ROBUST DESIGN OF SENSOR FUSION PROBLEM IN DISCRETE TIME

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Abstract. In this paper, we consider a robust sensor scheduling problem which estimates the state of an uncertain process based on measurements obtained by a given set of noisy sensors, where the measurements of sensors are subject to external interference uncertainties. We formulate this problem into a minimax optimal control problem, which is equivalent to a semi-infinite programming problem with a dynamic system. A discretization method is used to solve this problem, where the computation is very large scale in general. We propose an approximation method to reduce the computational complexity. For illustration, two numerical examples are solved.

1. Introduction. In many practical scenarios, such as optical communications, planetary probes, micro actuators, acoustics, earthquake monitoring, geological surveying, and neurobiology, a large amount of data is available for collection. On the basis of the data collected, we need to estimate the unknown signal as accurately as possible. This problem has received considerable attention in the open literature. The existence of a solution is considered in [7, 9] for the continuous time case, in [8] for the discrete time case and in [14] for the discrete-continuous case. The sensor scheduling problem space [17] is to determine which sensors to activate at any particular time to achieve a given performance. It is a discrete optimization problem and it is NP-hard. There exist many methods which consider the sensor scheduling problem, such as the optimal control method in [11], where only one sensor is used at any one time. It allows for a long sequence of sensors such that all possible sequences of sensors can be considered when searching for an optimal one. The branch and bound method is proposed in [3], where a lower bound dynamic system is obtained and used to calculate the lower bound during a tree search. In [6], a sensor scheduling problem is formulated, where several sensors are used at any one time. A computational method is developed for solving this problem based on a branch and bound method in conjunction with a gradient-based method. In [22], the sensor scheduling problem is formulated into an integer linear programming problem. Lower and upper bounds are developed and heuristic methods are proposed to solve...
this problem. In [12], a sensor scheduling problem is formulated to track multiple
targets. In [1], the theory of Tchebycheff systems is used to determine an improved
upper bound on the guaranteed minimum number of measurements.

The sensor fusion problem is considered in many papers in the literature, such as
[2, 15, 16, 18]. In [4], the problem is first shown to be equivalent to a deterministic
optimal control problem. Then, the control parametrization method [19, 20, 21]
and the control parametrization enhancing technique [14] can be used to solve this
problem. The discrete version of this problem is considered in [5].

The rest of the paper is organized as follows. The sensor fusion problem and
robust sensor fusion problem are formulated in Section 2. In Section 3, we propose
an approximate method to simplify the computation of the robust problem. For
illustration, two numerical examples are solved in Section 4.

2. Problem formulation.

2.1. Sensor fusion problem. Let \((\Omega, \mathcal{F}, P)\) be a given probability space. Suppose
that the unknown \(n\)-dimensional signal \(x(t)\) is driven by the following system:

\[
x(t + 1) = A(t)x(t) + B(t)V(t), \quad t \in I_1,
\]
\[
x(0) = x_0,
\]

where \(I_1 = \{0, 1, ..., T - 1\}\) and \(T\) is the terminal time. For each \(t \in I_1\), \(A(t) \in \mathbb{R}^{n \times n}\) and \(B(t) \in \mathbb{R}^{n \times d}\). The process \(\{V(t), t \in I_1\}\) is a sequence of independent
standard Gaussian random vectors with the mean \(0\) and the covariance being the
d-dimensional identity matrix. The initial state \(x_0\) is a random \(n\)-vector with mean
\(x_0\) and covariance matrix \(P_0\).

To estimate the unknown signal \(x(t)\), the measurement data is obtained by \(N\)
sensors, which are governed by

\[
y_i(t) = C_i(t)x(t) + D_i(t)W_i(t), \quad t \in I_2, \quad i = 1, \ldots, N,
\]

where \(I_2 = \{1, 2, ..., T\}\). For each \(t \in I_2\), \(C_i(t) \in \mathbb{R}^{m \times n}\), \(D_i(t) \in \mathbb{R}^{m \times m}\), \(y_i(t) \in \mathbb{R}^m\).

For each \(i = 1, ..., N\), \(\{W_i(t), t \in I_2\}\) is a sequence of independent standard \(m\)-
dimensional Gaussian random vectors with the mean \(0\) and the covariance being the
\(m\)-dimensional identity matrix.

Furthermore, we suppose that \(x_0\), \(\{V(t), t \in I_1\}\) and \(\{W_i(t), i = 1, ..., N, t \in I_2\}\)
are mutually independent.

Then, we can collect the measurement data \(\{y_i(t), t \in I_2, i = 1, ..., N\}\) from \(N\)
sensors. To estimate the unknown signal \(x(t)\), we first assign the weights \(\{w_i(t), t \in I_2, i = 1, ..., N\}\) to all the measurement data, that is,

\[
y(t) = \sum_{i=1}^{N} w_i(t)y_i(t)
\]
\[
= \sum_{i=1}^{N} w_i(t)[C_i(t)x(t) + D_i(t)W_i(t)],
\]

where for each \(t \in I_2\), the weight vector \(w(t) = (w_1(t), w_2(t), ..., w_N(t))^T\) belongs to the
following set:

\[
\Delta = \left\{(a_1, ..., a_N)^T : \sum_{i=1}^{N} a_i = 1, a_i \geq 0, i = 1, ..., N \right\}.
\]
For each $t$, the weight vector $\mathbf{w}(t) = (w_1(t), w_2(t), ..., w_N(t))^\top$ represents the importance of the sensor data at that time. Let 

$$\mathbf{w} = (\mathbf{w}^\top(1), \mathbf{w}^\top(2), ..., \mathbf{w}^\top(T))^\top$$

be the weight vector assignment strategy and let $\mathcal{W} = \Delta^T = \Delta \times \Delta \cdots \times \Delta$ be the set of all such strategies.

For each $\mathbf{w} \in \mathcal{W}$, we can obtain the measurement data $\mathbf{y}_w(t)$ by (3). Then the unbiased minimum variance estimate of the signal $\mathbf{x}(t)$ is given by the conditional expectation:

$$\hat{x}(t) = E\{\mathbf{x}(t)|\mathcal{F}_w\},$$

where

$$\mathcal{F}_w = \sigma\{\mathbf{y}_w(t), t \in I_2\}$$

denotes the smallest $\sigma$-algebra relative to which $\mathbf{y}$ is measurable.

It’s well known that the unbiased minimum variance estimate (5) is given by the Kalman filter as follows.

$$\hat{x}(t) = (\mathbf{I} - \Gamma_w(t)\Xi_w(t))\mathbf{A}(t-1)\hat{x}(t-1) + \Gamma_w(t)y(t),$$

$$\hat{x}(0) = \bar{x}_0,$$

where

$$\Gamma_w(t) = \mathbf{P}_w(t)\Xi_w(t)(\Xi_w(t)\mathbf{P}_w(t)\Xi_w^\top(t) + \mathbf{R}_w(t))^{-1},$$

$$\Xi_w(t) = \sum_{i=1}^{N} w_i(t)\mathbf{C}_i(t),$$

$$\mathbf{R}_w(t) = \sum_{i=1}^{N} w_i^2(t)\mathbf{D}_i(t)\mathbf{D}_i^\top(t).$$

$\mathbf{P}_w(t)$ is the error covariance matrix function before measurement defined by $E\{|\mathbf{x}(t) - \hat{x}^{-}(t)||\mathbf{x}(t) - \hat{x}^{-}(t)|\}^2$, where $\hat{x}^{-}(t)$ is the a priori estimate of the state given by

$$\hat{x}^{-}(t) = \mathbf{A}(t)\hat{x}(t).$$

$\mathbf{P}_w(t)$ is the error covariance matrix function after measurement defined by $E\{|\mathbf{x}(t) - \hat{x}(t)||\mathbf{x}(t) - \hat{x}(t)|\}^2$. $\mathbf{P}_w(t)$ and $\mathbf{P}_w^{-}(t)$ satisfy the difference equation:

$$\mathbf{P}_w(t) = (\mathbf{I} - \Gamma_w(t)\Xi_w(t))\mathbf{P}_w^{-}(t),$$

$$\mathbf{P}_w^{-}(t+1) = \mathbf{A}(t)\mathbf{P}_w(t)\mathbf{A}^\top(t) + \mathbf{B}(t)\mathbf{B}^\top(t),$$

$$\mathbf{P}_w(0) = \mathbf{P}_0.$$  

To minimize the error between $\hat{x}(t)$ and $\mathbf{x}(t)$, the cost function can be defined as

$$L(\mathbf{w}) = \sum_{t=0}^{T-1} Tr\{\mathbf{Q}(t)\mathbf{P}_w(t)\} + c Tr\{\mathbf{P}_w(T)\},$$

where $Tr\{\cdot\}$ is the trace operation, $c$ is a positive constant and for each $t$, $\mathbf{Q}(t)$ is an $n \times n$ positive semi-definite matrix.

For any weight vector $\mathbf{w} \in \mathcal{W}$, the cost function value is different. Hence, we should find the weight vector such that the cost function value is optimal, that is, 

**Problem 1.** Find a $\mathbf{w} \in \mathcal{W}$ such that the cost function (13) is minimized, subject to the dynamic constraints (12).
Problem [1] is the sensor fusion problem. It can be considered as an optimal control problem, where the system is governed by a discrete dynamic system, and the weight vector is treated as control vector. Then, the sequential quadratic programming (SQP) algorithm can be used to solve this problem.

2.2. Robust sensor fusion problem. Problem [1] is formulated in the case when the parameters of the sensors are measured exactly. However, these parameters are always estimated in some sense. In practical applications, these parameters are not deterministic due to internal and external uncertainties, such as noise, weather, etc.. Then, the measurement data of the sensors are not accurate and the solution of Problem [1] may become very poor.

To address this issue, we can consider the robust sensor fusion problem, where the parameters of the sensors can be stochastic. We can assume that the sensors are governed by

\[ y_i(t) = \tilde{C}_i(t, \theta)x(t) + \tilde{D}_i(t, \theta)W_i(t), \quad t \in I_2, \quad i = 1, \ldots, N, \tag{14} \]

where the sensor’s parameters \( C_i(t) \) and \( D_i(t) \) are replaced by \( \tilde{C}_i(t, \theta) \) and \( \tilde{D}_i(t, \theta) \), where \( \theta \) is the parameter for uncertainty. Without loss of generality, we can define \( \tilde{C}_i(t, 0) = C_i(t) \) and \( \tilde{D}_i(t, 0) = D_i(t) \).

The uncertainty parameter \( \theta \) is a \( r \)-dimensional vector. The magnitude of \( \theta \) is bounded by a vector \( \eta = (\eta_1, \ldots, \eta_r) \), that is, for each \( i, \theta_i \in [-\eta_i, \eta_i] \). Denote the set of all feasible \( \theta \) by \( \Theta \), that is,

\[ \Theta = [-\eta_1, \eta_1] \times [-\eta_2, \eta_2] \times \cdots \times [-\eta_r, \eta_r]. \]

For each \( \theta \in \Theta \), the measurement data is obtained by (14). Then, the estimate of the signal \( x(t) \) is given by

\[ \tilde{x}(t, \theta) = (I - \Gamma_w(t, \theta)\Xi_w(t, \theta))A(t)\tilde{x}(t-1, \theta) + \Gamma_w(t, \theta)y(t), \]
\[ \tilde{x}(0, \theta) = x_0, \tag{15} \]

where

\[ \Gamma_w(t, \theta) = P_w(t, \theta)\Xi_w(t, \theta)(\Xi_w(t, \theta)P_w(t, \theta)\Xi_w(t, \theta) + R_w(t, \theta))^{-1}, \tag{16} \]
\[ \Xi_w(t, \theta) = \sum_{i=1}^{N} w_i(t)\tilde{C}_i(t, \theta), \tag{17} \]
\[ R_w(t, \theta) = \sum_{i=1}^{N} w_i^2(t)\tilde{D}_i(t, \theta)\tilde{D}_i^T(t, \theta), \tag{18} \]

and \( P_w(t, \theta) \) is given by

\[ P_w(t, \theta) = (I - \Gamma_w(t, \theta))\Xi_w(t, \theta)P_w(t, \theta), \]
\[ P_w(t+1, \theta) = A(t)P_w(t, \theta)A^T(t) + B(t)B^T(t), \]
\[ P_w(0, \theta) = P_0. \tag{19} \]

Then, for each \( \theta \in \Theta \), the cost function is

\[ L(w, \theta) = \sum_{t=0}^{T-1} Tr\{Q(t)P_w(t, \theta)\} + cTr\{P_w(T, \theta)\}. \tag{20} \]
Since $\theta$ can take any value in $\Theta$, we should consider the worst case of the cost function (20) in $\Theta$, that is,

$$\max_{\theta \in \Theta} L(w, \theta) = \max_{\theta \in \Theta} \left( \sum_{t=0}^{T-1} Tr\{Q(t)P_w(t, \theta)\} + cTr\{P_w(T, \theta)\} \right).$$

(21)

Thus, we can formulate the robust sensor fusion problem into a minimax optimal control problem as follows.

**Problem 2.** Find a $w \in W$ such that the cost function (21) is minimized, subject to the dynamic constraints (19), where $c$ is a positive constant and $Q(t)$ is an $n \times n$ positive semi-definite matrix.

**Remark 1.** Problem 1 can be considered as a special case of Problem 2, if $\tilde{C}_i(t, 0) = C_i(t), \tilde{D}_i(t, 0) = D_i(t), \forall i = 1, \ldots, N, \eta = 0$.

3. **Method.** Problem 2 is a minimax optimal control problem, which is more complicated than Problem 1. It can be transformed into an equivalent semi-infinite programming problem as follows.

$$\min_{w, z} z \quad \text{s.t.} \quad L(w, \theta) \leq z, \forall \theta \in \Theta,$$

$$P_w(t, \theta) = (I - \Gamma_w(t, \theta))\Xi_w(t, \theta)P_w(t, \theta),$$

$$P_w(t+1, \theta) = A(t)P_w(t, \theta)A^T(t) + B(t)B^T(t),$$

$$P_w(0, \theta) = \tilde{P}_0.$$  

(22)

where $L(w, \theta)$ is given by (20).

Note that, due to the dynamic system in Problem (22), it is very complicated in general. A discretization method is always applied to solve semi-infinite programming problems, that is, the infinite set $\Theta$ is replaced by a sufficiently dense discrete set. Suppose that a sufficiently dense discrete set of $\Theta$ is denoted by $\Theta^d = \{\theta^1, \theta^2, \ldots, \theta^p\}$. Then, a discrete approximation of Problem (22) is

$$\min_{w, z} z \quad \text{s.t.} \quad L(w, \theta^i) \leq z, i = 1, \ldots, p,$$

$$P_w(t, \theta^i) = (I - \Gamma_w(t, \theta^i))\Xi_w(t, \theta^i)P_w(t, \theta^i),$$

$$P_w(t+1, \theta^i) = A(t)P_w(t, \theta^i)A^T(t) + B(t)B^T(t),$$

$$P_w(0, \theta^i) = \tilde{P}_0.$$  

(23)

where $L(w, \theta)$ is given by (20).

Problem (23) is a finite dimensional mathematical programming problem. Any suitable gradient based method can be applied to solve this problem. However, the size of $\Theta^d$ is large in general, and for each $\theta^i$, $P_w(t, \theta^i)$ must be computed by (19). Then, there are many constraints in Problem (23) and the computation is very expensive. In view of this, we develop an efficient method to solve Problem (22) as follows.

The parameter $\eta$ is generally not large in practice. In this case, it’s not necessary to perform the discretization of the set $\Theta$ and then Problem (22) can be simplified. This can be seen in the theorem below.
Theorem 3.1. Assume that $\eta$ is small. Then, for any $w \in W$, we have
\[
\max_{\theta \in \Theta} L(w, \theta) = \max_{\theta \in \bar{\Theta}^d} L(w, \theta) + o(\eta),
\] (24)
where
\[
\bar{\Theta}^d = \{ \theta : \theta = (b_1\eta_1, b_2\eta_2, \ldots, b_r\eta_r)^T, b_i = 1 \text{ or } b_i = -1, \forall i \}. \tag{25}
\]

Proof of Theorem 3.1. Since $\tilde{C}_i(t, 0) = C_i(t)$ and $\tilde{D}_i(t, 0) = D_i(t)$, then we have $L(w, 0) = L(w)$. We can rewrite $L(w, \theta)$ as
\[
L(w, \theta) = L(w, 0) + \frac{\partial L(w, 0)}{\partial \theta} \theta + o(\theta). \tag{26}
\]
Note that the first two terms of the right hand side of (26) are convex with respect to $\theta$ and the last term can be ignored when $\theta$ is sufficiently small. Since the maximizer of a convex function exists when $\theta$ is in the boundary set, for each $i$, we have
\[
\max_{\theta_i \in [-\eta_i, \eta_i]} L(w, \theta) = \max\{ L(w, 0) + \frac{\partial L(w, 0)}{\partial \theta_i} \eta_i + o(\eta_i),
\]
\[
L(w, 0) + \frac{\partial L(w, 0)}{\partial \theta_i} (-\eta_i) + o(\eta_i) \}
\]
\[
= \max\{ L(w, (\theta_1, \ldots, \theta_{i-1}, \eta_i, \theta_{i+1}, \ldots, \theta_r)^T) + o(\eta_i),
\]
\[
L(w, (\theta_1, \ldots, \theta_{i-1}, -\eta_i, \theta_{i+1}, \ldots, \theta_r)^T) + o(\eta_i) \}
\]
\[
\leq \max\{ L(w, (\theta_1, \ldots, \theta_{i-1}, \eta_i, \theta_{i+1}, \ldots, \theta_r)^T),
\]
\[
L(w, (\theta_1, \ldots, \theta_{i-1}, -\eta_i, \theta_{i+1}, \ldots, \theta_r)^T) \} + o(\eta_i).
\]
Since $i \in \{1, \ldots, r\}$ can be arbitrary, we obtain (24).

This completes the proof. \hfill \Box

By Theorem 3.1 if $\eta$ is small, Problem 2 can be approximated by a simplified form as follows.

Problem 3. Find a $w \in W$ and $z$, such that
\[
\min_{w, z} z
\]
\[
s.t. \sum_{t=0}^{T-1} Tr\{Q(t)P_w(t, \theta)\} + cTr\{P_w(T, \theta)\} \leq z, \quad \theta \in \bar{\Theta}^d,
\]
where $P_w(t, \theta)$ is the solution of the dynamic system (17).

It can be seen that the number of constraints in Problem 3 is reduced and the computation is simplified.

4. Illustrative example. In this section, we apply the proposed method to solve two robust sensor fusion problems as follows.

Example 1. Suppose that the unknown signal $x(t)$ is driven by the system (1), where the parameters are given by
\[
A(t) = \begin{bmatrix} 1 & -0.5 \\ 0 & 0.5t - 0.5 \end{bmatrix},
\]
\[
B(t) = \begin{bmatrix} 4 \\ 1.5 \end{bmatrix},
\]
where \( t \in \{0, 1, \ldots, T - 1\} \). The signal is measured by two sensors with the parameters in (14) given by

\[
\hat{C}_1(t, \theta) = \begin{bmatrix} 0.8 - 0.53t - 3.9\theta_1 & 0 \\ 0 & 0.8 + 0.13t - 4.3\theta_1 \end{bmatrix},
\]

\[
\hat{D}_1(t, \theta) = \begin{bmatrix} 1 + 0.13t - 2.8\theta_1 & 0 \\ 0 & 1 - 0.5t - 0.47\theta_1 \end{bmatrix},
\]

and

\[
\hat{C}_2(t, \theta) = \begin{bmatrix} 1 - 0.5t - 3.59\theta_2 & 0 \\ 2.4 - 0.15t - \theta_2 & 0.14t - 4.1\theta_2 \end{bmatrix},
\]

\[
\hat{D}_2(t, \theta) = \begin{bmatrix} 1 - 0.4t - 2.7\theta_2 & 0 \\ 0 & 1 - 0.12t - 0.57\theta_2 \end{bmatrix},
\]

where \( t \in \{1, 2, \ldots, T\} \), \( \theta_1 \in [-\eta_1, \eta_1] \) and \( \theta_2 \in [-\eta_2, \eta_2] \). The terminal time is \( T = 20 \), \( \eta_1 = 0.1 \) and \( \eta_2 = 0.4 \). The initial state is a deterministic vector \((0, 0)^T\). Then,

\[
P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

The cost function is given by

\[
L(w, \theta) = \sum_{t=0}^{20} Tr\{P_w(t, \theta)\}.
\]

First, we consider the sensor fusion problem, where the parameters of the sensors (2) are given by \( C_i(t, 0), D_i(t, 0), i = 1, 2 \). We can solve this problem and obtain the optimal solution \( w^* \), which is depicted in Figure 1(a). The corresponding cost function \( L(w^*, \theta) \) with respect to \( \theta \) is given in Figure 2(a) where the maximum value of \( L(w^*, \theta) \) is 649.4429.

Next, we consider the robust sensor fusion problem. We solve this problem and obtain the optimal solution \( \tilde{w}^* \), which is depicted in Figure 1(b). Then, the corresponding cost function value \( L(\tilde{w}^*, \theta) \) with respect to \( \theta \) is given in Figure 2(b) where the maximum value of \( L(\tilde{w}^*, \theta) \) is 56.0585. It can be seen that the maximum value is reduced remarkably. Thus, compared with the sensor fusion design, the robust sensor fusion design is more effective.

**Figure 1.** Optimal solutions in the first example.
Example 2. Suppose that the unknown signal \( x(t) \) is driven by the system (1), where the parameters are given by

\[
A(t) = \begin{bmatrix}
2 + 0.3t & -0.5 + \cos t & 0 \\
0 & 0.08t^2 - 0.5 & \cos t \\
2 \sin t^2 & 0 & 0.2 \sin 2t
\end{bmatrix},
B(t) = \begin{bmatrix}
3.02 + 0.3 \sin t \\
1.5 + 0.2t \\
1.5t + \cos t
\end{bmatrix},
\]

where \( t \in \{0,1,\ldots,T-1\} \). The signal is measured by three sensors with the parameters in (14) given by

\[
\tilde{C}_1(t, \theta) = \begin{bmatrix}
2 - 0.1t - 7\theta_1 & 0 & 0 \\
0 & -0.15t - 7\theta_1 & -0.01t - 5\theta_1 \\
0 & \sin \theta_1 & 0.22t - 4\theta_1
\end{bmatrix},
\]

\[
\tilde{D}_1(t, \theta) = \begin{bmatrix}
1 - 3\theta_1 & 0 & 0 \\
0 & 1 - 7\theta_1 & 0 \\
0 & 0 & 1 - 6\theta_1
\end{bmatrix},
\]

\[
\tilde{C}_2(t, \theta) = \begin{bmatrix}
2 - 0.12t - 7\theta_2 & 0 & 0 \\
0 & -0.15t - 7\theta_2 & -0.01t - 5\theta_2 \\
0 & \sin \theta_2 & 0.12t - 6\theta_2
\end{bmatrix},
\]

\[
\tilde{D}_2(t, \theta) = \begin{bmatrix}
1 - 2\theta_2 & 0 & 0 \\
0 & 1 - 4\theta_2 & 0 \\
0 & 0 & -\sin \theta_2 - 6\theta_2
\end{bmatrix},
\]

\[
\tilde{C}_3(t, \theta) = \begin{bmatrix}
2 - 0.15t - 7\theta_3 & 0 & 0 \\
0 & -0.15t - 6\theta_3 & -0.01t - 5\theta_3 \\
0 & \sin \theta_3 & 0.35t - 4\theta_3
\end{bmatrix},
\]

\[
\tilde{D}_3(t, \theta) = \begin{bmatrix}
1 - 2\theta_3 & 0 & 0 \\
0 & 1 - \theta_3 & 0 \\
0 & 0 & 1 - 5\theta_3
\end{bmatrix}.
\]

where \( t \in \{1,2,\ldots,T\}, \theta_1 \in [-\eta_1, \eta_1], \theta_2 \in [-\eta_2, \eta_2] \) and \( \theta_3 \in [-\eta_3, \eta_3] \). The terminal time is \( T = 10 \), \( \eta_1 = 0.1 \), \( \eta_2 = 0.2 \) and \( \eta_3 = 0.2 \). The initial state is a deterministic vector \((0,0,0)^T\). Then,

\[
P_0 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
The cost function is given by
\[ L(w, \theta) = \sum_{t=0}^{10} Tr\{P_w(t, \theta)\}. \]

First, we consider the sensor fusion problem, where the parameters of the sensors are given by \( C_i(t, 0), D_i(t, 0), i = 1, 2 \). We solve this problem and obtain the optimal solution \( w^* \), which is depicted in Figure 3(a). Then, the maximum value of \( L(w^*, \theta) \) can be computed as 439.2425 when \( \theta \in [-0.1, 0.1] \times [-0.2, 0.2] \times [-0.2, 0.2] \).

Next, we consider robust sensor fusion problem. We solve this problem and obtain the optimal solution \( \tilde{w}^* \), which is depicted in Figure 3(b). The corresponding maximum value of \( L(\tilde{w}^*, \theta) \) is 106.2902 when \( \theta \in [-0.1, 0.1] \times [-0.2, 0.2] \times [-0.2, 0.2] \).

It can be seen that the cost function value is reduced remarkably. Thus, compared with the sensor fusion design, the robust sensor fusion design is more effective.

5. **Conclusion.** In this paper, we have proposed a robust design of a sensor scheduling problem. The design problem can be formulated into a semi-infinite programming problem. To simplify the computation of this problem, we proposed a method by approximating the cost function by a convex function with respect to the uncertainty parameters, and the optimal solution of the cost function with respect to the uncertainty parameters can be obtained directly. Then, the complexity of this problem was reduced and the computation of the semi-infinite programming problem was simplified. We have demonstrated the efficiency and effectiveness by two numerical examples of robust problems.

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