Abstract

Existence, uniqueness and stability of kinetic and entropy solutions to the boundary value problem for the Kolmogorov-type genuinely nonlinear ultra-parabolic equation with a smooth source term is established. After this, we consider the case when the source term contains a small positive parameter and collapses to the Dirac delta-function, as this parameter tends to zero. In this case, the limiting passage from the original equation with the smooth source to the impulsive ultra-parabolic equation is fulfilled and rigorously justified. The proofs rely on the method of kinetic equation and on the compensated compactness techniques for genuinely nonlinear equations.

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Introduction

In this article, we study the second-order quasi-linear ultra-parabolic equation with partial diffusivity and a smooth distributed source supplemented by a set of initial, final and boundary conditions. The article consists of two major parts. The first part (Sections 1–8) is devoted to development of the existence and uniqueness theory for the boundary-value problem under consideration in suitable classes of kinetic and entropy solutions. The research in the second part (Sections 9–14) is related with the case when the source term contains a small positive parameter and collapses to the Dirac delta-function as this small parameter tends to zero. In this case, we focus on justification of the claim that the singular limit of the family of kinetic and entropy solutions of the original problem exists and resolves the boundary-value problem for the impulsive Kolmogorov-type equation.

Let us recall that ultra-parabolic equations with partial diffusivity arise in fluid dynamics, physics of particles, combustion theory, mathematical biology and financial mathematics [31, 37]. They describe, in particular, non-stationary transport of matter or energy in cases when effects of diffusion in some spatial directions are negligible as compared to convection. In line with the famous works of A. N. Kolmogorov [16–18], these equations are commonly called Kolmogorov-type equations. Worth noticing that they also have some other names in literature.

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For example, they are called the Graetz–Nusselt equations in description of partial diffusion of heat in fluid dynamics [30, 45, 47] or Fokker–Planck (Fokker–Planck–Kolmogorov) equations in description of stochastic diffusive processes modelling Brownian motion [8], [33] Chapter II, §21.

The nowadays theory of linear and quasi-linear ultra-parabolic equations is rather vast. Our study in this article is somewhat close to the works [1, 2, 11, 19, 52–54] devoted to well-posedness topics for boundary-value problems. Besides the presence of the source term, the peculiarity of the ultra-parabolic equation, considered in the present article, is that the flux in a purely convective direction (assigned to variable $s$) is non-monotonous. This circumstance makes the question of proper setting of initial and final conditions with respect to $s$ rather sophisticated. Our approach to overcoming this difficulty ascends to the works [7, 38]. More precisely, we formulate the initial and final conditions in the form of boundary entropy inequalities or, equivalently, boundary kinetic equalities, which eventually helps to establish the well-posedness results. The proofs in the article are heavily based on the genuine nonlinearity (non-degeneracy) condition imposed on the equation under study. We systematically use the techniques elaborated for genuinely nonlinear ultra-parabolic equations in [1, 21–23, 41, 42, 47].

In the second part of the article, we encounter with the ultra-parabolic equation, which involves a Dirac delta-function as the source term. Such equations are called impulsive equations. They also can be equivalently written as systems consisting of the equation with zero source term and additional impulsive condition. From the physical viewpoint, an impulsive source (or condition) reflects phenomena of instantaneous loading, i.e., drastic change of mass, energy, impulse, etc., at a moment. Studies of impulsive ordinary differential equations have long history and cover a wide range of topics in natural sciences (see, for example, monographs [4, 30, 46] and references therein). At the same time, the theory of impulsive partial differential equations is rather new and far from complete. Recently, some progress has been achieved in constructing well-posedness theory for impulsive hyperbolic, elliptic, parabolic, and abstract equations (see, for example, articles [5, 6, 13–15] and references therein). In the present article, we develop the idea from [56]. Namely, we deduce the equation with the Dirac delta-function-type source from the equation with the smooth distributed source. In other words, we fulfill and rigorously justify the limiting passage from the delayed ultra-parabolic equation to the impulsive ultra-parabolic equation as the time of delay tends to zero.

This paper develops and is essentially based on the results [21–27] due to the authors.

1 Formulation of the basic problem incorporating a distributed source

Let $\Omega$ be a bounded domain of spatial variables $x \in \mathbb{R}^d$ with a smooth boundary $\partial \Omega$ ($\partial \Omega \in C^2$). Let $t \in [0, T]$ and $s \in [0, S]$ be two independent time-like variables. Here $T$ and $S$ are given positive constants. Denote

\[
G_{T,S} := \Omega \times (0, T) \times (0, S),
\]

\[
\Xi^1 := \overline{\Omega} \times [0, S], \quad \Xi^2 := \overline{\Omega} \times [0, T],
\]

\[
\Gamma_t := \partial \Omega \times [0, T] \times [0, S],
\]

\[
\Gamma^0_1 := \overline{\Omega} \times \{t = 0\} \times [0, S], \quad \Gamma^0_T := \overline{\Omega} \times \{t = T\} \times [0, S],
\]

\[
\Gamma^0_0 := \overline{\Omega} \times [0, T] \times \{s = 0\}, \quad \Gamma^0_S := \overline{\Omega} \times [0, T] \times \{s = S\}.
\]
In this article, we study the following Cauchy — Dirichlet problem for the ultra-parabolic equation with partial diffusivity and a source term.

**Problem Π.** It is necessary to find a function \( u \): \( G_{T,S} \mapsto \mathbb{R} \) satisfying the quasi-linear ultra-parabolic Kolmogorov-type equation

\[
\partial_t u + \partial_s a(u) + \text{div}_x \varphi(u) = \Delta_x u + Z_\gamma(x,t,s,u), \quad (x,t,s) \in G_{T,S},
\]

the initial condition with respect to time-like variable \( t \)

\[
u|_{t=0} = u_0^{(1)}(x,s), \quad (x,s) \in \Sigma^1,
\]

the initial and final conditions with respect to time-like variable \( s \)

\[
u|_{s=0} \approx u_0^{(2)}(x,t), \quad u|_{s=S} \approx u_S^{(2)}(x,t), \quad (x,t) \in \Sigma^2,
\]

and the homogeneous boundary condition

\[
u|_{\Gamma_1} = 0.
\]

In the formulation of Problem Π, we suppose that the initial and final data \( u_0^{(1)}, u_0^{(2)}, u_S^{(2)} \), the nonlinearities \( a = a(\lambda), \varphi = (\varphi_1(\lambda), \ldots, \varphi_d(\lambda)) \), and the source term \( Z_\gamma = Z_\gamma(x,t,s,\lambda) \) are given and satisfy the conditions stated further.

In (1.1c) the relation sign \( \approx \) means that \( u_0^{(2)} \) and \( u_S^{(2)} \) may be unattained by a solution \( u \) on some parts of the sets \( \Gamma_0^2 \) and \( \Gamma_S^2 \), respectively. The fact whether \( \approx \) becomes equality (=), or not, is figured out a posteriori, i.e., after a solution of equation (1.1a) is constructed somehow.

**Conditions on \( u_0^{(1)} \& u_0^{(2)} \& u_S^{(2)} \).** The initial and final data meet the regularity requirements

\[
u_0^{(1)} \in C^{2+\alpha}(\Sigma^1), \quad u_0^{(2)}, u_S^{(2)} \in C^{2+\alpha}(\Sigma^2), \quad \alpha \in (0,1), \quad (1.2)
\]

and the consistency conditions

\[
u_0^{(1)} = 0 \text{ in a neighborhood of } \partial \Sigma^1, \quad (1.3)
\]

\[
u_0^{(2)} = 0, \quad u_S^{(2)} = 0 \text{ in a neighborhood of } \partial \Sigma^2. \quad (1.4)
\]

By \( C^{2+\alpha}(\Sigma) (\Sigma \subset \mathbb{R}^N) \) we standardly denote the space of twice-differentiable functions, whose second derivatives are Hölder-continuous with exponent \( \alpha \in (0,1) \), equipped with the norm

\[
\|\Phi\|_{C^{2+\alpha}(\Sigma)} = \|\Phi\|_{C^2(\Sigma)} + \sum_{|k|=2} \sup_{\zeta, \eta \in \Sigma} \frac{|D^{k}\Phi(\zeta) - D^{k}\Phi(\eta)|}{|\zeta - \eta|^{2+\alpha}}.
\]

**Conditions on \( a \& \varphi \& Z_\gamma \).** (i) Functions \( a, Z_\gamma \) and \( \varphi_i (i = 1, \ldots, d) \) meet the regularity requirements

\[
a \in C^{2}_{\text{loc}}(\mathbb{R}), \quad a(0) = 0, \quad Z_\gamma \in L^{\infty}(0,T; C^{1}_{\text{loc}}(\Sigma^1 \times \mathbb{R})), \quad \varphi_i \in C^{2}_{\text{loc}}(\mathbb{R}), \quad \varphi_i(0) = 0. \quad (1.5)
\]

(ii) Function \( a \) satisfies the genuine nonlinearity condition:

\[
\text{meas} \{ \lambda \in \mathbb{R} : \xi_1 + a'(\lambda)\xi_2 = 0 \} = 0 \text{ for every fixed } (\xi_1, \xi_2) \in S^1. \quad (1.6)
\]
(iii) Function $Z_\gamma$ satisfies the following growth condition:

there exist constants $b_\gamma^{(1)}$, $b_\gamma^{(2)} \geq 0$ such that for a.e. $(x, t, s) \in G_{T,S}$ and $\lambda \in \mathbb{R}$ the inequality

$$\lambda Z_\gamma(x, t, s, \lambda) \leq b_\gamma^{(1)} \lambda^2 + b_\gamma^{(2)}$$

holds.

In item (ii) and further in the article, by $S^1$ we denote the unit circle in $\mathbb{R}^2$ centered at the origin, $S^1 := \{(\xi_1, \xi_2) \in \mathbb{R}^2: \xi_1^2 + \xi_2^2 = 1\}$, and by meas $Q$ we denote the Lebesgue measure of any Lebesgue-measurable set $Q$.

**Remark 1.1.** According to the existing theory of genuinely nonlinear ultra-parabolic equations [41], condition (1.6) can be generalized in the following way:

set \( \{\lambda \in \mathbb{R}: \xi_1 + a'(\lambda)\xi_2 = 0\} \) has empty interior for each \((\xi_1, \xi_2) \in S^1\).

**Remark 1.2.** According to the well-known theory of quasi-linear parabolic equations [29, Chapter 1, Section 2], the growth condition (1.7) along with the regularity demands (1.2) provides the maximum principle for a solution (if any) of Problem $\Pi_\gamma$.

**Remark 1.3.** Note that, with $u|_{s=0} = u_0^{(2)}$ and $u|_{s=S} = u_S^{(2)}$ on $\Xi^2$ on the place of (1.1c), Problem $\Pi_\gamma$ becomes ill-posed. Indeed, since function $a = a(\lambda)$ is nonlinear and, in general, non-monotonous, it may be impossible to equate a solution $u$ of Problem $\Pi_\gamma$ to $u_0^{(2)}$ and $u_S^{(2)}$ on the entire sets $\Gamma_0^2$ and $\Gamma_S^2$. Therefore, we permit that a possible weak solution of Problem $\Pi_\gamma$ may deviate from $u_0^{(2)}$ and $u_S^{(2)}$ on $\Gamma_0^2$ and $\Gamma_S^2$, respectively. We set up a more loose non-classical condition (1.1c) following the original ideas presented in [7,33], see also [33 Sections 2.6–2.8].

In the formulation of Problem $\Pi_\gamma$ and in Conditions on $a&\varphi&Z_\gamma$ above, the label ‘$\gamma$’ is ‘dumb’ so far. Our first main result in this article consists of the proof of existence, uniqueness and stability of entropy and kinetic solutions to Problem $\Pi_\gamma$. Thus we generalize the results established in [23] onto the case of equations with the source term. In order to do this, we introduce into considerations and systematically study a strictly parabolic regularized formulation.

After this, in Sections 9–14 we consider the case when the source term has the specific form

$$Z_\gamma(x, t, s, \lambda) = K_\gamma(t, \tau)\beta(x, s, \lambda),$$

where $\tau \in (0, T)$ is a given fixed value and functions $K_\gamma$ and $\beta$ satisfy the following requirements.

**Conditions on $K_\gamma$&$\beta$.** (i) Function $K_\gamma$: $\mathbb{R}^2 \mapsto \mathbb{R}^+$ is defined by the formula

$$K_\gamma(t, \tau) = 1_{\{t \leq \tau\}} \frac{2}{\gamma} \omega\left(\frac{t - \tau}{\gamma}\right) \quad (\gamma > 0),$$

where $\omega: \mathbb{R} \to \mathbb{R}^+$ is a standard regularizing kernel having the properties

$$\omega \in C_0^\infty(\mathbb{R}), \quad \omega(t) \geq 0, \quad \omega(-t) = \omega(t) \quad \forall t \in \mathbb{R},$$

$$\text{supp}\omega \subset [-1, 1], \quad \int_\mathbb{R} \omega(t)\, dt = 1.$$
(ii) Function $\beta: \mathbb{R}_1 \times \mathbb{R}_\lambda \mapsto \mathbb{R}$ belongs to the space $C^1_0(\mathbb{R}_1 \times \mathbb{R}_\lambda)$ and satisfies the growth condition:
\[
\max_{\mathbb{R}_1 \times \mathbb{R}_\lambda} |\partial_\lambda \beta(x, s, \lambda)| \leq b_0 = \text{const} < +\infty,
\]
and the localization condition:
\[
\beta = 0 \text{ in some neighborhood of } \partial \mathbb{R}_1 \text{ for all } \lambda \in \mathbb{R},
\]
there is $b_1 = \text{const} > 0$ such that $\beta = 0$ for all $|\lambda| > b_1$.

**Remark 1.4.** On the strength of item (i) in Conditions on $K_\gamma \& \beta$, we easily deduce that
\[
K_\gamma(\cdot, \tau) \underset{\gamma \to 0^+}{\longrightarrow} \delta_{(t=\tau-0)} \text{ weakly$^*$ in } \mathcal{M}(\mathbb{R}),
\]
i.e.,
\[
\lim_{\gamma \to 0^+} \int_\mathbb{R} \phi(t) K_\gamma(t, \tau) \, dt = \phi(\tau - 0)
\]
for any integrable in a neighborhood of $\{t = \tau\} \subset \mathbb{R}$ function $\phi$ having the trace at the point $t = \tau$ from the left:
\[
\phi(\tau - 0) = \lim_{t \to \tau^-} \phi(t).
\]

In (1.10) and further in the article, $\mathcal{M}(\mathbb{R})$ is the normed space of Radon measures and $\delta_{(t=\tau)}$ is the Dirac delta-function on $\mathbb{R}$ concentrated at the point $t = \tau$.

As an example of proper $\omega$ in (1.9a), we can take the classical regularizing kernel
\[
\omega(t) = \begin{cases} 
C_*^{-1} \exp \left( \frac{-1}{1 - t^2} \right) & \text{for } 0 \leq |t| < 1, \\
0 & \text{for } |t| \geq 1,
\end{cases}
\]
where $C_* = \text{const} > 0$ ($C_* \approx 2, 2522$) is such that condition (1.9c)$_2$ holds.

**Remark 1.5.** Using the Lagrange mean value theorem and (1.9d), we find
\[
\beta(x, s, \lambda) = \left( \partial_\lambda \beta(x, s, \lambda) \right)_{|\lambda = \theta \lambda} \lambda^2 - \beta(x, s, 0) \lambda \leq b_0 \lambda^2 - \beta(x, s, 0) \lambda, \quad \forall (x, s) \in \mathbb{R}_1, \; \forall \lambda \in \mathbb{R},
\]
where $\theta \in (0, 1)$. Using (1.9a) – (1.9c), by rather simple evaluation from (1.12) we deduce that the source term $Z_\gamma = K_\gamma \beta$ satisfies the growth condition (1.7) with
\[
\beta^{(1)}_\gamma = 2 \gamma^{-1} \left\| \omega \right\|_{C[1-1]} \left( b_0 + 2^{1-1} \left\| \beta(\cdot, \cdot, 0) \right\|_{C(\mathbb{R}_1)} \right), \quad \beta^{(2)}_\gamma = \gamma^{-1} \left\| \omega \right\|_{C[1-1]} \left\| \beta(\cdot, \cdot, 0) \right\|_{C(\mathbb{R}_1)}.
\]

Thus, Conditions on $K_\gamma \& \beta$ are consistent with Conditions on $a \& \varphi \& Z_\gamma$.

Our second main result in this article consists of limiting passage as $\gamma \to 0^+$ in Problem $\Pi_\gamma$, with $Z_\gamma$ of the form (1.8). We prove that the family $\{u_\gamma\}$ of kinetic solutions to Problem $\Pi_\gamma$ has the unique limit $u_\gamma \in L^\infty(G_{T,S}) \cap L^2((0, T) \times (0, S); \dot{W}^1_\gamma(\Omega))$ in $L^1$-strong sense as $\gamma \to 0^+$, and that this limiting function $u_\gamma$ is the unique kinetic and entropy solution of the limiting problem. This limiting problem is formulated in Section 3 further. It is called Problem $\Pi_0$.  

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5
2 Parabolic Regularization

We construct a kinetic solution of Problem Π as a singular limit of the family of classical solutions $u_\varepsilon$ of the following strictly parabolic model.

**Problem Π.** For arbitrarily given boundary data satisfying Conditions on $u_0^{(1)} & u_0^{(2)} & u_S^{(2)}$, it is necessary to find a function $u_\varepsilon: G_{T,S} \mapsto \mathbb{R}$ satisfying the quasi-linear parabolic equation

\[ \partial_t u_\varepsilon + \partial_s a(u_\varepsilon) + \text{div}_x \varphi(u_\varepsilon) = \Delta_x u_\varepsilon + \varepsilon \partial^2_{ss} u_\varepsilon + Z_\gamma(x,t,s,u_\varepsilon), \quad (x,t,s) \in G_{T,S}, \]  

(2.1a)

the initial condition

\[ u_\varepsilon|_{t=0} = u_0^{(1)}(x,s), \quad (x,s) \in \Xi^1, \]  

(2.1b)

the initial and final conditions

\[ u_\varepsilon|_{s=0} = u_0^{(2)}(x,t), \quad u_\varepsilon|_{s=S} = u_S^{(2)}(x,t), \quad (x,t) \in \Xi^2, \]  

(2.1c)

and the homogeneous boundary condition

\[ u_\varepsilon|_{\Gamma_1} = 0. \]  

(2.1d)

Here $\varepsilon \in (0,1]$ is an arbitrarily fixed small parameter.

**Proposition 2.1.** Whenever Conditions on $u_0^{(1)} & u_0^{(2)} & u_S^{(2)}$ and Conditions on $a & \varphi & Z_\gamma$ hold, for any fixed $\varepsilon > 0$ there exists the unique classical solution $u_\varepsilon = u_\varepsilon(x,t,s)$ of Problem Π such that $u_\varepsilon \in H^{\alpha',\frac{\alpha''}{2}}(\overline{G_{T,S}}) \cap H^{2+\alpha'',1+\frac{\alpha''}{2}}(G_{T,S})$, where $\alpha', \alpha'' \in (0,1)$ depend on $\alpha$ and $\varepsilon$.

Moreover, the maximum principle

\[ \| u_\varepsilon \|_{L^\infty(\Xi_1 \times (0,t'))} \leq \inf_{\xi > u_0^{(1)}} \left( e^{\xi''} \max \left\{ \| u_0^{(1)} \|_{L^\infty(\Xi^1)}, \| u_0^{(2)} \|_{L^\infty(\Omega \times (0,t'))}, \| u_S^{(2)} \|_{L^\infty(\Omega \times (0,t'))}, \| \frac{\partial^2_{ss} u_\varepsilon}{\xi - b_1(2)} \| \right\} \right) \]  

(2.1d)

and the energy estimate

\[ \int_{G_{T,S}} (|\nabla_x u_\varepsilon|^2 + \varepsilon |\partial_s u_\varepsilon|^2) \, dx dt ds \leq C_1 \]  

(2.3)

hold. The constant $C_1$ does not depend on $\varepsilon$.

In the formulation of Proposition 2.1 by $H^{\alpha',\frac{\alpha''}{2}}(\overline{G_{T,S}})$ we standardly denote the spaces of Hölder-continuous functions of $(x,s)$ and $t$, with exponents $\alpha'$ and $\frac{\alpha''}{2}$, respectively. By $H^{2+\alpha'',1+\frac{\alpha''}{2}}(G_{T,S})$ we denote the space of differentiable functions $\phi$ on $G_{T,S}$ such that $\partial_t \phi, \partial_x \phi, \partial_s \phi, \partial^2_{xixj} \phi, \partial^2_{xss} \phi, \partial^2_{sxs} \phi \in H^{\alpha'',\frac{\alpha''}{2}}(G_{T,S})$, $i,j = 1, \ldots, d$.

**Proof of Proposition 2.1** directly follows from the well-known theory of quasi-linear parabolic equations of the second order [29, Chapter 1, Theorem 2.9; Chapter 5, Theorem 6.2].

In view of the forthcoming consideration of impulsive equation in Sec. 9 [14] it is suitable.
also to introduce the notion of weak solutions to Problem $Π_{γε}$ for the case of weaker restrictions on initial and final data than the ones that take place in Conditions on $u_0^{(1)} & u_0^{(2)} & u_S^{(2)}$.

Let us suppose that the initial and final data meet the weakened regularity requirements

$$u_0^{(1)} \in L^\infty(Ξ^1), \quad u_0^{(2)}, u_S^{(2)} \in C^{2+α}(Ξ^2), \quad α \in (0,1),$$

(2.4)

and the consistency conditions (1.4).

Let $u \in L^\infty(G_{T,S}) \cap C(0,T; W_2^2(Ξ^1))$ be an arbitrary extension of $u_0^{(1)}, u_0^{(2)}$ and $u_S^{(2)}$ into $G_{T,S}$ such that $u|_{Γ_1} = 0$.

**Remark 2.1.** Such $u$ exists due to the well-known Inverse Trace Theorem (see, for example, [34, Theorem 5.1] or [40, Theorem 2.22]) and requirements (2.4) and (1.4).

**Definition 2.1.** Function $u_ε \in L^\infty(G_{T,S}) \cap L^2(0,T; W_2^2(Ξ^1))$ is called a weak solution of Problem $Π_{γε}$, if it satisfies the following demands.

1. The equality $u_ε - \hat{u} = 0$ holds on $Γ_0^1 ∪ Γ_0^2 ∪ Γ_S^2 ∪ Γ_I$ in the trace sense.
2. The integral equality

$$\int_{G_{T,S}} (-u_ε \partial_t ϕ - a(u_ε) \partial_s ϕ - φ(u_ε) \cdot ∇_x ϕ) \cdot ∇_x u_ε + 2ε \partial_s u_ε ∂_s ϕ - Z_γ(x,t,s,u_ε)ϕ) \, dx \, dt \, ds = 0$$

holds for every $ϕ \in L^\infty(G_{T,S}) \cap W_2^1(Γ_{T,S})$.

(2.5)

**Proposition 2.2.** Whenever $u_0^{(1)}, u_0^{(2)}$ and $u_S^{(2)}$ satisfy the requirements (2.4) and (1.4), and Conditions on $a & φ & Z_γ$ hold, for any fixed $ε \in (0,1]$ there exists the unique weak solution $u_ε = u_ε(x,t,s)$ of Problem $Π_{γε}$ such that $u_ε \in W_2^{2,1}(G')$, where $G'$ is an arbitrary strictly interior subdomain of $G_{T,S}$ with a smooth boundary.

Moreover, the maximum principle (2.2) and the energy estimate (2.3) hold and $u_ε$ possesses the additional regularity property: $u_ε \in H^{α',2}(G_{T,S} \cup Ξ^2_S)$, where $Ξ^2_S$ is an arbitrary closed subset of $Ξ^2$ such that $Ξ^2 \cap \{t = 0\} = \emptyset$ and $∂ Ξ^2$ is smooth.

In the formulation of Proposition 2.2 exponent $α' \in (0,1)$ depends on $α$ and $ε$. By $W_2^{2,1}(G')$ we standardly denote the Sobolev space of measurable integrable functions $ϕ$ on $G'$ such that $\partial_i ϕ, \partial_{xi} ϕ, \partial_s ϕ, \partial^2_{xi} ϕ, \partial^2_{sxi} ϕ, \partial^2_{sx} ϕ, ϕ \in L^2(G'), i,j = 1,\ldots,d$.

**Proof of Proposition 2.2** follows from [28, Chapter 1, Theorem 2.9; Chapter 5, Theorem 1.1].

**Remark 2.2.** Clearly, a classical solution of Problem $Π_{γε}$ is a weak solution in the sense of Definition 2.1.

3 **Notions of kinetic and entropy solutions to Problem $Π_{γ}$**

In this section we set up notions of kinetic and entropy solutions to Problem $Π_{γ}$. To this end, let us introduce a few necessary concepts first.

Set

$$χ(λ; v) = \begin{cases} 1 & \text{for } 0 < λ < v, \\ -1 & \text{for } v < λ < 0, \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$
Definition 3.1. Let \( N \in \mathbb{N} \), \( L > 0 \), and \( \mathcal{O} \) be an open set in \( \mathbb{R}^N \). Let \( h \in L^\infty(\mathcal{O} \times (-L, L)) \) satisfy the inequality

\[
0 \leq h(z, \lambda) \text{sgn}(\lambda) \leq 1 \quad \text{for a.e.} \quad (z, \lambda) \in \mathbb{R}^{N+1}.
\]

We say that \( h \) is a \( \chi \)-function, if there exists a function \( v \in L^\infty(\mathcal{O}) \) such that

\[
h(z, \lambda) = \chi(\lambda; v(z)) \quad \text{for a.e.} \quad z \in \mathcal{O}.
\]

Lemma 3.1. ([44] Lemma 2.1.1.) The following two identities hold true (say, for \( \Psi \) locally Lipschitz continuous, i.e. \( \Psi' \in L^\infty_{\text{loc}}(\mathbb{R}) \)):

(i) \[
\int_{\mathbb{R}_L} \Psi'(\lambda) \chi(\lambda; v) \, d\lambda = \Psi(v) - \Psi(0), \quad \text{in particular,} \quad \int_{\mathbb{R}_L} \chi(\lambda; v) \, d\lambda = v;
\]

(ii) \[
\int_{\mathbb{R}_L} \Psi'(\lambda) |\chi(\lambda; v) - \chi(\lambda; w)| \, d\lambda = \text{sgn}(v - w)(\Psi(v) - \Psi(w)), \quad \text{in particular,} \quad \int_{\mathbb{R}_L} |\chi(\lambda; v) - \chi(\lambda; w)| \, d\lambda = |v - w|.
\]

Additionally,

(iii) \[|\chi(\lambda; v) - \chi(\lambda; w)| = |\chi(\lambda; v) - \chi(\lambda; w)|^2.\]

The following lemma establishes the link between sequences of \( \chi \)-functions and their limits.

Lemma 3.2. ([77]) Let \( \mathcal{O} \) be an open set of \( \mathbb{R}^N \) and \( h_n \in L^\infty(\mathcal{O} \times (-L, L)) \) be a sequence of \( \chi \)-functions converging weakly* to \( h \in L^\infty(\mathcal{O} \times (-L, L)) \). Set

\[
v_n(\cdot) = \int_{-L}^{L} h_n(\cdot, \lambda) \, d\lambda, \quad v(\cdot) = \int_{-L}^{L} h(\cdot, \lambda) \, d\lambda.
\]

Then the three assertions are equivalent to each other:

- \( h_n \) converges strongly to \( h \) in \( L^1_{\text{loc}}(\mathcal{O} \times (-L, L)) \),
- \( v_n \) converges strongly to \( v \) in \( L^1_{\text{loc}}(\mathcal{O}) \),
- \( h \) is a \( \chi \)-function.

In order to study topics regarding attainability of initial and final data, introduce the definition of one-sided essential limits.

Definition 3.2. Let \( \phi : [0, Y] \to \mathbb{R} \) be a measurable function, \( Y = \text{const} > 0 \). We say that \( K \in \mathbb{R} \) is the essential limit from the left of \( \phi \) at a point \( y_0 \in (0, Y) \) and write \( K = \text{ess lim}_{y \to y_0^-} \phi(y) \), if there exists a set \( E \subset (0, y_0) \) of full Lebesgue measure such that \( \lim_{y \to y_0^-} |\phi(y) - K| = 0 \).

Analogously, we say that \( K \in \mathbb{R} \) is the essential limit from the right of \( \phi \) at a point \( y_0 \in (0, Y) \) and write \( K = \text{ess lim}_{y \to y_0^+} \phi(y) \), if there exists a set \( E \subset (y_0, Y) \) of full Lebesgue measure such that \( \lim_{y \to y_0^+} |\phi(y) - K| = 0 \).
Now we are in a position to set up notions of kinetic and entropy solutions to Problem $\Pi_\gamma$.

**Definition 3.3.** Function $u \in L^\infty(G_{T,S}) \cap L^2((0,T) \times (0,S); \dot{W}_2^1(\Omega))$ is called a kinetic solution of Problem $\Pi_\gamma$, if it satisfies the kinetic equation

\[
\partial_t \chi(\lambda; u(x, t, s)) + a'(\lambda)\partial_u \chi(\lambda; u(x, t, s)) + \varphi'(\lambda) \cdot \nabla_x \chi(\lambda; u(x, t, s)) = Z_\gamma(x, t, s, \lambda) \partial_u \chi(\lambda; u(x, t, s)) = \\Delta_x \chi(\lambda; u(x, t, s)) + \delta_{(\lambda = 0)} Z_\gamma(x, t, s, \lambda) + \partial_u (m(x, t, s, \lambda) + n(x, t, s, \lambda)),
\]

\[(x, t, s, \lambda) \in G_{T,S} \times [-M_0, M_0], \quad (3.2a)\]

the kinetic initial condition

\[
\text{ess lim}_{t \to t_0^+} \int_{-M_0}^{M_0} \int_{\Xi^1} \left| \chi(\lambda; u(x, t, s)) - \chi(\lambda; u_0^{(1)}(x, s)) \right| \, dx \, ds \, d\lambda = 0, \quad (3.2b)
\]

the limiting relations

\[
\text{ess lim}_{s \to t_0^+} \int_{\Xi^2} \left| u(x, t, s) - u_0^{(2)}(x, t) \right| \, dx \, dt = 0, \quad (3.2c)
\]

\[
\text{ess lim}_{s \to S_0^-} \int_{\Xi^2} \left| u(x, t, s) - u_S^{(2)}(x, t) \right| \, dx \, dt = 0, \quad (3.2d)
\]

with some functions $u_0^{(2)}$, $u_S^{(2)} \in L^\infty(\Xi^2)$, and the kinetic boundary conditions

\[
a'(\lambda)\left( \chi(\lambda; u_0^{(2)}(x, t)) - \chi(\lambda; u_0^{(2)}(x, t)) \right) - \delta_{(\lambda = u_0^{(2)}(x, t))} \left( a(u_0^{(2)}(x, t)) - a(u_0^{(2)}(x, t)) \right) = \partial_u \mu_0^{(2)}(x, t, \lambda),
\]

\[(x, t, \lambda) \in \Xi^2 \times [-M_0, M_0], \quad (3.2e)\]

\[
a'(\lambda)\left( \chi(\lambda; u_S^{(2)}(x, t)) - \chi(\lambda; u_S^{(2)}(x, t)) \right) - \delta_{(\lambda = u_S^{(2)}(x, t))} \left( a(u_S^{(2)}(x, t)) - a(u_S^{(2)}(x, t)) \right) = -\partial_u \mu_S^{(2)}(x, t, \lambda),
\]

\[(x, t, \lambda) \in \Xi^2 \times [-M_0, M_0]. \quad (3.2f)\]

In (3.2a)–(3.2d) we have $m \in M^+(G_{T,S} \times [-M_0, M_0])$, $\mu_0^{(2)}$, $\mu_S^{(2)} \in M^+(\Xi^2 \times [-M_0, M_0])$, $n = \delta_{(\lambda = u)} |\nabla_x u|^2$, where $\delta_{(\lambda = u)}$ is the Dirac measure on $\mathbb{R}_\lambda$, concentrated at the point $\lambda = u(x, t, s)$, and $M_0$ is the constant defined in (2.2). Correspondingly, in (3.2a), $\delta_{(\lambda = 0)}$ is the Dirac measure on $\mathbb{R}_\lambda$ concentrated at the origin. By $M^+$ we denote the space of finite positive Radon measures. Functions $u_0^{(2)}$ and $u_S^{(2)}$ are $L^1(\Xi^2)$-strong traces of $u = u(x, t, s)$ on the planes $\{s = 0\}$ and $\{s = S\}$, respectively.

The kinetic equation (3.2a) and the kinetic boundary conditions (3.2c) and (3.2d) are understood in the sense of distributions: see further integral equality (4.20) in Section 4 and integral equality (6.21) in Section 6.

**Definition 3.4.** Function $u \in L^\infty(G_{T,S}) \cap L^2((0,T) \times (0,S); \dot{W}_2^1(\Omega))$ is called an entropy solution of Problem $\Pi_\gamma$, if it satisfies the entropy inequality

\[
\partial_t \eta(u) + \partial_u q_a(u) + \text{div}_x q_\nu(u) - Z_\gamma(x, t, s, u) \eta'(u) - \Delta_x \eta(u) \leq -\eta''(u)|\nabla_x u|^2, \quad (3.3a)
\]
the initial condition
\[
\text{ess lim}_{t \to 0^+} \int_{\Xi^t} \left| u(\mathbf{x}, t, s) - u_0^{(1)}(\mathbf{x}, s) \right| d\mathbf{x} ds = 0,
\]

the maximum principle
\[
\|u\|_{L^\infty(G_{t,s})} \leq M_0 < +\infty,
\] (3.3b)

the limiting relations (3.2c) and (3.2d) with some functions \(u_0^{tr,(2)}, u_S^{tr,(2)} \in L^\infty(\Xi^2)\), and the entropy boundary conditions
\[
q_0(u_0^{tr,(2)}(\mathbf{x}, t)) - q_0(u_0^{(2)}(\mathbf{x}, t)) - \eta'(u_0^{(2)}(\mathbf{x}, t))(a(u_0^{tr,(2)}(\mathbf{x}, t)) - a(u_0^{(2)}(\mathbf{x}, t))) \leq 0, \quad (\mathbf{x}, t) \in \Xi^2,
\] (3.3d)
\[
q_0(u_S^{tr,(2)}(\mathbf{x}, t)) - q_0(u_S^{(2)}(\mathbf{x}, t)) - \eta'(u_S^{(2)}(\mathbf{x}, t))(a(u_S^{tr,(2)}(\mathbf{x}, t)) - a(u_S^{(2)}(\mathbf{x}, t))) \geq 0, \quad (\mathbf{x}, t) \in \Xi^2.
\] (3.3e)

In (3.3a), (3.3d) and (3.3e), \(\eta \in C^2(\mathbb{R})\) is an arbitrary convex test-function: \(\eta'(z) \geq 0 \forall z \in \mathbb{R}\), and \((\eta, q_a, \varphi, q_\varphi)\) is a convex entropy flux triple:
\[
q_a'(z) = a'(z)\eta'(z), \quad q_\varphi'(z) = \varphi'(z)\eta'(z), \quad z \in \mathbb{R}.
\]

Entropy inequality (3.3a) is understood in the sense of distributions. Entropy boundary conditions (3.3d) and (3.3e) hold a.e. in \(\Xi^2\).

The following theorem on well-posedness of Problem \(\Pi_\gamma\) is the first main result of the article.

**Theorem 3.1.1. Existence, uniqueness and stability of kinetic solutions.** Under Conditions on \(u_0^{(1)}, u_0^{(2)}, u_0^{S,(2)}\) and Conditions on \(a, \varphi, Z, \gamma\), Problem \(\Pi_\gamma\) has the unique kinetic solution \(u = u(\mathbf{x}, t, s)\) in the sense of Definition 3.3.

Moreover, let \(u_1\) and \(u_2\) be two kinetic solutions corresponding to two given sets of data \((u_{1,0}^{(1)}, u_{1,0}^{(2)}, u_{1,S}^{(2)})\) and \((u_{2,0}^{(1)}, u_{2,0}^{(2)}, u_{2,S}^{(2)})\), respectively. Then the estimate
\[
\|u_1(\cdot, t, \cdot) - u_2(\cdot, t, \cdot)\|_{L^1(\Xi^1)} \leq e^{\mathcal{G}_\gamma(t)} \left[ \|u_{1,0}^{(1)} - u_{2,0}^{(1)}\|_{L^1(\Xi^1)} + \max_{\lambda \in [-M_1(t), M_1(t)]} a'(\lambda) \right] \int_0^t e^{-\mathcal{G}_\gamma(t')} \left( \|u_{1,0}^{(2)}(\cdot, t') - u_{2,0}^{(2)}(\cdot, t')\|_{L^1(\Omega)} + \|u_{1,S}^{(2)}(\cdot, t') - u_{2,S}^{(2)}(\cdot, t')\|_{L^1(\Omega)} \right) dt', \quad \forall t \in (0, T],
\] (3.4)

holds true, where
\[
\mathcal{G}_\gamma(t) = \int_0^t \max_{(\mathbf{x}, s, \lambda) \in \Xi^1 \times [-M_1(t), M_1(t)]} |\partial_\lambda Z_{\gamma}(\mathbf{x}, t', s, \lambda)| dt',
\] (3.5)

\[
M_1(t) = \max \left\{ M(t, \|u_{1,0}^{(1)}\|_{L^\infty(\Xi^1)}, \|u_{1,0}^{(2)}\|_{L^\infty(\Omega \times (0,t))}, \|u_{1,S}^{(2)}\|_{L^\infty(\Omega \times (0,t))}), M(t, \|u_{2,0}^{(1)}\|_{L^\infty(\Xi^1)}, \|u_{2,0}^{(2)}\|_{L^\infty(\Omega \times (0,t))}, \|u_{2,S}^{(2)}\|_{L^\infty(\Omega \times (0,t))}) \right\},
\] (3.6)

and \(M\) is defined in (2.2).

2. Existence, uniqueness and stability of entropy solutions. Under Conditions on
Moreover, let \( u_1 \) and \( u_2 \) be two entropy solutions corresponding to two given sets of data \((u_{1,0}, u_{1,0}, u_{1,1})\) and \((u_{2,0}, u_{2,0}, u_{2,1})\), respectively, then the estimate (3.2) holds true for them.

3. Equivalency of the notions of kinetic and entropy solutions. Function \( u = u(x,t,s) \) is an entropy solution of Problem \( \Pi_{\gamma} \) in the sense of Definition 3.4, if and only if it is a kinetic solution in the sense of Definition 3.3.

Estimate (3.2) manifests uniqueness and \( L^1 \)-strong stability (with respect to given initial and final data) of kinetic and entropy solutions.

The proof of assertion 1 of Theorem 3.1 is fulfilled further in Sections 4, 7, followed by the proofs of assertions 2 and 3 of Theorem 3.1 in Section 8.

4. Relative compactness of the family of classical solutions \( \{u_\varepsilon\}_{\varepsilon > 0} \) to Problem \( \Pi_{\gamma_\varepsilon} \) in \( L^1(G_{T,S}) \).

Derivation of the kinetic equation (3.2a)

Our first aim is to prove relative compactness of the family of weak solutions to Problem \( \Pi_{\gamma_\varepsilon} \). To this end, we use the Perthame — Souganidis and Lazar — Mitrović averaging compactness theorems [43, Theorem 6], [32, Theorem 7] (see also [44, Theorem 5.2.1]). We formulate the corollary of these theorems in the reduced form adapted for studying solutions of Problem \( \Pi_{\gamma_\varepsilon} \) as \( \varepsilon \to 0^+ \).

Proposition 4.1. (Corollary of the Perthame — Souganidis and Lazar — Mitrović Theorems.) Let functions \( f_n, f, g_{0,n}, k_n, k \in L^2(\mathbb{R}^d_t \times \mathbb{R}^+_s \times \mathbb{R}^+_\lambda) \) and Radon measures \( k_n, k \in \mathcal{M}(\mathbb{R}^d_x \times \mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}_\lambda) \) \( (n \in \mathbb{N}) \) satisfy the following relations:

(i) \( f_n \xrightarrow{_{n \to \infty}} f \) weakly in \( L^2(\mathbb{R}^d_t \times \mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}_\lambda) \);

(ii) \( g_{0,n} \xrightarrow{_{n \to \infty}} 0 \) strongly in \( L^2(\mathbb{R}^d_t \times \mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}_\lambda) \);

(iii) \( k_n \xrightarrow{_{n \to \infty}} k \) weakly* in \( \mathcal{M}(\mathbb{R}^d_x \times \mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}_\lambda) \), strongly in \( W^{-1,p'}_{\text{loc}}(\mathbb{R}^d_x \times \mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}_\lambda) \), \forall p' \in \left[1, \frac{d+3}{d+2}\right] ;

(iv) the triple \( \{f_n, g_{0,n}, k_n\} \) \( (n \in \mathbb{N}) \) resolves the kinetic equation

\[
\partial_t f_n + a'(\lambda)\partial_s f_n + \varphi'(\lambda) \cdot \nabla_x f_n - \Delta_x f_n = \partial_\lambda k_n + \partial_s \partial_\lambda g_{0,n}
\]

in the sense of distributions, where coefficients \( a, \varphi_1, \ldots, \varphi_d \) obey Conditions on \( a & \varphi & Z_\gamma \).

Then, for all compactly supported \( \psi \in L^2(\mathbb{R}_\lambda) \), the following limiting relation holds true:

\[
\int_{\mathbb{R}_\lambda} f_n(x,t,s,\lambda)\psi(\lambda) d\lambda \xrightarrow{_{n \to \infty}} \int_{\mathbb{R}_\lambda} f(x,t,s,\lambda)\psi(\lambda) d\lambda \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}^d_x \times \mathbb{R}^+_t \times \mathbb{R}^+_s).
\]

In item (iii) by \( W^{-1,p'}_{\text{loc}}(\mathbb{R}^d_x \times \mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}_\lambda) \) we denote the dual of the Sobolev space \( W_{p'}^{1}(\mathbb{R}^d_x \times \mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}_\lambda) \).

Now, using Propositions 2.2 and 1.1 and Lemmas 3.1 and 3.2 we establish the compactness result for \( \{u_\varepsilon\}_{\varepsilon > 0} \).
Lemma 4.1. The family of classical solutions \( \{u_\varepsilon\}_{\varepsilon > 0} \) of Problem \( \Pi_{\varepsilon} \) is relatively compact in \( L^1(G_{T,S}) \).

Proof. We justify Lemma 4.1 similarly to the claim of relative compactness of families of weak solution to the regularized problems outlined in [22, Section 3], [23, Section 3].

Recall that, due to Remark 2.2, the classical solution is the weak solution in the sense of Definition 2.1. In (2.5) take an arbitrary admissible test-function \( \phi \) vanishing in a neighborhood of the planes \( \{t = 0\}, \{t = T\}, \{s = 0\}, \{s = S\} \) and boundary \( \partial \Omega \), and integrate by parts with respect to \( t, s \) and \( x \) in the first three summands to get

\[
\int_{G_{T,S}} \left( \phi \partial_t u_\varepsilon + \phi \partial_s a(u_\varepsilon) + \phi \text{div}_x \phi(u_\varepsilon) + \nabla_x u_\varepsilon \cdot \nabla_x \phi + \varepsilon \partial_s u_\varepsilon \partial_s \phi - Z_\varepsilon(x, t, s, u_\varepsilon) \phi \right) \, dxdtds = 0. \tag{4.1}
\]

Taking \( \phi := \eta'(u_\varepsilon) \zeta \) with arbitrary \( \eta \in C^\infty(\mathbb{R}_\lambda) \) and \( \zeta \in C^\infty_0(G_{T,S}) \) in (4.1), using the chain rule and integration by parts in the first five summands, we arrive at the integral equality

\[
\int_{G_{T,S}} \left( - \eta(u_\varepsilon) \partial_t \zeta - q_0(u_\varepsilon) \partial_s \zeta - q_{\phi}(u_\varepsilon) \cdot \nabla_x \zeta - \eta(u_\varepsilon) \Delta_x \zeta + \varepsilon \partial_s u_\varepsilon \partial_s \zeta \\
+ \eta''(u_\varepsilon) (|\nabla_x u_\varepsilon|^2 + \varepsilon |\partial_s u_\varepsilon|^2) \zeta - Z_\varepsilon(x, t, s, u_\varepsilon) \eta'(u_\varepsilon) \zeta \right) \, dxdtds = 0. \tag{4.2}
\]

Using item (i) in Lemma 3.1, we rewrite (4.2) as the integral equality for the function \( \chi(\lambda; u_\varepsilon) \) having the form (3.1). More precisely, considering appropriate groups of terms in (4.2) one by one, we get

\[
\int_{G_{T,S}} \left( - \eta(u_\varepsilon) \partial_t \zeta - q_0(u_\varepsilon) \partial_s \zeta - q_{\phi}(u_\varepsilon) \cdot \nabla_x \zeta - \eta(u_\varepsilon) \Delta_x \zeta \right) \, dxdtds = \\
- \int_{G_{T,S}} \int_{\mathbb{R}_\lambda} \chi(\lambda; u_\varepsilon) \left( \partial_\lambda (\zeta \eta'(\lambda)) + a'(\lambda) \partial_\lambda (\zeta \eta'(\lambda)) + \varphi'(\lambda) \cdot \nabla_x (\zeta \eta'(\lambda)) + \Delta_x (\zeta \eta'(\lambda)) \right) \, d\lambda dxdtds, \tag{4.3}
\]

\[
\int_{G_{T,S}} \varepsilon \eta'(u_\varepsilon) \partial_s u_\varepsilon \partial_s \zeta \, dxdtds = \int_{G_{T,S}} \int_{\mathbb{R}_\lambda} \varepsilon (\chi(\lambda; u_\varepsilon) - 1_{(\lambda \geq 0)}) \partial_\lambda u_\varepsilon \partial_\lambda (\zeta \eta''(\lambda)) \, d\lambda dxdtds, \tag{4.4}
\]

\[
\int_{G_{T,S}} \eta''(u_\varepsilon) \left( |\nabla_x u_\varepsilon|^2 + \varepsilon |\partial_s u_\varepsilon|^2 \right) \zeta \, dxdtds = \left\langle \left( |\nabla_x u_\varepsilon|^2 + \varepsilon |\partial_s u_\varepsilon|^2 \right) \delta(\lambda = u_\varepsilon), \zeta \eta'' \right\rangle, \tag{4.5}
\]

\[- \int_{G_{T,S}} Z_\varepsilon(x, t, s, u_\varepsilon) \eta'(u_\varepsilon) \zeta \, dxdtds = \\
- \int_{G_{T,S}} \int_{\mathbb{R}_\lambda} Z_\varepsilon(x, t, s, u_\varepsilon) \left( \chi(\lambda; u_\varepsilon) - 1_{(\lambda \geq 0)} \right) \zeta \eta''(\lambda) \, dxdtds = \\
- \left\langle Z_\varepsilon(\cdot, \cdot, \cdot, u_\varepsilon), \chi(\cdot, u_\varepsilon) - 1_{(\lambda \geq 0)} \right\rangle \eta''. \tag{4.6}
\]

In (4.5), (4.6) and further in the paper, by \( \langle \cdot, \cdot \rangle \) we standardly denote the duality bracket between \( \mathcal{M}(\mathbb{R}_x^d \times \mathbb{R}_t^+ \times \mathbb{R}_s^+ \times \mathbb{R}_\lambda) \) and \( C_0(\mathbb{R}_x^d \times \mathbb{R}_t^+ \times \mathbb{R}_s^+ \times \mathbb{R}_\lambda) \).
Now, combining (4.2)–(4.6) we arrive at the desired integral equality:

\[- \int \int_{G_{T,S} \mathbb{R}_\lambda} \chi(\lambda; u_\varepsilon) \left( \partial_t (\zeta \eta'(\lambda)) + a'(\lambda) \partial_s (\zeta \eta'(\lambda)) + \varphi'(\lambda) \cdot \nabla_x (\zeta \eta'(\lambda)) + \Delta_x (\zeta \eta'(\lambda)) \right) d\lambda d\varepsilon dt ds + \int \int_{G_{T,S} \mathbb{R}_\lambda} g_\varepsilon(x, t, s, \lambda) \partial_s (\zeta \eta''(\lambda)) d\lambda d\varepsilon dt ds + \langle k_\varepsilon, \zeta \eta'' \rangle = 0, \quad (4.7)\]

where

\[g_\varepsilon(x, t, s, \lambda) := \varepsilon \left( \chi(\lambda; u_\varepsilon(x, t, s)) - \mathbf{1}_{\lambda > 0} \right) \partial_s u_\varepsilon(x, t, s),\]

\[k_\varepsilon(x, t, s, \lambda) := \left( (\nabla_x u_\varepsilon(x, t, s))^2 + \varepsilon |\partial_s u_\varepsilon(x, t, s)|^2 \right) \delta_{\lambda = u_\varepsilon(x, t, s)} - Z_\gamma(x, t, s, u_\varepsilon(x, t, s)) \left( \chi(\lambda; u_\varepsilon(x, t, s)) - \mathbf{1}_{\lambda > 0} \right), \quad k_\varepsilon \in M(\mathbb{R}_x^d \times \mathbb{R}_t^+ \times \mathbb{R}_s^+ \times \mathbb{R}_\lambda).\]

Notice that supp $\chi(\lambda; u_\varepsilon)$ lays in the layer $\{|\lambda| \leq M_0\}$, since the maximum principle (2.2) holds for $\{u_\varepsilon\}_{\varepsilon > 0}$. Therefore integration with respect to $\lambda$ in (4.7) is fulfilled, in fact, merely over $[-M_0, M_0]$. Since the linear span of the set $\{\zeta(x, t, s) \eta'(\lambda)\}$ is dense in $C^2_0(G_{T,S} \times \mathbb{R}_\lambda)$, equality (4.7) is equivalent to the kinetic equation

\[\partial_t \chi(\lambda; u_\varepsilon) + a'(\lambda) \partial_s \chi(\lambda; u_\varepsilon) + \varphi'(\lambda) \cdot \nabla_x \chi(\lambda; u_\varepsilon) - \Delta_x \chi(\lambda; u_\varepsilon) = \partial_\lambda k_\varepsilon - \partial_s \partial_\lambda g_\varepsilon\]

in the sense of distributions on $G_{T,S} \times [-M_0, M_0]$.

On the strength of the energy estimate (2.3), we have that

\[\|g_\varepsilon\|_{L^2(G_{T,S} \times [-M_0, M_0])} \leq \sqrt{2\varepsilon} C_1 \xrightarrow{\varepsilon \to 0^+} 0, \quad (4.8)\]

\[\|k_\varepsilon\|_{L^1_w(G_{T,S}; M[-M_0, M_0])} \leq C_1. \quad (4.9)\]

Here $L^1_w(G_{T,S}; M[-M_0, M_0])$ is the space of weakly measurable functions from $G_{T,S}$ into $M[-M_0, M_0]$ equipped with the norm

\[\|\nu\|_{L^1_w(G_{T,S}; M[-M_0, M_0])} = \int_{G_{T,S}} \|\nu(x, t, s, \cdot)\|_{M[-M_0, M_0]} d\lambda d\varepsilon dt ds.\]

Recall that the Dirac delta-function $\delta_{\lambda = u_\varepsilon(x, t, s)}$ (in the expression of $k_\varepsilon$) is a positive Radon measure having the unit mass, i.e., it is a probability measure. Clearly,

\[L^1_w(G_{T,S}; M[-M_0, M_0]) \hookrightarrow M(G_{T,S} \times [-M_0, M_0]).\]

Therefore there exist a subsequence $\{\varepsilon' \to 0^+\}$ and measure $k \in M(G_{T,S} \times [-M_0, M_0])$ such that

\[k_{\varepsilon'} \xrightarrow{\varepsilon' \to 0^+} k \text{ weakly* in } M(G_{T,S} \times [-M_0, M_0]). \quad (4.10)\]

On the strength of Sobolev’s embedding theorem [48, Chapter I, Section 8], the space of functionals $M(G_{T,S} \times [-M_0, M_0])$ is compactly embedded into $W^{-1, p'}_{\text{loc}}(G_{T,S} \times [-M_0, M_0])$ for any $p' \in [1, \frac{d+3}{d+2})$. Hence

\[k_{\varepsilon'} \xrightarrow{\varepsilon' \to 0^+} k \text{ strongly in } W^{-1, p'}_{\text{loc}}(G_{T,S} \times [-M_0, M_0]). \quad (4.11)\]
Finally, there exist a subsequence \( \{ \varepsilon'' \to 0^+ \} \) from \( \{ \varepsilon' \to 0^+ \} \) and \( f \in L^2(G_{T,S} \times [-M_0, M_0]) \) such that
\[
\chi(\lambda; u_{\varepsilon''}) \rightarrow f \text{ weakly in } L^2(G_{T,S} \times [-M_0, M_0]),
\]
and that \( f \) is the \( \chi \)-function:
\[
f(\mathbf{x}, t, s, \lambda) = \chi(\lambda; u(\mathbf{x}, t, s)).
\]
Lemma 4.1 is proved.

Furthermore, passing to the limit along the subsequence \( \varepsilon'' \to 0^+ \) in (4.17), we get
\[
- \int_{G_{T,S}} \int_{\mathbb{R}_\lambda} \chi(\lambda; u_{\varepsilon''}) \left( \partial_t (\zeta \eta'(\lambda)) + a'(\lambda) \partial_s (\zeta \eta'(\lambda)) + \varphi'(\lambda) \cdot \nabla_x (\zeta \eta'(\lambda)) + \Delta_x (\zeta \eta'(\lambda)) \right) d\lambda d\mathbf{x} dtds_{\varepsilon'' \to 0^+}
\]
due to (4.12) and (4.15),
\[
- \int_{G_{T,S}} \int_{\mathbb{R}_\lambda} g_{\varepsilon''}(\mathbf{x}, t, s, \lambda) \partial_s (\zeta \eta''(\lambda)) d\lambda d\mathbf{x} dtds_{\varepsilon'' \to 0^+} = 0
\]
due to (4.8), and
\[
\langle k_{\varepsilon''}, \zeta \eta'' \rangle \equiv \int_{G_{T,S}} \left( \eta''(u_{\varepsilon''}) (|\nabla_x u_{\varepsilon''}|^2 + \varepsilon'' |\nabla_s u_{\varepsilon''}|^2) - \eta'(u_{\varepsilon''}) \left( Z_\gamma (\mathbf{x}, t, s, u_{\varepsilon''}) \right) \right) \zeta d\mathbf{x} dtds_{\varepsilon'' \to 0^+}
\]
\[
\left\langle \lim_{\varepsilon'' \to 0^+} \delta_{(\lambda = u_{\varepsilon''})} \left( |\nabla_x u_{\varepsilon''}|^2 + \varepsilon'' |\nabla_s u_{\varepsilon''}|^2 \right), \eta''(\cdot) \zeta \right\rangle
\]
due to (4.5), (4.6), (4.12), (4.15) and item (i) in Lemma 3.1.
Denote
\[
n := \delta_{(\lambda = u)} |\nabla_x u|^2, \quad m := \lim_{\varepsilon'' \to 0^+} \left( \delta_{(\lambda = u_{\varepsilon''})} |\nabla_x u_{\varepsilon''}|^2 - \delta_{(\lambda = u)} |\nabla_x u|^2 + \delta_{(\lambda = u_{\varepsilon''})} \varepsilon'' |\nabla_s u_{\varepsilon''}|^2 \right).
\]
Notice that \( m, n \in \mathcal{M}^+(G_{T,S} \times [-M_0, M_0]) \).

Using (4.16)–(4.19), taking into account that integration in \( \lambda \) is fulfilled merely over \([-M_0, M_0]\) and that the linear span of the set \( \{ \zeta(x, t, s)\eta' philosophy \} \) is dense in \( C_0^2(G_{T,S} \times \mathbb{R}_\lambda) \), from (5.1) we derive the integral equality

\[
\int \int_{G_{T,S} - M_0} \chi(\lambda; u) \left( \partial_t \Phi + a'(\lambda) \partial_s \Phi + \varphi'(\lambda) \cdot \nabla_x \Phi + \Delta_x \Phi + \partial_t(\zeta(x, t, s, \lambda) \Phi) \right) d\lambda dx dt ds + \\
\langle \delta(\lambda=0)Z_\lambda(x, t, s, \cdot), \Phi \rangle = \langle m + n, \partial_\lambda \Phi \rangle \quad \forall \Phi \in C_0^2(G_{T,S} \times \mathbb{R}_\lambda). \quad (4.20)
\]

In the sense of distributions, (4.20) is equivalent to the kinetic equation (3.2a).

Thus, we have proved the following lemma.

**Lemma 4.2.** The limit \( \lim u \in L^\infty(G_{T,S} \times [-M_0, M_0]) \) of the subsequence \( \{ u_{\varepsilon''} \}_{\varepsilon'' \to 0^+} \) of classical solutions to Problem \( \Pi_{\gamma, \varepsilon} \), along with measures \( m, n \in \mathcal{M}^+(G_{T,S} \times [-M_0, M_0]) \), resolves the kinetic equation (3.2a) in the sense of distributions.

### 5 Traces of solutions of the kinetic equation (3.2a)

In this section, we establish existence of \( L^1 \)-strong one-sided traces \( u_0^{\text{tr},(1)}, u_{\tau-0}^{\text{tr},(1)}, u_{\tau+0}^{\text{tr},(1)}, u_0^{\text{tr},(2)}, u_{\tau-0}^{\text{tr},(2)}, u_{\tau+0}^{\text{tr},(2)} \) of solutions of the kinetic equation (3.2a) on the sections \( \Gamma_{0+}^1 = \mathcal{G} \times \{ t = 0^+ \} \times [0, S], \Gamma_{\tau-0}^1 = \mathcal{G} \times \{ t = \tau - 0 \} \times [0, S], \Gamma_{\tau+0}^1 = \mathcal{G} \times \{ t = \tau + 0 \} \times [0, S], \Gamma_{0+}^2 = \mathcal{G} \times [0, T] \times \{ s = 0^+ \} \), and \( \Gamma_{\tau-0}^2 = \mathcal{G} \times [0, T] \times \{ s = S - 0 \} \), respectively. Here \( \tau \in (0, T) \) is an arbitrarily fixed value.

At first, we prove that there exists the trace on \( \Gamma_{0+}^1 \). To this end, we use the theory elaborated in [1].

**Definition 5.1.** ([1] Definition 1.) We say that function \( f \in L^\infty(\mathbb{R}^d_x \times \mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}_\lambda) \) admits an averaged trace, if there exist function \( f_0 \in L^\infty(\mathbb{R}^d_x \times \mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}_\lambda) \) and set \( E \subset \mathbb{R}^+_t \) of full Lebesgue measure such that

\[
\lim_{t \to 0^+} \int_{E} \left| \int_{\mathbb{R}_\lambda} \left( f(x, t, s, \lambda) - f_0(x, s, \lambda) \right) \rho(\lambda) d\lambda \right| dx ds = 0 \quad (5.1)
\]

for any function \( \rho \in C_0^1(\mathbb{R}_\lambda) \) and any relatively compact set \( E \subset \mathbb{R}^d_x \times \mathbb{R}^+_s \).

**Remark 5.1.** In view of Definition 5.1, relation (5.1) can be equivalently written as follows:

\[
\text{ess lim}_{t \to 0^+} \int_{E} \left| \int_{\mathbb{R}_\lambda} \left( f(x, t, s, \lambda) - f_0(x, s, \lambda) \right) \rho(\lambda) d\lambda \right| dx ds = 0.
\]

**Proposition 5.1.** (Corollary of the Aleksić — Mitrović Strong Trace Existence Theorem [11] Theorem 4.) Let function \( f \in L^\infty(\mathbb{R}^d_x \times \mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}_\lambda) \) and locally finite Borel measure \( k \in \mathcal{M}(\mathbb{R}^d_x \times \mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}_\lambda) \) resolve the kinetic equation

\[
\partial_t f + a'(\lambda) \partial_s f + \varphi'(\lambda) \cdot \nabla_x f - \Delta_x f = \partial_\lambda k \quad (5.2)
\]

in the sense of distributions, where the coefficients \( a, \varphi_1, \ldots, \varphi_d \) obey Conditions on \( a \& \varphi \& Z \).

Additionally, let

\[
\text{supp} f \text{ lay in the layer } \{|\lambda| \leq M\}, \text{ with some } M = \text{const} > 0, \quad (5.3)
\]
and

\[ \partial_{x_i} f \in L^2_\omega(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}_s^+; \mathcal{M}(\mathbb{R})), \quad i = 1, \ldots, d. \]

Then there exists an average trace \( f_0 \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}_s) \) of function \( f \) on \( \mathbb{R}^d \times \{ t = 0+ \} \times \mathbb{R}_s^+ \times \mathbb{R}_\lambda \) in the sense of Definition 5.1.

**Remark 5.2.** It is worth to notice that the genuine nonlinearity condition (1.6) is excessive in Proposition 5.1. (See [1, Proof of Theorem 4].) Also remark that, in [1], the space \( L^2_\omega(\mathbb{R}^N; \mathcal{M}(\mathbb{R})) \) is denoted by \( L^2(\mathbb{R}^N; \mathcal{M}(\mathbb{R})) \).

Now we establish the following lemma.

**Lemma 5.1.** Let triple \( u \in L^\infty(G_{T,S}) \cap L^2((0, T) \times (0, S); \dot{W}^1_2(\Omega)), m \in \mathcal{M}^+(G_{T,S} \times [-M_0, M_0]) \) and \( n = \delta_{(\lambda=0)}|\nabla_x u|^2 \) resolve the kinetic equation (3.2a) in the sense of the integral equality (4.20). (As the matter of fact, we consider the case when \( u = \lim_{\varepsilon'' \to 0+} u_{\varepsilon''} \).

Then there exists \( u^{tr,(1)}_0 \in L^\infty(\Gamma^1), \) the trace of \( u \) on \( \Gamma^1_{0+} \), such that

\[ \text{ess lim}_{t \to 0+} \int_{Z^1} |u(x, t, s) - u^{tr,(1)}_0(x, s)| \, dx \, ds = 0, \quad (5.4) \]

\[ \text{ess lim}_{t \to 0+} \int_{Z^1} \int_{0-M_0}^{M_0} \left| \chi(\lambda; u(x, t, s)) - \chi(\lambda; u^{tr,(1)}_0(x, s)) \right| \, d\lambda \, dx \, ds = 0. \quad (5.5) \]

**Proof.** In (4.20) rewrite the last two terms in the left-hand side as follows:

\[
\int_{G_{T,S}} \int_{-M_0}^{M_0} \chi(\lambda; u) \partial_\lambda (Z_{\gamma}(x, t, s, \lambda) \Phi(x, t, s, \lambda))d\lambda \, dx \, dt \, ds + \langle \delta_{(\lambda=0)} Z_{\gamma}(x, t, s, \cdot), \Phi(x, t, s, \cdot) \rangle =
\]

\[
\int_{G_{T,S}} Z_{\gamma}(x, t, s, u(x, t, s)) \Phi(x, t, s, u(x, t, s)) \, dx \, dt \, ds =
\]

\[
- \int_{G_{T,S}} \left( \int_{-M_0}^{M_0} 1_{(\lambda \equiv u(x, t, s))} \partial_\lambda (Z_{\gamma}(x, t, s, \lambda) \Phi(x, t, s, \lambda))d\lambda \right) \, dx \, dt \, ds =
\]

\[
- \int_{G_{T,S}} \int_{-M_0}^{M_0} \left( 1_{(\lambda \equiv u(x, t, s))} Z_{\gamma}(x, t, s, \lambda) -
\int_{\lambda}^{\lambda'} 1_{(\lambda' \equiv u(x, t, s))} \partial_{\lambda'} Z_{\gamma}(x, t, s, \lambda') \, d\lambda' \right) \partial_\lambda \Phi(x, t, s, \lambda) \, d\lambda \, dx \, dt \, ds,
\]

\[ \forall \Phi \in C^0(G_{T,S} \times [-M_0, M_0]). \quad (5.6) \]

In this chain of equalities, at first, we use assertion (i) in Lemma 3.1, at second, we use the evident identity

\[ \phi(u(x, t, s)) = - \int_{\mathbb{R}} 1_{(\lambda \equiv u(x, t, s))} \phi'(\lambda) d\lambda \quad \forall \phi \in C^1(\mathbb{R}), \]

and, finally, we integrate by parts in \( \lambda \).
Substituting (5.6) into (4.20) we obtain the integral equality equivalent to (4.20). Consequently, the kinetic equation (3.2a) takes the following equivalent form:

\[ \partial_t \chi(\lambda; u(x, t, s)) + a'(-\lambda) \partial_s \chi(\lambda; u(x, t, s)) + \varphi'(\lambda) \cdot \nabla_x \chi(\lambda; u(x, t, s)) - \Delta_x \chi(\lambda; u(x, t, s)) = \partial_\lambda \left( m(x, t, s, \lambda) + n(x, t, s, \lambda) + 1_{(\lambda \geq u(x,t,s))} \omega_{\gamma}^1(x,t,s,\lambda) - \int_0^\lambda 1_{(\lambda' \geq u(x,t,s))} \partial_{\lambda'} \omega_{\gamma}^1(x,t,s,\lambda') \, d\lambda' \right). \]

(5.7)

Hence we conclude that the kinetic equation (3.2a) is the kinetic equation of the form (5.2) with \( f = \chi(\lambda; u) \) and

\[ k := k_* \equiv m + n + 1_{(\lambda \geq u)} \omega_{\gamma}^1 - \int_0^\lambda 1_{(\lambda' \geq u)} \partial_{\lambda'} \omega_{\gamma}^1 \, d\lambda'. \]

(5.8)

Clearly, \( \chi \) is bounded and satisfies (5.3), and \( k_* \) is a locally finite Borel measure belonging to \( \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}_+ \times \mathbb{R}_\lambda) \). Furthermore, using assertion (i) in Lemma 3.1 and Green’s theorem [35, Lemma 2.20], we establish the representation

\[ \langle \partial_{\lambda} \chi, \psi \rangle = - \int_{G_{T,S} - M_0} \int_{M_0} \chi(\lambda; u(x, t, s)) \partial_{\lambda} \psi(x, t, s, \lambda) \, d\lambda \, dx \, dt \, ds = \]

\[ \int_{G_{T,S}} \left[ (\partial_x \psi(x, t, s, \lambda)) \big|_{\lambda = u(x,t,s)} - \partial_x \psi(x, t, s, 0) \right] \, dx \, dt \, ds = \]

\[ \int_{G_{T,S}} \left[ \partial_x \psi(x, t, s, u) - \psi(x, t, s, u) \right] \partial_x u - \partial_x \psi(x, t, s, 0) \right] \, dx \, dt \, ds = \]

\[ - \int_{G_{T,S}} \psi(x, t, s, u) \partial_x u \, dx \, dt \, ds + \int_{G_{T,S}} (\Psi(\sigma, t, s, u(\sigma, t, s)) - \Psi(\sigma, t, s, 0)) \, n_i(\sigma) \, d\sigma \, dt \, ds = \]

\[ - \int_{G_{T,S}} \psi(x, t, s, u) \partial_x u \, dx \, dt \, ds, \quad \forall i = 1, \ldots, d, \quad \forall \psi \in C^1_c(G_{T,S}; C_0(\mathbb{R}_\lambda)), \]

(5.9)

where \( \Psi \) is the primitive of \( \psi \) with respect to \( \lambda \):

\[ \partial_\lambda \Psi(x, t, s, \lambda) = \psi(x, t, s, \lambda), \]

\( n_i \) is the \( i \)-th component of the unit outward normal to \( \partial \Omega \), \( d\sigma \) is an infinitesimal element of \( \partial \Omega \), and \( \partial_x \Psi(x, t, s, u) \) is the full derivative of \( \Psi \) with respect to \( x_i \), i.e.,

\[ \partial_{x_i} \Psi(x, t, s, u) = (\partial_x \Psi(x, t, s, \lambda)) \big|_{\lambda = u(x,t,s)} + \psi(x, t, s, u) \partial_x u. \]

The last equality in (5.9) is valid due to the property of finiteness of \( \psi \). More certainly, we have

\[ \Psi(\sigma, t, s, u(\sigma, t, s)) = \Psi(\sigma, t, s, 0) = 0 \quad \text{for} \ \sigma \in \partial \Omega. \]

Now using the Cauchy-Buniakowskii inequality, from (5.9) we derive the bound

\[ \left| \langle \partial_{\lambda} \chi, \psi \rangle \right| \leq \left| - \int_{G_{T,S}} \psi(x, t, s, u) \partial_x u \, dx \, dt \, ds \right| \leq C_u \| \psi \|_{L^2(G_{T,S}; C_0(\mathbb{R}_\lambda))} \| \partial_x u \|_{L^2(G_{T,S}; C_0(\mathbb{R}_\lambda))}, \quad \forall \psi \in C^1_c(G_{T,S}; C_0(\mathbb{R}_\lambda)). \]
Since \( C_0^1(G_{T,S}; C_0(\mathbb{R}^\lambda)) \) is dense in \( L^2(G_{T,S}; C_0(\mathbb{R}^\lambda)) \), this bound yields that
\[
\|\partial_x \chi\|_{L^2_v(G_{T,S}; M(\mathbb{R}^\lambda))} \leq C_u.
\]
Thus, all assumptions of Proposition \([5.1]\) hold for \( f = \chi \) and \( k = k_* \) (see \([5.8]\)). Therefore, the limiting relation \([5.1]\) is valid for \( \chi \), i.e., there exists \( f_0 \in L^\infty(G_{T,S} \times [-M_0, M_0]) \) such that
\[
\text{ess lim}_{t \to 0^+} \int_{\Xi^1} \left| \int_{-M_0}^{M_0} (\chi(\lambda; u(x, t, s)) - f_0(x, s, \lambda)) \rho(\lambda) \, d\lambda \right| \, dxd\sigma = 0, \quad \forall \rho \in C_0^1(\mathbb{R}^\lambda). \tag{5.10}
\]
On the strength of assertion (i) in Lemma \([3.1]\) taking \( \rho \equiv 1 \) on \([-M_0, M_0] \), from \((5.10)\) we immediately derive \((5.4)\) with
\[
u_0^{(1)}(x, s) := \int_{-M_0}^{M_0} f_0(x, s, \lambda) \, d\lambda. \tag{5.11}
\]
In turn, on the strength of assertion (ii) in Lemma \([3.1]\) from \((5.4)\) we deduce the limiting relation \((5.5)\). Lemma \([5.1]\) is proved. \(\square\)

**Remark 5.3.** In view of Lemma \([7.2]\) relations \((5.4), (5.5), (5.10), \) and \((5.11)\) yield that
\[
f_0(x, s, \lambda) = \chi(\lambda; \nu_0^{(1)}(x, s)).
\]

Next, we establish existence of traces on \( \Gamma^1_{\tau-0} \) and \( \Gamma^1_{\tau+0} \) for any fixed \( \tau \in (0, T) \).

**Lemma 5.2.** In the assumptions of Lemma \([5.1]\) for any fixed \( \tau \in (0, T) \) there exist \( u^{(1)}_{\tau-0} \), \( u^{(1)}_{\tau+0} \in L^\infty(\Xi^1) \) such that
\[
\text{ess lim}_{t \to \tau \pm 0} \int_{\Xi^1} \left| u(x, t, s) - u^{(1)}_{\tau+0}(x, s) \right| \, dxd\sigma = 0,
\]
\[
\text{ess lim}_{t \to \tau \pm 0} \int_{\Xi^1} \left| \chi(\lambda; u(x, t, s)) - \chi(\lambda; u^{(1)}_{\tau+0}(x, s)) \right| \, d\lambda dxd\sigma = 0.
\]

**Proof.** This lemma is quite analogous to justification of Lemma \([5.1]\). It is based on the natural modification of Definition \([5.1]\) and Proposition \([5.1]\) which consists in substitution of \( t = 0^+ \) by \( t = \tau-0 \) and \( t = \tau+0 \) in the formulations and relative proofs. After this substitution, we only need to thoroughly repeat track of the proof of Lemma \([5.1]\). \(\square\)

Existence of strong traces of \( u = \lim_{\varepsilon \to 0} u_{\varepsilon} \nu \) on \( \Gamma^1_{0+0} \) and \( \Gamma^2_{S-0} \) immediately follows from the result on existence of traces from \([23]\) Section 3.3.1, which, in turn, relies on the techniques initially introduced in \([28, 39, 40, 55]\), and then developed in \([21] \) Lemmas 2 and 4]. Thus we establish the following lemma.

**Lemma 5.3.** Let triple \( u \in L^\infty(G_{T,S}) \cap L^2((0, T) \times (0, S); \hat{W}^1_2(\Omega)) \), \( m \in \mathcal{M}^+(G_{T,S} \times [-M_0, M_0]) \) and \( n = \delta(\lambda = 0) |\nabla_x u|^2 \) resolve the kinetic equation \((3.2a)\) in the sense of the integral equality \((4.20)\). Let coefficients \( a, \varphi_1, \ldots, \varphi_d \) and function \( Z_\gamma \) satisfy Conditions on \( a \& \varphi \& Z_\gamma \).

Then there exist \( u^{(2)}_{0+0} \), \( u^{(2)}_{S-0} \in L^\infty(\Xi^2) \), the traces of \( u \) on \( \Gamma^2_{0+0} \) and \( \Gamma^2_{S-0} \), satisfying the limiting relations \((3.2a)\) and \((3.2b)\), respectively.

**Remark 5.4.** The demand \((1.6)\) of genuine nonlinearity is necessary in assumptions of Lemma \([5.3]\) and it cannot be discarded.
6 Kinetic initial and final conditions

Lemma 6.1. Let $u$ resolve the kinetic equation (3.2a) (along with measures $m, n \in M^+(G_{T,S} \times [-M_0, M_0])$) and be a strong limiting point of the family $\{u_\varepsilon\}_{\varepsilon > 0}$ of solutions to Problem $\Pi_{\gamma\varepsilon}$: $u = s_{\varepsilon \to 0} u_\varepsilon \varepsilon$. Then the kinetic initial condition (3.2b) holds true for $u$.

Proof. From the integral equality (2.5), the maximum principle (2.2) and the energy estimate Lemma 6.2, then apply the following lemma.

On the other hand, due to (2.2), values of the mappings $t \mapsto u_\varepsilon$ belong to the set $\{u \in W_2(\Xi)\}$. Hence, by the Rellich theorem, the set $\{u_\varepsilon\}_{\varepsilon \in (0,1]}$ is relatively compact in $C(0,T; W^{-1,2}(\Xi))$. Hence,

$$u_\varepsilon(\cdot, t, \cdot) \to u_0$$

strongly in $W^{-1,2}(\Xi)$ uniformly in $\varepsilon \in (0,1]$.

(6.1)

On the other hand, due to (2.2), values of the mappings $t \mapsto u_\varepsilon$ belong to the set

$$\{\phi \in L^2(\Xi): \text{ess sup}_{(x,s) \in \Xi} |\phi(x,s)| \leq M_0\},$$

which is a compact subset in $W^{-1,2}(\Xi)$ by the Rellich theorem. Therefore, by the Arcel theorem, the set $\{u_\varepsilon\}_{\varepsilon \in (0,1]}$ is relatively compact in $C(0,T; W^{-1,2}(\Xi))$. Hence,

$$u_{\varepsilon''}(\cdot, t, \cdot) \to u(\cdot, t, \cdot)$$

strongly in $W^{-1,2}(\Xi)$ uniformly on the segment $[0,T]$.

(6.2)

(Recall that $\{\varepsilon'' \to 0\}$ is the subsequence extracted in the proof of Lemma 4.1) Next, from (5.4) it immediately follows that

$$u(\cdot, t, \cdot) \to u_0^{tr,(1)}$$

strongly in $W^{-1,2}(\Xi)$.

(6.3)

From (6.1)–(6.3), by the triangle inequality we deduce that $u_0^{tr,(1)} = u_0^{(1)}$. Now, inserting $u_0^{(1)}$ on the place of $u_0^{tr,(1)}$ in (5.5), we conclude the proof of the lemma.

In order to derive the kinetic initial and final conditions (3.2c) and (3.2d), we will prove and then apply the following lemma.

Lemma 6.2. Under assumptions of Lemma 6.1 the limiting relations

$$\lim_{\varepsilon'' \to 0+} \int_{\Xi^2} \kappa(x,t)(-\varepsilon'' \partial u u_\varepsilon''(x,t,0)) \, dx \, dt = \int_{\Xi^2} \kappa(x,t) \left( a(u_0^{tr,(2)}(x,t)) - a(u_0^{(2)}(x,t)) \right) \, dx \, dt,$$

(6.4)

$$\lim_{\varepsilon'' \to 0+} \int_{\Xi^2} \kappa(x,t)(-\varepsilon'' \partial u u_\varepsilon''(x,t,S)) \, dx \, dt = \int_{\Xi^2} \kappa(x,t) \left( a(u_S^{tr,(2)}(x,t)) - a(u_S^{(2)}(x,t)) \right) \, dx \, dt$$

(6.5)

hold true for all test-functions $\kappa \in C^1_0(\Xi^2)$. 
Proof. At first, notice that the weak traces of $u_\varepsilon$ and $\partial_s u_\varepsilon$ are well-defined on $\Gamma^2_\tilde{s} = \tilde{\Omega} \times [0, T] \times \{s = \tilde{s}\}$ for all $\tilde{s} \in [0, S]$ and that the integral equality (2.3) is equivalent to the integral equality
\[
\int_{s'}^{s''} \int_{\Xi^2} \left( -u_\varepsilon \partial_t \phi - a(u_\varepsilon) \partial_s \phi - \varphi(u_\varepsilon) \cdot \nabla_x \phi + \nabla_x u_\varepsilon \cdot \nabla_x \phi + \varepsilon \partial_s u_\varepsilon \partial_s \phi - Z_\gamma(x, t, s, u_\varepsilon) \phi \right) \, dx \, dt \, ds
\]
\[+ \int_{\Xi^2} a(u_\varepsilon(x, t, s'')) \phi(x, t, s'') \, dx \, dt - \int_{\Xi^2} a(u_\varepsilon(x, t, s')) \phi(x, t, s') \, dx \, dt
\]
\[+ \int_{\Xi^2} (-\varepsilon \partial_s u_\varepsilon(x, t, s'')) \phi(x, t, s'') \, dx \, dt - \int_{\Xi^2} (-\varepsilon \partial_s u_\varepsilon(x, t, s')) \phi(x, t, s') \, dx \, dt = 0 \quad (6.6)
\]
for all $s', s'' \in [0, S]$ ($s' \leq s''$) and for all test-functions $\phi \in L^\infty(G_{T,S}) \cap W^1_2(G_{T,S})$ vanishing in the neighborhood of the planes $\{t = 0\}$, $\{t = T\}$ and the boundary $\partial \Omega$.

Existence of weak traces of $u_\varepsilon$ and $\partial_s u_\varepsilon$ on $\Gamma^2_\tilde{s}$ can be proved in a quite similar way as in [26, Proposition A.1]. Proof of the equivalence of (6.6) to (2.5) is quite similar to the proof in [3] Chapter 3, Section 1.1]. Therefore, we skip these proofs in this article.

Like in [20] and [10], let us introduce a family of functions $\{\rho^0_\delta : \mathbb{R}^+ \mapsto [0, 1]\}_{\delta > 0}$ such that
\[
\{\rho^0_\delta \} \subset C^2(\mathbb{R}^+), \quad \rho^0_\delta(s) = 0, \quad \forall s > \delta, \quad \rho^0_\delta(0) = 1, \quad |(\rho^0_\delta)'(\cdot)| \leq \frac{c}{\delta}
\]
with a constant $c > 0$ independent of $\delta$, (6.7)
and
\[
\lim_{\delta \to 0+} \int_0^\delta \Phi(s)(\rho^0_\delta)'(s) \, ds = -\Phi(0+) \quad (6.8)
\]
for any integrable in a neighborhood of $\{s = 0\} \subset \mathbb{R}^+$ function $\Phi$ having the trace on $s = 0$ from the right, $\Phi(0+) = \lim_{s \to 0+} \Phi(s)$.

In (6.6) fix $\delta > 0$, $s' = 0$, $s'' > \delta$ and take $\phi(x, t, s) = \kappa(x, t) \rho^0_\delta(s)$, where $\kappa \in C^1_0(\Xi^2)$ is an arbitrary test-function. On the strength of Proposition 2.2, Lemma 4.1 and properties (6.7), passing to the limit as $\varepsilon := \varepsilon'' \to 0+$, from (6.6) we derive the relation
\[
\int_0^\delta \rho^0_\delta(s) \int_{\Xi^2} \left( -\varepsilon \partial_s \kappa - \varphi(u) \cdot \nabla_x \kappa + \nabla_x u \cdot \nabla_x \kappa - Z_\gamma(x, t, s, u) \kappa \right) \, dx \, dt \, ds
\]
\[+ \int_0^\delta (\rho^0_\delta)'(s) \int_{\Xi^2} a(u) \kappa \, dx \, dt \, ds - \int_{\Xi^2} a(u^{(2)}_0) \kappa \, dx \, dt
\]
\[\lim_{\varepsilon'' \to 0+} \int_{\Xi^2} (-\varepsilon'' \partial_s u_{\varepsilon''}(x, t, 0)) \kappa(x, t) \, dx \, dt = 0. \quad (6.9)
\]

Further, (6.4) directly follows from (6.9) as $\delta \to 0+$, thanks to Lemma 5.3 and property (6.8). And, finally, (6.5) is deduced similarly by means of the test-function $\rho^0_\delta(s) = \rho^0_\delta(S - s)$, choosing $s' < S - \delta$ and $s'' = S$ in (6.6). This observation completes the proof. \qed

With the help of Lemma 6.2 we establish the following result.

Lemma 6.3. Under assumptions of Lemma 6.1 kinetic initial and final conditions (3.2c) and (3.2d) are valid.
Proof. In (6.6) fix \( s' = 0, s'' > \delta \), where \( \delta \) is an arbitrary rather small positive constant, and take the test-function

\[
\phi(x, t, s) := \eta'(u_\varepsilon(x, t, s)) \Theta(x, t) \rho_0^0(s),
\]

where \( \eta \in C^2(\mathbb{R}) \) is convex, \( \Theta \in C^1_c(\mathbb{R}^2) \) is nonnegative, and \( \rho_0^0 \) is the same as in the proof of Lemma 6.2. Due to sufficient regularity of \( u_\varepsilon \), such choice of \( \phi \) is legitime. Using integration by parts and the chain rule, from (6.6) we deduce the integral equality

\[
\int_0^\delta \int_{\mathbb{R}^2} \left( -\eta(u_\varepsilon) \rho_0^0 \partial_t \Theta - q_a(u_\varepsilon)(\rho_0^0)' \Theta - \rho_0^0 \mathbf{q}_\varphi(u_\varepsilon) \cdot \nabla_x \Theta - Z_\gamma(x, t, s, u_\varepsilon) \eta'(u_\varepsilon) \Theta \rho_0^0 + \eta''(u_\varepsilon) |\nabla_x u_\varepsilon|^2 \Theta \rho_0^0 + \rho_0^0 \nabla_x \eta(u_\varepsilon) \cdot \nabla_x \Theta + \varepsilon \eta''(u_\varepsilon) |\partial_s u_\varepsilon|^2 \Theta \rho_0^0 + \varepsilon (\rho_0^0)' \partial_s \eta(u_\varepsilon) \Theta \right) dxdtds - \int_{\mathbb{R}^2} q_a(u_0^{(2)}) \Theta dxdt - \int( -\varepsilon \partial_s u_\varepsilon(x, t, 0)) \eta'(u_0^{(2)}) \Theta dxdt = 0. \tag{6.10}
\]

Remark that

\[
\eta''(u_\varepsilon) \left( |\nabla_x u_\varepsilon|^2 + \varepsilon |\partial_s u_\varepsilon|^2 \right) \Theta \rho_0^0 \geq 0 \quad \text{in} \quad \mathbb{R}^2 \times (0, \delta), \tag{6.11}
\]

due to the choice of \( \eta, \Theta \) and \( \rho_0^0 \), and that

\[
\lim_{\varepsilon'' \to 0} \int_0^\delta \int_{\mathbb{R}^2} \left( -\eta(u_{\varepsilon''}) \rho_0^0 \partial_t \Theta - q_a(u_{\varepsilon''})(\rho_0^0)' \Theta - \rho_0^0 \mathbf{q}_\varphi(u_{\varepsilon''}) \cdot \nabla_x \Theta - Z_\gamma(x, t, s, u_{\varepsilon''}) \eta'(u_{\varepsilon''}) \Theta \rho_0^0 + \rho_0^0 \nabla_x \eta(u_{\varepsilon''}) \cdot \nabla_x \Theta + \varepsilon'' |\partial_s u_{\varepsilon''}|^2 \Theta \rho_0^0 \right) dxdtds - \int_0^\delta \int_{\mathbb{R}^2} \left( -\eta(u) \rho_0^0 \partial_t \Theta - q_a(u)(\rho_0^0)' \Theta - \rho_0^0 \mathbf{q}_\varphi(u) \cdot \nabla_x \Theta - Z_\gamma(x, t, s, u) \eta'(u) \Theta \rho_0^0 + \rho_0^0 \nabla_x \eta(u) \cdot \nabla_x \Theta \right) dxdtds \tag{6.12}
\]
due to Lemma 4.1 and the structure of function \( \rho_0^0 \).

On the strength of (6.11) and (6.12), from (6.10) we derive the inequality

\[
\lim_{\varepsilon'' \to 0} \int_{\mathbb{R}^2} (-\varepsilon'' \partial_s u_{\varepsilon''}(x, t, 0)) \eta'(u_0^{(2)}) \Theta dxdt - \int_0^\delta \int_{\mathbb{R}^2} \left( -\eta(u) \rho_0^0 \partial_t \Theta - q_a(u)(\rho_0^0)' \Theta - \rho_0^0 \mathbf{q}_\varphi(u) \cdot \nabla_x \Theta - Z_\gamma(x, t, s, u) \eta'(u) \Theta \rho_0^0 + \rho_0^0 \nabla_x \eta(u) \cdot \nabla_x \Theta \right) dxdtds + \int_{\mathbb{R}^2} q_a(u_0^{(2)}) \Theta dxdt \geq 0, \tag{6.13}
\]
as \( \varepsilon'' \to 0^+ \). Further, remark that

\[
\lim_{\varepsilon'' \to 0} \int_{\mathbb{R}^2} (-\varepsilon'' \partial_s u_{\varepsilon''}(x, t, 0)) \eta'(u_0^{(2)}) \Theta dxdt = \int_{\mathbb{R}^2} \left( a(u_{0}^{\text{tr.}}(2)) - a(u_0^{(2)}) \right) \eta'(u_0^{(2)}) \Theta dxdt \tag{6.14}
\]
due to Lemma 6.2,

\[
\lim_{\delta \to 0^+} \int_0^\delta \int_{\mathbb{R}^2} q_a(u)(\rho_0^0)' \Theta dxdtds = - \int_{\mathbb{R}^2} q_a(u_{0}^{\text{tr.}}(2)) \Theta dxdt \tag{6.15}
\]
due to property (6.8) of the derivative \( (\rho_0^0)' \).
On the strength of (6.14), (6.15), and the bounds \(|u| \leq M_0\) (a.e. in \(G_{T,S}\)) and \(0 \leq \rho_\delta^0 \leq 1\), as \(\delta \to 0^+\) from (6.13) we deduce

\[
\int_{\Xi^2} \left( \left( a(u_0^{tr,(2)} - a(u_0^{(2)})) \eta'(u_0^{(2)}) - (q_a(u_0^{tr,(2)}) - q_a(u_0^{(2)})) \right) \Theta \, dx \, dt \right) \geq 0. \tag{6.16}
\]

Quite analogously, we derive the integral inequality

\[
\int_{\Xi^2} \left( \left( a(u_S^{tr,(2)} - a(u_S^{(2)})) \eta'(u_S^{(2)}) - (q_a(u_S^{tr,(2)}) - q_a(u_S^{(2)})) \right) \Theta \, dx \, dt \leq 0. \tag{6.17}
\]

To this end, it suffices to substitute \(s' = 0, s'' > \delta\) and \(\phi = \eta'(u_\varepsilon)\Theta \rho_\delta^0\) by \(s' = S - \delta, s'' = S\) and \(\phi = \eta'(u_\varepsilon)\Theta \rho_\delta^S\) (with \(\rho_\delta^S(s) = \rho_\delta^0(S - s)\)), respectively, and to keep track of the above outline.

Now, let us prove that (6.16) is equivalent to (3.2e). This procedure is rather standard and simple. Introduce the linear functional \(\nu_0^{(2)}\) by the formula

\[
\left\langle \nu_0^{(2)}, \Phi \right\rangle_{\mathcal{M}(\Xi^2 \times \mathbb{R}_\lambda), C_0(\Xi^2 \times \mathbb{R}_\lambda)} := \int_{\Xi^2} \left( (a(u_0^{tr,(2)}(x,t)) - a(u_0^{(2)}(x,t))) \delta_{(-u_0^{(2)}(x,t))} \Phi(x,t) \right)_{\mathcal{M}(\mathbb{R}_\lambda,C_0(\mathbb{R}_\lambda))} - \int_{-M_0}^{M_0} a'(\lambda) \left( \left( \chi(\cdot; u_0^{tr,(2)}(x,t)) - \chi(\cdot; u_0^{(2)}(x,t)) \right) \Phi(x,t) \right) \, d\lambda \, dx \, dt \tag{6.18}
\]

for all \(\Phi \in C_0(\Xi^2 \times \mathbb{R}_\lambda)\). On the strength of the maximum principle (2.2), we have that \(|u_0^{tr,(2)}| \leq M_0\) a.e. on \(\Xi^2\). Hence, \(\text{supp} \nu_0^{(2)}\) lays in the layer \(\{-M_0 \leq \lambda \leq M_0\}\), and there exists \(C_3 = \text{const} > 0\) independent of \(\Phi\) such that

\[
\left| \left\langle \nu_0^{(2)}, \Phi \right\rangle \right| \leq C_3 \|\Phi\|_{L^1(\Xi^2; C[-M_0,M_0])}, \quad \forall \Phi \in L^1(\Xi^2; C[-M_0,M_0]),
\]

i.e. \(\nu_0^{(2)} \in L^w_\infty(\Xi^2; \mathcal{M}[-M_0,M_0])\).

Further, introduce the primitive \(\mu_0^{(2)}\) of \(\nu_0^{(2)}\) with respect to \(\lambda:\)

\[
\left\langle \mu_0^{(2)}, \Phi' \right\rangle_{\mathcal{M}(\mathbb{R}_\lambda), C_0(\mathbb{R}_\lambda)} = - \left\langle \nu_0^{(2)}, \Phi \right\rangle_{\mathcal{M}(\mathbb{R}_\lambda), C_0(\mathbb{R}_\lambda)} \quad \text{a.e. in } \Xi^2, \quad \forall \Phi \in C_0^1(\mathbb{R}_\lambda). \tag{6.19}
\]

Clearly, \(\mu_0^{(2)} \in L^\infty_w(\Xi^2; \mathcal{M}(\mathbb{R}_\lambda))\), and \(\text{supp} \mu_0^{(2)}\) lays in the layer \(\{-M_0 \leq \lambda \leq M_0\}\). Using assertion (i) in Lemma 3.1 and formulas (6.18) and (6.19), we rewrite (6.16) in terms of the mapping \(\mu_0^{(2)}\) as follows:

\[
\left\langle \partial_\lambda \mu_0^{(2)}, \Theta \eta'' \right\rangle \leq 0, \quad \forall \Theta \in C_0^1(\Xi^2), \quad \forall \eta \in C_0^2(\mathbb{R}_\lambda), \quad \eta \text{ is convex on } [-M_0,M_0]. \tag{6.20}
\]

In the sense of distributions, (6.20) is equivalent to the inequality \(\left\langle \mu_0^{(2)}, \Theta \eta'' \right\rangle \geq 0\). In turn, since \(\Theta \geq 0, \eta'' \geq 0\) and the linear span of the set \(\{\Theta \eta'': \Theta \in C_0^1(\Xi^2), \eta \in C_0^2(\mathbb{R}_\lambda)\}\) is dense in \(L^1(\Xi^2; C[-M_0,M_0])\), from (6.18)–(6.20) it follows that \(\mu_0^{(2)}\) is a positive Radon measure on \([-M_0,M_0]\) for a.e. \((x,t) \in \Xi^2\), i.e.,

\[
\mu_0^{(2)} \in L^\infty_w(\Xi^2; \mathcal{M}^+[M_0,M_0]),
\]

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and the integral equality

\[
\int_{\Xi^2} \left( \int_{-M_0}^{M_0} a'(\lambda) \left( \chi(\lambda; u_0^{\text{tr},(2)}) - \chi(\lambda; u_0^{(2)}) \right) \Phi \, d\lambda \right)
\]

\[
- \int_{\Xi^2} \left( a(u_0^{\text{tr},(2)}) - a(u_0^{(2)}) \right) \left( \delta_\lambda = u_0^{(2)}, \Phi \right)_{\mathcal{M}(\mathbb{R}, C_0(\mathbb{R}))} \, d\lambda dt =
\]

\[
- \int_{\Xi^2} \left( \mu_0^{(2)}(x, t, \cdot), \partial_x \Phi(x, t, \cdot) \right)_{\mathcal{M}(\mathbb{R}, C_0(\mathbb{R}))} \, d\lambda dt, \quad \forall \Phi \in L^1(\Xi^2; C_0^1(\mathbb{R})) \quad (6.21)
\]

holds true. In the sense of distributions, (6.21) is equivalent to (3.26). Quite analogously, we verify that (6.17) is equivalent to (3.26). Lemma (6.3) is proved.

**Remark 6.1.** Collecting altogether the results of Lemmas 4.1, 4.2, 6.1, and 6.3, we establish that there is an L^1-strong limiting point u of the family \{u_\varepsilon\}_{\varepsilon>0} of solutions to Problem \Pi_{\gamma \varepsilon} such that u is a kinetic solution of Problem \Pi_{\gamma} in the sense of Definition 3.3.

## 7 Stability and uniqueness of kinetic solutions

Let u_1 and u_2 be two kinetic solutions of Problem \Pi_{\gamma} corresponding to two given sets of data \((u_{1,0}, u_{1,2}, u_{1,s})\) and \((u_{2,0}, u_{2,2}, u_{2,s})\), respectively. In order to verify inequality (3.4), we revisit, modify, and somewhat improve the procedure outlined in [23, Section 4]. We start with fulfilling the normalization procedure for the difference of two kinetic equations for u_1 and u_2. More precisely, we prove the following lemma.

**Lemma 7.1.** For an arbitrary fixed \(t' \in (0, T]\) and for all nonnegative functions \(\xi \in C^2(\Omega)\) the renormalized inequality

\[
\int_{\Xi^1 \times (-M_1, M_1)} |\chi(\lambda; u_{1, t' - 0}(x, s)) - \chi(\lambda; u_{2, t' - 0}(x, s))|^2 \xi(x) \, dx ds d\lambda -
\]

\[
\int_{(0, t') \times \Xi^1 \times (-M_1, M_1)} \left( \varphi'(\lambda) \cdot \nabla_x \xi(x) + \Delta_x \xi(x) + \partial_x Z_\gamma(x, t, s, \lambda) \xi(x) \timesight.
\]

\[
|\chi(\lambda; u_1(x, t, s)) - \chi(\lambda; u_2(x, t, s))|^2 \, dx dt ds d\lambda \leq
\]

\[
\int_{\Xi^1 \times (-M_1, M_1)} |\chi(\lambda; u_{1, t' - 0}(x, s)) - \chi(\lambda; u_{2, t' - 0}(x, s))|^2 \xi(x) \, dx ds d\lambda +
\]

\[
\int_{(0, t') \times \Omega \times (-M_1, M_1)} a'(\lambda) \left( |\chi(\lambda; u_{1, t' - 0}(x, t)) - \chi(\lambda; u_{2, t' - 0}(x, t))| \right)^2 -
\]

\[
|\chi(\lambda; u_{1, t'}(x, t)) - \chi(\lambda; u_{2, t'}(x, t))|^2 \xi(x) \, dx dt d\lambda \quad (7.1)
\]

is valid, where \(M_1 = M_1(t')\) is given by (3.6).

**Proof.** The proof is divided into five steps.

**Step 1. The smoothing of the kinetic equation.** Introduce the regularizing kernel \(\omega \in C^\infty(\mathbb{R})\),
\[ \|\omega\|_{L^1(\mathbb{R})} = 1 \] that is a nonnegative smooth function with a compact support on \([0, 1]\). For any measurable function or enough regular distribution \( f: \mathbb{R}_+^d \times \mathbb{R}_+^+ \times \mathbb{R}_+^3 \times \mathbb{R}_+ \mapsto \mathbb{R} \) we denote

\[
\begin{align*}
f_{\varepsilon_0}(x, t, s, \cdot) &= \omega_{\varepsilon_0} * f(x, t, s, \cdot), \\
f_{\varepsilon_1}(x, t, \cdot, \lambda) &= \omega_{\varepsilon_1} * f(x, t, \cdot, \lambda), \\
f_{\varepsilon_2}(x, \cdot, s, \lambda) &= \omega_{\varepsilon_2} * f(x, \cdot, s, \lambda), \\
f_{\varepsilon_3}(\cdot, t, s, \lambda) &= \omega_{\varepsilon_3} * f(\cdot, t, s, \lambda),
\end{align*}
\]

where \( \varepsilon_0, \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) are small positive parameters, and

\[
\begin{align*}
\omega_{\varepsilon_0}(\lambda) &= \frac{\omega(\lambda \varepsilon_0)}{\varepsilon_0}, \\
\omega_{\varepsilon_1}(s) &= \frac{\omega(s \varepsilon_1)}{\varepsilon_1}, \\
\omega_{\varepsilon_2}(t) &= \frac{\omega(t \varepsilon_2)}{\varepsilon_2}, \\
\omega_{\varepsilon_3}(x) &= \frac{\omega(x \varepsilon_3)}{\varepsilon_3} \cdot \ldots \cdot \omega\left(\frac{x_d \varepsilon_3}{\varepsilon_3}\right).
\end{align*}
\]

Further we write \( f_{\alpha, \beta} \) instead of \((f_{\alpha})_{\beta}\) for \( \alpha, \beta = \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \) and denote \( f_{\varepsilon} = f_{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3} \) for the sake of brevity. Set

\[
\Phi(x, t, s, \lambda) = \omega_{\varepsilon_0}(\lambda - \lambda)\omega_{\varepsilon_1}((s - s)\omega_{\varepsilon_2}(t - t)\omega_{\varepsilon_3}(x - x)
\]

for \( \lambda \in \mathbb{R}_+, \bar{s} \in [\varepsilon_1, S], \bar{t} \in [\varepsilon_2, T] \) and \( \bar{x} \in \Omega_{\varepsilon_3} \) defined \( \{ \bar{x} \in \Omega : \text{dist}(\bar{x}', \partial \Omega) \geq \varepsilon_3 \} \). Notice that this is a legitimate test function for \((1.20)\).

Substituting \( \Phi(x, t, s, \lambda) \) on the place of a test function in \((1.20)\), we obtain the following equation for the smoothed \( \chi \)-function, where we write \( x, t, s, \lambda \) instead of \( \bar{x}, \bar{t}, \bar{s} \) and \( \bar{\lambda} \):

\[
\partial_t \chi_{\varepsilon}(\lambda; u) + a'(\lambda) \partial_s \chi_{\varepsilon}(\lambda; u) - \varphi(\lambda) \cdot \nabla_x \chi_{\varepsilon}(\lambda; u) - \Delta_x \chi_{\varepsilon}(\lambda; u) + Z_\gamma(x, t, s, \lambda) \partial_\lambda \chi_{\varepsilon}(\lambda; u)
\]

\[
- (Z_\gamma)_{\varepsilon_1, \varepsilon_2, \varepsilon_3}(x, t, s, 0) = \partial_\lambda (m_{\varepsilon} + n_{\varepsilon}) + R_{1,\varepsilon}^{(3)} + R_{2,\varepsilon}^{(3)} + R_{3,\varepsilon}^{(3)} + R_{4,\varepsilon}^{(3)}
\]

in \( \Omega_{\varepsilon_3} \times [\varepsilon_2, T] \times [\varepsilon_1, S] \times \mathbb{R}_+ \), (7.2)

where

\[
\chi_{\varepsilon}(\lambda; u) = \chi_{\varepsilon,0,\varepsilon_1,\varepsilon_2,\varepsilon_3}(\cdot; u(\cdot, \cdot, \cdot)),
\]

and the rest terms are given by the formulas

\[
\begin{align*}
R_{1,\varepsilon}^{(3)} &= a'(\lambda) \partial_\lambda \chi_{\varepsilon}(\lambda; u) - \partial_s (a'(\lambda)) \varepsilon, \\
R_{2,\varepsilon}^{(3)} &= \text{div}_x (\varphi(\lambda) \chi_{\varepsilon}(\lambda; u) - (\varphi'(\cdot, \lambda)) \varepsilon), \\
R_{3,\varepsilon}^{(3)} &= (\chi \partial_\lambda Z_\gamma) - \chi_{\varepsilon} \partial_\lambda Z_\gamma(x, t, s, \lambda), \\
R_{4,\varepsilon}^{(3)} &= \partial_\lambda (\chi_{\varepsilon}(\lambda; u) Z_\gamma(x, t, s, \lambda) - (\chi Z_\gamma)_{\varepsilon}).
\end{align*}
\]

Step 2. Renormalization of the smoothed kinetic equation. Now subtract \((7.2)\) with

\[
\chi_{\varepsilon}(\lambda; u) = \chi_{\varepsilon}(\lambda; u_2), \quad m_{\varepsilon} = m_{2\varepsilon}, \quad n_{\varepsilon} = n_{2\varepsilon}, \quad R_{k}^{(\varepsilon)} = R_{2k}^{(\varepsilon)} \quad (k = 1, 2, 3, 4)
\]

from \((7.2)\) with

\[
\chi_{\varepsilon}(\lambda; u) = \chi_{\varepsilon}(\lambda; u_1), \quad m_{\varepsilon} = m_{1\varepsilon}, \quad n_{\varepsilon} = n_{1\varepsilon}, \quad R_{k}^{(\varepsilon)} = R_{1k}^{(\varepsilon)} \quad (k = 1, 2, 3, 4)
\]

and multiply the both sides of the resulting equation by \( 2 (\chi_{\varepsilon}(\lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2)) \) to get

\[
\partial_\lambda (\chi_{\varepsilon}(\lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2))^2 + a'(\lambda) \partial_s (\chi_{\varepsilon}(\lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2))^2 +
\]

\[
\varphi'(\lambda) \cdot \nabla_x (\chi_{\varepsilon}(\lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2))^2 - \Delta_x (\chi_{\varepsilon}(\lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2))^2 +
\]

\[
2 |\nabla_x (\chi_{\varepsilon}(\lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2))^2| + Z_\gamma(x, t, s, \lambda) \partial_\lambda (\chi_{\varepsilon}(\lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2))^2 =
\]

\[
2 (\chi_{\varepsilon}(\lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2)) (\partial_\lambda ((m_{1\varepsilon} + n_{1\varepsilon}) - (m_{2\varepsilon} + n_{2\varepsilon})) + \sum_{k=1}^{4} (R_{1k}^{(\varepsilon)} - R_{2k}^{(\varepsilon)}))
\]

in \( \Omega_{\varepsilon_3} \times [\varepsilon_2, T] \times [\varepsilon_1, S] \times \mathbb{R}_+ \). (7.3)
Since \(u_1|_{\Gamma_1} = u_2|_{\Gamma_1} = 0\) by (1.14), equation (7.3) can be extended as the trivial identity beyond \(\partial \Omega_{\varepsilon_3}\) onto the whole space \(\mathbb{R}^d_\varepsilon\). Let \(B\) be an arbitrary open ball in \(\mathbb{R}^d_\varepsilon\) containing \(\overline{\Omega}\). In particular, we have dist(\(\partial \Omega, \partial B\)) > 0. Let \(\varepsilon \in C^2_0(B)\) be arbitrary and not necessarily finite in \(\Omega\). Multiply the both sides of (7.3) by \(\varepsilon\), integrate over \((\varepsilon_2, t') \times (\varepsilon_1, s') \times B \times \mathbb{R}_\lambda\) \((s' \in (0, S), t' \in [\varepsilon_2, T])\) and integrate by parts in \(x\) and \(\lambda\):

\[
\int_{B \times (\varepsilon_2, t') \times (\varepsilon_1, s') \times \mathbb{R}_\lambda} \bigg( \chi_0(\varepsilon(x; \lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2)) \bigg) dx dt ds d\lambda = \int_{B \times (\varepsilon_2, t') \times (\varepsilon_1, s') \times \mathbb{R}_\lambda} \bigg( \phi'(\lambda) \cdot \nabla_\lambda \zeta(x) + \Delta_\lambda \zeta(x) + \partial_\lambda \mathcal{Z}_\gamma(x, t, s, \lambda) \zeta(x) \bigg) \bigg| \chi_0(\varepsilon; \lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2) \bigg| dx dt ds d\lambda + \int_{B \times (\varepsilon_2, t') \times (\varepsilon_1, s') \times \mathbb{R}_\lambda} 2\zeta(x) |\nabla_\lambda(\chi_0(\varepsilon; \lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2))| dx dt ds d\lambda = 2\zeta(x) (\chi_0(\varepsilon; \lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2)) \sum_{k=1}^{4} \left( R_{1k}^{(\varepsilon)} - R_{2k}^{(\varepsilon)} \right) dx dt ds d\lambda. \quad (7.4)

**Step 3. Passage to the limit, as \(\varepsilon_0 \to 0^+\).** Further, for the sake of brevity, we denote \(f_{\varepsilon} = f_{\varepsilon_1, \varepsilon_2, \varepsilon_3}\) for \(f = \chi(\lambda; u_1), f = m_i, f = n_i, \) etc., and \(R_{ik}^{(\varepsilon)} = R_{ik}^{(\varepsilon_1, \varepsilon_2, \varepsilon_3)}\) \((k = 1, 2, 3, 4, i = 1, 2)\).

**Remark 7.1.** We start with the remark that the limiting passage as \(\varepsilon_0 \to 0^+\) in the left-hand side of (7.4) is simple, and in the limit we arrive at the same expressions with subscript \(\varepsilon\) on the places of \(\varepsilon_0\).

In the right hand side, due to the standard properties of the regularizing kernels \(\omega_{\varepsilon_0}, \omega_{\varepsilon_1}, \omega_{\varepsilon_2}\) and \(\omega_{\varepsilon_3}\), we have

\[
R_{ik}^{(\varepsilon)} \rightarrow_{\varepsilon_0 \to 0^+} R_{ik}^{(\varepsilon)} \equiv 0 \quad (i, k = 1, 2), \quad R_{13}^{(\varepsilon)} \rightarrow_{\varepsilon_0 \to 0^+} R_{13}^{(\varepsilon)}, \quad \chi_0(\varepsilon; \lambda; u_i) \rightarrow_{\varepsilon_0 \to 0^+} \chi_{\varepsilon}(\lambda; u_i) \quad (i = 1, 2)
\]

strongly in \(L^r_{loc}(B \times [\varepsilon_2, T] \times [\varepsilon_1, S] \times \mathbb{R}_\lambda), \forall r \in [1, +\infty)\). \quad (7.5)

Hence

\[
\int_{B \times (\varepsilon_2, t') \times (\varepsilon_1, s') \times \mathbb{R}_\lambda} 2\zeta(x)(\chi_{\varepsilon}(\lambda; u_1) - \chi_0(\varepsilon; \lambda; u_2)) \sum_{k=1}^{3} \left( R_{1k}^{(\varepsilon)} - R_{2k}^{(\varepsilon)} \right) dx dt ds d\lambda \rightarrow_{\varepsilon_0 \to 0^+} \int_{B \times (\varepsilon_2, t') \times (\varepsilon_1, s') \times \mathbb{R}_\lambda} 2\zeta(x)(\chi_{\varepsilon}(\lambda; u_1) - \chi_0(\varepsilon; \lambda; u_2)) \left( R_{13}^{(\varepsilon)} - R_{23}^{(\varepsilon)} \right) dx dt ds d\lambda. \quad (7.6)
\]

According to assertion (i) in Lemma 3.1 we have that

\[
\partial_\lambda \chi(\lambda; v) = \delta_{(\lambda=0)} - \delta_{(\lambda=v)}, \quad \forall v \in \mathbb{R} \quad (in \ \mathfrak{M}(\mathbb{R}_\lambda)). \quad (7.7)
\]
Using (7.7), integrating by parts in \( \lambda \), and passing to the limit, we get

\[
\int_{B \times (\varepsilon_2, t') \times (\varepsilon_1, s') \times \mathbb{R}_\lambda} 2\zeta(x) (\chi_{\varepsilon}(\lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2)) \left( R^{(\varepsilon)}_{14} - R^{(\varepsilon)}_{24} \right) dx dt ds d\lambda =
\]

\[
- \int_{B \times (\varepsilon_2, t') \times (\varepsilon_1, s') \times \mathbb{R}_\lambda} 2\zeta(x) \partial_\lambda (\chi_{\varepsilon}(\lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2)) \left( Z_\gamma(x, t, s, \lambda) (\chi_{\varepsilon}(\lambda; u_1) - \chi_{\varepsilon}(\lambda; u_2)) - (Z_\gamma(\chi(\lambda; u_1) - \chi(\lambda; u_2)) ) \right) dx dt ds d\lambda \xrightarrow{\varepsilon_0 \to 0+} 0
\]

\[
- \int_{B \times (\varepsilon_2, t') \times (\varepsilon_1, s')} 2\zeta(x) \left( \delta(\lambda - u_2) - \delta(\lambda - u_1) \right) dx dt ds d\lambda\]

\[
(Z_\gamma(\chi(\lambda; u_1) - \chi(\lambda; u_2)) ) dx dt ds. \quad (7.8)
\]

Notice that, since \( \lambda \mapsto \chi(\lambda; v) \) is the finite step-function in \( \mathbb{R}_\lambda \), the duality bracket \( \langle \cdot, \cdot \rangle_{M(\mathbb{R}_\lambda), C_0(\mathbb{R}_\lambda)} \) is well-defined, although the right-hand side in the bracket is not continuous in \( \lambda \). Next, integrate (7.2) over the interval \( (-\infty, \lambda_0) \) with respect to \( \lambda \). Since \( \chi(\lambda; u_i) \) and \( m_i + n_i \) vanish for \( \lambda < -M_1(T) \), we have

\[
m_{i\varepsilon}(x, t, s, \lambda_0) + n_{i\varepsilon}(x, t, s, \lambda_0) = \int_{-\infty}^{\lambda_0} \left( \partial_\lambda \chi_{\varepsilon}(\lambda; u_i) + a'(\lambda) \partial_\lambda \chi_{\varepsilon}(\lambda; u_i) + \varphi'(\lambda) \cdot \nabla_x \chi_{\varepsilon}(\lambda; u_i) - \Delta_x \chi_{\varepsilon}(\lambda; u_i) - \chi_{\varepsilon}(\lambda; u_i) \partial_\lambda Z_\gamma(x, t, s, \lambda) - Z_{\gamma\varepsilon}(x, t, s, 0) \right) dx dt ds d\lambda + \sum_{k=1}^{4} R^{(\varepsilon)}_{ik} d\lambda
\]

\[
Z_{\gamma}(x, t, s, \lambda_0) \chi_{\varepsilon}(\lambda_0; u_i) \quad (i = 1, 2). \quad (7.9)
\]

Passing to the limit in (7.9), we derive

\[
m_{i\varepsilon} + n_{i\varepsilon} \xrightarrow{\varepsilon_0 \to 0+} m_{i\varepsilon} + n_{i\varepsilon} \text{ strongly in } L^r_{\text{loc}} (B \times [\varepsilon_2, T] \times [\varepsilon_1, S] \times \mathbb{R}_{\lambda_0}), \quad \forall r \in [1, \infty), \quad (7.10)
\]

where

\[
m_{i\varepsilon}(x, t, s, \lambda_0) + n_{i\varepsilon}(x, t, s, \lambda_0) = \int_{-\infty}^{\lambda_0} \left( \partial_\lambda \chi_{\varepsilon}(\lambda; u_i) + a'(\lambda) \partial_\lambda \chi_{\varepsilon}(\lambda; u_i) + \varphi'(\lambda) \cdot \nabla_x \chi_{\varepsilon}(\lambda; u_i) - \Delta_x \chi_{\varepsilon}(\lambda; u_i) - \chi_{\varepsilon}(\lambda; u_i) \partial_\lambda Z_\gamma(x, t, s, \lambda) - Z_{\gamma\varepsilon}(x, t, s, 0) H(\lambda_0) \right) dx dt ds d\lambda
\]

\[
(\chi(\lambda; u_i) \partial_\lambda Z_\gamma \varepsilon(x, t, s, \lambda)) dx dt ds d\lambda + (\chi(\lambda; u_i) Z_\gamma \varepsilon(x, t, s, \lambda_0) - Z_{\gamma\varepsilon}(x, t, s, 0)) H(\lambda_0), \quad (i = 1, 2), \quad (7.11)
\]

\[H(\lambda_0) = 1_{(\lambda_0 > 0)} \quad \text{(the right-continuous Heaviside function).} \]

On the strength of the structure of function \( \chi \) (recall (3.11)), properties of the regularizing kernels \( \omega_{\varepsilon_1}, \omega_{\varepsilon_2} \) and \( \omega_{\varepsilon_3} \), Conditions on \( a, \varphi \) and \( Z_\gamma \), and representation (7.11), we conclude that the bound

\[
0 \leq m_{i\varepsilon} + n_{i\varepsilon} \leq C_4 = C_4(\varepsilon_1, \varepsilon_2, \varepsilon_3) \quad \forall (x, t, s, \lambda_0) \in B \times [\varepsilon_2, T] \times [\varepsilon_1, S] \times \mathbb{R}_{\lambda_0} \quad (7.12)
\]

holds true and that, for all \( (x, t, s) \in B \times [\varepsilon_2, T] \times [\varepsilon_1, S] \), the function

\[
\lambda_0 \mapsto m_{i\varepsilon}(x, t, s, \lambda_0) + n_{i\varepsilon}(x, t, s, \lambda_0) \quad (i = 1, 2) \quad (7.13)
\]
is absolutely continuous on $\mathbb{R}$ except for the point $\lambda_0 = 0$, where it suffers a finite jump.

Due to these properties of $m_i\hat{\varepsilon} + n_i\hat{\varepsilon}$, the limiting relation (7.10) and representation (7.7), we establish that

$$\int_{B \times (\varepsilon, t') \times (\varepsilon, s') \times \mathbb{R}} \left( -2\zeta(x) \partial_X (\chi_{\hat{\varepsilon}}(\lambda; u_1) - \chi_{\hat{\varepsilon}}(\lambda; u_2)) \right) (m_{1\hat{\varepsilon}} + n_{1\hat{\varepsilon}}) \ d\mathbf{x} d\mathbf{t} d\mathbf{s} d\lambda$$

$$\to 0$$

$$- \int_{B \times (\varepsilon, t') \times (\varepsilon, s')} 2\zeta(x) \left( (\delta_{\lambda=\varepsilon}) - (\delta_{\lambda=\varepsilon_u}) \right) \xi; (m_{1\hat{\varepsilon}} + n_{1\hat{\varepsilon}}) \ d\mathbf{x} d\mathbf{t} d\mathbf{s}.$$  

(7.14)

Now we are in a position to pass to the limit in (7.4), as $\varepsilon \to 0+$. Aggregating (7.6), (7.8), (7.14) and recalling Remark 7.7 from (7.1) we derive the integral equality

$$\int_{B \times (\varepsilon, t') \times (\varepsilon, s') \times \mathbb{R}} \left\{ \zeta(x) \left( \partial_t |\chi_{\hat{\varepsilon}}(\lambda; u_1) - \chi_{\hat{\varepsilon}}(\lambda; u_2)|^2 + d'(\lambda) \partial_s |\chi_{\hat{\varepsilon}}(\lambda; u_1) - \chi_{\hat{\varepsilon}}(\lambda; u_2)|^2 \right) - \right.$$

$$\left. (\varphi'(\lambda) \cdot \nabla x(x) + \Delta_x \zeta(x) + \zeta(x) \partial_X Z(\varepsilon(x, t, s, \lambda)) |\chi_{\hat{\varepsilon}}(\lambda; u_1) - \chi_{\hat{\varepsilon}}(\lambda; u_2)|^2 + 2\zeta(x) |\chi_{\hat{\varepsilon}}(\lambda; u_1) - \chi_{\hat{\varepsilon}}(\lambda; u_2)|^2 \right\} d\mathbf{x} d\mathbf{t} d\mathbf{s} d\lambda =$$

$$- \int_{B \times (\varepsilon, t') \times (\varepsilon, s')} 2\zeta(x) \left( (\delta_{\lambda=\varepsilon_u}) - (\delta_{\lambda=\varepsilon_u}) \right) \xi; (m_{1\hat{\varepsilon}} + n_{1\hat{\varepsilon}}) \ d\mathbf{x} d\mathbf{t} d\mathbf{s}$$

$$\int_{B \times (\varepsilon, t') \times (\varepsilon, s') \times \mathbb{R}} 2\zeta(x) \left( (\delta_{\lambda=\varepsilon_u}) - (\delta_{\lambda=\varepsilon_u}) \right) \xi; Z(t, s, \cdot) (\chi_{\hat{\varepsilon}}(\lambda; u_1) - \chi_{\hat{\varepsilon}}(\lambda; u_2)) -$$

$$(Z(t, \chi(\lambda; u_1) - \chi(\lambda; u_2))) \ d\mathbf{x} d\mathbf{t} d\mathbf{s} +$$

$$\int_{B \times (\varepsilon, t') \times (\varepsilon, s') \times \mathbb{R}} 2\zeta(x) (\chi_{\hat{\varepsilon}}(\lambda; u_1) - \chi_{\hat{\varepsilon}}(\lambda; u_2)) \left( R_{13}^{(\varepsilon_u)} - R_{23}^{(\varepsilon_u)} \right) d\mathbf{x} d\mathbf{t} d\mathbf{s} d\lambda.$$  

(7.15)

Step 4. Passage to the limit as $\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0+$. We pass to the limit in (7.15), as $\varepsilon \to 0+$, i.e., as $\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0+$ simultaneously, provided that ratios $\frac{\varepsilon_1}{\varepsilon_3}$ and $\frac{\varepsilon_2}{\varepsilon_3}$ tend to zero as well, for the technical reasons. Repeating arguments from [44] Chapter 4, proof of Theorem 4.3.1 (third step) with natural modifications, we deduce that

$$\lim_{\varepsilon \to 0+} \int_{B \times (\varepsilon, t') \times (\varepsilon, s')} 2\zeta(x) \left( (\delta_{\lambda=\varepsilon_u}) \right) \xi; m_{i\hat{\varepsilon}} + n_{i\hat{\varepsilon}} \ d\mathbf{x} d\mathbf{t} d\mathbf{s} = 0 \quad (i = 1, 2).$$  

(7.16)

(This limiting relation is analogous to [44 formula (4.3.5)].)

Since $\delta_{\lambda=\varepsilon_u}$, $m_i$ and $n_i$ are nonnegative measures and $\zeta$ is a nonnegative function, we have that

$$- \int_{B \times (\varepsilon, t') \times (\varepsilon, s')} 2\zeta(x) \left( (\delta_{\lambda=\varepsilon_u}) \right) \xi; m_{i\hat{\varepsilon}} + n_{i\hat{\varepsilon}} \ d\mathbf{x} d\mathbf{t} d\mathbf{s} \leq 0,$$  

(7.17)
\[
\int_{B \times (\varepsilon, t') \times (\varepsilon, s') \times \mathbb{R}^\lambda} 2\zeta(x) \left| \nabla_x (\chi_\varepsilon(\lambda; u_1) - \chi_\varepsilon(\lambda; u_2)) - 2\zeta(x) \left( \delta_{\lambda=0} \delta_{\lambda=1} - \delta_{\lambda=0} \delta_{\lambda=1} \right) \right|^2 d\mathbf{x} dt ds d\lambda = 0 \quad \forall \varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in (0, 1]. \quad (7.18)
\]

On the strength of the standard properties of the regularizing kernels \( \omega_{\varepsilon_1}, \omega_{\varepsilon_2} \) and \( \omega_{\varepsilon_3} \), we deduce the following limiting relations

\[
\int_{B \times (\varepsilon, t') \times (\varepsilon, s') \times \mathbb{R}^\lambda} 2\zeta(x) \left| \partial_s |\chi_\varepsilon(\lambda; u_1) - \chi_\varepsilon(\lambda; u_2)|^2 + a'(\lambda) \partial_s |\chi_\varepsilon(\lambda; u_1) - \chi_\varepsilon(\lambda; u_2)|^2 \right| d\mathbf{x} dt ds d\lambda \equiv
\]

\[
\int_{B \times (\varepsilon, t') \times (\varepsilon, s') \times \mathbb{R}^\lambda} \zeta(x) \left( |\chi_\varepsilon(\lambda; u_1)(x, t, s) - \chi_\varepsilon(\lambda; u_2)(x, t, s)|^2 \right|_{t'=t}^{t'=s} d\mathbf{x} ds d\lambda +
\]

\[
\int_{B \times (\varepsilon, t') \times (\varepsilon, s') \times \mathbb{R}^\lambda} \zeta(x) a'(\lambda) \left( |\chi_\varepsilon(\lambda; u_1)(x, t, s) - \chi_\varepsilon(\lambda; u_2)(x, t, s)|^2 \right|_{s'=s}^{s'=t} d\mathbf{x} dt d\lambda, \quad (7.19)
\]

\[
- \int_{B \times (\varepsilon, t') \times (\varepsilon, s') \times \mathbb{R}^\lambda} \left( \varphi'(\lambda) \cdot \nabla_x \zeta(x) + \Delta_x \zeta(x) + \zeta(x) \partial_s Z_\gamma(x, t, s, \lambda) \right) \times
\]

\[
\left| \chi_\varepsilon(\lambda; u_1) - \chi_\varepsilon(\lambda; u_2) \right|^2 d\mathbf{x} dt ds d\lambda \rightarrow_{\varepsilon \to 0+}
\]

\[
- \int_{B \times (\varepsilon, t') \times (\varepsilon, s') \times \mathbb{R}^\lambda} \left( \varphi'(\lambda) \cdot \nabla_x \zeta(x) + \Delta_x \zeta(x) + \zeta(x) \partial_s Z_\gamma(x, t, s, \lambda) \right) \times
\]

\[
\left| \chi(\lambda; u_1) - \chi(\lambda; u_2) \right|^2 d\mathbf{x} dt ds d\lambda, \quad (7.20)
\]

\[
- \int_{B \times (\varepsilon, t') \times (\varepsilon, s') \times \mathbb{R}^\lambda} 2\zeta(x) \left\{ \left( \delta_{\lambda=u_2} - \delta_{\lambda=u_1} \right) \chi_\varepsilon(\lambda; u_1) - \chi_\varepsilon(\lambda; u_2) \right\} \times
\]

\[
\left( Z_\gamma(\chi(\lambda; u_1) - \chi(\lambda; u_2)) \right) d\mathbf{x} ds d\lambda \rightarrow_{\varepsilon \to 0+} 0. \quad (7.21)
\]

Combining (7.15), (7.17), and (7.18), and then passing to the limit in the resulting inequality, on the strength of (7.16), (7.19), (7.20), and (7.21), we derive the integral inequality (7.1) with \( s' \) on the place of \( S \).

The last (fifth) step of the proof consists of the passage to the limit as \( s \to S - 0 \). This limiting
transition is quite straightforward thanks to existence of the strong traces $u_{1,S}^{\text{tr},(2)}$ and $u_{2,S}^{\text{tr},(2)}$ due to Lemma 5.3.

Lemma 7.1 is proved.

**Remark 7.2.** In view of relation (3.2b) and Lemma 5.2, it is legitimate to insert $u_{1,0}^{\text{tr}}(x,s)$ and $u_{1,t',s}$ on the places of $u_{1,0}^{\text{tr}}(x,s)$ and $u_{1,t',0}^{\text{tr}}(x,s)$, and to insert $u_{2,0}^{\text{tr}}(x,s)$ and $u_{2,t',0}^{\text{tr}}(x,s)$ in (7.1) and in the further considerations.

Next, we find the valuable relations between prescribed data (1.1c) and traces of kinetic solutions of Problem $\Pi_s$ on $\Gamma_0^2$ and $\Gamma_2^S$. Verification of these relations heavily relies on the kinetic boundary conditions (3.2e) and (3.2f).

**Lemma 7.2.** For an arbitrarily fixed $t' \in (0,\tau]$ and for all nonnegative test-functions $\xi \in C(\mathbb{R}^d)$ the inequalities

$$
\int_0^{t'} \int_0^{M_1(t')} \int_{\Omega} a'(\lambda) |\chi(\lambda; u_{1,0}^{\text{tr},(2)}(x,t)) - \chi(\lambda; u_{2,0}^{\text{tr},(2)}(x,t))|^2 \xi(x) \, d\lambda dx dt \leq \int_{-M_1(t')}^{M_1(t')} \int_0^{t'} \int_0^{M_1(t')} a'(\lambda) |\chi(\lambda; u_{1,0}^{(2)}(x,t)) - \chi(\lambda; u_{2,0}^{(2)}(x,t))|^2 \xi(x) \, d\lambda dx dt \quad (7.22)
$$

and

$$
\int_{-M_1(t')}^{M_1(t')} \int_0^{t'} \int_0^{M_1(t')} a'(\lambda) |\chi(\lambda; u_{1,S}^{\text{tr},(2)}(x,t)) - \chi(\lambda; u_{2,S}^{\text{tr},(2)}(x,t))|^2 \xi(x) \, d\lambda dx dt \leq \int_0^{t'} \int_0^{M_1(t')} \int_{\Omega} a'(\lambda) |\chi(\lambda; u_{1,S}^{(2)}(x,t)) - \chi(\lambda; u_{2,S}^{(2)}(x,t))|^2 \xi(x) \, d\lambda dx dt \quad (7.23)
$$

are valid. (Values $M_1(t')$ are given by (3.6).)

**Proof.** Assertion of Lemma 7.2 is just a minor modification of [23, Proposition 2]. Therefore justification is almost the same, as in [23], and we omit it.

Now, let us establish an important auxiliary inequality for the difference $u_1 - u_2$. The following lemma is somewhat similar to [23, Corollary 1].

**Lemma 7.3.** For an arbitrarily fixed $t' \in (0,\tau]$ and for all nonnegative test-functions
\( \xi \in C^2(\mathbb{R}^d) \) the inequality

\[
\int_{\Xi^1} |u_1(x, t', s) - u_2(x, t', s)| \xi(x) \, dx \, ds \\
- \int_{0}^{t'} \int_{\Xi^1} \sum_{i=1}^{d} \partial_i \xi(x)(\varphi_i(u_1(x, t, s)) - \varphi_i(u_2(x, t, s))) \text{sgn}(u_1(x, t, s) - u_2(x, t, s)) \, dx \, ds \, dt \\
- \int_{0}^{t'} \int_{\Xi^1} \Delta \xi(x)|u_1(x, t, s) - u_2(x, t, s)| \, dx \, ds \, dt
\]

\[
- \int_{0}^{t'} \int_{\Xi^1} (Z_\gamma(x, t, s, u_1(x, t, s)) - Z_\gamma(x, t, s, u_2(x, t, s))) \text{sgn}(u_1(x, t, s) - u_2(x, t, s)) \xi(x) \, dx \, ds \, dt \\
\]

\[
- \int_{0}^{t'} \int_{\Xi^1} \max_{\lambda \in [-M_1(t'), M_1(t')]} |\lambda'(\lambda)| \int_{\Omega} \left( |u^{(1)}_{1,0}(x, t) - u^{(1)}_{2,0}(x, s)| \right) \xi(x) \, dx \, dt
\]

holds true.

**Proof.** Firstly, we apply assertion (iii) and, after this, assertion (ii) of Lemma 3.1 to all terms in equalities (7.1), (7.22) and (7.23). Secondly, we combine these inequalities properly and take into account Remark 7.2. Thus we arrive at inequality (7.24), which completes the proof of the lemma.

Now we are in a position to establish estimate (3.4). Take \( \xi \equiv 1 \) in (7.24), which is a legal choice of test-function. Then estimate the last integral in the left-hand side using the Lagrange mean value theorem:

\[
\int_{0}^{t'} \int_{\Xi^1} (Z_\gamma(x, t, s, u_1(x, t, s)) - Z_\gamma(x, t, s, u_2(x, t, s))) \text{sgn}(u_1(x, t, s) - u_2(x, t, s)) \, dx \, ds \, dt \\
\]

\[
\int_{0}^{t'} \max_{(x,s,\lambda) \in \Xi^1 \times [-M_1(t'), M_1(t')]} |\partial_\lambda Z_\gamma(x, t, s, \lambda)| \int_{\Xi^1} |u_1(x, t, s) - u_2(x, t, s)| \, dx \, ds \, dt \quad (7.25)
\]
Combining this and (7.24) (with \( \xi \equiv 1 \)), we obtain the estimate

\[
\int_{\Xi^1} |u_1(x, t', s) - u_2(x, t', s)| \, dx \, ds \leq \int_{\Xi^1} |u_{1,0}^{(1)}(x, s) - u_{2,0}^{(1)}(x, s)| \, dx \, ds + \\
\int_{0}^{t'} \left\{ \max_{(x,s,\lambda) \in \Xi^1 \times [-M_1(t'), M_1(t')]} \left[ \partial_\gamma Z_\gamma(x, t, s, \lambda) \right] \left[ u_{1,0}^{(1)} - u_{2,0}^{(1)} \right] \, dx \, ds + \\
\max_{\lambda \in [-M_1(t'), M_1(t')]} |a'(\lambda)| \left( \int_{\Omega} \left[ |u_{1,0}^{(2)}(x, t) - u_{2,0}^{(2)}(x, t)| + |u_{1,t}^{(2)}(x, t) - u_{2,t}^{(2)}(x, t)| \right] \, dx \right) dt,
\]

\( \forall t' \in (0, T]. \) (7.26)

Applying Grönwall’s lemma to (7.26) and exchanging \( t \) and \( t' \), we arrive exactly at the estimate (3.4). Clearly, the uniqueness and stability of kinetic solutions to Problem \( \Pi_\gamma \) follow from this estimate.

Thus, the proof of assertion 1 of Theorem 3.1 is complete.

**Remark 7.3.** Since the kinetic solution \( u = u(x, t, s) \) of Problem \( \Pi_\gamma \) is unique, we conclude that the whole family \( \{u_\ell\}_{\ell \in (0, 1]} \) converges to \( u \) as \( \varepsilon \to 0+ \) strongly in \( L^2(G_{T,S}) \). Thus we have \( u = \lim_{\varepsilon \to 0} u_{\varepsilon} \), and the limiting relation \( u = \lim_{\varepsilon'' \to 0} u_{\varepsilon''} \) (see (4.11)) is a particular case.

### 8 Proof of assertion 2 and 3 of Theorem 3.1

The proof of equivalency of \( L^\infty \)-solutions of the kinetic equation (3.2a) to the entropy inequality (3.3a) is quite similar to justification of [9, Remark 2.5]. It is directly based on Lemma 3.1 and nonnegativity of the kinetic defect measure \( \mu \). The initial condition (3.2b) is equivalent to the initial condition (3.3c) thanks to assertion (ii) of Lemma 3.1. The kinetic boundary conditions (3.2e) and (3.2f) are equivalent to the entropy boundary conditions (3.3d) and (3.3e), respectively, due to assertion (i) of Lemma 3.1 and nonnegativity of the boundary kinetic defect measures \( \mu_{0}^{(2)} \) and \( \mu_{S}^{(2)} \). In fact, we have proved equivalency of (3.2e)–(3.2f) to (3.3d)–(3.3e) during justification of Lemma 6.3 (see formulas (6.16)–(6.21)).

Thus, the proof of assertion 3 of Theorem 3.1 is complete. Assertion 2 of Theorem 3.1 directly follows from assertions 1 and 3. Thus, Theorem 3.1 is proved.

### 9 Passage to the limit, as \( \gamma \to 0+ \):

formulation of the main results

Now suppose that the source term \( Z_\gamma \) in (1.1a) has the form (1.8) and Conditions on \( K_\gamma \) hold. That is, (1.1a) has the form

\[
\partial_t u + \partial_\alpha u + \text{div}_x \varphi(u) = \Delta_x u + K_\gamma(t, \tau) \beta(x, s, u).
\] (9.1)

On the strength of (1.10), for \( Z_\gamma = Z_\gamma(x, t, s, \lambda) \) we have

\[
Z_\gamma \to_{\gamma \to 0+} \beta \delta_{t=t-\tau=0} \text{ weakly}^* \text{ in } C^1(\Xi^1 \times \mathbb{R}_\lambda) \times \mathcal{M}(0, T).
\] (9.2)

Inserting directly \( \beta(x, s, u) \delta_{t=t-\tau=0} \) on the place of \( K_\gamma(t, \tau) \beta(x, s, u) \) in equation (9.1), we straightly set up the following formulation.
Problem $\Pi_0$. (The Cauchy — Dirichlet problem for the impulsive Kolmogorov-type equation.) For arbitrarily given initial and final data satisfying Conditions on $u_0^{(1)}$ & $u_0^{(2)}$ & $u_s^{(2)}$ and for arbitrarily given impulsive perturbation $\beta$ satisfying the demands of item (ii) in Conditions on $K$,$\gamma$,$\beta$, it is necessary to find a function $u: G_{T,S} \mapsto \mathbb{R}$ satisfying the quasi-linear ultra-parabolic equation

$$
\partial_t u + \partial_x a(u) + \text{div}_x \varphi(u) = \Delta_x u, \quad (x, t, s) \in G_{T,S} \setminus \{t = 0\},
$$

(9.3a)

the impulsive condition

$$
u(x, \tau + 0, s) = u(x, \tau - 0, s) + \beta(x, s, u(x, \tau - 0, s)), \quad (x, s) \in \Xi^1,
$$

(9.3b)

the initial and final conditions (1.1b) and (1.1c), and the homogeneous boundary condition (1.1d).

In this formulation, the nonlinearities $a = a(\lambda)$ and $\varphi = \varphi(\lambda) = (\varphi_1(\lambda), \ldots, \varphi_d(\lambda))$ satisfy the demands of items (i) and (ii) in Conditions on $a$&$\varphi$&$Z_\gamma$.

Remark 9.1. In the formulation of Problem $\Pi_0$, we have noticed that the system of equations (9.3a) and impulsive condition (9.3b) is equivalent to the equation

$$
\partial_t u + \partial_x a(u) + \text{div}_x \varphi(u) - \Delta_x u = \beta(x, s, u)\delta_{(t=\tau=0)}, \quad (x, t, s) \in G_{T,S},
$$

(9.4)

in the sense of distributions.

We introduce the notions of kinetic and entropy solutions to Problem $\Pi_0$ similarly to Definitions 3.3 and 3.4, with natural changes.

Definition 9.1. Function $u \in L^\infty(G_{T,S}) \cap L^2((0, T) \times (0, S); \hat{W}^1_2(\Omega))$ is called a kinetic solution of Problem $\Pi_0$, if it satisfies the kinetic equation

$$
\partial_t \chi(\lambda; u(x, t, s)) + a'(\lambda)\partial_x \chi(\lambda; u(x, t, s)) + \varphi'(\lambda) \cdot \nabla_x \chi(\lambda; u(x, t, s)) = \\
\Delta_x \chi(\lambda; u(x, t, s)) + \partial_\lambda(m(x, t, s, \lambda) + n(x, t, s, \lambda)),
$$

(9.5a)

with some measure $m \in \mathcal{M}^+(G_{T,S} \times \mathbb{R}_\lambda)$ such that $\text{supp} m \subset G_{T,S} \times [-M_3, M_3]$, and with $n = \delta_{(\lambda=\mu)}|\nabla_x u|^2$; the kinetic initial condition

$$
\text{esslim}_{t \to 0^+} \int_{-M_2}^{M_2} \int_{\Xi^1} \chi(\lambda; u(x, s, t)) - \chi(\lambda; u_0^{(1)}(x, s)) \, dx ds d\lambda = 0,
$$

(9.5b)

the kinetic boundary conditions

$$
da'(\lambda)(\chi(\lambda; u_0^{(2)}(x, t)) - \chi(\lambda; u_0^{(2)}(x, t))) - \\
\delta_{(\lambda=\mu_0^{(2)}(x, t))}(a(u_0^{(2)}(x, t)) - a(u_0^{(2)}(x, t))) = \partial_\lambda \mu_0^{(2)}(x, t, \lambda),
$$

(9.5c)

$$
da'(\lambda)(\chi(\lambda; u_s^{(2)}(x, t)) - \chi(\lambda; u_s^{(2)}(x, t))) - \\
\delta_{(\lambda=\mu_s^{(2)}(x, t))}(a(u_s^{(2)}(x, t)) - a(u_s^{(2)}(x, t))) = -\partial_\lambda \mu_s^{(2)}(x, t, \lambda),
$$

(9.5d)
with some measures $\mu_0^{(2)}$, $\mu_S^{(2)} \in \mathcal{M}^+(\Xi^2 \times \mathbb{R}_\lambda)$ such that $\text{supp}\, \mu_0^{(2)}$, $\text{supp}\, \mu_S^{(2)} \subset \Xi^2 \times [-M_3, M_3]$; and the kinetic impulsive condition

$$\int_{-M_3}^{M_3} \chi(\lambda; u(x, \tau+0, s)) \, d\lambda = \int_{-M_3}^{M_3} (1+\partial_\lambda \beta(x, s, \lambda)) \chi(\lambda; u(x, \tau-0, s)) \, d\lambda + \beta(x, s, 0), \quad (x, s) \in \Xi^1. \quad (9.5e)$$

Constants $M_2$ and $M_3$ arise from the maximum principle. They are defined by the formulas

$$M_2 := \max \left\{ \| u_0^{(1)} \|_{L^\infty(\Xi^1)}, \| u_0^{(2)} \|_{L^\infty(\Omega \times (0, \tau))}, \| u_S^{(2)} \|_{L^\infty(\Omega \times (0, \tau))} \right\} \quad (9.6)$$

and

$$M_3 := \max \left\{ M_2 + \| \beta \|_{C(\Xi^1 \times [-M_2, M_2])}, \| u_0^{(2)} \|_{L^\infty(\Omega \times (\tau, T))}, \| u_S^{(2)} \|_{L^\infty(\Omega \times (\tau, T))} \right\}. \quad (9.7)$$

Functions $u_0^{tr.(2)}$ and $u_S^{tr.(2)}$ are strong traces of a solution $u = u(x, t, s)$ (if any) of the kinetic equation $(9.5a)$ on the planes $\{s = 0\}$ and $\{s = S\}$, respectively. They are understood in the sense of limiting relations $(3.2c)$ and $(3.2d)$.

The kinetic equation $(9.5a)$ and the kinetic boundary conditions $(9.5c)$ and $(9.5d)$ are understood in the sense of distributions.

**Remark 9.2.** The system consisting of the kinetic equation $(9.5a)$ and the kinetic impulsive condition $(9.5e)$ can be equivalently written in the form of the kinetic equation with the singular term, incorporating delta-measure $\delta_{t=t=\tau=0}$, as follows:

$$\partial_\tau \chi(\lambda; u) + a'(\lambda) \partial_\lambda \chi(\lambda; u) + \varphi'(\lambda) \cdot \nabla_x \chi(\lambda; u) - \Delta_x \chi(\lambda; u) = \partial_\lambda \{ m + n + \delta_{t=t=\tau=0} \lambda \geq u \beta(x, s, u) \}. \quad (9.8)$$

Justification of this claim is given further in detail in Section 13.

**Definition 9.2.** Function $u \in L^\infty(G_{T,S}) \cap L^2((0, T) \times (0, S); \tilde{W}^1_2(\Omega))$ is called an entropy solution of Problem $\Pi_0$, if it satisfies the entropy inequality

$$\partial_\tau \eta(u) + \partial_s q_a(u) + \text{div}_x q_p(u) - \Delta_x \eta(u) \leq -\eta''(u) |\nabla_x u|^2, \quad (x, t, s) \in G_{T,S} \setminus \{ t = \tau \}, \quad (9.9a)$$

the maximum principle

$$\| u \|_{L^\infty(\Xi^1 \times (0, \tau))} \leq M_2, \quad \| u \|_{L^\infty(\Xi^1 \times (\tau, T))} \leq M_3, \quad (9.9b)$$

the initial condition

$$\text{ess sup}_{t \to 0^+} \int_{\Xi^1} \left| u(x, t, s) - u_0^{(1)}(x, s) \right| \, dx \, ds = 0, \quad (9.9c)$$

the entropy boundary conditions

$$q_a(u_0^{tr.(2)}(x, t)) - q_a(u_0^{(2)}(x, t)) - \eta'(u_0^{(2)}(x, t))(a(u_0^{tr.(2)}(x, t)) - a(u_0^{(2)}(x, t))) \leq 0, \quad (x, t) \in \Xi^2, \quad (9.9d)$$

$$q_a(u_S^{tr.(2)}(x, t)) - q_a(u_S^{(2)}(x, t)) - \eta'(u_S^{(2)}(x, t))(a(u_S^{tr.(2)}(x, t)) - a(u_S^{(2)}(x, t))) \geq 0, \quad (x, t) \in \Xi^2. \quad (9.9e)$$

and the impulsive condition $(9.3b)$. In $(9.9a)$, $(9.9d)$ and $(9.9e)$, $\eta \in C^2(\mathbb{R})$ is an arbitrary convex test-function, i.e., $\eta''(\lambda) \geq 0 \forall \lambda \in \mathbb{R}$, and $(\eta, q_a, q_p)$ is a convex entropy flux triple:

$$q_a'(\lambda) = a'(\lambda) \eta'(\lambda), \quad q_p'(\lambda) = \varphi'(\lambda) \eta'(\lambda), \quad \lambda \in \mathbb{R}.$$
We are going to fulfill and rigorously justify the limiting passage from Problem $\Pi_\gamma$ (with $Z_\gamma = K_\gamma \beta$) to Problem $\Pi_0$ as $\gamma \to 0+$, and to establish the well-posedness of Problem $\Pi_0$. More precisely, in Sections 10–14 further we prove the following theorem, which is the second main result of the article.

**Theorem 9.1. 1. (Convergence result.)** Let the source term $Z_\gamma$ in Problem $\Pi_\gamma$ have the form \[ [13] \], where functions $K_\gamma$ and $\beta$ satisfy Conditions on $K_\gamma \& \beta$. Let the nonlinearities $a = a(\lambda)$ and $\varphi = \varphi(\lambda)$ satisfy the demands of items (i) and (ii) in Conditions on $a \& \varphi \& Z_\gamma$ and the additional demands

\[
\max_{\lambda \in \mathbb{R}} |a'(\lambda)| \leq C_5, \quad \max_{\lambda \in \mathbb{R}} |\varphi'(\lambda)| \leq C_5, \quad C_5 = \text{const} < +\infty. \quad (9.10)
\]

Then there exists the unique limiting function $u_* \in L^\infty(G_{T,S}) \cap L^2((0,T) \times (0,S); \tilde{W}_2^1(\Omega))$ of the family $\{u_\gamma\}_{\gamma > 0}$ of kinetic and entropy solutions of Problem $\Pi_\gamma$ as $\gamma \to 0+$, such that

\[
u_\gamma \rightarrow u_* \quad \text{strongly in } L^1(G_{T,S}),
\]

\[
u_\gamma \rightarrow u_* \quad \text{weakly in } L^2((0,T) \times (0,S); \tilde{W}_2^1(\Omega)).
\]

Furthermore, $u_*$ is a kinetic and entropy solution of Problem $\Pi_0$ in the sense of Definitions [7,7] and [7,2].

**2. (Existence, uniqueness and stability of kinetic solutions to Problem $\Pi_0$.)** Let the nonlinearities $a = a(\lambda)$ and $\varphi = \varphi(\lambda)$ satisfy the demands of items (i) and (ii) in Conditions on $a \& \varphi \& Z_\gamma$. Whenever the impulsive perturbation $\beta = \beta(x,s,\lambda)$ belongs to $C^1_{\text{loc}}(\mathbb{R}_x \times \mathbb{R}_s)$ and initial and final data meet Conditions on $u_0^{(1)} \& u_0^{(2)} \& u_\gamma$, Problem $\Pi_0$ has the unique kinetic solution $u_* = u_*(x,t,s)$ in the sense of Definition [7,7].

Moreover, let $u_{1,1}$ and $u_{2,2}$ be two kinetic solutions of Problem $\Pi_0$ corresponding to two given sets of data $(\beta_1, u_{1,1}, u_{1,1}, u_{1,1})$ and $(\beta_2, u_{2,2}, u_{2,2}, u_{2,2})$, respectively, then the estimate

\[
\|u_{1,1}(\cdot,t,\cdot) - u_{2,2}(\cdot,t,\cdot)\|_{L^1(\Omega)} \leq
\|u_{1,0}^{(1)} - u_{2,0}^{(1)}\|_{L^1(\Omega)} + \|a'\|_{C([-M_4(T),M_4(T)])} \left(\|u_{1,0}^{(2)} - u_{2,0}^{(2)}\|_{L^1(\Omega \times (t,0))} + \|u_{1,1}^{(2)} - u_{2,1}^{(2)}\|_{L^1(\Omega \times (t,0))}\right) +
\left[1_{t > T} \left(\|\beta_1 - \beta_2\|_{C(\Omega \times [-M_5(\tau),M_5(\tau)])} + \max_{\Omega \times [-M_5(\tau),M_5(\tau)]} \|\beta'(x,s,\lambda)\|_{L^1(\Omega)}\right)\right]
\|u_{1,1}^{(2)} - u_{1,1}^{(2)}\|_{L^1(\Omega \times (0,t))} + \|u_{1,1}^{(2)} - u_{2,1}^{(2)}\|_{L^1(\Omega \times (0,t))}\right),
\]

\[ t \in (0,T], \tag{9.13} \]

holds true.

Here,

\[
M_4(t) := \max\left\{ M_5(\tau) + \|\beta_1\|_{C(\Omega \times [-M_5(\tau),M_5(\tau)])}, \|u_{1,0}^{(2)}\|_{L^\infty(\Omega \times (t,0))}, \|u_{1,1}^{(2)}\|_{L^\infty(\Omega \times (t,0))}\right\}
\]

\[ t \in (\tau,T] \tag{9.14} \]

and

\[
M_5(t) := \max\left\{ \|u_{1,0}^{(1)}\|_{L^\infty(\Omega)}, \|u_{1,0}^{(2)}\|_{L^\infty(\Omega \times (t,0))}, \|u_{1,1}^{(2)}\|_{L^\infty(\Omega \times (t,0))}\right\}
\]

\[ t \in (0,T] \tag{9.14} \]
3. (Equivalency of the notions of kinetic and entropy solutions.) Function \( u_\ast \) is an entropy solution of Problem \( \Pi_0 \) in the sense of Definition \( 9.2 \) if and only if it is a kinetic solution in the sense of Definition \( 9.1 \).

We are going to fulfill justification of Theorem \( 9.1 \) in the reverse order. That is, firstly, we prove assertion 3. Secondly, we prove assertion 2. Finally, we establish assertion 1 (convergence result), which is the most difficult part in the proof of Theorem \( 9.1 \).

Remark 9.3. From the arguments of Section \( 10 \) below, it becomes clear that the impulsive perturbation \( \beta = \beta(x, s, \lambda) \) and the nonlinearities \( a = a(\lambda) \) and \( \varphi = \varphi(\lambda) \) may not necessarily meet restrictions \( (1.9d) \) and \( (9.10) \) in assertions 2 and 3 of Theorem \( 9.1 \).

10 Proof of assertions 2 and 3 of Theorem \( 9.1 \)

The proof of assertion 3 of Theorem \( 9.1 \) is quite similar to the proof of assertion 3 of Theorem \( 3.1 \) in Section \( 8 \). Now let us show that assertion 2 of Theorem \( 9.1 \) directly follows from assertion 1 of Theorem \( 3.1 \).

We clearly notice that, in order to find a kinetic solution of Problem \( \Pi_0 \), it is necessary and sufficient to fulfill the following two steps. Firstly, in the subdomain \( \Omega \times (0, \tau) \times (0, T) \subset G_{T, S} \), we find the kinetic solution \( u_\ast \) of Problem \( \Pi_\gamma \), provided with the homogeneous right-hand side \( Z_\gamma \equiv 0 \) and the initial data \( (1.11) \). Secondly, in the subdomain \( \Omega \times (\tau, T) \times (0, S) \subset G_{T, S} \) we find the kinetic solution \( u_\ast \) of Problem \( \Pi_s \) provided with \( Z_\gamma \equiv 0 \) and the initial data

\[
\left. u_\ast (x, t, s) \right|_{t=\tau} = u_\ast (x, \tau - 0, s) + \beta(x, s, u_\ast (x, \tau - 0, s)), \quad (x, s) \in \Xi. \tag{10.1}
\]

In the right-hand side of \( (10.1) \), \( u_\ast (x, \tau - 0, s) \) is the trace on \( \{ t = \tau - 0 \} \) of the kinetic solution \( u_\ast \) obtained on the first step of the above described procedure.

On the strength of assertion 1 of Theorem \( 3.1 \) each of the problems considered on the first and second steps has the unique kinetic solution. Thus, Problem \( \Pi_0 \) has the unique kinetic solution in the sense of Definition \( 9.1 \). The maximum principle \( (9.9b) \) directly follows from \( (2.2) \), since we have \( b^{(1)} = b^{(2)} = 0 \) in \( (2.2) \) in the case when \( Z_\gamma \equiv 0 \).

Analogously, estimate \( (9.13) \) follows from estimate \( (3.4) \). More precisely, estimate \( (3.4) \) with \( Z_\gamma \equiv 0 \) reads as follows:

\[
\| u_{s1}(\cdot, t, \cdot) - u_{s2}(\cdot, t, \cdot) \|_{L^1(\Xi)} \leq \| u_{1,0}^{(1)} - u_{2,0}^{(1)} \|_{L^1(\Xi)} + \| a' \|_{C[-M_5(t), M_5(t)]} \left( \| u_{1,0}^{(2)} - u_{2,0}^{(2)} \|_{L^1(\Omega \times (0, t))} + \| u_{1,S}^{(2)} - u_{2,S}^{(2)} \|_{L^1(\Omega \times (0, t))} \right) \quad \text{for } t \in (0, \tau) \tag{10.2}
\]

and

\[
\| u_{s1}(\cdot, t, \cdot) - u_{s2}(\cdot, t, \cdot) \|_{L^1(\Xi)} \leq \| u_{s1}(\cdot, \tau - 0, \cdot) + \beta_1(\cdot, x, u_{s1}(\cdot, \tau - 0, \cdot)) - u_{s2}(\cdot, \tau - 0, \cdot) - \beta_2(\cdot, x, u_{s2}(\cdot, \tau - 0, \cdot)) \|_{L^1(\Xi)} + \| a' \|_{C[-M_4(t), M_4(t)]} \left( \| u_{1,0}^{(2)} - u_{2,0}^{(2)} \|_{L^1(\Omega \times (\tau, t))} + \| u_{1,S}^{(2)} - u_{2,S}^{(2)} \|_{L^1(\Omega \times (\tau, t))} \right) \quad \text{for } t \in [\tau, T]. \tag{10.3}
\]
We evaluate in the right-hand side of (10.3) to get

$$\|u_{1,0} - u_{2,0}\|_{L^1(\Xi^1)} + \|a'|_{C[-M_5(\tau),M_5(\tau)]}(\|u_{1,0}^{(1)} - u_{2,0}^{(1)}\|_{L^1(\Xi^1)} + \|u_{1,0}^{(2)} - u_{2,0}^{(2)}\|_{L^1(\Omega \times (0,\tau))})$$

$$\max_{\Xi^1 \times [-M_5(\tau),M_5(\tau)]} \left| \partial_{x_3} \beta_1(x,s,\lambda) \left[ \|u_{1,0}^{(1)} - u_{2,0}^{(1)}\|_{L^1(\Xi^1)} + \|a'|_{C[-M_5(\tau),M_5(\tau)]}(\|u_{1,0}^{(2)} - u_{2,0}^{(2)}\|_{L^1(\Omega \times (0,\tau))}) \right] \right) + \text{(meas } \Xi^1) \| \beta_1 - \beta_2 \|_{C(\Xi^1 \times [-M_5(\tau),M_5(\tau)])}. \quad (10.4)$$

Remark that\( \text{meas } \Xi^1 = S \text{meas } \Omega, [-M_5(t),M_5(t)] \subset [-M_5(\tau),M_5(\tau)] \forall t \in (0,\tau], \) and\([ -M_5(\tau),M_5(\tau)] \subset [-M_4(t),M_4(t)] \subset [-M_4(\tau),M_4(\tau)] \forall t \in [\tau,T). \) Taking this into account and combining (10.2), (10.3) and (10.4), we arrive at (9.13), which completes the proof of assertion 2 of Theorem 9.1

### 11 Uniform in \( \gamma \) estimates of the family \( \{u_{\gamma}\}_{\gamma > 0} \)

In order to pass to the limit as \( \gamma \to 0^+ \) in the ultra-parabolic equation (9.1) on the rigorous mathematical level, we need to establish some appropriate uniform in \( \gamma \) estimates of the family of kinetic solutions \( u_\gamma \) of Problem \( \Pi_{\gamma}, \gamma > 0. \) To this end, at first, we revisit the strictly regularized formulation, i.e., Problem \( \Pi_{\gamma,\epsilon} \) incorporating the source term (1.8), and build the refined energy estimate.

**Lemma 11.1.** Let the source term \( Z_\gamma \) in Problem \( \Pi_{\gamma,\epsilon} \) have the form (1.8), where functions \( K_\gamma \) and \( \beta \) satisfy Conditions on \( K_\gamma,\&, \beta. \) Let the nonlinearities \( a = a(\lambda) \) and \( \varphi = \varphi(\lambda) \) satisfy demands of items (i) and (ii) in Conditions on \( a\&, \varphi, Z_\gamma \) and the additional bounds (9.10).

Set

$$\gamma_0 := \min\{\tau, T - \tau\}. \quad (11.1)$$

Then the family of classical solutions \( \{u_{\gamma,\epsilon}\}_{\epsilon \in (0,\gamma_0]} \) of Problem \( \Pi_{\gamma,\epsilon} \) satisfies the energy estimate

$$\|u_{\gamma,\epsilon}(t')\|_{L^2(\Xi^1)} + \|\nabla_{x_3} u_{\gamma,\epsilon}\|_{L^2(\Xi^1 \times (0,t'))} + \sqrt{\epsilon}\|\partial_{x_3} u_{\gamma,\epsilon}\|_{L^2(\Xi^1 \times (0,t'))} \leq C_6 \forall t' \in (0,T], \quad (11.2)$$

where the positive constant \( C_6 \) does not depend on \( \gamma \) and \( \epsilon. \) Thus, (11.2) is the uniform bound on the family \( \{u_{\gamma,\epsilon}\}_{\epsilon \in (0,\gamma_0]} \).

The constant \( C_6 \) will be defined explicitly in the proof, see formula (11.17) below.

**Proof.** Justification of Lemma 11.1 is fulfilled by means of the standard techniques for derivation of energy estimates.

Notice that due to Conditions on \( u_0^{(1)}\&, u_0^{(2)}\&, u_S^{(2)} \) we can take \( \hat{u} \) belonging to \( C^{2+\alpha}(G_{T,S}). \) For example \( \hat{u} \) can be constructed explicitly in terms of \( u_0^{(1)}\&, u_0^{(2)}\& \) and \( u_S^{(2)} \) as the solution of the nonhomogeneous Dirichlet problem for Laplace’s equation (12, Section 2.2.4):

$$\Delta_{x,s,t}\hat{u} = 0 \text{ in } G_{T,S}, \quad \hat{u}|_{G_1} = 0, \quad \hat{u}|_{G_2^1} = u_0^{(1)}, \quad \hat{u}|_{G_2^2} = 0, \quad \hat{u}|_{G_3^2} = u_0^{(2)}, \quad \hat{u}|_{G_3} = u_S^{(2)}. \quad (12)$$
Rewrite (9.1) in the form
\[ \partial_t(u \gamma - \hat{u}) + \partial_s a(u \gamma) + \text{div}_x \varphi(u \gamma) - \Delta_x (u \gamma - \hat{u}) - \varepsilon \partial^2_x (u \gamma - \hat{u}) - K_\gamma(t, \tau)(\beta(x, s, u \gamma) - \beta(x, s, \hat{u})) = - \partial_t \hat{u} + \Delta_x \hat{u} + \varepsilon \partial^2_x \hat{u} + K_\gamma(t, \tau) \beta(x, s, \hat{u}). \] (11.3)

Multiply the both sides of (11.3) by \( u \gamma - \hat{u} \), integrate over \( \Xi^1 \times (0, t') \), where value \( t' \in (0, T] \) is taken arbitrarily, and consider terms in the resulting equality one by one. Applying integration by parts and Green’s formula, we derive

\[
\int_{\Xi^1} \int_0^{t'} (\partial_t(u \gamma - \hat{u}))(u \gamma - \hat{u}) \, dx \, ds \, dt =
\]

\[
\frac{1}{2} \int_{\Xi^1} |u \gamma(x, t', s) - \hat{u}(x, t', s)|^2 \, dx \, ds - \frac{1}{2} \int_{\Xi^1} |u_0^{(1)} - u_0^{(1)}|^2 \, dx \, ds =
\]

\[
\frac{1}{2} \int_{\Xi^1} |u \gamma(x, t', s) - \hat{u}(x, t', s)|^2 \, dx \, ds;
\]

\[
\int_{\Xi^1} \int_0^{t'} \partial_s a(u \gamma)(u \gamma - \hat{u}) \, dx \, ds \, dt = \int_{\Xi^1} \int_0^{t'} \partial_s \tilde{q}_a(u \gamma) \, dx \, ds \, dt - \int_{\Xi^1} \int_0^{t'} \partial_s a(u \gamma) \hat{u} \, dx \, ds \, dt =
\]

\[
\int_{\Xi^1} \int_0^{t'} \left( \tilde{q}_a(u_S^{(2)}) - \tilde{q}_a(u_0^{(2)}) - u_S^{(2)} a(u_S^{(2)}) + u_0^{(2)} a(u_0^{(2)}) \right) \, dx \, dt + \int_{\Xi^1} \int_0^{t'} a(u \gamma) \partial_s \hat{u} \, dx \, ds \, dt =
\]

\[
\int_{\Xi^1} \int_0^{t'} \left( \tilde{q}_a(u_S^{(2)}) - \tilde{q}_a(u_0^{(2)}) + A(u_S^{(2)}) - A(u_0^{(2)}) - u_S^{(2)} a(u_S^{(2)}) + u_0^{(2)} a(u_0^{(2)}) \right) \, dx \, dt
\]

\[
+ \int_{\Xi^1} \int_0^{t'} (a(u \gamma) - a(\hat{u})) \partial_s \hat{u} \, dx \, ds \, dt = \int_{\Xi^1} \int_0^{t'} (a(u \gamma) - a(\hat{u})) \partial_s \hat{u} \, dx \, ds \, dt;
\]

where \( \tilde{q}_a(\lambda) = \int_0^\lambda a'(\lambda) \lambda d\lambda, A(\lambda) = \int_0^\lambda a(\lambda) d\lambda \), and we have noticed that

\[
\tilde{q}_a(\lambda) + A(\lambda) - a(\lambda) \lambda = 0 \quad \forall \lambda \in \mathbb{R};
\]
\[
\int_0^{t'} \int_\Omega (\text{div}_x \varphi(u_{\gamma \varepsilon})) (u_{\gamma \varepsilon} - \hat{u}) \, dx \, ds \, dt =
\]
\[
\int_0^{t'} \int_\partial \Omega \tilde{q}_\varphi(u_{\gamma \varepsilon}(\sigma, t, s)) \cdot n(\sigma) \, ds \, dt - \int_0^{t'} \int_\partial \Omega \varphi(u_{\gamma \varepsilon}(\sigma, t, s)) \hat{u}(\sigma, t, s) \cdot n(\sigma) \, ds \, dt +
\]
\[
\int_0^{t'} \int_\Omega \varphi(u_{\gamma \varepsilon}) \cdot \nabla_x \hat{u} \, dx \, ds \, dt = \int_0^{t'} \int_\Omega \varphi(\hat{u}) \cdot \nabla_x \hat{u} \, dx \, ds \, dt
\]
\[
\int_0^{t'} \int_\Omega (\varphi(u_{\gamma \varepsilon}) - \varphi(\hat{u})) \cdot \nabla_x \hat{u} \, dx \, ds \, dt =
\]
\[
\int_0^{t'} \int_\Omega \tilde{q}_{\varphi}(\hat{u}(\sigma, t, s)) \cdot n(\sigma) \, ds \, dt =
\]
\[
\int_0^{t'} \int_\Omega (\varphi(u_{\gamma \varepsilon}) - \varphi(\hat{u})) \cdot \nabla_x \hat{u} \, dx \, ds \, dt,
\]
where \(\tilde{q}_\varphi(\lambda) = \int_0^\lambda \varphi'(\lambda) \lambda \, d\lambda\), \(n = n(\sigma)\) is the unit outward normal to \(\partial \Omega\), and we have noticed that \(\tilde{q}_\varphi(u_{\gamma \varepsilon})\), \(\tilde{q}_\varphi(\hat{u})\) and \(\hat{u}\) vanish on \(\partial \Omega\);

\[
- \int_0^{t'} \int_\Omega (\Delta_x(u_{\gamma \varepsilon} - \hat{u}))(u_{\gamma \varepsilon} - \hat{u}) \, dx \, ds \, dt = \int_0^{t'} \int_\Omega |\nabla_x(u_{\gamma \varepsilon} - \hat{u})|^2 \, dx \, ds \, dt;
\]

\[
- \int_0^{t'} \int_\Omega (\partial_s^2(u_{\gamma \varepsilon} - \hat{u}))\varepsilon(u_{\gamma \varepsilon} - \hat{u}) \, dx \, ds \, dt = \int_0^{t'} \int_\Omega \varepsilon|\partial_s(u_{\gamma \varepsilon} - \hat{u})|^2 \, dx \, ds \, dt.
\]

Using these representations, integrating by parts in \(s\), using Green’s formula with respect to \(x\) in the right-hand side, and properly arranging summands, from (11.3) we derive the first energy identity as follows:

\[
\frac{1}{2} \int_\Omega |u_{\gamma \varepsilon}(x, t', s) - \hat{u}(x, t', s)|^2 \, dx \, ds +
\]
\[
\int_0^{t'} \int_\Omega |\nabla_x(u_{\gamma \varepsilon} - \hat{u})|^2 \, dx \, ds \, dt + \varepsilon \int_0^{t'} \int_\Omega |\partial_s(u_{\gamma \varepsilon} - \hat{u})|^2 \, dx \, ds \, dt =
\]
\[
- \int_0^{t'} \int_\Omega (a(u_{\gamma \varepsilon}) - a(\hat{u})) \partial_s \hat{u} \, dx \, ds \, dt - \int_0^{t'} \int_\Omega (\varphi(u_{\gamma \varepsilon}) - \varphi(\hat{u})) \cdot \nabla_x \hat{u} \, dx \, ds \, dt +
\]
\[
\int_0^{t'} \int_\Omega (\beta(x, s, u_{\gamma \varepsilon}) - \beta(x, s, \hat{u})) (u_{\gamma \varepsilon} - \hat{u}) \, dx \, ds \, dt - \int_0^{t'} \int_\Omega \partial_t \hat{u}(u_{\gamma \varepsilon} - \hat{u}) \, dx \, ds \, dt -
\]
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\[
\int_0^{t'} \int_{\Xi^1} \nabla_x \hat{u} \cdot \nabla (u_{\gamma \epsilon} - \hat{u}) \, dx \, ds \, dt - \varepsilon \int_0^{t'} \int_{\Xi^1} \partial_s \hat{u} \partial_s (u_{\gamma \epsilon} - \hat{u}) \, dx \, ds \, dt + \\
\int_0^{t'} K_\gamma (t, \tau) \int_{\Xi^1} \beta(x, s, \hat{u}) (u_{\gamma \epsilon} - \hat{u}) \, dx \, ds \, dt, \quad t' \in (0, T].
\]

From the Lagrange mean value theorem it follows that

\[
|a(u_{\gamma \epsilon}) - a(\hat{u})| \leq \max_{\lambda \in \mathbb{R}} |a'(\lambda)||u_{\gamma \epsilon} - \hat{u}| \leq C_5 |u_{\gamma \epsilon} - \hat{u}|, \quad (11.5)
\]

\[
|\varphi(u_{\gamma \epsilon}) - \varphi(\hat{u})| \leq \max_{\lambda \in \mathbb{R}} |\varphi'(\lambda)||u_{\gamma \epsilon} - \hat{u}| \leq C_5 |u_{\gamma \epsilon} - \hat{u}|, \quad (11.6)
\]

and

\[
|\beta(x, s, u_{\gamma \epsilon}) - \beta(x, s, \hat{u})| \leq \left( \max_{(x, s, \lambda) \in \Xi^1 \times \mathbb{R}} |\partial_\lambda \beta(x, s, \lambda)| \right) |u_{\gamma \epsilon} - \hat{u}| \leq b_0 |u_{\gamma \epsilon} - \hat{u}|. \quad (11.7)
\]

Using these estimates and Young’s inequality, from (11.4) we derive the following inequality:

\[
\frac{1}{2} \|u_{\gamma \epsilon}(t') - \hat{u}(t')\|^2_{L^2(\Xi^1)} + \left( 1 - \frac{\delta_4}{2} \right) \int_0^{t'} \|\nabla_x (u_{\gamma \epsilon}(t) - \hat{u}(t))\|^2_{L^2(\Xi^1)} \, dt + \\
\left( 1 - \frac{\delta_5}{2} \right) \varepsilon \int_0^{t'} \|\partial_s (u_{\gamma \epsilon}(t) - \hat{u}(t))\|^2_{L^2(\Xi^1)} \, dt \leq \\
\int_0^{t'} \left( \frac{\delta_1}{2} C_5 + \frac{\delta_2}{2} C_5 + \frac{\delta_3}{2} C_5 + \frac{\delta_6}{2} K_\gamma (t, \tau) + b_0 K_\gamma (t, \tau) \right) \|u_{\gamma \epsilon}(t) - \hat{u}(t)\|^2_{L^2(\Xi^1)} \, dt + \\
\int_0^{t'} \left( \frac{C_5}{2 \delta_1} + \frac{1}{2 \delta_4} \right) \|\nabla_x \hat{u}(t)\|^2_{L^2(\Xi^1)} + \left( \frac{C_5}{2 \delta_1} + \frac{\varepsilon}{2 \delta_5} \right) \|\partial_s \hat{u}(t)\|^2_{L^2(\Xi^1)} + \\
\frac{1}{2 \delta_3} \|\partial_t \hat{u}(t)\|^2_{L^2(\Xi^1)} + \frac{1}{2 \delta_6} \|\beta\|^2_{C(\Xi^1 \times [-M_6, M_6])} (\text{meas } \Omega)S K_\gamma (t, \tau) \right) \, dt, \quad t' \in (0, T], \quad (11.8)
\]

where \(M_6 = \|\hat{u}\|_{C(\bar{\Omega}, T, S)}\) and \(\delta_k \in \mathbb{R} (k = 1, \ldots, 6)\) are arbitrary. Take \(\delta_1 = \delta_2 = \ldots = \delta_6 = 1\) and denote

\[
C_7(\gamma, t) := 2C_5 + 1 + (1 + 2b_0)K_\gamma (t, \tau)
\]

and

\[
C_8(\gamma, \varepsilon, t) := (C_5 + 1)\|\nabla_x \hat{u}(t)\|^2_{L^2(\Xi^1)} + (C_5 + \varepsilon) \|\partial_s \hat{u}(t)\|^2_{L^2(\Xi^1)} + \\
\|\partial_t \hat{u}(t)\|^2_{L^2(\Xi^1)} + \|\beta\|^2_{C(\Xi^1 \times [-M_7, M_7])} (\text{meas } \Omega)S K_\gamma (t, \tau)
\]

for the sake of conciseness. With this choice of \(\delta_k\) and notation, inequality (11.8) reduces to

\[
\|u_{\gamma \epsilon}(t') - \hat{u}(t')\|^2_{L^2(\Xi^1)} + \int_0^{t'} \|\nabla_x (u_{\gamma \epsilon}(t) - \hat{u}(t))\|^2_{L^2(\Xi^1)} \, dt + \varepsilon \int_0^{t'} \|\partial_s (u_{\gamma \epsilon}(t) - \hat{u}(t))\|^2_{L^2(\Xi^1)} \, dt \leq \\
\int_0^{t'} \left( C_7(\gamma, t)\|u_{\gamma \epsilon}(t) - \hat{u}(t)\|^2_{L^2(\Xi^1)} + C_8(\gamma, \varepsilon, t) \right) \, dt \quad \forall t' \in (0, T]. \quad (11.11)
\]
Here, discarding the second and the third terms in the left-hand side and applying Grönwall’s lemma, we establish the bound
\[
\|u_{\gamma}(t') - \hat{u}(t')\|_{L^2(\Xi^1)}^2 \leq \exp \left\{ \int_0^{t'} C_7(\gamma, t') \, dt \right\} \left( \int_0^{t'} C_8(\gamma, \varepsilon, t) \exp \left\{ - \int_0^t C_7(\gamma, \xi) \, d\xi \right\} \, dt \right). 
\] (11.12)

Combining (11.11) and (11.12) we obtain the estimate
\[
\|u_{\gamma}(t') - \hat{u}(t')\|_{L^2(\Xi^1)}^2 + \int_0^{t'} \|\nabla_x (u_{\gamma}(t) - \hat{u}(t))\|_{L^2(\Xi^1)}^2 \, dt + \varepsilon \int_0^{t'} \|\partial_s (u_{\gamma}(t) - \hat{u}(t))\|_{L^2(\Xi^1)}^2 \, dt \leq \int_0^{t'} \left( C_7(\gamma, t) \exp \left\{ \int_0^t C_7(\gamma, t''') \, dt''' \right\} \left( \int_0^t C_8(\gamma, \varepsilon, t''') \exp \left\{ - \int_0^{t''} C_7(\gamma, \xi) \, d\xi \right\} \, dt''\right) + C_8(\gamma, \varepsilon, t) \right) \, dt 
\forall t' \in (0, T]. 
\] (11.13)

In the right-hand side of (11.13) we easily see that
\[
0 \leq \int_0^t C_7(\gamma, t) \, dt'' \leq 2C_5T + 1 + 2b_0 \equiv C_9 \quad \forall \gamma \in (0, \gamma_0), \quad \forall t \in (0, T], 
\] (11.14)

\[
0 \leq \int_0^t C_8(\gamma, \varepsilon, t'') \, dt'' \leq (C_5 + 1) \left( \|\nabla_x \hat{u}\|_{L^2(0,T;L^2(\Xi^1))}^2 + \|\partial_s \hat{u}\|_{L^2(0,T;L^2(\Xi^1))}^2 \right) + \|\partial_t \hat{u}\|_{L^2(0,T;L^2(\Xi^1))}^2 + S \text{ meas } \Omega \|\beta\|_{C^1(\Xi^1 \times [-M_0, M_0])}^2 \equiv C_{10} \quad \forall \gamma \in (0, \gamma_0), \quad \forall \varepsilon \in (0, 1], \quad \forall t \in (0, T] 
\] (11.15)
due to (11.9), (11.10) and due to the fact that supp $K_\gamma(\cdot, \tau) \subset [0, T]$ for $\gamma \in (0, \gamma_0]$, and therefore $\int_0^T K_\gamma(t'', \tau) \, dt'' = 1$. Using (11.14) and (11.15), from (11.13) we derive the bound
\[
\|u_{\gamma}(t') - \hat{u}(t')\|_{L^2(\Xi^1)}^2 + \|\nabla_x (u_{\gamma}(t, \varepsilon) - \hat{u}(t, \varepsilon))\|_{L^2(\Xi^1 \times (0, t'))}^2 + \varepsilon \|\partial_s (u_{\gamma}(t, \varepsilon) - \hat{u}(t, \varepsilon))\|_{L^2(\Xi^1 \times (0, t'))}^2 \leq (C_9 e^{C_9} + 1)C_{10} \quad \forall \gamma \in (0, \gamma_0], \quad \forall \varepsilon \in (0, 1], \quad \forall t' \in (0, T]. 
\] (11.16)

Using the elementary inequality
\[
\frac{A_1 + A_2 + A_3}{3} \leq \sqrt[3]{\frac{A_1^2 + A_2^2 + A_3^2}{3}} \quad \forall A_1, A_2, A_3 \in \mathbb{R} 
\]
and the triangle inequality, from (11.16) we finally deduce (11.12) with
\[
C_6 = \sqrt{3C_{10}(C_9 e^{C_9} + 1) + \|\hat{u}\|_{L^\infty(0,T;L^2(\Xi^1))} + \|\nabla_x \hat{u}\|_{L^2(G_\tau; S)} + \|\partial_s \hat{u}\|_{L^2(G_\tau; S)}. 
\] (11.17)

Lemma 11.1 is proved. \qed

Recall that the kinetic equation (5.2a) with $Z_\gamma = K_\gamma \beta$ can be written in the form
\[
\partial_t \chi(\lambda; u_\gamma) + a'(\lambda) \partial_s \chi(\lambda; u_\gamma) + \varphi'(\lambda) \cdot \nabla_x \chi(\lambda; u_\gamma) - \Delta_x \chi(\lambda; u_\gamma) = \partial_\lambda(m_\gamma + n_\gamma + Z_\gamma), 
\]
\[
(x, t, s, \lambda) \in G_\tau; S \times \mathbb{R}_\lambda, 
\] (11.18)
This estimate completes the proof of Lemma 11.2.

Recall from Lemma 4.2 and Remark 7.3 that \( u_\gamma \) appears as the strong limiting point of the sequence \( \{u_{\gamma}\}_{\varepsilon \to 0} \). This fact along with Lemma 11.1 has the following implications.

**Lemma 11.2.** (i) The family \( \{u_{\gamma}\}_{\gamma \in (0, \gamma_0)} \) satisfies the energy estimate

\[
\|u_{\gamma}(t')\|_{L^2(\Omega)} + \|\nabla_x u_{\gamma}\|_{L^2(\Omega \times (0, t'))} \leq C_6 \quad \forall t' \in (0, T].
\]  

(ii) The families of measures \( m_\gamma \in \mathcal{M}^+(T\times \mathbb{R}_\lambda) \) and \( n_\gamma = \delta_{(\lambda=u_\gamma)}|\nabla_x u_{\gamma}|^2 \in \mathcal{M}^+(T\times \mathbb{R}_\lambda) \) are bounded uniformly in \( \gamma \). More precisely,

\[
\|m_\gamma\|_{\mathcal{M}(T\times \mathbb{R}_\lambda)} \leq 3C_6^2, \quad \|n_\gamma\|_{\mathcal{M}(T\times \mathbb{R}_\lambda)} \leq C_6^2 \quad \forall \gamma \in (0, \gamma_0).
\]  

(iii) There exists a constant \( C_{11} > 0 \), independent of \( \gamma \), such that

\[
\|Z_\gamma\|_{L^1(T\times \mathbb{R}_\lambda)} \leq C_{11} \quad \forall \gamma \in (0, \gamma_0).
\]

**Proof.** (i) Energy estimate (11.20) immediately follows from (11.2), Remark 7.3 and the lower semicontinuity property:

\[
\|\nabla_x u_{\gamma}\|_{L^2(\Omega \times (0, t'))} \leq \liminf_{\varepsilon \to 0^+} \|\nabla_x u_{\gamma_{\varepsilon}}\|_{L^2(\Omega \times (0, t'))}.
\]

(ii) For any \( \phi \in L^\infty(T\times \mathbb{R}_\lambda) \) we have

\[
|\langle n_{\gamma}, \phi \rangle| = \left| \int_{T\times \mathbb{R}_\lambda} |\nabla_x u_{\gamma}|^2 \phi(x, t, s, u_{\gamma}) \, dx \, dt \right| \leq 
\|\nabla_x u_{\gamma}\|_{L^2(T\times \mathbb{R}_\lambda)}^2 \|\phi\|_{L^\infty(T\times \mathbb{R}_\lambda)} \leq C_6^2 \|\phi\|_{L^\infty(T\times \mathbb{R}_\lambda)},
\]

and

\[
|\langle m_{\gamma}, \phi \rangle| = \left| \lim_{\varepsilon \to 0^+} \delta_{(\lambda=u_{\gamma_{\varepsilon}})} \left( |\nabla_x u_{\gamma_{\varepsilon}}|^2 + \varepsilon |\partial_s u_{\gamma_{\varepsilon}}|^2 \right) - \delta_{(\lambda=u_{\gamma})} |\nabla_x u_{\gamma}|^2, \phi \right| = 
\left| \int_{T\times \mathbb{R}_\lambda} \left( \lim_{\varepsilon \to 0^+} \left( |\nabla_x u_{\gamma_{\varepsilon}}|^2 + \varepsilon |\partial_s u_{\gamma_{\varepsilon}}|^2 \right) \phi(x, t, s, u_{\gamma_{\varepsilon}}) - |\nabla_x u_{\gamma}|^2 \phi(x, t, s, u_{\gamma}) \right) \, dx \, dt \right| \leq 
3C_6^2 \|\phi\|_{L^\infty(T\times \mathbb{R}_\lambda)}
\]

due to (11.20), (4.18) and (4.19). This immediately yields the bounds (11.21).

(iii) Simple evaluation gives

\[
\|Z_\gamma\|_{L^1(T\times \mathbb{R}_\lambda)} = 
\int_{T\times \mathbb{R}_\lambda} \left| \int_0^\lambda 1_{(\lambda \geq u_{\gamma})} K_{\gamma}(t, \tau, \beta(x, s, \lambda)) - \int_0^\lambda 1_{(\lambda' \geq u_{\gamma})} K_{\gamma}(t, \tau, \partial_{\lambda'} \beta(x, s, \lambda')) \, d\lambda' \right| \, dx \, dt \, d\lambda 
\leq 
\int_0^T \left( 2b_1 S(\text{meas } \Omega) \|\beta\|_{C^0(\mathbb{R}_\gamma \times [-b_1, b_1])} + 2b_1^2 S(\text{meas } \Omega) \|\partial_\lambda \beta\|_{C^0(\mathbb{R}_\gamma \times [-b_1, b_1])} \right) K_{\gamma}(t, \tau) \, dt =
2b_1 S(\text{meas } \Omega) \left( \|\beta\|_{C^0(\mathbb{R}_\gamma \times [-b_1, b_1])} + \|\partial_\lambda \beta\|_{C^0(\mathbb{R}_\gamma \times [-b_1, b_1])} \right) = C_{12}.
\]

This estimate completes the proof of Lemma 11.2.
Lemma 11.3. Let the source term $Z_\gamma$ in Problem $\Pi_{\gamma \varepsilon}$ have the form (18), where functions $K_\gamma$ and $\beta$ satisfy Conditions on $K_{\gamma, \lambda} \& \beta$. Let the nonlinearities $a = a(\lambda)$ and $\varphi = \varphi(\lambda)$ satisfy the demands of items (i) and (ii) in Conditions on $a(\gamma) \varphi(\gamma Z_\gamma)$. Set $\gamma_0 := \min\{\gamma, T - \tau\}$. Then the family of classical solutions $\{u_{\gamma \varepsilon}\}_{\gamma \in (0, \gamma_0 / 2]}$ of Problem $\Pi_{\gamma \varepsilon}$ satisfies the maximum principle

$$
\|u_{\gamma \varepsilon}(\cdot, t', \cdot)\|_{L^\infty(\Xi)} \leq e^{\xi t'} \max\{\|u_{0(\varepsilon)}^{(1)}\|_{L^\infty(\Xi)}, \|u_{0(\varepsilon)}^{(2)}\|_{L^\infty(\Xi)}, \|u_{a(\varepsilon)}^{(2)}\|_{L^\infty(\Xi)}\} \overset{\text{def}}{=} M_1(t')
$$

$$
\forall t' \in (0, T], \forall \gamma \in (0, \gamma_0 / 2], \forall \varepsilon \in (0, 1],
$$

(11.23)

where

$$
\xi_\varepsilon = \begin{cases}
0 & \text{for } b_1 \leq \|u_{0(\varepsilon)}^{(1)}\|_{L^\infty(\Xi)} - 1,
\frac{2}{2\tau - \gamma_0} \ln \frac{b_1 + 1}{\|u_{0(\varepsilon)}^{(1)}\|_{L^\infty(\Xi)}} & \text{for } b_1 > \|u_{0(\varepsilon)}^{(1)}\|_{L^\infty(\Xi)} - 1.
\end{cases}
$$

(11.24)

In particular, $\{u_{\gamma \varepsilon}\}_{\gamma \in (0, \gamma_0 / 2]}$ is uniformly in $\gamma$ and $\varepsilon$ bounded in $L^\infty(G_T, S)$.

Proof. Substitute $u_{\gamma \varepsilon}(x, t, s) = e^{\xi t} \omega_{\gamma \varepsilon}(x, t, s)$ into (11.1). Here $\omega_{\gamma \varepsilon} = \omega_{\gamma \varepsilon}(x, t, s)$ is a new dependent variable and $\xi$ is an arbitrarily fixed positive parameter. Upon division by $e^{\xi t}$, we get

$$
\partial_t \omega_{\gamma \varepsilon} + \partial_s \left(e^{-\xi t} a(\omega_{\gamma \varepsilon} e^{\xi t})\right) + \text{div}_x \left(e^{-\xi t} \varphi(\omega_{\gamma \varepsilon} e^{\xi t})\right) =
\Delta_x \omega_{\gamma \varepsilon} + \varepsilon \partial^2_s \omega_{\gamma \varepsilon} + K_\gamma(t, \tau) e^{-\xi t} \beta(\gamma, s, \omega_{\gamma \varepsilon} e^{\xi t}) - \xi \omega_{\gamma \varepsilon}.
$$

(11.25)

Also we have

$$
\omega_{\gamma \varepsilon} \big|_{t=0} = u_{0(\varepsilon)}^{(1)}, \quad \omega_{\gamma \varepsilon} \big|_{s=0} = u_{0(\varepsilon)}^{(2)} e^{-\xi t}, \quad \omega_{\gamma \varepsilon} \big|_{s=S} = u_{2s}^{(2)} e^{-\xi t}.
$$

Set $M_\varepsilon = \|\omega_{\gamma \varepsilon}\|_{L^\infty(G^{1,1}\cup R_{\text{g}}, R_{\text{g}})}$ and

$$
\omega_{\gamma \varepsilon}^{M_\varepsilon} = (\omega_{\gamma \varepsilon} - M_\varepsilon)^+ \overset{\text{def}}{=} \max\{\omega_{\gamma \varepsilon} - M_\varepsilon, 0\}.
$$

(11.26)

Remark that

$$
\omega_{\gamma \varepsilon}^{M_\varepsilon} \big|_{\partial G_T, S \setminus \{t=T\}} = 0,
$$

(11.27)

$$
\partial_t \omega_{\gamma \varepsilon}^{M_\varepsilon} = \begin{cases}
\partial_t \omega_{\gamma \varepsilon} & \text{for } \omega_{\gamma \varepsilon} > M_\varepsilon,
0 & \text{for } \omega_{\gamma \varepsilon} \leq M_\varepsilon,
\end{cases}
$$

(11.28)

$$
\partial_s \omega_{\gamma \varepsilon}^{M_\varepsilon} = \begin{cases}
\partial_s \omega_{\gamma \varepsilon} & \text{for } \omega_{\gamma \varepsilon} > M_\varepsilon,
0 & \text{for } \omega_{\gamma \varepsilon} \leq M_\varepsilon,
\end{cases}
$$

(11.29)

$$
\nabla_x \omega_{\gamma \varepsilon}^{M_\varepsilon} = \begin{cases}
\nabla_x \omega_{\gamma \varepsilon} & \text{for } \omega_{\gamma \varepsilon} > M_\varepsilon,
0 & \text{for } \omega_{\gamma \varepsilon} \leq M_\varepsilon,
\end{cases}
$$

(11.30)

and that $\omega_{\gamma \varepsilon}^{M_\varepsilon} \in W^{1,1}_{p}(G_T, S)$ for $p \in [1, +\infty)$ due to [35] Appendix A.1, Lemma 1.7.

Multiply (11.25) by $\omega_{\gamma \varepsilon}^{M_\varepsilon}$ and integrate on $\Xi^1 \times (0, t')$:

$$
\int_{0}^{t_1} \int_{\Xi^1} \omega_{\gamma \varepsilon}^{M_\varepsilon} \partial_t \omega_{\gamma \varepsilon}^{M_\varepsilon} d\boldsymbol{x} d\boldsymbol{s} dt +
\int_{0}^{t_1} \int_{\Xi^1} \omega_{\gamma \varepsilon}^{M_\varepsilon} \partial_s (e^{-\xi t} a(\omega_{\gamma \varepsilon} e^{\xi t})) d\boldsymbol{x} d\boldsymbol{s} dt +
\int_{0}^{t_1} \int_{\Xi^1} \omega_{\gamma \varepsilon}^{M_\varepsilon} \text{div}_x (e^{-\xi t} \varphi(\omega_{\gamma \varepsilon} e^{\xi t})) d\boldsymbol{x} d\boldsymbol{s} dt =
$$
\[
\int_0^{t'} \int_{\Xi^1} \omega_{\gamma e} \Delta \omega_{\gamma e} \, dx \, dsdt + \varepsilon \int_0^{t'} \int_{\Xi^1} \omega_{\gamma e}^4 \partial_s^2 \omega_{\gamma e} \, dx \, dsdt + \\
\int_0^{t'} \int_{\Xi^1} \omega_{\gamma e}^2 K_{\gamma}(t, \tau) e^{-\xi(t)} \beta(x, s, \omega_{\gamma e} e^{\xi(t)}) \, dx \, dsdt - \xi \int_0^{t'} \int_{\Xi^1} \omega_{\gamma e} \omega_{\gamma e} \, dx \, dsdt, \quad t' \in (0, T]. \tag{11.31}
\]

Let us consider each term in (11.31) separately. We have

\[
\int_0^{t'} \int_{\Xi^1} \omega_{\gamma e} \partial_s \omega_{\gamma e} \, dx \, dsdt \overset{(11.29)}{=} \int_0^{t'} \int_{\Xi^1} \omega_{\gamma e} \partial_t \omega_{\gamma e} \, dx \, dsdt = \frac{1}{2} \| \omega_{\gamma e}^M (t') \|_{L^2(\Xi^1)}; \tag{11.32}
\]

\[
\int_0^{t'} \int_{\Xi^1} \omega_{\gamma e} \partial_s (e^{-\xi(t)} a(\omega_{\gamma e} e^{\xi(t)})) \, dx \, dsdt = \int_0^{t'} \int_{\Xi^1} \omega_{\gamma e} a'(\omega_{\gamma e} e^{\xi(t)}) \partial_s \omega_{\gamma e} \, dx \, dsdt = \int_0^{t'} \int_{\Xi^1} \omega_{\gamma e} a'(\omega_{\gamma e} e^{\xi(t)}) \partial_s \omega_{\gamma e} \, dx \, dsdt = \int_0^{t'} \int_{\Xi^1} W(t, \omega_{\gamma e}(x, t, S)) - W(t, \omega_{\gamma e}(x, t, 0)) \, dx \, dt, \tag{11.33}
\]

where \( W(t, \lambda) = \int_0^\lambda \lambda a'((\lambda + M_a) e^{\xi(t)}) \, d\lambda; \)

\[
\int_0^{t'} \int_{\Xi^1} \omega_{\gamma e} \text{div}_x (e^{-\xi(t)} \varphi(\omega_{\gamma e} e^{\xi(t)})) \, dx \, dsdt = \int_0^{t'} \int_{\Xi^1} \omega_{\gamma e} \varphi'(\omega_{\gamma e} e^{\xi(t)}) \cdot \nabla_x \omega_{\gamma e} \, dx \, dsdt = \int_0^{t'} \int_{\Xi^1} \varphi'((\omega_{\gamma e}^M + M_a) e^{\xi(t)}) \cdot \nabla_x \omega_{\gamma e} \, dx \, dsdt = \int_0^{t'} \int_{\Xi^1} \Phi(t, \omega_{\gamma e}^M(\sigma, t, s)) \cdot n(\sigma) \, d\sigma \, dsdt = 0, \tag{11.34}
\]

where \( \Phi(t, \lambda) = \int_0^\lambda \lambda \varphi'((\lambda + M_a) e^{\xi(t)}) \, d\lambda; \)

\[
\int_0^{t'} \int_{\Xi^1} \omega_{\gamma e} \Delta \omega_{\gamma e} \, dx \, dsdt = \int_0^{t'} \int_{\Xi^1} \nabla_x \omega_{\gamma e} \cdot \nabla_x \omega_{\gamma e} \, dx \, dsdt = \int_0^{t'} \int_{\Xi^1} |\nabla_x \omega_{\gamma e}^M|^2 \, dx \, dsdt = -\| \nabla_x \omega_{\gamma e}^M \|^2_{L^2(\Xi^1 \times (0, t'))}; \tag{11.35}
\]
\[ \varepsilon \int_0^{t'} \int_\Omega \omega_{\gamma_\varepsilon} M^\ast \partial_x^2 \omega_{\gamma_\varepsilon} \, dxdtds = -\varepsilon \int_0^{t'} \int_\Omega \partial_s \omega_{\gamma_\varepsilon} M^\ast \partial_s \omega_{\gamma_\varepsilon} \, dxdtds \]

\[ = -\varepsilon \int_0^{t'} \int_\Omega |\partial_s \omega_{\gamma_\varepsilon} M^\ast|^2 \, dxdtds = -\varepsilon \|\partial_s \omega_{\gamma_\varepsilon} M^\ast\|^2_{L^2(\Omega \times (0,t'))}; \quad (11.36) \]

\[ \int_0^{t'} \int_\Omega \omega_{\gamma_\varepsilon} M^\ast K_\gamma(t, \tau)e^{-\xi t} \beta(x, s, \omega_{\gamma_\varepsilon} e^{\xi t}) \, dxdtds \]

\[ \int_0^{t'} \int_\Omega \omega_{\gamma_\varepsilon} M^\ast K_\gamma(t, \tau)e^{-\xi t} \beta(x, s, (\omega_{\gamma_\varepsilon}^M + M_s)e^{\xi t}) \, dxdtds; \quad (11.37) \]

and, finally,

\[ \xi \int_0^{t'} \int_\Omega \omega_{\gamma_\varepsilon}^M \omega_{\gamma_\varepsilon} \, dxdtds = \xi \int_0^{t'} \int_\Omega \omega_{\gamma_\varepsilon}^M (\omega_{\gamma_\varepsilon}^M + M_s) \, dxdtds. \quad (11.38) \]

Aggregating (11.31)–(11.38), we arrive at the integral equality

\[ \frac{1}{2} \|\omega_{\gamma_\varepsilon}^M(t')\|^2_{L^2(\Omega)} + \|\nabla_x \omega_{\gamma_\varepsilon} M^\ast\|^2_{L^2(\Omega \times (0,t'))} + \varepsilon \|\partial_s \omega_{\gamma_\varepsilon} M^\ast\|^2_{L^2(\Omega \times (0,t'))} + \int_0^{t'} \int_\Omega (\omega_{\gamma_\varepsilon}^M + M_s) \omega_{\gamma_\varepsilon}^M \xi \, dxdtds = \]

\[ \int_0^{t'} \int_\Omega \omega_{\gamma_\varepsilon}^M K_\gamma(t, \tau)e^{-\xi t} \beta(x, s, (\omega_{\gamma_\varepsilon}^M + M_s)e^{\xi t}) \, dxdtds. \quad (11.39) \]

Estimating the right-hand side from above, using the Lagrange mean-value theorem and the growth condition (1.9d), we get

\[ \int_0^{t'} \int_\Omega \omega_{\gamma_\varepsilon}^M K_\gamma(t, \tau)e^{-\xi t} \beta(x, s, (\omega_{\gamma_\varepsilon}^M + M_s)e^{\xi t}) \, dxdtds = \]

\[ \int_0^{t'} \int_\Omega \omega_{\gamma_\varepsilon}^M K_\gamma(t, \tau)e^{-\xi t} \left( \beta(x, s, (\omega_{\gamma_\varepsilon}^M + M_s)e^{\xi t}) - \beta(x, s, M_s e^{\xi t}) \right) \, dxdtds + \]

\[ \int_0^{t'} \int_\Omega \omega_{\gamma_\varepsilon}^M K_\gamma(t, \tau)e^{-\xi t} \beta(x, s, M_s e^{\xi t}) \, dxdtds \leq \]

\[ \int_0^{t'} b_0 K_\gamma(t, \tau) \|\omega_{\gamma_\varepsilon}^M(t)\|^2_{L^2(\Omega)} dt + \int_0^{t'} \int_\Omega \omega_{\gamma_\varepsilon}^M K_\gamma(t, \tau)e^{-\xi t} \beta(x, s, M_s e^{\xi t}) \, dxdtds. \quad (11.40) \]
Now, we use arbitrariness of $\xi$ and choose it such that the second integral in the right-hand side of (11.40) vanishes. Recall that supp $K_\gamma(\cdot, \tau) \subset [\tau - \gamma, \tau]$. Hence

$$K_\gamma(t, \tau) \equiv 0 \quad \text{for } 0 \leq t \leq t_0 := \tau - \frac{\gamma_0}{2}.$$ 

On the strength of condition (1.9e), in order to secure that $\beta(x, s, M_\varepsilon e^{\xi_0}) \equiv 0 \quad \forall (x, s) \in \Xi^1$, it is sufficient to take $\xi$ such that

$$M_\varepsilon e^{\xi_0} \geq b_1 + 1. \quad (11.41)$$

Remark that $M_\varepsilon$ depends on $\xi$, and we have

$$M_\varepsilon = \max \{ \| u_0^{(1)} \|_{L^\infty(\Xi^1)}, \| u_0^{(2)} e^{-\xi t} \|_{L^\infty(\Xi^2)}, \| u_S^{(2)} e^{-\xi t} \|_{L^\infty(\Xi^1)} \} \geq \| u_0^{(1)} \|_{L^\infty(\Xi^1)}.$$ 

Therefore, (11.41) holds true if $\| u_0^{(1)} \|_{L^\infty(\Xi^1)} \geq b_1 + 1$. The latter holds with $\xi = 0$ for $\| u_0^{(1)} \| \geq b_1 + 1$ and $\xi = \xi_* = \frac{1}{t_0} \ln \frac{b_1 + 1}{\| u_0^{(1)} \|_{L^\infty(\Xi^1)}}$ for $\| u_0^{(1)} \|_{L^\infty(\Xi^1)} < b_1 + 1$.

Thus, the second integral in the right-hand side of (11.40) vanishes, if $\xi$ is defined by (11.24). Further, notice that

$$\int_0^{t'} \int_{\Xi^1} (\omega_{\gamma_\varepsilon}^{M_\varepsilon} + M_\varepsilon \omega_{\gamma_\varepsilon}^{M_\varepsilon} \xi_\varepsilon) \, dx \, dt \, ds \geq 0$$

and discard this integral along with $\| \nabla \omega_{\gamma_\varepsilon}^{M_\varepsilon} \|_{L^2(\Xi^1 \times (0, t'))}^2$ and $\| \partial \omega_{\gamma_\varepsilon}^{M_\varepsilon} \|_{L^2(\Xi^1 \times (0, t'))}^2$ from the left-hand side of (11.39). As the result of above considerations, we derive the inequality

$$\| \omega_{\gamma_\varepsilon}^{M_\varepsilon}(t') \|_{L^2(\Xi^1)}^2 \leq \int_0^{t'} b_0 K_\gamma(t, \tau) \| \omega_{\gamma_\varepsilon}^{M_\varepsilon}(t') \|_{L^2(\Xi^1)}^2 \, dt, \quad t' \in (0, T). \quad (11.42)$$

Applying Grönwall’s lemma, we conclude that $\| \omega_{\gamma_\varepsilon}^{M_\varepsilon}(t') \|_{L^2(\Xi^1)}^2 = 0$, which implies that

$$u_{\gamma_\varepsilon}(x, t', s) \leq M_\varepsilon e^{\xi_\varepsilon t'} \leq e^{\xi_\varepsilon t'} \max \left\{ \| u_0^{(1)} \|_{L^\infty(\Xi^1)}, \| u_0^{(2)} \|_{L^\infty(\Xi^2)}, \| u_S^{(2)} \|_{L^\infty(\Xi^1)} \right\}$$

$$\forall (x, t', s) \in G_{T, S}, \quad \forall \gamma \in (0, \gamma_0/2], \quad \forall \varepsilon \in (0, 1]. \quad (11.43)$$

Analogously, we also establish that

$$u_{\gamma_\varepsilon}(x, t', s) \geq -M_\varepsilon e^{\xi_\varepsilon t'} \geq -e^{\xi_\varepsilon t'} \max \left\{ \| u_0^{(1)} \|_{L^\infty(\Xi^1)}, \| u_0^{(2)} \|_{L^\infty(\Xi^2)}, \| u_S^{(2)} \|_{L^\infty(\Xi^1)} \right\}. \quad (11.44)$$

Combining (11.43) and (11.44) we establish the maximum principle (11.23) (with $\xi_\varepsilon$ defined by (11.24)), which completes the proof of Lemma 11.3.

**Corollary 11.1.** The family of kinetic solutions $\{ u_\gamma \}_{\gamma \in (0, \gamma_0/2]}$ of Problem $\Pi_\gamma$ with $Z_\gamma = K_\gamma \beta$ satisfies the maximum principle

$$\| u_\gamma(\cdot, t', \cdot) \|_{L^\infty(\Xi^1)} \leq M_\gamma(t') \leq M_\gamma(T), \quad \forall t' \in (0, T], \quad \forall \gamma \in (0, \gamma_0/2]. \quad (11.45)$$

**Proof.** Bound (11.45) directly follows from (11.23) and the limiting relation $u_\gamma = s_{\varepsilon \to 0} u_\varepsilon$, see Remark 7.3.

**Corollary 11.2.** The family of $\chi$-functions $\{ \chi(\lambda; u_\gamma) \}_{\gamma \in (0, \gamma_0/2]}$ is uniformly in $\gamma$ bounded in $L^2(G_{T, S} \times [-M_\gamma(T), M_\gamma(T)])$. 

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Proof. From (11.45) and (3.1), it immediately follows that
\[ \|\chi(\cdot, u_{\gamma \varepsilon})\|_{L^2(G_{T,S} \times [-M_T(T), M_T(T)])} \leq \sqrt{TS} M_T(T) \text{ meas } \Omega, \]
which completes the proof. \(\Box\)

**Corollary 11.3.** The supports of the measures \(m_\gamma\) and \(n_\gamma\) lay in the layer \(\{-M_T(T) \leq \lambda \leq M_T(T)\}\) for all \(\gamma \in (0, \gamma_0/2]\).

**Proof.** The assertion directly follows from Corollary [11.1], Remark [7.3] and representation (4.19). \(\Box\)

### 12 Derivation of the impulsive kinetic equation

In this section, from the kinetic equation (3.2a) with \(Z_\gamma = K_\gamma \beta\) (or, equivalently, from (11.18)–(11.19)) we derive the limiting impulsive kinetic equation, as \(\gamma \to 0^+\). We expose this procedure in the form of the sequence of several lemmas.

**Lemma 12.1.** The family of kinetic solutions \(\{u_\gamma\}_{\gamma \in [0, \gamma_0/2]}\) of Problem \(\Pi_\gamma\) (in the sense of Definition 3.3) is relatively compact in \(L^1(G_{T,S})\).

**Proof.** Kinetic equation (3.2a) with \(Z_\gamma = K_\gamma \beta\) or, equivalently, kinetic equation (11.18) with \(Z_\gamma\) defined by (11.19), has the form of the kinetic equation in requirement (iv) in Proposition 4.1 with \(f_\nu = \chi(\lambda; u_{\gamma \nu})\), \(g_\nu \equiv 0\) and \(k_\nu = m_\gamma + n_\gamma + Z_\gamma\). Here \(\{\gamma_\nu\}_{\nu \in \mathbb{N}}\) is an arbitrary sequence such that \(\gamma_\nu \to 0^+\). On the strength of Corollaries 11.2 and 11.3 and assertions (ii) and (iii) of Lemma 11.2 there exist a subsequence of \(\{\gamma_\nu\}\), still denoted by \(\{\gamma_\nu\}\), a limiting function \(f_*\) and a limiting measure \(k_*\) such that
\[
\chi(\lambda; u_{\gamma_\nu}) \xrightarrow[\nu \to \infty]{} f_* \text{ weakly in } L^2(G_{T,S} \times [-M_T(T), M_T(T)]),
\]
\[
k_\nu \xrightarrow[\nu \to \infty]{} k_* \text{ weakly* in } \mathcal{M}(G_{T,S} \times [-M_T(T), M_T(T)]).\]

Since \(\mathcal{M}(G_{T,S} \times [-M_T(T), M_T(T)])\) is compactly embedded into \(W_{loc}^{-1,p'}(G_{T,S} \times [-M_T(T), M_T(T)])\) for any \(p' \in [1, \frac{d+3}{d+2}]\) due to Sobolev’s embedding theorem [48, Chapter I, Section 8], we also have that
\[
k_\nu \xrightarrow[\nu \to \infty]{} k_* \text{ strongly in } W_{loc}^{-1,p'}(G_{T,S} \times [-M_T(T), M_T(T)]).\]

On the strength of Proposition 4.1 from the limiting relations (12.1)–(12.3) it follows that
\[
\int_{-M_T(T)}^{M_T(T)} \chi(\lambda; u_{\gamma_\nu}) d\lambda \xrightarrow[\nu \to \infty]{} u_* := \int_{-M_T(T)}^{M_T(T)} f_* d\lambda \text{ strongly in } L^2(G_{T,S}).\]

On the strength of Lemma 3.2 and item (i) in Lemma 3.1 this yields that
\[
u_{\gamma_\nu} \xrightarrow[\nu \to \infty]{} u_* \text{ strongly in } L^2(G_{T,S}) \text{ (and in } L^1(G_{T,S}))\]
and that \(f_*\) is the \(\chi\)-function:
\[
f_*(x, t, s, \lambda) = \chi(\lambda; u_*(x, t, s)).\]

Lemma 12.1 is proved. \(\Box\)
Corollary 12.1. (Corollary of Lemmas 12.1, 11.2 and 11.3.) The limiting function \( u_* \) belongs to \( L^\infty(G_{T,S}) \cap L^2((0, T) \times (0, S); \hos)^\frac{1}{2}(\Omega) \) and satisfies the maximum principle
\[
-M_\gamma(T) \leq u_*(x, t, s) \leq M_\gamma(T) \quad \text{for a.e.} \ (x, t, s) \in G_{T,S}
\] (12.7)
and the energy estimate
\[
\|\nabla_x u_*\|_{L^2(G_{T,S})} \leq C_6.
\] (12.8)

Lemma 12.2. Let \( m_\gamma, n_\gamma \in \mathcal{M}^+(G_{T,S} \times \mathbb{R}_\lambda) \) be the measures having place in formulation of Problem \( \Pi_\gamma \), and function \( Z_\gamma \in L^1(G_{T,S} \times \mathbb{R}_\lambda) \) be defined by formula (11.19).

Then there exist subsequence \( \gamma_\nu \xrightarrow{\nu \to \infty} 0+ \) and measures \( m_* \in \mathcal{M}^+(G_{T,S} \times \mathbb{R}_\lambda) \) and \( Z_* \in \mathcal{M}(G_{T,S} \times \mathbb{R}_\lambda) \) such that
\[
m_\gamma + n_\gamma - \delta(\lambda = u_*)|\nabla_x u_*|^2 \xrightarrow{\nu \to \infty} m_* \quad \text{weakly* in} \ \mathcal{M}(G_{T,S} \times \mathbb{R}_\lambda),
\] (12.9)
\[
Z_\gamma \xrightarrow{\nu \to \infty} Z_* \quad \text{weakly* in} \ \mathcal{M}(G_{T,S} \times \mathbb{R}_\lambda),
\] (12.10)
\[
\text{supp } m_* \subset \overline{G}_{T,S} \times [-M_\gamma(T), M_\gamma(T)],
\] (12.11)
\[
\text{supp } Z_* \subset \Xi^1 \times \{t = \tau\} \times \{-b_1 \leq \lambda \leq b_1\}.
\] (12.12)

In particular,
\[
m_\gamma + n_\gamma + Z_\gamma \xrightarrow{\nu \to \infty} m_* + n_* + Z_* \quad \text{weakly* in} \ \mathcal{M}(G_{T,S} \times \mathbb{R}_\lambda),
\] (12.13)
where \( n_* = \delta(\lambda = u_*)|\nabla_x u_*|^2 \).

Proof. Firstly, on the strength of Alaoglu’s theorem, (12.9) follows from assertion (ii) of Lemma 11.2 and (12.10) follows from assertion (iii) of Lemma 11.2. Secondly, notice that measure
\[
\lim_{\nu \to \infty} (m_\gamma + n_\gamma - \delta(\lambda = u_*)|\nabla_x u_*|^2)
\]
is nonnegative, since \( m_\gamma \) is nonnegative for any \( \gamma \) and \( \lim_{\nu \to \infty} (n_\gamma - \delta(\lambda = u_*)|\nabla_x u_*|^2) \) is nonnegative due to lower semicontinuity property. Thus \( m_* \) is nonnegative. Finally, inclusion (12.11) follows from Corollary 11.3 and inclusion (12.12) follows from representation (11.19) and condition (1.9c).

Lemma 12.2 is proved. \( \square \)

Lemma 12.3. The limiting quadruple \((u_*, m_*, n_*, Z_*)\), defined in Lemmas 12.1 and 12.2, resolves the kinetic equation
\[
\partial_t \chi(\lambda; u_*) + a'(\lambda)\partial_s \chi(\lambda; u_*) + \varphi'(\lambda) \cdot \nabla_x \chi(\lambda; u_*) - \Delta_x \chi(\lambda; u_*) = \partial_\lambda (m_* + n_* + Z_*)
\] (12.14)
in the sense of distributions.

Proof. The proof is achieved by limiting passage in (11.18) as \( \gamma_\nu \xrightarrow{\nu \to \infty} 0+ \). This passage relies on the limiting relations (12.11), (12.13) and representation (12.6). \( \square \)

Lemma 12.4. Let the quadruple \( u_* \in L^\infty(G_{T,S}) \cap L^2((0, T) \times (0, S); \hos)^\frac{1}{2}(\Omega) \), \( m_* \in \mathcal{M}^+(G_{T,S} \times [-M_\gamma(T), M_\gamma(T)]) \), \( n_* = \delta(\lambda = u_*)|\nabla_x u_*|^2 \) and \( Z_* \in \mathcal{M}(G_{T,S} \times [-M_\gamma(T), M_\gamma(T)]) \) resolve the kinetic equation (12.14) in the sense of distributions. (As the matter of fact, we consider the case when \( u_* = \lim_{\nu \to \infty} u_{\gamma_\nu} \).)
Then there exists $u_{*0}^{(1)} \in L^\infty(\Xi^1)$ such that

$$\lim_{t \to 0^+} \int_{\Xi^1} \left| u_*(x, t, s) - u_{*0}^{(1)}(x, s) \right| \, dx \, ds = 0, \quad (12.15)$$

$$\lim_{t \to 0^+} \int_{M(t)} \int_{-M(t)} \chi(\lambda; u_*(x, t, s)) - \chi(\lambda; u_{*0}^{(1)}(x, s)) \, dx \, ds \, d\lambda = 0. \quad (12.16)$$

Furthermore, for any fixed $\tau_* \in (0, T)$ there exist both one-sided traces $u_{*\tau_* \pm 0}^{(1)} \in L^\infty(\Xi^1)$ such that

$$\lim_{t \to \tau_*^\pm 0} \int_{\Xi^1} \left| u_*(x, t, s) - u_{*\tau_* \pm 0}^{(1)}(x, s) \right| \, dx \, ds = 0, \quad (12.17)$$

$$\lim_{t \to \tau_*^\pm 0} \int_{-M(t)} \int_{-M(t)} \chi(\lambda; u_*(x, t, s)) - \chi(\lambda; u_{*\tau_* \pm 0}^{(1)}(x, s)) \, dx \, ds \, d\lambda = 0. \quad (12.18)$$

In particular, (12.17) and (12.18) are valid for $\tau_* = \tau$.

Proof relies on Lemma 12.3 and Corollary 12.1. It is just a minor modification of the proofs of Lemmas 5.1 and 5.2; therefore we skip it. \[\square\]

**Lemma 12.5.** Let $(u_*, m_*, n_*, Z_*)$ be the limiting point of the sequence $\{u_{\nu*}, m_{\nu*}, n_{\nu*}, Z_{\nu*}\}$ (as $\nu \to +\infty$) and resolve the kinetic equation (12.14) in the sense of distributions.

Then $u_* \in C(0, \tau; L^1(\Xi^1)) \cap C(\tau, T; L^1(\Xi^1))$. In particular, the initial condition

$$\lim_{t \to 0^+} \int_{\Xi^1} \left| u_*(x, t, s) - u_0^{(1)}(x, s) \right| \, dx \, ds = 0 \quad (12.19)$$

holds true.

Proof. We keep track of the proof of Lemma 6.1, with some necessary natural modifications.

Taking $\eta(u_{\nu*}) = \pm u_{\nu*}$ in (3.3a), we arrive at equation (11.1a) with $Z_{\nu*} = K_{\nu*} \beta$. In the sense of distributions, this equation is equivalent to the integral equality

$$\langle \partial_t u_{\nu*}(t), \phi \rangle_{W^{-1,2}(\Xi^1), W^1_2(\Xi^1)} =$$

$$\int_{\Xi^1} (a(u_{\nu*}) \partial_s \phi + \varphi(u_{\nu*}) \cdot \nabla_x \phi - \nabla_x u_{\nu*} \cdot \nabla_x \phi + K_{\nu*}(t) \beta(x, s, u_{\nu*}) \phi) \, dx \, ds = 0$$

$$\forall \phi \in \bar{W}^1_2(\Xi^1), \quad \forall t \in [0, T], \quad \forall \gamma_* > 0. \quad (12.20)$$

Fixing arbitrarily $\tau_0 \in (0, \tau)$, from (12.20), the maximum principle (11.45), the energy estimate (11.20) and the localization property $\text{supp} K_{\nu*} \subset (\tau - \gamma_* \nu, \tau)$, we easily derive the bound

$$\langle \partial_t u_{\nu*}(t), \phi \rangle_{W^{-1,2}(\Xi^1), W^1_2(\Xi^1)} \leq C_{13}(\tau_0) \| \phi \|_{W^1_2(\Xi^1)}$$

$$\forall \phi \in \bar{W}^1_2(\Xi^1), \quad \forall t \in [0, \tau_0], \quad \forall \gamma_* \in \left(0, \min \left\{ \tau_0, \frac{\gamma_0}{2} \right\} \right), \quad (12.21)$$

where

$$C_{13}(\tau_0) = \sqrt{8} \text{meas} \Omega \left( \| a \|_{C([-M_8(\tau_0), M_8(\tau_0)])} + \| \varphi \|_{C([-M_8(\tau_0), M_8(\tau_0)])} \right) + 2C_6.$$
This means that the family \( \{ \partial_t u_{\nu} \}_{\gamma_{\nu} \in (0, \min \{ \tau_0, \tau \})} \) is uniformly bounded in \( L^\infty(0, \tau_0; W^{-1,2}(\Xi^1)) \). Hence the family of functions \( \{ u_{\nu} : [0, \tau_0] \mapsto W^{-1,2}(\Xi^1) \}_{\gamma_{\nu} \in (0, \min \{ \tau_0, \tau \})} \) is equicontinuous, and we have

\[
 u_{\nu}(\cdot, t, \cdot) \longrightarrow u_{\nu}(\cdot, t_0, \cdot) \quad \text{strongly in } W^{-1,2}(\Xi^1)
\]

uniformly in \( \gamma_{\nu} \in \left(0, \min \left\{ \tau_0, \frac{\tau_0}{2} \right\}\right) \) for a.e. \( t_0 \in [0, \tau_0] \).

(12.22)

Here, in the case \( t_0 = 0 \), the right-sided limit is meant, and, in the case \( t_0 = \tau_0 \), the left-sided limit is meant.

On the other hand, due to (11.35), values of the mappings \( t \mapsto u_{\nu}(\cdot, t, \cdot) \) belong to the set

\[
 \mathfrak{F} \overset{\text{def}}{=} \{ \phi \in L^2(\Xi^1) : \text{ess sup}_{(x, s) \in \Xi^1} |\phi(x, s)| \leq M_T(\tau) \},
\]

which is a compact subset in \( W^{-1,2}(\Xi^1) \) by the Rellich theorem. Therefore, by the Arcel theorem, the set \( \{ u_{\nu} \}_{\gamma_{\nu} \in (0, \min \{ \tau_0, \tau \})} \) is relatively compact in \( C(0, \tau_0; W^{-1,2}(\Xi^1)) \). Hence

\[
 u_{\nu}(\cdot, t, \cdot) \longrightarrow u_*(\cdot, t, \cdot) \quad \text{strongly in } W^{-1,2}(\Xi^1) \quad \forall t_0 \in (0, \tau_0),
\]

(12.23)

Here we recall that \( \{ \gamma_{\nu} \to 0 \} \) is the subsequence extracted in the proof of Lemma 12.1. Next, from (12.15) and (12.17) it immediately follows that

\[
 u_*(\cdot, t, \cdot) \longrightarrow u_{* t_0}^{(1)} \quad \text{strongly in } W^{-1,2}(\Xi^1),
\]

(12.24)

\[
 u_*(\cdot, t, \cdot) \longrightarrow u_{* t_0}^{(1)} \quad \text{strongly in } W^{-1,2}(\Xi^1) \quad \forall t_0 \in (0, \tau_0),
\]

(12.25)

\[
 u_*(\cdot, t, \cdot) \longrightarrow u_{* t_0-0}^{(1)} \quad \text{strongly in } W^{-1,2}(\Xi^1).
\]

(12.26)

From (12.22)–(12.26), by the triangle inequality we deduce that

\[
 u_{* t_0}^{(1)}(x, s) = u_0^{(1)}(x, s), \quad u_{* t_0-0}^{(1)}(x, s) = u_*(x, \tau_0 - 0, s),
\]

(12.27)

\[
 u_{* t_0}^{(1)}(x, s) = u_*(x, t_0, s) \quad \forall t_0 \in (0, \tau_0).
\]

(12.28)

Using arbitrariness of \( \tau_0 \in (0, \tau) \), the localization property \( \text{supp } K_{\gamma_{\nu}} \subseteq (\tau - \gamma_{\nu}, \tau) \) and the fact that the set \( \mathfrak{F} \) does not depend on \( \tau_0 \), and taking the sequences \( \tau_0 \mu \rightarrow \tau - 0 \) and \( \gamma_{\mu} \in (0, \min \{ \tau_0, \frac{\tau_0}{2} \}) \), we deduce that equalities (12.27) hold true with \( \tau \) on the place of \( \tau_0 \). Inserting \( u_0^{(1)}(x, s) \) on the place of \( u_{* t_0}^{(1)}(x, s) \) in (12.15), \( u_*(x, t_0, s) \) and \( u_*(x, \tau_0, s) \) on the places of \( u_{* t_0}^{(1)}(x, s) \) and \( u_{* t_0-0}^{(1)}(x, s) \) in (12.25), we finally establish that \( u_* \in C(0, \tau; L^1(\Xi^1)) \).

Quite analogously, we verify that \( u_* \in C(\tau, T; L^1(\Xi^1)) \) and thus complete the proof of the lemma.

The following lemma finalizes derivation of the impulsive kinetic equation (9.8).

**Lemma 12.6.** The limiting measure \( \mathcal{Z}_* = w^k \lim_{\nu \to \infty} \mathcal{Z}_{\gamma_{\nu}} \) admits the representation

\[
 \mathcal{Z}_* = \delta_{(t = \tau - 0)} 1_{\lambda \geq u_*(x, t, s)} \beta(x, s, u_*(x, t, s)),
\]

(12.28)

which is understood in the sense of distributions and therefore can be equivalently written as follows:

\[
 \langle \mathcal{Z}_*, \phi \rangle = \int_{\Xi^1} \int_{\mathbb{R}_\lambda} 1_{\lambda \geq u_*(x, \tau - 0, s)} \beta(x, s, u_*(x, \tau - 0, s)) \phi(x, \tau - 0, s, \lambda) \, dx \, ds \, d\lambda \quad \forall \phi \in C_0(G_{\tau}; \mathbb{R} \times \mathbb{R}_\lambda).
\]

(12.29)
Proof. Let \( \phi \in C_0(G_{T,S} \times \mathbb{R}_\lambda) \) be an arbitrary test function. Set

\[
\Phi(x, t, s, \lambda) = \int_{-\infty}^{\lambda} \phi(x, t, s, \lambda') d\lambda',
\]
i.e., \( \Phi \) is the primitive of \( \phi \) with respect to \( \lambda \). On the strength of representation (11.19) and Conditions on \( K_\gamma & \beta \), the following chain of equalities holds true:

\[
\langle Z_{\gamma_0}, \phi \rangle = \int \int \frac{1}{(\lambda \geq u_{\gamma_0})} K_{\gamma_0}(t, \tau) \beta(x, s, \lambda) \phi(x, t, s, \lambda) d\lambda d\tau dt ds -
\]

\[
\int \int \frac{1}{(\lambda' \geq u_{\gamma_0})} K_{\gamma_0}(t, \tau) \partial_{\lambda'} \beta(x, s, \lambda') d\lambda' d\lambda d\tau dt ds -
\]

\[
\int K_{\gamma_0}(t, \tau) \int \beta(x, s, \lambda) \phi(x, t, s, \lambda) d\lambda d\tau dt ds -
\]

\[
\int K_{\gamma_0}(t, \tau) \int \beta(x, s, u_{\gamma_0}) \phi(x, t, s, u_{\gamma_0}) d\lambda d\tau dt ds -
\]

\[
- \int \frac{2}{\gamma_\nu} \omega \left( \frac{t - \tau}{\gamma_\nu} \right) \int \beta(x, s, u_{\gamma_0}(x, t, s)) \Phi(x, t, s, u_{\gamma_0}(x, t, s)) d\lambda d\tau dt ds \equiv I_{\gamma_\nu}. \quad (12.30)
\]

Let us change variable \( t \) for \( \zeta = \frac{t - s}{\gamma_\nu} \). Taking into account that \( \omega \) is even, for \( \gamma_\nu < \gamma_0 \) we have

\[
I_{\gamma_\nu} = - \int \int 2\omega(\zeta) \beta(x, s, u_{\gamma_0}(x, t - \zeta \gamma_\nu, s)) \Phi(x, t - \zeta \gamma_\nu, s, u_{\gamma_0}(x, t - \zeta \gamma_\nu, s)) d\lambda d\tau dt ds d\zeta. \quad (12.31)
\]

Recall that \( Z_\gamma = K_\gamma \beta \equiv 0 \) for \( t \in [0, \tau - \gamma] \). Therefore, \( u_{\gamma_\nu} \) and \( u_* \) are kinetic and entropy solutions of the same problem on the set \( \{ (x, t, s) \in \Omega \times [0, \tau - \gamma] \times [0, S] \} \), i.e., Problem \( \Pi_\gamma \) (or \( \Pi_0 \)). Since \( u_* = \lim_{\nu \to \infty} u_{\gamma_\nu} \), this implies that

\[
u_{\gamma_0}(x, t, s) = u_*(x, t, s) \text{ on } \Omega \times [0, \tau - \gamma_\nu] \quad (12.32)
\]

On the strength of Lemma [12.3] and identity (12.32), we conclude that

\[
u_{\gamma_0}(x, \tau - \zeta \gamma_\nu, s) \xrightarrow{\nu \to \infty} u_*(x, \tau - s, \zeta) \text{ a.e. in } \Omega \times [0, \zeta < 1]. \quad (12.33)
\]

Since \( \beta \) and \( \Phi \) are smooth and bounded, using the Lebesgue dominated convergence theorem [50, Theorem 1.4.48], we establish the limiting relation

\[
I_{\gamma_\nu} \xrightarrow{\nu \to \infty} - \int \int 2\omega(\zeta) \beta(x, s, u_*(x, \tau - s)) \Phi(x, \tau - s, u_*(x, \tau - s)) d\lambda d\tau dt ds d\zeta. \quad (12.34)
\]
Taking into account that \( \int_0^1 2\omega(\zeta)\,d\zeta = 1 \) and
\[
\Phi(x, \tau - 0, s, u_*(x, \tau - 0, s)) = -\int_{\mathbb{R}_\lambda} 1_{(\lambda \geq u_*(x, \tau - 0, s))}\phi(x, \tau - 0, s, \lambda)\,d\lambda
\]
and combining \([12.30]\), \([12.31]\) and \(Z_\ast = \omega^\ast \lim_{\nu \to \infty} Z_{\gamma_\nu} \), we arrive at the limiting relation \([12.29]\), which completes the proof of the lemma. \(\square\)

**Corollary 12.2. (Corollary of Lemmas 12.3 and 12.6.)** The limiting triple \((u_\ast, m_\ast, n_\ast)\), defined by Lemmas 12.1 and 12.2, resolves the kinetic equation \((9.5_e)\) in the sense of distributions.

**Corollary 12.3. (Corollary of Lemmas 12.3 and 12.6.)** The limiting triple \((u_\ast, m_\ast, n_\ast)\), defined by Lemmas 12.1 and 12.2, resolves the kinetic equation \((9.5_a)\) on \(\Omega \times ((0, \tau) \cup (\tau, T)) \times (0, S) \times \mathbb{R}_\lambda \) in the sense of distributions.

## 13 Derivation of the kinetic impulsive condition \((9.5_e)\)

In view of Lemma 12.5 and representation \([12.29]\), the impulsive kinetic equation \((9.8)\) is equivalent in the sense of distributions to the integral equality

\[
\int_{\Xi^1 \times (t_0, t_1) \times \mathbb{R}_\lambda} \chi(\lambda; u_\ast) (\partial_t \phi + a'(\lambda)\partial_t \phi + \varphi'(\lambda) \cdot \nabla_x \phi + \Delta_x \phi) \,dx \,dt \,ds \,d\lambda = \\
\int_{\Xi^1 \times \mathbb{R}_\lambda} (\chi(\lambda; u_\ast(x, t_1, s))\phi(x, t_1, s, \lambda) - \chi(\lambda; u_\ast(x, t_0, s))\phi(x, t_0, s, \lambda)) \,dx \,ds \,d\lambda + \\
\int_{\Xi^1 \times \mathbb{R}_\lambda} 1_{(\tau \in (t_0, t_1))} 1_{(\lambda \geq u_\ast(x, \tau - 0, s))}\beta(x, s, u_\ast(x, \tau - 0, s))\partial_t \phi(x, \tau - 0, s, \lambda) \,dx \,ds \,d\lambda + \\
\langle m_\ast + n_\ast, \partial_t \phi \rangle_{\mathcal{M}(\Xi^1 \times [t_0, t_1] \times \mathbb{R}_\lambda), C(\Xi^1 \times [t_0, t_1] \times \mathbb{R}_\lambda)} \tag{13.1}
\]

for all values \(t_0, t_1 \in [0, T]\), such that \(t_0 < t_1\), \(t_0, t_1 \neq \tau\), and for all test-functions \(\phi \in C^2(G_{T, S} \times \mathbb{R}_\lambda) \) vanishing in the neighborhood of \(\partial \Xi^1\) and for sufficiently large \(|\lambda|\), say, for \(|\lambda| \geq \max\{M_7(T), b_1\} + 1\). (Here \(M_7(T)\) is given by \((11.23)\) and \(b_1\) is given by \((1.9_e)\).

Taking \(\phi(x, t, s, \lambda) = \phi_1(x, s)\phi_2(\lambda)\), where \(\phi_2 \equiv 1\) on \([-M_7(T), M_7(T)]\), and passing to the limit as \(t_0 \to \tau - 0\) and \(t_1 \to \tau + 0\), from \((13.1)\) we derive the integral inequality

\[
\int_{\Xi^1} \phi_1(x, s) \int_{-M_7(T)}^{M_7(T)} (\chi(\lambda; u_\ast(x, \tau + 0, s)) - \chi(\lambda; u_\ast(x, \tau - 0, s))) \,d\lambda \,dx \,ds = \\
\int_{\Xi^1} \phi_1(x, s) \beta(x, s, u_\ast(x, \tau - 0, s)) \left( \int_{u_\ast(x, \tau - 0, s)}^{+\infty} \partial_\lambda \phi_2(\lambda) \,d\lambda \right) \,dx \,ds. \tag{13.2}
\]

Here we took into account that function \(\chi\) and measures \(m_\ast\) and \(n_\ast\) are supported on the segment \([-M_7(T) \leq \lambda \leq M_7(T)]\). In the right-hand side of \((13.2)\) notice that

\[
\int_{u_\ast(x, \tau - 0, s)}^{+\infty} \partial_\lambda \phi_2(\lambda) \,d\lambda = -\phi_2(u_\ast(x, \tau - 0, s)) + 1 \tag{13.3}
\]
and, using assertion (i) of Lemma 3.1, we represent
\[
\beta(x, s, u_*(x, \tau - 0, s)) = \int_{-M(T)}^{M(T)} \chi(\lambda; u_*(x, \tau - 0, s)) \partial_\lambda \beta(x, s, \lambda) d\lambda + \beta(x, s, 0). \tag{13.4}
\]

Combining (13.2)–(13.4), we arrive at the integral equality
\[
\int_{\Xi} \phi_1(x, s) \left( \int_{-M(T)}^{M(T)} \chi(\lambda; u_*(x, \tau + 0, s)) d\lambda \right) dx ds = \\
\int_{\Xi} \phi_1(x, s) \left( \int_{-M(T)}^{M(T)} (1 + \partial_\lambda \beta(x, s, \lambda)) \chi(\lambda; u_*(x, \tau - 0, s)) d\lambda + \beta(x, s, 0) \right) dx ds. \tag{13.5}
\]

Using assertion (i) of Lemma 3.1 again, we rewrite (13.5) in the equivalent form
\[
\int_{\Xi} \phi_1(x, s) u_*(x, \tau + 0, s) dx ds = \int_{\Xi} \phi_1(x, s) (u_*(x, \tau - 0, s) + \beta(x, s, u_*(x, \tau - 0, s))) dx ds. \tag{13.6}
\]

Since \( \phi_1 \) is arbitrary, (13.6) immediately yields the impulsive condition (9.3b) for \( u_* \).

In turn, (9.3b) yields that integration in (13.5) with respect to \( \lambda \) is fulfilled over interval \([-M_3, M_3]\). (\( M_3 \) is defined in (9.4)). Thus, we have established the following result.

**Lemma 13.1.** The limiting function \( u_* = \lim_{\nu \to \infty} u_{\nu} \) satisfies the impulsive condition (9.3b) and the kinetic impulsive condition (9.5c).

Notice that in this section we also have justified the claim of Remark 9.2 as a byproduct.

### 14 Derivation of the kinetic boundary conditions (9.5c) and (9.5d). Completion of the proof of Theorem 9.1

Recall the sequence \( \{u_{\nu}\}_{\nu \in \mathbb{N}} \) that was introduced in the proof of Lemma 12.1 so that the limiting relation (12.5) holds. Let us consider the traces \( u_{\nu,S}^{tr,(2)} \) and \( u_{\nu,S}^{tr,(2)} \) of \( u_{\nu} \) on the sets \( \Xi^2 \times \{s = 0+\} \) and \( \Xi^2 \times \{s = S - 0\} \), respectively. We start with the observation that, by the Tartar theorem [51, Chapter 3], there exist a subsequence still denoted by \( \{u_{\nu}\}_{\nu \to \infty}^{\nu = \nu_0 \to 0+} \) and two families of probability Radon measures \( \Lambda_{x,t}^{(0)} \) and \( \Lambda_{x,t}^{(S)} \) supported uniformly on \([-M_3, M_3]\) such that

\[
\phi(u_{\nu,0}^{tr,(2)}) \to \bar{\phi} \text{ weakly* in } L^\infty(\Xi^2), \quad \bar{\phi}(x, t) = \int_{\mathbb{R}_+} \phi(\lambda) d\Lambda_{x,t}^{(0)}(\lambda), \tag{14.1a}
\]

\[
\zeta(u_{\nu,S}^{tr,(2)}) \to \bar{\zeta} \text{ weakly* in } L^\infty(\Xi^2), \quad \bar{\zeta}(x, t) = \int_{\mathbb{R}_+} \zeta(\lambda) d\Lambda_{x,t}^{(S)}(\lambda). \tag{14.1b}
\]

for all \( \phi, \zeta \in C(\mathbb{R}_+) \).

Measures \( \Lambda_{x,t}^{(0)} \) and \( \Lambda_{x,t}^{(S)} \) are the Young measures associated with the subsequences \( \{u_{\nu,0}^{tr,(2)}\}_{\nu \to \infty}^{\nu = \nu_0 \to 0+} \) and \( \{u_{\nu,S}^{tr,(2)}\}_{\nu \to \infty}^{\nu = \nu_0 \to 0+} \), respectively. The mappings \((x, t) \mapsto \Lambda_{x,t}^{(0)} \) and...
(x, t) \mapsto A_{x,t}^{(S)} \) are weakly* measurable. More precisely, they belong to the space \( L^\infty_0(\Xi^2; \mathcal{M}(\mathbb{R}_\lambda)) \) (for details, see [35, Chapter 3, Definition 2.7]).

Now we are in a position to formulate and prove the following assertion.

**Lemma 14.1.** The limiting function \( u_* = \lim_{\nu \to \infty} u_{\gamma\nu} \) satisfies the entropy boundary conditions \((9.9c)\) and \((9.9e)\).

**Proof.** (1) Since the kinetic equation \((9.5a)\) on \( \Omega \times ((0, \tau) \cup (\tau, T)) \times (0, S) \times \mathbb{R}_\lambda \) is exactly the kinetic equation \((3.2a)\) with \( Z_\gamma \equiv 0 \), similarly to Lemma 5.3 we conclude that there exist \( u_{\nu0}^{tr,(2)}, u_{\nuS}^{tr,(2)} \in L^\infty(\Xi^2) \) — the traces of \( u_* \) on \( \Gamma_{\bar{S}}^2 \) and \( \Gamma_{\bar{S}-0}^2 \), respectively — satisfying the limiting relations

\[
\text{ess lim}_{s \to 0^+} \int_{\Xi^2} |u_*(x, t, s) - u_{\nu0}^{tr,(2)}(x, t)| \, dx \, dt = 0, \quad (14.2a)
\]

\[
\text{ess lim}_{s \to S-0} \int_{\Xi^2} |u_*(x, t, s) - u_{\nuS}^{tr,(2)}(x, t)| \, dx \, dt = 0. \quad (14.2b)
\]

(2) Firstly, we consider the case \( t \leq \tau \). Since supp \( K_\gamma(\cdot, \tau) \subset [\tau - \gamma, \tau] \), we deduce that

\[
u_*(x, t, s) = u_*(x, t, s) \quad \text{a.e. in} \quad \Omega \times (0, \tau - \gamma) \times (0, S). \quad (14.3)
\]

Along with Lemma 5.3, this implies that

\[
u_0^{tr,(2)}(x, t) = u_0^{tr,(2)}(x, t), \quad u_0^{tr,(2)}(x, t) = u_0^{tr,(2)}(x, t) \quad \text{for a.e.} \quad (x, t) \in \Omega \times (0, \tau - \gamma). \quad (14.4)
\]

Hence,

\[
u_0^{tr,(2)} \xrightarrow[\gamma \to 0^+]{} \nu_0^{tr,(2)} \quad \text{a.e. in} \quad \Omega \times (0, \tau), \quad (14.5a)
\]

\[
u_0^{tr,(2)} \xrightarrow[\gamma \to 0^+]{} \nu_0^{tr,(2)} \quad \text{a.e. in} \quad \Omega \times (0, \tau). \quad (14.5b)
\]

Since functions \( a = a(\lambda) \) and \( q_a = q_a(\lambda) \) are continuous, using \((14.5)\) we pass to the limit as \( \gamma \to 0^+ \) in the entropy boundary conditions \((3.3d)\) (with \( u_0^{tr,(2)} = u_0^{tr,(2)} \)) and \((3.3e)\) (with \( u_0^{tr,(2)} = u_0^{tr,(2)} \)) and derive the entropy boundary conditions \((9.9d)\) and \((9.9e)\) on \( \Omega \times (0, \tau). \)

(3) Secondly, we consider the case \( t > \tau \). Since supp \( K_\gamma(\cdot, \tau) \subset [\tau - \gamma, \tau] \), the inequality \((3.3a)\) has the form \((9.9a)\) on \( \Omega \times (\tau, T) \times (0, S) \) (with \( u = u_* \)). Taking into account existence of traces \( u_{\nu0}^{tr,(2)} \) and \( u_{\nuS}^{tr,(2)} \) on \( \Omega \times (\tau, T) \times \{s = 0^+\} \) and \( \Omega \times (\tau, T) \times \{s = S-0\} \), respectively, this inequality is equivalent in the sense of distributions to the integral inequality

\[
\frac{\int_T^T \int_0^S \int_\Omega \left[ \eta(u_\gamma) \partial_t \phi + a(u_\gamma) \partial_s \phi + q_a(u_\gamma) \cdot \nabla_x \phi + \eta(u_\gamma) \Delta_x \phi - \eta''(u_\gamma) |\nabla_x u_\gamma|^2 \phi \right] \, dx \, ds \, dt \right] - \int_\Omega \left[ q_a(u_{\nuS}^{tr,(2)}(x, t, S) - q_a(u_{\nu0}^{tr,(2)}(x, t, 0) \right] \, dx \, dt \geq 0, \quad (14.6)
\]

where \( \phi = \phi(x, t, s) \) is an arbitrary smooth nonnegative test-function vanishing in the neighborhood of the sections \( \{t = \tau\} \) and \( \{t = T\} \).
Due to sufficient arbitrariness of the test-function $\phi$, for $\eta(u_\gamma) = \pm u_\gamma$ inequality (14.6) reduces to the integral equality

$$\int T \int \int_0^\tau \int_0^\Omega \left[ u_\gamma \partial_t \psi + a(u_\gamma) \partial_x \psi + \varphi(u_\gamma) \cdot \nabla_x \psi + u_\gamma \Delta_x \psi \right] d\mathbf{x} ds dt$$

$$- \int T \int \int_0^\tau \int_0^\Omega \left[ a(u_\gamma^{(2)}) \psi(x, t, S) - a(u_\gamma^{(2)}) \psi(x, t, 0) \right] d\mathbf{x} dt = 0,$$  

(14.7)

where $\psi = \psi(x, t, s)$ is an arbitrary smooth (not necessarily nonnegative) test-function vanishing in the neighborhood of the sections $\{t = \tau\}$ and $\{t = T\}$. Substitute the test-function $\psi(x, t, s) = \theta(x, t) \rho_0^0(s)$ into (14.7), where $\theta$ is an arbitrary smooth test-function satisfying the above stated finiteness demands, and $\rho_0^0$ is defined by (6.7) and (6.8). Passing to the limit in (14.7) as $\gamma \to 0+$ along the subsequence $\{\gamma_\nu\}_{\nu \to \infty}$, using (12.5) and (14.1a) we establish that

$$\int T \int \int_0^\tau \int_0^\Omega \left[ u_\gamma \rho_0^0(s) \partial_t \theta + a(u_\gamma)(\rho_0^0)'(s) \theta + \varphi(u_\gamma) \rho_0^0(s) \cdot \nabla_x \theta + u_\gamma \rho_0^0(s) \Delta_x \theta \right] d\mathbf{x} ds dt$$

$$+ \int T \int \int_0^\tau \int_0^\Omega \left[ a(\lambda) d\Lambda_{x, t}^{(0)}(\lambda) \right] \theta d\mathbf{x} dt = 0.$$

From this, passing to the limit as $\delta \to 0+$, we derive the relation

$$\int T \int \int_0^\tau \int_0^\Omega \left[ a(\lambda) d\Lambda_{x, t}^{(0)}(\lambda) \right] \theta d\mathbf{x} dt = \int T \int \int_0^\tau \int_0^\Omega \left[ a(u_\gamma^{(2)}(x, t)) \right] \theta d\mathbf{x} dt,$$

or, equivalently,

$$\int \int_0^\tau \int_0^\Omega a(\lambda) d\Lambda_{x, t}^{(0)}(\lambda) = a(u_\gamma^{(2)}(x, t)) \quad \text{for a.e. } (x, t) \in \Omega \times (\tau, T),$$

(14.8a)

due to arbitrariness of $\theta$.

Quite analogously, using the test-function $\rho_0^5(s) = \rho_0^0(S - s)$ in (14.7) and limiting relations (12.5) and (14.1b), we derive the limiting relation

$$\int \int_0^\tau \int_0^\Omega a(\lambda) d\Lambda_{x, t}^{(5)}(\lambda) = a(u_\gamma^{(2)}(x, t)) \quad \text{for a.e. } (x, t) \in \Omega \times (\tau, T).$$

(14.8b)

(5) Now substitute $\phi(x, t, s) = \theta(x, t) \rho_0^0(s)$ into (14.6). Here $\theta$ is nonnegative, so that $\phi$ is a valid test-function. Passing to the limit in (14.6) as $\gamma \to 0+$ along the subsequence $\{\gamma_\nu\}_{\nu \to \infty}$, using (12.5) and (14.1a), we establish that

$$\int T \int \int_0^\tau \int_0^\Omega \left[ \eta(u_\gamma) \rho_0^0(s) \partial_t \theta + a(u_\gamma)(\rho_0^0)'(s) \theta + \varphi(u_\gamma) \rho_0^0(s) \cdot \nabla_x \theta + \eta(u_\gamma) \rho_0^0(s) \Delta_x \theta \right] d\mathbf{x} ds dt$$

$$+ \int T \int \int_0^\tau \int_0^\Omega \left[ a(\lambda) d\Lambda_{x, t}^{(0)}(\lambda) \right] \theta d\mathbf{x} dt \geq \liminf_{\nu \to \infty} \int T \int \int_0^\tau \int_0^\Omega \left[ \eta''(u_{\gamma_\nu}) \nabla_x u_{\gamma_\nu} \right] \rho_0^0(s) \theta d\mathbf{x} ds dt \geq 0.$$
From this, passing to the limit as $\delta \to 0+$, we derive the inequality
\[
\int_\tau^T \int_\Omega \left[ \int_{\mathbb{R}_{\lambda}} q_a(\lambda)d\Lambda^{(0)}_{x,t}(\lambda) \right] \theta dx dt \geq \int_\tau^T \int_\Omega q_a(u_{s0}^{tr,(2)}) \theta dx dt,
\]
or, equivalently,
\[
\int_{\mathbb{R}_{\lambda}} q_a(\lambda)d\Lambda^{(0)}_{x,t}(\lambda) \geq q_a(u_{s0}^{tr,(2)}(x,t)) \text{ for a.e. } (x,t) \in \Omega \times (\tau,T),
\]
due to arbitrariness and non-negativeness of $\theta$.

Quite analogously, using the test-function $\rho^S_\delta(s) = \rho^0_\delta(S-s)$ in \eqref{14.6} and the limiting relations \eqref{12.5} and \eqref{14.1b}, we derive the relation
\[
q_a(u_{sS}^{tr,(2)}(x,t)) \geq \int_{\mathbb{R}_{\lambda}} q_a(\lambda)d\Lambda^{(S)}_{x,t}(\lambda) \text{ for a.e. } (x,t) \in \Omega \times (\tau,T).
\]

(6) Passing to the limit in the entropy boundary condition \eqref{3.3d} (with $u_0^{tr,(2)} = u_{q\gamma}^{tr,(2)}$) on $\Omega \times (\tau,T)$ as $\gamma \to 0+$ along the subsequence $\{\gamma_\nu\}_\nu \to \infty$, using \eqref{12.5} and \eqref{14.1a}, we get
\[
\int_{\mathbb{R}_{\lambda}} q_a(\lambda)d\Lambda^{(0)}_{x,t}(\lambda) - q_a(u_0^{(2)}(x,t)) - \eta'(u_0^{(2)}(x,t)) \left( \int_{\mathbb{R}_{\lambda}} a(\lambda)d\Lambda^{(0)}_{x,t}(\lambda) - a(u_0^{(2)}(x,t)) \right) \leq 0
\]
for a.e. $(x,t) \in \Omega \times (\tau,T)$. \hspace*{1cm} \eqref{14.10}

Combining \eqref{14.10} with \eqref{14.8a} and \eqref{14.9a}, we establish \eqref{9.9d} on $\Omega \times (\tau,T)$.

Analogously, using \eqref{12.5}, \eqref{14.1b}, \eqref{14.8b} and \eqref{14.9b}, from \eqref{3.3c} we deduce \eqref{9.9e} on $\Omega \times (\tau,T)$, which completes the proof of the lemma.

\textbf{Lemma 14.2.} The limiting function $u_* = \lim_{\nu \to \infty} u_{\nu\gamma}$ satisfies the kinetic boundary conditions \eqref{9.5d} and \eqref{9.5c} with some nonnegative Radon measures $\mu^{(2)}_{s0}, \mu^{(2)}_{sS} \in \mathcal{M}^+((\mathbb{Z}^2 \times \mathbb{R}_{\lambda})$ such that $\text{supp } \mu^{(2)}_{s0}, \text{supp } \mu^{(2)}_{sS} \subset \mathbb{Z}^2 \times [-M_3, M_3]$.

\textit{Proof.} It is sufficient to notice that the kinetic boundary conditions \eqref{9.5c} and \eqref{9.5d} are equivalent to the entropy boundary conditions \eqref{9.9d} and \eqref{9.9e}, respectively. The proof of this claim is quite analogous to justification of equivalency of \eqref{3.2d}–\eqref{3.2e} to \eqref{3.3d}–\eqref{3.3c} in the proof of Lemma 6.3 (see formulas \eqref{6.16}–\eqref{6.21}).

Lemma 14.2 is proved.

\textit{Completion of the proof of Theorem 9.1.} Collecting altogether the results of Corollary 12.3 and Lemmas 12.5, 13.1 and 14.2, we conclude that the limiting function $u_*$ of the extracted in the beginning of Section 14 subsequence $\{u_{\gamma\gamma}\}_{\gamma \downarrow 0}$ is a kinetic solution of Problem $\Pi_0$ in the sense of Definition 9.1. Due to item 2 of Theorem 9.1, function $u_*$ is the unique kinetic solution of Problem $\Pi_0$. Therefore, the whole family $\{u_{\gamma}\}_{\gamma > 0}$ tends to $u_*$ strongly in $L^1(G_{T,S})$ and weakly in $L^2((0,T) \times (0,S); \tilde{W}^2_2(\Omega))$, as $\gamma \to 0+$.

Finally, due to item 3 of Theorem 9.1 function $u_*$ is the unique entropy solution of Problem $\Pi_0$ in the sense of Definition 9.2. This observation completes the proof of item 1 of Theorem 9.1. Thus, all assertions in Theorem 9.1 are proved.
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