A Generalization of the Casson Invariant

by

J. Gegenberg

Department of Mathematics and Statistics
University of New Brunswick
Fredericton, New Brunswick Canada E3B 5A3
[e-mail: lenin@math.unb.ca]

Abstract

A three dimensional supergravity theory which generalizes the super IG theory of Witten and resembles the model discussed recently by Mann and Papadopoulos is displayed. The partition function is computed, and is shown to be a three-manifold invariant generalizing the Casson invariant.
1 Three Dimensional Supergravity

The theory of supergravity which I will consider is closely related to that constructed recently by Mann and Papadopoulos [1]. This supergravity model is a generalization of Witten’s “super IG” theory [2] and of the “teleparallel theory” constructed by Carlip and the present author [3].

Although the supergravity theory can be written as supersymmetric Chern-Simons theory, it will be displayed here in a more useful form, wherein it resembles (first-order) 3-d Einstein gravity coupled to families of bosonic and fermionic vector fields.

The bosonic field content consists of three 1-form fields $E^a, B^a, C^a$ taking their values in the Lie algebra $so(3)$ and an $SO(3)$ connection 1-form $A^a$. The fermionic fields are a pair of spinorial 1-forms $\psi^i$, where the $so(3)$ indices $a, b, ... = 1, 2, 3$, and the $i, j, ... = 1, 2$. As we will see, the gravitational field is essentially determined by the $E^a$, while the the $\psi^i$ are gravitino fields. I use the same conventions in as in [1], namely that the gamma matrices $\gamma^a$ are pure imaginary so that

$$\epsilon_{\alpha \beta} \gamma^a \gamma^b = \epsilon_{\alpha \beta} \gamma^a \gamma^b,$$

(1.1)

$$\gamma^a \gamma^b = \delta^{ab} + i \epsilon^{abc} \gamma^c,$$

(1.2)

$$(\bar{\psi})_{\alpha} = \psi^i \epsilon_{\alpha \beta},$$

(1.3)

where $\alpha, \beta, ... = 1, 2$ are spinor indices.

The action for IISO(3) supergravity is

$$I = \int E^a \wedge F_a(A) + B^a \wedge D_A C_a + i \bar{\psi}^i \wedge D_A \psi^i$$

(1.4)

The operators $D_A, D_A$ are defined by

$$D_A B_a := dB_a + \frac{1}{2} \epsilon_{abc} A^b \wedge B^c,$$

(1.5)

$$D_A \psi^i := d\psi^i - \frac{i}{4} A^a \gamma_a \wedge \psi^i.$$

(1.6)

The action is invariant under the supersymmetry transformations

$$\delta E^a = \frac{i}{2} \bar{\zeta}^i \gamma^a \psi^i,$$

(1.7)

$$\delta \psi^i = D_A \zeta^i,$$

(1.8)
where the $\zeta^i$ are spinorial parameters. As well, the action is invariant under the IISO(3) gauge transformations:

$$
\delta B^a = D_A \rho^a + \frac{1}{2} \epsilon^{abc} B^b \tau^c, \quad (1.9)
$$

$$
\delta C^a = D_A \lambda^a + \frac{1}{2} \epsilon^{abc} \tau^c, \quad (1.10)
$$

$$
\delta E^a = D_A \zeta^a + \frac{1}{2} \epsilon^{abc} \left( E^b \tau^c + B^b \lambda^c + C^b \rho^c \right), \quad (1.11)
$$

$$
\delta A^a = D_A \tau^a. \quad (1.12)
$$

The equations of motion of the theory are

$$
F^a(A) := dA^a + \frac{1}{4} \epsilon^{abc} A_b \wedge A_c = 0, \quad (1.13)
$$

$$
D_A B^a = 0, \quad (1.14)
$$

$$
D_A C^a = 0, \quad (1.15)
$$

$$
\mathbf{D}_A \psi^i = 0, \quad (1.16)
$$

$$
D_A E^a + \frac{1}{2} \epsilon^{abc} B_b \wedge C_c - \frac{1}{4} \bar{\psi}^i \wedge \gamma^a \psi^i = 0. \quad (1.17)
$$

The solutions of the equations of motion can be interpreted as a three-dimensional $N = 2$ supergravity theory. The gravitino fields are the $\psi^i$, while the gravitational field consists of the spacetime triad $E^a$ and a compatible spin-connection $\omega^a$, determined by

$$
dE^a + \frac{1}{2} \epsilon^{abc} \omega_b \wedge E_c = 0, \quad (1.18)
$$

i.e. by

$$
\epsilon^{abc} \left[ (\omega_b - A_b) \wedge E_c - B_b \wedge C_c \right] + \frac{1}{2} \bar{\psi}^i \wedge \gamma^a \psi^i = 0, \quad (1.19)
$$

as can be seen from Eq. (1.17) and Eq. (1.18).

2 The Partition Function

The partition function for the quantum super-IISO(3) theory is

$$
Z = \int d\mu[\Phi] e^{-I[\Phi]}, \quad (2.1)
$$
where $d\mu[\Phi]$ is the measure in the configuration space of the fields $E, B, C, A, \psi$, denoted collectively by $\Phi$. The partition function splits up:

$$Z = \int d\mu[E, A] e^{-\int E^a \wedge F_a(A)} Z[B, C] Z[\psi],$$  \hspace{1cm} (2.2)

where

$$Z[B, C] := \int d\mu[B, C] e^{-\int B^a \wedge DC_a},$$  \hspace{1cm} (2.3)

$$Z[\psi] := \int d\mu[\psi] e^{-\int i\bar{\psi}^i \wedge D\psi^i}. \hspace{1cm} (2.4)$$

In the above, since the integral over $E$ has support only on regions of configuration space where $A$ is flat, the functional integrals $Z[B, C], Z[\psi]$ are evaluated at some flat $A$. \footnote{It turns out that the integral over $A$ cancels $Z[\psi]$ up to a sign, as in \[2\], while the $Z[B, C] = T(A)$, the Ray-Singer torsion for the flat connection $A$. Hence the partition function would be}

$$Z = \sum_\alpha (-1)^{t_\alpha} T(A_\alpha)$$ \hspace{1cm} (2.5)

where the index $\alpha$ is over the flat connections. The exponents $t_\alpha$ are 0 or 1.

The partition function for pure bosonic ISO(3) gravity is

$$Z_b = \sum_\alpha | T(A_\alpha) |^2,$$ \hspace{1cm} (2.6)

while for Witten's fermionic super-ISO(3) model, the partition function is

$$Z_f = \sum_\alpha (-1)^{t_\alpha}. \hspace{1cm} (2.7)$$

\footnote{The calculation of the partition function was performed using the method outlined in \[4\], \textit{i.e.} by parametrizing the fields $B, C, E$ and $\psi$ by their respective Hodge decompositions with respect to a flat SO(3) connection $\bar{A}$, and parametrizing $A$ itself by $A = \bar{A} + \star D_{\bar{A}} \star \alpha$, with $\alpha$ a closed so(3)-valued 2-form field. The $\star$ denotes the Hodge dual with respect to some Riemannian metric on the manifold $M$. The functional integral measure of each field contains as a factor the inverse of the volume of the appropriate gauge orbit, which cancels the corresponding functional integral over the pure-gauge (exact) Hodge component. The Jacobians and the Gaussian integrals over the bosonic fields give \textit{absolute values} of powers of functional determinants; while ‘Gaussian integrals’ over the fermionic fields have a specific sign \[5\].}
Witten [2] identifies the latter as the Casson invariant of the three-manifold. The partition function $Z$ for super-IISO(3) theory bears rather the same relation to $Z_b$ and $Z_f$ as a complex number does to its modulus squared and its argument, respectively. It is for this reason that the topological invariant $Z$ is interpreted as a new three-manifold invariant.

Work is currently in progress to understand the geometrical/topological meaning of $Z$.

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References

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