Separable and Equitable Hypergraphs

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Abstract

A k-hypergraph is separable if its vertices admit a certain labeling, and is equitable if the edges of the complete k-hypergraph admit a certain labeling. We show that these classes of hypergraphs are mutually exclusive. We conjecture a characterization of equitable hypergraphs and hence also of separable hypergraphs, and prove it for graphs, multipartite k-hypergraphs for all k, paving k-matroids and binary k-matroids for all k, and 3-matroids.

Keywords: hypergraph, matroid, combinatorial optimization, threshold graph

1 Introduction

A k-hypergraph is a pair \((V, H)\) where \(V\) is a finite set and \(H\) is a set of \(k\)-subsets of \(V\). The elements \(v \in V\) are vertices and the sets \(E \in H\) are edges. We avoid unnecessary trivialities and in all our statements, conjecture, lemmas, and theorems, assume \(1 \leq k < n := |V|\).

Here we study the following two classes of hypergraphs. First, a k-hypergraph is separable if there is a labeling \(x : V \to \mathbb{R}\) of vertices by real numbers such that, with \(x(E) := \sum_{v \in E} x(v)\),

\[
\mathcal{H} = \{ E \subseteq V : |E| = k, \ x(E) \geq 0 \} .
\]

For instance, the following 3-hypergraph \((V, \mathcal{H})\) is separable with a suitable labeling \(x\),

\[
V = [6], \quad \mathcal{H} = \{124, 134, 145, 146, 234, 245, 246, 345, 346\}, \quad x(v) = \begin{cases} -1, & v = 1, 2, 3; \\ 3, & v = 4; \\ -2, & v = 5, 6. \end{cases}
\]

Second, a k-hypergraph is equitable if there is a nonzero labeling \(y : \binom{V}{k} \to \mathbb{R}_+\) of \(k\)-subsets of \(V\) by nonnegative real numbers such that, with \(\overline{H} := \binom{V}{k} \setminus \mathcal{H}\) the complement of \(\mathcal{H}\),

\[
\sum_{E \in \mathcal{H}} y(E) = \sum_{F \in \overline{H}} y(F) \quad \text{for every} \ v \in V.
\]

For instance, the following 3-hypergraph \((V, \mathcal{H})\) is equitable with a suitable labeling \(y\),

\[
V = [6], \quad \mathcal{H} = \{135, 136, 145, 146, 235, 236, 245, 246\}, \quad y(G) = \begin{cases} 1, & G = 134, 135, 245, 256; \\ 0, & \text{otherwise}. \end{cases}
\]

As is shown in Lemma 2.1, the Farkas’ lemma in linear programming implies that these classes are mutually exclusive. We raise the following conjecture which proposes a characterization of equitable (and hence also of separable) hypergraphs (see Section 4 for the definition of multipartite hypergraphs and Section 5 for the definitions of paving and binary matroids).

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Conjecture 1.1 A $k$-hypergraph $(V, \mathcal{H})$ is equitable (so non separable) if and only if there are edges $E_1, E_2 \in \mathcal{H}$ and non edges $F_1, F_2 \in \overline{\mathcal{H}}$ such that $E_1 \cap E_2 = F_1 \cap F_2$ and $E_1 \cup E_2 = F_1 \cup F_2$.

Two more modest conjectures: (1) It holds for all $k$-matroids; (2) It holds for all 3-hypergraphs.

In fact, our results suggest that a stronger condition, which we propose in the following open question, might hold (for $k$-hypergraphs with $k \leq 3$ it coincides with the conjecture).

Question 1.2 In which classes, a $k$-hypergraph $(V, \mathcal{H})$ is equitable if and only if there are $E_1, E_2 \in \mathcal{H}$ and $v_1 \in E_1 \setminus E_2$, $v_2 \in E_2 \setminus E_1$ such that $E_1 \setminus \{v_1\} \cup \{v_2\}$, $E_2 \setminus \{v_2\} \cup \{v_1\} \in \overline{\mathcal{H}}$? (Equivalently, in which classes, a $k$-hypergraph $(V, \mathcal{H})$ is separable if and only if for every $E_1, E_2 \in \mathcal{H}$ and $v_1 \in E_1 \setminus E_2$, $v_2 \in E_2 \setminus E_1$, either $E_1 \setminus \{v_1\} \cup \{v_2\} \in \mathcal{H}$ or $E_2 \setminus \{v_2\} \cup \{v_1\} \in \mathcal{H}$)

We prove the following theorem (see Sections 3–5 for the precise statements):

Theorem 1.3 The condition in Question 1.2 holds for the following types of $k$-hypergraphs:

- All $k$-hypergraphs for $k = 1, 2$ and in particular all graphs;
- All multipartite $k$-hypergraphs for all $k$;
- All paving $k$-matroids and all binary $k$-matroids for all $k$ and all 3-matroids.

The dual of the $k$-hypergraph $(V, \mathcal{H})$ is the $(n - k)$-hypergraph $(V, \mathcal{H}^*)$ with set of edges $\mathcal{H}^* := \{V \setminus E : E \in \mathcal{H}\}$. As follows from Lemma 2.3 the theorem implies also the following.

Corollary 1.4 The condition in Question 1.2 also holds for all complements and all duals of hypergraphs appearing in Theorem 1.3. In particular, it holds for all hypergraphs with $n = k+1, k+2$ vertices and all duals of paving $k$-matroids, binary $k$-matroids, and 3-matroids.

A few notes are in order. First, the conjecture asserts (see proof of Lemma 2.2) that, if a hypergraph is equitable with a labeling $y$, then remarkably, it has in fact a labeling with only 0,1 values and only 4 non zeros. The property of admitting such a restricted, 0,1 valued labeling $y$ which satisfies (2), is superficially reminiscent of the recent notion of a null $k$-hypergraph from [3], which is one admitting a labeling $y$ satisfying (2) restricted to $y(E) = \pm 1$ for $E \in \mathcal{H}$ and $y(F) = 0$ for $F \in \overline{\mathcal{H}}$. But such a labeling $y$ has negative values, and indeed, already for $k = 3$, deciding null 3-hypergraphs is NP-complete [3], while deciding equitable $k$-hypergraphs can be done in polynomial time for any fixed $k$, see Section 6.

The class of separable hypergraphs is geometrically natural and extends the well studied class of threshold graphs [6] from $k = 2$ to any positive $k$. It is a rich class, as already for $k = 3$, deciding if a separable 3-hypergraph has a perfect matching (a set of edges whose disjoint union is $V$, see [4]), is NP-complete [8]. This is in contrast with the well known polynomial time solution of the perfect matching problem for all graphs. In fact, deciding a perfect matching in a separable 3-hypergraph is closely related to the classical NP-complete 3-partition problem, which is precisely to decide if there is a perfect matching in a 3-hypergraph given by $\mathcal{H} := \{E \subseteq V : |E| = 3, \ x(E) = 0\}$ for some given vertex labeling $x$, see [5].

The class of multipartite hypergraphs appearing in Theorem 1.3 extending bipartite graphs, is also rich, as deciding the existence of a perfect matching in a 3-partite hypergraph is precisely the classical NP-complete 3-dimensional matching problem, see [7] again.
Matroids form a class of hypergraphs which is central to combinatorial optimization [10]. The subclass of paving matroids is broad and in fact conjectured to contain almost all matroids [7], and the subclass of binary matroids is rich and well studied and includes in particular all graphic matroids, see Section 5 and [9]. The class of 3-matroids, in its oriented refinement, is already universal for semi-algebraic varieties in a well defined sense, see [1].

Finally, we note that $k$-hypergraphs are very complicated objects already for $k = 3$, and already quite simple problems over them are hard. For instance, deciding degree sequences is NP-complete for 3-hypergraphs [2] while polynomial time doable for graphs. Therefore it would be particularly interesting to verify Conjecture 1.1 at least for 3-hypergraphs.

The rest of the article is organized as follows. In Section 2 we give some preparatory lemmas used throughout. In Sections 3–5 we prove that the condition of Question 1.2, and hence also Conjecture 1.1 hold for graphs, multipartite hypergraphs, and paving, binary, and rank 3 matroids, respectively. Finally, in Section 6, we provide some remarks on the complexity of deciding if a $k$-hypergraph is separable. We show it can be done in polynomial time for each fixed $k$, polynomial time doable even for variable $k$ provided the condition of Question 1.2 holds, requires exponential time when the hypergraph is presented by an oracle, and can be done in polynomial time for binary matroids presented by independence oracles.

2 Preparation

First we prove that separable and equitable hypergraphs form mutually exclusive classes.

**Lemma 2.1** Every $k$-hypergraph $(V, \mathcal{H})$ is either separable or equitable but not both.

*Proof.* Define a $\binom{V}{k} \times V$ matrix $A$, and a vector $b \in \mathbb{R}^{\binom{V}{k}}$, by

$$A_{G,v} := \begin{cases} -1, & v \in G \in \mathcal{H}; \\ 1, & v \in G \in \overline{\mathcal{H}}; \\ 0, & v \notin G \in \binom{V}{k}, \end{cases} \quad b_G := \begin{cases} 0, & G \in \mathcal{H}; \\ -1, & G \in \overline{\mathcal{H}}. \end{cases}$$

Now, one version of the Farkas' lemma in linear programming (cf. [10, Corollary 7.1e]) asserts that one and only one of the following two systems of linear inequalities has a solution,

$$Ax \leq b, \quad x \in \mathbb{R}^V, \quad (3)$$

$$y^T A = 0, \quad y \in \mathbb{R}^{\binom{V}{k}}_+, \quad y^T b < 0. \quad (4)$$

It is easy to see that $x$ is a solution of (3) if and only if it is a labeling showing $(V, \mathcal{H})$ is separable, and $y$ is a solution of (4) if and only if it is a labeling showing $(V, \mathcal{H})$ is equitable. □

Next we prove that the conditions of Conjecture 1.1 and Question 1.2 are sufficient for a hypergraph to be equitable (for the latter we take $F_1 := E_1 \setminus \{v_1\} \cup \{v_2\}$, $F_2 := E_2 \setminus \{v_2\} \cup \{v_1\}$).

**Lemma 2.2** If in $k$-hypergraph $(V, \mathcal{H})$ there are edges $E_1, E_2 \in \mathcal{H}$ and non edges $F_1, F_2 \in \overline{\mathcal{H}}$ such that $E_1 \cap E_2 = F_1 \cap F_2$ and $E_1 \cup E_2 = F_1 \cup F_2$ then the $k$-hypergraph is equitable.

*Proof.* Define a labeling $y : \binom{V}{k} \to \mathbb{R}$ by $y(E_1) = y(E_2) = y(F_1) = y(F_2) = 1$ and $y(E) = 0$ for all other $E$. Then this labeling shows that the hypergraph is equitable, since for any $v \in V$: 
• if \( v \notin E_1 \cup E_2 \) then \( \sum \{ y(E) : v \in E \in \mathcal{H} \} = 0 = \sum \{ y(F) : v \in F \in \overline{\mathcal{H}} \} \);

• if \( v \in (E_1 \cup E_2) \setminus (E_1 \cap E_2) \) then \( \sum \{ y(E) : v \in E \in \mathcal{H} \} = 1 = \sum \{ y(F) : v \in F \in \overline{\mathcal{H}} \} \);

• if \( v \in E_1 \cap E_2 \) then \( \sum \{ y(E) : v \in E \in \mathcal{H} \} = 2 = \sum \{ y(F) : v \in F \in \overline{\mathcal{H}} \} \). □

The following lemma together with Theorem 1.3 implies Corollary 1.4.

**Lemma 2.3** A hypergraph \((V, \mathcal{H})\) is equitable if and only if \((V, \overline{\mathcal{H}})\) is if and only if \((V, \mathcal{H}^*)\) is. Also, a hypergraph \((V, \mathcal{H})\) has edges \(E_1, E_2\) and vertices \(v_1, v_2\) such that \(E_1 \setminus \{v_1\} \cup \{v_2\}\) and \(E_2 \setminus \{v_2\} \cup \{v_1\}\) are non edges if and only if so does \((V, \overline{\mathcal{H}})\) if and only if so does \((V, \mathcal{H}^*)\).

**Proof.** For the first statement, let \( y \) be a labeling of \( k \)-sets for which \((V, \mathcal{H})\) satisfies (2). Then clearly \((V, \overline{\mathcal{H}})\) also satisfies (2) with \( y \). Now define a labeling \( y^* \) of \((n - k)\)-sets by \( y^*(E) := y(V \setminus E) \). Note that summing (2) over all \( v \in V \) and dividing by \( k \) we obtain \( \sum \{ y(G) : G \in \mathcal{H} \} = \sum \{ y(G) : G \in \overline{\mathcal{H}} \} \). Note also that \( \mathcal{H}^* = \overline{\mathcal{H}} \). Then for each \( v \in V \),

\[
\sum \{ y^*(E) : v \in E \in \mathcal{H}^* \} = \sum \{ y(V \setminus E) : v \in E \in \overline{\mathcal{H}} \} = \sum \{ y(G) : v \notin G \in \mathcal{H} \}
\]

\[
= \sum \{ y(G) : v \notin G \in \mathcal{H} \} - \sum \{ y(G) : v \in G \in \mathcal{H} \},
\]

from which we conclude \( \sum \{ y^*(E) : v \in E \in \mathcal{H}^* \} = \sum \{ y^*(F) : v \in F \in \overline{\mathcal{H}}^* \} \). Therefore we find that \((V, \mathcal{H}^*)\) satisfies (2) with the labeling \( y^* \). The converses are proved in the same way.

For the second statement, if \( E_1, E_2 \in \mathcal{H}, F_1 := E_1 \setminus \{v_1\} \cup \{v_2\}, F_2 := E_2 \setminus \{v_2\} \cup \{v_1\} \in \overline{\mathcal{H}} \), so that \((V, \mathcal{H})\) has the desired property, then \( F_1, F_2 \in \overline{\mathcal{H}} \) and \( E_1 = F_1 \setminus \{v_2\} \cup \{v_1\}, F_2 = E_2 \setminus \{v_1\} \cup \{v_2\} \in \mathcal{H} \) so that \((V, \overline{\mathcal{H}})\) also has the desired property, and \( E_1^* := V \setminus E_1, E_2^* := V \setminus E_2 \in \mathcal{H}^*\) and \( F_1^* := E_1^* \setminus \{v_1\}, F_2^* := E_2^* \setminus \{v_2\} \in \overline{\mathcal{H}}^* \) so that \((V, \mathcal{H}^*)\) also has the desired property. The converses are again proved in the same way. □

In what follows, for a \( k \)-subset \( \{v_1, v_2, \ldots, v_k\} \), we also use the abridged form \( v_1 v_2 \cdots v_k \).

### 3 Graphs

We now resolve Question 1.2 for all \( k \)-hypergraphs with \( k = 1, 2 \). This will be used later on as well. In particular, it holds for graphs, that is, 2-hypergraphs. For a graph \((V, \mathcal{H})\) and vertex \( v \in V \) we let the *subgraph induced by* \( V \setminus \{v\} \) be \((V, \mathcal{H})(V \setminus \{v\}) := (V \setminus \{v\}, \{E : v \notin E \in \mathcal{H}\}) \).

A graph \((V, \mathcal{H})\) is *orderable* if there is an ordering \( v_1, v_2, \ldots \) of \( V \) with each \( v_j \) either *isolating*, meaning that \( \{v_i, v_j\} \notin \mathcal{H} \) for all \( i < j \), or *dominating*, meaning that \( \{v_i, v_j\} \in \mathcal{H} \) for all \( i < j \).

**Lemma 3.1** A graph, that is, a 2-hypergraph \((V, \mathcal{H})\), is separable if and only if it orderable.
Proof. In one direction, suppose \((V, \mathcal{H})\) is orderable and let \(v_1, v_2, \ldots\) be a suitable ordering. Define a labeling \(x\) of \(V\) by \(x(v_i) := -i\) if \(v_i\) is isolating and \(x(v_i) := i\) if \(v_i\) is dominating. It is easy to see that then \(\mathcal{H} = \{v_i, v_j : x(v_i) + x(v_j) \geq 0\}\) which shows that \((V, \mathcal{H})\) is separable.

For the second direction we use induction on \(n = |V| > 2\). For \(n = 3\) any graph is orderable so assume \(n \geq 4\). Let \(x\) be a labeling of \(V\) showing \((V, \mathcal{H})\) is separable. Assume first there is a vertex \(v \in V\) in \(\mathcal{H}\) attaining maximum \(|x(v)|\) and satisfying \(x(v) \geq 0\). The restriction of \(x\) to \(V \setminus \{v\}\) shows \((V, \mathcal{H})|V \setminus \{v\}\) is separable and hence by induction it is orderable with a suitable ordering. Now \(x(u) + x(v) \geq 0\) and hence \(uv \in \mathcal{H}\) for all \(u \neq v\) so adding \(v\) at the end of this order as dominating shows \((V, \mathcal{H})\) is orderable too. Otherwise pick any vertex \(v \in V\) attaining maximum \(|x(v)|\), which must satisfy \(x(v) < 0\). Then \(x(u) + x(v) < 0\) and hence \(uv \notin \mathcal{H}\) for all \(u \neq v\) so adding \(v\) at the end of this order as isolating shows again \((V, \mathcal{H})\) is orderable too. \(\blacksquare\)

**Theorem 3.2** For \(k = 1, 2\), a \(k\)-hypergraph \((V, \mathcal{H})\) is equitable if and only if there are edges \(E_1, E_2 \in \mathcal{H}\) and \(v_1 \in E_1 \setminus E_2, v_2 \in E_2 \setminus E_1\) such that \(E_1 \setminus \{v_1\} \cup \{v_2\}, E_2 \setminus \{v_2\} \cup \{v_1\} \in \overline{\mathcal{H}}\).

Proof. If there are such \(E_1, E_2, v_1, v_2\) then the hypergraph is equitable by Lemma \([22]\). So we need to prove that if \((V, \mathcal{H})\) is equitable then there are such suitable \(E_1, E_2, v_1, v_2\). For \(k = 1\) any 1-hypergraph is separable with the labeling \(x\) defined by \(x(v) := 0\) whenever \(\{v\} \in \mathcal{H}\) and \(x(v) := -1\) whenever \(\{v\} \notin \mathcal{H}\), so this statement trivially holds.

For \(k = 2\) we use induction on \(n = |V| > 2\). For \(n = 3\) any graph is separable so this trivially holds providing the induction base. So assume \(n \geq 4\). Suppose first there is some \(v \in V\) not contained in any \(E \in \mathcal{H}\). Since \((V, \mathcal{H})\) is equitable there is a labeling \(y\) of \(V\) satisfying \([20]\). This condition at \(v\) implies \(y(G) = 0\) for all \(G \in V\) with \(v \in G\). So the restriction of \(y\) to \(V - \{v\}\) shows that \((V, \mathcal{H})|V \setminus \{v\}\) is equitable. Then by induction on \(n\) there are suitable \(E_1, E_2 \subseteq V \setminus \{v\}\) and \(v_1, v_2 \in V \setminus \{v\}\). These are suitable for \((V, \mathcal{H})\) as well, and the induction on \(n\) follows. So we may assume every \(v \in V\) is contained in some \(E \in \mathcal{H}\).

A similar argument shows we may likewise assume every \(v \in V\) is contained in some \(F \in \overline{\mathcal{H}}\).

Now pick any \(v \in V\). If \((V, \mathcal{H})|V \setminus \{v\}\) is equitable then by induction there are good \(E_1, E_2 \subseteq V \setminus \{v\}, v_1, v_2 \in V \setminus \{v\}\). These are good for \((V, \mathcal{H})\) too, and the induction follows.

So suppose \((V, \mathcal{H})|V \setminus \{v\}\) is separable. Then by Lemma \([21]\) there is a vertex \(u \in V \setminus \{v\}\) such that either \(ua \notin \mathcal{H}\) for all \(a \in V \setminus \{v, u\}\) or \(ua \in \mathcal{H}\) for all \(a \in V \setminus \{v, u\}\).

Let \(u\) be such a vertex and assume \(ua \notin \mathcal{H}\) for all \(a \in V \setminus \{v, u\}\). The other case is similar and is omitted. Since \(u\) is in some edge in \(\mathcal{H}\) we must have \(uw \in \mathcal{H}\). Since \(v\) is in some non edge in \(\overline{\mathcal{H}}\) there is some \(w\) such that \(vw \in \overline{\mathcal{H}}\). Since \(w\) is in some edge in \(\mathcal{H}\) there is some \(z\) such that \(wz \in \mathcal{H}\). Since \(ua \notin \mathcal{H}\) for all \(a \in V \setminus \{v, u\}\) we have \(uz \in \overline{\mathcal{H}}\). Then the following,

\[E_1 := uw, \quad E_2 := wz, \quad v_1 := u, \quad v_2 := w,\]

satisfy the desired condition and the induction follows again. This completes the proof. \(\blacksquare\)

### 4 Multipartite hypergraphs

We write \(\bigcup_{i=1}^{k} V_i\) for the disjoint union of sets \(V_1, \ldots, V_k\), indicating in particular that they are pairwise disjoint. A \(k\)-partition hypergraph is a hypergraph \((\bigcup_{i=1}^{k} V_i, \mathcal{H})\) with a specified \(k\)-partition of its vertex set \(V := \bigcup_{i=1}^{k} V_i\) such that \(\mathcal{H} \subseteq \{E \subseteq V : |E \cap V_i| = 1, \quad i = 1, \ldots, k\}\).
We now resolve Question 4.1 for \( k \)-partite hypergraphs for all positive \( k \).

**Theorem 4.1** A \( k \)-partite hypergraph \( (w_i^{k=1} V_i, \mathcal{H}) \) is equitable if and only if there are edges \( E_1, E_2 \in \mathcal{H} \) and \( v_1 \in E_1 \setminus E_2, v_2 \in E_2 \setminus E_1 \) such that \( E_1 \setminus \{v_1\} \cup \{v_2\}, E_2 \setminus \{v_2\} \cup \{v_1\} \in \mathcal{H} \).

**Proof.** If there are such \( E_1, E_2, v_1, v_2 \) then the hypergraph is equitable by Lemma 2.2. So we need to prove that if \((V, \mathcal{H})\) is equitable then there are suitable \( E_1, E_2, v_1, v_2 \).

If \(|\mathcal{H}| \leq 1\) then \((V, \mathcal{H})\) is separable hence not equitable by Lemma 2.1 trivally proving the claim. So we may assume \(|\mathcal{H}| \geq 2\). Also, by Theorem 3.2 we may assume \( k \geq 3 \).

Suppose for a contradiction that \(|E_1 \cap E_2| = k - 1\) for all distinct \( E_1, E_2 \in \mathcal{H} \). Pick any two edges \( E_1 \neq E_2 \). Relabeling the \( V_i \) if necessary, we may assume that for some \( u_1, u_2 \in V_1 \) and some \( v_i \in V_i \) for \( 2 \leq i \leq k \) we have that \( E_j = \{u_j, v_2, \ldots, v_k\} \) for \( j = 1, 2 \). Then we claim that for some \( U \subseteq V_1 \) we have in fact \( \mathcal{H} = \{\{u, v_2, \ldots, v_k\} : u \in U\} \). Suppose \( E \) is any edge other than \( E_1, E_2 \). If for some \( 2 \leq i \leq k \) we have \( v_i \notin E \) then for \( j = 1, 2 \) we have that \(|E \cap E_j| = k - 1\) implies \( v_j \in E \), so \( u_1, u_2 \in E \) contradicting \(|E \cap V_1| = 1\). So the claim follows. But then \((V, \mathcal{H})\) is separable, contradicting it being equitable, as the following labeling shows:

\[
x(v) := \begin{cases}
1, & v = v_2, \ldots, v_k; \\
-(k - 1), & v \in U; \\
-k, & \text{otherwise}.
\end{cases}
\]

So there are edges \( E_1, E_2 \in \mathcal{H} \) with \(|E_1 \cap E_2| \leq k - 2\) and hence there are \( 1 \leq i < j \leq k \) and \( v_{1,i} \in (E_1 \setminus E_2) \cap V_i, \ v_{1,j} \in (E_1 \setminus E_2) \cap V_j, \ v_{2,i} \in (E_2 \setminus E_1) \cap V_i, \ v_{2,j} \in (E_2 \setminus E_1) \cap V_j \).

Define \( F_1 := E_1 \setminus \{v_{1,i}\} \cup \{v_{2,j}\} \) and \( F_2 := E_2 \setminus \{v_{2,j}\} \cup \{v_{1,i}\} \). Then \(|F_1 \cap V_j| = 2\) and \(|F_2 \cap V_i| = 2\) so \( F_1, F_2 \in \mathcal{H} \). Defining \( v_1 := v_{1,i}, v_2 := v_{2,j} \) we get the desired \( E_1, E_2, v_1, v_2 \). \( \square \)

## 5 Matroids

We now consider hypergraphs which are (sets of bases of) matroids. See [9] for a detailed development of the beautiful theory of matroids. A \( k \)-matroid is a \( k \)-hypergraph \((V, \mathcal{H})\) such that \( \mathcal{H} \neq \emptyset \) and for every \( E_1, E_2 \in \mathcal{H} \) and \( v_1 \in E_1 \setminus E_2 \) there is a \( v_2 \in E_2 \setminus E_1 \) such that \( E_1 \setminus \{v_1\} \cup \{v_2\} \in \mathcal{H} \) (compare this with the second condition in Question 4.1). Thus, in standard terminology, \( \mathcal{H} \) is the set of bases of a matroid of rank \( k \) on \( V \). For instance, both separable and equitable \( 3 \)-hypergraphs demonstrated in the introduction are \( 3 \)-matroids.

A subset \( I \subseteq V \) is independent in the matroid if \( I \subseteq E \) for some basis \( E \in \mathcal{H} \). If \( I \) is any independent set and \( E \in \mathcal{H} \) is any basis then \( I \) can be augmented from \( E \) to a basis, that is, \( I \cup J \in \mathcal{H} \) for some \( J \subseteq E \setminus I \). A loop in a \( k \)-matroid \((V, \mathcal{H})\) is a \( v \in V \) that is not contained in any \( E \in \mathcal{H} \). A coloop is a \( v \in V \) that is contained in every \( E \in \mathcal{H} \). If \( v \) is not a coloop then let \( \mathcal{H} \setminus v := \{E : v \notin E \in \mathcal{H}\} \), and if \( v \) is not a loop then let \( \mathcal{H} / v := \{E \setminus \{v\} : E \in \mathcal{H}\} \). If \( v \) is a coloop (hence not a loop) then let \( \mathcal{H} \setminus v := \mathcal{H} / v \), and if \( v \) is a loop (hence not a coloop) then let \( \mathcal{H} / v := \mathcal{H} \setminus v = H = \mathcal{H} \). The deletion of \((V, \mathcal{H})\) by \( v \) is \((V, \mathcal{H}) \setminus v := (V \setminus \{v\}, \mathcal{H} \setminus v)\), and is a \( k \)-matroid if \( v \) is not a coloop and a \((k - 1)\)-matroid if \( v \) is a coloop. The contraction of \((V, \mathcal{H})\) by \( v \) is \((V, \mathcal{H}) / v := (V \setminus \{v\}, \mathcal{H} / v)\), and is a \((k - 1)\)-matroid if \( v \) is not a loop and a \( k \)-matroid if \( v \) is a loop.
5.1 Paving matroids

A paving $k$-matroid is a $k$-matroid with the property that every $(k - 1)$-subset of $V$ is independent, that is, contained in some $E \in \mathcal{H}$. This is a broad and fundamental class of matroids, and in fact it is conjectured that almost all matroids are paving [7]. It is easy to verify that any deletion and any contraction of a paving matroid is again a paving matroid. For instance, the 3-matroid $(V, \mathcal{H})$ with $V = [4]$ and $\mathcal{H} = \binom{V}{3}$ is paving and separable with the zero labeling $x$, and the 3-matroid $(V, \mathcal{H})$ with the following data is paving and equable, $V = [5], \quad \mathcal{H} = \{123, 124, 134, 135, 145, 234, 235, 245\}$, $y(G) = \begin{cases} 1, & G = 125, 135, 245, 345; \\ 0, & \text{otherwise}. \end{cases}$

We now resolve Question 1.2 for paving $k$-matroids for all positive $k$.

**Theorem 5.1** A paving $k$-matroid $(V, \mathcal{H})$ is equitable if and only if there are edges $E_1, E_2 \in \mathcal{H}$ and vertices $v_1 \in E_1 \setminus E_2$, $v_2 \in E_2 \setminus E_1$ such that $E_1 \setminus \{v_1\} \cup \{v_2\}$, $E_2 \setminus \{v_2\} \cup \{v_1\} \in \mathcal{H}$.

**Proof.** If there are such $E_1, E_2, v_1, v_2$ then the matroid is equitable by Lemma 2.2. So we need to prove that if $(V, \mathcal{H})$ is equitable then there are suitable $E_1, E_2, v_1, v_2$.

We use induction on $k$. For $k = 1, 2$ this follows from Theorem 3.2 providing the base. Consider $k \geq 3$. We prove the claim for $k$ by induction on $n = |V| > k$. If $n = k + 1$ or $n = k + 2$ then the dual $(V, \mathcal{H}^*)$ is a 1-hypergraph or a 2-hypergraph and the statement follows from Theorem 3.2 and Lemma 2.3. Consider $n \geq k + 3$.

Suppose first that there is some $v \in V$ which is not contained in any $F \in \overline{\mathcal{H}}$. Then $v$ is not a coloop else $\mathcal{H} = \{E \in \binom{V}{k} : v \in E\}$ so the hypergraph is separable with the labeling $x(v) := k - 1$ and $x(u) := -1$ for all $u \neq v$, contradicting it being equitable. So $\mathcal{H} \setminus v = \{E : v \notin E \in \mathcal{H}\}$. Since $(V, \mathcal{H})$ is equitable there is a labeling $y$ of $\binom{V}{k}$ satisfying (2). This condition at $v$ implies $y(G) = 0$ for all $G \in \binom{V}{k}$ with $v \in G$. Therefore the restriction of $y$ to $(V \setminus \{v\})$ is an equitable paving $k$-matroid. Then by induction on $n$ there are suitable $E_1, E_2 \subseteq V \setminus \{v\}$ and $v_1, v_2 \in V \setminus \{v\}$. These are suitable for $(V, \mathcal{H})$ as well, and the induction on $n$ follows. So we may assume every $v \in V$ is contained in some $F \in \overline{\mathcal{H}}$.

Now pick any $v \in V$. As $(V, \mathcal{H})$ is paving, $v$ is not a loop, so $\mathcal{H} \setminus v = \{E \setminus \{v\} : v \in E \in \mathcal{H}\}$. Now, if $(V, \mathcal{H}) \setminus v$ is equitable then, since it is a paving $(k - 1)$-matroid, by induction on $k$ there are suitable $E_1, E_2 \subseteq V \setminus \{v\}$ and $v_1, v_2 \in V \setminus \{v\}$. Then $E_1 \cup \{v\}, E_2 \cup \{v\}, v_1, v_2$ are suitable for $(V, \mathcal{H})$, and the induction on $n$ follows. Assume then that $(V, \mathcal{H}) \setminus v$ is not equitable and hence by Lemma 2.1 is separable. Let $x$ be a suitable labeling of $V \setminus \{v\}$ and index the vertices in $V \setminus \{v\}$ so that $x(v_1) \leq \cdots \leq x(v_n)$. If $x(v_1) + \cdots + x(v_{k-2}) + x(v_n) < 0$ then $x(v_1) + \cdots + x(v_{k-2}) + x(v_1) < 0$ for all $k - 1 \leq i \leq n - 1$. Then $v_1 \cdots v_{k-2}v_1 \notin \mathcal{H}/v$ and hence $v_1 \cdots v_{k-2}v_1 \notin \mathcal{H}$ for all $i$. But $v_1 \cdots v_{k-2}v_1$ is independent in $(V, \mathcal{H})$ since it is paving, so must be contained in some basis, which is a contradiction. So $x(v_1) + \cdots + x(v_{k-2}) + x(v_n) \geq 0$ and hence for all $1 \leq i_1 < \cdots < i_{k-2} \leq n - 2$ we have that $x(v_{i_1}) + \cdots + x(v_{i_{k-2}}) + x(v_{i_{k-1}}) \geq 0$ and hence $v_{i_1} \cdots v_{i_{k-2}}v_{i_{k-1}} \in \mathcal{H}/v$ and $v_{i_1} \cdots v_{i_{k-2}}v_{i_{k-1}}v_{i_{k-2}}v_1 \in \mathcal{H}$. Letting $u := v_{n-1}$, this means that any $k$-subset of $V$ which contains $u, v$ is in $\mathcal{H}$.

Now, as proved above, there are $F_1, F_2 \subseteq \overline{\mathcal{H}}$ with $u \in F_1$ and $v \in F_2$. Since $(V, \mathcal{H})$ is paving, $F_1 \setminus \{u\}$ is independent in $\mathcal{H}$. Since $F_1 \cup \{v\}$ contains $u, v$, by what just proved it contains a basis in $\mathcal{H}$ from which $F_1 \setminus \{u\}$ can be augmented to a basis. This implies that $E_1 := F_1 \setminus \{u\} \cup \{v\} \in \mathcal{H}$. A similar argument shows that $E_2 := F_2 \setminus \{v\} \cup \{u\} \in \mathcal{H}$. Defining $v_1 := v, v_2 := u$ we get the desired $E_1, E_2, v_1, v_2$ and the inductions on $n$ and $k$ follow.
5.2 Three-matroids

We need two lemmas and some more matroid terminology as follows.

**Lemma 5.2** A $k$-matroid $(V, \mathcal{H})$ with a loop $v \in V$ is separable if and only if $(V, \mathcal{H}) \setminus v$ is.

*Proof.* Since $v$ is a loop we have $\mathcal{H} \setminus v = \mathcal{H}$. In one direction, suppose $(V, \mathcal{H})$ is separable with labeling $x : V \to \mathbb{R}$. Then its restriction to $V \setminus \{v\}$ shows $(V, \mathcal{H}) \setminus v$ is separable, since a $k$-subset $E \subseteq V \setminus \{v\}$ satisfies $x(E) \geq 0$ if and only if $E \in \mathcal{H} = \mathcal{H} \setminus v$. In the other direction, suppose $(V, \mathcal{H}) \setminus v$ is separable with labeling $x : V \setminus \{v\} \to \mathbb{R}$. Extend it to $V$ by setting $x(v) := -r$ where $r$ is a sufficiently large positive number. Then $(V, \mathcal{H})$ is separable since

$$\{E \subseteq V : |E| = k, \ x(E) \geq 0\} = \{E \subseteq V \setminus \{v\} : |E| = k, \ x(E) \geq 0\} = \mathcal{H} \setminus v = \mathcal{H}. \qed$$

Let $(V, \mathcal{H})$ be a matroid with no loops. A line of the matroid is a subset $\emptyset \neq L \subseteq V$ such that $uv$ is not independent for all distinct $u, v \in L$ and $uv$ is independent for all $u \in L$ and $v \in V \setminus L$. The line is nontrivial if $|L| \geq 2$. The set $V$ equals the disjoint union of the lines.

**Lemma 5.3** If a matroid $(V, \mathcal{H})$ with no loops has at least two nontrivial lines then there are $E_1, E_2 \in \mathcal{H}$ and $v_1 \in E_1 \setminus E_2, v_2 \in E_2 \setminus E_1$ such that $E_1 \setminus \{v_1\} \cup \{v_2\}$, $E_2 \setminus \{v_2\} \cup \{v_1\} \in \overline{\mathcal{H}}$.

*Proof.* Let $(V, \mathcal{H})$ be a $k$-matroid with no loops and two distinct nontrivial lines $L_1, L_2$. Pick any distinct $u_1, v_2 \in L_1$ and $u_2, v_1 \in L_2$. Since $u_1v_1$ is independent it is contained in some basis $E_1 \in \mathcal{H}$. Since $u_2v_2$ is independent it is contained in some basis $E_2 \in \mathcal{H}$. Now $E_1 \setminus \{v_1\} \cup \{v_2\} \in \overline{\mathcal{H}}$ since it contains $u_1v_2$ which is not independent. Similarly, $E_2 \setminus \{v_2\} \cup \{v_1\} \in \overline{\mathcal{H}}$ since it contains $u_2v_1$ which is not independent. The lemma follows. \qed

We now resolve Question 1.2 for all 3-matroids.

**Theorem 5.4** A 3-matroid $(W, \mathcal{H})$ is equitable if and only if there are edges $E_1, E_2 \in \mathcal{H}$ and vertices $v_1 \in E_1 \setminus E_2, v_2 \in E_2 \setminus E_1$ such that $E_1 \setminus \{v_1\} \cup \{v_2\}$, $E_2 \setminus \{v_2\} \cup \{v_1\} \in \overline{\mathcal{H}}$.

*Proof.* If there are such $E_1, E_2, v_1, v_2$ then the matroid is equitable by Lemma 2.2 So we need to prove that if $(W, \mathcal{H})$ is equitable then there are suitable $E_1, E_2, v_1, v_2$.

Deleting all loops if any one after the other we obtain a loopless 3-matroid $(V, \mathcal{H})$ with $n := |V| \geq 3$, which is equitable if and only if $(W, \mathcal{H})$ is by Lemma 5.2. And, if $E_1, E_2$ and $v_1, v_2$ are good for $(V, \mathcal{H})$, then they are also good for $(W, \mathcal{H})$. So it suffices to prove the claim for $(V, \mathcal{H})$. If $n = 3$ then $\mathcal{H} = \{V\}$ so $(V, \mathcal{H})$ is separable. If $n = 4$ or $n = 5$ then the dual $(V, \mathcal{H}^*)$ is a 1-hypergraph or a 2-hypergraph so the claim follows from Theorem 3.2.

Consider then $n \geq 6$. If $(V, \mathcal{H})$ has no nontrivial lines then every 2-subset of $V$ is independent, and therefore the matroid is paving and the claim follows from Theorem 5.1. If the matroid has at least two distinct nontrivial lines then the claim follows by Lemma 5.3.

So assume there is exactly one nontrivial line $L$. Let $U := V \setminus L$. Pick any $v \in L$. Since $v$ is not a loop, we have $\mathcal{G} := \mathcal{H} \setminus v = \{ab : abv \in \mathcal{H}\}$. Consider the graph $(U, \mathcal{G})$. If it is equitable then by Theorem 3.2 there are suitable $E_1, E_2 \subseteq U$ and $v_1, v_2 \in U$ and then $E_1 \cup \{v\}, E_2 \cup \{v\}$ and $v_1, v_2$ are suitable for $\mathcal{H}$ and we are done. So assume this graph is separable.

Consider any three distinct vertices $a, b, c \in U$ (if any) and let $i := |\{ab, ac, bc\} \cap \mathcal{G}|$ be the number of edges among $ab, ac, bc$ contained in $\mathcal{G}$. Suppose $i = 0$ and suppose for a
binary

Next suppose for a contradiction that \( i = 1 \), say \( ab \in \mathcal{G} \), which implies \( abv \in \mathcal{H} \). Since \( v \in L \) and \( c \notin L \) we have that \( cv \) is independent and so we must be able to augment it from \( abv \) to a basis. So we must have either \( acv \in \mathcal{H} \) or \( bcv \in \mathcal{H} \) which implies either \( ac \in \mathcal{G} \) or \( bc \in \mathcal{G} \) contradicting \( i = 1 \). So \( i \neq 1 \). Now suppose \( i \geq 2 \), say \( ab, bc \in \mathcal{G} \) which implies \( abv, bcv \in \mathcal{H} \). Pick any \( w \in L \setminus \{v\} \). Since \( bw \) is independent we must be able to augment it to a basis from \( bcv \), but \( vw \) is not independent so we find that this basis must be \( bcv \in \mathcal{H} \). Now, if \( abc \notin \mathcal{H} \) then taking \( E_1 := abv, E_2 := bcv, v_1 := v, v_2 := c \) the claim follows again.

We claim that indeed there must be some \( a, b, c \in U \) with \( i \geq 2 \) and \( abc \notin \mathcal{H} \) so that we are done. Assume, for a contradiction, that this is not the case. So, summarizing, we have that if \( i = 0 \) then \( abc \notin \mathcal{H} \), always \( i \neq 1 \), and we assume that \( abc \in \mathcal{H} \) whenever \( i = 2, 3 \).

Now, for any \( a \in U \) and any distinct \( w, z \in L \) we have that \( wz \) is not independent so \( awz \notin \mathcal{H} \). Also, for every distinct \( a, b \in U \) and \( w \in L \), we have that \( abw \notin \mathcal{H} \) if and only if \( abv \in \mathcal{H} \) and only if \( ab \in \mathcal{G} \). In particular, since \( \mathcal{H} \) is a matroid and hence nonempty, we see that \( \mathcal{G} \) cannot be empty, so has at least one edge and \( (U, \mathcal{G}) \) has \( r := |U| \geq 2 \) vertices.

Now, since \( (U, \mathcal{G}) \) is separable, by Lemma 3.1, it has an ordering \( u_1, \ldots, u_r \) of \( U \) such that each \( u_i \) is either isolating or dominating, with \( u_1 \) declared isolating. Now, if there are \( 1 < i < j \leq r \) with \( u_i \) dominating and \( u_j \) isolating then \( \{u_1 u_i, u_1 u_j, u_i u_j\} \cap \mathcal{G} = \{u_1 u_i\} \) which is impossible as shown above. So for some \( 1 \leq i < r \) we have that \( u_i \) is isolating for \( i \leq i \) and dominating for \( j > i \). Let \( U_0 := \{u_j : j \leq i\} \) and \( U_1 := \{u_j : j > i\} \) so that \( U = U_0 \cup U_1 \).

We claim that \( (V, \mathcal{H}) \) is separable, contradicting it being equitable. Consider the labeling

\[
x(u) := \begin{cases} 
-1, & u \in U_0; \\
3, & u \in U_1; \\
-2, & u \in L.
\end{cases}
\]

Consider any 3-subset \( S \subset V \). We show that \( S \in \mathcal{H} \) if and only if \( x(S) \geq 0 \). Suppose first that \( |S \cap L| = 0 \). If \( |S \cap U_1| = 0 \) then \( \left| \left( \begin{array}{c} S \cap L \end{array} \right) \cap \mathcal{G} \right| = 0 \) so \( S \in \overline{\mathcal{H}} \) and indeed \( x(S) = -3 < 0 \). If \( |S \cap U_1| \geq 1 \) then \( \left| \left( \begin{array}{c} S \cap L \end{array} \right) \cap \mathcal{G} \right| \geq 2 \) so \( S \in \mathcal{H} \) and indeed \( x(S) \geq 3 - 1 - 1 \geq 0 \). Next suppose that \( |S \cap L| = 1 \). If \( S \cap U \notin \mathcal{G} \) then \( S \in \mathcal{H} \) and indeed \( |S \cap U| \geq 1 \) hence \( x(S) \geq 3 - 1 - 2 \geq 0 \). If \( S \cap U \notin \mathcal{G} \) then \( S \in \mathcal{H} \) and indeed \( |S \cap U_1| = 0 \) hence \( x(S) = -2 - 1 - 1 < 0 \). Finally suppose that \( |S \cap L| \geq 2 \). Then \( S \in \overline{\mathcal{H}} \) and indeed \( x(S) \leq -2 - 2 + 3 < 0 \).

We conclude there are \( a, b, c \in U \) with \( i := |\{ab, ac, bc\} \cap \mathcal{G}| \geq 2 \) and \( abc \notin \mathcal{H} \), so there are suitable \( E_1, E_2 \) and \( v_1, v_2 \) as explained above. This completes the proof.

5.3 Binary matroids

We need some more matroid terminology. A circuit in a matroid \((V, \mathcal{H})\) is a subset \( C \subseteq V \) that is not independent but all its proper subsets are independent. If \( C_1, C_2 \) are two circuits with \( v \in C_1 \cap C_2 \) and \( u \in C_1 \setminus C_2 \) then there is another circuit \( C \) such that \( u \in C \subseteq (C_1 \cup C_2) \setminus \{v\} \). If \( E \in \mathcal{H} \) and \( v \in V \setminus E \) then there is a unique circuit \( C(E, v) \) satisfying \( C(E, v) \subseteq E \cup \{v\} \). A binary \( k \)-matroid is a \( k \)-matroid with the property that for any two distinct circuits \( C_1, C_2 \), their symmetric difference \( C_1 \Delta C_2 := (C_1 \setminus C_2) \cup (C_2 \setminus C_1) \) is a disjoint union of circuits. This is a broad and well studied class that includes in particular all graphic matroids, in which \( V \) is the set of edges of a graph and \( \mathcal{H} \) is the set of maximal forests in the graph, see [9].
We now resolve Question 1.2 for binary k-matroids for all positive k.

**Theorem 5.5** A binary k-matroid \((W, \mathcal{H})\) is equitable if and only if there are edges \(E_1, E_2 \in \mathcal{H}\) and vertices \(v_1 \in E_1 \setminus E_2, v_2 \in E_2 \setminus E_1\) such that \(E_1 \setminus \{v_1\} \cup \{v_2\}, E_2 \setminus \{v_2\} \cup \{v_1\} \in \overline{\mathcal{H}}\).

**Proof.** If there are such \(E_1, E_2, v_1, v_2\) then the matroid is equitable by Lemma 2.2. So we need to prove that if \((W, \mathcal{H})\) is equitable then there are suitable \(E_1, E_2, v_1, v_2\).

Deleting all loops if any one after the other we obtain a loopless k-matroid \((V, \mathcal{H})\) and \(n := |V| \geq k\), which is equitable if and only if \((W, \mathcal{H})\) is by Lemma 5.2. And, if \(E_1, E_2, v_1, v_2\) are good for \((V, \mathcal{H})\), then they are also good for \((W, \mathcal{H})\). So it suffices to prove the claim for \((V, \mathcal{H})\). If \(n = k\) then \(\mathcal{H} = \{V\}\) so \((V, \mathcal{H})\) is separable. If \(n = k + 1\) then the dual \((V, \mathcal{H}^*)\) is a 1-hypergraph so separable, see proof of Theorem 3.2, and hence so is \((V, \mathcal{H})\) by Lemma 2.3.

Consider \(n \geq k + 1\). If \((V, \mathcal{H})\) has at least two distinct nontrivial lines then by Lemma 5.3 there are suitable \(E_1, E_2, v_1, v_2\) and we are done. Suppose then there is at most one nontrivial line and let \(L\) be one with maximum \(l := |L| \geq 1\). Since \(|E \cap L| \leq 1\) for all \(E \in \mathcal{H}\) we have \(n \geq l + k - 1\). If \(n = l + k - 1\) then \(\mathcal{H} = \{E \in \binom{V}{k} : |E \cap L| = 1\}\) so the labeling \(x(u) := 1\) for \(u \in V \setminus L\) and \(x(u) := -(k - 1)\) for \(u \in L\) shows \((V, \mathcal{H})\) is separable. So assume \(n \geq l + k\). Pick any \(E \in \mathcal{H}\) with \(|E \cap L| = 1\). Then there is some \(v_2 \in V \setminus (E \cup L)\) and \(\{v_2\}\) must be a trivial line. Pick any \(v_1 \in V \setminus (E \cup \{v_2\})\). For \(i = 1, 2\) let \(C_i := C(E, v_i)\) and \(P_i := C_i \setminus \{v_i\}\). We claim \(P_1 \Delta P_2 \neq \emptyset\). Indeed, otherwise we get a contradiction, since then \(C_1 \Delta C_2 = \{v_1, v_2\}\) but \(C_1 \Delta C_2\) is a disjoint union of circuits since the matroid is binary, while \(v_1v_2\) is independent. So the claim is true, and we can assume, say, that there is some element \(u_1 \in P_1 \setminus P_2\).

Suppose there is also an element \(u_2 \in P_2 \setminus P_1\). Define the following sets,

\[
E_1 := E \sqcup \{v_1\} \setminus \{u_1\}, \quad E_2 := E \sqcup \{v_2\} \setminus \{u_2\}, \quad F_1 := E_1 \setminus \{v_1\} \sqcup \{v_2\}, \quad F_2 := E_2 \setminus \{v_2\} \sqcup \{v_1\}.
\]

Then, for \(i = 1, 2\), we have that \(E_i \in \mathcal{H}\) since \(u_i \in C_i\), and \(F_i \in \overline{\mathcal{H}}\) since \(C_{3-i} \subseteq F_i\).

Now suppose \(P_2 \not\subseteq P_1\). Since \(v_2v\) is independent for all \(v \neq v_2\) we have \(|C_2| \geq 3\) and hence \(|P_2| \geq 2\). Pick any two distinct vertices \(u_2, w_2 \in P_2\). Let \(G := E \sqcup \{v_1\} \setminus \{u_1\} \in \mathcal{H}\). Since \(C_2 \subseteq G \sqcup \{v_2\}\) we find that \(C(G, v_2) = C_2\) and therefore

\[
E' := G \sqcup \{v_2\} \setminus \{u_2\} = E \sqcup \{v_1, v_2\} \setminus \{u_1, u_2\} \in \mathcal{H}.
\]

For \(i = 1, 2\) let \(C'_i := C(E', u_i)\) and \(P'_i := C'_i \setminus \{u_i\}\). As we have seen, \(C'_2 = C_2\). Now consider \(C_1 \Delta C_2 = (P_1 \setminus P_2) \sqcup \{v_1, v_2\}\) which must be a disjoint union of circuits. Since \(u_2 \in C_1 \cap C_2\) and \(u_2 \notin (P_1 \setminus P_2) \sqcup \{v_1\} \subseteq E \sqcup \{v_1\}\) we conclude that \((P_1 \setminus P_2) \sqcup \{v_1\}\) contains no circuit for \(i = 1, 2\). So we must have \((P_1 \setminus P_2) \sqcup \{v_1, v_2\} = C'_1\). So \(v_1 \in P'_1 \setminus P'_2\) and \(w_2 \in P'_2 \setminus P'_1\). Define

\[
E'_1 := E \sqcup \{u_1\} \setminus \{v_1\}, \quad E'_2 := E \sqcup \{u_2\} \setminus \{v_2\}, \quad F'_1 := E'_1 \setminus \{u_1\} \sqcup \{u_2\}, \quad F'_2 := E'_2 \setminus \{u_2\} \sqcup \{u_1\}.
\]

Then \(E'_1, E'_2 \in \mathcal{H}\) since \(v_1 \in C'_1\) and \(w_2 \in C'_2\), and for \(i = 1, 2\) we have \(F'_i \in \overline{\mathcal{H}}\) since \(C_{3-i} \subseteq F'_i\).

So in either case there are two suitable bases and non bases and the claim follows. \(\Box\)
6 Remarks on complexity

We conclude with some remarks on the complexity of deciding if a hypergraph is separable.

First, for any fixed $k$, the number of $k$-subsets of $V$ is $O(n^k)$ and is polynomial in $n = |V|$. So, given the list of the edges in $\mathcal{H}$, it can be decided in polynomial time if $(V, \mathcal{H})$ is separable by writing down the system of linear inequalities (3) in Lemma 2.1 and deciding if it has a solution using linear programming which can be done in polynomial time, see e.g. [10].

Second, assume $k$ is a variable part of the input. Then the systems in (3) and (4) have exponential size and cannot be used. But suppose the condition of Question 1.2 is true. Let $m := |\mathcal{H}|$ and assume the hypergraph is given by a list of the $m$ edges in $\mathcal{H}$. Then, by checking for each of the $O(m^2)$ pairs of edges $E_1, E_2 \in \mathcal{H}$ and each of the $O(k^2)$ vertices $v_1 \in E_1 \setminus E_2$ and $v_2 \in E_2 \setminus E_1$ whether or not $E_1 \setminus \{v_1\} \cup \{v_2\}$, $E_2 \setminus \{v_2\} \cup \{v_1\} \in \overline{\mathcal{H}}$, we can decide in time polynomial in $m$ and $k$ if $(V, \mathcal{H})$ is separable.

Third, assume $k$ is again a variable part of the input, and the hypergraph, which may have a number of edges which is exponential in $k$ and $n$, is presented implicitly by an oracle that, queried on a $k$-set $E$, replies YES if $E \in \mathcal{H}$ and NO if $E \in \overline{\mathcal{H}}$. We claim that in general, exponentially many queries are needed to decide if $(V, \mathcal{H})$ is separable. To see this, suppose $n = 2k$ and $k \geq 2$. Suppose an algorithm trying to solve the decision problem makes less than $2^k - 1 \leq \frac{1}{2}(2k)$ queries. Then there is a pair of disjoint $k$-sets $F_1, F_2$ about which the algorithm did not query the oracle. Let $E_1, E_2 \neq F_1, F_2$ be two other disjoint $k$-sets. Consider the complete $k$-hypergraph $(V, \mathcal{H}_1)$ with $\mathcal{H}_1 := \binom{V}{k}$ and the $k$-hypergraph $(V, \mathcal{H}_2)$ with $\mathcal{H}_2 := \mathcal{H}_1 \setminus \{F_1, F_2\}$. Then $(V, \mathcal{H}_1)$ is separable with the identically zero labeling $x$, whereas $(V, \mathcal{H}_2)$ is equitable by Lemma 2.2 since $E_1, E_2 \in \mathcal{H}_2$, $F_1, F_2 \in \overline{\mathcal{H}_2}$, $E_1 \cap E_2 = \emptyset = F_1 \cap F_2$, and $E_1 \cup E_2 = V = F_1 \cup F_2$. But whether the oracle presents $(V, \mathcal{H}_1)$ or $(V, \mathcal{H}_2)$, it answers YES to all queries made by the algorithm, so the algorithm cannot distinguish between these two hypergraphs and cannot tell whether the one presented by the oracle is separable or equitable.

Finally, assume again $k$ is variable and $(W, \mathcal{H})$ is a binary $k$-matroid which is presented by an independence oracle that, queried on any subset $I \subseteq W$, replies YES if $I$ is independent and NO if it is not independent. The proof of Theorem 5.5 leads to an efficient algorithm as follows. Using $|W|$ queries we can identify all loops if any. We let $(V, \mathcal{H})$ be the matroid obtained by deleting them one after the other and let $n := |V| \geq k$. By Lemma 5.2 the original matroid is separable if and only if the deleted matroid is. Moreover, the oracle for $(W, \mathcal{H})$ is good also for $(V, \mathcal{H})$. So it suffices to solve the decision problem for the loopless $(V, \mathcal{H})$. If $n \leq k + 1$ then $(V, \mathcal{H})$ is separable. Assume $n \geq k + 2$. By asking the oracle about all $O(n^2)$ vertex pairs $\{u, v\} \subseteq V$ we can identify all the lines in $(V, \mathcal{H})$. If there are two nontrivial lines then $(V, \mathcal{H})$ is equitable by Lemma 5.3. Otherwise let $l := |L|$ be the largest cardinality of a line. Then, as shown in the proof of Theorem 5.5 if $n = l + k - 1$ then $(V, \mathcal{H})$ is separable, whereas if $n \geq l + k$ then $(V, \mathcal{H})$ is equitable. So we can decide if $(V, \mathcal{H})$ and hence also $(W, \mathcal{H})$ are separable using polynomially many queries to the oracle.

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