varphi_0: C_0 \to X$ to the closed fiber does not contract any component to the codimension one locus of $X_0$. It suffices to check this statement on $\check{U}_\tau = \varphi^{-1}(U_\tau) \subset C^*$. Thus we only have to verify that no irreducible component of $C_0 \cap \check{U}_\tau$ maps to the singular point of $V_e$. To this end let $\tilde{Z}_\tau = \varphi^{-1}(Z_\tau)$. The restriction of $\varphi$ to $\check{U}_\tau \setminus \tilde{Z}_\tau$ factors over a proper map to $\text{Spec}(O_{k^1,0}) \times_{k^1} (U_\tau \setminus Z_\tau) \subset X$. This map is finite except possibly over a finite set $T \subset X_0$. Hence Stein factorization produces another extension of $\varphi|_{\check{U}_\tau \setminus \tilde{Z}_\tau}$ over $\text{Spec } R$. This extension glues with $\varphi|_{C \setminus \varphi^{-1}(T)}$ to a map $\varphi': C' \to X$ because $\varphi$ is already finite on the gluing region. Moreover, since $Z_\tau$ contains
the inclusion of the toric boundary defines a natural log structure
the central fiber. Let $(\Gamma, E, h)$ be an $l$-marked rational tropical curve
with image contained in the 1-skeleton of $\mathcal{P}$. We assume that $(\Gamma, E, h)$ are transverse
in the sense that the map $\Phi$ in (4) before Corollary 2.5 is an isomorphism. We also assume
that all points of intersection of the $A_i$ with $h(\Gamma)$ are vertices of $\mathcal{P}$. Let $(C_0, \varphi_0, \varphi)$ be a
maximally degenerate, $l$-marked, rational stable map with dual intersection graph $(\Gamma, E, h)$ and with
$\varphi_0(x_i) \in \mathbb{Z}$, $\varphi(\text{LC}(A_i)).P_i \subset X$ for $j = 1, \ldots, l$, as given by Proposition 5.7. By
Corollary 3.8 the map $\mathbb{Z} \rightarrow \mathbb{A}^1$ is smooth at $\varphi_0(x_i)$ since we assumed that the unique point of
intersection of $h(E_i)$ with $A_i$ is a vertex of the polyhedral decomposition.

On any toric variety $X$ the inclusion of the toric boundary defines a natural log structure
$\mathcal{M}_X \rightarrow \mathcal{O}_X$, and a toric morphism is naturally a (log-) smooth morphism for these log
structures. Thus our degeneration defines a smooth morphism $\tau : X \rightarrow \mathbb{A}^1$ of log spaces.
Restricting to the central fiber now defines a smooth morphism to the standard log point
$$\pi_0 : X_0 \rightarrow O_0 := (\text{Spec} k, \mathbb{N} \times k^\times).$$
Recall also that the sheaf of logarithmic derivations $\Theta_X(\Sigma)$ of a toric variety defined by a fan
$\Sigma$ in $N_Q$ is canonically isomorphic to $N \otimes \mathbb{Z} \mathcal{O}_X(\Sigma)$. Hence
$$\Theta_{X/\mathbb{A}^1} = (N \oplus \mathbb{Z}) \otimes \mathcal{O}_X / (0 \oplus \mathcal{O}_X) = N \otimes \mathbb{Z} \mathcal{O}_X,$$
and then also $\Theta_{X_0/O_0} = N \otimes \mathbb{Z} \mathcal{O}_{X_0}$ by base change.
Next we want to lift \( \varphi_0 : C_0 \to X_0 \) to a log morphism. Consider the diagonal map \( \mathbb{N} \to \mathbb{N}^2 \) and the homothety \( \mathbb{N} \overset{c}{\to} \mathbb{N} \) and denote by \( S_e = \mathbb{N}^2 \oplus \mathbb{N} \) the monoid defined by push-out. Then \( S_e \) has generators \( a = ((1,0),0) \), \( b = ((0,1),0) \), \( c = ((0,0),1) \) with single relation \( a + b = c \cdot c \), and hence \( k[S_e] \simeq k[x,y,l]/(xy - l^c) \). Recall from Section 1 the notion of total marked weight \( w(\Gamma, E) = \prod_{E \in \Gamma^{[1]}} w_i(E) \cdot \prod_{i=1}^{d} w(E_i) \), and from the proof of Proposition 7.1 the subdivision \( \bar{\Gamma} \) of \( \Gamma \) with \( \bar{\Gamma}[0] = h^{-1}(\emptyset^{[0]} \rangle \).

**Proposition 7.1.** Assume that for every bounded edge \( E \subset \bar{\Gamma}^{[1]} \) the integral length of \( h(E) \) is a multiple of its weight \( w(E) \). Then there are exactly \( w(\Gamma, E) \) pairwise non-isomorphic pairs \( (\varphi_0 : C_0 \to X_0, x_0) \) with underlying marked stable map isomorphic to \((C_0, x_0, \varphi_0)\), with \( \varphi_0 \) strict wherever \( X_0 \to O_0 \) is strict, and such that the compositions \( C_0 \to X_0 \to O_0 \) are smooth and integral.

**Proof.** The fact that \( \varphi_0 \) lifts to a morphism of log spaces means that the pull-back log structure \( \varphi_0^* \mathcal{M}_{X_0} \to \mathcal{O}_{C_0} \) factors over a log structure \( \mathcal{M}_{C_0} \to \mathcal{O}_{C_0} \) that is smooth over \( \mathcal{O}_0 \). The locus of strictness of the morphism \( X_0 \to O_0 \) is the union of the big tori of the components of \( X_0 \). On this part there is no choice because strictness says \( \varphi_0^* \mathcal{M}_{X_0} = \mathcal{M}_{C_0} \), and this log structure is also smooth over \( O_0 \) trivially. The only non-nodal points \( P \in C_0 \) that do not map to this locus map to \( \bigcup_{E \in \Gamma^{[1]}} X_0(E) \), the degeneration of the toric boundary of the general fibers. On this part toric transversality implies that \( \varphi_0^* \mathcal{M}_{X_0,P} \to \mathcal{O}_{C_0,P} \) has precisely one smooth extension, namely the sum of \( (\pi \circ \varphi_0)^* \mathcal{M}_{O_0} \) and the log structure associated to the toric divisor \( X_{h(E)} \subset X_{h(\partial E)} \), see Proposition 1.1 in [KaF2], [LiJ2], [Mo], [We]. Due to its central importance in this paper and to avoid a technical discussion on how it relates to known formulations, we provide full details here.

Denote by \( t \) a linear coordinate on \( \mathbb{A}^1 \) and its lift to \( X \). Then \( \mathbb{N} \to \mathcal{M}_{\mathbb{A}^1} \subset \mathcal{O}_{\mathbb{A}^1} \), \( 1 \mapsto t \) is a chart for the log structure on \( \mathbb{A}^1 \). By toric transversality \( \varphi_0 \) \( P \) lies in the big torus of the \((n - 1)\)-dimensional toric variety \( X_{h(E)} \subset (X_0)^{\text{sing}} \). The total space has a singularity of type \( A_{e-1} \) along the big torus of \( X_{h(E)} \) where \( e \) is the integral length of \( h(E) \). Thus via a toric chart there exist \( x, y \in \mathcal{M}_{X, \varphi_0(P)} \subset \mathcal{O}_{X, \varphi_0(P)} \) such that \( xy = t^c \) and

\[
S_e = \mathbb{N}^2 \oplus \mathbb{N} \to \mathcal{O}_{X, \varphi_0(P)}, \quad ((a, b), c) \mapsto x^a y^b t^c
\]

is a chart for the log structure on \( X \) at \( P \). The homomorphism \( \pi^* \mathcal{M}_{\mathbb{A}^1} \to \mathcal{M}_X \) lifts to a map of charts via the inclusion \( \mathbb{N} \to \mathbb{N}^2 \oplus \mathbb{N} \) into the second factor. By the pre-log condition (Definition 4.3 (ii)) there exist generators \( z, w \in \mathcal{O}_{C_0, \varphi_0} \) of the maximal ideal with

\[
\varphi_0^*(x) = z^\mu, \quad \varphi_0^*(y) = w^\mu
\]

In particular, \( zw = 0 \). Denote by \( s_x, s_y, s_t \in \mathcal{M}_{X_0, \varphi_0} \) the restrictions of \( x, y, t \in \mathcal{M}_{X, \varphi_0} \) to the central fiber. Then \( s_x s_y = s_t^e \) holds in \( \mathcal{M}_{X_0, \varphi_0} \). All these choices do not depend on the extension \( \mathcal{M}_{C_0} \) and are made once and for all.
By \( \varphi_i(s_x), \varphi_i^*(s_y) \in (\mathcal{L}_i^* \mathcal{M}_{X_0})_{\overline{\mathcal{P}}} \) map to \( z^\mu, w^\mu \) under the structure homomorphism \((\mathcal{L}_i^* \mathcal{M}_{X_0})_{\overline{\mathcal{P}}} \to \mathcal{O}_{C_0, \overline{\mathcal{P}}}, \) respectively. Thus for any factorization \( \mathcal{M}_{C_0} \to \mathcal{O}_{C_0} \) of \( \mathcal{L}_i^* \mathcal{M}_{X_0} \to \mathcal{O}_{C_0} \) with the requested properties there exist unique \( s_z, s_w \in \mathcal{M}_{C_0, \overline{\mathcal{P}}} \) with \( \varphi_i^*(s_x) = s_z^\mu, \varphi_i^*(s_y) = s_w^\mu \) and \( s_z, s_w \) lifting \( z, w. \) Then \( s_x s_y = s_z^\mu \) implies \( (s_z s_w)^\mu = s_z^\mu. \) This is equivalent to \( s_z s_w = \zeta s_z^{\mu} \) for a well-defined \( \mu \)-th root of unity \( \zeta \in \mathbb{k}^\times. \) This step uses the integrality of \( e/\mu. \) Conversely, for any \( \zeta \in \mathbb{k}^\times \) with \( \zeta^\mu = 1 \) let \( \mathcal{M}_{C_0, \overline{\mathcal{P}}} \to \mathcal{O}_{C_0, \overline{\mathcal{P}}} \) be the log structure at \( \overline{\mathcal{P}} \) defined by the map

\[
S_{e/\mu} \to \mathcal{O}_{C_0, \overline{\mathcal{P}}}, \quad ((a, b), c) \mapsto \begin{cases} 
(\zeta^{-1} z)^a w^b, & c = 0 \\
0, & c \neq 0.
\end{cases}
\]

Denote by \( s_{((a,b),c)} \in \mathcal{M}_{C_0, \overline{\mathcal{P}}} \) the element belonging to \( ((a, b), c) \in S_{e/\mu}. \) The morphism to \( O_0 \) given by mapping \( s_i \) to \( s_{((0,0),1)} \) is smooth. In particular, its restriction to \( C_0 \setminus \{P\} \) is strict and hence equals \( \mathcal{L}_0^* \mathcal{M}_{X_0}|_{C_0 \setminus \{P\}} \). Mapping \( s_x, s_y \) to the \( \mu \)-th powers of \( s_z := \zeta s_{((1,0),0)}, s_w := s_{((0,1),0)} \), respectively, yields a compatible extension \( (\mathcal{L}_i^* \mathcal{M}_{X_0})_{\overline{\mathcal{P}}} \to \mathcal{M}_{C_0, \overline{\mathcal{P}}} \) with associated \( \mu \)-th root \( \zeta. \)

Our arguments so far produce a complete list of \( \prod_{E \in \Gamma[1]\setminus\Gamma[\infty]} w(E) \) log morphisms \( C_0 \to X_0 \) with the requested properties. It remains to show that together with the marked points this list comprises only

\[
w(\Gamma, E) = \prod_{E \in \Gamma[1]\setminus\Gamma[\infty]} w(E) \cdot \prod_{i=1}^l w(E_i)
\]

isomorphism classes. First, note that for any of our log structures \( \mathcal{M}_{C_0} \) on \( C_0 \) the sheaf \( \mathcal{M}_{C_0}/\mathcal{O}_{C_0} \) of finitely generated monoids is naturally a subsheaf of \( \mathcal{L}_0^* \mathcal{N}_{\mathcal{C}_0} \) for \( \mathcal{L}_0 : \mathcal{C}_0 \to \mathcal{C}_0 \) the normalization. At a nodal point \( P \in C_0 \) with chart \( S_{e/\mu} \to \mathcal{O}_{C_0, \overline{\mathcal{P}}} \) the image is generated by \( (e/\mu)s_1, (e/\mu)s_2 \) and \( s_1 + s_2 \) if \( s_1, s_2 \) are generators of \( \mathcal{L}_0^* \mathcal{N}_{\mathcal{C}_0, \overline{\mathcal{P}}} = \mathbb{N}^2. \) Away from the nodes it is an isomorphism. Second, if \( \kappa : \mathcal{C}_0 \to \mathcal{C}_0 \) commutes with \( \varphi_0 \) then it maps each irreducible component and each node to itself. In particular, \( \kappa \) induces the identity transformation on \( \mathcal{L}_0^* \mathcal{N}_{\mathcal{C}_0} \). Thus if \( \varphi_0' : C_0' \to X_0, \varphi_0'' : C_0'' \to X_0 \) are two log morphisms from our list and \( \kappa : C_0'' \to C_0' \) is an isomorphism with \( \varphi_0'' = \varphi_0' \circ \kappa \) then \( \kappa \) induces an isomorphism \( \mathcal{M}_{C_0'}/\mathcal{O}_{C_0'} \to \mathcal{M}_{C_0''}/\mathcal{O}_{C_0''}. \) This implies that \( \kappa \) is strict (KaFT, Lemma 3.3). Therefore it suffices to investigate the effect of the action of the \( k \)-rational points of the identity component \( \text{Auto}^0(C_0) \subseteq \text{Auto}(C_0) \) on our construction of \( \mathcal{M}_{C_0} \) by pull-back. Taking into account the marked points reduces further to the subgroup \( \text{Auto}^0(C_0, x_0) \subseteq \text{Auto}^0(C_0). \)

This action is non-trivial only on the components of \( C_0 \) having at most two special points, that is, corresponding to unmarked divalent vertices of \( \Gamma. \) In particular, \( \text{Auto}^0(C_0) \simeq \mathbb{G}_m^{[\Gamma[1]\setminus\Gamma[\infty]] - l}. \) We are thus reduced to considering the following situation. Let \( E \in \Gamma[1], \) assumed unmarked for the time being, split into edges \( E^1, \ldots, E^r \) in \( \Gamma \) with \( \partial E^1 = \{V_{i-1}, V_i\} \) for the bounded edges, and \( \partial E^1 = \{V_1\} \) in case \( E \) is unbounded (and then \( E_i \in \Gamma[1] \) iff \( i = 1 \)). Let \( P_i \in C_0 \) be the special point labeled by \( E^i \). If \( E^i \) is bounded then \( P_i \) is a nodal point. In this case the flags \( (E_i, V_{i-1}), (E_i, V_i) \) mark the two branches of \( C_0 \) at \( P_i \), for which we had chosen local coordinates \( w_i, z_i \in \mathcal{O}_{C_0, \overline{\mathcal{P}}} \) above. Observe that by their toric construction \( w_i, z_i \) extend to \( (C_0)_{V_{i-1}} \setminus \{P_{i-1}\} \cup (C_0)V_i \setminus \{P_{i+1}\} \). Thus \( z_i, w_i \) are unique up to constant and
the action of the identity component \( \text{Aut}^0(\mathcal{C}_0) \subset \text{Aut}(\mathcal{C}_0) \) on \( \mathcal{O}_{C_0, P_i} \) becomes linear in the following sense. Let \( \pi_i : \text{Aut}^0(\mathcal{C}_0)(k) \to \mathbb{G}_m(k) = k^* \) be the projection to the factor acting non-trivially on the \( i \)-th irreducible component \( (\mathcal{C}_0)_j \). Then for any \( \kappa \in \text{Aut}^0(\mathcal{C}_0)(k) \)

\[
\kappa^*z_r = z_r, \quad \kappa^*z_i = \pi_i(\kappa)^{-1} \cdot z_i, \quad \kappa^*w_{i+1} = \pi_i(\kappa) \cdot w_{i+1}, \quad i = 1, \ldots, r - 1,
\]

and, in the bounded case, \( \kappa^*w_1 = w_1 \). In particular, pulling back by such \( \kappa \) is compatible with our normalizations above iff \( \pi_i(\kappa)^\mu = 1 \) for all \( i \). Thus going inductively from \( i = 1 \) to \( r \) we may assume \( z_i, w_i \) chosen in such a way that the \( \mu \)-th root of unity \( \zeta_i \) associated to the log structure at \( P_i \) is equal to 1 for \( i = 1, \ldots, r - 1 \). On the other hand, under pulling back by \( \kappa \) the product \( \zeta_1 \cdots \zeta_r \) remains constant, and hence it is impossible to also normalize \( \zeta_r \). This shows that up to isomorphism there are only \( w(E) \) in the bounded case (1 in the unbounded case) rather than \( \prod_{i=1}^r w(E^i) = w(E)^r \) pairwise non-isomorphic choices of \( \mathcal{M}_{C_0} \to \mathcal{O}_{C_0} \) with the requested properties in a neighborhood of \( \bigcup_{i=1}^r (\mathcal{C}_0)_j \subset \mathcal{C}_0 \).

A similar argument holds for the marked edges \( E_i, i = 1, \ldots, l \). However, since \( \text{Aut}^0(\mathcal{C}_0, x_0) \)
acts trivially on the irreducible component \( (\mathcal{C}_0)_j \subset C_0 \) distinguished by the marked point one has to consider two chains of edges in \( \hat{\Gamma} \) according to the two connected components of \( E \setminus V \). Thus the total count for \( E \) keeps an additional factor of \( w(E) \), in agreement with our definition of total marked weight.

Now that we have a morphism of log spaces \( \varphi_0 : C_0 \to X_0 \) over \( O_0 \) we wish to lift it to the thickenings \( O_k := (\text{Spec} k[t]/(t^{k+1}), \mathcal{M}_{O_k}), \ k > 0 \), order by order. For the time being we forget the marked points and the incidence conditions. They will be taken care of in a separate discussion. The log structure on \( O_k \) is defined by asking the embedding \( O_k \to \mathbb{A}^1 \) to be strict, that is, by the chart \( \mathbb{N} \to O_{k,0}, 1 \mapsto t \). Finding \( \varphi_k \) involves both a lift \( C_k \to O_k \) of the domain and an extension of \( \varphi_0 \) to \( C_k \). In other words we are looking for diagrams of the form

\[
\begin{array}{ccc}
C_0 & \longrightarrow & C_k & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
O_0 & \longrightarrow & O_k & \longrightarrow & \mathbb{A}^1
\end{array}
\]

with given outer rectangle (formed by \( C_0, X, O_0 \) and \( \mathbb{A}^1 \)) and lower horizontal sequence. Denote such a diagram by \([C_k/O_k \to X]\). In the \( k \)-th step the answer involves the logarithmic normal sheaf \( N_{\varphi_0} := \varphi_0^* \Theta_{X/\mathbb{A}^1}/\Theta_{C_0/O_0} \). Note that this sheaf is locally free of rank \( n - 1 \), because pulling-back a differential with logarithmic pole along a divisor \( B \) under a cover with branch locus \( B \) preserves the logarithmic pole and hence \( \varphi_0^* \Omega^1_{X/\mathbb{A}^1} \to \Omega^1_{C_0/O_0} \) is surjective.

**Lemma 7.2.** Let \([\varphi_{k-1} : C_{k-1}/O_{k-1} \to X]\) be a lift of \([\varphi_0 : C_0/O_0 \to X]\) and assume \( C_0 \) is rational. Then the set of isomorphism classes of lifts \([\varphi_0 : C_k/O_k \to X]\) restricting to \( \varphi_{k-1} \) over \( O_{k-1} \) is a torsor under \( H^0(C_0, N_{\varphi_0}) \).

**Proof.** The obstruction group for the existence of a smooth lift \( C_k \to O_k \) of \( C_{k-1} \to O_{k-1} \) is \( \text{Ext}^2(\Omega^1_{C_{k-1}/O_{k-1}}, \mathcal{O}_{C_0}) \), see [KaK], 3.14 and [KaF1], Proposition 8.6. Since \( \Omega^1_{C_{k-1}/O_{k-1}} \) is locally free this group equals \( H^2(C_0, \Theta_{C_0/O_0}) \), which vanishes for dimension reasons. Hence \( C_k \to O_k \) exists.
Once a deformation of the domain is given, an extension of the morphism \( \varphi_{k-1} \) exists locally because \( X \to \mathbb{A}^1 \) is log-smooth \([\text{KaK}], 3.3\). We may thus cover \( C_k \) by finitely many open sets \( U \) such that a local extension \( \varphi'_k \) exists. The set of local extensions of \( \varphi_{k-1} \) to \( C_k \) form a torsor under the sections of \( \varphi_{k-1}^* \Theta_{X/\mathbb{A}^1} \), see \([\text{KaK}], \text{Proposition 3.9}\). Thus the differences between the \( \varphi'_k \) on intersections define a Čech 1-cocycle with values in \( \varphi_{k-1}^* \Theta_{X/\mathbb{A}^1} \). But \( \Theta_{X/\mathbb{A}^1} = N \otimes \mathcal{O}_X \) and hence \( H^1(C_k, \varphi_{k-1}^* \Theta_{X/\mathbb{A}^1}) = 0 \) by rationality of \( C_0 \). Thus there exists a correction of the \( \varphi'_k \) by a Čech 0-cochain to make them patch to a morphism \( \varphi_k \). This shows that the space of liftings \([\varphi_k : C_k/O_k \to X]\) of \( \varphi_{k-1} \) is non-empty. Note that unlike ordinary deformation theory we are not allowed to take the trivial deformation because of the requirement of log-smoothness.

To classify the set of isomorphism classes of such liftings knowing that at least one exists recall first that there is a one-to-one correspondence between extensions \( C_k/O_k \) of \( C_{k-1}/O_{k-1} \) and \( \mathcal{O}_{C_{k-1}} \)-module extensions \( E \) of \( \Omega^1_{C_{k-1}/O_{k-1}} \) by \( \mathcal{O}_{C_0} \) as follows \([\text{KaK}]\). Given \( E \) define \( \mathcal{O}_{C_k} \) by the fiber product of \( E \to \Omega^1_{C_{k-1}/O_{k-1}} \) and of the composition

\[
\mathcal{O}_{C_{k-1}} \xrightarrow{d} \Omega^1_{C_{k-1}/O_{k-1}} \to \Omega^1_{C_{k-1}/O_{k-1}}.
\]

This identifies \( \mathcal{O}_{C_k} \) with a subalgebra of the trivial extension \( \mathcal{O}_{C_{k-1}}[E] \). Define the extension \( \mathcal{M}_{C_k} \) of the log structure analogously by fiber product of \( E \to \Omega^1_{C_{k-1}/O_{k-1}} \) and of \( d \log \): \( \mathcal{M}_{C_{k-1}} \to \Omega^1_{C_{k-1}/O_{k-1}} \). As one easily checks the structure homomorphism \( \mathcal{M}_{C_{k-1}} \to \mathcal{O}_{C_{k-1}} \) lifts uniquely to \( C_k \). Conversely, any extension \( C_k/O_k \) gives rise to two diagrams with exact rows, one for the extension of scheme structure

\[
\begin{array}{cccccc}
0 & \to & t^k \mathcal{O}_{C_0} & \to & \mathcal{O}_{C_k} & \to & \mathcal{O}_{C_{k-1}} & \to & 0 \\
\downarrow & & \downarrow d & & \downarrow d & & & & \\
0 & \to & \mathcal{O}_{C_0} & \to & \Omega^1_{C_k/O_k} \otimes \mathcal{O}_{C_{k-1}} & \to & \Omega^1_{C_{k-1}/O_{k-1}} & \to & 0,
\end{array}
\]

and one for the extension of log structure

\[
\begin{array}{cccccc}
0 & \to & t^k \mathcal{O}_{C_0} & \to & \mathcal{M}_{C_k} & \to & \mathcal{M}_{C_{k-1}} & \to & 0 \\
\downarrow & & \downarrow \text{dlog} & & \downarrow \text{dlog} & & & & \\
0 & \to & \mathcal{O}_{C_0} & \to & \Omega^1_{C_k/O_k} \otimes \mathcal{O}_{C_{k-1}} & \to & \Omega^1_{C_{k-1}/O_{k-1}} & \to & 0.
\end{array}
\]

The two diagrams are identical after restricting the upper horizontal sequences to invertible elements. In any case, the one-to-one correspondence thus identifies the sheaf extension \( E \in \text{Ext}^1_{C_{k-1}}(\Omega^1_{C_{k-1}/O_{k-1}} \otimes \mathcal{O}_{C_0}) \) with the restriction to \( C_{k-1} \) of the sheaf of relative log-differentials \( \Omega^1_{C_k/O_k} \) of the corresponding extension \( C_k/O_k \) of log-spaces.

For the extension of morphism to \( X \) observe that an extension of \( \varphi_{k-1} : C_{k-1} \to X \) to \( C_k \) over \( \mathbb{A}^1 \) gives a factorization of \( \varphi_{k-1}^* \Omega^1_{X/\mathbb{A}^1} \to \Omega^1_{C_{k-1}/O_{k-1}} \) over \( \Omega^1_{C_k/O_k} \otimes \mathcal{O}_{C_{k-1}} = E \). It is not hard to see that this sets up a one-to-one correspondence between isomorphism classes of extensions of \( \varphi_{k-1} : C_{k-1} \to X \) to \( C_k \) over \( \mathbb{A}^1 \) and factorizations of \( \varphi_{k-1}^* \Omega^1_{X/\mathbb{A}^1} \to \Omega^1_{C_{k-1}/O_{k-1}} \)
over $\Omega^1_{C_k/O_k} \otimes \mathcal{O}_{C_{k-1}} = \mathcal{E}$. In other words, we need to classify diagrams

$$
\begin{array}{cccccc}
0 & \longrightarrow & \ker \varphi^*_k & \longrightarrow & \varphi^*_k \Omega^1_{X/\Lambda^1} & \longrightarrow & \Omega^1_{C_{k-1}/O_{k-1}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_{C_0} & \longrightarrow & \mathcal{E} & \longrightarrow & \Omega^1_{C_{k-1}/O_{k-1}} & \longrightarrow & 0
\end{array}
$$

with given upper horizontal sequence, up to isomorphism. Note that the left square is co-cartesian. A simple exercise in homological algebra now shows that such diagrams are given uniquely up to isomorphism by the left vertical homomorphism. The proof is finished by noting $\text{Hom}(\ker \varphi^*_k, \mathcal{O}_{C_0}) = \text{Hom}(\ker \varphi^*_0, \mathcal{O}_{C_0}) = H^0(C_0, \mathcal{N}_{\varphi_0})$ because $\mathcal{N}_{\varphi_0}$ is free.

It remains to take care of the incidence conditions given by the intersections with $Z_i$. To this end we first need to include deformations of the tuple of marked points $x_0 = (x_1, \ldots, x_l)$. In the $k$-th step this merely amounts to go over from $\Omega^1_{C_k/O_k}$ to the twisted module $\Omega^1_{C_k/O_k}(x)$, which brings in another $l$-dimensional freedom for the lift. We are then talking about lifting pairs $[\varphi_{k-1} : C_{k-1}/O_{k-1} \rightarrow X, x_{k-1}]$ to $O_k$. In each step we want the tuple of sections $x_{k-1} : \mathcal{O}_{k-1} \rightarrow X$ to factor over the tuple $Z$ of incidence varieties.

**Proposition 7.3.** Let $[\varphi_{k-1} : C_{k-1}/O_{k-1} \rightarrow X, x_{k-1}]$ be a lift of $[\varphi_0 : C_0/O_0 \rightarrow X, x_0]$ with $x_{k-1}$ factoring over $Z$, and assume $C_0$ is rational. Then up to isomorphism there is a unique lift $[\varphi_k : C_k/O_k \rightarrow X, x_k]$ to $O_k$ with $x_k$ factoring over $Z$.

**Proof.** For a closed point $y \in X$ denote by $T_{X/\Lambda^1, y} = \Theta_{X/\Lambda^1, y} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,y}/m_y$ the fiber of $\Theta_{X/\Lambda^1}$ at $y$, which is a $k$-vector space of dimension $n$, and similarly for $\Theta_{Z_i/\Lambda^1}$ and $\Theta_{C_0/O_0}$. By Lemma \ref{lemma} it suffices to prove that the transversality map

$$
H^0(\mathcal{N}_{\varphi_0}) \rightarrow \prod_i T_{X/\Lambda^1, \varphi_0(x_i)} / (T_{Z_i/\Lambda^1, \varphi_0(x_i)} + D\varphi_0(T_{C_0/O_0, x_i}))
$$

is an isomorphism. Now recall that $A_i$ intersects $h(E_i)$ in the relative interior ($E_i \in \Gamma^{[1]}$ was the $i$-th marked edge). Hence $A_i \cap h(E_i)$ is a divalent vertex $V$ of $\bar{\Gamma}$. In the discussion in Proposition \ref{prop} of the space of lines for this case we saw that $\varphi_0(C_V)$ is then an orbit closure of $G(Zu_i) \subset G(N)$ for $u_i \in N$ an integral generator of $L(h(E_i)) \cap N$. This yields a canonical identification $D\varphi_0(T_{C_0/O_0, x_i}) = ku_i \subset N_k := N \otimes_Z k$. Similarly the actions of $G(N)$ and of $G(L(A_i) \cap N)$ identify the other terms on the right-hand side of (9) as follows:

$$
T_{X/\Lambda^1, \varphi_0(x_i)} = N_k, \quad T_{Z_i/\Lambda^1, \varphi_0(x_i)} = L(A_i) \otimes_{\mathcal{O}} k \subset N_k.
$$

To describe $H^0(\mathcal{N}_{\varphi_0})$ in similar terms observe that the map $\varphi_0^* \Theta_{X/\Lambda^1} \rightarrow \mathcal{N}_{\varphi_0}$ induces a canonical surjection $N \otimes_Z \mathcal{O}_{C_V} \rightarrow \mathcal{N}_{\varphi_0} \otimes \mathcal{O}_{C_V}$, for any $V \in \bar{\Gamma}^{[0]}$. This gives a canonical surjection $N_k = \Gamma(N \otimes_Z \mathcal{O}_{C_V}) \rightarrow \Gamma(\mathcal{N}_{\varphi_0} \otimes \mathcal{O}_{C_V})$. To describe the kernel of this map consider the restriction of the exact sequence

$$
0 \rightarrow \Theta_{C_0/O_0} \rightarrow \varphi_0^* \Theta_{X/\Lambda^1} \rightarrow \mathcal{N}_{\varphi_0} \rightarrow 0
$$

to $C_V$. If $V$ is trivalent then $\Theta_{C_0/O_0}|_{C_V} \simeq \mathcal{O}_{C_V}(-1)$. Thus in this case $N_k \rightarrow \Gamma(\mathcal{N}_{\varphi_0} \otimes \mathcal{O}_{C_V})$ is an isomorphism because $H^i(\Theta_{C_0/O_0}|_{C_V}) = 0$ for $i = 0, 1$. In the divalent case $\Theta_{C_0/O_0}|_{C_V}$ is
trivial, hence spanned by the vector field generating the one-parameter subgroup with orbit \( \varphi_0(C_V) \). This shows \( \ker(\mathbb{N}_k \to \Gamma(N_{\varphi_0} \otimes OC_V)) = ku(V,E) \) for any edge \( E \) adjacent to \( V \).

Now if \( E \in \tilde{\Gamma}^{[1]} \) and \( \partial E = \{V, V'\} \) then the images of \( h, h' \in N_k \) glue to a section of \( H^0(N_{\varphi_0}) \) over \( CV \cup CV' \) if and only if \( h - h' \in L(h(E)) \otimes \mathbb{Q} \). In fact, because \( \varphi_0 \) is transverse to the toric prime divisor \( X_{h(E)} \subset X_V \) (or in \( X_{V'} \)) the canonical map \( TX_{h(E),\varphi_0}(P_E) \to N_{\varphi_0,P_E} \) is an isomorphism. Here \( P_E = CV \cap CV' \) is the nodal point corresponding to \( E \). Thus the claim follows by noting \( TX_{h(E),\varphi_0}(P_E) = (N_Q/L(h(E))) \otimes \mathbb{Q} \). We conclude that the kernel of

\[
\Phi : \text{Map}(\tilde{\Gamma}^{[0]}, N_k) \longrightarrow \prod_{E \in \tilde{\Gamma}^{[1]} \setminus \tilde{\Gamma}^{[0]}} N_k/ku(\partial - E, E), \quad h' \longmapsto (h'(\partial^+ E) - h'(\partial^- E))_E
\]

surjects onto \( H^0(N_{\varphi_0}) \). Dividing out \( \text{Map}(\tilde{\Gamma}^{[0]} \setminus \Gamma^{[0]}, N_k) \) to take into account the non-uniqueness caused by the divergent vertices, identifies \( H^0(N_{\varphi_0}) \) with the kernel of

\[
\Phi : \text{Map}(\Gamma^{[0]}, N_k) \longrightarrow \prod_{E \in \Gamma^{[1]} \setminus \Gamma^{[0]}} N_k/ku(\partial - E, E), \quad h' \longmapsto (h'(\partial^+ E) - h'(\partial^- E))_E.
\]

The proof is finished by noting that \( \Phi \) merged with \( \Box \) gives the lattice inclusion \( \square \) from Proposition 7.4.

**Corollary 7.4.** There exists a unique \( l \)-marked stable map \( (\varphi_\infty : C_\infty \to X, x_\infty) \) over \( O_\infty = \text{Spec} \mathbb{K}[\mathbb{I}] \) incident to \( Z_1, \ldots, Z_l \) and such that the induced log morphism of the central fibers agrees with \([\varphi_0 : C_0/O_0 \to X_0, x_0] \).

**Proof.** Forgetting the log structure from \([\varphi_k : C_k/O_k \to X_k, x_k] \) in the proposition gives an inverse system of maps from \( O_k \) to the stack of stable maps to \( X_k \). This stack has been constructed in \( \Box \). Going over to the limit provides a map from \( O_\infty \) to this stack, hence a stable map over \( O_\infty \).

Conversely, let \( (\varphi_\infty : C_\infty \to X, x_\infty) \) be a stable map over \( O_\infty \). On the involved spaces \( C_\infty, O_\infty, X, \mathbb{A}^1 \) take the log structures \( \mathcal{M}_{C_\infty}, \mathcal{M}_{O_\infty}, \mathcal{M}_X, \mathcal{M}_{\mathbb{A}^1} \) defined by the divisors \( C_0 \subset C_\infty, X_0 \subset X \), and the closed points \( O_0 \in O_\infty, 0 \in \mathbb{A}^1 \), respectively. Because \( \varphi_\infty \) maps the pair \( (\mathcal{C}_\infty, \mathcal{C}_0) \) to \( (X, X_0) \) it induces uniquely the commutative diagram of log spaces

\[
\begin{array}{ccc}
C_\infty & \longrightarrow & X \\
\downarrow & & \downarrow \\
O_\infty & \longrightarrow & \mathbb{A}^1.
\end{array}
\]

From \( \Box \) it is clear that \( C_\infty \to O_\infty \) is log smooth. By construction the restriction \( \varphi_0 : C_0 \to X_0 \) to the central fiber also fulfills the other properties listed in Proposition 7.4.

**Remark 7.5.** The same arguments work in the higher genus case for trivalent tropical curves of the correct genus and with deformation space of the expected dimension \( e + (n - 3)(1 - g) \) (not superabundant). In the superabundant case one obtains an obstruction bundle that is an input to the virtual intersection formalism.
8. The Main Theorem

Throughout this section fix the following data. (1) A complete fan $\Sigma$ in $\mathbb{N}_Q$. (2) A map $\Delta : N \setminus \{0\} \to \mathbb{N}$ with finite support and such that $\Delta(v) \neq 0 \Rightarrow \mathbb{Q}_{\geq 0}v \in \Sigma^{[1]}$. (3) A tuple $L = (L_1, \ldots, L_l)$ of linear subspaces of $\mathbb{N}_Q$ with $\text{codim} L_i = d_i + 1$ and such that $\sum_{i=1}^l d_i = e + n - 3$, $n = \dim_Q \mathbb{N}_Q$. To this data we are going to associate two numbers, one counting algebraic curves on a toric variety, the other counting tropical curves in $\mathbb{N}_Q$. The result is that both these numbers are well-defined and agree.

For the tropical count choose $A = (A_1, \ldots, A_l)$ with $A_i \in \mathbb{N}_Q/L_i$ in general position for $\Delta$, as provided by Proposition 2.4. Then any tropical curve of degree $\Delta$ matching $A$ fulfills the conditions (i)–(iii) of Definition 2.3. Denote by $T_{0, l, \Delta}$ the set of isomorphism classes of $l$-marked tropical curves of genus 0 and degree $\Delta$. Then the subset $T_{0, l, \Delta}(A) = \{ (\Gamma, E, h) \in T_{0, l, \Delta} \mid h(E_i) \cap A_i \neq \emptyset \}$ of curves matching $A$ is finite. In fact, by Proposition 2.4 there are only finitely many types of tropical curves of fixed genus and degree, while Proposition 2.4 says that for general $A$ any two tropical curves of the same type and matching $A$ are isomorphic. For $(\Gamma, E, h) \in T_{0, l, \Delta}(A)$ let $D(\Gamma, E, h, A)$ be the lattice index defined in Proposition 5.7 and

$$\hat{D}(\Gamma, E, h, A) := D(\Gamma, E, h, A) \cdot \prod_{i=1}^l \delta_i,$$

where $\delta_i$ is the index of $\mathbb{Z}u(\partial - E_i, E_i) + L(A_i) \cap \mathbb{N}$ in $(\mathbb{Q}u(\partial - E_i, E_i) + L(A_i)) \cap \mathbb{N}$ (see Remark 5.8). With these conventions we can define the count on the tropical side as follows.

**Definition 8.1.** Assume the affine constraint $A$ is general for $\Delta$. Then the number of tropical curves of degree $\Delta$ and genus 0 matching $A$ is

$$N_{0, \Delta}^{\text{trop}}(A) := \sum_{(\Gamma, E, h) \in T_{0, l, \Delta}(A)} w(\Gamma, E) \cdot \hat{D}(\Gamma, E, h, A).$$

For the count of algebraic curves consider the toric variety $X(\Sigma)$ defined by $\Sigma$. The toric prime divisors on $X(\Sigma)$ are denoted $D_v$ with $v \in N$ primitive generators of the rays of $\Sigma$. For a torically transverse (Definition 4.1) stable map $\varphi : C \to X(\Sigma)$ define a map $\Delta(\varphi) : N \setminus \{0\} \to \mathbb{N}$ as follows. For primitive $v \in N$ and $\lambda \in \mathbb{N}$ map $\lambda \cdot v$ to 0 if $\mathbb{Q}_{\geq 0}v \notin \Sigma^{[1]}$, and to the number of points of multiplicity $\lambda$ in $\varphi^* D_v$ otherwise. So $\Delta(\varphi)$ counts the number of intersection points with the toric prime divisors of any given multiplicity. Clearly $\Delta(\varphi)$ has finite support. Now define $M_{0, l, \Delta}$ as the set of isomorphism classes of $l$-marked stable maps $(\varphi : C \to X(\Sigma), x)$ of genus 0 with $\varphi$ torically transverse and such that $\Delta(\varphi) = \Delta$. Note that this space of stable maps disregards any curves intersecting the toric strata of codimension $\geq 2$ and hence generally is non-complete.

To bring in the incidence conditions let $P_1, \ldots, P_l \in X(\Sigma)$ be closed points in the big torus. Define $Z_i = Z_i(P_i) := \overline{G(L_i \cap N)P_i}$ the closure of the orbit through $P_i$ for the subgroup defined by $L_i \subset \mathbb{N}_Q$ and denote $Z = Z(P) = (Z_1, \ldots, Z_l)$. The set of isomorphism
classes of constrained curves is
\[ M_{0,l,\Delta}(Z) := \{ (\varphi : C \to X(\Sigma), x) \in M_{0,l,\Delta} \mid \varphi(x_i) \in \text{int}(Z_i) \}. \]
We will see in the Main Theorem that for general \( P_1, \ldots, P_l \) this set is finite. Because the \( M_{0,l,\Delta}(Z) \) fit into one algebraic family for varying incidence conditions \( P_1, \ldots, P_l \) its cardinality is then the same for any two general choices of points.

**Definition 8.2.** The number of rational curves in \( X(\Sigma) \) of degree \( \Delta \) and constrained by \( L \) is
\[ N_{0,\Delta}^{\text{alg}}(L) := \sharp M_{0,l,\Delta}(Z(P)). \]
for general \( P = (P_1, \ldots, P_l) \).

No signs or multiplicities enter in this definition, so \( N_{0,\Delta}^{\text{alg}}(L) \) is a true enumerative count.

**Theorem 8.3.** There exist \( P_1, \ldots, P_l \in X(\Sigma) \) such that \( M_{0,l,\Delta}(Z(P)) \) is finite, and hence \( N_{0,\Delta}^{\text{alg}}(L) \) is well defined. Moreover, it holds
\[ N_{0,\Delta}^{\text{alg}}(L) = N_{0,\Delta}^{\text{trop}}(A). \]

**Corollary 8.4.** \( N_{0,\Delta}^{\text{trop}}(A) \) depends only on \( \Delta \) and \( L \).

**Remark 8.5.** It is worthwhile to remark that \( N_{0,\Delta}^{\text{alg}}(L) \) is not in general a Gromov-Witten invariant in the traditional sense, and in particular, it may not be a symplectic invariant. This is due to the fact that our count disregards stable maps with components mapping to a toric divisor. For example, the results on \( \mathbb{P}^1 \times \mathbb{P}^1 \) differ from the results on the symplectomorphic even degree Hirzebruch surfaces \( F_{1k} \).

On the other hand, we believe that our numbers should have an interpretation as relative Gromov-Witten invariant for the pair \((X(\Sigma), D(\Sigma))\) where \( D(\Sigma) = X(\Sigma) \setminus \text{G}(N) \) is the union of the toric prime divisors. Unfortunately a theory of relative Gromov-Witten invariants so far exists only for smooth divisors and hence the verification of this claim remains elusive at this point for \( n \geq 2 \). Nevertheless, our statement is true provided a theory of relative Gromov-Witten invariants for pairs \((X, D)\) has the following expected property.

*If every stable map of the considered type, fulfilling the incidence conditions and without components mapping to \( D \), intersects \( D \) in the smooth locus and is infinitesimally rigid (unobstructed deformations) then the relative Gromov-Witten invariant coincides with the number of such curves.*

In fact, we will see in the proof of the theorem that for general \( P_1, \ldots, P_l \) the assumptions in this statement are fulfilled in our case.

**Proof of the theorem.** We will show that both numbers agree with the number of isomorphism classes of maximally degenerate curves together with log smooth structures, for a sufficiently fine toric degeneration of \( X(\Sigma) \).

1) **Construction of degeneration.** To define an appropriate toric degeneration let \( S \) be the intersection of \( A_1 \cup \ldots \cup A_l \) with the union of the images of all tropical curves in \( \Sigma_{0,l,\Delta}(A) \).
This set is finite for $\mathcal{T}_{0,l,\Delta}(A)$ is finite and the intersections with the $A_i$ are transverse. Apply Proposition 5.9 to the disjoint union of the finitely many tropical curves in $\mathcal{T}_{0,l,\Delta}(A)$, with the requirement that $S$ be contained in the set of vertices. This gives a polyhedral decomposition $\mathcal{P}$ of $N_\mathbb{Q}$ with asymptotic fan $\Sigma = \Sigma_{\mathcal{P}}$, with $S \subset \mathcal{P}^{[0]}$ and such that for all $(\Gamma, E, h) \in \mathcal{T}_{0,l,\Delta}(A)$

$$h(\Gamma^{[\mu]}) \subset \bigcup_{\Xi \in \mathcal{P}^{[\mu]}} \Xi, \quad \mu = 0, 1, \quad \text{and} \quad h(\Gamma) \cap A_i \subset \mathcal{P}^{[0]}, \quad j = 1, \ldots, l.$$ 

After rescaling we may assume $\mathcal{P}$ to be integral, that is $\mathcal{P}^{[0]} \subset N$. Then the image of each bounded edge of a tropical curve has an integral length, the number of integral points on this image minus one. After another rescaling we may assume that this integral length is a multiple of the weight of this edge, for any bounded edge of any tropical curve in $\mathcal{T}_{1,\Delta}(A)$. This gives our polyhedral decomposition $\mathcal{P}$ and an associated toric degeneration $\pi : X = X(\Sigma_{\mathcal{P}}) \to \mathbb{A}^1$ with general fiber isomorphic to $X(\Sigma)$ and reduced special fiber $X_0$.

As for the incidence conditions $Z_i \subset X(\Sigma)$ consider the subgroup $\tilde{G}_i := \mathbb{G}(LC(A_i) \cap (N \times \mathbb{Z})) \subset \mathbb{G}(N \times \mathbb{Z})$ associated to $A_i \subset N_\mathbb{Q}$. Identify $X(\Sigma)$ with any general fiber $\pi^{-1}(Q)$ and define $\tilde{Z}_i$ as the closure of the $\tilde{G}_i$-orbit through $P_i$. Then $\tilde{Z}_i \cap X(\Sigma) = Z_i$, so $\tilde{Z}_i$ is a degeneration of $Z_i$. If $v \in A_i \cap \mathcal{P}^{[0]}$ then the intersection of $\tilde{Z}_i$ with the irreducible component $X_v$ is the closure of a $\mathbb{G}(L_i \cap N)$-orbit of maximal dimension, that is, intersecting the big torus (Proposition 3.6). Along the big torus this intersection is even transverse (Corollary 3.8).

2) Interpretation of $N_{0,\Delta}^{trop}(A)$. The number of isomorphism classes of maximally degenerate curves in $X_0$ intersecting $\tilde{Z}_i \cap X_0$ and with dual intersection graph $(\Gamma, h) \in \mathcal{T}_{0,l,\Delta}(A)$ equals $\mathcal{D}(\Gamma, E, h, A)$ (Proposition 3.7). According to Remark 3.8 the image of each such stable map $\varphi_0 : C_0 \to X_0$ intersects $Z_i$ in $\delta_i$ points. Recall that the irreducible components of $C_0$ corresponding to the point of intersection $v \in h(E_i) \cap A_i$ is a copy of $\mathbb{P}^1$ mapping to a line in $X_v$ by a $w(E_i)$-fold covering, unbranched over $\text{int}(X_v)$. Thus each of the $\delta_i$ intersection points has $w(E_i)$ preimages on $C_0$. Any of these is a possible choice for the $i$-th marked point $x_{0i} \in C_0$. However, by the same token, choices lying in the same $\varphi_0$-fiber lead to isomorphic stable maps. Thus up to isomorphism there are $\tilde{D}(\Gamma, E, h, A) = \mathcal{D}(\Gamma, E, h, A) \cdot \prod_i \delta_i$ marked stable maps $(C_0, x, \varphi_0)$ with marked dual intersection graph $(\Gamma, E, h)$. Choose one of them.

According to Proposition 3.1 there are $w(\Gamma, E)$ pairwise non-isomorphic ways to lift $\varphi_0$ to a morphism of log smooth spaces $(C_0, M_{C_0}) \to (X_0, M_{X_0})$ relative the standard log point $(\text{Spec} k, N \times k^\times)$. Hence by definition $N_{0,\Delta}^{trop}(A)$ agrees with the total number of isomorphism classes of such maximally degenerate stable maps with $l$ marked points mapping to $\tilde{Z}_1, \ldots, \tilde{Z}_l$, and together with log structures.

3) Interpretation of $N_{0,\Delta}^{alg}(L)$. The results from Section 4 provide the link to curves on $X(\Sigma)$. Corollary 4.4 says that each log morphism $(C_0, M_{C_0}) \to (X, M_X)$ is induced by a unique family of stable maps $(C_\infty \to X, x_\infty)$ over $\text{Spec} k[\![t]\!]$, both incident to $\tilde{Z}_1, \ldots, \tilde{Z}_l$. For the relevant stack of $l$-marked stable maps to $X$ over $\mathbb{A}^1$ with incidences this means that there is an injection from the set of log morphisms $(C_0, M_{C_0}) \to (X, M_X, x_0)$ considered in Section 4.
to the set of prime components of this Deligne-Mumford stack at maximally degenerate curves mapping dominantly to \( \mathbb{A}^1 \). We claim that every such prime component arises in this way.

To this end assume that \( R \) is a discrete valuation ring with residue field \( \mathbb{k} \) and \( \text{Spec} \, R \to \mathbb{A}^1 \) is a morphism mapping the closed point to \( 0 \in \mathbb{A}^1 \). This map identifies the completion of \( R \) with \( \mathbb{k}[[t]] \). Let \( \varphi^* : C^* \to X \setminus X_0 \) together with \( l \) sections \( x_1^*, \ldots, x_l^* \) of marked points be a stable map defined over the quotient field \( \text{Spec} \, K \subset \text{Spec} \, R \). Corollary 6.3 says that possibly after a ramified base change and toric modification \( \tilde{X} \to X \) with centers on the central fiber, \( \varphi^* \) together with the \( l \) sections extends to a stable map \( (\varphi, x) \) defined over \( \text{Spec} \, R \), such that for every irreducible component \( X_v \subset X_0 \) the projection \( C \times_{X_0} X_v \to X_v \) is a torically transverse stable map. We will see towards the end of the proof that this base change and modification are indeed unnecessary. For the time being let us just make this transformation but keep the notations \( \mathcal{P}, X, \) etc. as above. Our goal is to show that the completion of \( \varphi \) at the closed point of \( \text{Spec} \, R \) is already in the list of stable maps constructed by Corollary above.

Let us first check that \( \varphi_0 \) is indeed a pre-log curve (Definition 4.3). In fact, if \( P \in C_0 \) is a closed point mapping to the singular locus of \( X_0 \) consider the homomorphism of complete local \( \mathbb{k}[t]-\)algebras \( \psi : \hat{\mathcal{O}}_{X_0(P)} \to \hat{\mathcal{O}}_{C,P} \). The following possibilities arise: (1) \( C_0 \) is smooth at \( P \); (2) \( C_0 \) has a node at \( P \) with both branches mapping to the same prime component of \( (X_0, \varphi_0(P)) \); (3) \( C_0 \) has a node at \( P \), but the two branches map to different prime components of \( (X_0, \varphi_0(P)) \). In the first case \( C \) is also smooth at \( P \), and by the maximal degeneracy condition \( \psi \) has the following form:

\[
\mathbb{k}[x, y, t, u_1, \ldots, u_{n-1}]/(xy - t^e) \longrightarrow \mathbb{k}[z, t], \quad \psi(x) = z^a g + th, \quad \psi(y) = tk,
\]

with \( g \) a unit. But then \( t^e = \psi(xy) = (z^a g + th) tk \), and comparing monomials shows that this case can not occur. A similar argument shows the impossibility of (2): By Lemma 8.6 (1) below, \( \hat{\mathcal{O}}_{C,P} \simeq \mathbb{k}[z, w, t]/(zw - \lambda t^b) \) with \( \lambda \in \{0, 1\} \) and \( b > 0 \). Then \( \psi \) takes the form

\[
\mathbb{k}[x, y, t, u_1, \ldots, u_{n-1}]/(xy - t^e) \longrightarrow \mathbb{k}[z, w, t]/(zw - \lambda t^b), \quad \psi(x) = z^a g + tk, \quad \psi(y) = z^b h + tl,
\]

with \( g, h \) units. Again this leads to a contradiction with \( \psi(xy) = t^e \). We are thus left with (3).

In this case Lemma 8.6 (2) below shows that the intersection numbers of the two branches of \( \varphi_0 \) with the singular locus of \( X_0 \) coincide. Hence Condition (ii) in Definition 4.3 is fulfilled.

Thus by Construction 4.4 the map from the dual intersection complex of \( C_0 \) to the 1-skeleton of \( \mathcal{P} \) defines a tropical curve \( h : \Gamma \to N_\mathbb{R} \) of genus 0 and degree \( \Delta \), and a corresponding subdivision \( \hat{\Gamma} \) with \( \hat{\Gamma}^{[0]} = h^{-1}(\mathcal{P}^{[0]}) \). It also comes with incidences with \( A_i \) as follows. Let \( V_i \subset \hat{\Gamma}^{[0]} \) be the vertex corresponding to the irreducible component of \( C_0 \) containing the \( i \)-th marked point. Because \( \varphi \circ x_i : \text{Spec} \, R \to X \) factors over the \( i \)-th incidence condition \( \tilde{Z}_i \) it follows \( \text{int}(X_{h(V_i)}) \cap \tilde{Z}_i \neq \emptyset \). Recalling that \( \tilde{Z}_i \) is an orbit closure for \( \mathbb{G}(LC(A_i) \cap (N \times \mathbb{Z})) \) Proposition 3.6 implies \( h(V_i) \subset A_i \).

We have now shown that \( (\Gamma, h, \mathbf{E}) \in \mathcal{I}_{0,1,\Delta}(A) \). In other words, it is one of the tropical curves already considered above, and hence it is general for \( A \) in the sense of Definition 2.3. In particular, all vertices of \( \Gamma \) are at most trivalent, the \( V_i \) are divalent and the map \( \Gamma^{[1]} \to \mathcal{P}^{[1]} \)
induced by $h$ is injective. For $n > 2$ the map $h$ itself is injective. For the following discussion let us make this assumption and leave the straightforward modifications for the case when $h$ is only an immersion to the interested reader. Then for every irreducible component $X_v \subset X_0$ there is at most one irreducible component $C_V \subset C_0$ mapping to $X_v$, namely if $v = h(V)$ for $V \in \Gamma[0]$, and $\varphi_0|_{C_V}$ intersects the toric boundary of $X_v$ in at most three points. Hence $\varphi_0|_{C_V}$ is a line for every $V \in \Gamma[0]$ and $\varphi_0$ is indeed maximally degenerate. Because it is of the correct genus and degree, and because it is incident to $\tilde{Z}_i \cap X_0$ for every $i$, it is one of the maximally degenerate curves constructed for $(\Gamma, h, E) \in \mathcal{F}_{0,t,\Delta}(A)$ above (for the refined degeneration $\tilde{X}$). The formal completion at the maximal ideal of $R$ thus defines a family of stable maps $(C_\infty \to X, x_\infty)$ over $k[[t]]$ isomorphic to one of the families constructed by log deformation theory before. \hfill \Box

In the proof we used the following algebraic results.

**Lemma 8.6.** 1) A $k[t]$-algebra of the form $k[z, w, t]/(zw - t^af)$ with $a > 0$ is isomorphic to $k[z', w', t]/(z'w' - \lambda t^b)$ with $\lambda \in \{0, 1\}$, $z' = zg$, $w' = wh$ for units $g, h$ and some $b \geq a$.

2) If $\phi : k[x, y, t, u_1, \ldots, u_{n-1}]/(xy - t^e) \to k[z, w, t]/(zw - t^a)$ is a $k[t]$-algebra homomorphism with $\phi(x) = z^\alpha \cdot g$, $\phi(y) = w^\beta \cdot h$, $g, h$ units, and $\alpha, \beta, a, e > 0$ then $\alpha = \beta$.

**Proof.** 1) This is well-known (and elementary to prove).

2) The composition with the inclusion

$$k[x, y, t]/(xy - t^e) \longrightarrow k[x, y, t, u_1, \ldots, u_{n-1}]/(xy - t^e)$$

reduces to the case $n = 1$. We may also assume $\alpha \leq \beta$ and, by (1), $f = 1$. In fact, if $f = 0$ then $xy$ maps to zero, but $xy = t^e$, so this case can not arise. Now $t^e = xy$ maps to a unit times $z^\alpha w^\beta = (zw)^{\alpha} w^{\beta-\alpha} = t^{\alpha} w^{\beta-\alpha}$. Both $t^e$ and $t^{\alpha} w^{\beta-\alpha}$ are part of the $k$-vector space basis of $k[z, w, t]/(zw - t^a)$ consisting of all monomials without $zw$-factor, and hence $\alpha \cdot e$ and $\beta = \alpha$. \hfill \Box

**Remark 8.7.** The proof of the theorem gives a more refined correspondence between the tropical and geometric counts, once a sufficiently fine degeneration $X \to \mathbb{A}^1$ has been chosen. First, a choice of tropical curve matching $A$ gives combinatorial information for the degeneration of stable maps, via intersection information on $X_0$. Then $\mathcal{D}(\Gamma, E, h, A)$ is the number of different degenerate unmarked stable maps. Next, there are $\delta_i$ points of intersection of the image with $Z_i$ compatible with the marked tropical curve. Hence $\mathcal{D}(\Gamma, E, h, A) = \mathcal{D}(\Gamma, E, h, A) \cdot \prod_i \delta_i$ is the number of different central fibers as marked stable maps. There are $w(\Gamma)$ ways of endowing such an unmarked stable map to $X_0$ with a log structure coming from a degeneration; taking into account the markings reduced the automorphism group to give $w(\Gamma, E) = w(\Gamma) \cdot \prod_i w(E_i)$ possibilities. Finally, there is a one-to-one correspondence between such marked stable maps with log structures and degenerations of marked stable maps of the considered type.
In two dimensions Mikhalkin gave a different definition of $N_{0,\Delta}^{\text{trop}}$ ([Mi], Definition 4.16), while his definition on the geometric side ([Mi], Definition 5.1) agrees with ours. The definitions on the tropical sides must therefore also coincide. We end this section with an independent proof of this statement. Note that point constraints determine the corresponding marking of a matching tropical curve uniquely, so in dimension two markings are redundant information.

Recall that Mikhalkin defines the multiplicity of a tropical curve $(\Gamma, h)$ as the product over the multiplicities $\text{mult}(\Gamma, h, V)$ of the trivalent vertices. If $E_1, E_2$ are two different edges emanating from a trivalent vertex $V$ then

$$\text{mult}(\Gamma, h, V) = w(E_1)w(E_2) \cdot \left| \det(u_{(V, E_1)}, u_{(V, E_2)}) \right|.$$  

By the balancing condition this number does not depend on the choices of $E_1$ and $E_2$. The claimed equivalence of the definition in [Mi] with ours follows readily from the following result.

**Proposition 8.8.** Suppose that $(\Gamma, h, E)$ is a genus zero tropical curve that is general for a tuple $P = (P_1, \ldots, P_l)$ of points in $N_\mathbb{Q} \cong \mathbb{Q}^2$. Then

$$w(\Gamma) \cdot \mathcal{D}(\Gamma, E, h, P) = \prod_{V \in \Gamma^{[0]}} \text{mult}(\Gamma, h, V).$$

**Proof.** The proof is by induction on the number of vertices. If there is only one vertex there are two point constraints. Letting $E_i$ be the edge with $P_i \in h(E_i)$ for the computation of $\text{mult}(V)$ shows that both sides agree trivially.

In the general case with $l$ marked points there are $l + 1$ unbounded edges, $l - 1$ vertices and $l - 2$ bounded edges. In particular, there must be one unmarked, unbounded edge $E$. Removing $E$ splits $\Gamma$ into two connected components, say $\Gamma_1$ and $\Gamma_2$. Let us first treat the case that both $\Gamma_i$ have vertices. In this case the restrictions of $(\Gamma, h, E)$ to the connected components define tropical curves $(\Gamma_i, h_i, E_i)$ themselves, and there is a splitting of $P$ into two tuples of points $P_1, P_2$ such that $(\Gamma_i, h_i, E_i)$ is general for $P_i$. Otherwise one $(\Gamma_i, h_i, E_i)$ moves in a one-parameter family with each member matching $P_i$, contradicting the generality of $(\Gamma, h, E)$.

Let $b_1, \ldots, b_{l-2} \in N$ be primitive generators of the lines spanned by the images of the bounded edges of $\Gamma$. We order the $b_i$ in such a way that the first $s - 2$ are the images of the bounded edges of $\Gamma_1$, $b_{s-1}$ and $b_s$ belong to the unbounded edges $E'$ and $E''$ of $\Gamma_1$ and $\Gamma_2$ with closure in $\Gamma$ intersecting $E$, and the last $l - 2 - s$ are the images of the bounded edges of $\Gamma_2$. Similarly, let $u_1, \ldots, u_t$ denote generators of the lines spanned by $h(E_1), \ldots, h(E_l)$ with the first $s$ coming from $\Gamma_1$ and the rest coming from $\Gamma_2$. Now $\mathcal{D} = \mathcal{D}(\Gamma_1, E_1, h_1, P_1)$ is the order of the cokernel of a map

$$\Phi' : \text{Map}(\Gamma_1^{[0]}, N) \to \prod_{i=1}^{s-2} N/\mathbb{Z}b_i \times \prod_{j=1}^{s} N/\mathbb{Z}u_j,$$
and \( \mathcal{O}_2 = \mathcal{O}(\Gamma_2, E_2, h_2, P_2) \) is the order of the cokernel of a map

\[
\Phi'' : \text{Map}(\Gamma_2[0], N) \longrightarrow \prod_{i=s+1}^{l-2} N/\mathbb{Z}b_i \times \prod_{j=s+1}^{l} N/\mathbb{Z}u_j.
\]

On the other hand, \( \mathcal{O}(\Gamma, E, h, P) \) is then the order of the cokernel of

\[
\Phi : \text{Map}(\Gamma_1[0], N) \times \text{Map}(\Gamma_2[0], N) \times N \longrightarrow B' \times B'' \times N/\mathbb{Z}b_{s-1} \times N/\mathbb{Z}b_s
\]

\[
(h', h'', Q) \longmapsto (\Phi'(h'), \Phi''(h''), h'((V')_i^0 - Q, h''((V'')_i^0 - Q).
\]

Here \( V' \in \Gamma_1[0], V'' \in \Gamma_2[1] \) are the unique vertices adjacent to \( E' \) and \( E'' \), respectively, and \( B', B'' \) denote the free abelian groups on the right-hand sides of the previous two displayed equations. The overlining indicates taking equivalence classes. It is then not hard to see that

\[
|\text{coker}(\Phi)| = |\text{coker}(\Phi')| \cdot |\text{coker}(\Phi'')| \cdot |\text{coker}(N \to N/\mathbb{Z}b_{s-1} \times N/\mathbb{Z}b_s)|.
\]

The last term on the right-hand side gives \( \det(b_{s-1}, b_s) \). Now it holds \( w(\Gamma) = w(\Gamma_1) \cdot w(\Gamma_2) \cdot w(E') \cdot w(E'') \) and hence, taking into account the induction hypothesis applied to \( \Gamma_i, h_i, E_i \),

\[
w(\Gamma) \cdot \mathcal{O}(\Gamma, E, h, P) = w(\Gamma_1) \mathcal{O}_1 \cdot w(\Gamma_2) \mathcal{O}_2 \cdot w(E') w(E'') \det(b_{s-1}, b_s)
\]

\[
= \prod_{V \in \Gamma_1[0]} \text{mult}(\Gamma_1, h_1, V) \cdot \prod_{V \in \Gamma_2[0]} \text{mult}(\Gamma_2, h_2, V) \cdot \text{mult}(\Gamma, h, \tilde{V}).
\]

Here \( \tilde{V} \in \Gamma[0] \) is the vertex separating \( \Gamma_1 \) and \( \Gamma_2 \) in \( \Gamma \). The expression in the last line equals \( \prod_{V \in \Gamma[0]} \text{mult}(\Gamma, h, V) \) as claimed.

In the case where one of \( \Gamma_i \) has no vertices, say \( \Gamma_2 \), by generality \( \Gamma_2 \) must contain a marked point, say the last one. If \( \Phi', B', V' \in \Gamma_1 \) are defined as above the definition for \( \Phi \) now reads

\[
\Phi : \text{Map}(\Gamma_1[0], N) \times N \longrightarrow B' \times N/\mathbb{Z}b_{l-2} \times N/\mathbb{Z}u_l
\]

\[
(h', Q) \longmapsto (\Phi'(h'), h'((V')_i^0 - Q, h''((V'')_i^0 - Q).
\]

Again this relates \( |\text{coker}(\Phi)| \) and \( |\text{coker}(\Phi')| \) by the determinant \( \det(b_{l-2}, u_l) \) of the two generators of the lines emanating from the image of the removed vertex. The computation is finished by the inductive assumption as before. Note that while the edge coming from \( \Gamma_2 \) is now unbounded, its weight still contributes to our definition of \( \Lambda_{0, \Delta}^{\text{trop}} \) because it contains a marked point.

\[ \square \]

9. Examples

The enumeration of tropical curves of given degree and fixed genus is a finite problem. However, the algorithmic problems involved in doing this efficiently are not well-studied except in dimension two, where Mikhalkin gave a beautiful algorithm via counting of lattice paths [M1].

In higher dimensions the combinatorics gets out of hands quickly, so that with a naive approach one is limited to tropical curves with few ends or to very special geometry. The following observation often drastically reduces the number of cases to be considered in higher dimensions. Define the distance of two edges in a graph as the minimal number of other
edges that one has to follow in any connecting path. So edges with a common vertex have distance 0.

**Proposition 9.1.** Let an affine constraint $\mathbf{A} = (A_1, \ldots, A_l)$ of codimension $(d_1, \ldots, d_l)$ be general for a type $\Delta$. Then for any tropical curve $(\Gamma, \mathbf{E}, h)$ matching $\mathbf{A}$ the distance between $E_i$ and $E_j$ is at least $d_i + d_j - n$, for any $i \neq j$.

**Proof.** In a path $E_i = E(0), E(1), \ldots, E(\delta), E(\delta + 1) = E_j$ connecting $E_i$ and $E_j$ and passing through vertices $V_\nu \in E(\nu) \cap E(\nu + 1)$ the $i$-th incidence condition restricts $h(V_\nu)$ to an $(n - d_i)$-dimensional subspace. Inductively, $h(V_\nu)$ is constraint to $n - d_i + \nu$ dimensions. Thus for the last vertex $h(V_\delta)$ to intersect a line of given direction $\mathbb{Q}u_{(V_\delta, E_i)}$ and meeting a general $d_j$-codimensional affine subspace $A_j$ requires $(n - d_i + \delta) + (n - d_j) \geq n$, that is, $\delta \geq d_i + d_j - n$. \[\square\]

Note that this observation can be refined by considering the constraints imposed by the $A_i$ at any vertex.

By this argument and tedious case by case considerations we checked the following examples.

**Example 9.2.** (1) **Curves of degree 2 in $\mathbb{P}^3$ through 4 points.** The expected number is zero because the dimension of the space of homogeneous quadratic polynomials in two variables is 3. Hence every such curve lies in a plane, while this is not true for 4 general points.

The statement on the tropical side says that there are no tropical curves of degree $2(1, 0, 0), 2(0, 1, 0), 2(0, 0, 1), 2(-1, -1, -1)$ through 4 general points. The notation means that there are 2 unbounded edges each in any of the directions $(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)$. Now there are very few marked trees fulfilling the requirement of Proposition 9.1 and it turns out none of these could be the domain of a tropical curve of the requested type. Alternatively, one can mimick the argument on the geometric side by showing that each tropical quadric lies in a tropical hyperplane, which does not contain 4 general points.

(2) **Curves of degree $(1, 1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ through 4 points.** Again there is no such curve because the projection to $\mathbb{P}^1 \times \mathbb{P}^1$ is the graph of an automorphism of $\mathbb{P}^1$, which varies only in a three-dimensional family.

Tropically we have degree $(\pm 1, 0, 0), (0, \pm 1, 0), 2(0, 0, \pm 1)$, so the projection to the first two coordinates is a tropical curve of degree $(\pm 1, 0), (0, \pm 1)$ in $\mathbb{Q}^2$. These form a three-dimensional family, and hence there is no curve of this type through 4 general points.

(3) **Curves of degree $(1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ through 4 points.** These are graphs of curves of degree 2 in $\mathbb{P}^2$, and it is not hard to check that generically there is exactly one of them. The corresponding computation on the affine side is for tropical curves of degree $(0, 0, \pm 1), 2(1, 0, 0), 2(0, 1, 0), 2(-1, -1, 0)$.

(4) **Curves of degree 2 in $\mathbb{P}^3$ through 8 lines, 5 of which non-general.** This example features higher dimensional constraints, several types of tropical curves and $\mathcal{D}(\Gamma, \mathbf{E}, h, \mathbf{A}) \neq \mathcal{D}(\Gamma, \mathbf{E}, h, \mathbf{A})$, that is, $\prod \delta_i \neq 1$. Consider quadrics in $\mathbb{P}^3$, so the degree is given by the vectors $(-1, 0, 0), (0, -1, 0), (0, 0, -1), (1, 1, 1)$, each with value 2, so $e + n - 3 = 8 + 3 - 3 = 8$.  

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We impose 8 constraints by lines
\[
\begin{align*}
L_1 &= (-2, -1, 0) + \mathbb{R} \cdot (1, \nu, 0), \\
L_2 &= (-1/2, -1/2, 0) + \mathbb{R} \cdot (\mu, -1, 0), \\
L_3 &= (1/2, 0, 100) + \mathbb{R} \cdot (-1, \lambda, 1), \\
L_4 &= (-3, 1, 0) + \mathbb{R} \cdot (0, 0, 1), \\
L_5 &= (-3, -1, 0) + \mathbb{R} \cdot (0, 0, 1), \\
L_6 &= (-1, -3, 0) + \mathbb{R} \cdot (0, 0, 1), \\
L_7 &= (1, -3, 0) + \mathbb{R} \cdot (0, 0, 1), \\
L_8 &= (3, 2, 0) + \mathbb{R} \cdot (0, 0, 1),
\end{align*}
\]
with integers $\nu \geq 2$, $\mu \geq 1$, $\lambda \geq 3$, $\lambda \neq 6$. Note the last five lines are parallel to each other.

The composition of a tropical curve $h : \Gamma \to \mathbb{R}^3$ of given degree and matching $A$ with the projection $(x, y, z) \mapsto (x, y)$ onto the $xy$-plane contracts the two unbounded edges in direction $(0, 0, -1)$. Hence it is a tropical plane quadric $\mathbb{T} : \mathbb{T} \to \mathbb{R}^2$ through the images of $L_4, \ldots, L_8$, the 5 points $P_4 = (-3, 1)$, $P_5 = (-3, -1)$, $P_6 = (-1, -3)$, $P_7 = (1, -3)$ and $P_8 = (3, 2)$. There is exactly one such plane quadric, depicted in the following figure.

The images of $L_1$, $L_2$, $L_3$ are lines $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$. The possible images of the first three marked points are denoted $a_i$, $b_j$, $c_k$. It is not hard to see that each of the $2 \cdot 3 \cdot 3 = 18$ possibilities $(a, b_j c_k)$ gives rise to a unique lift of $h$ to a tropical quadric in $\mathbb{R}^3$. Because on the considered region $L_3$ has a large value of the last coordinate the two contracted unbounded edges always lie over the selected $a_i$ and $b_j$, and these unbounded edges are the first and second marked edges.

It remains to compute the contribution to $\Lambda^{\top} \left( \mathbb{Z}^{\mathbb{T}} \right)$ for each of the 18 tropical curves thus obtained. We explain the calculation in the case $(a_1 b_3 c_j)$, $j = 1, 2, 3$. In this case, the vertices of the tropical curve are $V_1, V_2, V_3, V_4$ and lifts $A_1, B_3$ of $a_1, b_3$. $\mathcal{D}(\Gamma, E, h, A)$ is the index of the inclusion $\Phi$ of lattices defined in (Proposition 5.7). The domain of $\Phi$ is a copy of $\mathbb{Z}^3$ for each of the 6 vertices. Define primitive vectors $f_i \in \mathbb{Z}^3$ in the direction of the images of the bounded edges as in the figure below. Now the range of $\Phi$ is the free part of the following abelian group.

\[
\begin{align*}
\mathbb{Z}^3/\mathbb{Z}f_1 &\oplus \mathbb{Z}^3/\mathbb{Z}f_2 \oplus \mathbb{Z}^3/\mathbb{Z}f_3 \oplus \mathbb{Z}^3/\mathbb{Z}f_4 \oplus \mathbb{Z}^3/\mathbb{Z}f_5 \\
&\oplus \mathbb{Z}^3/\langle e_1 + \nu e_2, e_3 \rangle \oplus \mathbb{Z}^3/\langle \mu e_1 - e_2 + \nu e_3 \rangle \oplus \mathbb{Z}^3/\langle -e_1 + \lambda e_2 + e_3 \rangle + \mathbb{Z}f \rangle \\
&\oplus \mathbb{Z}^3/\langle e_1, e_3 \rangle \oplus \mathbb{Z}^3/\langle e_1, e_3 \rangle \oplus \mathbb{Z}^3/\langle e_2, e_3 \rangle \oplus \mathbb{Z}^3/\langle e_2, e_3 \rangle \oplus \mathbb{Z}^3/\langle (1, 1, 1), e_3 \rangle.
\end{align*}
\]
Here \( e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1) \). The vector \( f \) in the last summand is \( (0, 1, 0) \) for \((a_1b_3c_1)\), \((0, 1, 0)\) for \((a_1b_3c_2)\), and \((1, 1, 1)\) for \((a_1b_3c_3)\). This holds regardless of the positions of \( b_1 \) and \( c_1 \) relative to \( P_7 \) and of \( b_3 \) relative to \( P_4 \), which depend on the choices of \( \lambda \) and \( \mu \). The image of \((z_1, \cdots, z_6) \in (\mathbb{Z}^3)^{\oplus 6}\) under \( \Phi \) is

\[
(z_2 - z_1, z_3 - z_2, z_4 - z_2, z_1 - z_5, z_3 - z_6, \\
z_5 - r_1, z_6 - r_2, z - r_3, \\
z_6 - r_4, z_5 - r_5, z_1 - r_6, z_4 - r_7, z_4 - r_8),
\]

where \( r_i \in L_i \cap \mathbb{Z}^3 \) are arbitrary and each term is to be viewed in the quotient group. \( z \) in the last term is given by \( z = z_4 \) for \((a_1b_3c_1)\) or \((a_1b_3c_2)\), and \( z = z_3 \) for \((a_1b_3c_3)\). The contribution \( \tilde{D}(\Gamma, E, h, A) \) is given by the absolute value of the determinant of the \( 18 \times 18 \)-matrix representing the linear part of this map, times the order of the torsion group of the abelian group above. This gives \( \nu, \lambda \nu \) and \((1+\lambda)\nu\) for \((a_1b_3c_1)\), \((a_1b_3c_2)\), \((a_1b_3c_3)\), respectively.

\[
\begin{array}{cccccccccc}
 a_1b_1c_1 & a_1b_1c_2 & a_1b_1c_3 & a_1b_2c_1 & a_1b_2c_2 & a_1b_2c_3 & a_1b_3c_1 & a_1b_3c_2 & a_1b_3c_3 \\
 \mu \nu & \lambda \mu \nu & (1+\lambda)\mu \nu & (1+\mu)\nu & \lambda (1+\mu)\nu & (1+\lambda)(1+\mu)\nu & \nu & \lambda \nu & (1+\lambda)\nu
\end{array}
\]

If \( \lambda \) is odd \( \tilde{D}(\Gamma, E, h, A) = \tilde{D}(\Gamma, E, h, A) \) for all \((a_i b_j c_k)\); but if \( \lambda \) is even the last summand in the definition of the range of \( \Phi \) for the \((a_1b_3c_2)\)-case, that is, \( \mathbb{Z}^3/(\mathbb{Z}(-e_1 - \lambda e_2 + e_3) + \mathbb{Z}f) \), \( f = (1, 0, 1) \), is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). In fact, in this case \( \tilde{D}(\Gamma, E, h, A) = \lambda \nu/2, \prod_i \delta_i = 2 \) and \( \tilde{D}(\Gamma, E, h, A) = \lambda \nu \).

Finally, taking twice \((i = 1, 2)\) the sum of the terms in the table gives

\[
N_{0, \Delta}^{\text{trop}}(A) = 8\nu(\mu + 1)(\lambda + 1).
\]

This number can be checked by a direct algebraic-geometric computation, say for \( k = \mathbb{C} \). The five parallel lines define orbits of the form \((\alpha, \beta, \gamma t) \in (\mathbb{C}^*)^3, t \in \mathbb{C}^* \). Their closures are lines through \( P = [0, 0, 1, 0] \in \mathbb{P}^3 \). Projecting from this point defines a map \( \pi : \mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2 \)
contracting these lines to five points. For a general choice of lines through $P$ there is a unique quadric $Q \subset \mathbb{P}^2$ passing through them, which is smooth. The closure $\tilde{Q}$ of $\pi^{-1}(Q)$ is then a quadric cone with isolated singularity at $P$. The intersection of $\tilde{Q}$ with a hyperplane $H \subset \mathbb{P}^3$ defines a quadric curve intersecting the five given lines. Conversely, any quadric curve in $\mathbb{P}^3$ lies in a hyperplane, as already mentioned in the first example. Thus all quadric curves intersecting the five lines in $(\mathbb{C}^*)^3 \subset \mathbb{P}^3$ arise in this way.

Now the remaining three incidence curves $Z_1, Z_2, Z_3$ are rational curves of degrees $\nu, \mu + 1$ and $\lambda + 1$. Hence they intersect $\tilde{Q}$ in $2\nu, 2(\mu + 1)$ and $2(\lambda + 1)$ points, respectively, and all these points are disjoint for a general choice of $Z_1, Z_2, Z_3$. Each of the $2\nu \cdot 2(\mu + 1) \cdot 2(\lambda + 1)$ choices of three intersection points determines a unique hyperplane $H \subset \mathbb{P}^3$. Then $H \cap \tilde{Q}$ is the requested quadric curve intersecting $Z_1, \ldots, Z_8$. Thus

$$N_{0,\Delta}^{\text{alg}}(L) = 8\nu(\mu + 1)(\lambda + 1),$$

with $L = (L(A_1), \ldots, L(A_8))$, in agreement with the tropical computation. □

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