A HOMOTOPICAL SKOLEM–NOETHER THEOREM

AJNEET DHILLON AND PÁL ZSÁMBOKI

Abstract. The classical Skolem–Noether Theorem [Gir71] shows us (1) how we can assign to an Azumaya algebra $A$ on a scheme $X$ a cohomological Brauer class in $H^2(X, G_m)$ and (2) how Azumaya algebras correspond to twisted vector bundles. The Derived Skolem–Noether Theorem [Lie09] generalizes this result to weak algebras in the derived 1-category locally quasi-isomorphic to derived endomorphism algebras of perfect complexes. We show that in general for a co-family $C$ of presentable monoidal quasi-categories with descent over a quasi-category with a Grothendieck topology, there is a fibre sequence giving in particular the above correspondences. For a totally supported perfect complex $E$ over a quasi-compact and quasi-separated scheme $X$, the long exact sequence on homotopy splits, thus showing that the adjoint action induces isomorphisms $\pi_i(\Aut_{\text{Perf}} E, \id_E) \to \pi_i \Aut_{\text{Alg Perf}}(R\End E, \id_{R\End E})$ for $i \geq 1$. Further applications include complexes in Derived Algebraic Geometry, module spectra in Spectral Algebraic Geometry and ind-coherent sheaves and crystals in Derived Algebraic Geometry in characteristic 0.

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1. Introduction

1.1. The classical theory.

1.1.1. Skolem–Noether Theorem. [Gir71, V, Lemme 4.1] Let $X$ be a scheme (or a locally ringed topos), and $n$ a positive integer. Then there exists a short exact sequence of group sheaves on $X$:

$$1 \to \mathbf{G}_m \to \text{GL}_n \xrightarrow{\text{Ad}} \text{Aut}_{\text{Alg}} \text{End}_{\mathcal{O}} \mathcal{O}^\oplus_n \to 1.$$  

Taking deloopings, we get a fibre sequence of stacks on $X$:

(1) \[ \text{LB} \to \text{VB}_n \xrightarrow{\text{End}} \text{Az}_n. \]

Here, LB is the stack of line bundles, VB$_n$ is the stack of rank-$n$ vector bundles and Az$_n$ is the stack of rank-$n$ Azumaya algebras on $X$, that is forms of the endomorphism algebra End$_{\mathcal{O}} \mathcal{O}^\oplus_n$.

1.1.2. From Azumaya algebras to cohomological Brauer classes. Let $A$ be a rank-$n$ Azumaya algebra on an $X$-scheme $U$. Then it is classified by a map $U \xrightarrow{c_A} \text{Az}_n$. We can take the homotopy pullback square

By the fibre sequence (1), we get that $\mathcal{X}(A) \to U$ is a $\mathbf{G}_m$-gerbe on $U$, that is a $B\mathbf{G}_m$-bundle. We call it the gerbe of trivializations.

**Corollary 1.1.** The Azumaya algebra $A$ induces the cohomological Brauer class $[\mathcal{X}(A)] \in H^2(U, \mathbf{G}_m)$. 


1.1.3. Azumaya algebras and twisted sheaves. We have seen previously that Azumaya algebras give $\mathbb{G}_m$-gerbes. The most succinct way to express this is to notice that $\mathbb{G}_m$ is commutative (or $E_2$) and thus we can deloop further to get a fibre sequence of 2-stacks on $X$:

\[
\text{VB} \xrightarrow{\text{End}} \text{Az} \xrightarrow{} B^2 \mathbb{G}_m.
\]

That is, the map of stacks $\text{VB} \xrightarrow{\text{End}} \text{Az}$ is a $\mathbb{G}_m$-gerbe.

**Corollary 1.2.** Let $\mathcal{X}$ be a $\mathbb{G}_m$-gerbe on $U$. Then Azumaya algebras on $U$ with Brauer class $[\mathcal{X}]$ correspond to $\mathcal{X}$-twisted vector bundles on $U$, that is $B \mathbb{G}_m$-equivariant maps $\mathcal{X} \to \text{VB}$.

1.2. Derived Skolem–Noether Theorem. Let $X \to S$ be a morphism of schemes. By the classical Skolem–Noether Theorem, we see that the stack $B \text{PGL}_n(X/S)$ classifies families of twisted vector bundles of rank $n$. Lieblich has compatified this stack by letting twisted vector bundles degenerate into twisted totally supported perfect coherent sheaves [Lie09, Theorem 6.2.4]. To get a more algebraic description of these objects, Lieblich has proven the following result:

**Theorem 1.3** ([Lie09, Theorem 5.1.5]). Let $E$ be a perfect complex on a scheme $X$. Let $\mathcal{O}_E = \text{Coker}(\mathcal{O}_X \to \text{REnd} E)$. Then we have a short exact sequence of sheaves of groups on $X$

\[
1 \to \mathcal{O}_E^\times \xrightarrow{} \text{Aut} E \xrightarrow{\text{Ad}} \text{Aut} \text{REnd} E \to 1.
\]

Let $A$ be a weak algebra object in the monoidal 1-category $D(X)_{\otimes}$. Then $A$ is a pre-generalized Azumaya algebra if there exists:

1. an étale covering $U \to X$,
2. a totally supported perfect coherent sheaf $F$ on $U$ and
3. a quasi-isomorphism of weak algebras $A|_U \simeq \text{R End}(F)$.

We denote by $\mathcal{P} \mathcal{R}_X$ the category fibred in groupoids of pre-generalized Azumaya algebras on $X$. The stack of generalized Azumaya algebras $\mathcal{G} \mathcal{A} \mathcal{z}_X$ is the 1-stackification of $\mathcal{P} \mathcal{R}_X$. In other words, a generalized Azumaya algebra is 1-descent data of pre-generalized Azumaya algebras.

**Corollary 1.4** ([Lie09, Corollary 5.2.1.9 and §5.2.4]). Let $U \to X$ be a morphism of schemes. Then the following assertions hold:

1. Let $A$ be a generalized Azumaya algebra on $U$. Then it induces a gerbe of trivializations $\mathcal{X}(A) \in B^2 \mathbb{G}_m(U)$ and in particular a cohomological Brauer class $[\mathcal{X}(A)] \in H^2(U, \mathbb{G}_m)$.
2. Let $\mathcal{X}$ be a $\mathbb{G}_m$-gerbe on $U$. Then generalized Azumaya algebras with Brauer class $[\mathcal{X}]$ correspond to $\mathcal{X}$-twisted totally supported perfect coherent sheaves on $U$.

Note how these statements are about the derived 1-categories. Therefore, in this setting, one cannot detect higher descent conditions, which is the reason why 1-stackification is needed in the construction of the stack of generalized Azumaya algebras [Lie09 §5.2] unless higher homotopy groups vanish, for example when $X$ is a surface [Lie09 Proposition 6.4.1]. Using our Homotopical Skolem–Noether Theorem, we can show that the higher homotopy groups always vanish, so stackification is not needed in general:
Proposition 5.28. Let $X$ be a quasi-compact and quasi-separated scheme. Then the category fibred in groupoids of pre-generalized Azumaya algebras $\mathcal{P}\mathcal{R}_X$ is a 1-stack.

1.3. Derived Azumaya algebras.

1.3.1. $\mathcal{B}r \overset{?}{=} \mathcal{B}r'$. Let $X$ be a scheme. Two Azumaya algebras $A$ and $B$ on $X$ are Morita equivalent if there exists an equivalence of categories $\text{Mod}_A \simeq \text{Mod}_B$ between their categories of modules. The Morita equivalence class of an Azumaya algebra is its Brauer class. The set $\mathcal{B}r$ of Morita equivalence classes of Azumaya algebras admits a group structure via tensor product, thus we get the Brauer group.

One can see that taking cohomological Brauer classes induces an injection

$$\mathcal{B}r(X) \xrightarrow{\text{Az}[\mathcal{X}(A)]} \mathcal{H}^2(X, \mathbb{G}_m)_{\text{tors}} =: \mathcal{B}r'(X).$$

One can ask whether this is an isomorphism. A general positive result is the following:

Theorem 1.5 ([dJ, Theorem 1.1]). Let $X$ be a scheme equipped with an ample invertible sheaf. Then we have $\mathcal{B}r(X) = \mathcal{B}r'(X)$.

and a negative counterexample is the following:

Corollary 1.6 ([EHK01, Corollary 3.11]). Let $X = X_1 \cup X_2$ be two copies of $\text{Spec} \mathbb{C}[x^2, xy, y^2]$ glued along the nonsingular loci. Let $\mathcal{X}'$ be the $\mu_2$-gerbe obtained by gluing $(\mathbb{B} \mu_2)_{X_i}$, $i = 1, 2$ along the nontrivial involution $Q \mapsto Q \times_{\mu_2} P$ where $P = \mathbb{A}^2 \setminus \{(0, 0)\}$ with the antipodal action. Let $\mathcal{X} = \mathcal{X}' \times_{\mathbb{B} \mu_2} \mathbb{B}\mathbb{G}_m$. Then $[\mathcal{X}]$ is not represented by an Azumaya algebra.

This geometrical formulation has been explained to us by Siddharth Mathur. In the situation of the counterexample, there does not exist a twisted vector bundle $\mathcal{X} \to \text{VB}$. On the other hand, there does exist a twisted totally supported perfect complex $\mathcal{X} \to \text{Perf}$. Extending the notion of Azumaya algebras accordingly leads us to derived Azumaya algebras.

1.3.2. Derived Azumaya algebras. Let $\mathcal{Q}C^\mathbb{D}$ denote the symmetric monoidal $\infty$-stack of (unbounded) complexes of quasi-coherent modules. Let $X$ be a quasi-compact and quasi-separated scheme. Then an algebra $A \in \text{Alg} \mathcal{Q}C(X)$ is a derived Azumaya algebra, if there exists

1. an étale covering $U \to X$,
2. a totally supported perfect complex $E$ on $U$ and
3. a quasi-isomorphism of algebras $A|_U \simeq \mathcal{R}\text{End} E$.

Let $A$ be a derived Azumaya algebra. Then we denote by $\text{Mod}_A$ the dg-category of $A$-dg-modules. We say that two derived Azumaya algebras $A$ and $B$ are Morita equivalent if there exists an equivalence of dg-categories $\text{Mod}_A \simeq \text{Mod}_B$. The derived Brauer group $\text{dBr}$ is the group of Morita equivalence classes of derived Azumaya algebras.

1.3.3. $\text{dBr} = \text{dBr}'$. Let $\mathcal{D}g$ denote the $\infty$-stack of dg-categories on $X$. Let $\mathcal{D}g^{\text{Az}} \subseteq \mathcal{D}g$ denote the full substack on locally trivial dg-categories. Let $\text{Pic} \mathcal{Q}C \subseteq \mathcal{Q}C$ denote the full substack on invertible complexes. By the homotopical Eilenberg–Watts theorem [Lur16, Theorem 4.8.4.1], the functor $\text{Pic} \mathcal{Q}C \xrightarrow{\mathcal{L}=\mathcal{L}(\otimes)} \text{Aut}_{\mathcal{D}g} \mathcal{Q}C$ is an equivalence. Since a complex $E \in \mathcal{Q}C(X)$ is invertible if and only
if it is a shift of a line bundle, we get $Dg^{Az} \simeq B^2 G_m \times B \mathbb{Z}$. Therefore we get an injective map $dBr_{[A] \to [Mod_A]} \to dBr' := H^2(X, G_m) \times H^1(X, \mathbb{Z})$. One can again ask if this map is surjective.

**Theorem 1.7** ([Toe12 Corollary 4.8]). Let $\mathcal{M}$ be an fpf locally trivial dg-category on a quasi-compact and quasi-separated (derived) scheme $X$. Then there exists a derived Azumaya algebra $A$ such that $\mathcal{M} \simeq \text{Mod}_A$.

1.3.4. The case of Spectral Algebraic Geometry. We can develop the theory of derived Azumaya algebras in Spectral Algebraic Geometry too. In this setting, QC denotes the symmetric monoidal $\infty$-stack of quasi-coherent module spectra.

Here too we have a $dBr = dBr'$-type result:

**Theorem 1.8** ([AG14 Corollary 6.20]). Let $X$ be a quasi-compact and quasi-separated connective spectral scheme. Then every cohomological Brauer class on $X$ lifts to a derived Azumaya algebra.

While we were preparing this paper, Benjamin Antieau has told us about the paper [GL16], in which the Homotopical Skolem–Noether Theorem is proven in the case of affine spectral schemes [GL16 Proposition 5.15]. In what follows, we shall show that the result holds in a more general sense.

1.4. Homotopical Skolem–Noether Theorem – general case.

1.4.1. Towards the Homotopical Skolem–Noether Theorem. Recall that the key ingredient in identifying derived Brauer classes as cohomology classes was the equivalence $Dg^{Az} \simeq B \text{Pic QC}$ following from the Homotopical Eilenberg–Watts Theorem. In other words, it shows that the outer square in the following diagram is homotopy Cartesian:

$$
\begin{array}{ccc}
\text{Pic QC} & \xrightarrow{\phi^E} & \text{TPerf}^\infty \\
\downarrow \phi^h & & \downarrow \phi^h \\
X & \xrightarrow{\phi^h} & \text{Deraz} \\
\downarrow \phi^h & & \downarrow \phi^h \\
X & \xrightarrow{\phi^h} & Dg^{Az}.
\end{array}
$$

The Homotopical Skolem–Noether Theorem consists of showing that the inner squares are homotopy Cartesian too. Here, $\text{TPerf}^\infty \subseteq \text{QC}$ is the interior of the full substack on totally supported perfect complexes, $E \in \text{TPerf}(X)$ is a totally supported perfect complex on $X$ and $\text{Deraz} \subseteq \text{AlgQC}$ is the full substack on derived Azumaya algebras.

The Homotopical Skolem–Noether Theorem gives a long exact sequence on homotopy sheaves:

$$
\cdots \to \pi_{n+1}(\text{Deraz}, \text{REnd} E) \xrightarrow{\text{Mod}} \pi_n(\text{Pic QC}, \mathcal{O}) \xrightarrow{\phi^E} \pi_n(\text{TPerf}, E) \xrightarrow{\text{REnd} E} \pi_n(\text{Deraz}, \text{REnd} E) \xrightarrow{\text{Mod}} \pi_{n-1}(\text{Pic QC}, \mathcal{O}) \to \cdots
$$

In the case of complexes of quasi-coherent modules over a quasi-compact and quasi-separated scheme, this long exact sequence splits to give isomorphisms

$$(3) \quad \pi_n(\text{TPerf}, E) \xrightarrow{\text{REnd} E \cong} \pi_n(\text{Deraz, REnd} E)$$
for \( n > 1 \) and a short exact sequence

\[
1 \to G_m \to \text{Aut}_{T\text{Perf}(E)} \xrightarrow{\text{Ad}} \text{Aut}_{\text{Deraz}(\mathcal{R}\text{End}E)} \to 1.
\]

Note that (4) is the Derived Skolem–Noether Theorem. This together with the collection of isomorphisms (3) form the key step in showing that \( \mathcal{P} \mathcal{B}_X \) is a 1-stack.

1.4.2. Azumaya algebra objects in a presentable monoidal quasi-category. It turns out that the main result can be established on the level of generality of monoidal quasi-categories. In Section 2 we summarize the notions of algebras and modules and in Section 3 Morita Theory in Higher Algebra [Lur16].

Let \( K \) be a quasi-category equipped with a Grothendieck topology and \( \text{op} \mathcal{C} \to (\text{Assoc} \text{op}) \times K \) a family of presentable monoidal quasi-categories with descent over \( K \text{op} \). This gets us coCartesian fibrations over \( K \text{op} \):

1. \( \text{LTens} \mathcal{C} \) classifying families of presentable left-tensored quasi-categories,
2. \( \text{LTens} \mathcal{C}^\ast \) classifying families of presentable left-tensored quasi-categories with a section and
3. \( \text{Alg} \mathcal{C} \) classifying families of algebra objects in \( \mathcal{C} \).

These are equipped with

1. the forgetful functor \( \text{LTens} \mathcal{C}^\ast \to \text{LTens} \mathcal{C} \) that is a left fibration and
2. the pointed module functor \( \text{Alg} \mathcal{C} \xrightarrow{\text{Mod}_A \to \text{Mod}_A M} \text{LTens} \mathcal{C} \) that is fully faithful with a right adjoint.

Take \( X \in K \). By the Homotopical Morita Theorem (Lemma 4.21) a pair \( (\mathcal{M}, M) \in \text{LTens} \mathcal{C}(X) \) is in the essential image of \( \text{Mod}_A \) if and only if \( M \in \mathcal{M} \) is a dualizable generator, that is the map \( \text{Mod}_{\text{End} M} \to \mathcal{M} \) is an equivalence. We denote by \( \mathcal{C} \text{dgen} \subseteq \mathcal{C} \) the full subprestack on dualizable generators and \( \text{LTens} \text{dgen} \mathcal{C} \subseteq \text{LTens} \mathcal{C} \) the full subprestack on pairs \( (\mathcal{M}, M) \) where \( M \in \mathcal{M} \) is a dualizable generator.

Let \( A \in \text{Alg} \mathcal{C}(X) \) be an algebra object. Then \( A \) is an Azumaya algebra object if there exists

1. a \( \tau \)-covering \( U \to X \),
2. a dualizable generator \( M \in \mathcal{C} \text{dgen}(U) \) and
3. an equivalence \( A|U := \mathcal{C}(U) \otimes_{\mathcal{C}(X)} A \simeq \text{End} M \).

By the Homotopical Morita Theorem, we have an equivalence \( A|U \simeq \text{End} M \) of algebras for some \( M \in \mathcal{C} \text{dgen}(U) \) if and only if we have an equivalence \( \text{Mod}_A|U \simeq \mathcal{C}(U) \) of presentable quasi-categories left-tensored over \( \mathcal{C} \). We denote by \( \text{Az} \mathcal{C} \subseteq \text{Alg} \mathcal{C} \) the full subprestack on Azumaya algebras and \( \text{LTens} \text{Az} \mathcal{C} \subseteq \text{LTens} \text{dgen} \mathcal{C} \) the full subprestack on locally trivial pointed presentable left-tensored quasi-categories over \( \mathcal{C} \). Note that the equivalence \( \text{Alg} \mathcal{C} \xrightarrow{\text{Mod}_A} \text{LTens} \text{dgen} \mathcal{C} \) restricts to an equivalence \( \text{Az} \mathcal{C} \to \text{LTens} \text{Az} \mathcal{C} \).

1.4.3. Descent for presentable left-tensored quasi-categories with descent. Take \( T \in K \) and let \( \mathcal{M} \in \text{LTens} \mathcal{C}(T) \) be a presentable left-tensored quasi-category over \( \mathcal{C}(T) \). Then we say that \( \mathcal{M} \) has
\(\tau\)-descent if for all Čech nerves \(\Delta^ op \to K\) of \(\tau\)-coverings over \(T\), the canonical map
\[ \mathcal{M}/X \to \lim_{n \geq 0} (\mathcal{M}/U_n) \]
is an equivalence. We let \(\text{LTens}^\text{desc} \subseteq \text{LTens} \) denote the full presubstack on presentable left-tensored quasi-categories over \(\mathcal{C}\) with \(\tau\)-descent. We let \(\text{LTens}^{Az} \subseteq \text{LTens}^\text{desc} \) denote the full subpresubstack on locally trivial presentable left-tensored quasi-categories over \(\mathcal{C}\).

The following is the key criterion for the presubstacks of interest to have \(\tau\)-descent: we say that the family \(\mathcal{C}^\circ\) of presentable monoidal quasi-categories has \(\tau\)-descent if the following conditions hold:

1. The underlying Cartesian fibration \(\text{op} \mathcal{C} \to K\) has \(\tau\)-descent.
2. Base changes in \(\mathcal{C}^\circ\) commute with \(\tau\)-descent data. Then we have a fibre sequence in \(\text{Cart}^\text{op} \to K\), the base change \(\mathcal{M}/V \to \mathcal{M}/U\) is also a \(q\)-limit diagram.

**Theorem 4.15.** Let \(K\) be a quasi-category equipped with a Grothendieck topology \(\tau\). Let \(\mathcal{C}^\circ \to K^\text{op} \times \text{Assoc}^\circ\) be a coCartesian family of presentable monoidal quasi-categories with \(\tau\)-descent. Then the following assertions hold:

1. The family \(\text{op} \text{LTens}^\text{desc} \to K\) of presentable quasi-categories left-tensored over \(\mathcal{C}\) with \(\tau\)-descent is a \(\tau\)-stack.
2. For any object \(T \in K\) and associative algebra \(A \in \text{Alg}\mathcal{C}(T)\), the quasi-category \(\text{Mod}_A \mathcal{C}(T)\) left-tensored over \(\mathcal{C}(T)\) of right \(A\)-modules in \(\mathcal{C}(T)\) has \(\tau\)-descent.

**Corollary 4.18.** The presubstacks \(\text{Az} \mathcal{C}\) and \(\text{LTens}^{Az} \mathcal{C}\) also have \(\tau\)-descent.

To make it easier to show that \(\mathcal{C}^\circ\) has \(\tau\)-descent, we have proven the following criterion:

**Corollary 4.13.** Let \(K\) be a quasi-category equipped with a Grothendieck topology \(\tau\) and \(\mathcal{C}^\circ \to K^\text{op} \times \text{Assoc}^\circ\) a coCartesian fibration of presentable monoidal quasi-categories. Suppose that the following assertions hold:

1. The underlying Cartesian fibration \(\text{op} \mathcal{C} \to K\) has \(\tau\)-descent.
2. For all objects \(U \in K\), the presentable monoidal quasi-category \(\mathcal{C}(U) \in \text{Alg}(\text{Pr}^\text{L})\) has dualizable underlying presentable quasi-category.

Then base changes in \(\mathcal{C}\) commute with \(\tau\)-descent data.

1.4.4. The Homotopical Skolem–Noether Theorem.

**Theorem 4.20.** Let \(K\) be a quasi-category with final object \(S\), let \(\tau\) be a Grothendieck topology on it. Let \(\text{Cart}_{/K}^\tau\) denote the quasi-category of Cartesian fibrations over \(K\) with \(\tau\)-descent. Let \(\mathcal{C} \subseteq \text{Cart}_{/K}^\tau\) denote the full subcategory on right fibrations over \(K\) with \(\tau\)-descent, which is an \(\infty\)-topos. Let \(\mathcal{C}^\circ \to \text{Assoc}^\circ \times K^\text{op}\) be a family of presentable monoidal quasi-categories with \(\tau\)-descent. Then the following assertions hold:

1. We have a fibre sequence in \(\text{Cart}_{/K}^\tau\):
\[
\text{End}_{\mathcal{C}^\text{deg}}(\mathcal{C}, \mathcal{O}) \to \text{End}_{\mathcal{C}^\text{deg}}(\mathcal{C}, \mathcal{O}) \to \text{End}_{\mathcal{C}^\text{deg}}(\mathcal{C}, \mathcal{O})
\]
2. Let \(E \in \mathcal{C}^\text{deg}(S)\). Then we have a fibre sequence in \(\text{Cart}_{/K}^\tau\):
\[
\text{End}_{\mathcal{C}^\text{deg}}(\mathcal{C}, \mathcal{O}) \to \text{End}_{\mathcal{C}^\text{deg}}(\mathcal{C}, \mathcal{O}) \to \text{End}_{\mathcal{C}^\text{deg}}(\mathcal{C}, \mathcal{O}).
\]
(3) We have a long exact sequence of homotopy sheaves in $h\mathcal{D}$:

$$\cdots \to \pi_2(\mathop{\text{op}} \text{Az } \mathcal{C}, \text{End } E) \to \pi_1(\mathop{\text{op}} \text{Pic } \mathcal{C}, \mathcal{O}) \to \pi_1(\mathop{\text{op}} \text{C}_{\text{dgen}}, E) \to \pi_1(\mathop{\text{op}} \text{Az } \mathcal{C}, \text{End } E) \to$$

$$\to \pi_0(\mathop{\text{op}} \text{Pic } \mathcal{C}) \to \pi_0(\mathop{\text{op}} \text{C}_{\text{dgen}}) \to \pi_0(\mathop{\text{op}} \text{Az } \mathcal{C}) \to \pi_0(\mathop{\text{op}} \text{Lten}_{\text{Az } \mathcal{C}}) = \ast.$$ 

1.5. **Homotopical Skolem–Noether Theorem – applications.**

1.5.1. *Algebraic Geometry.* Let $X$ be a quasi-compact and quasi-separated scheme. We consider $K = \text{Fppf}_X$, and $\mathop{\text{op}} \text{C} = \mathcal{C}\text{QC}$. We have already seen $\text{Pic } \mathcal{C} \simeq B \mathbb{G}_m \times \mathbb{Z}$. A complex $E$ is dualizable if and only if it is a perfect complex. It is a generator if and only if it is totally supported. If $E$ is totally supported, then the multiplication map $\mathbb{G}_m \to \pi_1(\mathcal{C}, E)$ is injective. Therefore, the long exact sequence splits and we get the following result:

**Corollary 5.24** (Homotopical Skolem–Noether Theorem for schemes). Let $S$ be a quasi-compact and quasi-separated scheme. Let $\mathcal{C}^{\text{fppf}}_S$ denote the quasi-category of Cartesian fibrations on $\text{St}_S$ which satisfy fppf descent.

1. Let $\mathop{\text{op}} \text{TPerf}_S := \mathop{\text{op}}(\mathcal{C}^{\ast}_{\text{dgen}})$ denote the Cartesian fibration of totally supported perfect complexes on $S$, $\mathop{\text{op}} \text{Deraz}_S := \mathop{\text{op}} \text{Az } \mathcal{C}_S$ the Cartesian fibration of derived Azumaya algebras on $S$ and $\mathop{\text{op}} \text{Dg}^{\text{Az}}_S := \mathop{\text{op}} \text{Lten}_{\text{Az } \mathcal{C}_S}$ the Cartesian fibration of locally trivial presentable quasi-categories left-tensored over $\mathcal{C}_S^{\ast}$. Then the sequence in $(\mathcal{C}^{\text{fppf}}_S)$:

$$\mathop{\text{End}}(\mathop{\text{op}} \text{TPerf}_S^{\ast}, \mathcal{O}) \xrightarrow{\text{Mod}} \mathop{\text{End}}(\mathop{\text{op}} \text{Deraz}_S, \mathcal{O}) \xrightarrow{\mathcal{D}} \mathop{\text{End}}(\mathop{\text{op}} \text{Dg}^{\text{Az}}_S, \mathcal{D})$$

is a homotopy fibre sequence.

2. Let $E \in \text{TPerf}(S)$ be a totally supported perfect complex on $S$. Then the sequence in $(\mathcal{C}^{\text{fppf}}_S)$:

$$(\mathbb{G}_m \times \mathbb{Z}, \mathcal{O})^{\otimes E} \xrightarrow{\mathcal{D}} \mathop{\text{End}}(\mathop{\text{op}} \text{TPerf}_S^{\ast}, E) \xrightarrow{\mathcal{D}} \mathop{\text{End}}(\mathop{\text{op}} \text{Deraz}_S, \text{End } E)$$

is a homotopy fibre sequence.

3. We have isomorphisms of sheaves of groups

$$\pi_i(\mathop{\text{op}} \text{TPerf}_S, E) \cong \pi_i(\mathop{\text{op}} \text{Deraz}_S, \text{REnd } E)$$

for $i > 0$, a short exact sequence of sheaves of groups

$$1 \to \mathbb{G}_m \xrightarrow{a \mapsto} \text{Aut}_{\text{Perf }} E \xrightarrow{\text{Ad}} \text{Aut}_{\text{Deraz }}(\text{REnd } E) \to 1,$$

and an exact sequence of pointed sheaves of sets

$$\ast \to \mathbb{Z} = \pi_0(\mathbb{G}_m \times \mathbb{Z})^{\otimes E} \xrightarrow{\text{REnd}} \pi_0 \text{TPerf}_S \xrightarrow{\text{Mod}} \pi_0 \text{Deraz}_S \xrightarrow{\mathcal{D}} \pi_0 \text{Dg}^{\text{Az}}_S = \ast \to \ast.$$ 

1.5.2. **Homotopical Algebraic Geometry.** We can apply the Homotopical Skolem–Noether Theorem to Derived and Spectral Algebraic Geometry too:

**Corollary 5.39** (Homotopical Skolem–Noether theorem for Derived and Spectral Algebraic Geometry). Let $S$ be a derived or spectral affine scheme. Let $\mathcal{C}^{\text{fppc}}_S$ denote the quasi-category of Cartesian fibrations on $\text{St}_S$ which satisfy fppc descent.

1. Let $\mathop{\text{op}}(\text{Perf}^{\ast}_{\text{gen}}) := \mathop{\text{op}}(\mathcal{C}^{\ast}_{\text{dgen}})$ denote the right fibration of perfect generator complexes on $S$, $\mathop{\text{op}} \text{Deraz}_S := \mathop{\text{op}} \text{Az } \mathcal{C}_S$ the Cartesian fibration of derived Azumaya algebras on $S$ and $\mathop{\text{op}} \text{Dg}^{\text{Az}}_S :=$
twisted left crystals on $\mathcal{Y}$. Then the quasi-category $\text{Crys}^{\text{dr}}_{\text{coh}}(\mathcal{Y})$ of $T$-twisted left crystals on $\mathcal{Y}$ that are dualizable generators is equivalent to the quasi-category $\text{Deraz}^{T}(\mathcal{Y}_{\text{dr}})$ of derived Azumaya algebras on $\mathcal{Y}_{\text{dr}}$ with Brauer class $T$.

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2. Algebras and modules in Higher Algebra

In this section, we summarize notions of algebras of modules needed for Morita theory following [Lur16]. We will make free use of the theory of quasi-categories developed in [Lur09]. For a quick summary on stacks fibred in Kan complexes and quasi-categories, see our previous work [DZ22, §2.1].

Notation 2.1. Let $K$ be a simplicial set and $\mathcal{X} \rightarrow K$ a Cartesian fibration. Then we denote by $\mathcal{X}^\circ \subseteq \mathcal{X}$ its interior: for any vertex $T \in K$, the restriction $\mathcal{X}^\circ(T) \subseteq \mathcal{X}(T)$ of the inclusion of the fibre is the inclusion of the largest sub-Kan complex [DZ22, Definition 2.7].

2.1. Monoidal quasi-categories and algebras. Higher algebraic structure are encoded as higher categorical diagrams on $\infty$-operads. We shall now explain what this means.

Notation 2.2. Let $\text{Fin}_\ast$ denote the nerve of the category with
- objects the pointed finite sets $\langle n \rangle = \{*, 1, \ldots, n\}$ for $n \geq 0$. We denote $\langle n \rangle^\circ = \{1, \ldots, n\}$.
- morphism set $\text{Hom}_{\text{Fin}_\ast}(\langle m \rangle, \langle n \rangle) = \{\langle m \rangle \xrightarrow{\alpha} \langle n \rangle : \alpha(*) = *\}$. A morphism $\langle m \rangle \rightarrow \langle n \rangle$ can be thought of as a partially defined morphism $\langle m \rangle \rightarrow \langle n \rangle$. A map $\langle m \rangle \xrightarrow{f} \langle n \rangle$ is inert, if for each $i \in \langle n \rangle^\circ$, we have $|f^{-1}(i)| = 1$. Let $\mathcal{C} \xrightarrow{p} \text{Fin}_\ast$ be a morphism of simplicial sets. Then an edge $e$ in $\mathcal{C}^\circ$ is inert, if it is a $p$-coCartesian edge over an inert edge in $\text{Fin}_\ast$.

For each $n > 0$ and $i \in \langle n \rangle^\circ$, we fix the inert map $\langle n \rangle \xrightarrow{\rho^i} \langle 1 \rangle$ with
$$\rho^i(j) = \begin{cases} 1 & i = j \\ * & \text{else}. \end{cases}$$

Definition 2.3. Let $\mathcal{C} \xrightarrow{p} \text{Fin}_\ast$ be a morphism of simplicial sets. We denote $\mathcal{C}^\circ_{(1)}$ by $\mathcal{C}$. Then $p$ (or by abuse of notation: $\mathcal{C}^\circ$, or even $\mathcal{C}$) is a symmetric monoidal quasi-category, if it is a coCartesian fibration of $\infty$-operads [Lur16 Definition 2.1.2.13], that is

1. it is a coCartesian fibration, and
2. for each $n \geq 0$, the map $\mathcal{C}^\circ_{(n)} \xrightarrow{\rho^i_{(n)}} \mathcal{C}^{\times n}$ is a categorical equivalence.

Because of property (2), we denote by $C_1 \oplus \cdots \oplus C_n \in \mathcal{C}^\circ_{(n)}$ a preimage along $(\rho^i_{(n)})_{i=1}^n$ of $(C_1, \ldots, C_n) \in \mathcal{C}^{\times n}$.

The idea here is that coCartesian edges give the usual operations. A coCartesian edge over $\langle 2 \rangle \xrightarrow{1,2 \mapsto 1} \langle 1 \rangle$ gives a product operation $C \oplus D \mapsto C \otimes D$. It is a coCartesian edge, so it is unique up to homotopy. By the same uniqueness, for example, we get the homotopy commutative diagram
giving homotopies $C \otimes D \simeq D \otimes C$ natural in $C, D \in \mathcal{C}$.

**Definition 2.4.** Let $\mathcal{C} \overset{p}{\to} \text{Fin}$ be a symmetric monoidal quasi-category. Then a **commutative algebra object in $\mathcal{C}$** is a morphism of $\infty$-operads $\text{Fin}_n \xrightarrow{\Lambda} \mathcal{C}^\otimes \overset{[Lur16]}{\text{Definition 2.1.2.7}}$, that is, it is a section of $p$ which takes inert maps to inert maps. The **quasi-category of commutative algebras in $\mathcal{C}$** is the full subcategory $\text{CAlg}(\mathcal{C}) \subseteq \text{Fun}_{\text{Fin}_*}(\text{Fin}_*, \mathcal{C}^\otimes)$ on commutative algebras. We also denote $\text{Fin}_*$ by $\text{Comm}^\otimes$.

**Definition 2.5.** The associative $\infty$-operad denoted by $\text{Assoc}^\otimes$ is the $\infty$-operad $\text{Assoc}^\otimes \overset{\to}{\to} \text{Fin}^\otimes \overset{[Lur16]}{\text{Example 2.1.1.21 and Remark 4.1.1.4}}$. The data of $\text{Assoc}^\otimes$ is:

1. objects $\langle n \rangle$ for $n \geq 0$,
2. a morphism $\langle m \rangle \to \langle n \rangle$ is given by a map $\langle m \rangle \xrightarrow{\alpha} \langle n \rangle$ in $\text{Fin}_n$, and for each $i \in \langle n \rangle^\circ$, a linear ordering on the finite set $\alpha^{-1}(i)$, and
3. composition is given by lexicographical ordering.

The proof that this produces an $\infty$-operad can be found in [Lur16] Example 2.1.1.21 and Remark 4.1.1.4.

**Definition 2.6.** A **monoidal quasi-category** is a coCartesian fibration of $\infty$-operads $\mathcal{C}^\otimes \to \text{Assoc}^\otimes \overset{[Lur16]}{\text{Definition 4.1.10}}$. Observe that a coCartesian edge $\mathcal{C}^\otimes \overset{m}{\to} \mathcal{C}$ over the map $\{1 < 2\} \to \{1\}$ is a product map

$$\{C, D\} \mapsto C \otimes D,$$

and a coCartesian edge over the map $\langle 0 \rangle \to \langle 1 \rangle$ gives a unit object $\mathbf{1} \in \mathcal{C}$ for tensor product, where that this is a unit object on the left is shown by the homotopy commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}^\otimes & \xrightarrow{m} & \mathcal{C} \\
\xrightarrow{(1 \xrightarrow{1 \rightarrow 2} 2)} \downarrow & & \downarrow \text{id} \\
\mathcal{C} & \xrightarrow{\text{id}} & \mathcal{C}
\end{array}
$$

which follows from that coCartesian edges are unique up to homotopy. That is, we have a homotopy $\mathbf{1} \otimes C \simeq C$ natural in $C \in \mathcal{C}$.

**Definition 2.7.** Let $\mathcal{C}$ be a monoidal quasi-category. Then $C \in \mathcal{C}$ is **invertible**, if the endofunctor $\mathcal{C} \overset{C}{\to} \mathcal{C}$ is an equivalence. The **Picard quasi-category** $\text{Pic}(\mathcal{C}) \subseteq \mathcal{C}$ is the full subcategory on invertible objects.

**Definition 2.8.** Let $\mathcal{C}$ be a monoidal quasi-category. Then an **algebra object in $\mathcal{C}$** is a morphism of $\infty$-operads $\text{Assoc}^\otimes \overset{\Lambda}{\to} \mathcal{C}^\otimes$. The **quasi-category of algebras in $\mathcal{C}$** is the full subcategory $\text{Alg}(\mathcal{C}) \subseteq \text{Fun}_{\text{Fin}_*}(\text{Assoc}^\otimes, \mathcal{C}^\otimes)$ on algebras.

**Definition 2.9.** Following the procedure in 2.5 we construct an $\infty$-operad $\text{LM}^\otimes$ with a forgetful map $\text{LM}^\otimes \to \text{Assoc}^\otimes$. The simplicial set $\text{LM}^\otimes$ is obtained by taking the nerve of the category with
(1) objects \((\langle n \rangle, S)\) where \(S \subseteq \langle n \rangle^\ast\), and
(2) a map \((\langle n' \rangle, S') \to (\langle n \rangle, S)\) is given by a map \(\langle n' \rangle \to \langle n \rangle\) in \(\text{Assoc}^\circ\) such that
   (a) we have \(a(S' \cup \{s\}) \subseteq S \cup \{s\}\), and
   (b) for \(s \in S\), we have \(\alpha^{-1}(\{s\}) \cap S' = \{s'\}\), where \(s' = \max \alpha^{-1}(\{s\})\).

We denote \(a = (\langle 1 \rangle, \emptyset)\), \(m = (\langle 1 \rangle, \{1\}) \in \text{LM}^\circ\).

### 2.2. Left tensored quasi-categories and left modules.

#### Definition 2.10.
Let \(\mathcal{C}'\) be a monoidal quasi-category and \(\mathcal{M}\) a quasi-category. A *left-tensored structure of \(\mathcal{M}\) over \(\mathcal{C}'\) is a coCartesian fibration of \(\infty\)-operads \(\mathcal{C}' \to \text{LM}^\circ\) [Lur16 Definition 4.2.1.19] such that we have an equivalence of monoidal quasi-categories \(\mathcal{C}' \approx \mathcal{C}_a\), and an equivalence of quasi-categories \(\mathcal{M} \approx \mathcal{M}_m\).

We have a monomorphism
\[
\text{Assoc}^\circ \xrightarrow{(\langle n \rangle) \to (\langle n \rangle, \emptyset)} \text{LM}^\circ
\]
restricting along which we get the monoidal quasi-category structure \(\mathcal{C}_a^\circ := \mathcal{C} \times_{\text{LM}^\circ} \text{Assoc}^\circ\).

We also have to forgetful map
\[
\text{LM}^\circ \xrightarrow{(\langle n \rangle, S) \to (\langle n \rangle)} \text{Assoc}^\circ
\]
pulling back a monoidal quasi-category \(\mathcal{D}\) along which we get the *left-tensored structure of \(\mathcal{D}\) over itself.*

Let \(C_1, \ldots, C_n \in \mathcal{C}_a\) and \(M, N \in \mathcal{M}\). Then we let
\[
\text{Map}_\mathcal{C}(\{C_1, \ldots, C_n\} \otimes M, N) \subseteq \text{Map}_\mathcal{C}(\{C_1, \ldots, C_n, M, N\})
\]
denote the full subgroupoid over the map \((\langle n + 1 \rangle, \{n + 1\}) \xrightarrow{1 < \cdots < n} (\langle 1 \rangle, \{1\})\) in \(\text{LM}^\circ\).

#### Definition 2.11.
Let \(\mathcal{C} \to \text{LM}^\circ\) be a coCartesian fibration of \(\infty\)-operads which equips \(\mathcal{M} \approx \mathcal{C}_m\) with a left tensored structure over \(\mathcal{C}_a\). Let \(M, N \in \mathcal{M}\). Then a morphism object \(\text{Mor}_\mathcal{C}(M, N) \in \mathcal{C}_a\) is a representing object for the presheaf on \(\mathcal{C}_a\):
\[
C \mapsto \text{Map}_\mathcal{C}(\{C\} \otimes M, N).
\]
We say that \(\mathcal{M}\) is *enriched over \(\mathcal{C}_a\)*, if it has a morphism object for all \(M, N \in \mathcal{M}\).

Let’s show how to get an enriched composition map in a quasi-category \(\mathcal{M}\) enriched over \(\mathcal{C}_a\). Let \(L, M, N \in \mathcal{M}\). Then we have universal maps
\[
\alpha_{LM} \in \text{Map}_\mathcal{C}(\text{Mor}_\mathcal{C}(L, M) \otimes L, M),
\alpha_{MN} \in \text{Map}_\mathcal{C}(\text{Mor}_\mathcal{C}(M, N) \otimes M, N),
\alpha_{LN} \in \text{Map}_\mathcal{C}(\text{Mor}_\mathcal{C}(L, N) \otimes L, N).
\]
By the universal property of \(\alpha_{LN}\), there exists a map \(\text{Mor}_\mathcal{C}(M, N) \otimes \text{Mor}_\mathcal{C}(L, M) \xrightarrow{\zeta} \text{Mor}_\mathcal{C}(L, N)\) making the diagram
\[ \begin{align*}
\text{Mor}_\mathcal{C}(M, N) \otimes \text{Mor}_\mathcal{C}(L, M) \otimes L & \quad \xrightarrow{\text{id} \otimes \alpha_{LM}} \quad \text{Mor}_\mathcal{C}(M, N) \otimes M \\
\end{align*} \]

commutative.

**Definition 2.12.** Let \( \mathcal{M}^\otimes \xrightarrow{p} \mathcal{LM}^\otimes \) and \( \mathcal{N}^\otimes \xrightarrow{q} \mathcal{LM}^\otimes \) be coCartesian fibrations of \( \infty \)-operads, and

\[ \mathcal{M}^\otimes \times_{\mathcal{LM}^\otimes} \text{Assoc}^\otimes \xrightarrow{\alpha} \mathcal{C}^\otimes \xrightarrow{\beta} \mathcal{N}^\otimes \times_{\mathcal{LM}^\otimes} \text{Assoc}^\otimes \]

equivalences of \( \infty \)-operads. That is, \( \mathcal{M} \) and \( \mathcal{N} \) are left-tensored over the monoidal quasi-category \( \mathcal{C} \). Then an \( \mathcal{LM}^\otimes \)-functor \( \mathcal{M}^\otimes \xrightarrow{F} \mathcal{N}^\otimes \) is \((\mathcal{C},\mathcal{C})\)-linear, if

1. it takes \( p \)-coCartesian edges to \( q \)-coCartesian edges, and
2. we have \( F|_\mathcal{C} \mathcal{M}^\otimes \times_{\mathcal{LM}^\otimes} \text{Assoc}^\otimes = \beta \circ \alpha \).

We let \( \text{LinFun}(\mathcal{M}, \mathcal{N}) = \text{LinFun}_\mathcal{C}(\mathcal{M}, \mathcal{N}) \subseteq \text{Fun}_{\mathcal{LM}^\otimes}(\mathcal{M}^\otimes, \mathcal{N}^\otimes) \) denote the full subcategory on linear functors.

**Definition 2.13.** Let \( \mathcal{C}^\otimes \xrightarrow{p} \mathcal{LM}^\otimes \) exhibit \( \mathcal{C}_m = \mathcal{M} \) as a quasi-category left-tensored over the monoidal quasi-category \( \mathcal{C}_a^\otimes \). Then a left module in \( \mathcal{C} \) is a morphism of \( \infty \)-operads \( \mathcal{LM}^\otimes \xrightarrow{M} \mathcal{C}^\otimes \).

Let \( \text{Assoc}^\otimes \xrightarrow{\Lambda} \mathcal{C}^\otimes \) be an algebra in \( \mathcal{C}_a \). Then a left \( A \)-module in \( \mathcal{M} \) is a left module \( \mathcal{LM}^\otimes \xrightarrow{M} \mathcal{C}^\otimes \)

such that \( M|_{\text{Assoc}^\otimes} = A \). The quasi-category of left modules in \( \mathcal{C} \) is the full subcategory \( \text{LMod}(\mathcal{C}) \subseteq \text{Fun}_{\mathcal{LM}^\otimes}(\mathcal{M}^\otimes, \mathcal{C}^\otimes) \) on left modules. The quasi-category of left \( A \)-modules in \( \mathcal{M} \) is the full subcategory \( \text{LMod}_A(\mathcal{M}) \subseteq \text{LMod}(\mathcal{C}) \) on left \( A \)-modules. By abuse of notation, when \( \mathcal{C}^\otimes \) is clear from context, we will also denote this by \( A \text{-Mod} \) or \( \text{LMod}_A \).

Similarly, via the \( \infty \)-operad \( \text{RM}^\otimes \) we can define right modules \([\text{Lur16} \ \text{Variant 4.2.1.36}]\).

### 2.3. Bi-tensored quasi-categories and bimodules.

**Definition 2.14.** Let \( \mathcal{M} \) be a quasi-category, and \( \mathcal{C}_-, \mathcal{C}_+ \) monoidal quasi-categories. Then a bitensored structure of \( \mathcal{M} \) over \( \mathcal{C}_- \) on the left and \( \mathcal{C}_+ \) on the right is a coCartesian fibration of \( \infty \)-operads \( \mathcal{C}^\otimes \xrightarrow{\alpha} \text{BM}^\otimes \) [\text{Lur16} \ \text{Definition 4.3.1.17}] such that we have an equivalence of quasi-categories \( \mathcal{C}_m \simeq \mathcal{M} \), and equivalences of monoidal quasi-categories \( \mathcal{C}_{a_-} \simeq \mathcal{C}_- \), \( \mathcal{C}_{a_+} \simeq \mathcal{C}_+ \). Here, the \( \infty \)-operad \( \text{BM}^\otimes \) has

1. objects \( \langle (n), c_-, c_+ \rangle \) where \( c_-, c_+ \) are maps \( \langle (n) \rangle \xrightarrow{\alpha} [1] \), and
2. a morphism \( \langle (n'), c'_-, c'_+ \rangle \xrightarrow{\alpha} \langle (n), c_-, c_+ \rangle \) is a morphism \( \langle (n') \rangle \xrightarrow{\alpha} \langle (n) \rangle \) in \( \text{Assoc}^\otimes \) such that for \( j \in \langle (n) \rangle \) and \( \alpha^{-1}[j] = [i_1 > \cdots > i_k] \) we have
   (a) \( c'_-(i_1) = c_-(j) \),
   (b) \( c'_+(i_\ell) = c'_-(i_{\ell+1}) \) for \( \ell = 1, \ldots, k - 1 \), and
   (c) \( c'_+(i_k) = c_+(j) \).

We let \( a_- = (1, 0, 0), m = (1, 0, 1), \) and \( a_+ = (1, 1, 1) \). Unless specified otherwise, we let \( \mathcal{C}_- = \mathcal{C}_{a_-}, \) and \( \mathcal{C}_+ = \mathcal{C}_{a_+} \).
Remark 2.15. Let $((n), c_-, c_+)$ and $i \in [1, n]$. Then one can say that $c_-(i)$ says which algebra we’re acting with on the left, and $c_+(i)$ says which algebra we’re acting with on the right.

Definition 2.16. We let $\mathrm{BMod}(\mathcal{M}) = \mathrm{Alg}_B(\mathcal{M})$. Let $A \in \mathrm{Alg}(\mathcal{C}_-)$ and $B \in \mathrm{Alg}(\mathcal{C}_+)$. Then an $(A, B)$-bimodule object (in $\mathcal{M}$) is $M \in \mathrm{BMod}(\mathcal{M})$ such that $M|\mathrm{Alg}(\mathcal{C}_-) = A$ and $M|\mathrm{Alg}(\mathcal{C}_+) = B$. We let $A\mathrm{Mod}_{\mathcal{B}} \subseteq \mathrm{BMod}(\mathcal{M})$ denote the full subcategory of $(A, B)$-bimodule objects.

Construction 2.17. Let $\mathcal{C}_\otimes \to \mathcal{B}\mathcal{M}$ be a coCartesian fibration of $\infty$-operads. Then the quasi-category $\mathrm{LMod}(\mathcal{C}_m)$ of left module objects can be equipped with the structure of a quasi-category right-tensored over $\mathcal{C}_\cdot$ [Lur16, §4.3.2]. Heuristically, for $M \in \mathrm{LMod}_A$ and $C \in \mathcal{C}_\cdot$, the left $A$-module structure on $M \otimes C$ is given by

$$A \otimes (M \otimes C) \simeq (A \otimes M) \otimes C \xrightarrow{\alpha \otimes C} M \otimes C.$$

More precisely, we have a map $\mathrm{LM}^\otimes \times \mathrm{RM}^\otimes \xrightarrow{\Pr} \mathcal{B}\mathcal{M}^\otimes$ defined as follows.

- For objects $((m), S) \in \mathrm{LM}^\otimes$ and $((n), T) \in \mathrm{RM}^\otimes$, we have $$\Pr(((m), S), ((n), T)) = (X, \alpha, \beta),$$

where $$X = (\langle m \rangle \times T) \cup (S \times \langle n \rangle) \subseteq \langle m \rangle^\otimes \times \langle n \rangle^\otimes \equiv \langle mn \rangle^\otimes,$$

and for $(i, j) \in X$, we have $$c_-(i, j) = \begin{cases} 0 & j \in T, \\ 1 & j \notin T, \end{cases} \text{ and } c_+(i, j) = \begin{cases} 0 & i \notin S, \\ 1 & i \in S. \end{cases}$$

- Consider morphisms $((m), S) \xrightarrow{\alpha} ((m'), S')$ in $\mathrm{LM}^\otimes$ and $((n), T) \xrightarrow{\beta} ((n'), T')$ in $\mathrm{RM}^\otimes$. Let $(X, \alpha, \beta, c_\cdot) = \Pr(((m), S), ((n), T))$ and $(X', \alpha', \beta', c_\cdot) = \Pr(((m'), S'), ((n'), T')).$

1. The image of $\Pr(\alpha, \beta)$ in $\mathrm{Fin}$, is the map $X \xrightarrow{\gamma} X'$, such that for $(i, j) \in X$, we have

$$\gamma(i, j) = \begin{cases} (\alpha(i), \beta(j)) & \alpha(i) \in \langle m' \rangle^\otimes \text{ and } \beta(j) \in \langle n' \rangle^\otimes, \\ * & \text{else}. \end{cases}$$

2. Let $(i', j') \in X'$. We need to give $$\gamma^{-1}(i', j') = (\alpha^{-1}(i') \times \beta^{-1}(j')) \cap X$$

a linear ordering satisfying the conditions in Definition 2.14.

- Suppose that $i' \notin S'$. Then we have $\alpha^{-1}(i') \cap S = \emptyset$. As we have $X = (\langle m \rangle \times T) \cup (S \times \langle n \rangle)$, we get $j' \in T'$. Therefore, there exists a unique $j \in T$ such that $\beta(j) = j'$, and we get $\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\}$. We can give this the linear ordering induced by that on $\alpha^{-1}(i').$

- Similarly, if $j' \notin T'$, then we have $\gamma^{-1}(i', j') = \{i\} \times \beta^{-1}(j')$, which we can give the linear ordering induced by that on $\beta^{-1}(j').$
Suppose that $i' \in S'$ and $j' \in T'$. Then there exists a unique $i \in S$ resp. $j \in T$ such that $\alpha(i) = i'$ resp. $\beta(j) = j'$. Thus, by definition of $X$ we get

$$\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\} \sqcup \{i\} \times \beta^{-1}(j').$$

We can give this the unique linear ordering such that

- on $\alpha^{-1}(i') \times \{j\}$ it is induced by that on $\alpha^{-1}(i')$,
- on $\{i\} \times \beta^{-1}(j')$ it is induced by that on $\beta^{-1}(j')$, and
- for $i'' \in \langle m \rangle^\circ$ and $j'' \in \langle n \rangle^\circ$, we have $(i'', j) \preceq (i, j')$.

With this, we can define a quasi-category $\text{LMod}(\mathcal{C})$ and map $\text{LMod}(\mathcal{C}) \to \text{RM}^\circ$ as follows. For a map of simplicial sets $K \to \text{RM}^\circ$, we can regard $\text{LM}^\circ \times K$ as a simplicial set over $\text{BM}^\circ$ as the composite $\text{LM}^\circ \times K \xrightarrow{id \times f} \text{LM}^\circ \times \text{RM}^\circ \xrightarrow{\text{Pr}} \text{BM}^\circ$. Therefore, we can let

$$\text{Hom}_{\text{BM}^\circ}(K, \text{LMod}(\mathcal{C})) = \text{Hom}_{\text{BM}^\circ}(\text{LM}^\circ \times K, \mathcal{C}).$$

Let $\text{LMod}(\mathcal{C}) \subseteq \text{LMod}(\mathcal{C})$ denote the full subcategory on maps $\text{LM}^\circ \times \{X\} \to \mathcal{C}$ taking inert edges of $\text{LM}^\circ$ to inert edges of $\mathcal{C}$.

Note that the postcomposite of the canonical inclusion $\text{LM}^\circ \times \{m\} \to \text{LM}^\circ \times \text{RM}^\circ$ by $\text{Pr}$ is the canonical inclusion $\text{LM}^\circ \times \{m\} \cong \text{LM}^\circ \to \text{BM}^\circ$. This gives an isomorphism $\text{LMod}(\mathcal{C}) \times \text{RM}^\circ \{m\} \cong \text{LMod}(\mathcal{C})$.

Let $K \to \text{RM}^\circ$ be a map of simplicial sets. Then a map of simplicial sets $K \to \text{LMod}(\mathcal{C})$ over $\text{RM}^\circ$ is a map of simplicial sets $\text{LM}^\circ \times K \to \mathcal{C}$ over $\text{BM}^\circ$. Precomposing this by the canonical inclusion $\{m\} \times \text{RM}^\circ \to \text{LM}^\circ \times \text{RM}^\circ$, we get a map of simplicial sets $K \to \mathcal{C} \times_{\text{BM}^\circ} \text{RM}^\circ$ over $\text{RM}^\circ$. In sum, this gives a map of simplicial sets $\text{LMod}(\mathcal{C}) \to \mathcal{C} \times_{\text{BM}^\circ} \text{RM}^\circ$ over $\text{RM}^\circ$. The preimage

$$\text{LMod}(\mathcal{C}) \to \mathcal{C}^\circ$$

of this map along the canonical inclusion $\mathcal{C}^\circ = \mathcal{C} \times_{\text{BM}^\circ} \text{Assoc}^\circ$ gives an equivalence of categories $\text{LMod}(\mathcal{C}) \to \mathcal{C} \times_{\text{BM}^\circ} \text{RM}^\circ$ is a trivial Kan fibration [Lur16] Proposition 4.3.2.6).

The map $\text{LMod}(\mathcal{C}) \to \text{RM}^\circ$ is a coCartesian fibration of $\infty$-operads [Lur16, Proposition 4.3.2.5. 1)]. Therefore, it gives the quasi-category $\text{LMod}(\mathcal{C})$ a left-tensored structure over the monoidal quasi-category $\mathcal{C}^\circ$.

Moreover, precomposition by $\text{Pr}$ gives an equivalence of categories [Lur16, Theorem 4.3.2.7]

$$\text{BMod}(\mathcal{C}) \to \text{RMod}(\text{LMod}(\mathcal{C})).$$

**Theorem 2.18.** [Lur16, Theorem 4.8.4.1] Let $\mathcal{C}$ be a collection of simplicial sets containing $\text{N}(\Delta)^{op}$. Let $\mathcal{C}$ a monoidal quasi-category compatible with $\mathcal{L}$-indexed colimits, $\mathcal{M}$ a quasi-category left-tensored over $\mathcal{C}$ compatible with $\mathcal{L}$-indexed colimits, and $\text{Assoc} \xrightarrow{\Delta} \mathcal{C}$ an associative algebra object. Then the composite

$$\xymatrix{ \text{LinFun}_{\mathcal{C}}(\text{RMMod}_{\mathcal{C}} \mathcal{C}, \mathcal{M}) \ar[r]^{\text{Fun}(\text{LMod}_{\mathcal{C}} \text{RMMod}_{\mathcal{C}} \mathcal{C}, \text{LMod}_{\mathcal{C}} \mathcal{M}) \ar[l]_{\text{Fun}(\text{LMod}_{\mathcal{C}} \text{RMMod}_{\mathcal{C}} \mathcal{C}, \mathcal{M})}} & \text{LMod}_{\mathcal{C}} \mathcal{M} \ar[r]_{\text{Fun}(\text{LMod}_{\mathcal{C}} \text{RMMod}_{\mathcal{C}} \mathcal{C}, \mathcal{M})} & \mathcal{M}. \ar[l]_{\text{Fun}(\text{LMod}_{\mathcal{C}} \mathcal{M})} \ar[l]_{\text{Fun}(\text{LMod}_{\mathcal{C}} \mathcal{M})} \ar[l]_{\text{Fun}(\text{LMod}_{\mathcal{C}} \mathcal{M})} \ar[l]_{\text{Fun}(\text{LMod}_{\mathcal{C}} \mathcal{M})} \ar[l]_{\text{Fun}(\text{LMod}_{\mathcal{C}} \mathcal{M})} \ar[l]_{\text{Fun}(\text{LMod}_{\mathcal{C}} \mathcal{M})} \ar[l]_{\text{Fun}(\text{LMod}_{\mathcal{C}} \mathcal{M})} }$$

is an equivalence, with quasi-inverse mapping $M \in \text{LMod}_{\mathcal{C}} \mathcal{M}$ to $\text{RMMod}_{\mathcal{C}} \mathcal{C}$.

Observe that the right hand of the equivalence does not depend on $\mathcal{L}$, hence neither does the left.
Corollary 2.19 (Homotopical Eilenberg–Watts theorem). Let \( \text{Assoc}^\otimes \xrightarrow{B} \mathscr{C}^\otimes \) be another associative algebra object. Then the map
\[
\text{LinFun}_{\mathscr{C}}(\text{RMod}_A \mathscr{C}, \text{RMod}_B \mathscr{C}) \xrightarrow{F \mapsto F(A)} A \text{BMod}_B
\]
is an equivalence, with quasi-inverse mapping \( M \in A \text{BMod}_B \rightarrow \text{RMod}_A \mathscr{C} \xrightarrow{E \mapsto E \otimes_A M} \text{RMod}_B \mathscr{C} \).

2.4. Relative tensor product.

Definition 2.20. We let \( \text{Tens}^\otimes \) denote the generalized \( \infty \)-operad with

1. objects tuples \( (\langle n \rangle, [k], c_-, c_+) \) where \( \langle n \rangle \in \text{Assoc}^\otimes, [k] \in \Delta^\text{op}, \) and \( c_-, c_+ \) are maps of sets \( [1, n] \rightarrow [k] \) such that
   \[
c_-(i) \leq c_+(i) \leq c_-(i) + 1 \text{ for all } i \in [1, n],
   \]
and
2. a morphism \( (\langle n \rangle, [k], c_-, c+) \rightarrow (\langle n' \rangle, [k'], c'_-, c'_+) \) is a pair of a morphism \( \alpha : \langle n \rangle \rightarrow \langle n' \rangle \) in \( \text{Assoc}^\otimes \) and a morphism \( [k'] \xrightarrow{\lambda} [k] \) in \( \Delta^\text{op} \) such that for \( j \in [1, n'] \) with \( \alpha^{-1}[j] = \{i_1, \ldots, i_m\} \), we have
   (a) \( c_-(i_1) = \lambda(c'_-(j)) \),
   (b) \( c_+(i_\ell) = c_-(i_{\ell+1}) \) for \( \ell \in [1, m-1] \), and
   (c) \( c_+(i_m) = \lambda(c'_+(j)) \).

The forgetful functor \( \text{Tens}^\otimes \rightarrow \text{Fin}^\otimes \times \Delta^\text{op} \) is a family of \( \infty \)-operads \[\text{Lur}16\] Definition 2.3.2.10]. For \( k \geq 0 \), the fibre \( \text{Tens}^\otimes_{[k]} \) is the \( \infty \)-operadic colimit of the diagram

\[
\begin{array}{c c c c}
\text{Tens}^\otimes_{[0]} & \xrightarrow{\text{}} & \text{Tens}^\otimes_{[1]} & \xrightarrow{\text{}} & \text{Tens}^\otimes_{[k-1]} & \xrightarrow{\text{}} & \text{Tens}^\otimes_{[k]} \\
\text{Tens}^\otimes_{[0,1]} & \xleftarrow{\text{}} & \text{Tens}^\otimes_{[1,2]} & \xleftarrow{\text{}} & \text{Tens}^\otimes_{[k-1,k]} & \xleftarrow{\text{}} & \text{Tens}^\otimes_{[k,k+1]}
\end{array}
\]

\[\text{Lur}16\] Proposition 4.4.1.11]. In particular, for a monoidal quasi-category \( \mathscr{C}^\otimes \xrightarrow{q} \text{Assoc}^\otimes \), we have canonical equivalences

\[
\text{Alg}(\mathscr{C}) \rightarrow \text{Alg}_{\text{Tens}^\otimes_{[0]}(\mathscr{C})}, \quad \text{BMod}(\mathscr{C}) \rightarrow \text{Alg}_{\text{Tens}^\otimes_{[1]}(\mathscr{C})}, \quad \text{BMod}(\mathscr{C}) \times_{\text{Alg}(\mathscr{C})} \text{BMod}(\mathscr{C}) \rightarrow \text{Alg}_{\text{Tens}^\otimes_{[2]}(\mathscr{C})}
\]
where in the fibre product on the right, the left projection map is the algebra on the right and the right projection map is the algebra on the left.

We let \( \text{Tens}^\otimes \) be the strict pull-back of \( \text{Tens}^\otimes \rightarrow \Delta^\text{op} \) along the map \( \Delta^1 \xrightarrow{[0,2][1,2]} \Delta^\text{op} \). Let \( A, B, C \in \text{Alg}(\mathscr{C}) \) and \( M \in A \text{Mod}_B, N \in B \text{Mod}_C \). This data determines \( F_0 \in \text{Alg}_{\text{Tens}^\otimes_{[0]}(\mathscr{C})} \). Let \( K \in A \text{Mod}_C \). We say that \( F \in \text{Alg}_{\text{Tens}^\otimes_{[2]}(\mathscr{C})} \) exhibits \( K \) as the relative tensor product \( M \otimes_B N \), if

1. we have \( F|_{\text{Tens}^\otimes_{[2]}} = F_0 \),
2. we have \( F|_{\text{Tens}^\otimes_{[1]}} = K \), and
3. the diagram \( \text{Tens}^\otimes_{[2]} \xrightarrow{F} \mathscr{C}^\otimes \) is a \( q \)-operadic colimit \[\text{Lur}16\] Definition 3.1.1.2].

Let \( A \) be an algebra in \( \mathscr{C} \). Then the quasi-category \( A \text{BMod}_A \) can be equipped with a monoidal structure given by relative tensor product \[\text{Lur}16\] Proposition 4.4.3.12].
2.5. Endomorphism algebras and dualizable generators.

**Remark 2.21.** An element \((n), [k], c_-, c_+ \in \text{Tens}^\circ\) indexes an \(n\)-term expression of action of algebras \(A_0, \ldots, A_k\). For \(i \in \langle n \rangle^\circ\), we can act on the \(i\)-th element by \(A_{c_-(i)}\) on the left, and \(A_{c_+(i)}\) on the right.

**Definition 2.22.** Let \(\mathcal{C}\) be a monoidal quasi-category, \(\mathcal{M}\) a quasi-category left-tensored over \(\mathcal{C}\), and \(M \in \mathcal{M}\) an object. Then an object \(C \in \mathcal{C}\) equipped with a map \(C \otimes M \xrightarrow{\alpha} M\) is an endomorphism object of \(M\), denoted by \(\text{End}_{\mathcal{M}}(M)\) or \(\text{End}(M)\), if it represents the presheaf on \(\mathcal{C}\):

\[
C' \mapsto \text{Map}_{\mathcal{M}}(C' \otimes M, M).
\]

Let \(A\) be an algebra object in \(\mathcal{C}\). Then we say that a left \(A\)-module structure on \(M\) exhibits \(A\) as an endomorphism algebra of \(M\), if \(M \in \text{LMod}_{\mathcal{M}}\) with this left module structure represents the right fibration [Lur16, Corollary 4.7.1.42]

\[
\text{LMod}_{\mathcal{M}} \times_{\mathcal{M}} [M] \to \text{Alg}\mathcal{C}
\]

given by restriction.

It can be shown that if \(C \otimes M \xrightarrow{\alpha} M\) exhibits \(C \in \mathcal{C}\) as an endomorphism object of \(M\), then \(\alpha\) lifts to a module structure \(M \in \text{LMod}_{A\mathcal{M}}\), which exhibits \(A \in \text{Alg}\mathcal{C}\) as an endomorphism algebra of \(M\) [Lur16, §4.7.1], in a way that is unique up to homotopy.

**Definition 2.23.** Let \(\mathcal{C}\) be a monoidal quasi-category with neutral object \(O\), and \(C \in \mathcal{C}\) an object. Then an object \(C^\vee\) equipped with a map \(C^\vee \otimes C \xrightarrow{\text{ev}} O\) is a right dual of \(C\), if this data induce an adjunction

\[
\mathcal{C} \xlongleftarrow{\otimes} C^\vee \xrightarrow{\otimes} \mathcal{C}^\vee.
\]

In this case, the map \(C \otimes C^\vee \otimes C \xrightarrow{C \otimes \text{ev}} C\) shows that \(A := C \otimes C^\vee\) is an endomorphism object of \(C\). This equips \(A\) with an algebra structure, and \(C\) with a left \(A\)-module structure. We say that \(C\) is a dualizable generator, if the functor

\[
\text{A Mod} \xrightarrow{\otimes_{\mathcal{C}} A} \mathcal{C}
\]

is essentially surjective. We denote by \(\mathcal{C}_{\text{gen}} \subseteq \mathcal{C}\) the full subcategory of dualizable generators.

**Proposition 2.24.** Let \(\mathcal{C}\) be a monoidal quasi-category. Suppose that \(\mathcal{C}\) admits geometrical realizations, and the tensor product \(\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}\) commutes with geometrical realizations. Let \(C \in \mathcal{C}\) be a right dualizable object, and \(A = \text{End} C\). Then the following assertions hold.

1. The right \(A\)-module \(C^\vee\) is a right dual to the left \(A\)-module \(C\).
2. The functor \(\text{A Mod} \xrightarrow{\otimes_{\mathcal{C}} A} \mathcal{C}\) is fully faithful.

**Proof.** We can assume \(A = C \otimes C^\vee\). Then the map \(C^\vee \otimes C \xrightarrow{\text{ev}} O\) factors through the canonical map \(C^\vee \otimes C \xrightarrow{1} C^\vee \otimes_A C\) as \(C^\vee \otimes_A C \xrightarrow{\text{ev_A}} O\). We claim that the morphisms

\[
C^\vee \otimes_A C \xrightarrow{\text{ev_A}} O\text{ in }\mathcal{C},\text{ and}\nA \xrightarrow{\text{id}} C \otimes C^\vee\text{ in }\text{A Mod}_A
\]

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exhibit \( C^\vee \in \text{Mod}_A \) as a right dual to \( C \in A \text{Mod} \). That is, we need to show that the composites

\[
\begin{align*}
C & \xrightarrow{\sim} A \otimes_A C \xrightarrow{\text{id} \otimes_A C} C \otimes C^\vee \otimes_A C \xrightarrow{C \otimes A} C \otimes A \xrightarrow{\sim} C, \quad \text{and} \\
C^\vee & \xrightarrow{\sim} C^\vee \otimes_A A \xrightarrow{C^\vee \otimes A C} C \otimes C^\vee \xrightarrow{\text{ev} \otimes C^\vee} C \otimes C^\vee \xrightarrow{\sim} C^\vee
\end{align*}
\]

are homotopic to \( \text{id}_C \) and \( \text{id}_{C^\vee} \), respectively \cite[Proposition 4.6.2.1]{Lur16}. Since the forgetful functors on module categories are conservative \cite[Corollary 4.3.3.3]{Lur16}, it is enough to prove these assertions in \( \mathcal{C} \). We’ll show the first; the second is similar.

Let \( \Theta \xrightarrow{\sim} \mathcal{O} \otimes C \) denote the coevaluation map of the duality \((C, C^\vee)\) in \( \mathcal{C} \). We have the homotopy commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\Theta} & \mathcal{O} \otimes C \\
\downarrow & & \downarrow \\
A \otimes_A C & \xrightarrow{\text{id} \otimes_A C} & C \otimes C^\vee \otimes_A C \xrightarrow{C \otimes A} C \otimes A \\
\downarrow & & \downarrow \\
& & C \otimes \mathcal{T} \\
\downarrow & & \downarrow \\
& & C \\
& & \xrightarrow{\sim}
\end{array}
\]

The top composite is homotopic to \( \text{id}_C \) by assumption. Therefore so is the bottom composite, as required.

Therefore, we have an adjunction \( \text{Mod}_A \xrightarrow{\otimes_A C} \mathcal{C} \) with unit map \( \text{id} \otimes_A C \otimes C^\vee \) and counit map \( \otimes C^\vee \otimes_A C \xrightarrow{\text{ev}_A} \text{id} \). Since the unit map is an equivalence, the left adjoint \( \otimes_A C \) is fully faithful, as claimed.

\[\square\]

**Corollary 2.25.** Let \( \mathcal{C} \) be a monoidal quasi-category. Suppose that \( \mathcal{C} \) has geometric realizations, and the tensor product map \( \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) preserves geometric realizations. Let \( C \in \mathcal{C}_{\text{dgen}} \) be a dualizable generator. Then the functors \( \text{Mod}_A \xrightarrow{\otimes_A C} \mathcal{C} \) and \( \mathcal{C} \xrightarrow{\otimes C^\vee} \text{Mod}_A \) are mutually inverse equivalences.

### 3. The Morita Functor in Higher Algebra

To set up the Morita functor \( A \mapsto \text{Mod}_A \) sending an algebra \( A \) in a monoidal quasi-category \( \mathcal{C} \) to the quasi-category \( \text{Mod}_A \) of right \( A \)-modules that is left-tensored over \( \mathcal{C} \), we use classifying objects for monoidal quasi-categories and quasi-categories left-tensored by them. The endomorphism algebra functor will be the right adjoint of the pointed version \( A \mapsto (\text{Mod}_A, A) \) of the above functor. To make sure that this right adjoint exists, we will need to restrict to presentable underlying quasi-categories and make sure that the algebra and module structure maps respect colimits. In our main application, \( \mathcal{C} \) will be the monoidal quasi-category of cochain complexes over an \( S \)-scheme \( X \), and we will be interested in the endomorphism algebras \( A = \text{REnd}(E) \) for perfect complexes \( E \in \mathcal{C} \), and the dg-categories \( \text{Mod}_A \) of right \( A \)-modules. In this section, we shall follow \cite[§4.8]{Lur16}.
3.1. **Cartesian monoidal structures and monoid objects.** To get started, we will equip the quasi-category of quasi-categories $\text{Cat}_{\infty}$ with the symmetric monoidal structure given by taking products. This is a Cartesian monoidal structure, and thus unique up to equivalence.

Let $\mathcal{C}$ be an $\infty$-operad. A **lax Cartesian structure on** $\mathcal{C}$ **is a functor** $\mathcal{C} \to \mathcal{D}$ **into a quasi-category such that for all** $n \geq 0$ **and** $X \in \mathcal{C}_n$, **the collection of canonical maps** $\{\pi(C) \to \pi(C_i)\}_{i=1}^n$ **is a product diagram in** $\mathcal{D}$.

The lax Cartesian structure $\pi$ is a **weak Cartesian structure**, if

1. **the** $\infty$-operad $p$ **is a symmetric monoidal quasi-category, and**
2. **for any** $p$-coCartesian edge $f$ **over an active map of the form** $\langle n \rangle \to \langle 1 \rangle$ in $\text{Fin}_{\ast}$, **its image** $\pi(f)$ **is an equivalence in** $\mathcal{D}$.

The weak Cartesian structure $\pi$ is a **Cartesian structure**, if its restriction $\mathcal{C} \to \mathcal{D}$ **is an equivalence of quasi-categories**.

Let $\mathcal{C}$ **be a quasi-category with finite products. Then there exists a Cartesian structure** $\mathcal{C} \times \to \mathcal{C}$ **[Lur16, Proposition 2.4.1.5]**. Moreover, **this is the unique Cartesian symmetric monoidal structure on** $\mathcal{C}$ **up to equivalence** [Lur16, Corollary 2.4.1.8]. **A symmetric monoidal structure** $\mathcal{C} \otimes$ **on** $\mathcal{C}$ **is Cartesian**, if

1. **the unit object** $1 \in \mathcal{C}$ **is final,** and
2. **for any objects** $C, D \in \mathcal{C}$, **the diagram of canonical maps**
   $$ C \leftarrow C \otimes 1 \leftarrow C \otimes D \to 1 \otimes D \to D $$
   **is a product diagram in** $\mathcal{C}$.

Let $\mathcal{D}$ **be a quasi-category, and** $\mathcal{O}$ **an** $\infty$-operad. Then an **$\mathcal{O}$-monoid in** $\mathcal{D}$ **is a lax Cartesian structure of the form** $\mathcal{O} \to \mathcal{D}$. **We let** $\text{Mon}_{\mathcal{O}} \mathcal{D} \subseteq \text{Fun}(\mathcal{O}, \mathcal{D})$ **denote the full subcategory of monoid objects**.

Monoid objects give an alternative description of algebra objects in Cartesian symmetric monoidal quasi-categories: let $\mathcal{C} \to \mathcal{D}$ **be a Cartesian structure. Then the postcomposition map** $\text{Alg}_{\mathcal{O}} \mathcal{C} \to \text{Mon}_{\mathcal{O}} \mathcal{D}$ **is an equivalence** [Lur16, Proposition 2.4.2.5].

3.2. **Families of monoidal quasi-categories.** In this subsection we will construct classifying objects for families of algebras. Let $\mathcal{D} = \text{Cat}_{\infty}$ **and** $\pi$ **be the canonical Cartesian structure** $\text{Cat}_{\infty} \times \to \text{Cat}_{\infty}$. Let $\mathcal{C} \otimes \to \mathcal{O}$ **be a coCartesian fibration. Then it is classified by a functor** $\mathcal{O} \to \text{Cat}_{\infty}$. **The coCartesian fibration** $p$ **is an** $\mathcal{O}$-monoidal quasi-category if and only if the classifying map** $c_p$ **is a monoid object** [Lur16, Example 2.4.2.4]. Moreover, **the map** $\text{Alg}_{\mathcal{O}} \text{Cat}_{\infty} \to \text{Mon}_{\mathcal{O}} \text{Cat}_{\infty}$ **is an equivalence**. Thus we see that $\mathcal{O}$-monoidal quasi-categories are also classified by $\mathcal{O}$-algebra objects in $\text{Cat}_{\infty}$. 

Let $K$ **be a quasi-category, $\mathcal{O}$ an** $\infty$-operad, and $\mathcal{C} \otimes \to \mathcal{O} \times K$ **a coCartesian fibration. Then it is classified by a map** $K \to \text{Fun}(\mathcal{O}, \text{Cat}_{\infty})$. **By construction, the map** $c_p$ **maps into** $\text{Mon}_{\mathcal{O}} \text{Cat}_{\infty} \subseteq \text{Fun}(\mathcal{O}, \text{Cat}_{\infty})$ **if and only if** $p$ **is a coCartesian** $K$-**family of** $\mathcal{O}$-**monoidal categories, that is**

1. **in addition to** $p$ **being a coCartesian fibration,**
2. **for all** $k \in K$, **the fibre** $\mathcal{C}_k \otimes \to \mathcal{O}$ **is an** $\mathcal{O}$-monoidal quasi-category.

We let $\text{Cat}_{\infty}^{\text{Mon}} = \text{Mon}_{\text{Assoc}} \text{Cat}_{\infty}$. It classifies coCartesian families of monoidal quasi-categories.
3.3. **Families of associative algebra objects.** Let $K$ be a quasi-category, and $\mathcal{C}^\otimes \xrightarrow{p} \text{Assoc}^\otimes \times K$ a coCartesian family of monoidal quasi-categories classified. Then a section $A$ of $p$ is a family of associative algebra objects of $p$, if for every $k \in K$, the fibre $A_k$ is an associative algebra object of the monoidal quasi-category $p_k$, i.e. a morphism of $\infty$-operads.

**Notation 3.1.** Let $X \xrightarrow{q} B \times C$ be a map of simplicial sets. Then the simplicial set of partial sections of $q$ over $C$ is the simplicial sets $\Gamma_C(q)$ over $C$ defined by letting for a map of simplicial sets $L \to C$:

$$\text{Hom}_C(L, \Gamma_C(q)) = \text{Hom}_{B \times C}(B \times L, X).$$

Sometimes we will let $\Gamma_C(X) = \Gamma_C(q)$.

**Remark 3.2.** 1) Note that we have an adjunction

$$\text{Set}^{\Delta}_/C \xleftarrow{\perp} \text{Set}^{\Delta}_/(B \times C) \xrightarrow{\Gamma_C}.$$ 

2) In case $C = \ast$, we get the absolute section object $\Gamma^\ast X = \Gamma X$.

Let $\text{Alg} \subseteq \Gamma^\ast \text{Assoc}^\otimes$ be the full subcategory on associative algebra objects. Then by construction $\text{Alg}$ classifies associative algebra objects in $p$. The map $\text{Alg} \to K$ is a coCartesian fibration \cite[Lemma 4.8.3.13. 1)]{Lurie_Higher_Categories}. Now we want to classify pairs $(\mathcal{C}^\otimes, A)$ where $\mathcal{C}^\otimes$ is a monoidal quasi-category, and $A$ is an associative algebra object in $\mathcal{C}^\otimes$. Note that the identity map of $\text{Mon}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes}$ classifies the universal coCartesian family of monoidal quasi-categories $\tilde{\text{Mon}}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes} \xrightarrow{p_0} \text{Assoc}^\otimes \times \text{Cat}^\otimes$. Therefore, we have a strict fibre product diagram of simplicial sets

\begin{equation*}
\begin{tikzcd}
\mathcal{C}^\otimes \ar[r, hook] & \tilde{\text{Mon}}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes} \\
\text{Assoc}^\otimes \times K \ar[r, leftarrow, rightarrow, hook] & \text{Assoc}^\otimes \times \tilde{\text{Mon}}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes} \\
& \text{Assoc}^\otimes \times \text{Mon}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes} \\
\end{tikzcd}
\end{equation*}

It follows that, sections $A$ of $p$ correspond to maps

$$\text{Assoc}^\otimes \times K \to \tilde{\text{Mon}}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes} \quad \text{over} \quad \text{Assoc}^\otimes \times \text{Mon}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes}.$$ 

A map $\text{Assoc}^\otimes \times K \xrightarrow{A'} \tilde{\text{Mon}}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes}$ corresponds to a map $K \xrightarrow{A''} \text{Fun}(\text{Assoc}^\otimes, \tilde{\text{Mon}}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes})$. The map $A'$ is over $\text{Assoc}^\otimes \times \text{Mon}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes}$ if and only if the postcomposite of $A''$ with

$$\text{Fun}(\text{Assoc}^\otimes, \tilde{\text{Mon}}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes}) \xrightarrow{p_0} \text{Fun}(\text{Assoc}^\otimes, \text{Assoc}^\otimes \times \text{Mon}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes})$$

factors through the product of partially constant maps

$$\text{Mon}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes} \xrightarrow{c^\prime(\mathcal{C}^\otimes) = (\text{id}_{\text{Assoc}^\otimes}, \text{const}_{\mathcal{C}^\otimes})} \text{Fun}(\text{Assoc}^\otimes, \text{Assoc}^\otimes \times \text{Mon}_{\text{Assoc}^\otimes \times \text{Cat}^\otimes}).$$

Therefore, pairs $(\mathcal{C}^\otimes, A)$ of coCartesian families of monoidal quasi-categories and a section are classified by the strict fibre product of simplicial sets.
Most importantly, pairs \((\mathcal{C}^\otimes, A)\) of coCartesian families of monoidal quasi-categories and families of associative algebra objects are classified by the full subcategory \(\overline{\mathcal{Cat}}_\infty^{\text{Alg}} \subseteq \overline{\mathcal{Cat}}_\infty^{\text{Alg}}\) on pairs \((\mathcal{C}^\otimes, A)\) of monoidal quasi-categories and associative algebra objects.

**3.4. Compatibility with colimits.** Let \(K, L\) be simplicial sets, and \(\mathcal{C}^\otimes \xrightarrow{p} \overline{\mathcal{C}}^\otimes \times \mathcal{A}^{\text{assoc}}\) a coCartesian family of monoidal quasi-categories. Then we say that \(p\) is compatible with \(L\)-indexed colimits, if the following conditions are satisfied.

1. Let \(k \in K\) be a vertex. Then the fibre monoidal quasi-category \(\mathcal{C}^\otimes \xrightarrow{p_k} \overline{\mathcal{C}}^\otimes\) commutes with \(L\)-indexed colimits. That is,
   - (a) the underlying quasi-category \(\mathcal{C}_k\) has \(L\)-indexed colimits, and
   - (b) the tensor product functor \(\mathcal{C}_k \times \mathcal{C}_k \to \mathcal{C}_k\) commutes with \(L\)-indexed colimits componentwise.
2. Let \(k \xrightarrow{e} k'\) be an edge in \(K\). Then the induced functor on the underlying quasi-categories \(\mathcal{C}_k \to \mathcal{C}_k'\) commutes with \(L\)-indexed colimits.

Let \(\mathcal{L}\) be a collection of simplicial sets. Then we say that \(p\) commutes with \(\mathcal{L}\)-indexed colimits, if for all \(L \in \mathcal{L}\), \(p\) commutes with \(L\)-indexed colimits.

Let \(K \xrightarrow{e^0} \text{Mon}_{\mathcal{A}^{\text{assoc}}} \text{Cat}_\infty\) classify \(p\). Then we have the following.

1. For a vertex \(k \in K\), the fibre \(\mathcal{C}^\otimes \xrightarrow{p_k} \overline{\mathcal{C}}^\otimes\) is equivalent to the pullback of \(\text{Mon}_{\mathcal{A}^{\text{assoc}}} \text{Cat}_\infty \xrightarrow{p_0} \overline{\mathcal{C}}^\otimes \times \text{Mon}_{\mathcal{A}^{\text{assoc}}} \text{Cat}_\infty\) along the composite
   \[
   \overline{\mathcal{C}}^\otimes \times \{k\} \hookrightarrow \overline{\mathcal{C}}^\otimes \times K \xrightarrow{\text{id} \times e^0} \overline{\mathcal{C}}^\otimes \times \text{Mon}_{\mathcal{A}^{\text{assoc}}} \text{Cat}_\infty.
   \]
2. For an edge \(k \xrightarrow{e} k'\) in \(K\), the induced map \(\mathcal{C}_k \to \mathcal{C}_{k'}\) on the underlying quasi-categories, as an edge \(\Delta^1 \xrightarrow{e^0} \text{Cat}_\infty\), classifies the pullback of \(\mathcal{C}^\otimes \xrightarrow{p} \overline{\mathcal{C}}^\otimes \times K\) along the inclusion \(\Delta^1 \times \Delta^e \to \overline{\mathcal{C}}^\otimes \times K\). Therefore, the functor \(e^0\) is naturally equivalent to the composite
   \[
   \Delta^e \hookrightarrow K \xrightarrow{e^0} \text{Mon}_{\mathcal{A}^{\text{assoc}}} \text{Cat}_\infty \xrightarrow{\alpha(\Delta^0(1)) \mapsto \overline{\mathcal{C}}^\otimes} \text{Cat}_\infty.
   \]

This shows the following for a collection of simplicial sets \(\mathcal{L}\).

1. Let \(\text{Cat}^\text{Mon}_\infty(\mathcal{L}) = \text{Mon}_{\mathcal{A}^{\text{assoc}}} \text{Cat}_\infty \subseteq \text{Mon}_{\mathcal{A}^{\text{assoc}}} \text{Cat}_\infty\) be the largest subcategory with
   - (a) vertices classifying monoidal quasi-categories compatible with \(\mathcal{L}\)-indexed colimits, and
   - (b) edges classifying monoidal functors \(\mathcal{C}^\otimes \to \mathcal{D}^\otimes\) such that the restriction \(\mathcal{C} \to \mathcal{D}\) to underlying quasi-categories commutes with \(\mathcal{L}\)-indexed colimits.

Then \(\text{Mon}_{\mathcal{A}^{\text{assoc}}} \text{Cat}_\infty\) classifies families of monoidal quasi-categories compatible with \(\mathcal{L}\)-indexed colimits.
(2) Let
\[ \tilde{\text{Mon}}_{\mathbf{Assoc}} \text{Cat}_{\infty} = \text{Mon}_{\mathbf{Assoc}} \text{Cat}_{\infty} \times_{\text{Mon}_{\mathbf{Assoc}} \text{Cat}_{\infty}} \tilde{\text{Mon}}_{\mathbf{Assoc}} \text{Cat}_{\infty}. \]

Then the projection map \( \tilde{\text{Mon}}_{\mathbf{Assoc}} \text{Cat}_{\infty} \to \text{Mon}_{\mathbf{Assoc}} \text{Cat}_{\infty} \) is the universal family of monoidal quasi-categories compatible with \( \mathcal{L} \)-indexed colimits, that is it is classified by the identity map of \( \text{Mon}_{\mathbf{Assoc}} \text{Cat}_{\infty} \).

(3) Let
\[ \text{Cat}_{\infty}^{\text{Alg}}(\mathcal{L}) = \text{Cat}_{\infty}^{\text{Alg}} \times_{\text{Cat}_{\infty}^{\text{Mon}}} \text{Cat}_{\infty}^{\text{Mon}}(\mathcal{L}). \]

Then it classifies pairs \((C^\otimes, A)\) of families of monoidal quasi-categories compatible with \( \mathcal{L} \)-indexed colimits, and families of associative algebras on them.

3.5. **Families of left-tensored quasi-categories.** As in the case of monoidal quasi-categories, we can define and classify families of left tensored quasi-categories. Let \( K \) be a simplicial set. Then a coCartesian family of left-tensored quasi-categories over \( K \) is a

1. coCartesian fibration \( \mathcal{M}^\otimes \overset{\eta}{\to} \text{LM}^\otimes \times K \) such that
2. for all \( k \in K \), the fibre left-tensored quasi-category \( \mathcal{M}^\otimes_k \overset{\eta_k}{\to} \text{LM}^\otimes \) is a left-tensored quasi-category.

Note that the restriction
\[ \mathcal{M}^\otimes_a := \mathcal{M}^\otimes \times_{\text{LM}^\otimes} \text{Assoc}^\otimes \]

is a coCartesian family of monoidal quasi-categories. We say that \( \mathcal{M}^\otimes \) is a coCartesian family of quasi-categories over \( K \) left-tensored over \( \mathcal{M}^\otimes_a \).

Let \( \mathcal{L} \) be a collection of simplicial sets. Then we say that \( q \) commutes with \( \mathcal{L} \)-indexed colimits, if

1. for each \( k \in K \), the fibre left-tensored quasi-category \( \mathcal{M}^\otimes_k \overset{\eta_k}{\to} \text{LM}^\otimes \) commutes with \( \mathcal{L} \)-indexed colimits, that is
   a. the underlying quasi-categories \( \mathcal{M}^\otimes_m \) and \( \mathcal{M}^\otimes_a \) admit \( \mathcal{L} \)-indexed colimits, and
   b. the tensor product \( \mathcal{M}^\otimes_a \times \mathcal{M}^\otimes_a \to \mathcal{M}^\otimes_a \) and left action \( \mathcal{M}^\otimes_a \times \mathcal{M}^\otimes_m \to \mathcal{M}^\otimes_m \) functors commute with \( \mathcal{L} \)-indexed colimits, and
2. for each edge \( k \overset{\xi}{\to} k' \) in \( K \), the induced maps on underlying quasi-categories \( \mathcal{M}^\otimes_{k,m} \to \mathcal{M}^\otimes_{k',m} \) and \( \mathcal{M}^\otimes_{k,a} \to \mathcal{M}^\otimes_{k',a} \) commute with \( \mathcal{L} \)-indexed colimits.

Just as in the case of monoidal quasi-categories, left-tensored quasi-categories are classified by
\[ \text{Cat}_{\infty}^\text{Mod} := \text{Mon}_{\text{LM}} \text{Cat}_{\infty}, \]

and left-tensored quasi-categories compatible with \( \mathcal{L} \)-indexed colimits are classified by the largest subcategory \( \text{Cat}_{\infty}^\mathcal{L} = \text{Mon}_{\text{LM}} \text{Cat}_{\infty} \subseteq \text{Mon}_{\text{LM}} \text{Cat}_{\infty} \) such that

1. its vertices classify left-tensored quasi-categories compatible with \( \mathcal{L} \)-indexed colimits, and
2. its edges classify equivariant functors \( \mathcal{M}^\otimes \overset{f}{\to} \mathcal{N}^\otimes \) such that the restrictions to the underlying quasi-categories \( \mathcal{M}^\otimes_m \overset{f_m}{\to} \mathcal{M}^\otimes_m \) and \( \mathcal{M}^\otimes_a \overset{f_a}{\to} \mathcal{M}^\otimes_a \).
3.6. **Families of right module objects.** Let $K$ be a simplicial set, and $\mathcal{M} \xrightarrow{q} \mathcal{M}$ a coCartesian family of right-tensored quasi-categories. Then a section $\mathcal{M} \xrightarrow{M} \mathcal{M}$ is a family of right module objects, if for each $k \in K$, the restriction $\mathcal{M} \xrightarrow{M_k} \mathcal{M}$ is a right module object of $q_k$.

To classify right module objects in $\mathcal{M}$, we can apply the same construction as in the case of associative algebra objects (3.3): we let $\text{RMod}_\mathcal{M} \subseteq \Gamma_K(q)$ be the full subcategory on right module objects. Then the induced map $\text{RMod}(\mathcal{M}) \rightarrow K$ is a coCartesian fibration [Lur16, Lemma 4.8.3.13. 3].

Let $\mathcal{C} \xrightarrow{q} \text{Assoc} \times K$ denote the restriction $\mathcal{M}$. Then we get a restriction map

$$\text{RMod}_\mathcal{M} \xrightarrow{r} \text{Alg}_C.$$ 

In case $q$ commutes with $N(\Delta^\text{op})$-indexed colimits, the map $r$ is a coCartesian fibration [Lur16, Lemma 4.8.3.15].

Let $\text{Assoc} \times K \xrightarrow{A} \mathcal{C}$ be an associative algebra object. Then a family of right $A$-modules is a family of right modules $M$ such that $M(\mathcal{C}) = A$. Therefore, families of right $A$-modules are classified by

$$\text{RMod}_A \mathcal{C}_m = \text{Mod}_A \mathcal{C}_m = \text{RMod}_\mathcal{M} \times_{\text{Alg}_\mathcal{C}} K$$

where $K \rightarrow \text{Alg}_\mathcal{C}$ is the map classifying $A$. Note that by construction the projection map $\text{RMod}_A \mathcal{M}_m \rightarrow K$ is a coCartesian fibration.

Let $\mathcal{C} \xrightarrow{q} \text{Assoc} \times K$ be a coCartesian family of monoidal quasi-categories compatible with $N(\Delta^\text{op})$-indexed colimits, and $\text{Assoc} \times K \xrightarrow{A} \mathcal{C}$ be a family of associative algebra objects. Now we will relativize the construction of [2.17] to give the coCartesian family $\text{RMod}_A \mathcal{C}$ of right $A$-module objects in $q$ a left-tensored structure over $q$. Let $\text{Pr}_0$ denote the composite $\text{LM} \times \text{RM} \xrightarrow{\text{Pr}} \text{BM} \times U \rightarrow \text{Assoc}^\circ$ where $U$ is the forgetful functor. Then we can take the commutative diagram with strict Cartesian squares

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{q} & \mathcal{C} \\
\downarrow{q} & & \downarrow{q} \\
\text{Assoc} \times \text{LM} \times K & \xrightarrow{\text{Pr}_0} & \text{Assoc} \times K.
\end{array}$$

With this, we can let

$$\text{RMod}_A(\mathcal{C}) = \text{RMod}_A \mathcal{C}.$$ 

Since we assumed that $q$ commutes with $N(\Delta^\text{op})$-indexed colimits, so does $\bar{q}$, and therefore $\text{RMod}_A(\mathcal{C}) \xrightarrow{p} \text{LM} \times K$ is a coCartesian fibration. Note that for $k \in K$, the fibre $\text{RMod}_A(\mathcal{C}) \xrightarrow{p_k} \text{LM} \times K$ is the left-tensored quasi-category of right $A_k$-modules as defined in [2.17]. Therefore, the map $p$ is a coCartesian family of right-tensored quasi-categories. Let $\mathcal{C} \xrightarrow{p} \text{Assoc} \times K$ denote the pullback $p|(\text{Assoc} \times K)$. Then the inclusion $[m] \rightarrow \text{RM} \times K$ induces a morphism $\mathcal{C} \xrightarrow{\bar{q}} \mathcal{C}$ of coCartesian fibrations over $\text{Assoc} \times K$. Since its fibre over each $k \in K$ is a trivial Kan fibration [2.17], it is a categorical equivalence.
3.7. The Morita functor. Note that this construction is natural in $K$. That is, let $K' \xrightarrow{f} K$ be a morphism of simplicial sets. Then we have

$$\text{RMod}_{A/f}(\mathcal{E})^\otimes = \text{RMod}_{A}(\mathcal{E})^\otimes |(\text{id}_{\text{LM}} \times f).$$

In particular, we can give these left-tensored quasi-categories of right module objects as pullbacks along the classifying maps of algebras of the universal left-tensored quasi-category of right module objects. More precisely, let $\mathcal{L}$ be a collection of simplicial sets containing $N(\Delta^{op})$, and recall (3.3) that

$$\text{Cat}^\text{Alg}_{\mathcal{L}} \subseteq \text{Mon}^\mathcal{L}_{\text{Assoc}} \text{Cat}_\infty \times \text{Fun}(\text{Assoc}^\otimes, \text{Assoc} \times \text{Mon}^\mathcal{L}_{\text{Assoc}} \text{Cat}_\infty).$$

Therefore, it admits maps $\text{Cat}^\text{Alg}_{\mathcal{L}} \xrightarrow{\pi_1} \text{Mon}^\mathcal{L}_{\text{Assoc}}$ and $\text{Cat}^\text{Alg}_{\mathcal{L}} \times \text{Assoc}^\otimes \xrightarrow{\pi_2} \text{Mon}^\mathcal{L}_{\text{Assoc}} \text{Cat}_\infty$ induced by the projection maps. Using these, we can construct the universal pair $(\text{Cat}^\text{Alg}_{\mathcal{L}}, A^\text{univ})$ of a coCartesian family of monoidal quasi-categories compatible with $\mathcal{L}$-indexed colimits, and an associative algebra object in it, as given in the commutative diagram with strict Cartesian square

$$\text{Cat}^\text{Alg}_{\mathcal{L}} \xrightarrow{\pi_1 \times \text{id}} \text{Mon}^\mathcal{L}_{\text{Assoc}} \text{Cat}_\infty \times \text{Assoc}^\otimes.$$

Now we can form the universal coCartesian family

$$\text{RMod}_{A^\text{univ}}(\text{Cat}^\text{Alg}_{\mathcal{L}})^\otimes \to \text{LM}^\otimes \times \text{Cat}^\text{Alg}_{\mathcal{L}}$$

of left-tensored quasi-categories of right module objects compatible with $\mathcal{L}$-indexed colimits. It is classified by the Morita functor

$$\text{Cat}^\text{Alg}_{\mathcal{L}} \Theta \xrightarrow{\text{Mon}_{\text{LM}}} \text{Cat}_\infty.$$

The reason we call this functor the Morita functor is the following: Let $K$ be a simplicial set, $\mathcal{E} \xrightarrow{id} \text{Assoc}^\otimes \times K$ a coCartesian family of monoidal quasi-categories compatible with $\mathcal{L}$, and $\text{Assoc}^\otimes \times K \xrightarrow{\Delta} \mathcal{E}^\otimes$ an associative algebra object. Then the pair $(q, A)$ is classified by a map $K \xrightarrow{\Theta} \text{Cat}^\text{Alg}_{\mathcal{L}}$. Since as we said above, the formation of the coCartesian family of left-tensored quasi-categories of right module objects is natural, we get a diagram of homotopy Cartesian squares

$$\text{LM}^\otimes \times K \xrightarrow{\text{id} \times \Theta} \text{LM}^\otimes \times \text{Cat}^\text{Alg}_{\mathcal{L}} \xrightarrow{\text{id} \times \Theta} \text{LM}^\otimes \times \text{Mon}_{\text{LM}} \text{Cat}_\infty.$$
That is, the composite $K \xrightarrow{\varepsilon \cdot !} \text{Cat}^\text{Alg}_\infty(L) \xrightarrow{\Theta} \text{Mon}_{LM}^\text{Cat}_\infty$ classifies $\text{RMod}_A(\mathcal{C})^\circ$. In other words, we have

$$\Theta(\mathcal{C}^\circ, A) = \text{RMod}_A(\mathcal{C})^\circ$$

As we have seen (3.6), we have a canonical equivalence $\text{RMod}_A(\mathcal{C})^\circ \cong \text{Assoc}^\circ \times K \xrightarrow{\varepsilon^\circ} \mathcal{C}^\circ$ of coCartesian families of monoidal quasi-categories. Therefore, we get a homotopy commutative diagram of quasi-categories

$$\begin{array}{ccc}
\text{Cat}^\text{Alg}_\infty(L) & \xrightarrow{\Theta} & \text{Cat}^\text{Mod}_\infty(L) \\
\downarrow{\phi} & & \downarrow{\psi} \\
\text{Cat}^\text{Mon}_\infty(L) & & \\
\end{array}$$

It can be shown that [Lur16, Proposition 4.8.5.1]

1. the forgetful maps $\phi$ and $\psi$ are coCartesian fibrations, and
2. the Morita functor $\Theta$ carries $\phi$-coCartesian edges to $\psi$-coCartesian edges.

### 3.8. The endomorphism algebra functor.

Since we would like to study Azumaya algebra objects, we need an endomorphism algebra functor $E \mapsto \text{End}(E)$. As we will want this in the form of a morphism of stacks, we need a construction natural in the choice of the left-tensored quasi-category (in our main application, these will be the dg-categories of cochain complexes over schemes). Therefore, in the first place, we need a classifying object for triples $(C \otimes, M, M) \in \text{Cat}^\text{Alg}_\infty(L)$, where $C \otimes$ is a monoidal quasi-category, $M$ is a quasi-category left-tensored over $C$ compatible with $L$-indexed colimits, and $M \in M$.

Let $L$ be a collection of simplicial sets containing $N(\Delta^\text{op})$. Let $\mathcal{I}(L) \subseteq \mathcal{I}$ denote the smallest subcategory containing $\Delta^0$, which has $L$-indexed colimits. As in finite products commute with small colimits, the subcategory $\mathcal{I}(L)$ has finite products. Therefore, we can equip it with the Cartesian symmetric monoidal structure $\mathcal{I}(L)^\times$ (3.1). In particular, we have the trivial algebra $1 \in \text{Alg}\mathcal{I}(L)$ on $\Delta^0$. One can show that $(\mathcal{I}(L)^\times, 1) \in \text{Cat}^\text{Alg}_\infty(L)$ is an initial object [Lur16, Lemma 4.8.5.3].

We let $\mathcal{M} = \Theta(\mathcal{I}(L)^\times, 1) \in \text{Cat}^\text{Mod}_\infty(L)$. Let $\mathcal{M}^\circ$ be a left-tensored quasi-category compatible with $L$-indexed colimits. Then the restriction map

$$\text{Map}_{\text{Cat}^\text{Mod}_\infty(L)}(\mathcal{M}, \mathcal{M}^\circ) \xrightarrow{F \mapsto F(\Delta^0)} \mathcal{M}^\circ$$

is a trivial Kan fibration [Lur16, Remark 4.8.5.4]. Thus, objects in the undercategory $\text{Cat}^\text{Mod}_\infty(L)_{[\mathcal{M}]}$ correspond to triples $(\mathcal{C}^\circ, \mathcal{M}^\circ, M)$ of a monoidal quasi-category $\mathcal{C}^\circ$ compatible with $L$-indexed colimits, a quasi-category $\mathcal{M}^\circ$ left-tensored over $\mathcal{C}$ compatible with $L$-indexed colimits, and an object $M \in \mathcal{M}$, as required.

Since $(\mathcal{I}(L)^\times, 1) \in \text{Cat}^\text{Alg}_\infty(L)$ is an initial object, the forgetful functor $\text{Cat}^\text{Alg}_\infty(L)_{[\mathcal{M}]} \xrightarrow{U} \text{Cat}^\text{Alg}_\infty(L)$ is a trivial Kan fibration. Therefore, we can let $\text{Cat}^\text{Alg}_\infty(L) \xrightarrow{\Theta} \text{Cat}^\text{Mod}_\infty(L)_{[\mathcal{M}]}$ denote the
composite
\[ \text{Cat}_\infty^\text{Alg} \left( \mathcal{L} \right) \xrightarrow{U} \text{Cat}_\infty^\text{Alg} \left( \mathcal{L} \right)_{(\mathcal{L}^\otimes, 1)/} \xrightarrow{\Theta} \text{Cat}_\infty^\text{Mod} \left( \mathcal{L} \right)_{\text{Mor}}. \]

Informally, it carries \((\mathcal{C}^\otimes, A) \in \text{Cat}_\infty^\text{Alg} \left( \mathcal{L} \right)\) to \((\mathcal{C}^\otimes, \text{RMod}_A \mathcal{C}, A_A) \in \text{Cat}_\infty^\text{Mod} \left( \mathcal{L} \right)_{\text{Mor}}\). One can show that the functor \(\Theta\) is fully faithful [Lur16, Theorem 4.8.5.5].

Then endomorphism functor will be a right adjoint to \(\Theta\). To get it, we will need to assume that the underlying quasi-categories we’re dealing with are presentable. Let \(\hat{\text{Cat}}^\infty\) denote the quasi-category of big quasi-categories. Then we can use the same constructions as above to get classifying objects for higher algebraic structures with big underlying quasi-categories. We let \(\mathcal{L}\) denote the collection of all small simplicial sets. We let

1. \(\text{Pr}^\text{Mon} \subseteq \hat{\text{Cat}}^\infty_{\text{Mon}} \left( \mathcal{L} \right)\) denote the full subcategory on monoidal quasi-categories with presentable underlying quasi-category,
2. \(\text{Pr}^\text{Alg} = \hat{\text{Cat}}^\infty_{\text{Alg}} \left( \mathcal{L} \right) \times_{\hat{\text{Cat}}^\infty_{\text{Mon}} \left( \mathcal{L} \right)} \text{Pr}^\text{Mon}\), and
3. \(\text{Pr}^\text{Mod} \subseteq \hat{\text{Cat}}^\infty_{\text{Mod}} \left( \mathcal{L} \right)\) the full subcategory on presentable quasi-categories left-tensored over a presentable monoidal quasi-category.

Then the Morita functor \(\hat{\text{Cat}}^\infty_{\text{Alg}} \xrightarrow{\hat{\Theta}} \hat{\text{Cat}}^\infty_{\text{Mod}}\) restricts to a functor \(\text{Pr}^\text{Alg} \xrightarrow{\hat{\Theta}} \text{Pr}^\text{Mod}\) [Lur16, Corollary 4.2.3.7]. We also get the pointed version \(\text{Pr}^\text{Alg} \xrightarrow{\hat{\Theta}} \text{Pr}^\text{Mod}_{\text{Mor}}\) the same way. One can show that this functor \(\hat{\Theta}\) is fully faithful, and it admits a right adjoint [Lur16, Theorem 4.8.5.11]. The right adjoint maps \((\mathcal{C}^\otimes, \mathcal{M}^\otimes, M) \in \text{Pr}^\text{Mod}\) to \((\mathcal{C}^\otimes, \text{End}_{\mathcal{M}} M) \in \text{Pr}^\text{Alg}\).

3.9. Tensor products of quasi-categories. Let \(\mathcal{O}^\otimes\) be an \(\infty\)-operad, and \(\mathcal{K}\) a collection of simplicial sets. To finish, let us discuss another way to classify coCartesian families of \(\mathcal{O}\)-monoidal quasi-categories compatible with \(\mathcal{K}\)-indexed colimits, which will be useful when we study descent in the next section.

In §3.1, we have seen that the quasi-category of quasi-categories \(\text{Cat}_\infty\) can be equipped with the Cartesian symmetric monoidal structure, using which we can take the quasi-category of monoid objects \(\text{Mon}_\mathcal{O} \text{Cat}_\infty\), and that

1. by straightening-unstraightening classifies coCartesian families of \(\mathcal{O}\)-monoidal quasi-categories, and
2. is equipped with an equivalence \(\text{Mon}_\mathcal{O} \text{Cat}_\infty \rightarrow \text{Alg}_\mathcal{O} \text{Cat}_\infty\).

Then we have seen in §3.4-3.5 that the 2-full subcategory \(\text{Mon}_\mathcal{K}^\mathcal{O} \text{Cat}_\infty\) on objects classifying \(\mathcal{O}\)-monoidal quasi-categories compatible with \(\mathcal{K}\)-indexed colimits and edges classifying morphisms of \(\mathcal{O}\)-monoidal quasi-categories compatible with \(\mathcal{K}\)-indexed colimits by construction classifies coCartesian families of \(\mathcal{O}\)-monoidal quasi-categories compatible with \(\mathcal{K}\)-indexed colimits.

The alternative approach for the latter which we will now explain is to equip the full subcategory \(\text{Cat}_\infty(\mathcal{K}) \subseteq \text{Cat}_\infty\) on quasi-categories with \(\mathcal{K}\)-indexed colimits with a symmetric monoidal structure so that the equivalence \(\text{Alg}_\mathcal{O} \text{Cat}_\infty \rightarrow \text{Mon}_\mathcal{O} \text{Cat}_\infty\) restricts to an equivalence \(\text{Alg}_\mathcal{O} \text{Cat}_\infty(\mathcal{K}) \rightarrow \text{Mon}_\mathcal{K}^\mathcal{O} \text{Cat}_\infty\).

First of all, consider the following explicit construction \(\text{Cat}_\infty^\mathcal{O}\) of the Cartesian symmetric monoidal structure on \(\text{Cat}_\infty\):
(1) An object over \( \langle n \rangle \in \text{Fin}_n \) is an \( n \)-tuple \([X_1, \ldots, X_n]\) of quasi-categories.
(2) A morphism \([X_1, \ldots, X_n] \rightarrow [Y_1, \ldots, Y_m]\) over a morphism \( \langle n \rangle \xrightarrow{\sigma} \langle m \rangle \) is a collection of morphisms of quasi-categories \( \prod_{i:a(i)=j} X_i \xrightarrow{\eta_j} Y_j \) for all \( 1 \leq j \leq m \).

Let \( P \) be the collection of all small simplicial sets partially ordered by inclusion. Then we let \( \mathcal{M} \subseteq \text{Cat}^\infty_{\text{top}} \times N(P) \) be the 2-full subcategory on

1. pairs \( ([X_1, \ldots, X_n], \mathcal{K}) \) such that \( X_i \) has \( \mathcal{K} \)-indexed colimits for all \( 1 \leq i \leq n \), and
2. morphisms \( ([X_1, \ldots, X_n], \mathcal{K}) \xrightarrow{[\eta]} ([Y_1, \ldots, Y_m], \mathcal{K'}) \) such that \( \prod_{i:a(i)=j} X_i \xrightarrow{\eta_j} Y_j \) is compatible with \( \mathcal{K} \)-indexed colimits for all \( 1 \leq j \leq m \).

Then one can show that the map \( \mathcal{M} \rightarrow N(\text{Fin}_n) \times N(P) \) is a coCartesian fibration of symmetric monoidal quasi-categories [Lur16 Proposition 4.8.1.3]. For a collection of simplicial sets \( \mathcal{K} \), we let \( \text{Cat}^\infty(\mathcal{K})^\otimes = \mathcal{M} \times_{N(P)} [\mathcal{K}] \). Then one can show that the inclusion map \( \text{Cat}^\infty(\mathcal{K})^\otimes \rightarrow \text{Cat}^\infty \) is a lax monoidal functor [Lur16 Proposition 4.8.1.4]. This implies that the equivalence \( \text{Alg}_{\mathcal{O}} \text{Cat}^\infty \rightarrow \text{Mon}_{\mathcal{O}} \text{Cat}^\infty \) restricts to an equivalence \( \text{Alg}_{\mathcal{O}} \text{Cat}^\infty(\mathcal{K}) \rightarrow \text{Mon}_{\mathcal{O}} \text{Cat}^\infty(\mathcal{K}) \).

In particular, let \( \mathcal{K} \) denote the collection of all small simplicial sets. Then we have the symmetric monoidal quasi-category of big quasi-categories compatible with all small colimits \( \text{Cat}_{\leq}^\infty(\mathcal{K})^\otimes \). One can show that the full subcategory \( \text{Pr}^L \subseteq \text{Cat}_\leq^\infty(\mathcal{K})^\otimes \) is closed under tensor product [Lur16 Proposition 4.8.1.15]. Therefore, the inclusion \( \text{Pr}^L \rightarrow \text{Cat}_\leq^\infty(\mathcal{K}) \) can be lifted to a symmetric monoidal functor \( (\text{Pr}^L)^\otimes \rightarrow \text{Cat}_\leq^\infty(\mathcal{K})^\otimes \). This in turn gives the equivalence \( \text{Alg}_{\mathcal{O}} \text{Pr}^L \rightarrow \text{Mon}_{\mathcal{O}} \text{Pr}^L \) we wanted.

4. Homotopical Skolem–Noether theorem

Let \( K \) be a quasi-category equipped with a Grothendieck topology \( \tau \). Let \( \mathcal{C}^\otimes \rightarrow \text{Assoc}^\otimes \times K^\op \) be a coCartesian family of presentable monoidal quasi-categories. In Subsection 1 we introduce Azumaya algebra objects in \( \mathcal{C}^\otimes \) and give a criterion on \( \mathcal{C}^\otimes \) so that the prestack \( \text{Az}^\mathcal{C} \) of Azumaya algebra objects in \( \mathcal{C}^\otimes \) an the prestack \( \text{LTens}^\mathcal{A}^\text{Az} \mathcal{C} \) of locally trivial presentable left-tensored quasi-categories over \( \mathcal{C}^\otimes \) have \( \tau \)-descent. In Subsection 2 we prove our main result.

4.1. Descent for presentable left-tensored quasi-categories with descent.

Notation 4.1. Let \( K \) denote a quasi-category equipped with a Grothendieck topology \( \tau \) [Lur09 §6.2.2].

Let \( \mathcal{C}^\otimes \rightarrow \text{Assoc}^\otimes \times K^\op \) be a coCartesian family of presentable monoidal quasi-categories. Then it is classified by a map \( K^\op \xrightarrow{\mathcal{C}_*} \text{Pr}^\text{Alg} \times_{\text{Pr}^\text{Mon}} K^\op \). In our main example, the site \( K \) is the fppf site over a scheme, and \( \mathcal{C}^\otimes \) is the family of monoidal quasi-categories of cochain complexes of quasi-coherent sheaves.

Let \( \text{Alg}^\mathcal{C} = \text{Pr}^\text{Alg} \times_{\text{Pr}^\text{Mon}} K^\op \). In our main example, it classifies derived associative algebras over cochain complexes of quasi-coherent sheaves.

Definition 4.2. Take \( X \in K \) and let \( A \in \text{Alg}^\mathcal{C}(X) \) be an algebra object. Then \( A \) is an Azumaya algebra object if there exists

1. a \( \tau \)-covering \( U \rightarrow X \),
(2) a dualizable generator \( M \in C_{\text{dgen}}(U) \) and
(3) an equivalence \( \mathcal{A}|U := \mathcal{C}(U) \otimes_{C(X)} A \cong \text{End} M \).

We denote by \( \mathcal{A} \subseteq \text{Alg} \mathcal{C} \) the full subprestack on Azumaya algebras.

**Notation 4.3.** Let \( \text{LTens}_\mathcal{C} = \text{Pr}^{\text{Mod}} \times^{\text{PrMon}} K^{\text{op}} \). In our main example, it classifies presentable quasi-categories left-tensored over a monoidal quasi-category of cochain complexes of quasi-coherent sheaves. That is, it classifies lax dg-categories.

Let \( \text{LTens}_\mathcal{C} \subseteq \text{LTens}_\mathcal{C} \) denote the full subprestack on triples \((U, \mathcal{M}, M)\) of an object \( U \in K \), a quasi-category \( \mathcal{M} \) left-tensored over \( \mathcal{C}(U) \), and an object \( M \in \mathcal{M} \) such that \( M \) is a dualizable generator in \( \mathcal{M} \). Let \( \text{LTens}^{\mathcal{A}}_{\text{dgen}} \mathcal{C} \subseteq \text{LTens}_{\text{dgen}} \mathcal{C} \) denote the full subprestack on locally trivial pointed presentable left-tensored quasi-categories over \( \mathcal{C} \).

**Remark 4.4.** By the Homotopical Morita Theorem, we have an equivalence \( \mathcal{A}|U \cong \text{End} M \) of algebras for some \( M \in C_{\text{dgen}}(U) \) if and only if we have an equivalence \( \text{Mod}_{\mathcal{A}}|U \cong \mathcal{C}(U) \) of presentable quasi-categories left-tensored over \( \mathcal{C} \). Therefore, the equivalence \( \text{Alg} \mathcal{C} \xrightarrow{\text{Mod}} \text{LTens}_{\text{dgen}} \mathcal{C} \) restricts to an equivalence \( \mathcal{A} \mathcal{C} \to \text{LTens}^{\mathcal{A}}_{\text{dgen}} \mathcal{C} \).

Let \( \text{op} \text{LTens}_\mathcal{C} \xrightarrow{\text{op}} K \) denote the structure map. Let \( U \rightarrow X \) be a \( \tau \)-covering in \( K \). Let \( \Delta^\text{op}_+ \xrightarrow{U} K \) denote its \( \check{\text{C}} \)ech nerve \([\text{Lur}09, \text{below Proposition 6.1.2.11}]\). Then \( \text{op} \text{LTens}_\mathcal{C} \) has descent along \( f \), if the restriction map

\[
\Gamma_{\text{Cart}}(\bar{U}_* \text{op} \text{LTens}_\mathcal{C}) \xrightarrow{r} \Gamma_{\text{Cart}}(U_*, \text{op} \text{LTens}_\mathcal{C})
\]

is an equivalence of quasi-categories. This will not hold in general. But we can select a full subcategory \( \text{op} \text{LTens}^{\text{desc}} \mathcal{C} \subseteq \text{op} \text{LTens}_\mathcal{C} \) of presentable quasi-categories left-tensored over \( \mathcal{C} \) with \( \tau \)-descent, and we can show that

1. the restriction \( \text{op} \text{LTens}^{\text{desc}} \mathcal{C} \to K \) is a biCartesian fibration which satisfies \( \tau \)-descent, and
2. it has all the objects needed for Morita theory: for all \( U \in K \) and \( A \in \text{Alg} \mathcal{C}_U \), the right module quasi-category \( \text{Mod}_{\mathcal{A}} \mathcal{C}_U \) has \( \tau \)-descent.

First of all, consider the restriction map on the section quasi-categories

\[
\Gamma(\bar{U}_* \text{op} \text{LTens}_\mathcal{C}) \xrightarrow{r} \Gamma(U_*, \text{op} \text{LTens}_\mathcal{C}).
\]

Suppose that every section \( \Delta^\text{op}_+ \xrightarrow{k} \text{op} \text{LTens}_\mathcal{C} \) has a \( \text{q}^\text{op} \)-colimit \( \Delta^\text{op}_+ \xrightarrow{k} \text{op} \text{LTens}_\mathcal{C} \). Then \( r \) has a fully faithful left adjoint, which takes \( k \) to \( \bar{k} \) \([\text{Lur}09 \text{ Corollary 4.3.2.16 and Proposition 4.3.2.17}]\).

**Proposition 4.5.** Let \( K \) be a quasi-category, \( \mathcal{C}^{\text{op}} \to K^{\text{op}} \times \text{Assoc}^{\text{op}} \) a coCartesian family of presentable monoidal quasi-categories, and \( L \) a simplicial set. Then the family \( \text{LTens}_\mathcal{C} \xrightarrow{\mathcal{A}} K^{\text{op}} \) of presentable quasi-categories left-tensored over \( \mathcal{C} \) has all \( L \)-indexed \( q \)-limits.

**Corollary 4.6.** Over any diagram \( L^d \to K^{\text{op}} \), the restriction map

\[
\Gamma(L^d, \text{LTens}_\mathcal{C}) \xrightarrow{r} \Gamma(L, \text{LTens}_\mathcal{C})
\]

admits a fully faithful right adjoint.

**Proof of Proposition 4.5.** It is enough to show \([\text{Lur}09 \text{ Corollary 4.3.1.11}]\)
Proposition 5.5.3.13], the quasi-category \( \mathcal{L}\text{Mod}_\mathcal{C} \) has \( \mathcal{L} \)-indexed limits, and

(2) for all maps \( U \to V \) in \( \mathcal{K}^{\text{op}} \), the restriction map \( \mathcal{L}\text{ens}_\mathcal{C}(V) \xrightarrow{\mathcal{L}\text{ens}_\mathcal{C}(U)} \mathcal{L}\text{ens}_\mathcal{C}(U) \) preserves \( \mathcal{L} \)-indexed limits.

We have \( \mathcal{L}\text{ens}_\mathcal{C}(U) \cong \mathcal{L}\text{Mod}_\mathcal{C}(U)^\mathcal{L} \) \cite{Lur09} Remark 4.8.3.6]. As \( \mathcal{L} \) admits all limits \cite{Lur09} Proposition 5.5.3.13], the quasi-category \( \mathcal{L}\text{Mod}_\mathcal{C}(U)^\mathcal{L} \) also has limits \cite{Lur16} Corollary 4.2.3.3], which shows (1).

Since \( \mathcal{L} \text{Mod} \xrightarrow{\psi} \mathcal{L} \text{Mon} \) is biCartesian by Lemma 4.7 the restriction map \( \mathcal{L}\text{ens}_\mathcal{C}(V) \to \mathcal{L}\text{ens}_\mathcal{C}(U) \) admits a left adjoint \cite{Lur09} Corollary 5.2.2.5], and thus it preserves limits \cite{Lur09} Proposition 5.2.3.5], which shows (2).

\[\square\]

**Lemma 4.7.** The universal family \( \mathcal{L}\text{Mod} \xrightarrow{\psi} \mathcal{L} \text{Mon} \) of presentable left-tensored quasi-categories is biCartesian.

**Proof.** Let \( \mathcal{L} \) denote the collection of all small simplicial sets. Then the forgetful map \( \xrightarrow{\text{Cat}_\infty} \text{Alg}_\infty(\mathcal{L}) \xrightarrow{\psi} \text{Cat}_\infty(\mathcal{L}) \) is both Cartesian \cite{Lur16} Corollary 4.2.3.2] and coCartesian \cite{Lur16} Proposition 4.8.5.1].

Take a morphism \( \mathcal{D} \xrightarrow{i} \mathcal{D}' \) in \( \text{Alg}_\infty, \mathcal{M} \in \mathcal{L}\text{Mod}_\mathcal{D}^\mathcal{L}, \text{ and } \mathcal{M}' \in \mathcal{L}\text{Mod}_{\mathcal{D}'}^\mathcal{L} \). The \( \psi \)-Cartesian edge \( \mathcal{M} \to \mathcal{M}' \) is an equivalence on the underlying quasi-categories \cite{Lur09} Corollary 4.2.3.2], therefore \( \mathcal{D}, \mathcal{M}' \) has a presentable underlying quasi-category, and thus is an object of \( \mathcal{L}\text{Mod}_\mathcal{D}^\mathcal{L} \).

By Lemma 4.8 the base change \( \mathcal{D}' \boxtimes_{\mathcal{D}} \mathcal{M} \) is also presentable.

\[\square\]

**Lemma 4.8.** Let \( \mathcal{D} \to (\mathcal{D})^\mathcal{L} \) be a morphism of presentable monoidal quasi-categories. Let \( \mathcal{M}^\mathcal{L} \) be a presentable quasi-category left-tensored over \( \mathcal{D}^\mathcal{L} \). Then the base change \( \mathcal{D}' \boxtimes_{\mathcal{D}} \mathcal{M} \) is also presentable.

**Proof.** On the underlying quasi-categories, the \( \psi \)-coCartesian edge \( \mathcal{M} \to \mathcal{D}' \boxtimes_{\mathcal{D}} \mathcal{M} \) can be given by the bar construction, which is a colimit diagram \cite{Lur16} Theorem 4.4.2.8]. Since \( \mathcal{L} \) has colimits and the inclusion \( \mathcal{L} \) preserves colimits \cite{Lur16} Theorem 5.5.3.18], the base change \( \mathcal{D}' \boxtimes_{\mathcal{D}} \mathcal{M} \) has presentable underlying quasi-category too.

\[\square\]

**Corollary 4.9.** Over any diagram \( \mathcal{L}^\mathcal{L} \to \mathcal{K}^{\text{op}} \), the restriction map on the coCartesian section quasi-categories

\[\Gamma_{\text{coCart}}(\mathcal{L}^\mathcal{L}, \mathcal{L}\text{ens}_\mathcal{C}) \xrightarrow{r_{\text{coCart}}} \Gamma_{\text{coCart}}(\mathcal{L}, \mathcal{L}\text{ens}_\mathcal{C}).\]

admits a fully faithful right adjoint.

**Proof.** Let us denote \( \Gamma_{\text{coC}} = \Gamma_{\text{coCart}}(\mathcal{L}^\mathcal{L}, \mathcal{L}\text{ens}_\mathcal{C}), \Gamma_{\text{coc}} = \Gamma_{\text{coCart}}(\mathcal{L}, \mathcal{L}\text{ens}_\mathcal{C}), \Gamma = \Gamma(\mathcal{L}^\mathcal{L}, \mathcal{L}\text{ens}_\mathcal{C}) \) and \( \Gamma = \Gamma(\mathcal{L}, \mathcal{L}\text{ens}_\mathcal{C}) \). Then the inclusions \( \Gamma_{\text{coc}} \xrightarrow{i} \Gamma \) resp. \( \Gamma_{\text{coc}} \xrightarrow{i} \Gamma \) admit right adjoints \( \Gamma \xrightarrow{R} \Gamma_{\text{coc}} \) resp. \( \Gamma \xrightarrow{R} \Gamma_{\text{coc}} \) \cite{Lur09} Lemma 5.5.3.16]. Let us denote the right adjoint to the restriction map by \( \lim_{\text{coC}} \xrightarrow{i} \Gamma \). We claim that the composite \( \Gamma_{\text{coc}} \subseteq \Gamma \lim_{\text{coC}} \xrightarrow{R} \Gamma_{\text{coc}} \) is shown to be the right adjoint to \( r_{\text{coC}} \) by the composite \( \mathcal{M} \xrightarrow{u} \mathcal{M} \xrightarrow{R} \lim_r \mathcal{M}, \) where the two \( u \) denote unit maps for the adjunctions \( (i, R) \) resp. \( (r, \lim) \) applied to \( \mathcal{M} \in \Gamma_{\text{coC}} \). The claim can be checked on the commutative diagram
\[
\begin{array}{ccc}
\text{Map}_{\Gamma_{\text{coc}}} (\mathcal{R} \operatorname{lim} \mathcal{M}, \mathcal{R} \operatorname{lim} \mathcal{N}), & \xrightarrow{\mathcal{R}} & \text{Map}_r (\mathcal{R} \operatorname{lim} \mathcal{M}, \mathcal{R} \operatorname{lim} \mathcal{N}) \\
\xrightarrow{i} & & \xleftarrow{\mathcal{R} \circ \mathcal{R} \circ \mathcal{U}} \\
\text{Map}_{\Gamma_{\text{coc}}} (\mathcal{R} \operatorname{lim} \mathcal{M}, \mathcal{R} \operatorname{lim} \mathcal{N}) & \xrightarrow{\mathcal{R}} & \text{Map} (\mathcal{R} \operatorname{lim} \mathcal{M}, \mathcal{R} \operatorname{lim} \mathcal{N}) \\
\xrightarrow{\lim} & & \xleftarrow{\mathcal{R} \circ \mathcal{U}} \\
\text{Map} (\mathcal{R} \operatorname{lim} \mathcal{M}, \mathcal{R} \operatorname{lim} \mathcal{N}) & \xrightarrow{\mathcal{R}} & \text{Map} (\mathcal{R} \operatorname{lim} \mathcal{M}, \mathcal{R} \operatorname{lim} \mathcal{N})
\end{array}
\]

where \( \mathcal{N} \in \Gamma_{\text{coc}}. \)

\[\square\]

**Notation 4.10.** Recall that the restriction to \( \mathcal{U}_1 = X \)-map \( \Gamma_{\text{coc}}(\mathcal{U}_{\text{op}}^\bullet, \text{LTens} \mathcal{C}) \to \text{LTens} \mathcal{C}(X) \) is an equivalence of quasi-categories \[\text{DZ18}, \text{Lemma 2.5}.\] Take \( \mathcal{M} \). Recall that the restriction to \( \mathcal{M} \) its image by the right adjoint by \( \lim \).

If we start out from \( \mathcal{M} \in \text{LTens} \mathcal{C}(X) \), then we denote its image by the right adjoint by \( \mathcal{C}(\mathcal{U}_{\text{op}}^\bullet) \otimes_{\mathcal{C}(X)} \mathcal{M} \). Note that we get a unit map \( \mathcal{M} \to \lim(\mathcal{C}(\mathcal{U}_{\text{op}}^\bullet) \otimes_{\mathcal{C}(X)} \mathcal{M}) \).

**Definition 4.11.** Let \( K \) be a quasi-category equipped with a Grothendieck topology \( \tau \) and \( \mathcal{C}^\circ \to K_{\text{op}} \times \text{Assoc}^\circ \) a coCartesian fibration of presentable monoidal quasi-categories. Let \( \text{LTens} \mathcal{C} \to K_{\text{op}} \) be the corresponding family of presentable quasi-categories left-tensored over \( \mathcal{C}^\circ \). We say that

- base changes in \( q \) commute with \( \tau \)-descent data if for the Čech nerve \( \Delta_{\text{op}} \xrightarrow{\mathcal{U}} K \) of a \( \tau \)-covering, a \( q \)-limit diagram \( \mathcal{M} \in \Gamma_{\text{coc}}(\mathcal{U}_{\text{op}}^\bullet, \text{LTens} \mathcal{C}) \) and a map \( V \to \mathcal{U}_1 \) in \( K \), the base change \( \mathcal{C}(V) \otimes_{\mathcal{C}(\mathcal{U}_1)} \mathcal{M} \) is also a \( q \)-limit diagram.

We say that the family \( \mathcal{C}^\circ \) has \( \tau \)-descent if the following conditions hold:

1. The underlying Cartesian fibration \( \text{op} \mathcal{C} \xrightarrow{\mathcal{P}} K \) has \( \tau \)-descent.
2. Base changes in \( q \) commute with \( \tau \)-descent data.

Let \( T \in K \) be an object and \( \mathcal{M} \in \text{LTens} \mathcal{C}(T) \) a presentable quasi-category left-tensored over \( \mathcal{C}(T) \). We say that \( \mathcal{M} \) has \( \tau \)-descent, if for all \( \tau \)-coverings \( U \xrightarrow{f} X \) over \( T \) in \( K \), letting \( \Delta_{\text{op}} \xrightarrow{\mathcal{U}} K \) denote the Čech nerve of \( f \), the canonical map

\[
\mathcal{C}(X) \otimes_{\mathcal{C}(T)} \mathcal{M} \to \lim(\mathcal{C}(\mathcal{U}_{\text{op}}^\bullet) \otimes_{\mathcal{C}(X)} \mathcal{M})
\]

is an equivalence. We denote by \( \text{LTens}^{\text{desc}} \subseteq \text{LTens} \mathcal{C} \) the full subcategory on pairs \((T, \mathcal{M})\) of objects \( T \in K \) and presentable quasi-categories \( \mathcal{M} \) left-tensored over \( \mathcal{C}(T) \) with \( \tau \)-descent.

**Proposition 4.12.** Let \( K \) be a quasi-category equipped with a Grothendieck topology \( \tau \) and \( \mathcal{C}^\circ \to K_{\text{op}} \times \text{Assoc}^\circ \) a coCartesian fibration of presentable monoidal quasi-categories. Suppose that the underlying Cartesian fibration \( \text{op} \mathcal{C} \xrightarrow{\mathcal{P}} K \) has \( \tau \)-descent. Then the following assertions hold:

1. Suppose that for all morphisms \( V \to U \) in \( K \), the bimodule \( \mathcal{C}(V) \in \mathcal{C}(V) \operatorname{Mod}_{\mathcal{C}(U)} \) is left dualizable. Then base changes in \( q \) commute with \( \tau \)-descent data.
Suppose that base changes in \( q \) commute with \( \tau \)-descent data. Let \( T \in K \) be an object and \( \mathcal{M} \in \text{LTens} \mathcal{C}(T) \) a presentable quasi-category left-tensored over \( \mathcal{C}(T) \). Suppose that \( \mathcal{M} \in \text{LTens} \mathcal{C}(T) \) is left dualizable. Then \( \mathcal{M} \) has \( \tau \)-descent.

**Proof.** (1) Since \( \text{op} \mathcal{C} \to K \) has \( \tau \)-descent and \( \bar{U} \) is the Čech nerve of a \( \tau \)-covering, the diagram \( \Delta_+ \xrightarrow{\mathcal{C}(U)} \text{Pr}^{\text{Mon}} \) is a limit diagram. Therefore, the base change \( \mathcal{C}(V) \otimes_{\mathcal{C}(U)} \mathcal{M} \) is a \( q \)-limit diagram if and only if it is a limit diagram. Since \( \mathcal{C}(V) \in \text{Mod}_{\mathcal{C}(U)} \) is left dualizable, the functor \( \text{LTens} \mathcal{C}(U) \xrightarrow{\mathcal{C}(V) \otimes_{\mathcal{C}(U)}} \text{LTens} \mathcal{C}(V) \) has a left adjoint and thus commutes with small limits [Lur16, Proposition 4.6.2.1].

(2) Let \( \Delta_+ \xrightarrow{\bar{U}} K \) be the Čech nerve of a \( \tau \)-covering over \( T \). We need to show that the augmented simplicial diagram \( \mathcal{C}(\bar{U}) \otimes_{\mathcal{C}(T)} \mathcal{M} \) is a \( q \)-limit. Since \( p \) is a \( \tau \)-stack, the augmented simplicial diagram \( \mathcal{C}(\bar{U}) \) of presentable monoidal quasi-categories is a limit diagram. Therefore, it is enough to show that \( \mathcal{C}(\bar{U}) \otimes_{\mathcal{C}(T)} \mathcal{M} \) is a limit diagram. This follows from that \( \mathcal{M} \in \text{LTens} \mathcal{C}(T) \) is left dualizable.

\( \square \)

**Corollary 4.13.** Let \( K \) be a quasi-category equipped with a Grothendieck topology \( \tau \) and \( \mathcal{C}^\circ \to K^{\text{op}} \times \text{Assoc}^\circ \) a coCartesian fibration of presentable monoidal quasi-categories. Suppose that the following assertions hold:

1. The underlying Cartesian fibration \( \text{op} \mathcal{C} \xrightarrow{p} K \) has \( \tau \)-descent.
2. For all objects \( U \in K \), the presentable monoidal quasi-category \( \mathcal{C}(U) \in \text{Alg}(\text{Pr}^1) \) has dualizable underlying presentable quasi-category.

Then base changes in \( q \) commute with \( \tau \)-descent data.

**Proof.** This follows from the more general result Lemma 4.14.

\( \square \)

**Lemma 4.14.** Let \( A \in \text{Alg} \mathcal{C} \) be an algebra object in a monoidal quasi-category. Suppose that the underlying object \( A \in \mathcal{C} \) has left duality data

\[ \tilde{A} \otimes A \xrightarrow{\epsilon} 1, \quad 1 \xrightarrow{\epsilon} A \otimes \tilde{A}. \]

Then the object \( A \in A \text{Mod} \mathcal{C} \) has left duality data

\[ (\tilde{A} \otimes A) \otimes_A A \xrightarrow{\epsilon} \tilde{A} \otimes A \xrightarrow{\epsilon} 1, \quad A \xrightarrow{\epsilon \otimes A} A \otimes \tilde{A} \otimes A. \]

**Proof.** This can be checked directly.

\( \square \)

**Theorem 4.15.** Let \( K \) be a quasi-category equipped with a Grothendieck topology \( \tau \). Let \( \mathcal{C}^\circ \to K^{\text{op}} \times \text{Assoc}^\circ \) be a coCartesian family of presentable monoidal quasi-categories with \( \tau \)-descent. Then the following assertions hold.

1. The family \( \text{op} \mathcal{C} \xrightarrow{\text{LTens}^{\text{desc}}} K \) of presentable quasi-categories left-tensored over \( \mathcal{C} \) with \( \tau \)-descent is a \( \tau \)-stack.
2. For any object \( T \in K \) and associative algebra \( A \in \text{Alg} \mathcal{C}(T) \), the quasi-category \( \text{Mod}_A \mathcal{C}(T) \) left-tensored over \( \mathcal{C}(T) \) of right \( A \)-modules in \( \mathcal{C}(T) \) has \( \tau \)-descent.
Proof. (1) Let $\Delta^+ \xrightarrow{\Omega} K$ be the Čech nerve of a $\tau$-covering. We need to show that the restriction map

$$
\Gamma_{\text{coC}}(U_0^{\text{op}}, \text{LTens}^{\text{desc }} \mathcal{C}) \xrightarrow{r^{\text{desc}}} \Gamma_{\text{coC}}(U_0^{\text{op}}, \text{LTens}^{\text{desc }} \mathcal{C})
$$

is an equivalence. By Corollary 4.9, the restriction map on coCartesian sections

$$
\Gamma_{\text{coC}}(U_0^{\text{op}}, \text{LTens} \mathcal{C}) \xrightarrow{r_{\text{coC}}} \Gamma_{\text{coC}}(U_0^{\text{op}}, \text{LTens} \mathcal{C})
$$

has a fully faithful right adjoint $\lim_{\text{coC}}$. By Lemma 4.16, this map restricts to a fully faithful right adjoint of $r^{\text{desc}}$. In other words, for any descent diagram $\mathcal{M} \in \Gamma_{\text{coC}}(U_0^{\text{op}}, \text{LTens}^{\text{desc }} \mathcal{C})$, the counit map $r^{\text{desc}} \lim_{\text{coC}} \mathcal{M} \to \mathcal{M}$ is an equivalence. Let $\mathcal{N} \in \Gamma_{\text{coC}}(U_0^{\text{op}}, \text{LTens}^{\text{desc }} \mathcal{C})$ be an effective descent diagram. Since $\mathcal{N}_{-1}$ is a presentable quasi-category left-tensored over $\mathcal{C}(U_{-1})$ with descent, by definition the unit map $\mathcal{N} \to \lim_{\text{coC}} r^{\text{desc}} \mathcal{N}$ is an equivalence too. This shows that $r^{\text{desc}}$ is an equivalence of quasi-categories, as claimed.

(2) The presentable quasi-category $\text{Mod}_A \mathcal{C}(T)$ left-tensored over $\mathcal{C}(T)$ has a left dual [Lur16 Remark 4.8.4.8]. Therefore it has $\tau$-descent by Proposition 4.12 (2).

□

Lemma 4.16. Let $\mathcal{M} \in \Gamma_{\text{coC}}(U_0^{\text{op}}, \text{LTens}^{\text{desc }} \mathcal{C})$ be a descent diagram of presentable quasi-categories left-tensored over $\mathcal{C}$ with descent. Then its $q$-limit $\mathcal{M} \in \text{LTens} \mathcal{C}(U_{-1})$ also has descent.

Proof. Let’s consider the entire $q$-limit diagram $\mathcal{M} \in \Gamma_{\text{coC}}(U_0^{\text{op}}, \text{LTens} \mathcal{C})$. Let $X = U_{-1}$. Let $\Delta^+ \xrightarrow{\Omega} K$ be the Čech nerve of a $\tau$-covering with $\bar{\mathcal{V}}_{-1} = X$. We need to show that the unit map

$$
\mathcal{M}_{-1} \to \lim_{n \geq 0} \mathcal{C}(V_n) \otimes_{\mathcal{C}(X)} \mathcal{M}_{-1}
$$

is an equivalence. For $m, n \geq -1$, let $W_{mn} = U_m \times_X V_n \in K$, $\mathcal{M}_{mn} = \mathcal{C}(W_{mn}) \otimes_{\mathcal{C}(X)} \mathcal{M}_{-1}$, let

$$
\mathcal{M}_{-1} \xrightarrow{u^h_{-1}} \lim_{m \geq 0} \mathcal{M}_{m,-1} \text{ and } \mathcal{M}_{-1} \xrightarrow{u^v_{-1}} \lim_{n \geq 0} \mathcal{M}_{-1,n}
$$

denote unit maps, and let

$$
u^h_n = \mathcal{C}(\bar{V}_n) \otimes_{\mathcal{C}(X)} u^h_{-1} \text{ and } \nu^v_m = \mathcal{C}(\bar{U}_m) \otimes_{\mathcal{C}(X)} u^v_{-1}.
$$

Then we have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_{-1} & \xrightarrow{u^h_{-1}} & \lim_{m \geq 0} \mathcal{M}_{m,-1} \\
\downarrow u^v_{-1} & & \downarrow \lim_{n \geq 0} \nu^v_m \\
\lim_{n \geq 0} \mathcal{M}_{-1,n} & \xrightarrow{\lim_{m \geq 0} \nu^h_m} & \lim_{m,n \geq 0} \mathcal{M}_{mn}.
\end{array}
$$

□

Since $\mathcal{M}$ is a limit diagram, $u^h_{-1}$ is an equivalence. Since $\mathcal{M}_{m,-1} = \mathcal{M}_m$ has descent for $m \geq 0$, $\lim_{m \geq 0} \nu^v_m$ is an equivalence. Since $u^h_{-1}$ is an equivalence, for $n \geq 0$, as base changes in $q$ commute with $\tau$-descent data, the map $u^h_{-1}$ is also an equivalence. Therefore, $\lim_{n \geq 0} \nu^h_m$ is an equivalence. All this implies that $u^v_{-1}$ is an equivalence, which is what we needed to prove.
Corollary 4.17. The co-family \( \text{op} \, \text{LTens}^\text{desc}_\ast (\mathcal{C}) \) of pointed left-tensored quasi-categories with descent satisfies \( \tau \)-descent too.

**Proof.** During this proof, we will use the alternative join [Lur09 §4.2.1] to work with the classifying object \((\text{Pr}^\text{Mod})^\text{tri}\) and thus the subobject \(\text{LTens}^\text{desc}_\ast \mathcal{C} \) on pointed left-tensored quasi-categories with descent of its pullback \(\text{LTens}^\text{pos} \mathcal{C} \) along \( K^\text{op} \to \text{Pr}^\text{Mon} \). Let \( \Delta^+ \to K \) be the Čech nerve of a \( \tau \)-covering. We claim that the restriction map

\[
\Gamma_{\text{Cart}}(U^+_\cdot, \text{op} \, \text{LTens}^\text{desc}_\ast \mathcal{C}) \xrightarrow{r} \Gamma_{\text{Cart}}(U^+_\cdot, \text{op} \, \text{LTens}^\ast \mathcal{C})
\]

is a trivial fibration. Let \( K'' \subset K' \) be an inclusion of simplicial sets. The lifting problem

\[
\begin{array}{ccc}
K'' & \xrightarrow{r} & \Gamma_{\text{Cart}}(U^+_\cdot, \text{op} \, \text{LTens}^\text{desc}_\ast \mathcal{C}) \\
\cap & \searrow & \downarrow r \\
K' & \xrightarrow{r} & \Gamma_{\text{Cart}}(U^+_\cdot, \text{op} \, \text{LTens}^\ast \mathcal{C})
\end{array}
\]

is equivalent to the lifting problem

\[
(K'' \times \Delta^1) \cup (K' \times \Delta^0) \xrightarrow{r} \Gamma_{\text{Cart}}(U^+_\cdot, \text{op} \, \text{LTens}^\ast \mathcal{C})
\]

which has a solution as \( \text{op} \, \text{LMod} \mathcal{C} \) has \( \tau \)-descent, and thus \( r \) is a trivial fibration. This proves the claim.

\( \square \)

Corollary 4.18. The co-family \( \text{op} \, \text{LTens}^\text{desc}_\ast \mathcal{C} \) of presentable left-tensored quasi-categories with descent pointed by dualizable generators satisfies \( \tau \)-descent too.

**Proof.** By Corollary 4.17, the restriction map

\[
\Gamma_{\text{coc}}(U^\text{op}_\cdot, \text{LTens}^\text{desc}_\ast \mathcal{C}) \to \Gamma_{\text{coc}}(U^\text{op}_\cdot, \text{LTens}^\ast \mathcal{C})
\]

is a trivial Kan fibration. Therefore, it is enough to show that if for an effective descent datum \((\mathcal{M}, \mathcal{M}) \in \Gamma_{\text{coc}}(U^\text{op}_\cdot, \text{LTens}^\text{desc}_\ast \mathcal{C})\), for all \( n \geq 0 \) the object \( M_n \in \mathcal{M} \) is a dualizable generator, then the limit \( \mathcal{M}_{-1} \in \mathcal{M}_{-1} \) is a dualizable generator too. We have a counit map

\[
(\mathcal{C}, \text{Mod}_{\text{End} \mathcal{M}} \mathcal{C}, \text{End} \mathcal{M}) \to (\mathcal{C}, \mathcal{M}, \mathcal{M})
\]

natural in \((\mathcal{C}, \mathcal{M}, \mathcal{M}) \in \text{Pr}^\text{Mod}_\mathcal{M} \) [Lur16 Theorem 4.8.5.11] giving an edge \( v_{(\mathcal{C}, \mathcal{M}, \mathcal{M})} \) in the coCartesian section quasi-category \( \Gamma_{\text{coc}}(U^\text{op}_\cdot, \text{LTens}^\text{desc}_\ast \mathcal{C}) \). We claim that the edge \( v_{(\mathcal{C}, \mathcal{M}, \mathcal{M})} \) is actually in \( \Gamma_{\text{coc}}(U^\text{op}_\cdot, \text{LTens}^\text{desc}_\ast \mathcal{C}) \). This follows from Theorem 4.15(2) and Lemma 4.19. This shows that the map

\[
(\mathcal{C}(\mathcal{U}_{-1}), \text{Mod}_{\text{End} \mathcal{M}_{-1}} \mathcal{C}(\mathcal{U}_{-1})) \to (\mathcal{C}(\mathcal{U}_{-1}), \mathcal{M}_{-1}, \mathcal{M}_{-1})
\]
is the limit of the maps
\[
(C(U_n), \text{Mod}_{\text{End} M_n} C(U_n), \text{End} M_n) \xrightarrow{v(C(U_n), \text{End} M_n)^\circ} (C(U_n), \text{End} M_n, M_n).
\]
Therefore, since the maps \(v(C(U_n), \text{End} M_n)\) are equivalences for all \(n \geq 0\), so is their limit \(v(C(U_{-1}), \text{End} M_{-1})\), as we needed to show.

\[\square\]

**Lemma 4.19.** Let \(C^\circ \xrightarrow{f} (C')^\circ\) be a morphism of presentable monoidal quasi-categories. Let \(\mathcal{M}^\circ\) be a presentable quasi-category left-tensored over \(C^\circ\). Let \(M \in \mathcal{M}\) be an object. Then the base change
\[
(C' \otimes_C \text{Mod}_{\text{End} M} C, 1_{C'} \otimes \text{End} M) \xrightarrow{C' \otimes_C (\text{End} M)^M} (C' \otimes_C \mathcal{M}, 1_{C'} \otimes M)
\]
events \(1_{C'} \otimes \text{End} M \in \text{Alg}(C')\) as an endomorphism algebra of \(1_{C'} \otimes M \in C' \otimes C\).

**Proof.** Take a coCartesian edge
\[
\text{Mod}_{\text{End} M} C \to C' \otimes_C \text{Mod}_{\text{End} M} C
\]
in \(\text{Pr}^{\text{Mod}}\) over \(f\). Postcomposing it with the natural equivalence \(C' \otimes_C \text{Mod}_{\text{End} M} C \to \text{Mod}_{\text{End} M}(C')\) \([\text{Lur16}, \text{Theorem 4.8.4.6}]\), we get a coCartesian edge in \(\text{Pr}^{\text{Mod}}\) over \(f\) of the form
\[
\text{Mod}_{\text{End} M} C \to \text{Mod}_{\text{End} M}(C').
\]
Since the Morita functor \(\text{Pr}^{\text{Alg}} \to \text{Pr}^{\text{Mod}}\) takes the coCartesian edge \(\text{End} M \to 1_{C'}\), in \(\text{Pr}^{\text{Alg}}\) over \(f\) to the coCartesian edge
\[
\text{Mod}_{\text{End} M} C \to \text{Mod}_{1_{C'} \otimes \text{End} M}(C'),
\]
in \(\text{Pr}^{\text{Mod}}\) over \(f\) \([\text{Lur16}, \text{Proposition 4.8.5.1}]\), we get an equivalence
\[
C' \otimes_C \text{Mod}_{\text{End} M} C \simeq \text{Mod}_{1_{C'} \otimes \text{End} M}(C').
\]
Let
\[
\text{End} M \otimes M \to M
\]
be a map in \(\mathcal{M}\) exhibiting \(\text{End} M \in C\) as an endomorphism object of \(M\). Then the coCartesian edge
\[
\mathcal{M} \to C' \otimes_C \mathcal{M}
\]
in \(\text{Pr}^{\text{Mod}}\) over \(f\) takes it to a map
\[
(1_{C'} \otimes \text{End} M) \otimes (1_{C'} \otimes M) \to (1_{C'} \otimes M)
\]
in \(C' \otimes C\) exhibiting \(1_{C'} \otimes \text{End} M \in C'\) as an endomorphism object of \(1_{C'} \otimes M \in C' \otimes C\), giving an equivalence
\[
\text{Mod}_{1_{C'} \otimes \text{End} M} C' \simeq \text{Mod}_{\text{End}(1_{C'} \otimes M)} C'
\]
and thus concluding the proof.\[\square\]
4.2. Homotopical Skolem–Noether Theorem.

**Theorem 4.20** (Homotopical Skolem–Noether Theorem). Let $K$ be a quasi-category with final object $S$, let $	au$ be a Grothendieck topology on it. Let $\text{Cart}^\tau_K$ denote the quasi-category of Cartesian fibrations over $K$ with $\tau$-descent. Let $\mathcal{C} \subseteq \text{Cart}^\tau_K$ denote the full subcategory on right fibrations over $K$ with $\tau$-descent, which is an $\infty$-topos. Let $\mathcal{C} \rightarrow \text{Assoc} \times K^\text{op}$ be a family of presentable monoidal quasi-categories with $\tau$-descent. Then the following assertions hold:

1. We have a fibre sequence in $(\text{Cart}^\tau_K)^*$:
   $$(\text{Mod}(\text{AzC}, \text{O})) \rightarrow (\text{op} \text{AzC}, \text{End}(\text{O})) \rightarrow (\text{op} \text{LTensor} \text{AzC}, \text{Mod}(\text{O})).$$
2. Let $E \in \mathcal{C}$. Then we have a fibre sequence in $(\text{Cart}^\tau_K)^*$:
   $$(\text{Mod}(\text{Pic} \mathcal{C}, \text{O})) \rightarrow (\text{op} \text{Pic} \mathcal{C}, \text{End}(E)) \rightarrow (\text{op} \text{AzC}, \text{End}(E)).$$
3. We have a long exact sequence of homotopy sheaves in $h \mathcal{X}$:
   $$\cdots \rightarrow \pi_2(\text{op} \text{AzC}, \text{End}(E)) \rightarrow \pi_1(\text{op} \text{Pic} \mathcal{C}, \text{O}) \rightarrow \pi_1(\text{op} \mathcal{C} \Rightarrow \text{C} \Rightarrow \text{dgen}, E) \rightarrow \pi_0(\text{op} \text{Pic} \mathcal{C}) \rightarrow \pi_0(\text{op} \math{C} \Rightarrow \text{C} \Rightarrow \text{dgen}, E) \rightarrow \pi_0(\text{op} \text{LTensor} \text{AzC}) \rightarrow 0.$$

**Proof.** I) Consider the homotopy commutative diagram

![Diagram](image)

where

- $i$, is the inclusion $C \mapsto (\mathcal{C}, C)$,
- $\text{Mod}_*(A) = (\text{Mod}_A, A)$,
- $c \in \mathcal{C}$ is a Cartesian section with $c(S) = C_S$,
- $v(C)$ is the equivalence $\text{Mod}_{\text{End}C} \xrightarrow{\otimes \text{End}C} I_C$ as in Corollary 2.25, and
- $v(C)$ is its pointed version $\text{Mod}_{\text{End}C}$,

The forgetful map $\text{LTensor}_{\mathcal{C}} \rightarrow \text{LTensor}_{\mathcal{C}}$ is a left fibration, thus so is its restriction to connected components $\text{LTensor}_{\mathcal{C}} \rightarrow \text{LTensor}_{\mathcal{C}}$. Therefore, the square under $i$, which is strict Cartesian by construction, is homotopy Cartesian. We claim that $\text{Mod}_*(A) \rightarrow \text{LTensor}_{\mathcal{C}}$ is an equivalence. This will imply that the square under $\text{Mod}_*$ is homotopy Cartesian, and thus by the pasting lemma, the square under $\text{End}$ is homotopy Cartesian as well, as we needed to show.

By definition of dualizable generators and Theorem 4.15(2), we know that $\text{Alg}_\mathcal{C} \rightarrow \text{LTensor}_{\mathcal{C}}$ is an equivalence. Therefore, it is enough to show that, for any $U \in K$, an algebra $A \in \text{Alg}_\mathcal{C}(U)$ is
Azumaya if and only if \( \text{Mod}_A \in \text{LTens}^{\text{desc}} \mathcal{C}(U) \) is locally trivial. This in turn follows from Lemma 4.21.

II) Consider now the diagram

\[
\begin{array}{ccc}
\text{Pic} \mathcal{C} & \xrightarrow{\otimes E} & K \\
\downarrow{\varepsilon_{\text{End}, E}} & & \downarrow{\varepsilon_{\text{End}, E}} \\
\mathcal{C}^{\simeq}_{\text{dgen}} & \xrightarrow{\text{End}} & \text{Az} \mathcal{C},
\end{array}
\]

where \( \varepsilon_{\text{End}, E} \) is a Cartesian section such that \( \varepsilon_{\text{End}, E}(S) = \text{End} E \). We claim that the diagram is homotopy Cartesian. By the pasting lemma, and that by the homotopical Eilenberg–Watts theorem, we have \( \Omega(\text{LTens} \mathcal{C}, \mathcal{C}) = \text{Pic} \mathcal{C} \), it is enough to show that the square is homotopy commutative. For that, as the map \( \text{Az} \mathcal{C} \xrightarrow{\text{Mod}_{\text{dgen}}} \text{LTens}^{\text{desc}} \mathcal{C} \) is an equivalence, it will be enough to show that the diagram

\[
\begin{array}{ccc}
\text{Pic} \mathcal{C} & \xrightarrow{\otimes E} & K \\
\downarrow{\varepsilon_{\text{End}, E}} & & \downarrow{\varepsilon_{\text{End}, E}} \\
\mathcal{C}^{\simeq}_{\text{dgen}} & \subset & \text{LTens}^{\text{desc}} \mathcal{C}
\end{array}
\]

is homotopy commutative. That is, we need to supply a map

\[
\text{Pic} \mathcal{C} \times \Delta^1 \xrightarrow{f} \text{LTens}^{\text{desc}} \mathcal{C}
\]

such that \( f((\text{Pic} \mathcal{C} \times \Delta^0)) \) is the constant map \( c \) with value \((\mathcal{C}, E)\), and \( f((\text{Pic} \mathcal{C} \times \Delta^1)) \) is the map \( L \mapsto (\mathcal{C}, L \otimes E) \). The homotopical Eilenberg–Watts theorem supplies the map \( \mathcal{C} \xrightarrow{F \mapsto \otimes F} \text{Fun}_\mathcal{C}(\mathcal{C}, \mathcal{C}) \). This gives the map \( \text{EW} \) in the lifting problem

\[
\begin{array}{ccc}
\text{Pic} \mathcal{C} \times \Delta^0 & \xrightarrow{\otimes E} & \text{LTens}^{\text{desc}} \mathcal{C} \\
\downarrow{f} & & \downarrow{r} \\
\text{Pic} \mathcal{C} \times \Delta^1 & \xrightarrow{\otimes E} & \text{LTens}^{\text{desc}} \mathcal{C}
\end{array}
\]

where the restriction map \( r \) is a left fibration, thus we have a solution \( f \), which as \( E \) is a dualizable generator, maps into \( \text{LTens}^{\text{desc}} \mathcal{C} \), as required.
III) From the fibre sequences, we readily get a long exact sequence

\[ \cdots \to \pi_2^{(\text{op} \text{ Az}\, \mathcal{C}, \text{End} \, E)} \to \pi_1^{(\text{op} \text{ Pic}\, \mathcal{C}, \mathcal{O})} \to \pi_1^{(\text{op} \text{ Cgen}\, \mathcal{C}, \text{End} \, E)} \to \pi_1^{(\text{op} \text{ Az}\, \mathcal{C}, \text{End} \, E)} \to \]

\[ \to \pi_0^{(\text{op} \text{ Pic}\, \mathcal{C})} \to \pi_0^{(\text{op} \text{ Cgen}\, \mathcal{C})} \to \pi_0^{(\text{op} \text{ Az}\, \mathcal{C})} \to \pi_0^{(\text{op} \text{ LTens}\, \text{Az}\, \mathcal{C})} \].

Since we have \( \text{AZ} \, \mathcal{C} \simeq \text{LTens}^{\text{Az}} \, \mathcal{C} \), by definition of dualizable generators, we get that the map of homotopy sheaves \( \pi_0^{(\text{op} \text{ Az}\, \mathcal{C})} \to \pi_0^{(\text{op} \text{ LTens}\, \text{Az}\, \mathcal{C})} \) is surjective.

\[ \square \]

Lemma 4.21. Let \( \mathcal{C} \) be a monoidal quasi-category, and \( A \in \text{Alg} \, \mathcal{C} \). Then there exists \( E \in \text{Cgen} \) and \( A \simeq \text{End} \, E \) if and only if there exists \( \text{Mod}_A \simeq \mathcal{C} \) in \( \text{LTens} \, \mathcal{C} \).

Proof. \( \Rightarrow \): Suppose that there exists \( E \in \text{Cgen} \) and an equivalence \( A \xrightarrow{\phi} \text{End} \, E \) in \( \text{Alg} \, \mathcal{C} \). Then we get equivalences in \( \text{LTens} \, \mathcal{C} \)

\[ \text{Mod}_A \xrightarrow{\text{Mod}(\phi)} \text{Mod}_{\text{End} \, E} \xrightarrow{\text{End} \, E} \mathcal{C} \]

as needed.

\( \Leftarrow \): Suppose that there exists an equivalence \( \text{Mod}_A \xrightarrow{\phi} \mathcal{C} \) in \( \text{LTens} \, \mathcal{C} \). By the homotopical Eilenberg–Watts theorem, \( \phi \) is of the form \( \otimes_A E \) for some \( E \in \text{Mod} \). It will be enough to show that the action map \( A \otimes E \xrightarrow{\alpha} E \) exhibits \( A \) as an endomorphism object of \( E \in \mathcal{C} \). Let \( C \in \mathcal{C} \) and consider the diagram

\[ \text{Map}_\mathcal{C}(C, A) \xrightarrow{\otimes E} \text{Map}_\mathcal{C}(C \otimes E, A \otimes E) \xrightarrow{\otimes \circ} \text{Map}_\mathcal{C}(C \otimes E, E) \]

\[ \text{Map}_A(C \otimes A, A \otimes A) \xrightarrow{\mu \circ} \text{Map}_A(C \otimes A, A), \]

where \( A \otimes A \xrightarrow{\mu} A \) is the multiplication map. We know that the left triangle is homotopy commutative. We claim that the right square is homotopy commutative. That will conclude the proof as \( (\mu \circ) \circ (\otimes A) \) is an equivalence as \( \mu \) induces the counit of the induction \( \mathcal{C} \xrightarrow{\neg} \text{Mod}_A \), and \( \otimes_A E \) is an equivalence by assumption.

The claim follows from that the diagram

\[ A \text{Mod}_A \xrightarrow{\otimes_A E} \text{Mod}_A \]

\[ A \text{Mod} \xrightarrow{\otimes_A E} \mathcal{C}, \]

where the vertical maps are restriction maps, is commutative, and therefore the functor \( \otimes_A E \) takes \( A \in A \text{Mod}_A \) to \( E \in A \text{Mod} \).

\[ \square \]
5. Applications

In this section, we list a number of interesting families of monoidal quasi-categories we can apply the Homotopical Skolem–Noether Theorem to. In Subsection 1 we recall the the symmetric monoidal quasi-category of stable presentable quasi-categories [Lur16, §4.8.2] as all the presentable monoidal quasi-categories in our examples are stable. In Subsection 2 we recall the monoidal structure on the underlying quasi-category of a monoidal model category [Lur16, §4.1.7] as that will serve as our main device of getting examples. In Subsection 3 we apply our main result to Algebraic Geometry. This includes proving 1-descent for pre-generalized Azumaya algebras over a quasi-compact and quasi-separated scheme. In Subsection 4 we apply our main result to Derived and Spectral Algebraic Geometry. In Subsection 5 we apply our main result to ind-coherent sheaves and crystals.

5.1. The symmetric monoidal quasi-category of stable presentable quasi-categories.

5.1.1. Stable quasi-categories.

Definition 5.1. Let \( \mathcal{C} \) be a quasi-category. Then an object \( 0 \in \mathcal{C} \) is a zero object if it is both an initial object and a final object. We say that the quasi-category \( \mathcal{C} \) is pointed if it has a zero object. We say that the quasi-category \( \mathcal{C} \) is stable if the following assertions hold:

1. The quasi-category \( \mathcal{C} \) is pointed.
2. The quasi-category \( \mathcal{C} \) has finite limits and colimits.
3. A square

\[
\begin{align*}
C' & \longrightarrow C \\
D' & \longrightarrow D
\end{align*}
\]

in \( \mathcal{C} \) is a pushout diagram if and only if it is a pullback diagram.

Definition 5.2. Let \( \mathcal{C} \) be a stable quasi-category. For a nonnegative integer \( n \geq 0 \) we let \( \mathcal{C} \overset{\Sigma^n}{\longrightarrow} \mathcal{C} \) denote the \( n \)-th power of the suspension functor \( \mathcal{C} \overset{\Sigma}{\longrightarrow} \mathcal{C} \). For a nonpositive integer \( n \leq 0 \) we let \( \mathcal{C} \overset{\Omega^n}{\longrightarrow} \mathcal{C} \) denote the \((-n)\)-th power of the loop object functor \( \mathcal{C} \overset{\Omega}{\longrightarrow} \mathcal{C} \). We refer to these functors as translation functors.

Consider a diagram

\[
C \overset{f}{\rightarrow} D \overset{g}{\rightarrow} E \overset{h}{\rightarrow} C[1]
\]

in the homotopy category \( \text{Ho} \mathcal{C} \). Then we say that it is a distinguished triangle if there exists a diagram.
in $\mathcal{C}$ such that the following assertions hold:

1. The squares are pushout diagrams in $\mathcal{C}$.
2. The object $0,0' \in \mathcal{C}$ are zero objects.
3. The map $f$ represents $\bar{f}$ and the map $g$ represents $\bar{g}$.
4. Since $F$ is a suspension of $C$, there exists an equivalence $F \xrightarrow{h'} C[1]$ well-defined up to homotopy. Then the composite $h' \circ h$ represents $\bar{h}$.

**Theorem 5.3.** [Lur16, Theorem 1.1.2.14] Let $\mathcal{C}$ be a stable quasi-category. Then the homotopy category $Ho\mathcal{C}$ can be endowed with a triangulated category structure as follows:

1. Since $\mathcal{C}$ is pointed, for objects $C,D \in \mathcal{C}$ the zero map $C \to 0 \to D$ is a natural base point for the mapping space $\text{Map}(C,D)$.
2. The isomorphism $\pi_0 \text{Map}(C,D) \cong \pi_2(\Sigma^2 C,D)$ equips the set $\text{Hom}_{Ho\mathcal{C}}(C,D) \cong \pi_0 \text{Map}_{\mathcal{C}}(C,D)$ with an abelian group structure.
3. We have the translation functors and distinguished triangles defined above.

**Definition 5.4.** Let $\mathcal{C}$ be a stable quasi-category. Then we say that $\mathcal{C}$ is **compactly generated** if the triangulated category $Ho \mathcal{C}$ is compactly generated.

5.1.2. **Spectrum objects and the quasi-category of spectra.**

**Definition 5.5.** Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor between quasi-categories.

1. Suppose that $\mathcal{C}$ has pushouts. Then we say that $F$ is **excisive** if it takes pushout diagrams to pullback diagrams.
2. Suppose that $\mathcal{C}$ has a final object $* \in \mathcal{C}$. Then we say that $F$ is **reduced** if $F(*) \in \mathcal{D}$ is a final object.

Suppose that $\mathcal{C}$ has pushouts and a final object. Then we denote by $\text{Exc}_*(\mathcal{C}, \mathcal{D}) \subseteq \text{Fun}(\mathcal{C}, \mathcal{D})$ the full subcategory on excisive reduced functors.

Let $\mathcal{I}^{\text{fin}} \subseteq \mathcal{I}$ denote the smallest full subcategory that contains the final object $* \in \mathcal{I}$ and it is closed under small colimits. Let $\mathcal{I}_*^{\text{fin}} \subseteq \mathcal{I}_*$ denote the full subcategory on pointed objects of $\mathcal{I}_*^{\text{fin}}$. Let $\mathcal{C}$ be a quasi-category that has finite limits. Then the **quasi-category of spectrum objects of $\mathcal{C}$** is $\text{Sp}(\mathcal{C}) = \text{Exc}_*(\mathcal{I}^{\text{fin}}, \mathcal{C})$. The **quasi-category of spectra** is $\text{Sp} = \text{Sp}(\mathcal{I})$. We let $\text{Sp}(\mathcal{C}) \xrightarrow{\Omega^\infty} \mathcal{C}$ denote the functor given by substitution at $S^0 \in \mathcal{I}_*^{\text{fin}}$.

**Proposition 5.6.** [Lur16, Proposition 1.4.4.4 and Corollary 1.4.4.5] Let $\mathcal{C}$ be a presentable quasi-category. Then the following assertions hold:

1. The quasi-category $\text{Sp}(\mathcal{C})$ of spectrum objects of $\mathcal{C}$ is presentable.
2. The functor $\text{Sp}(\mathcal{C}) \xrightarrow{\Omega^\infty} \mathcal{C}$ has a left adjoint $\Sigma_+^\infty$. 

\[\begin{array}{ccc}
C & \xrightarrow{f} & D & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0' & \xrightarrow{g} & E & \xrightarrow{h} & F
\end{array}\]
(3) Let $\mathcal{D}$ be a presentable stable quasi-category. Then the precomposition map

$$\text{LFun}(\text{Sp}(\mathcal{C}), \mathcal{D}) \xrightarrow{\Sigma^\infty +} \text{LFun}(\mathcal{C}, \mathcal{D})$$

is an equivalence.

**Definition 5.7.** The sphere spectrum is $S = \Sigma^\infty(*) \in \text{Sp}$.

5.1.3. Idempotent objects and the symmetric monoidal structure on the quasi-category $\text{Pr}^{\text{St}}$ of presentable stable quasi-categories.

**Definition 5.8.** Let $\mathcal{C}$ be a monoidal quasi-category. Then an idempotent object in $\mathcal{C}$ is a morphism $1 \xrightarrow{c} \mathcal{C}$ such that the maps in $\mathcal{C}$:

$$C \simeq C \otimes 1 \xrightarrow{id_c \otimes c} C \otimes C \text{ and } C \simeq 1 \otimes C \xrightarrow{c \otimes id_c} C \otimes C$$

are equivalences.

**Proposition 5.9.** [Lur16, Proposition 4.8.2.4] Let $\mathcal{C}$ be a monoidal quasi-category and $1 \xrightarrow{c} \mathcal{C}$. Then the following assertions are equivalent:

1. The map $c$ is an idempotent object of $\mathcal{C}$.
2. Consider the endofunctor $\mathcal{C} \xrightarrow{C} \mathcal{C} \otimes -$. Then the natural transformation $id \xrightarrow{c} (C \otimes -)$ induced by $c$ exhibts $(C \otimes -)$ as a localization functor.

**Proposition 5.10.** [Lur16, Proposition 4.8.2.7] Let $\mathcal{C}$ be a symmetric monoidal quasi-category. Let $1 \xrightarrow{c} \mathcal{C}$ be an idempotent object of $\mathcal{C}$. Let $L$ denote the endofunctor $\mathcal{C} \xrightarrow{C} \mathcal{C} \otimes -$. Let $L\mathcal{C} \subseteq \mathcal{C}$ be the full subcategory on objects of the form $C_1 \oplus \cdots \oplus C_n$ where each $C_i$ is in $L\mathcal{C}$. Then the composite $L\mathcal{C} \otimes - \rightarrow \mathcal{C} \otimes - \rightarrow \text{Fin}_*$ of the inclusion map and the structure map gives $L\mathcal{C}$ a symmetric monoidal structure.

**Definition 5.11.** Let $A \in \text{CAlg} \mathcal{C}$ be a commutative algebra object in a symmetric monoidal quasi-category. Then it is idempotent if the multiplication map $A \otimes A \rightarrow A$ is an equivalence. Let $\text{CAlg}^{\text{idem}} \mathcal{C} \subseteq \text{CAlg} \mathcal{C}$ denote the full subcategory on idempotent commutative algebra objects.

**Proposition 5.12.** [Lur16, Proposition 4.8.2.9] Let $\mathcal{C} \otimes -$ be a symmetric monoidal quasi-category. Then the composite of canonical maps

$$\text{CAlg}^{\text{idem}} \mathcal{C} \rightarrow \text{CAlg} \mathcal{C} \rightarrow \text{CAlg}(\mathcal{C}) \\ 1 \rightarrow \mathcal{C}$$

is fully faithful with essential image the full subcategory on idempotent objects.

**Definition 5.13.** Let $(\mathcal{C}, C)$ be a pair of a presentable quasi-category $\mathcal{C}$ and an object $C \in \mathcal{C}$. We say that the pair $(\mathcal{C}, C)$ is idempotent if the colimit-preserving map $\mathcal{I} \xrightarrow{F} \mathcal{C}$ such that $F(*) = C$ is an idempotent object of $\text{Pr}^\infty$. In this case, Proposition 5.12 equips $\mathcal{C}$ with a symmetric monoidal structure with $C$ the unit object.

**Proposition 5.14.** [Lur16, Example 4.8.1.23] Let $\mathcal{C} \in \text{Pr}^\infty$ be a presentable quasi-category. Then we have an equivalence of presentable quasi-categories $\text{Sp} \otimes \mathcal{C} \simeq \text{Sp}(\mathcal{C})$.

**Corollary 5.15.** The pair $(\text{Sp}, S)$ is idempotent.
Proof. By Proposition 5.9 it is enough to show that the endofunctor \( P r^L \xrightarrow{L'(\epsilon) = Sp \otimes \epsilon} P r^L \) is a localization functor. By the Proposition it is equivalent to the endofunctor \( P r^L \xrightarrow{L'(\epsilon) = Sp(\epsilon)} P r^L \). By Proposition 5.6 (3) the map \( \Sigma^\infty \rightarrow \text{Sp} \) induces a natural transformation \( \text{id}_{P r^L} \rightarrow L' \) which exhibits \( L' \) as a localization functor. □

Proposition 5.16. [Lur16 1.4.2.21] Let \( \mathcal{C} \) be a quasi-category. Then it is stable if and only if the map \( \text{Sp} \xrightarrow{\Omega^\infty} \mathcal{C} \) is an equivalence.

Definition 5.17. The quasi-category of presentable stable quasi-categories is the full subcategory \( \text{Pr}^{\text{St}} \subseteq \text{Pr}^L \) on presentable stable quasi-categories. By Propositions 5.14 and 5.16 it is the essential image of the localization functor \( \text{Pr}^L \xrightarrow{L'(\epsilon) = Sp \otimes \epsilon} \text{Pr}^L \). By Corollary 5.15 it admits a symmetric model structure such that the following assertions hold:

1. The quasi-category \( \text{Sp} \) of spectra is a unit object of the symmetric monoidal quasi-category \( (\text{Pr}^{\text{St}})^\otimes \).
2. The natural inclusion map \( (\text{Pr}^{\text{St}})^\otimes \rightarrow (\text{Pr}^L)^\otimes \) is symmetric monoidal.

By Proposition 5.12 the quasi-category \( \text{Sp} \) of spectra admits a symmetric monoidal structure for which the sphere spectrum \( S \) is a unit object. We refer to the tensor product as the smash product of spectra.

Proposition 5.18. [Lur18 Proposition D.7.2.3] Let \( \mathcal{C} \) be a compactly generated stable quasi-category. Then it is a dualizable object of the symmetric monoidal quasi-category \( \text{Pr}^{\text{St}} \).

Remark 5.19. Since the natural inclusion map \( \text{Pr}^{\text{St}} \rightarrow \text{Pr}^L \) is symmetric monoidal, if \( \mathcal{C} \in \text{Pr}^{\text{St}} \) is dualizable, then it is also a dualizable object of \( \text{Pr}^L \).

5.2. The underlying quasi-category of a monoidal model category. Let \( X \) be a quasi-category. A system in \( X \) is a collection \( W \subseteq X_1 \) of morphisms which contains all equivalences, and it is stable under homotopy and composition. The collection of all systems on \( X \) forms a poset \( \text{Sys} X \). Therefore, we get a map \( \text{Cat}_{\infty}^{\otimes} \xrightarrow{X \mapsto N \text{Sys} X} \text{Cat}_{\infty} \), which classifies a Cartesian fibration \( W \text{Cat}_{\infty} \xrightarrow{q} \text{Cat}_{\infty} \). The objects of the quasi-category \( W \text{Cat}_{\infty} \) are pairs \( (X, W) \) where \( X \) is a quasi-category, and \( W \in \text{Sys} X \) is a system on \( X \). A mapping space \( \text{Map}_{W \text{Cat}_{\infty}}((X, W), (X', W')) \) can be identified with the full subcategory of the mapping space \( \text{Map}_{\text{Cat}_{\infty}}(X, X') \) on functors \( X \xrightarrow{f} X' \) such that \( f(W) \subseteq f(W') \).

The Cartesian fibration \( q \) is the forgetful map \( (X, W) \mapsto X \). It admits a section \( \text{Cat}_{\infty} \xrightarrow{G} W \text{Cat}_{\infty} \) sending a quasi-category \( X \) to the pair \( (X, W) \), where \( W \subseteq X_1 \) is the collection of weak equivalences in \( X \). The functor \( G \) admits a left adjoint \( \text{WCat}_{\infty} \xrightarrow{F(C, W) = [C^{-1}]W} \text{Cat}_{\infty} \), which moreover commutes with finite products [Lur16 Proposition 4.1.3.2]. We get the following universal property.

Proposition 5.20. Let \( (X, W) \in W \text{Cat}_{\infty} \) and let \( (X, W) \xrightarrow{u} G X[W^{-1}] \) denote the unit map. Then for any quasi-category \( Y \), the precomposition by \( u \) map

\[
\text{Fun}(X[W^{-1}], Y) \xrightarrow{u} \text{Fun}(X, Y)
\]
is fully faithful with essential image the collection of functors \( X \xrightarrow{f} Y \) such that \( f \) takes all morphisms in \( W \) to equivalences in \( Y \).

**Proof.** Since the right adjoint \( G \) is fully faithful, the precomposition by \( u \) map

\[
\text{Map}_{\text{Cat}_{\infty}}(X[W^{-1}], Y) \xrightarrow{\text{Map}_{\text{Cat}_{\infty}}(X, Y)} \text{Map}_{\text{Cat}_{\infty}}(X[W^{-1}], Y)
\]

is fully faithful with essential image the collection of functors \( X \xrightarrow{f} Y \) such that \( f \) takes all morphisms in \( W \) to equivalences in \( Y \). Since for a quasi-category \( Z \) the mapping space \( \text{Map}_{\text{Cat}_{\infty}}(Z, Y) \) is equivalent to the largest Kan complex in the functor quasi-category \( \text{Fun}(Z, Y) \), we get the statement about the essential image. Fully faithfulness follows from [Lur09, Proposition 3.1.3.3].

\( \square \)

Let \( A \) be a model category. Then the collection \( W \) of weak equivalences between cofibrant objects gives a system in the nerve \( N_{\text{A}}^c \) of the full subcategory on cofibrant objects. The underlying quasi-category of \( A \) is the localization \( \text{NA}^c[W^{-1}] \).

Suppose that the model structure on \( A \) is monoidal. Then there is an induced model structure on the full subcategory \( A^c \) on cofibrant objects. We get a monoidal structure on the nerve \( \text{NA}^c \), which in turn is classified by a monoid object in \( \text{Mon}_{\text{Assoc Cat}_{\infty}} \). Since the model structure on \( A \) is monoidal, the weak equivalences between cofibrant objects are preserved by tensor product, therefore the monoid object can be lifted to \( M \in \text{Mon}_{\text{Assoc WCat}_{\infty}} \). Since the left adjoint \( F \) preserves finite products, it preserves monoid objects. Therefore, the composite \( FM \) is a monoid object \( M \in \text{Mon}_{\text{Assoc Cat}_{\infty}} \). This equips the underlying quasi-category \( \text{NA}^c[W^{-1}] \) with a monoidal structure. We refer to this as the underlying monoidal quasi-category of the monoidal model category \( A \).

It comes equipped with a monoidal functor \( N(A^c)^\circ \xrightarrow{u} \text{NA}^c[W^{-1}]^\circ \) which satisfies the following universal property: for every monoidal quasi-category \( \mathcal{D}^\circ \), the precomposition by \( u \) functor

\[
\text{Fun}^\circ(\text{NA}^c[W^{-1}]^\circ, \mathcal{D}^\circ) \xrightarrow{\text{Map}_{\text{Cat}_{\infty}}(X, Y)} \text{Fun}^\circ(N(A^c)^\circ, \mathcal{D}^\circ)
\]

is fully faithful with essential image the full subcategory on monoidal functors \( N(A^c)^\circ \xrightarrow{f^\circ} \mathcal{D}^\circ \) which take morphisms in \( W \) to equivalences in \( \mathcal{D} \) [Lur16, Proposition 4.1.7.4].

**Construction 5.21** (The underlying presheaf of monoidal quasi-categories of a presheaf of monoidal categories with a system, and its extension via gluing). (1) Let \( K \) be a category, and let

\[
K \xrightarrow{k \mapsto (C(k)^\circ, W(k))} (\text{Mon}_{\text{Assoc WCat}_{\infty}})^{\text{op}}
\]

be a presheaf of monoidal categories with systems. Postcomposition with the opposite of the underlying quasi-category functor \( \text{Mon}_{\text{Assoc WCat}_{\infty}} \xrightarrow{F} \text{Mon}_{\text{Assoc Cat}_{\infty}} \) yields a presheaf of monoidal quasi-categories

\[
K \xrightarrow{k \mapsto C(k)^\circ[W(k)^{-1}]} (\text{Mon}_{\text{Assoc Cat}_{\infty}})^{\text{op}} = (\text{Cat}_{\infty}^{\text{Mon}})^{\text{op}}.
\]

(2) This functor can be extended to a colimit-preserving functor \( \mathcal{P}(K) \xrightarrow{\mathcal{D}^\circ} (\text{Cat}_{\infty}^{\text{Mon}})^{\text{op}} \) in a way that is unique up to homotopy [Lur09, Theorem 5.1.5.6]. This means the following. Take a presheaf
Let \( X \in \mathcal{P}(K) \). Then we have \( X \simeq \text{hocolim}_{i \in I} h_{k_i} \) for some diagram \( I \xrightarrow{k_i} K \) over some small simplicial set \( I \) [Lur09 Corollary 5.1.5.8]. The presheaf \( \mathcal{C}^\otimes \) satisfies
\[
\mathcal{C}(X)^\otimes = \text{holim}_{i \in I} \mathcal{C}(k_i)^\otimes
\]
in \( \text{Cat}_{\infty}^{\text{Mon}} \).

5.3. Algebraic Geometry.

**Construction 5.22.** Let \( S \) be a scheme. We will now apply Construction 5.21 to construct the co-Cartesian family \( \mathcal{Q}\mathcal{C}_S^\otimes \) of the monoidal quasi-categories of unbounded complexes of quasicoherent sheaves on the opposite of the quasi-category \( \text{St}_S \) of \( \infty \)-stacks on the big fppf site \( S_{\text{fppf}} \).

The quasi-category \( \text{St}_S \) is the localization at fppf-local equivalences of the quasi-category \( \mathcal{P}(\text{Aff}_S) \) of presheaves of spaces on the category \( \text{Aff}_S \) on affine \( S \)-schemes. For an affine \( S \)-scheme \( \text{Spec} A = T \in \text{Aff}_S \), the category \( \mathcal{C}(A) \) of unbounded complexes of \( A \)-modules can be equipped by the projective model structure, which is a combinatorial and monoidal model structure [Lur16 Propositions 7.1.2.8 and 7.1.2.11]. Moreover, for a morphism of \( S \)-algebras \( A \to B \) and a quasi-isomorphism \( f \) of \( \text{dg} \)-projective complexes of \( A \)-modules, the pullback \( f \otimes_A B \) is the derived pullback, and therefore it is also a quasi-isomorphism. This shows that we get a functor
\[
(\text{Aff}_S)^{\text{op}} \xrightarrow{A \mapsto [\mathcal{C}(A)^\otimes_{\text{dg-proj}} \text{qis}]} (A \mapsto B) \otimes_{A} B \xrightarrow{\otimes_{A} B} \text{Mon}^{\text{Assoc}} \text{WCat}_{\infty}.
\]

Since the projective model structure is combinatorial and monoidal, applying Construction 5.21 we get a functor \( \mathcal{P}(\text{Aff}_S) \xrightarrow{\mathcal{Q}\mathcal{C}_S^\otimes} \text{Pr}^{\text{Mon}} \) satisfying the following properties.

1. Let \( T = \text{Spec} A \) be an affine \( S \)-scheme. Then we have \( \mathcal{Q}\mathcal{C}_S(A)^\otimes \simeq \mathcal{C}(A)^{\text{dg-proj}}[\text{qis}^{-1}]^\otimes \).
2. Let \( T \in \text{St}_S \) be an \( \infty \)-stack over \( S \). Then it is some homotopy colimit \( T = \text{hocolim} T_i \) of affine \( S \)-schemes. We have \( \mathcal{Q}\mathcal{C}_S(T)^\otimes \simeq \text{holim} \mathcal{Q}\mathcal{C}_S(T_i)^\otimes \) in the quasi-category \( \text{Cat}_{\infty}^{\text{Mon}} \) of monoidal quasi-categories.

**Proposition 5.23.** Let \( S \) be a quasi-compact and quasi-separated scheme. Then the coCartesian family of presentable monoidal quasi-categories \( \mathcal{Q}\mathcal{C}_S^\otimes \xrightarrow{p} \text{St}_S^{\text{op}} \times \text{Assoc}^\otimes \) has fppf descent.

**Proof.** The Cartesian fibration of unbounded complexes of quasi-coherent sheaves is compactly generated [BB03 Theorem 3.1.1] satisfies fppf descent [Lur11a Corollary 6.13], [Lur16 Theorem 7.1.2.13]. Therefore, Proposition 5.18 and Corollary 4.13 show that \( p \) has fppf descent.

**Corollary 5.24 (Homotopical Skolem–Noether Theorem for schemes).** Let \( S \) be a quasi-compact and quasi-separated scheme. Let \( \text{Cart}_S^{\text{fppf}} \) denote the quasi-category of Cartesian fibrations on \( \text{St}_S \) which satisfy fppf descent.

1. Let \( \text{op} \text{TPerf}_S := \text{op} (\mathcal{Q}\mathcal{C}_S)_{\text{dg-proj}} \) denote the Cartesian fibration of totally supported perfect complexes on \( S \), \( \text{op} \text{Deraz}_S := \text{op} \mathcal{Q}\mathcal{C}_S \) the Cartesian fibration of derived Azumaya algebras on \( S \) and \( \text{op} \mathcal{D}^\text{Az}_S := \text{op} \text{LTensor}^\text{Az} \mathcal{Q}\mathcal{C}_S \) the Cartesian fibration of locally trivial presentable quasi-categories left-tensored over \( \mathcal{Q}\mathcal{C}_S^\otimes \). Then the sequence in \( (\text{Cart}_S^{\text{fppf}}, \otimes) \):
\[
(\text{op} \text{TPerf}_S, \otimes) \xrightarrow{\text{End}} (\text{op} \text{Deraz}_S, \otimes) \xrightarrow{\text{Mod}} (\text{op} \mathcal{D}^\text{Az}_S, \otimes)
\]
is a homotopy fibre sequence.

(2) Let \( E \in \text{TPerf}(S) \) be a totally supported perfect complex on \( S \). Then the sequence in \( (\text{Cart}_S^{\text{ppf}}) \):
\[
(\mathbf{B} G_m \times Z, \mathcal{O}) \xrightarrow{\otimes E} \text{End}^\text{op} \text{TPerf}_S, E) \xrightarrow{\text{End}} (\text{Deraz}_S, \text{End} E)
\]
is a homotopy fibre sequence.

(3) We have isomorphisms of sheaves of groups
\[
\pi_i \Omega(\text{Deraz}_S, \text{End} E) \cong \pi_i \Omega(\text{End} \text{Deraz}_S, \text{end} E)
\]
for \( i > 0 \), a short exact sequence of sheaves of groups
\[
1 \to G_m \xrightarrow{\text{ad}} \text{Aut}_{\text{End}} E \xrightarrow{\text{ad}} \text{Aut}_{\text{Deraz}}(\text{End} E) \to 1,
\]
and an exact sequence of pointed sheaves of sets
\[
\ast \to \pi_0(\mathbf{B} G_m \times Z) \xrightarrow{\otimes E} \pi_0 \text{TPerf}_S \xrightarrow{\text{End}} \pi_0 \text{Deraz}_S \xrightarrow{\text{Mod}} \pi_0 \text{Dg}^A_S \to \ast.
\]

**Proof.** By Proposition [5.23], the family \( QC_S^S \) satisfies the assumptions of Theorem [4.20]. Let \( T \) be an \( S \)-scheme and \( E \in \text{Perf}(T) \) a perfect complex on \( T \). Then \( T \) is a generator if and only if it is totally supported [Tho97, Lemma 3.14], [Lur16, Corollary 1.4.4.2]. Since being a dualizable generator is a local property, this shows that the full subcategories \( \text{TPerf}_T \) and \( (\text{QC}_S^S)_{\text{gen}} \) of \( \text{QC}_S^S \) agree. Moreover, \( E \) is an invertible element of \( \text{QC}(T) \) if and only if \( E \) is of the form \( \mathcal{L}[n] \) for an invertible sheaf \( \mathcal{L} \) and an integer \( n \in Z \). This shows that the inclusion \( \mathbf{B} G_m \times Z \to \text{Pic} \text{QC}_S \) is an equivalence. We have \( \pi_i(\mathbf{B} G_m \times Z) = 0 \) for \( i > 1 \), and if \( E \) is totally supported, then the map \( G_m \xrightarrow{\text{ad}} \text{Aut}_{\text{Perf}} E \) is injective. Finally, the map \( \pi_0(\mathbf{B} G_m \times Z) \xrightarrow{\otimes E} \pi_0 \text{TPerf}_S \) is injective, because the sheaf \( \pi_0 \mathbf{B} G_m \) is trivial, and for an integer \( n \in Z \), if we have \( E \cong E[n] \), then we get \( n = 0 \).

\[\square\]

**Remark 5.25.** Let \( E = \mathcal{O}_S^S \). Then \( E \) is totally supported. The short exact sequence
\[
1 \to G_m \to \text{Aut} E \to \text{Aut End} E \to 1
\]
is the one in the classical Skolem–Noether Theorem [Gir71, V, Lemme 4.1]:
\[
1 \to G_m \to GL_n \to PGL_n \to 1.
\]

**Remark 5.26.** Let’s show how this result implies Lieblich’s Derived Skolem–Noether Theorem [Lie09, Theorem 5.1.5]. Let \( T \) be an \( S \)-scheme, and \( E, F \) two nonzero perfect complexes on \( T \). The annihilator \( \text{Ann} E \) of \( E \) is the kernel of the scalar multiplication map \( \mathcal{O}_T \to \text{End} E \). It is the ideal sheaf of the support of \( E \). Note that this shows \( \text{Supp} E = \text{Supp} \text{End} E \). We let \( \mathcal{O}_E = \mathcal{O}_T / \text{Ann} E \).

We need to show that there exists a unique integer \( n \in Z \) such that the map of sheaves
\[
\pi_0 \text{Isom}_{\text{Perf}(T)}(E[n], F) \to \text{Isom}_{\text{Deraz}(T)}(\text{End} E, \text{End} F)
\]
is surjective, with each fibre being an \( \mathcal{O}_E^S \)-torsor, which in case \( E = F \) is split.

1) Suppose first that \( E \) is totally supported. Take a zigzag of weak equivalences of algebras \( \phi : \text{End} E \cong \text{End} F \). Then it determines a \( T \)-point of the homotopy fibre product and thus we get a dashed arrow in the following diagram:
That is, the equivalence of algebras $\phi$ is homotopically to the image by $\mathbb{R}\text{End}$ of an equivalence of perfect complexes $E \otimes \mathcal{L}[n] \cong F$. We get a preimage of the required form if we restrict to a trivializing cover of the invertible sheaf $\mathcal{L}$. The unicity of $n$ follows from the injectivity of the map $\mathbb{B}\mathbb{G}_m \times \mathbb{Z} \to \pi_0 \text{TPerf}_T$.

The rest of the statement follows from the short exact sequence

$$1 \to \mathbb{G}_m \to \pi_0 \text{Aut}_{\text{Perf}}(E) \to \pi_0 \text{Aut}_{\text{Deraz}}(\mathbb{R}\text{End} E) \to 1.$$  

II) In the general case, we can push forward from $\text{Supp} E = \text{Supp} \mathbb{R}\text{End} E = \text{Supp} \mathbb{R}\text{End} F = \text{Supp} F$.

**Application 5.27.** Let $X \xrightarrow{f} S$ be a proper and smooth morphism of algebraic spaces. In [Lie09], Lieblich compactifies the stack $f^*B\text{PGL}_n$ of families of principal $\text{PGL}_n$-bundles the following way. Using the version of the Skolem–Noether theorem

$$1 \to \mu_n \to \text{SL}_n \to \text{PGL}_n \to 1,$$

we get that the natural map $B\text{SL}_n /\!\!/\!\!\mu_n \to B\text{PGL}_n$ is an equivalence. Here, $B\text{SL}_n /\!\!/\!\!\mu_n$ is the rigidification, that is the target of the universal morphism $B\text{SL}_n \to B\text{SL}_n /\!\!/\!\!\mu_n$ which is invariant with respect to the $\mu_n$-action on $B\text{SL}_n$ given by scalar multiplication [ACV03] §5.1. Let $\mathcal{T}^\infty_{X/S}(n)$ denote the stack of totally supported sheaves with trivialized determinant and rank $n$ at every maximal point. Then one can show that the stack $f^*B\text{PGL}_n \to f^*(\mathcal{T}^\infty_{X/S}(n) /\!\!/\!\!\mu_n)$ is an open immersion [Lie09] Lemma 4.2.3].

To give another description of the objects classified by $f^*(\mathcal{T}^\infty_{X/S}(n) /\!\!/\!\!\mu_n)$, Lieblich introduces the notion of pre-generalized Azumaya algebras. These are perfect algebra objects $A$ of the derived category $D(X) = \text{Ho} \mathcal{D}(X)$ such that there exists a covering $U \to X$ and a totally supported perfect sheaf $F$ on $U$ such that $A|U \cong \mathbb{R}\text{End}(F)$. Then he considers the category fibred in groupoids $\mathcal{P}\mathcal{R}$ of pre-generalized algebras, where the isomorphisms are the weak algebra isomorphisms of $\mathcal{P}\mathcal{R}$. Since working in this truncated setting he can’t keep track of all the higher descent data, he needs to make the stack of generalized Azumaya algebras $\mathcal{A}$ the stackification of $\mathcal{P}\mathcal{R}$. Therefore, although he can show that the objects of $\mathcal{A}$ are the weak algebras of the form $\mathbb{R}\pi_\ast \mathbb{R}\text{End}(F)$ where $\mathcal{X} \xrightarrow{\pi} X$ is a $\mathbb{G}_m$-gerbe and $F$ is a totally supported perfect $\mathcal{X}$-twisted sheaf [Lie09] Proposition 5.2.1.12], he can only give a somewhat implicit description of the isomorphisms in $\mathcal{A}$ [Lie09].
Skolem–Noether theorem he proves \cite{Lie09} Theorem 5.1.5 can be viewed as the 1-truncation of our result. It implies that the natural map \( T^\infty_X(n) \xrightarrow{REnd} \mathcal{G}_X \) induces an equivalence \( T^\infty_X(n) \simeq \mathcal{G}_X \).

He shows that stackification is not needed in case \( X \xrightarrow{L} S \) is a smooth projective relative surface \cite{Lie09} Proposition 6.4.1. Our result implies that this holds in general.

**Proposition 5.28.** Let \( X \) be a quasi-compact and quasi-separated scheme. Then the category fibred in groupoids of pre-generalized Azumaya algebras \( \mathcal{PR}_X \) is a 1-stack.

**Proof.** Let \( \mathcal{PR}^\infty_X \subset \text{Alg Perf}_X \) denote the full substack of pre-generalized Azumaya algebras. We claim that \( \mathcal{PR}^\infty_X \) is a 1-stack. This will show that \( \mathcal{PR}^\infty_X \simeq \mathcal{PR}_X \) is a 1-stack.

Let \( A \) be a pre-generalized Azumaya algebra on \( X \). We need to show that \( B \text{Aut} A \subset \mathcal{PR}_X \) is 1-truncated. Letting \( X(A) \in B^2 \mathbb{G}_m(X) \) denote the gerbe of trivializations of \( A \), there exists an \( \mathcal{X} \)-twisted totally supported perfect sheaf \( F \) such that \( A \simeq R\text{End}(F) \) \cite{Lie09} Lemma 5.2.1.1]. Consider the sequence of canonical maps

\[
B \mathbb{G}_m \to B \text{Aut} F \xrightarrow{R\text{End}} B \text{Aut} A.
\]

Since being a fibration sequence of pointed \( \infty \)-stacks is local, and the twisted sheaf \( F \) is locally isomorphic to a sheaf, our result implies that this is a fibration sequence. Therefore, we have an exact sequence

\[
\pi_2 B \text{Aut} F \to \pi_2 B \text{Aut} A \to \pi_1 B \mathbb{G}_m \to \pi_1 B \text{Aut} F.
\]

As \( B \text{Aut} F \) is a 1-stack, we have \( \pi_2 B \text{Aut} F = 0 \). Moreover, the map \( \pi_1 B \mathbb{G}_m \to \pi_1 B \text{Aut} F \) is the scalar action \( \mathbb{G}_m \to \text{Aut} F \), which is injective. These two facts imply \( \pi_2 B \text{Aut} A = 0 \). For \( i > 2 \), we have an exact sequence

\[
0 = \pi_i B \text{Aut} F \to \pi_i B \text{Aut} A \to \pi_i B \mathbb{G}_m = 0,
\]

thus \( \pi_i B \text{Aut} A = 0 \). The claim is proven.

\[\Box\]

5.4. **Homotopical Algebraic Geometry.**

5.4.1. **Derived homotopical algebraic context.**

**Definition 5.29.** Let \( k \) be a commutative ring. Then the category \( \text{Mod}^\Delta_k \) of simplicial \( k \)-modules admits a symmetric monoidal model structure as follows \cite{GJ09} II, Example 6.2]:

1. Tensor product is defined levelwise: we have \( (A \otimes_k B)_n = A_n \otimes_k B_n \).
2. The forgetful functor \( \text{Mod}^\Delta_k \xrightarrow{U} \text{Set}_\Delta \) is a right Quillen adjoint where \( \text{Set}_\Delta \) is equipped with the Quillen model structure.

We shall call this the **Quillen model structure on simplicial \( k \)-modules.** We get an induced model structure on the category \( \text{CAlg}^\Delta_k \) of simplicial commutative \( k \)-algebras.

The **quasi-category** \( \text{CAlg}^\Delta_k \) of simplicial commutative \( k \)-algebras is the localization \( \text{CAlg}^\Delta_k[\text{weq}^{-1}] \) of the category of simplicial commutative \( k \)-algebras at the system of weak equivalences. The **quasi-category** \( \text{DAff}_k \) of derived affine schemes over \( k \) is the opposite quasi-category \( (\text{CAlg}^\Delta_k)^{op} \).
Let $A$ be a commutative dg $k$-algebra. Then the category $\text{Mod}^\text{dg}_A$ of dg $A$-modules admits a symmetric monoidal model structure as follows [BMR14, Theorem 3.3]:

1. Tensor product is induced by the tensor product on complexes.
2. A morphism of complexes is a weak equivalence if and only if it is a quasi-isomorphism.
3. A morphism of complexes is a fibration if and only if it is a degree-wise surjection.

We shall call this the \textit{projective model structure on dg $A$-modules}. A dg $k$-module $M \in \text{Mod}^\text{dg}_k$ is cofibrant if and only if it is \textit{dg-projective} [Hov02, Example 3.3], that is:

1. The $k$-modules $M_n$ are projective and
2. For all exact complexes $E \in \text{Mod}^\text{dg}_k$, the Hom complex $\text{Hom}(M,E)$ is also exact.

A bounded below complex of projective $k$-modules is dg-projective [Hov99, Lemma 2.3.6].

Let $M$ be a simplicial $k$-module. Then its Moore complex is the dg $k$-module

$$(CM)_n = M_n, \quad d = \sum_{i=0}^n (-1)^i d_i.$$ 

We denote by $DM \leq CM$ the subcomplex of degenerate simplices. The \textit{normalized Moore complex} is the dg $k$-module

$$NM = CM/DM.$$ 

Let $N$ be another simplicial $k$-module. Then the \textit{shuffle map}

$$CM \otimes_k CN \overset{\nabla}{\rightarrow} C(M \otimes_k N)$$

takes $m \otimes n \in CM_p \otimes_k NC_q$ to

$$\nabla(m \otimes n) = \sum_{(\mu,\nu)} \text{sign}(\mu, \nu)(s_{\nu} m) \otimes (s_{\mu} n)$$

where the summation is over $(p, q)$-\textit{shuffles}, that is permutations

$$(\mu, \nu) = (\mu_1 \ldots \mu_p, \nu_1, \ldots, \nu_q) \in \Sigma_{p+q}$$

where we have

$$\mu_1 < \cdots < \mu_p \quad \text{and} \quad \nu_1 < \cdots < \nu_q$$

and the associated degeneracy maps are

$$s_{\mu} = s_{\mu_p} \cdots s_{\mu_1} \quad \text{and} \quad s_{\nu} = s_{\nu_q} \cdots s_{\nu_1}.$$ 

Then the normalized Moore complex functor

$$\text{Mod}^\wedge_A \overset{N}{\rightarrow} \text{Mod}^\text{dg}_k$$

and the shuffle map give a lax monoidal right Quillen equivalence which is moreover lax symmetric monoidal [SS03 §4.2]. Let $A \in \text{CAlg}^\wedge_k$ be a commutative simplicial $k$-algebra. We denote by $\pi \bullet A$ the graded $k$-algebra induced by the commutative dg-algebra $NA$. 

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Construction 5.30. Let \( \text{Mod}_{k}^{dg} \xrightarrow{P} \text{Mod}_{k}^{dg} \) denote a dg-projective replacement functor. Consider the functor
\[
\text{DAff}^{\text{op}}_{k} \xrightarrow{A \mapsto (\text{Mod}_{\text{N}A}^{dg}, \text{qis})} \text{Mon}_{\text{Assoc WCat}}_{\infty}.
\]
By Construction 5.21 we get a functor \( \mathcal{P}(\text{DAff}_{k}) \xrightarrow{QC} \text{Cat}^{\text{Mon}}_{\infty} \).

5.4.2. Spectral homotopical algebraic context.

Definition 5.31. An \( \text{E}_{\infty} \)-ring is a commutative algebra object of the symmetric monoidal quasi-category \( \text{Sp} \) of spectra. The quasi-category \( \text{SAff} \) of affine spectral schemes is the opposite quasi-category \( (\text{CAlg Sp})^{\text{op}} \) of the quasi-category of \( \text{E}_{\infty} \)-rings. Let \( R \in \text{CAlg Sp} \) be an \( \text{E}_{\infty} \)-ring. Then we denote by \( \text{Spec} R \in \text{SAff} \) the object corresponding to \( R \). We denote by \( \text{SAff}_{R} \) the overcategory \( \text{SAff}/\text{Spec} R \).

Definition 5.32. An \( \text{E}_{1} \)-ring is an algebra object of the symmetric monoidal quasi-category \( \text{Sp} \) of spectra. Let \( R \in \text{Alg Sp} \) be an \( \text{E}_{1} \)-ring. For an integer \( n \in \mathbb{Z} \), let \( \pi_{n}R = \pi_{0} \text{Map}_{\text{Sp}}(S[n], R) \). These objects admit a natural abelian group structure by Theorem 5.3. Since the smash product on spectra commutes with colimits in each variable, we get equivalences \( S[n + m] \xrightarrow{\Delta} S[n] \otimes S[m] \). Let \( R \otimes R \xrightarrow{\mu} R \) be the multiplication map. We get maps
\[
\text{Map}_{\text{Sp}}(S[n], R) \times \text{Map}_{\text{Sp}}(S[m], R) \to \text{Map}_{\text{Sp}}(S[n] \otimes S[m], R \otimes R) \xrightarrow{\mu \circ \alpha} \text{Map}_{\text{Sp}}(S[n + m], R)
\]
endowing \( \pi_{\bullet}R = \bigoplus_{n} \pi_{n}R \) with a graded ring structure.

Proposition 5.33. [Lur16, Lemma 1.1.2.10] Let
\[
\begin{array}{ccc}
C & \xrightarrow{f} & 0 \\
\downarrow g & & \\
0' & \rightarrow & D
\end{array}
\]
be a diagram in a stable quasi-category \( \mathcal{C} \) representing an element \( \theta \in \text{Hom}_{\text{HoSp}}(C[1], D) \). Then the inverse \( -\theta \in \text{Hom}_{\text{HoSp}}(C[1], D) \) is represented by the transposed diagram
\[
\begin{array}{ccc}
C & \xrightarrow{g} & 0' \\
\downarrow f & & \\
0 & \rightarrow & D.
\end{array}
\]

Corollary 5.34. Let \( R \in \text{CAlg Sp} \) be an \( \text{E}_{\infty} \)-ring. Then the graded ring \( \pi_{\bullet}R \) is graded commutative.

Definition 5.35. Let \( X \in \text{Sp} \) be a spectrum. Then we say it is discrete if we have \( \pi_{n}X = 0 \) for all \( n \neq 0 \). The functor \( \text{Sp} \xrightarrow{\text{fin}} \text{Ab} \) restricts to an equivalence on the full subcategory \( \text{Disc Sp} \subseteq \text{Sp} \) of discrete spectra [Lur16, Proposition 1.4.3.6 (3)]. Since the map \( \mathcal{S}_{\text{t}}^{\text{fin}} \to \mathcal{S}_{\text{t}} \) with constant value the point is a zero object and the smash product \( \text{Sp} \otimes \text{Sp} \to \text{Sp} \) respects colimits in each variable, the equivalence \( \text{Disc Sp} \xrightarrow{\text{fin}} \text{Ab} \) is symmetric monoidal. In particular, we get a natural embedding
CAlg Ab → CAlg Sp of the category of discrete commutative rings into the quasi-category of E∞-rings.

Remark 5.36. Note that the unit object of CAlg Sp is the sphere spectrum S which is not discrete. In particular, this is not Z and thus SAff ∼= SAff₅ ∼= SAff₂.

Construction 5.37. Since the smash product on Sp gives a symmetric monoidal structure, for an E∞-ring R, relative tensor product equips the module category Mod⁰ R with a symmetric monoidal structure. Moreover the underlying quasi-category Mod⁰ R is stable [Lur16, Corollary 7.1.1.5] Therefore the Morita functor can be enhanced to give a functor CAlg Sp → MonCAlg PrSt [Lur16, Corollary 4.8.5.22]. This can be extended to a colimit-preserving functor ψ(SAff) QC −−→ (MonCAlg PrSt)op.

5.4.3. Homotopical Skolem–Noether theorem in homotopical algebraic geometry. In this subsubsection, scheme will mean either a derived or spectral scheme and stack will mean either a derived or spectral stack. Perfect stacks, introduced in [BFN10], constitute a broad class of stacks to which we can apply Theorem 4.20.

Definition 5.38. Let A f → B be a morphism of commutative simplicial or E∞ rings. Then we say that it is a flat morphism if the following conditions hold:

1. The morphism f induces an isomorphism of graded rings

\[ π₀B ⊗_{π₀A} π•A \rightarrow π•B \]

2. The morphism \( π₀(A) \xrightarrow{π₀f} π₀(B) \) is a flat morphism of rings.

A flat morphism is an étale morphism if moreover the morphism \( π₀(A) \rightarrow π₀(B) \) is an étale morphism of rings.

Let \( \{ A \xrightarrow{f} A_i : i \in I \} \) be a collection of morphisms of commutative simplicial or E∞ rings. Then we say that it is a flat covering if there exists a finite subset \( J \subseteq I \) such that the following conditions hold:

1. For each index \( j \in J \), the morphism \( f_j \) is a flat morphism.

2. The morphism

\[ π₀(A) \xrightarrow{⊕j π₀(f_j)} ⊕_k π₀(A_j) \]

is faithfully flat.

A flat covering is an étale covering if there exists a finite subset \( J \subseteq I \) such that moreover for each \( j \in J \) the morphism \( f_j \) is an étale morphisms.

These definitions define the fpqc resp. étale topology. Thus we can talk about fpqc resp. étale stacks. One can show that QC satisfies fpqc descent both in the derived [TV07 §3.1] and spectral [Lur11b, Proposition 2.7.14] context. We let S be an affine scheme and St₅ the quasi-category of étale stacks.

Corollary 5.39 (Homotopical Skolem–Noether theorem for derived and spectral algebraic geometry). Let S be an affine scheme. Let Cart₅fpqc denote the quasi-category of Cartesian fibrations on St₅ which satisfy fpqc descent.
(1) Let $\mathcal{P}(\text{Perf}^\gen_S) = \mathcal{P}(\text{QC}^\gen_S)_{\text{dgen}}$ denote the right fibration of perfect generator complexes on $S$, $\mathcal{P}(\text{Deraz}_S) := \mathcal{P}(\text{Az}QC^\gen_S)$ the Cartesian fibration of derived Azumaya algebras on $S$ and $D^\mathbf{Az}_S := \mathcal{P}(\text{LEns}^\mathbf{Az}_S)$ the Cartesian fibration of locally trivial presentable quasi-categories left-tensored over $QC^\gen_S$. Then the sequence in $(\text{Cart}^\text{fpqc}_S)_*$:

$$
\begin{array}{ccc}
(\mathcal{P}(\text{Perf}^\gen_S), \mathcal{O}) & \xrightarrow{\text{End}} & (\mathcal{P}(\text{Deraz}_S), \mathcal{O}) \\
& \xrightarrow{\text{Mod}} & (D^\mathbf{Az}_S, \mathcal{O})
\end{array}
$$

is a homotopy fibre sequence.

(2) Let $E \in \text{Perf}^\gen(S)$ be a perfect generator complex on $S$. Then the sequence in $(\text{Cart}^\text{fpqc}_S)_*$:

$$
\begin{array}{ccc}
(\mathcal{P}(\text{Pic} QC^\gen_S), \mathcal{O}) & \xrightarrow{\otimes_E} & (\mathcal{P}(\text{Perf}^\gen_S), E) \\
& \xrightarrow{\text{End}} & (\mathcal{P}(\text{Deraz}_S), \text{End} E)
\end{array}
$$

is a homotopy fibre sequence.

Proof. Let $T$ be a qcqs scheme. Then $QC(T)$ is compactly generated [BFN10, Proposition 3.19], thus dualizable by Proposition [5,18]. Therefore by Proposition 4.12 the family of monoidal quasi-categories $QC^\gen$ has fpqc descent, so we can apply Theorem 4.20.

□

Remark 5.40. In case we are in the derived or the connective spectral case, we have $D^\mathbf{Az}_S \simeq B^2 G_n \times B \mathbb{Z}$ [Toë12, Corollary 2.12], [AG14, Corollary 7.10].

Remark 5.41. For $E^\infty$-rings this result has previously appeared in [GL16, Propotision 5.15]. Moreover, in [GL16] Theorem 3.15 they show how the long exact sequence splits for algebraic Azumaya algebras, that is derived Azumaya algebras for which the associated graded algebras are also Azumaya.

5.5. Ind-coherent sheaves and crystals. In this subsection, we apply the Homotopical Skolem–Noether Theorem to the co-families of symmetric monoidal quasi-categories $\text{Ind Coh}$ and $\text{Crys}'$, which we introduce following [Cai13], [DG13], [GR14] and [GR17]. Let $k$ be a field of characteristic 0.

5.5.1. Finiteness conditions on prestacks.

Definition 5.42. The quasi-categories of $E^\infty$- and commutative dg $k$-algebras are equivalent [Lur16 Proposition 7.1.4.11]. We let $\text{CAlg}_k = \text{CAlg}_{\mathcal{O}}(k)$. Following this equivalence, we will refer to elements of $\text{CAlg}_k$ as commutative dg $k$-algebras and for $i \in \mathbb{Z}$ and $A \in \text{CAlg}_k$ we will write $H^iA = \pi_{-i}A$. We will say that a commutative dg $k$-algebra $A$ is connective if we have $H^iA = 0$ for $i > 0$. We let $\text{CAlg}_k^{\leq 0} \subseteq \text{CAlg}_k$ denote the full subcategory of connective commutative dg $k$-algebras. Then the quasi-category $\text{CAlg}^A_k$ of simplicial commutative $k$-algebras is equivalent to $\text{CAlg}^{\leq 0}_k$ [Lur18, Proposition 25.1.2.2]. The quasi-category of (derived) affine schemes is $\text{Aff} = \text{Aff}_k = (\text{CAlg}_k^{\leq 0})^{op}$. A connective commutative dg $k$-algebra $A$ corresponding to the affine scheme $\text{Spec} A \in \text{Aff}$. The quasi-category of prestacks is $\text{PreStk} = \text{PreStk}_k = \mathcal{P}(\text{Aff}_k)$.

Definition 5.43. Let $S = \text{Spec} A$ be an affine scheme and $n \in \mathbb{Z}_{\geq 0}$. Then we say that $S$ is $n$-coconnective if we have $H^iA = 0$ for $i < -n$. We denote by $\text{Aff}_{\leq n} \subseteq \text{Aff}$ the full subcategory on $n$-coconnective affine schemes. In particular, we say that $S$ is a classical affine scheme if it is 0-coconnective. We let
Aff_cl = Aff_{≤0}. We say that S is eventually coconnective if it is n-coconnective for some \( n ≥ 0 \). We let Aff_{<∞} = \( \cup_{n≥0} Aff_{≤n} \). We say that S is almost of finite type if the following assertions hold:

1. The commutative k-algebra \( H^nA \) is finitely generated.
2. For any \( i ∈ Z_{≥0} \), the \( H^nA \)-module \( H^nA \) is finitely generated.

We denote by Aff_{aff} ⊆ Aff the full subcategory on affine scheme almost of finite type. We say that S is of finite type if the following assertions hold:

1. The affine scheme S is eventually coconnective.
2. The affine scheme S is almost of finite type.

We let Aff_{ft} = Aff_{<∞} ∩ Aff_{aff}.

**Definition 5.44.** Take \( n ∈ Z_{≥0} \). Let \( ≤_n PreStk = \mathcal{P}(≤_n Aff) \). Let \( \mathcal{Y} \) be a prestack. Then we denote its restriction to \( ≤_n Aff^{op} \) by \( ≤_n \mathcal{Y} \). The left Kan extension functor \( ≤_n PreStk \xrightarrow{LKE} PreStk \) is a fully faithful left adjoint to the restriction functor. We say that \( \mathcal{Y} \) is n-coconnective if it is in the essential image of LKE. We let \( τ_{≤_n} \mathcal{Y} = LKE(≤_n \mathcal{Y}) \).

![Diagram](image)

In particular, we say that \( \mathcal{Y} \) is classical if it is 0-coconnective. We let \( cl PreStk = ≤_0 PreStk, cl \mathcal{Y} = ≤_0 \mathcal{Y} \) and \( τ_{cl} \mathcal{Y} = τ_{≤_0} \mathcal{Y} \). We say that \( \mathcal{Y} \) is eventually coconnective if it is n-coconnective for some \( n ≥ 0 \). We let \( PreStk_{<∞} = \cup_{n≥0} PreStk_{≤n} \). We say that \( \mathcal{Y} \) is convergent if it is the right Kan extension of its restriction to \( Aff_{<∞}^{op} \). We say that \( \mathcal{Y} \) is locally almost of finite type if it is the right Kan extension along the inclusion \( Aff_{<∞}^{op} ⊆ Aff^{op} \) of the left Kan extension along the inclusion \( Aff_{ft}^{op} ⊆ Aff_{<∞}^{op} \) of its restriction to \( Aff_{ft}^{op} \). We let \( PreStk_{aff} ⊆ PreStk \) denote the full subcategory on prestacks locally almost of finite type. We say that \( \mathcal{Y} \) is of finite type if it is the left Kan extension of its restriction to \( Aff_{ft}^{op} \). We let \( PreStk_{aff} ⊆ PreStk \) denote the full subcategory on prestacks locally of finite type.

5.5.2. **Open embeddings and proper morphisms between (derived) schemes.**

**Definition 5.45.** Let \( \mathcal{X} \xrightarrow{f} \mathcal{Y} \) be a morphism of prestacks. Then we say that f is affine schematic if for all affine schemes S and morphisms of prestacks \( S → \mathcal{Y} \) the fibre product \( \mathcal{X} ×_{\mathcal{Y}} S \) is representable by an affine scheme.

Suppose that f is affine schematic. We say that f is flat (resp. étale, Zariski, an open embedding) if for all \( S ∈ Aff_{/\mathcal{Y}} \) the morphism of affine schemes \( \mathcal{X} ×_{\mathcal{Y}} S → S \) is flat (resp. étale, Zariski, an open embedding). We say that f is a closed embedding if for all \( S ∈ Aff_{/\mathcal{Y}} \) the morphism of classical affine schemes \( cl(\mathcal{X} ×_{\mathcal{Y}} S) → clS \) is a closed embedding.

Let \( (\mathcal{X}_i \xrightarrow{f_i} \mathcal{Y} : i ∈ I) \) be a collection of affine schematic morphisms of prestacks. Then we say that \( (f_i)_{i∈I} \) is a covering if for all \( S ∈ Aff_{/\mathcal{Y}} \) the collection of morphisms of classical affine schemes \( cl(\mathcal{X}_i ×_{\mathcal{Y}} S) → clS \) is a covering.
Definition 5.46. Let \( \mathcal{Y} \) be a prestack. Then we say that it is a \textit{stack} if it satisfies étale descent. We denote by \( \text{Stk} \subset \text{PreStk} \) the full subcategory on stacks.

Definition 5.47. Let \( Z \) be a stack. Then we say that it is a \textit{(derived) scheme} if it satisfies the following assumptions:

1. The diagonal map \( Z \to Z \times Z \) is a closed embedding.
2. There exists a covering \( (S_i \to Z : i \in I) \) of open embeddings of affine schemes.

We refer to a collection \( (f_i : i \in I) \) such as in (2) as an \textit{affine atlas}. We denote by \( \text{Sch} \subset \text{Stk} \) the full subcategory on schemes.

Let \( Z \) be a scheme. We say that it is \textit{quasi-compact} if the classical scheme \( \text{cl}Z \) is quasi-compact.

Suppose that \( Z \) is quasi-compact. Then we say that it is \textit{almost of finite type} if it is locally almost of finite type. We denote by \( \text{Sch}_{aft} \subset \text{Sch} \) the full subcategory on schemes almost of finite type.

Let \( X \xrightarrow{f} Y \) be a morphism of schemes almost of finite type. We say that \( f \) is \textit{proper} if the morphism of classical schemes \( \text{cl}X \xrightarrow{\text{cl}f} \text{cl}Y \) is proper.

5.5.3. \textit{Ind-coherent sheaves.}

Definition 5.48. Let \( \mathcal{C} \) be a quasi-category. Then we say that \( \mathcal{C} \) is \textit{filtered} if for all finite simplicial sets \( K \) every map \( K \to \mathcal{C} \) has an extension along the inclusion \( K \to K^\circ \).

Let \( L \) be a simplicial set. Then we say that \( L \) is \textit{filtered} if there exists a filtered quasi-category \( \mathcal{C} \) and a categorical equivalence \( L \to \mathcal{C} \).

Let \( \mathcal{C} \) be a filtered quasi-category and \( \mathcal{C} \xrightarrow{F} \mathcal{D} \) a functor between quasi-categories. Then we say that \( F \) is \textit{continuous} if it commutes with filtered colimits. We denote by \( \text{Fun}_{\text{cont}}(\mathcal{C}, \mathcal{D}) \subset \text{Fun}(\mathcal{C}, \mathcal{D}) \) the full subcategory on continuous functors.

Proposition 5.49. \([\text{Lur}09, \text{Proposition 5.3.5.12}]\) Let \( \mathcal{C} \) be a small quasi-category. Let \( \text{Ind} \mathcal{C} \subset \mathcal{P}(\mathcal{C}) \) denote the full subcategory on presheaves \( \mathcal{C}^{\text{op}} \to \mathcal{I} \) which classify right fibrations \( \mathcal{C} \to \mathcal{C} \) such that the quasi-category \( \mathcal{C} \) is filtered. Then the Yoneda embedding \( \mathcal{C} \to \mathcal{P}(\mathcal{C}) \) factors through the inclusion \( \text{Ind} \mathcal{C} \to \mathcal{P}(\mathcal{C}) \). Let \( \mathcal{D} \) be a quasi-category such that it has filtered colimits. Then the precomposition with the Yoneda embedding map

\[
\text{Fun}_{\text{cont}}(\text{Ind} \mathcal{C}, \mathcal{D}) \to \text{Fun}(\mathcal{C}, \mathcal{D})
\]

is an equivalence.

Definition 5.50. Let \( \mathcal{C} \) be a quasi-category. We call \( \text{Ind} \mathcal{C} \) the \textit{ind-completion} of \( \mathcal{C} \).

Definition 5.51. Let \( X \) be a scheme almost of finite type. Then we denote by \( \text{Coh} X \subset \text{QC} X \) the full subcategory on complexes with bounded coherent cohomology. The inclusion map \( \text{Coh} X \to \text{QC} X \) induces a map \( \text{Ind} \text{Coh} X \to \text{QC} X \). The map \( \text{QC} X \otimes_{\text{QC} X} \text{QC} X \) induces a map \( \text{Ind} \text{Coh} X \otimes \text{Ind} \text{Coh} X \to \text{Ind} \text{Coh} \text{QC} X \times X \). Let \( X \xrightarrow{f} Y \) be a morphism of schemes almost of finite type. Then the direct image functor \( \text{QC} X \xrightarrow{f^*} \text{QC} Y \) induces a morphism \( \text{Ind} \text{Coh} X \xrightarrow{f^*} \text{Ind} \text{Coh} Y \).

Theorem 5.52. There exists a coCartesian family of presentable monoidal quasi-categories \( \text{Ind} \text{Coh}^{\otimes} \xrightarrow{p} \text{PreStk}^{\text{op}} \times \text{Assoc}^{\otimes} \) with the following properties:
(1) The family $p$ is classified by the right Kan extension of the map $\text{Sch}^{\text{op}}_{\text{aff}} \to \text{Pr}^{\text{Mon}}$ classifying its restriction.

(2) Let $X \xrightarrow{f} Y$ be a morphism of schemes almost of finite type. Then the Cartesian map $\text{Ind} \text{Coh} Y \xrightarrow{f^!} \text{Ind} \text{Coh} X$ is a composite $f_1^! f_2^*$ where

(a) $X \xrightarrow{\phi} \bar{X} \xrightarrow{f_2} Y$ is a decomposition of $f$ by an open embedding followed by a proper morphism.

(b) The map $f_1^!$ is a left adjoint of $(f_1)_!$.  

(c) The map $f_2^*$ is a right adjoint of $(f_2)_*$.

[GR17, II, Theorem 5.2.1.4]

(3) The underlying Cartesian fibration $\text{Ind} \text{Coh}^{\text{op}} \to \text{Sch}_{\text{aff}}$ has fppf descent [GR17, II, Corollary 5.3.3.7].

(4) The monoidal structure [GR17, II, Theorem 5.4.1.2] can be given as the composite

$$\otimes : \text{Ind} \text{Coh} X \otimes \text{Ind} \text{Coh} X \xrightarrow{\otimes} \text{Ind} \text{Coh}(X \times X) \xrightarrow{\Delta^!} \text{Ind} \text{Coh} X.$$ 

(5) Let $X$ be a scheme almost of finite type. Then the presentable quasi-category $\text{Ind} \text{Coh} X$ is self-dual [GR17, II, Theorem 5.4.2.5].

**Definition 5.53.** Let $\mathcal{Y} \xrightarrow{p} \text{Spec} k$ be a stack locally almost of finite type. The unit object of the symmetric monoidal quasi-category $\text{Ind} \text{Coh}^{\otimes}(\mathcal{Y})$ is the dualizing complex $\omega_{\mathcal{Y}} := p^!(k)$. The map $\text{Ind} \text{Coh}(\mathcal{Y}) \xrightarrow{\Psi_{\mathcal{Y}}} \text{QCoh}(\mathcal{Y})$ admits a symmetric monoidal left adjoint $\text{QCoh}(\mathcal{Y}) \xrightarrow{\Upsilon_{\mathcal{Y}}} \text{Ind} \text{Coh}(\mathcal{Y})$ [GR17, II, §6.3.3], which restricts to an equivalence on dualizable objects [GR17, II, Lemma 6.3.3.7]. Let $F = E \otimes \omega_{\mathcal{Y}} \in \text{Ind} \text{Coh}(\mathcal{Y})$ be a dualizable object. Then its support $\text{Supp}(F)$ is $\text{Supp}(E)$.

**Corollary 5.54** (Homotopical Skolem–Noether Theorem for $\text{Ind} \text{Coh}$). The following assertions hold:

(1) Then the following is a fibre sequence in $\text{Stk}_{\text{fl}}$:

$$\text{op} \text{TCoh} \xrightarrow{\text{End}} \text{Az Ind Coh} \xrightarrow{\text{Mod}} \text{op} \text{LTens}^{\text{Az}} \text{Ind Coh} \cong (B^2 \mathbb{G}_m \times \mathbb{B} \mathbb{Z}).$$

(2) Let $\mathcal{Y}$ be a stack locally of finite type and $E \in \text{TCoh}(\mathcal{Y})$ a totally supported complex with bounded coherent cohomology sheaves on $\mathcal{Y}$. Then the following is a fibre sequence in $(\text{Stk}_{\text{fl}})/\mathcal{Y}$:

$$((B \mathbb{G}_m) \times \mathbb{Z})_{\mathcal{Y}} \cong \text{op} \text{Pic Ind Coh}_{\mathcal{Y}} \xrightarrow{\text{End}} \text{op} \text{TCoh}_{\mathcal{Y}} \xrightarrow{\text{End}} \text{op} \text{Az Ind Coh}_{\mathcal{Y}}.$$

**Proof.** By Theorem 5.52, we can apply the abstract Homotopical Skolem–Noether Theorem, Theorem 4.20 to the co-family $\text{Ind} \text{Coh}^{\otimes}$. The rest of the Proof consists of identifying the output of the Homotopical Skolem–Noether Theorem.

Let $X \xrightarrow{p} \text{Spec} k$ be an affine scheme of finite type. Then by Remark 5.40, we have $(B \mathbb{G}_m \times \mathbb{Z})(X) \cong \text{Pic Qcoh}(X)$. The equivalence on dualizable objects $\text{Qcoh}(X)^d \xrightarrow{\text{End}} \text{Ind Coh}(X)^d$ restricts to an equivalence $\text{Pic Qcoh}(X) \cong \text{Pic Ind Coh}(X)$. Therefore every dualizable object of
Ind Coh°' is compact. By definition, a coherent complex $E \otimes \omega_X \in \text{Coh } X$ is totally supported if and only if the perfect complex $E \in \text{Perf } X$ is totally supported.

\[ \square \]

Remark 5.55. Let $\mathcal{Y}$ be a stack locally of finite type. Note that via the equivalence $\text{Perf}(\mathcal{Y})^\otimes \rightarrow \text{Coh}(\mathcal{Y})^\otimes$ the result of Corollary 5.34 is equivalent to Corollary 5.39 for Derived Algebraic Geometry. But it might offer a new look at derived Azumaya algebras, in particular in cases where $\omega_{\mathcal{Y}} \in \text{Ind Coh } (\mathcal{Y})$ is compact but $\mathcal{O}_{\mathcal{Y}} \in \text{QC } (\mathcal{Y})$ is not.

5.5.4. Crystals and D-modules.

Definition 5.56. Let $S$ be an affine scheme. The we say that it is a reduced scheme if it is a reduced classical scheme. We denote by $\text{red Aff} \subset \text{cl Aff}$ the full subcategory on reduced affine schemes and we let $\text{red PreStk} = \mathcal{P}(\text{red Aff})$.

The de Rham functor is the composite $dR : \text{PreStk} \rightarrow \text{red PreStk} \rightarrow \text{Pr}\text{Mon}$. Let $\mathcal{Y} \in \text{PreStk}$ be a prestack. Then the de Rham prestack $\mathcal{Y}_{dR}$ is the image $dR(\mathcal{Y})$.

Proposition 5.57. [GR14, Proposition 1.3.3] Let $\mathcal{Y} \in \text{PreStk}^{\text{left}}$ be a prestack locally almost of finite type. Then the de Rham prestack $\mathcal{Y}_{dR}$ is a classical prestack locally almost of finite type.

Definition 5.58. The left crystals functor is the composite $\text{Crys}^l : \text{PreStk} \rightarrow \text{Pr}\text{Mon}$.

The right crystals functor is the composite $\text{Crys}^r : \text{PreStk}^{\text{left}} \rightarrow \text{Pr}\text{Mon}$. They are related by a symmetric monoidal natural equivalence $(\text{Crys}^l | \text{PreStk}^{\text{left}}) \rightarrow \text{Crys}^r$ [GR17 II, §6.3.2], [GR14 Proposition 2.4.4].

Remark 5.59. Let $X$ be a classical scheme of finite type. Then there exist equivalences $\text{Crys}^l X \simeq \text{D-mod}^l X$ and $\text{Crys}^r X \simeq \text{D-mod}^r X$ [GR14 §5.5] where $\text{D-mod}^l X$ (resp. $\text{D-mod}^r X$) are the stable quasi-categories of left (resp. right) $D$-modules on $X$.

Corollary 5.60 (Homotopical Skolem–Noether Theorem for Crys). The following assertions hold:

1. Then the following is a fibre sequence in $\text{Stk}^{\text{left}}$:

   $\text{op}(\text{Crys}_{\text{Coh}}^{\text{r,gen}})^{\simeq} \text{End} (\text{Az Crys}^r)^{\simeq} \text{Mod} \text{op}\text{LTens}^\text{Az} (\text{Az Crys}^r)^{\simeq} \simeq (B^2 G_m \times B \mathbb{Z})_{dR}.$

2. Let $\mathcal{Y}$ be a stack locally of finite type and $E \in \text{Crys}_{\text{Coh}}^{\text{r,gen}}(\mathcal{Y})$ a generator complex with bounded coherent cohomology sheaves on $\mathcal{Y}$. Then the following is a fibre sequence in $(\text{Stk}^{\text{left}})_{/\mathcal{Y}_{dR}}$:

   $((B G_m \times \mathbb{Z})_{\mathcal{Y}_{dR}} \simeq \text{op Pic} \text{Crys}_{\mathcal{Y}}^{\text{r,gen}} \text{op}(\text{Crys}_{\text{Coh}}^{\text{r,gen}})^{\simeq} \text{End} (\text{Az Crys}^r)^{\simeq}.$

Proof. The underlying Cartesian fibration of $\text{Crys}^l$ satisfies fppf descent [GR14 Corollary 3.2.4]. For a scheme $X$ locally almost of finite type, the stable quasi-category $\text{Crys}^l(X)$ is compactly generated [GR14 Corollary 3.3.3], therefore it is dualizable [Lur18 Proposition D.7.2.3]. This shows by Proposition 4.12 that we can apply the abstract Homotopical Skolem–Noether Theorem. To finish, we need to identify its output.

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Let $X$ be a scheme of finite type. Then just as in Corollary\textsuperscript{5.54}, the dualizable objects in $\text{Crys}^l(X)$ coincide with the compact objects, which in turn coincide with the $D$-modules with coherent underlying complex \cite{DG13} \S5.1.17. Moreover the equivalence $\text{Crys}^l(X) \to \text{Crys}^r(X)$ restricts to the equivalence $((\mathbb{B} \mathbb{G}_m) \times \mathbb{Z})(X) \cong \text{Pic} \text{Crys}^l(X) \cong \text{Pic} \text{Crys}^r(X)$.

\[\Box\]

**Definition 5.61.** Let $\text{id}_{\text{PreStk}}^\text{u} : \text{PreStk} \to \text{dR}$ denote the unit of the adjunction $\text{PreStk} \rightleftarrows \text{red PreStk}$. Let $\mathcal{Y}$ be a prestack. Then a *twisting* on $\mathcal{Y}$ is a $\mathbb{G}_m$-gerbe $T$ on $\mathcal{Y}_{\text{dR}}$ equipped with a trivialization of the pullback $T|\mathcal{Y}$ along $u$.

Let $T$ be a twisting on $\mathcal{Y}$. Then via the symmetric monoidal structure on $\text{QC}$, the $\mathbb{G}_m$-gerbe $T$ acts on the quasi-category $\text{Crys}^l(\mathcal{Y})$ of left crystals. Therefore, we can form the quasi-category $\text{Crys}^{T/}(\mathcal{Y})$ of $T$-twisted left crystals on $\mathcal{Y}$.

**Corollary 5.62.** Let $\mathcal{Y}$ be a prestack and $T$ a twisting on $\mathcal{Y}$. Then the quasi-category $\text{dgen} \text{Crys}^{T/}(\mathcal{Y})$ of $T$-twisted left crystals on $\mathcal{Y}$ that are dualizable generators is equivalent to the quasi-category $\text{Deraz}^T(\mathcal{Y}_{\text{dR}})$ of derived Azumaya algebras on $\mathcal{Y}_{\text{dR}}$ with Brauer class $T$.

**Proof.** This is a formal consequence of the Homotopical Skolem–Noether Theorem for Derived Algebraic Geometry, Corollary\textsuperscript{5.39}.

\[\Box\]

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Email address: adhill3@uwo.ca

Western University, Canada

Email address: zsamboki@renyi.hu

Rényi Institute, Hungary