On the chromomagnetic expectation value $\mu_G^2$ and higher power corrections in heavy flavor mesons

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Abstract

The important parameter $\mu_G^2$ of the heavy quark expansion is analyzed including perturbative and power corrections. It is found that $\mu_G^2(2\text{ GeV})$ is known with a few percent accuracy. The perturbative corrections are computed and found small. A nonperturbative relation is suggested which allows to control the power corrections. We conclude that $\mu_G^2(1\text{ GeV}) = 0.35^{+0.03}_{-0.02} \text{ GeV}^2$. The two-loop expression for the effective “ME” radiation coupling $\alpha_s^{(\text{me})}(\omega)$ is given which improves reliability of the perturbative evolution of $\mu_G^2$ towards the low momentum scale. On the nonperturbative side, we advocate the utility of combining the heavy quark expansion with expanding around the “BPS”-type approximation for the meson wavefunction, which implies relations $\mu_G^2 \approx \mu_\pi^2$ and $-\rho_{LS}^3 \approx \rho_D^3$ as well as similar ones for the nonlocal correlators.

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The heavy quark expansion allows to quantify the effects of nonperturbative physics in beauty decays, often in a model-independent way starting from the first principles of QCD. The most informative predictions are obtained for observables where the Operator Product Expansion (OPE) applies, like the inclusive decay widths. The leading nonperturbative effects are described by the two heavy quark expectation values $\mu_\pi^2$ and $\mu_G^2$ of the kinetic and chromomagnetic operators, respectively, entering at the $\Lambda^2_{\text{QCD}}/m_b^2$ level \cite{1}. It turns out, in particular, that the extractions of $V_{cb}$ are sensitive to them.

The value of $\mu_G^2$ can be determined from the mass splitting between $B^*$ and $B$ mesons. The kinetic expectation value $\mu_\pi^2$ is a priori more uncertain. The set of heavy quark sum rules, however, significantly restricts $\mu_\pi^2$ in terms of $\mu_G^2$ \cite{2,3}. In particular, $\mu_\pi^2 - \mu_G^2$ is positive, yet unlikely to exceed 0.2 GeV$^2$.

In this paper we analyze the value of $\mu_G^2$. Throughout the paper it refers to the infinite mass limit. In QCD the heavy quark operators depend on the renormalization point. We consistently take this into account using the complete definition of these operators in the framework of the quantum field theory suggested earlier \cite{4}. To this end the perturbative correction to the hyperfine splitting is computed.

We also address the power corrections. A nonperturbative relation is suggested which looks well satisfied in QCD. It allows one to control a number of higher-order effects. We conclude that the associated uncertainty in $\mu_G^2$ is rather small.

1 Perturbative effects

The value of $\mu_G^2(\mu)$ is extracted in practice using the relation $\mu_G^2 = \frac{3}{2} m_b (M_{B^*} - M_B)$. There are perturbative corrections to this relation depending logarithmically on $\mu/m_b$. At finite $m_b$ there are also nonperturbative corrections suppressed by powers of $1/m_b$.

The bare heavy quark operator $O_G = \bar{Q} g_s \sigma_{\mu\nu} G^{\mu\nu} Q = -\bar{Q} g_s \vec{\sigma}\vec{B}Q$ is ultraviolet divergent in the non-Abelian theory. The normalization point $\mu$ is introduced via the upper cutoff in the integral over the antisymmetric small velocity (SV) heavy quark structure function $W_-(\varepsilon)$ expressing the sum rule for the matrix elements of $\mu_G^2$. In the standard heavy quark notation it takes the following form:

$$
\frac{\mu_G^2(\mu)}{3} = \int_0^\mu W_-(\varepsilon) \varepsilon^2 d\varepsilon = 2 \sum_{\varepsilon_n < \mu} \varepsilon_n^2 |\tau_{3/2}|^2 - 2 \sum_{\varepsilon_n < \mu} \varepsilon_n^2 |\tau_{1/2}|^2 , \quad (1)
$$

where $\tau$’s are the $P$-wave transition amplitudes and $\varepsilon_k$ are the corresponding excitation energies (for a review, see \cite{3}). The OPE gives the SV structure functions in
terms of the zero-recoil matrix elements of the momentum operators $\pi_j = \bar{Q} i D_j Q$:

$$\varepsilon^2 W(\varepsilon) \propto \sum_n \langle H_Q | \pi_j | n_p \rangle \langle n_p | \pi_l | H_Q \rangle \delta^3(\vec{p}) \delta(E_n - \varepsilon).$$

Since $[D_j, D_l] = -ig_s G_{jl}$ holds, the sum rule (1) is transparent [2]:

$$\bar{Q} \pi^2 Q = \sum_k \pi_k \pi_k,$$

and $\bar{Q} g_s B_l Q = i \delta_{ljk} \pi_j \pi_k$.

In QED the integral in Eq. (1) converges and defines the magnetic field strength $e \vec{B}_{em}(0)$ at the position of the static center. The magnetic spin interaction of an elementary heavy fermion is given precisely by this expectation value times the Dirac anomalous moment, $e \vec{B}_{em} \sigma \frac{1}{2m} \left(1 + \frac{\alpha}{2\pi} + \ldots\right)$. In a non-Abelian theory like QCD the integral diverges in the ultraviolet, and the expectation value of $g_s \vec{B}_{chr}(0)$ depends on the normalization point. Let us note that the adopted definition of the operator corresponds to the usual scheme with the two covariant derivatives taken at different points and connected by the $P$-exponent. The displacement lies on the (Euclidean) time axis and its magnitude is governed by $1/\mu$. More precisely,

$$\langle \bar{Q} \pi_j(x_0) \rho e^{i \int_0^x A_0(x) dx_0} \pi_k Q(0) \rangle = \int_0^\infty x_0 e^{-\mu x_0} d\mu \langle \bar{Q} \pi_j \pi_k Q(0) \rangle /\mu. \quad (2)$$

The heavy quark Hamiltonian has the well known form

$$\mathcal{H}_Q = m_Q - g_s A_0 + \frac{\vec{p}^2}{2m_Q} + c_G \frac{g_s \sigma_{jk} G_{jk}}{2m_Q} + O \left(\frac{1}{m_Q^2}\right). \quad (3)$$

To compute perturbatively the Wilson coefficient $c_G(\mu)$ we therefore can consider the zero-velocity heavy quark transition amplitude $T_{ij}(\omega)$ mediated by the currents $\bar{Q} i D_l Q$ and $\bar{Q} i D_j Q$, Fig. 1 (the corresponding OPE formalism is discussed in detail, e.g. in Ref. [3]). To select the matrix element of the chromomagnetic operator we evaluate it on a heavy quark state including scattering of an additional (transverse) gluon, and look for the component antisymmetric in $i, j$. This appears in the linear in the gluon momentum $\vec{q}$ approximation. For simplicity, we take $q_0 = 0$, and assume the initial quark is at rest. We also do not show explicitly the heavy quark spin indices (as if $Q$ is a scalar).

![Figure 1: The heavy quark forward scattering amplitude as a function of energy $\omega$. The solid blocks denote the momentum operator $\bar{Q} i D Q$.](image)
The tree level $O(\alpha_s^0)$ expression is obvious:

$$T_{ij}(\omega) = \frac{1}{-\omega + i\epsilon} g_s q_j \delta_{il} \bar{Q} \lambda^\epsilon Q$$

where $l$ and $c$ are the gluon polarization and color indices, respectively. The tree gluon QCD vertex is $g_s \bar{\psi} Q \gamma_\mu \lambda \psi Q = g_s \bar{\psi} Q \left( \frac{(p_1 + p_2)_\mu}{2m_Q} - \frac{\sigma_{\mu\nu}q_\nu}{2m_Q} \right) \lambda^\epsilon \psi Q$. Comparing its nonrelativistic expansion \( \bar{\psi} Q \gamma_\mu \lambda \psi Q \simeq (p_1 + p_2)_\mu \varphi_+^Q \varphi_Q + i\epsilon_{ijk} \varphi_+^Q \sigma_j \varphi_Q q_k \) with $T_{ij}(\omega)$ yields $c_{G_{\text{tree}}} = 1$.

To account for the strong interaction corrections we compute $T_{ij}(\omega)$ perturbatively assuming $-\omega \gg \Lambda_{\text{QCD}}$. The same corrections are computed for the $\bar{Q}Qg$ vertex in QCD projected onto the magnetic spin structure. The vertex yields the one-gluon matrix element of the effective heavy quark Lagrangian. The difference between the two determines the coefficient $c_G$. It is ultraviolet (UV) finite, as well as infrared (IR) finite even at $\vec{q} \to 0$. The latter limit significantly simplifies computations allowing for directly expanding the Feynman integrands over $q$. The resulting integrals are saturated in the domain of momenta between $\omega$ and $m_Q$. The diagrams one has to compute in the static theory are shown in Figs. 2.

![Figure 2: Examples of one-loop diagrams for the one-gluon matrix element of $T_{ij}(\omega)$. The remaining diagrams vanish for the chosen kinematics, or upon antisymmetrization over $i, j$. The gluon wavefunction renormalization is omitted.](image-url)

The individual diagrams, however, can be ultraviolet and infrared divergent. Since the whole integral for $c_G$ is well behaved, one can use any regularization in the infrared and the ultraviolet for computing separate diagrams; the only requirement is that it must be consistently the same. For example, one can compute $c_G$ in dimension $D = 4 + 2\epsilon$; then at $\epsilon > 0$ there are no IR singularity and the limit $\vec{q} \to 0$ is straightforward\[^1\]. On the other hand the point $\epsilon = 0$ is regular for $c_G$. However, since there are separate diagrams which diverge both in the UV and IR, dimensional regularization is not advantageous. Instead, we cut the integrals in the

\[^1\]At $D = 4$ the chromomagnetic moment in QCD has infrared divergence $\sim \ln 1/\vec{q}^2$. 

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UV at $k^2 = \Lambda^2$, and introduce the IR mass regulator in the gluon propagator $\frac{\delta_{\mu\nu}}{k^2}$. The latter allows us to use the limit $q^2 \to 0$ even at $D = 4$. Cancellation of the terms dependent on $\lambda^2$ and $\Lambda^2$ provides a useful cross-check.

The computation of the one-loop matrix element of $T_{ij}$ in the effective static theory results in the Feynman gauge in

$$
\langle Qg|g_s G_{ij}(\mu)|Q\rangle = g_s (\delta_{ij} q_j - \delta_{ji} q_i) \overline{Q} \frac{C}{2} Q \left\{ 1 - \frac{g_s^2}{16\pi^2} \frac{-C_A}{2} \left( 2 \ln \frac{\Lambda^2}{\mu^2} - 4 \right) \right\}.
$$

(5)

The QCD vertex takes the form

$$
\Gamma_\mu = \gamma_\mu - \frac{g_s^2}{16\pi^2} \left\{ \frac{-C_A}{2} \left[ 2 \ln \frac{\Lambda^2}{\lambda^2} - 8 \right] \gamma_\mu + \left[ C_F + \frac{-C_A}{2} \left( \ln \frac{m_Q^2}{\lambda^2} - 2 \right) \right] \gamma_\mu \right\},
$$

(6)

where $C_F = 4/3$ and $C_A = N_c$. We have omitted from both expressions the diagrams renormalizing the external gluon propagator, since they are the same in both cases. For magnetic structure the difference in the Abelian part amounts, as expected, to the Schwinger anomalous term $C_F \frac{\alpha_s}{2\pi}$. The non-Abelian difference is just

$$
-C_A \frac{\alpha_s}{2\pi} \ln \frac{m_Q^2}{\omega^2}.
$$

We note, however, that with the non-Abelian interaction the two theories would have different running strong coupling $g_s(k^2)$ below $m_Q$ if the bare coupling $g_s^{(0)}$ and the UV cutoff $\Lambda$ are taken the same. This is seen by evaluating the “charge” gluon vertex $\Gamma_0$ in the static theory. The difference originates from the part of the ‘Abelian’ vertex correction proportional to $C_A$ which is not canceled by the renormalization of the quark wavefunction – it yields pure $\ln \frac{\Lambda^2}{\mu^2}$. (The ‘non-Abelian’ vertex is absent from the charge interaction.) The total correction to the $\gamma_\mu$ structure in the vertex in QCD does not depend on the quark mass, Eq. (5) as it should be to respect gauge invariance. However, this holds only provided the UV cutoff is infinitely larger than all other mass scales. The renormalization of the static quark interaction differs by a constant since here the UV cutoff is much lower than $m_Q$. This means that the static quark gauge interaction requires a different counterterm. Alternatively, it can be expressed by saying that for static quarks $\Lambda_{\text{stat}}$ must be taken different from $\Lambda$ in full QCD. As follows from Eq. (6), at one loop one has $\Lambda_{\text{stat}} = \Lambda/e^2$ in the $C_A$ term, and $\frac{\alpha_s}{2\pi} C_A \ln \frac{m_Q^2}{\omega^2}$ gets replaced by $\frac{\alpha_s}{2\pi} C_A \left( \ln \frac{m_Q^2}{\omega^2} - 2 \right)$.

To calculate the matrix element of $\mu_{ij}^2(\mu)$ we need to integrate $\frac{1}{2\pi} \text{Im} T_{ij}(\omega)$. The literal expression (4) is not valid at $|\omega| \lesssim |\vec{q}|$ even in perturbation theory. However, the perturbative $T(\omega)$ has the proper analytic properties to any order. We then can represent the sum in Eq. (4) as an integral over the contours in the complex $\omega$ plane stretched away from small $\omega$, see Fig. 3 where the perturbative matrix element is given by Eq. (4). The integral is simple:

$$
\langle Qg|g_s G_{ij}(\mu)|Q\rangle = g_s (\delta_{ij} q_j - \delta_{ji} q_i) \overline{Q} \frac{C}{2} Q \left\{ 1 - \frac{g_s^2}{16\pi^2} \frac{-C_A}{2} \left( 4 \ln \frac{\Lambda^2}{e^2\mu} - 4 \right) \right\}.
$$

(7)
We thus get the final result

\[
c_G(\mu) = 1 + C_F \frac{\alpha_s}{2\pi} + C_A \frac{\alpha_s}{2\pi} \left( \ln \frac{\mu}{m_Q} + 2 \right).
\]

(8)

Figure 3: The complex plane of energy \( \omega \). Thick line at \( \omega > 0 \) shows the cut of \( T(\omega) \). The integral of \( \text{Im} T(\omega) \) can be taken over the circle \( |\omega| = \mu \).

It is convenient to absorb the constant term \( 2C_A \) in the non-Abelian part into the argument of the logarithm, it i.e. use \( (\ln \frac{\mu}{m_Q} + 2) = \ln \frac{\sqrt{\mu}}{m_Q} \). Moreover, the usual Abelian part \( \frac{\alpha_s}{2\pi} C_F \) depends on the normalization convention used for the heavy quark mass. The standard Schwinger coefficient in Eq. (8) refers to the pole mass, a choice disfavored in QCD. Using instead the running 'kinetic' mass \( m_Q(\mu) \) [4, 5] we find that for the chromomagnetic term in the Hamiltonian (3)

\[
c_G(\mu) = 1 + C_A \frac{\alpha_s}{2\pi} \ln \frac{e^\mu}{m_Q^2}.
\]

(9)

Alternatively, using the \( \overline{\text{MS}} \) mass this expression becomes

\[
c'_G(\mu) = \frac{1 + C_A \frac{\alpha_s}{2\pi} \ln \frac{e^\mu}{m_Q}}{2m_Q(m_Q/e^\mu)};
\]

(10)

however, the \( \overline{\text{MS}} \) mass of a heavy quark loses physical significance at the normalization scales below \( m_Q \) [4].

\section{2 Power corrections}

The perturbative relation

\[
c_G(\mu) \mu_G^2(\mu) = \frac{3}{2} m_b \cdot (M_{B^*} - M_B)
\]

holds only for asymptotically heavy \( b \) where power-suppressed effects die out. The \( 1/m_b^2 \) corrections to the hadron masses are given by two local heavy quark operators and four nonlocal correlators also describing the \( 1/m_Q \) corrections to the hadrons' wavefunctions. For \( M_{B^*} - M_B \) one has [4]

\[
M_{B^*} - M_B = \frac{2}{3} \frac{\mu_{BS}^2}{m_b} + \frac{1}{3} \frac{\rho_{LS}^3}{m_b^2} + \frac{1}{3} \frac{\rho_{1G}^3 + \rho_{A}^3}{m_b^3} + \mathcal{O} \left( \frac{1}{m_b^4} \right).
\]

(12)
The nonlocal correlators are saturated by the transitions into the $\mu$ where $\mu$\[7\]. In particular, for the first three moments we have around 1 GeV scale.

In actual QCD the heavy quark bound state is rather relativistic. This is quantified by the difference between the transition amplitudes $\tau_{3/2}$ and $\tau_{1/2}$ and between the masses of the $\frac{3}{2}$ and $\frac{1}{2} P$-wave states, which are the same in nonrelativistic systems [7]. In particular, for the first three moments we have

$$\frac{2 \sum_m \left| \tau_{3/2}^{(m)} \right|^2 - 2 \sum_n \left| \tau_{1/2}^{(n)} \right|^2}{2 \sum_m \left| \tau_{3/2}^{(m)} \right|^2 + \sum_n \left| \tau_{1/2}^{(n)} \right|^2} = \frac{1}{2q^2 - 1/2} \approx 0.7 \quad (13)$$

$$\frac{2 \sum_m \varepsilon_m \left| \tau_{3/2}^{(m)} \right|^2 - 2 \sum_n \varepsilon_n \left| \tau_{1/2}^{(n)} \right|^2}{2 \sum_m \varepsilon_m \left| \tau_{3/2}^{(m)} \right|^2 + \sum_n \varepsilon_n \left| \tau_{1/2}^{(n)} \right|^2} = \frac{2 \sum}{\bar{\Lambda}} \approx 0.7 \quad (14)$$

$$\frac{2 \sum_m \varepsilon_m^2 \left| \tau_{3/2}^{(m)} \right|^2 - 2 \sum_n \varepsilon_n^2 \left| \tau_{1/2}^{(n)} \right|^2}{2 \sum_m \varepsilon_m^2 \left| \tau_{3/2}^{(m)} \right|^2 + \sum_n \varepsilon_n^2 \left| \tau_{1/2}^{(n)} \right|^2} = \frac{\mu_G^2}{\mu_\pi^2} \approx 0.8 \quad (15)$$

with $\bar{\Lambda} \approx 700$ MeV and $\sum \approx 250$ MeV. The normalization scale dependent $q^2$, $\bar{\Lambda}$, $\mu_\pi^2$ and $\mu_G^2$ are taken at the scale around 1 GeV.

The magnitude of $\rho^3_{LS}$ can then be estimated using the next spin sum rule [2]:

$$- \rho^3_{LS} \approx \mu_G^2 \mu_{\text{hadr}} \approx 0.15 \text{ to } 0.2 \text{ GeV}^3 , \quad (16)$$

where $\mu_{\text{hadr}} \approx 500$ MeV is a characteristic mass scale for the $P$-wave excitations. The nonlocal correlators are saturated by the transitions into the $j^P = \frac{1}{2}^+$ for $\rho^3_{nG}$ and into $j^P = \frac{3}{2}^+$, $j^P = \frac{1}{2}^-$ for $\rho^3_A$ “radial” excitations; they are largely unknown.

We can estimate the necessary combination of the above spin-triplet $D=3$ parameters employing the empirical observation that the mass-square splitting between vector and pseudoscalar mesons is nearly a constant:

$$M^2_\rho - M^2_\pi \approx M^2_\rho - M^2_\pi \approx M^2_{D^*} - M^2_D \approx M^2_{B^*} - M^2_B \quad (17)$$

which extends even to strange charmed mesons. (A 12% decrease for $B$ fits well the expected perturbative renormalization.) It is related to the universal slope of the corresponding Regge trajectories, a yet poorly understood nonperturbative phenomenon of strong dynamics, perhaps related to a certain simplification in the large $N_c$ limit. This universality must be definitely violated for very heavy quarks due to hard gluons with momentum scaling with $m_Q$. Rather, it can be viewed as an inherent property of soft nonperturbative interactions responsible for physics around 1 GeV scale.

The universality implies the relation

$$- (\rho^3_{LS} + \rho^3_{nG} + \rho^3_A) \approx 2 \sum \mu_G^2 \approx 0.5 \text{ GeV}^3 , \quad (18)$$
where we have used \( \Lambda \simeq M_B - m_b \simeq 700 \text{ MeV} \). The above nonperturbative parameters are normalized at the scale around 1 GeV. With the estimate (16) we are led to an evaluation

\[-(\rho^3_{\pi\pi} + \rho^3_A) \approx 0.6 \text{ to } 0.7 \text{ GeV}^3 \quad (19)\]

The scale of these correlators lies in the expected range if one bears in mind the magnitude of other parameters \( \Lambda, \mu^2 \pi, \mu^2 G \) all being given by a mass 600 to 700 MeV to the corresponding power, though possibly on the upper side.

The sign of \( \rho^3_{\pi\pi} \) and \( \rho^3_A \) is not known \textit{a priori}. We note, however, that the so far observed nonperturbative phenomena in the heavy mesons fit well the approximation that the ground state \( B \) has nearly the “lowest Landau level” wavefunction, or is the BPS-saturated state. For instance, \( \mu^2 \pi - \mu^2 G \) is noticeably lower than \( \mu^2 G \). If this saturation was actually the case and \( \mu^2 \pi - \mu^2 G = 0 \), one would necessarily have \( \bar{\sigma}\bar{\pi} |B\rangle = 0 \), meaning that the asymptotic wavefunction of the light cloud annihilates two certain linear combinations of the total momentum operators, e.g. \( \mathcal{P}_z \) and \( \mathcal{P}_z - i\mathcal{P}_y \) for the state \( j_z = \frac{1}{2} \). Signifying vanishing of all \( \tau_1/2 \), this would also entail a series of relations which include vanishing of all the \( B \) meson correlators involving powers of \( \bar{\sigma}\bar{\pi} \). Say,

\[ -\rho^3_{LS} = \rho^3_D, \quad \rho^3_{\pi\pi} = -2\rho^3_{\pi\pi}, \quad \rho^3_A + \rho^3_{\pi\pi} = -\rho^3_{\pi\pi} - \rho^3_S \quad (20)\]

would hold. The first relation agrees with the estimate (16) (note that \( -\rho^3_{LS} \geq \rho^3_D \)). The correlators \( \rho^3_{\pi\pi} \) and \( \rho^3_S \), on the other hand, are positive, which shows consistency of the estimates.

Adopting relation (18) as the guideline, we end up with

\[ \mu^2_G = \frac{3}{4} \frac{S^{-1}(\mu; m_b)}{c^*} \cdot \left( M^2_B - M^2_{B^*} \right) \quad (21)\]

through order \( 1/m_b \). The effect of the \( 1/m_b \) corrections in Eq. (11) amounts to the apparent decrease of \( \mu^2_G \) by about 15\%, recovered in the analysis above. The magnitude of the shift is moderate and fits well the expectations [8].

### 3 Numerical estimates

Applying equations (9), (12) and (18) we arrive at the evaluation

\[ \mu^2_G(2.5 \text{ GeV}) \simeq 0.312 \text{ GeV}^2, \quad (22)\]

where the value of \( m_b(1 \text{ GeV}) = (4.57 \pm 0.05) \text{ GeV} \) was used and its evolution to the scale \( 1.5 \text{ GeV} \approx m_b/3 \) can be safely done using

\[ \frac{d m_b(\mu)}{d \mu} = -\left( \frac{16}{3} + \frac{4}{3 m_b} \right) \frac{\alpha_s^{(6)}(\mu)}{\pi}; \quad (23)\]
the effective dipole radiation strong coupling $\alpha_s^{(d)}$ is known to two loops $[3]$. The uncertainty in the mass affects $\mu_G^2$ here only at a percent level. Likewise, the perturbative effects are included up to the second loop; their uncertainty can hardly be significant.

The possibly largest uncertainty comes from the precise value of the $D = 6$ operators. It has been mentioned above that the 12% decrease in $M_B^2 - M_D^2$ compared to $M_B^2 - M_D^2$ is well described by the perturbative renormalization amounting at one loop to a factor of about 0.87. However, the two-loop result by Czarnecki and Grozin $[3]$ further suppresses $c^{(b)}_G/c^{(c)}_G$ down to about 0.8. Taken literally, this would imply even some overshooting, so that the power corrections governed by $-(\rho_5^2 + \rho_5^3 - \rho_5^{LS})$ have to make up for the extra suppression. This would imply the combination to further exceed the estimate $[8]$ by a factor as large as 1.5 to 2, being next to fatal for the expansions in $1/m_c$. However, we note that the major perturbative factor in the extra NLO enhancement of $M_B^2 - M_D^2$ comes from the increase in the renormalization of the chromomagnetic moment of the charm quark due to growing $\alpha_s$ at the charm scale. Introducing the proper Wilsonian cutoff around 1 GeV would effectively stop the increase of this short-distance coefficient for charm, and eliminate the apparent contradiction with the observed behavior without invoking a too large scale for higher-order power corrections.

To remain on the safe side we allow for an additional factor 0.75 to 1.4 in Eq. (18). Then we arrive at

$$\mu_G^2(2.5 \text{ GeV}) = (0.30 \text{ to } 0.33) \text{ GeV}^2. \quad (24)$$

The scale 2.5 GeV would be high enough to trust the perturbative expansion. In practical applications we need, however, to evaluate $\mu_G^2$ at the lower scale around 1 GeV. The perturbative evolution of $\mu_G^2$ obeys

$$\mu \frac{d\mu_G^2(\mu)}{d\mu} = -C_A \frac{\alpha_s^{(me)}(\mu)}{2\pi} \mu_G^2(\mu), \quad (25)$$

where $\alpha_s^{(me)}$, in analogy with the coupling in Eq. (23) can be called “ME”-radiation one, or “ME-coupling”. To first order it is the usual QCD $\alpha_s$. The complete two-loop calculation presented in paper $[4]$ combined with the one-loop result Eq. (8) is sufficient to determine $\alpha_s^{(me)}(\omega)$ to order $\alpha_s^2$, using renormalization invariance of observables. This yields

$$C_A \frac{\alpha_s^{(me)}(\omega)}{2\pi} = \gamma_m - 2C_A \left( \frac{11}{3} C_A - \frac{2}{3} n_f \right) \left( \frac{\alpha_s}{4\pi} \right)^2 + \mathcal{O} \left( \alpha_s^3 \right), \quad (26)$$

where $\gamma_m$ is the anomalous dimension of Ref. $[4]$, Eq. (15) and $n_f$ is the number of light flavors. Therefore, we have

$$\alpha_s^{(me)}(\omega) = \alpha_s \left( e^{-\frac{1}{4\pi} \omega} \right) \left[ 1 - \frac{25}{24} C_A \frac{\alpha_s}{\pi} \right], \quad C_A = N_c = 3. \quad (27)$$
As expected, the two-loop anomalous dimension of the physically defined chromomagnetic operator differs from the $\overline{\text{MS}}$ one already at order $\alpha_s^2$ (its conformal part is still the same to two loops).

The evolution equation Eq. (25) is immediately solved:

$$\mu^2 G(\mu) = \mu^2 G(\omega) \left[ \frac{\pi}{\alpha_s(\omega)} + \frac{\beta_1}{2\beta_0} \left( \frac{\pi}{\alpha_s(\mu)} + \frac{\beta_1}{2\beta_0} \right) \right],$$

where $\beta_0$ and $\beta_1$ are usual QCD one- and two-loop coefficients in the $\beta$-function, and the evolution of the $\text{ME}$ coupling to two loops is given by the same renormalization group equation.

The evolution equation (25) viewed perturbatively suggests that $\mu G(\mu)$ is enhanced toward lower normalization scale. Taking it at face value we get an enhancement factor $1.11$. This is larger than the literal one-loop difference coming from Eq. (8) due to the growing strong coupling which is accounted for in the evolution equation. One clearly cannot go with the perturbative evolution too low in $\mu$. For instance, at $\mu \to 0$ the value of $\mu_G$ goes to zero, constituting apparently only about $0.2 \text{GeV}^2$ at $\mu = 0.5 \text{GeV}$ [3, 7]. Nevertheless, it is justified to trust the above moderate enhancement. The physically defined coupling $\alpha_s^{(m)}$ is smaller than $\alpha_s$ and remains suppressed protecting perturbative corrections from blowing up. This is in contrast with the standard $\overline{\text{MS}}$ anomalous dimension which receives large positive two-loop correction.

Combining the above factors, we arrive at

$$\mu^2 G(1 \text{ GeV}) = \frac{3}{4} \left( M_{B^+}^2 - M_{B^0}^2 \right) \left[ 0.94 \pm 0.025 \begin{array}{l} +0.06 \pm 0.03 \end{array} \right].$$

Therefore, we conclude that the value of $\mu^2 G(1 \text{ GeV})$ most probably amounts to

$$\mu^2 G(1 \text{ GeV}) = 0.35^{+0.03}_{-0.02} \text{ GeV}^2;$$

the lower values can appear in an unfavorable scenario where power corrections in charm go out of theoretical control.

4 Discussion and outlook

The presented analysis allows us to determine the consistently defined chromomagnetic value $\mu_G^2$ normalized at the scale around $2 \text{ GeV}$ with minimal theoretical uncertainty. We conclude that the value $\mu_G^2(1 \text{ GeV}) \simeq 0.4 \text{ GeV}^2$ we have used so far was reasonably accurate, yet probably about 10% larger than it is in reality. In principle, as low a value as $0.30 \text{ GeV}^2$ cannot be rigorously excluded at present, for the price of endangering the $1/m_c$ expansions. The uncertainty in the perturbative effects can be further decreased carrying out the same program to two loops. A
complementary information on the $D=3$ nonperturbative parameters in the hadron mass expansion would be helpful to get more confidence in the evaluation of the power corrections and to shrink the error bars down to a few percent level.

We found the essentially non-Abelian effective “$\mathcal{ME}$”-coupling which does not have a counterpart in electrodynamics, either classical or quantum. It determines, for example, the fraction of the total momentum of the QCD degrees of freedom carried by the hard (short-distance) components of the light cloud; it is power suppressed, $\propto \mu_G^2/\mu^2$ [7]. Physics behind this phenomenon will be discussed elsewhere.

At the loop level the gluon self-interaction suppresses the effective “$\mathcal{ME}$”-coupling compared to the traditional $\overline{\text{MS}}$ coupling. The conformal part of it turns out similar to the one in the dipole radiation coupling [3], although the non-Abelian suppression is approximately twice larger then in the latter, $2\pi N_c \alpha_s / \pi$ vs. $(\pi^2/6 - 13/12) N_c \alpha_s / \pi$. The similarity further supports the \textit{a priori} expected advantage of using the radiation coupling as an effective perturbative expansion parameter. Following the OPE-based line of reasoning of Ref. [3] we find that the $1/\omega$ nonperturbative component in this coupling at large $\omega$ is associated with the $\text{LS}$ operator and is proportional to its anomalous dimension $\gamma_{\text{LS}}$. Implementing the approach of Ref. [10] one can show that this anomalous dimension exactly coincides with that of the chromomagnetic operator. Then we conclude that in $B$ mesons, for instance, at large $\omega$

$$\frac{\delta_{\text{np}} \alpha_s^{(me)}(\omega)}{\alpha_s^{(me)}(\omega)} \approx -\frac{\rho_{\text{LS}}^2(\omega)}{\omega \mu_G^2(\omega)} \approx \frac{0.5 \text{ GeV}}{\omega}. \quad (31)$$

This estimate is consistent with the possibility to perturbatively evolve $\mu_G^2$ down to a 1 GeV scale.

The precise expectation value of the kinetic operator $\mu_G^2$ is quite critical in a number of applications. Heavy quark sum rules ensure that the inequality $\mu_G^2(\mu) > \mu_G^2(\mu)$ holds at arbitrary normalization point [2, 3], and $\mu_G^2$ almost certainly must lie in the interval $0.4 \text{ GeV}^2 \leq \mu_G^2(1 \text{ GeV}^2) \leq 0.55 \text{ GeV}^2$. Moreover, using the spin sum rules [1] one has $\mu_G^2(\mu_0) - \mu_G^2(\mu) = 3\tilde{\varepsilon}^2 \cdot (\tilde{\gamma}^2(\mu_0) - 0.75)$ with $0.5 \text{ GeV} \lesssim \tilde{\varepsilon} < \mu$, and we expect $\tilde{\varepsilon} \approx 0.5 \text{ GeV}$ for the usual choice of $\mu = 1 \text{ GeV}$. (One should keep in mind that the value of the kinetic energy of actual $b$ quark in $B$ meson can be reduced due to often discarded, but probably noticeable $1/m_b$ effects [8, 11].)

We anticipate that combining the heavy quark expansion with an approximation assuming $\mu_G^2 - \mu_G^2 \ll \mu_G^2$, manifesting the proximity to the “BPS” regime for the ground state, can be useful in guiding us through understanding pattern, or even physics of higher-order power corrections in $B$ and $D$ mesons. Expanding around this approximation provides a new and effective nonperturbative parameter small enough to isolate a number of potentially large power corrections. For example, the anomalous dimensions of $c_G \rho_{\text{LS}}^3$ and $c_G^2 \rho_A^3$ vanish by the same token, and their mixing into $\rho_{\text{LS}}$ is easily fixed. I am grateful to M. Eides and A. Vainshtein for discussing this point.
usual spin-averaged heavy meson mass expansion replaced by

\[ m_b - m_c = M_B - M_D + \frac{\mu_p^2 - \mu_G^2}{2} \left( \frac{1}{m_c} - \frac{1}{m_b} \right) - \frac{\mu_D^2 - \mu_{LS}^2 + \mu_{\pi\pi}^2 + \mu_{\pi G}^2 + \mu_s^2}{4} \left( \frac{1}{m_c^2} - \frac{1}{m_b^2} \right) \]

\[ + \mathcal{O} \left( \frac{1}{m_Q^3} \right) \]

is possibly more stable in respect to higher orders being governed by the suppressed higher-dimension expectation values scaling like powers of \( \sqrt{\mu^2 - \mu_G^2} \). As has been pointed out [12], the uncertainty in the precise value of \( m_b - m_c \) is currently the main limiting factor in extracting \( |V_{cb}| \). The validity of the approximation can be experimentally cross checked by carefully analyzing the semileptonic \( b \to c \) transitions into excited states, in particular to the \( \frac{1}{2}^+ \) \( P \)-waves.

In the BPS regime, a number of other nonperturbative effects become more tractable. Say, the \( B \to D \) formfactor at zero recoil does not have \( 1/m_Q^2 \) corrections at all. The corrections still are present and significant in the \( B \to D^{*} \) zero recoil formfactor \( F(0) \), but their structure simplifies:

\[ F(0) = \xi_A^4 - (1+\chi) \frac{\mu_G^2}{6m_c^2} - \mathcal{O} \left( \frac{1}{m_Q^3} \right) \]

with \( \chi \mu_G^2 \) given by a sum of two other positive nonlocal correlators \( \rho_{\pi\pi}^2 + \rho_S^2 \)

\[ \chi \mu_G^2 = \int |x_0| \, d^4 x \frac{1}{4 M_B} \left[ \langle B | i T \{ \bar{b} \pi^2 b(x), \bar{b} \pi^2 b(0) \} | B \rangle' + \frac{1}{3} \langle B | i T \{ \bar{b} \sigma_j B_k b(x), \bar{b} \sigma_j B_k b(0) \} | B \rangle' \right], \]

similar to \( \rho_{\pi\pi}^2 \) and \( \rho_S^2 \) (see Eqs. (28) of Ref. [6]), but containing the extra factor of \( |x_0| \) in the integrand, or \( 1/(E_n - E_0) \) in the language of perturbation theory in quantum mechanics. We expect the approximation \( \rho_{\pi\pi}^2 + \rho_S^2 \approx (\rho_{\pi\pi}^2 + \rho_S^2)/\varepsilon_{rd} \) (actually, a rigorous upper bound [6]) to be reasonably accurate, with \( \varepsilon_{rd} \approx 600 \text{ MeV} \) the energy of the first \( j^P = \frac{1}{2}^+ \) radial excitation of the ground state. Taken literally this would suggest \( \chi \) to be quite large, around 2.

The limit \( \mu_p^2 - \mu_G^2 = 0 \) means that the heavy quark wavefunction minimizes the momentum square operator in a given chromomagnetic field. This happens for the lowest Landau level which is an example of a “BPS-saturated” state. It is worth noting that the newer relativistic quark models of heavy hadrons [13] properly implementing Lorentz transformations yield a good approximation to this limit. Complementary to this, the Block & Shifman QCD sum rule analysis of the IW function [14] strongly supports this by virtue of the spin sum rules. In usual quantum mechanical systems of electrons the BPS saturation is realized applying strong magnetic field. In mesons the chromomagnetic field is \textit{a priori} of the same order \( \Lambda_{QCD}^2 \) as the chromoelectric field, and is far from classical. In \( B \) mesons the approximate BPS

\[ \text{It is interesting to recall that the limit of a strong magnetic field in a quantum mechanical system of charged particles yields a physical realization of noncommuting space coordinates [15]. Actual } B \text{ mesons may well share some of their properties.} \]
saturation would rather be dictated by a very special internal structure of the light cloud, leading to the strong correlation between the spin and momentum operator. Such a correlation vanishes in a nonrelativistic system, and can be realized only in a deeply relativistic regime. It does not look probable that such a property, if confirmed experimentally, is purely accidental. Perhaps, it is related to a certain large parameter, like the number of space dimensions or the number of colors. It is intriguing to study such possible connections.

Our analysis of $\mu_2^2$ incorporating powerful constraints from a series of the heavy quark sum rules gives an indication that the higher-dimension expectation values are quite significant and probably lie at the higher end of the existing estimates. Likewise, the pattern of the excited heavy quark states seems to be at some variance with the routinely employed assumptions inherited from obsolete models. There are fresh ideas of how to improve the knowledge in less vulnerable ways. These will be addressed in forthcoming publications.

Acknowledgments: The author is pleased to thank M. Eides for illuminating conversations and I. Bigi for useful discussion of possible applications; help by A. Czarnecki, P. Nason and M. Shifman is gratefully acknowledged. I am thankful to A. Vainshtein for both criticism and constructive suggestions. This work was supported in part by the NSF under grant number PHY00-87419.

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